Tunneling Phase Diagrams in Anisotropic Multi-Weyl Semimetals

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Motivated by the exciting prediction of multi-Weyl topological semimetals stabilized by point group symmetries, tunneling in anisotropic multi-Weyl semimetals is studied. It is found that distant detectors for different ranges of an anisotropy parameter $\lambda$ and incident angle $\theta$ will measure different numbers of propagating transmitted modes. These findings are presented as phase diagrams that are valid for incoming waves with fixed wavenumber $k$. Energy is not held fixed for incoming waves but allowed to vary with choice of incident angle and wavenumber $k$. To gain a deeper understanding of this phenomenon they focus on the simplest case of anisotropic quadratic Weyl-semimetals and tunneling coefficients are analyzed analytically and numerically to confirm observations extracted from phase diagrams. Results show nonanalytical behavior, which is the hallmark of phase transitions. This serves as motivation to make a formal analogy with phase transitions known from statistical mechanics. Specifically, they argued that the long distance limit in the tunneling problem mimics the thermodynamic limit in statistical mechanics. A direct formal connection is found to the recently developed formalism of dynamical phase transitions. They propose that usage of this analogy with dynamical phase transitions can be fruitful in classifying transport properties of exotic semimetals.

1. Introduction

Recent theoretical and experimental advances have stimulated interest in classes of materials beyond the usual classification of metals, semiconductors, and insulators. The newly discovered classes of materials include topological materials such as topological insulators and semimetals such as 3D Dirac and Weyl semimetals, which are higher dimensional materials that share certain similarities to graphene.$^{[1,2]}$ The class of topological materials possesses nontrivial protected topological properties such as edge states that are at the origin of their exciting electronic and transport properties.$^{[1-6]}$ The characteristic feature of semimetals is that they are gapless but have negligible overlap between conduction and valence bands. That is, their conduction and valence bands touch at isolated points, nodal lines or nodal surfaces.$^{[7-26]}$ Most exciting are cases, where these band crossings are topologically protected. Different types of topological semimetals can then be distinguished based on the degeneracy of the band crossings and the crystal space group symmetries that protects these band crossings.$^{[7]}$ The qualitative behavior of the energy dispersion near the band crossings is also very important for transport and can be considered as an additional discriminating characteristic of these materials. All together, these properties give rise to a wide class of distinct semimetals such as Dirac$^{[1,2,27-53]}$ or Weyl semimetals,$^{[2,27-30,54-86]}$ semimetals with higher order band crossings such as multi-Weyl- and Dirac semimetals,$^{[33,51,54,71,80,87-90]}$ nodal line, and surface semimetals.$^{[7-26,91]}$

However, there is still a large zoo of semimetals that have been predicted based on symmetry considerations but have not been exhaustively studied. In particular, we would like to point out to the recently predicted anisotropic cubic Dirac semi-metal.$^{[92]}$ We have recently investigated theoretically a wealth of interesting tunneling phenomena in cubic Dirac semimetals due to the richness of its band structure, which allows a linear energy dispersion in the $z$-direction and cubic energy dispersion and rotational symmetry, or lack of it in the $x$–$y$ plane. In general, Dirac or Weyl band crossings in bulk solids can be accompanied by different energy dispersion in different directions, as imposed by their crystal symmetries. At these band crossings, in general, the dispersion along the principle rotation axis is linear, whereas the dispersion along the perpendicular plane could be either linear, quadratic, or cubic. Tremendous amount of efforts need to be deployed to understand the unique electronic properties of such a large zoo of semi-metals and some work has already been done in this direction.$^{[89,90,91-96]}$

