A Takahashi type characterization of stationary spacelike submanifolds in a spherical Robertson-Walker spacetime

D. Ferreira, E.A. Lima Jr., F.J. Palomo and A. Romero

Abstract

A natural one codimension isometric embedding of each \((n+1)\)-dimensional spherical Robertson-Walker (RW) spacetime \(I \times f S^n\) in \((n+2)\)-dimensional Lorentz-Minkowski spacetime \(L^{n+2}\) permits to contemplate \(I \times f S^n\) as a rotation Lorentzian hypersurface of \(L^{n+2}\). After a detailed study of such Lorentzian hypersurfaces, we can see any \(k\)-dimensional spacelike submanifold of such an RW spacetime as a spacelike submanifold of \(L^{n+2}\). Then, we use that situation to study \(k\)-dimensional stationary (i.e., of zero mean curvature vector field) spacelike submanifolds of the RW spacetime. In particular, we prove a wide extension of the Lorentzian version of the classical Takahashi theorem, giving a characterization of stationary spacelike submanifolds of \(I \times f S^n\) when contemplating them as spacelike submanifolds of \(L^{n+2}\).

1 Introduction

For any isometric immersion \(\Psi: M^k \to \mathbb{R}^m_s\) of a Riemannian manifold \(M^k\) in an \(m\)-dimensional semi-Euclidean space of arbitrary signature, \(\mathbb{R}^m_s\), the position vector field is closely related to the extrinsic geometric of the immersion by means of the well-known Beltrami formula

\[ \Delta \Psi = kH, \]

where \(\Delta\) is the Laplacian operator on \(M^k\) and \(H\) is the mean curvature vector field. This elegant and simple formula permits translate geometric notions on \(H\) into analytic ones on \(\Psi\). For instance, \(M^k\) is stationary if and only if the components of \(\Psi\) are harmonic on \(M^k\). Conversely, assumptions on \(\Psi\) involving \(\Delta\) are also translated to conditions on \(H\), for instance, an isometric immersion \(\Psi: M^k \to \mathbb{R}^m_s\) is said of finite type when the position vector field \(\Psi\) is written as \(\Psi = c + \Psi^1 + \cdots + \Psi^k\) where \(c \in \mathbb{R}^m_s\) and the components of each

---

\(^1\)The first author was partially supported by CAPES, Brazil. The second authors is partially supported by CNPq, Brazil, PQ-2 Grant 309668/2021-2. The third author was partially supported by Spanish MICINN project PID2020-118452GB-I00. The fourth named author was partially supported by the Spanish MICINN and ERDF project PID2020-116126GB-I00. The third and the fourth authors by the Andalusian and ERDF project A-FQM-494-UGR18. Research partially supported by the “María de Maeztu” Excellence Unit IMAG, reference CEX2020-001105-M, funded by MCIN-AEI-10.13039-501100011033.
map $\Psi^i$ are eigenfunctions on $M^k$ with the same eigenvalue $\lambda$. In this terms, Takahashi proved in 1966 that an isometric immersion $\Psi : M^k \to \mathbb{E}^m$ of a Riemannian manifold $M^k$ in Euclidean space $\mathbb{E}^m$ satisfies

$$\Delta \Psi + \lambda \Psi = 0,$$

where $\lambda \in \mathbb{R} \setminus \{0\}$ if and only if $\Psi$ is a minimal immersion in some hypersphere $S^{m-1}(\sqrt{k/\lambda})$ in $\mathbb{E}^m$, [16]. The extension of this result to spacelike submanifolds in Lorentz-Minkowski spacetime $\mathbb{L}^m$ was obtained by Markvorsen, [11, Thm. 1], as follows: an spacelike immersion $\Psi : M^k \to \mathbb{L}^m$ satisfies $\Delta \Psi + \lambda \Psi = 0$, where $\Delta$ is the Laplacian of the induced Riemannian metric on $M^k$, with $\lambda > 0$ if and only if the image of $\Psi$ is contained in a De Sitter spacetime $S^1_1(\sqrt{k/\lambda})$ and the spacelike submanifold $\Psi : M^k \to S^1_1(\sqrt{k/\lambda})$ is stationary.

Obviously, these kind of results only works for submanifolds in pseudo-Euclidean spaces. However, the main purpose of this work is to prove an extension of this Takahashi type result to spacelike submanifolds in arbitrary spherical Robertson-Walker spacetimes. Consequently, in order to do that, we have to contemplate, in a natural way, each spacetime of this family as a Lorentzian hypersurface of Lorentz-Minkowski spacetime $\mathbb{L}^m$. Thus, a spacelike submanifold of that spacetime becomes a spacelike submanifold of $\mathbb{L}^m$.

Recall that a spherical Robertson-Walker spacetime is the Lorentzian warped product $I \times_f S^n(1)$ where $f \in C^\infty(I)$ with $f > 0$. It is well-known that a De Sitter spacetime of arbitrary radius $R$, $S^{n+1}_1(R)$, can be seen as the Robertson-Walker spacetime $\mathbb{R} \times_f S^n(1)$, where $f(t) = R \cosh(t/R)$, [13]. Once we contemplate $\mathbb{R} \times_f S^n(1)$ as a rotation hypersurface of $\mathbb{L}^{n+2}$, as De Sitter spacetime naturally does, the main purpose of this work is to prove an extension of Markvorsen theorem obtaining a Takahashi type result for spacelike submanifolds in an arbitrary spherical Robertson-Walker spacetime.

The content of this article is organized as follows. In Section 2, a family of rotation Lorentzian hypersurfaces $Q(r)$ of $\mathbb{L}^{n+2}$ is studied. Each Lorentzian hypersurface depends on the choice of $r \in C^\infty(J)$, where $J$ is an open interval of $\mathbb{R}$, with $0 \in J$, that satisfies $r(t) > 0$ and $|r'(t)| < 1$ for all $t \in J$. In what follows, such a function will be called admissible. The admissible condition assures that $Q(r)$ is actually a Lorentzian hypersurface of $\mathbb{L}^{n+2}$, Lemma 2.1. It is shown that each $Q(r)$ is quasiumbilical, Proposition 2.7. The case totally umbilical is characterized in terms of a differential equation involving the function $r$, Remark 2.8. whose solutions give rise to De Sitter spacetimes of arbitrary sectional curvature, Remark 2.9. From an intrinsic point of view, each Lorentzian hypersurface $Q(r)$ admits a timelike conformal and closed vector field, Proposition 2.6. Each $Q(r)$ may be seen then as a generalized Robertson-Walker spacetime [15] whose fibre has positive constant sectional curvature, Proposition 2.7. Indeed it can be realized as the image of the isometric embedding of $I \times_f S^n$ in $\mathbb{L}^{n+2}$ constructed in [11].

