1. INTRODUCTION

Albert Einstein [1] found some radiative solutions to the field equations of general relativity and deduced the existence of gravitational waves, which could be interpreted as ripples in the curvature of spacetime propagating with the speed of light. In a subsequent paper [2], he computed the total power emitted in the form of gravitational waves by an isolated “Newtonian” source, and found that this power depends quadratically on the variations of the quadrupole moment of the source in a celebrated formula known today as Einstein’s quadrupole formula. The fact that gravitational waves from an isolated source are dominantly quadrupolar is a consequence of the equivalence principle, which implies the conservation (or linear variation in time) of the source’s monopole and dipole moments as the consequence of the equations of motion.

Existence of gravitational radiation was confirmed by Eddington [3] who showed that the propagation at the speed of light of weak gravitational waves on a flat background has an intrinsic meaning (independent of the choice of a coordinate system), and by Bondi [4] who proved that it is possible to construct a detector of gravitational waves which is heated when the wave passes through it. Surely gravitational radiation thus carries energy. Furthermore this energy is extracted from the mass-energy of the source (as measured from infinity), which has been shown by Bonnor [5] and Bondi, van der Burg and Metzner [6] (see also Sachs [7]) to be a monotonically decreasing function of time during the
emission. Before emission this mass-energy reduces to the mass of Arnowitt, Deser and Misner which represents the total mass-energy of the source plus the contribution of the waves to be emitted. Some theorems due notably to Shoen and Yau have established the positivity of both the ADM and Bondi mass of an isolated system. This proves in particular that the mass of the source bounds from above the total amount of energy which it can emit in the form of radiation. All these theoretical works show that the classic vision of a wave carrying energy, which is extracted from its source and can be put down an antenna, is legitimate in the case of gravitational radiation.

On the observational side gravitational radiation is known to exist thanks to the discovery by Hulse and Taylor of the binary pulsar PSR 1913+16. Five years of accurate timing observations of this pulsar have yielded the conclusion (by Taylor, Fowler and Mc Culloch) that the orbital period of the binary system of the pulsar and its companion is steadily decreasing, and therefore that by Kepler’s third law the system is loosing energy. There is exact agreement with the value of the flux of energy by gravitational radiation as predicted by the Einstein quadrupole formula. This effect can also be seen as due to the reaction forces acting on the orbit in reaction to the emission of gravitational waves. Radiation reaction forces in general relativity and the associated energy balance equation have been computed by Chandrasekhar and Esposito, Burke and Thorne, and many others authors. None of these computations, however, gave a complete description of the dynamics of gravitating systems, nor were applicable to systems containing strongly self-gravitating bodies such as neutron stars. The first complete derivation of the dynamics of the binary pulsar, up to the level where radiation reaction effects appear, was obtained by Damour and Deruelle (see the lecture of T. Damour in this volume).

Experimental research on gravitational radiation is presently very active with the development of new technologies for bar detectors, and most importantly with the construction of two large scale laser interferometric detectors for observation of the waves in the frequency bandwidth between $\approx 10$ Hz and $\approx 1000$ Hz: the American LIGO detector, and the Franco-Italian VIRGO detector. These detectors should observe at least one type of source, the inspiralling compact binary. This source is composed of two neutron stars or black holes spiralling very rapidly around each other in the last rotations preceding their final coalescence. Orbital velocities are much larger than in the binary pulsar for instance, which will coalesce with its companion in few hundreds of millions years.

For very relativistic sources like inspiralling compact binaries, the precision given by the Einstein quadrupole formula for the energy in the waves is unsufficient. Similarly unsufficient is the precision of the radiation reaction force of Burke and Thorne for instance. What is required is a relativistic or post-Newtonian formalism (involving an expansion when the speed of light goes to infinity) for both the emission and reaction of waves from isolated sources with substantially large internal velocities. However as we do not know what are the
sources which will actually be detected by VIRGO and LIGO, it is important
to have a formalism which is sufficiently general (not limited, for instance, to
inspiralling compact binaries).

Pioneering work for extending the Einstein quadrupole formula (and for-

mulas alike) to post-Newtonian order is due to Epstein and Wagoner [38]. Note
however that the post-Newtonian expansion is limited to the near-zone of the
source (see Section 2.1). Thus, for studying the radiation field one needs a pri-
or to supplement the post-Newtonian expansion by some another method. The
post-Minkowskian method (expansion when the gravitational constant $G$ goes
to zero) is adequate for that purpose. Pioneering work on post-Minkowskian
expansions is due to Bertotti and Plebanski [32], Bonnor and collaborators
[5, 40, 43], and Thorne and collaborators [44, 45]. Notably Bonnor [5] intro-
duced the idea of considering the post-Minkowskian expansion outside the
source where a multipolar decomposition of the field simplifies the resolution
of the field equations.

Our aim in this lecture is to discuss, with emphasis on recent developme-
nts, a particular post-Newtonian (and post-Minkowskian) formalism for the gen-
eration of gravitational waves by a relativistic source, and the reaction of the
waves on the source. Part of this formalism is issued from the line of research
initiated by Bonnor and collaborators [5, 40, 43]. Initial work was inspired
by Thorne’s review on multipole expansions [46]. The formalism is sufficien-
tly mature to tackle the problem of inspiralling compact binaries, and we pre-
sent at the end of the lecture the results which have been obtained so far in this ap-
lication. The work on this post-Newtonian formalism started with the present
author’s PhD thesis (advised by T. Damour and published in [47]). The list of
references is provided at the end of the lecture [47–57]. A very useful auxiliary
technical development was brought by T. Damour and B. Iyer [58]. Applica-
tions to binary systems (and, most importantly, inspiralling compact binaries)
were done in collaboration with G. Schäfer [59, 61], T. Damour and B. Iyer
[61], and partly with C. Will and A. Wiseman [62, 63] who used also their own
method based on the Epstein and Wagoner approach [38, 64]. A. Wiseman
[63, 66] and L. Kidder et al [67, 68] made also applications of the formalism.

The present lecture is strongly biased by the author’s point of view, so we
would like to refer here to other articles for different approaches and methods,
or simply for different points of views. See in particular the Les Houches book
of 1982 on Gravitational Radiation [69] (chapters by Thorne, Damour, Walker,
Choquet-Bruhat, Friedrich, Ashtekar), the chapters of Thorne and Damour
in the book on 300 years of gravitation [70] (see in particular the discus-
sions on approximation methods), the review by Damour in the Cargèse book on
Gravitation in Astrophysics [71] (see especially the discussion on the various
quadrupole laws and equations), the chapters by Iyer and Schmidt in Advances
in Gravitation and Cosmology [72], several reviews by Thorne [73, 75], Schutz
[74, 77] and Bonazzola and Marck [78], more devoted to detectors and the
sources of waves, and one review by Will [71] on the post-Newtonian approach
to gravitational radiation. See also the lectures of J. Čičák and T. Damour in
this volume.

We will proceed didactically starting in Section II with several useful definitions of regions around an isolated source, some recalls from the quadrupole formalism, and a general outline of the method. Sections III and IV are devoted respectively to some relevant properties of radiative gravitational fields and the generation of waves by the source. Section V deals with the radiation reaction effects occurring within the source. Finally Section VI applies the results of this wave generation formalism to inspiralling compact binaries.

2. BASICS AND OUTLINE OF THE METHOD

2.1. Regions around an isolated source

All over this lecture, we consider a general isolated source of gravity described by a matter stress-energy tensor with components \(T^{\mu\nu}(x,t)\) in some coordinate system \((x,t)\) covering the source (Greek indices take the values 0, 1, 2, 3, and Latin 1, 2, 3). Let \(a\) be the radius of a sphere which totally encloses the source.

We shall make the restriction that this isolated source is slowly moving, in the sense that the typical internal velocity inside the source divided by the speed of light \(c\), i.e. the typical ratio \(T^{0i}/T^{00}\), defines a small parameter \(\varepsilon \approx 1/c\) referred to as the slow motion or post-Newtonian parameter. We assume further that the typical stresses inside the source divided by the energy density in the source, i.e. the typical ratio \(T^{ij}/T^{00}\), is of order of the square of the parameter \(\varepsilon\). Finally we consider a purely gravitational problem, so that the source is self-gravitating (its internal motions are driven by gravitational forces). In this case the ratio \(GM/c^2a\), where \(M\) is the total mass-energy (or ADM mass) of the source, is also of order \(\varepsilon^2\). This implies that the gravitational field is weak (and of order \(\varepsilon^2\)) everywhere inside and outside the source.

Note that the parameter \(\varepsilon\) is required to be small, but not exceedingly small. For instance it could be as large as say 0.2 or 0.3. This is in order to allow for the very interesting case of inspiralling compact binaries where \(\varepsilon\) can reach 0.3 during the last rotations. Thus we are here considering truly relativistic sources, whose accurate description necessitates controlling many relativistic corrections (how many depending on the exact range of magnitude of \(\varepsilon\)). However, what we do not consider in this lecture is the case of a fully relativistic source, which is made of massless particles moving at the speed of light, or, to give another example, a cosmic string whose internal stress is of the same order as its energy density.

A useful characterization of slowly moving sources is that their spatial extension is small (and of order \(\varepsilon\)) as compared to one typical wavelength of the radiation. Indeed the wavelength is given by \(\lambda = cP\) where \(P\) is a period of motion in the source. But \(a \approx vP\) with \(v \approx \varepsilon c\), thus \(cP \approx a/\varepsilon\) and therefore \(a/\lambda \approx \varepsilon\) which is indeed the statement above. Note that since \(GM/ac^2 \approx \varepsilon^2\) we have also \(GM/\lambda c^2 \approx \varepsilon^3\).
It is convenient for slowly moving sources to introduce an interior domain $D_i$, sometimes called also the near zone, defined by $D_i = \{(x, t), \, |x| < r_i\}$, where the radius $r_i$ is such that $r_i \approx \varepsilon \lambda$ and $r_i > a$. The near zone $D_i$ is small with respect to the wavelength of the radiation and covers entirely the source. This is possible only for a slowly moving source. We choose $r_i$ to be strictly larger than $a$ (instead of being $a$ itself) for later convenience.

The near-zone is the domain where one can confidently use the post-Newtonian expansion. The real precision of some post-Newtonian expression will be exactly given, in $D_i$, by the formal order in $\varepsilon$ of the neglected terms. Note that all powers of $1/c$ must be taken into account in finding the magnitude of a term in terms of $\varepsilon \approx 1/c$, including the ones which arise from the temporal gradients $\partial_0 = c^{-1} \partial / \partial t$ (which are really of order $\varepsilon$ with respect to the spatial gradients $\partial_i$). This is clear from the definition of the near zone, where the field is quasi-static and propagation effects are small. These considerations are also familiar from electromagnetism.

Having clarified the concept of near-zone $D_i$, it is now necessary to introduce an exterior domain defined by $D_e = \{(x, t), \, |x| > r_e\}$, where the radius $r_e$ is chosen to be strictly between $a$ and $r_i$, i.e. $a < r_e < r_i$. This choice, which can always be done for slowly-moving sources, is in order that the intersection between $D_i$ and $D_e$ (the exterior part of the near zone $D_i \cap D_e$) exists.

Included in the exterior domain $D_e$, we also consider the so-called exponentially far wave zone, namely the domain $D_w = \{(x, t), \, |x| > r_w\}$, where $r_w$ is a radius such that $r_w \approx \lambda e^{\lambda^2/GM}$. In $D_w$, where the observer will be located, one can expand the field in powers of the distance of the source, and keep the leading order term in this expansion. The error done in assuming this will be negligible for future astrophysical sources of radiation.

As is well-known from the study of the Schwarzschild solution, the actual space-time light cones at large distances from the black hole deviate by a logarithmic term from the “flat” light cones associated with some coordinate systems more suitable near the source (for instance a harmonic coordinate system, or a Schwarzschild-like one). The definition of $D_w$ as being the exponentially far wave zone is chosen precisely in order that this effect becomes appreciable there. Our exponentially far zone corresponds to the distant wave zone in the terminology of Thorne [46], who introduces also a local wave zone starting several wavelength apart from the source. We shall not need here to make this distinction. It will be sufficient to consider a minimal splitting into three regions $D_i, D_e$ and $D_w$. See Thorne [46] for physical discussions of the effects which arise in different regions, and for a more general situation of a possibly strong-field source and a non-flat gravitational background.

Let us comment that all the radius $a$, $r_i$, $r_e$ and $r_w$ which have been introduced will in fact never appear explicitly in the results presented below. We shall only use the existence of the various domains as determined by these radius, for instance the fact that the intersection $D_i \cap D_e$ is not empty. This will permit us to prove some statements about the construction of solutions, for instance that one can licitly match the outer field to the inner field of the
source in $D_i \cap D_c$, but the precise locations of the radius are unimportant (these radii are anyway loosely defined).

2.2. The quadrupole formalism

By quadrupole formalism we mean the lowest order formalism, in the post-Newtonian expansion $\varepsilon \to 0$, for the generation of gravitational radiation from the source, and also for the reaction of the radiation onto the source. Although this may seem to be a little paradoxical, the quadrupole formalism can thus be viewed as a Newtonian formalism. Relativistic corrections to the formalism are referred to as post-Newtonian corrections accordingly to which post-Newtonian order in the equations of motion of the source (beyond the usual Newtonian acceleration) is needed in order to reduce all accelerations with consistent accuracy. For instance, the first post-Newtonian (in short 1PN) formalism retains all terms in the radiation field and in the reaction which can be computed consistently using the 1PN equations of motion, which include the corrections $\varepsilon^2$ beyond the Newtonian force (using the standard practice that the $n$th post-Newtonian order refers to the order $\varepsilon^{2n}$).

