OPTIMALITY OF TWO INEQUALITIES FOR EXONENTS OF
DIOPHANTINE APPROXIMATION

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Dedicated to the 50th birthday of Yann Bugeaud

Abstract. We investigate two inequalities of Bugeaud and Laurent, each involving
triples of classical exponents of Diophantine approximation associated to \( \xi \in \mathbb{R}^n \). We
provide a complete description of parameter triples that admit equality for suitable \( \xi \),
which turns out rather surprising. For \( n = 2 \) our results agree with work of Laurent.
Moreover, we establish lower bounds for the Hausdorff and packing dimensions of the
involved \( \xi \), and in special cases we can show they are sharp. Proofs are based on the
variational principle in parametric geometry of numbers, we enclose sketches of associ-
ated combined graphs (templates) where equality is feasible. A twist of our construction
provides refined information on the joint spectrum of the respective exponent triples.

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1. Introduction

1.1. Classical exponents of approximation. Let \( n \geq 1 \) be an integer and \( \xi = \left( \xi_1, \ldots, \xi_n \right) \in \mathbb{R}^n \) with \( \{1, \xi_1, \ldots, \xi_n\} \) linearly independent over \( \mathbb{Q} \). Let the (possibly
infinite) exponents of approximation \( \omega \) and \( \hat{\omega} \) resp. be defined as the suprema of reals \( u \) so that the system

\[
1 \leq x \leq X, \quad \max_{1 \leq i \leq n} |x_\xi_i - y_i| \leq X^{-u}
\]

has a solution in integer vectors \((x, y_1, \ldots, y_n)\) for arbitrarily large and all large \( X \), re-
respectively. Let \( \omega^*, \hat{\omega}^* \) be the supremum of \( v \) so that

\[
1 \leq \max_{1 \leq i \leq n} |a_i| \leq X, \quad |a_0 + a_1 \xi_1 + \cdots + a_n \xi_n| \leq X^{-v}
\]

has a solution in integers \( a_i \) for arbitrarily large \( X \) and all large \( X \), respectively. By
variants of Dirichlet’s Theorem, for any \( \xi \in \mathbb{R}^n \) we have

\[
\infty \geq \omega \geq \hat{\omega} \geq \frac{1}{n}, \quad \infty \geq \omega^* \geq \hat{\omega}^* \geq n.
\]
1.2. Two inequalities by Bugeaud and Laurent. Let \( n \geq 2 \) be an integer. Bugeaud and Laurent [2] established that every \( \xi \) as above satisfies the estimates

\[
\omega \geq \frac{(\hat{\omega}^* - 1)\omega^*}{((n-2)\hat{\omega}^* + 1)\omega^* + (n-1)\hat{\omega}^*}
\]

and

\[
\omega^* \geq \frac{(n-1)\omega + \hat{\omega} + n - 2}{1 - \hat{\omega}}.
\]

In view of (2) they imply Khintchine's transference inequalities [12]

\[
\omega^* \geq n\omega + n - 1, \quad \omega \geq \frac{\omega^*}{(n-1)\omega^* + n}.
\]

These are known to be sharp for any parameter pairs induced by \( \omega \in [1/n, \infty] \) resp. \( \omega^* \in [n, \infty] \). Moreover, combining (BL1), (BL2) implies non-trivial relations between \( \omega \) and \( \hat{\omega} \), and likewise between \( \omega^* \) and \( \hat{\omega}^* \). For \( n = 2 \), they become

\[
\omega \geq \frac{\hat{\omega}^2}{1 - \hat{\omega}}, \quad \omega^* \geq \hat{\omega}^2 - \hat{\omega}^*,
\]

already known by Jarník [11] and are again sharp for any parameter pairs induced by \( \omega \in [1/n, \infty] \) resp. \( \omega^* \in [n, \infty] \). For \( n > 2 \), the implied relations turn out to be no longer best possible. Marnat and Moshchevitin [15] settled the sharp estimates conjectured by Schmidt and Summerer [26] along their discussion of the "regular graph", who gave proofs for \( n = 3 \) themselves. See also Schmidt and Summerer [25], Moshchevitin [16] and Rivard-Cooke’s thesis [17].

As noticed by German and Moshchevitin [9], the inequalities (BL1) resp. (BL2) split into pairs of inequalities respectively given by

\[
1 + \omega^{-1} \geq \frac{1}{1 + \omega^{-1}}, \quad 1 + \omega^* \geq \frac{1}{1 + \omega^*}.
\]

The respective left inequalities originate in Schmidt and Summerer [24], the right ones in German [7], with alternative proofs given later in [8], [9] resp. [27]. In particular, in case equality in (BL1) or (BL2), both respective splitting inequalities must be identities as well. Thus our results below also establish optimality of these splitting inequalities for certain parameter ranges (shown for the resp. right inequalities already in [27]).

2. Main result: Description of equality cases

For \( n = 2 \) all estimates (BL1), (BL2) are sharp by a result of Laurent [13], as already pointed out in [2]. For general \( n \geq 2 \), when \( \omega^* \geq n, \hat{\omega}^* = n \) resp. \( \omega \geq 1/n, \hat{\omega} = 1/n \) inequalities (BL1) resp. (BL2) simplify to (3) and are sharp, as noticed in Section 1.2. Other than that, when \( n > 2 \) the optimality of these estimates was very open. The main purpose of this note is to give a comprehensive description of equality cases.

Let us start with (BL1). We explicitly determine the submanifold generated by the intersection of the spectrum of \((\omega^*, \hat{\omega}^*, \omega) \subseteq (\mathbb{R} \cup \{\infty\})^3\) induced by \( \xi \in \mathbb{R}^n \) with the...
hypersurface in \((\mathbb{R} \cup \{\infty\})^3\) induced by equality in (BL1). For given \(n \geq 2\) and a real parameter \(x \geq n\), define
\[
\rho_1(n, x) = \frac{x}{(n-1)x + n},
\]
and
\[
\rho_2(n, x) = \frac{(2n-4)x^2 + (2n-1 - \sqrt{4n-4)x + 1})x - \sqrt{4n-4)x + 1 + 1}{2((n-2)x^2 + (2n^2 - 6n + 3)x + n^2 - 2n)}.
\]
By taking limits we put \(\rho_1(n, \infty) = 1/(n-1)\) and \(\rho_2(n, \infty) = 1/(n-2)\), where we consider \(1/0 = +\infty\). Observe \(\rho_1(n, n) = \rho_2(n, n) = 1/n\). Our results will involve Hausdorff and packing dimension, see [5] for an introduction.

**Theorem 2.1.** Let \(n \geq 2\) be an integer. Then precisely for triples \((w^*, \hat{w}^*, w) \subseteq (\mathbb{R} \cup \{\infty\})^3\) that can be parametrized as

\[
(6) \quad w^* \in [n, \infty], \quad w \in [\rho_1(n, w^*), \rho_2(n, w^*)], \quad \hat{w}^* = \frac{w^*(w+1)}{w^* - (n-2)ww^* - (n-1)w}
\]

there is \(\xi \in \mathbb{R}^n\) which induces equality in (BL1) and

\[
(7) \quad \omega^* = w^*, \quad \hat{\omega}^* = \hat{w}^*, \quad \omega = w.
\]

In fact, for each admissible parameter triple \((w^*, \hat{w}^*, w)\) as in ([6], the set of \(\xi\) inducing (11) has Hausdorff dimension at least \(n-2\) and packing dimension at least \(n-2 + 1/n\).

The identity in (6) is a reformulation of equality in (BL1) and its explicit statement is purely conventional. The lower bound \(\rho_1\) reflects identity in the right estimate of (3), so it is as small as it can possibly be. On the other hand, the surprising bound \(\rho_2\) is strictly smaller than the bound \((w^* - n + 1)/n\) from the left estimate in (3), unless in trivial cases. Thus, by the optimality of Khintchine’s estimates, when \(w^* \in (n, \infty]\) and \(w \in (\rho_2, (w^* - n + 1)/n]\), there are \(\xi\) whose associated exponents satisfy \(\omega^* = w^*, \omega = w\) but there is no \(\xi\) with additional equality in (BL1). We remark that if \(w = \rho_2\), then as \(w^* \to \infty\) also \(\hat{w}^* \to \infty\), and \(w \to 1/(n - 2)\). We point out that for \(\xi\) as in the theorem inducing (7), the remaining exponent \(\hat{\omega}\) can be determined as

\[
(8) \quad \hat{\omega} = \frac{1 + w^{* - 1}}{1 + w^{-1}}
\]
by (5). Consequently, if \(w = \rho_2\) then \(\hat{\omega} \to 1/(n - 1)\) as \(w^* \to \infty\). As \(w^* \to n\) from above, both \(\rho_1, \rho_2\) tend to \(1/n\) as it needs to be. For refinements of the metrical claim see Section 4.2 below.

For \(n = 2\) the claim agrees with the findings of Laurent [13] discussed above. For \(n = 2\) and \(\omega = \rho_2\) we have equality in the inequalities of (4), thereby we obtain a regular graph as mentioned in Section 1.2. For \(n > 2\), no regular graph can appear upon identity in (BL1), unless in the trivial case \(w^* = \hat{w}^* = n, w = 1/n\).

Our results suggest that solutions to the following questions are in reach.

**Problem 1.** For given \(\omega^*\) and \(\omega > \rho_2(n, \omega^*)\) find sharp estimates improving (BL1). Ideally, determine the spectrum of \((\omega^*, \hat{w}^*, \omega) \subseteq (\mathbb{R} \cup \{\infty\})^3\).
We turn towards the dual estimates (BL2). For \( n \geq 2 \) and \( x \geq 1/n \), define
\[
\tau_1(n, x) = nx + n - 1,
\]
and
\[
\tau_2(n, x) = \frac{x^2}{2} + \left( n - \frac{1}{2} + \frac{\sqrt{x(x + 4n - 4)}}{2} \right) x + \frac{\sqrt{x(x + 4n - 4)}}{2} + n - 2.
\]
By taking limits we extend it to \( \tau_j(n, \infty) = \infty \) for \( j = 1, 2 \). Our result reads as follows.

Theorem 2.2. Let \( n \geq 2 \) be an integer. Then precisely for triples \((w, \hat{w}, w^*) \subseteq (\mathbb{R} \cup \{\infty\})^3\) that can be parametrized by the properties
\[
\begin{align*}
w \in [1/n, \infty], & \quad w^* \in [\tau_1(n, w), \tau_2(n, w)], & \quad \hat{w} = \frac{w^* - (n - 1)w - n + 2}{1 + w^*} \quad \text{(9)} \end{align*}
\]
there is \( \xi \in \mathbb{R}^n \) which induces equality in (BL2) and
\[
\begin{align*}
\omega = w, & \quad \hat{\omega} = \hat{w}, & \quad \omega^* = w^* \quad \text{(10)}
\end{align*}
\]
For each suitable parameter triple \((w, \hat{w}, w^*)\), the set of associated \( \xi \) inducing (10) has packing dimension at least \( 1/2 \), and positive Hausdorff dimension if \( w < \infty \).

Analogous remarks as for Theorem 2.1 apply. The lower bound \( \tau_1 \) reflects equality in the left estimate of (3), whereas \( \tau_2 \) is strictly smaller than the value induced by equality in the right inequality of (3), unless if \( \omega = \omega^* = n \). Let \( w^* = \tau_2 \). Then, as \( w \to \infty \) we have \( \hat{w} \to 1 \) and \( w^* \to \infty \), where the latter agrees with the obvious estimate \( \omega^* \geq \omega \). Note that 1 is the largest value \( \hat{\omega} \) can attain. For \( \xi \) as in Theorem 2.2, the missing exponent \( \hat{\omega}^* \) can again be evaluated as
\[
\hat{\omega}^* = \frac{1 + w^*}{1 + w} \quad \text{(11)}
\]
by (5). If \( \omega^* = \tau_2(n, \omega) \) then again as \( w \to \infty \) also \( \hat{\omega}^* \to \infty \), and iff \( n = 2 \) this leads to the regular graph. We formulate the analogous questions to Problem 1.

Problem 2. For \( \omega^* > \tau_2(n, \omega) \), find sharp estimates improving (BL2). Ideally, determine the spectrum of \((\omega, \hat{\omega}, \omega^*) \subseteq (\mathbb{R} \cup \{\infty\})^3\).

As partial results to Problems 1, 2 for any triples satisfying
\[
\begin{align*}
w^* & \in [n, \infty], \quad w \in [\rho_1(n, w^*), \rho_2(n, w^*)], \quad \hat{w}^* \in \left[ n, \frac{w^*(w + 1)}{w^* - (n - 2)ww^* - (n - 1)w} \right] \quad \text{(12)}
\end{align*}
\]
resp.
\[
\begin{align*}
w & \in [1/n, \infty], \quad w^* \in [\tau_1(n, w), \tau_2(n, w)], \quad \hat{w} \in \left[ \frac{1}{n}, \frac{w^* - (n - 1)w - n + 2}{1 + w^*} \right] \quad \text{(13)}
\end{align*}
\]
for suitable \( \xi \in \mathbb{R}^n \) we still get (7) resp. (10). Clearly the ranges for \( \hat{\omega}^*, \hat{\omega} \) are optimal. See Section 5.7 for the proof. Problems 1, 2 can be considered partial problems towards finding the entire spectrum in \( \mathbb{R}^{2n+2} \) of all extremal values of successive minima exponents, which is wide open for \( n > 2 \). See Section 3.1 for details.
We outline the rest of the paper. In Section 3 we introduce parametric geometry of numbers in the notion of Schmidt and Summerer [25] and formulate a special case the variational principle by Das, Fishman, Simmons, Urbański [3], [4] within this framework. We append relations to notions of Roy [19], [20] and Schmidt, Summerer [28]. In Section 4 we reformulate Theorems 2.1, 2.2 into this language, and refine the metrical claims. In Sections 5, 6 these claims and (12), (13) are proved using the prerequisites from [3], [4]. It is worth noting that everything except from the metrical claims can be alternatively obtained from Roy [19], [20] in place of [3], [4], see Section 3.2 for details. Finally we close with some remarks interconnecting our work with [2], [20] in Section 7.

