Palatini $f(R)$ cosmology and Noether symmetry

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We study Palatini $f(R)$ cosmology using Noether symmetry approach for the matter dominated universe. In order to construct a point-like Lagrangian in the flat FRW space-time, we use the dynamical equivalence between $f(R)$ gravity and scalar-tensor theories. The existence of Noether symmetry of the cosmological $f(R)$ Lagrangian helps us to find out the form of $f(R)$ and the exact solutions for cosmic scale factor. We show that this symmetry always exist for $f(R) \sim R^n$ and the Noether constant is a function of the Newton’s gravitational constant and the current matter content of the universe.

1. INTRODUCTION

The accelerated expansion of the universe which has revealed from the observations of type Ia supernova [1], is one of the biggest problems that theoretical physics faces nowadays. The equations of motion of GR cannot explain such an acceleration by standard sources of matter and energy and the mechanism that drives this observed expansion is yet unclear. In an attempt to justify this puzzle, some certain modifications of classical cosmology have been proposed. It should be noted that before this some modifications had been applied to GR standard too. For example the scalar-tensor theory of gravity had been proposed for improving GR from the Mach principle’s point of view [2]. However, the situation is different now and this cosmic speed up may be a serious signal of a failure of GR. Some authors follow the idea that dark energy sources such as cosmological constant or quintessence could be responsible for this acceleration [3, 4, 5, 6, 7, 8]. On the other hand, there exist some ideas that propose modification of the Hilbert-Einstein Lagrangian arguing that there could be nonlinear terms of Ricci scalar in the action. The theories constructed in this way are known as $f(R)$ gravity. These theories propose a geometric origin for cosmic speed up without any need for sources of dark energy. For a given $f(R)$ Lagrangian, as the GR case, there exist two ways for constructing the gravitational theory, namely, metric and Palatini formalisms. But unlike the GR case, this two formalisms are not equivalent, the former leads to a system of fourth-order partial differential equations for the metric whereas the later leads to second-order equations. It has been shown that many different $f(R)$ Lagrangians lead to correct cosmic acceleration [9, 10, 11], on the other hand, there exist some arguments against Palatini $f(R)$ gravity [12, 13]. In [12] author has discussed some aspects of the violation of the equivalence principle in these theories for nonlinear $f(R)$.

In this letter we consider the matter dominated universe in the flat FRW space-time, by using the Palatini formalism of $f(R)$ gravity and by following the so called Noether symmetry approach [14, 15, 16]. We look for $f(R)$ cosmological models which are consistent with Noether symmetry. Metric $f(R)$ cosmology has been examined by this approach [17] but the Palatini formalism hasn’t been considered in the literature yet. It is important to note that for applying the Noether symmetry approach, we should make a point-like lagrangian for the cosmological model. In the case of metric formalism, using the method of lagrange multipliers, the Ricci scalar and the cosmic scale factor can be considered as two independent dynamical variables and then by choosing a suitable lagrange multiplier, the cosmological lagrangian takes the point-like form. But in the case of Palatini formalism one cannot use this procedure. Varying with respect to Ricci scalar leads to an inappropriate lagrange multiplier since the time derivatives of variables appear in the denominator of the lagrange multiplier. So one cannot construct a canonical effective lagrangian. In this letter, in order to construct a point-like lagrangian, we use the dynamical equivalence between Palatini $f(R)$ gravity and Brans-Dicke theory.

The letter is organized as follows. In Sect. 2, we search for Noether symmetries of the action which leads to explicit forms for $f(R)$. Sect. 3 is devoted to find out the exact solutions for cosmic scale factor and the discussion of the various sub-cases.