We wish to make another step in a direction that will help classify properties of the large zoo of semimetals. Specifically, we wish to borrow some ideas from statistical mechanics, where physical systems are classified according to phases and phase transitions. Phase transitions in classical and quantum systems...
are usually characterized by nonanalytic behavior of one or more physical quantities. A sudden jump in the free energy, hereby, is classified as a first order phase transition, and higher derivatives are classified as second order phase transitions—a classification that goes all the way back to Paul Ehrenfest. Recently it has been pointed out that in close analogy with thermodynamic phase transitions, dynamical systems can also exhibit similar nonanalytic behavior at a critical time, rather than with a critical temperature or other parameter. Such behavior has been coined a dynamical phase transition and crucially differs from the statistical mechanics case in that it can be extracted from the time evolution operator rather than a statistical ensemble density matrix. From the mathematical point of view, this similarity can be traced to the correspondence between the time evolution operator and the Gibbs ensemble density matrix through a Wick rotation. Thus, classical thermodynamic phase transitions and the dynamical phase transitions seem to be tightly related. However, experimental support to this assertion had to await recent advances in experimental techniques that opened the door to a lot of research activities in this field, including experimental observations of such transitions as well as the identification of exactly solvable models.

In our present work we will show that similar nonanalytic behavior is also present in the tunneling coefficients of anisotropic multi-Weyl semimetals. The usual thermodynamic limits with large system size \( N \), here, is replaced by the large distance limit in the transmission region beyond a localized scattering potential. Our paper is organized as follows. In Section 2, we give a brief mathematical formulation of our model Hamiltonian with linear energy dispersion along the \( z \)-axis and \( n \)th order dispersion in the \( x-y \) plane. In Section 3 we study the tunneling problem through a square barrier and in particular investigate how the number of propagating solutions depend of the incident angle \( \theta \) and an anisotropy parameter \( \lambda \). We then investigate a formal connection between tunneling phase diagrams and phase transitions. In Section 4 we consider tunneling in the case of a quadratic energy dispersion in the \( x-y \) plane for both isotropic and anisotropic realizations and confirm nonanalytic behavior near phase boundaries and classify the transitions as second order phase transitions. Finally, in Section 5 we present our conclusion and summarize our main findings.

2. Model

In a seminal paper Fang et al. have made a prediction of multi-Weyl semimetals that are stabilized via point group symmetries. Motivated by this work we consider the general class of Hamiltonians that are of Weyl-type and have a square barrier and in particular investigate how the number of propagating solutions depend of the incident angle \( \theta \) and an anisotropy parameter \( \lambda \). We then investigate a formal connection between tunneling phase diagrams and phase transitions. In Section 4 we consider tunneling in the case of a quadratic energy dispersion in the \( x-y \) plane for both isotropic and anisotropic realizations and confirm nonanalytic behavior near phase boundaries and classify the transitions as second order phase transitions. Finally, in Section 5 we present our conclusion and summarize our main findings.

focus on the general case before restricting our attention to the case \( n = 2 \) later. We should also note that the model for \( n = 3 \) is similar to the cubic Dirac semimetal that was recently predicted based on first principles calculations in ref. [54] and where some of its tunneling properties were discussed in ref. [112]. We also note that tunneling properties for a similar class of models with \( v_x = v_y \)—have been investigated recently.

Therefore, for the purposes of this article we will focus on the new physics that is related to the anisotropy parameter \( \lambda = v_x/v_y \). Hence, for simplicity we will consider the case of a material that is sufficiently thin that the linearly dispersed \( z \)-direction is frozen out and we can safely set \( k_z = 0 \) (see the appendix in ref. [112] for details). The Hamiltonian can then be expressed in unitless form only in terms of the anisotropy parameter \( \lambda \) as

\[
H_{\text{flat}} = \begin{pmatrix}
0 & \frac{\lambda+1}{2} k_x^2 + \frac{\lambda-1}{2} k_y^2 \\
\frac{\lambda+1}{2} k_x^2 + \frac{\lambda-1}{2} k_y^2 & 0
\end{pmatrix}
\]  

(2)

3. Tunneling Phase Diagrams

We may now consider the system subjected to square-shaped barrier region—this can be a potential in the \( x \)-direction given by

\[
V(x) = \begin{cases}
V_0, & -\frac{1}{2} < x < \frac{1}{2} \\
0, & \text{elsewhere}
\end{cases}
\]  

(3)

as shown in Figure 1.

