Liouville type of theorems for the Euler and the Navier-Stokes equations

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Abstract
We prove Liouville type of theorems for weak solutions of the Navier-Stokes and the Euler equations. In particular, if the pressure satisfies $p \in L^1(0, T; L^1(\mathbb{R}^N))$ with $\int_{\mathbb{R}^N} p(x, t)dx \geq 0$, then the corresponding velocity should be trivial, namely $v = 0$ on $\mathbb{R}^N \times (0, T)$. In particular, this is the case when $p \in L^1(0, T; H^1(\mathbb{R}^N))$, where $H^1(\mathbb{R}^N)$ the Hardy space. On the other hand, we have equipartition of energy over each component, if $p \in L^1(0, T; L^1(\mathbb{R}^N))$ with $\int_{\mathbb{R}^N} p(x, t)dx < 0$.

Similar results hold also for the magnetohydrodynamic equations.

1 Introduction
We are concerned on the Navier-Stokes equations (the Euler equations for $\nu = 0$) on $\mathbb{R}^N$, $N \in \mathbb{N}, N \geq 2$.

$$(NS)_{\nu} \begin{cases}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p + \nu \Delta v, & (x, t) \in \mathbb{R}^N \times (0, \infty) \\
\text{div } v = 0, & (x, t) \in \mathbb{R}^N \times (0, \infty) \\
v(x, 0) = v_0(x), & x \in \mathbb{R}^N
\end{cases}$$

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where \( v(x, t) = (v^1(x, t), \ldots, v^N(x, t)) \) is the velocity, \( p = p(x, t) \) is the pressure, and \( \nu \geq 0 \) is the viscosity. Given \( a, b \in \mathbb{R}^N \), we denote by \( a \otimes b \) the \( N \times N \) matrix with \((a \otimes b)_{ij} = a_i b_j\). For two \( N \times N \) matrices \( A \) and \( B \) we denote \( A : B = \sum_{i,j=1}^N A_{ij} B_{ij} \). Given \( m \in \mathbb{N} \cup \{0\}, q \in [1, \infty], \) we denote

\[
W_{\sigma}^{m,q}(\mathbb{R}^N) := \{ v \in [W^{m,q}(\mathbb{R}^N)]^N, \ \text{div} \ v = 0 \},
\]

where \( W^{m,q}(\mathbb{R}^N) \) is the standard Sobolev space on \( \mathbb{R}^N \), and the derivatives in \( \text{div} (\cdot) \) are in the sense of distribution. In particular, \( H_{\sigma}^m(\mathbb{R}^N) := W_{\sigma}^{m,2}(\mathbb{R}^N) \) and \( L_q(\mathbb{R}^N) := W_q^{0,q}(\mathbb{R}^N) \). The Schwartz class of functions, which consists of rapidly decreasing smooth functions, is denoted by \( S \) with its dual \( S' \). Let \( \varphi \in S(\mathbb{R}^N) \) with \( \int_{\mathbb{R}^N} \varphi(x) \, dx \neq 0 \) be given. We set \( \varphi_t(x) = t^{-N} \varphi(t^{-1}x), t > 0 \). Then, the Hardy space \( \mathcal{H}^q(\mathbb{R}^N), 0 < q \leq 1, \) is defined by

\[
\mathcal{H}^q(\mathbb{R}^N) = \left\{ f \in S' \mid \mathcal{M}_\varphi f(x) := \sup_{t > 0} |f * \varphi_t(x)| \in L^q(\mathbb{R}^N) \right\}
\]

with the norm \( \|f\|_{\mathcal{H}^q} := \|\mathcal{M}_\varphi f\|_{L^q} \). It is well-known that the definition is independent of the choice of \( \varphi \in S \) with \( \int_{\mathbb{R}^N} \varphi(x) \, dx \neq 0 \)(see [10]). A property of \( \mathcal{H}^q(\mathbb{R}^N) \), which will be used later is the fact about its dual

\[
[\mathcal{H}^q(\mathbb{R}^N)]' = C^\gamma(\mathbb{R}^N), \quad \gamma = N \left( \frac{1}{p} - 1 \right), \tag{1.1}
\]

if \( 0 < p < 1 \), where \( C^\gamma(\mathbb{R}^N) \) is the homogeneous Hölder space. In \( \mathbb{R}^N \) we define weak solutions of the Navier-Stokes(Euler) equations as follows.

