Time Dependence of Advection Dominated Accretion Flow with a Toroidal Magnetic Field

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ABSTRACT
The present study examines self-similarity evolution of advection dominated accretion flow (ADAF) in the presence of a toroidal magnetic field. In this research, it was assumed that the angular momentum transport is due to viscous turbulence and \( \alpha \)-prescription was used for kinematics coefficient of viscosity. The flow does not have a good cooling efficiency and so, a fraction of energy accretes with matter on central object. The effect of a toroidal magnetic field on such systems in a dynamical behavior was investigated. In order to solve the integrated equations which govern the dynamical behavior of the accretion flow, self-similar solution was used. The solution provides some insights into the dynamics of quasi-spherical accretion flow and avoids many of the strictures of the steady self-similar solutions. The solutions show that the behavior of physical quantities in a dynamical ADAF are different from steady accretion flow and a disk with polytropic approach. The effect of the toroidal magnetic field is considered with additional variable \( \beta = \frac{p_{\text{mag}}}{p_{\text{gas}}} \), where \( p_{\text{mag}} \) and \( p_{\text{gas}} \) are the magnetic and gas pressure, respectively. Also, by considering the effect of advection in these systems, the advection parameter \( f \) was introduced that stands for a fraction of energy that accretes by matter to the central object. The solution indicates a transonic point in the accretion flow for all selected amounts of \( f \) and \( \beta \). Also, by adding strength of the magnetic field and the degree of advection, the radial-thickness of the disk decreased and the disk compressed. The model implies that the flow has differential rotation and is sub-Keplerian at small radii and is super-Keplerian in large radii and that different result was obtained using a polytropic accretion flow. The obtained \( \beta \) parameter was used a function of position that increases by increasing radii. Also, the behavior of ADAF in a large toroidal magnetic field implies that different result was obtained using steady self-similar models in large magnetic field.

Key words: accretion, accretion disks, magnetohydrodynamics: MHD

1 INTRODUCTION
During recent years one type of accretion disks has been studied, in which it is assumed that the energy released through viscous processes in the disk may be trapped within the accreting gas. This kind of flow is known as advection-dominated accretion flow (ADAF). The basic ideas of such ADAF models have been developed by a number of researchers (e.g., Ichimaru 1977; Rees et al. 1982; Narayan & Yi 1994, 1995; Abramowicz et al. 1995; Ogilvie 1998; Akizuki & Fukue 2006; hereafter AF06).

It is thought that accretion disks, whether in star-forming regions, in X-ray binaries, in cataclysmic variables, or in the centers of active galactic nuclei, are likely to be threaded by magnetic fields. Consequently, the role of magnetic fields on ADAF has been analyzed in detail by a number of investigators (Bisnovatyi-kogan & Lovelace 2001; Shadmehri 2004; AF06; Ghanbari et al. 2007, Abbassi et al. 2008). The existence of the toroidal magnetic fields have been proven in the outer regions of YSO discs (Greaves et al. 1997; Aitken et al. 1993; Wright et al. 1993) and in the Galactic center (Novak et al. 2003; Chuss et al. 2003). Thus, considering the accretion disks with a toroidal magnetic field have been studied by several authors (AF06; Begelman & Pringle 2007; Abbassi et al. 2008; Khesali & Faghei 2008 and references within; hereafter KF08). KF08 considered dynamic behavior of a polytropic accretion flow in presence of a toroidal magnetic field. In a dynamic approach they showed the radial behavior of the physical quantities were different with results achieved by those who considered the accretion flow in a steady self-similar methods (Shadmehri 2004; AF06; Ghanbari et al 2007; Abbassi et al. 2008). For example, KF08 presented that ratio of the magnetic pres-
sure to the gas pressure is not constant and varies by radius. The results of KF08 were assembled on polytropic equation that implies the accreting gas has a good cooling efficiency, while results of some authors have shown that the behavior of physical quantities are very sensible to fraction of the energy that traps within the accreting gas (AF06). So, in the present study it is intended to investigate dynamic behavior of an ADAF in presence of a toroidal magnetic field. The paper is organized as follows. In section 2, the general problem of constructing a model for quasi-spherical magnetized accretion dominated accretion flow will be defined. In section 3, self-similar model for solving the integrated equations which govern the dynamic behavior of the accreting gas was utilized. The summary of the model will appear in section 4.

