AMENABLE REPRESENTATIONS AND DYNAMICS OF THE UNIT SPHERE IN AN INFINITE-DIMENSIONAL HILBERT SPACE

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ABSTRACT. We establish a close link between the amenability property of a unitary representation \( \pi \) of a group \( G \) (in the sense of Bekka) and the concentration property (in the sense of V. Milman) of the corresponding dynamical system \((S_\pi, G)\), where \( S_H \) is the unit sphere the Hilbert space of representation. We prove that \( \pi \) is amenable if and only if either \( \pi \) contains a finite-dimensional subrepresentation or the maximal uniform compactification of \( S_\pi \) has a \( G \)-fixed point. Equivalently, the latter means that the \( G \)-space \((S_\pi, G)\) has the concentration property: every finite cover of the sphere \( S_\pi \) contains a set \( A \) such that for every \( \epsilon > 0 \) the \( \epsilon \)-neighbourhoods of the translations of \( A \) by finitely many elements of \( G \) always intersect. As a corollary, amenability of \( \pi \) is equivalent to the existence of a \( G \)-invariant mean on the uniformly continuous bounded functions on \( S_\pi \). As another corollary, a locally compact group \( G \) is amenable if and only if for every strongly continuous unitary representation of \( G \) in an infinite-dimensional Hilbert space \( H \) the system \((S_H, G)\) has the property of concentration.

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1. Introduction

Let \( \pi \) be a unitary representation of a group \( G \) in a Hilbert space \( H \). Then in particular \( G \) acts on the unit sphere \( S_H \) of the space of representation. The resulting topological dynamical system \((S_H, G, \pi)\) (which we will also denote by \((S_\pi, G)\)) is

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thus a pretty common object in mathematics, and examining its properties from the
dynamics viewpoint could be worth while.

Dynamical systems of this kind have received plenty of attention for finite-dimension-
al representations: in this case one can assume without loss in generality that $G$
is a (compact) Lie group, and actions of Lie groups on finite-dimensional spheres
are being studied intensely, cf. e.g. [3]. However, for $\mathcal{H}$ infinite-dimensional much
less appears to be known. One obvious reason for that is the non-compactness
of an infinite-dimensional sphere — indeed, the main body of concepts and results in
present-day topological dynamics have substance for (locally) compact phase spaces
only, cf. [1, 3, 26, 34]. But what if one compactifies the dynamical system $(S_{\mathcal{H}}, G)$?

If we equip $S_{\mathcal{H}}$ with the additive uniform structure (determined by the norm), then
$G$ acts on the sphere by uniform isomorphisms. Denote by $\sigma S_{\mathcal{H}}$ the maximal uniform,
or Samuel, compactification of the uniform space $S_{\mathcal{H}}$, that is, the Gelfand spectrum
of the $C^*$-algebra of all bounded uniformly continuous functions on the sphere. Every
uniform isomorphism of $S_{\mathcal{H}}$ determines a unique self-homeomorphism of $\sigma S_{\mathcal{H}}$, and
in this way $G$ acts on the compactum $\sigma S_{\mathcal{H}}$. Now we are in the realm of abstract
topological dynamics, where a traditional question of importance is: does a given
compact $G$-flow contain fixed points?

The existence of a fixed point in $\sigma S_{\mathcal{H}}$ can be expressed in terms of the original
system $(S_{\mathcal{H}}, G)$. Following Milman [20, 21] and only slightly extending a setting for
his definition, let us introduce the following concept.

**Definition 1.1.** Let $X = (X, U_X)$ be a uniform space, and let $F$ be a family of
uniformly continuous self-maps of $X$.

A subset $A \subseteq X$ is called essential (for $F$) if for every entourage of the diagonal
$V \in U_X$ and every finite collection of transformations $f_1, f_2, \ldots, f_n \in F$, $n \in \mathbb{N}$, one has

$$\bigcap_{i=1}^n f_i V[A] \neq \emptyset,$$

where $V[A] = \{x \in X : \text{for some } a \in A, (a, x) \in V\}$ is the $V$-neighbourhood of $A$.

One says that the pair $(X, F)$ has the **property of concentration** if every finite cover
$\gamma$ of $X$ contains an essential set $A \in \gamma$. ▲

The property of concentration implies — and, if $F$ is a group, is equivalent to — the
existence of a common fixed point for $F$ in $\sigma X$ (equivalently, in every $F$-equivariant
uniform compactification of the uniform space $X$), cf. Proposition 2.1 below.

An important observation was made by Gromov and Milman [12] (cf. also [20]):
in a number of situations, the concentration property of a dynamical system of the
form $(X, F)$, where $X$ is a uniform space infinite-dimensional in some clear sense, is
just another manifestation of the phenomenon of concentration of measure on high-
dimensional structures [4, 18, 21, 22, 29, 31]. Among the results proved by Gromov
and Milman [12, 20] there are the following three.

- If $G$ is abelian and $\dim \mathcal{H} = \infty$, then the pair $(S_{\mathcal{H}}, G)$ has the property of
  concentration.
- If $G$ is compact and $\mathcal{H}$ infinite-dimensional, then $(S_{\mathcal{H}}, G)$ has the property of
  concentration.
The pair \((S^\infty, U(\infty))\), where \(S^\infty\) is the unit sphere of \(l_2\) and \(U(\infty) = \cup_{i=1}^\infty U(n)\), has the property of concentration.

These results had led to the following natural question, which, though not included by Gromov and Milman in their original paper \([12]\), was later advertised by Milman in \([20, 21]\): does the pair \((S_H, U(H))\) have the property of concentration for an infinite-dimensional Hilbert space \(H\)?

The answer to the question is ‘No,’ and a very simple counter-example was constructed already in 1988 by Imre Leader \([17]\), see Example 3.9 below. Unfortunately, the example was never published and, in particular, the present author only learned about its existence after his note \([27]\) had appeared. The existence of such an example suggests a deeper reading of the above question: which groups of unitary transformations have the concentration property on spheres and which do not, and why? Some light on the point at issue was thrown by the present author in \([27]\), where the following was proved.

- A discrete group \(G\) is amenable if and only if the dynamical system \((S_H, G, \pi)\) has the property of concentration for every unitary representation \(\pi\) of \(G\) in an infinite-dimensional Hilbert space \(H\). (Equivalently: for the left regular representation \(\pi_2\).)

It is evident that if \(H\) is a subgroup of \(G\) and \((S_H, G)\) has the property of concentration, then so does \((S_H, H)\). Therefore, for example, the pair formed by the unit sphere of the space \(L_2(F_2)\) and the full unitary group of this space does not have the property of concentration, where \(F_2\) denotes the free group on two generators.

It remained still unclear whether or not the above result could be extended to locally compact groups. Indeed attempts to link properties of the \(G\)-flow \((\sigma S_H, G, \pi)\) with topologo-algebraic properties of a non-discrete group \(G\) encounter the following difficulty: in general, the extended action of \(G\) on the Samuel compactification \(\sigma S_H\) is no longer continuous, and thus the dynamical system \((\sigma S_H, G, \pi)\) does not even ‘remember’ the topology of \(G\).

This observation suggests that the concentration property of the system \((S_H, G, \pi)\) has to do not with the amenability of the acting group \(G\) as such, but rather with the amenability of the representation \(\pi\) as defined by Bekka \([2]\). Adopting this viewpoint (suggested by Pierre de la Harpe after he got acquainted with our e-print \([27]\)) turns out to be very fruitful.

**Definition 1.2.** According to Bekka \([2]\), a unitary representation \(\pi\) of a group \(G\) in a Hilbert space \(H\) is amenable if there exists a \(G\)-invariant state \(\phi\) on the algebra of bounded operators \(L(H)\). It means that \(\phi \in L(H)^*, \phi \geq 0, \phi(1) = 1\), and \(\phi(\pi(g)T \pi(g)^{-1}) = \phi(T)\) for every \(T \in L(H)\) and every \(g \in G\). Such a functional \(\phi\) is called a \(G\)-invariant mean. ▲

This concept unifies several previous theories of amenability, and in particular a locally compact group \(G\) is amenable if and only if every strongly continuous representation of \(G\) is amenable \([2]\). Notice also that amenability of a unitary representation does not depend on the topology of \(G\).

Amenability of a representation turns out to be a necessary prerequisite for the concentration property.
Corollary 3.3. Let \( \pi \) be a unitary representation of a group \( G \) in a Hilbert space \( \mathcal{H} \). If the dynamical system \((\mathcal{S}_\mathcal{H}, G, \pi)\) has the concentration property, then the representation \( \pi \) is amenable.

We are of course more interested in trying to reverse this statement à la Gromov and Milman. As Example 6.1 shows, the \( G \)-flow \((\mathcal{S}_\mathcal{H}, G, \pi)\) need not have the concentration property even if a representation \( \pi \) of a group \( G \) in an infinite-dimensional space \( \mathcal{H} \) is amenable. Nevertheless, excluding ‘trivially amenable’ representations leads to the following result.

Theorem 6.4. Let \( \pi \) be a unitary representation of a group \( G \) in an infinite-dimensional Hilbert space \( \mathcal{H} \). If every subrepresentation of \( \pi \) having finite codimension is amenable (that is, \( \pi \) is not of the form \( \pi_1 \oplus \pi_2 \), where \( \pi_1 \) is finite-dimensional and \( \pi_2 \) is non-amenable), then the dynamical system \((\mathcal{S}_\mathcal{H}, G, \pi)\) has the concentration property.

The proof is again based on the technique of concentration of measure on high-dimensional structures.

Now we are able to derive a number of definitive results linking the amenability property of a unitary representation \( \pi \) with the concentration property of the system \((\mathcal{S}_\mathcal{H}, G, \pi)\). We begin with a description of subgroups of the full unitary group \( U(\mathcal{H}) \) whose action on the unit sphere has the concentration property.

Theorem 7.1. Let \( \pi \) be a unitary representation of a group \( G \) in a Hilbert space \( \mathcal{H} \). The system \((\mathcal{S}_\mathcal{H}, G, \pi)\) has the concentration property if and only if

- either \( \pi \) has a non-zero invariant vector, or
- \( \dim \mathcal{H} = \infty \) and every subrepresentation of \( \pi \) having finite codimension is amenable.

We can now extend our criterion of amenability from discrete groups [27] to all locally compact ones.

Theorem 7.4. A locally compact group \( G \) is amenable if and only if for every strongly continuous unitary representation \( \pi \) of \( G \) in an infinite-dimensional Hilbert space \( \mathcal{H} \), the dynamical system \((\mathcal{S}_\mathcal{H}, G, \pi)\) has the concentration property.

In their turn, amenable representations can be characterized in terms of the concentration property.

