Existence and Hölder continuity conditions for self-intersection local time of Rosenblatt process

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\textbf{ABSTRACT}
We consider the existence and Hölder continuity conditions for the self-intersection local time of Rosenblatt process. Moreover, we study the cases of intersection local time and collision local time, respectively.

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1. Introduction
Fractional Brownian motion (fBm) on $\mathbb{R}$ with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $B^H = \{B^H_t, \ t \geq 0\}$ with covariance function given by
\[
\mathbb{E}[B^H_tB^H_s] = \frac{1}{2} \left[ t^{2H} + s^{2H} - |t-s|^{2H} \right].
\]
Note that $B^H_t$ is a classical standard Brownian motion. Let $D = \{(r,s): 0 < r < s < t\}$. The self-intersection local time (SLT) of fBm was first investigated in [1] and formally defined as
\[
\zeta_t(y) := \int_D \delta(B^H_s - B^H_r - y)drds,
\]
where $y \in \mathbb{R}$, $B^H$ is a fBm and $\delta$ is the Dirac delta function. Intuitively, $\zeta_t(0)$ measures the amount of time that the process spends intersecting itself on the time interval $[0,t]$ and has been an important topic of the theory of stochastic process.

After [1], the topic of SLT of fBm has received a great deal of attention. In particular, Hu and Nualart [2] showed the existence of $\zeta_t(y)$ whenever $H < 1$. Moreover, $\zeta_t(y)$ is
Hölder continuous in time of any order strictly less than $1 - H$ which can be derived from [3]. These results arouse the interest of many scholars in this research direction. The corresponding results are not only enriched from one-dimensional case to multi-dimensional case, but also extend the SLT itself to its derivative, including [4–12] and references therein.

The Gaussian property of fBm makes its characteristic function present the form of exponential decay, which guarantees the finiteness of the moment of SLT. However, for non-Gaussian processes, there is no such good characteristic function. This makes the corresponding results of local time not rich. To the best of our knowledge, the study of stable motion only appears in [13], Rosenblatt process case is only discussed in [14, 15]. In particular [15] have studied the path properties of local time for the Rosenblatt process, which gives us the motivation to study the SLT of the Rosenblatt process in this paper.

In addition, inspired by the study of intersection local time (ILT) of Gaussian process in [16–18] and the study of collision local time (CoLT) of Gaussian process in [19, 20], we also consider the ILT and CoLT of Rosenblatt process in this paper.

We formally define the SLT of Rosenblatt process at $y$ as

$$\hat{a}_{t}(y) = \int_{D} \delta(X_{s}^{H} - X_{r}^{H} - y) dr ds,$$

where $X^{H}$ is a Rosenblatt process (the definition given in Section 2) with parameter $H \in (1/2, 1)$ and $\delta$ is the Dirac delta function.

Set

$$f_{\varepsilon}(x) = \frac{1}{\sqrt{2\pi \varepsilon}} e^{-\frac{|x|^2}{2\varepsilon}} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ipx} e^{-\frac{|p|^2}{2 \varepsilon}} dp.$$  

Since the Dirac delta function $\delta$ can be approximated by $f_{\varepsilon}(x)$, we can approximate $\hat{a}_{t}(y)$ by

$$\hat{a}_{t, \varepsilon}(y) = \int_{D} f_{\varepsilon}(X_{s}^{H} - X_{r}^{H} - y) dr ds, \quad \text{as} \quad \varepsilon \to 0.$$  

(1.1)

If $\hat{a}_{t, \varepsilon}(y)$ converges to a random variable in $L^{p}$ as $\varepsilon \to 0$, we denote the limit by $\hat{a}_{t}(y)$ and call it the SLT of $X^{H}$. Then we can obtain the Hölder continuous both in time and space.

**Theorem 1.1.** If $\frac{1}{2} < H < 1$, then $\hat{a}_{t}(y)$ exists in $L^{p}$, for all $p \geq 1$. Moreover, $\hat{a}_{t}(y)$ is Hölder continuous in $y$ of any order strictly less than $\min\{1, \frac{1}{H} - 1\}$ and Hölder continuous in $t$ of any order strictly less than $1 - H$, that is

$$\left| \mathbb{E}\left[ (\hat{a}_{t}(x) - \hat{a}_{t}(y))^{\mu} \right] \right| \leq c |x - y|^{|\mu|},$$

where $\lambda < \min\{1, \frac{1}{H} - 1\}$ and

$$\left| \mathbb{E}\left[ (\hat{a}_{t}(y) - \hat{a}_{t}(y))^{\nu} \right] \right| \leq c |t - \tilde{t}|^{|\nu|},$$

where $\beta < 1 - H$.

**Remark 1.1.** The condition $H > 1/2$ is a necessary condition, which is introduced in the definition of Rosenblatt process and guarantees the finiteness of integral (3.6) in the
proof of Theorem 1.1. Moreover, for \((d \geq 2)\)-dimensional Rosenblatt processes, we can define its SLT. However, the \(L^p\) existence may not be found, because the finiteness of its \(p\)-th moment needs condition \(Hd < 1\), which is contrary to \(H > 1/2\).

Similarly, we can define the ILT of Rosenblatt process with parameter \(H \in (1/2, 1)\). Define

\[
\tilde{x}_t(y) := \int_{[0, t]} \delta(X_s^{H_1} - \tilde{X}_s^{H_2} - y) \, ds,
\]

where \(X_s^{H_1}\) and \(\tilde{X}_s^{H_2}\) are two independent Rosenblatt processes. We can approximate \(\tilde{x}_t(y)\) by

\[
\tilde{x}_{t, \varepsilon}(y) = \int_{0}^{t} \int_{0}^{t} f_c(X_s^{H_1} - \tilde{X}_s^{H_2} - y) \, ds \, dr,
\]

as \(\varepsilon \to 0\).

Then we obtain the \(L^p\) existence and Hölder continuity of \(\tilde{x}_t(y)\).

Theorem 1.2. If \(\frac{H_1 + H_2}{H_1 + H_2} < 1\), then \(\tilde{x}_t(y)\) exists in \(L^p\), for all \(p \geq 1\). Moreover, \(\tilde{x}_t(y)\) satisfies the Hölder continuity:

\[
\|\mathbb{E}[(\tilde{x}_t(x) - \tilde{x}_t(y))]\| \leq c |x - y|^\tilde{\lambda},
\]

where \(\tilde{\lambda} < \min\{1, \frac{H_1 + H_2}{2H_1H_2} - \frac{1}{2}\}\) and

\[
\|\mathbb{E}[(\tilde{x}_t(y) - \tilde{x}_t(y))]\| \leq c |t - \tilde{t}|^\tilde{\beta},
\]

where \(\tilde{\beta} < 1 - \frac{H_1 + H_2}{H_1 + H_2}\).

Moreover, we can also define the CoLT of Rosenblatt process with parameter \(H \in (1/2, 1)\). Let

\[
\tilde{x}_t(y) := \int_{[0, t]} \delta(X_s^{H_1} - \tilde{X}_s^{H_2} - y) \, ds,
\]

where \(X_s^{H_1}\) and \(\tilde{X}_s^{H_2}\) and are two independent Rosenblatt processes. \(\tilde{x}_t(y)\) can also be defined as the limit of

\[
\tilde{x}_{t, \varepsilon}(y) = \int_{[0, t]} f_c(X_s^{H_1} - \tilde{X}_s^{H_2} - y) \, ds, \quad \varepsilon \to 0.
\]

We now give the \(L^p\) existence and Hölder continuity of \(\tilde{x}_t(y)\) as follows.

