Hodge structure of fibre integrals associated to the affine
hypersurface in a torus

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Abstract. We calculate the fibre integrals of the affine hypersur-
face in a torus in the form of their Mellin transforms. Especially,
our method works efficiently for an affine hypersurface defined by a
so called “simpliciable” polynomial. The relations between poles of
Mellin transforms of fibre integrals, the mixed Hodge structure of
the cohomology of the hypersurface, the hypergeometric differential
equation, and the Euler characteristic of fibres are clarified.

0 Introduction

In this note we propose a simple method to calculate concretely fibre integrals associated to the
affine hypersurface in a torus. We establish an expression of the position of poles of the Mellin
transform with the aid of the mixed Hodge structure of an hypersurface $Z_f$ defined by a $\Delta-$regular
polynomial explained by V. Batyrev [1]. The trial to relate the asymptotic behaviour of a fibre
integral with the Hodge structure of the fibre variety goes back to [13] where Varchenko established
the equivalence of the asymptotic Hodge structure and the mixed Hodge structure in the sense of
Deligne-Steenbrink for the case of plane curves and (semi-)quasihomogeneous singularities.

Later on, several authors ([5], [8], [9], [10]) have pursued studies on the the asymptotic behaviour
of fibre integrals in making use of the Mellin transforms. Their main idea consists in the fact,
that it is possible to visualize the asymptotic behaviour (i.e. the filtration) of fibre integrals by
means of the poles of Mellin transform. Especially in the case of complete intersection singularities,
the advantage of this method is quite clear. Let us remark also that not only poles of the Mellin
transform but its zeros play role in the calculus of the global monodromy of the fibre integrals (e.g.
see Proposition 5.3). The relation between the poles of the Mellin transform and the mixed Hodge
structure has been explained for examples of isolated complete intersections of space curve type in
[11].

In this note, we illustrate the clarity of this approach in taking the example of a hypersurface in
a torus defined by so called simpliciable polynomial (see Definition 2). It serves as an introduction
to the author’s main paper in preparation [12] where he establishes similar results for the fibre
integrals of the “simpliciable” complete intersection singularities.

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version of the calculus.

1 Hodge structure of the cohomology group of a
hypersurface in a torus

In this section we review fundamental notions on the Hodge structure of the cohomology group
of a hypersurface in a torus after [1], [2].
Let $\Delta$ be a convex $n$–dimensional convex polyhedron in $\mathbb{R}^n$ with all vertices in $\mathbb{Z}^n$. Let us define a ring $S_\Delta \subset \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$ of the Laurent polynomial ring as follows:

\[
S_\Delta := \mathbb{C} \cup \bigcup_{\vec{a} \in \Delta, \exists k \geq 1} \mathbb{C} \cdot x^{\vec{a}}.
\]

We denote by $\Delta(f)$ the convex hull of the set $\vec{a} \in \text{supp}(f)$ and call it the Newton polyhedron of a Laurent polynomial $f(x)$. We introduce the following Jacobi ideal:

\[
J_{f, \Delta} = \langle x_1 \frac{\partial f}{\partial x_1}, \ldots, x_n \frac{\partial f}{\partial x_n} \rangle \cdot S_{\Delta(f)}.
\]

Let $\tau$ be a $\ell$–dimensional face of $\Delta(f)$ and define

\[
f^\tau(x) = \sum_{\vec{a} \in \tau \cap \text{supp}(f)} a_{\vec{a}} x^{\vec{a}},
\]

where $f(x) = \sum_{\vec{a} \in \text{supp}(f)} a_{\vec{a}} x^{\vec{a}}$. The Laurent polynomial $f(x)$ is called $\Delta$ regular, if $\Delta(f) = \Delta$ and for every $\ell$–dimensional face $\tau \subset \Delta(f)$ ($\ell > 0$) the polynomial equations:

\[
f^\tau(x) = x_1 \frac{\partial f^\tau}{\partial x_1} = \cdots = x_n \frac{\partial f^\tau}{\partial x_n} = 0,
\]

have no common solutions in $T^n = (\mathbb{C}^\times)^n$.

**Proposition 1.1** Let $f$ be a Laurent polynomial such that $\Delta(f) = \Delta$. Then the following conditions are equivalent.

(i) The elements $x_1 \frac{\partial f}{\partial x_1}, \ldots, x_n \frac{\partial f}{\partial x_n}$ gives rise to a regular sequence in $S_{\Delta(f)}$

(ii) \[\dim \left( \frac{S_\Delta}{J_{f, \Delta}} \right) = n! \text{vol}(\Delta).\]

(iii) $f$ is $\Delta$–regular.

It is possible to introduce a filtration on $S_\Delta$, namely $\vec{a} \in S_k$ if and only if $\vec{a} \in \Delta$. Consequently we have an increasing filtration:

\[
C \equiv \{0\} = S_0 \subset S_1 \subset \cdots \subset S_n \subset \cdots,
\]

that induces a decreasing filtration on $\frac{S_\Delta}{J_{f, \Delta}}$:

\[
F^n \left( \frac{S_\Delta}{J_{f, \Delta}} \right) \subset F^{n-1} \left( \frac{S_\Delta}{J_{f, \Delta}} \right) \subset \cdots \subset F^0 \left( \frac{S_\Delta}{J_{f, \Delta}} \right).
\]

This is called the Hodge filtration of $\frac{S_\Delta}{J_{f, \Delta}}$. It is worthy to remark here that the Hodge filtration ends up with $n$–th term.

Let us remind us of the notion of Ehrhart polynomial:

**Definition 1** Let $\Delta$ be an $n$–dimensional convex polytope. Denote the Poincaré series of graded algebra $S_\Delta$ by

\[
P_\Delta(t) = \sum_{k \geq 0} \ell(k\Delta) t^k,
\]

\[
Q_\Delta(t) = \sum_{k \geq 0} \ell^*(k\Delta) t^k,
\]

where $\ell(k\Delta)$ and $\ell^*(k\Delta)$ count the number of lattice points in $k\Delta$.
where $\ell(k\Delta)$ (resp. $\ell^*(k\Delta)$) represents the number of integer points in $k\Delta$. (resp. interior integer points in $k\Delta$.) Then
\[
\Psi_{\Delta}(t) = \sum_{k=0}^{n} \psi_k(\Delta)t^k = (1-t)^{n+1}P_{\Delta}(t),
\]
\[
\Phi_{\Delta}(t) = \sum_{k=0}^{n} \varphi_k(\Delta)t^k = (1-t)^{n+1}Q_{\Delta}(t),
\]
are called Ehrhart polynomials which satisfy
\[
t^{n+1}\Psi_{\Delta}(t^{-1}) = \Phi_{\Delta}(t).
\]
Further, the main object of our study will be the cohomology group of the hypersurface $Z_f := \{x \in \mathbb{T}^n; f(x) = 0\}$. We have an important isomorphism on the Hodge filtration of $PH^{n-1}(Z_f)$.

