CATEGORICAL RECONSTRUCTION OF CRYSTALS

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0. Introduction

This paper is split into two sections, each with a different flavour, to study crystal bases by two different means. Our goal is to endow some crystal bases with the structure of a bialgebra. Conceptually, we may think of these algebraic structures as quantum groups over the hypothetical field with one element.

In the first section, we study the algebraic structures that are induced on certain $U_q(\mathfrak{sl}_2)$-modules by the Clebsch-Gordan decomposition, specifically that of the space spanned by matrix coefficients of the irreducible $\mathfrak{sl}_2$ representations. This space, which can also be viewed as the space of algebraic functions on the group $SL(2; \mathbb{C})$, forms a bialgebra whose comodules are precisely representations of $\mathfrak{sl}_2$. The main aim of this section is to see whether this structure remains when we move to crystal bases, and whether we obtain a similar classification result. We find that we retain much, but not all, of this structure and that crystal bases can not be classified in the same way. Much of this work is done for crystal bases over general quantum groups, although, in the case of $\mathfrak{sl}_2$, our results can be explicitly computed.

The second section has more of a categorical feel. After initially failing to classify crystal bases as comodules over an abstract algebra in the category of crystals, we turn to category theory and the theory of monadic (and comonadic) functors. This gives the classification of crystal bases as coalgebras over a comonadic functor, which we link back to the attempts from the first section. We finish by encoding the monoidal structure of the category of crystals into our comonadic functor, giving some form of bimonadic functor. This is done by applying an extention of the Barr-Beck Theorem to monadic functors on monoidal categories. All of this work is done for a crystal bases over an arbitrary quantum group.

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Part 1. Crystal Bialgebras

1. Quantum Groups and Crystals

1.1. Quantum Groups. The following constructions of general quantum groups can be seen in Kashiwara’s paper \[6\] and in Jantzen’s book \[5\] p. 51.

Suppose a Lie algebra \( \mathfrak{g} \) is defined by the following data:

i) A weight lattice \( \Phi \), a free \( \mathbb{Z} \)-module, with simple roots \( \alpha_i \in \Phi \) for \( i \) in an indexing set \( I \) that form a basis of the root lattice \( \Psi \) (with respect to some Cartan subalgebra) contained in \( \Phi \);

ii) A symmetric bilinear form \( (\cdot, \cdot) : \Phi \times \Phi \rightarrow \mathbb{Q} \) such that \((\alpha_i, \alpha_i) \in 2\mathbb{N} \), \((\alpha_i, \alpha_j) \leq 0 \) for \( i, j \in I \), \( i \neq j \);

iii) Simple coroots \( \lambda_i \in \Phi^* \) is defined by the following data:

\[ h_i, h_j = 0, \quad [e_i, f_i] = \delta_{ij} h_i, \quad [h_i, e_j] = \lambda_i(\alpha_j)e_j, \quad [h_i, f_j] = -\lambda_i(\alpha_j)f_j, \]

and for \( i \neq j \),

\[ (ad e_i)^{1-\lambda_i(\alpha_j)} e_j = 0, \quad (ad f_i)^{1-\lambda_i(\alpha_j)} f_j = 0, \]

where \( ad \) is the adjoint map \( (ad x)(y) = [x, y] \).

Note that, in the case of \( \mathfrak{sl}_2 \), the weight lattice \( \Phi \) is \( \mathbb{Z} \) with simple root 2.

Definition Take \( q \) to be a general nonzero element of our base field \( k \) which is not a root of unity. It will be convenient to think of \( q \) as an independent indeterminant and work over \( k(q) \). We may then define the quantised enveloping algebra \( U_q(\mathfrak{g}) \) to be the algebra generated over our field \( k \) by \( e_i, f_i, q(\lambda) \) for \( i \in I, \lambda \in \Phi^* \), with the defining relations

\[
\begin{align*}
q(\lambda) &= 1, \\
q(\lambda_1)q(\lambda_2) &= q(\lambda_1 + \lambda_2), \\
q(\lambda)e_i q(\lambda)^{-1} &= q^{\lambda(\alpha_i)} e_i, \\
q(\lambda)f_i q(\lambda)^{-1} &= q^{-\lambda(\alpha_i)} f_i, \\
e_i f_i - f_i e_i &= \delta_{ij} \frac{t_j - t_i^{-1}}{q_i - q_j} \quad \text{where} \quad q_i = q^{\langle \alpha_i, \alpha_i \rangle/2} \\
\text{and} \quad t_i &= q^{\langle \alpha_i, \alpha_i \rangle/2} \lambda_i, \quad \text{(Serre relations)}
\end{align*}
\]

for \( i \neq j \)

\[
\sum_{k=0}^{b} (-1)^k e_i^{(k)} e_j^{(b-k)} f_i^{(k)} f_j^{(b-k)} = 0 \quad \text{where} \quad b = 1 - \lambda_i(\alpha_j)
\]

and

\[
[k]_i! = \frac{q_i^k}{q_i - q_i^{-1}}, \quad [k]_i! = [1]_i [2]_i \ldots [k]_i, \quad e_i^{(k)} = e_i^k/[k]_i!.
\]

Remark In the simple case of \( \mathfrak{sl}_2 \), \( U_q(\mathfrak{sl}_2) \) (as seen in \[6\] and \[7\] p. 122)) is the \( k \)-algebra generated by \( e, f, t, t^{-1} \) with defining relations

\[
tet^{-1} = q^2 e, \quad tft^{-1} = q^{-2} f, \quad ef - fe = \frac{t - t^{-1}}{q - q^{-1}}.
\]
We can see that the subalgebras of $U_q(\mathfrak{g})$ generated by $e_i, f_i, t_i, t_i^{-1}$, denoted $U_q(\mathfrak{g})_i$, are isomorphic to $U_q(\mathfrak{sl}_2)$. So we may build these quantum groups up from quantised $\mathfrak{sl}_2$, which we will often refer to as our main example. A lot of the motivation behind this paper comes from $\mathfrak{sl}_2$, but we will adapt our results to more general cases.

1.2. The Category of Crystals. We begin by constructing a category of crystals, as Kashiwara does in his paper [6]. These crystals arise from the notion of crystal bases of (integrable) $U_q(\mathfrak{sl}_2)$ representations, but we shall only allude briefly to these origins. For more details, we direct the reader to [6] and [5].

**Definition** A pointed set is a set with a distinct point or element, which we shall denote $0$, $A^* = A \cup \{0\}$ with unions $A^* \cup B^* = (A \cup \{0\}) \cup (B \cup \{0\}) := (A \cup B) \cup \{0\}$ and products $A^* \times B^* = (A \times B) \cup \{0\}$. A morphism between pointed sets $A^*$ and $B^*$ is a map from $A$ to $B \cup \{0\}$ extended to map $0 \mapsto 0$. This defines a category of pointed sets, which we shall denote $\text{Set}^*$. 

**Definition** We define the objects of the category of crystals, denoted $\text{Crys}$, to be pointed sets $B$ equipped with maps

\[
\begin{align*}
\text{wt} & : B \to \Phi \\
\varepsilon_i & : B \to \mathbb{Z} \sqcup \{-\infty\} \\
\phi_i & : B \to \mathbb{Z} \sqcup \{-\infty\} \\
\tilde{e}_i & : B \to B \\
\tilde{f}_i & : B \to B
\end{align*}
\]

for all $i \in I$ such that, for a crystal $B$ and $b \in B$,

- if $\tilde{e}_i(b) \neq 0$ then $\phi_i(b) = \lambda_i(\text{wt}(b)) + \varepsilon_i(b)$,
- if $\tilde{f}_i(b) \neq 0$ then $\varepsilon_i(\tilde{f}_i(b)) = \varepsilon_i(b) + 1$,

\[
\begin{align*}
\phi_i(\tilde{e}_i b) & = \phi_i(b) + 1, \\
\text{wt}(\tilde{e}_i b) & = \text{wt}(b) + \alpha_i,
\end{align*}
\]

- if $\tilde{f}_i(b) \neq 0$ then $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$,
- if $\phi_i(b) = -\infty$ then $\tilde{e}_i b = \tilde{f}_i b = 0$.

again, with the assumption that $-\infty + n = -\infty$ for any $n \in \mathbb{Z}$. For crystals $B_1, B_2$, we say that a map $\psi : B_1 \to B_2$ is a morphism of crystals if, for $b \in B_1$,

\[
\begin{align*}
\psi(0) & = 0, \\
\text{if } \psi(b) \neq 0 & \text{ then } \varepsilon_i(\psi(b)) = \varepsilon_i(b), \\
\phi_i(\psi(b)) & = \phi_i(b), \\
\text{wt}(\psi(b)) & = \text{wt}(b), \\
\text{if } \psi(b) \neq 0 & \text{ and } \psi(\tilde{e}_i b) \neq 0 \text{ then } \psi(\tilde{e}_i b) = \tilde{e}_i \psi(b), \\
\text{and if } \psi(b) \neq 0 & \text{ and } \psi(\tilde{f}_i b) \neq 0 \text{ then } \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b).
\end{align*}
\]

The motivation behind defining this category comes from the study of crystal bases. These are defined to be local bases of integrable $U_q(\mathfrak{g})$-modules with certain properties. See [6] for more details.

**Definition** We will call a crystal finite if its underlying pointed set is of finite cardinality. We say that a crystal is irreducible if it has no nontrivial proper subcrystals.
Objects in this category have some nice combinatorial properties, as well as a graph structure. We obtain a crystal graph from each crystal $B$ whose vertices are the nonzero points in $B$ with arrows labeled by $i \in I$, $b \xrightarrow{i} b'$ if and only if $b' = f_i b$. Crystal graphs are made up of disjoint unions of connected components. It is clear that a crystal base is irreducible if and only if its crystal graph is connected.

We may classify $U_q(\mathfrak{g})$-modules up to isomorphism by a result of G. Lusztig [6].

**Proposition 1.1.** (G. Lusztig [6]) All $U_q(\mathfrak{g})$-modules are semisimple, and all irreducible $U_q(\mathfrak{g})$-modules are isomorphic to some module $V(\alpha)$ indexed by $\alpha \in \Phi_+ = \{ \alpha \in \Phi \mid \lambda_i(\alpha) \geq 0 \text{ for any } i \in I \}$.

It is known that all $V(\alpha)$ have unique crystal bases, and hence have corresponding crystals in our category, generated by a highest weight element $u_\alpha$ for $\alpha \in \Phi$. We shall denote these by $B(\alpha)$ (see [6] for more details). Since $U_q(\mathfrak{g})$ is semisimple, we see that any integrable $U_q(\mathfrak{g})$-module has a unique crystal base arising as a disjoint union of these $B(\alpha)$. We shall call such crystals the crystals arising from integrable $U_q(\mathfrak{g})$-modules.

