HYERS–ULAM STABILITY OF DERIVATIONS AND LINEAR FUNCTIONS

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ABSTRACT.

1. INTRODUCTION AND PRELIMINARIES

In this paper \( \mathbb{N}, \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{R} \) denotes the set of the natural (positive integer), the integer, the rational and the real numbers, respectively.

The stability theory of functional equations basically deals with the following question: Is it true that an ‘approximate’ solution of a functional equation ‘can be approximated’ by a solution of the functional equation in question? This problem was raised by S. M. Ulam (see [13]) and answered (affirmatively) by D. H. Hyers concerning the additive Cauchy equation see [5]. Since 1941 this result has been extended and generalized in several ways, see e.g., Hyers–Isac–Rassias [6] and the references therein. Of course, the question of stability can be raised not only concerning the Cauchy equation but also in connection with other equations.

The aim of this paper is to examine the stability of a system of equations that defines derivations as well as linear functions.

Definition 1.1. A function \( f : \mathbb{R} \to \mathbb{R} \) is called an \textit{additive} function if,

\[
(1.1) \quad f(x + y) = f(x) + f(y)
\]

holds for all \( x, y \in \mathbb{R} \). Furthermore, we say that an additive function \( f : \mathbb{R} \to \mathbb{R} \) is a \textit{derivation} if

\[
(1.2) \quad f(xy) = xf(y) + yf(x)
\]

is fulfilled for all \( x, y \in \mathbb{R} \).

From (1.2) \( f(1) = 0 \) follows, whence every derivation vanishes at the rationals. Furthermore, it is known that there exist not identically zero derivations, see Kuczma [9].

It is easy to see from the above definition that every derivation \( f : \mathbb{R} \to \mathbb{R} \) satisfies the equation

\[
(1.3) \quad f(x^k) = kx^{k-1}f(x) \quad (x \in \mathbb{R} \setminus \{0\})
\]

for arbitrarily fixed \( k \in \mathbb{Z} \setminus \{0\} \). Furthermore, the converse is also true, in the following sense: if \( k \in \mathbb{Z} \setminus \{0,1\} \) is fixed and an additive function \( f : \mathbb{R} \to \mathbb{R} \) satisfies (1.3), then \( f \) is a derivation, see e.g., Kurepa [10] and Kannappan–Kurepa [8].
Motivated by a problem of I. Halperin (1963), Jurkat [7] and independently, Kurepa [10] proved that every additive function \( f : \mathbb{R} \to \mathbb{R} \) satisfying
\[
 f \left( \frac{1}{x} \right) = \frac{1}{x^2} f(x) \quad (x \in \mathbb{R} \setminus \{0\})
\]
has to be linear.

In [12] A. Nishiyama and S. Horinouchi investigated additive functions \( f : \mathbb{R} \to \mathbb{R} \) satisfying the additional equation
\[
 f(x^n) = c x^k f(x^m) \quad (x \in \mathbb{R} \setminus \{0\}),
\]
where \( c \in \mathbb{R} \) and \( n, m, k \in \mathbb{Z} \) are arbitrarily fixed. This approach is obviously the common generalization of the above mentioned results. In the second part of the paper we will deal with the stability of this last system of functional equations. Our main results could serve as a generalization of the theorems of [12]. However, the aim of the paper is not only to prove a stability theorem. In the so-called mixed theory of information it is usual to consider a functional equation that characterizes the inset measure of information, see Maksa [11]. While solving this equation one obtains an additive function satisfying also the equation
\[
 f \left( \frac{1}{x} \right) = -\frac{1}{x^2} f(x) \quad (x \in \mathbb{R} \setminus \{0\}).
\]
Clearly, this is a particular case of equation (1.3) with \( k = -1 \). Therefore it is rather natural to expect that the investigation of the stability of the above mentioned equation for the inset measure of information should be preceded by the verification of the stability of the characterization of derivations by Kannappan and Kurepa. Thus our results can be applied when we investigate the stability of a functional equation characterizing the inset measure of information.

In what follows we will list some preliminary definitions and statements that will be used during the proof of our main result. These can be found e.g., in Kuczma [9].

Let \( p \in \mathbb{N} \). A function \( f : \mathbb{R}^p \to \mathbb{R} \) is called \( p \)-additive if, for every \( i \in \{1, 2, \ldots, p\} \) and for every \( x_1, \ldots, x_p, y_i \in \mathbb{R} \)
\[
f(x_1, \ldots, x_i-1, x_i + y_i, x_{i+1}, \ldots, x_p)
= f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_p) + f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_p),
\]
i.e., \( f \) is additive in each of its variables \( x_i \in \mathbb{R}, i = 1, \ldots, p \). A 2–additive function is called biadditive.