Theorem 1.3. If \(\frac{H_1 + H_2}{H_1 + H_2} < \frac{1}{2}\), then \(\tilde{x}_t(y)\) exists in \(L^p\), for all \(p \geq 1\). Moreover, \(\tilde{x}_t(y)\) satisfies the Hölder continuity:

\[
\|\mathbb{E}[(\tilde{x}_t^{(k)}(x) - \tilde{x}_t^{(k)}(y))]\| \leq c |x - y|^\tilde{\lambda},
\]

where \(\tilde{\lambda} < \min\{1, \frac{H_1 + H_2}{2H_1H_2} - \frac{1}{2}\}\) and

\[
\|\mathbb{E}[(\tilde{x}_t(y) - \tilde{x}_t(y))]\| \leq c |t - \tilde{t}|^\tilde{\beta},
\]

where \(\tilde{\beta} < 1 - \frac{2H_1H_2}{H_1 + H_2}\).
It should be noted that our results are not applicable to the case of derivative of SLT (ILT, CoLT) for Rosenblatt process, because the Rosenblatt process can be rewritten as the second chaos (see (2.3)), its characteristic function is similar to the chi-squared distribution, showing polynomial decay. If the derivative case is considered, we need to add the term of polynomial divergence in the integrand of (3.4), which makes the finiteness of the integral cannot be guaranteed.

Moreover, we think the related results can be extended to the general cases, such as the case of Hermite process (with parameter $q \geq 1$, see in (2.1)). Although there are few references about the local time of Hermite process for $q > 2$, only reference [21] considers the existence for the local time of the solution of Hermite sheet wave equation, our generalization from fBm ($q = 1$) and Rosenblatt process ($q = 2$) to Hermite process ($q > 2$) is a very intuitive extension. These processes have many same properties, such as the stationarity of increment, self-similarity. So we believe that our results can be extended to Hermite processes in the future, if Hermite processes are understood as $q$-th chaos and the corresponding spectral representation is obtained.

The paper has the following structure. Section 2 contains some necessary preliminaries on Rosenblatt process. Section 3 is to prove the main results. To be exact, we will split this section into three subsections to prove the three theorems given in Section 1. Throughout this paper, if not mentioned otherwise, the letter $c$, with or without a subscript, denotes a generic positive finite constant and may change from line to line.

2. Preliminaries

In this section, we first give the definition of Rosenblatt process described in [22, 23], then give the result of spectral representation for Rosenblatt process obtained in [15].

The Hermite process (include Rosenblatt process) is an interesting class of self-similar processes with long-range dependence, it is given as limits of the so-called non-central limit theorem studied in [24, 25]. Let us briefly recall the general context.

Denote by

$$H_j(x) = (-1)^j e^{x^2} \frac{d^j}{dx^j} e^{-x^2}, \quad j = 1, 2, ...$$

the Hermite polynomial of order $j$ with $H_0(x) = 1$, and let the Borel function $g : \mathbb{R} \to \mathbb{R}$ satisfy $\mathbb{E}[g(\xi_0)] = 0$, $\mathbb{E}|g(\xi_0)|^2 < \infty$ and $g(x) = \sum_{j=0}^{\infty} c_j H_j(x)$, $c_j = \frac{1}{j!} \mathbb{E}[g(\xi_0)H_j(\xi_0)]$. The Hermite rank of $g$ is defined by

$$q = \min\{j : c_j \neq 0\}.$$ 

Clearly, $q \geq 1$ since $\mathbb{E}[g(\xi_0)] = 0$.

Let $g$ be a function of Hermite rank $q$ and let $\{\xi_n, n \in \mathbb{N}\}$ be a stationary centered Gaussian sequence with $\mathbb{E}|\xi_n|^2 = 1$, which exhibits long-range dependence in the sense that the correlation function satisfies

$$r(n) := \mathbb{E}(\xi_0 \xi_n) = n^{2H-2} L(n),$$

where $q \geq 1$ is an integer, $H \in (\frac{1}{2}, 1)$, and $L$ is a slowly varying function at infinity. Then, the non-central limit theorem implies that the stochastic processes $\frac{1}{n^q} \sum_{j=1}^{[nt]} g(\xi_j)$
([nt] denotes the integer part of nt) converges, as \( n \to \infty \), in the sense of finite dimensional distributions to the process

\[
X_{t}^{H,q} = c(H,q) \int_{\mathbb{R}^q} \int_0^t \left( \prod_{j=1}^q (s - y_j)^- \right) ds B(dy_1) \cdots B(dy_q),
\]

(2.1)

where \( x_+ = \max\{x,0\} \) and the above integral is a Wiener-Itô multiple integral with respect to the standard Brownian motion \( \{B(y), y \in \mathbb{R}\} \) excluding the diagonals \( \{y_i = y_j\}, i \neq j \). \( c(H,q) \) is a positive normalization constant depending only on \( H \) and \( q \) such that \( \mathbb{E}(X_{1}^{H,q})^2 = 1 \). The process \( \{X_{t}^{H,q}, t \geq 0\} \) is called the Hermite process of order \( q \), it is \( H \)-self-similar and has stationary increments. The class of Hermite processes includes \( \text{fBm} \ (q = 1) \), which is the only Gaussian process in this class. Since they are non-Gaussian (\( q \geq 2 \)) and self-similar with stationary increments, the Hermite processes can also be an input in models where self-similarity is observed in empirical data which appears to be non-Gaussian. When \( q = 2 \), the process (2.1) is known as the Rosenblatt process.

The Rosenblatt process \( \{X_{t}^{H} := X_{t}^{H,2}, t \geq 0\} \) admits the following stochastic representation (spectral representation):

\[
X_{t}^{H} = \int_{\mathbb{R}^2} H_{t}(x,y)Z_{G}(dx)Z_{G}(dy),
\]

(2.2)

where the double Wiener-Itô integral is taken over \( x \neq y \), \( Z_{G}(dx) \) is a complex-valued random white noise with control measure \( G \) satisfying \( G(dx) = |x|^{-H} dx \), and

\[
H_{t}(x,y) = \frac{e^{t(x+y)} - 1}{i(x + y)}
\]

is a complex valued Hilbert-Schmidt kernel with \( H_{t}(x,y) = H_{t}(y,x) = H_{t}(\overline{x,y}) \) and

\[
\int_{\mathbb{R}^2} |H_{t}(x,y)|^2 G(dx)G(dy) < \infty.
\]

In particular, by the spectral theorem, we can rewrite \( X_{t}^{H} \) as an element in the second chaos

\[
X_{t}^{H} \overset{\text{law}}{=} \sum_{j=1}^{\infty} \lambda_j (Y_j^2 - 1),
\]

(2.3)

where \( \{Y_j, j \geq 1\} \) is a sequence of independent standard Gaussian random variable and \( \{\lambda_j, j \geq 1\} \) are the eigenvalues of the self-adjoint operator \( A : L_G^2(\mathbb{R}) \to L_G^2(\mathbb{R}) \),

\[
(Af)(x) = \int_{\mathbb{R}} H_{t}(x, -y)f(y)G(dy) = \int_{\mathbb{R}} H_{t}(x, -y)f(y)|x|^{-H} dy.
\]

Moreover, we need condition \( \sum_{j=1}^{\infty} \lambda_j^2 < \infty \) to make sure (2.3) converges.
3. Proof of the main results

In this section, the proof of Theorems 1.1, 1.2 and 1.3 are taken into account. We will divide this section into three parts and give the proof of the corresponding theorem in each part.