**Definition 1.1** We say the pair \( (v, p) \in L^1(0, T; L^2_\sigma(\mathbb{R}^N)) \times L^1(0, T; S'(\mathbb{R}^N)) \) is a weak solution of \((NS)_\nu\) on \( \mathbb{R}^N \times (0, T) \) if

\[
- \int_0^T \int_{\mathbb{R}^N} v(x, t) \cdot \phi'(x) \xi(t) \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} v(x, t) \otimes v(x, t) : \nabla \phi(x) \xi(t) \, dx \, dt = \int_0^T < p(t), \text{div} \phi > \xi(t) dt + \nu \int_0^T \int_{\mathbb{R}^N} v(x, t) \cdot \Delta \phi(x) \xi(t) \, dx \, dt
\]

for all \( \xi \in C^1_0(0, T) \) and \( \phi = [C^\infty_0(\mathbb{R}^N)]^N \), where \( < \cdot, \cdot > \) denotes the dual pairing between \( S \) and \( S' \).
The definition is weaker than the standard Leray-Hopf weak solution for the Navier-Stokes equations, since we are concerned also on possible weak solutions of the Euler equations, the right function space of whose existence is not yet known. Below we denote

\[ E_j(t) = \frac{1}{2} \int_{\mathbb{R}^N} (v^j(x,t))^2 dx, \quad j = 1, \ldots, N \]

which will be called the \(j\)-th component of the total energy,

\[ E(t) = \frac{1}{2} \int_{\mathbb{R}^N} |v(x,t)|^2 dx = E_1(t) + \cdots + E_N(t). \]

Let us introduce the function class,

\[ L^{1}_+(0,T;L^1(\mathbb{R}^N)) = \left\{ f \in L^1(0,T;L^1(\mathbb{R}^N)), \int_{\mathbb{R}^N} f(x,t) dx \geq (\leq) 0 \text{ a.e. } t \in (0,T) \right\}. \]

**Theorem 1.1** Let \((v,p)\) be a weak solution to \((NS)_{\nu}\) with \(\nu \geq 0\).

(i) (Liouville type of property) Suppose

\begin{equation}
\text{either } p \in L^1_+(0,T;L^1(\mathbb{R}^N)), \text{ or } p \in L^1(0,T;H^q(\mathbb{R}^N)) \quad (1.3)
\end{equation}

for some \(q \in (0,1]\). Then,

\begin{equation}
v(x,t) = 0 \text{ a.e. in } \mathbb{R}^N \times (0,T), \quad (1.4)
\end{equation}

(ii) (Equipartition of energy) Suppose \(p \in L^1_+(0,T;L^1(\mathbb{R}^N))\). Then,

\begin{equation}
E_1(t) = \cdots = E_N(t) = -\frac{1}{2} \int_{\mathbb{R}^N} p(x,t) dx, \quad (1.5)
\end{equation}

and

\begin{equation}
\int_{\mathbb{R}^N} v^j(x,t)v^k(x,t) dx = 0 \quad \forall j, k \in \{1, \cdots, N\} \quad \text{with } j \neq k \quad (1.6)
\end{equation}

for almost every \(t \in (0,T)\).
Remark 1.1 Let us recall that \( \int_{\mathbb{R}^N} f(x)dx = 0 \), if \( f \in \mathcal{H}^1(\mathbb{R}^N) \), where \( \mathcal{H}^1(\mathbb{R}^N) \) is the Hardy space in \( \mathbb{R}^N \) (see [10]), and
\[
L^1(0,T;\mathcal{H}^1(\mathbb{R}^N)) \subset L^1_+(0,T;L^1(\mathbb{R}^N)).
\]
The part (i) of the above theorem says that the cancelation property of the pressure is not allowed for nontrivial solutions of the Navier-Stokes and the Euler equations. Note that the condition (1.3) with \( q = 1 \) is already far beyond the natural scaling of the usual regularity criterion on the pressure for the Navier-Stokes equations,
\[
p \in L^q(0,T;L^r(\mathbb{R}^N)), \quad \frac{2}{q} + \frac{N}{r} \leq 2.
\]
(see [2, 1]), and our conclusion is not just the solution is regular, but it is trivially zero.

