Distinguishing locally finite trees

Svenja Hüning * Wilfried Imrich † Judith Kloas * Hannah Schreiber *
Thomas W. Tucker ‡

October 5, 2018

Abstract

The distinguishing number $D(G)$ of a graph $G$ is the smallest number of colors that is needed to color the vertices of $G$ such that the only color preserving automorphism is the identity. For infinite graphs $D(G)$ is bounded by the supremum of the valences, and for finite graphs by $\Delta(G) + 1$, where $\Delta(G)$ is the maximum valence. Given a finite or infinite tree $T$ of bounded finite valence $k$ and an integer $c$, where $2 \leq c \leq k$, we are interested in coloring the vertices of $T$ by $c$ colors, such that every color preserving automorphism fixes as many vertices as possible. In this sense we show that there always exists a $c$-coloring for which all vertices whose distance from the next leaf is at least $\lceil \log_b k \rceil$ are fixed by any color preserving automorphism, and that one can do much better in many cases.

1 Introduction

This paper is concerned with automorphism breaking of finite and infinite trees of bounded valence by vertex colorings. 1977 Babai [2] showed that the vertices of every $k$-regular tree, where $k \geq 2$ is an arbitrary cardinal, can be colored with two colors such that only the identity automorphism preserves the coloring. For trees that are not regular such a 2-coloring need not be possible. This raises the question of how many colors are needed to break all automorphisms of a given tree $T$, that is, of finding the smallest cardinal $d$ to which there exists a $d$-coloring of the vertices of $T$ that is only preserved by the identity automorphism.

Another question is to find, for a given $c \geq 2$, $c$-colorings of the vertices of a given tree $T$ such that the color-preserving automorphisms fix subtrees of $T$ that are maximal in some sense. This is the problem, which we consider here.

Our note is related to [3], where both questions were answered for connected graphs of maximum valence 3. That paper, in turn, was motivated by the general problem of determining the distinguishing number of graphs and the Infinite Motion Conjecture of Tucker.

The distinguishing number $D(G)$ of a graph $G$ is the smallest cardinal number $d$ such that there exists a $d$-coloring of the vertices of $G$ which is only preserved by the identity automorphism. We also say that such a coloring breaks $\text{Aut}(G)$ and that $G$ is $d$-distinguishable. These concepts were introduced 1996 by Albertson and Collins [1] and have spawned a series of related papers. In two of them, by Collins and Trenk [3] and Klavžar, Wong and Zhu [7], it is shown...
that the distinguishing number $D(G)$ of any finite connected graphs $G$ of maximal valence $k$ is at most $k$, unless $G$ is $K_k$, $K_{k,k}$ or $C_5$. Then $D(G) = k + 1$. In [8] this was extended to infinite graphs. In that case $D(G)$ is bounded by the supremum of the valences of the vertices.

Despite the fact that the distinguishing number can be arbitrarily large, it is 1 for asymmetric graphs. As almost all finite graphs are asymmetric, this means that almost all graphs have distinguishing number 1. Furthermore, almost all finite graphs that are not asymmetric have just one non-identity automorphisms, which is an involution\footnote{Florian Lehner, private communication.}. One can break it by coloring one selected vertex black and all others white. Clearly such graphs are 2-distinguishable.

For infinite graphs we have the Infinite Motion Conjecture [9]. It says that all connected, locally finite, infinite graphs are 2-distinguishable if every non-identity automorphism moves infinitely many vertices. Despite the fact that it is true for many classes of graphs, for example for graphs of subexponential growth, see Lehner [8], the conjecture is still open. Until recently it was not even clear whether it holds for graphs of maximum valence 3, but in [6] it was shown that all connected infinite graphs of maximal valence 3 are 2-distinguishable and that no motion assumption is needed.

In the case of connected finite graphs $G$ of maximum valence 3 it is even enough to require that every automorphism moves at least three vertices to ensure 2-distinguishability, unless $G$ is the cube or the Petersen graph. In fact, with the exception of $K_4$, $K_{3,3}$, the cube and the Petersen graph, all connected finite graphs $G$ of maximum valence 3 admit a 2-coloring where every automorphism that preserves the coloring fixes all vertices, with the exception of at most one pair of interchangeable vertices. We say the 2-coloring fixes all but this pair of vertices. For example, consider three copies of $K_{1,3}$, select a vertex of valence 1 in each copy and identify them. The resulting tree has distinguishing number 3, and it is easy to find a 2-coloring that fixes all but two vertices.

In this note we generalize this to trees $T$ of maximal valence $k$ and an arbitrary number of colors $c$. We wish to find $c$-colorings that fix as many vertices as possible, that is, we wish to maximize the sets of vertices that are fixed by each color preserving automorphism. As this is hard to control, we look for the smallest number $r(c,k)$ such that there exists a $c$-coloring of $T$ that fixes in the worst case at least all vertices of $T$ whose distance from the next leaf is at least $r(c,k) = \lceil \log_c k \rceil$. This is made more precise in Section 5 where one can see that in a lot of cases this number will be significantly smaller.

2 Preliminaries

2.1 Graph representation

Let $G = (V(G), E(G))$ be a connected graph, $V(G)$ its set of vertices and $E(G)$ its set of edges. To simplify the notation we write $V$ and $E$ if the graph is clear by the context. If the vertices of $G$ have maximal valence 3, then $G$ is called subcubic. As usual we call the number of edges on a shortest path between two vertices $u$ and $v$ be the distance $d(u, v)$ between $u$ and $v$.