2 BASIC EQUATIONS

In this section, we derive the basic equations which describe the physics of accretion flow with a toroidal magnetic field. We use the spherical coordinates (r, θ, φ) centred on the accreting object and make the following standard assumptions:

(i) The accreting gas is a highly ionized gas with infinitive conductivity;
(ii) The magnetic field has only an azimuthal component;
(iii) The gravitational force on a fluid element is characterized by the Newtonian potential of a point mass, Ψ = −GMr/r, with G representing the gravitational constant and Ms standing for the mass of the central star;
(iv) The equations written in spherical coordinates are considered in the equatorial plane θ = π/2 and terms with any θ and φ dependence are neglected, hence all quantities will be expressed in terms of spherical radius r and time t;
(v) For simplicity, the self-gravity and general relativistic effects have been neglected.

Under the assumptions and the approximation of quasi-spherical symmetry and the ideal magnetohydrodynamics treatment, the dynamics of a magnetized accretion flow is described by the following equations:

the continuity equation
\[ \frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho v_r \right) = 0, \]

the radial force equation
\[ \rho \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{GM_s}{r^2} = \frac{r \Omega^2}{r} - \frac{B_\phi}{4\pi r \rho} \frac{\partial (r B_\phi)}{\partial r}, \]

the azimuthal force equation
\[ \rho \left[ \frac{\partial}{\partial t} \left( r^2 \Omega \right) + v_r \frac{\partial}{\partial r} \left( r^2 \Omega \right) \right] = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ \nu r^2 \Omega \frac{\partial}{\partial r} \right], \]

the energy equation
\[ \frac{1}{\gamma - 1} \left[ \frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} \right] + \frac{\gamma}{\gamma - 1} \frac{1}{r \partial \theta} \frac{\partial}{\partial \theta} \left( r^2 v_r \right) = f \nu r^2 \left( \frac{\partial \Omega}{\partial r} \right)^2 \]

and the field freezing equation
\[ \frac{\partial B_\phi}{\partial t} + \frac{1}{r \partial r} (r v_r B_\phi) = 0, \]

Here \( \rho \) is the density, \( v_r \) the radial velocity, \( \Omega \) the angular velocity, \( M_s \) the mass of the central object, \( p \) the gas pressure, \( B_\phi \) the toroidal component of magnetic field, \( \nu \) the kinematic viscosity coefficient and it is given, as in Narayan & Yi (1995), by an \( \alpha \)-model
\[ \nu = \frac{\alpha \rho_{gas} \Omega K}{\beta K} \]

where \( \Omega_K = (GM_*/r^3)^{1/2} \) is the Keplerian angular velocity. The parameters \( \gamma \) and \( \alpha \) are assumed to be constant and \( f \) measures the degree to which the flow is advection-dominated (Narayan & Yi 1994), and is assumed to be constant.

3 SELF-SIMILAR SOLUTIONS

3.1 Analysis

Self-similar models have proved very useful in astrophysics because the similarity assumption reduces the complexity of the partial differential equations. Even greater simplification is achieved in the case of spherical symmetry since the governing equations then reduce to comparatively simple ordinary differential equations. We introduce a similarity variable \( \eta \) and assume that each physical quantity is given by the following form:

\[ r = r_0(t) \eta \]

\[ \rho(r, t) = \rho_0(t) R(\eta) \]

\[ p(r, t) = p_0(t) P(\eta) \]

\[ v_r(r, t) = v_0(t) V(\eta) \]

\[ \Omega(r, t) = \Omega_0(t) \Omega(\eta) \]

\[ B_\phi(r, t) = b_0(t) B(\eta). \]

By assuming power-law time dependent of \( r_0(t) = at^n \), where \( n = 2/3 \), we find the following relations:

\[ r_0(t) = at^{2/3} \]

\[ p_0(t)/\rho_0(t) = \frac{GM_*/a}{t^{-2/3}} \]

\[ v_0(t) = \frac{GM_*/a}{\sqrt{t^{-1/3}}} \]

\[ \Omega_0(t) = \sqrt{\frac{GM_*/a^3}{t^{-1}}} \]

\[ b_0^2(t)/8\pi \rho_0(t) = \frac{GM_*/a}{t^{-2/3}}. \]

The above results imply that \( p_0(t) \) and \( b_0(t) \) are dependent on timely behavior of \( \rho_0(t) \). So, for specifying time dependent of \( \rho_0(t) \), and then \( p_0(t) \) and \( b_0(t) \), we introduce the mass accretion rate \( \dot{M} \)

\[ \dot{M} = -4\pi r^2 \rho v_r. \]

Similar to equations (7)-(12) for the mass accretion rate we can write

\[ \dot{M}(r, t) = \dot{M}_0(t) \dot{r}(\eta). \]

