NONLINEAR METASTABILITY FOR A PARABOLIC SYSTEM OF REACTION-DIFFUSION EQUATIONS

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Abstract. We consider a system of reaction-diffusion equations in a bounded interval of the real line, with emphasis on the metastable dynamics, whereby the time-dependent solution approaches its steady state in an asymptotically exponentially long time interval as the viscosity coefficient $\varepsilon > 0$ goes to zero. To rigorously describe such behavior, we analyze the dynamics of solutions in a neighborhood of a one-parameter family of approximate steady states, and we derive an ODE for the position of the internal interfaces.

Key words: Metastability, slow motion, internal layers, reaction-diffusion systems.

AMS subject classifications: 35B36, 35B40, 35K45, 35K57

1. Introduction

In this paper we study the metastable dynamics of solutions for a class of parabolic system of the form

$$
\partial_t u = F^\varepsilon[u],
$$

where $F^\varepsilon$ is a nonlinear differential operator of reaction-diffusion type that depends singularly on the parameter $\varepsilon$, meaning that $F^0[u]$ is of lower order. The unknown $u$ belongs to the space $[L^2(I)]^n$ for some bounded interval $I \subset \mathbb{R}$. System (1.1) is complemented with initial datum $u_0(x) \in [L^2(I)]^n$ and appropriate boundary conditions.

Roughly speaking, a metastable dynamics appears when the time dependent solutions of an evolutive PDE exhibit a first transient phase where they are close to some non-stationary state before converging, in an extremely long time, to their asymptotic limits, namely the solution to $F^\varepsilon[u] = 0$. In other words, a dynamics is said to be metastable if there exists a first time scale of order $O(1)$ in time where a pattern of internal interfaces is formed; once this structure is formed, it persists for an exponentially long time interval, whose size usually depends on the parameter $\varepsilon$. Last step of the dynamics is an (exponentially slow) motion of the solutions towards their asymptotic configuration.

As a consequence, two different time scales emerge: a first transient time phase of order $O(1)$, where the solutions develop a layered structure, and a subsequent long time phase characterizes by the slow convergence of such solutions to some stable configuration.

Slow motion of internal interfaces has been investigated for a number of different partial differential equations. To name just some of these results, we recall here [30], a pioneering
article in the study of slow dynamics for viscous scalar conservation laws; here the authors consider the viscous Burgers equation in a bounded interval of the real line, namely
\[
\partial_t u + \varepsilon \partial_x^2 u - \partial_x f(u),
\]
complemented with Dirichlet boundary conditions, proving that the eigenvalues of the linearized operator around an equilibrium configurations are real, negative, and have the following distribution with respect to $\varepsilon$:
\[
\lambda_1^\varepsilon = \mathcal{O}(\varepsilon^{-1/\varepsilon}) \quad \text{and} \quad \lambda_k^\varepsilon < -\frac{c_0}{\varepsilon} < 0 \quad \forall k \geq 2.
\]
The presence of a first eigenvalue small with respect to $\varepsilon$ implies that the convergence towards the equilibrium is very slow, when $\varepsilon$ is small: indeed, the large time behavior of solutions is described by terms of order $e^{\lambda_1^\varepsilon t}$.

Starting from this result, there are several papers concerning slow motion of internal shock layer for viscous conservation laws, see for example [32], [33], [42] and [34]. Slow dynamics of interfaces has been examined also for convection-reaction-diffusion equations (see [11] and [45]), for relaxation system, with the contribution [44], and for phase transition problem described by the Cahn-Hilliard equation in [2, 41].

In this paper we mean to analyze the slow motion of internal interfaces for a system of reaction diffusion equations. Depending on the assumptions on the system, reaction-diffusion type systems can be used to describe different models in mathematics and physics: the principle areas of application, together with a few of the many possible references, are the following: neurophysics and biophysics [29, 39], chemical physics [24, 38], phase changes [14], population genetics [9] and mathematical ecology [35]. Most of the above applications deal with a system of equations, one of which is a reaction diffusion equation for the unknown $u$, with the source term depending on $u$ and another variable $v$, that satisfies a different equation coupled to $u$.

The main example we have in mind in this paper is the case of a single equation, that is the one dimensional Allen-Cahn equation
\[
(1.2) \quad \partial_t u = \varepsilon \partial_x^2 u - f(u), \quad x \in I,
\]
which is the Allen-Cahn equation (1.2) with appropriate Dirichlet or Neumann boundary conditions and initial datum $u_0(x)$. The function $f : \mathbb{R} \to \mathbb{R}$ is the derivative of a symmetric double well potential $W(u)$ with two equal non-degenerate minima. To fix the ideas, we may assume $W(u) = \frac{(u^2 - 1)^2}{4}$, which has two minima in $u = \pm 1$.

Equation (1.2) was firstly introduced by S.M. Allen and J.W. Cahn in [6] to model the interface motion between different crystalline structures in alloys; in this context $u$ represents the concentration of one of the two components of the alloy and the parameter $\varepsilon$ is the interface width. A possible way to derive the Allen-Cahn equation is to compute the gradient of the Ginzburg-Landau energy functional
\[
(1.3) \quad \mathcal{I}(u) = \int_0^1 \left( W(u) + \frac{1}{2} \varepsilon |\partial_x u|^2 \right) dx
\]
into the space $L^2(I)$, where $W(u)$ is defined as $\partial_u W(u) = f(u)$. This point of view implies that we are assuming the field $u$ to be governed by a gradient equation on the form $\partial_t u = -\delta_u \mathcal{I}$ (see [25]), that is we assume the solutions to the Allen-Cahn equation (1.2) to
evolve according to the $L^2$ gradient flow of the energy functional (1.3). Precisely, there are two different effects: the reaction term $f(u)$ pushes the solution towards the two minima $u = \pm 1$, while the diffusion term $\varepsilon \partial_x^2 u$ tends to regularize and smoothen the solution. In the small viscosity limit, i.e. $\varepsilon \to 0$, two different phases appear, corresponding to regions where the solution is close to $\pm 1$, and the width of the transition layers between these two phases is of order $\varepsilon$. In [21] it is proven that, in the limit $\varepsilon \to 0$, the solutions only attains the values $\pm 1$ and the interface evolves according to the motion by mean curvature.

The steady state solutions to (1.2) have been studied in several papers: for example, [22] for solutions with a single transition layer in the whole line (and [23] for the same problem in a bounded interval with Dirichlet boundary conditions); the problem of the stability of the steady state has been considered in [19] in the case of finite domains with Neumann boundary conditions, but the most complete result concerning existence, uniqueness and stability for both simple and multi-layered solutions to (1.2) in a one dimensional bounded domain has been proven in [8].

For the time dependent problem, it is proven in [26] that, given an initial datum $u_0$ that changes sign once inside the interval, the solution of the evolutive equation develops into a layered function $u^\xi$, where $\xi \in \mathbb{R}^N$ represents the position of the interfaces, being $N$ the number of the layers. Moreover, the study of the eigenvalue problem obtained by linearizing around $u^\xi$ is crucial in the study of the motion of the layers ([17, 18]): it is possible to show that there exist exactly $N$ exponentially small eigenvalues, with $N$ equals to the number of the layers, while the rest of the spectrum is bounded away from zero uniformly with respect to $\varepsilon$ (see, for example [8]).

