Quaternion-based generalization of conformal maps

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Abstract. For the conformal map in two dimensions, the local rescaling factor of metric amounts to that of surface element. To avoid the restriction on the functional form of the metric in $d$ dimensions ($d > 2$), we generalize the conformal map in such a way that the volume element (quadratic form), not the metric (quadratic form), is invariant up to local rescaling. For $d = 4$, quaternions are available to formulate such a generalized conformal map, as complex analysis is useful for the conventional conformal map.

1. Introduction

Conformal symmetry, which is fundamental in physics, is realized in various fields, such as condensed matter physics, relativity, and string theory [1, 2, 3]. In two dimensions, there are infinite numbers of local conformal maps (angle-preserving maps on the complex plane). In $d$ ($d > 2$) dimensions, however, there are $\frac{1}{2}(d+1)(d+2)$ independent solutions. The restriction on the number of the independent solutions is favorable or unfavorable, depending on the physical situations. Here we attempt to remove the restriction. The starting point is to note that in two dimensions, the distance $dx^2 + dy^2$ and the area element $dx \wedge dy$ transform in the same way under the holomorphic map. Actually, we have

$$\begin{cases}
    dx^2 + dy^2 \mapsto du^2 + dv^2 = W \left( dx^2 + dy^2 \right), & W = |f'(z)|^2, \\
    dx \wedge dy \mapsto du \wedge dv = J \left( dx \wedge dy \right), & J = |f'(z)|^2,
\end{cases}$$

under the map $f : \mathbb{C} \cong \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $f(z, \bar{z}) = u(x, y) + iv(x, y)$, provided that the maps $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfy the Cauchy-Riemann relation $u_x = v_y, u_y = -v_x$, or $f_{\bar{z}} = 0$ ($\bar{z} = x - iy$) so that $f(z, \bar{z})$ can be regarded as a function of $z$ only. In higher dimensions, however, the coincidence of the Weyl factor $W$ and the Jacobian $J$ in the volume element does not hold in general. Nonetheless, it seems reasonable to use the volume element, rather than the distance (metric) [4], so as to lift the restriction on the degree of freedom of the independent solutions.

The aim of this paper is to generalize conformal maps using the volume element, imposing the non-negativity of the Jacobian $J$ (as is analogous with the non-negativity of the Weyl factor $W$ in the ordinary conformal map), to find out that in four dimensions, some sufficient condition for $J \geq 0$ is closely related to the Cauchy-Riemann-Fueter relation [5], “analyticity” in quaternionic functions, as is analogous with the Cauchy-Riemann relation, analyticity in complex functions. Sec. 2 is concerned with a typical conformal map in two dimensions, together with the analogous
map in four dimensions. In Sec. 3, we deal with a general map in four dimensions, giving a sufficient condition for $J \geq 0$. Sec. 4 is devoted to conclusion.

2. Generalization of conformal maps

In this section, we first give a typical (and instructive) example of conformal map in two dimensions. Then we make its generalization in four dimensions, emphasizing the role of Jacobian for the volume element. Let $\phi : S_2 \ni (\psi, \theta) \mapsto (t, x) \in \mathbb{R}^2$ be a one-to-one map

$$t \pm x = \tan \left( \frac{\psi \pm \theta}{2} \right),$$

where $S_n$ (for $n \in \mathbb{N}$), which is the generalization of $S_2$, is given by

$$S_n = \{ (\theta_0, \theta_1, \ldots, \theta_{n-1}) \in \mathbb{R}^n \mid \sum_{i=0}^{n-1} |\theta_i| < \pi \}.$$  

Noticing that $S_2$ can be rewritten as

$$S_2 = S_2^{(+)} \cap S_2^{(-)}, \quad S_2^{(\pm)} := \{ (\theta_0, \theta_1) \in \mathbb{R}^2 \mid -\pi < \theta_0 \pm \theta_1 < \pi \},$$

we find that the condition of $(\psi, \theta) \in S_2$ guarantees that $-\infty < t \pm x < \infty$, which indicates that the map $\phi$ is bijective. Considering that the Minkowski distance $ds^2 := dt^2 - dx^2$ is transformed under the map of eq. (1) as

$$ds^2 = \frac{1}{\omega^2} (d\psi^2 - d\theta^2), \quad \omega = \cos \psi + \cos \theta$$

we can regard eq. (1) as a conformal map. As was pointed out in Sec. 1, it should be remarked that the Weyl factor $\omega$ can also be given using the Jacobian $J$ by

$$dt \wedge dx = J \, d\psi \wedge d\theta, \quad J = \frac{1}{\omega^2}.$$  

In $d$ ($> 2$) dimensions, the conventional conformal map is so introduced that the distance satisfies an axisymmetric property, so that eq. (1) is generalized as $t \pm r = \tan \left( \frac{\psi \pm \theta}{2} \right)$, where $r = \sqrt{\sum_{i=1}^{d-1} x_i^2}$. In this case, we obtain $ds^2 = dt^2 - (dr^2 + r^2 \, d\Omega^2) = \frac{1}{\omega^2} \left( d\psi^2 - d\theta^2 - \sin^2 \theta \, d\Omega^2 \right)$ [6], with $\omega$ remaining the same as in eq. (3). Nonetheless, we deal with the Jacobian, rather than the metric. For $d = 3$, the transformation analogous to eq. (1) may be written as

$$\begin{cases}
  x_0 + x_1 - x_2 = \tan \left( \frac{\theta_0 + \theta_1 - \theta_2}{2} \right), \\
  x_0 - x_1 + x_2 = \tan \left( \frac{\theta_0 - \theta_1 + \theta_2}{2} \right), \\
  x_0 - x_1 - x_2 = \tan \left( \frac{\theta_0 - \theta_1 - \theta_2}{2} \right),
\end{cases}$$

for $(\theta_0, \theta_1, \theta_2) \in S_3$ and $(x_0, x_1, x_2) \in \mathbb{R}^3$. In this case, however, the Jacobian can be given by choosing three factors from the Jacobian for the corresponding transformation in $d = 4$, with the fourth angular variable $\theta_3 \to 0$. In this sense, we concentrate on the case of $d = 4$.

