Functional renormalization group approach to conventional theory of superfluidity and beyond

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Abstract. Fermionic functional renormalization group (FRG) is applied to describe the superfluid phase transition of the two-component fermionic system with attractive contact interaction. Connection between the fermionic FRG approach and the Bardeen-Cooper-Schrieffer (BCS) theory with its Gorkov and Melik-Barkhudarov (GMB) correction is made clear, and the FRG flow of the fermion self-energy is also studied to go beyond the BCS+GMB theory. The superfluid transition temperature and the associated chemical potential are calculated in the region of the negative scattering length using fermionic FRG.

1. Introduction
Superfluidity in many-fermion systems is one of the central problems in condensed matter, atomic, nuclear and particle physics. Examples include liquid superfluid 3He, electron superconductivity, cold atoms, nucleon superfluidity, color superconductivity, etc. [1]. We report our recent study on the application of fermionic functional renormalization group (f-FRG) method to the superfluid phase transition of two-component fermionic systems [2]. The purpose of this study is to develop a series of approximations without introducing the auxiliary bosonic field to make a firm connection between the non-perturbative FRG approach and the conventional Bardeen-Cooper-Schrieffer (BCS) theory with its Gorkov and Melik-Barkhudarov (GMB) correction. Furthermore, we wish to explore the role of the RG flow of the fermion self-energy to go beyond BCS+GMB theory. We note that such analyses can be best achieved by fermionic FRG without bosonization, because f-FRG can provide a systematic and unbiased study of interacting fermions [3, 4].

2. Fermionic FRG Formalism
We consider non-relativistic two-component fermions $\psi = (\psi_\uparrow, \psi_\downarrow)$ with a contact interaction:

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau \int d^3x \left[ \bar{\psi} \left( \partial_{\tau} - \frac{\nabla^2}{2m} - \mu \right) \psi + g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\uparrow \psi_\downarrow \right],$$

where $\beta(=1/T)$, $\mu$, $m$ and $g$ are the inverse temperature, the chemical potential, the mass and the bare coupling constant, respectively. The classical action [1] is symmetric under global $U(1)$ phase rotation, $SU(2)$ spin rotation, and space-time translation.
\[
\partial_k \begin{array}{c}
\psi_R
\end{array} = \partial_k \begin{array}{c}
\psi_R
\end{array} + \partial_k \begin{array}{c}
\tilde{\psi}_R
\end{array} = \tilde{\partial}_k \begin{array}{c}
\tilde{\psi}_R
\end{array} + \text{terms involving self-energies and four-point vertices}
\]

**Figure 1.** Flow equation of the self energy \(\Sigma_k\) and the four-point vertex \(\Gamma_k^{(4)}\).

Following the idea of the FRG method \[5\], we define a scale-dependent average effective action \(\Gamma_k[\tilde{\psi}, \psi]\) as a one-particle-irreducible (1PI) effective action of \(S[\tilde{\psi}, \psi] + \tilde{\psi}R_k\psi\), where \(R_k\) is an infrared (IR) regulating function. We consider the following Ansatz of \(\Gamma_k\):

\[
\Gamma_k[\tilde{\psi}, \psi] = \int_p^{(T)} \tilde{\psi}(p) (G^{-1}(p) - \Sigma_k(p)) \psi(p) + \int_{p,q,q'}^{(T)} \Gamma_k^{(4)}(p; q, q') \tilde{\psi}_\uparrow(p/2 + q) \tilde{\psi}_\downarrow(p/2 - q) \psi_\uparrow(p/2 - q') \psi_\downarrow(p/2 + q'),
\]

where \(G^{-1}(p) = ip^0 + p^2/2m - \mu\) is the inverse propagator with \(p = (p^0, \mathbf{p})\), \(\Sigma_k(p)\) is the self-energy, and \(\Gamma_k^{(4)}(p; q, q')\) is the four-point vertex. Also, we adopt an abbreviated notation, \(\int_p^{(T)} \equiv \int \frac{d^3p}{(2\pi)^3} T \sum_p\). In this study, we neglect the momentum dependence of the self-energy and define its constant part by \(\sigma_k \equiv \Re \Sigma_k(\pm \pi T, 0)\). IR regulator \(R_k(p)\) is chosen to suppress one-particle excitations around the Fermi surface with the excitation energy smaller than \(k^2/2m\):