**Remark** In the case where $\mathfrak{g} = \mathfrak{sl}_2$, each irreducible $U_q(\mathfrak{sl}_2)$-module $V(n)$ has a corresponding crystal base $\{ u^{(n)}_k \}_{0 \leq k \leq n}$ which corresponds to a unique crystal, denoted by $B(n) = \{ x^i y^{n-i} \mid 0 \leq i \leq n \}$ in this paper (here we identify $u^{(n)}_i$ with $x^i y^{n-i}$). This has crystal structure defined by

$$
\begin{align*}
\tilde{f}(x^i y^{n-i}) &= x^{i+1} y^{n-i-1}, \\
\tilde{e}(x^i y^{n-i}) &= x^{i-1} y^{n-i+1} \\
\varepsilon(x^i y^{n-i}) &= i, \\
\phi(x^i y^{n-i}) &= n - i, \\
\text{wt}(x^i y^{n-i}) &= n.
\end{align*}
$$

So, for example, the crystal base of an irreducible $U_q(\mathfrak{sl}_2)$-module would have crystal graph

Definition We say that a morphism $\psi : B_1 \rightarrow B_2$ of crystals is strict if for all $b \in B_1$ and for all $i \in I$, $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$ and $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.

**Remark** It will later be useful to restrict the morphisms in our category to strict ones so we may use the following result.

**Lemma 1.2.** (Schur’s Lemma for Strict Morphisms) A nonzero strict morphism between irreducible crystals is an isomorphism.

**Proof.** It is clear from the definition of a strict morphism that the image and kernel of such a morphism $\psi : B_1 \rightarrow B_2$ are subcrystals of $B_1, B_2$ respectively. The proof then follows from the usual Schur’s Lemma argument. \hfill \square

In the $\mathfrak{sl}_2$ case, finite crystals are easily classified based on their highest weights and the lengths of their connected components. To make this precise, let us denote by $T_\lambda = \{ t_\lambda \}$ the singleton crystal with $\text{wt}(t_\lambda) = \lambda$ for $\lambda \in \mathbb{Z}$. Then we have:

**Proposition 1.3.** All finite irreducible $\mathfrak{sl}_2$ crystals are of the form $B(n) \otimes T_\lambda$ for some $n \in \mathbb{N}, \lambda \in \mathbb{Z}$.

**Proof.** Let $C$ be an irreducible finite crystal and let $c \in C$ be nonzero. Then, as $C$ is connected, for each $c' \in C$ there is $n \in \mathbb{N}$ such that $c' = \tilde{f}^n c$ or $c' = \tilde{e}^n c$. Also, there is a maximal natural number $N = \varepsilon(b) \in \mathbb{N}$ such that $\tilde{e}^N c \neq 0$. We say that $c_0 := \tilde{f}^N c \in C$ is of highest weight with weight $\lambda := \text{wt}(c_0)$. If $|C \setminus \{ 0 \}| = l + 1$ then we can see that $C = \{ \tilde{f}^k c_0 \mid k = 0, 1, \ldots, l \} \cup \{ 0 \}$ and $\text{wt}(\tilde{f}^k c_0) = \lambda - 2k$. Then there is an isomorphism

$$
B(l) \otimes T_{\lambda-l} \rightarrow C, \quad x^i y^j \otimes t_{\lambda-l} \mapsto \tilde{f}^j c_0
$$

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noting that \( \text{wt}(x^iy^j \otimes t_\lambda - l) = (\lambda - l) + l - 2i = \lambda - 2i = \text{wt}(\hat{f}^i c_0) \). In fact, these irreducible finite crystals are uniquely determined by their size and the weight of their highest weight element. Thus we can classify all irreducible finite crystals as \( B(n) \otimes T_\lambda \) for \( n \in \mathbb{N} \) and \( \lambda \in \mathbb{Z} \). \( \square \)

The category of crystals is endowed with a tensor product as constructed in [6]. This tensor product arises naturally from the one constructed for crystal bases by Kashiwara.

**Definition** The tensor product of crystals is defined such that, for crystals \( B_1 \) and \( B_2 \), \( B_1 \otimes B_2 := \{ b_1 \otimes b_2 \mid b_1 \in B_1, b_2 \in B_2 \} \) with

\[
\begin{align*}
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} 
\tilde{e}_ib_1 \otimes b_2 & \text{if } \phi_i(b_1) \geq \varepsilon_i(b_2) \\
 b_1 \otimes \tilde{e}_ib_2 & \text{if } \phi_i(b_1) < \varepsilon_i(b_2),
\end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} 
\tilde{f}_ib_1 \otimes b_2 & \text{if } \phi_i(b_1) > \varepsilon_i(b_2) \\
 b_1 \otimes \tilde{f}_ib_2 & \text{if } \phi_i(b_1) \leq \varepsilon_i(b_2),
\end{cases} \\
\varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \lambda_i(\text{wt}(b_1))), \\
\phi_i(b_1 \otimes b_2) &= \max(\phi_i(b_1) + \lambda_i(\text{wt}(b_1)), \phi_i(b_2)), \\
\text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2).
\end{align*}
\]

**Remark** In the \( \mathfrak{sl}_2 \) case, for the crystals \( B(n), B(m) \) this can be visualised as follows:

\[
B(n) \overset{\circ}{} \overset{\rightarrow}{} \overset{\circ}{} \cdots \overset{\rightarrow}{} \overset{\circ}{} \overset{\rightarrow}{} \\
B(m) \overset{\circ}{} \overset{\rightarrow}{} \overset{\circ}{} \cdots \overset{\rightarrow}{} \overset{\circ}{} \overset{\rightarrow}{}
\]

In Henriques and Kamnitzer’s paper on *Crystals and Coboundary Categories* [3] they describe a **commuter for crystals** \( \sigma_{B_1 \otimes B_2} : B_1 \otimes B_2 \to B_2 \otimes B_1 \) for crystals \( B_1, B_2 \). This provides a way of commuting tensor products of crystals which we shall make use of later. In the case of \( \mathfrak{sl}_2 \), this has an explicit description. If \( \zeta : B \to B \) exchanges the highest and lowest weight elements of a crystal, essentially reversing the crystal graph, then \( b \otimes b' \mapsto \zeta(\zeta(b') \otimes \zeta(b)). \)

1.3. **The Clebsch-Gordan Formula.** In the case of \( \mathfrak{sl}_2 \), since \( U(\mathfrak{sl}_2) \) forms a bialgebra, \( V(n) \otimes V(m) \) is a \( U(\mathfrak{sl}_2) \)-module for every pair \( m, n \in \mathbb{N} \) and so must decompose into simple modules. This decomposition is determined by the Clebsch-Gordan Formula:

**Lemma 1.4.** (The Clebsch-Gordan Formula) [7, p. 105 & p. 175] *For all \( n, m \in \mathbb{N} \), there is an isomorphism of \( U(\mathfrak{sl}_2) \)-representations*

\[
V(n) \otimes V(m) \cong \bigoplus_{k=0}^{\min(m,n)} V(m + n - 2k).
\]

*Furthermore, the same Clebsch-Gordan formula holds in the case of \( U_q(\mathfrak{sl}_2) \)-representations.*

We can see a similar Clebsch-Gordan decomposition for \( \mathfrak{sl}_2 \) crystals.
Proposition 1.5. (Clebsch-Gordan for $\mathfrak{sl}_2$ Crystals) For all $m, n \in \mathbb{N}$ there is an isomorphism

$$B(m) \otimes B(n) \cong \bigoplus_{k=0}^{\min(m,n)} B(m + n - 2k) = \bigoplus_{|m-n|<k<m+n, m+n\equiv k(mod2)} B(k).$$

Proof. Suppose $m \geq n$. Consider $y^m \otimes x^k y^{n-k} \in B(m) \otimes B(n)$ for $k = 0, 1, \ldots, n$. From the above we have:

$$\tilde{e}(y^m \otimes x^k y^{n-k}) = 0$$

$$\tilde{f}(y^m \otimes x^k y^{n-k}) = \begin{cases} x^i y^{m-i} \otimes x^k y^{n-k} & \text{if } i < m-k \\ x^{m-k} y^k \otimes x^{i+2k-m} y^{m+n-2k-i} & \text{if } m-k \leq i \leq (m+n) - 2k \\ 0 & \text{otherwise} \end{cases}$$

Thus $y^m \otimes x^k y^{n-k}$ is a highest weight vector in a string of length $(m+n) - 2k$. So we have an inclusion of crystals $\bigcup_{k=0}^{n} B(m + n - 2k) \hookrightarrow B(m) \otimes B(n)$. But

$$|\bigcup_{k=0}^{n} B(m + n - 2k)| = \sum_{k=0}^{n} |B(m + n - 2k)| = \sum_{k=0}^{n} m + n - 2k + 1 = (m+1)(n+1) = |B(m) \otimes B(n)|$$

So, by the pigeonhole principal, this map of crystals is surjective, hence is an isomorphism. Similarly, if $m < n$, we may consider $y^m \otimes x^k y^{n-k} \in B(m) \otimes B(n)$ for $k = 0, 1, \ldots, m$ and we obtain the same result.

As a result of the above, we have $B(1) \otimes B(n) \cong B(n-1) \sqcup B(n+1)$ for $n \geq 1$. This gives us the following:

Proposition 1.6. There is an embedding

$$B(n) \hookrightarrow B(1)^{\otimes n}$$

for all $n \in \mathbb{N}$.

