SINGULAR ANALYSIS OF THE STRESS CONCENTRATION IN THE NARROW REGIONS BETWEEN THE INCLUSIONS AND THE MATRIX BOUNDARY

CHANGXING MIAO AND ZHIWEN ZHAO

Abstract. We consider the Lamé system arising from high-contrast composite materials whose inclusions (fibers) are nearly touching the matrix boundary. The stress, which is the gradient of the solution, always concentrates highly in the narrow regions between the inclusions and the external boundary. This paper aims to provide a complete characterization in terms of the singularities of the stress concentration by accurately capturing all the blow-up factor matrices and making clear the dependence on the Lamé constants and the curvature parameters of geometry. Moreover, the precise asymptotic expansions of the stress concentration are also presented in the presence of a strictly convex inclusion close to touching the external boundary for the convenience of application.

1. Introduction

In composite structure comprising of a matrix and inclusions (fibers), it is common for inclusions in close proximity to each other or close to touching the matrix boundary. The mathematical problem can be described by the Lamé system. The principal quantity of interest from the perspective of engineering is the stress, which is the gradient of a solution to the Lamé system. It is well known that the stress may blow up as the distance $\varepsilon$ between inclusions or between the inclusion and the matrix boundary approaches to zero. Especially the latter exhibits more complex singular behavior due to the interaction from the boundary data.

This work is devoted to the investigation of the blow-up phenomena occurring in high-contrast fiber-reinforced composites, which is stimulated by the great work of Babuška et al. [6], where the Lamé system was utilized and they observed numerically that the size of the strain tensor stays bounded independent of the distance $\varepsilon$ between adjacent inclusions. Subsequently, Bonnetier and Vogelius [14] considered the scalar equation in the conductivity problem and proved that the gradient of a solution to the conductivity equation with piecewise constant coefficients remains bounded for two touching disks. Li and Vogelius [38] then studied the general divergence form second order elliptic equation with piecewise smooth coefficients and obtained the boundness of the gradient with its upper bound independent of $\varepsilon$ for the inclusions of arbitrary smooth shape. In subsequent work [37], Li and Nirenberg further extended the $\varepsilon$-independent gradient estimates in [38] to general divergence form second order elliptic systems, especially covering systems of elasticity. Their results also demonstrate the numerical observation in [6]. Dong and Li [15] recently made clear the dependence on the elliptic coefficients and the

Date: May 6, 2022.
distance between two disks in the optimal gradient estimates and established more
interesting higher-order derivative estimates for the isotropic conductivity problem.
This, in particular, answered open problems (b) and (c) proposed by Li and Vo-
gelius \cite{38}. However, for the inclusions of general smooth shape and second order
elliptic systems of divergence form, it remains to be solved. We refer to page 894
of \cite{37} for these open problems. In addition, Kim and Lim \cite{27} made use of the sin-
gle and double layer potentials with image line charges to establish an asymptotic
expression for the solution to the conductivity problem in the presence of core-shell
structure with circular boundaries. Calo, Efendiev and Galvis \cite{15} obtained an as-
symptotic formula of a solution to the second-order elliptic equations of divergence
form in the presence of high-conductivity inclusions or low-conductivity inclusions.

The elliptic coefficients considered in aforementioned work are assumed to be
away from 0 and $\infty$. The situation will become very different when the elliptic
constants are allowed to deteriorate. For example, the scalar conductivity problem
with finite conductivity $k$ turns into the perfect conductivity problem as the contrast
$k$ degenerates to be $\infty$. It has been shown in various literature that the electric field,
which is the gradient of a solution to the perfect conductivity equation, generally
blows up as the distance $\varepsilon$ between inclusions or between the inclusions and the
matrix boundary goes to zero. Its blow-up rate has been proved to be $\varepsilon^{-1/2}$ in
two dimensions \cite{4,5,7,11,28,40,41}, $(\varepsilon |\ln\varepsilon|)^{-1}$ in three dimensions \cite{11,12,30,36},
and $\varepsilon^{-1}$ in dimensions greater than three \cite{11}, respectively. Further, more precise
characterization in terms of the singularities of the electric field concentration have
been established by Ammari et al. \cite{3}, Bonnetier and Triki \cite{13}, Kang et al. \cite{22–24},
Li et al. \cite{29,33}. Recently, the study on the singular behavior of the gradient has
been generalized to the nonlinear $p$-Laplacian \cite{19,20} and the Finsler $p$-Laplacian
\cite{16,17}. Their method of barriers is purely nonlinear, which differs from the ones
utilized in the linear case.

However, there is significant difficulty in extending the results in the perfect
conductivity problem to the full elasticity. For example, the maximum principle
does not hold for the systems. A delicate iterate technique was then built in \cite{31}
to overcome this difficulty, where Li et al. \cite{31} obtained $C^k$ estimates for a class
of elliptic systems. After that, this iterate scheme was applied to the investigation
on the blow-up behavior of the gradient of a solution to the Lamé system with
partially infinity coefficients. Bao, Li and Li \cite{9,10} firstly established the pointwise
upper bounds on the gradient for two strictly convex inclusions. Their results
indicated that the gradient blow-up rate under vectorial case is consistent with
that under scalar case above. The subsequent work \cite{32} provided a lower bound on
the gradient, which demonstrates the optimality of the blow-up rate in dimensions
two and three. It is worth emphasizing that Kang and Yu \cite{26} recently introduced
singular functions constructed by nuclei of strain to give a precise description for
the stress concentration in two dimensions. The mathematical approaches in \cite{26}
are based on the layer potential techniques and the variational principle, which
are different from the iterate scheme above. Besides the aforementioned interior
estimates, there is another direction of research to establish the boundary estimates
\cite{8,35}. For one strictly convex inclusion close to touching the matrix boundary,
Bao, Ju and Li \cite{8} obtained the optimal upper and lower bounds on the gradient.
In particular, the lower bound on the gradient in \cite{8} was constructed by finding
a blow-up factor which is a linear functional in relation to the boundary data.
Subsequently, Li and Zhao [35] extended to the general \( m \)-convex inclusions and found that some special boundary data with \( k \)-order growth will strengthen the singularities of the stress. This is a novel blow-up phenomenon induced by the boundary data. Although the optimal blow-up rate of the stress is derived in [8, 35], the dependence on the Lamé constants and the curvature parameters of geometry is not explicit. Moreover, they only captured a single blow-up factor in the construction of the lower bound of the gradient but not the whole blow-up factor matrices, which determine essentially whether the blow-up will occur or not. By contrast with the results in [8, 35], the novelty of this paper lies in capturing all the blow-up factor matrices and meanwhile showing the explicit dependence on the Lamé coefficients and the curvature parameters of geometry in the optimal upper and lower bounds on the gradient. Further, these blow-up factor matrices can be applied to establish the corresponding asymptotic expansions of the gradient.

The outline of this paper is as follows. Section 2 is to describe the elasticity problem and state the main results. In Section 3, we consider a general boundary value problem (3.3) and give the leading term for the gradient of the solution. Section 4 is dedicated to the proofs of Theorems 2.1 and 2.5. The asymptotic behavior of the stress concentration for a strictly convex inclusion close to touching the matrix boundary is analyzed in Section 5.

2. Problem formulation and Main results

2.1. The elasticity problem. In this paper, the Lamé system in linear elasticity is considered. Let \( D \subset \mathbb{R}^d (d \geq 2) \) be a bounded domain with \( C^{2, \alpha} \) boundary, and \( D_1^* \) be a convex open set in \( D \) with \( C^{2, \alpha} \) boundary, \( 0 < \alpha < 1 \), which touches the external boundary \( \partial D \) at one point. By a translation and rotation of the coordinates, if necessary, let

\[ \partial D_1^* \cap \partial D = \{0\} \in \mathbb{R}^d, \quad D_1^* \subset \{(x', x_d) \in \mathbb{R}^d | x_d > 0\}. \]

Here and below, we denote \((d - 1)\)-dimensional variables and domains by adding superscript prime. After moving upward \( D_1^* \) along \( x_d \)-axis by an arbitrarily small constant \( \varepsilon > 0 \), we get

\[ D_1^* := D_1^* + (0', \varepsilon). \]

For simplicity, let

\[ D_1 := D_1^*, \quad \Omega := D \setminus D_1. \]

Suppose that \( \Omega \) and \( D_1 \) are filled with two different isotropic and homogeneous elastic materials with different Lamé constants \((\lambda, \mu)\) and \((\lambda_1, \mu_1)\), respectively. The elasticity tensors \( C^0 \) and \( C^1 \) for the background \( \Omega \) and the inclusion \( D_1 \) can be written, respectively, as

\[ C^0_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \]

and

\[ C^1_{ijkl} = \lambda_1 \delta_{ij} \delta_{kl} + \mu_1 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \]

where \( i, j, k, l = 1, 2, \ldots, d \) and \( \delta_{ij} \) denotes the kronecker symbol: \( \delta_{ij} = 0 \) for \( i \neq j \), \( \delta_{ij} = 1 \) for \( i = j \). For a given boundary data \( \varphi = (\varphi^1, \varphi^2, \ldots, \varphi^d)^T \), we consider the
following Dirichlet problem for the Lamé system with piecewise constant coefficients

$$\begin{cases}
\nabla \cdot ((\chi_\Omega C^0 + \chi_{D_1} C^1) e(u)) = 0, & \text{in } D, \\
u = \varphi, & \text{on } \partial D, 
\end{cases}$$

(2.1)

where $u = (u^1, u^2, \ldots, u^d)^T : D \to \mathbb{R}^d$ represents the elastic displacement field, $e(u) = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right)$ denotes the elastic strain, $\chi_\Omega$ and $\chi_{D_1}$ are the characteristic functions of $\Omega$ and $D_1$, respectively. As shown in [9], if the strong ellipticity condition

$$\mu > 0, \quad d\lambda + 2\mu > 0, \quad \mu_1 > 0, \quad d\lambda_1 + 2\mu_1 > 0$$

holds, then there exists a unique solution $u \in H^1(D; \mathbb{R}^d)$ of the Dirichlet problem (2.1) for given $\varphi \in H^1(D; \mathbb{R}^d)$.

Denote the linear space of rigid displacement in $\mathbb{R}^d$ by

$$\Psi := \{ \psi \in C^1(\mathbb{R}^d; \mathbb{R}^d) \mid \nabla \psi + (\nabla \psi)^T = 0 \}.$$ 

A base of $\Psi$ is written as

$$\{ \psi_\alpha \}_{\alpha = 1}^{d(d+1)/2} := \{ e_i, \quad x_k e_j - x_j e_k \mid 1 \leq i \leq d, \quad 1 \leq j < k \leq d \},$$

(2.2)

where $\{ e_1, e_2, \ldots, e_d \}$ is the standard basis in $\mathbb{R}^d$. For the convenience of computations, we adopt the following order with respect to $\psi_\alpha$: $\psi_\alpha = e_\alpha$ if $\alpha = 1, 2, \ldots, d$; $\psi_\alpha = x_d e_{\alpha - d} - x_{\alpha - d} e_d$ if $\alpha = d + 1, \ldots, 2d - 1$; if $\alpha = 2d, \ldots, \frac{d(d+2)}{2}$ ($d \geq 3$), there exist two indices $1 \leq i_a < j_a < d$ such that $\psi_\alpha = (0, \ldots, 0, x_{i_a}, 0, \ldots, 0, -x_{j_a}, 0, \ldots, 0)$.

It has been proved in the Appendix of [9] that for fixed $\lambda$ and $\mu$,

$$u_{\lambda_1, \mu_1} \to u \quad \text{in } H^1(D; \mathbb{R}^d), \quad \text{as } \min \{ \mu_1, d\lambda_1 + 2\mu_1 \} \to \infty,$$

where $u_{\lambda_1, \mu_1}$ is the solution of (2.1) and $u \in H^1(D; \mathbb{R}^d)$ verifies

$$\begin{cases}
\mathcal{L}_{\lambda, \mu} u := \nabla \cdot (C^0 e(u)) = 0, & \text{in } \Omega, \\
u = C^\alpha \psi_\alpha, & \text{on } \partial D_1, \\
\int_{\partial D_1} \frac{\partial u}{\partial n_\alpha} \cdot \psi_\alpha = 0, & \alpha = 1, 2, \ldots, \frac{d(d+1)}{2}, \\
u = \varphi, & \text{on } \partial D. 
\end{cases}$$

(2.3)

Here the free constants $C^\alpha$, $\alpha = 1, 2, \ldots, \frac{d(d+1)}{2}$ are determined by the third line of (2.3) and the co-normal derivative is given by

$$\frac{\partial u}{\partial n_\alpha} := (\nabla e(u)) \cdot \nu = \lambda (\nabla \cdot u) \nu + \mu (\nabla u + (\nabla u)^T) \nu,$$

where $\nu$ is the unit outer normal of $\partial D_1$ and the subscript $+$ denotes the limit from outside the domain. We here would like to remark that the existence, uniqueness and regularity of weak solutions to problem (2.3) have also been established in the Appendix of [9]. Furthermore, the $H^1$-solution $u$ of problem (2.3) was improved to be of $C^1(\overline{\Omega}; \mathbb{R}^d) \cap C^1(\overline{D_1}; \mathbb{R}^d)$ for any $C^{2,\alpha}$-domain.

Suppose that $\partial D_1$ and $\partial D$ near the origin are, respectively, the graphs of two $C^{2,\alpha}$ functions $\varepsilon + h_1$ and $h$ with respect to $x'$. To be specific, for some positive constant $R$, independent of $\varepsilon$, let $h_1$ and $h$ satisfy that

(H1) $\kappa_1|x'|^m \leq h_1(x') - h(x') \leq \kappa_2|x'|^m$, if $x' \in B_{2R}',$

(H2) $|\nabla_j h_1(x')|, |\nabla^2_j h(x')| \leq \kappa_3|x'|^{m-j}$, if $x' \in B_{2R}', j = 1, 2,$

(H3) $\|h_1\|_{C^{2,\alpha}(B_{2R}')} + \|h\|_{C^{2,\alpha}(B_{2R}')} \leq \kappa_4.$
where \( m \geq 2 \) is an integer and \( \kappa_i, i = 1, 2, 3, 4 \) are all positive constants independent of \( \varepsilon \). We additionally assume that \( h_1 - h \) is even with respect to each \( x_i \) in \( B_R^j \), \( i = 1, \ldots, d - 1 \). It is worth emphasizing that the inclusions under conditions (H1)–(H3) actually contain the strictly convex inclusions, which were extensively studied in previous work [31,10,20].

For \( z' \in B^j_R \) and \( 0 < t \leq 2R \), define the thin gap
\[
\Omega_t(z') := \left\{ x \in \mathbb{R}^d \mid h(x') < x_d < \varepsilon + h_1(x'), \ |x' - z'| < t \right\}.
\]
For notational simplicity we adopt the abbreviated notation \( \Omega_t \) to represent \( \Omega_t(0') \).

\[\Gamma_t^+ := \{ x \in \mathbb{R}^d \mid x_d = \varepsilon + h_1(x'), \ |x'| < t \}, \quad \Gamma_t^- := \{ x \in \mathbb{R}^d \mid x_d = h(x'), \ |x'| < t \}.\]

Introduce a scalar auxiliary function \( \bar{v} \in C^2(\mathbb{R}^d) \) such that \( \bar{v} = 1 \) on \( \partial D_1 \), \( \bar{v} = 0 \) on \( \partial D \),
\[
\bar{v}(x', x_d) := \frac{x_d - h(x')}{\varepsilon + h_1(x') - h(x')}, \quad \text{in } \Omega_{2R}, \quad \text{and } \| \bar{v} \|_{C^2(\Omega_1 \Omega_N)} \leq C. \tag{2.4}
\]

Denote
\[
\delta(x') := \varepsilon + h_1(x') - h(x'), \quad f(\bar{v}) := \frac{1}{2} \left( \bar{v} - \frac{1}{2} \right)^2 - \frac{1}{8}. \tag{2.5}
\]

For \( x \in \Omega_{2R} \), we define a family of vector-valued auxiliary functions as follows:
\[
\bar{u}_0 = \varphi(x', h(x'))(1 - \bar{v}(x', x_d)) + \mathcal{F}_0, \tag{2.6}
\]
and
\[
\bar{u}_\alpha(x', x_d) = \psi_\alpha \bar{v}(x', x_d) + \mathcal{F}_\alpha, \quad \alpha = 1, 2, \ldots, \frac{d(d+1)}{2}, \tag{2.7}
\]
where \( \psi_\alpha, \alpha = 1, 2, \ldots, \frac{d(d+1)}{2} \), are defined in (2.2) and
\[
\mathcal{F}_0 = -\frac{\lambda + \mu}{\mu} f(\bar{v}) \varphi^d(x', h(x')) \sum_{i=1}^{d-1} \partial_{x_i} \delta e_i - \frac{\lambda + \mu}{\lambda + 2\mu} f(\bar{v}) \sum_{i=1}^{d-1} \varphi^d(x', h(x')) \partial_{x_i} \delta e_d. \tag{2.8}
\]
and, for \( \alpha = 1, 2, \ldots, \frac{d(d+1)}{2} \),
\[
\mathcal{F}_\alpha = \frac{\lambda + \mu}{\mu} f(\bar{v}) \psi^d_{\alpha} \sum_{i=1}^{d-1} \partial_{x_i} \delta e_i + \frac{\lambda + \mu}{\lambda + 2\mu} f(\bar{v}) \sum_{i=1}^{d-1} \psi^d_{\alpha} \partial_{x_i} \delta e_d. \tag{2.9}
\]

We here would like to point out that the correction terms \( \mathcal{F}_\alpha, \alpha = 1, 2, \ldots, d \) were captured in the previous work [34]. In this paper, we further find the correction term for a more general boundary value problem [33], see Theorem 3.1 below.

2.2. Main results. To begin with, we first recall the following three types of boundary data introduced in [35], which are classified according to the parity. To be specific, let \( \varphi = (\varphi^1, \ldots, \varphi^d) \neq 0 \) on \( \Gamma^R_+ \) satisfying that for \( x \in \Gamma^R_- \),

(A1) for \( i = 1, 2, \ldots, d, j = 1, \ldots, d - 1 \), \( \varphi^i(x) \) is an even function of each \( x_j \);

(A2) if \( d = 2 \), for \( i = 1, 2 \), \( \varphi^i(x) \) is odd with respect to \( x_j \); if \( d \geq 3 \), for \( i = 1, \ldots, d - 1 \), \( \varphi^i(x) \) is odd with respect to some \( x_j \), \( j \in \{1, \ldots, d - 1\} \), and \( \varphi^d(x) \) is odd with respect to \( x_1 \) and even with respect to each \( x_j, j = 2, \ldots, d - 1 \);
(A3) if $d = 2$, $\varphi^1(x)$ is odd with respect to $x_1$, and $\varphi^2(x) = 0$; if $d \geq 3$, for $i = 1, \ldots, d - 1$, $\varphi^i(x)$ is odd with respect to $x_i$, and $\varphi^d(x)$ is odd with respect to $x_1$ and $x_2$, respectively.

For the convenience of notations, introduce the blow-up rate indices as follows: for $i = 0, 2$ and $i = k, k + 1, k \geq 2$,

$$
\rho_i(d, m; \varepsilon) := \begin{cases} 
\frac{d+i-1}{m} - 1, & m > d + i - 1, \\
|\ln \varepsilon|, & m = d + i - 1, \\
1, & m < d + i - 1.
\end{cases}
$$

Under these three types of boundary data, Li and Zhao [35] obtained the following results.

**Theorem A (Corollary 1.6 of [35]).** Assume that $D_1 \subset D \subset \mathbb{R}^d (d \geq 2)$ are defined as above, conditions (H1)–(H3) hold. Let $u \in H^1(D; \mathbb{R}^d) \cap C^1(\overline{D}; \mathbb{R}^d)$ be the solution of (2.3). Assume that one of (A1), (A2) and (A3) holds. If $\varphi \in C^2(\partial D; \mathbb{R}^d)$ satisfies the $k$-order growth condition,

$$
|\varphi(x)| \leq \eta |x|^k, \quad \text{on } \Gamma_R,
$$

for some integer $k > 0$ and a positive constant $\eta$. Then for a sufficiently small $\varepsilon > 0$,

$$
|\nabla u(x)| \leq \frac{C}{\varepsilon + |x|^m} \left[ \eta \rho_A(\varepsilon) + \frac{\|\varphi\|_{C^2(\partial D)}}{\rho_0(d, m; \varepsilon)} + |x^i| \left( \eta \rho_B(\varepsilon) + \frac{\|\varphi\|_{C^2(\partial D)}}{\rho_2(d, m; \varepsilon)} \right) \right]
$$

$$
+ \frac{\eta |x|^k}{\varepsilon + |x|^m} + C\|\varphi\|_{C^2(\partial D)}, \quad x \in \Omega_R,
$$

where

$$
\rho_A(\varepsilon) = \begin{cases} 
\rho_k(d, m; \varepsilon)/\rho_0(n, m; \varepsilon), & \text{for case (A1)}, \\
1/\rho_0(d, m; \varepsilon), & \text{otherwise},
\end{cases}
$$

$$
\rho_B(\varepsilon) = \begin{cases} 
\rho_{k+1}(d, m; \varepsilon)/\rho_2(n, m; \varepsilon), & \text{for case (A2)}, \\
1/\rho_2(d, m; \varepsilon), & \text{otherwise}.
\end{cases}
$$

From (2.12), we see that the singular behavior of $|\nabla u|$ is determined by the following three parts: $\rho_A(\varepsilon) \varepsilon^{-1}$, $\rho_B(\varepsilon) \varepsilon^{1/m - 1}$ and $\varepsilon^{k/m - 1}$ $(m > k)$. We point out that the blow-up rate $\rho_A(\varepsilon) \varepsilon^{-1}$ is achieved at the $(d - 1)$-dimensional ball $\{\|x\| = \sqrt{\varepsilon}\} \cap \Omega$, while, the latter two blow-up rates are generated on the cylinder surface $\{\|x\| = \sqrt{\varepsilon}\} \cap \Omega$. Furthermore, it follows from (2.12) that

(a) if condition (A1) holds, then for $m \leq d$, $k \geq 1$ or $m \geq d + k$, $k \geq 1$, $|\nabla u|$ blows up at the rate of $\frac{\rho_k(d, m; \varepsilon)}{\rho_0(d, m; \varepsilon)}$, while, for $d + k > m > d$, $k > 1$, the blow-up rate of $|\nabla u|$ is $\frac{1}{\varepsilon^{1/k - \rho_0(d, m; \varepsilon)}}$;

(b) if condition (A2) holds, then for $m \leq d$, $k \geq 1$, $|\nabla u|$ blows up at the rate of $\frac{1}{\varepsilon^{1/m - \rho_2(d, m; \varepsilon)}}$, while, for $m > d$, $k \geq 1$, its blow-up rate is $\frac{\rho_{k+1}(d, m; \varepsilon)}{\rho_2(d, m; \varepsilon)}$. In particular, when $m = d + k$, $k = 1$ or $m > d + k$, $k \geq 1$, we have $\frac{\rho_{k+1}(d, m; \varepsilon)}{\rho_2(d, m; \varepsilon)} = \varepsilon^{k/m - 1}$;

(c) if condition (A3) holds, then we obtain that for $m \leq d$, $k \geq 1$, the blow-up rate of $|\nabla u|$ is $\frac{1}{\varepsilon^{1/m - \rho_2(d, m; \varepsilon)}}$, for $m \geq d + k$, $k \geq 1$, $|\nabla u|$ blows up at the rate of $\frac{1}{\varepsilon^{1/m - \rho_2(d, m; \varepsilon)}}$; for $m \geq d + k$, $k \geq 1$, its blow-up rate is $\frac{1}{\varepsilon^{1/k - \rho_2(d, m; \varepsilon)}}$. Especially when $m = d + k$, $k \geq 1$, we get $\frac{1}{\varepsilon^{1/k - \rho_2(d, m; \varepsilon)}} = \frac{1}{\varepsilon^{1/k - \rho_2(d, m; \varepsilon)}}$. Especially when $m = d + k$, $k \geq 1$, we get $\frac{1}{\varepsilon^{1/k - \rho_2(d, m; \varepsilon)}} = \frac{1}{\varepsilon^{1/k - \rho_2(d, m; \varepsilon)}}$.
To show the optimality of the blow-up rates summarized in (a)–(c) above, we select three special examples from conditions (A1)–(A3) to establish the optimal upper and lower bounds on the blow-up rate of the gradient and meanwhile extract the accurate information of the boundary data $\varphi$, the Lamé constants $\lambda$ and $\mu$, and the curvature parameters $\tau_1$ and $\tau_2$ from the constant $C$ in (2.12). Specifically, suppose that for $x \in \Gamma_R$,

1. $\varphi^i = -\eta |x|^k$, $i = 1, 2, ..., d$,
2. $\varphi^i = 0$, $i = 1, ..., d - 1$, $\varphi^d = \eta x_1|x_1|^{k-1}$,
3. $\varphi^i = \eta x_i|x_i|^{k-1}$, $i = 1, ..., d - 1$, $\varphi^d = 0$,

where $\eta$ is a positive constant and $k$ is a positive integer. To ensure that $\varphi \in C^2(\partial D)$, if condition (E1) holds, we consider $k \geq 2$; if condition (E2) or (E3) holds, we consider $k \geq 1, k \neq 2$.