**Theorem 1.2** ([1]) For the primitive part $PH^{n-1}(Z_f)$ of $H^{n-1}(Z_f)$, the following isomorphism holds;
\[
\frac{F^{i+1}PH^{n-1}(Z_f)}{F^{i}PH^{n-1}(Z_f)} \cong Gr^{n-i}_F \left( \frac{S_{\Delta}}{J_{f,\Delta}} \right) = \frac{F^{i}(\frac{S_{\Delta}}{J_{f,\Delta}})}{F^{i+1}(\frac{S_{\Delta}}{J_{f,\Delta}})}.
\]
Furthermore
\[
dim Gr^{n-i}_F \left( \frac{S_{\Delta}}{J_{f,\Delta}} \right) = \sum_{q \geq 0} h^{i,q}(PH^{n-1}(Z_f)) = \psi_{n-i}(\Delta),
\]
for $i \leq n - 1$.

As for the weight filtration, we have the following characterization. We understand the notion of the stratum of the support of the algebra $\frac{S_{\Delta}}{J_{f,\Delta}}$ in identifying a polynomial $x^{\bar{\alpha}} \in S_{\Delta}$ with $\bar{\alpha} \in \mathbb{Z}^n$. We call $(n-j)$-dimensional stratum of $supp(S_{\Delta})$ the set of those points $\bar{t}$ from $k\Delta$, $k = 1, 2, \cdots$ such that $\frac{1}{t}$ is located on the $(n-j)$-dimensional face of $\Delta$ and not on any $(n-j-1)$-dimensional face $\Delta' \subset \Delta$.

**Theorem 1.3** The weight filtration on $PH^{n-1}(Z_f)$ is defined as a decreasing filtration
\[
0 = W_{n-2} \subset W_{n-1} \subset \cdots \subset W_{2n-2} = PH^{n-1}(Z_f),
\]
such that $W_{n+i-1} \cong \{ \text{the integer points located on the strata with dimension } \geq (n-i) \text{ of } supp(\frac{S_{\Delta}}{J_{f,\Delta}}) \text{ but not on the } (n-i-1)-\text{dimensional stratum.} \}$ for $0 \leq i \leq n - 2$.

This theorem is an easy consequence of the Theorem 8.2 [1]. First we notice that the following exact sequence takes place,
\[
0 \to H^n(\mathbb{T}) \to H^n(\mathbb{T} \setminus Z_f) \xrightarrow{Res} H^{n-1}(Z_f) \to 0.
\]
The Poincaré residue mapping $Res$ gives a morphism of mixed Hodge structure of the Hodge type $(-1, -1)$,
\[
Res(F^j H^n(\mathbb{T} \setminus Z_f)) = F^{j-1} H^{n-1}(Z_f), \quad Res(W_j H^n(\mathbb{T} \setminus Z_f)) = W_{j-2} H^{n-1}(Z_f).
\]
Thus we have,
\[
0 \to W_{n+i} H^n(\mathbb{T}) \to W_{n+i} H^n(\mathbb{T} \setminus Z_f) \xrightarrow{Res} W_{n+i-2} H^{n-1}(Z_f) \to 0,
\]
for \( i = 2, \ldots, n - 1 \) where
\[
W_{2n-1} H^n(T) = \cdots = W_{n-1} H^n(T) = 0,
\]
and \( \dim W_2 H^n(T) = 1 \). In view of the equality (1.5) the Poincaré residue mapping \( Res \) gives an isomorphism
\[
Res : W_{n+i} H^n(T \setminus Z_f) \rightarrow W_{n+i-2} H^{n-1}(Z_f),
\]
for \( i = 1, \ldots, n - 1 \). The algebraic structure of the space \( W_{n+i} H^n(T \setminus Z_f), i = 1, \ldots, n - 1 \) has already been established by Theorem 8.2 [1].

Further in the course of this paper we identify the element \( x^\alpha \in S_\Delta \) with \( \frac{x^\alpha \, dx}{d\alpha} \) representing an element of \( H^{n-1}(Z_f) \).

\section{Preliminary combinatorics} \label{sec:combinatorics}

Let us consider a polynomial
\[
f(x) = \sum_{1 \leq i \leq M} x^{\vec{\alpha}(i)}
\]
with \( M \geq N + 1 \). Here \( \vec{\alpha}(i) \) denotes the multi-index
\[
\vec{\alpha}(i) = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}^N.
\]
In the case when \( M > N \) we associate to \( f(x) \) another polynomial in \( M - 1 \) variables \( f^\sigma(x, x') \)
\[
f^\sigma(x, x') = \sum_{i=1}^{M-N-1} x'_i x_1^{\vec{\alpha}(\sigma(i))} + \sum_{j=M-N}^M x_j^{\vec{\sigma}(\sigma(j))}
\]
with \( \sigma \in S_M \), the permutation group of \( M \) elements. Here we used the notation of the multi-index:
\[
\vec{\alpha}(\sigma(i)) = (\alpha_1^{\sigma(i)}, \ldots, \alpha_N^{\sigma(i)}) \in \mathbb{Z}^N.
\]
In this situation, the expression \( u(f^\sigma(x, x') + s) \) is a polynomial depending on \( M + 1 \) variables \((x_1, \ldots, x_N, x'_1, \ldots, x'_{M-N-1}, s, u)\). Further we shall assume
\[
supp(f^\sigma) \cap \text{int}(\Delta(f^\sigma)) = \emptyset,
\]
for each \( \sigma \) under question. Here \( \Delta(f^\sigma) \) denotes the Newton polyhedron of \( f^\sigma(x, x') \).

\textbf{Remark 1} A polynomial that depends on \( M + 1 \) variables and contains \( M + 1 \) monomials is called of Delsarte type. Jean Delsarte proposed to study algebraic cycles on the hypersurface defined by a polynomial of this class.

