INCOMPRESSIBLE MAGNETOHYDRODYNAMIC FLOW WITH ZERO RESISTIVITY

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Abstract. We prove the existence of both local and global smooth solutions to the Cauchy problem in \( \mathbb{R}^3 \) for the incompressible magnetohydrodynamics (MHD) system. We also prove that the solution to the incompressible MHD system can be obtained as the incompressible limit of the corresponding compressible system. We apply methods in extracting weak dissipative structure which were suggested by Lei-Liu-Zhou [9].

1. Introduction

We prove the existence of both local and global smooth solutions to the equations of magnetohydrodynamics (MHD) with zero resistivity in \( \mathbb{R}^3 \):

\[
\begin{align*}
t &+ (u \cdot \nabla)u = \lambda \Delta u - \nabla p + (\nabla \times H) \times H, \\
H_t &+ (u \cdot \nabla)H = (H \cdot \nabla)u, \\
\text{div}(u) &\equiv \text{div}(H) = 0 
\end{align*}
\]

Here \( u = (u^1, u^2, u^3) \) and \( H = (H^1, H^2, H^3) \) are the unknown functions of \( x \in \mathbb{R}^3 \) and \( t \geq 0 \) representing velocity and magnetic field, \( p = p(x,t) \) is the pressure function and \( \lambda > 0 \) is the viscosity of the fluid. The system (1.1)-(1.3) is solved subject to initial conditions

\[
(u(\cdot, 0), H(\cdot, 0)) = (u_0, H_0).
\]

and the incompressibility assumption

\[
\text{div}(u_0) = \text{div}(H_0) = 0.
\]

The above system (1.1)-(1.3) can be derived by combining the Navier-Stokes equations for incompressible flow with Maxwell’s equations in free space and the ideal Ohm’s law (see Cabannes [3] for details). The subject of MHD was first initiated by Alfven [1] in 1940’s, since then it has become one of the most challenging topics in fluid dynamics. In the fully viscous case, we further have the resistivity term \( \nu \Delta B \) on the right side of (1.2), so that the system becomes

\[
\begin{align*}
t &+ (u \cdot \nabla)u = \lambda \Delta u - \nabla p + (\nabla \times H) \times H, \\
H_t &+ (u \cdot \nabla)H = (H \cdot \nabla)u + \nu \Delta H, \\
\text{div}(u) &\equiv \text{div}(H) = 0. 
\end{align*}
\]

where \( \nu > 0 \) is the electrical resistivity constant. For initial data with arbitrary large energy, Duvaut and Lions [6] and Lassener [8] showed that there exists at least
one global Leray-type weak solution to (1.6)-(1.8) in a bounded domain. Under the smallness assumption of initial data, Chen, Tan and Wang [5] prove the global existence of strong solutions to (2.1)-(2.3), while in the framework of Besov spaces, Miao and Yuan [12] showed global existence of strong solutions with small initial data in Besov space $\dot{B}^{n/p}_{p,r} - 1/p$ for $1 \leq p < \infty$ and $1 \leq r \leq \infty$.

On the other hand, when the resistivity $\nu$ becomes zero (system (1.1)-(1.4)), there is not much known result in the literature. The only result found which is related to incompressible MHD with $\nu = 0$ is Chae [4] who proved the nonexistence of asymptotically self-similar singularity of the system (1.1)-(1.4). The goal of the present paper is thus to explore the subject and to investigate both the local and global existence of $H^2$-solutions to (1.1)-(1.4) as well as the incompressible limit of the corresponding compressible system which was studied by Suen in [13].

Our main results can be stated as follows. We begin with the following local existence theorem which can be proved by standard energy estimates:

**Theorem 1.1 (Local Existence)** Let $\lambda > 0$ and $\tilde{H} \in \mathbb{R}^3$ be given. Suppose that the initial data $u_0, H_0 - \tilde{H} \in H^2(\mathbb{R}^3)$ satisfies the condition (1.5). Then there exists a positive time $T$, which depends only on $\lambda$, $||u_0||_{H^2}$ and $||H_0 - \tilde{H}||_{H^2}$, such that the initial value problem for (1.1)-(1.4) has a unique classical solution in the time interval $[0, T)$ which satisfies

$$
\partial_j t D^\alpha_x u \in L^\infty([0, T); H^{2-j-|\alpha|}) \cap L^2([0, T); H^{2-2j-|\alpha|+1}),
$$

for all $j, \alpha$ satisfying $2j + |\alpha| \leq 2$.

