Singularities on the 2-Dimensional Moduli Spaces of Stable Sheaves on K3 Surfaces

Nobuaki Onishi and Kota Yoshioda

0. Introduction

Let \( X \) be a K3 surface over \( \mathbb{C} \). Mukai introduced a lattice structure \( \langle \; , \; \rangle \) on \( H^*(X, \mathbb{Z}) := \bigoplus_i H^{2i}(X, \mathbb{Z}) \) by

\[
\langle x, y \rangle := -\int_X x^\vee \wedge y = \int_X (x_1 \wedge y_1 - x_0 \wedge y_2 - x_2 \wedge y_0),
\]

where \( x_i \in H^{2i}(X, \mathbb{Z}) \) (resp. \( y_i \in H^{2i}(X, \mathbb{Z}) \)) is the \( 2i \)-th component of \( x \) (resp. \( y \)) and \( x^\vee = x_0 - x_1 + x_2 \). It is now called the Mukai lattice. For a coherent sheaf \( E \) on \( X \), we can attach an element of \( H^*(X, \mathbb{Z}) \) called the Mukai vector

\[
v(E) := ch(E)\sqrt{td_X} = ch(E)(1 + \rho_X),
\]

where \( ch(E) \) is the Chern character of \( E \), \( td_X \) the Todd class of \( X \) and \( \rho_X \) the fundamental cohomology class of \( X \) \((f_X^*\rho_X = 1)\).

**Definition 0.1.** \([Y3]\) We fix an ample divisor \( H \) on \( X \) and an element \( G \in K(X) \otimes \mathbb{Q} \) with \( \text{rk} G > 0 \).

(i) Let \( E \) be a torsion free sheaf on \( X \). \( E \) is \( G \)-twisted semi-stable (resp. stable) with respect to \( H \), if

\[
\frac{\chi(G, F(nH))}{\text{rk}(F)} \leq \frac{\chi(G, E(nH))}{\text{rk}(E)}, \quad n \gg 0
\]

for \( 0 \leq F \leq E \) (resp. the inequality is strict).

(ii) For a \( w \in H^*(X, \mathbb{Q}_{\text{alg}}) := \mathbb{Q} \oplus \text{NS}(X) \otimes \mathbb{Q} \otimes \mathbb{Q}_X \) with \( \text{rk} w > 0 \), we define the \( w \)-twisted semi-stability as the \( G \)-twisted semi-stability, where \( G \in K(X) \otimes \mathbb{Q} \) satisfies \( v(G) = w \).

Matsuki and Wentworth \([M-W]\) constructed the moduli space of \( w \)-twisted semi-stable sheaves \( E \) with \( v(E) = w \). We denote it by \( \overline{M}_H^w(v) \). If \( w = v(O_X) \), then the \( v(O_X) \)-twisted stability is nothing but the usual Gieseker’s semi-stability. Hence we denote \( \overline{M}_H^{v(O_X)}(v) \) by \( M_H(v) \). Assume that \( v \) is an isotropic Mukai vector. In \([A]\), Abe considered the singularities of \( M_H(v) \). Replacing \( M_H(v) \) by \( \overline{M}_H(v) \), we shall generalize Abe’s results:

**Theorem 0.1.**

1. \( \overline{M}_H^w(v) \) is normal.

2. For a suitable choice of \( \alpha \) with \([\alpha^2]\) \( \ll 1 \), there is a surjective morphism \( \phi_\alpha : \overline{M}_H^{v+\alpha}(v) \to \overline{M}_H^w(v) \) which becomes a minimal resolution of the singularities.

3. Let \( x \) be a point of \( \overline{M}_H^w(v) \) corresponding to the \( S \)-equivalence class \( \bigoplus_{i=0}^n E_i \otimes \alpha_i \), where \( E_i, 0 \leq i \leq n \) are \( v \)-twisted stable sheaves. Then the matrix \((-\langle v(E_i), v(E_j)\rangle)_{i,j=0}^n\) is of affine type \( \tilde{A}_n, \tilde{D}_n, \tilde{E}_n \).

Assume that \( \alpha_0 = 1 \). Then the singularity of \( \overline{M}_H^w(v) \) at \( x \) is a rational double point of type \( A_n, D_n, E_n \) according as the type of the matrix \((-\langle v(E_i), v(E_j)\rangle)_{i,j=1}^n\).

Moreover we shall show that the Weyl chamber of the corresponding finite Lie algebra appears as a parameter space of \( \alpha \).

If the matrix is of type \( \tilde{A}_n \), then the assertion (1) is due to Abe \([A] \; \text{Thm. 3.3}\). Moreover if \( n = 1, 2 \), then he showed the assertion (3). The assertion (2) is also contained in \([A] \; \text{Thm. 3.3}\). The main point of the proof is due to Matsuki and Wentworth \([M-W]\) or Ellingsrud and Göttsche \([E-G]\). The \( v \)-twisted stability naturally appears in the study of the Fourier-Mukai transforms. In \([Y3], [Y5]\), we studied the Fourier-Mukai transform and showed that the Fourier-Mukai transform preserves the \( v \)-twisted semi-stability under suitable assumptions. So the \( v \)-twisted semi-stability is important and this is our original motivation to study the moduli space of \( v \)-twisted semi-stable sheaves. Another motivation is the following: For the GIT quotients related to the moduli spaces of vector bundles on curves with additional structures, the wall crossing behaviors have been studied by several authors. In particular, Thaddeus \([T]\) described the wall crossing behavior as a sequence of blow-ups and blow-downs and used it to show the Verlinde formula.
For the rank two case, Ellingsrud and Göttsche [E-G] studied the similar variation problem for the moduli space of stable sheaves on a K3 surface. In this case, Mukai’s elementary transformation appears. For all these examples, the exceptional locus of the blowing-up is irreducible. So it is interesting to construct an example with a reducible exceptional locus, and a rational double point will be a simple and interesting example to consider.

Our main idea to study the exceptional locus is the same as the one in [Y1] to study the Brill-Noether locus of sheaves, under the assumption that \(w = v + \alpha, |(\alpha^2)| \ll 1\) belongs to a special chamber. By using the Weyl group action on the parameter space, we can give a set-theoretic description of the exceptional locus for general cases. Finally we shall prove that \(\overline{M}_H(v)\) is normal. In section 3, we give some examples of singular moduli spaces by using the surjectivity of the period map.

1. Definitions

Let \(L\) be a lattice (or a \(\mathbb{Q}\)-vector space with a bilinear form) with a weight 2 Hodge structure: \(L \otimes \mathbb{C} = \bigoplus_{p+q=2} L^{p,q}\). We set \(L_{alg} := L \cap L^{1,1}\). The Mukai lattice \(H^*(X, \mathbb{Z})\) has a Hodge structure:

\[
H^{2,0}(H^*(X, \mathbb{C})) = H^{2,0}(X),
\]

\[
H^{1,1}(H^*(X, \mathbb{C})) = H^{0,0}(X) \oplus H^{1,1}(X) \oplus H^{2,2}(X),
\]

\[
H^{0,2}(H^*(X, \mathbb{C})) = H^{0,2}(X).
\]

Then \(H^*(X, \mathbb{Z})_{alg} = \mathbb{Z} \oplus \text{Pic}(X) \oplus \mathbb{Z}\rho_X\).

1.1. Twisted stability. Let \(G\) be an element of \(K(X) \otimes \mathbb{Q}\) with \(rk G > 0\). We fix an ample divisor \(H\) on \(X\). For a coherent sheaf \(E\) on \(X\), we define the \(G\)-twisted rank, degree, and Euler characteristic of \(E\) by

\[
rk_G(E) := \text{rk}(G^\vee \otimes E),
\]

\[
deg_G(E) := (c_1(G^\vee \otimes E), H),
\]

\[
\chi_G(E) := \chi(G^\vee \otimes E).
\]

We shall rewrite the condition (1.3) on the twisted stability. By the Riemann-Roch theorem, we get that

\[
\frac{\chi(G, E(nH))}{\text{rk}(G) \text{rk}(E)} - \frac{\chi(G, F(nH))}{\text{rk}(G) \text{rk}(F)} = \frac{n}{\text{rk}(G)} \left( \frac{\deg_G(E)}{\text{rk}(E)} - \frac{\deg_G(F)}{\text{rk}(F)} \right) + \left( \frac{\chi_G(E)}{\text{rk}(G)} - \frac{\chi_G(F)}{\text{rk}(F)} \right)
\]

\[
= n \left( \frac{c_1(E, H)}{\text{rk}(E)} - \frac{c_1(F, H)}{\text{rk}(F)} \right) + \left( \frac{\chi(E)}{\text{rk}(E)} - \frac{\chi(F)}{\text{rk}(F)} \right)
\]

\[
+ \frac{c_1(E)}{\text{rk}(E)} \cdot \frac{c_1(G)}{\text{rk}(F)}.
\]

Let \(\varphi : \text{Pic}(X) \otimes \mathbb{Q} \to H^1\) be the orthogonal projection. Then the twisted stability depends only on \(\varphi(c_1(G)/\text{rk}(G)) \in H^1\) and it is nothing but the twisted stability due to Matsuki-Wentworth [M-W].

Definition 1.1. A polarization \(H\) is general with respect to \(v\), if the following condition holds:

\((*)\) for every \(\mu\)-semi-stable sheaf \(E\) with \(v(E) = v\), if \(F \subset E\) satisfies \((c_1(F), H)/\text{rk} F = (c_1(E), H)/\text{rk} E\), then \(c_1(F)/\text{rk} F = c_1(E)/\text{rk} E\).

If \(H\) is general with respect to \(v\), then the \(w\)-twisted semi-stability does not depend on the choice of \(w\). The following theorem was proved in [L-W].

Theorem 1.1. [M-W] Let \(w\) be an element of \(H^*(X, \mathbb{Q})_{alg}\) such that \(\text{rk} w > 0\). Then there is a coarse moduli scheme \(\overline{M}_H(v)\) of \(S\)-equivalence classes of \(w\)-twisted semi-stable sheaves \(E\) with \(v(E) = v\). \(\overline{M}_H(v)\) is a projective scheme.

Definition 1.2. We denote the open subscheme of \(\overline{M}_H(v)\) consisting of \(w\)-twisted stable sheaves by \(M_H^w(v)\).

If \(w = v(O_X)\), then we denote \(\overline{M}_H(v)\) (resp. \(M_H^w(v)\)) by \(M_H(v)\) (resp. \(M_H^w(v)\)).

Proposition 1.2. \(\overline{M}_H(v) \neq \emptyset\) if \(v^2 \geq -2\).

Proof. We may assume that \(v\) is primitive. If \(H\) is general with respect to \(v\), then [Y2, Thm. 8.1] implies that \(\overline{M}_H(v) \neq \emptyset\). By the study of the chamber structure (cf. [Y2, sect. 1, Prop. 4.2]), we get our claim. \(\square\)
1.2. Line bundles on $\mathcal{M}^w_H(v)$. Throughout this note, $v := r + \xi + a\rho_X$, $\xi \in \text{Pic}(X)$ is a primitive isotropic Mukai vector with $r > 0$.

We define a homomorphism which preserves the Hodge structure and the metric:

$$
\delta : H^2(X, \mathbb{Q}) \to H^*(X, \mathbb{Q}) \quad \mapsto \quad D + \frac{D, \xi}{r^2}\rho_X.
$$

We denote $\delta(D)$ by $\tilde{D}$. Then we have an orthogonal decomposition:

$$
H^*(X, \mathbb{Q}) = (\mathbb{Q}v \oplus \mathbb{Q}\rho_X) \perp \delta(H^2(X, \mathbb{Q})).
$$

Let $\theta^\alpha_v : v^\perp \to H^2(M^{w+\alpha}_H(v), \mathbb{Z})$ be the Mukai homomorphism defined by

$$
\theta^\alpha_v(x) := \frac{1}{\rho} \left[ p_{M^{w+\alpha}_H(v)} \ast \left( (cH) \sqrt{\text{td}(X_x)} \right) \right],
$$

where $E$ is a quasi-universal family of similitude $\rho$. If $x \in (v^\perp / \mathbb{Z}v) \otimes \mathbb{Q}$, then we have a $\mathbb{Q}$-line bundle $L(x)$ on $M^{w+\alpha}_H(v)$ such that $c_1(L(x)) = \theta^\alpha_v(x)$. For $\tilde{H} = H + \{(H, \xi)/\rho_X, L(r\tilde{H})$ is the Donaldson’s determinant line bundle and J. Li [4] showed that canonically $L(r\tilde{H})$ extends to a line bundle on $\mathcal{M}^w_H(v)$. We also denote this extension by $L(r\tilde{H})$. Then $L(r\tilde{H})$ is a nef and big line bundle and we have a contraction map from the Gieseker moduli space to the Uhlenbeck moduli space. Hence $L(r\tilde{H})$ is important.