Following the line of a spectral analysis, the phenomenon of metastability for the Allen-Cahn equation has been studied, among others, in [15, 16] and [27]; in particular, in [15], the authors prove that the dynamics of the solutions to (1.2) is the following: starting from two-phase initial data, a pattern of interfacial layers develops in a relatively short time far from the stable equilibrium, defined as the minimizer of the energy functional $I(u)$. Once the solution has reached this state, it changes extremely slow, with a time scale proportional to $\varepsilon^{1/\varepsilon}$. To rigorously describe such metastable dynamics, the authors build-up a family of functions $u^\xi(x)$ which approximates a metastable state with $N$ transition layers, being $\xi = (\xi_1, \ldots, \xi_N)$ the $N$ layer positions; they subsequent linearize the original system around an element of the family, obtaining a system of ordinary differential equations for the shock positions $\xi_i$.

As already stressed, the key of the analysis is the spectral analysis of the linearized operator arising from the linearization around $u^\xi(x)$: precisely, it is shown that there exist exactly $N$ small (with respect to $\varepsilon$) eigenvalues, while all the other eigenvalues are negative and bounded away from zero uniformly with respect to $\varepsilon$. The existence of, at least, one first small eigenvalue implies that the convergence towards the equilibrium can be extremely slow, depending on the size of $\varepsilon$.

Another important contribution in the study of the metastable dynamics for reaction-diffusion equations is the work of F. Otto and M.G. Reznikoff [40], where the authors describes the slow motion of PDEs with gradient flow structure, with a particular attention to the one-dimensional Allen-Cahn equation: the idea here is to translate informations on the energy into informations on the dynamics of the solutions; in particular, the authors
state and prove a Theorem that gives sufficient condition to be imposed on the energy so that the associated gradient flow exhibits a metastable behavior.

A fundamental contribution in the study of the metastability for the Allen-Chan equation in dimension greater or equal to one, is the reference [4]: concerning spectral properties, it is expected an infinity of critical eigenvalues associated with the interface (see [3] for a precise result in this direction). The study of the eigenvalue problem provides, as usual, informations on the stability of the interfaces, as shown in [20].

Results relative to metastability for systems of reaction-diffusion equations appear to be rare (we recall here, [28, 31]), the main difficulty stemming from the fact that a spectral analysis in the case of a system of equations needs much more care. A recent contribution is the reference [12], where the authors study the slow dynamics of solutions of a system of reaction-diffusion equations by means of the study of the evolution of localized energy (see also [13]).

The existence, the asymptotic stability of the steady state and the study of the time dependent problem for different type of reaction-diffusion systems in different space dimensions have been extensively studied. To name some of these results, see, for example, [36] for a two component reaction-diffusion system in one dimensional space (see also [37] for the internal layer problem), or [1] for a couple of scalar reaction diffusion equations with $x \in \mathbb{R}^2$.

In this context, the aim of this paper is to apply the general technique firstly performed in [34] to rigorous describe the slow motion of a pattern of internal layers for a system of reaction-diffusion equations; precisely, we are interested in the case of a single shock layer, and we want to describe the dynamics after this is formed. Heuristically, in the case of a single equation of reaction-diffusion type, the solution with a single transition layer evolves as follow: since the layer interacts with its reflections with respect to the boundary points of the interval $I = (a, b)$, if the layer is closer to $a$, then it is attracted by its reflection with respect to $b$ and it moves towards $b$. Once the layer have reached a neighborhood of $x = b$ it suddenly disappears, and, for large times, the solution converges to one of the stable patternless solutions $u = u_\pm$, where $u_\pm$ are the minima of the energy $I(u)$.

The fact that the dynamics yields to a reduction of the number of the layers up to their totally disappearance, agrees with the fact that, if $u$ is the solution of the equation, then the number $n(t)$ of the zeroes of $u$ is non increasing in time (see [7] as a reference).

In order to describe such dynamics for a system of equations, the strategy we use is the following:

1. We build-up a one parameter family of functions $\{U^\xi(x; \xi)\}$, whose elements are approximate steady states for (1.1) in a sense that will be specified later, and where the parameter $\xi$ usually describe the location of the interface.

2. In order to study the dynamics of solutions up to the formation of the internal shock layer, we linearize the original system around the family $\{U^\xi\}$, by writing the solution $u$ as the sum of an element of the family and a small perturbation $v$. 
3. Under suitable hypotheses, we state and prove a result (see Theorem 3.2) concerning the study of the coupled system for the perturbation $v$ and the parameter $\xi$, and describing the metastable behavior of the system under consideration.

In [34], the authors firstly utilize such a technique to describe the slow motion of the solutions to a general class of parabolic systems: the key of their analysis is the linearization around a one-parameter family of approximate steady states (parametrized by $\xi(t)$, describing the motion along the family), and the subsequent analysis of the system obtained for the couple $(\xi, v)$, where $v$ is the perturbative term. Dealing with this system brings into the analysis of the specific form of the quadratic terms arising from the linearization that, for a certain class of parabolic system (as, for example, system of viscous conservation laws) involve a dependence on the space derivative of the solution. This is the reason why the authors consider an approximation of the complete nonlinear equations for the couple $(\xi, v)$, obtained by disregarding the quadratic terms in $v$, so that the dependence on the space derivative is canceled out.

The aim of this present paper is to show that, in the specific case of parabolic systems of reaction-diffusion type, it is possible to perform a study of the complete system for the couple $(\xi, v)$.

Indeed, in [34] it is stated that "... in the case of parabolic systems of reaction-diffusion type, we expect that a result analogous to Theorem 2.1 (see [34, Theorem 2.1]) could be proved, under the assumption of an a priori $L^\infty$ bound on the perturbation. Differently, when a nonlinear first order space derivative term is present (as is the case of viscous conservation laws), the quadratic terms involve a dependence on the space derivative of the solution and a rigorous result needs an additional bound, which we are not presently able to achieve".

More precisely, when linearizing a reaction-diffusion type system around an approximate steady state (step 2) the nonlinear terms arising from the linearization do not involve space derivatives, so that it is possible to obtain an $L^2$ estimate for the perturbation $v$, without disregarding the nonlinear terms, as stated in [34].

We close this Introduction describing in details the contribution of the paper. In Section 2, we adapt the general framework firstly developed in [34] to a system of reaction-diffusion equations; we build up a one-parameter family of approximate steady state $\{U^\varepsilon(x; \xi)\}$ and we linearize the original system around an element of the family. By using an adapted version of the projection method, we obtain a coupled system for the perturbation $v$, defined as the difference $v := u - U^\varepsilon$, and the parameter $\xi$, describing the slow motion of the interface; in particular, the equation for $\xi$ is obtained in such a way the first component of the perturbation (the one that has a very slow decay) is canceled out. Finally, in Section 3, we analyze the complete system for the couple $(\xi, v)$, and we state and prove the following Theorem, that gives a precise estimate for the $L^2$ norm of the perturbation.