**Theorem 1.2.** Let \( f : \mathbb{R}^p \to \mathbb{R} \) be a continuous \( p \)-additive function. Then there exists a constant \( c \in \mathbb{R} \) such that
\[
f(x_1, x_2, \ldots, x_p) = cx_1 x_2 \ldots x_p
\]
for all \( x_1, x_2, \ldots, x_p \in \mathbb{R} \).

**Theorem 1.3.** Let \( f : \mathbb{R}^p \to \mathbb{R} \) be a \( p \)-additive function, bounded above, or below on a set \( T \subset \mathbb{R}^p \), which has positive Lebesgue–measure. Then \( f \) is continuous.
Given a function $F : \mathbb{R}^p \to \mathbb{R}$, by the diagonalization (or trace) of $F$ we understand the function $f : \mathbb{R} \to \mathbb{R}$ arising from $F$ by putting all the variables (from $\mathbb{R}$) equal:

$$f(x) = F(x, \ldots, x). \quad (x \in \mathbb{R})$$

We will also refer to the definition of the difference operator $\Delta_h$ with the span $h \in \mathbb{R}$, which is given for a function $f : \mathbb{R} \to \mathbb{R}$ by the formula

$$\Delta_h f(x) = f(x + h) - f(x) \quad (x \in \mathbb{R}).$$

The superposition of several difference operators will be denoted shortly by

$$\Delta_{h_1 h_2 \ldots h_p} f = \Delta_{h_1} \Delta_{h_2} \ldots \Delta_{h_p} f,$$

where $p \in \mathbb{N}$ and $h_1, h_2, \ldots, h_p \in \mathbb{R}$.

**Lemma 1.4.** Let $F : \mathbb{R}^p \to \mathbb{R}$ be a symmetric $p$-additive function, and let $f : \mathbb{R} \to \mathbb{R}$ be the diagonalization of $F$. For every $n \in \mathbb{N}$, $n \geq p$ and for every $x, h_1, \ldots, h_n \in \mathbb{R}$ we have

$$\Delta_{h_1 \ldots h_n} f(x) = \begin{cases} \ p! F(h_1, \ldots, h_p), & \text{if } n = p \\
0, & \text{if } n \geq p. \end{cases}$$

We remark that according to Theorem 15.1.1 in Kuczma [9], we have for all $p \in \mathbb{N}$,

$$\Delta_{h_1 \ldots h_p} f(x) = \sum_{\varepsilon_1, \ldots, \varepsilon_p = 0}^1 (-1)^{p-(\varepsilon_1+\ldots+\varepsilon_p)} f(x + \varepsilon_1 h_1 + \ldots + \varepsilon_p h_p).$$

We will also make use of a result of Kannappan–Kurepa [8].

**Theorem 1.5.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be additive functions and $n, m \in \mathbb{Z} \setminus \{0\}$, $n \neq m$. Suppose that

$$f(x^n) = x^{n-m} g(x^m)$$

holds for all $x \in \mathbb{R} \setminus \{0\}$. Then the functions $F, G : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = f(x) - f(1)x \quad \text{and} \quad G(x) = g(x) - g(1)x \quad (x \in \mathbb{R})$$

are derivations and $nF(x) = mG(x)$ is fulfilled for all $x \in \mathbb{R}$.

2. **INEQUALITIES FOR ADDITIVE FUNCTIONS**

**Lemma 2.1.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be additive functions, $n, m \in \mathbb{Z} \setminus \{0\}$, $n \neq m$ suppose furthermore that either $n = -m$ or $\text{sign}(n) = \text{sign}(m)$ and assume that there exists an interval $I \subset \mathbb{R}$ with positive length such that

$$|f(x^n) - x^{n-m} g(x^m)| \leq K$$

holds for all $x \in \mathbb{R} \setminus \{0\}$ with a certain $K \in \mathbb{R}$. Then there exist derivations $F, G : \mathbb{R} \to \mathbb{R}$ such that $nF(x) = mG(x)$ $(x \in \mathbb{R})$ and

$$f(x) = F(x) + f(1)x \quad (x \in \mathbb{R}),$$

$$g(x) = G(x) + g(1)x \quad (x \in \mathbb{R}).$$
Proof. Firstly, we will show that inequality (2.1) implies that there exists $L \in \mathbb{R}$ so that
\begin{equation}
|f(x^n) - x^{n-m}g(x^m)| \leq L |x^n|
\end{equation}
is fulfilled for all $x \in \mathbb{R} \setminus \{0\}$.

