Rateless Lossy Compression via the Extremes

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Abstract

We begin by presenting a simple lossy compressor operating at near-zero rate: The encoder merely describes the indices of the few maximal source components, while the decoder’s reconstruction is a natural estimate of the source components based on this information. This scheme turns out to be near-optimal for the memoryless Gaussian source in the sense of achieving the zero-rate slope of its distortion-rate function. Motivated by this finding, we then propose a scheme comprising of iterating the above lossy compressor on an appropriately transformed version of the difference between the source and its reconstruction from the previous iteration. The proposed scheme achieves the rate distortion function of the Gaussian memoryless source (under squared error distortion) when employed on any finite-variance ergodic source. It further possesses desirable properties we respectively refer to as infinitesimal successive refinability, ratelessness, and complete separability. Its storage and computation requirements are of order no more than $\frac{n^2}{\log n}$ per source symbol for $\beta > 0$ at both the encoder and decoder. Though the details of its derivation, construction, and analysis differ considerably, we discuss similarities between the proposed scheme and the recently introduced SPARC of Venkataramanan et al.

Index Terms

Complete separability, extreme value theory, infinitesimal successive refinability, order statistics, rate distortion code, rateless code, spherical distribution, uniform random orthogonal matrix.

I. INTRODUCTION

Consider an independent and identically distributed (i.i.d.) standard normal source $X^n = (X_1, X_2, \ldots, X_n)$. It is well known [8] that the maximum value concentrates on $\sqrt{2 \log n}$, i.e., $\max_{1 \leq i \leq n} X_i = \sqrt{2 \log n}$. This fact suggests a simple lossy source coding scheme for the Gaussian source under quadratic distortion. The encoder sends the index of the maximum value and the decoder reconstructs $\hat{X}_n$ according to

$$\hat{X}_i = \begin{cases} \sqrt{2 \log n} & \text{if } X_i \text{ is the maximum} \\ \frac{\sqrt{2 \log n}}{n-1} & \text{otherwise}. \end{cases}$$

(1)

For the meager $\log n$ nats that it requires, this simple scheme achieves essentially optimum distortion (in a sense made concrete in Section [1]) and has obviously modest storage and computational requirements. We can generalize this scheme by describing the indices of the $k_n$ largest values, and the scheme still achieves optimum distortion for its operating rate. Note that this scheme can be considered a special case of a permutation code [3], where the encoder sends a rough ordering of the source. It can perform as well as the best entropy-constrained scalar quantizer
(ECSQ) but can not achieve the optimum distortion-rate function at general positive rates [9]. In [3], the authors mentioned the $k_n = 1$ case explicitly as being asymptotically optimum under the expected distortion criterion. Our focus is more on the excess distortion probability than the expected distortion. Furthermore, we establish a more general result where $k_n$ grows sub-linearly in $n$.

We generalize this idea to a scheme we refer to as Coding with Random Orthogonal Matrices (CROM), which achieves the distortion-rate function at all rates. Let $A$ be a random $n$ by $n$ matrix uniformly drawn from the set of all $n$ by $n$ orthogonal matrices, i.e., for any $n$-dimensional vector $Y^n$, the random vector $AY^n$ is uniformly distributed on the sphere with radius $\|Y^n\|$. Since a uniform random vector on the sphere resembles a Gaussian random vector, we can expect the behavior of $A(X^n - \hat{X}^n)$ to be similar to that of an i.i.d. Gaussian random vector. Therefore, we can apply the above scheme again to describe a lossy version of it, using another $\log n$ nats, and so on. In this paper, we show that this iterative scheme achieves the Gaussian rate distortion function for any finite variance ergodic source under quadratic distortion, while enjoying additional properties such as a strong notion of successive refinability and relatively low complexity.

One nice property of CROM is ratelessness. Similar to the rateless codes in the channel coding setting, CROM is able to reconstruct a source with partial messages while the optimum distortion for that rate is achieved. More precisely, suppose the decoder received first fraction $\nu$ of the messages for some $0 < \nu < 1$, then it can reconstruct a source with a distortion $D_G(\nu R)$. Thanks to the ratelessness, the encoder does not have to determine the rate ahead of encoding. However, unlike in many rateless channel coding settings, CROM requires that the bits observed are the first fraction $\nu$ bits, rather than that number of bits gleaned from any set of locations along the stream.

Much work has been dedicated to reducing the complexity of rate-distortion codes (c.f. [7], [10], [13] and references therein). In particular, Venkataramanan et al. proposed the sparse regression code (SPARC) that achieves the Gaussian distortion-rate function with low complexity [20], [21]. SPARC and CROM have similarities, which we discuss in detail.

The paper is organized as follows. In Section II we present the simple zero-rate scheme and the sense in which it is optimal for Gaussian sources. CROM is described, along with some of its properties and performance guarantees, in Section III. We compare our scheme with SPARC in Section IV. We also discuss dual channel coding results in Section V. Section VI provides proofs of our main results and we conclude the paper in Section VII.

**Notation:** Both $X^n$ and $X$ denote an $n$-dimensional random vector $(X_1, X_2, \ldots, X_n)$. We let $X_{(i)}$ denote the $i$-th largest element of $X^n$. We denote an $n$ by $n$ random orthogonal matrix by $A$, and a non-random orthogonal matrix by $A$. We denote the distortion rate-function of the memoryless Gaussian source by $D_G(R)$. Finally, we use nats instead of bits and $\log$ pertains to the natural base unless specified otherwise.

## II. Optimum Zero-Rate Gaussian Source Coding Scheme

In this section, we propose a simple zero-rate lossy compressor which is essentially optimal for the i.i.d. standard Gaussian source under quadratic distortion. Before that, let us be more rigorous regarding our notion of “zero-rate optimum source coding” for a Gaussian source under squared error distortion. Consider a scheme using a number of
nats for the lossy description of the source which is sub-linear in the block length $n$, i.e., the rate of the scheme $R_n$ converges to zero. Suppose the scheme achieves a distortion $D_n(\epsilon)$, where the target excess distortion probability is $\epsilon$, i.e.,

$$\Pr\left[\frac{1}{n}\|X^n - \hat{X}^n\|^2 > D_n(\epsilon)\right] < \epsilon.$$  \quad (2)

We further define $D(n, 0, \epsilon)$ to be the minimum distortion achievable over all possible strictly zero-rate schemes when the target excess distortion probability is $\epsilon$. For the i.i.d. standard Gaussian source under squared error distortion, the best reconstruction is the all zero vector $0 = (0, 0, \ldots, 0)$, and therefore

$$D(n, 0, \epsilon) \triangleq \inf \left\{ D : \Pr\left[\frac{1}{n}\|X^n\|^2 > D\right] < \epsilon. \right\}.$$  \quad (3)

It is not hard to show that

$$D(n, 0, \epsilon) = 1 + \sqrt{\frac{2}{n}} Q^{-1}(\epsilon) + O\left(\frac{1}{n}\right).$$  \quad (4)

Finally, we say that a sequence of zero-rate schemes achieves the zero-rate optimum if

$$\lim_{n \to \infty} \frac{D_n(\epsilon) - D(n, 0, \epsilon)}{R_n} = D'_G(0)$$  \quad (5)

for all $\epsilon > 0$, where $D'_G(0) = -2$ is the slope of the Gaussian distortion-rate function at zero rate. Equivalently,

$$D_n(\epsilon) = 1 - 2R_n + \sqrt{\frac{2}{n}} Q^{-1}(\epsilon) + o(R_n).$$  \quad (6)