3.1. Proof of Theorem 1.1

3.1.1. Existence in $L^p$

We first prove that $\hat{\alpha}_{t,e}(y)$ converges in $L^p$ for $p \geq 1$. Since $|e^{-ipy}| = 1$, we only need to consider the finiteness of $\left| \mathbb{E}(\hat{\alpha}_{t,e}(0)) \right|^p$.

By (1.1), $\hat{\alpha}_{t,e}(0)$ can be written as

$$\hat{\alpha}_{t,e}(0) = \int_0^t \int_0^\beta f_i(X_i^H - X_i^H)drds$$

$$= \frac{1}{2\pi} \int_0^\beta \int_0^\beta e^{-ipx^2} e^{ip(x_i^H - x_i^H)} dpdrds =: \hat{\alpha}_{t,e},$$

where $D = \{0 < r < s < t\}$ and $i = \sqrt{-1}$.

For any integer $n \geq 1$,

$$\left| \mathbb{E}(\hat{\alpha}_{t,e})^n \right| \leq \frac{1}{(2\pi)^n} \int_0^\beta \int_0^\beta \left| e^{-i\sum_{i=1}^n p_i^2} \right| \left| \mathbb{E}e^{i\sum_{i=1}^n p_i(x_i^H - x_i^H)} \right| dpdrds. \quad (3.1)$$

We use the method of sample configuration as in [8]. Fix an ordering of the set $\{r_1, s_1, r_2, s_2, \ldots, r_n, s_n\}$, and let $l_1 < l_2 < \cdots < l_{2n}$ be a relabeling of the set $\{r_1, s_1, r_2, s_2, \ldots, r_n, s_n\}$. Let $u_1 \cdots u_{2n-1}$ be the proper linear combinations of the $p_i$’s so that

$$\mathbb{E}\left[e^{i\sum_{i=1}^n p_i(x_i^H - x_i^H)}\right] = \mathbb{E}\left[e^{i\sum_{i=1}^{2n-1} u_i(x_i^H - x_i^H)}\right].$$

Therefore, we can rewrite (3.1) as

$$\left| \mathbb{E}(\hat{\alpha}_{t,e})^n \right| \leq c \int_{E^n} \int_{\mathbb{R}^n} \mathbb{E}e^{i\sum_{i=1}^{2n-1} u_i(x_i^H - x_i^H)} dpdt, \quad (3.2)$$

where $E^n = \{0 < l_1 < \cdots < l_{2n} < t\}$.

By (2.2), we have

$$\sum_{i=1}^{2n-1} u_i(x_i^H - x_i^H) = \int_{\mathbb{R}} \sum_{i=1}^{2n-1} u_i \frac{e^{i\ell_{i+1}(x+y)} - e^{i\ell_i(x+y)}}{i(x+y)} Z_0(dx)Z(dy),$$

where the integral is taken over $x \neq y$. Define an operator $A_{\Delta, u}$,

$$(A_{\Delta, u}f)(x) = \int_{\mathbb{R}} \sum_{i=1}^{2n-1} u_i \frac{e^{i\ell_{i+1}(x+y)} - e^{i\ell_i(x+y)}}{i(x+y)} f(y)|y|^{-H} dy.$$  

Thus, similar to (2.3), we have

$$\sum_{i=1}^{2n-1} u_i(x_i^H - x_i^H) \overset{L^2}{=} \sum_{j=1}^{\infty} \lambda_j \left(Y_j^2 - 1\right), \quad (3.3)$$
where \( \{Y_j, j \geq 1\} \) is a sequence of independent standard Gaussian random variable and \( \{\lambda_j, j \geq 1\} \) are the eigenvalues of operator \( A_{\Delta, u} \). \( Y_j^2 \) is a chi-squared distribution, and its characteristic function is \( \mathbb{E}e^{itY_j^2} = (1 - 2it)^{-1/2} \). This gives
\[
|\mathbb{E}e^{i\sum_{j=1}^n p(X_n^j - X_0^j)}| = \prod_{j=1}^n \frac{e^{-i\lambda_j}}{\sqrt{1 - 2it\lambda_j}} = \prod_{j=1}^n (1 + 4\lambda_j^2)^{-1/4}.
\]
Substituting the above equation into (3.2),
\[
|\mathbb{E}(\bar{\phi}_{i, u})^n| \leq c \int_{\mathbb{R}^n} \prod_{j=1}^n (1 + 4\lambda_j^2)^{-1/4} dp d\ell.
\]  
Next, using the methods of Lemma 2.2 in [15], we obtain that
\[
\lambda_j \geq c_H \max_{1 \leq i \leq 2n-1} \{ |u_i| |\ell_{i+1} - \ell_i| \} \bar{\mu}_j^2
\]  
where \( \bar{\mu}_u \sim \bar{c}_H n^{-\frac{d}{2}}, \) and \( c_H, \bar{c}_H > 0 \) are constants that only depends on \( H \).

In fact, let
\[
B_{\Delta, u} := c_H K_{H/2} M_g K_{H/2} : L^2(\mathbb{R}) \to L^2(\mathbb{R}),
\]
where \( g(x) = \sum_{i=1}^{2n-1} u_i \mathbf{1}_{[\ell_i, \ell_{i+1})}(x) \), \( M_g \) is the multiplication operator \( (M_g f)(x) = g(x)f(x) \) and \( K_{H/2} \) is a convolution operator defined via the Fourier transform \( (K_{H/2} f)(x) = |x|^{-H/2} \hat{f}(x) \). Then we define operator
\[
(Tf)(x) = |x|^{H/2} f(x).
\]
Note that, \( T : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) is an isometric isomorphism. Thus, the operator \( A_{\Delta, u} \) is isometrically isomorphic to \( V_{\Delta, u} = TA_{\Delta, u} T^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) and satisfies
\[
V_{\Delta, u} f(x) = |x|^{H/2} \int_{\mathbb{R}} \sum_{i=1}^{2n-1} u_i \frac{e^{i\ell_{i+1}(x-y)} - e^{i\ell_i(x-y)}}{i(x-y)} f(y) |y|^{-H/2} dy.
\]
The Fourier transform of \( g \) is
\[
(\mathcal{F}g)(x) = \left( \sum_{i=1}^{2n-1} u_i \mathbf{1}_{[\ell_i, \ell_{i+1})} \right)(x) = \sum_{i=1}^{2n-1} u_i \frac{e^{-i\ell_{i+1}x} - e^{-i\ell_ix}}{-ix},
\]
which gives
\[
\mathcal{F}^3(K_{H/2} M_g K_{H/2} f)(x) = 2\pi \mathcal{F}(K_{H/2} M_g K_{H/2} f)(-x) = 2\pi |x|^{-H/2} \int_{\mathbb{R}} \sum_{i=1}^{2n-1} u_i \frac{e^{-i\ell_{i+1}x} - e^{-i\ell_ix}}{-ix} \hat{f}(-y) |y|^{-H/2} dy
\]
\[
= |x|^{-H/2} \int_{\mathbb{R}} \sum_{i=1}^{2n-1} u_i \frac{e^{-i\ell_{i+1}x} - e^{-i\ell_ix}}{-ix} \left( \mathcal{F}^3 f \right)(y) |y|^{-H/2} dy
\]
\[
= (V_{\Delta, u} \mathcal{F}^3 f)(x).
\]
Hence, \( A_{\Delta, u} \) and \( c_H K_{H/2} M_g K_{H/2} =: B_{\Delta, u} \) are unitarily equivalent and they have the same eigenvalues.
By the Lemma 5.4 in [15], we can find that
\[ \mu_j(M_{1,[\ell_i,\ell_{i+1}]}, K_{H/2}M_{1,[\ell_i,\ell_{i+1}]}) \geq c_H|\ell_{i+1} - \ell_i|^{H/2} \tilde{\mu}_j, \]
the eigenvalue \( \mu_j \) of \( B_{\Delta u} \) is greater than
\[ \mu_j(K_{H/2}M_gK_{H/2}) \geq \mu_j \left( (M_gK_{H/2}M_g)^2 \right) \]
\[ = (\mu_j(M_gK_{H/2}M_g))^2 \]
\[ \geq c_H \max_{1 \leq i \leq 2n-1} |u_i||\ell_{i+1} - \ell_i|^{H} \tilde{\mu}_j^2, \]
where \( \tilde{\mu}_j \sim j^{-H/2} \).