In this work we want to study the tunneling properties of electrons in this system on general grounds. In its simplest form \( V(x) \) is just an ordinary potential barrier. But let us for now remain more general before we restrict ourselves to this case. Let us assume an incoming wave with wavevector \( k = k(\cos \theta, \sin \theta) \) then our model in free space leads to an energy dispersion given by

\[
E = \pm \frac{k_0}{\sqrt{2}} \sqrt{1 + \lambda^2 + (\lambda^2 - 1) \cos(2n\theta)}
\]  

(4)

A plot of the positive energy solution for \( n = 2, 3 \) is shown in Figure 2, which displays \( 2n \)-fold rotational symmetry.

We will find that it is the anisotropic character of the dispersion that can lead to some interesting tunneling phenomena. To gain a better understanding of this situation the eigenvalue problem may now be recast as a nonlinear eigenvalue problem for \( k_x \) at a fixed energy \( E \)—assuming that we want to consider a tunneling problem for a barrier perpendicular to the \( x \)-direction. For the example for \( n = 2 \) one finds the nonlinear eigenvalue problem

\[
(\lambda k_x^2 \sigma_z - 2k_x k_y \sigma_y - \lambda k_y^2 \sigma_z - E)\psi = 0
\]  

(5)

Solving such a nonlinear generalized eigenvalue equations for \( k_x \) at a fixed energy allows us to determine what modes \( k_x \) and corresponding eigenvectors can contribute to the tunneling problem in the different regions. That is, the solutions for a zero potential allow us to determine what modes can in principle be generated by the interaction with a barrier—whether it is a potential barrier, a region with magnetic field or otherwise. Of particular interest are modes with real valued \( k_x \) because those survive the
asymptotic limit $x \to \infty$, that is they could be measured by a distant detector since they do not decay exponentially (exponentially growing modes are not allowed for $x \to \pm \infty$ because they are unphysical). Below in Figure 3 we show a diagram that displays how many real valued solutions there are for $k_x$ at an energy given by Equation (4) for various values of $n$.

We find that there are different ranges for the anisotropy parameter $\lambda$ and the incident angle $\theta$ that allow for a different number of real valued solutions. We interpret the transitions between different regions as phase boundaries in a phase diagram for a distant detector. This suggests an interesting structure for tunneling processes, which we will interpret later. We should stress that this diagram—regardless of the specific type of barrier, whether it is a potential or something else—tells us how many modes can appear in an asymptotic limit after scattering. It is therefore a general feature of specific to multi-Weyl semimetals. One should
also note the crucial dependence on the anisotropy parameter—the case of an isotropic system $\lambda = \pm 1$ is trivial with only two allowed modes regardless of incident angle.

Before we dig any deeper in our interpretation of these diagrams, let us clarify why we use the notion of phase diagram for a distant detector. This can be made slightly more precise by making use of an analogy to dynamical phase transitions. It has been argued in ref. [99] that expressions like

$$P_\alpha = \left| \langle \psi_n | U(t) \psi \rangle \right|^2$$

bear a formal resemblance to partition functions (skipping all technical details and subtleties like time ordered exponentials, etc. The time evolution operator $U(t) = e^{iHt}$ is related to the Gibbs ensemble density matrix $\rho(\beta) = e^{-\beta H}$ via a Wick rotation). Nonanalytic behavior that can be found in the $P_\alpha$ is at the heart of the analogy that was used to coin the term dynamical phase transition. Similarly, in the case of a tunneling problem for a distant detector one is interested in terms having the same form as Equation (6)—for the situation $t \to \infty$—rather than finite $t$. It therefore seems reasonable to expect similar nonanalytic behavior, which can be related to a phase transition—in this case it would be a phase transition that differs slightly from the time dependent case. In the coming sections we will see that indeed we will find nonanalytic properties of tunneling coefficients as suggested by our diagrams. We argue, however, that in our case

Figure 3. Shown is a tunneling phase diagram for the Hamiltonian (1) that shows how many real solutions for $k_x$ exist for certain parameter combinations for incident angle $\theta$ and anisotropy parameter $\lambda$.

(a) Tunneling phase diagram for the $n = 2$ case.

(b) Tunneling phase diagram for the $n = 3$ case.

