Physical nonlinear aspects of classical and quantum q-oscillators

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Abstract

The classical limit of quantum q-oscillators suggests an interpretation of the deformation as a way to introduce non linearity. Guided by this idea, we considered q-fields, the partition function, and compute a consequence on specific heat and second order correlation function of the q-oscillator which may serve for experimental checks for the non linearity.
1. Introduction

Recently the concept of quantum groups was developed in connection with the solution of nonlinear equations [1-5]. The special case of the quantum Heisenberg-Weyl group which produced the notion of quantum q-oscillator was studied by Biedenharn [6] and Macfarlane [7]. They introduced q-oscillators starting from generalized commutation relations containing an extra dimensionless parameter q. The physical sense of the quantum q-oscillator up to now was not clarified. The attempt to connect quantum q-oscillators with relativistic oscillator model has been discussed in [8]. The attempt to introduce q-oscillators and quantum group SU(2) in the frame of the generalized James-Cummings model has been discussed in [9]. Nevertheless these attempts are based on pure mathematical properties of q-oscillators. The aim of our work is to show that the q-oscillator is a special case of a nonlinear quantum oscillator.

The nature of q-oscillators of the electromagnetic field is clarified by the nonlinearity of field vibrations. The partition function for both classical and quantum q-oscillator is calculated.

The spirit of the electromagnetic field q-quantum oscillator reminds us of such a model of electrodynamics as the nonlinear Born-Infeld model [10]. In addition to clarify the physical nature of quantum q-oscillator as the usual quantum oscillator with a "nonlinearity" of special type, the aim of this work is to study the classical q-oscillator. This turns out to be the classical nonlinear oscillator with the frequency depending on the amplitude by the particular functional dependence. We will show that in this sense the classical q-oscillator acquires a dependence on the amplitude of the physical pendulum. Assuming that the electromagnetic field be described by the q-oscillator (both -quantum and classical ones) we will study the physical consequences of this suggestion, namely how the parameters of the q-oscillators change the specific heat formula. This formula reflects the linear properties of the vibration of the vacuum at small amplitudes. It is clear that at very high intensities there must be influence of the nonlinearity of the vacuum vibrations. We will calculate the corrections to the partition function and to the specific heat formula.

As for notations, we shall denote by $\hbar$ the deforming parameter, to avoid introducing a new symbol with respect to those already widely used in the literature [1-7]. We apologize with reader for this. We ask him to keep in
mind that Planck’s constant is set equal to 1 everywhere in paper.

2. Quantum q-oscillator

Let us introduce the usual creation and annihilation oscillator operators \( a \) and \( a^\dagger \) obeying bosonic commutation relations

\[
[a, a^\dagger] = 1. \quad (2.1)
\]

The classical dynamical variables to which \( a \) and \( a^\dagger \) correspond oscillate with a frequency \( \omega = 1 \). It is known that the operators \( a, a^\dagger, 1 \) form the Lie algebra of Heisenberg-Weyl group. So, the linear harmonic oscillator may be connected with the generators of pure Heisenberg-Weyl Lie group. In view of the commutation relation (2.1) the usual scheme for generating the states of the harmonic oscillator is based on the properties of the Hermitean number operator \( \hat{n} = a^\dagger a \)

\[
[a, \hat{n}] = a, \quad [a^\dagger, \hat{n}] = -a^\dagger. \quad (2.2)
\]

Thus constructing the vacuum state \(|0\rangle\) obeying the equation

\[
a|0\rangle = 0 \quad (2.3)
\]

and the excited states

\[
|n\rangle = \frac{a^\dagger^n}{\sqrt{n!}}|0\rangle \quad (2.4)
\]

which are eigenstates of the number operator \( \hat{n} \)

\[
\hat{n}|n\rangle = n|n\rangle, n \in \mathbb{Z}^+ \quad (2.5)
\]

the matrix representation of the operators \( a \) and \( a^\dagger \) in the basis (2.4) have the known expressions

\[
a = \begin{pmatrix}
0 & \sqrt{1} & 0 & \ldots \\
0 & 0 & \sqrt{2} & 0 \\
0 & 0 & 0 & \sqrt{3} \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

\[
a^\dagger = \begin{pmatrix}
0 & 0 & 0 & \ldots \\
\sqrt{1} & 0 & 0 & \ldots \\
0 & \sqrt{2} & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix} \quad (2.6)
\]
while the number operator \( \hat{n} \) is described by the matrix
\[
\hat{n} = \begin{pmatrix}
0 & 0 & 0 & \ldots \\
0 & 1 & 0 & \ldots \\
0 & 0 & 2 & \ldots \\
& & & \ddots \\
& & & & \ddots 
\end{pmatrix}.
\]
(2.7)

The Hamiltonian for such a system is defined as
\[
H = \omega \frac{a^\dagger a + aa^\dagger}{2}, \quad \omega > 0.
\]
(2.8)

The q-oscillators may be introduced by generalizing the matrices (2.6) and (2.7) with the help of the q-integer numbers \( n_q \),
\[
n_q = \frac{\sinh n}{\sinh \bar{h} q}, \quad q = e^{\bar{h}}.
\]
(2.9)

Here \( \bar{h} \) and \( q \) are dimensionless c-numbers, which appear at this purely mathematical level. When \( \bar{h} = 0, q=1 \) and the q-integer \( n_q \) coincides with \( n \). Then, replacing the integers in (2.6) and (2.7) by q-integers we obtain matrices which define the annihilation and creation operators of the quantum q-oscillator,
\[
a_q = \begin{pmatrix}
0 & \sqrt{1_q} & 0 & \ldots \\
0 & 0 & \sqrt{2_q} & \ldots \\
0 & 0 & 0 & \sqrt{3_q} \\
& & & \ddots \\
& & & & \ddots 
\end{pmatrix}
\]
\[
a^\dagger_q = \begin{pmatrix}
0 & 0 & 0 & \ldots \\
0 & \sqrt{1_q} & 0 & \ldots \\
0 & 0 & \sqrt{2_q} & \ldots \\
& & & \ddots \\
& & & & \ddots 
\end{pmatrix}
\]
\[
\hat{n}_q = \begin{pmatrix}
0 & 0 & 0 & \ldots \\
0 & 1_q & 0 & \ldots \\
0 & 0 & 2_q & \ldots \\
& & & \ddots \\
& & & & \ddots 
\end{pmatrix}.
\]
(2.10)