One of the reason we consider the $v$-twisted stability is the following proposition.

**Proposition 1.3.** $L(r\tilde{H})$ is an ample line bundle on $\mathcal{M}^w_H(v)$.

**Proof.** We recall the construction of $\mathcal{M}^w_H(v)$ in [5, 6]. Let $E$ be a $v$-twisted stable sheaf with $v(E) = v$. We set $N := \chi(E, E(nH))$. Let $Q := \text{Quot}^v_{E(-nH)\oplus N/X/C}$ be a quasi-scheme parametrizing all quotients $E(-nH)\oplus N 
\to F$ such that $v(F) = v$ and $O^\oplus E(-nH) 
\to Q$ the universal quotient. Let $Q^{ss}$ be an open subscheme of $Q$ consisting of $q \in Q$ such that

(i) $Q_q$ is $v$-twisted semi-stable,

(ii) $\text{Hom}(E, E(nH))$ is an isomorphism,

(iii) $\text{Ext}^i(E, Q_q(nH)) = 0$, $i > 0$.

Then we have an isomorphism $O^\oplus_{Q^{ss}} \to p_{Q^{ss}}(Q \otimes p_X^*(E(-nH))^\vee)$. $Q$ has a natural action of $GL(N)$. We set $L_m := \text{det}(O^\oplus_{Q^{ss}})$. We can observe that $L_m$ is also $GL(N)$-linearized. By the construction of $Q$, $L_{n+m}$, $m > 0$ gives an embedding of $Q$ to a Grassmann variety. Thus $L_{m+n}$, $m > 0$ is ample. Let $T := \text{det}(O^\oplus_{Q^{ss}})$ be the $GL(N)$-linearized line bundle induced by the standard action of $GL(N)$ on $O^\oplus_{Q^{ss}}$. The center $\mathbb{C}^N$ acts trivially and the action on $L_m$ is the multiplication by $\chi(E, E(nH))$. By a simple calculation, we see that $\chi(E, E(nH)) = r^2m^2(H^2)/2$. Hence $L := L_m \otimes L_{\circ}^{-n-m}$ and $L' := L_m \otimes (n+m)^2$ have $PGL(N)$-linearizations. By the construction of the moduli space, $\mathcal{M}^w_H(v)$ is described as a GIT quotient $Q^{ss} \to \mathcal{M}^w_H(v)$, where $n > 0$ and $L'$ is the linearization. Since $L_{Q^{ss}} = L'_{Q^{ss}}$ as $PGL(N)$-line bundles, $L_{Q^{ss}}$ descends to an ample line bundle on $\mathcal{M}^w_H(v)$.

We note that

$$
L = L_m \otimes L_{\circ}^{-n-m} = \text{det}(Q \otimes p_X^*(L)^\vee),
$$

where $L = m^2E(-n+m)H - (n+m)^2E(-nH)$ in $K(X)$. Since $\text{det}(Q \otimes p_X^*(L)^\vee) = O_Q^{ss}$,

$$
det(\text{det}(Q \otimes p_X^*(L)^\vee)) = \text{det}(Q \otimes p_X^*(L'^\vee)),
$$

where $L' = L - (m^2 - (n+m)^2)E$. Since $v(L') = rmn(m+n)\tilde{H}$, we get our claim. \hfill $\square$

**Corollary 1.4.**

1. If $\xi \in \mathbb{Q}H$, then $\mathcal{M}^w_H(v) = \mathbb{M}^w_H(v)$. Hence $L(r\tilde{H})$ is an ample line bundle on $\mathcal{M}^w_H(v)$.

2. Let $\mathcal{M}_n$ (resp. $\mathcal{M}_n$) be the moduli space of polarized (resp. quasi-polarized) K3 surfaces $(X, H)$ with $(H^2) = 2n$. We set $v := r + dH + a\rho_X$, $d^2(H^2) = 2ra$. Then we have a morphism of the moduli spaces $\mathcal{M}_n \to \mathcal{M}_n: (X, H) \mapsto (M_H(v), L(\tilde{H}))$ where $a = ra/d^2$ and $n'$ is determined by the primitive class in $\text{Q}(L(\tilde{H})) \cap \text{Pic}(\mathcal{M}_H(v))$.

In particular, if $v := r + H + a\rho_X$ satisfies $\text{gcd}(r, a) = 1$, then $M_H(v)$ is compact and $\tilde{H} = H + 2a\rho_X$ gives a canonical primitive polarization of $M_H(v)$. $M_H(v)$.

**Remark 1.1.** If $\text{gcd}(r, d) = 1$, then $M_H(v)$ consists of $\mu$-stable locally free sheaves for a general $X$. For a special $X$, $M_H(v)$ may consist of properly $\mu$-semi-stable sheaves. Indeed let $X \to \mathbb{P}^1$ be an elliptic K3 surface with a section $\sigma$. We set $H := \sigma + 3f$, where $f$ is a fiber of $\pi$. If $\text{Pic}(X) = \mathbb{Z}\sigma \oplus \mathbb{Z}f$, then $H$ is an
ample divisor with \((H^2) = 4\). We set \(v = 2 + H + \rho_X\). Then \((v^2) = 0\) and every member of \(M_H(v)\) is given by
\[
E := \ker(O_X(\sigma + f) \oplus O_X(2f) \xrightarrow{e^v} \mathbb{C}_s),
\]
s \(\in X\).

For the Mukai homomorphism, Mukai [Mu2] showed the following.

**Theorem 1.5** (Mukai). Assume that \(M_H^{v+\alpha}(v)\) is compact. Then \(\theta^v\) is surjective and the kernel is \(Zv\).

Moreover \(\theta^v : v/\mathbb{Z}v \to H^2(M_H^{v+\alpha}(v), \mathbb{Z})\) is a Hodge isometry.

By (1.4) and (1.5), we have a sequence of Hodge isometries:
\[
H^2(X, \mathbb{Q}) \to \delta(H^2(X, \mathbb{Q})) \to (v/\mathbb{Z}v) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^2(M_H^{v+\alpha}(v), \mathbb{Q}).
\]

Then since \(\hat{H} = \delta(H) \in v^+\), we have an isometry
\[
\delta(H^+)_{\text{alg}} \to ((\hat{H}^+ \cap v^+)/\mathbb{Z}v)_{\text{alg}} \otimes \mathbb{Q}.
\]
In particular, \(((\hat{H}^+ \cap v^+)/\mathbb{Z}v)_{\text{alg}}\) is negative definite.

### 1.3. Chamber structure

We shall study the dependence of \(\overline{M}_H^w(v)\) on \(w\). By (1.3), we may assume that \(w = v + \alpha, \alpha \in \delta(H^+)_{\text{alg}}\). Let \(u\) be a Mukai vector such that \(0 < \text{rk} u < \text{rk} v\), \(\langle v, u \rangle \leq 0\), \(\langle u^2 \rangle = -2\) and \(\langle u, \hat{H} \rangle = 0\). We define a wall \(W_u \subset \delta(H^+)_{\text{alg}} \otimes_{\mathbb{Q}} \mathbb{R}\) with respect to \(v\) by
\[
W_u := \{\alpha \in \delta(H^+)_{\text{alg}} \otimes \mathbb{R} \mid \langle v + \alpha, u \rangle = 0\}.
\]

For a properly \(v + \alpha\)-twisted stable sheaf \(E\) with \(v(E) = v\), we consider the Jordan-Hölder filtration
\[
0 < F_1 < F_2 < \cdots < F_s = E
\]
of \(E\) with respect to the \(v + \alpha\)-twisted stability, that is, \(E_i := F_i/F_{i-1}\) is a \(v + \alpha\)-twisted stable sheaf with
\[
\langle c_1(E_i), H \rangle / \text{rk} E_i = (c_1(E), H) / \text{rk} E = (\xi, H)/r,
\]
\[
\langle v + \alpha, v(E_i) \rangle / \text{rk} E_i = \langle v + \alpha, v \rangle / \text{rk} E.
\]

We set \(u_i := v(E_i)\). Then we see that \(\langle u_i, \hat{H} \rangle = (c_1(E_i), H) - (\xi, H)r_i/r = 0\) and \(\langle v + \alpha, u_i \rangle = 0\).

**Lemma 1.6.** \(\langle v, u_i \rangle \leq 0\) and \(\langle u_i^2 \rangle = -2\) for some \(i\).

**Proof.** Since \(u_i / \text{rk} u_i - v / \text{rk} v \in \hat{H}^+ \cap \rho^\perp_X\) and \((\hat{H}^+ \cap \rho^\perp_X)_{\text{alg}}\) is negative semi-definite, \(\langle (u_i / \text{rk} u_i - v / \text{rk} v)^2 \rangle \leq 0\). Then \(\langle u_i^2 \rangle \leq 2 \langle u_i, v \rangle / \text{rk} u_i / \text{rk} v\). Since \(\sum_i \langle u_i, v \rangle = \langle v, v \rangle = 0\), we get \(\langle u_i, v \rangle \leq 0\) for some \(i\). In particular \(\langle u_i^2 \rangle < 0\) provided that \(\langle u_i, v \rangle < 0\). If \(\langle u_i, v \rangle = 0\), then \(u_i \in \hat{H}^+ \cap v^+.\) Since \(\text{rk} u_i < \text{rk} v\), we get \(u_i \not\in \mathbb{Z}v\). Then (1.11) implies that \(\langle u_i^2 \rangle < 0\). Since \(\langle u_i^2 \rangle \geq -2\), we conclude that \(\langle u_i^2 \rangle = -2\).

Therefore \(\alpha \in W_{u_i}\). We set
\[
U := \left\{ u \in H^*(X, \mathbb{Z})_{\text{alg}} \mid \begin{array}{l}
\langle u^2 \rangle = -2, \langle u, v \rangle \leq 0, \langle \hat{H}, u \rangle = 0,
0 < \text{rk} u < \text{rk} v
\end{array} \right\}.
\]

For a fixed \(v\) and \(H\), \(U\) is a finite set.

**Lemma 1.7.** If \(\alpha\) does not lie on any wall \(W_u, u \in U\), then \(\overline{M}_H^{v+\alpha}(v) = M_H^{v+\alpha}(v)\). In particular, \(\overline{M}_H^{v+\alpha}(v)\) is a K3 surface.

**Definition 1.3.** Let \(C\) be a connected component of \(\delta(H^+)_{\text{alg}} \otimes_{\mathbb{Q}} \mathbb{R} \setminus \cup_{u \in U} W_u\). We call \(C\) a chamber.

As is proved in [MV], we get

**Proposition 1.8.** The \(v + \alpha\)-twisted stability does not depend on the choice of \(\alpha \in C\). If \(\beta\) belongs to the closure of \(C\), then we have a morphism \(\overline{M}_H^{v+\alpha}(v) \to \overline{M}_H^{v+\beta}(v)\) for \(\alpha \in C\). In particular, we have a morphism \(\phi_\alpha : \overline{M}_H^{v+\alpha}(v) \to \overline{M}_H(v)\) for \(|\alpha^2| < 1\).

Let \(T \subset v^\perp\) be a sufficiently small neighborhood of 0. Then \(W_u\) intersects \(T\) if and only if \(\langle v, u \rangle = 0\). Since we are interested in the neighborhood of \(v\), we may assume that the defining equation of a wall \(W_u\) belongs to the subset
\[
U' := \{ u \in U \mid \langle v, u \rangle = 0 \}.
\]

By the same argument as above, we get the following.

**Lemma 1.9.** Let \(E\) be a properly \(v\)-twisted semi-stable sheaf with \(v(E) = v\) and
\[
0 < F_1 < F_2 < \cdots < F_s = E
\]
the Jordan-Hölder filtration of \(E\) with respect to the \(v\)-twisted stability. Then \(\langle v(F_i/F_{i-1})^2 \rangle = -2\).
1.4. Reflexion. For an \( \alpha \in \delta(H^+_{alg}) \) with \( |(\alpha^2)| \ll 1 \), let \( F \) be a \( v + \alpha \)-twisted stable torsion free sheaf such that

(i) \( (v(F))^2 = -2 \).

(ii) \( (v(F), \tilde{H})/\text{rk} F = (c_1(F), H)/\text{rk} F - (\xi, H)/r = 0 \) and

(iii) \( (v, v(F)) = (\alpha, v(F)) = 0 \).

By (i), \( F \) is a rigid torsion free sheaf, and hence \( F \) is locally free.