Remark 1.2 We also recall the relation between the pressure and velocity for the Navier-Stokes and Euler equations:
\[
p(x,t) = \sum_{j,k=1}^{N} [R_jR_k(v^j(\cdot,t)v^k(\cdot,t))] (x), \quad (1.7)
\]
where \( R_j, j = 1, \cdots, N \), is the Riesz transforms in \( \mathbb{R}^N \), defined by
\[
R_j(f)(x) = C_N \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{y_j}{|y|^{n+1}} f(x-y)dy, \quad C_N = \frac{\Gamma \left( \frac{N+1}{2} \right)}{\pi^{(N+1)/2}}.
\]
Thus, we find that (1.3) with \( q = 1 \) is guaranteed if
\[
v \otimes v \in L^1(0,T;\mathcal{H}^1(\mathbb{R}^N)). \quad (1.8)
\]
In reality the pressure for the Leray weak solutions of the N-dimensional \( (N=2,3) \) Navier-Stokes equations has the property that \( v \in L^2(0,T;H^1_\sigma(\mathbb{R}^N)) \cap L^\infty(0,T;L^2_\sigma(\mathbb{R}^N))([8]) \), which implies
\[
v \otimes v \in L^1(0,T;L^q(\mathbb{R}^N)) \quad (1.9)
\]
for all \( q \in [1,\frac{N}{N-2}] \) if \( N \geq 3 \), while \( q \in [1,\infty) \) if \( N = 2 \). On the other hand, we note that the local smooth solution \( v \in C([0,T);H^m_{\sigma}(\mathbb{R}^N)), m > N/2 + 1, \)
constructed by Kato([9]), has the property $v \in C([0, T); L^q_\sigma(\mathbb{R}^N))$ for all $q \in [2, \infty]$ due to the embedding $H^m(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, combined with the interpolation between $L^2(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$. Hence, for $v \in C([0, T); H^m_\sigma(\mathbb{R}^N))$, $m > N/2 + 1$, we have

$$v \otimes v \in C([0, T); L^q(\mathbb{R}^N)) \quad \forall q \in [1, \infty]. \quad (1.10)$$

It would be interesting to recall the related known properties of the pressure for the Leray weak solutions, which are proved in [3] (see also [9]):

$$D^2 p \in L^1(0, T; \mathcal{H}^1(\mathbb{R}^N)),$$

$$\nabla p \in L^2(0, T; \mathcal{H}^1(\mathbb{R}^N)) \cap L^1(0, T; L_{N-1}^N(\mathbb{R}^N)),$$

$$p \in \begin{cases} L^1(0, T; L_{N-2}^\infty(\mathbb{R}^N)), \quad N \geq 3 \\ L^1(0, T; C_0(\mathbb{R}^2)), \quad N = 2 \end{cases}$$

where $L^{q,r}(\mathbb{R}^N)$ is the Lorentz space, and $C_0(\mathbb{R}^2)$ is the class of continuous functions vanishing near infinity.

**Remark 1.3** One immediate corollary of the above theorem is that the following pressureless Navier-Stokes(Euler) system is locally ill-posed if $v_0 \in H^m(\mathbb{R}^N), m > N/2 + 1,$

$$\begin{aligned}
\frac{\partial v}{\partial t} + (v \cdot \nabla)v &= \nu \Delta v, \quad (x, t) \in \mathbb{R}^N \times (0, \infty) \\
\text{div } v &= 0, \quad (x, t) \in \mathbb{R}^N \times (0, \infty) \\
v(x, 0) &= v_0(x), \quad \text{div } v_0 = 0, \quad x \in \mathbb{R}^N 
\end{aligned}$$

since we need to have $v(\cdot, t) = 0$ for $t > 0$ from the fact $\int_{\mathbb{R}^N} p(x, t)dx = 0$ for all $t > 0$.

**Remark 1.4** In [5] the equipartition of energy over each component has been derived for steady Euler equations in a different context, using different definition of weak solutions. A completely different type of approach to the Liouville type of theorems for the Navier-Stokes equations is studied in [7].

## 2 Proof of the Main Theorem

**Proof of Theorem 1.1**

(a) the case $p \in L^1(0, T; L^1(\mathbb{R}^N))$ : Let us consider a cut-off function $\sigma \in$
\( C_0^\infty(\mathbb{R}^N) \) such that

\[
\sigma(x) = \sigma(|x|) = \begin{cases} 
1 & \text{if } |x| < 1 \\
0 & \text{if } |x| > 2,
\end{cases}
\]

and \( 0 \leq \sigma(x) \leq 1 \) for \( 1 < |x| < 2 \). Then, given \( R > 0 \), we set

\[
\varphi_R(x) = \frac{x_1^2}{2\sigma\left(\frac{x}{R}\right)}.
\]

Let \( \xi \in C_0^1(0,T) \). We choose the vector test function \( \phi \) in (1.2) as

\[
\phi = \nabla \varphi_R(x) = \left(x_1\sigma_R(x) + \frac{x_1^2}{2} \partial_1 \sigma_R(x), \frac{x_1^2}{2} \partial_2 \sigma_R(x), \ldots, \frac{x_1^2}{2} \partial_N \sigma_R(x)\right).
\]