Let $\Omega^* := D \setminus \overline{D}_R$. For $\alpha, \beta = 1, 2, ..., \frac{d(d+1)}{2}$, define

$$a^*_{\alpha\beta} := \int_{\Omega^*} (C^0 e(u^*_\alpha), e(u^*_\beta)) dx, \quad Q^*_\alpha[\varphi] := \int_{\partial D_1^*} \frac{\partial u^*_\alpha}{\partial \nu} \cdot \psi_\alpha,$$  

where $\varphi \in C^2(\partial D; \mathbb{R}^d)$ and $u^*_\alpha \in C^2(\Omega^*; \mathbb{R}^d)$, $\alpha = 0, 1, ..., \frac{d(d+1)}{2}$, respectively, verify

$$\begin{align*}
L_{\lambda,\mu} u^*_0 = 0, \quad &\text{in } \Omega^*, \\
0 = 0, \quad &\text{on } \partial D_1^* \setminus \{0\}, \\
u^*_0 = \varphi(x), \quad &\text{on } \partial D, \\
\end{align*}$$

$$\begin{align*}
L_{\lambda,\mu} u^*_\alpha = 0, \quad &\text{in } \Omega^*, \\
u^*_\alpha = \psi_\alpha, \quad &\text{on } \partial D_1^* \setminus \{0\}, \\
u^*_\alpha = 0, \quad &\text{on } \partial D.
\end{align*}$$

We would like to remark that the definitions of $a^*_{\alpha\beta}$ and $Q^*_\alpha[\varphi]$ are valid only in some cases, see Lemmas 4.2 and 4.3 below. Suppose that for some $\kappa_5 > 0$,

$$\kappa_5 \leq \mu, d\lambda + 2\mu \leq \frac{1}{\kappa_5}. \quad (2.17)$$

Introduce the Lamé constants $L^*_d, \alpha = 1, 2, ..., \frac{d(d+1)}{2}$ as follows:

$$(L^*_1, L^*_2, L^*_3) = (\mu, \lambda + 2\mu, \lambda + 2\mu), \quad d = 2, \quad \quad (2.18)$$

$$(L^*_d, ..., L^*_d, L^*_d, ..., L^*_d, ..., L^*_d, ..., L^*_d, ..., L^*_d) = (\mu, ..., \mu, \lambda + 2\mu, ..., \lambda + 2\mu, 2\mu, ..., 2\mu), \quad d \geq 3. \quad \quad (2.19)$$

Before stating our main results, we first introduce the blow-up factor matrices as follows:

$$A^* = \begin{pmatrix}
a^*_1 & \cdots & a^*_d \\
\vdots & \ddots & \vdots \\
a^*_d & \cdots & a^*_d
\end{pmatrix}, \quad B^* = \begin{pmatrix}
a^*_1 & \cdots & a^*_{1 \frac{d(d+1)}{2}} \\
\vdots & \ddots & \vdots \\
a^*_d & \cdots & a^*_{\frac{d(d+1)}{2}}
\end{pmatrix}.$$
Without loss of generality, we set \( \phi \) \( \nabla \) from which it suffices to study the singular behavior of the gradient throughout this paper. Notice that following the standard elliptic theory (see [1,2]),

\[
\text{gap } \Omega \setminus \Omega_R \to 0 \quad \text{as } \epsilon \to 0.
\]

Our first main result is concerned with the optimal upper and lower bounds on the blow-up rate of the gradient in the shortest segment \( \{x' = 0'\} \cap \Omega \) between the inclusion and the matrix boundary.
Theorem 2.1. Assume that $D_1 \subset D \subset \mathbb{R}^d$ ($d \geq 2$) are defined as above, conditions (H1)–(H3) hold, and $\varphi \in C^2(\partial D; \mathbb{R}^d)$. Let $u \in H^1(D; \mathbb{R}^d) \cap C^1(\Omega; \mathbb{R}^d)$ be the solution of (2.3). Then for a sufficiently small $\varepsilon > 0$, $x \in \{x' = 0\} \cap \Omega$, (i) if condition (E1), (E2) or (E3) holds for $m \leq d$, then

$$
|\nabla u| \gtrsim \begin{cases}
\frac{\max_{1 \leq \alpha \leq d} \frac{d-1}{\kappa_1} |L_2^{-1}| \det F_1^{\alpha}[\varphi]|}{\varepsilon \rho_0(d, m; \varepsilon)}, & d - 1 \leq m \leq d, \\
\frac{\max_{1 \leq \alpha \leq d} |\det F_3^{\alpha}[\varphi]|}{\varepsilon \rho_0(d, m; \varepsilon)}, & m < d - 1,
\end{cases}
$$

and, there exists some integer $1 \leq \alpha_0 \leq d$ such that $\det F_1^{\alpha_0}[\varphi] \neq 0$ and $\det F_3^{\alpha_0}[\varphi] \neq 0$,

$$
|\nabla u| \gtrsim \begin{cases}
\frac{\max_{1 \leq \alpha \leq d} \frac{d-1}{\kappa_1} |L_2^{-1}| \det F_3^{\alpha}[\varphi]|}{\varepsilon \rho_0(d, m; \varepsilon)}, & d - 1 \leq m \leq d, \\
\frac{\max_{1 \leq \alpha \leq d} |\det F_3^{\alpha}[\varphi]|}{\varepsilon \rho_0(d, m; \varepsilon)}, & m < d - 1;
\end{cases}
$$

(ii) if condition (E1) holds for $m \geq d + k$, then

$$
\frac{\eta_{d+1}}{\kappa_1} \frac{\rho_k(d, m; \varepsilon)}{\epsilon \rho_0(d, m; \varepsilon)} \lesssim |\nabla u| \lesssim \frac{\eta_{d+1}}{\kappa_2} \frac{\rho_k(d, m; \varepsilon)}{\epsilon \rho_0(d, m; \varepsilon)},
$$

where $\kappa_i$, $i = 1, 2$ are defined in condition (H1), $L_2^\alpha$, $\alpha = 1, 2, \ldots, d$ are defined by (2.1)–(2.10), $\rho_i(d, m; \varepsilon)$, $i = 0, k$ are defined in (2.10), the blow-up factor matrices $D^\alpha$, $F^\alpha$, $A_1^\alpha$, $\alpha = 1, 2, \ldots, d$ and $F_3^{\alpha}[\varphi]$, $\alpha = 1, 2, \ldots, \frac{d(d+1)}{2}$ are defined by (2.20)–(2.23), respectively.

Remark 2.2. Our results in Theorems 2.1 and 2.6 not only answer the optimality of the blow-up rate of the gradient in all dimensions, but also improve the results in Theorems 6.10 and 6.1 of [35] by accurately capturing the blow-up factor matrices and revealing the explicit dependence on the Lamé constants $L_3^\alpha$ and the curvature parameters $\kappa_1$ and $\kappa_2$.

Remark 2.3. The assumed condition $\det F_1^{\alpha_0} \neq 0$ or $\det F_3^{\alpha_0}[\varphi] \neq 0$ implies that $\varphi \neq 0$ on $\partial D$. Otherwise, if $\varphi = 0$ on $\partial D$, then it follows from integration by parts that $Q^*_{\alpha_0}[\varphi] = \int_{\partial D} \frac{\partial \varphi}{\partial n_\alpha}[\varphi] = 0$. This yields that $\det F_1^{\alpha_0}[\varphi] = 0$ and $\det F_3^{\alpha_0}[\varphi] = 0$, which provides a contradiction. Although it is difficult to demonstrate the assumed condition $\det F_1^{\alpha_0}[\varphi] \neq 0$ or $\det F_3^{\alpha_0}[\varphi] \neq 0$ for any given boundary data $\varphi$, it is an interesting problem to analyze these blow-up factor matrices by numerical computations and simulations.

Remark 2.4. In Theorem 2.1, for the purpose of constructing the lower bound on the gradient in the case of $m < d - 1$, the blow-up factor matrix $F^\alpha[\varphi]$, as a whole, is required to be non-zero, which means that there exists at least one non-zero element in the column vector $(Q^*[\varphi], Q^*[\varphi], \ldots, Q^*[\varphi])^T$. This weakens the assumed condition (Phi) of Theorem 1.10 in [35] that for $m < d - 1$, there exists some integer $1 \leq k_0 \leq d$ such that $Q^*_k[\varphi] \neq 0$ and $Q^*_k[\varphi] = 0$ for all $\beta \neq k_0$.

The next theorem aims to establish the optimal gradient estimates on the cylinder surface $\{|x'| = \sqrt{\varepsilon}\} \cap \Omega$. Similarly as before, we first introduce some blow-up factor matrices. For $\alpha = d + 1, \ldots, \frac{d(d+1)}{2}$, after replacing the elements of $\alpha$-th column in the matrix $D^\alpha$ by column vector $(Q^*_{d+1}[\varphi], \ldots, Q^*_d[\varphi])^T$, we obtain the
new matrix $F^*_2[\varphi]$ as follows:

$$F^*_2[\varphi] =:\begin{pmatrix}
    a^*_{d+1,d+1} & \cdots & Q^*_d[\varphi] & \cdots & a^*_{d+1,d(d+1)} \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    a^*_d & \cdots & Q^*_d[\varphi] & \cdots & a^*_d d(d+1)
\end{pmatrix}. \tag{2.24}
$$

Then our second main theorem is stated as follows:

**Theorem 2.5.** Assume that $D_1 \subset D \subseteq \mathbb{R}^d \ (d \geq 2)$ are defined as above, conditions (H1)-(H3) hold, and $\varphi \in C^2(\partial D; \mathbb{R}^d)$. Let $u \in H^1(D; \mathbb{R}^d) \cap C^1(\overline{D}; \mathbb{R}^d)$ be the solution of (2.3). Then for a sufficiently small $\varepsilon > 0$, $x \in \{x' = (\nabla \varepsilon, 0, \ldots, 0)\} \cap \Omega$,

(i) if condition (E1), (E2) or (E3) holds for $d < m < d + k$, $k > 1$, then

$$|\nabla u| \lesssim \begin{cases}
    \max_{d+1 \leq \alpha \leq d(d+1)} \frac{|L^*_d|^{-1}|Q^*_\alpha[\varphi]|^{\frac{d+1}{d} \kappa_2}}{1+\kappa_1 \varepsilon^{-1-k/m} \rho_2(d,m;\varepsilon)}, & d + 1 \leq m < d + k, \\
    \max_{d+1 \leq \alpha \leq d(d+1)} \frac{|Q^*_\alpha[\varphi]| \rho_2(d,m;\varepsilon)}{(1+\kappa_1) \det F^*_2[\varphi]^{\frac{1}{1+\kappa_1}} \varepsilon^{-1-k/m}}, & d < m < d + 1,
\end{cases}
$$

and, under the condition of $Q^*_{d+1}[\varphi] \neq 0$ and $F^*_{d+1}[\varphi] \neq 0$,

$$|\nabla u| \gtrsim \begin{cases}
    \frac{\eta}{(1+\kappa_2) \kappa_2^{d+1}} \frac{1}{\varepsilon^{1-k/m}}, & d + 1 \leq m < d + k, \\
    \frac{\eta}{(1+\kappa_1) \kappa_1^{d+1}} \frac{1}{\varepsilon^{1-k/m}}, & d < m < d + 1,
\end{cases}
$$

(ii) if condition (E2) holds, then for $m > d + k$, $k \geq 1$, $k \neq 2$ or $m = d + k$, $k = 1$,

$$\eta \frac{(d+1)^{\frac{d+1}{d}}}{1+\kappa_2} \frac{1}{\varepsilon^{1-k/m}} \lesssim |\nabla u| \lesssim \eta \frac{(d+1)^{\frac{d+1}{d}}}{1+\kappa_2} \frac{1}{\varepsilon^{1-k/m}},$$

and, for $m = d + k$, $k > 2$,

$$\frac{\eta \kappa_1^{d+1}}{(1+\kappa_2) \kappa_2^{d+1}} \frac{1}{\varepsilon^{1-k/m}} \rho_2(d,m;\varepsilon) \lesssim |\nabla u| \lesssim \frac{\eta \kappa_1^{d+1}}{(1+\kappa_1) \kappa_1^{d+1}} \frac{1}{\varepsilon^{1-k/m}} \rho_2(d,m;\varepsilon).$$

(iii) if condition (E3) holds, then for $m > d + k$, $k \geq 1$, $k \neq 2$ or $m = d + k$, $k = 1$,

$$\frac{\eta}{1+\kappa_2} \frac{1}{\varepsilon^{1-1/m}} \lesssim |\nabla u| \lesssim \frac{\eta}{1+\kappa_1} \frac{1}{\varepsilon^{1-k/m}}, \tag{2.25}
$$

and, for $m = d + k$, $k > 2$,

$$\max_{d+1 \leq \alpha \leq d(d+1)} \frac{\kappa_2^{d+1}}{1+\kappa_1} \frac{|L^*_d|^{-1}|Q^*_\alpha[\varphi]| + \eta}{\varepsilon^{1-k/m}},$$

and, under the condition of $Q^*_{d+1}[\varphi] \neq 0$,

$$|\nabla u| \gtrsim \frac{1}{\varepsilon^{1-k/m}} \kappa_1^{d+1} |Q^*_{d+1}[\varphi]| \frac{1}{(1+\kappa_2)L^{d+1}_d},$$

where $\kappa_i, i = 1, 2$ are defined in condition (H1), the blow-up factors $Q^*_\alpha[\varphi]$ and the Lamé constants $L^*_d$, $\alpha = d + 1, \ldots, \frac{d(d+1)}{2}$ are, respectively, defined by (2.15)
and \((2.18)-(2.19)\), \(\rho_i(d, m; \varepsilon), i = 2, k + 1\) are defined in \((2.10)\), the blow-up factor matrices \(D^*\) and \(P_1^*\[\varphi]\), \(\alpha = d + 1, \ldots, \frac{d(d + 1)}{2}\) are defined by \((2.20)\) and \((2.21)\), respectively.

Remark 2.6. The optimal gradient estimates in Theorem \(2.4\) address the remaining optimality of the blow-up rate in Remark 6.5 of [35], which makes complete the optimality of the blow-up rate in Remark 2.6.

Remark 2.6. The optimal gradient estimates in Theorem \(2.4\) address the remaining optimality of the blow-up rate in Remark 6.5 of [35], which makes complete the optimality of the blow-up rate on the cylinder surface \(\{|x'| = \sqrt{\varepsilon}\} \cap \Omega\). In addition, it is worth mentioning that as shown in \((2.25)\), if condition (E3) holds, the leading singularity of \(|\nabla u|\) only arises from \(|\nabla u_0|\) in the case of \(m > d + k, k \geq 1, k \neq 2\) or \(m = d + k, k = 1\). This is different from the blow-up phenomenon under condition (E1) or (E2).

By applying the proofs of Theorems \(2.1\) and \(2.5\) with a slight modification, we obtain the following two corollaries. Before stating the first corollary, we first introduce two notations \(H^*_A(m, d, k; \varphi)\) and \(H^*_B(m, d, k; \varphi)\). To be specific, define

\[
H^*_A(m, d, k; \varphi) := \begin{cases}
\eta \kappa_2^{-\frac{d+1}{m}} \kappa_1^{-\frac{d+k}{m}}, & m \geq d + k - 1, \text{ for case (A1),} \\
\max_{1 \leq \alpha \leq d} \kappa_2^{\frac{d+1}{m}} (L^*_d)^{-1} |Q^*_\alpha[\varphi]|, & m \geq d + k - 1, \text{ for case (A2) or (A3),} \\
\max_{1 \leq \alpha \leq d} \kappa_2^{\frac{d+1}{m}} (L^*_d)^{-1} |Q^*_\alpha[\varphi]|, & d + 1 \leq m < d + k - 1, \\
\max_{1 \leq \alpha \leq d} \kappa_2^{\frac{d+1}{m}} (L^*_d)^{-1} |det P^*_{\alpha}[\varphi]|, & d - 1 \leq m < d + 1, \\
\max_{1 \leq \alpha \leq d} \kappa_2^{\frac{d+1}{m}} (L^*_d)^{-1} |det P^*_{\alpha}[\varphi]|, & m < d - 1,
\end{cases}
\]

and

\[
H^*_B(m, d, k; \varphi) := \begin{cases}
\eta \kappa_2^{-\frac{d+1}{m}} \kappa_1^{-\frac{d+k}{m}}, & m \geq d + k, \text{ for case (A2),} \\
\max_{d+1 \leq \alpha \leq d+\frac{d+1}{2}} \kappa_2^{-\frac{d+1}{m}} (L^*_d)^{-1} |Q^*_\alpha[\varphi]|, & m \geq d + k, \text{ for case (A1) or (A3),} \\
\max_{d+1 \leq \alpha \leq d+\frac{d+1}{2}} \kappa_2^{-\frac{d+1}{m}} (L^*_d)^{-1} |Q^*_\alpha[\varphi]|, & d + 1 \leq m < d + k, \\
\max_{d+1 \leq \alpha \leq d+\frac{d+1}{2}} |det P^*_{\alpha}[\varphi]|, & d - 1 \leq m < d + 1, \\
\max_{d+1 \leq \alpha \leq d+\frac{d+1}{2}} |det P^*_{\alpha}[\varphi]|, & m < d - 1.
\end{cases}
\]

Then the first corollary is listed as follows:

**Corollary 2.7.** Assume that \(D_1 \subseteq D \subseteq \mathbb{R}^d (d \geq 2)\) are defined as above, conditions (H1)–(H3) hold. Let \(u \in H^3(D; \mathbb{R}^d) \cap C^1(\Omega; \mathbb{R}^d)\) be the solution of \((2.3)\). Assume that (A1), (A2) or (A3) holds. If \(\varphi \in C^2(\partial D; \mathbb{R}^d)\) satisfies the \(k\)-order growth condition \((2.11)\), then for a sufficiently small \(\varepsilon > 0\), \(x \in \Omega_R\),

\[
|\nabla u| \lesssim \frac{1}{\varepsilon + \kappa_1 |x'|^m} \left( H^*_A(m, d, k; \varphi) \rho_A(\varepsilon) + H^*_B(m, d, k; \varphi) \rho_B(\varepsilon) |x'| + \eta |x'|^k \right),
\]

where \(\rho_A(\varepsilon)\) and \(\rho_B(\varepsilon)\) are, respectively, defined by \((2.13)-(2.14)\), \(H^*_A(m, d, k; \varphi)\) and \(H^*_B(m, d, k; \varphi)\) are defined by \((2.20)-(2.27)\), respectively.
Remark 2.8. In contrast to [21], we improve its result by making clear the dependence on the blow-up factor matrices, the Lamé constants $L^2_\beta$ and the curvature parameters $\kappa_1$ and $\kappa_2$ as shown in $\mathcal{H}_A(m, d, k; \varphi)$ and $\mathcal{H}_B(m, d, k; \varphi)$.

Our results can also be extended to the inclusions with partially flat boundaries as follows:

(S1) $\kappa_1 \text{dist}^m(x', \Sigma') \leq h_1(x') - h(x') \leq \kappa_2 \text{dist}^m(x', \Sigma')$, if $x' \in B_{2R} \setminus \Sigma'$,

(S2) $|\nabla^1_x h_1(x')|, |\nabla^1_x h(x')| \leq \kappa_3 \text{dist}^{m-1}(x', \Sigma')$, if $x' \in B_{2R}$, $j = 1, 2$.

For the convenience of notation, let $a \approx b$ represent that both $a \lesssim b$ and $a \gtrsim b$ hold.

For $i = 0, 2, k, k + 1$ and $j = 1, 2$, denote

$$
G^i_j(|\Sigma'|) := |\Sigma'|^\frac{d+1-i}{d+i} + |\Sigma'| \frac{d-2+i}{d+i} \kappa_j \epsilon^\frac{1}{k} + \kappa_j \frac{d-1+i}{k} \epsilon^\frac{1}{k} \rho_i(d, m; \epsilon). \tag{2.28}
$$

With this notation, we state the second corollary as follows:

Corollary 2.9. Assume that $D_1 \subset D \subseteq \mathbb{R}^d (d \geq 2)$ are defined as above, conditions (S1)–(S2) and (H3) hold with $\Sigma' = B'_2(0')$ for some $0 < r < R$. Let $u \in H^1(D; \mathbb{R}^d) \cap C^1(\overline{\Omega}; \mathbb{R}^d)$ be the solution of (2.28). Then for a sufficiently small $\epsilon > 0$, $x \in \{x' = (r, 0, \ldots, 0)'\} \cap \Omega$,

(i) if condition (E1) holds,

$$
\left(\frac{G^i_k(|\Sigma'|)}{G^i_{k+1}(\Sigma')} + |\Sigma'|^\frac{i}{d+i} \right) \frac{\eta}{\epsilon} \lesssim |\nabla u| \lesssim \left(\frac{G^i_k(|\Sigma'|)}{G^i_{k+1}(\Sigma')} + |\Sigma'|^\frac{i}{d+i} \right) \frac{\eta}{\epsilon},
$$

(ii) if condition (E2) holds,

$$
\left(\frac{|\Sigma'|^\frac{i}{d+i} G^i_{k+1}(\Sigma')} {G^i_k(\Sigma')} + |\Sigma'|^\frac{i}{d+i} \right) \frac{\eta}{\epsilon} \lesssim |\nabla u| \lesssim \left(\frac{|\Sigma'|^\frac{i}{d+i} G^i_{k+1}(\Sigma')} {G^i_2(\Sigma')} + |\Sigma'|^\frac{i}{d+i} \right) \frac{\eta}{\epsilon},
$$

(iii) if condition (E3) holds,

$$
|\nabla u| \approx \frac{\eta|\Sigma'|^\frac{i}{d+i}}{\epsilon},
$$

where $G^i_j(|\Sigma'|)$, $i = 0, 2, k, k + 1$, $j = 1, 2$ are defined by (2.28).

Remark 2.10. In fact, (2.28) can be rewritten as $G^i_j(|\Sigma'|) = |\Sigma'|^\frac{i}{d+i} + O(\epsilon)$, which implies that if condition (E1), (E2) or (E3) holds, then for $x \in \{x' = (r, 0, \ldots, 0)'\} \cap \Omega$, the results in Corollary 2.9 have an unified expression as follows:

$$
|\nabla u| \approx \frac{\eta|\Sigma'|^\frac{i}{d+i}}{\epsilon}.
$$

Remark 2.11. It is worth mentioning that in view of decomposition (3.2) below, we conclude from Corollary 2.9 that if condition (E1) holds, the singular behavior of the gradient is determined by the first part $\sum_{\alpha=1}^d C^\alpha \nabla u_\alpha$ and the third part $\nabla u_0$ together; if condition (E2) holds, the major singularity of the gradient lies in the second part $\sum_{\alpha=d+1}^d C^\alpha \nabla u_\alpha$ and the third part $\nabla u_0$; if condition (E3) holds, the singular behavior of the gradient is determined only by the third part $\nabla u_0$. These fact indicates that when the inclusions with partially flat boundaries are closely located at the matrix boundary, $\nabla u_0$ possesses the largest blow-up rate $\epsilon^{-1}$ under these three cases all the time, which is different from the blow-up phenomenon with $m$-convex inclusions as seen in Theorems 2.11 and 2.13. Moreover, this blow-up phenomena also differs from that in the interior estimates of [21], where Hou, Ju
and Li proved that there appears no blow-up of the gradient in the presence of two nearby inclusions with partially flat boundaries for any given boundary data.

3. Preliminary

As shown in [8,35], the solution of (2.3) can be decomposed as follows:

\[ u = \sum_{\alpha=1}^{d} C^{\alpha} u_{\alpha} + \sum_{\alpha=d+1}^{d(d+1)} C^{\alpha} u_{\alpha} + u_{0}, \quad \text{in } \Omega, \]

where the free constants \( C^{\alpha}, \alpha = 1, 2, ..., \frac{d(d+1)}{2} \), will be determined later by utilizing the third line of (2.3), and \( u_{\alpha} \in C^{1}(\overline{\Omega}; \mathbb{R}^{d}) \cap C^{2}(\Omega; \mathbb{R}^{d}), \alpha = 0, 1, 2, ..., \frac{d(d+1)}{2} \), verify

\[
\begin{align*}
\mathcal{L}_{\lambda,\mu} u_{0} &= 0, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial D_{1}, \\
u &= \varphi, \quad \text{on } \partial D, \\
u &= \psi_{\alpha}, \quad \text{on } \partial D_{1}, \\
u &= 0, \quad \text{on } \partial D,
\end{align*}
\]

respectively. Therefore,

\[ \nabla u = \sum_{\alpha=1}^{d} C^{\alpha} \nabla u_{\alpha} + \sum_{\alpha=d+1}^{d(d+1)} C^{\alpha} \nabla u_{\alpha} + \nabla u_{0}, \quad \text{in } \Omega. \] (3.2)

With this, we transfer the original problem to the estimates of the free constants \( C^{\alpha}, \alpha = 1, 2, ..., \frac{d(d+1)}{2} \) and the gradients \( \nabla u_{\alpha}, \alpha = 0, 1, 2, ..., \frac{d(d+1)}{2} \). Moreover, decomposition (3.2) splits \( \nabla u \) into three parts, that is, \( \sum_{\alpha=1}^{d} C^{\alpha} \nabla u_{\alpha}, \sum_{\alpha=d+1}^{d(d+1)} C^{\alpha} \nabla u_{\alpha} \) and \( \nabla u_{0} \). Then it suffices to compare the singularities of these three parts and then identify the largest blow-up rate of them.

For the purpose of studying the singular behavior of \( \nabla u_{\alpha}, \alpha = 0, 1, 2, ..., \frac{d(d+1)}{2} \), we first consider a general boundary value problem as follows:

\[
\begin{align*}
\mathcal{L}_{\lambda,\mu} v &:= \nabla \cdot (C^{0} e(v)) = 0, \quad \text{in } \Omega, \\
v &= \psi(x), \quad \text{on } \partial D_{1}, \\
v &= \phi(x), \quad \text{on } \partial D,
\end{align*}
\]

where \( \psi \in C^{2}(\partial D_{1}; \mathbb{R}^{d}) \) and \( \phi \in C^{2}(\partial D; \mathbb{R}^{d}) \) are two given vector-valued functions.