Let $|a, b| \subseteq I$. Since the rationals are dense in $\mathbb{R}$, for every $x \in \mathbb{R} \setminus \{0\}$ we can find $r(x) \in \mathbb{Q}$ (a rational number depending only on $x$) such that $a < r(x)x < b$. If we replace $x$ by $r(x)x$ in (2.1), we obtain that
\begin{equation}
|f(x^n) - x^{n-m}g(x^m)| \leq K |r(x)|^{-n},
\end{equation}
where we used the fact that every additive function is $\mathbb{Q}$–homogeneous. Since $a < r(x)x < b$,
\[
\min \{ |a|, |b| \} < r(x)x < \max \{ |a|, |b| \}.
\]
In case $n > 0$, we obtain from this that
\[
(\min \{ |a|, |b| \})^{-n} |x|^n > |r(x)|^{-n}
\]
and in case $n < 0$ we get that
\[
(\max \{ |a|, |b| \})^{-n} |x|^n > |r(x)|^{-n}.
\]
Therefore, if we define
\[
L = \begin{cases} (\min \{ |a|, |b| \})^{-n} K, & \text{if } n > 0 \\ (\max \{ |a|, |b| \})^{-n} K, & \text{if } n < 0 \end{cases}
\]
we get inequality (2.4).

At this point of the proof we have to distinguish several cases. First suppose that $n, m > 0$. Without the loss of generality $n > m$ can be assumed.

Define the function $H$ on $\mathbb{R}^n$ by
\[
H(x_1, \ldots, x_n) = f(x_1 \cdot \ldots \cdot x_n) - \frac{1}{n} x_1 \cdot \ldots \cdot x_{n-m} g(x_{n-m+1} \cdot \ldots \cdot x_n) - \frac{1}{n} x_2 \cdot \ldots \cdot x_{n-m+1} g(x_{n-m+2} \cdot \ldots \cdot x_n x_1) - \ldots - \frac{1}{n} x_n x_1 \cdot \ldots \cdot x_{n-m-1} g(x_{n-m} \cdot \ldots \cdot x_2).
\]
Due to the additivity of the functions $f$ and $g$, the function $H$ is a symmetric and $n$–additive function, and its trace
\[
H(x, \ldots, x) = f(x^n) - x^{n-m}g(x^m). \quad (x \in \mathbb{R})
\]
In view of inequality (2.1), this yields that
\[
|H(x, \ldots, x)| = |f(x^n) - x^{n-m}g(x^m)| \leq L |x^n|, \quad (x \in \mathbb{R} \setminus \{0\})
\]
that is, the trace of the function $H$ can be dominated by the term $L |x^n|$. On the other hand, Lemma 1.4. states that the function $H$ is uniquely determined by its trace via the formula
\[
H(h_1, \ldots, h_n) = \frac{1}{n!} \Delta_{h_1 \ldots h_n} H(x, \ldots, x), \quad (x, h_1, \ldots, h_n \in \mathbb{R})
\]
This yields that the function $H$ is bounded on a subset of $\mathbb{R}^n$ which has positive Lebesgue–measure. Thus, by Theorem 1.3., the function $H$ is continuous on $\mathbb{R}^n$. Therefore, especially,
\[
H(x, \ldots, x) = cx^n
\]
holds for all \( x \in \mathbb{R} \) with a certain \( c \in \mathbb{R} \). From this we get that \( H(1, \ldots, 1) = c \), on the other hand by the definition of the function \( H \), \( H(1, \ldots, 1) = f(1) - g(1) \) follows. All in all, 

\[
(2.5) \quad (f(1) - g(1)) x^n = H(x, \ldots, x) = f(x^n) - x^{n-m} g(x^m). \quad (x \in \mathbb{R} \setminus \{0\})
\]

Define the functions \( F, G : \mathbb{R} \to \mathbb{R} \) by

\[
F(x) = f(x) - f(1)x \quad \text{and} \quad G(x) = g(x) - g(1)x, \quad (x \in \mathbb{R} \setminus \{0\})
\]

then from (2.5) we get that

\[
F(x^n) = x^{n-m}G(x^m)
\]

for all \( x \in \mathbb{R} \setminus \{0\} \). Using Theorem 1.5., this yields that the functions \( F \) and \( G \) are derivations and \( nF(x) = mG(x) \) holds for all \( x \in \mathbb{R} \). This means that equations (2.2) and (2.3) hold in case \( n, m > 0 \).

