Statistical approach of modulational instability beyond Gaussian approximation

Anca Visinescu, Dan Grecu
Department of Theoretical Physics, Horia Hulubei National Institute of Physics and Nuclear Engineering, 30 Reactorului, P.O.B. MG-6, 077125 Magurele, Romania, EU
E-mail: avisin@theory.nipne.ro

Abstract. A statistical description of the modulation instability is discussed for the NLS equation. The kinetic equation for the two-point correlation function is obtained by using a decoupling procedure beyond the usual Gaussian approximation for the averaged values of products of four and six field variables. The induced evolution generated by a plane wave perturbation are calculated. They comprise both an evolution of the background and of the second harmonics.

1. Introduction

The nonlinear Schrödinger type equations are generic equations describing the evolution of (quasi) monochromatic waves in weak nonlinear media. A general feature of (quasi) monochromatic plane waves with an amplitude dependent dispersion relation (a Stokes wave) is their instability with respect to slow modulations of their amplitudes.

The modulation instability (MI) is one of the most frequent instability in nature. It was first discussed by Benjamin and Feir [1] in the context of hydrodynamics and by Bespalov and Talanov [2] in plasma physics. Is studied in electrodynamics and nonlinear optics also. In a very simple way the MI is the interaction of a strong carrier wave of frequency $\omega$ and small sidebands of frequencies $\omega_{1,2} = \omega \pm \Omega$ with the fulfillment of resonance condition on the corresponding wave vectors $k_1 + k_2 = 2k$ also. All these approaches can be termed as deterministic, as they generally study the evolution of small perturbations of the amplitude of the carrier wave [3], [4], [5], [6], [7]. The early history of the phenomenon was recently reviewed by Zakharov and Ostrovsky [3].

An alternative and complementary approach was based on the study of wave-wave energy transfer within a broad spectrum in a nearly homogeneous medium, especially in hydrodynamics [8], [9]. A bridge between the deterministic and the random school was elaborated by Alber [10]. In his approach the field variable is satisfying a deterministic nonlinear evolution equation (Davey-Stewardson equation in this case), but the attention is focused on a statistical quantity, the two-point correlation function, and no on the solutions of the evolution equation. Writing an evolution equation for the two-point correlation function, a linear stability analysis was investigated, based on the Wigner-Moyal transform [11], and the instability regions were determined for various backgrounds of the unperturbed initial state. A similar approach was successfully applied on the case of incoherent light propagation in nonlinear media [12],...
[13], theory of surface gravity waves in deep oceans [14], wave propagation in non-stationary inhomogeneous plasma [15], dynamics of charged beams in accelerators [16], dynamics of Bose-Einstein condensate [17]. Recently in a series of papers the SAMI was studied for several NLS type equations (discrete nonlinear lattices, derivative NLS equation, Manakov’s system, cylindrical/spherical NLS equation) [18]-[20].

In the present paper starting from the NLS equation the kinetic equation for the two-point correlation functions was obtained using a decoupling procedure beyond the usual Gaussian decoupling for averaged values of products of four and six field variable. The induced evolutions generated by a plane wave perturbation are calculated. The induced second harmonic component of Wigner’s function and of the second harmonic component of the density were written down.

2. SAMI for NLS. Basic equation.

We consider the NLS equation
\[ i\partial_t \Psi + \frac{1}{2} \partial_x^2 \Psi + \alpha |\Psi|^2 \Psi = 0 \] (1)
and the following 2-point correlation function
\[ W(1,2) = \langle \Psi(x_1,t)\Psi^*(x_2,t) \rangle = \langle \Psi_1 \Psi_2^* \rangle . \]
It has symmetry properties \( W(2,1) = (W1,2)^* \). A kinetic equation for the time evolution of \( W(1,2) \) can be derived in the usual way [10]
\[ i\frac{\partial W(1,2)}{\partial t} + \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) W(1,2) + 2\alpha(n(1) - n(2))W(1,2) + \alpha P(1,2) = 0, \] (2)
where \( P(1,2) \) is a higher order 2-point correlation function, namely
\[ P(1,2) = \langle (|\Psi_1|^2 - |\Psi_2|^2)\Psi_1 \Psi_2^* \rangle = -2(n(1) - n(2))W(1,2). \]
In the case of a Gaussian decoupling of higher order products of four \( \Psi \) variables one has
\[ \langle |\Psi(1)|^2\Psi(1)\Psi^*(2) \rangle \simeq 2n(1)W(1,2) \]
and consequently \( P(1,2) \equiv 0 \).