Introduce a vector-valued auxiliary function as follows:

\[
\begin{align*}
\tilde{v} &= \psi(x', \varepsilon + h_{1}(x')) \tilde{v} + \phi(x', h(x')) (1 - \tilde{v}) \\
&\quad + \frac{\lambda + \mu}{\mu} f(\tilde{v})(\psi^{d}(x', \varepsilon + h_{1}(x')) - \phi^{d}(x', h(x')) \sum_{i=1}^{d-1} \partial_{x_{i}} \delta e_{i} \\
&\quad + \frac{\lambda + \mu}{\lambda + 2\mu} f(\tilde{v}) \sum_{i=1}^{d-1} \partial_{x_{i}} \delta(\psi^{d}(x', \varepsilon + h_{1}(x')) - \phi^{d}(x', h(x'))) e_{d},
\end{align*}
\]

where \( \tilde{v} \) is defined by (2.4), \( \delta \) and \( f(\tilde{v}) \) are defined in (2.5). For the remaining term, write

\[
\begin{align*}
\mathcal{R}_{\delta}(\psi, \phi) &= ||\psi(x', \varepsilon + h_{1}(x')) - \phi(x', h(x'))||_{C^{2}(\partial D_{1})}^{\frac{\mu + 2}{\mu}} + \delta \left( ||\psi||_{C^{2}(\partial D_{1})} + ||\phi||_{C^{2}(\partial D_{2})} \right) \\
&\quad + ||\nabla x'_{\varepsilon}(\psi(x', \varepsilon + h_{1}(x')) - \phi(x', h(x')))||.
\end{align*}
\] (3.5)
Theorem 3.1. Assume as above. Let \( v \in H^1(\Omega; \mathbb{R}^d) \) be a weak solution of (3.3). Then for a sufficiently small \( \varepsilon > 0 \),
\[
\nabla v = \nabla \bar{v} + O(1)\mathcal{R}_\delta(\psi, \phi), \quad \text{in} \ \Omega_R,
\]
where \( \delta \) is defined in (2.5), the leading term \( \bar{v} \) and the remaining term \( \mathcal{R}_\delta(\psi, \phi) \) are defined by (3.4), respectively.

For readers’ convenience, the proof of Theorem 3.1 is left in the Appendix.

To end this section, we recall some properties of the tensor \( \mathbb{C}^0 \). For the isotropic elastic material, set
\[
\mathbb{C}^0 := (C^0_{ijkl}) = (\lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})), \quad \mu > 0, \quad d\lambda + 2\mu > 0,
\]
whose components \( C^0_{ijkl} \) possess the symmetry property as follows:
\[
C^0_{ijkl} = C^0_{klij} = C^0_{klji}, \quad i, j, k, l = 1, 2, \ldots, d. \tag{3.6}
\]
For every pair of \( d \times d \) matrices \( A = (A_{ij}) \) and \( B = (B_{ij}) \), we introduce the following notations:
\[
(C^0 A)_{ij} = \sum_{k,l=1}^n C^0_{ijkl} A_{kl}, \quad \text{and} \quad (A, B) = A : B = \sum_{i,j=1}^n A_{ij} B_{ij}.
\]
Obviously,
\[
(C^0 A, B) = (A, C^0 B).
\]
Using (3.6), we get that \( \mathbb{C}^0 \) verifies the ellipticity condition: for every \( d \times d \) real symmetric matrix \( \xi = (\xi_{ij}) \),
\[
\min\{2\mu, d\lambda + 2\mu\}|\xi|^2 \leq (\mathbb{C}^0 \xi, \xi) \leq \max\{2\mu, d\lambda + 2\mu\}|\xi|^2,
\]
where \( |\xi|^2 = \sum_{ij} \xi_{ij}^2 \). Especially,
\[
\min\{2\mu, d\lambda + 2\mu\}|A + A^T|^2 \leq (\mathbb{C}^0 (A + A^T), (A + A^T)). \tag{3.7}
\]
Additionally, it is well known that for any open set \( O \) and \( u, v \in C^2(O; \mathbb{R}^d) \),
\[
\int_O (\mathbb{C}^0 e(u), e(v)) \, dx = -\int_O (\mathcal{L}_\mu u) \cdot v + \int_{\partial O} \frac{\partial u}{\partial \nu_0} \cdot v. \tag{3.8}
\]

4. Proofs of Theorems 2.1 and 2.5

Taking \( \psi = \psi_\alpha, \phi = 0, \alpha = 1, 2, \ldots, \frac{d(d+1)}{2} \), or \( \psi = 0, \phi = \varphi \) in Theorem 3.1, we have

Corollary 4.1. Assume as above. Let \( u_\alpha \in H^1(\Omega; \mathbb{R}^d), \alpha = 1, 2, \ldots, \frac{d(d+1)}{2} \) be a weak solution of (3.3). Then, for a sufficiently small \( \varepsilon > 0 \), \( x \in \Omega_R \),
\[
\nabla u_\alpha = \nabla \bar{u}_\alpha + O(1) \begin{cases} \|\varphi\|_{C^2(\partial D)}, & \alpha = 0, \\ \delta \frac{\|\varphi\|_{C^2(\partial D)}}{m}, & \alpha = 1, 2, \ldots, d, \\ 1, & \alpha = d + 1, \ldots, \frac{d(d+1)}{2}, \end{cases} \tag{4.1}
\]
and
\[
\|\nabla u_\alpha\|_{L^\infty(\Omega \setminus \Omega_R)} = O(1) \begin{cases} \|\varphi\|_{C^2(\partial D)}, & \alpha = 0, \\ 1, & \alpha = 1, 2, \ldots, \frac{d(d+1)}{2}. \end{cases}
\]
where \( \tilde{u}_\alpha, \alpha = 0, 1, 2, \ldots, \frac{d(d+1)}{2} \), are defined in \((2.6) - (2.7)\). \(O(1)\) denotes some quantity satisfying \(|O(1)| \leq C\) for some \(\varepsilon\)-independent \(C\).

Recalling decomposition \((3.2)\), it remains to provide a precise calculation for the free constants \(C^\alpha, \alpha = 1, 2, \ldots, \frac{d(d+1)}{2}\) in the following. Observe that by utilizing the third line of \((2.3)\), it follows from \((3.2)\) that

\[
\sum_{\alpha=1}^{d(d+1)} C^\alpha a_{\alpha\beta} = Q_\beta[\varphi], \quad \beta = 1, 2, \ldots, \frac{d(d+1)}{2},
\]

where, for \(\alpha, \beta = 1, 2, \ldots, \frac{d(d+1)}{2}\),

\[a_{\alpha\beta} := -\int_{\partial D_1} \frac{\partial u_\alpha}{\partial \nu_0} \cdot \psi_\beta, \quad Q_\alpha[\varphi] := \int_{\partial D_1} \frac{\partial u_\alpha}{\partial \nu_0} \cdot \psi_\alpha.
\]

In light of \((4.2)\), we need to calculate each element \(a_{\alpha\beta}\) and every blow-up factor \(Q_\beta[\varphi]\) for the purpose of estimating the free constants \(C^\alpha, \alpha = 1, 2, \ldots, \frac{d(d+1)}{2}\).

### 4.1. Estimates and Asymptotics of \(Q_\alpha[\varphi]\), \(\alpha = 1, 2, \ldots, \frac{d(d+1)}{2}\)

To begin with, the unit outer normals of \(\partial D_1\) and \(\partial D\) near the origin are, respectively, written by

\[
\nu := (\nu_1, \nu_2, \ldots, \nu_d) = \left( -\frac{\nabla \cdot \nu h_1}{\sqrt{1 + |\nabla \cdot \nu h_1|^2}}, \frac{1}{\sqrt{1 + |\nabla \cdot \nu h_1|^2}} \right),
\]

and

\[
\nu := (\nu_1, \nu_2, \ldots, \nu_d) = \left( \frac{\nabla \cdot \nu h}{\sqrt{1 + |\nabla \cdot \nu h|^2}}, \frac{-1}{\sqrt{1 + |\nabla \cdot \nu h|^2}} \right),
\]

**Lemma 4.2.** Assume as above. Then for a sufficiently small \(\varepsilon > 0\),

(i) for \(\alpha = 1, 2, \ldots, d\), if condition \((E1)\) holds,

\[
\left\{ \begin{array}{l}
\kappa_2^{-\frac{d+k-1}{m}} \leq \frac{Q_\alpha[\varphi]}{\varepsilon^{\eta_{d+1}(d,m,\varepsilon)}} \leq \kappa_1^{-\frac{d+k-1}{m}}; \\
Q_\alpha[\varphi] = Q_\alpha^*[\varphi] + O(1)\varepsilon^{\frac{d+k-1}{m(m+c+\varepsilon)}},
\end{array} \right.
\]

and, if condition \((E2)\) or \((E3)\) holds,

\[
Q_\alpha[\varphi] = Q_\alpha^*[\varphi] + O(1)\varepsilon^{\frac{d+k-2}{m(m+c+\varepsilon)}};
\]

(ii) for \(\alpha = d+1\), if condition \((E2)\) holds,

\[
\left\{ \begin{array}{l}
\kappa_2^{-\frac{d+k}{m}} \leq \frac{Q_{d+1}[\varphi]}{\varepsilon^{\eta_{d+1}(d,m,\varepsilon)}} \leq \kappa_1^{-\frac{d+k}{m}}; \\
Q_{d+1}[\varphi] = Q_{d+1}^*[\varphi] + O(1)\varepsilon^{\frac{d+k-1}{m(m+c+\varepsilon)}},
\end{array} \right.
\]

and, if condition \((E1)\) or \((E3)\) holds,

\[
Q_{d+1}[\varphi] = Q_{d+1}^*[\varphi] + O(1)\varepsilon^{\frac{d+k-1}{m(m+c+\varepsilon)}};
\]

(iii) for \(\alpha = d+2, \ldots, \frac{d(d+1)}{2} - 1\), \(d \geq 3\), if condition \((E1)\), \((E2)\) or \((E3)\) holds,

\[
Q_\alpha[\varphi] = Q_\alpha^*[\varphi] + O(1)\varepsilon^{\frac{d+k-1}{m(m+c+\varepsilon)}},
\]

where \(Q_\alpha^*[\varphi]\), \(\alpha = 1, 2, \ldots, \frac{d(d+1)}{2}\) are defined in \((2.15)\).
Proof of Lemma 4.2. Step 1. Proofs of (4.5)–(4.6). We divide into two subparts to prove (4.5)–(4.6) in the following.

Step 1.1. By definition, we have

\[ Q_{\alpha}[\varphi] = \sum_{i=1}^{d} \int_{\partial D_1} \frac{\partial u_{0i}}{\partial \nu_0} | + \cdot \psi_{\alpha} \]

Then in view of (4.3), it follows from Corollary 4.1 that

\[ \text{Step 1.2.} \text{From (3.8), we deduce that for } (a) \text{ for } \alpha = 1, \ldots, d - 1, \]

\[ Q_{\alpha}[\varphi] \lesssim \eta \int_{|x'| R} \frac{|x'|^k}{\varepsilon + \kappa_1 |x'|^m} \lesssim \eta \int_{0}^{R} \frac{s^{d+k-2}}{\varepsilon + \kappa_2 s^m} \lesssim \frac{\eta}{\kappa_1^{\frac{m}{m+1}}} \rho_k(d, m; \varepsilon), \]

and

\[ Q_{\alpha}[\varphi] \gtrsim \eta \int_{|x'| R} \frac{|x'|^k}{\varepsilon + \kappa_2 |x'|^m} \gtrsim \eta \int_{0}^{R} \frac{s^{d+k-2}}{\varepsilon + \kappa_2 s^m} \gtrsim \frac{\eta}{\kappa_2^{\frac{m}{m+1}}} \rho_k(d, m; \varepsilon); \]

(b) for \( \alpha = d, \)

\[ Q_{d}[\varphi] = \int_{\partial D_1} \left[ \sum_{(i,j) \neq (d,d)} \left( \lambda \partial_{x_j} u_{0i}^{d} \nu_d + \mu (\partial_{x_d} u_{0i}^{d} + \partial_{x_d} \bar{u}_{0d}) \nu_d \right) \right. \]

\[ + \left. (\lambda + 2 \mu) \partial_{x_d} (u_{0d}^{d} - \bar{u}_{0d}) \nu_d \right] + \int_{\partial D_1} (\lambda + 2 \mu) \partial_{x_d} \bar{u}_{0d} \nu_d \]

\[ = - (\lambda + 2 \mu) \int_{\partial D_1 \cap \Gamma_{R}^+} \frac{\varphi^{d}(x', h(x'))}{\varepsilon + h_1(x') - h(x')} \nu_d + O(1)\|\varphi\|_{C^2(\partial D)}, \]

which yields that

\[ Q_{d}[\varphi] \lesssim (\lambda + 2 \mu) \eta \int_{|x'| < R} \frac{|x'|^k}{\varepsilon + \kappa_1 |x'|^m} \lesssim \frac{(\lambda + 2 \mu) \eta}{\kappa_1^{\frac{m}{m+1}}} \rho_k(d, m; \varepsilon), \]

and

\[ Q_{d}[\varphi] \gtrsim (\lambda + 2 \mu) \eta \int_{|x'| < R} \frac{|x'|^k}{\varepsilon + \kappa_2 |x'|^m} \gtrsim \frac{(\lambda + 2 \mu) \eta}{\kappa_2^{\frac{m}{m+1}}} \rho_k(d, m; \varepsilon). \]

Step 1.2. From (3.8), we deduce that for \( \alpha = 1, 2, \ldots, d, \)

\[ Q_{\alpha}[\varphi] - Q_{\alpha}^{*}[\varphi] = \int_{\partial D} \frac{\partial (u_{0\alpha} - u_{0\alpha}^*)}{\partial \nu_0} | + \cdot \varphi(x), \]
where \( u^*_\alpha \) and \( u_\alpha \) are defined by (2.16) and (3.1), respectively.

For \( 0 < t \leq 2R \), denote \( \Omega^*_t := \Omega^* \cap \{|x'| < t\} \). Introduce a scalar auxiliary function \( \tilde{v}^* \in C^2(\mathbb{R}^d) \) such that \( \tilde{v}^* = 1 \) on \( \partial D^*_1 \setminus \{0\} \), \( \tilde{v}^* = 0 \) on \( \partial D \), and

\[
\tilde{v}^*(x', x_d) = \frac{x_d - h(x')}{h_1(x') - h(x')} \quad \text{in } \Omega^*_2R, \quad ||\tilde{v}^*||_{C^2(\Omega^* \setminus \Omega^*_R)} \leq C.
\]

Define a family of auxiliary functions as follows:

\[
\bar{u}^*_\alpha = \psi_\alpha \tilde{v}^* + \mathcal{F}^*_\alpha, \quad \alpha = 1, 2, \ldots, \frac{d(d + 1)}{2},
\]

where

\[
\mathcal{F}^*_\alpha = \frac{\lambda + \mu}{\mu} f(\tilde{v}^*) \psi^d_\alpha \sum_{i=1}^{d-1} \partial_x \delta e_i + \frac{\lambda + \mu}{\lambda + 2\mu} f(\tilde{v}^*) \sum_{i=1}^{d-1} \psi^i_\alpha \partial_x \delta e_d.
\] (4.10)

Making use of conditions \((H1)\)–\((H2)\), we obtain that for \( x \in \Omega^*_R \),

\[
|\nabla x'(\bar{u}^*_\alpha - \bar{u}^*_\alpha)| \leq \frac{C}{|x'|}, \quad |\partial x_d(\bar{u}^*_\alpha - \bar{u}^*_\alpha)| \leq \frac{Cf}{|x'|m(\varepsilon + |x'|^m)}.
\] (4.11)

By applying Corollary 4.1 to \( u^*_\alpha \) defined in (4.16), we have

\[
|\nabla (u^*_\alpha - \bar{u}^*_\alpha)| \leq C|x'|^{m-2}, \quad x \in \Omega^*_R.
\] (4.12)

For \( 0 < r < R \), write

\[
C_r := \left\{ x \in \mathbb{R}^d \mid |x'| < r, \frac{1}{2} \min_{|x'| \leq r} h(x') \leq x_d \leq \varepsilon + 2 \max_{|x'| \leq r} h_1(x') \right\}.
\] (4.13)

We next estimate the difference \(|Q_\alpha[\varphi] - Q^*_\alpha[\varphi]|\) step by step. Observe that \( u_\alpha - u^*_\alpha \) solves

\[
\left\{
\begin{array}{ll}
\mathcal{L}_{\lambda, \mu}(u_\alpha - u^*_\alpha) = 0, & \text{in } D \setminus (D_1 \cup D^*_1), \\
u_\alpha - u^*_\alpha = \psi_\alpha - u^*_\alpha, & \text{on } \partial D_1 \setminus D^*_1, \\
u_\alpha - u^*_\alpha = u_\alpha - \psi_\alpha, & \text{on } \partial D^*_1 \setminus (D_1 \cup \{0\}), \\
u_\alpha - u^*_\alpha = 0, & \text{on } \partial D.
\end{array}
\right.
\]

To start with, we estimate \(|u_\alpha - u^*_\alpha| \) on \( \partial(D_1 \cup D^*_1) \setminus \{e^\gamma\} \), where \( 0 < \gamma < 1/2 \) to be determined later. Recalling the definition of \( u^*_\alpha \), it follows from the standard boundary and interior estimates of elliptic systems that

\[
|\partial x_d u^*_\alpha| \leq C, \quad \text{in } \Omega^* \setminus \Omega^*_R.
\]

Then for \( x \in \partial D_1 \setminus D^*_1 \),

\[
|(u_\alpha - u^*\alpha)(x', x_d)| = |u^*_\alpha(x', x_d - \varepsilon) - u^*\alpha(x', x_d)| \leq C\varepsilon.
\] (4.14)

Utilizing (4.1), we obtain that for \( x \in \partial D^*_1 \setminus (D_1 \cup \{e^\gamma\}) \),

\[
|(u_\alpha - u^*\alpha)(x', x_d)| = |u^*_\alpha(x', x_d) - u_\alpha(x', x_d + \varepsilon)| \leq C\varepsilon^{1-m}\gamma.
\] (4.15)

From (4.11) and (4.11)–(4.12), we deduce that for \( x \in \Omega^*_R \cap \{ |x'| = \varepsilon^\gamma \} \),

\[
|\partial x_d(u_\alpha - u^*_\alpha)| \leq |\partial x_d(u_\alpha - \bar{u}_\alpha)| + |\partial x_d(\bar{u}_\alpha - u^*_\alpha)| + |\partial x_d(u^*_\alpha - \bar{u}^*_\alpha)| \leq C \left( \frac{1}{\varepsilon^{2m\gamma - 1}} + 1 \right).
\]
This, together with the fact that \(\bar{u}_\alpha - \bar{u}^*_\alpha = 0\) on \(\partial D\), reads that
\[
\|(u_\alpha - u^*_\alpha)(x', x_d)\| = |(u_\alpha - u^*_\alpha)(x', x_d) - (u_\alpha - u^*_\alpha)(x', h(x'))| \\
\leq C(\varepsilon^{1-m\gamma} + \varepsilon^{m\gamma}).
\] (4.16)

Let \(\gamma = \frac{1}{2m}\). Then it follows from (4.14)–(4.16) that
\[
|u_\alpha - u^*_\alpha| \leq C\varepsilon^{\frac{1}{2}}, \quad \text{on} \quad \partial(D \setminus (D_1 \cup D_1^{\varepsilon} \cup C_{\varepsilon^{2m}}^{\frac{1}{2m}-\gamma})),
\]
which, in combination with the maximum principle for Lamé system in [39], yields that
\[
|u_\alpha - u^*_\alpha| \leq C\varepsilon^{\frac{1}{2}}, \quad \text{in} \quad D \setminus (D_1 \cup D_1^{\varepsilon} \cup C_{\varepsilon^{2m}}^{\frac{1}{2m}-\gamma}).
\] (4.17)

Then from the standard interior and boundary estimates for Lamé system, it follows that for any \(0 < \tilde{\gamma} < \frac{1}{2m}\),
\[
|\nabla(u_\alpha - u^*_\alpha)| \leq C\varepsilon^{m\tilde{\gamma}}, \quad \text{in} \quad D \setminus (D_1 \cup D_1^{\varepsilon} \cup C_{\varepsilon^{2m}}^{\frac{1}{2m}-\gamma}),
\]
which indicates that
\[
|A^{\text{out}}| := \left| \int_{\partial D \cap C_{\varepsilon^{2m}}^{\frac{1}{2m}-\gamma}} \frac{\partial (u_\alpha - u^*_\alpha)}{\partial \nu_0} \right| \cdot \varphi(x) \leq C\varepsilon^{m\tilde{\gamma}}\|\varphi\|_{L^\infty(\partial D)},
\] (4.18)

where \(0 < \tilde{\gamma} < \frac{1}{2m}\) will be determined later.

We next estimate the remainder as follows:
\[
A^{\text{in}} := \int_{\partial D \cap C_{\varepsilon^{2m}}^{\frac{1}{2m}-\gamma}} \frac{\partial (u_\alpha - u^*_\alpha)}{\partial \nu_0} \right|_+ \cdot \varphi(x)
\]
\[
= \int_{\partial D \cap C_{\varepsilon^{2m}}^{\frac{1}{2m}-\gamma}} \frac{\partial (\bar{u}_\alpha - \bar{u}^*_\alpha)}{\partial \nu_0} \right|_+ \cdot \varphi(x) + \int_{\partial D \cap C_{\varepsilon^{2m}}^{\frac{1}{2m}-\gamma}} \frac{\partial (w_\alpha - w^*_\alpha)}{\partial \nu_0} \right|_+ \cdot \varphi(x)
\]
\[
= : A^{\tilde{\alpha}} + A^{\bar{\alpha}},
\]
where \(w_\alpha = u_\alpha - \bar{u}_\alpha, w^*_\alpha = u^*_\alpha - \bar{u}^*_\alpha\). On one hand, if \(\alpha = 1, ..., d - 1\), then
\[
A^{\tilde{\alpha}} = \int_{\partial D \cap C_{\varepsilon^{2m}}^{\frac{1}{2m}-\gamma}} \left\{ \lambda \sum_{i=1}^{d} \partial_{x_i}(\bar{u}_\alpha - \bar{u}^*_\alpha)\nu_i \varphi^i + \mu \partial_{x_d}(\bar{u}_\alpha - \bar{u}^*_\alpha)\nu_d \varphi^d \\
+ \lambda \sum_{i=1}^{d} \partial_{x_i}(\bar{u}_\alpha - \bar{u}^*_\alpha)\nu_i \varphi^i + \mu \partial_{x_d}(\bar{u}_\alpha - \bar{u}^*_\alpha)\nu_d \varphi^d \right\} \\
+ \int_{\partial D \cap C_{\varepsilon^{2m}}^{\frac{1}{2m}-\gamma}} \mu \partial_{x_d}(\bar{u}_\alpha - \bar{u}^*_\alpha)\nu_d \varphi^d
\]
\[
= : A^{1}_{\tilde{\alpha}} + A^{2}_{\tilde{\alpha}}.
\]

Combining (29), (4.1) and (4.10)–(4.11), we obtain
\[
|A^{1}_{\tilde{\alpha}}| \leq \int_{|x'| < \varepsilon^{\frac{1}{2m}-\gamma}} C|x'|^{k-1} \leq C\varepsilon^{(\frac{1}{2m}-\gamma)(d+k-2)},
\] (4.19)
while, for the second term \(A^{2}_{\tilde{\alpha}}\), we have
(i) if condition (E1) holds, then for \( m < d + k - 1 \),
\[
|A_u^2| \leq \int_{|x'| < \varepsilon^{\frac{1}{d+m-\gamma}}} C|x'|^{k-m} \leq C\varepsilon^{\left(\frac{1}{d+m-\gamma}\right)(d+k-1-m)}; \quad (4.20)
\]

(ii) if condition (E2) or (E3) holds, then in view of the fact that the integrand is odd and the integrating domain is symmetric, we have
\[ A_u^2 = 0. \]

On the other hand, if \( \alpha = d \), then
\[
A_u = \int_{\partial D \cap \mathbb{C} + \frac{1}{d+m-\gamma} \varepsilon} \left\{ \alpha \sum_{i=1}^{d-1} \partial x_i (\bar{u}_d^i - \bar{u}_d^{*i}) \nu_i \varphi^i + \mu \sum_{i=1}^{d-1} \partial x_i (\bar{u}_d^i - \bar{u}_d^{*i}) (\nu_\alpha \varphi^i + \nu_i \varphi^\alpha) + \mu \sum_{i=1}^{d-1} \partial x_i (\bar{F}_d^i - \bar{F}_d^{*i}) \nu_i \varphi^i \right\} \\
= : A_u^1 + A_u^2.
\]

By the same argument as in (4.19) – (4.20), we have
\[
|A_u^1| \leq C\varepsilon^{\left(\frac{1}{d+m-\gamma}\right)(d+k-2)}; \quad (4.21)
\]

and
\[
A_u^2 = \begin{cases} 
O(1)\varepsilon^{\left(\frac{1}{d+m-\gamma}\right)(d+k-1-m)}, & \text{if condition (E1) holds, } m < d + k - 1, \\
0, & \text{if condition (E2) or (E3) holds.} 
\end{cases} \quad (4.22)
\]

We now proceed to estimate \( |A_{w_\alpha}| \). An immediate consequence of Corollary 4.1 gives that for \( 0 < |x'| \leq R \),
\[
|\nabla w_\alpha| + |\nabla w_\alpha^*| \leq C, \quad \alpha = 1, 2, ..., d. \quad (4.23)
\]

Since
\[
A_w = \int_{\partial D \cap \mathbb{C} + \frac{1}{d+m-\gamma} \varepsilon} \left\{ \alpha \sum_{i,j=1}^{d} \partial x_j (w_\alpha^i - w_\alpha^{*i}) \nu_j \varphi^j + \mu \sum_{i,j=1}^{d} \partial x_j (w_\alpha^i - w_\alpha^{*i}) (\nu_\alpha \varphi^i + \nu_i \varphi^\alpha) + \mu \sum_{i,j=1}^{d} \partial x_j (\bar{F}_d^i - \bar{F}_d^{*i}) \nu_i \varphi^i \right\},
\]
than we deduce from (4.23) that
\[
|A_w| \leq \int_{|x'| < \varepsilon^{\frac{1}{d+m-\gamma}}} C|x'|^k \leq C\varepsilon^{\left(\frac{1}{d+m-\gamma}\right)(d+k-1)}. \quad (4.24)
\]

By using (1.18) – (1.22) and (1.24), we derive that for \( \alpha = 1, 2, ..., d \),
(i) if condition (E1) holds, by picking \( \bar{\gamma} = \frac{d+k-1-m}{2m(d+k-1)} \), then
\[
|Q_\alpha[\varphi] - Q_\alpha^*[\varphi]| \leq C\varepsilon^{\frac{d+k-1-m}{2m(d+k-1)}}, \quad m < d + k - 1;
\]
(ii) if condition (E2) or (E3) holds, by picking \( \bar{\gamma} = \frac{d+k-2}{2m(d+k-1)} \), then
\[
|Q_\alpha[\varphi] - Q_\alpha^*[\varphi]| \leq C\varepsilon^{\frac{d+k-2}{2m(d+k-1)}}.
\]
Proofs of (4.7)–(4.8). We next divide into two substeps to complete the proofs of (4.7)–(4.8).