This definition is reminiscent of the finite block length result in lossy compression [12], [14], where the authors showed the minimum distortion $D(n, R, \epsilon)$ among all possible schemes for given rate $R$, target excess distortion probability $\epsilon$, and block length $n$ is

$$D(n, R, \epsilon) = D_G(R) + \sqrt{\frac{2}{n}} Q^{-1}(\epsilon) + O\left(\frac{\log n}{n}\right).$$  \quad (7)

We are now ready to propose the simple zero-rate optimum source coding scheme. Let $X^n = (X_1, X_2, \ldots, X_n)$ be an i.i.d. standard normal random process. The encoder simply sends the index of the maximum value, $m = \arg \max_{1 \leq i \leq n} X_i$, and the decoder reconstructs $\hat{X}^n$ as

$$\hat{X}_i = \begin{cases} \alpha_n & \text{if } i = m \\ -\frac{\alpha_n}{n-1} & \text{otherwise}, \end{cases}$$  \quad (8)

where $\alpha_n > 0$ is naturally chosen as $\mathbb{E}[X_{(1)}] \approx \sqrt{2 \log n}$. Note that the encoder only describes the index of the maximum entry but not its value. This scheme works because the unsent value of the maximum entry concentrates on the specific value near $\sqrt{2 \log n}$, i.e., $\max_{1 \leq i \leq n} X_i \approx \sqrt{2 \log n}$, which is a well-known fact from extreme value theory [8].

The rate of this scheme is $R_n = \frac{\log n}{n}$ nats per symbol, and it is not hard to show that the distortion is reduced by $2\log n$ (plus lower order terms), which is twice the rate we are using. Therefore, it is natural to suspect that such a scheme is zero-rate optimum.
We can generalize this scheme to send more than one index: The encoder sends the indices of the $k_n$ largest values of $X^n$, and the decoder reconstructs $\hat{X}^n$ as

$$\hat{X}_i = \begin{cases} 
\alpha_n & \text{if } X_i \text{ is one of the } k_n \text{ largest values of } X^n \\
-k_n \alpha_n & \text{otherwise.}
\end{cases}$$

(9)

Here we will choose $k_n = \lceil \log \beta n \rceil$ for some $\beta > 0$ and $\alpha_n$ to be roughly the expected value of the $k_n$-th largest value of $X^n$, i.e., $\alpha_n \approx E[X_{(k_n)}]$. Clearly this scheme has rate $R_n = \frac{1}{n} \log \left( \frac{n}{k_n} \right)$ where $\lim_{n \to \infty} R_n = 0$. The following theorem shows that this scheme is optimal at zero rate.

**Theorem 1:** For any $\beta \geq 0$ and $k_n = \lceil \log \beta n \rceil$, there is an $\alpha_n > 0$ such that the above scheme achieves the zero-rate optimum. More precisely, the scheme achieves

$$\Pr \left[ \frac{1}{n} \|X^n - \hat{X}^n\|^2 > D_n \right] \leq \epsilon,$$

(10)

where

$$D_n = 1 - 2R_n + \sqrt{\frac{2}{n}} Q^{-1}(\epsilon) + O \left( \frac{k_n \log \log n}{n} \right).$$

(11)

Since $R_n = O \left( \frac{k_n \log n}{n} \right)$, we can say that the above scheme is zero-rate optimum. The proof is given in Section VIB. We note that the encoding and decoding can be done in almost linear time. Moreover, essentially no extra information needs to be stored except the single real number $\alpha_n$.

### III. CODING WITH RANDOM ORTHOGONAL MATRICES

#### A. Preliminaries

Before presenting the scheme, we briefly review some key ingredients: random orthogonal matrices and spherical distributions.

Let $O(n)$ be the set of all $n$ by $n$ orthogonal matrices. We write $A \sim \text{Unif}(O(n))$ to denote that $A$ is a random $n$ by $n$ orthogonal matrix uniformly drawn from $O(n)$. This uniform distribution is with respect to Haar measure, c.f. [11]. Such a random matrix can be constructed in an efficient manner [6]. Note that $B \times A$ is also uniform on $O(n)$ for any orthogonal matrix $B \in O(n)$, i.e., $BA \sim \text{Unif}(O(n))$. Now, let us recall the definition of a radially symmetric random vector and its relation with uniform random orthogonal matrices.

**Definition 1:** An $n$-dimensional random vector $X^n$ has a spherical distribution if and only if $X^n$ and $AX^n$ has the same distribution for all orthogonal matrices $A \in O(n)$.

One nice property of a spherically distributed random vector $X^n$ is that its characteristic function is radially symmetric [19], i.e., $\phi(t) = E \left[ \exp(it^T X^n) \right] = g(\|t\|)$ for some $g(\cdot)$. Therefore, it is enough to consider the norm $\|X^n\|_2$ for a spherically distributed random vector $X^n$. It is clear to see that an i.i.d. Gaussian random vector has a spherical distribution. The following lemma shows how to symmetrize a vector with a uniform random orthogonal matrix.
Lemma 2: Suppose $A$ is a uniform random orthogonal matrix on $O(n)$. For any random vector $X^n$, the random vector $AX^n$ has a spherical distribution.

The lemma is direct consequence of the respective definitions of a uniform random orthogonal matrix and a spherical distribution.

B. Coding with Random Orthogonal Matrices

For notational convenience, define $g_k : \mathbb{R}^n \to \{0,1\}^n$ to be the function that finds the $k$ largest values of the input. Specifically, if $z^n = g_k(x^n)$, then $z_i = 1$ if and only if $x_i$ is one of the $k$ largest entries of $x^n$ and $z_i = 0$ otherwise. Let $A_1, A_2, \ldots, A_{L_n+1} \in O(n)$ be orthogonal matrices, $\alpha_1, \alpha_2, \ldots, \alpha_{L_n}$ be scalars, and assume that $k_n$ is a positive integer smaller than $n$. We are ready to describe the iterative scheme.

Algorithm 1 CROM

Set $X^{(1)} = A_1X^n$.

for $i = 1$ to $L_n$

Let $m^{(i)} = g_{k_n}(X^{(i)})$.

Let $U^{(i)} = (U_1^{(i)}, U_2^{(i)}, \ldots, U_n^{(i)})$ where

$$U_j^{(i)} = \begin{cases} \frac{n-k_n}{nk_n} & \text{if } m_j^{(i)} = 1 \\ -\sqrt{\frac{k_n}{n(n-k_n)}} & \text{otherwise.} \end{cases}$$

(12)

Let $X^{(i+1)} = A_{i+1}(X^{(i)} - \alpha_iU^{(i)})$.

end for

Send $(m^{(1)}, m^{(2)}, \ldots, m^{(L_n)})$.