Next, by extracting the weak dissipation on the magnetic field $B$, we proceed to obtain higher order estimates which are sufficient for proving global existence of solutions to (1.1)-(1.4). The method we use in finding the weak dissipative structure of the MHD system was reminiscent of Lei-Liu-Zhou [9] for the incompressible viscoelastic flow. The results can be summarized as follows:

**Theorem 1.2 (Global Existence)** Let $\lambda > 0$ and $\tilde{H} \in \mathbb{R}^3$ be given. Suppose that the initial data $u_0, H_0 - \tilde{H} \in H^2(\mathbb{R}^3)$ satisfies the condition (1.5). Then there exists a positive constant $C$, which depends only on $\lambda, \tilde{H}, ||u_0||_{H^2}$ and $||H_0 - \tilde{H}||_{H^2}$, such that if the initial data further satisfies

$$
||u_0||_{H^2} + ||H_0 - \tilde{H}||_{H^2} < C,
$$

then the corresponding solution $(u, H - \tilde{H})$ described in Theorem 1.1 above exists on all of $\mathbb{R}^3 \times [0, \infty)$. 

Finally, we consider the approximate compressible MHD system and prove that solutions for (1.1)-(1.4) can be obtained as the incompressible limit of the corresponding compressible system when the Mach number tends to zero. Details will be given in section 4 and we include the theorem below for expository purpose:

**Theorem 1.3 (Incompressible Limit)** Let $\lambda > 0$ and $\tilde{H} \in \mathbb{R}^3$ be given. There exists a positive constant $\tilde{C} > 0$ such that the global classical solution for system (1.1)-(1.4) as described by Theorem 1.2 can be viewed as the incompressible limit
of the corresponding compressible system (defined later in section 4) if (4.5), (4.7) -(4.9) and (4.10)-(4.12) hold and the incompressible initial data satisfies (1.5) and

\[ \|u_0\|_{H^3} + \|H_0 - \tilde{H}\|_{H^3} < \tilde{C}. \]

The rest of the paper is organized as follows. We begin the proof of Theorem 1.1 in section 2 with several \textit{a priori} estimates for approximate solutions, and Theorem 1.1 will then follow by standard Galerkin method. In section 3 we proceed to obtain higher order estimates for local solutions as described in Theorem 1.1 by extracting weak dissipative structure from the system via a method suggested in [9], thereby proving Theorem 1.2. Finally in section 4 we outline the corresponding approximate compressible MHD system and show that solutions for (1.1)-(1.4) can be obtained as the incompressible limit of the corresponding compressible system when the Mach number tends to zero.

We make use of the following well-known Sobolev-type inequalities (see Ziemer [15] Theorem 2.1.4, Remark 2.4.3, and Theorem 2.4.4, also Alinhac [2]). First, given \( r \in [2,6] \) there is a constant \( C(r) \) such that for \( w \in H^1(\mathbb{R}^3) \),

\[ \|w\|_{L^r(\mathbb{R}^3)} \leq C(r) \left( \|w\|_{L^2(\mathbb{R}^3)}^{(6-r)/2r} \|\nabla w\|_{L^2(\mathbb{R}^3)}^{(3r-6)/2r} \right). \]  
(1.12)

Also, for \( v \in W^{3,2}(\mathbb{R}^3) \),

\[ \|v\|_{L^\infty(\mathbb{R}^3)} \leq C\|v\|_{L^2(\mathbb{R}^3)}^{\frac{3}{5}} \|D^2_x v\|_{L^2(\mathbb{R}^3)}^{\frac{2}{5}} \]  
(1.13)

and

\[ \|v\|_{L^4(\mathbb{R}^3)} \leq C\|v\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|D^2_x v\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}. \]  
(1.14)

Also, for simplicity, if \( X \) is a Banach space we will abbreviate \( X^3 \) by \( X \).

