Third-Order Asymptotics of Variable-Length Compression Allowing Errors

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Abstract

This study investigates the fundamental limits of variable-length compression in which prefix-free constraints are not imposed (i.e., one-to-one codes are studied) and non-vanishing error probabilities are permitted. Due in part to a crucial relation between the variable-length and fixed-length compression problems, our analysis requires a careful and refined analysis of the fundamental limits of fixed-length compression in the setting where the error probabilities are allowed to approach either zero or one polynomially in the blocklength. To obtain the necessary refinements, we employ tools from moderate deviations and strong large deviations. Finally, we provide the third-order asymptotics of the variable-length compression with non-vanishing error probabilities and show that unlike several other information-theoretic problems in which the third-order asymptotics are known, for the problem of interest here, the third-order term depends on the permissible error probability.

I. INTRODUCTION

Characterizing fundamental limits of coding problems is the central goal in information theory. The class of variable-length compression problems (i.e., fixed-to-variable length coding problems) constitute a classical and important family of information-theoretic problems in view of their multitude of practical applications. Han [1] considered the problem of variable-length compression with prefix-free constraints allowing a small error probability. He then derived the first-order optimal coding rate for a general source when the error probability is required to vanish. Later, Koga and Yamamoto [2] derived the first-order optimal coding rate in the regime of non-vanishing error probabilities. In the particular case of a stationary memoryless source $X$, their work [2] showed that

$$L_{\text{prefix}}^*(\varepsilon | X^n) = n (1 - \varepsilon) H(X) + o(n) \quad (1)$$

as $n \to \infty$ for fixed $0 < \varepsilon < 1$, where $L_{\text{prefix}}^*(\varepsilon | X^n)$ denotes the minimum of average codeword lengths of binary prefix-free codes for $n$ i.i.d. copies $X^n$ of $X$ in which the error probability is at most $\varepsilon$, and $H(X)$ stands for the entropy of $X$ measured in bits. Hence, in general, the strong converse property (cf. [3]) fails to hold in variable-length compression problems.

In this paper, we consider variable-length compression problems without prefix-free constraints. In the zero-error setting, this class of fixed-to-variable length codes is often known as one-to-one codes. While the redundancy\(^1\) of a prefix-free code is always nonnegative, the redundancy of a one-to-one code can be negative (cf. [4], [5]).\(^2\) In fact, Szpankowski and Verdú [7] proved an asymptotic expansion of the smallest redundancies of one-to-one codes for a stationary memoryless source $X$. They showed that for finitely supported non-equiprobable $X$,

$$L^*(0 | X^n) = n H(X) - \frac{1}{2} \log n + O(1) \quad (2)$$

as $n \to \infty$, where $L^*(0 | X^n)$ denotes the minimum of average codeword lengths of one-to-one codes for $X^n$. Furthermore, Szpankowski [6] refined the remainder term $+O(1)$ in (2) when $X$ is a Bernoulli source, and clarified necessary and sufficient conditions on $X$ for which the dominant term within the $+O(1)$ remainder term converges or oscillates. On the other hand, in the regime of non-vanishing error probabilities, Kostina, Polyanskiy, and Verdú [8] derived the second-order optimal coding rate of this fundamental limit for a stationary memoryless source. They [8] showed that

$$L^*(\varepsilon | X^n) = n (1 - \varepsilon) H(X) - \sqrt{n V(X)} e^{-\Phi^{-1}(\varepsilon)^2/2} + O(\log n) \quad (3)$$

as $n \to \infty$ for fixed $0 \leq \varepsilon \leq 1$, provided that the absolute central third moment of the information density $- \log P_X(X)$ is finite, where $L^*(\varepsilon | X^n)$ stands for the minimum of average codeword lengths of non-prefix-free codes for $X^n$ in which the error probability is at most $\varepsilon$, the quantity $V(X)$ stands for the varentropy of $X$ measured in bits squared per source symbol (cf. [9]), and $\Phi^{-1}(\cdot)$ stands for the inverse of the Gaussian cumulative distribution function. It is clear that (3) is consistent with (2) when $\varepsilon = 0$.

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\(^1\)The redundancy of a fixed-to-variable length code is defined as “its average codeword length minus the entropy of a source.”

\(^2\)Hence, the redundancy of a one-to-one code is termed the anti-redundancy (cf. [6]).
A. Contributions of This Study

In this study, we consider refinements of (2) and (3) simultaneously. In particular, we generalize Szpankowski and Verdú’s work [7] from the zero-error setting (i.e., $\varepsilon = 0$) to the almost lossless setting (i.e., $\varepsilon > 0$). More importantly, we refine the $+O(\log n)$ remainder term in Kostina et al.’s second-order asymptotic result [8]. We show that this term equals $-(1-\varepsilon) \log n)/2$. To do so, we use a crucial relation between variable-length and fixed-length codes and their fundamental limits (cf. [10]). In other words, we derive higher-order asymptotics of the variable-length compression problem by leveraging that of the fixed-length compression problem. In this strategy, we have to consider the fixed-length compression problem in which error probability approaches zero or one polynomially in the blocklength. To deal with these sequences of error probabilities that tend to the boundary of $[0,1]$, we apply techniques from moderate deviations and strong large deviations (cf. [11]–[13]). The resulting higher-order asymptotics of the fixed-length compression problem yields our desired third-order asymptotic expansion of the fundamental limit of the variable-length compression problem. Somewhat interestingly, unlike several other information-theoretic problems in which the third-order asymptotics are known, for the problem of interest here, the third-order term depends on the permissible error probability. Finally, we believe that the new mathematical results derived here (cf. Lemmas 8 and 10) may be of independent interest in information theory and beyond.

B. Related Works

1) Higher-Order Asymptotics of Fixed-Length Compression: In view of the recent developments of the second- and third-order asymptotics of coding problems [14]–[19], given a source $X$ with countable source alphabet $\mathcal{X}$, it is well-known that

$$
\log M'(n, \varepsilon) = n H(X) - \sqrt{n} V(X) \Phi^{-1}(\varepsilon) - \frac{1}{2} \log n + O(1)
$$

(4)

as $n \to \infty$ for fixed $0 < \varepsilon < 1$, provided that the absolute central third moment of the information density $-\log P_X(X)$ is finite, where $log$ denotes the logarithm to the base-2 and $M'(n, \varepsilon)$ stands for the smallest cardinality a set $\mathcal{A} \subseteq \mathcal{X}^n$ in which the $P_{X^n}$-probability of $\mathcal{A}$ is at least $1 - \varepsilon$. In his seminal work, Strassen [14] derived the fourth-order asymptotics of the fixed-length compression with non-vanishing error probabilities, i.e.,

$$
\log M'(n, \varepsilon) = n H(X) - \sqrt{n} V(X) \Phi^{-1}(\varepsilon) - \frac{1}{2} \log \left(2\pi e \Phi^{-1}(\varepsilon)^2 n V(X)\right) + \log \log n - \sqrt{V(X)} S(X) (1 - \Phi^{-1}(\varepsilon)^2) + O(1)
$$

(5)

as $n \to \infty$ for fixed $0 < \varepsilon < 1$, provided that the information density $-\log P_X(X)$ is a nonlattice random variable (r.v.), where $S(X)$ stands for the skewness of $-\log P_X(X)$. Equation (5) was derived by applying the Edgeworth expansion to the information spectrum of the source $X$ [20]. The Edgeworth expansion is a higher-order asymptotic expansion that goes beyond the central limit theorem (cf. [11], [12]). Recently, Hayashi [19] investigated the fourth-order asymptotics of various information-theoretic problems.

2) Moderate Deviations Analysis: In information theory, there are two main types of coding theorems that provide refinements to capacity results, theorems concerning error exponents and second-order asymptotics. The former evaluates the exponential decay of error probabilities when coding rates are fixed; the latter evaluates the deviations from the first-order fundamental limits (which are typically of order $1/\sqrt{n}$) when error probabilities are fixed. The moderate deviations analysis of coding problems lie in between these two asymptotic regimes. Moderate deviations examines the interplay between the sub-exponential decay of the logarithm of the error probability and the deviation from the first-order fundamental limits which are typically of order $\kappa_n/\sqrt{n}$ where the positive sequence $\kappa_n = o(1) \cap o(\sqrt{n})$ as $n \to \infty$. See [21, Section I] for earlier works on moderate deviations in information theory. Most notably, in the channel coding problem, Altuğ and Wagner [21] investigated the sub-exponential rate of decay of the error probabilities when the coding rate approaches the capacity slower than that in the study of the second-order asymptotics [14]–[19]. Chubb, Tan, and Tomamichel [22] extended the moderate deviations result in classical channel coding to classical communications over quantum channels.

3) Exact Asymptotics of Error Probabilities: The study of strong large deviations [13, Theorem 3.7.4] and [12, Chapter VIII.4], or exact asymptotics, is a refinement of the large deviations principle. While the rate function in the large deviations principle characterizes the exponential decay of the complementary cumulative distribution function of a sum of independent r.v.'s, the theorems in the study of strong large deviations further characterize its sub-exponential decay, and such sub-exponential terms are often referred to as pre-factors. The classical error exponent analysis of channel coding theorems has been refined in the context of the exact asymptotics of the error probability (cf. [23]–[27]).

3The $+ \log \log n$ term in the right-hand side of (5) arises from the change of the base of logarithms of $V(X)$ inside the logarithm function, because the original statement in [14] is calculated in terms of the natural logarithm rather than the binary logarithm.

4In the technical parts of [19], the present authors were not able to verify the correctness of the use of the Edgeworth expansion for lattice distributions; see [12, Chapter VI.3].
4) **Variable-Length Slepian–Wolf Coding:** The Slepian–Wolf prefix-free coding problems are studied by Kuzuoka and Watanabe [28] and by He, Lastras-Montaño, Yang, Jagmohan, and Chen [29]. Specifically, He et al.’s result [29] can be thought of as a moderate deviations result, i.e., they evaluated the second-order coding rate of variable-length Slepian–Wolf coding when error probabilities vanish but the rate of decay is not exponential in the blocklength. Applying Bernstein’s inequality, Kuzuoka [30] discovered an alternative proof of the achievability result in He et al.’s work [29], and the alternative proof is more amenable for combining large deviations analysis and the usual techniques used in information spectrum methods [20].

**C. Paper Organization**

The rest of this paper is organized as follows: Section II introduces basic definitions and notations in this study. Section III revisits previous works summarized in (2) and (3), and states our main result as their integration. Section IV proves our main result by presenting several technical lemmas. Section V investigates moderate deviations and strong large deviations analyses for the fixed-length compression. Section VI concludes this study.

