Investigating stability of a class of black hole spacetimes under Ricci flow

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Abstract
We prove the linear stability of Schwarzschild–Tangherlini spacetimes and their anti-de Sitter counterparts under Ricci flow for a special class of perturbations. This is useful in the choice of suitable initial conditions in numerical Ricci-flow-based algorithms for obtaining new solutions to the Einstein equation when the cosmological constant is zero or negative. The Ricci flow is a first-order renormalization group (RG) flow in string theory, and its solutions are believed to approximate string field theory processes in certain cases. Thus, this result offers insights into the off-shell stability of these Euclidean black hole geometries in string theory, as well as in the Euclidean path integral approach to quantum gravity.

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1. Introduction

There have been numerous studies of the classical stability of a black hole and black brane spacetimes under gravitational perturbations. It is well known that Schwarzschild black holes are classically stable [1]. The classical stability of the higher dimensional Schwarzschild–Tangherlini and AdS–Schwarzschild–Tangherlini black holes for a class of perturbations has been shown by Gibbons and Hartnoll [2] and a stability result under all classes of perturbations obtained by Ishibashi and Kodama [3].

There are also many distinct motivations for studying the off-shell stability of black hole spacetimes. What is meant by the term ‘off-shell’ varies depending on the context. The subject of this paper is a study of the off-shell stability (in a certain sense) of Schwarzschild–Tangherlini and AdS–Schwarzschild–Tangherlini black holes. More precisely, we study the linear stability of these black hole spacetimes under Ricci flow. We discuss the various...
motivations for this study, and in the process, we review the various notions of off-shell stability and their connections to physics.

1.1. Euclidean path integral formulation of quantum gravity

One of the earliest motivations comes from the Euclidean path integral formulation of quantum gravity. Classical configurations are stationary points of the gravity action, and off-shell configurations mediate in a quantum tunnelling from one classical configuration to another. In this context, off-shell perturbations of a classical configuration that make the action negative lead to instabilities at least in the semiclassical approximation to quantum gravity. This has motivated a study of the off-shell linear (in)stability of the Euclidean Schwarzschild instanton—this instanton was shown by Gross, Perry and Yaffe (GPY) [4] to have an unstable off-shell mode. Computationally, GPY find a normalizable eigenmode of the Lichnerowicz Laplacian (2.6) for the Euclidean Schwarzschild metric with a negative eigenvalue. The perturbation described by this negative mode is off-shell (the perturbed geometry is not a solution to the Einstein equation in this linearized approximation). This perturbation is therefore evidence of the off-shell instability of the Euclidean Schwarzschild instanton in semiclassical gravity. It was argued by Reall [5] that this ‘quantum’ instability is in fact the same as the classical Gregory–Laflamme instability of the uncharged black $p$-brane whose metric splits into the four-dimensional Schwarzschild metric and the flat metric in $p$ dimensions [6]. Gregory and Laflamme assume a special ansatz for the perturbation (periodic along the $p$ directions) such that the classical unstable mode of the $p$-brane is the same as the negative eigenmode of the Lichnerowicz Laplacian on the four-dimensional Schwarzschild instanton obtained after compactifying $p$ directions. Arguments about the existence of a GPY-type negative mode for the Lichnerowicz Laplacian of higher dimensional black holes can be found in [7]. These results have close connections to the linear stability of these black holes under Ricci flow—the operator describing the linearized flow of the perturbation of the black hole spacetime is the Lichnerowicz Laplacian (or related to it). Thus, instability under Ricci flow can be due to the Lichnerowicz operator having a negative mode. This would then also have implications for the classical stability of corresponding $p$ branes.

1.2. Numerical Ricci flow-based algorithm for obtaining new solutions to the Einstein equation

The Ricci–de Turck flow ((2.1), with $\alpha' = 2$) is a geometric flow equation well studied in mathematics and was used to successfully resolve the Poincare conjecture. Ricci-flat metrics are fixed points of this flow, which is a nonlinear parabolic PDE. Recently, a numerical algorithm has been proposed for obtaining new solutions to the vacuum Einstein equation—this aims to use the Ricci flow through static spacetimes to converge to new Ricci-flat solutions of the Einstein equation [8, 9]. The algorithm involves a numerical simulation of the Ricci flow on static spacetimes starting from fine-tuned initial data—the hope being that if the initial data are appropriately chosen, the Ricci flow will converge to a Ricci flat fixed point. Without careful choice of initial data, the Ricci flow could become singular, or may not converge to a Ricci-flat fixed point. This problem is illustrated by a numerical simulation by Headrick and Wiseman [8] of the Ricci flow of a (Euclidean) Schwarzschild black hole in a radial box. If the ratio of horizon radius to box radius is less than $2/3$, this black hole is the ‘small’ black hole and its Lichnerowicz operator has a negative mode (as discussed before). If this ratio is greater than $2/3$ (the large black hole), this negative mode is not present. The negative mode of the small black hole is not an artifact of putting the black hole in a box. When the box
radius is taken to infinity, this mode persists and is precisely the GPY mode. In [8], the small black hole perturbed by this unstable mode is chosen to be the initial data for a numerical Ricci flow simulation. Surprisingly, the authors find that the numerical simulation yields different results depending on whether the perturbation is added to the small black hole metric, or subtracted from it. In one case, the metric flows to the large black hole and in the other case, it becomes singular. Thus, the moral of this example seems to be that for the algorithm to work; the initial metric cannot be an arbitrary static metric. For example, one could choose a perturbation of a known Ricci-flat metric, which leads to an instability, causing a flow away from the known Ricci-flat metric. Thus, knowledge of both the number of unstable modes of Ricci-flat static metrics and the exact form of the unstable mode is required for choosing the right initial data that will lead to convergence of the Ricci flow. This instability under Ricci flow is an ‘off-shell’ instability in a sense, since the perturbed metric does not solve the Einstein equation. The algorithm in [9] could be extended to finding new solutions to Einstein equation with cosmological constant by using flow (2.3) instead of the Ricci flow.

This is one of the main motivations for this paper, where we examine the linear stability of Schwarzschild–Tangherlini black holes (and their AdS analogs) under Ricci flow or flow (2.3). As we also show in section 4, this is related to the problem of eigenmodes of the Lichnerowicz Laplacian on these spacetimes.

