A New Hydrodynamic Spherical Accretion Exact Solution and Its Quasi-spherical Perturbations

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Abstract

We present an exact $\gamma = 5/3$ spherical accretion solution that modifies the Bondi boundary condition of $\rho \rightarrow \text{constant}$ as $r \rightarrow \infty$ to $\rho \rightarrow 0$ as $r \rightarrow \infty$. This change allows for simple power-law solutions on the density and infall velocity fields, ranging from a cold empty freefall condition where pressure tends to zero, to a hot hydrostatic equilibrium limit with no infall velocity. As in the case of the Bondi solution, a maximum accretion rate appears. As in the $\gamma = 5/3$ case of the Bondi solution, no sonic radius appears, this time however, because the flow is always characterized by a constant Mach number. This number equals 1 for the case of the maximum accretion rate, diverges toward the cold empty state, and becomes subsonic toward the hydrostatic equilibrium limit. It can be shown that in the limit $r \rightarrow 0$, the Bondi solution tends to the new solution presented, extending the validity of the Bondi accretion value to cases where the accretion density profile does not remain at a fixed constant value out to infinity. We then explore small deviations from sphericity and the presence of angular momentum through an analytic perturbative analysis. Such perturbed solutions yield a rich phenomenology through density and velocity fields in terms of Legendre polynomials, which we begin to explore for simple angular velocity boundary conditions having zeros on the plane and pole. The new solution presented provides complementary physical insight into accretion problems in general.

Unified Astronomy Thesaurus concepts: Hydrodynamics (1963); Gravitation (661); Accretion (14); Bondi accretion (174)

1. Introduction

It is well established that the infall of gas in the central gravitational potential of accreting objects is the mechanism powering a vast range of astrophysical phenomena across many orders of magnitude in mass and scale (e.g., Hawley et al. 2015). From the jets originating around young stellar objects (YSOs), the powerful emissions of gamma-ray bursts (GRBs) to the active galactic nuclei harboring accreting supermassive black holes (SMBHs) in their centers, it is accretion that ultimately powers the observed outflows.

The study of the hydrodynamics of astrophysical accretion processes started with the seminal work of Bondi (1952), where spherical symmetry is assumed about a point mass, together with the boundary condition that a density tends to a constant value for large radii. This model, together with its relativistic extension by Michel (1972), has provided valuable insight into the magnitude of accretion rates onto central objects and the physical scalings of this process with the mass of the accretor and the densities and temperatures of the accreting material. The above, despite the lack of explicit analytical expressions for the infall velocity and density fields in the Bondi model.

In this paper, we modify the Bondi boundary conditions to a density profile that tends to zero at infinity, which for a $\gamma = 5/3$ equation of state and a Keplerian potential allows for a simple analytic power-law steady-state solution. Hence, this solution might be of particular relevance to systems where accretion density profiles do not quickly converge to constant values outward of the Bondi radius, but exhibit falling density profiles for large dynamical ranges.

In Section 2, we present a new exact analytic spherically symmetric accretion model for $\gamma = 5/3$, explore some of its scalings and implications, and present a comparison to the corresponding Bondi model and to a couple of observed accretion density profiles in nearby X-ray emitting elliptical galaxies from the sample of Pšek et al. (2022). Our new solution is then used in Section 3 as the basis for a first-order perturbative analysis preserving axial symmetry, but to first order on departures from sphericity with the polar angle and considering the inclusion of angular momentum, also to first order. Then, Section 4 briefly presents two particular examples of the latter, and Section 5 a first numerical exploration of the new spherical solution obtained and of its simplest polar angle deviations from sphericity. Section 6 presents our conclusions. Appendix A shows the fits of our new spherical accretion model to the full sample of observed accretion density profiles from Pšek et al. (2022), and finally, Appendix B presents a perturbative analysis for deviations from $\gamma = 5/3$ for the spherically symmetric model.

2. Spherical $\gamma = 5/3$ Accretion Model

We begin with the steady-state equations of conservation of mass and radial and angular momentum for gas distribution in the presence of a Newtonian gravitational potential produced by a point mass $M$. Assuming axial symmetry and using a spherical coordinate system with $\theta$ the angle between the positive vertical direction and the position vector $r$ and $\phi$ an
azimuthal angle we have
\[ \frac{1}{r^2} \frac{\partial (r^2 \rho V)}{\partial r} = -\frac{1}{r \sin \theta} \frac{\partial (\rho U \sin \theta)}{\partial \theta}, \quad (1) \]
\[ V \frac{\partial V}{\partial r} + U \frac{\partial V}{\partial \theta} - \frac{U^2}{r} - \frac{W^2}{r} = -\frac{1}{\rho r^2} \frac{\partial P}{\partial r} - \frac{GM}{r^2}, \quad (2) \]
\[ V \frac{\partial U}{\partial r} + U \frac{\partial U}{\partial \theta} + \frac{U V}{r} - \frac{W^2 \cot \theta}{r} = -\frac{1}{\rho r^2} \frac{\partial P}{\partial \theta}, \quad (3) \]
\[ V \frac{\partial W}{\partial r} + \frac{U \partial W}{r \partial \theta} + \frac{V W}{r} + \frac{U W \cot \theta}{r} = 0, \quad (4) \]
where \( \rho(r, \theta), P(r, \theta), V(r, \theta), U(r, \theta), W(r, \theta) \) are the gas density and pressure, and the \( r, \theta, \) and \( \phi \) velocities, respectively (e.g., Binney & Tremaine 1987). Assuming a barotropic equation of state \( P = K \rho^\gamma \) Equations (2) and (3) become
\[ V \frac{\partial V}{\partial r} + U \frac{\partial V}{\partial \theta} - \frac{U^2}{r} - \frac{W^2}{r} = -K \rho^{\gamma-1} \frac{\partial \rho}{\partial r} - \frac{GM}{r^2}, \quad (5) \]
\[ V \frac{\partial U}{\partial r} + U \frac{\partial U}{\partial \theta} + \frac{U V}{r} - \frac{W^2 \cot \theta}{r} = -K \rho^{\gamma-2} \frac{\partial \rho}{\partial \theta}, \quad (6) \]
We now write the above equations in dimensionless form introducing the variables \( \bar{\rho} = \rho/\bar{\rho}, \bar{V} = V/c, \bar{U} = U/c, \bar{V} = W/c, \) and \( R = r/c, \) where \( \bar{\rho} \) is a reference density at a certain point, \( c^2 = K \rho^{-1}, \) the speed of sound at this same reference point, and \( r = GM/c^2. \) Equations (1), (5), (6), and (4) now read as
\[ \frac{\partial (R^2 \bar{\rho} \bar{V})}{\partial R} = -R \frac{\partial (\sin \theta \phi \bar{U})}{\partial \theta}, \quad (7) \]
\[ R \frac{\partial \bar{V}}{\partial R} + U \frac{\partial \bar{V}}{\partial \theta} - \bar{U}^2 - \bar{W}^2 = -R \bar{\rho}^{-2} \frac{\partial \bar{\rho}}{\partial R} - \frac{1}{R}, \quad (8) \]
\[ R \frac{\partial \bar{U}}{\partial R} + U \frac{\partial \bar{U}}{\partial \theta} + \bar{V} \bar{U} - \bar{W}^2 \cot \theta = -R \bar{\rho}^{-2} \frac{\partial \bar{\rho}}{\partial \theta}, \quad (9) \]
\[ R \frac{\partial \bar{V}}{\partial \theta} + U \frac{\partial \bar{V}}{\partial \theta} + \bar{V} \bar{W} + \bar{U} \bar{W} \cot \theta = 0. \quad (10) \]
Equations (7)-(10) now define the general problem. The first step is to obtain a solution for the unperturbed state, which will be one of spherical accretion and zero angular momentum, \( \partial \rho/\partial \theta = \bar{U} = \bar{V} = 0, \) and hence a solution to
\[ \frac{d (R^2 \bar{\rho} \bar{V})}{d R} = 0, \quad (11) \]
\[ R \frac{d \bar{V}}{d R} = -R \bar{\rho}^{-2} \frac{d \bar{\rho}}{d R} - \frac{1}{R}, \quad (12) \]
where at this point we have assumed \( \bar{\rho} \) and \( \bar{V} \) are both functions of \( R \) alone. Although this problem is generally treated in terms of the Bondi (1952) solution, we change the Bondi boundary condition of \( \bar{\rho} = \) constant as \( R \rightarrow \infty \) for \( \bar{\rho} = 0 \) as \( R \rightarrow \infty. \) While the assumptions of strict spherical symmetry, zero angular momentum, and an adiabatic index of exactly \( \gamma = 5/3, \) will never be realized in any actual astrophysical setting, we introduce them here to obtain an exact solution, which can then be used to study perturbative modifications introduced by the inclusion of small amounts of angular momentum and angular dependences in velocity and density fields, as well as small variations about \( \gamma = 5/3. \)

We seek a new solution to Equations (11) and (12) by imposing an assumption of radial power laws for the velocity and density fields, and then try to find such functions guided by the structure of these two equations. The last term of Equation (12), the Newtonian point mass potential, fixes the other two terms of this same equation so that they must necessarily also scale with \( R^{-1}. \) For the case of the velocity term, the scaling imposed by the potential uniquely determines the velocity power-law scaling as \( \bar{V}(R) \propto R^{-1/2}. \) Going back to the mass conservation equation, Equation (11), the above velocity scaling now uniquely fixes the density scaling as \( \bar{\rho}(R) \propto R^{-3/2}. \) Now, the second term in Equation (12) will also be a power law, with a scaling of \( R^{3(1-\gamma)/2} \). This last will only correspond to the \( R^{-1} \) scaling imposed by the Newtonian potential in Equation (12) precisely and uniquely for the value of \( \gamma = 5/3 \) we are interested in. The velocity and density scalings found above are both negative and hence we see that they satisfy our required boundary conditions.