For $d = 4$, we have two maps $\phi_+, \phi_- : (-\pi, \pi)^4 \to \mathbb{R}^4$, which are analogous to eq. (1), by

$$\phi_\pm : \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \mapsto \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2} (U_\pm)^{-1} \left( I \otimes \tan \right) U_\pm \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix},$$

where $U_\pm := \begin{pmatrix} U_{00}^\pm & 0 & 0 & 0 \\ 0 & U_{11}^\pm & 0 & 0 \\ 0 & 0 & U_{22}^\pm & 0 \\ 0 & 0 & 0 & U_{33}^\pm \end{pmatrix}$, and $U_{ij}^\pm := \begin{pmatrix} \cos \frac{\theta_i \pm \theta_j}{2} \\ \sin \frac{\theta_i \pm \theta_j}{2} \end{pmatrix}$.
where $U_+$ and $U_-$, which are not only orthogonal but also symmetric matrices, are given by

$$U_+ = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}, \quad U_- = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \end{pmatrix}.$$  \hfill (6)

For notational convenience, we have represented $(\tan \chi_0 \tan \chi_3) \in \mathbb{R}^4$ as $(\mathbb{1} \otimes \tan) \begin{pmatrix} \chi_0 \\ \chi_3 \end{pmatrix}$. Notice that $U_+$ and $U_-$ are related to (see also Appendix A)

$$U_+ U_- U_+ = U_+ U_- U_+ = \eta,$$  \hfill (7)

where $\eta = \text{diag}(-1, 1, 1, 1)$, a diagonal matrix. As in case of eq. (2), $S_4$ can be rewritten as

$$S_4 = S_4^{(+)} \cap S_4^{(-)}; \quad S_4^{(-)} = \{(\theta_0, \theta_1, \theta_2, \theta_3) \in \mathbb{R}^4 \mid -\pi < \theta_i^{(\pm)} < \pi \quad (i = 0, 1, 2, 3)\},$$

where $\theta_i^{(\pm)}$s (for $i = 0, 1, 2, 3$) are related to $\theta_i$'s through the relation

$$(\theta_0^{(\pm)} \theta_1^{(\pm)} \theta_2^{(\pm)} \theta_3^{(\pm)}) = 2 (\theta_0 \theta_1 \theta_2 \theta_3) U_{\pm}.$$

Under the map $\phi_{\pm}$, the Jacobian $J$ is given by

$$dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 = J d\theta_0 \wedge d\theta_1 \wedge d\theta_2 \wedge d\theta_3, \quad J = \frac{1}{\omega_{\pm}},$$  \hfill (9)

where $\omega_{\pm} = 2^2 \det \left[\mathbb{1} \otimes \cos\right] U_{\pm} \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$. Before proceeding further, we give some properties concerning $\omega_{\pm}$. For all $i, j \in \{0, 1, 2, 3\}$, we have [7]

$$\begin{cases} \omega_{\pm} \rightarrow \omega_{\pm} & (\theta_i \leftrightarrow \theta_j) \\
\omega_+ \leftrightarrow \omega_- & (\theta_i \rightarrow -\theta_i). \end{cases}$$

Moreover, the product $\omega_+ \omega_-$ can be written using a symmetric polynomial with respect to $\cos \theta_0, \ldots, \cos \theta_3$ as

$$\omega_+ \omega_- = \Psi(\cos \theta_0, \cos \theta_1, \cos \theta_2, \cos \theta_3),$$

where the functional form of the polynomial $\Psi(\chi_0, \chi_1, \chi_2, \chi_3)$ is given by

$$\Psi(\chi_0, \chi_1, \chi_2, \chi_3) = 4(\chi_0 \chi_1 + \chi_2 \chi_3)(\chi_0 \chi_2 + \chi_3 \chi_1)(\chi_0 \chi_3 + \chi_1 \chi_2) - \prod_{i=0}^{3} (s - 2 \chi_i),$$  \hfill (10)

with $s = \sum_{i=0}^{3} \chi_i$. It may be interesting to point out that each factor in the right-hand side of eq. (10) is either invariant or anti-invariant (change in overall sign) under the linear transform $U : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ (for $U = U_+, U_- \ldots$). To be more concrete, we have

$$\begin{cases} \chi_0 \chi_1 + 2 \chi_2 \chi_3, \chi_0 \chi_2 + 2 \chi_3 \chi_1, \chi_0 \chi_3 + 2 \chi_1 \chi_2 \in A_+, \\
s - 2 \chi_0 \in A_-, \\
s - 2 \chi_i \in I_- \quad (i = 1, 2, 3), \end{cases}$$