3. Parametric geometry of numbers

3.1. Parametric functions and their extremal values. Let us interpret the simultaneous approximation problem (1) as a parametric successive minima problem. For a parameter, let

\[ K(q) \subseteq \mathbb{R}^{n+1} \]

be the box of points \((z_0, z_1, \ldots, z_n)\) that satisfy

\[ |z_0| \leq e^{nq}, \quad \max_{1 \leq i \leq n} |z_i| \leq e^{-q}. \]

Further let \(\Lambda = \Lambda_\xi\) be the lattice consisting of all points of the form \((x, \xi x - y_1, \ldots, \xi x - y_n) : x, y_i \in \mathbb{Z}\). Denote by \(\lambda_1(q), \ldots, \lambda_{n+1}(q)\) the successive minima of \(K(q)\) with respect to \(\Lambda\), to obtain functions of \(q\). For \(1 \leq j \leq n+1\), derive \(\varphi_j(q) = \log \lambda_j(q) / q\) and define the lower and upper limits

\[ \underline{\varphi}_j = \liminf_{q \to \infty} \varphi_j(q), \quad \overline{\varphi}_j = \limsup_{q \to \infty} \varphi_j(q). \]

These quantities lie within the interval \([-n, 1]\). Schmidt and Summerer [25, (1.8), (1.9)] observed they are connected to exponents of Section 1.1 via the transference identities

(T1) \[(1 + \omega)(n + \varphi_1) = (1 + \hat{\omega})(n + \underline{\varphi}_1) = n + 1, \]

and

(T2) \[(1 + \omega^*)(1 - \overline{\varphi}_{n+1}) = (1 + \hat{\omega}^*)(1 - \underline{\varphi}_{n+1}) = n + 1. \]

Hereby we mean \(\omega = \infty\) if \(\underline{\varphi}_1 = -n\) and likewise for other identities. See [3, Corollary 8.5] for a generalization. In terms of \(\overline{\varphi}_j, \underline{\varphi}_j\), Khintchine’s estimates (3) simply read

(14) \[ n \underline{\varphi}_1 + \overline{\varphi}_{n+1} \geq -\frac{\varphi_1}{n}, \quad \frac{\varphi_1}{n} \leq -\frac{\overline{\varphi}_{n+1}}{n}. \]

The deduction from (T1), (T2) is carried out in Remark (b) in [24]. Equivalent formulations of (4) obtained similarly can be found in [25, (1.20), (1.20’)]. Moreover (T1), (T2) imply that (BL1), (BL2) are respectively equivalent to

(SS1) \[ n \underline{\varphi}_1 + \overline{\varphi}_{n+1} \leq -\frac{\varphi_1}{n} \cdot \left( \frac{n + 1}{n - 1} + \frac{2}{n - 1} \overline{\varphi}_{n+1} \right) \]

and

(SS2) \[ n \overline{\varphi}_{n+1} + \underline{\varphi}_1 \geq -\frac{\varphi_1}{n} \cdot \left( \frac{n + 1}{n - 1} + \underline{\varphi}_{n+1} + \frac{2}{n - 1} \underline{\varphi}_1 \right). \]
This was again already observed by Schmidt and Summerer [25] who provided independent proofs of (SS1), (SS2) based on parametric geometry of numbers, and thereby new proofs (BL1), (BL2). Other proofs of (BL1), (BL2) that again rely on parametric geometry of numbers came as a byproduct in [22], see [22, (23)] and [22, Remark 6]. The latter proofs from [22] give some information on $(\varphi_1(q), \ldots, \varphi_{n+1}(q))$ as a function $[0, \infty) \to [-n, 1]^{n+1}$, i.e. on the combined graph defined in Section 3.2 below, in case of equality. This observation inspired this note.

In recent years, much work has been done on the joint spectrum of exponents, that is the subset of $\mathbb{R}^{2n+2}$ that occurs as $(\varphi_1(q), \ldots, \varphi_{n+1}, \varphi_1, \ldots, \varphi_{n+1}) \in \mathbb{R}^{2n+2}$ when $\xi \in \mathbb{R}^n$ runs through all vectors that are $\mathbb{Q}$-linearly independent together with $\{1\}$. See for example [17], [18], [20], [21], [25], [27], [28], in particular the detailed exposition in Section 1 of [21]. In view of the equivalent claims (SS1), (SS2), our claims in Section 2 contribute to this area by providing new results on the projection to the three-dimensional spaces with coordinate variables $(\varphi_1, \varphi_{n+1}, \varphi_{n+1})$ and $(\varphi_1, \varphi_1, \varphi_{n+1})$, respectively. See Theorems 4.1, 4.2 below. We remark that for $n = 2$, a complete description of the joint spectrum via a system of inequalities was given in [21, Theorem 11.5], thereby containing implicitly the work of Laurent [13], Schmidt, Summerer [28] as well as the case $n = 2$ of our Theorems 2.1, 2.2. However, the explicit deductions from [21] seem cumbersome.

3.2. $n$-templates and the variational principle. From the functions $\varphi_j(q)$ and $\lambda_j(q)$ associated to $\xi \in \mathbb{R}^n$ as defined in Section 3.1, we derive $L_j(q) = q \varphi_j(q) = \log \lambda_j(q)$. These functions are piecewise linear with slopes among $\{-n, 1\}$, and by Minkowski’s Second Convex Body Theorem their sum is uniformly bounded

$$\sum_{j=1}^{n+1} L_j(q) \leq C(n), \quad q \in [0, \infty).$$

The values $\underline{\varphi}_j, \overline{\varphi}_j$ are just the extremal average slopes of the $L_j$ in a start segment, i.e.

$$\varphi_j = \liminf_{q \to \infty} \frac{L_j(q)}{q}, \quad \overline{\varphi}_j = \limsup_{q \to \infty} \frac{L_j(q)}{q}, \quad (1 \leq j \leq n+1).$$

We call $L_\xi(q) = (L_1(q), \ldots, L_{n+1}(q))$ on $q \in [0, \infty)$ the combined graph associated to $\xi \in \mathbb{R}^n$. We approximate it by easier systems $P = (P_1(q), \ldots, P_{n+1}(q))$ without error term as in (15), but where we may glue consecutive functions $P_j$ on intervals where they differ by a small amount. Let $Z(j) = \{j, j-1-n\}$, $P_0(q) = -\infty$, $P_{n+2}(q) = +\infty$.

Then following [4, Definition 5.1], an elegant formal description of the family of functions we consider can be stated as follows.

**Definition 1.** We call a continuous, piecewise-linear map $P : [0, \infty) \to \mathbb{R}^{n+1}$ an $n$-template if the component functions $P_j(q)$ satisfy (i)–(iv) below.
We have

\[(17) \quad \sum_{j=1}^{n+1} P_j(q) = 0, \quad q \in [0, \infty).\]

(ii) \(P_1(q) \leq P_2(q) \leq \cdots \leq P_{n+1}(q)\) for all \(q \in [0, \infty)\)

(iii) For any \(q \in [0, \infty)\) and \(1 \leq j \leq n+1\), if \(P_j\) is differentiable at \(q\) then

\[-n \leq P_j'(q) \leq 1.\]

(iv) For \(j = 0, 1, \ldots, n+1\) and every interval where \(P_j < P_{j+1}\) holds, the function

\[F_j(q) = P_1(q) + P_2(q) + \cdots + P_j(q)\]

is convex and has slopes \(F_j'\) in \(Z(j)\).

In place of (i) it would suffice to demand \(P_1(0) = 0\). Indeed \(\mathbf{P}(q)\) is an \(n\)-template iff \(\mathbf{P}(q)/n\) is a balanced \(n \times 1\)-template in [3, 4]. Each \(P_j\) is one-sided differentiable on \(q \in [0, \infty)\) with slopes within the finite set \(\{1\} \cup \{-k/(n+1-k) : 0 \leq k \leq n\}\). We further remark that generalizing 3-systems defined by Schmidt, Summerer [28] naturally to \((n+1)\)-systems for \(n \geq 1\), leads to special cases of \(n\)-templates, with slopes of the \(P_j\) restricted to \(\{-n, 1\}\) as for \(L_j\). More precisely, equipping the space of functions \([0, \infty) \to \mathbb{R}^{n+1}\) with the supremum norm, the closure of the set of \((n+1)\)-systems in [28] becomes our set of \(n\)-templates. By (i), the sum of the slopes at points of differentiability vanishes as well. In the special case of \((n+1)\)-systems this means one component decays with slope \(-n\) while the remaining \(P_j\) rise with slope +1. We further want to notice that upon applying some affine map, our \(n\)-templates correspond to the generalized \((n+1)\)-systems defined by Roy [20, Definition 5.1]. So an equivalent formulation of \(n\)-templates can be derived from a twist of [20, Definition 4.1]. This also implies that the results in [19], [20], [21], remain basically valid and we will use them occasionally below.

As noticed in [4], the following special case of [4, Theorem 5.2] is already covered by Schmidt and Summerer [25] and Roy [20, Corollary 4.7].

**Theorem 3.1.** For any \(\xi \in \mathbb{R}^n\), its associated combined graph has uniformly bounded distance from some suitable \(n\)-template \(\mathbf{P}\), i.e.

\[(18) \quad \|L_\xi(q) - \mathbf{P}(q)\| = \max_{1 \leq j \leq n+1} |P_j(q) - L_j(q)| \leq C(n), \quad q \in [0, \infty).\]

Conversely, for any \(n\)-template \(\mathbf{P}\), there is \(\xi \in \mathbb{R}^n\) inducing (18) for some effective \(C(n)\).

In particular, if \(\mathbf{P}\) is as in the theorem for \(\xi\), then by (16) we see

\[(19) \quad \limsup_{q \to \infty} \frac{P_j(q)}{q} = \varphi_j, \quad \liminf_{q \to \infty} \frac{P_j(q)}{q} = \underline{\varphi}_j.\]

Theorem 3.2 below refines the latter claim of Theorem 3.1 by adding metrical information. For \(\mathbf{P}\) an \(n\)-template, define its local contraction rate at \(q \in [q_0, \infty]\) by

\[\delta(\mathbf{P}, q) = \kappa - 1, \quad \kappa := \max \{j : P_j(q) < 1\}.\]
This agrees with the definition of $\delta(f, I)$ in [4, (5.10)] for $n \times 1$ templates, i.e. our $n$-templates. See also [14]. Derive the average contraction rate in the interval $[q_0, q]$ by

$$\Delta(P, q) = \frac{1}{q - q_0} \cdot \int_{q_0}^{q} \delta(P, u) \, du$$

and the upper and lower contraction rates by

$$\underline{\delta}(P) = \lim_{q \to \infty} \inf \Delta(P, q), \quad \overline{\delta}(P) = \lim_{q \to \infty} \sup \Delta(P, q).$$

Denote by $\dim_H$ resp. $\dim_P$ the Hausdorff resp. packing dimensions, and call a family $F$ of templates closed under finite perturbation if when $P, Q$ are $n$-templates and $P \in F$ and $\|P - Q\| < \infty$, then also $Q \in F$. Then a consequence of the variational principle [4, Theorem 5.3] can be stated as follows.

**Theorem 3.2.** Let $F$ be a family of $n$-templates closed under finite perturbation. Denote by $M = M(F)$ the set of all $\xi \in \mathbb{R}^n$ whose associated combined graphs $L_{\xi}(q)$ satisfy $\|L_{\xi}(q) - P(q)\| < \infty$ for some (thus all) $P \in F$. Then

$$\dim_H(M) = \sup_{P \in F} \underline{\delta}(P), \quad \dim_P(M) = \sup_{P \in F} \overline{\delta}(P).$$

In particular, for any given $n$-template $P$, taking $F$ the ”finite perturbation hull” of $\{P\}$ and deriving $M(F) = M(\{P\})$ as above, we see

$$\dim_H(M) \geq \underline{\delta}(P), \quad \dim_P(M) \geq \overline{\delta}(P).$$

As remarked in [3], there is no equality in general as $\delta$ is sensitive to perturbations.

4. **Reformulating claims in terms of $\varphi_j, \varphi_j$**

4.1. **Equivalent formulations of Theorems 2.1, 2.2** For $n \geq 2$ an integer and $x \in [-n, 1]$ write

$$g_n(x) = \frac{(3 - 2n)x + 1 - 2n + \sqrt{(1 - x)((4n - 5)x + 4n^2 - 4n + 1)}}{2(n - 1)^2}.$$  

An equivalent formulation of Theorem 2.1 in terms of the quantities $\varphi_j, \varphi_j$ that we will prefer to prove reads as follows.