2. Local Existence: Proof of Theorem 1.1

In this section we show the local existence of solution to the system (1.1)-(1.3) and hence prove Theorem 1.1. Using the Galerkin method originally used for the standard Navier-Stokes equation [14] and later modified for viscoelastic flow [10], we can construct the approximate solutions to the momentum equation of \( u \), and then substitute this approximate \( u \) into the magnetic field equation to obtain the appropriate solutions of \( B \). To prove the convergence of the approximate solutions, we only need \textit{a priori} estimates which will be given by a sequence of lemmas.

Without loss of generality, we take \( \tilde{H} = (1,1,1) \) and define \( B = H - \tilde{H} \). Then system (1.1)-(1.3) can be expressed as the following equivalent form:

\[ u_t + (u \cdot \nabla)u = \lambda \Delta u - \nabla p + (\nabla \times B) \times B + (\nabla \times B) \times \tilde{H}, \]  
(2.1)

\[ B_t + (u \cdot \nabla)B = (B \cdot \nabla)u + (\tilde{H} \cdot \nabla)u, \]  
(2.2)

\[ \text{div}(u) = \text{div}(B) = 0. \]  
(2.3)

We begin with the following energy-balanced law:
Lemma 2.1 Assume that the hypotheses and notations of Theorem 1.1 are in force. Then
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \left( |u|^2 + |B|^2 \right) dx + \int_{\mathbb{R}^3} \lambda |\nabla u|^2 dx = 0. \tag{2.4}
\]

Proof. Multiply (2.1) by \(u\) and integrate with respect to \(x\),
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot u dx + \lambda \int_{\mathbb{R}^3} |\nabla u|^2 dx
= \int_{\mathbb{R}^3} (B \cdot \nabla) B \cdot u dx - \int_{\mathbb{R}^3} (\nabla \times B) \times \tilde{H} dx. \tag{2.5}
\]
Similarly, we multiply (2.2) by \(B\) and integrate to obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |B|^2 dx + \int_{\mathbb{R}^3} (u \cdot \nabla) B \cdot B dx - \int_{\mathbb{R}^3} (B \cdot \nabla) u \cdot B dx
= \int_{\mathbb{R}^3} (\tilde{H} \cdot \nabla) u \cdot B dx. \tag{2.6}
\]
Using (2.3), we have
\[
\int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot u dx = \int_{\mathbb{R}^3} (B \cdot \nabla) B \cdot B dx = 0,
\int_{\mathbb{R}^3} (B \cdot \nabla) B \cdot u dx + \int_{\mathbb{R}^3} (B \cdot \nabla) u \cdot B dx = - \int_{\mathbb{R}^3} \text{div}(B) B \cdot u dx = 0,
\]
and
\[- \int_{\mathbb{R}^3} (\nabla \times B) \times \tilde{H} dx + \int_{\mathbb{R}^3} (\tilde{H} \cdot \nabla) u \cdot B dx
= - \int_{\mathbb{R}^3} (\tilde{H} \cdot \nabla)(u \cdot B) dx + \int_{\mathbb{R}^3} \text{div}[u(B \cdot \tilde{H})] dx = 0.
\]
Adding (2.5) and (2.6) and using the above results, (2.4) follows. \(\square\)

Next we derive the following \(H^2\)-estimate for \(u\) and \(B\) which are crucial in proving Theorem 1.1:

Lemma 2.2 Assume that the hypotheses and notations of Theorem 1.1 are in force. Then there exist \(\alpha > 0\) and \(M > 0\) depends on \(\lambda\) such that
\[
\frac{d}{dt} \left[ ||D_x^2 B(\cdot, t)||_{L^2}^2 + ||\nabla B(\cdot, t)||_{L^2}^2 + ||\nabla u(\cdot, t)||_{L^2}^2 + ||u_t(\cdot, t)||_{L^2}^2 \right]
+ \left[ ||\nabla u(\cdot, t)||_{L^2(\mathbb{R}^3)}^2 + ||u_t(\cdot, t)||_{L^2(\mathbb{R}^3)}^2 \right]
\leq M \left[ ||D_x^2 B(\cdot, t)||_{L^2}^2 + ||\nabla B(\cdot, t)||_{L^2}^2 + ||\nabla u(\cdot, t)||_{L^2}^2 + ||u_t(\cdot, t)||_{L^2}^2 \right]^\alpha \tag{2.7}
\]