for $U = U_+, U_- \ldots$. Moreover, the product $\omega_+ \omega_-$ can be written using a symmetric polynomial with respect to $\cos \theta_0, \ldots, \cos \theta_3$ as

$$\omega_+ \omega_- = \Psi(\cos \theta_0, \cos \theta_1, \cos \theta_2, \cos \theta_3),$$

where the functional form of the polynomial $\Psi(\chi_0, \chi_1, \chi_2, \chi_3)$ is given by

$$\Psi(\chi_0, \chi_1, \chi_2, \chi_3) = 4(\chi_0 \chi_1 + \chi_2 \chi_3)(\chi_0 \chi_2 + \chi_3 \chi_1)(\chi_0 \chi_3 + \chi_1 \chi_2) - \prod_{i=0}^{3} (s - 2 \chi_i),$$  \hfill (10)

with $s = \sum_{i=0}^{3} \chi_i$. It may be interesting to point out that each factor in the right-hand side of eq. (10) is either invariant or anti-invariant (change in overall sign) under the linear transform $U : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ (for $U = U_+, U_- \ldots$). To be more concrete, we have

$$\begin{cases} \chi_0 \chi_1 + 2 \chi_2 \chi_3, \chi_0 \chi_2 + 2 \chi_3 \chi_1, \chi_0 \chi_3 + 2 \chi_1 \chi_2 \in I_+, \\
s - 2 \chi_0 \in A_-, \\
s - 2 \chi_i \in I_- \quad (i = 1, 2, 3), \end{cases}$$
Table 1. Linear polynomials in $I_\pm, A_\pm$, and quadratic polynomials in $I_\pm$. The linear polynomials in $I_\pm, A_\pm$ can be derived from $U_\pm U_\mp = U_\mp \eta$ by eq. (7). Since $U_\pm$ is a unitary matrix, it follows that $\sum_{i=0}^{3} \chi_i^2$ is a quadratic polynomial in $I_\pm$. Other quadratic polynomials in $I_\pm$ can be derived from the statement that $(p, q \in I_\pm) \implies (pq, p \pm q \in I_\pm)$. For example, $\chi_0\chi_1 + 2\chi_2\chi_3 \in I_+$ can be derived from the identity $4(\chi_0\chi_1 + 2\chi_2\chi_3) = (-\chi_0 - \chi_1 + 2 + (\chi_0 + \chi_1 + 2\chi_2 + \chi_3)^2 = 2 \sum_{i=0}^{3} \chi_i^2$.

| $U$ | Linear invariant | Linear anti-invariant | Quadratic invariant |
|-----|-----------------|----------------------|-------------------|
| $U_+$ | $-\chi_0 + \chi_1 - \chi_2 + \chi_3$ | $-\chi_0 - \chi_1 + \chi_2 + \chi_3$ | $\chi_0\chi_1 + 2\chi_2\chi_3$ |
|     | $+\chi_0 + \chi_1 - \chi_2 + \chi_3$ | $-\chi_0 - \chi_1 + \chi_2 + \chi_3$ | $\chi_0\chi_1 + 2\chi_2\chi_3$ |
| $U_-$ | $-\chi_0 + \chi_1 - \chi_2 + \chi_3$ | $\chi_0 - \chi_1 - \chi_2 + \chi_3$ | $\chi_0\chi_1 + \chi_2\chi_3$ |
|     | $-\chi_0 + \chi_1 - \chi_2 + \chi_3$ | $\chi_0 - \chi_1 + \chi_2 - \chi_3$ | $\chi_0\chi_1 - \chi_3$ |

where $I_\pm, A_\pm \in \mathcal{P}$ (where $\mathcal{P}$ represents the set of polynomials with respect to $\chi_0, \ldots, \chi_3$) represent the sets such that (for more details, see Table 1)

$$\begin{align*}
I_\pm := \{p(\chi_0, \ldots, \chi_3) \in \mathcal{P} \mid p(\theta_0^{(\pm)}, \ldots, \theta_3^{(\pm)}) = +p(\theta_0, \ldots, \theta_3)\}, \\
A_\pm := \{p(\chi_0, \ldots, \chi_3) \in \mathcal{P} \mid p(\theta_0^{(\pm)}, \ldots, \theta_3^{(\pm)}) = -p(\theta_0, \ldots, \theta_3)\}.
\end{align*}$$

with $\theta_0^{(\pm)}$ given by eq. (8). As is analogous to $\Psi(\chi_0, \chi_1, \chi_2, \chi_3)$ in eq. (10), we have symmetric polynomials $\Psi(\chi_0, \chi_1)$ and $\Psi(\chi_0, \chi_1, \chi_2)$ for $d = 2$ and $d = 3$, respectively (see Appendix B).