Then, (1.2) becomes

\[
0 = \int_0^T \int_{\mathbb{R}^N} (v_1(x,t))^2 \sigma_R(x) \xi(t) \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^N} p(x,t) \sigma_R(x) \xi(t) \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^N} (v_1(x,t))^2 \left[2x_1 \partial_1 \sigma_R(x) + \frac{x_1^2}{2} \partial_1^2 \sigma_R(x)\right] \xi(t) \, dx \, dt \\
+ 2 \sum_{j=2}^N \int_0^T \int_{\mathbb{R}^N} v_1(x,t) v_j(x,t) \left[x_1 \partial_j \sigma_R(x) + \frac{x_1^2}{2} \partial_1 \partial_j \sigma_R(x)\right] \xi(t) \, dx \, dt \\
+ \sum_{j,k=2}^N \int_0^T \int_{\mathbb{R}^N} v_j(x,t) v_k(x,t) x_1^2 \partial_j \partial_k \sigma_R(x) \xi(t) \, dx \, dt \\
+ \int_0^T \int_{\mathbb{R}^N} p(x,t) \left[2x_1 \partial_1 \sigma_R(x) + \frac{x_1^2}{2} \Delta \sigma_R(x)\right] \xi(t) \, dx \, dt \\
:= I_1 + \cdots + I_6.
\]

Note that the first term of the left hand side and the second term of the right hand side in (1.2) vanish, since

\[
\int_0^T \int_{\mathbb{R}^N} v(x,t) \cdot \nabla \varphi_R(x) \xi'(t) \, dx \, dt = 0,
\]

\( (2.1) \).
and
\[ \int_0^T \int_{\mathbb{R}^N} v(x, t) \cdot \nabla (\Delta \varphi_R(x)) \xi(t) \, dx \, dt = 0 \]

for \( v \in L^1(0, T; L^2_{\sigma}(\mathbb{R}^N)) \) by the definition of divergence free condition in the sense of distribution. We pass \( R \to \infty \) in (2.1). Since \( v \in L^\infty(0, T; L^2_{\sigma}(\mathbb{R}^N)) \) by hypothesis,

\[
\left| I_1 - \int_0^T \int_{\mathbb{R}^N} (v_1(x, t))^2 \xi(t) \, dx \, dt \right| \leq \int_0^T \int_{\mathbb{R}^N} (v_1(x, t))^2 |\xi(t)| |1 - \sigma_R(x)| \, dx \, dt \\
\leq \sup_{0<t<T} |\xi(t)| \int_0^T \int_{|x|>R} (v_1(x, t))^2 \, dx \, dt \quad (2.2)
\]

Since
\[ g_R(t) := \int_{|x|>R} (v_1(x, t))^2 \, dx \to 0 \quad \text{as} \quad R \to \infty \]

for almost every \( t \in (0, T) \), and

\[ |g_R(t)| \leq g(t) := \int_{\mathbb{R}^N} (v_1(x, t))^2 \, dx \]

with \( g \in L^1(0, T) \), we can apply the dominated convergence theorem in (2.2) to get
\[ \int_0^T \int_{|x|>R} (v_1(x, t))^2 \, dx \, dt \to 0 \quad \text{as} \quad R \to \infty, \]

and hence
\[ I_1 \to \int_0^T \int_{\mathbb{R}^N} (v_1(x, t))^2 \xi(t) \, dx \, dt \quad \text{as} \quad R \to \infty. \quad (2.3) \]

Since \( p \in L^1(0, T; L^1(\mathbb{R}^N)) \), we have
\[
\left| I_2 - \int_0^T \int_{\mathbb{R}^N} p(x, t) \xi(t) \, dx \, dt \right| \leq \sup_{0<t<T} |\xi(t)| \int_0^T \int_{\mathbb{R}^N} |p(x, t)| |1 - \sigma_R(x)| \, dx \, dt \\
\leq \sup_{0<t<T} |\xi(t)| \int_0^T \int_{|x|>R} |p(x, t)| \, dx \, dt \to 0
\]
as \( R \to \infty \) by the dominated convergence theorem. Hence,
\[ I_2 \to \int_0^T \int_{\mathbb{R}^N} p(x, t) \xi(t) \, dx \, dt \quad \text{as} \quad R \to \infty \quad (2.4) \]
We will show below that \( I_3, \cdots, I_6 \to 0 \) as \( R \to \infty \). In view of (1.8) Next, we note that for \( m \geq 1 \) and \( j, k \in \{1, \cdots, N\} \)

\[
\left| \int_0^T \int_{\mathbb{R}^N} \xi(t)v^j(x,t)v^k(x,t)x_1^mD^m\sigma_R(x)dxdt \right|
\]

\[
\leq \frac{1}{R^m} \sup_{1 < s < 2} |\sigma^{(m)}(s)| \int_0^T \int_{R<|x|<2R} |\xi(t)||v(x,t)||x|^m dxdt
\]

\[
\leq 2^m \sup_{1 < s < 2} |\sigma^{(m)}(s)| \sup_{0 < t < T} |\xi(t)| \int_0^T \int_{R<|x|<2R} |v(x,t)|^2 dxdt
\]

\[
\to 0 \quad (2.5)
\]

