A robust Kalman-Bucy filtering problem

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Abstract. A generalized Kalman-Bucy model under model uncertainty and a corresponding robust problem are studied in this paper. We find that this robust problem is equivalent to an estimate problem under a sublinear operator. By Girsanov transformation and the minimax theorem, we prove that this problem can be reformulated as a classical Kalman-Bucy filtering problem under a new probability measure. The equation which governs the optimal estimator is obtained. Moreover, the optimal estimator can be decomposed into the classical optimal estimator and a term related to model uncertainty.

Key words. Kalman-Bucy filter, model uncertainty, robust, minimum mean square estimator, minimax theorem, sublinear operator,

1 Introduction

It is well-known that Kalman and Bucy [10] built the fundamental results of the filtering problem for linear systems, which are the foundation of modern filtering theory (see Bensoussan [2], Liptser and Shiryaev [12] et al). In more details, Kalman and Bucy considered that the signal process \((x_t) \in \mathbb{R}^n\) and the observation process \((m_t) \in \mathbb{R}^m\) satisfy the following linear system:

\[
\begin{aligned}
\dot{x}_t &= (F_t x_t + f_t)dt + dw_t, \\
x(0) &= x_0, \\
\dot{m}_t &= (G_t x_t + g_t)dt + dv_t, \\
m(0) &= 0
\end{aligned}
\] (1.1)
on a complete probability space \((Ω, F, P)\). For the given observation information \(Z_t = σ\{m(s), \ 0 ≤ s ≤ t\}\), the optimal estimator \(\hat{x}_t\) of the signal \(x_t\) solves the minimum mean square estimation
\[
\min_{ζ ∈ L^2_Z(Ω, P)} E_P\|x_t - ζ\|^2
\]
where \(L^2_Z(Ω, P)\) is the set of all the square integral \(Z_t\)-measurable random variables.

In this paper, we suppose that there exists model uncertainty for the system \((1.1)\). Specifically, we don’t know the true probability \(P\) and only know that it falls in a set of probability measures \(P\). For this case, it is naturally to consider the worst-case minimum mean square estimation:
\[
\min_{ζ} \sup_{P ∈ P} E_P\|x_t - ζ\|^2 \tag{1.2}
\]
which is to minimize the maximum expected loss over a range of possible models, an idea that goes back at least as far as Wald [17]. Recently this type of estimator has been utilized by Borisov [3] and [4], who studied the filtering of finite state Markov processes with uncertainty of the transition intensity and the observation matrices.

In our context, we adopt the \(k\)-ignorance model in Chen and Epstein [5] to formulate the model uncertainty (readers can refer to Section 2 for more details). Under this formulation, \(\sup_{P ∈ P} E[\cdot]\) is a sublinear operator which is denoted by \(E(\cdot)\). Thus, the problem \((1.2)\) can be reformulated as
\[
\min_{ζ} E(\|x_t - ζ\|^2).
\]
The main results about the estimation problem under sublinear operators are obtained in Sun, Ji [13] and Ji, Kong, Sun [14]. Sun and Ji [13] studied this estimation problem for bounded random variables. Ji, Kong and Sun [14] deleted the boundedness assumption and generalized the corresponding results to the case in which the random variables belong to the space \(L^{2+}_Z(Ω, P)\) where \(ε\) is a constant such that \(ε ∈ (0, 1)\). The \(k\)-ignorance model is one of the so-called drift-uncertainty models (see [7, 8] for more general uncertainty models).

Under some mild conditions, we prove that the optimal estimator \(\hat{x}\) and the optimal probability measure \(P^θ\) exist. It results that we only need to consider the classical Kalman-Bucy filtering problem under the probability measure \(P^θ\). Moreover, the optimal estimator \(\hat{x}_t\) can be decomposed to two parts. One part is the optimal estimator of the signal process under the probability measure \(P\) and the other part contains a parameter \(θ^*\) (see Corollary 3.5 for details). It is worth pointing out that Allan, Cohen [1] studied this type problem under nonlinear expectations. They reformulated the problem as an optimal control problem and analyzed the corresponding value function by HJB equation which is a different approach from ours.