3. Quaternionic analysis

In this section, we deal with a general (differentiable) transformation $f_n : \mathbb{R}^4 \supset \Omega \to \mathbb{R}$ (for $n = 0, 1, 2, 3$). Under the transformation, the Jacobian $J$ is given by

$$df_0 \wedge df_1 \wedge df_2 \wedge df_3 = J \, dt \wedge dx \wedge dy \wedge dz,$$

where $f_n = f_n(t, x, y, z)$. It may be convenient to introduce two quaternions $f, q \in \mathbb{H}$ such that

$$\begin{align*}
f &= f_0 + i f_1 + j f_2 + k f_3, \\
q &= t + ix + jy + kz,
\end{align*}$$

where $i^2 = j^2 = k^2 = ijk = -1$. Quaternions can be regarded as a pair of complex numbers in the sense that

$$\begin{align*}
f &= g + j h, \\
q &= v + j w,
\end{align*}$$

where $g = f_0 + i f_1, h = f_2 - i f_3$, and $v = t + ix, w = y - iz$. Under the decomposition, we obtain

$$\begin{align*}
dg \wedge d\bar{g} \wedge dh \wedge d\bar{h} &= 4 \, df_0 \wedge df_1 \wedge df_2 \wedge df_3, \\
dv \wedge d\bar{v} \wedge dw \wedge d\bar{w} &= 4 \, dt \wedge dx \wedge dy \wedge dz,
\end{align*}$$

...
where $\bar{z}$ for $z \in \mathbb{C}$ represents the complex conjugate of $z$. Hence, eq. (11) can be rewritten as
\[
dq \wedge d\bar{g} \wedge dh \wedge d\bar{h} = J dv \wedge dv \wedge dw \wedge dw.
\] (12)
Considering that $\phi \in \{ g, \bar{g}, g, \bar{w} \}$ can be regarded as a function of $v, \bar{v}, w, \bar{w}$, we can write $d\phi$ as $d\phi = \phi_v dv + \phi_{\bar{v}} d\bar{v} + \phi_w dw + \phi_{\bar{w}} d\bar{w}$ (where the subscript represents the derivative as $\phi_v = \partial \phi / \partial v$). As a consequence, $J$ can be composed into six terms using the Jacobian $\frac{\partial(f, g)}{\partial(x,y)} = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ as
\[
J = \sum_{n=1}^{6} J_n, \quad J_n = \sigma_n \frac{\partial(g, h)}{\partial(t_n, x_n)} \frac{\partial(\bar{g}, \bar{h})}{\partial(y_n, z_n)},
\]
where $\sigma_n, (t_n, x_n), (y_n, z_n)$ are summarized in Table 2. Considering the relation $(y_n, z_n) = (\bar{t}_n, \bar{x}_n)$ for $n = 1, 2, 3, 4$, together with the sign of $\sigma_n$, we find that $J_1, J_2 \geq 0$ and $J_3, J_4 \leq 0$, although the sign of $J_5 + J_6$ is indefinite. Thus, $J$ can be classified as
\[
J = J_+ + J_- + J_{\text{ind}}, \quad \begin{cases} J_+ = J_1 + J_2, \\ J_- = J_3 + J_4, \\ J_{\text{ind}} = J_5 + J_6, \end{cases}
\] (13)
depending on the sign of $J_n$. A quadratic map is exemplified in the decomposition of $J$ as in eq. (13).

**Example 3.1** Let $\phi : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be given by
\[
\phi : \left( \begin{array}{c} \theta_0 \\ \theta_1 \\ \theta_2 \\ \theta_3 \end{array} \right) \mapsto \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \\ x_3 \end{array} \right) = \frac{1}{4} (U_+)^{-1} \begin{pmatrix} (\theta_0^{(+)})^2 \\ (\theta_1^{(+)})^2 \\ (\theta_2^{(+)})^2 \\ (\theta_3^{(+)})^2 \end{pmatrix},
\] (14)
where $U_+$ and $\theta_i^{(+)}$ are given by eqs. (6) and (8), respectively. In this case, $J$ and $J_i$ $(i = 1, 2, \ldots, 6)$ are calculated as
\[
J = \prod_{i=0}^{3} \theta_i^{(+)}, \quad \begin{cases} J_1 = (\theta_1 + \theta_2 \theta_3)^2, \\ J_2 = (\theta_0 - \theta_2 \theta_3)^2, \\ J_3 = -\frac{1}{4} (\theta_0^2 + \theta_1^2 - \theta_2^2 - \theta_3^2)^2, \\ J_4 = -\frac{1}{4} (\theta_0^2 - \theta_1^2 - \theta_2^2 + \theta_3^2)^2. \end{cases}
\]
Notice that $J$ in this case is not necessarily non-negative. This should be compared with the map $\phi_{\pm}$ in eq. (5), where the Jacobian $J$ is always non-negative, as is shown in eq. (9).

**Table 2.** Decomposition of $J$ as $J = \sum_{n=1}^{6} J_n$, where $J_n = J_n (n = 1, 2, 3, 4)$ and $J_5 = J_6$ are satisfied, which guarantees $J \in \mathbb{R}$. While $J_1, J_2 \geq 0$ and $J_3, J_4 \leq 0$, the sign of $J_5 + J_6$ is not definite.

| $n$ | $\sigma_n$ | $(t_n, x_n)$ | $(y_n, z_n)$ | $S$ such that $J_n \in S$ | Sign of $\text{Re} (J_n)$ |
|-----|-------------|-------------|-------------|-----------------------------|-----------------------------|
| 1   | 1           | $(v, w)$    | $(\bar{v}, \bar{w})$ | $\mathbb{R}$ | Non-negative |
| 2   | 1           | $(\bar{v}, \bar{w})$ | $(v, w)$ | $\mathbb{R}$ | Non-negative |
| 3   | -1          | $(v, \bar{w})$ | $(v, w)$ | $\mathbb{R}$ | Non-positive |
| 4   | -1          | $(\bar{v}, w)$ | $(v, \bar{w})$ | $\mathbb{R}$ | Non-negative |
| 5   | -1          | $(v, \bar{v})$ | $(w, \bar{w})$ | $\mathbb{C}$ | Indefinite |
| 6   | -1          | $(w, \bar{w})$ | $(v, \bar{v})$ | $\mathbb{C}$ | Indefinite |
As was shown in the above example, the sign of $J$ is not definite. Here we proceed to obtain a sufficient condition to guarantee the non-negativity of $J$. Such a condition is given by imposing the condition of $J_\rightarrow$ → 0.

