I. INTRODUCTION

In the presence of an applied magnetic field a type-II superconductor may exhibit a vortex state, characterized by a certain density of order parameter singularities (vortices). In equilibrium, and in the absence of important disorder effects, the vortices generically form an ordered “Abrikosov lattice”. An applied current produces a force on the vortices, and the resulting vortex motion leads to dissipation, which is one of the main concerns for many applications such as superconducting qubits and superconducting digital memory, high-field magnets, THz radiation sources, or resonant cavities in particle accelerator. A deepened understanding of the nature of vortex motion may enhance device performances enabling many intriguing applications.

Vortex structures driven by an external current in systems without pinning sites have been studied. Of particular interest is the phase slip line state where a set of parallel vortex rows are created and each vortex follows in its own “assigned” channel. This phenomenon originates from an effective attraction to the other vortices due to a suppression of the order parameter created behind each moving vortex. This mechanism was analysed in the context of the viscosity and the instability of the vortex motion using both Bardeen-Stephen and Larkin-Ovchinnikov theory. However, finite size effects, in particular boundaries which are always present in real superconducting devices, have received little attention in this context so far.

In this work, we use time-dependent Ginzburg-Landau (TGDL) theory to investigate the flux flow properties of clean two dimensional superconductors driven by externally sourced currents, in finite geometries. We find a low current regime where the vortex lattice moves as a whole, with each vortex moving at the same velocity. At larger sourced current, vortex motion is organized into channels, with vortices in channels nearer to the sample edges moving faster than those farther away from sample edges, opposite to the Poiseuille flow of basic hydrodynamics where the velocity is lowest at the boundaries. At intermediate currents, a stick-slip motion of the vortex lattice occurs in which vortices in the channel at the boundary break free from the Abrikosov lattice, accelerate, move past their neighbors and then slow down and reattach to the vortex lattice at which point the stick-slip process starts over. These effects could be observed experimentally, e.g. using fast scanning microscopy techniques.

II. FORMALISM

We use TDGL equations to describe the dynamics of a complex superconducting order parameter $\Delta = |\Delta|e^{i\phi}$ in a two dimensional strip geometry in the presence of both an external current created by a source and drain of particles whose strength is denoted by $Q$ and an external magnetic field $B$ directed perpendicular to the superconducting film. The electromagnetic field is expressed by the vector potential $A$ and the scalar potential $\Theta$. We denote the charge density as $\rho$ and current density as $J$ and choose unit as $\hbar = c = e = 1$. The equations are:

$$\frac{1}{\hbar} \left( \partial_t + 2i\Psi \right) \Delta = \frac{1}{\xi^2} \Delta |\alpha - \beta| \Delta^2$$

$$+ \left| \nabla - 2iA \right|^2 \Delta,$$

$$J = \sigma [-\nabla \Psi - \partial_t A] + \sigma \tau_s \Re[\Delta^\star (-i\nabla - 2A) \Delta],$$

$$\rho = \frac{\Psi - \Theta}{4\pi \lambda_T},$$

$$\partial_t \rho + \nabla \cdot J = Q,$$

$$\nabla^2 \Theta = -4\pi \rho.$$
Here $D$ is the normal state diffusion constant, $\Psi$ is the electrochemical potential per electron charge, $\xi = \sqrt{6D/\tau_s}$, and the superconducting coherent length is given as $\xi_0 = \xi/\sqrt{\alpha/\beta}$, where $\tau_s$ is the spin-flip scattering time. $\alpha$ and $\beta$ are system dependent constants, which set the magnitude of the order parameter. $\sigma$ and $\lambda_{TF}$ are the normal state conductivity and the Thomas-Fermi static charge screening length, respectively. We measure length in units of $\xi$ and time in units $\xi^2/D$ (since $\xi$ is the unit of length, we write this as $D^{-1}$). We choose the parameters as follows: $\alpha = \beta = 1$, $\tau_s = 6D/\xi^2$, $\lambda_{TF}/\xi = 1$, $\sigma/(D/\xi^2) = 1$, and set the length and width of the two dimensional strip to be $L = 50\xi$. We confirmed that this particular parameter choice does not affect the general conclusions of this paper. We choose the Coulomb gauge $\nabla \cdot A = 0$.32

