TOPOLOGICAL AND AFFINE STRUCTURE OF COMPLETE FLAT MANIFOLDS

MICHAŁ SADOWSKI

Abstract. The results of the paper concern the topological structure of complete riemannian manifolds with cyclic holonomy groups and low-dimensional orientable complete flat manifolds. We also discuss related results such as the affine classification of orientable complete flat 4-manifolds, an algebraic criterion of an affine equivalence and the relationship between holonomy homomorphisms and some algebraic and geometric invariants.

1. Introduction

The aim of this paper is to collect some results concerning topological and affine structure of complete flat manifolds (cf-manifolds). We recall particularly important known results in Section 2. The others seem to be new. Complete flat manifolds play a particular role in geometry. On one hand they are natural generalizations of euclidean spaces, having the same local properties. On the other hand the study of some complicated questions, arising in differential geometry and related fields, often starts with the examination of them in the case of manifolds of constant curvature. A cf-manifold is the orbit space $M = \tilde{M}/\Gamma$ of a properly discontinuous, isometric, and free action of a discrete group $\Gamma$ on an euclidean space $\tilde{M}$. The holonomy homomorphism of $M$ is the map $\Phi$ carrying $\gamma \in \Gamma$ onto linear part of $\gamma$ and $\Phi(\Gamma)$ is the holonomy group of $M$. The linear isometry $\Phi(\gamma)$ can be written as $\Phi_X(\gamma) \times \Phi_U(\gamma)$, where $\Phi_X(\gamma)$ acts on the universal covering space $\tilde{X}$ of the totally geodesic submanifold $X$ of $M$, homotopy equivalent to $M$, and $\Phi_U(\gamma)$ acts on the orthogonal complement of $\tilde{X}$. If $M$ is compact, then $M = X$ and $\Phi_X = \Phi$. In this case we say that $M$ is a Bieberbach manifold. If $M$ is noncompact, then $M$ is determined by the Bieberbach group $\Gamma$ and the vertical holonomy homomorphism $\Phi_U$ so that the theory of cf-manifolds can be treated as the theory of orthogonal representations of Bieberbach groups. The manifold $M$ is the total space of a flat riemannian vector bundle $\xi : M \to X$ whose structure group can be reduced to $\Phi_U(\Gamma)$. The main difference between the noncompact case and the compact one is that $\Phi(\Gamma)$ is not always finite and $\Phi(\Gamma)$ is not a topological invariant of $M$ (cf. Section 4 below).

The main difficulty in the classification of Bieberbach manifolds is that it is based on the classification of conjugacy classes of finite subgroups of $\text{GL}(n, \mathbb{Z})$. This is a hard problem in integral representation group theory, solved for cyclic groups of prime order $p$, for cyclic groups of order $p^2$, and for small values of $n$ only. Bieberbach manifolds are described when the holonomy groups of them are cyclic groups of prime order and when the dimensions of them are smaller than 7. The

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complexity of the problem is also connected with the fact that the number $\nu_f(n)$ of $n$-dimensional closed flat manifolds increases rapidly with $n$. It is known that $\nu_f(2) = 2$, $\nu_f(3) = 10$ ([29] Section 3.5), $\nu_f(4) = 74$, $\nu_f(5) = 1060$ ([5]), and $\nu_f(6) = 38746$ ([5]).

In this paper we study cf-manifolds in two particular important cases: when they have cyclic holonomy groups and when their dimensions are smaller than 5. These particular cases are starting points in the investigation of more general ones. In the affine classification it is convenient to use a criterion of affine equivalence formulated in Section 4. We also answer some natural questions concerning algebraic and geometric invariants that are used in the paper. Flat manifolds were investigated in many books and papers. Only few of them deal with the noncompact case (see e.g. [10], [29], [16], [22], [25], [26], [28]). Related results can be found in papers dealing with flat vector bundles, for instance in [11], [7], [9], [10], [12], and [15].

Throughout this paper the following notation will be used. The universal covering space of a topological space $Y$ will be denoted by $\tilde{Y}$. If $H$ is a group, $h_1, ..., h_k \in H$, then $(h_1, ..., h_k)$ will denote the subgroup of $H$ generated by $h_1, ..., h_k$. $\text{Diff}(M)$ is the group of diffeomorphisms of a smooth manifold $M$ and $\text{Aff}(V)$ is the group of affine diffeomorphisms of $V$. By $\text{Hol}(X)$ we denote the holonomy group of a riemannian manifold $X$. The symbol $\Pi_{ab}$ stands for projection of a Bieberbach group $\Gamma$ onto its abelianization $H_1(\Gamma)$ and $E_\xi$ for the total space of a vector bundle $\xi$. Real and complex $d$-dimensional trivial vector bundles will be denoted by $\Theta^d_{\mathbb{R}}$ and $\Theta^d_{\mathbb{C}}$ respectively.

