A direct imaging method for the exterior and interior inverse scattering problems

Deyue Zhang, Yue Wu, Yinglin Wang and Yukun Guo

Abstract

This paper is concerned with the inverse acoustic scattering problems by an obstacle or a cavity with a sound-soft or a sound-hard boundary. A direct imaging method relying on the boundary conditions is proposed for reconstructing the shape of the obstacle or cavity. First, the scattered fields are approximated by the Fourier-Bessel functions with the measurements on a closed curve. Then, the indicator functions are established by the superpositions of the total fields or their derivatives to the incident point sources. We prove that the indicator functions vanish only on the boundary of the obstacle or cavity. Numerical examples are also included to demonstrate the effectiveness of the method.

Keywords: direct imaging, inverse obstacle scattering, inverse cavity scattering, Fourier-Bessel expansion

1 Introduction

The inverse scattering problems are of significant importance in diverse applications such as radar sensing, sonar detection and biomedical imaging (see, e.g. [8]). In archetypal inverse scattering problems, the target objects are illuminated by an incident wave coming from their exterior and the corresponding scattering data is measured from the outside as well. Determining the unknown scatterer from such externally accessed information constitutes the exterior inverse scattering problems. In the last three decades, a huge number of computational attempts have been made to solve the exterior inverse scattering problems of identifying impenetrable obstacles or penetrable medium. Typical numerical strategies developed for the exterior inverse scattering problems include the decomposition methods, iteration schemes, recursive linearization based algorithms and the sampling approaches (see, e.g. [1, 4, 7, 8]). We also refer to [3, 5, 9] for some recent studies on the unique recovery issues in inverse scattering theory. In addition, inverse scattering problems without the phase information receive great interests recently. Some uniqueness results and numerical methods on the exterior inverse scattering problems with phaseless data can be found in [2, 6, 10, 17, 18, 20, 30, 33, 34].

As the interior counterparts of the aforementioned exterior problems, the interior inverse scattering problems for recovering the shape of cavities rely on the signals due to interior emitters and
sensors, which arise in many practical areas of non-destructive testing and reservoir exploration [15, 26]. In contrast to the exterior inverse scattering problems, the interior inverse scattering problems are usually more complicated to tackle because of the repeated reflections of the inescapably trapped scattering waves. For the numerical reconstruction algorithms concerning interior inverse scattering problems, we refer to [12, 23, 26, 27, 29, 39, 40] for the linear sampling method, the regularized Newton iterative method, the decomposition method, the factorization method and the reciprocity gap functional method. There have also been some mathematical study on the inverse cavity problems. We refer to [12, 23, 26, 27, 28, 40] for some uniqueness results with full data (both the intensity and phase). A recent result on uniqueness of the inverse cavity scattering with phaseless data was established in [38] by utilizing the reference ball technique in conjunction with the superposition of incident point sources.

In this paper, we consider the incident point sources and deal with the reconstruction of the shape of a sound-soft (or sound-hard) obstacle (or cavity). We propose a novel imaging scheme to determine the shape of unknown scatterer with known boundary conditions. Motivated by the Fourier-Bessel method for solving the Cauchy problems for the Helmholtz equation [22, 36, 37], we first make the approximation of the scattered fields by the expansion of Fourier-Bessel functions from the measurements on a closed curve. Then, by utilizing the a priori boundary conditions of the obstacle or the cavity, we introduce the associated indicator functions with the superposition of the approximated total fields or their derivatives with respect to the incident point sources. It is proved that the indicator functions vanish only on the boundary of the obstacle or cavity. Therefore, in the last step, profile of the underlying obstacle/cavity can be recognized as the zeros of the indicator functions depicted over a suitably chosen imaging domain. Mathematical analysis of the stability is presented to justify the theoretical foundations of the method. Numerical examples are also included to demonstrate the effectiveness of the method.

In our opinion, the interesting novelties of the proposed method can be highlighted as follows. First, the proposed method can be viewed as a direct imaging method since the forward solver or the iteration process is not needed. Hence it is convenient to implement without time-consuming computations. Second, the unified framework of inversion scheme can be well applied to both the exterior and interior problems with sound-soft or sound-hard boundary conditions, whereas the most existing methods for exterior problems produce drastically low-quality reconstructions for imaging the cavity. Third, rigorous mathematical justifications are provided to characterize the approximation properties with respect to the noisy data, which essentially guarantee the robustness of the algorithm. Finally, to our best knowledge, this is the first attempt in the literature towards combining the idea of direct imaging and the technique of Fourier-Bessel expansion for solving inverse scattering problems.

Throughout this paper, we assume that $D \subset \mathbb{R}^2$ is an open and simply connected domain with $C^2$ boundary $\partial D$. Let $k > 0$ be the wave number and $H_n^{(1)}$ be the Hankel function of the first kind of order $n$. For a generic point $z \in \mathbb{R}^2$, the incident field $u^i$ due to the point source located at $z$ is given by the fundamental solution to the Helmholtz equation

$$u^i(x; z) := \frac{i}{4} H_0^{(1)}(k|x - z|)$$

where

$$\begin{cases} z \in \mathbb{R}^2 \setminus \overline{D}, \ x \in \mathbb{R}^2 \setminus \overline{D} \cup \{z\}, & \text{for exterior problem,} \\
\ & \\
\ z \in D, \ x \in D \setminus \{z\}, & \text{for interior problem.} \end{cases}$$
The total field \( u \) is the superposition of the incident field \( u^i \) and the scattered field \( u^s \), namely, \( u = u^i + u^s \). We shall also employ \( u(x; z) = u^i(x; z) + u^s(x; z) \) to indicate the dependence of these wave fields on the point source location \( z \) and the wave number \( k \). To characterize the physical properties of distinct scatterers, the boundary operator \( \mathcal{B} \) is introduced by

\[
\mathcal{B}u = \begin{cases} 
  u, & \text{for a sound-soft obstacle/cavity,} \\
  \frac{\partial u}{\partial \nu}, & \text{for a sound-hard obstacle/cavity,}
\end{cases}
\]

where \( \nu \) is the unit outward normal to \( \partial D \).

The rest of this paper is arranged as follows. In the next section, we present the imaging method for the inverse obstacle scattering problem, including the model problem, uniqueness, indicator function as well as the Fourier-Bessel approximation. Then the interior counterparts for the inverse cavity scattering problem are discussed in Section 3. Next, numerical validations and discussions of the proposed method are illustrated in Section 4. Finally, some concluding remarks are given in Section 5.

## 2 Inverse obstacle scattering problem

We begin this section with the mathematical formulations of the model exterior scattering problem. The obstacle scattering problem can be formulated as: find the scattered field \( u^s \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \overline{D}) \) satisfying the following boundary value problem:

\[
\begin{align*}
\Delta u^s + k^2 u^s &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \\
\mathcal{B}(u^i + u^s) &= 0 \quad \text{on } \partial D, \\
\frac{\partial u^s}{\partial r} -iku^s &= o\left(r^{-1/2}\right), \quad r = |x| \to \infty,
\end{align*}
\]

where the Sommerfeld radiation condition \( (4) \) holds uniformly in all directions \( x/|x| \). The existence of a solution to the direct scattering problem \( (2) \) to \( (4) \) is well known (see, e.g., [8]).

With these preparations, the inverse obstacle scattering problem can be stated as the following.

**Problem 2.1** (Inverse obstacle scattering). Let \( D \subset B_\rho = \{ x \in \mathbb{R}^2 : |x| < \rho \} \) be the impenetrable obstacle with boundary condition \( \mathcal{B} \) and \( \Gamma \subset \mathbb{R}^2 \setminus \overline{D} \) be a curve. Given the near-field data

\[ \{ u(x; z) : x \in \partial B_\rho, \; z \in \Gamma \}, \]

for a fixed wave number \( k \), determine the boundary \( \partial D \) of the obstacle.

We refer to Figure 1 for an illustration of the geometry setting of Problem 2.1

### 2.1 Uniqueness and the indicator functions

From Theorem 2.1 in [8], we know that the near-field data \( \{ u(x; z) : x \in \partial B_\rho, \; z \in \Gamma \} \) can uniquely determine the boundary \( \partial D \) of the obstacle. The following theorem plays an important role in our numerical algorithm.
Figure 1: An illustration of the inverse obstacle scattering problem.

**Theorem 2.1.** Let $D$ be a sound-soft obstacle. Assume that $\Lambda$ is the boundary of domain $D'$ and the total fields satisfy that
\[
\int_\Gamma \int_\Lambda |u(x; z)| ds(x) ds(z) = 0. \tag{5}
\]
Then we have $D' = D$ and $\Lambda = \partial D$.

**Proof.** Assume that $D' \neq D$ and $D_0 = D \cap D'$. Without loss of generality, we assume that $D^* = D \setminus D_0$ is nonempty. From (5), it is readily to see that $u(x; z) = 0$ for all $x \in \Lambda, z \in \Gamma$. Then, we see that for every $z \in \Gamma$,
\[
\Delta u(\cdot; z) + k^2 u(\cdot; z) = 0 \quad \text{in } D^*,
\]
\[
u(\cdot; z) = 0 \quad \text{on } \partial D^*.
\]
This implies that $u(\cdot; z)$ is a Dirichlet eigenfunction for the negative Laplacian in the domain $D^*$ with eigenvalue $k^2$. Further, from Theorem 2.1 in [20], we know that the functions $u(\cdot; z)$ are linearly independent for distinct $z \in \Gamma$, which leads to a contraction since there are only finitely many linearly independent Dirichlet eigenfunctions $u_n$ (see [8] Theorem 5.1]). Therefore, $D' = D$ and $\Lambda = \partial D$.