The quadrupole formalism for the wave generation expresses the gravitational field $h_{ij}^{\text{TT}} = (g_{ij} - \delta_{ij})^{\text{TT}}$ (where $g_{ij}$ denotes the spatial part of the covariant metric and $\delta_{ij}$ the Kronecker symbol) in a transverse and traceless (TT) coordinate system as

$$h_{ij}^{\text{TT}} = \frac{2G}{c^4R} P_{ijkm}(N) \left\{ \frac{d^2 Q_{km}}{dT^2} (T - R/c) + O(\varepsilon) \right\} + O\left(\frac{1}{R^2}\right). \quad (1)$$

The coordinate system $(X, T)$ is centered on the source, with $R = |X|$ the distance to the source and $N = X/R$ the unit direction from the source to the observer (later we shall need to be more precise about the exact definition of this coordinate system). The retarded time at the observer position in $D_w$ is $T - R/c$, and terms of order $1/R^2$ in the distance of the source are neglected (in addition to the post-Newtonian terms of order $\varepsilon$). In front of (1) appears

$$P_{ijkm}(N) = (\delta_{ik} - N_i N_k)(\delta_{jm} - N_j N_m) - \frac{1}{2}(\delta_{ij} - N_i N_j)(\delta_{km} - N_k N_m) \quad (2)$$

which is the TT projection operator onto the plane orthogonal to the direction $N$. The quadrupole moment of the source takes the familiar Newtonian form

$$Q_{ij}(t) = \int d^3x \: \rho(x, t) \left( x^i x^j - \frac{1}{3} \delta^{ij} x^2 \right), \quad (3)$$

where $\rho$ denotes the Newtonian mass density of the source (in some coordinate system which we shall also need to specify later). The quadrupole moment (3) is taken to be tracefree ($\delta_{ij} Q_{ij} = 0$). This is not important in the waveform (1) because at the quadrupolar level the TT operator in front cancels any possible trace in the moment, but it will be very convenient in higher orders to choose
all multipole moments to be symmetric and tracefree (STF) with respect to all their indices.

By differentiating with respect to time the waveform (1)-(3), squaring the result using standard formulas and integrating over all the directions \( \mathbf{N} \), one obtains the total power in the gravitational waves emitted by the source,

\[
\mathcal{L} = \frac{G}{5c^5} \left\{ \frac{d^3Q_{ij}}{dT^3} \frac{d^3Q_{ij}}{dT^3} + O(\varepsilon^2) \right\}.
\] (4)

This result is the famous Einstein quadrupole formula [2], which neglects post-Newtonian terms of order \( \varepsilon^2 \). It was derived originally by Einstein under the very restrictive assumption that the motion of the source is non-gravitational. Then Landau and Lifschitz [80, 81] showed that the quadrupole formula applies also to a gravitationally bound system, for instance a Newtonian binary star system. Many improvements in the derivation of this formula have been obtained later, notably by Fock [82] who made use of the idea of matching between the inner field of the source and some exterior field in \( D_i \cap D_e \). See [50] for a completely satisfactory derivation of the quadrupole formula (and its first relativistic corrections) using Fock’s idea. The notation \( \mathcal{L} \) in (4) stands for the total luminosity of the source in gravitational waves, by analogy with the total luminosity of a star in electromagnetic waves.

It is straightforward using the expression of the quadrupole moment of the source to rewrite the quadrupole formula (4) in the form of a balance equation relating the mechanical energy loss in the source to the total flux of energy in the waves. Indeed we can easily rewrite (4) in the form

\[
-\mathcal{L} + \frac{G}{5c^5} \frac{d}{dT} \left[ Q_{ij}^{(3)} Q_{ij}^{(2)} - Q_{ij}^{(4)} Q_{ij}^{(1)} \right] = \int d^3x \dot{v}_i F_{\text{reac}}^i + O(\varepsilon^7). \] (5)

The superscript \( (n) \) denotes \( n \) time derivatives. The second term in the left-hand-side of (5), which has been added to the energy flux, is a total time derivative and therefore its time average is numerically small in the case of quasi-periodic motion (see e.g. [28] for a discussion). On the other hand, we can interpret the right-hand-side as the power extracted within the source by the radiation reaction force density \( F_{\text{reac}}^i \), where \( F_{\text{reac}}^i \) is given explicitly by

\[
F_{\text{reac}}^i = \rho \partial_i V_{\text{reac}} + O(\varepsilon^7); \quad V_{\text{reac}}(x,t) = -\frac{G}{5c^5} \left\{ x^i x^j Q_{ij}^{(5)}(t) + O(\varepsilon^2) \right\}. \] (6)

This expression for the radiation reaction force represents the gravitational analogue of the damping force of electromagnetism, and was obtained by Burke and Thorne [20]-[23]. Note that gravitational radiation reaction is inherently gauge-dependent, so the expression of the force depends on the coordinate system which is used (here, a Burke-Thorne coordinate system whose definition can be found for instance in Chapter 36 of Misner, Thorne and Wheeler [83]). Chandrasekhar and Esposito [14] have independently obtained another (and more intricate) expression of the reaction force, and Miller [84] has shown
how one can transform one expression into the other by means of a coordinate transformation (see Schäfer [85] for a comparison between various expressions of the reaction force).

When dealing with relativistic sources having \( \varepsilon \lesssim 0.3 \) (say), the post-Newtonian terms which are neglected in the quadrupole formulas both for the wave generation (3)-(4) and radiation reaction (5)-(6) will become important. Our aim in this lecture is to compute these terms.

### 2.3. Outline of the method

Generalizing the previous quadrupole formalism for application to relativistic sources entails solving the mathematical problem of the resolution of the Einstein field equations with due taking into account of all the nonlinear effects entering some high-order post-Newtonian approximation we are considering.

The general method we use can be decomposed into several different steps. As a first step one investigates the gravitational field generated by the isolated source in all the exterior domain \( D_e \) (which comprises the far wave zone \( D_w \)). The field in \( D_e \) is a solution of the vacuum field equations and is computed using the combination of two methods:

(i) a formal infinite post-Minkowskian approximation method for the metric field (or perturbation around Minkowski’s space-time, or also expansion in powers of Newton’s constant \( G \)) — this method is valid wherever the field is weak;

(ii) a decomposition of each coefficients of the latter post-Minkowskian expansion into multipole moments (or simply expansion in spherical harmonics), which is valid only in the exterior \( D_e \).

The precise assumptions underlying the methods (i)-(ii) can be found in Section 1.2 of [47]. An iterative resolution (à la Bonnor [5]) of the field equations permits us in principle to control, and if necessary to compute, all the nonlinearities of the exterior field. Spherical harmonics are expressed using the symmetric and tracefree (STF) notation which is especially convenient when dealing with the decomposition of tensors. This leads also to transparent definitions of the multipole moments. See Thorne [46] and the appendices A and B of [47] for many technical formulas and discussion.

The second step consists of

(iii) expanding the latter solution in the far wave zone \( D_w \) (distance \( R \to \infty \)) in order to find the observable moments of the radiative field which are actually measured in the far away detector.

This step involves constructing a suitable (Bondi-type) coordinate system valid in \( D_w \) and correcting in particular for the logarithmic deviation of the true light cones with respect to the flat light cones. The construction to all orders in the post-Minkowskian expansion of such a coordinate system can be found in [45]. This yields the operational link between the observable moments and the moments parametrizing the multipole decomposition (ii). Computations of the observable moments with increasing precision can be found in the sections
II.C of [52], IV.C of [54], V of [56], and in [57].

Finally it remains to bridge the field in $D_e$ to the inner field of the source which is solution in $D_i$ of the non-vacuum Einstein field equations. To do so we

(iv) compute the post-Newtonian expansion of the external post-Minkowskian field and match it in the overlapping region $D_i \cap D_e$ to an inner solution obtained by direct post-Newtonian iteration of the field equations in $D_i$.

This is an application of the method of matched asymptotic expansions [86]. See [22] for a discussion on how useful is this method for bridging the near zone and the exterior zone. After matching, the field is determined everywhere inside and outside the source as a functional of the source’s parameters. This allows the determination both of the exterior multipole moments as integrals over the source, and of the radiation reaction forces acting inside the source. Matchings have been performed in the sections III of [50], IV of [51] and III of [54] to find the multipole moments, and in the sections VI of [49] and in [55] to find the reaction forces.

3. PROPERTIES OF RADIATIVE GRAVITATIONAL FIELDS

3.1. The field in the exterior domain $D_e$

In $D_e$ we have the vacuum field equations $R_{\mu\nu}[g] = 0$. It is advantageous to use instead of the covariant metric $g_{\mu\nu}$ (with determinant $g$) the densitized metric deviation $h_{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$ as a basic field variable, where $\eta^{\mu\nu} = \text{diag}(-1,1,1,1)$ is the Minkowski metric. The convention is that the indices on $h_{\mu\nu}$ (but not on $g^{\mu\nu}$ which is the inverse of $g_{\mu\nu}$) are lowered using the flat metric $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$. With this convention one defines for instance $h_{\mu\nu} = \eta_{\mu\rho}h^{\rho\sigma}h_{\sigma\nu}$ and $h = \eta_{\mu\nu}h^{\mu\nu}$. Imposing the harmonic (or De Donder) gauge condition for the coordinates $x^\mu$ (considered as forming four scalars $x^0, \cdots, x^3$) yields $0 = \Box_g x^\mu = -g^{\sigma\rho}\Gamma_{\rho\sigma}^{\mu} = -(-g)^{-1/2}\partial_\nu h^{\mu\nu},$ hence

$$\partial_\nu h^{\mu\nu} = 0 .$$

(7)

The vacuum field equations in these coordinates then read

$$\Box h^{\mu\nu} = \Lambda^{\mu\nu}(h) ,$$

(8)

where $\Box \equiv \Box_g = \eta^{\mu\nu}\partial_\mu\partial_\nu$ denotes the Minkowski (flat) d’Alembertian operator, and where $\Lambda^{\mu\nu}(h)$ can be viewed as a source for the gravitational field in vacuum which originates from the gravitational field itself. We have $\partial_\nu \Lambda^{\mu\nu} = 0$ as a consequence of the Bianchi identities, which is implied here by the harmonic coordinates condition.

The source $\Lambda^{\mu\nu}(h)$ encompasses all the nonlinearities of the field equations, and is at least quadratic in $h$ and its first and second space-time derivatives. For instance the quadratic part of $\Lambda^{\mu\nu}(h)$, which is such that $\Lambda^{\mu\nu}(h) = N^{\mu\nu}(h, h) +$
\(O(h^3)\), is explicitly given by \((t^{\mu\nu} = \frac{1}{2}(t^{\mu\nu} + t^{\nu\mu})\)

\[
N^{\mu\nu}(h, h) = -h^{\rho\sigma} \partial_\rho \partial_\sigma h^{\mu\nu} + \frac{1}{2} \partial^\mu h^{\rho\sigma} \partial_\rho \partial^\nu h^{\sigma} - \frac{1}{4} \partial^\mu h \partial^\nu h^{\rho\sigma} - 2\partial^\mu h \partial_\sigma \partial^\nu h^{\rho\sigma} + \partial_\sigma h^{\mu\rho}(\partial^\nu h_\rho + \partial_\rho h^{\nu\sigma})
\]

\[
+ \eta^{\mu\nu} \left[ -\frac{1}{4} \partial_\lambda h_{\rho\sigma} \partial^\lambda h^{\rho\sigma} + \frac{1}{8} \partial_\rho h \partial^\nu h + \frac{1}{2} \partial_\rho h \partial_\sigma h^{\nu\lambda} \partial^\rho h^{\sigma\lambda} \right].
\]

The cubic part of \(\Lambda^{\mu\nu}(h)\) can be found e.g. in the section I.B of [54].

The main technique we use in this section is to look for the solution of the field equations in the form of the post-Minkowskian expansion

\[
h^{\mu\nu} = G h_1^{\mu\nu} + G^2 h_2^{\mu\nu} + \cdots + G^n h_n^{\mu\nu} + \cdots,
\]

where we denote by \(G\) an ordering (or book-keeping) parameter which shall be identified later with Newton’s constant. The mathematical status of the post-Minkowskian expansion is known in vacuum (as is the case here). This expansion is asymptotic, in the sense that any solution of the perturbation equations (written in (11)-(12) below) comes from the Taylor expansion when \(G \to 0\) of a family of exact solutions [87]. Therefore we do not lose any generality in assuming the expansion (10), provided that this expansion is carried out up to infinity.

Substituting the asymptotic expansion (10) into the vacuum Einstein equations (7)-(8), and equalizing all the coefficients of the \(G^n\)'s in both sides of the equations, one obtains an infinite \((n = 1, \cdots, \infty)\) set of perturbation equations to be solved for all the \(h_n^{\mu\nu}\)'s. These equations take the form of wave equations in the ordinary sense supplemented by the condition of harmonic coordinates,

\[
\Box h_n^{\mu\nu} = \Lambda_n^{\mu\nu}(h_1, \cdots, h_{n-1}) ; \quad \partial_\nu h_n^{\mu\nu} = 0,
\]

where the important point is that since \(\Lambda^{\mu\nu}\) is at least quadratic in \(h\) (see (9)), all the nonlinear sources \(\Lambda_n^{\mu\nu}\) depend for any given \(n\) only on the previous iterations \(h_m^{\mu\nu}\) with \(1 \leq m \leq n - 1\). For instance we have \(\Lambda_2^{\mu\nu} = N^{\mu\nu}(h_1, h_1)\) where \(N^{\mu\nu}\) is given by (9).