since the action of \( \hat{n}_q \) on eigenstates \( |n> \) is given by
\[
\hat{n}_q|n> = \frac{\sinh n\bar{h}}{\sinh \bar{h}}|n>
\]
(2.11)
The above matrices obey the commutation relation

\[ [a_q, \hat{n}] = a_q, \quad [a_q^\dagger, \hat{n}] = -a_q^\dagger \]  \tag{2.12} 

but the commutation relations of the operators \( a_q \) and \( a_q^\dagger \) do not coincide with the boson commutation relations. Equation (2.1) is replaced by

\[ [a_q, a_q^\dagger] = F(\hat{n}) \]  \tag{2.13} 

where the function \( F(\hat{n}) \) has the form

\[ F(\hat{n}) = f^2(\hat{n} + 1) - f^2(\hat{n}) \]  \tag{2.14} 

For \( \hbar = 0 \) (2.12) reduces to (2.1).

In addition to the above commutation relation there exists the reordering relation

\[ a_qa_q^\dagger - qa_q^\dagger a_q = q^{-\hat{n}} \]  \tag{2.15} 

which usually is taken as the definition of q-oscillators.

It is worthy noting that the operators \( a_q \) and \( a_q^\dagger \) can be expressed in terms of the operators \( a \) and \( a^\dagger \) [11-13]

\[ a_q = af(\hat{n}), \quad a_q^\dagger = f(\hat{n})a^\dagger, \]  \tag{2.16} 

where (see Eq. (2.9))

\[ f(\hat{n}) = \sqrt{\frac{\hat{n}_q}{\hat{n}}} \]  \tag{2.17} 

We have also

\[ \hat{n}_q = a_q^\dagger a_q \]  \tag{2.18} 

and

\[ [a_q, \hat{n}_q] = F(\hat{n})a_q \quad [a_q^\dagger, \hat{n}_q] = -a_q^\dagger F(\hat{n}) \]  \tag{2.19} 

In the Schrödinger representation the evolution operator of the harmonic oscillator

\[ U(t) = \exp \left[ -i\omega \frac{(a^\dagger a + aa^\dagger)t}{2} \right] \]  \tag{2.20} 

gives the possibility to find out explicitly linear integrals of motion which depend on time [14,15],

\[ A(t) = U(t)aU^{-1}(t) = e^{i\omega t}a \]
\[ A^\dagger(t) = U(t)a^\dagger U^{-1}(t) = e^{-i\omega t}a^\dagger . \]  \tag{2.21}
The matrices of the integrals of motion (2.21) in Fock basis may be obtained from the equations

\[ A(t)|n\rangle = e^{i\omega t}\sqrt{n}|n-1\rangle, \quad A^\dagger(t)|n\rangle = e^{-i\omega t}\sqrt{n+1}|n+1\rangle \] (2.22)

Let us now introduce the Hamiltonian

\[ \hat{H} = \omega a_q a_q^\dagger + a_q^\dagger a_q \] (2.23)

for which the evolution operator takes the form

\[ U_q(t) = \exp \left[-i\omega t\left(\frac{a_q a_q^\dagger + a_q^\dagger a_q}{2}\right)\right]. \] (2.24)

We have for the integrals of motion

\[ A_q(t) = U_q a_q U_q^{-1}, \quad A^\dagger_q = U_q a_q^\dagger U_q^{-1} \] (2.25)

the following matrix expressions,

\[ A_q(t) = \begin{pmatrix} 0 & \sqrt{T_q} e^{i(1_q - 0_q)\omega t} & 0 & \cdots \\ 0 & 0 & \sqrt{T_q} e^{i(2_q - 1_q)\omega t} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \]

\[ A^\dagger_q(t) = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{T_q} e^{-i(1_q - 0_q)\omega t} & 0 & 0 & \cdots \\ 0 & \sqrt{T_q} e^{-i(2_q - 1_q)\omega t} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \] (2.26)

These operators are the generalizations of the linear integrals of motion (2.21) to the case of nonlinear Hamiltonian.

In general we can prove the following string of relations

\[ U^{-1} a_q = a_q U^{-1}_1 \]
\[ U^{-1} a_q^\dagger = a_q^\dagger U^{-1}_1 \]
\[ U a_q = a_q U_{-1} \]
\[ U a_q^\dagger = a_q^\dagger U_{-1} \] (2.27)
where $U_{\pm p}$ is defined implicitly by expressing $U$ as a function of the operator $\hat{n}$,

$$U_{\pm p}(\hat{n}) \equiv U(\hat{n} \pm \hat{p}) \quad (2.28)$$

The integrals of motion $A_q(t), A^\dagger_q(t)$, which coincide with the operators $a_q, a^\dagger_q$ at $t = 0$, may be expressed in terms of the matrices (2.6) and (2.7) by means of the relations (2.26) and (2.27).

3. Classical q-variables

In this section we will use the inverse of the procedure proposed by Dirac to quantize a classical system, i.e. we consider a ”dequantization procedure”. We will obtain therefore a dynamical system on the classical phase space $(q, p)$, or $(\alpha, \alpha^\ast)$, from the quantum system of the previous section, the latter then being the Dirac quantized version of the former.

