New interpretation of the extended geometric deformation in isotropic coordinates

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Abstract We study the particular case in which extended geometric deformation does consists of consecutive deformations of temporal and spatial components of the metric, in Schwarzschild-like and isotropic coordinates. In the latter, we present two inequivalent ways to perform this two-step GD. This was done in such a way that the method may be applied to different seed solutions. As an example, we use Tolman IV as seed solution, in order to obtain two inequivalent physical solutions with anisotropy in the pressures in Schwarzschild-like coordinates. In the isotropic sector, we obtained four different solutions with anisotropy in the pressures that satisfies physical acceptability conditions, using Gold III as seed solution.

1 Introduction

It is known that general relativity (GR) is the most successful theory describing the gravitational interaction. It explores the relation between the geometry of space-time and the energy–matter content of the universe. This can be seen from the Einstein field equations [1], which is a nonlinear system of equations in partial derivatives. Gravitational systems, such as Relativistic Compact Stars or Black Holes, are modeled by solutions to these equations. Now, due to the complicated form of Einstein’s equations, finding analytical solutions describing general matter configurations could be very difficult, or even impossible in many cases. For this reason, Einstein’s equations are solved using numerical method for more complicated (and also more realistic) systems. The known analytical solutions are obtained by assuming some simplifications. These simplifications are related to restrictions of the geometry of the space-time, or the matter distributions.

A common choice for the geometry of space time is to assume spherical symmetric systems, which leads to huge simplifications of Einstein equations. On the matter side, the major part of the known analytical solutions are for isotropic perfect fluid, which is one of the simplest cases known. Historically, since Tolman [2] presented his seminal work, there are different interior solutions in terms of the isotropic perfect fluid solutions that have been found [3–5]. However, anisotropy in the pressures of interior solutions, it is a desirable property in order to describe realistic compact objects [6–11]. Indeed, the local anisotropy in the pressures is expected to be present in matter distributions, since it could be caused by

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a series of phenomena that could be present in compact objects, as for example the viscosity or the intense magnetic fields.

Over the last years, one popular way to obtain analytical solutions with physical relevance to Einstein equations has been the use of the minimal geometric deformation (MGD) approach, which allows us to decouple gravitational sources in GR. The method was first introduced in the context of Randall–Sundrum Brane World (RSBW) [12–19] (see also [20]) and later on was introduced in GR [21]. This method has been widely used to study different scenarios in GR like compact objects [22–28], black holes [29–33], wormholes, cosmology [34] or the coupling of Einstein equations with matter fields [35,36], among other possibilities. Moreover, it has been studied in dimensions different than four [32,37–40], for theories beyond GR (for example \( f(R) \) gravity [41], \( f(G) \) [42,43], \( f(R,T) \) gravity [44], Lovelock gravity [45], a more recent application to RSBW [46]) and other gravitational theories as Brans–Dicke gravity [47,48]. Noticing that interior solutions related with self-gravitating objects obtained by MGD, must have anisotropy in the pressures, we can then understand MGD as a way to obtain realistic anisotropic systems [49–51]. In the context of MGD, the word “minimal” is related to the fact that we are considering only the geometric deformation of the spatial component. Other formulations of the MGD method include the study of systems with cylindrical symmetry [52] and its generalization to axial symmetry [53].

In [54], we propose two new MGD-inspired methods, to solve Einstein’s equations in isotropic coordinates, in order to obtain new physical anisotropic solutions. Now, as it should be, the physics do not depend on the chosen a coordinate system. However, studying MGD in these coordinates could be interesting since the coordinate transformation between the standard and isotropic is not always well defined (see [55]). Then, it is possible to find solutions in isotropic coordinates whose transformation to Schwarzschild-like coordinates is not well defined. Or equivalently, solutions that cannot be written in analytical form in Schwarzschild-like coordinates. Therefore, the MGD-inspired methods in [54] could be useful to study solutions of Einstein’s equations in these cases. They may also be more useful in cosmology and astrophysics [56]. It is worth mentioning that in general, there is no gravitational decoupling in isotropic coordinates. Despite the large number of applications of the MGD method, probably one of the biggest limitations it is that we are restricted to study sources that only modify the spatial component of the metric. Therefore, the method cannot be applied to decouple some self-gravitational configurations, for example, the Einstein–Maxwell configuration.

The extended geometric deformation (EGD) presented in [57], allow us to decouple two spherically symmetric and static gravitational sources by geometric deformation of both, spatial and temporal components of the metric. In this case, the decoupling of the two sources is only possible when there is an interchange of energy between both sources, which is not a requirement in the MGD case. It has been recently studied in [39,58,59] that the extended version of the method also allows to extend solutions to the anisotropic domain and study even more complicated contribution of modified theories of gravity of the GR. This shows the great potential of the geometric deformation to study self-gravitational systems. It is evident that the extended case contains a new unknown function to be determined. Therefore, we have to give more information than standard MGD in order to solve the system (see [60–86]) for more applications of the geometric deformation method).

In this work, we are interested in studying the extended version of MGD in Standard-like and isotropic coordinates. In particular, we will consider consecutive geometric deformations of radial and temporal components of the metric. We will show that in general, these deformations do not commute. However the solutions are always contained in EGD. In Sect. 1, we will briefly resume the EGD method in Schwarzschild-like coordinates. In Sect. 2, we will
study consecutive geometric deformations in Schwarzschild-like coordinates and we will apply the results to the known Tolman IV solution. In Sect. 3, we will present the extended version of the geometric deformation in isotropic coordinates, and in Sect. 4, we will consider consecutive spatial and temporal deformations of the metric in isotropic coordinates. In this section, we will also present some examples of solutions taking as a seed the Gold III solution for perfect fluid. Finally, in Sect. 5, we present a discussion on the obtained results.

2 Extended geometric deformation

Let us begin by writing the line element in standard coordinates, also known as Schwarzschild-like coordinates

\[ ds^2 = e^{\tilde{\nu}(r)} \, dt^2 - \frac{1}{\tilde{\mu}(r)} \, dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right). \]  

(1)

It can be shown that Einstein’s equations

\[ R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = -8\pi T_{\mu\nu}, \]

(2)

take the following form

\[ 8\pi T_0^0 = \frac{1}{r^2} - \frac{\tilde{\mu}}{r^2} - \frac{\tilde{\mu}'}{r}, \]

(3)

\[ -8\pi T_1^1 = -\frac{1}{r^2} + \tilde{\mu} \left( \frac{1}{r^2} + \frac{\tilde{\nu}'}{r} \right), \]

(4)

\[ -8\pi T_2^2 = \frac{\tilde{\mu}}{4} \left( 2\tilde{\nu}'' + \tilde{\nu}'^2 + \frac{2\tilde{\nu}'}{r} \right) + \frac{\tilde{\mu}'}{4} \left( \tilde{\nu}' + \frac{2}{r} \right), \]

(5)

where the prime indicates derivatives respect to variable \( r \). Finally, the conservation equation, derived from the latter system of equations, is

\[ \nabla_{\mu} T^{\mu\nu} = 0, \]

(6)

whose radial component leads to the equilibrium equation of the matter distribution

\[ 8\pi (T_1^1)' + \frac{16\pi}{r} (T_1^1 - T_2^2) + 4\pi \tilde{\nu}' (T_1^1 - T_0^0) = 0. \]

(7)

The starting point of this method is to assume that the energy-momentum tensor has the specific form

\[ T_{\mu\nu} = T_{\mu\nu}^0 + \alpha \, \theta_{\mu\nu}, \]

(8)

where \( \alpha \) is a coupling constant. In this work, for simplicity we will assume that \( T_{\mu\nu}^0 \) is the matter–energy content associated with a perfect fluid

\[ T_{\mu\nu}^0 \equiv T_{\mu\nu}^{PF} = (\rho + P) \, u_\mu \, u_\nu - P \, g_{\mu\nu}, \]

(9)

with the fluid 4-velocity given by \( u^\mu = e^{-\tilde{\nu}/2} \delta^\mu_0 \).

Now, we can consider a perfect fluid solution of Einstein equations \( (\alpha = 0) \), with a line element written in standard coordinates as (1). We can define

\[ \tilde{\mu}(r) = 1 - \frac{8\pi}{r} \int_0^r x^2 \rho \, dx = 1 - \frac{2m(r)}{r}, \]

(10)
which is the standard expression for the mass function in GR. The next step is to take account of the anisotropy introduced by the gravitational source $\theta_{\mu\nu}$ in our system. This will be done by assuming that the contribution of $\theta_{\mu\nu}$, in the perfect fluid solution (1), is encoded in the deformations $h$ and $f$ of the temporal and radial metric components, respectively

$$\t = \xi + \alpha h,$$

$$\tm = \mu + \alpha f.$$  \hfill (11) \hfill (12)

In this case it is easy to check that, using Eqs. (8), (11) and (12), Einstein’s equations (3)-(5) splits in two systems. The first one coincides with Einstein’s equations system for a perfect fluid

$$8\pi \rho = \frac{1}{r^2} - \frac{\mu}{r^2} - \frac{\mu'}{r},$$

$$8\pi P = -\frac{1}{r^2} + \mu \left( \frac{1}{r^2} + \frac{\xi'}{r} \right),$$

$$8\pi P = \frac{\mu}{4} \left( 2\xi'' + \xi'^2 + \frac{2\xi'}{r} \right) + \frac{\mu'}{4} \left( \xi' + \frac{2}{r} \right),$$

with the corresponding conservation equation

$$P' + \frac{\xi'}{2} (\rho + P) = 0.$$  \hfill (16)

This expression turns out to be Eq. (6) with the energy momentum tensor associated to a perfect fluid (9) (Eq. (8) with $\alpha = 0$).