When \(n = 1\), the corresponding source is zero, \(\Lambda_1^{\mu\nu} = 0\), and therefore the coefficient of \(G\) in the expansion (10), which corresponds to the linearized approximation, satisfies

\[
\Box h_1^{\mu\nu} = 0 ; \quad \partial_\nu h_1^{\mu\nu} = 0.
\]

An elementary retarded solution in \(D_e\) of the source-free wave equation \(\Box \varphi = 0\) reads \(\varphi = F(t - r/c)/r\), where \(F\) is an arbitrary function of the retarded time \(t - r/c\) (with \(r = |x|\) the distance to the spatial origin of coordinates located in the source). The most general retarded solution is readily obtained by differentiating with respect to space the elementary solution an arbitrary number of times, say \(\ell : \varphi_{i_1\cdots i_\ell} = \partial_{i_1} \cdots \partial_{i_\ell} \{F(t - r/c)/r\}.\) We shall extensively use
below the notation $\partial_L \equiv \partial_{i_1} \cdots \partial_{i_\ell}$ to denote a product of $\ell$ space derivatives, where $L = i_1 i_2 \cdots i_\ell$ represents a multi-index composed of $\ell$ indices. With this notation $\varphi_L = \partial_L \{F(t-r/c)/r\}$. Thorne\cite{46} has shown that the most general solution $h_1^{\mu\nu}$ of the equations (12) can always be written, modulo an infinitesimal gauge transformation preserving the harmonic gauge condition, in terms of two and only two types of arbitrary functions $F_L$, which can be chosen to be symmetric and tracefree (STF) with respect to the $\ell$ indices they carry (i.e. $F_L$ is symmetric under the permutation of any pair of its indices, and $\delta_{i_1 i_2} F_L = 0$). We shall denote by $M_L(t)$ and $S_L(t)$ these two types of STF tensorial functions of time and we shall call them the “algorithmic” multipole moments of mass type ($M_L$, having $\ell = 0, 1, \cdots$) and of current type ($S_L$, having $\ell = 1, 2, \cdots$). (The adjective “algorithmic” refers to the algorithmic construction below of the external field.) In the time-independent case (for stationary systems), these multipole moments are identical, aside from normalization, to the Geroch\cite{88} and Hansen\cite{89} multipole moments (see the proof in\cite{90}).

The moments $M_L(t)$ and $S_L(t)$ are for the moment arbitrary except that if the harmonic gauge condition in (12) is to be satisfied, the lowest order moments $M_1$ and $S_1$ (mass monopole and dipole moments), and $S_1$ (current dipole moment) are necessarily constant. This corresponds to the laws of conservation of the total mass, of the position of the center of mass, and of the total angular momentum (the linear momentum which is the time derivative of $M_i$ is zero because we consider the case of a source which was at rest in the infinite past). Furthermore we shall often choose $M_1 = 0$ by translating the origin of the coordinates to the center of mass. The mass monopole $M$ is the total ADM mass\cite{8} of the source.

We shall write the result for $h_1^{\mu\nu}$ in the form

$$G h_1^{00} = -\frac{4}{c^2} V_1^{\text{ext}}; \quad G h_1^{0i} = -\frac{4}{c^3} V_i^{\text{ext}}; \quad G h_1^{ij} = -\frac{4}{c^4} V_{ij}^{\text{ext}},$$

(13)

where we have introduced for later convenience some external potentials (valid in $D_e$) defined by

$$V_1^{\text{ext}} = \sum_{\ell=0}^{\infty} \frac{(-)^\ell}{\ell!} \partial_L \left[ \frac{1}{r} M_{(1)}(t-r/c) \right],$$

$$V_i^{\text{ext}} = -G \sum_{\ell=1}^{\infty} \frac{(-)^\ell}{\ell!} \partial_{L-1} \left[ \frac{1}{r} M_{(1)_{L-1}}(t-r/c) \right],$$

$$-G \sum_{\ell=1}^{\infty} \frac{(-)^\ell}{\ell!} \frac{\ell}{\ell+1} \varepsilon_{iab} \partial_{a_{L-1}} \left[ \frac{1}{r} S_{b_{L-1}}(t-r/c) \right],$$

$$V_{ij}^{\text{ext}} = G \sum_{\ell=2}^{\infty} \frac{(-)^\ell}{\ell!} \partial_{L-2} \left[ \frac{1}{r} M_{(2)_{ij(L-2)}}(t-r/c) \right],$$

$$+G \sum_{\ell=2}^{\infty} \frac{(-)^\ell}{\ell!} \frac{2\ell}{\ell+1} \partial_{a_{L-2}} \left[ \frac{1}{r} \varepsilon_{ab(i)S_{j(L-2)}^{(1)}}(t-r/c) \right].$$

(14)
The \( \ell \)-dependent coefficients in these expressions are chosen so that \( M_\ell(t) \) and \( S_\ell(t) \) reduce in the limit \( \varepsilon \to 0 \) corresponding to a Newtonian source to the usual expressions of the Newtonian moments of the source \[46\]. For instance,

\[
M_{ij}(t) = Q_{ij}(t) + O(\varepsilon^2) ,
\]

where \( Q_{ij} \) is the Newtonian mass quadrupole \( \[3\] \). A complete and self-contained derivation of the result \( \[13\]-\[14\] \) for the linearized metric can be found in Section 2 of \[47\].

3.2. Computing the field nonlinearities in \( D_e \)

Suppose that starting from the linearized metric \( \[13\]-\[14\] \) one has succeeded in solving the field equations \( \[11\] \) for all the nonlinear coefficients \( h^\mu_\nu_\mu \) up to some given order \( n-1 \) \( (1 \leq m \leq n-1) \). Each of these coefficients \( h^\mu_\nu_\mu \) will be in the form of a multipole expansion like \( h^\mu_\nu_\mu \). How can we find the solution of the equations for the next-order coefficient \( h^\mu_\nu_\mu \) ? The solution naturally involves the inverse retarded d’Alembertian operator \( \Box^{-1}_R \) defined by

\[
(\Box^{-1}_R f)(x, t) = -\frac{1}{4\pi} \iiint \frac{d^3x'}{|x-x'|} f(x', t - \frac{1}{c}|x-x'|) ,
\]

which satisfies \( \Box(\Box^{-1}_R f) = f \). However in trying to solve the wave equation in \( \[11\] \) directly with this operator one encounters a problem. Namely the source term \( \Lambda^\mu_\nu_\mu \) is like the previous coefficients \( h^\mu_\nu_\mu \) in the form of a multipole expansion which is valid only in \( D_e \), while the retarded integration in \( \[16\] \) extends over all space, including the inner domain \( D_i \). At the spatial origin of the coordinates \( r \to 0 \) located in \( D_i \), the multipole expansion is singular (see the equations \( \[14\] \)). This problem is of course not a physical problem but simply a mathematical one, as we are simply interested in finding a solution of the equations \( \[11\] \) which is legitimate in \( D_e \).

As a cure of this problem we use a mathematical trick which allows us to find the solution \( h^\mu_\nu_\mu \) of the equations \( \[11\] \) in the form of a multipole expansion, while keeping the range of integration in the retarded integral as it is in \( \[15\] \). This trick consists of replacing the real source \( \Lambda^\mu_\nu_\mu \) by a fictitious source obtained by multiplying \( \Lambda^\mu_\nu_\mu \) by \( r^B \) where \( B \) is a complex number. Assuming that the real part of \( B \) is large enough (and assuming that the multipole expansion is actually finite), one “removes” in this way the singularity at the origin \( r \to 0 \). There is no problem with the behaviour when \( r \to +\infty \) if one assumes that the source was stationary in the remote past. Then one can apply to this fictitious source the operator \( \[16\] \) as it stands. The result is in general infinite at \( B = 0 \), but the point is that the retarded integral of the fictitious source has been proved in Section 3 of \[47\] to admit (after analytic continuation) a Laurent expansion near \( B \to 0 \) whose finite part, defined to be the coefficient of the zeroth power of \( B \) in the expansion, is precisely the solution of the wave
For the equation in (11) we are looking for. Denoting the finite part at $B = 0$ by the symbol $FP_{B=0}$ we pose

$$p_{\mu \nu}^{\rho} = FP_{B=0} \Box^{-1}_R (r^B \Lambda_{L}^{\mu \nu}).$$

(17)

Then $p_{\mu \nu}^{\rho}$ satisfies $\Box p_{\mu \nu}^{\rho} = \Lambda_{L}^{\mu \nu}$ and is in the form of a multipole expansion like the previous coefficients $h_{\mu \nu}^{\rho}$. However the second equation in (11) which is the harmonic gauge condition is still to be satisfied. We shall refer here to [47] for the computation from the object $p_{\mu \nu}^{\rho}$ (or rather from its non-zero divergence $w_{\mu}^{\rho} \equiv \partial_{\nu} p_{\mu \nu}^{\rho}$) of a second object $q_{\mu \nu}^{\rho}$ satisfying at once $\Box q_{\mu \nu}^{\rho} = 0$ and $\partial_{\nu} q_{\mu \nu}^{\rho} = -w_{\mu}^{\rho}$ (see Eqs.(4.13) in [47] for the defining expression of $q_{\mu \nu}^{\rho}$). With these definitions a solution of both the wave equation and the gauge condition in (11) reads

$$h_{\mu \nu}^{\rho} = p_{\mu \nu}^{\rho} + q_{\mu \nu}^{\rho}.$$  

(18)

This completes the construction of the $n$th coefficient of the post-Minkowskian expansion (10), and therefore by induction of a whole infinite asymptotic expansion solving the vacuum field equations in $D_e$.

A priori, the above construction seems to be particular, however it has been proved in Section 4.2 of [47] to represent in fact the most general solution of the field equations in $D_e$ (modulo an arbitrary change of coordinates in $D_e$). The conclusion is that the general radiative field outside an isolated system depends on two and only two sets of time-varying multipole moments $M_L(t)$ and $S_L(t) : h_{\mu \nu}^{\rho} = h_{\mu \nu}^{\rho}[M_L, S_L]$. As a side result but whose proof is in fact crucial in the above construction [47], it is shown that the general structure of the post-Newtonian expansion ($\varepsilon \to 0$) of the exterior field in $D_i \cap D_e$ is of the form $\varepsilon^k (\ln \varepsilon)^p$, where $k$ and $p$ are positive integers. This result can be extended by a matching argument to the whole near zone $D_e$. Using some energy and momenta balance arguments the result could also (probably) be extended to the post-Newtonian expansion of the radiative field in $D_w$. The exact level at which the first logarithms of $\varepsilon$ appear is investigated in [49] (see also [91]–[94]).

In principle the construction of the field in $D_e$ could be implemented by an algebraic computer programme. However more theoretical work should be done first on the occurrence and properties of high-order tail integrals in high-order post-Minkowskian approximations (a preliminary investigation at the cubic level can be found in [57]).

For practical computations one needs some explicit formulas expressing the retarded integral (16) when the function $f$ has a definite multipolarity $\ell$, say $f(x,t) = \hat{n}_L f(r,t)$ where $\hat{n}_L$ denotes the STF part of the product of unit vectors $n_L = n_i_1 \cdots n_i_\ell$ ($n_i = x_i / r$). As reviewed in [46] and Appendix A of [47], the STF tensors $\hat{n}_L = \hat{n}_L(\theta, \phi)$ can be used equivalently to the usual spherical harmonics $Y_{\ell m}(\theta, \phi)$ for the multipole decomposition. The relevant formulas for this are (6.3)–(6.4) in [47], namely

$$\Box^{-1}_R (\hat{n}_L f(r,t)) = \int_{-\infty}^{t-r/c} ds \partial_L \left\{ \frac{g \left( \frac{t-r/c-s}{2}, s \right) - g \left( \frac{t+r/c-s}{2}, s \right)}{r} \right\},$$

(19)
where the function $g$ is related to $f$ by

$$g(\rho, s) = \rho^\ell \int_\alpha^\rho dx \frac{(\rho - x)^\ell}{\ell!} \left(\frac{2}{x}\right)^{\ell-1} f(\rho, s + x/c)$$  \hspace{1cm} (20)

(where $\alpha$ is arbitrary). The formulas (19)-(20) are correct only when the source $f(r, t)$ tends to zero when $r \to 0$ more rapidly than some power of $r$ (specified in [47]). This will always be the case of the fictitious source $B f = r^B f$ which was used in the previous construction of the exterior field (taking $Re(B)$ initially large enough and analytically continue the result). Notice the nice correspondence in (19) between the STF tensor $\hat{n}_L$ in the source of the wave equation and the STF derivative operator $\hat{\partial}_L$ (STF part of the product of derivatives $\partial_i \cdots \partial_{\alpha}$) in the corresponding solution.

Here are some formulas derived from (19)-(20) in Section 4 of [49] and which are particularly useful in the study of the quadratic nonlinearities, in which case one needs to integrate some source terms of the form $f(r, t) = r^{-k}F(t - r/c)$ with $k \geq 2$. When $k = 2$ we find

$$\Box^{-1}_R \left[ \hat{n}_L r^{-2}F(t - r/c) \right] = \frac{(-)^\ell}{2!} \int_r^{+\infty} d\lambda F(t - \lambda/c) \times \hat{\partial}_L \left\{ \frac{(\lambda - r)^\ell \ln(\lambda - r) - (\lambda + r)^\ell \ln(\lambda + r)}{r} \right\},$$  \hspace{1cm} (21)

which involves a non-local integral extending over the whole past of the source (see below for the associated physical effects). On the contrary the case $3 \leq k \leq \ell + 2$ yields a local result:

$$\Box^{-1}_R \left[ \hat{n}_L r^{-k}F(t - r/c) \right]_{|_{B=0}} = -\frac{2^{k-3}(k - 3)!(\ell + 2 - k)!}{(\ell + k - 2)!} \times \hat{\partial}_L \sum_{i=0}^{k-3} \frac{(\ell + i)!}{2^i!(\ell - i)!} \frac{F(k - 3 - i)(t - r/c)}{c^{k - 3 - i}r^{1+i}}.$$  \hspace{1cm} (22)

In these two cases the result is in fact finite at $B = 0$ and therefore we have dropped the Finite Part symbol. When $k \geq \ell + 3$ the result is no longer finite (there is a simple pole at $B = 0$) and the expression of the finite part is more complicated. See [57] for other formulas enabling one to investigate the cubic nonlinearities of the field equations.