Let \( \mathcal{E} \) be a coherent sheaf on \( X \times X \) which is defined by an exact sequence

\[
0 \to \mathcal{E} \to p_1^*(F^\vee) \otimes p_2^*(F) \xrightarrow{\xi} \mathcal{O}_\Delta \to 0,
\]

where \( p_i : X \times X \to X \), \( i = 1, 2 \) are projections. We consider the Fourier-Mukai transform induced by \( \mathcal{E} \):

\[
\mathcal{F}_\mathcal{E} : \quad \mathcal{D}(X) \to \mathcal{D}(X), \quad x \mapsto \mathcal{R}p_{2*}(p_1^*(x) \otimes \mathcal{E}),
\]

where \( \mathcal{D}(X) \) is the bounded derived category of \( X \). Up to shift, the inverse of \( \mathcal{F}_\mathcal{E} \) is given by

\[
\mathcal{F}^{-1}_\mathcal{E} : \quad \mathcal{D}(X) \to \mathcal{D}(X), \quad y \mapsto \mathcal{R}\text{Hom}_{p_1}(s, p_2^*(y)).
\]

Definition 1.4. Let \( E \) be a coherent sheaf on \( X \).

(i) We denote the \( i \)-th cohomology sheaf of \( \mathcal{F}_\mathcal{E}(E) \) (resp. \( \mathcal{F}_\mathcal{E}^{-1}(E) \) (resp. \( \mathcal{F}^{-1}_\mathcal{E}(E) \)).

(ii) \( E \) satisfies WIT \( _i \) with respect to \( \mathcal{F}_\mathcal{E} \) (resp. \( \mathcal{F}^{-1}_\mathcal{E} \)), if \( \mathcal{F}^{-1}_\mathcal{E}(E) = 0 \) (resp. \( \mathcal{F}^{-1}_\mathcal{E}(E) = 0 \)) for \( j \neq i \).

The Fourier-Mukai transform \( \mathcal{F}_\mathcal{E} \) induces an isometry of the Mukai lattice \( \mathcal{F}_\mathcal{E} : \mathcal{H}^*(X, \mathbb{Z}) \to \mathcal{H}^*(X, \mathbb{Z}) \).

Let \( R_{v(F)} : \mathcal{H}^*(X, \mathbb{Z}) \to \mathcal{H}^*(X, \mathbb{Z}) \) be the reflection defined by the \((-2)\)-vector \( v(F) \):

\[
R_{v(F)}(u) = u + \langle u, v(F) \rangle v(F), \quad u \in \mathcal{H}^*(X, \mathbb{Z}).
\]

Then we see that \( \mathcal{F}_\mathcal{E} = -R_{v(F)} \). Thus the Fourier-Mukai transform \( \mathcal{F}_\mathcal{E} \) is the geometric realization of the reflection \( R_{v(F)} \).

Lemma 1.10. Let \( G \) be a \( v + \alpha \)-twisted semi-stable sheaf such that \( \text{deg}(G)/\text{rk} G = \text{deg}(F)/\text{rk} F \) and \( \chi(E + A, G) \geq 0 \), where \( E, A \in K(X) \otimes \mathbb{Q} \) satisfy \( v(E) = v, v(A) = \alpha \). Then

\[
\text{Ext}^2(\mathcal{E}_{(x)} \times X, G) = 0
\]

for all \( x \in X \).

Proof. Assume that there is a non-zero homomorphism \( \varphi : G \to \mathcal{E}_{(x)} \times X \). Then we have a non-zero homomorphism \( \psi : G \to F \). Since \( G \) is a \( v + \alpha \)-twisted semi-stable sheaf with \( \text{deg}(G)/\text{rk} G = \text{deg}(F)/\text{rk} F \) and \( F \) is \( v + \alpha \)-twisted stable, we get that \( 0 \leq \chi(E + A, G)/\text{rk} G \leq \chi(E + A, F)/\text{rk} F = 0 \). Hence \( \chi(E + A, G) = 0 \) and \( \psi \) is surjective. Thus \( \im \varphi \) contains \( F \). On the other hand, by the construction of \( \mathcal{E}_{(x)} \times X, \mathcal{E}_{(x)} \times X \) does not contain \( F \). Therefore \( \text{Hom}(G, \mathcal{E}_{(x)} \times X) = 0 \). By the Serre duality, we get \( \text{Ext}^2(\mathcal{E}_{(x)} \times X, G) = 0 \).

Lemma 1.11. Let \( E, A \in K(X) \otimes \mathbb{Q} \) be as in Lemma [1.11]. Let \( G \) be a \( v + \alpha \)-twisted semi-stable sheaf such that \( \text{deg}(G)/\text{rk} G = \text{deg}(F)/\text{rk} F \) and \( \chi(E + A, G) = 0 \). Then the evaluation map \( \phi : \text{Hom}(F, G) \otimes F \to G \) is injective and coker \( \phi \) is a \( v + \alpha \)-twisted semi-stable sheaf.

Proof. By the \( v + \alpha \)-twisted semi-stability of \( E \) and \( F \), we see that \( \text{deg}(\im \phi)/\text{rk}(\im \phi) = \text{deg}(F)/\text{rk} F \) and \( \chi(E + A, \im \phi) = 0 \). Hence we get \( \text{deg}(\ker \phi)/\text{rk}(\ker \phi) = \text{deg}(F)/\text{rk} F \) and \( \chi(E + A, \ker \phi) = 0 \). Assume that \( \ker \phi \neq 0 \). By the \( v + \alpha \)-twisted semi-stability of \( \text{Hom}(F, G) \otimes F \), \( \ker \phi \) is a \( v + \alpha \)-twisted semi-stable. Then we see that \( \ker \phi \cong F^{\otimes k} \), which implies that \( \text{Hom}(F, \ker \phi) \neq 0 \). On the other hand, \( \phi \) induces an isomorphism \( \text{Hom}(F, G) \otimes F \to \text{Hom}(F, G) \). Hence we have \( \text{Hom}(F, \ker \phi) = 0 \), which is a contradiction. Therefore \( \ker \phi = 0 \).

Proposition 1.12. We set \( \alpha^\pm := \pm ev(F) + \alpha \), where \( 0 < \epsilon \ll 1 \).

1. Let \( E \) be a \( v + \alpha^- \)-twisted semi-stable sheaf with \( v(E) = v \). Then WIT\( _1 \) holds for \( E \) with respect to \( \mathcal{F}_\mathcal{E} \) and \( \mathcal{F}^{-1}_\mathcal{E}(E) \) is a \( v + \alpha^- \)-twisted semi-stable sheaf.

2. Conversely, for a \( v + \alpha^+ \)-twisted semi-stable sheaf \( E \) with \( v(E) = v \), WIT\( _1 \) holds with respect to \( \mathcal{F}^{-1}_\mathcal{E} \) and \( \mathcal{F}^{-1}_\mathcal{E}(E) \) is a \( v + \alpha^- \)-twisted semi-stable sheaf.

3. Moreover \( \mathcal{F}_\mathcal{E} \) preserves the S-equivalence classes. Hence we have an isomorphism

\[
\overline{M}_{H}^{v + \alpha^-}(v) \to \overline{M}_{H}^{v + \alpha^+}(v).
\]
Proof. We take an element \( A \in K(X) \otimes \mathbb{Q} \) such that \( v(A) = \alpha \). We note that \( F \) is \((v + \varepsilon v(F) + \alpha)\)-twisted stable for \( 0 \leq \varepsilon \ll 1 \). We first prove (1). We note that \( E \) is \( v + \alpha \)-twisted semi-stable. By the definition of \( \mathcal{E} \), we get an exact sequence

\[
\begin{align*}
0 & \longrightarrow p_{2*}(\mathcal{E} \otimes p_1^* (E)) \longrightarrow \text{Hom}(F, E) \otimes F \longrightarrow E \\
& \quad \longrightarrow R^1p_{2*}(\mathcal{E} \otimes p_1^* (E)) \longrightarrow \text{Ext}^1(F, E) \otimes F \longrightarrow 0 \\
& \quad \longrightarrow R^2p_{2*}(\mathcal{E} \otimes p_1^* (E)) \longrightarrow \text{Ext}^2(F, E) \otimes F \longrightarrow 0.
\end{align*}
\]

Since \( \deg F / \text{rk} F = \deg E / \text{rk} E \) and \( \chi(E - \varepsilon F + A, F) / \text{rk} F = -2\varepsilon / \text{rk} F < 0 = \chi(E - \varepsilon F + A, E) / \text{rk} E \), the \( v + \alpha \)-twisted semi-stability of \( E \) and \( F \) imply that \( \text{Ext}^2(F, E) = \text{Hom}(E, F)^\vee = 0 \). Thus \( R^2p_{2*}(\mathcal{E} \otimes p_1^* (E)) = 0 \). Since \( E \) is \( v + \alpha \)-twisted semi-stable, Lemma 1.11 implies that \( \text{Hom}(F, E) \otimes F \to E \) is injective, and hence \( p_{2*}(\mathcal{E} \otimes p_1^* (E)) = 0 \). Therefore \( \text{WIT}_1 \) holds for \( E \) and \( \mathcal{F}^1_E \) is a \( v + \alpha \)-twisted semi-stable sheaf with \( v(\mathcal{F}^1_E) = 0 \). Assume that \( \mathcal{F}^1_E \) is not \( v + \alpha \)-twisted semi-stable. Then there is an exact sequence

\[
\begin{align*}
0 & \to G_1 \to \mathcal{F}^1_E \to G_2 \to 0
\end{align*}
\]

such that \( G_1 \) is a \( v + \alpha \)-twisted semi-stable sheaf with \( \deg G_1 = \chi(E + A, G_1) = 0 \) and \( G_2 \) is a \( v + \alpha \)-twisted semi-stable sheaf with \( \chi(E + \varepsilon F + A, G_2) < 0 \). By Lemma 1.11, we get \( \mathcal{F}^1_E(G_1) = 0 \). Since \( \chi(E + \varepsilon F + A, G_2) < 0 \) and \( \chi(E + \varepsilon F + A, F) = 2\varepsilon > 0 \), \( \mathcal{F}^1_E(G_2) = \text{Hom}(F, G_2) \otimes F = 0 \). Therefore \( \text{WIT}_1 \) holds for \( G_1, G_2 \), and we get an exact sequence

\[
\begin{align*}
0 & \to \mathcal{F}^1_E(G_1) \to E \to \mathcal{F}^1_E(G_2) \to 0.
\end{align*}
\]

Since \( \chi(F, \mathcal{F}^1_E(G_2)) = -\chi(F, G_2) > 0 \), we get a contradiction.

(2) Conversely, let \( E \) be a \( v + \alpha \)-twisted semi-stable sheaf with \( v(E) = v \). Then we have an exact sequence

\[
\begin{align*}
0 & \longrightarrow \text{Hom}_{p_1}(\mathcal{O}_\Delta, p_2^* (E)) \longrightarrow \text{Hom}(F, E) \otimes F \longrightarrow \text{Hom}_{p_1}(\mathcal{E}, p_2^* (E)) \\
& \quad \longrightarrow \text{Ext}^1_{p_1}(\mathcal{O}_\Delta, p_2^* (E)) \longrightarrow \text{Ext}^1(F, E) \otimes F \longrightarrow \text{Ext}_{p_1}(\mathcal{E}, p_2^* (E)) \\
& \quad \longrightarrow \text{Ext}^2_{p_1}(\mathcal{O}_\Delta, p_2^* (E)) \longrightarrow \text{Ext}^2(F, E) \otimes F \longrightarrow \text{Ext}^2_{p_1}(\mathcal{E}, p_2^* (E)).
\end{align*}
\]

By Lemma 1.10, \( \mathcal{F}^2_E(E) = \text{Ext}^2_{p_1}(\mathcal{E}, p_2^* (E)) = 0 \). It is easy to see that

\[
\begin{align*}
\text{Hom}_{p_1}(\mathcal{O}_\Delta, p_2^* (E)) &= 0, \\
\text{Ext}^1_{p_1}(\mathcal{O}_\Delta, p_2^* (E)) &= 0, \\
\text{Ext}^2_{p_1}(\mathcal{O}_\Delta, p_2^* (E)) &= E.
\end{align*}
\]

Since \( \chi(E + \varepsilon F + A, F) = \chi(E + \varepsilon F + A, E) = 0 \), the \( v + \alpha \)-twisted semi-stability of \( E \) and \( F \) imply that \( \mathcal{F}^2_E(E) = \text{Hom}(F, E) \otimes F = 0 \). Therefore \( \text{WIT}_1 \) holds with respect to \( \mathcal{F}_E \) and \( \mathcal{F}^1_E \) is \( v + \alpha \)-twisted semi-stable. Assume that \( \mathcal{F}^1_E \) is not \( v + \alpha \)-twisted semi-stable. Then there is an exact sequence

\[
\begin{align*}
0 & \to G_1 \to \mathcal{F}^1_E \to G_2 \to 0
\end{align*}
\]

such that \( G_1 \) is a \( v + \alpha \)-twisted stable sheaf with \( \deg G_1 = 0 \) and \( G_2 \) is a \( v + \alpha \)-twisted semi-stable sheaf. Since \( \chi(E + \varepsilon F + A, G_1) > 0 > \chi(E + \varepsilon F + A, F) / \text{rk} F \), \( \text{Ext}^2(F, G_1) = \text{Hom}(G_1, F)^\vee = 0 \). Thus \( R^2p_{2*}(\mathcal{E} \otimes p_1^* (G_1)) = 0 \). Since \( G_2 \) is a \( v + \alpha \)-twisted semi-stable sheaf with \( \deg G_2 = \chi(E + A, G_2) = 0 \), Lemma 1.11 implies that \( \text{Hom}(F, G_2) \otimes F \to G_2 \) is injective, and hence \( p_{2*}(\mathcal{E} \otimes p_1^* (G_2)) = 0 \). Therefore \( \text{WIT}_1 \) holds for \( G_1 \) and \( G_2 \) with respect to \( \mathcal{F}_E \). Since \( \chi(E + \varepsilon F + A, \mathcal{F}^1_E(G_1)) > 0 \), we get a contradiction.