Thus, we see that imposing only spherical symmetry, steady-state accretion, a Newtonian point mass potential and power-law solutions for both the density and velocity fields, uniquely determines \( \bar{\rho}(R) \propto R^{-3/2}, \) \( \bar{V}(R) \propto R^{-1/2}, \) and \( \gamma = 5/3. \) No other power-law solution to the steady-state spherically symmetric accretion problem exists for a Keplerian potential. Under this scheme, \( \gamma = 5/3 \) becomes a constraint, which happens to correspond to a value of the adiabatic index often considered in a variety of astrophysical scenarios.

Our new solution is hence clearly less general than the Bondi solution with respect to the thermodynamic range over which it is valid; while a broad range of adiabatic indices is consistent with the Bondi framework, the solution presented here is only valid for \( \gamma = 5/3, \) although perturbative deviations from this value can be formally treated, see Appendix B. The ansatz of power-law solutions introduced follows from analytic simplicity considerations. However, the particular unique power-law scalings found are determined by the physics of the assumptions introduced, having taken a stationary adiabatic spherically symmetric accretion and a Keplerian potential.

Therefore, Equations (11) and (12) allow a closed analytic solution for the natural choice of \( \gamma = 5/3, \) using only radial power laws described by
\[ \bar{V} = \bar{V}_0 R^{-1/2}, \quad (13) \]
\[ \bar{\rho} = \bar{\rho}_0 R^{-3/2}, \quad (14) \]
where \( \bar{V}_0 \) and \( \bar{\rho}_0 \) are two constants satisfying Equation (12):
\[ \bar{V}_0^2 = 2 - 3 \bar{\rho}_0^{2/3}. \quad (15) \]
Thus, the full boundary conditions at infinity are for the fluid at rest with zero density and zero speed of sound. This fixes the Bernoulli equation with constant \( = 0, \) as in the Bondi solution. It is clear that Equation (11) can be integrated directly to yield the dimensionless mass accretion rate as
\[ \dot{M} = 4\pi \bar{\rho}_0 \bar{V}_0 \phi. \quad (16) \]

Equation (4) in Bondi (1952) is the Bernoulli equation for the problem, a first integral of the momentum equation solved for nonzero pressure and density at infinity. Here we do not make use of the Bernoulli relation, nor impose the existence of a sonic radius (or its nonexistence a priori either), but rather solve the conservation equations directly. Therefore, the
solution of Equations (13) and (14) does not lie within the Bondi framework and constitutes a new solution.\(^3\)

We see from Equation (15) that the solution will only exist for the interval \(0 < \rho_0^{2/3} < 2/3\), over which the \(\gamma_0\) constant determines the amplitude of the inward velocity flow, and hence implies that this constant is negative, and will vary within the range of \(2 > \gamma_0 > 0\). As \(\rho_0 \to 0\) pressure tends to zero, we reach an empty state in Keplerian freefall. As \(\rho_0^{2/3} \to 2/3\), we approach a dense hydrostatic equilibrium configuration where the infall velocity tends to zero. The accretion rate is clearly zero at these two limits, and has a maximum at some intermediary value of \(\rho_0^{2/3}\) which is hence seen as the sole parameter of the spherical solution. We can use Equation (16) to write Equation (15) in terms of \(\rho_0^{2/3}\) and \(M\) only as

\[
M^2 = (4\pi \rho_0^{2/3})^2 (2 - 3\rho_0^{2/3}). \tag{17}
\]

The above equation can now be differentiated \(w.r.t.\) \(\rho_0\) to obtain the maximum value of the accretion rate. This will occur at \(\rho_0^{2/3} = \gamma_0^{3/2} = 1/2\) at a value of \(M = \pi\). Figure 1 provides a plot of \(M\) as a function of \(\rho_0\) in the range of interest. In physical units we obtain

\[
M_{c,ph} = \frac{\pi D}{\rho_0^{2/3}} (GM)^2, \tag{18}
\]

which can be written in terms of the barotropic constant of the problem \(K\) as

\[
\frac{\dot{M}}{\rho_0^{2/3}} = \pi (GM)^2 \left(\frac{3}{5K}\right)^{3/2}. \tag{19}
\]

Thus, we can see that in terms of physical units, the problem is fully determined by the constant \(K\), which can be fixed by specifying a pair of density and speed of sound values at any given radius.

Notice that the framing of the solution of Equations (13) and (14) in the context of accretion determines the sign of the \(\gamma_0\) constant as negative. However, as can be seen from Equation (15), this sign could just as well be positive and the resulting ejection solution will still solve exactly the conservation equations under the assumptions of spherical symmetry and \(\gamma = 5/3\). Such a solution could represent the asymptotic consequence of a wind expanding in a vacuum and clearly also deserves attention and further study. Still, in this first exploration of the problem we choose to concentrate on accretion scenarios and defer the study of the ejection branch of the solution, its temporal, thermodynamic, and nonspherical perturbations to future works.

We complete the study of the spherically symmetric adiabatic accretion solution with an evaluation of the Mach number of the flow, which can be derived by writing \(V\) from Equation (13) in physical units by multiplying by \(\bar{c}\), and dividing by the local physical speed of sound \(c = (5K/3)^{1/3}\rho_0^{-1/3}\)

with \(\rho\) written as \(\rho = \rho(\bar{R}) \times \bar{p}\) we obtain

\[
M = \frac{\rho_0^{2/3} - 1/3}{V_0}. \tag{20}
\]

We see that for the maximum accretion rate at \(\rho_0^{2/3} = V_0^{1/3} = 1/2\), the Mach number of the flow is precisely \(M = 1\). In going toward the hotter and denser subsonic configurations toward the \(\rho_0 = 2/3\) of hydrostatic equilibrium, the Mach number drops gradually to zero, while for less dense supersonic cold models with \(\rho_0^{2/3} < 1/2\), the Mach number gradually diverges on approaching \(\rho_0^{2/3} = 0\) as the speed of sound goes to zero. A plot of this is given in Figure 2. While in the Bondi solution, a sonic radius exists for \(\gamma = 5/3\), and the monoatomic adiabatic value is a singular point where the sonic radius goes to zero, in the power-law solutions presented here there is no sonic radius as \(M\) becomes a constant with radius.

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\(^3\) When going through Bondi (1952) one must be cautious of the typos in Equations (12) and (13) which should read as \(\gamma = (\gamma - 1)(\lambda x^2)^{-1/2}\) and \(\zeta = (\lambda x^2 n)^{-1/2}\), respectively.
with all sub-maximum accretion models being either supersonic or subsonic at all radii, depending on whether the value of $\varrho_0$ is chosen above or below the critical one corresponding to maximal accretion at $\mathcal{M} = 1$. Hence, for the new solution presented the flow is never transonic and no shocks will develop.

The spherical accretion solution of Equations (13) and (14) could arise from the inward propagation of a rarefaction wave into gas initially at rest with a density profile decreasing to zero at infinity. This has to be explored through both analytical and numerical dynamical studies so as to assess the initial condition parameter space in both the velocity and density fields to constrain which regions might result in the $\gamma = 5/3$ constant Mach number solution presented, as a consequence of the dynamics of the propagation of the rarefaction wave, the details of the initial conditions, or a combination of both. The above is beyond the scope of the first presentation of the new spherical accretion presented here and will be treated in subsequent studies.

That the solution presented here does not form part of the Bondi framework becomes evident since we have explicitly shown exact non-transonic solutions to the spherically symmetric accretion problem with radially constant values of $\mathcal{M}$. As the local speed of sound is given by $(dP/d\varrho)^{1/2}$, and given the barotropic equation of state used of $P \propto \varrho^\gamma$, the speed of sound will scale as $c^2 \propto \varrho^{-1}$, and for $\gamma = 5/3$ we get $c \propto \varrho^{1/3}$. For the exact solution of the conservation equations presented in Equations (14) and (14), as $\varrho \propto R^{-3/2}$, we get $c \propto R^{-1/2}$, exactly the same radial scaling of the velocity flow in Equation (13), forcing a radially constant Mach number. Thus, the Mach number of the flow is a constant with radius, at a value fixed once $\varrho_0$ is chosen, which in turn determines $\varrho_0$ through Equation (15).

Notice that we have in no way either imposed at the onset or forced the lack of a sonic radius at any point in the development leading to the new solution found. This unexpected feature of our solution follows from the new exact power-law solutions to both the mass conservation and radial Euler equations we found. We do not know if this particular feature holds only for the solution we have found, or if there might exist other more complex non-power-law solutions where also no sonic radius exists and the Mach number appears as a radial constant. It is entirely possible that just as we have found a simple exact spherical accretion solution where no sonic radius appears and $\mathcal{M}$ is constant, there might exist more complex non-power-law solutions where this feature is maintained. However, as shown and discussed in Appendix B, even perturbative departures from $\gamma = 5/3$ result in unequal changes to the local speed of sound and the flow velocity scalings, implying the presence of a sonic radius for even the slightest deviation from $\gamma = 5/3$.

We can now compare our $\mathcal{M} = 1$ maximum accretion rate solution explicitly to the maximum accretion rate of the Bondi solution (usually referred to simply as the Bondi solution) for $\gamma = 5/3$, where the density field satisfies the following implicit relation:

$$ (1/2)^4 + 3R^4 \varrho_B^{8/3} = 3R^4 \varrho_B^2 + 2R^3 \varrho_B^2, $$

which is Equation (15) in Bondi (1952) for $\gamma = 5/3$, where $\varrho_B$ is the dimensionless density in units of the density at infinity assumed by Bondi. Notice that substituting an $R^{-3/2}$ density scaling in the above equation will result in a divergence in the first term on the right-hand side, showing once again that the solution of Equations (13)–(15) lies outside of the Bondi framework.

