PERIOD-DOUBLING CONTINUED FRACTIONS ARE ALGEBRAIC IN CHARACTERISTIC 2

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Abstract. Considering an arbitrary pair of distinct and non constant polynomials, $a$ and $b$ in $\mathbb{F}_2[t]$, we build a continued fraction in $\mathbb{F}_2((1/t))$ whose partial quotients are only equal to $a$ or $b$. In a previous work of the first author and Han (to appear in Acta Arithmetica), the authors considered two cases where the sequence of partial quotients represents in each case a famous and basic 2-automatic sequence, both defined in a similar way by morphisms. They could prove the algebraicity of the corresponding continued fractions for several pairs $(a, b)$ in the first case (the Prouhet-Thue-Morse sequence) and gave the proof for a particular pair for the second case (the period-doubling sequence). Recently Bugeaud and Han (arXiv:2203.02213) proved the algebraicity for an arbitrary pair in the first case. Here we give a short proof for an arbitrary pair in the second case.

1. Introduction

Let $\mathbb{F}_2((1/t))$ be the field of power series in $1/t$, where $t$ is a formal indeterminate, over the finite field $\mathbb{F}_2$. A non-zero element is $\alpha = t^n + \sum_{i<n} a_i \cdot t^i$ where $n$ belongs to $\mathbb{Z}$ and $a_i = 0$ or 1. An absolute value on $\mathbb{F}_2((1/t))$ is defined by $|0| = 0$ and $|\alpha| = |t|^n$ where $|t| > 1$ is a fixed given real number.

So $\mathbb{F}_2((1/t))$ is the completion of $\mathbb{F}_2[t]$ for this absolute value. Every irrational (resp. rational) element in $\mathbb{F}_2((1/t))$ can be expanded in an infinite (resp. finite) continued fraction: $\alpha = [a_0; a_1, a_2, ...]$ where the $a_i$ are in $\mathbb{F}_2[t]$ and $\deg(a_i) > 0$ for $i > 0$. For a basic introduction on fields of power series and continued fractions, the reader may consult [6].

The origin of the question discussed here is due to G.-N. Han and the first author. In [1], the authors considered two basic sequences, formed by an infinite word with two letters $a$ and $b$, belonging to the family of 2-automatic sequences (see [1] p.173 and p.176). Both sequences are obtained in a similar way, as fixed point of a morphism. For the first one, the $(a; b)$-Prouhet-Thue-Morse sequence, denoted by $t$, the morphism $\tau$ is defined by $\tau(a) = ab$ and $\tau(b) = ba$, and we have $t = \tau^\infty(a) = ababaabbaab...$

Note that this famous sequence was considered very long ago and has been the starting point of various studies. For the second one, the $(a; b)$-period-doubling sequence, denoted by $p$, the morphism $\sigma$ is defined by $\sigma(a) = ab$ and $\sigma(b) = aa$, and we have

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In both cases, we consider a pair \((a, b)\) of distinct and non-constant polynomials in \(\mathbb{F}_2[t]\), and we can associate with each sequence an infinite continued fraction in \(\mathbb{F}_2((1/t))\), \(\text{CF}(t)\) and \(\text{CF}(p)\), where the sequence of partial quotients is derived from the sequences \(\alpha_t\) and \(\alpha_p\):

\[
\alpha_t = \text{CF}(t) = [a; b, a, b, a, a, \ldots]
\]

and

\[
\alpha_p = \text{CF}(p) = [a; b, a, a, b, a, a, \ldots].
\]

It was proved in [4] that, for pairs \((a, b)\) with \(\deg a + \deg b \leq 7\), \(\alpha_t\) was a root of a polynomial of degree 4, with five coefficients in \(\mathbb{F}_2[t]\) depending on \((a, b)\). In a recent work, extending the case of \(\text{CF}(t)\), Bugeaud and Han [3] obtained the same result for all pairs \((a, b)\) and they gave the explicit formulas for the 5 coefficients as polynomials in \(a\) and \(b\).

Also in [4], it was proved that for \((a, b) = (t^3, t^2 + t + 1)\) we have

\[
(t^5 + t^3 + t^2) \cdot \alpha_p^4 + (t^8 + t^6 + t^5 + t^3) \cdot \alpha_p^3 + (t^5 + t^4 + t^3) \cdot \alpha_p^2 + 1 = 0.
\]

In this note, we give a short proof of the general case for the sequence \(p\) with the following theorem, thus confirming Conjecture 1.5 from [4].