The last claim (3) will easily follow from the above arguments. We omit the proof. \( \square \)

The following is proved in [Y1].

Proposition 1.13. Keep notation as above. Assume that \( v + \alpha^- \) does not lie on walls. Then \( \mathcal{F}_E \) induces an isometry \( R_{v(F)} : v^- \to v^- \) and the following diagram is commutative.

\[
\begin{array}{ccc}
v^- & \xrightarrow{R_{v(F)}} & v^+ \\
\downarrow \vartheta^+ & & \downarrow \vartheta^- \\
H^2(M^{v+\alpha^-}_{H}(v), \mathbb{Z}) & \longrightarrow & H^2(M^{v+\alpha^-}_{H}(v), \mathbb{Z})
\end{array}
\]

Remark 1.2. If \( \alpha \) belongs to exactly one wall \( W_u, u \in \mathcal{U} \), then there is a \( v + \alpha \)-twisted stable sheaf \( F \) with \( v(F) = u \). So we can apply Propositions 1.12 and 1.13 to this \( F \).
2. Resolution of the singularities of $\overline{M}_g^\nu(v)$

2.1. Exceptional locus of the resolution. Assume that there is a point $x$ of $\overline{M}_g^\nu(v)$ representing a properly $v$-twisted semi-stable sheaf. Assume that $x$ is represented by an $S$-equivalence class $\bigoplus_{i=0}^{n} E_i^x$, where $E_i$ is a $v$-twisted stable sheaf such that $\langle v(E_i), \hat{H} \rangle = \langle v(E_i), v \rangle = 0$ and $E_i \neq E_j$ for $i \neq j$. We set $v_i := v(E_i)$.

Lemma 2.1. $\mathbb{Z}v_0 + \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ is a negative semi-definite lattice of affine type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_n$. More precisely, $(-\langle v_i, v_j \rangle)_{i,j=0}^n$ is the Cartan matrix of the affine Lie algebra $\tilde{A}_n, \tilde{D}_n, \tilde{E}_n$. In particular, $v_0, v_1, \ldots, v_n$ are linearly independent.

Proof. We note that

$$
\begin{align}
\langle v_i^2 \rangle &= -2, \\
\langle v_i, v_j \rangle &\geq 0, \quad i \neq j, \\
\langle v, v_i \rangle &= \langle \hat{H}, v_i \rangle = 0.
\end{align}
$$

If there is a decomposition $\bigoplus_{i=0}^{n} E_i^x$, where $E_i$ is a $v$-twisted stable sheaf such that $\langle v(E_i), \hat{H} \rangle = \langle v(E_i), v \rangle = 0$ and $E_i \neq E_j$ for $i \neq j$. We set $v_i := v(E_i)$. Then we have the following lemma.

Lemma 2.2.

(2.2) $(\mathbb{Z}v_0 + \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n) \subset (\mathbb{Z}v_0' + \mathbb{Z}v_1' + \cdots + \mathbb{Z}v_n')$.

Proof. We set

$$
\begin{align}
S_1 &:= \{ i \mid v_i \in (\mathbb{Z}v_0 + \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n)^\perp \}, \\
S_2 &:= \{ 0, 1, \ldots, n' \} \setminus S_1.
\end{align}
$$

Then $v = \sum_{i \in S_1} a_i v'_i + \sum_{i \in S_2} a'_i v'_i$. Assume that $i \in S_2$. Since $0 = \langle v'_i, v \rangle = \sum_{j} a_j \langle v'_i, v_j \rangle$, $\langle v'_i, v_j \rangle < 0$ for some $j$. Then $\chi(E'_i, E_j) > 0$, which implies that there is a non-zero homomorphism $E'_i \to E_j$ or $E_j \to E'_i$. Since $E'_i$ and $E_j$ are $v$-twisted stable sheaves such that $\chi(E, E'/(nH)) = \chi(E, E'/(nH))/rk E'_i = \chi(E, E'/(nH))/rk E_i$ for all $n$, we get $E'_i \cong E_j$. Thus $v'_i \in (v_0, v_1, \ldots, v_n)$. Then we get that $\langle \sum_{i \in S_1} a'_i v'_i + \sum_{i \in S_2} a_i v_i \rangle = 0$, and hence $\sum_{i \in S_1} a'_i v'_i + \sum_{i \in S_2} a_i v_i \in \mathbb{Z}v$. Since $\sum_{i \in S_1} a'_i v'_i = 0$ or $v$. If $\sum_{i \in S_1} a'_i v'_i = v$, then Lemma 2.1 implies that $S_2 = \{ 0, 1, \ldots, n' \}$ and $a_i = a_i'$, which implies that $x' = x$. Since $x \not= x'$, we get that $S_1 = \{ 0, 1, \ldots, n' \}$. Thus our claim holds.

We shall study the fiber of $\phi_\alpha : M_g^{\nu+\alpha}(v) \to \overline{M}_g^\nu(v)$ at $x$. By the classification of the (extended) Dynkin diagram, we may assume that $a_0 = 1$. Then $v_i, i = 1, 2, \ldots, n$ become a fundamental root system of the corresponding finite Lie algebra $\mathfrak{g}$ under the change of the sign of the bilinear form.

Lemma 2.3.

(2.4) $U' = (U' \cap \bigoplus_{i=0}^{n} \mathbb{Z}v_i)^\perp \bigcap (U' \cap \bigoplus_{i=0}^{n} \mathbb{Z}v_i)$

and

$$
U' \cap (\bigoplus_{i=0}^{n} \mathbb{Z}v_i) = \{ u \in \bigoplus_{i=0}^{n} \mathbb{Z}v_i | \langle u^2 \rangle = -2, 0 < rk u < rk v \}
$$

(2.5) $= \Psi_+ \prod_{i=0}^{n} (v - \Psi_+),$ 

where $\Psi_+ := \{ u = \sum_{i=1}^{n} b_i v_i | \langle u^2 \rangle = -2, b_i \geq 0 \}$ is the set of positive roots of $\mathfrak{g}$.

Proof. For $u \in U'$, we set $w = v - u$. Since $\langle v, u \rangle = 0$, we get $\langle u^2 \rangle = -2$. Since $0 < rk u < rk v$, we have $rk w > 0$. By Proposition 1.2, there are $v$-twisted semi-stable sheaves $F$ and $G$ with $v(F) = u$ and $v(G) = w$. Applying Lemma 2.2 to $F \oplus G$, we see that $u = \sum_{i=0}^{n} b_i v_i \in \bigoplus_{i=0}^{n} \mathbb{Z}v_i$, $a_i \geq b_i \geq 0$ or $u \in (\bigoplus_{i=0}^{n} \mathbb{Z}v_i)^\perp$ according as $F \oplus G$ is $S$-equivalent to $\bigoplus_{i=0}^{n} E_i^x$ or not. Thus the first claim holds. If $b_0 = 0$, then $u \in \Psi_+$ and if $b_0 = 1$, then $w \in \Psi_+$. Thus $u = v - w \in v - \Psi_+$. Therefore the second assertion also holds.

\]
Therefore the wall \( W_u \) corresponds to the wall defining the Weyl chamber. More precisely, \( W_u \cap ((\bigoplus_{i=0} \mathbb{Z}v_i)/\mathbb{Z}v) \otimes \mathbb{R} \) is the corresponding wall. We define the fundamental Weyl chamber:

\[
D := \{ \alpha \in \delta(H^1)_{\text{alg}} \otimes_{\mathbb{Q}} \mathbb{R} \mid \langle v_i, \alpha \rangle > 0, i > 0 \}.
\]

For a small \( \alpha \in D \), we describe the exceptional set \( \phi^{-1}_\alpha(x) \). The method is the same as in \([Y]\).

**Lemma 2.4.** Assume that \( \alpha, \beta \in \delta(H^1)_{\text{alg}} \) belongs to \( D \) and \( |\langle \alpha^2 \rangle| \ll 1 \). Let \( F \) be a \( v + \alpha \)-twisted semi-stable sheaf such that \( v(F) = v_0 + \sum_{j > 0} b_j v_j \), \( 0 \leq b_j \leq a_j \).

(1) If \( v(F) \neq v \), then \( F \) is \( v + \alpha \)-twisted stable and \( F \) is \( S \)-equivalent to \( E_0 \oplus (\bigoplus_{j > 0} E_{j}^{\otimes b_j}) \) with respect to the \( v \)-twisted stability.

(2) For a non-zero homomorphism \( \phi : E_i \rightarrow F \), \( i > 0 \), \( \phi \) is injective and \( F' := \text{coker} \phi \) is a \( v + \alpha \)-twisted stable sheaf.

(3) If there is a non-trivial extension

\[
(2.7) \quad 0 \rightarrow E_i \rightarrow F'' \rightarrow F \rightarrow 0
\]

and \( b_i + 1 \leq a_i \), then \( F'' \) is \( v + \alpha \)-twisted stable.

**Proof.** We take elements \( E, A \in K(X) \otimes \mathbb{Q} \) such that \( v(E) = v, v(A) = \alpha \). Since \( \langle \alpha^2 \rangle \ll 1 \), \( F \) is \( v \)-twisted semi-stable. Assume that \( F \) is \( S \)-equivalent to \( \bigoplus_{j>0} F_j^{\otimes b_j} \) with respect to the \( v \)-twisted stability. Since \( F \oplus (\bigoplus_{j > 0} F_j^{\otimes b_j}) \) is \( S \)-equivalent to \( \bigoplus_{j>0} E_j^{\otimes b_j} \oplus (\bigoplus_{j > 0} E_j^{\otimes (a_j - b_j)}) \), by Lemma 2.3 and Lemma 2.4, we get that \( F \) is \( S \)-equivalent to \( E_0 \oplus (\bigoplus_{j > 0} E_j^{\otimes b_j}) \) with respect to the \( v \)-twisted stability. Since \( \chi(E + A, F) \geq \chi(E + A, E) = 0 \) and \( \chi(E + A, E_i) < 0 \) for all \( i > 0 \), there is no proper subsheaf \( E' \) such that \( \chi(E + A, E', (nH)) / \text{rk} E' \) for all \( n \). Thus the claim (1) holds.

We next prove (2). Since \( E_i \) is \( v \)-twisted stable and \( F \) is \( v \)-twisted semi-stable, \( \phi \) is injective and \( F' \) is a \( v \)-twisted semi-stable sheaf. By (1), \( F' \) is \( S \)-equivalent to \( E_0 \oplus (\bigoplus_{j > 0} E_j^{\otimes c_j}) \) with respect to the \( v \)-twisted stability, where \( v(F') = v_0 + \sum_{j>0} c_j v_j \). If \( F' \) is not \( v + \alpha \)-twisted stable, then there is a quotient sheaf \( F'' \rightarrow G \) such that \( \text{deg}_E G = 0 \) and \( \chi(E + A, G) / \text{rk} G < \chi(E + A, F') / \text{rk} F' \). Since \( \langle \alpha^2 \rangle \ll 1 \) and \( \chi(E + A, E_i) < 0 \) for all \( i > 0 \), we see that \( G \) is \( v \)-twisted semi-stable and is \( S \)-equivalent to \( \bigoplus_{j > 0} E_j^{\otimes c_j} \) with respect to the \( v \)-twisted stability. Hence we get that \( \chi(E + A, G) < 0 \), which implies that \( F' \) is \( v + \alpha \)-twisted stable. Therefore \( F'' \) is \( v + \alpha \)-twisted stable.

