Inflation at the maxima of symmetric potentials

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ABSTRACT

We construct a two-stage inflationary model which can accommodate early inflation at a scale $\Lambda_1$ as well as a second stage of inflation at $\Lambda_2$ with a single scalar field $\phi$. We use a symmetric potential, valid in a frictionless world, in which the two inflationary periods have exactly the same scale, i.e. $\Lambda_1 = \Lambda_2$. However, we see today $\Lambda_1 \gg \Lambda_2$ due to the friction terms (expansion of the universe and interaction with matter). These type of models can be motivated from supergravity. Inflation occurs close to the maxima of the potential. As a consequence both inflations are necessarily finite. This opens the interesting possibility that the second inflation has already or is about to end. A first inflation is produced when fluctuations displace the inflaton field from its higher maximum rolling down the potential as in new inflation. Instead of rolling towards a global minimum the inflaton approaches a lower maximum where a second inflation takes place.

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1 Introduction

The idea that the universe underwent an early inflationary expansion is now widely accepted [1]. This era of inflation makes plausible certain initial conditions for standard cosmology and provides a mechanism for structure formation. More speculatively the idea that the universe is at present undergoing inflation (usually denoted by the term quintessence) is the subject of much current interest [2]. Several models have been proposed where typically the potential energy of a scalar field, in general different from the one producing early inflation, is dominating the dynamics of the universe. Usually the potential is an inverse power of the field decreasing monotonically towards zero. In the present work we are interested in studying a model which accommodates two stages of inflation by the evolution of a single scalar field [3]. Here, however, we look at the possibility that both inflations are produced when the inflaton is close to the maxima of the potential. The fact that both inflations occur at the maxima implies that they are necessarily finite. This opens the interesting possibility where the second inflation has already or is about to end. This possibility is not ruled out by existing data and could be testable with far more, higher accuracy, supernovae on the Hubble diagram [4].

We take the position that inflation occurs not only at the maxima of the potential but also that these two stages of inflation, the initial and the present day, have the same value \( V(M_1) = V(M_2) \). Where \( M_1 \) and \( M_2 \) denote the first and second maxima of the potential, respectively. The reason why we see today \( V(M_1) \gg V(M_2) \) is due to the existence of friction terms. The friction terms are given by the expansion of the universe and by the interaction of the inflaton field with matter.

In what follows we develop this idea by using an analogy with a problem from classical mechanics. We discuss a simple toy model illustrating the main points. The resulting potential, when rewritten for the inflaton, could also be obtained from supergravity (see Appendix).

![Figure 1: The potential energy \( U(x) \) of a particle in classical mechanics. In the absence of friction if we leave the particle at the point \( A \) with vanishing velocity it will eventually reach \( B \), with \( U(A) = U(B) \), also with vanishing velocity. In the limit when \( A \to M_1 \) it will take an infinite amount of time for the particle to reach \( B \to M_2 \).](image)
Let us consider a potential $U(x)$ as shown in Fig. 1. When there is no friction the equation of motion for a particle of mass $m = 1$ is given by

$$\ddot{x} + U'(x) = 0$$

where $\dot{x} \equiv dx/dt$ and $U' \equiv dU/dx$. The conserved quantity of eq. (1) is just the energy $E = \dot{x}^2/2 + U(x)$. We study the problem of a particle that leaves with vanishing velocity somewhere from the left of the minimum, let us say $A$ and reaches a maximum height $B$ some time later with $U(A) = U(B) = E$. If we fix the origin of time at the minimum of $U(x)$ then the particle leaves $A$ in the past reaching $B$ sometime in the future. As $A$ becomes close to the maximum at $M_1$ the particle spends longer close to the maxima. In the limit when $A \rightarrow M_1$ it takes an infinite amount of time for the particle to reach $M_2$. The particle would spend most of the time leaving $M_1$ and trying to reach $M_2$. As a result the kinetic energy is negligible close to the maxima; the potential energy dominates. We call this the limiting solution. The maximum at $M_1$ is located at $x = 0$ thus we require $x(t = -\infty) = 0$ and $x(t = +\infty)$ locates the maximum at $M_2$. As a concrete example let us consider the potential

