AN ASYMPTOTIC ANALYSIS OF SEPARATING POINTLIKE AND $C^\beta$-CURVELIKE SINGULARITIES

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Abstract. In this paper, we present a theoretical analysis of separating images consisting of pointlike and $C^\beta$-curvelike structures, where $\beta \in (1, 2]$. Our approach is based on $l_1$-minimization, in which the sparsity of the desired solution is exploited by two sparse representation systems. It is well known that for such components wavelets provide an optimally sparse representation for point singularities, whereas $\alpha$-shearlet type with $\alpha = \frac{3}{2}$ might be best adapted to the $C^\beta$-curvilinear singularities. In our analysis, we first propose a reconstruction framework with theoretical guarantee on convergence, which is extended to use general frames instead of Parseval frames. We then construct a dual pair of bandlimited $\alpha$-shearlets which possesses a good time and frequency localization. Finally, we apply the result to derive an asymptotic accuracy of the reconstructions. In addition, we show that it is possible to separate these two components as long as $\alpha < 2$, i.e., bandlimited $\alpha$-shearlets which range from wavelet to shearlet type do not coincide with wavelets in the sense of isotropic fashion.

1. Introduction

In the era of data analytics, the task of geometric separation is of interest for various applications. For instance, astronomers might want to extract stars from filaments, or neurologists might want to separate neurons from dendrites. This task arises due to the fact that image data are often the superposition of different geometric components. In fact, numerous publications have been contributed to this area both in the mathematical and engineering communities [21, 24, 29, 37, 41, 42].

To turn image separation into a uniquely solvable problem, the first steps are assumptions onto the shape of the components to be recovered. Recently, the methodology of compressed sensing allows the efficient reconstruction of sparse or approximately sparse data from highly incomplete linear measurements by $l_1$-minimization or thresholding [27, 28, 33]. The key ingredient is to choose two appropriate dictionaries, each one sparsely representing the corresponding component but failing to sparsely represent the other. In recent decades, there have been various applications of compressed sensing techniques, including deblurring and deconvolution [8, 34], image inpainting [13, 30, 36], data compression [31, 32], as well as geometric separation [2, 7, 12, 18, 26]. Along the way, the task of separating pointlike and curvelike structures was first introduced in [12] with a theoretical recovery guarantee by using wavelet and curvelet Parseval frames. Indeed, wavelets are well adapted for pointlike phenomena, whereas curvelets provide optimal representation for images with edges.

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However, the limitation is that curvelets use rotation which ignores the discrete lattice structures. Later, shearlet systems, originally introduced in [11] and then followed by other types [9, 17, 35, 36], make use of shearing instead of rotation. They have been shown to share similar optimal approximation behavior with curvelets, but they allow the unified treatment of the continuum and digital realm leading to faithful implementations compared with other known sparse representation systems like wavelets [1], ridgelets [4], curvelets [5], bandlets [14] and contourlets [25].

In addition, a class of shearlets appeared using flexible scaling to adapt the system according to the smoothness of the data. Among them, universal shearlets [36] form Parseval frames by changing scaling parameters at each scale which are best adapted to only high frequency part, whereas compactly supported $\alpha$-shearlets [39, 40] with fixed scaling parameter provide superior localization but fail to form Parseval frames. Thus, although Parseval frames play a crucial role both in applications and theoretical studies, this property is not always achieved. Overall, each of them has its own advantages and disadvantages depending on the approach chosen in applications.

In this paper, we consider the problem of separating pointlike singularities and $C^\beta$-curvilinear singularities, where $\beta \in (1, 2]$. For our analysis, we construct a dual pair of bandlimited $\alpha$-shearlets with flexible scaling which adaptively match their decompositions to the smoothness of observed data, i.e., they provide optimal sparse representation of $C^\beta$-curvilinear singularities, $\beta = 2/\alpha$. They range from wavelets ($\alpha = 1$) to shearlets ($\alpha = 2$). This problem is more involved than when restricting to Parseval frame pairs of wavelets-curvelets [12], or wavelets-shearlets [2, 18]. In our analysis, we provide a theoretical guarantee for geometric separation using general frames not restricting to Parseval frames. Here we also answer the question, if separation is still possible in case of wavelets, i.e., $\alpha \to 2$.

1.1. Our contributions. Our contributions in this paper consist in three main points. First, we present a theoretical guarantee for the problem of separating two geometric components using two general frames (Theorem 3.6). Second, in Subsection 4.3 we construct a pair of bandlimited $\alpha$-shearlet dual frames which provide an optimal sparse representation for $C^\beta$-curvilinear singularities. Finally, we derive an asymptotic geometric separation result for separating pointlike and $C^\beta$-curvelike singularities (Theorem 5.1). Also, we show that the proposed algorithm successfully reconstructs sub-images by $l^1$-minimization if the bandlimited $\alpha$-shearlets do not coincide with wavelets in sense of isotropic fashion.

2. Formulation of the problem

2.1. Notation and basic definitions. We first start with some basic notions and definitions.

For $f, F \in L^1(\mathbb{R}^2)$ we define the Fourier transform and inverse Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x)e^{-2\pi ix^T\xi}dx,$$
$$\check{F}(x) = \int_{\mathbb{R}^2} F(\xi)e^{2\pi i x^T\xi}d\xi,$$

with the usual extension to $L^2(\mathbb{R}^2)$. 
A frame for a separable Hilbert space $H$ is a countable family $\Phi = \{ \varphi_i \}_{i \in I}$ in $H$ for which there exist constants $0 < A \leq B < +\infty$ such that
\[ A \| f \|^2 \leq \sum_{i \in I} |\langle f, \varphi_i \rangle|^2 \leq B \| f \|^2, \quad \forall f \in H. \] (1)

The constants $A$ and $B$ are called the lower and upper frame bounds, respectively. If $A = B$, this system is called an $A$-tight frame. In addition, if $A = B = 1$, then it is called a Parseval frame.

Slightly abusing notion, we use $\Phi$ again to denote the synthesis operator $\Phi : \ell^2(I) \to H$, $\Phi(\{c_i\}_{i \in I}) = \sum_{i \in I} c_i \varphi_i$.

We denote by $\Phi^*$ the analysis operator $\Phi^* : H \to \ell^2(I)$, $\Phi^*(f) = (\langle f, \varphi_i \rangle)_{i \in I}$.

The frame operator $S$ associated with a frame $\Phi = \{ \varphi_i \}_{i \in I}$ is defined by $S = \Phi \Phi^* : H \to H$, $Sf = \sum_{i \in I} \langle f, \varphi_i \rangle \varphi_i$.

Given a frame $\Phi = \{ \varphi_i \}_{i \in I}$ for $H$, there exists a sequence $\Phi^d = \{ \varphi^d_i \}_{i \in I}$ in $H$ such that
\[ f = \Phi(\Phi^d)^* f = \sum_{i \in I} \langle f, \varphi^d_i \rangle \varphi_i, \quad \forall f \in H, \] (2)
and
\[ f = (\Phi^d) \Phi^* f = \sum_{i \in I} \langle f, \varphi_i \rangle \varphi^d_i, \quad \forall f \in H. \] (3)

If both equations hold, $\Phi^d$ is called an alternative dual frame (or simply a dual frame) of $\Phi$. Respectively, we call $\Phi^d$ an analysis pseudo-dual of the frame $\Phi$, if only (2) holds and a synthesis pseudo-dual in case only (3) holds. In addition, the inequality (1) implies that $S$ is a self-adjoint, invertible operator on $H$. This leads to a special dual frame $\{ S^{-1} \varphi_i \}_{i \in I}$ called the canonical dual frame of $\Phi = \{ \varphi_i \}_{i \in I}$ with frame bounds $(B^{-1}, A^{-1})$. In our analysis, we denote by $\mathbb{D}_\Phi$ the set of all synthesis pseudo-dual of the frame $\Phi$, i.e.,
\[ \mathbb{D}_\Phi = \left\{ \Phi^d = \{ \varphi^d_i \}_{i \in I} \in H \mid f = \Phi^d \Phi^* f = \sum_{i \in I} \langle f, \varphi_i \rangle \varphi^d_i, \forall f \in H \right\}. \] (4)

Obviously, $\mathbb{D}_\Phi \neq \emptyset$ since for each frame $\Phi$ the frame $S^{-1} \Phi = \{ S^{-1} \varphi_i \}_{i \in I}$ is dual to $\Phi$ and therefore in $\mathbb{D}_\Phi$. Consequently, if $\Phi$ is a Parseval frame we have $\Phi \in \mathbb{D}_\Phi$. For more details on frame theory we refer to [3, 16, 20].

2.2. Outline. The rest of the paper is organized as follows. First we formulate the general image separation problem in Section 2. Next, in Section 3 we provide a theoretical machinery which guarantees the success of Algorithm 1 based on notions of joint concentration and cluster coherence of two general frames. We then present the model of pointlike and curvelike singularities and construct radial wavelets as well as a pair of dual bandlimited $\alpha$-shearlets in Section 4. In Section 5 we finally present an asymptotic separation result of the proposed component models by $l^1-$minimization. We close with a conclusion and outlook to further applications in Section 6.
2.3. **General component separation.** Given an image \( f \), we assume that \( f \) can be composed of two geometric components, i.e.,

\[
    f = \mathcal{P} + \mathcal{C},
\]

where \( \mathcal{P}, \mathcal{C} \) are two unknown components which we want to recover. Since the unknowns are twice as many as the equations, the task of component separation is ill-posed without additional assumptions. However, we often have more information about the components \( \mathcal{P}, \mathcal{C} \). Compressed sensing techniques enable us to exactly recover these components which are sparse in appropriate dictionaries. Here, we assume that \( \mathcal{P} \) is smooth away from point discontinuities and \( \mathcal{C} \) is smooth away from curvilinear singularities. In our analysis, we use microlocal analysis to give a heuristic understanding of why separation might be possible. The core ingredient is based on the idea that important coefficients are clustered geometrically in phase space.

2.4. **Recovery via \( l_1 \)-minimization.** We consider the following algorithm proposed in the past which used Parseval frames for separating two components. We extend it to the case of two general frames \( \{\Phi_1\}_{i \in I}, \{\Phi_2\}_{j \in J} \) with frame bounds \((A_1, B_1)\) and \((A_2, B_2)\) respectively.

**Algorithm 1: Image separation**

**Data:** observed image \( f \), two frames \( \{\Phi_1\}_{i \in I}, \{\Phi_2\}_{j \in J} \).

**Compute:** \((\mathcal{P}^*, \mathcal{C}^*)\), where

\[
    (\mathcal{P}^*, \mathcal{C}^*) = \arg \min_{f_1, f_2} \|\Phi_1^* f_1\|_1 + \|\Phi_2^* f_2\|_1, \quad \text{subject to} \quad f_1 + f_2 = f.
\]

**Result:** recovered components. \( \mathcal{P}^*, \mathcal{C}^* \).

We would like to remark that here we minimize the \( l^1 \) norm of the analysis coefficients when expanding the components in two frames to exploit their geometric features underlying the image. The success of the algorithm (6) is proven later under prior information that each geometric component is captured by the corresponding frame.

3. **Theoretical guarantee for component separation**

3.1. **Joint concentration analysis.** The notion of joint concentration was first introduced in [12], which used Parseval frames to propose an analyzing tool for deriving the theoretical guarantee. There, the joint concentration associated with two Parseval frames \( \Phi_1, \Phi_2 \) and sets of indexes \( \Lambda_1, \Lambda_2 \) is defined by

\[
    \kappa(\Lambda_1, \Lambda_2) = \sup_{f \in \mathcal{H}} \frac{\|I_{\Lambda_1} \Phi_1^* f\|_1 + \|I_{\Lambda_2} \Phi_2^* f\|_1}{\|\Phi_1^* f\|_1 + \|\Phi_2^* f\|_1}.
\]

For an extension, we modify the joint concentration associated with two frames instead of Parseval frames based on the idea that each frame can sparsely represent each component but can not sparsely represent the other. This philosophy plays a central role in the success of the proposed algorithm.