Proof. For simplicity, we first define
\[
X(t) = ||\Delta B(\cdot, t)||_{L^2}^2 + ||\nabla B(\cdot, t)||_{L^2}^2 + ||\nabla u(\cdot, t)||_{L^2}^2 + ||u_t(\cdot, t)||_{L^2}^2.
\]
We multiply the momentum equation (2.1) by $\Delta u$ and integrate,
\[
\lambda \|\Delta u(\cdot, t)\|_{L^2}^2 \\
\leq M \|\Delta u(\cdot, t)\|_{L^2} \|u(\cdot, t)\|_{L^\infty} \|\nabla u(\cdot, t)\|_{L^2} \\
+ \|B(\cdot, t)\|_{L^\infty} \|\nabla B(\cdot, t)\|_{L^2} + \|\nabla B(\cdot, t)\|_{L^2} \\
+ M \|\Delta u(\cdot, t)\|_{L^2} \|u_t(\cdot, t)\|_{L^2}
\]
\[
\leq M \|\Delta u(\cdot, t)\|_{L^2} \|u(\cdot, t)\|_{L^2} \frac{\lambda}{2} \|\nabla u(\cdot, t)\|_{L^2} \\
+ M \|\Delta u(\cdot, t)\|_{L^2} \|B(\cdot, t)\|_{L^2} \frac{\lambda}{2} \|\nabla B(\cdot, t)\|_{L^2} \\
+ M \|\Delta u(\cdot, t)\|_{L^2} \|\nabla B(\cdot, t)\|_{L^2} + M \|\Delta u(\cdot, t)\|_{L^2} \|u_t(\cdot, t)\|_{L^2},
\]
where the last inequality follows from (1.13). Therefore, using (2.4) from Lemma 2.1 and the definition of $X(t)$, there exists some $\alpha > 0$ that
\[
\|D_x^2 u(\cdot, t)\|_{L^2} \leq MX(t)^\alpha. \tag{2.8}
\]
Similarly, we differentiate (2.1) with respect to $t$, multiply it by $u_t$ and integrate to obtain
\[
\frac{1}{2} \frac{d}{dt} \|u_t(\cdot, t)\|_{L^2}^2 + \frac{\lambda}{2} \|\nabla u_t(\cdot, t)\|_{L^2}^2 \\
\leq M \|u_t(\cdot, t)\|_{L^2} \|u(\cdot, t)\|_{L^2} \frac{\lambda}{2} \|D_x^2 u(\cdot, t)\|_{L^2} \\
+ M \|B_t(\cdot, t)\|_{L^2} \|B(\cdot, t)\|_{L^2} \frac{\lambda}{2} \|D_x^2 B(\cdot, t)\|_{L^2} \\
\leq MX(t)^\alpha + M \|B_t(\cdot, t)\|_{L^2} \|D_x^2 B(\cdot, t)\|_{L^2} \tag{2.9}
\]
Next, we multiply (2.1) by $u_t$, integrate with respect to $x$, apply (1.14) and (2.8) to get
\[
\frac{\lambda}{2} \frac{d}{dt} \|\nabla u(\cdot, t)\|_{L^2}^2 + \|u_t(\cdot, t)\|_{L^2}^2 \\
\leq M \|\nabla u_t(\cdot, t)\|_{L^2} \left[ \|B(\cdot, t)\|_{L^2} + \|u(\cdot, t)\|_{L^2} \right] \\
\leq M \|\nabla u_t(\cdot, t)\|_{L^2} \left[ \|B(\cdot, t)\|_{L^2} \|B(t)\|_{L^2} + \|u(\cdot, t)\|_{L^2} \|\nabla u(\cdot, t)\|_{L^2} \right] \\
\leq M \|\nabla u_t(\cdot, t)\|_{L^2} X(t)^\alpha. \tag{2.10}
\]
Adding (2.9) and (2.10) and absorbing terms,
\[
\frac{d}{dt} \left[ \|u_t(\cdot, t)\|_{L^2}^2 + \|\nabla u(\cdot, t)\|_{L^2}^2 \right] + \|u_t(\cdot, t)\|_{L^2}^2 + \|\nabla u_t(\cdot, t)\|_{L^2}^2 \\