**Theorem 2.2.** Let $D$ be a sound-hard obstacle. Assume that $\Lambda$ is the boundary of domain $D'$ and the total fields satisfy that
\[
\int_\Gamma \int_\Lambda \left| \frac{\partial u(x; z)}{\partial \nu(x)} \right| ds(x) ds(z) = 0. \tag{6}
\]
Then we have $D' = D$ and $\Lambda = \partial D$.

**Proof.** By (6), we have $\frac{\partial u(x; z)}{\partial \nu(x)} = 0$ for all $x \in \Lambda, z \in \Gamma$, and that $D$ and $D'$ produce the same scattered fields, and then the far-field patterns coincide. Further, the mixed reciprocity relation [8]...
Theorem 3.24 implies that for all $z \in \Gamma$ and $\tilde{x} \in S^1 = \{ x \in \mathbb{R}^2 : |x| = 1 \}$,
\[
v_D^s(z; -\tilde{x}) = v_{D'}^s(z; -\tilde{x}),
\]
where $v_D^s(x; d)$ and $v_{D'}^s(x; d)$ are the scattered fields generated by the obstacle $D$ and $D'$ according to the incident plane wave $v^i(x; d) = e^{ikx \cdot d}$. This means the far field patterns coincide. And thus from Theorem 5.6 in [8], we have $D = D'$ and $\Lambda = \partial D$.

Now, we introduce an indicator function for the sound-soft obstacle
\[
I_s(x) = \int_\Gamma |u(x; z)| ds(z). \tag{7}
\]
From Theorem 2.1, we only need to find a closed curve $\Lambda$ such that $I_s = 0$ on $\Lambda$.

For a sound-hard obstacle, we take a fixed $z_0 \in \Gamma$, and let $\nu(x; z_0)$ be a unit vector such that $\nu(x; z_0) \cdot \nabla u(x; z_0) = 0$. Then, we have $\nabla u(x; z) \cdot \nu(x; z_0) = 0$ on $\partial D$ for every $z \in \Gamma$. Therefore, the indicator function for the sound-hard obstacle is as follows
\[
I_h(x) = \int_\Gamma |\nabla u(x; z) \cdot \nu(x; z_0)| ds(z). \tag{8}
\]
By Theorem 2.2, one needs to find a closed curve $\Lambda$ such that $I_h = 0$ on $\Lambda$.

The approximation on $u(x; z)$ with the near-field data $u(x; z)|_{\partial B_R}$ will be given in the next subsection.

### 2.2 The Fourier-Bessel approximation

In this subsection we consider the approximation of the scattered fields $u^s(x; z)$ for every $z$ by the expansion of Fourier-Hankel functions
\[
u_N^s(x; z) = \sum_{n=-N}^{N} \hat{u}_n(z) \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kR)} e^{in\theta}, \quad r > R, \tag{9}
\]
with the Fourier coefficients
\[
\hat{u}_n(z) = \frac{1}{2\pi} \int_0^{2\pi} u^s(R, \theta; z)e^{-in\theta} d\theta,
\]
where the polar coordinates $(r, \theta) : x = r(\cos \theta, \sin \theta)$ is used and $B_R = \{ x \in \mathbb{R}^2 : |x| < R \} \subset D$.

Here similar to Chapter 5 in [8] we make the assumption that the scattered fields can be extended analytically to $D \setminus B_R$.

By the orthogonality of the Fourier basis functions, the coefficients $\hat{u}_n(z)$ can be given by
\[
\hat{u}_n(z) = \frac{H_n^{(1)}(kR)}{2\pi H_n^{(1)}(k\rho)} \int_0^{2\pi} u^s(\rho, \theta; z)e^{-in\theta} d\theta, \tag{10}
\]
where $u^s(\cdot; z)|_{\partial B_R} = u(\cdot; z)|_{\partial B_R} - u^i(\cdot; z)|_{\partial B_R}$ is the measured scattered data.
Taking the ill-posedness of underlying inverse problem into account, we have the following expansion of Fourier-Hankel functions

\[ u^s,\delta_N(x; z) = \sum_{n=-N}^{N} \hat{u}^\delta_n(z) \frac{H^{(1)}_n(kr)}{H^{(1)}_n(k\rho)} e^{in\theta}, \quad R < r < \rho, \]  

where

\[ \hat{u}^\delta_n(z) = \frac{1}{2\pi} \int_0^{2\pi} u^s,\delta(\rho, \theta; z)e^{-in\theta} d\theta, \]

and \( u^s,\delta \in L^2(\partial B_\rho) \) are measured noisy data satisfying \( \|u^s,\delta - u^s\|_{L^2(\partial B_\rho)} \leq \delta \|u^s\|_{L^2(\partial B_\rho)} \) with \( 0 < \delta < 1 \). For simplicity, we shall also use \( \langle \cdot, \cdot \rangle_\Lambda \) for the usual inner product on \( L^2(\Lambda) \) with \( \Lambda \) being either \( \partial B_\rho \) or \( \partial B_R \) accordingly.

In what follows, \( \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \) is the Gamma function, \( J_n \) and \( Y_n \) denote respectively the Bessel functions and Neumann functions of order \( n \), and \( \lfloor \cdot \rfloor \) signifies the rounding function which truncates the variable down to the nearest integer. To consider the error estimate, we need the following results.

**Lemma 2.1.** For \( z > 0 \) and \( n \in \mathbb{N} \) such that \( n > (ez + 1)/2 \), we have

\[ \frac{1}{2} \leq \frac{\pi z^n |H^{(1)}_n(z)|}{3 \cdot 2^{n-1} \Gamma(n)} \leq e^z. \]  

**Proof.** By the integral representations

\[ J_n(z) = \frac{2(\frac{z}{2})^n}{\sqrt{\pi \Gamma(n+1/2)}} \int_0^1 (1 - t^2)^{n-1/2} \cos(zt) dt, \quad n \geq 0, \]

and

\[ Y_n(z) = \frac{2(\frac{z}{2})^n}{\sqrt{\pi \Gamma(n+1/2)}} \left( \int_0^1 (1 - t^2)^{n-1/2} \cos(zt) dt - \int_0^\infty e^{-zt} (1 + t^2)^{n-1/2} dt \right), \quad n \geq 0, \]

and the fact that for \( z > 0 \) and \( n \geq 1 \),

\[ \int_0^1 (1 - t^2)^{n-1/2} \cos(zt) dt \leq \int_0^1 (1 - t^2)^{n-1/2} dt \]

\[ = \int_0^{\pi/2} \cos^{2n} t dt \]

\[ = \frac{\pi}{2} \left( \frac{(2n-1)!!}{(2n)!!} \right) \]

\[ = \frac{\pi}{2} \left( \frac{(2n-1)!!}{(2n)!!} \right)^{1/2} \]

\[ \leq \frac{\pi}{2} \left( \frac{(2n-1)!!}{(2n)!!} \right)^{1/2} \]

\[ = \frac{\pi}{2\sqrt{2n+1}}, \]

\[ (13) \]

6
it is readily to see that for $z > 0$ and $n \geq 1$,

$$|J_n(z)| \leq \frac{2(\frac{z}{2})^n}{\sqrt{\pi} \Gamma(n + 1/2)} \frac{\pi}{2\sqrt{2n + 1}}$$  \hspace{1cm} (14)$$

and

$$|Y_n(z)| \leq \frac{2(\frac{z}{2})^n}{\sqrt{\pi} \Gamma(n + 1/2)} \left( \frac{\pi}{2\sqrt{2n + 1}} + \int_0^{\infty} e^{-zt}(1 + t^2)^{n-1/2}dt \right).$$  \hspace{1cm} (15)$$

Moreover, by the Legendre duplication formula $\sqrt{\pi} \Gamma(2z) = 2^{2z-1} \Gamma(z) \Gamma(z + 1/2)$, we deduce that

$$\int_0^{\infty} e^{-zt}(1 + t^2)^{n-1/2}dt \geq \int_0^{\infty} e^{-zt} t^{2n-1}dt$$

$$= \frac{1}{2^n} \int_0^{\infty} e^{-t} t^{2n-1}dt$$

$$= \frac{\Gamma(2n)}{z^{2n}}$$

$$= \frac{2^{2n-1} \Gamma(n) \Gamma(n + 1/2)}{\sqrt{\pi} z^{2n}},$$  \hspace{1cm} (16)$$

and

$$\int_0^{\infty} e^{-zt}(1 + t^2)^{n-1/2}dt \leq \int_0^{\infty} e^{-zt} (1 + t)^{2n-1}dt$$

$$= e^z \int_0^{\infty} e^{-z(t+1)} (1 + t)^{2n-1}d(t + 1)$$

$$\leq e^z \int_0^{\infty} e^{-zt} t^{2n-1}dt$$

$$= e^z \frac{\Gamma(2n)}{z^{2n}}.$$

(17)