In Section 3 we prove that a spacelike immersion $\Psi : M^k \to Q(r)$ is stationary if and only if

$$\Delta \Psi + q\Psi_0 \mathbf{P} = 0,$$

where $\mathbf{P}$ is the vector field along the immersion $\Psi$ given by $\mathbf{P} = (r(\Psi_0)r'(\Psi_0), \Psi_1, \ldots, \Psi_{n+1})$
and
\[ q_{\Psi_0} = \frac{[k - (r''(\Psi_0)r(\Psi_0) + r'(\Psi_0)^2 - 1)\|\nabla\Psi_0\|^2]}{r(\Psi_0)^2(1 - r'(\Psi_0)^2)} \in C^\infty(M^k), \]

Proposition 3.1. We also include several applications of Proposition 3.1 to physically realistic spherical Robertson-Walker spacetimes, Corollary 3.6.

It should be noticed that if \( r(t) = \sqrt{1 + t^2} \) for all \( t \in \mathbb{R} \), \( Q(r) \) is the unitary De Sitter spacetime \( S^{n+1}_1 \) and then a spacelike immersion \( \Psi: M^k \rightarrow Q(r) \subset \mathbb{L}^{n+2} \) is stationary if and only if \( \Delta \Psi + k \Psi = 0 \), as proved in [11]. On the other hand, if \( r(t) = 1 \) for all \( t \in \mathbb{R} \), \( Q(1) \) is the Lorentzian product manifold \( \mathbb{R} \times S^n \), this result specializes to: a spacelike immersion \( \Psi: M^k \rightarrow Q(1) \subset \mathbb{L}^{n+2} \) is stationary if and only if (Example 3.3)

\[ \Delta \Psi_0 = 0, \quad \Delta \Psi_i + (k + \|\nabla \Psi_0\|^2) \Psi_i = 0, \quad i = 1, 2, \ldots, n + 1. \]

Finally, we prove that given an admissible function \( r(t) \) and \( \Psi: M^k \rightarrow \mathbb{L}^{n+2} \) any spacelike immersion with \( q_{\Psi_0} > 0 \). If \( \Delta \Psi + q_{\Psi_0} P = 0 \) holds, then \( \Psi \) realizes a spacelike immersion through \( Q(r) \) and \( \Psi: M^k \rightarrow Q(r) \) is stationary, Theorem 3.7. Summarizing, in Proposition 3.1 and Theorem 3.7 we have proved that

Let \( r(t) \) be an admissible function and \( \Psi: M^k \rightarrow \mathbb{L}^{n+2} \) any spacelike immersion with \( q_{\Psi_0} > 0 \). Then, the following assertions are equivalent

1. \( \Delta \Psi + q_{\Psi_0} P = 0 \).

2. \( \Psi \) realizes a stationary spacelike immersion in \( Q(r) \).

Thus, this theorem is a wide extension of the corresponding result for De Sitter spacetime in [11].

2 Preliminaries

Let \( \mathbb{L}^{n+2} \) be the \((n + 2)\)-dimensional Lorentz-Minkowski spacetime, that is, \( \mathbb{R}^{n+2} \) endowed with the Lorentzian metric

\[ \langle \ , \rangle = -(dt)^2 + (dx_1)^2 + (dx_2)^2 + \ldots + (dx_{n+1})^2, \quad (1) \]

where \((t, x_1, x_2, \ldots, x_{n+1}) = (t, x) \in \mathbb{R} \times \mathbb{E}^{n+1} \) are the usual coordinates of \( \mathbb{R}^{n+2} \) and \( \mathbb{E}^{n+1} \) denotes the \((n + 1)\)-dimensional Euclidean space.

Fix an open interval \( J \subset \mathbb{R} \), with \( 0 \in J \). Consider \( r \in C^\infty(J) \) that satisfies \( r(t) > 0 \) and \( |r'(t)| < 1 \) for all \( t \in J \). In what follows, such a function will be called admissible. Let us denote

\[ Q(r) := \left\{ (t, x) \in J \times \mathbb{E}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = \|x\|^2 = r(t)^2 \right\} \subset \mathbb{L}^{n+2}. \quad (2) \]
Lemma 2.1. For every admissible function \( r \in C^\infty(J) \), \( Q(r) \) is a Lorentzian hypersurface of \( \mathbb{L}^{n+2} \).

Proof. Consider the smooth map \( F : J \times \mathbb{E}^{n+1} \to \mathbb{R} \) given by \( F(t, x) = \|x\|^2 - r(t)^2 \). We have \( Q(r) = F^{-1}(\{0\}) \) and 0 is a regular value for \( F \). On the other hand, a direct computation shows that for every point \((t, x) \in Q(r)\), the tangent space of \( Q(r) \) at \((t, x)\) is given by

\[
T_{(t,x)}Q(r) = \left\{ (a, v) \in \mathbb{R} \times \mathbb{R}^{n+1} : -r(t)r'(t)a + \sum_{i=1}^{n+1} v_i x_i = 0 \right\} = (r(t)r'(t), x)\perp
\]

where \( \perp \) denotes the orthogonal subspace in \( \mathbb{L}^{n+2} \) of the spacelike vector \((r(t)r'(t), x)\) for every \((t, x) \in Q(r)\).

Remark 2.2. If we choose \( r(t) = \sqrt{R^2 + t^2} \), \( t \in \mathbb{R} \), for some constant \( R > 0 \), then the corresponding \( Q(r) \) is the \((n+1)\)-dimensional De Sitter spacetime \( S^{n+1}_1(R) \) of constant sectional curvature \( 1/R^2 \). On the other hand, if \( r = R > 0 \) then \( Q(r) \) is the Lorentzian cylinder \( \mathbb{R} \times S^n(R) \).

Note that each hypersurface \( t = t_0 \) of \( Q(r) \) is spacelike and the family of such spacelike hypersurfaces of \( Q(r) \) defines an \( n \)-dimensional foliation \( \mathcal{D}_r \) on \( Q(r) \). Moreover,

Lemma 2.3. For any admissible functions \( r, s \in C^\infty(J) \), let us consider the corresponding Lorentzian hypersurfaces \( Q(r), Q(s) \subset \mathbb{L}^{n+2} \). Then, the diffeomorphism

\[
\Phi : Q(r) \to Q(s), \quad (t, x) \mapsto \left( t, \frac{s(t)}{r(t)} x \right)
\]

carries homothetically each leaf \( t = t_0 \) of \( \mathcal{D}_r \) onto the leaf \( t = t_0 \) of \( \mathcal{D}_s \).