Following the Wigner-Moyal transform [11] we introduce the center of mass coordinate \( X = \frac{1}{2}(x_1 + x_2) \) and the relative coordinate \( x = x_1 - x_2 \) and perform a Fourier transform of \( W(1,2) \) with respect to the relative coordinate
\[ \rho(X,k,t) = \frac{1}{2\pi} \int e^{-ikx} < \Psi(X + \frac{x}{2}) \Psi(X - \frac{x}{2}) > dx; \]
\( \rho(X,k,t) \) is usually called the Wigner function. It is easily seen to be a real function.

Denoting \( n(X,t) = \langle |\Psi(X,t)|^2 \rangle, \) (\( n(X,t) \) is the pulse intensity in optics and the fluid density in hydrodynamics), it is easily seen that
\[ n(X,t) = \int \rho(X,k,t) dk. \]
Then we can perform the Fourier transform of the exact kinetic equation (2). Taking into account that the Fourier transform of \( P(1,2) \) is a purely imaginary quantity \( i\Pi(X,k,t) \), with \( \Pi(X,k,t) \) a real one, we get
\[ \frac{\partial \rho(X,k,t)}{\partial t} + k \frac{\partial \rho}{\partial X} + 4\alpha n(X,t) \sin \left( \frac{1}{2} \frac{\partial}{\partial X} \frac{\partial}{\partial k} \right) \rho(X,k,t) + \alpha \Pi(X,k,t) = 0, \] (3)
Here the \( \sin(...) \) operator is defined by its Taylor expansion and the arrows are indicating the direction in which the derivatives are acting.

The next step is to write a kinetic (approximate) equation for \( P \). We have

\[
i \frac{\partial P(1, 2)}{\partial t} = \left( i \frac{\partial |\Psi_1|^2}{\partial t} - i \frac{\partial |\Psi_2|^2}{\partial t} \right) \Psi_1 \Psi_2^* + < (|\Psi_1^2| - |\Psi_2|^2) i \frac{\partial (\Psi_1 \Psi_2^*)}{\partial t} > - 2 \left( i \frac{\partial n(1)}{\partial t} - i \frac{\partial n(2)}{\partial t} \right) W(1, 2) - 2(n(1) - n(2)) i \frac{\partial W(1, 2)}{\partial t}.
\]

We transform this using
- the conservation of number density

\[
i \frac{\partial |\Psi|^2}{\partial t} = \frac{1}{2} \left( \frac{\partial^2 \Psi^\ast}{\partial x^2} - \Psi \frac{\partial^2 \Psi}{\partial x^2} \right) = \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial \Psi^\ast}{\partial x} - \Psi \frac{\partial \Psi}{\partial x} \right),
\]
- the equation

\[
i \frac{\partial (\Psi_1 \Psi_2^*)}{\partial t} = - \frac{1}{2} \left( \frac{\partial^2 \Psi_1}{\partial x_1^2} - \frac{\partial^2 \Psi_2}{\partial x_2^2} \right) \Psi_1 \Psi_2^* - \alpha (|\Psi_1|^2 - |\Psi_2|^2) \Psi_1 \Psi_2^*;
\]
- and the equations for \( \frac{\partial n}{\partial t}, \frac{\partial W}{\partial t} \).