**Definition 1.** Let $v \in V$. The ball of radius $n$ with center $v$ is defined as the set $B(n, v) = \{u \in V \mid d(u, v) \leq n\}$. The sphere of radius $n$ around $v$ is the set $S(n, v) = \{u \in V \mid d(u, v) = n\}$, that means it is the set of all vertices of distance $n$ from $v$, see Figure 1.

In this paper, we mostly represent a graph $G$ by an arrangement of spheres with a common center $v \in V(G)$, and say that $G$ is rooted in $v$. In that context, a down-neighbor, a cross-neighbor and an up-neighbor of $u \in S(n, v)$ is a neighbor $w$ of $u$ that it is in $S(n - 1, v)$,
\( S(n, v) \) or \( S(n+1, v) \) respectively; see Figure 1. The corresponding edges are called down-edges, cross-edges and up-edges.

**Definition 2.** Two vertices \( z, z' \) are siblings if they have the same down-neighbor. We call a vertex \( w \) an only child of a vertex \( u \) if \( w \) is the only neighbor of \( u \) of valence 1.

![Diagram of a graph decomposed into spheres centered at a vertex](image)

**Figure 1:** Decomposition of a graph into spheres centered at a vertex \( v \). Each vertex in \( S(i, v) \) has distance \( i \) from \( v \).

### 2.2 Ends and rays

A subgraph of a graph \( G = (V(G), E(G)) \) is a graph \( H = (V(H), E(H)) \) such that \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). If \( E(H) \) contains all edges between vertices of \( V(H) \) that are also in \( E(G) \), we say \( H \) is an induced subgraph of \( G \) and denote it by \( (V(H), E(H)) \). If \( S \subseteq V(G) \), then \( G \setminus S \) is the subgraph of \( G \) induced by the vertices of the set \( V(G) \setminus S \).

**Definition 3.** A ray is an infinite graph \( R = (V, E) \), with \( V = \{v_0, v_1, v_2, \ldots\} \) and \( E = \{v_0v_1, v_1v_2, v_2v_3, \ldots\} \) where the \( v_i \) are pairwise different. If a ray \( R_1 = (V_1, E_1) \) is a subgraph of a ray \( R_2 = (V_2, E_2) \), then \( R_1 \) is called a tail of \( R_2 \).

Two rays \( R_1 \) and \( R_2 \) in a graph \( G = (V(G), E(G)) \) are equivalent, in symbols \( R_1 \sim R_2 \), if for every finite set \( S \subseteq V(G) \) there exists a connected component of \( G \setminus S \) containing a tail of both \( R_1 \) and \( R_2 \). One can show that \( \sim \) is an equivalence relation. The equivalence classes of \( \sim \) are called ends of \( G \).

If \( T = (V, E) \) is an infinite tree, then the set of vertices in \( V \) such that \( T - v_i = (V \setminus v_i) \) contains at least two infinite components is denoted by \( V_C = \{v_1, v_2, \ldots\} \). We call the induced subgraph \( \langle V_C \rangle \) the trunk of \( T \) and denote it by \( T_C \).

### 2.3 Automorphisms and Colorings

An isomorphism \( \varphi : V(G) \to V(H) \) between two graphs \( G \) and \( H \) is a bijection such that \( uv \in E(G) \) if and only if \( \varphi(u)\varphi(v) \in E(H) \). The isomorphisms of \( G \) onto itself are called automorphisms. They form a group \( A(G) \). The stabilizer of a vertex \( v \in V(G) \) is the set \( A(G)_v = \{\alpha \in A(G) | \alpha(v) = v\} \).

**Definition 4.** A \( d \)-coloring \( c : V \to \{1, \ldots, d\} \) of a graph \( G \) is a map that gives each vertex of \( G \) a color \( i \in \{1, \ldots, d\} \). We say that an automorphism preserves a coloring \( c \), if \( c(v) = c(\varphi(v)) \) for all \( v \in V \). Otherwise, we say that the coloring breaks the automorphism.
The set $A(G)_c$ of all automorphisms preserving $c$ forms a group. If $v \in V(G)$ and $A(G)_c \subseteq A(G)_v$, we say that $c$ fixes $v$. As mentioned in the introduction our aim is to construct graph colorings that fix as many vertices as possible. The minimal number of colors needed to fix all vertices is called the distinguishing number.

**Definition 5.** The **distinguishing number** $D(G)$ of a graph $G$ is the the smallest $d$ for which there exists a $d$-coloring $c$ of $G$ such that the only automorphism preserving $c$ is the identity.

**Definition 6.** The **center** of a finite graph is a vertex for which the greatest distance from $v$ to any other vertex of the graph is minimal.

In a finite tree $T$ the center is either a single vertex or an edge. If the center of $T$ is a vertex $w$, then $A(T)_w = A(T)$. If the center is an edge $uv$, any automorphism $\alpha \in A(T)$ satisfies either $\alpha(v) = v$ and $\alpha(u) = u$ or $\alpha(v) = u$ and $\alpha(u) = v$. We define a subtree $T_w$ of the tree $T$ rooted in $v$, with $w \in S(n,v)$, as the tree induced by $w$ and all the vertices in $S(m,v)$, $m > n$, for which there exists a path to $w$ in $T$ not containing $v$.

### 3 Coloring locally finite infinite trees

In this section we prove that every locally finite infinite tree with maximal valence $k$ has distinguishing number $k - 1$. For the proof we need the following two lemmas.