Let us introduce new variables \( T_1, \ldots, T_{M+1} \):
\[
T_1 = u x'_1 x_1^{\vec{\alpha}(1)}, T_2 = u x'_2 x_2^{\vec{\alpha}(2)}, \ldots
\]
\[
T_{M-N-1} = u x'_{M-N-2} x_2^{\vec{\alpha}(M-N-1)}, T_{M-N} = u x_2^{\vec{\alpha}(M-N)}, \ldots, T_{M+1} = us.
\]
To express the situation in a compact form, we use the following notations:
\[
\Xi := \{ x_1, \ldots, x_N, x'_1, \ldots, x'_{M-N-1}, u, s \},
\]
\[ \text{Log } T := \langle \log T_1, \ldots, \log T_{M+1} \rangle = \langle \tau_1, \ldots, \tau_{M+1} \rangle, \]

\[ \text{Log } \Xi := \langle \log x_1, \ldots, \log x_N, \log x'_1, \ldots, \log x'_{M-N-1}, \log u, \log s \rangle. \]

In making use of these notations, we have the relation

\[ \tau_1 = \log u + \log x'_1 + < \alpha(1), \log x >, \ldots, \]
\[ \tau_{M-N-1} = \log u + \log x'_{M-N-1} + < \alpha(M-N-1), \log x >, \]
\[ \tau_{M-N} = \log u + < \alpha(M-N), \log x >, \ldots, \tau_{M+1} = \log u + \log s. \]

We can rewrite the relation (2.8) with the aid of a matrix \( L^\sigma \in \text{End} (\mathbb{Z}^{M+1}) \), as follows:

\[ \text{Log } T = L^\sigma \cdot \text{Log } X. \]

where

\[
L^\sigma = \begin{bmatrix}
\alpha_1^{(1)} & \ldots & \alpha_N^{(1)} & 1 & 0 & \ldots & 0 & 0 & 1 \\
\alpha_1^{(2)} & \ldots & \alpha_N^{(2)} & 0 & 1 & \ldots & 0 & 0 & 1 \\
\vdots & \ldots & \vdots & 0 & 0 & 1 & \ldots & 0 & 1 \\
\ldots & \ldots & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 \\
\alpha_1^{(M-N-1)} & \ldots & \alpha_N^{(M-N-1)} & 0 & 0 & \ldots & 1 & 0 & 1 \\
\alpha_1^{(M-N)} & \ldots & \alpha_N^{(M-N)} & 0 & 0 & \ldots & 0 & 0 & 1 \\
\vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_1^{(M)} & \ldots & \alpha_N^{(M)} & 0 & 0 & \ldots & 0 & 0 & 1 \\
0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\
\end{bmatrix}
\]

Further we shall assume that the determinant of the matrix \( L^\sigma \) is positive. This assumption is always satisfied without loss of generality, if we permute certain column vectors of the matrix, which evidently corresponds to the change of positions of variables \( x \). We denote the determinant by \( \gamma^\sigma = \text{det}(L^\sigma) \). The row vectors of \( L^\sigma \) will be denoted by \( e_1^\sigma, \ldots, e_{M+1}^\sigma \). Later we will make use of the notation of variables \( X := (X_1, \ldots, X_{M-1}) := (x_1, \ldots, x_N, x'_1, \ldots, x'_{M-N-1}) \) and that of the polynomial \( f^\sigma(x, x') = f^\sigma(X) \).

**Definition 2** We call that polynomial \( f(x) \) is simpliciable if for every \( \sigma \in S_M, \text{det}(L^\sigma) = \gamma^\sigma \neq 0 \).

For \( \tau \subset \Delta(f^\sigma) \) we denote by \( \Sigma(\tau) \) a \((\dim \tau + 1)\)-dimensional simplex consisting all segments connecting \( \{0\} \) and a point of \( \tau \). Let us define a graded algebra

\[ S_\tau := \bigcup_{\alpha \in \Sigma(\tau), k \geq 1} C X^\alpha. \]

and a polynomial

\[ f_{\sigma, \tau}(X) := \sum_{\alpha \in \text{supp}(f^\sigma) \cap \tau} X^\alpha \]

**Lemma 2.1** If \( f(x) \) is a simpliciable polynomial, then \( f^\sigma(X) \) is \( \Delta(f^\sigma) \)-regular.

**Proof** The condition \( \text{det}(L^\sigma) = \gamma^\sigma \neq 0 \) yields that \( X_1 \frac{\partial f^\sigma}{\partial X_1}, X_2 \frac{\partial f^\sigma}{\partial X_2}, \ldots, X_{M-1} \frac{\partial f^\sigma}{\partial X_{M-1}} \) form a regular sequence in \( S_\tau \) for any face \( \tau \subset \Delta(f^\sigma) \). Q.E.D.
3 Mellin transforms

In this section we proceed to the calculation of the Mellin transform of the fibre integrals associated to the hypersurface $Z_{f^*+s} = \{X \in T^{M-1}; f^*(X) + s = 0\}$ defined by a simpliciable polynomial.

First of all we consider the fibre integral taken along the fibre $\gamma(s) \in H_{M-2}(Z_{f^*+s})$ as follows,

\[
I_{X^J, \partial \gamma}(s) := \int_{\gamma(s)} \frac{X^J dX}{df^*(X)} = \frac{1}{2\pi i} \int_{\partial \gamma(s)} \frac{X^J dX}{(f^*(X) + s)X^1}
\]

where $\partial \gamma(s) \in H_{M-1}(T^{M-1} \setminus Z_{f^*+s})$ is a cycle obtained after the application of $\partial$, Leray’s coboundary operator. Here $X^1 = X_1 \cdots X_{M-1}$, $X^J = X_1^j \cdots X_{M-1}^j$. See the book by V.A. Vassiliev on ramified integrals for the Leray’s coboundary operator.