We study a system that is periodic in $y$ and with open superconductor-vacuum boundary conditions in $x$:16,33,34

$$\Delta(y + L) = \Delta(y),$$  \hspace{1cm} (6)
$$(-i\nabla_x - A_x) \Delta|_{x=0, L} = 0,$$  \hspace{1cm} (7)
$$(-\nabla \Psi)_{x=0, L} = 0.$$  \hspace{1cm} (8)

The advantage of the periodic boundary condition is that it avoids the need to consider complications, irrelevant here, related to creation and destruction of vortices at sample edges.35

We consider very thin films where the sample thickness is much less than the London penetration depth so that the magnetic field can be taken to be spatially uniform and described by the Coulomb-gauge vector potential $A = (0, B(x - L/2), 0)$. We choose magnetic field values such that the number of vortices is commensurate with the system size so that each vertical line of vortices in Fig. 1 has the same number of vortices to avoid additional complications that arise when there is an additional vortex which could go into any one of the channels.

The external current is introduced by the source term $Q$ in Eq. (4), which defines source and drain regions of width $\xi$ over the entire length of the open boundaries. Specifically we take $Q$ to be independent of $y$ and $Q(x, y) \neq 0$ only for $0 < x < \xi$ and $L - \xi < x < L$. Thus, this current flow is homogeneous in the entire sample (except the source and drain sections).

The simulations are performed using a finite element method in space implemented in FEniCS35 and we discretize the time derivative by a finite difference approximation in which the order parameter is updated in time steps of $D\Delta t = 0.5$. We have verified that decreasing the time step does not change the results. The initial conditions (at time $t = 0$) of all variables except $\Delta$ are set to be zero, while $\Delta(x, y, t = 0)$ is chosen to describe the superconducting state with some random fluctuation over the entire 2D sample.38

In our calculation protocol we first set the current to zero and evolve the system; once a stable vortex lattice is formed we introduce the current, and then analyse the flow after a steady state is reached. To be specific, non-zero $Q$ is introduced at $Dt = 1000$, when a rigid vortex structure has already been formed and an analysis of vortex flow created by the sourced current is performed after $Dt = 1500$.

### III. RESULTS

In this section we present the main features of our results. Before doing so we summarize the expected physics. In a magnetic field, the superconducting order parameter $|\Delta|$ in the upper panel and the corresponding $y$-averaged order parameter, denoted as $|\Delta|_y$ against $x$ in the lower panel, calculated with $\xi^2B = 0.043$ for zero applied current ($Q/(D^2/\xi^6) = 0$) in (a), (c) and nonzero applied current ($Q/(D^2/\xi^6) = 0.55$) in (b), (d). The configuration depicted in (b) is a snapshot showing one instant in time of a dynamic state. The corresponding video is available in the SM as video14.36

![Graph showing results](image)

**FIG. 1.** False color representation of the magnitude of the superconducting order parameter $|\Delta|$ in the upper panel and the corresponding $y$-averaged order parameter, denoted as $|\Delta|_y$ against $x$ in the lower panel, calculated with $\xi^2B = 0.043$ for zero applied current ($Q/(D^2/\xi^6) = 0$) in (a), (c) and nonzero applied current ($Q/(D^2/\xi^6) = 0.55$) in (b), (d). The configuration depicted in (b) is a snapshot showing one instant in time of a dynamic state. The corresponding video is available in the SM as video14.36
\[ F = J \times B/c \] directed perpendicular to the current and the magnetic field, and experiences a viscosity \( \eta \), which is within Bardeen-Stephen picture given by\(^3\)

\[ \eta = \Phi_0 \sigma H_{c2} \sim \Delta^2, \quad (9) \]

where \( \sigma \) is the conductivity appearing in the TDGL equations and \( \Phi_0 \) is the flux quantum. \( H_{c2} \) is the upper critical field which is proportional to the inverse of the square of the superconducting order parameter in the clean limit studied here. The system we study has a translation invariance in the \( y \) direction so that in the presence of a current one possible solution is a uniformly translating vortex lattice in which the velocity is determined from the Lorentz force and the viscosity. As we shall see, in our system, other solutions are possible, corresponding to non-uniform vortex motion. In all of our investigations, the non-uniformly moving state exhibits the same row structure as in the equilibrium and uniformly moving states, but vortices in different rows may move with different velocities.