Substituting (3.5) into (3.4),
\[ |E(\tilde{x}_{t,i})^\alpha| \leq c \int_{E^n} \prod_{i=1}^{\infty} \left( 1 + c_H \max_{1 \leq i \leq 2n-1} |u_i|^2 |\ell_{i+1} - \ell_i|^{2H} \tilde{\mu}_j^4 \right)^{-\frac{1}{4}} dpd\ell \]
\[ \leq c \int_{E^n} \prod_{i=1}^{\infty} \left( 1 + c_H \max_{i \in A} |u_i|^2 |\ell_{i+1} - \ell_i|^{2H} \tilde{\mu}_j^4 \right)^{-\frac{1}{4}} dpd\ell \]
\[ \leq c \int_{E^n} \prod_{i \in A} \exp \left( -c_H \max_{i \in A} |u_i|^2 |\ell_{i+1} - \ell_i|^H \right) dpd\ell \]
\[ \leq c_H \left| J \right| \int_{E^n} \prod_{i \in A} \left[ \prod_{i \in A} \exp \left( -c_H |u_i|^2 |\ell_{i+1} - \ell_i| \right) \right]^{\frac{1}{H}} du d\ell, \]
where \( A \) is a subset of \( \{1, 2, \ldots, 2n-1\} \) such that the set \( \{u_i, i \in A\} \) spans \( \{p_1, p_2, \ldots, p_n\} \), \( |J| \) is the Jacobian determinant of changing variables \( (p_1, p_2, \ldots, p_n) \) to \( \{u_i, i \in A\} \), we use the inequality (see in page 518 line –5 of [15])
\[ \prod_{j=1}^{\infty} \left( 1 + c_H \max_i |u_i|^2 |\ell_{i+1} - \ell_i|^{2H} \tilde{\mu}_j^4 \right)^{-\frac{1}{4}} \leq c \exp \left( -c_H \max_i |u_i|^2 |\ell_{i+1} - \ell_i| \right) \]
with \( H \in \left( \frac{1}{2}, 1 \right) \), in the third inequality.

It is easy to see that
\[ \int_{\mathbb{R}} \exp \left( -c_H |u|^2 |\Delta \ell_{i+1}| \right) du \leq c_H |\Delta \ell_{i+1}|^{-H}. \]

Thus, the integral with respect to \( du \) in (3.6) is
\[ \int_{E^n} \left[ \prod_{i \in A} \exp \left( -c_H |u_i|^2 |\ell_{i+1} - \ell_i| \right) \right]^{\frac{1}{H}} du \leq c_{H,n} \prod_{i \in A} |\ell_{i+1} - \ell_i|^{-H}. \]

This gives
\[ |E(\tilde{x}_{t,i})^\alpha| \leq c_{H,n} \int_{E^n} \prod_{i \in A} |\ell_{i+1} - \ell_i|^{-H} d\ell, \]
(3.8)
which is finite since \( H < 1 \).
Next, we need to prove that \( \{ \hat{\alpha}_{t, \varepsilon}, \varepsilon > 0 \} \) is a Cauchy sequence. For any \( \varepsilon, \eta > 0 \),

\[
|E(\hat{\alpha}_{t, \varepsilon} - \hat{\alpha}_{t, \eta})|^n \leq \frac{1}{(2\pi)^n} \int_{E^n} \int_{\mathbb{R}^n} \prod_{j=1}^{\infty} \frac{|e^{\frac{1}{2}\varepsilon_j^2} - e^{\frac{1}{2}\eta_j^2}|}{(1 + 4\lambda_j^2)^{-\frac{1}{2}}} dp d\ell.
\]

By the dominated convergence theorem and

\[
\int_{E^n} \int_{\mathbb{R}^n} \prod_{j=1}^{\infty} (1 + 4\lambda_j^2)^{-\frac{1}{2}} dp d\ell < \infty.
\]

So, we can see that \( \hat{\alpha}_{t, \varepsilon} \) exists in \( L^p \), for all \( p \geq 1 \), under the condition \( 1/2 < H < 1 \).

### 3.1.2. Hölder continuity in space variable

For the Hölder continuity of \( \hat{\alpha}_t(y) \) in the variable \( y \), define \( x, y \in \mathbb{R} \),

\[
|E(\hat{\alpha}_{t, \varepsilon}(x) - \hat{\alpha}_{t, \varepsilon}(y))|^n \leq \frac{1}{(2\pi)^n} \int_{E^n} \int_{\mathbb{R}^n} \prod_{j=1}^{\infty} \left( |e^{\frac{1}{2}\varepsilon_j^2} - e^{\frac{1}{2}\eta_j^2}| \right) \prod_{j=1}^{\infty} (1 + 4\lambda_j^2)^{-\frac{1}{2}} dp d\ell.
\]

Note that

\[
|e^{\frac{1}{2}\varepsilon_j^2} - e^{\frac{1}{2}\eta_j^2}| \leq c\lambda |x - y|^\lambda |p_j|^\lambda, \quad \lambda \in [0, 1].
\]

Fix \( j \), and let \( j_1 \) be the smallest value such that \( u_{j_1} \) contains \( p_j \) as a term and then choose \( j_2 \) to be the smallest value strictly larger than \( j_1 \) such that \( u_{j_2} \) does not contain \( p_j \) as a term. Then \( p_j = u_{j_1} - u_{j_1 - 1} = u_{j_2 - 1} - u_{j_2} \). We can see that, with the convention that \( u_0 = u_{2n} = 0 \),

\[
|p_j|^\lambda = |u_{j_1} - u_{j_1 - 1}|^\lambda |u_{j_2 - 1} - u_{j_2}|^\lambda \leq c\left(|u_{j_1}|^\lambda + |u_{j_1 - 1}|^\lambda\right) \left(|u_{j_2 - 1}|^\lambda + |u_{j_2}|^\lambda\right).
\]

Thus,

\[
\prod_{i=1}^{\infty} |p_i|^\lambda \leq c \prod_{i=1}^{\infty} \left(|u_i|^\lambda + |u_{i-1}|^\lambda\right) \leq c \sum_{S_1} \prod_{i=1}^{\infty} \left(|u_i|^\lambda + |u_{i-1}|^\lambda\right),
\]

where

\[
S_1 = \{ \gamma_i, \bar{\gamma}_i : \gamma_i \in \{0, 1\}, \gamma_i + \bar{\gamma}_i = 1, \quad i = 1, \ldots, 2n \}
\]

and

\[
S_2 = \{ x_i : x_i \in \{0, 1, 2\}, \quad i = 1, \ldots, 2n - 1 \}.
\]
Then
\[
|\mathbb{E}(\mathcal{A}_{t,v}(x) - \mathcal{A}_{t,v}(y))| \leq c |x - y|^\lambda \sum_{S_2} \int_{\mathbb{R}^n} \prod_{i=1}^{2n-1} |u_i|^{\frac{1}{2'H} j_i} \exp \left( \prod_{j=1}^{\infty} (1 + 4\lambda_j^2)^{-\frac{1}{2}} dp \right).
\]