Theorem 7.5. A unitary representation \( \pi \) of a group \( G \) in a Hilbert space \( \mathcal{H} \) is amenable if and only if

- either \( \pi \) contains a finite-dimensional subrepresentation, or
- the \( G \)-space \((\mathcal{S}_\mathcal{H}, G, \pi)\) has the concentration property.

One of the applications is to the ‘Lévy-type integral’ for functions on the unit sphere in an infinite-dimensional Hilbert space. (Cf. [11, 21].)

Theorem 7.6. Let \( \pi \) be a unitary representation of a group \( G \) in a Hilbert space \( \mathcal{H} \). The following conditions are equivalent.
• The space $C^b_u(S_H)$ of all bounded uniformly continuous functions on the unit sphere $S_H$ admits a $G$-invariant mean.
• The representation $\pi$ is amenable.

Also in Section 4 we establish a number of dynamical corollaries listed in our C.r. note [27] without proofs.

2. Fixed points and concentration property

Let $X = (X, U_X)$ be a uniform space. The Samuel compactification, or else the maximal uniform compactification, of $X$ is a Hausdorff compact space $\sigma X$ together with a uniformly continuous mapping $i_X: X \to \sigma X$ such that every uniformly continuous mapping $f$ of $X$ to an arbitrary compact Hausdorff space $K$ factors through $i_X$, that is, there exists a continuous mapping $\bar{f}: \sigma X \to K$ with $f = \bar{f} \circ i_X$. (Recall that every compact space supports a unique compatible uniform structure.) In particular, it follows easily that the image $i_X(X)$ is everywhere dense in $\sigma X$.

The Samuel compactification $\sigma X$ is the completion of the uniform space $(X, C^*(X))$, where $C^*(X)$ is the finest totally bounded uniform structure on $X$ contained in $U_X$. The uniformity $C^*(X)$ is at the same time the coarsest uniformity making each bounded uniformly continuous complex-valued function on $(X, U_X)$ uniformly continuous on $(X, C^*(X))$.

The Stone-Čech compactification, $\beta X$, of a Tychonoff topological space $X$ is a special case of the Samuel compactification recovered if $X$ is equipped with the finest compatible uniformity.

In particular, every uniformly continuous mapping $f: X \to X$ determines a unique continuous mapping $\bar{f}: \sigma X \to \sigma X$. If $f$ is a uniform automorphism of $X$, then $\bar{f}$ is a self-homeomorphism of $\sigma X$.

Let $F$ be a family of uniformly continuous self-maps of a uniform space $X$. A compactification $(K, i)$ of $X$ (that is, a pair formed by a compact space $K$ and a uniformly continuous mapping $i: X \to K$ with an everywhere dense image) is called $F$-equivariant if for each $f \in F$ there exists a (necessarily unique) continuous mapping $\bar{f}: K \to K$ satisfying $\bar{f} \circ i = i \circ f$. It follows that the Samuel compactification of $X$ is $F$-equivariant.

The Samuel compactification can also be described as the Gelfand spectrum of the commutative $C^*$-algebra $C^b_u(X) \cong C(\sigma X)$ of all bounded uniformly continuous complex-valued functions on a uniform space $X$ equipped with the supremum norm. Since every uniformly continuous mapping $f: X \to X$ gives rise to a unital $C^*$-algebra endomorphism $f^*$ of $C^b_u(X)$, one can talk of $F$-invariant means (in the $C^*$-algebraic terminology, states) on $C^b_u(X)$.

**Proposition 2.1.** For a family $F$ of automorphisms of a uniform space $X = (X, U_X)$ the following are equivalent.

1. The pair $(X, F)$ has the property of concentration.
2. For every finite subfamily $F_1 \subseteq F$, the pair $(X, F_1)$ has the property of concentration.
3. The pair $(\sigma X, F)$ has the property of concentration.
4. The family $F$ has a common fixed point in the Samuel compactification of $X$. 
5. The family $F$ has a common fixed point in every $F$-equivariant uniform compactification of $X$.
6. There exists an $F$-invariant multiplicative mean on the space $C^b_u(X)$.

If $F$ is a family of uniformly continuous self-maps of $X$, then \( \mathcal{(1)} \iff \mathcal{(2)} \iff \mathcal{(3)} \iff \mathcal{(4)} \)

Proof. \( \mathcal{(1)} \Rightarrow \mathcal{(2)} \): obvious.

\( \mathcal{(2)} \Rightarrow \mathcal{(3)} \): Let $\gamma$ be a finite cover of $\sigma X$. For every finite $F_1 \subseteq F$ denote by $\gamma_{F_1}$ the (non-empty, finite) collection of all $F_1$-essential elements of $\gamma$. Clearly, whenever $F_1 \subseteq F_2$, one must have $\gamma_{F_2} \subseteq \gamma_{F_1}$. The compactness (or rather finiteness) considerations lead one to conclude that
\[
\bigcap_{F_1 \subseteq F, \ |F_1| < \infty} \gamma_{F_1} \neq \emptyset, \quad (2.1)
\]
thus finishing the proof: every element $A$ of the above intersection is $F$-essential.

\( \mathcal{(1)} \Rightarrow \mathcal{(3)} \): if $\gamma$ is a finite cover of $\sigma X$, then at least one of the sets $A \cap X$, $A \in \gamma$, is $F$-essential in $X$, and it follows that $A$ is $F$-essential in $\sigma X$.

\( \mathcal{(3)} \Rightarrow \mathcal{(4)} \): emulates a proof of Proposition 4.1 and Theorem 4.2 in [20].

There exists a point $x^* \in \sigma X$ whose every neighbourhood is essential: assuming the contrary, one can cover the compact space $\sigma X$ with open $F$-inessential sets and select a finite subcover containing no $F$-essential sets, a contradiction.

We claim that $x^*$ is a common fixed point for $F$. Assume it is not so. Then for some $f \in F$ one has $fx^* \neq x^*$. Choose an entourage, $W$, of the unique compatible uniform structure on $\sigma X$ with the property $W^2[x^*] \cap W^2[fx^*] = \emptyset$. Since $f$ is uniformly continuous, there is a $W_1 \subseteq W$ with $(x, y) \in W_1 \Rightarrow (fx, fy) \in W$ for all $x, y$. Since $f(W_1[x^*]) \subseteq W[f(x^*)]$, we conclude that $W_1[x^*]$ is $F$-inessential (with $V = W$), a contradiction.

\( \mathcal{(4)} \Rightarrow \mathcal{(3)} \): Let $(K, i)$ be an $F$-equivariant compactification of $X$. There exists a unique continuous $j: \sigma X \to K$ with $j \circ i_X = i$, and it follows easily that $j$ is an $F$-equivariant mapping, that is, $j \circ \tilde{f} = \tilde{f} \circ j$ for every $f \in F$. The image of an $F$-fixed point $x^*$ under $j$ is an $F$-fixed point in $K$.

\( \mathcal{(3)} \Rightarrow \mathcal{(4)} \): trivial.

\( \mathcal{(3)} \Leftrightarrow \mathcal{(5)} \): fixed points in the Gelfand space $\sigma X$ of the commutative $C^*$-algebra $C^b_u(X)$ correspond to $F$-invariant multiplicative means (states) on $C^b_u(X)$.

\( \mathcal{(4)} \Rightarrow \mathcal{(5)} \): If $\gamma$ is a finite cover of $\sigma X$, then there is an $A \in \gamma$ containing an $F$-fixed point, and such an $A$ is clearly $F$-essential.

\( \mathcal{(5)} \Rightarrow \mathcal{(4)} \): this is the only implication where we assume $F$ to be a group.

Without loss in generality and replacing $X$ with its separated reflection if necessary, one can assume that $X$ is a separated uniform space (that is, $\mathcal{U} = \Delta_X$): indeed, the Samuel compactifications of a uniform space $X$ and of its separated reflection are canonically homeomorphic. Thus, we can identify $X$ (as a topological, not uniform space!) with an everywhere dense subspace of $\sigma X$.

If now $\gamma$ is a finite cover of $X$, then the closures of all $A \in \gamma$ taken in $\sigma X$ cover the latter space, and so there is an $A \in \gamma$ with $\text{cl}_{\sigma X}(A)$ containing an $F$-fixed point $x^* \in \sigma X$. We claim that $A$ is $F$-essential in $X$.

To prove this, we need a simple fact of general topology. Let $B_1, \ldots, B_n$ be subsets of a uniform space $X$ satisfying the condition $V[B_1] \cap \cdots \cap V[B_n] = \emptyset$ for some
entourage $V \in \mathcal{U}_X$. Then the closures of $B_i$, $i = 1, 2, \ldots, n$ in the Samuel compactification $\sigma X$ have no point in common: $\text{cl}_{\sigma X}(B_1) \cap \cdots \cap \text{cl}_{\sigma X}(B_n) = \emptyset$.

Let $\rho$ be a uniformly continuous bounded pseudometric on $X$ subordinated to the entourage $V$ in the sense that $(x, y) \in V$ whenever $x, y \in X$ and $\rho(x, y) < 1$. For each $i = 1, \ldots, n$ and $x \in X$ set $d_i(x) = \inf \{\rho(b, x) : b \in B_i\}$. The real-valued functions $d_i$ are uniformly continuous (indeed $\rho$-Lipschitz-1) and bounded on $X$, and therefore extend to (unique) continuous functions $\tilde{d}_i$ on $\sigma X$. If there existed a common point, $x^*$, for the closures of all $B_i$ in $\sigma X$, then all $\tilde{d}_i$ would vanish at $x^*$ and consequently for any given $\epsilon > 0$ there would exist an $x \in X$ with $d_i(x) < \epsilon$, and in particular $x \in V[B_i]$, for all $i$. However, for every $x \in X$, there is an $i$ with $x \notin V[B_i]$.

Now assume that $\cap_{f \in F_1} f(V[A]) = \emptyset$ for some $V \in \mathcal{U}_X$, where $F_1$ is a finite subfamily of $F$. Every $f \in F_1$ has a uniformly continuous inverse $f^{-1} \in F$, and there is an entourage $V_1 \in \mathcal{U}_X$ with $(x, y) \in V_1 \Rightarrow (f^{-1}x, f^{-1}y) \in V$ for all $x, y \in X$ and $f \in F_1$. Let $x \in V_1[f(A)]$, that is, $(x, f(a)) \in V_1$ for some $a \in A$. Then $(f^{-1}(x), a) \in V$, that is, $y = f^{-1}(x) \in V[A]$, and consequently $x = f(y) \in f(V[A])$. We conclude: $V_1[f(A)] \subseteq f(V[A])$ for each $f \in F_1$, and therefore $\cap_{f \in F_1} V_1[f(A)] = \emptyset$ as well.