### A. Equilibrium

First, we perform simulations without an externally sourced current, \( Q = 0 \). We initialize our simulation as described in section II and allow the system to evolve to a time independent state. The time evolution takes around \( 600D^{-1} \). The result is shown in Fig. 1 (a), which presents a false-color plot of the magnitude of the order parameter \(|\Delta|\) against position. As expected, an Abrikosov vortex lattice is formed, with the lattice aligned to the boundaries. The vortex positions are visible as regions of very small order parameter amplitude, and the row structure alluded to above is evident. The periodic boundary conditions along \( y \) mean that there is a continuous family of equilibrium solutions translated by arbitrary amounts along \( y \).

Remarkably, in the equilibrium case there are essentially no boundary effects except for the alignment of the lattice and that the vortices avoid the sample boundaries because the supercurrent is reflected at the boundary. The order parameter amplitude in the region away from the vortices shows no significant variation with \( x \) as in Fig. 1 (c), where we plot the \( y \)-averaged (flow direction) order parameter of panel (a), denoted as \(|\Delta|_y\).

### B. Nonzero current drive

At small applied currents our numerically computed solution is just a uniformly translating vortex lattice. However above a critical current the lattice structure is somewhat disrupted. The parallel chain structure remains, but the vortices in the chain closest to the sample edge flow at a higher velocity. This is illustrated in Fig. 2 which shows a series of false color plots of the order parameter amplitude presented at time intervals \( 20D^{-1} \) apart. Two vortices, one in the left most row and one in the second row from the left are highlighted by red and blue color respectively. The difference in velocities is evident.

To quantify the dependence of the differential vortex velocity on the magnitude of the source term \( Q \) in Eq. (4), we present in Fig. 3 a plot of the difference of the time averaged velocities of two tagged particles, \( v_1 \) of a vortex in the left most row and \( v_2 \) of a vortex in the adjacent row (e.g. the red and blue vortices in Fig. 2, respectively) as a function of \( Q \). We see that the difference in velocities remains zero up to a critical current \( Q \approx 0.4D^2/\xi^6 \). For larger \( Q \) the time averaged velocity difference increases linearly with \( Q - Q_c \) until at a yet larger \( Q \approx 0.6D^2/\xi^6 \) the system is reset: the number of vortex channels changes from 6 to 4 due to a strong suppression of the order parameter close to the open boundary (see the SM video17, 18, 19). In the newly established 4 channel structure, the same characteristic as for the intermediate regime \( 0.4 < Q/(D^2/\xi^6) < 0.6 \) is observed as shown in Fig. 3 by a linear increase as a function of \( Q/(D^2/\xi^6) \) after the jump in the curve.
FIG. 3. Differences of the time averaged velocities of two tagged particles, \( v_1 \) of a vortex in the left most row and \( v_2 \) of a vortex in the adjacent row (e.g. the red and blue vortices in Fig. 2, respectively) as a function of the strength of the source term \( Q \). The vortex speeds are extracted from the simulations averaging from \( Dt = 1500 \) to \( Dt = 1980 \). The color of the diamonds in the figure corresponds to the parameters shown in Fig. 7. All corresponding videos are available in the SM. (see also TABLE I in Appendix A).

FIG. 4. Ratio of the averaged order parameter amplitude between \( |\Delta|_1 \) in the left most row (edge) and \( |\Delta|_3 \) in the middle left row (bulk) as a function of the strength of the source term \( Q \). The \( x \)-dependent order parameter \( |\Delta| \) are calculated by averaging over the \( y \)-direction and over the same time duration as for the vortex speed shown in Fig. 3. The color of the diamonds in the figure corresponds to the parameters shown in Fig. 7. All corresponding videos are available in the SM. (see also TABLE I in Appendix A).