Step 2.1. If condition (E2) holds for $m \geq d + k$, then it follows from Corollary 4.1 and (4.3) again that

\[
Q_{d+1}[\varphi] = \sum_{i=1}^{d} \int_{\partial D_i} \frac{\partial u_{0i}}{\partial \nu_i} \cdot \psi_{d+1}
\]

\[
= \int_{\partial D_1} \left[ \sum_{i,j=1}^{d} \left( \lambda \partial_{x_i} u_{0i} \nu_j + \mu (\partial_{x_i} u_{0i} + \partial_{x_i} u_{0i}) \nu_j \right) x_d - \sum_{i=1}^{d} \lambda \partial_{x_i} u_{0i} \nu_d x_1 \right] - \int_{\partial D_1} (\lambda + 2\mu) \partial_{x_d} u_{0d} \nu_d x_1
\]

\[
= (\lambda + 2\mu) \int_{\partial D_1} \frac{\varphi^d(x', h(x')) x_1}{\varepsilon + \kappa_1 |x'|^m} + O(1) \|\varphi\|_{C^{2}(\partial D)}
\]

where we used the fact that $|x_d| = |e + h_1(x')| \leq C(\varepsilon + |x'|^m)$ on $\Gamma_R^+$. This yields that

\[
Q_{d+1}[\varphi] \lesssim (\lambda + 2\mu) \eta \int_{|x'| < R} \frac{|x_1|^{k+1}}{\varepsilon + \kappa_1 |x'|^m} \lesssim (\lambda + 2\mu) \eta \int_{0}^{R} \frac{d^{d+k-1}}{\varepsilon + \kappa_1 s^m} \lesssim \frac{(\lambda + 2\mu) \eta}{\kappa_1 m} \rho_{k+1}(d, m; \varepsilon),
\]

and

\[
Q_{d+1}[\varphi] \gtrsim (\lambda + 2\mu) \eta \int_{|x'| < R} \frac{|x_1|^{k+1}}{\varepsilon + \kappa_2 |x'|^m} \gtrsim \frac{(\lambda + 2\mu) \eta}{\kappa_2 m} \rho_{k+1}(d, m; \varepsilon).
\]

Step 2.2. First, using (M8), we have

\[
Q_{d+1}[\varphi] - Q_{d+1}^*[\varphi] = \int_{\partial D} \frac{\partial (u_{d+1} - u_{d+1}^{*})}{\partial \nu_0} |_{+} \cdot \varphi(x),
\]

where $u_{d+1}^*$ and $u_{d+1}$ are, respectively, defined by (2.10) and (3.1) with picking $\alpha = d + 1$.

Similarly as above, it follows from (H1)–(H2) that for $x \in \Omega_R^*$,

\[
|\nabla x' (u_{d+1} - u_{d+1}^{*})| \leq C, \quad |\partial_{x_d} (u_{d+1} - u_{d+1}^{*})| \leq \frac{C\varepsilon}{|x'|^{m-1}(|x'|^m + \varepsilon)}. \tag{4.25}
\]

Applying Corollary 4.1 to $u_{d+1}^*$, we derive

\[
|\nabla (u_{d+1}^* - u_{d+1}^{*})| \leq C, \quad x \in \Omega_R^*. \tag{4.26}
\]

Observe that $u_{d+1} - u_{d+1}^{*}$ satisfies

\[
\begin{cases}
\mathcal{L}_{\lambda, \mu}(u_{d+1} - u_{d+1}^{*}) = 0, & \text{in } D \setminus (D_1 \cup D_1^*), \\
u_{d+1} - u_{d+1}^{*} = \psi_{d+1} - u_{d+1}^{*}, & \text{on } \partial D_1 \setminus D_1^*, \\
u_{d+1} - u_{d+1}^{*} = u_{d+1} - \psi_{d+1}, & \text{on } \partial D_1 \setminus (D_1 \cup \{0\}), \\
u_{d+1} - u_{d+1}^{*} = 0, & \text{on } \partial D.
\end{cases}
\]
We first give the estimation of $|u_{d+1} - u^*_d|$ on $\partial(D_1 \cup D_1^*) \setminus C_\gamma$, where $C_\gamma$ is defined in (4.13) and $\frac{1}{2m-1} < \gamma < 1/2$ to be chosen later. Analogously as before, by using the standard boundary and interior estimates of elliptic systems, we obtain that for $x \in \partial D_1 \setminus D_1^*$,

$$|(u_{d+1} - u^*_d)(x', x_d)| = |u^*_d(x', x_d - \varepsilon) - u^*_d(x', x_d)| \leq C\varepsilon. \quad (4.27)$$

Utilizing (4.11), we get that for $x \in \partial D_1^* \setminus (D_1 \cup C_\gamma)$,

$$|(u_{d+1} - u^*_d)(x', x_d)| = |u_{d+1}(x', x_d) - u_{d+1}(x', x_d + \varepsilon)| \leq C\varepsilon^{1-(m-1)\gamma}. \quad (4.28)$$

From Corollary 4.1 and (4.29)-(4.26), it follows that for $x \in \Omega_R \cap \{|x'| = \varepsilon\}$,

$$|\partial_x(u_{d+1} - u^*_d)| \leq |\partial_x(u_{d+1} - \tilde{u}_{d+1})| + |\partial_x(u^*_d - \tilde{u}^*_d)| + |\partial_x(u^*_d - \tilde{u}^*_d)| \leq C\varepsilon^{1-(m-1)\gamma}, \quad \text{for } x \in \Omega_R \cap \{|x'| = \varepsilon\}. \quad (4.29)$$

Take $\gamma = \frac{1}{m}$. From (4.27), (4.29), we have

$$|u_{d+1} - u^*_d| \leq C\varepsilon^{1/2}, \quad \text{on } \partial(D \setminus (D_1 \cup D_1^* \cup C_{\varepsilon^{1/m}})),$$

which, together with the maximum principle for the Lamé system in (39), reads that

$$|u_{d+1} - u^*_d| \leq C\varepsilon^{1/2}, \quad \text{on } D \setminus (D_1 \cup D_1^* \cup C_{\varepsilon^{1/m}}).$$

Then using the standard interior and boundary estimates for the Lamé system, we get that for any $\frac{m-1}{m} < \tilde{\gamma} < \frac{1}{m}$,

$$|\nabla(u_{d+1} - u^*_d)| \leq C\varepsilon^{m\tilde{\gamma} - \frac{m-1}{m}}, \quad \text{on } \partial D \setminus C_{\varepsilon^{1/m-\gamma}},$$

from which we obtain

$$|A^{out}| := \int_{\partial D \setminus C_{\varepsilon^{1/m-\gamma}}} \left| \frac{\partial(u_{d+1} - u^*_d)}{\partial v_0} \right| \cdot \varphi(x) \leq C\varepsilon^{m\tilde{\gamma} - \frac{m-1}{m}}, \quad (4.30)$$

where $\frac{m-1}{m} < \tilde{\gamma} < \frac{1}{m}$ will be given in the following.

It remains to estimate the residual part $A^{in}$ as follows:

$$A^{in} := \int_{\partial D \cup C_{\varepsilon^{1/m-\gamma}}} \frac{\partial(u_{d+1} - u^*_d)}{\partial v_0} \cdot \varphi(x)$$

$$= \int_{\partial D \cup C_{\varepsilon^{1/m-\gamma}}} \frac{\partial(w_{d+1} - w^*_d)}{\partial v_0} \cdot \varphi(x) + \int_{\partial D \cup C_{\varepsilon^{1/m-\gamma}}} \frac{\partial(\tilde{u}_{d+1} - \tilde{u}^*_d)}{\partial v_0} \cdot \varphi(x)$$

$$= A_w + A_\tilde{u},$$

where $w_{d+1} = u_{d+1} - \tilde{u}_{d+1}, w^*_d = u^*_d - \tilde{u}^*_d$. Analogously as above, it follows from Corollary 4.1 that

$$|\nabla w_{d+1}(x)| \leq C, \quad |\nabla w^*_d(x)| \leq C, \quad x \in \Omega_R^\varepsilon. \quad (4.31)$$
By definition,

\[ A_w = \int_{\partial D \cap C_{\frac{1}{m}, -\gamma}} \left\{ \lambda \sum_{i,j=1}^{d} \partial_{x_i} (w^i_{d+1} - w^i_{d+1}) \nu_j \varphi^j(x) + \mu \sum_{i,j=1}^{d} \left[ \partial_{x_i} (w^i_{d+1} - w^i_{d+1}) + \partial_{x_i} (w^i_{d+1} - w^i_{d+1}) \right] \nu_j \varphi^j(x) \right\}. \]

Then from (4.31) and (4.32), we get

\[ |A_w| \leq \int_{\partial D \cap C_{\frac{1}{m}, -\gamma}} C|x'|^k \leq C \varepsilon^\left(\frac{1}{d+k-2}\right) \quad (4.32) \]

We next estimate the second term \( A_\bar{u} \). Recalling the definitions of \( F_{d+1} \) and \( F_{d+1}^* \), we obtain

\[ |\nabla (F_{d+1} - F_{d+1}^*)| \leq C, \]

which reads that

\[ \left| \int_{\partial D \cap C_{\frac{1}{m}, -\gamma}} \frac{\partial (F_{d+1} - F_{d+1}^*)}{\partial \nu_0} \right| \cdot \varphi(x) \leq C \varepsilon^\left(\frac{1}{d+k-2}\right). \]

Then

\[ A_\bar{u} = \int_{\partial D \cap C_{\frac{1}{m}, -\gamma}} \frac{\partial (\bar{u}_{d+1} - \bar{u}_{d+1}^*)}{\partial \nu_0} \right| \cdot \varphi(x) = \int_{\partial D \cap C_{\frac{1}{m}, -\gamma}} \left( \frac{\partial (\psi_{d+1} \bar{v} - \psi_{d+1} \bar{v}^*)}{\partial \nu_0} \right) \cdot \varphi(x) + \frac{\partial (F_{d+1} - F_{d+1}^*)}{\partial \nu_0} \right| \cdot \varphi(x) \]

Denote

\[ A_{\bar{u}}^1 = \int_{\partial D \cap C_{\frac{1}{m}, -\gamma}} \frac{\partial (\psi_{d+1} \bar{v} - \psi_{d+1} \bar{v}^*)}{\partial \nu_0} \right| \cdot \varphi(x). \]

We further decompose \( A_{\bar{u}}^1 \) into two parts as follows:

\[ A_{\bar{u}}^{11} = \int_{\partial D \cap C_{\frac{1}{m}, -\gamma}} \sum_{i=1}^{d} \left[ \lambda x_i \partial_{x_i} (\bar{v} - \bar{v}^*) \nu_i \varphi^i + \mu \partial_{x_i} (x_i \bar{v} - x_i \bar{v}^*) (\nu_i \varphi^i + \nu_i \varphi^i) \right] \]

\[ - \int_{\partial D \cap C_{\frac{1}{m}, -\gamma}} \sum_{i=1}^{d-1} \left[ \lambda x_i \partial_{x_i} (\bar{v} - \bar{v}^*) \nu_i \varphi^i + \mu \partial_{x_i} (x_i \bar{v} - x_i \bar{v}^*) (\nu_i \varphi^i + \nu_i \varphi^i) \right], \]

\[ A_{\bar{u}}^{12} = - \int_{\partial D \cap C_{\frac{1}{m}, -\gamma}} (\lambda + 2\mu) x_1 \partial_{x_1} (\bar{v} - \bar{v}^*) \nu_1 \varphi^1. \]

With regard to \( A_{\bar{u}}^{11} \), in view of the fact that \( |x_d| = |h(x')| \leq C|x'|^m \) on \( \Gamma_{R'} \), we get that if condition (E1), (E2) or (E3) holds,

\[ |A_{\bar{u}}^{11}| \leq \int_{\partial D \cap C_{\frac{1}{m}, -\gamma}} C|x'|^k \leq C \varepsilon^\left(\frac{1}{d+k-2}\right) \quad (4.33) \]
For the second term $A_{ii}^{12}$, we have
\[ A_{ii}^{12} = \begin{cases} O(1)\varepsilon^{(\frac{d-k}{d+1})(d+k-m)}, & \text{if condition (E2) holds, } m < d + k, \\ 0, & \text{if condition (E1) or (E3) holds,} \end{cases} \] 
where we used the fact that the integrand is odd with respect to $x_1$ in the case when condition (E1) or (E3) holds. Then from \((3.3)\) and \((3.4)\)–\((3.6)\), we obtain that

(a) if condition (E1) or (E3) holds, by taking $\bar{\gamma} = \frac{m+2k-2}{m+2k-1}$, then
\[ |Q_{d+1}[\varphi] - Q_{d+1}^{*}[\varphi]| \leq C\varepsilon^{\frac{d+k-1}{m+2k-1}}; \]
(b) if condition (E2) holds, by taking $\bar{\gamma} = \frac{d+k-1}{m+2k-1}$, then
\[ |Q_{d+1}[\varphi] - Q_{d+1}^{*}[\varphi]| \leq C\varepsilon^{\frac{d+k-m}{m+2k-1}}. \]

Consequently, we complete the proofs of \((1.7)\)–\((1.8)\).

**Step 3.** Proof of \((4.1)\). Observe that for $i = 1, 2, \ldots, d$ and $j = 1, \ldots, d - 1, i \neq j$, (i) if condition (E1) holds, then $\varphi^i(x', h(x')) x_j$ is odd with respect to $x_j$;
(ii) if condition (E2) holds, then $\varphi^i(x', h(x')) x_j = 0$ for $i = 1, \ldots, d - 1$, and $\varphi^i(x', h(x')) x_j$ is odd with respect to $x_1$ for $j = 2, \ldots, d - 1, d \geq 3$;
(iii) if condition (E3) holds, then $\varphi^i(x', h(x')) x_j$ is odd with respect to $x_i$ for $i = 1, \ldots, d - 1$, and $\varphi^i(x', h(x')) x_j = 0$.

Then following the same argument as in **Step 2**, we deduce that for $\alpha = d + 2, \ldots, \frac{d(d+1)}{2}$, $d \geq 3$,
\[ |Q_{\alpha}[\varphi] - Q_{\alpha}^{*}[\varphi]| \leq C\varepsilon^{\frac{d+k-1}{m+2k-1}}. \]
That is, \((4.9)\) holds.

\[ \square \]

### 4.2. Estimates and asymptotics of $a_{\alpha\beta}$, $\alpha = 1, 2, \ldots, \frac{d(d+1)}{2}$

Multiplying the first line of \((3.1)\) by $u_{\beta}$, it follows from \((3.8)\) that
\[ a_{\alpha\beta} = \int_{\Omega} (C^0 e(u_\alpha), e(u_\beta)), \quad \alpha, \beta = 1, 2, \ldots, \frac{d(d+1)}{2}. \]

**Lemma 4.3.** Assume as above. Then for a sufficiently small $\varepsilon > 0$,

(i) if $\alpha = 1, 2, \ldots, d$, then
\[ \kappa_2^{\frac{d+1}{d}} \leq \frac{a_{\alpha\alpha}}{\varepsilon^{\frac{d+1}{2} - \frac{2\alpha}{d+1}}}, \quad m \geq d - 1, \]
\[ a_{\alpha\alpha} = a_{\alpha\alpha}^* + O(\varepsilon^{\frac{d+1}{2} - \frac{2\alpha}{d+1}}), \quad m < d - 1; \]  
(ii) if $\alpha = d + 1, \ldots, \frac{d(d+1)}{2}$, then
\[ \kappa_2^{\frac{d+1}{d}} \leq \frac{a_{\alpha\alpha}}{\varepsilon^{\frac{d+1}{2} - \frac{2\alpha}{d+1}}}, \quad m \geq d + 1, \]
\[ a_{\alpha\alpha} = a_{\alpha\alpha}^* + O(\varepsilon^{\frac{d+1}{2} - \frac{2\alpha}{d+1}}), \quad m < d + 1; \]  
(iii) if $d = 2$, for $\alpha, \beta = 1, 2, \alpha \neq \beta$, then
\[ a_{12} = a_{21} = O(1) |\ln \varepsilon|, \]
and if $d \geq 3$, for $\alpha, \beta = 1, 2, \ldots, d$, $\alpha \neq \beta$, then
\[ a_{\alpha\beta} = a_{\alpha\beta}^* + O(1) \varepsilon^{\frac{d+2}{2} - \frac{d+1}{2d}}, \]
and if $d \geq 2$, for $\alpha = 1, 2, \ldots, d$, $\beta = d + 1, \ldots, \frac{d(d+1)}{2}$, then
\[
a_{\alpha \beta} = a_{\beta \alpha} = a_{\alpha \beta}^* + O(1)\epsilon^{\min\left(\frac{1}{2}, \frac{d+1}{2d-1}\right)},
\]
(4.39)
and if $d \geq 3$, for $\alpha, \beta = d + 1, \ldots, \frac{d(d+1)}{2}$, $\alpha \neq \beta$, then
\[
a_{\alpha \beta} = a_{\beta \alpha} = a_{\alpha \beta}^* + O(1)\epsilon^{\min\left(\frac{1}{2}, \frac{d+1}{2d-1}\right)},
\]
(4.40)
where $a_{\alpha \beta}^*$, $\alpha, \beta = 1, 2, \ldots, \frac{d(d+1)}{2}$ are defined in (2.15).

**Proof.** **Step 1.** Proof of (4.35). Fix $\bar{\gamma} = \frac{1}{12m}$. For $\alpha = 1, 2, \ldots, d$, we begin with a decomposition for $a_{\alpha \alpha}$ as follows:
\[
a_{\alpha \alpha} = \int_{\Omega_{\alpha \gamma}} (C^0(e(u_\alpha), e(u_\alpha))) + \int_{\Omega_{\alpha \gamma} \setminus \Omega_R} (C^0(e(u_\alpha), e(u_\alpha))) + \int_{\Omega_R \setminus \Omega_{\alpha \gamma}} (C^0(e(u_\alpha), e(u_\alpha))) =: I + II + III.
\]
(4.41)
In light of the definitions of $\bar{u}_\alpha$ and $C^0$, it follows from a straightforward calculation that
\[
(C^0(e(\bar{u}_\alpha), e(\bar{u}_\alpha))) = (\lambda + \mu)(\partial_{x_\alpha} \bar{v})^2 + \mu \sum_{i=1}^{d} (\partial_{x_i} \bar{v})^2 + (C^0(e(F_\alpha), e(F_\alpha))) + 2(C^0(e(\bar{\psi}_\alpha \bar{v}), e(F_\alpha))), \quad \alpha = 1, 2, \ldots, d,
\]
which, in combination with Corollary 4.1, gives that
\[
I = \int_{\Omega_{\alpha \gamma}} (C^0(e(\bar{u}_\alpha), e(\bar{u}_\alpha))) + 2 \int_{\Omega_{x_\gamma}} (C^0(e(u_\alpha - \bar{u}_\alpha), e(\bar{u}_\alpha)))
+ \int_{\Omega_{\alpha \gamma}} (C^0(e(u_\alpha - \bar{u}_\alpha), e(u_\alpha - \bar{u}_\alpha)))
= L^2_d \int_{|x'| < \bar{\gamma}} \frac{dx'}{\varepsilon + h_1(x') - h(x')} + O(1)\epsilon^{(d-1)\bar{\gamma}},
\]
(4.42)
where $F_\alpha$ is defined in (2.17) and $L^2_d$ is defined by (2.18) and (2.19).

To estimate the latter two terms, we first analyze the difference $|\nabla (u_\alpha - u_\alpha^*)|$ in $D \setminus (\overline{D_1} \cup D_1^* \cup C_{\bar{\gamma}})$. For $\varepsilon^\gamma \leq |z'| \leq R$, by carrying out a change of variable as follows:
\[
\begin{align*}
x' = z', \quad x_d = |z'|^m y_d,
\end{align*}
\]
we rescale $\Omega_{|z'|^m \leq |z'|} \setminus \Omega_{|z'|}$ and $\Omega_{|z'|^m \leq |z'|} \setminus \Omega_{|z'|}$ into $Q_1$ and $Q_1^*$ of nearly unit-size squares (or cylinders), respectively. Denote
\[
U_\alpha(y) := u_\alpha(z' + |z'|^m y', |z'|^m y_d), \quad \text{in } Q_1,
\]
and
\[
U_\alpha^*(y) := u_\alpha^*(z' + |z'|^m y', |z'|^m y_d), \quad \text{in } Q_1^*.
\]
Then we conclude from the standard interior and boundary estimates of elliptic systems that
\[
|\nabla^2 U_\alpha| \leq C, \quad \text{in } Q_1, \quad \text{and } |\nabla^2 U_\alpha^*| \leq C, \quad \text{in } Q_1^*.
\]
By utilizing an interpolation with (4.17), we get
\[
|\nabla (U_\alpha - U_\alpha^*)| \leq C\varepsilon^{\frac{1}{2} - \frac{1}{2}} \leq C\varepsilon^{\frac{1}{2}}.
\]

Then rescaling it back to \( u_\alpha - u_\alpha^* \) and in view of \( \varepsilon' \leq |z'| \leq R \), we have
\[
|\nabla (u_\alpha - u_\alpha^*)(x)| \leq C\varepsilon^{\frac{1}{2}}|z'|^{-m} \leq C\varepsilon^{\frac{1}{2}}, \quad x \in \Omega^*_R + |z'|m \setminus \Omega^*_R,
\]
which implies that for \( \alpha = 1, 2, ..., d \),
\[
|\nabla (u_\alpha - u_\alpha^*)| \leq C\varepsilon^{\frac{1}{2}}, \quad \text{in } D \setminus (D_1 \cup D_1^* \cup C_\varepsilon).
\]

Then from (4.43), we obtain
\[
\begin{align*}
\text{II} & = \int_{D_1 \setminus (D_1 \cup D_1^* \cup \Omega_R)} (C^0(e(u_\alpha), e(u_\alpha))) + O(1)\varepsilon \\
& = \int_{D_1 \setminus (D_1 \cup D_1^* \cup \Omega_R)} ((C^0(e(u_\alpha^*), e(u_\alpha^*))) + 2(C^0(e(u_\alpha - u_\alpha^*), e(u_\alpha^*)))) \\
& \quad + \int_{D_1 \setminus (D_1 \cup D_1^* \cup \Omega_R)} (C^0(e(u_\alpha - u_\alpha^*), e(u_\alpha - u_\alpha^*))) + O(1)\varepsilon \\
& = \int_{\Omega \setminus \Omega^*_R} (C^0(e(u_\alpha), e(u_\alpha))) + O(1)\varepsilon^{\frac{1}{2}},
\end{align*}
\]

where we used the fact that \( |\nabla u_\alpha| \) remains bounded in \((D_1^* \setminus (D_1 \cup \Omega_R)) \cup (D_1 \setminus D_1^*)\) and the volumes of \((D_1^* \setminus (D_1 \cup \Omega_R)) \) and \((D_1 \setminus D_1^*)\) are of order \( O(\varepsilon) \).