$$U(x) = E \cos^2(x).$$

It is easy to check that the limiting solution is

$$x(t) = (2 \arctan[\tanh(t \sqrt{E/2})] + \pi/2),$$

where $x(t = -\infty) = 0$, $x(t = +\infty) = \pi$ and $\dot{x}(t = -\infty) = \dot{x}(t = +\infty) = 0$. The potential $U(x)$ is already illustrated in Fig. 1. If we could lower the r.h.s. branch of this potential we could use this mechanical problem as an analogy to construct a model with two stages of inflation. Actually this can be done as follows. Instead of the symmetric potential $U(x)$ let us consider a new potential $V(x)$, which we call the asymmetric potential, illustrated in Fig. 2. Now the maximum at $M_2$ is much smaller than the maximum at $M_1$. It is clear that we need a friction term in the corresponding Eq. (1) to stop the particle precisely at $M_2$. Instead of eq. (1) we would have

$$\ddot{x} + c\dot{x} + V'(x) = 0.$$  

Eq. (4) is similar to eq. (1) if we replace $U(x)$ by the ”potential” $U'(t) = V'(x) + c\dot{x}$. Furthermore, eq. (4) gives a conserved quantity $E = \dot{x}^2/2 + U(t)$ where $U(t) = V(x) + \int_{x_i}^{x} c\dot{x}dx = V(x) + \int_{t_i}^{t} \dot{x}^2dt$ since $E = \dot{x}(\ddot{x} + c\dot{x} + V'(x)) = 0$. Notice that $E$ is in general not the energy since at a given point $(t,x(t))$ it depends on the history of the trajectory through the integral $\int_{t_i}^{t} \dot{x}^2dt$ in $U(t)$ and only in the case $c = 0$ (i.e. no friction) $E$ is the conserved energy. However, since $E$ is conserved even for $c \neq 0$ one has a maximum of the potential $U(t_i) = U(t_f)$ at $\dot{x}(t_i) = \dot{x}(t_f) = 0$, i.e. the height of the potential is the same, and the two maximum values of the potential $V(x)$ are then given by

$$V(x_i) = U(t_i) \quad \quad V(x_f) = U(t_f) - \int_{x_i}^{x_f} c\dot{x}dx.$$  

(5)
For $\int_{x_i}^{x_f} c \dot{x} dx > 0$ we can easily have $V(x_f) < V(x_i) = U(t_i) = U(t_f)$. Using the conservation of $E$ we can write the friction term in the following equivalent forms

$$\int_{x_i}^{x_f} c \dot{x} dx = \int_{t_i}^{t_f} c \dot{x}^2 dt = \int_{t_i}^{t_f} c(2E - U) dt. \quad (6)$$

If we use the potential in eq. (2) we can integrate the friction term giving $\int_{x_i}^{x_f} c \dot{x} dx = c\sqrt{2E} (\cos(x_i) - \cos(x_f))$ and taking $x_i = 0, x_f = \pi$ the potential at the second maximum is $V(x_f) = U(t_f) - 2c\sqrt{2E} = E - 2c\sqrt{2E}$ which is smaller than $V(x_i) = E$. The limiting solution eq. (3) solves eqs. (1) and (4) and imposing this solution to the potential $V(x)$ determines the friction term or equivalently if we know the friction term we can determine the energy scale at the maxima with $M_2 \ll M_1$ where the particle stops.

Figure 2: A particle leaves the maximum at $M_1$ with vanishing velocity. It will just reach $M_2$ also with vanishing velocity if there is a friction term which stops the particle precisely at $M_2$.

In inflationary models of the "new" type one typically starts with a very flat potential and inflation occurs close to the maximum at $\phi = 0$, where $\phi$ is the inflaton field. There could be a previous "primordial" stage of inflation probably of the chaotic type setting the initial conditions for new inflation. For simplicity in what follows we will call this new inflationary epoch a first stage or simply first inflation characterized by a scale $\Lambda_1$. This scenario is illustrated in Fig. 2 with $x(t) \to \phi(t)$. Here we study the possibility of a second stage of inflation at a scale $\Lambda_2(t_2)$, where $\Lambda_2(t_2) \ll \Lambda_1(t_1)$. The mechanical analogy indicates that the second inflation will occur also close to a maximum.