**Theorem 1.1.** For every $v_0 \in [L^2(I)]^n$ such that $|v_0|_{L^2} \leq c\varepsilon$ and for every $t \leq T^\varepsilon$, there holds for the solution $v$

$$
|v - z|_{L^2}(t) \leq C \left( |\Omega^\varepsilon|_{L^\infty} + \exp \left( \int_0^t \lambda_1^\varepsilon(\tau) \, d\tau \right) |v_0|_{L^2}^2 \right),
$$
where the function $z$ is defined as

$$z(x, t) := \sum_{k \geq 2} v_k(0) \exp \left( \lambda_k(\xi) \right) \phi_k(x; \xi(t)).$$

Precisely, the solution $v$ is decomposed as the sum of two functions, $v = z + R$, where $z$ is given explicitly in terms of the wave speed measure $\xi(t)$ and the error $R$ is estimated. This estimate can be used to decouple the original system and leads us to the statement of the Corollary 3.4 providing the following estimate for the parameter

$$|\xi(t) - \bar{\xi}| \leq |\xi_0|e^{-\beta^\varepsilon t},$$

where $\bar{\xi}$ indicates the asymptotic value for $\xi$, and $\beta^\varepsilon \to 0$ as $\varepsilon \to 0$. This formula explicitly shows how the motion of the interface towards its equilibrium configuration is much slower as $\varepsilon$ becomes smaller.

The main difference with respect to the previous papers concerning metastability for reaction-diffusion equations is that the eigenvalue analysis of the linearized problem around the steady solution does not show how to understand the transition from the metastable state to the final stable state.

In this paper, Theorem 3.2 and Corollary 3.4 give a good qualitative explanation of this transition, providing an explicit estimate for the size of the parameter $\xi$, together with an expression for the speed rate of convergence of the time dependent solution towards the equilibrium. In particular, the two different phases of the dynamics are explicitly described and separated.

Also, we here present a general strategy that can be applied to the case of a generic system of reaction-diffusion type.

2. The metastable dynamics

Given $\ell > 0, I := (-\ell, \ell)$ and $n \in \mathbb{N}$, we consider a system of reaction-diffusion equations

$$\partial_t u = \varepsilon \partial^2_x u - f(u), \quad u(t, 0) = u_0(x)$$

for the unknown $u(x, \cdot) : [0, +\infty) \to [L^2(I)]^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n, f = (f_1(u), \ldots, f_n(u))$. System (2.1) is also complemented with appropriate boundary conditions.

We require that there exists a $C^2$ potential function $W : \mathbb{R}^n \to \mathbb{R}$ such that $f(u) = \nabla W(u)$. Moreover we assume that $W$ has two distinct global minima $\pm u^*$ such that

i. $W(\pm u^*) = \nabla W(\pm u^*) = 0$,

ii. $W(u) > 0$ for $-u_1^* < u_1 < u_1^*, \ldots, -u_n^* < u_n < u_n^*$, where $u^* = (u_1^*, \ldots, u_n^*)$.

Hence we require the function $W$ to be a double well energy with equal minima at $u = \pm u^*$. Under these hypotheses, it is known (see, among others, [3, 43]) that the only possible stable equilibrium solutions to (2.1) are minimizers of the energy functional

$$I(u) = \int_I \left( \frac{1}{2} \varepsilon |\partial_x u|^2 + W(u) \right) dx.$$ 

As usual, we define

$$\langle u, v \rangle := \int_{-\ell}^{\ell} u(x) \cdot v(x) \, dx, \quad u, v \in [L^2(I)]^n$$
the scalar product in $[L^2(I)]^n$, where $\cdot$ denotes the usual scalar product in $\mathbb{R}^n$.

We are interested in studying the behavior of $u$ in the vanishing viscosity limit $\varepsilon \to 0$, in order to show that, when $\varepsilon \sim 0$, a metastable behavior occurs for the solutions to (2.1). For later use, let us define the nonlinear differential operator

$$\mathcal{F}^\varepsilon[u] := \varepsilon \partial_x^2 u - f(u).$$

The strategy we use here is analogous to the one performed in [34]; given a one-dimensional open interval $J$, we define the one-parameter family $\{U^\varepsilon(\cdot; \xi) : \xi \in J\}$, $U^\varepsilon = (U_1^\varepsilon, \ldots, U_n^\varepsilon)$, whose elements can be considered as an approximation for the stationary solution to (2.1) with a single internal layer, in the sense that $\mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)] \to 0$ as $\varepsilon \to 0$.

Precisely, we assume that the term $\mathcal{F}^\varepsilon[U^\varepsilon]$ belongs to the dual space of the continuous functions space $[C(I)]^n$ and that there exists a family of smooth positive functions $\Omega^\varepsilon = \Omega^\varepsilon(\xi)$, uniformly convergent to zero as $\varepsilon \to 0$, such that there holds

$$\langle \psi(\cdot), \mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)] \rangle \leq \Omega^\varepsilon(\xi) |\psi|_\infty \quad \forall \psi \in [C(I)]^n,$$

for any $\xi \in J$. Hence, we ask for each element of the family to satisfy the stationary problem up to an error that is small in $\varepsilon$, and that is measured by $\Omega^\varepsilon$.

Additionally, we require that there exists $\xi \in J$ such that $U^\varepsilon(x; \xi)$ is a stable stationary solution, that is $\mathcal{F}^\varepsilon[U^\varepsilon(x; \xi)] = 0$. In particular, the parameter $\xi$ describes the dynamics of the solutions along the family $\{U^\varepsilon\}$, and characterizes the slow motion of the interface; hence, the convergence of $\xi(t)$ towards $\xi$ describes the convergence of a layered solution into a patternless steady state.

In order to describe the dynamics of solutions localized far away from the equilibrium configuration, we linearize the original system (2.1) around an element of the family $\{U^\varepsilon\}$ by looking for a solution to the initial value problem (2.1) in the form

$$u(x, t) = U^\varepsilon(x; \xi(t)) + v(x, t),$$

with $\xi = \xi(t) \in J$ and the perturbation $v = v(x, t) \in [L^2(I)]^n$ to be determined. Substituting into (2.1), we obtain

$$\partial_t v = L^\xi v + \mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)] - \partial_\xi U^\varepsilon(\cdot; \xi) \frac{d\xi}{dt} + \mathcal{Q}^\varepsilon[v, \xi],$$

where

$$L^\xi v := d\mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)] v,$$

$$\mathcal{Q}^\varepsilon[v, \xi] := \mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi) + v] - \mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)] - d\mathcal{F}^\varepsilon[U^\varepsilon(\cdot; \xi)] v.$$

Precisely, in the specific case considered here, $L^\xi$ has the following expression

$$L^\xi v := \varepsilon \partial_x^2 v - f'(U^\varepsilon) v.$$

Hence, when $n = 1$, it is easy to check that $L^\xi$ is self-adjoint and therefore its spectrum is composed by real eigenvalues. In the general case $n > 1$, the self-adjointness of the operator is not guaranteed, and we have to consider the chance of having complex eigenvalues.

