OPERADS, GROTHENDIECK TOPOLOGIES AND DEFORMATION THEORY

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0. INTRODUCTION

0.1. Gerstenhaber’s papers in the Annals showed that deformation theory of associative algebras over a field is ”controlled” by Hochschild cohomology. The passage from deformations to cohomology is realised by means of Hochschild cochains. This approach has two main drawbacks:

1. It is impossible to generalize it to the case of algebras over a ring, or more generally, to sheaves of algebras over a scheme. This is because neither deformations are described by cochains, nor cohomologies can be computed using bar-resolution.

2. Although deformations and cohomology are invariant objects they are connected by choosing some specific resolution.

0.2. Our aim in the present work is to define an appropriate cohomology theory and to find an invariant way to pass from deformations to cohomology.

The initial idea, which goes back, probably, to Quillen, is to describe deformations of an algebraic object (e.g. associative algebra) by means of ”resolving” it by free objects of the same type (in our example, free associative algebras). In principal, all the results can be formulated already on the level of our initial algebra $A$ and a free algebra $B$ mapping onto it. However, the picture is much easier to grasp, when we consider the category of all algebras over $A$. These algebras form a site, and cohomologies that we are looking for are just cohomologies of certain sheaves on this site. This is the main idea of the paper.

An advantage of this approach is that we can treat in the same framework algebras of all types, i.e. algebras over an arbitrary operad.

0.3. Let us describe briefly the contents of the paper.

In sections 1 and 2 we describe the formalism of operads, algebras over operads and modules over them. Our presentation is inspired by some ideas of A. Beilinson [6] and is very close to that of [1]. The essential difference is that we are using the language of pseudo-monoidal categories.

In section 3 we introduce the site $C(A)$ and study the connection between the category of sheaves on this site and the category of $A$-modules. In particular, we introduce the notion of the cotangent complex of an algebra. The site $C(A)$ was introduced first in [1]. However, one can think of deformation theory (e.g. Theorem 4.2) as giving a hint how to define correct cohomologies: just look at deformations of the corresponding object.
In section 4 we study deformations of an algebra over an operad. Cohomology classes that arise in deformation theory are realised as classes of certain torsors and gerbes.

Finally, in section 5 we give an application of the theory presented in the paper. We prove the Poincaré-Birkhoff-Witt theorem for Lie algebras which are flat modules over the ground ring. The idea of such an approach to the Poincaré-Birkhoff-Witt theorem belongs to J.Bernstein and has been already realized in [6].

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1. Pseudo-monoidal categories and operads

1.0. We are working over a fixed ground ring $R$ and all categories are assumed to be $R$-linear. If $C$ is a category, $C^{o}$ will denote the opposite category.

1.1 Pseudo-monoidal categories. Let $C$ be a category. A pseudo-monoidal structure on it is a collection of functors for each finite set $S$ :

$$F_{C}^{S} : (C^{o})^{\times S} \times C \to R\text{-modules}$$

equipped with the following additional data (composition maps):

If $S \to T$ is a surjection of finite sets, for each element $t \in T$, we are given a natural transformation between two functors $(C^{o})^{\times S} \times C \to R\text{-modules}$ :

$$\bigotimes_{t \in T} F_{C}^{S_{t}}(\times_{s \in S_{t}} A_{s}, B_{t}) \otimes F_{C}^{T}(\times_{t \in T} B_{t}, D) \to F_{C}^{S}(\times_{s \in S} A_{s}, D),$$

where $S_{t}$ denotes here the preimage of $t \in T$) with $F_{C}^{1}(A, B) = Hom(A, B)$ (subscript 1 means one element set) such that these natural transformations are compatible with respect to compositions of partitions in the obvious sense.

1.1.1 Example. If $C$ is a strictly symmetric monoidal category, we endow it with a pseudo-monoidal structure by setting

$$F_{C}^{S}(\times_{s \in S} A_{s}, B) = Hom(\otimes_{s \in S} A_{s}, B)$$

where the above natural transformations are just composition maps.

1.1.2 Variant. As in usual monoidal categories, one can require existence of a unit object in a pseudo-monoidal category. This means that there must be an object $1 \in C$ such that for each finite set $S$, for each $\times_{s \in S} A_{s} \in C^{\times S}$ and for each $B \in C$, $F_{C}^{S \cup 1}(1 \times A_{s}, B)$ is canonically isomorphic to $F_{C}^{S}(\times_{s \in S} A_{s}, B)$. Such pseudo-monoidal categories will be called unital.