2. Holonomy representations and vector bundles

The aim of the first part of this section is to reformulate some known results in a more convenient for our purposes form. The results will be expressed in terms of vector bundles associated with coverings. To describe these bundles consider a covering map $\Pi: \tilde{X} \to X$ and its covering transformation group $H$. Let $\mathbb{F}$ be the field of reals or the field of complex numbers, and let $\rho: H \to GL(s, \mathbb{F})$ be an $s$-dimensional representation of $H$. Take the diagonal action $h(x, u) = (hx, \rho(h)u)$ on $\tilde{X} \times \mathbb{F}^s$ and the orbit space $\Pi[\rho] = (\tilde{X} \times \mathbb{F}^s)/H$. Let $p: \Pi[\rho] \to X$ be the map determined by the projection $p_0: \tilde{X} \times \mathbb{F}^s \ni (x, u) \to x \in \tilde{X}$. Then the triple $(\Pi[\rho], X, p)$ is a vector bundle associated to the principal bundle $\Pi$ with typical fiber $\mathbb{F}^s$. In the sequel we identify this bundle with $\Pi[\rho]$. We often use the following.

**Proposition 2.1.** Let $M$ be a connected $m$-dimensional cf-manifold and let $X$ be a closed totally geodesic submanifold of $M$, homotopy equivalent to $M$. Denote $\Phi(\Gamma)$ by $H$, $\dim X$ by $n$, and $m - n$ by $s$. Then there are a riemannian covering $\Pi: \tilde{X} \to X$ and an orthogonal representation $\rho: H \to O(s)$ such that $M = \Pi[\rho]$. The projection $p: \Pi[\rho] \to X$ is affine. The action of $\gamma \in \Gamma$ on $\tilde{X}$ can be written as $(t_{\rho(\gamma)} \circ \Phi_M(\gamma)) \times \Phi_U(\gamma)$, where $\Phi_M(\gamma): \tilde{X} \to \tilde{X}$ and $\Phi_U(\gamma): \tilde{X}^\perp \to \tilde{X}^\perp$.

There are different proofs of Proposition 2.1. For instance it follows from the arguments used in the proof of Theorem 3.3.3 in [29, ch. 3]. The homomorphism $\Phi_U$ is the vertical holonomy homomorphism of $M$.

**Corollary 2.1.** If the holonomy group $H$ of $M$ is abelian, then the bundle $p: M \to X$ is isomorphic to

$$\bigoplus_j \Pi[\rho_j] \oplus \bigoplus_k \Pi[\lambda_k],$$
where $\rho_j$ are 1-dimensional complex representations of $H$ and $\lambda_k$ are 1-dimensional real representations of $H$.

Proposition 2.1 and the following observation show that cf-manifolds correspond to orthogonal representations of Bieberbach groups.

**Proposition 2.2.** If $X = \overline{X}/\Gamma$ is a Bieberbach manifold, $\rho : \Gamma \to O(s)$ is an orthogonal representation of $\Gamma$, and the action of $\gamma \in \Gamma$ on $\overline{\Gamma} \times \mathbb{R}^s$ is given by the formula

$$\gamma(x, u) = (\gamma x, \rho(\gamma)u)$$

then the orbit space $M(\rho) = (\overline{X} \times \mathbb{R}^s)/\Gamma$ is a complete flat manifold homotopy equivalent to $X$.

If $\rho' : \Gamma \to O(s)$ is another orthogonal representation of $\Gamma$, then it is natural to ask when $M(\rho)$ is diffeomorphic (affinely diffeomorphic) to $M(\rho')$. We consider this question in the next sections. In particular, we give an algebraic criterion of an affine equivalence.

The following three simple lemmas are known.

**Lemma 2.1.** (cf. [13, ch. 16, § 11]). If $Y$ is a topological space, $\xi : M \to Y$ is a real vector bundle and $\zeta : V \to Y$ is a complex vector bundle, then

$$(r \circ c)([\xi]_\mathbb{R}) = 2[\xi]_\mathbb{R} \quad \text{and} \quad (c \circ r)([\zeta]) = [\zeta \oplus \zeta^*].$$

The maps $r$ and $c$ carrying $[\zeta]_C \in KU(X)$ onto $[\zeta]_\mathbb{R} \in KO(X)$ and $[\zeta]_\mathbb{R} \in KO(X)$ onto $[\zeta \oplus \zeta]_C \in KU(X)$ are called the restriction and the complexification.