3.3. The field in the far wave zone $D_w$

The study of “asymptotics” (namely what is the structure of space-time at large distances from the source?) is an active research field in general relativity since the work of Bondi et al [3] and Sachs [7] showing that it is possible to find some solutions of the field equations with bounded sources in the form of a power series in the inverse of the distance $R$ to the source (with a null coordinate $u$.
remaining constant). The reformulation of this approach in geometrical terms by Penrose [95, 96] led to the concept of an asymptotically simple spacetime, which is by definition a spacetime sharing with Minkowski’s spacetime some (mathematically precise) global and local asymptotic properties, and which we would like to be associated with any radiating isolated system (see [97]–[99] for reviews, and the lecture of J. Bičák in this volume).

The harmonic coordinate system used for the previous construction of the field in $D_e$ was very convenient (because the equations for all components of the field take the form of wave equations), but it has one drawback: it is not of the “Bondi-type” because the expansion of the field when $r \to +\infty$ with $t - r/c = \text{const}$ involves besides the normal powers of $r^{-1}$ some logarithms of $r$. This was noticed long ago by Fock [82]. Madore [100] proved that the harmonic-coordinates linearized metric (13)-(14) cannot be the first approximation of a Bondi-type expansion at infinity. The origin of the problem is in the deviation of the flat light cones in harmonic coordinates with respect to the true null cones. The logarithms of the distance present in $D_w$ imply that the $n$th post-Minkowskian approximation there becomes larger than the $(n - 1)$th one. However this is not a real problem because the logarithms of $r$ are not genuine – they can be removed by a change of coordinates in $D_w$ (in contrast with the logarithms of $c$ in the post-Newtonian expansion which are probably present in all coordinate systems covering $D_i$).

The construction of Bondi-type (sometimes also called radiative) coordinates $X^\mu$ from the harmonic coordinates $x^\mu$ has been achieved in the present framework to all orders in the post-Minkowskian expansion [48]. A “radiative” post-Minkowskian metric was constructed first in the manner of the previous section, but correcting at each post-Minkowskian order the flat cones associated with the coordinate system in order that they agree (at least asymptotically) with the real light cones $u = \text{const}$. Then it was shown (section 4 of [48]) that the radiative metric so constructed indeed differs from the general harmonic-coordinates metric by a mere coordinate transformation. As a corollary it was proved that the Penrose definition of asymptotic simplicity (in a form given by Geroch and Horowitz [101]) is formally satisfied for any isolated radiating source. Note that this proof holds only in the case where the source is stationary in the past, i.e. when there exists a finite initial instant $-T$ at which the source starts vibrating and emitting the radiation. If this assumption is relaxed, the definition is probably no longer valid [102]–[104] (at least in its original form).

The main interest of the construction of Bondi-type coordinates in [48] is that it is explicit and can be performed to all orders in $G$. To leading order in $G$ one finds that the retarded time $T - R/c$ in Bondi-type coordinates ($T - R/c$ and the null coordinate $u$ coincide in the limit $R \to \infty$) differs from the harmonic coordinate time $t - r/c$ by

$$T - \frac{R}{c} = t - \frac{r}{c} - \frac{2GM}{c^3} \ln \left( \frac{r}{cb} \right) + O(G^2) .$$

(23)
This is simply the known logarithmic deviation of light cones in harmonic coordinates. The constant $b$ is an arbitrary gauge-dependent time scale. The correction of order $G^2$ will be given below.

The Bondi-type coordinates are well-adapted to the study of the field in $D_w$, and most importantly permit a clear definition of two sets of multipole moments $U_L(t)$ and $V_L(t)$ parametrizing the radiative field at infinity, and which are directly “measured” in a detector located in $D_w$. We shall refer to these moments as the “observable” or “radiative” moments, respectively of mass-type ($U_L$) and of current-type ($V_L$). These moments are distinct from the algorithmic moments $M_L$ and $S_L$. Their definition goes as follows. For all practical purposes the distance of the source is so large (say $R = 100$ Mpc for inspiralling binaries) that one can consider only the leading-order term $1/R$ in the Bondi expansion. Like in (1) the wave is entirely characterized by the TT projection of the spatial metric. It is a standard result (see e.g. [46]) that this projection can be decomposed uniquely into multipole moments according to

$$h_{ij}^{TT}(X,T) = \frac{4G}{c^2 R} P_{ijkm}(N) \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \left\{ N_{L-2} U_{km,L-2}(T - R/c) - \frac{2\ell}{(\ell + 1)c} N_{aL-2} t_{ab}(k V_m b_{L-2}(T - R/c)) + O\left(\frac{1}{R^2}\right) \right\} + O\left(\frac{1}{R^2}\right). \quad (24)$$

The TT operator $P_{ijkm}$ is given by (2), and $t_{(km)} = \frac{1}{2} (t_{km} + t_{mk})$. The multipole decomposition (24) is the defining expression for the observable moments $U_L$ and $V_L$ (which are functions of the retarded time $T - R/c$). The total power, or luminosity, contained in the waveform (24) is given by

$$L = \sum_{\ell=2}^{\infty} \frac{G}{c^{2\ell+1}} \left\{ \frac{(\ell + 1)(\ell + 2)}{(\ell - 1)\ell!(2\ell + 1)!!} U_L^{(1)} U_L^{(1)} + \frac{4\ell(\ell + 2)}{(\ell - 1)(\ell + 1)!!c^2} V_L^{(1)} V_L^{(1)} \right\}. \quad (25)$$

The $\ell$-dependent coefficients in (24)-(25) are adjusted so that the observable moments $U_L$ and $V_L$ agree at the linearized order with the $\ell$th time derivatives of the algorithmic moments $M_L$ and $S_L$ (beware of the fact that $U_L$ and $V_L$ do not have the usual dimensions of multipole moments). Comparing (24) with (13)-(14) (and taking into account a sign difference due to the use in (13)-(14) of the metric perturbation $\sqrt{-g} g^{ij} - \delta^{ij}$ instead of $g_{ij} - \delta_{ij}$ in (24)) we have

$$U_L(T) = d^\ell M_L/dT^\ell + O(G),$$
$$V_L(T) = d^\ell S_L/dT^\ell + O(G), \quad (26)$$

where the terms $O(G)$ symbolize the nonlinear corrections (which can be computed in principle to all orders using [48]). On the other hand we have seen
in (15) that the mass quadrupole moment $M_{ij}$ agrees in the Newtonian limit with the quadrupole $Q_{ij}$ given by the integral over the source (3). Thus, from (26) and (15), we see that in the Newtonian limit the connection between the observable moment at infinity $U_{ij}$ and the actual source moment $Q_{ij}$ is realized (nonlinear corrections in (26) vanish in the Newtonian limit). Of course this yields simply the quadrupole formalism reviewed in Section 2.

What we shall do in the next subsection is to generalize the relationships linking the observable multipole moments to the algorithmic moments in order to include explicitly the nonlinear corrections of order $G$ (at least) appearing in (26). The computation of the algorithmic moments themselves as explicit integrals over the source generalizing (15) to post-Newtonian order will be dealt with in the section 4. It is clear that knowing both the relations between the observable moments $U_L, V_L$ and the moments $M_L, S_L$, and between the moments $M_L, S_L$ themselves and the source, one solves the problem at hand.

3.4. Nonlinear effects in the radiation field

It follows from their construction [48] that the observable moments $U_L$ and $V_L$ are necessarily given by some nonlinear functionals of the moments $M_L$ and $S_L$ (reducing to (26) at the linearized order). A very useful information can be inferred on dimensional grounds. Indeed the general form of the relations (26) can be very easily proved to be

$$U_L(T) = M_L^{(f)}(T) + \sum_{n \geq 2} \frac{G^{n-1}}{c^{3(n-1)+2k}} X_{nL}(T),$$

$$\varepsilon a i i_{l-1} V_{nL-2}(T) = \varepsilon a i i_{l-1} S_{nL-2}^{(l-1)}(T) + \sum_{n \geq 2} \frac{G^{n-1}}{c^{3(n-1)+2k}} Y_{nL}(T) \quad (27)$$

(ending the current moment with its natural Levi-Civita symbol), where the functions $X_{nL}$ and $Y_{nL}$ represent some nonlinear functionals of $n$ moments $M_L$ and $S_L$ whose general structure reads

$$X_{nL}(T) = \sum \int_{-\infty}^{T} dv_1 \cdots \int_{-\infty}^{T} dv_n \mathcal{X}(T, v_1, \cdots, v_n) M_{L_1}^{(a_1)}(v_1) \cdots S_{L_n}^{(a_n)}(v_n). \quad (28)$$

The (a priori) quite complicated kernel $\mathcal{X}$ has an index structure made of Kronecker and Levi-Civita symbols, and depends only on variables having the dimension of time. Accordingly the powers of $c^{-1}$ in each $n$th nonlinear terms of (27) are such that the dimensions of the moments $U_L$ and $V_L$ are respected. They turn out (see e.g. the section IV.C of [54]) to be of the type $3(n-1) + 2k$ where $k$ is a positive integer representing a number of contractions between the indices on the moments composing the term in question. This information is useful when determining the type of nonlinear terms which arise in a formula like (26) below.
As expressed in (28) the functionals $X_{nL}$ and $Y_{nL}$ are given by some highly non-local in time (or “hereditary”) functionals of the moments $M_L$ and $S_L$. This illustrates the fact that the gravitational field in higher approximations depends on all the integrated past (or “history”) of the source [39], [105]–[109]. The propagation of the radiation proceeds not only along the characteristic surfaces of spacetime or light cones, but also inside them. One can view this phenomenon as the propagation at the speed of light of a front wave spreading by continuous scattering a secondary wave propagating in average at a speed less than $c$. This wave is referred generally to as the tail of the wave. Perhaps a better name would be the wake of the wave. An interesting particular case is that of the scattering onto the static spacetime curvature generated by the total mass-energy $M$ of the source itself. Often the name of tail of the wave is reserved for this specific effect. In that case the tail results from the nonlinear interaction between the varying multipole moments describing the radiation field, primarily the quadrupole moment, and the static mass monopole moment $M$ [41]–[43], [106, 110, 111, 49, 52, 60]. A signature of the presence of wave tails in the gravitational signals received from inspiralling compact binaries should be detectable by VIRGO and LIGO [112, 113].

In order to determine more explicitly the relations (27) one must implement to nonlinear order the coordinate transformation from harmonic to radiative coordinates. This has been done to quadratic order in Section II of [52]. To see how this works let us first consider the leading-order $1/r$ part when $r \to \infty$ (with $t - r/c$ fixed) of the linearized metric $h^{\mu\nu}_1$ given by (13)-(14). One can write

$$h^{\mu\nu}_1 = \frac{1}{r} \left\{ -\frac{4M}{c^2} \delta_0^{\mu} \delta_0^{\nu} + z^{\mu\nu}(t - r/c, n) \right\} + O \left( \frac{1}{r^2} \right),$$

(29)

where we have isolated the static term depending on $M$ from the non-static contribution $z^{\mu\nu}$ given by

$$z^{00}(t, n) = -4 \sum_{\ell \geq 2} \frac{\mathcal{M}^{\ell}_{L}(t)}{c^{\ell+2} \ell!} M^{(\ell)}_{2L-1}(t),$$

$$z^{ij}(t, n) = -4 \sum_{\ell \geq 2} \frac{\mathcal{M}^{\ell}_{L-2}(t)}{c^{\ell+2} \ell!} \varepsilon_{ij} M^{\ell}_{iL-2}(t),$$

$$z^{ij}(t, n) = -4 \sum_{\ell \geq 2} \frac{\mathcal{M}^{\ell}_{L-1}(t)}{c^{\ell+2} \ell!} \varepsilon_{iab} n_{aL-1} S_{bL-1}^{(t)}(t),$$

(30)

Following the construction of the exterior field in Section 3.1 we insert (29) into the quadratically nonlinear source (3) and arrive at the equation to be satisfied.
by $h^\mu_2$ at the leading-order in $1/r$,

$$
\Box h^\mu_2 = \frac{1}{r^2} \left\{ \frac{4M}{c^4} z^{(2)\mu\nu} + \frac{k^\mu k^\nu}{c^2} \Pi \right\} (t - r/c, \mathbf{n}) + O \left( \frac{1}{r^3} \right),
$$

(31)

where $k^\mu = (1, \mathbf{n})$ denotes the outgoing null direction and where $\Pi$ is proportional to the power $dE_1/dt d\Omega$ (per unit of steradian) contained in the linearized wave $h_1$:

$$
\Pi = \frac{1}{2} z^{(1)\mu \nu} z^{(1)}_{\mu \nu} - \frac{1}{4} z^{(1)\mu} z^{(1)}_{\mu} = \frac{16 \pi}{c^3} \frac{dE_1}{dt d\Omega}.
$$

(32)