**Theorem 4.1.** Let $n \geq 2$ be an integer. Then precisely for triples $(t, \mu, \sigma)$ satisfying

$$t \in [0, 1], \quad \mu \in [g_n(t), -\frac{t}{n}], \quad \sigma = (1 - n)\frac{t + n\mu}{n + 1 + 2t + (n - 1)\mu}$$

there is $\xi \in \mathbb{R}^n$ inducing equality in (SS1) and with associated values

$$\varphi_{n+1} = t, \quad \varphi_{n+1} = \sigma, \quad \varphi_1 = \mu.$$

For given $(t, \mu, \sigma)$ obeying the restrictions (21), the set of $\xi$ with these properties

$$\Theta = \Theta_{t, \mu}^{(n)} = \{\xi \in \mathbb{R}^n : \varphi_{n+1} = t, \quad \varphi_{n+1} = \sigma, \quad \varphi_1 = \mu\}$$

in fact has Hausdorff dimension at least $n - 2$ and packing dimension at least $n - 2 + 1/n$.  

Write \( \mu_0 := g_n(t) \) in the sequel. The deduction of Theorem 2.1 from Theorem 4.1 relies purely on (11), (12). By equivalence of (3) and (14) the upper bound \(-t/n\) is a consequence of the bound \( \rho_1 \) and again the best we can hope for. For \( \rho_2 \), we further use the defining equation

\[
(1 + t + (n - 1)\mu_0)^2 - (1 - t)(1 - \mu_0) = 0
\]

of \( \mu_0 \). Moreover, by (11), (12) we may write

\[
t = 1 - \frac{n + 1}{1 + \omega^*} = \frac{\omega^* - n}{1 + \omega^*}, \quad \mu = \frac{n + 1}{\omega + 1} - n = \frac{1 - n\omega}{\omega + 1}.
\]

Inserting for \( t \) and \( \mu = \mu_0 \) in (23) leads to a quadratic equation for \( \omega \) in \( \omega^* \) with (the correct) solution \( \rho_2 \), we omit the elementary calculation. Clearly, the argument can be read in reverse direction and the claims are indeed equivalent. We remark that if \( \mu = \mu_0 \), as \( \phi_{n+1} \to 1 \) we compute \( \phi_1 \to -2/(n - 1) \), and one can show \( \phi_1 \to -1/n \).

We refine the metrical claim of Theorem 4.1 in Section 4.2.

Keep \( g_n \) defined in (20). An equivalent formulation of Theorem 2.2 is the following.

**Theorem 4.2.** Let \( n \geq 2 \) be an integer. Then precisely for triples \((s, \nu, \gamma)\) satisfying

\[
s \in [-n, 0], \quad \nu \in \left[-\frac{s}{n}, g_n(s)\right], \quad \gamma = (1 - n) \frac{s + n\nu}{n + 1 + 2s + (n - 1)\nu}
\]

there is \( \xi \in \mathbb{R}^n \) inducing equality in (SS2) and with associated values

\[
\varphi_1 = s, \quad \varphi_1 = \gamma, \quad \varphi_{n+1} = \nu.
\]

For given \((s, \nu, \gamma)\) obeying the restrictions (24), the corresponding set

\[
\Sigma_{s, \nu}^{(n)} = \{ \xi \in \mathbb{R}^n : \varphi_1 = s, \quad \varphi_1 = \gamma, \quad \varphi_{n+1} = \nu \}
\]

has packing dimension at least 1/2, and positive Hausdorff dimension as soon as \( s > -n \).

The equivalence is derived similarly as for Theorem 4.1 we omit details. If \( \nu = \nu_0 := g_n(s) \), as \( \varphi_1 \to -n \) we calculate \( \varphi_1 \to -(n - 1)/2 \) and \( \varphi_{n+1} \to 1 \), in fact \( \varphi_{n+1} \to 1 \) holds. See again Section 4.2 for refined metrical claims.

### 4.2. Refinements of the metrical claims.

For simplicity we state our metrical claims in the language of Section 4.1 only. Corresponding results in terms of classical exponents can be inferred from (11), (12), which in particular imply the equivalences

\[
t \to 0^+ \iff \omega^* \to n^+, \quad t \to 1^- \iff \omega^* \to \infty, \quad \mu = -\frac{t}{n} \iff \omega = \rho_1, \quad \mu = \mu_0 \iff \omega = \rho_2,
\]

and

\[
s \to 0^- \iff \omega \to \frac{1}{n}, \quad s \to -n^+ \iff \omega \to \infty, \quad \nu = -\frac{s}{n} \iff \omega^* = \tau_1, \quad \nu = \nu_0 \iff \omega^* = \tau_2.
\]

We start with refining Theorem 4.1. Let

\[
A = A_{t, \mu}^{(n)} = \frac{1 - t}{2t + (n - 1)\mu} \cdot \frac{1}{n + 1} \cdot \left( 3t + 2(n - 1)\mu - n + \frac{n(1 + t + (n - 1)\mu)^3}{1 - t} \right) \geq 0,
\]
with $A > 0$ as soon as $t < 1$, and derive $B = B_{t,\mu}^{(n)}$ and $C = C_{t,\mu}^{(n)}$ as
\begin{equation}
B = n - \frac{(2 - A)(n + 1)}{n + 1 + 2t + (n - 1)\mu}, \quad C = n - 2 + A\frac{n + 1}{n + 1 + (n - 1)t + n(n - 1)\mu}.
\end{equation}
Then $A \in [0, 2]$ and $n \geq B \geq n-2+A, n \geq C \geq n-2+A$ for every $t, \mu$ as in Theorem 4.1
Recall the Hausdorff dimension of a set never exceeds its packing dimension [5].

**Theorem 4.3.** The dimensions of the sets $\Theta = \Theta_{t,\mu}^{(n)}$ in (22) are bounded from below by
\begin{equation}
\dim_H(\Theta) \geq n - 2 + A, \quad \dim_P(\Theta) \geq \max\{B, C\}.
\end{equation}
There is equality (at least) if $t > 0$ and $\mu = \mu_0$. Moreover, $\max\{B, C\} \geq n - 2 + \frac{1}{n}$.

We conjecture equalities in (27) as soon as $t > 0$ and $\mu \in [\mu_0, -t/n]$, but only rigorously prove it for $\mu = \mu_0$ in the last paragraph of Section 5.8. The restriction $t > 0$ is necessary and probably equality does not extend to $\mu = -t/n$ either, as we explain below. We discuss special cases. If $\mu$ attains its maximum value $\mu = -t/n$, then $A_{t, -t/n}^{(n)}$ decreases from $1 + (n + 1)^{-1}$ to 1 as $t$ rises in $(0, 1)$, and we infer
\begin{equation}
\lim_{t \to 0^+} \dim_P(\Theta_{t, -t/n}^{(n)}) \geq \lim_{t \to 0^+} \dim_H(\Theta_{t, -t/n}^{(n)}) \geq n - 1 + \frac{1}{n + 1}, \quad \lim_{t \to 1^-} \dim_H(\Theta_{t, -t/n}^{(n)}) \geq n - 1.
\end{equation}
As $t \to 0^+$ then $B_{t, -t/n}, C_{t, -t/n}$ tend to $n - 1 + 1/(n + 1)$ as well, suggesting equality in all estimates. We can pass to $t = 0, t = 1$ by considering limiting graphs, so in particular
\begin{equation}
\dim_P(\Theta_{t, -t/n}^{(n)}) \geq \dim_H(\Theta_{t, -t/n}^{(n)}) \geq n - 1, \quad t \in [0, 1],
\end{equation}
The lower limit for the Hausdorff dimension is sharp, see the last paragraph of this section. If $\mu = \mu_0$, evaluating the limits of $A_{t, \mu_0}^{(n)}, B_{t, \mu_0}^{(n)}, C_{t, \mu_0}^{(n)}$ as $t \to 0^+$ and $t \to 1^-$ from Theorem 4.3 we get
\begin{equation}
\lim_{t \to 0^+} \dim_P(\Theta_{t, \mu_0}^{(n)}) = \lim_{t \to 0^+} \dim_H(\Theta_{t, \mu_0}^{(n)}) = n - 2 + \frac{3}{n + 1},
\end{equation}
and
\begin{equation}
\lim_{t \to 1^-} \dim_H(\Theta_{t, \mu_0}^{(n)}) = n - 2, \quad \lim_{t \to 1^-} \dim_P(\Theta_{t, \mu_0}^{(n)}) = n - 2 + \frac{1}{n}.
\end{equation}
On the other hand, if $t = 0$ equivalent to $\overline{\varphi}_j = \overline{\varphi}_j = 0$ for all $1 \leq j \leq n + 1$, we have
\begin{equation}
\dim_P(\Theta_{0, 0}^{(n)}) = \dim_H(\Theta_{0, 0}^{(n)}) = n,
\end{equation}
since then $\omega^* = n, \omega = 1/n$, which holds for almost all $\xi \in \mathbb{R}^n$. More generally, we expect the maps $(t, \mu) \mapsto \dim_H(\Theta_{t, \mu}^{(n)})$ and $(t, \mu) \mapsto \dim_P(\Theta_{t, \mu}^{(n)})$ to be discontinuous on the curve $\mu = -t/n$ where $\sigma = 0$. In this case our constructions in the proof below can be refined to obtain larger dimensions than in Theorem 4.3.

We turn towards Theorem 4.2. Let
\begin{equation}
D = D_{s,\mu}^{(n)} := n - \frac{s - s^2}{2s + (n - 1)\mu} \geq 0
\end{equation}
and $D > 0$ as soon as $s > -n$, and derive $E = E_{s, \nu}^{(n)}$, $F = F_{s, \nu}^{(n)}$ via

$$E = n - (n - D)\frac{(n + 1)(s + (n - 1)\nu + 1)}{(1 - s)(n + 1 + s - \nu)}, \quad F = D\frac{n + 1}{n + 1 + 2s + (n - 1)\nu}.$$  

Then $D \in [0, n]$ and $n \geq E \geq D, n \geq F \geq D$ for all $s, \nu$ in the parameter range.

**Theorem 4.4.** The dimensions of the sets $\Sigma = \Sigma_{s, \nu}^{(n)}$ in (25) are bounded by

$$\dim_H(\Sigma_{s, \nu}^{(n)}) \geq D, \quad \dim_P(\Sigma_{s, \nu}^{(n)}) \geq \max\{E, F\}.$$  

There is equality (at least) if $s < 0$ and $\nu = \nu_0$. Moreover, $\max\{E, F\} \geq \frac{1}{2}$.

We again conjecture equalities if $s < 0$ and $\nu \in (-s/n, \nu_0]$, but can guarantee it only for $\nu = \nu_0$, see Section 6.3. Again $s < 0$ and $\nu \neq -s/n$ are vital. If $\nu = \nu_0$ then as $s \to 0$ all limits $D, E, F$ become $n - 2 + 3/(n + 1)$ so we get

$$\lim_{s \to 0^-} \dim_H(\Sigma_{s, \nu_0}^{(n)}) = \lim_{s \to 0^-} \dim_P(\Sigma_{s, \nu_0}^{(n)}) = n - 2 + \frac{3}{n + 1}.$$  

As $s$ increases in $[-n, 0]$, the bound $D_{s, \nu_0}^{(n)}$ for the Hausdorff dimensions decays whereas $\max\{E_{s, \nu_0}^{(n)}, F_{s, \nu_0}^{(n)}\}$ increases. In the other extremal case $\nu = -s/n$, the values $D_{s, -s/n}^{(n)}$ are strictly increasing with $s$ and evaluating limits we get

$$\lim_{s \to 0^-} \dim_H(\Sigma_{s, \nu}^{(n)}) \geq \lim_{s \to 0^-} D_{s, -s/n}^{(n)} = \lim_{s \to 0^-} E_{s, -s/n}^{(n)} = \lim_{s \to 0^-} F_{s, -s/n}^{(n)} = n - 1 + \frac{1}{n + 1},$$  

with right hand side in (0, 1). As $s \to -n$, we explain below that we must have

$$\lim_{s \to -n^+} \dim_H(\Sigma_{s, \nu}^{(n)}) = \lim_{s \to -n^+} D_{s, \nu}^{(n)} = 0, \quad \nu \in [-s/n, \nu_0],$$  

but we verify strictly positive dimensions as soon as $s < -n$. Furthermore

$$\lim_{s \to -n^+} \dim_P(\Sigma_{s, \nu_0}^{(n)}) = \lim_{s \to -n^+} F_{s, \nu_0}^{(n)} = \frac{1}{2},$$  

and since $D, E, F$ decrease in $s, \nu$, we infer the lower bound 1/2 in Theorem 4.4. If $\nu = -s/n$ the limits as $s \to -n$ of both $E, F$ become $n - 1 + 1/(n + 1)$, so

$$\lim_{s \to -n^+} \dim_P(\Sigma_{s, -s/n}^{(n)}) \geq n - 1 + \frac{1}{n + 1}.$$  

Moreover, as soon as $s > -n$, the bounds for the dimensions become positive, for every $\nu \in [-s/n, \nu_0]$. Again we have

$$\dim_H(\Sigma_{0, \nu_0}^{(n)}) = \dim_P(\Sigma_{0, \nu_0}^{(n)}) = n$$  

and we expect the maps $(s, \nu) \mapsto \dim_H(\Sigma_{s, \nu}^{(n)})$ and $(s, \nu) \mapsto \dim_P(\Sigma_{s, \nu}^{(n)})$ to be discontinuous on the curve $\nu = -s/n$ where $\gamma = 0$.