**Lemma 2.2.** (cf. [13, ch. 8, Theorem 2.6]). Let $Y$ be a topological space homotopy equivalent to a finite $n$-dimensional CW-complex.

a) If $\xi$ and $\xi'$ are two real vector bundles over $Y$, $\dim_\mathbb{R} \xi = \dim_\mathbb{R} \xi' \geq n + 1$, and $[\xi]_\mathbb{R} = [\xi']_\mathbb{R}$, then $\xi \cong \xi'$.

b) If $\zeta$ and $\zeta'$ are two complex vector bundles over $Y$, $\dim_\mathbb{R} \zeta = \dim_\mathbb{R} \zeta' \geq n$, and $[\zeta]_C = [\zeta']_C$, then $\zeta \cong \zeta'$.

**Lemma 2.3.** Let $X^n$ be a closed flat manifold. Assume that $\xi : M \to X$ is an $s$-dimensional real flat vector bundle and $\zeta : V \to X$ is a $q$-dimensional complex flat vector bundle. Then

a) $M$ and $V$ have the structures of complete flat manifolds,

b) the image of the total Chern class $c(\zeta)$ in $H^*(X, \mathbb{Q})$ is equal to 1,

c) $c(\zeta) = 1$,

d) $[\zeta] - [\Theta^s_\mathbb{R}] \in \text{Tors} KU(X)$,

e) $[\xi] - [\Theta^s_\mathbb{R}] \in \text{Tors} KO(X)$.

The proof of a) can be found in [20] (see also [22]). Parts b) and c) are consequences of the flatness of $\zeta$, which implies that Chern forms representing Chern classes of $\zeta$ are equal to 0 (see e.g. [13, ch. 11] and [21, Appendix C, Corollary 2]). Parts d) and e) follow from the fact that $ch : KU(X) \otimes \mathbb{Q} \to H^{ev}(X, \mathbb{Q})$ is a monomorphism (cf. [17, ch. 5, Theorem 3.25], [8, ch. 1, Section B, § 4]), from the equality $ch([\zeta] - [\Theta^s_\mathbb{R}]) = 0$, and from Lemma 2.1.

Now we formulate a particular case of a result of Wilking ([28 Corollaries 6.4 and 6.5]). It is a generalization of the second Bieberbach theorem.
Theorem 2.1. The set $E_{\text{Diff}}^c(m)$ of diffeomorphism classes of $m$-dimensional $c\!\!f$-manifolds is finite and each element of $E_{\text{Diff}}^c(m)$ contains a $c\!\!f$-manifold with finite holonomy group.

An affine variant of Theorem 2.1 is false because the holonomy groups of affinely diffeomorphic flat manifolds are isomorphic.

3. Complete flat manifolds with cyclic holonomy groups

The aim of this section is to give a topological classification of complete flat manifolds whose holonomy groups are cyclic and whose fundamental groups are isomorphic to a fixed group $\Gamma$.

Definition 3.1. A flat riemannian manifold with cyclic holonomy group will be called a fch-manifold. A nolcyc bundle is a nonorientable line bundle $L : EL \to X$ such that $EL$ is a complete fch-manifold and $X$ is a totally geodesic submanifold of $EL$.

The main result here are the following.

Theorem 3.1. Let $M$, $X$, $\Pi[\rho]$, $\Gamma$, and $s$ be as in Proposition 2.1 and let $e(\gamma) = \det \Phi_U(\gamma)$. Assume that $\Phi(\Gamma)$ is a cyclic group.

a) If $\Pi[\rho]$ is an orientable bundle, then $\Pi[\rho] \cong \Theta_R^s$.

b) If $\Pi[\rho]$ is a nonorientable bundle, then $\Pi[\rho] \cong \Theta_R^{s-1} \oplus \Pi[e]$.

Theorem 3.2. Let $M$ be a complete, connected, $m$-dimensional riemannian manifold with cyclic holonomy group and let $X$ be a closed $n$-dimensional flat manifold homotopy equivalent to $M$. Suppose that $L$ is a nolcyc bundle over $X$ and $\dim M > \dim X$. Then $M$ is diffeomorphic either to $X \times \mathbb{R}^{m-n}$ or to the total space of $(X \times \mathbb{R}^{m-n-1}) \oplus L$.

Theorem 3.2 shows that there are exactly two diffeomorphism classes of complete, noncompact fch-manifolds having the same fundamental group. It reduces the classification of complete fch-manifolds to the classification of compact ones. Recall that the classification of Bieberbach manifolds with a fixed cyclic holonomy group $C$ is a difficult problem solved only when $C$ has prime order ([2] [3]).