**Theorem 1.1.** Let \(a, b\) be two distinct non-constant elements in \(\mathbb{F}_2[t]\). Let \(\alpha_p = [\sigma^\infty(a)] \in \mathbb{F}_2((1/t))\). Define \(P(x) \in \mathbb{F}_2(t)[x]\) to be

\[
P(x) = Ax^4 + Bx^3 + Cx^2 + 1
\]

with

\[
A = ab + b^2 + 1, \quad B = ab(a + b), \quad C = ab.
\]

Then \(P(\alpha_p) = 0\).

**Remark 1.** An elementary proof shows that \(P(x)\) has no solution in \(\mathbb{F}_2(t)\). Consequently, since \(\alpha_p\) is not quadratic, \(P(x)\) is irreducible and \(\alpha_p\) has degree 4 over \(\mathbb{F}_2(t)\).

**Remark 2.** The theorem remains true when we replace \(\mathbb{F}_2\) by any other field \(K\) of characteristic 2.

**Remark 3.** In fact we have proven that the continued fraction \(\alpha_p\) as a series in the ring \(\mathbb{F}_2((1/a, 1/b))\) is algebraic over \(\mathbb{F}_2(a, b)\).

In Section 2 we recall notation and formulas for continued fractions. In Section 3 we give the proof of Theorem 1.1 and in a last section we make some comments about the link with Riccati differential equations.

### 2. Notation and Basic Formulas for Continued Fractions

We use the same notation as in [6], which we recall in this section.

Let \(W = w_1, w_2, \ldots, w_n\) be a sequence of variables over a ring \(A\). We set \(|W| = n\) for the length of the word \(W\). We define the following operators for the word \(W\).

\[
W' = w_2, w_3, \ldots, w_n \text{ or } W' = \emptyset \text{ if } |W| = 1.
\]

\[
W'' = w_1, w_2, \ldots, w_{n-1} \text{ or } W'' = \emptyset \text{ if } |W| = 1.
\]
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\[ W^* = w_n, w_{n-1}, \ldots, w_1. \]

We consider the finite continued fraction associated to \( W^* \) to be

\[ [W] = [w_1, \ldots, w_n] = w_1 + \frac{1}{w_2 + \frac{1}{w_3 + \frac{1}{\ddots + \frac{1}{w_n}}}} \]

The continued fraction \([W]\) is a quotient of multivariate polynomials in the variables \( w_1, w_2, \ldots, w_n \). These polynomials are called continuants built on \( W^* \). They are defined inductively as follows:

Set \([\emptyset]\) = 1. If the sequence \( W^* \) has only one element, then we have

\[ [W] = [w_1] = w_1. \]

Hence, with the above notation, the continuants can be computed recursively on the length \(|W^*|\), by the following formula

\[ [W] = [w_1] [W^*'] = w_1 [W^*'] \]

Thus, with this notation, for any finite word \( W^* \), the finite continued fraction \([W]\) satisfies

\[ [W] = \frac{[W]}{[W']} \]

It is easy to prove by induction that \([W^*] = [W] \). For any finite sequences \( A \) and \( B \) of variables over \( A \), defining \( A, B \) as the concatenation of the sequences, by induction on \(|A|\), we also have the following generalization of (2.2)

\[ (A, B) = (A') (B') \]

Using induction of \(|W|\), we have the following classical identity

\[ (W) (W')'' - (W') (W'') = (-1)^{|W|} \]

3. Proof of Theorem

For \( n \geq 0 \), set

\[ W_n = (\sigma^n(a))''. \]

We have \( W_0 = \emptyset, W_1 = a, W_2 = aba, \) etc., and \(|W_n| = 2^n - 1\).

We will prove that \( P([W_n]) \) converges to 0. For this, we need to following two lemmas.

**Lemma 3.1.** For all \( n \geq 0 \),

\[ W_{n+1} = W_n, \varepsilon_n, W_n \]

where \( \varepsilon_n = a \) if \( n \) is even and \( \varepsilon_n = b \) if \( n \) is odd. In consequence, for \( n \geq 0 \), we have

\[ W_n = W^*_n. \]

**Proof.** We prove by induction. Suppose identity (3.1) holds for \( n \), then

\[ W_{n+2} = \sigma(\sigma^{n+1}(a))'' \]

\[ = \sigma(W_{n+1})a \]

\[ = \sigma(W_n), \sigma(\varepsilon_n), \sigma(W_n)a \]

\[ = W_{n+1}, \varepsilon_{n+1}, W_{n+1}. \]
In the last equality, we use the fact that \( \sigma(\varepsilon_n) = a\varepsilon_{n+1} \).