We justify (34). If $s \to -n$ then $\omega \to \infty$ by (11), and the sets $W_w^{(n)} := \{\xi \in \mathbb{R}^n : \omega(\xi) \geq w\}$ have Hausdorff dimension $(n + 1)/(w + 1)$ (see Jarník [1]) which tends to 0 as $w \to \infty$. On the other hand, $t \to 1$ gives $\omega^* \to \infty$ and the according sets $W_w^{(n)} := \{\xi \in \mathbb{R}^n : \omega^*(\xi) \geq w^*\}$ have Hausdorff dimension at least $n - 1$ with equality for $w^* = \infty$, see [1]. This shows the bound in (25) is optimal and explains the dimension drop.
from Theorem 4.3 to Theorem 4.4. Our results complement metrical findings in [3], [4]. For example the set of ”dually infinite singular vectors” 
\[ S^{(n)}_{\infty} := \{ \xi \in \mathbb{R}^n : \hat{\omega}^*(\xi) = \infty \} \]
satisfies \( \dim_H(S^{(n)}_{\infty}) = n - 2, \dim_P(S^{(n)}_{\infty}) = n - 1 \), see [3] Section 1.2. Since \( t \to 1 \) and \( \mu = \mu_0 \) imply \( \hat{\omega}^* \to \infty \) by a comment below Theorem 2.1, in this sense the limiting Hausdorff dimension in [20] is as large as possible, whereas the packing limit is not.

5. Proof of Theorem 2.1

We have seen that Theorem 2.1 is equivalent to Theorem 4.1. We split the proof of the latter in existence and non-existence part.

5.1. Generalized existence result. For the existence part, we show the following more general claim in the course of Sections 5.1-5.6 that also includes Theorem 4.3.

**Theorem 5.1.** Let \( n \geq 2 \) and \( t \in [0, 1] \). Derive

\[
\mu_0 = \mu_0(n, t) = \frac{(3 - 2n)t + 1 - 2n + \sqrt{(1 - t)(4n - 5)t + 4n^2 - 4n + 1}}{2(n - 1)^2}.
\]

Then \( \mu_0 \leq -t/n \) and for every \( \mu \in [\mu_0, -t/n] \) there exists a non-empty set \( \Theta^* = \Theta^{*(n)}_{t, \mu} \subseteq \mathbb{R}^n \) consisting of \( \xi = \xi_{t, \mu} \in \mathbb{R}^n \) and

\[
(36) \quad \xi_{n+1} = t, \quad \xi_{n} = (1 - n) \frac{t + n\mu}{n + 1 + 2t + (n - 1)\mu}, \quad \xi_{1} = \cdots = \xi_{n-1} = \mu,
\]

and

\[
(37) \quad \xi_1 = \cdots = \xi_{n-1} = \frac{n\mu + t}{n + 1 + t - \mu},
\]

and

\[
(38) \quad \xi_n = -\frac{1}{n} + \frac{n + 1}{n} \cdot \frac{1}{1 + n + (n - 1)t + n(n - 1)\mu}
\]

hold. In particular, (36) implies equality in (SS1) for any \( \xi \in \Theta^* \). The dimensions of \( \Theta^{*(n)}_{t, \mu} \) satisfy the lower bounds (27), with equality if \( t > 0 \) and \( \mu = \mu_0 \).

The identity \( \xi_1 = (n\mu + t)/(n + 1 + t - \mu) \) in (37) agrees with (38) when using (11), so it is in fact necessary in our setting. If \( \mu = \mu_0 \) then the values in (37), (38) coincide, thus \( \xi_1 = \cdots = \xi_{n-1} = \xi_n \). Moreover, \( \xi_{n+1} = t \) and \( \xi_1 = \mu_0 \) and equality in (SS1) directly imply all claims (36), (37), (38), in particular by (11) we may express \( \hat{\omega} \) as remarked in Section 2. Due to the additional conditions (37), (38) we see \( \Theta^{*(n)}_{t, \mu} \subseteq \Theta^{(n)}_{t, \mu} \) with \( \Theta^{(n)}_{t, \mu} \) from Theorem 4.1, so the latter and Theorem 4.3 are indeed implied. We will make use of the following calculations.

**Proposition 1.** Let \( n \geq 2 \) be an integer. For any \( t \in [0, 1] \) and \( \mu_0 = \mu_0(n, t) \) defined in (35) we have

\[
-\frac{t}{n} \geq \mu_0 \geq -\frac{t^2 + (2n + 1)t}{n^2 - t} \geq -\frac{2}{n - 1} t.
\]
Proof. We check the most challenging middle inequality first. Define
\[ F_n(x, y) = (1 + x + (n - 1)y)^2 - (x - 1)(y - 1) = 0. \]
We use that \( \mu_0 \) is solution to the quadratic equation
\[ F_n(t, \mu_0) = (1 + t + (n - 1)\mu_0)^2 - (t - 1)(\mu_0 - 1) = 0, \]
equivalent to (38). Taking \( A_n(t) = -(t^2 + (2n + 1)t)/(n^2 - t) \) we see that for the resulting identity
\[ G_n(t) = F_n(t, A_n(t)) = 0 \]
then leads to a quartic polynomial with solutions \( t = 0, t = 1 \) and a double solution \( t = -n \notin [0, 1] \). Hence by the continuity of \( \mu_0(n, t), A_n(t) \) and \( F_n(t) \) in \( t \) we see that \( F_n(t, A_n(t)) \) and \( \mu_0(n, t) - A_n(t) \) do not change sign on \([0, 1]\) (as otherwise \( \mu_0(n, t) = A_n(t) \) for some \( t \in (0, 1) \) but then \( F_n(t, \mu) = F_n(t, A_n(t)) = G_n(t) = 0 \), contradiction to \( G_n \) not having zeros in \((0, 1)) \). Thus either \( \mu_0 \geq A_n(t) \) or \( \mu_0 \leq A_n(t) \) for all \( t \in [0, 1]. \)

To check that the first case occurs, it is convenient to use a different approach. The claim middle inequality, or \( \mu_0 \geq A_n(t) \), are equivalent to
\[ H_n(t) = \frac{(3 - 2n)t + 1 - 2n + \sqrt{(1 - t)(4n - 5)t + 4n^2 - 4n + 1}}{2(n - 1)^2} + \frac{t^2 + (2n + 1)t}{n^2 - t} \geq 0. \]
Then we calculate \( H_n(0) = H_n(1) = 0 \) and
\[ H_n'(t) = \frac{2n + 1 + 2t}{n^2 - t} + \frac{(t^2 + (2n - 1)t)}{(n^2 - t)^2} + \frac{3}{2(n - 1)^2} - \frac{n}{(n - 1)^2} - \frac{2n^2 - 4n + 3 + (4n - 5)t}{2(n - 1)^2\sqrt{(1 - t)(4n - 5)t + 2(n - 1)^2}} \]
and it is easily seen that \( H_n' \) has a pole at \( t = 1 \) and the limit is \(-\infty\) as \( t \) approaches \( 1 \) from below. Thus \( H_n(t) \) and hence \( \mu(n, t) - A_n(t) \) are non-negative for \( t \in [1 - \epsilon, 1] \), and by the above findings actually for all \( t \in [0, 1]. \)

For the most left inequality, we can proceed very similarly with \( B_n(t) = -t/n \) instead of \( A_n(t) \). Then \( I_n(t) = F_n(t, B_n(t)) = 0 \) has solutions \(-n, 0 \) for \( t \) and a similar argument shows \( \mu_0(n, t) - B_n(t) \) does not change sign on \([0, 1]\), and by a similar derivative argument we can again check the expression is never positive. Finally the most right inequality can be verified straightforward using \( t \in [0, 1]. \) \( \square \)

By Theorem 5.1, it suffices to construct for given \( t \in [0, 1], \mu \in [\mu_0, -t/n] \) an \( n \)-template \( P = P_{t, \mu} \) inducing the upper and lower limit values \( \underline{\varphi}_j, \overline{\varphi}_j \) as in Theorem 5.1. We may assume \( 0 < t < 1 \) in the sequel. This follows from the compactness of the spectrum of \( (\underline{\varphi}_1, \ldots, \overline{\varphi}_{n+1}) \subseteq \mathbb{R}^{2n+2} \) settled in [21] and the continuous dependency of the bounds for \( \sigma, \mu \) from \( t \) in the theorem. The case \( t = 0 \) implies \( \underline{\varphi}_j = \overline{\varphi}_j = 0 \) for all \( j \), which yields the trivial \( n \)-template \( P_j(q) = 0 \) for \( 1 \leq j \leq n + 1, q \in [0, \infty) \) anyway. It satisfies Theorem 5.1 and puts the origin in the spectrum of \( (\underline{\varphi}_1, \ldots, \overline{\varphi}_{n+1}) \subseteq \mathbb{R}^{2n+2}. \) We should note that Roy’s spectrum differs from ours, but it is easily seen that compactness is preserved when switching between the formalisms. In the sequel, we call \( q \) a switch point of \( P \) if some \( P_j \) is not differentiable at \( q \), i.e. \( P_j \) has a local maximum or minimum.
5.2. **Preperiod of** $P_{t, \mu}$. As customary when applying the variational principle, we want to define an $n$-template with a periodic pattern. First we describe how to obtain the initial state of the repeating construction. For given $t \in (0, 1)$ and $\mu \in [\mu_0, -t/n]$ with $\mu_0$ as in (35), the goal is the following scenario: At some $q_0 > 0$ we have

$$
(40) \quad \frac{P_1(q_0)}{q_0} = \ldots = \frac{P_{n-1}(q_0)}{q_0} = \mu, \quad \frac{P_n(q_0)}{q_0} = \theta, \quad \frac{P_{n+1}(q_0)}{q_0} = t,
$$

which is the starting point of Figure 1 below. Here according to (17) we put

$$
(41) \quad \theta = -(t + (n-1)\mu).
$$

Moreover, at $q_0$ we want the function $P_{n+1}$ to decay with slope $-n$ and $P_1, \ldots, P_n$ rise with slope $+1$. It follows easily from the prescribed range for $\mu$ and Proposition 1 that

$$
(42) \quad -n \leq -\frac{2}{n-1} \leq \mu_0 \leq \mu \leq -\frac{t}{n} \leq \theta \leq t \leq 1, \quad t \in [0, 1],
$$

so indeed the values $P_j(q_0)/q_0$ belong to the required interval $[-n, 1]$, and their ordering is as in (40).

We describe how the initial data (40) can be achieved. Take any $q_0 > 0$. We start at $q = 0$ with $P_1(0) = \ldots = P_n(0) = 0$. Let $P_{n+1}$ rise with slope $+1$ until some switch point $q' \in (0, q_0]$, and then decay with slope $-n$ until $q_0$, where $q'$ is chosen so that $P_{n+1}(q_0) = t q_0$. By equating

$$
P_{n+1}(q_0) = t q_0 = q' - n(q_0 - q'),
$$

we see

$$
q' = \frac{n + t}{n + 1} q_0.
$$

To get the desired value for $P_n(q_0)$, we let $P_n$ together with $P_1, \ldots, P_{n-1}$ initially decay with slope $-1/n$ up to some switch point $q''$ where $P_n$ starts rising with slope $1$ up to $q_0$, with $q''$ chosen so that $P_n(q_0) = \theta q_0$. One determines

$$
q'' = \frac{(1-\theta)n}{n + 1} q_0 \leq q',
$$

where the inequality holds due to (42). We remark that $q'' = 0$ if $t = 1$. We take the remaining first $n - 1$ successive minima functions $P_1, \ldots, P_{n-1}$ identical in $[0, q_0]$, so that they are determined in $[0, q_0]$ by the vanishing sum property (17) and the description of $P_n, P_{n+1}$ above. This means $P_1, \ldots, P_{n-1}$ decay with slope $-1/n$ in $[0, q'']$, with slope $-2/(n-1)$ in $[q'', q']$ and finally rise with slope $+1$ in $[q', q_0]$. This concludes the preperiod.

