Monte Carlo Evaluation of Non-Abelian Statistics

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(Dated: November 9, 2018)

PACS numbers: 73.43.-f, 05.30.-d

We develop a general framework to (numerically) study adiabatic braiding of quasiholes in fractional quantum Hall systems. Specifically, we investigate the Moore-Read (MR) state at $\nu = 1/2$ filling factor, a known candidate for non-Abelian statistics, which appears to actually occur in nature. The non-Abelian statistics of MR quasiholes is demonstrated explicitly for the first time, confirming the results predicted by conformal field theories.

The quantum statistics of a system of identical particles describe the effect of adiabatic particle interchange on the many-body wave function. All fundamental particles belong to one of two classes: those that have their wave function unaffected by particle interchange (bosons) and those whose wave function gets a minus sign under permutation (fermions). In two dimensions, it is known that a number of exotic types of statistics can exist for particle-like collective excitations. For example, elementary excitations of the Laughlin fractional quantum Hall (FQH) states exhibit “fractional” statistics: The phase of the wave function is rotated by an odd fraction of $\pi$ when two Laughlin quasiparticles (or quasiholes) are interchanged \cite{1,2}. Even more exotic statistics can exist when a system with several excitations fixed at given positions is degenerate \cite{3}. In such a case, adiabatic interchange (braiding) of excitations can nontrivially rotate the wave function within the degenerate space. In general, these braiding operations need not commute, hence the statistics are termed “non-Abelian”. Remarkably, the Moore-Read (MR) state, a state which is commonly believed \cite{4} to describe observed FQH plateaus at $\nu = 5/2$ and $7/2$ (which correspond respectively to half filling of electrons or holes in the first excited Landau level), is thought to have such non-Abelian elementary excitations \cite{5}. Other possible physical realizations of non-Abelian statistics have also been proposed \cite{6}. States of this type have been suggested to be attractive for quantum computation \cite{7}.

In Ref. \cite{2}, in order to establish the nature of the statistics of the Laughlin quasiholes, a Berry’s phase calculation was performed that explicitly kept track of the wavefunction phase as one quasihole was transported around the other. Although approximations were involved in this calculation, it nonetheless established quite convincingly the fractional nature of the statistics. Unfortunately, it has not been possible to generalize this calculation to explicitly investigate statistics of the MR quasiholes \cite{5}. Although there has been much study of the statistics of the MR quasiholes in the framework of conformal field theories (CFT), it would be desirable to perform a direct calculation analogous to that of Ref. \cite{2}. The purpose of this paper is to provide such a calculation, albeit numerically. Furthermore, the approach developed here is readily applicable to other FQH systems which are not easily accessible to analytic investigations.

The evolution operator of a many-body system described by a Hamiltonian $H(\lambda)$ is in principle determined by the Schrödinger equation. In general, $H(\lambda)$ itself can change in time through dependence on some varying parameter $\lambda(t)$. In such a case, let us define $\varphi_i(t)$ at a given time to be an orthonormal basis for a particular degenerate subspace, requiring that this basis is locally smooth as a function of $t$. If $\lambda$ is varied adiabatically (and so long as the subspace does not cross any other states), then the time-evolution operator maps an orthonormal basis of the subspace at one time onto an orthonormal basis at another. A solution of the Schrödinger equation, $\psi_i(t) = U_{ij}(t)\varphi_j(t)$, is simply given by

$$ (U^{-1}\dot{U})_{ij} = \langle \varphi_i | \dot{\varphi}_j \rangle \equiv A_{ij}(t). $$

Since the matrix $A$ is anti-Hermitian, $U(t)$ is guaranteed to be unitary if its initial value $U(0)$ is unitary. Note that if we vary $\lambda$ so that the Hamiltonian returns to its initial value at time $t$, i.e., $H(\lambda(t)) = H(\lambda(0))$, the corresponding transformation of the degenerate subspace can be nontrivial, i.e., $\psi_i(t) \neq \psi_i(0)$.