It is clear from Equation (21) that as $R \to 0$, the first term on the right-hand side can be dropped, as the second term on the right-hand side will dominate. The three remaining terms must hence balance, leading to $\varrho_B \propto R^{-3/2}$. Mass conservation through Equation (11) now implies $\varrho_B \propto R^{-1/2}$, fulfilling the match to the scalings of Equations (13) and (14).

Introducing the mass conservation equation for the Bondi solution for $\gamma = 5/3$ of $R^2 \varrho_B \varrho_0 = 1/4$, allows evaluating the scaling constants of the small radius asymptote of the Bondi solution as $(1/2)^3/2$ and $(1/2)^1/2$ for the density and velocity field respectively, exactly the values of our new solution. Hence, we see that the small radius limit of the $\gamma = 5/3$ Bondi solution exactly corresponds to the solution we have presented.

A full comparison of the density and velocity fields of the $\gamma = 5/3$ Bondi solution to our new one is presented in Figure 3, showing clearly the convergence proven above for small radii. It is also evident from Figure 3, unavoidable since the boundary conditions at infinity for our solution and the Bondi case are very different, that both solutions strongly diverge as the radius grows larger than the Bondi radius. Given that the accretion rate is a constant with radius, we see that the well-established scalings of the Bondi accretion rate with density and the speed of sound, do not require the formal validity of the Bondi density field, which remains at a finite value out to infinity, and are in fact realized also for the solution presented, where the density falls to zero as $R^{-3/2}$.

Notice that the convergence proven for the maximum accretion classical Bondi solution at $\gamma = 5/3$ to the maximum accretion rate of our new $\gamma = 5/3$ solution applies only for the limit as $R \to 0$. Indeed, it is crucial to appreciate that the Bondi solution implies more than just a scaling between a reference density and the speed of sound and a resulting accretion rate; it implies also particular functional forms for the density and velocity profiles of the accreting material. In particular, the density profile of the Bondi solutions is such that a constant asymptotic value is rapidly approached outward of the reference Bondi radius. This might be a reasonable description of some actual astrophysical situations, but not of others. For example, the observed accretion density profiles in X-ray luminous elliptical galaxies reported by Allen et al. (2006) show no convergence toward finite values out to many tens and in some cases even hundreds of Bondi radii. In fact, the abovementioned accretion density profiles are clearly falling consistently as the radius increases, out to the last observed point, much more in line with the new solution presented here where the accretion density profile tends to zero at large radii, than with the expectations of the classical Bondi models.

In terms of a quantitative comparison of the new spherically symmetric accretion solution presented here and observations, the works of Wong et al. (2014), Russell et al. (2015), and Runge & Walker (2021) inferring de-projected hot X-ray accretion density profiles in NGC 3115, M81, and NGC 1600, respectively, are relevant. M81 and NGC 1600 show substantial active galactic nuclei (AGN) activity tending to flatten the observed accretion density profiles outside of the Bondi radius (Runge & Walker 2021), which have power-law slopes of $-0.41 \pm 0.04$ and $-0.65 \pm 0.04$, respectively, for radial ranges larger than the Bondi radii. Again, accretion density profiles do not tend to a constant out to many tens of
their Bondi radii, with the expectation that in the absence of AGN activity, these two density profiles would be steeper. Finally, NGC 3115 shows a decreasing power-law density profile beyond the Bondi radius with a slope of $-1.34^{+0.25}_{-0.2}$, consistent with the expectations of the new model presented here of $\rho \propto R^{-3/2}$. Interestingly, this last galaxy is a quiescent system with no AGN activity to perturb the accretion density profile.

Similarly, accretion density profiles for hot X-ray emitting gas using Chandra observations of SMBHs for a large sample of 20 nearby galaxies are reported by Plšek et al. (2022), all of which show clearly falling radial profiles beyond the Bondi radius, with power-law slopes close to the expectations of the new model presented here. Figure 4 shows the inferred volumetric electron density profiles from Plšek et al. (2022) for NGC 1316 and NGC 4649, points with error bars, together with $R^{-3/2}$ fits to the observed profiles. In all cases for the innermost points, the reported relative radial errors are 100%, for which reason the innermost points have been ignored in the fits.

It is clear that the fits are very accurate representations of the data over the close to two orders of magnitudes in radius...
covered by the observations. Also, the Bondi radii of the two galaxies shown in the figure are $7.4^{+1.3}_{-1.2}$ pc for NGC 1316 and $111 \pm 19$ pc for NGC 4649, as inferred by Plšek et al. (2022).

Thus, a Bondi fit to the observed profiles would appear close to a horizontal line for the case of NGC 1316 over the radial range covered by the data, and beyond 1 kpc a solution rapidly convergent to a constant value for the case of NGC 4649. The rapid flattening of the Bondi profile outward of the Bondi radius, as can be seen in Figure 3, shows that once constrained by the inferred values of the Bondi radii, the two cases shown above will yield significantly poorer fits to classical Bondi profiles than to the expected $R^{-3/2}$ density scalings of the solution in Equations (13) and (14).

Appendix A presents corresponding fits to the remaining 18 galaxies in Plšek et al. (2022), showing in most cases excellent agreement to the density radial scaling of our new solution of $R^{-3/2}$. Even in cases where this scaling is not an excellent fit, it remains a much better representation than the essentially flat expectations of the classical Bondi solution, for the radial ranges covered. It appears that for hot accretion density profiles of central galactic SMBHs, observationally inferred profiles are much more consistent with our new solution than with the classical Bondi one. Indeed, we are not aware of even a single astrophysical hot accretion density profile that is observed to flatten to a constant value outward of the Bondi radius.

This, however, does not necessarily invalidate mass accretion rates calculated under the Bondi model for the above observed profiles. The reason being that although the Bondi mass accretion rate is always proportional to $\rho_\infty c_\infty^2$, in the particular case of $\gamma = 5/3$ the density at infinity is no longer a critical parameter for the accretion rate in the Bondi case. Since as $c^2 = dP/d\rho$, it is easy to see that the Bondi accretion rate will in general scale with $\rho^{2/3}c^{-\gamma/2}$, which for the particular case of $\gamma = 5/3$ reduces to a constant given by the entropy of the gas (as also happens in our case, see Equation (19)), proportional to $\log(P/\rho^\gamma)$. Thus, for $\gamma = 5/3$ the density at infinity for the Bondi model ceases to be a critical parameter and the accretion rate remains constant regardless of the particular value of the density at infinity, provided the gas remains at constant entropy. Hence, in concordance with the discussion following Equation (21), mass accretion rates generally found in the literature, which are calculated using density and speed of sound estimates at the Bondi radius, are consistent with the solution presented here (which converges to the Bondi one for small radii), even if the large-scale accretion density profiles do not correspond to the detailed Bondi density accretion profile.

It is clear that the comparison presented above is approximate, in as much as in the physical systems described a number of effects beyond the simplifying assumptions giving rise to the exact solution of Equations (13) and (14) are relevant. Some of these can be dealt with through careful treatment of the observations used, such as the presence of nonthermal components in the X-ray emitting gas arising from low-mass X-ray binaries or AGN jet activity. Indeed, all the references mentioned above include careful spectral analysis to identify only the thermal component of the X-ray-emitting gas when producing de-projected density profiles. Toward the central regions, in some cases thermal instability might set in, leading to a multiphase gas structure, as evident through emission line diagnostics. About two-thirds of the galaxies in the Plšek et al. (2022) sample show such evidence, or the central region of NGC 1600 as studied by Runge & Walker (2021) where cold and hot phases are directly inferred through spectroscopic fitting to the Chandra data used. It is reassuring that the data discussed clearly show density profiles consistent with the solution of Equations (14) substantially beyond these inner regions. Similarly, other physical processes at play in the actual observed cases include the nonadiabatic character of the gas flows. Fortunately, as shown by, e.g., Sun & Yang (2021), given the strong density dependences of both heating (from irradiation by a central X-ray emission) and cooling rates, this lack of adiabaticity will again be confined to the very central regions, typically at parsec scales much smaller than the radial ranges fitted in Figure 4 and Appendix A.

As with the cases discussed above, many other of the more complex physical processes likely at play in a real situation will naturally intensify toward smaller radii, e.g., general relativistic departures from the Newtonian potential assumed in the solution presented here. It is, therefore, encouraging that the generally good agreement of the density profile of Equation (14), and indeed its much better agreement with the data than the close to horizontal expectation of a classical Bondi profile for the Bondi radii inferred in the cases presented, applies to the large-scale regions of the observations discussed.

We are aware that the comparisons in density profiles presented here and in Appendix A cannot be regarded as any more than approximate, given the existence of a range of physical processes in the actual cases that have not been included in the idealized model presented. We do not intend the comparisons presented to be proof of the accurate validity of the model presented, although the quite good fits obtained in many cases (see Appendix A), particularly beyond the more problematic central regions, and the fact that these fits are clearly much better than a similarly approximate Bondi profile, give us confidence that the solution presented captures some of the physics at play, to a greater degree than what happens with a classical Bondi profile, which flattens to a constant density on crossing the Bondi radius.