The paper is organized as follows. In section 2, a generalized robust Kalman-Bucy filtering problem is introduced. The main results are given in section 3. In section 4, we summarize the main conclusions of this paper and list some auxiliary theorems in section 5.

2 Problem formulation

Let \((Ω, F, P)\) be a complete probability space on which two independent \(n\)-dimensional and \(m\)-dimensional independent Brownian motions \(w(·)\) and \(v(·)\) are defined. Assume that \(F = \{F_t, 0 ≤ t ≤ T\}\) is the \(P\)-
augmentation of the natural filtration of \( w(\cdot) \) and \( v(\cdot) \), where \( \mathcal{F} = \mathcal{F}_T \) and \( \mathcal{F}_0 \) contains all \( P \)-null sets of \( \mathcal{F} \). The means of \( w(\cdot) \) and \( v(\cdot) \) are zero and the covariance matrices are \( Q(\cdot) \) and \( R(\cdot) \) respectively. The matrix \( R(\cdot) \) is uniformly positive definite. Denote by \( \mathbb{R}^n \) the \( n \)-dimensional real Euclidean space and \( \mathbb{R}^{n \times k} \) the set of \( n \times k \) real matrices. Let \((\cdot, \cdot)\) (resp. \( \| \cdot \| \)) denote the usual scalar product (resp. usual norm) of \( \mathbb{R}^n \) and \( \mathbb{R}^{n \times k} \). The scalar product (resp. norm) of \( M = (m_{ij}), N = (n_{ij}) \in \mathbb{R}^{n \times k} \) is denoted by \((M, N) = tr\{MNT\}\) (resp. \( \|M\| = \sqrt{\langle M, M \rangle} \)), where the superscript \( ^\top \) denotes the transpose of vectors or matrices. For a \( \mathbb{R}^n \)-valued vector \( x = (x_1, \ldots, x_n)^\top \), \( |x| := (|x_1|, \ldots, |x_n|)^\top \); for two \( \mathbb{R}^n \)-valued vectors \( x \) and \( y \), \( x \leq y \) means that \( x_i \leq y_i \) for \( i = 1, \ldots, n \). Denote by \( L^2_\mathbb{F}(0, T; \mathbb{R}^n) \) the space of \( \mathbb{F} \)-adapted \( \mathbb{R}^n \)-valued stochastic processes on \([0, T]\) such that

\[
\mathbb{E}_P \left[ \int_0^T |f(r)|^2 dr \right] < \infty.
\]

Throughout this paper, 0 denotes the matrix/vector with appropriate dimension whose all entries are zero.

Under the probability measure \( P \), the signal process \((x_t) \in L^2_\mathbb{F}(0, T; \mathbb{R}^n) \) and the observation process \((m_t) \in L^2_\mathbb{F}(0, T; \mathbb{R}^m) \) satisfy

\[
\begin{aligned}
dx_t &= (F_tx_t + f_t)dt + dw_t, \\
x(0) &= x_0, \\
dm_t &= (G_tx_t + g_t)dt + dv_t, \\
m(0) &= 0
\end{aligned}
\]

where \( F_t \in \mathbb{R}^{n \times n}, G_t \in \mathbb{R}^{n \times n}, f_t \in \mathbb{R}^n, g_t \in \mathbb{R}^m \) are bounded, continuous function in \( t \) and \( x_0 \in \mathbb{R}^n \) be a given constant vector. Set \( Z_t = \sigma\{m(s); 0 \leq s \leq t\} \). Then the filtration \( Z = \{Z_t, 0 \leq t \leq T\} \) represents the observable information. By the Kalman-Bucy filtering theory (see Bensoussan [2], Kalman, Bucy [10], Liptser, Shiryaev [12] and Xiong [17]), the optimal estimate of \( x_t \) under probability measure \( P \) is governed by