With regard to the last term III of (4.41), we first decompose it as follows:

\[
\begin{align*}
\text{III}_1 & = \int_{(\Omega_R \setminus \Omega^*_R) \setminus (\Omega^*_R \setminus \Omega^*_\varepsilon^+)} (C^0(e(u_\alpha), e(u_\alpha))), \\
\text{III}_2 & = \int_{\Omega^*_R \setminus \Omega^*_\varepsilon^+} (C^0(e(u_\alpha - u_\alpha^*), e(u_\alpha - u_\alpha^*))) + 2 \int_{\Omega^*_R \setminus \Omega^*_\varepsilon^+} (C^0(e(u_\alpha - u_\alpha^*), e(u_\alpha^*))), \\
\text{III}_3 & = \int_{\Omega^*_R \setminus \Omega^*_\varepsilon^+} (C^0(e(u_\alpha^*), e(u_\alpha^*))).
\end{align*}
\]

Since the thickness of \((\Omega_R \setminus \Omega^*_R) \setminus (\Omega^*_R \setminus \Omega^*_\varepsilon^+)\) is \( \varepsilon \), then we deduce from Corollary 4.11 that
\[
|\text{III}_1| \leq C\varepsilon \int_{\varepsilon^5 < |z'| < R} \frac{dx'}{|x'|^{2m}} \leq C \begin{cases} 
\varepsilon, & m < \frac{d-1}{2}, \\
\varepsilon|\ln \varepsilon|, & m = \frac{d-1}{2}, \\
\varepsilon^{\frac{m+1-d}{m}}, & m > \frac{d-1}{2},
\end{cases}
\]
(4.45)

On the other hand, a consequence of (4.12) and (4.43) gives
\[
|\text{III}_2| \leq C\varepsilon^{\frac{1}{2}}.
\]
(4.46)
Using (4.12) again, we derive

\[
\text{III}_3 = \int_{\Omega_R^* \setminus \Omega_R^c} (\mathcal{C}^0 e(\bar{u}_\alpha^*), e(\bar{u}_\alpha^*)) + 2 \int_{\Omega_R^* \setminus \Omega_R^c} (\mathcal{C}^0 e(u_\alpha^* - \bar{u}_\alpha^*), e(\bar{u}_\alpha^*)) \\
+ \int_{\Omega_R^* \setminus \Omega_R^c} (\mathcal{C}^0 e(u_\alpha^* - \bar{u}_\alpha^*), e(u_\alpha^* - \bar{u}_\alpha^*)) \\
= \mathcal{L}_d \int_{\varepsilon < |x'| < \varepsilon + h} \frac{dx'}{h_1(x') - h(x')} - \int_{\Omega_R^* \setminus \Omega_R^c} (\mathcal{C}^0 e(u_\alpha^*), e(u_\alpha^*)) \\
+ M^*_d + O(1) \varepsilon \min\left(\frac{1}{h_1 - h}, \frac{1}{\varepsilon + h_1 - h}\right),
\]

where

\[
M^*_d = \int_{\Omega_R^* \setminus \Omega_R^c} (\mathcal{C}^0 e(\bar{u}_\alpha^*), e(\bar{u}_\alpha^*)) + \int_{\Omega_R^* \setminus \Omega_R^c} (\mathcal{C}^0 e(u_\alpha^* - \bar{u}_\alpha^*), e(u_\alpha^* - \bar{u}_\alpha^*)) \\
+ \int_{\Omega_R^*} [2(\mathcal{C}^0 e(u_\alpha^* - \bar{u}_\alpha^*), e(\bar{u}_\alpha^*)) + 2(\mathcal{C}^0 e(\bar{v}_\alpha^*), e(\mathcal{F}_\alpha^*))) \\
+ \int_{\Omega_R^*} (\lambda + \mu)(\partial_{x_\alpha} \bar{v}_\alpha^*)^2 + \mu \sum_{i=1}^{d-1}(\partial_{x_i} \bar{v}_\alpha^*)^2, \alpha = 1, \ldots, d - 1, \\
+ \int_{\Omega_R^*} \mu \sum_{i=1}^{d-1}(\partial_{x_i} \bar{v}_\alpha^*)^2, \alpha = d. 
\]

Then from (4.42) and (4.44) - (4.47), we get

\[
a_{\alpha\alpha} = \mathcal{L}_d^3 \left( \int_{\varepsilon < |x'| < \varepsilon + h_1(x')} \frac{dx'}{h_1(x') - h(x')} + \int_{|x'| < \varepsilon} \frac{dx'}{h_1(x') - h(x')} \right) \\
+ M^*_d + O(1) \varepsilon \min\left(\frac{1}{h_1 - h}, \frac{1}{\varepsilon + h_1 - h}\right).
\]

We now divide into two cases to calculate \(a_{\alpha\alpha}\) as follows:

(i) if \(m \geq d - 1\), then

\[
\int_{\varepsilon < |x'| < h} \frac{1}{h_1 - h} = \int_{|x'| < \varepsilon} \frac{1}{\varepsilon} \\
= \frac{1}{\varepsilon} + \int_{\varepsilon < |x'| < \varepsilon + h_1(h_1 - h)} \frac{1}{h_1 - h} \\
= \frac{1}{\varepsilon} + \int_{|x'| < \varepsilon} \frac{1}{\varepsilon} + \int_{\varepsilon < |x'| < \varepsilon + h_1(h_1 - h)} \frac{1}{h_1 - h} \\
= \frac{1}{\varepsilon} + O(1) \left\{ \begin{array}{ll} \varepsilon, & m < \frac{d-1}{2}, \\
\varepsilon \ln |x'|, & m = \frac{d-1}{2}, \\
\varepsilon^{10m + d-1}, & m > \frac{d-1}{2}, \end{array} \right.
\]

which implies that

\[
a_{\alpha\alpha} \lesssim \mathcal{L}_d^3 \int_{|x'| < R} \frac{1}{\varepsilon + h_1 x'} \lesssim \mathcal{L}_d^3 \int_{|x'| < R} \frac{1}{\varepsilon + \kappa_1 |x'| m} \\
\lesssim \mathcal{L}_d^3 \left( \frac{R^{d-2}}{\varepsilon + \kappa_1 s m} \lesssim \frac{\mathcal{L}_d^3}{\kappa_1 m} \rho_0(d, m; \varepsilon), \right.
\]

and

\[
a_{\alpha\alpha} \gtrsim \mathcal{L}_d^3 \int_{|x'| < R} \frac{1}{\varepsilon + \kappa_2 |x'| m} \gtrsim \frac{\mathcal{L}_d^3}{\kappa_2 m} \rho_0(d, m; \varepsilon); \quad (4.50)
\]
(ii) if \( m < d - 1 \), then

\[
\begin{align*}
\alpha \beta &= L_d^\alpha \left( \int_{|x'| < R} \frac{dx'}{h_1 - h} \right. \\
&\quad + M_d^\alpha + O(1)\varepsilon^{-\min\left(\frac{d-1}{2}, \frac{d-1}{m-1}\right)} \\
&\left. = L_d^\alpha \int_{\Omega_R^\alpha} (\partial_{x_i} \tilde{u}_\alpha)^2 + M_d^\alpha + O(1)\varepsilon^{-\min\left(\frac{d-1}{2}, \frac{d-1}{m-1}\right)} \right) \\
&\quad = a^\alpha_\alpha + O(1)\varepsilon^{-\min\left(\frac{d-1}{2}, \frac{d-1}{m-1}\right)}.
\end{align*}
\]

Consequently, substituting (4.50)–(4.52) into (4.49), we complete the proof of (4.35).

**Step 2.** Proof of (4.36). According to the definition of \( \psi_\alpha \), we see that for \( d + 1 \leq \alpha \leq \frac{d(d+1)}{2} \), there exist two indices \( 1 \leq i_\alpha < j_\alpha \leq d \) such that \( \psi_\alpha = (0, \ldots, 0, x_{i_\alpha}, 0, \ldots, 0, -x_{i_\alpha}, \ldots, 0, -x_{j_\alpha-1}) \). Especially when \( d + 1 \leq \alpha \leq 2d - 1 \), we have \( i_\alpha = \alpha - d, j_\alpha = d \) and thus \( \psi_\alpha = (0, \ldots, 0, x_d, 0, \ldots, 0, -x_{\alpha-d}) \). Similar to (4.41), for \( \alpha = d + 1, \ldots, \frac{d(d+1)}{2} \), we split \( a_{\alpha\alpha} \) as follows:

\[
\begin{align*}
I &= \int_{\Omega_R^\alpha} (\mathcal{C}^\alpha e(u_\alpha), e(u_\alpha)), \\
II &= \int_{\Omega_R \setminus \Omega_R^\alpha} (\mathcal{C}^\alpha e(u_\alpha), e(u_\alpha)), \\
III &= \int_{\Omega_R \setminus \Omega_R^\alpha} (\mathcal{C}^\alpha e(u_\alpha), e(u_\alpha)),
\end{align*}
\]

where \( \gamma = \frac{1}{12m} \). A direct computation shows that for \( \alpha = d + 1, \ldots, \frac{d(d+1)}{2} \),

\[
(\mathcal{C}^\alpha e(\bar{u}_\alpha), e(\bar{u}_\alpha)) = \mu(x_{i_\alpha}^2 + x_{j_\alpha}^2) \sum_{k=1}^d (\partial_{x_k} \tilde{v})^2 + (\lambda + \mu)(x_{i_\alpha} \partial_{x_{i_\alpha}} \tilde{v} - x_{i_\alpha} \partial_{x_{j_\alpha}} \tilde{v})^2 \\
+ 2(\mathcal{C}^\alpha e(\psi_\alpha \tilde{v}), e(F_\alpha)) + (\mathcal{C}^\alpha e(F_\alpha), e(F_\alpha)),
\]

where the correction term \( F_\alpha \) is defined by (2.59). This, together with Corollary 4.1 reads that

\[
I = \frac{L_d^\alpha}{d-1} \int_{|x'| < R} \frac{|x'|^2}{\varepsilon + h_1(x') - h(x')} \varepsilon dx' + O(1)\varepsilon^{-\frac{d-1}{2m}}.
\]

where \( L_d^\alpha \) is defined in (2.18)–(2.19).

Observe that by applying (4.27)–(4.29) with \( \gamma = \frac{1}{2(m-1)} \), we get that for \( \alpha = d + 1, \ldots, \frac{d(d+1)}{2} \),

\[
|u_\alpha - u_\alpha^*| \leq C\varepsilon^\frac{1}{2}, \quad \text{in} \quad D \setminus \left(D_1 \cup D_1^* \cup C_{\varepsilon^{-\frac{1}{2(m-1)}}}\right).
\]

Proceeding as before, making use of (4.54), the rescale argument, the interpolation inequality and the standard elliptic estimates, we deduce that for \( \alpha = d + 1, \ldots, \frac{d(d+1)}{2} \),

\[
|\nabla (u_\alpha - u_\alpha^*)| \leq C\varepsilon^\frac{1}{2}, \quad \text{in} \quad D \setminus \left(D_1 \cup D_1^* \cup C_{\varepsilon^{-\frac{1}{2(m-1)}}}\right).
\]
Following the same argument as in (4.41), we deduce from (4.55) that for $\alpha = d + 1, \ldots, \frac{d(d+1)}{2}$,

$$II = \int_{\Omega^*_{\gamma}} (\mathcal{C}^0 \psi(u^*_\alpha), \psi(u^*_\alpha)) + O(1)\varepsilon^\frac{1}{2}. \quad (4.56)$$

With regard to the last term $III$, similarly as before, we split it as follows:

$$III = \int_{(\Omega_R \setminus \Omega^*_{\gamma}) \setminus (\Omega_R^* \setminus \Omega^*_{\gamma})} (\mathcal{C}^0 \psi(u^*_\alpha), \psi(u^*_\alpha)) + \int_{\Omega^*_{\gamma}} (\mathcal{C}^0 \psi(u^*_\alpha), \psi(u^*_\alpha))$$

$$+ 2 \int_{\Omega^*_{\gamma}} [(\mathcal{C}^0 e(u^*_\alpha - u^*_\alpha), e(u^*_\alpha)) + (\mathcal{C}^0 e(u^*_\alpha - \bar{u}^*_\alpha), e(u^*_\alpha))]$$

$$+ \int_{\Omega^*_{\gamma}} [(\mathcal{C}^0 e(u^*_\alpha - u^*_\alpha), e(u^*_\alpha - u^*_\alpha)) + (\mathcal{C}^0 e(u^*_\alpha - \bar{u}^*_\alpha), e(u^*_\alpha - \bar{u}^*_\alpha))].$$

Analogously as above, in light of the fact that the thickness of $(\Omega_R \setminus \Omega^*_{\gamma}) \setminus (\Omega_R^* \setminus \Omega^*_{\gamma})$ is $\varepsilon$, we deduce from (4.20) and (4.55) that

$$III = \frac{L^d_{\alpha}}{d-1} \int_{\varepsilon^\gamma < |x'| < R} \frac{|x'|^2}{h_1(x') - h_2(x')} dx' - \int_{\Omega^*_{\gamma}} (\mathcal{C}^0 e(u^*_\alpha), e(u^*_\alpha))$$

$$+ M_d^{\alpha} + O(1)\varepsilon^{\min\{\frac{d}{d-1}, 1\}}, \quad (4.57)$$

where

$$M_d^{\alpha} = \int_{\Omega^*_{\gamma}} (\mathcal{C}^0 e(\bar{u}^*_\alpha), e(\bar{u}^*_\alpha)) + \int_{\Omega^*_{\gamma}} (\mathcal{C}^0 e(u^*_\alpha - \bar{u}^*_\alpha), e(u^*_\alpha - \bar{u}^*_\alpha))$$

$$+ \int_{\Omega^*_{\gamma}} [(2\mathcal{C}^0 e(u^*_\alpha - u^*_\alpha), e(\bar{u}^*_\alpha)) + 2\mathcal{C}^0 e(\psi_a \bar{v}^*), e(F_a^\gamma)] + (\mathcal{C}^0 e(F_a^\gamma), e(F_a^\gamma))]$$

$$+ \int_{\Omega^*_{\gamma}} \left[ \frac{\mu(x^2_{a-d} + x^2_{a-d})}{d(k-1)} \sum_{k=1}^{d-1} (\partial_{x_a} \bar{v}^*)^2 + \mu(\lambda + \mu)(x_a \partial_{x_a} \bar{v}^*)^2 \right]$$

$$- 2(\lambda + \mu)(x_a \partial_{x_a} \bar{v}^*)^2 \partial_{x_a} \bar{v}^*] \right) + \left( \alpha = d + 1, \ldots, \frac{d(d+1)}{2}, \right)$$

Then from (4.43) and (4.50)–(4.57), we get that

(a) if $m \geq d + 1$, then

$$a_{\alpha \alpha} = \frac{L^d_{\alpha}}{d-1} \left( \int_{\varepsilon^\gamma < |x'| < R} \frac{|x'|^2}{h_1(x') - h_2(x')} + \int_{|x'| < \varepsilon} \frac{|x'|^2}{\varepsilon + h_1(x') - h_2(x')} \right)$$

$$+ M_d^{\alpha} + O(1)\varepsilon^{\min\{\frac{d}{d-1}, 1\}}, \quad \alpha = d + 1, \ldots, \frac{d(d+1)}{2}. \quad (4.58)$$
Note that
\[
\int_{|x'|<R} \frac{|x'|^2}{\varepsilon + h_1 - h} + \int_{|x'|<\varepsilon'} \frac{|x'|^2}{\varepsilon + h_1 - h}
= \int_{|x'|<R} \frac{|x'|^2}{\varepsilon + h_1 - h} + \int_{\varepsilon<|x'|<R} \frac{\varepsilon|x'|^2}{(h_1 - h)(\varepsilon + h_1 - h)}
= \int_{|x'|<R} \frac{|x'|^2}{\varepsilon + h_1 - h} + O(1) \begin{cases} \varepsilon, & m < \frac{d+1}{2}, \\ \varepsilon \ln \varepsilon, & m = \frac{d+1}{2}, \\ \varepsilon^{-\frac{10m-d+1}{12m}}, & m > \frac{d+1}{2}, \end{cases}
\]
(4.59)
Then substituting (4.59) into (4.58), we get
\[
a_{\alpha\alpha} \lesssim \frac{L_d^2}{d - 1} \int_{|x'|<R} \frac{|x'|^2}{\varepsilon + h_1 (x') - h(x')} \lesssim \frac{L_d^2}{d - 1} \int_{|x'|<R} \frac{|x'|^2}{\varepsilon + \kappa_1 |x'|^m}
\lesssim \frac{L_d^2}{d - 1} \int_{0}^{R} \frac{s^d}{\varepsilon + \kappa_1 s^m} \lesssim \frac{L_d^2}{d - 1} \frac{C_2(d, m; \varepsilon)}{\kappa_1^m}
\]
and
\[
a_{\alpha\alpha} \gtrsim \frac{L_d^2}{d - 1} \int_{|x'|<R} \frac{|x'|^2}{\varepsilon + \kappa_2 |x'|^m} \gtrsim \frac{L_d^2}{d - 1} \frac{C_2(d, m; \varepsilon)}{\kappa_2^m}
\]
(b) if \(m < d + 1\), for \(\alpha = d + 1, \ldots, 2d - 1\), we obtain
\[
a_{\alpha\alpha} = L_d^\alpha \left( \int_{\varepsilon<|x'|<R} \frac{x_{\alpha-d}^2}{h_1 - h} + \int_{|x'|<\varepsilon'} \frac{x_{\alpha-d}^2}{\varepsilon + h_1 - h} \right) + M_d^{*\alpha} + O(1)\varepsilon^{\min\{\frac{1}{2}, \frac{d+1}{2d+1}\}}
\]
\[
= L_d^\alpha \left( \int_{|x'|<R} \frac{x_{\alpha-d}^2}{h_1 - h} - \int_{|x'|<\varepsilon'} \frac{\varepsilon x_{\alpha-d}^2}{(h_1 - h)(\varepsilon + h_1 - h)} \right)
+ M_d^{*\alpha} + O(1)\varepsilon^{\min\{\frac{1}{2}, \frac{d+1}{2d+1}\}}
= L_d^\alpha \int_{\Omega_R} |x_{\alpha-d} \partial_x \bar{\sigma}|^2 + M_d^{*\alpha} + O(1)\varepsilon^{\min\{\frac{1}{2}, \frac{d+1}{2d+1}\}}
= a_{\alpha\alpha}^* + O(1)\varepsilon^{\min\{\frac{1}{2}, \frac{d+1}{2d+1}\}},
\]
while, for \(\alpha = 2d, \ldots, \frac{(d+1)\alpha}{2}\), \(d \geq 3\),
\[
a_{\alpha\alpha} = L_d^\alpha \left( \int_{\varepsilon<|x'|<R} \frac{x_{\alpha}^2 + x_{\beta}^2}{h_1 - h} + \int_{|x'|<\varepsilon'} \frac{x_{\alpha}^2 + x_{\beta}^2}{\varepsilon + h_1 - h} \right)
+ M_d^{*\alpha} + O(1)\varepsilon^{\min\{\frac{1}{2}, \frac{d+1}{2d+1}\}}
\]
\[
= L_d^\alpha \int_{|x'|<R} \frac{x_{\alpha}^2 + x_{\beta}^2}{h_1 - h} + M_d^{*\alpha} + O(1)\varepsilon^{\min\{\frac{1}{2}, \frac{d+1}{2d+1}\}}
= L_d^\alpha \int_{\Omega_R} (x_{\alpha}^2 + x_{\beta}^2) |\partial_x \bar{\sigma}|^2 + M_d^{*\alpha} + O(1)\varepsilon^{\min\{\frac{1}{2}, \frac{d+1}{2d+1}\}}
= a_{\alpha\alpha}^* + O(1)\varepsilon^{\min\{\frac{1}{2}, \frac{d+1}{2d+1}\}}.
\]
Hence, (4.36) is established by combining the results above.
**Step 3.** Proofs of (4.37)–(4.40). Set \(\bar{\gamma} = \frac{1}{12m}\) again. By symmetry, we only need to calculate \(a_{\alpha\beta}\) under the case of \(\alpha < \beta\). Analogously as above, for \(\alpha, \beta = \)
Corollary 4.1, (4.43) and (4.55) that $R^2$.

For the term II, we further decompose it as follows:

$$a_{\alpha\beta} = \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (C^0 e(u_\alpha), e(u_\beta)) + \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (C^0 e(u_\alpha), e(u_\beta)) + \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (C^0 e(u_\alpha), e(u_\beta))$$

$$= I + II + III.$$

First of all, similar to (4.44), we get

$$I = \int_{D_1 \cup D_2 \cup D_3} (C^0 e(u_\alpha), e(u_\beta)) + O(1) \varepsilon$$

$$= \int_{D_1 \cup D_2 \cup D_3} [(C^0 e(u_\alpha^*), e(u_\beta^*)) + (C^0 e(u_\alpha - u_{\alpha}^*), e(u_{\beta} - u_{\beta}^*))]$$

$$+ \int_{D_1 \cup D_2 \cup D_3} [(C^0 e(u_\alpha^*), e(u_{\beta} - u_{\beta}^*)) + (C^0 e(u_\alpha - u_{\alpha}^*), e(u_{\beta}^*))]$$

$$= \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (C^0 e(u_\alpha^*), e(u_{\beta}^*)) + O(1) \varepsilon^{\frac{3}{2}}. \quad (4.60)$$

With regard to the second term II, we further decompose it as follows:

$$II_1 = \int_{(\Omega_\delta \setminus \Omega_\epsilon^*) \setminus (\Omega_\delta \setminus \Omega_\epsilon^*)} (C^0 e(u_\alpha^*), e(u_{\beta}^*)) + \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (C^0 e(u_\alpha - u_{\alpha}^*), e(u_{\beta} - u_{\beta}^*))$$

$$+ \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (C^0 e(u_\alpha - u_{\alpha}^*), e(u_{\beta}^*)) + \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (C^0 e(u_\alpha^*), e(u_{\beta} - u_{\beta}^*)),$n

$$II_2 = \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (C^0 e(u_\alpha^*), e(u_{\beta}^*)).$$

Based on the fact that the thickness of $(\Omega_\delta \setminus \Omega_\epsilon^*) \setminus (\Omega_\delta \setminus \Omega_\epsilon^*)$ is $\varepsilon$, we deduce from Corollary 4.1, 4.3 and 4.55 that

$$II_1 = O(1) \varepsilon^{\frac{3}{2}}. \quad (4.61)$$

For the term II_2, we divide into two cases to discuss as follows:

1. if $d = 2$, $\alpha = 1, \beta = 2$, then from (4.12) we get

$$II_2 = \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (C^0 e(\bar{u}_1^*), e(\bar{u}_2^*)) + \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (C^0 e(u_1^* - \bar{u}_1^*), e(u_2 - \bar{u}_2^*))$$

$$+ \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (C^0 e(u_1^* - \bar{u}_1^*), e(\bar{u}_2^*)) + \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (C^0 e(\bar{u}_1^*), e(u_2^* - \bar{u}_2^*))$$

$$= \int_{\Omega_\delta \setminus \Omega_\epsilon^*} (\lambda + \mu) \partial x_1 v^* \partial x_2 v^* + O(1)$$

$$= O(1) |\ln \varepsilon|; \quad (4.62)$$
(2) if \( \alpha, \beta = 1, 2, \ldots, d, \alpha < \beta \), if \( \alpha = 1, 2, \ldots, d, \beta = d+1, \ldots, \frac{d(d+1)}{2}, \alpha < \beta \), or if \( \alpha = 1, 2, \ldots, d+1, \ldots, \frac{d(d+1)}{2}, \alpha < \beta \), then from (4.12) and (4.26) we obtain

\[
\Pi_2 - \int_{\Omega^*} (C^0 e(u^*_\alpha), e(u^*_\beta)) = - \int_{\Omega^*} (C^0 e(u^*_\alpha), e(u^*_\beta))
\]

\[
= \int_{\Omega_{\alpha\gamma}} (C^0 e(u^*_\alpha), e(u^*_\beta)) + \int_{\Omega_{\beta\gamma}} (C^0 e(u^*_\alpha), e(u^*_\beta))
\]

\[
+ \int_{\Omega_{\alpha\gamma}} (C^0 e(u^*_\beta), e(u^*_\beta)) + \int_{\Omega_{\beta\gamma}} (C^0 e(u^*_\beta), e(u^*_\beta))
\]

\[
= \int_{\Omega_{\alpha\gamma}} (C^0 e(\psi^*_{\alpha}), e(\psi^*_{\beta})) + O(1) \left\{ \begin{array}{l}
eq e^{(d-1)\bar{\gamma}}, \\
\alpha = 1, 2, \ldots, d, \beta = 1, 2, \ldots, \frac{d(d+1)}{2}, \alpha < \beta, \\
\alpha, \beta = d + 1, \ldots, \frac{d(d+1)}{2}, \alpha < \beta,
\end{array} \right. \]  

(4.63)

where we utilized the fact that

\[
(C^0 e(u^*_\alpha), e(u^*_\beta)) = (C^0 e(\psi^*_{\alpha}), e(\psi^*_{\beta})) + (C^0 e(F^*_{\alpha}), e(F^*_{\beta}))
\]

\[
+ (C^0 e(\psi^*_{\alpha}), e(F^*_{\beta})) + (C^0 e(F^*_{\alpha}), e(\psi^*_{\beta})).
\]

Denote \( E_{\alpha\beta}(\bar{v}^*) = (C^0 e(\psi^*_{\alpha}), e(\psi^*_{\beta})) \). Then it follows from a direct calculation that

(i) if \( \alpha, \beta = 1, 2, \ldots, d, \alpha < \beta \), then

\[
E_{\alpha\beta}(\bar{v}^*) = (\lambda + \mu) \partial_{x_{i\alpha}} \bar{v}^* \partial_{x_{\beta}} \bar{v}^*;
\]  

(4.64)

(ii) if \( \alpha = 1, 2, \ldots, d, \beta = d + 1, \ldots, \frac{d(d+1)}{2}, \) then there exist two indices \( 1 \leq i_\beta < j_\beta \leq d \) such that \( \psi^*_{\beta} = (0, \ldots, 0, x_{j_\beta} \bar{v}^*, 0, \ldots, 0, -x_{i_\beta} \bar{v}^*, 0, \ldots, 0) \). In the case of \( i_\beta \neq \alpha, j_\beta \neq \alpha \), we have

\[
E_{\alpha\beta}(\bar{v}^*) = (\lambda + \mu) \partial_{x_{i\alpha}} \bar{v}^* (x_{j_\beta} \partial_{x_{i\beta}} \bar{v}^* - x_{i_\beta} \partial_{x_{j_\beta}} \bar{v}^*),
\]  

(4.65)

and in the case of \( i_\beta = \alpha, j_\beta \neq \alpha \), then

\[
E_{\alpha\beta}(\bar{v}^*) = \mu x_{j_\beta} \sum_{k=1}^{d} (\partial_{x_{k}} \bar{v}^*)^2 + (\lambda + \mu) \partial_{x_{i\alpha}} \bar{v}^* (x_{j_\beta} \partial_{x_{i\beta}} \bar{v}^* - x_{i_\beta} \partial_{x_{j_\beta}} \bar{v}^*),
\]  

(4.66)

and in the case of \( i_\beta \neq \alpha, j_\beta = \alpha \), then

\[
E_{\alpha\beta}(\bar{v}^*) = - \mu x_{i_\beta} \sum_{k=1}^{d} (\partial_{x_{k}} \bar{v}^*)^2 + (\lambda + \mu) \partial_{x_{i\alpha}} \bar{v}^* (x_{j_\beta} \partial_{x_{i\beta}} \bar{v}^* - x_{i_\beta} \partial_{x_{j_\beta}} \bar{v}^*);
\]  

(4.67)

