The Theory of Nearly Incompressible Magnetohydrodynamic Turbulence: Homogeneous Description

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Abstract. The theory of nearly incompressible magnetohydrodynamics (NI MHD) was developed to understand the apparent incompressibility of the solar wind and other plasma environments, particularly the relationship of density fluctuations to incompressible manifestations of turbulence in the solar wind and interstellar medium. Of interest was the identification of distinct leading-order incompressible descriptions for plasma beta $\beta \gg 1$ and $\beta \approx 1$ or $\ll 1$ environments. In the first case, the “dimensionality” of the MHD description is 3D whereas for the latter two, there is a collapse of dimensionality in that the leading-order incompressible MHD description is 2D in a plane orthogonal to the large-scale or mean magnetic field. Despite the success of NI MHD in describing fluctuations in a low-frequency plasma environment such as the solar wind, a basic turbulence description has not been developed. Here, we rewrite the NI MHD system in terms of Elsässer variables. We discuss the distinction that emerges between the three cases. However, we focus on the $\beta \sim 1$ or $\ll 1$ regimes since these are appropriate to the solar wind and solar corona. In both cases, the leading-order turbulence model describes 2D turbulence and the higher-order description corresponds to slab turbulence, which forms a minority component. The Elsässer $\beta \sim 1$ or $\ll 1$ formulation exhibits the nonlinear couplings between 2D and slab components very clearly, and shows that slab fluctuations respond in a passive scalar sense to the turbulently evolving majority 2D component fluctuations. The coupling of 2D and slab fluctuations through the $\beta \sim 1$ or $\ll 1$ NI MHD description leads to a very natural emergence of the “Goldreich-Sridhar” critical balance scaling parameter, although now
with a different interpretation. Specifically, the critical balance parameter shows that the energy flux in wave number space is a consequence of the intensity of Alfvén wave sweeping versus passive scalar convection by leading-order 2D Elsässer fluctuations, with critical balance being achieved when Alfvén wave sweeping balances passive scalar convection by leading-order 2D Elsässer fluctuations. Besides yielding predictions of 2D and slab spectra for Elsässer fluctuations, NI MHD shows that density fluctuations are advected by the majority or dominant incompressible velocity fluctuations. In the case of \( \beta \sim 1 \) or \( \ll 1 \), the density spectrum is Kolmogorov in the perpendicular wave number, thus providing a possible explanation for the observed extended Kolmogorov-like power law spectrum for electron density fluctuations in the interstellar medium.

1. Introduction

Developed in the early 1990s [1–13] for homogenous flows, and subsequently extended to large-scale inhomogeneous flows [14–17], NI MHD is a formulation of the MHD equations in a weakly compressible or nearly incompressible regime. The early work focused on the apparent incompressibility of the solar wind and other plasma environments, and the relationship of density fluctuations to incompressible manifestations of turbulence in the solar wind and interstellar medium, such as a \( k^{-5/3} \) Kolmogorov-like density spectrum [18–21]. An important prediction of NI MHD was that solar wind turbulence, for which the plasma beta is \( O(1) \) or \( \ll 1 \), is a superposition of dominant 2D and minority slab components [10; 11].

Density fluctuations observed in the supersonic solar wind have typical amplitudes that deviate no more than 10% from the mean solar wind density [8; 22; 23]. Magnetic field and velocity fluctuations are frequently well described by incompressible turbulence [23], exhibiting power-law distributions in energy, density, temperature, magnetic field with a Kolmogorov exponent \( -5/3 \) or occasionally an Iroshnikov-Kraichnan exponent \( -3/2 \). Solar wind observations suggest that compressible MHD often converges to an incompressible state, passing through a nearly incompressible regime in which weakly compressible corrections are related to the final incompressible state. The inverse square of the “turbulent Mach number,” i.e., a Mach number defined by the fluctuating velocity rather than the bulk flow velocity and mean sound speed, is an important scaling parameter in compressible MHD since it is typically very small. The turbulent Mach number therefore introduces a highly singular term into the MHD equations that leads to the generation of high-frequency (magneto)acoustic fluctuations. However, because of the three fundamental wave speeds in MHD, it is necessary to consider three plasma beta \( \beta \) regimes \( (\beta = P/(B^2/2\mu_0)) \), where \( P \) is the thermal pressure, \( B = |\mathbf{B}| \), \( B \) the magnetic field, and \( \mu_0 \) the magnetic permeability). Zank and Matthaeus [11] (hereafter ZM93) considered \( \beta \ll 1 \), \( \beta \sim 1 \), and \( \beta \gg 1 \) and showed that the description of solar wind turbulence is very different in the \( \beta \gg 1 \) and the \( \beta \sim 1 \) or \( \ll 1 \) regimes.

2. The equations of NI MHD

We consider here the NI MHD equations for a homogeneous plasma only. The inhomogeneous development was done by Hunana and Zank [17] and subsequently extended in [24; 25]. We present the equations for the \( \beta \gg 1 \) and \( \beta \sim 1 \) cases (the
The \( \beta \ll 1 \) equations are essentially the same as the \( \beta \sim 1 \) case, where the latter assumes the presence of a large constant magnetic field \( B_0 \). A related analysis has been given in [26] for the \( \beta \gg 1 \) case. The NI MHD description comprises a core set of incompressible equations given by the normalized equations (11) - (13) and (57) - (59) in ZM93 for the \( \beta \gg 1 \) and \( \beta \sim 1 \) regimes respectively. These equations are derived by requiring that all fast-scale magnetoacoustic variation to vanish. The elimination of fast-time scale variation is accomplished through the principle of bounded derivatives [27]. The bounding of the time derivatives yields the incompressible MHD equations as secular conditions that ensure the elimination of all fast-scale variation [5; 7]. The \( \beta \gg 1 \) regime yields a leading-order incompressible MHD description that is fully 3D, whereas the leading-order description for the \( \beta \ll 1, \beta \sim 1 \) regimes is reduced to two dimensions in the plane perpendicular to the mean magnetic field. Higher-order nearly incompressible fluctuations for all \( \beta \) regimes comprise 3D propagating magnetosonic and sound waves, and Alfvén waves propagating parallel to the magnetic field. When \( \beta \ll 1 \) or \( \sim 1 \), nearly incompressible transverse fluctuations propagate parallel to the magnetic field. The weakly compressible equations are given by the normalized equations (69) - (72) in ZM93. The higher-order corrections can be both compressible and incompressible. ZM93 predicted that in the \( \beta \ll 1 \) or \( \sim 1 \) regimes, solar wind turbulence is a superposition of 2D and slab turbulence, dominated by the 2D component. Numerical simulations [28] are “tantalizingly consistent” with the predictions of a superposition of dominant 2D and slab turbulence by [11].

We use the notation of ZM93 in using an “\( \infty \)” superscript to denote MHD variables that satisfy the core incompressible equations (i.e., the sound speed is “infinite”), and the superscript “*” or subscript “1” to indicate higher-order corrections. ZM93 showed that the incompressible pressure and the NI pressure correction enter at the same order in the square of the turbulent sonic or Alfvénic (depending on the plasma beta regime) Mach number, as does the fluctuating density (however see [17] for the inhomogeneous case where the density enters linearly in the turbulent sonic Mach number). The incompressible magnetic field component \( B^{\infty} \) is of order the turbulent sonic Mach number and the NI correction \( B^* \) enters at the next order. The NI velocity \( u_1 \) is also of order the turbulent sonic Mach number. Zank et al. [24] renormalize the \( \beta \sim 1 \) form of the NI MHD equations derived in ZM93. This renormalization is easily extended to the \( \beta \gg 1 \) regime. Accordingly, we can express the NI MHD equations in the \( \beta \gg 1 \) and \( \beta \sim 1 \) regimes in the following dimensional fashion,