**II. Preliminaries**

**A. Random Variables and Discrete Memoryless Sources**

In this subsection, we introduce basic notions in probability theory, a discrete memoryless source and its information measures. Let \((Ω, F, P)\) the underlying probability space, and \(Z\) a real-valued r.v. Denote by \(P_Z := P \circ Z\) the probability distribution induced by \(Z\). We say that \(Z\) is a **lattice r.v.** if it is discrete and there exists a positive constant \(d\) such that \(\mathcal{D}(Z) := \{z_1 - z_2 \mid P_Z(z_1) P_Z(z_2) > 0\}\) is a subset of \(\mathbb{Z}\). Otherwise, we say that \(Z\) is a **nonlattice r.v.** For a lattice r.v. \(Z\), its **maximal span** is defined by the maximum of positive constants \(d\) satisfying \(\mathcal{D}(Z) \subset \mathbb{Z}\).

Given a real-valued r.v. \(Z\) and a real number \(0 < \varepsilon < 1\), define the \(\varepsilon\)-**cutoff transformation action of** \(Z [8, Equation (13)]\) by

\[
\langle Z\rangle_{\varepsilon} := \begin{cases} 
Z & \text{if } Z < \eta, \\
B Z & \text{if } Z = \eta, \\
0 & \text{if } Z > \eta,
\end{cases}
\]

where \(B\) is the Bernoulli r.v. with parameter \(1 - \beta\) in which \(B \perp Z\), and two real parameters \(\eta \in \mathbb{R}\) and \(0 \leq \beta < 1\) are chosen so that

\[
\mathbb{P}\{Z > \eta\} + \beta \mathbb{P}\{Z = \eta\} = \varepsilon.
\]

Consider a countably infinite alphabet \(X\) and an \(X\)-valued r.v. \(X\). In this study, i.i.d. copies \(\{X_i\}_{i=1}^{\infty}\) of \(X\) play the role of a discrete memoryless source, and we simply call \(X\) the **source**. A source \(X\) is said to be **finitely supported** if the support \(\text{supp}(X) := \{x \in X \mid P_X(x) > 0\}\) is finite. We say that \(X\) is a **lattice source** if \(P_X(X)\) is a lattice r.v., where log stands for the logarithm to the base-2. On the other hand, we say that \(X\) is a **nonlattice source** if \(P_X(X)\) is a nonlattice r.v. For a lattice source \(X\), denote by \(d_X\) the maximal span of \(\log P_X(X)\). For convenience, we set \(d_X = 0\) if \(X\) is a nonlattice source.

Define the following quantities:

\[
H(X) := \sum_{x \in \text{supp}(X)} P_X(x) \log \frac{1}{P_X(x)},
\]

\[
H_\alpha(X) := \frac{1}{1 - \alpha} \log \left( \sum_{x \in \text{supp}(X)} P_X(x)^\alpha \right),
\]

\[
V(X) := \sum_{x \in \text{supp}(X)} P_X(x) \left( \log \frac{1}{P_X(x)} - H(X) \right)^2,
\]

\[
S(X) := \mathbb{E} \sum_{x \in \text{supp}(X)} P_X(x) \left( -\log P_X(x) - H(X) \right)^3.
\]

Throughout this study, assume that \(V(X) > 0\), i.e., assume that \(X\) is not uniformly distributed on a finite subalphabet \(\mathcal{A} \subset X\). Similar to a notion in probability theory (cf. [12, Chapter VIII.2]), we define the following condition on a source \(X\).

**Definition 1.** We say that a source \(X\) satisfies Cramér’s condition if \(H_\alpha(X)\) is finite for some \(0 < \alpha < 1\).

**Remark 1.** The Rényi entropy \(H_\alpha(X)\) can be thought of as the cumulant generating function of the information density \(-\log P_X(X)\), i.e., we readily see that

\[
(1 - \alpha) H_\alpha(X) = \log \mathbb{E}[2^{(1-\alpha)\log P_X(X)}].
\]
Namely, Cramér’s condition on $X$ ensures the existence of the $k$-th moment $\mathbb{E}[\log^k P_X(X)]$ for every $k \geq 1$, i.e., the quantities $H(X)$, $V(X)$, and $S(X)$ are finite in this case. Note that there exists a source $X$ such that $H(X)$, $V(X)$, and $S(X)$ are finite but Cramér’s condition fails to hold (see, e.g., [31, Example 5]). On the other hand, since $H_n(X) \leq \log |\text{supp}(X)|$ for every $\alpha \geq 0$, it is easy to see that $X$ satisfies Cramér’s condition, provided that $X$ is finitely supported.

B. Gaussian Distributions

Define the Gaussian probability density function and the Gaussian cumulative distribution function as

$$
\varphi(u) := \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \tag{13}
$$

$$
\Phi(u) := \int_{-\infty}^{u} \varphi(t) \, dt \tag{14}
$$

for $u \in \mathbb{R}$, respectively. Moreover, define

$$
f_G(s) := \begin{cases} 
\varphi(\Phi^{-1}(s)) & \text{if } 0 < s < 1, \\
0 & \text{if } s = 0 \text{ or } s = 1,
\end{cases} \tag{15}
$$

$$
g_G(s) := \begin{cases} 
\varphi(\Phi^{-1}(s)) \Phi^{-1}(s) & \text{if } 0 < s < 1, \\
0 & \text{if } s = 0 \text{ or } s = 1,
\end{cases} \tag{16}
$$

for $0 \leq s \leq 1$, where $\Phi^{-1}(\cdot)$ denotes the inverse function of $\Phi(\cdot)$. It is known that

$$
\Phi^{-1}(s) \sim -\frac{\sqrt{2 \ln s}}{s} \tag{17}
$$

$$
f_G(s) \sim s \frac{\sqrt{2 \ln s}}{s} \tag{18}
$$

as $s \to 0^+$ (cf. [32, Lemma 5.2]), where $\ln$ stands for the natural logarithm. Thus, we see that

$$
\lim_{s \to 0^+} f_G(s) \Phi^{-1}(s) = -\lim_{s \to 1^-} f_G(s) \Phi^{-1}(s) = \lim_{s \to 0^+} 2 s \ln s = 0, \tag{19}
$$

implying that the definitions of $g_G(s)$ at $s = 0$ and at $s = 1$ are consistent with the limits as $s \to 0^+$ and as $s \to 1^-$, respectively. The following lemma shows a higher-order asymptotic expansion of $\Phi^{-1}(\cdot)$ beyond that presented in (17).

**Lemma 1** ([33]). It holds that

$$
\Phi^{-1}(s)^2 = 2 \ln \frac{1}{2 \sqrt{\pi} s} - \ln \frac{1}{2 \sqrt{\pi} s} + O \left( \frac{\ln \ln(1/s)}{\ln(1/s)} \right) \tag{20}
$$

as $s \to 0^+$.

The following lemma is useful to integrate polynomials of $\Phi^{-1}(\cdot)$.

**Lemma 2.** Given $0 \leq a < b \leq 1$, it holds that

$$
\int_a^b \Phi^{-1}(s) \, ds = f_G(a) - f_G(b), \tag{21}
$$

$$
\int_a^b \Phi^{-1}(s)^2 \, ds = (b - a) - g_G(b) + g_G(a). \tag{22}
$$

**Proof of Lemma 2:** Elementary calculations yield all formulas, and we omit the details here.

C. Asymptotic Notations

In this paper, we use the following asymptotic notations to express our asymptotic expansions in source coding problems. Let $I_n$ be a sequence of real intervals, and $I = \bigcup_n I_n$. Consider two sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ of real-valued functions on $I$, and a sequence $\{a_n\}_{n=1}^{\infty}$ of positive numbers. For fixed $t \in I$, we say that $f_n(t) = g_n(t) + O(a_n)$ as $n \to \infty$ if

$$
\limsup_{n \to \infty} \left| \frac{f_n(t) - g_n(t)}{a_n} \right| < \infty, \tag{23}
$$

and that $f_n(t) = g_n(t) + o(a_n)$ as $n \to \infty$ if

$$
\lim_{n \to \infty} \left| \frac{f_n(t) - g_n(t)}{a_n} \right| = 0. \tag{24}
$$
In particular, we say that \( f_n(t) = g_n(t) + O(a_n) \) uniformly on \( I_n \) as \( n \to \infty \) if
\[
\limsup_{n \to \infty} \frac{1}{a_n} \sup_{t \in I_n} |f_n(t) - g_n(t)| < \infty,
\]
and that \( f_n(t) = g_n(t) + o(a_n) \) uniformly on \( I_n \) as \( n \to \infty \) if
\[
\lim_{n \to \infty} \frac{1}{a_n} \sup_{t \in I_n} |f_n(t) - g_n(t)| = 0.
\]

In this study, these uniform convergence properties on a sequence of intervals are used in the moderate deviations analysis to investigate higher-order asymptotics of the fixed-length compression problem in which the error probabilities are asymptotically close to zero or one for sufficiently large codeword lengths; see Section IV-A for details.

III. VARIABLE-LENGTH COMPRESSION

A. Variable-Length Compression Allowing Errors—Revisited

In this subsection, we revisit the previous results stated in (2) and (3) formally. Consider compressing a discrete memoryless source \( X \) into a finite-length binary string. Let
\[
\{0, 1\}^\ast := \{\emptyset\} \cup \left( \bigcup_{n=1}^\infty \{0, 1\}^n \right)
\]
be the set of finite-length binary strings containing the empty string \( \emptyset \). Denote by \( \ell : \{0, 1\}^\ast \to \mathbb{N} \cup \{0\} \) the length function of a binary string, i.e., \( \ell(\emptyset) = 0 \), \( \ell(0) = \ell(1) = 1 \), \( \ell(00) = \ell(01) = \ell(10) = \ell(11) = 2 \), etc.

**Definition 2.** An \((L, \varepsilon)-code\) for a source \( X \) is a pair of a stochastic encoder \( F : X \to \{0, 1\}^\ast \) and a stochastic decoder \( G : \{0, 1\}^\ast \to X \) such that
\[
\mathbb{E}[\ell(F(X))] \leq L,
\]
\[
\mathbb{P}\{X \neq G(F(X))\} \leq \varepsilon.
\]

Given a permissible probability of error \( 0 \leq \varepsilon \leq 1 \), denote by \( L^\ast(\varepsilon \mid X) \) the infimum of \( L > 0 \) such that an \((L, \varepsilon)-code\) exists for \( X \). We recall the definition of \( \langle \cdot \rangle_\varepsilon \) in (6). It is known (cf. [8, Equation (26)]) and [34, Lemmas 1 and 5]) that
\[
L^\ast(\varepsilon \mid X^n) = \mathbb{E}[\langle \log \varsigma_n^{-1}(X^n) \rangle_\varepsilon],
\]
where \( \varsigma_n : \{1, 2, 3, \ldots \} \to X^n \) is an arbitrary bijection satisfying\(^5\)
\[
P_{X^n}(\varsigma_n(1)) \geq P_{X^n}(\varsigma_n(2)) \geq P_{X^n}(\varsigma_n(3)) \geq \cdots,
\]
and \( \lfloor \cdot \rfloor := \max\{z \mid z \leq \cdot \} \) denotes the floor function. However, note that the right-hand side of (30) is not single-letterized, and we are interested to determine asymptotic expansions of \( L^\ast(\varepsilon \mid X^n) \) as \( n \to \infty \) in a computable form.