We explicitly demonstrate that this is the case for the MR state with at least four quasiholes. The analysis is done in spherical geometry \cite{5}: $N$ electrons are positioned on a sphere of unit radius, with their coordinates given by $(u_1, v_1), \ldots, (u_N, v_N)$, using the spinor notation (i.e., $u = e^{i\phi/2} \cos \theta/2$ and $v = e^{-i\phi/2} \sin \theta/2$ in terms of the usual spherical coordinates). A monopole of charge $2S = 2N + n - 3$ in units of the flux quanta $\Phi_0 = hc/e$ is placed in the center of the sphere, giving rise to $2n$ quasiholes which are put at $(\tilde{u}_1, \tilde{v}_1), \ldots, (\tilde{u}_{2n}, \tilde{v}_{2n})$. Using gauge $\tilde{A} = (\Phi_0 S/2\pi) \hat{\phi} \cot \theta$, the MR wave function \cite{5} is then given by

$$ \psi_{\text{pf}} = \text{Pf} \Lambda_{ij}^{(a, b, \ldots)(\alpha, \beta, \ldots)} \prod_{i<j} (u_i v_j - v_i u_j)^2, $$

where $\text{Pf} \Lambda_{ij}^{(a, b, \ldots)(\alpha, \beta, \ldots)}$ is the Pfaffian \cite{8} of the $N \times N$
antisymmetric matrix \[ A_{ij}^{(a,b,...)(\alpha,\beta,...)} = (u_i v_j - v_i u_j)^{-1} \times \]
\[
\left( (u_i v_a - v_i u_a)(u_j v_b - v_j u_b) \times (u_i v_b - v_i u_b)(u_j v_a - v_j u_a) \times \cdots \times (i \leftrightarrow j) \right).\n\]
Pfaffian wave functions \[ 2 \] were first constructed in Ref. \[ 3 \] as CFT conformal blocks. This MR state is the exact ground state for a special three-body Hamiltonian \[ 10 \] and is also thought to pertain for realistic two-body interactions in the first excited Landau level \[ 4 \]. The presence of quasiholes in the ground state is dictated by the incommensuration of the flux with the electron number. Physically, the MR state can be thought of as p-wave BCS pairing of composite fermions (CF’s) at zero net field with quasiholes being the vortex excitations \[ 3, 11, 12 \]. Each quasihole has charge \( 1/2 \) and corresponds to half a quantum of flux (because of the paired order parameter \[ 3 \]). Eq. (2) describes a space with degeneracy \( 2^{n-1} \).\n
Consider an orthonormal basis \( \varphi_i \), with \( i = 1, \ldots, 2^{n-1} \), for the subspace with \( 2n \) quasiholes, which is locally smooth when parameterized by the quasihole coordinates. In order to determine the braiding statistics, we find the transformation \( \varphi_i \to U_{ij} \varphi_j \) under the evolution operator after two of the quasiholes are interchanged while the others are held fixed. The unitary matrix \( U_{ij} \) is obtained by first solving Eq. (1) and then projecting the final basis onto the initial one. (Since we require \( \varphi_i \) to be only locally smooth, the basis itself can nontrivially rotate after the quasiholes return to their original positions). Eq. (1) is integrated numerically: The differential equation is discretized and the wave-function overlaps (the right-hand side of the equation) are evaluated using the Metropolis Monte Carlo method. The computational errors are easily evaluated by varying the number of operations. We aim the calculation at addressing the following questions: (1) What is the Berry’s phase accumulated upon quasihole interchange due to the enclosed magnetic flux and due to the relative statistics? (2) What is the transformation matrix for the ground-state subspace corresponding to the braiding operations? In the following, we will first describe the numerical method, then present the results, and compare them to CFT predictions \[ 2, 13 \].