as \( R \to \infty \) by the dominated convergence theorem, which shows that \( I_3, I_4 \) and \( I_5 \) converge to zero as \( R \to \infty \). Similarly,

\[
\left| \int_0^T \int_{\mathbb{R}^N} \xi(t)p(x,t)x_1^mD^m\sigma_R(x)dxdt \right|
\]

\[
\leq \frac{1}{R^m} \sup_{1 < s < 2} |\sigma^{(m)}(s)| \int_0^T \int_{R<|x|<2R} |\xi(t)||p(x,t)||x|^m dxdt
\]

\[
\leq 2^m \sup_{1 < s < 2} |\sigma^{(m)}(s)| \sup_{0 < t < T} |\xi(t)| \int_0^T \int_{R<|x|<2R} |p(x,t)| dxdt
\]

\[
\to 0 \quad (2.6)
\]

as \( R \to \infty \) by the dominated convergence theorem, which shows that \( I_6 \) converges to zero as \( R \to \infty \), since \( p \in L^1(0, T; L^1(\mathbb{R}^N)) \). Therefore, after passing \( R \to \infty \) in (2.1), we are left with

\[
\int_0^T \int_{\mathbb{R}^N} \xi(t) \left[(v^1(x,t))^2 + p(x,t)\right] dxdt = 0 \quad \forall \xi \in C^1_0(0, T).
\]

Hence,

\[
\mathcal{E}_1(t) = -\frac{1}{2} \int_{\mathbb{R}^N} p(x,t)dx \quad \text{for almost every } t \in (0, T).
\]

Similarly, if we choose the vector test function \( \phi \) in (1.2) as

\[
\phi(x) = \nabla \left( \frac{x_j^2}{2} \sigma_R(x) \right), \quad j \in \{1, \cdots, N\}
\]
then we could obtain
\[
\int_0^T \int_{\mathbb{R}^N} \xi(t) \left[ (v^j(x,t))^2 + p(x,t) \right] dx dt = 0 \quad \forall \xi \in C^1_0(0,T),
\]
and hence
\[
\mathcal{E}_j(t) = -\frac{1}{2} \int_{\mathbb{R}^N} p(x,t) dx \quad \text{for almost every } t \in (0,T)
\]
for all \( j \in \{1, \cdots, N\} \). This proves (1.5). In order to prove (1.6) we choose the test function
\[
\phi(x) = \nabla (x_1 x_2 \sigma_R(x)) = (x_2 \sigma_R(x) + x_1 x_2 \partial_1 \sigma_R(x), x_1 \sigma_R(x) + x_1 x_2 \partial_2 \sigma_R(x), x_1 x_2 \partial_3 \sigma_R(x), \cdots, x_1 x_2 \partial_N \sigma_R(x)).
\]
Then, we have
\[
0 = 2 \int_0^T \int_{\mathbb{R}^N} v^1(x,t)v^2(x,t)\sigma_R(x)\xi(t) dx dt
+ \int_0^T \int_{\mathbb{R}^N} \left[ (v^1(x,t))^2 + 2x_2 \partial_1 \sigma_R(x) + x_1 x_2 \partial_1^2 \sigma_R(x) \right] \xi(t) dx dt
+ \int_0^T \int_{\mathbb{R}^N} \left[ (v^2(x,t))^2 + 2x_1 \partial_1 \sigma_R(x) + x_1 x_2 \partial_2^2 \sigma_R(x) \right] \xi(t) dx dt
+ \int_0^T \int_{\mathbb{R}^N} v^1(x,t)v^2(x,t) \left[ x_1 \partial_1 \sigma_R(x) + x_2 \partial_2 \sigma_R(x) + 2x_1 x_2 \partial_1 \partial_2 \sigma_R(x) \right] \xi(t) dx dt
+ 2 \sum_{j=3}^N \int_0^T \int_{\mathbb{R}^N} v^1(x,t)v^j(x,t) \left[ x_2 \partial_j \sigma_R(x) + x_1 x_2 \partial_1 \partial_j \sigma_R(x) \right] \xi(t) dx dt
+ 2 \sum_{j=3}^N \int_0^T \int_{\mathbb{R}^N} v^2(x,t)v^j(x,t) \left[ x_1 \partial_j \sigma_R(x) + x_1 x_2 \partial_1 \partial_j \sigma_R(x) \right] \xi(t) dx dt
+ 2 \sum_{j,k=3}^N \int_0^T \int_{\mathbb{R}^N} v^j(x,t)v^k(x,t) x_1 x_2 \partial_j \partial_k \sigma_R(x) \xi(t) dx dt
+ \int_0^T \int_{\mathbb{R}^N} p(x,t) \left[ 2x_2 \partial_1 \sigma_R(x) + 2x_1 \partial_2 \sigma_R(x) + x_1 x_2 \Delta \sigma_R(x) \right] \xi(t) dx dt
:= J_1 + \cdots + J_8.
\]
Similarly to the previous proof we deduce

