On the entropy associated with the interior of a black hole

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ABSTRACT

The investigation about the volume of a black hole is closely related to the quantum nature of the black hole. The entropy is a significant concept for this. A recent work by Majhi and Samanta [Phys. Lett. B 770 (2017) 314] after us presented a similar conclusion that the entropy associated with the volume is proportional to the surface area of the black hole, but the proportionality coefficient is different from our earlier result. In this paper, we clarify the difference and show that their calculation is unrelated to the interior of the black hole.
1 Introduction

Recently, Christodoulou and Rovelli (CR) suggested a definition of “volume” for a collapsed black hole [1], which bridges between the interior of black holes and thermodynamics [2]. The calculation of entropy associated with the volume is vital for building this connection. We firstly calculated such entropy statistically and found that the entropy is proportional to the surface area of the black hole [2]. In particular, the thermodynamics corresponding to the volume entropy is balanced by the contribution from the volume. Thus, the first law of black hole thermodynamics is not modified, unlike the situation in which the cosmological constant is considered [3–5]. We also found that the semi-classical result is not enough to interpret the black hole entropy statistically, which is the motivation for us to continue the calculation at Planck level with the aid of spatial noncommutativity. Thus, we found an example that a black hole has infinite volume but wrapped by finite surface [6], which supports the conjecture [7, 8] that the black hole entropy can be independent of the black hole interior.

While Majhi and Samanta reinvestigated this volume entropy after us, they got a different result and pointed out that the difference was due to our “improper” treatments [9]. We have to say that all our “improper” treatments were also adopted in the calculation made by Majhi and Samanta. Firstly, their vital result is the expression of the energy, i.e. Eq. (39) in their paper [9], which derives from the relativistic dispersion relation or the primary constraint in their paper. In fact, this constraint condition is consistent with WKB approximation. In this paper, we will present this point and discuss the feasibility of this approximation although we had discussed this in our earlier paper [2]. The second point is that when we calculated the entropy from the free energy, we didn’t make the derivative with respect to the inverse temperature for the CR volume. Actually, we once calculated this term and found that it gives a very small constant, so in our earlier paper, we wrote such words “Temporarily ignoring the exotic feature of the CR volume” when we calculated the entropy statistically from the free energy. In this paper, we will present this point in detail. The final point pointed out by Majhi and Samanta is that the Stefan-Boltzmann law is not valid in the final stage of black hole evaporation, which had been discussed in detail in our earlier paper. Actually, this is the reason that we introduce the spatial noncommutativity to continue to finish the calculation, i.e. see our paper [6]. Similarly, the work made by Majhi and Samanta had the same question and cannot extend to the final stage of evaporation.

So, since their treatment is consistent with ours, why does the expression of the entropy associated with the volume obtained by Majhi and Samanta is different from ours? Because their calculation is not related to the interior (volume) of the Schwarzschild black hole. This is the reason that we decide to write this paper, not only for clarifying our calculation, but also for clarifying the fact that the entropy obtained in Ref. [9] is not that corresponding to the interior of a Schwarzschild black hole as they claimed. Throughout this paper, we use units with $G = c = \hbar = k_B = 1$. 
2 CR volume

We start with the CR definition [1] for a collapsed black hole, which is based on finding the maximal space-like hypersurface \( \Sigma \) bounded by a given surface \( S \). For example, given a 2-dimensional sphere \( S \) in flat spacetime by \( t = 0 \) and \( r^2 = x^2 + y^2 + z^2 = R^2 \), the volume surrounded by the hypersurface \( t = t(r) \) becomes

\[
V = \int_0^R 4\pi r^2 \sqrt{1 - \left( \frac{dt}{dr} \right)^2} \, dr,
\]

which gives \( V = 4\pi R^3/3 \) for a sphere by maximizing the hypersurface with \( t = \) constant. A similar discussion can be applied to collapsed matter described by the Eddington-Finkelstein coordinates

\[
d^2 s^2 = -f(r) dv^2 + 2dvdr + r^2 d\varphi^2 + r^2 \sin^2 \varphi d\phi^2,
\]

where \( f(r) = 1 - 2M/r \) and the advanced time \( v = t + \int dr/f(r) = t + r + 2M \ln |r - 2M| \). The collapsed matter forms a Schwarzschild black hole in the end with its event horizon at \( r = 2M \) serving as the required surface that bounds many space-like hypersurfaces. Any one hypersurface \( \Sigma \) can be coordinatized by \( \lambda, \varphi, \phi \) [1], and the line element of the induced metric on it can be expressed as

\[
d^2 s^2_{\Sigma} = [-f(r) \dot{v}^2 + 2\dot{v}\dot{r}] d\lambda^2 + r^2 d\varphi^2 + r^2 \sin^2 \varphi d\phi^2
\]

where the dot represents a partial derivative with regard to the parameter \( \lambda \), and \(-f(r) \dot{v}^2 + 2\dot{v}\dot{r} > 0 \) for a spacelike hypersurface. Its volume takes the usual form

\[
V_{\Sigma} = \int d\lambda d\varphi d\phi \sqrt{r^4 [-f(r) \dot{v}^2 + 2\dot{v}\dot{r}] \sin^2 \varphi}
\]

\[
= 4\pi \int d\lambda \sqrt{r^4 [-f(r) \dot{v}^2 + 2\dot{v}\dot{r}]}.
\]

The maximization is obtained at \( r = 3M/2 \) by the method of auxiliary manifold [1] or the method of maximal slicing of mathematical relativity [2]. Carrying out the integration with maximization condition [1,10,12], it is found the CR volume at late time

\[
V_{CR} \sim 3\sqrt{3} \pi M^2 v.
\]

This interesting result shows that volume is determined by advanced time, which depends on the future behavior, e.g. evaporation [13].