Secondly assume that \( n, m < 0 \). In this case let us replace \( x \) by \( \frac{1}{x} \) in inequality (2.1) to obtain

\[
|f(x^{-n}) - x^{-(n)-(-m)}g(x^{-m})| \leq L|x|^{-n}. \quad (x \in \mathbb{R} \setminus \{0\})
\]

Since \(-n\) and \(-m\) are positive integers, the results of the previous case can be applied. Therefore there exist derivations \( F, G : \mathbb{R} \to \mathbb{R} \) so that \( nF(x) = mG(x) \) holds for all \( x \in \mathbb{R} \) and

\[
f(x) = F(x) + f(1)x
\]

and

\[
g(x) = G(x) + g(1)x
\]

holds for all \( x \in \mathbb{R} \).

Suppose now that \( n = -m \). Then inequality (2.1) yields that

\[
(2.6) \quad |f(x^{-m}) - x^{-2m}g(x^m)| \leq L|x|^{-m}
\]

holds for all \( x \in \mathbb{R} \setminus \{0\} \). If we replace \( x \) by \( x^{1/m} \) \((x > 0)\) then inequality (2.6) yields that

\[
(2.7) \quad \left| f\left(\frac{1}{x}\right) - \frac{1}{x^2}g(x) \right| \leq \frac{L}{|x|}
\]

is fulfilled for all \( x > 0 \). Replace in this inequality \( x \) by \( x(x+1) \), then

\[
\left| f\left(\frac{1}{x(x+1)}\right) - \frac{1}{x^2(x+1)^2}g(x(x+1)) \right| \leq \frac{L}{|x(x+1)|}
\]

is fulfilled for all \( x > 0 \). After using the additivity of the function \( f \),

\[
\left| f\left(\frac{1}{x}\right) - f\left(\frac{1}{x+1}\right) - \frac{1}{x^2(x+1)^2}g(x(x+1)) \right| \leq \frac{L}{|x(x+1)|} \quad (x > 0)
\]
Lemma 2.3. Let \( \kappa \in \mathbb{R} \) and \( n, m \in \mathbb{Z} \setminus \{0, 1\} \), \( n \neq m \), and assume that there exists an interval \( I \subset \mathbb{R} \) such that

\[
|f(x^n) - \kappa x^{n-m} f(x^m)| \leq K
\]
This last two inequalities and the triangle inequality imply that

$$\forall x \in \mathbb{R} \setminus \{0\} \text{ with a certain } K \in \mathbb{R}. \text{ Then there exists a derivation }$$

$$F : \mathbb{R} \rightarrow \mathbb{R} \text{ for which } (n - \kappa m)F(x) = 0 \text{ and }$$

$$f(x) = F(x) + f(1)x$$

holds for all $$x \in \mathbb{R}$$.

**Proof.** In view of Lemma 2.1. we have to only deal with the case $$\text{sign}(n) \neq \text{sign}(m)$$ and $$n \neq -m$$. Furthermore, due the proof the previous lemma, inequality (2.8) is equivalent to the following inequality

$$|f(x^n) - \kappa x^{n-m} f(x^m)| \leq L|x^n|$$

for all $$x \in \mathbb{R} \setminus \{0\}$$, where $$L$$ is a certain real constant. Let us substitute $$x^n$$ in place of $$x$$ into this inequality,

$$|f(x^{n^2}) - \kappa x^{n(n-m)} f(x^{nm})| \leq L|x^{n^2}|. \quad (x \in \mathbb{R} \setminus \{0\})$$

Additionally, the above inequality with the substitution $$x^m$$ yields

$$|f(x^{nm}) - \kappa x^{m(n-m)} f(x^m)| \leq L|x^{nm}|. \quad (x \in \mathbb{R} \setminus \{0\})$$

This last two inequalities and the triangle inequality imply that

$$|f \left( x^{n^2} \right) - \kappa^2 x^{n^2-m^2} f \left( x^{m^2} \right) |$$

$$\leq |f(x^{n^2}) - \kappa x^{n(n-m)} f(x^{nm})| + |\kappa x^{n(n-m)} f(x^m)| \cdot |f(x^{nm}) - \kappa x^{m(n-m)} f(x^m)|$$

$$= L \left| x^{n^2} \right| + L \left| \kappa x^{n(n-m)} x^{nm} \right| = L (1 + |\kappa|) \cdot x^{n^2}$$

holds for all $$x \in \mathbb{R} \setminus \{0\}$$. Let us observe that $$n^2, m^2 > 0$$ and the results of Lemma 2.1. can be applied (with the choice $$g(x) = \kappa f(x)$$) to obtain that

$$f(x) = F(x) + f(1)x$$

holds for all $$x \in \mathbb{R}$$, where $$F : \mathbb{R} \rightarrow \mathbb{R}$$ is a derivation which also satisfies $$(n - \kappa m)F(x) = 0$$ for arbitrary $$x \in \mathbb{R}$$. \( \square \)

> From this lemma the following statement can be concluded immediately.