The above observation from uniform topology implies that $\cap_{f \in F} \text{cl}_{\sigma X}(f(A)) = \emptyset$. Since extensions of $f$ to $\sigma X$ are homeomorphisms, $\text{cl}_{\sigma X}(f(A)) = f(\text{cl}_{\sigma X}(A))$, and consequently $\cap_{f \in F_1} f(\text{cl}_{\sigma X}(A)) = \emptyset$, a contradiction because the intersection contains $x^*$.

\textbf{Remark 2.2.} As pointed out in [13] on a similar occasion, the condition of uniform equicontinuity of $F$, imposed in [12, 20, 21], is superfluous.

\textbf{Example 2.3.} Here is a simple example showing that in general [14] \( \not\Rightarrow \) [2] if $F$ consists just of uniformly continuous mappings. Let $X = (-1, 0) \cup (0, 1)$ with the additive uniform structure, and set $F = \{f_1, f_2\}$, where $f_1(x) = |x|$ and $f_2(x) = -|x|$. Then $\sigma X = [-1, 1]$ and $x^* = 0$ is an $F$-fixed point in $X$. At the same time, the pair $(X, F)$ does not have the property of concentration, as can be seen from considering the cover $\gamma = \{(-1, 0), (0, 1)\}$, both elements of which are $F$-inessential.

\textbf{Remark 2.4.} If one however replaces the condition [14] in the Definition [13] of the concentration property with the condition

$$\bigcap_{i=1}^{n} V[f_i(A)] \neq \emptyset,$$

then all six conditions in Proposition [2.1] become equivalent for an arbitrary family $F$ of uniformly continuous self-maps of $X$. Perhaps, using the concept of concentration property so modified is a sensible thing to do.

\textbf{Example 2.5.} A unitary representation $\pi$ of a group $G$ has almost invariant vectors if for every finite $F \subseteq G$ and every $\epsilon > 0$ there is a $\xi$ in the space of representation $H_\xi$ with $\|\xi\| = 1$ and $\|g \cdot \xi - \xi\| < \epsilon$ for every $g \in F$. As can be easily seen at the level of definitions, if a representation $\pi$ has almost invariant vectors, then the system $(S_H, G, \pi)$ has the concentration property.

The converse is not true. The simplest example possible is the representation of $\mathbb{Z}_2 \cong \{1, -1\}$ in $l_2$ by scalar multiplication. Non-existence of almost invariant
vectors is manifest, yet according to the results by Gromov and Milman cited in the Introduction, the system \((S_\infty, \mathbb{Z}_2)\) has the concentration property. ▲

The following result will be employed in Sections 3 and 4.

**Proposition 2.6.** Let \(\pi_i, i = 1, 2,\) be unitary representations of a group \(G\) in Hilbert spaces \(\mathcal{H}_i\). Let \(\pi = \pi_1 \oplus \pi_2\) be the direct sum representation. Then the following are equivalent.

1. The dynamical system \((S_{\mathcal{H}_1 \oplus \mathcal{H}_2}, G, \pi_1 \oplus \pi_2)\) has the concentration property.
2. At least one of the systems \((S_{\mathcal{H}_i}, G, \pi_i), i = 1, 2\) has the concentration property.

**Proof.** Since the unit spheres \(S_i = S_{\mathcal{H}_i}, i = 1, 2,\) are contained in \(S_{\mathcal{H}}\) in a canonical way both as uniform subspaces and \(G\)-subspaces, it follows that the compactifications \(\sigma S_i, i = 1, 2,\) are compact \(G\)-subflows of \(\sigma S_{\mathcal{H}}\). This establishes \((2) \Rightarrow (1)\).

Now assume that both systems \((S_{\mathcal{H}_i}, G, \pi_i), i = 1, 2\) do not have the concentration property. Then there exist finite covers \(\gamma_j\) of \(S_{\mathcal{H}_j}, j = 1, 2,\) a finite collection \(g_1, \ldots, g_n\) of elements of \(G\), and an \(\epsilon > 0\) having the property that for every \(A \in \gamma_j, j = 1, 2:\)

\[
\cap_{i=1}^n \pi_j(g_i)(\mathcal{O}_\epsilon(A)) = \emptyset.
\]

(Here and below it is convenient to assume that the \(\epsilon\)-neighbourhood of an \(A\) in the sphere, \(\mathcal{O}_\epsilon(A)\), is formed with respect to the geodesic distance.)

For every \(j = 1, 2\) and each \(A \in \gamma_j\) set

\[
\tilde{A} = \left\{ x \in S_{\mathcal{H}} : \|\pi_j(x)\| \geq \frac{\sqrt{2}}{2} \text{ and } \frac{\pi_j(x)}{\|\pi_j(x)\|} \in A \right\}.
\]

The collection \(\cup_{j=1,2} \{\tilde{A} : A \in \gamma_j\}\) covers \(S_{\mathcal{H}}\), and it is easy to see that for each \(j = 1, 2\) and each \(A \in \gamma_j\) one has

\[
\cap_{i=1}^n \pi_j(g_i)(\mathcal{O}_\delta(\tilde{A})) = \emptyset
\]

whenever \(\delta\) is sufficiently small, for example \(\delta < \min\{\epsilon/3, \pi/8\}\). □

3. From concentration property to amenability

**Proposition 3.1.** Let \(\pi\) be a representation of a group \(G\) in a Hilbert space \(\mathcal{H}\). If the space \(\mathcal{C}_u^0(S_{\mathcal{H}})\) of all bounded uniformly continuous functions on the unit sphere \(S_{\mathcal{H}}\) admits a \(G\)-invariant mean, then the representation \(\pi\) is amenable.

**Proof.** Let \(\psi : \mathcal{C}_u^0(S_{\mathcal{H}}) \to \mathbb{C}\) denote a \(G\)-invariant mean on the unit sphere \(S_{\mathcal{H}}\), that is, a positive functional of norm 1, taking the function 1 to 1 and such that \(\psi(gf) = \psi(f)\) for all \(g \in G\) and all \(f \in \mathcal{C}_u^0(S_{\mathcal{H}})\), where \(gf(x) := f(\pi(g)x)\). For every bounded linear operator \(T\) on \(\mathcal{H}\) define a function \(f_T : S_{\mathcal{H}} \to \mathbb{C}\) by

\[
S_{\mathcal{H}} \ni \xi \mapsto f_T(\xi) := (T\xi, \xi) \in \mathbb{C}.
\]

Then \(f_T\) is bounded (by \(\|T\|\)) and Lipschitz with constant \(2\|T\|:\)

\[
f_T(\xi) - f_T(\eta) = (T\xi, \xi) - (T\eta, \eta)
\]

\[
= (T\xi, \xi - \eta) + (T(\xi - \eta), \eta)
\]

\[
\leq 2\|T\| \cdot \|\xi - \eta\|.
\]

Therefore, \( f_T \in C^b_u(S_\mathcal{H}) \). Set

\[
\phi(T) := \psi(f_T).
\]

(3.3)

The following properties of the mapping \( \phi: L(H) \to \mathbb{C} \) are obvious.

1. \( \phi \) is linear. [If \( T, S \in L(H) \) and \( \lambda, \mu \in \mathbb{C} \), then for every \( \xi \in S_\mathcal{H} \) one has \( f_{\lambda T + \mu S}(\xi) = ((\lambda T + \mu S)\xi, \xi) = \lambda f_T(\xi) + \mu f_S(\xi) \), and consequently \( \phi(\lambda T + \mu S) = \psi(f_{\lambda T + \mu S}) = \lambda \psi(f_T) + \mu \psi(f_S) = \lambda \phi(T) + \mu \phi(S) \).]

2. \( \phi \) is positive. [If \( T \geq 0 \), then \( f_T(\xi) = (T\xi, \xi) \geq 0 \) for all \( \xi \), therefore \( \phi(T) = \psi(f_T) \geq 0 \).]

3. \( \phi(1) = 1 \). [\( f_1(\xi) = (\xi, \xi) \equiv 1 \).]

4. \( \phi \) is \( G \)-invariant. [Let \( g \in G \) and \( T \in L(H) \). Then

\[
\begin{align*}
f_{\pi(g)T\pi(g)^{-1}}(\xi) &= (\pi(g)T\pi(g)^{-1}(\xi), \xi) \\
&= (T\pi(g)^{-1}\xi, \pi(g)^{-1}\xi) \\
&= f_T(\pi(g)^{-1}\xi) \\
&= g^{-1}(f_T)(\xi).
\end{align*}
\]

Therefore,

\[
\begin{align*}
\phi(\pi(g)T\pi(g)^{-1}) &= \psi(f_{\pi(g)T\pi(g)^{-1}}) \\
&= \psi(g^{-1}f_T) \\
&= \psi(f_T) \\
&= \phi(T).
\end{align*}
\]

(3.4)

The conditions (1)-(3) imply that \( \phi \) is also bounded of norm 1 (cf. [28], Prop. 1.5.1), and by (4) \( \phi \) is a \( G \)-invariant mean on \( L(H) \), as required.

Remark 3.2. The above result will be inverted in the concluding Section (Thm. 7.6), leading to a new equivalent definition of amenable representations, very much in the classical spirit.

\[
\square
\]

Corollary 3.3. Let \( \pi \) be a representation of a group \( G \) in a Hilbert space \( \mathcal{H} \). If the dynamical system \( (S_\mathcal{H}, \pi, G) \) has the concentration property, then the representation \( \pi \) is amenable.

Proof. Let \( x^* \) be a \( G \)-fixed point in the Samuel compactification \( \sigma S_\mathcal{H} \). For every \( f \in C^b_u(S_\mathcal{H}) \) set \( \psi(f) = \tilde{f}(x^*) \), where \( \tilde{f} \) is the unique continuous extension of \( f \) to \( \sigma S_\mathcal{H} \). Clearly, \( \psi \) is a \( G \)-invariant mean on \( C^b_u(S_\mathcal{H}) \).

\[
\square
\]

Remark 3.4. The converse statement is false: indeed, every finite-dimensional representation is amenable [4], but for such a representation the concentration property is equivalent to the existence of a non-zero invariant vector.

As can be seen from results of Gromov and Milman, the assumption of infinite-dimensionality of the space of representation is essential for deriving the concentration property of the sphere. This observation will be reinforced in the subsequent sections of our article. Strictly speaking, Corollary 3.3 cannot be inverted even for infinite-dimensional Hilbert spaces (Example [11] below). However, if one dismisses ‘trivially amenable’ representations (that is, those whose amenability stems from the existence
of a single finite-dimensional representation), then the concentration property of the sphere is back, cf. Theorem 6.4. ▲

Corollary 3.5. Let $G$ be a locally compact group. Denote by $\pi_2$ the left regular representation of $G$, and by $S_2$ the unit sphere in the space $L_2(G)$. If the system $(S_2, G, \pi_2)$ has the concentration property, then $G$ is an amenable LC group.