The value of the potential at the second inflationary period $t_2$ from a symmetric point of view is exactly the same as that of the first inflationary epoch at $t_1$, i.e. $U(t_1) = U(t_2)$, but because we live in an asymmetric world (a world with friction terms) we see that $V(t_2) = \Lambda_2^4 \ll V(t_1) = \Lambda_1^4$. We propose that it is only due to the dynamics including friction terms that we are now seeing a very small inflationary scale compared to the first one.
2 The Cosmological Model

The inflaton field equation of motion and Friedman’s equation are, as usual, given by

\[ \ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \]  
\[ 3H^2 = \rho, \]  

(7)

(8)

where we have set the reduced Planck mass \( M = 2.44 \times 10^{18} \text{GeV} \) to unity and \( \rho \) is the total energy density and we are considering a flat universe and an homogenous scalar inflaton field. In analogy with eq.(4) we will assume a symmetric potential \( U(t) \) such that \( U(t_i) = U(t_f) = 0. \) From eq. (7) and \( U'(t) = 3H\dot{\phi} + V'(\phi) \) one has

\[ U(t) = V(\phi) + \int_{\phi_i}^\phi 3H\dot{\phi}d\phi = V(\phi) + \int_{t_i}^t 3H\dot{\phi}^2 dt \]  

(9)

where the integration constant has been fixed by demanding \( V(t_i) = U(t_i) \) and we have used \( d\phi = \dot{\phi}dt. \) From eq.(7) we can define a conserved quantity \( E_\phi = \dot{\phi}^2/2 + V(\phi) \) and at the points \( \dot{\phi}(t_a) = 0 \) one has the maximum of the potential \( U(t_a). \) The derivative of \( H \) is

\[ \dot{H} = -\frac{1}{2}(\dot{\phi}^2 + \rho_a(1 + w_a)) \]  

(10)

where we have assumed \( \rho = \rho_\phi + \rho_a \) with \( \rho_\phi = \dot{\phi}^2/2 + V(\phi) \) and \( \rho_a \) is the energy density of matter or radiation. From eq.(10) the friction term gives

\[ \int_{\phi_i}^\phi 3H\dot{\phi}d\phi = -\int_{t_i}^t (2H\dot{H} + H\rho_a(1 + w_a))dt = 3\Delta H^2 - \Delta \rho_a \]  

(11)

with \( dH^2/dt = 2H\dot{H}, \rho_a = -3H\rho_a(1 + w_a) \) and \( \Delta H^2 = H_i^2 - H_a^2 \geq 0, \Delta \rho_a = \rho_{ai} - \rho_a(t) \geq 0. \) At the second stage of inflation the potential is simply given by

\[ V(t_2) = U(t_2) - \int_{t_i}^t 3H\dot{\phi}^2 dt = V(t_1) - 3\Delta H^2 + \Delta \rho_a \]  

(12)

where we have used in the last equality that \( U(t_2) = U(t_1) = V(t_1). \) Of course eq.(12) is self-consistent and it is no surprise since \( 3\Delta H^2 - \Delta \rho_a \simeq V(t_1) - V(t_2) \) for inflationary epochs where \( E_k(t_i) \ll V(t_i), i = 1, 2. \) The new point of view is that eq.(12) predicts a second stage of inflation at a much lower scale (seen from our asymmetric world) and that the value of this scale can be determined by the scale of the first inflationary period and the friction term.

2.1 A Toy Model

Let us now study a toy model in the absence of matter or radiation. This model is unrealistic for the reasons given at the end of the section, however, we believe the model illustrates in a very
simple way the main points raised in this work. We take the ansatz for the symmetric potential as \( U = A + B \cos^2[\alpha \phi] \) with \( A, B > 0 \) and \( \alpha \) constants to be determined. The period for \( \phi \) is taken as \( 0 \leq \alpha \phi \leq \pi \) with \( \phi(t_i) = -\infty \) and \( \phi(t_f = \infty) = \pi / \alpha \). The maxima of the potential are at \( \phi(t_i) = 0 \), \( \alpha \phi(t_f) = \pi \), where \( \dot{\phi}(t_i) = \dot{\phi}(t_f) = 0 \), fixing \( A + B \equiv \Lambda_1^2 \) we can determine \( \dot{H} \) and \( H \) giving

\[
\dot{H} = -\frac{\dot{\phi}^2}{2} = -B \sin^2[\alpha \phi] \tag{13}
\]

\[
H = \frac{1}{2 \sqrt{3}} \left( A(1 + \cos[\alpha \phi]) + 2B \right) \tag{14}
\]