The Mellin transform of $I_{X^J, \partial \gamma}(s)$ is defined by the following integral:

\[
M_{X^J, \partial \gamma}(z) := \int_{\Pi} (-s)^z I_{X^J, \partial \gamma}(s) \frac{ds}{s}.
\]

Here $\Pi$ stands for a cycle in $C$ that avoids the poles of $I_{X^J, \partial \gamma}(s)$. We assume that on the set $\partial_1 \cup_{s \in \Pi} \partial \gamma(s)$, $\Re(f^*(X) + s) > 0$. We denote by $L_q(J, z)$ the inner product of $(J, z, 1)$ with the $q$–th column vector of $(L^\sigma)^{-1}$. Let us deform the integral (3.2) in making use of the definition (3.1):

\[
M_{X^J, \partial \gamma}(z) = \int_{\mathbb{R}_- \times \partial_1 \cup_{s \in \Pi}} e^{u(f^*(X) + s)} X^J u(-s) \frac{du}{u} \wedge \frac{dX}{X^1} \wedge \frac{ds}{s}
\]

\[
= \frac{1}{\gamma^\sigma} \int_{(L^\sigma)^{-1} \times \mathbb{R}_- \times \partial_1 \cup_{s \in \Pi}} e^{\Psi(T)} \prod_{q=1}^M T_q L_q(J, z) \prod_{q=1}^M \frac{dT_q}{T_q},
\]

with

\[
\Psi(T) = T_1(X, u) + \cdots + T_M(X, u) + T_{M+1}(s, u) = u(f^*(X) + s)
\]

where each term $T_i(X, u)$ $(1 \leq i \leq M)$ represents a monomial term of variables $X, u$ of the polynomial (3.4) while $T_{M+1}(s, u) = su$. By virtue of the simple structure of the matrix $L^\sigma$ (2.10), we can consider the simplexes of $\mathbb{R}^{M-1}$ defined as $\{A_{ij}^q, 1 \leq q \leq M + 1\}$ where we identify $\tilde{\sigma}_q^q \in \mathbb{Z}^{M+1}$ with that of $\mathbb{Z}^{M-1}$ after ignoring the last two entries. It means that we identify $\tilde{\sigma}_q^q$ with the $i$–th row vector of the matrix $L^\sigma$ of which one removes the last two columns $^t(0, 0, \cdots, 0, 1), ^t(1, 1, \cdots, 1) \in \mathbb{Z}^{M+1}$. The chain $\partial_1 \cup_{s \in \Pi} \mathbb{R}_-$ can be deformed in $C^M$ so far as it does not encounter the singularity of the integrand.

**Proposition 3.1** 1) The Mellin transform $M_{X^J, \partial \gamma}(z)$ of the fibre integral associated to the simpliciable polynomial $f^*(X)$ has the following form.

\[
M_{X^J, \partial \gamma}(z) = g(z) \prod_{q=1}^M \Gamma(L_q(J, z)), 1 \leq q \leq M + 1,
\]

where $g(z)$ is a rational function in $e^{\frac{z}{\gamma^\sigma}}$ with $\gamma^\sigma = (M - 1)! \cdot \text{vol}(\Delta(f^*))$. The linear function in $(J, z)$,

\[
L_q(J, z) = ^t(J, z, 1)w_q^\sigma = \frac{<\tilde{\sigma}_q^q, J > + B_q^\sigma z + C_q^\sigma}{\gamma^\sigma},
\]

where $w_q^\sigma$ is the $q$–th column vector of the matrix $(L^\sigma)^{-1}$.
2) The $M + 1$ linear functions $L_q(J, z)$ are classified into the following three groups.

\[(3.7)_1 \quad L_{M+1}(J, z) = \frac{B_M^\sigma}{\gamma^\sigma} z = \frac{\gamma^\sigma}{\gamma^\sigma} z = z.\]

For $q$ such that $\bar{w}_q^\sigma = B_q^\sigma(\bar{v}_q^\sigma, 1, -1)$ for some $\bar{v}_q^\sigma \in \mathbb{Z}^{M-1}$, and $B_q^\sigma \neq 0$,

\[(3.7)_2 \quad L_q(J, z) = \frac{B_q^\sigma(\langle \bar{v}_q^\sigma, J \rangle + z - 1)}{\gamma^\sigma}.\]

For $q$ such that $\bar{w}_q^\sigma = (\bar{v}_q^\sigma, 0, 0)$ for some $\bar{v}_q^\sigma \in \mathbb{Z}^{M-1}$, and $B_q^\sigma = 0$,

\[(3.7)_3 \quad L_q(J, z) = \frac{\langle \bar{v}_q^\sigma, J \rangle}{\gamma^\sigma}.\]

Here the case $(3.7)_3$ corresponds to such $q$ that $\dim \tau_q^\sigma < M - 1$.

3) $|B_q^\sigma| = (M - 1)! \text{vol}(\tau_q^\sigma)$.

4) For $J \in \tau_q^\sigma \cap \Delta(f^\sigma)$, with $\dim \tau_q^\sigma = M - 1$, $\tau_q^\sigma \neq \Delta(f^\sigma)$,

\[\langle \bar{v}_q^\sigma, J \rangle = 1.\]

\[\langle \bar{v}_{M+1}^\sigma, J \rangle = 0.\]

**Proof** 1) The definition of the $\Gamma -$ function sounds as follows:

\[\int_{\mathbb{R}_-} e^T (-T)^\sigma \frac{dT}{T} = (1 - e^{2\pi i \sigma}) \int_{\mathbb{R}_-} e^T (-T)^\sigma \frac{dT}{T} = (1 - e^{2\pi i \sigma}) \Gamma(\sigma),\]

for the unique nontrivial cycle $\mathbb{R}_-$ turning around $T = 0$ that begins and returns to $\mathbb{R}_- \to -\infty$.

We apply it to the integral (3.3) and get (3.5). We consider an action on the chain $C_a = \mathbb{R}_-$ or $\mathbb{R}_-$ on the complex $T_a$ plane, $\lambda : C_a \to \lambda(C_a)$ defined by the relation,

\[\int_{\lambda(C_a)} e^{T_a} T_a^{\sigma_a} \frac{dT_a}{T_a} = \int_{(C_a)} e^{T_a} (e^{2\pi \sqrt{-1}} T_a)^{\sigma_a} \frac{dT_a}{T_a}.\]

By means of this action the chain $L_\ast(\mathbb{R}_- \times \gamma^\Pi)$ turns out to be homologous to,

\[\sum_{j_1^{(\rho)} \cdots j_{M+1}^{(\rho)}} m_{j_1^{(\rho)} \cdots j_{M+1}^{(\rho)}} \lambda_{j_1^{(\rho)}}(\mathbb{R}_-) \prod_{a' = 2}^{M+1} \lambda_{a'}^\sigma(\mathbb{R}_-),\]

with $m_{j_1^{(\rho)} \cdots j_{M+1}^{(\rho)}} \in \mathbb{Z}$. This explains the presence of the factor $g(z) = \sum_{j_1^{(\rho)} \cdots j_{M+1}^{(\rho)}} e^{2\pi \sqrt{-1} j_1^{(\rho)} L_1(J, z, \xi)} \prod_{a' = 2}^{M+1} e^{2\pi \sqrt{-1} j_{a'}^{(\rho)} L_{a'}(J, z, \xi)} (1 - e^{2\pi \sqrt{-1} L_{a'}(J, z, \xi)})$ except for the $\Gamma -$ function factors.

