Nekhoroshev’s law for the destability of nonlinear normal modes in the Fermi-Pasta-Ulam-Tsingou chains

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Nonlinear normal modes are periodic orbits that survive in nonlinear chains, whose destabilization plays a crucial role in the dynamics of many-body Hamiltonian systems. A lot of work has been done on the dependence of perturbation threshold for destability on system size $N$. Particularly, in the large $N$ limit, a power-law relationship between the threshold and the size has been reported. Here we focus on how the destability time of nonlinear modes depends on the perturbation parameters and observe whether they have the same behavior in different models. To this end, as illustrating examples, the destabilization dynamics of the $N/2$ mode in both the Fermi-Pasta-Ulam-Tsingou (FPUT) -$\alpha$ and -$\beta$ chains under fixed boundary conditions were numerically studied. We find that the destability time as a function of perturbation parameters is well covered by the Nekhoroshev’s estimation for both models. Based on this estimation, we obtain the dependence of the destability threshold on the system size and compare it with the theoretical results obtained by applying the Floquet theory. We show that in the FPUT-$\alpha$ model, except for the small size, the theoretical and numerical results are identical, while in the FPUT-$\beta$ model, the numerical results are slightly smaller than the theoretical estimation, but the overall trend agrees well. Finally, the differences in the results of these two models were discussed.

I. INTRODUCTION

Since the 1950s, the pioneering numerical experiments of Fermi, Pasta, Ulam, and Tsingou (FPUT) [1] have stimulated a huge amount of researches about the nonlinear dynamics of many-body Hamiltonian systems [2, 3]. The problem of stability of nonlinear normal modes (NNMs) in one-dimensional nonlinear chains is one of them [4–15]. The NNMs are periodic orbits that survive in nonlinear chains, which are named by different terms, e.g., one-mode solutions [7], simple periodic orbits [8], and one-dimensional bushes [11, 12]. Their spatial patterns are the same as those of the linear normal modes, but their time dependence of the displacements of all particles are identical and correspond to nonlinear oscillations [10]. In particular, the dynamics of NNMs are decoupled from all other modes under appropriate initial conditions, that is, if only one of them is initially excited, its evolution will not transfer energy to any other modes.

In general, these NNMs are mainly found through the following two methods. The one is by analyzing the selection rules derived from the equations of motion in Fourier space [7, 9], and the other is to analyze the discrete symmetry of the chain model by using the group theory [10–13]. Then the stability of the NNMs can be detected numerically through integrating the Newtonian dynamics of the whole system [14] or studied theoretically by analyzing the variational dynamics of the NNMs by the Floquet theory [8]. As a result, the destability threshold as a function of system size $N$ is obtained. Especially in the large $N$ limit, it has been reported that the destability threshold decreases in a power-law manner with the increase of $N$; i.e., $N^{-\theta}$, and $\theta = 1$, or 2 is decided by both the boundary conditions and the wave number of the NNMs [10]. It has been suggested that there is a certain relationship between the destability of specific NNMs and the emergence of global chaos in the system [15]. For instance, if $\theta = 1$ for a specific mode, then its destability is related to the transition from weak to strong chaos. On the other hand, if $\theta = 2$ for a specific mode, it is connected with the onset of weak chaos as a result of the breakdown of the FPUT recurrences (detailed discussions see [3]). Whereas it also has been proposed that the destability of the NNMs leads to the formation of stochastic layers, but their thickness and extension in phase space remain small at small perturbations [7]. Obviously, the dependence of destability threshold on $N$ is not enough to determine the role of the NNMs in the structure of the phase space of nonlinear many-body systems, which can not reflect the details of nonlinear mode diffusion in phase space. Thus, the destability dynamics of these nonlinear modes need to be further studied in detail.