1.3. Off-shell stability in string theory

Finally we discuss yet another unrelated motivation for studying the stability of the Schwarzschild–Tangherlini black holes under Ricci flow. The Ricci flow arises naturally as a first-order world-sheet renormalization group (RG) flow in closed string theory. It has been conjectured that off-shell processes in string field theory such as tachyon condensation are approximated qualitatively by solutions to world-sheet RG flows in string theory (see [12] for a review). There is some evidence for this—for example, the tachyon condensation process leading to the geometry change $\mathbb{C}/\mathbb{Z}_n \to \mathbb{C}$ is described in [13], and the exact solution to the Ricci (first-order RG) flow which describes this geometry change is one of the Kähler–Ricci solitons of Cao [14] (this solution also appears in different coordinates in [15]). One of the arguments in support of this conjecture is that in many cases, in the CFT describing the fixed point of the RG flow (a vacuum or on-shell geometry), it is possible to construct operators that are relevant perturbations of the fixed point (causing an RG flow) and also, tachyonic. This suggests an investigation of the off-shell (in)stability of a vacuum geometry by studying its (in)stability under a suitable RG flow—for example, the Ricci flow. This proposal was suggested in [16] where the linear stability of Euclidean AdS space (hyperbolic space $\mathbb{H}^n$) under Ricci flow was proved. We continue this program here by investigating the stability of black hole geometries under Ricci flow. The Euclidean Schwarzschild–Tangherlini geometry is a fixed point of the first-order world-sheet RG (Ricci) flow. Of course, the Lorentzian black hole geometry has a curvature singularity, and therefore, the perturbative sigma model analysis of this target space geometry ought to break down at the singularity. However, it is useful to recall that the standard $\beta$ function and RG flow computations for world-sheet sigma models assume that the metric is Riemannian (Wick-rotated Lorentzian metric). In fact, the first-order RG flow is not well behaved when considered as a PDE on general Lorentzian spacetimes. Thus, the RG flow should be thought of as a flow through a class of spacetimes for which Wick-rotation makes sense (static or at least stationary)—such as the spacetime

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4 This problem is not faced in a similar simulation of the Kähler–Ricci flow on the third del-Pezzo surface as in [10]. In this case, a theorem of Tian and Zhu [11] guarantees that starting from any initial Kähler metric on this surface (obeying a certain condition), the flow converges exponentially to a Kähler–Einstein metric.
outside the horizon for the Schwarzschild–Tangherlini black holes\(^5\). Analogous to classical stability calculations, it would be interesting to analyze the stability of the Schwarzschild metric treating the horizon as a boundary. If the metric is unstable under first-order RG flow, then the natural question to pursue is what the end-point of the flow is.

In this paper, we prove the linear stability of (the spacetime outside the horizon of) the Schwarzschild–Tangherlini black holes and their AdS counterparts under Ricci flow for a special class of static perturbations. We have already seen several motivations for this stability investigation. We also show that given an arbitrary static perturbation, these black holes do not have any instability in the flow of the trace of the perturbation. In section 2, we discuss the Ricci flow and a linearized analysis of the evolution of a perturbation of a Ricci-flat geometry under Ricci flow. We also describe how the evolution of perturbations of any Einstein metric under Ricci flow can be studied. We then discuss details such as gauge-fixing for simplifying the analysis of the flow of the perturbation. In section 3, we restrict to a special class of perturbations and discuss the flow equation for the perturbation of Schwarzschild–Tangherlini black holes (and their AdS counterparts) in this case. Section 4 contains the analysis of the flow equation by assuming an ansatz, and proving that there are no normalizable unstable perturbations of this form for both classes of black holes. Sections 3 and 4 are the analogs of the classical stability results of Gibbons and Hartnoll [2], and we try to follow a similar notation wherever possible. Section 5 provides a more rigorous argument (without the need to choose a special ansatz) that shows that perturbations of compact support do not grow under the linearized flow. This is analogous to the result of Wald [17] in the proof of the classical stability of the Schwarzschild black hole. Finally, in the last section, we summarize our results and discuss how to extend them to investigate the stability of the Schwarzschild–Tangherlini black holes under all classes of perturbations.

2. A linearized analysis of stability

The Ricci flow, which is a subject of active research in mathematics, is also the simplest lowest order (in square of string length \(\alpha'\)) RG flow of the world-sheet sigma model for closed strings. In this context, the Ricci flow is the flow of the metric of the target space with respect to the RG flow parameter \(\tilde{\tau}\). As mentioned earlier, this flow is not well behaved when considered as a flow through general Lorentzian spacetimes. It should be thought of as a flow through Riemannian geometries which are Wick-rotated Lorentzian spacetimes. Both in physics and in mathematics, we are interested in a flow of metrics mod diffeomorphisms\(^6\). Therefore, all flows related to each other by \(\tilde{\tau}\)-dependent diffeomorphisms generated by a vector field \(V\) are equivalent; we write the generic flow in this class—the \textit{Ricci–de Turck} flow, as

\[
\frac{\partial \tilde{g}_{ab}}{\partial \tilde{\tau}} = -\alpha' (\tilde{R}_{ab} + \tilde{\nabla}_a V_b + \tilde{\nabla}_b V_a)
\]  

(2.1)

The Ricci flow in two and three dimensions on compact manifolds is now well understood—with curvature conditions on an initial geometry, much is known about the limiting geometry under Ricci flow [18, 19]. This is not the case for solutions to the Ricci flow in higher dimensions, or on noncompact manifolds. Stability results for geometries under small perturbations are therefore useful in such cases (for some stability results for geometries under Ricci flow, see [16, 20] and references therein).

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5 Since the first-order (Ricci) flow preserves isometries, if the initial spacetime is static, we are guaranteed that the solution will be in this class.

6 In the mathematics literature, \(\alpha' = 2\).
The background metrics whose linear stability problem we are interested in are a class of static metrics on $D$-dimensional spacetimes, given by
\[ ds^2 = -f(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 d\tilde{s}_d^2 \] (2.2)
where $d\tilde{s}_d^2$ is a Riemannian metric on a $d = (D - 2)$-dimensional ‘base’ manifold $B$. Further these metrics are Einstein, and are thus solutions to the Einstein equations with a cosmological constant.

The flow of Lorentzian spacetimes is not a well-posed problem in general even in a linearized approximation. We will restrict ourselves to the class of static perturbations of the spacetimes—for this class, the (linearized) flow is well posed.

It is more convenient to study the stability of a geometry under a geometric flow when it is a fixed point of the flow. However, Einstein metrics with a non-zero Ricci tensor (i.e. non-zero cosmological constant) are not fixed points of the Ricci flow, so we describe a technique which can be used to study the stability of such metrics. This is useful in extending the algorithms in [9] to obtain new Einstein spacetimes with non-zero cosmological constant. Further, it is expected that at least some Einstein metrics may be fixed points of string theory RG flows with background fields (AdS$_3$ is a fixed point of the RG flow with a $B$ field). In the absence of a full knowledge of string theory $\beta$ functions with background fields (like the RR field), we can still discuss a notion of stability of Einstein metrics under Ricci flow (i.e. is the Einstein metric an attractor on the space of solutions to the Ricci flow?). The result is expected to be indicative of stability under an appropriate RG flow at least with respect to metric perturbations.

Einstein metrics have a simple evolution under Ricci flow—the metric either expands or contracts uniformly (depending on the sign of the cosmological constant) by a conformal factor. One can study if perturbations of the Einstein space decay, and if the perturbed geometry approaches the Einstein space under Ricci flow (up to overall scale). This notion of stability is called the geometric stability of the Einstein space with respect to the Ricci flow.