The following theorem is a known second-order asymptotic result for this problem.

**Theorem 1** (Kostina, Polyanskiy, and Verdú [8, Theorem 4]). Given a fixed \( 0 \leq \varepsilon \leq 1 \) and a source \( X \), it holds that
\[
L^\ast(\varepsilon \mid X^n) = n(1 - \varepsilon) H(X) - \sqrt{n V(X)} f_\varepsilon(\varepsilon) + O(\log n)
\]
as \( n \to \infty \), provided that \( \mathbb{E}[\log^3 P_X(X)] \) is finite.

In [8], Theorem 1 was proved by establishing the one-shot bounds\(^6\)
\[
\mathbb{E}\left[ \log \frac{1}{P_{X^n}(X^n)} \right] - \log(1 + n H(X)) - \log \varepsilon \leq L^\ast(\varepsilon \mid X^n) \leq \mathbb{E}\left[ \log \frac{1}{P_{X^n}(X^n)} \right] \]  
(33)

and the asymptotic expansion\(^7\)
\[
\mathbb{E}\left[ \log \frac{1}{P_{X^n}(X^n)} \right] = n H(X) - \sqrt{n V(X)} f_\varepsilon(\varepsilon) + O(1)
\]
as \( n \to \infty \). Roughly speaking, this proof strategy converts the analysis of \( \log \varsigma_n^{-1}(X^n) \) to that of the information density \( -\log P_{X^n}(X^n) \); see (30). On the other hand, this proof strategy does not allow us to obtain a more precise expression of the remainder term \( +O(\log n) \) of Theorem 1. For an asymptotic relation (in an almost sure sense) between \( \log \varsigma_n^{-1}(X^n) \) and

\(^5\)Namely, the bijection \( \varsigma_n \) plays the role of a decreasing rearrangement of \( P_{X^n}(\cdot) \).

\(^6\)When \( \varepsilon = 0 \), the lower bound specializes Alon and Orlitsky’s bound [5], and the upper bound specializes Wyner’s bound [4].

\(^7\)This asymptotic expansion was proven by Berry–Esseen-type bounds (cf. [11, Chapter XVI.5] and [12, Chapter V.4]).
− \log P_{X^n}(X^n) up to the \( +o(\kappa_n, \log n) \) term with any slowly divergent positive sequence \( \{\kappa_n\}_{n=1}^{\infty} \), we refer the reader to the study of pointwise redundancy studied by Kontoyiannis and Verdú [9, Section IV].

In the particular case of \( \varepsilon = 0 \) and finitely supported \( X \), Theorem 1 can be refined as follows:

**Theorem 2** (Szpankowski and Verdú [7, Theorem 4]). For a finitely supported source \( X \), it holds that

\[
L^*(0 \mid X^n) = n H(X) - \frac{1}{2} \log n + O(1)
\]  
(35)

as \( n \to \infty \).

In [7], Theorem 2 was proven via complex analysis so-called the analytic poissonization/depoissonization (cf. [35]) and Stirling’s formula to approximate multinomial coefficients.

**Remark 2.** Recently, the present authors [34] investigated the problem considering variable-length compression allowing errors in the case when some side-information \( Y \) of the source \( X \) is available at both encoder and decoder. For this problem, we introduced two error formalisms: the average and maximum error criteria, where the averaging and maximization are taken with respect to \( Y \). We showed that the first-order optimal coding rates are the same under both error criteria, and the second-order optimal coding rates differ under these error criteria. In particular, this difference can be characterized by the law of total variance for the conditional information density \( \log P_{X|Y}(X \mid Y) \).

### B. Main Result—Higher-Order Asymptotics of Variable-Length Compression

The following theorem is a refinement of Theorems 1 and 2 and constitutes the main result of the paper.

**Theorem 3.** Let \( 0 < \varepsilon \leq 1 \) be fixed. If the source \( X \) satisfies Cramér’s condition, then

\[
L^*(\varepsilon \mid X^n) = n (1 - \varepsilon) H(X) - \sqrt{n V(X)} f_0(\varepsilon) - \frac{1 - \varepsilon}{2} \log n + O(1)
\]  
(36)

as \( n \to \infty \). On the other hand, if \( \varepsilon = 0 \) and \( X \) is finitely supported, then (35) holds.

**Remark 3.** Note that the third-order term in (36), being \(-(1 - \varepsilon)/2) \log n\), differs from the usual third-order terms for other information-theoretic problems such as channel and source coding (cf. [18]), in which these terms do not depend on \( \varepsilon \).

We prove Theorem 3 in the next section, which contains also an alternative proof of Theorem 2. Our proof strategy relies more heavily on information-spectrum methods [20] compared to the original proof in [7, Section V].

The proof outline of Theorem 3 is as follows: Since every codeword length is a nonnegative integer, it is known that

\[
L^*(0 \mid X^n) = \sum_{k=1}^{\infty} \mathbb{P}\{\log \varsigma_n^{-1}(X^n) \geq k\}
\]  
(37)

(cf. [9, Section III]). Given \( 0 < \varepsilon \leq 1 \), this identity can be readily extended as

\[
L^*(\varepsilon \mid X^n) = \sum_{k=1}^{\tilde{\xi}_n} \mathbb{P}\{\log \varsigma_n^{-1}(X^n) \geq k\} - \varepsilon \tilde{\xi}_n,
\]  
(38)

where the integer \( \tilde{\xi}_n = \tilde{\xi}_n(\varepsilon, X) \) is chosen so that

\[
\mathbb{P}\{\log \varsigma_n^{-1}(X^n) \geq \tilde{\xi}_n\} \geq \varepsilon,
\]  
(39)

\[
\mathbb{P}\{\log \varsigma_n^{-1}(X^n) > \tilde{\xi}_n\} < \varepsilon.
\]  
(40)

Here, the complementary cumulative distribution function \( \mathbb{P}\{\log \varsigma_n^{-1}(X^n) > k\} \) can be thought of as the minimum average probability of error for \( n \)-to-\( k \) binary block codes for the source \( X^n \). Namely, the average codeword length \( L^*(\varepsilon \mid X^n) \) of variable-length compression can be analyzed via the fundamental limits of fixed-length compression.\(^8\)

### IV. Proofs of Theorem 3

In this section, we prove Theorem 3 by presenting some technical lemmas.

\(^8\)This relation was mentioned by S. Verdú in his Shannon Lecture [10].
A. Moderate Deviations and Strong Large Deviations of Fixed-Length Compression

Before investigating higher-order asymptotic expansions of the variable-length compression problem, we now consider the fixed-length compression problem for a stationary memoryless source $X^n$. An $(n,M,\varepsilon)$-code for the source $X$ consists of an encoder $f : X^n \rightarrow \{1,2,\ldots,M\}$ and a decoder $g : \{1,2,\ldots,M\} \rightarrow X^n$ such that

$$
P\{X^n \neq g(f(X^n))\} \leq \varepsilon.
$$

(41)

Denote by $M^*(n,\varepsilon)$ the minimum of $M \in \mathbb{N}$ such that an $(n,M,\varepsilon)$-code exists for the source $X$. In other words, it is defined as

$$
M^*(n,\varepsilon) = \min_{\mathcal{A} \subset X} |\mathcal{A}|.
$$

(42)

The following lemma is a result of judiciously combining the use of a moderate deviations theorem and a strong large deviations theorem [11]–[13].

**Lemma 3.** Suppose that $X$ satisfies Cramér’s condition stated in Definition 1. Let $(\varepsilon_n)_{n=1}^{\infty}$ be a real sequence on $(0,1)$. If

$$
\frac{1}{n^r} \leq \varepsilon_n \leq 1 - \frac{1}{n^r}
$$

for some positive real $r$ and for sufficiently large $n$, then\(^9\)

$$
\log M^*(n,\varepsilon_n) = nH(X) - \sqrt{n}V(X)\Phi^{-1}(\varepsilon_n) - \frac{1}{2}\log n - \frac{\Phi^{-1}(\varepsilon_n)^2}{2\ln 2} + O(1)
$$

(44)

as $n \to \infty$.

**Proof of Lemma 3:** See Section V.

**Remark 4** (Refinements to the source dispersion term in the moderate deviations regime). Define

$$
D^*(n,\varepsilon) := \frac{\log M^*(n,\varepsilon) - nH(X)}{\sqrt{n}V(X)}.
$$

(45)

It is well-known that $D^*(n,\varepsilon) \to \Phi^{-1}(1-\varepsilon)$ as $n \to \infty$ for fixed $0 < \varepsilon < 1$. More precisely, it is clear from (4) that

$$
D^*(n,\varepsilon) = \Phi^{-1}(1-\varepsilon) - \frac{\log n}{2\sqrt{n}V(X)} + O\left(\frac{1}{\sqrt{n}}\right).
$$

(46)

as $n \to \infty$ for fixed $0 < \varepsilon < 1$. By Lemma 3, Equation (46) can be extended to the case when $\varepsilon$ approaches to either zero or one polynomially in $n$ as follows: Given an arbitrary positive real number $r$, it follows from Lemmas 1 and 3 that

$$
D^*(n, n^{-r}) = \Phi^{-1}(1-n^{-r}) - \frac{(1 + 2r)\log n}{2\sqrt{n}V(X)} + O\left(\frac{1}{\sqrt{n}}\right),
$$

(47)

$$
D^*(n, 1 - n^{-r}) = \Phi^{-1}(n^{-r}) - \frac{(1 + 2r)\log n}{2\sqrt{n}V(X)} + O\left(\frac{1}{\sqrt{n}}\right)
$$

(48)

as $n \to \infty$. To asymptotically expand $\Phi^{-1}(1-n^{-r})$ and $\Phi^{-1}(n^{-r})$ in (47) and (48), respectively, we see from Lemma 1 that

$$
D^*(n, n^{-r})^2 = \Phi^{-1}(1-n^{-r})^2 + O\left(\frac{\log^{3/2} n}{\sqrt{n}}\right) = 2r \ln n - n - \frac{\pi^2}{2n} - \ln(2n) + O\left(\frac{\ln \ln n}{n}\right)
$$

(49)

as $n \to \infty$; and analogously, we get

$$
D^*(n, 1 - n^{-r})^2 = \Phi^{-1}(n^{-r})^2 + O\left(\frac{\log^{3/2} n}{\sqrt{n}}\right) = 2r \ln n - n - \frac{\pi^2}{2n} - \ln(2n) + O\left(\frac{\ln \ln n}{n}\right)
$$

(50)

as $n \to \infty$. Thus, we have obtained expressions for the higher-order optimal coding rates of the fixed-length compression problem when the error probabilities vanish polynomially in the blocklength $n$. In the classical-quantum channel coding problem, the evaluation of the pre-factors in moderate deviations theorem was posed in [22, Section V] as a future work. Note that the remainder terms $+O((\ln \ln n)/(\ln n))$ in the right-hand sides of (49) and (50) can be further refined by using higher-order asymptotic expansions of $u \to \Phi(u)$ as $u \to -\infty$ beyond that stated in Lemma 1; see [33] and [36, Chapter VII.7].