In this work, we revisit the destability dynamics of the NNMs, focusing on how the destability time depends on the perturbation strength, and exploring whether they have the same behavior in different models. Toward this end, we take the Fermi-Pasta-Ulam-Tsingou (FPUT) -$\alpha$ and -$\beta$ models as examples to systematically study the destability dynamics of the $N/2$ mode which is jointly owned by these two models under fixed boundary conditions. Comparing the destability dynamics of the NNMs in these two models is mainly motivated by the fact that the FPUT-$\alpha$ model is vary different from the FPUT-$\beta$ model in many aspects, such as the thermal expansion property [16], the process of energy equiapartition [17], and the behavior of energy transport [18].
the following sections, we first introduce the models and the $N/2$ mode solutions in the Sec. II, then give our main numerical results in Sec. III. Theoretical analysis is provided in Sec. IV, followed by the conclusions in Sec. V.

II. THE MODELS AND THE $N/2$ MODE SOLUTIONS

A. The models

We consider a nonlinear lattice consisting of $N$ particles of unit mass with nearest-neighbor interaction. Its Hamiltonian is

$$H = \sum_{j=1}^{N} \left[ \frac{p_j^2}{2} + \frac{(x_j - x_{j-1})^2}{2} + \frac{\theta_n}{n} (x_j - x_{j-1})^n \right], \quad (1)$$

where $p_j$ and $x_j$ are, respectively, the momentum and the displacement from the equilibrium position of the $j$th particle, $\theta_n$ denotes the nonlinear coupling strength, and $n$ is the order of the anharmonic interaction. Here, we only focus on the two cases: $n = 3$ for the FPUT-$\alpha$ model ($\theta_3 \mapsto \alpha$); and $n = 4$ for the FPUT-$\beta$ model ($\theta_4 \mapsto \beta$). Based on different considerations, Refs. [14] and [19] consistently show that the perturbation strength of the the FPUT-$\alpha$ model is

$$\lambda = \alpha^2 \varepsilon, \quad (2)$$

where $\varepsilon$ is the energy per particle. For the FPUT-$\beta$ chain, the dimensionless perturbation strength is

$$\lambda = \beta \varepsilon. \quad (3)$$

From the Hamiltonian (1), we can derive the equations of motion as

$$\ddot{x}_j = (x_{j+1} - x_j) - (x_j - x_{j-1}) + \theta_n \left[(x_{j+1} - x_j)^{n-1} - (x_j - x_{j-1})^{n-1}\right]. \quad (4)$$

For convenience of discussion, we introduce the normal mode through

$$x_j(t) = \sqrt{\frac{2}{N}} \sum_{k=1}^{N} Q_k(t) \sin \left(\frac{jk\pi}{N}\right), \quad (5)$$

for the fixed boundary conditions, i.e., $x_0 = p_0 = x_N = p_N = 0$ (there are $N - 1$ moving particles), where $Q_k$ is the amplitude of the $k$th normal mode. Inserting Eq. (5) into Eq. (1), the Hamiltonian can be rewritten as a function of the normal modes, then we can drive the motion equations of the normal modes as

$$\ddot{Q}_k = -\omega_k^2 Q_k - \frac{\theta_n}{(2N)^{n/2}} \sum_{k_2, \ldots, k_n} \omega_{k_2} \cdots \omega_{k_n} C_{k, k_2, \ldots, k_n} Q_{k_2} \cdots Q_{k_n}, \quad (6)$$

where

$$\omega_k = 2 \sin \left(\frac{\pi k}{2N}\right), \quad 1 \leq k \leq N - 1, \quad (7)$$

is the frequency of the $k$th normal mode, and

$$C_{k, k_2, \ldots, k_n} = \sum_{\pm} \delta_{k \pm k_2 \pm \cdots \pm k_n, 0} - \sum_{\pm} \delta_{k \pm k_2 \pm \cdots \pm k_n, 2N} - \sum_{\pm} \delta_{k \pm k_2 \pm \cdots \pm k_n, -2N + \cdots}$$

represents the selection rule of the interaction among the normal modes, and $\delta$ is the Kronecker delta function.