Regarding a first quantitative comparison to numerical experiments, Sun & Yang (2021) model the hydrodynamical accretion of optically transparent hot gas onto a central black hole for low luminosity AGNs, assuming spherical symmetry and an adiabatic index of $\gamma = 5/3$, and obtain density profiles consistent with the expectations of the new model presented here, see their Figure 7. Along these lines, in Section 5 we present the first numerical simulations of the new spherical solution presented here, showing it to be stable.

The original idea of Bondi was to model a small star accreting material from a large cloud of constant-density material, formally taken as infinite in size. Thus, the star presents only a small local perturbation on the structure of the cloud, and the accretion density profile rapidly converges to the asymptotic constant density of the infinite cloud. As can be seen from Figure 3, the Bondi accretion density profile changes qualitatively on crossing the Bondi radius from an inner $\rho \propto R^{-3/2}$ region to an outer one with a fast convergence to the flat asymptote, after a few tens of Bondi radii, further changes in the density profile are below the percent level. The accretion of hot gas by an SMBH is a very distinct situation, where no infinite gas cloud of constant density is evident. The presence of the central black hole is in no way a minor perturbation on the structure of a constant-density distribution and there is no large radius constant-density distribution to be seen. Indeed, it is clear that in the absence of the central black hole, the gas
distribution would be significantly different. Thus, the accretion density profile is determined by the central potential out to large radii, and this density profile continues to fall outside of the Bondi radius by many orders of magnitude out to a large radial dynamical range and down to densities smaller than the values at the Bondi radius by many orders of magnitude. The many observational studies quoted above make it clear that the mass at the Bondi radius by many orders of magnitude out to a large range to define reality than the classical Bondi model, at least for the cases of hot accretion onto SMBHs.

We complete the description of the spherically symmetric solution of Equations (13)–(15) in Appendix B at the end of the paper where we show that with the exception of the region about \( M = 1 \), small deviations about \( \gamma = 5/3 \) result only in small changes in the power-law indices describing the steady-state spherical solution of Equations (13) and (15).

### 3. Deviations from Spherical Symmetry

We now use the solution derived in the previous section as a ground state for a first-order perturbation analysis of deviations from sphericity in the density distribution, angular momentum, and velocity fields. Given the simplicity of the accretion solution of Equations (13)–(15), this development will be much simpler than the corresponding exploration of nonspherical deviations for the Bondi model (e.g., Foglizzo & Ruffert 1997; Foglizzo 2001).

We shall return to the system of Equations (7)–(10), assuming the generic solution

\[
\mathcal{V} = \mathcal{V}_0 R^{-1/2} + \epsilon \mathcal{V}(R) \mathcal{V}_0(\theta),
\]

\[
\mathcal{U} = \epsilon \mathcal{U}(R) \mathcal{U}_0(\theta),
\]

\[
\mathcal{W} = \epsilon \mathcal{W}(R) \mathcal{W}_0(\theta),
\]

\[
\varphi = \varphi_0 R^{-3/2} (1 + \varphi(R) \varphi_0(\theta)),
\]

where \( \epsilon < 1 \), a separation of variables ansatz for the nonspherical perturbation (e.g., Bender & Orszag 1978). The assumption of a barotropic adiabatic \( \gamma = 5/3 \) equation of state would be invalidated by the presence of shocks, viscosity, dissipation, or crossing streamlines, neither of which are present in our model or appear in any of the solutions found. The lack of spherical symmetry or the number of spatial dimensions defining the flow solution implies no thermodynamic constraints in and of themselves, and hence in no way affect the validity of the \( \gamma = 5/3 \) assumption. On this, see, e.g., Chapter 4 in Clarke & Carswell (2007) or Gruzinov (2022) for a recent example of the use of \( \gamma = 5/3 \) in the study of astrophysical accretion outside of spherical symmetry, or Mancino et al. (2022), who consider \( \gamma = 5/3 \) while studying spherical accretion onto a black hole.

When introducing the above ansatz into Equations (7)–(10), the \( \varphi^{-1/2} \) nonlinearities in Equations (8) and (9) will be treated through a series expansion keeping only terms to first order in \( \epsilon \). The order \( \epsilon^0 \) terms will collectively sum to zero in the four equations, as they satisfy the unperturbed spherical system, while all terms to order \( \epsilon^2 \) and higher appearing in Equations (7)–(10) will be dismissed. The first point to notice is that the azimuthal velocity, \( \mathcal{V}_0 \), does not appear in the mass conservation equation, Equation (7), and enters only quadratically in Equations (8) and (9). Hence, to first order in \( \epsilon \), Equations (7)–(9) form a system of three equations in the three perturbed nonspherical components of the radial and polar velocity and density fields. Then, Equation (10) constrains \( \mathcal{W} \).

We shall now solve the system of Equations (7)–(9) for the perturbations in \( \mathcal{V}(R) \mathcal{V}_0(\theta), \mathcal{U}(R) \mathcal{U}_0(\theta) \) and \( \varphi(R) \varphi_0(\theta) \), leaving the simpler case of the constraint on \( \mathcal{W} \) through Equation (10) for later.

Equation (9) yields

\[
\epsilon \mathcal{V}_0 \mathcal{U}_0 R^{1/2} \frac{d \mathcal{U}_R}{d R} + \epsilon \mathcal{V}_0 \mathcal{U}_R R^{-1/2} = -\epsilon \theta^{2/3} R^{-1/2} \frac{d \varphi}{d \theta},
\]

which can be simplified to

\[
\frac{R^{1/2}}{\mathcal{U}_R} \frac{d (\mathcal{U}_R)}{d R} = -\frac{\theta^{2/3}}{\mathcal{V}_L \mathcal{U}_0} \frac{d \varphi}{d \theta}.
\]

The above equation yields two conditions:

\[
C_3 \varphi_R = R^{1/2} \frac{d (\mathcal{U}_R)}{d R},
\]

and

\[
\theta^{2/3} \frac{d \varphi}{d \theta} = -C_3 \mathcal{V}_0 \mathcal{U}_R ,
\]

where \( C_3 \) is a separation constant.

Equation (8) yields

\[
\epsilon \mathcal{V}_0 \mathcal{V}_0 \left( \frac{R^{1/2}}{d \mathcal{V}_R} \frac{d \mathcal{V}_R}{d R} - \frac{\mathcal{V}_R}{2 R^{1/2}} \right) = -\epsilon \theta^{2/3} \frac{d (\varphi R / R^{3/2})}{d R}.
\]

which can be simplified to

\[
\mathcal{V}_0 \mathcal{V}_0 \frac{d (\mathcal{V}_R / R^{1/2})}{d R} = -\theta^{2/3} \frac{d (\varphi R / R)}{d R}.
\]

The above equation yields two conditions:

\[
\frac{d (\mathcal{V}_R / R^{1/2})}{d R} = C_2 \frac{d (\varphi R / R)}{d R},
\]

and

\[
\varphi_R = \left( \frac{C_2 \mathcal{V}_0}{\theta^{2/3}} \right) \mathcal{V}_R,
\]

where \( C_2 \) is a separation constant.

Finally, Equation (7) yields

\[
\epsilon \mathcal{V}_0 \mathcal{V}_0 \frac{d (\mathcal{V}_R / R^{1/2})}{d R} + \epsilon \mathcal{V}_0 \mathcal{V}_0 \frac{d \varphi_R}{d R} = -\epsilon \varphi_0 \mathcal{U}_R \frac{d (\mathcal{U}_R \sin \theta)}{R^{1/2} \sin \theta} \frac{d \theta}{d \theta}.
\]

Although the above equation is not immediately susceptible to the separation of variables, use of Equation (33) allows replacing \( \mathcal{V}_0 \) for \( \varphi_R \) and hence yields the following two conditions:

\[
\mathcal{V}_0 \frac{d \varphi_R}{d R} = \frac{\theta^{2/3}}{C_2 \mathcal{V}_0} \frac{d (\mathcal{V}_R / R^{1/2})}{d R} = C_2 \mathcal{U}_R \frac{d \mathcal{U}_R}{d R} = \frac{C_2 \mathcal{U}_R}{R^{1/2}},
\]

and

\[
\frac{d \mathcal{U}_R}{d \theta} + \cot \theta \mathcal{U}_R = -C_1 \mathcal{U}_\theta ,
\]
where $C_1$ is a separation constant. We are hence left with a purely radial system in exact derivatives of Equations (28), (32), and (35), and a purely angular system in exact angular derivatives of Equations (29), (33), and (36), which we shall deal with first. Equation (29) shall be used to replace $\mathcal{U}_0$ in Equation (36) for derivatives of $\vartheta_0$, yielding an equation in this last variable alone:

$$\frac{d^2 \vartheta_0}{d\theta^2} + \cot \theta \frac{d \vartheta_0}{d\theta} - \left( \frac{C_1 C_3 V_0}{\vartheta_0^{2/3}} \right) \vartheta_0 = 0. \quad (37)$$

This last equation has as solutions Legendre polynomials of $\cos \theta$ of the first and second kinds, and hence opens a very rich phenomenology capable of modeling a wide variety of density configurations, and given the constraints linking $\mathcal{U}_0$ and $V_0$ to $\vartheta_0$, an extensive range of flow patterns. Considering, for example, odd polynomials, or the addition of odd and even polynomials, will yield asymmetric velocity fields. This last might even suggest the existence of an intrinsic component to observed morphological and energetic jet asymmetries such as might even suggest the existence of an intrinsic component to fields. This last equation has as solutions Legendre polynomials of $\cos \theta$ of the first and second kinds, and hence opens a very rich phenomenology capable of modeling a wide variety of density configurations, and given the constraints linking $\mathcal{U}_0$ and $V_0$ to $\vartheta_0$, an extensive range of flow patterns. Considering, for example, odd polynomials, or the addition of odd and even polynomials, will yield asymmetric velocity fields. This last might even suggest the existence of an intrinsic component to observed morphological and energetic jet asymmetries such as the ones often observed in FR II systems (e.g., Hardcastle et al. (1999) or Liu et al. (2020)). As the Legendre polynomials form an orthogonal basis, any given axisymmetric flow configuration can be reproduced with the velocity solutions presented here, with the model then yielding a corresponding self-consistent density configuration. Similarly, any desired density configuration within the limits of the quasi-spherical approximation can be obtained as a sum of solutions to Equation (37), with the model then yielding a corresponding self-consistent flow configuration.