Theorem 3.1 is a consequence of the following algebraic property of the deck groups of fch-manifolds.

Lemma 3.1. If $\Phi(\Gamma)$ is a finite cyclic group, then there are $a \in H_1(\Gamma)$ and a subgroup $B$ of $H_1(\Gamma)$ such that $H_1(\Gamma) = \langle a \rangle \oplus B$, $\langle a \rangle \cong \mathbb{Z}$, $\Psi(\langle a \rangle) = \Phi(\Gamma)$, and $\text{Tors} H_1(\Gamma) \subset B \subset \text{ker } \Psi$.

Theorem 3.1 implies that if the bundle $M \to X$ is orientable, then $M$ is diffeomorphic to $X \times \mathbb{R}^{m-n}$. The case when the bundle $M \to X$ is nonorientable is more difficult. It follows from the fact that any two nolcyc bundles $\lambda$ and $X'$, with the same base space $X$, belong to the same orbit of the action of $\text{Diff}(X)$ on $X$.

Theorem 3.1 is a generalization of a result of Thorpe stating that $X$ is parallelizable or $TX$ is isomorphic to the direct sum of a trivial bundle and a line bundle (cf. [27]). Using Theorem 3.1 it is easy to verify a more general version of the last statement.

Theorem 3.3. Let $M$ be a complete fch-manifold.

a) If $M$ is orientable, then $M$ is parallelizable.

b) If $M$ is nonorientable, then $TM \cong \Theta_R^{m-1} \oplus \lambda$ for some nolcyc-bundle $\lambda$ over $M$. 
Corollary 3.1. If $M$ is a complete flat manifold and $\text{Hol}(M)$ is a cyclic group of odd order, then $M$ is parallelizable.

4. Affinely equivalent complete flat manifolds

The aim of this section is to describe algebraic invariants corresponding to affine equivalence classes of noncompact cf-manifolds. For details and for the proofs we refer to [23].

Theorem 4.1. Let $M$ and $M'$ be two $m$-dimensional cf-manifolds with isomorphic fundamental groups. Let $X \subseteq M$ and $X' \subseteq M'$ be totally geodesic submanifolds of $M$ and $M'$ homotopy equivalent to $M$. Assume that $n = \dim X$, $\tilde{M} = \tilde{X} \times U$ and $\tilde{M}' = \tilde{X}' \times U'$. Let $\Phi_U : \pi_1(X) \to \text{O}(m-n)$, $\Phi_U' : \pi_1(X') \to \text{O}(m-n)$ be the vertical holonomy homomorphisms of $M$ and $M'$. Then the following conditions are equivalent.

(a) $M$ is affinely diffeomorphic to $M'$,
(b) there is an isomorphism $f : \pi_1(X) \to \pi_1(X')$ and a linear isomorphism $L : U \to U'$ such that

$$\Phi_U'(f(\gamma)) = L \circ \Phi_U(\gamma) \circ L^{-1}$$

for $\gamma \in \pi_1(X)$.

Let $m$, $\Gamma$, and $n$ be as in the formulation of Theorem 4.1. For a fixed discrete group $G$ consider the set $\mathcal{I}(\Gamma, G, m)$ of all pairs $(\epsilon, \rho)$, where $\epsilon : \Gamma \to G$ is an epimorphism and $\rho : G \to \text{O}(s)$ is a representation.

Definition 4.1. Two elements $(\epsilon, \rho)$ and $(\epsilon', \rho')$ of $\mathcal{I}(\Gamma, G, m)$ are equivalent if there are $f \in \text{Aut}(\Gamma)$ and a linear isomorphism $L : \mathbb{R}^s \to \mathbb{R}^s$ such that

$$L(\rho) \circ \epsilon = \rho' \circ \epsilon' \circ f,$$

where $L(\rho)(g) = L \circ \rho(g) \circ L^{-1}$.

Let Inv($\Gamma$, $G$, $m$) be the set of equivalence classes of the elements of $\mathcal{I}(\Gamma, G, m)$, let Epi($\Gamma$, $G$) be the set of epimorphisms from $\Gamma$ to $G$, and let Rep($G$, $s$) be the set of conjugacy classes of representations of $G$ in $\mathbb{R}^s$. Applying Theorem 4.1 we have.

Theorem 4.2. If $\Gamma$ is a Bieberbach group, then there is a bijection $\nu : \text{Inv}(\Gamma, G, m) \to \text{E}_{\text{Aff}}^\text{cf}(\Gamma, G, m)$.