In order to integrate Eq. (1) numerically, the quasihole interchange is performed in a finite number of steps. If \( U^{(l)} \) is the value of the transformation matrix at the \( l \)th step, then at the next step
\[
U^{(l+1)} = U^{(l)}[1 + A^{(l)}/2][1 - A^{(l)}/2]^{-1},
\]
where \( A^{(l)} = \langle \varphi_i^{(l)} + \varphi_i^{(l+1)} - \varphi_i^{(l)} \rangle/2 \). Our choice of the finite-element scheme \[ 3 \] will become clear later. In practice, in general we do not know an orthonormal basis for the MR states \[ 2 \] in an analytic form, but we can numerically orthonormalize a set of \( 2^{n-1} \) linearly-independent Pfaffian wave functions \( \psi_{\text{Ph}} \). Let \( B_{ij}^{(l)} = [\psi_{\text{Ph}}^{(l)}, \psi_{\text{Ph}}^{(l+1)}] \) denote the normalized overlaps of different states. (It is implied here and throughout the paper that \( [\psi_{\text{Ph}}^{(k)}, \psi_{\text{Ph}}^{(l)}] = [\psi_{\text{Ph}}^{(k)}|\psi_{\text{Ph}}^{(l)}]/\|\psi_{\text{Ph}}^{(k)}\|\|\psi_{\text{Ph}}^{(l)}\| \) is evaluated numerically.) We then easily show that
\[
A^{(l)} = [V^{(l)}]^\dagger V^{(l+1)}[V^{(l+1)}]/2 - \text{H.c.},
\]
where \( W^{(l)}_{ij} = [\psi_{\text{Ph}}^{(l)}, \psi_{\text{Ph}}^{(l+1)}] \) and \( V^{(l)} \) is defined by
\[
|V^{(l)}|B^{(l)}V^{(l)} = \mathbb{1},
\]
constructing an orthonormal basis \( \varphi_i^{(l)} = V_{ji}^{(l)} \psi_{\text{Ph}}^{(l)} \). We require \( V^{(l)} \) to be locally smooth as a function of the quasihole coordinates: The basis can continuously transform while the quasiholes are moved, but, e.g., sudden sign flips are not allowed.

According to Eq. (4), \( A^{(l)} \) is anti-Hermitian, so that the transformation \( U^{(l+1)} \) is guaranteed to be unitary if \( U^{(l)} \) is unitary. This explains our choice \[ 3 \] for discretizing Eq. (1). Another feature preserved by our numerical scheme is that making a step forward, \( \psi_{\text{Ph}}^{(l)} \to \psi_{\text{Ph}}^{(l+1)} \), followed by a step backward, \( \psi_{\text{Ph}}^{(l+1)} \to \psi_{\text{Ph}}^{(l)} \), results in a trivial transformation. We start at \( U^{(0)} = \mathbb{1} \) and find \( U^{(n_s)} \) after performing \( n_s + 1 \) steps for braiding of two quasiholes (\( n_s \) is increased to convergence). Because \( \psi_{\text{Ph}}^{(n_s)} \) is some nontrivial linear combination of \( \psi_{\text{Ph}} \), we, finally, have to project the transformation onto the initial basis: \( U^{(n_s)} \to U^{(n_s)}O^T \), where \( O = [V^{(0)}]^\dagger \Omega V^{(n_s)} \) and \( \Omega_{ij} = [\psi_{\text{Ph}}^{(0)}, \psi_{\text{Ph}}^{(n_s)}] \). The resulting unitary transformation matrix \( U \) then gives a representation of the braid group for quasihole interchanges. In the following, we describe our numerical experiments.

The space describing \( 2n = 2 \) MR quasiholes is nondegenerate, so non-Abelian statistics cannot occur. There is, nevertheless, a Berry’s phase accumulated from wrapping these quasiholes around each other. Our calculation of this phase for the MR state is analogous to the one performed in Ref. \[ 2 \] for the Laughlin state, except that our calculation is numerical and therefore requires no mean-field approximation. Let us first briefly recall results for the Laughlin wave function at filling factor \( \nu = 1/p \). In the disk geometry, the Berry’s phase \( \chi \) corresponding to taking a single quasihole around a loop is given by \( 2\pi \) for each enclosed electron, i.e., \( \chi = 2\pi \langle N \rangle \), where \( \langle N \rangle \) is the expectation number of enclosed electrons \[ 2 \]. Therefore, when another quasihole is moved.
inside the loop, the phase $\chi$ drops by $2\pi/p$ which implies fractional statistics of the quasiholes. In spherical geometry \[8\], the same result holds unless the south and north poles (which have singularities in our choice of gauge) are located on different sides of the loop. In the latter case, the Berry’s phase is given by $\chi = \pi(N_{\text{in}} - N_{\text{out}})$, where $N_{\text{in(out)}}$ is the number of electron inside (outside) the loop. If a single Laughlin quasihole is then looped around the equator, its Berry’s phase vanishes, but if another quasihole is placed above or below, the phase becomes $\chi = \pm \pi/p$. We check our Monte Carlo method by reproducing these results numerically. The charge of the MR ($\nu = 1/2$) quasihole is $e/4$, so that by analogy with the Laughlin state one might naively expect that the Berry’s phase for looping one quasihole around the equator with another fixed above or below it is given by the Berry’s phase for looping one quasihole around the equator (north pole), for various $N$ as in Fig. 1. The straight dashed lines in the upper half are $0.5(N + 1/4)(1 - \cos \xi)$, corresponding to the expectation of the number of electrons enclosed by the loop. The $+1/4$ accounts for the charge pushed out by one of the quasiholes. For $\chi < 0$, the dashed lines are $-0.5(N + 1/4) \cos \xi$, i.e., one half of the number of electrons inside minus one half the number outside the loop. Open symbols, corresponding to a similar calculation with one quasihole moving and the other fixed at the center of the circle, almost overlay the filled symbols, confirming the trivial relative statistics.