In this first exploration of the problem, and guided by the simple bipolar infall/outflow geometries observed in the numerical experiments of Aguayo-Ortiz et al. (2019), Tejeda et al. (2020), and Waters et al. (2020), we choose to consider here only solutions of the type

$$-C_1 C_3 V_0 \vartheta_0^{2/3} = 6, \quad (38)$$

which will yield a simple geometry with $\mathcal{U}_0$ satisfying the boundary conditions $\mathcal{U}_0(0) = \mathcal{U}_0(\pi/2) = 0$,

$$\vartheta_0 = \vartheta_1 (3 \cos^2 \theta - 1), \quad (39)$$

$$V_0 = -\vartheta_1 \left( \frac{\vartheta_0^{2/3}}{C_2 V_0} \right) (3 \cos^2 \theta - 1), \quad (40)$$

$$\mathcal{U}_0 = \vartheta_1 \left( \frac{6 \vartheta_0^{2/3}}{C_3 V_0} \right) \cos \theta \sin \theta, \quad (41)$$

which solve the angular system in terms of an amplitude constant $\vartheta_1$. We stress that the particular choice of Legendre polynomial adopted at this point is arbitrary and is motivated merely by mathematical convenience and the qualitative correspondence of the resulting velocity flow patterns to those of the previous numerical experiments mentioned above.

We now turn to the radial system of Equations (28), (32), and (35), which can be solved by taking radial power laws for the unknown functions

$$V_R = R^{\alpha}, \quad U_R = R^\beta, \quad \vartheta_R = R^\delta,$$  

where $\alpha$, $\beta$, and $\delta$ are real. Equation (28) becomes

$$(\beta + 1) R^\beta = C_3 R^{\delta - 1/2}, \quad (43)$$

which yields the conditions

$$\beta = \delta - 1/2, \quad (44)$$

and

$$\beta + 1 = C_3. \quad (45)$$

Equation (32) becomes

$$(\alpha - 1/2) R^{\alpha - 3/2} = C_2(\delta - 1) R^{\delta - 2}, \quad (46)$$

which yields the conditions

$$\alpha = \delta - 1/2 \Rightarrow \alpha = \beta, \quad (47)$$

and

$$C_2 = 1. \quad (48)$$

Lastly, Equation (35) becomes

$$\delta V_0 R^{\delta - 1} - \frac{\vartheta_0^{2/3}}{V_0} (\alpha + 1/2) R^{\alpha - 1/2} = C_1 R^{\delta - 1/2} \quad (49)$$

which yields constraints on the power indices consistent with those obtained previously, and

$$\delta V_0 - \frac{\vartheta_0^{2/3}}{V_0} (\alpha + 1/2) = C_4. \quad (50)$$

We can now solve for the power indices in terms of the unique free parameter of the unperturbed solution $\vartheta_0$, by replacing $\alpha$ for $\delta - 1/2$ and $C_1$ for its value through condition (38) into the above equation to obtain an equation in $\delta$ alone:

$$\delta (\delta + 1/2) \left( 1 - \frac{\vartheta_0^{2/3}}{V_0^{2/3}} \right) = -6 \frac{\vartheta_0^{2/3}}{V_0^{2/3}}, \quad (51)$$

which can be written as

$$\delta (\delta + 1/2) = A, \quad (52)$$

where $A = 3 \vartheta_0^{2/3} / (2 \vartheta_0^{2/3} - 1)$, having the solutions

$$\delta = -\frac{1}{4} \pm \left( \frac{1}{16} + A \right)^{1/2}, \quad (53)$$

which completes the full solution in terms of the unique parameter of the unperturbed model $\vartheta_0$ as

$$\vartheta = \vartheta_0 R^{3/2} (1 + \epsilon \vartheta_1 R^\beta [3 \cos^2 \theta - 1]), \quad (54)$$

$$V = \vartheta_0 R^{-1/2} - \epsilon \vartheta_1 R^{-1/2} \left( \frac{\vartheta_0^{2/3}}{V_0} \right) [3 \cos^2 \theta - 1], \quad (55)$$

$$U = \epsilon \vartheta_1 R^{-1/2} \left( \frac{6 \vartheta_0^{2/3}}{[\delta + 1/2] V_0} \right) \cos \theta \sin \theta. \quad (56)$$

Equation (53) has real roots only for the subsonic regime of $\vartheta_0^{2/3} > 1/2$, where one root is larger than zero, and the small interval $0 < \vartheta_0^{2/3} < 0.02$, where both roots are smaller than zero. The above ranges of $\vartheta_0$ hence define validity limits for the perturbative analysis presented for the radial system. Figure 5 provides a plot of the $\alpha$ radial power-law index as a function of $\vartheta_0^{2/3}$ for the “+” sign in Equation (53), for the subsonic region of parameter space. We see a mild variation, with the exception of a highly localized divergence toward $\mathcal{M} = 1$. Notice that for $\alpha > -1/2$, $\delta > 0$, and hence the spherically symmetric components of the velocity and density fields dominate over
one, the limit of the validity of the mathematical development is reached and there is no certainty that the solution remains a good description of the physical situation being modeled.

However, as discussed in the following section, it is interesting that within the range of the validity of the density series expansion we see that the velocity field remains a qualitative match to the numerical velocity fields of Aguayo-Ortiz et al. (2019) and Waters et al. (2020) even after the nonspherical term in the velocity field dominates over the spherical one. Indeed, a first series of numerical experiments presented in Section 5 confirms that the convergence steady-state solution to initial conditions given by the equatorial infall and polar outflow modes of Equations (53)–(56), preserves the qualitative structure of the flow, although quantitative changes appear in the details as one goes beyond the validity regime of the linear approximation.

For the positive root of $\delta$, toward the hydrostatic equilibrium limit, small $\gamma_0$ values can yield an outer region where the second term in Equation (55) dominates over the radial infall of the first term in this equation (as $\gamma_0 < 0$), and lead to the development of a bipolar outflow highly reminiscent of the results of the numerical studies mentioned above.

Regarding the physical conditions that might be responsible for the boundary conditions required for any of the many options allowed by the model, any small amplitude, large angular scale departure from spherical symmetry in density, and/or velocity could be described by a sum of Legendre functions solving Equation (37). We chose not to focus on any specific astrophysical situation as a way to stress the generality of the model presented, which we believe can be of relevance over a variety of scenarios. However, in the context of the hot X-ray gas accreting onto SMBHs discussed in Section 2 and Appendix A, the presence of a small degree of angular momentum would imply the appearance of large angular scale, small amplitude nonspherical modes in the accreting material. This would become particularly relevant toward the innermost regions, probably playing a part in the initial jet launching dynamics, as suggested by the equatorial infall/polar outflow solutions discussed in the following section.

Since the mass accretion rate is not modified, the material in the polar outflows is hence supplied by an infall rate above the one of the spherically symmetric solution. As seen also in the choked accretion results of Aguayo-Ortiz et al. (2019) and Tejeda et al. (2020), it is the non-spherically symmetric accretion of material above the spherically symmetric accretion rate that results in the bipolar outflows obtained. Thus, our present results qualitatively maintain the choked accretion phenomenology described in the two references above.

4. Particular Examples

We now present a particular example of a model close to the hydrostatic equilibrium limit of $\delta^2/3 = 2/3$, taking $\delta^2/3 = 0.66533$, which results in $\gamma_0 = 0.004$ and $\delta = 2.22$, $\alpha = 1.72$, for the “+” sign in Equation (53).

A plot of the velocity flow of this model is shown in the left panel of Figure 6, superimposed on the logarithm of the density field shown as a color scale plot. The equatorial infall and polar outflow structure are evident at large radii, together with the highly spherical configuration of the density field. This last, however, is not exactly spherical, it is the slight prolateness present that sources the non-radial terms in radial and angular velocities.
The velocity flow shown in the left panel in Figure 6 is qualitatively equivalent to those that arise from the numerical simulations of Aguayo-Ortiz et al. (2019), Waters et al. (2020), and Tejeda et al. (2020), showing an inner region tending to spherical accretion in both the density and velocity fields, and a more external equatorial infall and polar outflow one. A first quantitative comparison of our analytic solution to numerical simulations appears in Section 5. In the case shown here, a small value of $\nu_0$ results in strong polar outflows with radial velocities that become larger than the local escape velocities well within the validity regime of the density quasi-spherical approximation.

It is clear that the velocity outflow described above is far from the narrow collimated jets of astrophysical objects, where other physics such as magnetic fields and rotation clearly play a role. However, the model described could present a complementary ingredient in helping to initially turn an infall into an outflow, or to describe the much wider-angle components inferred sometimes to accompany some jet phenomena (e.g., Sato et al. (2021) or Duque et al. (2022)).

As can be seen from Equation (55), the radial velocity of the ejected material is a strong function of the angle, decreasing gradually at first for small angles, but then at an increasing rate, reminiscent of the structured jets inferred in some GRB studies (e.g., the Gaussian angular velocity profiles of jets treated in Lamb et al. 2021 or the structured jets of Kethirigamaraju et al. 2018), or of the transition toward an isotropic phase envisioned for GRBs (e.g., Waxman 2004).