(c) Tunneling phase diagram for the $n = 4$ case.

(d) Tunneling phase diagram for the $n = 5$ case.
a long distance limit (distance from the barrier) takes the place of
a thermodynamic large N limit. The usefulness of this analogy can be
seen through the following theorem: The number of real valued modes
as a means to characterize the system is only sensible in this long
distance limit. Indeed, outside of this limit the boundaries in the
atom 3 are meaningless—much like the phase boundaries in thermal transitions without the thermody-
namic limit. Furthermore, contributions to wavefunctions from
complex valued k modes in the transmission region decay expo-
nentially and therefore don’t contribute to the scattering matrix.

4. Tunneling in the Case of \( n = 2 \)

To actually understand our phase diagrams better, we will now
consider the simplest case \( n = 2 \) with a step-wise potential barrier
such that the Hamiltonian for the different regions in Figure 1 is
given as

\[ H_j(k_x, k_y) = \left( \varepsilon_j k_x^2 - \varepsilon_j k_y^2 - 2ik_j k_y \right) \left( \varepsilon_j k_x^2 - \varepsilon_j k_y^2 + 2ik_j k_y \right) \]

(7)

where \( \varepsilon_j = \delta_{j1} V \). We focus on this case \( n = 2 \) rather than \( n > 2 \)
because it allows for relatively tractable expressions and carries
most of the important insights. It therefore makes for a good
minimal model of the physics we capture in this work. We note
for anyone that might be interested in the richer case \( n = 3 \) that
the some of the discussion would be almost identical to the
study\(^{112}\) (just focusing on a single Weyl subspace of the
Dirac Hamiltonian).

In order to solve the eigenvalue problem we can separate vari-
ables and write the eigenspinors as plane waves in the y-direction.
This is due to the fact that \( \{ H_j, k_j \} = 0 \) requires the conservation
of momentum along the y-direction, then we can write \( \Psi(x, y) =
\exp(i y \phi) \psi(x) \). The associated eigenspinors are solutions of the
non-linear eigenvalue equation

\[ \left[ \lambda \left( k_x^2 - k_y^2 \right) \sigma_y - 2k_x k_y \sigma_x + (\varepsilon_j - E_j) \right] \psi = 0 \]

(8)

The eigenvalues in regions 1 and 3 (compare Figure 1) are
found to be

\[ k_{x,1} = -k_{c,1} = k \cos \theta \]

(9)

\[ k_{x,2} = -i k \sqrt{\frac{\delta}{2 \lambda}} \cos(\phi, \lambda) \]

(10)

where we have set

\[ f(\theta, \lambda) = (3\lambda^2 - 4) \cos(2\theta) - \lambda^2 + 4 \]

(11)

The number of real modes, that is, modes that can be measured
by a distant detector—therefore solely depends on the sign of
\( f(\theta, \lambda) \).

Asymptotically the tunneling problem for the different colored
regions in Figure 3a can be interpreted as in the Figure 4.

We find that in blue colored regions of Figure 3a an incoming
wave vector \( k \) can scatter into two different transmitted modes
with wavevectors \( (k_{x,1}^+, k_y) \) and \( (k_{x,2}^+, k_y) \) (see Figure 4b). That is,
we get transmitted waves at two different angles after the barrier.
Similarly, in the brown colored regions there is only one mode
that survives in the long distance limit (see Figure 4a). There-
fore, the phase barriers in Figure 3a can be interpreted as critical
incident angles (at fixed incoming wavevector amplitude \( k \))
for which it becomes possible to obtain two transmitted modes.

Similarly to regions 1 and 3 (see Figure 1) for completeness we
mention that in the region 2 we find the following solutions

\[ q_{x,1} = -q_{x,1} = q \cos \phi \]

(12)

\[ q_{x,2} = -q_{x,2} = -i \frac{q}{\sqrt{2 \lambda}} \sqrt{f(\phi, \lambda)} \]

(13)

where \( q = \left( \frac{2(\varepsilon_j - E_j)}{\sqrt{\lambda^2 - 1}} \right)^2 \) and energy \( E \) given as in Equa-
tion (4). Furthermore, one may recall that for an incoming
wave in region 1, \( k_y = k \sin \theta \) and by definition \( q_y = q \sin \phi \).
From momentum conservation in $\gamma$-direction we then have $\phi = \sin^{-1}(\frac{x}{L} \sin \theta)$.