1.1.3. A pseudo-monoidal functor between two pseudo-monoidal categories $C_{1}$ and $C_{2}$ is a (covariant) functor $G : C_{1} \to C_{2}$ equipped with a natural transformation $F_{C_{1}}^{S} \to F_{C_{2}}^{S} \circ G$ which is compatible with composition maps for each finite set $S$. Pseudo-monoidal natural transformations are defined in a similar way.

1.2 Operads. Operad is by definition a pseudo-monoidal category $O$ which has essentially one object.
1.2.1. Equivalently, one can view operads as the following linear algebra data:

1. A collection of $R$-modules for each finite set $S$ denoted by $O(S)$ (thought of as $F^C_S(A^S, A)$ for $A \in C$)
2. A distinguished element $1 \in O(1)$
3. If $S \to T$ is a surjection of finite sets there is a composition map

$$O(T) \bigotimes_{t \in T} O(S_t) \to O(S)$$

that satisfies

1. The composition $1 \otimes O(S) \to O(1) \otimes O(S) \to O(S)$ is the identity map for each finite set $S$.
2. Composition maps are compatible with compositions of partitions.

1.2.2 Variant. We define unital operads as unital pseudo-monoidal categories having at most one isomorphism class of objects distinct from $1$. It is not difficult to work out this definition also in linear algebra terms.

1.2.3. If $R \to R'$ is a ring homomorphism, to any operad over $R$ one can assign an operad over $R'$ by taking tensor products over $R$ with $R'$. We will denote them by same letters when no confusion can occur.

1.3 Algebras over an operad. In order to simplify the exposition we will consider only algebras of $R$-modules. In principal, one can define algebras over an operad in any strictly symmetric monoidal category and develop deformation theory for them.

An $O$-algebra (of $R$-modules), or equivalently, an algebra over $O$ is by definition a pseudo-monoidal functor $O \to R$-modules. Morphisms between $O$-algebras are defined to be pseudo-monoidal natural transformations between such functors.

1.3.1. In linear algebra terms, an $O$-algebra is an $R$-module $A$ which for each finite set $S$ is endowed with a map

$$O(S) \otimes A^\otimes S \to A$$

such that if $S \to T$ is a surjection of finite sets the square

$$
\begin{array}{ccc}
O(T) \bigotimes_{t \in T} O(S_t) \otimes A^\otimes S & \longrightarrow & O(S) \otimes A^\otimes S \\
\downarrow & & \downarrow \\
O(T) \otimes A^\otimes T & \longrightarrow & A
\end{array}
$$

is commutative.

1.3.2 Examples.

1. Set for each finite set $S$, $O(S) = R$. This is called $O_{\text{com,ass}}$. Algebras over it are commutative and associative algebras.

2. Set for each finite set $S$, $O(S) = R^\text{all bijections:}\{1,2,\ldots,|S|\} \to S$ with an obvious definition of composition maps. This operad (denoted $O_{\text{ass}}$) corresponds to associative algebras.
3. One can define in the same manner unital operads $O^{\text{comm,ass,1}}, O^{\text{ass,1}}$ and they will correspond to unital algebras.

4. In a similar way one defines operads $O^{\text{Lie}}, O^{\text{Poisson}}$, etc.

1.3.3 Variant. A pseudo-monoidal functor from an operad $O$ to the category of graded $R$-modules (morphisms in this last category are homogeneous of degree 0) will be called a graded $O$-algebra.

1.4 Free $O$-algebras.

In this subsection we fix an operad $O$.

Lemma. The forgetful functor $(O\text{-algebras} \rightarrow R\text{-modules})$ admits a left adjoint.

Proof. For an $R$-module $V$ we will construct an $O$-algebra $\text{Free}(V)$:

Construction.

$$\text{Free}(V) = \bigoplus_i (\text{Free}_i(V)),$$

where $\text{Free}_i(V) = (V^\times T \otimes O(T))_{ST}$

Here $T = \{1, 2, ..., i\}$ and $S_T$ is the group of permutations of the set $T$.

It is not difficult to see that $V \rightarrow \text{Free}(V)$ is the adjoint functor we looked for.

Remark. Free $O$-algebras satisfy usual properties; e.g. if $V$ is projective as an $R$-module then any surjection onto $\text{Free}(V)$ admits a section.

2. Modules over an algebra over an operad

2.0. If $A$ is an algebra over an operad $O$ we will introduce the notion of a module over it. Our definition is motivated by a suggestion of A. Beilinson. As it was said earlier we are dealing only with algebras of $R$-modules and hence modules over them will also lie in the category of $R$-modules, although they can be defined in a more general context.