Proof. It is enough to apply the following result of Bekka ([2], Thm. 2.2): the left regular representation $\pi_2$ of a LC group $G$ is amenable if and only if $G$ is amenable.

It is instructive to look at the direct proof as well. Let $x^*$ be a $G$-fixed point in $\sigma S_2$. For every Borel set $A \subseteq G$ and each $f \in S_2$ set $z_A(f) = \|\chi_A \cdot f\|_2$, where $\chi_A$ denote, as usual, the characteristic function of $A$, and the dot stands for the multiplication (equivalence classes of) functions. Since the mapping $f \mapsto \|f\|_2$ is 2-Lipschitz on $S_2$, so is the function $z_A : S_2 \rightarrow \mathbb{R}$. Being also bounded, $z_A \in C^b(S_2)$. Denote by $\tilde{z}_A$ the unique continuous extension of $z_A$ to the Samuel compactification of the sphere, and set $m(A) = \tilde{z}_A(x^*)$. Then $m$ is a finitely-additive left-invariant normalized measure on Borel subsets of $G$, vanishing on locally null sets, and consequently $G$ is amenable. □

Corollary 3.6. The unit sphere of an infinite-dimensional Hilbert space $\mathcal{H}$ admits no left invariant means on bounded uniformly continuous functions with respect to the full unitary group $U(\mathcal{H})$.

Proof. It is enough to make an obvious remark: an $U(\mathcal{H})$-invariant mean on $C^b_u(S_\mathcal{H})$ is invariant with respect to the action of every group $G$ represented in $\mathcal{H}$ by unitary operators. Since $\mathcal{H}$ is infinite-dimensional, one can find a non-amenable discrete group $G$ of the same cardinality as is the density character of $\mathcal{H}$, and to realize $\mathcal{H}$ as $l_2(G)$. (For example, take as $G$ the free group of rank equal to the density character of $\mathcal{H}$.) Now one can apply Prop. 3.1. □

Remark 3.7. The above result means, in essence, that there exists no Lévy-type integral of uniformly continuous functions on the sphere that is invariant under the action of the full unitary group. (Cf. [11, 21].) Soon we will see (Theorem 7.6) that the existence of a $G$-invariant Lévy-type integral on the unit sphere $S_\mathcal{H}$ is in fact equivalent to the amenability of the representation of $G$ in $\mathcal{H}$. ▲

Corollary 3.8. Let $\mathcal{H}$ be a Hilbert space. The pair $(S_\mathcal{H}, U(\mathcal{H}))$ does not have the concentration property.

Proof. If $\mathcal{H}$ is infinite-dimensional, the statement follows from Corol. 3.3 or 3.6. If $\dim \mathcal{H} < \infty$, the unitary group $U(n)$ possesses no non-zero invariant vectors, and there is no concentration property in a trivial way. □

Example 3.9. The first counter-example showing the absence of the concentration property of the system $(S^\infty, U(l_2))$ was obtained in 1988 by Imre Leader [17] (unpublished). This example is remarkably simple — indeed, as simple as one can probably ever get. Here we reproduce it upon a kind permission from the author.

Let $S^\infty$ denote the unit sphere in the space $l_2(\mathbb{N})$. For a $\Gamma \subseteq \mathbb{N}$, let $p_\Gamma$ stand for the orthogonal projection of $l_2(\mathbb{N})$ onto its subspace $l_2(\Gamma)$. Denote by $E$ (respectively,
Let \( F \) the set of all even (resp., odd) natural numbers, and let
\[
A = \left\{ x \in S^\infty : \|P_E x\| \geq \sqrt{2}/2 \right\},
\]
\[
B = \left\{ x \in S^\infty : \|P_E x\| \leq \sqrt{2}/2 \right\}.
\]

Clearly, \( A \cup B = S^\infty \). At the same time, both \( A \) and \( B \) are inessential. To see this, let \( E_1, E_2, E_3 \) be three arbitrary disjoint infinite subsets of \( \mathbb{N} \), and let \( \phi_i : \mathbb{N} \to \mathbb{N} \) be bijections with \( \phi_i(E) = E_i, i = 1, 2, 3 \). Let \( g_i \) denote the unitary operator on \( l_2(\mathbb{N}) \) induced by \( \phi_i \). Now
\[
g_i(A) = \left\{ x \in S^\infty : \|p_{E_i} x\| \geq \sqrt{2}/2 \right\},
\]
and consequently
\[
O_\epsilon(g_i(A)) \subseteq \left\{ x \in S^\infty : \|p_{E_i} x\| \geq \left( \sqrt{2}/2 \right) - \epsilon \right\}.
\]
Thus, as long as \( \epsilon < \sqrt{2}/2 - \sqrt{3}/3 \), we have
\[
\bigcap_{i=1}^3 O_\epsilon(g_i(A)) = \emptyset.
\]
The set \( B \) is treated in exactly the same fashion.

Another such counter-example can be obtained through combining the proof of Corol. 3.3 with von Neumann’s proof of non-amenability of \( F_2 \) (cf. e.g., [24], ex. 0.6).

In this form the link between amenability and concentration property becomes very transparent, while the construction remains remarkably similar to that in Leader’s example 3.9.

Example 3.10. Let \( a, b \) be free generators of \( F_2 \), and let \( \pi = \pi_2 \) be the left regular representation of \( F_2 \) in \( \mathcal{H} = l_2(F_2) \); we will write \( xf \) for \( \pi(x)(f) \). Denote by \( W_n \) the collection of all words whose irreducible representation starts with \( a^n, n \in \mathbb{Z} \). Set
\[
A_1 = \{ f \in S_2 : \|\chi_{W_0} \cdot f\| \leq 1/3 \},
\]
\[
A_2 = \{ f \in S_2 : \|\chi_{W_0} \cdot f\| \geq 1/3 \},
\]
and
\[
F = \{ a, a^2, a^3, a^4, b \}.
\]
Clearly, \( S_2 = A_1 \cup A_2 \). Both \( A_1 \) and \( A_2 \) are \( F \)-inessential. Indeed, if \( f \in A_1 \), then
\[
\|\chi_{W_0} \cdot bf\| \geq \|\chi_{F_2 \setminus W_0} \cdot f\| \geq 2/3
\]
and consequently
\[
O_{1/12}(A_1) \cap O_{1/12}(bA_1) = \emptyset.
\]
If \( f \in A_2 \), there is an \( i \in \{ 1, 2, 3, 4 \} \) such that \( \|\chi_{W_{-1}} \cdot f\| < 1/6 \), and consequently
\[
\|\chi_{W_0} \cdot a^if\| < 1/6,
\]
meaning that
\[
O_{1/12}(A_2) \cap \bigcap_{i=1}^4 O_{1/12}(a^iA_2) = \emptyset.
\]

4. Dynamical corollaries

Let $G$ be a topological group and let $X$ be a topological $G$-space, that is, a topological space equipped with a continuous action of $G$. The maximal $G$-compactification of $X$ is a compact $G$-space $\alpha_G(X)$ together with a morphism of $G$-spaces (that is, a $G$-equivariant continuous mapping) $i: X \to \alpha_G(X)$ such that any morphism from $X$ to a compact $G$-space uniquely factors through $i$ [33, 34, 19]. Necessarily, the image $i(X)$ is everywhere dense in $\alpha_G(X)$, though, somewhat surprisingly, the mapping $i$ need not be a homeomorphic embedding — in fact, it can be even a constant mapping for a nontrivial $G$-space $X$, cf. [19].

By $G/H \overset{\text{r}}\rightarrow$ we denote the left factor-space $G/H$ of a topological group $G$ by a closed subgroup $H$, equipped with the uniformity whose basis is formed by entourages of the form $V \overset{\text{r}}\rightarrow = \{(xH, yH) : xy^{-1} \in V\}$, where $V$ is a neighbourhood of $e_G$. One can show, using results from [19], that in general the uniform space $G/H \overset{\text{r}}\rightarrow$ need not be separated and can even induce the indiscrete topology.

If $H = \{e_G\}$, then we obtain the uniform space $G_0$, which is always separated and induces the topology of the group $G$.

We say that a function $f: G/H \rightarrow \mathbb{R}$ is $\overset{\text{r}}\rightarrow$-uniformly continuous ($\overset{\text{r}}\rightarrow$-u.c.), if it satisfies the condition: for every $\epsilon > 0$, there is a $V \ni e$ with
\[
\forall x, y \in G, \ xy^{-1} \in V \Rightarrow |f(xH) - f(yH)| < \epsilon
\] (4.1)

Remark 4.1. We deliberately avoid using the ‘right/left uniformly continuous’ terminology, because the mathematical community seems to be divided into two groups of roughly the same size, one of them calling the $\overset{\text{r}}\rightarrow$-u.c. functions ‘right’ uniformly continuous, the other ‘left’ uniformly continuous; references to the both kinds of usage are given in [25].

Our system of notation, suggested in [25, 26], has a mnemonic advantage: the symbol $\overset{\text{r}}\rightarrow$ (in TEX, \overset{\text{r}}\rightarrow) reminds of the position of the inversion symbol in the expression $xy^{-1}$. The functions satisfying the property
\[
\forall x, y \in G, \ x^{-1}y \in V \Rightarrow |f(xH) - f(yH)| < \epsilon
\] (4.2)
are naturally called $\overset{\text{r}}\rightarrow$-uniformly continuous. ▲

Notice that bounded $\overset{\text{r}}\rightarrow$-uniformly continuous functions on $G/H$ are identified in an obvious way with bounded $\overset{\text{r}}\rightarrow$-u.c. functions on $G$ that are constant on each left coset $xH, \ x \in G$. Their totality forms a $G$-invariant $C^*$-subalgebra of $\overset{\text{r}}\rightarrow C_u(G_0)$, which we will denote by $\overset{\text{r}}\rightarrow C_u(G/H_0)$.

The following result must be known, but it is difficult to find an exact reference.

Proposition 4.2. The maximal $G$-compactification of the left topological $G$-space $G/H$ coincides with the Samuel compactification of $G/H_0$.

Proof. Since the left regular representation of $G$ in $\overset{\text{r}}\rightarrow C_u(G_0)$ (defined by $(gf)(x) = f(gx)$) is well known (and easily checked) to be strongly continuous [32, 33, 1, 26], so is the subrepresentation of $G$ in $\overset{\text{r}}\rightarrow C_u(G/H_0)$. Now it follows from a result of Teleman [23] that the action of $G$ on the Gelfand spectrum of $\overset{\text{r}}\rightarrow C_u(G/H_0)$ is continuous, that is,
σ(G/H_r) is a topological G-space. The uniformly continuous mapping of compactification G/H_r → σ(G/H_r) has everywhere dense image and is G-equivariant.

