Diffusion Limit and the optimal convergence rate of the Vlasov-Poisson-Fokker-Planck system

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Abstract

In the present paper, we study the diffusion limit of the classical solution to the Vlasov-Poisson-Fokker-Planck (VPFP) system with initial data near a global Maxwellian. We prove the convergence and establish the optimal convergence rate of the global strong solution to the VPFP system towards the solution to the drift-diffusion-Poisson system based on the spectral analysis with precise estimation on the initial layer.

Key words. Vlasov-Poisson-Fokker-Planck system, spectral analysis, diffusion limit, convergence rate.

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1 Introduction

The Vlasov-Poisson-Fokker-Planck (VPFP) system can be used to model the time evolution of dilute charged particles governed by the electrostatic force coming from their (self-consistent) Coulomb interaction. The collision term in the kinetic equation is the Fokker-Planck operator that describes the Brownian force. In general, the rescaled VPFP system defined on \( \mathbb{R}^3 \times \mathbb{R}^3 \) takes the form

\[
\begin{align*}
\frac{\partial F_\epsilon}{\partial t} + \frac{1}{\epsilon} v \cdot \nabla_x F_\epsilon + \frac{1}{\epsilon} \nabla_x \Phi_\epsilon \cdot \nabla_v F_\epsilon &= \frac{1}{\epsilon^2} \nabla_v \cdot (\nabla_v F_\epsilon + vF_\epsilon), \\
\Delta_x \Phi_\epsilon &= \int_{\mathbb{R}^3} F_\epsilon dv - 1, \\
F_\epsilon(0, x, v) &= F_0(x, v),
\end{align*}
\]

where \( \epsilon > 0 \) is a small parameter related to the mean free path, \( F_\epsilon = F_\epsilon(t, x, v) \) is the number density function of charged particles, and \( \Phi_\epsilon(t, x) \) denotes the electric potential, respectively. Throughout this paper, we assume \( \epsilon \in (0, 1) \).
In this paper, we study the diffusion limit of the strong solution to the rescaled VPFP system (1.1)–(1.3) with initial data near the normalized Maxwellian $M(v)$ given by

$$M = M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}, \quad v \in \mathbb{R}^{3}.$$  

Our motivation is to prove the convergence and establish the convergence rate of strong solutions $(F_\varepsilon, \Phi_\varepsilon)$ to (1.1)–(1.3) towards $(NM, \Phi)$, where $(N, \Phi)(t, x)$ is the solution of the following drift-diffusion-Poisson (DDP) system:

$$\partial_t N + \nabla_x \cdot (\nabla_x N + N \nabla_x \Phi) = 0, \quad (1.4)$$  

$$\Delta_x \Phi = N - 1, \quad (1.5)$$  

$$N(0, x) = N_0(x) = \int_{\mathbb{R}^3} F_0 dv. \quad (1.6)$$

Here, $N = N(t, x)$ stands for densities of charged particles in the ionic solution and $\Phi(t, x)$ is the self-consistent electric potential.

The DDP system (1.4)–(1.6) provides a continuum description of the evolution of charged particles via macroscopic quantities, for example, the particle density, the current density etc., which have cheaper costs for numerics. Such continuum models can be (formally) derived from kinetic models by coarse graining methods, like the moment method, the Hilbert expansion method and so on [19, 25, 26]. Concerning the mathematical analysis, the initial value problem and the initial boundary value problem of the DDP system have been extensively studied, we refer to [1, 3, 4, 5, 14, 19, 20].

The existence and uniqueness of solutions to the initial value problem or the initial boundary value problem of the VPFP system have been investigated in the literature. We refer to [7, 30, 33, 18] for results on the classical solutions and to [8, 11, 12, 32] for weak solutions and their regularity. Concerning the long-time behavior of the VPFP system, we refer to [0, 9, 12, 13]. The diffusion limit of the weak solution to the VPFP system has been studied extensively in the literature (cf. [15, 16, 17, 27, 29, 34]). In [15, 16, 29], the authors proved the convergence of suitable solutions to the single species VPFP system towards a solution to the drift-diffusion-Poisson model in the whole space. In [31], the authors established a global convergence result of the renormalized solution to the multiple species VPFP system towards a solution to the Poisson-Nernst-Planck system. The spectrum structure and the optimal decay rate of the classical solution to the VPFP system were investigated in [22].

However, in contrast to the works on weak solution [15, 16, 17, 27, 29, 34], the diffusion limit of the classical solution to the VPFP system (1.1)–(1.3) has not been given despite of its importance. On the other hand, the convergence rate of the weak solution to the VPFP system towards the solution to the DDP system hasn’t been obtained in [15, 16, 17, 27, 29, 34]. Therefore, it is natural to establish an explicit convergence rate of the solution to the VPFP system towards its diffusion limit.

First of all, the VPFP system (1.1)–(1.2) has a stationary solution $(F_*, \Phi_*) = (M(v), 0)$. Hence, we define the perturbation of $F_*$ as

$$F_\varepsilon = M + \sqrt{M} f_\varepsilon.$$  

Then Cauchy problem of the VPFP system (1.1)–(1.3) can be rewritten as

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_\varepsilon - \frac{1}{\varepsilon} v \sqrt{M} \cdot \nabla_x \Phi_\varepsilon - \frac{1}{\varepsilon^2} L f_\varepsilon = \frac{1}{\varepsilon} G(f_\varepsilon), \quad (1.7)$$  

$$\Delta_x \Phi_\varepsilon = \int_{\mathbb{R}^3} f_\varepsilon \sqrt{M} dv, \quad (1.8)$$  

$$f_\varepsilon(0, x, v) = f_0(x, v) = \frac{F_0 - M}{\sqrt{M}}, \quad (1.9)$$
In general, the convergence is not uniform near $t < \mu < 0$ the initial layer of the VPFP system. But we can show that if the initial data

$$f_0 = n_0 \sqrt{M},$$

Then Cauchy problem of the DDP system (1.4)–(1.6) becomes

$$\begin{align*}
\partial_t n - \Delta_x n + n &= -\nabla_x \cdot (n \nabla_x \Phi), \\
\Delta_x \Phi &= n, \\
n(0, x) &= n_0 = \int_{\mathbb{R}^3} f_0 \sqrt{M} \, dv.
\end{align*}$$

The aim of this paper is to prove the convergence and establish the convergence rate of strong solutions of (1.7)–(1.9) towards (1.10)–(1.12). In general, the convergence is not uniform near $t = 0$. The breakdown of the uniform convergence near $t = 0$ is the initial layer of the VPFP system. But we can show that if the initial data $f_0 = n_0 \sqrt{M}$, then the uniform convergence is up to $t = 0$.

We denote $L^2(\mathbb{R}^3)$ a Hilbert space of complex-value functions $f(x)$ on $\mathbb{R}^3$ with the inner product and the norm

$$(f,g) = \int_{\mathbb{R}^3} f(x)\overline{g(x)} \, dx, \quad \|f\| = \left(\int_{\mathbb{R}^3} |f(x)|^2 \, dx\right)^{1/2}.$$ 

The nullspace of the operator $L$, denoted by $N_0$, is a subspace spanned by $\sqrt{M}$. Let $P_0$ be the projection operators from $L^2(\mathbb{R}^3)$ to the subspace $N_0$ with

$$P_0 f = (f, \sqrt{M}) \sqrt{M}, \quad P_1 = I - P_0.$$ 

Corresponding to the linearized operator $L$, we define the following dissipation norm:

$$\|f\|^2_{L^2} = \|\nabla_x f\|^2 + \|v f\|^2.$$ 

This norm is stronger than the $L^2$-norm because

$$\|f\|^2_{L^2} \geq 2\|\nabla_x f\|\|v f\| \geq -2(\nabla_x f, v f) = 3\|f\|^2.$$ 

From [23], the linearized operator $L$ is non-positive and locally coercive in the sense that there is a constant $0 < \mu < 1$ such that

$$(L f, f) \leq -\|P_0 f\|^2, \quad (L f, f) \leq -\mu \|P_1 f\|^2_{L^2}. $$

**Notations:** Before state the main results in this paper, we list some notations. For any $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ and $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3$, denotes

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}, \quad \partial^\beta = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \partial_{x_3}^{\beta_3}.$$ 

The Fourier transform of $f = f(x,v)$ is denoted by

$$\hat{f}(\xi, v) = \mathcal{F} f(\xi, v) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x,v) e^{-ix \cdot \xi} \, dx.$$ 

In the following, denote by $\| \cdot \|_{L^2}^2$ the norm of the function space $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, and denote by $\| \cdot \|_{L^2}$ and $\| \cdot \|_{L^2}$ the norms of the function spaces $L^2(\mathbb{R}^3)$ and $L^2(\mathbb{R}^3)$ respectively. For any integer $k \geq 1$, we denote by $\| \cdot \|_{H^k}$ and $\| \cdot \|_{H^k}$ the norms of the function spaces $H^k(\mathbb{R}^3)$ and $H^k(\mathbb{R}^3)$ respectively.

Now we are ready to state main results in this paper.
Theorem 1.1. There exist small positive constants $\delta_0$ and $\varepsilon_0$ such that if the initial data $f_0$ satisfies $\|f_0\|_{H^2(L^2)} + \|\sqrt{M} f_0\|_{L^1} \leq \delta_0$, then for any $\varepsilon \in (0,\varepsilon_0)$, there exists a unique function $f_\varepsilon(t)$ to the VPFP system (1.17)–(1.19) satisfying
\[
\|f_\varepsilon(t)\|_{H^2(L^2)} + \|
abla_x \Phi_\varepsilon(t)\|_{H^2} \leq C\delta_0 e^{-\frac{t}{\varepsilon}}, \quad t > 0,
\]
where $\nabla_x \Phi_\varepsilon(t) = \nabla_x \Delta_x^{-1}(f_\varepsilon, \sqrt{M})$ and $C > 0$ is a constant independent of $\varepsilon$.

There exists a small constant $\delta_0 > 0$ such that if $\|f_0\|_{H^2(L^2)} + \|\sqrt{M} f_0\|_{L^1} \leq \delta_0$, then the DDP system (1.12)–(1.14) admits a unique global solution $u(t)$ satisfying
\[
\|u(t)\|_{H^2} + \|
abla_x u(t)\|_{H^2} \leq C\delta_0 e^{-\frac{t}{\varepsilon}},
\]
where $\nabla_x u(t) = \nabla_x \Delta_x^{-1}u(t)$ and $C > 0$ is a constant.

Theorem 1.2. Let $(f_\varepsilon, \Phi_\varepsilon) = (f_\varepsilon(t, x), \Phi_\varepsilon(t, x))$ be the global solution to the VPFP system (1.17)–(1.19), and let $(n, \Phi) = (n, \Phi)(t, x)$ be the global solution to the DDP system (1.12)–(1.14). Then, there exist small positive constants $\delta_0$ and $\varepsilon_0$ such that if the initial data $f_0$ satisfies $\|f_0\|_{H^2(L^2)} + \|\nabla_x f_0\|_{H^2(L^2)} + \|\sqrt{M} f_0\|_{H^2(L^2)} + \|\nabla_x f_0\|_{L^1} \leq \delta_0$, then for any $\varepsilon \in (0,\varepsilon_0)$,
\[
\|f_\varepsilon(t) - n(t)\sqrt{M}\|_{H^2(L^2)} + \|
abla_x \Phi_\varepsilon(t) - \nabla_x \Phi(t)\|_{H^2} \leq C\delta_0 \left( e^{-\frac{t}{\varepsilon}} + e^{-\frac{Ct}{\varepsilon^2}} \right),
\]
where $a > 0$ and $C > 0$ are two constants independent of $\varepsilon$.

Moreover, if the initial data $f_0 = n_0\sqrt{M} \in N_0$ and $\|n_0\|_{H^2} + \|n_0\|_{L^1} \leq \delta_0$, then we have
\[
\|f_\varepsilon(t) - n(t)\sqrt{M}\|_{H^2(L^2)} + \|
abla_x \Phi_\varepsilon(t) - \nabla_x \Phi(t)\|_{H^2} \leq C\delta_0 e^{-\frac{t}{\varepsilon}}.
\]

The results in Theorem 1.2 on the convergence rate of diffusion limits of the VPFP system is proved based on the spectral analysis [22] and the ideas inspired by [2, 13, 23]. Indeed, we first develop a non-local implicit function theorem to show the existence of the eigenvalue $\lambda_0(\xi, \varepsilon)$ of the linear VPFP operator $B_\varepsilon(\xi)$ defined by (2.4) for $\varepsilon|\xi|$ small and establish the expansions of the eigenvalue $\lambda_0(\xi, \varepsilon)$ and the corresponding eigenfunction $\psi_0(\xi, \varepsilon)$ (See Lemma 2.7 and Theorem 2.9)
\[
\begin{align*}
\lambda_0(\xi, \varepsilon) &= -\varepsilon^2(1 + |\xi|^2) + O(\varepsilon^4(1 + |\xi|^2)^2), \\
P_0\psi_0(\xi, \varepsilon) &= \frac{|\xi|}{\sqrt{1 + |\xi|^2}} \sqrt{M} + O(\varepsilon|\xi|), \\
P_1\psi_0(\xi, \varepsilon) &= -i\varepsilon \sqrt{1 + |\xi|^2} \left( v \cdot \frac{\xi}{|\xi|} \right) \sqrt{M} + O(\varepsilon^2(1 + |\xi|^2)).
\end{align*}
\]

Based on the spectral analysis of $B_\varepsilon(\xi)$, we can decompose the semigroup $e^{B_\varepsilon(\xi)t/\varepsilon^2}$ into the fluid part $S_1$ and the remainder part $S_2$, where the remainder part $S_2$ satisfies the decay rate $e^{-Ct/\varepsilon^2}$, and the fluid part $S_1$ takes the form (See Theorem 2.10)
\[
S_1(t, \xi, \varepsilon)f = e^{\frac{t}{\varepsilon^2} \lambda_0(\xi, \varepsilon)} \left( \frac{f, \psi_0(\xi, \varepsilon)}{\xi} \right) \psi_0(\xi, \varepsilon).
\]
2 Spectral analysis

In this section, we are concerned with the spectral analysis of the operator \( B_\varepsilon(\xi) \) defined by (2.4), which will be applied to study diffusion limit of the solution to the VPFP system (1.7)–(1.9).

