PARTIAL NORMALITY CLASSES OF WEIGHTED CONDITIONAL TYPE OPERATORS ON $L^2(\Sigma)$

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Abstract. In this paper, some various partial normality classes of weighted conditional expectation type operators on $L^2(\Sigma)$ are investigated. Also, some applications of weak hyponormal weighted conditional type operators are presented.

1. Introduction

The notion of conditional expectation is rightfully thought of as belonging to the theory of probability. In such a text, it is set against a background of a probability space $(\Omega, \mathcal{F}, P)$ and $\sigma$-subalgebra ($\sigma$-field as it is commonly called in probability texts) $\mathcal{G}$ of $\mathcal{F}$. If $X$ denotes an integrable random variable, then the conditional expected value of $X$ given $\mathcal{G}$ is the random variable $E[X|\mathcal{G}]$ such that

1. $E[X|\mathcal{G}]$ is $\mathcal{G}$-measurable,
2. $E[X|\mathcal{G}]$ satisfies the functional relation
   \[ \int_{\mathcal{G}} E[X|\mathcal{G}] dP = \int_{\mathcal{G}} X dP, \forall G \in \mathcal{G}. \]

Any number of standard texts will illustrate concisely the probabilistic formulation and interpretation of the function $E[X|\mathcal{G}]$, and the reader is invited to consult such references as [2]. Our main interests, however, reside in the view of conditional expectation as an operator between the $L^p$-spaces, specially between $L^2$-spaces.

Among the earlier investigations along these lines is that of Shu-Teh Chen Moy in his seminal 1954 paper [10]. Set within the familiar framework of a probability space $(\Omega, \mathcal{F}, P)$, Moy obtains necessary and sufficient conditions for a linear transformation $T$ between function spaces to be of the form $TX = E[gX|\mathcal{G}]$, where $\mathcal{G} \subset \mathcal{F}$ is a $\sigma$-subalgebra and $g$ is a nonnegative measurable function with bounded conditional expected value. The function $E[gX|\mathcal{G}]$ can best be described as the weighted conditional expected value of $X$. Moreover, Conditional expectations have been studied in an operator theoretic setting, by, for example, R. G. Douglas, [4], de Pagter and Grobler [7], P.G. Dodds, C.B. Hujsmans and B. De Pagter, [3], J. Herron, [8], Alan Lambert [9] and Rao [11][12], as positive operators acting on $L^p$-spaces or Banach function spaces. The combination of conditional expectation and multiplication operators appears more often in the service of the study of other operators rather than being the object of study in and of themselves.

In [5][6] we investigated some classic properties of multiplication conditional expectation operators $M_wEM_u$ on $L^p$ spaces. In this paper we will be concerned with

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characterizing weighted conditional expectation type operators on $L^2(\Sigma)$ in terms of membership in the various partial normality classes and some applications of them.

2. Preliminaries

Let $(X, \Sigma, \mu)$ be a complete $\sigma$-finite measure space. For any $\sigma$-subalgebra $A \subseteq \Sigma$, the $L^2$-space $L^2(X, A, \mu_{|A})$ is abbreviated by $L^2(A)$, and its norm is denoted by $\| \cdot \|_2$. All comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set. The support of a measurable function $f$ is defined as $S(f) = \{ x \in X : f(x) \neq 0 \}$. We denote the vector space of all equivalence classes of almost everywhere finite valued measurable functions on $X$ by $L^0(\Sigma)$.

For a $\sigma$-subalgebra $A \subseteq \Sigma$, the conditional expectation operator associated with $A$ is the mapping $f \mapsto E^A f$, defined for all non-negative, measurable function $f$ as well as for all $f \in L^2(\Sigma)$, where $E^A f$, by the Radon-Nikodym theorem, is the unique $A$-measurable function satisfying

$$\int_A f \, d\mu = \int_A E^A f \, d\mu, \quad \forall A \in A.$$ 

As an operator on $L^2(\Sigma)$, $E^A$ is idempotent and $E^A(L^2(\Sigma)) = L^2(A)$. If there is no possibility of confusion, we write $E(f)$ in place of $E^A(f)$. This operator will play a major role in our work and we list here some of its useful properties:

- If $g$ is $A$-measurable, then $E(fg) = E(f)g$.
- $|E(f)|^2 \leq E(|f|^2)$.
- If $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$.
- $|E(fg)| \leq (E(|f|^2))^{\frac{1}{2}} (E(|g|^2))^{\frac{1}{2}}$, (Hölder inequality).
- For each $f \geq 0$, $S(f) \subseteq S(E(f))$.