(iii) if \( \alpha, \beta = d+1, \ldots, \frac{d(d+1)}{2}, \alpha < \beta \), then there exist four indices \( 1 \leq i_\alpha < j_\alpha \leq d \) and \( 1 \leq i_\beta < j_\beta \leq d \) such that \( \psi^*_{\alpha} = (0, \ldots, 0, x_{j_\alpha} \bar{v}^*, 0, \ldots, 0, -x_{i_\alpha} \bar{v}^*, 0, \ldots, 0) \) and \( \psi^*_{\beta} = (0, \ldots, 0, x_{j_\beta} \bar{v}^*, 0, \ldots, 0, -x_{i_\beta} \bar{v}^*, 0, \ldots, 0) \). In view of the fact that \( \alpha < \beta \), we get \( j_\beta \leq j_\alpha \). In the case of \( i_\alpha \neq i_\beta, j_\alpha \neq j_\beta, i_\alpha \neq j_\beta \), we get

\[
E_{\alpha\beta}(\bar{v}^*) = (\lambda + \mu) (x_{j_\alpha} \partial_{x_{i\alpha}} \bar{v}^* - x_{i_\alpha} \partial_{x_{j_\alpha}} \bar{v}^*) (x_{j_\beta} \partial_{x_{i\beta}} \bar{v}^* - x_{i_\beta} \partial_{x_{j_\beta}} \bar{v}^*),
\]  

(4.68)
and in the case of $i_\alpha = i_\beta$, $j_\alpha \neq j_\beta$, 

\[
E_{i_\alpha \beta}(\bar{v}^*) = \mu x_{i_\beta} x_{j_\beta} \sum_{k=1}^{d} (\partial_{x_k} \bar{v}^*)^2 \\
+ (\lambda + \mu)(x_{j_\alpha} \partial_{x_{j_\alpha}} \bar{v}^* - x_{i_\alpha} \partial_{x_{i_\alpha}} \bar{v}^*) (x_{j_\beta} \partial_{x_{j_\beta}} \bar{v}^* - x_{i_\beta} \partial_{x_{i_\beta}} \bar{v}^*),
\]

(4.69)

and in the case of $i_\alpha \neq i_\beta$, $j_\alpha = j_\beta$, 

\[
E_{i_\alpha \beta}(\bar{v}^*) = \mu x_{i_\beta} x_{j_\beta} \sum_{k=1}^{d} (\partial_{x_k} \bar{v}^*)^2 \\
+ (\lambda + \mu)(x_{j_\alpha} \partial_{x_{j_\alpha}} \bar{v}^* - x_{i_\alpha} \partial_{x_{i_\alpha}} \bar{v}^*) (x_{j_\beta} \partial_{x_{j_\beta}} \bar{v}^* - x_{i_\beta} \partial_{x_{i_\beta}} \bar{v}^*),
\]

(4.70)

and in the case of $i_\beta < j_\beta = i_\alpha < j_\alpha$, 

\[
E_{i_\alpha \beta}(\bar{v}^*) = -\mu x_{i_\beta} x_{j_\beta} \sum_{k=1}^{d} (\partial_{x_k} \bar{v}^*)^2 \\
+ (\lambda + \mu)(x_{j_\alpha} \partial_{x_{j_\alpha}} \bar{v}^* - x_{i_\alpha} \partial_{x_{i_\alpha}} \bar{v}^*) (x_{j_\beta} \partial_{x_{j_\beta}} \bar{v}^* - x_{i_\beta} \partial_{x_{i_\beta}} \bar{v}^*),
\]

(4.71)

Hence, utilizing the parity of integrand and the symmetry of integral region and in view of the fact that

\[
\left| \int_{h(x')}^{h_1(x')} x_d dx_d \right| \leq \frac{1}{2}|(h_1 + h)(x')(h_1 - h)(x')| \leq C|x'|^{2m}, \quad \text{in } B'_R,
\]

we deduce from (4.63)--(4.71) that

\[
\Pi_2 = \int_{\Omega_R} (\mathbb{C}^0 e(u^*_\alpha), e(u^*_\beta)) \\
\begin{cases}
\varepsilon^{\frac{d-2}{2m}}, & d \geq 3, \alpha, \beta = 1, 2, ..., d, \alpha < \beta, \\
\varepsilon^{\frac{d}{2m}}, & d \geq 2, \alpha = 1, 2, ..., d, \beta = d + 1, ..., \frac{d(d+1)}{2}, \\
\varepsilon^{\frac{d}{2m}}, & d \geq 3, \alpha, \beta = d + 1, ..., \frac{d(d+1)}{2}, \alpha < \beta.
\end{cases}
\]

(4.72)

Consequently, from (4.61)--(4.62) and (4.72) we get that

\[
\Pi = O(1) \ln \varepsilon, \quad d = 2, \alpha = 1, \beta = 2,
\]

(4.73)

and

\[
\Pi = \int_{\Omega_R} (\mathbb{C}^0 e(u^*_\alpha), e(u^*_\beta)) \\
\begin{cases}
\varepsilon^{\min\left\{\frac{d-2}{2m}, \frac{d-1}{2m}\right\}}, & d \geq 3, \alpha, \beta = 1, 2, ..., d, \alpha < \beta \\
\varepsilon^{\min\left\{\frac{d}{2m}, \frac{d-2}{2m}\right\}}, & d \geq 2, \alpha = 1, 2, ..., d, \beta = d + 1, ..., \frac{d(d+1)}{2}, \\
\varepsilon^{\min\left\{\frac{d}{2m}, \frac{d}{2m}\right\}}, & d \geq 3, \alpha, \beta = d + 1, ..., \frac{d(d+1)}{2}, \alpha < \beta.
\end{cases}
\]

(4.74)
With regard to the last part III, following the same argument as in (4.63), we have

\[ III = \int_{\Omega_\gamma} \mathbb{C}^0 e(u_\alpha) e(u_\beta) \]
\[ = \int_{\Omega_\gamma} \mathbb{C}^0 e(\psi_\alpha \bar{v}) e(\psi_\beta \bar{v}) \]
\[ + O(1) \left\{ \begin{array}{l}
\varepsilon^{(d-1)\bar{\gamma}}, \quad \alpha = 1, 2, \ldots, d, \beta = 1, 2, \ldots, \frac{d(d+1)}{2}, \alpha < \beta, \\
\varepsilon^{d\bar{\gamma}}, \quad \alpha = d + 1, \ldots, \frac{d(d+1)}{2}, \alpha < \beta.
\end{array} \right. \]  \tag{4.75}

Proceeding as in (4.64)–(4.71) with \( \bar{v} \) substituting for \( \bar{v}^* \), we conclude from (4.75) that

\[ III = O(1) \left\{ \begin{array}{l}
|\ln \varepsilon|, \quad d = 2, \alpha = 1, \beta = 2, \\
\varepsilon^{\frac{d-2}{2m}}, \quad d \geq 3, \alpha, \beta = 1, 2, \ldots, d, \alpha < \beta, \\
\varepsilon^{\frac{d-2}{2m}}, \quad d \geq 2, \alpha = 1, 2, \ldots, d, \beta = d + 1, \ldots, \frac{d(d+1)}{2}, \\
\varepsilon^{\frac{d-2}{2m}}, \quad d \geq 3, \alpha, \beta = d + 1, \ldots, \frac{d(d+1)}{2}, \alpha < \beta.
\end{array} \right. \]  \tag{4.76}

Then combining (4.60), (4.73)–(4.74) and (4.76), we obtain that

\[ a_{12} = O(1)|\ln \varepsilon|, \quad d = 2, \]
\[ a_{\alpha \beta} - a_{\alpha \beta}^* = O(1) \left\{ \begin{array}{l}
\varepsilon^{\min\left(\frac{1}{2}, \frac{d-2}{2m}\right)}, \quad d = 3, \alpha, \beta = 1, 2, \ldots, d, \alpha < \beta, \\
\varepsilon^{\min\left(\frac{1}{2}, \frac{d-2}{2m}\right)}, \quad d \geq 2, \alpha = 1, 2, \ldots, d, \beta = d + 1, \ldots, \frac{d(d+1)}{2}, \\
\varepsilon^{\min\left(\frac{1}{2}, \frac{d-2}{2m}\right)}, \quad d \geq 3, \alpha, \beta = d + 1, \ldots, \frac{d(d+1)}{2}, \alpha < \beta.
\end{array} \right. \]

\[ \square \]

4.3. The proofs of Theorems 2.1 and 2.5. To begin with, we state a Lemma with its proof seen in [8], which will be used to prove Theorems 2.1 and 2.5 in the following.

Lemma 4.4. There exists a positive universal constant \( C \), independent of \( \varepsilon \), such that

\[ \sum_{i,j=1}^{\frac{d(d+1)}{2}} a_{ij} \xi_i \xi_j \geq \frac{1}{C}, \quad \forall \xi \in \mathbb{R}^{\frac{d(d+1)}{2}}, |\xi| = 1. \]  \tag{4.77}

In this section, \( a \preceq b \) denotes \( \frac{a}{b} = 1 + o(1) \), where \( b \neq 0 \) and \( \lim_{\varepsilon \to 0} o(1) = 0 \). This means that \( b \) is approximately equal to \( a \). Before proving our main results, we first introduce some notations. Denote

\[ X = \left( C^1, C^2, \ldots, C^{\frac{d(d+1)}{2}} \right)^T, \quad Y = \left( Q_1[\varphi], Q_2[\varphi], \ldots, Q_{\frac{d(d+1)}{2}}[\varphi] \right)^T, \]
and

\[
A = \begin{pmatrix}
  a_{11} & \cdots & a_{1d} \\
  \vdots & \ddots & \vdots \\
  a_{d1} & \cdots & a_{dd}
\end{pmatrix},
B = \begin{pmatrix}
  a_{1d+1} & \cdots & a_{1(d+1)} \\
  \vdots & \ddots & \vdots \\
  a_{d(d+1)} & \cdots & a_{d(d+1)}
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
  a_{d+11} & \cdots & a_{d+1d} \\
  \vdots & \ddots & \vdots \\
  a_{d(d+1)} & \cdots & a_{d(d+1)}
\end{pmatrix},
D = \begin{pmatrix}
  a_{d+1d+1} & \cdots & a_{d+1(d+1)} \\
  \vdots & \ddots & \vdots \\
  a_{d(d+1)} & \cdots & a_{d(d+1)}
\end{pmatrix}.
\]

(4.78)

Let

\[
F = \begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}.
\]

(4.79)

Then (4.72) can be written as

\[
FX = Y.
\]

(4.80)

The proofs of Theorems 2.1 and 2.5. Step 1. Let \(m \geq d + 1\). Applying the Cramer’s rule to (4.80), it follows from Lemmas 4.2 and 4.3 that

\[
C^\alpha \simeq \frac{\prod_{i \neq \alpha} a_{i\alpha} Q_\alpha[\varphi]}{\prod_{i=1}^{d+1} a_{ii}} \frac{Q_\alpha[\varphi]}{a_{\alpha\alpha}},
\]

(4.81)

which yields that

(i) if condition (E1) holds, then

\[
|C^\alpha| \lesssim \begin{cases}
  \frac{d-1}{\xi_2} \frac{\rho_1(d,m;\epsilon)}{\rho_2(d,m;\epsilon)}, & \alpha = 1, \ldots, d, m \geq d + k - 1, \\
  \frac{2^{d+3}}{\xi_2} \left[ Q_\alpha^*[\varphi] \right] \frac{1}{\rho_1(d,m;\epsilon)}, & \alpha = 1, \ldots, d, d + 1 \leq m < d + k - 1, \\
  \frac{d+1}{\xi_2} \frac{[Q_\alpha^*[\varphi]]^*}{\rho_1(d,m;\epsilon)}, & \alpha = d + 1, \ldots, \frac{d(d+1)}{2}, m \geq d + 1,
\end{cases}
\]

(4.82)
and

\[
|C^\alpha| \gtrless \begin{cases} 
\frac{\eta_k m^{-1}}{\rho_k(d,m;\varepsilon)} & \alpha = 1, 2, ..., d, m \geq d + k - 1, \\
\frac{\eta_{k+1} m^{-1}}{\rho_{k+1}(d,m;\varepsilon)} & \alpha = d + 1, m \geq d + k + 1,
\end{cases}
\]

(4.82)

(ii) if condition (E2) holds, then

\[
|C^\alpha| \lesssim \begin{cases} 
\frac{\kappa_k m^{-1}}{\rho_k(d,m;\varepsilon)} & \alpha = 1, 2, ..., d, m \geq d + 1, \\
\frac{\eta_{k+1} m^{-1}}{\rho_{k+1}(d,m;\varepsilon)} & \alpha = d + 1, m \geq d + k,
\end{cases}
\]

(4.83)

and

\[
|C^\alpha| \gtrsim \begin{cases} 
\frac{\kappa_k m^{-1}}{\rho_k(d,m;\varepsilon)} & \alpha = 1, 2, ..., d, m \geq d + 1, \\
\frac{\eta_{k+1} m^{-1}}{\rho_{k+1}(d,m;\varepsilon)} & \alpha = d + 1, m \geq d + k + 1,
\end{cases}
\]

(4.84)

(iii) if condition (E3) holds, then

\[
|C^\alpha| \lesssim \begin{cases} 
\frac{\kappa_k m^{-1}}{\rho_k(d,m;\varepsilon)} & \alpha = 1, 2, ..., d, m \geq d + 1, \\
\frac{\eta_{k+1} m^{-1}}{\rho_{k+1}(d,m;\varepsilon)} & \alpha = d + 1, m \geq d + k + 1,
\end{cases}
\]

(4.85)

and

\[
|C^\alpha| \gtrsim \begin{cases} 
\frac{\kappa_k m^{-1}}{\rho_k(d,m;\varepsilon)} & \alpha = 1, 2, ..., d, m \geq d + 1, \\
\frac{\eta_{k+1} m^{-1}}{\rho_{k+1}(d,m;\varepsilon)} & \alpha = d + 1, m \geq d + k + 1,
\end{cases}
\]

(4.86)

Step 2. Let \(d - 1 \leq m < d + 1\). Denote

\[
F_1^\alpha[\varphi] := \begin{pmatrix} Q_{\alpha}[\varphi] & a_{\alpha d+1} & \cdots & a_{\alpha d(d+1)} \\
Q_{d+1}[\varphi] & a_{d+1 d+1} & \cdots & a_{d+1 d(d+1)} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{d+1}[\varphi] & a_{d+1 d+1 d+1} & \cdots & a_{d+1 d(d+1) d(d+1)} \\
\end{pmatrix}, \quad \alpha = 1, 2, ..., d
\]

(4.87)

and, for \(\alpha = d+1, \ldots, d(d+1)\), by substituting the column vector \((Q_d[\varphi], \ldots, Q_{d(d+1)}[\varphi])^T\) for the elements of \(\alpha\)-th column in the matrix \(\mathbb{D}\) defined by (4.78), we get the new
Then it follows from the Cramer’s rule and Lemmas \[ \text{(4.2)} \] \ and \[ \text{(4.3)} \] that if condition (E1), (E2) or (E3) holds for \( d - 1 \leq m < d + 1 \),

\[
\begin{align*}
C^\alpha &\approx \left\{ \frac{\prod_{1 \leq a \leq d} \det \mathbb{F}_1^\alpha[\varphi]}{\det \mathbb{D}} \right\} \frac{1}{\rho_0(d, m; \varepsilon)}, \quad \alpha = 1, 2, ..., d, \\
&\approx \frac{d(d+1)}{2} \frac{\det \mathbb{F}_2^\alpha[\varphi]}{\det \mathbb{D}^*} \frac{1}{\rho_0(d, m; \varepsilon)}, \quad \alpha = d + 1, \ldots, \frac{d(d+1)}{2}.
\end{align*}
\]

Using Lemmas \[ \text{(4.2)} \] \ and \[ \text{(4.3)} \] again, we derive

\[
\begin{align*}
\det \mathbb{F}_1^\alpha[\varphi] &= \det \mathbb{F}_1^\alpha[\varphi] + O(1)^{\min\left(\frac{d(d+1)}{12m}, \frac{d(d+1)}{2d+1}\right)}, \quad \alpha = 1, 2, ..., d, \\
\det \mathbb{F}_2^\alpha[\varphi] &= \det \mathbb{F}_2[\varphi] + O(1)^{\min\left(\frac{d+1}{2m}, \frac{d+1}{2d+1}\right)}, \quad \alpha = d + 1, ..., \frac{d(d+1)}{2}.
\end{align*}
\]

Therefore, substituting \[ \text{(4.90)} \] \ into \[ \text{(4.89)} \], we deduce that for \( \alpha = 1, 2, ..., d \),

\[
\begin{align*}
\kappa_1 \frac{\det \mathbb{F}_1^\alpha[\varphi]}{\det \mathbb{D}^*} \frac{1}{\rho_0(d, m; \varepsilon)} \lesssim |C^\alpha| \lesssim \kappa_2 \frac{\det \mathbb{F}_2^\alpha[\varphi]}{\det \mathbb{D}^*} \frac{1}{\rho_0(d, m; \varepsilon)},
\end{align*}
\]

and

\[
C^\alpha \approx \frac{\det \mathbb{F}_2^\alpha[\varphi]}{\det \mathbb{D}^*}, \quad \alpha = d + 1, ..., \frac{d(d+1)}{2}.
\]

We now claim that \( \det \mathbb{D}^* > 0 \). Taking \( \xi = (\xi_1, \xi_2, ..., \xi_{d(d+1)}) \in \mathbb{R}^{d(d+1)} \) with \( \xi_1 = \cdots = \xi_d = 0 \) and \( |\xi| = 1 \) in \[ \text{(4.77)} \], we deduce from Lemma \[ \text{(4.3)} \] that

\[
\sum_{i,j=1}^{d(d+1)} a_{ij} \xi_i \xi_j = \sum_{i,j=d+1}^{d(d+1)} a_{ij} \xi_i \xi_j + O(\varepsilon^{\min\left(\frac{d(d+1)}{12m}, \frac{d+1}{2d+1}\right)}) \geq \frac{1}{C},
\]

where the constant \( C \) is independent of \( \varepsilon \). Therefore,

\[
\sum_{i,j=d+1}^{d(d+1)} a_{ij} \xi_i \xi_j \geq \frac{1}{C} > 0,
\]

which implies that the matrix \( \mathbb{D}^* \) is positive definite and thus \( \det \mathbb{D}^* > 0 \).

**Step 3.** Let \( m < d - 1 \). For \( \alpha = 1, 2, ..., \frac{d(d+1)}{2} \), by replacing the elements of \( \alpha \)-th column of the matrix \( \mathbb{F} \) defined in \[ \text{(4.79)} \] by column vector \((Q_1[\varphi], Q_2[\varphi], ..., Q_{\frac{d(d+1)}{2}}[\varphi])^T\),...
we obtain the new matrix $F_{3}^{α}[ϕ]$ as follows:

$$F_{3}^{α}[ϕ] =:\begin{pmatrix} a_{11} & \cdots & Q_{1}[ϕ] & \cdots & a_{1} & a_{1} & d+1 \frac{1}{2} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \ddots \\
a_{d+1} & \cdots & Q_{d} & \cdots & a_{d+1} & \cdots & a_{d+1} \end{pmatrix} \tag{4.95}$$

Then a consequence of Lemmas 4.2 and 4.3 yields that

$$F_{3}^{α}[ϕ] = F_{3}^{α}[ϕ] + O(1)\varepsilon \min\left\{ \frac{d-k-1}{d+2}, \frac{d-k-1}{12m} \right\}, \quad α = 1, 2, \ldots, d + \frac{1}{2},$$

and

$$F = F^{α} + O(1)\varepsilon \min\left\{ \frac{d-k-1}{d+2}, \frac{d-k-1}{12m} \right\}.$$ 

Similarly as before, we see from the Cramer’s rule that if condition (E1), (E2) or (E3) holds for $m < d - 1$,

$$C^{α} = \frac{\det F_{3}^{α}[ϕ]}{\det F} \sim \frac{\det F_{3}^{α}[ϕ]}{\det F^{α}}, \quad α = 1, 2, \ldots, d + \frac{1}{2}. \tag{4.96}$$

We now demonstrate that $\det F^{α} > 0$. Proceeding as before, it follows from Lemmas [4,3] and [4,4] that

$$\sum_{i,j=1}^{d+1} a_{ij} \xi \xi_j = \sum_{i,j=1}^{d+1} a_{ij}^{*} \xi \xi_j + O(\varepsilon \min\left\{ \frac{d-k-1}{d+2}, \frac{d-k-1}{12m} \right\}) \geq \frac{1}{C}, \quad \forall \xi \in \mathbb{R}^{d+1}, |\xi| = 1.$$ 

This reads that

$$\sum_{i,j=1}^{d+1} a_{ij}^{*} \xi \xi_j \geq \frac{1}{C}.$$ 

Thus $F^{α}$ is a positive definite matrix, which means that $\det F^{α} > 0$.

**Step 4.** In this step we aim to establish the optimal upper and lower bounds on the blow-up rate of the gradient in the shortest segment $\{x^{'} = 0\} \cap \Omega$.

(i) Let condition (E1), (E2) or (E3) hold for $m \leq d$. In light of decomposition \[3,2], we deduce from Corollary [4,1] and (4.93)–(4.96) that for $x \in \{x^{'} = 0\} \cap \Omega$,

$$|∇u| \leq \sum_{α=1}^{d} |C^{α}| |∇u_α| + \sum_{α=d+1}^{d+1} |C^{α}| |∇u_α| + |∇u_d|$$

$$\lesssim \left\{ \begin{array}{ll}
\max_{1 \leq α \leq d} \frac{d-k-1}{d+2} \frac{|C^{α}_{d+1}|}{|C^{α}_{d+1}|} \frac{1}{\varepsilon^{k} (d,m; ε)} & \text{for } d - 1 \leq m \leq d,
\max_{1 \leq α \leq d} \frac{|\det \mathbb{F}_{3}^{α}[ϕ]|}{|\det \mathbb{F}|} \frac{1}{\varepsilon} & \text{for } m < d - 1,
\end{array} \right.$$
and

$$|\nabla u| \geq \sum_{\alpha=1}^{d} C^\alpha \nabla u_{\alpha} - \sum_{\alpha=d+1}^{d+1} C^\alpha \|\nabla u_{\alpha}\| - |\nabla u_{0}|$$

$$\geq \sum_{\alpha=1}^{d} C^\alpha \partial_{x_d} u_{\alpha} - C$$

$$\geq \begin{cases} \frac{d-1}{d-1} \frac{|\det A_{\alpha}^{\gamma}\alpha|}{|\det D^{\gamma}\alpha|} \frac{1}{\varepsilon \rho_0(d,m;\varepsilon)} \|\nabla u_0\|_{L_2} & d - 1 \leq m \leq d, \\ \frac{1}{\det F^{\gamma}\alpha} \frac{1}{\varepsilon \rho_0(d,m;\varepsilon)} & m < d - 1, \end{cases}$$

where we utilized the fact that for \( x \in \{ x' = 0' \} \cap \Omega, \)

$$|\partial_{x_d} u_{\alpha}^0| \geq |\partial_{x_d} \tilde{u}_{\alpha}^0| - |\partial_{x_d}(u_{\alpha}^0 - \tilde{u}_{\alpha}^0)| \geq \frac{1}{C\varepsilon},$$

and

$$|\partial_{x_d} u_{\alpha}^0| \leq |\partial_{x_d} \mathcal{F}_{\alpha}^0| + |\partial_{x_d}(u_{\alpha}^0 - \tilde{u}_{\alpha}^0)| \leq C, \quad \alpha = 1, 2, \ldots, d, \alpha \neq \alpha_0.$$

(ii) Let condition \((E1)\) hold for \( m \geq d + k \). Observe that for \( x \in \{ x' = 0' \} \cap \Omega, \) we deduce from Corollary \(4.1\) that \( |\nabla u_0| \leq C \), and \( |\nabla u_{\alpha}| \leq C, \alpha = d+1, \ldots, \frac{d(d+1)}{2}. \)

On one hand, using \((4.81)\), we have

$$|\nabla u| \leq \sum_{\alpha=1}^{d} \|C^\alpha\| |\nabla u_{\alpha}| + \sum_{\alpha=d+1}^{d+1} \|C^\alpha\| |\nabla u_{\alpha}| + |\nabla u_{0}|$$

$$\leq \frac{\eta m_\alpha^k}{\kappa_1} \rho_k(d,m;\varepsilon) \frac{1}{\varepsilon \rho_0(d,m;\varepsilon)}.$$

On the other hand, it follows from \((4.82)\) that

$$|\nabla u| \geq \sum_{\alpha=1}^{d} C^\alpha \nabla u_{\alpha} - \sum_{\alpha=d+1}^{d+1} C^\alpha \|\nabla u_{\alpha}\| - |\nabla u_{0}|$$

$$\geq \sum_{\alpha=1}^{d} C^\alpha \partial_{x_d} u_{\alpha} - C \geq \frac{\eta m_\alpha^k}{\kappa_1} \rho_k(d,m;\varepsilon) \frac{1}{\varepsilon \rho_0(d,m;\varepsilon)},$$

where we used the fact that for \( x \in \{ x' = 0' \} \cap \Omega, \)

$$|\partial_{x_d} u_{d}^0| \geq |\partial_{x_d} \tilde{u}_{d}^0| - |\partial_{x_d}(u_{d}^0 - \tilde{u}_{d}^0)| \geq \frac{1}{C\varepsilon},$$

and

$$|\partial_{x_d} u_{d}^0| \leq |\partial_{x_d} \mathcal{F}_{d}^0| + |\partial_{x_d}(u_{d}^0 - \tilde{u}_{d}^0)| \leq C, \quad \alpha = 1, \ldots, d - 1.$$

**Step 5.** This step is devoted to the establishments of the optimal upper and lower bounds on the blow-up rate of the gradient on the cylinder surface \( \{ |x'| = \sqrt{\varepsilon} \} \cap \Omega. \)