To each mode one can associate a harmonic energy

$$E_k(t) = \frac{1}{2} \left[ \dot{Q}_k^2(t) + \omega_k^2 Q_k^2(t) \right]; \quad (8)$$

and a phase, $\varphi_k$, defined via

$$Q_k = \sqrt{E_k(0)/\omega_k^2} \cos \varphi_k, \quad P_k = \sqrt{2E_k(0)} \sin \varphi_k. \quad (9)$$

It is seen that, if $\theta_n = 0$, the system is integrable: All normal modes oscillate independently and $E_k$ is constant for each $k$. The normal modes are instead coupled when $\theta_n \neq 0$.

B. The $N/2$ mode solution for the FPUT-$\alpha$ chain

For the FPUT-$\alpha$ chain, if only the mode with $k = N/2$ was initially excited, from Eq. (6) we obtain

$$\ddot{Q}_{N/2} = -\omega_{N/2}^2 Q_{N/2} \quad (10)$$

since $C_{N/2, k_2, k_3} = 0$ for any $k_2$ and $k_3$, and its solution is

$$Q_{N/2} = A \cos \left(\omega_{N/2} t + \varphi_{N/2}\right), \quad (11)$$

where $A$ and $\varphi_{N/2}$ are, respectively, the amplitude and the initial phase governed by the initial condition; i.e., Eq. (9).

C. The $N/2$ mode solution for the FPUT-$\beta$ chain

Similarly, for the FPUT-$\beta$ chain, if only the $N/2$ mode was excited, from Eq. (6) we have

$$\ddot{Q}_{N/2} = -\omega_{N/2}^2 Q_{N/2} - \frac{3\beta}{2N} \omega_{N/2}^4 Q_{N/2}^3. \quad (12)$$

It is well known that the solution of Eq. (12) can be written, with the Jacobi elliptic cosine function; i.e., $cn$, in the form [20]

$$Q_{N/2} = A \sqrt{N} \text{cn} \left(\Omega t, \Gamma^2\right), \quad (13)$$
shows the 2 mode, the equations of motion (14) to study numerically the destabilization dynamics of this system will be unstable when the intrinsic nonlinear parameter that controls the threshold. Figure 2 shows the numerical results (plotted by the scatter points) of $T$ as a function of $\lambda$ for a given initial condition $\varphi_{\lambda,0} = 0$. The solid lines are fitting curves based on the Nekhoroshev’s estimation as below

$$\Delta E_\lambda = E_\lambda(0) - E_\lambda(T) = \mu \varepsilon,$$

where $\mu$ is a free parameter that controls the threshold. Figure 2 shows the numerical results (plotted by the scatter points) of $T$ as a function of $\lambda$ for a given initial condition $\varphi_{\lambda,0} = 0$. The solid lines are fitting curves based on the Nekhoroshev’s estimation as below

$$T \sim \exp \left[ (\lambda - \lambda_c)^{-\eta} \right],$$

where the exponent $\eta$ is generally depend on the freedom of system [22] (see Fig. 6); and the vertical dashed lines are corresponding to $\lambda_c$ which gives the destabilization threshold of the $N/2$ mode for a certain $N$. We see that the $T$ follows the Nekhoroshev’s law very well. In our simulations, the variation of the perturbation strength is realized by changing the nonlinear coupling coefficient with fixed energy density $\varepsilon = 0.01$, and this strategy is used throughout for all the numerical results presented. Besides, $\mu = 0.01$ is adopted for controlling the threshold value. Though assuming the threshold value is artificial, it does not influence the whole behavior of $T$ vs $\lambda$, especially the estimation of $\lambda_c$, which has been checked by $\mu = 0.001$ (results are not shown here).