We may now analyze the tunneling properties of our system.

### 4.1. Isotropic Case $\lambda = 1$

First recall that Figure 3 shows that there are two parameter regions that have to be treated separately in an analytical discussion. The blue region in the diagram has been found to host four real modes, while the light brown region hosts only two real modes and two complex modes. For simplicity and to gain some analytical insights let us first consider the light brown region, which only has one transmitted mode that survives the asymptotic limit $x \to \infty$. More specifically we will restrict our discussion in this section to the isotropically dispersed case $\lambda = 1$. This is because most analytic progress can be made for this case and the discussion is not obscured by mathematical complications.

The more general case will only be treated numerically. In the case $\lambda = 1$, we find that the wavefunctions $\Phi_i(x)$ in region $i$ are given as

$$\Phi_1(x) = e^{ik_{1,m}x} \left( X_{1,1}^+ + r_1 e^{ik_{1,m}x} X_{1,1}^- + r_2 e^{ik_{1,m}x} X_{2,2}^- \right)$$

(14)

$$\Phi_2(x) = b_1 e^{ik_{1,m}x} \left( X_{1,1}^- + b_1 e^{ik_{1,m}x} X_{2,2}^+ \right)$$

(15)

$$\Phi_3(x) = t_1 e^{ik_{1,m}x} \left( X_{1,1}^+ + t_2 e^{ik_{1,m}x} X_{2,2}^- \right)$$

(16)

where we have used the short-hand notation

$$X_{\pm,m} = \frac{k_{\pm,m}}{k_{\pm,m} - ik \sin \theta}$$

(17)

and $m = 1, 2$ is labeling different wave vectors. Hereby, $r_1$, $b_1$, and $t_1$ are reflection, barrier amplitudes, and transmission coefficients, respectively, that have to be determined by the boundary conditions at the barrier edges. We note that from all the possible momentum solutions shown in Equations (9) and (12) we chose the appropriate ones that are physically allowed to contribute as follows. In region 1 ($x < -L/2$), we need to have finite $\Phi_1$ as $x \to -\infty$ and we find that $k_{\pm,2}$ is a propagating wave and therefore is allowed. Next we see that $k_{\pm,3}$ corresponds to a solution that grows exponentially as $x \to -\infty$ and so it is not physically allowed. In region 2 ($-\frac{L}{2} < x < \frac{L}{4}$) there is no physical restrictions and hence all possible plane wave contributions can be included. Last, for region 3 ($x > \frac{L}{4}$), where only modes propagation away from the barrier are allowed and terms need to be finite for $x \to \infty$, we find that $k_{\pm,1}$ and $k_{\pm,2}$ fulfill this condition.

To determine the coefficients $r_1$, $b_1$, and $t_1$, we use the boundary conditions at the interfaces $x = \pm L/2$. The continuity of the spinor wavefunctions and their first derivatives at each junction interface read as

$$\Phi_1\left( \frac{-L}{2} \right) = \Phi_2\left( \frac{-L}{2} \right), \quad \Phi_2\left( \frac{L}{2} \right) = \Phi_1\left( \frac{L}{2} \right)$$

(18)

Once the different coefficients for the wavefunction are determined we may then determine the currents in the different regions. The current in $x$-direction is found as

$$J_x = i \left[ \left( \frac{\partial}{\partial x} \right)^{\uparrow} - \frac{\partial}{\partial x} \Psi \right] - 2 \left( \frac{\partial}{\partial y} \Psi \right)$$

(20)

One may then use the transmitted current $J_{xT}$ and incident current $J_{xI}$ to define a transmission and reflection coefficients. Here, one has to keep in mind that in the limit $x \to \pm \infty$ not all terms in the wavefunction will survive. Specifically, for the wavefunction in region 1 the term $r_2$ will not survive the asymptotic limit because it decays rapidly away from the barrier. Similarly, for the wavefunction in region 3 the term $t_2$ will not survive the asymptotic limit. Hence, we can easily define transmission and reflection coefficients in the asymptotic limit as the following ratios

$$R_i = \left| \frac{J_{xI}}{J_{xT}} \right|^2, \quad T_i = \left| \frac{J_{xT}}{J_{xI}} \right|^2$$

(21)

where $J_{xI}$ is the $x$-component of the incoming current, $J_{xT}$ the reflected current in the asymptotic limit and $J_{xT}$ the transmitted current in the asymptotic limit.