2.1. Let $C$ be a pseudo-monoidal category and let $\Upsilon$ be another category. We say that $C$ acts on $\Upsilon$ if for each finite set $S$ and $s \in S$ we are given a functor

$$F^Y_{S,s'} : (C^o)^{(S-s')} \times \Upsilon^o \times \Upsilon \rightarrow R\text{-modules}$$

with $F^Y_{1,1}(v, v') = \text{Hom}(v, v')$ such that for $S = \bigcup_{t \in T} S_t$, $s' \in S_{t'}$, for each $\times B_t \in C^{\times T-t'}$ and for each $v \in \Upsilon$ we are given a natural transformation between two functors $(C^o)^{(S-s')} \times \Upsilon^o \times \Upsilon \rightarrow R\text{-modules}$ (composition maps): from the functor

$$\bigotimes_{t \in T-t'} F^C_{S_t,s \times A_s,B_t}(\times A_s,B_t) \otimes F^Y_{S_{t'},s',s}(\times A_{s'},v',v) \otimes F^Y_{T,t'}(\times B_t,v,v'')$$

to the functor $F^Y_{S_{s'},s'}(\times A_{s'},v',v'')$

2.1.1 Examples.

1. Any pseudo-monoidal category $C$ acts on itself
2. Let $C$ be as in 1.1.1 and let $\mathcal{Y}$ be a category equipped with an action of $C$. Then when we consider $C$ as a pseudo-monoidal category it will act on $\mathcal{Y}$ in a natural way:

$$F_{S, s'}^{\mathcal{Y}}(\times_{s \in S - s'} A_s, v_1, v_2) = Hom(\times_{s \in S - s'} A_s(v_1), v_2)$$

2.1.2. For two pairs $C_1, \mathcal{Y}_1$ and $C_2, \mathcal{Y}_2$ of a pseudo-monoidal category and a category which it acts upon, one defines notions of pseudo-monoidal functors and pseudo-monoidal natural transformations between pseudo-monoidal functors as in 1.1.2.

2.1.3 Variant. One can modify the above definitions to the case of unital pseudo-monoidal categories. Essentially, one needs that the unit object $1 \in C$ “acts identically” on $\mathcal{Y}$. In what follows there will be no difference between operads and unital operads.

2.2. Let now $O$ be an operad and let $\mathcal{Y}$ act on a category $\mathcal{Y}$. We say that $\mathcal{Y}$ is a model over $O$ if it has essentially one object.

2.2.1. A model $\mathcal{Y}$ can be thought of as the following linear algebra data (analogously to 1.2):

1. A collection of $R$-modules $\mathcal{Y}(S, s')$ for each finite set $S$ and $s' \in S$ (they correspond to $F_{S, s'}^{\mathcal{Y}}(A^{\times(S - s')} \times v \times v)$ for $A \in O, v \in \mathcal{Y}$)
2. A distinguished element $1 \in \mathcal{Y}(1, 1)$
3. Composition maps for each finite set $S$ and $s' \in S$

$$\mathcal{Y}(T, t') \otimes O(S_t) \otimes \mathcal{Y}(S_{t'}, s') \to \mathcal{Y}(S, s')$$

2.3. Let now $O$ be an operad and let $\mathcal{Y}$ be a model over it. A pseudo-monoidal functor from the pair $(O, \mathcal{Y})$ to the pair $(R$-modules, $R$-modules) is called an $A$-module of type $\mathcal{Y}$ for $A$ defined by the underlying functor $O \to R$-modules. Morphisms between $A$-modules of type $\mathcal{Y}$ are defined to be natural transformations between such functors. For any fixed $\mathcal{Y}$ such modules form an abelian category.

2.3.1. Again, we can describe $A$-modules of type $\mathcal{Y}$ in linear algebra terms: An $A$-module is an $R$-module $M$ equipped with a system of maps for each finite set $S$ and $s' \in S$

$$\mathcal{Y}(S, s') \otimes A^{\otimes(S - s')} \otimes M \to M$$

such that if $S \to T$ is a surjection of finite sets, the following diagram becomes commutative:

$$\begin{array}{ccc}
\mathcal{Y}(T, t') \otimes O(S_t) \otimes \mathcal{Y}(S_{t'}, s') \otimes A^{\otimes(S - s')} \otimes M & \longrightarrow & \mathcal{Y}(S, s) \otimes A^{\otimes(S - s')} \otimes M \\
\downarrow & & \downarrow \\
\mathcal{Y}(T, t') \otimes A^{\otimes(T - t')} \otimes M & \longrightarrow & M
\end{array}$$
2.3.2 Example. When we put $\Upsilon = O$, our definition coincides with that of [2]: $\Upsilon(S, s') = O(S)$, and for $O = O_{\text{Lie}}$ (resp. $O_{\text{ass,comm}}$) we get usual $A$-modules (resp. Lie-algebra representations), whereas for $O = O_{\text{ass}}$ we get $A$-bimodules. In the case of the corresponding unital operads we get modules acted on identically by the unit.

