CONTINUOUS FIELDS OF POSTLIMINAL $C^*$-ALGEBRAS

ALDO J. LAZAR

ABSTRACT. We discuss a problem of Dixmier [3, Problem 10.10.11] on continuous fields of postliminal $C^*$-algebras and the greatest liminal ideals of the fibers.

1. INTRODUCTION

In [3, Problem 10.10.11] Dixmier asked the following question: given a continuous field $((A(t), \Theta))$ of postliminal $C^*$-algebras over some space $T$ and with $B(t)$ the greatest liminal ideal of $A(t)$ and $\Theta' := \{x \in \Theta | x(t) \in B(t), t \in T\}$ is $((B(t)), \Theta')$ a continuous field of $C^*$-algebras? We shall call a continuous field of postliminal $C^*$-algebras for which the answer to this question is affirmative a tame continuous field.

An example of a continuous field that is not tame can be constructed over $T := \mathbb{N} \cup \{\infty\}$. We let $A(n), n \in \mathbb{N}$, be the unitization of $K(H)$, the algebra of all compact operators over an infinite dimensional Hilbert space $H$, and $A(\infty) := \mathbb{C}I_H$, $I_H$ being the identity operator on $H$. $\Theta$ consists of all the fields $x$ such that $x(n) = \lambda_n I_H + a_n$, $\{\lambda_n\}$ being a sequence in $\mathbb{C}$ that converges to some $\lambda \in \mathbb{C}$, $\{a_n\}$ being a sequence in $K(H)$ that converges to $\{0\}$, and $x(\infty) = \lambda I_H$. Then $((A(t))_{t \in T}, \Theta)$ is a continuous field of postliminal $C^*$-algebras. Now the largest liminal ideal of $A(n)$ is $B(n) = K(H)$ and the largest liminal ideal of $A(\infty)$ is $B(\infty) = A(\infty) = \mathbb{C}I_H$. Clearly $x \in \Theta$ satisfies $x(t) \in B(t)$ for every $t \in T$ only if $x(\infty) = 0$ and the continuous field is not tame.

In the next section we shall show that the continuous fields of postliminal $C^*$-algebras in a certain class that properly includes the locally trivial continuous fields...
are always tame. Afterwards we shall exhibit an example of a continuous field of postliminal $C^*$-algebras such that all its fibers are mutually isomorphic and its restriction to any open subset of the base space is not tame.

We shall use the terminology and the notation for continuous fields as introduced in [3, Chapter 10]. The preference to work with continuous fields rather than Banach bundles as it is more common nowadays is motivated by the fact that Dixmier’s original question was expressed in these terms. The closed unit ball of the Banach space $X$ is denoted $X_1$.

2. Results

The main ingredient in the proof of Proposition 1 that follows is Michael’s selection theorem [4, Theorem 3.2]: a multivalued map $\varphi$ from a paracompact space $T$ to the family of the non-void closed convex subsets of a Banach space $X$ that is lower semicontinuous admits a continuous selection, i.e., there is a continuous function $f: T \to X$ such that $f(t) \in \varphi(t)$ for every $t \in T$. Moreover, if $F$ is a closed subset of $T$ and $g: F \to X$ is a continuous selection for $\varphi|_F$ then one may choose $f$ so $f|_F = g$ is satisfied. Recall that $\varphi$ is called lower semicontinuous if for each open subset $U$ of $X$ the set \{ $t \in T$ | $\varphi(t) \cap U \neq \emptyset$ \} is open.

**Proposition 1.** Let $T$ be a paracompact space or a locally compact Hausdorff space and $X$ a Banach space. Denote by $\mathcal{M}$ the space of all the closed unit balls of the closed subspaces of $X$ endowed with the Hausdorff metric. Suppose $t \to X(t)_1$, $t \in T$, is a continuous map into $\mathcal{M}$, $X(t)$ being a closed subspace of $X$. With $\Gamma$ the space of all the continuous functions $\varphi: T \to X$ such that $\varphi(t) \in X(t)$, $t \in T$, $((X(t)), \Gamma)$ is a continuous field of Banach spaces.