2. NOETHER SYMMETRY APPROACH

Let us begin by introducing the action of the Palatini $f(R)$ theories

\[ S = \frac{1}{2k} \int d^4x \sqrt{-g} f(\tilde{R}) + S_m(g_{\mu\nu}, \psi_m) \]  

(1)

Here $f(\tilde{R})$ is a function of $\tilde{R} = g^{\mu\nu} R_{\mu\nu}(\tilde{\Gamma})$, where $\tilde{\Gamma}^a_{\mu\nu}$ is the connection. The matter action $S_m$ depends on the matter fields $\psi_m$ and $g_{\mu\nu}$. Varying Eq. (1) with respect to the metric we obtain

\[ f'(\tilde{R}) \tilde{R} - \frac{1}{2} f(\tilde{R}) g_{\mu\nu} = k T_{\mu\nu} \]  

(2)

where $f'(\tilde{R}) = df/d\tilde{R}$. The trace of Eq. (2) is

\[ f'(\tilde{R}) \tilde{R} - 2 f(\tilde{R}) = kT \]  

(3)

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(3)
Therefore we obtain a Lagrangian related to the action (6) takes the point-like form. One can easily verify that, in a time-ordered cosmology, the matter Lagrangian can be chosen as \( L_m = -\rho_m a^{-3} \) where \( a \) is the cosmic scale factor and \( \rho_m \) is a suitable constant related to matter content.

Using the flat Friedmann-Robertson-Walker metric, the scalar curvature takes the form \( R = -6(\dot{\phi}^2 + \ddot{\phi}) \), where the dot denotes the derivative with respect to time. In order to apply the Noether symmetry approach, one can easily verify that, in a FRW manifold, the Lagrangian related to the action (6) takes the point-like form

\[
\mathcal{L} = 12a^2 \dot{\varphi} \ddot{\varphi} + 6 \dot{a}^2 a + 6a \ddot{\varphi}^2 - V(\varphi) a^3 - 2k \rho_m a
\]

in which we have used the redefinition \( \Phi = \varphi^2 \). The equations of motion for \( a \) and \( \varphi \) are respectively

\[
\ddot{\varphi} + \varphi (H^2 + 2) + 2 \dot{\varphi} H + \frac{\dot{\varphi}^2}{2 \varphi} + \frac{V(\varphi)}{4 \varphi} = 0
\]

\[
\ddot{\varphi} + \varphi (H^2 + 2) + 3 \dot{\varphi} H + \varphi H^2 = 0
\]

here \( H \) is the Hubble parameter. Finally, as a result of general covariance, the energy function associated with the Lagrangian (7) vanishes, that is

\[
E_\mathcal{L} = 2 \dot{\varphi} H + \varphi H^2 + \frac{\dot{\varphi}^2}{\varphi} + \frac{V(\varphi)}{6 \varphi} + \frac{2k \rho_m}{\varphi a^3} = 0
\]

Now, let us introduce the lift vector field \( X \) as an infinitesimal generator of the Noether symmetry in the tangent space \( TQ = \{ a, \dot{a}, \dot{\varphi}, \varphi \} \) related to the configuration space \( Q = \{ a, \varphi \} \) as follows

\[
X = A \frac{\partial}{\partial a} + B \frac{\partial}{\partial \varphi} + \dot{A} \frac{\partial}{\partial a} + \dot{B} \frac{\partial}{\partial \varphi}
\]

where \( A \) and \( B \) are unknown functions of the variables \( a \) and \( \varphi \). The existence condition for the symmetry, \( L_X \mathcal{L} = 0 \), leads to the following system of partial differential equations

\[
2 \varphi A + a B + \varphi^2 \frac{\partial A}{\partial \varphi} + a \varphi \frac{\partial A}{\partial a} + a^2 \frac{\partial B}{\partial a} + a \varphi \frac{\partial B}{\partial \varphi} = 0
\]

\[
\varphi A + 2a B + 2a \varphi \frac{\partial A}{\partial a} + 2a^2 \frac{\partial B}{\partial a} = 0
\]

\[
3a^2 V(\varphi) A + B \frac{dV}{d\varphi} a^3 = 0
\]

From Eq. (15) we have

\[
A = \left[ -\frac{V'(\varphi)}{3V(\varphi)} \right] B a
\]