**Lemma 1.** Let $T$ be a tree with maximal valence $k$. If $T$ has a vertex $v$ of valence $1 \leq \text{val}(v) \leq k - 1$ then $D(A(T)_v) \leq k - 1$. In other words, there is a $(k - 1)$-coloring $c$ of $T$ that breaks all automorphisms of $A(T)_v$.

**Proof** Let $v$ be a vertex of valence $1 \leq \text{val}(v) \leq k - 1$. We color $v$ with an arbitrary color and all its neighbors with different colors. This coloring fixes $B(1,v)$ in $A(T)_v$. We prove that for all $i > 0$ there is a $k - 1$-coloring of $B(i,v)$ that breaks all automorphisms of $A(B(i,v))_v$. To show this, it suffices to extend a given coloring $c(i)$ of $B(i,v)$ with $k - 1$ colors to $B(i + 1,v)$. We do this as follows. Every vertex in $S(i,v)$ has at most $k - 1$ up-neighbors. Each of them can be colored with a different color. If we continue to color the tree in this way, we obtain a coloring that breaks all automorphisms of $A(T)_v$ but the identity. □

The second result that we need is König's Lemma, see for example [4].

**Lemma 2** (König’s Lemma). Let $V_0, V_1, V_2, \ldots$ be an infinite sequence of non-empty, disjoint sets. Let $V = \cup_{i \geq 0}$ denote their union. If $G = (V, E)$ is a graph such that for all $v \in V_n$, $n \geq 1$, there exits a vertex $f(v) \in V_{n - 1}$ adjacent to $v$, then there exists an infinite path $v_0 v_1 \ldots$ in $G$ with $v_n \in V_n$ for all $n \geq 0$.

Note that parts of the next result (namely the finite case) can be extracted from a recent classification result from [5]. Here we provide a direct and constructive proof which also generalizes to trees with maximal valence $k$.

**Theorem 1.** Every locally finite, infinite tree with maximal valence $k$ has distinguishing number $D(G) \leq k - 1$.

**Proof** Let $T$ be a locally finite infinite tree. It is well known that such a tree without vertices of valence 1 is 2-distinguishable, see for example [10]. Hence, we can assume that $T$ has at least one vertex of valence 1.
By König’s Lemma, $T$ has at least one end. We first treat the case, where $T$ has exactly one end, say $e$. Let $R \in e$ be a ray with origin $v_0$, where $v_0$ is a vertex of valence 1.

We color all vertices of $R$ with color 1. Thus every automorphism that stabilizes $R$ also fixes all vertices of $R$. Consider a vertex $z$ in $R$ and its $j$ neighbors that are not in $R$. Since $j \leq k - 2$ we can color these vertices with different colors without using color 1. We proceed by doing this for all neighbored vertices of $R$. Let $v$ be one neighbor of $z$ not in $R$ and let $T_v$ be the component of $T - z$ that contains $v$. Now, color $T_v$ as in Lemma 1. Continue by coloring the neighbors of all vertices in $R$ in the same way. If $w$ is another vertex of $T$ of valence 1, then the unique ray $R_w$ in $e$ with origin $w$ contains at least one vertex which is not colored with 1. Hence, independently of how the coloring of $T$ will be finished, any color preserving automorphism $\alpha$ of $T$ will satisfy $\alpha(v_0) \neq w$. Because this holds for all vertices of valence 1 that are different from $v_0$ we infer that $\alpha(v_0) = v_0$ and that $\alpha$ fixes every vertex of $R$. Because the vertices in $R$ are fixed also the neighbors and finally all finite trees connected to $R$ are fixed.

We now treat the case when $T$ has two or more ends. Consider the trunk $T^C$ of $T$ which is a locally finite, infinite tree without leaves. Since it is unique any automorphism of $T$ leaves $T^C$ invariant. By [10] we know that $T^C$ is 2-distinguishable. Any vertex $z$ of $T^C$ has at most $k - 2$ neighbors not in $T^C$ that can be fixed by coloring them with different colors. Again, for each of these neighbors $v_i$ of $z$ we consider the underlying tree $T_{v_i}$ of $T - z$ that contains $v_i$ as before and apply Lemma 1.

The following example shows that Theorem 1 cannot be generalized to finite trees.

**Example 2.** Consider a finite tree with center $v$ where every vertex is of valence 3 or 1 and every leaf has the same distance from $v$. The distinguishing number of such trees is 3. For any 2-coloring of such a tree, there remains a pair of leaves that is indistinguishable, see Figure 2.

An other easy example is the $K_{1,3}$.

![Figure 2](image-url)  

Figure 2: Example of a finite tree $T$ with $D(T) = 3$. With a 2-coloring, two leaves are interchangeable.

### 4 Finite trees

We begin with the analog of Theorem 1 for finite trees.

**Lemma 3.** Let $T$ be a finite tree with maximal valence $k$. Then there is a $k - 1$-coloring, which fixes all vertices, with the possible exception of two sibling leaves.

**Proof** If the center of $T$ consists of a single vertex $v$, assign $v$ an arbitrary color and color its neighbors $v_1, v_2, \ldots$ with as many different colors as possible. Only in the case where $v$ has valence $k$ we have to use one color twice. As before let $T_{v_i}$ be the component of $T - v$ containing $v_i$. We extend the coloring of $T_{v_i}$ as in Lemma 1 for all $i$. In the case where $v$ has $k$ neighbors it
is possible that there is an automorphism that interchanges the two subtrees, say $T_{v_m}$ and $T_{v_n}$ (with $v_m$ and $v_n$ of the same color). We take an endpoint $a$ of $T_{v_m}$ and change its color so that $T_{v_m}$ and $T_{v_n}$ are indistinguishable.