**Proof.** The only evidence we must provide is that for $t_0 \in T$ and $x_0 \in X(t_0)$ there exists $\varphi \in \Gamma$ such that $\varphi(t_0) = x_0$. Clearly we may suppose $x_0 \neq 0$. Set $y_0 := x_0/\|x_0\|$. We claim that $t \to X(t)_1$ is lower semicontinuous as a multivalued map from $T$ to $X$. To see this let $U$ be an open subset of $X$, $s \in \{ t \in T \mid U \cap X(t)_1 \neq \emptyset \}$, and $z \in U \cap X(s)_1$. There are an open ball of $X$ of center $z$ and of radius $\varepsilon > 0$ contained in $U$ and a neighborhood $V$ of $s$ in $T$ such that $d(X(s)_1, X(t)_1) < \varepsilon$ for all $t \in V$, $d$ being the Hausdorff metric. Thus for each $t \in V$ there is $w_t \in X(t)_1$ for which $\|z - w_t\| < \varepsilon$. It follows that $V \subset \{ t \in T \mid U \cap X(t) \neq \emptyset \}$ and we conclude
that \( \{ t \in T \mid U \cap X(t) \neq \emptyset \} \) is open. We obtained that the map \( t \to X(t)_1 \) is indeed lower semicontinuous.

Suppose now that \( T \) is paracompact. By Michael’s selection theorem mentioned above there exists a continuous map \( \varphi' : T \to X \) such that \( \varphi'(t) \in X(t) \) for every \( t \in T \) and \( \varphi'(t_0) = y_0 \). The map \( \varphi \) defined by \( \varphi(t) := \|x_0\| \varphi'(t) \) suits the requirements.

Let now \( T \) be locally compact Hausdorff. Let \( W \) be a compact neighborhood of \( t_0 \). Again by Michael’s selection theorem there is a continuous map \( \varphi' : W \to X \) such that \( \varphi'(t) \in X(t) \) for every \( t \in W \) and \( \varphi'(t_0) = y_0 \). Let now \( f : T \to [0, 1] \) be a continuous function such that \( f(t_0) = 1 \) and \( f(t) = 0 \) for \( t \notin \text{Int}(W) \). The function \( \varphi : T \to X \) defined by

\[
\varphi(t) := \begin{cases} 
\|x_0\|f(t)\varphi'(t), & \text{if } t \in W, \\
0, & \text{if } t \notin W.
\end{cases}
\]

is continuous, satisfies \( \varphi(t) \in X(t) \) for \( t \in T \) and \( \varphi(t_0) = x_0 \).

\[ \square \]

A continuous field of Banach spaces over a paracompact or a locally compact Hausdorff space \( T \) isomorphic to a continuous field of Banach spaces as described in Proposition 1 will be called wieldy. Obviously, a trivial continuous field of Banach spaces is wieldy. A continuous field of Banach spaces over \( T \) is called locally wieldy if \( T \) has an open cover \( \{ U_\alpha \} \) such that its restriction to each \( U_\alpha \) is wieldy. By using the regularity of the base space as was done in the last paragraph of the previous proof one gets the following Proposition.

**Proposition 2.** The conclusion of Proposition 1 holds for locally wieldy continuous fields of Banach spaces.

Of interest for us are the continuous fields of \( C^* \)-algebras and it is natural to ask if a wieldy continuous field of \( C^* \)-algebras has to be locally trivial. There is some indication in \[2, Theorem 4.3\] that this may be the case when the fibers are nuclear and separable. On the other hand, \[2, Theorem 3.3\] provides an example of a wieldy continuous field of (non-separable) nuclear \( C^* \)-algebras that is not locally trivial.
Theorem 3. Suppose \(((A(t)), \Gamma)\) is a locally wieldy continuous field of postliminal \(C^*\)-algebras over a paracompact or locally compact Hausdorff space \(T\). Let \(B(t)\) be the largest liminal ideal of \(A(t)\) and \(\Gamma' := \{x \in \Gamma \mid x(t) \in B(t), t \in T\}\). Then \(((B(t)), \Gamma')\) is a locally wieldy continuous field of \(C^*\)-algebras over \(T\).