\[
\frac{\partial u^{\infty}}{\partial t} + u^{\infty} \cdot \nabla u^{\infty} = -\frac{1}{\rho_0} \nabla \left( P^{\infty} + \frac{B^{\infty 2}}{2\mu_0} \right) + \frac{1}{\mu_0 \rho_0} B^{\infty} \cdot \nabla B^{\infty};
\]

\[
\frac{\partial B^{\infty}}{\partial t} + u^{\infty} \cdot \nabla B^{\infty} = B^{\infty} \cdot \nabla u^{\infty},
\]
\[
\frac{\partial \mathbf{u}^\infty}{\partial t} + \mathbf{u}^\infty \cdot \nabla \mathbf{u}^\infty = -\frac{1}{\rho_0} \nabla \left( \rho^\infty + \frac{1}{2\mu_0} B^\infty_2 \right) + \frac{1}{\mu_0 \rho_0} \mathbf{B}^\infty \cdot \nabla \mathbf{B}^\infty; \quad (4)
\]
\[
\frac{\partial \mathbf{B}^\infty}{\partial t} + \mathbf{u}^\infty \cdot \nabla \mathbf{B}^\infty = \mathbf{B}^\infty \cdot \nabla \mathbf{u}^\infty, \quad (5)
\]
\[
\frac{\partial \rho^*}{\partial t} + \mathbf{u}^\infty \cdot \nabla \rho^* + \rho_0 \nabla \cdot \mathbf{u}_1 = 0; \quad (6)
\]
\[
\frac{\partial \mathbf{u}_1}{\partial t} + \mathbf{u}^\infty \cdot \nabla \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{u}^\infty = -\frac{1}{\rho_0} \nabla P^* - \frac{1}{\mu_0 \rho_0} \nabla \left( \mathbf{B}_0 \cdot \mathbf{B}^* + \mathbf{B}^\infty \cdot \mathbf{B}^* \right)
\]
\[
+ \frac{1}{\mu_0 \rho_0} \mathbf{B}_0 \cdot \nabla \mathbf{B}^* + \frac{1}{\mu_0 \rho_0} \mathbf{B}^* \cdot \nabla \mathbf{B}^\infty; \quad (7)
\]
\[
\frac{\partial P^*}{\partial t} + \mathbf{u}^\infty \cdot \nabla P^* + \gamma \rho_0 \nabla \cdot \mathbf{u}_1 = -\frac{\partial P^\infty}{\partial t} - \mathbf{u}^\infty \cdot \nabla P^\infty; \quad (8)
\]
\[
\frac{\partial \mathbf{B}^*}{\partial t} + \mathbf{u}^\infty \cdot \nabla \mathbf{B}^* = \mathbf{B}^\infty \cdot \nabla \mathbf{u}_1 - \left( \mathbf{B}_0 + \mathbf{B}^\infty \right) \cdot \nabla \cdot \mathbf{u}_1
\]
\[- \mathbf{u}_1 \cdot \nabla \mathbf{B}^\infty. \quad (9)
\]

However, the variables \( \mathbf{u}^\infty, \mathbf{B}^\infty \), etc. are functions of \((x, y, z)\) when \( \beta \gg 1 \), and functions of \((x, y)\), \(B_0 \hat{z}\), when \( \beta \sim 1 \). Notice that equations (7) - (10) are not linearized but instead describe the driving of the higher-order corrections by the dynamics of the leading-order incompressible flow variables. The incompressible flow \( \mathbf{u}^\infty \), in particular, introduces an important passive scalar transport component to the dynamics of the higher-order plasma variables. The variables \( \rho^* \) and \( \mathbf{u}_1 \) possess both an incompressible and a compressible component. Furthermore, as shown by equation (9) for example, the incompressible flow acts as a source of higher-order fluctuations, including compressible fluctuations, and therefore represents a generalization of the Lighthill mechanism for the generation of sound. Consequently, incompressible fluctuations can be a possible source of density fluctuations. Since the NI corrections enter at \( O(M^2) \), \( M \) being either the turbulent sonic or Alfvénic Mach number, turbulent fluctuations in a \( \beta \sim 1 \) or \( \ll 1 \) plasma are primarily 2D rather than slab. In the solar wind, the observed turbulent Mach number ordering implies a crude partitioning of energy between 2D and slab components.
in the ratio 80 : 20 [10], with some observational confirmation [29]. The partitioning of fluctuation energy into 2D and slab also implies a strong variance anisotropy in the solar wind fluctuations since the parallel variance is greatly reduced compared to the perpendicular variance, consistent with that observed (e.g., [30; 31]).

We may introduce Elsässer variables for the leading-order or majority incompressible and minority NI components through

\[ z^\infty_{\pm} = u^\infty_{\pm} \pm \frac{b^\infty}{\sqrt{\mu_0\rho_0}} \quad \text{and} \quad z^*_{\pm} = u_1^* \pm \frac{B^*}{\sqrt{\mu_0\rho_0}}. \quad (11) \]

In so doing, we may rewrite the coupled incompressible core equations (1) - (3) \((\beta \gg 1)\) as

\[ \frac{\partial z^\infty_{\pm}}{\partial t} + z^\infty_{\mp} \cdot \nabla z^\infty_{\pm} = -\frac{1}{\rho_0} \nabla \left( P^\infty + \frac{1}{2\mu_0} B^\infty_0^2 \right), \quad (12) \]

and (4) - (6) \((\beta \sim 1)\) as

\[ \frac{\partial z^\infty_{\pm}}{\partial t} + z^\infty_{\mp} \cdot \nabla \bot z^\infty_{\pm} = -\frac{1}{\rho_0} \nabla \bot \left( P^\infty + \frac{1}{2\mu_0} B^\infty_0^2 \right), \quad (13) \]

where, as a reminder, \(\nabla \bot\) refers to a coordinate system orthogonal to the mean magnetic field that is oriented along the \(\hat{z}\)-axis.

For the \(\beta \sim 1\) or \(\ll 1\) regimes, we assume explicitly a strong mean magnetic field \(B_0\), which introduces a small turbulent Alfvénic Mach number \(M_{A0}\). For \(\beta \sim 1\), \(M_{A0}\) and the corresponding sonic Mach number based on the sound speed are of the same order. For the \(\beta \gg 1\) regime, although not explicitly present, we do not rule out the presence of a large-scale mean magnetic field \(B_0\), assuming only that the mean Alfvén velocity is of the same order as the fluctuating plasma velocity and the fluctuating magnetic field Alfvén velocity i.e., if we assume there exists \(B_0\) and express \(B^\infty = B_0 + b^\infty\), then we may rewrite (11) as

\[ z^\infty_{\pm} = u^\infty_{\pm} \pm \frac{b^\infty}{\sqrt{\mu_0\rho_0}} \pm \frac{B_0}{\sqrt{\mu_0\rho_0}} = u^\infty_{\pm} \pm V_{A0} = z_{f\pm}^\infty \pm V_{A0}. \quad (14) \]

Subsequently, the \(\beta \gg 1\) equations (12) assume, for the homogeneous case, the familiar form

\[ \frac{\partial z_{f\pm}^\infty}{\partial t} + V_{A0} \cdot \nabla z_{f\pm}^\infty + z_{f\mp}^\infty \cdot \nabla z_{f\pm}^\infty = -\frac{1}{\rho_0} \nabla \left( P^\infty + \frac{1}{2\mu_0} B^\infty_0^2 \right). \quad (15) \]

Equation (15) is the “standard” form of the incompressible MHD equations used typically to study turbulence in a variety of settings, including the solar wind and interstellar medium. However, we reemphasize that (15) is appropriate only under the condition \(\beta \gg 1\).