Under transformations of equations (7), (8) and (10), equation (19) becomes
Now, we consider a set of solutions that \( \dot{M}(t) \) is a constant (KF08), thus we can write
\[
\rho(t) = \left( \dot{M}/\sqrt{GM_\star a^2} \right) t^{-1}
\]
that implies
\[
\rho(t) = \left( \dot{M}_0/\sqrt{GM_\star a^2} \right) t^{-5/3}
\]
and
\[
\dot{b}_0^2(t)/8\pi = \left( \dot{M}_0/\sqrt{GM_\star a^2} \right) t^{-5/3}.
\]
Substituting equations (6)-(12) and (13)-(17) into the basic equations (1)-(6), the similarity equations are obtained as
\[
\begin{align*}
-M + \left( V - \frac{2\eta}{3} \right) \frac{dV}{d\eta} + \frac{P}{R} \frac{dP}{d\eta} + \frac{1}{\eta} &= 0,
\end{align*}
\]
\[
\begin{align*}
\frac{V}{3} + \left( V - \frac{2\eta}{3} \right) \frac{dV}{d\eta} + \frac{1}{\eta} &= \frac{1}{\gamma - 1 \left[ \frac{5}{4} \frac{P}{R} \left( V - \frac{2\eta}{3} \right) \frac{dP}{d\eta} \right] + \frac{\gamma}{\gamma - 1 \eta^2} \frac{d}{d\eta} \left( \eta^2 V \right)}
\end{align*}
\]
\[
\begin{align*}
&= \alpha_f \frac{P}{R} \frac{d\eta}{d\eta} \left( \frac{d\eta}{d\eta} \right)^2,
\end{align*}
\]
\[
\begin{align*}
&= 0.
\end{align*}
\]
To investigate existence of transonic point, the square of the sound velocity is introduced that subsequently can be expressed as
\[
v_s^2 = \frac{p}{\rho} = \frac{GM_\star P}{a R} t^{-2/3}
\]
Here, \( S = (P/R)^{1/2} \) the sound velocity in self-similar flow, which is rescaled in the course of time. The Mach number referred to the reference frame is defined as (Fukue & Fukue 1983)
\[
\mu \equiv \frac{v_r - v_F}{v_s} = \frac{V - \eta \eta}{S}
\]
where
\[
v_F = \frac{dr}{dt} = \frac{r}{t}
\]
is the velocity of the reference frame which is moving outward as time goes by. The Mach number introduced so far, represents the instantaneous and local Mach number of the unsteady self-similar flow. We will consider transonic points of accretion flow in next subsection.

In order to consider the strength of the magnetic field in the plasma, the \( \beta \) parameter is introduced that is ratio of the magnetic to the gas pressures
\[
\beta(r,t) = \frac{B_0^2(r,t)/8\pi}{P(r,t)} = \frac{B_0^2(\eta)}{P(\eta)}.
\]

In completing this section, we also summarize the main results here. Solving equations (1), (10), (11), and (19) under transformations (12)-(15) in non-magnetically state, makes it clear that time behavior of physical quantities in the non-magnetically and the magnetically disk are the same. This result is one of the strictures of time-dependent self-similar solution. on the other hand, the fact that timely-dependent behavior of the magnetic and gas pressures becomes same is one of limits the self-similarity solution. On the other hand, the physical quantities with a same physical dimension have similar behaviors in self similar solution.