**Definition 3.1.** Let \( \Phi_1, \Phi_2 \) be two frames. We define the joint concentration \( \bar{\kappa} = \bar{\kappa}(\Lambda_1, \Lambda_2) \) with respect to sets of coefficients \( \Lambda_1, \Lambda_2 \) by

\[
    \bar{\kappa}(\Lambda_1, \Lambda_2) = \sup_{\mathcal{P}, \mathcal{C} \in \mathcal{H}} \frac{\|I_{\Lambda_1} \Phi_1^* \mathcal{P}\|_1 + \|I_{\Lambda_2} \Phi_2^* \mathcal{C}\|_1}{\|\Phi_1^* \mathcal{C}\|_1 + \|\Phi_2^* \mathcal{P}\|_1}.
\]
Definition 3.2. Fix $\delta > 0$. Given a Hilbert space $\mathcal{H}$ with a frame $\Phi, f \in \mathcal{H}$ is \(\delta\)-relatively sparse in $\Phi$ with respect to $\Lambda$ if $\|1_{\Lambda^c} \Phi^* f\|_1 \leq \delta$, where $\Lambda^c$ denotes $X \setminus \Lambda$.

Under the assumption of joint concentration and $\delta$-relative sparsity of the components $\mathcal{P}$ and $\mathcal{C}$, we can guarantee the success of (6), as the next proposition shows.

Proposition 3.3. Let $\Phi_1, \Phi_2$ be two frames with frame bounds $(A_1, B_1), (A_2, B_2)$, respectively. For $\delta_1, \delta_2 > 0$, we assume that $f = \mathcal{P} + \mathcal{C}$ where $\mathcal{P}, \mathcal{C}$ is $\delta_1, \delta_2$-relatively sparse in $\Phi_1$ and $\Phi_2$ with respect to $\Lambda_1$ and $\Lambda_2$, respectively. Let $(\mathcal{P}^*, \mathcal{C}^*)$ solve (6) and we have $\bar{\kappa}(\Lambda_1, \Lambda_2) < \frac{1}{2}$, then

$$\|\mathcal{P}^* - \mathcal{P}\|_2 + \|\mathcal{C}^* - \mathcal{C}\|_2 \leq \frac{2\max\{B_1, B_2\}(\delta_1 + \delta_2)}{1 - 2\bar{\kappa}(\Lambda_1, \Lambda_2)}. \quad (7)$$

Proof. For convenience, we set $\bar{\kappa} := \bar{\kappa}(\Lambda_1, \Lambda_2), \delta = \delta_1 + \delta_2,$ and $err := \mathcal{P}^* - \mathcal{P} = \mathcal{C} - \mathcal{C}^*$. Note here that we have $\mathcal{P}^* + \mathcal{C}^* = \mathcal{P} + \mathcal{C} = f$. By the upper frame bounds of $\Phi_1, \Phi_2$, we obtain

$$\|\mathcal{P}^* - \mathcal{P}\|_2 + \|\mathcal{C}^* - \mathcal{C}\|_2 \leq \max\{B_1, B_2\}\left(\|\Phi_1^* (\mathcal{P}^* - \mathcal{P})\|_1 + \|\Phi_2^* (\mathcal{C}^* - \mathcal{C})\|_1\right)$$

$$\leq \max\{B_1, B_2\}\left(\|\Phi_1^* (err)\|_1 + \|\Phi_2^* (err)\|_1\right). \quad (8)$$

Thus, we have

$$S := \|\Phi_1^* (err)\|_1 + \|\Phi_2^* (err)\|_1 \leq \|1_{\Lambda_1} \Phi_1^* (err)\|_1 + \|1_{\Lambda_2} \Phi_2^* (err)\|_1 + \|1_{\Lambda_1} \Phi_1^* (\mathcal{P}^* - \mathcal{P})\|_1 + \|1_{\Lambda_2} \Phi_2^* (\mathcal{C}^* - \mathcal{C})\|_1$$

$$\leq \bar{\kappa}S + \|1_{\Lambda_1} \Phi_1^* \mathcal{P}^*\|_1 + \|1_{\Lambda_2} \Phi_2^* \mathcal{C}^*\|_1 + \|1_{\Lambda_1} \Phi_1^* \mathcal{P}\|_1 + \|1_{\Lambda_2} \Phi_2^* \mathcal{C}\|_1$$

$$\leq \bar{\kappa}S + \|1_{\Lambda_1} \Phi_1^* \mathcal{P}^*\|_1 + \|1_{\Lambda_2} \Phi_2^* \mathcal{C}^*\|_1 + \delta_1 + \delta_2$$

$$= \bar{\kappa}S + \delta + \|1_{\Lambda_1} \Phi_1^* \mathcal{P}\|_1 + \|1_{\Lambda_1} \Phi_1^* \mathcal{P}^*\|_1 + \|1_{\Lambda_2} \Phi_2^* \mathcal{C}\|_1 + \|1_{\Lambda_2} \Phi_2^* \mathcal{C}^*\|_1.$$

We now exploit that $(\mathcal{P}^*, \mathcal{C}^*)$ is a minimizer of (6). Therefore, we obtain

$$\|\Phi_1^* \mathcal{P}^*\|_1 + \|\Phi_2^* \mathcal{C}^*\|_1 \leq \|\Phi_1^* \mathcal{P}\|_1 + \|\Phi_2^* \mathcal{C}\|_1.$$

Thus, the triangle inequality yields

$$S \leq \bar{\kappa}S + \delta + \|\Phi_1^* \mathcal{P}\|_1 + \|\Phi_2^* \mathcal{C}\|_1 - \|1_{\Lambda_1} \Phi_1^* \mathcal{P}\|_1 - \|1_{\Lambda_2} \Phi_2^* \mathcal{C}\|_1$$

$$\leq \bar{\kappa}S + \delta + \|\Phi_1^* \mathcal{P}\|_1 + \|\Phi_2^* \mathcal{C}\|_1 + \|1_{\Lambda_1} \Phi_1^* (err)\|_1 + \|1_{\Lambda_2} \Phi_2^* (err)\|_1 - \|1_{\Lambda_1} \Phi_1^* \mathcal{P}\|_1$$

$$- \|1_{\Lambda_1} \Phi_1^* \mathcal{P}\|_1 + \|1_{\Lambda_1} \Phi_1^* \mathcal{P}\|_1$$

$$\leq \bar{\kappa}S + 2\delta + \bar{\kappa}S = 2\bar{\kappa}S + 2\delta.$$

This implies

$$S \leq \frac{2\delta}{1 - 2\bar{\kappa}},$$

and the claim follows in combination with (5). \(\square\)

3.2. Theoretical guarantee via Cluster coherence. Typically, the information on each component is encoded by a particular choice of the clusters $\Lambda_1, \Lambda_2$ through the number of non-zero coefficients. By choosing such clusters we might get the successful recovery by Proposition 3.3 but it seems hard to achieve the joint concentration. The notion of cluster coherence was first introduced in [12] in an attempt to transfer the theoretical guarantee based on the notion of joint concentration to another analyzing tool that enables us to check in practice. They used Parseval frames in their definition.
In our paper, we modify this ansatz and extend it to the case of generic frames instead of Parseval frames.

**Definition 3.4.** Given two frames $\Phi_1 = (\Phi_{1i})_{i \in I}$ and $\Phi_2 = (\Phi_{2j})_{j \in J}$ with frame bounds $(A_1, B_1), (A_2, B_2)$, the cluster coherence $\mu_c(\Lambda, \Phi_1; \Phi_2)$ of $\Phi_1$ and $\Phi_2$ with respect to the index set $\Lambda \subset I$ is defined by

$$\mu_c(\Lambda, \Phi_1; \Phi_2) = \max_{j \in J} \sum_{i \in \Lambda} |\langle \phi_{1i}, \phi_{2j} \rangle|.$$ 

Formulated in this way, the notion of cluster coherence encodes the geometric difference between the components in a way that can be checked in practice. The following lemma allows to bound the joint concentration from above by the cluster coherence.

**Lemma 3.5.** We have

$$\kappa_1(A_1, A_2) \leq \inf_{\Phi_1^d \in \mathcal{D}_{\Phi_1}, \Phi_2^d \in \mathcal{D}_{\Phi_2}} \max \{ \mu_c(\Lambda_1, \Phi_1; \Phi_2^d), \mu_c(\Lambda_2, \Phi_2; \Phi_1^d) \}.$$ 

**Proof.** For $\mathcal{P}, C \in \mathcal{H}, \Phi_1^d \in \mathcal{D}_{\Phi_1}, \Phi_2^d \in \mathcal{D}_{\Phi_2}$, we have $\mathcal{P} = \sum_{i \in I} \langle \mathcal{P}, \phi_{1i} \rangle \phi_{1i}^d$, $C = \sum_{i \in I} \langle C, \phi_{1i} \rangle \phi_{1i}^d$. In the other words, $\mathcal{P} = \Phi_2^d \Phi_1^* \mathcal{P}$, $C = \Phi_1^d \Phi_1^* C$. Now we set $\alpha_1 = \Phi_1^d C$, $\alpha_2 = \Phi_2^d \mathcal{P}$, we then obtain $\mathcal{P} = \Phi_2^d \alpha_2$, $C = \Phi_1^d \alpha_1$. Therefore, we have

$$\|\mathbb{I}_{\Lambda_1} \Phi_1^* \mathcal{P} \|_1 + \|\mathbb{I}_{\Lambda_2} \Phi_2^* C \|_1 = \|\mathbb{I}_{\Lambda_1} \Phi_1^* \Phi_2^d \alpha_2 \|_1 + \|\mathbb{I}_{\Lambda_2} \Phi_2^d \alpha_1 \|_1 \leq \sum_{i \in \Lambda_1} \left( \sum_j |\langle \Phi_{1i}, \Phi_{2j}^d \rangle | \right) \| \alpha_2 \|_1 + \sum_{j \in \Lambda_2} \left( \sum_i |\langle \Phi_{2j}, \Phi_{1i}^d \rangle | \right) \| \alpha_1 \|_1 \leq \mu_c(\Lambda_1, \Phi_1; \Phi_2^d) \| \alpha_2 \|_1 + \| \alpha_1 \|_1 \leq \max \{ \mu_c(\Lambda_1, \Phi_1; \Phi_2^d), \mu_c(\Lambda_2, \Phi_2; \Phi_1^d) \} (\| \alpha_2 \|_1 + \| \alpha_1 \|_1) \leq \max \{ \mu_c(\Lambda_1, \Phi_1; \Phi_2^d), \mu_c(\Lambda_2, \Phi_2; \Phi_1^d) \} (\| \Phi_2^d \mathcal{P} \|_1 + \| \Phi_1^* C \|_1).$$ 

Thus, we obtain $\kappa_1(A_1, A_2) \leq \max \{ \mu_c(\Lambda_1, \Phi_1; \Phi_2^d), \mu_c(\Lambda_2, \Phi_2; \Phi_1^d) \}$, $\forall \Phi_1^d \in \mathcal{D}_{\Phi_1}, \forall \Phi_2^d \in \mathcal{D}_{\Phi_2}$. This completes the proof. \hfill $\square$

We can now present our theoretical guarantee for the procedure (6) to be convergent using two generic frames instead of Parseval frames.

**Theorem 3.6.** Let $\Phi_1, \Phi_2$ be two frames with frame bounds $(A_1, B_1), (A_2, B_2)$, respectively. For $\delta_1, \delta_2 > 0$, we suppose that $f \in \mathcal{H}$ can be decomposed as $f = \mathcal{P} + C$ so that each component $\mathcal{P}, C$ is $\delta_1, \delta_2$-relatively sparse in $\Phi_1$ and $\Phi_2$ with respect to $\Lambda_1$ and $\Lambda_2$, respectively. Let $(\mathcal{P}^*, C^*)$ solve (6). If we have $\mu_c(A_1, A_2) := \inf_{\Phi_1^d \in \mathcal{D}_{\Phi_1}, \Phi_2^d \in \mathcal{D}_{\Phi_2}} \max \{ \mu_c(\Lambda_1, \Phi_1; \Phi_2^d), \mu_c(\Lambda_2, \Phi_2; \Phi_1^d) \} < \frac{1}{2}$, then

$$\|\mathcal{P}^* - \mathcal{P} \|_2 + \|\mathcal{C}^* - \mathcal{C} \|_2 \leq \frac{2 \max \{ B_1, B_2 \} (\delta_1 + \delta_2)}{1 - 2 \mu_c(A_1, A_2)}.$$ 

**Proof.** The proof follows directly from Proposition 3.3 and Lemma 3.5. \hfill $\square$

We would like to remark that in case $\Phi_1, \Phi_2$ are Parseval frames we have $\mu_c(A_1, A_2) \leq \max \{ \mu_c(\Lambda_1, \Phi_1; \Phi_2), \mu_c(\Lambda_2, \Phi_2; \Phi_1) \}$ since $\Phi_1 \in \mathcal{D}_{\Phi_1}, \Phi_2 \in \mathcal{D}_{\Phi_2}$. This consequence is exactly the result used in several papers [2, 10, 12, 18, 21] which chose Parseval frames as sparsifying systems. Thus, our theoretical guarantee based on the notion of cluster
coherence [9] is more general than are shown in aforementioned papers since we can choose an other dual instead of using itself. Although it is not easy to construct a dual, this theoretical guarantee actually shows that we can use synthesis pseudo-dual’s properties instead of its explicit construction. The nice property we need in our analysis is that dual frame elements have good time-frequency localization which leads to small cluster coherence.