Corollary 4.1. If $|G| < \infty$, then

$$\text{card } \text{E}_{\text{Aff}}^\text{cf}(\Gamma, G, m) \leq \text{card } \text{Epi}(\Gamma, G) \text{ card Rep}(G, s).$$

Let $E^\text{cf}_{\text{Aff}}(\Gamma, m)$ be the set of affine diffeomorphism classes of $m$-dimensional complete flat manifolds with the same fundamental group $\Gamma$ and let $n$ be as above.

Theorem 4.3. If $m \geq n + 2$ and $H_1(\Gamma, \mathbb{Z})$ is infinite, then $E^\text{cf}_{\text{Aff}}(\Gamma, m)$ is uncountable.

Proposition 4.1. Let $\Gamma$ be a Bieberbach group. Then there are infinitely many affine equivalence classes of cf-manifolds whose fundamental groups are isomorphic to $\Gamma$ and whose holonomy groups are finite.
5. Topological and affine classification of low-dimensional orientable cf-manifolds

The aim of this section is to describe topological and affine equivalence classes of cf-4-manifolds. For simplicity we deal with the orientable case. A nonorientable case is somewhat more complicated and will be considered elsewhere. For the classification of cf-m-manifolds ($m \leq 3$) we refer to (29). We shall use the fact that there are 10 affine diffeomorphism classes of closed flat 3-manifolds $X_1, \ldots, X_{10}$ (see e.g. (29) Theorems 3.5.5 and 3.5.9] for the description of them). For the classification of closed 4-manifolds we refer to [11] or [1].

Theorem 5.1. There are 14 diffeomorphism classes of orientable, noncompact cf-4-manifolds. They are represented by:

\[ \mathbb{R}^4, \quad S^1 \times \mathbb{R}^3, \quad T^2 \times \mathbb{R}^2, \quad TK, \quad \text{and} \quad \Lambda^3 X_j, \quad j = 1, \ldots, 10. \]

Here $K$ is the Klein bottle and $TK$ is the tangent bundle of $K$. To deal with affine classification of cf-4-manifolds we need some definitions. Consider the action of $\text{SL}(2, \mathbb{Z})$ on $T^2$, induced by the standard action of $\text{GL}(2, \mathbb{Z})$ on $\mathbb{R}^2$, and the arising orbit space $T^2/\text{GL}(2, \mathbb{Z})$. Let $\rho$ be the equivalence relation in $T^2$ such that

\[ (e^{i\alpha}, e^{i\beta})\rho(e^{i\alpha'}, e^{i\beta'}) \]

if and only if $e^{i\alpha} = e^{i\epsilon_1 \alpha'}$ and $e^{i\beta} = e^{i\epsilon_2 (\beta' - k\alpha')}$ for some $(\epsilon_1, \epsilon_2) \in \{-1, 1\}^2$ and some $k \in \mathbb{Z}$.

Theorem 5.2. Affine equivalence classes of orientable noncompact cf-4-manifolds, not diffeomorphic to $T^2 \times \mathbb{R}^2$, $TK$, or $S^1 \times \mathbb{R}^3$, are represented by $\mathbb{R}^4$ and $\Lambda^3 X_j, j = 1, \ldots, 10$. Affine equivalence classes of cf-manifolds, diffeomorphic to $T^2 \times \mathbb{R}^2$, $TK$, or $S^1 \times \mathbb{R}^3$ correspond to the elements of $T^2/\text{GL}(2, \mathbb{Z})$, $T^2/\rho$, and $S^1/z \sim -z$.

6. Diffeomorphism classes of some cf-manifolds

In this chapter we discuss the problem of the topological classification of cf-manifolds in a more general context than in Chapter 5. We consider cf-manifolds homotopy equivalent to some low-dimensional Bieberbach manifolds. We also deal with stable diffeomorphism classes of some cf-manifolds.

Definition 6.1. Two manifolds $M_1$ and $M_2$ are stably diffeomorphic if there is a positive integer $k$ such that $M_1 \times \mathbb{R}^k$ is diffeomorphic to $M_2 \times \mathbb{R}^k$.

Given a Bieberbach manifold $X$ and $f_0 \in \text{Aut} (\pi_1(X))$, there is a diffeomorphism $f \in \text{Diff}(X)$ such that $f_* = f_0$ (3 ch. 2, Theorem 5.3), (29) ch. 3, Theorem 3.2.2]). This induces an action of $\text{Aut} (\pi_1(X))$ on $K\text{O}(X)$. Let $[\xi] \in [X]$ be the class of a flat bundle $\xi$ over $X$ in $\text{Tors} K\text{O}(X)/\text{Aut}(\pi_1(X))$. The investigation of stable diffeomorphism classes of cf-manifolds is based on the following consequence of a result of Mazur (cf. (19) Theorem 2]).