Next, we will explore the dependence of $\lambda_c$ on $N$. In fact, the initial excitation phase of the $N/2$ mode has a significant influence on the destabilization threshold. In order to eliminate this effect, the destabilization thresholds for different system sizes are the averages which were done over 24 phases uniformly distributed in $[0, 2\pi]$. Figure 3 presents the numerical results of $\lambda_c$ as a function of $N$. We see that the numerical results are slightly larger than the theoretical values, i.e., Eq. (31), at very small sizes, but the numerical and theoretical results are consistent with each other as the increase of $N$ (see details of theo-
FIG. 3. The destability threshold $\lambda_c$ of the N/2 mode as a function of the system size $N$ in log-log scale for the FPUT-$\alpha$ model. The red solid line is Eq. (31). The blue dotted line is an asymptote for large $N$, i.e., the first term of Eq. (31).

FIG. 4. The destability time $T$ versus perturbation strength $\lambda$ in log-log scale for the FPUT-$\beta$ model with different system size $N$, and the solid lines are fitting curves based on the Nekhoroshev’s estimation, i.e., expression (16); and the vertical dashed lines correspond to $\lambda_c$. It is clearly seen that the destability time follows the Nekhoroshev’s law as well.

FIG. 5. The destability threshold $\lambda_c$ of the N/2 mode as a function of the system size $N$ in log-log scale for the FPUT-$\beta$ model. The red solid line is Eq. (38). The blue dotted line is an asymptote for large $N$, i.e., the first term of Eq. (38).

are fitting curves based on the expression (16); and the vertical dashed lines also give the destability threshold $\lambda_c$. It is clearly seen that the destability time follows the Nekhoroshev’s law as well.

Figure 5 displays the results of $\lambda_c$ as a function of $N$ for the FPUT-$\beta$ model. It is noted that the numerical results are slightly less than the theoretical values, but the overall trend is consistent with the theoretical prediction (see the red solid line), i.e., Eq. (38).

Comparing the results of the FPUT-$\alpha$ model (Fig. 3) and the FPUT-$\beta$ model (Fig. 5), we clearly see that except for the qualitative difference of the law of $T$ in the range of small size, the behavior in the large size is qualitatively identical; i.e., $\lambda_c \propto N^{-1}$. It should be noted, however, that the emergence of this unified law is that the perturbation strength of the FPUT-$\alpha$ model is defined as $\lambda = \alpha^2 \varepsilon$ rather than $\lambda = \alpha \sqrt{\varepsilon}$ in this study.

Figure 6 shows the dependence of $\eta$ on $N$ for both the FPUT-$\alpha$ model and the FPUT-$\beta$ model. It can be seen that in the range of system size we studied, $\eta$ decreases monotonically with the increase of size. In the FPUT-$\alpha$ model, $\eta$ decreases in a power-law manner as a whole, while in the FPUT-$\beta$ model, $\eta$ first decays rapidly and then tends to power-law decay. Besides, the value of $\eta$ in the FPUT-$\beta$ model is larger than that in the FPUT-$\alpha$ model, which means that the curve near the vertical line in Fig. 2 is steeper than that of the corresponding parameters in Fig. 4. That is to say, $\lambda_c$ in the FPUT-$\alpha$ model is more robust to the choice of the threshold of destability time; i.e., $\mu$, because the abscissa near $\lambda_c$ changes very little when the ordinate varies in a large range.
where the coefficients $Q_\alpha$ and $Q_\beta$ are fitting.

FIG. 6. The index $\eta$ of the Nekhoroshev’s estimation, i.e., expression (16) as a function of the system size $N$ in log-log scale for the FPUT-$\alpha$ and $-\beta$ models. The solid curves are fitting.

IV. THEORETICAL ANALYSIS FOR STABILITY OF THE N/2 MODE SOLUTIONS

To study the stability of the $N/2$ mode, we suppose that only this mode is initially excited. Let $\Delta Q_k$ be the error on the mode $Q_k$, from Eq. (6), we obtain:

$$\Delta \dot{Q}_k = -\omega_k^2 \Delta Q_k - \frac{(n - 1)\theta_n \omega_k}{(2N)^{n/2 - 1}} \omega_k \Delta Q_k - \sum_{k_n} \omega_n C_{k,k_k \cdots k_n} \Delta Q_{k_n},$$

(17)

where the coefficients

$$C_{k,k_k \cdots k_n} = \sum_{\pm} \delta_{k\pm k_n} \cdots \delta_{k_n\pm 0} \omega_n C_{k,k_k \cdots k_n},$$

Equation (17) governs the evolution behavior of the error of all the modes.