However, now there might be a color preserving automorphism that moves $a$. This is only possible if there is another vertex of valence 1, say $b$, that has a common neighbor with $a$. This is the only pair of that kind in $T$.

We still have to consider the case where the center of $T$ is an edge, say $uv$. In that case we color $u$ and $v$ with different colors, and consider the components $T_u$ and $T_v$ of $T - uv$ that contain $u$ or $v$, respectively. We now obtain a distinguishing $k - 1$-coloring by applying Lemma 1 to $T_u$ and $T_v$.

It is known from Babai [2] that each (infinite) homogeneous tree of valence $k > 2$ is 2-distinguishable. This result cannot be adapted to the finite case, but note that we can fix all vertices except the leaves by a 2-coloring.

**Lemma 4.** Let $T$ be a finite tree where every vertex has valence 1 or $k$. Then there exist a 2-coloring of $T$ that fixes all vertices except (some of) the leaves.

**Proof** Suppose the center of $T$ consists of the single vertex $v$. This vertex is then automatically fixed by any automorphism. Let $v$ be a white vertex and color all $k$ neighbors of $v$ black.

![Figure 3: Example of the coloring algorithm given in the proof of Lemma 4 with $k = 4$.](image)

To avoid that they can be changed we assign different colorings to the next $k - 1$ up-neighbors. Since the number of different 2-colorings of $k - 1$ indistinguishable vertices is $k$, each of the black vertices can be fixed. We proceed with this coloring process until we reach the leaves. Then, all vertices except the leaves are fixed. See Figure 3 for an example.

If the center of $T$ is an edge, say $uv$, we color $u$ white, $v$ black and continue with a coloring for the subtrees $T_u$ respectively $T_v$ as before. 

\[ \square \]

## 5 Main Theorem

Given a tree $T$ with maximal valence $k$, we now construct a $c$-coloring of $T$ which fixes all vertices of $T$ that are sufficiently far away from the next leaf, respectively all vertices if there are no leaves.

**Main Theorem.** Let $T$ be a finite or infinite tree of maximal valence $k < \infty$. Suppose we are given $c$ colors, $2 \leq c \leq k$, to color the vertices of $T$ and that $r$ is an integer that satisfies

\[
3 < k \leq 2^{r-1} \quad \text{for } c = 2, \quad \text{or} \quad \\
3 < k \leq c^r (c - 1) + 2 \quad \text{for } c > 2.
\]
Then there exists a $c$-coloring of $T$ which fixes all vertices of $T$ whose distance from the next leaf is at least $r$.

Furthermore, $r = 0$ for $c = k$ and $r = 1$ for $c = k - 1$.

The proof of the main theorem is based on the following proposition, which actually contains more information than the main theorem. In the proposition we let $T$ be a tree rooted in a vertex $v$. If $T$ is finite, then we define $T_u$ as the vertex $u$ together with the components of $T - u$ that do not contain $v$. In the infinite case $T_u$ consists of the vertex $u$ together with all finite subtrees of $T - u$, see Figure 4.

![Figure 4: After removal of $u$, all remaining finite trees are attached to $u$ in $T_u$.](image)

**Proposition 3.** Let $T$ be a finite or infinite tree of maximal valence $k < \infty$. We assume $T$ to be rooted in a fixed vertex $v$ (the center of the tree in the finite case) and choose a number $c \geq 2$. Then, for every pair $c, k$, there exists a number $r(c, k)$ and a $c$-coloring of $T$ that fixes all vertices $u \in V(T)$, for which there exists a leaf $w$ in $T_u$ such that $d(u, w) \geq r(c, k)$. If $k \in \{0, 1, 2\}$ or $c \geq k$, then $r(c, k) = 0$, which means that the entire graph is fixed. If $k \geq 3$ and $c = k - 1$, then $r(c, k) = 1$. Otherwise

$$r(c, k) := \begin{cases} \log_2(\max\{3, k - 2\}) + 1 & \text{for } k \geq 4 \text{ and } c = 2, \\ \log_c\left(\max\left\{3, \left\lceil \frac{k - 2}{c - 1} \right\rceil \right\} \right) & \text{for } k \geq 4 \text{ and } 2 < c \leq k - 2. \end{cases} \quad (1)$$

For the values of $[r(c, k)]$ compare Table 1, which is based on Lemma 3, Proposition 3, and summarized in the Main Theorem. In particular, it shows that, whenever the entry in the table is 1, we find a $c$-coloring that fixes all vertices of the tree with the possible exception of leaves.

The strategy to prove this proposition the following. First we introduce an explicit coloring algorithm and then we show two of its properties (see Lemmas 5 and 6).

We start by considering a finite tree $T$ with maximal valence $4 \leq k < \infty$. We assume that we have $2 \leq c \leq k - 2$ colors. For simplicity we write $\{0, 1, 2, 3, ..., c - 1\}$ for the $c$ different colors (in the following figures, 0 is white, 1 black and 2 gray).

Furthermore, a $c$-coloring of a set of siblings optimal if the maximal number of vertices with the same color is minimal. Moreover, a vertex $u$ is said to satisfy the distance condition if there exists a leaf $w$ in $T_u$ of distance $d(u, w) \geq r(c, k)$. 

7
| \(c\) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2   | 0 | 1 | 3 | 3 | 4 | 4 | 4 | 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 3   | - | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 4   | - | - | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 5   | - | - | - | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6   | - | - | - | - | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7   | - | - | - | - | - | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 1: Values of \(\lceil r(c,k) \rceil\) according to Proposition 3.