Now, suppose that the source $X$ is nonlattice as defined in Section II-A. In [7, Theorem 3], it is stated that

$$
L^*(0 \mid X^n) \geq nH(X) - \frac{1}{2} \log n - \frac{1}{2} \log(8\pi e V(X)) + o(1)
$$

(51)

\(^9\)The remainder term $+O(1)$ in (44) depends only on $X$ and $r$, i.e., it is independent of the sequence $(\varepsilon_n)_{n=1}^{\infty}$. 

as \( n \to \infty \). To prove (51), citing Strassen’s seminal paper [14], Szpankowski and Verdú [7, Equations (50)–(52)] used\(^\text{10}\)
\[
[\log M^*(n, e)] = n H(X) - \sqrt{n} \Phi(V(X)) \Phi^{-1}(e) + \frac{1}{2} \log n - \frac{1}{2} \log \left(2 \pi e^{\Phi^{-1}(e)} V(X)\right) - \frac{\sqrt{V(X) S(X) (1 - \Phi^{-1}(e)^2)}}{6n} + o(1) \tag{52}
\]
as \( n \to \infty \), which is an asymptotic expansion not for \((n, M, e)\)-codes defined in the first paragraph of Section IV-A but for \(n\)-to-\(k\) binary block codes (cf. [32, Chapter 1]). In addition, Szpankowski and Verdú [7, Equations (64)–(65)] stated a certain estimate on the remainder term \( +o(1) \) in (52), and then employed the monotone convergence theorem to ensure that the \( +o(1) \) term is finitely integrable with respect to \( e \in (0, 1) \). Finally, they stated that the definite integral of the \( +o(1) \) term over the interval \((0, 1)\) still vanishes as \( n \to \infty \). Thus, substituting (52) into [7, Equations (44)–(48)] (see Remark 5 to follow), Szpankowski and Verdú obtained the relation in (51).

However, the present authors could not verify the correctness of all the steps in the proof strategy in [7, Section V]. In particular, Equation (52) does not immediately follow from Strassen’s seminal result in (5) because of the additional ceiling operation on \( \log M^*(n, e) \). Indeed, the right-hand sides of (5) and (52) are not equivalent, and the ceiling operation will, in general, result in some oscillatory terms in the fourth-order term in the asymptotic expansion of \( [\log M^*(n, e)] \). In addition, unfortunately, there appears to be a gap in how Szpankowski and Verdú [7] justified that the \( o(1) \) term in (52) is finitely integrable with respect to \( e \in (0, 1) \). More precisely, the finite integrability property is ensured by a certain bound [7, Equations (64)–(65)], which was stated by citing Strassen’s paper [14]. However, Strassen used the central limit theorem to get the corresponding asymptotic estimates [14, Equations (2.17)–(2.19)]; these estimates fail to hold uniformly in \( e \in (0, 1) \) due to the fact that \( \Phi^{-1}(\cdot) \) is not uniformly continuous on \((0, 1)\).

To circumvent these issues, we applied different tools—such as the moderate deviations results in Section V-C—to analyze the fixed-length compression problem. This results in a new asymptotic expansion for fixed-length compression in Lemma 3 that is also amenable to integration over the error probability parameter (over a certain range) to obtain a third-order asymptotic expansion for the variable-length compression problem.

B. On the Cutoff Operation for Logarithm of Integer-Valued Random Variable

Due to the cutoff function in the expectation on the right-hand side of (30), it is difficult to deal with the exact fourth-order term of \( L'(e \mid X^n) \), i.e., the \( +O(1) \) term of \( L'(e \mid X^n) \). Instead, the following two lemmas investigate \( \mathbb{E}[\langle \log \varsigma^{-1}_n(X^n)\rangle_e] \), i.e., the expectation of \( \langle \log \varsigma^{-1}_n(X^n)\rangle_e \) in the absence of the floor function.

**Lemma 4.** Given \( 0 \leq e \leq 1 \), it holds that
\[
\mathbb{E}[\langle \log \varsigma^{-1}_n(X^n)\rangle_e] = \int_e^1 \log M^*(n, s) \, ds + o(1) \tag{53}
\]
as \( n \to \infty \).

**Proof of Lemma 4:** See Appendix A.

**Remark 5.** In [7, Equations (44)–(48)], Szpankowski and Verdú showed that
\[
\mathbb{E}[\langle \log \varsigma^{-1}_n(X^n)\rangle] = \int_0^1 [\log M^*(n, s)] \, ds - 1, \tag{54}
\]
and the proof of Lemma 4 is similar to that of this identity. Note that \( [\log M^*(n, e)] \) denotes the infimum of integers \( k \) such that an \( n\)-to-\( k\) binary block code, which the error probability is at most \( e \), exists. Namely, it is slightly different from the fixed-length compression problem described in Section IV-A.

**Lemma 5.** Let \( 0 < e \leq 1 \) be fixed. If the source \( X \) satisfies Cramér’s condition, then
\[
\mathbb{E}[\langle \log \varsigma^{-1}_n(X^n)\rangle_e] = n(1 - e) H(X) - \sqrt{n} \Phi(V(X)) f_0(e) - \frac{1 - e}{2} \log n + O(1) \tag{55}
\]
as \( n \to \infty \). On the other hand, if \( X \) is finitely supported, then
\[
\mathbb{E}[\log \varsigma^{-1}_n(X^n)] = n H(X) - \frac{1}{2} \log n + O(1) \tag{56}
\]
as \( n \to \infty \).

**Proof of Lemma 5:** Firstly, suppose that \( X \) satisfies Cramér’s condition. Since \( e \mapsto \log M^*(n, e) \) is nonnegative and nonincreasing on \((0, 1)\), we readily see that
\[
\int_e^{1-e^{-1}} \log M^*(n, s) \, ds \leq \int_e^1 \log M^*(n, s) \, ds \leq \int_e^{1-\epsilon^{-1}} \log M^*(n, s) \, ds + \frac{1}{n} \log M^*(n, e) \tag{57}
\]
\(^\text{10}\)Equation (52) can also be found in [9, Equation (36)].
for \( n \geq (1 - \varepsilon)^{-1} \). Now, Lemma 3 implies that

\[
K_n(X) = O(1)
\] (58)
as \( n \to \infty \), where

\[
K_n(X) \coloneqq \sup_{n^{-1} \leq \varepsilon \leq 1 - n^{-1}} \left| \log M^*(n, \varepsilon) - \left( n (1 - \varepsilon) H(X) - \sqrt{n V(X)} \Phi^{-1}(\varepsilon) - \frac{1}{2} \log n - \frac{\Phi^{-1}(\varepsilon)^2}{2} \right) \right|.
\] (59)

Then, it follows from Lemma 2 and (58) that

\[
\int_{\varepsilon}^{1 - n^{-1}} \log M^*(n, s) \, ds \leq n \left( 1 - \frac{1}{n} - \varepsilon \right) H(X) - \sqrt{n V(X)} \left( f_G(\varepsilon) - f_G(n^{-1}) \right)
- \frac{1 - n^{-1} - \varepsilon}{2} \log n - \frac{1 - n^{-1} - \varepsilon - g_G(1 - n^{-1}) + g_G(\varepsilon)}{2 \ln 2} + K_n(X)
= n (1 - \varepsilon) H(X) - \sqrt{n V(X)} f_G(\varepsilon) - \frac{1 - \varepsilon}{2} \log n + O(1)
\] (60)
as \( n \to \infty \). Analogously, we get

\[
\int_{\varepsilon}^{1 - n^{-1}} \log M^*(n, s) \, ds \geq n \left( 1 - \frac{1}{n} - \varepsilon \right) H(X) - \sqrt{n V(X)} \left( f_G(\varepsilon) - f_G(n^{-1}) \right)
- \frac{1 - n^{-1} - \varepsilon}{2} \log n - \frac{1 - n^{-1} - \varepsilon - g_G(1 - n^{-1}) + g_G(\varepsilon)}{2 \ln 2} - K_n(X)
= n (1 - \varepsilon) H(X) - \sqrt{n V(X)} f_G(\varepsilon) - \frac{1 - \varepsilon}{2} \log n + O(1)
\] (61)

Since \( \log M^*(n, \varepsilon) = O(n) \) as \( n \to \infty \), combining (57), (60), and (61), we obtain (55) of Lemma 5.

Finally, suppose that \( X \) is finitely supported. Similar to (57), we get

\[
\int_{n^{-1}}^{1 - n^{-1}} \log M^*(n, s) \, ds \leq \int_{n^{-1}}^{1} \log M^*(n, s) \, ds
\leq \int_{n^{-1}}^{1 - n^{-1}} \log M^*(n, s) \, ds + 2 \log |\text{supp}(X)|.
\] (62)

Since every finitely supported source \( X \) satisfies Cramér’s condition, it follows from Lemmas 2 and 3 that

\[
\int_{n^{-1}}^{1 - n^{-1}} \log M^*(n, s) \, ds = n \left( 1 - \frac{2}{n} \right) H(X) - \frac{1 - 2 n^{-1}}{2} \log n - \frac{1 - 2 n^{-1} + 2 g_G(n^{-1})}{2 \ln 2} + O(1)
= n H(X) - \frac{1}{2} \log n + O(1)
\] (63)
as \( n \to \infty \). Combining (62) and (63), we obtain (56) of Lemma 5. This completes the proof of Lemma 5. ■

We now use the above to complete the proof of Theorem 3. We see from (30) that

\[
\mathbb{E}[\log \varsigma_n^{-1}(X^n)] - L^*(\varepsilon \mid X^n) = \mathbb{E}[\{\log \varsigma_n^{-1}(X^n)\}_{\varepsilon}].
\] (64)

where \( \{u\} := u - [u] \) denotes the fractional part of \( u \in \mathbb{R} \). Thus, since \( 0 \leq \{u\} < 1 \) for every \( u \in \mathbb{R} \), it is clear that the gap between \( L^*(\varepsilon \mid X^n) \) and \( \mathbb{E}[\{\log \varsigma_n^{-1}(X^n)\}_{\varepsilon}] \) is at most 1 bit. Therefore, Lemma 5 implies Theorem 3, completing the proof. ■

V. Higher-Order Asymptotics of Fixed-Length Compression

In this section, we prove Lemma 3 by employing refinements of the central limit theorem and certain variants of the moderate deviations and strong large deviations theorems. In the next three subsections, we introduce these fundamental results.