A detailed discussion and verification of most of these properties may be found in [13].

Let $f \in L^0(\Sigma)$, then $f$ is said to be conditionable with respect to $E$ if $f \in D(E) := \{ g \in L^0(\Sigma) : E(|g|) \in L^0(A) \}$. Throughout this paper we take $u$ and $w$ in $D(E)$.

Every operator $T$ on a Hilbert space $\mathcal{H}$ can be decomposed into $T = U|T|$ with a partial isometry $U$, where $|T| = (T^*T)^{\frac{1}{2}}$. $U$ is determined uniquely by the kernel condition $\mathcal{N}(U) = \mathcal{N}(|T|)$. Then this decomposition is called the polar decomposition. The Aluthge transformation of $T$ is the operator $\hat{T}$ given by $\hat{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$.

The plan for the reminder of this paper is to present characterizations of weighted conditional expectation type operators in some various normality classes. Here is a brief review of what constitutes membership for an operator $T$ on a Hilbert space in each class:

(i) $T$ is normal if $T^*T = TT^*$.
(ii) $T$ is hyponormal if $T^*T \geq TT^*$.
(iii) For $0 < p < \infty$, $T$ is $p$-hyponormal if $(T^*T)^p \geq (TT^*)^p$.
(iv) $T$ is $\infty$-hyponormal if it is $p$-hyponormal for all $p$. 


(v) $T$ is weakly hyponormal if $|\hat{T}| \geq |T| \geq |\hat{T}^*|$.  
(vi) $T$ is normaloid if $\|T\|^n = \|T^n\|$ for all $n \in \mathbb{N}$.

3. Some classes of weighted conditional expectation type operators

In the first we reminisce some theorems that we have proved in [6].

**Theorem 3.1.** The operator $T = M_w EM_u$ is bounded on $L^2(\Sigma)$ if and only if
\[
\left( E|w|^2 \right)^{\frac{1}{2}} \left( E|u|^2 \right)^{\frac{1}{2}} \in L^\infty(\mathcal{A}),
\]
and in this case its norm is given by $\|T\| = \|(E(|w|^2))^{\frac{1}{2}}(E(|u|^2))^{\frac{1}{2}}\|_{\infty}$.

**Lemma 3.2.** Let $T = M_w EM_u$ be a bounded operator on $L^2(\Sigma)$ and let $p \in (0, \infty)$. Then
\[
(T^*T)^p = M_{\bar{u}}(E(|u|^2))^{p-1}(E(|w|^2))^p EM_u
\]
and
\[
(TT^*)^p = M_w(E(|u|^2))^{p-1}(E(|w|^2))^p EM_{\bar{w}},
\]
where $S = S(E(|u|^2))$ and $G = S(E(|w|^2))$.

**Theorem 3.3.** The unique polar decomposition of bounded operator $T = M_w EM_u$ is
\[
|T|(f) = \left( \frac{E(|u|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S \bar{u} E(uf)
\]
and
\[
U(f) = \left( \frac{\chi_S \cap G}{E(|u|^2)E(|u|^2)} \right)^{\frac{1}{2}} w E(uf),
\]
for all $f \in L^2(\Sigma)$.

**Theorem 3.4.** The Aluthge transformation of $T = M_w EM_u$ is
\[
\hat{T}(f) = \chi_S E(\bar{w}u) \bar{u} E(uf), \quad f \in L^2(\Sigma).
\]

From now on, we consider the operators $M_w EM_u$ and $EM_u$ are bounded operators on $L^2(\Sigma).$ In the sequel some necessary and sufficient conditions for normality, hyponormality, $p$-hyponormality, . . . will be presented.