5.1 Coloring Algorithm

We first assume that the center of \(T\) consists of a single vertex \(v\). We root \(T\) in \(v\) and color \(v\) with color 0. Let \(n' \in \mathbb{N}\) be the maximal radius such that \(S(n',v) \neq \emptyset\). Now consider the following steps, each of which maybe processed several times.

**Step 0:** If there exist indices \(\ell \in \{1,\ldots,n'\}\) such that there still exist uncolored vertices in \(S(\ell,v)\), then let \(n\) be the minimum of these indices and continue with Step 1.

Otherwise stop the coloring algorithm.

**Step 1:** If there exists an uncolored vertex in \(S(n,v)\) call it \(u\) and go to Step 2. Otherwise go back to Step 0.

**Step 2:** If the vertex \(u\) does not fulfill the distance condition, color \(u\) with 0 and go back to Step 1. Otherwise continue with Step 3.

**Step 3:** Note that the vertex \(u\) satisfies the distance condition. Consider \(u\) and all of its uncolored siblings \(\{v_1,\ldots,v_r\}\) which fulfill the distance condition. If this set is not empty, then color them optimally. If there are no indistinguishable vertices within \(\{u,v_1,\ldots,v_r\}\) with the same color, go back to Step 1. Otherwise continue the coloring in the following way (which is still part of Step 3):

**Main Line Coloring:** Let \(V_j\) be the set of vertices among \(\{u,v_1,\ldots,v_r\}\) with color \(j\), where at least two vertices have color \(j\), i.e. \(|V_j| \geq 2\) (vertices with a color that appears only once are clearly fixed).

We can assume that for all \(v \in V_j\), and for all \(j\), there exists a leaf \(w\) in \(T_v\) of distance \(d(v,w) \geq r(c,k)\). Let \(V_j = \{v_{j1},v_{j2},\ldots,v_{jm_j}\}\) for each of the colors \(j = 1,\ldots,\ell \leq c\) and do the following:

Consider (one of) the longest path(s) \(R_i\) from \(v_{ji}\) to a leaf in \(T_{v_{ji}}\) for each of these vertices \(v_{ji}, i = 1,\ldots,m_j\). We call the chosen paths \(R_i\) main lines. To distinguish \(\{v_{j1},v_{j2},\ldots,v_{jm_j}\}\) we color the main lines \(R_i\) with pairwise different finite sequences of colorings.

For example, if \(c = 2\), we color the paths with different “reverse binary colorings”. That means \(R_1\) will be colored with 00000..., \(R_2\) with 10000..., \(R_3\) with 01000..., \(R_4\) with 11000... and so on. If \(c = 3\), we use “reverse ternary colorings”, meaning that \(R_1\) will be colored with 00000..., \(R_2\) with 10000..., \(R_3\) with 20000..., \(R_4\) with 01000... and so on, see Figure 5.

In general we use a reversed coloring based on the number system with base \(c\). Having done this for all \(j \in \{1,\ldots,\ell\}\) continue with Step 4.

---

2In this step we always look after the lowest sphere where there are uncolored vertices, such that in the end we can guarantee that we colored all vertices.
Figure 5: Coloring of main lines for \( c = 3 \).

**Step 4:** We consider all vertices \( \{w_1, \ldots, w_r\} \) produced in Step 3 that are part of one of the main lines and have valence \( \geq 3 \). That means we consider vertices that have at least one second up-neighbor, which is not in the main line. Consider a vertex \( w_i \). For simplicity we denote its up-neighbors (it has at least two) by \( v_1, v_2, \ldots, v_\ell \). By construction only one vertex is in a main line, let it be \( v_1 \). Therefore, \( v_1 \) is already colored, whereas \( v_2, \ldots, v_\ell \) are uncolored. Let \( a \) be the color of \( v_1 \). We consider two cases:

1. **Case 1:** \( \ell = 2 \).

   Then color \( v_2 \) with \( (a + 1) \mod c \), see Figure 6 for an example, where \( c = 3 \).

   ![Figure 6: How to color unique siblings of vertices in a main line for \( c = 3 \).](image)

   If \( c > 2 \) continue with Step 4 for the next \( w_i \). Otherwise do the following:

2. **Case 2:** \( 3 \leq \ell \leq k - 1 \).

   If the subtree \( T_{v_2} \) contains only vertices of valence one or two (we say if there is no branching), color all vertices of \( T_{v_2} - v_2 \) with 1. Now, consider the case in which there is a branching in \( T_{v_2} \). Color all vertices up to the branching in \( T_{v_2} - v_2 \) with 1. If there are 2 or 3 sibling vertices in that branching, color them all with 1. If the branching consists of four or more vertices, then assign them an optimal coloring, see Figure 7. Restart Step 4 for the next \( w_i \).

   ![Figure 7: Coloring of secondary lines for \( c = 2 \).](image)

   **Case 2:** \( 3 \leq \ell \leq k - 1 \).
Color \(v_2, v_3, \ldots, v_\ell\) with an optimal coloring considering two additional restrictions:

First, we do not use the color \(a\). Second, if there exists \(j \in \{2, \ldots, \ell\}\) such that \(v_j\) has the color \(b \in \{0, \ldots, c-1\} \setminus \{a\}\), then there exists \(j' \in \{2, \ldots, \ell\}, j \neq j'\), such that \(v_{j'}\) also has color \(b\). This means that a color is always used at least twice, see Figure 8. Note that \(v_1\) is the only vertex among the siblings whose color appears just once.