Finally we prove (3). By our assumption, \( \chi(E + A, F'') \geq \chi(E + A, E) = 0 \). If \( F'' \) is not \( v + \alpha \)-twisted stable, then there is a quotient sheaf \( F'' \rightarrow G \) such that \( \text{deg}_E G = 0 \) and \( \chi(E + A, G) / \text{rk} G < \chi(E + A, F'') / \text{rk} F'' \). Then we see that \( G \) is \( v \)-twisted semi-stable and is \( S \)-equivalent to \( \bigoplus_{j > 0} E_j^{\otimes c_j} \) with respect to the \( v \)-twisted stability. Then there is a quotient \( G \rightarrow E_j, j > 0 \). By (2.7), we have an exact sequence

\[
(2.8) \quad 0 = \text{Hom}(F, E_j) \rightarrow \text{Hom}(F'', E_j) \rightarrow \text{Hom}(E_i, E_j).
\]

We consider the map \( \psi : E_i \rightarrow F'' \rightarrow G \rightarrow E_j \). Then \( \psi \) is an isomorphism, which implies that the extension (2.7) splits.

By Lemma 2.4 (1), we get the following corollary.

**Corollary 2.5.** We set \( w := v_0 + \sum_{j > 0} b_j v_j \), \( 0 \leq b_j \leq a_j \). If \( w \neq v \), then \( \nabla(M_H^{v+a}(w)) = M_H^{v+a}(w) \). In particular, if \( (w^2) = -2 \), then \( M_H^{v+a}(w) \) is not empty and consists of one element.

**Corollary 2.6.** Let \( F \) be a \( v + \alpha \)-twisted stable sheaf with \( v(F) = v_0 + \sum_{j > 0} b_j v_j \), \( 0 \leq b_j \leq a_j \).

(1) If \( v(F) = v \), then \( \dim \text{Hom}(E_i, F) \leq 1 \).

(2) If \( v(F) \neq v \), then \( \dim \text{Hom}(E_i, F) = \max\{-\langle v(F), v_i \rangle, 0\} \).

**Proof.** We set \( \text{dim} \text{Hom}(E_i, F) = k \). By the Riemann-Roch theorem, \( k \geq -\langle v(F), v_i \rangle \). Hence if \( k = 0 \), then our claims (1), (2) hold. Assume that \( k > 0 \). By Lemma 2.4, \( \phi : \text{Hom}(E_i, F) \otimes E_i \rightarrow F \) is injective and \( F' := \text{coker} \phi \) is \( v + \alpha \)-twisted stable. If \( v(F) = v \), then \( -2 \leq \langle v(F'), v_i \rangle = -2k^2 \). Hence \( k \leq 1 \). If \( v(F) \neq v \), then \( \langle v(F'), v_i \rangle = -2 \) and hence \( -2 \leq \langle v(F), v_i \rangle = -2 - 2k(\langle v(v), v_i \rangle) \). Then \( k + \langle v(F), v_i \rangle \leq 0 \), which implies that \( k = -\langle v(F), v_i \rangle \).

**Corollary 2.7.** We set \( w = v_0 + \sum_{j > 0} b_j v_j \), \( 0 \leq b_j \leq a_j \). If \( (w^2) = \langle w - v_i \rangle = -2 \), then we have an isomorphism \( M_H^{v+a}(w) \rightarrow M_H^{v+a}(w - v_i) \) sending \( F \) to \( \text{coker}(E_i \rightarrow F) \).

**Proof.** By our assumption, we see that \( \langle w, v_i \rangle = -1 \). By Corollary 2.4, \( \text{Hom}(E_i, F) = \mathbb{C} \). By Lemma 2.4 (2), \( F' := \text{coker}(E_i \rightarrow F) \) is a \( v + \alpha \)-twisted stable sheaf with \( v(F') = w - v_i \). Conversely for a \( v + \alpha \)-twisted stable sheaf \( F' \) with \( v(F') = w - v_i \), we get \( \langle w - v_i, v_i \rangle = 1 \), and hence by Corollary 2.4 and Lemma 2.4 (3), the non-trivial extension of \( F' \) by \( E_i \) gives a \( v + \alpha \)-twisted stable sheaf \( F \) with \( v(F) = w \).

\[\Box\]
We set
\[
C_i := \left\{ (E, U) \left| \begin{array}{l}
E \in M^{r+\alpha}_H(v), U \subset \text{Hom}(E_i, E) \\
\dim U = 1 \\
\end{array} \right. \right\} = \{E_i \subset E | E \in M^{r+\alpha}_H(v)\}.
\]

$C_i$ is the moduli space of twisted coherent systems.

**Proposition 2.8.**

1. $C_i \subset \phi^{\alpha-1}_i(x)$.
2. $C_i \cong \mathbb{P}^1$ and the natural map $\pi : C_i \to M^{r+\alpha}_H(v)$ is a closed immersion. In particular, $C_i$ is not empty.

**Proof.** We set $F := \text{coker}(E_i \to E)$. Then $E$ is $S$-equivalent to $E_i \oplus F$ with respect to the $v$-twisted stability.

By Lemma 2.4 (1), $F$ is $S$-equivalent to $E_i \oplus (a_i-1) \oplus \bigoplus_j E_j \oplus a_j$, and hence the first claim holds. We next show the assertion (2). We note that the Zariski tangent space of $C_i$ at $E_i \to E$ is
\[
\text{Ext}^1(E_i \to E, E) = \text{Ext}^1(F, E)
\]
and the obstruction for the infinitesimal lifting belongs to
\[
\ker(\text{Ext}^2(E_i \to E, E) \to \text{Ext}^2(E, E) \to \mathcal{H}_2(X, \mathcal{O}_X)).
\]
We shall first show that $C_i$ is smooth at $E_i \subset E$. Since $\text{Ext}^2(E_i \to E, E) = \text{Ext}^2(F, E) = \text{Hom}(E, F)\psi$, it is sufficient to show that $\text{Hom}(E, F) = \mathbb{C}$. By the exact sequence
\[
0 \to E_i \to E \to F \to 0,
\]
we get an exact sequence
\[
0 \to \text{Hom}(E, F) \to \text{Hom}(E_i, F) \to \text{Hom}(E_i, E).
\]
If $\text{Hom}(E_i, F) \neq 0$, then we get $\dim \text{Hom}(E_i, E) \geq 2$, which contradicts Corollary 2.4. Hence $\text{Hom}(E, F) \cong \text{Hom}(E_i, F)$. By Lemma 2.4 (2), $F$ is simple. Therefore $\text{Hom}(E, F) = \mathbb{C}$. Since the homomorphism
\[
\text{Ext}^1(E_i \to E, E) \to \text{Ext}^1(F, E)
\]
between the Zariski tangent spaces is injective, $C_i \to M^{r+\alpha}_H(v)$ is a closed immersion, provided that $C_i \neq \emptyset$.

We next show that $C_i \neq \emptyset$ and isomorphic to $\mathbb{P}^1$. Since $\langle (v-v_i)^2 \rangle = -2$, Corollary 2.4 implies that $M^{r+\alpha}_H(v)$ consists of exactly one $v + \alpha$-twisted stable sheaf $F$. By Corollary 2.6 (2), $\text{Hom}(E_i, F) = 0$. Thus $\text{Ext}^2(F, E_i) = \text{Hom}(E_i, F)\psi = 0$. Since $\text{Hom}(F, E_i) = 0$, we get $\text{Ext}^1(F, E_i) \cong \mathbb{C}\psi^2$. Let $E$ be a coherent sheaf which is defined by a non-trivial extension
\[
0 \to E_i \to E \to F \to 0.
\]
By Lemma 2.4 (3), $E$ is $v + \alpha$-twisted stable. Therefore $C_i \neq \emptyset$ and $C_i \cong \mathbb{P}^1$.

**Proposition 2.9.** We identify $C_i$ with its image $\pi(C_i)$. Then we have

1. $\phi^{\alpha-1}_i(x) = \cup_{i=1}^{n} C_i$
2. $\cup_{i=1}^{n} C_i$ is a simple normal crossing divisor.

(2) $(C_i, C_j) = (v_i, v_j)$. In particular the dual graph of $C_i$, $1 \leq i \leq n$ is of type $A_n, D_n, E_n$.

**Proof.** By Proposition 2.3 (1), $\phi^{\alpha-1}_i(x) \supset \cup_{i=1}^{n} C_i$. Since $\alpha \in D$, we get $\phi^{\alpha-1}_i(x) = \cup_{i=1}^{n} C_i$. We shall study the configuration of $C_i$, $1 \leq i \leq n$. Here we give a geometric argument based on Lemma 2.4. We shall prove the following assertions:

(i) $C_i \cap C_j \neq \emptyset$ if and only if $\langle v_i, v_j \rangle = 1$.

(ii) If $C_i \cap C_j \neq \emptyset$, then $\#(C_i \cap C_j) = 1$.

(iii) If $C_i \cap C_j \neq \emptyset$, then $C_i$ and $C_j$ intersect transversely.

(iv) $C_i \cap C_j \cap C_k = \emptyset$ for three curves $C_i, C_j, C_k$.

(i) Assume that $C_i \cap C_j \neq \emptyset$ and take a point $E \in C_i \cap C_j$. Then $E$ fits in an exact sequence
\[
0 \to E_i \oplus E_j \to E \to F' \to 0
\]
By Lemma 2.4 (2), $F'$ is a $v + \alpha$-twisted stable sheaf. Since $\delta(H^\vee)_\text{alg}$ is negative definite, $-2 \leq \langle v(F')^2 \rangle < 0$. Thus $\langle (v(F')^2) \rangle = -2$. Then $-2 = \langle v(F)^2 \rangle = -4 + 2\langle v(E_i), v(E_j) \rangle$, which implies that $\langle v(E_i), v(E_j) \rangle = 1$. Conversely if $\langle v(E_i), v(E_j) \rangle = 1$, then $\langle (v - (v_i + v_j)^2 \rangle = -2$. Hence there is a $v + \alpha$-twisted semi-stable sheaf $F'$ with $v(F') = v - (v_i + v_j)$. By Lemma 2.4 (1), $F'$ is $v + \alpha$-twisted stable. By Corollary 2.6 (2), $\text{Hom}(E_i, F') = 0$. Hence $\dim \text{Ext}^1(E_i, F') = \langle v_i, v - (v_i + v_j) \rangle = 1$. We also have $\dim \text{Ext}^1(E_j, F') = 1$. We take an extension
\[
0 \to E_i \oplus E_j \to E \to F' \to 0
\]
whose extension class is given by \((e_i, e_j) \in \text{Ext}^1(E_i, F') \oplus \text{Ext}^1(E_j, F')\), \(e_i, e_j \neq 0\). Then Lemma \(2.4\) (3) implies that \(E\) is a \(v + \alpha\)-twisted stable sheaf with \(v(E) = v\). Therefore \(C_i\) and \(C_j\) intersect at \(E\).

(ii) Assume that \(C_i \cap C_j \neq \emptyset\). Then every member of \(C_i \cap C_j\) fits in an extension \(2.18\). Since \(M_H^{n+\alpha}(v - (v_i + v_j)) = \{F'\}\) and \(E\) does not depend on the choice of \((e_i, e_j)\), we get \(#(C_i \cap C_j) = 1\).

(iii) Assume that \(C_i\) and \(C_j\) intersect at \(E\). Since \(\text{Hom}(E_i \oplus E_j, F') = 0\) and \(F'\) is \(v + \alpha\)-twisted stable, \(\text{Ext}^2(F', E) = \text{Hom}(E, F')^\vee = \mathbb{C}\). Then we see that the natural homomorphism

\[
\text{Ext}^1(E/E_i, E) \oplus \text{Ext}^1(E/E_j, E) \to \text{Ext}^1(E, E)
\]

of tangent spaces is an isomorphism, and hence \(C_i\) and \(C_j\) intersect transversely.

(iv) If \(C_i \cap C_j \cap C_k \neq \emptyset\), then \(\langle v_i, v_j \rangle = \langle v_k, v_i \rangle = 1\), which implies that \(\langle (v_i + v_j + v_k)^2 \rangle = 0\). Since \(\delta(H^+)\) is negative definite, this is impossible.