IV. INTERPRETATION

We now argue that the velocity differential arises from a position dependence of the viscosity. We see from Fig. 1 (b), (d) that at high drive current the typical value of \(|\Delta|_1 \) far from a vortex is smaller near the sample edge than in the middle, implying via Eq. (9) that the vortices closest to the sample edge experience a lower viscosity. As clearly seen from the comparison between Fig. 1 (c) and (d), in the presence of an externally sourced current, \(|\Delta|_y \) in the region away from the vortices is reduced as it goes to the sample edge. To study the dependence of the reduction of the order parameter on the strength of the source term \( Q \) in Eq. (4), we define the time and \( y \)-direction averaged order parameter amplitude, denoted as \(|\Delta| \) in each vortex row and plot in Fig 4 the ratio of \(|\Delta| \) assigned in the two rows, \(|\Delta|_1 \) in the left most row and \(|\Delta|_3 \) in middle left row. The ratio \(|\Delta|_1/|\Delta|_3 \) monotonously decreases both before and after \( Q \approx 0.6D^2/\xi^6 \) where the number of vortex channels changes from 6 to 4, which implies that the viscosity difference between the left most and middle left row increases as the sourced current increases within the systems of the same number of vortex channels.

Now let us consider the equation of motion in \( y \)-direction for a vortex in the left-most row, supposing that the other vortices in the system are in an ordered, uniformly translating, state:

\[ F_{VL}(y) + \frac{J\Phi_0}{c} = \eta \dot{y}. \]  

Here \( F_{VL} \) is the force arising from all of the other vortices. The current term may be related to the velocity of the uniformly translating part of the vortex lattice, denoted \( V \) as \( \frac{J\Phi_0}{c} = \bar{\eta}V \), where \( \bar{\eta} \) is the average viscosity relevant to these vortices. In the frame co-moving with the uniformly translating state Eq. (10) becomes

\[ F_{VL}(y) + (\bar{\eta} - \eta)V = \eta \dot{y}. \]  

In other words, in the co-moving frame the vortices at the edge of the sample experience two forces—one set by the mean velocity and the viscosity difference, and one from the lattice of uniformly moving vortices. This latter force is periodic under translation by the vortex lattice vector and has a maximum value. At small current, both the difference in viscosity and the mean velocity are small, so this force is insufficient to overcome the force from the vortex lattice. The solution in the co-moving frame is then \( y = 0 \) with \( y \) shifted from its equilibrium position by an amount proportional to the force. As the current is increased, the \((\bar{\eta} - \eta)V \) term increases, the shift in mean position increases, and when the position exceeds...
the point of maximum lattice force the solution in the co-moving frame becomes time dependent.

A. Anti-Poiseuille flow

As is discussed above, in the non-uniformly moving state \((0.4 < Q/(D^2/\xi^6))\) in Fig. 7, the vortices move faster at the edge of the sample because the vortices closest to the edge experience a lower viscosity due to the reduction of the order parameter, shown in Eq. (9). To quantify the relation between the vortex velocity and the reduction of the order parameter, here we perform a simple analysis by neglecting vortex-vortex interaction and the corresponding force arising from it, \(F_{\text{VL}} \approx 0\), which enables us to consider the dynamics of each vortex row independently, and may be reasonable in the large current regime of our present study. In this case, Eq. (10) can be applied to all vortex rows, which is given as \(J\frac{\partial v}{\partial x} = \eta \dot{y}\). Since the current \(J\) is homogeneous over almost the entire two-dimensional sample in our simulations (except the source and drain sections), together with Eq. (9), we can get a simple prediction for the vortex speed and the order parameter:

\[
\dot{y} \propto \Delta^{-2}. \quad (12)
\]

In order to check the relation Eq. (12), we summarize representative results in Fig. 5, where we plot time averaged vortex speeds \(v\) (for an analysis of the time dependent velocity \(\dot{y}\), see section IV C) and \(x\)-dependent inverse of the square of the averaged order parameter \(\dot{\Delta}^{-2}\). Overall, we find good agreement between the simple prediction Eq. (12) and the full numerical simulations for \(v\). The slight disagreement between \(v/(\xi D)\) (blue round dot) and \(\dot{\Delta}^{-2}\) (blue curve) may be due to the negligence of the intricate vortex-vortex interactions.