In the presence of a large mean magnetic field (\(\beta \sim 1\)), the dominant fluctuating component is 2D and propagation effects are completely absent at leading-order, as illustrated in (13). Equation (13) indicates that all core modes have zero frequency and the interactions are purely “nonlinear” i.e., there is no mediation by Alfvénic wave packets [32; 33].
On rewriting equations (7) - (10), we obtain

\[
\frac{\partial z^{\pm}}{\partial t} + V_{A0} \cdot \nabla z^{\pm} + z^{\infty \pm} \cdot \nabla z^{\pm} + z^{\pm} \cdot \nabla z^{\infty \pm} = -\frac{1}{\rho_0} \nabla \left( P^* + \frac{1}{\mu_0} B_0 \cdot B^* + \frac{1}{\mu_0} B^\infty \cdot B^* \right) \mp \left( V_{A0} + \frac{B^\infty}{\sqrt{\mu_0 \rho_0}} \right) \nabla \cdot u_1. \tag{16}
\]

Note that the last term on the right-hand-side of (16) vanishes if we restrict our attention to incompressible (transverse) NI modes.

Evidently, the NI Els"esser modes propagate at the Alfvén speed \( \pm V_{A0} \) along the mean magnetic field. A very important difference between the \( \beta \gg 1 \) and \( \beta \sim 1 \) or \( \ll 1 \) NI MHD descriptions is that the dominant nonlinear interaction for the \( \beta \sim 1 \) or \( \ll 1 \) minority slab (see below) fluctuations is due to the coupling of minority Alfvénic fluctuations \( z^{\pm} \) to the dominant convected 2D fluctuating component \( z^{\infty \pm} \), i.e., the \( \beta \sim 1 \) or \( \ll 1 \) NI Els"esser variables respond to the core \( z^{\infty \pm} \) fluctuations in a passive sense. This is quite unlike the nature of nonlinear interactions in the \( \beta \gg 1 \) regime since equations (15) show that both nonlinear \( z^{\infty \pm} \cdot \nabla z^{\infty \pm} \) and propagation \( \mp V_{A0} \cdot \nabla z^{\infty \pm} \) effects are present at the leading order. For the \( \beta \sim 1 \) or \( \ll 1 \) minority slab component, nonlinear couplings \( z^{\pm} \cdot \nabla z^{\pm} \) and \( z^{\pm} \cdot \nabla z^{\pm} \) enter only at the next higher order. Thus, the nonlinear cascade of energy in the low-frequency inertial range for slab fluctuations in a \( \beta \sim 1 \) or \( \ll 1 \) plasma is governed primarily by the dynamical time scale of the dominant advected 2D component. We show below that this conclusion is analogous in a sense to the conclusions of Shebalin et al. [34] who argued that the nonlinear cascade proceeded via the three-wave coupling of counter-propagating Alfvén waves and a zero frequency mode. However, the Shebalin et al. analysis [34] is appropriate only to the \( \beta \gg 1 \) regime, is based on a weak turbulence model, and assumed that the mean magnetic field lay within the 2D plane. The clear separation of the 2D and slab components described by equation (16) shows that the 2D core nonlinear time scale rather than the Alfvénic time scale dominates for a \( \beta \sim 1 \) or \( \ll 1 \) plasma, and therefore determines the basic spectral characteristics of the minority slab component. Since the spectral time scale is associated with the dominant 2D fluctuating component, it is not a resonant wave coupling that describes how \( z^{\pm} \) behaves primarily; instead, \( z^{\pm} \) responds as a passive scalar to the \( z^{\infty \pm} \) fluctuating field.

In the analysis below, we do not neglect the nonlinear terms that enter (16) at the next order. The higher-order nonlinear terms \( z^{\pm} \cdot \nabla z^{\pm} \) enter the NI expansion (7) - (10) with the inclusion of terms such as \( u_1 \cdot \nabla u_1, B^* \cdot \nabla B^*, u_1 \cdot \nabla B^*, \) etc. A straightforward extension of ZM93 in which such terms are retained shows that these terms contribute higher-order but equivalent terms as those already contained in (16) and the only structurally new terms introduced are the nonlinear terms \( z^{\pm} \cdot \nabla z^{\pm} \). The “richest” NI evolution equation should therefore include the nonlinear terms in (16) despite it being formally of higher order. Thus, the nonlinear NI transport equation is given by

\[
\frac{\partial z^{\pm}}{\partial t} + V_{A0} \cdot \nabla z^{\pm} + z^{\infty \pm} \cdot \nabla z^{\pm} + z^{\pm} \cdot \nabla z^{\infty \pm} + z^{\pm} \cdot \nabla z^{\infty \pm} = -\frac{1}{\rho_0} \nabla \left( P^* + \frac{1}{\mu_0} B_0 \cdot B^* + \frac{1}{\mu_0} B^\infty \cdot B^* \right) \mp \left( V_{A0} + \frac{B^\infty}{\sqrt{\mu_0 \rho_0}} \right) \nabla \cdot u_1. \tag{17}
\]
3. Properties of the $\beta \sim 1$ 2D incompressible MHD equations

We now restrict our attention to the $\beta \sim 1$ regime that is relevant to most of the solar wind. Following Zank et al. [24], we analyze equations (13) using a von Karman-Howarth approach [26; 35–37]. The nonlinear time scales are given by

$$\tau^{\pm}_\infty^{-1} \equiv \frac{\langle z^{\infty_\pm 2} \rangle^{1/2}}{\lambda^{\pm}_\perp},$$

(18)

[38–42], allowing us to approximate the quadratic nonlinearity in (13) as

$$\text{NL}^{\infty_\pm} \equiv z^{\infty_\pm} \frac{\langle z^{\infty_\pm 2} \rangle^{1/2}}{\lambda^{\pm}_\perp}.$$  

(19)

The length scale $\lambda^{\pm}_\perp$ is the correlation length in the 2D "forward" and "backward" modes. The total energy in fluctuations is

$$\langle z^{\infty 2} \rangle \equiv \frac{\langle z^{\infty + 2} \rangle + \langle z^{\infty - 2} \rangle}{2},$$

(20)

where $\langle \bullet \rangle$ is an ensemble-averaging operator. For 2D turbulence we assume that

$$\frac{\langle z^{\infty - 2} \rangle^{1/2}}{\lambda^{+}_\perp} = \frac{\langle z^{\infty + 2} \rangle^{1/2}}{\lambda^{-}_\perp} \equiv \frac{\langle z^{2} \rangle^{1/2}}{2\lambda^{\infty}_\perp},$$

(21)

which defines $\lambda^{\infty}_\perp$. The ensemble-averaged 1-point energy containing transport equation for the total energy is then

$$\frac{d}{dt} \langle z^{\infty 2} \rangle = -\frac{\langle z^{\infty 2} \rangle^{3/2}}{\lambda^{\infty}_\perp}. $$

(22)

We assume that the steady energy transfer flux $\Pi(k)$ ($k$ is the wave number vector) is proportional to the time scale for the decay of the transfer function correlations $\tau_3$ (also known as the triple correlation time scale) and that the energy-containing eddies determine the rate of spectral energy transfer in decaying turbulence. We can then adopt the phenomenological steady energy transfer rate model

$$\epsilon = \Pi(k) = \tau_3 \frac{\langle z^{2} \rangle}{\tau^{nl}},$$

(23)

where $\epsilon$ is the dissipation rate [43]. The nonlinear dynamical time scale is denoted by $\tau^{nl}$ and, from (22) can be expressed as $\lambda^{\infty}_\perp / \langle z^{\infty 2} \rangle^{1/2}$. By assuming isotropy in the 2D plane orthogonal to the mean $\mathbf{B}$ and writing $\langle z^{\infty 2} \rangle = \int E^{\infty} dk_\perp$, it follows from (23) that

$$E^{\infty} = \epsilon^{2/3}_\infty k^{-5/3}_\perp,$$  

(24)
where \( k_\perp = |k_\perp| = |(k_x, k_y)| \) is the length of the wave number vector in the 2D plane orthogonal to \( B_0\hat{z} \), and \( \epsilon_\infty \) is the dissipation rate for 2D incompressible MHD turbulence. Thus, the energy spectrum of the dominant component in the \( \beta \sim 1 \) or \( \ll 1 \) NI MHD theory, when expressed in Elsässer variables, is of a Kolmogorov form in the 2D perpendicular wave number \( k_\perp \), i.e., \( \propto k_\perp^{-5/3} \).