Although the results described above occur within the validity regime of the density approximation, on reaching the point where the perturbative velocity component becomes comparable to the spherically symmetric one, the formal validity regime of the scheme developed is reached. Hence, we cannot be certain that the extrapolation of the model is valid out to the regime where the radial infall velocity becomes an outflow.

Given the qualitative agreement of the solution presented to the various numerical experiments described above, we can present a first quantitative comparison in terms of the most general feature of the solution shown above. It is clear from Equation (55) that as $R \rightarrow \infty$, the radial velocity will have a zero, where $3\cos^2 \theta = 1$, $\theta = 54^\circ 74$. This can be compared to the published numerical results of Aguayo-Ortiz et al. (2019; Figure 4, bottom right) and Waters et al. (2020) (e.g., Figure 1 leftmost panel) where the angle at which the radial velocity equals zero at the outer edge of their simulation domains (and hence a lower value for the final asymptotic value of this parameter) is of $51^\circ 3$ and $52^\circ 2$, respectively. We hence see that the numerical results listed above are not inconsistent with the value of the angle for which the radial velocity of Equation (55) goes to zero for large radii. Indeed, in the first numerical experiments presented in Section 5 we see that numerical convergence beyond the range of the validity of the linear approximations in the previous section yields solutions that maintain the qualitative structure of the model presented in Sections 3 and 4.

Lastly, we look at the minus sign in Equation (53), which now fixes a value of $\delta = -2.72$, for the same choice of $\theta_0^2/3 = 0.66533$ as taken above. This solution is qualitatively different from the previous one for a number of reasons. First, we see from the denominator in Equation (56) that the sign of the angular velocity will change, meaning that instead of focusing the flow toward the poles, this will happen toward the equator. Also, as the radial powers of the perturbative terms are now smaller than the ones of the spherically symmetric terms in Equations (54) and (55), the unperturbed solution will dominate for large radii, and the nonspherical terms become relevant at small, rather than large values.
In the right panel of Figure 6 we present the velocity flow of this last example, together with the logarithm of the density field; this last, as in the case of the left panel of this figure, deviates only marginally from spherical symmetry. The dominance of the unperturbed spherical solution of Equations (13) and (15) is evident toward the outer regions of the figure, as is the change in the inflow solution for small radii. This transition occurs well within the density validity limit of the solution for small radii, denoted by the exclusion rectangle at the center. For small radii, we see the appearance of a radial outflow of limited extent, and of the funneling of the accreted material onto the equatorial plane, at a strongly centrally divergent velocity.

The above results are interesting in light of recent indications of super-Eddington accretion in a variety of astrophysical scenarios. The discovery of over 200 quasars at very early redshifts $z > 6$ (e.g., Mortlock et al. 2011; Bañados et al. 2018) implies black hole growth rates well above the Eddington limit (e.g., Volonteri et al. 2015). At stellar scales, X-ray binaries (e.g., Okuda 2002) or ultraluminous X-ray sources inferred to contain black holes (e.g., Winter et al. 2006) have also been interpreted as evidence for super-Eddington accretion. At a fundamental level, exceeding the Eddington limit is a natural consequence of the breaking of spherical symmetry in the accretion flow (e.g., Paczynski & Abramowicz 1982; Massonneau et al. 2023). As such, the accretion model presented in the right panel of Figure 6, does precisely that; a large-scale spherically symmetric accretion flow is redirected into an equatorial infall covering a very small solid angle, presenting a minimal interaction cross section to any photon flux coming from the central object.

As happens also with the case shown in the left panel of Figure 6, the solution described lies within the validity regime of the density approximation beyond the central exclusion rectangle. However, the formal validity regime of the velocity approximation ends before the point where the radial velocity changes sign. Thus, the equatorial focusing described is only a mild effect within the full validity regime of the solution. Over said range, this equatorial focusing is seen as a monotonically growing effect, highly suggestive of this becoming dominant at smaller radii, which unfortunately cannot be traced reliably with the perturbative scheme developed.

The examples shown above might seem to be artificially close to the hydrostatic equilibrium $\rho_0^{2/3} = 2/3$ limit. However, during sufficiently early phases of massive stellar collapse, for example, the system can be chosen arbitrarily close to hydrostatic equilibrium. As can be seen from Equation (55), as $V_0 \to 0$ toward the hydrostatic equilibrium limit, the polar positive outflow at $\theta = 0$ diverges. Note also that the model presented is equally applicable to accretion onto a black hole, provided the region of interest remains at radii much larger than the Schwarzschild radius of the problem, and hence within the Newtonian validity regime.

5. First Numerical Explorations

In this final section, we present a numerical setup to be used in the exploration of the results of the previous sections, showing at this point the results of only the first of many computational experiments relevant to the new spherical accretion solution described in Section 2 and its nonspherical perturbations as described in Section 3.

We perform numerical simulations using the hydrodynamical code AZTEKAS (Aguayo-Ortiz & Mendoza 2021), which solves the inviscid Euler equations in a conservative form, coupled with an ideal gas equation of state providing a polytropic relation for the pressure. A detailed description of the code and extensive validation against standard analytical hydrodynamical tests can be found in Aguayo-Ortiz et al. (2018) and Tejeda & Aguayo-Ortiz (2019). All the simulations presented here use a high-resolution shock-capturing scheme along with a second-order piecewise linear reconstructor and an HLL approximate Riemann solver. The time evolution is computed using a second-order total variation diminishing Runge–Kutta time integrator, with a Courant factor of 0.5.

We first reproduce the new spherically symmetric accretion solution of Equations (13) and (14), by choosing a $\gamma^{2/3} = 0.6$, corresponding to a case in the subsonic region, and using $\gamma = 5/3$. A Keplerian potential due to a central point mass $M$ is assumed and the equations are all written in dimensionless variables such that $G = M = 1$, following also the dimensionless variables used in Section 2. 1D spherically symmetric simulations were performed using a domain consisting of 64, 128, or 256 numerical cells distributed over an exponential radial grid with $R \in [0.25, 10]$, as in Aguayo-Ortiz et al. (2019). Initial conditions were chosen by taking the analytic solution of Equations (13) and (14). Both the inner and outer boundary conditions were fixed at the values given by the analytic solution at all times, while the solution for the rest of the domain was completely free to evolve and settle to whichever radial profiles it naturally converged to. Convergence is defined as the point where the accretion rate, averaged over the entire volume, has a relative change between subsequent time steps of less than one part in $10^6$.

The results of one such simulation are given in Figure 7, where the top panels show the density and velocity radial profiles at convergence, in the left and right panels, respectively, for the three resolutions mentioned above represented by blue crosses, together with the initial conditions and the analytic solution of Equations (13) and (14) represented by a solid red line. It is clear that the numerical convergence solution very closely matches the new analytic solution. Indeed, the bottom panels of this figure give the logarithm of the deviation between the convergence numerical solution and the exact analytic solution, for runs using radial resolutions of 64, 128, and 256, from the top to the bottom of these panels. Numerical convergence is evident, as the fractional error rapidly falls as the resolution increases. Cases with $\gamma^{2/3} < 0.5$, corresponding to the supersonic region of the spherical solution, were also tested and give similar results, a quick convergence to the analytic model, with even greater robustness than for the subsonic case. As in that case perturbations do not propagate outward, the inner boundary conditions can be left free. These tests show the stability of the new solution of Equations (13) and (14), as numerical instabilities are seen not to propagate or grow as the numerical solution converges quite rapidly to the exact solution of these equations. As both the density and infall velocity radial profiles converge to the analytic exact solution of Equations (13) and (14), given the equation of state taken, the flat profile of $M$ is also verified in the simulations.

Given that the radial accretion model of Equations (13) and (14) is an exact solution of the conservation equations of the problem, the excellent agreement of the numerical solution to
these equations is not a validation of the analytical model, but of the numerical procedure implemented. Indeed, we are not aware of any other exact spherical accretion solution for the case of $\gamma = 5/3$ that can be used to test numerical codes, as numerical experiments testing against the classical Bondi solution typically fail on reaching exactly $\gamma = 5/3$ (e.g., Waters et al. 2020).

Lastly, we go to a 2D numerical setup under cylindrical symmetry using 256 radial cells and 128 angular ones, to explore the nonspherical perturbations to the exact radial solution, the results of Section 3. Again taking a subsonic case, this time $\varpi^{2/3} = 0.56$, the initial conditions were taken from Equations (54), (55), (56), and (53). In this case, we used $\epsilon_{\psi_0} = 0.0005$, so as to remain close to the validity of the linear analysis at small radii, the plus sign was taken in Equation (53), leading to the equatorial infall/radial outflow solution. The results are given in Figure 8, where the left panel shows the convergence numerical solution in terms of flow lines and

Figure 7. Left: the top panel shows numerical convergence density field solutions for the initial conditions as given by the exact solution of Equations (13) and (14), for a subsonic case of $\varpi^{2/3} = 0.6$ (blue crosses), together with the exact solution (red line). The bottom panel gives the logarithm of the relative error between the exact solution and numerical converge 1D solutions for radial resolutions of 64, 128, and 256, showing the numerical convergence of the simulations. Right: here we show the corresponding velocity field results, for the same simulations shown in the left panel.