Substituting (16) into (13), we find that \( A = f(\varphi) a^n \) and \( -V'/3V = \frac{-2n}{1+2n} \varphi^{-1} \), where \( n \) is an arbitrary number. By substituting these results in (14) we obtain

\[
f(\varphi) = \beta \varphi^{n-1}
\]

where \( \beta \) is a constant. These results satisfy Eq. (12) for any arbitrary \( n \). From Eqs. (16) and (17) we have

\[
A = \beta a^n \varphi^{n-1}, B = \left( \frac{2n+1}{2n} \right) \beta a^{n-1} \varphi^n
\]

\[
V(\varphi) = \lambda \varphi^{\frac{2n+1}{2n}} = \Lambda \Phi^{\frac{2n+1}{2n}}
\]

where \( \lambda \) is a constant. In conclusion, the Noether symmetry for the Lagrangian (7) exists and the vector field \( X \) is determined by (18) and (11). We can rewrite Eq. (19) as follows

\[
\ddot{f}(\tilde{R}) - f(\tilde{R}) = \lambda [f'(\tilde{R})]^{\frac{2n+1}{2n}}
\]

and we can solve this equation for finding the form of \( f(\tilde{R}) \). There exist two series of solutions for this equation

\[
f(\tilde{R}) = g(n) \tilde{R}^{\frac{3n}{n-1}}
\]

\[
f(\tilde{R}) = a \tilde{R} - \lambda a^{\frac{2n+1}{2n}}
\]

where \( g(n) = \left[ 27n^3 \left( \frac{1}{2n+1} \right)^{\frac{2n+1}{2n}} \right]^{\frac{3n}{n-1}} (n-1) \). Solution (22) represents the Hilbert-Einstein action with cosmological constant. For this action metric and Palatini formalisms coincide. The existence of Noether symmetry means that there exists a constant of motion. The constant of motion for Lagrangian (6), \( \Sigma \), is

\[
\Sigma = A \frac{\partial \mathcal{L}}{\partial a} + B \frac{\partial \mathcal{L}}{\partial \varphi} = -\frac{6 \beta}{n} a^{n+1} \varphi^n \frac{d(\varphi a)}{dt}
\]
3. SOLUTIONS

3.1 CASE 1 $f(\tilde{R}) = g(n)\tilde{R}^{\frac{2n}{n-1}}$

From Eq. (3) we have

$$\tilde{R}f'\tilde{R} - 2f(\tilde{R}) = -k\rho m_0 a^{-3}$$

(24)

This equation can be regarded as an equation for $\tilde{R}$ in terms of the cosmic scale factor, that is

$$\tilde{R} = G(n)a^{\frac{1+n}{n}}$$

(25)

where $G(n) = \left[\frac{k\rho m_0 (1-n)}{g(n)(n+2)}\right]^\frac{1}{n}$. In this case the Noether constant is

$$\Sigma = -\frac{3\beta}{2n^2} \gamma(n) n a^{\frac{(n-1)}{2}}$$

(26)

in which:

$$\gamma(n) = \frac{3g(n)n}{n-1} G(n)^{\frac{2n+1}{n-1}}$$

(27)

we can use Eq. (26) to find out the time dependence of the cosmic scale factor, that is

$$a(t) = (a_0 + \sigma(n)t)^{\frac{2n}{n-1}}$$

(28)

where:

$$\sigma(n) = -\frac{n^2 \Sigma}{3\beta(n)} a^{\frac{(n-1)}{2}}$$

(29)

Now we want to consider that whether $a(t)$ obtained from Noether symmetry, satisfies the field equation of Palatini $f(\tilde{R})$ cosmology or not. The modified Friedmann equation in Palatini $f(\tilde{R})$ is

$$(H + \frac{f'}{2f})^2 = \frac{1}{6} \frac{k(\rho + 3p) + f}{f'}$$

(30)

where $\rho$ and $p$ are the energy density and the pressure of the cosmic fluid respectively. Now by using Eqs. (25) and (30), after putting $p = 0$, we have

$$a^2 a^{\frac{(n-1)}{n}} = \eta(n)$$

(31)

where:

$$\eta(n) = -\frac{2n^2 k\rho m_0}{3\gamma(n)} + \frac{2G(n)n(n-1)}{9}$$

(32)

so $a(t)$ (Eq. (28)) satisfies Eq. (31) if $\eta(n) = 4\sigma(n)^2$. This equation can be considered as a constraint for $\lambda$, $\beta$, $k$ and $n$, so $n$ can be an arbitrary value. Thus for any $n$ there exist Noether symmetry and one can easily, by using the constraint equation, show that the Noether constant of motion is related to the gravitational constant and matter content of the universe, that is

$$\Sigma^2 \sim (c[k\rho m_0 \frac{2n^2 + 6n + 1}{3n}] + d[k\rho m_0 \frac{2n + 2}{n+1}])$$

(33)

where $c$ and $d$ are functions of $n$.