The 2D isotropic spectral result can be generalized by using \( \epsilon^\pm = \Pi^\pm(k) = \frac{\tau_3^\pm}{\tau_{nl}^2} \left( \frac{\epsilon_\infty^\pm}{\epsilon_\infty^0} \right)^{2} \), where \( \epsilon^\pm \) refers to the dissipation rates for forward and backward fluctuating Elsässer variables [39; 44]. Use of the nonlinear time scales (18) for \( \tau_3^\pm \) then yields the Kolmogorov spectra for the intensities \( \left< \epsilon^\pm \right> = \int E_{k_\perp}^{\epsilon^\pm} dk_\perp \) as

\[
E_\infty^\pm = \left( \frac{\epsilon_\infty^\pm}{\epsilon_\infty^0} \right)^{4/3} k_\perp^{-5/3} \] with \( E_\infty^+/E_\infty^- = \left( \frac{\epsilon_\infty^+}{\epsilon_\infty^-} \right)^2 \).

The core 2D MHD equations (4) - (6) admit a class of exact nonlinear solutions that are the analogue of hydrodynamic vortices [45–47].

On introducing the vector potential \( \mathbf{A}^\infty \) by \( \mathbf{B}^\infty \equiv \nabla \times \mathbf{A}^\infty = \nabla \times A^\infty \hat{z} \), the induction equation (6) shows that \( \mathbf{A}^\infty \) is a passive scalar in the 2D flow, satisfying

\[
\frac{\partial \mathbf{A}^\infty}{\partial t} + \mathbf{u}^\infty \cdot \nabla \mathbf{A}^\infty = 0. \tag{25}
\]

The flow vorticity is expressed as \( \xi^\infty \equiv \nabla \times \mathbf{u}^\infty \) and a flux function \( \Psi^\infty \) can be introduced through \( \mathbf{u}^\infty \equiv \hat{z} \times \nabla \Psi^\infty \). It then follows that

\[
\frac{\partial \mathbf{A}^\infty}{\partial t} + \{\Psi^\infty, \mathbf{A}^\infty\} = 0, \tag{26}
\]

where \( \{a,b\} \equiv (\partial a/\partial x)(\partial b/\partial y) - (\partial a/\partial y)(\partial b/\partial x) \) denotes a Poisson bracket. Since the current \( J^\infty \) is defined by \( \mu_0 \mathbf{J}^\infty = \nabla \times \mathbf{B}^\infty \), for 2D flows we find that \( \xi^\infty \parallel \mathbf{J}^\infty \) and

\[
\frac{\partial \xi^z}{\partial t} + \mathbf{u}^\infty \cdot \nabla_\perp \xi^z = \frac{1}{\rho} \mathbf{B}^\infty \cdot \nabla \mathbf{J}^\infty. \tag{27}
\]

Since \( \xi^z = \nabla^2 \Psi^\infty \) and \( \mu_0 J^z_\infty = -\nabla_\perp^2 A^\infty \), we can rewrite (27) as

\[
\frac{\partial \xi^z}{\partial t} + \{\Psi^\infty, \xi^z\} = \frac{1}{\rho} \{A^\infty, J^z_\infty\}; \tag{28}
\]

\[
\frac{\partial}{\partial t} \left( \nabla_\perp^2 \Psi^\infty \right) + \{\Psi^\infty, \nabla_\perp^2 \Psi^\infty\} = \frac{1}{\mu_0 \rho} \{A^\infty, \nabla_\perp^2 \}.
\]

For non-propagating structures, equations (26) and (28) become

\[
\{\Psi^\infty, A^\infty\} = 0;
\]

\[
\{\Psi^\infty, \nabla_\perp^2 \Psi^\infty\} = \frac{1}{\mu_0 \rho} \{A^\infty, \nabla_\perp^2 \}, \tag{29}
\]

the first condition of which ensures that \( A^\infty = f(\Psi^\infty) \) for an arbitrary function \( f \). The second condition of (29) implies that

\[
\nabla_\perp^2 \Psi^\infty = -\frac{1}{\rho} f' J^z_\infty + f(\Psi^\infty), \tag{30}
\]
where $f_1$ is another arbitrary function. The simplest spherically symmetric solution of (30) is $A^\infty = \Psi^\infty$ (which implies $f' = 1$). For $J_z^\infty$ linear in $A^\infty$, say $\mu_0 J_z^\infty = k^2 A^\infty$ for some constant $k$, (30), written in polar coordinates, reduces to Bessel’s equation of order 0,

$$\frac{d^2 A^\infty}{dx^2} + \frac{1}{x} \frac{d A^\infty}{dx} + A^\infty = 0,$$

(31)

after setting $x = \alpha r$, where $\alpha = k/\sqrt{\mu_0 \rho}$. Solutions are therefore $A^\infty(r) = CJ_0(k/\sqrt{\mu_0 \rho} r)$, where $J_0$ is Bessel’s function of order 0. The parameter $k$ is chosen to ensure that $A(r_0) = 0$. This then yields the vortical solution

$$A^\infty(r) = \begin{cases} 
CJ_0 \left( \frac{k}{\sqrt{\mu_0 \rho}} r \right) & r < r_0 \\
0 & r \geq r_0
\end{cases},$$

(32)

Here $r$ is the polar coordinate distance with origin at the center of the vortex structure, $r_0$ is the edge of the vortex, and $C$ and $k$ are constants. The vorticity $\xi^\infty \equiv \nabla \times u^\infty$ and current $\mu_0 J^\infty \equiv \nabla \times B^\infty$ are aligned and parallel to the large-scale magnetic field $B_0$. Since the contours of $A^\infty$ determined by (32) correspond to closed magnetic field lines, the vortex solutions (32) are a subclass of magnetic islands in the 2D plane perpendicular to the mean field $B_0$.

The existence of 2D magnetic islands or vortex structures is an explicit prediction of $\beta \sim 1$ or $\ll 1$ NI MHD. Observations of magnetic islands on a variety of scales in the supersonic solar wind have been reported, ranging from hourly time scales [48–50] to minute time scales [46] to ion kinetic scales [51]. In the last case, the plasma beta of the solar wind environment in which vortex/magnetic island structures were embedded was $\beta < 1 \sim 0.5 - 0.7$.

4. Properties of the NI transport equation for $z^*$
Consider now the minority fluctuating component described by the NI corrections (7) - (10). It is illuminating to consider a “linearized” form of equations (7) - (10) in which the nonlinear coupling terms between the core and nearly incompressible corrections, such as $u^\infty \cdot \nabla u_1$, etc., are neglected. We neglect also the incompressible source terms. Equations (7) - (10) then reduce to a linear system in the NI variables

$$\frac{\partial \rho^*}{\partial t} + \rho_0 \nabla \cdot u_1 = 0;$$

(33)

$$\frac{\partial u_1}{\partial t} = -\frac{1}{\rho_0} \nabla P^* - \frac{1}{\mu_0 \rho_0} (\nabla \times B^*) \times B_0;$$

(34)

$$\frac{\partial B^*}{\partial t} = B_0 \nabla \cdot u_1 - B_0 \nabla \cdot u_1 = \nabla \times (u_1 \times B_0);$$

(35)

$$\frac{\partial P^*}{\partial t} + \gamma p_0 \nabla \cdot u_1 = 0.$$

(36)

Strictly speaking, equations (33) - (36) do not simply represent a linearization of the compressible MHD equations since terms of similar order such as $P^\infty$ and $B^\infty$ are
excluded. On assuming that $B_0 = B_0 \hat{z}$, we obtain the wave equation [52]

$$\frac{\partial^2 \Delta}{\partial t^2} = C_{s0}^2 \nabla^2 \Delta + V_{A0}^2 \nabla^2 (\Delta - \Gamma),$$