Then from Stirling’s approximation,

$$n! > \sqrt{2\pi n} \left( \frac{n}{e} \right)^n,$$

we know that for $n > N_z := (ez + 1)/2$,

$$\frac{\Gamma(2n)}{z^{2n}} \geq \frac{\sqrt{2\pi(2n-1)}}{z} \left( \frac{2n-1}{ez} \right)^{2n-1}$$

$$\geq e^{\sqrt{2\pi(2n-1)}} \frac{2n-1}{2n-1}$$

$$\geq e^{\sqrt{2\pi}} \frac{2n+1}{2n+1}$$

$$\geq \frac{2\pi}{\sqrt{2n+1}}.$$  \hspace{1cm} (18)$$
Now, by (14), (15), (17) and (18), we deduce that for $n > N_z$,
\[
|H_n^{(1)}(z)| = |J_n(z) + iY_n(z)|
\leq |J_n(z)| + |Y_n(z)|
\leq \frac{2z^n}{\sqrt{\pi} \Gamma(n+1/2)} \left( \frac{\pi}{2^n + 1} + \sqrt{\frac{e^{-z^2}}{2^n + 1} \Gamma(2n)} \right)
\leq \frac{2z^n}{\sqrt{\pi} \Gamma(n+1/2)} \left( \frac{\pi}{2^n + 1} \right)
\leq \frac{3e^{z^n}}{2} \frac{2^{n-1} \Gamma(n+1/2)}{\sqrt{\pi} z^{2n}}
= e^{z^n} 3 \frac{2^{n-1} \Gamma(n)}{\pi z^n}.
\] (19)

Similarly, from the integral representation of $Y_n(z)$, (13), (16) and (18), it can be seen that for $n > N_z$,
\[
\left| H'_n(z) \right| \geq |Y_n(z)|
\geq \frac{2z^n}{\sqrt{\pi} \Gamma(n+1/2)} \left( \sqrt{\frac{e^{-z^2}}{2^n + 1} \Gamma(2n)} - \frac{\pi}{2^n + 1} \right)
\geq \frac{3z^n}{4 \sqrt{\pi} 2^{n-1} \Gamma(n+1/2) \pi z^{2n}} \frac{\Gamma(2n)}{2^n + 1}
= \frac{1}{2} 3 \cdot 2^{n-1} \Gamma(n) \frac{1}{\pi z^n}.
\] (20)

Finally, combining (19) and (20), we arrive at estimate (12) and this completes the proof.

Lemma 2.2 ([32, Lemma 3.3]). Let $n \in \mathbb{Z}$, and $t, t_1, t_2 \in \mathbb{R}^+$ satisfy $t_1 \leq t_2$. Then the following estimates hold
\[
\left| H_n^{(1)}(t) \right| \leq |H_n^{(1)}(t_1)|.
\] (21)

From Lemmas 2.1 and 2.2, we obtain the following approximation result.

Theorem 2.3. Let $\gamma = \text{dist}(\partial B_R, \partial D)$ and $N \geq N_0 := \frac{e^{\gamma + 1}}{2}$. Then there exist positive constants $C_1, C_2, \ldots, C_6$, such that
\[
\left\| u^s - u^s_N \right\|_{L^2(\partial B_R)} \leq C_1 \gamma_1^{-N} + C_2 N^{1/2} \delta,
\] (22)
\[
\left\| u^s - u^s_N \right\|_{L^2(\partial D)} \leq C_3 \gamma_2^{-N} + C_4 \gamma_3^N \delta,
\] (23)
\[
\left\| \partial_\nu u^s - \partial_\nu u^s_N \right\|_{L^2(\partial D)} \leq C_5 N \gamma_2^{-N} + C_6 N \gamma_3^N \delta.
\] (24)
where \( \tau_1 = \frac{\rho}{R}, \tau_2 = \frac{R+\rho}{R}, \tau_3 = \frac{\rho}{R+\gamma} \). Moreover, for \( 0 < \delta < \tau_1^{-N_0} \), take \( N = \left\lfloor \frac{1}{\ln \tau_1 \ln \frac{1}{\delta}} \right\rfloor \), then the following results hold

\[
\left\| u^s - u^s_N \right\|_{L^2(\partial B_{\rho})} \leq C_1 \delta + \frac{C_2}{\sqrt{\ln \tau_1}} \delta^{1/2}, \\
\left\| u^s - u^s_N \right\|_{L^2(\partial D)} \leq (C_3 + C_4) \delta^{\frac{\ln \tau_1}{\ln \tau_2}}, \\
\left\| \partial_\nu u^s - \partial_\nu u^s_N \right\|_{L^2(\partial D)} \leq \frac{C_5 + C_6}{\ln \tau_1} \delta^{\frac{\ln \tau_1}{\ln \tau_2}} |\ln \delta|.
\]

(25)

(26)

(27)

**Proof.** From (9) and (12), it can be seen that

\[
\left\| u^s - u^s_N \right\|_{L^2(\partial B_{\rho})}^2 = 2\pi \rho \sum_{|n| > N} \left| \frac{H_n^{(1)}(k\rho)}{H_n^{(1)}(kR)} \right|^2 |u_n(z)|^2 \\
\leq 2\pi \rho \sum_{|n| > N} 4e^{2k\rho} \left( \frac{R}{\rho} \right)^{2|n|} |u_n(z)|^2 \\
\leq 8\pi \rho e^{2k\rho} \left( \frac{R}{\rho} \right)^{2N} \left\| u^s \right\|_{L^2(\partial B_{\rho})}^2.
\]

Further, by (9)-(11), we have

\[
\left\| u^s_N - u^s_N^\delta \right\|_{L^2(\partial B_{\rho})}^2 = \frac{1}{2\pi \rho} \sum_{n = -N}^N \left| \langle u^s - u^s_N^\delta, e^{in\theta} \rangle_{\partial B_{\rho}} \right|^2 \\
\leq (2N + 1) \delta^2 \left\| u^s \right\|_{L^2(\partial B_{\rho})}^2.
\]

And thus,

\[
\left\| u^s - u^s_N \right\|_{L^2(\partial B_{\rho})} \leq \left\| u^s - u^s_N \right\|_{L^2(\partial B_{\rho})} + \left\| u^s_N - u^s_N^\delta \right\|_{L^2(\partial B_{\rho})} \\
\leq C_1 \tau_1^{-N} + C_2 N^{1/2} \delta,
\]

where \( \tau_1 = \frac{\rho}{\pi}, C_1 = 2\sqrt{2\pi \rho e^{kp}} \left\| u^s \right\|_{L^2(\partial B_{\rho})} \) and \( C_2 = \sqrt{3} \left\| u^s \right\|_{L^2(\partial B_{\rho})} \). This leads to estimate (22).
From \((12)\) and \((21)\), it is obvious that for \(x \in \partial D\),
\[
\left| u^s(x; z) - u^s_N(x; z) \right| \leq \sum_{|n| > N} |\hat{u}_n(z)| \left| \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kR)} \right| \\
\leq \sum_{|n| > N} |\hat{u}_n(z)| \left| \frac{H_n^{(1)}(k(R + \gamma))}{H_n^{(1)}(kR)} \right| \\
\leq \sum_{|n| > N} 2e^{k(R + \gamma)} \left( \frac{R}{R + \gamma} \right)^{|n|} |\hat{u}_n(z)| \\
\leq 2e^{k(R + \gamma)} \tau_2^{-N} \left( \sum_{|n| \geq 1} \tau_2^{-2|n|} \right)^{1/2} \left( \sum_{|n| > N} |\hat{u}_n(z)|^2 \right)^{1/2} \\
\leq \tilde{C}_3 \tau_2^{-N} \| u^s \|_{L^2(\partial D)}.
\]
where \(\gamma = \text{dist}(\partial B_R, \partial D)\), \(\tau_2 = \frac{R + \gamma}{R}\) and \(\tilde{C}_3 = 2\sqrt{2}e^{k(R + \gamma)}(\tau_2^2 - 1)^{-1/2}\). This implies
\[
\| u^s - u^s_N \|_{L^2(\partial D)} \leq \tilde{C}_3 \tau_2^{-N}, \tag{28}
\]
where \(\tilde{C}_3 = \tilde{C}_3 \| u^s \|_{L^2(\partial B_R)} \| \partial D \|^{1/2}\) and \(\| \partial D \|\) denotes the length of \(\partial D\). By using estimates \((12)\) and \((21)\), we have that for \(x \in \partial D\),
\[
\left| u^s_N(x; z) - u_N(x; z) \right| \leq \frac{1}{2\pi \rho} \sum_{n = -N}^N \left| \frac{H_n^{(1)}(kr)}{H_n^{(1)}(k\rho)} \right| \left| \left( \begin{array}{c} u^s \- u^s_N \end{array} \right) , e^{i\theta} \right|_{\partial B_\rho} \\
\leq \frac{\| u^s \|_{L^2(\partial B_\rho)}}{\sqrt{2\pi \rho}} \sum_{n = -N}^N \left| \frac{H_n^{(1)}(k(R + \gamma))}{H_n^{(1)}(k\rho)} \right| \\
\leq \frac{\| u^s \|_{L^2(\partial B_\rho)}}{\sqrt{2\pi \rho}} \sum_{n = -N}^N 2e^{k(R + \gamma)} \left( \frac{\rho}{R + \gamma} \right)^{|n|} \\
\leq \frac{5}{3} \frac{\| u^s \|_{L^2(\partial B_\rho)}}{\sqrt{2\pi \rho}} \delta \sum_{n = -N}^N \left( \frac{2\rho}{R + \gamma} \right)^{|n|} \\
\leq \tilde{C}_4 \delta \tau_3^{-N + 1} \| u^s \|_{L^2(\partial B_\rho)},
\]
where \(\tau_3 = \frac{\rho}{R + \gamma}\) and \(\tilde{C}_4 = 2\sqrt{2}e^{k(R + \gamma)}(\tau_3^{-1})^{-1/2}\). This, together with \((28)\), yields
\[
\left\| u^s - u_N^s \right\|_{L^2(\partial D)} \leq \left\| u^s - u_N^s \right\|_{L^2(\partial D)} + \left\| u_N^s - u_N^s \right\|_{L^2(\partial D)} \\
\leq \tilde{C}_3 \tau_2^{-N} + C_4 \delta \tau_3^{-N},
\]
where \(\tilde{C}_4 = \tilde{C}_4 \tau_3^{-1/2}\), which implies \((23)\).