However, by means of varying $\Upsilon$ the above definition allows to get modules with an additional structure.

2.3.3 Variant. A pseudo-monoidal functor from the pair $(O, \Upsilon)$ to the pair (graded $R$-modules, graded $R$-modules) will be called a graded module over the corresponding graded algebra. We have the functor $T = \text{shift of grading}$ on the category of graded modules over a graded algebra.

2.4 Free modules. We will introduce the notion of a free module over an algebra over an operad. In particular, this will lead to the [2] construction of the universal enveloping algebra of an algebra over an operad.

2.4.1.

Lemma. The forgetful functor from $A$-modules of type $\Upsilon$ to $R$-modules admits a left adjoint.

Proof. Let $U$ be an $R$-module. We will construct an $A$-module $F(U)$, the free $A$-module on $U$, such that the functor $U \to F(U)$ is the adjoint functor we need. Let us observe first, that it is sufficient to construct $F(R)$ because then we can set $F(U) = F(R) \otimes_R U$.

Construction. Set $F'(R) = \bigoplus_{i=0,1,...} F'_i(R)$, where

$$F'_i(R) = (\Upsilon(\{0,1,\ldots,i\}, 0) \otimes A^i)_{s'}. $$

For any $i$ and $j$ we have maps

$$\Upsilon(\{0,1,\ldots,i\}, 0) \otimes \Upsilon(\{0,1,\ldots,j\}, 0) \to \Upsilon(\{0,1,\ldots,i + j\}, 0)$$

which induce on $F'(R)$ a structure of an associative algebra with a unit. Consider now for all $i, j$ (non-commutative) diagrams of the type:

$$\begin{array}{ccc}
\Upsilon(T, t') \otimes O(S_i) \otimes \Upsilon(S_{t'}, s') \otimes A^{\otimes S - s'} \otimes F'(R) & \longrightarrow & \Upsilon(S, s) \otimes A^{\otimes S - s'} \otimes F'(R) \\
\downarrow & & \downarrow \\
\Upsilon(T, t') \otimes A^{\otimes t - 1} \otimes F'(R) & \longrightarrow & F'(R)
\end{array}$$

for $(S, s') = (\{0,1,\ldots,i\}, 0)$ and $(T, t') = (\{0,1,\ldots,j\}, 0)$. We define $F(R)$ as a quotient of $F'(R)$ by the ideal generated by the images of $\phi_1 - \phi_2$, where $\phi_1$ and $\phi_2$ are two diagonal (\sigma) maps in the above diagrams. It is easy to see then, that $F(R)$ constructed in this way satisfies the properties we need.

2.4.2. Put now $U = R$ and let us denote by $P_A$ the corresponding $A$-module $F(R)$: $\text{Hom}_A(P_A, M) \simeq M$ as an $R$-module, for any $A$-module $M$. Then $P_A$ has a natural structure of an associative algebra with a unit, since $P_A \simeq \text{End}_A(P_A)$, and the category of $A$-modules of type $\Upsilon$ is naturally equivalent to the category of right $P_A$-modules. For $\Upsilon = O$ the algebra $P_A$ is the universal enveloping algebra of $A$ in the terminology of [2].
2.4.3. Let now $B$ be another $O$-algebra and let us have a homomorphism from $B$ to $A$. Then the obvious restriction functor from the category of $A$-modules to the category of $B$-modules admits a left adjoint, called the induction functor. Its existence is obvious from the equivalence of categories of 2.4.2 and the fact that we have an associative algebra homomorphism $P_B \rightarrow P_A$.

2.5 Derivations. From now on we will consider modules over an algebra over an operad with $\mathcal{Y} = O$ and we will call them just $A$-modules.