3.2 CASE 2 $f(\tilde{R}) = \alpha\tilde{R} - \lambda a^\frac{2n}{n-1}$

If $\alpha = 1$, then the action is the Hilbert-Einstein action with cosmological constant $\Lambda = \lambda/2$. It is important to note that in this case, metric and Palatini formalisms coincide. By using Eq. (24) we have

$$R = 2\lambda + \frac{k\rho m_0}{a^3}$$

(34)

and the constant of motion is

$$\Sigma = -\frac{6\beta}{n}(aa^{n+1})$$

(35)

so the scale factor is

$$a(t) = (a_0 + \sigma(n)(n+2)t)^{\frac{2n}{n-1}}$$

(36)

in which $\sigma(n) = -\frac{n\Sigma}{6\beta}$. For $n = -2$ we have $a(t) = a_0 e^{\delta t}$, where $\delta = \frac{2\lambda}{\beta}$. The modified Friedmann equation in this case is $a^2 = \frac{\lambda a^2 k\rho m_0}{3\gamma}$. The solution for $a(t)$ which is obtained from Noether symmetry, Eq. (36), satisfies the generalized Friedmann equation only if $n = -1/2$, $\lambda = 0$ and $\sigma^2 = \frac{1}{2} k\rho m_0$. So in this case $a(t) = (a_0 + \frac{\Sigma}{\beta^2} t^{2/3})$, which is the special case of Eq. (28) when $n = -1/2$. Also it is easy to verify that the Noether constant is related to gravitational constant and matter density of the universe, that is $\Sigma^2 \sim k\rho m_0$.

4. CONCLUDING REMARKS

In this letter, we have considered the Palatini $f(\tilde{R})$ cosmology by a general method. The approach is based on the seek for Noether symmetry which allows to fix the form of $f(\tilde{R})$ in a physically motivated manner. In order to construct a point-like Lagrangian, we have used the dynamical equivalence between $f(\tilde{R})$ gravity and Brans-Dicke theory, since in the framework of Palatini $f(\tilde{R})$ gravity, one cannot fix the form of $f(\tilde{R})$ by using the Lagrange multipliers method [17].

We have shown that this symmetry always exist for $f(\tilde{R}) \sim R^n$ and the Noether constant is a function of the Newton’s gravitational constant and the current matter content of the universe. It is interesting to note that in the metric formalism, the Noether symmetry exists for some limited forms of $f(\tilde{R})$ such as $\tilde{R} + 2\Lambda$, $\tilde{R} + \alpha \tilde{R}^2$ and $R^{3/2}[16]$ in the vacuum case. We know that in vacuum, Palatini $f(\tilde{R})$ gravity is equivalent to GR with cosmological constant, so for any arbitrary function of $\tilde{R}$, the Noether symmetry exists. On the other hand, in the matter dominated universe the Noether symmetry exists only for $f(\tilde{R}) \sim R^{3/2}$ in the metric formalism[17] but in Palatini approach according to Eq. (21), our calculation shows that this symmetry exists considering any arbitrary power of $\tilde{R}$ in the lagrangian.
A further point which has to be stressed is that Noether symmetry approach don’t allow \( f(R) \) forms such as \( f(R) = R + \epsilon \phi(R) \), which have attracted many interest in the literature as the suitable forms \([20, 21]\).

However, this is a satisfactory result in the sense of the results that have been obtained in \([12]\), which imply that any \( f(R) \) in Palatini formalism, except linear one, can violates the equivalence principle. It is also important to mention that one cannot follow this approach in the metric-affine formalism of \( f(R) \) gravity, since there is no equivalence with Brans-Dicke theories when the matter action depends on the connection \([18]\).

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