From the system (1.7)–(1.9), we have the following linearized VPFP system:

\[
\begin{aligned}
\epsilon^2 \partial_t f_\epsilon &= B_\epsilon f_\epsilon, \quad t > 0, \\
f_\epsilon(0, x, v) &= f_0(x, v),
\end{aligned}
\]  

(2.1)

where

\[
B_\epsilon f = L f - \epsilon v \cdot \nabla_x f - \epsilon v \cdot \nabla_x (-\Delta_x)^{-1} P_0 f.
\]  

(2.2)

Take Fourier transform to (2.1) in \( x \) to get

\[
\begin{aligned}
\epsilon^2 \partial_t \hat{f}_\epsilon &= B_\epsilon(\xi) \hat{f}_\epsilon, \quad t > 0, \\
\hat{f}_\epsilon(0, \xi, v) &= \hat{f}_0(\xi, v),
\end{aligned}
\]  

(2.3)

where

\[
B_\epsilon(\xi) = L - i\epsilon v \cdot \xi - i\epsilon \frac{v \cdot \xi}{|\xi|^2} P_0, \quad \xi \neq 0.
\]  

(2.4)

Following [24], we decompose \( L \) as follows:

\[
\begin{align*}
Lf &= -Af + Kf, \\
Af &= -\Delta v f - \frac{|v|^2}{4} f - \frac{3}{2} 1_{|v| \geq R} f, \\
Kf &= \frac{3}{2} 1_{|v| \leq R} f,
\end{align*}
\]  

(2.5)

where \( R > 0 \) is chosen to be large enough and \( 1_{|v| \leq R} \) is the indicator function of the domain \( |v| \leq R \). The operators \( A \) and \( K \) are self-adjoint on \( L^2 \), and \( K \) is a \( A \)–compact operator, that is, for any sequences both \( \{u_n\} \) and \( \{Au_n\} \) bounded in \( L^2 \), \( \{Ku_n\} \) contains a convergent subsequence in \( L^2 \) (cf. Lemma 2.3 in [24]). For \( R > 0 \) sufficiently large, there are two constants \( \nu_0, \nu_1 > 0 \) such that

\[
(Af, f) \geq \nu_1 \|f\|^2_{L^2_\xi} \geq \nu_0 \|f\|^2.
\]  

(2.6)

Also, we define

\[
A_\epsilon(\xi) = -A - i\epsilon (v \cdot \xi).
\]  

(2.7)

Introduce a weighted Hilbert space \( L^2_\xi(\mathbb{R}^3) \) for \( \xi \neq 0 \) as

\[
L^2_\xi(\mathbb{R}^3) = \{ f \in L^2(\mathbb{R}^3) \mid \|f\|_\xi = \sqrt{(f, f)_\xi} < \infty \},
\]

with the inner product defined by

\[
(f, g)_\xi = (f, g) + \frac{1}{|\xi|^2} (P_0 f, P_0 g).
\]

Since \( P_0 \) is a self-adjoint projection operator, it follows that \( (P_0 f, P_0 g) = (P_0 f, g) = (f, P_0 g) \) and hence

\[
(f, g)_\xi = (f, g + \frac{1}{|\xi|^2} P_0 g) = (f + \frac{1}{|\xi|^2} P_0 f, g).
\]  

(2.8)

By (2.8), we have for any \( f, g \in L^2_\xi(\mathbb{R}^3) \cap D(B_\epsilon(\xi)) \),

\[
(B_\epsilon(\xi)f, g)_\xi = (B_\epsilon(\xi)f, g + \frac{1}{|\xi|^2} P_0 g) = (f, B_\epsilon(-\xi)g)_\xi.
\]  

(2.9)
Moreover, $B_{\epsilon}(\xi)$ is a dissipate operator in $L^2_{\xi}(\mathbb{R}^3)$:

$$\text{Re}(B_{\epsilon}(\xi)f, f)_{\xi} = (Lf, f) \leq 0.$$  \hfill (2.10)

We can regard $B_{\epsilon}(\xi)$ as a linear operator from the space $L^2_{\xi}(\mathbb{R}^3)$ to itself because

$$\|f\|^2 \leq \|f\|_{\xi}^2 \leq (1 + |\xi|^{-2})\|f\|^2, \quad \xi \neq 0.$$  

The $\sigma(A)$ denotes the spectrum of the operator $A$. The discrete spectrum of $A$, denoted by $\sigma_d(A)$, is the set of all isolated eigenvalues with finite multiplicity. The essential spectrum of $A$, $\sigma_{ess}(A)$, is the set $\sigma(A) \setminus \sigma_d(A)$.

We denote $\rho(A)$ to be the resolvent set of the operator $A$.

By (2.9), (2.10), (2.11) and (2.12), we have the following two lemmas.

**Lemma 2.1.** (1) The operator $B_{\epsilon}(\xi)$ generates a strongly continuous contraction semigroup on $L^2_{\xi}(\mathbb{R}^3)$, which satisfies

$$\|e^{tB_{\epsilon}(\xi)}f\|_{\xi} \leq \|f\|_{\xi}, \quad \forall t > 0, f \in L^2_{\xi}(\mathbb{R}^3).$$  \hfill (2.11)

(2) The operator $A_{\epsilon}(\xi)$ generates a strongly continuous contraction semigroup on $L^2(\mathbb{R}^3)$, which satisfies

$$\|e^{tA_{\epsilon}(\xi)}f\| \leq e^{-\nu_0 t}\|f\|, \quad \forall t > 0, f \in L^2(\mathbb{R}^3).$$  \hfill (2.12)

**Proof.** Since the operator $B_{\epsilon}(\xi)$ is a densely defined closed operator on $L^2_{\xi}(\mathbb{R}^3)$, and both $B_{\epsilon}(\xi)$ and $B_{\epsilon}(\xi)^* = B_{\epsilon}(-\xi)$ are dissipative on $L^2_{\xi}(\mathbb{R}^3)$ by (2.10), it follows that $B_{\epsilon}(\xi)$ generates a strongly continuous contraction semigroup on $L^2_{\xi}(\mathbb{R}^3)$. This proves (1). Similarly, we can prove (2) by (2.6). □

**Lemma 2.2.** The following conditions hold for all $\xi \neq 0$ and $\epsilon \in (0, 1)$.

(1) $\sigma_{ess}(B_{\epsilon}(\xi)) \subset \{\lambda \in \mathbb{C} | \text{Re}\lambda \leq -\nu_0\}$ and $\sigma(B_{\epsilon}(\xi)) \cap \{\lambda \in \mathbb{C} | -\nu_0 < \text{Re}\lambda \leq 0\} \subset \sigma_d(B_{\epsilon}(\xi))$.

(2) If $\lambda$ is an eigenvalue of $B_{\epsilon}(\xi)$, then $\text{Re}\lambda < 0$ for any $\epsilon \xi \neq 0$ and $\lambda = 0$ iff $\epsilon \xi = 0$.

**Proof.** By Lemma 2.1, $A_{\epsilon}(\xi)$ is invertible for $\text{Re}\lambda > -\nu_0$, and hence $\sigma(A_{\epsilon}(\xi)) \subset \{\lambda \in \mathbb{C} | \text{Re}\lambda \leq -\nu_0\}$. Since $K$ is $A_{\epsilon}(\xi)$-compact and $i(v \cdot \xi)|\xi|^{-2}P_0$ is compact, namely, $B_{\epsilon}(\xi)$ is a compact perturbation of $A_{\epsilon}(\xi)$, it follows that $\sigma_{ess}(B_{\epsilon}(\xi)) = \sigma_{es}(A_{\epsilon}(\xi))$ and $\sigma(B_{\epsilon}(\xi))$ in the domain $\text{Re}\lambda > -\nu_0$ consists of discrete eigenvalues with possible accumulation points only on the line $\text{Re}\lambda = -\nu_0$. This proves (1). By a similar as Proposition 2.2.8 in 31, we can prove (2) and the detail is omitted for simplicity. □

Now denote by $T$ a linear operator on $L^2(\mathbb{R}^3)$ or $L^2_{\xi}(\mathbb{R}^3)$, and we define the corresponding norms of $T$ by

$$\|T\| = \sup_{\|f\| = 1} \|Tf\|, \quad \|T\|_{\xi} = \sup_{\|f\|_{\xi} = 1} \|Tf\|_{\xi}.$$  

Obviously,

$$(1 + |\xi|^{-2})^{-1/2}\|T\| \leq \|T\|_{\xi} \leq (1 + |\xi|^{-2})^{1/2}\|T\|.$$  \hfill (2.13)

First, we consider the spectrum and resolvent sets of $B_{\epsilon}(\xi)$ for $\epsilon|\xi| > r_0$ with $r_0 > 0$ a constant. To this end, we decompose $B_{\epsilon}(\xi)$ into

$$\lambda - B_{\epsilon}(\xi) = \lambda - A_{\epsilon}(\xi) - K + i\epsilon \frac{v \cdot \xi}{|\xi|^2} P_0$$

$$= \left( I - K(\lambda - A_{\epsilon}(\xi))^{-1} + i\epsilon \frac{v \cdot \xi}{|\xi|^2} P_0(\lambda - A_{\epsilon}(\xi))^{-1} \right) (\lambda - A_{\epsilon}(\xi)).$$  \hfill (2.14)

Then, we have the estimates on the right hand terms of (2.14) as follows.

**Lemma 2.3** [24, 22]. There exists a constant $C > 0$ so that the following holds
1. For any $\delta > 0$, if $\text{Re}\lambda \geq -\nu_0 + \delta$, we have
\[
\|K(\lambda - A_\epsilon(\xi))^{-1}\| \to 0, \quad \text{as } |\text{Im}\lambda| + \epsilon|\xi| \to \infty.
\] (2.15)

2. For any $\delta > 0$ and $r_0 > 0$, if $\text{Re}\lambda \geq -\nu_0 + \delta$ and $|\xi| \geq r_0$, we have
\[
\|(v \cdot \xi)|\xi|^{-2}P_0(\lambda - A_\epsilon(\xi))^{-1}\| \leq C(\delta^{-1} + 1)(r_0^{-1} + 1)(|\xi| + |\lambda|)^{-1}.
\] (2.16)

By (2.14) and Lemma 2.3, we have the spectral gap of the operator $B_\epsilon(\xi)$ for $|\xi| > r_0$.

**Lemma 2.4 (Spectral gap).** Fix $\epsilon \in (0, 1)$. For any $r_0 > 0$, there exists $\alpha = \alpha(r_0) > 0$ such that for $|\xi| \geq r_0$,
\[
\sigma(B_\epsilon(\xi)) \subset \{ \lambda \in \mathbb{C} | \text{Re}\lambda \leq -\alpha \}.
\] (2.17)

**Proof.** Let $\lambda \in \sigma(B_\epsilon(\xi)) \cap \{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\nu_0 + \delta \}$ with $\delta > 0$. We first show that $\sup_{|\xi| \geq r_0} |\text{Im}\lambda| < +\infty$. By (2.13), (2.15) and (2.16), there exists a large constant $r_1 = r_1(\delta) > 0$ such that for $\text{Re}\lambda \geq -\nu_0 + \delta$ and $|\xi| \geq r_1$,
\[
\|K(\lambda - A_\epsilon(\xi))^{-1}\| \leq 1/4, \quad \|(v \cdot \xi)|\xi|^{-2}P_0(\lambda - A_\epsilon(\xi))^{-1}\| \leq 1/4.
\] (2.18)

This implies that $I + K(\lambda - A(\xi))^{-1} + i(v \cdot \xi)|\xi|^{-2}P_0(\lambda - A_\epsilon(\xi))^{-1}$ is invertible on $L^2(\mathbb{R}^3)$, which together with (2.14) yield that $\lambda - B_\epsilon(\xi)$ is also invertible on $L^2(\mathbb{R}^3)$ and satisfies
\[
(\lambda - B_\epsilon(\xi))^{-1} = (\lambda - A_\epsilon(\xi))^{-1}
\left( I - K(\lambda - A_\epsilon(\xi))^{-1} + i\frac{v \cdot \xi}{|\xi|^2}P_0(\lambda - A_\epsilon(\xi))^{-1} \right)^{-1}.
\] (2.19)

Therefore $\{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\nu_0 + \delta \} \subset \rho(B_\epsilon(\xi))$ for $|\xi| \geq r_1$.