The usual harmonic oscillator vibrating with unit frequency may be described in terms of the following variables

$$\alpha = \frac{q + ip}{\sqrt{2}} \quad \alpha^\ast = \frac{q - ip}{\sqrt{2}}, \quad (3.1)$$

with non-zero Poisson bracket

$$\{\alpha, \alpha^\ast\} = -i \quad (3.2)$$

and Hamiltonian function

$$H = \alpha\alpha^\ast. \quad (3.3)$$

The flow associated with equations of motion is

$$\alpha = Ae^{i(t+\varphi)} \quad \alpha^\ast = Ae^{-i(t+\varphi)} \quad (3.4)$$

where

$$A = \alpha(0)\alpha(0)^\ast, \quad \varphi = \frac{1}{2i} \ln \frac{\alpha(0)}{\alpha(0)^\ast} \quad (3.5)$$
For a frequency different from 1, the Hamiltonian is

\[ H = \frac{p^2 + \omega^2 q^2}{2} \quad \omega > 0 \quad (3.6) \]

In terms of complex variables, we have now

\[ \alpha = \left( \frac{ip}{\sqrt{\omega}} + \sqrt{\omega} q \right) \frac{1}{\sqrt{2}} \quad \alpha^* = \left( \frac{-ip}{\sqrt{\omega}} + \sqrt{\omega} q \right) \frac{1}{\sqrt{2}} \]

\[ H(\alpha, \alpha^*) = \omega \alpha \alpha^* \]

\[ \{\alpha, \alpha^*\} = -i \quad (3.7) \]

From (2.16) we define the classical q-oscillator in terms of these new variables \((\alpha, \alpha^*)\), namely

\[ \alpha_q = \sqrt{\frac{\sinh \bar{h} \alpha \alpha^*}{\alpha \alpha^* \sinh \bar{h}}} \quad \alpha_q^* = \sqrt{\frac{\sinh \bar{h} \alpha \alpha^*}{\alpha \alpha^* \sinh \bar{h}}} \quad \alpha^* \quad (3.8) \]

Their Poisson bracket can be computed and expressed in terms of themselves, obtaining

\[ \{\alpha_q, \alpha_q^*\} = -i \frac{\hbar}{\sinh \bar{h}} \sqrt{1 + |\alpha_q|^4 \sinh \bar{h}^2} \quad (3.9) \]

so that we will consider a new system described by such q-variables with Hamiltonian function

\[ H(\alpha_q, \alpha_q^*) = \omega \alpha_q \alpha_q^* \quad (3.10) \]

The equations of motion are then

\[ \dot{\alpha}_q = -i \frac{\hbar}{\sinh \bar{h}} \omega \sqrt{1 + |\alpha_q|^4 \sinh \bar{h}^2} \alpha_q \quad (3.11) \]

and complex conjugated, with solutions

\[ \alpha_q(t) = \alpha_q(0) \exp \left[ -i \omega \hbar \frac{\alpha_q(0)}{\sinh \bar{h}} \sqrt{1 + |\alpha_q(0)|^4 \sinh \bar{h}^2} \right] \quad (3.12) \]

and its conjugate.

All this can be commented stating that we have performed a non canonical transformation, i.e. deformed the Poisson bracket structure, while preserving the form of the Hamiltonian function and note that the above Poisson bracket has the standard bracket as a limit when \(\hbar\) goes to 0.
Such a system can be, of course, re-written in terms of \((\alpha, \alpha^*)\) variables: in these coordinates the original Poisson bracket is unchanged while the Hamiltonian function is

\[ H_q(\alpha, \alpha^*) = \omega \frac{\sinh \hbar \alpha \alpha^*}{\sinh \hbar} \]  

(3.13)

This new dynamical system has a phase portrait which is the same as the usual linear harmonic oscillator. In facts, the equations of motion for the latter system, which evolves under the Hamiltonian \(H\) are

\[ \dot{\alpha} = -i\omega \alpha \quad \dot{\alpha}^* = i\omega \alpha^* \]  

(3.14)

and for the system with the Hamiltonian \(H_q\) are

\[ \dot{\alpha} = -i\omega_q \alpha \quad \dot{\alpha}^* = i\omega_q \alpha^* \]  

(3.15)

with

\[ \omega_q = \omega \frac{\hbar}{\sinh \hbar} \cosh \hbar \alpha \alpha^* . \]  

(3.16)

We notice that \(\alpha \alpha^*\) is a constant of the motion for both systems. The main difference between such systems is that the frequency for the second one depends on the orbit while for the first one is constant.

This leads to interpreting q-oscillators as systems carrying a particular non-linearity. What in the harmonic oscillator is a constant \(\omega\) characterizing the evolution along any orbit, becomes here a function constant on each orbit separately, i.e. a constant of the motion for the undeformed harmonic oscillator. From the formulae (2.16) it follows a possible physical interpretation of the q-oscillator. The amplitude of the vibrations of the harmonic oscillator is not connected with the phase of the vibrations which changes linearly in time. For the q-oscillator the frequency depends on its amplitude. The formula (2.16) reflects this classical nonlinear phenomenon for the quantum oscillator. Such a situation can be generalized to any linear system.

Let us have, indeed, a linear system

\[ \dot{\mathbf{X}} = A \mathbf{X} \]  

(3.17)

where \(\mathbf{X} \in \mathbb{R}^n\) and \(A\) a \(n \times n\) matrix with constant entries. It can be integrated

\[ \mathbf{X}(t) = e^{tA} \mathbf{X}(0) . \]  

(3.18)
Its integrals of the motion can be expressed in terms of the initial conditions \( X(0) = X_0 \). Replacing \( A \) with \( B(X) \) defined by

\[
[B(X)]^i_j = A^i_j f^j_i ,
\]

(3.19)

where \( f^j_i \) is any constant of the motion for the original linear system, the new system, which is non-linear,

\[
\dot{X} = B(X)X
\]

(3.20)

can be integrated via exponentiation, because for any initial condition \( B(X(0)) \) is a matrix constant along the trajectory originated from it.

When the starting linear system is Hamiltonian, it turns out to be completely integrable. Therefore, eq. (3.20) is a generalization of our q-deformed oscillator. By employing various constants of the motion we could accommodate deformations with many parameters. There are other ways to get non linear systems out of linear ones, for instance by using a generalized reduction procedure [16]. In the future we shall analyze and compare among them these procedures that use deformation or reduction to get non linear systems.