\[
\begin{aligned}
d\bar{x}_t &= (F_t\bar{x}_t + f_t)dt + P_tG_t^\top R_t^{-1}dI_t, \\
\bar{x}(0) &= x_0,
\end{aligned}
\]

and the variance of estimate error \( P_t = E_P[(x_t - \bar{x}_t)(x_t - \bar{x}_t)^\top] \) is governed by

\[
\begin{aligned}
\frac{dP_t}{dt} &= F_tP_t + P_tF_t^\top - P_tG_t^\top R_t^{-1}G_tP_t + Q_t, \\
P(0) &= 0
\end{aligned}
\]

where \( \bar{x}_t = E_P(x_t|Z_t) \) and \( I_t = m_t - \int_0^t (G_s \bar{x}_s + g_s)ds \) is the so called innovation process which is a Brownian motion adapted to \( Z \). Set \( I_t = \sigma\{I(s); 0 \leq s \leq t\} \). Then the filtration \( \{I_t\}_{0 \leq t \leq T} \) equals to \( Z \).

Now we give the \( k \)-ignoramce model which is proposed by Chen and Epstein [4]. For a fixed \( \mathbb{R}^n \)-valued nonnegative constant vector \( \mu \), denote by \( \Theta \) the set of all \( \mathbb{R}^n \)-valued progressively measurable processes \((\theta_t)\) with \(|\theta_t| \leq \mu \). Define

\[
\mathcal{P} = \{P^\theta \frac{dP^\theta}{dP} = f^\theta_T \text{ for } \theta \in \Theta\}
\]

\[3\]
where
\[ f_\theta^T = \exp \left( \int_0^T \theta_t^T dw_t - \frac{1}{2} \int_0^T \|\theta_t\|^2 dt \right). \]

Due to the boundness of \( \theta \), the Novikov’s condition holds (see Karatzas and Shreve \[11\]). Therefore, \( P^\theta \) defined by \[2.4\] is a probability measure and the processes \((w^\theta_t)\) and \((v_t)\) are Brownian motions under this probability measure \( P^\theta \) by Girsanov theorem. Moreover, the probability measure \( P^\theta \) is equivalent to the probability measure \( P \) with the Radon Nikodym derivative \( \exp(\int_0^T \theta_s dw_s - \frac{1}{2} \int_0^T \theta_s^2 ds) \). The k-ignornace model describes an agent who is uncertain about the drift of the underlying Brownian motion and allows any drift between \(-\mu\) and \(\mu\).

Taking into account the k-ignornace model, we generalize the Kalman-Bucy filtering problem \[2.1\] to the following minimax problem. Under every probability measure \( P^\theta \in \mathcal{P} \), consider
\[
\begin{align*}
\left \{ 
\begin{array}{l}
\quad dx_t = (F_t x_t + f_t + \theta_t) dt + dw^\theta_t, \\
\quad x(0) = x_0, \\
\quad dm_t = (G_t x_t + g_t) dt + dv_t, \\
\quad m(0) = 0
\end{array}
\right. 
\end{align*}
\]
(2.5)

where \( w^\theta_t = w_t - \int_0^t \theta_s ds \) and study the minmax problem
\[
\inf_{\zeta \in L_{\mathcal{Z}_t}^{2+\epsilon}(\Omega, P, \mathbb{R}^n)} \sup_{P^\theta \in \mathcal{P}} E_{P^\theta} \|x_t - \zeta\|^2 
\] (2.6)

where \( \epsilon \) is a constant such that \( 0 < \epsilon < 1 \) and \( L_{\mathcal{Z}_t}^{2+\epsilon}(\Omega, P, \mathbb{R}^n) \) is the set of all the \( \mathbb{R}^n \)-valued \( (2 + \epsilon) \) integrable \( \mathcal{Z}_t \)-measurable random variables.