Constructing the solution of (31) using the method of Section 3.1 necessitates knowing the retarded integral of a source with multipolarity $\ell$, and behaving in $r$ like $1/r^2$. The relevant formula is (21). Working out this formula to leading-order in $1/r$ (or $\ln r/r$ in fact) we get [52]

$$
\Box^{-1} \left[ \hat{r}_L r^{-2} F \left( t - \frac{r}{c} \right) \right] = \frac{c\hat{r}_L}{2r} \int_0^{+\infty} dy F \left( t - \frac{r}{c} - y \right) \left[ \ln \left( \frac{cy}{2r} \right) + \sum_{i=1}^{\ell} \frac{2}{\ell} \right]

+ O \left( \frac{\ln r}{r^2} \right),
$$

(33)

showing explicitly how some logarithms of $r$ are generated at the quadratic nonlinear level. All these logarithms can be get rid off by implementing the coordinate transform of [48]. With this approximation one finds

$$
X^\mu = x^\mu + G \xi^\mu + G^2 \lambda^\mu + O(G^3),
$$

(34)

where the vectors $\xi^\mu$ and $\lambda^\mu$ are given by

$$
\xi^\mu = -\frac{2M}{c^2} \delta^\mu_0 \ln \left( \frac{r}{cb} \right),

\lambda^\mu = \Box^{-1} \left[ \frac{k^\mu}{2c^2} \int_{-\infty}^{t-r/c} dv \Pi(v, \mathbf{n}) \right].
$$

(35)

To order $G$ the coordinate transformation (34)-(35) implies (23). Both the vectors $\xi^\mu$ and $\lambda^\mu$ correct for the deviation of the curved light cones from the flat cones in harmonic coordinates. After coordinate transformation the metric is of the Bondi-type and its $1/R$ term at infinity is readily obtained. By comparison of the result with the expression of the waveform (24) one obtains the looked-for observable moments in terms of the moments $M_L, S_L$. To quadratic order and keeping only the contributions of non-local (or hereditary) integrals one finds [52]

$$
U_L(T) = M_L^{(\ell)}(T) + \frac{2GM}{c^3} \int_0^{+\infty} dy \ln \left( \frac{y}{2b} \right) M_L^{(\ell+2)}(T - y)

+ \frac{Gc^{\ell+1} \ell!}{2(\ell + 1)(\ell + 2)} \int_{-\infty}^{T} dv \Pi_L(v) + G\mathcal{I}_L(T) + O(G^2),
$$
\[ V_L(T) = S^{(0)}_L(T) + \frac{2GM}{c^3} \int_0^{+\infty} dy \ln \left( \frac{y}{2b} \right) S^{(\ell+2)}_L(T-y) \]

\[ + GJ_L(T) + O(G^2) \, , \tag{36} \]

where \( \Pi_L \) denotes the STF coefficient in factor of \( \hat{n}_L \) in the STF spherical harmonics decomposition of \( \Pi \) (namely \( \Pi = \sum \hat{n}_L \Pi_L \)), and where we denote by \( I_L \) and \( J_L \) some instantaneous (by opposition to hereditary) contributions which are not controlled at this stage (but see below).

The expressions (36) display two different types of non-local integrals. The first type, arising in both \( U_L \) and \( V_L \) and having \( M \) in factor, represents the contribution of the backscattering of the waves emitted in the past onto the static background curvature associated with the mass of the source (tail effect).

The constant \( b \) in the logarithmic kernel of the tail integrals is the same as in (23) and (35). The second type of non-local integral, arising only in \( U_L \) (at this order), is due to the re-radiation of gravitational waves by the distribution of stress-energy of linear waves. We shall refer to this non-local integral as the non-linear memory integral. The qualitatively different nature of the tail and non-linear memory effects is more clearly understood when taking the limit \( T \to +\infty \) corresponding to very late times after the source has ceased to emit radiation (in the sense that the \((\ell+1)\)th time derivatives of the moments \( M_L \) and \( S_L \) tend to zero). In this limit the tail integrals (as well as the instantaneous terms \( I_L \) and \( J_L \)) tend to zero while the memory integral tends to a finite limit (hence its name):

\[ U_L(+\infty) = M^{(\ell)}_L(+\infty) + \frac{Gc^{\ell+1} \ell!}{2(\ell+1)(\ell+2)} \int_{-\infty}^{+\infty} dv \Pi_L(v) + O(G^2) \, . \tag{37} \]

The nonlinear memory term was first noticed by Payne [114] in a particular context. It appeared in the form \( [20] \) in the present author’s habilitation thesis [115] (and later in [52]). Then the memory effect was obtained by Christodoulou [116] using asymptotic techniques, and Thorne [117] pointed out that the effect can be recovered simply by adding to his formulas for the linear memory [118] the contribution of gravitons. Wiseman and Will [119] evaluated the term in the quadrupole approximation and found a result equivalent to the ones of [115, 52]. The nonlinear memory term is however not very important in the case of inspiralling compact binaries as compared with the terms associated with tails (and even tails of tails).

It can be readily verified that in a post-Newtonian expansion \( \varepsilon \to 0 \) the tail integral present in the observable moment \( U_L \) dominates the memory integral (see section III.A in [52]). Indeed the tail arises dominantly at order 1.5PN (this is clear from the factor \( c^{-3} \) in front of this term), while the nonlinear memory term is smaller, being at least of order 2.5PN. For instance, one finds that in the post-Newtonian expansion \( \varepsilon \to 0 \), the mass quadrupole observable moment \( \dot{U}_{ij} \) (which is the most important moment to obtain) reads, up to
1.5PN order, as

\[ U_{ij}(T) = M_{ij}^{(2)}(T) + \frac{2GM}{c^3} \int_{0}^{+\infty} dy \left[ \ln \left( \frac{y}{2b} \right) + \frac{11}{12} \right] M_{ij}^{(4)}(T-y) + O(\varepsilon^5). \]  

(38)

The constant 11/12 in the integral is actually in factor of an instantaneous term (depending only on \( T \) through the third time derivative \( M_{ij}^{(3)}(T) \)). It is computed in the appendix B of [52].

Pushing the post-Newtonian expansion beyond the 2.5PN order controlled in (36) requires investigating some cubic nonlinearities in the radiation field. To 3PN order there appears an effect due to the “tails of tails”, which are the tails generated by backscattering of the (quadratically nonlinear) tails themselves onto the background curvature associated with \( M \). This effect corresponds to the trilinear interaction between two mass monopoles \( M \) and (say) the quadrupole \( M_{ij} \) describing the linear radiation. The dominant tails of tails are computed in [57]. Let us give now the expression of the observable mass quadrupole moment up to the 3PN order (\( \langle ij \rangle \) denotes the STF projection):

\[ U_{ij}(T) = M_{ij}^{(2)}(T) + \frac{2GM}{c^3} \int_{0}^{+\infty} dy \left[ \ln \left( \frac{y}{2b} \right) + \frac{11}{12} \right] M_{ij}^{(4)}(T-y) + \frac{2}{c^3} \int_{0}^{+\infty} dy \left[ \ln \left( \frac{y}{2b} \right) + \frac{57}{70} \ln \left( \frac{y}{2b} \right) + \sigma \right] M_{ij}^{(5)}(T-y) + O(\varepsilon^7). \]  

(39)

The various terms are easily recognizable on this expression. Besides the dominant tail term of order \( \varepsilon^3 \approx c^{-3} \), there is the lowest order contribution due to the nonlinear memory which is the integral appearing at order \( \varepsilon^5 \) (it depends quadratically on the quadrupole moment [115, 119, 52]), and there is the tail of tail term which is given by the last integral of order \( \varepsilon^6 \) (one assumes that \( b \) enters \([15]\) in the combination \( (r/cb)^2 \)). The \( \alpha, \beta, \gamma, \lambda, \) and \( \sigma \) are some purely numerical constants in factor of instantaneous terms (like the 11/12), and which are computed in [57]. Note that the precise numerical value of these constants is not very important physically (they partly reflect our use of harmonic coordinates), but their computation using the formulas of Section 3.2 (and other formulas) is in general long and tedious.

The observable multipole moments other than the quadrupole admits some expressions similar to (39), but thanks to the fact that the multipolarity of a contribution scales with \( 1/c \) in the waveform (24) and with \( 1/c^2 \) in the energy loss [25], the needed accuracy for these moments is less than for the quadrupole.
We quote here the expressions of the mass octupole $U_{ijk}$ and current quadrupole $U_{ij}$ with an accuracy sufficient to include the dominant tails only:

\[
U_{ijk}(T) = M_{ijk}^{(3)}(T) + 2 \frac{GM}{c^3} \int_{0}^{+\infty} dy \left[ \ln \left( \frac{y}{2b} \right) + \frac{97}{60} \right] M_{ijk}^{(5)}(T - y) + O(\varepsilon^5),
\]

\[
V_{ij}(T) = S_{ij}^{(2)}(T) + 2 \frac{GM}{c^3} \int_{0}^{+\infty} dy \left[ \ln \left( \frac{y}{2b} \right) + \frac{7}{6} \right] S_{ij}^{(4)}(T - y) + O(\varepsilon^5).
\]

(40)

The computation of the constants $97/60$ and $7/6$ can be found in Appendix C of [54]. We assume in (40) that our coordinate system is mass-centered, $M_i = 0$.

4. GENERATION OF GRAVITATIONAL WAVES

4.1. Post-Newtonian iteration of the field in $D_i$

We now come to grips with the last step of the present approach, which is to find the expressions of the multipole moments $M_L$ and $S_L$ as explicit integrals over the distribution of stress-energy of the matter fields (and the gravitational field) in the source. This problem is evidently crucial in this approach since without a satisfactory solution the formulas (36)-(40) giving the observables measured far from the source would remain devoided of meaning, and the precise interpretation of the gravitational signals received by the future detectors would be impossible.

To be consistent with the formula (39) for instance, we should obtain the mass quadrupole moment $M_{ij}$ with a precision corresponding to the 3PN order, thereby generalizing the Newtonian relation (15) to take into account all the relativistic corrections up to the very high order $\varepsilon^6$. As we shall see the 3PN precision in this moment is not available presently, but $M_{ij}$ is known with the precision which precedes immediately, namely the 2.5PN or $\varepsilon^5$ precision [54, 56].

The non-vacuum Einstein field equations in harmonic coordinates read

\[
\Box h_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu} + \Lambda_{\mu\nu}(h),
\]

\[
\partial_\nu h^{\mu\nu} = 0,
\]

(41)

where $T_{\mu\nu}$ represents the stress-energy distribution of the matter fields (with $T_{\mu\nu}$ the usual stress-energy tensor), and where the other notations are defined in Section 3.1. In the near-zone $D_i$ the field equations can be solved by means of a direct post-Newtonian iteration when $\varepsilon \to 0$. We shall perform it without making any assumption concerning the stress-energy distribution $T_{\mu\nu}$ (it could be for instance that of a perfect fluid, or it could describe point
particles without interactions), except that it should have a compact support and correspond to a source whose parameter $\varepsilon$ is of the order 0.3 (say) at most.

It is useful to introduce as basic variables describing the source some densities of mass $\sigma$, current $\sigma_i$ and stresses $\sigma_{ij}$ defined in terms of the contravariant components of the stress-energy tensor $T^{\mu\nu}$ (with $T^{ii} = \Sigma \delta_{ij} T^{ij}$):

$$\sigma = \frac{T^{00} + T^{ii}}{c^2},$$
$$\sigma_i = \frac{T^{0i}}{c},$$
$$\sigma_{ij} = T^{ij}.$$  \hspace{1cm} (42)

In particular $\sigma$ is related to the Tolman mass density valid for stationary systems \[50\]. Here the powers of $c^{-1}$ are such that these densities have a finite non-zero limit when $\varepsilon \to 0$ ($T^{\mu\nu}$ has the dimension of an energy density). The usefulness of these definitions lies in the fact that they simplify appreciably the first post-Newtonian approximation (this was pointed out in Section II of \[50\]), and therefore, as we shall see, the subsequent approximations built on it. See the lecture of T. Damour in this volume for the derivation of the 1PN approximation using the source densities (42), and for applications to the relativistic N-body problem.

From the densities (42) we introduce next the following retarded potentials

$$V = -4\pi G \Box^{-1}_R \sigma,$$
$$V_i = -4\pi G \Box^{-1}_R \sigma_i,$$
$$W_{ij} = -4\pi G \Box^{-1}_R \left[ \sigma_{ij} + \frac{1}{4\pi G} \left( \partial_i V \partial_j V - \frac{1}{2} \delta_{ij} \partial_k V \partial_k V \right) \right],$$  \hspace{1cm} (43)

where $\Box^{-1}_R$ is the flat retarded d’Alembertian operator \[12\]. The scalar and vectorial potentials $V$ and $V_i$ reduce in the limit $\varepsilon \to 0$ to the Newtonian potential $U$ and the usual “gravitomagnetic” potential $U_i$. The tensorial potential $W_{ij}$ is more complicated but is generated in the limit $\varepsilon \to 0$ by the sum of the matter stress density in the source and the stresses associated with the Newtonian gravitational field itself (the gravitational stresses are of the same order as the matter stresses when $\varepsilon \to 0$). The sources of the potentials $V$ and $V_i$ are of compact support, while the source of $W_{ij}$ is non compact. Thanks to the equations of continuity and of motion for the source densities (42) these potentials satisfy some simple differential identities to Newtonian order: $\partial_t V + \partial_i V_i = O(\varepsilon^2)$ and $\partial_t V_i + \partial_j W_{ij} = O(\varepsilon^2)$. Beware of the fact that these potentials are different from the external potentials introduced in (13)-(14).