The integration constant in \( H \) is fixed by demanding that at \( t_i \) we have \( U(t_i) = V(t_i) = 3H^2(t_i) \) since \( \dot{\phi}(t_i) = 0 \). The resulting potential is then simply given by \( V = 3H^2 + H \). By imposing that the minimum of the potential \( V \) is zero, i.e. \( V \geq 0 \), and defining the scale at \( t_f = \infty \) as \( V(t_f) \equiv \Lambda_2^2 \) we can determine the constants \( A, B, \alpha \) in terms of \( \Lambda_1, \Lambda_2 \) giving \( A = \Lambda_1^2(\Lambda_2^2 - \Lambda_2^2) \), \( B = \Lambda_1^2\Lambda_2^2 \) and \( \alpha = \sqrt{6}\Lambda_1\Lambda_2 / (\Lambda_1^2 - \Lambda_2^2) \). In terms of these scales one has \( U = \Lambda_1^4(1 - (\Lambda_2^2 / \Lambda_1^2) \sin^2[\alpha \phi]) \) and

\[
\phi = \frac{1}{\sqrt{6}} \frac{\Lambda_1^2 - \Lambda_2^2}{\Lambda_1\Lambda_2} \left( 2 \arctan[\tanh(\Lambda_1\Lambda_2\alpha(t + t_o))] + \frac{\pi}{2} \right) \tag{15}
\]

\[
\dot{\phi} = \sqrt{-2\dot{H}} = \sqrt{2}\Lambda_1\Lambda_2 \sin[\alpha \phi] \tag{16}
\]

\[
V = \frac{1}{4} \left( \Lambda_1^2 - \Lambda_2^2 + (\Lambda_1^2 + \Lambda_2^2) \cos[\alpha \phi] \right)^2 \tag{17}
\]

\[
H = \frac{1}{2 \sqrt{3}} \left( \Lambda_1^2 + \Lambda_2^2 + (\Lambda_1^2 - \Lambda_2^2) \cos[\alpha \phi] \right). \tag{18}
\]

In Fig. 3 we show the symmetric potential \( U(t) \), the asymmetric potential \( V(\phi) \) as well as the acceleration of the scale factor of the universe \( \ddot{a} / a = H^2 + \dot{H} \) as functions of \( \phi(t) \). The initial time is taken at the origin \( \phi(t_i = -\infty) = 0 \) with the maximum of the potential \( V(0) = U(0) = \Lambda_1^4 \) and in the limiting solution eq. (15) one has at \( t_f = \infty, \alpha \phi = \pi \) and \( V(t_f) = \Lambda_2^2 \ll U(t_f) = \Lambda_1^4 \). The acceleration of the universe is positive around the maxima of the potential. Notice also in Fig. 3 the cyclic nature of the potential \( U \) and \( V \).

In a more realistic situation the inflaton leaves not from the maximum at \( \Lambda_1 \) but from a slightly displaced position. The potential is shown in Fig. 3. A mechanism setting the field away from the maximum at \( \phi = 0 \) is provided by its fluctuations. We have that

\[
\delta \phi \approx \frac{H(t \to -\infty)}{2\pi} \approx \frac{\Lambda_1^2}{2\pi \sqrt{3}}. \tag{19}
\]

Depending on the initial conditions the scalar field approaches \( \Lambda_2 \) ending in oscillations around one of the minima. The time evolution of \( \phi \) and state parameter \( w = p / \rho \) illustrated in Fig. 4 correspond to a field which is unable to reach the maximum at \( \Lambda_2 \) ending in oscillations around the first minimum. For a larger initial kinetic energy the field would be able to overcome the maximum at \( \Lambda_2 \) ending at the second minimum. The beginning and duration of the second
Figure 3: We show $U(t), V(\phi)$ and the acceleration $\ddot{a}/a = H^2 + \dot{H}$ as a function of $\phi$ (dotted, solid and dashed lines, respectively). Notice that $U$ has symmetric maxima while $V$ develops a smaller maximum at $\Lambda_2$ with $\alpha\phi = \pi$. Acceleration occurs around the maxima of the potential.

Figure 4: The inflaton leaves from close to $\Lambda_1$ where it has been displaced due to its fluctuations $\delta\phi \approx H(t \to -\infty)/2\pi \approx \Lambda_1^2/2\pi\sqrt{3}$. After some time it approaches the second maximum at $\Lambda_2$ ending in oscillations around the minimum of the potential.
inflation depend on the initial conditions with which the universe was prepared. We believe this example illustrates in a simple way the main points raised in this work. However it is not realistic for several reasons: one can easily show that the end of inflation gives \( H(t_{\text{end}}) \approx \Lambda_2^2 \) which is a very low value for a realistic \( \Lambda_2 \). This could be ameliorated by relaxing the condition that the potential vanishes at the minima. For a negative potential at the minima there could also be a second inflation followed by a big crunch. Also there should be a mechanism of particle production at the end of the first inflation. This has been originally discussed in terms of gravitational particle production \([6]\) and subsequently criticized \([5]\) as an inefficient mechanism. A more efficient one being preheating \([5]\), where another scalar field is used to reheat the universe. It is also possible to invoke the action of a curvaton field to produce the reheating of the universe while the inflaton is in charge of inflation only \([7]\).