As for $r_0 \leq |\xi| \leq r_1$, by (2.15) and (2.16), there exists $\beta = \beta(r_0, r_1, \delta) > 0$ such that if $\text{Re}\lambda \geq -\nu_0 + \delta$, $|\text{Im}\lambda| > \beta$ and $|\xi| \in [r_0, r_1]$, then (2.18) still holds and thus $\lambda - B_\epsilon(\xi)$ is invertible. This implies that $\{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\nu_0 + \delta, |\text{Im}\lambda| > \beta \} \subset \rho(B_\epsilon(\xi))$ for $r_0 \leq |\xi| \leq r_1$. Thus, we conclude
\[
\sigma(B_\epsilon(\xi)) \cap \{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\nu_0 + \delta \} \subset \{ \lambda \in \mathbb{C} | \text{Re}\lambda \geq -\nu_0 + \delta, |\text{Im}\lambda| \leq \beta \}, \quad |\xi| \geq r_0.
\] (2.20)

Next, we prove that $\sup_{|\xi| \geq r_0} \text{Re}\lambda < 0$. Base on the above argument, it is sufficient to prove that $\text{Re}\lambda < 0$ for $|\xi| \in [r_0, r_1]$ and $|\text{Im}\lambda| \leq \beta$. If not, there exists $\lambda_n \in \sigma(B_\epsilon(\xi_n))$ with $\epsilon(\xi_n) \in [r_0, r_1]$ and $f_n \in L^2(\mathbb{R}^3)$ with $\|f_n\|_{\xi_n} = 1$ such that
\[
L_{\xi_n}f_n - i(v \cdot \xi_n)f_n - \frac{i(v \cdot \xi_n)}{|\xi_n|^2}P_0f_n = \lambda_n f_n, \quad \text{Re}\lambda_n \to 0.
\]

Taking the inner product $(\cdot, \cdot)_{\xi_n}$ between the above equation and $f_n$ and choosing the real part, we have
\[
\text{Re}\lambda_n \|f_n\|_{\xi_n} = (L_{\xi_n}f_n, f_n) \leq -\mu \|P_{\epsilon}f_n\|^2.
\]

This implies that $\lim_{n \to \infty} \|P_{\epsilon}f_n\| = 0$ and hence $\lim_{n \to \infty} \|P_{\epsilon}f_n\|_{\xi_n} = 1$. Since $P_0$ is a compact operator, there exists a subsequence $n_j$ of $n$ and a function $f_0 \neq 0$ in $N_0$ such that $P_0f_{n_j} \to f_0$ in $L^2$ as $j \to \infty$. Thus we have $f_{n_j} \to f_0$ in $L^2$ as $j \to \infty$. Due to the fact that $\epsilon(\xi_n) \in [r_0, r_1]$, $|\text{Im}\lambda_n| \leq \beta$ and $\text{Re}\lambda_n \to 0$, there exists a subsequence of $(\xi_n, \lambda_n)$, $(\xi_0, \lambda_0)$ with $\epsilon(\xi_0) \in [r_0, r_1]$, $\text{Re}\lambda_0 = 0$ such that $(\xi_{n_j}, \lambda_{n_j}) \to (\xi_0, \lambda_0)$. It follows that $B_\epsilon(\xi_0)f_0 = \lambda_0 f_0$ and thus $\lambda_0$ is an eigenvalue of $B_\epsilon(\xi_0)$ with $\text{Re}\lambda_0 = 0$, which contradicts $\text{Re}\lambda < 0$ for $|\xi| \neq r_0$ established by Lemma 2.3. This proves the lemma.

Then, we investigate the spectrum and resolvent sets of $B_\epsilon(\xi)$ for $|\xi| \leq r_0$. To this end, we decompose $\lambda - B_\epsilon(\xi)$ as follows
\[
\lambda - B_\epsilon(\xi) = \lambda P_0 + \lambda P_1 - Q_\epsilon(\xi) + i\epsilon P_0(v \cdot \xi)P_1 + i\epsilon P_1(v \cdot \xi) \left( 1 + \frac{1}{|\xi|^2} \right)P_0,
\] (2.21)

where
\[
Q_\epsilon(\xi) = L - i\epsilon P_1(v \cdot \xi)P_1.
\] (2.22)
Lemma 2.5. Let $\xi \neq 0$ and $Q_\epsilon(\xi)$ defined by (2.22). We have

1. If $\lambda \neq 0$, then
   \[
   \|\lambda^{-1}P_1(v \cdot \xi)\left(1 + \frac{1}{|\xi|^2}\right)P_0\|_\xi \leq C(|\xi| + 1)|\lambda|^{-1}. \tag{2.23}
   \]

2. If $\text{Re}\lambda > -1$, then the operator $\lambda P_1 - Q_\epsilon(\xi)$ is invertible on $N_0^+$ and satisfies
   \[
   \|\lambda P_1 - Q_\epsilon(\xi)\|^{-1} \leq (\text{Re}\lambda + 1)^{-1}, \tag{2.24}
   \]
   \[
   \|P_0(v \cdot \xi)P_1(\lambda P_1 - Q_\epsilon(\xi))^{-1}P_1\|_\xi \leq C(\text{Re}\lambda + 1)^{-1}|\xi|\left(1 + \frac{|\lambda|}{1 + \epsilon|\xi|}\right)^{-1}. \tag{2.25}
   \]

Proof. By using
   \[
   \|\lambda^{-1}P_1(v \cdot \xi)\left(1 + \frac{1}{|\xi|^2}\right)P_0f\|_\xi \leq C|\lambda|^{-1}\left(|\xi| + \frac{1}{|\xi|}\right)\|P_0f\| \leq C|\lambda|^{-1}(1 + 1)|f||_\xi,
   \]
   we prove (2.23).

Then, we show that for any $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > -1$, the operator $\lambda P_1 - Q_\epsilon(\xi) = \lambda P_1 - L + i\epsilon P_1(v \cdot \xi)P_1$ is invertible from $N_0^+$ to itself. Indeed, by (1.13), we obtain for any $f \in N_0^+ \cap D(L)$ that
   \[
   \text{Re}[\lambda P_1 - L + i\epsilon P_1(v \cdot \xi)P_1]f, f = \text{Re}\lambda\langle f, f \rangle - \langle Lf, f \rangle \geq (1 + \text{Re}\lambda)||f||^2,
   \]
   which implies that the operator $\lambda P_1 - Q_\epsilon(\xi)$ is a one-to-one map from $N_0^+$ to itself so long as $\text{Re}\lambda > -1$, and its range $\text{Ran}[\lambda P_1 - Q_\epsilon(\xi)]$ is a closed subspace of $L^2(\mathbb{R}^3)$. It then remains to show that the operator $\lambda P_1 - Q_\epsilon(\xi)$ is also a surjective map from $N_0^+$ to $N_0^+$, namely, $\text{Ran}[\lambda P_1 - Q_\epsilon(\xi)] = N_0^+$. In fact, if it does not hold, then there exists a function $g \in N_0^+ \setminus \text{Ran}[\lambda P_1 - Q_\epsilon(\xi)]$ with $g \neq 0$ so that for any $f \in N_0^+ \cap D(L)$ that
   \[
   (\lambda P_1 - L + i\epsilon P_1(v \cdot \xi)P_1)g = (f, i\lambda P_1 - L - i\epsilon P_1(v \cdot \xi)P_1)g = 0,
   \]
   which yields $g = 0$ since the operator $i\lambda P_1 - L - i\epsilon P_1(v \cdot \xi)P_1$ is dissipative and satisfies the same estimate as (2.26). This is a contradiction, and thus $\text{Ran}[\lambda P_1 - Q_\epsilon(\xi)] = N_0^+$. The estimate (2.24) follows directly from (2.26).

By (2.24) and $\|P_0(v \cdot \xi)P_1f\|_\xi \leq C(|\xi| + 1)||P_1f||$, we have
   \[
   \|P_0(v \cdot \xi)P_1(\lambda P_1 - Q_\epsilon(\xi))^{-1}P_1f\|_\xi \leq C||\xi|(\text{Re}\lambda + 1)^{-1}||f||. \tag{2.27}
   \]

Meanwhile, we can decompose the operator $P_0(v \cdot \xi)P_1(\lambda P_1 - Q_\epsilon(\xi))^{-1}P_1$ as
   \[
   P_0(v \cdot \xi)P_1(\lambda P_1 - Q_\epsilon(\xi))^{-1}P_1 = \frac{1}{\lambda}P_0(v \cdot \xi)P_1 + \frac{1}{\lambda}P_0(v \cdot \xi)P_1Q_\epsilon(\xi)(\lambda P_1 - Q_\epsilon(\xi))^{-1}P_1.
   \]

This together with (2.24) and the fact $\|P_0(v \cdot \xi)P_1Q_\epsilon(\xi)f\|_\xi \leq C||\xi||(1 + \epsilon||\xi||)||P_1f||$ give
   \[
   \|P_0(v \cdot \xi)P_1(\lambda P_1 - Q_\epsilon(\xi))^{-1}P_1f\|_\xi \leq C||\xi||\lambda|^{-1}||\text{Re}\lambda + 1||^{-1} + (1 + \epsilon||\xi||)||f||. \tag{2.28}
   \]

The combination of the two cases (2.24) and (2.27) yields (2.25). \hfill \square

By (2.21) and Lemmas 2.2–2.5, we are able to analyze the spectral and resolvent sets of the operator $B_\epsilon(\xi)$ as follows.

Lemma 2.6. Fix $\epsilon \in (0, 1)$. The following facts hold.

1. For all $\xi \neq 0$, there exists $y_0 > 0$ such that
   \[
   \{\lambda \in \mathbb{C} \mid \text{Re}\lambda \geq -\frac{y_0}{2}, |	ext{Im}\lambda| \geq y_0\} \cup \{\lambda \in \mathbb{C} \mid \text{Re}\lambda > 0\} \subset \rho(B_\epsilon(\xi)). \tag{2.29}
   \]
2. For any $\delta > 0$, there exists $r_0 = r_0(\delta) > 0$ such that if $\epsilon(1 + |\xi|) \leq r_0$, then
\[
\sigma(B_\epsilon(\xi)) \cap \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq -\frac{1}{2} \} \subset \{ \lambda \in \mathbb{C} \mid |\lambda| \leq \delta \}.
\] (2.30)

Proof. By Lemma 2.23, we have for $\Re \lambda > -1$ and $\lambda \neq 0$ that the operator $\lambda P_0 + \lambda P_1 - Q_\epsilon(\xi)$ is invertible on $L^2(H)_{\xi}$ and it satisfies
\[
(\lambda P_0 + \lambda P_1 - Q_\epsilon(\xi))^{-1} = \lambda^{-1} P_0 + (\lambda P_1 - Q_\epsilon(\xi))^{-1} P_1,
\] (2.31)
because the operator $\lambda P_0$ is orthogonal to $\lambda P_1 - Q_\epsilon(\xi)$. Therefore, we can re-write (2.24) as
\[
\lambda - B_\epsilon(\xi) = (I + Y_\epsilon(\lambda, \xi))(\lambda P_0 + \lambda P_1 - Q_\epsilon(\xi)),
\]
\[
Y_\epsilon(\lambda, \xi) = i \epsilon \lambda^{-1} P_1(v \cdot \xi)(1 + \frac{1}{|\xi|^2}) P_0 + i \epsilon P_0(v \cdot \xi) P_1(\lambda P_1 - Q_\epsilon(\xi))^{-1} P_1.
\]

By Lemma 2.4, there exists $R_0 > 0$ large enough so that if $\Re \lambda \geq -i \eta_0$, and $|\lambda| \geq y_0$, then $\lambda - B_\epsilon(\xi)$ is invertible on $L^2(H)_{\xi}$ and satisfies (2.19). Thus, $\{ \lambda \in \mathbb{C} \mid \Re \lambda \geq -i \eta_0 \} \subset \rho(B_\epsilon(\xi))$ for $|\xi| \geq R_0$.

For the case $|\epsilon| \leq R_0$ such that $|\epsilon(1 + |\xi|)| \leq R_0 + 1$, by (2.25) and (2.26) we can choose $y_0 > 0$ such that it holds for $\Re \lambda \geq -1/2$ and $|\Im \lambda| \geq y_0$ that
\[
\|\epsilon \lambda^{-1} P_1(v \cdot \xi)(1 + |\xi|^{-2}) P_0\| \leq \frac{1}{4} \|eP_0(v \cdot \xi) P_1(\lambda P_1 - Q_\epsilon(\xi))^{-1} P_1\| \leq \frac{1}{4}.
\] (2.32)
This implies that the operator $I + Y_\epsilon(\lambda, \xi)$ is invertible on $L^2(H)_{\xi}$ and thus $\lambda - B_\epsilon(\xi)$ is invertible on $L^2(H)_{\xi}$ and satisfies
\[
(\lambda - B_\epsilon(\xi))^{-1} = (\lambda^{-1} P_0 + (\lambda P_1 - Q_\epsilon(\xi))^{-1} P_1)(I + Y_\epsilon(\lambda, \xi))^{-1}.
\] (2.33)
Therefore, $\rho(B_\epsilon(\xi)) \supset \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq -1/2, |\Im \lambda| \geq y_0 \}$ for $|\xi| \leq R_0$. This and Lemma 2.1 lead to (2.29).

Assume that $|\lambda| > |\delta|$ and $\Re \lambda \geq -1/2$. Then, by (2.25) and (2.26) we can choose $r_0 = r_0(\delta) > 0$ so that estimates (2.29) still hold for $|\epsilon(1 + |\xi|)| \leq r_0$, and the operator $\lambda - B_\epsilon(\xi)$ is invertible on $L^2(H)_{\xi}$. Therefore, we have $\rho(B_\epsilon(\xi)) \supset \{ \lambda \in \mathbb{C} \mid |\lambda| > \delta, \Re \lambda \geq -1/2 \}$ for $|\epsilon(1 + |\xi|)| \leq r_0$, which gives (2.30).