At this stage it is necessary to adopt a certain decoupling procedure for the higher correlation functions appearing in the resulting equation. We consider

\[
< \frac{\partial^2 \Psi_1}{\partial x_1^2} \frac{\partial \Psi_1^\ast}{\partial x_2} > \simeq 2 < \frac{\partial \Psi_1^\ast}{\partial x_2} \frac{\partial^2 \Psi_1}{\partial x_1^2} > W(1, 2),
\]

\[
< \frac{\partial^2 \Psi_1}{\partial x_1^2} \frac{\partial \Psi_1^\ast}{\partial x_1} > \simeq < \frac{\partial \Psi_1^\ast}{\partial x_1} \frac{\partial^2 \Psi_1}{\partial x_1^2} > W(1, 2) + n(1) \frac{\partial^2}{\partial x_1^2} W(1, 2),
\]

and for the products of six \( \Psi \) functions the simple decoupling

\[
< (|\Psi_1|^2 - |\Psi_2|^2) \Psi_1 \Psi_2^* > \simeq 2(n(1) - n(2)) [P(1, 2) + 2(n(1) - n(2)) W(1, 2)] + (n(1) - n(2))^2 W(1, 2).
\]

We remark that this decoupling of the product of six \( \Psi \)-variables is a very simple one, and better decouplings are possible; the following results are mainly dependent on this simple decoupling procedure.

Then the kinetic equation for \( P(1, 2) \) takes a very simple form

\[
i \frac{\partial P(1, 2)}{\partial t} + \alpha (n(1) - n(2))^2 W(1, 2) = 0.
\]

and its Fourier transform is

\[
\frac{\partial \Pi}{\partial t} + 4 \alpha n(X, t) \sin \left( \frac{1}{2} \frac{\partial}{\partial X} \frac{\partial}{\partial k} \right) \rho(X, k, t) \sin \left( \frac{1}{2} \frac{\partial}{\partial k} \frac{\partial}{\partial X} \right) n(X, t) = 0.
\]
The quantity \( \Pi(X, k, t) \) can now be eliminated from the equation (3) satisfied by \( \rho(X, k, t) \). We finally remain with

\[
\frac{\partial}{\partial t} \left[ \frac{\partial \rho}{\partial t} + k \frac{\partial \rho}{\partial X} + 4\alpha n(X, t) \sin \left( \frac{1}{2} \frac{\partial}{\partial X} \frac{\partial}{\partial k} \right) \rho \right] = 4\alpha n(X, t) \sin \left( \frac{1}{2} \frac{\partial}{\partial X} \frac{\partial}{\partial k} \right) \rho \sin \left( \frac{1}{2} \frac{\partial}{\partial k} \frac{\partial}{\partial X} \right) n(X, t).
\]

(5)

3. Linear stability analysis

The initial (unperturbed) state is assumed to be stationary, homogeneous and isotropic. Consequently the two-point correlation function in the initial state will be a function of \( |x|, W_0(|x|) \), and its Fourier transform \( \rho_0(k) \) is an even function of \( k \).

A linear stability analysis can be done assuming that the perturbed state \( \rho(X, k, t) \) can be written as

\[
\rho(X, k, t) = \rho_0(k) + \epsilon \rho_1(X, k, t) + \epsilon^2 \rho_2(X, k, t)
\]

\[
n(X, t) = n_0 + \epsilon n_1(X, t) + \epsilon^2 n_2(X, t)
\]

(6)

where

\[
n_0 = \int_{-\infty}^{+\infty} \rho_0(k) dk, \quad n_1(X, t) = \int_{-\infty}^{+\infty} \rho_1(X, k, t) dk, \ldots
\]

Here \( \epsilon \ll 1 \) and moreover the present discussion is valid for small time values where \( \epsilon \epsilon^\Omega_t \ll 1 \), \( \Omega_t \) being the instability increment. In the first order in \( \epsilon \) we have

\[
\frac{\partial \rho_1}{\partial t} + k \frac{\partial \rho_1}{\partial X} + 4\alpha n_1(X, t) \sin \left( \frac{1}{2} \frac{\partial}{\partial X} \frac{\partial}{\partial k} \right) \rho_0(k) = 0,
\]