For the construction of a shearlet dual, some works [19, 22] have shown desirable properties such as well localization and highly directional sensitivity. In the next section, we use the approach from [19] to construct a dual pair of bandlimited $\alpha$-shearlets for our analysis.

4. Component separation

4.1. Mathematical model of components. Consider the image separation problem (5), we now introduce the model of components. In our analysis, we assume that the pointlike part $\mathcal{P}$ is modeled as

$$\mathcal{P}(x) = \sum_{i=1}^{K} c_i |x - x_i|^{-\lambda_i},$$  

where two sequences of constants $\{\lambda_i\}_{i=1}^{K}, \{c_i\}_{i=1}^{K}$ satisfy $0 < \lambda_i < 2, 0 < c_i, \forall i = 1, 2, \ldots, K$. We choose $\lambda_i < 2$ to bound the energy of the component in frequency domain as the scale goes finer. The choice of $\{\lambda_i\}_{i=1}^{K}$ depends on each problem of image separation which make components comparable.

For the curvilinear singularities, we first recall the Schwartz functions or the rapidly decreasing functions

$$\mathcal{S}(\mathbb{R}^2) = \left\{ f \in C^\infty(\mathbb{R}^2) \mid \forall K, N \in \mathbb{N}_0, \sup_{x \in \mathbb{R}^2} (1 + |x|^2)^{-N/2} \sum_{|\alpha| \leq K} |D^\alpha f(x)| < \infty \right\}.$$  

Let $\sigma : [0, 1] \to \mathbb{R}^2$ be a closed $C^\beta$ curve, $\beta \in (1, 2]$, with non vanishing curvature everywhere. We first consider the model of curvilinear singularity $\mathcal{C}$ as

$$\mathcal{C} = \int_{\mathbb{R}^2} \delta_{\sigma(t)} dt,$$  

where $\delta_x$ denotes the usual Dirac delta distribution located at $x$. It is well known that the class of $\alpha$-shearlets using $\alpha$-scaling can sparsely represent such curvilinear structures [39, 40]. For the sake of simplicity, we restrict our model to the case of line singularity. The reader should be aware of the fact that by Tubular neighborhood theorem we can extend it to the general case. Intuitively, we can use the technique in [12] to first partition the curve $\sigma$ into small pieces and then bend them to the form of a line singularity, see [12, Section 6] for details.

Similarly as introduced in several papers [15, 36], we model the line distribution $w\mathcal{L}$ acting on Schwartz functions by

$$\langle w\mathcal{L}, f \rangle = \int_{\rho}^\rho w(x_1)f(x_1, 0)dx_1, \ f \in \mathcal{S}(\mathbb{R}^2),$$

where $w$ is a weighted function such that $0 \not\equiv w \in C^\infty(\mathbb{R})$, supp $w \subset [-\rho, \rho]$, for some $\rho > 0$, and $0 \leq w(x) \leq 1, \forall x \in [-\rho, \rho]$. For such a setting, we now approach the question if it possible to separate point-singularities and the line singularities using wavelets and bandlimited $\alpha$-shearlet types which interpolate from wavelet type ($\alpha = 2$) to shearlet type ($\alpha = 1$). In our analysis, we prove that we can separate
them as long as bandlimited \( \alpha \)-shearlets do not coincide with the wavelet type, i.e., \( 2 > \alpha \geq 1 \).

Among well-known systems, wavelets provide an optimal sparse representation to the pointwise singularities, whereas shearlets are efficient for curvilinear structures. In what follows, we introduce the construction of these two sparsifying systems.

4.2. Wavelet frames. To sparsely present \( \mathcal{P} \), we choose radial wavelets which form a Parseval frame with perfectly isotropic generating elements. We modify the construction of radial wavelets as follows.

Let \( \Xi \) be a Schwartz function on \( \mathbb{R}^2 \) such that \( \text{supp} \hat{\Xi} \subset [-\frac{1}{16}, \frac{1}{16}], 0 \leq \hat{\Xi}(\theta) \leq 1 \) for \( \theta \in \mathbb{R} \) and \( \hat{\Xi}(\theta) = 1 \) for \( \theta \in [-\frac{1}{32}, \frac{1}{32}] \). We now define the low-pass function \( \Omega(\xi) \) and the window function \( W(\xi) \) for \( j \in \mathbb{N} \) and \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \),

\[
\hat{\Omega}(\xi) := \hat{\Xi}(\xi_1)\hat{\Xi}(\xi_2), \quad (13)
\]

\[
W(\xi) := \sqrt{\hat{\Omega}^2(2^{-2\xi}) - \hat{\Omega}^2(\xi)}, \quad W_j(\xi) := W(2^{-2j}\xi). \quad (14)
\]

By definition, \( \sup W \subset [-\frac{1}{4}, \frac{1}{4}]^2 \setminus [-\frac{1}{32}, \frac{1}{32}]^2 \) and \( W_j \) is compactly supported in the corona

\[
A_j := [-2^{2j-2}, 2^{2j-2}]^2 \setminus [-2^{2j-5}, 2^{2j-5}]^2, \quad \forall j \geq 1. \quad (15)
\]

In addition, we obtain the partition of unity property

\[
\hat{\Omega}^2(\xi) + \sum_{j \geq 0} W_j^2(\xi) = 1, \quad \forall \xi \in \mathbb{R}^2. \quad (16)
\]

The radial wavelets \( \Psi = \{\psi_{j,m}(\cdot)\}_{j,m} \cup \{\Omega(\cdot - m)\}, j \in \mathbb{N}_0, m \in \mathbb{Z}^2 \) which form a Parseval frame are then defined by their Fourier transforms

\[
\hat{\psi}_{j,m}(\xi) = 2^{-2j}W_j(\xi)e^{2\pi i T_m / 2^j}, j \in \mathbb{N}_0, m \in \mathbb{Z}^2.
\]

Since the low frequency part is not of interest to us in our analysis, we therefore simply write \( \Psi = \{\psi_{j,m}(\cdot)\}_{j,m} \) at some points.

4.3. A pair of bandlimited \( \alpha \)-shearlet dual frames. This section is devoted to the construction of a pair of bandlimited \( \alpha \)-shearlet dual frames which possess many desirable properties. We choose shearlets as it is widely accepted that shearlets in general provide optimal sparse representation for images which are governed by curvilinear structures \( [6, 13, 2] \). Motivated by \([19]\), we modify the construction of the shearlet frame pair and extend it to the case of \( \alpha \)-scaling instead of parabolic scaling since \( \alpha \)-shearlet type with \( \alpha = \frac{2}{3}, \alpha \in [1, 2) \), might be best adapted to \( C^\beta \) curvilinear singularities. In addition, we modify the Fourier domain decomposition to make it comparable to the wavelet frames.

We first define the scaling and shearing matrix by

\[
A_{\alpha,h} := \begin{bmatrix} 2^\alpha & 0 \\ 0 & 2^\alpha \end{bmatrix}, \quad S_h := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad (17)
\]

\[
A_{\alpha,v} := \begin{bmatrix} 2^\alpha & 0 \\ 0 & 2^\alpha \end{bmatrix}, \quad S_v := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}. \quad (18)
\]
Theoretical analysis for pointlike and $C^\beta$ curvelike singularities

where $\alpha \in [1, 2)$ is the scaling parameter. Let $v \in C^\infty(\mathbb{R})$ be a bump function such that $\text{supp } v \subset \left[-\frac{3}{2}, \frac{3}{2}\right]$ and

$$\sum_{l=-2}^{2} |v(\omega - l)|^2 = 1, \text{ for } \omega \in \left[-\frac{3}{2}, \frac{3}{2}\right].$$

Consequently, the following holds for $j \geq 0, \omega \in \left[-\frac{3}{2}, \frac{3}{2}\right]$:

$$\sum_{l=-[2 \cdot 2^{(2-\alpha)j}]}^{[2 \cdot 2^{(2-\alpha)j}]} |v(2^{(2-\alpha)j}\omega - l)|^2 = 1. \quad (19)$$

Next, we define the cone functions $V_h, V_v$ by

$$V_h(\xi) := v\left(\frac{\xi_2}{\xi_1}\right), \quad V_v(\xi) := v\left(\frac{\xi_1}{\xi_2}\right). \quad (20)$$

horizontal frequency cone and the vertical frequency cone

$$C_h := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| \geq \frac{1}{8}, \frac{\xi_2}{\xi_1} \leq \frac{3}{2} \right\}, \quad (21)$$

$$C_v := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \geq \frac{1}{8}, \frac{\xi_1}{\xi_2} \leq \frac{3}{2} \right\}, \quad (22)$$

and low frequency part

$$C_0 := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1|, |\xi_2| \leq 1 \right\}. \quad (23)$$

A system of shearlets is then defined by

$$\Phi = \left\{ \tilde{\phi}_{j,l,k}^\alpha(x), \tau = \{h, v\}, j \in \mathbb{N}, -[2 \cdot 2^{(2-\alpha)j}] \leq l \leq [2 \cdot 2^{(2-\alpha)j}], k \in \mathbb{Z}^2 \right\},$$

where $\tilde{\phi}_{j,l,k}^\alpha(\xi) = W_j(\xi) V_l \left( \xi^T A_{\alpha,j}^{-1} S_{l}^{-1} \right) e^{2\pi i \xi^T A_{\alpha,j}^{-1} S_{l}^{-1} k}, \tau = \{h, v\}$. For an illustration, Fig. 1 shows the tiling of the frequency domain induced by shearlets.

By the definition $\tilde{\phi}_{j,l,k}^\alpha$ has compact support in the trapezoidal region

$$\text{supp } \tilde{\phi}_{j,l,k}^\alpha = \left\{ \xi \in \mathbb{R}^2 : \xi_2 \in [-2^{2j-2}, 2^{2j-2}] \setminus [-2^{2j-5}, 2^{2j-5}], \frac{\xi_1}{\xi_2} - l 2^{-(2-\alpha)} \right\}, \quad (24)$$

$$\text{supp } \tilde{\phi}_{j,l,k}^\alpha = \left\{ \xi \in \mathbb{R}^2 : \xi_1 \in [-2^{2j-2}, 2^{2j-2}] \setminus [-2^{2j-5}, 2^{2j-5}], \frac{\xi_2}{\xi_1} - l 2^{-(2-\alpha)} \right\}. \quad (25)$$

For convenience, we also define two following compact sets

$$C_h^\alpha := \left\{ \xi \in \mathbb{R}^2 : \frac{1}{32} \leq |\xi_1| \leq \frac{1}{4}, \frac{|\xi_2|}{|\xi_1|} \leq \frac{3}{2} \right\} \supset \text{supp } \tilde{\phi}_{j,l,k}^\alpha(\xi^T S_{h}^j A_{\alpha,h}^j), \quad (26)$$

$$C_v^\alpha := \left\{ \xi \in \mathbb{R}^2 : \frac{1}{32} \leq |\xi_2| \leq \frac{1}{4}, \frac{|\xi_1|}{|\xi_2|} \leq \frac{3}{2} \right\} \supset \text{supp } \tilde{\phi}_{j,l,k}^\alpha(\xi^T S_{v}^j A_{\alpha,v}^j). \quad (27)$$
The key ingredient for the construction of a pair of dual shearlet frames is derived from the following Parseval frames for sub-spaces \( L^2(\mathbb{C}_h)^\vee, L^2(\mathbb{C}_v)^\vee \) which can be then aggregated to form a frame for \( L^2(\mathbb{R}^2) \).