For \( n \geq 1 \), define \( u_n = \langle W_n \rangle \) and \( v_n = \langle W'_n \rangle \), so that

\[
[W_n] = \frac{u_n}{v_n}.
\]

**Lemma 3.2.** For all \( n \geq 1 \), we have

\[
u_{n+1} = \varepsilon_n u_n^2, \\
v_{n+1} = \varepsilon_n u_n v_n + 1.\]

**Proof.** We use **Lemma 3.1** identity (2.4), and the fact that we are in characteristic 2.

\[
u_{n+1} = \langle W_{n+1} \rangle \\
= \langle W_n, \varepsilon_n, W_n \rangle \\
= \langle W_n \rangle \langle \varepsilon_n, W_n \rangle + \langle W''_n \rangle \langle W_n \rangle \\
= \langle W_n \rangle \langle \varepsilon_n (W_n) + (W'_n) \rangle + \langle W''_n \rangle \langle W_n \rangle \\
= \langle W_n \rangle \langle \varepsilon_n (W_n) + (W'_n) \rangle + \langle W''_n \rangle \langle W_n \rangle \\
= \varepsilon_n u_n v_n + (W'_n) \langle W_n \rangle + \langle W''_n \rangle \langle W_n \rangle \\
= \varepsilon_n u_n v_n + 1.
\]

In the last step, we also use (2.5), and the symmetry of \( W_n \).

**Proof of Theorem 1.1.** For \( n \geq 1 \), set

\[
X_n = Au_n^4 + Bu_n^3 v_n + Cu_n^2 v_n^2 + v_n^4,
\]

so that \( P(u_n/v_n) = X_n/v_n^4 \).

Using **Lemma 3.2** and noticing that \( \varepsilon_n^2 + ab = \varepsilon_n(a + b) \), we obtain

\[
X_n + 1 = \varepsilon_n^4 u_n^4 (X_n + 1) + \varepsilon_n^3 u_n^4 (a + b) (1 + ab u_n^2).
\]

We observe that

\[
X_1 + 1 = a^3(a + b) = \varepsilon_0 u_1^2(a + b),
\]

this allows us, using **3.2** and the fact that \( \varepsilon_n \varepsilon_{n-1} = ab \), to get by induction that for all \( n \geq 1 \)

\[
X_n + 1 = \varepsilon_n u_n^2(a + b).
\]

Therefore

\[
P\left( \frac{u_n}{v_n} \right) = \frac{\varepsilon_n u_n^2(a + b) + 1}{v_n^4}.
\]
This is a quotient of polynomials in $F_2[t]$, and it can be easily seen from Lemma 3.2 that the degree in $t$ of the denominator minus that of the numerator tends to plus infinity as $n$ goes to infinity. And therefore the sequence converges to 0 in $F_2((1/t))$. □

4. Link with Riccati differential equations

In both cases, the Prouhet-Thue-Morse and the period-doubling continued fraction, a particular choice of the pair $(a, b)$ in $F_2[t]^2$, brings us back to an investigation undertaken 45 years ago. Indeed, in these two cases, taking $(a, b) = (t, t + 1)$, all the partial quotients of these continued fractions are $t$ or $t + 1$. In 1977, Baum and Sweet [2] considered all the infinite continued fractions in $F_2((1/t))$ whose partial quotients have all degree one.

They considered the subset $D$ of irrational continued fractions $\alpha$ in $F_2((1/t))$ such that

$$\alpha = [0; a_1, a_2, \ldots, a_n, \ldots]$$

where $\deg(a_i) = 1$ (i.e., $a_i = t$ or $t + 1$) for $i \geq 1$. Let $P$ the subset of $F_2((1/t))$ containing all $\alpha$ such that $|\alpha| < 1$. Hence $P$ contains $D$. We have:

**Theorem 4.1.** [Baum-Sweet, 1977] An element $\alpha \in P$ is in $D$ if and only if $\alpha$ satisfies

$$\alpha^2 + t\alpha + 1 = (1 + t)\beta^2$$

for some $\beta \in P$.