The points 2)- 5) are reduced to the linear algebra. For example 3) can be shown, if one remembers the definition of $M$ minors of the matrix $L'$ calculated in removing the $M-$th column.

4) If $J \in \tau_q^\sigma$, the vector $\bar{e}_q^\sigma$ is orthogonal to $(\bar{v}_{M+1}^\sigma, 1, -1)$ for $i \neq q$ and $\langle \bar{e}_q^\sigma, B_q^\sigma(\bar{v}_{M+1}^\sigma, 1, -1) \rangle = \gamma^\sigma$. 

The result on the $M$–th and $(M+1)$–st element is explained by the fact that $\bar{e}_{M+1}^q = (0, \cdots, 0, 1, 1)$ is orthogonal to $(\bar{e}_{M+1}^q, 1, -1)$ for $1 \leq q \leq M$.

**Q.E.D.**

Let us denote the set of such indices $q$ with strictly positive (resp. strictly negative) $B_q^+$ by $I^+ \subset \{1, \cdots, M+1\}$, (resp. by $I^- \subset \{1, \cdots, M+1\}$). The set of indices $q$ for which $B_q^+ = 0$ will be denoted by $I^0$. With these notations, one can formulate the following,

**Corollary 3.2** 1) The Newton polyhedron admits the following representation, $\Delta(f^\sigma) = \{\bar{t} \in R^M; \langle \bar{t}_q, \bar{t} \rangle \geq 1 \text{ for } q \in I^+, \langle \bar{t}_q, \bar{t} \rangle \leq 1 \text{ for } q \in I^-, \langle \bar{t}_q, \bar{t} \rangle \geq 0 \text{ for } q \in I^0 \}$.

2) We denote by $\chi(Z_{f^\sigma+1})$ the Euler–Poincaré characteristic of the hypersurface $Z_{f^\sigma+1} = \{X \in T^{M-1}, f^\sigma(X) + 1\}$ here under the constant 1 we understand a generic value for $f^\sigma(X)$. The following equality holds,

$$\sum_{q \in I^+} B_q^+ = (M-1)! \text{vol}_{M-1}(\Delta(f^\sigma(X) + 1)) = (-1)^M \chi(Z_{f^\sigma+1}).$$

3) $\sum_{q=1}^{M+1} B_q^+ = 0$. In other words,

$$\sum_{q \in I^-} B_q^- = -\left(\sum_{q \in I^+} B_q^+\right).$$

**Proof** 1) After the definition of vectors $\bar{e}_1^q, \cdots, \bar{e}_M^q$ we can argue as follows. If $\bar{t}$ does not belong to the hyperplane $\langle \bar{e}_1^q, \cdots, \bar{e}_M^q \rangle$, then $\langle \bar{t}_q, \bar{t} \rangle = 1 + \frac{\bar{t}_q}{B_q^+}$. In the case when $q \in I^+$ (resp. $q \in I^-$) $\langle \bar{t}_q, \bar{t} \rangle \geq 1$ (resp. $\langle \bar{t}_q, \bar{t} \rangle < 1$) that is equivalent to say that all the points $\bar{t}$ of the Newton polyhedron $\Delta(f^\sigma)$ satisfy $\langle \bar{t}_q, \bar{t} \rangle \geq 1$ for $q \in I^+$ (resp. $\langle \bar{t}_q, \bar{t} \rangle \leq 1$ for $q \in I^-$. If $\bar{t} \in \langle \bar{e}_1^q, \cdots, \bar{e}_M^q \rangle$, then $\langle \bar{t}_q, \bar{t} \rangle = 1$. For $q \in I^0$, $\Delta(f^\sigma) \subset \{\bar{t} \in \langle \bar{e}_1^q, \cdots, \bar{e}_M^q \rangle \geq 0\}$, because $\langle \bar{t}_q, \bar{t} \rangle = 1$ for $\bar{t} \notin \langle \bar{e}_1^q, \cdots, \bar{e}_M^q \rangle$. As all possible cases are exhausted by $I^+, I^-, I^0, |I^+| + |I^-| + |I^0| = M + 1$. This yields the statement. 2) Apply the Theorem by [3, 4] on the Euler characteristic. 3) The $(M+1)$–st column vector of $L$ is orthogonal to the $M$–th row vector of $L^{-1}$, $(B_1^+, \cdots, B_{M+1}^+)$. 

**Corollary 3.3** Under the above situation, the Mellin inverse of $M_{X,\gamma}(s)$ with properly chosen periodic function $g(z)$ with period $\gamma$:

$$I_{X,\gamma}(s) = \int_{\Pi} g(z) \prod_{a \in I^+} \Gamma(\mathcal{L}_a(J, z)) \prod_{a \in I^-} \Gamma(1 - \mathcal{L}_a(J, z)) s^{-z} dz,$$

defines a convergent analytic function in $-\pi < \arg s < \pi$.

**Proof** In applying the Stirling’s formula

$$\Gamma(z + 1) \sim (2\pi z)^{\frac{1}{2}} z^z e^{-z}, \quad \Re z \to +\infty,$$

to the integrand of (3.11), we take into account the relation (3.10). Here we remind us of the formula $\Gamma(z) \Gamma(1 - z) = \frac{\sin \pi z}{\pi z}$. As for the choice of the periodic function $g(z)$ one makes use of Nörlund’s technique [4]. In this way we can choose such $g(z)$ that the integrand is of exponential decay on $\Pi$.