Figure 8. Left: final convergence 2D numerical results, velocity fields, and density field for the initial conditions as given by the linear perturbed solution of Equations (53)-(56), for $\varpi^{2/3} = 0.56$, $\epsilon_{\psi_0} = 0.0005$, and taking the plus sign in Equation (53). The qualitative character of the initial equatorial infall/radial outflow solution is preserved. Right: relative differences between the linear perturbed analytical solution and the convergence numerical solution in density, radial velocity, and angular velocity (top to bottom). The top two panels represent curves for three values of the angle, $\theta = 0^\circ$, $\theta = 54^\circ.74$, and $\theta = 90^\circ$ (purple, blue, and green lines, respectively). Given the symmetry conditions, the angular velocity vanishes for both $\theta = 0^\circ$ and $\theta = 90^\circ$, and the bottom panel shows angular velocity difference curves for $\theta = 0^\circ$, $\theta = 54^\circ.74$, and $\theta = 90^\circ$ (purple, blue, and green curves, respectively).
density field. This final condition is visibly different from the initial conditions used, in that the stagnation point in the radial velocity along the polar direction has shifted outward a little, but otherwise, the general structure of the solution is preserved (e.g., compare to Figure 6, left panel). Some adjustments on approaching the stagnation point and beyond are to be expected, as when the radial velocity goes to zero, the linear approximation leading to Equations (53)–(56) breaks down. Crucially, the overall character of the solution remains unchanged, validating the infall/outflow character of the perturbed analytic solution.

In more detail, the right panel of Figure 8 gives the logarithms of the relative differences between the analytic solution of Equations (53)–(56) and the final numerical convergence solutions, once the spherical component, the solution of Equations (13) and (14) has been subtracted, so as to yield a comparison only of the nonspherical components. The right panel gives the quantities mentioned for three values of the angle, \( \theta = 0^\circ, \theta = 54^\circ.74, \) and \( \theta = 90^\circ \) (purple, blue, and green lines respectively) for the two top panels. As given the symmetry conditions the angular velocity vanishes for both \( \theta = 0^\circ \) and \( \theta = 90^\circ \), the three angles chosen for the bottom panel where \( \theta = 34^\circ.74, \theta = 54^\circ.74, \) and \( \theta = 74^\circ.74, \theta = 54^\circ.74 \) is the critical angle where the linear analytic solution yields \( \mathcal{V} = 0 \) for large radii.

We can see that for most of the density and velocity radial profiles shown, the numerical convergence solution differs from the input approximate one by factors of less than order one. Although for some angles and some radii this is not the case and such differences become larger, as can be seen from the overall flow pattern resulting in the left panel of Figure 8, the general equatorial infall and polar outflow character of the steady-state solution is maintained. Thus, the changes are quantitative but not qualitative, the general structure is preserved even after the end of the validity regime of the linear approximation.

We note that the cases presented here cover a very small region of the parameter space to be tested numerically; deviations from \( \gamma = 5/3 \), temporal perturbations, dynamical cases exploring the convergence to the analytic cases from various initial conditions, more general angular variations, and various combinations of the above need to be explored in order to better understand the full richness of the solutions presented in the previous sections.

Here we present only a brief preliminary series of tests showing the numerical stability of the exact spherical solution, and that even after crossing the validity regimes of the linear perturbation study presented in Section 3, the perturbed solution converges to a steady state preserving the qualitative features of the perturbative solution described in Section 3. A subsequent series of papers will explore the very large number of clearly relevant variants in detail.

6. Conclusions

We have presented a new, exact steady-state spherically symmetric zero angular momentum hydrodynamic accretion model for \( \gamma = 5/3 \) in a Keplerian potential. It differs from classical Bondi accretion in that the condition at infinity has been changed from \( \rho \rightarrow \rho_\infty \) to \( \rho \rightarrow 0 \), which allows simple power-law solutions for the density and the infall velocity in the spherically symmetric case. These solutions are characterized by having a Mach number, \( \mathcal{M} \), that is constant for all radii, at supersonic values toward the cold, empty limit, and at subsonic ones toward the hot, dense hydrostatic equilibrium one. A maximum accretion rate appears precisely at \( \mathcal{M} = 1 \). It is interesting that in the limit as \( R \rightarrow 0 \), the \( \gamma = 5/3 \) Bondi solution actually converges to the new solution we present here. This new solution is stable to small temporal deviations, and also to small changes in the adiabatic index about \( \gamma = 5/3 \).

Having an exact solution in closed form for the spherical accretion problem allows its use to explore deviations from sphericity with the polar angle in velocity and density fields and the inclusion of angular momentum, through an analytic perturbative analysis. This yields an ample spectrum of solutions given by Legendre polynomials for the density field, through which the velocity fields follow.

As a first exploration we show simple bipolar configurations having equatorial infall and polar outflows, where interestingly, one branch can naturally yield polar outflows for radii larger than a critical value, still within the consistency region of the perturbative analysis performed on the density field. Although the formal validity limit of the perturbative solution in the velocity field is reached before said point, it is interesting that even beyond this point the analytical solution is qualitatively consistent with recent independent numerical experiments. For the same solution, a second branch alters the spherical symmetry solution only at small radii, and does so by redirecting the large-scale spherical infall into a wedge along the equatorial plane, also well within the validity regime in the treatment of the density field. This effect is minor within the validity regime of the velocity approximation, and grows to become dominant on approaching the velocity validity threshold, suggesting the possibility of an interesting mechanism for super-Eddington accretion.

A first series of numerical experiments confirm the temporal stability of the new spherically symmetric solution. Simulations in 2D allowing for polar angle deviations from sphericity deviate quantitatively from the simple analytic linear angular perturbation solutions described in Section 3, which consider only one term in the Legendre series, while maintaining the overall equatorial accretion and radial outflow structure of such solutions.

Given the infinite range of behaviors that can be modeled using the Legendre polynomials solving Equation (37), it is reasonable to expect the framework presented might be of interest in a range of astrophysical settings.

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Appendix A
Comparison of Spherical Solution to Observed Accretion Profiles from Plšek et al. (2022)

Here we show in Figures 9 and 10 the remaining 18 density accretion profiles from Plšek et al. (2022) along with \( R^{-3/2} \) fits to the data, the expected density scaling from the new solution.
Inferred electron density profiles about the central SMBHs of 12 galaxies as presented by Plšek et al. (2022) (points with error bars). The solid lines give $R^{-3/2}$ fits, as expected by the new solution of Equations (13) and (14). The Bondi radii of the galaxies shown are $63^{+11}_{-12}$ pc for IC 4296, $87^{+26}_{-24}$ pc for NGC 507, $16^{+3}_{-2}$ pc for NGC 708, $37^{+15}_{-13}$ pc for NGC 1399, $170^{+50}_{-30}$ pc for NGC 1407, $590 \pm 50$ pc for NGC 1600, $80^{+13}_{-12}$ pc for NGC 4261, $42 \pm 4$ pc for NGC 4374, $100^{+16}_{-15}$ pc for NGC 4472, $270 \pm 30$ pc for NGC 4486, $15^{+2}_{-2}$ pc for NGC 4552, and $37 \pm 5$ pc for NGC 4636, as reported by Plšek et al. (2022). Classical Bondi fits would look very close to horizontal over the radial range shown, given the constraint of the inferred values for the Bondi radii.
of Equations (13) and (14). As explained in Section 2, the first point in each profile is reported as having a 100% error, as described by the above authors this innermost point is a model extrapolation rather than a direct measurement, and hence was excluded from the fits. In about half the cases, such as NGC 4261, NGC 4552, or the two galaxies included in Section 2, the fits are very good, and the radial scaling predicted by the new model presented quite accurately reproduces the data. In about a third of the cases (e.g., NGC 4472, NGC 4696, or NGC 5813), the observed profiles still show the predicted scaling over a substantial intermediary radial range, but show also inner and/or outer flattening with respect to the single power-law expectations of the solution of Equations (13) and (14), signaling a validity regime for the solution presented. Toward the inner regions, this could be due to the effects of AGN activity, as described in, e.g., Runge & Walker (2021), while toward the outer regions, the presence of radio lobes or a regime change from the inner accretion profile to an outer unperturbed medium could explain the deviations mentioned, among many other effects not considered in the very simple spherically symmetric accretion model presented. Finally, in three cases, NGC 507, NGC 5044, and NGC 6166, the fit presented is not a good match to the data, which shows not only inner and/or outer flattening with respect to a single power-law fit, but also a power-law scaling in the intermediary regions incompatible with the $R^{-3/2}$ of the fits.

Further insight into the possible causes determining the existence or otherwise of extended radial ranges compatible with the $R^{-3/2}$ accretion density profiles can be gained by considering correlations with other observed properties for these systems. As a first such exploration, we can look at the presence or absence of Hα + [N II] line emitting gas extended over central regions larger than 2 kpc. This can be an indication of the presence of thermally unstable atmospheres (Plšek et al. 2022) and hence more complex physical conditions falling outside the simple model presented in Section 2. Half of the sample, 10 systems, are reported by Plšek et al. (2022) as presenting such extended emission, and half as not presenting this feature. We take a first-order appraisal of the degree to which the $R^{-3/2}$ density profile represents the data, where fits are classified poor (P) when at least two data points (excluding the innermost) lie at a horizontal distance of more than 1.5σ from the $R^{-3/2}$ density profile and as good (G) otherwise. Also, from Plšek et al. (2022) we know which systems show Hα + [N II] line emitting gas over an extent larger than 2 kpc, (E), and which do not (N). The full list of 20 galaxies is now IC 4296 (E,G), NGC 507 (N,P), NGC 708 (E,P), NGC 1316 (E,G), NGC 1399 (N,G), NGC 1407 (N,G), NGC 1600 (N,G), NGC 4261 (N,G), NGC 4374 (N,G), NGC 4472 (N,P), NGC 4486 (E,P), NGC 4552 (E,G), NGC 4636 (N,G), NGC 4649 (N,G), NGC 4696 (E,P), NGC 4778 (N,P), NGC 5044 (E,P), NGC 5813 (E,G), NGC 5846 (E,G), and NGC 6166 (E,P), where we note for each system the two above classifications. It is interesting that cases presenting extended Hα + [N II] are almost twice as likely to be classified as poor fits (5/10) than cases not presenting such extended emission (3/10).