The second system of equations reads

$$8\pi \theta_{0}^0 = \frac{-f}{r^2} - \frac{f'}{r},$$

$$8\pi \theta_{1}^1 + Z_1 = -f \left( \frac{1}{r^2} + \frac{\t'}{r} \right),$$

$$8\pi \theta_{2}^2 + Z_2 = -\frac{f}{4} \left( 2\t'' + \t'^2 + \frac{2\t'}{r} \right) - \frac{f'}{4} \left( \t' + \frac{2}{r} \right),$$

and the conservation equation associated with the source is

$$\left( \theta_{1}^1 \right)' - \frac{\t'}{2} \left( \theta_{0}^0 - \theta_{1}^1 \right) - \frac{2}{r} \left( \theta_{2}^2 - \theta_{1}^1 \right) = 0,$$  \hfill (20)

with

$$Z_1 = \frac{\mu h'}{r},$$

$$4Z_2 = \mu \left( 2h'' + \alpha h'^2 + \frac{2h'}{r} + 2\xi' h' \right) + \mu' h'.$$  \hfill (22)

In order to find a solution of Einstein’s equations for an energy-momentum tensor of the form (8), we have to solve the systems (13)–(15) and (17)–(19). In the case when we start with a known perfect fluid solution, then is only necessary to solve the second system. Now, in both cases there are more unknown functions than equations, so additional information is required in order to solve the system. Specifically, it is necessary to impose two conditions in order to solve the second system. This information can be given in the form of an equation of state or any other expression that relates the physical variables of the system under study.
Now, it is important to mention that the sources \( T^{(PF)}_{\mu\nu} \) and \( \theta^{\mu\nu} \) can be decoupled only if there exists an interchange of energy between them. This can be easily seen from the conservation equations

\[
\nabla_\mu (T^{PF})^\mu_\nu = -\frac{\hbar'}{2}(P + \rho)\delta^1_\nu, \tag{23}
\]

and

\[
\nabla_\mu \theta^{\mu\nu} = \frac{\hbar'}{2}(P + \rho)\delta^1_\nu. \tag{24}
\]

At this point is clear that EGD is a powerful tool to study more complicated solutions of Einstein’s field equations, than the ones obtained with the MGD method. Nevertheless, find a solution for the equations (17)–(19) could be very complicated depending on the system under study.

3 The two-step GD in Schwarzschild-like coordinates

In this section, we analyze a possible way to simplify the problem of finding solutions for Eqs. (17)–(19). This simplification consist in considering deformations of the metric components not simultaneous, but consecutive.

Let us first notice that there are two straightforward limits of the EGD method. The first one is obtained by taking \( h = 0 \), it corresponds with usual MGD and in this case, (17)–(19) may be written as

\[
8\pi \left( \theta^f \right)_0^0 = -\frac{f}{r^2} - \frac{f'}{r}, \tag{25}
\]

\[
8\pi \left( \theta^f \right)_1^1 = -f \left( \frac{1}{r^2} + \frac{\xi'}{r} \right), \tag{26}
\]

\[
8\pi \left( \theta^f \right)_2^2 = -\frac{f}{4} \left( 2\xi'' + \xi'^2 + 2\frac{\xi'}{r} \right) - \frac{f'}{4} \left( \xi' + \frac{2}{r} \right). \tag{27}
\]

The second limit corresponds to the case where \( f = 0 \), in which case the system of Eqs. (17)–(19) is given by

\[
8\pi \left( \theta^h \right)_0^0 = 0, \tag{28}
\]

\[
8\pi \left( \theta^h \right)_1^1 + Z_1 = 0, \tag{29}
\]

\[
8\pi \left( \theta^h \right)_2^2 + Z_2 = 0. \tag{30}
\]

This corresponds to a purely temporal deformation of the metric. Notice that this limit of the EGD cannot be thought as a simple modification of the time scale due to the \( r \)-dependence on the deformation function. In contrast with the pure spatial metric deformation, the \( (\theta^h)_{\mu\nu} \) cannot be interpreted as a physical source of matter due to Eq. (28), only the total energy momentum has a physical interpretation. The “source” \( (\theta^h)_{\mu\nu} \) can be understood as a mathematical “artifact” in order to obtain new solutions, or to connect inequivalent solutions of Einstein’s equations. Analyzing the Ricci invariants, see Appendix 6, it can be shown that all the solutions obtained by EGD (and therefore by any of their limits) are different from the seed solutions. Therefore, geometric deformation of the spatial or temporal component of the metric, or both of them, leads in general, to inequivalent solutions. For example, it can be
check that Einstein metric for an isotropic perfect fluid and Schwarzschild interior solution can be related with a pure temporal deformation.

Therefore, both limits can be used as a method to find solutions of Einstein’s field equations, by deformation of the spatial or temporal component of the metric, respectively. Then, instead solving directly the system (17)–(19), we can obtain solutions by taking consecutive deformations of the metric. Specifically, we can choose a seed solution and obtain a new one with a deformation in the spatial (temporal) component of the metric. We may now use this result as seed solution to find another one by performing a deformation of the temporal (spatial) component of the metric. In this way, we are able to obtain solutions with deformations in both, spatial and temporal components of the metric. In consequence, these are solutions of (17)–(19), which is a more general system and it may be more difficult to solve. Let us named this particular process two-step geometric deformation (two-step GD).

It is evident that there are two different ways to perform the two-step GD. For reasons of simplification, we will refer to them as the left and right paths, corresponding to the case where the spatial and temporal deformation is made first, respectively. We represent this in the figure. It is easy to verify that solutions obtained by the left and right paths, separately, are not only inequivalent but they correspond to different set of solutions. Regularity conditions ensures that solutions obtained by the left path, cannot be obtained by the right path, and vice versa. For this reason, it is interesting to study both left and right paths. Moreover, it can be shown (see Appendix 6), that from the left and right paths of the two-step GD, the first four theorems presented [87], which corresponds to transformations between perfect fluid spheres to perfect fluid spheres (different from the original ones), may be obtained. This gives a simple example of the inequivalence between the right and left paths.

We have already mentioned that two-step GD is a particular case of EGD, in fact, it corresponds to solutions with a source term of the form

\[ \theta_{\mu\nu} = \left( \theta^f \right)_{\mu\nu} + \frac{\beta}{\alpha} \left( \theta^h \right)_{\mu\nu}, \]  

(31)

where \( \left( \theta^f \right)_{\mu\nu} \) and \( \left( \theta^h \right)_{\mu\nu} \) are responsible of deformations of the spatial and temporal component of the metric, respectively (See Fig. 1). This particular decomposition of the source \( \theta_{\mu\nu} \) is not required for EGD. Thus, there are solutions that could be obtained by the EGD and not by the two-step GD. It is also important to mention that the source \( \theta_{\mu\nu} \) will depend on the path. Thus, from now on, we will denote as \( \left( \theta_L \right)_{\mu\nu} \) and \( \left( \theta_R \right)_{\mu\nu} \) the sources for the left path and right path, respectively.

In order to give an example let us consider Tolman IV solution of Einstein equations as seed solution

\[ e^\xi = B^2 \left( 1 + \frac{r^2}{A_1^2} \right), \]  

\[ \mu = \frac{\left( 1 - \frac{r^2}{C_2^2} \right) \left( 1 + \frac{r^2}{A_1^2} \right)}{\left( 1 + \frac{2r^2}{A_1^2} \right)} \],  

\[ \rho = \frac{3A_1^4 + A_1^2(3C^2 + 7r^2) + 2r^2(C^2 + 3r^2)}{8\pi C^2(A_1^2 + 2r^2)^2}, \]  

\[ P = \frac{C^2 - A_1^2 - 3r^2}{8\pi C^2(A_1^2 + 2r^2)}. \]  

(32)  

(33)  

(34)  

(35)
where $A_1, B$ and $C$ are constants that can be determine by the matching conditions. In this work, we will match all the internal solutions with the Schwarzschild vacuum solution.

### 3.1 The left path

If we consider first the geometric deformation of the radial component ($h_L = 0$) of Tolman IV solution, which is the known MGD, we have to solve the following system

\[
8\pi \left( \theta_L^f \right)_0^0 = -\frac{f_L}{r^2} - \frac{f_L'}{r},
\]

\[
8\pi \left( \theta_L^f \right)_1^1 = -f_L \left( \frac{1}{r^2} + \frac{\xi'}{r} \right),
\]

\[
8\pi \left( \theta_L^f \right)_2^2 = -\frac{f_L}{4} \left( 2\xi'' + \xi'^2 + 2\frac{\xi'}{r} \right) - \frac{f_L'}{4} \left( \xi' + \frac{2}{r} \right).
\]

Taking now the mimic constrain $(\theta_L^f)_1^1 = p$, we obtain from (37) that

\[
f_L = \frac{(A_1^2 + 3r^2 - C^2)r^2(A_1^2 + r^2)}{C^2(A_1^2 + 2r^2)(A_1^2 + 3r^2)}.
\]

Is easy to see that, once we obtain the deformation function $f_L$, we are able to compute $(\theta_L^f)_0^0$ and $(\theta_L^f)_2^2$. Therefore, the new anisotropic solution is characterized by the metric

\[
e^\nu = e^{\bar{k}},
\]

\[
\bar{\mu} = \mu + \alpha f_L,
\]

\[
\bar{\rho} = \rho + \alpha (\theta_L^f)_0^0,
\]

\[
\bar{P}_r = (1 - \alpha) p,
\]

\[
\bar{P}_t = p - \alpha (\theta_L^f)_2^2
\]

which is the anisotropic solutions of Einstein’s equations obtained in [88].