5.3. **Period of** $P_{t, \mu_0}$: **Special case** $\mu = \mu_0$. We take $\mu = \mu_0$ throughout this section and consequently implicitly consider $\theta = \theta_0$ derived from (11) with this choice. We remark that by our choice of $\mu_0$, we verify that $t, \mu = \mu_0, \theta = \theta_0$ in (11) are linked by the quadratic identity

$$
(43) \quad \theta^2 - 2\theta - \mu t + t + \mu = (\theta - 1)^2 - (\mu - 1)(t - 1) = 0,
$$

which reflects (23).
The figure shows the first period $[q_0, q_1]$ of the iterative construction, starting from $q_0$ where (10) holds, up to some $q_1 > q_0$ to be determined below. We continue the slopes as from the preperiod at $q_0$ to the right, i.e., slope $-n$ for $P_{n+1}$ and $+1$ for $P_1, \ldots, P_n$. At some point $\tilde{q}_1 > q_0$ the functions $P_n, P_{n+1}$ will intersect. By equating

$$P_{n+1}(\tilde{q}_1) = tq_0 - n(\tilde{q}_1 - q_0) = \theta q_0 + (\tilde{q}_1 - q_0) = P_n(\tilde{q}_1),$$

and a brief computation, upon inserting (41), this point is given by

$$\tilde{q}_1 = \frac{n + 1 + 2t + (n-1)\mu}{n+1} \cdot q_0$$

and induces the quotients

$$\frac{P_n(\tilde{q}_1)}{\tilde{q}_1} = \frac{P_{n+1}(\tilde{q}_1)}{\tilde{q}_1} = (1-n)\frac{t + n\mu}{n + 1 + 2t + (n-1)\mu}.$$  

Note that the right hand side of (45) is the desired slope for $\tilde{q}_n$ and $\bar{q}_n$ in (36). At $\tilde{q}_1$ the functions $P_n, P_{n+1}$ exchange slopes, so $P_n$ starts decaying with slope $-n$ and $P_{n+1}$ rising with slope $+1$. Then at some point $\tilde{q}_2 > \tilde{q}_1$ the rising functions $P_1 = \cdots = P_{n-1}$ will intersect $P_n$. From

$$P_n(\tilde{q}_2) = t\tilde{q}_0 - n(\tilde{q}_2 - q_0) = \mu q_0 + (\tilde{q}_2 - q_0) = P_1(\tilde{q}_2) = \cdots = P_{n-1}(\tilde{q}_2)$$

we derive

$$\tilde{q}_2 = \frac{t - \mu + n + 1}{n+1} q_0. $$

To the right of this switch point $\tilde{q}_2$, let the slopes of $P_1, \ldots, P_{n-1}$ become $-2/(n-1)$ whereas the slope of $P_n$ becomes $+1$. Observe $P_{n+1}$ also still increases with slope $+1$. Since we can assume $t < 1$, then at some point $q_1 \geq \tilde{q}_1$ we will have $P_{n+1}(q_1)/q_1 = t$, concretely from imposing $P_{n+1}(q_1) = \theta q_0 + (q_1 - q_0) = q_1 t$ we calculate

$$q_1 = \frac{\theta - 1}{t - 1} q_0 = \frac{\mu - 1}{\theta - 1} q_0,$$

where the right equality uses $\mu = \mu_0, \theta = \theta_0$ and reflects (13). With a little effort it can be checked, and follows from the below calculations, that actually $q_1 \geq \tilde{q}_2$. Indeed by some rearrangements this is equivalent to

$$t^2 + (2n + 1 - \mu)t + n^2 \mu \geq 0 \iff \mu \geq -\frac{t^2 + (2n + 1)t}{n^2 - t},$$

which is confirmed in Proposition 1. Moreover, at the same point $q_1$ we further have

$$\frac{P_n(q_1)}{q_1} = \frac{\mu q_0 + (q_1 - q_0)}{q_1} = \theta, \quad \frac{P_1(q_1)}{q_1} = \cdots = \frac{P_{n-1}(q_1)}{q_1} = \mu.$$

The left identity is equivalent to (48), the right follows consequently from $P_{n+1}(q_1)/q_1 = t$, the vanishing sum property (17) and $P_1(q_1) = \cdots = P_{n-1}(q_1)$. Finally we let $q_1$ be a switch point where $P_{n+1}$ starts decaying with slope $-n$, whereas $P_1, \ldots, P_{n-1}$ start rising with slope $+1$. Note $P_n$ is differentiable at $q_1$ with slope $+1$ and clearly $q_1 > q_0$ if $t > 0$, which we can assume.

Since the slopes as well as the right-sided derivatives of all $P_j$ at $q_1$ are then identical to the data at $q_0$, at $q_1$ we have precisely the same conditions as at $q_0$. Thus, up to a
scaling factor $q_1/q_0$ in each step, we can extend the construction from $[q_0, q_1]$ periodically ad infinitum. Together with the preperiod, this will give an $n$-template on $[0, \infty)$.

Figure 1: Sketch period of $P_{t, \mu_0}$ (special case $\mu = \mu_0$)

5.4. Period of $P_{t, \mu}$: The general case. Now let $\mu$ be arbitrary in $[\mu_0, -t/n]$. We start the period construction as in the special case $\mu = \mu_0$: After the preperiod we derive at some $q_0 > 0$ where for $\theta = -(t + (n - 1)\mu$ we are given

$$\frac{P_1(q_0)}{q_0} = \cdots = \frac{P_{n-1}(q_0)}{q_0} = \mu, \quad \frac{P_n(q_0)}{q_0} = \theta, \quad \frac{P_{n+1}(q_0)}{q_0} = t.$$ 

We describe the period $[q_0, q_1]$. Starting from $q_0$, we let $P_{n+1}$ decay with slope $-n$ and the others rise with slope $+1$ until at $\tilde{q}_1$ the functions $P_n, P_{n+1}$ meet and exchange slopes. Then at some point $\tilde{q}_2 > \tilde{q}_1$ the functions $P_1 = \cdots = P_{n-1}$ meet $P_n$. The calculations for $\tilde{q}_1, \tilde{q}_2$ and $P_n(\tilde{q}_2)$ are precisely as in the special case in Section 5.3 for general $\mu$ and implied $\theta$ throughout.

Now at $\tilde{q}_2$ we make a twist to our construction above by prescribing that all $P_1, \ldots, P_n$ decay with slope $-1/n$ in $[\tilde{q}_2, \tilde{q}_3]$ for some $\tilde{q}_3 \geq \tilde{q}_2$ to be determined, while $P_{n+1}$ keeps increasing with slope $+1$. At some point $q_1 > q_0$ we will have $P_{n+1}(q_1)/q_1 = t$ again. It is obvious that $q_1 > \tilde{q}_1$ and since $\mu \geq \mu_0$ it follows from the argument in the special case and $\mu \geq \mu_0$ that even $q_1 \geq \tilde{q}_2$. The identity (45) and the left identity in (48) hold for the same reason as in the special case, in particular $q_1$ is evaluated as for $\mu = \mu_0$. 
We will choose \( \tilde{q}_3 \leq q_3 \) and in the interval \([\tilde{q}_3, q_3]\) we let \( P_n \) rise with slope +1 and \( P_1, \ldots, P_{n-1} \) decay with slope \(-2/(n-1)\). (In particular, if \( \tilde{q}_2 = \tilde{q}_3 \) then the graph is as for \( \mu = \mu_0 \)). We claim that upon a proper choice of \( \tilde{q}_3 \in [\tilde{q}_2, q_3] \), at \( q_3 \) again we will have

\[
\frac{P_1(q_1)}{q_1} = \cdots = \frac{P_{n-1}(q_1)}{q_1} = \mu, \quad \frac{P_n(q_1)}{q_1} = \theta, \quad \frac{P_{n+1}(q_1)}{q_1} = t.
\]

Assume this is shown. Then the period is finished and again we repeat the construction upon scaling by \( q_1/q_0 \) in each step.

To evaluate \( \tilde{q}_3 \), we have to satisfy the identity

\[
\theta q_1 = P_n(\tilde{q}_2) - \frac{1}{n}(\tilde{q}_3 - \tilde{q}_2) + (q_1 - \tilde{q}_3),
\]

thus inserting for \( \theta \) we get

\[
\tilde{q}_3 = \frac{n}{n+1} \left[ P_n(\tilde{q}_2) + \frac{1}{n}\tilde{q}_2 + (1 + t + (n-1)\mu)q_1 \right].
\]

Inserting from (46), (47), (48) we get

\[
\tilde{q}_3 = \frac{n}{n+1} \left[ t + n + \frac{1-n}{n}(t-\mu+n+1) + \frac{(1+t+(n-1)\mu)^2}{1-t} \right] \cdot q_0
\]

\[
= \frac{1+t+(n-1)\mu}{n+1} \cdot \frac{1+n+(n-1)(t+n\mu)}{1-t} \cdot q_0.
\]

We now finally check analytically that indeed \( \tilde{q}_3 \in [\tilde{q}_2, q_3] \) as claimed for any choice \( n \geq 2 \), \( t \in (0, 1) \) and \( \mu \in [\mu_0, -t/n] \).

Writing \( q_1/q_0 = (1-\theta)/(1-t) \) and \( \tilde{q}_3/q_0 = (1-\theta)(1+n(1-\theta)/(1-t))/(n+1) \), the claim \( \tilde{q}_3 \leq q_1 \) can be rearranged to \( \theta \geq -t/n \) which is true by (42). For the estimate \( \tilde{q}_2 \leq \tilde{q}_3 \), we check by inserting that there is equality \( \tilde{q}_2 = \tilde{q}_3 \) at the minimum value \( \mu = \mu_0 \), which agrees with the special case from the section above. Since \( \tilde{q}_2 \) decreases with slope \(-1/(n+1)\) as a function of \( \mu \) by (47), to conclude it suffices to check that for fixed \( t \) the value \( \tilde{q}_3 \) in (49) increases as a function of \( \mu \). We calculate

\[
\frac{d}{d\mu} \tilde{q}_3 = \frac{(n-1)(2n-1)t + 2n + 1 + 2n(n-1)\mu}{(n+1)(1-t)} q_0.
\]

Since \( t \in [0, 1] \), for this to be positive we require

\[
(2n-1)t + 2n + 1 + 2n(n-1)\mu \geq 0.
\]

Using \( \mu \geq \mu_0 \geq -\frac{2}{n-1}t \) by Proposition [1] and \( t \in [0, 1] \) this can be readily verified. This completes the construction of first period \([q_0, q_1]\) for general \( \mu \). The periodic continuation to an \( n \)-template on \([0, \infty)\) via adjacent scaled copies is performed as in Section 5.3.
5.5. **Evaluation of extremal values.** Since the maximum slope $P_{n+1}(q)/q$ of $P_{n+1}$ for $q \in [q_0, q_1]$ is clearly attained at the interval ends where it takes the value $t$, and similarly the minimal slopes for $P_1 = \cdots = P_{n-1}$ at the interval ends equal to $\mu$, we see that

$$\lim \inf_{q \to \infty} \frac{P_j(q)}{q} = \mu, \quad (1 \leq j \leq n-1), \quad \lim \sup_{q \to \infty} \frac{P_{n+1}(q)}{q} = t.$$ 

Moreover, obviously the expression $P_{n+1}(q)/q$ for $q \in [q_0, q_1]$ is minimal at its local minimum $\tilde{q}_1$, and similarly the slope of $P_n$ attains its maximum within $[q_0, q_1]$ at $\tilde{q}_1$. From Theorem 3.1 and (16), (19), (45) we conclude (36).

Finally, for (37), (38) we show that within $q \in [q_0, q_1]$, the values $P_j(q)/q = \cdots = P_{n-1}(q)/q$ take their maxima at $\tilde{q}_2$, and $P_n(q)/q$ its minimum at $\tilde{q}_3$. It is clear that the extrema in question are taken in the interval $I = [\tilde{q}_2, \tilde{q}_3]$, since outside the slopes take the extremal values $-n$ and 1. We show that $P_n(\tilde{q}_2)/\tilde{q}_2 = (n\mu + t)/(n + 1 + t - \mu)$ exceeds the slope $-1/n$ of $P_1, \ldots, P_n$ in $I$. Then the values $P_j(q)/q$ for $j = 1, 2, \ldots, n$ decrease within $I$ and the claim follows. To show

$$\frac{n\mu + t}{n + 1 + t - \mu} > \frac{1}{n}.$$
we rearrange to the equivalent form \( n + 1 > (n - 1)t + (n^2 + 1)\mu^2 \) which is true since as \( t \in [0, 1] \) and \( \mu \leq 0 \) by Proposition 1 we have
\[
n + 1 > n - 1 \geq (n - 1)t \geq (n - 1)t + (n^2 + 1)\mu^2.
\]
The claim (37) follows directly. For (38), from (46), (47), (49) we calculate
\[
\mathcal{L}_n = \frac{P_n(\tilde{q}_3)}{\tilde{q}_3} = \frac{P_n(\tilde{q}_2) - \tilde{q}_1^2}{\tilde{q}_3} = \frac{1}{n} + \frac{n + 1}{n} \cdot \frac{1 + t + (n - 1)\mu}{1 + t + (n - 1)\mu^2}
\]
Dividing numerator and denominator by \( \theta = 1 + t + (n - 1)\mu \) yields the claimed expression after a brief rearrangement. We again conclude with Theorem 3.1 and (16), (19). Finally, inserting for \( \mathcal{L}_1, \mathcal{L}_{n+1}, \mathcal{L}_{n+1} \) from (35), a calculation verifies equality in (SS1).