4. Two dimensional q-oscillators

In this section we will analyze a possible q-deformation of the two dimensional oscillator. There exists two essentially different possibilities to introduce the q-deformed creation and annihilation operators. They are based on our interpretation of the q-oscillators as describing nonlinear vibrations. If two modes in the linear limit have different frequencies it is natural to deform the annihilation and creation operators for these two modes independently of each other, by simply using the formulae for one-dimensional q-oscillator discussed above. But if we want to take into account the isotropy of two dimensional oscillator and to conserve the \( U(2) \) dynamical symmetry even in the nonlinear vibration regime we might deform the two dimensional oscillator in such a way that both constituent oscillators have the same frequency in the equal initial conditions for this regime too. Taking these remarks into account let us introduce two sets of operators

\[
a_{q\pm} = a_{\pm} f(\hat{n}), \quad a_{q\pm}^\dagger = f(\hat{n}) a_{q\pm}^\dagger
\]

(4.1)
Here the number operator $\hat{n}$ is defined as follows

$$\hat{n} = n_+ + n_-$$  \hspace{1cm} (4.2)

where

$$n_\pm = a_\pm^\dagger a_\pm$$  \hspace{1cm} (4.3)

and the operators $a_\pm, a_\pm^\dagger$ obey the usual boson commutation relations

$$[a_\pm, a_\pm^\dagger] = 1$$  \hspace{1cm} (4.4)

all the others being zero. The commutation relations for the two components of the two-dimensional q-oscillator are then as follows

$$[a_{q+}, a_{q-}] = 0, \quad [a_{q+}, a_{q-}^\dagger] = a_+ a_+^\dagger [f^2(\hat{n} + 1) - f^2(\hat{n})]$$  \hspace{1cm} (4.5)

The other commutation relations are obtained from the conjugation of the above. The non-zero commutation relation of the annihilation and creation operators describing the same component is of the form

$$[a_{q\pm}, a_{q\pm}^\dagger] = a_\pm^\dagger a_\pm (f^2(\hat{n} + 1) - f^2(\hat{n})) + f^2(\hat{n} + 1).$$  \hspace{1cm} (4.6)

Due to the nonlinear interaction of vibrations the operators $a_{q+}$ and $a_{q-}^\dagger$ do not commute, as well as $a_{q-}$ and $a_{q-}^\dagger$. In the limit, when $\hbar$ goes to zero (q goes to 1) the written commutation relations lead to the usual boson commutation relations since in this limit the function $f$ goes to 1. The four operators

$$a_\pm^\dagger a_+, a_\pm^\dagger a_-, a_\pm^\dagger a_-, a_\pm^\dagger a_+$$  \hspace{1cm} (4.7)

form the generators of the U(2) group representation. This representation is realized in the space of states of two dimensional harmonic oscillator. We deformed the operator $a_+$ and $a_-$ and they became the operators $a_{q+}$ and $a_{q-}$. But the deformation given by the formulae (4.1) is such that the operators

$$a_{q+}^\dagger a_{q+}, a_{q-}^\dagger a_{q-}, a_{q-}^\dagger a_{q+}, a_{q+}^\dagger a_{q-}$$  \hspace{1cm} (4.8)

realize the same representation of the same undeformed group U(2) in the space of the same two dimensional harmonic oscillator states. This happens because these new generators differ from the former ones only by the coefficients which depend on the Casimir operator of the rotation group. So, for
any irreducible subspace these coefficients are simply c-numbers and the operators (4.10) close on the Lie algebra of the group U(2) in these subspaces. It can be shown that the following re-ordering relation holds

\[ a_q + a_q^\dagger + a_q - a_q^\dagger - q(a_q^\dagger a_q + a_q^\dagger a_q^\dagger) = q^{-\hat{n}} \]  \hspace{1cm} (4.9)

Had we \( q \)-deformed the two components of the oscillator as independent ones, namely as those appearing in the previous section 2, the right hand side of the Eq.(4.11) would be replaced by the term \( q^{-\hat{n}} + q^{-\hat{n}} \). This is a different instance of the non linearity of the \( q \)-oscillators.

If we deal with many more oscillators, we can use mixtures of these \( q \)-deformation procedures according to which linear transformations we want to survive after the deformation process. Let us clarify this statement. We now study transformations between two-dimensional \( q \)-oscillators which are analogous to canonical transformations of linear oscillators, \( (a_\pm, a_\pm^\dagger) \) and \( (b_\pm, b_\pm^\dagger) \), by requiring that

\[ a_\pm a_\pm^\dagger + a_\mp a_{\mp}^\dagger = b_\pm b_\pm^\dagger + b_{\mp} b_{\mp}^\dagger \]  \hspace{1cm} (4.10)

The resulting linear transformations are

\[ a_\pm = A_\pm b_\pm + B_\pm b_\mp \]  \hspace{1cm} (4.11)
\[ a_\pm^\dagger = A_\pm^\ast b_\pm^\dagger + B_\pm^\ast b_{\mp}^\dagger \]  \hspace{1cm} (4.12)

with \( A_\pm \) and \( B_\pm \) complex numbers such that the matrix

\[ \begin{pmatrix} A_+ & B_+ \\ A_- & B_- \end{pmatrix} \]

belongs to U(2). Substituting the above formulae in (4.1) and (4.2) we have consequently

\[ a_{q\pm} = (A_\pm b_\pm + B_\pm b_\mp) f(\hat{n}) \]  \hspace{1cm} (4.13)
\[ a_{q\pm}^\dagger = f(\hat{n})(A_\pm^\ast b_\pm^\dagger + B_\pm^\ast b_{\mp}^\dagger) \]  \hspace{1cm} (4.14)

Here the number operator \( \hat{n} \) is written as follows

\[ \hat{n} = \hat{n}_{b_\pm} + \hat{n}_{b_{\mp}} \]  \hspace{1cm} (4.15)

where

\[ \hat{n}_{b_\pm} = b_\pm^\dagger b_\pm \]  \hspace{1cm} (4.16)
\[
\hat{n}_b^- = b_b^\dagger b^- .
\]

If we consider a dynamical system described by the Hamiltonian,
\[
H = \frac{1}{2}(a_{q+}^\dagger a_{q+} + a_{q-}^\dagger a_{q-} + a_{q+}a_{q+}^\dagger + a_{q-}a_{q-}^\dagger) .
\]

Written in terms of the number operator, \(H\) acquires the following form
\[
H = \frac{1}{2}[\hat{n}f^2(\hat{n} - 1) + (\hat{n} + 2)f^2(\hat{n} + 1)]
\]
which is manifestly invariant under the previous transformations. In the Fock basis \(|n_+ > |n_- >\) the eigenvalues of \(H\) are immediately computed as \(\hat{n}\) has positive integers eigenvalues and each of them has degeneracy \(n + 1\).