Let the Einstein metric have a Ricci tensor $R_{ab} = c(d + 1)g_{ab}$. Given the Ricci–de Turck flow (2.1), we consider the flow
\[ \frac{\partial g_{ab}}{\partial \tau} = -\alpha' \left[ R_{ab} - c(d + 1)g_{ab} \right] \] (2.3)
whose solutions are related to those of (2.1) by
\[ \tilde{\tau} = \frac{1}{\alpha' c} e^{\alpha' c(d + 1) \tau}, \quad \tilde{g}_{ab} = e^{\alpha' c(d + 1) \tau} g_{ab}. \] (2.4)

Flow (2.3) has the Einstein metrics with $R_{ab} = c(d + 1)g_{ab}$ as its fixed points. First we study the linear stability of the Einstein space under this flow. Then, by the rescalings (2.4) this leads to a result on the geometric stability of the space under Ricci flow (2.1). Ricci-flow-based numerical algorithms such as [9] to obtain solutions to the vacuum Einstein equations can be easily generalized to flow (2.3) in order to obtain new solutions to the Einstein equation with non-zero cosmological constant.

The linear stability problem under flow (2.3) can be set up as follows: given a background metric $g_{ab}$, a perturbed metric $\tilde{g}^a_{\ b} = g_{ab} + h_{ab}$. One studies the evolution of the perturbed metric $\tilde{g}^a_{\ b}$ in a linearized approximation. The linearized flow for the perturbation $h$ is
\[ \frac{\partial h_{ab}}{\partial \tau} = \frac{\alpha'}{2} \left[ -(\Delta_L h)_{ab} + \nabla_a \nabla_b H - \nabla_a (\nabla^c h_{cb}) - \nabla_b (\nabla^c h_{ca}) + 2\nabla_a V_b + 2\nabla_b V_a + 2c(d + 1)h_{ab} \right]. \] (2.5)
Here and in what follows, all covariant derivatives are taken with respect to the background metric \( g \). \( H = g^{ab} h_{ab} \) is the trace of the perturbation.

\[
(\Delta_L h)_{ab} = -\Delta h_{ab} + 2 R^d_{ab} h^d_{\cd} + R^c_{bc} h_{ac} + R^c_{bc} h_{ab}
\]  

(2.6)
is the Lichnerowicz Laplacian acting on symmetric two-tensors (all curvature tensors being those of the background metric). The convention we follow for the Lichnerowicz Laplacian is that of the physics literature, and differs from the mathematics one by a negative sign.

We can choose \( V \) so that we get rid of the divergence terms in (2.5), namely

\[
\frac{\partial h_{ab}}{\partial \tau} = -\frac{\alpha'}{2} [(\Delta_L h)_{ab} - 2 c (d+1) h_{ab}] = (L h)_{ab};
\]

(2.7)

We now need to ‘fix gauge’ in order to simplify our problem. The following results are useful for gauge-fixing in problems involving linearized stability analysis of Einstein spaces.

1. Let \( h_{ab} \) be a trace-free perturbation, i.e. \( g^{ab} h_{ab} = 0 \). \( g^{ab} (\Delta_L h)_{ab} = -\Delta H = 0 \), and therefore \( (L h)_{ab} \) is also trace free.
2. If the background metric \( g \) is Einstein, and \( \nabla^a h_{ab} = 0 \) (i.e. the perturbation is transverse), then \( \nabla^a (\Delta_L h)_{ab} = 0 \) and therefore, \( \nabla^a (L h)_{ab} = 0 \).

**Proof.** Consider \( \nabla^a (\Delta_L h)_{ab} \). Let \( h_{ab} \) be a transverse perturbation:

\[
\nabla^a (\Delta_L h)_{ab} = 2 \nabla^a (R^e_{\ cd}) h^d_{\ e} + 2 R^c_{\ abd} \nabla^a (h^d_{\ c})

+ \nabla^a (R_{ca}) h^c_{\ a} + \nabla^a (R_{cb}) h^c_{\ b} - \nabla^a (\Delta h)_{ab}.
\]

(2.8)

Then,

\[
\nabla^a (\nabla^c h)_{ab} = g^{ad} g^{ce} (\nabla_d \nabla_e h)_{ab} - R^{ad} \nabla_d h_{ab} + R^{de} \nabla_e h_{db} - R^{de} \nabla_e h_{db} = \nabla_e (R^{de}_{\ bc}) h_{ab} - \nabla^c (R_{dabc}) h^d_{\ e} - \nabla^c (R_{dabc}) h^d_{\ e} + R^{de} \nabla_e h_{db} - R_{dabc} \nabla^e h_{ad}.
\]

(2.9)

Now using the Bianchi identity \( \nabla_a R^a_{\ bcd} = \nabla_d R_{cb} - \nabla_b R_{cd} \), and (2.8) and (2.9), we obtain that

\[
\nabla^a (\Delta_L h)_{ab} = (\nabla_d R_{cb} + \nabla_e R_{db} - \nabla_b R_{cd}) h^d_{\ e}.
\]

(2.10)

This means that when the background metric \( g \) is Einstein, \( \nabla^a (\Delta_L h)_{ab} = 0 \) (in fact, this is true whenever the right-hand side of (2.10) is zero even if \( g \) is not Einstein). It obviously follows that \( \nabla^a (L h)_{ab} = 0 \). It therefore follows that if \( h_{ab} \) is transverse and traceless (TT), then so is \( (L h)_{ab} \).

3. Consider a perturbation of the form \( h_{ab} = \nabla_a W_b + \nabla_b W_a \), a ‘pure divergence’. Inserting this on the right-hand side of (2.5) with \( V = 0 \), when the background metric is Einstein, we get zero. Pure divergence perturbations do not flow; they are ‘zero modes’ of the linearized flow. Thus, there is no loss of generality in taking the perturbation \( h \) to be transverse. This is similar to gauge-fixing in relativity, but the difference is that the trace of the perturbation cannot be gauged away in Ricci flow stability problems.

A choice of gauge where the perturbation is transverse was used in the proof of linear stability of Euclidean AdS space (\( \mathbf{H}^n \)) under (2.3) (and consequently under Ricci flow) in [16].
We briefly sketch some steps in this computation that are similar to section 5 of our paper. The strategy is to define ‘energy integrals’ on $M = \mathbb{H}^n$, given by

$$E^{(K)} = \int_M |(L^Kh)_{ab}|^2 dV,$$

and prove an upper bound on these integrals under the flow (in terms of their initial values). Here the notation $(L^2h)_{ab}$ indicates, for example, $(LLh)_{ab}$. In fact, it is possible to prove that these integrals go to zero as $\tau \to \infty$. This is then used to prove that a certain Sobolev norm of the perturbation goes to zero as well in this limit, where the Sobolev norm $\|h\|_{k,2}$ is defined by

$$\|h\|_{k,2}^2 = \int_M |h_{ij}|^2 dV + \int_M |\nabla_p h_{ij}|^2 dV + \cdots + \int_M |\nabla_p \cdots \nabla_p h_{ij}|^2 dV.$$  \hspace{1cm} (2.12)

$|h_{ij}|^2$, for example, is the square of the (pointwise) tensor norm of the perturbation, i.e. $h^{ij}h_{ij}$. The Sobolev norm goes to zero under the flow for all $k$. The final step in the proof uses a Sobolev inequality on $\mathbb{H}^n$ that implies when this Sobolev norm goes to zero, the perturbation and all its derivatives go to zero pointwise in $M$.7 This program is hard to implement to prove the stability of other Einstein metrics partly because the bound on the energy integrals (2.11) was only possible due to the simple form of Riemann curvature for $\mathbb{H}^n$. Also, Sobolev inequalities are not known for most other Einstein manifolds. Nevertheless, the analysis given in section 5 of our paper bears some similarities to the steps above, where we derive and use a very simple Sobolev inequality.