\[
R_k(p) = \left[ \frac{k^2}{2m} \text{sgn}(\xi(p)) - \xi(p) \right] \theta\left( \frac{k^2}{2m} - |\xi(p)| \right) \text{ with } \xi(p) = p^2/2m - \mu - \sigma_0.
\]

\(\Gamma_k\) obeys the flow equation

\[
\partial_k \Gamma_k = \frac{1}{2} \int_p^{(T)} \text{Tr} \left[ \frac{\partial_k R_k(p)}{\Gamma_k^{(2)}(p) + R_k(p)} \right].
\]

Our primary goal is to calculate the ratios \(T_c/\varepsilon_F\) and \(\mu/\varepsilon_F\) as a function of the dimensionless constant \(1/(k_F a_s)\): Here \(a_s\) is the scattering length between fermions, \(k_F \equiv (3\pi^2 n)^{1/3}\) with \(n\) being the fermion number density and \(\varepsilon_F \equiv k_F^2/2m\). For a second-order phase transition, the critical point \(T = T_c\) can be determined from the Thouless criterion \[6\] by looking at the divergence of the fermion-fermion scattering matrix at the total momentum \(p = 0\):

\[
\left[ \Gamma_{k=0}^{(4)}(p = 0) \right]^{-1} = 0 \quad \text{at } T = T_c.
\]

Note that the number density \(n\) is related to \(T\) and \(\mu\) through the number equation:

\[
n = \langle \bar{\psi} \psi \rangle = 2 \int_p^{(T)} \frac{-1}{G^{-1}(p) - \sigma_0}. \quad (5)
\]

3. Flow equation of f-FRG for self-energy and four-point vertex

Taking the vertex expansion \[2\] of the scale dependent 1PI effective action \(\Gamma_k[\tilde{\psi}, \psi]\), a closed set of equations for the self-energy and four-point vertex can be derived from \[3\]. They are given diagrammatically in Fig. 1. In order to simplify the flow equation of four-point vertex, we neglect its momentum dependence and regard it as an effective coupling constant. At the lowest momenta, our flow equations can be approximated as

\[
\partial_k \sigma_k = \Re \tilde{\partial}_k \int_l^{(T)} \frac{\Gamma_k^{(4)}(\pm \pi T + t^0, l)}{G^{-1}(l) - \sigma_k + R_k(l)}.
\]
are shown in Fig.3. Fig.4 and Fig.5 show our numerical results of the critical temperature (the self-energy correction is neglected. The flows of the self-energy for (this actually coincides with the usual Random Phase Approximation). After applying the

\[ T \]  

is nothing but the GMB correction of the critical temperature:  

\[ T = \exp\left(\frac{1}{4} - \frac{1}{4} \frac{\pi^2}{3}\right) \approx 0.589 T \]

By taking into account the PH loop contribution (evaluating it at zero temperature) and still neglecting the self-energy, the transition temperature is reduced by a factor \( 4e_0^2 \exp(\pi/2k_Fa_s) \approx 2.2 \). This is nothing but the GMB correction of the critical temperature:  

\[ T_c^{\text{GMB}} = T_c^{\text{BCS}/2.2} \]