Next, let us require the linear operator $L^\xi$ to have a discrete spectrum composed by semi-simple eigenvalues $\{\lambda^\xi_k(\xi)\}_{k \in \mathbb{N}}$. Moreover, assume that for any $\xi \in (-\ell, \ell)$ the first
eigenvalue \( \lambda_1^\varepsilon \) is real and such that \( \lim_{\varepsilon \to 0} \lambda_1^\varepsilon(\xi) = 0 \), uniformly with respect to \( \xi \); additionally, we suppose there hold
\[
(2.4) \quad \lambda_1^\varepsilon(\xi) - \text{Re} \lambda_2^\varepsilon(\xi) \geq C, \quad \text{Re} \lambda_k^\varepsilon(\xi) \leq -C \quad \text{for } k \geq 2,
\]
for some constant \( C > 0 \) independent on \( k, \varepsilon \) and \( \xi \).

**Remark 2.1.** The assumption (2.4) states that there is only one critical eigenvalue, and it is compatible with the case of a single transition layer we are considering in this paper. In the general case of \( N \geq 1 \) layers, it was shown in [15] that there exist exactly \( N \) small (with respect to \( \varepsilon \)) eigenvalues.

If we now denote with \( \phi_k^\varepsilon = \phi_k^\varepsilon(\cdot; \xi) \) and \( \psi_k^\varepsilon = \psi_k^\varepsilon(\cdot; \xi) \) the eigenfunctions of \( L_\xi^\varepsilon \) and of the adjoint operator \( L_\xi^\varepsilon \) respectively, we can define
\[
v_k = v_k(\xi; t) := \langle \psi_k^\varepsilon(\cdot; \xi), v(\cdot, t) \rangle.
\]
Since we have assumed the first eigenvalue of the linearized operator to be small in \( \varepsilon \), i.e. \( \lambda_1^\varepsilon \to 0 \) as \( \varepsilon \to 0 \), a necessary condition for the solvability of (2.3) is that the first component of the solution \( v_1 \) has to be zero. This request translates into an equation for the parameter \( \xi(t) \), chosen in such a way that the unique growing terms in the perturbation \( v \) are canceled out. Precisely, we require
\[
\frac{d}{dt} \langle \psi_1^\varepsilon(\cdot; \xi(t)), v(\cdot, t) \rangle = 0 \quad \text{and} \quad \langle \psi_1^\varepsilon(\cdot; \xi_0), v_0(\cdot) \rangle = 0.
\]
Using equation (2.3), and since \( \langle \psi_1^\varepsilon, L_\xi^\varepsilon v \rangle = \lambda_1^\varepsilon \langle \psi_1^\varepsilon, v \rangle = 0 \), we obtain a scalar differential equation for the variable \( \xi \), describing the dynamics in a neighborhood of the approximate family, that is
\[
(2.5) \quad \alpha^\varepsilon(\xi, v) \frac{d\xi}{dt} = \langle \psi_1^\varepsilon(\cdot; \xi), \mathcal{F}[U_\varepsilon(\cdot; \xi)] + \mathcal{Q}[v, \xi] \rangle,
\]
where
\[
\alpha^\varepsilon(\xi, v) := \langle \psi_1^\varepsilon(\cdot; \xi), \partial_\xi U_\varepsilon(\cdot; \xi) \rangle - \langle \partial_\xi \psi_1^\varepsilon(\cdot; \xi), v \rangle,
\]

Together with the condition on the initial datum \( \xi_0 \)
\[
\langle \psi_1^\varepsilon(\cdot; \xi_0), v_0(\cdot) \rangle = 0.
\]
For the sake of simplicity, we renormalize the first eigenfunction \( \psi_1^\varepsilon \) in such a way
\[
\langle \psi_1^\varepsilon(\cdot; \xi), \partial_\xi U_\varepsilon(\cdot; \xi) \rangle = 1.
\]
Such constraint can be imposed if we ask for
\[
|\langle \psi_1^\varepsilon(\cdot; \xi), \partial_\xi U_\varepsilon(\cdot; \xi) \rangle| \geq c_0 > 0,
\]
where \( c_0 \) is independent on \( \xi \). The last assumption gives a (weak) restriction on the choice of the family of functions \( \{U_\varepsilon\} \); indeed, we need the manifold to be never transverse to the first eigenfunction \( \psi_1^\varepsilon \).

Going further, in the regime of small \( v \), we have
\[
\frac{1}{\alpha^\varepsilon(\xi, v)} = 1 + \langle \partial_\xi \psi_1^\varepsilon, v \rangle + R[v],
\]
where the reminder $R^\varepsilon$ is of order $o(|v|)$, and it is defined as
\[ R^\varepsilon[v] := \frac{(\partial_\xi \psi_1^\varepsilon(\cdot; \xi), v)^2}{1 - (\partial_\xi \psi_1^\varepsilon(\cdot; \xi), v)}. \]

Hence, we may rewrite the nonlinear equation for $\xi$ as
\[ \frac{d\xi}{dt} = \theta^\varepsilon(\xi) \left(1 + \langle \partial_\xi \psi_1^\varepsilon, v \rangle \right) + \rho^\varepsilon[\xi, v], \]
where
\[ \theta^\varepsilon(\xi) := \langle \psi_1^\varepsilon, \mathcal{F}[U^\varepsilon] \rangle, \]
\[ \rho^\varepsilon[\xi, v] := \frac{1}{\alpha^\varepsilon(\xi, v)} \left( \langle \psi_1^\varepsilon, Q^\varepsilon \rangle + \langle \partial_\xi \psi_1^\varepsilon, v \rangle^2 \right). \]

Equation (2.6) is an equation of motion for the parameter, describing the dynamics of $\xi(t)$ around the family of approximate steady states; since an element of the family $\{U^\varepsilon(x; \xi)\}_{\xi \in J}$ is not an exact steady state to (2.1), the dynamics walks away from $U^\varepsilon$ with a speed dictated by (2.6).

Hence, at a first approximation, for small perturbations $v \sim 0$, the leading order term in the equation for $\xi(t)$, given by $\theta^\varepsilon(\xi)$, characterizes the speed of the solution during its motion towards its equilibrium configuration. From the definition of $\theta^\varepsilon$ we can formally see that such convergence is much slower as $\varepsilon$ becomes smaller, since $\theta^\varepsilon$ is going to zero as $\varepsilon \to 0$.