3 Entropy associated with CR volume

Based on the above statement about CR volume, we calculated the statistical entropy in this volume with the phase-space [14] labeled by position \( \{ \lambda, \varphi, \phi \} \) and momentum \( \{ p_{\lambda}, p_{\varphi}, p_{\phi} \} \). The total number of quantum states arises from integrating \( d\lambda d\varphi d\phi dp_{\lambda} dp_{\varphi} dp_{\phi}/(2\pi)^3 \)
over the complete phase space. The integration is carried out by considering a massless scalar field $\Phi$ in the spacetime with the metric

$$ds^2 = -dT^2 + [ -f(r)\dot{\nu}^2 + 2\dot{\nu}\dot{r}]d\lambda^2 + r^2 d\varphi^2 + r^2 \sin^2 \varphi d\phi^2,$$

(3.6)

which comes from Eq. (2.2) by the transformation $dv = \frac{1}{\sqrt{-f}}dT + d\lambda$ and $dr = \sqrt{-f}dT$.

But Majhi and Samanta took the phase space labeled by \{r, $\nu$\} and momentum \{$p_r, p_\nu$\} and considered the particle moving in such a background metric

$$ds^2_{\text{ansatz}} = -dt^2 + r^4 (-f(r)dv^2 + 2dvdr).$$

(3.7)

They claimed that in such ansatz, the non-static property could be avoided.

The first and most important problem which requires to be clarified: is the metric given in Eq. (3.7) related to the interior of the Schwarzschild black hole? Obviously, it is unrelated. It is noted that the effective metric $ds^2_{\text{eff}} = r^4 (-f(r)dv^2 + 2dvdr)$ is only an auxiliary manifold taken for the calculation of the maximization of the volume (2.4) [1]. So it is not the maximal hypersurface. The maximal hypersurface is expressed in Eq. (2.3) with $r = 3M/2$. As for the sphere in flat spacetime, its volume can be defined according to the maximization of Eq. (2.1). When the motion of particles needs to be discussed in this sphere, can it be taken in such background $ds^2_{\text{eff}} = r^4 (-dt^2 + dr^2)$? Of course, it can not!

Then, is the metric (3.7) static as claimed by Majhi and Samanta? It is difficult to give a positive answer. In the statistical calculation for the modes of the scalar field, the interior spacetime has to be used. The metric (3.6) is obviously equivalent to the interior metric (2.2), but the metric (3.7) is not. If ones want to used the metric (3.7) to represent the interior of the black hole, a transformation has to be made to connect the metric (3.7) with Eddington-Finkelstein coordinates, which will be found that the coordinate $t$ will not be independent on the coordinates $r$ and $\nu$. Thus, the background given by the ansatz is not static. In general, the non-static characteristics is essential for the black hole interior, and cannot be taken out by the transformation of coordinates. So, either the metric (3.7) is not related to the interior of the black hole, which is the case made in Ref. [2], or it is not static, which will lead to the failure of the Hamiltonian analysis. Moreover, the metric (3.7) is strange, because both the coordinates $t$ and $\nu$ are both timelike.

The second problem is that the azimuthal coordinates and their conjugated momenta were not considered in phase space by Majhi and Samanta. This is improper, since the scalar field is moving in the spacetime including the azimuthal directions. If the azimuthal degrees of freedom is ignored in the construction of phase space, to say the least, how is the integral result $4\pi$ in the expression (2.4) of CR volume incorporated into the entropy associated with the volume? In particular, the spherical symmetry is represented by these azimuthal coordinates, which will constrain the possible modes for the scalar field and lead to a smaller statistical result than having no such consideration. This is the reason why our result is smaller than that obtained by Majhi and Samanta.

Now it is clear that the calculation made by Majhi and Samanta is not related to the interior of the black hole and is incomplete even in their ansatz. In what follows, we will clarify our calculation, and at first interpret the feasibility of WKB approximation. The main reason to make the approximation is that our calculation is to be carried out
at $v >> M$ and $r = 3M/2$, which is essentially the hypersurface of $T=\text{constant}$ and is unaffected by the non-static nature of the metric. Actually, the WKB approximation is held only if the evolution of spacetime is slow enough near the maximal hypersurface. It is better to understand this point by maximal slicing \[15\] \[17\] in which the evolution avoids the singularity, but the physical process should be equivalent since in essence the complete theory should slice-independent. As well-known, in that case, at the final stage of the hypersurfaces’ evolution, a phenomenon called “collapse of the lapse” happens, which means that near the maximal hypersurface, the proper time between two neighboring hypersurfaces tends to zero and no evolution happens there. Thus, the hypersurface is nearly static and nearly no lapse into the next one. So slow evolution is just the requirement by the WKB approximation.