Similar to (3.6),
\[
\int_{\mathbb{R}^n} \prod_{i=1}^{2n-1} |u_i|^{\frac{1}{2'H} j_i} \prod_{j=1}^{\infty} (1 + 4\lambda_j^2)^{-\frac{1}{2}} dp
\]
\[
\leq c \int_{\mathbb{R}^n} \prod_{i=1}^{2n-1} |u_i|^{\frac{1}{2'H} j_i} \prod_{j=1}^{\infty} (1 + c_{H,n} \max_{1 \leq i \leq 2n-1} |u_i|^{\frac{1}{2'} | \ell_{i+1} - \ell_i |^{2'H} \mu_j})^{-\frac{1}{2}} dp \]
\[
\leq c \int_{\mathbb{R}^n} \prod_{i=1}^{2n-1} |u_i|^{\frac{1}{2'H} j_i} \exp \left( -c_{H,n} \max_{1 \leq i \leq 2n-1} |u_i|^{\frac{1}{2'} | \ell_{i+1} - \ell_i |} \right) dp
\]
\[
\leq c \int_{\mathbb{R}^n} \prod_{i=1}^{2n-1} \left( |u_i|^{\frac{1}{2'H} j_i} \exp \left( -\frac{c_{H,n}}{2n} |u_i|^{\frac{1}{2'} | \ell_{i+1} - \ell_i |} \right) \right) \prod_{i \in A} \left( |u_i|^{\frac{1}{2'H}} \exp \left( -\frac{c_{H,n}}{2n} |u_i|^{\frac{1}{2'} | \ell_{i+1} - \ell_i |} \right) \right) du
\]
\[
\leq c_{H,n} \prod_{i \in A^C} |\ell_{i+1} - \ell_i|^{-\frac{i_n}{2'H}} \prod_{i \in A} |\ell_{i+1} - \ell_i|^{-\frac{i_n'H}{2'H - H}},
\]
where \( A^C \) denotes the complement of \( A \) in \( \{1, 2, \ldots, 2n - 1\} \), \(|J|\) is the Jacobian determinant of changing variables \((p_1, p_2, \ldots, p_n)\) to \(\{u_i, i \in A\}\) and we use the inequality
\[
|u_i|^{\frac{1}{2'H} j_i} \exp \left( -\frac{c_{H,n}}{2n} |u_i|^{\frac{1}{2'} | \ell_{i+1} - \ell_i |} \right) \leq c_{H,n} |\ell_{i+1} - \ell_i|^{-\frac{i_n}{2'H}}
\]
in the second last inequality.

Thus,
\[
|\mathbb{E}(\mathcal{A}_{t,v}(x) - \mathcal{A}_{t,v}(y))| \leq c_{H,n} |x - y|^\lambda \sum_{S_2} \int_{\mathbb{R}^n} \prod_{i \in A^C} |\ell_{i+1} - \ell_i|^{-\frac{i_n}{2'H}} \prod_{i \in A} |\ell_{i+1} - \ell_i|^{-\frac{i_n'H}{2'H - H}} d\ell,
\]
which is finite with \( H(\lambda + 1) < 1 \), since \( \alpha_i \in \{0, 1, 2\} \).

So, we need the condition \( H(\lambda + 1) < 1 \) to make sure
\[
|\mathbb{E}(\mathcal{A}_{t,v}(x) - \mathcal{A}_{t,v}(y))| \leq c_n |\mathbb{E}(\mathcal{A}_{t,v}(x) - \mathcal{A}_{t,v}(x))| + c_n |\mathbb{E}(\mathcal{A}_{t,v}(x) - \mathcal{A}_{t,v}(y))| + c_n |\mathbb{E}(\mathcal{A}_{t,v}(x) - \mathcal{A}_{t,v}(y))| + c_{n,H,t} |x - y|^h.
\]

Hence, from the Kolmogorov continuity criterion, the Hölder continuity of \( \mathcal{A}_{t,v}(y) \) in the space variable of any order \( \lambda \) strictly less than \( \min\{1, \frac{1}{H} - 1\} \).
3.1.3. H"older continuity in time variable

For the H"older continuity of $\hat{\alpha}_t(y)$ in the time variables $t$. Without loss of generality, we assume that $t < \hat{t}$ and let $\hat{D} = \{(r,s) : 0 < r < s < \hat{t}\}$.

Then

$$\left|\mathbb{E}(\hat{\alpha}_{t,e}(y) - \hat{\alpha}_{t,e}(y))\right| \leq \frac{1}{(2\pi)^n} \int_{(\hat{D}\setminus D)^n} \int_{\mathbb{R}^n} \left|\mathbb{E}e^{\sum_{i=1}^n p_i(X_i^H - X_i)}\right| dpdrds$$

$$\leq c \int_{[t,\hat{t}]} \int_{[s_1,\hat{t}]} \int_{[s_1,\hat{t}]} \int_{\mathbb{R}^n} \left|\mathbb{E}e^{\sum_{i=1}^n p_i(X_i^H - X_i)}\right| dpdrds$$

$$\leq c \int_{D^n} \prod_{i=1}^n 1_{[t,\hat{t}]}(s_i) \int_{\mathbb{R}^n} \left|\mathbb{E}e^{\sum_{i=1}^n p_i(X_i^H - X_i)}\right| dpdrds$$

$$\leq c |\hat{t} - t|^{\eta \beta} \left( \int_{\mathbb{R}^n} \left|\mathbb{E}e^{\sum_{i=1}^n p_i(X_i^H - X_i)}\right| dp \right)^{\frac{1}{1-\beta}} drds$$

$$= c |\hat{t} - t|^{\eta \beta} \Lambda,$$

where we use the Hölder’s inequality in the last inequality with $\beta < 1 - H$.

Using the similar methods as in (3.8), $\Lambda$ is bounded by

$$\left|\mathbb{E}(\hat{\alpha}_{t,e})\right| \leq c_{H,n} \left( \int_{\mathbb{R}^n} \prod_{i \in A} |\ell_{i+1} - \ell_i|^{-H/(1-\beta)} d\ell \right)^{1-\beta}$$

since $1 - H/(1 - \beta) > 0$.

This gives

$$\left|\mathbb{E}(\hat{\alpha}_t(x) - \hat{\alpha}_{t,e}(x))\right| \leq c_n |\mathbb{E}(\hat{\alpha}_t(x) - \hat{\alpha}_{t,e}(x))| + c_n |\mathbb{E}(\hat{\alpha}_{t,e}(x) - \hat{\alpha}_{t,e}(x))|$$

$$+ c_n |\mathbb{E}(\hat{\alpha}_{t,e}(x) - \hat{\alpha}_{t,e}(x))|$$

$$\leq c_{n,H_1,H_2,t} |\hat{t} - t|^{\eta \beta}$$

with $\beta < 1 - H$. This completes the proof.

3.2. Proof of Theorem 1.2

In this subsection, we will consider the case of ILT of Rosenblatt process.