Proposition 6.1. Let $M_1$, $M_2$ be two cf-m-manifolds homotopy equivalent to the same Bieberbach manifold $X$ and let $\xi_j : M_j \to X, j = 1, 2$, be the arising flat bundles. Assume that $m > 2 \text{dim } X$. Then the following conditions are equivalent

a) $M_1$ and $M_2$ are diffeomorphic,

b) $[\xi_1]_0 = [\xi_2]_0$.

As an immediate consequence of Proposition 6.1 we have
Corollary 6.1. Let $M_1, M_2$ be two cf-$m$-manifolds homotopy equivalent to the same Bieberbach manifold $X$ and let $\xi_j : M_j \to X, j = 1, 2,$ be the arising flat bundles. Then the following conditions are equivalent

\begin{itemize}
  \item[a)] $M_1$ and $M_2$ are stably diffeomorphic,
  \item[b)] $[\xi_1]_K = [\xi_2]_K$.
\end{itemize}

Corollary 6.2. Diffeomorphism classes of complete flat $(n + 1)$-manifolds, homotopy equivalent to a fixed Bieberbach $n$-manifold $X$, correspond to the elements of $H^1(X, \mathbb{Z})/\text{Aut}(\pi_1(X))$.

Let $g : \mathbb{R}^2 \ni (x, y) \to (-x, y + 1)$, let $\mu$ be the Möbius bundle:

$$E\mu = \mathbb{R}^2/(g) \ni [x, y] \to [y] \in \mathbb{R}/\mathbb{Z} = S^1,$$

let $P_j : S^1 \times S^1 \ni (z_1, z_2) \to z_j, j = 1, 2, \mu_j = P_j^* \mu$, and let $a, b$ be the generators of the deck group of the Klein bottle $K$ defined by the formulas

$$a(x, y) = (x + 1, y) \quad \text{and} \quad b(x, y) = (-x, y + \frac{1}{2}).$$

Consider the generators $\alpha, \beta$ of $H^1(K, \mathbb{Z})$, dual to the images of $a, b$ in $H_1(K, \mathbb{Z})$, and line bundles $\lambda_1, \lambda_2$ such that $w_1(\lambda_1) = \alpha$ and $w_1(\lambda_2) = \beta$. We have

Proposition 6.2. Let $M$ be a cf-$m$-manifold homotopy equivalent to $S^1$. If $m \geq 2$, then $M$ is diffeomorphic to $S^1 \times \mathbb{R}^{m-1}$ or $E\mu \times \mathbb{R}^{m-2}$.

Theorem 6.1. a) If $m \geq 5$, then the diffeomorphism classes of cf-$m$-manifolds homotopy equivalent to $T^2$ are represented by $T^2 \times \mathbb{R}^{m-2}$, $E(\mu_1 \oplus \Theta^m_R)$, $E(\mu_1 \oplus \mu_2 \oplus \Theta^{m-4}_R)$, and $E(\mu_1 \oplus \mu_2 \oplus \mu_1 \mu_2 \oplus \Theta^{m-5}_R)$.

b) Diffeomorphism classes of cf-$4$-manifolds homotopy equivalent to $T^2$ are represented by $T^2 \times \mathbb{R}^2$, $E(\mu_1 \oplus \Theta^2_R)$, and $E(\mu_1 \oplus \mu_2)$.

Theorem 6.2. a) If $m \geq 5$, then the diffeomorphism classes of cf-$m$-manifolds homotopy equivalent to the Klein bottle $K$ are represented by $K \times \mathbb{R}^{m-2}$, $E(\lambda_1 \oplus \Theta^{m-3}_R)$, $E(\lambda_2 \oplus \Theta^{m-3}_R)$, $E(\lambda_1 \oplus \lambda_2 \oplus \Theta^{m-4}_R)$, and $E(\lambda_2 \oplus \lambda_1 \lambda_2 \oplus \Theta^{m-5}_R)$.

b) Diffeomorphism classes of cf-$4$-manifolds homotopy equivalent to the Klein bottle are represented by $K \times \mathbb{R}^2$, $E(\lambda_1 \oplus \Theta^2_R)$, $E(\lambda_2 \oplus \Theta^2_R)$, $E(\lambda_1 \oplus \lambda_2)$, and $E(\lambda_2 \oplus \lambda_1 \lambda_2)$.