**Proposition 3.1** In the limit of $J_\rightarrow$ → 0, $J$ turns out to be non-negative, despite the indefiniteness of the sign of $J_{\text{ind}}$, that is

$$\lim_{J_\rightarrow \rightarrow 0} J \geq 0. \quad (15)$$

**Proof.** Recalling that $J_3$ and $J_4$ are non-positive, and that $(g_n, z_n) = (\bar{t}_n, \bar{x}_n)$ for $n = 3, 4$, we find that

$$J_\rightarrow \rightarrow 0 \iff \begin{cases} J_3 \rightarrow 0 \\ J_4 \rightarrow 0 \end{cases} \iff \begin{cases} g_v h_{\bar{w}} - g_{\bar{a}} h_v \rightarrow 0 \\ g_v h_w - g_a h_{\bar{a}} \rightarrow 0 \end{cases}$$

In this limit, $J_5$ reduces to $\frac{h_v h_{\bar{a}}}{h_w h_{\bar{w}}} |g_{\bar{a}} h_w - g_w h_{\bar{a}}|^2$, assuming, for the time being, $h_w h_{\bar{a}} \neq 0$, so that

$$J_{\text{ind}} = J_5 + J_\rightarrow \rightarrow 1 - \left( \frac{h_v h_{\bar{a}}}{h_w h_{\bar{w}}} \right) |g_{\bar{a}} h_w - g_w h_{\bar{a}}|^2,$$

where (c.c.) represents the complex conjugate. In the meanwhile, $J_1 \rightarrow \frac{h_v h_{\bar{a}}}{h_w h_{\bar{w}}} |g_{\bar{a}} h_w - g_w h_{\bar{a}}|^2$ and $J_2 \rightarrow \frac{h_v h_{\bar{a}}}{h_w h_{\bar{w}}} |g_{\bar{a}} h_w - g_w h_{\bar{a}}|^2$, resulting in the relation that

$$J_\rightarrow + J_{\text{ind}} \rightarrow \left| \frac{h_v}{h_{\bar{a}}} - \frac{\bar{h}_v}{h_w} \right|^2 |g_{\bar{a}} h_w - g_w h_{\bar{a}}|^2 \geq 0.$$ 

So far, we have shown eq. (15) under the condition of $h_w h_{\bar{a}} \neq 0$. In what follows, we show eq. (15) even in the case of $h_w h_{\bar{a}} = 0$.

(i) $h_w = 0$. In this case, $J_\rightarrow$ → 0 implies that either $g_w \rightarrow 0$ or $h_{\bar{a}} \rightarrow 0$.

(a) $g_w \rightarrow 0$. In this case, $J_5$ (and hence $J_6$) turns out to be vanishing, leading to $J \rightarrow J_1 + J_2 \geq 0$.

(b) $h_{\bar{a}} \rightarrow 0$. In this case, $J_5 \rightarrow -g_v \bar{g}_w h_w \bar{h}_w$, $J_6 \rightarrow -g_w \bar{g}_v h_w \bar{h}_v$, and $J_1 \rightarrow g_w \bar{g}_a h_v \bar{h}_v, J_2 \rightarrow g_v \bar{g}_w h_w \bar{h}_w$, so that $J \rightarrow J_1 + J_2 + J_5 + J_6 = |g_w \bar{h}_v - g_v \bar{h}_w|^2 \geq 0$.

(ii) $h_{\bar{a}} = 0$. As in the case of $h_w = 0$ above, it is found that $J$ turns out to be non-negative in the limit of $J_\rightarrow$ → 0. \[ \square \]

**Remark 3.1** The decomposition of $J$ into the terms whose sign is non-negative, non-positive, or indefinite, as in eq. (13), is not uniquely determined. For example, $J$ can be decomposed as

$$J = J'_+ + J'_- + J'_{\text{ind}},$$

$$\begin{cases} J'_+ = + \frac{\partial (g, h)}{\partial (v, w)} \bigg|_{(v, w)}^2 + \frac{\partial (g, h)}{\partial (v, \bar{w})} \bigg|_{(v, \bar{w})}^2, \\ J'_- = - \frac{\partial (g, h)}{\partial (v, w)} \bigg|_{(v, w)}^2 - \frac{\partial (g, h)}{\partial (v, \bar{w})} \bigg|_{(v, \bar{w})}^2, \\ J'_{\text{ind}} = \left[ \frac{\partial (g, \bar{h})}{\partial (v, w)} \frac{\partial (h, h)}{\partial (v, \bar{w})} \right]_{(v, w)} + \text{(c.c.)} \bigg|_{(v, \bar{w})} \end{cases}.$$
Even in this case, the relation analogous to eq. (15) holds in the form that

$$\lim_{J' \to 0} J \geq 0.$$  \hspace{1cm} (16)

This is because under the substitution \((g, v) \leftrightarrow (\bar{g}, \bar{v}) \text{ [or} (h, w) \leftrightarrow (\bar{h}, \bar{w})\)], we have

\((J, J_-) \leftrightarrow (J, J'_-),\)

(the invariance of \(J\) comes from its definition, eq. (12)), so that the relation of eq. (15) implies eq. (16), and vice versa.