Another strategy is to split a general perturbation into a trace-free part and a part proportional to the trace; $h_{ab} = H_{ab} + H_d r + 2g_{ab}$ ($H_{ab}$ denotes the trace-free part). Then the flow given by (2.7) naturally splits into separate flows for the trace-free part and the trace.

We can attempt to prove that the trace-free part and the trace decay under their respective flows, making the background geometry linearly stable. Finally, one can also split a perturbation more explicitly into a transverse traceless (TT) piece, a trace and the traceless part of a divergence piece, as $h_{ab} = h_{ab}^{TT} + H_{ab} + \nabla_a Y_b + \nabla_b Y_a - \frac{\nabla c}{\Delta r} g_{ab}$. Here $h_{ab}^{TT}$ is transverse and traceless (TT). We would then have to study the flows of $h_{ab}^{TT}, Y_a$ and $H$. Due to the fact that $(Lh_{ab}^{TT})$ is also TT, the flow of $h_{ab}^{TT}$ decouples from the other flows. In this paper, we are unable to address the full stability problem as it is computationally difficult. In the next section, we describe the class of perturbations for which we are able to obtain analytical stability results.

3. The flow for a special class of perturbations

Let us take the background metric to be of the form

$$ds^2 = - f(r) dt^2 + g(r) dr^2 + r^2 d\tilde{s}_d^2.$$  \hspace{1cm} (3.1)

Here $d\tilde{s}_d^2$ is the Riemannian metric on a $d$-dimensional compact ‘base manifold’ $B$. We will consider the background metric to be a static solution of the vacuum Einstein equation with a cosmological constant,

$$R_{ab} = c(d + 1)g_{ab},$$  \hspace{1cm} (3.2)

where we use Latin letters above for spacetime indices. This implies that the base manifold is also Einstein, with

$$\tilde{R}_{ab} = \epsilon(d - 1)\tilde{g}_{ab},$$  \hspace{1cm} (3.3)

Note that even if the Sobolev norm goes to zero, $h$ or its derivatives could still be non-zero on a set of measure zero; therefore, we need a Sobolev inequality to argue that they go to zero pointwise in $M$.  \hspace{1cm} 7
our convention being that the Greek letters label the coordinate indices on the base manifold. 

\[ \epsilon = \pm 1 \text{ or } \epsilon = 0 \] (in fact, if the spacetime is Ricci flat, \( B \) has to have positive curvature, so \( \epsilon > 0 \) [21]). Further, \( f(r)g(r) = 1 \) and \( f(r) = (\epsilon - (\alpha/r)^{d-1} - cr^2) \).

We write the perturbed metric as

\[ g_{ab} = g_{ab}^{(\text{background})} + h_{ab} \] (3.4)

where \( a \) and \( b \) run over \( d + 2 \) indices. As discussed in the previous section, we can decompose the perturbation into a traceless part and trace:

\[ h_{ab} = H_{ab} + \frac{1}{d + 2} g_{ab} H \] (3.5)

where \( H = g^{ab} h_{ab} \) and \( g^{ab} H_{ab} = 0 \).

Then, the flow of the trace \( H \) is easily obtained by taking the trace of (2.7) and is as follows.

3.1. Flow of the trace of an arbitrary static perturbation

\[ \frac{\partial H}{\partial \tau} = \frac{\alpha'}{2} [\Delta H + 2c(d + 1)H]. \] (3.6)

According to our conventions, the Laplacian operator on the Einstein spacetime (3.1) is \( \Delta = \nabla^a \nabla_a \). In the study of the trace of the perturbation, we will restrict to perturbations which are static, i.e. time independent. As mentioned in the introduction, this ensures that the flow equation for the perturbation is a well-behaved PDE. We can then study the flow of the trace without further restrictions on the perturbation. This analysis is presented in sections 4 and 5.

The next step would be to study the flow of the trace-free part \( H_{ab} \)—however, for a general static perturbation, this equation is difficult to analyze. Recalling that we can write \( h_{ab} = h_{ab}^{TT} + \frac{H}{d+2} g_{ab} + \nabla a Y_b + \nabla b Y_a - \frac{\gamma Y}{d+2} g_{ab} \) and that the flow of \( h_{ab}^{TT} \) decouples from the other flows, we can study the flow of static TT perturbations as a first step. Even this problem is hard in all generality. For computational reasons, we focus on static TT perturbations \( h_{ab}^{TT} \) that satisfy \( h_{ab}^{TT} = h_{ab}^{TT} = 0 \) for any spacetime index. These are the class of 'tensor' perturbations on \( S^d \)—the term 'tensor' perturbation refers to the fact that the perturbation transforms like a tensor of rank 2 on \( S^d \) when we do a coordinate transformation on \( S^d \). The full flow (2.7) is consistent with this restriction to TT 'tensor' perturbations (and is not consistent with restricting merely to trace-free perturbations satisfying \( h_{ab}^{TT} = h_{ab}^{TT} = 0 \)). The Lichnerowicz Laplacian acting on \( h_{ab}^{TT} \) can be written for this restricted class of perturbations and is the same as that obtained by Gibbons and Hartnoll [2] in the classical stability analysis of these black holes, except that the perturbations we consider are time independent. Under these restrictions, we can write the flow of \( h_{ab}^{TT} \).

3.2. Flow of TT 'tensor' perturbation on \( S^d \)

\[ \frac{\partial h_{ab}^{TT}}{\partial \tau} = \frac{\alpha'}{2} \left[ -\frac{1}{r^2} (\Delta_L h^{TT})_{ab} + f(r) \frac{d^2}{dr^2} h_{ab}^{TT} \right. \]

\[ \left. + \left( f'(r) - \frac{4 - d}{r} f(r) \right) \frac{d}{dr} h_{ab}^{TT} + \frac{4d}{r^2} h_{ab}^{TT} + 2c(d + 1) h_{ab}^{TT} \right]. \] (3.7)

To study the flow of a general TT perturbation, we will need to also study the flow of TT perturbations that are ‘vector’ and ‘scalar’ perturbations on \( S^d \) (i.e. transforming as rank 1 and rank 0 tensors respectively, under coordinate transformation in \( S^d \)). We discuss how to address the stability problem for a wider class of perturbations in the last section.
4. Perturbations of Schwarzschild–Tangherlini black holes and their AdS analogs

4.1. Schwarzschild–Tangherlini black holes

In (3.1), we now specialize to the case when \( c = 0 \). In this case, we must have \( \epsilon = 1 \). We will take \( B = S^d \). \( f(r) = (1 - (\alpha/r)^{d-1}) \) and we have the Schwarzschild–Tangherlini black holes.