To Newtonian order one has $h^{00} = -4V/c^2 + O(\varepsilon^4)$, $h^{0i} = O(\varepsilon^3)$ and $h^{ij} = O(\varepsilon^4)$, that we can substitute into the right-hand-side of the wave equation in (41), where with this approximation the nonlinear term $\Lambda^{\mu\nu}(h)$ can be replaced by its quadratic nonlinear piece given by (3). Applying $\Box^{-1}_R$ to the result...
(without any finite part of course because the source is regular in \( D_i \)), and neglecting consistently all the small terms, one obtains the first post-Newtonian approximation in the form [54]

\[
\begin{align*}
    h^{00}_{\text{in}} &= -\frac{4}{c^2} V + \frac{4}{c^4} (W_{ii} - 2V^2) + O(\varepsilon^6), \\
    h^{0i}_{\text{in}} &= -\frac{4}{c^3} V_i + O(\varepsilon^5), \\
    h^{ij}_{\text{in}} &= -\frac{4}{c^4} W_{ij} + O(\varepsilon^6).
\end{align*}
\]

Actually this approximation is slightly more accurate than the 1PN approximation because it takes into account the terms of order \( \varepsilon^4 \) not only in the 00 component of the metric, but also in its \( ij \) components. We indicate by the subscript “in” that the metric is the inner metric valid in \( D_i \), which differs by a coordinate transformation from the exterior metric of Section 3 (see below). Substituting the expressions (44) back into the right-hand-side of the field equation in (41), where now the nonlinear term \( \Lambda^{\mu\nu}(h) \) should include besides the quadratically nonlinear piece (9) a cubically nonlinear piece (given in Section I.B of [54]), neglecting consistently the higher order terms and then inverting the result by means of \( \Box^{-1} \), one gets

\[
\begin{align*}
    h^{\mu\nu}_{\text{in}} &= \Box^{-1} \left[ \frac{16\pi G}{c^4} T^{\mu\nu} + \bar{\Lambda}^{\mu\nu}(V, W) \right] + O(\varepsilon^8, \varepsilon^7, \varepsilon^8),
\end{align*}
\]

where the remainder means that the neglected terms are \( O(\varepsilon^8) \), \( O(\varepsilon^7) \) and \( O(\varepsilon^8) \) in the components \( h^{00}_{\text{in}}, h^{0i}_{\text{in}} \) and \( h^{ij}_{\text{in}} \), respectively. The quantities \( T^{\mu\nu} \) and \( \bar{\Lambda}^{\mu\nu}(V, W) \) are given by the corresponding quantities in the right-hand-side of (41), but where one retains only the terms in the post-Newtonian expansion up to the accuracy of the remainder in (44). These are given as explicit combinations of derivatives of \( V \), \( V_i \) and \( W_{ij} \). For instance the effective nonlinear source \( \bar{\Lambda}^{\mu\nu}(V, W) \) reads

\[
\begin{align*}
    \bar{\Lambda}^{00}(V, W) &= -\frac{14}{c^4} \partial_k V \partial_k V + \frac{16}{c^5} \left\{-V \partial_i^2 V - 2V_k \partial_i \partial_k V + W_{km} \partial_{km} V + \frac{5}{8} (\partial_i V)^2 + \frac{1}{2} \partial_k V_m (\partial_i V_m + 3\partial_m V_k) \right. \\
    &\quad \left. + \partial_k V \partial_i V_k + 2\partial_k V \partial_k W_{mm} - \frac{7}{2} V \partial_k V \partial_k V \right\}, \\
    \bar{\Lambda}^{0i}(V, W) &= \frac{16}{c^5} \left\{ \partial_k V (\partial_i V_k - \partial_k V_i) + \frac{3}{4} \partial_i V \partial_i V \right\}, \\
    \bar{\Lambda}^{ij}(V, W) &= \frac{4}{c^6} \left\{ \partial_i V \partial_j V - \frac{1}{2} \delta_{ij} \partial_k V \partial_k V \right\} + \frac{16}{c^6} \left\{ 2\partial_{(i} V \partial_{j)} V_j - \partial_i V_k \partial_j V_k - \partial_i V_k \partial_j V_k + 2\partial_{(i} V_k \partial_{j)} V_{k} - \frac{3}{8} \delta_{ij} (\partial_i V)^2 - \delta_{ij} \partial_k V \partial_i V_k \right\}.
\end{align*}
\]
$$\left\{ V_m, \frac{1}{2}\delta_{ij}\partial_k V_m(\partial_k V_m - \partial_m V_k) \right\}.$$ \hspace{1cm} (46)

See the section II.A of [54] for the derivation of these formulas.

4.2. The multipole moments as integrals over the source

The expressions of the multipole moments $M_L$ and $S_L$ will follow from the comparison between the inner metric (45)-(46) valid in $D_i$ and the re-expansion when $\varepsilon \to 0$ of the exterior metric valid in $D_e$. The comparison (and in fact matching) is done in the overlapping region $D_e \cap D_i$ which always exists for slowly-moving sources. This “matching” region is where one can simultaneously expand the field when $\varepsilon \to 0$ and decompose it into multipole moments. The matching requirement is simply that the two inner and outer post-Newtonian asymptotic expansions should be (term by term) identical in $D_e \cap D_i$, after the performing of a suitable coordinate transformation. Let this coordinate transformation be

$$x^\mu_{\text{ext}} = x^\mu_{\text{in}} + \varphi^\mu(x_{\text{in}}),$$ \hspace{1cm} (47)

where $x^\mu_{\text{ext}}$ and $x^\mu_{\text{in}}$ denote the coordinates which were used in $D_e$ and $D_i$ respectively (with a slight change of notation with respect to Section 3).

Recall that the exterior metric is constructed starting with the linearized metric $h_{\mu\nu}^{\text{lin}}$ whose components are defined by some external potentials $V_{\text{ext}}, V_{\text{ext}}^i$ and $V_{\text{ext}}^{ij}$ given explicitly as some infinite multipole expansions parametrized by $M_L$ and $S_L$ (see (13)-(14)). The problem is to relate via the matching requirement (47) these exterior potentials to the inner potentials $V, V_i$ and $W_{ij}$ used in $D_i$. In fact we do so first with some intermediate accuracy. The method is to look for a numerical equality in $D_i \cap D_e$ between exterior and inner potentials, and then transform this equality into a matching equation (i.e. an equation relating two mathematical expressions of the same nature) by replacing the inner potentials by their multipole expansions in $D_e$. The matching equation is valid formally “everywhere”, and it can be used to get the functional relationships linking the multipole moments to the source.

For instance it is found that the matching equations relating the potentials $V_{\text{ext}}$ and $V_{\text{ext}}^i$ to the multipole expansions $\mathcal{M}(V)$ and $\mathcal{M}(V_i)$ of the compact-support inner potentials $V$ and $V_i$ (where $\mathcal{M}$ refers to the multipole expansion) read as

$$V_{\text{ext}} = \mathcal{M}(V) + c\partial_t \varphi^0 + O(\varepsilon^4),$$

$$V_{\text{ext}}^i = \mathcal{M}(V_i) - \frac{e^3}{4} \partial_t \varphi^0 + O(\varepsilon^2),$$ \hspace{1cm} (48)

where $\varphi^0$ denotes the zero component of the gauge transformation vector in (47). The equation for $V_{\text{ext}}$ is valid to first post-Newtonian order, but the equation for $V_{\text{ext}}^i$ is valid only to Newtonian order (it can be proved that $\varphi^0$ is of order $\varepsilon^3$). The matching equations (48) were obtained in Section II of [50].
The multipole expansion of a retarded potential with \textit{compact-support source} is known from the work of Campbell, Macek and Morgan [120]. It has been re-calculated in the appendix B of [50] using STF spherical harmonics. Using this formula Damour and Iyer [58] improved much the work [120] dealing with the case of linearized gravity. The result (with obvious notation for the index \(\mu\)) is

\[
M(\mathcal{V}_\mu) = G \sum_{\ell \geq 0} \frac{(-1)^\ell}{\ell!} \partial_\ell \left[ \frac{1}{r} \mathcal{V}_\mu^L \left( t - \frac{r}{c} \right) \right],
\]

where the multipole moments which parametrize the expansion are given by

\[
\mathcal{V}_\mu^L(t) = \int d^3\hat{x}_L \int_{-1}^{1} dz \, \delta_\ell(z) \sigma_\mu(x, t + z|x|/c).
\]

We denote by \(\hat{x}_L\) the STF part of the product of \(\ell\) spatial positions \(x_L = x_{i_1} \cdots x_{i_\ell}\). The function \(\delta_\ell(z)\) takes into account the physical delays due to the propagation of the waves inside the compact-support source. It is given by

\[
\delta_\ell(z) = \frac{(2\ell + 1)!!}{2^{\ell+1}\ell!} (1 - z^2)^\ell; \quad \int_{-1}^{1} dz \, \delta_\ell(z) = 1
\]

\((\delta_\ell\text{ can be simply related via the Rodrigues formula to the Legendre polynomial of order } \ell)\). With the two matching equations (48) and the explicit formulas (50)-(51), it is a simple question of algebraic manipulation (using STF techniques), and analytic post-Newtonian expansion (using the formula (B.14) of [50]), to deduce the multipole moments \(M_L\) and \(S_L\) entering the left-hand-sides of (48). In particular the mass moment \(M_L\) is obtained with post-Newtonian accuracy as

\[
M_L(t) = \int d^3x \left\{ \hat{x}_L \sigma + \frac{|x|^2 \hat{x}_L}{2c^2(2\ell + 3)} \partial_t^2 \sigma - \frac{4(2\ell + 1)\hat{x}_{i_L}}{c^2(\ell + 1)(2\ell + 3)} \partial_t \sigma_i \right\} + O(\varepsilon^4).
\]

This expression generalizes to 1PN order the result (15) of the Newtonian formalism. The striking feature about this expression is that the integrand has a compact support, although it includes the contribution of the gravitational field to 1PN order, which is expected to extend over the whole three-dimensional space. However, the gravitational contribution to 1PN order turns out to be entirely contained into the mass density \(\sigma\). As the expression (52) has been derived rigorously within the post-Newtonian framework, it can also be said (when combined with the previous results in \(D_w\)) to bring a satisfactory proof of the lowest-order quadrupole formula itself.

To the next order (2PN) things are more delicate precisely because of the explicit contributions of the gravitational field which come into play, making the expressions of the mass multipole moments a priori non compact. This problem has been solved in [54], where it was found necessary to obtain first the expression of the multipole expansion \(\mathcal{M}(\mathcal{W}_{ij})\) of the inner potential \(\mathcal{W}_{ij}\)
defined in (43), and whose source we recall is non-compact. Thus the formulas (50)-(51) cannot be employed in this case, but instead we have obtained (in Section III.C of [54])

\[
\mathcal{M}(W_{ij}) = G \sum_{\ell \geq 0} \frac{(-1)^\ell}{\ell!} \partial L \left[ \frac{1}{r} W_{ij}^L \left(t - \frac{r}{c}\right) \right] + \text{FP}_{B=0} \left[ r^B \left( -\partial_i \mathcal{M}(V) \partial_j \mathcal{M}(V) + \frac{1}{2} \delta_{ij} \partial_k \mathcal{M}(V) \partial_k \mathcal{M}(V) \right) \right],
\]

(53)

where the multipole moments are given by

\[
W_{ij}^L(t) = \text{FP}_{B=0} \int d^3x \frac{x_i}{|x|} \hat{x}_L \times \int_{-1}^1 dz \delta(z) \left[ \sigma_{ij} + \frac{1}{4\pi G} \left( \partial_i V \partial_j V - \frac{1}{2} \delta_{ij} \partial_k V \partial_k V \right) \right] (x, t + z|x|/c).
\]

(54)

The first term in (53) together with the moments (54) look like exactly what we would obtain by using blindly the formulas (50)-(51), even though the potential is non compact. However there is an important difference which is the presence in the moments (54) of the finite part procedure which was used in the definition of the exterior metric (see (17)). The role of this finite part is to deal with the infinite support of the integral (54), which would be divergent otherwise (because of the presence in the integrand of \( \hat{x}_L \) behaving like \(|x|^{\ell}\) at spatial infinity). Notice that the finite part is conveyed all the way through the analysis, and is seen \textit{a posteriori} to be in fact necessary to ensure the convergence of the integrals. Thus no \textit{ad hoc} prescription has been used to obtain the moments (54) in a well-defined form mathematically, even in the case of a non-compact supported source (see the proof in [54]). As for the second term in (53), it ensures as is also seen a posteriori that \( \mathcal{M}(W_{ij}) \) satisfies the correct wave equation (deduced from (43)) outside the source. This second term involves the multipole expansion \( \mathcal{M}(V) \) of \( V \) in conformity with the retarded integral operator \( \text{FP}_{B=0} \mathcal{R}^{-1} r^B \) which is defined only when acting on multipole expansions.