A. Refinements of the Central Limit Theorem

Consider i.i.d. copies \( \{Z_i\}_{i=1}^\infty \) of a real-valued r.v. \( Z \) with zero mean. Suppose that the variance of \( Z \),

\[
\sigma^2 := \mathbb{E}[Z^2],
\] (65)
is positive and finite. Now, we want to characterize the distribution function defined by\(^{11}\)

\[
F_n(z) := \mathbb{P}\left( \sum_{i=1}^n Z_i \leq z \sigma \sqrt{n} \right)
\] (66)

\(^{11}\)By convention, we also define its left limit as \( F_n(z^-) := \lim_{u \to z^-} F_n(u) \).
for each $z \in \mathbb{R}$. The \textit{central limit theorem} states that
\begin{equation}
F_n(z) = \Phi(z) + o(1)
\end{equation}
uniformly on $\mathbb{R}$ as $n \to \infty$. In this study, to examine higher-order asymptotics of source coding problems, we shall control the error term in (67) more precisely. The following lemma is known as the Edgeworth expansion.

\textbf{Lemma 6} ([11, Chapter XVI.4] or [12, Chapter VI.3]). Suppose that $\mathbb{E}[|Z|^3]$ is finite. If $Z$ is nonlattice, then
\begin{equation}
F_n(z) = \Phi(z) + \frac{\mu_3 (1 - z^2) \varphi(z)}{6 \sigma^3 \sqrt{n}} + o(n^{-1/2})
\end{equation}
uniformly on $\mathbb{R}$ as $n \to \infty$.

\section*{B. Moderate Deviations}

Recall the notations used in the previous subsection. Given a real number $0 < \varepsilon < 1$, choose $\zeta_n(\varepsilon) \in \mathbb{R}$ so that
\begin{equation}
\zeta_n(\varepsilon) := \inf\{z \in \mathbb{R} \mid F_n(z) \geq 1 - \varepsilon\}.
\end{equation}
By (67), one readily sees
\begin{equation}
\Phi(\zeta_n(\varepsilon)) = 1 - \varepsilon + o(1)
\end{equation}
uniformly on $(0, 1)$ as $n \to \infty$. We will, however, require a statement similar to (70) when $\varepsilon$ is a sequence $\{\varepsilon_n\}_{n=1}^\infty$ with limit infimum and limit supremum respectively equal to zero and one. The sequence $\{\varepsilon_n\}_{n=1}^\infty$ should also have the property that its subsequences approach zero or one polynomially fast. In fact, we will require a stronger statement that also quantifies the “rate of convergence”. To this end, we shall use the following version of the \textit{moderate deviations theorem}.

\textbf{Lemma 7} ([12, Chapter VIII.2]). Suppose that the moment-generating function $\mathbb{E}[e^{tZ}]$ is finite for some neighborhood of $t = 0$ (i.e., Cramér’s condition on $Z$). Given a nonnegative real sequence $\{z_n\}_{n=1}^\infty$ satisfying $z_n = O(n^{1/6})$ as $n \to \infty$, it holds that
\begin{equation}
1 - F_n(z_n) = \left(1 - \Phi(z_n)\right) \exp\left(\frac{\varepsilon_n^3 \mu_3}{6 \sqrt{n} \sigma^3}\right) + O\left(\frac{\varphi(z_n)}{\sqrt{n}}\right),
\end{equation}
\begin{equation}
F_n(-z_n) = \Phi(-z_n) \exp\left(-\frac{\varepsilon_n^3 \mu_3}{6 \sqrt{n} \sigma^3}\right) + O\left(\frac{\varphi(z_n)}{\sqrt{n}}\right)
\end{equation}
as $n \to \infty$.

Using Lemma 7, we refine the $+o(1)$ term in (70) as follows:

\textbf{Lemma 8}. Let $\{\varepsilon_n\}_{n=1}^\infty$ be a real sequence satisfying
\begin{equation}
\frac{1}{n^r} \leq \varepsilon_n \leq \frac{1}{2}
\end{equation}
for some positive constant $r$. Suppose that the moment generating function $\mathbb{E}[e^{tZ}]$ is finite for some neighborhood of $t = 0$. Then, it holds that
\begin{equation}
1 - \Phi(\zeta_n(\varepsilon_n)) = \varepsilon_n \exp\left(O\left(\frac{\varepsilon_n \Phi^{-1}(\varepsilon_n)^2}{n}\right)\right) + O\left(\frac{f_3(\varepsilon_n)}{\sqrt{n}}\right),
\end{equation}
\begin{equation}
\Phi(\zeta_n(1 - \varepsilon_n)) = (1 - \varepsilon_n) \exp\left(O\left(\frac{(1 - \varepsilon_n) \Phi^{-1}(\varepsilon_n)^2}{n}\right)\right) + O\left(\frac{f_3(\varepsilon_n)}{\sqrt{n}}\right)
\end{equation}
as $n \to \infty$.

\textit{Proof of Lemma 8}: See Appendix B.

\textbf{Remark 6}. Given a finitely supported nonlattice source $X$, let $Z = -\ln P_X(X)$. Then, for fixed $0 < \varepsilon < 1$, asymptotic expansions
\begin{equation}
\Phi(\zeta_n(\varepsilon)) = 1 - \varepsilon + O(n^{-1/2}),
\end{equation}
as $n \to \infty$ were investigated by Strassen [14, Equation (2.21)] based on Lemma 6. For a precise analysis of the Berry–Esseen bound used to derive (76), we refer the reader to Kontoyiannis and Verdú’s work [9, Section V]. On the other hand, Lemma 8 exhibits similar asymptotic expansions when either $\varepsilon_n \to 0^+$ or $\varepsilon_n \to 1^-$ along certain subsequences polynomially as $n \to \infty$. 
C. Strong Large Deviations

In this subsection, we introduce strong large deviations theorems for \( \sigma \)-finite measures that are not necessarily probability measures.\(^{12}\) Let \((\Omega, \mathcal{F}, \mu)\) be a \( \sigma \)-finite measure space, and \( f : \Omega \to \mathbb{R} \) a Borel-measurable function. Denote by \( \mu_f := \mu \circ f^{-1} \) the measure on \( \mathbb{R} \) induced by \( f \). Define the cumulant generating function as

\[
\Lambda_{\mu_f}(s) := \ln \left( \int_{\mathbb{R}} e^{st} \mu_f(dt) \right),
\]

and the Fenchel–Legendre transform of \( \Lambda_{\mu_f}(s) \) by

\[
\Lambda_{\mu_f}^*(a) := \sup_{s \in \mathbb{R}} \left( as - \Lambda_{\mu_f}(s) \right).
\]

Let \( \mathcal{D}_{\mu_f} := \{ s \mid \Lambda_{\mu_f}(s) < \infty \} \) and \( \text{int}(\mathcal{D}_{\mu_f}) \) its interior. Similar to \([13, \text{Lemma 2.2.5 and Exercise 2.2.24}]\), it can be verified by Hölder’s inequality and the dominated convergence theorem for the Lebesgue integrals with respect to a \( \sigma \)-finite measure that \( \Lambda_{\mu_f}(s) \) is of class \( \mathcal{C}^\infty \) in \( s \in \text{int}(\mathcal{D}_{\mu_f}) \) and especially, it holds that for each \( s \in \text{int}(\mathcal{D}_{\mu_f}) \),

\[
\Lambda_{\mu_f}'(s) = a \implies \Lambda_{\mu_f}^*(a) = as - \Lambda_{\mu_f}(s).
\]

Similar to Section II-A, one can consider the notion of \( \mu_f \) being a lattice measure. We say that \( \mu_f \) is a lattice measure if \( \mu_f \) is discrete\(^{13}\) and there exists a positive constant \( d \) such that \( f(t_1) - f(t_2) \) is a multiple of \( d \) whenever \( \mu_f(t_1)\mu_f(t_2) > 0 \). Otherwise, we say that \( \mu_f \) is a nonlattice measure. For a lattice measure \( \mu_f \), its maximal span \( d_f \) is defined by the maximum of those \( d \). For convenience, we set \( d_f = 0 \) if \( \mu_f \) is nonlattice. Then, given a positive parameter \( s \), define

\[
\nu_s(f) := \begin{cases} \frac{df}{e^{sf} - 1} & \text{if } \mu_f \text{ is lattice}, \\ \frac{df}{s} & \text{if } \mu_f \text{ is nonlattice}. \end{cases}
\]

Now, consider \( n \) Borel-measurable functions \( f_1, \ldots, f_n \) in which \( \mu \circ (f_1, \ldots, f_n)^{-1} = \mu_{f_1} \times \cdots \times \mu_{f_n} \) and \( \mu_{f_i} = \mu_f \) for each \( 1 \leq i \leq n \).\(^{14}\) The following lemma states a strong large deviations result known as the Bahadur–Rao theorem.

**Lemma 9** ([13, Theorem 3.7.4] and [12, Chapter VIII.4]) for probability measures \( \mu \). Let \( a = \Lambda_{\mu_f}^*(s) \) for some positive \( s \in \text{int}(\mathcal{D}_{\mu_f}) \). Then, it holds that\(^{15}\)

\[
\mu \left\{ \frac{n}{i=1} f_i > an \right\} = \frac{e^{-n\Lambda_{\mu_f}^*(a)}}{\sqrt{2\pi n \Lambda_{\mu_f}''(s)}} \nu_s(f) + o(1) \quad (\text{as } n \to \infty),
\]

\[
\mu \left\{ \frac{n}{i=1} f_i = an \right\} = \frac{e^{-n\Lambda_{\mu_f}^*(a)}}{\sqrt{2\pi n \Lambda_{\mu_f}''(s)}} df + o(1) \quad (\text{as } n \to \infty).
\]

In [13, Theorem 3.7.4] and [12, Chapter VIII.4], Lemma 9 is stated in the case when \( \mu \) is a probability measure, and its proof can be readily extended to \( \sigma \)-finite measures \( \mu \). We give a proof sketch of Lemma 9 in Appendix C.

The following lemma is a variant of Lemma 9.