Proof. Set \(\alpha(s) := s + 1/2 - (1/4 - 2s)^{1/2}\) for \(s \in [0, 1/8]\). Let \(U\) be an open subset of \(T\) over which \(((A(t)), \Gamma)\) is wieldy. There is no loss of generality if we suppose that over \(U\) all the fibers \(A(t)\) are \(C^*\)-subalgebras of a certain \(C^*\)-algebra and \(t \to A(t)_1\) is continuous for the Hausdorff metric. If \(t', t'' \in U\) satisfy \(d(A(t')_1, A(t'')_1) < s(\leq 1/80)\) then it follows from Theorem 2.7 and Lemma 1.2 of [5] that \(d(B(t')_1, B(t'')_1) < 3s + 2\alpha(s)\). Hence \(t \to B(t)_1\) is continuous on \(U\) and the conclusion follows from Proposition 2. 

\(\square\)

It is very likely that in the statement of the theorem one may replace 'liminal' by 'uniformly liminal' or by 'Fell \((I_0)\)'. I intend to look at this by checking how the multiplicities of the irr. representations (Archbold) behave under the homeomorphism of Phillips.

Question 4. Let \(((A(t)), \Gamma)\) be a wieldy continuous field of postliminal \(C^*\)-algebras over \(T\). Can one choose a non-trivial continuous trace ideal \(B(t), t \in T\), of \(A(t)\) such that with \(\Gamma' := \{x \in \Gamma \mid x(t) \in B(t), t \in T\}\), \(((B(t)), \Gamma')\) is a continuous field of \(C^*\)-algebras?

Given a continuous field \(((A(t)), \Gamma)\) of postliminal \(C^*\)-algebras over a locally compact Hausdorff space \(T\), let \(A\) be the \(C^*\)-algebra defined by this continuous field as in [3, 10.4.1]; obviously it is a postliminal \(C^*\)-algebra. Let \(B\) be its greatest liminal ideal. If the image of \(B\) in \(A(t)\) by the evaluation map is the greatest liminal ideal of \(A(t), t \in T\), then it is easily seen that the given field is tame. Conversely, suppose that \(((A(t)), \Gamma)\) is tame and let \(C\) be the \(C^*\)-algebra defined by the continuous field of the greatest liminal ideals. Then \(C = B\) the greatest liminal ideal of \(A\). Indeed, it is clear that \(C\) is a liminal ideal of \(A\) so \(C \subset B\).

Let now \(x \in B\). With \(t \in T\), \(\rho \in \hat{A}(t)\), we have that \(y \to \rho(y(t)), y \in A\), is an irreducible representation of \(A\) hence \(\rho(x(t))\) is a compact operator over the space of the representation. We conclude by [3, 4.2.6] that \(x(t) \in C(t)\). Thus \(B \subset C\).
3. An Example

As mentioned in the Introduction, we are going to construct in this section a continuous field of postliminal $C^*$-algebras over $[0,1]$ whose fibers are mutually isomorphic and which has the additional property that none of its restrictions to the relatively open subsets of $[0,1]$ is tame.

First we proceed to prepare two $C^*$-algebras that will serve as building blocks of the fibers. Let $\mathbb{N} = \bigcup_{p=1}^{\infty} S_p$ where the sets $\{S_p\}$ are mutually disjoint and each $S_p = \{n_1^p < n_2^p < \ldots\}$ is infinite. Let $H$ be a separable Hilbert space with an orthonormal basis $\{\xi_k\}_{k=1}^{\infty}$ and $\mathcal{B}(H)$ the $C^*$-algebra of all the bounded operators on $H$. Denote by $e^0_{ij}$ the partial isometry that maps $\xi_j$ to $\xi_i$ and vanishes on each $\xi_k$ with $k \neq j$. The $C^*$-subalgebra of $\mathcal{B}(H)$ generated by $\{e^0_{ij} \mid i, j = 1, 2, \ldots\}$ is the ideal of all compact operators and we shall denote it by $A_0$. Put now $e^1_{ij} := \sum_{m=1}^{\infty} e^0_{n_i^m n_j^m}$, $i, j = 1, 2, \ldots$ where the series converges in the strong operator topology. Then $\{e^1_{ii}\}_{i=1}^{\infty}$ are mutually orthogonal projections, $\sum_{i=1}^{\infty} e^1_{ii} = 1_H$, $e^1_{ij}$ is a partial isometry from $e^1_{jj}(H)$ onto $e^1_{ii}(H)$, $(e^1_{ij})^* = e^1_{ji}$ and

$$e^1_{ij} e^1_{rs} = \begin{cases} e^1_{is}, & j = r \\ 0, & j \neq r. \end{cases}$$