(37)

where $\Delta \equiv \nabla \cdot \mathbf{u}_1$ and $\Gamma \equiv \partial u_{1z} / \partial z$. The $z$-component of the NI vorticity $\nabla \times \mathbf{u}_1$, $\xi \equiv -\partial u_{1x} / \partial y + \partial u_{1y} / \partial x$, satisfies

$$\frac{\partial^2 \xi}{\partial t^2} = V_{A0}^2 \frac{\partial^2 \xi}{\partial z^2},$$

(38)

showing that $\xi$ propagates along the mean magnetic field at the Alfvén speed $V_{A0}$. By contrast, the gradient in $\mathbf{u}_1$ along the magnetic field direction $\Gamma$ is coupled to the compressive component of the NI velocity $\mathbf{u}_1$, propagating at the sound speed according to

$$\frac{\partial^2 \Gamma}{\partial t^2} = C_{s0}^2 \frac{\partial^2 \Delta}{\partial z^2}. $$

(39)

Equations (37) - (39) completely determine $\mathbf{u}_1$. However, it is more useful to introduce the $x$- and $y$-components of the vorticity

$$\eta \equiv \frac{\partial u_{1z}}{\partial y} - \frac{\partial u_{1y}}{\partial z}, \quad \zeta \equiv \frac{\partial u_{1x}}{\partial z} - \frac{\partial u_{1z}}{\partial x},$$

since

$$\frac{\partial^2 \eta}{\partial t^2} = V_{A0}^2 \left( \frac{\partial^2 \eta}{\partial z^2} - \frac{\partial^2 \Delta}{\partial y \partial z} \right), \quad \frac{\partial^2 \zeta}{\partial t^2} = V_{A0}^2 \left( \frac{\partial^2 \zeta}{\partial z^2} - \frac{\partial^2 \Delta}{\partial x \partial z} \right).$$

(40)

The $x$- and $y$-vortical components are compressive and propagate obliquely to the mean magnetic field. The NI fluctuations are therefore a superposition of incompressible and compressible modes. If we restrict our attention to incompressible NI fluctuations satisfying $\Delta = 0$ i.e., $\nabla \cdot \mathbf{u}_1 = 0$, then equations (40) are identical to (38),

$$\frac{\partial^2 \eta}{\partial t^2} = V_{A0}^2 \frac{\partial^2 \eta}{\partial z^2}, \quad \frac{\partial^2 \zeta}{\partial t^2} = V_{A0}^2 \frac{\partial^2 \zeta}{\partial z^2}. $$

(41)

Incompressible NI modes therefore all propagate at the Alfvén speed in the parallel direction. The minority incompressible fluctuating component is therefore an admixture of counter-propagating Alfvén modes i.e., slab turbulence, that are coupled nonlinearly to the zero-frequency, non-propagating 2D fluctuations described by the core equations (4) - (6).

Consider now the higher-order nonlinearity in (17) when the passive scalar terms are neglected. Since $\mathbf{z}^{\pm} \cdot \nabla \mathbf{z}^{\pm}$ enters formally at the next order in the NI expansion, we may express $\mathbf{z}^{\pm} = \mathbf{z}_1^{\pm} + \mathbf{z}_2^{\pm}$, where $\mathbf{z}_2^{\pm}$ is of higher order such that $O(\mathbf{z}_2^{\pm}) = O(\mathbf{z}_1^{\mp} \cdot \nabla \mathbf{z}_1^{\pm})$. Then, neglecting the passive scalar terms and the RHS of (17), we have

$$\frac{\partial \mathbf{z}_1^{\pm}}{\partial t} + V_{A0} \cdot \nabla \mathbf{z}_1^{\pm} = 0;$$

(42)

$$\frac{\partial \mathbf{z}_2^{\pm}}{\partial t} + V_{A0} \cdot \nabla \mathbf{z}_2^{\pm} = -\mathbf{z}_1^{\mp} \cdot \nabla \mathbf{z}_1^{\pm}. $$

(43)
Equation (42) is a linear wave equation with solutions \( z_1^{*\pm} \propto \exp[i(k \cdot x - i\omega^\pm t)] \) provided the frequency \( \omega \) and wave number \( k \) are related via the dispersion relation \( \omega^\pm = \mp V_{A0} \cdot k \).

Equation (43) is a wave equation too with a source term. On expressing

\[
\begin{align*}
\sum_k A_k^{\pm} e^{i k \cdot (x \pm V_{A0} t)}
\end{align*}
\]

and substituting in the RHS of (43), secular terms are absent provided the resonance conditions

\[
k'' = k' + k''; \quad k'' \cdot V_{A0} = -k' \cdot V_{A0} + k'' \cdot V_{A0},
\]

(44)

hold. Condition (44) is identical to the resonance condition found by \cite{34} (see also \cite{53}) from their perturbation analysis of the incompressible 3D MHD (\( \beta \gg 1 \)) equations.

The distinction here from \cite{34} is that the resonance condition holds only for the NI Elsässer variables i.e., the slab component, and the perturbation ordering is a natural consequence of the NI expansion. Equation (44) implies of course that

\[
k' \cdot V_{A0} = 0 \quad \iff \quad k' \perp B_0,
\]

(45)

meaning that the nonlinear cascade of energy for slab turbulence proceeds via the coupling of forward and backward propagating Alfvénic fluctuations with 2D zero frequency modes. This is consistent with the standard picture of the cascade process in MHD turbulence \cite{34, 54}. At the NI level, the nonlinear cascade of energy therefore drives 2D modes, which are then transferred to the dissipation range and heat the plasma. Consequently, the NI MHD 2D core equations acquire a source term corresponding to the nonlinear loss term in the slab NI description, thereby coupling the leading-order 2D and higher-order NI correction descriptions.

Slab turbulence spectra can be derived from (17). The forward and backward intensities \( \langle z^{*\pm 2} \rangle \) satisfy the 1-point transport equations

\[
\frac{\partial}{\partial t} \langle z^{*\pm 2} \rangle + V_{A0} \cdot \nabla \langle z^{*\pm 2} \rangle \simeq -2 \frac{\langle z^{*\pm 2} \rangle \langle z^{\infty \pm 2} \rangle^{1/2}}{\lambda^{\pm}_\perp} - 2 \frac{\langle z^{*\pm 2} \rangle^{1/2} \langle z^{\pm 2} \rangle^{1/2}}{\lambda^{\pm}_s},
\]

(46)

after invoking orthogonality between the NI and core Elsässer variables i.e., \( \langle z^{*\pm} \cdot z^{\infty \pm} \rangle \simeq 0 \). Consider the total energy

\[
\langle z^2 \rangle = \frac{\langle z^{*+ 2} \rangle + \langle z^{*- 2} \rangle}{2}.
\]

On exploiting the Kolmogorov phenomenology, we have instead of (23),

\[
\epsilon_s = \Pi(k) = \frac{\langle z^2 \rangle}{\tau_{nl}}.
\]

(47)