In order to obtain another formulation of Theorem 4.1, we will now use formal differentiation in $F_2((1/t))$. Continued fractions and formal differentiation in power series fields are related, these are the main tools in Diophantine approximation (see [3], p.221-225). We recall that in characteristic 2, which is the case considered here, an element has derivative zero if and only if this element is a square. Moreover the derivative of an element is a square (hence the second derivative is zero). We have the following theorem.

**Theorem 4.2.** An element $\alpha \in P$ is in $D$ if and only if $\alpha$ satisfies

$$(\alpha \cdot t(t + 1))' = \alpha^2 + 1.$$

**Proof.** If

$$\alpha^2 + t\alpha + 1 = (1 + t)\beta^2$$

then by derivation we get

$$(ta)' = \beta^2.$$

Consequently, we have

$$\alpha^2 + 1 = ta + (1 + t) (ta)' = (\alpha \cdot t(t + 1))'.$$

Conversely, if

$$(\alpha \cdot t(t + 1))' = \alpha^2 + 1.$$

then

$$\alpha^2 + t\alpha + 1 = (\alpha \cdot t(t + 1))' + t\alpha = (t + 1)(\alpha \cdot t)' = (t + 1)\beta^2,$$

for some $\beta \in P$ since $(ta)'$ is a square in $P$. □
Thus the Riccati differential equation
\[(R0) \quad (x \cdot t(t + 1))' = x^2 + 1\]
is characteristic of infinite continued fraction expansions in $\mathbb{F}_2((1/t))$, having all partial quotients of degree one. Indeed, there exists a direct proof, without Theorem 4.1, showing that a solution in $\mathbb{F}_2((1/t))$ of (R0) is irrational and has all partial quotients of degree one (see [5], p. 225).

Now we turn to elements in $\mathbb{F}_2((1/t))$ which are algebraic over $\mathbb{F}_2(t)$ of degree $d \geq 1$. If $\alpha$ is such an element, we have (see [5], p. 221)
\[(4.1) \quad \alpha' = a_0 + a_1 \alpha + \cdots + a_{d-1} \alpha^{d-1}\]
where the coefficients $a_i$ belong to $\mathbb{F}_2(t)$.

Concerning the algebraic continued fraction $\alpha_t = \text{CF}(t) \in \mathbb{F}_2((1/t))$ associated with the Prouhet-Thue-Morse sequence, we have observed that it satisfies the Riccati differential equation (see [3], Proposition 2.4):
\[(R) \quad (ab(a + b)x)' = (ab)'(1 + x^2).\]
In fact, the continued fraction $\text{CF}(p)$ (also algebraic of degree 4), associated with the period-doubling sequence satisfies the same Riccati differential equation.

**Proposition 4.3.** The continued fraction $\alpha_p = \text{CF}(p)$ satisfies the Riccati differential equation (R)

**Proof.** We know from Theorem 4.1 that
\[(4.2) \quad A\alpha_p^4 + B\alpha_p^3 + C\alpha_p^2 + 1 = 0\]
with
\[A = ab + b^2 + 1, \quad B = ab(a + b), \quad C = ab.\]
Hence, by straightforward derivation of (4.2), we get
\[B\alpha_p' + B'\alpha_p = A'\alpha_p^2 + C'.\]
Therefore, observing that $A' = C' = (ab)'$, we obtain
\[(ab(a + b)\alpha_p)' = (B\alpha_p)' = A'\alpha_p^2 + C' = (ab)'(1 + \alpha_p^2). \quad \Box\]

Note that for the pair $(a, b) = (t, t + 1)$, equation (R) reduces to (R0), the one stated in Theorem 4.2. This was the way, taking in consideration differential aspects for $\text{CF}(t)$ and $\text{CF}(p)$, knowing the equation (R0) in the basic case, and inspired by the particular case published in [4], that allowed us to guess the coefficients for the algebraic equation satisfied by $\text{CF}(p)$.

We have observed that the two particularly simple sequences, the Prouhet-Thue-Morse sequence and the period-doubling sequence, are close in many different ways, and not only in their generation process. This leads to the intuition that other 2-automatic sequences, defined on an alphabet of two letters, could have a representation as an algebraic continued fraction in $\mathbb{F}_2((1/t))$ (not necessarily of degree 4), as it happens for the two famous ones considered up to now.
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