\[ |J_1 - 2 \int_0^T \int_{\mathbb{R}^N} v^1(x,t)v^2(x,t)\xi(t) \, dx \, dt| \]
\[ \leq 2 \int_0^T \int_{\mathbb{R}^N} |v^1(x,t)v^2(x,t)||\xi(t)||1 - \sigma_R(x)| \, dx \, dt \]
\[ \leq 2 \sup_{0 < t < T} |\xi(t)| \int_0^T \int_{|x| > R} |v(x,t)|^2 \, dx \, dt \to 0 \]

by the dominated convergence theorem, and

\[ J_1 \to 2 \int_0^T \int_{\mathbb{R}^N} v^1(x,t)v^2(x,t)\xi(t) \, dx \, dt. \]

Using the computations similar to (2.5) and (2.6), we also find that

\[ \sum_{k=2}^{13} |J_k| \to 0 \quad \text{as } R \to \infty. \]

Hence, taking \( R \to \infty \) in (2.7), we are left with

\[ 0 = \int_0^T \int_{\mathbb{R}^N} v^1(x,t)v^2(x,t)\xi(t) \, dx \, dt \quad \forall \xi \in C^1_0(0,T), \]

and therefore

\[ \int_{\mathbb{R}^N} v^1(x,t)v^2(x,t) \, dx = 0 \quad \text{for almost every } t \in (0,T). \]

Similarly, choosing the test function

\[ \phi(x) = \nabla(x_j x_k \sigma_R(x)), \quad j, k \in \{1, \cdots, N\}, j \neq k \]

and repeating the above argument, we could drive

\[ \int_{\mathbb{R}^N} v^j(x,t)v^k(x,t) \, dx = 0 \quad \text{for almost every } t \in (0,T). \]

for all \( j, k \in \{1, \cdots, N\} \) with \( j \neq k. \)

(b) the case \( p \in L^1(0,T; \mathcal{H}^q(\mathbb{R}^N)), 0 < q \leq 1 \): The borderline case \( p = 1 \)
is contained in the part (a) above (see Remark 1.1), and we assume here $0 < p < 1$. In order to derive the Liouville type of property in this case it suffice to show that $I_2, I_6 \to 0$ as $R \to \infty$ in (2.1). This can be shown by the following estimates for $m \in \mathbb{N} \cup \{0\}$

$$\left| \int_0^T \xi(t) \cdot \left< p(\cdot, t), x_1^m D^m \sigma_R(x) \right> dt \right|$$

$$\leq \sup_{0 \leq t \leq T} \int_0^T |\xi(t)||p(t)||_{H^q} dt \cdot x_1^m D^m \sigma_R(x)\|_{C^\gamma}$$

$$\leq \frac{C}{R^\gamma} \|\xi\|_{L^{\infty}(0,T)} \|p(t)\|_{L^1(0,T;H^q)} \to 0,$$

where $\gamma = N(1/q - 1) > 0$, where we used the duality, $[H^q(\mathbb{R}^N)]' = C^\gamma(\mathbb{R}^N)$, and the simple estimate,

$$\|x_1^m D^m \sigma_R(x)\|_{C^\gamma} \leq \frac{C}{R^\gamma}.$$

□

**Remark after the proof** A natural question is if there exists an initial data $v_0 \in H^m_a(\mathbb{R}^N), m > N/2 + 1$, with $\text{div} v_0 = 0$ such that the corresponding initial pressure satisfies

$$p_0 = \sum_{j,k=1}^N R_j R_k (v_0^j v_0^k) \in L^1(\mathbb{R}^N), \tag{2.8}$$

but $E_j(0) \neq E_k(0)$ for some $j, k \in \{1, \cdots, N\}, j \neq k$. If this is possible, then it implies that function $t \mapsto \|p(t)\|_{L^1}$ is discontinuous at $t = 0+$ for the local classical solution $(v(\cdot, t), p(\cdot, t))$ constructed by Kato(4) with such initial data. Using the Fourier transform, Professor P. Constantin showed that there exists no such initial data(4). Actually similar conclusion can be derived by slight change of the above proof as follows. From the relation (2.8) we have

$$- \int_{\mathbb{R}^N} p_0(x) \Delta \psi dx = \sum_{j,k=1}^N \int_{\mathbb{R}^N} v_0^j(x) v_0^k(x) \partial_j \partial_k \psi(x) dx \forall \psi \in C^2_0(\mathbb{R}^N).$$
Similarly to the above proof, choosing $\psi(x) = x_j x_k \sigma_R(x)$, and then passing $R \to \infty$, we have

$$- \int_{\mathbb{R}^N} p_0(x)dx = \int_{\mathbb{R}^N} (v^j_0(x))^2dx \quad \forall j = 1, \cdots, N$$

for $j = k$, while

$$\int_{\mathbb{R}^N} v^j_0(x)v^k_0(x)dx = 0 \quad \forall j, k = 1, \cdots, N,$$

for $j \neq k$.