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**Fig. 1** The two-step GD diagram
Now, we can choose this result as a seed solution and perform a deformation in the temporal component of the metric. Then we have to solve the following system of equations

\[ 8\pi (\theta^h_L)_0 = 0, \]
\[ 8\pi (\theta^h_L)_1 = -\frac{\tilde{\mu} h'_L}{r}, \]
\[ 8\pi (\theta^h_L)_2 = -\frac{\mu}{4} \left( 2h''_L + \alpha h'^2 + \frac{2h'_L}{r} + 2\xi h'_L \right) - \frac{\mu h'_L}{4}. \]

Imposing now the constraint \((\tilde{\theta}^h)^1 = \tilde{\mu} \tilde{\rho},\) it can be check from (46) that

\[ h_L = (1 - \alpha) \left( \frac{3}{4} \frac{r^2}{C^2} - \frac{1}{4} \ln (A^2_1 + 2r^2) - \frac{1}{8} \frac{\ln (A^2_1 + 2r^2)}{C^2} \right). \]

As before, once we have obtained the deformation function \(h_L,\) we are able to compute \((\theta^h_L)_2^2.\)

The final solutions with both deformations of the metric components is given by

\[
\begin{align*}
\bar{\varphi} &= B^2 \left( 1 + \frac{r^2}{A^2_1} \right) \bar{F}_1 (r), \\
\bar{\mu} &= \tilde{\mu} = \mu + \alpha f_L, \\
\bar{\rho} &= \tilde{\rho} = \rho + \alpha (\theta^f_L)_0, \\
\bar{\rho}_r &= \tilde{\rho}_r - \beta (\theta^h_L)_1 = (1 - \alpha)(1 - \beta \tilde{\mu}) p, \\
\bar{\rho}_t &= \tilde{\rho}_t - \beta (\theta^h_L)_2 = p - \alpha \left( (\theta^f_L)_2 + \frac{\beta}{\alpha} (\theta^h_L)_2 \right),
\end{align*}
\]

where \(\bar{F}_1 = e^{\theta h^t}.\) From the matching conditions we obtain

\[
\begin{align*}
A_1 &= \frac{R}{M} \sqrt{(R - 3M)M}, \\
C &= \sqrt{A^2_1 + 3R^2}, \\
B^2 &= \left( 1 - \frac{2M}{R} \right) \left( 1 + \frac{R^2}{A^2_1} \right)^{-1} e^{-\beta g_L(R)}. \end{align*}
\]

Now, to present an example of a physically acceptable solution we choose the following values for the free parameters \(R = 5, \) \(M = 1, \) \(\alpha = 0.3\) and \(\beta = 0.1.\) Well-behavior of this solution is evident from Figs. (2) and (3), where pressures and energy density were plotted, respectively

3.2 The right path

Let us now consider first the deformation of the temporal component \((f = 0)\) of Tolman IV solutions. In this case, the system that we must solve is

\[
\begin{align*}
8\pi (\theta^h_R)_0 &= 0, \\
8\pi (\theta^h_R)_1 &= -\frac{\mu h'}{r}, \\
8\pi (\theta^h_R)_2 &= -\frac{\mu}{4} \left( 2h'' + \alpha h'^2 + \frac{2h'_R}{r} + 2\xi h'_R \right) - \frac{\mu h'_R}{4}. \end{align*}
\]
Fig. 2  Radial and tangential pressures for the left path in Schwarzschild like vs the radial coordinate

Fig. 3  Energy density for the left path in Schwarzschild like vs the radial coordinate
In order to solve it, we will impose a constraint of the same form that the one in left path \((\theta_R^h)^1_1 = p\mu\). Then, it is easy to see that

\[
h_R = \frac{3}{4} \frac{r^2}{C^2} - \frac{1}{4} \ln(A_1^2 + 2r^2) - \frac{1}{8} \frac{\ln(A_1^2 + 2r^2)A_1^2}{C^2}.
\]

(60)

Thus, the new anisotropic solution is given by

\[
e^{\tilde{\nu}} = e^{\xi(r) + \alpha h(r)} = B^2 \left(1 + \frac{r^2}{A_1^2}\right) F_1(r),
\]

(61)

\[
\tilde{\mu} = \mu,
\]

(62)

\[
\tilde{\rho} = \rho,
\]

(63)

\[
\tilde{P}_r = (1 - \alpha \mu) P,
\]

(64)

\[
\tilde{P}_t = P - \alpha (\theta_R^h)^2_2
\]

(65)

with \(F_1(r) = e^{\alpha h}\).

As before, we can take this result as seed solution, and then make a deformation of the spatial component of the metric. In this case, the system of equations that we have to solve is given by

\[
8\pi (\theta_R^f)^0_0 = -\frac{f_R}{r^2} - \frac{r f_R'}{r},
\]

(66)

\[
8\pi (\theta_R^f)^1_1 = -f_R \left(\frac{1}{r^2} + \frac{\tilde{\nu}'}{r}\right),
\]

(67)

\[
8\pi (\theta_R^f)^2_2 = -\frac{f_R}{4} \left(2\tilde{\nu}'' + \tilde{\nu}' + 2\tilde{\nu}'' + \frac{\tilde{\nu}'}{r}\right) - \frac{f_R'}{4} \left(\tilde{\nu}' + \frac{2}{r}\right),
\]

(68)

Now, considering the mimic constraint, \((\theta_R^f)^1_1 = \tilde{P}_r\) as in the left path (61) is easy to see that

\[
f_R = -\frac{(A_1^2 - C^2 + 3r^2)(A_1^2 + r^2) J_1(r)}{C^2(A_1^2 + 2r^2) J_2(r)},
\]

(69)

with

\[
J_1(r) = A_1^2 C^2 \alpha - A_1^2 \alpha r^2 + C^2 \alpha r^2 - \alpha r^4 - A_1^2 C^2 - 2C^2 r^2,
\]

(70)

\[
J_2(r) = A_1^4 \alpha r^2 - A_1^2 C^2 \alpha r^2 + 4A_1^2 \alpha r^4 - C^2 \alpha r^4 + 3\alpha r^6 + A_1^4 C^2 + 5A_1^2 C^2 r^2 + 6C^2 r^4
\]

(71)

Therefore, the final solution with both deformations in the metric is given by

\[
e^{\tilde{\nu}} = e^{\tilde{\nu}},
\]

(72)

\[
\tilde{\mu} = \mu + \beta f_R,
\]

(73)

\[
\tilde{\rho} = \rho + \beta (\theta_R^f)^0_0,
\]

(74)

\[
\tilde{P}_r = \tilde{P}_r - \beta (\theta_R^f)^1_1 = (1 - \beta)(1 - \alpha \mu) P,
\]

(75)

\[
\tilde{P}_t = \tilde{P}_t - \beta (\theta_R^f)^2_2 = P - \alpha \left(\frac{(\theta_R^h)^2_2}{\alpha (\theta_R^f)^2_2}\right)
\]

(76)

where \(F_1(r) = e^{\alpha R}\). The matching conditions for this case leads to

\[
A_1 = \frac{R}{M \sqrt{(R - 3M)M}},
\]

(77)

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Finally, as before, to give an example of a physically acceptable solution we choose the following values $R = 5$, $M = 1$, $\alpha = 0.3$ and $\beta = 0.1$ and plot the pressures and energy density in the Figs. (4) and (5), respectively. On the other hand, we compare the radial pressures and energy densities obtained by the right and left path in Figs. (6) and (7), respectively.

This anisotropic solution was obtained after two consecutive geometric deformations of the temporal and radial components of Tolman IV. It can be check that turning off any of the coupling constants $\alpha$ or $\beta$, related to the first and second deformation, respectively, results in the expected sector of solutions as it is shown in the diagram.

4 Extended geometric deformation in isotropic coordinates

Let us consider the interior of static and spherical symmetric matter distributions using the line element in isotropic coordinates

$$d\tilde{s}^2 = e^{\tilde{v}(r)} dr^2 - \frac{1}{\tilde{\omega}} (dr^2 + r^2 d\Omega^2),$$

(80)

Be aware that, for simplicity, from now on, we will use $r$ for the isotropic coordinates and $r_1$ for the Schwarzschild coordinates.
Fig. 5  Energy density for the right path in Schwarzschild like vs the radial coordinate

Fig. 6  Radial pressures obtained by the left and right paths
in which Einstein’s equations take the following form

\begin{align*}
8\pi T_0^0 &= \tilde{\omega}'' - \frac{5}{4} \frac{\tilde{\omega}^{'2}}{\tilde{\omega}} + \frac{2}{r} \tilde{\omega}', \\
-8\pi T_1^1 &= \frac{1}{4} \frac{\tilde{\omega}^{'2}}{\tilde{\omega}} - \frac{1}{2} \tilde{\omega}' \tilde{\nu}' + \frac{\tilde{\nu}' \tilde{\omega} - \tilde{\omega}'}{r}, \\
-8\pi T_2^2 &= \frac{1}{2} \left( \frac{\tilde{\omega}^{'2}}{\tilde{\omega}} - \tilde{\omega}'' \right) + \left( \frac{\tilde{\nu}''}{2} + \frac{\tilde{\nu}'^2}{4} + \frac{\tilde{\nu}'}{2r} \right) \tilde{\omega} - \tilde{\omega}' \tilde{\omega}' \frac{2}{2r}.
\end{align*}

(81) \quad (82) \quad (83)

This coordinates seems to be more general than the Schwarzschild like ones (1), since there is always possible to transform the line element from the standard form to the isotropic one by

\begin{align*}
    r &= K_1 \exp \left\{ \int \frac{\mu^{-1/2} dr_1}{r_1} \right\},
\end{align*}

(84)

where \( K_1, \mu^{-1} \) and \( r_1 \) are the radial component and the coordinate associated to the line element in standard coordinates (1). However, it is not always possible to perform the reverse process. Therefore, there is a chance to obtain solutions to Einstein’s equations that cannot be found using the Schwarzschild like coordinates.

If we consider that \( \tilde{\omega} = (\tilde{A}(r))^2 \), Eqs. (81)–(83) read

\begin{align*}
8\pi T_0^0 &= -3(\tilde{A}')^2 + 2\tilde{A} \tilde{A}'' + \frac{4}{r} \tilde{A} \tilde{A}', \\
-8\pi T_1^1 &= (\tilde{A}')^2 - \tilde{A} \tilde{A}' \tilde{\nu}' + \tilde{\nu}' \frac{\tilde{A}^2}{r} - \frac{2\tilde{A} \tilde{A}'}{r}, \\
-8\pi T_2^2 &= (\tilde{A}')^2 - \tilde{A} \tilde{A}'' + \left( \frac{\tilde{\nu}''}{2} + \frac{(\tilde{\nu}')^2}{4} + \frac{\tilde{\nu}'}{2r} \right) \tilde{A}^2 - \frac{\tilde{A} \tilde{A}'}{r}.
\end{align*}

(85) \quad (86) \quad (87)
We notice from (81)–(83) (or (85)–(87)) that the system will not decouple if we choose an energy momentum tensor of the form (8) and consider the particular ansatz for the metric (12) of the MGD method. We have discussed this issue on [54] and we have shown how this conditions can be used to obtain new internal analytical and physical solutions of Einstein’s equations in isotropic coordinates. In fact, we have proposed two different and inequivalent algorithms that allow us to obtain new solutions in isotropic coordinates, starting from a seed solution. This was done considering only (12) which in Schwarzschild coordinates it is known as MGD.