With this sub-problem of obtaining \( \mathcal{M}(W_{ij}) \) solved we can write down similarly to (48) a matching equation relating the external potential \( V_{ij}^{\text{ext}} \) to the multipole expansion \( \mathcal{M}(W_{ij}) \). Together with (48) this permits us to relate the nonlinear source in the exterior field, which to this post-Newtonian order is shown to be exactly \( \tilde{\mathcal{M}}_{\mu\nu}(V^{\text{ext}}, W^{\text{ext}}) \), where \( \tilde{\mathcal{M}}_{\mu\nu} \) is given by (46) and where \( W_{ij}^{\text{ext}} \) is defined from \( V_{ij}^{\text{ext}} \), to the multipole expansion of the nonlinear source, that is \( \tilde{\mathcal{M}}_{\mu\nu}(\mathcal{M}(V), \mathcal{M}(W)) \), modulo the terms associated with the coordinate transformation. This relation being established one can write a matching equation valid to higher post-Newtonian order. The nonlinear part of the coordinate
transformation is found to cancel out, so that it remains simply a linear gauge transformation associated with the vector $\varphi^\mu$. Finally a reasoning similar to the one followed to obtain (53)-(54) leads to the linear piece $h_1^{\mu\nu}$ of the external field (which is a functional of the algorithmic moments $M_L$ and $S_L$) in the form of the explicit multipole expansion

$$Gh_1^{\mu\nu}[M_L, S_L] = -\frac{4G}{c^4} \sum_{\ell \geq 0} \frac{(-)^\ell}{\ell!} \partial_L \left[ \frac{1}{r} F^{\mu\nu}_L \left( t - \frac{r}{c} \right) \right] + \partial \varphi^{\mu\nu} + O(\varepsilon^7), \quad (55)$$

where $\partial \varphi^{\mu\nu} = \partial^\mu \varphi^\nu + \partial^\nu \varphi^\mu - \eta^{\mu\nu} \partial^\lambda \varphi^\lambda$ is the linear gauge transformation, which depends on some (reducible) multipole moments given by

$$F^{\mu\nu}_L(t) = FP_{B=0} \int d^3x|x|^B \hat{x}_L \int_{-1}^1 dz \delta_\ell(z) \tau^{\mu\nu}(x, t + z|x|/c). \quad (56)$$

The integrand of these multipole moments is nothing but the total stress-energy momentum of the matter fields and the gravitational field (with 2PN accuracy), namely

$$\tau^{\mu\nu} = \bar{\tau}^{\mu\nu} + \frac{c^4}{16\pi G} \bar{\Lambda}^{\mu\nu}(V, W), \quad (57)$$

which is given, thanks (in particular) to (46), as a fully explicit functional of the inner potentials $V, V_i$ and $W_{ij}$. Like in (54) there is a Finite Part symbol in front of the integral which is seen (a posteriori) to make the non-compact-support multipole moments to be mathematically well-defined.

The result (55)-(57) (proved in Section III.D of [54]) is satisfying because one recovers finally the expression of the total pseudotensor of the matter and gravitational fields in the source, which is exactly the one which appears in the right-hand-side of the Einstein field equation (41), except that here it is expanded when $\varepsilon \to 0$ up to 2PN order. Notice that although the range of integration extends up to infinity, it is the post-Newtonian expansion valid in $D_i$ of the pseudotensor which is to be used in (56). This is thanks to the finite part procedure and the properties of analytic continuation (see [54] for the reason why this works).

The form of the result (55)-(57) is therefore identical to what we would obtain by writing from the field equation (41) the solution directly as the retarded integral $\square_R^{-1}$ of the total pseudotensor (this operation is perfectly licit), but then by writing incorrectly the multipole expansion of that solution applying the formulas (49)-(51). This second operation is illicit because the formulas apply only to compact-supported sources. As a result the multipole moments would be given by divergent integrals. This was the approach initially followed by Epstein and Wagoner [38] to 1PN order, and then generalized formally by Thorne [46] to all post-Newtonian orders. The error is not too severe to 1PN order because one actually recovers the correct result by discarding the infinite surface terms in the Epstein-Wagoner moments. This has been proved using the compact-support formula (52) (see the appendix A of [50]), and this has
also been checked in applications to binary systems \[59\]. However in higher-orders the neglect of infinite surface terms in the Epstein-Wagoner moments becomes physically uncorrect, because it is then equivalent to the neglect of physical effects in the waveform, namely the tail and other nonlinear effects computed in Section 3.4. This is clear from the fact that by (55)-(57) one sees that the multipole moments which are given by (the finite part of) integrals over the total stress-energy pseudotensor are simply the moments \( M_\ell \) and \( S_\ell \), which differ from the observable moments \( U_\ell \) and \( V_\ell \) in the wave field \( \mathcal{F}_\ell \) by all the nonlinear effects in \( D_w \).

Recently Will and Wiseman \[64\] have succeeded in curing the defects of the Epstein-Wagoner approach. As a result they obtain a manifestly convergent and finite procedure for calculating the gravitational radiation from isolated systems to post-Newtonian order.

It still remains to compute from (55)-(57) the (irreducible) moments \( M_\ell \) and \( S_\ell \), which means to decompose in the right-hand-side of (55) the moments \( F_\mu\nu^\ell \) into irreducible STF pieces. Apart from the Finite Part symbol in front of the moments and the post-Newtonian remainder in (55), this technical problem is the same as in the case of linearized gravity in which case it has been solved by Damour and Iyer \[58\]. The mass multipole moment \( M_\ell \) to 2PN order is therefore deduced, and, after further transformations done in Section IV.B of \[54\], we end up with

\[
M_\ell(t) = \text{FP}_{B=0} \int d^3x |x|^B \left\{ \hat{x}_L \left[ \sigma + \frac{4}{c^4} (\sigma_{\mu\nu} \Upsilon - \sigma P_{\mu\nu}) \right] \right. \\
+ \frac{|x|^2 \hat{x}_L}{2c^2(2\ell + 3)} \frac{\partial^2 \sigma}{\partial t^2} - \frac{4(2\ell + 1)\hat{x}_i \hat{x}_L}{c^2(2\ell + 1)(2\ell + 3)} \frac{\partial_t}{\partial t} \left[ \left( 1 + \frac{4U}{c^2} \right) \sigma_i \right] \\
+ \frac{1}{\pi Gc^2} \left[ \partial_k U [\partial_i U_k - \partial_k U_i] + \frac{3}{4} \partial_i U \partial_t U \right] \\
+ \frac{|x|^4 \hat{x}_L}{8c^4(2\ell + 3)(2\ell + 5)} \frac{\partial^4 \sigma_i}{\partial t^4} - \frac{2(2\ell + 1)|x|^2 \hat{x}_i \hat{x}_L}{c^4(2\ell + 1)(2\ell + 3)(2\ell + 5)} \frac{\partial_t^2 \sigma_i}{\partial t^2} \\
+ \frac{2(2\ell + 1)}{c^4(2\ell + 1)(2\ell + 2)(2\ell + 5)} \hat{x}_\ell \hat{x}_L \partial_t^2 \left[ \sigma_{ij} + \frac{1}{4\pi G} \partial_i U \partial_j U \right] \\
+ \frac{1}{\pi Gc^4} \hat{x}_L \left[ -P_{ij} \frac{\partial^2 \sigma_j}{\partial U^2} - 2U_i \partial_i \partial_j U + 2\partial_i U_j \partial_j U_i \\
- \frac{3}{2} (\partial_t U)^2 - U \partial_t^2 U \right] \right\} + O(\varepsilon^5). \tag{58}
\]

The potentials \( U, U_i \) and \( P_{ij} \) are the Newtonian-like potentials to which tend the potentials \( V, V_i \) and \( W_{ij} \) in the limit \( \varepsilon \to 0 \). As we see there are many explicit contributions to 2PN order of the non-compact-supported distribution of the gravitational field. The expression (58) has been generalized recently to include the next 2.5PN correction \[56\]. Similarly one deduces also the expression
of the current multipole moment $S_L$, but with 1PN accuracy only:

$$S_L(t) = \text{FP}_{B=0} \varepsilon_{abc}^{\ell} \int d^3|x| |x|^b \left\{ \hat{x}_{L-1}^{\ell+a} \left( 1 + \frac{4}{c^2} U \right) \sigma_b + \frac{|x|^2 \hat{x}_{L-1}^{\ell+a}}{2c^2(2\ell + 3)} \partial_t^2 \sigma_b \right. $$

$$+ \frac{1}{\pi G c^2} \hat{x}_{L-1}^{\ell+a} \left[ \partial_k U \partial_b U \partial_k - \partial_k U \partial_b U \right] $$

$$- \frac{(2\ell + 1)\hat{x}_{L-1}^{\ell+ac} \partial_t}{c^2(\ell + 2)(2\ell + 3)} \partial_t \left[ \sigma_b c + \frac{1}{4\pi G} \partial_b U \partial_c U \right] \} + O(\varepsilon^4) . \tag{59}$$

The first correct expression for the 1PN current multipole moment was in fact obtained earlier by Damour and Iyer [51], making extensive use of distribu-
tional kernels to deal with the quadratic nonlinearities of the gravitational
field. Their result has the advantage over (59) to explicitly express the 1PN
current moments $S_L$ in terms of compact-support integrals. Its extension to
higher orders would require to generalize the quadratic kernels they introduce
to cubic and higher kernels. [In a sense this generalization has been per-
formed in Section III.C of [61], where the result (58) has been transformed in an ex-
plicitly compact-support integral.] See Section IV.B of [54] for the com-
plete equivalence of (59) with the result of [51].

5. GRAVITATIONAL RADIATION REACTION EFFECTS

On physical grounds one expects that it should be possible to any post-Newton-
ian order to interpret the results concerning the outgoing radiation field in $D_w$
(and the energy, linear momentum and angular momentum it carries) as due
to some radiation reaction force acting locally inside the source (in $D_i$). Indeed
this is known to be correct at the Newtonian level [16], [24]–[30]. In Sec-
tion 2.2 is reviewed the Burke and Thorne [20]–[23] expression for the reaction
force at this level. Other expressions of the force in other coordinate sys-
tems were obtained by several authors, notably Chandrasekhar and Espos-
it [16] (see [85] for a discussion). Recall that the (Newtonian) radiation reaction
force is directly responsible for the observed acceleration of the orbital mo-
tion of the binary pulsar PSR 1913+16 [12, 35].

Unfortunately the problem of radiation reaction onto the source is not so
well understood as the problem of the generation of radiation in the regions far
from the source (sections 3-4). Here we shall report the results of the extension
to first post-Newtonian order (and even 1.5PN order) of the lowest-order Burke
and Thorne radiation reaction force (6). These results confirm to this order the
physical expectation that all effects of the reaction force are contained in the
fluxes of energy and momenta carried out by the waves at infinity, but clearly
some more work should be done on this problem.

The novel feature when one goes from the Newtonian reaction force (6) to
the first post-Newtonian one is that the reaction potential is no longer com-
poised of a single scalar (depending on the mass-type multipole moments), but
involves also a vectorial component depending on the variations of the current-type multipole moments. The vectorial component of the reaction could be important in some astrophysical situations like rotating neutron stars undergoing gravitational instabilities. It was first noticed in the physically restricting case where the dominant quadrupolar radiation from the source is suppressed [12],

From the general case investigated in [53, 55] it follows in particular that the expressions of the scalar and vectorial reactive potentials which generalize to 1PN order the Burke-Thorne potential (6) are given by

\[ V_{\text{react}}(\mathbf{x}, t) = -\frac{G}{c^5} x_{ij} M^{(5)}_{ij}(t) - \frac{G}{c^7} \left[ \frac{1}{189} x_{ijk} M^{(7)}_{ijk}(t) - \frac{1}{70} |\mathbf{x}|^2 x_{ij} M^{(7)}_{ij}(t) \right] + O(\varepsilon^8), \]

\[ V_{\text{react}}^i(\mathbf{x}, t) = \frac{G}{c^5} \left[ \frac{1}{21} x_{ijk} M^{(6)}_{jk}(t) - \frac{4}{45} \varepsilon_{ijk} x_{jm} S^{(5)}_{km}(t) \right] + O(\varepsilon^6). \]  

One recognizes in the first term of \( V_{\text{react}} \) the Burke-Thorne scalar potential for the mass quadrupole moment \( M_{ij} \). The vector potential \( V_{\text{react}}^i \) depends notably on the current quadrupole moment \( S_{ij} \). It is crucial that \( M_{ij} \) in the Burke-Thorne term should be given with relative 1PN precision in order to be consistent with the accuracy of the approximation. The correct expression for that moment is (52) above (taken for \( \ell = 2 \)). The other moments in (60) take their usual Newtonian forms at this approximation. Notice that the expressions (60) are valid within the full nonlinear theory. They are shown in [53] to arise from a specific time-asymmetric component of the nonlinear gravitational field in \( D_e \). The matching of these expressions to the field of the source in \( D_i \) is done in [53].

The energy, linear momentum and angular momentum which are extracted from the source by the 1PN radiation reaction force associated with the scalar and vectorial potentials (60) are to be computed locally (in \( D_i \)) using the equations of motion of the source. This computation has been done recently [54] with a result in perfect agreement with the corresponding 1PN fluxes of energy and momenta already known from their computations in the far zone \( D_w \) [46, 50]. Namely,

\[ \frac{dE}{dt} = -\frac{G}{c^5} \left\{ \frac{1}{5} M^{(3)}_{ij} M^{(3)}_{ij} + \frac{1}{c^2} \left\{ \frac{1}{189} M^{(4)}_{ijk} M^{(4)}_{ijk} + \frac{16}{45} S^{(3)}_{ij} S^{(3)}_{ij} \right\} \right\} + O(\varepsilon^8), \]

\[ \frac{dP_i}{dt} = -\frac{G}{c^7} \left\{ \frac{2}{63} M^{(4)}_{ijk} M^{(3)}_{jk} + \frac{16}{45} \varepsilon_{ijk} M^{(3)}_{jm} S^{(3)}_{km} \right\} + O(\varepsilon^9), \]

\[ \frac{dS_i}{dt} = -\frac{G}{c^5} \varepsilon_{ijk} \left\{ \frac{2}{5} M^{(2)}_{jm} M^{(3)}_{km} + \frac{1}{c^2} \left\{ \frac{1}{63} M^{(3)}_{jmn} M^{(4)}_{kmn} + \frac{32}{45} S^{(2)}_{jm} S^{(3)}_{km} \right\} \right\} + O(\varepsilon^8), \]  

where \( E, P_i \) and \( S_i \) denote some local (instantaneous) quantities which are
given by some compact-support integrals over the matter in the source, and agree to 1PN order with the usual notions of energy, linear momentum and angular momentum of the source. These formulas show the validity to 1PN order of the (generally postulated) balance equations between the decreases of energy and momenta in the source and the corresponding fluxes at infinity. Of particular interest (notably because this is a purely 1PN effect) is the decrease of the linear momentum $dP_i/dt$, which corresponds to a net recoil of the center of mass of the source in reaction to the emission of waves. Numerous authors \cite{10,122,123,46} had computed before this effect as a flux of linear momentum at infinity.