The unit vector $U^{(i)}$ indicates the $k_n$ largest values of $X^{(i)}$, and $\alpha_i$’s are scaling factors which depend on the norm of $X^{(i)}$ and will be specified later. Since $A_{i+1}^{-1} = A_i^T$, the inverse of the recursion is $X^{(i)} = A_{i+1}^T X^{(i+1)} + \alpha_iU^{(i)}$ for all $i$. This implies

$$X^n = \alpha_1A_1^TU^{(1)} + \alpha_2A_1^TA_2^TU^{(2)} + \cdots + \alpha_i(A_1^TA_2^T \cdots A_i^T)U^{(i)} + X^{(i+1)}. \quad (13)$$

Therefore, when the decoder receives $(m^{(1)}, m^{(2)}, \ldots, m^{(i)})$ for some $i \leq L_n$, it outputs the reconstruction

$$\hat{X}^{(i)} = \alpha_1A_1^TU^{(1)} + \alpha_2A_1^TA_2^TU^{(2)} + \cdots + \alpha_i(A_1^TA_2^T \cdots A_i^T)U^{(i)}.$$ \quad (14)

The decoder can sequentially generate reconstructions using the relation $\hat{X}^{(i+1)} = \hat{X}^{(i)} + \alpha_i(A_1^TA_2^T \cdots A_i^T)U^{(i+1)}$.

Note that the decoder can compute $\hat{X}^{(i)}$ efficiently according to

$$\hat{X}^{(i)} = A_1^T \left( \alpha_1U^{(1)} + A_2^T \left( \alpha_2U^{(2)} + \cdots + \alpha_iA_1^T U^{(i)} \right) \right). \quad (15)$$

Since we need $\log \left( \binom{n}{k_n} \right)$ nats to store (send) $m^{(i)}$, rate $R$ corresponds to $L_n = \frac{nR}{\log \left( \binom{n}{k_n} \right)}$ number of iterations. We are ready to state our main theorem asserting that Algorithm 1 achieves the Gaussian distortion-rate function.
Theorem 3: Suppose $X^n$ is emitted by a stationary ergodic source of marginal second moment $\sigma^2$. For any $\beta \geq 0$, let $k_n = \lfloor (\log n)^\beta \rfloor$ and suppose the rate is $R > 0$. If we take
\[
\alpha_i = \sqrt{\frac{n\sigma^2}{\beta} \left(1 - e^{-\frac{2}{\beta}R} + e^{-\frac{2}{\beta}R} \gamma_i\right) \left(e^{-\frac{2}{\beta}R} - e^{-\frac{2}{\beta}R} \gamma_i\right)},
\]
then there exist scalars $\gamma_i$ and orthogonal matrices $A_1, \ldots, A_{L_n+1} \in \mathbb{O}(n)$ such that Algorithm 1 satisfies
\[
\lim_{n \to \infty} \Pr \left[ \frac{1}{n} \left\| X^n - \hat{X}^{(i)} \right\|^2 > \sigma^2 \left(e^{-\frac{2}{\beta}R} + e^{-\frac{2}{\beta}R} \gamma_i\right)^2 \right] = 0.
\]
For stationary $X^n$, (17) holds for any $\gamma_n \equiv \gamma > 0$. If $X^n$ is i.i.d. distributed and $\mathbb{E} [\|X_i\|^3] < \infty$, then we can find $\gamma_n$ such that
\[
\gamma_n = O \left(\frac{\log \log n}{\log n}\right).
\]
The proof of Theorem 3 is given in Section VI-C with full details regarding the choice of $\gamma_n$.

C. Discussion

1) Role of Orthogonal Matrices: It is known that an i.i.d. Gaussian random vector has a spherical distribution and the variance of its norm is small. Therefore, if a random vector $X^n$ has a spherical distribution and the variance of its norm is small enough, $X^n$ can be thought of as an approximately i.i.d. Gaussian random vector. In the proof of CROM, we employ a randomization argument. Specifically, we assume that $A_1, A_2, \ldots, A_{L_n+1}$ are drawn i.i.d. $\text{Unif}(\mathbb{O}(n))$ and show that equation (17) holds when the probability is averaged over this ensemble of random matrices. The source at $i$-th iteration $X^{(i)} = A_i(X^{(i-1)} - \alpha_{i-1}U^{(i-1)})$ has spherical distribution by Lemma 2 and we can therefore expect $X^{(i)}$ to be a near Gaussian source, where we indirectly show that the norm of $X^{(i)}$ has small variance. This shows that multiplying by uniformly distributed random matrices can be thought of as a way to not only symmetrize but also Gaussianize the random vector so that we can apply the idea of Theorem 1 iteratively.

A similar idea can be found in the work of Asnani et al. [1]. The authors showed that any coding scheme for a Gaussian network source coding problem can be adapted to perform well for other network source coding problems that are not necessarily Gaussian but have the same covariances. The key idea of the paper is applying an orthogonal transformation to the sources which basically “Gaussianizes” them so that the coding scheme for Gaussian sources are applicable in the transform domain.

2) Storage and Computational Complexity: Unlike the zero-rate scheme of Section II, this scheme requires the storage of matrices (and scalars). Since $L_n = \lfloor \frac{nR}{\log (\frac{1}{\epsilon_n})} \rfloor = O \left(\frac{n}{(\log n)^{\beta+1}}\right)$, both the encoder and decoder must keep $O \left(\frac{n^3}{\log^{\beta+1} n}\right)$ real values to store matrices $A_1, A_2, \ldots, A_{L_n}$. In terms of computation, the encoder finds the $k_n$ largest entries of an $n$ dimensional vector and performs a matrix-vector multiplication for each iteration. The dominant cost is $O(n^2)$, the cost of matrix-vector multiplication. Therefore, the overall computational complexity is of order $O \left(\frac{n^3}{\log^{\beta+1} n}\right)$.
3) **Infinitesimal Successive Refinability:** Suppose the decoder gets only the first \(i\) messages \((m^{(1)}, m^{(2)}, \ldots, m^{(i)})\). Note it needs to have seen only the first \(n \frac{i}{L_n}\) R nats for that. With this partial message set, the decoder is able to reconstruct \(\hat{X}^{(i)}\) which achieves a distortion
\[
\sigma^2 \left( e^{-\frac{i}{L_n} R} + e^{\frac{i}{L_n} R} \gamma_n \right)^2,
\]
where the theorem guarantees \(e^{\frac{i}{L_n} R} \gamma_n\) is arbitrarily negligible for large enough \(n\). In other words, the decoder essentially achieves a distortion \(\sigma^2 e^{-2 \frac{i}{L_n} R}\), which is the Gaussian distortion-rate function at rate \(\frac{i}{L_n} R\). Evidently, CROM can be viewed as a successive refinement coding scheme with \(L_n\) stages. Since we have a growing number of stages (in \(n\)), the rate increment at each stage is negligible (i.e., sub-linear number of additional nats per stage) and this is a key difference from classical successive refinement problems where the number of stages is fixed. Note that Theorem 3 implies that the probability of excess distortion beyond the relevant point on the distortion-rate curve at any of the successive refinement stages is negligible. Therefore, if the source is i.i.d. Gaussian, our coding scheme simultaneously achieves every point on the optimum distortion-rate curve. This *infinitesimal successive refinability* can be considered a strengthened version of successive refinement. In other words, to implement and operate CROM, the value of the rate \(R\) need not be known or set in advance, a point we will expound in Section III-C4.