We will see how we can generalize the first and second algorithms from [54], in order to introduce both deformations.

4.1 First Algorithm

As we mention before, Einstein equations in isotropic coordinates are not decoupled when considering a deformation of the metric component of the form (see [54] for details)

\[
\tilde{\omega} = \omega + \alpha f, \quad \tilde{\nu} = \xi + \alpha h.
\]

However, let us notice that if we consider a specific combination of equations (81)–(83) given by

\[
8\pi (P_r + \rho + 2P_t) = \frac{1}{2} \xi^2 \omega + \xi'' \omega + \frac{2\xi' \omega}{r} - \frac{1}{2} \omega' \xi',
\]

is easy to verify that (8), (88) and (89) implies that Eq. (90) splits into two equations

\[
8\pi (P_r + \rho + 2P_t) = \frac{1}{2} \xi^2 \omega + \xi'' \omega + \frac{2\xi' \omega}{r} - \frac{1}{2} \omega' \xi',
\]

\[
8\pi (-\theta_1^1 + \theta_0^0 - 2\theta_2^0) + Z(r) = F(r),
\]

with

\[
Z(r) = h'' \omega + \frac{1}{2} \alpha h'^2 \omega + \xi' h' \omega + \frac{2}{r} h' \omega - \frac{1}{2} \omega' h',
\]

\[
F(r) = \frac{1}{4} \left(2\tilde{\nu}^2 f + 4\tilde{\nu}'' f + \frac{8\tilde{\nu}' f}{r} - 2 f'\tilde{\nu}' \right),
\]

related to the perfect fluid and the source, respectively. At this point, two restrictions has to be imposed in order to obtain \( f \) and \( h \). Moreover, let us notice that from Eqs. (81)–(83) it can be check that

\[
8\pi \theta_0^0 = \frac{1}{\omega + \alpha f} \left\{ \left[ \omega'' + \frac{2}{r} \omega' - 8\pi T_0^0 \right] f + \left[ \frac{2}{r} \omega - \frac{5}{2} \omega' \right] f' + f'' \omega + \alpha \left[ \frac{2}{r} f' f - \frac{5}{4} f'^2 + f f'' \right] \right\},
\]

\[
8\pi \theta_1^1 = -\frac{1}{(\omega + \alpha f)} \left\{ 8\pi T_1^1 + \frac{2}{r} \tilde{\nu}' \omega - \frac{1}{2} \omega' \tilde{\nu}' - \frac{\omega'}{r} \right] f + \left[ \frac{1}{2} \omega' - \frac{1}{2} \omega \tilde{\nu}' - \frac{1}{2} \omega \right] f'
- \left[ \frac{1}{2} h' \omega' - \frac{h' \omega}{r} \right] \omega + \alpha \left[ \frac{1}{4} f'^2 - \frac{1}{2} \tilde{\nu}' \omega' f + \frac{\tilde{\nu}' f^2 - f' f}{r} \right] \right\},
\]

\[
8\pi \theta_2^2 = -\frac{1}{(\omega + \alpha f)} \left\{ 8\pi T_2^2 - \frac{1}{2} \omega'' + 2\omega \left( \frac{\tilde{\nu}'}{2} + \frac{(\tilde{\nu}')^2}{4} + \frac{\tilde{\nu}'}{2r} - \frac{\omega'}{2r} \right) f + \left[ \omega' - \frac{\omega}{2r} \right] f' \right\}.
\]
First let us assume that the sets \( \{ \xi, A, T_{\mu\nu}^{PF} \} \) and \( \{ \nu, f, H_{\mu\nu} \} \) are both Einstein equations systems. There is a third system given by \( \text{deformations functions } f \) and \( h \), we can obtain \( \theta_0^0, \theta_1^1 \) and \( \theta_2^2 \).

### 4.2 Second algorithm

Let us consider the system of Eqs. (85–87) and let us assume that the system with \( \alpha \neq 0 \) is characterized by

\[
\tilde{A}(r) = A(r) + \alpha f(r), \quad \tilde{v}(r) = \xi(r) + ah(r).
\]

In this algorithm, there are two possible ways to generalize the procedure presented in [54]: First let us assume that the sets \( \{ \xi, A, T_{\mu\nu}^{PF} \} \) and \( \{ \xi, f, H_{\mu\nu} \} \) are solutions of Einstein equations. Then the system of equations is decomposed in

\[
8\pi \rho = -3(A')^2 + 2AA'' + \frac{4}{r}AA',
\]

\[
8\pi P = (A')^2 - AA'\xi' + \frac{\xi^2 A^2}{r} - \frac{2AA'}{r},
\]

\[
8\pi P = (A')^2 - AA'' + \left( \frac{\xi''}{2} + \frac{(\xi')^2}{4} + \frac{\xi'}{2r} \right) A^2 - \frac{AA'}{r},
\]

and

\[
8\pi H_0^0 = -3(f')^2 + 2ff'' + \frac{4}{r}ff',
\]

\[
-8\pi H_1^1 = (f')^2 - ff'\xi' + \frac{\xi^2 f^2}{r} - \frac{2ff'}{r},
\]

\[
-8\pi H_2^2 = (f')^2 - ff'' + \left( \frac{\xi''}{2} + \frac{(\xi')^2}{4} + \frac{\xi'}{2r} \right) f^2 - \frac{ff'}{r},
\]

which are both Einstein equations systems. There is a third system given by

\[
8\pi \theta_0^0 = -6A'f' + 2(Af'' + A''f) + \frac{4}{r}(Af' + A'f),
\]

\[
-8\pi \theta_1^1 + Z_1(r) = 2A'f' - (Af' + A'f) \left( \tilde{v}' + \frac{2}{r} \right) + 2\tilde{v}' \frac{fA}{r} + \alpha^3 h' \left( \frac{f^2}{r} - ff' \right),
\]

\[
-8\pi \theta_2^2 + Z_2(r) = 2A'f' - (Af'' + A''f) + 2 \left( \frac{\tilde{v}''}{2} + \frac{(\tilde{v}')^2}{4} + \frac{\tilde{v}'}{2r} \right) Af,
\]

\[
+ \alpha^2 \frac{f^2}{2} \left( \frac{h'' A^2}{2} + \frac{\xi' h' A^2}{2} + \frac{h' A^2}{2r} \right) - \frac{(Af' + A'f)}{r},
\]

In the second possibility, we can start by assuming that the sets \( \{ \xi, A, T_{\mu\nu}^{PF} \} \) and \( \{ v, f, H_{\mu\nu} \} \) are solutions of Einstein equations, and this lead us to the following systems.
of equations

\[
8\pi \rho = -3(A')^2 + 2AA'' + \frac{4}{r}AA', \tag{109}
\]

\[
8\pi P = (A')^2 - AA'\xi' + \xi' \frac{A^2}{r} - \frac{2AA'}{r}, \tag{110}
\]

\[
8\pi P = (A')^2 - AA'' + \left(\frac{\xi''}{2} + \frac{(\xi')^2}{4} + \frac{\xi'}{2r}\right)A^2 - \frac{AA'}{r}, \tag{111}
\]

and

\[
8\pi H_0^0 = -3(f')^2 + 2ff'' + \frac{4}{r}ff', \tag{112}
\]

\[
-8\pi H_1^1 = (f')^2 - ff'\tilde{\nu}' + \tilde{\nu}' \frac{f^2}{r} - \frac{2ff'}{r}, \tag{113}
\]

\[
-8\pi H_2^2 = (f')^2 - ff'' + \left(\frac{\tilde{\nu}''}{2} + \frac{(\tilde{\nu}')^2}{4} + \frac{\tilde{\nu}'}{2r}\right)f^2 - \frac{ff'}{r}. \tag{114}
\]

We also have a third system of equations given by

\[
8\pi \Theta_0^0 = -6A'f' + 2(Af'' + A''f) + \frac{4}{r}(Af' + A'f), \tag{115}
\]

\[
-8\pi \Theta_1^1 + Z_1(r) = 2A'f' - (Af' + A'f) \left(\tilde{\nu}' + \frac{2}{r}\right) + 2\tilde{\nu}' \frac{fA}{r}, \tag{116}
\]

\[
-8\pi \Theta_2^2 + Z_2(r) = 2A'f' - (Af'' + A''f) + 2 \left(\frac{\tilde{\nu}''}{2} + \frac{(\tilde{\nu}')^2}{4} + \frac{\tilde{\nu}'}{2r}\right)Af - \frac{(Af' + A'f)}{r}, \tag{117}
\]

where

\[
Z_1(r) = AA'h' + \frac{1}{r}h'A^2, \tag{118}
\]

\[
Z_2(r) = -\frac{h''A^2}{2} - \frac{\tilde{\nu}'h'A^2}{2} - \frac{h'A^2}{2r}. \tag{119}
\]

Finally, in order to avoid the appearance of singularities on the surface of the distribution, we must impose the well-known matching conditions between the interior and the exterior space-time geometries. The inner region is defined by the metric (80) and it could being obtained using any of the two inequivalent algorithms. We will consider that the outer region is described by the vacuum Schwarzschild solution

\[
ds^2 = \left(1 - \frac{2M}{r_1}\right)dt^2 - \left(1 - \frac{2M}{r_1}\right)^{-1}dr_1^2 + r_1^2d\Omega^2, \tag{120}
\]

where \(M\) denote the total mass of the distribution.