Having done this for all \(w_1, \ldots, w_k\) check whether there are indistinguishable vertices of the same color satisfying the distance condition (as an example consider the case that there is a branching in the subtree \(T_{v_2}\) when \(c = 2\) or within \(\{v_2, v_3, \ldots, v_\ell\}\)), then repeat the Main Line Coloring method and apply Step 4 to these vertices. If there are no colored indistinguishable vertices, continue with Step 1.

![Figure 8: Examples of how to color the four (left) or the six (right) up-neighbors of a vertex in some main line, with \(c \geq 3\).](image)

This completes the algorithm for the case that the center of \(T\) is a single vertex.

Now, assume that the center of \(T\) is not a vertex but an edge \(uv\). Let \(T_u\) be the tree containing \(u\) in \(T - v\) and \(T_v\) be the tree containing \(v\) in \(T - u\). Color \(u\) with 1, \(v\) with 0, and proceed with the coloring of \(T_u\) and \(T_v\) as explained above for trees whose center consists of a single vertex.

### 5.2 Proof of the Main Theorem

**Lemma 5.** Let \(V\) be a set of \(t\) vertices. Assume they are colored with an optimal coloring consisting of \(j\) colors, with the additional restriction that every color appears at least twice. Then, the maximal number \(p\) of vertices with the same color in \(V\) is \(\max \{3, \left\lceil \frac{t}{j} \right\rceil \}\).

**Proof** If \(t\) is even and \(j \geq \frac{t}{2}\), then there are enough colors such that the vertices are colored pairwise differently, meaning that \(p = 2\). If \(t\) is even but \(j < \frac{t}{2}\), we are forced to use every possible color, and every color has to be used at least twice. Thus, the restriction is automatically fulfilled and \(p = \left\lceil \frac{t}{j} \right\rceil\).

Now, consider the case when \(t\) is odd. We first ignore one vertex and proceed as in the even case for the remaining \(t - 1\) vertices. The ignored vertex then has to get the same color as one of the remaining vertices due to the additional restriction. That means

\[
\text{if } j \geq \frac{t-1}{2}, \text{ then } p = 2 + 1 = 3, \text{ and } \\
\text{if } j < \frac{t-1}{2}, \text{ then } p = \left\lceil \frac{t-1}{j} \right\rceil + 1.
\]

The +1 arises from the vertex that was first ignored. It is needed in the case where each color is used equally often. It is easy to verify that \(\left\lceil \frac{t}{j} \right\rceil \geq \left\lceil \frac{t-1}{j} \right\rceil + 1\) for every possible pair of \(t\) and \(j\). Therefore an upper bound for \(p\) is always \(\max \{3, \left\lceil \frac{t}{j} \right\rceil \}\).

\(\square\)
Lemma 6. Let $T$ be a finite tree with maximal valence $0 < k < \infty$. If $T$ is colored as described in the Coloring Algorithm where $c \geq 2$, then the largest number of sibling vertices fulfilling the distance condition and having the same color that can appear is $\max\{3, \left\lceil \frac{k-3}{c-1} \right\rceil \}$.

**Proof** Within our Coloring Algorithm there are three situations in which sibling vertices with the same color fulfilling the distance condition might appear.

**Situation 1:** If we apply an optimal coloring to $m$ vertices, we obtain a maximum of $\left\lceil \frac{m}{c-1} \right\rceil$ vertices with the same color. For a tree with maximal valence $k$ it is bounded by $s_1 = \left\lceil \frac{k}{c} \right\rceil$.

**Situation 2:** In Step 4 case 2 of the Coloring Algorithm, we use an optimal coloring with $c - 1$ colors such that each color appears at least twice. By applying Lemma 5 we obtain a maximum number $s_2 = \max\{3, \left\lceil \frac{k-2}{c-1} \right\rceil \}$ of sibling vertices with the same color in that case.

**Situation 3:** Consider the case $c = 2$ in Step 4 Case 1 of the algorithm where we are in the branching situation. There are at most $s_3 = \max\{3, \left\lceil \frac{k-1}{2} \right\rceil \}$ vertices with the same color. We see that $s_2 \geq s_3$ for $c = 2$ and $k \geq 4$.

It remains to compute the maximum of $s_1$ and $s_2$. Straight forward calculations show that

$$k < 2c \iff \frac{k - 2}{c - 1} < \frac{k}{c} < 2.$$ 

Therefore, for $k < 2c$ the maximum is 3, while for $k \geq 2c$ it is $\frac{k - 2}{c - 1}$. Thus, the maximum of $s_1$ and $s_2$ is $\max\{3, \left\lceil \frac{k-3}{c-1} \right\rceil \}$.

We are now able to prove Proposition 3.

**Proof** [Proposition 3] The cases $k \in \{0, 1, 2\}$ and $c \geq k$ are trivial. For $c = k - 1$, we refer to Lemma 6 in Section 4. So, assume that we have $2 \leq c \leq k - 2$ and $k \geq 4$.

First, we consider a finite tree $T$ and apply the Coloring Algorithm. Assume that the center of $T$ consists of a single vertex $v$. This vertex is fixed by each automorphism of $T$. Thus, it remains to show that for all $n \in \mathbb{N}$ the vertices with the same color in $S(n, v)$ which satisfy the distance condition are indistinguishable due to our algorithm.

The distance $r(c, k)$ is built upon the maximal number of indistinguishable vertices with the same color that can appear in the same sphere and the given Main Line Coloring in the algorithm. The aim of these main lines is to fix indistinguishable siblings that have the same color and the vertices on these main lines themselves.