Now we establish the asymptotic expansions of the eigenvalues and eigenfunctions of $B_\epsilon(\xi)$ for $|\xi|$ sufficiently small. Firstly, we consider a 1-D eigenvalue problem:
\[
B_\epsilon(s)e =: \left( L - i \epsilon s v_1 - i \epsilon \frac{v_1}{s} P_0 \right) e = \beta e, \quad s \in \mathbb{R}.
\] (2.34)
Let $e$ be the eigenfunction of (2.34), we rewrite $e$ in the form $e = e_0 + e_1$, where $e_0 = P_0 e = C_0 \sqrt{M}$ and $e_1 = (I - P_0) e = P_1 e$. The eigenvalue problem (2.34) can be decomposed into
\[
\lambda e_0 = -i \epsilon s P_0[v_1(e_0 + e_1)],
\]
\[
\lambda e_1 = Le_1 - i \epsilon s P_1[v_1(e_0 + e_1)] - i \epsilon \frac{v_1}{s} e_0.
\] (2.36)
From Lemma 2.5 and 2.6, we obtain that for any $\Re \lambda > -1$,
\[
e_1 = i \epsilon (L - \lambda P_1 - i s P_1 v_1 P_1)^{-1} P_1 \left( s v_1 e_0 + \frac{v_1}{s} e_0 \right).
\] (2.37)
Substituting (2.35) into (2.36) and taking inner product the resulted equation with $\sqrt{M}$ gives
\[
\lambda C_0 = \epsilon^2 (1 + s^2) (R(\lambda, \epsilon s) v_1 \sqrt{M}, v_1 v_1 \sqrt{M}) C_0.
\] (2.38)
where
\[
R(\lambda, s) = (L - \lambda P_1 - i s P_1 v_1 P_1)^{-1}.
\]
Denote
\[
D_0(z, s, \epsilon) = z - (1 + s^2) (R(\epsilon^2 z, \epsilon s) v_1 v_1 \sqrt{M}),
\] (2.39)
Lemma 2.7. There are two constants $r_0, r_1 > 0$ such that the equation $D_0(z, s, \epsilon) = 0$ has a unique solution $z = z(s, \epsilon)$ for $\epsilon(1 + |s|) \leq r_0$ and $|z + 1 + s^2| \leq r_1(1 + s^2)$, which is a $C^\infty$ function of $s$, $\epsilon$ and satisfies

$$z(s, 0) = -(1 + s^2), \quad \partial_z z(s, 0) = 0. \tag{2.40}$$

In particular, $z(s, \epsilon)$ satisfies the following expansion:

$$z(s, \epsilon) = -(1 + s^2) + O(\epsilon^2(1 + s^2)^2). \tag{2.41}$$

Proof. By (2.39) and the fact Lemma 2.7. There are two constants $10$

In particular,

$$z(s) = -(1 + s^2). \tag{2.42}$$

Define

$$D(z, s, \epsilon) = (1 + s^2)R_{11}(\epsilon^2 z, \epsilon s),$$

with $R_{11}(\epsilon^2 z, \epsilon s) = (R(\epsilon^2 z, \epsilon s)v_1\sqrt{M}, v_1\sqrt{M})$. It is straightforward to verify that a solution of $D_0(z, s, \epsilon) = 0$ for any fixed $s$ and $\epsilon$ is a fixed point of $D(z, s, \epsilon)$.

Since for any $(z, s, \epsilon) \in \mathbb{C} \times \mathbb{R} \times \mathbb{R}$,

$$|\partial_z R_{11}(\epsilon^2 z, \epsilon s)| \leq C\epsilon^2, \quad |\partial_{z}s R_{11}(\epsilon^2 z, \epsilon s)| \leq C(|\epsilon z| + |s|),$$

it follows that

$$|D(z, s, \epsilon) - z(s)| = (1 + s^2)\left|R_{11}(\epsilon^2 z, \epsilon s) - R_{11}(0, 0)\right| \leq C(1 + s^2)|(|\epsilon z| + |\epsilon s|) \leq r_1(1 + s^2),$$

$$|D(z_1, s, \epsilon) - D(z_2, s, \epsilon)| \leq C\epsilon^2(1 + s^2)|z_1 - z_2| \leq \frac{1}{2}|z_1 - z_2|,$$

for $|z - z(s)| \leq r_1(1 + s^2)$ and $\epsilon(1 + |s|) \leq r_0$ with $r_0, r_1 > 0$ sufficiently small.

Hence by the contraction mapping theorem, there exists a unique fixed point $z(s, \epsilon) : \epsilon \in B(0, r_0/(1 + |s|)) \rightarrow B(z(s), r_1(1 + s^2))$ such that $D(z(s, \epsilon), s, \epsilon) = z(s, \epsilon)$ and $z(s, 0) = z(s)$. This is equivalent to that $D_0(z(s, s), s, \epsilon) = 0$. Since $D_0(z, s, \epsilon)$ is $C^\infty$ with respect to $z$, $s$ and $\epsilon$, it follows that $z(s, \epsilon)$ is a $C^\infty$ function with respect to $s$ and $\epsilon$.

Next, we estimate the derivative of $z(s, \epsilon)$. By (2.39) and using the fact

$$L\sqrt{M} = 0, \quad L(v_j\sqrt{M}) = -v_j\sqrt{M}, \quad LP_1(v_j^2\sqrt{M}) = -2P_1(v_j^2\sqrt{M}), \quad j = 1, 2, 3,$$

with $P_1(v_j^2\sqrt{M}) = (v_j^2 - |v_j|^2/3)\sqrt{M}$, we have

$$\partial_z D_0(z, s, 0) = 1,$$

$$\partial_z D_0(z, s, 0) = i(1 + s^2)s(P_1(v_j^2\sqrt{M}), v_1\sqrt{M}) = 0.$$

It follows that

$$\partial_z z(s, 0) = \partial_z D_0(z(s), s, 0) = 0. \tag{2.43}$$

Combining (2.42) and (2.43), we prove (2.40). Finally, by a direct computation we obtain

$$\partial_z D_0(z, s, \epsilon) = 1 + O(1)\epsilon^2(1 + s^2), \quad \partial_z D_0(z, s, \epsilon) = O(1)\epsilon(1 + s^2)^2,$$

for $\epsilon(1 + |s|) \leq r_0$ and $|z + 1 + s^2| \leq r_1(1 + s^2)$. This implies that

$$\partial_z z(s, \epsilon) = \frac{\partial_z D_0(z, s, \epsilon)}{\partial_z D_0(z(s), s, 0)} = O(1)\epsilon(1 + s^2)^2, \quad \epsilon(1 + |s|) \leq r_0,$$

which together with (2.40) leads to (2.41). This proves the lemma. \qed
With the help of Lemma 2.7, we have the eigenvalue \( \beta_0(s, \epsilon) \) and the corresponding eigenfunction \( e_0(s, \epsilon) \) of \( B_\epsilon(s) \) defined by (2.34) as follows.

**Lemma 2.8.** (1) There exists a small constant \( r_0 > 0 \) such that \( \sigma(B_\epsilon(s)) \cap \{ \lambda \in \mathbb{C} \mid \Re \lambda > -1/2 \} \) consists of one point \( \beta_0(s, \epsilon) \) for \( \epsilon(1 + |s|) \leq r_0 \). The eigenvalue \( \beta_0(s, \epsilon) \) is a \( C^\infty \) function of \( s \) and \( \epsilon \), and admit the following asymptotic expansion for \( \epsilon(1 + |s|) \leq r_0 \):

\[
\beta_0(s, \epsilon) = -\epsilon^2(1 + s^2) + O(\epsilon^4(1 + s^2)^2). \tag{2.44}
\]

(2) The corresponding eigenfunction \( e_0(s, \epsilon) = e_0(s, \epsilon, v) \) is \( C^\infty \) in \( s \) and \( \epsilon \) satisfying

\[
\begin{cases}
P_0 e_0(s, \epsilon) = \frac{s}{\sqrt{1 + s^2}}\sqrt{M} + O(\epsilon s), \\
P_1 e_0(s, \epsilon) = -i\epsilon\sqrt{1 + s^2}v_1\sqrt{M} + O(\epsilon^2(1 + s^2)).
\end{cases} \tag{2.45}
\]

**Proof.** The eigenvalue \( \beta_0(s, \epsilon) \) and the eigenfunction \( e_0(s, \epsilon) \) can be constructed as follows. We take \( \beta_0 = \epsilon^2 z(s, \epsilon) \) with \( z(s, \epsilon) \) being the solution of the equation \( D_0(z, s, \epsilon) = 0 \) given in Lemma 2.7, and define the corresponding eigenfunction \( e_0(s, \epsilon) \) by

\[
e_0(s, \epsilon) = a_0(s, \epsilon)\sqrt{M} + i\epsilon \left( s + \frac{1}{s} \right) a_0(s)\left(L - \epsilon^2 z(s, \epsilon) - i\epsilon v_1 v_1^{-1}v_1\sqrt{M}, \right.
\]

where \( a_0(s, \epsilon) \) is a complex value function determined later. We can normalize \( e_0(s, \epsilon) \) by taking

\[
\left( e_0(s, \epsilon), \overline{e_0(s, \epsilon)} \right)_s = (e_0, \overline{e_0}) + \frac{1}{s^2}(P_0 e_0, P_0\overline{e_0}) = 1.
\]

The coefficient \( a_0(s, \epsilon) \) is determined by the normalization condition as

\[
a_0(s, \epsilon)^2 \left( 1 + \frac{1}{s^2} + \epsilon^2 \left( s + \frac{1}{s} \right)^2 D_1(s, \epsilon) \right) = 1, \tag{2.47}
\]

where \( D_1(s, \epsilon) = (R(\beta_0, \epsilon)s v_1\sqrt{M}, R(\beta_0, -\epsilon)s v_1\sqrt{M}) \). Substituting (2.44) into (2.47), we obtain

\[
a_0(s, \epsilon) = \frac{s}{\sqrt{1 + s^2}} \left[ 1 + O(\epsilon\sqrt{1 + s^2}) \right]. \tag{2.48}
\]

Combining (2.46) and (2.48), we can obtain the expansion of \( e_0(s, \epsilon) \) given in (2.45). This completes the proof of the lemma. \( \square \)

Now we consider a 3-D eigenvalue problem:

\[
B_\epsilon(\xi)\psi = \left( L - i\epsilon v \cdot \xi - i\frac{v \cdot \xi}{|\xi|^2} P_0 \right) \psi = \lambda \psi, \quad \xi \in \mathbb{R}^3. \tag{2.49}
\]

With the help of Lemma 2.8, we have the eigenvalue \( \lambda_0(|\xi|, \epsilon) \) and the corresponding eigenfunction \( \psi_0(\xi, \epsilon) \) of \( B_\epsilon(\xi) \) defined by (2.49) as follows.

**Theorem 2.9.** (1) There exists a small constant \( r_0 > 0 \) such that \( \sigma(B_\epsilon(\xi)) \cap \{ \lambda \in \mathbb{C} \mid \Re \lambda > -1/2 \} \) consists of one point \( \lambda_0(|\xi|, \epsilon) \) for \( \epsilon(1 + |\xi|) \leq r_0 \). The eigenvalue \( \lambda_0(|\xi|, \epsilon) \) is a \( C^\infty \) function of \( |\xi| \) and \( \epsilon \) for \( \epsilon(1 + |\xi|) \leq r_0 \):

\[
\lambda_0(|\xi|, \epsilon) = \beta_0(|\xi|, \epsilon) = -\epsilon^2(1 + |\xi|^2) + O(\epsilon^4(1 + |\xi|^2)^2), \tag{2.50}
\]

where \( \beta_0(s, \epsilon) \) is a \( C^\infty \) function of \( s \) and \( \epsilon \) given in Lemma 2.8.

(2) The eigenfunction \( \psi_0(\xi, \epsilon) = \psi_0(\xi, \epsilon, v) \) satisfies

\[
\begin{cases}
P_0 \psi_0(\xi, \epsilon) = \frac{|\xi|}{\sqrt{1 + |\xi|^2}}\sqrt{M} + O(|\xi|), \\
P_1 \psi_0(\xi, \epsilon) = -i\epsilon\sqrt{1 + |\xi|^2} \left( \frac{v \cdot \xi}{|\xi|^2} \right) \sqrt{M} + O(\epsilon^2(1 + |\xi|^2)).
\end{cases} \tag{2.51}
\]
Proof. Let $\mathcal{O}$ be a rotational transformation in $\mathbb{R}^3$ such that $\mathcal{O} : \frac{\xi}{|\xi|} \rightarrow (1, 0, 0)$. We have

$$\mathcal{O}^{-1} \left( L - i e v \cdot \xi - i \frac{v}{|\xi|^2} P_0 \right) \mathcal{O} = L - i e s u_1 - i \frac{v_1}{s} P_0.$$  \hfill (2.52)

From Lemma 2.8 we have the following eigenvalue and eigenfunction for (2.49):

$$(L - i e v \cdot \xi - i \frac{v}{|\xi|^2} P_0) \psi_0(\xi, \epsilon) = \lambda_0(|\xi|, \epsilon) \psi_0(\xi, \epsilon),$$

where $\lambda_0(|\xi|, \epsilon) = \beta_0(|\xi|, \epsilon), \quad \psi_0(\xi, \epsilon) = \mathcal{O} \psi_0(|\xi|, \epsilon)$. This proves the theorem. \hfill \qed

By virtue of Lemmas 2.3, 2.4, and Theorem 2.9 we can make a detailed analysis on the semigroup $S(t, \xi, \epsilon) = e^{-t \mathcal{B}_r(\xi)}$ in terms of an argument similar to that of Theorem 3.4 in [21], we give the details of the proof in Appendix.

**Theorem 2.10.** Let $\epsilon > 0$ be given in Theorem 2.9. For any fixed $\epsilon \in (0, \epsilon_0)$, the semigroup $S(t, \xi, \epsilon) = e^{-t \mathcal{B}_r(\xi)}$ with $\xi \neq 0$ can be decomposed into

$$S(t, \xi, \epsilon) = S_1(t, \xi, \epsilon) f + S_2(t, \xi, \epsilon) f, \quad f \in L^2(\mathbb{R}^3), \quad t > 0,$$

where

$$S_1(t, \xi, \epsilon) f = e^{-t \lambda_0(|\xi|, \epsilon) t} \left( f, \frac{1}{\xi} \psi_0(\xi, \epsilon) \right) \psi_0(\xi, \epsilon) 1_{\{e^{(1 + |\xi|)} \leq \epsilon \}},$$

with $(\lambda_0(|\xi|, \epsilon), \psi_0(\xi, \epsilon))$ being the eigenvalue and eigenfunction of the operator $\mathcal{B}_r(\xi)$ given in Theorem 2.9 for $e^{(1 + |\xi|)} \leq \epsilon$, and $S_2(t, \xi, \epsilon)$ satisfies for two constants $a > 0$ and $C > 0$ independent of $\xi$ and $\epsilon$

$$\|S_2(t, \xi, \epsilon) f\| \leq C e^{-\frac{t}{|\xi|^2}} \|f\|.$$  \hfill (2.55)

**3 Fluid approximations of semigroup**

In this section, we give the first and second order fluid approximations of the semigroup $e^{-t \mathcal{B}_r}$, which will be used to prove the convergence and establish the convergence rate of the solution to the VPFP system towards the solution to the DDP system.