Now, using (80) and (120) the matching condition takes the following form

\[
e^{\nu\Sigma} = \left(1 - \frac{2M}{r_1}\right)^{1/2}, \tag{121}
\]

\[
r_{\Sigma} \left(\frac{2}{r_{\Sigma}} - \frac{w'}{w}\right)_{\Sigma} = \left(1 - \frac{2M}{r_1}\right)^{1/2}, \tag{122}
\]

\[P_r(r_{\Sigma}) = 0,\]
were the subscript $\Sigma$ indicates that the quantity is evaluated at the boundary of the distribution. Then it is possible to obtain an expression for the total mass of the distribution given by

$$
M = \frac{r^2}{2w^{1/2}} \left[ 1 - \frac{r^2}{4} \left( \frac{2}{r} \frac{w'}{w} \right)^2 \right].
$$

(123)

5 The two-step GD in isotropic coordinates

In this section, we will present a way to simplify the problem of finding solutions to Einstein equations in isotropic coordinates, taking into account consecutive deformations on the spatial and temporal components of the metric. Let us notice that from the extended version of the two algorithms in isotropic coordinates, it is possible to take the same limits that the EGD in Schwarzschild like coordinates. This is, we can take $f \neq 0$ with $h = 0$ or $f = 0$ with $h \neq 0$. These correspond to

- For the first algorithm, and we have that the first limit corresponds to $h = 0$ which is the case studied in [54], where we may obtain $f$ in terms of $F(r)$ from (92) as

$$
f = e^v (v')^2 r^4 \left( 2 \int F(r) e^{-v} \frac{dv}{v'} dr + C \right).
$$

(124)

Therefore, once we have obtained $f$, for a specific function $F(r)$, then $(\theta^h)_1$, $(\theta^h)_2$ and $(\theta^h)_3$ are given by expressions (95), (96) and (97) with $h = 0$, respectively.

The second limit corresponds to the case where $f = 0$, in which the system (95)–(97) leads to

$$
8\pi (\theta^h)_0 = 0,
$$

(125)

$$
8\pi (\theta^h)_1 = \left[ \frac{\omega'}{2} - \frac{\omega}{r} \right] h',
$$

(126)

$$
8\pi (\theta^h)_2 = -\omega \left[ \frac{h''}{2} + \frac{v' h'}{2} + \frac{h'}{2r} + \frac{1}{4} \omega h'^2 \right].
$$

(127)

Notice that we also have to impose an additional condition in order to obtain the deformation function $h$ and with it, expressions for $(\theta^h)_1$, $(\theta^h)_2$, $(\theta^h)_3$.

- For the second algorithm, the two different possibilities gives the same limits. For the first limit $h = 0$, we have that the system of equations (103) remains the same, while (128) can be written as

$$
8\pi \Theta^0_0 = -6 A' f' + 2 (A f'' + A' f) + \frac{4}{r} (A f' + A' f),
$$

(128)

$$
-8\pi \Theta^1_1 = 2 A' f' - (A f' + A' f) \left( v' + \frac{2}{r} f \right) + 2 v' \frac{f A}{r},
$$

(129)

$$
-8\pi \Theta^2_2 = 2 A' f' - (A f'' + A' f) + 2 A f \left( \frac{v''}{2} + \frac{(v')^2}{4} + \frac{v'}{2r} \right) - (A f' + A' f) \frac{r}{r}.
$$

(130)

The second limit corresponds to the case when $f = 0$. In that case, the system of equations can be written as

$$
H^0_0 = H^1_1 = H^2_2 = 0,
$$

(131)
and

\[ 8\pi \Theta^0_0 = 0, \quad (132) \]

\[ -8\pi \Theta^1_1 + Z_1(r) = 0, \quad (133) \]

\[ -8\pi \Theta^2_2 + Z_2(r) = 0, \quad (134) \]

Then, we can follow the same idea mentioned before, with Tolman solution in standard-like coordinates and define the two-step GD in isotropic coordinates. Let us perform a consecutive deformations of the temporal and radial components of the line element in isotropic coordinates in order to obtain solutions with \( h \neq 0 \) and \( f \neq 0 \) with local anisotropy in the pressures.

We will consider as seed solution the one given by Gold III [89]

\[ e^\nu = D \left( \frac{g - 1}{g + 1} \right), \quad (135) \]

\[ \frac{1}{\omega} = B \left( \frac{g + a}{g} \right)^2, \quad (136) \]

\[ P(r) = \frac{b}{2\pi B} \frac{1}{(g + 1)^2} \left[ \frac{g}{(g + 1)^2} - br^2 \right], \quad (137) \]

\[ \rho(r) = \frac{b}{2\pi B} \frac{1}{(g + 1)^2} \left[ \frac{3g(g - 1)}{(g^2 - 1)^{1/2}} - br^2(3 - 2g) \right], \quad (138) \]

\[ g(r) = \cosh a + br^2, \quad (139) \]

where \( D, B, a \) and \( b \) are constants. It can be seen that this solution will be regular at the origin if

\[ B = \frac{(e^{2a} + 1)^2}{(e^a + 1)^4}. \quad (140) \]

As we have shown before, we found [54] two different algorithms to obtain solutions of Einstein equations in isotropic coordinates considering a geometric deformation of the spatial component of the metric, inspired in MGD. Therefore, let us study the extended version for both cases separately.

5.1 Using the first algorithm

5.1.1 Left path

As we show in [54], if we choose

\[ F(r) = \frac{16DGb^2v'}{r(g + 1)^2}, \quad (141) \]

where \( G \) is a proportionality constant, then regularity at the origin implies

\[ B = \frac{3}{3 + 8aDb^2G} \frac{(e^{2a} + 1)^2}{(1 + e^a)^4}, \quad (142) \]

and it can be seen that

\[ f_L = \frac{16Db^2r^6}{(g + 1)^2} \left[ \tilde{C} - \frac{G}{3r^6} \right], \quad (143) \]

\( \tilde{C} \) Springer
where \( \tilde{C} = 2K + C \) and \( K \) is an integration constant. Therefore, it can be check that

\[
\tilde{H}_1 = \frac{6}{r} + \frac{8b^2r^2}{g'}(1-g) + \frac{32DGb^2}{rf(g+1)^2},
\]

\[
\tilde{H}_2 = H_1^2 + H_1' \left[ 5r + \frac{8b^2r^2}{g'}(1-g) \right] \frac{16DGb^2}{(g+1)^2r},
\]

(144)

(145)

The new solution is given by

\[
e^\nu = D \left( \frac{g-1}{g+1} \right),
\]

(146)

\[
\tilde{\omega} = \omega + \alpha f_L,
\]

(147)

\[
\tilde{P}_r(r) = P(r) - \alpha (\theta_L^f)_1^1,
\]

(148)

\[
\tilde{\rho}(r) = \rho(r) + \alpha (\theta_L^f)_0^0,
\]

(149)

\[
\tilde{P}_f(r) = P(r) - (\theta_L^f)_2^2.
\]

(150)

This is the anisotropic solution that was found by the geometrical deformation of the radial component of Gold III, following the first algorithm in [54].

Now, let us consider this result as seed solution, and let us perform a geometric deformation of the temporal component of the metric.

\[
\tilde{\nu} \to \nu = \xi + \beta \tilde{h}_L(r).
\]

(151)

It can be seen from equations of motion (81)-(83) that

\[
8\pi (\theta_L^h)_0^0 = 0,
\]

(152)

\[
8\pi (\theta_L^h)_1^1 = \left[ \frac{\tilde{\omega}'}{2} - \frac{\tilde{\omega}}{r} \right] \tilde{h}_L',
\]

(153)

\[
8\pi (\theta_L^h)_2^2 = -\omega \left[ \frac{\tilde{h}_L''}{2} + \frac{\tilde{\omega}'' \tilde{h}_L'}{2} + \frac{\tilde{h}_L'}{2r} + \frac{1}{4} \beta \tilde{h}_L^2 \right],
\]

(154)

where there are four unknown functions. In order to solve the system, let us consider a constraint of the form

\[
(\theta_L^h)_1^1 = - \left[ \frac{\omega'}{2} - \frac{\omega}{r} \right] \chi(r),
\]

(155)

therefore is easy to verify that

\[
h_L(r) = -8\pi \tilde{F}(r) + \tilde{C} \text{ with } \tilde{F}'(r) = \chi(r),
\]

(156)

where \( \tilde{C} \) is an integration constant, and \( (\theta_L^h)_2^2 \) is given by expression (154).

The new anisotropic solution is

\[
e^{\tilde{\nu}} = e^{\nu + \beta \tilde{h}} = D \left( \frac{g-1}{g+1} \right) e^{\beta (-8\pi \tilde{F}(r) + \tilde{C})},
\]

(157)

\[
\frac{1}{\tilde{\omega}} = \frac{1}{\omega + \alpha f_L},
\]

(158)

\[
\tilde{\rho}(r) = \rho + \alpha (\theta_L^f)_0^0(r),
\]

(159)

\[
\tilde{P}_r(r) = P(r) - \alpha (\theta_L^f)_1^1 - \beta (\theta_L^h)_1^1,
\]

(160)
Fig. 8 Radial and tangential pressures for the left path in isotropic coordinates vs the radial coordinate

\[
\tilde{P}_l(r) = P(r) - \alpha(\theta_L^f)^2 - \beta(\theta_L^h)^2.
\]  

(161)

Now, choosing \( \chi(r) \) in such a way that

\[
h_L(r) = Kr^n,
\]  

(162)

and imposing \( a = 2, b = 1, \tilde{a} = D\alpha = 0.01, \tilde{b} = K\beta = 0.001, n = 3 \) and \( C = 1 \) in order to show a specific example, we found that \( r_\Sigma = 0.9895, r_1 \Sigma = 0.8612, M = 0.1418 \) and \( K = -202.3 \) using matching conditions. The pressures and energy density are plotted in figures (8)-(9). It can be check that this solution satisfies the conditions for physical acceptability given in the appendix. This solution was obtained from known Gold III solution by the two-step geometric deformation.

5.1.2 Right path

If we consider a geometric deformation of the temporal component of the metric of Gold III

\[
\tilde{\nu} \rightarrow \nu = \xi + \alpha h_R(r),
\]  

(163)

It can be seen from the equations of motion (81)–(83) that

\[
8\pi (\vartheta_R^h)^0 = 0,
\]  

(164)

\[
8\pi (\vartheta_R^h)^1 = \left[ \frac{\omega'}{2} - \frac{\omega}{r} \right] h'_R,
\]  

(165)

\[
8\pi (\vartheta_R^h)^2 = -\omega \left[ \frac{h''_R}{2} + \frac{\tilde{\nu}' h'_R}{2r} + \frac{h'_R}{2r} + \frac{1}{4} \alpha (h'_R)^2 \right],
\]  

(166)
where there are four unknown functions. In order to solve the system, let us consider a constraint of the form

$$ (\theta^h_R)_1^1 = \left[ \frac{\omega'}{2} - \frac{\omega}{r} \right] \chi(r), $$

therefore is easy to verify that

$$ h_R(r) = -8\pi F(r) + \tilde{C} $$

with $F'(r) = \chi(r)$.