Proof. We suppose, as an inductive hypothesis, that, for some $k \in \mathbb{N}$, $B(1)^{\otimes k} \cong B(k) \sqcup C_k$ for some crystal $C_k$. Here, by $A^{\otimes k}$ we mean the tensor product $A \otimes A \otimes \ldots \otimes A$ of $k$ copies of $A$. We see that the base case, when $k = 1$, is trivial with $C_1 = \emptyset$. So

$$(B(1)^{\otimes k+1} = B(1) \otimes (B(1)^{\otimes k})$$

$$\cong B(1) \otimes (B(k) \sqcup C_k)$$

$$= (B(1) \otimes B(k)) \sqcup (B(1) \otimes C_k)$$

$$\cong (B(k-1) \sqcup B(k+1)) \sqcup (B(1) \otimes C_k)$$

$$= B(k+1) \sqcup (B(k-1) \sqcup (B(1) \otimes C_k))$$

$$= B(k+1) \sqcup C_{k+1},$$

where $C_{k+1} = B(k-1) \sqcup (B(1) \otimes C_k)$. So, inductively, we have that $B(1)^{\otimes n} \cong B(n) \sqcup C_n$ for all $n \in \mathbb{N}$. Thus there is an injection, for each $n \in \mathbb{N}$,

$$B(n) \hookrightarrow B(n) \sqcup C_n \cong B(1)^{\otimes n}$$

which is given by

$$x^i y^j \mapsto \underbrace{x \otimes x \otimes \ldots \otimes x}_{i \text{ times}} \otimes \underbrace{y \otimes y \otimes \ldots \otimes y}_{j \text{ times}}.$$
Remark Using this Clebsch-Gordan decomposition for crystals we may also compute Henriques’
and Kamnitzer’s commutator of crystals explicitly. Let \( x^r y^s \times x^j y^l \in B(n) \otimes B(m) \). Then
\[
\sigma_{B(n) \otimes B(m)}(x^r y^s \times x^j y^l) = \zeta(x^r y^s) \otimes \zeta(x^j y^l) = \zeta(x^r y^s \times x^j y^l).
\]
If \( j \leq r \) then \( x^r y^s \times x^j y^l \) corresponds to \( x^{s+j-r} y^s \) in an isomorphic copy of \( B(n+m-2j) \). Then,
applying \( \zeta \) we obtain \( x^{s+j-r} y^s \in B(n+m-2j) \). If \( i \leq s \) then this corresponds to \( x^{i+j-r} y^s \times x^j y^l \)
in \( B(m) \otimes B(n) \), and if \( i > s \) then this corresponds to \( x^r y^s \times x^j y^l \) in \( B(m) \otimes B(n) \). Now
if \( j > r \) then \( x^r y^s \times x^j y^l \) corresponds to \( x^{s+j-r} y^s \) in a copy of \( B(n+m-2r) \). Then \( \zeta(x^{s+j-r} y^s) = x^{s+j-r} y^r \in B(n+m-2r) \).
So if \( i \leq s \) this corresponds to \( x^r y^s+i-j \times x^r y^l \) and if \( i > s \) then
this corresponds to \( x^r y^l+i-s \times x^r y^l+i-r \). Thus the commutator of crystals gives
\[
\sigma_{B(n) \otimes B(m)} : x^i y^j \times x^r y^s \mapsto \begin{cases} 
\begin{align*}
& x^{i+j-r} y^{s+i} \times x^j y^l & \text{if } j \leq r, i \leq s, \\
& x^{s+i-j} y^s \times x^i y^l & \text{if } j \leq r, i > s, \\
& x^{r+i-j} y^{i+s} \times x^r y^l & \text{if } j > r, i \leq s, \\
& x^r y^l+i-s & \times x^r y^l+i-r & \text{if } j > r, i > s.
\end{align*}
\end{cases}
\]
More generally, we have a way of determining how the tensor product of such crystals decomposes
as stated in Kashiwara’s paper [6].

Proposition 1.7. (Decomposition of Tensor Product of Crystals, [6]) There is an isomorphism of
 crystals
\[
B(\alpha) \otimes B(\beta) \cong \bigsqcup B(\alpha + \mathfrak{w}(b))
\]
where the disjoint union ranges over all \( b \in B(\beta) \) such that \( \varepsilon_i(b) \leq \lambda_i(\alpha) \) for all \( i \in I \). Most
importantly, \( \varepsilon_i(u_{\beta}) = 0 \leq \lambda_i(\alpha) \) for each \( i \), so \( B(\alpha + \beta) \) appears as a term in this decomposition
since \( \alpha \in \Phi_+ = \{ \alpha \in \Phi | \lambda_i(\alpha) \geq 0 \text{ for any } i \in I \} \).

Definition For a crystal \( B \), we may construct a crystal \( B^\vee \), as in [6], obtained by reversing arrows
in its crystal graph. More precisely, \( B^\vee = \{ b^\vee \mid b \in B \} \) with
\[
\varepsilon_i(b^\vee) = (\tilde{f}_i(b))^\vee, \quad \phi_i(b^\vee) = (\tilde{e}_i(b))^\vee,
\]
\[
\mathfrak{w}(b^\vee) = -\mathfrak{w}(b), \quad \varepsilon_i(b^\vee) = \phi_i(b), \quad \phi_i(b^\vee) = \varepsilon_i(b).
\]
Then, for \( \alpha \in \Phi \) we will define \( B(-\alpha) := B(\alpha)^\vee \), and we will use this as the notion of a dual \( B(\alpha)^* \)
to \( B(\alpha) \).

Then both \( B(\alpha) \otimes B(-\alpha) \) and \( B(-\alpha) \otimes B(\alpha) \) contain in their decompositions the trivial crystal,
respectively \( \{ u_{\alpha} \otimes u_{-\alpha} \} \) and \( \{ u_{-\alpha} \otimes u_{\alpha} \} \) where \( u_{-\alpha} = u_{\alpha}^\vee \). For crystals \( B_1, B_2 \) we can see that
\( (B_1 \otimes B_2)^\vee \cong B_2^\vee \otimes B_1^\vee \), thus we may deduce that \( B(-\alpha+\beta) \) appears as a term in the decomposition
of \( B(-\alpha) \otimes B(-\beta) \). In the case of \( \mathfrak{sl}_n \), we simply have \( B(n)^\vee \cong B(n) \).

1.4. Multiplication on the \( \mathfrak{sl}_n \)-crystal \( \bigsqcup_{n \in \mathbb{N}} B(n) \). As a result of the Clebsch-Gordan formula, we may define a multiplication on the \( U(\mathfrak{sl}_2) \)-module \( \oplus_{n \in \mathbb{N}} V(n)^* \) as follows. For \( n \in \mathbb{N} \) we consider the simple \( (n+1) \)-dimensional \( U(\mathfrak{sl}_2) \)-module \( V(n) \) with usual basis \( \{ u_k^{(n)} \}_{0 \leq k \leq n} \) of \( h \)-eigenvectors
(see [7, p. 101]), and its dual \( V(n)^* \) with dual basis \( \{ \tilde{u}_k^{(n)} \}_{0 \leq k \leq n} \). By the Clebsch-Gordan formula,
we have an injection
\[
\phi : V(m+n) \hookrightarrow \bigoplus_{k=0}^{\min(m,n)} V(m+n-2k) \cong V(m) \otimes V(n)
\]
Explicitly,

\[ \phi(u_k^{m+n}) = \phi\left( \frac{1}{k!} f^k u_0^{m+n} \right) \]

\[ = \frac{1}{k!} f^k \phi(u_0^{m+n}) \]

\[ = \frac{1}{k!} f^k (u_0^{(m)} \otimes u_0^{(n)}) \]

\[ = \frac{1}{k!} \sum_{i=1}^{k} \binom{k}{i} (f^i u_0^{(m)} \otimes f^{k-i} u_0^{(n)}) \]

\[ = \sum_{i=1}^{k} \frac{1}{i!(k-i)!} (i! u_i^{(m)} \otimes (k-i)! u_{k-i}^{(n)}) \]

\[ = \sum_{i=1}^{k} u_i^{(m)} \otimes u_{k-i}^{(n)} \]

\[ = \sum_{i} \sum_{j} \delta_{i+j,k} u_i^{(m)} \otimes u_j^{(n)}. \]

This gives a transpose map

\[ \phi^* = \mu_1 : V(n)^* \otimes V(m)^* \to (V(n) \otimes V(m))^* \to V(n+m)^*, \]

\[ \tilde{u}^{(n)}_i \otimes \tilde{u}^{(m)}_j \mapsto \tilde{u}^{(m+n)}_{i+j}. \]

If we let \( x^iy^j \) denote \( \tilde{u}^{(i+j)}_i \) as above for \( i, j \in \mathbb{N} \), that is, \( \tilde{u}^{(n)}_i = x^i y^{n-i} \), then this multiplication becomes

\[ \mu_1 : x^i y^j \otimes x^k y^l \mapsto x^{i+k} y^{j+l}. \]

This multiplication agrees with the multiplication on the affine plane, \( k[x, y] \), for our field \( k \).

Similarly, for \( n \in \mathbb{N} \), we may consider the simple \( U_q(\mathfrak{sl}_2) \)-module \( V(n) \) and its dual \( V(n)^* \). Again, we give these respectively the usual basis \( \{ u_k^{(n)} \}_{0 \leq k \leq n} \) of \( t \)-eigenvectors (see [7, p. 127]) and dual basis \( \{ \tilde{u}_k^{(n)} \}_{0 \leq k \leq n} \). We have the injection

\[ \phi : V(m+n) \mapsto \bigoplus_{k=0}^{\text{min}(m,n)} \cong V(m) \otimes V(n), \]

\[ u_0^{(m+n)} \mapsto u_0^{(m)} \otimes u_0^{(n)}. \]

Since \( \Delta(f^k) = \sum_{s=0}^{k} q^{s(k-s)} \binom{k}{s} (f^s t^{-(k-s)} \otimes f^{k-s}) \)
we have \( f^k(u_0^{(m)} \otimes u_0^{(n)}) = \sum_{s=0}^{k} q^{r-m}(k-r) u_s^{(m)} \otimes u_{k-s}^{(n)}. \)
Here we have adopted the following notation for $q$-binomial coefficients:

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]! := \prod_{k=1}^{n} [k], \quad \left[ \frac{n}{k} \right] = \frac{[n]!}{[k]![n-k]!}.$$ 

Thus we may write the above map explicitly as

$$\phi(u_k^{(m+n)}) = \phi\left( \frac{1}{[k]!} f^k u_0^{(m+n)} \right)$$

$$= \frac{1}{[k]!} f^k \phi(u_0^{(m+n)})$$

$$= \frac{1}{[k]!} f^k (u_0^{(m)} \otimes u_0^{(n)})$$

$$= \sum_{i=1}^{k} q^{(s-m)(k-s)} [i]! [k-i]! (\phi u_i^{(m)} \otimes u_{k-i}^{(n)})$$

$$= \sum_{i=1}^{k} q^{(s-m)(k-s)} u_i^{(m)} \otimes u_{k-i}^{(n)}$$

$$= \sum_i \sum_j \delta_{i+j,k} q^{(s-m)(k-s)} u_i^{(m)} \otimes u_j^{(n)}.$$ 

This gives a transpose map

$$\phi^\ast = \mu_q : V(n)^* \otimes V(m)^* \to (V(n) \otimes V(m))^* \to V(n + m),$$

$$\tilde{u}_i^{(m)} \otimes \tilde{u}_j^{(n)} \mapsto q^{-(m-i)j} \tilde{u}_{i+j}^{(m+n)}.$$ 

If we again let $x^iy^j$ denote $\tilde{u}_i^{(i+j)}$ for $i,j \in \mathbb{N}$ this multiplication becomes

$$\mu_q : x^iy^j \otimes x^s y^t \mapsto q^{-ij} x^{i+s} y^{j+t}.$$ 

Similarly to before, this multiplication agrees with that of the quantum affine plane, $k_q[x,y]$, for our field $k$. That is, the quotient algebra of the free algebra on $x$ and $y$ by the relation $xy = qyx$, $k_q[x,y] := k(x,y)/(yx - qxy)$. So $x$ and $y$ no longer commute but instead $q$-commute. When $q = 1$, this agrees with $\mu_1$ as before.