Let us introduce a Sobolev space of function $f = f(x, \nu)$ by $H^k_\nu = \{ f \in L^2(\mathbb{R}^3 \times \mathbb{R}^3_\nu) \mid \|f\|_{H^k_\nu} < \infty \}$ ($L^2_\nu = H^0_\nu$) with the norm $\| \cdot \|_{H^k_\nu}$ defined by

$$\|f\|_{H^k_\nu} = \left( \int_{\mathbb{R}^3} (1 + |\xi|^2)^k \|f\|_\xi^2 d\xi \right)^{1/2}$$

$$= \left( \int_{\mathbb{R}^3} (1 + |\xi|^2)^k \left( \|f\|_\xi^2 + \frac{1}{|\xi|^2} \left| (\hat{f}, \sqrt{M}) \right|^2 \right) d\xi \right)^{1/2},$$

where $\hat{f} = \hat{f}(\xi, \nu)$ denotes the Fourier transform of $f(x, \nu)$ with respect to $x \in \mathbb{R}^3$. Note that it holds

$$\|f\|_{H^k_\nu}^2 = \|f\|_{H^0_\nu}^2 + \|\nabla_x \Delta_x^{-1} (f, \sqrt{M})\|_{H^2_\nu}^2.$$  \hfill (3.1)

For any $f_0 \in L^2(\mathbb{R}^3_\nu)$, set

$$e^{-t \mathcal{B}_r} f_0 = (F^{-1} e^{-t \mathcal{B}_r} \nu)(\mathcal{F} f_0).$$

By Lemma 2.1 we have

$$\|e^{-t \mathcal{B}_r} f_0\|_{H^k_\nu} = \int_{\mathbb{R}^3} (1 + |\xi|^2)^k \|e^{-t \mathcal{B}_r(\xi)} f_0\|_\xi^2 d\xi \leq \int_{\mathbb{R}^3} (1 + |\xi|^2)^k \|f_0\|_\xi^2 d\xi = \|f_0\|_{H^k_\nu}.$$  \hfill (3.2)
By combining (3.7)–(3.11), we obtain (3.5).

Indeed, it follows from (5.1) that for \( \psi \)

\[
\text{Lemma 3.1.} \quad \text{For any} \quad f \quad \text{and therefore,} \quad \text{where} \quad \epsilon \quad \text{This means that the linear operator} \quad \epsilon^{-2} B_\epsilon \quad \text{generates a strongly continuous contraction semigroup} \quad e^{\frac{t}{\epsilon} B_\epsilon} \quad \text{in} \quad H^k_p, \quad \text{and therefore,} \quad f(t, x, v) = e^{\frac{t}{\epsilon} B_\epsilon} f_0 \quad \text{is a global solution to the linearized VPFP system (2.7) for any} \quad f_0 \in H^k_p.

Now we are going to establish the first and second order fluid approximations of the semigroup \( e^{\frac{t}{\epsilon} B_\epsilon} \). First of all, we introduce the following linearized DDP system to (1.12)–(1.14):

\[
\begin{aligned}
\partial_t n &= Dn, \quad n(0, x) = n_0(x), \\
\end{aligned}
\]

where

\[
D = -I + \Delta_x.
\]

It is easy to verify that the operator \( D \) generates a strongly continuous contraction semigroup \( e^{tD} \) in \( H^k_x \). Thus, \( n(t, x) = e^{tD} n_0 \) is a global solution to the linearized DDP system \( e^{\frac{t}{\epsilon} B_\epsilon} \) for any \( n_0 \in H^k_x \).

We give a prepare lemma which will be used to study the fluid dynamical approximations of the semigroup \( e^{\frac{t}{\epsilon} B_\epsilon} \).

\[
\text{Lemma 3.1.} \quad \text{For any} \quad f_0 \in N_0, \quad \text{we have}
\]

\[
\| S_2(t, \xi, \epsilon) f_0 \| \leq C \left( \epsilon (1 + |\xi|) 1_{\{ |\epsilon (1 + |\xi|) \leq r_0 \}} + 1_{\{ |\epsilon (1 + |\xi|) \geq r_0 \}} \right) e^{-\frac{t}{\epsilon^2}} \| f_0 \|_{\xi}.
\]

\[
\text{Proof.} \quad \text{Define the projection} \quad P_\epsilon(\xi) \quad \text{by}
\]

\[
P_\epsilon(\xi) f = \left( f, \psi_0(\xi, \epsilon) \right)_\xi \psi_0(\xi, \epsilon), \quad \forall f \in L^2(\mathbb{R}^3_x),
\]

where \( \psi_0(\xi, \epsilon) \) is the eigenfunctions of \( B_\epsilon(\xi) \) for \( \epsilon (1 + |\xi|) \leq r_0 \) defined by (2.5).

By Theorem 2.10 we can assert that

\[
S_1(t, \xi, \epsilon) = S(t, \xi, \epsilon) 1_{\{ |\epsilon (1 + |\xi|) \leq r_0 \}} P_\epsilon(\xi).
\]

Indeed, it follows from (5.4) that for \( \epsilon (1 + |\xi|) \leq r_0 \),

\[
e^{\frac{t}{\epsilon} B_\epsilon(\xi)} P_\epsilon(\xi) f = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\lambda t} (\lambda - B_\epsilon(\xi))^{-1} P_\epsilon(\xi) f d\lambda = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\lambda t} (\lambda - \lambda_0(|\xi|, \epsilon))^{-1} P_\epsilon(\xi) f d\lambda = e^{\frac{t}{\epsilon^2} \lambda_0(|\xi|, \epsilon)} (f, \psi_0(\xi, \epsilon))_\xi \psi_0(\xi, \epsilon) = S_1(t, \xi, \epsilon) f.
\]

By (5.6), we can decompose \( S_2(t, \xi, \epsilon) \) into

\[
S_2(t, \xi, \epsilon) = S_{21}(t, \xi, \epsilon) + S_{22}(t, \xi, \epsilon),
\]

where

\[
S_{21}(t, \xi, \epsilon) = S(t, \xi, \epsilon) 1_{\{ |\epsilon (1 + |\xi|) \leq r_0 \}} (I - P_\epsilon(\xi)),
\]

\[
S_{22}(t, \xi, \epsilon) = S(t, \xi, \epsilon) 1_{\{ |\epsilon (1 + |\xi|) \geq r_0 \}}.
\]

It holds that

\[
\| S_{2j}(t, \xi, \epsilon) f_0 \| \leq C e^{-\frac{t}{\epsilon^2}} \| f_0 \|_{\xi}, \quad j = 1, 2.
\]

Since for any \( f_0 \in N_0 \),

\[
f_0 - P_\epsilon(\xi) f_0 = (f_0, h_0(\xi))_\xi h_0(\xi) - (f_0, \psi_0(\xi, \epsilon))_\xi \psi_0(\xi, \epsilon),
\]

where

\[
h_0(\xi) = \frac{|\xi|}{\sqrt{1 + |\xi|}} \sqrt{M},
\]

it follows from (2.6) and the the fact \( S_{21}(t, \xi, \epsilon) = S_{21}(t, \xi, \epsilon)(I - P_\epsilon(\xi)) \) that

\[
\| S_{21}(t, \xi, \epsilon) f_0 \| \leq C (1 + |\xi|) 1_{\{ |\epsilon (1 + |\xi|) \leq r_0 \}} e^{-\frac{t}{\epsilon^2}} \| f_0 \|_{\xi}.
\]

By combining (3.7)–(3.11), we obtain (3.5).
Lemma 3.2. For any integer \( k \geq 0 \) and any \( f_0 \in H_p^{k+1} \), we have
\[
\left\| e^{B_t} f_0 - e^{tD n_0 \sqrt{M}} \right\|^2_{L_2^p(L_2^q)} \leq C \left( e^{-\frac{t}{2}} + e^{-\frac{2t}{2}} \right) \left\| f_0 \right\|_{H_p^{k+1}},
\]
where \( n_0 = (f_0, \sqrt{M}) \), and \( C > 0 \) are two constants independent of \( \epsilon \). Moreover, if \( f_0 \in N_0 \), then
\[
\left\| e^{B_t} f_0 - e^{tD n_0 \sqrt{M}} \right\|^2_{H_p^k} \leq C e^{-\frac{t}{2}} \left\| f_0 \right\|_{H_p^{k+1}}.
\]

Proof. For simplicity, we only prove the case of \( k = 0 \). By Theorem 2.10 and taking \( \epsilon \leq r_0/2 \) with \( r_0 > 0 \) given in Theorem 2.10, we have
\[
\left\| e^{B_t} f_0 - e^{tD n_0 \sqrt{M}} \right\|^2_{L_2^p(L_2^q)} = \int_{\mathbb{R}^3} \left\| e^{B_t}(\xi) \right\|^2 d\xi = \int_{\mathbb{R}^3} \left\| e^{D(\xi)} \hat{n}_0 \right\|^2 d\xi \leq 3 \int_{1+|\xi| \leq \frac{2}{\epsilon}} \left\| S_1(t, \xi, \epsilon) f_0 - e^{tD(\xi)} \hat{n}_0 \right\|^2 d\xi + 3 \int_{|\xi| \geq \frac{2}{\epsilon}} \left\| e^{tD(\xi)} \hat{n}_0 \right\|^2 d\xi = I_1 + I_2 + I_3,
\]
where \( D(\xi) = -(1 + |\xi|^2) \) is the Fourier transform of the operator \( D \).

We estimate \( I_j \), \( j = 1, 2, 3 \) as follows. Since from (2.5) and (2.56) and (2.51),
\[
S_1(t, \xi, \epsilon) \hat{f}_0 = e^{-(1+|\xi|^2)+O(\epsilon^2(1+|\xi|^2)^2)} \left( \hat{n}_0 \sqrt{M} + O(\epsilon \sqrt{1+|\xi|^2}) \right),
\]
it follows that
\[
I_1 = 3 \int_{1+|\xi| \leq \frac{2}{\epsilon}} \left\| e^{-(1+|\xi|^2)+O(\epsilon^2(1+|\xi|^2)^2)} \left( \hat{n}_0 \sqrt{M} + O(\epsilon \sqrt{1+|\xi|^2}) \right) - e^{-(1+|\xi|^2) t} \hat{n}_0 \sqrt{M} \right\|^2 d\xi \leq C \epsilon^2 \int_{1+|\xi| \leq \frac{2}{\epsilon}} \left( r_0^2 (1 + |\xi|^2)^2 |\hat{n}_0|^2 + (1 + |\xi|^2) \| \hat{f}_0 \|^2 \right) d\xi \leq C \epsilon^2 e^{-t} \| \hat{f}_0 \|^2_{L_2^p(L_2^q)}.
\]
By (2.55), we have
\[
I_2 \leq C \int_{\mathbb{R}^3} e^{-\frac{t}{2}} \| \hat{f}_0 \|^2 d\xi \leq C e^{-\frac{t}{2}} \left( \| f_0 \|^2_{L_2^p(L_2^q)} + \| \nabla_x \Phi_0 \|^2_{L_2^p(L_2^q)} \right),
\]
For \( I_3 \), it holds that
\[
I_3 = 3 \int_{1+|\xi| \geq \frac{2}{\epsilon}} e^{-2(1+|\xi|^2) t} |\hat{n}_0|^2 d\xi \leq C e^{-\frac{t}{2}} \| f_0 \|^2_{L_2^p(L_2^q)}.
\]
Therefore, it follows from (3.6)–(3.11) that
\[
\left\| e^{B_t} f_0 - e^{tD n_0 \sqrt{M}} \right\|^2_{L_2^p(L_2^q)} \leq C \left( e^{2e^{-t}} + e^{-2\frac{t}{2}} \right) \| f_0 \|^2_{H_p^k}.
\]
Proof. For simplicity, we only prove the case of $k = 1$. Thus, by (3.15), (3.20) and (3.21) we obtain (3.13). This proves the lemma.

We estimate $I_2$ as follows. Since for any $f_0 \in L^2_m$, by Lemma 3.1, we have

$$I_2 \leq C \int_{1+|\xi| \geq \frac{M}{2}} e^2(1 + |\xi|^2) e^{-\frac{2}{\epsilon} |\xi|_e t} f_0^2 d\xi \leq C e^2 e^{-\frac{2}{\epsilon} |\xi|_e t} \|f_0\|^2_{H^2(L^2)}.$$  \hspace{1cm} (3.20)

Moreover, we can bound $I_3$ by

$$I_3 = 3 \int_{1+|\xi| \geq \frac{M}{2}} e^{-2(1+|\xi|^2) t} |\xi|_e^2 d\xi \leq C e^2 e^{-\frac{2}{\epsilon} t} \|f_0\|^2_{H^2(L^2)}.$$  \hspace{1cm} (3.21)

Thus, by (3.15), (3.20) and (3.21) we obtain (3.13). This proves the lemma. \hfill \Box

We have the second order fluid approximation of the semigroup $e^{tB_1}$ as follow.

**Lemma 3.3.** For any integer $k \geq 0$ and any $f_0 \in H^{k+1}_p$ satisfying $P_0 f_0 = 0$,

$$ \left\| \frac{1}{\epsilon} e^{tB_1 f_0} + e^{tD} \text{div}_x m_0 \sqrt{M} \right\|_{H^p_0} \leq C \left( e^{-\frac{1}{2} t} + \frac{1}{\epsilon} e^{-\frac{2}{\epsilon} t} \right) \|f_0\|_{H^{k+1}_2},$$  \hspace{1cm} (3.22)

where $m_0 = (f_0, v \sqrt{M})$, and $a > 0$, $C > 0$ are two constants independent of $\epsilon$.