(i) Let condition \((E1), (E2)\) or \((E3)\) hold for \( d < m < d + k, k > 1. \) In view of \((4.81), (4.83), (4.85)\) and \((4.93)-(4.94), \) it follows from decomposition \((3.2)\) and
Therefore, from (4.97)–(4.99), we have

$$|\nabla u| \leq \sum_{\alpha=d+1}^{d(d+1)} |C\alpha||\nabla u_\alpha| + \left( \sum_{\alpha=1}^{d} |C\alpha||\nabla u_\alpha| + |\nabla u_0| \right)$$

$$\leq \left\{ \begin{array}{ll}
\max_{d+1 \leq \alpha \leq d(d+1)} \frac{|Q_d^\alpha[\varphi]|}{(1+\kappa_1) \varepsilon^{1-1/m} \rho_2(d,m;\varepsilon)}, & d + 1 \leq m < d + k, \\
\max_{d+1 \leq \alpha \leq d(d+1)} \frac{1}{\det \mathcal{F}_d^{\alpha}[\varphi]} \frac{1}{\varepsilon^{1-1/m}}, & d < m < d + 1.
\end{array} \right.$$  

Due to the fact that for $x \in \{ x' = (\sqrt{\varepsilon},0,\ldots,0)' \} \cap \Omega$,

$$|\partial_{x_d} u_{d+1}^d| \geq |\partial_{x_d} \bar{u}_{d+1}^d| - |\partial_{x_d}(u_{d+1}^d - \bar{u}_{d+1}^d)| \geq \frac{1}{C\varepsilon^{1-1/m}},$$

and

$$|\partial_{x_d} u_{\alpha}| \leq |\partial_{x_d} \mathcal{F}_d^\alpha| + |\partial_{x_d}(u_\alpha - \bar{u}_\alpha)| \leq C, \quad \alpha = d + 2, \ldots, \frac{d(d+1)}{2}, d \geq 3,$$

we obtain

$$\sum_{\alpha=d+1}^{d(d+1)} C\alpha \nabla u_\alpha \geq \sum_{\alpha=d+1}^{d(d+1)} C\alpha \partial_{x_d} u_{\alpha}^d$$

$$\geq C^{d+1} \partial_{x_d} u_{d+1}^d - \left\{ \begin{array}{ll}
0, & d = 2, \\
\sum_{\alpha=d+2}^{d(d+1)} C\alpha \partial_{x_d} u_{\alpha}^d, & d \geq 3
\end{array} \right.$$  

while for $x \in \{ x' = (\sqrt{\varepsilon},0,\ldots,0)' \} \cap \Omega$,

$$|\nabla u_0| \leq \frac{C|x'|^k}{\varepsilon + \kappa_1 |x'|^m} \leq \left\{ \begin{array}{ll}
\varepsilon^{1-1/m}, & m > k, \\
1, & m \leq k.
\end{array} \right.$$  

and

$$\sum_{\alpha=1}^{d} C\alpha||\nabla u_\alpha| \leq \left\{ \begin{array}{ll}
\frac{\rho_0(d,m;\varepsilon)}{\varepsilon \rho_0(d,m;\varepsilon)}, & \text{if condition (E1) holds}, \\
\frac{1}{\rho_0(d,m;\varepsilon)}, & \text{if condition (E2) or (E3) holds}.
\end{array} \right.$$  

Therefore, from (4.97)–(4.99), we have

$$|\nabla u| \geq \sum_{\alpha=d+1}^{d(d+1)} C\alpha \nabla u_\alpha - \sum_{\alpha=1}^{d} C\alpha||\nabla u_\alpha| - |\nabla u_0|$$

$$\geq \left\{ \begin{array}{ll}
\sum_{\alpha=d+1}^{d(d+1)} C\alpha \partial_{x_d} u_{\alpha}^d - \sum_{\alpha=1}^{d} C\alpha||\nabla u_\alpha| - |\nabla u_0| & \text{if condition (E1) holds}, \\
\sum_{\alpha=d+1}^{d(d+1)} C\alpha \partial_{x_d} u_{\alpha}^d - \sum_{\alpha=1}^{d} C\alpha||\nabla u_\alpha| - |\nabla u_0| & \text{if condition (E2) or (E3) holds}.
\end{array} \right.$$
(ii) Let condition (E2) hold for \( m > d + k, \ k = 1 \). Similarly as before, we derive that for \( x \in \{ x' = (\sqrt{\varepsilon}, 0, \ldots, 0)' \} \cap \Omega \),

$$
\left| \nabla u \right| \leq \sum_{\alpha = d+1}^{d(d+1)} |C^\alpha||\nabla u_\alpha| + \left( \sum_{\alpha = 1}^{d} |C^\alpha||\nabla u_\alpha| + |\nabla u_0| \right)
$$

which, in combination with (4.98) – (4.99), yields that

$$
\left| \nabla u \right| \geq \left[ \sum_{\alpha = d+1}^{d(d+1)} C^\alpha \nabla u_\alpha \right] - \left\{ \sum_{\alpha = d+2}^{d(d+1)} C^\alpha \partial_{x_\alpha} u_\alpha^d \right\}, \quad d = 2,
$$

$$
\left| \nabla u \right| \geq \left\{ \sum_{\alpha = d+1}^{d(d+1)} C^\alpha \partial_{x_\alpha} u_\alpha^d \right\} - \left[ \sum_{\alpha = d+1}^{d(d+1)} C^\alpha ||\nabla u_\alpha| - |\nabla u_0| \right]
$$

(iii) Let condition (E3) hold for \( m > d + k, \ k \geq 1, \ k \neq 2 \) or \( m = d + k, \ k = 1 \). From (4.11), we see that for \( x \in \{ x' = (\sqrt{\varepsilon}, 0, \ldots, 0)' \} \cap \Omega \),

$$
\left| \nabla u_0 \right| \geq \frac{\varphi(x', h(x'))}{\varepsilon + \kappa_1 |x'|^m} \leq \frac{\eta |x_1|^k}{\varepsilon + \kappa_1 |x|^m} \leq \frac{\eta}{1 + \kappa_1 \varepsilon^{1-k/m}}
$$

$$
\left| \nabla u_0 \right| \geq \frac{\varphi^1(x', h(x'))}{\varepsilon + \kappa_2 |x'|^m} \geq \frac{\eta |x_1|^k}{\varepsilon + \kappa_2 |x|^m} \geq \frac{\eta}{1 + \kappa_2 \varepsilon^{1-k/m}}
$$

This, together with (4.85), reads that

$$
\left| \nabla u \right| \leq \left| \nabla u_0 \right| + \sum_{\alpha = 1}^{d(d+1)} |C^\alpha||\nabla u_\alpha| \lesssim \frac{\eta}{1 + \kappa_1 \varepsilon^{1-k/m}}
$$

and

$$
\left| \nabla u \right| \geq \left| \nabla u_0 \right| - \sum_{\alpha = 1}^{d(d+1)} |C^\alpha||\nabla u_\alpha| \gtrsim \frac{\eta}{1 + \kappa_2 \varepsilon^{1-k/m}}
$$

(iv) Now we proceed to consider the remaining two cases: condition (E2) holds in the case of \( m > d + k, \ k \geq 1, \ k \neq 2 \) or \( m = d + k, \ k = 1 \); condition (E3) holds in the case of \( m = d + k, \ k > 2 \). Similarly as above, it follows from (4.83), (4.85) and Corollary (4.1) that for \( x \in \{ x' = (\sqrt{\varepsilon}, 0, \ldots, 0)' \} \cap \Omega \), if condition (E2) holds for
\[ m > d + k, \ k \geq 1, \ k \neq 2 \text{ or } m = d + k, \ k = 1, \]
\[
|\nabla u| \leq (|C^{d+1}| |\nabla u_{d+1}| + |\nabla u_0|) + \sum_{\alpha=1, \alpha \neq d+1}^{d(d+1)} |C^{\alpha}||\nabla u_{\alpha}|
\]
\[
\leq \eta \left( \frac{\nu^{d+1}}{\kappa_2 ^{d+1}} \frac{d+1}{1 + \kappa_1} + 1 \right) + \frac{1}{\varepsilon^{1-k/m}}.
\]
and, if condition (E3) holds for \( m = d + k, \ k > 2, \)
\[
|\nabla u| \leq \left( \sum_{\alpha=d+1}^{d(d+1)} |C^{\alpha}||\nabla u_{\alpha}| + |\nabla u_0| \right) + \sum_{\alpha=1}^{d} |C^{\alpha}||\nabla u_{\alpha}|
\]
\[
\leq \frac{\max_{d+1 \leq \alpha \leq d(d+1)} \frac{\nu^{d+1}}{\kappa_2 ^{d+1}} |C^{\alpha}| |\nabla u_{\alpha}|}{1 + \kappa_1} + \frac{1}{\varepsilon^{1-k/m}}.
\]

On the other hand, utilizing (4.84) and (4.86), we deduce that
\[
\left| \sum_{\alpha=d+1}^{d(d+1)} C^{\alpha} \nabla u_{\alpha} + \nabla u_0 \right| \geq \left| \sum_{\alpha=d+1}^{d(d+1)} C^{\alpha} \partial_{x_d} u_{d+1}^{d} + \partial_{x_d} u_0^{d} \right|
\]
\[
\geq |C^{d+1} \partial_{x_d} u_{d+1}^{d} + \partial_{x_d} u_0^{d}| - \left\{ \begin{array}{ll}
0, & d = 2, \\
\sum_{\alpha=d+2}^{d(d+1)} C^{\alpha} \partial_{x_d} u_{\alpha}^{d}, & d \geq 3
\end{array} \right.
\]
\[
\geq \frac{1}{\varepsilon^{1-k/m}} \left\{ \begin{array}{ll}
\frac{\nu^{d+1}}{\kappa_2 ^{d+1}} + \frac{d+1}{1 + \kappa_2} + 1, & \text{if condition (E2) holds for } m = d + k, \ k = 1
\end{array} \right.
\]
\[
\text{or } m > d + k, \ k \geq 1, \ k \neq 2,
\]
\[
\geq \frac{d+1}{\kappa_2 ^{d+1}} \frac{d+1}{1 + \kappa_2} + 1, & \text{if condition (E3) holds for } m = d + k, \ k > 2,
\]
which, in combination with (4.39), yields that
\[
|\nabla u| \geq \left| \sum_{\alpha=d+1}^{d(d+1)} C^{\alpha} \nabla u_{\alpha} + \nabla u_0 \right| - \sum_{\alpha=1}^{d} |C^{\alpha}||\nabla u_{\alpha}|
\]
\[
\geq \frac{1}{\varepsilon^{1-k/m}} \left\{ \begin{array}{ll}
\frac{\nu^{d+1}}{\kappa_2 ^{d+1}} + \frac{d+1}{1 + \kappa_2} + 1, & \text{if condition (E2) holds for } m > d + k, \ k > 2
\end{array} \right.
\]
\[
\text{or } m = d + k, \ k = 1,
\]
\[
\geq \frac{d+1}{\kappa_2 ^{d+1}} \frac{d+1}{1 + \kappa_2} + 1, & \text{if condition (E3) holds for } m = d + k, \ k > 2,
\]
where in the case of condition (E3) we used the fact that \( \varphi^d = 0 \) on \( \Gamma_{R_{\varepsilon}}. \)
5. Asymptotic expansions of the stress concentration with strictly convex inclusions

For future application, in this section we aim to establish the asymptotic expansions of the stress concentration when one strictly convex inclusion approaches the matrix boundary closely. Suppose that $h_1$ and $h$ satisfy

$$(Q1) \quad h_1(x') > h(x'), \text{ if } x' \in B'_2 \setminus \{0\},$$

$$(Q2) \quad h_1(0') = h(0') = \nabla x' h_1(0') = \nabla x' h(0') = 0,$$

$$(Q3) \quad \nabla^2_x (h_1(0') - h(0')) \geq \tau_0 I,$$

$$(Q4) \quad ||h_1||_{C^{3,1}(B'_2)} + ||h||_{C^{3,1}(B'_2)} \leq \tau,$$

where $\tau_0$ and $\tau$ are two positive constants independent of $\varepsilon$ and $I$ denotes the $(d-1) \times (d-1)$ identity matrix. Our main result in this section is as follows:

**Theorem 5.1.** Assume that $D_1 \subset D \subseteq \mathbb{R}^d$ ($d \geq 2$) are defined as above, conditions $(Q1)$–$(Q4)$ hold, and $\varphi \in C^2(\partial D; \mathbb{R}^d)$. Let $u \in H^1(D; \mathbb{R}^d) \cap C^1(\Omega; \mathbb{R}^d)$ be the solution of $(4.78)$. Then for a sufficiently small $\varepsilon > 0$, $x \in \Omega_R$,

(i) for $d = 2$, if $\nabla x' \varphi(0) = 0$, then

$$\nabla u = \frac{2}{\varepsilon} \sum_{\alpha=1}^{2} \det F^{* \alpha} \left[ \varphi \right] \sqrt{\varepsilon \tau_1} \frac{1 + O(\varepsilon \frac{\pi}{\tau_1})}{\sqrt{2\pi \mathcal{L}^2_2}} \nabla \bar{u}_\alpha + \frac{2}{\varepsilon} \sum_{\alpha=1}^{2} \det F^{* \alpha} \left[ \varphi \right] \sqrt{\varepsilon \tau_2} \frac{1 + O(\varepsilon \frac{\pi}{\tau_2})}{\sqrt{2\pi \mathcal{L}^2_3}} \nabla \bar{u}_\alpha + O(1)||\varphi||_{C^2(\partial D)};$$

(ii) for $d = 3$, then

$$\nabla u = \sum_{\alpha=1}^{3} \det F^{* \alpha} \left[ \varphi \right] \sqrt{\varepsilon \tau_1} \frac{1 + O(\varepsilon \frac{\pi}{\tau_1})}{\sqrt{2\pi \mathcal{L}^3_2}} \nabla \bar{u}_\alpha + \frac{3}{\varepsilon} \sum_{\alpha=1}^{3} \det F^{* \alpha} \left[ \varphi \right] \sqrt{\varepsilon \tau_2} \frac{1 + O(\varepsilon \frac{\pi}{\tau_2})}{\sqrt{2\pi \mathcal{L}^3_3}} \nabla \bar{u}_\alpha + O(1)||\varphi||_{C^2(\partial D)};$$

(iii) for $d \geq 4$, then

$$\nabla u = \sum_{\alpha=1}^{\frac{d(d+1)}{2}} \det F^{* \alpha} \left[ \varphi \right] \sqrt{\varepsilon \tau_1} \frac{1 + O(\varepsilon \frac{\min\{\frac{1}{2}, \frac{d-3}{2}\}}{\tau_1})}{\sqrt{2\pi \mathcal{L}^3_3}} \nabla \bar{u}_\alpha + O(1)||\varphi||_{C^2(\partial D)},$$

where $\tau_1$ and $\tau_2$ are the eigenvalues of $\nabla^2_{x'} (h_1 - h)(0')$, $\mathcal{L}_d^2$, $\alpha = 1, 2, \ldots, d$ are defined by $(2.18)$–$(2.19)$, the blow-up factor matrices $\mathbb{D}$, $F^*_1[\varphi]$, $\alpha = 1, 2, \ldots, d$, $F^*_2[\varphi]$, $\alpha = d + 1, \ldots, \frac{d(d+1)}{2}$, $F^*_3[\varphi]$, $\alpha = 1, 2, \ldots, \frac{d(d+1)}{2}$, are defined by $(4.78)$, $(4.82)$ and $(4.85)$, respectively, the geometry constants $G^{* \alpha}_d$, $d = 2, 3$ are defined by $(5.11)$ below.

**Remark 5.2.** In contrast to the previous work [8], we improve the gradient estimates there to be the precise asymptotic expressions in Theorem 5.1 here. We additionally point out that the geometry constants $G^{* \alpha}_d$, $d = 2, 3$ depend not on the distance parameter $\varepsilon$ and the length $R$ of the narrow region $\Omega_R$, which is critical to numerical computations and simulations in future investigations. Although our asymptotic results can be generalized to more general $m$-convex inclusions, we restrict ourselves to the setup above for the convenience of computation and presentation.

To begin with, a direct application of Lemma 4.2 yields the following corollary.
Corollary 5.3. Assume as in Theorem 5.1. Then for a sufficiently small $\varepsilon > 0$,
(i) for $\alpha = 1, 2, \ldots, d$,

$$Q_\alpha[\varphi] = Q^*_\alpha[\varphi] + O(1) \begin{cases} \varepsilon^{\frac{1}{2}}, & d = 2, \\ \varepsilon^{\frac{d-2}{2d}}, & d \geq 3 \end{cases};$$  \quad (5.1)

(ii) for $\alpha = d + 1, \ldots, \frac{d(d+1)}{2}$,

$$Q_\alpha[\varphi] = Q^*_\alpha[\varphi] + O(1) \begin{cases} \varepsilon^{\frac{1}{4}}, & d = 2, \\ \varepsilon^{\frac{d-1}{2d}}, & d \geq 3 \end{cases},$$  \quad (5.2)

where $Q^*_\alpha[\varphi], \alpha = 1, 2, \ldots, \frac{d(d+1)}{2}$ are defined in (2.15).

Lemma 5.4. Assume as in Theorem 5.1. Then for a sufficiently small $\varepsilon > 0$, if $\alpha = 1, 2, \ldots, d$,
(i) for $d = 2$,

$$a_{\alpha\alpha} = \frac{\sqrt{2\pi L^2}}{\sqrt{\tau_1}} \frac{1}{\sqrt{\varepsilon}} + K_2^{\alpha} + O(1)\varepsilon^{\frac{1}{12}};$$

(ii) for $d = 3$,

$$a_{\alpha\alpha} = \frac{2\pi L^2}{\sqrt{\tau_1 \tau_2}} |\ln \varepsilon| + K_3^{\alpha} + O(1)\varepsilon^{\frac{1}{12}},$$

where $K_2^{\alpha}, d = 2, 3$ are defined by (5.10). $\tau_1$ and $\tau_2$ are the eigenvalues of $\nabla^2_x (h_1 - h)(0')$.

Remark 5.5. We here would like to point out that Li, Li and Yang [33] were the first to give the precise computation of the energy for the perfect conductivity problem in the presence of two strictly convex inclusions.

Proof. Analogous to [1, 49], we obtain that for $\alpha = 1, 2, \ldots, d$,

$$a_{\alpha\alpha} = L_d^\alpha \left( \int_{\varepsilon^{1/4} < |x'| < R} \frac{dx'}{h_1(x') - h(x')} + \int_{|x'| < \varepsilon^{1/4}} \frac{dx'}{h_1(x') - h(x')} \right) + M_d^{\alpha} + O(1)\varepsilon^{\min\left(\frac{1}{2}, \frac{d-1}{2d}\right)},$$  \quad (5.3)

where $M_d^{\alpha}$ is defined by (1.38). According to the assumed conditions (Q1)–(Q4), after a rotation of the coordinates if necessary, we have

$$h_1(x') - h(x') = \sum_{i=1}^{d-1} \frac{\tau_i}{2} x_i^2 + \sum_{|\alpha|=3} C_\alpha x'^\alpha + O(|x'|^4),$$

where $\text{diag}(\tau_1, \ldots, \tau_{d-1}) = \nabla^2_x (h_1 - h)(0'), C_\alpha$ are some constants, $\alpha$ is an $(d - 1)$-dimensional multi-index. In fact, $\tau_1, \ldots, \tau_{d-1}$ are the relative principal curvatures of
\[ \partial D_1 \] and \[ \partial D \] at the origin. Observe that
\[
\int_{|x'|<\varepsilon} \frac{1}{\varepsilon + h_1 - h} - \int_{|x'|<\varepsilon} \frac{1}{\varepsilon + \sum_{i=1}^{d-1} \frac{\tau_i}{2} x_i^2}
\]
\[
= \int_{|x'|<\varepsilon} \frac{1}{\varepsilon + \sum_{i=1}^{d-1} \frac{\tau_i}{2} x_i^2} \left[ \left( 1 + \sum_{|\alpha|=3} C_\alpha x^{\alpha} \right) \left( \varepsilon + \sum_{i=1}^{d-1} \frac{\lambda_i}{2} x_i^2 \right)^{-1} - 1 \right]
\]
\[
= - \int_{|x'|<\varepsilon} \sum_{|\alpha|=3} C_\alpha x^{\alpha} + \int_{|x'|<\varepsilon} O(1) = O(\varepsilon^{1+\alpha}), \tag{5.4}
\]
where in the last line the Taylor expansion was utilized in virtue of the smallness of \( R \) and we used the fact that \( \sum_{|\alpha|=3} C_\alpha x^{\alpha} \) is odd and the integrating domain is symmetric. Analogously, we get
\[
\int_{|x'|<\varepsilon R<|x'|<R} \frac{1}{\varepsilon + h_1 - h} - \int_{|x'|<\varepsilon R<|x'|<R} \frac{1}{\varepsilon + \sum_{i=1}^{d-1} \frac{\tau_i}{2} x_i^2} = C^* + O(\varepsilon^{1+\alpha}), \tag{5.5}
\]
where the constant \( C^* \) depends on \( d, R, \tau_1, \tau_2 \), but not on \( \varepsilon \). Observe that
\[
\int_{|x'|<\varepsilon} \frac{1}{\varepsilon + \sum_{i=1}^{d-1} \frac{\tau_i}{2} x_i^2} + \int_{|x'|<\varepsilon} \frac{1}{\varepsilon + \sum_{i=1}^{d-1} \frac{\tau_i}{2} x_i^2} \varepsilon
\]
\[
= \int_{|x'|<\varepsilon} \frac{1}{\varepsilon + \sum_{i=1}^{d-1} \frac{\tau_i}{2} x_i^2} \left( \varepsilon + \sum_{i=1}^{d-1} \frac{\tau_i}{2} x_i^2 \right) \left( \varepsilon + \sum_{i=1}^{d-1} \frac{\tau_i}{2} x_i^2 \right)
\]
\[
= \int_{|x'|<\varepsilon} \frac{1}{\varepsilon + \sum_{i=1}^{d-1} \frac{\tau_i}{2} x_i^2} + O(1)\varepsilon^{1+\alpha}.
\]
This, in combination with (5.4)–(5.5), leads to that
\[
\int_{|x'|<\varepsilon} \frac{dx'}{\varepsilon + h_1(x') - h(x')} + \int_{|x'|<\varepsilon} \frac{dx'}{\varepsilon + h_1(x') - h(x')}
\]
\[
= \int_{|x'|<\varepsilon} \frac{1}{\varepsilon + \sum_{i=1}^{d-1} \frac{\tau_i}{2} x_i^2} + C^* + O(\varepsilon^{1+\alpha}). \tag{5.6}
\]
Then, for \( d = 2 \), we have
\[
\int_{|x|<\varepsilon} \frac{1}{\varepsilon + \frac{\tau_1}{2} x_1^2} = \int_{-\infty}^{+\infty} \frac{1}{\varepsilon + \frac{\tau_1}{2} x_1^2} - \int_{|x|>\varepsilon} \frac{1}{\varepsilon + \frac{\tau_1}{2} x_1^2} + \int_{|x|>\varepsilon} \frac{\varepsilon}{\varepsilon + \frac{\tau_1}{2} x_1^2}
\]
\[
= \frac{\sqrt{2\pi}}{\sqrt{\tau_1}} \frac{1}{\varepsilon} - \frac{4}{\tau_1 R} + O(1)\varepsilon, \tag{5.7}
\]
while, for \( d = 3 \),
\[
\int_{|x|<\varepsilon} \frac{1}{\varepsilon + \sum_{i=1}^{2} \frac{\tau_i}{2} x_i^2} = \frac{8}{\sqrt{\tau_1 \tau_2}} \int_0^\frac{\pi}{2} \int_0^{R(\theta)} \frac{t}{\varepsilon + t^2} \, dt \, d\theta
\]
\[
= \frac{4}{\sqrt{\tau_1 \tau_2}} \int_0^\frac{\pi}{2} \left( \ln \varepsilon + \ln(R(\theta)^2) + \ln \left( 1 + \frac{\varepsilon}{R(\theta)^2} \right) \right) \, d\theta
\]
\[
= \frac{2\pi}{\sqrt{\tau_1 \tau_2}} \ln \varepsilon + \frac{8}{\sqrt{\tau_1 \tau_2}} \int_0^\frac{\pi}{2} \ln R(\theta) \, d\theta + O(\varepsilon), \tag{5.8}
\]
where \( R(\theta) := \frac{R}{\sqrt{2}} (\tau_1^{-1} \cos^2 \theta + \tau_2^{-1} \sin^2 \theta)^{-1/2} \) and in the last line we used the Taylor expansion of \( \ln(1 + x) \) for \( |x| < 1 \). Consequently, in view of (5.3), it follows
from (5.6)–(5.8) that for \( \alpha = 1, 2, ..., d \),

\[
a_{\alpha \alpha} = \begin{cases} 
\frac{\sqrt{\pi} \varepsilon L_2^{\alpha}}{2\pi L_2^{\alpha}} + \frac{1}{\varepsilon} + K_2^{\alpha} + O(1)\varepsilon, & d = 2, \\
\frac{\sqrt{\pi} \varepsilon L_2^{\alpha}}{2\pi L_2^{\alpha}} \ln \varepsilon + K_3^{\alpha} + O(1)\varepsilon, & d = 3,
\end{cases}
\]  

(5.9)

where

\[
K_d^{\alpha} = \begin{cases} 
\mathcal{L}_2^{\alpha} \left( C^{\alpha} - \frac{1}{\tau_1 R} \right), & d = 2, \\
\mathcal{L}_3^{\alpha} \left( C^{\alpha} + \frac{4}{\sqrt{\pi \tau_1}} \int_0^R \ln R(\theta) d\theta \right), & d = 3.
\end{cases}
\]  

(5.10)

We now demonstrate that the geometry constants \( K_d^{\alpha} \), \( d = 2, 3 \) are independent of the length parameter \( R \) of the thin gap \( \Omega_R \). Otherwise, assume that there exist \( \varepsilon \)-independent constants \( K_d^{\alpha}(R_1) \) and \( K_d^{\alpha}(R_2) \), \( R_1 \neq R_2 \), such that (5.1) holds. Then

\[
K_d^{\alpha}(R_1) - K_d^{\alpha}(R_2) = O(1)\varepsilon^{d-1},
\]

which yields that \( K_d^{\alpha}(R_1) = K_d^{\alpha}(R_2) \).

\[\Box\]

The proof of Theorem 5.7 Denote

\[
G_d^{\alpha} = \begin{cases} 
\frac{\sqrt{\pi} \varepsilon L_2^{\alpha}}{2\pi L_2^{\alpha}}, & d = 2, \\
\frac{\sqrt{\pi} \varepsilon L_2^{\alpha}}{2\pi L_2^{\alpha}}, & d = 3, \\
\alpha = 1, 2, ..., d.
\end{cases}
\]  

(5.11)

For \( \alpha = 1, 2, ..., d \), it follows from Lemma 5.4 that

\[
1 = \mathcal{A}_{\alpha \alpha} = \begin{cases} 
\frac{\sqrt{\pi} \varepsilon L_2^{\alpha}}{2\pi L_2^{\alpha}} \left( 1 - \frac{2\varepsilon L_2^{\alpha}}{\sqrt{\pi} \varepsilon L_2^{\alpha}} \right), & d = 2, \\
\frac{\sqrt{\pi} \varepsilon L_2^{\alpha}}{2\pi L_2^{\alpha}} \left( 1 - \frac{2\varepsilon L_2^{\alpha}}{\sqrt{\pi} \varepsilon L_2^{\alpha}} \right), & d = 3.
\end{cases}
\]  

(5.12)

In light of (5.8), (5.10), (5.11), (5.14), (5.14) and (5.1), we deduce that if \( d = 2, 3 \),

\[
\begin{align*}
\det F_1^{\alpha}[\varphi] &= \det F_1^{\alpha}[\varphi] - \det F_1^{\alpha}[\varphi] - \det F_1^{\alpha}[\varphi] + \det F_1^{\alpha}[\varphi], \\
\det F_2^{\alpha}[\varphi] &= \det F_2^{\alpha}[\varphi] - \det F_2^{\alpha}[\varphi] - \det F_2^{\alpha}[\varphi] + \det F_2^{\alpha}[\varphi],
\end{align*}
\]  

(5.13)

and, if \( d \geq 4 \),

\[
\begin{align*}
\det F_3^{\alpha}[\varphi] &= \det F_3^{\alpha}[\varphi] - \det F_3^{\alpha}[\varphi] - \det F_3^{\alpha}[\varphi] + \det F_3^{\alpha}[\varphi], \\
\det F_4^{\alpha}[\varphi] &= \det F_4^{\alpha}[\varphi] - \det F_4^{\alpha}[\varphi] - \det F_4^{\alpha}[\varphi] + \det F_4^{\alpha}[\varphi],
\end{align*}
\]  

(5.14)
where the blow-up factor matrices $D, F_1^\alpha[\varphi], \alpha = 1, 2, \ldots, d, F_2^\alpha[\varphi], \alpha = d+1, \ldots, \frac{d(d+1)}{2}$, $F_3^\alpha[\varphi], \alpha = 1, 2, \ldots, \frac{d(d+1)}{2}$, are defined by (4.78), (4.87)–(4.88) and (4.95), respectively.