### 3 Conclusions

We have studied a model of inflation which can accommodate two inflationary eras. We use a symmetric potential, valid in a frictionless world, in which the two inflationary periods have exactly the same scale. However, we see in our world a second stage of inflation with a much smaller energy scale as the first one due to friction terms (expansion of the universe and interaction with other fields). Both stages of inflation are derived by the potential energy of a single scalar field. The new feature is that inflation occurs close to the maxima of the potential where the kinetic energy is negligible. As a consequence both inflations are of finite duration. It is then possible that the second inflation has already or is about to end which should be testable by substantially increasing the number and accuracy of supernovae on the Hubble diagram. We show an explicit example using a toy model. This model is not realistic but nicely illustrates the main ideas presented in this work. In the limiting solution, the scalar field takes an infinite amount of time to reach the second, smaller, maximum. In a more realistic case the scalar field is displaced from the higher maximum by its fluctuations ending in oscillations around a minimum of the potential. Finally, we have been able to show that a potential of the type Eq.(17) could be derived from supergravity. In supergravity the only natural scale is the Planck scale and there are arguments to explain the origin of the first scale of inflation \([8]\) while the second stage of inflation is derived from the first one by considering friction terms.

### Appendix

Let us consider the supergravity potential for one chiral superfield with scalar component \( z \) and without D-terms \([9]\)

\[
V = e^K \left[ F^* (K_{zz})^{-1} F - 3 |W|^2 \right],
\]

(20)
where $F = \frac{\partial W}{\partial z} + \left(\frac{\partial K}{\partial z^*}\right) W$, \quad $K_{zz^*} = \frac{\partial^2 W}{\partial z \partial z^*}$. The reduced Planck mass $M \sim 2.4 \times 10^{18}$ GeV has been set equal to one. The superpotential and Kähler potential denoted $W$ and $K$ respectively. Here we are interested in models where $W$ and $K$ are given by polynomial expressions such as $W = \sum_{n=0}^{\infty} a_n z^n$, and $K = \sum_{n=1}^{\infty} b_n (z z^*)^n$ where $a_n$ and $b_n$ are real coefficients. In general this structure leads to expressions that contain cos-form potentials for the angular field $\phi$ which is a real field defined from $z$ in the following way $z = e^{i\phi}$. By using the above ansätze for the superpotential and Kähler potential, it is straightforward to show that the supergravity potential can be written in the form

$$V = e^K \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ \frac{(n + K_1)(m + K_1)}{K_2} - 3 \right] a_n a_m z^n z^m z^*,$$

(21)

where $K_i$ denote the sums $K_1 = \sum_{n=1}^{\infty} n b_n (z z^*)^n$ and $K_2 = \sum_{n=1}^{\infty} n^2 b_n (z z^*)^n$. Let us now insert the radial and angular fields $z = e^{i\phi}$ in eq. (21). The potential is then given by [10]

$$V = e^K \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ \frac{(n + K_1)(m + K_1)}{K_2} - 3 \right] a_n a_m \chi^n \chi^m \cos[(n - m)\phi],$$

(22)

It is easy to show that Eq. (22) can give rise to potentials of the type Eq. (17). Let us write the superpotential and Kähler potential in the form $W = a_0 + a_1 z + a_2 z^2$ and $K = z z^* = \chi^2$. Assuming that the $\chi$ field has relaxed to its v.e.v., $\chi_0$ and eliminating e.g., $a_1$ we get

$$V(\phi) = c_1 (c_2 + \cos[\phi])^2,$$

(23)

where $c_1 = 2e^{\chi_0^2} \sqrt{(\chi_0^2 - 1) a_0 a_2}$ and

$$c_2 = \frac{((\chi_0^2 - 2) a_0 + (\chi_0^4 + 2) a_2) \sqrt{(\chi_0^2 - 3) a_0^2 - 2\chi_0^2 (\chi_0^2 - 1) a_0 a_2 + \chi_0^4 (\chi_0^2 + 4) a_2^2}}{\sqrt{(\chi_0^2 - 2)^2 a_0^2 - 2(\chi_0^2 - 2\chi_0^4 + 2\chi_0^2 + 2)a_0 a_2 + (\chi_0^4 + 2)^2 a_2^2}}.$$

(24)

It is then reasonable that a model of the type Eq. (17) could arise from a sugra particle physics model.

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