**Theorem 3.5.** Let $T = M_w EM_u$, then
(a) If $(E(|u|^2))^{\frac{1}{2}} \bar{w} = u(E(|w|^2))^{\frac{1}{2}}$, then $T$ is normal.
(b) If $T$ is normal, then $|E(u)|^2 E(|w|^2) = |E(w)|^2 E(|u|^2)$.

**Proof.** (a) Applying lemma 3.2 we have
\[
T^*T - TT^* = M_u E(|u|^2) EM_u - M_w E(|u|^2) EM_{\bar{w}}.
\]
So for every $f \in L^2(\Sigma)$,
\[
\langle T^*T - TT^*(f), f \rangle = \]
\[ \int_X E(|w|^2) E(u f) \hat{w} f - E(|u|^2) E(w f) w f \, d\mu \]
\[ = \int_X |E(u (E(|w|^2)) \hat{w} f)|^2 - |E((E(|u|^2)) \hat{w} f)|^2 \, d\mu. \]

This implies that if
\[ (E(|u|^2)) \hat{w} = u (E(|w|^2)), \]
then for all \( f \in L^2(\Sigma), \langle T^* T - TT^* (f), f \rangle = 0, \) thus \( T^* T = T T^*. \)

(b) Suppose that \( T \) is normal. For all \( f \in L^2(\Sigma) \) we have
\[ \int_X |E(u (E(|w|^2)) \hat{w} f)|^2 - |E((E(|u|^2)) \hat{w} f)|^2 \, d\mu = 0. \]

Let \( A \in \mathcal{A} \), with \( 0 < \mu(A) < \infty \). By replacing \( f \) to \( \chi_A \), we have
\[ \int_A |E(u (E(|w|^2)) \hat{w} f)|^2 - |E((E(|u|^2)) \hat{w} f)|^2 \, d\mu = 0 \]
and so
\[ \int_A |E(u)|^2 E(|w|^2) - |E(w)|^2 E(|u|^2) \, d\mu = 0. \]
Since \( A \in \mathcal{A} \) is arbitrary, then \( |E(u)|^2 E(|w|^2) - |E(w)|^2 E(|u|^2) \) is normal.

**Corollary 3.6.** The operator \( EM_u \) is normal if and only if \( u \in L^\infty(\mathcal{A}) \).

**Theorem 3.7.** Let \( T = M_w EM_u \) and let \( p \in (0, \infty) \).

(a) The followings are equivalent.

(i) \( T \) is hyponormal.

(ii) \( T \) is \( p \)-hyponormal.

(iii) \( T \) is \( \infty \)-hyponormal.

(b) If \( u(E(|w|^2)) \hat{w} - (E(|u|^2)) \hat{w} \geq 0 \), then \( T \) is hyponormal.

(c) If \( T \) is hyponormal, then \( |E(u)|^2 E(|w|^2) - |E(w)|^2 E(|u|^2) \geq 0 \).

**Proof.** (a) Applying Lemma 3.2 we obtain that \( (T^* T)^p \geq (TT^*)^p \) if and only if
\[ M_{\chi_{\mathcal{E} \subseteq \mathcal{E}}(E(|u|^2))^{p-1}(E(|w|^2))^{p-1}}(M_{\hat{w} E(|w|^2)} EM_u - M_{w E(|u|^2)} EM_{\hat{w}}) \geq 0. \]

This inequality holds if and only if
\[ T^* T - TT^* = M_{\hat{w} E(|w|^2)} EM_u - M_{w E(|u|^2)} EM_{\hat{w}} \geq 0, \]
where we have used the fact that \( T_1 T_2 \geq 0 \) if \( T_1 \geq 0, T_2 \geq 0 \) and \( T_1 T_2 = T_2 T_1 \) for all \( T_i \in \mathcal{B}(\mathcal{H}) \), the set of all bounded linear operators on Hilbert space \( \mathcal{H} \). Since \( 0 < p < \infty \) is arbitrary, then (i), (ii) and (iii) are equivalent.