### 3 Remarks on the MHD equations

In this section we extend the previous results on the system $(NS)_\nu$ to the magnetohydrodynamic equations in $\mathbb{R}^N$, $N \geq 2$.

$$(MHD)_{\mu,\nu} \begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla)v = (b \cdot \nabla)b - \nabla(p + \frac{1}{2}|b|^2) + \nu \Delta v, \\ \frac{\partial b}{\partial t} + (v \cdot \nabla)b = (b \cdot \nabla)v + \mu \Delta b, \\ \text{div } v = \text{div } b = 0, \\ v(x,0) = v_0(x), \quad b(x,0) = b_0(x) \end{cases}$$

where $v = (v_1, \cdots, v_N)$, $v_j = v_j(x,t)$, $j = 1, \cdots, N$, is the velocity of the flow, $p = p(x,t)$ is the scalar pressure, $b = (b_1, \cdots, b_N)$, $b_j = b_j(x,t)$, is the magnetic field, and $v_0$, $b_0$ are the given initial velocity and magnetic field, satisfying $\text{div } v_0 = \text{div } b_0 = 0$, respectively. Let us begin with the definition of the weak solutions of $(MHD)_{\mu,\nu}$.

**Definition 3.1** We say the triple $(v, b, p) \in [L^1(0,T; L^2_{\sigma}(\mathbb{R}^N))]^2 \times L^1(0,T; S'(\mathbb{R}^N))$ is a weak solution of $(MHD)_{\mu,\nu}$ on $\mathbb{R}^N \times (0,T)$, if

$$- \int_0^T \int_{\mathbb{R}^N} v(x,t) \cdot \phi(x)\xi'(t)dxdt - \int_0^T \int_{\mathbb{R}^N} v(x,t) \otimes v(x,t) : \nabla \phi(x)\xi(t)dxdt$$

$$= - \int_0^T \int_{\mathbb{R}^N} b(x,t) \otimes b(x,t) : \nabla \phi(x)\xi(t)dxdt + \int_0^T <p(t), \text{div } \phi > \xi(t)dt$$

$$+ \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} |b(x,t)|^2 \text{div } \phi(x)\xi(t)dxdt + \nu \int_0^T \int_{\mathbb{R}^N} v(x,t) \cdot \Delta \phi(x)\xi(t)dxdt,$$

$$\tag{3.1}$$
and
\[
- \int_0^T \int_{\mathbb{R}^N} b(x, t) \cdot \phi(x) \xi'(t) \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} v(x, t) \otimes b(x, t) : \nabla \phi(x) \xi(t) \, dx \, dt \\
= - \int_0^T \int_{\mathbb{R}^N} b(x, t) \otimes v(x, t) : \nabla \phi(x) \xi(t) \, dx \, dt + \mu \int_0^T \int_{\mathbb{R}^N} b(x, t) \cdot \Delta \phi(x) \xi(t) \, dx \, dt
\]
(3.2)
for all \( \xi \in C^1_0(0, T) \) and \( \phi = [C^\infty_0(\mathbb{R}^N)]^N \).

We have the following theorem.

**Theorem 3.1** Suppose \((v, b, p)\) is a weak solution to \((MHD)_{\mu, \nu}\) with \(\mu, \nu \geq 0\) on \(\mathbb{R}^N \times (0, T)\) satisfying

\[
\text{either} \quad p \in L^1_+(0, T; L^1(\mathbb{R}^N)), \quad \text{or} \quad p \in L^1(0, T; \mathcal{H}^q(\mathbb{R}^N))
\]
(3.3)

for some \(q \in (0, 1]\). Then, for \(N \geq 3\), we have

\[
v(x, t) = b(x, t) = 0 \quad \text{a.e. in } \mathbb{R}^N \times (0, T),
\]
(3.4)

while for \(N = 2\)

\[
v(x, t) = 0, \quad \text{a.e. in } \mathbb{R}^2 \times (0, T),
\]
(3.5)

\[
\int_{\mathbb{R}^2} (b^1(x, t))^2 \, dx = \int_{\mathbb{R}^2} (b^2(x, t))^2 \, dx
\]
(3.6)

almost everywhere in \((0, T)\), and \(b(x, t)\) is a weak solution of the heat equation

\[
\partial_t b = \mu \Delta b.
\]