Proof. Clearly \( \Phi \) is a diffeomorphism and \( \Phi^{-1}(t, x) = (t, \frac{r(t)}{s(t)} x) \). If \( \Phi_{t_0} \) denotes the restriction of \( \Phi \) to the leaf \( t = t_0 \) then \( \Phi_{t_0}(t_0, x) = (t_0, \frac{s(t_0)}{r(t_0)} x) \), ending the proof.

Remark 2.4. According to the previous result, every \( Q(r) \) is diffeomorphic to \( S^{n+1}_1(1) \) and, therefore, diffeomorphic to \( \mathbb{R} \times S^n \).

Remark 2.5. Each \( Q(r) \) is a rotation hypersurface of \( \mathbb{L}^{n+2} \) in the terminology of [5]. In fact, for a given admissible function \( r \in C^\infty(J) \), let us consider the curve \( \gamma : J \to \mathbb{L}^{n+2} \), given by \( \gamma(t) = (t, r(t), 0, \ldots, 0) \). Note that the first assumption on \( r \) implies that the image of \( \gamma \) does not meet the timelike axis \( x_j = 0, j = 1, \ldots, n+1 \). On the other hand, the second one means that \( \gamma \) is timelike. Denote now by \( O^1(n+2) \) the group of linear isometries of \( \mathbb{L}^{n+2} \) and by \( G \) the subgroup \( \{1\} \times O(n+1) \) of \( O^1(n+2) \). Then, we have

\[
Q(r) = \{ (A \circ \gamma)(t) : A \in G, t \in J \}.
\]
Thus, $Q(r)$ is a rotation hypersurface of $\mathbb{L}^{n+2}$ with profile curve $\gamma$ and rotation axis $x_j = 0, j = 1, ..., n + 1$. Note that if $E_t$ is the orthogonal hyperplane to the rotation axis through the point $(t, 0, ..., 0)$, then $E_t$ is spacelike, and therefore, identifiable to the Euclidean space $\mathbb{E}^{n+1}$. Observe also that the slice $Q(r) \cap E_t$ is an $n$-dimensional round sphere in $E_t$ with radius $r(t)$.

Each Lorentzian manifold $Q(r)$ is time orientable. Indeed, the tangent vector field $T \in X(Q(r))$ given by

$$T(t, x) = \frac{1}{\sqrt{1 - r'(t)^2}} \left(1, \frac{r'(t)}{r(t)} x\right)$$

for every $(t, x) \in Q(r)$, satisfies $\langle T, T \rangle = -1$ everywhere on $Q(r)$. Precisely, $T$ is the normalization of a timelike conformal symmetry as the following result shows,

**Proposition 2.6.** On each Lorentzian hypersurface $Q(r)$ of $\mathbb{L}^{n+2}$, we have that $K \in X(Q(r))$, given by

$$K(t, x) = \frac{1}{\sqrt{1 - r'(t)^2}} (r(t), r'(t)x),$$

for all $(t, x) \in Q(r)$, satisfies

$$\nabla_V K = \frac{r'(t)}{\sqrt{1 - r'(t)^2}} V$$

for any $V \in T_{(t,x)}Q(r)$ and any $(t, x) \in Q(r)$. Thus, the vector field $K$ on $Q(r)$ is conformal with $\mathcal{L}_K \langle \cdot, \cdot \rangle = 2 \rho \langle \cdot, \cdot \rangle$, where $\rho(t, x) = r'(t) / \sqrt{1 - r'(t)^2}$ and the metrically equivalent 1-form to $K$ is closed.

It should be notice that (7) gives that the distribution $\text{Span}\{K\} \perp$ on $Q(r)$ is integrable and each leaf $t = t_0$ is a totally umbilical spacelike hypersurface of $Q(r)$ with constant mean curvature. Note that $t = t_0$ is totally geodesic if and only if $r'(t_0) = 0$.

The Lorentzian hypersurface $Q(r)$ of $\mathbb{L}^{n+2}$ admits a unit spacelike normal vector field $N \in X^+(Q(r))$, given at $(t, x) \in Q(r)$ by

$$N(t, x) = \frac{1}{r(t)\sqrt{1 - r'(t)^2}} (r'(t)r(t), x).$$

Let us denote now by $\nabla^0$ and $\nabla$ the Levi-Civita connections of $\mathbb{L}^{n+2}$ and $Q(r)$, respectively. For any $V, W \in \mathfrak{X}(Q(r))$, the Gauss and Weingarten formulae of $Q(r) \subset \mathbb{L}^{n+2}$ are respectively written as

$$\nabla^0_V W = \nabla_V W + \langle AV, W \rangle N, \quad (9)$$

$$\nabla^0_V N = -A(V), \quad (10)$$

where $A$ is the Weingarten operator with respect to $N$. The following result gives explicitly the Weingarten operator on each $Q(r)$. 

5
Lemma 2.7. The Weingarten operator $A$ with respect to $N$ of the Lorentzian hypersurface $Q(r) \subset \mathbb{L}^{n+2}$ is given by

$$A(V) = \alpha(t) V + \beta(t) \langle T(t, x), V \rangle T(t, x)$$

(11)

for all $V \in T(t,x)Q(r)$, where

$$\alpha(t) = \frac{-1}{r(t) \sqrt{1 - r'(t)^2}}, \quad \beta(t) = \frac{r''(t)r(t) + r'(t)^2 - 1}{r(t)(1 - r'(t)^2)^{3/2}}.$$  

(12)

Proof. Write $V = (a, v) \in T(t_0, x_0)Q(r)$, i.e., $a r'(t_0)r(t_0) = \langle x_0, v \rangle$, and consider a curve $s \mapsto (t(s), x(s))$ in $Q(r)$ such that $(t(0), x(0)) = (t_0, x_0)$ and $(t'(0), x'(0)) = (a, v)$.