(b) By lemma 3.2 we have
\[ T^* T - TT^* = M_{\hat{w} E(|w|^2)} EM_u - M_{w E(|u|^2)} EM_{\hat{w}}. \]
So for every \( f \in L^2(\Sigma) \),
\[
\langle T^*T - TT^*(f), f \rangle = \\
= \int_X |E(u(E(|w|^2))^{\frac{1}{2}} f)|^2 - |E((E(|u|^2))^{\frac{1}{2}} \bar{w} f)|^2 d\mu.
\]
This implies that, if \( u(E(|w|^2))^{\frac{1}{2}} - (E(|u|^2))^{\frac{1}{2}} \bar{w} \geq 0 \),
then \( T \) is hyponormal.

(c) Let \( T \) be hyponormal. For all \( f \in L^2(\Sigma) \) we have
\[
\int_X |E(u(E(|w|^2))^{\frac{1}{2}} f)|^2 - |E((E(|u|^2))^{\frac{1}{2}} \bar{w} f)|^2 d\mu \geq 0.
\]
Let \( A \in \mathcal{A} \), with \( 0 < \mu(A) < \infty \). By replacing \( f \) to \( \chi_A \), we have
\[
\int_A |E(u(E(|w|^2))^{\frac{1}{2}} f)|^2 - |E((E(|u|^2))^{\frac{1}{2}} \bar{w} f)|^2 d\mu \geq 0
\]
and so
\[
\int_A |E(u)|^2 E(|w|^2) - |E(w)|^2 E(|u|^2) d\mu \geq 0.
\]
Since \( A \in \mathcal{A} \) is arbitrary, then \( |E(u)|^2 E(|w|^2) \geq |E(w)|^2 E(|u|^2) \). \( \square \)

**Corollary 3.8.** Let \( T = EM_u \), and \( p \in (0, \infty) \). Then the followings are equivalent.

(i) \( T \) is normal.

(ii) \( T \) is hyponormal.

(iii) \( T \) is \( p \)-hyponormal.

(iv) \( T \) is \( \infty \)-hyponormal.

(v) \( u \in L^\infty(\mathcal{A}) \).

**Theorem 3.9.** Let \( T = M_u EM_u \), then

(a) If \( |E(uw)|^2 \geq E(|u|^2)E(|w|^2) \), then \( T \) is \( p \)-quasihyponormal.

(b) If \( T \) is \( p \)-quasihyponormal, then \( |E(uw)|^2 \geq E(|u|^2)E(|w|^2) \) on \( \sigma(E(u)) \cap G \).

(c) If \( S(w) = S(u) = X \), then \( T \) is \( p \)-quasihyponormal if and only if \( |E(uw)|^2 \geq E(|u|^2)E(|w|^2) \).

**Proof.** (a) By Lemma 3.2, it is easy to check that
\[
T^* (T^*T)^p T = M_{\bar{u}E(|u|^2)}E(|w|^2)^{p-1}E\chi_{\sigma(E(|u|^2)))} E|E(uw)|^2 EM_u;
\]
\[
T^* (TT^*)^p T = M_{\bar{u}E(|u|^2)}E(|w|^2)^{p+1}E\chi_{\sigma(E(|u|^2)))} EEM_u.
\]
It follows that \( T^* (T^*T)^p T \geq T^* (TT^*)^p T \) if
\[
M_{E(|u|^2)}E(|w|^2)^{p-1}E\chi_{\sigma(E(|u|^2)))} M_{|E(uw)|^2 - E(|u|^2)E(|w|^2)} EM_u \geq 0.
\]
By the same argument in Theorem 3.7, this inequality holds if \( M(|E(uw)|^2 - E(|w|^2)E(|u|^2)) \geq 0 \); i.e. \( |E(uw)|^2 - E(|w|^2)E(|u|^2) \geq 0 \).