It is obvious how to define the variables for the classical 2-dimensional \(q\)-oscillators. For instance
\[
\alpha_{q\pm} = \alpha_{\pm} f(n)
\]
where \((\alpha_{\pm}, \alpha_{\pm}^*)\) are variables for the usual harmonic oscillators and
\[
n = n_+ + n_-
\]
with
\[
n_{\pm} = |\alpha_{\pm}|^2 .
\]
The computation of the non-zero Poisson brackets then gives the following result
\[
\{\alpha_{q\pm}, \alpha_{q\mp}^*\} = -i\alpha_{\pm}\alpha_{\mp}^* (\hbar n) \cosh n\hbar - \sinh n\hbar
\]
\[n^2 \sinh \hbar \]
and
\[
\{\alpha_{q\pm}, \alpha_{q\pm}^*\} = \frac{-i}{n \sinh \hbar} [(1 - \frac{n_{\pm}}{n}) \sinh n\hbar + \hbar n_{\pm} \cosh n\hbar] .
\]

Thus, it is clear that those transformations which preserve the term responsible for the non linearity will be compatible with the \(q\)-deformation. More explicitly, this means that these transformations commute with the deformation procedure.

5. \(q\)-fields
We take into consideration the usual expansions of classical and quantum fields on an interval of length $L = 1$ in terms of usual harmonic oscillators. For simplicity we start considering a classical real massless scalar field $\phi(x, t)$ and its conjugate $\pi(x, t)$, satisfying also $\phi = \pi$

$$\phi(x, t) = \sum_{j} e^{ik_j x} q_j(t) \quad \pi(x, t) = \sum_{j} e^{-ik_j x} p_j(t) \ . \quad (5.1)$$

Notice that here $q_j$ and $p_j$ are complex variables which anyhow can be expressed in terms of harmonic oscillators as, for instance in [17] so that also in this case the usual Poisson bracket holds

$$\{q_i, p_j\} = \delta_{ij} \quad k_j = 2\pi j \ . \quad (5.2)$$

The use of such fields is aimed at a study of some aspects of electrodynamics with massless q-photons. Following [17], we have

$$\phi(x, t) = \sum_{j} e^{ik_j x} \frac{1}{\sqrt{2\omega_j}} (\alpha_{-j}^* + \alpha_{j})$$
$$\pi(x, t) = -i \sum_{j} e^{-ik_j x} \sqrt{\omega_j} 2 (\alpha_{-j} - \alpha_{j}^*)$$
$$\omega_j = \vert 2\pi j \vert \ . \quad (5.3)$$

These variables $(\alpha_j, \alpha_j^*)$ with $j = \pm$ integers can be written in terms of usual oscillator variables for positive and negative frequencies $(\alpha_{j\pm}, \alpha_{j\pm}^*)$ with $j$ positive integers, namely

$$\alpha_j = \frac{\alpha_{j+} - i\alpha_{j-}}{\sqrt{2}} \ , \quad (5.4)$$
$$\alpha_{-j} = \frac{\alpha_{j+} + i\alpha_{j-}}{\sqrt{2}} \quad (5.5)$$

and their complex conjugates. The Poisson brackets of the usual oscillators are of the form

$$\{\alpha_{j\pm}, \alpha_{j'\pm}\} = 0 \ , \quad (5.6)$$
$$\{\alpha_{j\pm}, \alpha_{j'\pm}^*\} = 0 \ , \quad (5.7)$$

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and
\[ \{ \alpha_{j\pm}, \alpha_{j'\pm}^* \} = -i \delta_{jj'} . \quad (5.8) \]

We have
\[ \{ \phi(x, t), \pi(x', t) \} = \delta(x - x') . \quad (5.9) \]

We are now in a position to define a pair of classical q-fields, noting that there is not a unique definition due to the fact that the frequency depends on \( \alpha_j \alpha_j^* \).

By using previous equations, we write
\[
\phi_q(x, t) = \sum_j e^{i2\pi jx} \frac{1}{\sqrt{\omega_{jq}}} \frac{\alpha_{jq}^* + \alpha_{jq}}{\sqrt{2}} 
\]
\[
\pi_q(x, t) = -i \sum_j e^{-i2\pi jx} \sqrt{\omega_{jq}} \frac{-\alpha_{jq}^* + \alpha_{-jq}}{\sqrt{2}} 
\]
where
\[
\omega_{jq} = |2\pi j| \hbar \frac{\cosh(\alpha_j \alpha_j^* + \alpha_{-j} \alpha_{-j}^*) \hbar}{\sinh \hbar} \quad (5.11)
\]
and \( \alpha_{jq} \) and \( \alpha_{jq}^* \) are given as functions of \( \alpha_{j\pm} \) and its complex conjugate; for instance,
\[
\alpha_{jq} = \frac{\alpha_{j+} - i \alpha_{j-}}{\sqrt{2}} \sqrt{\frac{\sinh(|\alpha_{j+}|^2 + |\alpha_{j-}|^2) \hbar}{(|\alpha_{j+}|^2 + |\alpha_{j-}|^2) \sinh \hbar}} . \quad (5.12)
\]

The computation of the Poisson brackets for these fields gives results which are different from the usual ones, as it was expected, up to corrections of the order of \( \hbar^2 \). The above fields have been constructed allowing a coupling between positive and negative frequency modes belonging to the same index \( j \).