and looking for plane wave solutions

\[
\rho_1(X, k, t) = g_1(k)e^{i(QX - \Omega t)} + g_1^*(k)e^{-i(QX - \Omega^* t)}
\]

\[
n_1(X, t) = N_1 \left( e^{i(QX - \Omega t)} + e^{-i(QX - \Omega^* t)} \right),
\]

\[
N_1 = \int g_1(k) dk,
\]

one obtains the known dispersion relation of the Gaussian decoupling case [19]

\[
1 + 2\alpha \frac{Q}{Q^*} \int \frac{\rho_0(k + \frac{Q}{2}) - \rho_0(k - \frac{Q}{2})}{k - \frac{Q}{2}} dk = 0.
\]

(7)

It can be solved for different initial distributions \( \rho_0(k) \). We consider both the case of a \( \delta \)-distribution \( \rho_0(k) = n_0 \delta(k) \), and a Lorentzian one, \( \rho_0(k) = \frac{n_0}{\pi k^2 + p^2} \), where \( n_0 \) is the mean density of the unperturbed state. One obtains \( \Omega = i\Omega_t \) (purely imaginary) \( \Omega_t = Q\sqrt{2\alpha n_0 - \frac{Q^2}{4}} \) for the \( \delta \)-distribution, and \( \Omega_t = \sqrt{2\alpha n_0 - \frac{Q^2}{4}} - p \) for the Lorentzian distribution. The instability is associated with the positive imaginary part of \( \Omega \), and for a \( \delta \)-distribution the instability domain is restricted to the long wave length region

\[
0 \leq Q \leq 2\sqrt{2\alpha n_0}
\]
\( \Omega_i = 0 \) for \( Q = 0 \) and \( Q = 2\sqrt{2\alpha n_0} \) and has a maximum \( 2\alpha n_0 \) at \( Q_0 = 2\sqrt{\alpha n_0} \).

For the Lorentzian distribution the instability domain is more restricted

\[ 0 \leq Q \leq 2\sqrt{2\alpha n_0 - p^2} \]

and disappears for \( p^2 \geq 2\alpha n_0 \).

In the second order in \( \epsilon \) we have

\[
\frac{\partial}{\partial t} \left[ \frac{\partial \rho_2}{\partial t} + k \frac{\partial \rho_2}{\partial X} + 4\alpha n_2(X, t) \sin \left( \frac{1}{2} \frac{\partial}{\partial X} \frac{\partial}{\partial k} \right) \rho_0(k) \right] = \\
= \frac{\partial}{\partial t} \left[ 4\alpha n_1(X, t) \sin \left( \frac{1}{2} \frac{\partial}{\partial X} \frac{\partial}{\partial k} \right) \rho_1(X, k, t) \right] + \frac{4\alpha^2 n_1(X, t) \sin \left( \frac{1}{2} \frac{\partial}{\partial X} \frac{\partial}{\partial k} \right) \rho_0(k) \sin \left( \frac{1}{2} \frac{\partial}{\partial X} \frac{\partial}{\partial k} \right) n_1(X, t),
\]

which is a driven non-homogeneous linear partial differential equation for \( \rho_2(X, k, t) \). With the r.h.s. depending only on \( \rho_1(X, k, t) \) and \( n_1(X, t) \).

We evaluate the r.h.s. assuming \( \Omega = \Omega_i \), \( N_1 \) real, and

\[
\rho_1(X, k, t) = e^{\Omega_i t} \left( g_1(k)e^{iQX} + g_1^*(k)e^{-iQX} \right) \\
n_1(X, t) = e^{\Omega_i t} N_1 \left( e^{iQX} + e^{-iQX} \right) \\
g_1 = 2\alpha N_1 \frac{\rho_0(k + \frac{Q}{2}) - \rho_0(k - \frac{Q}{2})}{\Omega - kQ}.
\]