Hence the $C^*$-subalgebra $K_1$ of $\mathcal{B}(H)$ generated by $\{e^1_{ij} \mid i, j = 1, 2, \ldots\}$ is isomorphic to $A_0$ and $A_0 \cap K_1 = \{0\}$. We have

$$(1) \quad e^1_{ij} e^0_{rs} = \begin{cases} e^0_{n_i^m n_j^m}, & \text{if } r = n_j^m \text{ for some } m, \\ 0, & \text{otherwise} \end{cases}$$

and

$$(2) \quad e^0_{rs} e^1_{ij} = \begin{cases} e^0_{n_i^m n_j^m}, & \text{if } s = n_i^m \text{ for some } m, \\ 0, & \text{otherwise.} \end{cases}$$

$A_1 := A_0 + K_1$ is a postliminal $C^*$-algebra since $A_0$ and $K_1 \sim A_1/A_0$ are postliminal $C^*$-algebras (actually liminal $C^*$-algebras in this case). Each $x \in A_1$ admits a unique decomposition $x = x_0 + x_{K_1}$ with $x_0 \in A_0$ and $x_{K_1} \in K_1$. The map $x \rightarrow x_{K_1}$ is a homomorphism hence $\|x_{K_1}\| \leq \|x\|$ and $\|x_0\| \leq 2\|x\|$.

From (1) and (2) it follows that the sequence $\{\sum_{i=1}^{m} e^1_{ii}\}_{m=1}^{\infty}$ is an increasing approximate unit for $A_1$ consisting of projections.
We suppose now that the $C^*$-subalgebras $K_1, \ldots, K_{i-1}$ of $B(H)$ have been defined such that $K_p, 1 \leq p \leq l-1$, is spanned by

$$e_{ij}^p := \sum_{m=1}^{\infty} e_{nm}^{p-1} n_{jm}, i, j = 1, 2, \ldots.$$  

It follows that $\{e_{ij}^p\}_{i=1}^{\infty}$ are mutually orthogonal projections, $\sum_{i=1}^{\infty} e_{ii}^p = 1_H$, and $e_{ij}^p$ is a partial isometry from $e_{jj}^p(H)$ onto $e_{ii}^p(H)$. With $A_p := A_{p-1} + K_p, 1 \leq p \leq l-1$, we have $A_{p-1} \cap K_p = \{0\}$, $A_p$ is a postliminal $C^*$-subalgebra of $B(H)$, $A_{p-1}$ is an ideal of $A_p$ and $\{\sum_{i=1}^{m} e_{ii}^p\}_{m=1}^{\infty}$ is an increasing approximate unit of $A_p$ consisting of projections. Every element $x \in A_p$ admits a unique decomposition $x = x_{p-1} + x_{K_p}$ where $x_{p-1} \in A_{p-1}, x_{K_p} \in K_p$. Moreover, $x \rightarrow x_{K_p}$ is a homomorphism hence $\|x_{K_p}\| \leq \|x\|$ and $\|x_{p-1}\| \leq 2\|x\|$.

We define now

$$e_{ij}^l := \sum_{m=1}^{\infty} e_{nm}^{l-1} n_{jm}, i, j = 1, 2, \ldots.$$  

Then $\{e_{ij}^l\}_{i=1}^{\infty}$ are mutually orthogonal projections, $\sum_{i=1}^{\infty} e_{ii}^l = 1_H$, and $e_{ij}^l$ is a partial isometry from $e_{jj}^l(H)$ onto $e_{ii}^l(H)$. We obtain $(e_{ij}^l)^* = e_{ji}^l$ and

$$e_{ij}^l e_{rs}^l = \begin{cases} e_{is}^l, & \text{if } j = r, \\ 0, & \text{otherwise,} \end{cases}$$

hence the $C^*$-subalgebra $K_l$ of $B(H)$ generated by $\{e_{ij}^l \mid i, j = 1, 2, \ldots\}$ is isomorphic to $A_0$. We also have