### 3.2 Asymptotic behavior

In this subsection, the asymptotic behavior of the equations (22), (26)-(30), and (34) at \( \eta \to 0 \) and \( \gamma < 5/3 \) is investigated, the asymptotic solutions are given by
\[
R(\eta) \sim R_0 \eta^{-3/2}
\]
\[
P(\eta) \sim P_0 \eta^{-5/2}
\]
\[
V(\eta) \sim V_0 \eta^{-1/2}
\]
\[
\omega(\eta) \sim \omega_0 \eta^{-3/2}
\]
\[
B(\eta) \sim B_0 \eta^{-1/2}
\]
\[
\dot{m}(\eta) \sim -4\pi R_0 V_0
\]
\[
\beta(\eta) \sim (B_0^2/P_0) \eta^{3/2}
\]
in which
\[
R_0 = \frac{3}{8\pi} \alpha_f \dot{m} s_a \left( \frac{\gamma - 1}{\gamma - 5/3} \right) \left( \frac{g_2}{g_3} \right)
\]
\[
P_0 = \frac{\dot{m} n_a}{6\pi a}
\]
\[
V_0 = \frac{2}{3\alpha f} \left( \frac{\gamma - 5/3}{\gamma - 1} \right) \left( \frac{g_2}{g_1} \right)
\]
\[
\omega_0 = \frac{2}{3\alpha f} \left( \frac{\gamma - 5/3}{\gamma - 1} \right) \left( \frac{g_2}{g_1} \right)^{1/2}
\]
\[
B_0 = \frac{\beta_0 \dot{m} n_a}{6\pi a}
\]
where
\[
1 - \frac{5f}{2} \left( \frac{\gamma - 1}{\gamma - 5/3} \right)
\]
\[
\frac{3f}{2} \alpha f \left( \frac{\gamma - 1}{\gamma - 5/3} \right)
\]
\[
\gamma_0 = -1 + \sqrt{1 + 2g_1^2 g_2^2}
\]
\[
\beta_0 = \beta_a / \eta^{3/2}.
\]
The achieved results for asymptotic behavior of physical quantities show that the physical quantities of accretion flow are very sensitive to parameters of $\alpha$, $\gamma$, $f$, $\beta_{in}$, and $\dot{m}_{in}$. The $\beta_{in}$ and $\dot{m}_{in}$ are amounts of $\beta$ and $\dot{m}$ at $\eta_{in}$ that $\eta_{in}$ is a point near of the center. The affects of the viscous parameter $\alpha$ and the advection parameter $f$ on accretion flow are plotted in figure 1. The angular velocity profiles indicate that by increasing the viscous parameter $\alpha$, the angular velocity of accretion flow decreases, because we increase the viscous torque by increasing parameter $\alpha$. Also increasing the advection parameter $f$ decreases the angular velocity that is qualitatively consistent with AF06. Figure 1 shows the radial infall velocity increases by adding $\alpha$ and $f$ that are similar to the results of AF06 and KF08. Also the density profiles represent density decreases by adding $f$ and $\alpha$.

3.3 Numerical solutions

If the value of $\eta_{in}$ is guessed, that is a point very near to the center, the equations can be integrated from this point to the outward through the use of the above expansion. Examples of such solutions are presented in figures 2, 3, and 4. The profiles in figure 2 are plotted for different $\beta_{in}$, the profiles in figure 3 are plotted for different $f$ and in figure 4 transonic behavior of the accreting gas for different amount of $f$ and $\beta_{in}$ is considered. The delineated quantities ($\log(\eta^{3/2} R)$, $\log(-\eta^{1/2} V)$, ...) in figures 2, 3, and 4 are constant in steady self-similar solutions (Narayan & Yi 1994; Narayan & Yi 1995; Shadmehri 2004; AF06; Ghanbari et. al. 2007; Abbassi et al. 2008), while here, they vary by position.

Figure 2 informs us that density and the radial thickness of disk decreases by adding strength of the toroidal magnetic field, these results are well consistent with KF08. Also, by decreasing amount of magnetic field, the behavior of density becomes similar to non-magnetic case (Ogilvie 1999). The behavior of the gas pressure in KF08 had polytropic behavior and this selection caused the gas pressure follow the density behavior, while here we see behavior of the gas pressure does not follow the density behavior. Also, by adding the $\beta$ parameter, the radial infall velocity increases; such property is qualitatively consistent with AF07 and KF08. This is due to the magnetic tension terms, which dominate the magnetic pressure term in the radial momentum equation that assist the radial infall motion. The profiles of the angular velocity imply that the disk is sub-Keplerian in inner part of the disk and is super-Keplerian in outer part of it, while in polytropic accreting flow (KF08) and non-magnetic accretion flow (Ogilvie 1999) the angular velocity is sub-Keplerian in all radii (KF08). Similar to the results KF08 the $\beta$ parameter, the ratio of the magnetic pressure to the gas pressure, is a function of position and arises from inner to outer that the result is well consistent with observational evidence obtained by some authors (Aitken et al. 1993; Wright et al. 1993; Greaves et al. 1997). While the $\beta$ parameter in steady self-similar solution becomes constant at all radii (AF06) that is one of restriction of steady self-similar solution. Figure 3 is plotted for different amounts of the advection parameter $f$. The advection parameter $f$ has slight effect on the toroidal magnetic field, the parameter of $\beta$, and the Mach number, however has outstanding effect on the density, the gas pressure, the radial infall velocity, and the angular velocity. The density and the radial thickness of disk decrease by more advecting of accreting gas that is same at all part of the disk, the result can be achieved by assuming of $f$ as a constant amount. Also we see by increasing the amount of the advection parameter $f$, the gas pressure decreases. By increasing $f$, the radial infall velocity increases and the angular velocity decreases. The results are qualitatively consistent with the results of AF06.