4) **(Near) Ratelessness:** In the channel coding setting, it is well-known that rateless coding schemes, including Raptor codes, achieve the capacity of erasure channels. In this setting, the rate \(R\) does not have to be specified in advance, and the receiver is able to decode a message upon observing sufficiently many packets (or bits), regardless of their order. As we have discussed above, CROM has a similar property that it does not need to be specified a rate \(R\) in advance of the code design. This is because \(\frac{1}{L_n}\) is a function of \(n\) only, and therefore \(\alpha_i\)'s are independent to \(R\). Furthermore, we will see in the proof that \(\gamma_n\) depends only on \(n\). If the source is i.i.d. \(\mathcal{N}(0, \sigma^2)\), the decoder can achieve a distortion \(D_G(\nu R)\) upon observing fraction \(\nu\) of the message bits. This is similar to a rateless code in channel coding because the decoder can achieve the optimum as soon as it collects sufficiently many of the message bits. However, CROM decoder needs its observed bits to be a contiguous sequence at the beginning of the message bit stream while it is enough to have any combination of channel output observations in the rateless channel coding setting.

5) **Complete Separability:** In the classical separation scheme, the source encoder must know the channel capacity \(C\) in order to design the source coding scheme with rate \(R(D) < C\) where the source encoder often does not have this prior knowledge. However, if the source is Gaussian, the proposed scheme achieves the optimum distortion without channel information. Let \(C_0\) be a sufficiently large constant and say the encoder uses the proposed scheme with rate \(R = C_0\). When the decoder receives the first \(C/C_0\) fraction of message bits and performs the reconstruction, we achieve the distortion \(D\) that satisfies \(R_G(D) = C\) due to the *infinitesimal successive refinability*. Since we can achieve the optimum performance using a simple scheme while the source encoder is blind to the capacity of the link, we can call this property *complete separability.*
Another interesting example is a relay network without a direct link, as described in Figure 1, where the source is i.i.d. Gaussian. Both the links from the encoder to the relay node and the relay node to the decoder are noiseless with capacity $C_1$ and $C_2$ respectively and assume that $C_1 > C_2$. If the encoder knows the capacity of both links, then the problem is equivalent to the successive refinement problem. However, consider the case where the encoder only knows $C_1$. If the encoder is optimized only for the first link, the relay node has to decode the whole message and compress it again with rate $C_2$. However, if we use CROM, the relay node can simply send the first $\frac{C_1}{C_2}$ fraction of messages to the decoder and the decoder will be able to have optimal reconstruction with respect to its own link capacity.

6) Convergence Rate: After the $i$-th iteration, the decoder can achieve a distortion

$$
\sigma^2 \left( e^{-\frac{i}{L_n}R} + \gamma_n e^{\frac{i}{L_n}R} \right)^2 = \sigma^2 \left( e^{-2\frac{i}{L_n}R} + 2\gamma_n e^{2\frac{i}{L_n}R} \gamma_n^2 \right)
$$

$$\leq \sigma^2 \left( e^{-2\frac{i}{L_n}R} + 2\gamma_n + e^{2R}\gamma_n^2 \right). \tag{20}
$$

Recall that the Gaussian distortion-rate function at rate $\frac{i}{L_n}R$ is $\sigma^2 \exp \left(-2\frac{i}{L_n}R\right)$, and therefore the gap between the achieved distortion and $D_G \left( \frac{i}{L_n}R \right)$ is uniformly bounded by $2\sigma^2\gamma_n + \sigma^2 e^{2R}\gamma_n^2$ at all stages. Recall that if the source is i.i.d. with bounded $\mathbb{E} [ |X|^3 ]$, we can choose $\gamma_n$ to vanish in the order of $\frac{\log \log n}{\log n}$.

IV. COMPARISON TO SPARC

Recall that CROM can be viewed as a nonzero-rate generalization of the zero-rate scheme introduced in Section \ref{sec:zero-rate-scheme}. On the other hand, SPARC implements the idea of describing a codeword with a linear combination of sub-codewords. Though the derivations of these two schemes were based on different ideas, they share several similarities. In this section, we outline the similarities and differences.

A. Sparse Linear Regression Codes

Let us briefly review SPARC. Let $X^n$ be the first $n$ components of an ergodic source with mean 0 and variance 1. Define $L$ sub-codebooks $C_1, C_2, \ldots, C_L$, where each sub-codebook has $M$ sub-codewords. Sub-codewords are generated independently according to the standard normal distribution. Parameters $M$ and $L$ are chosen to be $M_L = e^{nR}$, where $R$ is the rate of the scheme, and define constants $c_1, c_2, \ldots, c_L$ appropriately. Then, the following algorithm exhibits the main structure of the sparse linear regression code (SPARC), which was presented in [20] and shown to achieve the Gaussian distortion-rate function for any ergodic source (under appropriate choice of parameters).
Algorithm 2 SPARC

Set $X^{(1)} = X^n$.

for $i = 1$ to $L$ do

Let $U^{(i)} = \arg\max_{U^n \in C} U^n >$ and $m^{(i)}$ be the index of $U^{(i)}$.

Let $X^{(i+1)} = X^{(i)} - c_i U^{(i)}$.

end for

Send $(m^{(1)}, m^{(2)}, \ldots, m^{(Ln)})$.

Note that there is another version of SPARC [21] where encoding is not done sequentially but is done by exhaustive search. Since we are focusing on efficient lossy compressors, we only consider the SPARC described in Algorithm 2 throughout the paper.

B. Comparison to SPARC

In SPARC, the codebook consists of $L$ sub-codebooks where each sub-codebook has $M$ codewords. Our proposed iterative scheme is similar to SPARC with $L = nR \log n$ and $M = n$; finding the sub-codeword that achieves the maximum inner product can be viewed as finding the maximum entries after multiplying the matrix in our iterative scheme.

There are, however, two main differences. The first is that our scheme finds the $k_n$ largest values at each iteration. This implies that one iteration of our proposed encoding scheme is equivalent to $k_n$ iterations of SPARC’s encoding. The second is that the sub-codewords of our proposed scheme are uniformly drawn from the surface of a sphere as $\alpha_1 A_2 U^{(1)}$ is uniformly distributed on the sphere of radius $\alpha_1$. This is reminiscent of the result of Lapidoth [15], which showed that the Gaussian distortion-rate function can be achieved using a random codebook whose codewords are uniformly drawn from the unit sphere. Note that the sub-codewords of CROM are not drawn independently. Due to the orthogonality of the matrix, there is a certain structural relationship among the possible sub-codewords. For example, if $k_n = 1$, all sub-codewords are orthogonal to each other. This guarantees that sub-codewords are evenly distributed on the unit sphere.

C. Trade-off Between Complexity and Convergence Rate

In Section III-C2, we have seen that CROM requires $O\left(\frac{n^2}{\log \frac{n}{\log n}}\right)$ operations per symbol, for an arbitrarily chosen $\beta > 0$. The gap between the distortion and $D_G(R)$ is $\frac{\log \log n}{\log n}$. In SPARC, the gap between the distortion and $D_G(R)$ is $\frac{\log \log M}{\log M}$. In order to calibrate with CROM, we can set $M = n$. However, $ML$ operation per symbol is required for SPARC encoding where $ML = e^{nR}$, and therefore the number of operations for SPARC is $O\left(\frac{n^2}{\log n}\right)$.

D. Proofs

As we discussed in Section IV-B, sub-codewords in CROM is drawn from the surface of the sphere while sub-codewords in SPARC are drawn according to the i.i.d. Gaussian distribution. Under this difference, we would like to introduce some dualities. For example, consider the following lemma used in the proof of SPARC.
Lemma 4: [20] Lemma 1] Let $Z_1, \ldots, Z_N$ be independent random vectors with i.i.d. standard Gaussian elements. Then for any random vector $B$ supported on the $n$ dimensional unit sphere and independent of the $Z_i$’s, the inner products $\{<Z_i, B>\}_{i=1}^N$ are i.i.d. standard Gaussian random variables that are independent of $B$.