**Proof.** For simplicity, we only prove the case of $k = 0$. By Theorem 2.10 and taking $\epsilon \leq r_0/2$ with $r_0 > 0$ given in Lemma 2.8, we have

$$ \left\| \frac{1}{\epsilon} e^{tB_1 f_0} + e^{tD} \text{div}_x m_0 \sqrt{M} \right\|^2_{L^2_t(L^2_x)} \leq 3 \int_{1+|\xi| \leq \frac{M}{2}} \left\| \frac{1}{\epsilon} S_1(t, \xi, e) f_0 + e^{tD}(\hat{m}_0 \cdot \xi) \sqrt{M} \right\|^2 d\xi \leq 3 \int_{R^3} \left\| \frac{1}{\epsilon} S_2(t, \xi, e) f_0 \right\|^2 d\xi + 3 \int_{1+|\xi| \geq \frac{M}{2}} \left| e^{tD}(\hat{m}_0 \cdot \xi) \right|^2 d\xi \leq I_1 + I_2 + I_3.$$  \hspace{1cm} (3.23)

We estimate $I_j$, $j = 1, 2, 3$ as follows. Since for any $f_0 \in L^2_x(L^2_v)$ satisfying $P_0 f_0 = 0$,

$$S_1(t, \xi, e) f_0 = e^{-(1+|\xi|^2) t + O\epsilon^2(1+|\xi|^2)^2 t} \left[ -e(\hat{m}_0 \cdot \xi) \sqrt{M} + O\epsilon^2(1+|\xi|^2) \right],$$

it follows that

$$I_1 = 3 \int_{1+|\xi| \leq \frac{M}{2}} \left\| e^{-(1+|\xi|^2) t + O\epsilon^2(1+|\xi|^2)^2 t} \left[ (\hat{m}_0 \cdot \xi) \sqrt{M} - O\epsilon(1+|\xi|^2) \right] \right\|^2 d\xi$$

$$\leq C e^2 \int_{1+|\xi| \leq \frac{M}{2}} e^{-(1+|\xi|^2) t + O\epsilon^2(1+|\xi|^2)^2 t} \left[ \left( \frac{3}{2}(1+|\xi|^2)^2 + (1+|\xi|^2)^2 \|f_0\|^2 \right) \right] d\xi \leq C e^{2 t} e^{-t} \|f_0\|^2_{H^2(L^2)}.$$  \hspace{1cm} (3.24)
By (2.35) and $P_0f_0 = 0$, we have

$$ I_2 \leq C \int_{\mathbb{R}^3} \frac{1}{\epsilon^2} e^{-2\epsilon t} \|f_0\|^2 d\xi \leq C \frac{1}{\epsilon^2} e^{-2\epsilon t} \|f_0\|_{L^2(L_2)}^2. \tag{3.25} $$

For $I_3$, it holds that

$$ I_3 = 3 \int_{1+|\xi| \geq \frac{1}{\epsilon^2}} e^{-2(1+|\xi|)^2} |(\tilde{m}_0 \cdot \xi)|^2 d\xi \leq C e^{-\frac{2}{\epsilon^2} t} \|f_0\|_{H^1_2(L_2)}. \tag{3.26} $$

Therefore, it follows from (3.25) and (3.26) that

$$ \left\| \frac{1}{\epsilon} e^{\frac{\epsilon}{2} B_t} f_0 + e^{tD} \text{div}_x m_0 \sqrt{M} \right\|_{L^2_2(L_2)}^2 \leq C \left( e^{2t-1} e^{-t} + \frac{1}{\epsilon^2} e^{-2\epsilon t} \right) \|f_0\|_{H^1_2(L_2)}. \tag{3.27} $$

Similarly, we have

$$ \left\| \frac{1}{\epsilon} \nabla_x \Delta_x^{-1} (e^{\frac{\epsilon}{2} B_t} f_0, \sqrt{M}) - \nabla_x \Delta_x^{-1} e^{tD} \text{div}_x m_0 \right\|_{L^2_2(L_2)}^2 \leq 3 \int_{1+|\xi| \leq \frac{1}{\epsilon^2}} \left| \frac{\epsilon}{1+|\xi|^2} (S_1(t, \xi, \epsilon) \tilde{f}_0, \sqrt{M}) - \frac{\xi}{|\xi|^2} e^{tD(\xi)} (\tilde{m}_0 \cdot \xi) \right|^2 d\xi + \frac{1}{\epsilon^2} e^{-2\epsilon t} \|f_0\|_{H^1_2(L_2)}. \tag{3.28} $$

Combining (3.27) and (3.28), we obtain (3.22). This proves the lemma.

4 Diffusion limit

In this section, we study the diffusion limit of the solution to the nonlinear VPFP system (1.7)–(1.9) based on the fluid approximations of the solution to the linear VPFP system given in Section 3.

Since the operators $B_\epsilon$ and $A$ generate contraction semigroups in $H^k_0$ and $H^k_\infty$ ($k \geq 0$) respectively, the solution $f_\epsilon(t)$ to the VPFP system (1.7)–(1.9) and the solution $n(t)$ to the DDP system (1.12)–(1.14) can be represented by

$$ f_\epsilon(t) = e^{\frac{\epsilon}{2} B_t} f_0 + \int_0^t \frac{1}{\epsilon} e^{\frac{\epsilon}{2} B_s} G(f_\epsilon(s)) ds, \tag{4.1} $$

$$ n(t) = e^{tD} n_0 - \int_0^t e^{(t-s)D} \text{div}_x (n \nabla_x \Phi)(s) ds, \tag{4.2} $$

where

$$ n_0 = \int_{\mathbb{R}^3} f_0 \sqrt{M} dv. $$

By taking inner product between $\sqrt{M}$ and $f_\epsilon$, we obtain

$$ \partial_t f_\epsilon + \frac{1}{\epsilon} \text{div}_x m_\epsilon = 0, \tag{4.3} $$

where

$$ n_\epsilon = (f_\epsilon, \sqrt{M}), \quad m_\epsilon = (f_\epsilon, v \sqrt{M}). $$

Taking the microscopic projection $P_1$ to $L^2$, we have

$$ \partial_t (P_1 f_\epsilon) + \frac{1}{\epsilon} P_1 (v \cdot \nabla_x P_1 f_\epsilon) - \frac{1}{\epsilon} v \sqrt{M} \cdot \nabla_x \Phi_\epsilon = \frac{1}{\epsilon^2} L(P_1 f_\epsilon) $$
Similarly, taking the inner product between $\partial_t P_1 f_\epsilon$ as

$$P_1 f_\epsilon = L^{-1}[\epsilon^2 \partial_t (P_1 f_\epsilon) + \epsilon P_1 (v \cdot \nabla_x P_1 f_\epsilon) - \epsilon P_1 G] - \epsilon v \sqrt{M} \cdot (\nabla_x n_\epsilon - \nabla_x \Phi_\epsilon).$$

(4.5)

Substituting (1.5) into (1.3), we obtain

$$\partial_t n_\epsilon + \epsilon \partial_t \text{div}_x m_\epsilon + n_\epsilon - \Delta_x n_\epsilon = -\text{div}_x (v \cdot \nabla_x P_1 f_\epsilon - P_1 G, \sqrt{M}).$$

(4.6)

Define the energy $E(f_\epsilon)$ and the dissipation $D(f_\epsilon)$ by

$$E(f_\epsilon) = \|f_\epsilon(t)\|^2_{H_2^t(L^2)} + \|\nabla_x \Phi_\epsilon(t)\|^2_{H_2^t},$$

$$D(f_\epsilon) = \frac{1}{\epsilon^2} \|P_1 f_\epsilon\|^2_{H_2^t(L^2)} + \|P_0 f_\epsilon\|^2_{H_2^t(L^2)} + \|\nabla_x \Phi_\epsilon\|^2_{H_2^t},$$

where $\nabla_x \Phi_\epsilon = \nabla_x \Delta_\epsilon^{-1}(f_\epsilon, \sqrt{M})$.

First, we establish the following energy estimate for the solution $f_\epsilon$ to the system (1.7)–(1.9).

**Lemma 4.1.** For any $\epsilon \ll 1$, there exists a small constant $\delta_0 > 0$ such that if $\|f_0\|_{H_2^t(L^2)} + \|(f_0, \sqrt{M})\|_{L^2} \leq \delta_0$, then the system (1.7)–(1.9) admits a unique global solution $f_\epsilon(t) = f_\epsilon(t, x, v)$ satisfying the following energy estimate:

$$E(f_\epsilon(t)) + \int_0^t D(f_\epsilon(s)) ds \leq C\delta_0^2,$$

(4.7)

where $C > 0$ is a constant independent of $\epsilon$. Moreover, $f_\epsilon(t)$ has the following time-decay rate:

$$E(f_\epsilon(t)) \leq C\delta_0^2 e^{-t}.$$

(4.8)

**Proof.** First, we establish the macroscopic energy estimate of $f_\epsilon$. Taking the inner product between $\partial_x^\alpha n_\epsilon$ and $\partial_x^\alpha \Phi_\epsilon$ with $|\alpha| \leq 3$, we have

$$\frac{d}{dt} \|\partial_x^\alpha n_\epsilon\|^2_{L^2} + 2 \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha \text{div}_x m_\epsilon \partial_x^\alpha n_\epsilon dx + \frac{3}{2} \left( \|\partial_x^\alpha n_\epsilon\|^2_{L^2} + \|\partial_x^\alpha \nabla_x n_\epsilon\|^2_{L^2} \right) \leq C \|\partial_x^\alpha \nabla_x P_1 f_\epsilon\|^2_{L^2} + C E(f_\epsilon) D(f_\epsilon).$$

(4.9)

Similarly, taking the inner product between $-\partial_x^\alpha \Phi_\epsilon$ and $\partial_x^\alpha \Phi_\epsilon$ with $|\alpha| \leq 3$ we obtain

$$\frac{d}{dt} \|\partial_x^\alpha \nabla_x \Phi_\epsilon\|^2_{L^2} + 2 \frac{d}{dt} \int_{\mathbb{R}^3} \partial_x^\alpha m_\epsilon \partial_x^\alpha \nabla_x \Phi_\epsilon dx + \frac{3}{2} \left( \|\partial_x^\alpha \nabla_x \Phi_\epsilon\|^2_{L^2} + \|\partial_x^\alpha \nabla_x \Phi_\epsilon\|^2_{L^2} \right) \leq C \left( \|\partial_x^\alpha P_1 f_\epsilon\|^2_{L^2} + \|\partial_x^\alpha \nabla_x P_1 f_\epsilon\|^2_{L^2} \right) + C E(f_\epsilon) D(f_\epsilon).$$

(4.10)

Taking the summation of (1.9) + (1.10) with $|\alpha| \leq 3$, we obtain

$$\frac{d}{dt} \left( \|n_\epsilon\|^2_{H_2^t} + \|\nabla_x \Phi_\epsilon\|^2_{H_2^t} \right) + \frac{3}{2} \left( \|\nabla_x n_\epsilon\|^2_{H_2^t} + 2 \|n_\epsilon\|^2_{H_2^t} + \|\nabla_x \Phi_\epsilon\|^2_{H_2^t} \right) + \frac{d}{dt} \sum_{|\alpha| \leq 3} \epsilon \left( 2 \int_{\mathbb{R}^3} \partial_x^\alpha \text{div}_x m_\epsilon \partial_x^\alpha n_\epsilon dx + 2 \int_{\mathbb{R}^3} \partial_x^\alpha m_\epsilon \partial_x^\alpha \nabla_x \Phi_\epsilon dx \right) \leq C E(f_\epsilon) D(f_\epsilon) + C \|P_1 f_\epsilon\|^2_{H_2^t(L^2)}.$$ 

(4.11)

Next, we deal with the microscopic energy estimate of $f_\epsilon$. By taking the inner product between $\partial_x^\alpha f_\epsilon$ and $\partial_x^\alpha \Phi_\epsilon$ with $|\alpha| \leq 4$, we have

$$\frac{1}{2} \frac{d}{dt} \left( \|\partial_x^\alpha f_\epsilon\|^2_{L^2} + \|\partial_x^\alpha \nabla_x \Phi_\epsilon\|^2_{L^2} \right) - \frac{1}{\epsilon^2} \int_{\mathbb{R}^3} (L \partial_x^\alpha f_\epsilon) \partial_x^\alpha f_\epsilon dv dx$$
\[
\frac{1}{2e} \int_{\mathbb{R}^3} \partial_{x}^\alpha (v \cdot \nabla_x \Phi \cdot f_\epsilon) \partial_{x}^\beta P_1 f_\epsilon \, dx \, dv - \frac{1}{\epsilon} \int_{\mathbb{R}^3} \partial_{x}^\alpha (\nabla_x \Phi \cdot \nabla_v f_\epsilon) \partial_{x}^\beta P_1 f_\epsilon \, dx \, dv
\]
\[
\leq \frac{\mu}{2e^2} \| \partial_{x}^\alpha P_1 f_\epsilon \|_{L^2(\mathbb{R}^3)}^2 + CE(f_\epsilon) D(f_\epsilon),
\]
(4.12)
which leads to
\[
\frac{d}{dt} \left( \| f_\epsilon \|_{H^2(\mathbb{R}^3)}^2 + \| \nabla_x \Phi \|_{H^2}^2 \right) + \frac{\mu}{e^2} \| P_1 f_\epsilon \|_{H^2(\mathbb{R}^3)}^2 \leq CE(f_\epsilon) D(f_\epsilon).
\]
(4.13)
Taking the summation of C.1 and C.13 with C.1 > 0 large and \( \epsilon > 0 \) small to get
\[
\frac{d}{dt} \left( \| f_\epsilon \|_{H^2(\mathbb{R}^3)}^2 + C_1 \| n_\epsilon \|_{H^2}^2 + (C_1 + 1) \| \nabla_x \Phi \|_{H^2}^2 \right) + \frac{d}{dt} C_1 \sum_{|\alpha| \leq 3} \epsilon \left( 2 \int_{\mathbb{R}^3} \partial_{x}^\alpha \text{div}_x m_\epsilon \partial_{x}^\beta n_\epsilon \, dx + 2 \int_{\mathbb{R}^3} \partial_{x}^\alpha m_\epsilon \partial_{x}^\beta \nabla_x \Phi \, dx \right)
\]
\[
+ \frac{\mu}{e^2} \| P_1 f_\epsilon \|_{H^2(\mathbb{R}^3)}^2 + \frac{3}{2} C_1 \left( \| \nabla_x n_\epsilon \|_{H^2}^2 + 2 \| n_\epsilon \|_{H^2}^2 + \| \nabla_x \Phi \|_{H^2}^2 \right)
\]
\[
\leq CE(f_\epsilon) D(f_\epsilon).
\]
(4.14)
Assume that \( E(f_\epsilon) \leq C \delta_0 \). It follows from (4.14) that for \( \epsilon > 0 \) and \( \delta_0 > 0 \) sufficiently small, there exist two functionals \( E_1(f_\epsilon) \sim E(f_\epsilon) \) and \( D_1(f_\epsilon) \sim D(f_\epsilon) \) such that
\[
\frac{d}{dt} E_1(f_\epsilon) + D_1(f_\epsilon) \leq 0, \quad D_1(f_\epsilon) \geq E_1(f_\epsilon).
\]
(4.15)
Since
\[
\| \nabla_x \Phi \|_{H^2}^2 = \int_{\mathbb{R}^3} \frac{1}{|\xi|^2} (1 + |\xi|^2)^2 (\hat{f}_0, \sqrt{M})^2 d\xi
\]
\[
\leq C \sup_{|\xi| \leq 1} \| (\hat{f}_0, \sqrt{M}) \|^2 \int_{|\xi| \leq 1} \frac{1}{|\xi|^2} d\xi + \int_{|\xi| \geq 1} (1 + |\xi|^2)^2 \| (\hat{f}_0, \sqrt{M}) \|^2 d\xi
\]
\[
\leq C \left( \| f_0 \|_{H^2(\mathbb{R}^3)}^2 + \| (f_0, \sqrt{M}) \|_{L^2}^2 \right),
\]
we can prove (4.17) and (4.18) by (4.15) for \( \epsilon > 0 \) and \( \delta_0 > 0 \) sufficiently small. This completes the proof of the lemma.