4.2. Flow of trace for any static perturbation

We first study the flow of the trace of any arbitrary static perturbation of the region from the horizon \( r = \alpha \) to infinity of these black holes. We impose Dirichlet conditions on the perturbation at the two ‘boundaries’: the horizons \( r = \alpha \) and \( r = \infty \) (we could instead have chosen Neumann boundary conditions at the horizon—the analysis that follows is unaltered by this choice). The flow of the trace \( H \) is given by (3.6). Assume the ansatz

\[
\dot{H} = R(r) e^{-d/2 \Delta} H(x^a) e^{d/2 \Delta T},
\]

where

\[
\Delta = \tilde{\Delta} H = \lambda T \dot{H}.
\]

\( \tilde{\Delta} = \tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \) is the Laplacian on \( B \) with respect to the base metric. Then we have

\[
f(r) R''(r) + f'(r) R'(r) + \left( \frac{-d^2}{4r^2} f(r) + \frac{d}{2r^2} f'(r) - \frac{\lambda T}{r^2} - \Omega_T \right) R(r) = 0,
\]

where primes indicate derivative with respect to \( r \). Finally, we define the ‘tortoise coordinate’ \( \tilde{r}_* \) by \( d\tilde{r}_* = f(r) dr \) and so \( -\infty < \tilde{r}_* < \infty \) when \( \alpha < r < \infty \). We can rewrite (4.3) in the Schrödinger form as

\[
-\frac{d^2}{d\tilde{r}_*^2} - f(r) \left( \frac{-d^2}{4r^2} f(r) + \frac{d}{2r^2} f(r) - \frac{\lambda T}{r^2} - \Omega_T \right) R(r) = 0.
\]

We are therefore interested in normalizable (in this case, square integrable) functions \( R(\tilde{r}_*) \) that correspond to the zero modes of the Schrödinger potential:

\[
V_T(\tilde{r}_*) = -f(r) \left( \frac{-d^2}{4r^2} f(r) + \frac{d}{2r^2} f(r) - \frac{\lambda T}{r^2} - \Omega_T \right).
\]

Clearly, if \( R(\tilde{r}_*) \to 0 \) as \( \tilde{r}_* \to \pm \infty \) (which is true since we have imposed Dirichlet boundary conditions on the perturbation) and if \( V_T \neq 0 \) for all \( -\infty < \tilde{r}_* < \infty \), there are no normalizable zero modes (this can be easily seen by multiplying both sides of (4.4) by \( R(\tilde{r}_*) \) and integrating over the range of \( \tilde{r}_* \)). We can write \( V_T = V_1 + \Omega_T f(r) \), where

\[
V_1 = -f(r) \left( \frac{-d^2}{4r^2} f(r) + \frac{d}{2r^2} f(r) - \frac{\lambda T}{r^2} \right).
\]

Now \( \Omega_T f(r) > 0 \) for \( -\infty < \tilde{r}_* < \infty \), so if \( V_1 > 0 \), no normalizable zero modes are possible. We now explore the conditions on \( \lambda_T \) such that \( V_1 > 0 \). Since \( 0 < f(r) < 1 \) for the range of \( r \) we are interested in,

\[
V_1 > 0 \iff \left( \frac{d^2}{4r^2} f(r) - \frac{d}{2r^2} f'(r) - \frac{\lambda T}{r^2} \right) > 0.
\]

Now, \( d \geq 2 \) for Schwarzschild–Tangherlini black holes. In this case, by substituting the explicit form of \( f(r) \) into the above inequality, and also observing that \( f'(r) > 0 \), we get

\[
\left( \frac{d^2}{4r^2} f(r) - \frac{d}{2r^2} f'(r) - \frac{\lambda T}{r^2} \right) > 0 \iff \left( \frac{d^2}{4} - \frac{d}{2} \lambda_T \right) > 0.
\]
Therefore, there will be no unstable normalizable modes of the trace of a static perturbation for \( \lambda_T < \frac{(d-2)}{4} \). \( \lambda_T \) is the eigenvalue of the scalar Laplacian on the base manifold \( B \) (according to our convention, this Laplacian is \( \Delta = \bar{g}^{\alpha \beta} \nabla_\alpha \nabla_\beta \), acting on smooth functions). The base manifold for Schwarzschild–Tangherlini black holes is \( S^d \), for which the spectrum of the scalar Laplacian is known. This spectrum is non-positive (with our conventions) and therefore there are no unstable normalizable modes of the trace of the perturbation.

### 4.3. Flow of static TT ‘tensor’ perturbations

We now examine the flow of the special class of TT perturbations described in section 3 which are rank-2 ‘tensor’ perturbations on \( S^d, H_{\alpha \beta} \). This is flow (3.7) with \( c = 0 \). We assume an ansatz of the form \( h_{TT}^{\alpha \beta} = \bar{h}_{TT}^{\alpha \beta}(\tilde{r}) \bar{r}^d \phi(r) e^{\frac{\phi}{\Omega}} \), where \( (\Delta_L \bar{h}^{TT})_{\alpha \beta} = \tilde{\lambda} \bar{h}^{TT}_{\alpha \beta} \).

As usual, perturbations with \( \Omega > 0 \) are unstable modes. We then have the following PDE:

\[
-\frac{d}{dr} \left( f \frac{d \phi}{dr} \right) + \left( \frac{d}{2} \right) \frac{d}{dr} \left( \frac{f}{r} \phi \right) + \left( \left( \frac{f d^2}{4 r^2} \right) + \frac{\tilde{\lambda}}{r^2} - \frac{2 f'(r)}{r} - \frac{(2 d - 2) f}{r^2} \right) \phi = -\Omega \phi.
\]

(4.9)

Rewriting the above equation in the Schrödinger form using the tortoise coordinate \( r_* \), we have

\[
-\frac{d^2}{dr_*^2} \phi + \bar{V}(r) \phi = 0,
\]

(4.10)

\[
\bar{V}(r) = f(r) \left( \frac{\tilde{\lambda}}{r^2} + \frac{(d - 4) f'(r)}{2 r} + \frac{(d^2 - 10 d + 8)}{4} \frac{f}{r^2} \right).
\]

(4.11)

This equation, and the flow of the TT modes, is similar to that in [2] (with some important differences: \( \Omega > 0 \), which labels the unstable mode now appears in the potential, and we are interested in zero eigenvalues of this potential). As before in the case of the flow of the trace, \( \Omega f(r) > 0 \) for \( -\infty < r_* < \infty \), so if

\[
\bar{V}_1 = f(r) \left( \frac{\tilde{\lambda}}{r^2} + \frac{(d - 4) f'(r)}{2 r} + \frac{(d^2 - 10 d + 8)}{4} \frac{f}{r^2} \right) > 0,
\]