The new anisotropic solution obtained by the geometric deformation of temporal component of Gold III [89] is given by

1. $e^{\tilde{\nu}} = e^{\nu + \alpha h} = D \left( \frac{g}{g + 1} \right) e^{\alpha(-8\pi F(r) + \tilde{C})}$,

2. $\frac{1}{\omega} = B \left( g + 1 \right)^{2}$,

3. $\tilde{\rho}(r) = \rho(r)$,

4. $\tilde{P_r}(r) = P(r) - \alpha(\theta^h_R)_1^1$,

5. $\tilde{P_t}(r) = P(r) - \alpha(\theta^h_R)_2^2$.

We may consider now this result as seed solution and let us do a geometric deformation on the radial component of the line element. Let us name $\alpha$ and $\beta$ the parameters of the first and second deformation, respectively. Therefore

$$ \tilde{\omega} \to \omega + \beta f_R $$. 
It can be shown that, following once again the first algorithm of [54], we can impose the constraint

\[(\theta_R^f)^1 + (\theta_R^f)^0 - 2(\theta_R^f)^2 = \frac{\tilde{F}}{8\pi},\]  

(175)

and we get

\[f_R = e^{\tilde{\nu}}(\tilde{\nu}')^2 r^4 \left( 2\int \frac{\tilde{F}(r)e^{-\tilde{\nu}}}{\tilde{\nu}^3 r^4} dr + C \right).\]  

(176)

As before, we may choose

\[\tilde{F}(r) = \frac{16DGb^2\tilde{\nu}'}{r(g + 1)^2},\]  

(177)

with G a proportionality constant, then regularity at the origin implies

\[B = \frac{3}{3 + 8\alpha Db^2G (1 + e^a)^4},\]  

(178)

and it can be seen that

\[f_R = e^{\tilde{\nu}}(\tilde{\nu}')^2 r^4 \left( C - \frac{G}{3r^6} \right),\]  

(179)

where C and G are constants. Therefore, we may obtain \((\tilde{\theta}^f)^0, (\tilde{\theta}^f)^1\) and \((\tilde{\theta}^f)^2\) as usual.

The new anisotropic solution is given by

\[e^{\tilde{\nu}} = e^{\nu + \alpha h_R} = D \left( \frac{g - 1}{g + 1} \right) e^{\alpha (-8\pi F(r) + \tilde{C})},\]  

(180)

\[\tilde{\omega} = \omega + \beta f_R,\]  

(181)

\[\tilde{P}_1(r) = P(r) - \alpha (\theta_R^f)^1 - \beta (\theta_R^f)^1,\]  

(182)

\[\tilde{\rho}(r) = \rho(r) + \beta (\theta_R^f)^0,\]  

(183)

\[\tilde{P}_1(r) = P(r) - \alpha (\theta_R^f)^2 - \beta (\theta_R^f)^2.\]  

(184)

Now choosing \(A(r)\) in such a way that

\[h(r) = \tilde{K} r^n,\]  

(185)

It can be seen that imposing \(a = 2, b = 1, n = 4, \tilde{a} = G\alpha = 0.01, \tilde{\beta} = D\alpha = 0.001\)

and \(C = 1\), from the matching conditions, we found that \(r_\Sigma = 1.0535, r_1\Sigma = 0.9064, M = 0.149\) and \(\tilde{K} = -179.07\). The pressures, energy density and acceptability conditions are plotted in figures (10)–(11). This solution was obtained by two-step GD of Gold III perfect fluid solution.

5.2 Using the second algorithm

5.2.1 Left path

As before, we will take a deformation of the spacial component of the metric. Then we need to solve the following systems of equations; the first one

\[8\pi (H_L^f)^0_0 = -3(f_L^f)^2 + 2ff'' + \frac{4}{r}f_L f_L',\]  

(186)
Fig. 10  Radial and tangential pressures for the right path in isotropic coordinates vs the radial coordinate

Fig. 11  Energy density for the right path in isotropic coordinates vs the radial coordinate
The last system can be solved for $f_L$. In this case, the system that we need to solve is given by the second one

\[-8\pi (H^f_L)^1_1 = (f' L)^2 - f_L f'_L \nu' + \nu' f^2 L \nu - \frac{2 f_L f'_L}{r}, \tag{187} \]

\[-8\pi (H^f_L)^2_2 = (f' L)^2 - f_L f''_L + \left( \frac{\nu''}{2} + \frac{(\nu')^2}{4} + \frac{\nu'}{2r} \right) f'_L - \frac{f_L f'_L}{r}, \tag{188} \]

the second one

\[-8\pi (\Theta^f_L)^0_0 = -6 A' f'_L + 2 (A f''_L + A'' f_L) + \frac{4}{r} (Af'_L + A' f_L), \tag{189} \]

\[-8\pi (\Theta^f_L)^1_1 = 2 A' f'_L - (A f''_L + A' f_L) \left( \nu' + \frac{2}{r} \right) + 2 \nu' \frac{f L A}{r}, \tag{190} \]

\[-8\pi (\Theta^f_L)^2_2 = 2 A' f'_L - (A f''_L + A'' f_L) - \frac{(Af'_L + A' f_L)}{r} + 2 \left( \frac{\nu''}{2} + \frac{(\nu')^2}{4} + \frac{\nu'}{2r} \right) A f_L. \tag{191} \]

The last system can be solved for $f_L$ as

\[f_L = - \frac{r^4 (\nu')^2 e^\nu}{A} \left( \int \frac{8\pi ((\Theta^f_L)^0_0 - 2(\Theta^f_L)^2_2 - (\Theta^f_L)^1_1)}{r^4 (\nu')^3 e^\nu} \, dr + C \right). \tag{192} \]

expression that can be simplified by imposing the constraint

\[8\pi ((\Theta^f_L)^0_0 - 2(\Theta^f_L)^2_2 - (\Theta^f_L)^1_1) = - \frac{D(\nu')^2 e^\nu}{r^3}, \tag{193} \]

as

\[f_L = \frac{1}{g(g + 1)} \left( 16r^6 DC \sqrt{B} b^2 - \frac{1}{6} \right). \tag{194} \]

Then, with $f_L$ it is easy to compute all the components of $\Theta^f_L$ and $H^f_L$. Now we can write the solution of the Einstein’s equation as

\[e^\tilde{\nu} = e^\nu, \tag{195} \]

\[A = A' + \alpha f_L, \tag{196} \]

\[\tilde{\rho} = \rho + \alpha [((\Theta^f_L)^0_0 + \alpha (H^f_L)^0_0], \tag{197} \]

\[\tilde{P}_r = p + \alpha [((\Theta^f_L)^1_1 + \alpha (H^f_L)^1_1], \tag{198} \]

\[\tilde{P}_t = p + \alpha [((\Theta^f_L)^2_2 + \alpha (H^f_L)^2_2]. \tag{199} \]

Let us take this solution and perform a deformation of the temporal component of the metric. In this case, the system that we need to solve is given by

\[8\pi (\Theta^h_L)^0_0 = 0, \tag{200} \]

\[8\pi (\Theta^h_L)^1_1 = \tilde{A} \tilde{A}' h'_L + \frac{1}{r} h'_L \tilde{A}^2, \tag{201} \]

\[8\pi (\Theta^h_L)^2_2 = - \frac{h''_L \tilde{A}^2}{2} - \frac{\nu' h'_L \tilde{A}^2}{2} - \frac{h'_L \tilde{A}^2}{2r}. \tag{202} \]

Now, it is necessary to impose a constraint to solve this system. For simplicity, we will choose

\[8\pi (\Theta^h_L)^1_1 = K r^n \left( \tilde{A} \tilde{A}' + \frac{1}{r} \tilde{A}^2 \right), \tag{203} \]
where $K$ and $n$ are constants. In order to avoid a singularity in the center of the distribution $n \geq 1$. In this case, it is an straightforward calculation to obtain

$$h_L = E + K r^{n+1}. \quad (204)$$

Then, with $h_L$, we can write the final solution that it does include both perturbations of the Gold III metric components as

$$e^{\tilde{\nu}} = e^{\nu + \beta h_L}, \quad (205)$$

$$\tilde{\tilde{A}} = A + \alpha f_L, \quad (206)$$

$$\tilde{\tilde{\rho}} = \rho + \alpha \left[ (\Theta^h_{L})^0_0 + \alpha (H^h_L)^0_0 \right], \quad (207)$$

$$\tilde{P}_r = p - \alpha \left[ (\Theta^h_{L})^1_1 + \frac{\beta}{\alpha} (\Theta^h_{L})^1_1 + \alpha (H^h_L)^1_1 \right], \quad (208)$$

$$\tilde{P}_t = p - \alpha \left[ (\Theta^h_{L})^2_2 + \frac{\beta}{\alpha} (\Theta^h_{L})^2_2 + \alpha (H^h_L)^2_2 \right], \quad (209)$$

where to ensure the regularity at the center of the distribution

$$B = \frac{9(e^{2a} + 1)^4}{(3 + 3e^{4a} + 6e^{3a} + (2 \alpha + 6) e^{2a} + 6e^a)^2}. \quad (210)$$

Now, to show an example of the behavior of the obtained solution, we choose the following values for the constants, $E = K = 1.00, C = 0, \alpha D = 0.70, \beta = -0.01, n = 3, a = 3,b = 1$. Then, the matching conditions leads to $r_{\Sigma} = 0.53, M = 0.02$ and $D = 1.07$. The result is shown in Figures (12)–(13)