Using (11), we have,

$$A(V) = -\frac{d}{ds} \bigg|_{s=0} N(t(s), x(s)) =$$

$$- a \frac{r'(t_0)}{r(t_0)} \beta(t_0) \left( r'(t_0)r(t_0), x_0 \right)$$

$$+ \alpha(t_0) \left( a [r''(t_0)r(t_0) + r'(t_0)^2], v \right).$$

(13)

First of all, suppose that $\langle T(t_0, x_0), V \rangle = 0$. Then, from $ar(t_0) = r'(t_0)\langle x_0, v \rangle = ar'(t_0)^2r(t_0)$ we obtain $a = 0$. Hence, equation (13) reduces to

$$A(V) = \alpha(t_0)V.$$

Now, for the case $V = T(t_0, x_0)$, we have from (12) and (13),

$$A(T(t_0, x_0)) = \frac{-r'(t_0)}{r(t_0) \sqrt{1 - r'(t_0)^2}} \beta(t_0) \left( r'(t_0) r(t_0), x_0 \right)$$

$$- \beta(t_0) \sqrt{1 - r'(t_0)^2} \left( 1, 0 \right) + \alpha(t_0) T(t_0, x_0) =$$

$$= - \beta(t_0) \left( \frac{r'(t_0)^2}{\sqrt{1 - r'(t_0)^2}} + \sqrt{1 - r'(t_0)^2}, \frac{r'(t_0)}{r(t_0) \sqrt{1 - r'(t_0)^2}} \right) x_0$$

$$+ \alpha(t_0) T(t_0, x_0) = (\alpha(t_0) - \beta(t_0)) T(t_0, x_0),$$

which ends the proof. 

Remark 2.8. Now we are in position to analyze the extrinsic geometry of $Q(r)$. First, let us observe that, obviously, $Q(r)$ is not totally geodesic for any admissible function $r$. On the other hand, Lemma 2.7 says that $Q(r)$ is quasiumbilical (6). Observe that thanks to (11) we have $A(X) = \alpha(t)X$ for any $X \in T(t,x)Q(r)$ orthogonal to $T(t,x)$ and $A(T) = (\alpha(t) - \beta(t))T$. Therefore, $Q(r)$ is totally umbilical if and only if $\beta(t) = 0$ everywhere on $Q(r)$, i.e., if and only if the function $r$ satisfies the differential equation
\[ r(t)r''(t) + r'(t)^2 = 1 \] or equivalently \( \frac{a^2}{t^2} r(t)^2 = 2 \). Consequently, \( r(t)^2 = t^2 + at + b \), where \( b = r(0)^2 > 0 \) since \( r > 0 \), and \( a = 2r(0)r'(0) \) with \( a^2 < 4b \), making use of \( |r'| < 1 \). Summarizing, we have obtained that \( Q(r) \) is totally umbilical if and only if
\[ r(t) = \sqrt{t^2 + at + b}, \] where the constants satisfy \( b > 0 \) and \( a^2 < 4b \). \hfill (14)

**Remark 2.9.** If \( Q(r) \) is totally umbilical then \( \alpha(t, x) = \frac{-2}{\sqrt{4b-a^2}} \), and, therefore, the Weingarten operator is \( A = \frac{-2}{\sqrt{4b-a^2}} I \), where \( I \) denotes the identity operator, everywhere. In this case, the Gauss equation of \( Q(r) \) in \( \mathbb{L}^{n+2} \) gives for the curvature tensor \( R \) of \( Q(r) \), the following expression
\[ R(X,Y)Z = (\frac{4}{4b-a^2}) \{ \langle Y,Z \rangle X - \langle X,Z \rangle Y \}, \] i.e., \( Q(r) \) has sectional curvature \( \frac{4}{4b-a^2} \), indeed, \( Q(r) \) is, up to a translation, the \((n+1)\)-dimensional De Sitter spacetime \( S_{n+1}(\sqrt{\frac{4b-a^2}{2}}) \).

From Lemma 2.7, the mean curvature function of \( Q(r) \) with respect to \( N, H := \frac{1}{n+1} \text{trace}(A) \), satisfies
\[ H(t, x) = \alpha(t) - \frac{\beta(t)}{n+1} = -\frac{n(1 - r'(t)^2) + r''(t)r(t)}{(n+1)r(t)(1 - r'(t)^2)^{3/2}}. \hfill (15) \]

**Remark 2.10.** In the case \( n = 1 \), formula (15) agrees, up the sign of \( H \) due to our choice of \( N \), with [10, eq. (8)]. Therefore, if \( H \) is constant then
\[ \frac{r(t)}{\sqrt{1 - r'(t)^2}} + r(t)^2 H = \text{constant}, \] for any \( t \in J \), i.e., we have [10 eq. (9)]. However, for \( n > 1 \) no extension of this fact holds, and we have to make a different reasoning.

Now, taking into account
\[ \alpha'(t) = -\frac{r'(t)}{r(t)} \beta(t), \]
we get from (15)

**Corollary 2.11.** The Lorentzian hypersurface \( Q(r) \) of \( \mathbb{L}^{n+2} \) has constant mean curvature if and only if there exists \( c \in \mathbb{R} \) such that
\[ \beta(t) = \frac{c}{r^{n+1}(t)}, \hfill (16) \]
equivalently
\[ \frac{r(t)^n (r''(t)r(t) + r'(t)^2 - 1)}{(1 - r'(t)^2)^{3/2}} = c \hfill (17) \]
for all \( t \in J \), where \( c \) is a constant. In this case,
\[ \alpha(t) = \frac{c}{(n+1)r^{n+1}(t)} + H. \hfill (18) \]
Remark 2.12. Clearly, for the choice \( c = 0 \) we have the totally umbilical case. On the other hand, for each \( c < 0 \) we have that \( r = \sqrt{-c} \) is a solution giving \( Q(r) = \mathbb{R} \times S^n(r) \) which has constant mean curvature but it is not totally umbilical.

Extending [8] we are going to explore which Lorentzian hypersurfaces of the family \( Q(r) \) have the property that the principal curvature in the axial direction is a constant multiple of the common value of the principal curvatures in the rotational directions.

Definition 2.13. The Lorentzian hypersurface \( Q(r) \) of \( L^{n+2} \) has proportional principal curvatures when there exists \( \lambda \in \mathbb{R} \) such that

\[
\lambda \alpha(t) = \alpha(t) - \beta(t),
\]

for all \( t \in J \). If \( Q(r) \) has proportional principal curvatures, then it is totally umbilical when \( \lambda = 1 \) and, using (15), it has zero mean curvature when \( \lambda = -n \).

As a direct consequence of formulas (12) we have,

Proposition 2.14. The Lorentzian hypersurface \( Q(r) \) of \( L^{n+2} \) has proportional principal curvatures if and only if

\[
r(t)r''(t) = \lambda(1 - r'(t)^2)
\]

for all \( t \in J \).