Let
\[
\rho_d(\varepsilon) := \begin{cases} 
\sqrt{\varepsilon} \ln \varepsilon^{-2}, & d = 2, \\
\ln \varepsilon^{-1}, & d = 3.
\end{cases}
\]

Then combining (5.12)–(5.15), it follows from Cramer’s rule that

(i) if $d = 2, 3$, for $\alpha = 1, 2, \ldots, d$,
\[
C_\alpha = \prod_{i \neq \alpha} a_{ii} \det F_1^\alpha[\varphi] \prod_{i=1}^d a_{ii} \det D 
= \begin{cases} 
\det F_2^\alpha[\varphi] \sqrt{\varepsilon} \ln \varepsilon^{-2} (1 + O(\varepsilon^{-1})), & d = 2, \\
\det F_3^\alpha[\varphi] 2\pi L_1^\alpha \sqrt{\ln \varepsilon^{-1}} (1 + O(\varepsilon^{-1})), & d = 3,
\end{cases}
\]

and, for $\alpha = d+1, \ldots, \frac{d(d+1)}{2}$,
\[
(5.16)
\]

(ii) if $d \geq 4$, for $\alpha = 1, 2, \ldots, \frac{d(d+1)}{2}$,
\[
C_\alpha = \det F_3^\alpha[\varphi] \sqrt{\ln \varepsilon^{-1}} (1 + O(\varepsilon^{-1}))),
\]

Consequently, substituting (4.1) and (5.16)–(5.18) into (3.2), we obtain that Theorem 5.1 holds.

\[\Box\]

6. Appendix: The proof of Theorem 3.1

In the following, the iterate technique developed in [31] will be used to prove Theorem 3.1. For the sake of discussion and presentation, we choose $\psi = 0$ on $\partial D_1$ in (3.3). To begin with, we decompose the solution of (3.3) as follows:
\[
v = \sum_{j=1}^d v_j, \quad v_j = (v_j^1, v_j^2, \ldots, v_j^d)^T,
\]
where $v_j, j = 1, 2, \ldots, \frac{d(d+1)}{2}$, satisfy that $v_j^j = 0, j \neq i$, and $v_j$ is a solution to the following problem
\[
\begin{align*}
\mathcal{L}_{\lambda, \mu} v_j := \nabla \cdot (C^0 e(v_j)) &= 0, & \text{in } \Omega, \\
v_j &= 0, & \text{on } \partial D_1, \\
v_j &= (0, \ldots, 0, \phi^j, 0, \ldots, 0)^T, & \text{on } \partial D.
\end{align*}
\]

Then
\[
\nabla v = \sum_{j=1}^d \nabla v_j.
\]
For $x \in \Omega_{2R}$, denote
\[ \tilde{v}_j := \phi^j(x', h(x'))(1 - \tilde{v})e_j - \frac{\lambda + \mu}{\lambda + 2\mu} f(\tilde{v})\phi^j(x', h(x')) \partial_{x_j} \delta e_d, \quad j = 1, \ldots, d - 1, \]
and
\[ \tilde{v}_d := \phi^d(x', h(x'))(1 - \tilde{v})e_d - \frac{\lambda + \mu}{\mu} f(\tilde{v})\phi^d(x', h(x')) \sum_{i=1}^{d-1} \partial_{x_i} \delta e_i. \]
Then
\[ \tilde{v} = \sum_{j=1}^{d} \tilde{v}_j, \]
where $\tilde{v}$ is defined by (3.4) with $\psi = 0$ on $\partial D_1$. It follows from a direct computation that for $\tilde{v} = 0$ on $\partial D_1$, it can be seen from (3.4) that $\tilde{v}_j$ satisfies
\[ |\mathcal{L}_{\lambda, \mu} \tilde{v}_j| \leq \left| \frac{C|\phi^j(x', h(x'))|}{\delta^{\alpha/m}} \right| + C \left| \frac{|\nabla x' \phi^j(x', h(x'))|}{\delta} \right| + C\|\phi^j\|_{C^2(\partial D)}. \quad (6.2) \]
For $x \in \Omega_{2R}$, write
\[ w_j := v_j - \tilde{v}_j, \quad j = 1, 2, \ldots, d. \]
Therefore, $w_j$ satisfies
\[ \begin{cases} \mathcal{L}_{\lambda, \mu} w_j = -\mathcal{L}_{\lambda, \mu} \tilde{v}_j, & \text{in } \Omega_{2R}, \\ w_j = 0, & \text{on } \Gamma_{2R}^\pm. \end{cases} \quad (6.3) \]
We next divide into two subparts to prove Theorem 3.1. For the convenience of presentation, in the following we utilize $\|\phi^j\|_{C^1}$ and $\|\phi^j\|_{C^2}$ to denote $\|\phi^j\|_{C^1(\partial D)}$ and $\|\phi^j\|_{C^2(\partial D)}$, respectively.

**Part 1.** Let $v_j \in H^1(\Omega; \mathbb{R}^d)$ be the weak solution of (6.1), $j = 1, 2, \ldots, d$. Then
\[ \int_{\Omega_{2R}} |\nabla w_j|^2 dx \leq C\|\phi^j\|^2_{C^2}, \quad j = 1, 2, \ldots, d. \quad (6.4) \]
Firstly, we extend $\phi \in C^2(\partial D; \mathbb{R}^d)$ to $\phi \in C^2(\overline{\Omega}; \mathbb{R}^d)$ satisfying that for $j = 1, 2, \ldots, d$, $\|\phi^j\|_{C^2(\overline{\Omega})} \leq C\|\phi^j\|_{C^2(\partial D)}$. Pick a smooth cutoff function $\rho \in C^2(\overline{\Omega})$ such that $0 \leq \rho \leq 1$, $|\nabla \rho| \leq C$ in $\overline{\Omega}$, and
\[ \rho = 1 \text{ in } \Omega_{2R}, \quad \rho = 0 \text{ in } \overline{\Omega} \setminus \Omega_{2R}. \quad (6.5) \]
For $x \in \Omega$, denote
\[ \hat{v}_j(x) = (0, \ldots, 0, [\rho(x)\phi^j(x', h(x')) + (1 - \rho(x))\phi^j(x)](1 - \tilde{v}(x)), 0, \ldots, 0)^T. \]
Especially,
\[ \hat{v}_j(x) = (0, \ldots, 0, \phi^j(x', h(x'))(1 - \tilde{v}(x)), 0, \ldots, 0)^T, \quad \text{in } \Omega_R. \]
From (6.1) and (6.2), we obtain
\[ \|\hat{v}_j\|_{C^2(\overline{\Omega})} \leq C\|\phi^j\|_{C^2}, \quad j = 1, 2, \ldots, d. \quad (6.6) \]
For $j = 1, 2, \ldots, d$, write $\hat{w}_j := v_j - \hat{v}_j$ in $\Omega$. Consequently, $\hat{w}_j$ satisfies
\[ \begin{cases} \mathcal{L}_{\lambda, \mu} \hat{w}_j = -\mathcal{L}_{\lambda, \mu} \hat{v}_j, & \text{in } \Omega, \\ \hat{w}_j = 0, & \text{on } \partial \Omega. \end{cases} \quad (6.7) \]
By a direct calculation, we deduce that for $i = 1, \ldots, d - 1$, $x \in \Omega_{2R}$,

$$
|\partial_{x_i} \hat{\nu}_j| \leq C|\phi^j(x', h(x'))|\delta^{-1/m} + \|\phi^j\|_{C^1},
$$

(6.8)

$$
|\partial_{x_d} \hat{\nu}_j| = |\phi^j(x', h(x'))|\delta^{-1}, \quad \partial_{x_d} \hat{\nu}_j = 0.
$$

(6.9)

Multiplying equation (6.7) by $\hat{\nu}_j$ and integrating by parts, we have

$$
\int_{\Omega} (C^0 e(\hat{\nu}_j), e(\hat{\nu}_j)) \, dx = \int_{\Omega} \hat{\nu}_j (\mathcal{L}_{\lambda, \mu} \hat{\nu}_j) \, dx.
$$

(6.10)

From the Poincaré inequality, we have

$$
\|\hat{\nu}_j\|_{L^2(\Omega_{2R})} \leq C\|\nabla \hat{\nu}_j\|_{L^2(\Omega_{2R})}.
$$

(6.11)

The first Korn’s inequality, in combination with (3.7) and (6.10), gives that

$$
\int_{\Omega} |\nabla \hat{\nu}_j|^2 \, dx \leq 2 \int_{\Omega} |e(\hat{\nu}_j)|^2 \, dx
$$

$$
\leq C \int_{\Omega_{2R}} \hat{\nu}_j (\mathcal{L}_{\lambda, \mu} \hat{\nu}_j) \, dx + C \int_{\Omega_{2R}} \hat{\nu}_j (\mathcal{L}_{\lambda, \mu} \hat{\nu}_j) \, dx.
$$

(6.12)

On one hand, from (6.6) and (6.11) we deduce that

$$
\int_{\Omega_{2R}} \hat{\nu}_j (\mathcal{L}_{\lambda, \mu} \hat{\nu}_j) \, dx \leq \|\phi^j\|_{C^2} \int_{\Omega_{2R}} |\hat{\nu}_j| \, dx \leq C\|\phi^j\|_{C^2} \|\nabla \hat{\nu}_j\|_{L^2(\Omega_{2R})}.
$$

(6.13)

On the other hand, observe that the Sobolev trace embedding theorem allows us to write

$$
\int_{\Omega} |\hat{\nu}_j| \, dx \leq C \left( \int_{\Omega_{2R}} |\nabla \hat{\nu}_j|^2 \, dx \right)^{1/2},
$$

(6.14)

where $C$ is a positive constant independent of $\varepsilon$. In view of (6.8), we deduce

$$
\int_{\Omega_{2R}} |\nabla x^j \hat{\nu}_j|^2 \, dx \leq C \int_{|x'| < R} \delta \left( \frac{|\phi^j(x', h(x'))|^2}{\delta^{2/m}} + \|\phi^j\|_{C^1}^2 \right) \, dx' \leq C\|\phi^j\|_{C^1}^2.
$$

(6.15)

Then combining (6.9) and (6.14)–(6.15), we obtain

$$
\int_{\Omega_{2R}} \hat{\nu}_j (\mathcal{L}_{\lambda, \mu} \hat{\nu}_j) \, dx \leq C \sum_{k + l \leq 2d} \int_{\Omega_{2R}} \hat{\nu}_j \partial_{x_k x_l} \hat{\nu}_j \, dx
$$

$$
\leq C \int_{\Omega_{2R}} |\nabla \hat{\nu}_j| |\nabla x^j \hat{\nu}_j| \, dx + \int_{|x'| = R, \ h(x') < x d < h_1(x')} C |\nabla x^j \hat{\nu}_j| \, dx
$$

$$
\leq C\|\nabla \hat{\nu}_j\|_{L^2(\Omega_{2R})} \|\nabla x^j \hat{\nu}_j\|_{L^2(\Omega_{2R})} + C\|\phi^j\|_{C^1} \|\nabla \hat{\nu}_j\|_{L^2(\Omega_{2R})}
$$

$$
\leq C\|\phi^j\|_{C^1} \|\nabla \hat{\nu}_j\|_{L^2(\Omega)}.
$$

This, together with (6.12)–(6.13), yields that

$$
\|\nabla \hat{\nu}_j\|_{L^2(\Omega)} \leq C\|\phi^j\|_{C^2}.
$$
Due to the fact that
\[ w_j = \hat{w}_j + \begin{cases} \frac{\lambda + \mu}{s + 2\mu} f(\tilde{v})\phi^j(x', h(x')) \partial_{x_i} \delta e_i, \quad &j = 1, \ldots, d - 1, \\
\frac{\lambda + \mu}{\mu} f(\tilde{v})\phi^j(x', h(x')) \sum_{i=1}^{d-1} \partial_{x_i} \delta e_i, \quad &j = d, \end{cases} \text{ in } \Omega_{2R}, \]

it follows that \((6.4)\) holds.

**Part 2.** Proof of
\[ \int_{\Omega_s(z')} |\nabla w_j|^2 \, dx \leq C \delta^4(|\phi^j(z', h(z'))|^2 \delta^{-4/m} + |\nabla x^j h(z', h(z'))|^2) \]
\[ + C \delta^{d+2} \|\phi^j\|_{C^2(\partial D)}^2, \quad j = 1, 2, \ldots, d. \tag{6.16} \]

For \(0 < t < s < R\) and \(|z'| \leq R\), let \(\eta \in C^2(\Omega_{2R})\) be a smooth cutoff function satisfying that \(\eta(x') = 1\) if \(|x' - z'| < t\), \(\eta(x') = 0\) if \(|x' - z'| > s\), \(0 \leq \eta(x') \leq 1\) if \(t \leq |x' - z'| \leq s\), and \(|\nabla x^j \eta(x')| \leq \frac{2}{s-t}\). Multiplying equation \((6.3)\) by \(w_j \eta^2\), it follows from integration by parts that
\[ \int_{\Omega_s(z')} (C_0 e(w_j), e(w_j \eta^2)) \, dx = \int_{\Omega_s(z')} w_j \eta^2 \left( L_{\lambda, \mu} \hat{v}_j \right) \, dx. \tag{6.17} \]

On one hand, from \((2.17)\), \((5.7)\) and the first Korn’s inequality, we deduce
\[ \int_{\Omega_s(z')} (C_0 e(w_j), e(w_j \eta^2)) \, dx \geq \frac{1}{C} \int_{\Omega_s(z')} |\eta \nabla w_j|^2 - C \int_{\Omega_s(z')} |\nabla \eta|^2 |w_j|^2. \tag{6.18} \]

On the other hand, applying Hölder inequality and Cauchy inequality to the right hand side of \((6.17)\), we deduce
\[ \left| \int_{\Omega_s(z')} w_j \eta^2 \left( L_{\lambda, \mu} \hat{v}_j \right) \, dx \right| \leq \frac{C}{(s-t)^2} \int_{\Omega_s(z')} |w_j|^2 \, dx + C(s-t)^2 \int_{\Omega_s(z')} |L_{\lambda, \mu} \hat{v}_j|^2 \, dx, \]

which, in combination with \((6.17)-(6.18)\), gives the following iteration formula:
\[ \int_{\Omega_s(z')} |\nabla w_j|^2 \, dx \leq \frac{C}{(s-t)^2} \int_{\Omega_s(z')} |w_j|^2 \, dx + C(s-t)^2 \int_{\Omega_s(z')} |L_{\lambda, \mu} \hat{v}_j|^2 \, dx. \]

For \(|z'| \leq R\), \(\delta < s \leq \vartheta(\kappa_1, \kappa_3) \delta^{1/m}, \vartheta(\kappa_1, \kappa_3) = \frac{1}{2^{m+1} \kappa_3 \max\{1, \kappa_1^{m-1}\}}\), using conditions \((H1)\) and \((H2)\), we obtain that for \((x', x_d) \in \Omega_s(z')\),
\[ |\delta(x') - \delta(z')| \leq |h_1(x') - h_1(z')| + |h(x') - h(z')| \]
\[ \leq (|\nabla x^j h_1(x_{d_j})| + |\nabla x^j h(x_{d_j})|)|x' - z'| \]
\[ \leq \kappa_3 |x' - z'| ([x_{d_j}]^m - 1 + [x_{d_j}]^{m-1}) \]
\[ \leq \delta(z') \frac{s^{m-1} + |z'|^{m-1}}{2}, \tag{6.19} \]

which implies that
\[ \frac{1}{2} \delta(z') \leq \delta(x') \leq \frac{3}{2} \delta(z'), \quad \text{in } \Omega_s(z'). \tag{6.20} \]

In view of the fact that \(w_j = 0\) on \(\Gamma_R\), it follows from \((6.20)\) that
\[ \int_{\Omega_s(z')} |w_j|^2 \, dx \leq C \delta^2 \int_{\Omega_s(z')} |\nabla w_j|^2. \tag{6.21} \]
Hence, applying Claim that for we rescales \( \Omega \delta \geq \epsilon > 0 \), we have

\[
\int_{\Omega_t(z')} |\mathcal{L}_{\lambda, \mu} \varphi_j|^2 \leq C \delta^{-1} s^{d-1} (|\phi_j(z', h(z'))|^2 \delta^{2-4/m} + |\nabla_{x'} \phi_j(z', h(z'))|^2)
+ C \delta^{-1} s^{d+1} \|\phi_j\|^2_{C^2}.
\]

(6.22)

Write

\[
F(t) := \int_{\Omega_t(z')} |\nabla w_j|^2.
\]

Then combining (6.21) and (6.22), we deduce that

\[
F(t) \leq \left( \frac{c \delta}{s-t} \right)^2 F(s) + C(s-t)^2 s^{d+1} \delta^{-1} \|\phi_j\|^2_{C^2(\partial D)}
+ C(s-t)^2 \delta^{-1} s^{d-1} (|\phi_j(z', h(z'))|^2 \delta^{2-4/m} + |\nabla_{x'} \phi_j(z', h(z'))|^2),
\]

which, together with \( s = t_{i+1}, t = t_i, t_i = \delta + 2 \epsilon \delta, i = 0, 1, 2, \ldots, \left[ \frac{\delta(t_1, t_2)}{4c \delta^{1-1/m}} \right] + 1, \)
leads to that

\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i+1)^{d+1} \delta^{d+2} \|\phi_j\|^2_{C^2}
+ C(i+1)^{d-1} \delta^d (|\phi_j(z', h(z'))|^2 \delta^{2-4/m} + |\nabla_{x'} \phi_j(z', h(z'))|^2). \]

(6.23)

Hence, applying \( \left[ \frac{\delta(t_1, t_2)}{4c \delta^{1-1/m}} \right] + 1 \) iterations to (6.23), it follows from (6.4) that for a sufficiently small \( \epsilon > 0 \),

\[
F(t_0) \leq C \delta^d (|\phi_j(z', h(z'))|^2 \delta^{2-4/m} + |\nabla_{x'} \phi_j(z', h(z'))|^2) + \delta^2 \|\phi_j\|^2_{C^2}.
\]

Part 3. Claim that for \( j = 1, 2, \ldots, d, z \in \Omega_R, \)

\[
|\nabla w_j(z)| \leq C(|\phi_j(z', h(z'))|\delta^{1-2/m} + |\nabla_{x'} (\phi_j(z', h(z')))|) + C \delta \|\phi_j\|_{C^2}. \]

(6.24)

Making use of a change of variables in the thin gap \( \Omega_\delta(z') \) as follows:

\[
\begin{aligned}
x' - z' = \delta y', \\
x_d = \delta y_d,
\end{aligned}
\]

we rescales \( \Omega_\delta(z') \) into \( Q_1 \), where, for \( 0 < r \leq 1, \)

\[
Q_r = \left\{ y \in \mathbb{R}^d \left| \frac{1}{\delta} h(\delta y' + z') < y_d < \frac{\epsilon}{\delta} + \frac{1}{\delta} h_1(\delta y' + z'), |y'| < r \right. \right\}.
\]

Denote the top and bottom boundaries of \( Q_r \), respectively, by

\[
\hat{\Gamma}_r^+ = \left\{ y \in \mathbb{R}^d \left| y_d = \frac{\epsilon}{\delta} + \frac{1}{\delta} h_1(\delta y' + z'), |y'| < r \right. \right\},
\]

and

\[
\hat{\Gamma}_r^- = \left\{ y \in \mathbb{R}^d \left| y_d = \frac{1}{\delta} h(\delta y' + z'), |y'| < r \right. \right\}.
\]

Analogously as in (6.19), we obtain that for \( x \in \Omega_\delta(z'), \)

\[
|\delta(x') - \delta(z')| \leq 2^{m-1} \kappa_3 (\delta^{m-1} + |z'|^{m-1})
\leq 2^m \kappa_3 \max\{1, \kappa_1^{m-1}\} \delta^{2-1/m},
\]
From (6.2) and (6.16), it follows that for $Q$.

Therefore, $W$.

Since $W$.

This, in combination with the fact that $R$ is a small positive constant, leads to that $Q_1$ is of nearly unit size as far as applications of Sobolev embedding theorems and classical $L^p$ estimates for elliptic systems are concerned.

Let

$$W_j(y', y_d) := w_j(\delta y' + z', \delta y_d), \quad \tilde{V}_j(y', y_d) := \tilde{v}_j(\delta y' + z', \delta y_d).$$

Therefore, $W_j(y)$ verifies

$$\left\{ \begin{array}{ll}
\mathcal{L}_{\lambda, \mu} W_j &= -\mathcal{L}_{\lambda, \mu} \tilde{V}_j, \quad \text{in } Q_1, \\
W_j &= 0, \quad \text{on } \tilde{\Gamma}^\pm.
\end{array} \right.$$  

Since $W_j = 0$ on $\tilde{\Gamma}^\pm$, then we see from Poincaré inequality that

$$\|W_j\|_{H^1(Q_1)} \leq C \|\nabla W_j\|_{L^2(Q_1)}.$$  

This, together with the Sobolev embedding theorem and classical $W^{2,p}$ estimates for elliptic systems, yields that for some $p > d$,

$$\|\nabla W_j\|_{L^\infty(Q_{1/2})} \leq C \|W_j\|_{W^{2,p}(Q_{1/2})} \leq C(\|\nabla W_j\|_{L^2(Q_1)} + \|\mathcal{L}_{\lambda, \mu} \tilde{V}_j\|_{L^\infty(Q_1)}).$$  

Consequently, rescaling back to $w_j$ and $\tilde{v}_j$, we have

$$\|\nabla w_j\|_{L^\infty(\Omega_{3/2}(z'))} \leq C \left( \delta^{1-d/2} \|\nabla w_j\|_{L^2(\Omega_3(z'))} + \delta^2 \|\mathcal{L}_{\lambda, \mu} \tilde{v}_j\|_{L^\infty(\Omega_3(z'))} \right).$$  

(6.25)

From (6.22) and (6.16), it follows that for $z' \in B_{2R}^*$,

$$\delta^{-\frac{2}{d}} \|\nabla w_j\|_{L^2(\Omega_3(z'))} + \delta \|\mathcal{L}_{\lambda, \mu} \tilde{v}_j\|_{L^\infty(\Omega_3(z'))} \leq C \left( |\phi^j(z', h(z'))| \delta^{1-2/m} + \|\nabla x' \phi^j(z', h(z'))\| \right) + C \delta \|\phi^j\|_{C^2}.$$  

This, in combination with (6.25), yields that (6.24) holds. The proof is complete.

Acknowledgements. C. Miao was supported by the National Key Research and Development Program of China (No. 2020YFA0712900) and NSFC Grant 11831004. Z. Zhao was partially supported by CPSF (2021M700358).

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(C.X. Miao) 1. Beijing Computational Science Research Center, Beijing 100193, China.

2. Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing, 100088, China.

Email address: miao_changxing@iapcm.ac.cn

(Z.W. Zhao) Beijing Computational Science Research Center, Beijing 100193, China.

Email address: zwzhao3658163.com