A. Analysis for the FPUT-$\alpha$ Chain

For the FPUT-$\alpha$ chain, Eq. (17) can be rewritten as

$$\Delta \dot{Q}_k = -\omega_k^2 \Delta Q_k - \sum_{k_3} \omega_{k_3} C_{k,k_3 \cdots k_3} \Delta Q_{k_3},$$

(18)

where the coefficient

$$C_{k,k_3 \cdots k_3} = \sum_{\pm} \delta_{k\pm k_3 \cdots k_3} \omega_{k_3} C_{k,k_3 \cdots k_3},$$

(19)

From Eq. (19), we note that the coefficient is not vanish only when $k_3 = N/2 - k$ or $k_3 = N/2 + k$ for $0 < k < N/2$ while $k_3 = k - N/2$ or $k_3 = k - 3N/2$ for $N/2 < k < N$. After a careful analysis, the evolution of the error controlled by Eq. (18) has the following three types.

- **The one-mode equation**: If $k = N/2$, from Eq. (18) we have

$$\Delta \dot{Q}_{N/2} = -\omega_{N/2}^2 \Delta Q_{N/2}. $$

(20)

The solution of Eq. (20) can be easily obtained:

$$\Delta Q_{N/2} = \frac{2\alpha}{N} \omega_{N/2} Q_{N/2} C,$$

(21)

where $\omega_{N/2} = \sqrt{2}$ has been substituted. Namely, the error of the $N/2$ mode is decoupled with other modes and evolves in the form of oscillation.

- **The two-mode coupled equations**: If $k = N/4$ or $k = 3N/4$, Eq. (18) gives

$$\begin{cases}
\Delta \dot{Q}_{N/4} = -\omega_{N/4}^2 \Delta Q_{N/4} - \frac{2\alpha}{N} \omega_{N/4} Q_{N/4} C, \\
\Delta \dot{Q}_{3N/4} = -\omega_{3N/4}^2 \Delta Q_{3N/4} - \frac{2\alpha}{N} \omega_{3N/4} Q_{3N/4} C,
\end{cases}$$

(22)

where $C = \omega_{N/4} \Delta Q_{N/4} + \omega_{3N/4} \Delta Q_{3N/4}$. The coupled equations are of Hill type and the stability of the coupled modes can be easily obtained by searching the stability chart [23].

- **The four-mode coupled equations**: The above three modes $k = N/4$, $N/2$, and $3N/4$ divide the space of $k$ into four intervals: $(0, N/4)$, $(N/4, N/2)$, $(N/2, 3N/4)$, and $(3N/4, N)$. We have verified that in the four cases one obtains the same final results. Then, let us consider the case $k \in (0, N/4)$. From Eq. (18), we acquire

$$\begin{cases}
\Delta \dot{Q}_k = -\omega_k^2 \Delta Q_k - \frac{2\alpha}{N} \omega_k Q_k C_1, \\
\Delta \dot{Q}_{N/4 - k} = -\omega_{N/4 - k}^2 \Delta Q_{N/4 - k} - \frac{2\alpha}{N} \omega_{N/4 - k} Q_{N/4 - k} C_2, \\
\Delta \dot{Q}_{3N/4 + k} = -\omega_{3N/4 + k}^2 \Delta Q_{3N/4 + k} - \frac{2\alpha}{N} \omega_{3N/4 + k} Q_{3N/4 + k} C_3, \\
\Delta \dot{Q}_{N - k} = -\omega_{N - k}^2 \Delta Q_{N - k} - \frac{2\alpha}{N} \omega_{N - k} Q_{N - k} C_4,
\end{cases}$$