Using (2.6), we can rewrite the equation for $v$ as
\[ \partial_t v = H^\varepsilon(x; \xi) + (\mathcal{L}_\xi^\varepsilon + \mathcal{M}_\xi^\varepsilon)v + \mathcal{R}^\varepsilon[v, \xi], \]
where
\[ H^\varepsilon(\cdot; \xi) := \mathcal{F}[U^\varepsilon(\cdot; \xi)] - \partial_\xi U^\varepsilon(\cdot; \xi) \theta^\varepsilon(\xi), \]
\[ \mathcal{M}_\xi^\varepsilon v := -\partial_\xi U^\varepsilon(\cdot; \xi) \theta^\varepsilon(\xi) \langle \partial_\xi \psi_1^\varepsilon, v \rangle, \]
\[ \mathcal{R}^\varepsilon[v, \xi] := Q^\varepsilon[v, \xi] - \partial_\xi U^\varepsilon(\cdot; \xi) \rho^\varepsilon[\xi, v], \]
obtaining the couple system (2.6)--(2.7) for the parameter $\xi$ and the perturbation $v$.

**Example 2.2. The case $n=1$.** The main example we have in mind is the initial boundary value problem for Allen-Cahn equation in a one-dimensional interval $I = (-\ell, \ell)$, that is
\[ \begin{cases} \partial_t u = \varepsilon \partial_x^2 u - f(u) & x \in I, t \geq 0 \\ \partial_x u(\pm \ell, t) = 0 & t \geq 0 \\ u(x, 0) = u_0(x) & x \in I \end{cases} \]

We assume $f(u) = W'(u)$, where $W(u)$ is a double well function with equal minima in $u = \pm 1$. We require
\[ W(\pm 1) = W'(\pm 1) = 0, \quad W(u) > 0 \quad \text{for} \quad -1 < u < 1 \]
A standard example is given by $W(u) = \frac{(u^2 - 1)^2}{4}$, which has two minima in $u = \pm 1$.

Given a reference state $U^\varepsilon(x; \xi)$ such that the nonlinear term $\mathcal{F}[U^\varepsilon] = \varepsilon \partial_x U^\varepsilon - f(U^\varepsilon)$ is going to zero as $\varepsilon \to 0$ in the sense of (2.2), if we linearize around $U^\varepsilon$ we obtain the following equation for the perturbation
\[ \partial_t v = \varepsilon \partial_x^2 v - f'(U^\varepsilon)v - \partial_x U^\varepsilon \frac{d\xi}{dt} + \mathcal{F}[U^\varepsilon] + Q^\varepsilon[\xi, v], \]
where $Q^\varepsilon[\xi, v] := -f''(U^\varepsilon)v^2 - f'''(U^\varepsilon)v^3$. In particular, since we are assuming the perturbation $v$ to be small, i.e. $v \ll 1$, it follows that $|Q^\varepsilon| \leq C|v|^2$.

We note that, in this case, the parameter $\xi$ can be chosen as the position of the (unique) interface, so that its motion characterizes the slow dynamics of the shock layer towards the equilibrium.

**The case $n=2$.** Let us consider a system of two reaction-diffusion equations, namely

\[
\begin{align*}
\partial_t u &= \varepsilon \partial_x^2 u - f_1(u, v) \\
\partial_t v &= \varepsilon \partial_x^2 v - f_2(u, v)
\end{align*}
\]

and let $U^\varepsilon = (U^\varepsilon, V^\varepsilon)$ be an approximate steady state of the problem, meaning that

\[
F^\varepsilon[U^\varepsilon] := \left(\begin{array}{c}
\varepsilon \partial_x^2 U^\varepsilon - f_1(U^\varepsilon, V^\varepsilon) \\
\varepsilon \partial_x^2 V^\varepsilon - f_2(U^\varepsilon, V^\varepsilon)
\end{array}\right)
\]

is small in $\varepsilon$ in the sense of (2.2). By substituting

\[
(u, v)(x, t) = (w, z)(x, t) + (U^\varepsilon(x; \xi(t)), V^\varepsilon(x; \xi(t)))
\]

into (2.9), and by expanding $f_{1,2}(w + U^\varepsilon, z + V^\varepsilon)$, we obtain the equation (2.3) for the perturbation $v = (w, z)$, where

\[
L^\varepsilon \xi v := \left(\begin{array}{c}
\varepsilon \partial_x^2 w - \partial_u f_1(U^\varepsilon, V^\varepsilon)w - \partial_v f_1(U^\varepsilon, V^\varepsilon)z \\
\varepsilon \partial_x^2 z - \partial_u f_2(U^\varepsilon, V^\varepsilon)w - \partial_v f_2(U^\varepsilon, V^\varepsilon)z
\end{array}\right),
\]

and the nonlinear term is explicitly given by

\[
Q^\varepsilon[\xi, v] := \left(\begin{array}{c}
-\frac{1}{2} \partial_u^2 f_1(U^\varepsilon, V^\varepsilon)w^2 - \frac{1}{2} \partial_v^2 f_1(U^\varepsilon, V^\varepsilon)z^2 - \partial_u v f_1(U^\varepsilon; V^\varepsilon)wz \\
-\frac{1}{2} \partial_u^2 f_2(U^\varepsilon, V^\varepsilon)w^2 - \frac{1}{2} \partial_v^2 f_2(U^\varepsilon, V^\varepsilon)z^2 - \partial_u v f_2(U^\varepsilon; V^\varepsilon)wz
\end{array}\right),
\]

where the third order terms are omitted since $v \ll 1$. Again, such formula explicitly shows that $|Q^\varepsilon| \leq C|w, v|^2$.

From this example one can easily check that, when $n > 1$, the operator $L^\varepsilon \xi$ is not necessarily self-adjoint.

For a general $n \in \mathbb{N}$, the nonlinear term $Q^\varepsilon[\xi, v]$ is defined as

\[
Q^\varepsilon[\xi, v] := \left(\begin{array}{c}
-\sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{u_i u_j} f_1(U^\varepsilon)u_i u_j \\
\cdot \\
\cdot \\
\cdot \\
-\sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{u_i u_j} f_n(U^\varepsilon)u_i u_j
\end{array}\right)
\]
so that $|Q|_{L^2} \leq C|v|^2_{L^2}$ for some constant $C$ depending on $f$ and $|U^\varepsilon|$.

3. Nonlinear metastability

Let us consider the system \([2.6]-[2.7]\) for the couple \((\xi, v)\)

\[
\begin{aligned}
\frac{d\xi}{dt} &= \theta^\varepsilon(\xi)(1 + \langle \partial_\xi \psi_1^\varepsilon, v \rangle) + \rho^\varepsilon[\xi, v], \\
\partial_t v &= H^\varepsilon(\xi) + (L^\varepsilon_\xi + M^\varepsilon_\xi)v + R^\varepsilon[\xi, v]
\end{aligned}
\]

complemented with initial conditions

\[
(3.2) \quad \xi(0) = \xi_0 \in (-\ell, \ell) \quad \text{and} \quad v(x, 0) = v_0(x) = u_0 - U^\varepsilon(;\xi_0) \in [L^2(I)]^n.
\]

In what follow, we mean to analyze solutions to (3.1). The main difference with respect to the work of C. Mascia and M. Strani [34], is that here the nonlinear terms $R^\varepsilon$ and $\rho^\varepsilon$ depend only from the perturbation $v$, and not from its derivatives, so that an estimate for the $L^2$ norm of the solution can be performed under appropriate smallness assumption on the initial datum. Before stating our result, we recall the hypotheses we require on the terms of the system

**H1.** There exists a family of smooth positive functions $\Omega^\varepsilon$ such that

\[
|\langle \psi(\cdot), F^\varepsilon[U^\varepsilon(\cdot;\xi)] \rangle| \leq \Omega^\varepsilon(\xi) |\psi|_{L^\infty} \quad \forall \psi \in [C(I)]^n,
\]

with $\Omega^\varepsilon$ converging to zero as $\varepsilon \to 0$, uniformly with respect to $\xi$.

