QUATERNIONIC HAMILTON EQUATIONS

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Abstract. The classical Hamilton equations are reinterpreted by means of complex analysis, in a non standard way. This suggests a natural extension of the Hamilton equations to the quaternionic case, extension which coincides with the one introduced in [2] by a completely different approach.

1. Introduction

In the last years, a better understanding of the geometrical framework of the classical hamiltonian dynamics led to some interesting generalizations. This is for instance the case of the Nambu dynamics ([7], [10], [6]), where the evolution of the system in a \( n \)-dimensional manifold is related to \( (n-1) \) hamiltonian functions, or of the Liouville dynamics, whose properties are studied in [5].

Very recently, in [2] the authors proposed a different extension of the classical hamiltonian dynamics to spaces which are endowed by three symplectic structures, having the same commutation rules as the quaternionic imaginary units. Roughly speaking, this new hyper–hamiltonian dynamics is the sum of three different hamiltonian dynamics, each one corresponding to a single symplectic structure and to an arbitrary hamiltonian function. The hyper–hamiltonian dynamics retains most of the appealing features of the classical hamiltonian one, included a variational formulation ([2], [3]).

Here the reader will not be faced with any new application of the hyper–hamiltonian dynamics, nor with any discussion about its properties. The aim of this paper is to point out that the extension given in [2], which is natural from the geometrical point of view, is also natural from the point of view of analysis. More precisely, if the classical hamiltonian dynamics is reinterpreted in the light of the complex analysis, and then extended by replacing the complex analysis with the quaternionic one, what one gets is again the same hyper–hamiltonian dynamics introduced in [2].

It should be stressed that the former step does not consist in writing the Hamilton equations by using complex coordinates, but rather in having the linear notion underlying complex holomorphy (i.e. complex linearity) coming into play. This is done in section 3.

When the complex numbers are replaced by the quaternions, it is well known that the complex holomorphy correspondingly changes in the so called quaternionic regularity ([1], [8]). This is done in section 4 where the results in [4] are used to tune the translation process to our needs.

Finally, in section 5 the linear quaternionic regularity replaces the complex linearity into the equation governing the dynamics. To conclude, it is shown that the resulting equations coincide with the one already obtained in [2] by a completely

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different approach. Section 2 is devoted to introduce the right context where the previous program can be worked out. The real needs are metrics and complex structures, which calls for Kähler manifolds instead of just symplectic one, as it may be expected.

**Notations.** The symbols $\mathbb{C}$ and $\mathbb{H}$ denote the complex numbers and the quaternions; their standard bases are $1, i$ and $1, i_1, i_2, i_3$ respectively. Repeated indexes, like in $i_\alpha \partial / \partial x^\alpha$, allude to a summation on $\alpha = 1, 2, 3$; when misunderstandings are possible, explicit summations or some extra information are added. Finally, if $X$ is a vector field over a differential manifold $M$, by $X \upharpoonright t$ one intends its contraction with the tensor field $t$.

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### 2. General setting

Let consider a Riemannian manifold $M$ with a metric $g$. On $M$, take a $g$–invariant complex structure $J$, namely a linear map $TM \rightarrow TM$ such that:

$$J^2 = -I \quad \text{and} \quad g(JX, JY) = g(X, Y) \quad \forall X, Y.$$  

The manifold $M$ is said to be a Kähler manifold when the 2–form defined by:

$$\omega(X, Y) = g(JX, Y)$$

is closed. Since $\omega$ is nondegenerate and anti–symmetric, it defines a symplectic structure on $M$. Given a function $h : M \rightarrow \mathbb{R}$, the Hamilton equations may be introduced:

$$X \upharpoonright \omega = dh.$$  

Of course one can define Hamilton equations on any symplectic manifold, regardless its symplectic structure originates from a metrics or it doesn’t. However, this restriction is essential for all the discussion in the next section.