Then the r.h.s. of the non-homogeneous linear partial differential equation (8) contains
- a time dependent, spatial uniform term
  \[ A_0(k)e^{2\Omega_i t} = 2 \left( G_0(k) - \Omega_i Im G_1(k) \right) e^{2\Omega_i t} \]
- and a time and space dependent term
  \[ \left( A_2(k)e^{2iQX} + A_2^*e^{-2iQX} \right) e^{2\Omega_i t} \]
  \[ A_2(k) = - \left( G_0(k) + i\Omega_i G_1(k) \right). \]

Here we denoted

\[
G_0(k) = \alpha^2 N_1^2 \left[ \rho_0(k + Q) - 2\rho_0(k) + \rho_0(k - Q) \right] \\
G_1(k) = 4\alpha N_1 \left[ g_1(k + \frac{Q}{2}) - g_1(k - \frac{Q}{2}) \right].
\]

This allow us to seek \( \rho_2(X, k, t) \) of the form

\[
\rho_2(X, k, t) = \rho^{(0)}_2(k, t) + \left( \rho^{(2)}_2(k, t)e^{2iQX} + c.c. \right)
\]

where \( \rho^{(0)}_2(k, t) \) - real, \( \rho^{(2)}_2(k, t) \)-complex and

\[
n_2(X, t) = n^{(0)}_2(t) + \left( n^{(0)}_2(t)e^{2iQX} + c.c. \right)
\]
\( n_2^{(0)}(t) \) - modulation of the uniform background, \( n_2^{(2)}(t) \) - amplitude of the generated second harmonic, \( n_2^{(0)}(t) \) - real, \( n_2^{(2)}(t) \) - eventually complex.

\[
n_2^{(0)}(t) = \int \rho_2^{(0)}(k,t) dk, \quad n_2^{(2)}(t) = \int \rho_2^{(2)}(k,t) dk.
\]

\( \rho_2^{(0)}(k,t) \) and \( \rho_2^{(2)}(k,t) \) will satisfy the equations

\[
\frac{\partial^2 \rho_2^{(0)}(k,t)}{\partial t^2} = A_0(k)e^{2\Omega_i t},
\]

\[
\frac{\partial}{\partial t} \left\{ \frac{\partial \rho_2^{(2)}(k,t)}{\partial t} + 2iQk\rho_2^{(2)} + 2i\alpha n_2^{(2)} \left( \rho_0(k + \frac{Q}{2}) - \rho_0(k - \frac{Q}{2}) \right) \right\} = A_2(k)e^{2\Omega_i t}.
\]

These have to be solved with the initial conditions

\[
\rho_2^{(0)}(t = 0) = \rho_2^{(2)}(t = 0) = 0
\]

and assuming that in the limit \( \Omega_i \to 0 \) the solutions are non-singular.

We consider

\[
\rho_2^{(0)}(k,t) = a_0(k) + b_0(k)t + \frac{A_0(k)}{4\Omega_i^2} e^{2\Omega_i t},
\]

\[
a_0(k) = -\frac{A_0(k)}{4\Omega_i^2}, \quad b_0(k) = -\frac{1}{\Omega_i} G_0(k).
\]

Then

\[
\rho_2^{(0)}(k,t) = G_0(k) e^{2\Omega_i t} - \frac{(1 + 2\Omega_i t)}{2\Omega_i^2} - Im G_1(k) e^{2\Omega_i t} - \frac{1}{2\Omega_i}, \quad (10)
\]

\[
n_2^{(0)}(t) = \frac{e^{2\Omega_i t}}{2\Omega_i^2} \int G_0(k) dk - \frac{e^{2\Omega_i t} - 1}{2\Omega_i} \int G_1(k) dk
\]

and both conditions mentioned above are satisfied.