It is easy to verify that \( \mathcal{E}(\cdot) = \sup_{P^\theta \in \mathcal{P}} E_{P^\theta}[\cdot] \) is a sublinear operator. Thus, the problem \[2.6\] can be represented as following: given the observation information \( \{\mathcal{Z}_t\} \), we want to find the optimal estimator \( \hat{x}_t \) for the signal \( x_t \) for \( t \in [0, T] \) such that
\[
\mathcal{E}\|x_t - \hat{x}_t\|^2 = \inf_{\zeta \in K_t} \mathcal{E}\|x_t - \zeta\|^2
\] (2.7)

where
\[
K_t = \{ \zeta : \Omega \rightarrow \mathbb{R}^n ; \zeta \in L_{\mathcal{Z}_t}^{2+\epsilon}(\Omega, P, \mathbb{R}^n) \}.
\]

Ji, Kong and Sun \[14\] has explored the minimum mean square estimator of random variables under sublinear operators and obtained the existence and uniqueness results of the optimal estimator. In the next section, we will utilize the results in \[14\] to solve the problem \[2.7\].

**Remark 2.1** The optimal solution \( \hat{x}_t \) of problem \[2.7\] is called minimum mean square estimator. It is also regarded as a minimax estimator in statistical decision theory. If \( \theta \equiv 0 \), then \( \mathcal{P} \) contains only the probability measure \( P \) and the sublinear operator \( \mathcal{E}(\cdot) \) degenerates to linear expectation operator. In this case, it is well-known that the minimum mean square estimator \( \hat{x}_t \) is just the conditional expectation \( E_P(x_t|\mathcal{Z}_t) \).
3 Main results

In this section, we calculate the minimum mean square estimator $\hat{x}_t$ of the problem (2.7) for $t \in [0, T]$. Without loss of generality, all the statements in this section are only proved for the one dimensional case.

Lemma 3.1 The set \( \{ dP^\theta : P^\theta \in \mathcal{P} \} \subset L^{1+\frac{1}{2}}(\Omega, \mathcal{F}, P) \) is \( \sigma(L^{1+\frac{1}{2}}(\Omega, \mathcal{F}, P), L^{1+\frac{1}{2}}(\Omega, \mathcal{F}, P)) \)-compact and the set \( \mathcal{P} \) is convex.

Proof. Since \( \theta \) is bounded, by Theorem (4.1) in Appendix, the set \( \{ dP^\theta : P^\theta \in \mathcal{P} \} \) is uniformly normed bounded in \( L^{1+\frac{1}{2}}(\Omega, \mathcal{F}, P) \). From the Theorem 4.1 of Chapter 1 in Simons [16], we know that the set \( \{ dP^\theta : P^\theta \in \mathcal{P} \} \) is \( \sigma(L^{1+\frac{1}{2}}(\Omega, \mathcal{F}, P), L^{1+\frac{1}{2}}(\Omega, \mathcal{F}, P)) \)-compact.

The convexity of \( \mathcal{P} \) is proved in Chen and Epstein [3] (see Theorem 2.1 (c)).

By Lemma 3.1 we can apply the minimax theorem (see Theorem (4.2) to (2.6) which leads to the following theorem.

Theorem 3.2 For a given \( t \in [0, T] \), there exists a \( \theta^* \in \Theta \) such that
\[
\inf_{\zeta \in K_t} \mathcal{E}[x_t - \zeta]^2 = \inf_{\zeta \in K_t} \sup_{P^\theta \in \mathcal{P}} E_{P^\theta}[(x_t - \zeta)^2] = \inf_{\zeta \in K_t} E_{P^{\theta^*}}[(x_t - \zeta)^2].
\]

Proof. Choose a sequence \( \{ \theta_n \}, n = 1, 2, \ldots \) such that
\[
\lim_{n \to \infty} \inf_{\zeta \in K_t} E_{P^{\theta_n}}[(x_t - \zeta)^2] = \sup_{\zeta \in K_t} \inf_{P^\theta \in \mathcal{P}} E_{P^\theta}[(x_t - \zeta)^2].
\]