Remark 2.15. A similar family of differential equations to (19) appears in [8] where the authors study hypersurfaces of revolution with proportional principal curvatures in Euclidean spaces, giving the corresponding solutions in [8, Thm. 1]. In our setting, the solutions of (19) for some choices of \( \lambda \in \mathbb{R} \) do not provide Lorentzian hypersurfaces of \( L^{n+2} \). Indeed for the choice \( \lambda = -1 \) the set of such solutions of (19) is given by

\[
r(t) = \frac{1}{b} \sinh(bt + c)
\]

where \( b \neq 0, c \in \mathbb{R} \) and \( t > -c/b \). Since the condition \( r'(t)^2 < 1 \), for all \( t \in J \) does not hold, the corresponding hypersurface does not inherit a Lorentzian metric from \( L^{n+2} \).

We end this section showing that each Lorentzian hypersurface \( Q(r) \) can be realized as the image by an isometric embedding of certain \((n+1)\)-dimensional Robertson-Walker spacetime in \( L^{n+2} \). Let \((S^n, g)\) be the \((n \geq 2)\)-dimensional unit (round) sphere endowed with the usual Riemannian metric of constant sectional curvature 1, and let \( I \subset \mathbb{R} \) be an open interval with metric \(-ds^2\). Consider the product manifold \( I \times S^n \) endowed with the Lorentzian metric

\[
g^f := -\pi^*_I(ds^2) + f(\pi)^2\pi^*(g),
\]

where \( f > 0 \) is a smooth function on \( I \), and \( \pi_I, \pi \) denote the projections onto \( I \) and \( S^n \), respectively. The resulting Lorentzian manifold is a warped product, in the sense of [13, Def. 7.33] with base \( I \), fibre \((S^n, g)\) and warping function \( f \), that is denoted by \( I \times_f S^n \) and called an \((n+1)\)-dimensional Robertson-Walker (RW) spacetime with spherical fibre. Without loss of generality, we will assume \( 0 \in I \) in all that follows.
Given an \((n + 1)\)-dimensional RW spacetime \(I \times f \mathbb{S}^n\), let \(h \in C^\infty(I)\) be given by \(h'(s) = \sqrt{1 + f'(s)^2} > 0\), for all \(s \in I\) and \(h(0) = 0\). Take \(J := h(I)\) and \(r := f \circ h^{-1} > 0\) that satisfies
\[
    r'(t) = \frac{f'(s)}{\sqrt{1 + f'(s)^2}} \quad \text{and} \quad r''(t) = \frac{f''(s)}{(1 + f'(s)^2)^2};
\]  
for any \(t = h(s)\). In particular we have \(|r'(t)| < 1\) for all \(t(= h(s)) \in J\). Thus, the map \(\psi : I \times f \mathbb{S}^n \to \mathbb{L}^{n+2}\) given by
\[
    \psi(s, p) = (h(s), f(s)p),
\]  
for any \((s, p) \in I \times f \mathbb{S}^n\) is an isometric embedding \([1]\). Now, we can observe that \(\psi(I \times f \mathbb{S}^n) = Q(r)\).

**Remark 2.16.** A Lorentzian manifold \((M, g)\) admits an isometric embedding in an \(N\)-dimensional Lorentz-Minkowski spacetime \(\mathbb{L}^N\) if and only if \((M, g)\) is a stably causal spacetime \([3\text{ p. 63}]\) and admits \(\tau \in C^\infty(M)\) such that \(g(\nabla \tau, \nabla \tau) \leq -1\) \([12\text{ Thm. 1.1}]\). Clearly, the function \(\tau : I \times f \mathbb{S}^n \to \mathbb{R}\), given by \(\tau(t, p) = t\) is smooth and its gradient satisfies \(\nabla \tau = -\partial / \partial t\). Therefore, the spacetime \(I \times f \mathbb{S}^n\) is stably causal \([4]\). Moreover, \(g(\nabla \tau, \nabla \tau) = -1\) everywhere, thus \(I \times f \mathbb{S}^n\) lies under the assumptions of \([12\text{ Thm. 1.1}]\) and hence, it is isometrically embeddable in \(\mathbb{L}^N\), indeed, formula \([22]\) asserts that, in this case, \(N = n + 2\).

**Remark 2.17.** Given \(f \in C^\infty(I), f > 0\), consider now on \(I \times \mathbb{S}^n\) the Riemannian metric \(g_f := \pi_1^*(ds^2) + f(\pi)^2 \pi^*(g)\), (compare with \([20]\)). Thus, we have a Riemannian warped product \((I \times \mathbb{S}^n, g_f)\). Assume \(|f'| < 1\) and let \(h \in C^\infty(I)\) given by \(h'(s) = \sqrt{1 - f'(s)^2} > 0\) for all \(s \in I\) and \(h(0) = 0\). Take as above \(J := h(I)\) and \(r := f \circ h^{-1} > 0\). In this case, \(r'(t) = f'(s)/\sqrt{1 - f'(s)^2}\), where \(h(s) = t\). Hence, the condition \(|r'| < 1\) does not hold here as in the Lorentzian case. Now, the map \(\varphi : I \times \mathbb{S}^n \to \mathbb{E}^{n+2}\) defined by \(\varphi(s, p) = (h(s), f(s)p)\), is an isometric embedding of \((I \times \mathbb{S}^n, g_f)\) in \(\mathbb{E}^{n+2}\). Moreover \(\varphi(I \times \mathbb{S}^n) = \{(t, x) \in J \times \mathbb{E}^{n+1} : \|x\|^2 = r(t)^2\}\) is a rotation hypersurface in \(\mathbb{E}^{n+1}\), \([5]\).

### 3 Stationary spacelike submanifolds of \(Q(r)\)

First of all, recall that for a spacelike immersion \(\Psi : M^k \to \mathbb{L}^{n+2}\), \(\Psi := (\Psi_0, \Psi_1, \ldots, \Psi_{n+1})\), the Beltrami equation for \(\Psi\) looks as follows
\[
    \Delta \Psi = k \mathbf{H},
\]  
where \(\mathbf{H}\) is the mean curvature vector field of \(\Psi\), \(\Delta\) the Laplacian of \(M^k\) and \(\Delta \Psi := (\Delta \Psi_0, \Delta \Psi_1, \ldots, \Delta \Psi_{n+1})\).