In [2] the Hamilton equations are extended to cover the case where $M$ is an hyper–Kähler manifold. Instead of a single one, $TM$ has now three $g$–invariant complex structures such that:

$$J^2_1 = J^2_2 = -I \quad J_1 J_2 = -J_2 J_1 \quad J_3 = J_1 J_2 ,$$

and all the 2–forms $\omega_\alpha$ associated to $J_\alpha$ via (2.1) are closed (see e.g. [3]). The hyper–hamiltonian mechanics introduced in [2] then depends on the choice of three distinct hamiltonian functions $h_\alpha$, its dynamics being defined by the following equations:

$$X \upharpoonright \Omega = \frac{1}{(2n - 1)!} \sum_\alpha dh_\alpha \wedge (\omega_\alpha)^{2n-1} .$$

where $\Omega$ is the volume form on $M$, whose dimension is $4n$, and $(\omega_\alpha)^{2n-1}$ denotes the wedge product of the differential form $\omega_\alpha$ by itself for $2n - 1$ times.

A decomposition formula for the vector field $X$ is then proved in [2]:

$$X = X_1 + X_2 + X_3 \quad \text{where} \quad X_\alpha \upharpoonright \omega_\alpha = dh_\alpha \quad \forall \alpha ,$$

suggesting the most general hyper–hamiltonian dynamics is the sum of three independent hamiltonian dynamics.
3. Hamilton equations and complex analysis

Consider first classical Hamiltonian mechanics. By making use either of the metrics $g$ and of the complex structure $J$ on $TM$, one can rewrite the Hamilton equations (2.2) in the form:

$$g(X,Y) = dh(JY) \quad \forall Y.$$ 

The key point is now to give a closer look at the 1–form lying at the right hand side, namely:

$$\theta(Y) = dh(JY)$$

from the point of view of complex analysis. First of all, notice that $M$ is an almost complex manifold in a natural way. Indeed defining

$$(zY)_p := tY_p + xJpY_p$$

where $z = t + ix \in \mathbb{C}$ makes $T_pM$ a complex linear space, for every $p \in M$. As a consequence, it makes sense to talk about complex linear maps on it, and one easily realizes that $\theta$ is the only real valued 1–form for which the complex valued 1–form $\theta + i \, dh$ is in fact a complex linear map. Such a completion property deserves the ad hoc notation:

$$\hat{i} \, dh = \theta.$$ 

The obvious identity $i \, dh = d(ih)$ then suggests the following approach. First of all think of the pure imaginary complex function:

$$H = ih$$

as the Hamiltonian function describing the system, and then rewrite the Hamilton equations in the equivalent form:

$$(3.1) \quad X \cdot g = \hat{d}H.$$ 

Until now, nothing really new. However, everyone can now guess how to extend the approach from complex numbers to quaternions. Assume $M$ is an hyper–Kähler manifold, and just think of the Hamiltonian as a function:

$$H : M \to \mathbb{H}$$

with pure imaginary values (summation on $\alpha$ is implicit). Then think of $\hat{d}H$ as the completion of its differential with respect to the natural notion of holomorphy in the framework of quaternionic analysis: the so called regularity ([8], [4]). This will be done in the next section, where the following formula:

$$\hat{d}H(Y) = dh^\alpha(J_\alpha Y)$$

will be proved.

Equation (3.1) makes sense also in this new framework, and selects a unique vector field $X$. The important point is that, maybe surprisingly, $X$ is exactly the same vector field defined by (2.4). This is the main result of the paper, and it will be proved in the last section.
4. Regularity and completion to a regular form

In 1935 Fueter proposed a definition of right regularity for a function \( \varphi : \mathbb{H} \to \mathbb{H} \), by adapting the Cauchy–Riemann equations to the increased number of independent imaginary units in \( \mathbb{H} \), namely (4.1):

\[
\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x^\alpha} i_\alpha = 0 \quad q = t + x^\alpha i_\alpha \in \mathbb{H}.
\]

By putting the units \( i_\alpha \)'s on the left, one obtain the so called left regular functions, whose theory, clearly, is completely equivalent to the one of right regular functions. To a large extent, the quaternionic analysis based on this notion mimics the richness of complex analysis (see (8)), though the non commutativity of \( \mathbb{H} \) introduces some important differences: for instance, the product or the composition of regular functions need not to be regular.