5.6. Deduction of metrical results. We bound the Hausdorff and packing dimensions of the set \( \Theta_{t,\mu}^{(n)} \) in Theorem 5.1 as in (27). First assume \( 0 < t < 1 \) again where our construction is well-defined. Since any set \( \Theta_{t,\mu}^{(n)} \) is contained in \( \Theta_{t,\mu} \) from Theorem 4.1 clearly (27) follows. We determine the contraction rates for the \( n \)-template \( P = P_{t,\mu} \) constructed above. We evaluate the local contraction rates within the period interval \([q_0, q_1]\) as

\[
\delta(P, q) = \begin{cases} n, & \text{if } q \in [q_0, \tilde{q}_1], \\
n - 1, & \text{if } q \in [\tilde{q}_1, \tilde{q}_3], \\
n - 2, & \text{if } q \in [\tilde{q}_3, q_1]. \end{cases}
\]

Denoting for \( j \geq 1 \) the \( j \)-th period interval \( I_j = [q_{j-1}, q_j] \), this is true accordingly in \( I_j \). From the variational principle we directly conclude that the Hausdorff and packing dimensions cannot be less than \( n - 2 \). For the precise calculation, we observe that the local rate decays within each interval \( I_j \). We readily conclude that the lower limit is attained when considering intervals \([q_0, q_N]\), and in fact by periodicity the resulting average contraction rate in these intervals is independent of \( N \geq 1 \). So from the variational principle Theorem 3.2 we get
\[
\dim_H(\Theta_{t,\mu}^{(n)}) \geq \delta(P) = \frac{\int_{q_0}^{q_1} \delta(P, q) \, dq}{q_1 - q_0} = \frac{n(\tilde{q}_1 - q_0) + (n - 1)(\tilde{q}_3 - \tilde{q}_1) + (n - 2)(q_1 - \tilde{q}_3)}{q_1 - q_0}
\]
\[
= \frac{(n - 2)\tilde{q}_1 + \tilde{q}_1 + \tilde{q}_3 - nq_0}{q_1 - q_0} = n - 2 + \frac{\tilde{q}_1 + \tilde{q}_3 - 2q_0}{q_1 - q_0}.
\]

Inserting for \( \tilde{q}_1, \tilde{q}_3, q_1 \) from (11), (18), (19) we verify \( \dim_H(\Theta_{t,\mu}^{(n)}) \geq n - 2 + A \) as in (27). For the upper limit, we consider intervals \([q_0, \tilde{q}_1, N]\) and \([q_0, \tilde{q}_3, N]\) for large \( N \), where \( \tilde{q}_i, N \) denotes for \( i = 1, 2, 3 \) the value corresponding to \( \tilde{q}_i \in [q_0, q_1] \) in the interval \([q_{N-1}, q_N]\). In particular \( q_{N} \leq \tilde{q}_{1, N} \leq \tilde{q}_{3, N} \leq q_{N+1} \) for all \( N \) and the contraction rates in the subintervals \([q_{N}, q_{1, N}], [\tilde{q}_{1, N}, \tilde{q}_{3, N}] \) and \([\tilde{q}_{3, N}, q_{N+1}] \) take the values as for \( N = 0 \) in (50). Hence
\[
\dim_P(\Theta_{t,\mu}) \geq \delta(P) \geq \max\{S, T\}
\]
where \( S \) and \( T \) are respectively the average limit contraction rates in the intervals \([q_0, \tilde{q}_{N,1}]\) and \([q_0, \tilde{q}_{N,3}]\) respectively as \( N \to \infty \). In fact it is not hard to check equality \( \delta(P) = \max\{S, T\} \). To conclude, we show \( S \geq B, T \geq C \) with \( B, C \) as in (26). Since we identified
\(\hat{\delta}(P)\) as the average contraction rate in any interval \([q_0, q_N]\) and \(q_0 = o(q_N)\) as \(N \to \infty\), we evaluate

\[
S = \lim_{N \to \infty} \frac{\int_{q_0}^{\hat{q}_{1,N}} \delta(P, q) \, dq}{\hat{q}_{1,N} - q_0} = \lim_{N \to \infty} \frac{\hat{\delta}(P)(q_N - q_0) + n(\hat{q}_{1,N} - q_N)}{\hat{q}_{1,N} - q_0}
\]

\[
= \lim_{N \to \infty} \frac{\delta(P)q_N + n(\hat{q}_{1,N} - q_N)}{\hat{q}_{1,N} - q_0} = n - (n - \hat{\delta}(P)) \cdot \lim_{N \to \infty} \frac{q_N}{\hat{q}_{1,N}}.
\]

Now since \(q_N/\hat{q}_{1,N}\) is independent of \(N\), inserting \(\hat{\delta}(P) \geq n - 2 + A\) we infer

\[
S \geq n - \frac{q_0}{\hat{q}_1}(n - \hat{\delta}(P)) \geq n - (2 - A) \frac{q_0}{\hat{q}_1} = n - \frac{(2 - A)(n + 1)}{n + 1 + 2t + (n - 1)\mu} = B.
\]

For \(T\) a similar calculation shows

\[
T = \lim_{N \to \infty} \frac{\int_{q_0}^{\hat{q}_{3,N}} \delta(P, q) \, dq}{\hat{q}_{3,N} - q_0}
\]

\[
= \lim_{N \to \infty} \frac{\int_{q_0}^{q_{N+1}} \delta(P, q) \, dq - \int_{\hat{q}_{3,N}}^{q_{N+1}} \delta(P, q) \, dq}{\hat{q}_{3,N} - q_0}
\]

\[
= \lim_{N \to \infty} \frac{(q_{N+1} - q_0)\hat{\delta}(P) - (n - 2)(q_{N+1} - \hat{q}_{3,N})}{\hat{q}_{3,N} - q_0}
\]

\[
= \lim_{N \to \infty} \frac{q_{N+1}\hat{\delta}(P) - (n - 2)(q_{N+1} - \hat{q}_{3,N})}{\hat{q}_{3,N}}
\]

\[
= n - 2 + \left(\hat{\delta}(P) - n + 2\right) \lim_{N \to \infty} \frac{q_{N+1}}{\hat{q}_{3,N}}
\]

\[
= n - 2 + \left(\hat{\delta}(P) - n + 2\right)\frac{q_1}{\hat{q}_3} = n - 2 + A\frac{q_1}{\hat{q}_3}
\]

\[
= n - 2 + A \frac{(n + 1)(1 + t + (n - 1)\mu)}{(1 - t)(1 + t + (n - 1)\mu) + n(1 + t + (n - 1)\mu)^2}
\]

\[
= n - 2 + A \frac{n + 1}{n + 1 + (n - 1)t + n(n - 1)\mu} = C,
\]

as claimed, where we inserted for \(q_1, \hat{q}_3\) from \([18], [19]\) in the last line. Finally, for \(t = 0\) the claim \([27]\) is trivial by \([30]\), and we can extend the formula to \(t = 1\) by considering a limiting \(n\)-template, compare with \([23\text{ }\text{Section } 2]\), we omit details. The proof of Theorem \([5.1]\) is complete.

### 5.7. Extending the range of \(\hat{\omega}^*\)

We sketch how to alter the graphs in Figure 2 to get a prescribed value for \(\hat{\omega}^*\) as in the interval of \([12]\). We keep \(q_0, \hat{q}_1\) and the graph from \(P_{t,\mu}\) in \([q_0, \hat{q}_1]\) unchanged. We alter the formulas for \(\hat{q}_2, \hat{q}_3, q_1\), still satisfying \(\hat{q}_1 \leq \hat{q}_2 \leq \hat{q}_3 \leq q_1\), and introduce a new point \(\hat{r}\) between \(\hat{q}_1\) and \(\hat{q}_2\). We let \(P_n, P_{n+1}\) decay with slope \(-(n - 1)/2\) in \([\hat{q}_1, \hat{r}]\) and then starting at \(\hat{r}\) we let \(P_{n+1}\) rise with slope +1 and \(P_n\) decay with slope \(-n\) until it meets \(P_1 = \cdots = P_{n-1}\). The construction in \([\hat{q}_2, q_1]\) remains
basically as in $P_{t,\mu}$ in Figure 2. For given

$$\eta \in \left[0, (1 - n) \frac{t + n\mu}{n + 1 + 2t + (n - 1)\mu}\right],$$

appropriate choices of $\tilde{r}, \tilde{q}_2, \tilde{q}_3, q_1$ induce an $n$-template $P_{t,\mu,\eta}$ satisfying (36) apart from $\varphi_{n+1}$ altered to $\varphi_{n+1} = \eta$. Thus by (12) we obtain any $\hat{\omega}^*$ as in (12) (remark: (37), (38) are not preserved). We omit the calculations and only want to illustrate qualitatively the graph in Figure 3 below. We omit metrical claims derived from Theorem 3.2 as well.

We prove the theorem. For $\mu > -t/n$ we cannot even have $\varphi_{n+1} = t, \varphi_1 = \mu$ due to the reverse inequality $\varphi_1 \leq -\varphi_{n+1}/n$ in (14). It remains to contradict $\mu < \mu_0$ upon the assumptions $\varphi_{n+1} = t, \varphi_1 = \mu$ and equality in (SS1) of the theorem. Keep in mind for the
sequent the equivalence in the claims of Theorem \[5.2\] i.e. upon \(\varphi_{n+1} = t, \varphi_1 = \mu\), equality in \(SS1\) is equivalent to \(\varphi_{n+1}\) taking the value \(\sigma\) in \(51\).

**Step 1**: We show that equality in \(SS1\) implies that essentially the situation as in the interval \([q_0, q_1] \subseteq [q_0, q_1]\) in Figure 2 (or Figure 1) occurs for arbitrarily large \(q_0\). For this we basically rephrase an argument within the proof of \[22\] Theorem 3.2]: Since \(\varphi_{n+1} = t\), for any \(\varepsilon > 0\) there are arbitrarily large \(q_0\) with \(|L_{n+1}(q_0) - t q_0| \leq \varepsilon q_0\). Choose large \(q_0\) with this property. For simplicity of notation, we omit \(\varepsilon\) and use \(o\) notation in the sequel, so we write \(L_{n+1}(q_0) = t q_0 + o(q_0)\) and mean that in fact we consider a sequence of \(q_0\) values with this property that tends to infinity. We may assume that at \(q_0\) there is a local maximum of \(L_{n+1}\). Consider the next point \(q_0 + \tilde{q}\) where \(L_n, L_{n+1}\) meet to the right of \(q_0\), i.e. \(\tilde{q} > 0\) minimal so that \(L_n(q_0 + \tilde{q}) = L_{n+1}(q_0 + \tilde{q})\). By definition of \(\varphi_{n+1}\) clearly

\[
L_{n+1}(q_0 + \tilde{q}) \geq (\varphi_{n+1} - \varepsilon)(q_0 + \tilde{q}) = (\sigma - o(1))(q_0 + \tilde{q}).
\]

Since \(L_n\) has slope at most 1, we infer

\[
L_n(q_0) \geq L_n(q_0 + \tilde{q}) - \tilde{q} = L_{n+1}(q_0 + \tilde{q}) - \tilde{q} \geq (\sigma - 1)\tilde{q} + \sigma q_0 - o(q_0 + \tilde{q}).
\]

Together with the bounded sum property \((15)\), we infer

\[
L_1(q_0) \leq - \frac{L_n(q_0) + L_{n+1}(q_0)}{n - 1} + O(1) \leq - \frac{(t + \sigma)q_0 + (\sigma - 1)\tilde{q} + o(q_0 + \tilde{q})}{n - 1}.
\]

We estimate \(\tilde{q}\). Since \(L_{n+1}\) decays with slope \(-n\) in \([q_0, q_0 + \tilde{q}]\) and \(L_{n+1}(q_0 + \tilde{q})/(q_0 + \tilde{q})\) is at least \(\sigma + o(1)\) by definition of \(\varphi_{n+1} = \sigma\), we have

\[
L_{n+1}(q_0 + \tilde{q}) = L_{n+1}(q_0) - n\tilde{q} \geq (\sigma - o(1))(q_0 + \tilde{q}).
\]

Inserting \(L_{n+1}(q_0) = t q_0 + o(1)q_0\) we get

\[
\tilde{q} \leq \left( \frac{t - \sigma}{n + \sigma} + o(1) \right) q_0.
\]

Since \(\sigma \leq 1\), by \((53)\) when dividing by \(q_0\) we get

\[
\frac{L_1(q_0)}{q_0} \leq - \frac{t + \sigma + (\sigma - 1)\frac{t - \sigma}{n + \sigma}}{n - 1} + o(1).
\]

As we can assume \(\varphi_1 \leq L_1(q_0)/q_0 + o(1)\), after some rearrangement when taking limits we may drop the \(o(1)\) terms, and find the corresponding inequality

\[
\varphi_1 \leq - \frac{\varphi_{n+1} + \varphi_{n+1} + (\varphi_{n+1} - 1)\frac{t - \sigma}{n + \sigma}}{n - 1}
\]

to be equivalent to \(SS1\). This means that in case of equality in \(SS1\), there must be (asymptotic) equality in all inequalities above. So as \(q_0\) as above tends to infinity, by \[(52), (54)\] we must have

\[
L_{n+1}(q_0) = t q_0 + o(q_0), \quad L_n(q_0) = (\sigma - 1)\frac{t - \sigma}{n + \sigma} + o(q_0)
\]
and further from equality in (53) we infer

\[ L_j(q_0) = \frac{t + \sigma + (\sigma - 1) \frac{\ell - q}{\varepsilon} - \sigma}{n - 1} q_0 + o(q_0), \quad 1 \leq j \leq n - 1. \]

With some calculation, we check that when dropping the remainder terms, the expression for \( L_n(q_0)/q_0 \) agrees with the value \( \theta \) from (41), and \( L_1(q_0)/q_0 \) with \( \mu \). Upon identifying \( \tilde{q} + q = \tilde{q}_1 \), this indeed verifies that essentially the combined graph in the interval \([\tilde{q}_0, \tilde{q}_1]\) must look like in Figure 2 from the construction.

Step 2: we show that if \( \mu < \mu_0 \) we cannot extend the graph of Figure 2 from \([q_0, \tilde{q}_1]\) to the right of \( \tilde{q}_1 \) without violating the requirements of a combined graph, thereby we get a contradiction. Let \( r > \tilde{q}_1 \) be the first coordinate of the next meeting point of \( L_n, L_{n+1} \) to the right of \( \tilde{q}_1 \), i.e., the smallest solution for \( L_n(r) = L_{n+1}(r) \) with \( r > \tilde{q}_1 \). Write \( I = [\tilde{q}_1, r]. \) Now we distinguish two cases.