Once we have identified \(I \times f \mathbb{S}^n\) with \(Q(r)\), for a suitable function \(r\), any spacelike immersion \(\Psi\) from \(M^k\) to \(I \times f \mathbb{S}^n\) may be naturally seen as \(\Psi : M^k \to Q(r) \subset \mathbb{L}^{n+2}\).
Thus, let us consider $\sigma$ and $\tilde{\sigma}$ the second fundamental forms of $\Psi : M^k \to \mathbb{L}^{n+2}$ and $\tilde{\Psi} : M^k \to Q(r)$, respectively. From (23) we have
\[
\sigma(X,Y) = \tilde{\sigma}(X,Y) + \langle AX,Y \rangle N,
\]
for all $X,Y \in \mathcal{X}(M^k)$, where $N$, given in (8), is the unit normal vector field of $Q(r)$ in $\mathbb{L}^{n+2}$ and $A$ its corresponding Weingarten operator given by (11). Therefore, the respective mean curvature vector fields $H$ and $\tilde{H}$ are related by
\[
H = \tilde{H} + \frac{1}{k} \sum_{i=1}^{k} \langle AX_i, X_i \rangle N,
\]
where $\{X_i\}$ is a local orthonormal basis of tangent vector fields to $M^k$. Then, a direct computation from Lemma 2.7 gives that $\tilde{H} = 0$ if and only if
\[
\Delta \tilde{\Psi} = \left( k \alpha(\Psi_0) + \beta(\Psi_0) \|T^\top\|^2 \right) N.\]
Now, for any local orthonormal tangent frame $\{X_1,\ldots,X_k\}$, we have,
\[
\|T^\top\|^2 = \frac{1}{1 - r'(\Psi_0)^2} \sum_{i=1}^{k} \left( \left( 1, \frac{r'(\Psi_0)}{r(\Psi_0)} \Psi_1, \ldots, \frac{r'(\Psi_0)}{r(\Psi_0)} \Psi_{n+1} \right), (X_i(\Psi_0), \ldots, X_i(\Psi_{n+1})) \right)^2
\]
\[
= \frac{1}{1 - r'(\Psi_0)^2} \sum_{i=1}^{k} \left( -X_i(\Psi_0) + \frac{r'(\Psi_0)}{r(\Psi_0)} \Psi_1 X_i(\Psi_1) + \cdots + \frac{r'(\Psi_0)}{r(\Psi_0)} \Psi_{n+1} X_i(\Psi_{n+1}) \right)^2
\]
\[
= \frac{1}{1 - r'(\Psi_0)^2} \sum_{i=1}^{k} \left( -X_i(\Psi_0) + r'(\Psi_0)^2 X_i(\Psi_0) \right)^2
\]
\[
= (1 - r'(\Psi_0)^2) \sum_{i=1}^{k} (X_i(\Psi_0))^2
\]
\[
= (1 - r'(\Psi_0)^2) \|\nabla \Psi_0\|^2.
\]
Therefore, we get
\[
k \alpha(\Psi_0) + \beta(\Psi_0) \|T^\top\|^2 = \alpha(\Psi_0) \left[ k - \left( r''(\Psi_0)r(\Psi_0) + r'(\Psi_0)^2 - 1 \right) \|\nabla \Psi_0\|^2 \right].
\]
We summarize previous computations as follows,
Proposition 3.1. A spacelike immersion $\Psi: M^k \to Q(r)\subset \mathbb{L}^{n+2}$ is stationary if and only if
\[
\Delta \Psi = \alpha(\Psi_0) \left[ k - \left( r''(\Psi_0) r(\Psi_0) + r'(\Psi_0)^2 - 1 \right) \|\nabla \Psi_0\|^2 \right] N. \tag{28}
\]
Equivalently, if and only if
\[
\Delta \Psi + q_{\Psi_0} P = 0, \tag{29}
\]
where $P$ is the vector field along the immersion $\Psi$ given by
\[
P := \left( r(\Psi_0)r'(\Psi_0), \Psi_1, \ldots, \Psi_{n+1} \right) = -\frac{1}{\alpha(\Psi_0)} N \circ \Psi \tag{30}
\]
and
\[
q_{\Psi_0} := \alpha(\Psi_0)^2 \left[ k - \left( r''(\Psi_0) r(\Psi_0) + r'(\Psi_0)^2 - 1 \right) \|\nabla \Psi_0\|^2 \right]. \tag{31}
\]
Remark 3.2. The previous result gives rise to the following system of elliptic partial differential equations for the components of the spacelike immersion $\Psi: M^k \to \mathbb{L}^{n+1}$,
\[
\begin{aligned}
\Delta \Psi_0 + q_{\Psi_0} r(\Psi_0) r'(\Psi_0) &= 0, \\
\Delta \Psi_i + q_{\Psi_0} \Psi_i &= 0, \quad i = 1, 2, \ldots, n+1.
\end{aligned} \tag{32}
\]
Note that the second family of equations are stationary Schrödinger equations.

Example 3.3. Assume $r(t) = 1$ for all $t \in \mathbb{R}$. In this case $Q(1)$ is isometric to the Lorentzian product manifold $\mathbb{R} \times S^n$. As a direct consequence of Proposition 3.1 a spacelike immersion $\Psi: M^k \to Q(1) \subset \mathbb{L}^{n+2}$ is stationary if and only if
\[
\Delta \Psi_0 = 0, \quad \Delta \Psi_i + (k + \|\nabla \Psi_0\|^2) \Psi_i = 0, \quad i = 1, 2, \ldots, n+1. \tag{32}
\]
Assume now $r(t) = \sqrt{1 + t^2}$ for all $t \in \mathbb{R}$. In this case $Q(r)$ is isometric to the unitary De Sitter spacetime $S_1^{n+1}$. A spacelike immersion $\Psi: M^k \to Q(r) \subset \mathbb{L}^{n+2}$ is stationary if and only if
\[
\Delta \Psi + k \Psi = 0. \tag{33}
\]
Proposition 3.4. Given a stationary spacelike immersion $\Psi: M^k \to Q(r)$, if the function $r(\Psi_0)^2$ on $M^k$ attains a local maximum value at $x \in M^k$ then $q_{\Psi_0}(x) > 0$.

Proof. In fact, this follows from $\sum_{j=1}^{n+1} \Psi_j^2 = r(\Psi_0)^2$ that gives
\[
\frac{1}{2} \Delta r(\Psi_0)^2 + q_{\Psi_0} r(\Psi_0)^2 = \sum_{j=1}^{n+1} \|\nabla \Psi_j\|^2 > 0,
\]
using Remark 3.2. \qed
**Corollary 3.5.** Let $\Psi: M^k \to Q(r)$ be a stationary compact spacelike immersion. Assume $q_{\Psi_0} > 0$ and $r' \leq 0$ or $r' \geq 0$. Then, $\Psi: M^k \to Q(r)$ factors through a slice $\Psi_0 = \text{cte}$ with $r'(\Psi_0) = 0$. Therefore, $q_{\Psi_0}$ is a positive constant and $\Psi: M^k \to Q(r)$ realizes a minimal immersion in a totally geodesic slice of $Q(r)$, which is isometric to an $n$-dimensional round sphere of radius $r(\Psi_0)$. In particular, there is no compact stationary spacelike submanifold in $Q(r)$ which factors through a slab with $r' < 0$ or $r' > 0$.