By definition (1.2), we let

$$\tilde{\alpha}_{t,e} := \tilde{\alpha}_{t,e}(0) = \int_0^t \int_0^t f_\epsilon(X_s^H - \tilde{X}_r^H)drds$$

$$= \frac{1}{2\pi} \int_{[0,\hat{t}]} \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} e^{\epsilon p(X_i^H - \tilde{X}_r^H)} dpdrds,$$

where $X^H_1$ and $\tilde{X}^H_2$ are two independent Rosenblatt processes.
3.2.1. Existence in $L^p$

For positive integer $n \geq 1$,

$$|\mathbb{E}(\tilde{X}_{t,s})^n| \leq \frac{1}{(2\pi)^n} \int_{[0,t]^n} \int_{\mathbb{R}^n} \left| \mathbb{E}e^{i\sum_{i=1}^n p_i(X_i^{H_1}-\tilde{X}_i^{H_2})} \right| dpdrds. \quad (3.9)$$

Define two operators $A_{s,p}$ and $\tilde{A}_{r,p}$,

$$(A_{s,p}f)(x) = \int_{\mathbb{R}} \sum_{i=1}^n p_i \frac{e^{is_i(x+y)}}{t(x+y)} f(y)|y|^{-H_1} dy$$

and

$$(\tilde{A}_{r,p}f)(x) = \int_{\mathbb{R}} \sum_{i=1}^n p_i \frac{e^{ir_i(x+y)}}{t(x+y)} f(y)|y|^{-H_2} dy.$$ 

Similar to (3.3), we have

$$\sum_{i=1}^n p_i(X_{t_j}^{H_1}-\tilde{X}_{t_j}^{H_2}) \overset{law}{=} \sum_{j=1}^\infty \left( \lambda_j(Y_j^2 - 1) - \tilde{\lambda}_j(\tilde{Y}_j^2 - 1) \right),$$

where $\{Y_j, j \geq 1\}$ and $\{\tilde{Y}_j, j \geq 1\}$ are two sequences of independent standard Gaussian random variable, $\{\lambda_j, j \geq 1\}$ and $\{\tilde{\lambda}_j, j \geq 1\}$ are the eigenvalues of operator $A_{s,p}$ and $\tilde{A}_{r,p}$, respectively. Then

$$\left| \mathbb{E}e^{i\sum_{i=1}^n p_i(X_i^{H_1}-\tilde{X}_i^{H_2})} \right| = \prod_{j=1}^{\infty} \frac{e^{-i\lambda_j}}{\sqrt{1-2i\lambda_j}} \prod_{j=1}^{\infty} \frac{e^{i\tilde{\lambda}_j}}{\sqrt{1+2i\tilde{\lambda}_j}}$$

$$= \prod_{j=1}^{\infty} (1+4\lambda_j^2)^{-\frac{1}{4}} \prod_{j=1}^{\infty} (1+4\tilde{\lambda}_j^2)^{-\frac{1}{4}}.$$ 

Substituting the above equation into (3.9),

$$|\mathbb{E}(\tilde{X}_{t,s})^n| \leq \frac{1}{(2\pi)^n} \int_{[0,t]^n} \int_{\mathbb{R}^n} \prod_{j=1}^{\infty} (1+4\lambda_j^2)^{-\frac{1}{4}} \prod_{j=1}^{\infty} (1+4\tilde{\lambda}_j^2)^{-\frac{1}{4}} dpdrds$$

$$\leq c \int_{[0,t]^n} \left( \int_{\mathbb{R}^n} \prod_{j=1}^{\infty} (1+4\lambda_j^2)^{-\frac{1}{4}} \frac{d\mu_j^{H_1+H_2}}{m_i^{H_1+H_2}} \right) drds$$

$$\times \left( \int_{\mathbb{R}^n} \prod_{j=1}^{\infty} (1+4\tilde{\lambda}_j^2)^{-\frac{1}{4}} \frac{d\mu_j^{H_1+H_2}}{m_i^{H_1+H_2}} \right) drds,$$ 

where $D = \{0 < s_1 < s_2 < \cdots < s_n < t\}$. 

(3.10)
Similar to the case of SLT, note that \( A_{s,u} \) and \( c_{H_1}K_{H_1/2}M_gK_{H_1/2} \) are unitarily equivalent and have the same eigenvalues, where

\[
\tilde{g}(x) = \sum_{i=1}^{n} p_i \mathbf{1}_{[0,s_j]}(x) = \sum_{i=1}^{n} (p_i - p_{i-1}) \mathbf{1}_{[s_{i-1},s_j]}(x)
\]

and \( 0 = s_0 < s_1 < \cdots < s_n \) with the convention \( p_0 = 0 \).

By (3.5), the \( j \)-th singular value \( \lambda_{ji} \) of \( A_{s,p} \) satisfies

\[
\lambda_{ji} = \mu_{ji}(K_{H_1/2}M_gK_{H_1/2}) \geq c_{H_1} \max_{1 \leq j \leq n} |p_j - p_{j-1}| |s_j - s_{j-1}|^{H_1} \mu_{ji}^2,
\]

where \( \mu_{ji} / c_{H_1}n^{-\frac{m_j}{2}} \), and \( c_{H_1}, \tilde{c}_{H_1} > 0 \) are constants that only depend on \( H_1 \).

This gives

\[
\int_{\mathbb{R}^n} \prod_{j_1=1}^{\infty} (1 + 4\lambda_{j_1}^2)^{-\frac{3}{4}} dp \leq c \int_{\mathbb{R}^n} \prod_{j_1=1}^{\infty} \left( 1 + c_{H_1} \max_{1 \leq j \leq n} |p_j - p_{j-1}| |s_j - s_{j-1}|^{H_1} \mu_{ji}^4 \right)^{-\frac{1}{4}} dp
\]

\[
\leq c_n \int_{\mathbb{R}^n} \prod_{j_1=1}^{\infty} \left( 1 + c_{H_1} \max_{1 \leq j \leq n} |s_j - s_{j-1}|^{H_1} \xi_j \right)^{-\frac{1}{4}} d\xi,
\]

(3.11)

where we make the change of variables \( \xi_j = \sum_{i=j}^{n} p_i \) for \( j = 1, 2, \ldots, n \) with the convention \( \xi_{n+1} = 0 \) and use \( \max_j |2\tilde{c}_{j} - \tilde{c}_{j-1} - \tilde{c}_{j+1}| \geq c_n \max_j |\tilde{c}_j| \) in the last inequality.