Returning to the study of crystals, from the corresponding Clebsch Gordan formula for $\mathfrak{sl}_2$ crystals we have two maps, for $n \in \mathbb{N}$,

$$B(0) \hookrightarrow B(n)^* \otimes B(n), B(n) \otimes B(n)^* \to B(0).$$

If we look at the chain of maps

$$B(n) \cong B(n) \otimes B(0) \hookrightarrow B(n) \otimes B(n)^* \otimes B(n) \to B(0) \otimes B(n) \cong B(n)$$

$$x^iy^j \mapsto x^iy^j \otimes x^0 \mapsto x^iy^j \otimes y^0 \otimes x^0 \mapsto 0 \mapsto 0$$

we see that for $n \neq 0$, this composition is the zero map and so is not the identity on $B(n)$. We see a similar result with the composition

$$B(n) \cong B(0) \otimes B(n) \hookrightarrow B(n) \otimes B(n)^* \otimes B(n) \to B(n) \otimes B(0) \cong B(n).$$
Thus we do not have duality between crystals $B(n)$ and $B(n)^*$ as we would hope. We can, however, still use Clebsh Gordan to define a multiplication map on $\bigsqcup_{n \in \mathbb{N}} B(n)$ as in the non crystal case.

We define the multiplication map $\mu_0$ by

$$\mu_0 : \bigsqcup_{n \in \mathbb{N}} B(n) \otimes \bigsqcup_{n \in \mathbb{N}} B(n) \to \bigsqcup_{n \in \mathbb{N}} B(n)$$

$$B(n) \otimes B(m) \cong \bigsqcup_{k=0}^{\min(m,n)} B(m + n - 2k) \to B(m + n).$$

Which, explicitly, gives the map

$$\mu_0 : x^i y^j \otimes x^r y^s \mapsto \begin{cases} 
  x^{i+r} y^s & \text{if } j = 0 \\
  x^i y^{j+s} & \text{if } r = 0 \\
  0 & \text{if } j \neq 0 \neq r.
\end{cases}$$

This is easily seen when we look at the diagram shown earlier of the crystal graph of $B(n) \otimes B(m)$. Note that this agrees with the multiplication on the monic monomials lying in the quotient space $k_0[x, y] := k(x, y)/(yx)$. This can be thought of as the limit as $q$ tends to 0 of $k_q[x, y]$, which arises in the quantum case.

We may also define the unit map $\eta$ by

$$\eta : B(0) \to \bigsqcup_{n \in \mathbb{N}} B(n)$$

$$x^0 y^0 \mapsto x^0 y^0$$

which gives the following commutative diagrams:

So we obtain:

**Proposition 1.8.** The crystal $\bigsqcup_{n \in \mathbb{N}} B(n)$ forms an algebra object in Crys given by the triple $(\bigsqcup_{n \in \mathbb{N}} B(n), \mu_0, \eta)$

Note that the analogue to a base field $k$ in this setting is the trivial crystal $B(0)$, the smallest possible crystal weighted at 0.
2. The Crystal Coalgebra $B$

2.1. The Bialgebra $O(SL_2(\mathbb{C}))$. Recall that the algebraic group $SL_2(\mathbb{C})$ of $2 \times 2$ matrices over $\mathbb{C}$ of determinant 1 has coordinate algebra $O(SL_2(\mathbb{C})) := \mathbb{C}[a, b, c, d]/(ad - bc - 1)$ which exhibits a bialgebra structure whose comodules are precisely the modules of the algebra $U(\mathfrak{sl}_2)$ (see [1] p. 154 for further details).

As a result of the Peter-Weyl Theorem for $SL_2(\mathbb{C})$, there is an isomorphism of $U(\mathfrak{sl}_2)$-modules

$$O(SL_2(\mathbb{C})) \cong \bigoplus_{n \in \mathbb{N}} V(n) \otimes V(n)^*.$$  

Thus we will consider the crystal equivalent of this module, $B := \bigsqcup_{n \in \mathbb{N}} B(n) \otimes B(n)^*$, with the hope that we may exhibit some crystals as an analogue of comodules over a bialgebra-like structure.

2.2. The Structure of $B$. We begin by considering the $\mathfrak{sl}_2$ case. By the crystal version of the Clebsch-Gordan decomposition, there is an isomorphism $\vartheta : B(n) \otimes B(m) \to \bigsqcup_{k=0}^{\min(m,n)} B(n+m-2k)$ which we may express explicitly as

$$\vartheta : x^i y^j \otimes x^r y^s \mapsto \begin{cases} x^i y^{j+s-r} & \text{if } r \leq j \\ x^{i+r-j} y^s & \text{if } r > j. \end{cases}$$

Since $V(n)^+$ and its dual are isomorphic for all $n$, we may treat $B(n)^*$ as another copy of $B(n)$. Thus, using our isomorphism $\vartheta : B(n) \otimes B(m) \to \bigsqcup_{k=0}^{\min(m,n)} B(n+m-2k)$, we can construct a multiplication $\mu$ on $B := \bigsqcup_{n \in \mathbb{N}} B(n) \otimes B(n)^*$ as an extension of the composition

$$\begin{array}{rcl}
B(n) \otimes B(n)^* \otimes B(m) \otimes B(m)^* & \xrightarrow{id \otimes \sigma_{B(n)^*} \otimes B(m) \otimes id} & B(n) \otimes B(m) \otimes B(n)^* \otimes B(m)^* \\
\bigsqcup_{k=0}^{\min(m,n)} B(n+m-2k) \otimes \bigsqcup_{k=0}^{\min(m,n)} B(n+m-2k)^* & \xrightarrow{\vartheta \otimes \vartheta} & \bigcup_{k=0}^{\min(m,n)} B(n+m-2k) \otimes B(n+m-2k)^*
\end{array}$$

to a map $B \otimes B \to B$, where $\sigma$ is the crystal commuter as previously described (see [3]). This comes with a unit

$$\eta : B(0) \to \bigsqcup_{n \in \mathbb{N}} B(n) \otimes B(n)^*, \ x^0 y^0 \mapsto x^0 y^0 \otimes x^0 y^0$$

which gives us the same commutative diagrams required for an algebra structure. So we have:

**Proposition 2.1.** There is an algebra structure on $B$ in the category $Crys$.

We also have a map

$$\iota : B(0) \hookrightarrow \bigsqcup_{k=0}^n B(2n-2k) \cong B(n)^* \otimes B(n)$$

which allow us to produce a comultiplication map $\Delta$ arising from the composition
We can see that
\[ x^i y^j \otimes y^n \otimes x^n \otimes x^n \otimes x^r y^s \]
\[ \Delta \otimes \text{id} \]
\[ x^i y^j \otimes y^n \otimes x^n \otimes x^n \otimes x^r y^s \]
\[ \Delta \]
\[ x^i y^j \otimes x^r y^s \]

hence we have the commutative diagram
\[
\begin{array}{c}
B \otimes B \otimes B \otimes B \otimes B \\
\Delta \otimes \text{id} \ \
\end{array}
\]

For this comultiplication to have a counit \( \varepsilon \), we would need the following commutative diagram:
\[
\begin{array}{c}
B(0) \otimes B \otimes B \otimes B \\
\varepsilon \otimes \text{id} \\
\end{array}
\]

That is, we would require that in general
\[ x^0 y^0 \otimes x^n \otimes x^r y^s = \varepsilon(x^j y^j \otimes y^n) \otimes x^n \otimes x^r y^s = x^0 y^0 \otimes x^j y^j \otimes x^r y^s, \]
\[ x^i y^j \otimes y^n \otimes x^0 y^0 = x^i y^j \otimes y^n \otimes \varepsilon(x^n \otimes x^r y^s) = x^i y^j \otimes x^r y^s \otimes x^0 y^0, \]
which is not possible for any \( \varepsilon \). Thus we have just a coalgebra structure without a counit. Hence we have:

**Proposition 2.2.** There is a non-counital coalgebra structure on \( B \).

In the non crystal case, when considering the \( \mathfrak{sl}_2 \) representation \( \mathcal{O}(SL_2(\mathbb{C})) \), we see a bialgebra structure. So we would hope to see a commutative diagram of the form
\[
\begin{array}{c}
B \otimes B \otimes B \otimes B \otimes B \otimes B \\
\Delta \otimes \Delta \\
\end{array}
\]

\[
\begin{array}{c}
B \otimes B \otimes B \otimes B \otimes B \otimes B \\
\text{id} \otimes \sigma \otimes \text{id} \\
\mu \otimes \mu \\
\end{array}
\]

\[
\begin{array}{c}
B \otimes B \\
\Delta \\
\end{array}
\]
These maps do not agree for strictly positive $n$ diagram. Hence $\Delta$ we see when we crystallise its lack of nice braiding properties - it fails to satisfy the usual hexagon axiom. This is discussed in not remain when we reduce to crystal bases. This is in part due to the commuter of crystals and $\Delta$ making

Then, similar to before, we have maps

$$\vartheta : B(\alpha) \otimes B(\beta) \rightarrow \bigsqcup_b B(\alpha + \text{wt}(b)),$$

$$\vartheta : B(-\alpha) \otimes B(-\beta) \rightarrow \bigsqcup_b B(-\alpha + \text{wt}(b)),$$

$$\iota : B(0) \leftrightarrow B(-\alpha) \otimes B(\alpha) \text{ and } B(0) \cong B(0) \otimes B(-0),$$

hence we may define unit, multiplication and comultiplication maps.

$$\eta : B(0) \cong B(0) \otimes B(-0) \rightarrow \bigsqcup_{\alpha \in \Phi} B(\alpha) \otimes B(-\alpha) = B$$

$$\mu : B \otimes B \rightarrow B$$

$$\Delta : B \rightarrow B \otimes B$$

$$B(\alpha) \otimes B(-\alpha) \otimes B(\beta) \otimes B(-\beta)$$

$$\cong$$

$$B(\alpha) \otimes B(\beta) \otimes B(-\alpha) \otimes B(-\beta)$$

$$\cong$$

$$\bigsqcup_b B(\alpha + \text{wt}(b)) \otimes \bigsqcup_b B(-\alpha + \text{wt}(b))$$

$$\bigsqcup_b B(\alpha + \text{wt}(b)) \otimes B(-\alpha + \text{wt}(b))$$

These maps do not agree for strictly positive $n$ and $m$, so we do not obtain the desired commutative diagram. Hence $\Delta$ cannot be a morphism of algebras. So the bialgebra structure of $O(SL_2(\mathbb{C}))$ does not remain when we reduce to crystal bases. This is in part due to the commuter of crystals and its lack of nice braiding properties - it fails to satisfy the usual hexagon axiom. This is discussed in further detail in Henriques’ and Kamnitzner’s paper [4]. This result highlights the loss of structure we see when we crystallise $\mathfrak{sl}_2$ representations.