For the case of normal incidence $\theta = 0$ the problem permits a simple analytic solution and the transmission coefficient $T_1$ is given by

$$t_1 = \frac{2ikq e^{-\alpha L}}{(k - q)(k + q) \sinh(2q) + 2ikq \cosh(2q)}$$

(22)

which in the limit $L \to \infty$ becomes zero. This is a result that was previously noted.\textsuperscript{113}

Below we study the more general case of arbitrary incident angle (still the isotropic case) numerically. First we plot the transmission and reflection coefficients as functions of barrier thickness $L$, which is shown in Figure 5.

We find that the transmission coefficient $T_1$ exhibits the well-known and expected Fabry–Perot resonances. The same resonances can be seen as dips in the reflection amplitude $R_1$, due to the requirement, $T_1 + R_1 = 1$, which follows from the conservation law for the probability current.

In a similar fashion we are able to see the behavior of the transmission amplitude $T_1$ as function of angle. This is shown in Figure 6.

We find for normal incidence $\theta = 0$ that there is a dip in the transmission amplitude that gets closer to zero as we increase the width $L$ of the barrier. This effect is sometimes referred to as anti Klein tunneling\textsuperscript{113} and is something that we can see easily from our analytical solution for normal incidence Equation (22). We stress that this kind of effect does not appear in the case of Weyl semimetals,\textsuperscript{114} graphene\textsuperscript{110} and cubic Dirac semi-metals that show Klein tunneling.\textsuperscript{112} Since this effect appears at all choices of energy it also fundamentally differs from the behavior of normal conductors and quadratic band touching.
4.2. General Case

For the general anisotropic case an analytical discussion would go along the same lines as in previous section but is riddled with various technical pitfalls—for instance wavefunctions in blue colored and brown colored regions of Figure 3 have different expressions with different asymptotics, etc. To avoid discussing these complications, which do not offer any useful insights, we rather make use of a simulation package KWANT\(^\textsuperscript{[113]}\) to compute tunneling amplitudes. The other reason we use this package is because it makes direct use of a description involving scattering matrices and therefore makes the relation of the phase diagram in Figure 3 with conventional phase transitions more lucid (see Section 3 for details). More precisely, we use numerical methods to express the scattering matrix and then extract transmission/reflection amplitudes at a given energy. Our first step in numerically solving the Schrödinger equation is to discretize the Hamiltonian and express it in terms of a tight binding model with on-site potentials \(V_{nm}\) and hopping parameters \(t_{nm}\). In our particular case, the system is infinite in the \(y\)-direction, which allows us to use the translational invariance and treat \(k_y\) as a parameter. This means that for any fixed \(k_y\), we obtain the band structure as function of \(k_x\). More explicitly, the discretization we chose is a square lattice tight binding model given by

\[
H = \sum_{n,m} \left[ V_{nm} c_{n,m}^\dagger c_{n,m} + \sum_{\delta = \pm 1} \lambda_{\delta} c_{n,\delta a}^\dagger c_{n+\delta a,m} \right]
\]

with nearest neighbor (NN) hopping matrices

\[
t_{1,0} = -t_{0,1} = -\lambda \sigma_x
\]

and next nearest neighbor (NNN) hopping matrices

\[
t_{1,-1} = -t_{1,1} = \frac{1}{2} \sigma_y
\]

the remaining hoppings are neglected. Hereby, \(\sigma\) are Pauli matrices, \(\sigma_x = (0,1,0,0,0,0)\) is a vector of annihilation operators and \(c_{n,m}^\dagger\) is the corresponding vector of creation operators. The labels \(\uparrow, \downarrow\) correspond to the pseudospin degree of freedom in Equation (2).