As a simple application of Prop. 3.1, we use the map \(\phi\) in eq. (14). In this case, eq. (15) dictates that

$$\lim_{\theta_0^2+\theta_1^2+\theta_2^2 \to 0} \prod_{i=0}^{3} \theta_i^{(+)} \geq 0,$$

which, as first sight, does not seem to be so trivial an inequality (see Appendix C).

At the end of this section, we discuss the formal analogy between the condition of \(J_- = 0\) and the "quaternionic analyticity", which can be regarded as the generalization of the Cauchy-Riemann relation in complex analysis. The quaternionic analyticity [5] is given by the vanishing of the trace of \(2 \times 2\) matrices \(M_1\) and \(M_2\) as

\[
\text{tr } M_1 = \text{tr } M_2 = 0 \quad \text{(Cauchy-Riemann-Fueter relation),} \hspace{1cm} (18)
\]

where \(M_1 = \left( \begin{array}{cc} g & g_v \\ h & \bar{h} \end{array} \right) \) and \(M_2 = \left( \begin{array}{cc} \bar{g} & \bar{g}_v \\ \bar{h} & h \end{array} \right) \); whereas the condition of \(J_- = 0\) can be written as the vanishing of the determinant of \(M_1\) and \(M_2\), that is

\[
\det M_1 = \det M_2 = 0 \quad \text{("quaternionic conformality")}. \hspace{1cm} (19)
\]

The relation of eq. (18) is a sufficient condition for harmonicity of \(f\) as \(\nabla^2 f = 0\) \((\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_w^2)\) or equivalently \(g_{zz} + g_{ww} = h_{zz} + h_{ww} = 0\), while the relation of eq. (19) is a sufficient condition for non-negativity of \(J\) as \(J \geq 0\). Both relations are compatible.

4. Conclusion
In generalizing the conformal map, we have dealt with a volume element, rather than a distance (metric), although in two dimensions, they transform in the same way. We have imposed the non-negativity of the Jacobian \(J\) in the volume element, as is the non-negativity of the Weyl factor in the metric in the ordinary conformal theory. In four dimensions, we give a sufficient condition for \(J \geq 0\), as is shown in Prop. 3.1. In an analogous way where the non-negativity of the Weyl factor in the two-dimensional metric is guaranteed by the Cauchy-Riemann relation in complex analysis, the non-negativity of the Jacobian in four-dimensional volume element is closely related to the Cauchy-Riemann-Fueter relation in quaternionic analysis, in the sense that the latter is given by the vanishing of the trace of the Jacobi matrices \(M_1\) and \(M_2\) [see eq. (18)], while the former is given by the vanishing of their determinant [see eq. (19)].

In dimensions higher than four, it would be interesting to examine the analogous relation between the "conformality" and the analyticity, where the Clifford algebra analysis, for example, would be of great use.

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Appendix A. Decomposition of $U_+$ and $U_-$

Decompose $U_+$ and $U_-$ as $U_+ = \frac{1}{2} (\mathbb{1} + \sum_{i=1}^{3} e_i)$ and $U_- = \frac{1}{2} (\mathbb{1} + \sum_{i=1}^{3} \tilde{e}_i)$, where $\mathbb{1} = \text{diag}(1,1,1,1)$. Then $e_i$ and $\tilde{e}_i$ (for $i = 1, 2, 3$) can be written as

$$
\begin{align*}
& e_1 = \sigma_0 \otimes \sigma_1, \\
& e_2 = \sigma_1 \otimes \sigma_0, \\
& e_3 = \sigma_1 \otimes (-\sigma_1), \\
& \tilde{e}_1 = \sigma_3 \otimes (-\sigma_1), \\
& \tilde{e}_2 = \sigma_1 \otimes (-\sigma_3), \\
& \tilde{e}_3 = i\sigma_2 \otimes i\sigma_2,
\end{align*}
$$

where $\sigma_0 = \text{diag}(1,1)$ and $\sigma_i$ (for $i = 1, 2, 3$) represent the Pauli matrices. The commutation relations between $e_i$’s and $\tilde{e}_j$’s and between themselves are given by

$$
\begin{align*}
& e^2 \equiv e_1^2 = e_2^2 = e_3^2 = -e_1 e_2 e_3 = 1, \\
& \tilde{e}_1^2 = \tilde{e}_2^2 = \tilde{e}_3^2 = -\tilde{e}_1 \tilde{e}_2 \tilde{e}_3 = 1,
\end{align*}
$$

and

$$
\begin{align*}
& [e_i, \tilde{e}_i] = 0 \ (i = 1, 2, 3), \\
& \{e_i, \tilde{e}_j\} = 0 \ (i \neq j),
\end{align*}
$$

where $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$. From the first relations in eq. (A.2), we obtain the commutativity $[e_i, \tilde{e}_j] = 0 = [\tilde{e}_i, \tilde{e}_j]$ for all $i, j \in \{1, 2, 3\}$. As an application of eq. (A.2), we calculate $U_+ U_- U_+$. A slightly lengthy calculation yields

$$
U_+ U_- U_+ = \frac{1}{2} \left( \mathbb{1} + \sum_{i=1}^{3} e_i \tilde{e}_i \right).
$$

Under the representation of eq. (A.1), it follows that $1 + \sum_{i=1}^{3} e_i \tilde{e}_i = 2\eta$, where $\eta = \text{diag}(-1,1,1,1)$, which amounts to the Minkowski metric, and plays a role in quaternionic conjugate $C : \mathbb{H} \ni q \mapsto \bar{q} \in \mathbb{H}$ by $\bar{q} = q_0 - iq_1 - jq_2 - kq_3$ for $q = q_0 + iq_1 + jq_2 + kq_3 \ (q_0, \ldots, q_3 \in \mathbb{R})$.