We may now generalise this construction of $B$ to a general Lie algebra $\mathfrak{g}$. We will now define

$$B := \bigsqcup_{\alpha \in \Phi} B(\alpha) \otimes B(-\alpha).$$

Then, similar to before, we have maps

$$\vartheta : B(\alpha) \otimes B(\beta) \rightarrow \bigsqcup_b B(\alpha + \text{wt}(b)),$$

$$\vartheta : B(-\alpha) \otimes B(-\beta) \rightarrow \bigsqcup_b B(-\alpha + \text{wt}(b)),$$

$$\iota : B(0) \leftrightarrow B(-\alpha) \otimes B(\alpha) \text{ and } B(0) \cong B(0) \otimes B(-0),$$

hence we may define unit, multiplication and comultiplication maps.

$$\eta : B(0) \cong B(0) \otimes B(-0) \rightarrow \bigsqcup_{\alpha \in \Phi} B(\alpha) \otimes B(-\alpha) = B$$

$$\mu : B \otimes B \rightarrow B$$

$$\Delta : B \rightarrow B \otimes B$$

$$B(\alpha) \otimes B(-\alpha) \otimes B(\beta) \otimes B(-\beta)$$

$$\cong$$

$$B(\alpha) \otimes B(\beta) \otimes B(-\alpha) \otimes B(-\beta)$$

$$\cong$$

$$\bigsqcup_b B(\alpha + \text{wt}(b)) \otimes \bigsqcup_b B(-\alpha + \text{wt}(b))$$

$$\bigsqcup_b B(\alpha + \text{wt}(b)) \otimes B(-\alpha + \text{wt}(b))$$

**Proposition 2.3.** These maps give an algebra and a non-counital coalgebra structure on $B$ as before.

As we saw in the $\mathfrak{sl}_2$ case, these algebra and coalgebra structures do not interact as we would hope. Again, we do not have a bialgebra but instead have just a non-counital coalgebra $B$. Thus we still may ask what comodules of $B$ look like.
2.3. $\mathcal{B}$ Comodules. By a $\mathcal{B}$-comodule, we mean a pointed set $C$ with a coaction map $\Delta_C : C \to \mathcal{B} \otimes C$ satisfying the appropriate commutative diagrams in the category of pointed sets. We may refer to these as the comodules of $\mathcal{B}$ in the category of pointed sets.

With respect to this coalgebra crystal, we may exhibit $B(\alpha)$ as comodules. We define the coaction

$$\Delta_\alpha : B(\alpha) \cong B(\alpha) \otimes B(0) \to B(\alpha) \otimes B(-\alpha) \otimes B(\alpha) \to \mathcal{B} \otimes B(\alpha)$$

$$b \mapsto (b \otimes u_{-\alpha}) \otimes u_\alpha.$$

This gives:

**Proposition 2.4.** All crystals that arise from integrable $\mathfrak{g}$-modules have a $\mathcal{B}$-comodule structure.

In fact, in the case of $\mathfrak{sl}_2$, we can say a little more

**Corollary 2.5.** All finite irreducible $\mathfrak{sl}_2$ crystals, and hence all $\mathfrak{sl}_2$ crystals whose connected components are finite, are comodules over $\mathcal{B}$.

**Proof.** If we again let $T_\lambda = \{t_\lambda\}$ be the singleton crystal with weight $\lambda \in \mathbb{Z}$ then we have a comodule structure on $B(n) \otimes T_\lambda$ extending $\Delta_n$,

$$\Delta_{n,\lambda} : B(n) \otimes T_\lambda \to \mathcal{B} \otimes B(n) \otimes T_\lambda, \quad x^i y^j \otimes t_\lambda \mapsto (x^i y^j \otimes y^n) \otimes x^n \otimes t_\lambda.$$

As we have seen, in the case of $\mathfrak{sl}_2$ crystals, all finite irreducible crystals are of the form $B(n) \otimes T_\lambda$ for some integer $\lambda$. Thus we can say a little more.

Suppose $C$ is a $\mathcal{B}$-comodule with coaction $\Delta_C : C \to \mathcal{B} \otimes C$. Let $c \in C$, and suppose $\Delta_C(c) = (b_1 \otimes b_2) \otimes c'$ for $c' \in C$, $b_1 \otimes b_2 \in \mathcal{B}$. Suppose further that $\Delta_C(c') = (b_1' \otimes b_2') \otimes c''$ for $c'' \in C$, $b_1' \otimes b_2' \in \mathcal{B}$. Then, by definition, we have, for some $\alpha \in \Phi$,

$$b_1 \otimes u_{-\alpha} \otimes b_2 \otimes c' = (\Delta \otimes \text{id}) \circ \Delta_C(c) = (\text{id} \otimes \Delta_C) \circ \Delta_C(c) = b_1 \otimes b_2 \otimes b_1' \otimes b_2' \otimes c''$$

So we know that $b_2 = u_{-\alpha} = b_2'$, $b_1' = u_\alpha$ and $c' = c''$. So, for a general $c \in C$, there is $c' \in C$, $n \in \mathbb{N}$ and $b \in B(n)$ such that

$$\Delta_C(c) = b \otimes u_{-\alpha} \otimes c', \quad \Delta(c') = u_\alpha \otimes u_{-\alpha} \otimes c'.$$

For a comodule $C$, denote by $\hat{C}$ the subcomodule

$$\hat{\mathcal{C}} = \{c \in C \mid \Delta_C(c) = u_\alpha \otimes u_{-\alpha} \otimes c \text{ for some } \alpha \in \Phi\}.$$

For each $c \in \hat{\mathcal{C}}$, the set $\{c\}$ is a simple subcomodule, and every nontrivial subcomodule of $C$ contains one of these singleton subcomodules. Thus the simple subcomodules are precisely these singleton subcomodules $\{c\}$ for $c \in \hat{\mathcal{C}}$. We again see that $\mathcal{B}$ is not semisimple. In this context, we mean that comodules such as these are semisimple if they are a union of irreducible comodules. Thus the coalgebra $\mathcal{B}$ is not semisimple. This contrasts with the coordinate algebra $\mathcal{O}(SL_2(\mathbb{C}))$, which is semisimple as a coalgebra. Thus it appears that we may not be able to classify all $\mathcal{B}$-comodules in the same way.

Indeed, consider again the case of $\mathfrak{sl}_2$. Suppose that the categories of comodules of $\mathcal{B}$ and of crystals coincide. Consider a comodule $C = \{a, b, c\}$ with coaction defined by $\Delta_C(a) = \Delta_C(b) = y \otimes y \otimes c$, $\Delta_C(c) = x \otimes y \otimes c$. This has subcomodules $\{a, c\}$ and $\{b, c\}$, and so by our assumption these would have crystal structures. Then they are disjoint unions of connected components with respect to their crystal graphs. But these comodules intersect at the subcomodule $\{c\}$, and so this must be a
connected component of both crystals. Thus we must have that \{a\}, the complement to \{c\}, is also a crystal and hence a comodule. But this is not true. Hence we may not classify all comodules as crystals in this way.

**Remark** All of the above maps are (strict) morphisms of crystals, so in fact such crystals are comodules of \(B\) in the subcategory of crystals whose irreducible components are finite. Of course, it is then a trivial fact that the comodules of \(B\) in this subcategory of crystals form the entire category, and so give a classification of the category. We, however, seek to consider a larger class of comodules, namely the comodules of \(B\) in the category of pointed sets.
Part 2. A Functorial Approach to Crystals

3. The Barr-Beck Theorem

3.1. Monads and Comonads. We begin by defining the generalised notions of algebras and coalgebras in the setting of functors on categories. For more details see Borceaux’s *Handbook of Categorical Algebra* 2 [I] p. 189-197.

**Definition** A monad on a category $C$ is a triple $T = (T, \eta, \mu)$ where $T: C \to C$ is a functor and $\eta: \text{id}_C \Rightarrow T$, $\mu: T \circ T \Rightarrow T$ are natural transformations such that the following diagrams commute:

\[
\begin{array}{ccc}
T \circ T \circ T & \xrightarrow{\mu \ast \text{id}} & T \circ T \\
\text{id} \ast \mu & \downarrow & \mu \\
T \circ T & \xrightarrow{\mu} & T
\end{array}
\quad \begin{array}{ccc}
\text{id}_C \circ T & \xrightarrow{\eta \ast \text{id}} & T \circ T \circ T \\
\mu & \downarrow & \text{id} \\
T \circ T & \xrightarrow{\mu} & T \circ \text{id}_C
\end{array}
\]

Note that these commutative diagrams are almost the same as those required for an algebra. So monads are essentially the functorial equivalent of algebras. As algebras have modules, monads have what are known as *algebras* over them.

**Definition** For a category $C$ with monad $T = (T, \eta, \mu)$ as above, an algebra on this monad is a pair $(C, \xi)$ where $C$ is an object in the category and $\xi: T(C) \to C$ is a morphism in the category such that we have the following commutative diagrams:

\[
\begin{array}{ccc}
T \circ T(C) & \xrightarrow{\mu_C} & T(C) \\
T(\xi) & \downarrow & \xi \\
T(C) & \xrightarrow{\xi} & C
\end{array}
\quad \begin{array}{ccc}
C & \xrightarrow{\eta_C} & T(C) \\
\xi & \downarrow & \xi \\
C & \xrightarrow{\xi} & C
\end{array}
\]

A morphism of algebras $f: (C, \xi) \to (C', \xi')$ is a morphism $f: C \to C'$ in the category such that $f \circ \xi = \xi' \circ T(f)$.

Note that these notions are extremely similar to the notions of modules and their morphisms. These algebras in $C$ over a monad $T$ form a category, denoted $C^T$, known as the Eilenberg-Moore category of the monad.

Naturally there is a dual notion, called a comonad.