**H2.** Let \(\{\lambda^\varepsilon_k(\xi)\}_{k \in \mathbb{N}}\) be the sequence of eigenvalues of the linear operator $L^\varepsilon_\xi$. Assume that for any $\xi \in I$, the first eigenvalue $\lambda^\varepsilon_1$ is simple, real and such that $\lambda^\varepsilon_1(\xi) \to 0$ as $\varepsilon \to 0$ uniformly with respect to $\xi$. Moreover, there hold

\[
\lambda^\varepsilon_1(\xi) - \text{Re} \lambda^\varepsilon_2(\xi) \geq C, \quad \text{Re} \lambda^\varepsilon_k(\xi) \leq -C \quad \text{for } k \geq 2,
\]

for some constant $C > 0$ independent on $k$, $\varepsilon$ and $\xi$.

**H3.** Given $\xi \in I$, let $\phi^\varepsilon_k(\cdot; \xi)$ and $\psi^\varepsilon_k(\cdot; \xi)$ be a sequence of eigenfunction for the operators $L^\varepsilon_\xi$ and $L^\varepsilon_r$ respectively, we assume

\[
(3.3) \quad \sum_j \langle \partial_\xi \psi^\varepsilon_k, \phi^\varepsilon_j \rangle^2 = \sum_j \langle \psi^\varepsilon_k, \partial_\xi \phi^\varepsilon_j \rangle^2 \leq C.
\]

for all $k$ and for some constant $C$ independent on the parameter $\xi$.

**Remark 3.1.** Hypothesis **H2** is a crucial hypothesis concerning the distribution of the eigenvalues of the linearized operator around an (approximate) steady state. For example, in the case of the Allen-Cahn equation \([2.8]\) with a single transition layer and with Neumann boundary conditions, it is proven ([10] and [15] for a linearization around an approximate steady state) that the spectrum is composed by real eigenvalues with the following distribution

\[
\lambda^\varepsilon_1 = \mathcal{O}(e^{-c/\varepsilon}) \quad \text{and} \quad \lambda^\varepsilon_k \leq -c \quad \forall k \geq 2,
\]

where the constant $c > 0$ is independent on $\varepsilon$.

For systems of reaction-diffusion types the situation is more delicate (see, as an example, [37]), and a spectral analysis is needed in order to verify hypothesis **H2**. Conversely, when rigorous results are not to be realizable, it could be possible to obtain numerical evidence of the spectrum of the linearized operator.
Theorem 3.2. Let the couple \((\xi, v)\) be the solution to initial-value problem (3.1). If the hypotheses H1-2-3 are satisfied, then, for every \(\varepsilon\) sufficiently small, for every \(v_0 \in [L^2(I)]^n\) such that \(|v_0|_{L^2} \leq c\varepsilon\) and for every \(t \leq T^\varepsilon\), there holds for the solution \(v\)

\[
|v - z|_{L^2} (t) \leq C \left( |\Omega^\varepsilon|_{L^\infty} + \exp \left( \int_0^t \lambda_1^\varepsilon (\tau) \, d\tau \right) |v_0|_{L^2}^2 \right),
\]

where the function \(z\) is defined as

\[
z(x, t) := \sum_{k \geq 2} v_k(0) \exp (\lambda_k^\varepsilon (\xi) \, d\tau) \phi_k^\varepsilon (x; \xi(t)).
\]

Moreover, the time \(T^\varepsilon\) is of order \(\ln |\Omega^\varepsilon|_{L^\infty}^{-1} \sup_{\xi \in I} \lambda_1^\varepsilon (\xi)^{-1}\), hence diverging to \(+\infty\) as \(\varepsilon \to 0\).

Proof. Setting \(v(x, t) = \sum_j v_j(t) \phi_j^\varepsilon (x, \xi(t))\), we obtain an infinite-dimensional differential system for the coefficients \(v_j\)

\[
(3.4) \quad \frac{dv_k}{dt} = \lambda_k^\varepsilon (\xi) v_k + \langle \psi_k^\varepsilon, F \rangle + \langle \psi_k^\varepsilon, G \rangle
\]

where, omitting the dependencies for shortness,

\[
F := H^\varepsilon + \sum_j v_j \left\{ M_\xi \phi_j^\varepsilon - \partial_\xi \phi_j^\varepsilon \frac{d\xi}{dt} \right\} = H^\varepsilon - \theta^\varepsilon \sum_j (a_j + \sum_\ell b_{j\ell} v_\ell) v_j,
\]

\[
G := Q^\varepsilon - \left( \sum_j \partial_\xi \phi_j^\varepsilon v_j + \partial_\xi U^\varepsilon \right) \left\{ \frac{\langle \psi_1^\varepsilon, Q^\varepsilon \rangle}{1 - \langle \partial_\xi \psi_1^\varepsilon, v \rangle} \phi_1^\varepsilon + \theta^\varepsilon \frac{\langle \partial_\xi \psi_1^\varepsilon, v \rangle^2}{1 - \langle \partial_\xi \psi_1^\varepsilon, v \rangle} \right\} = Q^\varepsilon - N^\varepsilon.
\]

The coefficients \(a_j, b_{jk}\) are given by

\[
a_j := \langle \partial_\xi \psi_1^\varepsilon, \phi_j^\varepsilon \rangle \partial_\xi U^\varepsilon + \partial_\xi \phi_j^\varepsilon, \quad b_{j\ell} := \langle \partial_\xi \psi_1^\varepsilon, \phi_\ell^\varepsilon \rangle \partial_\xi \phi_j^\varepsilon.
\]

Convergence of the series is guaranteed by assumption (3.3). Now let us set

\[
E_k(s, t) := \exp \left( \int_s^t \lambda_k^\varepsilon (\xi(\tau)) \, d\tau \right).
\]

Note that, for \(0 \leq s < t\), there hold

\[
E_k(s, t) = \frac{E_k(0, t)}{E_k(0, s)} \quad \text{and} \quad 0 \leq E_k(s, t) \leq e^{\Lambda_k(t-s)}.
\]