Set \( f_n = dP^{\theta_n} \). By Komlós theorem in the appendix of Pham [15], we have that there exist a subsequence \( \{ f_{n_k} \}_{k \geq 1} \) and a \( f^* \in L^1(\Omega, \mathcal{F}, P) \) such that
\[
\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} f_{n_k} = f^*, \text{ P-a.s.}
\]

Let \( g_m = \frac{1}{m} \sum_{k=1}^{m} f_{n_k} \). Then \( g_m \xrightarrow{P-a.s.} f^* \) and
\[
\sup_{P^\theta \in \mathcal{P}} \inf_{\zeta \in K_t} E_{P^\theta}[(x_t - \zeta)^2] = \lim_{n \to \infty} \inf_{\zeta \in K_t} E_{P^\theta_n}[(x_t - \zeta)^2] \leq \lim_{m \to \infty} \inf_{\zeta \in K_t} E_{P^\theta}[(x_t - \zeta)^2] \tag{3.1}
\]

By Theorem (4.1) for any given constants \( p > 1 \) and \( m \), \( E_P(g_m)^K \leq M \) where \( K = (1 + \frac{2}{\nu})p \) and \( M = \exp(K^2/2\|\mu\|^2T) \). Then, we have \( \{ |g_m|^{1+\frac{1}{2}} : m = 1, 2, \ldots \} \) is uniformly integrable. Therefore, \( g_m \xrightarrow{L^{1+\frac{1}{2}}(\Omega, \mathcal{F}, P)} f^* \) and \( f^* \in L^{1+\frac{1}{2}}(\Omega, \mathcal{F}, P) \). According to the convexity and weak compactness of the set \( \{ dP^\theta : P^\theta \in \mathcal{P} \} \), there exists a \( \theta^* \) such that \( dP^{\theta^*} = f^* \) and the following relations hold
\[
\sup_{P^\theta \in \mathcal{P}} \inf_{\zeta \in K_t} E_{P^\theta}[(x_t - \zeta)^2] \geq \inf_{\zeta \in K_t} E_{P^{\theta^*}}[(x_t - \zeta)^2] = \inf_{\zeta \in K_t} E_{P^\theta}(f^*(x_t - \zeta)^2) = \inf_{\zeta \in K_t} E_{P^\theta}[(x_t - \zeta)^2] \geq \lim_{m \to \infty} \inf_{\zeta \in K_t} E_{P^\theta}(g_m(x_t - \zeta)^2) \geq \sup_{P^\theta \in \mathcal{P}} \inf_{\zeta \in K_t} E_{P^\theta}[(x_t - \zeta)^2]
\]
where the second ’ $\geq'$ is based on the upper semi-continuous property. It follows that
\[
\sup_{p^* \in P} \inf_{\zeta \in K_t} E_{p^*}[ (x_t - \zeta)^2 ] = \inf_{\zeta \in K_t} E_{p^*^*}[ (x_t - \zeta)^2 ].
\]

By the minimax theorem (Theorem 4.2), we obtain
\[
\sup_{p^* \in P} \inf_{\zeta \in K_t} E_{p^*}[ (x_t - \zeta)^2 ] = \inf_{\zeta \in K_t} E_{p^*^*}[ (x_t - \zeta)^2 ] = \inf_{\zeta \in K_t} E_{p^*^*}[ (x_t - \zeta)^2 ].
\]

It leads to that
\[
\inf_{\zeta \in K_t} E_{p^*^*}[ (x_t - \zeta)^2 ] = \inf_{\zeta \in K_t} E_{p^*^*}[ (x_t - \zeta)^2 ].
\]
\[
\boxdot
\]