This is clearly only a first attempt at correlating observational indicators of perturbed physical conditions in these systems with the degree to which the density profile of the new solution presented fits the observed accretion profiles, the small present sample of high-quality observations does not warrant

Figure 10. Inferred electron density profiles about the central SMBHs of the six remaining galaxies of the 20 as presented by Plšek et al. (2022) (points with error bars). The solid lines give $R^{-3/2}$ fits, as expected by the new solution of Equations (13) and (14). The Bondi radii of the galaxies shown are $34^{+2}_{-2}$ pc for NGC 4696, $37^{+4}_{-4}$ pc for NGC 4778, $9.9^{+5.1}_{-3.3}$ pc for NGC 5044, 30 ± 4 pc for NGC 5813, $48^{+4}_{-3}$ pc for NGC 5846, and $44^{+3}_{-3}$ pc for NGC 6166, as reported by Plšek et al. (2022). Classical Bondi fits would look very close to horizontal over the radial range shown, given the constraint of the inferred values for the Bondi radii.
any further refinements of these explorations at this point, which also fall beyond the intent of our present study.

The figure captions provide the inferred values of the Bondi radii for each galaxy, also from Pišek et al. (2022), in all cases values much smaller than the middle range of the radial values shown, often smaller than the lower radial range. Hence, classical Bondi fits to the accretion density profiles shown would look very close to horizontal lines over most of the radial ranges shown in the log–log plots given. Therefore, once the inferred Bondi radii are fixed for all galaxies, even for the cases where the new proposed fit is poor, this remains a much better fit than a classical Bondi profile.

**Appendix B**

**Variations in the Spherical Solution About \( \gamma = 5/3 \).**

We now explore small variations about \( \gamma = 5/3 \) to test for the stability of the solution in Equations (13)–(15) to departures from the adiabatic value for which said solution was developed. We now take \( \gamma = 5/3 + \mu \), where we shall assume \( \mu << 1 \).

The spherically symmetric solution will be modified to

\[
V = V_0 R^{-\mu-1/2}, \quad \rho = \rho_0 R^{-\nu-3/2}. \tag{B1}
\]

For the above solution, it is clear that the mass conservation equation, Equation (11), will still be satisfied identically, as the power-law departures for \( \gamma = 5/3 \) in \( V \) and \( \rho \) will cancel, leaving the product \( \mathcal{V} \rho \) unchanged.

The momentum conservation equation, however, Equation (12), will no longer be satisfied identically, and is only satisfied to leading order in the deviations with respect to the \( \gamma = 5/3 \) solution, under the assumption of \( \mu < 1 \) and \( \nu < 1 \). Equation (12) becomes

\[
(\nu - 1/2) V_0^2 R^{2 \nu - 2} = (\nu + 3/2) \rho_0^{\mu + 2/3} R^{-2 \nu + 3 \nu / 2 - 2} - R^{-2}. \tag{B3}
\]

We shall now introduce a series expansion for \( R^{4+\epsilon} = R^4 + \epsilon R^4 \text{ln}(R) + O(\epsilon^2) \) to develop the radial powers in the above equation, yielding

\[
(\nu - 1/2) V_0^2 R^{2 \nu - 2} + 2 \nu (\nu - 1/2) V_0^2 R^{-2} \text{ln}(R) = (\nu + 3/2) \rho_0^{\mu + 2/3} R^{-2} - (2 \nu / 3 + 3 \mu / 4) (\nu + 3/2) \rho_0^{\nu + 2/3} R^{-2} \text{ln}(R). \tag{B4}
\]

The terms without \( \text{ln}(R) \) in the above equation yield

\[
(\nu - 1/2) V_0^2 = (\nu + 3/2) \rho_0^{\mu + 2/3} - 1, \tag{B5}
\]

while the terms containing \( \text{ln}(R) \) yield to first order in \( \mu \) and \( \nu \):

\[
\nu V_0^2 = (\nu + 9 \mu / 4) \rho_0^{\mu + 2/3}. \tag{B6}
\]

Equations (B5) and (B6) can now be used to obtain the departure in the power laws of the solution of Equations (B1) and (B2) to our original spherical solution, \( \nu \), as a function of the departure from \( \gamma = 5/3 \), \( \mu \), and the parameter of the spherical solution, \( \rho_0 \):

\[
\nu = \frac{9 \mu \rho_0^{2/3 + \mu}}{8(1 - 2 \rho_0^{2/3 + \mu})}. \tag{B7}
\]

Notice that as the departure from \( \gamma = 5/3 \) goes to zero, \( \mu \to 0 \), the departure from the original solution also goes to zero, as is also the case toward the empty freefall solution of \( \varrho = 0 \), where in the absence of hydrodynamical effects the freefall solution is recovered, regardless of the value of the adiabatic index. Condition (15) is modified slightly in this case to yield

\[
V_0^2 = 2 - 3 \rho_0^{2/3 + \mu}. \tag{B8}
\]

With the exception of a divergence in \( \nu \) close to the \( \mathcal{M} = 1 \) value of \( \rho_0^{2/3} = 1/2 \), small deviations from \( \gamma = 5/3 \) result in small deviations in the power laws of the original spherically symmetric solution. Hence, the power-law character of the solution in Equations (13) and (14) is generally preserved with only small variations when changing \( \gamma \) slightly away from the idealized \( 5/3 \) value. As expected, \( \mu \to 0 \) recovers the original solution for \( \gamma = 5/3 \). We see that the spherically symmetric solution of Equations (13) and (14) is robust to small changes in the adiabatic index.

We can now follow the same procedure described in Section 2 to calculate the Mach number of the spherically symmetric solution, but this time using the solutions for slight deviations from \( \gamma = 5/3 \), \( \gamma = 5/3 + \mu \), Equations (B1) and (B2), instead of Equations (13) and (14). Assuming both \( \mu << 1 \) and \( \nu < 1 \), to first order in these variables we obtain

\[
\mathcal{M} = V_0 \rho_0^{(1/3 + \mu/2)} R_0^{1/3 + 4 \mu / 3}. \tag{B9}
\]

using the result of Equation (B7) we obtain to first order

\[
\mathcal{M} = V_0 \rho_0^{(1/3 + \mu/2)} R_0^{1/3 + 4 \mu / 3}. \tag{B10}
\]

Clearly, when \( \gamma \to 5/3 \) both \( \mu \to 0 \) and \( \nu \to 0 \), and the result of Equation (20) for the constant \( \mathcal{M} \) of the solution described in Section 3 is recovered, showing the consistency of the above analysis.

The above result is interesting because it shows that the radial constancy of the Mach number obtained for \( \gamma = 5/3 \) is lost when considering even perturbative deviations in the adiabatic index away from \( 5/3 \), while retaining the \( \rho \to 0 \) as \( r \to \infty \) condition. While the power-law character of the density and velocity fields in the spherical solution is robust to small changes in \( \gamma \) (provided one is not close to \( \rho_0^{2/3} = 1/2 \)), the lack of a sonic radius is not. As shown in Equation (B10), even small deviations away from \( \gamma = 5/3 \) result in a radial profile in the Mach number of the solution that is no longer flat. This profile can be slightly increasing, or slightly decreasing with radius, depending on the signs of \( 1-2 \rho_0^{2/3} \) and of \( \mu \). When the two signs are different, an outer subsonic region results, which transitions to an inner supersonic one at a sonic radius, much like the classical Bondi solution. However, when both signs are equal, the reverse ensues, an outer \( \mathcal{M} > 1 \) region, a sonic radius, and an inner \( \mathcal{M} < 1 \) one. This last feature does not imply any inward deceleration, the velocity flow in all cases remains that of Equation (B1), which for the assumption of \( \mu << 1 \) and \( \nu << 1 \), under which the results in this appendix were derived, implies a flow that always accelerates inward, much like the one of the \( \gamma = 5/3 \) solution described in Section 3. The transition from \( \mathcal{M} > 1 \) at large radii to the opposite at the small ones mentioned above, arises merely from the fact that the speed of sound increases at a faster rate toward small radii than the flow velocity.

For the exact solution to the conservation equations with \( \gamma = 5/3 \) in the spherical case of Equations (13) and (14), the radial power-law scalings of the local speed of sound and the...
infall velocities are equal, yielding the constant Mach number discussed in Section 3. However, even for perturbative departures from $\gamma = 5/3$, both radial dependences discussed above change in slightly different manners. As can be seen in Equation (B10), in this case, the Mach number will no longer be constant and a sonic radius will appear. Still, the radial variation in $M$ will remain small, provided $\mu$ is small and $\theta_{0}/3$ is not close to $1/2$. If either of these last two conditions are not met, the analysis in this appendix will no longer be valid, but from the developments leading to Equation (B7), we can see that when $\theta_{0}/3$ approaches $1/2$, even for small $\mu$, large deviations will appear in the velocity and density fields of the solution, and hence also in the radial profile of $\mathcal{M}$, when moving away from $\gamma = 5/3$. Thus, the constant $\mathcal{M}$ of the solution to Equations (13) and (14) is only a feature of the idealized $\gamma$ exactly equal to $5/3$ case. Still, within the validity regime of the above perturbative analysis, the radial profile of $\mathcal{M}$ remains in general much flatter than in the case of the classical Bondi solution.

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