**Lemma 4.2.** For any \( \epsilon \ll 1 \), there exists a small constant \( \delta_0 > 0 \) such that if \( \| f_0 \|_{H^2(\mathbb{R}^3)} + \| \nabla_v f_0 \|_{H^2(\mathbb{R}^3)} + \| v |f_0| \|_{H^2(\mathbb{R}^3)} + \| (f_0, \sqrt{M}) \|_{L^2} \leq \delta_0 \), then the solution \( f_\epsilon(t) = f_\epsilon(t, x, v) \) to the system (4.1) satisfies
\[
\| f_\epsilon \|_{H^2(\mathbb{R}^3)}^2 + \| \nabla_v f_\epsilon \|_{H^2(\mathbb{R}^3)}^2 + \| v f_\epsilon \|_{H^2(\mathbb{R}^3)}^2 + \| \nabla_x \Phi \|_{H^2}^2 \leq C \delta_0^2 e^{-t},
\]
where \( C > 0 \) is a constant independent of \( \epsilon \).

**Proof.** By taking \( P_1 \) to (4.17) and noting that \( P_1 L f_\epsilon = L P_1 f_\epsilon \), we have
\[
\partial_t P_1 f_\epsilon + \frac{1}{\epsilon} v \cdot \nabla_x P_1 f_\epsilon - \frac{1}{\epsilon^2} \Delta_x P_1 f_\epsilon - \frac{1}{\epsilon^2} \frac{|v|^2}{4} P_1 f_\epsilon - \frac{3}{\epsilon^2} \frac{1}{2} P_1 f_\epsilon
\]
\[
= \frac{1}{\epsilon} v \cdot \nabla_x \Phi \cdot f_\epsilon - \frac{1}{\epsilon} v \cdot \nabla_x P_0 f_\epsilon + \frac{1}{\epsilon} P_0 (v \cdot \nabla_x P_1 f_\epsilon)
\]
\[
+ \frac{1}{\epsilon} v \cdot \nabla_x \Phi \cdot f_\epsilon + \frac{1}{\epsilon} \nabla_x \Phi \cdot \nabla_v f_\epsilon.
\]
(4.17)
Taking the inner product between \( |v|^2 \partial_x^\alpha P_1 f_\epsilon \) and \( \partial_x^\alpha \) with \( |\alpha| \leq 3 \), we obtain
\[
\frac{d}{dt} \| |v|^2 \partial_x^\alpha P_1 f_\epsilon \|_{H^2(\mathbb{R}^3)}^2 + \frac{1}{\epsilon^2} \left( \| v \nabla_x P_1 f_\epsilon \|_{H^2(\mathbb{R}^3)}^2 + \frac{1}{4} \| |v|^2 P_1 f_\epsilon \|_{H^2(\mathbb{R}^3)}^2 \right)
\]
\[
\leq C \left( \| \nabla_x f_\epsilon \|_{H^2(\mathbb{R}^3)}^2 + \| \nabla_x \Phi \|_{H^2}^2 \right) + \frac{C}{\epsilon^2} \| |v|^2 P_1 f_\epsilon \|_{H^2(\mathbb{R}^3)}^2
\]
Lemma 4.3. There exists a small constant $\delta_0 > 0$ such that if $\|n_0\|_{H^1} + \|n_0\|_{L^1} \leq \delta_0$, then the system (1.12)–(1.14) admits a unique global solution $n(t)$ satisfying
\[
\|n(t)\|_{H^1} + \|\nabla_x \Phi(t)\|_{H^1} \leq C\delta_0 e^{-\frac{t}{2}},
\] (4.21)
where $\nabla_x \Phi(t) = \nabla_x \Delta_x^{-1} n(t)$ and $C > 0$ is a constant.

Theorem 1.1 is directly follows from Lemmas 4.1.

With the help of Lemmas 4.1–4.3, we can prove Theorem 1.2 as follows.

**Proof of Theorem 1.2**

Define
\[
Q_\epsilon(t) = \sup_{0 \leq s \leq t} \left( e^{-\frac{s}{2}} + e^{-\frac{t-s}{\epsilon}} \right)^{-1} \| f_\epsilon(s) - n(s) \sqrt{M} \|_{H^1},
\] (4.22)
where the norm $\| \cdot \|_{H^1}$ is defined by (4.1).

We claim that
\[
Q_\epsilon(t) \leq C\delta_0, \quad \forall t > 0.
\] (4.23)

It is easy to verify that the estimate (4.21) follows from (4.22).

By (4.1) and (4.2), we have
\[
\| f_\epsilon(t) - n(t) \sqrt{M} \|_{H^1} \leq \left\| e^{\frac{t-s}{\epsilon}} f_0 - e^{\frac{(t-s)}{\epsilon}D_0} \sqrt{M} \right\|_{H^1} + \int_0^t \left\| e^{\frac{t-s}{\epsilon}B} G(f_\epsilon) - e^{\frac{(t-s)}{\epsilon}D} \text{div}_x (n \nabla_x \Phi) \sqrt{M} \right\|_{H^1} ds
=: I_1 + I_2.
\] (4.24)

By (4.12), we can bound $I_1$ by
\[
I_1 \leq C\delta_0 \left( e^{-\frac{t}{2}} + e^{-\frac{t}{2\epsilon}} \right).
\] (4.25)
To estimate $I_2$, we decompose

$$I_2 \leq \int_0^t \left\| \frac{1}{\epsilon} e^{\frac{t-s}{\epsilon^2} B} G(f_s) - e^{(t-s)D} \text{div}_x (n \nabla_x \Phi_e) \sqrt{M} \right\|_{H^2_x} ds$$

$$+ \int_0^t \left\| e^{(t-s)D} [\text{div}_x (n \nabla_x \Phi_e) - \text{div}_x (n \nabla_x \Phi)] \sqrt{M} \right\|_{H^2_x} ds$$

$$=: I_3 + I_4.$$ (4.26)

Since $(G(f_s), \sqrt{M}) = 0$ and $(G(f_s), v \sqrt{M}) = n_r \nabla_x \Phi_e$, it follows from (3.22) that

$$I_3 \leq C \int_0^t \left( \epsilon(t-s)^{-\frac{1}{2}} e^{-\frac{t-s}{\epsilon^2}} + \frac{1}{\epsilon} e^{-\frac{\epsilon(t-s)}{\epsilon^2}} \right) \|G(f_s)\|_{H^2_x(L^2)} ds$$

$$\leq C \delta_0^2 \int_0^t \left( \epsilon(t-s)^{-\frac{1}{2}} e^{-\frac{t-s}{\epsilon^2}} + \frac{1}{\epsilon} e^{-\frac{\epsilon(t-s)}{\epsilon^2}} \right) e^{-\frac{t-s}{\epsilon^2}} ds, (4.27)$$

where we have used (refer to (4.10))

$$\|G(f_s)\|_{H^2_x(L^2)} \leq C (\|v|f_s\|_{H^2_x(L^2)} + \|\nabla_v f_s\|_{H^2_x(L^2)}) \|\nabla_x \Phi_e\|_{H^2_x} \leq C \delta_0^2 \epsilon^{-t}.$$

Denote

$$J_1 = \int_0^t \frac{1}{\epsilon} e^{-\frac{\epsilon(t-s)}{\epsilon^2}} e^{-\frac{t-s}{\epsilon^2}} ds. (4.28)$$

It holds that

$$J_1 = \left( \int_0^{t/2} + \int_{t/2}^t \right) \frac{1}{\epsilon} e^{-\frac{\epsilon(t-s)}{\epsilon^2}} e^{-\frac{t-s}{\epsilon^2}} ds$$

$$\leq \int_0^{t/2} \frac{1}{\epsilon} e^{-\frac{\epsilon(t-s)}{\epsilon^2}} ds + e^{-\frac{t}{2}} \int_{t/2}^t \frac{1}{\epsilon} e^{-\frac{\epsilon(t-s)}{\epsilon^2}} ds$$

$$\leq C \left( e^{-\frac{t}{2\epsilon}} + e^{-\frac{t}{4\epsilon^2}} \right). (4.29)$$

Thus, it follows from (4.27) to (4.29) that

$$I_3 \leq C \delta_0^2 \epsilon \left( e^{-\frac{t}{2\epsilon}} + e^{-\frac{t}{4\epsilon^2}} \right). (4.30)$$

For any vector $F = (F_1, F_2, F_3) \in H^k_x$, we have

$$\|e^{(t-s)D} \text{div}_x F \sqrt{M}\|_{H^2_x}^2 \leq C \int_{\mathbb{R}^3} e^{-2(1+|\xi|^2)\frac{t}{1} + |\xi|^2 \frac{k+1}{1}} \hat{F}^2 d\xi \leq C t^{-1} \epsilon^{-t} \|F\|_{H^2_x}^2.$$}

This together with (4.22), (4.21) and (4.13) imply that

$$I_4 \leq \int_0^t (t-s)^{-\frac{1}{2}} e^{-\frac{t-s}{\epsilon^2}} \left( \|n_r - n\|_{H^2_H} \|\nabla_x \Phi_e\|_{H^2_x} + \|n\|_{H^2_x} \|\nabla_x \Phi_e - \nabla_x \Phi\|_{H^2_x} \right) ds$$

$$\leq C \delta_0 Q_0(t) \int_0^t (t-s)^{-\frac{1}{2}} e^{-\frac{t-s}{\epsilon^2}} \left( e^{-\frac{t}{\epsilon^2}} + e^{-\frac{t}{2\epsilon^2}} \right) ds. (4.31)$$

Denote

$$J_2 = \int_0^t (t-s)^{-\frac{1}{2}} e^{-\frac{t-s}{\epsilon^2}} e^{-\frac{t}{2\epsilon^2}} ds. (4.32)$$

Since

$$\frac{1}{\epsilon} \sqrt{8e^{-\frac{t}{\epsilon^2}}} \leq C e^{-\frac{t}{2\epsilon^2}},$$
we have
\[ J_2 \leq C\epsilon e^{-\frac{\epsilon}{2}} \left( \int_{0}^{t/2} + \int_{t/2}^{t} \right) (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds \leq C\epsilon e^{-\frac{\epsilon}{2}}. \]  

(4.33)

Thus, it follows from (4.31)–(4.33) that
\[ I_4 \leq C\delta_0 Q_{\epsilon}(t) e^{-\frac{\epsilon}{2}}. \]  

(4.34)

By combining (4.21)–(4.26), (4.30) and (4.34), we obtain
\[ Q_{\epsilon}(t) \leq C\delta_0 + C\delta_0^2 + C\delta_0 Q_{\epsilon}(t), \]  

(4.35)

where \( C > 0 \) is a constant independent of \( \epsilon \). By taking \( \delta_0 > 0 \) small enough, we can prove (4.29). By (5.13) and a similar argument as above, we can prove (1.27).

\[ \square \]

5 Appendix

We give the proof of Theorem 2.10 in the following.