As a test of this numerical approach and our discretization in particular, we verified that the numerically computed \(E(k_x)\) agrees with the analytical result \(E = \sqrt{4\lambda^2 (\cos k_y - \cos k_y) + (2 \sin k_y \sin k_y)^2}\) (for \(n = 2\) in Equation (2)) in the low energy regime that we will restrict ourselves to. That this agreement is indeed good can be seen in Figure 7.

Our numerical results agree with our analytical prediction in Figure 3a that for \(\lambda < \sqrt{2}\) an incoming mode can be scattered to a unique mode with the same angle. In the other case \(\lambda > \sqrt{2}\) the transmission to a second mode is possible as it can be seen in Figure 7. To understand why this happens, we need to recall that since the Hamiltonian is not isotropic, changing the incident angle will lead to a change in the energy (at a fixed wavenumber \(k\)) and thus, for a critical angle \(\theta\), a change in number of allowed modes may occur. An example of how this effect manifests itself in a transmission amplitude computed at a fixed incoming wavevector \(k = k(\cos \theta, \sin \theta)\) can be seen in Figure 8.

We first note that the transmission to the same mode is several magnitudes larger than the transmission to the second mode. We also find that although we are in a tunneling regime, full transmission is still possible. Indeed, two maxima with unit transmis-
Different bands for different values $\lambda$ at the same value $k_y = 0.65$ (in units of $1/a$ and $a$ is the lattice constant of the tight binding model we used). The dashed line is obtained for $\lambda = \sqrt{2}$. Possible scattering to two modes $\lambda > \sqrt{2}$ (left curves) and scattering to a unique mode for $\lambda < \sqrt{2}$ (right curves).

Figure 8. a) The transmission $T_{11}$ from one mode to the same mode as function of the incoming incidence angle. b) The transmission $T_{12}$ from the incoming mode to another mode. The dashed line is the angle for which a second mode opens up ($\lambda = 1.67$). As parameters we used the potential $v = 0.0661/a^2$ and $L = 2a$, $k = 0.067/a$, where $a$ is the lattice constant of the tight binding model. For these conditions the energy $E$ is not constant (wavenumber $k$ is kept constant) but it is chosen to be smaller than the barrier potential $E < V$ for all incidence angles (tunneling regime).

Figure 7. Different bands for different values $\lambda$ at the same value $k_y = 0.65$ (in units of $1/a$ and $a$ is the lattice constant of the tight binding model we used). The dashed line is obtained for $\lambda = \sqrt{2}$. Possible scattering to two modes $\lambda > \sqrt{2}$ (left curves) and scattering to a unique mode for $\lambda < \sqrt{2}$ (right curves).

In conclusion we have studied tunneling in nonisotropic multi-Weyl semimetals and found them to have interesting tunneling phenomena such as Klein tunneling, anti-Klein tunneling and the possibility for an ordinary potential to scatter electrons into two or more transmitted modes with different directions of wave propagation. We found that this type of phenomenon occurs only after certain critical values of the anisotropy parameter and incident angle. We summarized this observation in Figure 3, which we were able to identify as a kind of phase diagram—invoking an interesting analogy to a time dependent phase transition. This analogy allowed us to coin the term tunneling phase transition because we found that—much like an ordinary phase transition—phase boundaries in Figure 3 coincide with nonanalytical behavior in experimental observables (in our case the tunneling coefficients in Figure 8), which is the hallmark of any phase transition.

Unlike the case of the more conventionally studied phase transitions, the nonanalytical behavior was not caused by a large number of particles $N \to \infty$ limit but rather by a long distance limit $L \to \infty$. We anticipate that these observations can serve as motivations to look for similar phenomena in other systems. After all, our work shows that the large $N$ limit is not unique in being able to cause behavior akin to phase transitions of matter. Rather, similar behavior can be expected in other systems with limiting behavior.

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Conflict of Interest

The authors declare no conflict of interest.
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