**Corollary 2.4.** Let $$f : \mathbb{R} \rightarrow \mathbb{R}$$ be an additive function and $$r \in \mathbb{Q} \setminus \{0, 1\}$$. Assume that there exists an interval $$I \subset \mathbb{R}$$ with positive length such that

$$|f \left( x^r \right) - r x^{r-1} f(x)| \leq K$$

holds for all $$x \in I$$ with a certain $$K \in \mathbb{R}$$. Then there exists a derivation $$F : \mathbb{R} \rightarrow \mathbb{R}$$ such that

$$f(x) = F(x) + f(1)x$$

is fulfilled for all $$x \in \mathbb{R}$$.

### 3. Stability of Derivations and Linear Functions

As a starting point of the proof of the main result of this section the theorem of Hyers will be used. Originally this statement was formulated in terms of functions that are acting between Banach spaces, see Hyers [5]. However, we will use this theorem only in the particular case when the domain and the range are the set of reals. In this setting the proposition of Hyers’ theorem is the following.
Theorem 3.1. Let $\varepsilon \geq 0$ and suppose that the function $f : \mathbb{R} \to \mathbb{R}$ fulfills the inequality
\[ |f(x + y) - f(x) - f(y)| \leq \varepsilon \]
for all $x \in \mathbb{R}$. Then there exists an additive function $a : \mathbb{R} \to \mathbb{R}$ such that
\[ |f(x) - a(x)| \leq \varepsilon \]
holds for arbitrary $x \in \mathbb{R}$.

Applying this theorem and our results in the previous section, we can establish our main result.

Theorem 3.2. Let $\varepsilon_1, \varepsilon_2 \geq 0$, $\kappa \in \mathbb{R}$, $n, m \in \mathbb{Z} \setminus \{0\}$, $n \neq m$ and assume that the function $f : \mathbb{R} \to \mathbb{R}$ fulfills the inequalities
\[ |f(x + y) - f(x) - f(y)| \leq \varepsilon_1 \]
and
\[ |f(x^n) - \kappa x^{n-m} f(x^m)| \leq \varepsilon_2 \]
for all $x \in \mathbb{R} \setminus \{0\}$. Then there exist a derivation $F : \mathbb{R} \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that
\[ (n - \kappa m)F(x) = 0 \]
and
\[ |f(x) - [F(x) + \lambda x]| \leq \varepsilon_1 \]
are satisfied for all $x \in \mathbb{R}$.

Proof. Due the theorem of Hyers, inequality (3.1) immediately implies that there exists an additive function $a : \mathbb{R} \to \mathbb{R}$ satisfying
\[ |f(x) - a(x)| \leq \varepsilon_1 \]
for all $x \in \mathbb{R}$. In view of inequality (3.2) this implies that
\[ |a(x^n) - \kappa x^{n-m} a(x^m)| \]
\[ \leq |a(x^n) - f(x^n)| + |\kappa x^{n-m}| \cdot |a(x^m) - f(x^m)| + |f(x^n) - \kappa x^{n-m} f(x^m)| \]
\[ \leq \varepsilon_1 + |\kappa x^{n-m}| \varepsilon_1 + \varepsilon_2 = (1 + |\kappa x^{n-m}|) \varepsilon_1 + \varepsilon_2 \]
is fulfilled for all $x \in \mathbb{R} \setminus \{0\}$. Thus the expression $|a(x^n) - \kappa x^{n-m} a(x^m)|$ is bounded on a real interval with non-void interior. Therefore Lemma 2.3. yields that there exists a derivation $F : \mathbb{R} \to \mathbb{R}$ such that $(n - \kappa m)F(x) = 0$ and
\[ a(x) = F(x) + a(1)x \]
holds for all $x \in \mathbb{R}$. This, together with (3.4), implies (3.3) with $\lambda = a(1)$. \hfill $\square$

Let us note that our result serves as a stability theorem for linear functions if $n \neq \kappa m$. 

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