**Definition** A comonad on a category $C$ is a triple $U = (U, \varepsilon, \Delta)$ where $U: C \to C$ is a functor and $\varepsilon: U \Rightarrow \text{id}_C$, $\Delta: U \Rightarrow U \circ U$ are natural transformations such that the following diagrams commute:

\[
\begin{array}{ccc}
U \circ U \circ U & \xrightarrow{\Delta \ast \text{id}} & U \circ U \\
\text{id} \ast \Delta & \downarrow & \Delta \\
U \circ U & \xrightarrow{\Delta} & U
\end{array}
\quad \begin{array}{ccc}
\text{id}_C \circ U & \xrightarrow{\varepsilon \ast \text{id}} & U \circ U \\
\text{id} \ast \varepsilon & \downarrow & \Delta \\
U \circ U & \xrightarrow{\Delta} & U \circ \text{id}_C
\end{array}
\]

A coalgebra on this monad is a pair $(D, \zeta)$ where $D$ is an object in the category and $\zeta: D \to U(D)$ is a morphism in the category such that we have the following commutative diagrams:
A morphism of coalgebras \( g : (D, \zeta) \to (D', \zeta') \) is a morphism \( g : D \to D' \) in the category such that \( U(g) \circ \zeta = \xi' \circ g \). These coalgebras in \( \mathcal{C} \) over a comonad \( \mathbb{U} \) form a category, which we shall denote \( \mathcal{C}_\mathbb{U} \).

Suppose we have a pair of adjoint functors \( F : \mathcal{C} \to \mathcal{D} \), \( G : \mathcal{D} \to \mathcal{C} \) with \( F \dashv G \). Then, for an object \( A \) in \( \mathcal{C} \), \( \operatorname{Hom}_\mathcal{D}(F(A), F(A)) \cong \operatorname{Hom}_\mathcal{C}(A, GF(A)) \), so we take the morphism \( \eta_A : A \to GF(A) \) corresponding to the identity on \( F(A) \) under this isomorphism. This defines a natural transformation \( \eta : \text{id}_{\mathcal{C}} \to G \circ F \). Similarly, for an object \( B \) in \( \mathcal{D} \), \( \operatorname{Hom}_\mathcal{C}(G(B), G(B)) \cong \operatorname{Hom}_\mathcal{D}(FG(B), B) \). So we define a morphism \( \varepsilon_B : FG(B) \to B \) corresponding to the identity on \( G(B) \). This defines a natural transformation \( \varepsilon : F \circ G \to \text{id}_\mathcal{D} \). Then \( T = (T := G \circ F, \eta, \mu) \) defines a monad where \( \mu \) is the horizontal composition \( \mu = \text{id}_{\mathcal{C}} * \varepsilon * \text{id}_F \). Similarly, \( \mathbb{U} = (U := F \circ G, \varepsilon, \Delta) \) forms a comonad where \( \Delta := \text{id}_F * \eta * \text{id}_G \). Furthermore, we have comparison functors \( K^\mathbb{T} : \mathcal{D} \to \mathcal{C}^\mathbb{T} \), \( J_\mathbb{U} : \mathcal{C} \to \mathcal{D}_\mathbb{U} \) defined, respectively, by

\[
K^\mathbb{T}(A) = (G(A), G(\varepsilon_A)) \\
K^\mathbb{T}(f) = G(f) \\
J_\mathbb{U}(B) = (F(B), F(\eta_B)) \\
J_\mathbb{U}(g) = F(g)
\]

for all objects \( A \) in \( \mathcal{D} \) and \( B \) in \( \mathcal{C} \) and for all morphisms \( f \) in \( \mathcal{D} \) and \( g \) in \( \mathcal{C} \). These comparison functors allow us to give objects \( A \) in \( \mathcal{D} \) and objects \( B \) in \( \mathcal{C} \) the respective structures of algebras over \( \mathbb{T} \) and coalgebras over \( \mathbb{U} \).

### 3.2. The Barr-Beck Theorem.

**Definition** A functor \( G : \mathcal{D} \to \mathcal{C} \) is called monadic if there exists a monad \( \mathbb{T} = (T, \eta, \mu) \) on \( \mathcal{C} \) and an equivalence of categories \( J : \mathcal{D} \to \mathcal{C}^\mathbb{T} \) such that \( F \circ J \) is isomorphic as a functor to \( G \), where \( F : \mathcal{C}^\mathbb{T} \to \mathcal{C} \) is the forgetful functor. Again, see [11, p. 212] for more details. Equivalently ([9]), a functor \( G : \mathcal{D} \to \mathcal{C} \) is monadic if it has a left adjoint \( F : \mathcal{C} \to \mathcal{D} \), and so the pair form a monad \( \mathbb{T} = (T := G \circ F, \eta, \mu) \) on \( \mathcal{C} \), and if the comparison functor \( K^\mathbb{T} : \mathcal{D} \to \mathcal{C}^\mathbb{T} \) is an equivalence of categories.

**Definition** Dually, a functor \( G : \mathcal{D} \to \mathcal{C} \) is comonadic if it has a left adjoint \( F : \mathcal{C} \to \mathcal{D} \), and so form a comonad \( \mathbb{U} = (U := F \circ G, \varepsilon, \Delta) \) on \( \mathcal{D} \), and if the comparison functor \( J_\mathbb{U} : \mathcal{C} \to \mathcal{D}_\mathbb{U} \) is an equivalence of categories.

So monadicity allows us to classify a category as algebras over a monad, and similarly comonadicity classifies a category as coalgebras over a comonad. We seek to use this to try to classify crystals as coalgebras over a comonad using the following result, sometimes known as Beck’s Monadicity Theorem, that gives criterion for when a functor is monadic (or comonadic).
Theorem 3.1. (The Barr-Beck Theorem \[1\] p. 212) A functor $G : \mathcal{D} \to \mathcal{C}$ is monadic if and only if

i) $G$ has a left adjoint $F$;

ii) $G$ reflects isomorphisms. That is, if $G(f)$ is an isomorphism then $f$ is an isomorphism for all morphisms $f$;

iii) If a pair $f, g : A \to B$ are morphisms in $\mathcal{D}$ such that $G(f), G(g)$ have a split coequaliser $d : G(B) \to D$ in $\mathcal{C}$ then $f, g$ have a coequaliser $c : B \to C$ in $\mathcal{D}$ such that $G(c) = d, G(C) = D$.

A dual version of the Barr-Beck theorem then characterises comonadic functors as follows.

Theorem 3.2. A functor $G : \mathcal{D} \to \mathcal{C}$ is comonadic if and only if

i) $G$ has a left adjoint $F$;

ii) $G$ reflects isomorphisms;

iii) If a pair $f, g : A \to B$ are morphisms in $\mathcal{D}$ such that $G(f), G(g)$ have a split equaliser $h : H \to G(A)$ in $\mathcal{C}$ then $f, g$ have an equaliser $e : E \to A$ in $\mathcal{D}$ such that $G(e) = h, G(E) = H$.

4. Classification of Crystals

4.1. The Crystal Functor. From here we shall restrict our study to the crystals in $\text{Crys}$ that arise from integrable $g$-modules. That is, crystals that are disjoint unions of $B(\alpha)$ for $\alpha \in \Phi$. We will also consider only strict morphisms of crystals. We shall refer to this subcategory as $\text{Crys}_g$.

Since all objects in the category of crystals are pointed sets with additional structure, there is a forgetful functor $F : \text{Crys}_g \to \text{Set}_\bullet$ which sends a crystal to its underlying pointed set, forgetting this additional structure.

For us to apply Barr-Beck, we seek an adjunction between the category of crystals and the category of pointed sets using this forgetful functor. Consider the pairing

$$G : \text{Ob}(\text{Set}_\bullet) \to \text{Ob}(\text{Crys}_g)$$

$$X \mapsto \bigsqcup_{\alpha \in \Phi} \bigsqcup_{f \in \text{Hom}(F(B(\alpha)), X)} B(\alpha)_f$$

where $B(\alpha)_f$ are distinct copies of $B(\alpha)$ indexed by the functions $f$. Let functions $\psi \in \text{Hom}_{\text{Set}_\bullet}(X, Y)$ be mapped to the morphisms $G(\psi)$ gained from extending the isomorphisms $B(\alpha)_f \to B(\alpha)_{\psi \circ f}$ (or mapping $B(\alpha)_f$ to $0$ if $\psi \circ f = 0$). We can see that this describes a functor as $G(id_X) = id_G(X)$ and for pointed sets $X, Y, Z$ and morphisms $\psi_1 : X \to Y$, $\psi_2 : Y \to Z$, $G(\psi_2 \circ \psi_1) = G(\psi_2) \circ G(\psi_1)$.

Let $\alpha \in \Phi$ and let $Y$ be a set. Then $B(\alpha)$ is a connected component so, by our crystal version of Shur’s Lemma (Lemma 1.2), a nonzero (strict) morphism of crystals $f : B(\alpha) \to G(Y)$ is an isomorphism between $B(\alpha)$ and an isomorphic copy $B(\alpha)_f$ for $f' : F(B(\alpha)) \to Y$. So such a map $f : B(\alpha) \to G(Y)$ picks out a unique nonzero map of pointed sets $f' : F(B(\alpha)) \to Y$. This defines an isomorphism between the nonzero elements of $\text{Hom}_{\text{Crys}_g}(B(\alpha), G(Y))$ and the nonzero elements of $\text{Hom}_{\text{Set}_\bullet}(FB(\alpha), X)$. Thus we get an isomorphism

$$\text{Hom}_{\text{Crys}_g}(B(\alpha), G(Y)) \cong \text{Hom}_{\text{Set}_\bullet}(FB(\alpha), Y)$$
by sending the zero map to the zero map.

For a general crystal $X$, let $X_i$ be the irreducible components for $i \in I$, an indexing set. Then the $X_i$ are isomorphic to $B(\alpha_i)$ for some $\alpha_i \in \Phi$. So there is an isomorphism
\[
\rho_{X,Y} : \text{Hom}_{\text{Crys}}(X,G(Y)) = \text{Hom}_{\text{Crys}}(\bigsqcup_{i \in I} X_i, G(Y))
\]
\[
\cong \prod_{i \in I} \text{Hom}_{\text{Crys}}(X_i, G(Y))
\]
\[
\cong \prod_{i \in I} \text{Hom}_{\text{Set}}(F(X_i), Y)
\]
\[
\cong \text{Hom}_{\text{Set}}(F(\bigsqcup_{i \in I} X_i), Y)
\]
\[
= \text{Hom}_{\text{Set}}(F(X), Y).
\]

Since any morphism of crystals $f : X \rightarrow X'$ is either an isomorphisms or the zero map on each irreducible component, we clearly have the commutative square
\[
\begin{array}{ccc}
\text{Hom}_D(F(X), Y) & \xrightarrow{- \circ F(f)} & \text{Hom}_D(F(X'), Y) \\
\rho_{X,Y} & \downarrow & \rho_{X',Y} \\
\text{Hom}_C(X, G(Y)) & \xrightarrow{- \circ f} & \text{Hom}_C(X', G(Y))
\end{array}
\]
for any pointed set $Y$. So it remains to check the naturality condition for a function $g : Y \rightarrow Y'$ between pointed sets and for an irreducible crystal $B(\alpha)$. Then for $\psi : F(B(\alpha)) \rightarrow Y$, $\rho_{X,Y'}(g \circ \psi)$ is the isomorphism $B(\alpha) \cong B(\alpha)_{g \circ \psi}$, as is the map $G(g) \circ \rho_{X,Y}(\psi)$. Thus we have the commutative square
\[
\begin{array}{ccc}
\text{Hom}_D(F(X), Y) & \xrightarrow{g \circ -} & \text{Hom}_D(F(X), Y') \\
\rho_{X,Y} & \downarrow & \rho_{X,Y'} \\
\text{Hom}_C(X, G(Y)) & \xrightarrow{G(g) \circ -} & \text{Hom}_C(X, G(Y'))
\end{array}
\]
So the isomorphism is natural. Thus:

**Proposition 4.1.** There is an adjunction $F \dashv G$ between the category of pointed sets and the category of crystals.