In the case of binary systems, Iyer and Will \cite{124,125} have obtained, assuming the validity of the balance equations for energy and angular momentum, the radiation reaction force in the equations of motion of the binary to 1PN relative order (which corresponds to the 3.5PN order beyond the Newtonian acceleration because the reaction effects arise at 2.5PN order). Their result is valid for a large class of coordinate systems. They also show \cite{125} that the reactive potentials (60) when specialized to binary systems correspond in their formalism to a unique and consistent choice of a coordinate system (namely, a “Burke-Thorne-extended” coordinate system). This represents a non-trivial check of the validity of the reactive potentials (60).

To the next 1.5PN order in the reactive potentials there appears the same phenomenon as in the radiation field in $D_w$, namely the occurrence of hereditary effects associated with tails. This is required if the balance equation for the energy is to remain valid at this order. That this is indeed the case has been proved in \cite{13} where the dominant hereditary contribution in the post-Newtonian expansion of the equations of motion of the source (in $D_i$) was shown to arise at the 4PN order beyond the Newtonian acceleration, that is precisely at the 1.5PN relative order in the reaction force. Therefore, one can say that locality in time is lost at the 4PN order (or order $\varepsilon^8$) in the equations of motion of a self-gravitating source. This hereditary contribution can be shown in a particular gauge to modify the scalar reactive potential in (60) by the tail integral \cite{49}

$$\delta V_{\text{reac}} = -\frac{G^2}{5c^8} x_{ij} \int_0^{\infty} dy \left[ \ln \left( \frac{y}{2b} \right) + \frac{11}{12} \right] M^{(7)}_{ij}(t-y). \quad (62)$$

This tail contribution in the reaction is recovered in Section III.D of \cite{52} using a more systematic method, and the constant $11/12$ is computed.

The following picture can be given to this phenomenon. The waves emitted by the source at all epochs in the past are scattered onto the spacetime curvature generated by the mass $M$. The secondary waves which are produced converge back onto the source at our epoch and modify its present dynamics. The total reaction force is thus the vectorial sum of the reaction force due to the present emitted radiation (this is (60)), and of the reaction force due to the incoming tail radiation (see (62)). It is shown in Section III.D of \cite{52} that the tail-induced reactive potential $\delta V_{\text{reac}}$ is consistent with the tail integral in
in the sense that it gives rise to a term of order $\varepsilon^8$ in the energy balance equation in (61) which is in perfect agreement with the contribution of the tail integral to the flux of energy in $D_w$.

6. APPLICATION TO INSPIRALLING COMPACT BINARIES

Inspiralling compact binaries are very relativistic (but not fully relativistic) binary systems of compact objects (neutron stars or black holes) in their late phase of evolution which precedes immediately the final coalescence, during which the two objects spiral very rapidly around each other under the influence of the gravitational radiation reaction forces.

It has been recognized in recent years that a very precise prediction from general relativity, taking into account many relativistic (or post-Newtonian) corrections, will be necessary in order that all the potential information contained in the signals from inspiralling binaries can be deciphered by the future detectors LIGO and VIRGO (126–133). The demanded precision can be calculated using black-hole perturbation techniques in the special case where the mass of one object is very small as compared with the other mass (134–140). Thus, inspiralling compact binaries offer us the ideal application of the previous relativistic formalism — and constitute presently our main motivation for its development.

The information contained in the signals of inspiralling compact binaries will be extracted from the noisy output of the detector by a technique of matched filtering, necessitating the a priori knowledge of the signal. Crucial to a successful data analysis will be the knowledge of the instantaneous phase of the binary, namely the angle between the separation of the two bodies and a fixed direction in the orbital plane.

The time evolution of the orbital phase should in principle be determined from the gravitational radiation reaction forces acting locally on the orbit. However, as these forces are only known to 1PN order (124, 125, 53, 55) and 1.5PN order (49), one relies in practice on an energy balance equation similar to the one in (61), namely

$$\frac{dE}{dt} = -\mathcal{L},$$

where in both sides one uses some very precise expressions of the (binding) energy $E$ of the binary and its total luminosity $\mathcal{L}$ in gravitational waves (the angular momentum balance equation is not necessary because the orbit has been circularized by the radiation reaction forces). The fact that one is obliged presently to postulate the validity of the energy balance equation (63) to higher order than 1.5PN (for instance 2.5PN as postulated below) is certainly a weak point in the analysis, which shall have to be improved in future work.

The binding energy $E$ of the binary which appears in the left-hand-side of (63) results from the equations of motion of the binary which have been
obtained to 2.5PN order by Damour and Deruelle \cite{31,34,35} in their study of the binary pulsar PSR 1913+16. For inspiralling compact binaries one simply needs to specialize these equations to the case where the orbit is circular (apart from the gradual inspiral). The 2.5PN–accurate binding energy reads

$$E = -\frac{c^2}{2} M \nu x \left\{ 1 - \frac{1}{12} (9 + \nu) x - \frac{1}{8} \left( 27 - 19 \nu + \frac{\nu^2}{3} \right) x^2 + O(x^3) \right\} . \quad (64)$$

(See also the appendix B of \cite{56} for a derivation for this formula from the Damour-Deruelle equations of motion.) The total mass of the binary is $M = m_1 + m_2$, and $\nu = m_1 m_2 / M^2$ represents a particular mass ratio which is such that $0 < \nu \leq 1/4$, with $\nu = 1/4$ corresponding to the case of two equal masses and $\nu \to 0$ corresponding to the test-mass limit for one body. For convenience we denote by $x = (GM \omega / c^3)^{2/3}$ a small dimensionless post-Newtonian parameter of order $\varepsilon^2$, where $\omega$ is the instantaneous angular frequency of the orbit ($\omega = 2\pi / P$, where $P$ is the orbital period).

On the other hand, the 2.5PN–accurate luminosity $\mathcal{L}$ of the binary is obtained by application of the formalism above. Namely, one uses the decomposition of $\mathcal{L}$ in terms of observable moments $U_L$ and $V_L$ as given by (25). Then one relates the observable moments to the moments $M_L$ and $S_L$ by the equations (36)-(40). Actually to 2.5PN order the mass quadrupole moment comprises the dominant tail integral and the quadrupole-quadrupole interaction terms which include the nonlinear memory integral, but the tail of tail integral appearing in (39) is negligible. Nonlinear effects are rather easy to compute for binary systems thanks to the study done in \cite{60}, where the required formula is proved to apply to a binary which is actually spiralling (or decaying). Indeed this is not obvious a priori because we are considering hereditary effects which depend on the whole integrated past of the binary (see the appendix B in \cite{64}). In a second stage one must use the expressions of the moments $M_L$ and $S_L$ as integrals over the source. These are given by (58) and (59) (to 2.5PN order the next-order term in $M_L$ is a priori also needed, but in fact this term is instantaneous and does not contribute in the energy loss to this order, see below). A hard task, carried out to 2PN order in \cite{61} and \cite{64}, is to reduce all these integrals for compact binary systems. This entails using delta functions to describe the compact objects (but this can probably be justified using the results of \cite{31,34}). Notably the main cubically nonlinear term in (58) (term with $P_{ij} \partial_{ij} U$) deserves a special treatment. This term has been computed by two different methods in \cite{61} (see Section III.C and Appendix B there). The final result for $\mathcal{L}$ reads

$$\mathcal{L} = \frac{32 c^5}{5G} \nu^2 x^5 \left\{ 1 - \left( \frac{1247}{336} + \frac{35}{12} \nu \right) \frac{x}{x^3/2} \right. \left. + \left( - \frac{44711}{9072} + \frac{9271}{504} \nu + \frac{651}{18} \nu^2 \right) x^2 \right\} .$$
This expression was already known to Newtonian order by Landau and Lifchitz [80, 81], and to 1PN order by Wagoner and Will [141] using the Epstein-Wagoner moments [38] (in which infinite surface terms are to be discarded). It has been re-calculated to 1PN order in [59] using the well-defined compact-support moment (52). The 1.5PN order is the $4\pi$ term which is the contribution of the dominant tail integral present in (38). The coefficient $4\pi$ is specific to binary systems moving in circular orbits, and was obtained first by Poisson [135] using a black-hole perturbation technique valid in the test-mass limit $\nu \to 0$ (this happens to give the answer, which does not depend on $\nu$, even for arbitrary mass ratios). The computation of the tail integral in (38) in the case of binary systems was done in [60, 66] with result yielding the $4\pi$ term. The 2PN term computed in [61] is entirely due to relativistic corrections in the moments of the source, most importantly the mass quadrupole moment (58) (issued from [54]). Will and Wiseman [64] computed independently this term using the Epstein-Wagoner moments which give the correct result, as proved in [54], because there are no tails at this order (provided that infinite surface terms are discarded). (But see [64] for improvements of the Epstein-Wagoner approach, and for how to compute tails within this approach.) The common result of these 2PN computations was summarized in [62], where it was also pointed out that the effects of intrinsic rotation of the two objects are in general negligible as compared to the gravitational effects. Finally the 2.5PN term has been added in [56] and shown to be due exclusively to the mass octupole and current quadrupole tail integrals in (40), and to the post-Newtonian corrections in the dominant mass quadrupole tail integral in (38) or (39). The fact that only the tail integrals give a contribution in $\mathcal{L}$ to 2.5PN order is special to the case of a circular orbit [56]. In the non-circular case the instantaneous terms would also contribute. Incidentally notice that the nonlinear memory integral is instantaneous in the energy loss, and therefore gives no contribution in (65). Thus the nonlinear memory effect has rather poor observational consequences for inspiralling binaries.

The tail of tail integral given by the last term in (39) gives a contribution to $\mathcal{L}$ at the first order which is neglected in (53), namely the order 3PN or $x^3$. This contribution to $\mathcal{L}$ is computed in [57] (it involves a logarithmic term in the frequency). However the whole 3PN term in (55) will include also many contributions coming from the relativistic corrections in the moments of the source. These contributions are not under control presently. The next 3.5PN order is expected like the 1.5PN and 2.5PN orders to be due entirely to higher-order multipolar tails, and to post-Newtonian corrections in low-order multipolar tails. The still higher 4PN order includes many effects and seems presently difficult to reach within the post-Newtonian theory.

There is a case, however, where $\mathcal{L}$ is known up to 4PN order. This is the test-mass limit case $\nu \to 0$, which has been computed by black-hole pertur-
bation techniques. Tagoshi and Nakamura [137] first obtained numerically the coefficients in \( L \) up to 4PN order. Then Sasaki [138] showed how to solve in an iterative way the Regge-Wheeler equation for the perturbations of the Schwarzschild black-hole, and finally, based on this, Tagoshi and Sasaki [139] obtained analytically the coefficients in \( L \) up to 4PN order. (See also [142, 143] for the case of a rotating black hole.) The result (65) of post-Newtonian theory is in perfect agreement in the limit \( \nu \to 0 \) with the truncation to 2.5PN order of the result of perturbation theory (tails of tails computed in [57] also perfectly agree). As the post-Newtonian and perturbation theories are so different, the agreement is very satisfying, although mandatory of course.

Finally it remains to substitute the 2.5PN expressions of both \( E \) and \( L \) into the energy balance equation (63) (assumed to be true at the 2.5PN level), in order to obtain the instantaneous phase \( \phi \) of the binary as a function of time. The phase is \( \phi = \int \omega dt \) where \( \omega \) is the orbital frequency, from which one deduces \( \phi = - \int (\omega/L)dE \). The result is simpler if we use instead of the local time \( t \) flowing in the observer’s frame (\( t = T - R/c \) with the notation of Section 3) the adimensional time variable [61]

\[
\Theta = \frac{c^3 \nu}{5GM}(t_c - t),
\]

where \( t_c \) denotes the instant of coalescence (at which the frequency goes formally to infinity). In terms of this time variable the orbital phase is obtained as

\[
\phi(t) = \phi_0 - \frac{1}{\nu} \left\{ \Theta^{5/8} + \left( \frac{3715}{8064} + \frac{55}{96} \nu \right) \Theta^{3/8} - \frac{3\pi}{4} \Theta^{1/4} \right. \\
\left. + \left( \frac{9275495}{14450688} + \frac{284875}{258048} \nu + \frac{1855}{2048} \nu^2 \right) \Theta^{1/8} \\
\left. - \left( \frac{38645}{172032} + \frac{15}{2048} \nu \right) \pi \ln \Theta + O(\Theta^{-1/8}) \right\},
\]

where \( \phi_0 \) is some constant phase determined by initial conditions (for instance when the signal enters the frequency bandwidth of the detector). This expression is valid only in the post-Newtonian regime where \( \Theta^{-1} = O(\epsilon^8) \).

Note that the determination of the phase (67) does not constitute by itself the complete answer to the problem because the waveform (described by two independent polarization states \( h_+ \) and \( h_\times \)) is also to be computed. Thus the phase \( \phi(t) \), and the orbital frequency \( \omega(t) = \dot{\phi}(t) \), have to be inserted into the waveform of the binary, which is itself to be known with the best possible post-Newtonian precision (though the latter precision does not need to be so high as in the determination of the phase; see, e.g., [131]). It is known [60] that the waveform introduces in the phase evolution some contributions which add to those coming purely from the energy balance equation (63), but these extra contributions arise at an order which is higher than 2.5PN. For the computation of the waveform to 2PN order see [63].
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