From the formula \(H_n^{(1)}(t) = -H_{n+1}^{(1)}(t) + nH_n^{(1)}(t)/t\), we have we have
\[
\left| \nabla H_n^{(1)}(kr) \right| \leq k \left| H_{n+1}^{(1)}(kr) \right| + \frac{|n|}{r} \left| H_n^{(1)}(kr) \right|,
\]
which, in conjunction with $|\nabla e^{i\theta}| = |\nu|$, implies

$$
\left| \nabla \left( H_n^{(1)}(kr)e^{i\theta} \right) \right| \leq k \left| H_n^{(1)}(kr) \right| + \frac{2|\nu|}{r} \left| H_n^{(1)}(kr) \right|.
$$

(29)

Now, by estimates [12], [21] and [29], we derive that for $x \in \partial D$,

$$
\left| \frac{\partial}{\partial \nu} u^s(x;z) - \frac{\partial}{\partial \nu} u_N^s(x;z) \right| \leq \sum_{|n| > N} |\tilde{u}_n(z)| \left| \nabla \left( H_n^{(1)}(kr)e^{i\theta} \right) \right| H_n^{(1)}(kr)
\leq \sum_{|n| > N} |\tilde{u}_n(z)| \left( k \left| H_n^{(1)}(kr) \right| + \frac{2|\nu|}{r} \left| H_n^{(1)}(kr) \right| \right)
\leq \frac{8e^{k(R+\gamma)}}{R + \gamma} \sum_{|n| > N} |n| \left( \frac{R}{R + \gamma} \right)^{|n|} |\tilde{u}_n(z)|
\leq \tilde{C}_5 N \tau_2^{-N} \|u^s\|_{L^2(\partial \Omega_B)}.
$$

where

$$
\tilde{C}_5 = \frac{8\sqrt{2}e^{k(R+\gamma)}}{R + \gamma} \left( \sum_{|n| \geq 1} n^2 \tau_2^{-2|n|} \right)^{1/2}.
$$

This implies

$$
\left\| \frac{\partial}{\partial \nu} u^s(x;z) - \frac{\partial}{\partial \nu} u_N^s(x;z) \right\|_{L^2(\partial \Omega_D)} \leq \tilde{C}_5 N \tau_2^{-N},
$$

(30)

where $C_5 = \tilde{C}_5 \|u^s\|_{L^2(\partial \Omega_B)} |\partial D|^{1/2}$. Again, estimates [12], [21] and [29], we see that for $x \in \partial D$,

$$
\left| \frac{\partial}{\partial \nu} u_N^s(x;z) - \frac{\partial}{\partial \nu} u_N^{s,\delta}(x;z) \right| \leq \frac{1}{2\pi \rho} \sum_{n = -N}^{N} \left| \nabla \left( H_n^{(1)}(kr)e^{i\theta} \right) \right| \left| (u^s - u^{s,\delta} e^{i\theta}) \right|_{\partial B_D}
\leq \frac{1}{2\pi \rho} \sum_{n = -N}^{N} \left( k \left| H_n^{(1)}(kr) \right| + \frac{2|\nu|}{r} \left| H_n^{(1)}(kr) \right| \right) \delta \|u^s\|_{L^2(\partial B_D)}
\leq \frac{8e^{k(R+\gamma)} \|u^s\|_{L^2(\partial B_D)}}{2\pi \rho} \delta \sum_{n = -N}^{N} |n| \left( \frac{\rho}{R + \gamma} \right)^{|n|}
\leq \tilde{C}_6 \delta \tau_3^{N+1} \|u^s\|_{L^2(\partial B_D)},
$$

where $\tilde{C}_6 = \frac{8\sqrt{2}e^{k(R+\gamma)}}{\sqrt{\pi^2 |\rho| (R + \gamma)^3}}$. This, together with [30], yields

$$
\left\| \partial_{\nu} u^s - \partial_{\nu} u_N^{s,\delta} \right\|_{L^2(\partial \Omega_D)} \leq \left\| \partial_{\nu} u^s - \partial_{\nu} u_N^{s} \right\|_{L^2(\partial \Omega_D)} + \left\| \partial_{\nu} u_N^{s,\delta} \right\|_{L^2(\partial \Omega_D)}
\leq C_5 N \tau_2^{-N} + C_6 \delta \tau_3^{N},
$$

where $C_6 = \tilde{C}_6 \tau_3 \|u^s\|_{L^2(\partial B_D)} |\partial D|^{1/2}$, which implies [24].
For $0 < \delta < \tau_1^{-N_0}$, from $N = \left[ \frac{1}{\ln \tau_1} \ln \frac{1}{\delta} \right]$, we see $N \geq N_0$. Substituting $N = \left[ \frac{1}{\ln \tau_1} \ln \frac{1}{\delta} \right]$ into (22), (23), (24), and using $\ln t < t$ for $t > 1$, we derive the estimates (25), (26) and (27). The proof is complete.

With the approximation of the scattered fields $u^s(x; z)$ by $u^s_N(x; z)$ in (11), we now introduce the approximating indicator functions.

(i) The sound-soft case. The approximating indicator function for (7) is defined by

$$I_{s,N}(x) = \int_{\Gamma} \left| u^s_N(x; z) + u^t(x; z) \right| ds(z)$$

(ii) The sound-hard case. The approximating indicator function for (8) is defined by

$$I_{h,N}(x) = \int_{\Gamma} \left| \nabla \left( u^s_N(x; z) + u^t(x; z) \right) \cdot \nu^\delta(x; z_0) \right| ds(z),$$

where $\nu^\delta(x; z_0)$ is a unit vector satisfying $\nu^\delta(x; z_0) \cdot \nabla(u^s_N(x; z_0) + u^t(x; z_0)) = 0$ and

$$\left| \nabla \left( u^s_N(x; z_0) + u^t(x; z_0) \right) \right| = \max_{z \in \Gamma} \left\{ \left| \nabla \left( u^s_N(x; z) + u^t(x; z) \right) \right| \right\}. $$

Based on Theorem 2.3, the following estimates for the indicator functions hold.

**Theorem 2.4.** Let $\tau_1, \tau_2$ and $N_0$ be defined in Theorem 2.3. Then for $0 < \delta < \tau_1^{-N_0}$ and $N = \left[ \frac{1}{\ln \tau_1} \ln \frac{1}{\delta} \right]$, there exist positive constants $C_s$ and $C_h$, such that

$$I_{s,N}(x) \leq C_s \delta^\alpha,$$

$$I_{h,N}(x) \leq C_h \delta^\alpha \ln \delta,$$

where $\alpha = \ln \tau_2 / \ln \tau_1$.

**Proof.** From the proof of (23) in Theorem 2.3 and $\tau_3 = \tau_1 / \tau_2$, it can be seen that for $x \in \partial D$,

$$\left| u^s(x; z) - u^s_N(x; z) \right| \leq C_7 \left( \tau_2^{-N} \| u^s \|_{L^2(\partial B_R)} + \delta \tau_3^N \| u^s \|_{L^2(\partial B_R)} \right),$$

where $C_7 > 0$ is a constant. By taking $N = \left[ \frac{1}{\ln \tau_1} \ln \frac{1}{\delta} \right]$, we know that for $x \in \partial D$,

$$\left| u^s(x; z) - u^s_N(x; z) \right| \leq C_7 \left( \delta^{\frac{\ln \tau_2}{\ln \tau_1}} \| u^s \|_{L^2(\partial B_R)} + \delta^{1-\frac{\ln \tau_2}{\ln \tau_1}} \| u^s \|_{L^2(\partial B_R)} \right)$$

$$= C_7 \left( \| u^s \|_{L^2(\partial B_R)} + \| u^s \|_{L^2(\partial B_R)} \right) \delta^{\frac{\ln \tau_2}{\ln \tau_1}}.$$ 

And thus, for $x \in \partial D$,

$$I_{s,N}(x) = \int_{\Gamma} \left| u^s_N(x; z) - u^s(x; z) + u^t(x; z) \right| ds(z)$$

$$= \int_{\Gamma} \left| u^s_N(x; z) - u^s(x; z) \right| ds(z)$$

$$\leq C_s \delta^\alpha,$$
where
\[ \alpha = \frac{\ln \tau_2}{\ln \tau_1}, \quad C_s = C_s C_0, \quad C_0 := \int_G (\|u^\ast(\cdot; z)\|_{L^2(\partial B_R)} + \|u^\ast(\cdot; z)\|_{L^2(\partial B_s)}) \, ds(z). \]

This leads to the estimate \(34\).