2.5.1. Let $A$ be an algebra over an operad $O$ and let $M$ be an $A$-module. An $R$-linear map $\phi : A \rightarrow M$ is said to be a derivation (from $A$ to $M$) if for each finite set $S$ the diagram

$$
\begin{array}{c}
O(S) \otimes A^\otimes S \\
\sum_{s \in S} (\phi \otimes s \otimes id) \rightarrow \\
\sum_{s \in S} (O(S, s) \otimes A^\otimes S - s \otimes M) \\
\downarrow \\
A \rightarrow M
\end{array}
$$

is commutative.

The set of all derivations from $A$ to $M$ will be denoted by $\Omega(A, M)$

2.5.2. Suppose that $A$ is a free algebra $A = Free(V)$. Then $\Omega(A, M) = Hom_R(V, M)$ for any $A$-module $M$.

3. Sheaves and cohomology

3.0. Starting with an $O$-algebra $A$, we will construct a site $C(A)$. This definition appeared first in [1] where Quillen proved that cohomologies of certain sheaves on this site provide correct cohomology theories for $A$-modules (in the case of commutative, associative and Lie algebras).

3.1. Let us consider the category $C(A)$ consisting of $O$-algebras $B$ with a homomorphism $B \rightarrow A$. This category possesses a fibered product and we introduce a Grothendieck topology on it by declaring epimorphisms to be the covering maps.

Thus we can consider sheaves on $C(A)$ and their cohomologies.

3.2 Sheaves $\mathcal{S}_M$.

Let an $M$ be an $A$-module. Then it is also a module over each algebra $B \in C(A)$. We define a sheaf $\mathcal{S}_M$ on $C(A)$ by setting

$$
\Gamma(B, \mathcal{S}_M) = \Omega(B, M)
$$

Sheaf axioms are easily verified.

3.2.1. The following remark is due to essentially to [1]: Each $\mathcal{S}_M$ is representable by $A_M \in C(A)$, equal to $A \oplus M$ with the natural algebra structure on it, in other words

$$
\Gamma(B, \mathcal{S}_M) = \text{algebra homomorphisms } B \rightarrow A \oplus M.
$$

It is also not difficult to see that the functor $M \rightarrow \mathcal{S}_M$ is fully faithful.

3.3. Let us mention several properties of the category $C(A)$.
3.3.1. If $E \to D$ is a covering in $C(A)$, then the functor $C(A)_D$ (objects of $C(A)$ over $D$) $\to$ descent data on $E$ with respect to $D$ is an equivalence of categories.

3.3.2. If $V$ is a projective $R$-module with an $R$-module map to $A$, then $Free(V) \in C(A)$ and the functor $\mathfrak{S} : \Gamma(Free(V), \mathfrak{S})$ is exact. This follows e.g. from Remark 1.4.

3.3.3.

**Proposition.** The functor $\mathfrak{S} : (A\text{-modules} \to \text{sheaves})$ admits right and left adjoint functors, $R$ and $L$ respectively. We have $R \circ \mathfrak{S} \simeq L \circ \mathfrak{S} \simeq Id_{A\text{-mod}}$.

**Proof.** For each $X \in C(A)$ consider the sheaf $Const_X$ defined by

$$\Gamma(Y, Const_X) = \bigoplus_{Hom(Y,X)} R.$$  

The sheaf $Const_X$ is defined uniquely by the following property: $Hom(Const_X, S) \cong \Gamma(X, S)$ functorially in $S$—a sheaf over $C(A)$. Let $B = Free(V)$. Then

$$Hom(Const_{Free(V)}, \mathfrak{S}_M) = \Gamma(Free(V), \mathfrak{S}_M) = \Omega(Free(V), M) = Hom_R(V, M).$$

This fact together with 3.3.2 imply that the functor $\mathfrak{S}$ is exact.

In order to construct the functor $L$, it is sufficient to define the values of $L$ on sheaves of the form $Const_B$ for free algebras $B = Free(V)$, because any sheaf over $C(A)$ is a quotient of a direct sum of such sheaves. However, the above calculation shows that for these sheaves we can put $L(Const_{Free(V)}) = F(V)$ in the notation of 2.4.

To construct the functor $R$ we put for a sheaf $\mathcal{F}$, $R(\mathcal{F}) = Hom(\mathfrak{S}(P_A), \mathcal{F})$ with the obvious structure of a right module over $P_A$. Then we use 2.4.2.

The fact that $R \circ \mathfrak{S} \simeq L \circ \mathfrak{S} \simeq Id_{A\text{-mod}}$ follows from the full faithfulness of the functor $\mathfrak{S}$.