The Mach number profiles in figure 4 imply that the flow of outer part for all selected amounts of the magnetic field become super sonic. We can see this result in polytropic accretion flow by KF08. The advection parameter decreases the amount of the Mach number slightly.

The profiles of physical quantities in figure 2 imply that they have the power of law dependency to $\eta$ in magnetical domination ($\beta_{in} > 1$). So, by fitting a power function on data in magnetical domination ($\beta_{in} = 10$), we can write

\begin{align}
R(\eta) & \propto \eta^{-1.66} \\
P(\eta) & \propto \eta^{-2.58} \\
V(\eta) & \propto \eta^{-0.01} \\
\omega(\eta) & \propto \eta^{-1.25} \\
B(\eta) & \propto \eta^{-0.83} \\
\beta(\eta) & \propto \eta^{-0.92} \\
\mu(\eta) & \propto \eta^{0.93} \\
\dot{m}(\eta) & \propto \eta^{-0.33}.
\end{align}

The achieved results are different with steady magnetical dominated accretion flow (Meier 2005, Shadmehri & Khajenabi 2005).

4 SUMMARY AND DISCUSSION

In the paper, the equations of time-dependent of advection dominated accretion flow with a toroidal magnetic field have been solved by semi-analytical similarity methods. The flow is not able to radiate efficiency, so we substituted the energy equation instead of polytropic equation that KF08 had used. A solution was found for the case $\gamma < 5/3$ that has differential rotation and viscous dissipation. The flow avoids many of the strictures of steady self-similar solutions (Narayan & Yi 1994; AF06; Ghanbari et al. 2007; Abbassi et al. 2008). Thus, the radial-dependence of calculated physical quantities in this approach are different from steady self-similar solution.

Increase of the advection parameter $f$ and the parameter $\beta_{in}$ will separately increase the infall radial velocity and decrease the angular velocity. The flow has differential rotation and is sub-Keplerian in inner part and is super-Keplerian in large radii in which the behavior is seen in some astrophysical objects such as M81, M87 and Milky Way (Sofue 1998; Ford & Tsvetanov 1999). The solution showed that the flow for all selected amounts of $f$ and $\beta_{in}$ becomes super sonic in large radii and sub-sonic in small radii that are qualitatively consistent with the results of KF08. The parameter of $\beta$ is a function of position that raises from inner to outer and states the magnetic field is more important in large radii. It is also consistent with observational evidences in the outer regions of YSO discs (Greaves et al. 1997; Aitken et al. 1993; Wright et al. 1993; Greaves et al. 1997).
et al. 1993; Wright et al. 1993) and in the Galactic center (Novak et al. 2003; Chuss et al. 2003).

Here, latitudinal dependence of physical quantities is ignored, while some authors showed that latitudinal dependence is important in the structure of a disk (Narayan & Yi 1995; Ghanbari et al. 2007). Latitudinal behavior of such disks can be investigated in other studies. Also we did not consider relativity effect, if the central object is relativistic, the gravitational field should be changed. Furthermore, in a realistic model the advection parameter $f$ is a function of position and time, other researchers can consider such disks.

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Figure 1. Numerical coefficient $\omega_0$ (dotted lines), $R_0$ (solid lines) and $V_0$ (dashed lines) as functions of advection parameter $f$ or the viscous parameter $\alpha$. The ratio of specific heats is set to be $\gamma = 1.5$ and the inner mass accretion rate is $\dot{m}_{\text{in}} = 0.001$.

Figure 2. Time-dependent self-similar solution for $\gamma = 1.5$, $\alpha = 0.5$, $f = 1.0$, and $\dot{m}_{\text{in}} = 0.001$. The lines represent $\beta_{\text{in}} = 0.1, 0.5, 1.0, 10$ that $\beta_{\text{in}}$ is value of $\beta$ in $\eta_{\text{in}}$. 
Figure 3. Time-dependent self-similar solution for $\gamma = 1.5$, $\alpha = 0.5$, $\beta = 1.0$, and $\dot{m}_{in} = 0.001$. Lines represent $f = 0.1, 0.5, 1.0$.

Figure 4. Left panel: Mach number profiles for $\gamma = 1.5$, $\alpha = 0.5$, $f = 1.0$, and $\dot{m}_{in} = 0.001$. Right panel: Mach number profiles for $\gamma = 1.5$, $\alpha = 0.5$, $\beta = 1.0$, and $\dot{m}_{in} = 0.001$. 