For the integrand function in (3.11), similar to (3.7), we have

\[
\prod_{j_1=1}^{\infty} \left( 1 + c_{H_1} \max_{1 \leq j \leq n} |s_j - s_{j-1}|^{H_1} |\xi_j| \right)^{-\frac{1}{4}} \leq c \exp \left( -c_{H_1} \max_{1 \leq j \leq n} |s_j - s_{j-1}| |\xi_j|^{\frac{1}{4}} \right).
\]

(3.12)

Substituting (3.12) into (3.11),

\[
\int_{\mathbb{R}^n} \prod_{j_1=1}^{\infty} (1 + 4\lambda_{j_1}^2)^{-\frac{3}{4}} dp \leq c \int_{\mathbb{R}^n} e^{-c_{H_1} \max_{1 \leq j \leq n} |s_j - s_{j-1}| |\xi_j|^{\frac{1}{H_1}}} d\xi
\]

\[
\leq c \int_{\mathbb{R}^n} \left( e^{-c_{H_1} \sum_{j=1}^{n} |s_j - s_{j-1}| |\xi_j|^{\frac{1}{H_1}}} \right)^{\frac{1}{2}} d\xi
\]

\[
\leq c_{H_1,n} \prod_{j=1}^{n} |s_j - s_{j-1}|^{-H_1}.
\]

(3.13)

Similarly, we have the same inequality with respect to \( \tilde{\lambda}_{j_1} \),

\[
\int_{\mathbb{R}^n} \prod_{j_1=1}^{\infty} (1 + 4\tilde{\lambda}_{j_1}^2)^{-\frac{3}{4}} dp \leq c_{H_1,n} \prod_{j=1}^{n} |s_j - s_{j-1}|^{-H_2}.
\]

(3.14)

Together (3.10), (3.13), and (3.14), we have

\[
\left| \mathbb{E} \left( \tilde{X}_{H_1}(k) \right) \right|^n \leq c_n \left( \int_{[0,t]} \prod_{j=1}^{n} |r_j - r_{j-1}|^{-H_1 H_2/H_1+H_2} dr \right)^2,
\]

(3.15)

which is finite under condition \( H_1 H_2/H_1+H_2 < 1 \).
Next, we need to prove that \( \{ \tilde{x}_{t, \varepsilon} \} \) is a Cauchy sequence. For any \( \varepsilon, \eta > 0 \),

\[
\left| \mathbb{E}(\tilde{x}_{t, \varepsilon} - \tilde{x}_{t, \eta})^n \right| \leq \frac{1}{2\pi} \int_{[0, t]^{2n}} \prod_{j=1}^{n} (1 + 4\lambda_{j1}^2)^{-\frac{1}{4}} \prod_{j=1}^{n} (1 + 4\lambda_{j2}^2)^{-\frac{1}{4}} dp dr ds.
\]

By the dominated convergence theorem and

\[
\int_{[0, t]^{2n}} \prod_{j=1}^{n} (1 + 4\lambda_{j1}^2)^{-\frac{1}{4}} \prod_{j=1}^{n} (1 + 4\lambda_{j2}^2)^{-\frac{1}{4}} dp dr ds < \infty,
\]

with \( \frac{H_1 H_2}{H_1 + H_2} < 1 \). So, we can see that \( \tilde{x}_{t, \varepsilon} \) exists in \( L^p \), for all \( p \geq 1 \), under the condition \( H_1 H_2 < H_1 + H_2 \).

### 3.2.2. Hölder continuity in space variable

For \( x, y \in \mathbb{R} \),

\[
\left| \mathbb{E}(\tilde{x}_{t, \varepsilon}(x) - \tilde{x}_{t, \varepsilon}(y))^n \right| \\
\leq \frac{1}{2\pi} \int_{[0, t]^{2n}} \prod_{j=1}^{n} (1 + 4\lambda_{j1}^2)^{-\frac{1}{4}} \prod_{j=1}^{n} (1 + 4\lambda_{j2}^2)^{-\frac{1}{4}} dp dr ds.
\]

Note that

\[
|e^{-ipx} - e^{-ipy}| \leq c_{\lambda} |x - y|^\lambda |p|^\lambda, \quad \lambda \in [0, 1].
\]

Then

\[
\left| \mathbb{E}(\tilde{x}_{t, \varepsilon}(x) - \tilde{x}_{t, \varepsilon}(y))^n \right| \\
\leq c |x - y|^\lambda \int_{[0, t]^{2n}} \prod_{j=1}^{n} |p_j|^\lambda \exp \prod_{j=1}^{n} (1 + 4\lambda_{j1}^2)^{-\frac{1}{4}} \prod_{j=1}^{n} (1 + 4\lambda_{j2}^2)^{-\frac{1}{4}} dp dr ds
\]

\[
\leq c_n \int_{\tilde{D}^{2n}} \left( \prod_{j=1}^{n} |p_j|^\lambda \prod_{j=1}^{n} (1 + 4\lambda_{j1}^2)^{-\frac{1}{4}} dp \right)^{\frac{H_2}{n+H_2}}
\times \left( \prod_{j=1}^{n} \prod_{j=1}^{n} (1 + 4\lambda_{j1}^2)^{-\frac{1}{4}} dp \right)^{\frac{H_1}{n+H_2}} dr ds,
\]

where \( \tilde{D} = \{ 0 < s_1 < s_2 < \cdots < s_n < t \} \).

Making the change of variables \( \tilde{\xi}_j = \sum_{i=j}^{n} p_i \) for \( j = 1, 2, \ldots, n \) with the convention \( \tilde{\xi}_{n+1} = 0 \). Then by (3.11) and (3.12),
\[
\int_{\mathbb{R}^n} \prod_{i=1}^{n} |p_i|^{\frac{1}{2}} \left(1 + 4\lambda_j^2\right)^{-\frac{1}{2}} dp \leq c \int_{\mathbb{R}^n} \prod_{j=1}^{n} |\xi_j - \xi_{j+1}|^{\frac{1}{2}} e^{-c\lambda_j \max_{1 \leq s \leq n} |s_j - s_{j-1}|^{\frac{1}{2}}} d\xi
\]
\[
\leq c \sum_{S_1} \int_{\mathbb{R}^n} \prod_{j=1}^{n} |\xi_j|^{\frac{1}{2}} e^{-c\lambda_j \max_{1 \leq s \leq n} |s_j - s_{j-1}|^{\frac{1}{2}}} d\xi
\]
\[
\leq c \int_{\mathbb{R}^n} \prod_{j=1}^{n} |\xi_j|^{\frac{1}{2}} \left(e^{-c\lambda_j \sum_{j=1}^{n} |s_j - s_{j-1}|^{\frac{1}{2}}} \right)^{\frac{3}{2}} d\xi
\]
\[
\leq c_{H_1, n} \prod_{j=1}^{n} |s_j - s_{j-1}|^{-H_1(\lambda_j + 1)},
\]

where in the second inequality we use
\[
\prod_{j=1}^{n} |\xi_j - \xi_{j+1}|^{\frac{1}{2}} \leq c \prod_{j=1}^{n} (|\xi_j|^{\frac{1}{2}} + |\xi_{j+1}|^{\frac{1}{2}})
\]
\[
= \sum_{S_1} \prod_{j=1}^{n} (|\xi_j|^{\frac{1}{2}} |\xi_{j+1}|^{\frac{1}{2}})
\]
\[
\leq \sum_{S_2} \prod_{j=1}^{n} (|\xi_j|^{\frac{1}{2}}),
\]

with
\[
S_1 = \{\gamma_j, \overline{\gamma_j} : \gamma_j \in \{0, 1\}, \ \gamma_j + \overline{\gamma_j} = 1, \ j = 1, ..., n\}
\]

and
\[
S_2 = \{x_j : x_j \in \{0, 1, 2\}, \ j = 1, ..., n\}.
\]

Similarly, we have the same inequality for \(\tilde{\lambda}_j\) as (3.16). Thus,
\[
|\mathbb{E}(\tilde{x}_{t, e})|^n \leq c_n |x - y|^2 \left(\sum_{S_2} \int_{0 < r_1 < \cdots < r_n < t} \prod_{j=1}^{n} (r_j - r_{j-1})^{-\frac{n+1}{H_1 + H_2} (1 + \lambda)} dr \right)^2.
\]