**Lemma 4.1.** The system \( \Phi^*_\alpha = \{ \phi^{\alpha,\ell}_{j,k}, \ell \in \mathbb{N}_0, -[2 \cdot 2^{(2-\alpha)j}], k \in \mathbb{Z}^2 \} \) forms a Parseval frame for \( L^2(\mathbb{C}_\ell)^\vee, \ell = \{ h, v \} \) where \( L^2(\mathbb{C}_\ell)^\vee = \{ f \in L^2(\mathbb{R}^2) : \text{supp } \hat{f} \subset \mathbb{C}_\ell \} \).

**Proof.** We only consider \( \ell = h \), since the other case is done similarly. Indeed, by (16) and (19) we have

\[
\sum_{j \in \mathbb{N}_0} |\phi^{\alpha,\ell}_{j,k}(\xi)|^2 = \sum_{j \in \mathbb{N}_0} |W_j(\xi)|^2 = 1, \quad \forall \xi \in \mathbb{C}_h.
\]

Here we note that low-pass function \( \Omega(\xi) = 0, \forall \xi \in \mathbb{C}_h \). By using Parseval’s identity and the observation that \( \text{supp } \hat{\phi}^{\alpha,\ell}_{j,k}(\xi) S^h \hat{A}^j_{\alpha,h} \subset \mathbb{C}_h^* \subset [-\frac{1}{2}, \frac{1}{2}]^2 \), we concludes the claim by standard arguments. \( \square \)

We will construct a pair of dual shearlet frames by carefully patching together three Parseval frames \( \Phi^*_\alpha, \ell = \{ 0, h, v \} \), where the construction of translation-invariant Parseval frame \( \Phi^0 \) for \( L^2(\mathbb{C}_0) \) is well-known. We first define corresponding cones of the dual

\( \mathbb{C}_h^d := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| \geq \frac{1}{4}, \left| \frac{\xi_2}{\xi_1} \right| \leq \frac{4}{3} \right\} \),

(28)

\( \mathbb{C}_v^d := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \geq \frac{1}{4}, \left| \frac{\xi_1}{\xi_2} \right| \leq \frac{4}{3} \right\} \),

(29)

and low frequency part

\( \mathbb{C}_0^d := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1|, |\xi_2| \leq \frac{2}{3} \right\} \).

(30)

For an illustration, we refer to Figure 2.
Next, we choose \( \chi, \gamma \) in \( C^\infty(\mathbb{R}^2) \), \( \iota = 0, h, v \) by the following lemma.

**Lemma 4.2.** There exist \( \chi, \gamma \) in \( C^\infty(\mathbb{R}^2) \), \( \iota = 0, h, v \) such that the following properties hold:

1. \( \text{supp}\ \chi, \gamma \subset C_\iota, \ i = \{0, h, v\}, \sum_{i \in \{0, h, v\}} \chi_i(\xi) = 1, \ \forall \xi \in \mathbb{R}^2. \)

2. \( \chi_i(\xi^T S_i A_i^j), \gamma_i(\xi^T S_i A_i^j) \in C^\infty(C^*_\iota) \) with norms

\[
\|\chi_i((\cdot)^T S_i A_i^j)\|_{C^N(C^*_\iota)}, \ |\gamma_i((\cdot)^T S_i A_i^j)\|_{C^N(C^*_\iota)} \leq C_N,
\]

where constants \( C_N \) are independent of \( j \).

**Proof.** Let \( \chi_0 \in C^\infty, \text{supp}\ \chi = C_0 = \{\xi \in \mathbb{R}^2 : |\xi| \leq 1\} \) and \( \chi_0(\xi) = 1, \ \forall \xi = (\xi_1, \xi_2) : |\xi| \leq \frac{3}{4}. \) We now define

\[
\chi_h := g_h(\xi_1)h_h\left(\frac{\xi_2}{\xi_1}\right), \ \ \chi_v(\xi_1, \xi_2) := g_h(\xi_2)h_h\left(\frac{\xi_1}{\xi_2}\right),
\]

where \( g_h, h_h \) are real-valued functions such that \( g_h, h_h \in C^\infty(\mathbb{R}), \text{supp}\ g_h = [\frac{1}{3}, \infty), g_h|_{[\frac{1}{3}, \frac{1}{2}]} = 1. \) and \( \text{supp}\ h_h \subset [-\frac{3}{2}, \frac{3}{2}], h_h|_{[-\frac{3}{2}, \frac{3}{2}]} = 1. \)

Next, we define

\[
\gamma_h(\xi) := g_v(\xi_1)h_v\left(\frac{\xi_2}{\xi_1}\right), \ \ \gamma_v(\xi) = g_v(\xi_2)\left[1 - h_v\left(\frac{\xi_2}{\xi_1}\right)\right],
\]

where \( g_v = [\frac{1}{4}, \infty), g_v|_{[\frac{1}{4}, \infty)} = 1, \) and \( \text{supp}\ h_v = [-\frac{4}{3}, \frac{4}{3}], h_v|_{[-\frac{4}{3}, \frac{4}{3}]} = 1. \) We now choose \( \gamma_0(\xi) = 1 - \chi_h(\xi)\gamma_h(\xi) - \chi_v(\xi)\gamma_v(\xi), \ \forall \xi \in \mathbb{R}^2. \) We prove that \( \chi, \gamma \) satisfy desired properties. Indeed,

1. Obviously, \( \text{supp}\ \chi \subset C_\iota, \ i = \{0, h, v\}. \) By definition, \( \text{supp}\ \gamma_h \subset C^d_h, \) and \( \text{supp}\ \gamma_v \subset C^d_v \) since \( 1 - h_v\left(\frac{\xi_2}{\xi_1}\right) = 0 \) for \( \xi = (\xi_1, \xi_2) : \left|\frac{\xi_2}{\xi_1}\right| > \frac{4}{3}. \)
Now it remains to show that supp \( \gamma_0 \subset C_0 \). For this, we need to prove \( \chi_h(\xi)\gamma_h(\xi) + \chi_v(\xi)\gamma_v(\xi) = 1 \) for \( \xi, |\xi| > \frac{2}{3} \). Indeed, we first observe

\[
h_h(\frac{\xi_0}{\xi_1}) \cdot h_v(\frac{\xi_2}{\xi_1}) = h_v(\frac{\xi_0}{\xi_1}), \forall \xi \in \mathbb{R}^2,
\]

and

\[
h_h(\frac{\xi_1}{\xi_2}) \cdot [1 - h_v(\frac{\xi_2}{\xi_1})] = 1 - h_v(\frac{\xi_0}{\xi_1}), \forall \xi \in \mathbb{R}^2.
\]

Next we consider four cases

Case 1: \( |\xi_1| \geq \frac{2}{3}, |\xi_2| < \frac{1}{2} \). Since \( |\xi_1| > \frac{1}{2}, \left| \frac{\xi_2}{\xi_1} \right| > \frac{4}{3}, \left| \frac{\xi_0}{\xi_1} \right| < \frac{3}{4} \), we obtain \( g_h(\xi_1) = g_v(\xi_1) = 1 \). Thus, \( \chi_h(\xi)\gamma_h(\xi) + \chi_v(\xi)\gamma_v(\xi) = 1 \).

Case 2: \( |\xi_1| \geq \frac{2}{3}, |\xi_2| \geq \frac{1}{2} \). We have \( g_h(\xi_1) = g_v(\xi_1) = g_h(\xi_2) = g_v(\xi_2) = 1 \). This implies

\[
\chi_h(\xi)\gamma_h(\xi) + \chi_v(\xi)\gamma_v(\xi) = h_h(\frac{\xi_2}{\xi_1})h_v(\frac{\xi_2}{\xi_1}) + h_v(\frac{\xi_1}{\xi_2})\left[1 - h_v(\frac{\xi_2}{\xi_1})\right] = 1.
\]

Case 3: \( |\xi_2| \geq \frac{2}{3}, |\xi_1| < \frac{1}{2} \). Since \( |\xi_2| \geq \frac{1}{2}, \left| \frac{\xi_2}{\xi_1} \right| > \frac{4}{3}, \left| \frac{\xi_0}{\xi_1} \right| < \frac{3}{4} \), we obtain \( g_h(\xi_2) = g_v(\xi_2) = 1 \), \( h_h(\frac{\xi_2}{\xi_1}) = 1, h_v(\frac{\xi_2}{\xi_1}) = 0 \). Thus, \( \chi_h(\xi)\gamma_h(\xi) + \chi_v(\xi)\gamma_v(\xi) = 0 + 1 = 1 \).

Case 4: \( |\xi_2| \geq \frac{2}{3}, |\xi_1| \geq \frac{1}{2} \). Since \( g_h(\xi_1) = g_v(\xi_1) = g_h(\xi_2) = g_v(\xi_2) = 1 \), we derive

\[
\chi_h(\xi)\gamma_h(\xi) + \chi_v(\xi)\gamma_v(\xi) = h_h(\frac{\xi_2}{\xi_1})h_v(\frac{\xi_2}{\xi_1}) + h_v(\frac{\xi_1}{\xi_2})\left[1 - h_v(\frac{\xi_2}{\xi_1})\right] = 1.
\]

Thus, we obtain \( \text{supp} \gamma_0 \subset C_0^d \).

On the other hand, by definition we have \( \gamma_0(\xi) + \chi_h(\xi)\gamma_h(\xi) + \chi_v(\xi)\gamma_v(\xi) = 1, \forall \xi \in \mathbb{R}^2 \). In addition, \( \text{supp} \gamma_0 \subset C_0^d = \{ \xi \in \mathbb{R}^2 : |\xi| \leq \frac{2}{3} \} \) and \( \chi_0 \equiv 1, \forall \xi \in C_0^d \), we obtain \( \chi_0(\xi)\gamma_0(\xi) = \gamma_0(\xi), \forall \xi \in \mathbb{R}^2 \). Thus, \( \sum_{t \in \text{supp} \chi_h} \chi_t(\xi)\gamma_t(\xi) = 1 \). This concludes the claim.

iii) First we consider

\[
\chi_h(\xi^T S^l_{h} A^j_{\alpha,h}) = \chi_h(2^{2j} \xi_1, 2^{\alpha j} (\xi_2 + l \xi_1)) = g_h(2^{2j} \xi_1) h_h(2^{-\alpha j} \xi_2) h_v(\frac{\xi_2}{\xi_1} + l).
\]

Since we have \( g_h(2^{2j} \xi_1) \equiv 1, \forall \xi \in C_0^*, \) for \( j \geq 2 \). Obviously, \( \chi_h(\xi^T S^l_{h} A^j_{\alpha,h}) \in C^\infty(C_0^*) \) with \( \| \chi_h(\cdot)^T S^l_{h} A^j_{\alpha,h} \|_{C^\infty(C_0^*)} \leq C_N \). The proof works analogously to case of \( \chi_v(\xi^T S^l_{v} A^j_{\alpha,v}) \), \( \gamma_h(\xi^T S^l_{h} A^j_{\alpha,h}) \) It remains to prove for \( \gamma_v(\xi^T S^l_{v} A^j_{\alpha,v}) \) we have

\[
\gamma_v(\xi^T S^l_{v} A^j_{\alpha,v}) = g_v(2^{2j} \xi_2) \left[1 - h_v(\frac{2^{2j} \xi_2}{2^{\alpha j} (l \xi_1 + \xi_2)})\right].
\]

The observation \( h_v = [-\frac{4}{3}, \frac{4}{3}] \) implies \( h_v(\frac{2^{2j} \xi_2}{2^{\alpha j} (l \xi_1 + \xi_2)}) = 0 \), for \( \left| \frac{2^{2j} \xi_2}{2^{\alpha j} (l \xi_1 + \xi_2)} \right| > \frac{4}{3} \).