Once we find the optimal $\theta^*$, the problem (2.5) and (2.6) can be reformulated under the probability measure $P^{\theta^*}$ by Theorem 3.2. In more details, on the filtered probability space $(\Omega, F, \{F_t\}_{0 \leq t \leq T}, P^{\theta^*})$, the processes $(x_t)_{0 \leq t \leq T}$ and $(m_t)_{0 \leq t \leq T}$ satisfy
\[
\begin{align*}
\frac{dx_t}{dt} &= (F_t x_t + f_t + \theta_t^*) dt + dw_t^\theta, \\
x(0) &= x_0, \\
\frac{dm_t}{dt} &= (G_t x_t + g_t) dt + dv_t, \\
m(0) &= 0.
\end{align*}
\]

We solve the classical minimum mean square estimation problem under $P^{\theta^*}$
\[
E_{p^{\theta^*}} \| x_t - \hat{x}_t \|^2 = \inf_{\zeta \in K_t} E_{p^{\theta^*}} \| x_t - \zeta \|^2.
\]

Assumption 3.3 The process $(\theta_t^*)_{0 \leq t \leq T}$ is adapted to the filtration $\{Z_t\}_{0 \leq t \leq T}$.

Under Assumption 3.3, we study the above model (3.2) and the following problem: to obtain the optimal estimator $\hat{x}_t$ such that
\[
E_{p^{\theta^*}} \| x_t - \hat{x}_t \|^2 = \inf_{\zeta \in K_t} E_{p^{\theta^*}} \| x_t - \zeta \|^2.
\]

where $K_t = \{ \zeta : \Omega \to \mathbb{R}^n; \ z \in L^2(\Omega, P^{\theta^*}, \mathbb{R}^n) \}$.

The problem (3.4) is a classical linear partially observable system with a fixed parameter $\theta^*$. This estimate problem is to characterize the conditional distribution $P^{\theta^*}(x_t \in A | Z_t)$, where $A$ is a Borel set in $\mathbb{R}^n$. Then, we are in the realm of Kalman-Bucy filtering and it is well known (Kalman and Bucy [10], Liptser and Shiryaev [12]) that the conditional distribution is again Gaussian and the conditional mean $\hat{x}_t = E_{p^{\theta^*}(x_t | Z_t)}$ solves the following equation:
\[
\begin{align*}
\frac{d\hat{x}_t}{dt} &= (F_t \hat{x}_t + f_t + \theta_t^*) dt + P_t G_t R_t^{-1} d\hat{I}_t, \\
\hat{x}_t(0) &= x_0
\end{align*}
\]

where the variance of the error equation $P_t = E_{p^{\theta^*}}[ (x_t - \hat{x}_t)^2 ]$ is the same as (2.3) and $\hat{I}_t = m_t - \int_0^t G_s \hat{x}_s + g_s ds, 0 \leq t \leq T$ is $Z_t$-measurable Brownian motion. Thus, the optimal estimator $\hat{x}_t$ of the problem (3.4) has been obtained.

Next, we expound that this solution $\hat{x}_t$ is also the optimal estimator of the problem (2.7) at time $t \in [0, T]$. 6
Theorem 3.4 Under Assumption 3.3, the solution \( \hat{x}_t \) governed by (3.5) is also the optimal solution of the problem (2.7) at time \( t \in [0, T] \).

Proof. Note that
\[
\inf_{\zeta \in K_t} \sup_{P \in \mathcal{P}} E_{P^*}(x_t - \zeta)^2 = \inf_{\zeta \in K_t} E_{P^*}(x_t - \zeta)^2 \geq \inf_{\zeta \in K_t} E_{P^*}(x_t - \zeta)^2.
\]
(3.6)
Since \( F_t, G_t, f_t \) and \( g_t \) are bounded continuous functions in \( t \) and \( \theta^* \) is bounded, by Theorem 3.3, \( \hat{x}_t \) is not only square integrable but also \( (2 + \epsilon) \) integrable under probability measure \( P^{\theta^*} \). Then, the solution \( \hat{x}_t \) to (3.5) also belongs to \( K_t \). It yields that \( \hat{x}_t \) is the optimal solution of problem (2.7) at time \( t \in [0, T] \).