To give an idea of the effects of a q-deformation, we give explicitly the behaviour of the Poisson bracket with \( \hbar \to 0 \) between conjugated fields, which have been q-deformed in another way that allows for much simpler computations. Such new fields are defined as follows
\[
\phi_q(x, t) = \sum_j e^{i2\pi jx} \frac{\alpha_{jq}^* + \alpha_{jq}}{\sqrt{2\omega_{jq}}} 
\]
\[
\pi_q(x, t) = -i \sum_j e^{-i2\pi jx} \sqrt{\omega_{-jq}} \alpha_{-jq} - \sqrt{\omega_{jq}} \alpha_{jq}^* \quad (5.13)
\]

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where
\[ \omega_{jq} = |2\pi j| \frac{\hbar}{\sinh \hbar} \cosh(|\alpha_j|^2 \hbar) \]  
(5.15)
and the q-deformed oscillators are each other independent and defined as in (3.8). That is we are allowing only the self coupling. We can compute the Poisson brackets between such fields at equal times and obtain
\[ \{\phi_q(x, t), \pi_q(x', t)\} = \delta(x - x')(1 - \frac{\hbar^2}{6}) + \hbar^2 (c_1 + c_2) \]  
(5.16)
where
\[ c_1 = \frac{1}{4} \sum_{j > 0} [(e^{i2\pi j (x-x')}(|\alpha_j|^4 + |\alpha_{-j}|^4)) + \text{c.c.}] \]  
(5.17)
and
\[ c_2 = \frac{1}{2} \sum_{j > 0} [(e^{i2\pi j (x+x')}(\alpha_j^2|\alpha_j|^4 - \alpha_{-j}^2|\alpha_{-j}|^4)) + \text{c.c.}] . \]  
(5.18)

It can be shown that
\[ \{\phi_q(x, t), \phi_q(x', t)\} = O(\hbar^2) \]  
(5.19)
\[ \{\pi_q(x, t), \pi_q(x', t)\} = O(\hbar^2) \]  
(5.20)
All this gives an account of the effect of q-nonlinearity as far as the principle of locality is concerned. The corrections destroy the δ-function behaviour of the Poisson bracket between conjugated fields and are proportional to the fourth power of the amplitudes of nonlinear modes vibrations. We shall come back to these aspects in the future. The quantized fields are defined from eqs. (5.13-14) replacing \( c \)-numbers \( (\alpha_{jq}, \alpha_{jq}^*) \) with the creation and annihilation operators \( (a_{jq}, a_{jq}^\dagger) \) with \( \omega_{jq} \) becoming also an operator \( \hat{\omega}_{jq} \), namely
\[ \hat{\phi}_q(x, t) = \sum_j e^{i2\pi j x} (\frac{1}{\sqrt{2\hat{\omega}_{jq}}} a_{jq} + \frac{1}{\sqrt{2\hat{\omega}_{-jq}}} a_{jq}^\dagger) \]  
(5.21)
and
\[ \hat{\pi}_q(x, t) = \sum_j e^{i2\pi j x} (\sqrt{\frac{\hat{\omega}_{-jq}}{2}} a_{-jq} - \sqrt{\frac{\hat{\omega}_{jq}}{2}} a_{jq}^\dagger) \]  
(5.22)
with
\[ \hat{\omega}_{jq} = |2\pi j| \hbar \frac{\cosh((a_{jq}^\dagger a_{jq}) \hbar)}{\sinh \hbar} \]  
(5.23)
Such a definition of quantized q-fields just follows the usual prescription, particularly for what concerns the \( j = 0 \) term in the sum and holds since the operators \( \hat{\omega}_j \) are invertible for \( j \neq 0 \). Their commutation relations at equal time can be calculated and as it was easily expected, we have that such fields are not local.

It should be noticed that we have not addressed the problem of deformation of the differential Maxwell equations, i.e. we are not exhibiting a q-deformed d’Alembertian. As a matter of fact, at this stage we do not even know whether our q-fields are solutions of some differential equation. These and other related aspects will be taken up in a forthcoming paper.

6. Partition and time-correlation functions

The partition function for a single q-oscillator

\[
Q(\beta, \hbar) = \sum_{n=0}^{\infty} \exp(-\beta n_q)
\]

(6.1)

is a function of the temperature, through the variable \( \beta = \omega/kT \), and the parameter \( \hbar \). In particular, \( \hbar \) is very small and the series representing \( Q \) can be represented in analytic forms. We will consider first the limiting expression of \( Q \) obtained by letting \( \hbar \to 0 \) for fixed \( \beta \) thus obtaining a series representation of the form,

\[
Q(\beta, \hbar) = Q_0 + \frac{1}{2} \hbar^2 \frac{d^2}{d\hbar^2} Q + ...
\]

(6.2)

where \( Q_0(\beta) = Q(\beta, 0) = 1/(1 - e^{-\beta}) \) represents the partition function of the ideal harmonic oscillator. In particular, we will see that the above series does not converge for \( \beta < \hbar \). In view of this, we will discuss the limit of \( Q \) for \( \hbar \to 0 \) for given value of the ratio \( \zeta = \beta / \sinh \hbar \). The quantum partition function obtained in this second case is well approximated by the classical one.
6.0.1 Partition function for given $\beta$

When we keep $\beta$ fixed and let $\bar{h}$ go to zero we can represent $Q$ as a power series in $\bar{h}$ by relying on the power expansion of $n_q$,

$$ n_q = n + \frac{h^2}{3!}(n^3 - n) + \frac{h^4}{5!}\left(n^5 - \frac{10}{3}n^3 + \frac{7}{3}n\right) + \cdots $$ \hspace{1cm} (6.3)

Then,

$$ Q(\beta, \bar{h}) = \sum_{n=0}^{\infty} \exp(-\beta n + \beta n) \exp(-\beta n) $$

$$ \approx Q_0(\beta) \left(1 + W^2 \beta^3 (Q_3 - Q_1)\right) $$

$$ = \exp\left(\beta \frac{\sinh (h \partial / \partial \zeta)}{- \sinh \bar{h}} - \beta \partial / \partial \zeta\right) Q_0(\zeta)|_{\zeta=\beta} $$

$$ \approx Q_0(\beta) \left(1 + W^2 \beta^3 (Q_3 - Q_1) + W^4 \left(\frac{1}{3!} \beta^6 (Q_6 - 2Q_4 + Q_2) + \frac{1}{5!} \beta^5 (Q_5 - \frac{10}{3}Q_3 + \frac{7}{3}Q_1)\right)\right) $$ \hspace{1cm} (6.4)

where $W = \bar{h}/\beta$ and

$$ Q_0(\beta) = \frac{1}{1 - e^{-\beta}} $$

$$ Q_n = Q_0^{-1} \frac{d^n}{d\beta^n} Q_0(\beta) $$

$$ Q_1 = 1 - Q_0 $$

$$ Q_2 = 1 - 3Q_0 + 2Q_0^2 $$

$$ Q_3 = 1 - 7Q_0 + 12Q_0^2 - 6Q_0^3 $$ \hspace{1cm} (6.5)