Therefore \(\cup_i C_i\) is simple normal crossing and \((C_i, C_j) = \langle v_i, v_j \rangle\).

\[\square\]

**Lemma 2.10.** Let \(\{C_i\} \in H_2(M_H^{n+\alpha}(v), \mathbb{Z})\) be the fundamental class of \(C_i\) and \(\text{PD}([C_i]) \in H^2(M_H^{n+\alpha}(v), \mathbb{Z})\) the Poincaré dual of \([C_i]\). Then

\[
\text{PD}([C_i]) = \theta^n_v(-v_i).
\]

**Proof.** Let \(M_H^{n+\alpha}(v) = \bigcup \lambda U_\lambda\) be an analytic open covering and \(\mathcal{F}_\lambda\) a local universal family on \(U_\lambda \times X\). Then \(\text{Hom}_{\mathcal{P}_\mathcal{U}(\mathcal{P}_X(\mathcal{E}_i), \mathcal{F}_\lambda)} = \text{Ext}^1_{\mathcal{P}_\mathcal{U}(\mathcal{P}_X(\mathcal{E}_i), \mathcal{F}_\lambda)} = 0\) and \(C_i \cap U_\lambda\) is the scheme-theoretic support of \(\text{Ext}^1_{\mathcal{P}_\mathcal{U}(\mathcal{P}_X(\mathcal{E}_i), \mathcal{F}_\lambda)}\) where \(px : U_\lambda \times X \to X\) and \(p_{U_\lambda} : U_\lambda \times X \to U_\lambda\) are projections. For a sufficiently large integer \(n\), let \(V_\lambda := p_{U_\lambda}(\mathcal{F}_\lambda \otimes p_X^*(\mathcal{O}_X(nH)))\) is a locally free sheaf on \(U_\lambda\). Then we can glue \(\{\mathcal{F}_\lambda \otimes p_X^*(\mathcal{V}_\lambda^\wedge)\}\) together and we get a quasi-universal family \(\mathcal{F}\) on \(M_H^{n+\alpha}(v) \times X\). By using the Grothendieck Riemann-Roch theorem, we get that the Poincaré dual of \([C_i]\) is \(\theta^n_v(-v_i)\).

\[\square\]

**Remark 2.1.** By Lemma 2.10, the non-emptiness of \(C_i\) also follows from the fact that \(\theta^n_v(v_i) \neq 0\). Since \(\theta^n_v\) is an isometry, the configuration of \(C_i\) is also described by the configuration of \(v_i\). In particular, we get a different proof of Proposition 2.5.

Since \(\phi^{-1}_\alpha(x)\) is a union of \((-2)\)-curves, we get the following proposition.

**Proposition 2.11.** \(\phi_\alpha : M_H^{n+\alpha}(v) \to \overline{M}_H^\alpha(v)\) is surjective and \(\overline{M}_H^\alpha(v)\) contains a \(v\)-twisted stable sheaf.

**Proof.** Let \(x\) be a point of \(\overline{M}_H^\alpha(v)\) and assume that \(x\) corresponds to a properly \(v\)-twisted semi-stable sheaf. Then \(\phi_\alpha^{-1}(x) \neq \emptyset\) with \(\dim \phi_\alpha^{-1}(x) = 1\). Since \(\phi_\alpha : \phi_\alpha^{-1}(M_H^{n+\alpha}(v)) \to M_H^{n+\alpha}(v)\) is an isomorphism and \(\overline{M}_H^\alpha(v) \setminus M_H^{n+\alpha}(v)\) is a finite set, we get our claims.

\[\square\]

The remaining of this section is an appendix.

**Lemma 2.12.** Assume that \(\alpha \in D, |\langle \alpha^2 \rangle| \ll 1\) satisfies that

\[
\langle v_i, \alpha \rangle/\text{rk} v_i > \langle v + \sum_{j>0} a_j v_j, \alpha \rangle/\text{rk}(v + \sum_{j>0} a_j v_j)
\]

for all \(i > 0\). Let \(F\) be a \(v + \alpha\)-twisted semi-stable sheaf such that \(v(F) = v + \sum_{i>0} b_j v_j, 0 \leq b_j \leq a_j\). Then

1. \(F\) is \(v + \alpha\)-twisted stable.
2. Let \(F'\) be a coherent sheaf which fits in a non-trivial extension

\[
0 \to E_i \to F' \to F \to 0
\]

and \(b_i + 1 \leq a_i\). Then \(F'\) is \(v + \alpha\)-twisted stable.

3. For a subextension \(E_i, i > 0\) of \(F, F' := F/E_i\) is \(v + \alpha\)-twisted stable.

**Proof.** We take \(E, A \in K(X) \oplus \mathbb{Q}\) with \(v(E) = v, v(A) = \alpha\). Let

\[
0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F
\]

be the Jordan-Hölder filtration of \(F\) with respect to the \(v\)-twisted stability. For \(F_i/F_{i-1}\) with \(F_i/F_{i-1} \notin \{E_0, E_1, \ldots, E_n\}\), a similar argument to the proof of Lemma 2.2 shows that \(\text{Ext}^1(F_i/F_{i-1}, E_j) = 0\) for all \(j\). Hence replacing the filtration, we may assume that \(F_i/F_{i-1} \notin \{E_0, E_1, \ldots, E_n\}\) for \(0 \leq i \leq k\) and \(F_i/F_{i-1} \notin \{E_0, E_1, \ldots, E_n\}\) for \(i > k\). Then we get \(F = F_k \oplus F/F_k\) and \(-2 \leq \langle v(F) \rangle^2 = \langle v(F_k) \rangle^2 + \langle v(F/F_k) \rangle^2\). Assume that \(F_k \neq 0\), that is, \(k > 0\). Then we see that (i) \(v(F_k) = v\) and \(\langle v(F/F_k) \rangle^2 = -2\), or (ii) \(\langle v(F_k) \rangle^2 = -2\) and \(v(F/F_k) \in \mathbb{Z}v\). If \(v(F_k) = v\), then \(\langle v(F_k), \alpha \rangle = 0\) and \(\langle v(F/F_k), \alpha \rangle = \sum b_j \langle v_j, \alpha \rangle > 0\), which contradicts the \(v + \alpha\)-twisted semi-stability of \(F\). Hence the case (i) does not occur. If the case (ii) occurs, then since \(\langle v(F_k) \rangle^2 = -2\), we get that \(\langle v(F), v(F_i/F_{i-1}) \rangle = \langle v(F_k), v(F_i/F_{i-1}) \rangle \neq 0\) for some \(i \leq k\). On the other hand, by our choice of \(v(F)\), we get \(\langle v(F), v(F_i/F_{i-1}) \rangle = 0\) for all \(i \leq k\). Hence the case (ii) does not occur. Therefore \(F_k = 0\). Then we see that \(F\) is \(S\)-equivalent to \(E_0 \oplus (\bigoplus_{j>0} F_j^{\oplus (a_j+b_j)})\).
By the Grothendieck Riemann-Roch theorem, the Poincaré dual of the scheme-theoretic support of \(H\) where

\[(2.29)\]

\[\chi(E + A, 2E - E_0)/\text{rk}(2E - E_0) \leq \chi(E + A, F)/\text{rk} F.\]

In the same way as in the proof of Lemma 2.4 (1), we see that \(F\) is a \(v + \alpha\)-twisted stable sheaf. Thus (1) holds. The proof of (2) and (3) are the same as in the proof of Lemma 2.4.

**Remark 2.2.** If \(\langle v_i, \alpha \rangle = \langle v_j, \alpha \rangle\) for all \(1 \leq i, j \leq n\), then (2.21) is satisfied.

In the same way as in the proof of Corollary 2.7, we get the following.

**Corollary 2.13.** Assume that \(\alpha \in D, |\langle \alpha^2 \rangle| \ll 1\) satisfies (2.21). We set \(w = v + \sum_{j > 0} b_j v_j, 0 \leq b_j \leq a_j\).

1. If \(w \neq v\), then \(M_H^{v+\alpha}(w) = M_H^{v+\alpha}(w)\).
2. If \(\langle w^2 \rangle = \langle (w - v_1)^2 \rangle = -2\), then we have an isomorphism \(M_H^{v+\alpha}(w) \to M_H^{v+\alpha}(w - v_1)\) sending \(F \in M_H^{v+\alpha}(w)\) to \(\text{coker}(E_i \to F)\).

**Remark 2.3.** Assume that \(\alpha \in D, |\langle \alpha^2 \rangle| \ll 1\) satisfies (2.21). We note that \(H_2(\phi^{-1}_\alpha(x), \mathbb{C}) \to H_2(M_H^{v+\alpha}(v), \mathbb{C})\) is injective. We can regard \(H_2(\phi^{-1}_\alpha(x), \mathbb{C})\) as the Cartan subalgebra of \(\mathfrak{g}\). In order to get \(\mathfrak{g}\), we set

\[(2.26)\]

\[\Psi := \{u | u = \sum_{i=1}^{n} b_i v_i, \langle u^2 \rangle = -2\}.

Let \(P(w, w - v_i)\) be the subscheme of \(M_H^{v+\alpha}(w) \times M_H^{v+\alpha}(w - v_i)\) consisting of pairs \((E, F) \in M_H^{v+\alpha}(w) \times M_H^{v+\alpha}(w - v_i)\) which fits in an exact sequence

\[(2.27)\]

\[0 \to E_i \to E \to F \to 0.

Then we can show that \(P(w, w - v_i)\) is isomorphic to \(\mathbb{P}^1\) or a point. As in [2.4], we see that there is an action of \(\mathfrak{g}\) on \(H_2(M_H^{v+\alpha}(v), \mathbb{C}) \oplus \bigoplus_{u \in \Psi} H_0(M_H^{v+\alpha}(v + u), \mathbb{C})\) and we have an isomorphism of \(\mathfrak{g}\)-module:

\[(2.28)\]

\[H_2(\phi^{-1}_\alpha(x), \mathbb{C}) \oplus \bigoplus_{u \in \Psi} H_0(M_H^{v+\alpha}(v + u), \mathbb{C}) \cong \mathfrak{g}.

For a homology class \([x] \in H_{2+\langle w^2 \rangle}(M_H^{v+\alpha}(w), \mathbb{C})\), the action of Chevalley generators \(e_i, f_i, h_i, 1 \leq i \leq n\) are given by

\[e_i : [x] \mapsto p_{M_H^{v+\alpha}(w+v_i)}((M_H^{v+\alpha}(w+v_i) \times [x]) \cap P(w+v_i, w))\]

\[(2.29)\]

\[f_i : [x] \mapsto (-1)^{t(w)} p_{M_H^{v+\alpha}(w-v_i)}([x] \times M_H^{v+\alpha}(w-v_i)) \cap P(w, w-v_i))\]

\[h_i : [x] \mapsto -\langle w, v_i \rangle [x],\]

where \(p_{M_H^{v+\alpha}(w+kv)} : M_H^{v+\alpha}(w) \times M_H^{v+\alpha}(w-v_i) \to M_H^{v+\alpha}(w+kv), k = 0, -1\) are projections and \(t(w) = (\dim M_H^{v+\alpha}(w-v_i) - \dim M_H^{v+\alpha}(w))/2 = -((w,v_i) + 1)\).

2.2. Other chambers.

**Definition 2.1.** Let \(W\) be the Weyl group generated by reflections \(R_{v_i}, i = 1, 2, \ldots, n\).

\(W\) is the Weyl group of \(\mathfrak{g}\). By Lemma 2.10 and Proposition 1.13 (also see Remark 1.3), we get the following.

**Proposition 2.14.** If \(\alpha \in w(D), w \in W\), then \(\text{PD}([C_i]) = \theta^\alpha(-w(v_i)).\)

We shall give a set-theoretic description of \(C_i\). Let \(M_H^{v+\alpha}(v) = \cup _\lambda U_\lambda\) be an analytic open covering and \(F_\lambda\) a local universal family on \(U_\lambda \times X\). If \(\text{rk}(w(v_i)) = 0\), then we set \(w(v_i) := \eta + b \eta X\), where \(\eta \in \text{Pic}(X)\) satisfies \((\eta^2) = -2\). Then \(0 = \langle \hat{H}, w(v_i) \rangle = (H, \eta), \) which contradicts the ampleness of \(H\). Hence \(\text{rk}(w(v_i)) \neq 0\). Let \(F_i\) be a \(v + \alpha\)-twisted semi-stable sheaf with \(v(F_i) = \pm w(v_i)\). We first assume that \(\text{rk}(w(v_i)) > 0\), that is, \(v(F_i) = w(v_i)\). Since \(\langle \alpha, w(v_i) \rangle = (w^{-1}(\alpha), v_i) > 0\), \(\chi(E + A, F_i) = -\langle v + \alpha, v(F_i) \rangle = -\langle \alpha, v(F_i) \rangle < 0\).