**Q.E.D.**

**Example** Let us illustrate the above procedures by a simple example.

$$f(x) = x_1^5 + x_1^2 x_2 + x_1 x_3^2 + x_4^4.$$
We have 4 possibilities to add a new variable $x_1'$ so that the polynomial (3.12) becomes a simplicial.

$$f^{\sigma_1}(x, x') = x_1'x_1^5 + x_1^3x_2 + x_1x_2^2 + x_2^2,$$

$$f^{\sigma_2}(x, x') = x_1^5 + x_1^3x_2 + x_1x_2^2 + x_2^2,$$

$$f^{\sigma_3}(x, x') = x_1^5 + x_1^3x_2 + x_1x_2^2 + x_1'x_1^2 + x_1'x_2^2 + x_2^2,$$

$$f^{\sigma_4}(x, x') = x_1^5 + x_1^3x_2 + x_1x_2^2 + x_1'x_1^2 + x_1'x_2^2 + x_1'x_2^2.$$  

Let us calculate $L^{\sigma_3}$ and $(L^{\sigma_3})^{-1}$.

$$L^{\sigma_3} = \begin{bmatrix} 5 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$(L^{\sigma_3})^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -4 & 0 & 1 & 0 \\ 2 & -5 & 0 & 3 & 0 \\ 1 & -6 & 7 & -2 & 0 \\ 8 & -20 & 0 & 5 & 7 \\ -8 & 20 & 0 & -5 & 0 \end{bmatrix}.$$  

We have

$$L_1(J, z) = \frac{3i_1 + 2i_2 + i_3 + 8(z - 1)}{7}, L_2(J, z) = \frac{-4i_1 - 5i_2 - 6i_3 - 20(z - 1)}{7}, L_3(J, z) = \frac{7i_3}{7},$$

$$L_4(J, z) = \frac{i_1 + 3i_2 - 2i_3 + 5(z - 1)}{7}, L_5(J, z) = \frac{7z}{7}.$$  

Let us denote by $\bar{e}_1 = (5, 0, 0), \bar{e}_2 = (2, 1, 0), \bar{e}_3 = (1, 2, 1), \bar{e}_4 = (0, 4, 0), \bar{e}_5 = (0, 0, 0)$. Then we have

$$vol(\tau_5) = 3!vol(\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4) = 7.$$  

Similarly $vol(\tau_4) = 5, vol(\tau_3) = 0, vol(\tau_2) = 20, vol(\tau_1) = 8$. Remark $\tau_1 + \tau_3 + \tau_4 + \tau_5 = \tau_2$ (a subdivision of simplex into three simplices) which yields $7 + 8 + 5 = 20$. The face not affected (see Definition below) by $\sigma_3$ is that spanned by $\bar{e}_1, \bar{e}_2, \bar{e}_4$.

## 4 Hodge structure of the fibre integrals

Now we can state the relationship between the Hodge structure of the $PH^{M-2}(Z_{f'})$ and the poles of the Mellin transform after suitable period function multiplication $\prod_{\alpha \in \Gamma} \Gamma(L_\alpha(J, z)) \prod_{\beta \in \Gamma} \Gamma(1 - L_\beta(J, z))$. Here we remind us of the relation $\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}$. We will misuse the expression “the poles of the Mellin transform” in meaning those of $\prod_{\alpha \in \Gamma} \Gamma(L_\alpha(J, z)) \prod_{\beta \in \Gamma} \Gamma(1 - L_\beta(J, z))$.

**Theorem 4.1** 1) For $X^J \in Gr^p_p Gr^q_{M-2} PH^{M-2}(Z_{f'})$, $0 \leq p \leq M - 1$, the following properties hold

(a) $0 < \langle \bar{v}^p_q, J \rangle < M - 1 - p$ for $q^0 \in I^0$

(b) $M - 1 - p < \langle \bar{v}^p_q, J \rangle < (M - 1 - p)(1 + \frac{s_q^p}{B^q})$ for $q^+ \in I^+$
Theorem 4.2

1) For i.e. vertices free of \( \sigma \) the face not affected by \( \sigma \), if not affected by \( \sigma \),

\[ 1 - (M - 1 - p)(1 + \max_{q \in I^+} \frac{\gamma_q}{B_q}) < z < 2 + p - M. \]

If not \( \langle \vec{v}_q, J \rangle = 0 \).

b) The maximal pole of the Mellin transform satisfies;

\[ 1 - (M - 1 - p)(1 + \max_{q \in I^+} \frac{\gamma_q}{B_q}) < z < 2 + p - M. \]

Here the pole is not necessarily a simple pole.

2) For \( X^J \in Gr_P^N Gr^{w}_{N-1} PH^{M-2}(Z_f) \), \( 0 \leq p \leq M - 1 \), the following properties hold.

a) There exists unique index \( q \in I^+ \) such that:

\[ \langle \vec{v}_q, J \rangle = M - 1 - p \]

b) The maximal pole of the Mellin transform is the simple pole

\[ z = 2 + p - M. \]

3) For \( X^J \in Gr_P^N Gr^{w}_{N-1} PH^{M-2}(Z_f) \), \( 1 \leq r \leq M - 3 \), \( 0 \leq p \leq M - 1 \), the following properties hold.

a) There exist \( r \) indices \( q_1, \ldots, q_r \in I^+ \) such that:

\[ \langle \vec{v}_{q_1}, J \rangle = \langle \vec{v}_{q_2}, J \rangle = \cdots = \langle \vec{v}_{q_r}, J \rangle = M - 1 - p, \]

but no such \( r + 1 \) pair of indices \( q_1, \ldots, q_{r+1} \).

b) The maximal pole of the Mellin transform satisfies;

\[ z = 2 + p - M, \]

which is of order \( \leq r + 1 \) i.e. there can be cancellation of poles.

The defect number \( (r + 1) - \{ \text{order of poles} \} \) will be described in §5.

Proof of the theorem can be achieved by a combination of Theorems 1.2, 1.3 and the Proposition 3.1, Corollary 3.2. We remember here that the \( \Gamma(z) \) has simple poles at \( z = 0, -1, -2, \ldots \).

The above theorem mentions about how the Hodge structure of \( PH^{M-2}(Z_f) \) influences on the poles of the Mellin transform. How about the original Hodge structure \( PH^{N-1}(Z_f) \)? To state this relationship, we need to introduce the following notion.

Definition 3 The face \( \tau \in \Delta(f) \) is called “not affected by \( \sigma \)” in \( S_M \) if \( \tau \in \Delta(f^\sigma) \) after the extension of \((i_1, \ldots, i_N) \in \tau \subset R^N \) into \( R^M \) transforming it into the vector \((i, 0) = (i_1, \cdots, i_N, 0, \cdots, 0, 0) \in R^M. \)

The face not affected by \( \sigma \) for the polynomial (2.2) is a face (or its sub-face) spanned by the vertices

\[ \sum_{j=M-N}^{M} x_i^{\alpha(\sigma(j))} \]

i.e. vertices free of \( x_i^\sigma \).