(4.12)

no normalizable eigenfunctions with zero eigenvalue are possible for the potential \( \bar{V} \). From the explicit form of \( f(r) \), it is clear that

\[
\bar{V}_1(r) > 0 \iff \tilde{\lambda} > \frac{(d^2 - 10 d + 8)}{4}.
\]

(4.13)

Recall that \( \tilde{\lambda} \) is the eigenvalue of the Lichnerowicz operator on the base manifold, \( (\Delta_L \bar{h}^{TT})_{\alpha \beta} = \tilde{\lambda} \bar{h}^{TT}_{\alpha \beta} \). For the Schwarzschild–Tangherlini black holes, the base manifold is \( S^d \), and one can write down the precise form of the Lichnerowicz operator, \( (\Delta_L \bar{h}^{TT})_{\alpha \beta} = -(\Delta_{TT} h^{TT})_{\alpha \beta} + 2 \bar{h}^{TT}_{\alpha \beta} \), and the spectrum of the Laplacian \( \Delta_r \) on symmetric two-tensors \( S^d \) is non-positive. In fact, the space of symmetric two-tensors on \( S^d \) is spanned by a canonical set of symmetric, transverse, trace-free spherical harmonics obeying

\[
(\Delta_{TT} h)_{\alpha \beta} = -[k(k + d - 2)].
\]

(4.14)

for integers \( k \geq 2 \) (see p 30 of [2] and also [22] for the spectrum of the Laplacian on symmetric two-tensors in \( S^d \)). Thus, \( \tilde{\lambda} \geq 2d \).

\( h_{TT}^{\alpha \beta} \) are given in terms of the symmetric tensor spherical harmonics of rank 2. It is known that there are no such tensors for \( d = 2 \) [22]. Therefore, while the flow of the trace was relevant for four-dimensional Schwarzschild black holes, these perturbations are present only for the higher dimensional black holes.
From our analysis above, it is clear that unstable normalizable modes of this TT perturbation are only possible when the potential $\tilde{V}$ is not positive. A necessary condition therefore is that $\tilde{\lambda} \leq -\frac{(d^2-10d+8)}{4}$. However, $\tilde{\lambda} \geq 2d$, and it is easy to see that this condition cannot be satisfied. Therefore, the Schwarzschild–Tangherlini black holes are stable under the class of TT perturbations we have considered.

4.4. AdS–Schwarzschild–Tangherlini black holes

We set the cosmological constant $c = -1/L^2$. Also let $B = S^d$. Then $f(r) = (1 - (\alpha/r)^{d-1} + r^2/L^2)$.

4.5. Flow of the trace for any static perturbation

Considering the flow of the trace $H$ of an arbitrary static perturbation (3.6) as before, and assuming the same ansatz (4.2), the equation for $R(r)$ in tortoise coordinates is

$$-rac{d^2}{dr_*^2} f(r) \left( -\frac{d^2}{4r^2} f(r) + \frac{d}{2r^2} f(r) - \frac{d}{2r^2} f'(r) + \frac{\tilde{\lambda}_T}{r^2} - \Omega_T - \frac{2}{L^2}(d+1) \right) R = 0.$$  \hspace{1cm} (4.15)

There are no physically reasonable solutions when the potential $V_T = -f(r) \left( -\frac{d^2}{2r^2} f(r) + \frac{d}{2r^2} f'(r) - \frac{\tilde{\lambda}_T}{r^2} \right)$ is positive (either the perturbation or its first derivative is not normalizable). As in the previous analysis, by putting the explicit form of $f(r)$ above, we deduce that a necessary condition for $V_T$ to be negative is that \( \frac{(d^2-10d+8)}{4} \leq \tilde{\lambda}_T \), which is the same condition as in the zero cosmological constant case. Thus, a necessary condition for the existence of the normalizable unstable modes of the trace is that $\tilde{\lambda}_T > \frac{d(d-2)}{4}$, which is not fulfilled by the spectrum of the Laplacian on $S^d$, and such modes do not exist.

4.6. Flow of static TT ‘tensor’ perturbations

We can also repeat the analysis for the flow of the TT perturbations that are rank-2 tensor perturbations on $S^d$, given by (3.7). We assume the ansatz (4.9) as in the zero cosmological constant case. We get the equation in the Schrödinger form to be (4.10) with

$$\tilde{V}(r) = f(r) \left( \frac{\tilde{\lambda}_T}{r^2} + \frac{(d-4)f'(r)}{2r} + \frac{(d^2-10d+8)}{4} \frac{f}{r^2} + \frac{2}{L^2}(d+1) + \Omega_T \right).$$  \hspace{1cm} (4.16)

As before, a necessary condition for the potential $\tilde{V}$ to be negative is the same as the zero cosmological constant case; $\tilde{\lambda} \leq -\frac{(d^2-10d+8)}{4}$. This is never possible for the base manifold being $S^d$, and therefore the AdS–Schwarzschild–Tangherlini black holes are also stable under the class of perturbations considered. We could attempt a similar analysis for de Sitter black holes. However, in this case, the perturbations are confined to the region between the black hole and cosmological horizons. The conditions under which the potential (4.16) with $L^2$ replaced by $-L^2$ is positive in this case are not easy to read off, and will depend on the relative magnitudes of the two horizon radii.

5. A more rigorous argument

In this section, we present a more rigorous discussion of the stability of the black holes considered in the previous section, either for the flow of the trace of any static perturbation, or the flow of the static, TT perturbations that are tensor perturbations on $S^d$. The first step in
our study was the choice of ansatz—of the form (4.1) for the flow of the trace (3.6); and of the form (4.9) for the flow of the traceless part (3.7). The question that naturally arises is as follows: How general is this ansatz? Clearly there is no loss of generality in expressing the dependence of the perturbation on angular coordinates on the base $S^d$ in terms of suitable scalar or tensor spherical harmonics as we have done. However, our ansatz for the $\tau$ dependence of the perturbation involves the assumption that every unstable perturbation that is a solution of the flow (either of the trace or TT tensor part) can be expressed as a superposition of solutions with $\tau$ dependence of the form $e^{i\Omega_1 \tau}$ where $\Omega_1 > 0$ is real. A further assumption we are implicitly making in such a stability analysis is that if we cannot find normalizable modes satisfying our ansatz, then there can be no perturbations of compact support, for example, that are growing in $\tau$. This need not be true, as we could conceive of perturbations of compact support constructed as a linear superposition of the unnormalizable modes (these would consequently grow in $\tau$). Generically, operators such as the Laplacian or the Lichnerowicz Laplacian have a spectrum with a continuous component on noncompact manifolds, and the corresponding eigentensors are not normalizable. The absence of such unstable normalizable modes does not guarantee the stability of the spacetime under initially well-behaved perturbations. We therefore present a more rigorous stability result below to resolve this issue. In the context of classical stability analysis of black holes, such a rigorous argument, was given by Wald [17]. Our argument bears some similarities to it—however, we analyze a (degenerate) parabolic PDE, not a hyperbolic PDE as in the classical stability analysis, and thus there are differences in the proof.