Then \(\text{Ext}^2(F_i, E) = 0\) for all \(E \in M_H^{v+\alpha}(v)\). If \(E\) is \(v\)-twisted stable, then \(\text{Hom}(F_i, E) = 0\). Hence \(L_\lambda := \text{Ext}^1_{\text{Pic}_\lambda}(p_X^*(F_i), F_\lambda)\) is a torsion sheaf of pure dimension 1 whose support is contained in \(\phi^{-1}_\alpha(x)\). By the Grothendieck Riemann-Roch theorem, the Poincaré dual of the scheme-theoretic support of \(L_\lambda\) is \(\theta^\alpha(-w(v_i)) = \text{PD}([C_i])\). Since \(H^0(M_H^{v+\alpha}(v), C_M^{v+\alpha}(v)(C_i)) = \mathbb{C}\), we get that

\[(2.30)\]

\[C_i = \{E \in M_H^{v+\alpha}(v) | \text{Ext}^1(F_i, E) \neq 0\} = \{E \in M_H^{v+\alpha}(v) | \text{Hom}(F_i, E) \neq 0\}.\]
If \( \chi(E + A, F_i) > 0 \), that is, \( \operatorname{rk}(w(v_i)) < 0 \), then we see that \( \operatorname{Hom}(F_i, E) = 0 \) for all \( E \in M_H^{v+\alpha}(v) \) and \( \operatorname{Ext}^1(F_i, E) = 0 \) for a \( v \)-twisted stable sheaf \( E \). Then we also have \( \operatorname{Ext}^2(F_i, E) = 0 \). Hence we see that \( \operatorname{Ext}^1_{\mu_\lambda}(p'_{\lambda}(F), \mathcal{F}_\lambda) = 0 \) and \( \operatorname{Ext}^2_{\mu_\lambda}(p'_{\lambda}(F), \mathcal{F}_\lambda) \) is a torsion sheaf of pure dimension 1. Hence we get that
\[
C_i = \{ E \in M_H^{v+\alpha}(v) | \operatorname{Ext}^2(F_i, E) \neq 0 \}
= \{ E \in M_H^{v+\alpha}(v) | \operatorname{Hom}(E, F_i) \neq 0 \}.
\]
Therefore we get the following proposition.

**Proposition 2.15.** Assume that \( \alpha \in w(D), w \in W \). Then \( w(v_i) \neq 0 \) for \( 1 \leq i \leq n \). Let \( F_i \) be a \( v+\alpha \)-twisted semi-stable sheaf with \( v(F_i) = \pm w(v_i) \) according to the sign of \( \operatorname{rk}(w(v_i)) \). If \( \operatorname{rk}(w(v_i)) > 0 \), then
\[
C_i = \{ E \in M_H^{v+\alpha}(v) | \operatorname{Hom}(E, F_i) \neq 0 \}.
\]
If \( \operatorname{rk}(w(v_i)) < 0 \), then
\[
C_i = \{ E \in M_H^{v+\alpha}(v) | \operatorname{Hom}(E, F_i) \neq 0 \}.
\]

### 2.3. Normality of \( \overline{M}_H(v) \).

**Proposition 2.16.** \( \overline{M}_H(v) \) is normal.

**Proof.** We take \( \alpha \in D \) with \( |(\alpha^2)| < 1 \). Let \( T \) be a smooth curve and we consider a flat family of polarized K3 surfaces \( \pi : (X, H) \rightarrow T \) such that

(i) \((X_t, H_t) = (X, H), t \in T \),

(ii) there are families of Mukai vectors \( v \in R^* \pi_* Z, a \in R^* \pi_* Q \) with \( v_t = v, a_t = \alpha \) and

(iii) \( \operatorname{rk}(\pi(X_t)) \leq 3 \) for a point \( t \in T \),

where \((X_t, H_t) := (X \times k(t), H \times k(t)) \) and \( k(t) \) is the residue field at \( t \in T \). Replacing \( T \) by a suitable covering of \( T \), we may assume that there is a section of \( \pi \) and a locally free sheaf \( E \) on \( X \) with \( v(\mathcal{E}_t) = v_t, t \in T \). We consider the relative quotient-scheme \( Q := \operatorname{Quot}_{\mathcal{E}_t(-nH_t)\otimes \mathbb{N} / X \times \mathbb{P}^1} \rightarrow T \) parametrizing all quotients \( \mathcal{E}_t(-nH_t)\otimes \mathbb{N} \rightarrow F, t \in T \) with \( v(F) = v_t, \) where \( N := \chi(\mathcal{O}(nH_t)) \). We denote the universal quotient sheaf by \( \mathcal{F} \). We set
\[
Q_{ss} := \{ q \in Q | \mathcal{F}_q := \mathcal{F} \otimes k(q) \text{ is a } v_t\text{-twisted semi-stable with respect to } H_t \}.
\]
For \( n \gg 0 \), we have a relative coarse moduli space \( \overline{M}_{X/T,H}(v) := Q_{ss}/\operatorname{PGL}(N) \rightarrow T \). Since \( T \) is defined over a field of characteristic 0, \( \overline{M}_{X/T,H}(v)_t = \overline{M}_{H_t}(v_t) \) (cf. [MFK, Thm. 1.1]). We also have a relative moduli space \( \overline{M}_{X/T,H}^{v+\alpha}(v) \rightarrow T \). Replacing \( T \) by an open subscheme, we may assume that \( \overline{M}_{X/T,H}(v)_t \) consists of \( v_t + a_t \)-twisted stable sheaves on \( X_t \) for all \( t \in T \) and there is no walls between \( v_t \) and \( v_t + a_t \). Then \( \overline{M}_{X/T,H}(v) \rightarrow T \) is a smooth morphism ([Mu1, Thm. 1.17]) and we have a morphism \( \Phi : \overline{M}_{X/T,H}^{v+\alpha}(v) \rightarrow \overline{M}_{X/T,H}(v) \).

**Claim 2.1.** \( \overline{M}_{X/T,H}(v) \) is normal.

**Proof of Claim 2.1.** It is sufficient to show that \( Q_{ss} \) is normal. By Serre’s criterion, we shall show that \( Q_{ss} \) is Cohen-Macaulay and \( Q_{ss} \) is regular in codimension 1. We first prove that \( Q_{ss} \) is Cohen-Macaulay. Let \( Q_{spl} \) be the open subscheme of \( Q_{ss} \) parametrizing simple sheaves:
\[
Q_{spl} := \{ q \in Q_{ss} | \mathcal{F}_q \text{ is simple } \}.
\]
Then \( Q_{spl} \rightarrow T \) is a smooth morphism ([Mu1, Thm. 1.17]). By the usual deformation theory of sheaves and Lemma 2.17 below, \( Q_{ss} \) is a locally complete intersection scheme for all \( t \). In particular \( Q_{ss} \) is Cohen-Macaulay. Since \( Q_{ss} \) is smooth at the generic point, \( Q_{ss} \) is reduced. Let \( x_t \) be the local parameter of \( T \) at \( t \). By Lemma 2.18 below, \( x_t \) is a regular element, which implies that \( Q_{ss} \) is flat over \( T \). Then \( Q_{ss} \) is also Cohen-Macaulay.

We next show that \( Q_{ss} \) is regular in codimension 1. It is sufficient to show that \( \dim(Q_{ss} \setminus Q_{spl}) \leq \dim Q_{ss} - 2 \). By Lemma 2.17, \( \dim(Q_{ss} \setminus Q_{spl}) \leq \dim(Q_{ss} \setminus Q_t) \leq \dim Q_{ss} - 1 \) for all \( t \in T \). For a point \( t \in T \) with \( \rho(x_t) \leq 3 \), by a direct computation, we see that \( \dim(Q_{ss} \setminus Q_{spl}) \leq \dim Q_{ss} - 2 \). Since \( Q_{spl} \) is an open subscheme of \( Q_{ss} \), we get that \( \dim(Q_{ss} \setminus Q_{spl}) \leq \dim Q_{ss} - 2 \). Therefore our claim holds.

Since \( \Phi \) is a birational morphism, we get
\[
\Phi_* (\mathcal{O}_{\overline{M}_H^{v+\alpha}(v)}) = \mathcal{O}_{\overline{M}_{X/T,H}^{v+\alpha}(v)}.
\]
Let \( \varphi : \overline{M}_H(v)_{nor} \rightarrow \overline{M}_H(v) \) be the normalization of \( \overline{M}_H(v) \) and \( \phi^*_\alpha : M^{v+\alpha}(v) \rightarrow \overline{M}_H(v)_{nor} \) the morphism with \( \varphi \circ \phi^*_\alpha = \phi_{\alpha} \). Since \( \overline{M}_H(v)_{nor} \) has at worst rational double points as its singularities, we get \( R^1 \phi^*_\alpha \mathcal{O}_{M^{v+\alpha}(v)} = 0 \). Since \( \varphi \) is a finite morphism, by the Leray spectral sequence, we get
\[
R^1 \phi^*_\alpha \mathcal{O}_{M^{v+\alpha}(v)} = \varphi_* (R^1 \phi^*_\alpha \mathcal{O}_{M^{v+\alpha}(v)}) = 0.
\]
Since $\overline{M}_{\mathcal{X}/T,\mathcal{R}}(v)$ and $\overline{M}_{\mathcal{X}/T,\mathcal{R}}(\mathcal{V})$ are flat over $T$, by using the base change theorem, we get

\[(2.38) \quad \Phi_*(\mathcal{O}_{\overline{M}_{\mathcal{X}/T,\mathcal{R}}(\mathcal{V})}) \otimes k(t_0) = \phi_{a_0}(\mathcal{O}_{\overline{M}_{\mathcal{X}/T,\mathcal{R}}(\mathcal{V})}).\]

Combining this with (2.33), we get $\phi_{a_0}(\mathcal{O}_{\overline{M}_{\mathcal{X}/T,\mathcal{R}}(\mathcal{V})}) = \mathcal{O}_{\overline{M}_H(v)}$, which implies that $\overline{M}_H(v)$ is normal. \qed

**Lemma 2.17.** We set

\[(2.39) \quad Q^s_i := \{ q \in Q^s_i \mid F_q \text{ is } v_i + a_i\text{-twisted semi-stable} \}.

Then $\dim Q^s_i (Q^s_i \setminus Q^s_i) = \dim Q^s_i - 1$.

**Proof.** Let $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F_q$ be the Harder-Narasimhan filtration of $F_q$ with respect to the $v_i + a_i$-twisted semi-stability. We set $v_i := v(F_i/F_{i-1})$. Then $\langle \hat{H}, v_i \rangle = \langle v, v_i \rangle = 0$ and $\langle v^2 \rangle = -2(\text{rk} v_i)^2$. We shall compute the dimension of an open subscheme of the flag-scheme $F(v_1, v_2, \ldots, v_s)$ over $T$ parametrizing filtrations $0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = F_q$, $q \in Q^s_i$ such that $F_i/F_{i-1}, 1 \leq i \leq s$ are $v_i + a_i$-twisted semi-stable sheaves with $v(F_i/F_{i-1}) = v_i$. By the arguments in [Y3, sect. 3.3], we get

\[(2.40) \quad \dim F(v_1, v_2, \ldots, v_s) - \dim GL(N) = \sum_{i>j} \langle v_i, v_j \rangle + \sum_i (-2(\text{rk} v_i)^2).

By the equality

\[(2.41) \quad 2 \sum_{i>j} \langle v_i, v_j \rangle + 2 \sum_i (-2(\text{rk} v_i)^2) = \sum_{i>j} \langle v_i, v_j \rangle + \sum_i \langle v_i^2 \rangle = \langle v^2 \rangle = 0,

we get $\dim F(v_1, v_2, \ldots, v_s) - \dim PGL(N) = 1$. Hence our claim holds. \qed

**Lemma 2.18.** Let $(A, m)$ and $(B, n)$ be Noetherian local rings and $f : A \to B$ a local homomorphism. Let $x \in m$ be a non-zero divisor of $A$ satisfying

1. $xB = p_1 \cap p_2 \cap \cdots \cap p_n$ for some prime divisors $p_1, p_2, \ldots, p_n$ of $B$, that is, $B/xB$ is reduced,
2. $B_{p_i}, 1 \leq i \leq n$ are flat over $A$.

Then $x$ is also a non-zero divisor of $B$.

**Proof.** We set

\[(2.42) \quad K := \{ a \in B \mid x^na = 0 \text{ for some positive integer } n \}.

$K$ is an ideal of $B$ and

\[(2.43) \quad \{ a \in B \mid xa \in K \} = K.