Theorem 4.2 1) For \( x^J \in Gr_P^N Gr^{w}_{N-1} PH^{N-1}(Z_f) \), \( 0 \leq p \leq N \), for which \((i, 0) \) lies in \( \text{supp}(\frac{S_{\Delta(f^\sigma)}}{I^{i_\sigma, \Delta(f^\sigma)}}) \) not affected by \( \sigma \), the following properties hold:

a) \[ 0 < \langle \vec{v}_q^\sigma, (i, 0) \rangle < N - p \] for \( q^\sigma \in I^0, \)
\[ N - p < \langle \vec{v}_q^\sigma, (i, 0) \rangle < (N - p)(1 + \frac{\sigma}{B_q^\sigma}) \text{ for } q \in I^+, \]
\[ (N - p)(1 + \frac{\sigma}{B_q^\sigma}) < \langle \vec{v}_q^\sigma, (i, 0) \rangle < N - p \text{ for } q \in I^-, \]
if not \( \langle \vec{v}_q^\sigma, (i, 0) \rangle = 0 \), or \( \langle \vec{v}_q^\sigma, (i, 0) \rangle = 0 \).

b) The maximal pole of the Mellin transform satisfies;
\[ 1 - (N - p)(1 + \max_{q \in I^+} \frac{\sigma}{B_q^\sigma}) < z < 1 - N + p. \]

Here the pole is not necessarily a simple pole.

The proof is straightforward if one applies Theorem 4.1 to \( \Delta(f) \). We consider the \( N \)-dimensional face \( \tau_q^\sigma \subset \mathbb{Z}^N \) that is a \( N \)-dimensional simplex contained in \( \Delta(f) \). One can verify that there exist \((i, 0) \in \text{supp} \left( \frac{S_{\Delta(f)}(\tau)}{J^{\Delta(f)}(\tau)} \right) \) such that \( x^i \in Gr_{F^\sigma}Gr_{N-1}^\sigma PH^{N-1}(Z_f), 0 \leq p \leq N - 1 \) for the cases \( N = 2, 3, 4 \) by means of polyhedra realizing the formulae 5.11, [2].

We remark the following simple combinatorial fact.

**Proposition 4.3** For every \( x^i \in Gr_{F^\sigma}Gr_{N-1}^\sigma PH^{N-1}(Z_f) \), there exists an element \( \sigma \in S_M \) such that \( x^i \) is not affected by \( \sigma \). That is to say there exists \( \sigma \in S_M \) such that \( x^i \in S_{\Delta(f)} \cap S_{\Delta(f^\sigma)} \).

## 5 Hypergeometric group associated to the fibre integrals

Let us introduce two differential operators of order \( \Delta^\sigma := (M - 1)!vol_{M-1}(\Delta(f^\sigma(X) + 1)) = |\chi(Z_{f^\sigma + 1})| = |I^+| = |I^-|:

\[
(5.1) \quad P^\sigma_f(\partial_s) = \prod_{q \in I^+} \prod_{j=0}^{B_q^\sigma-1} (L_q(J, -\partial_s) + j)
\]
\[
(5.2) \quad Q^\sigma_f(\partial_s) = \prod_{q \in I^-} \prod_{j=0}^{B_q^\sigma-1} (-L_q(J, -\partial_s) - j),
\]
where \( I^+ \), \( I^- \) are those sets of indices introduced in §3. We have the following theorem as a corollary to the Proposition 3.1.

**Theorem 5.1** The fibre integral \( I_{X^+, \gamma}^\sigma(s) \) is annihilated by the operator
\[
(5.3)_1 \quad R^\sigma_f(\partial_s) = P^\sigma_f(\partial_s) - s^{\gamma^\sigma} Q^\sigma_f(\partial_s),
\]
that is to say
\[
(5.4) \quad [P^\sigma_f(\partial_s) - s^{\gamma^\sigma} Q^\sigma_f(\partial_s)] I_{X^+, \gamma}^\sigma(s) = 0.
\]

It is worthy to remark that the operator \( R^\sigma_f(\partial_s) \) is a push-forward of the Pochhammer hypergeometric operator of order \( \Delta^\sigma \),
\[
(5.3)_2 \quad P^\sigma_f(\gamma^\sigma \partial_t) - t Q^\sigma_f(\gamma^\sigma \partial_t),
\]
by the Kummer covering $t = s^{\gamma}$. In certain cases, the operator (5.3) turns out to be reducible. Let us introduce the following set of rational numbers.

$$C^+(J) = \bigcup_{q \in I^+} \bigcup_{0 \leq j \leq B_q^{-1}} \left\{ \frac{j}{B_q} - \frac{\langle \bar{v}_q \alpha, J > - 1 \rangle}{\gamma^j} \right\}.$$

$$C^-(J) = \bigcup_{q \in I^{-1}} \bigcup_{1 \leq j \leq B_q^{-1}} \left\{ \frac{j}{B_q} - \frac{\langle \bar{v}_q \alpha, J > - 1 \rangle}{\gamma^j} \right\}.$$

$$C^0(J) = C^+(J) \cap C^-(J).$$

We define a positive integer $\Delta^\alpha = \sharp[C^+(J) \setminus C^0(J)] = \sharp[C^-(J) \setminus C^0(J)]$. Then ”the irreducible part” of (5.3) (i.e. after the division by operators with rational function solution of type $t^{\alpha}$, $\alpha \in C^0(J)$) can be defined as

$$\bar{R}^\Delta_{\gamma}(\theta_t) = \prod_{\alpha + \epsilon C^+(J) \setminus C^0(J)} (\theta_t + \alpha^+) - t \prod_{\alpha - \epsilon C^-(J) \setminus C^0(J)} (\theta_t + \alpha^- + 1),$$

as an operator of order $\Delta^\alpha$ up to multiplication by a constant to the variable ”$t$”.

We consider solutions $u_{t,m}(t), 1 \leq \ell \leq \Delta^\alpha$, to the equation

$$(5.5) \quad \bar{R}^\Delta_{\gamma}(\theta_t) u_{t,m}(t) = 0,$$

with the asymptotic behaviour

$$(5.5)_1 \quad u_{t,m}(t) \cong t^{\rho_\Delta} \sum_{\mu=0}^m (\log t)^\nu A_{t,\nu}(t).$$

Here $0 \leq m \leq m_\ell$, $\sum_\ell (m_\ell + 1) = \Delta^\alpha$, hence holomorphic in the neighbourhood of $t = 0$. Similarly, we consider the asymptotic behaviour at $t = \infty$ of the solutions to (5.5)

$$v_{t,k}(t) \cong \left( \frac{1}{t} \right)^{\rho_\Delta} \sum_{\mu=0}^k (\log t)^\nu B_{t,\nu}(1/t).$$

Here $0 \leq k \leq k_\ell$, $\sum_\ell (k_\ell + 1) = \Delta^\alpha$, hence holomorphic in the neighbourhood of $1/t = 0$. Here $m_\ell + 1$ (resp. $k_\ell + 1$) denotes the multiplicity of $-\rho_\Delta$ (resp. $-\rho_\Delta$) in the set $C^+(J) \setminus C^0(J)$ (resp. $C^-(J) \setminus C^0(J)$).