Similarly, following the proof of \(24\) in Theorem 2.3, we have that for \(x \in \partial D\),
\[
\left| \nabla \left( u^\ast(x; z) - u_N^{s, \delta}(x; z) \right) \right| \leq C_8 \left( \|u^\ast\|_{L^2(\partial B_R)} + \|u^\ast\|_{L^2(\partial B_s)} \right) \delta^\alpha |\ln \delta|,
\]
where \(C_8 > 0\) is a constant. This means that for \(x \in \partial D\),
\[
\int_G \left| \nabla \left( u_N^{s, \delta}(x; z) - u^\ast(x; z) \right) \cdot \nu(x; z_0) \right| \, ds(z) \leq C_8 C_0 \delta^\alpha |\ln \delta|. \tag{36}
\]

For \( \eta = (\eta_1, \eta_2)^T \in \mathbb{R}^2 \), we introduce the notation \(\eta^\perp = (\eta_2, -\eta_1)^T \in \mathbb{R}^2\). For convenience, denote \(\xi_\delta(x; z) := \nabla \left( u_N^{s, \delta}(x; z) + u^\ast(x; z) \right)\) and \(\xi(x; z) := \nabla \left( u^\ast(x; z) + u^\ast(x; z) \right)\). Since for \(x \in \partial D\),
\[
\nu^\delta(x; z_0) \cdot \nabla \left( u_N^{s, \delta}(x; z_0) + u^\ast(x; z_0) \right) = 0 \quad \text{and} \quad \nu(x; z_0) \cdot \nabla \left( u^\ast(x; z_0) + u^\ast(x; z_0) \right) = 0,
\]
it is readily to see that
\[
\nu^\delta(x; z_0) = \frac{\xi^\perp_\delta(x; z_0)}{|\xi^\perp_\delta(x; z_0)|}, \quad \nu(x; z_0) = \frac{\xi^\perp(x; z_0)}{|\xi^\perp(x; z_0)|}
\]
and
\[
\left| \nu^\delta(x; z_0) - \nu(x; z_0) \right| \leq \frac{2 |\xi^\perp_\delta(x; z_0) - \xi^\perp(x; z_0)|}{|\xi^\perp_\delta(x; z_0)|}
\]
\[
= \frac{2 |\xi_\delta(x; z_0) - \xi(x; z_0)|}{|\xi^\perp_\delta(x; z_0)|}
\]
\[
= \frac{2 \left| \nabla \left( u_N^{s, \delta}(x; z_0) - u^\ast(x; z_0) \right) \right|}{\left| \nabla \left( u_N^{s, \delta}(x; z_0) + u^\ast(x; z_0) \right) \right|}.
\]

Further, from \(33\), we see that for \(x \in \partial D\),
\[
\int_G \left| (\nu^\delta(x; z_0) - \nu(x; z_0)) \cdot \nabla \left( u_N^{s, \delta}(x; z) + u^\ast(x; z) \right) \right| \, ds(z)
\]
\[
\leq 2 \int_G \frac{\left| \nabla \left( u_N^{s, \delta}(x; z_0) - u^\ast(x; z_0) \right) \right|}{\left| \nabla \left( u_N^{s, \delta}(x; z_0) + u^\ast(x; z_0) \right) \right|} \left| \nabla \left( u_N^{s, \delta}(x; z) + u^\ast(x; z) \right) \right| \, ds(z)
\]
\[
\leq 2 C_8 C_0 \delta^\alpha |\ln \delta|,
\]
which, together with \(32\) and \(36\), yields that for \(x \in \partial D\),
\[
\int_G \left| \nabla \left( u_N^{s, \delta}(x; z) + u^\ast(x; z) \right) \cdot \nu^\delta(x; z_0) \right| \, ds(z)
\]
\[
\leq \int_G \left| \nabla \left( u_N^{s, \delta}(x; z) + u^\ast(x; z) \right) \cdot (\nu^\delta(x; z_0) - \nu(x; z_0)) \right| \, ds(z)
\]
\[
+ \int_G \left| \nabla \left( u_N^{s, \delta}(x; z) - u^\ast(x; z) \right) \cdot \nu(x; z_0) \right| \, ds(z)
\]
\[
\leq 3 C_8 C_0 \delta^\alpha |\ln \delta|.
\]
This implies estimate (35) holds.

3 Inverse cavity scattering problem

We begin this section with the precise formulations of the model cavity scattering problem. The interior scattering problem for cavities can be formulated as: to find the scattered field \( u^s \in H^1(D) \) which satisfies the following boundary value problem:

\[
\Delta u^s + k^2 u^s = 0 \quad \text{in } D, \\
\mathcal{B} u = 0 \quad \text{on } \partial D,
\]

(37) (38)

where \( u = u^i + u^s \) denotes the total field. Provided that \( k^2 \) is not a Dirichlet eigenvalue for \(-\Delta\) in \( D \), it is well known that the direct scattering problem (37)–(38) admits a unique solution \( u^s \in H^1(D) \) (see, e.g. [4, 8]). From now on we assume that \( k^2 \) is not a Dirichlet eigenvalue for \(-\Delta\) in \( D \).

Following [38], we introduce the following definition of admissible curve.

**Definition 3.1.** (Admissible curve) An open curve \( \Gamma \) is called an admissible curve with respect to domain \( \Omega \) if

(i) \( \Omega \subset D \) is simply-connected;
(ii) \( \partial \Omega \) is analytic homeomorphic to \( S^1 \);
(iii) \( k^2 \) is not a Dirichlet eigenvalue of \(-\Delta\) in \( \Omega \);
(iv) \( \Gamma \subset \partial \Omega \) is a one-dimensional (1D) analytic manifold with nonvanishing measure.

**Remark 3.1.** We would like to remark that this requirement for the admissibility of \( \Gamma \) can be easily fulfilled. For example, since the first zero of \( J_0 \) is approximately \( 2.4048 \), \( \Omega \) can be chosen as a disk whose radius is less than \( 2.4048/k \) and \( \Gamma \) is chosen as an arbitrary corresponding semicircle.

Now, we introduce the interior inverse scattering problem for incident point sources.

**Problem 3.1** (Inverse cavity scattering). Let \( D \) be the impenetrable cavity with boundary condition \( \mathcal{B} \). Assume that \( \Gamma \) and \( \Gamma_R \) are admissible curves with respect to \( \Omega \) and \( B_R \), respectively, such that \( \Omega \subset B_R \) and \( B_R \subset D \). Given the total field data

\[
\{u(x; z) : x \in \partial B_R, z \in \Gamma\},
\]

for a fixed wavenumber \( k \), determine the boundary \( \partial D \) of the cavity.

We refer to Figure 2 for an illustration of the geometry setting of Problem 3.1.

3.1 Uniqueness and the indicator functions

From [26 Theorem 2.1] and [27 Theorem 3.1], we know that the total field data \( \{u(x; z) : x \in \partial B_R, z \in \Gamma\} \) can uniquely determine the boundary \( \partial D \) of the cavity. Analogous to the inverse obstacle scattering problem, we have the following uniqueness results and the indicator functions.

**Theorem 3.1.** Let \( D \) be a sound-soft cavity. Assume that \( \Lambda \) is the boundary of domain \( D' \). Suppose that the total fields satisfy that

\[
\int_{\Gamma} \int_{\Lambda} |u(x; z)| ds(x) ds(z) = 0.
\]

(39)
Then we have $D' = D$ and $\Lambda = \partial D$.

Proof. See Theorem 2.1 for a similar proof.

**Theorem 3.2.** Let $D$ be a sound-hard cavity. Assume that $\Lambda$ is the boundary of domain $D'$. Suppose that the total fields satisfy that

$$\int_{\Gamma} \int_{\Lambda} \left| \frac{\partial u(x; z)}{\partial \nu(x)} \right| ds(x)dz = 0.$$  

Then we have $D' = D$ and $\Lambda = \partial D$.

Proof. By (40) and the definition of admissible curve $B_R$, we have $\frac{\partial u(x; z)}{\partial \nu(x)} = 0$ for all $x \in \Lambda, z \in \Gamma$, and that $D$ and $D'$ produce the same scattered fields. Further, from the reciprocity relation [28, Theorem 2.1], we see that for all $z \in \Gamma$ and $x \in D_0 = D \cap D'$,

$$u^s_D(z; x) = u^s_{D'}(z; x).$$

Further, the definition of admissible curve $\Gamma$ and the reciprocity relation lead to

$$u^s_D(x; z) = u^s_{D'}(x; z), \ \forall x, z \in D_0.$$  

Then, by a similar discussion in Theorem 5.6 in [8], we have $D = D'$ and $\Lambda = \partial D$. 

Now, the indicator functions for a sound-soft or a sound-hard cavity are as follows

$$I_s(x) = \int_{\Gamma} |u(x; z)|ds(z),$$  

Figure 2: An illustration of the interior inverse scattering problem.
\[ I_b(x) = \int_\Gamma |\nabla u(x; z) \cdot \nu(x; z_0)| \mathrm{d}s(z), \quad (42) \]

where \( z_0 \in \Gamma \) and \( \nu(x; z_0) \cdot \nabla u(x; z_0) = 0 \). In the next subsection, we will approximate \( u(x; z) \) by the Fourier-Bessel functions with the measurements on \( \partial B_R \).

### 3.2 The Fourier-Bessel approximation

We assume the scattered fields can be extended analytically to \( B_\rho \supseteq D \) and approximate the scattered fields \( u^s(x; z, k) \) for every \( z \in \Gamma \) by the expansion of Fourier-Bessel functions

\[ u^s_N(x; z) = \sum_{n=-N}^{N} \hat{u}_n(z) \frac{J_n(kr)}{J_n(k\rho)} e^{in\theta}, \quad r < \rho, \quad (43) \]

with the Fourier coefficients

\[ \hat{u}_n(z) = \frac{1}{2\pi} \int_0^{2\pi} u^s(\rho, \theta; z)e^{-in\theta} \mathrm{d}\theta, \]

where \( J_n(t) \) is the Bessel function of the first kind of order \( n \), under the polar coordinates \((r, \theta) : x = r(\cos \theta, \sin \theta)\). The parameters \( \hat{u}_n(z) \) can be determined by the measurements \( u^s(x; z, k) = u(x; z) - u^i(x; z) \) on \( \partial B_R \) by

\[ \hat{u}_n(z) = \frac{J_n(k\rho)}{2\pi J_n(kR)} \int_0^{2\pi} u^s(R, \theta; z)e^{-in\theta} \mathrm{d}\theta. \quad (44) \]

Due to measured noisy data \( u^{s, \delta} \in L^2(\partial B_R) \) satisfying \( \|u^{s, \delta} - u^s\|_{L^2(\partial B_R)} \leq \delta \|u^s\|_{L^2(\partial B_R)} \) with \( 0 < \delta < 1 \), the expansion of Fourier-Bessel functions is as follows

\[ u^{s, \delta}_N(x; z) = \sum_{n=-N}^{N} \hat{u}_n^{\delta}(z) \frac{J_n(kr)}{J_n(k\rho)} e^{in\theta}, \quad R < r < \rho, \quad (45) \]

where

\[ \hat{u}_n^{\delta}(z) = \frac{1}{2\pi} \int_0^{2\pi} u^{s, \delta}(R, \theta; z)e^{-in\theta} \mathrm{d}\theta. \]

Now, we give the following approximation results.