Notice that equation (4.1) is in fact a condition on the differential of \( d\varphi \). Indeed, \( \varphi \) is right regular if and only if \( d\varphi \) belongs to a special class of 1–forms \( T\mathbb{H} \to \mathbb{H} \), namely the \( \theta \)'s which satisfy the condition:

\[
\theta(1) + \sum_\alpha \theta(i_\alpha) i_\alpha = 0.
\]

As pointed out in the introduction, the interest here is essentially in a more general type of 1–forms, that is:

\[
\theta : T\mathbb{M} \to \mathbb{H}.
\]

where \( \mathbb{M} \) is an hyper–Kähler manifold. What does it mean regularity in this framework, is a question which may be answered using the results in (4).

In that paper, the notion of regularity is extended to functions acting between general quaternionic spaces, and the extension is shown to be strictly dependent on if \( \mathbb{H} \) acts on the involved spaces either on the left or on the right. For instance, in \( \mathbb{H} \) is proved that the above considered notions of regularity make sense only when \( \mathbb{H} \) acts differently on the two spaces. In particular, the needs for the right regularity are a domain which is a left space, and a codomain which is a right one. Now, the way \( \mathbb{H} \) acts on the domain \( T\mathbb{M} \) of \( \theta \) is in fact implicit in our definition of hyper–Kähler manifold. Indeed, due to the commutation rules (2.3), the position:

\[
(qY)_{p} = tY_{p} + x^\alpha (i_\alpha)_{p} Y_{p} \quad \text{where} \quad q = t + x^\alpha i_\alpha \in \mathbb{H}
\]

makes every \( T_{p}\mathbb{M} \) a left quaternionic space. Hence, as pointed out above, the codomain \( \mathbb{H} \) has to be considered as a right space over itself, and the only applicable notion of regularity is the right one.

Following (4), the correct definition of regularity for \( \theta \) is then:

\[
\theta(Y) + \sum_\alpha \theta(i_\alpha Y) i_\alpha = 0 \quad \forall Y.
\]

That (4.3) is really an extension of (4.2) depends on the quaternionic anti–linearity of its left–hand member (4). Because of that indeed, when \( \mathbb{M} = \mathbb{H} \) the equality in (4.3) holds for any \( Y \in T\mathbb{H} \) if and only if it holds for \( Y = 1 \). Before going further, a remark is in order about the invariance of the notion of regularity. Looking at (4), this notion seems to depend on the choice of the imaginary units \( i_\alpha \)'s. Now, in \( \mathbb{H} \) there is an \( S^2 \) of imaginary units, and any choice of a triple of units, having the right commutation properties, is an admissible one; a natural quaternionic object must be independent on that choice. The following lemma says this is the case for the notion of regularity.
Lemma 4.1. If the triple \( j_1, j_2 \) and \( j_3 \) of quaternions is chosen in such a way that:
\[
j_1^2 = j_2^2 = 1 \quad j_3 = j_1 j_2,
\]
then, for any quaternion valued 1–form \( \theta \), the following identity holds:
\[
\sum_\alpha \theta(i_\alpha Y)i_\alpha = \sum_\alpha \theta(j_\alpha Y)j_\alpha \quad \forall Y.
\]

Proof. Define a \( 3 \times 3 \) real matrix \( C \) by means of:
\[
j_\alpha = C_\alpha^\beta i_\beta \quad \forall \alpha.
\]
The commutation rules in the hypothesis say that \( C^T C = I \). The conclusion follows from straightforward computations. \( \square \)

In the next proposition the regularity will be exploited to introduce the notion of completion to a regular form, for 1–forms \( \xi \) which take their values in the imaginary quaternions, namely:
\[
(4.4) \quad \xi : TM \to \mathbb{H} \quad \text{with} \quad \xi + \overline{\xi} = 0.
\]

Proposition 4.2. If \( \xi \) satisfies condition \( (4.4) \), then there is a unique real valued 1–form \( \hat{\xi} \in T^*M \) such that \( \xi + \hat{\xi} \) is a regular 1–form. Moreover, the completion \( \xi \mapsto \hat{\xi} \) is a real linear operation and, if \( \xi = \xi^\alpha i_\alpha \) with \( \xi^\alpha \in T^*M \), then:
\[
\hat{\xi}(Y) = \xi^\alpha(i_\alpha Y) \quad \forall Y.
\]

The proof is a straightforward consequence of the following lemma, where regularity of the general 1–form \( \theta \) is rewritten using its components.