Case 1: The functions \( L_{n-1} \) and \( L_n \) do not meet in \( I \), i.e. \( L_n(q) > L_{n-1}(q) \) for \( q \in I \). Then it is clear from the theory of combined graphs/\( n \)-templates that, up to \( o(q) \), the graph in \( I \) must look as follows: there is some switch point \( u \in I \) so that in \([\tilde{q}_1, u]\) the function \( L_{n+1} \) must rise with slope 1 and decay in \([u, r]\) with slope \(-n\), whereas for some \( v = u + o(u) \) very close to \( u \) the opposite happens for \( L_n \), i.e. \( L_n \) decays with slope \(-n\) in \([\tilde{q}_1, v]\) and increases with slope \(+1\) in \([v, r]\). (The functions \( L_1, \ldots, L_{n-1} \) all rise with average slope \(+1 - o(1)\) in the entire interval \( I \).) We justify this claim, but for brevity omit full rigour: First note that since \( L_{n-1} \) and \( L_n \) do not meet in the interior of \( I \), any "serious" local minimum of \( L_n(q) \) at some \( q \in I \) induces a local maximum of \( L_{n+1}(\ell) \) at some \( \ell = q + O(1) \). This can be seen by passing to a close \( n \)-template as in (18) and the convexity condition in Definition 1 and (15). Now since \( L_n \) and \( L_{n+1} \) move apart in a neighborhood to the right of \( \tilde{q}_1 \), the function \( L_n \) must change slope to \(+1\) somewhere in \( I \), and by the above argument in proximity \( L_{n+1} \) must change slope to \(-n\). So clearly there is at least one switch point in the interior of \( I \) where \( L_n, L_{n+1} \) exchange slopes as above. Assume there was another "serious" switch point \( \tilde{\ell} \) in the interior of \( I \) where \( L_n \) changes slope. Then at \( \tilde{\ell}, L_n \) starts to decay with slope \(-n\) and \( L_{n+1} \) must start to rise with slope \(+1\) by (15) and we assume these slopes continue to the right on a subinterval of \( I \) of substantial length. Since \( \tilde{\ell} \) is in the interior of \( I \), then in an associated close \( n \)-template satisfying (18), the function \( P_{n+1} \) would have a local minimum which is not a local maximum of \( P_n \). This contradicts the convexity condition of templates again. (Instead of passing to templates in the last step, we can alternatively argue with the first two successive minima of the dual lattice point problem.) This confirms our claim.

In particular, the average slope of \( L_n \) and \( L_{n+1} \) in \( I \) is \(- (n - 1)/2 + o(1) \) < 0. Since clearly \( L_{n+1}(q) \geq 0 \) everywhere, hence at \( r \) we get \( L_{n+1}(r)/r < L_n(\tilde{q}_1) / \tilde{q}_1 = \varphi_{n+1} + o(1) \), contradiction to \( \varphi_{n+1} \leq L_{n+1}(r)/r - o(1) \) unless \( r - \tilde{q}_1 \) is very small. However, it is clear that we may assume this is not the case. For example, we may pass to \( n \)-templates again, or start with \( \varepsilon > 0 \) and restrict to points \( r \) with \( r > (1 + \varepsilon)\tilde{q}_1 \) and then use the above argument. We omit the technical details.

Case 2: The functions \( L_{n-1} \) and \( L_n \) meet in \( I \). Starting at its local minimum \( \tilde{q}_1 \) where it meets \( L_n \), the function \( L_{n+1} \) rises with slope \(+1\). However, since \( \varphi_{n+1} = t \), this happens
at most until a point \( y \) on the first axis where \( L_{n+1}(y)/y = t + o(1) \). Then \( L_{n+1} \) decays with slope \(-n\) until it meets \( L_n \) at \( r > y > \tilde{q}_1 \). Since \( L_{n-1} \) and \( L_n \) meet in \( I \) and no slope can exceed \(+1\), it is clear that

\[
L_n(r) \leq L_{n-1}(q_0) + (r - q_0).
\]

Recall that in the construction for \( \mu = \mu_0 \), the value \( y \) was as large as possible (up to \( o(1) \)) since \( L_{n+1} \) indeed went up until \( y \) with \( L_1(y)/y = t \), and there was equality in (55), since at the meeting point \( \tilde{q}_2 \) of \( L_{n-1} \) and \( L_n \) the slope of \( L_n \) changed from \(-n\) to \(+1\) and remained \(+1\) until it met \( L_{n+1} \). Further identifying our \( y \) with \( q_1 \) from that proof, for \( \mu = \mu_0 \) we had the minimum possible value \( L_{n+1}(r)/r = \sigma \) at \( r \). So it is geometrically obvious that if we start with \( \mu < \mu_0 \) (which also implies larger values of \( \sigma \) and \( \theta \)), even in the most disadvantageous case of maximal \( y \) and equality in (55), at the smallest point \( z > y \) where \( L_{n+1}(z)/z = \sigma \), we will have \( L_{n+1}(z) \geq (1 + \varepsilon)L_n(z) \) with some \( \varepsilon > 0 \) depending on \( \mu, \mu_0 \). We omit the explicit calculation. This means that \( r > (1 + \varepsilon_0)z \) and \( L_{n+1} \) continues to decay with slope \(-n\) in \([z, r]\) until it meets \( L_n \) at \( r \), for some \( \varepsilon_0 > 0 \). Thus obviously \( \varphi_{n+1}(r) = L_{n+1}(r)/r \leq (1 - \varepsilon_1)L_{n+1}(z)/z = (1 - \varepsilon_1)\sigma \) for some \( \varepsilon_1 > 0 \). However, this contradicts the definition of \( \sigma = \varphi_{n+1} \). This completes the proof of Theorem 5.2.

We finally observe that the equality in the dimension formulas (27) for \( t > 0 \) and \( \mu = \mu_0 \) follows from the proof above. Our argument shows that then the combined graph must indeed be composed from consecutive periods as in Figure 1, up to \( o(q) \) as \( q \to \infty \).

Take the family \( \mathcal{F} \) of \( n \)-templates with these properties, which is closed under finite perturbations in view of the error term. Then it is not hard to see that the suprema of \( \delta(Q), \delta(Q) \) over \( Q \in \mathcal{F} \) are attained for \( Q = P_{t,\mu} \) as constructed (since \( o(q) \) has a negligible effect in the limit and by changing slopes of some \( P_i \) locally in intervals where consecutive functions \( P_i \) are glued, we may only decrease the local contraction rate. We skip details). Application of the variational principle yields the claim. Note that we lose the case \( t = 0 \), where (27) indeed fails as pointed out in Section 4.2 since then \( q_0 = q_1 \) in our construction by (18), so the period collapses to a singleton and we get no \( n \)-template.

6. Proof of Theorem 2.2

By equivalence of Theorems 2.2, 1.2 we again may just prove Theorem 1.2 and we show the following more general existence claim that includes Theorem 1.4.

**Theorem 6.1.** Let \( n \geq 2 \) and \( s \in [-n, 0] \). Derive \( \nu_0 = g_n(s) \) with \( g_n \) as in (20). Then \( \nu_0 \geq -s/n \) and for every \( \nu \in [-s/n, \nu_0] \) there exists a non-empty set \( \Sigma = \Sigma_{s, \nu}^{(n)} \) consisting of \( \xi = \xi_{s, \nu} \in \mathbb{R}^n \) whose associated quantities \( \varphi_j, \overline{\varphi}_j \) satisfy

\[
\varphi_1 = s, \quad \overline{\varphi}_1 = \varphi_2 = (1 - n) \frac{s + n\nu}{n + 1 + 2s + (n - 1)\nu}, \quad \overline{\varphi}_3 = \cdots = \overline{\varphi}_{n+1} = \nu,
\]

and

\[
\varphi_3 = \cdots = \varphi_{n+1} = \frac{n\nu + s}{n + 1 + s - \nu},
\]
Moreover, if inequalities $n < s < \nu$ construct $\varphi$ so again it is necessary in our framework. We again have $\Sigma^{(n)}_{s,\nu}$ induces equality in (SS2). The dimensions of $\Sigma^{(n)}_{s,\nu}$ are bounded as in (33), with equality if $s < 0$ and $\nu = \nu_0$.

The identity $\varphi_{n+1} = (n\nu + s)/(n + 1 + s - \nu)$ in (57) agrees with (11) when using (12), so again it is necessary in our framework. We again have $\Sigma^{(n)}_{s,\nu} \subseteq \Sigma^{(n)}_{s,\nu}$ from Theorem 4.2 Moreover, if $\nu = \nu_0$, then $\varphi_2 = \varphi_3 = \cdots = \varphi_{n+1}$ and additional equality in (SS2) induces all values $\varphi_i$ as in the theorem. Again we construct suitable $n$-templates $P_{s,\nu}$ in order to apply Theorem 3.1. The construction is dual in some sense. We omit certain computations that are similar to the proof of Theorem 5.1. We start with the dual version of Proposition 1.

Proposition 2. For any $s, \nu_0$ as in Theorem 6.1 we have

$$- \frac{2}{n-1} s \geq \nu_0 \geq - \frac{s^2 + (2n + 1)s}{n^2 - s} \geq \frac{s}{n}.$$

We skip the proof as it works very similar as in Proposition 1. We explain how we construct $n$-templates $P_{s,\nu}$ with the desired properties. We may again assume strict inequalities $-n < s < 0$ by compactness of the spectrum.

6.1. Preperiod of $P_{s,\nu}$. Similarly to Theorem 5.1 here for some $g_0 > 0$ we want

$$P_3(q_0)/q_0 = \cdots = P_{n+1}(q_0)/q_0 = \nu, \quad P_2(q_0)/q_0 = \vartheta, \quad P_1(q_0)/q_0 = s,$$

where again $\vartheta$ is determined from $s, \nu$ in view of (17) via

$$\vartheta = -(s + (n - 1)\nu).$$

Moreover, at $q_0$ the functions $P_1, P_2$ rise with slope +1 while $P_3, \ldots, P_{n+1}$ decay with slope $-2/(n-1)$. By Proposition 2 we again check $-n \leq s \leq \vartheta \leq -s/n \leq \nu \leq \nu_0 \leq 1$ for any $s \in [-n, 0]$. So the slopes are well-defined and the ordering (58) is correct. Moreover, $s = -n$ is equivalent to $\vartheta = \nu = 1$.

By choice of $\vartheta$, for $\nu = \nu_0 = g_n(s)$ we again have

$$\vartheta^2 - 2\vartheta - \nu s + \nu + s = (\vartheta - 1)^2 - (\nu - 1)(s - 1) = 0.$$

To obtain (58), starting at $q = 0$ we let $P_1$ decay with slope $-n$ up to a switch point $q' \in (0, q_0]$ where it starts increasing with slope +1 until $q_0$. Hereby $q'$ is determined via the property $P_1(q_0) = sq_0$, giving $q' = ((1 - s)/(n + 1))q_0$. In $[0, q']$ we let all $P_2, \ldots, P_{n+1}$ rise with slope +1. At the switch point $q'$ we start letting $P_2$ decay with slope $-n$ up to some point $q'' \geq q'$ while the other functions all rise with slope +1 in $[q', q'']$. Then starting from $q''$ we let $P_2$ rise with slope +1 so that $P_3, \ldots, P_{n+1}$ have slopes $-2/(n-1)$ in $[q'', q_0]$. A suitable choice of $q''$ will lead to $P_2(q_0)/q_0 = \vartheta$, and by $P_1(q_0)/q_0 = s$, the
vanishing sum property \(17\) and \(P_3(q_0) = \cdots = P_{n+1}(q_0),\) actually all conditions in \(58\) are satisfied. Concretely \(q'' = ((2 - s - \vartheta)/(n + 1))q_0\) is derived from
\[
q' - n(q'' - q') + (q_0 - q'') = \vartheta q_0
\]
and inserting for \(q',\) and indeed \(q' \leq q''\) since this is equivalent to \(\vartheta \leq 1\) which is trivial.

6.2. Period of \(P_{s,\nu}\) and conclusion. It is convenient to give a reverse construction of the period, i.e. start from \(q_1 > q_0\) where the properties
\[
(60) \quad \frac{P_3(q_1)}{q_1} = \cdots = \frac{P_{n+1}(q_1)}{q_1} = \nu, \quad \frac{P_2(q_1)}{q_1} = \vartheta, \quad \frac{P_1(q_1)}{q_1} = s,
\]
are satisfied and calculate back to derive the same conditions \(58\) at \(q_0,\) using our choice of \(\vartheta.\) It is clear that ultimately we can change the direction back to positive and repeat the period \([q_0, q_1],\) blown up by the constant factor \(q_1/q_0\) in each step, ad infinitum again.