**Lemma 3.1.** For \( z > 0 \) and \( n \in \mathbb{N} \) such that \( n > \max\{\frac{3}{10} z^2 - 1, 1\} \), we have

\[ 1 \leq \frac{2^n n! |J_n(z)|}{z^n} \leq 1. \quad (46) \]

**Proof.** From the definition of the Bessel functions of the first kind

\[
J_n(z) = \sum_{p=0}^{\infty} (-1)^p \frac{(-\frac{z}{2})^{n+2p}}{p!(n+p)!} = \frac{z^n}{2^n n!} \left(1 - \frac{(\frac{z}{2})^2}{n+1} + \sum_{p=2}^{\infty} (-1)^p \frac{(\frac{z}{2})^{2p}}{p!(n+1)\cdots(n+p)}\right),
\]

16
we see that for \( n \geq \frac{3}{10} z^2 - 1 \),
\[
J_n(z) \geq \frac{z^n}{2^n n!} \left( 1 - \frac{(\frac{z}{2})^2}{n + 1} \right) \geq \frac{1}{6} \frac{z^n}{2^n n!}
\]
and
\[
J_n(z) \leq \frac{z^n}{2^n n!}.
\]
The proof is complete.

**Theorem 3.3.** Let \( \mu = \text{dist}(\partial B, \partial D) \) and \( N \geq N_0 := \left\lceil \max \left\{ \frac{3}{10} (k \rho)^2 - 1, 1 \right\} \right\rceil \). Then there exist positive constants \( K_1, \cdots, K_6 \), such that
\[
\| u^s - u_{N}^{s, \delta} \|_{L^2(\partial B_R)} \leq K_1 \sigma_1^{-N} + K_2 N^{1/2} \delta, \tag{47}
\]
\[
\| u^s - u_{N}^{s, \delta} \|_{L^2(\partial D)} \leq K_3 \sigma_2^{-N} + K_4 \sigma_3^N \delta, \tag{48}
\]
\[
\| \partial_\nu u^s - \partial_\nu u_{N}^{s, \delta} \|_{L^2(\partial D)} \leq K_5 N \sigma_2^{-N} + K_6 N \sigma_3^N \delta, \tag{49}
\]
where \( \sigma_1 = \frac{\rho}{R}, \sigma_2 = \frac{\rho}{R} - \mu, \sigma_3 = \frac{\rho - \mu}{R} \). Moreover, for \( 0 < \delta < \sigma_1^{-N_0} \), take \( N = \left\lceil \frac{1}{\ln \sigma_1} \ln \frac{1}{\delta} \right\rceil \), then the following results hold
\[
\| u^s - u_{N}^{s, \delta} \|_{L^2(\partial B_R)} \leq K_1 \delta + \frac{K_2}{\ln \sigma_1} \delta^{1/2}, \tag{50}
\]
\[
\| u^s - u_{N}^{s, \delta} \|_{L^2(\partial D)} \leq (K_3 + K_4) \frac{\ln \sigma_1}{\ln \sigma_1} \delta, \tag{51}
\]
\[
\| \partial_\nu u^s - \partial_\nu u_{N}^{s, \delta} \|_{L^2(\partial D)} \leq K_5 N \frac{\ln \sigma_1}{\ln \sigma_1} \delta \ln \delta. \tag{52}
\]

**Proof.** From (43) and (46), it can be seen that
\[
\| u^s - u_{N}^{s, \delta} \|_{L^2(\partial B_R)}^2 = 2\pi R \sum_{|n| > N} \left| J_n(k R) \right|^2 \left| \hat{u}_n(z) \right|^2 \\
\leq 2\pi R \sum_{|n| > N} 36 \left( \frac{R}{\rho} \right)^{2|n|} \left| \hat{u}_n(z) \right|^2 \\
\leq 72\pi R \left( \frac{R}{\rho} \right)^{2N} \| u^s \|_{L^2(\partial B_R)}^2.
\]
Further, by (43)-(45), we have
\[
\| u_{N}^{s, \delta} - u_{N}^{s, \delta} \|_{L^2(\partial B_R)}^2 = \frac{1}{2\pi R} \sum_{n=-N}^{N} \left| \langle u^s - u_{N}^{s, \delta} e^{i n \theta} \rangle_{\partial B_R} \right|^2 \\
\leq (2N + 1) \delta^2 \| u^s \|_{L^2(\partial B_R)}^2.
\]
And thus,
\[
\|u^*-u^{*,\delta}_N\|_{L^2(\partial B_R)} \leq \|u^*-u^{*,\delta}_{N}\|_{L^2(\partial B_R)} + \|u^{*,\delta}_{N}\|_{L^2(\partial B_R)} \\
\leq K_1 \sigma_1^{-N} + K_2 N^{1/2} \delta,
\]
where \(\sigma_1 = \frac{R}{R}, K_1 = 6\sqrt{2\pi R} \|u^*\|_{L^2(\partial B_R)}\) and \(K_2 = \sqrt{3} \|u^*\|_{L^2(\partial B_R)}\). This leads to estimate \(47\).

From \(40\), it is readily to see that for \(x \in \partial D\),
\[
|u^*(x; z) - u^{*,\delta}_N(x; z)| \leq \sum_{|n| > N} \left| \frac{J_n(kr)}{J_n(kR)} \right| \hat{u}_n(z) \]
\[
\leq 6 \sum_{|n| > N} \left| \frac{r}{\rho} \right| |\hat{u}_n(z)| \]
\[
\leq 6 \sum_{|n| > N} \left| \frac{\rho - \mu}{\rho} \right| |\hat{u}_n(z)| \]
\[
\leq 6 \sigma_2^{-N} \left( \sum_{|n| \geq 1} \sigma_2^{-|n|} \right)^{1/2} \left( \sum_{|n| > N} |\hat{u}_n(z)|^2 \right)^{1/2} \]
\[
\leq \tilde{K}_3 \sigma_2^{-N} \|u^*\|_{L^2(\partial B_R)},
\]
where \(\mu = \text{dist}(\partial B_\rho, \partial D)\), \(\sigma_2 = \frac{\rho - \mu}{\rho}\) and \(\tilde{K}_3 = 6\sqrt{2}/\sqrt{\sigma_2^2 - 1}\). This implies
\[
\|u^* - u^{*,\delta}_N\|_{L^2(\partial D)} \leq K_3 \sigma_2^{-N}, \tag{53}
\]
where \(K_3 = \tilde{K}_3 \|u^*\|_{L^2(\partial B_\rho)} |\partial D|^{1/2}\). Similarly, by using \(45\) and \(46\), we have that for \(x \in \partial D\),
\[
|u^{*,\delta}_N(x; z) - u^{*,\delta}_N(x; z)| \leq \frac{1}{2 \pi R} \sum_{n = -N}^{N} \left| \frac{J_n(kr)}{J_n(kR)} \right| \|u^* - u^{*,\delta}_N, e^{i\theta}\|_{\partial B_R} \]
\[
\leq \frac{1}{\sqrt{2 \pi R}} \sum_{n = -N}^{N} \left| \frac{J_n(kr)}{J_n(kR)} \right| \|u^*\|_{L^2(\partial B_R)} \]
\[
\leq \|u^*\|_{L^2(\partial B_R)} \delta \sum_{n = -N}^{N} \frac{1}{\sqrt{2 \pi R}} \left( \frac{\rho - \mu}{R} \right)^{|n|} \]
\[
\leq \tilde{K}_4 \delta \sigma_3^{N+1},
\]
where \(\sigma_3 = \frac{\rho - \mu}{R}\) and \(\tilde{K}_4 = \frac{6\sqrt{2}\|u^*\|_{L^2(\partial B_R)}}{3\pi R(\sigma_3 - 1)}\). This, together with \(53\), yields
\[
\|u^* - u^{*,\delta}_N\|_{L^2(\partial D)} \leq \|u^* - u^{*,\delta}_N\|_{L^2(\partial D)} + \|u^{*,\delta}_N - u^{*,\delta}_N\|_{L^2(\partial D)} \]
\[
\leq K_3 \sigma_2^{-N} + K_4 \delta \sigma_3^{N},
\]
where $K_4 = \tilde{K}_4 \sigma_3 |\partial D|^{1/2}$, which implies (48).