We now need to include the passive scalar coupling of the 2D core variable \( \langle z^{\infty 2} \rangle \) to \( \langle z^2 \rangle \). The nonlinear coupling terms enter at the next higher order, as expressed in equation (17), and this corresponds to \( \tau_{nl} \) in (47).
We should not assume that $\langle z^2 \rangle$ is isotropic\(^1\), so we define
\[
\langle z^2 \rangle = \frac{1}{2\pi} \int E^\ast (k) d^3 k = \frac{1}{2\pi} \int \int \int E^\ast (k_\perp, k_\parallel) dk_\perp dk_\parallel
\]
\[
= \int \int E^\ast (k_\perp, k_\parallel) k_\perp d k_\perp d k_\parallel \sim E^\ast (k_\perp, k_\parallel) k_\perp^2 k_\parallel,
\]
on assuming isotropy in the 2D plane. Consider three cases. (i) If we assume $\tau^{-1}_3 = \tau_n^{-1} = \langle z^2 \rangle^{1/2} k_\perp$, then, from (47), we obtain
\[
E^\ast (k_\perp, k_\parallel) k_\perp^2 = e_3^2 k_\perp^{-2/3} k_\parallel^{-1},
\]
which is a slight generalization of the Kolmogorov spectrum. (ii) If instead we assume that $\tau^{-1}_3 = \tau^{-1}_A = V_A k_\parallel$ and $\tau_n^{-1} = \langle z^2 \rangle^{1/2} k_\perp$, we recover the extended Iroshnikov-Kraichnan (IK) spectrum
\[
E^\ast (k_\perp, k_\parallel) k_\perp^2 = (e_3 V_A)^{1/2} k_\perp^{-1} k_\parallel^{-1/2}.
\]
(iii) We do however need to incorporate the dominant passive scalar interaction of $\langle z^2 \rangle$ with $\langle z^\infty \rangle$. As before,
\[
\langle z^\infty \rangle = \frac{1}{2\pi} \int \tilde{E}^\ast (k_\perp) dk_\perp = \int \tilde{E}^\ast (k_\perp) k_\perp d k_\perp \sim \tilde{E}^\ast (k_\perp) k_\perp^2 \equiv E^\ast (k_\perp),
\]
on assuming isotropy of $\langle z^\infty \rangle$ in the 2D plane and relating $\tilde{E}^\ast (k_\perp)$ to $E^\ast (k_\perp)$ given by equation (24). Following the suggestion of [43; 55; 56], we introduce the triple correlation time as the sum of the relevant inverse time scales, of which the dominant ones are the passive scalar term $\tau_3^{-1} \equiv \langle z^\infty \rangle^{1/2} k_\perp = e_3^{1/3} k_\perp^{-5/6} k_\perp^{3/2}$ and the Alfvén term $\tau_A^{-1} = V_A k_\parallel$, i.e.,
\[
\tau_3^{-1} = V_A k_\parallel + e_3^{1/3} k_\perp^{2/3} = V_A k_\parallel \left(1 + \left(\frac{e_3}{V_A}\right)^{1/3} k_\perp^{2/3} k_\parallel^{-1}\right).
\]
In (52), the term
\[
\left(\frac{e_3}{V_A}\right)^{1/3} k_\perp^{2/3} k_\parallel^{-1} \equiv k_A^{-1} k_\perp^{2/3} k_\parallel^{-1},
\]
is the critical balance parameter identified by [57], although here with a quite different physical interpretation. In our case ($\beta \sim 1$ or $\ll 1$), if the term (53) is of order 1 ("critical balance"), then the energy flux in wave number space is a consequence of a
\(^1\) To put the analysis here into perspective, suppose that $\langle z^2 \rangle$ is isotropic. In this case, $\langle z^2 \rangle = (4\pi)^{-1} \int E^\ast (k) d^3 k = (4\pi)^{-1} \int \int E^\ast (k) k^2 \sin \theta d \theta d \phi = \int E^\ast (k) k^2 dk \equiv \int E^\ast (k) dk \sim E^\ast (k)$, where $E^\ast (k)$ is the omni-directional or reduced spectrum. We utilize equation (47). (i) Suppose $\tau_3^{-1} = \tau_n^{-1} = \langle z^2 \rangle^{1/2} k$, which implies $E^\ast (k) = e_3^{2/3} k^{-2/3}$ or $E^\ast (k) = e_3^{3/2} k^{-5/3}$, which is the Kolmogorov spectrum. (ii) If $\tau_3^{-1} = \tau_A^{-1} = V_A k$, then $E^\ast (k) = (e_3 V_A)^{1/2} k^{-7/2}$ or $E^\ast (k) = (e_3 V_A)^{3/2} k^{3/2}$, i.e., the Iroshnikov-Kraichnan spectrum. However, there is no good reason to suppose that NI turbulence should be isotropic.
balance of Alfvén wave sweeping and passive scalar convection by leading-order 2D $z^\infty$ fluctuations.

The phenomenological steady-state energy transfer rate equation (47), yields the NI or slab spectrum as

$$E^*(k_\parallel, k_\perp)k_\perp^2 = (\epsilon_A V_A)^{1/2} k_\perp^{-1} k_\parallel^{-1/2} \left( 1 + \left( \frac{\epsilon_{\infty}}{V_A^3} \right)^{1/3} k_\perp^{2/3} k_\parallel^{-1} \right)^{1/2}.$$  (54)

If $\epsilon_{\infty} V_A^{-1} k_\perp^{-1} k_\parallel^{-1} \sim$ const. or $\ll 1$, we recover the IK spectrum (50) from (54). Conversely, if $\epsilon_{\infty} V_A^{-1} k_\perp^{-1} k_\parallel^{-1} \gg 1$, then $E^*(k_\perp, k_\parallel)k_\parallel^2 = \epsilon_{\infty}^{1/2} \epsilon_{\infty}^{1/6} k_\perp^{-2/3} k_\parallel^{-1}$, and we obtain the Kolmogorov spectrum from (54).

Since NI MHD is a superposition of the core 2D MHD equations and the NI corrections, the total turbulent energy spectrum may be expressed as $\langle z^2 \rangle \equiv \langle z^\times2 \rangle + \langle z^{\times2} \rangle$. On using $\langle z^\times2 \rangle = \int E^\times(k_\parallel)\delta(k_\parallel)d k_\perp d k_\parallel$, the total spectrum (core plus NI) is given by

$$E(k_\perp, k_\parallel)k_\parallel^2 = C_\infty^2 \epsilon_{\infty}^{2/3} k_\perp^{-2/3} k_\parallel^{-1} + C_\star (\epsilon_A V_A)^{1/2} k_\perp^{-1} k_\parallel^{-1/2} \left( 1 + \left( \frac{\epsilon_{\infty}}{V_A^3} \right)^{1/3} k_\perp^{2/3} k_\parallel^{-1} \right)^{1/2},$$  (55)

where $C_\infty$ and $C_\star$ are constants reflecting the assumed ratio of 2D to slab turbulence. The majority 2D component always contributes a Kolmogorov power law spectrum $k_\perp^{-5/3}$, whereas the minority component contributes a “broken power law” distribution along both the $k_\parallel$ or $k_\perp$ directions. The break in the spectrum is determined by the parameter $k_A$ and the values of $k_\parallel$ or $k_\perp$ relative to it. The total spectrum, being the sum of the dominant 2D and minority slab components, is therefore not a power law necessarily but can possess a somewhat complex “broken power law” structure.

5. The fluctuating density variance determined from NI MHD

NI MHD allows us describe the basic physics and dynamics of density fluctuations in a turbulent flow, including the determination of the fluctuating density variance spectrum. For incompressible NI fluctuations satisfying $\nabla \cdot \mathbf{u}_1 = 0$, the homogeneous density transport equation (7) describes the passive scalar advection of density fluctuations,

$$\frac{\partial \rho^*}{\partial t} + \mathbf{u}^\infty \cdot \nabla \rho^* = 0.$$  (56)

The density fluctuations are purely entropic and the passive scalar behavior is due to convection by the dominant core incompressible velocity fluctuations for all plasma beta regimes. This is in marked contrast to the conjecture by Lithwick and Goldreich [58] that entropic density modes are “passively mixed by the cascade of shear Alfvén waves” - see also the related discussion by [59]. The transport equation for the density fluctuation variance $\langle \rho^{*2} \rangle$ is given by

$$\frac{d}{dt} \langle \rho^{*2} \rangle = - \langle \mathbf{u}^\infty \cdot \nabla \rho^{*2} \rangle \approx - \langle \mathbf{u}^\infty \rangle^{1/2} \langle \rho^{*2} \rangle \frac{\ell_u}{\ell_u},$$  (57)
where we have introduced a correlation length $\ell_u$ appropriate to the core 2D incompressible velocity fluctuations $u^\infty$. Since $\langle u^\infty^2 \rangle = \left( \langle u^\infty^2 \rangle + \langle u^\infty^2 \rangle \right) / 2$ and the residual energy $E^\infty_D = \langle z^\infty^2 \rangle = \langle B^\infty^2 \rangle / (\mu \rho_0)$, we have

$$\langle u^\infty^2 \rangle = \frac{\langle z^\infty^2 \rangle + E^\infty_D}{2}. \quad (58)$$