Accordingly, we have

$$ Q(\beta, \bar{h}) \approx Q_0(\beta) (1 - W^2 \beta^3 Q_0(0) - 1)^2 $$ \hspace{1cm} (6.6)
Consequently, the average energy $E$ and the specific heat $C$ are respectively given by

$$
E = -\frac{Q'}{Q} = E_0 + \hbar^2 \frac{d}{d\beta} (\beta Q_0(Q_0 - 1)^2) \\
C = C_0 - \beta^2 \hbar^2 \frac{d^2}{d\beta^2} (\beta Q_0(Q_0 - 1)^2)
$$  \hspace{1cm} (6.7)

It is worth discussing the behavior of the above expression of $Q$ for given $W$ and $\beta \to 0$. Since for small $\beta$ $Q_0 \approx 1/\beta$, then

$$
Q(\beta) \approx Q_0(1 - W^2 + W^4 + ...) \\
E(\beta) \approx E_0 - 2W^2 \\
C(\beta) \approx C_0 - 6W^2
$$  \hspace{1cm} (6.8)

Accordingly, the above representation of $Q$ is valid only for $\beta^2 > 6\hbar^2$, that is for $W \leq 1/\sqrt{6}$.

### 6.0.2 Partition function for given $\beta/\hbar$

For given $\zeta = \beta/\hbar$ we can rely on the following inequalities for the partition function

$$
\int_0^\infty e^{-\beta E_q(x)} dx < Q(\beta, \hbar) < 1 + \int_0^\infty e^{-\beta E_q(x)} dx
$$  \hspace{1cm} (6.9)

where $E_q(x)$ is a monotone function of $x$ which coincides with $n_q$ for $x = n$. The above integral, which represents the classical limit for the partition function, can be represented as a combination of Weber function $E_0$ and Neumann function $N_0$ [18],

$$
\int_0^\infty e^{-\beta E_q(x)} dx = -\frac{\pi}{4\hbar} [N_0(\zeta) + E_0(\zeta)]
$$  \hspace{1cm} (6.10)

being $\zeta = \beta/\sinh \hbar$. In particular,

$$
E_0(\zeta) = -\sum_{n=0}^{\infty} \frac{(-1)^n (\zeta/2)^{2n+1}}{\Gamma^2(n+3/2)}
$$  \hspace{1cm} (6.11)
It is noteworthy that for fixed $\zeta$ there exists a limiting value $\bar{\hbar}$ of $\hbar$ such that for $\hbar < \bar{\hbar}$ the partition function is so large to coincide with the classical one. In fact, for $\zeta$ very small we have

$$-\frac{\pi}{4\hbar}[E_0(\zeta) + N_0(\zeta)] \approx \frac{1}{2\Gamma^2(1/2)\zeta} + O(\zeta^{-2})$$

(6.12)

Hence, for $\zeta < .1$ the classical and quantum partition functions are almost coincident for $\zeta > 10\bar{\hbar}$.

In conclusion, while the quantum partition function of the q-oscillator is well approximated by the classical limit for $\hbar$ going to zero, this is not true for the classical harmonic oscillator.

Next, we consider the average energy $E$ for which we can write the following inequalities

$$E_\leq \leq E \leq E_\geq$$

(6.13)

where the lower ($E_\leq$) and upper ($E_\geq$) bounds are given by

$$E_\leq = \frac{-\frac{1}{\zeta\beta} + \int_0^\infty E_q(x)e^{-\beta E_q(x)}dx}{1 + \int_0^\infty e^{-\beta E_q(x)}dx}$$

$$E_\geq = \frac{\frac{1}{\zeta\beta} + \int_0^\infty E_q(x)e^{-\beta E_q(x)}dx}{\int_0^\infty e^{-\beta E_q(x)}dx}$$

(6.14)

Plugging (6.10) into the above expressions yields

$$\frac{(2 \sinh \hbar W)/ (\pi e) - E'_0(\zeta) - N'_0(\zeta)}{-4\hbar/\pi + E_0(\zeta) + N_0(\zeta)} < E \sinh \hbar < \frac{(2 \sinh \hbar W)/ (\pi e) - E'_0(\zeta) - N'_0(\zeta)}{E_0(\zeta) + N_0(\zeta)}$$

(6.15)

When $\hbar$ is so small to satisfy the following inequalities

$$\frac{2\hbar}{\pi} \ll E_0(W^{-1}) + N_0(W^{-1})$$

$$\frac{2 \sinh \hbar W}{\pi e} \ll -E'_0(W^{-1}) - N'_0(W^{-1})$$

(6.16)

we can approximate $E$ by

$$E = -\frac{1}{\sinh \hbar} \frac{E'_0 + N'_0}{E_0 + N_0}$$

(6.17)
Next, letting $\beta \to 0$ into the above equation we have

$$\beta E \approx -\frac{1}{\ln \beta} \quad (6.18)$$

Consequently, the specific heat decreases for $T \to \infty$ as

$$C \propto \frac{1}{\ln T} \quad (6.19)$$

Thus the behavior of the specific heat of the q-oscillator is different from the behavior of the usual oscillator in the high temperature limit. This property may serve for an experimental check of the existence of vibrational non linearity of the q-oscillator fields.

### 6.0.3 Time-correlation function

For characterizing adequately the thermalized q-oscillator it is worth discussing the relative second-order correlation function $\gamma_q(\beta, \hbar; t)$. Defining

$$\tilde{Q}(\beta, \hbar; t) = \sum_{n=0}^{\infty} \exp(-\beta f^2(n) + itF(n - 1)),$$  

we have

$$\gamma_q(z, \hbar; t) = Tr(\rho a_q^\dagger(t)a_q(0)) = -\frac{1}{\tilde{Q}(\beta, \hbar)} \frac{\partial \tilde{Q}(\beta, \hbar; t)}{\partial \beta}. \quad (6.21)$$

To compute $\gamma_q(\beta, \hbar; t)$ for $\hbar \to 0$ we can use the approximate expression of $\tilde{Q}(\beta, \hbar; t)$

$$\tilde{Q}(\beta, \hbar; t) \approx e^{it} \left[ Q_0(\beta) - \hbar^2 \left( \frac{\langle n^3 \rangle - \langle n \rangle}{6} + it \frac{\langle n \rangle - \langle n^2 \rangle}{2} \right) \right], \quad (6.22)$$

where

$$\langle n^k \rangle = \sum_{n=0}^{\infty} n^k e^{-\beta n}, \quad k = 0, 1, \ldots \quad (6.23)$$

It is worthwhile noting that $\gamma_q$ depends on the hierarchy of photon distribution momenta $\langle n \rangle, \langle n^2 \rangle, \ldots$. This circumstance marks the main
difference between q-oscillators and standard ones. In fact, the dependence of the correlation function on the intensity is a general property of q-oscillators, holding true also when the q-oscillator is in a generalized coherent state. This property could be used for testing experimentally the possibility that electromagnetic fields behave like q-oscillators. In fact, accurate measurements of the dependence of the field correlations (second and higher orders) on the field intensity could help in finding upper bounds for the small quantity $\bar{\hbar}$.