On the other hand, recall Lemma 3, which asserts that any random vector multiplied by uniform random orthogonal matrix has a spherical distribution.

E. Successive Refinability

That SPARC possesses the successive refinability property was briefly mentioned by the authors. On the other hand, we have seen that CROM has a somewhat stronger (infinitesimal) notion of successive refinability in Section III-C3. Moreover, we have pointed out that CROM has uniform convergence rates, uniformly and simultaneously on all points on the rate distortion curve, in Section III-C6.

V. CHANNEL CODING DUAL

In [18], we can find a dual result in the Gaussian channel coding problem. In this section, we briefly review the idea of [18] (with slightly changed notation). Consider the AWGN channel $Y_i = X_i + Z_i$ where $Z^n$ is an i.i.d. standard normal random vector. Suppose the number of messages is $n$, i.e., the rate of scheme is $R_n = \frac{\log n}{n}$ nats per channel use. Based on message $m \in \{1, 2, \ldots, n\}$, the encoder simply sends $X^n$ where $X_m = (1 + \epsilon_n)\sqrt{2 \log n}$ and $X_i = -(1 + \epsilon_n)\sqrt{2 \log n}$ if $i \neq m$. Then, the decoder finds the index of the maximum value of $Y^n$ and recovers the message, i.e., $\hat{m} = \arg \max_{1 \leq i \leq n} Y_i$. The average power that the encoder used is $P_n = 2(1 + \epsilon_n)^2 \frac{1}{n-1} \log n$.

We will specify $\epsilon_n$ such that $\lim_{n \to \infty} \epsilon_n = 0$.

Before considering the probability of error $P_e^{(n)}$, let us introduce the useful lemma.

Lemma 5: Let $Z^n$ be an i.i.d. standard normal random vector, then

$$\Pr \left[ \max_{1 \leq i \leq n} Z_i > \sqrt{2 \log n} \right] \leq \frac{1}{\sqrt{\log n}}.$$  \hspace{1cm} (22)

Proof:

$$\Pr \left[ \max_{1 \leq i \leq n} Z_i > \sqrt{2 \log n} \right] = 1 - \Phi(\sqrt{2 \log n})^n$$  \hspace{1cm} (23)

$$= 1 - (1 - Q(a))^n$$  \hspace{1cm} (24)

$$\leq nQ(a)$$  \hspace{1cm} (25)

$$\leq n \frac{1}{\sqrt{2 \log n}} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{2 \log n}{2} \right)$$  \hspace{1cm} (26)

$$\leq \frac{1}{\sqrt{\log n}}.$$  \hspace{1cm} (27)

where $\Phi(x)$ is a standard normal cumulative distribution function and $Q(x) = 1 - \Phi(x)$. We used the fact that $Q(x) \leq \frac{1}{2} f(x)$ where $f(x)$ is a probability density function of standard normal random variable. \hfill \(\blacksquare\)

Now we are ready to bound $P_e^{(n)}$. Without loss of generality, we can assume that $m = 1$.

$$P_e^{(n)} = \Pr [Y_1 < \max_{2 \leq i \leq n} Y_i]$$  \hspace{1cm} (28)
\[
\Pr \left[ (1 + \epsilon_n) \sqrt{2 \log n} + Z_1 < -(1 + \epsilon_n) \frac{\sqrt{2 \log n}}{n-1} + \max_{2 \leq i \leq n} Z_i \right]
\]
(29)

\[
= \Pr \left[ \frac{n}{n-1} (1 + \epsilon_n) \sqrt{2 \log n} + Z_1 < \max_{2 \leq i \leq n} Z_i \right]
\]
(30)

\[
\leq \Pr \left[ \sqrt{2 \log n} < \max_{2 \leq i \leq n} Z_i \right] + \Pr \left[ \frac{1 + n\epsilon_n}{n-1} \sqrt{2 \log n} + Z_1 < 0 \right]
\]
(31)

\[
\leq \frac{1}{\sqrt{\log n}} + \Pr \left[ \frac{1 + n\epsilon_n}{n-1} \sqrt{2 \log n} + Z_1 < 0 \right].
\]
(32)

If we choose \( \epsilon_n \) such that \( \frac{1 + n\epsilon_n}{n-1} = (\log n)^{-1/3} \), then \( \frac{1 + n\epsilon_n}{n-1} \sqrt{2 \log n} \) goes to infinity as \( n \) grows. Therefore,

\[
\lim_{n \to \infty} P^{(n)}_n = 0.
\]
(33)

Since \( P_n \) converges to zero as \( n \) grows, we can approximate the capacity by \( C(P_n) = \frac{1}{2} \log(1 + P_n) \approx \frac{P_n}{2} = (1 + \epsilon_n)^{2 \log n} \). It is clear that \( \frac{R_n}{C(P_n)} \) converges to one as \( n \) grows, i.e.,

\[
\lim_{n \to \infty} \frac{R_n}{C(P_n)} = 1.
\]
(34)

This is reminiscent of the definition of a zero-rate optimal scheme in the source coding problem. We can say that this scheme is zero-rate optimal in the channel coding setting. We further note that the encoding and decoding can be done in almost linear time, and essentially no extra information needs to be stored.

However, unlike CROM, we could not find an iterative scheme building on this zero-rate one that achieves reliable communication at a positive rate. The main challenge is that the tail behavior on the left side is very different from the right side. In the source coding problem, a small maximum value yields an error, while it is a large maximum value that yields an error in the channel coding problem. The probability density function of the maximum of Gaussian random variables decays double-exponentially as \( x \) decreases, when \( x < \sqrt{2 \log n} \). This allows a small cumulative error for our iterative scheme. On the other hand, the probability density function decays only exponentially as \( x \) increases when \( x > \sqrt{2 \log n} \), and the cumulative error does not remain negligible when we employ the scheme iteratively. We believe that for similar reasons a channel coding analog of SPARC with efficient encoding would not work.

VI. PROOFS

A. Extreme Value of Gaussian Random Variables

Before providing proofs, consider the following lemma which shows the probabilistic bound of \( Z_{(i)} \) when \( Z^n \) is an i.i.d. standard normal random vector.

**Lemma 6:** Let \( \epsilon_n > 0 \). If positive integers \( n \) and \( i \) satisfy \( 0 \leq \frac{1}{n-i+1} \log \frac{n-i+1}{\epsilon_n} \leq 1 \), then

\[
\Pr \left[ Z_{(i)} < \Phi^{-1} \left( 1 - \frac{1}{n-i+1} \log \frac{n-i+1}{\epsilon_n} \right) \right] \leq \epsilon_n,
\]
(35)

where \( \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \) is a standard normal cumulative distribution function.