From the formula $J'_n(t) = J_{n-1}(t) - nJ_n(t)/t$, it can be seen that

$$|\nabla J_n(kr)| \leq k |J'_n(kr)| \leq k |J_{n-1}(kr)| + \frac{|n|}{r} |J_n(kr)|,$$

which, together with $|\nabla e^{i\theta}| = |n|/r$, implies

$$|\nabla (J_n(kr)e^{i\theta})| \leq k |J_{n-1}(kr)| + \frac{2|n|}{r} |J_n(kr)|. \quad (54)$$

Now, by using estimate (46), we deduce that for $x \in \partial D$,

$$\left| \frac{\partial}{\partial \nu} u^s(x; z) - \frac{\partial}{\partial \nu} u_N^s(x; z) \right| \leq \sum_{|n| > N} |\hat{u}_n(z)| \left| \frac{\nabla (J_n(kr)e^{i\theta})}{J_n(kr)} \right| \leq \sum_{|n| > N} |\hat{u}_n(z)| \left( k \left| \frac{J_{n-1}(kr)}{J_n(kr)} \right| + \frac{2|n|}{r} \left| \frac{J_n(kr)}{J_n(kr)} \right| \right) \leq \frac{24}{R} \sum_{|n| > N} |n| \left( \frac{\rho - \mu}{\rho} \right)^{|n|} |\hat{u}_n(z)| \leq \frac{24}{R} N \sigma_2^{-N} \left( \sum_{|n| \geq 1} n^2 \sigma_2^{-2|n|} \right)^{1/2} \left( \sum_{|n| > N} |\hat{u}_n(z)|^2 \right)^{1/2} \leq \tilde{K}_5 N \sigma_2^{-N} \|	ext{u}^s\|_{L^2(\partial B_r)}.$$

where $\tilde{K}_5 = \frac{24}{R} \left( \sum_{|n| \geq 1} n^2 \sigma_2^{-2|n|} \right)^{1/2}$. This implies

$$\left\| \frac{\partial}{\partial \nu} u^s(x; z) - \frac{\partial}{\partial \nu} u_N^s(x; z) \right\|_{L^2(\partial D)} \leq \tilde{K}_5 N \sigma_2^{-N}, \quad (55)$$

where $K_5 = \tilde{K}_5 \|	ext{u}^s\|_{L^2(\partial B_r)} |\partial D|^{1/2}$. Further, from estimates (46) and (54), we see that for $x \in \partial D$,

$$\left| \frac{\partial}{\partial \nu} u_N^s(x; z) - \frac{\partial}{\partial \nu} u_N^{\delta}(x; z) \right| \leq \frac{1}{2\pi R} \sum_{n=-N}^{N} \left| \frac{\nabla (J_n(kr)e^{i\theta})}{J_n(kr)} \right| \left| \langle \text{u}^s - u_N^{\delta}, e^{i\theta} \rangle_{\partial B_R} \right| \leq \frac{1}{\sqrt{2\pi R}} \sum_{n=-N}^{N} \left( k \left| \frac{J_{n-1}(kr)}{J_n(kr)} \right| + \frac{2|n|}{r} \left| \frac{J_n(kr)}{J_n(kr)} \right| \right) \delta \|	ext{u}^s\|_{L^2(\partial B_R)} \leq \frac{\|\text{u}^s\|_{L^2(\partial B_R)} \delta}{\sqrt{2\pi R}} \sum_{n=-N}^{N} \frac{24|n|}{R} \left( \frac{\rho - \mu}{R} \right)^{|n|} \leq \frac{12\sqrt{2} \|\text{u}^s\|_{L^2(\partial B_R)} \delta N}{R \sqrt{\pi R}} \sum_{n=-N}^{N} \left( \frac{\rho - \mu}{R} \right)^{|n|} \leq \tilde{K}_6 \delta N \sigma_3^{N+1},$$

19
where $\bar{K}_6 = \frac{24\sqrt{2}\|u^s\|_{L^2(\partial D)}}{R\sqrt{\pi R_0(\sigma_3-1)}}$. This, together with (55), yields

$$
\|\partial_\nu u^s - \partial_\nu u^s_N\|_{L^2(\partial D)} \leq \|\partial_\nu u^s - \partial_\nu u^s_N\|_{L^2(\partial D)} + \|\partial_\nu u^s_N - \partial_\nu u^s\|_{L^2(\partial D)} \\
\leq K_5 N \sigma_2^{-N} + K_6 N \sigma_3 N,
$$

where $K_6 = \bar{K}_6 \sigma_3 |\partial D|^{1/2}$, which implies (49).

For $0 < \delta < \sigma_1^{-N_0}$, from $N = \left[\frac{1}{\ln \sigma_1} \ln \frac{1}{\delta}\right]$, we see $N \geq N_0$. Substituting $N = \left[\frac{1}{\ln \sigma_1} \ln \frac{1}{\delta}\right]$ into (47), (48), and using $\ln t < t$ for $t > 1$, $\sigma_3 < \sigma_1$, we derive the estimates (50) and (51). The proof is complete.

With the approximation of the scattered fields $u^s(x; z)$ by $u^s_N(x; z)$ in (45), we achieve the approximating indicator functions $I_{s,N}(x)$ and $I_{h,N}(x)$, which have the same forms as those in (31) - (32). And by analogous arguments in Theorem 3.3, we have the following stability result on the indicator functions for the inverse cavity scattering.

**Theorem 3.4.** Let $\sigma_1, \sigma_2$ and $N_0$ be defined in Theorem 3.3. Then for $0 < \delta < \sigma_1^{-N_0}$ and $N = \left[\frac{1}{\ln \sigma_1} \ln \frac{1}{\delta}\right]$, there exist positive constants $K_s$ and $K_h$, such that

$$
I_{s,N}(x) \leq K_s \delta^\beta,
$$

$$
I_{h,N}(x) \leq K_h \delta^\beta |\ln \delta|,
$$

where $\beta = \ln \sigma_2/\ln \sigma_1$.

### 4 Numerical algorithm and examples

Based on the indicator functions in the previous sections, the direct imaging scheme for reconstructing the shape of scatterer is given in the following Algorithm.

| Algorithm: Reconstruction by the Fourier-Bessel approximated imaging scheme |
|---|
| **Step 1** | Given a monochromatic frequency or a spectrum of frequencies, collect the corresponding noisy near-field data due to the point sources and the obstacle/cavity with sound-soft or sound-hard boundary condition. |
| **Step 2** | Choose an approximate truncation $N$ and compute the approximate scattered field by the Fourier-Hankel expansion (41) or Fourier-Bessel expansion (43). |
| **Step 3** | Select a suitable imaging mesh $T$ covering the target scatterer. In terms of the boundary condition, for each imaging point $x \in T$, evaluate the perturbed indicator functions $I_{s,N}(x)$ or $I_{h,N}(x)$, according to (31) and (32) for the obstacle or (41) and (42) for the cavity. |
| **Step 4** | The boundary of the scatterer can be recovered as the zeros of the monochromatic indicator functions or their superpositions with respect to the frequencies. |

In this section, we present several numerical examples of the inverse obstacle/cavity scattering problems in two dimensions to illustrate the effectiveness and applicability of our algorithm. In our
numerical experiments, the boundary of the model scatterer is represented by the parametric form

\[ x(t) = (x_1(t), x_2(t)), \quad 0 \leq t < 2\pi \]

which is a simple smooth and closed curve. We compute the synthetic near-field data by solving

the forward problems via the boundary integral equation method \[8\]. The exact boundary curves

of the scatterers are listed in the parametric form and plotted in Figure 3.

With the purpose of testing the stability of the proposed method, we further add some random
noise to the computed scattered data by

\[ u^{s,\delta} = u^s + \delta r_1 |u^s| e^{i\pi r_2} \]

where \( r_1, r_2 \) are two uniformly distributed random number ranging from −1 to 1, and \( \delta > 0 \) denotes
the relative noise level.

For the ease of visualization and comparison, the reciprocals (multiplicative inverse) of the
normalized indicator functions are depicted in the following figures. Consequently, the profile of
the scatterer can be qualitatively reconstructed as the significant peak levels in the images. As
a rule of thumb, truncation of the Fourier-Bessel expansion is chosen as \( N = \lceil \ln \delta \rceil + 1 \) for the
exterior problem and \( N = \lceil 1.5 \ln \delta \rceil + 1 \) for the interior problem.

4.1 Exterior problems

In this part for the exterior problems, we present the results for imaging the obstacles with
sound-soft and sound-hard boundary conditions. We collect the scattered data on the measurement
circle centered at the origin with radius 2.2. In each example, 12 point sources and 128 receivers
are equally placed on the measurement circle. These sources and receivers are marked by the small
red and blue points, respectively. Hereafter, the boundary of exact obstacle is denoted by the black
dashed lines. The imaging domain is chosen as \([-1.5, 1.5]^2\) with \(150 \times 150\) equally spaced imaging
grid.
Example 4.1. The goal of the first example is to reconstruct the sound-soft obstacles. Here wavenumbers $k = 3$ and $k = 6$ are used. The reconstructions are shown in Figure 4 and it can be seen that the boundary curves are well recovered by the proposed imaging method.

Example 4.2. In the second example, we present the reconstructions of sound-hard obstacles. We respectively choose $k = 4$ and $k = 5$ for the single-frequency indicator and $k = 3 + \ell/2, (\ell = 0, 1, \cdots, 6)$ for the multi-frequency indicator. Moreover, for the multi-frequency case, the final reconstruction is produced by the superposition of each normalized single-frequency indicators. The results are plotted in Figure 5.

It deserves noting that, no matter what the boundary type is, the sharp corners facing outside are superiorly reconstructed. This is because these corners are relatively more closer to the sources and receivers. In other words, they could be more adequately illuminated by the point sources in certain sense. We would like to remark that, if the incident wave is alternatively chosen as the plane wave, then the same indicator functions can be defined and our algorithm procedure remains valid. The plane wave incidence would make the scattered information be captured in a more balanced manner and consequently improve the reconstructions. Since the case of plane wave is beyond the current study, the numerical results with plane wave will be reported in a future work.
Figure 5: Reconstruction of sound-hard obstacles with 2% noise. Top row: $k = 4$; Middle row: $k = 5$; Bottom row: superposition with multiple frequencies $k = 3, 3.5, \cdots, 6$. 
4.2 Interior problems

Similar to the preceding examples for the exterior problems, we present the reconstruction results for the cavities in this subsection. Both the sources and receivers are located on an interior detection curve, which is chosen as the circle centered at the origin with radius 0.5. Without otherwise specified, all the parameters used here are the same as the exterior problems. The imaging domain is chosen as $[-1.5, 1.5]^2$ with 150 $\times$ 150 equally spaced imaging grid excluding the interior of measurement circle. For the sake of comparison, we also present the reconstructions of previous circle, kite- and starfish-shaped scatterers.