For a \( \delta \)-distribution

\[
Im g_1(k) = -2\alpha N_1 \frac{\Omega_i}{Q^2} \frac{\delta(k + \frac{Q}{2}) - \delta(k - \frac{Q}{2})}{k^2 + \Omega_i^2}
\]

\[
\int G_0(k) dk = 0, \quad \int Im G_1(k) dk = 0.
\]

and consequently

\[
n_2^{(0)} = 0.
\]

For reasons to be seen later we consider

\[
\rho_2^{(2)}(k,t) = f(t)G_0(k) + g_2^{(2)}(e^{2\Omega_i t} - 1)
\]

\[
f(0) = 0.
\]

Then because \( \int G_0(k) dk = 0 \) for any distribution of \( \rho_0(k) \) we have

\[
n_2^{(2)}(t) = N_2(e^{2\Omega_i t} - 1)
\]
\[ N_2 = \int g_2^{(2)}(k) dk. \]

Introducing into the equation satisfied by \( \rho_2^{(2)}(k, t) \) after several manipulations we get

\[ \left( \frac{d^2 f}{dt^2} - 2\Omega_i \frac{df}{dt} + e^{2\Omega_i t} \right) \left( \frac{1}{2\Omega_i} \int \frac{G_0(k)}{\Omega_i + iQk} dk + \left( N_2 + \frac{i}{2} \int \frac{G_1(k)}{\Omega_i + iQk} dk \right) e^{2\Omega_i t} \right) = 0. \]

For a \( \delta \)-distribution

\[ i\alpha \int \frac{\delta(k + \frac{Q}{2}) - \delta(k - \frac{Q}{2})}{\Omega_i + iQk} dk = -\frac{1}{2}, \]

\[-\frac{i}{2} \int \frac{G_1(k)}{\Omega_i + iQk} dk = \frac{6\alpha^2 N_1^2 Q^2}{\Omega_i^2 + Q^4}, \]

but

\[-\frac{1}{2\Omega_i} \int \frac{G_0(k)}{\Omega_i + iQk} dk = \alpha^2 N_1^2 n_0 \frac{Q^4}{\Omega_i^2 + Q^4}, \]

is singular when \( \Omega_i \to 0 \). To eliminate this singularity we determine \( f(t) \) from

\[ \frac{d^2 f}{dt^2} - 2\Omega_i \frac{df}{dt} + e^{2\Omega_i t} = 0. \]

One obtains \( \Omega_i < 1 \)

\[ f(t) = 2e^{2\Omega_i t} \sin \sqrt{1 - \Omega_i^2} t. \]

Finally we get

\[ \rho_2^{(2)}(k, t) = 2e^{2\Omega_i t} \sin \sqrt{1 - \Omega_i^2} t + g_2^{(2)}(k) \left( e^{2\Omega_i t} - 1 \right), \]

for the induced second harmonic component of Wigner’s function and

\[ N_2 = \frac{6\alpha^2 N_1^2 Q^2}{\Omega_i^2 + Q^4}, \]

for the induced second harmonic component of the density.

4. Conclusions

The main results of the present paper can be summarized as follows:

1) A statistical approach of the modulation instability was realized, going beyond the usual Gaussian approximation. The exact kinetic equation (2) for the two-point correlation function \( W(1, 2) \) was supplemented with an approximate kinetic equation (4) for the higher order correlation function \( P(1, 2) \). This was obtained using the usual Gaussian approximation for the decoupling of products of four \( \Psi \)-functions, but a special decoupling procedure for products of six \( \Psi \)-functions. We expect that an improved decoupling procedure will lead to better results. Finally an improved kinetic equation (5) for the Fourier transform \( \rho(X, k, t) \) is obtained.

2) A linear stability analysis of eq. (5) around a stationary initial state is performed. In first order the dispersion relation of the usual Gaussian approximation is obtained and this was not changed in higher orders. The second order represents a driven non-homogeneous partial differential equation for \( \rho_2(X, k, t) \) (eq. (8)), and the solution (9) is written as a superposition of a modulated \( \rho_2^{(0)}(k, t) \) of the initial Wigner function \( \rho_0(k) \) and a new generated second harmonic \( \left( \rho_2^{(2)}(k, t)e^{2iQX} + cc \right) \). These were determined for a zero initial condition and assuming that in the limit \( \Omega_i \to 0 \) the solutions are non-singular.

3) We emphasize again that these results were obtained using a simple decoupling of the products of six \( \Psi \)-functions, and probably will be essentially changed for a better decoupling. This fact is for the moment under investigation.
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