**Proof:** Since \( \Phi(Z_1), \Phi(Z_2), \ldots, \Phi(Z_n) \) are i.i.d. uniform random variables, \( \Phi(Z_{(i)}) \) can be considered as the \( i \)-th largest value of an \( n \) dimensional i.i.d. uniform random vector. The probability density function of \( \Phi(Z_{(i)}) \) is
\[
\frac{n^i}{(n-i)!(i-1)!}x^{n-i}(1-x)^{i-1}. \text{ Therefore,}
\]
\[
\Pr \left[ Z(i) < \Phi^{-1} \left( 1 - \frac{1}{n-i+1} \log \frac{n^{i-1}}{\epsilon_n} \right) \right] = \Pr \left[ \Phi \left( Z(i) \right) < 1 - \frac{1}{n-i+1} \log \frac{n^{i-1}}{\epsilon_n} \right] = \int_0^{1 - \frac{1}{n-i+1} \log \frac{n^{i-1}}{\epsilon_n}} n! (n-i)! (i-1)! x^{n-i}(1-x)^{i-1} \, dx
\]
\[
\leq \int_0^{1 - \frac{1}{n-i+1} \log \frac{n^{i-1}}{\epsilon_n}} n! (n-i)! (i-1)! x^{n-i} \, dx
\]
\[
= \frac{n!}{(n-i+1)! (i-1)!} \left( 1 - \frac{1}{n-i+1} \log \frac{n^{i-1}}{\epsilon_n} \right)^{n-i+1}
\]
\[
\leq n^{i-1} \exp \left( - \log \frac{n^{i-1}}{\epsilon_n} \right) = \epsilon_n.
\]

This concludes the proof.

**B. Proof of Theorem 7**

In the proof, we use \( \alpha = \alpha_n \) for simplicity. By the definition of \( \hat{X}^n \), we have

\[
\|X^n - \hat{X}^n\|^2 = \|X^n\|^2 + \frac{2k_n \alpha}{n-k_n} \sum_{i=1}^n X_i - \frac{2n \alpha}{n-k_n} \sum_{i=1}^k X(i) + \frac{nk_n}{n-k_n} \alpha^2.
\]

Let \( \gamma_n \) and \( \delta_n \) be positive real numbers where we specify their values later. Then,

\[
\Pr \left[ \|X^n - \hat{X}^n\|^2 > n(1+\gamma_n - \delta_n) \right]
\]
\[
= \Pr \left[ \|X^n\|^2 + \frac{2k_n \alpha}{n-k_n} \sum_{i=1}^n X_i - \frac{2n \alpha}{n-k_n} \sum_{i=1}^k X(i) + \frac{nk_n}{n-k_n} \alpha^2 - n(1+\gamma_n - \delta_n) \right]
\]
\[
\leq \Pr \left[ \|X^n\|^2 > n(1+\gamma_n) \right] + \Pr \left[ \left( \frac{2k_n \alpha}{n-k_n} \sum_{i=1}^n X_i - \frac{2n \alpha}{n-k_n} \sum_{i=1}^k X(i) + \frac{nk_n}{n-k_n} \alpha^2 > -n\delta_n \right) \right].
\]

Consider the first term of (44). Let \( \gamma_n = \sqrt{\frac{2}{n} Q^{-1}} \left( \epsilon_n - \frac{15}{\sqrt{n}} - \frac{2}{n} \right) \), then we have

\[
\Pr \left[ \|X^n\|^2 > n(1+\gamma_n) \right] \leq Q \left( \sqrt{n} \gamma_n \right) + \frac{15}{\sqrt{n}}
\]
\[
= \epsilon_n - \frac{2}{n}.
\]

In (45), we used Berry-Esseen theorem [4]:

\[
\sup_x \left| \Pr \left[ \frac{\sum_{i=1}^n (X_i^2 - 1)}{\sigma \sqrt{n}} > x \right] - Q(x) \right| < \frac{\rho}{\sqrt{n}},
\]

where \( \sigma^2 = \mathbb{E} [X^4] - \mathbb{E} [X^2]^2 = 2 \) and \( \rho = \mathbb{E} [X^6] = 15 \).

Consider the second term of (44).

\[
\Pr \left[ \frac{2k_n \alpha}{n-k_n} \sum_{i=1}^n X_i - \frac{2n \alpha}{n-k_n} \sum_{i=1}^k X(i) + \frac{nk_n}{n-k_n} \alpha^2 > -n\delta_n \right]
\]
Let \( \alpha = \sqrt{\frac{n-k_n}{k_n}} \delta_n = p_n - q_n \), where

\[
p_n = \Phi^{-1} \left( 1 - \frac{1}{n-k_n + 1} \log n^{k_n} \right)
\]

\[
q_n = \frac{1}{\sqrt{n}} Q^{-1} \left( 1 \right).
\]

Then, we have

\[
\Pr \left[ \frac{2k_n}{n-k_n} \sum_{i=1}^{n} X_i - \frac{2n\alpha}{n-k_n} \sum_{i=1}^{k_n} X_{(i)} + \frac{n k_n}{n-k_n} \alpha^2 > -n \delta_n \right]
\]

\[
= \Pr \left[ \frac{1}{k_n} \sum_{i=1}^{k_n} X_{(i)} - \frac{1}{n} \sum_{i=1}^{n} X_i < p_n - q_n \right]
\]

\[
\leq \Pr \left[ \frac{1}{k_n} \sum_{i=1}^{k_n} X_{(i)} < p_n \right] + \Pr \left[ \frac{1}{n} \sum_{i=1}^{n} X_i > q_n \right]
\]

\[
\leq \Pr \left[ X(k_n) < p_n \right] + \Pr \left[ \frac{1}{n} \sum_{i=1}^{n} X_i > q_n \right].
\]

By Lemma 6 \( \Pr \left[ X(k_n) < p_n \right] \leq \frac{1}{n} \). Since \( \frac{1}{n} \sum_{i=1}^{n} X_i \) has a Gaussian distribution with zero mean and variance \( \frac{1}{n} \),

\[
\Pr \left[ \frac{1}{n} \sum_{i=1}^{n} X_i > q_n \right] = \frac{1}{n}.
\]

Therefore,

\[
\Pr \left[ \frac{2k_n}{n-k_n} \sum_{i=1}^{n} X_i - \frac{2n\alpha}{n-k_n} \sum_{i=1}^{k_n} X_{(i)} + \frac{n k_n}{n-k_n} \alpha^2 > -n \delta_n \right] \leq \frac{2}{n}.
\]

With (46), we have

\[
\Pr \left[ \left\| X^n - \hat{X}^n \right\|^2 > n(1 + \gamma_n - \delta_n) \right] \leq \epsilon.
\]

Now, let consider the bound on \( 1 + \gamma_n - \delta_n \). It is clear that the inequality \( \sqrt{\frac{2 \log \frac{n-k_n+1}{k_n \log^3 n}}{1 + 2 \log \frac{n-k_n+1}{k_n \log^3 n}} \sqrt{2 \pi n - k_n + 1}} \frac{k_n \log^3 n}{k_n \log^3 n} \) holds for large enough \( n \), and therefore

\[
Q \left( \sqrt{\frac{2 \log \frac{n-k_n+1}{k_n \log^3 n}}{1 + 2 \log \frac{n-k_n+1}{k_n \log^3 n}} \sqrt{2 \pi n - k_n + 1}} \frac{k_n \log^3 n}{k_n \log^3 n} \right) \geq \frac{k_n \log n}{n-k_n+1},
\]

which implies

\[
p_n = Q^{-1} \left( \frac{k_n}{n-k_n+1} \log n \right) \geq \sqrt{\frac{2 \log \frac{n-k_n+1}{k_n \log^3 n}}{1 + 2 \log \frac{n-k_n+1}{k_n \log^3 n}} \sqrt{2 \pi n - k_n + 1}} \frac{k_n \log^3 n}{k_n \log^3 n}.
\]
On the other hand, it is not hard to show that
\[ q_n = \frac{1}{\sqrt{n}} Q^{-1} \left( \frac{1}{n} \right) \leq \sqrt{\frac{2 \log \frac{3}{n}}{n}}. \]  
(62)