3.3.4. As always, the functor $L$ is right exact and the functor $R$ is left exact and we can consider their left (resp. right) derived functors $L^\cdot L$ (resp. $R^\cdot R$). (The functor $L$ can be derived e.g. because the sheaves $Const_{Free(V)}$ for projective $V$ are projective (!) in the category of sheaves of $R$-modules.)

3.4. Let us apply the functor $L^\cdot L$ to the sheaf $R_A$. We obtain an object $T_A$ in $D(A)$ (the derived category of $A$-modules). $T_A$ is called the cotangent complex of $A$.

$$RHom(T_A, M) \simeq R\Gamma(A, \mathfrak{S}_M)$$

3.4.1. The fact that $L$ is right exact implies that

$$H^i(T_A) = \begin{cases} 0, & i > 0 \\ I_A, & i = 0 \end{cases}$$

where $I_A$ is the $A$-module representing the functor $M \to \Omega(A, M)$.

3.4.2 Examples.
1. If $O$ is the Lie operad $O^{Lie}$, $I_A$ canonically identifies with the augmentation ideal of the universal enveloping algebra.

2. If $O = O^{ass,1}$, $I_A \simeq I = \ker(A \otimes A \to A)$.

3. If $O = O^{com,ass,1}$, $I_A \simeq I/I^2$, with $I$ as above.

4. Deformations

4.0. Results of this section are partially contained in [1],[3] and [4]. We decided to present them, since the formalism developed in the preceding sections seems to be a convenient tool for passage from deformations to cohomology. By definition, we put $H^i(A, M) = R\text{Hom}^i(TA, M)$.

4.1. Let $A$ be an $O$-algebra (over $R$). An $i$-th level deformation of $A$ is an $O$-algebra $A_i$ over $R[t]/t^{i+1}$ with an isomorphism $\phi : A_i/t \cdot A_i \simeq A$ and such that $\text{Tor}_1^{R[t]/t^{i+1}, R[t]}(A_i, R) = 0$.

In other words, we need that $\ker(t : A_i \to A_i) = \text{im}(t^i : A_i \to A_i)$ identifies under a natural map with $A$.

4.1.1. The category of $i$-th level deformations (morphisms are compatible with $\phi$’s) is a groupoid denoted by $\text{Deform}^i(A)$. For each $i$ there are functors from $\text{Deform}^{i+1}(A)$ to $\text{Deform}^i(A)$ (taking modulo $t^{i+1}$). If $A_i \in \text{Deform}^i(A)$, the fiber $\text{Deform}^{i+1}_{A_i}(A)$ of $\text{Deform}^{i+1}(A)$ over $A_i$ will be called the category of prolongations of $A_i$.

4.2. Theorem.

(1) The category of 1-st level deformations is equivalent to the category $T(\mathcal{S}_A)$ of $\mathcal{S}_A$-torsors. In particular, $\pi_o(C_1) \simeq H^1(A, A)$.

(2) If $A_{i+1} \in \text{Deform}^{i+1}_{A_i}(A)$, $\text{Aut}(A_{i+1})$ as of an object of this category is canonically isomorphic to $\Omega(A, A)$.

(3) To each $A_i \in \text{Deform}^i(A)$ we can associate a gerbe $G_{A_i}$ over $C(A)$ bounded by $\mathcal{S}_A$ in such a way that $G_{A_i}$ is canonically equivalent to $\text{Deform}^{i+1}_{A_i}(A)$.

In particular, this means that to each $A_i$ we can assign an element in $H^2(A, A)$ that vanishes if and only if $\text{Deform}^{i+1}_{A_i}(A)$ is nonempty. And if $\text{Deform}^{i+1}_{A_i}(A)$ is nonempty, its $\pi_o$ is a torsor over $H^1(A, A)$.

Proof.

(1) The functor $T : C(A) \to T(\mathcal{S}_A)$ is given by:

$$\Gamma(B, T(A_1)) = O - \text{algebras homomorphisms over } R:B \to A_1$$

Using 3.3.1 it is easy to show that it is an equivalence of categories.

(2) This is a direct verification.

(3) We define the gerbe as follows:
$G_{A_i}(B)$ is the groupoid of $R[t]/t^{i+2} \cdot R[t] - O$ algebras $B_{i+2}$ with an isomorphism $B_{i+1}/t^{i+1} \cdot B_{i+1} A_i$ such that

$$\ker(t^{i+1} : B_{i+1} \to B_{i+1}) = \text{im}(t : B_{i+1} \to B_{i+1})$$

and identifies under a natural morphism with $A_i$.

Functors $G_{A_i}(B) \to G_{A_i}(D)$ for maps $D \to B$ are given by taking fibre products.