On Fig.2 the flow of \( -\partial_k/\Gamma_k^{(4)}(0) \) is plotted with and without particle-hole (PH) loop when the self-energy correction is neglected. The flows of the self-energy for \( (k_Fa_s)^{-1} = -2, -1, 0 \) at \( T_c \) are shown in Fig.3, Fig.4 and Fig.5 show our numerical results of the critical temperature \( T_c^{\text{BCS}}/\varepsilon_F \) and the chemical potential \( (\mu/\varepsilon_F) \) as a function of the inverse scattering length \( (k_Fa_s)^{-1} < 0)\),
intermediate energy scales the BCS theory is weakened due to the existence of matter, and such screening occurs around \( k \) only around the region \( T \) becomes stronger towards the unitary regime, but the effect is rather limited because \( \sigma_k \) on the other hand, it almost stops for small \( k \): This is due to the fact that the approximate particle-hole symmetry valid for small \( k \) protects the shift of the Fermi level. Fig. 3 shows that the magnitude of the self-energy \( \sigma_0 \) becomes larger as the coupling becomes strong, while the saturation of \( \sigma_k \) at \( k \sim k_F \) takes place for any coupling strength.

As can be seen from Fig. 4, PH correlation is the dominant source of the reduction of \( T_c/\varepsilon_F \), and, as already discussed, its physical origin is the screening of the coupling strength. We should note that this does not imply the smallness of the self-energy correction. Since the flow of self-energy stops for \( k \lesssim k_F \), it does not affect low-energy dynamics after shifting the Fermi level away from the chemical potential. The self-energy correction still leads to further small reduction of \( T_c/\varepsilon_F \), since it makes \( \varepsilon_F \) (or the number density \( n \)) bigger, as the coupling strength becomes stronger towards the unitary regime, but the effect is rather limited because \( \sigma_0 - \sigma_k \) around \( k \sim k_F \) cannot be large due to the particle-hole symmetry, as discussed above.

Fig. 3 shows that, the effects of the PH correlation to \( \mu/\varepsilon_F \) are opposite of what we find for \( T_c/\varepsilon_F \). This is because the critical temperature \( T_c \) of PP approximation is higher than that of the PP+PH result for the same chemical potential \( \mu \). Therefore, the number of fermions in the PP case is larger than that of PP+PH. When the self-energy correction is taken into account, the number density \( n \) and hence the Fermi energy \( \varepsilon_F \) increase for given \( \mu \), therefore a considerable decrease of \( \mu/\varepsilon_F \) arises.

Although it is beyond our scope to predict a quantitatively correct \( T_c \) in the unitary regime, it is still instructive to compare our result at unitarity, \( (T_c/\varepsilon_F, \mu/\varepsilon_F) = (0.237, 0.367) \), with the previous FRG results using auxiliary bosonic field, \( (T_c/\varepsilon_F, \mu/\varepsilon_F) \simeq (0.248, 0.51) \) [7]. For

Figure 2. Derivative of \( 1/\Gamma_k^{(4)}(0) \) at \( T/\mu = 0.06 \) with and without the PH loop included.

Figure 3. Self-energy \( \sigma_k \) as a function of \( k \) for \((k_F a_s)^{-1} = -2, -1, 0 \) at \( T = T_c \).
5. Summary

We developed a fermionic FRG to describe the superfluid phase transition for two-component fermions with a contact interaction. By making vertex expansion of the 1PI effective action $\Gamma_k[\bar{\psi}, \psi]$ up to the four-point vertex and solving RG flow equations, we determined the critical temperature $T_c$ in the regime of negative scattering lengths. In order to clarify the relation between the FRG approach and the BCS+GMB theory and to go beyond, we have taken into account the PP and PH correlations together with the self-energy correction, and the role of each is analyzed in details in their respective flow equation.

Resultant value of $T_c/\varepsilon_F$ does not receive large correction from the self-energy except for the unitary regime ($1/(k_Fa_s) \to 0$). On the other hand, $\mu/\varepsilon_F$ shows a large reduction by the self-energy correction due to the increase of the Fermi energy given $\mu$. It shows that the self-energy correction is still comparable with the chemical potential even for $(k_Fa_s)^{-1} \lesssim -1$, however, its effect on $T_c/\varepsilon_F$ is almost negligible. Extrapolation to unitarity gives $T_c/\varepsilon_F = 0.237$ and $\mu/\varepsilon_F = 0.367$.

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