(23)

where

$$C_1 = \omega_{N/4 - k} \Delta Q_{N/4 - k} + \omega_{3N/4 + k} \Delta Q_{3N/4 + k},$$

$$C_2 = \omega_{k} \Delta Q_{k} + \omega_{N - k} \Delta Q_{N - k},$$

$$C_3 = \omega_{k} \Delta Q_{k} - \omega_{N - k} \Delta Q_{N - k},$$

$$C_4 = \omega_{N/4 - k} \Delta Q_{N/4 - k} - \omega_{3N/4 + k} \Delta Q_{3N/4 + k}.$$

To study the stability of the four-mode coupled equations, let $X(t)$ be the $(8 \times 8)$ fundamental matrix of the system of Eqs. (23) that satisfy the initial condition $X(0) = I$, where $I$ is the $(8 \times 8)$ identity matrix. With
the aid of $Q_\infty = \sqrt{N\varepsilon} \cos(\tau), \text{ where } \tau = \sqrt{2}t$, then the system is equivalent to the following matrix equation:

$$\frac{d}{d\tau}X(\tau) = KX(\tau) + \gamma L(\tau)X(\tau)$$

(24)

where $\gamma = 2\alpha\sqrt{\varepsilon}$, and

$$K = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\omega^2_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\omega^2_{2k} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\omega^2_{3k} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\omega^2_{N-k}
\end{pmatrix},$$

$$L(\tau)$$ is also a matrix whose elements that differ from zero are

$$l_{23} = l_{41} = -\omega_k\omega_{2k} \cos(\tau),$$
$$l_{25} = l_{61} = -\omega_k\omega_{2k} \cos(\tau),$$
$$l_{47} = l_{84} = -\omega_{2k}\omega_{N-k} \cos(\tau),$$
$$l_{67} = l_{85} = \omega_{2k}\omega_{N-k} \cos(\tau),$$

respectively.

The usual way to study stability of a given periodic dynamical regime is the Floquet’s method [24]. Following the approach presented in [5], the solution of Eq. (24) has the form:

$$X(\tau) = \sum_{i=0}^{\infty} \gamma^i X_i(\tau)$$

(25)

with all $X_i(\tau)$ of class $C^\infty$. It is assumed that the series $\sum_{i=0}^{\infty} \gamma^i [X_i(\tau)]_{ik}$ are all uniformly convergent with respect to $\tau$. This guarantees the derivation term to term with respect to $\tau$.

Then, inserting Eq. (25) in Eq. (24), we obtain

$$\frac{d}{d\tau}X_0(\tau) = KX_0(\tau)$$

(26)

with $X_0(0) = I$ and, for $n \geq 1$, the recurrence relation

$$\frac{d}{d\tau}X_n(\tau) = KX_n(\tau) + L(\tau)X_{n-1}(\tau)$$

(27)

with $X_n(0) = 0$.

From Eq. (26) and Eq. (27), one gets:

$$X_0(\tau) = e^{K\tau}X_0(0) = e^{K\tau}$$

(28)

and the recurrence relation

$$X_n(\tau) = \int_0^\tau X_0(\tau - \tau')L(\tau')X_{n-1}(\tau').$$

(29)

In particular, for $\tau = 2\pi$, the period of matrix $L(\tau)$, the characteristic numbers $\chi$ of the system described by Eq. (24), which are also eigenvalues of matrix $X(2\pi)$. If $|\chi_j| < 1$ for $j \in [1, 8]$, then the system is stable. On the contrary, as long as one of them exceeds 1, the system is unstable.

In principle, the stability analysis of the system of Eq. (24) should be made for each $k \in (0, N/4)$, which determines the elements of $K$ and $L(\tau)$. However, from our numerical results (see Fig. 1), we clearly see that the adjacent modes; i.e., $k = N/2 \pm 1$ become unstable first. Hence, the destability threshold of the system will be estimated by setting $k = 1$, such that $K$ and $L(\tau)$ are settled.