Example 4.3. In this example, we consider the recovery of sound-soft cavities with wavenumbers $k = 3$ and $k = 5$ with $\delta = 0.05$. The reconstructions are illustrated in Figure 6. One can observe that the circle and the starfish are relatively well recovered while two wings of the kite are far less accurate. Physically speaking, this phenomenon is probably due to the fact that the energy of point sources decays rapidly with respect to the increasing of propagating distance and the wavenumber. Hence, the scattered signals become much weaker when any portion of cavity recedes from the sources. As a result, one should not expect to obtain a large amount of information from the far side of cavity and thus the corresponding reconstruction inevitably deteriorates. Similar effects and discussions can be applied to the other examples regardless of boundary conditions.

Example 4.4. In the last example, we study the reconstruction of cavities with Neumann boundary condition. The imaging results are given in Figure 7. It is worthwhile noting that the sound-hard
indicators are in general more sensitive to the noise-contaminated data than the sound-soft ones. In our view, the main reason is the incorporation of additional evaluation on the normal derivatives in the formulation of sound-hard indicators.

5 Conclusion

In this work, we propose a fast computational scheme for solving the inverse acoustic scattering problems. By the a priori sound-soft or sound-hard boundary condition of the scatterer and the Fourier-Bessel approximation of the scattered field, some novel indicator functions are proposed. Then the shape of target obstacle or cavity can be recovered by locating the zeros of associated imaging functions. Theoretical analysis is given to justify the rationale behind the algorithm and simulation experiments are conducted to validate its applicability.

This is an initial attempt in developing a Fourier-Bessel based imaging framework for tackling the inverse scattering problems. Clearly, several computational issues deserve further investigation, for example, is there any optimal choice for the truncation \( N \) in the Fourier-Bessel expansion? How to select an appropriate frequency spectrum for a specific inverse scattering problem? Can we improve the accuracy and stability of the reconstructions for sound-hard scatterers? Concerning the future work, feasible extensions include the application of the imaging method to three-dimensional problems or more complicated physical configurations such as mixed boundary conditions. Further extensions to the scenarios of electromagnetic or elastic waves are also interesting topics.

Acknowledgments

The work of D. Zhang, Y. Wu and Y. Wang were supported by NSFC grant 12171200. The work of Y. Guo was supported by NSFC grant 11971133 and the Fundamental Research Funds for the Central Universities.

References

[1] Bao G, Li P, Lin J and Triki F 2015 Inverse scattering problems with multi-frequencies Inverse Problems 31 093001

[2] Bao G and Zhang L 2016 Shape reconstruction of the multi-scale rough surface from multi-frequency phaseless data Inverse Problems 32 085002

[3] Blästen, E and Liu H 2020 Recovering piecewise constant refractive indices by a single far-field pattern. Inverse Problems 36 085005

[4] Cakoni F and Colton D 2006 Qualitative Methods in Inverse Scattering Theory (Berlin: Springer-Verlag)

[5] Cao X, Diao H, Liu H and Zou J 2020 On nodal and generalized singular structures of Laplacian eigenfunctions and applications to inverse scattering problems J. Math. Pures Appl. 143(9) 116–161

[6] Chen Z and Huang G 2017 Phaseless imaging by reverse time migration: acoustic waves Numer. Math. Theor. Meth. Appl. 10 1–21

25
Figure 7: Reconstruction of sound-hard cavities with 2% noise. Top row: $k = 4$; Middle row: $k = 5$; Bottom row: superposition with multiple frequencies $k = 3, 3.5, \cdots, 6$. 
7 Colton D and Kress R 2018 Looking back on inverse scattering theory. *SIAM Review* **60**(4) 779–807
8 Colton D and Kress R 2019 *Inverse Acoustic and Electromagnetic Scattering Theory 4th ed.* (Cham: Springer-Verlag)
9 Diao H, Cao X and Liu H 2021 On the geometric structures of transmission eigenfunctions with a conductive boundary condition and applications. *Comm. Partial Differential Equations* **46** 630–679
10 Dong H, Zhang D and Guo Y 2019 A reference ball based iterative algorithm for imaging acoustic obstacle from phaseless far-field data *Inverse Problems and Imaging* **13**(1), 177–195
11 Grebenkov D and Nguyen B 2013 Geometrical structure of Laplacian eigenfunctions *SIAM Rev.* **55** 601–667
12 Hu Y, Cakoni F and Liu J 2014 The inverse scattering problem for a partially coated cavity with interior measurements *Appl. Anal.* **93** 936–956
13 Ivanyshyn O and Kress R 2010 Identification of sound-soft 3D obstacles from phaseless data *Inverse Probl. Imaging* **4** 131–149
14 Ivanyshyn O and Kress R 2011 Inverse scattering for surface impedance from phaseless far field data *J. Comput. Phys.* **230** 3443–3452
15 Jakubik P and Potthast R 2008 Testing the integrity of some cavity-the Cauchy problem and the range test *Appl. Numer. Math.* **58** 899–914
16 Ji X, Liu X and Zhang B 2019 Target reconstruction with a reference point scatterer using phaseless far field patterns *SIAM J. Imaging Sciences* **12**(1) 372–391
17 Ji X, Liu X and Zhang B 2019 Phaseless inverse source scattering problem: phase retrieval, uniqueness and direct sampling methods *Journal of Computational Physics: X* **1** 100003
18 Klibanov M V and Romanov V G 2017 Uniqueness of a 3-D coefficient inverse scattering problem without the phase information *Inverse Problems* **33** 095007
19 Li J and Liu H 2015 Recovering a polyhedral obstacle by a few backscattering measurements *J. Differential Equat.* **259** 2101–2120
20 Li J, Liu H and Wang Y 2017 Recovering an electromagnetic obstacle by a few phaseless backscattering measurements *Inverse Problems* **33** 035001
21 Li J, Liu H and Zou J 2009 Strengthened linear sampling method with a reference ball *SIAM J. Sci. Comput.* **31**(6) 4013–4040
22 Liu M, Zhang D, Zhou X and Liu F 2017 The Fourier-Bessel method for solving the Cauchy problem connected with the Helmholtz equation *J. Comput. Appl. Math.* **311** 183–193
23 Liu X 2014 The factorization method for cavities *Inverse Problems* **30** 015006
24 Maretzke S and Hohage T 2017 Stability estimates for linearized near-field phase retrieval in X-ray phase contrast imaging *SIAM J. Appl. Math.* **77** 384-408
[25] Qin H and Cakoni F 2011 Nonlinear integral equations for shape reconstruction in the inverse interior scattering problem *Inverse Problems* **27** 035005

[26] Qin H and Colton D 2012 The inverse scattering problem for cavities *Appl. Numer. Math.* **62** 699–708

[27] Qin H and Colton D 2012 The inverse scattering problem for cavities with impedance boundary condition *Adv. Comput. Math.* **36** 157–174

[28] Qin H and Liu X 2015 The interior inverse scattering problem for cavities with an artificial obstacle *Appl. Numer. Math.* **88** 18–30

[29] Sun Y, Guo Y and Ma F 2016 The reciprocity gap functional method for the inverse scattering problem for cavities *Appl. Anal.* **95**(6) 1327–1346

[30] Zhang B and Zhang H 2017 Recovering scattering obstacles by multi-frequency phaseless far-field data *J. Comput. Phys.* **345** 58–73

[31] Zhang B and Zhang H 2018 Fast imaging of scattering obstacles from phaseless far-field measurements at a fixed frequency *Inverse Problems* **34** 104005

[32] Zhang D and Guo Y 2015 Fourier method for solving the multifrequency inverse source problem for the Helmholtz equation *Inverse Problems* **31** 035007

[33] Zhang D and Guo Y 2018 Uniqueness results on phaseless inverse scattering with a reference ball *Inverse Problems* **34** 085002

[34] Zhang D and Guo Y 2021 Some recent developments in the unique determinations in phaseless inverse acoustic scattering theory *Electronic Research Archive* **29**(2) 2149–2165

[35] Zhang D, Guo Y, Li J and Liu H 2018 Retrieval of acoustic sources from multi-frequency phaseless data *Inverse Problems* **34** 094001

[36] Zhang D, Sun F, Ma Y and Guo Y 2020 A Fourier-Bessel method with a regularization strategy for the boundary value problems of the Helmholtz equation. *Journal of Computational and Applied Mathematics*, **368** 112562

[37] Zhang D, Sun W 2016 Stability analysis of the Fourier-Bessel method for the Cauchy problem of the Helmholtz equation, *Inverse Probl. Sci. Eng* **24** (4) 583–603

[38] Zhang D, Wang Y, Guo Y and Li J 2020 Uniqueness in inverse cavity scattering problems with phaseless near-field data, *Inverse Problems* **36** 025004

[39] Zeng F, Cakoni F and Sun J 2011 An inverse electromagnetic scattering problem for a cavity *Inverse Problems* **27** 125002

[40] Zeng F, Suarez P and Sun J 2013 A decomposition method for an interior inverse scattering problem, *Inverse Probl. Imaging* **7**(7) 291–303

28