**Lemma 10.** Let \( \{I_n\}_{n=1}^{\infty} \) be a sequence of real intervals, and \( I = \bigcup_n I_n \). Consider a real function \( a_n(.) \) on \( I \) for each \( n \in \mathbb{N} \). Suppose that \( a = \Lambda_{\mu_f}^*(s) \) for some positive \( s \in \text{int}(\mathcal{D}_{\mu_f}) \). If

\[
a_n(t) = o(n) \quad \text{uniformly on } I_n \text{ as } n \to \infty,
\]

then there exist \( r_f^{(1)}(n,t) = o(1) \) and \( r_f^{(2)}(n,t) = O(1) \) uniformly on \( I_n \) as \( n \to \infty \) such that

\[
\mu \left\{ \frac{n}{i=1} f_i > an + a_n(t) \right\} = \frac{e^{-K_f(n,s,t)}}{\sqrt{2\pi n \Lambda_{\mu_f}''(s)(1 + o(1))}} \nu_s(f) + o(1),
\]

\[
\mu \left\{ \frac{n}{i=1} f_i = an + a_n(t) \right\} = \frac{e^{-K_f(n,s,t)}}{\sqrt{2\pi n \Lambda_{\mu_f}''(s)(1 + o(1))}} df + o(1).
\]

\(^{12}\)For strong large deviations for finite measures, refer to [19, Section VIII] or [37, Footnote 8]. In this study, we consider \( \sigma \)-finite measures to deal with a countably infinite source alphabet \( \mathcal{X} \), because the results on finite measures are applicable only for finite source alphabets.

\(^{13}\)A measure \( \nu \) is said to be discrete if there exists a measurable set \( \mathcal{E} \) such that it is countable and \( \nu(\mathcal{E}^c) = 0 \), where \( \mathcal{E}^c \) denotes the complement of \( \mathcal{E} \).

\(^{14}\)When \( \mu \) is a probability measure, this implies that \( f_1, \ldots, f_n \) are i.i.d. copies of a real-valued r.v. \( f \).

\(^{15}\)When \( \mu_f \) is lattice, then the remainder terms \( +o(1) \) can be refined as \( +o(n^{-1}) \); see [12, Chapter VIII.4].
uniformly on $I_n$ as $n \to \infty$, where the exponent part $K_f(n,s,t)$ is given as

$$K_f(n,s,t) = n \Lambda_{\beta_Y}^*(a) + s a_n(t) + \frac{a_n(t)^2}{2 n \Lambda''_{\beta_Y}(s)} (1 + o(1))$$

uniformly on $I_n$ as $n \to \infty$.

Proof of Lemma 10: See Appendix D.

Remark 7. Lemma 10 is a minor extension of [19, Lemma 3]. The main differences vis-à-vis [19, Lemma 3] are that $\mu$ is not only finite but $\sigma$-finite and the asymptotic expansion in (86) is refined.

D. Proof of Lemma 3

Denote by $\omega(X) := -\ln P_{X^n}(X^n)$ the information density of $X^n$, where $\omega(X) := \omega_1(X)$. Consider a $\sigma$-finite measure $\mu$ in which $\mu_X$ is the counting measure on supp$(X)$ and $\mu_X(X \setminus$ supp$(X)) = 0$. Now, define $\nu_X := \mu \circ (-\omega(X))^{-1}$. Since $H_n(X) < \infty$ for some $0 < \alpha < 1$, we observe that $s \mapsto \Lambda_{\nu_X}(s)$ is infinitely differentiable at $s = 1$. Then, a direct calculation shows

$$\Lambda_{\nu_X}^*(1) = -(\ln 2) H(X),$$

$$\Lambda_{\nu_X}^{**}(1) = (\ln 2)^2 V(X),$$

Choose the nonnegative number $\eta_n(\epsilon_n, X)$ so that

$$F^+_n(\epsilon_n, X) := P\{\epsilon_n(X) \leq \eta_n(\epsilon_n, X)\} \geq 1 - \epsilon_n,$$

$$F^-_n(\epsilon_n, X) := P\{\epsilon_n(X) < \eta_n(\epsilon_n, X)\} < 1 - \epsilon_n,$$

respectively. It follows from (42) that

$$M^*(n, \epsilon_n) = \mu\{\epsilon_n(X) < \eta_n(\epsilon_n, X)\} + \left[ \frac{(1 - \epsilon_n) - F^-_n(\epsilon_n, X)}{F^n(\epsilon_n, X) - F^-_n(\epsilon_n, X)} \right] \mu\{\epsilon_n(X) = \eta_n(\epsilon_n, X)\},$$

yielding that

$$\mu\{\epsilon_n(X) < \eta_n(\epsilon_n, X)\} < M^*(n, \epsilon_n) \leq \mu\{\epsilon_n(X) \leq \eta_n(\epsilon_n, X)\}. (92)$$

Fix a positive number $r$ arbitrarily. Define

$$\lambda_n(\epsilon_n, X) := \frac{\eta_n(\epsilon_n, X) - n (\ln 2) H(X)}{(\ln 2) V(X)}.$$ (93)

By Taylor’s theorem for $s \mapsto \Phi^{-1}(s)$ around $s = 1 - \epsilon_n$, we observe that

$$\lambda_n(\epsilon_n, X) = \Phi^{-1}(1 - \epsilon_n) - \frac{(1 - \epsilon_n) - \Phi(\lambda_n(\epsilon_n, X))}{f_G(s_n(\epsilon_n, X))},$$ (94)

where $0 < s_n(\epsilon_n, X) < 1$ is given by

$$s_n(\epsilon_n, X) := \theta_n(\epsilon_n, X) (1 - \epsilon_n) + (1 - \theta_n(\epsilon_n, X)) \Phi(\lambda_n(\epsilon_n, X))$$ (95)

for some $0 \leq \theta_n(\epsilon_n, X) \leq 1$. Substituting (94) into (93), we see that

$$\frac{\eta_n(\epsilon_n, X)}{(\ln 2)} = n H(X) - \sqrt{n} V(X) \left( \Phi^{-1}(\epsilon_n) + \frac{(1 - \epsilon_n) - \Phi(\lambda_n(\epsilon_n, X))}{f_G(s_n(\epsilon_n, X))} \right).$$ (96)

By (18) and Lemma 8, there exist positive constants $A = A(X, r)$ and $B = B(X, r)$, depending only on $X$ and $r$, such that

$$\left| \frac{(1 - \epsilon_n) - \Phi(\lambda_n(\epsilon_n, X))}{f_G(s_n(\epsilon_n, X))} \right| \leq \exp \left( \frac{A}{n} + r \ln n \right) \left( 1 - \exp \left( - \frac{A}{n} \right) \right) + \frac{B}{\sqrt{n}}$$ (97)

for sufficiently large $n$. Hence, it follows from (96) and (97) that

$$\frac{\eta_n(\epsilon_n, X)}{(\ln 2)} = n H(X) - \sqrt{n} V(X) \Phi^{-1}(\epsilon_n) + O(1)$$ (98)

as $n \to \infty$. Now, note from (87) and (88) that

$$\eta_n(\epsilon_n, X) = -n \Lambda_{\nu_X}^*(1) - \sqrt{n} \Lambda''_{\nu_X}(1) \Phi^{-1}(\epsilon_n) + O(1)$$ (99)

16This identity is a consequence of the Neyman–Pearson lemma.
as \( n \to \infty \). Therefore, applying (84) of Lemma 10 with

\[
\begin{align*}
s &= 1, \\
f_i &= \ln P_{X_i}(X_i) \quad \text{(for } i = 1, \ldots, n), \\
a_n(\varepsilon_n) &= \sqrt{n \Lambda_{\xi_n}''(1)} \Phi^{-1}(\varepsilon_n) + O(1) \quad \text{(as } n \to \infty),
\end{align*}
\]

we see that

\[
\mu\{t_n(X) < \eta_n(\varepsilon_n, X)\}
\]

\[
= \mu\left\{ \sum_{i=1}^{n} \ln P_{X_i}(X_i) > n \Lambda_{\xi_n}'(1) + \sqrt{n \Lambda_{\xi_n}''(1)} \Phi^{-1}(\varepsilon_n) + O(1) \right\}
\]

\[
= \frac{\nu(X) + o(1)}{\sqrt{2\pi n \Lambda_{\xi_n}''(1)(1 + o(1))}} \exp\left(-\frac{1}{2 n \Lambda_{\xi_n}''(1)} \left(n \Lambda_{\xi_n}''(1) \Phi^{-1}(\varepsilon_n) + O(1)\right)^2\right)
\]

\[
= \frac{\nu(X) + o(1)}{\sqrt{2\pi n (\ln 2)^2 V(X)(1 + o(1))}} \exp\left(\eta_n(\varepsilon_n, X) - \frac{\Phi^{-1}(\varepsilon_n)^2}{2} + O(1)\right)
\]

\]

\[
(103)
\]

as \( n \to \infty \), where \( \nu(X) \) is given as

\[
\nu(X) := \begin{cases} 
\frac{(\ln 2) d_X}{2^{d_X} - 1} & \text{if } X \text{ is a lattice source,} \\
1 & \text{if } X \text{ is not a lattice source,}
\end{cases}
\]

\[
(104)
\]

and \( d_X \) is defined in Section II-A. Analogously, we get from (85) of Lemma 10 that

\[
\mu\{t_n(X) = \eta_n(\varepsilon_n, X)\} = \frac{(\ln 2) d_X + o(1)}{\sqrt{2\pi n (\ln 2)^2 V(X)(1 + o(1))}} \exp\left(\eta_n(\varepsilon_n, X) - \frac{\Phi^{-1}(\varepsilon_n)^2}{2} + O(1)\right)
\]

\[
(105)
\]

as \( n \to \infty \). Combining (92), (103), and (105), we obtain

\[
\ln M^*(n, \varepsilon_n) = \eta_n(\varepsilon_n, X) - \frac{\Phi^{-1}(\varepsilon_n)^2}{2} - \frac{1}{2} \ln \left(2\pi n (\ln 2)^2 V(X)(1 + o(1))\right) + O(1)
\]

\[
= \eta_n(\varepsilon_n, X) - \frac{1}{2} \ln n - \frac{\Phi^{-1}(\varepsilon_n)^2}{2} + O(1)
\]

\[
(106)
\]

as \( n \to \infty \). This completes the proof of Lemma 3.

\[\square\]

VI. CONCLUDING REMARKS

In this study, we investigated the third-order asymptotics of the problem of variable-length compression allowing errors. Our main result is stated in Theorem 3, which shows that the first-, second-, and third-order coding rates depend on the permissible probability of error \( 0 \leq \varepsilon \leq 1 \). This observation differs from the third-order asymptotics of the fixed-length compression problem as stated in (4), which shows that the first- and third-order coding rates do not depend on \( \varepsilon \). To derive Theorem 3, we leveraged certain moderate deviations and strong large deviations results for the fixed-length compression problem in Lemma 3. This proof strategy, together with a connection between the variable- and fixed-length compression problems (see Section IV-B), shows a novel utility of a combination of moderate deviations and strong large deviations analyses in information theory. Finally, we believe that the mathematical results in Lemma 8 and Lemma 10 may be of independent interest.