In the other words, for \( j \geq 2, \gamma_v(\xi^T S^l_{v} A^j_{\alpha,v}) \neq 1 \) only for \( l \geq 2^{(2-\alpha)j} \). By a direct computation, for \( j \geq 2, l \geq 2^{(2-\alpha)j} \), we obtain \( G(\xi) := 2^{(2-\alpha)j} \frac{\xi_2}{l \xi_1 + \xi_2} \in C^\infty(C_0^*) \) and
\[
\left| \frac{\partial^N}{\partial \xi^N} G(\xi) \right| \leq C_N, \text{ where } C_N \text{ are constants independent of } j. \text{ This finishes the proof.} \]

Let us now go back to the starting point of constructing a dual pair of shearlet types. We define two representation systems of bandlimited \( \alpha \)-shearlet associated with those functions in Lemma 4.2 by
\[
\Phi_\alpha = \left\{ \tilde{\chi}_\ell \ast \phi_{\alpha, l, k}^{\alpha} (x), \ell = \{h, v\}, j \in \mathbb{N}_0, -\left\lfloor 2 \cdot 2^{(2-\alpha)j} \right\rfloor \leq 1 \leq \left\lfloor 2 \cdot 2^{(2-\alpha)j} \right\rfloor, k \in \mathbb{Z}^2 \right\}
\]
\[
\bigcup \left\{ \chi_0 \ast \phi_\ell^0 (x), k \in \mathbb{Z}^2 \big\},
\]
and
\[
\Phi_\alpha^d = \left\{ \tilde{\gamma}_\ell \ast \phi_{\alpha, l, k}^{\alpha} (x), \ell = \{h, v\}, j \in \mathbb{N}_0, -\left\lfloor 2 \cdot 2^{(2-\alpha)j} \right\rfloor \leq 1 \leq \left\lfloor 2 \cdot 2^{(2-\alpha)j} \right\rfloor, k \in \mathbb{Z}^2 \right\}
\]
\[
\bigcup \left\{ \gamma_0 \ast \phi_\ell^0 (x), k \in \mathbb{Z}^2 \big\},
\]
where \( \Phi_0 = \left\{ \phi_\ell^0 (x) \right\}_{k \in \mathbb{Z}^2} \) forms a Parseval frame for \( L^2(\mathbb{C}_0) \).

As mentioned before, these two systems are a natural extension of shearlets by using flexible scaling to accommodate the smoothness of the data. They consist of a countable collection of well-localized shearlet elements at various locations, scales, and orientations. In addition, we combine two Parseval frames by using \( \chi_\ell, \gamma_\ell \) instead of \( \gamma_\ell, \tilde{\chi}_\ell \) used in [19] which might lead to bad behavior when we consider higher derivatives of shearlet elements.

For convenience, let us find the following index set of shearlets
\[
\Delta := \left\{ (j, l, k, \ell) \mid j \geq 0, l \in \mathbb{Z}, |l| \leq \left\lfloor 2 \cdot 2^{(2-\alpha)j} \right\rfloor, k \in \mathbb{Z}^2, \ell \in \{h, v\} \right\}. \tag{33}
\]

**Lemma 4.3.** The systems \( \Phi_\alpha, \Phi_\alpha^d \) form a pair of shearlet frames for \( L^2(\mathbb{R}^2) \). Consequently, we have \( \Phi_\alpha^d \in \mathbb{D}_\Phi_\alpha \).

**Proof.** First we prove that \( \Phi_\alpha \) forms a frame for \( L^2(\mathbb{R}^2) \). Indeed, the observation \( \text{supp } \chi_\ell \subseteq \mathbb{C}_\ell, \ell = \{0, h, v\} \) implies \( \text{supp } \chi_\ell \hat{f} \subseteq \mathbb{C}_\ell, \ell = \{0, h, v\} \). Now we exploit that \( \Phi_0^0 \) and \( \Phi_\alpha^d = \left\{ \phi_{\alpha, j, l, k}^{\alpha} \right\}_{j \in \mathbb{N}_0, -\left\lfloor 2 \cdot 2^{(2-\alpha)j} \right\rfloor \leq l \leq \left\lfloor 2 \cdot 2^{(2-\alpha)j} \right\rfloor, k \in \mathbb{Z}^2}, \ell = \{h, v\} \) constitute Parseval frames for \( \mathbb{C}_\ell, \ell = \{0, h, v\} \) and obtain
\[
\sum_{\{(j, l, k, \ell) \in \Delta \}} |\langle f, \Phi_\alpha \rangle|^2 = \sum_{\{(j, l, k, \ell) \in \Delta \}} |\langle \hat{f}, \chi_\ell \phi_{\alpha, l, k}^{\alpha} \rangle|^2 + \sum_{k \in \mathbb{Z}^2} |\langle \hat{f}, \chi_0 \phi_\ell^0 \rangle|^2
\]
\[
= \sum_{\{(j, l, k, \ell) \in \Delta \}} |\langle \chi_\ell \hat{f}, \tilde{\phi}_{\alpha, l, k}^{\alpha} \rangle|^2 + \sum_{k \in \mathbb{Z}^2} |\langle \chi_0 \hat{f}, \phi_\ell^0 \rangle|^2
\]
\[
= \sum_{\ell \in \{0, h, v\}} \|\chi_\ell \hat{f}\|_2^2 \leq \sum_{\ell \in \{0, h, v\}} \|\chi_\ell\|_\infty^2 \cdot \|\hat{f}\|_2^2. \tag{34}
\]

In addition,
\[
3 \sup_{\ell \in \{0, h, v\}} \|\gamma_\ell\|_\infty^2 \cdot \sum_{\{(j, l, k, \ell) \in \Delta \}} |\langle f, \Phi_\alpha \rangle|^2 = 3 \sup_{\ell \in \{0, h, v\}} \|\gamma_\ell\|_\infty^2 \cdot \sum_{\ell \in \{0, h, v\}} \|\chi_\ell \hat{f}\|_2^2
\]
\[
\geq \|\sum_{\ell \in \{0, h, v\}} \chi_\ell \gamma_\ell \hat{f}\|_2^2 = \|\hat{f}\|_2^2. \tag{35}
\]
Combine (34) and (35) we obtain

$$
\frac{1}{3 \sup_{\iota \in \{0, h, v\}} \| \gamma_\iota \|_\infty^2} \cdot \| f \|_2^2 \leq \sum_{\{ (j, l, k, \iota) \in \Delta \} \cup \{ \iota = 0, k \in \mathbb{Z}^2 \}} \| \langle f, \Phi_\alpha \rangle \|_2^2 \leq \sum_{\iota \in \{0, h, v\}} \| \chi_\iota \|_\infty^2 \cdot \| f \|_2^2. \quad (36)
$$

Similarly, we have

$$
\frac{1}{3 \sup_{\iota \in \{0, h, v\}} \| \chi_\iota \|_\infty^2} \cdot \| f \|_2^2 \leq \sum_{\{ (j, l, k, \iota) \in \Delta \} \cup \{ \iota = 0, k \in \mathbb{Z}^2 \}} \| \langle f, \Phi_\alpha^d \rangle \|_2^2 \leq \sum_{\iota \in \{0, h, v\}} \| \gamma_\iota \|_\infty^2 \cdot \| f \|_2^2. \quad (37)
$$

It remains to show that $\Phi_\alpha, \Phi_\alpha^d$ form a pair of dual frames. Due to Plancherel’s theorem and $\Phi_0, \Phi_\iota, \iota = \{h, v\}$ being Parseval frames for $L^2(C_\iota), \iota = \{0, h, v\}$, we have

$$
\sum_{\{ (j, l, k, \iota) \in \Delta \} \cup \{ \iota = 0, k \in \mathbb{Z}^2 \}} \langle f, \Phi_\alpha \rangle \hat{\Phi}_\alpha^d = \sum_{(j, l, k, \iota) \in \Delta} \langle \hat{f}, \chi_\iota \hat{\phi}_{j,l,k} \rangle \hat{\gamma}_\iota \hat{\phi}_{j,l,k} + \sum_{k \in \mathbb{Z}^2} \langle \hat{f}, \chi_0 \hat{\phi}_k^0 \rangle \hat{\gamma}_0 \hat{\phi}_k^0 = \sum_{(j, l, k, \iota) \in \Delta} \langle \chi_\iota \gamma_\iota \hat{f}, \hat{\phi}_{j,l,k} \rangle \hat{\phi}_{j,l,k} + \sum_{k \in \mathbb{Z}^2} \langle \chi_0 \gamma_0 \hat{f}, \hat{\phi}_k^0 \rangle \hat{\phi}_k^0 = \sum_{\iota \in \{0, h, v\}} \chi_\iota \gamma_\iota \hat{f} = \hat{f}.
$$

This finishes the proof. \qed

Shearlets have been studied extensively so far due to their highly directional performance and smooth digital grid, whereas wavelets are best adapted to anisotropic features like point singularities. Bandlimited $\alpha$-shearlets allow for a unified treatment of wavelets and shearlets. If $\alpha = 1$ we obtain the bandlimited shearlet frame by using parabolic scaling. Different from the literature [19], we here rescaled the parameter $j$ to $2j$. Therefore, the spatial footprints of shearlets are of size $2^{-j}$ times $2^{-2j}$ instead of $2^{-j/2}$ times $2^{-j}$. Also, if $\alpha$ approaches 2 the elements of $\alpha$-shearlets scale in an isotropic fashion. Thus, $\Phi_\alpha = \{ \phi_{j,l,k}^\alpha \}_{(j,l,k,\iota) \in \Delta} \cup \{ \phi_k^0 \}_{k \in \mathbb{Z}^2}$ can be viewed as a special instance of wavelets. Consider the shearlet system $\Phi_\alpha$ the tiling of the frequency domain on the vertical cone $C_v$ is illustrated in Figure 4.3.
The reader should be aware of the fact that we can construct a system satisfying a milder condition \( \Phi \), i.e., an element of \( D_\Phi \), instead of forming a dual. In fact, if there exists a well-localized synthesis pseudo-dual the success of the proposed algorithm is guaranteed.

5. Multi-scale component separation

For \( \alpha \in [1, 2) \), we first fix a constant \( \epsilon \) such that
\[
0 < \epsilon < \frac{2 - \alpha}{4}. \tag{38}
\]

Using the window function \( W_j \) by \( 14 \), we define a class of frequency filters \( F_j \) defined by its Fourier transform
\[
\hat{F}_j(\xi) := W_j(\xi) = W(\xi/2^j), \quad \forall j \geq 1, \xi \in \mathbb{R}^2, \tag{39}
\]
and in low frequency part
\[
\hat{F}_{low}(\xi) := \Omega(\xi), \quad \xi \in \mathbb{R}^2,
\]
where \( \Omega \) is defined in \( 13 \). Using these filters to decompose \( f \) into sub-images by \( f_j = F_j * f \), the original image \( f \) is then recovered from its pieces \( f_j \) by \( 16 \)
\[
f = F_{low} * F_{low} * f + \sum_{j \in \mathbb{N}_0} F_j * f_j. \tag{40}
\]

We intend to apply Algorithm 1 for each sub-image \( f_j \) which is assumed to be \( P_j + w_L \), where \( P_j = F_j * P, w_L = F_j * w_L \). The whole signal \( f \) is then reconstructed by \( 40 \).

These filters allow us to consider the pointlike and curvelike part at different scales. The separation problem is meaningful at each scale if the components are comparable in sense of their energy. A main issue is the balance of the energy of each component \( 12, 21 \), i.e., we may assume that \( \{\gamma_i\}_{i=1}^K, \{c_i\}_{i=1}^K \) are chosen satisfying \( \|P_j\|_2 \approx \|C_j\|_2 \) at each scale \( j \). In that case, we can choose \( \max_{i=1,2,\ldots,K} \{\lambda_i\} = \frac{3}{2} \). However, we will not restrict to this energy balancing condition in our analysis. We later show that the success of the proposed algorithm is guaranteed as long as the energy of components...
are not too small, i.e., they satisfy a lower bound at each scale. This guarantees the performance even in case of different energies.

We would like to emphasize that there are very few elements interacting significantly with the discontinuities, i.e., few elements of the wavelet and shearlet expansions are enough to provide accurate approximations of the components. This allows us to define the following clusters which will be then exploited to derive small error approximation as well as the cluster coherence of

\[
\Lambda_{1j} = \left\{ (j', m) \mid j' \in \mathbb{Z}, |j' - j| \leq 1, m = (m_1, m_2) \in \mathbb{Z}^2, \sqrt{m_1^2 + m_2^2} \leq 2^j \right\},
\]

and

\[
\Lambda_{2j} = \left\{ (j', l, k, v) \mid j' \in \mathbb{Z}, |j' - j| \leq 1, k = (k_1, k_2) \in \mathbb{Z}^2, |k_2 - lk_1| \leq 2^j \right\},
\]

where \( \Delta \) is defined in (33). Intuitively, the geometry underlying these two components should be different so that a chosen frame pair can provide sufficient small cluster sparsity and cluster coherence, which guarantee the success of Algorithm 1. Indeed, although points and curves can overlap in spatial domain, their wavefront set might be quite different in phase space consisting of a pair of positions and directions. Here the wavefront set of a distribution \( f \) is the set of positions and orientations at which \( f \) is not smooth. This difference intuitively enables us to separate the components by microlocal analysis, see [12] for a more comprehensive discussion.