It may be interesting to point out that the elements $e_1, e_2, e_3$ and $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ form a group, whose order is 32. Denote by $G$ the group as

$$
G = \langle e_1, e_2, e_3, \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \mid \text{eq. (A.2)} \rangle.
$$

Then all the elements of $G$ are given by

$$
G = \bigcup_{\mu, \nu = 0}^{3} \left\{ e_\mu \tilde{e}_\nu, -e_\mu \tilde{e}_\nu \right\} \quad (e_0 := 1, \tilde{e}_0 := 1),
$$

so that the order of $G$ is given by $2 \times 4^2$. Since $e_\mu \tilde{e}_\nu$ (for $\mu, \nu \in \{0, 1, 2, 3\}$) are linearly independent over $\mathbb{R}$, any $x \in M_4(\mathbb{R})$ (where $M_4(\mathbb{R})$ represents the set of $4 \times 4$ real matrices) can be written as a linear combination of $e_\mu$’s and $\tilde{e}_\nu$’s in the form

$$
x = \sum_{\mu, \nu = 0}^{3} c_{\mu \nu} e_\mu \tilde{e}_\nu \quad (c_{\mu \nu} \in \mathbb{R}).
$$

This expansion can be rearranged using the 16-dimensional real Clifford algebra $Cl_{2,2}$ as

$$
x = \alpha \mathbb{1} + \beta \gamma_5 + \sum_{\mu = 0}^{3} (\alpha_\mu \gamma_\mu + \beta_\mu \gamma_5 \gamma_\mu) + \sum_{0 \leq \mu < \nu \leq 3} \alpha_{\mu \nu} \gamma_\mu \gamma_\nu \quad (\alpha, \beta, \alpha_\mu, \beta_\nu, \alpha_{\mu \nu} \in \mathbb{R}),
$$

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where \( \gamma_0 = e_1, \gamma_1 = \tilde{e}_2, \gamma_2 = e_1 \tilde{e}_3, \gamma_3 = e_2 \tilde{e}_3 \), with \( \gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) (for \( e_3 \tilde{e}_3 \)). Actually, \( \gamma_\mu \) and \( \gamma_\nu \) satisfy \( \{ \gamma_\mu, \gamma_\nu \} = 2 g_{\mu\nu} \mathbb{I} \) (for \( \mu, \nu = 0, 1, 2, 3 \)), where \( g_{00} = g_{11} = 1, g_{22} = g_{33} = -1 \) and \( g_{\mu\nu} = 0 \) for \( \mu \neq \nu \).

In connection with quaternions, consider the three following elements in \( G \):

\[
\begin{align*}
\Delta_1 &= -e_2 \tilde{e}_3, \\
\Delta_2 &= -e_3 \tilde{e}_1, \\
\Delta_3 &= +e_1 \tilde{e}_2,
\end{align*}
\]

which can be identified with the quaternionic imaginary units \((i, j, k)\) in that

\[
\Delta_1^2 = \Delta_2^2 = \Delta_3^2 = \Delta_1 \Delta_2 \Delta_3 = -\mathbb{I}.
\]

Notice that we have introduced, for later convenience, the minus sign in the right-hand side of \( \Delta_1 \) and \( \Delta_2 \) in eq. (A.3), although the minus sign is irrelevant to the relation of eq. (A.4). Let the map \( \rho : \mathbb{H} \to M_4(\mathbb{R}) \) by

\[
\rho(q_0 + iq_1 + jq_2 + kq_3) = q_0 \mathbb{I} + \sum_{i=1}^{3} q_i \Delta_i \quad (q_0, q_1, q_2, q_3 \in \mathbb{R}).
\]

Then \( \rho(q) \) for \( q \in \mathbb{H} \) turns out to be the (faithful) representation matrix for the right multiplication \( R_q : x \mapsto qx \) (for \( x \in \mathbb{H} \)). Recalling that the quaternions are associative, so that \( R_{qq'} = R_q R_{q'} \) for all \( q, q' \in \mathbb{H} \), we reconfirm that the map \( \rho \) is isomorphic to \( \mathbb{H} \), that is, \( \rho(qq') = \rho(q) \rho(q') \) for all \( q, q' \in \mathbb{H} \).