In practice, Eq. (25) has to be truncated. In our study, the equation is only retained to the term of $\gamma^3$. Namely,

$$X(2\pi) = X_0(2\pi) + \gamma X_1(2\pi) + \gamma^2 X_2(2\pi) + \gamma^3 X_3(2\pi) + O(\gamma^4),$$

(30)

by assuming $0 < \gamma \ll 1$. The iterative operation of $(8 \times 8)$ matrix is much more complicated, it has been performed with the aid of the MATHEMATICA. Finally, we obtain

$$\lambda_c = \frac{0.33}{N} \frac{0.86}{N^2} \frac{0.48}{N^3} + O\left(\frac{1}{N^4}\right),$$

(31)

which is a theoretical estimation of the destability threshold for the FPUT-\(\alpha\) chain studied here.

B. Analysis for the FPUT-\(\beta\) Chain

For the FPUT-\(\beta\) chain, if the $N/2$ mode was initially excited, from Eq. (17), we obtain

$$\Delta \dot{Q}_k = -\omega^2_k \Delta Q_k$$

$$-\frac{3\beta}{2N} \omega^2_k \frac{\omega^2_k}{\omega^2_{\infty}} \sum_{k_4} \omega_{k_4} C_{k, k_4, k_4, k_4} \Delta Q_{k_4},$$

(32)

where $C_{k, k_4, k_4, k_4}$ is not vanish only when $k_4 = k$ or $k_4 = N - k$. Thus, Eq. (32) reduce to

$$\Delta \dot{Q}_k = -\omega^2_k \left(1 + \frac{6\beta}{N} \frac{\omega^2_k}{\omega^2_{\infty}}\right) \Delta Q_k,$$

(33)

which shows that all the modes are decoupled, where $Q_{\infty}$ is given by Eq. (13).

We now consider the expansions of the elliptic function $\alpha_\infty$ in terms of trigonometric function [23]. By performing the change of variable $\tau = (\pi\Omega/2K(\Gamma))t$, where $K(\Gamma)$ is the complete elliptic integral of the first kind, we obtain, up to $\Gamma^6$ terms,

$$\Delta \dot{Q}_k = -\omega^2_k \left[\frac{1}{2} + \frac{q}{4} + \frac{3q^2}{64} + \frac{q^3}{128}ight] \Delta Q_k$$

$$+ (q + \frac{q^2}{2} + \frac{85q^3}{256}) \cos(2\tau)$$

$$+ \frac{q^2}{8} \sin(4\tau) + \frac{3q^3}{256} \cos(6\tau),$$

(34)
where $q = \Gamma^2$. Let $X(\tau)$ is the $(2 \times 2)$ fundamental matrix of the system of Eq. (34) that satisfies the initial condition $X(0) = I$. Then the system is equivalent to the following matrix equation:

$$
\frac{d}{d\tau}X(\tau) = EX(\tau) + qF(\tau)X(\tau) + q^2G(\tau)X(\tau) + q^3H(\tau)X(\tau)
$$

(35)

where

$$
E = \begin{pmatrix}
0 & 1 \\
-\frac{1}{2} & 0
\end{pmatrix},
$$

$$
F(\tau) = \begin{pmatrix}
0 & 0 \\
\left[4 + \cos(2\tau)\right] & 0
\end{pmatrix},
$$

$$
G(\tau) = \begin{pmatrix}
0 & 0 \\
\left[4 + \cos(2\tau) + \frac{\cos(4\tau)}{4}\right] & 0
\end{pmatrix},
$$

$$
H(\tau) = \begin{pmatrix}
0 & 0 \\
\left[\frac{1}{16} + \frac{85\cos(2\tau)}{32} + \cos(4\tau) + \frac{3\cos(6\tau)}{32}\right] & 0
\end{pmatrix}.
$$