(b) Suppose that \( T \) is \( p \)-quasihyponormal. Then for all \( f \in L^2(A) \), we have

\[
(T^*(T^*T)^pT - T^*(TT^*)^pTf, f)
\]

\[
= \int_X (E(|u|^2))^{p-1} \chi_S(E(|w|^2))^p (|E(uw)|^2 - E(|w|^2)E(|u|^2))E(u)^2|f|^2d\mu \geq 0.
\]

Thus

\[
(E(|u|^2))^{p-1} \chi_S(E(|w|^2))^p (|E(uw)|^2 - E(|w|^2)E(|u|^2))E(u)^2 \geq 0,
\]

and hence we obtain \( |E(uw)|^2 \geq E(|w|^2)E(|u|^2) \) on \( \sigma(E(u)) \cap G \).

(c) It follows from (a) and (b). \( \square \)

**Corollary 3.10.** Let \( T = EM_u \), \( S(u) = X \) and \( p \in (0, \infty) \). Then the following cases are equivalent.

(i) \( T \) is hyponormal.

(ii) \( T \) is \( p \)-hyponormal.

(iii) \( T \) is \( \infty \)-hyponormal.

(iv) \( T \) is \( p \)-quasihyponormal.

(v) \( u \in L^\infty(A) \).

**Example 3.11.** Let \( X = [0, 1] \times [0, 1] \), \( d\mu = dx \, dy \), \( \Sigma \) the Lebesgue subsets of \( X \) and let \( \mathcal{A} = \{ A \times [0, 1] : A \text{ is a Lebesgue set in } [0, 1] \} \). Then, for each \( f \) in \( L^2(\Sigma) \), \( (Ef)(x, y) = \int_0^1 f(x, t)dt \), which is independent of the second coordinate. This example is due to A. Lambert and B. Weinstock [7]. Now, if we take \( u(x, y) = y \bar{y} \) and \( w(x, y) = \sqrt{(4+x)y} \), then \( E(|u|^2)(x, y) = \frac{1}{4+y} \) and \( E(|w|^2)(x, y) = \frac{4}{4+y} \). So, \( E(|u|^2)(x, y)E(|w|^2)(x, y) = 2 \) and \( |E(uw)|^2(x, y) = 64 \left( \frac{4+y}{4+y+12+y} \right) \). Direct computations show that \( E(|u|^2)(x, y)E(|w|^2)(x, y) \leq |E(uw)|^2(x, y) \). Thus, by Theorem 3.9 the weighted conditional type operator \( M_u EM_u \) is \( p \)-quasihyponormal.

**Theorem 3.12.** Let \( T = M_u EM_u \), then

(a) If \( |E(uw)| = E(|u|^2)(E(|w|^2))^{\frac{1}{2}} \) on \( S = S(E(|u|^2)) \), then \( T \) is weakly hyponormal.

(b) If \( T \) is weakly hyponormal, then \( |E(uw)| = E(|u|^2)(E(|w|^2))^{\frac{1}{2}} \) on \( S(E(u)) \).

**Proof.** (a) For every \( f \in L^2(\Sigma) \) by Theorem 3.3 and 3.4 we have

\[
|\hat{T}(f)| = |(\hat{T})^*(f) = |E(uw)| \chi_S(E(|u|^2)) \bar{u} E(uf),
\]

where \( S = S(E(|u|^2)) \).
So, $T$ is weakly hyponormal if and only if $|T| = |\hat{T}|$. For every $f \in L^2(\Sigma)$,

$$\langle |T|(f) - |\hat{T}|(f), f \rangle = \int_X \left( \frac{E(|u|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S u f E(u f) - |E(u w)| \chi_S E(|u|^2) \overline{\chi} u f E(u f) d\mu$$

$$\int_X \left( \frac{E(|u|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S |E(u f)|^2 - |E(u w)| \chi_S E(|u|^2) \overline{\chi} |E(u f)|^2 d\mu,$$

this implies that if $|E(u w)| = E(|u|^2)(E(|u|^2))^{\frac{1}{2}}$ on $S$, then $|T| = |\hat{T}|$.