We present our main separation result.

**Theorem 5.1.** Consider a wavelet frame \( \Psi = \{ \psi_{j,m} \}_{(j,m)} \cup \{ \Omega(\cdot - m) \}, j \in \mathbb{N}_0, m \in \mathbb{Z}^2 \) and a bandlimited \( \alpha \)-shearlet frame \( \Phi_\alpha = \{ \phi_{j,k}^{\alpha,\iota} \}_{(j,k,\iota)} \cup \{ \phi_k^{0} \}_{k \in \mathbb{Z}^2}, \alpha \in \{ 1, 2 \} \). Then separation based on scale \( j \) is achieved by (6) in the limit of large \( j \). Namely, we have

\[
\frac{\| P_{j}^* - P_{j} \|_2}{\| P_{j} \|_2} + \frac{\| C_{j}^* - wL_{j} \|_2}{\| wL_{j} \|_2} = o(2^{-Nj}) \rightarrow 0, \quad j \rightarrow \infty,
\]

for all \( N \geq 0 \), where \( (P_{j}^*, C_{j}^*) \) is the solution of (6) and \( (C_{j}, T_{\alpha,j}) \) are purported components.

We delay the proof until later after we have introduced our main tools in Subsections 5.1 and 5.2. We will use Theorem 3.6 for the proof of this theorem. For this, we need to show that clusters \( \Lambda_{1j}, \Lambda_{2j} \) provide sparse representation of pointlike and curvelike singularities in the sense that the sparsity \( \delta = \delta_{1,2} + \delta_{2,2} \) is negligible. In addition, the cluster coherence \( \mu_c = \mu_c(\Lambda_{1j}, \Psi; \Phi^{\alpha}) + \mu_c(\Lambda_{2j}, \Phi_\alpha; \Psi) \) is also sufficiently small. Those conditions ensure the success of Algorithm 1 as a result of Theorem 3.6. We would like to remark that in our analysis we need the fast decay of the dual frame rather than its explicit construction.

As a consequence, Theorem 5.1 implies that it is possible to separate pointlike and curvelike structures using wavelets and \( \alpha \)-shearlets if \( \alpha \)-shearlets do not share the isotropic feature with wavelets, i.e., \( \alpha < 2 \). Indeed, traditional wavelets which are based on isotropic dilations are not very effective when dealing with multivariate data. In contrast, shearlets show a greater ability to capture anisotropic features by applying a different dilation factor along the two axes. Here the parameter \( \alpha \) measures the degree of anisotropy. If \( \alpha = 2 \) the scaling matrices \( A_{\alpha, h, v} = \{ h, v \} \) become isotropic dilations which show similar behavior as wavelets in the sense of directional sensitivity.
Thus, and similarly \( N \) estimates hold for any arbitrary integer \( C \).

Then there exists a universal constant \( C_N \) independent of \( j \) such that the following estimates hold for any arbitrary integer \( N = 1, 2, \ldots \)

\[
(i) \quad |\psi_{j,m}(x)| \leq C_N \cdot 2^{j} \cdot \langle |2^{j} x_1 + m_1| \rangle \langle |2^{j} x_2 + m_2| \rangle \langle |2^{j} x_2 + m_2| \rangle \langle |2^{j} x_2 + m_2| \rangle - N.
(ii) \quad |\chi_{v} \ast \phi_{j,l}^{h}(x)| \leq C_N \cdot 2^{(\delta + \delta)j/2} \cdot \langle |2^{j} x_1 + k_1| \rangle \langle |2^{j} x_2 + l_2^{\alpha j} x_1 + k_2| \rangle \langle |2^{j} x_2 + l_2^{\alpha j} x_1 + k_2| \rangle \langle |2^{j} x_2 + l_2^{\alpha j} x_1 + k_2| \rangle - N.
(iii) \quad |\gamma_{v} \ast \phi_{j,l}^{h}(x)| \leq C_N \cdot 2^{(\delta + \delta)j/2} \cdot \langle |2^{j} x_1 + k_1| \rangle \langle |2^{j} x_2 + l_2^{\alpha j} x_1 + k_2| \rangle \langle |2^{j} x_2 + l_2^{\alpha j} x_1 + k_2| \rangle \langle |2^{j} x_2 + l_2^{\alpha j} x_1 + k_2| \rangle - N.
\]

and similarly

\[
(i) \quad |\psi_{j,m}(x)| \leq C_N \cdot 2^{j} \cdot \langle |2^{j} x_1 + m_1| \rangle \langle |2^{j} x_2 + m_2| \rangle \langle |2^{j} x_2 + m_2| \rangle \langle |2^{j} x_2 + m_2| \rangle - N.
(ii) \quad |\chi_{v} \ast \phi_{j,l}^{h}(x)| \leq C_N \cdot 2^{(\delta + \delta)j/2} \cdot \langle |2^{j} x_1 + k_1| \rangle \langle |2^{j} x_2 + l_2^{\alpha j} x_1 + k_2| \rangle \langle |2^{j} x_2 + l_2^{\alpha j} x_1 + k_2| \rangle \langle |2^{j} x_2 + l_2^{\alpha j} x_1 + k_2| \rangle - N.
(iii) \quad |\gamma_{v} \ast \phi_{j,l}^{h}(x)| \leq C_N \cdot 2^{(\delta + \delta)j/2} \cdot \langle |2^{j} x_1 + k_1| \rangle \langle |2^{j} x_2 + l_2^{\alpha j} x_1 + k_2| \rangle \langle |2^{j} x_2 + l_2^{\alpha j} x_1 + k_2| \rangle \langle |2^{j} x_2 + l_2^{\alpha j} x_1 + k_2| \rangle - N.
\]

Proof. i) With the variable \( \zeta = 2^{-j} \xi \), we have

\[
|\psi_{j,m}(x)| = \left| \int_{\mathbb{R}^2} 2^{-2j} W_j(\xi) e^{2\pi i T_m/2^{j}} e^{2\pi i T_x} d\xi \right| = \left| \int_{\mathbb{R}^2} 2^{2j} W_j(\xi) e^{2\pi i T(2^{j} x + m)} d\xi \right|.
\]

By integration by parts for \( N_1, N_2 = 1, 2, \ldots \), with respect to \( \zeta, \zeta_2 \), respectively, we have

\[
|\psi_{j,m}(x)| = \left| \int_{\mathbb{R}^2} 2^{2j} (2^{j} x_1 + m_1)^{-N_1} \frac{\partial^{N_1}}{\partial \zeta_1^{N_1}} [W(\zeta)] e^{2\pi i T(2^{j} x + m)} d\xi \right| = \left| \int_{\mathbb{R}^2} 2^{2j} (2^{j} x_1 + m_1)^{-N_1} (2^{j} x_2 + m_2)^{-N_2} \frac{\partial^{N_1+N_2}}{\partial \zeta_1^{N_1} \partial \zeta_2^{N_2}} [W(\zeta)] e^{2\pi i T(2^{j} x + m)} d\xi \right| \leq 2^{2j} |2^{j} x_1 + m_1|^{-N_1} |2^{j} x_2 + m_2|^{-N_2} \int_{\mathbb{R}^2} \frac{\partial^{N_1+N_2}}{\partial \zeta_1^{N_1} \partial \zeta_2^{N_2}} [W(\zeta)] d\xi,
\]

and similarly

\[
|\psi_{j,m}(x)| \leq 2^{2j} |2^{j} x_i + m_i|^{-N_i} \int_{\mathbb{R}^2} \frac{\partial^{N_i}}{\partial \zeta_i^{N_i}} [W(\zeta)] d\xi,
\]

for \( i = 1, 2 \). Note that the boundary terms vanish since \( W(\zeta) \) has compact support. Thus,
In addition, there exist constants 
and 

By a similar approach as in i) the decay of each shearlet element is then estimated

Thus, we obtain

Since 

there exists a constant 

independent of 

such that

Thus, we obtain

In addition, there exist constants 

for each 


and

Finally, we obtain

This shows the first claim.

ii) With the variable 

we obtain

By a similar approach as in i) the decay of each shearlet element is then estimated by

The same fact is true with the roles of 

interchanged. It should be mentioned that here we use Lemma 4.2 ii) to estimate the constant 

iii) By using i) and ii) the change of variables 

we easily verify
the claim. Indeed, for $\nu = v$ we have
\[
|\langle \psi'_{j,m}, \tilde{\chi}_v * \phi_{j,l,k}^{a,v} \rangle| \leq \int_{\mathbb{R}^2} |\psi_{j,m}(x)||\tilde{\chi}_v * \phi_{j,l,k}^{a,v}(x)| dx \\
\leq C \cdot 2^{j(j+1)/2} \int_{\mathbb{R}^2} \langle |2^j x_1 + m_1| \rangle^{-N} \langle |2^j x_2 + m_2| \rangle^{-N} \\
\cdot \langle |2^{j+1} x_1 + k_1| \rangle^{-N} \langle |2^{j+1} x_2 + 2^{j+1} x_1 + k_2| \rangle^{-N} dx \\
\leq C \cdot 2^{-2(\alpha+j)/2} \int_{\mathbb{R}^2} \langle |y_1 + m_1| \rangle^{-N} \langle |y_2 + m_2| \rangle^{-N} dy_1 dy_2 \\
\leq C' \cdot 2^{-2(\alpha+j)/2}.
\]

For the other cases, the proofs are similar.

Such decay estimates in Lemma 5.2 are particularly useful for estimating the cluster coherence introduced next as well as the cluster sparsity in Subsection 5.2.

**Proposition 5.3.** Consider wavelet frame $\Psi$ and bandlimited $\alpha$-shearlet frame $\Phi_\alpha$, we have
\[
\mu_c(\Lambda_{1,j}, \Psi; \Phi_\alpha) \to 0, \ j \to +\infty.
\]

**Proof.** By definition, we have
\[
\mu_c(\Lambda_{1,j}, \Psi; \Phi_\alpha) = \max_{(j,l,k) \in \Delta} \sum_{(j',m) \in \Lambda_{1,j}} |\langle \psi'_{j',m}, \tilde{\chi}_v(x) * \phi_{j,l,k}^{a,v} \rangle|.
\]
Without loss of generality, we assume that the maximum is attained at $\psi_{j,l,k}^{a,v}$.