We shall prove that $K = 0$. By (ii), $K_{p_i} = 0$ for all $i$. Since $(K + xB)/xB$ is a sub $B$-module of $B/xB$ and $B/xB$ is reduced, we get that $(K + xB)/xB = 0$. By (2.43), $K = xK$. By Nakayama’s lemma, we get $K = 0$. \qed

3. Examples

In this section, we shall give some examples of $\overline{M}_H(v)$ with one singular point. Let $L := (-E_8) \oplus 2 \oplus U \oplus 3$ be the K3 lattice, where $U$ is the hyperbolic lattice.

**Lemma 3.1.** Let $N$ be an even lattice of signature $(1, s)$ which has a primitive embedding $N \hookrightarrow L$. We set $\Delta(N) := \{ C \in N \mid (C^2) = -2 \}$. Assume that there is a primitive element $H$ such that $(H^2) > 0$ and $(H, C) \neq 0$ for all $C \in \Delta(N)$. Then there is a K3 surface $X$ and an isometry $f : L \to H^2(X, \mathbb{Z})$ such that $f(N) = \text{Pic}(X)$ and $f(H)$ is ample.

**Proof.** By the surjectivity of the period map, there is a K3 surface $X$ such that $\text{Pic}(X) = N$. We set $\Delta(X)^+ := \{ C \in \text{Pic}(X) \mid C \text{ is a } (-2)\text{-curve} \}$. By the Picard-Lefschetz reflections, we can find a Hodge isometry $\phi : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ such that $\phi(f(H)), C) > 0$ for all $C \in \Delta(X)^+$. Replacing $f$ by $\phi \circ f$, we can choose $f(H)$ to be ample. \qed

**Lemma 3.2.** Let $(a_{ij})_{i,j=0}^n$ be a Cartan matrix of affine type $A_n, D_n, E_n$. Let $N_1 := (\oplus_{i=0}^n \mathbb{Z}\beta_i) \oplus \mathbb{Z} \sigma$, be a lattice such that $(\sigma^2) = 0, (\sigma, \beta_0) = 1, (\sigma, \beta_i) = 0, i > 0$ and $(\beta_i, \beta_j) = -a_{i,j}$. Assume that there is a primitive embedding $N_1 \hookrightarrow L$. Then there is a positive integer $d$ and a primitive sublattice $N := (\oplus_{i=0}^n \mathbb{Z} \xi_i, ( , ))$ of $L$ such that $(\xi_i, \xi_j) = -a_{i,j} + 2d$.

**Proof.** Since the signature of $N_1$ is $(1, n + 1)$, there is a vector $x \in N_1^1$ such that $2d := (x^2) > 0$. We set $\xi_i := \beta_i + x$. Then $\oplus_{i=0}^n \mathbb{Z} \xi_i$ is a primitive sublattice of $L$ with $(\xi_i, \xi_j) = (\beta_i, \beta_j) + (x^2) = -a_{i,j} + 2d$. \qed

**Remark 3.1.** For $A_n, D_n, n \leq 18$ or $E_n$, there is a primitive sublattice $N_1$ of $L$ (cf. [S-N]).
Let $C = (a_{i,j})_{i,j=0}^n$ be a Cartan matrix of affine type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_n$ and $Q := (\oplus_{i=0}^n \mathbb{Z}u_i, (\ ,\ ))$ the associated root lattice, that is, $(a_i, a_j) = a_{i,j}$. Then there is a vector $\delta := \sum_{i=0}^n a_i \alpha_i, a_i \in \mathbb{Z}$ such that

\[(3.1)\quad Q^\perp := \{x \in Q \mid (x, y) = 0 \text{ for all } y \in Q\} = \mathbb{Z}\delta.
\]

By the classification of the Cartan matrix of affine type, we may assume that $a_0 = 1$.

Let $N := (\oplus_{i=0}^n \mathbb{Z}u_i, (\ ,\ ))$ be a primitive sublattice of $L$ (Lemma 3.2) such that

\[(3.2)\quad (\xi, \xi) = -a_{ij} + 2ra,
\]

where $r$ and $a$ are positive integers with $d = ra$. We set

\[(3.3)\quad H := \sum_{i=0}^n a_i \xi_i,
\]

**Lemma 3.3.**

1. $(H, \xi_j) = 2ra(\sum_{i=0}^n a_i)$ for all $j$. In particular, $(H^2) = 2ra(\sum_{i=0}^n a_i)^2 > 0$.
2. $H^\perp := \{\xi \in N \mid (H, \xi) = 0\}$ is negative definite and $H^\perp \cap \Delta(N) = 0$.

**Proof.** By (3.1), $\sum_{i=0}^n a_i a_{i,j} = 0$ for all $j$. Then we see that

\[(3.4)\quad (H, \xi_j) = \sum_{i=0}^n a_i (\xi_i, \xi_j) = \sum_{i=0}^n a_i (-a_{ij} + 2ra) = 2ra(\sum_{i=0}^n a_i)
\]

for all $j$. Thus the claim (1) holds. We next show the claim (2). By (3.4), we see that

\[(3.5)\quad H^\perp = \{\sum_{i=0}^n d_i \xi_i \mid d_i \in \mathbb{Z}, \sum_{i=0}^n d_i = 0\} = \bigoplus_{i=0}^{n-1} \mathbb{Z}(\xi_i - \xi_{i+1}).
\]

We define a homomorphism

\[(3.6)\quad \varphi : H^\perp \to Q = \bigoplus_{i=0}^n \mathbb{Z}u_i
\]

by sending $\xi_i - \xi_{i+1} \in H^\perp$ to $a_i - a_{i+1} \in Q$. Obviously $\varphi$ is injective and

\[(3.7)\quad \text{im } \varphi = \bigoplus_{i=0}^{n-1} \mathbb{Z}(u_i - u_{i+1}) = \{\sum_{i=0}^n d_i u_i \mid d_i \in \mathbb{Z}, \sum_{i=0}^n d_i = 0\}.
\]

By (3.2), we see that

\[(3.8)\quad (\xi_i - \xi_{i+1}, \xi_j - \xi_{j+1}) = -(a_i - a_{i+1}, a_j - a_{j+1}).
\]

Hence $\varphi$ changes the sign of the bilinear forms. In order to prove our claim, it is sufficient to show the following assertions:

(a) $\text{im } \varphi$ is positive definite.
(b) There is no vector $x \in \text{im } \varphi$ with $(x^2) = 2$.

By (3.3), $\delta$ does not belong to $\text{im } \varphi$, which implies that $\text{im } \varphi$ is positive definite. We next prove the claim (b). Since $a_0 = 1$, we can take $\{\delta, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ as a $\mathbb{Z}$-basis of $Q$. Let $\mathfrak{g}$ be the finite simple Lie algebra whose root lattice is $\oplus_{i=1}^n \mathbb{Z}u_i$. Assume that an element $x = l\delta + \sum_{i=1}^n m_i \alpha_i, l, m_i \in \mathbb{Z}$ satisfies that $(x^2) = 2$. Then $\sum_{i=1}^n m_i \alpha_i$ becomes a root of $\mathfrak{g}$. Hence $m_i \geq 0$ for all $i$, or $m_i \leq 0$ for all $i$. Since $\theta := \sum_{i=1}^n a_i \alpha_i$ is the highest root of $\mathfrak{g}$, $\sum_{i=0}^n a_i > |\sum_{i=1}^n m_i|$, and hence we get $l(|\sum_{i=0}^n a_i| + \sum_{i=1}^n m_i) \neq 0$. Thus $x$ does not belong to $\text{im } \varphi$.

Applying Lemma 3.3 to the lattice $N$ (see (3.2)), we see that there is a polarized K3 surface $(X, H)$ such that the Picard lattice of $X$ is $N$ with $H = \sum_{i=0}^n a_i \xi_i$. We set

\[(3.9)\quad v_i := r + \xi_i + ap\xi_i, \quad 0 \leq i \leq n,
\]

Then we get that

\[(3.10)\quad \begin{cases} (v_i, v_j) = -a_{ij} \\ (v, v_j) = 0 \\ (H, v_j) = 0. \end{cases}
\]

Let $E_i$ be a $v$-twisted semi-stable sheaf with $v(E_i) = v_i$ (Proposition 1.2). If $E_i$ is properly $v$-twisted semi-stable, then $\text{rk Pic}(X) > \text{rk } N$, which is a contradiction. Hence $E_i$ is $v$-twisted stable for all $i$. Thus $M_H(v)$ has a rational double point of type $(a_{i,j})_{i,j=1}^n$. In particular, Remark 3.4 implies that there is a moduli space $M_H(v)$ which has a rational double point of type $A_n, D_n, n \leq 18$, or $E_n$. 


4. Appendix

Finally we treat the wall crossing phenomenon under a wall \( W_u \) with \( u \in \mathcal{U} \setminus \mathcal{U}' \). Assume that \( \alpha \) belongs to exactly one wall \( W_u \). Let \( F \) be a \( v + \alpha \)-twisted semi-stable sheaf with \( v(F) = u \). By our assumption, \( F \) must be \( v + \alpha \)-twisted stable. We set \( \alpha^\pm := \pm v + \alpha \). We consider the Fourier-Mukai transform \( \mathcal{F}_E \) in (4.18). Then we see that \( \text{Ext}^1(F, E) = 0 \) for \( E \in M_H^v(\alpha)(v) \): Indeed, if \( \text{Ext}^1(F, E) \neq 0 \), then we have a non-trivial extension \( 0 \to E \to G \to F \to 0 \) and we see that \( G \) is a \( v + \alpha \)-twisted stable. On the other hand, \( (v(G))^2 < -2 \), which is a contradiction. Hence \( \text{Ext}^1(F, E) = 0 \). Then, we have an isomorphism

\[
M_H^{v+\alpha}(v) \leftrightarrow M_H^{v+\alpha}(v')
\]

(4.1)

where \( v' = R_v(F)(v) \). We also have an isomorphism

\[
M_H^{v+\alpha}(v') \leftrightarrow M_H^{v+\alpha}(v)
\]

(4.2)

where \( v' = R_v(F)(v) \). We consider the Fourier-Mukai transform \( \mathcal{F}_E \). We also have an isomorphism \( M_H^{v+\alpha}(v) \to M_H^{v+\alpha}(v) \) and under this identification, we get \( \theta_v^{-} = \theta_v^{+} \).

References

[A] Abe, T., A remark on the 2-dimensional moduli spaces of vector bundles on K3 surfaces, Math. Res. Lett. 7 (2000), 463–470

[E-G] Ellingsrud, G., Göttsche, L., Variation of moduli spaces and Donaldson invariants under change of polarization, J. Reine Angew. Math. 467 (1995), 1–49

[Li] Li, J., Compactification of moduli of vector bundles over algebraic surfaces, Collection of papers on geometry, analysis and mathematical physics, World Sci. Publishing, River Edge, NJ, (1997), 98–113

[M-W] Matsuki, K., Wentworth, R., Mumford-Thaddeus principle on the moduli space of vector bundles on an algebraic surface, Internat. J. Math. 8 (1997), 97–148

[Mu1] Mukai, S., Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, Invent. math. 77 (1984), 101–116

[Mu2] Mukai, S., On the moduli space of bundles on K3 surfaces I, Vector bundles on Algebraic Varieties, Oxford, 1987, 341–413

[MFK] Mumford, D., Fogarty, J., Kirwan, F., Geometric invariant theory, Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) 34. Springer-Verlag, Berlin, 1994.

[Na] Nakajima, H., Varieties associated with quivers, Representation theory of algebras and related topics (Mexico City, 1994), 139–157, CMS Conf. Proc., 19, Amer. Math. Soc., Providence, RI, 1996.

[S-N] Shimada, I., Zhang, D.-Q., Classification of extremal elliptic K3 surfaces and fundamental groups of open K3 surfaces, Nagoya Math. J. 161 (2001), 23–54

[T] Thaddeus, M., Stable pairs, linear systems and the Verlinde formula, Invent. Math. 117 (1994), 317–353

[Y1] Yoshioka, K., Some examples of Mukai’s reflections on K3 surfaces, J. reine angew. Math. 515 (1999), 97–123

[Y2] Yoshioka, K., Moduli spaces of stable sheaves on abelian surfaces, math.AG/0009003 Math. Ann. 321 (2001), 817–884

[Y3] Yoshioka, K., Twisted stability and Fourier-Mukai transform I, Compositio Math. to appear

[Y4] Yoshioka, K., Twisted stability and Fourier-Mukai transform II, preprint

[Y5] Yoshioka, K., A note on Fourier-Mukai transform, math.AG/0112267

Department of mathematics, Faculty of Science, Kobe University, Kobe, 657, Japan

E-mail address: onishi@math.kobe-u.ac.jp, yoshioka@math.kobe-u.ac.jp