From equalities (3.4) and since there holds \(v_1 \equiv 0\), there follows

\[
v_k(t) = v_k(0) E_k(0, t)
\]

\[
+ \int_0^t \left\{ \langle \psi_k^\varepsilon, H^\varepsilon \rangle - \theta^\varepsilon (\xi) \sum_j \left( \langle \psi_k^\varepsilon, a_j \rangle + \sum_\ell \langle \psi_k^\varepsilon, b_{j\ell} \rangle v_\ell \right) v_j \right\} E_k(s, t) \, ds
\]

\[
+ \int_0^t \left\{ \langle \psi_k^\varepsilon, Q^\varepsilon \rangle - \langle \psi_k^\varepsilon, N^\varepsilon \rangle \right\} E_k(s, t) \, ds
\]
for \( k \geq 2 \). Now let us define \( \Lambda_k^\varepsilon := \sup_{\xi \in I} \lambda_k^\varepsilon(\xi) \) and let us introduce the function

\[
z(x, t) := \sum_{k \geq 2} v_k(0) E_k(0, t) \phi_k^\varepsilon(x; \xi(t)),
\]

which satisfies the estimate \( |z|_{L^2} \leq |v_0|_{L^2} e^{\Lambda_k^\varepsilon t} \). From the representation formulas for the coefficients \( v_k \), there holds

\[
|v - z|_{L^2}^2 \leq \sum_{k \geq 2} \left( \int_0^t \left( |\langle \psi_k^\varepsilon, F \rangle| + |\langle \psi_k^\varepsilon, G \rangle| \right) E_k(s, t) \, ds \right)^2.
\]

Moreover, since

\[
|\theta^\varepsilon(\xi)| \leq C \Omega^\varepsilon(\xi) \quad \text{and} \quad |\langle \psi_k^\varepsilon, H^\varepsilon \rangle| \leq C \Omega^\varepsilon(\xi) (1 + |\langle \psi_k^\varepsilon, \partial_\xi U^\varepsilon \rangle|)
\]

for some constant \( C > 0 \) depending on the \( L^\infty \)-norm of \( \psi_k^\varepsilon \), there holds

\[
\sum_{k \geq 2} \left( \int_0^t |\langle \psi_k^\varepsilon, F \rangle| E_k(s, t) \, ds \right)^2 \leq C \sum_{k \geq 2} \left( \int_0^t \Omega^\varepsilon(\xi) \left( 1 + |\langle \psi_k^\varepsilon, \partial_\xi U^\varepsilon \rangle| + |\langle \psi_k^\varepsilon, \partial_\xi U^\varepsilon \rangle| \sum_j |\langle \partial_\xi \psi_j^\varepsilon, \phi_j^\varepsilon \rangle| v_j \right)
\]

\[
+ \sum_j |\langle \partial_\xi \phi_j^\varepsilon, \phi_j^\varepsilon \rangle| v_j | + \sum_j |\langle \psi_j^\varepsilon, \partial_\xi \phi_j^\varepsilon \rangle| v_j | + \sum_j |\langle \psi_j^\varepsilon, \partial_\xi \psi_j^\varepsilon \rangle| v_j | \right) E_k(s, t) \right)^2
\]

\[
\leq C \sum_{k \geq 2} \left( \int_0^t \Omega^\varepsilon(\xi) \left( 1 + |v|_{L^2}^2 \right) E_k(s, t) \, ds \right)^2.
\]

On the other side, concerning the nonlinear terms, there holds

\[
\sum_{k \geq 2} \left( \int_0^t |\langle \psi_k^\varepsilon, G \rangle| E_k(s, t) \, ds \right)^2 \leq C \sum_{k \geq 2} \left( \int_0^t |\langle \psi_k^\varepsilon, Q^\varepsilon \rangle| + \frac{|\langle \psi_k^\varepsilon, Q^\varepsilon \rangle|}{1 - |\langle \partial_\xi \psi_1^\varepsilon, v \rangle|} \left( \sum_j |\langle \partial_\xi \phi_j^\varepsilon, \psi_j^\varepsilon \rangle| v_j + |\langle \psi_j^\varepsilon, \partial_\xi U^\varepsilon \rangle| \right)
\]

\[
+ \Omega^\varepsilon(\xi) \left( \frac{|\langle \partial_\xi \psi_1^\varepsilon, v \rangle|}{1 - |\langle \partial_\xi \psi_1^\varepsilon, v \rangle|} \left( \sum_j |\langle \partial_\xi \phi_j^\varepsilon, \psi_j^\varepsilon \rangle| v_j + |\langle \psi_j^\varepsilon, \partial_\xi U^\varepsilon \rangle| \right) \right) E_k(s, t) \right)^2.
\]

Moreover, since \( |Q^\varepsilon|_{L^2} \leq C |v|_{L^2}^2 \), we have

\[
|\langle \psi_k^\varepsilon, Q^\varepsilon \rangle| \leq C |v|_{L^2}^2,
\]

\[
|\sum_k \langle \psi_k^\varepsilon, \partial_\xi \phi_j^\varepsilon \rangle v_j | \leq C |v|_{L^2},
\]

\[
|\langle \psi_k^\varepsilon, \partial_\xi U^\varepsilon \rangle| \leq \frac{|\langle \psi_k^\varepsilon, Q^\varepsilon \rangle|}{1 - |\langle \partial_\xi \psi_1^\varepsilon, v \rangle|} \leq \frac{C |v|_{L^2}^2}{1 - C |v|_{L^2}^2} \leq 2C |v|_{L^2}^2;
\]

\[
|\theta^\varepsilon(\xi)| \leq C \Omega^\varepsilon(\xi) |v|_{L^2}^2.
\]
so that we end up with
\[ \sum_{k \geq 2} \left( \int_0^t |(\psi_k^\varepsilon, \mathcal{G})| E_k(s, t) \, ds \right)^2 \leq C \sum_{k \geq 2} \left( \int_0^t |v|_{L^2}^2 (1 + \Omega^\varepsilon(\xi)) E_k(s, t) \, ds \right)^2. \]

Since \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \), we infer
\[ |v - z|_{L^2} \leq \sum_{k \geq 2} \int_0^t \left\{ \Omega^\varepsilon(\xi) \left( 1 + |v|_{L^2}^2 \right) + |v|_{L^2}^2 \right\} E_k(s, t) \, ds \]
\[ \leq C \int_0^t \left\{ \Omega^\varepsilon(\xi) \left( 1 + |v|_{L^2}^2 \right) + |v|_{L^2}^2 \right\} \sum_{k \geq 2} E_k(s, t) \, ds. \]

The assumption on the asymptotic behavior of the eigenvalues \( \lambda_k^\varepsilon \) can now be used to bound the series. Indeed, there holds
\[ \sum_{k \geq 2} E_k(s, t) \leq E_2(s, t) \sum_{k \geq 2} \frac{E_k(s, t)}{E_2(s, t)} \leq C (t - s)^{-1/2} E_2(s, t). \]

As a consequence, we infer
\[ |v - z|_{L^2} \leq C \int_0^t \Omega^\varepsilon(\xi)(t - s)^{-1/2} E_2(s, t) \, ds \]
\[ + \int_0^t \left\{ |v - z|_{L^2}^2 + |z|_{L^2}^2 \right\} (t - s)^{-1/2} E_2(s, t) \, ds. \]