5.2.2 Right path

Performing the temporal deformation of the metric first, we have that the system that we need to solve is

$$8\pi (\Theta^h_{R})^0_0 = 0, \quad (211)$$

$$8\pi (\Theta^h_{R})^1_1 = AA'h'_R + \frac{1}{r} h'_R A^2, \quad (212)$$

$$8\pi (\Theta^h_{R})^2_2 = -\frac{h''_R A^2}{2} - \frac{v'h'_R A^2}{2} - \frac{h'_R A^2}{2r}. \quad (213)$$

Then, as before, if we impose the constraint

$$8\pi (\Theta^h_{R})^1_1 = K r^n \left( AA' + \frac{1}{r} A^2 \right), \quad (214)$$

it is possible to solve the system of equations for $h_R$ to obtain

$$h_R = \tilde{E} + \tilde{K} r^{n+1}. \quad (215)$$

In that case, the new solution can be written as

$$e^{\tilde{\nu}} = e^{\nu + a h_R}, \quad (216)$$

$$\tilde{\tilde{A}} = A, \quad (217)$$

$$\tilde{\tilde{\rho}} = \rho + \alpha (\Theta^h_{R})^0_0, \quad (218)$$

$$\tilde{P}_r = p + \alpha (\Theta^h_{R})^1_1, \quad (219)$$
Fig. 12  Radial and tangential pressures for the left path of the second algorithm in isotropic coordinates vs the radial coordinate

Fig. 13  Energy density for the left path of the second algorithm in isotropic coordinates vs the radial coordinate
Now, we can take this solution as the starting point to perform a spatial deformation of this metric. In this case, we need to solve the following system of equations

\[
8\pi (\Theta^f_R)_0^0 = -6\ddot{A}' f_R' + 2(\dddot{A} + \dddot{f}_R + \dddot{f}_R) + \frac{4}{r}(\dddot{A}' f_R' + \dddot{f}_R), \tag{221}
\]

\[
-8\pi (\Theta^f_R)_1^1 = 2\dddot{A}' f_R' - (\dddot{A} f_R' + \dddot{f}_R) \left(\ddot{v}' + \frac{2}{r}\right) + 2\ddot{v} f_R \dddot{A}, \tag{222}
\]

\[
-8\pi (\Theta^f_R)_2^2 = 2\dddot{A}' f_R' - (\dddot{A} f_R' + \dddot{f}_R) - \frac{(\dddot{A} + \dddot{f}_R)}{r} + 2 \left(\frac{\ddot{v}'}{2} + \frac{(\ddot{v})^2}{4} + \frac{\dddot{v}'}{2r}\right) \dddot{A} f_R. \tag{223}
\]

The solution of this system for \( f_R \) has the same form of the left path, but substituting \( \ddot{v} \) for \( \dddot{v} \). Then, if we impose a constraint with the form

\[
8\pi ((\Theta^f_R)_0^0 - 2(\Theta^f_R)_2^2 - (\Theta^f_R)_1^1) = -\frac{\tilde{D}(\ddot{v})^3 e^{\ddot{v}}}{r^3}, \tag{224}
\]

it not difficult to show that

\[
f_R = \frac{1}{g(g + 1)} \left(16\pi^6 DC\sqrt{B}b^2 - \frac{1}{6}\right) e^{\alpha K r^2 + 1} \left(1 + \frac{\alpha K \sqrt{g^2 - 1} r^{n-1}}{2b}\right)^2. \tag{225}
\]

The final solution with both deformations on the metric components is given by

\[
e^{\ddot{v}} = e^{v + \alpha h_R}, \tag{226}
\]

\[
\dddot{A} = A + \beta f_R, \tag{227}
\]

\[
\dddot{\rho} = \rho + \beta \left( (\Theta^f_R)_0^0 + \frac{\alpha}{\beta} (H^f_R)_0^0 \right), \tag{228}
\]

\[
\dddot{P}_r = p - \beta \left( (\Theta^f_R)_1^1 + \frac{\alpha}{\beta} (H^f_R)_1^1 \right), \tag{229}
\]

\[
\dddot{P}_t = p - \beta \left( (\Theta^f_R)_2^2 + \frac{\alpha}{\beta} (H^f_R)_2^2 + \alpha (H^f_R)_2^2 \right), \tag{230}
\]

where, to ensure the regularity at the center of the distribution

\[
B = \frac{9(e^{2a} + 1)^4}{(3 + 3e^{2a} + 6e^{3a} + (2\beta + 6)e^{2a} + 6e^{2a})^2}. \tag{231}
\]

As before, to show an example of the behavior of this solution, we choose the following values for the constants \( \tilde{E} = \tilde{K} = 1.00, C = 0, \alpha D = -0.01, \beta = 0.70, n = 5, a = 3, b = 1. \) Then the matching condition leads to \( r_{\Sigma} = 0.53, M = 0.02 \) and \( D = 1.07 \). The result is shown in Figs. (14)–(15)

6 Conclusions

The extended geometric deformation represents a powerful tool to study gravitational systems in General Relativity, including possible corrections to more general gravitational theories. It allows us to obtain solutions with deformations in the temporal and spatial components of the metric. However, as it is expected, to solve the remaining system in order to obtain
Fig. 14 Radial and tangential pressures for the right path of the second algorithm in isotropic coordinates vs the radial coordinate

Fig. 15 Energy density for the right path of the second algorithm in isotropic coordinates vs the radial coordinate
the metric deformations it is, in general, a difficult task. In this work, we have presented a different approach in order to obtain solutions with EGD for static and spherical symmetric solutions in Schwarzchild-like and isotropic coordinates. This approach is based on performing consecutive, non-simultaneous deformations of the metric components. For reasons of simplification, we denote it, two-step geometric deformation. Now, in order to ensure the decoupling of the sources, it is necessary to assume that $\theta_{\mu \nu}$ can be decomposed in two parts, where each part is responsible for only one deformation (temporal or spatial) of the metric. For this reason, the set of solutions that can be obtained by the two-step GD, represents a subset of the set of solutions of the EGD. On the other hand, we find that the order in which the deformations of the metric are done, it is relevant for the final solution. We named the two possible cases: left and right paths. It is worth mentioning that by restricting the two-step GD to consider only isotropic solutions of Einstein’s equations, we were able to reproduce the transformations presented in Theorems 1, 2, 3, and 4 of [87].

It is evident that the physics do not depend on the coordinate system. However, as it is mentioned in [55] and [54], isotropic coordinates seem to be more general than Schwarzschild-like ones. It can be checked that it is always possible to transform the line element from the standard form to the isotropic one by (84), while the reverse process is not always true. Therefore, we may obtain solutions to Einstein’s equations that cannot be found by using Schwarzschild-like coordinates. Now, while in the latter the two-step GD is constructed as limits of the EGD, in the case of isotropic coordinates it is more complicated. In fact, we discuss in [54] that, in general, Einstein’s equations cannot be decoupled, and we proposed two inequivalent MGD inspired algorithms to obtain solutions in this case. In the first algorithm, the solutions are obtained even when there is no decoupling of the sources, while in the second one, the sources can be decoupled but with some restrictions in the energy momentum tensor. In this work, we have extended both algorithms to include temporal and spatial deformations of the metric. Then, we construct the corresponding procedure to perform the two-step GD. For the second algorithm, we found two possible extensions that include both deformations of the metric. Now, from the two-step GD’s point of view, there is no difference between them.

Among all the possible applications of the EGD method, one of particular interest is that it allows us to study gravitational systems, either taking into account contributions from theories beyond GR or the coupling of the Einstein equations with other fields (Maxwell, Klein Gordon, etc.). Due to the particular restrictions over the energy momentum tensor that is required for the two-step GD, the application of this approach to the gravitational systems mentioned before is not clear. Thus, a more detailed analysis it is required in each particular case. This feature becomes even more evident on the algorithms in isotropic coordinates.

In order to verify the two-step GD, we have chosen seed solutions of Einstein equations, which for simplification reasons, are perfect fluid solutions. In Schwarzchild-like coordinates, this solution was Tolman IV. Via the two-step GD, we were able to obtain two inequivalent solutions with anisotropy in the pressures by consecutive deformation of radial and temporal components. The inequivalence of these solutions can be checked from the figures and therefore, the deformation of both components of the metric does not commute. Moreover, solutions obtained by this approach are not equivalent to the simultaneous case, which is presented in [88], also using also Tolman IV as seed solution. This can be verified using the same values for the free parameters. However, solutions obtained by two-step GD, as we mentioned before, are contained in the simultaneous case. In isotropic coordinates, we have chosen Gold III as seed solution. We have executed the two-step GD for each algorithm presented in [54]. For each algorithm we obtained two solutions that correspond to the left and right path. Once again, it can be seen from the figures that the solutions obtained with each path are not equivalent. Finally, we want to emphasize that, even though
all the solutions presented in this work do satisfy the acceptability conditions, our main goal is to present the two-step GD for the Schwarzschild-like and isotropic coordinates and not to study any particular matter distribution.

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**Data availability** This manuscript has no associated data or the data will not be deposited. [Authors comment: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.]

**Appendix A: Physical acceptability conditions**

Solving Einstein’s equations does not ensure that the solution will describe any physical system. Indeed, among all the known solutions of Einstein’s equations, only a part of them fulfill the physically acceptable conditions (see for example [5]).

Then, in order to ensure that the solutions of Einstein’s equations are physically acceptable, we must verify if the following conditions are satisfied

- \( P_r, P_t \) and \( \rho \) are positive and finite inside the distribution.
- \( \frac{dP_r}{dr}, \frac{dP_t}{dr} \) and \( \frac{d \rho}{dr} \) are monotonically decreasing.
- Dominant energy condition: \( \frac{P_r}{\rho} \leq 1 \), \( \frac{P_t}{\rho} \leq 1 \).
- Causality condition: \( 0 < \frac{dP_r}{d \rho} < 1 \), \( 0 < \frac{dP_t}{d \rho} < 1 \).
- The local anisotropy of the distribution should be zero at the center and increasing towards the surface.