The vortex flow behaves oppositely to the Poiseuille flow of water in a pipe. Instead of moving slower towards the boundary, the vortices tend to speed up. We therefore call this type of flow Anti-Poiseuille flow. The quantitative details of the flow behavior of course depend on the geometry and parameter regime considered. To quantify this effect, we note that in our simulations the behavior can be fitted by an exponential function \(v = C_0 + C_1 \cosh(C_2 \Delta x)\), where \(\Delta x\) is the distance from the center of a sample and \(C_{0/1/2}\) are constants. Specifically, we fit the vortex speeds by \(C_0 + C_1 \cosh(C_2 x - L/2)/\xi\). In this language \(C_1 < 0\) would correspond to vortices moving slower approaching the boundary, while \(C_1 > 0\) means that vortices move faster at the edges (Anti-Poiseuille flow), with \(|C_1|\) and \(|C_2|\) quantifying the strength of the influence of the boundary.

B. Resistivity

To connect to experimentally accessible transport quantities we consider the resistances across the vortex channels \(R = \Delta \Theta/Q\), where \(\Delta \Theta\) is the difference of the scalar potential left and right to a given channel averaged over the \(y\)-direction. In a Bardeen-Stephen picture, that difference of the scalar potential is proportional to the number of vortices in each channel and their velocities. In Fig. 6, we show the time and \(y\)-direction averaged scalar potential depending on \(x\). Since the external current is constant in our system, we can consider differences of the scalar potential \(r = \Delta \Theta\) to be directly proportional to the resistance between each vortex channel. Fig. 5 clearly shows that faster vortex flows (blue dot) result in proportionally higher resistance \(r\) (purple square) in accord with the Bardeen-Stephen picture. Following the same arguments as for the vortex velocity we therefore find \(r \propto \dot{\Delta}^{-2}\).
C. Intermediate drive: stick-slip motion

Fig. 7 shows how the difference between the position along the y-direction of the left most $y_1$ and the adjacent $y_2$ vortices evolves as a function of time $Dt$. For weaker drive ($Q/(D^2/\xi^6) = 0.4$) the two vortices move together ($(y_1 - y_2)/\xi$ is time independent). For the strongest drive ($Q/(D^2/\xi^6) = 0.5875$) the two vortices move at different approximately constant velocities, but at intermediate drive a “stick-slip” behavior is evident, in which the relative position exhibits a series of approximately linear increase in the difference of the positions of neighboring vortices separated by plateaus where the separation is time independent.

In panel (b) we present the difference of the corresponding time derivatives. For the smallest $Q/(D^2/\xi^6) = 0.4$, the difference in time dependent velocities $(\dot{y}_1 - \dot{y}_2)/(\xi D)$ is almost zero at all times meaning that the lattice moves as a whole, while for the largest $Q/(D^2/\xi^6) \geq 0.55$, the velocity difference $(\dot{y}_1 - \dot{y}_2)/(\xi D)$ is essentially constant. For the intermediate $Q$ the relative velocity alternates in time between a large and a small value.