It only remains to prove the maximality of σ(G/H_r) as a G-equivariant compactification of G/H. Let X be a compact G-space, and let φ: G/H → X be a continuous G-equivariant mapping. It determines a morphism of C^*-algebras, φ^*, from C(X) to C_b^u(G/H_r) via

\[ C(X) \ni f \mapsto [(xH) \mapsto \tilde{f}(xH) := f(\phi(xH))] \in C_b^u(G/H_r). \tag{4.3} \]

The dual continuous mapping f~: σ(G/H_r) → X between the Gelfand spaces of the corresponding C^*-algebras is G-equivariant and its restriction to G/H is easily seen to coincide with f. The proof is thus finished.

**Corollary 4.3.** The pair (G/H_r, G) has the concentration property if and only if α_G(G/H) has a fixed point.

The superscripts ‘u’ and ‘s’ will denote the uniform (respectively strong) operator topology on the unitary group. Since the sphere S_H is both uniformly and as a U(H)-space isomorphic to (U(H)_u/St_ξ)^r, where ξ ∈ S_H is any and St_ξ is the stabilizer of ξ, we obtain:

**Corollary 4.4.** The maximal U(H)_u-compactification of the unit sphere of a Hilbert space H has no fixed points.

**Remark 4.5.** One should compare this result with Stoyanov’s theorem [30]: the maximal U(l_2)_s-compactification of S^∞ coincides with the unit ball of l_2 with the weak topology, and thus has a fixed point. Another way to reformulate Stoyanov’s result is of course this: the homogeneous space (U(H)_s/St_ξ)^r is uniformly isomorphic to the sphere S_H equipped with the restriction of the additive uniform structure of H_w, where the latter denotes the Hilbert space with its weak topology.

A topological group G is called extremely amenable (e.a.) [24, 25, 26] if every continuous action of G on a compact space has a fixed point. This property is equivalent to the existence of a fixed point in the greatest ambit S(G) of G [1, 33, 32, 26], that is, the Samuel compactification of G_r.

**Corollary 4.6.** A topological group G is extremely amenable if and only if the pair (G_r, G) has the concentration property.

**Remark 4.7.** There are very few known examples of extremely amenable topological groups: those due to Herer and Christensen [15], Gromov and Milman [12], Glasner [8] (who notes that examples of the same kind were independently discovered by Furstenberg and B. Weiss but never published), and the present author [25]. Nevertheless, they include some very natural topological groups having importance in Analysis, for example the group U(∞) equipped with the Hilbert-Schmidt metric [12] (and therefore the groups U(∞)_u and U(l_2)_s), and the group Homeo_+(I) of orientation-preserving homeomorphisms of the closed interval [23].

Recall that the Calkin group is the topological factor-group of U(l_2)_u by the closure of U(∞) (in the uniform topology). This closure, \( \overline{U(\infty)} \), is a normal subgroup of U(l_2), consisting of all operators of the form I + T, where T is compact. Denote by T the
subgroup of $U(l_2)$ consisting of all scalar multiples of the identity $\lambda \mathbb{I}$, where $\lambda \in \mathbb{C}$ and $|\lambda| = 1$. The group $\mathbb{T} \cdot \overline{U(\infty)}$ forms the largest proper closed normal subgroup of $U(l_2)_u$ \cite{14}. (Actually, the statement remains true even without the word 'closed' \cite{14}.) The topological factor-group $U(l_2)_u/(\mathbb{T} \cdot \overline{U(\infty)})$ is called the projective Calkin group.

**Corollary 4.8.** The projective Calkin group admits an effective minimal action on a compact space.

*Proof.* By Proposition 4.2, the action of $U(H)_u$ upon $\sigma(S_H)$ is continuous for every Hilbert space $H$, that is, $\sigma(S_H)$ forms a compact $U(H)_u$-space. According to a result by Gromov and Milman that we cited in the Introduction \cite{12, Example 5.1}, if a compact group $G$ acts by isometries on the unit sphere $S^\infty$ of $l_2$, then the pair $(S^\infty, G)$ has the concentration property. It means that there exists a $\mathbb{T}$-fixed point $x_1 \in \sigma(S^\infty)$. Denote by $\mathfrak{X}$ the closure of the $U(l_2)$-orbit of $x_1$ in $\sigma(S^\infty)$. It is a compact $U(l_2)_u$-subspace of $\sigma(S^\infty)$. Since $\mathbb{T}$ is the centre of $U(l_2)$, every point of $\mathfrak{X}$ is $\mathbb{T}$-fixed. (In particular, it follows that $\mathfrak{X}$ is a proper subspace of $\sigma(S^\infty)$.)

It is a well-known and easy consequence of Zorn’s lemma that every compact $G$-flow contains a minimal subflow (that is, a non-empty compact $G$-subspace such that the orbit of each point is everywhere dense in it, see e.g. \cite{14}). Denote by $\mathcal{M}$ any minimal subflow of $\mathfrak{X}$. Since $U(l_2)$ has no fixed points in $\sigma(S^\infty)$ (Corol. 4.4), it follows that every minimal $U(l_2)$-subflow of $\sigma(S^\infty)$ is nontrivial, that is, contains more than one point. In particular, this applies to $\mathcal{M}$.

By force of the extreme amenability of the group $U(\infty)_u$ (combine the results of \cite{14} and \cite{13}), there is a $U(\infty)$-fixed point, $x^*$, in $\mathcal{M}$. It follows from the continuity of the action that $x^*$ is also a fixed point for $\overline{U(\infty)}$, that is, the stabilizer $\mbox{St}_{x^*}$ contains $\overline{U(\infty)}$. Since every point of $\mathcal{M}$ is $\mathbb{T}$-fixed, it follows that $x^*$ is fixed under the action of the group $\mathbb{T} \cdot \overline{U(\infty)}$.

The stabilizers of elements of the orbit of $x^*$ under the action of $U(l_2)$ are conjugate to $\mbox{St}_{x^*}$. Since $\mathbb{T} \cdot \overline{U(\infty)}$ is normal in $U(l_2)_u$, every such stabilizer contains $\mathbb{T} \cdot \overline{U(\infty)}$. Because of minimality of $\mathcal{M}$, the $U(l_2)$-orbit of $x^*$ is everywhere dense in $\mathcal{M}$, and we conclude: all points of $\mathcal{M}$ are fixed under the action of $\mathbb{T} \cdot \overline{U(\infty)}$. It implies that the action of $U(l_2)_u$ on $\mathcal{M}$ factors through an action of the projective Calkin group $U(l_2)_u/(\mathbb{T} \cdot \overline{U(\infty)})$, and the latter action is continuous. Moreover, it is also minimal.

Denote by $\mathcal{K}$ the set of all $u \in U(l_2)_u$ leaving each element of $\mathcal{M}$ fixed. This is a closed normal subgroup of $U(l_2)_u$, containing $\overline{U(\infty)}$, and since it is proper (in view of minimality and nontriviality of $\mathcal{M}$), it must be contained in $\mathbb{T} \cdot \overline{U(\infty)}$ and consequently coincide with it. It means that the action of the projective Calkin group is on the compact space $\mathcal{M}$ is minimal and effective, and the statement is proved. \hfill $\square$

**Remark 4.9.** Contrary to what was in effect claimed in \cite{12}, Remark 3.5, the concentration of measure on finite permutation groups \cite{13} (cf. also \cite{31}) does not lead to the extreme amenability of the infinite symmetric group $S_\infty$. In fact, the group $S_\infty$ of all (finite) permutations of a countably infinite set $\omega$, equipped with the topology of pointwise convergence on the (discrete) set $\omega$, acts effectively on its universal minimal compact flow and, in particular, admits continuous actions on compacta without fixed points (\cite{23}, Th. 6.5). This result, combined with a theorem of Gaughan \cite{4}
that every Hausdorff group topology on $S_\infty$ contains the topology of pointwise convergence, immediately implies that there is no Hausdorff group topology making $S_\infty$ into an extremely amenable topological group.

In particular, $S_\infty$ cannot be made into a Lévy group in the sense of [12, 20, 21]. In other words, the concentration of measure on the family of finite symmetric groups $S_n$ cannot be observed with respect to a right-invariant metric generating a group topology on $S_\infty$.

The Hamming distance on finite groups of permutations $S_n$, $n \in \mathbb{N}$ is given by

$$d(\sigma_1, \sigma_2) = |\{ i : \sigma_1(i) \neq \sigma_2(i) \}|.$$  \hspace{1cm} (4.4)

While the group $S_\infty$ can be represented as the union of an increasing chain of finite permutation groups $S_n$, the above observation essentially says that there is no ‘coherent’ way of putting together the normalised Hamming distances so as to obtain a right-invariant metric on $S_\infty$.

In [12], Remark 3.5, it was suggested to consider with that purpose the function

$$\varphi(\sigma, \eta) = \begin{cases} 
\frac{d(\sigma, \eta)}{\max\{d(\sigma, e), d(\eta, e)\}}, & \text{if } \sigma \neq \eta, \\
0, & \text{otherwise},
\end{cases} \hspace{1cm} (4.5)$$

and then to choose a metric, $\hat{d}$, on $S_\infty$, determining the topology of the latter group and Lipschitz equivalent to $\varphi$ with Lipschitz constant 2.

Such a metric of course does exist. However, what matters for the concentration property and the existence of fixed points in compactifications, is not the topology of a topological group $G$ per se, but the uniform structure $U_\cdot$ of $G$. Let us show that the uniform structure generated by $\hat{d}$ has the property that the right translations of $S_\infty$ do not form a right equicontinuous family, and therefore this uniform structure does not coincide with the uniform structure $U_\cdot$ of any group topology on $S_\infty$. (Notice that if $(x, y) \in V_\cdot$ and $g \in G$, then $(xg)(yg)^{-1} = xgg^{-1}y^{-1} = xy^{-1} \in V$, that is, $(xg, yg) \in V_\cdot$ as well, hence the equicontinuity property for right translations.)