So, we need the condition \(\frac{H_1 H_2}{H_1 + H_2} (2\lambda + 1) < 1\) to make sure
\[
|\mathbb{E}(\tilde{x}_t(x) - \tilde{x}_t(y))|^n
\]
\[
\leq c_n |\mathbb{E}(\tilde{x}_t(x) - \tilde{x}_{t, e}(x))|^n + c_{n, e} |\mathbb{E}(\tilde{x}_{t, e}(x) - \tilde{x}_{t, e}(y))|^n
\]
\[
+ c_{n, e} |\mathbb{E}(\tilde{x}_{t, e}(y) - \tilde{x}(y))|^n
\]
\[
\leq c_{n, H_1, H_2, t} |x - y|^n\lambda.
\]

Hence, from the Kolmogorov continuity criterion, the Hölder continuity of \(\tilde{x}_t(y)\) in the space variable of any order \(\lambda\) strictly less than \(\min\{1, \frac{H_1 H_2}{2H_1 H_2} - \frac{1}{2}\}\).
3.2.3. Hölder continuity in time variable

Without loss of generality, we assume that $t < \tilde{t}$. Then

$$|\mathbb{E}(\tilde{x}_{t,\epsilon}(y) - \tilde{x}_{\tilde{t},\epsilon}(y))^{n}|$$

$$\leq \frac{1}{(2\pi)^n} \left( \int_{|t,\tilde{t}|^n} + \int_{[0,\tilde{t}]^n \times |t,\tilde{t}|^n} \right) \prod_{j=1}^{\infty} (1 + 4\lambda_{j_1}^2)^{-\frac{1}{2}} \prod_{j_2=1}^{\infty} (1 + 4\lambda_{j_2}^2)^{-\frac{1}{2}} dp dr ds$$

$$=: II_1 + II_2.$$

By (3.15), we have

$$II_1 \leq c_n \left( \sum_{1 \leq j_1 < \ldots < j_n \leq t} \frac{n! (t - r_{j-1})}{n!} dr \right)^2$$

$$\leq c_n \left( \tilde{t} - t \right)^{2n} \left( 1 - \frac{H_1 H_2}{H_1 + H_2} \right)^2$$

$$\leq c_n (\tilde{t} - t)^{2n} \left( 1 - \frac{n! n_1}{n_1 + n_2} \right),$$

and similarly we get

$$II_2 \leq c_n (\tilde{t} - t)^{2n} \left( 1 - \frac{n! n_1}{n_1 + n_2} \right).$$

This gives

$$|\mathbb{E}(\tilde{x}_t(x) - \tilde{x}_{\tilde{t}}(x))^{n}| \leq c_n |\mathbb{E}(\tilde{x}_t(x) - \tilde{x}_{t,\epsilon}(x))^{n}| + c_n |\mathbb{E}(\tilde{x}_{t,\epsilon}(x) - \tilde{x}_{\tilde{t},\epsilon}(x))^{n}|$$

$$+ c_n |\mathbb{E}(\tilde{x}_{\tilde{t},\epsilon}(x) - \tilde{x}_{\tilde{t}}(x))^{n}|$$

$$\leq c_n |\mathbb{E}(\tilde{x}_{t,\epsilon}(x) - \tilde{x}_{\tilde{t},\epsilon}(x))^{n}|$$

$$\leq c_n |\mathbb{E}(\tilde{x}_t(x) - \tilde{x}_{\tilde{t}}(x))^{n}| \leq c_n (\tilde{t} - t)^{n_1 n_2}$$

with $\tilde{\beta} < 1 - \frac{H_1 H_2}{H_1 + H_2}$. This completes the proof.

3.3. Proof of Theorem 1.3

In this section, we will prove the existence and the Hölder continuity for the case of CoLT. Compared with cases of SLT and ILT, the integral structure of CoLT here is relatively simple. So we only need to use the similar method to the previous two subsections.

3.3.1. Existence in $L^p$

By the definition of CoLT in (1.3), for any integer $n \geq 1$,

$$|\mathbb{E}(\tilde{x}_{t,\epsilon}(y))^{n}| \leq \frac{1}{(2\pi)^n} \left[ \int_{[0,\tilde{t}]^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \sum_{j=1}^{n} |p_j|^2} \mathbb{E} e^{\sum_{j=1}^{n} p_j (x_{t_1} - \tilde{x}_{t_1}^{(j)})} \right] dp ds$$

$$\leq \frac{1}{(2\pi)^n} \left[ \int_{[0,\tilde{t}]^n} \int_{\mathbb{R}^n} \prod_{j=1}^{\infty} (1 + 4\lambda_{j_1}^2)^{-\frac{1}{2}} \prod_{j_2=1}^{\infty} (1 + 4\lambda_{j_2}^2)^{-\frac{1}{2}} dp ds \right].$$

Using the same method as in (3.13), we have
\[
\int_{\mathbb{R}^n} \prod_{j=1}^{\infty} (1 + 4\lambda_j^2)^{-\frac{1}{4}} \prod_{j=1}^{\infty} (1 + 4\tilde{\lambda}_j^2)^{-\frac{1}{4}} dp \\
\leq c \left( \int_{\mathbb{R}^n} \prod_{j=1}^{\infty} (1 + 4\lambda_j^2)^{-\frac{1}{4}} dp \right)^{\frac{H_1}{n_1+n_2}} \left( \int_{\mathbb{R}^n} \prod_{j=1}^{\infty} (1 + 4\tilde{\lambda}_j^2)^{-\frac{1}{4}} dp \right)^{\frac{H_2}{n_1+n_2}}
\leq c \prod_{j=1}^{n} |s_j - s_{j-1}|^{\frac{2H_1H_2}{n_1+n_2}}.
\]

Thus,

\[
|\mathbb{E}(\tilde{\alpha}_{t,\varepsilon})|^n \leq c \int_{[0,\varepsilon]^n} \prod_{j=1}^{n} |s_j - s_{j-1}|^{\frac{2H_1H_2}{n_1+n_2}} ds \\
\leq c \int_{\{0<s_1<\ldots<s_n<t\}} \prod_{j=1}^{n} |s_j - s_{j-1}|^{\frac{2H_1H_2}{n_1+n_2}} ds,
\]

which is finite under condition \(\frac{H_1H_2}{n_1+n_2} < \frac{1}{2}\). It is easy to see that \(\{\tilde{\alpha}_{t,\varepsilon}, \varepsilon > 0\}\) is a Cauchy sequence and we get \(\tilde{\alpha}_{t,\varepsilon}\) exists in \(L^p\), for all \(p \geq 1\), under the condition \(\frac{H_1H_2}{n_1+n_2} < \frac{1}{2}\).

### 3.3.2. Hölder continuity in space and time variables

According to Equation (3.19), we find that the result of Hölder continuity can be obtained by replacing \(\frac{H_1H_2}{n_1+n_2} \) with \(\frac{2H_1H_2}{H_1+H_2} \) in the case of ILT. Then similar to (3.17) and (3.18), we can obtain that

\[
|\mathbb{E}(\tilde{\alpha}_t(x) - \tilde{\alpha}_t(y))|^n \leq c_{n,H_1,H_2} |x-y|^n \lambda, \quad \lambda < \min\{1, \frac{H_1+H_2}{H_1+H_2} - \frac{1}{2}\};
\]

and

\[
|\mathbb{E}(\tilde{\alpha}_t(x) - \tilde{\alpha}_t(x))|^n \leq c_{n,H_1,H_2} |t-t'|^{n \beta}, \quad \beta < 1 - \frac{2H_1H_2}{H_1+H_2},
\]

This completes the proof.

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