Lemma 4.3. Assume \( \theta = \theta^0 + \theta^\alpha i_\alpha \), where \( \theta^0 \) and each \( \theta^\alpha \) lie in \( T^*M \) (namely are real valued 1–forms). Then \( \theta \) is regular if and only if:
\[
\theta^0(Y) = \theta^\alpha(i_\alpha Y) \quad \forall Y.
\]

Proof. Just write down the sum in \( (4.3) \), separating the contributions along 1 and each \( i_\alpha \). The four conditions obtained in this way are all equivalent: the statement expresses the vanishing along 1. \( \square \)

Remark 4.4. Like the notion of regularity, the notion of completion is also independent on the choice of the imaginary units \( i_\alpha \).

5. Quaternionic Hamilton equations

Coming back to dynamics, take an hamiltonian function:
\[
H : M \to \mathbb{H} \quad \text{with} \quad \overline{H} + H = 0.
\]
on the hyper–Kähler manifold \( M \). Proposition 4.2 applies to the 1–form \( dH \), and its completion may be used to introduce the Hamilton like equation:
\[
(5.1) \quad X \lrcorner g = \mathring{d}H.
\]

Remark 5.1. In some special situations, the quaternionic Hamilton equations coincide with the classical ones. This is the case when the hamiltonian function is directed along a single pure imaginary quaternion, namely:
\[
H(p) = h(p) u \quad h \text{ real valued}, \quad u^2 = -1.
\]
Indeed, by means of Proposition 4.2 and the remark thereafter:
\[ \hat{d}H(Y) = dh(uY) \quad \forall Y, \]
so that equation (5.1) becomes:
\[ X \omega_u = dh. \]
Here, of course, \( \omega_u \) is the symplectic form associated to the complex structure \( J_u Y = uY \).

To complete the program announced in the introduction, it remains to show that both the equations (2.4) and (5.1) define the same vector field \( X \). This is a consequence of the following result and formula 2.5.

**Proposition 5.2.** Assume \( H = h^\alpha i_\alpha \). Then the vector field \( X \) satisfies equation (5.1) if and only if it decomposes as:
\[ X = X_1 + X_2 + X_3 \]
where:
\[ X_\alpha \omega_\alpha = dh^\alpha \quad \forall \alpha. \]

**Proof.** Using Proposition 4.2, condition (5.1) rewrites as:
\[ g(X, Y) = dh^\alpha (i_\alpha Y) \quad \forall Y. \]
The decomposition suggested in the statement clearly gives rise to a vector field \( X \) which satisfies the previous condition. On the other hand, there is only one vector field which has this property.

Summing up, two possible extensions of classical hamiltonian mechanics have been considered. One is the hyper–hamiltonian mechanics introduced in [2], whose dynamics is defined by (2.4), and which is natural from the point of view of symplectic geometry. The other one is the quaternionic hamiltonian mechanics introduced in this note, whose dynamics is defined by (5.1), and which is natural from the point of view of complex and quaternionic analysis. Though they arise from apparently unrelated approaches, the two extension coincide.

Moreover, in the quaternionic approach the exact meaning of the sum in (2.5) also becomes clear. The intrinsic object is the hamiltonian function \( H \), whereas the \( h^\alpha \) are just its components with respect to a given basis in the space of imaginary quaternions. Changing the base, the components of \( H \) change, and so do the vector fields \( X_\alpha \): however, its sum is intrinsic inasmuch as it does not depend on any choice.

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