$$e_{ij}^l e_{rs}^{l-1} = \begin{cases} e_{ms}^{l-1}, & \text{if } r = n_s^l \text{ for some } m, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$e_{rs}^{l-1} e_{ij}^l = (e_{ji}^l e_{sr}^{l-1})^* = \begin{cases} e_{rs}^{l-1}, & \text{if } s = n_j^l \text{ for some } m, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, if $x \in K_l$ and $y \in K_{l-1}$ then $xy, yx \in K_{l-1}$. Suppose now that $x \in K_l, z \in A_{l-2}$. Then

$$xz = \lim_{m \rightarrow \infty} x \left( \sum_{i=1}^{m} e_{ii}^{l-1} \right) z = \lim_{m \rightarrow \infty} \left( x \sum_{i=1}^{m} e_{ii}^{l-1} \right) z.$$  

We established that $x \sum_{i=1}^{m} e_{ii}^{l-1} \in K_{l-1}$ for every $m$ hence $xz \in A_{l-2}$. Similarly, $zx \in A_{l-2}$ and we gather that $A_{l-2} = A_{l-2} + K_{l-1}, k \geq 2$, is an ideal in $A_l := A_{l-1} + K_l$ which is a $C^*$-subalgebra of $B(H)$ by [3, 1.8.4]. We want now to prove $A_{l-1} \cap K_l = \{0\}$. Denote by $B_l^m$ the finite dimensional $C^*$-algebra generated by
\( \{ e_{ij}^l \mid 1 \leq i, j \leq m \} \). Then \( K_1 = \bigcup_{m=1}^{\infty} B_1^m \). From

\[
\| \sum_{i,j=1}^{m} \alpha_{ij} e_{ij} - \sum_{i,j=1}^{m} \alpha_{ij} e_{ij}^{s} \| = \sum_{i,j=1}^{m} \alpha_{ij} e_{ij}^{l} \|\]

for every \( s \) we get \( B_1^m \cap A_{l-1} = \{0\} \) for every \( m \). Thus, the quotient map \( A_l \to A_l/A_{l-1} \) is isometric on each \( B_1^m \) hence it is isometric on \( K_1 \) and we conclude that \( A_{l-1} \cap K_l = \{0\} \). \( A_l \) is a postliminal \( C^* \)-algebra since \( A_{l-1} \) and \( A_l/A_{l-1} \sim K_1 \) are postliminal \( C^* \)-algebras.

In this manner we construct inductively an increasing sequence \( \{ A_l \}_{l=0}^{\infty} \) of postliminal \( C^* \)-subalgebras of \( B(H) \) such that \( A_{l-1} \) is an ideal in \( A_l \). It follows that

\[
A := \bigcup_{l=0}^{\infty} A_l \]

is a postliminal \( C^* \)-algebra of \( B(H) \) whose greatest liminal ideal is \( A_0 \).

Set now \( C_1 := K_1, C_2 := C_1 + K_2 \). \( C_1 \) is an ideal in \( C_2 \) hence \( C_2 \) is closed by [2, 1.8.4]. We have \( C_1 \cap K_2 = \{0\} \) and \( A_0 \cap C_2 = \{0\} \) since \( (A_0 + K_1) \cap K_2 = \{0\} \) and \( A_0 \cap K_1 = \{0\} \). We define inductively \( C_l := C_{l-1} + K_l \). Then \( C_{l-1} \) is an ideal of the \( C^* \)-algebra \( C_l, C_{l-1} \cap K_l = \{0\} \) and \( A_0 \cap C_l = \{0\} \). From [3], [4] and [5] we find that \( e_{ij}^l \to e_{ij}^{l-1}, 1 \leq i, j \geq 1 \) yields an isomorphism \( \varphi_p \) of \( C_p \) onto \( A_{p-1} \). Obviously \( \varphi_{p+1} \) extends \( \varphi_p \) hence one gets an isomorphism \( \varphi \) from \( C := \bigcup_{p=1}^{\infty} C_p \) onto \( A \) that extends each \( \varphi_p \). Let now \( x \in A, x = \lim_{p \to \infty} x_p \) with \( x_p \in A_p, p \geq 1 \). Then \( x_p = x_0^p + x_p^p \) where \( x_0^p \in A_0 \) and \( x_p^p \in C_p \). From \( \| x_0^p - x_0^q \| \leq 2 \| x_p - x_q \| \)