The times of linear increase in panel (a) with the peaks in panel (b) correspond to when the left most vortices passes the adjacent while the plateau in panel (a), (b) corresponds to the left most and the adjacent vortex moving with the same speeds as in the small sourced current regime $Q/(D^2/\xi^6) \leq 0.4$ (see the SM video10, 12). At each region of the linear increase in panel (a) and the peak in (b), a shift of the left most and the adjacent vortices channels occurs. Or put into the language of the stick-slip phenomenology: initially the left most vortex sticks to the lattice, then it accelerates and passes (slips) across the adjacent vortex, and slows down to stick again to the newly established Abrikosov lattice, which is repeated as a converged non-equilibrium stationary state. This behaviour relies on an inter-play between the vortex-vortex interactions and the large driving cur-
rent (creating the reduced order parameter) as well as the boundary. The former tries to keep the Abrikosov vortex lattice while the latter two enforce the Anti-Poiseuille flow. By increasing \( Q/(D^2/\xi^6) \), the plateaus are disappearing in panel (a) and in (b) the number of peak is increased with the decrease of the peak height, which shows that vortex-vortex interaction is becoming gradually almost negligible and the speed difference between the left most and the adjacent vortices converges to a constant. So for increasing the strength of the source the stick-slip phenomenology is increasingly washed out.

V. CONCLUSION

Using time-dependent Ginzburg Landau theory, we study dynamics of vortex flows in 2D superconducting thin films with an external current sourced through the sample. We find that at small sourced currents the vortex lattice moves as a whole where all vortices move at the same velocity, while at larger sourced currents a vortex motion is assigned into channels where vortices in channels closer to the sample edge move faster than those farther way from the sample edge. We show that the velocity differential in the larger current regime arises from a position dependence of the viscosity depicted as by a Bardeen-Stephen picture and express the equation of motion for the vortex in the channel next to the boundary. Further analysis with neglecting intricate vortex-vortex interactions yields a simple prediction for the position dependent vortex velocity, which demonstrates a good agreement between this simple prediction and our full numerical calculations within the time-averaged analysis. The behaviour of vortex flow is found to be opposite from what is typical in Poiseuille-like flow, namely the velocity of the flow increases towards the boundary. In addition, detailed real-time analysis of the vortices’ motions for the larger current regime reveals a stick-slip motion of the vortex lattice as the sourced current is increased. Initially, the vortices in the channel at the edges stick to the Abrikosov lattice, then they start to accelerate, move past their neighbours, followed by slowing down to again stick to another reestablished Abrikosov lattice, which is periodically continuing as a stationary state of the system. This intricate motion relies on the interplay between the vortex-vortex interactions and the large driving current at a boundary (creating the reduced order parameter).

Experimentally, this effect may be observed by a local probe that can count the rate at which vortices cross the probe to map the vortex velocity in space along the edge. A scanning superconducting quantum interference device (SQUID) can be used for low Ginzburg-Landau parameter superconductors, \( \kappa = \Lambda/\xi \) (where \( \Lambda \) and \( \xi \) are the penetration depth and coherent length, respectively), or one could use an ultra-fast STM.

Although for all of the simulations in the present study, a choice of the boundary condition, there is no reason to believe that general features shown in this paper cannot be observed in real metal-superconductor boundary, which naturally creates the same scenario shown in this paper since the order parameter continuously goes to zero at the metal-superconductor boundary.

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Appendix A: Short explanation of the videos

In this section, we show the correspondence between the video number and a strength of the source term \( Q \) in the SM. This is summarized in TABLE I.

| video | quantity \( Q/(D^2/\xi^6) \) |
|-------|--------------------------|
| 1     | \( \Delta \) 0.05        |
| 2     | \( \Delta \) 0.10        |
| 3     | \( \Delta \) 0.15        |
| 4     | \( \Delta \) 0.20        |
| 5     | \( \Delta \) 0.25        |
| 6     | \( \Delta \) 0.30        |
| 7     | \( \Delta \) 0.35        |
| 8     | \( \Delta \) 0.40        |
| 9     | \( \Delta \) 0.425       |
| 10    | \( \Delta \) 0.45        |
| 11    | \( \Delta \) 0.475       |
| 12    | \( \Delta \) 0.50        |
| 13    | \( \Delta \) 0.525       |
| 14    | \( \Delta \) 0.55        |
| 15    | \( \Delta \) 0.575       |
| 16    | \( \Delta \) 0.5875      |
| 17    | \( \Delta \) 0.60        |
| 18    | \( \Delta \) 0.625       |
| 19    | \( \Delta \) 0.65        |