Now, we are ready to bound \( D_n = 1 + \gamma_n - \delta_n \). Since \( R_n = \frac{1}{n} \log \left( \frac{n}{k_n} \right) \), we have
\[
D_n = 1 + \gamma_n - \frac{k_n}{n - k_n} (p_n - q_n)^2 
= 1 + \sqrt{\frac{2}{n} Q^{-1} \left( \frac{15}{\sqrt{n}} - \frac{2}{n} \right)} - \frac{k_n}{n - k_n} \left( Q^{-1} \left( \frac{k_n}{n - k_n + \log n} \right) - \frac{1}{n} Q^{-1} \left( \frac{1}{n} \right) \right)^2 
\leq 1 + \sqrt{\frac{2}{n} Q^{-1} (\epsilon - 15 \sqrt{n} - 2)} - \frac{k_n}{n - k_n} \left( 2 \log \frac{n - k_n + 1}{k_n \log \frac{n}{n}} - \frac{2}{n} \log \frac{2}{n} \right)^2 
\leq 1 + \sqrt{\frac{2}{n} Q^{-1} (\epsilon - 15 \sqrt{n} - 2)} + O \left( \frac{1}{n} \right) - \frac{k_n}{n - k_n} \left( \epsilon - 15 \sqrt{n} - 2 \right) 
= 1 + \sqrt{\frac{2}{n} Q^{-1} (\epsilon - 15 \sqrt{n} - 2)} + O \left( \frac{k_n \log \log n}{n} \right). 
\]
(66)

This concludes the proof.

C. Proof of Theorem 3

Throughout the proof, we will let \( \sigma^2 = 1 \) and use \( L \) instead of \( L_n \) for simplicity. Also, instead of choosing specific orthogonal matrices \( A_1, \ldots, A_{L+1} \), we employ a randomization argument. More precisely, we assume that \( A_1, A_2, \ldots, A_{i+1} \) are drawn i.i.d. \( \text{Unif}(\mathcal{O}(n)) \) and show that equation (17) holds when the probability is averaged over this ensemble of random matrices. Let \( S_i = \|X^{(i)}\| \) and \( X^{(i)} = S_i B^{(i)} \) where \( B^{(i)} \) is uniformly distributed on the \( n \)-dimensional unit sphere and independent to \( S_i \). Since matrices are i.i.d., random variables \( B^{(1)}, \ldots, B^{(L+1)} \) are also independent. Recall (13) and (14), we have \( \|X^n - \tilde{X}^{(i)}\| = \|X^{(i+1)}\| = S_{i+1}^2 \), and this implies that the distortion after the \( i \)-th iteration coincides with \( S_{i+1}^2 \) divided by \( n \). We further let \( \tilde{S} \) be a chi-distributed random variable with degrees of freedom \( n \) and independent to all \( B^{(i)} \)'s, i.e., \( \tilde{S}^2 \sim \chi^2(n) \). Using union bound, we can obtain an upper bound on the excess distortion probability.

\[
\Pr \left[ \frac{1}{\sqrt{n}} S_{i+1} > e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \text{ for some } 0 \leq i \leq L \right] 
\leq \Pr \left[ \frac{1}{\sqrt{n}} S_1 > 1 + \gamma_n \text{ or } \frac{1}{\sqrt{n}} \tilde{S} > \sqrt{1 + \gamma_n} \right] 
\leq \sum_{i=1}^{L} \Pr \left[ \frac{1}{\sqrt{n}} S_{i+1} > e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n, \frac{1}{\sqrt{n}} S_j \leq e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \text{ for all } j < i, \text{ and } \frac{1}{\sqrt{n}} \tilde{S} \leq \sqrt{1 + \gamma_n} \right] 
\leq \Pr \left[ \frac{1}{\sqrt{n}} S_1 > 1 + \gamma_n \text{ or } \frac{1}{\sqrt{n}} \tilde{S} > \sqrt{1 + \gamma_n} \right] 
\leq \sum_{i=1}^{L} \Pr \left[ \frac{1}{\sqrt{n}} S_{i+1} > e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n, \frac{1}{\sqrt{n}} S_i \leq e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n, \text{ and } \frac{1}{\sqrt{n}} \tilde{S} \leq \sqrt{1 + \gamma_n} \right] 
\]
(70)

From the definition of \( X^{(i+1)} \), we have
\[
S_{i+1}^2 = \left\| X^{(i+1)} \right\|^2 = \left\| X^{(i)} - \sigma_i U^{(i)} \right\|^2 
\]
(71)
\[ \| X^{(i)} \|^2 + \alpha_i^2 = 2 \alpha_i \left( -\sqrt{\frac{k_n}{n(n-k_n)}} 1 + \sqrt{\frac{n}{(n-k_n)k_n}} m^{(i)} \right)^T X^{(i)}, \]  

(73)

where \((m^{(i)})^T X^{(i)}\) is a sum of \(k_n\) largest value of \(X^{(i)}\). Let \(V_i = \left( -\sqrt{\frac{k_n}{n(n-k_n)}} 1 + \sqrt{\frac{n}{(n-k_n)k_n}} m^{(i)} \right)^T B^{(i)}\), then \(V_i\) and \(S_i\) are independent. We can now rewrite (73) as

\[ S_{i+1}^2 = S_i^2 + \alpha_i^2 - 2 \alpha_i S_i V_i, \]  

(74)

It is not hard to show that \(S_i^2 + \alpha_i^2 - 2 \alpha_i S_i V_i\) is an increasing function in \(S_i\) when \(\frac{1}{\sqrt{n}} S_{i+1} > e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n\) and \(\frac{1}{\sqrt{n}} S_i \leq e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n\). Therefore,

\[ S_{i+1}^2 = S_i^2 + \alpha_i^2 - 2 \alpha_i S_i V_i \leq n \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right)^2 + \alpha_i^2 - 2 n \alpha_i \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right) V_i, \]  

(75)

which is equivalent to

\[ \frac{n \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right)^2 + \alpha_i^2 - 2 n \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right)^2}{2 \sqrt{n} \alpha_i \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right)} > V_i. \]  

(76)

This implies

\[ \Pr \left[ \frac{1}{\sqrt{n}} S_{i+1} > e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n, \frac{1}{\sqrt{n}} S_i \leq e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n, \text{ and } \frac{1}{\sqrt{n}} \tilde{S} \leq \sqrt{1 + \gamma_n} \right] \]

\[ = \Pr \left[ \frac{n \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right)^2 + \alpha_i^2 - 2 n \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right)^2}{2 \sqrt{n} \alpha_i \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right)} > V_i \text{ and } \frac{1}{\sqrt{n}} \tilde{S} \leq \sqrt{1 + \gamma_n} \right]. \]  

(77)

Recall that we took

\[ \alpha_i = \sqrt{n \left( 1 - e^{-\frac{1}{2} R} \right) \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right) \left( e^{-\frac{1}{2} R} - e^{\frac{1}{2} R} \gamma_n \right)}, \]  