APPENDIX A

PROOF OF LEMMA 4

Now, choose a positive integer \( \xi_n = \xi_n(n, X) \) so that

\[
\begin{align*}
\mathbb{P}\{s_n^{-1}(X^n) \geq \xi_n\} &\geq \varepsilon, \\
\mathbb{P}\{s_n^{-1}(X^n) > \xi_n\} &< \varepsilon.
\end{align*}
\]

(107)

(108)
for each $0 < \varepsilon \leq 1$, and $\xi_n = 2^{nH_0(X)}$ if $\varepsilon = 0$, where

$$H_0(X) := \begin{cases} \log |\text{supp}(X)| & \text{if supp}(X) \text{ is finite,} \\ \infty & \text{if supp}(X) \text{ is infinite.} \end{cases} \quad (109)$$

It is clear that $\xi_n = 2^{nH_m(X)}$ if $\varepsilon = 1$, where $H_m(X) := -\log \max_{x \in X} P_X(x)$ denotes the min-entropy. Since $\mathbb{E}[\log \xi_n^{-1}(X^n)] = 0$ if $\varepsilon \geq 1 - 2^{-nH_m(X)}$, it suffices to consider the case when $0 \leq \varepsilon < 1 - 2^{-nH_m(X)}$. Then, a direct calculation shows that

$$\mathbb{E}[\log \xi_n^{-1}(X^n)] = \int_0^\infty P(\log \xi_n^{-1}(X^n) > s) \, ds \leq \int_0^\infty \mathbb{P}(s < \log \xi_n^{-1}(X^n) \leq \log \xi_n) \, ds$$

$$= \int_0^\infty \mathbb{P}(\log \xi_n^{-1}(X^n) > s) \, ds - \int_{\log \xi_n}^{\infty} \mathbb{P}(\log \xi_n^{-1}(X^n) > t) \, dt - (\log \xi_n) \mathbb{P}(\xi_n^{-1}(X^n) > \xi_n)$$

$$= \int_0^\infty \mathbb{P}(\log \xi_n^{-1}(X^n) > s) \, ds - \int_{\log \xi_n}^{\infty} \mathbb{P}(\log \xi_n^{-1}(X^n) > t) \, dt$$

$$- \sum_{k=\xi_n+1}^{\infty} \int_{\log k}^{\log(k+1)} \mathbb{P}(\log \xi_n^{-1}(X^n) > u) \, du - (\log \xi_n) \mathbb{P}(\xi_n^{-1}(X^n) > \xi_n)$$

$$= \int_0^\infty \mathbb{P}(\log \xi_n^{-1}(X^n) > s) \, ds - \sum_{k=1+\xi_n}^{\infty} (\log k) \mathbb{P}(\xi_n^{-1}(X^n) = k)$$

$$= \sum_{k=2}^{\infty} (\log k) \mathbb{P}(\xi_n^{-1}(X^n) = k)$$

$$= \int_{1-2^{-nH_m(X)}}^{\infty} \log M^*(n,s) \, ds$$

$$= \int_{1-2^{-nH_m(X)}}^{\infty} \log M^*(n,s) \, ds + (\log \xi_n - \mathbb{P}(\xi_n^{-1}(X^n) > \xi_n)) \log M^*(n,\varepsilon)$$

$$\leq \int_{1-2^{-nH_m(X)}}^{\infty} \log M^*(n,s) \, ds + \mathbb{P}(\xi_n^{-1}(X^n) = \xi_n) \log M^*(n,\varepsilon)$$

$$\leq \int_{1-2^{-nH_m(X)}}^{\infty} \log M^*(n,s) \, ds + 2^{-nH_m(X)} \log M^*(n,\varepsilon),$$

where

- (a) follows from the fact that
  $$\mathbb{E}[Z] = \int_0^\infty \mathbb{P}(Z > z) \, dz \quad (111)$$
  for every nonnegative-real-valued r.v. $Z$,
- (b) follows from (108),
- (c) follows from the fact that $M^*(n,s) = k - 1$ if $\mathbb{P}(\xi_n^{-1}(X^n) > k) < s < \mathbb{P}(\xi_n^{-1}(X^n) \geq k),$
- (d) follows from (107), and
- (e) follows from the fact that
  $$2^{-nH_m(X)} = \mathbb{P}(\xi_n^{-1}(X^n) = 1) \geq \mathbb{P}(\xi_n^{-1}(X^n) = k)$$
  for every integer $k \geq 1.$
Analogously, we see that
\[ \mathbb{E}[\langle \log s_n^{-1}(X^n) \rangle_e] \geq \int_{\varepsilon}^{1} \log M^*(n, s) \, ds - 2^{-nH_n(X)^*} \log M^*(n, \varepsilon) \]  
(113)
Combining (110) and (113), we obtain Lemma 4.

\[ \blacksquare \]

\section*{APPENDIX B

\textbf{Proof of Lemma 8}}

Let \( r \) be a positive constant and \( \{e_n\}_{n=1}^{\infty} \) a real sequence on \((0, 1)\). When \( e_n \) is bounded away from either zero for sufficiently large \( n \), then Lemma 8 is equivalent to (70). Therefore, it suffices to assume that
\[ \frac{1}{n^r} \leq e_n \leq \frac{1}{3} \]  
(114)
Firstly, we shall prove (74). Given a positive constant \( c \), choose an \( n_0 \geq 1 \) so that
\[ \frac{c f_3(e_n)}{\sqrt{n}} \leq \frac{1}{2} \]  
(115)
for every \( n \geq n_0 \). Define
\[ z_n^- := -\Phi^{-1}\left(1 - \frac{c f_3(e_n)}{\sqrt{n}}\right) e_n \]  
(116)
\[ z_n^+ := -\Phi^{-1}\left(1 + \frac{c f_3(e_n)}{\sqrt{n}}\right) e_n \]  
(117)
for each \( n \geq n_0 \). Then, it follows from Lemma 1 that
\[ 0 \leq z_n^- \leq 2r \ln n + 2 \ln 2, \]  
(118)
\[ 0 \leq z_n^+ \leq 2r \ln n \]  
(119)
for sufficiently large \( n \). Thus, it follows from Lemma 7 that
\[ 1 - F_n(z_n^-) = \left(1 - \frac{c f_3(e_n)}{\sqrt{n}}\right) e_n \exp \left(\frac{\mu_3(z_n^-)^3}{6\sqrt{n} \sigma^3}\right) + O\left(\frac{\phi(z_n^-)}{\sqrt{n}}\right) \]  
(120)
\[ 1 - F_n(z_n^+) = \left(1 + \frac{c f_3(e_n)}{\sqrt{n}}\right) e_n \exp \left(\frac{\mu_3(z_n^+)^3}{6\sqrt{n} \sigma^3}\right) + O\left(\frac{\phi(z_n^+)}{\sqrt{n}}\right) \]  
(121)
as \( n \to \infty \). Hence, by taking \( c \) sufficiently large, we see that \( F_n(z_n^+) \leq 1 - e_n \leq F_n(z_n^-) \) for all sufficiently large \( n \) (depending on the choice of \( c \)).\footnote{The remainder terms in (120) and (121) are independent of the constant \( c \); see [12, Chapter VIII.2].}
By the definition of \( \zeta_n(e) \) in (69), since every distribution function is nondecreasing, this implies that
\[ z_n^- \leq \zeta_n(e_n) \leq z_n^+ \]  
(122)
for sufficiently large \( n \). Therefore, we have from Lemma 7, (116)–(119), and (122) that
\[ 1 - F_n(\zeta_n(e_n)) = \left(1 - \Phi(\zeta_n(e_n))\right) \exp \left(\frac{\mu_3 \zeta_n(e_n)^3}{6\sqrt{n} \sigma^3}\right) + O\left(\frac{\varphi(\zeta_n(e_n))}{\sqrt{n}}\right) \]  
(123)
as \( n \to \infty \). Combining (120)–(123), there exists a positive constant \( A = A(Z, r) \), depending only on \( Z \) and \( r \), such that
\[ \left(1 - \frac{c f_3(e_n)}{\sqrt{n}}\right) e_n \exp \left(\frac{\mu_3 \left((z_n^-)^3 - (z_n^+)^3\right)}{6\sigma^3 \sqrt{n}}\right) - \frac{A}{\sqrt{n}} f_3 \left(1 - \frac{c f_3(e_n)}{\sqrt{n}}\right) e_n \leq 1 - \Phi(\zeta_n(e_n)) \leq \left(1 + \frac{c f_3(e_n)}{\sqrt{n}}\right) e_n \exp \left(\frac{\mu_3 \left((z_n^+)^3 - (z_n^-)^3\right)}{6\sigma^3 \sqrt{n}}\right) + \frac{A}{\sqrt{n}} f_3 \left(1 + \frac{c f_3(e_n)}{\sqrt{n}}\right) e_n \]  
(124)
for sufficiently large \( n \). By the mean value theorem, we see that
\[ (z_n^-)^3 - (z_n^+)^3 = \frac{6 c e_n f_3(e_n) \Phi^{-1}(\gamma_n^{(1)})^2}{\sqrt{n} f_3(\gamma_n^{(1)})}, \]  
(125)
for some \( \gamma_n^{(1)} = \gamma_n^{(1)}(e_n, X) \) satisfying
\[ \left(1 - \frac{c f_3(e_n)}{\sqrt{n}}\right) e_n \leq \gamma_n^{(1)} \leq \left(1 + \frac{c f_3(e_n)}{\sqrt{n}}\right) e_n . \]  
(126)
Since \( r_n^1 = e_n (1 + O(n^{-1/2})) \) as \( n \to \infty \) (noting that \( f_G(s) \) is a bounded function on \([0,1]\)), it follows from (125) that
\[
(z_n^-)^3 - (z_n^+)^3 = \frac{6 c e_n \Phi^{-1}(e_n)^2}{\sqrt{n}} \left( 1 + O\left( \frac{1}{\sqrt{n}} \right) \right)
\] (127)
as \( n \to \infty \). Substituting (127) into (124), we obtain
\[
\left( 1 - \frac{c f_G(e_n)}{\sqrt{n}} \right) e_n \exp \left( \frac{\mu_3 c e_n \Phi^{-1}(e_n)^2}{2 \sigma^3 n} \right) - \frac{A}{\sqrt{n}} f_G \left( 1 - \frac{c f_G(e_n)}{\sqrt{n}} \right) e_n
\leq 1 - \Phi(\zeta_n(e_n)) \leq \left( 1 + \frac{c f_G(e_n)}{\sqrt{n}} \right) e_n \exp \left( - \frac{\mu_3 c e_n \Phi^{-1}(e_n)^2}{2 \sigma^3 n} \right) + \frac{A}{\sqrt{n}} f_G \left( 1 + \frac{c f_G(e_n)}{\sqrt{n}} \right) e_n
\] (128)
for sufficiently large \( n \). On the other hand, it follows by Taylor’s theorem for \( s \mapsto f_G(s) \) around \( s = e_n \) that
\[
f_G \left( 1 - \frac{c f_G(e_n)}{\sqrt{n}} \right) e_n = f_G(e_n) + O \left( \frac{e_n g_G(e_n)}{\sqrt{n}} \right)
\] (129)
for some \( \gamma_n^{(2)} = \gamma_n^{(2)}(\varepsilon, X) \) satisfying
\[
\left( 1 - \frac{c f_G(e_n)}{\sqrt{n}} \right) e_n \leq \gamma_n^{(2)} \leq e_n.
\] (130)
Since \( \gamma_n^{(2)} = e_n (1 + O(n^{-1/2})) \) as \( n \to \infty \), it follows from (129) that
\[
f_G \left( 1 - \frac{c f_G(e_n)}{\sqrt{n}} \right) e_n = f_G(e_n) + O \left( \frac{e_n g_G(e_n)}{\sqrt{n}} \right)
\] (131)
as \( n \to \infty \). Analogously, we get
\[
f_G \left( 1 + \frac{c f_G(e_n)}{\sqrt{n}} \right) e_n = f_G(e_n) + O \left( \frac{e_n g_G(e_n)}{\sqrt{n}} \right)
\] (132)
as \( n \to \infty \). Substituting (131) and (132) into (128), we have that there exists a positive constant \( B = B(Z, r) \), depending only on \( Z \) and \( r \), such that
\[
e_n \exp \left( \frac{\mu_3 c e_n \Phi^{-1}(e_n)^2}{2 \sigma^3 n} \right) - \frac{(A + 2 c e_n) f_G(e_n)}{\sqrt{n}} + \frac{B e_n g_G(e_n)}{n}
\leq 1 - \Phi(\zeta_n(e_n)) \leq e_n \exp \left( - \frac{\mu_3 c e_n \Phi^{-1}(e_n)^2}{2 \sigma^3 n} \right) + \frac{(A + 2 c e_n) f_G(e_n)}{\sqrt{n}} - \frac{B e_n g_G(e_n)}{n}
\] (133)
for sufficiently large \( n \), which proves (74) of Lemma 8 together with (18).