Define
\[
R^t_s = P^{-1}_t[\Phi(t, s)Q(s) - \int_s^t A(t, r)G(r, s)Q(s)dr]
\]
where \( A(t, r) = P_t R^1_s \exp \int_s^t (F_t - P_t G_t R^1_s)dr \) is the impulse response of the classical Kalman-Bucy filter and \( \Phi(t, s) = \exp \int_s^t F_t dr \). Then applying an analysis similar to Theorem 1 in Davis and Varaiya [6], we obtain the following Corollary.

Corollary 3.5 With equations (2.2) and (3.5), the optimal estimator \( \hat{x}_t \) in this case for any time \( t \in [0, T] \) can be expressed as
\[
\hat{x}_t = \tilde{x}_t + \int_0^t P_t R^t_s \theta^*_s ds.
\]
(3.7)
where \( \tilde{x}_t \) is defined by equation (2.2).

4 Appendix

For the convenience of the reader, we put the following three theorems used in the paper in this appendix.

Theorem 4.1 (Girsanov [9]) We suppose that \( \phi(t, \omega) \) satisfies the following conditions:
(1) \( \phi(t, \omega) \) are measurable in both variables;
(2) \( \phi(t, \omega) \) is \( \mathcal{F}_t \)-measurable for fixed \( t \);
(3) \( \int_0^T |\phi(t, \omega)|^2 dt < \infty \) almost everywhere; and \( 0 < c_1 \leq |\phi(t, \omega)| \leq c_2 \) for almost all \( (t, \omega) \), then \( \exp(\alpha \zeta^*_t(\phi)) \)

is integrable and for \( \alpha > 1 \)
\[
\exp \left[ \frac{(\alpha^2 - \alpha)}{2}(t - s)c_1^2 \right] \leq E[\exp(\alpha \zeta^*_t(\phi))] \leq \exp \left[ \frac{(\alpha^2 - \alpha)}{2}(t - s)c_2^2 \right]
\]
(4.1)
where \( \zeta^*_t(\phi) = \int_0^t \phi(u, \omega)dw_u - \frac{1}{2} \int_0^t \phi^2(u, \omega)du \).

Theorem 4.2 (Pham [15] Theorem B.1.2) Let \( \mathcal{X} \) be a convex subset of a normed vector space, and \( \mathcal{Y} \) be a convex subset of a normed vector space \( E \), compact for the weak topology \( \sigma(E, E') \). Let \( f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) be a function satisfying:
(1) \( x \mapsto f(x, y) \) is a continuous and convex on \( \mathcal{X} \) for all \( y \in \mathcal{Y} \);
(2) \( y \mapsto f(x, y) \) is a concave on \( \mathcal{Y} \) for all \( x \in \mathcal{X} \).

Then, we have
\[
\sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} f(x, y) = \inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} f(x, y).
\]
Theorem 4.3 (Yong and Zhou [18]) Given a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)$, on which a standard $\mathbb{R}$-valued Brownian motion $W(\cdot)$ is defined. $\mathcal{F}_t = \sigma\{W_s, s \leq t\}$. The process $(X_t)$ satisfies:

$$dX_t = f(t, X_t)dt + l(t, X_t)dW_t,$$

$$X_0 = x.$$ 

Supposed that $f(t,x)$ and $l(t,x)$ satisfy the following conditions:

(L): Lipschitz condition: $|f(t,x) - f(t,y)| + |l(t,x) - l(t,y)| \leq K|x-y|$, $K \geq 0$ is a constant;

(B): $\sup_{t \in [0,T]} (|f(t,0)| + |l(t,0)|) < \infty$

and $E_P(|X_0|^p) < \infty$, $p \geq 2$, then there exists constant $C_p$ such that

$$E_P(\sup_{0 \leq s \leq t} |X_s|^p) \leq C_p(1 + |X_0|^p)(1 + t^p)e^{C_p(t^{p/2} + t^p)}$, $t \geq 0.$$

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