we conclude that the Cauchy sequence \( \{ x_0^p \}_{p=1}^{\infty} \) converges to some \( x^0 \in A_0 \) hence \( \{ x_p^p \}_{p=1}^{\infty} \) converges to some \( x^C \in C \). Now \( x \to x_p^p \) is a homomorphism for each \( p \) therefore \( x \to x^C \) is a homomorphism and \( \| x^C \| \leq \| x \| \) and \( \| x^0 \| \leq 2 \| x \| \). The quotient map \( A \to A/A_0 \) is isometric on each \( C_p \) hence it is isometric on \( C \). It follows that \( A_0 \cap C = \{0\} \) and the decomposition \( x = x^0 + x^C \) is unique.

Now we can begin constructing the continuous field of \( C^* \)-algebras we need. Let \( \{ r_n \} \) be an enumeration of the set of rational numbers in \( [0,1] \). For an irrational number \( t \in [0,1] \) we define \( A(t) := c_0(A) \) that is, the direct sum of \( A \) with itself \( \aleph_0 \) times. \( A(r_n) \) is a \( C^* \)-subalgebra of \( c_0(A) \) that is also a direct sum of copies of \( A \) except that at the \( n \)-th spot we insert \( C \) instead of \( A \). Obviously all the fibers are mutually isomorphic postliminal \( C^* \)-algebras. The \( * \)-algebra \( \Gamma \) of the continuous vector fields consists of all the continuous functions \( x : [0,1] \to c_0(A) \) such that \( x(t) \in A(t) \) for every \( t \in [0,1] \).
To show that \((A(t))_{t \in [0,1]} \cap \Gamma\) so defined is a continuous field we must check that 
\(\{x(t) \mid x \in \Gamma\} = A(t)\) for \(t \in [0,1]\). To this end let \(t_0 \in [0,1]\) and \(\{a_n\} \in A(t_0)\).

For \(n \in \mathbb{N}\) let \(f_n : [0,1] \to [0,1]\) be a continuous function such that \(f_n(t_0) = 1\) and \(f_n(r_n) = 0\) if \(r_n \neq t_0\). Define \(x(t) := \{f_n(t)a_n\}, t \in [0,1]\). Then \(x\) is a continuous function from \([0,1]\) to \(c_0(A)\) such that \(x(t) \in A(t)\) for \(t \in [0,1]\) i.e. \(x \in \Gamma\). Moreover \(x(t_0) = \{a_n\}\) and we have proved that we constructed a continuous field of \(C^*\)-algebras.

The greatest liminal ideal \(B(t)\) of \(A(t)\) is \(c_0(A_0)\) when \(t\) is irrational. The greatest liminal ideal \(B(r_n)\) of \(A(r_n), n \in \mathbb{N}\), is again a direct sum whose components are all equal to \(A_0\) except the one at the \(n\)-th place that it is equal to \(K_1\). Thus if \(x \in \Gamma\) satisfies \(x(t) \in B(t)\) for every \(t \in [0,1]\) then the \(n\)-th component of \(x(r_n)\) must vanish since \(A_0 \cap C = \{0\}\). It follows that for the restriction of our continuous field of \(C^*\)-algebras to any relatively open subset \(U\) of \([0,1]\) the family \((B(t))_{t \in U}\) together with \(\{x \in \Gamma \mid x(t) \in B(t), t \in U\}\) does not form a continuous field of \(C^*\)-algebras.

References

1. E. Christensen and M.-D. Choi, Completely order isomorphic and close \(C^*\)-algebras need not be *-isomorphic, Bull. London Math. Soc. 15 (1983), 604–610.
2. E. Christensen, A. M. Sinclair, R. R. Smith, S. A. White and W. Winter, Perturbations of nuclear \(C^*\)-algebras, Acta Math. 208 (2012), 93–150.
3. J. Dixmier, \(C^*\)-algebras, North-Holland, Amsterdam, 1977.
4. E. Michael, Continuous selections. I, Ann. of Math. 63 (1956), 361–382.
5. J. Phillips, Perturbations of \(C^*\)-algebras, Indiana Univ. Math. J. 23 (1974), 1167–1176.

School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel
E-mail address: aldo@post.tau.ac.il