**Proof.** From Remark 3.2, we have

$$\int_{M^k} q_{\Psi_0} r(\Psi_0) r'(\Psi_0) d\mu_g = 0,$$

where $d\mu_g$ denotes the canonical measure associated to the induced metric. Therefore, we get $r'(\Psi_0) = 0$ and Remark 3.2 implies that $\Delta \Psi_0 = 0$. The compactness of $M^k$ shows that $\Psi_0 = \text{cte}$ and $r'(\Psi_0) = 0$ implies that the corresponding slice is totally geodesic in $Q(r)$. Now, equation (29) reduces to

$$\Delta \Psi_i + \frac{k}{r(\Psi_0)^2} \Psi_i = 0, \quad i = 1, 2, \ldots, n + 1,$$

and Takahashi classic result [16] ends the proof. \qed

In order to provide a physical interpretation to the assumptions in Corollary 3.5, let us recall that for a given reference frame $U$ in a spacetime $\overline{M}$, the observers in $U$ are spreading out (resp. coming together) if $\text{div}(U) > 0$ (resp. $\text{div}(U) < 0$) [14, p. 58]. Thus, for the observers in $\overline{M}$ their universe is expanding (resp. contracting). In the case $\overline{M} = I \times_f \mathbb{S}^n$ and $U = \partial_t$, the co-moving reference frame, we have $\text{div}(\partial_t) = n f'/f$. Therefore, the Robertson-Walker spacetime $I \times_f \mathbb{S}^n$ is expanding (resp. contracting) (for co-moving observers) if $f'(t) > 0$ (resp. $f'(t) < 0$) for all $t \in I$. On the other hand, a spacetime $\overline{M}$ obeys the Null Convergent Condition if its Ricci tensor satisfies $\overline{\text{Ric}}(X, X) \geq 0$ for all null tangent vector $X$. This assumption is a necessary mathematical condition that holds from the physical fact that gravity attracts on average. Moreover, it also holds that if the spacetime obeys the Einstein equation (with zero cosmological constant) for suitable stress-energy tensors. In the case $\overline{M} = I \times_f \mathbb{S}^n$, the Null Convergence Condition holds if and only if

$$f^2(\log f)'' \leq 1, \quad (34)$$

(see [2], for instance).

Taking into account that the spherical Robertson-Walker spacetime $I \times_f \mathbb{S}^n$ is isometric to $Q(r)$ by means of (22). Therefore, from equations in (21), we get

$$r''(t)r(t) + r'(t)^2 - 1 = \frac{f''(s)f(s) - f'(s)^2 - 1}{(1 + f'(s)^2)^2}$$

and the Null Convergence Condition implies that $q_{\Psi_0} \geq 0$ holds for every stationary spacelike immersion in $Q(r)$. Moreover, we would like to point out that $\overline{M} = I \times_f \mathbb{S}^n$ is Einstein
if and only if
\[ r''(t)r(t) + r'(t)^2 - 1 = \frac{f''(s)f(s) - f'(s)^2 - 1}{(1 + f'(s)^2)^2} = 0. \]
(see [2], for instance). In particular, \( Q(r) \) must be totally umbilical in \( \mathbb{L}^{n+2} \) (see Remark 2.8).

Summarizing, for every \( k \)-dimensional stationary spacelike immersion in \( Q(r) \), satisfying the Null Convergence Condition, we have
\[ q_{\Psi_0} \geq k \alpha(\Psi_0)^2 \]
with equality if \( Q(r) \) is an Einstein manifold.

As a consequence of previous discussion, we have,

**Corollary 3.6.** There is no stationary compact spacelike submanifold in a spherical expanding or contracting Robertson-Walker spacetime \( I \times fS^n \) satisfying the Null Convergence Condition.

**Proof.** Let us argue by contradiction, suppose there exists a stationary compact spacelike immersion satisfying the above mentioned conditions. Now Corollary 3.5 can be applied since \( f' \) is signed and \( f''(s)f(s) - f'(s)^2 \leq 1 \), therefore, we should have \( f'(\Psi_0) = 0 \), which is a contradiction. \( \square \)

At this point, we would like to recall the semi-Riemannian version [11, Theorem 1] of the Takahashi result [16].

Let \( \Psi: M^k \rightarrow \mathbb{L}^{n+2} \) be a spacelike immersion, then \( \Delta \Psi + \lambda \Psi = 0 \), with \( \lambda > 0 \), holds if and only if \( \Psi \) realizes an immersion with zero mean curvature vector field in the De Sitter spacetime of radius \( \sqrt{k/\lambda} \). In particular, the equation \( \Delta \Psi + \lambda \Psi = 0 \) implies \( \sum_{i=1}^{n+1} \Psi_i^2 = \Psi_0^2 + k/\lambda \).

From Remark 2.9, we know that De Sitter spacetime of radius \( \sqrt{k/\lambda} \) equals \( Q(r) \) for \( r(t) = \sqrt{t^2 + k/\lambda} \). In this case, for a spacelike immersion \( \Psi: M^k \rightarrow \mathbb{L}^{n+2} \), formulas (29) reduce to \( \Delta \Psi + \lambda \Psi = 0 \) and Takahashi result shows, in particular, that (29) imply \( \Psi(M^k) \subset S^{n+1}_1(\sqrt{k/\lambda}) \) for \( r(t) = \sqrt{t^2 + k/\lambda} \).

Now the following question arises in a natural way: let \( r(t) \) be an admissible function defined on an open interval \( J \subset \mathbb{R} \). Let \( \Psi: M^k \rightarrow \mathbb{L}^{n+2} \) be any spacelike immersion and consider the function \( q_{\Psi_0} \) given as in (31). Suppose that \( \Psi \) satisfies \( \Delta \Psi + q_{\Psi_0} \mathbf{P} = 0 \).

Is it possible to deduce that \( \Psi(M^k) \subset Q(r) \)?

Note that if the answer is affirmative, Proposition 3.1 show that \( M^k \) is stationary in \( Q(r) \).