Hereafter, we also put $k = N/2 - 1$, by searching for the general solution of the approximate equation by Eq. (25), we obtain the trace of solution $X(\tau)$:

$$
Tr(X(\tau)) \approx (2 + \frac{\pi^4}{4N^2} + \frac{\pi^5}{8N^3})
$$

$$
+ q\left(-\frac{\pi^3}{4N} + \frac{\pi^4}{16N^2} + \frac{7\pi^5}{96} + \frac{\pi^7}{96}\right) \frac{1}{N^2}
$$

$$
+ q^2\left(-\frac{3\pi^2}{16} + \frac{15\pi^3}{64N} + \frac{7\pi^4}{256} + \frac{\pi^6}{384}\right) \frac{1}{N^2}
$$

$$
+ q^3\left(-\frac{25\pi^5}{512} + \frac{5\pi^7}{1536}\right) \frac{1}{N^3}.
$$

We take the strategy given in [10], that is, by computing the discriminant

$$
D_1 = (a_1 + 2)(a_1 - 2) = 0
$$

(36)

explicitly via these coefficients of the characteristic polynomial, where $a_1$ is the first coefficient in Newton formula:

$$
a_1 = Tr(X(\pi)).
$$

We get

$$
q = \frac{2\pi}{3N} + \frac{13\pi^2}{24N^2} + \left(\frac{189}{512} + \frac{7\pi^2}{432}\right) \frac{\pi^3}{N^3} + O\left(\frac{1}{N^4}\right),
$$

(37)

and, with the help of Eq. (15), finally, we obtain a theoretical estimation for the FPUT-\(\beta\) model as

$$
\lambda_c = \frac{4\pi}{9N} + \frac{5\pi^2}{4N^2} + \frac{(67823 + 224\pi^2)\pi^3}{20736N^3} + O\left(\frac{1}{N^4}\right).
$$

(38)

V. CONCLUSIONS

In summary, we have studied the destabilization dynamics of the $N/2$ mode for both the FPUT-\(\alpha\) and -\(\beta\) models with fixed boundary conditions systematically. We found that the destabilization time, $T$, as a function of the perturbation strength, $\lambda$, follows the Nekhoroshev’s estimation; i.e., $T \propto \exp\left(\frac{\lambda - \lambda_c}{-\eta}\right)$, for both models, $\lambda = \alpha^2\varepsilon$ and $\lambda = \beta\varepsilon$ were defined, respectively. Then the exponent $\eta$ and the destabilization threshold $\lambda_c$ were accurately obtained by nonlinear fitting, and then the dependence of $\eta$ and $\lambda_c$ on $N$ were given. It was noted that $\eta$ decreases slowly in a power-law with the increase of $N$ for the FPUT-\(\alpha\) and -\(\beta\) models, the only difference is that the value of the former is smaller than that of the latter.

In addition, the destabilization threshold $\lambda_c$, as a function of the system size $N$, up to $O(N^{-4})$, was also analytically obtained by using the Floquet theory for both the FPUT-\(\alpha\) model; i.e., Eq. (31), and the FPUT-\(\beta\) model; i.e., Eq. (38). We have shown that the numerical results of the FPUT-\(\alpha\) model are slightly larger than the theoretical predictions under very small size (4 $\leq N < 10$), but the theoretical results are in good agreement with the numerical results as the increase of the size. As a contrast, in the FPUT-\(\beta\) model, the overall numerical results are slightly smaller than the theoretical values, but the trends of the theoretical and the numerical are completely consistent in the whole range of the size studied. In the large $N$ limit, $\lambda_c$ varies with $N$ for these two models are same, that is, $\lambda_c \propto N^{-1}$. However, in the range of small size, there is a qualitative difference: the function curve, $\lambda_c(N)$, of the FPUT-\(\alpha\) model is concave upward, while that of the FPUT-\(\beta\) model is convex downward. We conjecture that this difference may be caused by the different symmetries of the interaction potential functions, which will be further studied.

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