It is easy to check that $G_{A_i}$ is indeed a gerbe bounded by $\mathcal{I}_{A_i}$ over $\mathcal{C}(A)$ and that its fiber over $A$ is equivalent to $\text{Deform}_{A_i}^{i+1}(A)$.

4.2.1. To summarize, we have shown that the cohomology groups $H^i(A, A)$ "control" the deformation theory of $A$.

4.3. For the remainder of this paper we restrict ourselves to the case $O = O^{\text{ass}}.1$.

4.3.1. When $A$ is flat as an $R$-module, we have a theorem of Quillen [1]:

**Theorem.** $H^i(T_A) = 0$ for $i \neq 0$.

By the cohomology long exact sequence of the triple

$$0 \to I_A \to A \otimes A \to A \to 0$$

we get that in this case $H^i(A, M) = \text{Ext}_{A \otimes A}^{i+1}(A, M)$ for any $A$-bimodule $M$ and $i \geq 1$.

**Variants.**

4.3.2. If $A$ is an augmented algebra we can look for its deformations in the class of augmented algebras. Then Theorem 4.2 remains valid after replacing $H^i(A, A)$ by $H^i(A, A_+)$, where $A_+$ denotes the augmentation ideal of $A$.

4.3.3. Suppose now that $A$ is a graded algebra. Then we will consider the site $C(A)$ that corresponds to graded algebras over $A$. If now $M$ is a graded $A$-bimodule, we introduce graded cohomology groups as $(H^i(A, M))_j = H^i(A, T^j(M))$, with $T$ being the translation functor of 2.3.3.

A graded deformation of $A$ of $i$-th level is an algebra $A_i$ as above endowed with a grading such that $\text{deg}(t) = 1$. Then the Theorem 4.2 is restated in the following way:

1. Isomorphism classes of first level deformations are in 1-1 correspondence with the elements of $(H^1(A, A))_{-1}$
2. The automorphism group of a prolongation of an $i$-th level deformation is naturally isomorphic to $\Omega(A, A)_{-i-1}$
3. The obstruction to the existence of a prolongation of a given $i$-th level deformation lies in $(H^2(A, A))_{-i-1}$
4. The set of isomorphism classes of prolongations of a given $i$-th level deformation is a torsor over $(H^1(A, A))_{-i-1}$

In the graded case Theorem 4.3.1 states that

$$(H^i(A, M))_j = (\text{Ext}_{A \otimes A}^{i+1}(A, M))_j$$

for any graded $A$-bimodule $M$. 
4.3.3'. Suppose now that we have a family \( A_i \) of graded deformations which are prolongations of one another. In this case we can form an algebra

\[
A_t = \text{elements of finite degree in } \lim_{\leftarrow} (A_i)
\]

This is an algebra over \( R[t] \). Consider its fiber at \( t = 1 : A_1 = A_t/(t - 1) \cdot A_t \). This algebra will carry a natural filtration and the associated graded algebra \( gr(A_1) \) will be canonically isomorphic to \( A \).

4.3.4. One, of course, can consider a combination of graded and augmented situations. Statement of Theorem 4.2 will change correspondingly.

5. The Poincaré-Birkhoff-Witt theorem

5.0. In this section we will give an application of the theory presented above. We will prove the Poincaré-Birkhoff-Witt theorem for Lie algebras over a ring \( R \) which are flat as \( R \)-modules.

5.1. Let us recall main definitions.

5.1.1. Let \( g \) be an \( R \)-module. Its \( i \)-th exterior power \( \wedge^i(g) \) is defined to be the subspace of \( g \otimes i \) spanned by tensors of the form \( \text{Alt}(g_1, g_2, \ldots, g_i) \), where \( \text{Alt} \) means alternating sum.

5.1.2. \( S^i(g) \) will denote the symmetric algebra of \( g \) which is by definition the quotient of the tensor algebra \( T^i(g) \) by the ideal generated by \( \wedge^2(g) \). It is a graded algebra with graded components denoted by \( S^i(g) \).

5.1.3. A Lie algebra structure on \( g \) is a map \([,] : \wedge^2(g) \to g\) such that the map 
\[
[,] \circ (id \otimes [,] - [,] \otimes id) : \wedge^3(g) \to g
\]
vanishes.

5.1.4. For a Lie algebra \( g \) its universal enveloping algebra \( U(g) \) is the quotient of the tensor algebra \( T^\cdot(g) \) by the ideal generated by \( \omega - [\omega][\omega] \) for \( \omega \in \wedge^2(g) \). \( U(g) \) carries a natural filtration and there is a canonical surjection \( S^\cdot(g) \to gr(U(g)) \).