(b) If $|T| = |\hat{T}|$, then for all $f \in L^2(\Sigma)$ we have

$$\int_X \left( \frac{E(|u|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S |E(u)|^2 - |E(u w)| \chi_S E(|u|^2) \overline{\chi} |E(u)|^2 d\mu = 0.$$ 

Let $A \in \mathcal{A}$, with $0 < \mu(A) < \infty$. By replacing $f$ to $\chi_A$, we have

$$\int_A \left( \frac{E(|u|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S |E(u)|^2 - |E(u w)| \chi_S E(|u|^2) \overline{\chi} |E(u)|^2 d\mu = 0.$$ 

Since $A \subset \mathcal{A}$ is arbitrary, then

$$\left( \frac{E(|u|^2)}{E(|u|^2)} \right)^{\frac{1}{2}} \chi_S |E(u)|^2 - |E(u w)| \chi_S E(|u|^2) \overline{\chi} |E(u)|^2 = 0.$$ 

Hence $|E(u w)| = E(|u|^2)(E(|u|^2))^{\frac{1}{2}}$ on $S(E(u))$. □

**Corollary 3.13.**

(a) If $T = EM_u$ and $S(E(u)) = S(E(|u|^2))$, then $T$ is weakly hyponormal if and only if $|E(u)| = E(|u|^2)$ on $S(E(u))$.

(b) If $T = M_w E$, then $T$ is weakly hyponormal if and only if $w \in L^\infty(\mathcal{A})$.

### 4. Some Applications

From now on, we shall denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_{jp}(T)$, $\sigma_{ap}(T)$, $r(T)$ the spectrum of $T$, the point spectrum of $T$, the approximate point spectrum, the joint point spectrum of $T$, the spectral radius of $T$, respectively. The spectrum of an operator $T$ is the set

$$\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible} \}.$$ 

A complex number $\lambda \in \mathbb{C}$ is said to be in the point spectrum $\sigma_p(T)$ of the operator $T$, if there is a unit vector $x$ satisfying $(T - \lambda)x = 0$. If in addition, $(T^* - \lambda)x = 0$, then $\lambda$ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of $T$. The approximate point spectrum of $T$ is the set of those $\lambda$ such that $T - \lambda I$ is not bounded below. Also, the spectral radius of $T$ is defined by $r(T) = \sup\{ |\lambda| : \lambda \in \sigma(T) \}$.

If $A, B \in \mathcal{B}(\mathcal{H})$, then it is well known that

$$\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\},$$

$$\sigma_p(AB) \setminus \{0\} = \sigma_p(BA) \setminus \{0\},$$
\[ \sigma_{ap}(AB) \setminus \{0\} = \sigma_{ap}(BA) \setminus \{0\}, \]
\[ \sigma_{jp}(AB) \setminus \{0\} = \sigma_{jp}(BA) \setminus \{0\}. \]

J. Herron showed that if \( EM_u : \mathcal{L}^2(\Sigma) \to \mathcal{L}^2(\Sigma) \), then \( \sigma(EM_u) = \text{ess range}(E(u)) \cup \{0\} \). So we conclude that
\[ \sigma(M_u EM_u) \setminus \{0\} = \text{ess range}(E(uw)) \cup \{0\}. \]

Let \( A_\lambda = \{ x \in X : E(u)(x) = \lambda \} \), for \( 0 \neq \lambda \in \mathbb{C} \). Suppose that \( \mu(A_\lambda) > 0 \). Since \( \mathcal{A} \) is \( \sigma \)-finite, there exists an \( \mathcal{A} \)-measurable subset \( B \) of \( A_\lambda \) such that \( 0 < \mu(B) < \infty \), and \( f = \chi_B \in \mathcal{L}^p(\mathcal{A}) \subseteq \mathcal{L}^p(\Sigma) \). Now
\[ EM_u(f) - \lambda f = E(u)\chi_B - \lambda \chi_B = 0. \]
This implies that \( \lambda \in P_T(EM_u) \).