Under this situation, we define characteristic polynomials of the exponents of solutions to (5.5) at $t = 0$

$$(X_0,J)(t) = \prod_{\ell=1}^{\tilde{\Delta}^\alpha}(t - e^{2\pi i \rho_\Delta J - 1}) = \prod_{\alpha + \epsilon C^+(J) \setminus C^0(J)} (t - e^{2\pi i \gamma J - 1 \alpha^+}),$$

and $t = \infty$

$$(X_{\infty,J})(t) = \prod_{\ell=1}^{\tilde{\Delta}^\alpha}(t - e^{2\pi i \rho_\Delta J - 1}) = \prod_{\alpha - \epsilon C^-(J) \setminus C^0(J)} (t - e^{2\pi i \gamma J - 1 \alpha^-}).$$

Especially in the case $C^0 = \emptyset$, we have the following simple formulae.
\textbf{Corollary 5.2} The characteristic polynomials defined above can be calculated in the following way.

\begin{align}
X_{0,i}(t) &= \prod_{\sigma \in I^*} \left( t^{B^\sigma_\gamma} - e^{-2\pi(1-\langle \sigma^\gamma, J \rangle)\frac{\sigma^\gamma}{\pi}} \right), \\
X_{\infty,i}(t) &= \prod_{\sigma \in I^*} \left( t^{-B^\sigma_\gamma} - e^{-2\pi(1-\langle \sigma^\gamma, J \rangle)\frac{\sigma^\gamma}{\pi}} \right).
\end{align}

For the polynomials introduced in (5.6)$_1$, (5.6)$_2$, we introduce two vectors \((A_1, A_2, \cdots, A_{A^*}), (B_1, B_2, \cdots, B_{A^*}) \in \mathbb{C}^{A^*}\), after the following relation:

\[ X_{0,i}(t) = t^{A_1} + A_2t^{A^*} + \cdots + A_{A^*}, \]

\[ X_{\infty,i}(t) = t^{B_1} + B_2t^{A^*} + \cdots + B_{A^*}. \]

Let us denote by \(\omega^i, i = 0, 1, 2, \cdots, \gamma^* - 1\) the non-zero singular points of the equation (5.4) i.e. \(\{ \sigma \in \mathbb{C}; \prod_{\sigma \in I^*} B_\sigma - (\prod_{\delta \in I^*} B_\delta)\sigma^\gamma = 0 \} \).

\textbf{Proposition 5.3} A representation of the hypergeometric group (global monodromy group) of the solutions to (5.5) is given by

\begin{align}
M_0 &= h_0^\omega, M_\omega = h_1 = (h_0h_\infty)^{-1}, M_\infty = h_\infty, M_\omega = h_\infty h_1 h_\infty (i = 1, 2, \cdots, \gamma^* - 1),
\end{align}

for the matrices

\[ h_0 = \begin{pmatrix}
0 & 0 & \cdots & 0 & -A_{A^*} \\
1 & 0 & \cdots & 0 & -A_{A^* - 1} \\
0 & 1 & \cdots & 0 & -A_{A^* - 2} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -A_1
\end{pmatrix}, \]

\[ (h_\infty)^{-1} = \begin{pmatrix}
0 & 0 & \cdots & 0 & -B_{A^*} \\
1 & 0 & \cdots & 0 & -B_{A^* - 1} \\
0 & 1 & \cdots & 0 & -B_{A^* - 2} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & -B_1
\end{pmatrix}. \]

where \(M_\omega\) denotes the monodromy action around the point \(\omega^i \in \mathbb{CP}^1_s\).

\textbf{proof} The monodromies of the solutions annihilated by \(R^\gamma_\alpha(\sigma)\) are given by \(h_0\), \(h_1, h_\infty\) after \(\mathbf{H}\) at \(t = 0\), \(t = 1, \infty\). Let us think of a \(\gamma^*\)-leaf covering \(\mathbb{CP}^1_s\) of \(\mathbb{CP}^1_s\) that corresponds to the Kummer covering \(s^{\gamma^*} = t\). In lifting up the path around \(t = 1\) the first leaf of \(\mathbb{CP}^1_s\), the monodromy \(h_1\) is sent to the conjugation with a path around \(t = \infty\). That is to say we have \(M_\omega = h_\infty^{-1} h_1 h_\infty\). For other leaves the argument is similar. \textbf{Q.E.D.}

In combining the above result with that of Theorem 4.1, 3), we get the following.

\textbf{Corollary 5.4} For \(X^J \in Gr^P_{\alpha^+} Gr^w_{M-2+r} PH^{M-2}(Z_{J^*}), 1 \leq r \leq M - 2, 0 \leq p \leq M - 1\) the size of a Jordan cell of the monodromies \(M_\alpha\) with unit eigenvalue arising from the term of the form (5.5)$_1$ with \(\alpha^+ = \rho^+_J\) is \(r + 1 - \# \{ \alpha^+ \in C^0(J); \alpha^+ \in \mathbb{Z} \} \).
proof It is enough to remember the following relation for a cycle $C$ avoiding $z + \alpha = 0$:

$$(r + 1)! \int_C \frac{s^{-z}}{(z + \alpha)^{r+1}} dz = \int_C s^{-z} \left( \frac{d}{dz} \frac{1}{z + \alpha} \right) dz$$

$$= \int_C \frac{1}{(z + \alpha)^{r}} s^{-z} \left( \frac{d}{dz} s^{-z} \right) dz = \int_C \frac{1}{(z + \alpha)^{r}} s^{-z} (\log s)^r dz = 2\pi \sqrt{-1} s^{\alpha} (\log s)^r.$$

If the set $C^0(J)$ is empty, the order of the poles of the Mellin transform for $X^J \in Gr_F^{g} Gr_{M-2+r}^{w}$ $PH^{M-2}(Z_{f\alpha})$ is $r + 1$ after Theorem 4.1, 3$)a)$. If $C^0(J)$ is not empty, the order of poles is reduced by $\sharp \{\alpha \in C^0(J); \alpha \in Z\}$. Q.E.D.

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