5.2 The Poincaré-Birkhoff-Witt theorem.

Theorem. Let \( g \) be a Lie algebra over \( R \) which is flat as an \( R \)-module. Then the canonical epimorphism \( S(g) \to gr(U(g)) \) is an isomorphism.

5.2.1. We will prove this theorem by constructing a graded deformation as in 4.3.4 of \( S(g) \) in the class of augmented algebras. This idea is borrowed from [5], where the Poincaré-Birkhoff-Witt theorem is proved for Koszul algebras also by means of deformation theory.

By 4.3.2 and 4.3.4 we know that in this case the deformations of \( A = S(g) \) are controled by \( Ext_{A \otimes A}(A, A_+) \)'s.
5.2.2.

**Proposition.** Let \( g \) be a flat \( R \)-module. Then

\[
\begin{align*}
(\text{Ext}^2_{A \otimes A})_{-j} &= \begin{cases} 0, & j > 1 \\ \text{Hom}_R(\wedge^2(g), g) & j = 1 \end{cases} \\
(\text{Ext}^3_{A \otimes A})_{-j} &= \begin{cases} 0, & j > 2 \\ \text{Hom}_R(\wedge^3(g), g) & j = 2 \end{cases}
\end{align*}
\]

This proposition easily follows from the fact that for flat \( g \)-modules the Koszul complex

\[
\cdots \rightarrow S(g) \otimes \wedge^i(g) \otimes S(g) \rightarrow \cdots \rightarrow S(g) \otimes \wedge^2(g) \otimes S(g) \rightarrow S(g) \otimes g \otimes S(g) \rightarrow S(g)
\]

is exact.

5.3.

**Proof of the theorem.** We put \( A = S(g) \). This is a graded augmented algebra and by a deformation we will mean a graded deformation in the class of augmented algebras.

**Step 1.** Let us build a first level deformation of \( A \) that corresponds to the cohomology class of our \( [\cdot, \cdot] : \wedge^2(g) \rightarrow g \) via 4.3.4 and 5.2.2. It is canonical up to a unique isomorphism since \( \Omega(A, (A_+)^{-1}) = 0 \).

Untwisting of the passage (first level deformations \( \rightarrow \) cohomology) shows that there exists a canonical \( R \)-linear isomorphism \( \phi : g \rightarrow ((A_1)^+)_1 \) such that

\[
\phi(x) \cdot \phi(y) - \phi(y) \cdot \phi(x) = t\phi([x, y])
\]

**Step 2.** A direct verification shows that the obstruction to the existence of a prolongation of this deformation as a map \( \wedge^3(g) \rightarrow g \) is given by the expression

\[
[\cdot, \cdot] \circ (id \otimes [\cdot, \cdot]) - [\cdot, \cdot] \otimes id
\]

which in turn vanishes by the Jacobi identity. So a prolongation exists and is unique up to a unique isomorphism by 4.3.4. and 5.2.2.

**Step 3.** Again by 4.3.4 and 5.2.2 for any \( i \geq 2 \) an \( i \)-th level deformation of \( A \) can be prolonged in a unique up to a unique isomorphism way and so we find ourselves in the situation of 4.3.3'.

**Step 4.**

**Claim.** There exists a canonical \( R \)-linear map \( \phi' : g \rightarrow ((A_t)^+)_1 \) such that

\[
\phi'(x) \cdot \phi'(y) - \phi'(y) \cdot \phi'(x) = t\phi'([x, y])
\]

Indeed, since \( \text{deg}(t) = 1 \), there is a unique way to lift \( \phi \) of Step 1 to a map \( g \rightarrow ((A_t)^+)_1 \).

Then, by the definition of \( U(g) \) there exists a map \( \phi' : U(g) \rightarrow A_1 \) that prolongs the map \( \phi' \) above.
Step 5. Let us consider the associated graded map $gr(\phi') : gr(U(\mathfrak{g})) \to gr(A_1)$ and let us also consider the composition

$$S(\mathfrak{g}) \overset{5.1.4}{\to} gr(U(\mathfrak{g})) \overset{\phi'}{\to} gr(A_1) \overset{\sim}{\to} S(\mathfrak{g})$$

This composition is easily seen to be the identity map. This implies that

1. $\phi'$ is an isomorphism between $U(\mathfrak{g})$ and $A_1$
2. The Poincaré-Birkhoff-Witt theorem.

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