Cases $\nu = h, \tilde{j} = \{j-1, j+1\}$ are done similarly. By Lemma 5.2 iii), there exist a constant $C > 0$ such that
\[
\mu_c(\Lambda_{1,j}, \Psi; \Phi) \leq \#\{(j, m) \in \Lambda_{1,j}\} \cdot C \cdot 2^{-2(\alpha+j)/2} \\
\leq C \cdot 2^{2j \cdot j} \cdot 2^{-2(\alpha+j)/2} \\
= C \cdot 2^{-2(\alpha-4\epsilon+j)/2} \to 0, \ j \to +\infty,
\]
where the last estimate is due to $2 \cdot \alpha - 4\epsilon > 0$ by (38). This finishes the proof. □

**Proposition 5.4.** Consider wavelet frame $\Psi = \{\psi, j,k\}_{(j,k)}$ and band limited $\alpha$-shearlet frame $\Phi_\alpha = \{\phi_{j,l,k}^{a,v}\}_{(j,l,k) \in \Delta}$, we have
\[
\mu_c(\Lambda_{2,j}, \Phi_\alpha; \Psi) \to 0, \ j \to +\infty.
\]

**Proof.** By definition, we have
\[
\mu_c(\Lambda_{2,j}, \Phi_\alpha; \Psi) = \max_{j \in \mathbb{N}, \nu \in \mathbb{Z}^2} \sum_{(j,l,k) \in \Lambda_{2,j}} |\langle \tilde{\chi}_v(x) * \phi_{j,l,k}^{a,v}, \psi_{j,m} \rangle|.
\]
We assume that the maximum is attained at $\psi_{j,m}$. By Lemma 5.2 i) and ii), we have
\[
\mu_c(\Lambda_{2,j}, \Phi_\alpha; \Psi) \leq C' \cdot 2^{2(\alpha+1)/2} \cdot \sum_{|l| \leq 1, \nu, \nu' \leq 1} \int_{\mathbb{R}^2} \langle |2^j x_1 + m'_1| \rangle^{-N} \langle |2^j x_2 + m'_2| \rangle^{-N} dx \\
\cdot \langle |2^{j+1} x_1 + k_1| \rangle^{-N} \langle |2^{j+1} x_2 + 2^{j+1} x_1 + k_2| \rangle^{-N} dx.
\]
Without loss of generality, we consider only the case $j' = j$, the cases $j' = j-1, j+1$ follow similarly.
Indeed, by the change of variable \((y_1, y_2) = (2^{2j}x_1, 2^{2j}x_2)\), we obtain

\[
\mu_c(\Lambda_{2j}, \Phi_\alpha; \Psi) \leq C_N 2^{-(2-\alpha)j/2} \sum_{l \in \{-1,0,1\}, k \in \mathbb{Z}^2, |k_2 - k_l| < 3 \beta} \int_{\mathbb{R}^2} \langle |y_1 + m'_1| \rangle^{-N} \langle |y_2 + m'_2| \rangle^{-N} \cdot \langle |2^{-(2-\alpha)j}y_1 + k_1| \rangle^{-N} \langle |y_2 + l2^{\alpha j}x_1 + k_2| \rangle^{-N} dy.
\]

(45)

In addition, there exists a constant \(C'\) such that

\[
\sum_{k \in \mathbb{Z}^2} \langle |2^{-(2-\alpha)j}y_1 + k_1| \rangle^{-N} \langle |y_2 + l2^{\alpha j}x_1 + k_2| \rangle^{-N} \leq C'.
\]

(46)

Combining (45) with (46), we obtain

\[
\mu_c(\Lambda_{2j}, \Phi_\alpha; \Psi) \leq C'_N \cdot 2^{-(2-\alpha)j/2} \int_{\mathbb{R}^2} \langle |y_1 + m'_1| \rangle^{-N} \langle |y_2 + m'_2| \rangle^{-N} dy
\]

\[
\leq C''_N \cdot 2^{-(2-\alpha)j/2} \xrightarrow{j \to +\infty} 0.
\]

(47)

This concludes the claim. \(\square\)

5.2. Sparse representation error.

**Proposition 5.5.** For all \(N = 1, 2, \ldots\), we have \(\delta_{1j} = o(2^{-Nj}).\)

**Proof.** We remark that it is sufficient to consider \(P(x) = \frac{1}{|x|^{\lambda}}, \lambda \in (0, 2).\) The general case is then done by combining all single cases and triangle inequality.

We first observe that \(P(x) = \frac{1}{|x|^{\lambda}}.\) Thus,

\[
\delta_{1j} = \sum_{(j',m) \in \Lambda_{1,j}} \langle \psi_{j',m}, P \rangle
\]

\[
= \sum_{|j' - j| \leq 1, |m| > 2^j} \int_{\mathbb{R}^2} 2^{-2j} W_{j'}(\xi) e^{2\pi i \xi \cdot m} W_j(\xi) \frac{1}{|\xi|^{2-\lambda}} d\xi.
\]

Since \(W_{j'}(\xi) W_j(\xi) = 0\) for \(|j' - j| > 1\) due to their supports. Without loss of generality we can assume that \(j' = j,\) the other cases are estimated analogously.

By the change of variables \(\eta = (\eta_1, \eta_2) = 2^{-2j}(\xi_1, \xi_2),\) we obtain

\[
\delta_{1j} \lesssim 2^{2(\lambda - 1)j} \sum_{|m| > 2^j} \int_{\mathbb{R}^2} \frac{1}{|\eta|^{2-\lambda}} W^2(\eta) e^{2\pi i \eta \cdot m} d\eta.
\]

(48)

Using integration by parts for \(N = 1, 2, \ldots,\) with respect to \(\eta_1, \eta_2,\) respectively, yields

\[
\delta_{1j} \lesssim 2^{2(\lambda - 1)j} \sum_{m_1, m_2 \neq 0, |m| > 2^j} \int_{\mathbb{R}^2} |m_1|^{-N} |m_2|^{-N} \frac{\partial^{2N}}{\partial \eta_1 \partial \eta_2} \left[ \frac{1}{|\eta|^{2-\lambda}} W(\eta) e^{2\pi i \eta \cdot m} d\eta \right]
\]

\[
\leq C_N 2^{2(\lambda - 1)j} \sum_{|m| > 2^j} |m_1|^{-N} |m_2|^{-N}
\]

\[
\leq C_N 2^{2(\lambda - 1)j} \sum_{m_1 \in \mathbb{Z}} \sum_{|m_2| \geq 2^j} |m_1|^{-N} |m_2|^{-N}
\]

\[
\leq C_N 2^{2(\lambda - 1)j} 2^{-(N-1)(\lambda-1)},
\]
Similarly, we obtain

\[ \forall N \in \mathbb{N}. \text{ Note that the boundary terms vanish due to the compact support of } W(\zeta). \text{ This concludes the claim.} \]

**Proposition 5.6.** We have \( \delta_{2j} = o(2^{-N_j}), \text{ for } \forall N = 1, 2, \ldots \)

**Proof.** By definition, we have

\[ \delta_{2j} = \sum_{(j',l,k) \notin \Lambda_{2j}} |\langle w \mathcal{L}_{j',l,k}, \mathcal{X}_i * \phi_{j',l,k}^{\alpha,i} \rangle| \]

\[ = \sum_{\{j',l,k\} \in \Delta, ||l|| > 1} |\langle \hat{w} \mathcal{L}_{j',l,k}, \chi_i * \phi_{j',l,k}^{\alpha,i} \rangle| + \sum_{\{j',l,k,v\} \in \Delta, ||l|| \leq 1, |k_2 - l_2| > 2^{j'} |\langle \hat{w} \mathcal{L}_{j',l,k}, \chi_i * \phi_{j',l,k}^{\alpha,i} \rangle| \]

\[ = I_1 + I_2, \quad (49) \]

where

\[ I_1 = \sum_{\{j',l,k\} \in \Delta, ||l|| > 1} |\langle \hat{w} \mathcal{L}_{j',l,k}, \chi_i * \phi_{j',l,k}^{\alpha,i} \rangle|, \quad I_2 = \sum_{\{j',l,k,v\} \in \Delta, ||l|| \leq 1, |k_2 - l_2| > 2^{j'} |\langle \hat{w} \mathcal{L}_{j',l,k}, \chi_i * \phi_{j',l,k}^{\alpha,i} \rangle| \]

Since \( W_j(\xi) W_j(\xi) = 0 \) for \( |j' - j| > 1 \) due to the different support we only consider \( j' = j \) without loss of generality. Before starting with \( I_1 \), we put

\[ t^v = (t_1^v, t_2^v) := A_{-j}^{-1} S_{-l}^{k} = (2^{-\alpha j} k_1, 2^{-2j} (k_2 - l_1)), \quad (50) \]

\[ t^h = (t_1^h, t_2^h) := A_{-h}^{-1} S_{-l}^{k} = (2^{-2j} (k_1 - l_2), 2^{-2j} k_2). \quad (51) \]

We have

\[ |\langle w \mathcal{L}_{j}, \mathcal{X}_i * \hat{\phi}_{j,l,k}^{\alpha,i} \rangle| = |\langle \hat{w} \mathcal{L}_{j}, \chi_i * \hat{\phi}_{j,l,k}^{\alpha,i} \rangle| = \left| \int_{\mathbb{R}^2} \hat{w}(\xi_1) \chi_i(\xi) W_j(\xi) \hat{\phi}_{j,l,k}^{\alpha,i}(\xi) \right| \]

\[ = \left| \int_{\mathbb{R}^2} e^{2\pi i t_1 \xi_1} \int_{\mathbb{R}^2} \hat{w}(\xi_1) \chi_i(\xi) W_j(\xi) \hat{\phi}_{j,l,0}^{\alpha,i} \xi_1 \xi_2 \right| \]

\[ = \left| \int_{\mathbb{R}^2} e^{2\pi i t_1 \xi_1} \int_{\mathbb{R}^2} \hat{w}(\xi_1) \Theta_{j,l}^{\alpha,i}(\xi) \xi_1 \xi_2 \right|. \quad (52) \]

where \( \Theta_{j,l}^{\alpha,i}(\xi) := \chi_i(\xi) W_j(\xi) \hat{\phi}_{j,l,0}^{\alpha,i}(\xi) \).

Applying repeated integration by parts twice with respect to \( \xi_i, i = 1, 2 \), we have

\[ |\langle w \mathcal{L}_{j}, \hat{\phi}_{j,l,k}^{\alpha,i} \rangle| = \left| \int_{\mathbb{R}^2} e^{2\pi i t_1 \xi_1} \left[ \int_{\mathbb{R}^2} \frac{\partial^2}{\partial \xi_1^2} [\hat{w}(\xi_1) \Theta_{j,l}^{\alpha,i}(\xi)] d\xi_1 d\xi_2 \right] \left[ \frac{1}{(2\pi i t_1)^2} e^{2\pi i t_1 \xi_1} d\xi_1 \right] \right| d\xi_2 \]

\[ \leq \frac{C}{|t_1|^2} \left| \int_{\mathbb{R}^2} e^{2\pi i t_1 \xi_1} \left[ \int_{\mathbb{R}^2} \frac{\partial^2}{\partial \xi_1^2} [\hat{w}(\xi_1) \Theta_{j,l}^{\alpha,i}(\xi)] e^{2\pi i t_1 \xi_1} d\xi_1 d\xi_2 \right] \right| \]

\[ \leq C|t_1|^{-2} \left| \int_{\mathbb{R}^2} \frac{\partial^2}{\partial \xi_1^2} [\hat{w}(\xi_1) \Theta_{j,l}^{\alpha,i}(\xi)] d\xi_1 d\xi_2. \quad (52) \]

Similarly, we obtain

\[ |\langle w \mathcal{L}_{j}, \hat{\phi}_{j,l,k}^{\alpha,i} \rangle| \leq C|t_1|^{-2} \left| \int_{\mathbb{R}^2} \frac{\partial^4}{\partial \xi_1^4} [\hat{w}(\xi_1) \Theta_{j,l}^{\alpha,i}(\xi)] d\xi_1 d\xi_2 \right| \]

\[ \leq C|t_1|^{-2} \left| \int_{\mathbb{R}^2} \frac{\partial^4}{\partial \xi_1^4 \partial \xi_2^4} [\hat{w}(\xi_1) \Theta_{j,l}^{\alpha,i}(\xi)] d\xi_1 d\xi_2. \quad (53) \]

\[ |\langle w \mathcal{L}_{j}, \hat{\phi}_{j,l,k}^{\alpha,i} \rangle| \leq C|t_1|^{-2} \left| \int_{\mathbb{R}^2} \frac{\partial^4}{\partial \xi_1^4 \partial \xi_2^4} [\hat{w}(\xi_1) \Theta_{j,l}^{\alpha,i}(\xi)] d\xi_1 d\xi_2. \quad (54) \]
Therefore, we have
\[ |t_1'|^2 \langle \omega L_j, \phi_{j,l,k}^{\alpha,\iota} \rangle \leq C \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\partial^2}{\partial \xi_1^2} [\hat{w}(\xi_1)\Theta_{j,l}(\xi)] \right| d\xi_1 d\xi_2 \]
\[ |t_2'|^2 \langle \omega L_j, \phi_{j,l,k}^{\alpha,\iota} \rangle \leq C \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\partial^2}{\partial \xi_1^2} [\hat{w}(\xi_1)\Theta_{j,l}(\xi)] \right| d\xi_1 d\xi_2 \]
\[ |t_1'|^2 |t_2'|^2 \langle \omega L_j, \phi_{j,l,k}^{\alpha,\iota} \rangle \leq C \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\partial^4}{\partial \xi_1^2 \partial \xi_2^2} [\hat{w}(\xi_1)\Theta_{j,l}(\xi)] \right| d\xi_1 d\xi_2. \]