In view of the Lipschitz equivalence of $\hat{d}$ and $\varphi$, the following sets form a basis of entourages of the diagonal for the uniformity generated by $\hat{d}$ as $\epsilon$ runs over all positive reals:

$$V_\epsilon := \{ (\sigma, \eta) \in S_\infty : \varphi(\sigma, \eta) < \epsilon \}.$$  \hspace{1cm} (4.6)

Equicontinuity of right translations means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $(\sigma, \eta) \in V_\delta$ and $\theta \in S_\infty$, one has $(\sigma \theta, \eta \theta) \in V_\epsilon$. Let $n$ be even, and set

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \ldots & n-1 & n \\ 2 & 1 & 4 & 3 & 6 & 5 & \ldots & n & n-1 \end{pmatrix},$$

$$\eta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \ldots & n-1 & n \\ 1 & 2 & 4 & 3 & 6 & 5 & \ldots & n & n-1 \end{pmatrix}.$$  \hspace{1cm} (4.7)

One has $\varphi(\sigma, \eta) = 2/n$ and thus, by choosing $n$ sufficiently large, we can make the pair $(\sigma, \eta)$ belong to any entourage $V_\delta$, $\delta > 0$. At the same time, $\varphi(\sigma \eta, \eta^2) = \varphi \left( \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, e \right) = 2/2 = 1$, that is, the right translation of every entourage of the form $V_\delta$ is not a subset of $V_1$, however small $\delta > 0$ be. ▲
5. SOME LEMMATA ON THE GEOMETRY OF SPHERES

Recall that a probabilistic metric space is a triple, \((X, \rho, \mu)\), formed by a metric space \((X, \rho)\) and a normalised \((\mu(X) = 1)\) Borel measure on \(X\).

For a subset \(A \subseteq X\) we denote by \(\mathcal{O}_\epsilon(A)\) the \(\epsilon\)-neighbourhood of \(A\) in \(X\).

The concentration function, \(\alpha = \alpha_X\), of a probabilistic metric space \(X\) is defined for each \(\epsilon > 0\) by

\[
\alpha_X(\epsilon) = 1 - \inf \left\{ \mu(\mathcal{O}_\epsilon(A)) : A \subseteq X \text{ is Borel and } \mu(A) \geq \frac{1}{2} \right\}
\]

and \(\alpha_X(0) = 1/2\). It is a decreasing function in \(\epsilon\).

The following observations are straightforward yet useful. (See e.g. Lemma 3.2 in [20].)

**Lemma 5.1.** Let \(X\) be a probabilistic metric space with the concentration function \(\alpha\). Let \(A \subseteq X\) and \(\epsilon > 0\).

1. If \(\mu(A) > \alpha(\epsilon)\), then \(\mu(\mathcal{O}_\epsilon(A)) > 1/2\).
2. If \(\mu(A) > \alpha(\epsilon/2)\), then \(\mu(\mathcal{O}_\epsilon(A)) \geq 1 - \alpha(\epsilon/2)\). \(\square\)

A family \((X_n)_{n=1}^{\infty}\) of probabilistic metric spaces is called a Lévy family if for each \(\epsilon > 0\), \(\alpha_X(\epsilon) \to 0\) as \(n \to \infty\), and a normal Lévy family (with constants \(C_1, C_2 > 0\)) if for all \(n\) and \(\epsilon > 0\)

\[
\alpha_X(\epsilon) \leq C_1 e^{-C_2 \epsilon^2 n}.
\]

By \(\mu_n\) we will denote the (unique) normalized rotation-invariant Borel measure on the \(n\)-dimensional Euclidean sphere \(S^n\). The distances between points on the spheres will be geodesic distances. In such a way, the sphere \(S^n\) becomes a probabilistic metric space. The family of spheres \(S^{n+1}\), \(n \in \mathbb{N}\) is normal Lévy with constants \(C_1 = \sqrt{\pi/8}\) and \(C_2 = 1/2\).

For the major concepts, examples, and results of the theory of concentration of measure on high-dimensional structures, see [3, [2, [18, [20, [21, [22, [24, [29, [31].

**Lemma 5.2.** There are absolute constants \(C_1, C_2 > 0\) with the following property. Let \(\mathcal{H}\) be a real Hilbert space, let \(n \in \mathbb{N}\) and \(\epsilon > 0\). Let \(P_1\) and \(P_2\) be two rank \(n\) orthogonal projections in \(\mathcal{H}\), satisfying

\[
\|P_1 - P_2\|_1 < n\epsilon,
\]

where \(\|\cdot\|_1\) denotes the trace class operator norm. Denote by \(S_i\) the unit sphere in the space \(P_i(\mathcal{H})\), \(i = 1, 2\). Then

\[
\mu_n \left( \{ x \in S_1 : \|x - P_2 x\| < \epsilon \} \right) \geq 1 - C_1 \exp(-C_2 \epsilon^2 n).
\]

**Proof.** Let \(0 < \delta < 1\) be arbitrary and fixed. Let for every natural number \(n\), \(P_1^{(n)}\) and \(P_2^{(n)}\) be two rank \(n\) projections in \(\mathcal{H}\). Assume for a while that, as \(n \to \infty\),

\[
\mu_n \left( \{ x \in S_1 : \|x - P_2 x\| < \delta \} \right) = O(1) \exp(-O(1)n),
\]

that is, for some positive constants \(C_1, C_2 > 0\) and all \(n\) one has

\[
\mu_n \left( \{ x \in S_1 : \|x - P_2 x\| < \delta \} \right) \leq C_1 \exp(-C_2 n),
\]
where \( S_1^{(n)} \) denotes the unit sphere in the space \( P_1^{(n)}(\mathcal{H}) \), \( i = 1, 2 \).

Then a standard argument (cf. e.g. \([21]\), p. 276) implies that, for \( n \) sufficiently large, there is an orthonormal basis in \( P_1^{(n)}(\mathcal{H}) \) formed by elements \( e_1, e_2, \ldots, e_n \in S_1^{(n)} \) each at a distance \( \geq \delta \) from \( P_2^{(n)}(\mathcal{H}) \). Indeed, fix a \( \xi \in S_1^{(n)} \) and denote by \( \nu \) the Haar probability measure on \( SO(n) \). Then
\[
\nu \{ u \in SO(n): \| u\xi - P_2(u\xi) \| \geq \delta \} = \mu_n \left( \left\{ x \in S_1^{(n)}: \| x - P_2 x \| \geq \delta \right\} \right) \geq 1 - C_1 \exp(-C_2n),
\]
(5.6)
therefore for any finite subset \( F \subseteq S_1^{(n)} \)
\[
\nu \{ u \in SO(n): \| x - P_2 x \| \geq \delta \text{ for all } x \in uF \} \geq 1 - |F| \cdot C_1 \exp(-C_2n). \tag{5.7}
\]
If the size \( |F| \) of \( F \) grows slower than \( C_1 \exp(C_2n) \), for example if \( |F| \) is polynomial in \( n \), then for \( n \) satisfying the condition \( n < C_1 \exp(C_2n) \) we can find a rotation \( u \) taking \( F \) outside of the \( \delta \)-neighbourhood of \( P_2^{(n)}(\mathcal{H}) \) in \( S_1^{(n)} \). Applying this observation to an arbitrary orthonormal basis of \( P_1^{(n)}(\mathcal{H}) \) as \( F \), we obtain an orthonormal system \( e_1, e_2, \ldots, e_n \in S_1^{(n)} \) with the desired property.

Now extend the collection of \( e_i, i = 1, 2, \ldots, n \) to an orthonormal basis \( (e_i)_{i \leq \alpha} \) of \( \mathcal{H} \). One has
\[
\left\| P_1^{(n)} - P_2^{(n)} \right\|_1 = \text{Tr} \left( \left| P_1^{(n)} - P_2^{(n)} \right| \right) \geq \sum_{i < \alpha} \left| (P_1^{(n)} - P_2^{(n)})(e_i), e_i \right| \geq \sum_{i=1}^n 1 - (P_2^{(n)}(e_i), e_i) \geq n \left( 1 - \sqrt{1 - \delta^2} \right). \tag{5.8}
\]
If now \( \epsilon > 0 \) and \( \delta > \sqrt{2\epsilon} \), then \( 1 - \delta^2 < (1 - \epsilon)^2 \) and the latter expression in (5.8) is greater than \( n \epsilon \).

The above argument establishes the following: for each pair of positive constants \( C_1, C_2 > 0 \), if \( n \) is so large that \( n < C_1 \exp(C_2n) \) and \( P_1, P_2 \) are two rank \( n \) projections in \( \mathcal{H} \) satisfying \( \| P_1 - P_2 \|_1 < n \epsilon \), then the measure of the set of points \( x \in S_1 \) that are at a distance \( < \sqrt{2\epsilon} \) from \( P_2^{(n)}(\mathcal{H}) \) is greater than \( C_1 \exp(-C_2n) \).

Since
\[
\alpha_{3n+1}(\epsilon) \leq \sqrt{\pi/8} \exp(-\epsilon^2 n/2) \tag{5.9}
\]
and therefore
\[
\alpha_{3n+1}(\sqrt{2\epsilon}) \leq \sqrt{\pi/8} \exp(-n \epsilon), \tag{5.10}
\]
one has in particular \( \mu_n \left( \left\{ x \in S_1: \| x - P_2 x \| < \sqrt{2\epsilon} \right\} \right) > \alpha_{3n}(\sqrt{2\epsilon}) \) whenever \( n \) is so large that
\[
n < \sqrt{\pi/8} \exp(-\epsilon(n - 1)). \tag{5.11}
\]
Now observe that the set
\[
\mathcal{O}_2 := \left\{ x \in S_1: \| x - P_2 x \| < 2\sqrt{2\epsilon} \right\}
\]
contains the open $\sqrt{2\epsilon}$-neighbourhood of the set
\[ O_1 := \{ x \in \mathbb{S}_1 : \| x - P_2 x \| < \sqrt{2\epsilon} \}. \]

According to Lemma 5.1, (2), one has
\[
\mu_n \left( \{ x \in \mathbb{S}_1 : \| x - P_2 x \| < \sqrt{2\epsilon} \} \right) \geq 1 - \alpha_{2n}(\sqrt{2\epsilon}) \\
\geq 1 - \alpha_{2n+1}(\sqrt{2\epsilon}) \\
\geq 1 - \frac{\pi}{8} e^{-\frac{1}{4}n\epsilon} \tag{5.12}
\]
whenever $n$ satisfies (5.11). Replacing $\epsilon$ in both formulae with $\epsilon^2/2$ yields the following:
\[
\mu_n \left( \{ x \in \mathbb{S}_1 : \| x - P_2 x \| < \epsilon \} \right) \geq 1 - \frac{\pi}{8} e^{-\frac{1}{4}n\epsilon^2} \tag{5.13}
\]
whenever $n$ is so large that
\[
n < \sqrt{\frac{\pi}{8}} \exp \left( - \frac{1}{2} \epsilon^2 (n - 1) \right) \tag{5.14}
\]
and $P_1, P_2$ are two orthogonal projections of rank $n$ satisfying $\| P_1 - P_2 \|_1 < n\epsilon$. And this is the desired result in slight disguise. \hfill \Box