Now, setting \( N(t) := \sup_{s \in [0, t]} E_1(s, 0) \, |(v - z)(s)|_{L^2} \), we obtain
\[ E_1(t, 0) |v - z|_{L^2} \leq C \int_0^t N^2(s) (t - s)^{-1/2} E_2(s, t) E_1(0, s) \, ds \]
\[ + C \int_0^t \Omega^\varepsilon(\xi)(t - s)^{-1/2} E_2(s, t) E_1(s, 0) \, ds \]
\[ + C \int_0^t |v_0|_{L^2}^2 (t - s)^{-1/2} E_2(s, t) E_1(s, 0) \, ds \]
\[ \leq C_1 N^2(t) E_1(0, t) + C_2 \left( \Omega^\varepsilon|_{L^\infty} + |v_0|_{L^2}^2 E_1(t, 0) \right), \]

where we used
\[ \int_0^t e^{\Lambda_2^\varepsilon s} \, ds = \frac{1}{\Lambda_2^\varepsilon} (e^{\Lambda_2^\varepsilon s} - 1) \leq \frac{1}{|\Lambda_2^\varepsilon|}, \]
\[ \int_0^t (t - s)^{-1/2} E_2(s, t) \, ds \leq \int_0^t (t - s)^{-1/2} e^{\Lambda_2^\varepsilon (t-s)} \, ds \leq \frac{1}{|\Lambda_2^\varepsilon|^{1/2}} \]

and \( C_1 \) and \( C_2 \) depend on \( \Lambda_2^\varepsilon \). Hence, as soon as
\[ 4C_1 C_2 \left( E_1(0, t) \, |\Omega^\varepsilon|_{L^\infty} + |v_0|_{L^2}^2 \right) < 1 \]
we obtain the following \( L^2 \)-estimate for the difference \( v - z \)
\[ |v - z|_{L^2} \leq \left( |\Omega^\varepsilon|_{L^\infty} + |v_0|_{L^2}^2 \right). \]
Condition (3.5) is a condition on the final time $T^\varepsilon$. Indeed, (3.5) can be rewritten as
\[ e^{\Lambda_1^\varepsilon t} \leq C \ln \left( \frac{1 - |v_0|_{L^2}^2}{|\Omega^\varepsilon|_{L^\infty}} \right) \]
that is, $T^\varepsilon$ can be chosen of order $\ln |\Omega^\varepsilon|_{L^\infty}^{-1} |\Lambda_1^\varepsilon|^{-1}$.

\[ \square \]

**Remark 3.3.** In the case of the Allen-Cahn equation, $|\Lambda_1^\varepsilon|^{-1} \sim e^{1/\varepsilon}$, so that the order of the final time $T^\varepsilon$ coincides with the corresponding expression determined in [15].

Estimate (3.6) can be used to decoupled the system (3.1); this leads to the following consequence of Theorem 3.2.

**Corollary 3.4.** Let hypotheses H1-2-3 be satisfied and let us also assume
\[ (\xi - \bar{\xi}) \theta^\varepsilon(\xi) < 0 \quad \text{for any } \xi \in J, \xi \neq \bar{\xi} \quad \text{and} \quad \theta^\varepsilon(\bar{\xi}) < 0. \]
Then, for $\varepsilon$ and $|v_0|_{L^2}$ sufficiently small, the solution $(\xi, v)$ converges exponentially fast to $(\bar{\xi}, 0)$ as $t \to +\infty$.

**Proof.** For any initial datum $\xi_0$, the variable $\xi(t)$ is such that
\[ \frac{d\xi}{dt} = \theta^\varepsilon(\xi)(1 + r) + \rho^\varepsilon(\xi, v), \quad \text{with} \quad |r| \leq C \{ |v_0|_{L^2} (e^{-\varepsilon t} + 1) + |\Omega^\varepsilon|_{L^\infty} \} \]
and
\[ |\rho^\varepsilon(\xi, v)| \leq C |v|_{L^2}^2 \leq |z|^2_{L^2} + R \leq |v_0|_{L^2} e^{-\varepsilon t} + |v_0|_{L^2}^2 + |\Omega^\varepsilon|_{L^\infty}. \]
Hence, recalling that $|v_0|_{L^2} \leq c \varepsilon$, in the regime of small $\varepsilon$ the solution $\xi(t)$ has similar decay properties of the solution to the following reduced equation
\[ \frac{d\eta}{dt} = \theta^\varepsilon(\eta), \quad \eta(0) = \xi(0). \]
As a consequence, $\xi$ converges exponentially fast to $\bar{\xi}$ as $t \to +\infty$. More precisely there exists $\beta^\varepsilon > 0$, $\beta^\varepsilon \to 0$ as $\varepsilon \to 0$, such that
\[ |\xi(t) - \bar{\xi}| \leq |\xi_0| e^{-\beta^\varepsilon t}, \]
for any $t$ under consideration. Concerning the perturbation $v$, from (3.4), we deduce
\[ v_k(t) = v_k(0) \exp \left( \int_0^t \lambda_k^\varepsilon d\tau \right) + \int_0^t \langle \psi_k^\varepsilon, F \rangle(s) \exp \left( \int_s^t \lambda_k^\varepsilon d\tau \right) ds \]
\[ \quad + \int_0^t \langle \psi_k^\varepsilon, G \rangle(s) \exp \left( \int_s^t \lambda_k^\varepsilon d\tau \right) ds. \]
By the Jensen’s inequality, we infer the estimate
\[ |v|_{L^2}^2(t) \leq C \left\{ |v_0|_{L^2}^2 e^{2\lambda_1^\varepsilon t} + t \int_0^t \left( |F|_{L^2}^2(s) + |G|_{L^2}^2(s) \right) e^{2\lambda_1^\varepsilon (t-s)} ds \right\}. \]
Let $\nu^\varepsilon > 0$ and $\mu^\varepsilon > 0$ be such that $|F|_{L^2}(t) \leq C e^{-\nu^\varepsilon t}$ and $|G|_{L^2}(t) \leq C e^{-\mu^\varepsilon t}$; then, if $\nu^\varepsilon, \mu^\varepsilon \neq |\Lambda_1^\varepsilon|$, there holds
\[ |v|_{L^2}^2(t) \leq C \left\{ |v_0|_{L^2}^2 e^{2\lambda_1^\varepsilon t} + t \left( e^{-2\nu^\varepsilon t} + e^{-2\mu^\varepsilon t} + e^{2\lambda_1^\varepsilon t} \right) \right\}, \]
showing the exponential convergence to 0 of the perturbation $v$. \[ \square \]
Formula (3.9) shows the slow motion of the interface for small $\varepsilon$. Precisely,

$$\xi(t) \sim \bar{\xi} + |\xi_0|e^{-\beta\varepsilon t},$$

showing that the layer is converging exponentially slowly towards its asymptotic configuration, and this motion is much slower as $\varepsilon$ becomes smaller. After approaching $\bar{\xi}$, as already pointed out in the Introduction, the layer suddenly disappears and we have convergence of the solution to a patternless equilibrium configuration.

Also, since for $t \to +\infty$ the approximate steady state $U^\varepsilon$ is converging to one of the stable steady states of the system, the perturbation $v$ is converging to zero, as pointed out in (3.10).
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