(78)

and it can be easily shown that

\[ \frac{n \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right)^2 + \alpha_i^2 - 2 n \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right)^2}{2 \sqrt{n} \alpha_i \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right)} = \frac{2 \alpha_i^2}{2 \sqrt{n} \alpha_i \left( e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \right)} \]

\[ \leq \sqrt{\left( 1 - e^{-\frac{1}{2} R} \right) \frac{e^{-\frac{1}{2} R} - e^{\frac{1}{2} R} \gamma_n}{e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n}} \]

\[ \leq \sqrt{\left( 1 - e^{-\frac{1}{2} R} \right) \frac{1 - \gamma_n}{1 + \gamma_n}}. \]  

(81)

Thus, we have

\[ \Pr \left[ \frac{1}{\sqrt{n}} S_{i+1} > e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n, \text{ and } \frac{1}{\sqrt{n}} S_i \leq e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n, \text{ and } \frac{1}{\sqrt{n}} \tilde{S} \leq \sqrt{1 + \gamma_n} \right] \]

\[ \leq \Pr \left[ \sqrt{\left( 1 - e^{-\frac{1}{2} R} \right) \frac{1 - \gamma_n}{1 + \gamma_n}} > V_i \text{ and } \frac{1}{\sqrt{n}} \tilde{S} \leq \sqrt{1 + \gamma_n} \right] \]  

(82)

\[ \leq \Pr \left[ \sqrt{n \left( 1 - e^{-\frac{1}{2} R} \right)} \left( 1 - \gamma_n \right) > \tilde{S} V_i \right] \]  

(83)

\[ \leq \Pr \left[ \frac{2 n R}{L} \left( 1 - \gamma_n \right) > \tilde{S} V_i \right]. \]  

(84)
Since \( B^{(i)} \) is uniformly distributed on a unit sphere and it is independent of \( \tilde{S} \), we have \( S B^{(i)} \overset{d}{=} Z \) where \( Z \) is an \( n \) dimensional i.i.d. standard normal random vector. Furthermore,

\[
\tilde{S} V_i \overset{d}{=} \left( -\sqrt{\frac{k_n}{n(n-k_n)}} \mathbf{1} + \sqrt{\frac{n}{(n-k_n)k_n}} \mathbf{m}^{(1)} \right)^T Z
\]  

(85)

\[
= \frac{nk_n}{n-k_n} \left( \frac{1}{k_n} \sum_{i=1}^{k_n} Z(i) - \frac{1}{n} \sum_{i=1}^{n} Z_i \right).
\]  

(86)

If we have \( \gamma_n \geq 1 - (p_n - q_n)^2 \frac{L}{2nR} \frac{n k_n}{n - k_n} \), where \( p_n \) and \( q_n \) were defined as (51) and (52), then we can use a result from the proof of Theorem 1. I.e.,

\[
\Pr \left[ \frac{1}{\sqrt{n}} S_{i+1} > e^{-\frac{1}{2}R} + e^{\frac{1}{2}R} \gamma_n, \text{ and } \frac{1}{\sqrt{n}} S_i \leq e^{-\frac{1}{2}R} + e^{\frac{1}{2}R} \gamma_n, \text{ and } \frac{1}{\sqrt{n}} \tilde{S} \leq \sqrt{1 + \gamma_n} \right]
\]

\[
\leq \Pr \left[ \sqrt{\frac{2nR}{L}} (1 - \gamma_n) > \tilde{S} V_i \right]
\]

(87)

\[
\leq \Pr \left[ p_n - q_n > \frac{1}{k_n} \sum_{i=1}^{k_n} Z(i) - \frac{1}{n} \sum_{i=1}^{n} Z_i \right]
\]

(88)

\[
\leq \frac{2}{n}
\]

(89)

Recall that \( p_n \geq \sqrt{2 \log \frac{n-k_n+1}{k_n \log n}} \), and \( q_n \leq \sqrt{\frac{2 \log \frac{n}{k_n+1}}{n}} \). Therefore, it is easy to check that

\[
1 - (p_n - q_n)^2 \frac{L}{2nR} \frac{n k_n}{n - k_n} \leq 1 - \left( \sqrt{2 \log \frac{n-k_n+1}{k_n \log n}} - \sqrt{\frac{2 \log \frac{n}{k_n+1}}{n}} \right)^2 \frac{k_n}{2 \log \left( \frac{n}{k_n+1} \right)}
\]

(90)

\[
= O \left( \frac{\log \log n}{\log n} \right).
\]

(91)

Firstly, if \( \gamma_n \) is equal to any constant \( \gamma > 0 \), due to the stationarity of the source, we have

\[
\lim_{n \to \infty} \Pr \left[ \frac{1}{\sqrt{n}} S_1 > 1 + \gamma_n \text{ or } \frac{1}{\sqrt{n}} \tilde{S} > \sqrt{1 + \gamma_n} \right] = 0.
\]

(92)

Therefore,

\[
\lim_{n \to \infty} \Pr \left[ \frac{1}{\sqrt{n}} S_{i+1} > e^{-\frac{1}{2}R} + e^{\frac{1}{2}R} \gamma_n \text{ for some } 0 \leq i \leq L \right]
\]

\[
= \lim_{n \to \infty} \Pr \left[ \frac{1}{\sqrt{n}} S_1 > 1 + \gamma_n \text{ or } \frac{1}{\sqrt{n}} \tilde{S} > \sqrt{1 + \gamma_n} \right] + \lim_{n \to \infty} \frac{2}{n} L_n
\]

(93)

\[
= 0.
\]

(94)

Suppose the source is i.i.d. distributed with \( \mathbb{E} \left[ X_1^3 \right] < \infty \), then we can let \( \gamma_n = O \left( \frac{\log \log n}{\log n} \right) \) such that

\[
\gamma_n \geq 1 - \left( \sqrt{2 \log \frac{n-k_n+1}{k_n \log n}} - \sqrt{\frac{2 \log \frac{n}{k_n+1}}{n}} \right)^2 \frac{k_n}{2 \log \left( \frac{n}{k_n+1} \right)}
\]

(95)

and still have

\[
\lim_{n \to \infty} \Pr \left[ \frac{1}{\sqrt{n}} S_1 > 1 + \gamma_n \text{ or } \frac{1}{\sqrt{n}} \tilde{S} > \sqrt{1 + \gamma_n} \right] = 0.
\]

(96)
We would like to point out that the right hand side of (95) is independent to the choice of $R$. Finally, it is clear that
\[
\lim_{n \to \infty} \Pr \left[ \frac{1}{\sqrt{n}} S_{i+1} > e^{-\frac{1}{2} R} + e^{\frac{1}{2} R} \gamma_n \text{ for some } 0 \leq i \leq L \right] = \lim_{n \to \infty} \Pr \left[ \frac{1}{\sqrt{n}} S_1 > 1 + \gamma_n \right] + \lim_{n \to \infty} \frac{2 \gamma_n}{n} \tag{97}
\]
\[
= 0. \tag{98}
\]
This concludes the proof.

VII. CONCLUSIONS

Our starting point (and inspiration for the subsequent main scheme and result) was an extremely simple scheme that achieves the optimum zero-rate distortion for the Gaussian source. We then generalized it to CROM, a lossy source coding scheme that simultaneously achieves the distortion-rate function of the Gaussian memoryless source for all rates when operating on any ergodic source. The merit of CROM over classical random coding schemes is its low storage and computational complexity, as well as the fact that the encoding can be oblivious to the rate desired while the decoding is essentially sequential (sub-linear lookahead) and simultaneously achieves all points on the distortion-rate curve.

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