Remark 5.3. Of course, in general one does not expect all points of $\mathbb{S}_1$ to be at a distance $< \epsilon$ from $\mathcal{H}_2$. Consider the projections $P_A, P_B$ in the space $l_2$, where $A, B$ are two distinct subsets of the index set $\mathbb{N}$ having the same finite cardinality $n$. If $i \in A \setminus B$, then $e_i \in \mathbb{S}_1$ and $d(e_i, \mathcal{H}_2) = 1$. At the same time, $P_A$ and $P_B$ can be chosen as to satisfy the condition $\| P_A - P_B \|_1 < n\epsilon$ with $\epsilon > 0$ is as small as desired, as the ‘Følner ratio’ $|A\Delta B|/|A \cup B| \to 0$. \hfill \blacktriangle

Lemma 5.4. Let $P_1$ and $P_2$ be projections of the same finite rank $n$ in a (real or complex) Hilbert space $\mathcal{H}$. Then there exist one-dimensional projections $e_i^j, j = 1, 2, i = 1, 2, \ldots, n$, such that
\[
P_j = \lor_{i=1}^n e_i^j, \quad j = 1, 2 \tag{5.15}
\]
and
\[
e_i^j \perp e_k^m \quad \text{whenever } j, k \in \{1, 2\}, i, m \in \{1, 2, \ldots, n\}, \text{ and } i \neq m. \tag{5.16}
\]

Proof. Let $x$ be an eigenvector of $P_1 + P_2$ corresponding to an eigenvalue $\lambda$. Then $\lambda = 1 \pm \cos \theta$, where $\theta \in [0, \pi/2]$ is the angle between one-dimensional subspaces spanned by $P_1 x$ and $P_2 x$. The space $(P_1 \lor P_2)(\mathcal{H})$ has an orthogonal basis formed by eigenvectors of $P_1 + P_2$ which can be written in the form $x_i^\pm, i = 1, 2, \ldots, n$, where $x_i^+$ corresponds to the eigenvalue $\lambda_i^+ = 1 \pm \cos \theta_i, \theta_i$ being as above; if $\theta_i = 0$, then we only consider $\lambda_i^+ = 2$ and $x_i^+$. Since $P_j(x_i^+), j = 1, 2$ span the same subspace of dimension 2 (respectively 1 where $\theta_i = 0$) as the vectors $x_i^+$ and $x_i^-$ do (respectively, the vector $x_i^+$ does), it follows that $P_j(x_i^+) \perp P_k(x_m^+)$ whenever $i \neq m$. Now let $e_i^j$ be the projection onto the one-dimensional subspace spanned by $P_j(x_i^+)$. \hfill \Box
**Lemma 5.5.** There are absolute constants $C'_1, C'_2 > 0$ with the property that, under the assumptions and using the notation of Lemma 5.2, there exists an isometry $r: S_1 \to S_2$ with

$$
\mu_n \left( \{ x \in S_1 : \| x - r(x) \| < \epsilon \} \right) \geq 1 - C'_1 \exp(-C'_2\epsilon^2 n).
$$

(5.17)

**Proof.** Let the projections $e^i_j$, $j = 1, 2$, $i = 1, 2, \ldots, n$ be as in Lemma 5.4. For every $i = 1, 2, \ldots, n$ denote by $r_i$ the (unique) isometric isomorphism from the one-dimensional range of $e^1_i$ to that of $e^2_i$. (That is, the reflection across the linear span of $x^+_i$, using the notation from the proof of Lemma 5.4.) Since $P_2P_1x^+_i = (\lambda - 1)P_2x^+_i \in \text{span}(P_2x^+_i)$, one concludes that if $x \in \text{span}(P_1x^+_i)$ and $\| x \| = 1$, then

$$
\| r_i(x) - x \| \leq \sqrt{2} \text{dist}(x, P_2x^+_i) = \sqrt{2} \text{dist}(x, H_2).
$$

(5.18)

The orthogonal sum of linear operators $r = \bigoplus_{i=1}^n r_i$ is an isometry between $H_1$ and $H_2$. The equation (5.18) implies that for each $x \in H_1$, $\| x \| = 1$, one has

$$
\| r(x) - x \| \leq \sqrt{2} \text{dist}(x, H_2) \leq \sqrt{2} \text{dist}(x, S_2),
$$

(5.19)

and the proof is finished by applying Lemma 5.2. □

**Remark 5.6.** As an immediate corollary of the statement in the real case, Lemma 5.5 remains true in a complex Hilbert space $\mathcal{H}$ as well. ▲

### 6. From amenability to concentration property

From the previous work [12, 20, 21, 27] we know that if a group $G$ is compact or discrete amenable, and $\pi$ is a unitary representation of $G$ in an infinite-dimensional Hilbert space, then the dynamical system $(S_\pi, G)$ has the property of concentration. Our present aim is to push this result further as far as possible. A plausible-looking conjecture might be that the conclusion remains in force if $\pi$ is just an amenable representation of a group in an infinite-dimensional Hilbert space. However, this is not true.

**Example 6.1.** Let $G = F_2$ be the free group on two generators. Denote by $\pi_1$ an irreducible unitary representation of $F_2$ in a finite-dimensional Hilbert space $H_1$, and let $\pi_2$ denote the left regular representation of $F_2$ in the space $H_2 = l_2(F_2)$. Let $\pi = \pi_1 \oplus \pi_2$ be the direct sum representation of $F_2$ in the Hilbert space $\mathcal{H} = H_1 \oplus H_2$. Since $\pi$ contains a finite-dimensional subrepresentation $\pi_1$, it is amenable ([2], Th. 1.3, (i)+(ii)). Both $\pi_1$ and $\pi_2$ do not have the concentration property (cf. Ex. 3.10). Now it is enough to apply Proposition 2.4. ▲

Nevertheless, the above situation — in which amenability of a representation only stems from the presence of a single finite-dimensional subrepresentation — is, in fact, the only one where the concentration property is not to be found. If a representation is amenable in a ‘nontrivial way,’ the concentration property of spheres rebounds. The rest of this section will be devoted to establishing the corresponding result (Theorem 6.4), which not only generalizes all the previously obtained results in this direction, but is, in a sense, ‘at the end of the road.’
Lemma 6.2. Let $F$ be a finite collection of unitary operators on a (real or complex) Hilbert space $\mathcal{H}$. Suppose that for every $\epsilon > 0$ and every natural $k$ there is an orthogonal projection $P$ in $\mathcal{H}$ of rank $n \geq k$ such that

$$\|gPg^{-1} - P\|_1 < n\epsilon$$

for all $g \in F$. Then the system $(\mathcal{S}_{\mathcal{H}}, F)$ has the concentration property.

Proof. Choose a sequence of orthogonal projections $P_n$, $n \in \mathbb{N}$, having the properties:

1. $r_n = \text{rank } P_n \to \infty$, and
2. $\|gP_ng^{-1} - P_n\|_1 < r_n/n$ for all $g \in F$ as $n \to \infty$.

Denote by $S_n = S^{r_n}$ the unit sphere in the space $P_n(\mathcal{H})$, and for each $g \in F$ denote by $S^g_n = gS_n$ the unit sphere in the $r_n$-dimensional space $gP_n(\mathcal{H}) \equiv gP_ng^{-1}(\mathcal{H})$.

Now let $\gamma = \{A_1, A_2, \ldots, A_m\}$ be an arbitrary finite cover of the unit sphere $S_{\mathcal{H}}$. Clearly, for at least one $i$ such that $1 \leq i \leq m$ the set of natural numbers

$$\{n \in \mathbb{N} : \mu_{S_n}(A_i \cap S_n) \geq 1/m\} \quad (6.2)$$

is infinite. We claim that the set $A_i = A_i$ is then $F$-essential.

Proceeding to a subsequence of the selected sequence of projections if necessary, we can assume without loss in generality that $\mu_{S_n}(A_i \cap S_n) \geq 1/m$ for all $n \in \mathbb{N}$. For every $g \in F$ the measure of the set $gA \cap S^g_n$ in the latter sphere is the same as the measure of $A \cap S_n$, and therefore $\geq 1/m$ for all $n$.

Let an $\epsilon > 0$ be arbitrary. According to Lévy’s concentration of measure property of spheres, the measure of every set of the form $gO_\epsilon(A) \cap S^g_n$ in $S^g_n$, where $g \in F \cup \{e\}$, is $1 - O(1) \exp(-(O(1)\epsilon^2n))$.

An application of Lemma 5.5 and Remark 5.6 to $P = P_1$ and $gPg^{-1} = P_2$ yields that for every $g$ as above the measure of the set $gO_{2\epsilon}(A) \cap S_n$ in $S_n$ is $1 - O(1) \exp(-(O(1)\epsilon^2n))$. Indeed, there is an isometry $i_g : S^g_n \to S_n$ with the property that for all points $x$ of $S^g_n$ apart from a set of measure $O(1) \exp(-(O(1)\epsilon^2n))$ one has $\|x - i_g(x)\| < \epsilon$. Consequently, the set $gO_{2\epsilon}(A) \cap S_n$ contains the set $i_g(gO_\epsilon(A) \cap S^g_n)$. The measure of the latter set in $S_n$ is $1 - O(1) \exp(-(O(1)\epsilon^2n))$, because so is the measure of $gO_\epsilon(A) \cap S^g_n$ in $S^g_n$ and the isometry $i_g$ between the spheres is automatically a measure-preserving map.

We conclude that if $n$ is sufficiently large, then the sets $gO_{2\epsilon}(A)$, $g \in F$, have a non-empty common part, and indeed the measure of its intersection with the sphere $S_n$ is $1 - O(1)|F| \exp(-(O(1)\epsilon^2n))$. This finishes the proof.$\square$

Lemma 6.3. Let $\pi$ be a unitary representation of a group $G$ in a Hilbert space $\mathcal{H}$. If $\pi$ is not of the form $\pi_1 \oplus \pi_2$, where $\pi_1$ is finite-dimensional and $\pi_2$ is non-amenable, then for every finite subset $F \subseteq G$, every $\epsilon > 0$ and every natural $k$ there is an orthogonal projection $P$ in $\mathcal{H}$ of rank $n \geq k$ such that

$$\|\pi(g)P\pi(g)^{-1} - P\|_1 < n\epsilon$$

for all $g \in F$.

Proof. If $\pi$ admits finite-dimensional subrepresentations of arbitrarily high dimension, then the desired projections can be constructed in an obvious way. Otherwise, using the assumption of the lemma, one can assume without loss in generality till the end of the proof that $\pi$ contains no nontrivial finite-dimensional subrepresentations.
According to the Følner property of amenable representations as established by Bekka [2, Th. 6.2], for every finite $F \subseteq G$ and each $\epsilon > 0$ there is an orthogonal projection $P = P_{F,\epsilon}$ in $\mathcal{H}$ of finite rank such that

$$\|\pi(g)P\pi(g)^{-1} - P\|_1 < \|P\|_1 \epsilon$$

(6.4)

for all $g \in F$.