**4.2. Classifying Crystals.** If we hope to apply the Barr-Beck theorem to our pair of adjoint functors, we must check that the crystal functor, $G$, reflects isomorphisms and preserves split equalisers. In fact, we shall show a stricter result that $G$ preserves all equalisers.

**Proposition 4.2.** $G$ reflects isomorphisms.

**Proof.** Suppose $f : A \rightarrow B$ is a morphism in $\text{Set}_*$ such that $G(f)$ is an isomorphism in $\text{Crys}_\ast$. Recall that $G(f)$ is the morphism of crystals that extends the isomorphisms $B(\alpha) \rightarrow B(\alpha)_{f \circ g}$.
where \( g : FB(\alpha) \to A \). Suppose \( f(a) = f(b) \) for \( a, b \in A \). Then there are maps \( \hat{a} : \{u_0\} \to A \), \( u_0 \mapsto a \), \( \hat{b} : \{u_0\} \to A, u_0 \mapsto b \), which correspond to components \( B(0)\hat{a}, B(0)\hat{b} \) of \( G(A) \). But \( f \circ \hat{a} = f \circ \hat{b} \), so \( G(f)(B(0)\hat{a}) = G(f)(B(0)\hat{b}) \). So, as \( G(f) \) is an isomorphism, \( B(0)\hat{a} = B(0)\hat{b} \), so \( \hat{a} = \hat{b} \), so \( a = b \). Now let \( c \in B \) and let \( \hat{c} : \{u_0\} \to B, u_0 \mapsto c \). Then \( B(0)\hat{c} \) is a component of \( G(B) \). So, as \( G(f) \) is an isomorphism, \( B(0)\hat{c} = B(0)f_\varphi \) for some \( \varphi : \{u_0\} \to A \). Then \( c = \hat{c}(u_0) = f(\varphi(0)) \) is in the image of \( f \). So \( f \) is bijective, and hence an isomorphism. \( \square \)

**Proposition 4.3.** \( G \) preserves equalisers.

**Proof.** Now let \( f, g : A \to B \) be a pair of maps in \( \text{Set}_\bullet \). Let \( H := \{x \in G(A) \mid G(f)(x) = G(g)(x)\} \). As we saw before, if \( G(f) \) and \( G(g) \) agree on a single element of a connected component of \( G(A) \) then they agree on the whole connected component. So \( H \) is a subcrystal of \( G(A) \). Let \( E = \{x \in A \mid f(x) = g(x)\} \) be the subset of \( A \) with injection \( e : E \to A \), which is an equaliser of \( f, g \). Then

\[
B(\alpha) \varphi \subset H \iff B(\alpha)_{f \circ \varphi} = B(n)_{g \circ \varphi} \iff f \circ \varphi = g \circ \varphi \\
\iff \text{Image}(\varphi) \subset H \iff \varphi : FB(\alpha) \to H \\
\iff B(\alpha) \varphi \subset G(E).
\]

So \( H = G(E) \). As in the \( sl_2 \) case we see that \( h : H \hookrightarrow G(A) \) is an equaliser of \( G(f), G(g) \) with \( G(h) = e, G(H) = E \). \( \square \)

Thus we have:

**Theorem 4.4.** The comonad \( U = (U = F \circ G, \eta, \mu) \) gives an equivalence of categories \( U_\bullet : \text{Crys}_U \to \text{Set}_\bullet \) between the category of \( \mathfrak{g} \) crystals and the category of algebras over the comonad \( U \). Thus we have classified all crystals as coalgebras over the comonad \( U \).

**Proof.** As a result of the above, we have satisfied the conditions of the Barr-Beck Theorem. \( \square \)

Explicitly, we see that

\[
FG : A \mapsto \bigsqcup_{\alpha \in \Phi} \bigsqcup_{f \neq 0} F(B(\alpha)_f)
\]

with

\[
\eta_B(\alpha) : B(\alpha) \mapsto \bigsqcup_{\beta \in \Phi} \bigsqcup_{f \neq 0} B(\beta)_f,
\]

\[
b \mapsto (b)_{\text{id}_{FB(\alpha)}} \in F(B(\alpha)_{\text{id}_{FB(\alpha)}})
\]

and

\[
\varepsilon_A : \bigsqcup_{\alpha \in \Phi} \bigsqcup_{f \neq 0} F(B(\alpha)_f) \to A,
\]

\[
(b)_f \mapsto f(b)
\]

so

\[
\Delta_A : \bigsqcup_{\alpha \in \Phi} \bigsqcup_{f \neq 0} F(B(\alpha)_f) \mapsto \bigsqcup_{\beta \in \Phi} \bigsqcup_{g \neq 0} F(B(\beta)_g)
\]

\[
(b)_f \mapsto (b)_{x \mapsto (x)_f} \in F(B(\alpha)_{x \mapsto (x)_f})
\]
where we have the maps $B(\alpha) \to FG(A)$, $x \mapsto (x)f$. For notational purposes, let us denote these maps $s_f$. From here we can explicitly see the coalgebra structure of each $B(\alpha)$ over $FG$ is given by a map

$$\zeta : F(B(\alpha)) \to FG(F(B(\alpha))), \ b \mapsto (b)id_{F(B(\alpha))}$$

which extends to the coalgebra structure of a general crystal $X = \bigsqcup_{i \in I} B(\alpha_i)$ as follows:

$$\zeta : F(X) \to FG(F(X)), \ b \mapsto (b)_{1\in I}(B(\alpha_i)) \text{ for } b \in F(B(\alpha_i)).$$

4.3. **Recovering the Crystal Structure.** Given a pointed set $A$ with a coalgebra structure $(A, \zeta_A)$ over our comonad $U = FG$, we know from the above that $A$ carries a crystal structure that has been forgotten by the forgetful functor $F$. In fact, there is a way of recovering this crystal structure from the coalgebra structure. We regain the Kashiwara operator $\tilde{f}$ (and similarly $\tilde{e}$) via the following composition:

$$A \xrightarrow{\zeta_A} FG(A) \xrightarrow{\tilde{f}} FG(A) \to A$$

where the last arrow is the map $(b)f \mapsto f(b)$. We also regain the weight function via

$$A \to FG(A) \to \Phi$$

where the last arrow is the map $(b)f \mapsto wt(b)$.

4.4. **The link with $B$.** For coalgebras $C, C'$, if we have a morphism of $\psi : C \to C'$ of coalgebras then we may give a $C$-comodule $(M, \Delta_M)$ a $C'$-comodule structure via the composition of maps $(\psi \otimes \text{id}) \circ \Delta_M : M \to C \otimes M \to C' \otimes M$. Thus we may push forward $C$-comodules to $C'$-comodules via $\psi$.

We have already seen that $\text{Crys}_g$ forms a subcategory of the category of $B$-comodules from Proposition 2.4. So we may ask whether there is a morphism between our comonad $FG$ and our coalgebra $B$ that gives $FG$-coalgebras a $B$-comodule structure as above. For this to make sense, we must first view $B$ as a functor on $Set_*$.

Since $B$ has an underlying pointed set, we may consider just the pointed set $B$. Then we have a tensor functor $H := B \otimes - : Set_* \to Set_*$ that associates

$$A \mapsto B \otimes A, \ f \mapsto \text{id}_B \otimes f$$

for pointed sets $A$ and morphisms $f$. This comes with natural transformation acting as comultiplication on the comonad $\Delta \otimes \text{id} : H \to H \circ H$. As before, we do not have a counit map $\varepsilon$. We have already seen that this satisfies the appropriate commutative diagram to obtain a comonad, $H$, without counit. It is clear that a pointed set $A$ is a comodule over $B$ with coaction $\zeta : A \to B \otimes A$ if and only if $A$ is a coalgebra over the comonad $H$ with map $\zeta : A \to H(A)$.

**Proposition 4.5.** The $B$-comodule structure on each $B(\alpha)$, and hence by extension every object in $\text{Crys}_g$, arises as a result of pushing forward the $U$-coalgebra structure via a natural transformation $\theta : U \Rightarrow B \otimes -$.

**Proof.** We consider the collection of morphisms $\theta_A : FG(A) \to H(A)$, for pointed sets $A \in \text{Ob}(Set_*)$, defined by

$$\theta_A : \bigsqcup_{\alpha \in \Phi} \bigsqcup_{f \in \text{Hom}(F(B(\alpha)), A)} F(B(\alpha))_f \to \bigsqcup_{\alpha \in \Phi} B(\alpha) \otimes B(\alpha)^* \otimes A$$

$$\qquad \qquad (b)_f \mapsto (b \otimes u_{-\alpha}) \otimes f(u_{\alpha})$$
for \((b)_f \in F(B(\alpha)_f)\) indexed by \(f \in \text{Hom}(FB(\alpha),A)\). Then, for pointed sets \(A,A'\) with a morphism \(g : A \rightarrow A'\) between them, \(\theta_A \circ FG(f)((b)_f) = \theta_{A'}((b)_{gf}) = (b \otimes u_{-\alpha}) \otimes g \circ f(u_\alpha)\) and \((\text{id} \otimes g) \circ \theta_A((b)_{gf}) = (\text{id} \otimes g)(b \otimes u_{-\alpha}) \otimes f(u_\alpha) = (b \otimes u_{-\alpha}) \otimes g \circ f(u_\alpha)\), where \((b)_f \in F(B(\alpha)_f)\) is indexed by \(f \in \text{Hom}(FB(\alpha),A)\). Thus we have the commutative diagram

\[
\begin{array}{ccc}
FG(A) & \xrightarrow{FG(g)} & FG(A') \\
\theta_A \downarrow & & \theta_{A'} \downarrow \\
H(A) & \xrightarrow{FG(g)} & H(A')
\end{array}
\]

So \(\theta : FG \Rightarrow H\) defines a natural transformation. Thus we can give each \(B(\alpha)\), and by extension every object in \(\text{Crys}_g\), the structure of a \(B\)-comodule via \(\theta\). It remains to check that this coaction agrees with the coaction we already have.

\[
F(B(\alpha)) \rightarrow FG(F(B(\alpha))), \ b \mapsto (b)_{\text{id}_{F(B(\alpha))}}
\]

\[
\downarrow \theta
\]

\[
F(B(\alpha)) \rightarrow H(B(\alpha)) = B \otimes B(\alpha), \ b \mapsto (b \otimes u_{-\alpha}) \otimes u_\alpha
\]

Indeed, this is the same coaction as we previously had. \(\square\)

Thus we have a connection between our initial attempt at a coalgebra, \(B\), and our comonad classifying crystals, \(U\).