The proof of Proposition 6.2 is easy. To describe the idea of the proofs of the other results denote $T^2$ and $K$ by $X$. The 3 and 4-dimensional case follows from a direct argument. By Lemma 6.4 the isomorphism classes of flat vector bundles over $X$ whose dimension is greater than 2 correspond to their images in $\text{Tors} KO(X)$. Using the Atiyah-Hirzebruch $KO$-spectral sequence of the fibration $X \to S^1$ one can check that the map

$$W : \text{Tors} KO(X) \to H^1(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \cong \mathbb{Z}_2^3,$$

carrying the class of the bundle $\xi$ onto $(w_1(\xi), w_2(\xi))$, is a bijection. Now it suffices to find the orbit space of the action of $\pi_1(X)$ on $H^1(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$.

7. Holonomy homomorphisms and geometric invariants

In this section we express characteristic classes of some flat bundles in terms of their holonomy homomorphisms. We also discuss how to calculate cohomology groups containing some invariants arising in this paper. By Corollary 2.1 any cf-manifold with abelian vertical holonomy group is the total space of the direct sum of complex line bundles $L_1, \ldots, L_k$ and real line bundles $\lambda_1, \ldots, \lambda_l$. These line bundles are determined by their Chern classes and Stiefel-Whitney classes, respectively.
Theorem 7.1. Let $\lambda : E\lambda \to X$ be a real flat line bundle over a closed flat manifold $X$. Assume that $\Phi = \Phi_X \times \Phi_U$ is the holonomy homomorphism of $\lambda$ and $\Phi_X \times \Phi_U = (\Psi_X \times \Psi_U) \circ \Pi_{ab}$. Let $P : H_1(X, \mathbb{Z}) \to H_1(X, \mathbb{Z}_2)$ be the projection and let $\mu : O(1) \to \mathbb{Z}_2$ be the isomorphism. Then

$$u_1(\lambda) \circ P = \mu \circ \Psi_U.$$ 

To state an analogous description of the Chern classes write the first homology group of a Bieberbach manifold $X = \overline{X}/\Gamma$ as $\mathbb{Z}^{b_1(X)} \oplus S$, where $S = \text{Tors} H_1(X, \mathbb{Z})$. Let $k_X$ be the order of the holonomy group of $X$ and let $S_2$ be the torsion subgroup of $H^2(X, \mathbb{Z})$. The set $\text{FLB}_C(X)$, of isomorphism classes of flat complex line bundles over $X$, is a commutative group with tensor product as a group operation. It is known that $c_1 : \text{FLB}_C(X) \to H^2(X, \mathbb{Z})$ is a monomorphism (cf. [13] ch. 16, Theorem 3.4) and $c_1(\text{FLB}_C(X)) = S_2$ ([13] Theorem 6.1]). For every $\Psi \in \text{Hom} (S, \mathbb{Z}_{k_X})$ the formula

$$\Psi^X (x) = \begin{cases} \Psi(x) & \text{for } x \in S \\ 0 & \text{for } x \in \mathbb{Z}^{b_1(X)} \end{cases}$$

defines $\Psi^X \in \text{Hom} (H_1(X, \mathbb{Z}), \mathbb{Z}_{k_X}) = H^1(X, \mathbb{Z}_{k_X})$. Consider the coboundary homomorphism $\delta : H^1(X, \mathbb{Z}_{k_X}) \to H^2(X, \mathbb{Z})$ induced by the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{k_X} \mathbb{Z} \xrightarrow{j} \mathbb{Z}_{k_X} \to 0,$$

where $\lambda(x) = k_X x$ and $j$ is the canonical projection.

Lemma 7.2. Let $X$ be a Bieberbach manifold and let $k_X, \Psi^X, \delta$ be as above. Let $L$ be a complex flat line bundle over $X$ with holonomy homomorphism $\Phi_L$. Take the factorization

$$\Phi_L : \Gamma \xrightarrow{\Pi_{ab}} H_1(\Gamma) \xrightarrow{\Psi} U(1)$$

of $\Phi_L$. Then

a) $\delta_S : \text{Hom} (S, \mathbb{Z}_{k_X}) \to \text{Tors} H^2(X, \mathbb{Z})$ carrying $\Psi \in \text{Hom} (S, \mathbb{Z}_{k_X})$ onto $\delta(\Psi^X)$ is an isomorphism,

b) $c_1(L) = \delta_S(\Phi_L|_{S})$.

The proof of Lemma 7.1 is an easy exercise. We do not know a reference to the statement and proof of Lemma 7.2.

Corollary 7.1. a) $c_1(\text{FLB}_C(X)) \cong \text{Hom} (S, \mathbb{Z}_{k_X})$.

b) If $\text{Tors} H_1(X, \mathbb{Z}) = \{0\}$, then all complex flat line bundles over $X$ are trivial.

c) If $\text{Tors} H_1(X, \mathbb{Z}) = \{0\}$ and $M$ is a cf-manifold homotopy equivalent to $X$, having abelian vertical holonomy group, then $M$ is diffeomorphic to the total space of the direct sum of some real line bundles over $X$.