**Theorem 3.7.** Let \( r(t) \) be an admisible function and \( \Psi: M^k \rightarrow \mathbb{L}^{n+2} \) any spacelike immersion with \( q_{\Psi_0} > 0 \). If
\[ \Delta \Psi + q_{\Psi_0} \mathbf{P} = 0 \] (35)
holds, where \( \mathbf{P} \) is the vector field along \( \Psi \) given in (30), then \( \Psi \) realizes a stationary spacelike immersion in \( Q(r) \).
Proof. We have
\[ kH + q_{\Psi_0} \mathbf{P} = 0, \tag{36} \]
that implies \( \mathbf{P} \) is normal everywhere along the spacelike immersion \( \Psi: M^k \to \mathbb{L}^{n+2} \).

Now, for every \( v \in T_x M^k, v = (v_0, v_1, ..., v_{n+1}) \), we get
\[
\nabla^0_v \mathbf{P} = \left( \left[ r'(\Psi_0(x))^2 + r(\Psi_0(x))r''(\Psi_0(x)) \right] v_0, v_1, ..., v_{n+1} \right) \\
= v + v_0 \left[ r'(\Psi_0(x))^2 + r(\Psi_0(x))r''(\Psi_0(x)) - 1 \right] \frac{\partial}{\partial t} |_{\Psi(x)}. \tag{37} \]

On the other hand, Weingarten equation gives
\[
\nabla^0_v \mathbf{P} = -A_\mathbf{P}(v) + \nabla^\perp_v \mathbf{P}. \tag{38} \]

From the well-known formula \( \nabla \langle \Psi, a \rangle = a^\top \), for every \( a \in \mathbb{L}^{n+1} \), we get
\[
\left( \frac{\partial}{\partial t} |_{\Psi(x)} \right)^\top = \nabla \langle \Psi, e_0 \rangle = -\nabla \Psi_0.\]

Therefore, equations (37) and (38) imply
\[
A_\mathbf{P} = -\text{Id} + \left[ r'(\Psi_0)^2 + r(\Psi_0) r''(\Psi_0) - 1 \right] d\Psi_0 \otimes \nabla \Psi_0. \tag{39} \]

In particular, we have
\[
\text{trace}(A_\mathbf{P}) = -k + \left[ r'(\Psi_0)^2 + r(\Psi_0) r''(\Psi_0) - 1 \right] \| \nabla \Psi_0 \|^2 = -\frac{q_{\Psi_0}}{\alpha(\Psi_0)^2}. \tag{40} \]

From (31), we get
\[
\| \mathbf{H} \|^2 = \frac{\text{trace}(A_\mathbf{H})}{k} = \frac{q_{\Psi_0}^2}{k^2} \left( -r(\Psi_0)^2 r'(\Psi_0)^2 + \sum_{j=1}^{n+1} \Psi_j^2 \right), \tag{41} \]
and from (31) and (40)
\[
\text{trace}(A_\mathbf{H}) = k\| \mathbf{H} \|^2 = \frac{q_{\Psi_0}^2}{k \alpha(\Psi_0)^2}. \]

Therefore, we obtain
\[
-r(\Psi_0)^2 r'(\Psi_0)^2 + \sum_{j=1}^{n+1} \Psi_j^2 = \frac{1}{\alpha(\Psi_0)^2}, \]
which implies that \( \Psi(M^k) \subset Q(r) \). Finally, Lemma 3.1 is now called to end the proof.
References

[1] M.M. Akbar, Embedding FLRW geometries in pseudo-Euclidean and anti-de Sitter spaces, Physical Review D, 95 (2017), 064058(1–10).

[2] J.A. Aledo, R.M. Rubio and J.J. Salamanca, Complete spacelike hypersurfaces in generalized Robertson–Walker and the null convergence condition: Calabi–Bernstein problems, RACSAM, 111 (2017), 115–128.

[3] J.K. Beem, P.E. Ehrlich and K.L. Easley, Global Lorentzian Geometry, second edition, Monographs and Textbooks in Pure and Applied Mathematics, 202, Marcel Dekker, 1996.

[4] A.N. Bernal and M. Sánchez, Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes, Commun. Math. Phys., 257 (2005) 43–50.

[5] M. do Carmo and M. Dajczer, Rotation hypersurfaces in spaces of constant curvature, Trans. Amer. Math. Soc., 227 (1983), 685–709.

[6] B.-Y. Chen, Geometry of Submanifolds, Marcel Dekker, New York, 1973.

[7] B.-Y. Chen, On the total curvature of immersed manifolds IV: Spectrum and total mean curvature, Bull. Inst. Math. Acad. Sinica 7 (1979) 301–311.

[8] V. Coll and M. Harrison, Hypersurfaces of revolution with proportional principal curvatures, Advances in Geometry, 13 (2013), 485–496.

[9] D. Ferreira, E.A. Lima Jr. and A. Romero, Complete stationary spacelike surfaces in an n-dimensional Generalized Robertson-Walker spacetime, Mediterranean J. Math. (2022), (to appear).

[10] R. López, Timelike surfaces with constant mean curvature in Lorentz three-space, Tohoku Math. J., 52 (2000), 515–532.

[11] S. Markvorsen, A Characteristic Eigenfunction for Minimal Hypersurfaces in Space Forms, Math Z., 202, (1989), 375–382.

[12] O. Müller and M. Sánchez, Lorentzian manifolds isometrically embeddable in $\mathbb{L}^N$, Trans. Amer. Math. Soc., 363, (2011), 5367–5379.

[13] B. O’Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.

[14] R. Sachs and H. Wu, General Relativity for Mathematicians, Graduate Texts in Math. 48, Springer, New York, 1977.

[15] M. Sánchez, On the Geometry of Generalized Robertson Walker Spacetimes: Geodesics, Gen. Relat. Gravitation, 30 (1998), 915–932.
[16] T. Takahashi, Minimal immersion of Riemannian manifolds, *J. Math. Soc. Japan.*, 18 (1966), 380–385.

Danilo Ferreira and Eraldo A. Lima Jr  
Departamento de Matemática,  
Universidade Federal da Paraíba,  
58051-900 João Pessoa, PB, Brazil  
danilodfs.math@gmail.com  
eraldo.lima@academico.ufpb.br

Francisco J. Palomo  
Departamento de Matemática Aplicada  
Universidad de Málaga, 29071 Málaga, Spain  
fpalomo@uma.es

Alfonso Romero  
Departamento de Geometría y Topología,  
Universidad de Granada, 18071 Granada, Spain  
aromero@ugr.es