**Appendix B. Analogous symmetric polynomial in \( d = 2, 3 \)**

For \( d = 2, 3 \), the polynomial analogous to eq. (10) is given by

\[
\begin{align*}
\Psi(\chi_0, \chi_1) &= \chi_0 + \chi_1 \\
\Psi(\chi_0, \chi_1, \chi_2) &= \chi_0^2 + \chi_1^2 + 2\chi_0\chi_1\chi_2 - 1
\end{align*}
\]

(B.1)

where the \( \Psi \)'s are introduced in such a way that they satisfy

\[
\begin{align*}
\Psi(\cos \theta_0, \cos \theta_1) &= 2 \cos \left( \frac{\theta_0 + \theta_1}{2} \right) \cos \left( \frac{\theta_0 - \theta_1}{2} \right), \\
\Psi(\cos \theta_0, \cos \theta_1, \cos \theta_2) &= 2^2 \cos \left( \frac{\theta_0 + \theta_1 + \theta_2}{2} \right) \cos \left( \frac{-\theta_0 + \theta_1 + \theta_2}{2} \right) \cos \left( \frac{\theta_0 - \theta_1 + \theta_2}{2} \right) \cos \left( \frac{\theta_0 + \theta_1 - \theta_2}{2} \right).
\end{align*}
\]

More generally, \( \Psi(\chi_0, \ldots, \chi_{n-1}) \) represents the minimum polynomial of \( \chi_0, \ldots, \chi_{n-1} \) such that \( \Psi(\cos \theta_0, \ldots, \cos \theta_{n-1}) \geq 0 \) for \( \sum_{i=0}^{n-1} |\theta_i| \leq \pi \) (with \( \theta_0, \ldots, \theta_{n-1} \in S_n \)). By the construction of \( \Psi \)'s, it is easy to show that

\[
\begin{align*}
\Psi(\chi_0, \chi_1, 1) &= \Psi^2(\chi_0, \chi_1), \\
\Psi(\chi_0, \chi_1, \chi_2, 1) &= \Psi^2(\chi_0, \chi_1, \chi_2).
\end{align*}
\]

(B.2)

To reveal the relations of eq. (B.2), it may be convenient to rewrite the \( \Psi \) for \( d = 3, 4 \) as

\[
\begin{align*}
\Psi(\chi_0, \chi_1, \chi_2) &= \left( \sum_{i=0}^{2} \chi_i - 1 \right)^2 - 2 \prod_{i=0}^{2} (1 - \chi_i), \\
\Psi(\chi_0, \chi_1, \chi_2, \chi_3) &= \left( \sum_{i=0}^{3} x_i^2 + 2 \prod_{i=0}^{2} (1 - \chi_i) \right)^2 - 4 \prod_{i=0}^{2} (1 - \chi_i^2).
\end{align*}
\]
The polynomial $\Psi = \Psi(x, y, z)$ appears in the formula for the infinite series of the product of three Legendre polynomials $P_n(x)$'s as [8]

$$\sum_{n=0}^{\infty} (-1)^n (2n + 1) P_n(x) P_n(y) P_n(z) = \begin{cases} 0 & (\Psi > 0), \\ \frac{2}{\sqrt{\Psi}} & (\Psi < 0), \end{cases} \quad (B.3)$$

while $\omega_{\pm}$ in eq. (9) is realized in the analogous infinite series of four Legendre polynomials of the form [7]

$$\sum_{n=0}^{\infty} (-1)^n (2n + 1) \prod_{i=0}^{3} P_n(\cos \theta_i) = \begin{cases} 0 & (\omega_+ > 0, \omega_- > 0), \\ \frac{2}{\sqrt{\omega_+} \sqrt{\omega_-}} 2F1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\omega_-}{\omega_+}\right) & (\omega_+ < 0, \omega_- < 0), \\ \frac{2}{\sqrt{\omega_+} \sqrt{-\omega_-}} 2F1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{-\omega_-}{\omega_+}\right) & (\omega_+ > 0, \omega_- < 0), \\ \frac{2}{\sqrt{\omega_+} \sqrt{-\omega_-}} 2F1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{-\omega_-}{-\omega_+}\right) & (\omega_+ < 0, \omega_- > 0), \end{cases} \quad (B.4)$$

where $2F1(\alpha, \beta; \gamma; z)$ represents the hypergeometric function, and $2F1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$ is related to $K(k)$ (complete elliptic integral of the first kind) through the relation

$$K(k) := \int_0^\frac{\pi}{2} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \, d\theta = \frac{\pi}{2} 2F1\left(1, \frac{1}{2}; \frac{1}{2}; 1; k^2\right).$$

Notice that eq. (B.4), whose validity can be verified by numerical calculation, reduces to eq. (B.3) by taking the limit of $\theta_3 \to 0$ in eq. (B.4), from which $P_n(\cos \theta_3) \to 1$ (for all $n \in \mathbb{N}$) and $\omega_{\pm} \to \Psi(\cos \theta_0, \cos \theta_1, \cos \theta_2)$. In this sense, eq. (B.4) is the generalization of eq. (B.3).

**Appendix C. Derivation of eq. (17)**

To show eq. (17) in a direct way, it is convenient to start with the following relation

$$\prod_{i=0}^{3} \theta_i^{(+) \rightarrow (+) \to 0} = \prod_{i=0}^{3} (s - 2\theta_i) = \left(s = \sum_{i=0}^{3} \theta_i \right) = \left[ (\theta_2 + \theta_3)^2 - (\theta_0 - \theta_1)^2 \right] \left[ (\theta_0 + \theta_1)^2 - (\theta_2 - \theta_3)^2 \right].$$

Then it is found that

$$\lim_{\theta_0^2 + \theta_1^2 \to 0} \prod_{i=0}^{3} \theta_i^{(+) \rightarrow (+) \to 0} = \lim_{\theta_0^2 + \theta_1^2 \to 0} \theta_2^2 + \theta_3^2 \left[ 2 (\theta_0 \theta_1 + \theta_2 \theta_3) \right]^2,$$

which turns out to be non-negative.

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