After separation of angular variables, the flow of either the trace or TT tensor perturbation is given in terms of a function $\Phi(r_*, \tau)$ (where the tortoise coordinate $r_*$ is defined by $dr_* = dr/f$) by an equation of the form

$$\frac{\partial}{\partial \tau} \Phi(r_*, \tau) = -\mathcal{L} \Phi(r_*, \tau),$$

where

$$\mathcal{L} = \frac{1}{f} \left( -\frac{\partial^2}{\partial r_*^2} + V \right).$$

$V$ is a ‘potential’ of the form $V_1$ or $\tilde{V}_1$ we saw in the previous section. In the $r$ coordinate,

$$\mathcal{L} = -\frac{\partial}{\partial r} \left( f \frac{\partial}{\partial r} \right) + \frac{V}{f}.$$

Then, we are interested in the situation when $V > 0$. We saw that for both the flow of the trace and TT tensor perturbation of the (AdS) Schwarzschild–Tangherlini black holes, this is always true for the potential. We will assume $\Phi(r_*, \tau)$ to be a smooth ($C^\infty$) function of $r_*$ (so that the perturbed geometry is smooth). We will also assume that it is of compact support in $r_*$. In fact, our analysis applies to a wider class of perturbations. Specifically, we only need to impose a Dirichlet (or Neumann) boundary condition on the perturbation at $r_* = -\infty$ (treating the horizon as a boundary) and suitable decay conditions as $r_* \to \infty$ so that $\Phi$ and its derivatives up to order 2 go to zero in this limit. This ensures that boundary terms generated while integrating by parts in the following analysis vanish.

The question we address is whether there exist growing solutions $\Phi(r_*, \tau)$ to (5.1) when $V > 0$. We first define the following ‘energies’:

$$E_0 = \int_a^\infty \Phi(r, \tau)^2 \, dr;$$

$$E_1 = \int_a^\infty (\mathcal{L}\Phi(r, \tau))^2 \, dr.$$
Next, we observe that
\begin{equation}
\frac{d}{d\tau} E_0 = -2 \int_a^\infty \Phi(\mathcal{L}\Phi) \, dr;
\end{equation}
\[= 2 \int_a^\infty \Phi \frac{\partial}{\partial r} \left( f \frac{\partial}{\partial r} \right) \Phi \, dr - 2 \int_a^\infty \frac{V}{f} \Phi^2 \, dr. \tag{5.5}\]
Note that for the black hole potentials, \( \frac{V}{f} \) is finite as \( r \to \alpha \). Integrating the first integral on the right-hand side by parts, we get no boundary terms, since the perturbation is of compact support. Therefore, since \( f \geq 0 \) and \( V > 0 \),
\begin{equation}
\frac{d}{d\tau} E_0 = -2 \int_a^\infty f \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \right) \Phi \, dr - 2 \int_a^\infty \frac{V}{f} \Phi^2 \, dr \leq 0. \tag{5.6}\end{equation}
By replacing \( \Phi \) with \( \mathcal{L}\Phi \) in the above argument, we see that \( \frac{d}{d\tau} E_1 \leq 0 \) as well. Therefore, \( E_0(\tau) \) and \( E_1(\tau) \) are bounded from above by their initial values \( E_0(0) \) and \( E_1(0) \) which we assume are finite. By the Cauchy–Schwarz inequality,
\begin{equation}
\left| \int_a^\infty \Phi \mathcal{L}\Phi \, dr \right| \leq \sqrt{E_0(\tau) E_1(\tau)} \leq \sqrt{E_0(0) E_1(0)}. \tag{5.7}\end{equation}
Now define the following integrals:
\begin{align}
I_0 &= \int_{r_*=-\infty}^{r_*=\infty} \Phi(r_*, \tau)^2 \, dr_*; \\
I_1 &= \int_{r_*=-\infty}^{r_*=\infty} \left( \frac{\partial}{\partial r_*} \Phi(r_*, \tau) \right)^2 \, dr_*.
\end{align}
We note that
\begin{align}
I_1 &= \int_{-\infty}^{\infty} \left( \frac{\partial}{\partial r_*} \Phi(r_*, \tau) \right)^2 \, dr_* \\
&= \int_{-\infty}^{\infty} f \mathcal{L}\Phi \, dr_* - \int_{-\infty}^{\infty} V \Phi^2 \, dr_*; \\
&\leq \int_{-\infty}^{\infty} f \mathcal{L}\Phi \, dr_* = \int_a^\infty \Phi \mathcal{L}\Phi \, dr, \tag{5.9}\end{align}
where we did integration by parts and discarded boundary terms as the perturbation has compact support. Thus, from (5.7) and (5.9), we conclude that \( I_1(\tau) \) is bounded from above by a \( \tau \)-independent constant \( C_1 \) depending on the initial values of the energies. Our analysis so far applies to both Schwarzschild–Tangherlini and AdS–Schwarzschild–Tangherlini black holes, provided the integrals \( E_0, E_1, I_0 \) and \( I_1 \) are finite at initial \( \tau \). Now there is a slight departure in the analysis for the two types of black holes in the next step.

We now wish to prove that \( I_0 \) is also bounded from above by a \( \tau \)-independent constant \( C_0 \).

5.1. For Schwarzschild–Tangherlini black holes
From (5.6), we conclude that for \( \tau \geq 0 \)
\begin{equation}
E_0(0) - E_0(\tau) = \int_{-\infty}^{\infty} f(\Phi(r_*, 0)^2 - \Phi(r_*, \tau)^2) \, dr_* \geq 0. \tag{5.10}\end{equation}
Since \( 0 \leq f \leq 1 \), we note that
\begin{equation}
\Phi(r_*, 0)^2 - \Phi(r_*, \tau)^2 \geq f(\Phi(r_*, 0)^2 - \Phi(r_*, \tau)^2) \tag{5.11}\end{equation}
for all \( r_* \). Therefore,
\[
\int_{-\infty}^{\infty} (\Phi(r_*, 0)^2 - \Phi(r_*, \tau)^2) \, dr_* \geq \int_{-\infty}^{\infty} f(\Phi(r_*, 0)^2 - \Phi(r_*, \tau)^2) \, dr_* \geq 0.
\]
Thus, \( I_0(\tau) \) is bounded from above by its initial value which we denote by \( C_0 \).

5.2. For AdS–Schwarzschild–Tangherlini black holes

From (5.6) we conclude that for \( \tau \geq 0 \),
\[
E_0(0) - E_0(\tau) = \int_{-\infty}^{\infty} (1 - (\alpha/r)^{d-1} + r^2/L^2)(\Phi(r_*, 0)^2 - \Phi(r_*, \tau)^2) \, dr_* \geq 0
\]
\[
\Rightarrow \int_{-\infty}^{\infty} (1 + r^2/L^2)\Phi(r_*, 0)^2 - \Phi(r_*, \tau)^2) \, dr_* \geq 0,
\]
as \((1 - (\alpha/r)^{d-1} + r^2/L^2) \leq (1 + r^2/L^2)\) for all \( r \) in the range of interest.