**Appendix B: The Ricci invariants**

Let us verify that solutions obtained by EGD, and therefore, by any of its limits, are in general inequivalent to the seed solution considered. In order to see this, let us notice that the Ricci invariants can be written in terms of the trace-free Ricci tensor

\[
S^\mu_{\nu} = R^\mu_{\nu} - \delta^\mu_{\nu} R/4, \tag{232}
\]

where \( R^\mu_{\nu} \) and \( R \) are the Ricci tensor and the scalar curvature, respectively. Besides the scalar curvature, Ricci invariants are defined as

\[
r_1 = \frac{1}{4} S^\mu_{\nu} S^\nu_{\mu}, \tag{233}
\]
\[
r_2 = -\frac{1}{8} S^\mu_{\nu} S^\rho_{\mu} S^\nu_{\rho}, \tag{234}
\]
\[
r_3 = S^\mu_{\nu} S^\rho_{\mu} S^\sigma_{\rho} S^\nu_{\sigma}. \tag{235}
\]

Let us assume that \( \{\tilde{\nu}, \tilde{\mu}\} \) and \( \{\nu, \mu\} \) represent two solutions of the of Einstein’s equations for an spherically symmetric fluid with a line element of the form (1). Defining \( \tilde{\nu} \) and \( \tilde{\mu} \) as

\[
\tilde{\mu} = \mu + \alpha f(r), \quad \tilde{\nu} = \nu + \beta h(r), \tag{236}
\]
then it can be shown that the scalar curvature and the trace-free Ricci tensor satisfies the following relations

$$\tilde{R} = R + \alpha \xi_1(r) + \beta \xi_2(r) + \alpha \beta \xi_3(r).$$  \hfill (237)

and

$$\tilde{S}_i^\mu = S_i^\mu + \frac{\delta_i^\mu}{2} \left[ \alpha \left( \frac{\xi_1}{2} - \frac{2f}{r} \left( \frac{f}{r} + f' \right) \right) + \frac{\beta}{2} \left( \xi_2 + \alpha \xi_3 \right) \right]$$

$$+ \frac{\delta_i^\mu}{2} \left[ \alpha \left( \frac{\xi_1}{2} - \frac{2f}{r} \left( \nu' + \frac{1}{r} \right) \right) + \beta \left( \frac{\xi_2}{2} - 2\mu h' \right) \right]$$

$$+ \alpha \beta \left( \frac{\xi_3}{2} - \frac{2f h'}{r} \right) + \frac{\delta_i^\mu}{2} \left[ \alpha \left( \frac{f}{r} \left( \nu' + \frac{2}{r} \right) + \frac{f'}{r} - \frac{\xi_1}{2} \right) \right]$$

$$+ \beta \left( \frac{\mu h'}{r} - \frac{\xi_2}{2} \right) + \alpha \beta \left( \frac{f h'}{r} - \frac{\xi_3}{2} \right).$$ \hfill (238)

where

$$\xi_1(r) = \frac{1}{2} \left\{ f \left[ (\nu')^2 + 2\nu' + \frac{4}{r} \left( \nu' + \frac{1}{r} \right) \right] + f' \left( \nu' + \frac{4}{r} \right) \right\},$$

$$\xi_2(r) = \frac{1}{2} \left\{ \mu \left[ 2h'' + 2h'\nu' + \beta (h')^2 + \frac{4}{r} h' \right] + \mu h' \right\},$$

$$\xi_3(r) = \frac{1}{2} \left\{ f \left[ 2h'' + 2h'\nu' + \beta (h')^2 + \frac{4}{r} h' \right] + f' h' \right\}.$$

Now, using (1), the Ricci invariants can be written as

$$\tilde{r}_1 = \frac{1}{4} \left[ (\tilde{S}_0^0)^2 + (\tilde{S}_1^1)^2 + 2(\tilde{S}_2^2)^2 \right],$$ \hfill (239)

$$\tilde{r}_2 = -\frac{1}{8} \left[ (\tilde{S}_0^0)^3 + (\tilde{S}_1^1)^3 + 2(\tilde{S}_2^2)^3 \right],$$ \hfill (240)

$$\tilde{r}_3 = (\tilde{S}_0^0)^4 + (\tilde{S}_1^1)^4 + 2(\tilde{S}_2^2)^4.$$

Thus, from (238), it is evident that in general

$$\tilde{r}_1 \neq r_1, \quad \tilde{r}_2 \neq r_2, \quad \tilde{r}_3 \neq r_3,$$ \hfill (242)

and therefore the two solutions of Einstein’s equations, represented by \{\tilde{v}, \tilde{\mu}\} and \{v, \mu\}, are inequivalent. It is easy to see, from equations (237) and (238), that the solutions are different even when we take \(\alpha = 0\) or \(\beta = 0\), which corresponds to the to two limits of the EGD method mentioned in Section (2). Since the right and left paths of the two-step GD produce different results for the deformations functions \(f\) and \(g\), then the Ricci invariant of the final solutions obtained with each path will be in general different. This indicates that the right path and left paths leads, in general, to inequivalent solutions. For the case of the isotropic coordinates the analysis is analogous.
Appendix C: Two-step GD and the BVW theorems

In this appendix we will present the unnoticed relation that exists between the two-step GD (and therefore, EGD) with the first four theorems presented by the authors in [87]. Let us consider as seed solution a perfect fluid sphere represented by the set of functions \( \{v, \mu, \rho, P\} \).

Left path

Starting with the pure spatial deformation (MGD) and imposing the condition \( (\phi f^L)_1^1 = (\phi f^L)_2^2 \), implies that the system (36)-(38) leads to the following differential equation

\[
2 f_L \left[ r^2 B'' - (r B)' \right] + f'_L r (r B)' = 0,
\]

where \( B(r)^2 = \exp v \). The solution to (243) can be written as

\[
f_L = \frac{r^2}{(r B')^2} \exp \int \frac{4B'}{(r B)'} dr.
\]

The new solution is given by the set of functions \( \{ \bar{v}, \bar{\mu}, \bar{\rho}, \bar{P} \} \), where

\[
\bar{\mu} = \mu + \alpha f_L,
\]

\[
\bar{\rho} = \rho + \alpha (\phi f^L)_0^0,
\]

\[
\bar{P} = P - \alpha (\phi f^L)_1^1.
\]

This correspond to the transformation in the first theorem in [87]. Now, let us consider \( \{v, \bar{\mu}, \bar{\rho}, \bar{P}\} \) as a seed solution and perform a pure temporal deformation with the constraint \( (\phi h^L)_1^1 = (\phi h^L)_2^2 \), then the system (45)–(47) leads to the following differential equation

\[
\frac{\bar{\mu}}{r} [r (Z''_L + Z'_L v) - Z'_L] + \frac{\bar{\mu}' Z'_L}{2} = 0,
\]

where \( Z_L(r)^2 = \exp \beta h_L \). The solution in this case is

\[
Z_L(r) = A + B \int \frac{r dr}{e^v \sqrt{\bar{\mu}}},
\]

The final solution is characterized by the set \( \{ \bar{v}, \bar{\mu}, \bar{\rho}, \bar{P}\} \)

\[
\bar{v} = v + \beta h_L,
\]

\[
\bar{\mu} = \mu + \alpha f_L,
\]

\[
\bar{\rho} = \rho + \alpha (\phi f^L)_0^0,
\]

\[
\bar{P} = P - \alpha \left( (\phi f^L)_1^1 + \frac{\beta}{\alpha} (\phi h^L)_1^1 \right).
\]

The complete left path that goes from \( \{v, \mu, \rho, P\} \) to \( \{\bar{v}, \bar{\mu}, \bar{\rho}, \bar{P}\} \) corresponds to the transformation in the third theorem in [87].

Right path

As in the left path, let us begin by taking \( \{v, \mu, \rho, P\} \) as seed solution. Then the pure temporal deformation of the metric subject to the constraint \( (\phi h^R)_1^1 = (\phi h^R)_2^2 \), leads to the following
differential equation

\[ \mu \frac{Z_R''(r)}{r} + Z_R'(R) - Z_R'(r) + \frac{\mu' Z_R(r)}{2} = 0, \tag{254} \]

notice that it has the same form of Eq (248) but changing \( \tilde{\mu} \) with \( \mu \). Therefore, it can be check that

\[ Z_R(r) = C + D \int \frac{r dr}{e^\nu \sqrt{\mu}}. \tag{255} \]

The new solution is given by \( \{ \tilde{\nu}, \tilde{\mu}, \tilde{\rho}, \tilde{P} \} \) where

\[ \tilde{\nu} = \nu + \alpha h_L, \tag{256} \]
\[ \tilde{\mu} = \mu + \beta f_R, \tag{257} \]

The transformation from \( \{ \nu, \mu, \rho, P \} \) to \( \{ \tilde{\nu}, \tilde{\mu}, \tilde{\rho}, \tilde{P} \} \), given by the pure temporal deformation, corresponds to the second theorem in [87]. Now we will take \( \{ \tilde{\nu}, \tilde{\mu}, \tilde{\rho}, \tilde{P} \} \) as seed solution and perform a pure spatial deformation, subject to the constraint \( (\theta^f_R)_1 = (\theta^f_R)_2 \). In this case the system (36)–(38) leads to the following differential equation

\[ 2 f_R \left[ r^2 \tilde{B}'' - (r \tilde{B})' \right] + f''_R r (r \tilde{B})' = 0, \tag{258} \]

where \( \tilde{B}(r)^2 = \exp \tilde{\nu} \). Then

\[ f_R = \frac{r^2}{(r \tilde{B})'} \exp \int \frac{4 \tilde{B}'}{(r \tilde{B})'} dr. \tag{259} \]

and the final solution of the right path can be written as

\[ \tilde{\nu} = \tilde{\nu} = \nu + \alpha h_R, \tag{260} \]
\[ \tilde{\mu} = \mu + \beta f_R, \tag{261} \]
\[ \tilde{\rho} = \rho + \beta (\theta^f_R)_0, \tag{262} \]
\[ \tilde{P} = P - \beta \left( (\theta^f_R)_1 + \frac{\alpha}{\beta} (\theta^h_R)_1 \right). \tag{263} \]

The complete right path that goes from \( \{ \nu, \mu, \rho, P \} \) to \( \{ \tilde{\nu}, \tilde{\mu}, \tilde{\rho}, \tilde{P} \} \) corresponds to the transformation in the fourth theorem in [87].

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