5. The Structure of \(U\)

In this section we shall look at the generalisation of bialgebras to the setting of monadic functors through the study of monoidal functors.

5.1. Monoidal Functors. In the setting of functors, the notion of a bimonad is not obvious. The subtlety comes from the lack of symmetry when composing functors - there is no natural twist \(A \circ B \Rightarrow B \circ A\) for functors \(A,B\) on a category \(C\). Recall that, for a coalgebra \(H\) (over, say, the category of vector spaces) the categories of modules and comodules of \(H\) inherit a monoidal structure. In this paper, we wish to generalise the property of bialgebras that allows us to encode the monoidal structure of the categories of modules and comodules, as seen in the previous section. To generalise this, we consider monoidal functors.

**Definition** A comonadic functor \(T\) on a category \(C\) is said to be **monoidal** (or a bicomonad), as discussed in Brugières and Virelizier’s paper [2], if there is a natural tranformation

\[
\chi_{A,B} : T(A) \otimes T(B) \Rightarrow T(A \otimes B)
\]

and a morphism \(\text{I} \rightarrow T(\text{I})\), where \(\text{I}\) is taken to be the identity of the tensor product, satisfying the following compatibility conditions for the monad structure:

\[
\begin{array}{ccc}
T(A) \otimes T(B) & \xrightarrow{\chi_{A,B}} & T(A \otimes B) \\
\Delta_A \otimes \Delta_B \downarrow & & \Delta_{A \otimes B} \downarrow \\
TT(A) \otimes TT(B) & \xrightarrow{T(\chi_{A,B})} & T(T(A) \otimes T(B)) & \xrightarrow{T(\chi_{A,B})} & TT(A \otimes B)
\end{array}
\]
\[
\begin{array}{cccc}
T(A \otimes I) & \hookrightarrow & T(A) \otimes T(I) & \hookrightarrow & T(A) \otimes I & \quad T(I \otimes A) & \hookrightarrow & T(I) \otimes T(A) & \hookrightarrow & I \otimes T(A)
\end{array}
\]

Dually, we may define opmonoidal monadic functors (or comonads) with natural transformations \(\chi_{A,B} : T(A \otimes B) \rightarrow T(A) \otimes T(B)\) and morphisms \(T(I) \rightarrow I\) satisfying analogous compatibility conditions.

For a comonadic functor, the property of being monoidal gives a monoidal structure to the category of coalgebras. Because of this, we may think of the functor as bimonadic (or, more precisely, bicomonadic). The coaction on a tensor product of two coalgebras is given by the following composition:

\[
A \otimes B \rightarrow T(A) \otimes T(B) \rightarrow T(A \otimes B)
\]

where the former arrow is given by the respective coactions of \(A\) and \(B\), and the latter given by \(\chi\). We see analogous results for the category of algebras over an opmonoidal monadic functor. These opmonoidal monads are the standard notion of a bimonadic functor, introduced by Moerdijk in [8] (as a Hopf monad) and later studied by Bruguieres and Virelizier in [2], although here we shall consider the dual notion of monoidal comonads (called bicomonads in [3]).

When the category of coalgebras over a comonadic functor already has a monoidal structure, we may ask when this can be reconstructed from a monoidal functor. The following two results give sufficient conditions for a functor to be monoidal, and for the resulting comparison functor \(J_U\) to be an equivalence of monoidal categories. The latter, an extension of the Barr-Beck Theorem, is not unique to this paper. Similar (stronger) results have been proven, for example in [8].

**Lemma 5.1.** Suppose \(G : D \rightarrow C\) is a comonadic functor, and so \(U = (U = FG, \Delta, \epsilon)\) is a monad with equivalence of categories \(J_U : C \rightarrow D_U\). Suppose further that the forgetful functor \(F\) is a strong monoidal functor, with isomorphism \(\kappa_F : F(\bullet \otimes \bullet) \xrightarrow{\simeq} F(\bullet) \otimes F(\bullet)\), and that there are natural morphisms \(\tilde{\chi}_{A,B} : G(A) \otimes G(B) \rightarrow G(A \otimes B), I \rightarrow G(I)\) satisfying

\[
\begin{align*}
& \eta_X \otimes \eta_Y & GF(X) \otimes GF(Y) \\
X \otimes Y \quad & \quad \eta_{X \otimes Y} \\
\eta_{X \otimes Y} & GF(X \otimes Y) \quad G(\kappa_F) \quad \tilde{\chi}_{F(X),F(Y)}
\end{align*}
\]

\[
G(A \otimes I) \quad G(A) \otimes G(I) \quad G(A) \otimes I \quad G(I \otimes A) \quad G(I \otimes G(A)) \quad I \otimes G(A)
\]

Then \(U\) is monoidal with \(\chi\) defined by \(FG(A \otimes B) \rightarrow F(GA \otimes GB) \xrightarrow{\simeq} FGA \otimes FGB\).

**Proof.** By the naturality of \(\eta\) there is a commutative square

\[
\begin{array}{cc}
GA \otimes GB & \xrightarrow{\tilde{\chi}} & G(A \otimes B) \\
GF(GA \otimes GB) & \xrightarrow{GF(\tilde{\chi})} & GFG(A \otimes B)
\end{array}
\]

By the naturality of \(\eta\) there is a commutative square

\[
\begin{array}{cc}
GA \otimes GB & \xrightarrow{\tilde{\chi}} & G(A \otimes B) \\
GF(GA \otimes GB) & \xrightarrow{GF(\tilde{\chi})} & GFG(A \otimes B)
\end{array}
\]
which, when added to the commutative square in our assumptions, gives a commutative diagram

\[
\begin{array}{c}
G(A) \otimes G(B) \xrightarrow{\eta_{GA} \otimes \eta_{GB}} G(A \otimes B) \\
GFG(A) \otimes GFG(B) \xrightarrow{\tilde{\chi}} G(FG(A) \otimes FG(B)) \xrightarrow{GF(\tilde{\chi}) \circ G(\kappa_F^{-1})} GFG(A \otimes B)
\end{array}
\]

Using the naturality of \(\kappa_F\), the images under \(F\) of this diagram and the latter two from the assumptions gives the monoidal structure arising from \(\chi\).

\[\square\]

**Theorem 5.2.** Let \(\mathcal{C}, \mathcal{D}\) be monoidal categories. Suppose \(G: \mathcal{D} \to \mathcal{C}\) satisfies the following conditions:

i) \(G\) has a left adjoint \(F\) which is a strong monoidal functor;

ii) \(G\) reflects isomorphisms;

iii) If a pair \(f, g: A \to B\) are morphisms in \(\mathcal{D}\) such that \(G(f), G(g)\) have a split equaliser \(h: H \to G(A)\) in \(\mathcal{C}\) then \(f, g\) have an equaliser \(e: E \to A\) in \(\mathcal{D}\) such that \(G(e) = h, G(E) = H\);

iv) There are natural morphisms \(\tilde{\chi}_{A,B}: G(A) \otimes G(B) \to G(A \otimes B)\), \(I \to G(I)\) satisfying the following commutative diagrams.

\[
\begin{array}{c}
X \otimes Y \xrightarrow{\eta_X \otimes \eta_Y} GF(X) \otimes GF(Y) \\
GF(X \otimes Y) \xrightarrow{G(\kappa_F)} G(F(X) \otimes F(Y)) \\
G(A \otimes I) \leftarrow G(A) \otimes G(I) \rightarrow G(A) \otimes I \leftarrow G(I) \otimes G(A) \rightarrow I \otimes G(A)
\end{array}
\]

Then \(G: \mathcal{D} \to \mathcal{C}\) is comonadic with \(\mathcal{D}_U\) a monoidal category and \(J_U: \mathcal{C} \to \mathcal{D}_U\) an equivalence of monoidal categories.

\[\text{Proof.}\] As we have seen in Lemma 5.1, conditions (i)-(iv) are sufficient to give \(\mathcal{D}_U\) a monoidal structure. It then remains to show that \(J_U\) is a strong monoidal functor. Indeed, \(F\) is strong monoidal by assumption, so it remains to check that the coactions coincide. That is, we require the commutativity of

\[
\begin{array}{c}
F(A \otimes B) \xrightarrow{\Delta} F(A) \otimes F(B) \\
FGF(A \otimes B) \xrightarrow{\Delta \otimes \Delta} FGF(A) \otimes FGF(B) \\
\end{array}
\]

which is the image under \(F\) of our assumed commutative diagram. Since \(J_U\) is an equivalence of categories, our result follows. \[\square\]

We now apply these results to our specific functors. In the context of our category of pointed sets, \(\text{Set}_\bullet\), \(I\) is just the pointed singleton set \(\{\ast\}_\bullet\). It is clear that the forgetful functor \(F\) essentially
preserves tensor products. In this case, we can give our comonadic functor \( U = FG \) a monoidal structure as follows:

\[
\tilde{\chi}_{A,B} : G(A) \otimes G(B) = \bigsqcup_{\alpha \in \Psi} \bigsqcup_{\beta, \gamma \in \Gamma} B(\alpha)f \otimes B(\beta)g
\]

\[
\rightarrow G(A \otimes B) = \bigsqcup_{\gamma \in \Psi} \bigsqcup_{h : FB(\gamma) \rightarrow A \times B} B(\gamma)h,
\]

with

\[
\eta_{X} \otimes \eta_{Y} \\
\eta_{X \otimes Y} \\
(b \otimes b')_{s_{f} \otimes s_{g}} \\
\rightarrow \tilde{\chi}_{F(X), F(Y)} \\
G(\kappa_{F}) \\
(b \otimes b')_{s_{f} \otimes s_{g}} \\
\rightarrow (b \otimes v_{0})_{(f, s)} \\
(b)_{f} \otimes (v_{0})_{s} \\
(b)_{f} \otimes * \\
\rightarrow (b)_{f}
\]

and similarly for the third diagram. Hence we have our result.

**Proposition 5.3.** The monoidal structure of \( \text{Crys}_{g} \) given by the tensor product of crystals can be seen as a result of the monoidal structure of \( U \).

**Proof.** We check the compatibility conditions:

Thus we have managed to encode the tensor structure of \( \text{Crys}_{g} \) into our comonad \( T \).
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