Let $n' \in \mathbb{N}$ be the smallest index such that there exist indistinguishable sibling vertices with the same color $v_1, \ldots, v_m$, $m \geq 2$, in $S(n', v)$ which fulfill the distance condition.

Consider $v_1, \ldots, v_m$ and their main lines $R_i$ given by Step 3 of the algorithm. We see that if each $v_i$ has distance at least $\log_c m$ to a leaf in $T_{v_i}$, then the main lines $R_i$ are colored pairwise differently. Due to Lemma 6 this distance is bounded by $\log_c \max\{3, \left\lceil \frac{k-2}{c-1} \right\rceil \}$. Since this coincides with our distance condition regarding the given $r(c, k)$, for any color-preserving automorphism $\alpha$ of $T$ with $\alpha(v_i) = v_j$ and $\alpha(R_i) = R_j$ for some $i, j \in \{1, \ldots, m\}$, we see that $i = j$. This means that the main lines are not interchangeable.

Next we argue why a main line $R_i$ cannot be mapped to any other string of some $T_{v_j}$. By a string we mean a path, that has at most one vertex of each sphere. Therefore we show that there is no automorphism $\alpha$ such that $\alpha(T_{v_i}) = T_{v_j}$ for any $i, j \in \{1, \ldots, m\}$.

Without loss of generality we consider $T_{v_i}$ and $T_{v_j}$, $i \neq j$ and assume that there exists at least one vertex in $R_i$ resp. $R_j$ with at least two up-neighbors (otherwise $T_{v_i}$ and $T_{v_j}$ are stabilized by the Main Line Coloring method).

11
Case 1: There exists a vertex $w_0$ in $R_i$ with one up-neighbor $w_1$ in $R_i$ and at least two more up-neighbors, called $w_2, \ldots, w_\ell$, $\ell \geq 3$. Let us assume that there exists a color-preserving automorphism $\alpha$ of $T$ that maps $T_{v_j}$ to $T_{v_i}$. By the Main Line Coloring method we know that $\alpha(R_i) \neq \alpha(R_j)$. Thus, there exists a vertex $\tilde{w} \in R_j$ (in the same sphere as $w_1, \ldots, w_\ell$) and $k \in \{2, \ldots, \ell\}$ such that $\alpha(\tilde{w}) = w_k$. See Figure 9.

Due to Step 4, Case 2, in the algorithm, $\tilde{w}$ does not have a sibling vertex with the same color, whereas $w_k$ for sure has. So, $\alpha$ cannot map $\tilde{w}$ to $w_k$, and we have a contradiction. We conclude that $\alpha$ does not exists.

Case 2: There exists a vertex $w_0$ in $R_i$ with one up-neighbor $w_1$ in $R_j$ and exactly one additional up-neighbor, called $w_2$.

Let us again assume that there exists a color-preserving automorphism $\alpha$ of $T$ that maps $T_{v_j}$ to $T_{v_i}$. By main line coloring we know that $\alpha(w_1) \notin R_j$. Thus there exits a vertex $\tilde{w} \in R_j$ such that $\alpha(w_2) = \tilde{w}$.

Case 2.1: Let $c \geq 3$. If $\alpha$ swaps $\tilde{w}$ and $w_2$, then $\alpha$ also swaps $w_1$ and the unique sibling of $\tilde{w}$. But if $\tilde{w}$ and $w_2$ have the same color, their siblings do not, because of the shifting of the colors modulo $c$ in Case 1 of Step 4 of the algorithm. Thus, such an $\alpha$ does not exist. Note, here it is important that we assume that $c \geq 3$. For $c = 2$, we only have the color pairs $(0, 1)$ and $(1, 0)$ which cannot be distinguished.

Case 2.2: Let $c = 2$. If the subtree $T_{w_2}$ contains only vertices of valence one or two (no branching), all vertices of $T_{w_2}$ (except maybe $w_2$) are colored with 1. In contrast to this, at least the last vertex of $R_j$ (the leaf) is colored with 0. This is due to the given $r(2, k)$, see Figure 8 (here we need the +1 in the case where $r(c, k) = \log_2(\max\{3, k-2\}) + 1$). Thus, such an $\alpha$ does not exist.

Now, assume there is a branching in $T_{w_2}$. By our algorithm, we have avoided that there exist single vertices of a certain color in such a branching (see the left and the right picture in Figure 7) while in every branching in the main line of $R_j$ there exist a vertex (the vertex in the
main line itself) with color \( d \) that is the only vertex with this color in that branching.\(^3\)

All in all, we showed that \( v_1, \ldots, v_m \) are fixed. Now, we can use a final inductive argument. Assume that all vertices with the same color up to the sphere \( S(n, v) \) are fixed. Consider indistinguishable sibling vertices that fulfill the distance condition in the sphere \( S(n + 1, v) \). At this point, we can apply the same argument as above to ensure that they have to be fixed. Note that here it is important to assume that everything below these vertices is already fixed.

If the center of \( T \) consists of an edge \( uv \), color \( u \) with 0 and \( v \) with 1. Then \( u \) and \( v \) are fixed by the properties of a tree and we can apply the Coloring Algorithm to the remaining vertices in the two subtrees containing \( u \) and \( v \) after removing the edge \( uv \) rooted in, respectively, \( u \) and \( v \). Then, we use the same reasoning as above. This completes the proof for the finite case.