Suppose it is not in general possible to choose such a $P$ of an arbitrarily high finite rank. In such a case, there are a finite $\Phi \subseteq G$ and an $\epsilon' > 0$ such that for each finite $F \supseteq \Phi$ and each $\epsilon < \epsilon'$ an arbitrary projection $P$ satisfying (6.3) has rank $\|P\|_1 \leq N$, where $N$ is a fixed natural number. Moreover, one can assume without loss in generality that the equality is always achieved for a suitable $P = P_{F,\epsilon}$.

Notice that on the collection of all projections of fixed finite rank $N$ the trace class metric and the operator metric are both Lipschitz equivalent to the Hausdorff distance between unit spheres in the range spaces of the projections. (The equivalence of the trace class and operator metrics follows from the obvious inequality $\text{Tr}(|P_1 - P_2|) \leq 2N\|P_1 - P_2\|$ for every two projections $P_1, P_2$ of rank $N$. For the equivalence of the operator and Hausdorff metrics, see [23], 3.12 and 3.4.(h).) The space of all projections of rank $N$ is thus uniformly isomorphic to the Grassmannian $\text{Gr}_N(\mathcal{H})$ and forms a complete uniform space. (Cf. [23], 3.12.(c) and 3.4.(e).)

For every $F$ and $\epsilon$ as above denote by $\mathcal{P}_{F,\epsilon}$ the non-empty set of all projections of rank exactly $N$ satisfying (6.3). Now equip the set $\mathcal{P}_\omega(G) \times \mathbb{R}_+$ of all pairs $(F, \epsilon)$ as above with the product partial order making it into a directed set. The diameters of $\mathcal{P}_{F,\epsilon}$ cannot converge to zero over $\mathcal{P}_\omega(G) \times \mathbb{R}_+$. Otherwise the sets $\mathcal{P}_{F,\epsilon}$ would form a Cauchy prefilter having a limit point $P$, which is again a projection of rank $N$ satisfying the property (6.3) for every finite $F \subseteq G$ and each $\epsilon > 0$. In other words, $P$ commutes with every $g \in G$, that is, $P(\mathcal{H})$ is the space of a nontrivial finite-dimensional subrepresentation of $\pi$, leading to a contradiction. We conclude that for some $\delta > 0$, the set of pairs $(F, \epsilon)$ satisfying

$$\text{diam} (\mathcal{P}_{F,\epsilon}) \geq \delta$$

(6.5)

is cofinal in $\mathcal{P}_\omega(G) \times \mathbb{R}_+$.

It remains to notice that, if an arbitrary finite set $F \subseteq G$ is fixed, then having at one’s disposal, for every $\epsilon > 0$, a pair of projections $P_1$ and $P_2$ of the same finite rank $N$ satisfying

$$\|\pi(g)P_i\pi(g)^{-1} - P_i\| < \epsilon$$

(6.6)

for all $g \in F$ and $i = 1, 2$, and also the condition

$$\|P_1 - P_2\| \geq \delta$$

(6.7)

for a fixed $\delta > 0$ enables one to produce a new projection $P'$ of rank $\geq N + 1$ satisfying

$$\|\pi(g)P'\pi(g)^{-1} - P'\| < \epsilon_1,$$

(6.8)

where $\epsilon_1 \to 0$ as $\epsilon \to 0$, thus obtaining a contradiction with the presumed maximality of $N$.

Indeed, choose two sequences of projections $P_i^{(1)}$ and $P_i^{(2)}$ of rank $N$ having the properties $\|P_i^{(1)} - P_i^{(2)}\| \geq \delta$ and $\|\pi(g)P_i^{(j)}\pi(g)^{-1} - P_i\| \to 0$ as $i \to \infty$ for all $g \in F$.
Let $d > N + 1$.

Without loss of generality and proceeding to a subsequence if necessary, one can assume that $\|k e_1^i - k e_2^i\| \to c_k$ for each $k$ as $i \to \infty$, where $0 \leq c_1 \leq c_2 \leq \cdots \leq c_N$. Let $d$ be the smallest integer $\leq N$ with the property $c_d > 0$; clearly, such a $d$ exists. Let

$$P_i = \bigvee_{k=1}^{d-1} k e_1^i \lor \bigvee_{k=d}^N k e_1^i \lor \bigvee_{k=d}^N k e_2^i.$$ 

The rank of the projection $P_i$ is $2N - d + 1 > N$. The space $P_i(\mathcal{H})$ is the direct sum of subspaces $\mathcal{H}_i$, $i = 1, 2, \cdots, N$, where $\mathcal{H}_i = \text{span}(k e_1^i)$ for $k < d$ and $\mathcal{H}_i = \text{span}(k e_1^i, k e_2^i)$ for $k \geq d$.

For every $k \geq d$ and every $g \in F$, the Hausdorff distance between the unit spheres $S_{\mathcal{H}_k}$ and $\pi(g) S_{\mathcal{H}_k} \pi(g)^{-1}$ approaches zero as $i \to \infty$, because (i) the geodesic distances between $k e_1^i$ and $k e_2^i$ are bounded from below by some positive constant and at the same time do not exceed $\pi/2$, and (ii) the distances between $k e_1^i$ and $\pi(g) k e_1^i \pi(g)^{-1}$, $i = 1, 2, \cdots, N$, $j = 1, 2$ converge to zero as $i \to \infty$. For $k < d$ the same is true in a trivial sort of way. Since the spheres $S_{\mathcal{H}_k}$, $k = 1, 2, \cdots, n$ are pairwise orthogonal and the same is true of the spheres $\pi(g) S_{\mathcal{H}_k} \pi(g)^{-1}$, one concludes that the Hausdorff distance between the unit sphere $S_i$ in the space $P_i(\mathcal{H})$ and the unit sphere $\pi(g) S_i \pi(g)^{-1}$ approaches zero as $i \to \infty$. Since the rank of $P_i$ is bounded above by $2N$, this means $\|P_i - g P_i g^{-1}\|_1 \to 0$, as required.

\[\square\]

**Theorem 6.4.** Let $\pi$ be a unitary representation of a group $G$ in an infinite-dimensional Hilbert space $\mathcal{H}$. If every subrepresentation of $\pi$ having finite codimension is amenable (that is, $\pi$ is not of the form $\pi_1 \oplus \pi_2$, where $\pi_1$ is finite-dimensional and $\pi_2$ is non-amenable), then the dynamical system $(S_{\mathcal{H}}, G, \pi)$ has the concentration property.

**Proof.** Combine Lemmas 6.3 and 6.2. \[\square\]

### 7. Amenability vis-à-vis Concentration Property

Now we can deduce all our main results. To begin with, we obtain a description of subgroups of the full unitary group $U(\mathcal{H})$ whose action on the unit sphere has the concentration property.

**Theorem 7.1.** Let $\pi$ be a unitary representation of a group $G$ in a Hilbert space $\mathcal{H}$. The system $(S_{\mathcal{H}}, G, \pi)$ has the concentration property if and only if

- either $\pi$ has a non-zero invariant vector, or
- $\dim \mathcal{H} = \infty$ and every subrepresentation of $\pi$ of finite codimension is amenable.

**Proof.** $\Rightarrow$: If $\dim \mathcal{H} < \infty$, then under our assumption $\pi$ clearly has a non-zero invariant vector. Otherwise, suppose there exists a non-amenable subrepresentation $\pi_1$ of $\pi$ having finite codimension. If the finite-dimensional subrepresentation $\pi_1^\perp$ has no
non-zero invariant vectors, then it does not have the concentration property and the same is true of $\pi$ according to Proposition 2.6. \(\Rightarrow\): Immediate from Theorem 6.4. \(\Box\)

The following particular cases are of some interest.

**Corollary 7.2.** If a unitary representation of a group $G$ in a Hilbert space $\mathcal{H}$ is amenable and has no finite-dimensional subrepresentations, then the system $(\mathcal{S}_H, G, \pi)$ has the concentration property. \(\Box\)

**Corollary 7.3.** If $\pi$ is an amenable irreducible unitary representation of a group $G$ in an infinite-dimensional Hilbert space $\mathcal{H}$, then the system $(\mathcal{S}_H, G, \pi)$ has the concentration property. \(\Box\)

Now we are able to extend our criterion of amenability stated in [27] from discrete groups to locally compact ones.

**Theorem 7.4.** A locally compact group $G$ is amenable if and only if for every strongly continuous unitary representation $\pi$ of $G$ in an infinite-dimensional Hilbert space $\mathcal{H}$, the dynamical system $(\mathcal{S}_H, G, \pi)$ has the concentration property.

**Proof.** $\Rightarrow$: every strongly continuous unitary representation of an amenable locally compact group is amenable [2], and therefore so are all its subrepresentations, and Theorem 7.1 applies. $\Leftarrow$: if $G$ is finite, there is nothing to prove, otherwise $L_2(G)$ is infinite-dimensional and Corollary 3.3 applies together with our assumption. \(\Box\)

Conversely, we can characterize amenable representations in terms of the concentration property.

**Theorem 7.5.** A unitary representation $\pi$ of a group $G$ in a Hilbert space $\mathcal{H}$ is amenable if and only if

- either $\pi$ contains a finite-dimensional subrepresentation, or
- the $G$-space $(\mathcal{S}_H, G, \pi)$ has the concentration property.

**Proof.** $\Rightarrow$: If $\pi$ has no finite-dimensional subrepresentations, then Corollary 7.2 applies. $\Leftarrow$: Combine Corollary 3.3 with the following: a unitary representation is amenable if it contains a finite-dimensional subrepresentation ([2], Thm. 1.3). \(\Box\)

As an application of our techniques, we show that amenability of a representation of a group $G$ is equivalent to the existence of a Lévy-type $G$-invariant integral for functions on the sphere in the space of representation. Namely, we are able to invert Proposition 3.1 and obtain a new equivalent definition of an amenable representation very much in the classical spirit of amenability.

**Theorem 7.6.** Let $\pi$ be a unitary representation of a group $G$ in a Hilbert space $\mathcal{H}$. The following conditions are equivalent.

- The space $C^b_u(\mathcal{S}_H)$ of all bounded uniformly continuous functions on the unit sphere $\mathcal{S}_H$ admits a $G$-invariant mean.
- The representation $\pi$ is amenable.

**Proof.** $\Rightarrow$: Prop. 3.1. $\Leftarrow$: If $\pi$ contains a finite-dimensional subrepresentation $\pi_1 = \pi|_{\mathcal{H}_1}$, then the desired $G$-invariant mean is obtained by integrating the (restriction of) an $f \in C^b_u(\mathcal{S}_H)$ over the unit sphere of $\mathcal{H}_1$. If $\pi$ contains no finite-dimensional subrepresentations, then, according to Corollary 7.2 and Proposition 2.1.(6), there even exists a multiplicative $G$-invariant mean on $C^b_u(\mathcal{S}_H)$. \(\Box\)
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