Consider three cases. (i) $E^\infty_D = 0$ implies $\langle u^\infty^2 \rangle = \langle B^\infty^2 \rangle / (\mu \rho_0)$ and hence $\langle u^\infty^2 \rangle = \langle z^\infty^2 \rangle / 2$. (ii) The turbulent kinetic energy dominates i.e., $E^\infty_D \simeq \langle u^\infty^2 \rangle$ and $\langle B^\infty^2 \rangle / (\mu \rho_0) = 0$, which implies $\langle u^\infty^2 \rangle = \langle z^\infty^2 \rangle / 2$. Finally, (iii) if the magnetic energy dominates, then $\langle u^\infty^2 \rangle \simeq 0$, in which case there is no passive scalar convection of the entropic density fluctuation. This result is interesting because, as shown by Adhikari et al. [60], solar wind turbulence beyond $\sim 2$ AU becomes increasingly dominated by the magnetic energy, and by $\sim 10$ AU, the normalized residual energy $\sigma_D = -1$ (see also [37] and references therein). Thus, within $\sim 2$ AU, density turbulence in the solar wind evolves dynamically as a passive scalar, being mixed by the dominant 2D turbulent velocity component. Beyond 2 AU, the density turbulence “freezes” into a non-evolving statistical state, until about 10 AU when pickup ion driven turbulence [37; 60; 61] begins to generate turbulent velocity fluctuations in the outer heliosphere. In this region, the density turbulence “thaws” and begins to evolve dynamically again. This is discussed further below when we extend these results to the inhomogeneous solar wind.

For either $E^\infty_D = 0$ or kinetic energy dominated MHD turbulence, equation (57) can be approximated as

$$\frac{d}{dt} \langle \rho^2 \rangle \simeq -\langle z^2 \rangle^{1/2} \langle \rho^2 \rangle^{1/2} = -\frac{\epsilon_\rho}{2}, \quad (59)$$

where $\epsilon_\rho$ is the density dissipation rate. Consider again $\beta \sim 1$ or $\ll 1$ so that the core incompressible majority description is 2D. For 2D isotropic density turbulence, $\langle \rho^2 \rangle = \int E_\rho d\|k\| = E_\rho(k_\perp)k_\perp$. On using $E^\infty(k_\perp) = \epsilon^2_\infty k_\perp^{-5/3}$, it follows that

$$E_\rho(k_\perp) = \epsilon_\rho \epsilon^2_\infty k_\perp^{-5/3}, \quad (60)$$

i.e., the density variance spectrum is Kolmogorov. Alternatively, if we do not assume isotropy, we may, as before, express

$$\langle \rho^2 \rangle = \int \int E_\rho(k_\perp, k_\parallel) k_\perp d\|k\| d\|k\| \simeq E_\rho(k_\perp, k_\parallel) k_\perp^2 k_\parallel. \quad (61)$$

On using $\langle z^\infty^2 \rangle = E^\infty(k_\perp)k_\perp^2 = E^\infty(k_\perp)k_\perp$ as above, together with $\epsilon_\rho = \langle z^\infty^2 \rangle^{1/2} k_\perp E_\rho(k_\perp, k_\parallel) k_\perp^2 k_\parallel$, we obtain

$$E_\rho(k_\perp, k_\parallel) k_\perp^2 = \epsilon_\rho \epsilon^2_\infty k_\perp^{-2/3} k_\parallel^{-1}. \quad (62)$$

The results (60) and (62) show that the density variance spectrum is of a Kolmogorov form for plasma beta regimes $\sim 1$ or $\ll 1$, providing a possible explanation for the observed interstellar electron density spectrum [18–21].

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6. Conclusions

Zank et al. [24] presented a detailed development of the theory of nearly incompressible MHD turbulence for homogeneous and inhomogeneous flows. Here we summarize the [24] results for homogeneous NI MHD, extending the results to the $\beta \gg 1$ regime and contrasting these with the $\beta \sim 1$ and $\ll 1$ results. We re-expressed the original dimensionless form of the NI MHD equations in dimensional units. For $\beta \gg 1$, NI MHD describes a majority incompressible component that is fully 3D and can admit low-frequency Alfvén waves together with a NI component that is essentially slab in the incompressible limit. By contrast, in the case of plasma beta values $\beta \sim O(1)$ or $\ll 1$, we find that NI MHD describes a superposition of majority 2D and minority slab fluctuations. This regime appears to be appropriate to the solar wind, unlike the $\beta \gg 1$ case. We therefore focused more attention on the $\beta \sim 1$ regime. Our conclusions are enumerated as follows.

(i) For $\beta \gg 1$, the dominant turbulence component is described by the “standard” 3D incompressible MHD equations. If a mean magnetic field $B_0$ is present, the majority component of the Elsässer form of the 3D incompressible MHD equations includes the usual Alfvén velocity propagation term. This system is very well studied, although it is frequently forgotten that it is valid only in the $\beta \gg 1$ regime.

(ii) For $\beta \sim 1$ or $\ll 1$, the majority fluctuating component is fully 2D in planes perpendicular to the large-scale mean magnetic field, being advected by the large-scale background flow. The Alfvén velocity is absent, ensuring that the majority component interaction occurs only between 2D modes, ensuring that the spectrum for the inertial range of the energy density for Elsässer modes is Kolmogorov in the perpendicular wave number $k_\perp = |\mathbf{k}_\perp|$, i.e., $\propto k_\perp^{-5/3}$.

(iii) For $\beta \sim 1$ or $\ll 1$, 2D magnetic islands or vortex structures are a nonlinear component of the dominant core 2D incompressible MHD fluctuations.

(iv) For all plasma beta regimes, the homogeneous NI transport equations show that the component of vorticity parallel to the mean magnetic field satisfies the 1D Alfvén wave equation. The remaining NI modes are magnetosonic. By considering only the incompressible NI corrections, we show that all three components of the vorticity satisfy a 1D Alfvén wave equation, illustrating that the incompressible NI corrections correspond to a minority slab component. NI MHD in the $\beta \sim 1$ or $\ll 1$ limits is therefore a superposition of 2D plus slab fluctuations.

(v) For $\beta \sim 1$ or $\ll 1$, the NI Elsässer formulation shows that the slab components are mixed passively by the coupling of the higher-order NI corrections to the leading order 2D incompressible fluctuations, this being the dominant nonlinear interaction. The nonlinear interaction of counter-propagating Alfvén wave packets, which are included in the higher-order NI transport equation, is via the generation of zero frequency 2D modes [34]. This nonlinear interaction is a source of 2D fluctuations in the leading-order 2D incompressible description, serving to couple the NI description back to the majority component for the $\beta \sim 1$ or $\ll 1$ regimes.

(vi) In considering the spectral characteristics of the slab component for $\beta \sim 1$ or $\ll 1$, we identified the inverse triple correlation time as the sum of the inverse passive scalar time scale associated with the leading-order 2D Elsässer variables and the
inverse Alfvén time scale. The “critical balance” parameter emerges from the triple correlation time as the term that determines whether the slab energy flux in wave number space is dominated by either passive scalar convection by leading-order 2D Elsässer fluctuations or Alfvén wave sweeping. This is a quite different physical interpretation of the critical balance parameter than that of Goldreich and Sridhar [57] and does not emerge as a conjecture.