The same argument proves (75) of Lemma 8, completing the proof of Lemma 8.

\[
\text{APPENDIX C}
\]

\[\text{PROOF OF LEMMA 9}\]

After some algebra, we get
\[
\mu \left( \sum_{i=1}^{n} f_i \geq a \right) = e^{-m \Lambda_{y_f}(a)} \int_0^\infty e^{-st} \sqrt{\frac{m \Lambda_{y_f}(s)}} dF_n(t),
\] (134)
where \( F_n \) is a distribution function of the r.v. \( W_n \) given by
\[
W_n = \frac{1}{\sqrt{n \Lambda'_{y_f}(s)}} \sum_{i=1}^{n} (U_i - a),
\] (135)
and \( U_1, \ldots, U_n \) are i.i.d. r.v.’s with generic distribution \( \tilde{\mu}_f \) constructed by the Radon–Nikodym derivative
\[
\frac{d\tilde{\mu}_f}{d\mu_f}(t) = e^{st - \Lambda_{y_f}(s)}.
\] (136)

Then, Lemma 9 can be proven by applying Lemma 6 to the distribution \( F_n \) in (134).
APPENDIX D
PROOF OF Lemma 10

For the sake of brevity, we write
\[ \alpha_n = \alpha_n(t) := a n + a_1(t) \sqrt{n} + a_2(t) + \delta_n(t). \] (137)

As in [13, Exercise 2.2.24], we observe that \( \Lambda'_{\mu_f}(\bar{a}) \) is of class \( C^\infty \) on \( \bar{a} \in \text{int}(D_{\mu_f}) \), where \( D_{\mu_f} := \{ \Lambda'(\bar{s}) | \bar{s} \in \text{int}(D_{\mu_f}) \} \).

Thus, since \( s \in \text{int}(D_{\mu_f}) \), it follows from (79), (83), and (137) that \( \alpha_n/n \in \text{int}(D_{\mu_f}) \) for sufficiently large \( n \). Henceforth, we assume that \( n \) is large enough. Noting this fact, denote by \( s_n = s_n(t) \) the root of the equation \( n \Lambda'(\bar{s}) = \alpha_n \) with respect to \( \bar{s} \in \text{int}(D_{\mu_f}) \).

Since \( \Lambda'(\cdot) \) is of class \( C^\infty \) on \( \text{int}(D_{\mu_f}) \), and since (83) implies that \( \alpha_n \to a \) uniformly on \( I_n \) as \( n \to \infty \) we see that \( s_n \to s \) as \( n \to \infty \) uniformly on \( I_n \). In addition, it follows from Taylor’s theorem for \( v \mapsto \Lambda'(v) \) around \( v = s \) that

\[ \frac{\alpha_n}{n} = \Lambda'(s_n) = a + \Lambda''(s_n)(s_n - s) + O((s_n - s)^2) \] (138)

uniformly on \( I_n \) as \( n \to \infty \), which is equivalent to

\[ s_n = s + \frac{a_1(t) \sqrt{n} + a_2(t) + \delta_n(t)}{n \Lambda'(s)} + O((s_n - s)^2) \] (139)

uniformly on \( I_n \) as \( n \to \infty \). Therefore, since \( s_n \to s \) as \( n \to \infty \) uniformly in \( t \in I_n \), we observe that

\[ s_n - s = \left( \frac{a_1(t) \sqrt{n} + a_2(t) + \delta_n(t)}{n \Lambda'(s)} \right) (1 + o(1)) \] (140)

as \( n \to \infty \) uniformly on \( I_n \). On the other hand, it follows from (79), (139), and Taylor’s theorem for \( v \mapsto \Lambda'(v) \) around \( v = s \) that there exist real sequences \( r_f^{(1)}(n,t) = o(1) \) and \( r_f^{(2)}(n,t) = O(1) \) uniformly on \( I_n \) as \( n \to \infty \) such that

\[ \frac{a_n s_n - \Lambda'(s_n)}{n} = \Lambda'(s_n) \]

\[ \begin{align*}
&= \Lambda'(s) + \Lambda''(s)(s_n - s) + \frac{\Lambda'''(s)}{2} (s_n - s)^2 + r_f^{(1)}(n,t) (s_n - s)^3 \\
&= a s - \Lambda'(a) + a (s_n - s) + \frac{\Lambda'''(s)}{2} (s_n - s)^2 + r_f^{(1)}(n,t) (s_n - s)^3 \\
&= \left( a - \frac{\alpha_n}{n} \right) s - \Lambda'(a) + a (s_n - s) - \frac{\alpha_n s_n}{n} \left( s_n - s \right) + \frac{\Lambda'''(s)}{2} (s_n - s)^3 + r_f^{(1)}(n,t) (s_n - s)^3 \\
&= \left( a - \frac{\alpha_n}{n} \right) s - \Lambda'(a) + a (s_n - s) - \frac{\alpha_n s_n}{n} \left( \frac{a_1(t) \sqrt{n} + a_2(t) + \delta_n(t)}{n \Lambda'(s)} \right) \\
&+ \left( \frac{a_1(t) \sqrt{n} + a_2(t) + \delta_n(t)}{n \Lambda'(s)} \right) (1 + r_f^{(1)}(n,t))^2 + r_f^{(2)}(n,t) \left( \frac{a_1(t) \sqrt{n} + a_2(t) + \delta_n(t)}{n \Lambda'(s)} \right)^3 (1 + r_f^{(1)}(n,t))^3 \\
&= \left( a - \frac{\alpha_n}{n} \right) s - \Lambda'(a) + a (s_n - s) - \frac{\alpha_n s_n}{n} \left( \frac{a_1(t) \sqrt{n} + a_2(t) + \delta_n(t)}{n \Lambda'(s)} \right) \\
&+ \left( \frac{a_1(t) \sqrt{n} + a_2(t) + \delta_n(t)}{n \Lambda'(s)} \right) \tilde{r}_f(n,t)^2 + r_f^{(2)}(n,t) \left( \frac{a_1(t) \sqrt{n} + a_2(t) + \delta_n(t)}{n \Lambda'(s)} \right)^3 (1 + r_f^{(1)}(n,t))^3,
\end{align*} \] (141)

where \( \tilde{r}_f(n,t) := 2 r_f^{(1)}(n,t) + r_f^{(2)}(n,t) \). This is equivalent to

\[ \Lambda'' \left( \frac{\alpha_n}{n} \right) = K_f(n,t), \] (142)

where \( K_f(n,t) \) is defined as

\[ K_f(n,s,t) := n \Lambda'(a) + s a_n(t) + \frac{\alpha_n(t)}{2 n \Lambda'(s)} \left( 1 + r_f^{(1)}(n,t) \left( 2 + r_f^{(1)}(n,t) \right) + r_f^{(2)}(n,t) \left( 1 + r_f^{(1)}(n,t) \right)^3 \frac{2 \alpha_n(t)}{n \Lambda'(s)^2} \right). \] (143)

Finally, it follows from Taylor’s theorem for \( v \mapsto \Lambda''(v) \) around \( v = s \) that there exists a real sequence \( r_f^{(3)}(n,t) = O(1) \) uniformly on \( I_n \) as \( n \to \infty \) such that

\[ \Lambda''(s_n) = \Lambda''(s) + \Lambda'''(s)(s_n - s) + r_f^{(3)}(n,t) (s_n - s)^2 \]

\[ = \Lambda''(s) (1 + R_f(n,t)), \] (144)
where \( R_f(n,t) \) is defined as

\[
R_f(n,t) := \frac{\Lambda^{(r)}_{nt}(s)}{\Lambda^{(r)}_{nt}(s)} \left( \frac{a_1(t) \sqrt{n} + a_2(t) + \delta_n(t)}{n \Lambda^{(r)}_{nt}(s)} + \frac{r_f^{(3)}(n,t) (1 + r_f^{(1)}(n,t))^2}{n \Lambda^{(r)}_{nt}(s)} \right)^2. \tag{145}
\]

By (83), we readily see that \( R_f(n,t) = o(1) \). Applying the above asymptotic results to Lemma 9, we obtain Lemma 10, as desired.

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