Now, let \( T \) be an infinite, locally finite tree. We first assume that \( T \) has at least two ends. In that case consider the trunk \( T^C \) of \( T \) which is a locally finite, infinite tree without leaves. Since it is unique any automorphism of \( T \) leaves \( T^C \) invariant. By \([10]\) we know that \( T^C \) is 2-distinguishable. Now, for every vertex \( u \) in the trunk we consider the subtree \( T_u \) that contains, as explained above, the vertex \( u \) together with all finite subtrees of \( T - u \). We apply the Coloring Algorithm to each of these subtrees and fix all vertices which fulfill the distance condition.

If \( T \) has only one end, then there exists a vertex \( v \) with valence 1. Let \( R \) be the ray with root \( v \). We color all vertices of \( R \) with 0. For each vertex in \( R \) we color the maximal \( k - 2 \) neighbors not in \( R \) with an optimal coloring with the \( c - 1 \) colors different from 0. Thus, there are at most \( \lceil \frac{k-2}{c-1} \rceil \) vertices with the same color. If there are indistinguishable vertices we apply the Coloring Algorithm from above to fix all vertices which fulfill the distance condition. \( \square \)

**Proof** [Main Theorem] If \( c = k \), we have enough colors to fix all vertices of the tree, so \( r = 0 \). If \( c = k - 1 \), we refer to Theorem [1] and Lemma [3] in Section [4]. Assume now that \( c = 2 \) and that we have an integer \( r \) satisfying \( 3 < k \leq 2^{r-1} \). Then \( \log_2 k \leq r - 1 \). Since \( 3 < k \), we conclude that \( \log_2(\max\{3, k - 2\}) + 1 < r \). The claim of the Theorem follows then by Proposition [3]. Consider now the remaining case that \( c > 2 \) and \( r \) is an integer such that \( 3 < k \leq c'(c - 1) + 2 \). Then we have \( \frac{k-2}{c-1} \leq c' \). Since \( c' \) is an integer, it follows that \( \log_c\lceil \frac{k-2}{c-1} \rceil \leq r \). If \( \lceil \frac{k-2}{c-1} \rceil \geq 3 \) the claim follows by Proposition [3]. Furthermore we have \( \log_c \frac{1}{c} < r \), since \( 3 < c'(c - 1) + 2 \), which implies \( 1 < r \). Thus \( \log_c 3 < r \) and the claim follows again by Proposition [3]. \( \square \)

We end with an example that shows that the given constant \( r(c, k) \) in (1) is tight in some special cases.

**Example 4.** Consider a tree as in Figure [11] with maximal valence \( k = 10 \), which we would like to color with \( c = 3 \) colors. Without a coloring the vertices in the first sphere are indistinguishable. Using an optimal coloring for these vertices there are \( \lceil \frac{10}{3} \rceil = 4 \) vertices \( v_1, \ldots, v_4 \) with the same color. Thus in our algorithm they are starting points of main lines of length two and we can distinguish them in that way, see Figure [11]. Clearly, it is not possible to distinguish \( v_1, \ldots, v_4 \) by using only 3-colors in the next sphere, but we can fix them by coloring all vertices of the next two spheres. That is what the upper bound \( r(3, 10) = \log_3 \max\{3, \frac{10-2}{2} \} \approx 1.26 \) yields, which means we need at least distance 2.

One easily deduces that the number of vertices which can be distinguished by the given coloring in Proposition [3] depends on the structure of the graph. Especially it depends on the number and the distribution of the leaves. Of course there are enough cases where it is possible to distinguish considerably more vertices through a coloring than provided by our theorem (e.g. if a lot of vertices are fixed by the structure of the tree). But for a general result one has to consider the critical cases as in Example 4 where we have seen that the bound is tight.

---

\(^3\)Coloring the neighbors of a vertex in \( R_j \) we excluded the use of the color of this particular vertex.
We always considered the center of a finite tree as a fixed starting vertex in our Coloring Algorithm. But one also could take another vertex of the graph, that is fixed by all automorphisms. In some cases this may lead to colorings, that fix more vertices of the graph than in the case where we start the coloring algorithm in the center.

Acknowledgement. The authors acknowledges the support of the Austrian Science Fund (FWF) W1230 and of the Award 317689 of the Simons Foundation.

References

[1] M. O. Albertson and K. L. Collins, Symmetry breaking in graphs, *Electron. J. Combin.* 3 (1996), R18.

[2] L. Babai, Asymmetric trees with two prescribed valences, *Acta Math. Acad. Sci. Hung.* 29 (1977), 193–200.

[3] K.L. Collins and A.N. Trenk, The distinguishing chromatic number, *Electron. J. Combin.* 13 (2006) R16.

[4] R. Diestel, *Graph theory*, 3rd ed., Graduate Texts in Mathematics 173, Springer, Berlin, 2006.

[5] W. Imrich, S. Hüning, J. Kloas, H. Schreiber and T. Tucker Distinguishing graphs of maximum valence 3, submitted, https://arxiv.org/abs/1709.05797 (2017).

[6] W. Imrich, S. Klavžar and V. Trofimov, Distinguishing infinite graphs, *Electron. J. Combin.* 14 (2007), R36.

[7] S. Klavžar, T.-L. Wong and X. Zhu, Distinguishing labelings of group action on vector spaces and graphs, *J. Algebra* 303 (2006) 626–641.

[8] F. Lehner, Distinguishing graphs with intermediate growth, *Combinatorica* 36 (2016), 333–347.

[9] T. Tucker, Distinguishing maps, *Electron. J. Combin.* 18 (2011), P50.

[10] M.E. Watkins and X. Zhou, Distinguishability of locally finite trees, *Electron. J. Combin.* 14 (2007), R29.