(vii) Use of the triple correlation time scale and the leading order 2D spectrum allows us to derive the anisotropic slab energy spectrum \( E^*(k_\perp, k_\parallel) k_\perp^2 \) (equation (54)), in which the critical balance parameter enters by demarcating the wave number regime that has a Kolmogorov spectrum from the region with an IK spectrum. The complete wave number spectrum can be expressed as a superposition of the majority 2D and minority slab spectra (equation (55)). The composite Elsässer energy spectrum is anisotropic and does not possess a simple power law in either \( k_\perp \) or \( k_\parallel \), sometimes exhibiting a somewhat complex “broken power law” structure.

(viii) Although NI density fluctuations enter at different orders in the turbulent Mach number for the homogeneous and inhomogeneous formulations (\( O(M^2) \) and \( O(M) \) respectively), NI MHD shows that density fluctuations behave as passive scalars in response to advection by the majority incompressible velocity fluctuations. This is in contrast to the conjecture [58] that ascribes the mixing of density fluctuations to shear Alfvén waves. For \( \beta \sim 1 \) or \( \ll 1 \), the predicted isotropic and anisotropic wave number spectra for the fluctuating density variance are \( \propto k_\perp^{-5/3} \) (equation (60)) or \( \propto k_\perp^{-2/3} k_\parallel^{-1} \) (equation (62)) respectively. If, at some point, the majority 2D component of the magnetic energy dominates the kinetic energy, i.e., \( \sigma_D^2 \sim -1 \), the density turbulence “freezes” into a non-evolving statistical state. The predicted density spectra are a possible explanation for the observed Kolmogorov-like interstellar density spectrum.

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References
[1] Klainerman S and Majda A 1981 Communications in Pure Applied Mathematics 34 481–524
[2] Klainerman S and Majda A 1982 Communications in Pure Applied Mathematics 35 629-651
[3] Montgomery D, Brown M R and Matthaeus W H 1987 J. Geophys. Res. 92 282–284
[4] Matthaeus W H and Brown M R 1988 Physics of Fluids 31 3634–3644
[5] Zank G P and Matthaeus W H 1990 Physical Review Letters 64 1243–1246
[6] Zank G P, Matthaeus W H and Klein L W 1990 Geophys. Res. Lett. 17 1239–1242
[7] Zank G P and Matthaeus W H 1991 Physics of Fluids 3 69–82
[8] Matthaeus W H, Klein L W, Ghosh S and Brown M R 1991 J. Geophys. Res. 96 5421–5435
[9] Zank G P and Matthaeus W H 1992 J. Plasma Phys. 48 85
[10] Zank G P and Matthaeus W H 1992 J. Geophys. Res. 97 17189
[11] Zank G P and Matthaeus W H 1993 Phys. Fluids 5 257–273
[12] Bayly B J, Levermore C D and Passot T 1992 Physics of Fluids 4 945–954
[13] Ghosh S and Matthaeus W H 1992 Physics of Fluids 4 148–164
[14] Bhattacharjee A, Ng C S and Spangler S R 1998 ApJ 494 409
[15] Hunana P, Zank G P and Shaikh D 2006 Phys. Rev. E 74 026302
[16] Hunana P, Zank G P, Heerikhuisen J and Shaikh D 2008 J. Geophys. Res. 113 A11105
[17] Hunana P and Zank G P 2010 ApJ 718 148–167
[18] Armstrong J W, Cordes J M and Rickett B J 1981 Nature 291 561–564
[19] Armstrong J and Woo R 1980 Rep. IOM 3331-80-070
[20] Spangler S R and Armstrong J W 1990 Low Frequency Astrophysics from Space (Lecture Notes in Physics, Berlin Springer Verlag vol 362) ed Kassim N E and Weiler K W pp 155–164
[21] Armstrong J W, Rickett B J and Spangler S R 1995 ApJ 443 209–221
[22] Bavassano B and Bruno R 1995 J. Geophys. Res. 100 9475–9480
[23] Bruno R and Carbone V 2013 Living Reviews in Solar Physics 10
[24] Zank G P, Adhikari L, Hunana P, Shiota D, Bruno R and Telloni D 2017 ApJ 835 147
[25] Adhikari L, Zank G P, Hunana P, Shiota D, Bruno R, Hu Q and Telloni D 2017 ApJ 841-85
[26] Zank G P, Dosch A, Hunana P, Florinski V, Matthaeus W H and Webb G M 2012 ApJ 745 35
[27] Kreiss H O 1980 Communications on Pure and Applied Mathematics 33 399–439
[28] Ghosh S and Parashar T N 2015 Physics of Plasmas 22 042302
[29] Bieber J W, Wanner W and Matthaeus W H 1996 J. Geophys. Res. 101 2511–2522
[30] Belcher J W and Davis Jr L 1971 J. Geophys. Res. 76 3534–3563
[31] Parashar T, Oughton S, Matthaeus W and Wan M 2016 ApJ 600 763–775
[32] Fyfe D and Montgomery D 1976 Journal of Plasma Physics 16 181–191
[33] Fyfe D, Montgomery D and Joyce G 1977 Journal of Plasma Physics 17 369–398
[34] Shebalin J V, Matthaeus W H and Montgomery D 1983 Journal of Plasma Physics 29 525–547
[35] von Karman T and Howarth L 1938 Proc. R. Soc. London, Ser. A 164 192–215
[36] Matthaeus W H, Oughton S, Pontius Jr D H and Zhou Y 1994 J. Geophys. Res. 99 19267
[37] Zank G P, Matthaeus W H and Smith C W 1996 J. Geophys. Res. 101 17093–17108
[38] Pouquet A, Frisch U and Leorat J 1976 J. Fluid Mech. 77 321–354
[39] Dobrowolny M, Mangeney A and Veltri P 1980 Physical Review Letters 45 144–147
[40] Grappin R, Frisch U, Pouquet A and Leorat J 1982 Astron. Astroph. 105 6–14
[41] Grappin R, Leorat J and Pouquet A 1983 Astron. Astroph. 126 51–58
[42] Matthaeus W H and Zhou Y 1989 Phys. Fluids B 1 1929–1931
[43] Zhou Y, Matthaeus W H and Dmitruk P 2004 Reviews of Modern Physics 76 1015–1035
[44] Marsch E 1990 Turbulence in the solar wind (Springer Verlag Berlin) p 43 Reviews of Modern Astronomy
[45] Kadomtsev B B and Pogutse O P 1973 Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki 65 575–589
[46] Verkhoglyadova O P, Dasgupta B and Tsurutani B T 2003 Nonlinear Processes in Geophysics 10 335–343
[47] Alexandrova O 2008 Nonlinear Processes in Geophysics 15 95–108
[48] Kharabrova O V, Zank G P, Li G, le Roux J A, Webb G M, Dosch A and Malandraki O E 2015 ApJ 808 181 (Preprint 1504.06616)
[49] Khararova O V, Zank G P, Li G, le Roux J A, Webb G M, Malandraki O E and Zharkova V V 2015 Journal of Physics Conference Series 642 02033
[50] Khararova O V, Zank G P, Li G, le Roux J A and Webb G M 2016 ApJ 827 122
[51] Lion S, Alexandrova O and Zaslavsky A 2016 ApJ 824 47 (Preprint 1602.07213)
[52] Lighthill M J 1960 Philosophical Transactions of the Royal Society of London Series A 252 397–430
[53] Montgomery D 1989 in Lecture Notes on Turbulence: Magnetohydrodynamic Turbulence (World Scientific)
[58] Lithwick Y and Goldreich P 2001 ApJ 562 279-296 (Preprint astro-ph/0106425)
[59] Chandran B D G, Quataert E, Howes G G, Xia Q and Pongkitiwanichakul P 2009 ApJ 707 1668–1675 (Preprint 0908.0757)
[60] Adhikari L, Zank G P, Bruno R, Telloni D, Hunana P, Dosch A, Marino R and Hu Q 2015 ApJ 805 63
[61] Adhikari L, Zank G P, Hu Q and Dosch A 2014 ApJ 793 52