Under WKB approximation, the Klein-Gordon equation in curved spacetime gives the energy expression for the scalar field,

$$E^2 - \frac{1}{-f(r)v^2 + 2\dot{v}r}p_\lambda^2 - \frac{1}{r^2}p_\varphi^2 - \frac{1}{r^2\sin^2\varphi}p_\phi^2 = 0,$$

(3.8)

which is just the primary constraint $p^2 = 0$ for massless field obtained using the Hamiltonian analysis of Ref. [9], but the Eq. (3.6) while not Eq. (3.7) is taken as the required spacetime metric. With the expression of energy, the number of quantum states and free energy can be calculated with the standard statistical method [14]. So, the entropy associated with the CR volume is expressed as

$$S_{\text{CR}} = \beta^2 \frac{\partial F}{\partial \beta} = \frac{\pi^2V_{\text{CR}}}{45\beta^3} - \frac{\pi^2}{180\beta^4} \frac{\partial V_{\text{CR}}}{\partial \beta},$$

(3.9)

where the free energy $F = -\frac{\pi^2V_{\text{CR}}}{180\beta^4}$ [2]. With the CR volume in Eq. (2.5) and $\beta = T^{-1} = 8\pi M$, the second term is constant and the value is approximately $10^{-7}$ which is less and less than the first term. This is the reason why we didn’t include the second term in our earlier paper. Here, $\beta = T^{-1} = 8\pi M$ is an important assumption that is guaranteed by the condition that the system is in equilibrium. It is consistent with our calculation with the Stefan-Boltzmann law. This assumption could also be confirmed and understood by the first law of black hole thermodynamics which was given in our earlier paper [2].

As presented above, the calculation of CR volume is made in the case of $v >> M$ in which the black hole has formed by the collapse of the matter and is static\[1\] for external observers. Thus, ones can construct the first law of thermodynamics, which defines an equilibrium state that means that the inside and outside of the black hole separated by the horizon is in equilibrium. In particular, it is noted that the interior of the black hole is static when our statistical calculation is made at the hypersurface of $T=\text{constant}$. The avoidance of non-static nature make the definition of the temperature in the interior feasible. Since the exterior temperature is the Hawking temperature for the observer at infinity, the interior temperature can also be taken as the Hawking temperature for the same observer by the requirement of equilibrium. In particular, such assumption remains the validity of the first law and provides an interpretation for the vacuum pressure at horizon from the perspective of thermodynamics which will be discussed in the next section.

\[1\]More rigorously, it should be quasi-static since the black hole emits Hawking radiation and is gradually evaporating.
So the volume entropy is found

\[ S_{\text{CR}} \sim \frac{3\sqrt{3}}{45 \times 8^3} M^2 = \frac{3\sqrt{3}}{90 \times 8^4 \pi} A, \]

where \( A = 16\pi M^2 \) is the event horizon area for a Schwarzschild black hole.

4 Thermodynamic significance

The entropy \( S_{\text{CR}} \) Eq. (3.10) associated with \( V_{\text{CR}} \) remains insufficient to interpret \( S_H \) because the prefactor in front of \( A \) is much smaller than 1/4, but the thermodynamics associated with the volume has to be considered in the first law [18, 19]. In the original calculation of Hawking [13], the concept of particles is used in the asymptotically flat region far away from a black hole where particles can be unambiguously defined. While the particle flux carries away positive energy, an accompany flux of negative energy ones falls into the hole across the horizon, which can only be understood by the zero point fluctuations of the local energy density in a quantum theory. This phenomena is called vacuum polarization [20] which causes a quantum pressure \( P = 1/(90 \times 8^4 \pi^2 M^4) \) at the horizon [21, 24]. It gives \( PdV_{\text{CR}} \sim 10^{-5}dM \) that nicely balances out \( TdS_{\text{CR}} \sim 10^{-5}dM \) on the level of \( 10^{-5} \) and supports the introduction of the volume \( V_{\text{CR}} \) thermodynamically interpretable together with the quantum pressure of vacuum polarization. It presents not only the connection of volume with quantum properties of the gravitational field, but also the robustness of the first law of black hole thermodynamics. At the same time, this in turn shows that little information can be leaked out from radiation emission except those about the vacuum polarization and thus indirectly confirms the thermal result of Hawking radiations. But the result of Majhi and Samanta indicated that the first law has to be broken, since the terms from the volume in their expression cannot be balanced out from the two sides of the equation for the first law. Thus, from this perspective of semi-classical calculation, their result is also not credible.

In conclusion, our result about the entropy associated with the CR volume is self-consistent in the semiclassical region, and the reason of the difference from the calculation made by Majhi and Samanta is that they choose an erroneous ansatz for the spacetime metric, in which they calculated an entropy. But its connection with the interior of the Schwarzschild black hole is unjustified.

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