Again first consider the special case \(\nu = \nu_0 = g_n(s).\) Let \(q_0, q_1\) be related via
\[
(61) \quad q_1 = \frac{s - 1}{\vartheta - 1}q_0 = \frac{\vartheta - 1}{\nu - 1}q_0.
\]
The case \(\nu = \vartheta = 1\) is equivalent to \(s = -n\) which we excluded. We determine \(\tilde{q}_1 < q_1\) from intersecting the continuation of \(P_1\) to the left decreasing with slope \(-n\) with the likewise continuation of \(P_2\) increasing with slope +1. From equating \(P_1\) and \(P_2\) at \(\tilde{q}_1\) we get
\[
P_1(\tilde{q}_1) = sq_1 + n(q_1 - \tilde{q}_1) = \vartheta q_1 - (q_1 - \tilde{q}_1) = P_2(\tilde{q}_1).
\]
After some calculation and using \(58, \ 59\) we derive
\[
(62) \quad \tilde{q}_1 = \frac{n + 1 + 2s + (n - 1)\nu}{n + 1} q_1, \quad \frac{P_1(\tilde{q}_1)}{\tilde{q}_1} = (1 - n)\frac{s + n\nu}{n + 1 + 2s + (n - 1)\nu},
\]
and we recognize the right hand side as the value from \(56.\) When moving to the left from \(\tilde{q}_1,\) we let \(P_1, P_2\) exchange slopes at \(\tilde{q}_1\) up to a point \(\tilde{q}_2 < \tilde{q}_1\) where \(P_2\) intersects \(P_3 = \cdots = P_{n+1}\) that rise with slope +1 in \([\tilde{q}_2, q_1].\) From
\[
P_3(\tilde{q}_2) = \cdots = P_{n+1}(\tilde{q}_2) = \nu q_1 - (q_1 - \tilde{q}_2) = sq_1 + n(q_1 - \tilde{q}_2) = P_2(\tilde{q}_2)
\]
we calculate
\[
(63) \quad \tilde{q}_2 = \frac{n + 1 + s - \nu}{n + 1} q_1, \quad \frac{P_2(\tilde{q}_2)}{\tilde{q}_2} = \frac{n\nu + s}{n + 1 + s - \nu}.
\]
We further check by Proposition \(2\) and it follows from the construction below, that \(\tilde{q}_2 > q_0.\) At the switch point \(\tilde{q}_2,\) when going to the left we change the slope of \(P_2\) to +1 and the slopes of \(P_3, \ldots, P_{n+1}\) according to \(17\) to \(-2/(n - 1),\) recalling \(P_1\) still has slope +1. Since \(P_1\) rises with slope +1 left of \(\tilde{q}_1,\) at some point \(q^* < \tilde{q}_1\) we will have \(P_1(q^*) = sq^*\). We will check that \(q^* = q_0\) is the value as in \(61.\) Moreover, from \(61\) we verify
\[
P_2(q_0) = \nu q_1 - (q_1 - q_0) = \vartheta q_0.
\]
Since clearly $P_3(q_0) = \cdots = P_{n+1}(q_1)$ from (17) we conclude the remaining claims of (58), proving our assertion.

Finally, for general $\nu \in [-s/n, \nu_0]$, we again split the interval $[q_0, \tilde{q}_2]$ into $[q_0, \tilde{q}_3]$ and $[\tilde{q}_3, \tilde{q}_2]$ for some $q_0 \leq \tilde{q}_3 \leq \tilde{q}_2$ and let $P_2, \ldots, P_{n+1}$ all decay with slope $-1/n$ in $[\tilde{q}_3, \tilde{q}_2]$, and in $[q_0, \tilde{q}_3]$ we take the slopes $-2/(n-1)$ for $P_3, \ldots, P_{n+1}$ and $+1$ for $P_2$, i.e. as in the interval $[q_0, \tilde{q}_2]$ when $\nu = \nu_0$. The value $\tilde{q}_3$ is again determined so that the imposed assumptions (58) at $q_0$ are met. Similar to Theorem 5.1 we get

$$\tilde{q}_3 = \frac{1 + s + (n-1)\nu}{n+1} \cdot \frac{1 + n + (n-1)(s+\nu)}{1-s} \cdot q_1.$$ 

We omit details of the calculation. This finishes the period and gives rise to an $n$-template.

We again easily verify the claimed upper and lower limits $\varphi_j, \overline{\varphi}_j$ of the theorem. Inserting $\overline{\varphi}_1 = s, \overline{\varphi}_{n+1} = \nu$ and for $\overline{\varphi}_1$ from (56), a calculation verifies equality in (SS2).

Extending the interval for $\hat{\omega}$ as in (13) works similarly as in Section 5.7 by splitting the interval $[\tilde{q}_2, \tilde{q}_1]$ suitably to attain given $\overline{\varphi}_1$ within a corresponding range, we skip details.

To estimate the Hausdorff and packing dimensions in Theorem 6.1 we evaluate the local contraction rates of $P = P_{s,\nu}$ within the period interval $[q_0, q_1]$ as

$$\delta(P, q) = \begin{cases} n, & \text{if } q \in [q_0, \tilde{q}_2], \\ 1, & \text{if } q \in [\tilde{q}_2, \tilde{q}_1], \\ 0, & \text{if } q \in [\tilde{q}_1, q_1]. \end{cases}$$
By a similar argument as in Theorem 5.1 we see that to find the lower limit we may consider the average contraction rate within the interval \([q_0, q_1]\) and find
\[
\dim_H(\Sigma_{s,n}) \geq \delta(P) = \frac{\int_{q_1}^{q_1} \delta(P, q) \, dq}{q_1 - q_0} = \frac{n(q_2 - q_0) + (\tilde{q}_1 - \tilde{q}_2)}{q_1 - q_0}.
\]
Inserting for the ratios \(\tilde{q}_1/q_1, \tilde{q}_2/q_1, q_0/q_1\) from (61), (62), (63) the bound becomes \(D\) in (31). Non-existence in Theorem 4.2.

We finally estimate the packing dimension. Define \(\tilde{q}_{1,N}, \tilde{q}_{2,N}\) within \([q_N, q_{N+1}]\) corresponding to \(\tilde{q}_1, \tilde{q}_2\) in \([q_0, q_1]\) likewise as in the proof of Theorem 5.1. Then
\[
\dim_P(\Sigma_{s,n}) \geq \delta(P) \geq \max\{U, V\}
\]
where \(U\) resp. \(V\) are the average limit contraction rates in the intervals \([q_0, \tilde{q}_{2,N}]\) resp. \([q_0, \tilde{q}_{1,N}]\) as \(N \to \infty\). We show \(U \geq E, V \geq F\) with \(E, F\) from (32). Since we identified \(\delta(P)\) as the average contraction rate in any interval \([q_0, q_N]\) and \(q_0 = o(q_N)\) as \(N \to \infty\), we evaluate
\[
U = \lim_{N \to \infty} \frac{\int_{q_0}^{q_N} \delta(P, q) \, dq}{\tilde{q}_{2,N} - q_0} = \lim_{N \to \infty} \frac{\delta(P)(q_N - q_0) + n(q_{2,N} - q_N)}{\tilde{q}_{2,N} - q_0} = \lim_{N \to \infty} \frac{\delta(P)q_N + n(\tilde{q}_{2,N} - q_N)}{\tilde{q}_{2,N}} = n + (\delta(P) - n) \cdot \frac{q_N}{\tilde{q}_{2,N}}
\]
and since \(q_N/\tilde{q}_{2,N}\) is independent of \(N\), inserting \(\delta(P) \geq D\) this equals
\[
U \geq n + \frac{q_0}{\tilde{q}_2} (\delta(P) - n) \geq n + (D - n) \frac{q_0}{\tilde{q}_2} = n + \left(\frac{n + 1}{1 - s}\right) \frac{1}{s + n + 1 + s - \nu} = E,
\]
where we used (61), (62), (63) to evaluate \(q_0/\tilde{q}_2\). For \(V\) a similar calculation shows
\[
V = \lim_{N \to \infty} \frac{\int_{q_0}^{q_N} \delta(P, q) \, dq}{\tilde{q}_{1,N} - q_0} = \lim_{N \to \infty} \frac{\int_{q_0}^{q_{N+1}} \delta(P, q) \, dq - \int_{q_1}^{q_{N+1}} \delta(P, q) \, dq}{\tilde{q}_{1,N} - q_0} = \lim_{N \to \infty} \frac{\delta(P)q_{N+1}}{\tilde{q}_{1,N}} = \frac{\delta(P)}{\tilde{q}_1} \frac{q_1}{\tilde{q}_1} \geq D \frac{q_1}{\tilde{q}_1} = D \frac{n + 1}{n + 1 + 2s + (n - 1)\nu} = F,
\]
as claimed, where we used (62) in the last line. Theorem 6.1 is proved.

6.3. Non-existence in Theorem 4.2. To complete the proof of Theorem 4.2 the following remains to be proved.

Theorem 6.2. With the notation of Theorem 4.2, for \(\nu \notin [-s/n, \nu_0]\) the set \(\Sigma_{s,n}^{(n)}\) is empty, i.e. there is no \(\xi\) inducing equality in (SS2) and with \(\varphi_1 = s, \varphi_{n+1} = \nu\).
OPTIMALITY OF TWO INEQUALITIES

For \( \nu < -s/n \) again we get a contradiction to (14). So it remains to exclude \( \nu > \nu_0 \). This can be done very similarly as excluding \( \mu < \mu_0 \) in Theorem 5.2 by some dual setup. Again using the method from the proof of [22, (7)] one can show that for arbitrarily large \( q \), up to \( o(q) \) as \( q \to \infty \), we must have a situation as in the interval \([\tilde{q}_1, q_1]\) in Figure 4. Finally, for \( \nu > \nu_0 \) we again derive a contradiction when considering the next meeting point of \( L_1, L_2 \) to the left of \( \tilde{q}_1 \). We leave the details to the reader. Again we can deduce equality in the dimension formulas for \( s < 0, \nu = \nu_0 \) since then the entire combined graph must essentially be built up from consecutive periodical patterns as in Figure 4.

7. Final comments relating to work of Bugeaud, Laurent and Roy

According to the comments below [2, Theorem 3], sharpness of (BL1), (BL2) are respectively equivalent to optimality of certain systems of inequalities.

Identify \( \xi \in \mathbb{R}^n \) with its projective image in \( P^n(\mathbb{R}) \). Let \( 0 \leq d \leq n - 1 \) an integer. We denote by \( \omega_d = \omega_d(\xi) \) the supremum of the real numbers \( u \) for which there exist infinitely many rational linear subvarieties \( L \subseteq P^n(\mathbb{R}) \) such that \( \dim(L) = d \) and \( d(\xi, L) \leq H(L)^{-1-u} \), where \( H(L) \) is the Weil height of any system of Plücker coordinates of \( L \) and \( d(A, B) \) denotes the distance of two projective sets \( A, B \subseteq P^n(\mathbb{R}) \). Then \( \omega_d \) corresponds to the classical exponent \( \omega \), and \( \omega_{n-1} \) to our \( \omega^* \). Translating [20, Proposition 3.1] to our formalism, any \( \omega_d \) can be written as an expression involving certain \( \varphi_j(q) \) via

\[
\frac{1}{1 + \omega_d} = \limsup_{q \to \infty} \frac{n - d - \sum_{j=d+2}^{n+1} \varphi_j(q)}{n + 1}, \quad 0 \leq d \leq n - 1,
\]

upon the convention \( \omega_d = \infty \) if the right hand side becomes 0.

Analyzing the proof of (BL2) in [2], identity is equivalent to identities

\[
\omega_1 = \omega + \hat{\omega} \frac{1}{1 - \hat{\omega}} \quad \text{(65)}
\]

and

\[
\omega_{d+1} = \frac{(n - d)\omega_d + 1}{n - d - 1}, \quad 1 \leq d \leq n - 2. \quad \text{(66)}
\]

For general \( \xi \in \mathbb{R}^n \), the left hand sides are bounded below by the right hand sides in (65), (66) for every \( 0 \leq d \leq n - 2 \), so we have inequalities. A special case of a result by Roy [20, Theorem 2.3] implies that for any reasonable choice of \( \omega_0 = \omega \), there exists \( \xi \in \mathbb{R}^n \) that simultaneously satisfies all identities in (66) (for \( d = 0 \) as well). In fact, Maurat [14] evaluated Hausdorff and packing dimensions of the corresponding sets of \( \xi \in \mathbb{R}^n \) with the aid of Theorem 3.2. However, the condition (65) on \( \hat{\omega} \) remains. Indeed, the restrictions from Theorem 2.2 show that we cannot have this additional identity in certain cases. We mention that if (65) and an extension of (66) also valid for \( d = 0 \) hold, then we could conclude \( \hat{\omega} = 1/n \). In particular, if \( \omega > 1/n \) then our vectors \( \xi \) in Theorem 2.2 do not
have these properties as it can be checked that they induce \( \hat{\omega} > 1/n \). Similarly, looking at the proof of (BL1) in [2] we check that equality happens if and only if
\[
(67) \quad \omega_{d-1} = \frac{d\omega_d + 1}{\omega_d + d - 1}, \quad 1 \leq d \leq n - 2,
\]
and
\[
(68) \quad \omega_{n-2} = \frac{(\hat{\omega}^* - 1)\omega^*}{\omega^* + \hat{\omega}^*}.
\]
Again in general there are just inequalities in (67) and (68), with the right hand sides not exceeding the left. Again (67) can be satisfied for \( \xi \) by Roy’s [20, Theorem 2.3], however we are still left with a condition on \( \hat{\omega}^* \). Again, if we impose (68) and an extension of (67) valid for \( d = n - 1 \) as well, we conclude \( \hat{\omega}^* = n \), so for \( \omega^* > n \) the examples in Theorem 2.1 do not satisfy these properties.

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