There are different methods allowing to calculate first and second cohomology groups of a Bieberbach manifold $X$. One can use a general approach based on the Smith normal form of an integer matrix (cf. [13], [21]). We discuss another simple approach that can be applied if the holonomy group of $X$ is a cyclic group of order $k$. In this case $X$ is affinely diffeomorphic to the mapping torus $M(g)$ of an affine diffeomorphism $g : T^{n-1} \to T^n$ such that $g^k = id$.

Theorem 7.1. Let $G$ be a group isomorphic to $\mathbb{Z}_k$ acting on $A \cong \mathbb{Z}^{n-1}$ and let $g_0$ be a generator of $G$. Assume that $g$ is an affine diffeomorphism of $T^{n-1}$ such that $g_* = g_0$ and $M(g)$ is the mapping torus of $g$. Then

$$\text{Tors} H_1(M(g), \mathbb{Z}) \cong H^1(G, A) \cong \text{Tors} A_G.$$
Here \( A_G = A/\text{im}(g_0 - \text{id}) \). The cohomology group \( H^1(G, A) \) is of interest in its own right. To see this denote by \( n_{CC}(G, V) \) the number of connected components of the fixed point set of a smooth action of \( G \) on a manifold \( V \) homotopy equivalent to \( T^{n-1} \). Identifying \( \pi_1(T^{n-1}) \) with \( A \) we can treat \( A \) as a \( \mathbb{Z}[G] \)-module. If \( G \) is a \( p \)-group, then
\[
n_{CC}(G, V) = \text{card } H^1(G, A)
\]
(see Theorem A.10). Recall that a \( G \)-lattice is a \( G \)-module that is also a free abelian group of finite rank.

**Theorem 7.2.** Let \( G \) be a finite group and let \( A \) be a \( G \)-lattice. Assume that \( m = |G| \) and \( q \) is a positive integer relatively prime to \( m \). Then
\[
\text{card } H^1(G, A) = \text{card } (A \otimes \mathbb{Z}_m)^G \ m^{-\text{rank}_{\mathbb{Z}_q}(A \otimes \mathbb{Z}_q)}.
\]

**Corollary 7.2.** Let \( p \) and \( q \) be two different prime numbers and let \( A \) be a \( \mathbb{Z}_p \)-lattice. Then
\[
\text{card } H^1(G, A) = p^{\text{dim}_{\mathbb{Z}_p}(A \otimes \mathbb{Z}_p)^G} - \text{dim}_{\mathbb{Z}_q}(A \otimes \mathbb{Z}_q)^G.
\]

Corollary 7.2 is particularly convenient because it reduces the calculation of \( \text{card } H^1(G, A) \) to the determination of the number of solutions of systems of linear equations in finite fields \( \mathbb{Z}_p \) and \( \mathbb{Z}_q \). Applying Theorem 6.1 and Lemma 7.2 we have

**Corollary 7.3.** Let \( X \) be a closed flat manifold, whose holonomy group is isomorphic to \( \mathbb{Z}_k \), let \( q \) be an affine diffeomorphism of \( T^{n-1} \) such that \( X \) is diffeomorphic to \( M(g) \), and let \( A \) denote \( \pi_1(T^{n-1}) \) with the induced \( \mathbb{Z}_k \)-action on it. Then
\[
\text{Tors } H^2(X, \mathbb{Z}) \cong H^1(\mathbb{Z}_k, A) \quad \text{and} \quad H^1(X, \mathbb{Z}_2) \cong \mathbb{Z}_2^{b_1(X)} \otimes H^1(\mathbb{Z}_k, A) \otimes \mathbb{Z}_2.
\]

Now we give a convenient criterion of the triviality of \( H^1(G, A) \).

**Proposition 7.1.** Let \( G \) be a finite group, let \( A \) be a \( G \)-lattice, and let \( q \) be a prime number such that \( (|G|, q) = 1 \). Assume that for every prime divisor \( p \) of \( |G| \) we have
\[
\text{dim}_{\mathbb{Z}_p}(A \otimes \mathbb{Z}_p)^G = \text{dim}_{\mathbb{Z}_q}(A \otimes \mathbb{Z}_q)^G.
\]
Then \( H^1(G, A) = \{0\} \).

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E-mail address: msa@delta.math.univ.gda.pl

DEPARTMENT OF MATHEMATICS GDAŃSK UNIVERSITY 80-952 GDAŃSK, WITA STWOSZA 57 POLAND