These imply
\[ (1 + |t_1'|^2 + |t_2'|^2 + |t_1'|^2 |t_2'|^2) \langle \omega L_j, \phi_{j,l,k}^{\alpha,\iota} \rangle = \langle |t_1'|^2 \rangle^2 \langle t_2'|^2 \rangle \langle \omega L_j, \phi_{j,l,k}^{\alpha,\iota} \rangle \]
\[ \leq C \int_{\mathbb{R}} \left| \hat{w}(\xi_1)\Theta_{j,l}(\xi) \right| + \left| \frac{\partial^2}{\partial \xi_1^2} [\hat{w}(\xi_1)\Theta_{j,l}(\xi)] \right| + \left| \frac{\partial^2}{\partial \xi_1^2} [\hat{w}(\xi_1)\Theta_{j,l}(\xi)] \right| \]
\[ + \left| \frac{\partial^4}{\partial \xi_1^2 \partial \xi_2^2} [\hat{w}(\xi_1)\Theta_{j,l}(\xi)] \right| d\xi = C \int_{\mathbb{R}} \Gamma(\xi_2) d\xi, \]
where \( \Gamma(\xi_2) := \int_{\mathbb{R}} \left| \hat{w}(\xi_1)\Theta_{j,l}(\xi) \right| + \left| \frac{\partial^2}{\partial \xi_1^2} [\hat{w}(\xi_1)\Theta_{j,l}(\xi)] \right| + \left| \frac{\partial^2}{\partial \xi_1^2} [\hat{w}(\xi_1)\Theta_{j,l}(\xi)] \right| + \left| \frac{\partial^4}{\partial \xi_1^2 \partial \xi_2^2} [\hat{w}(\xi_1)\Theta_{j,l}(\xi)] \right| d\xi_1. \]

Now we investigate the term \( \Gamma \) further. By the definition, we have \( \hat{w}_{j,l}^{\alpha,\iota} \) has compact support in the trapezoidal regions by \( \{24\}, \{25\} \). This implies that for any \( \xi \in \text{supp } \Theta_{j,l}^{\iota} \), with \( \iota = v \), or \( \iota = h \), \( |l| > 1 \), there exist constants \( C_1, C_2 > 0 \) such that
\[ \xi_1 \in I_{j,l} := [-C_2 l 2^{\alpha j}, -C_1 l 2^{\alpha j}] \cup [C_2 l 2^{\alpha j}, C_1 l 2^{\alpha j}]. \]

Thus, we obtain the decay estimate of \( \hat{w}^{(n)}(\xi_1), \xi_1 \in I_{j,l}, n = 0, 1, 2 \), by
\[ |\hat{w}^{(n)}(\xi_1)| \leq C_N \langle 2^{\alpha j} \rangle^{-N}, \forall N \in \mathbb{N}. \] (55)

For an illustration, we refer to Fig. 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{shearlet_support.png}
\caption{Frequency support of shearlets with \( \iota = v \), \( |l| > 1 \) (gray) and shearlets with \( \iota = h \) (black).}
\end{figure}
By a direct computation, we also have \( \Theta^{(n)}_{j,k} \leq C_n, n = 0, 1, 2 \). Combining this with (55), we obtain \( \int_R \Gamma(\xi_2) d\xi_2 \leq C_N \langle t_1^2 \rangle^{-N}, \forall N \in \mathbb{N}_0 \). Thus,

\[
\left\langle w \mathcal{L}_j, \phi^{0, t}_{j,l,k} \right\rangle \leq C_N \langle t_1^2 \rangle^{-N} \langle t_2^2 \rangle^{-2}.
\]

This implies

\[
I_1 \leq C_N \langle t_1^2 \rangle^{-N} \sum \langle t_1^2 \rangle^{-2} \langle t_2^2 \rangle^{-2}.
\]  

(56)

In addition, we have

\[
\sum_{k_1, k_2 \in \mathbb{Z}} \langle t_1^2 \rangle^{-2} \langle t_2^2 \rangle^{-2} = \sum_{k_1, k_2 \in \mathbb{Z}} \langle 2^{-2j} (k_1 - l k_2) \rangle^{-2} \langle 2^{-\alpha j} k_2 \rangle^{-2} = \sum_{k_1, k_2 \in \mathbb{Z}} \langle 2^{-2j} k_1 \rangle^{-2} \langle 2^{-\alpha j} k_2 \rangle^{-2} \leq \int_R \langle 2^{-2j} t_1 \rangle^{-2} dt_1 \int_R \langle 2^{-\alpha j} t_2 \rangle^{-2} dt_2 \leq C 2^{(2+\alpha)2j}.
\]  

(57)

Similarly, we obtain

\[
\sum_{k_1, k_2 \in \mathbb{Z}} \langle t_1^2 \rangle^{-2} \langle t_2^2 \rangle^{-2} \leq C 2^{(2+\alpha)2j}.
\]  

(58)

In addition, we have

\[
|l| \leq [2 \cdot 2^{(2-\alpha)j}].
\]  

(59)

Combining (56), (57), (58), and (59) we obtain

\[
I_1 \leq C_N \langle t_1^2 \rangle^{-N} \langle t_2^2 \rangle^{-N}.
\]  

(60)

We now turn our attention to the term \( I_2 \). We have

\[
|w \mathcal{L}_j(x)| = \left| \int_R w(y_1) F_j(x - (y_1, 0)) dy_1 \right| \\
\leq \left| \int_R |w(y_1)| \cdot 2^{4j} |W(2^{2j}(x - (y_1, 0)))| dy_1 \right| \\
\leq \int_R |w(y_1)| \cdot C_N 2^{4j} \langle |2^{2j} x_2| \rangle^{-N} \langle |2^{2j} (y_1 - x_1)| \rangle^{-N} dy_1 \\
= C_N 2^{4j} \langle |2^{2j} x_2| \rangle^{-N} \langle |w * (2^{2j} \cdot |\rangle^{-N} (x_1) \\
= C_N 2^{4j} \langle |2^{2j} x_2| \rangle^{-N} \tau_{N,j} (x_1),
\]

(61)
where \( \tau_{N,j}(x_1) := [w * (2^j \cdot \cdot)]^N(x_1) \), \( x = (x_1, x_2) \in \mathbb{R}^2 \). Combining (61) with Lemma [5,2] we obtain

\[
I_2 \leq C_N \sum_{l \in \{-1,0,1\}, \|k_2 - l k_1\| > 2^j} \int_{\mathbb{R}^2} 2^{4j} \langle |2^{2j} x_2| \rangle^{-N} \tau_{N,j}(x_1) 2^{(\alpha + 2)j} \langle |2^{2j} x_1 - k_1| \rangle^{-N} \cdot \langle |2^{\alpha j} x_1 + 2^{2j} x_2 - k_2| \rangle^{-N} dx
\]

\[
= C_N 2^{(6-\alpha)j/2} \sum_{l \in \{-1,0,1\}, \|k_2 - l k_1\| > 2^j} \int_{\mathbb{R}^2} \langle |x_2| \rangle^{-N} \tau_{N,j}(2^{-\alpha j} x_1 + k_1) \langle |x_1| \rangle^{-N} \cdot \langle |l x_1 + l k_1 + x_2 - k_2| \rangle^{-N} dx
\]

\[
\leq C_N 2^{(6-\alpha)j/2} \sum_{l \in \{-1,0,1\}, \|k_2 - l k_1\| > 2^j} \int_{\mathbb{R}} \tau_{N,j}(2^{-\alpha j} (x_1 + k_1)) \langle |x_1| \rangle^{-N} \cdot \langle |l x_1 + l k_1 - k_2| \rangle^{-N} dx_1.
\]

The last equality comes from the fact that there exists a constant \( C_N > 0 \) satisfying

\[
\int_{\mathbb{R}} \langle |x| \rangle^{-N} \langle |x + a| \rangle^{-N} dx \leq C_N \langle |a| \rangle^{-N}, \quad \forall a \in \mathbb{R}.
\]

In addition, we have

\[
\sum_{k_1 \in \mathbb{Z}} \tau_{N,j}(2^{-\alpha j} (x_1 + k_1)) = \sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} |w(y_1)| \langle |2^{2j} (y_1 - 2^{-\alpha j} (x_1 + k_1)) | \rangle^{-N} dy_1
\]

\[
= \sum_{k_1 \in \mathbb{Z}} \int_{\mathbb{R}} |w(y_1)| \langle |2^{(\alpha - 2)j} (k_1 + x_1 - 2^{\alpha j} y_1) | \rangle^{-N} dy_1
\]

\[
\leq \int_{\mathbb{R}} |w(y_1)| \left( \sum_{k_1 \in \mathbb{Z}} \langle |k_1 + x_1 - 2^{\alpha j} y_1| \rangle^{-N} \right) dy_1
\]

\[
\leq C \int_{\mathbb{R}} |w(y_1)| \left( \int_{\mathbb{R}} \langle |t + x_1 - 2^{\alpha j} y_1| \rangle^{-N} dt \right) dy_1
\]

\[
= C \int_{\mathbb{R}} |w(y_1)| \left( \int_{\mathbb{R}} \langle |t| \rangle^{-N} dt \right) dy_1 \leq C_N.
\]

Therefore, we obtain

\[
I_2 \leq C_N 2^{(6-\alpha)j/2} \sum_{l \in \{-1,0,1\}, k_2 \in \mathbb{Z}, \|k_2\| > 2^j} \int_{\mathbb{R}} \langle |x_1| \rangle^{-N} \langle |l x_1 + k_2| \rangle^{-N} dx_1
\]

by (62)

\[
\leq C_N 2^{(-\alpha)j/2} \sum_{k_2 \in \mathbb{Z}, \|k_2\| > 2^j} \langle |k_2| \rangle^{-N} dx_1
\]

\[
\leq C_N 2^{(6-\alpha)j/2} \int_{t > 2^j} \langle |t| \rangle^{-N} \leq C_N 2^{(6-\alpha)j/2} 2^{-(N-1)j}, \quad \forall N \in \mathbb{N}_0.
\]

Finally, combining (60), (63) with (49) concludes the claim. \qed
5.3. Proof of main theorem. Now we are well prepared to provide a proof for Theorem 5.1.

Proof. Applying Theorem 3.6 and Propositions 5.3, 5.4, 5.5, 5.6 we obtain
\[ \|P^* - P_j\|_2 + \|C^* - wL_j\|_2 = o(2^{-Nj}) \rightarrow 0, \quad j \rightarrow \infty. \] (64)

Since (64) holds for arbitrarily large number \( N \in \mathbb{N} \) we are left to estimate the lower bounds of \( \|P_j\|_2 \) and \( \|wL_j\|_2 \). Without loss of generality, we assume that \( P = \frac{1}{|x|^\lambda}, 0 < \lambda < 2 \). The more general case follows by combining translation invariance with many uses of the triangle inequality. By Plancherel’s theorem and \( \hat{(\frac{1}{|x|^\lambda})} = \frac{1}{|\xi|^{2-\lambda}} \), we have
\[ \|P_j\|_2^2 = \|F_j \ast \frac{1}{|x|^\lambda}\|_2^2 = \int_{A_j} W^2_j(\xi) \frac{1}{|\xi|^{2(2-\lambda)}} d\xi \gtrsim c \cdot 2^{(2-2\lambda)2j}. \]

In addition,
\[ \|wL_j\|_2^2 = \int_{A_j} \hat{w}^2(\xi_1) W^2_j(\xi) d\xi \gtrsim \int_{\xi_1 \in \mathbb{R}} \hat{w}^2(\xi_1) \int_{2^{j-5}}^{2^{j-2}} Cd\xi_2 \gtrsim C 2^{2j}. \]

This finishes the proof. \( \Box \)

6. Conclusions and future directions

Our main results, Theorem 3.6 and Theorem 5.1, show that the ground truth components are under reasonable assumptions perfectly recovered at fine scales by Algorithm 1. For a practical multiscale approach, we first apply the proposed algorithm for each filtered subband image at each scale and then recover the whole signal using the reconstruction formula (40). These theorems might be extended in several ways. Here we focus on separating pointlike and curvelike structures, but our approach holds for other types of geometric components as well as any frames providing sparse representation. We can therefore use wavelets, curvelets, Gabor frames, as well as other sparsifying systems for representing components to be separated. A similar asymptotic separation result can be achieved by using such sparsifying systems if they provide sufficient small cluster sparsity and cluster coherence.

In our future work, we will extend our theory to the case of multiple-component separation. Our results might also be applicable to images that are corrupted by noise. In addition, other algorithms such as thresholding methods [26] or \( l_1 \)-minimization with an additional regularization term [18] which is effective for image separation in empirical work can be considered in our future study.

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