For $t = 0$ the correlation function gives the mean value of $\bar{n}_q$, i.e.

$$\bar{n}_q = \gamma_q(\beta, \hbar; 0).$$

Such a distribution function is expressed in terms of Eq. (6.21) in which $\bar{Q}(\beta, \hbar; t)$ is taken at $t=0$.

q-deformed Bose distribution $\bar{n}$ can be obtained by the same method starting from the Hamiltonian operator (2.23) with $\omega = 1$ and not from $H = a_q^\dagger a_q$ and one obtains

$$\bar{n} = \bar{n}_0 - \frac{\hbar^2}{6} \left[ \frac{5}{2} (\bar{n})_0 - (\bar{n})^2_0 + \frac{3}{2} ((\bar{n}^3)_0 - \bar{n}_0 (\bar{n}^2)_0) + \frac{3}{2} ((\bar{n}^4)_0 - \bar{n}_0 (\bar{n}^3)_0) \right],$$

in which $\bar{n}_0$ is the usual Bose distribution function and

$$\bar{n}^k_0 = 2 \sinh \frac{\beta}{2} \sum_{n=0}^{\infty} n^k e^{-\beta(n+1/2)}. \quad (6.26)$$

7. Conclusions

We conclude this paper by pointing out some ideas of the work. As we understand now the one-dimensional q-oscillator is nothing else than a nonlinear oscillator with a very specific type of the nonlinearity. Namely, its frequency depends on its energy as hyperbolic cosine of the energy. Thus, classical motion of such nonlinear oscillator is described as motion of q-oscillator. So, in this case the frequency of the vibrations (or the velocity of motion) increases exponentially with the energy. After standard quantization we obtain quantum q-oscillator which is nothing else than a nonlinear quantum oscillator with specific anharmonicity described by the infinite power series in energy. From the discussion above it is now clear that there
are different approaches to generalise this picture. In the frame of nonlinearities, even when the frequency of vibrations depends only on the energy we could choose other functional dependencies introducing functions different from hyperbolic cosine. We could call these oscillators f-oscillators where \( f \) is now the function determining the dependence of the frequency on the energy. This function is related to the Poisson brackets of new \( \alpha_f \)-coordinates where \( \alpha_f = \alpha_f(\alpha\alpha^*) \) and the \( \alpha \) is the harmonic oscillator coordinate. Of course, this function has to contain a dependence on the parameter \( \hbar \) such that for \( \hbar \to 0 \) the function \( f \to 1 \).

The quantum \( f \)-oscillator obtained by canonical quantization of the above is again a nonlinear oscillator with a different dependence of \( \omega \) on the energy. We could also introduce other nonlinearities by making the frequency to depend on other constants of the motion, different from the energy. This gives the possibility to undertake a classification of such nonlinearities and especially it is worthy for multidimensional oscillator. Here we may introduce a deformed nonlinear motion considering the frequencies of the oscillators depending either on their own energies or on energies of some groups of other oscillators. So, to preserve the U(N)-symmetry of multidimensional isotropic oscillator we may introduce the frequency dependence on energy to be a function of total energy of all the oscillators. If each oscillator is deformed as independent one we destroy the U(N)-symmetry of the initially isotropic multidimensional oscillator.

In the quest for adequate field theories, we limited ourselves to consider some possible effects in systems which can be represented as a collection of single oscillators. In particular, we showed how the specific heat formula is changed by the q-non-linearity. In addition, the second-order correlation function depends on the average occupation number of the q-oscillator. This property marks the main difference with the standard oscillator.

In spite of mathematical beauty of q-deformation procedure, (and \( f \)-deformation too) the most important issue is to envisage possible experimental tests of the possibility of describing electrodynamic phenomena by means of these q-oscillators. At a first thought, non-linear optics experiments appear as potential candidates for obtaining upper bounds for the value of \( \hbar \). However, these experiments require the development of a q-field formulation of QED. For example, if a laser beam behaves like a collection of q-oscillators interacting with the matter in the standard way, an optical rectification experiment carried out at different laser intensities could be
used for testing the deviation of the boson spectrum from the ideal equispaced pattern. As an instance, apparatus presently developed for carrying out accurate interferometric experiments (f.i. LIGO in USA and VIRGO in Italy-France) on gravitational waves, in view of their ability to reducing the photon noise limit by averaging over periods of the order of one year could in future become ideal candidates for performing these tests.

Another class of experiments could be based on the dependence of time-correlation function of q-oscillator on the intensity. The interpretation of these experiments does not depend critically on the development of a q-field QED. In fact, these measurements can be carried out in vacuum. For example, a continuous laser beam could be split in two beams of different intensities and sent to two spectra analyzers. Measuring the widths of the two spectra it would be possible to put an upper limit to the deformation parameter $\bar{h}$. These measurements could last for years thus guaranteeing a signal-to-noise ratio adequate for appreciating very small values of the parameter.

The non-linearity may also produce squeezing [19] or may be related to non locality and lattice structure in space and time. The q-oscillator and squeezing has been discussed in [20]. The non locality in time has been studied in [21] where a q-deformed linear version of Klein-Gordon equation has been suggested. We would like to point at the recent work [22] where similar ideas are discussed. The connection of q-oscillator with anharmonic oscillator has been also discussed in [23]. We will present a more detailed analysis of possible generalizations and experimental consequences in other future publications.

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