FIG. 2: For $\chi > 0$ ($\chi < 0$) filled symbols show the phase accumulated by interchanging two quasiholes around a circle with opening angle $\xi$ centered on the equator (north pole), for various $N$ as in Fig. 1. The straight dashed lines in the upper half are $0.5(N + 1/4)(1 - \cos \xi)$, corresponding to the expectation of the number of electrons enclosed by the loop. The $+1/4$ accounts for the charge pushed out by one of the quasiholes. For $\chi < 0$, the dashed lines are $-0.5(N + 1/4) \cos \xi$, i.e., one half of the number of electrons inside minus one half the number outside the loop. Open symbols, corresponding to a similar calculation with one quasihole moving and the other fixed at the center of the circle, almost overlay the filled symbols, confirming the trivial relative statistics.

We now turn to $2n = 4$ MR quasiholes, which is the simplest case when statistics can be non-Abelian (the ground state has degeneracy 2). While the above results for 2 quasiholes are anticipated by the plasma analogy \[10\], one may need deeper CFT \[8\] \[12\] arguments in order to understand the following findings. In the calculation, we first fix all quasiholes on the equator and then interchange an adjacent pair of them around a circle with different opening angles $\xi$ centered on the equator. Parameterizing a unitary matrix $U$ by

$$U = e^{i\chi} \begin{pmatrix} e^{i\eta} \cos \beta/2 & ie^{-i\eta/2} \sin \beta/2 \\ ie^{i\eta/2} \sin \beta/2 & e^{-i\eta} \cos \beta/2 \end{pmatrix},$$

we plot in Figs. 3 and 4 the results (in a convenient basis) for the transformation $U_1$ corresponding to the braiding
Furthermore, we find numerically that $U$ and transforming interchanges of pairs ordered along the equator) by rotating and projecting the initial basis and correspondingly transforming $U_1$. It is then easy to show that $U_1 = U_3$ and $U_2 = U_4$ due to the form $\chi$ of the wave function. Furthermore, we find numerically that $U_2 \approx F^0 U_1 F$, where $F = (\sigma_z - \sigma_x)/\sqrt{2}$, $\sigma$'s being the usual Pauli matrices. This approximation is good within a few percent for smaller systems and is even better for larger ones.

According to Fig. 1, we see that apart from the Abelian phase $\chi$, $U_1$ can be approximated by $U_1 \approx \text{diag}(1 + i, 1 - i)/\sqrt{2}$, with the disagreement becoming smaller for larger systems. Using $F$, we can then construct all other matrices $U_i$. After performing the above approximations, we find that the unitary transformations corresponding to the braid operators realize the right-handed spinor representation of $SO(2n) \times U(1)$ (restricted to $\pi/2$ rotations around the axes) as predicted in Ref. 13 using CFT. In addition to the usual relations required of a representation of the braid group on the plane, on the sphere the generators must obey an additional relation. For the case of $2n = 4$, for example, we expect to have $U_1 U_2 U_3 U_2 U_1 = 1$. One can easily show that (for general $n$) the relevant representation of the braid group predicted in Ref. 13 satisfies this additional relationship up to an Abelian phase. (The failure of the Abelian phase to satisfy this law is related to the gauge singularities, and will be discussed elsewhere.)

In summary, we formulated a numerical method to study braiding statistics of FQH excitations and applied it to perform the first direct calculation of the non-Abelian statistics in the MR state. Our findings confirm results previously drawn within the CFT framework.

We have enjoyed helpful discussions with B. I. Halperin, C. Nayak, F. von Oppen, and N. Read. This work was supported in part by NSF Grant DMR 99-81283. The computations were performed on the O2000 supercomputer CPU farm at Boston University.

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