**Proof of Theorem 2.10** For simplicity, we assume that \( 0 < \epsilon \leq r_0/2 \). By Theorem 2.7 in [28], it is sufficient to prove (2.55) for \( f \in D(B_{c}(\xi)^2) \) because the domain \( D(B_{c}(\xi)^2) \) is dense in \( L^2_{\pi}(\mathbb{R}^3_{\xi}) \). By Corollary 7.5 in [28], the semigroup \( e^{-tB_{c}(\xi)} \) can be represented by
\[ e^{-tB_{c}(\xi)} f = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} e^{\lambda t} (\lambda - B_{c}(\xi))^{-1} f d\lambda, \quad f \in D(B_{c}(\xi)^2), \quad \kappa > 0. \]  

(5.1)

First of all, we investigate the formula (5.1) for \( \epsilon(1 + |\xi|) \leq r_0 \). By (5.33) we have
\[ (\lambda - B_{c}(\xi))^{-1} = [\lambda^{-1} P_0 + (\lambda P_1 - Q_{\epsilon}(\xi))^{-1} P_1] - Z_{\epsilon}(\lambda, \xi), \]  

(5.2)

with the operator \( Z_{\epsilon}(\lambda, \xi) \) defined by
\[ Z_{\epsilon}(\lambda, \xi) = [\lambda^{-1} P_0 + (\lambda P_1 - Q_{\epsilon}(\xi))^{-1} P_1][I + Y_{\epsilon}(\lambda, \xi)]^{-1} Y_{\epsilon}(\lambda, \xi), \]  

(5.3)
\[ Y_{\epsilon}(\lambda, \xi) = i\epsilon\lambda^{-1} P_1 (v \cdot \xi)(1 + \frac{1}{|\xi|^2}) P_0 + i\epsilon P_0 (v \cdot \xi) P_1 (\lambda P_1 - Q_{\epsilon}(\xi))^{-1} P_1. \]  

(5.4)

By a similar argument as Lemma 3.1 in [28], we conclude from (2.24), (5.3) that the operator \( Q_{\epsilon}(\xi) \) generates a strongly continuous contraction semigroup on \( \mathcal{N}^0_{\pi} \), which satisfies for any \( t > 0 \) and \( f \in \mathcal{N}^0_{\pi} \) that
\[ \|e^{tQ_{\epsilon}(\xi)} f\| \leq e^{-\epsilon t} \|f\|. \]  

(5.5)

In addition, for any \( x > -1 \) and \( f \in \mathcal{N}^0_{\pi} \), it holds that
\[ \int_{-\infty}^{x+\infty} \| (x + iy) P_1 - Q_{\epsilon}(\xi) \| f^2 dy \leq \pi(x + 1)^{-1} \|f\|^2. \]  

(5.6)

Substituting (5.2) into (5.1), we have the following decomposition of the semigroup \( e^{-tB_{c}(\xi)} \)
\[ e^{-tB_{c}(\xi)} f = P_0 f + e^{-tQ_{\epsilon}(\xi)} P_1 f - \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} e^{\lambda t} Z_{\epsilon}(\lambda, \xi) f d\lambda, \quad \epsilon(1 + |\xi|) \leq r_0. \]  

(5.7)

To estimate the last term on the right hand side of (5.7), let us denote
\[ U_{\kappa,N} = \frac{1}{2\pi i} \int_{-N}^{N} e^{\frac{\kappa+i|\xi|}{2}} Z_{\epsilon}(\kappa + iy, \xi) f dy, \]  

(5.8)

where the constant \( N > 0 \) is chosen large enough so that \( N > \gamma_0 \) with \( \gamma_0 \) defined in Lemma 2.6. Since \( Z_{\epsilon}(\lambda, \xi) \) is analytic on the domain \( \text{Re}\lambda > -1/2 \) with only finite singularities at \( \lambda = \lambda_0(|\xi|, \epsilon) \in \sigma(B_{c}(\xi)) \) and \( \lambda = 0 \), we
can shift the integration (5.3) from the line \( \text{Re}\lambda = \kappa > 0 \) to \( \text{Re}\lambda = -1/2 \). Then by the Residue Theorem, we obtain

\[
U_{\kappa,N} = \text{Res}\left\{ e^{\frac{\lambda}{\kappa} Z_\epsilon(\lambda, \xi)f; \lambda_0(\|\xi\|, \epsilon)} \right\} + \text{Res}\left\{ e^{\frac{i\pi}{\kappa} Z_\epsilon(\lambda, \xi)f; 0} \right\} + U_{-\frac{1}{2},N} + J_N,
\]

(5.9)

where \( \text{Res}\{f(\lambda); \lambda_j\} \) means the residue of \( f(\lambda) \) at \( \lambda = \lambda_j \) and

\[
J_N = \frac{1}{2\pi i} \left( \int_{-\frac{1}{2} + iN}^{\frac{1}{2} + iN} - \int_{-\frac{1}{2} - iN}^{\frac{1}{2} - iN} \right) e^{\frac{\lambda}{\kappa} Z_\epsilon(\lambda, \xi)f} d\lambda.
\]

The right hand side of (5.9) is estimated as follows. By Lemma 2.5 it is easy to verify that

\[
\|J_N\|_\xi \to 0, \quad \text{as} \quad N \to \infty.
\]

(5.10)

Let

\[
\lim_{N \to \infty} U_{-\frac{1}{2},N}(t) = U_{-\frac{1}{2},\infty}(t) = \int_{-\frac{1}{2} + i\infty}^{\frac{1}{2} + i\infty} e^{\frac{\lambda}{\kappa} Z_\epsilon(\lambda, \xi)f} d\lambda.
\]

(5.11)

Since it follows from Lemma 2.5 that \( Y_\epsilon(-\frac{1}{2} + iy, \xi) \|_\xi \leq 1/2 \) for \( y \in \mathbb{R} \) and \( \epsilon(1 + |\xi|) \leq r_0 \) with \( r_0 > 0 \) being sufficiently small, the operator \( I - Y_\epsilon(-\frac{1}{2} + iy, \xi) \) is invertible on \( L^2_\xi(\mathbb{R}^3_\xi) \) and satisfies \( \|I - Y_\epsilon(-\frac{1}{2} + iy, \xi)\|_\xi \leq 2 \) for \( y \in \mathbb{R} \) and \( \epsilon(1 + |\xi|) \leq r_0 \). Thus, we have for any \( f, g \in L^2_\xi(\mathbb{R}^3_\xi) \)

\[
|\langle (U_{-\frac{1}{2},\infty}(t) f, g) \rangle| \leq e^{-\frac{\pi}{\kappa}} \int_{-\infty}^{+\infty} \| \langle Z_\epsilon(\lambda, \xi) f, g \rangle \rangle dy
\]

\[
\leq C\epsilon(1 + |\xi|) e^{-\frac{\pi}{\kappa}} \int_{-\infty}^{+\infty} \left( \| \langle \lambda P_\lambda - Q_\epsilon(\xi) \rangle \rangle P_\lambda f \| + |\lambda| \| P_\lambda f \| \right) dy
\]

\[
\times \left( \| \langle \lambda P_\lambda - Q_\epsilon(\xi) \rangle \rangle P_\lambda g \| + |\lambda| \| P_\lambda g \| \right) dy, \quad \lambda = -\frac{1}{2} + iy.
\]

This together with (5.9) and

\[
\int_{-\infty}^{+\infty} \| (x + iy)^{-\frac{1}{2}} P_\lambda f \|_\xi^2 dy = \pi |x|^{-1} \| P_\lambda f \|_\xi^2
\]

imply that \( |\langle (U_{-\frac{1}{2},\infty}(t) f, g) \rangle| \leq Cr_0 \cdot e^{-\frac{\pi}{\kappa}} \| f \|_\xi \| g \|_\xi \), and

\[
\|U_{-\frac{1}{2},\infty}(t)\|_\xi \leq Cr_0 e^{-\frac{\pi}{\kappa}}.
\]

(5.12)

Since \( \lambda_0(\|\xi\|, \epsilon), 0 \in \rho(Q_\epsilon(\xi)), \) and

\[
Z_\epsilon(\lambda, \xi) = \lambda^{-1} P_\lambda + (\lambda P_\lambda - Q_\epsilon(\xi))^{-1} P_\lambda - (\lambda - B_\epsilon(\xi))^{-1},
\]

we can obtain

\[
\text{Res}\{e^{\lambda Z_\epsilon(\lambda, \xi)f; \lambda_0(\|\xi\|, \epsilon)} = -\text{Res}\{e^{\lambda (\lambda - B_\epsilon(\xi))^{-1}} f; \lambda_0(\|\xi\|, \epsilon)}
\]

\[
= -e^{\lambda_0(\|\xi\|, \epsilon)} \left( f, \psi_0(\xi, \epsilon) \right)_\xi \psi_0(\xi, \epsilon),
\]

(5.13)

\[
\text{Res}\{e^{\lambda Z_\epsilon(\lambda, \xi)f; 0} \} = \text{Res}\{e^{\lambda} P_0 f; 0 \} = P_0 f.
\]

(5.14)

Therefore, we conclude from (5.7) and (5.8) – (5.14) that

\[
e^{\frac{i\pi}{\kappa} B_\epsilon(\xi)f} = e^{\frac{i\pi}{\kappa} Q_\epsilon(\xi) P_\lambda f + U_{-\frac{1}{2},\infty}(t) + e^{\frac{i\pi}{\kappa} \lambda_0(\|\xi\|, \epsilon)} \left( f, \psi_0(\xi, \epsilon) \right)_\xi \psi_0(\xi, \epsilon),
\]

(5.15)

for \( \epsilon(1 + |\xi|) \leq r_0 \).
Next, we turn to investigate the formula (5.1) for \( \epsilon(1 + |\xi|) > r_0 \). It holds that \( \epsilon|\xi| > r_0/2 \). By (2.19), we have

\[
(\lambda - B_\epsilon(\xi))^{-1} = (\lambda - A_\epsilon(\xi))^{-1} + H_\epsilon(\lambda, \xi),
\]

with

\[
H_\epsilon(\lambda, \xi) = (\lambda - A_\epsilon(\xi))^{-1}[I - G_\epsilon(\lambda, \xi)]^{-1}G_\epsilon(\lambda, \xi),
\]

\[
G_\epsilon(\lambda, \xi) = (K - i\epsilon(v \cdot \xi))\xi^{-1}P_0(\lambda - A_\epsilon(\xi))^{-1}.
\]

By Lemma 2.3 and a similar argument as Lemma 2.4, we can obtain

\[
\|e^{tA_\epsilon(\xi)}f\| \leq e^{-\nu_0 t}\|f\|.
\]

Substituting (5.16) into (5.1) yields

\[
\sup_{|\xi| > r_0/2} \|I - G_\epsilon(\lambda, \xi)\|^{-1} \leq C.
\]

Moreover, by Lemma 2.3 and a similar argument as Lemma 2.4, we can obtain

\[
\|V_{-\sigma_0, \infty}(t)\| \leq C e^{-\tau_0} (\lambda - \sigma_0 - iy)^{-1}.
\]

By (5.17) and (5.25), we have for any \( f, g \in L^2(\mathbb{R}_+^3) \)

\[
\|V_{-\sigma_0, \infty}(t)f, g\| \leq C e^{-\epsilon t}(\lambda - \sigma_0)^{-1}f\|g\|.
\]

\[
\epsilon(1 + |\xi|) > r_0 \quad \text{and} \quad \epsilon|\xi| > r_0/2.
\]

By (5.18), (5.25) and (5.20), we have for any \( f \in L^2(\mathbb{R}_+^3) \) that

\[
\int_{-\infty}^{+\infty} \|x + iy - A_\epsilon(\xi))^{-1}f\|_2^2 dy \leq \pi(x + \nu_0)^{-1}\|f\|_2^2.
\]

Substituting (5.16) into (5.1) yields

\[
e^{i\epsilon t}A_\epsilon(\xi)f = e^{i\epsilon t}A_\epsilon(\xi)f + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{i\epsilon t}H_\epsilon(\lambda, \xi)f d\lambda,
\]

Similarly, in order to estimate the last term on the right hand side of (5.24), let us denote

\[
V_{\kappa, N} = V_{-\sigma_0, N} + I_N,
\]

for sufficiently large constant \( N > 0 \) as in (5.8). Since the operator \( H_\epsilon(\lambda, \xi) \) is analytic on the domain \( \text{Re}\lambda \geq -\sigma_0 \) for the constant \( \sigma_0 = \alpha(r_0/2)/2 \) with \( \alpha(r) > 0 \) given by Lemma 2.4, we can again shift the integration of (5.22) from the line \( \text{Re}\lambda = \kappa > 0 \) to \( \text{Re}\lambda = -\sigma_0 \) to obtain

\[
V_{\kappa, N} = V_{-\sigma_0, N} + I_N,
\]

with

\[
I_N = \frac{1}{2\pi i} \left( \int_{-\sigma_0 + iN}^{-\kappa + iN} - \int_{-\sigma_0 - iN}^{-\kappa - iN} \right) e^{i\epsilon t}H_\epsilon(\lambda, \xi)f d\lambda.
\]

By Lemma 2.3, it holds that

\[
\|I_N\| \to 0 \quad \text{as} \quad N \to \infty.
\]

Moreover, by Lemma 2.3 and a similar argument as Lemma 2.4, we can obtain

\[
\sup_{|\xi| > r_0/2} \|I - G_\epsilon(\lambda, \xi)\|^{-1} \leq C.
\]

By (5.17), (5.26) and (5.20), we have for any \( f, g \in L^2(\mathbb{R}_+^3) \)

\[
\|V_{-\sigma_0, \infty}(t)f\| \leq C e^{-\epsilon t}(\lambda - \sigma_0)^{-1}f\|g\|.
\]

From (5.26) and the fact \( \|f\|_2^2 \leq \|f\|_2^2 \leq (1 + 4\epsilon^2 r_0^{-2})\|f\|_2^2 \) for \( \epsilon(1 + |\xi|) > r_0 \), we have

\[
\|V_{-\sigma_0, \infty}(t)\| \leq C e^{-\epsilon t}(\lambda - \sigma_0)^{-1}.
\]
Therefore, we conclude from (5.21) and (5.22)–(5.27) that
\[ e^{\frac{1}{\epsilon}S_2(t, \xi, \epsilon)f} = e^{\frac{1}{\epsilon}A_1(t)f} + V_{-\sigma_0, \infty}(t), \quad \epsilon(1 + |\xi|) > r_0. \] \hspace{1cm} (5.28)

The combination of (5.17) and (5.28) gives rise to (2.56) with \( S_1(t, \xi, \epsilon)f \) and \( S_2(t, \xi, \epsilon)f \) defined by
\[ S_1(t, \xi, \epsilon)f = e^{\frac{1}{\epsilon}A_1(t)f} \left( f, \psi_0(\xi, \epsilon) \right) \psi_0(\xi, \epsilon)1_{\{\epsilon(1 + |\xi|) \leq r_0\}}, \]
\[ S_2(t, \xi, \epsilon)f = \left( e^{\frac{1}{\epsilon}Q(\xi)}P_1 f + U_{-\sigma_0, \infty}(t) \right)1_{\{\epsilon(1 + |\xi|) \leq r_0\}} \]
\[ + \left( e^{\frac{1}{\epsilon}A_1(t)f} + V_{-\sigma_0, \infty}(t) \right)1_{\{\epsilon(1 + |\xi|) > r_0\}}. \]

In particular, \( S_2(t, \xi, \epsilon)f \) satisfies (2.56) in terms of (5.8), (5.19), (5.12) and (5.27). \( \square \)

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