Therefore, we have the following inequality:
\[
\int_{-\infty}^{\infty} \Phi(r_*, \tau)^2 \, dr_* \leq \int_{-\infty}^{\infty} (1 + r^2/L^2)\Phi(r_*, \tau)^2 \, dr_*
\]
\[
\leq \int_{-\infty}^{\infty} (1 + r^2/L^2)\Phi(r_*, 0)^2 \, dr_*.
\]

Assume that the integral \( \int_{-\infty}^{\infty} (1 + r^2/L^2)\Phi(r_*, 0)^2 \, dr_* = C_{\text{AdS}}^0 \) is finite. Then \( I_0 \) is bounded from above by the \( \tau \)-independent constant \( C_{\text{AdS}}^0 \).

The rest of the analysis applies to both classes of black holes. Now we expand \( \Phi(r_*, \tau) \) in terms of its Fourier modes as
\[
\Phi(r_*, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikr_*} \hat{\Phi}(k, \tau) \, dk;
\]
\[
\Rightarrow \int_{-\infty}^{\infty} (1 + k^2)^{-1/2} e^{ikr_*} \hat{\Phi}(k, \tau) \, dk.
\]
Finally, using the Cauchy–Schwarz inequality, we obtain
\[
|\Phi(r_*, \tau)|^2 = \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [(1 + k^2)^{-1/2} e^{ikr_*}][1 + k^2]^{1/2} \hat{\Phi}(k, \tau) \, dk \right)^2 \geq \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} (1 + k^2)^{-1} e^{2ikr_*} \, dk \right) \left( \int_{-\infty}^{\infty} (1 + k^2) |\hat{\Phi}(k, \tau)|^2 \, dk \right).
\]

Now, using elementary contour integration techniques:
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} (1 + k^2)^{-1} e^{2ikr_*} \, dk = \frac{1}{2} e^{-2|r_*|}.
\]
Using the Plancherel theorem:
\[
\int_{-\infty}^{\infty} (1 + k^2) |\hat{\Phi}(k, \tau)|^2 \, dk = \int_{-\infty}^{\infty} |\Phi(r_*, \tau)|^2 \, dr_* + \int_{-\infty}^{\infty} \left| \frac{\partial \Phi(r_*, \tau)}{\partial r_*} \right|^2 \, dr_*.
\]
Thus, we have the following inequality (a simple case of a Sobolev inequality, where the right-hand side is the square of a Sobolev norm):

\[
|\Phi(r_*, \tau)|^2 \leq \int_{-\infty}^{\infty} |\Phi(r_*, \tau)|^2 \, dr_* + \int_{-\infty}^{\infty} \left| \frac{\partial \Phi(r_*, \tau)}{\partial r_*} \right|^2 \, dr_*.
\]  

(5.19)

We have already seen that the integrals \( I_0(\tau) \) and \( I_1(\tau) \) are bounded from above by the \( \tau \)-independent constants \( C_0 \) and \( C_1 \) respectively (we replace \( C_0 \) with \( C_{0,AdS} \) below in the AdS case). It follows that

\[
|\Phi(r_*, \tau)|^2 \leq C_0 + C_1.
\]  

(5.20)

Therefore, we have shown that the pointwise norm of the perturbation \( \Phi(r_*, \tau) \) stays bounded along the linearized flow. While we have not proved that this norm goes to zero, this clearly clarifies some of the issues we raised about the choice of ansatz (4.1) or (4.9) for the trace or the TT perturbation, respectively. There are no trace modes or TT ‘tensor’ perturbations of compact support growing in \( \tau \) for either the Schwarzschild–Tangherlini or AdS–Schwarzschild–Tangherlini black holes. Both in Wald’s paper on the classical stability of the Schwarzschild black hole [17] and in the proof of linear stability of \( H^n \) under Ricci flow, it is possible to also bound the higher derivatives of the perturbation under the flow. We are unable to use techniques similar to those in [16] to bound higher derivatives as the operator \( L \) given by (5.2) is not self-adjoint with respect to the measure \( dr_* \).

6. Summary and discussion

The results presented in this paper are the beginning of a program to study the linear stability of Schwarzschild–Tangherlini black holes (and their AdS counterparts) under Ricci flow. As discussed in the introduction, there are diverse motivations from physics for such a study. Such stability results also offer insights into the Ricci flow on noncompact manifolds, which is not as well understood as the Ricci flow on compact manifolds. We briefly summarize the results of this paper.

We study the evolution of static perturbations of the spacetime outside the horizon of the (AdS) Schwarzschild–Tangherlini spacetimes under a linearized Ricci flow (or a flow related to it by rescalings).

(i) We are able to show that there is no instability under the flow of the trace of an arbitrary static perturbation. This is done in two steps: we assume a separation of variables ansatz in section 4 and show that the flow equation for the trace has no unstable normalizable modes. In section 5, we go beyond such a specific choice of ansatz and show that the pointwise norm of a solution of compact support in \( r_* \) stays bounded under the linearized flow of the trace.

(ii) For static TT perturbations that obey \( h^{TT}_{a \bar{a}} = h^{TT}_{\bar{a} \bar{a}} = 0 \), where \( a \) is any spacetime index (i.e. perturbations that behave as rank-2 tensors on \( S^d \)), we show that there is no instability under the flow. This is done by adopting a specific ansatz in section 4, and showing in section 5 (as for the trace) that the pointwise norm of a perturbation of compact support in \( r_* \) stays bounded under the flow.

We now discuss how to widen our study to a more general class of static perturbations. The strategy is to first split the perturbation explicitly into a TT part, a part proportional to the trace and the traceless part of a divergence as \( h_{ab} = h^{TT}_{ab} + \frac{\Delta}{\Delta + 1} \Delta g_{ab} + \nabla_a Y_b + \nabla_b Y_a - \frac{\nabla_a Y_b + \nabla_b Y_a}{\Delta + 1} g_{ab} \). Then, we consider its evolution under either (2.5) or (2.7) and attempt to decouple the flows of the various parts by choosing \( V_a \) appropriately, for example, in (2.5). The flow of \( h^{TT}_{ab} \) decouples...
and can be studied separately. The analysis of this flow is simplified by a result of Kodama and Sasaki (see p 139 in [23]). The result implies that any covariant linear differential equation on the spacetime that is at most second order (like the flow of $h_{TT}$) is decomposed into equations for perturbations $h_{ab}$ that behave as scalar, vector and rank-2 tensor on $S^d$ respectively (this is also true for the classical stability analysis in [3]). Thus, we can then analyze the flow of each type of TT perturbation separately. We have analyzed the rank-2 tensor type in this paper. We hope to analyze the scalar- and vector-type TT perturbations in future work. We have already concluded that there is no instability in the flow of the trace. We hope that a systematic analysis as outlined will lead to a better understanding of the nature of the unstable modes of the (AdS) Schwarzschild–Tangherlini spacetimes under Ricci flow or in quantum gravity.

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