If there exists \( f \in \mathcal{L}^p(\Sigma) \) such that \( f\chi_C \neq 0 \mu \text{-a.e.} \), for \( C \in \Sigma \) of positive measure and \( E(uf) = \lambda f \) for \( 0 \neq \lambda \in \mathbb{C} \), then \( f = \frac{E(u)}{\lambda} \), which means that \( f \) is \( \mathcal{A} \)-measurable. Therefore \( E(uf) = E(u)f = \lambda f \) and \( (E(u) - \lambda)f = 0 \). This implies that \( C \subseteq A_\lambda \) and so \( \mu(A_\lambda) > 0 \). Hence
\[ \sigma_p(EM_u) = \{ \lambda \in \mathbb{C} : \mu(A_\lambda) > 0 \}. \]
Thus
\[ \sigma_p(M_u EM_u) \setminus \{0\} = \{ \lambda \in \mathbb{C} : \mu(A_{\lambda,w}) > 0 \} \setminus \{0\}, \]
where \( A_{\lambda,w} = \{ x \in X : E(uw)(x) = \lambda \} \).

For each natural number \( n \), we define
\[ \triangle_n(T) = \frac{\triangle_{n-1}T}{\triangle_1(T)} \quad \triangle_1(T) = \theta(T) = \hat{T}. \]
We call \( \triangle_n(T) \) the \( n \)-th Aluthge transformation of \( T \). It is proved that \( r(T) = \lim_{n \to \infty} \|\triangle_n(T)\| \) in [14].

**Theorem 4.1.** Let \( T = M_u EM_u \). Then
(a) \( \hat{T} \) is normaloid.
(b) \( T \) is normaloid if and only if
\[ \|E(uw)\|_\infty = \|(E(|u|^2))^{1/2}(E(|w|^2))^{1/2}\|_\infty. \]

**Proof.** (a) By Theorem 3.1 we have \( \|\hat{T}\| = \|E(uw)\|_\infty \). By Theorem 3.4 we conclude that for every natural number \( n \) we have \( \triangle_n(T) = \triangle(T) = \hat{T} \). Hence \( r(\hat{T}) = r(T) = \|\hat{T}\| = \|E(uw)\|_\infty \). So \( \hat{T} \) is normaloid.

(b) By conditional type Holder inequality, boundedness of \( T \) and Theorem 3.1 we have
\[ r(T) = \|E(uw)\|_\infty \leq \|(E(|u|^2))^{1/2}(E(|w|^2))^{1/2}\|_\infty = \|T\|. \]
Hence \( T \) is normaloid if and only if
\[ \|E(uw)\|_\infty = \|(E(|u|^2))^{1/2}(E(|w|^2))^{1/2}\|_\infty. \]
Theorem 4.2. Let $T = M_wEM_u$ be weakly hyponormal with $ker T \subset ker T^*$, then $T = \hat{T}$.

Proof. Direct computations show that $\hat{T}$ is normal. So, by Theorem 2.6 of [1] we have $T = \hat{T}$.

Theorem 4.3. If $|E(uw)| = E(|u|^2)(E(|w|^2))^{\frac{1}{2}}$ on $S = S(E(|u|^2))$, then

(a) $\sigma_{fp}(M_wEM_u) \setminus \{0\} = \sigma_p(M_wEM_u) \setminus \{0\} = \{\lambda \in \mathbb{C} : \mu(A_{\lambda,w}) > 0\} \setminus \{0\}$.

(b) $\sigma(M_wEM_u) \setminus \{0\} = \{\lambda : \bar{\lambda} \in \sigma_{ap}(M_{\bar{w}}EM_{\bar{w}})\} \setminus \{0\}$.

(c) If $\lambda$ is an isolated point in $\sigma(M_wEM_u)$, then $\lambda \in \sigma_p(M_wEM_u)$.

Proof. By using Theorem 3.12 and Theorems 3.2 and 3.7 of [1] we conclude (a), (b) and (c).

Corollary 4.4. If $|E(uw)| = E(|u|^2)(E(|w|^2))^{\frac{1}{2}}$ on $S = S(E(|u|^2))$, then

$$\sigma_{ap}(M_wEM_u) \setminus \{0\} = \sigma(M_wEM_u) \setminus \{0\} = ess\ range(E(uw)) \setminus \{0\}.$$
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