**Proof** The method of proof is similar to that of Theorem 1.1 with slight changes. We will be brief, describing only essential points. We choose the vector test function \(\phi = \nabla(\nabla^j \sigma_R(x))\) with \(j \in \{1, \cdots, N\}\) in (3.1). Then, in the case \(p \in L^1(0, T; L^1(\mathbb{R}^N))\), we obtain that

\[
- \int_0^T \int_{\mathbb{R}^N} (v^j(x, t))^2 \sigma_R(x) \xi(t) \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} (b^j(x, t))^2 \sigma_R(x) \xi(t) \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} |b(x, t)|^2 \sigma_R(x) \xi(t) \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} p(x, t) dx \sigma_R(x) \xi(t) \, dx \, dt + o(1),
\]
(3.7)
where $o(1)$ denotes the sum of the terms vanishing as $R \to \infty$. Taking $R \to \infty$ in (3.7), and summing over $j = 1, \cdots, N$, we find that

$$- \int_0^T \int_{\mathbb{R}^N} |v(x,t)|^2 \xi(t)dxdt - \frac{N-2}{2} \int_0^T \int_{\mathbb{R}^N} |b(x,t)|^2 \xi(t)dxdt = N \int_0^T \int_{\mathbb{R}^N} p(x,t)dx \xi(t)dxdt. \tag{3.8}$$

If $p \in L^1_+(0,T;L^1(\mathbb{R}^N))$, choosing $\xi \in C^1_0(0,T)$ with $\xi(t) \geq 0$ in (3.8), then $N \geq 3$ implies $v(x,t) = b(x,t) = 0$ and $p(x,t) = 0$ for almost every $(x,t) \in \mathbb{R}^N \times (0,T)$. If $N = 2$, instead, then $v(x,t) = 0$ and $p(x,t) = 0$ for almost every $(x,t) \in \mathbb{R}^N \times (0,T)$. Then, (3.2) becomes

$$- \int_0^T \int_{\mathbb{R}^N} b(x,t) \cdot \phi(x) \xi(t)dxdt = \mu \int_0^T \int_{\mathbb{R}^N} b(x,t) \cdot \Delta \phi(x) \xi(t)dxdt$$

for all $\phi \in [C^2_0(\mathbb{R}^N)]^N$ and $\xi \in C^1_0(0,T)$, which shows that $b(x,t)$ is a weak solution of $\partial_t b = \mu \Delta b$. Moreover, after passing $R \to \infty$, the equation (3.7) with $j = 1$ becomes

$$\int_0^T \int_{\mathbb{R}^2} [(b^2(x,t))^2 - (b^1(x,t))^2] \xi(t)dxdt = 0, \quad \forall \xi \in C^1_0(0,T)$$

and hence we obtain (3.6). In the case $p \in L^1(0,T;\mathcal{H}^q(\mathbb{R}^N))$, $0 < q < 1$, following the same argument as in the part (b) of proof of Theorem 1.1 in the previous section, one can derive

$$- \int_0^T \int_{\mathbb{R}^N} |v(x,t)|^2 \xi(t)dxdt - \frac{N-2}{2} \int_0^T \int_{\mathbb{R}^N} |b(x,t)|^2 \xi(t)dxdt = 0$$

instead of (3.8), from which our conclusion follows. □

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References

[1] L. C. Berselli and G. P. Galdi, *Regularity criteria involving the pressure for the weak solutions to the Navier-Stokes equations*, Proc. Amer. Math. Soc., 130, no. 12, (2002), pp. 3585-3595.

[2] D. Chae and J. Lee, *Regularity criterion in terms of pressure for the Navier-Stokes equations*, Nonlinear Anal., 46, (2001), pp. 727-735.

[3] R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl., (9), 72, no. 3, (1993), pp. 247-286.

[4] P. Constantin, *private communication*.

[5] Q. Jiu and Z. Xin, *On strong convergence to 3D steady vortex sheets*, J. Diff. Eqns, 239, no. 2, (2007), pp. 448-470.

[6] T. Kato, *Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$*, J. Funct. Anal. 9, (1972), pp. 296-305.

[7] G. Koch, N. Nadirashvili, G. Seregin and V. Šverák, *Liouville theorems for the Navier-Stokes equations and applications*, arXiv preprint no. AP0706.3599 (to appear in Acta Math.).

[8] J. Leray, *Essai sur le mouvement d’un fluide visqueux emplissant l’espace*, Acta Math., 63, (1934), pp.193-248.

[9] P.-L. Lions, *Mathematical Topics in Fluid Mechanics*, Vol. 1, Incompressible Models, Clarendon Press, Oxford (1996).

[10] E. Stein, *Harmonic Analysis*, Princeton Univ. Press, Princeton, New Jersey (1993).