\leq M \left[ X(t)^\alpha + \|B_t(\cdot, t)\|_{L^2} \|D_x^2 B(\cdot, t)\|_{L^2} \right]. \tag{2.11}
\]
We are going to estimate the terms $\|B_t(\cdot, t)\|_{L^2}$, $\|\nabla B(\cdot, t)\|_{L^2}$ and $\|D_x^2 B(\cdot, t)\|_{L^2}$.
For $\|B_t(\cdot, t)\|_{L^2}$, using the magnetic field equation (2.2),
\[
\|B_t(\cdot, t)\|_{L^2} \leq M \|B(\cdot, t)\|_{L^\infty} \|\nabla u(\cdot, t)\|_{L^2} \\
+ \|u(\cdot, t)\|_{L^\infty} \|\nabla B(\cdot, t)\|_{L^2} + \|\nabla u(\cdot, t)\|_{L^2} \\
\leq M \|B(\cdot, t)\|_{L^2} \|D_x^2 B(\cdot, t)\|_{L^2} \|\nabla u(\cdot, t)\|_{L^2} + M \|\nabla u(\cdot, t)\|_{L^2} \|D_x^2 B(\cdot, t)\|_{L^2} \|\nabla B(\cdot, t)\|_{L^2} + M \|\nabla u(\cdot, t)\|_{L^2},
\]
so using (2.4) and (2.8),
\[ \|B(t, t)\|^2_{L^2} \leq MX(t)^\alpha. \] (2.12)
For the term \( \|\nabla B(t, t)\|_{L^2} \), we differentiate (2.2) with respect to \( x \) and multiply it by \( \nabla B \),
\[ \frac{1}{2} \frac{d}{dt}\|\nabla B(t, t)\|^2_{L^2} \leq M\|\nabla B(t, t)\|_{L^2}\|\nabla u(t, t)\|_{L^\infty} + M\|\nabla B(t, t)\|_{L^2}\|D^2_u u(t, t)\|_{L^2} \]
\[ + M\|\nabla B(t, t)\|_{L^2}\|B(t, t)\|_{L^2}\|D^2_u u(t, t)\|_{L^2} \]
\[ \leq M\|\nabla B(t, t)\|_{L^2}\|\nabla u(t, t)\|_{L^2}\|\nabla_B u(t, t)\|_{L^2} \]
\[ + M\|\nabla B(t, t)\|_{L^2}\|B(t, t)\|_{L^2}\|D^2_B B(t, t)\|_{L^2}\|D^2_u u(t, t)\|_{L^2} \]
\[ + M\|\nabla B(t, t)\|_{L^2}\|D^2_B u(t, t)\|_{L^2} \]
\[ \leq MX(t)^\alpha \left[ \|D^3_B u(t, t)\|_{L^2}^2 + 2\right]. \] (2.13)
and similarly for \( \|D^2_B B(t, t)\|_{L^2} \),
\[ \frac{1}{2} \frac{d}{dt}\|D^2_B B(t, t)\|^2_{L^2} \leq M \left[ \|D^2_B B(t, t)\|_{L^2}\|\nabla u(t, t)\|_{L^\infty} \right] \]
\[ + M \left[ \|D^2_B B(t, t)\|_{L^2}\|D^2_u u(t, t)\|_{L^2}\|B(t, t)\|_{L^2} \right] \]
\[ + M \left[ \|D^2_B B(t, t)\|_{L^2}\|D^2_u B(t, t)\|_{L^2}\|\nabla_B B(t, t)\|_{L^2} \right] \]
\[ + M\|D^2_B B(t, t)\|_{L^2}\|D^2_u B(t, t)\|_{L^2} \]
\[ \leq MX(t)^\alpha \left[ \|D^3_u u(t, t)\|_{L^2}^2 + \|D^3_u u(t, t)\|_{L^2} + \|D^3_u u(t, t)\|_{L^2}^2 \right]. \] (2.14)
In view of (2.13) and (2.14), it suffices to estimate \( \|D^3_u u(t, t)\|_{L^2} \). Using (2.1),
\[ \|D^3_u u(t, t)\|_{L^2} \leq M \left[ \|\nabla u(t, t)\|_{L^2} + \|D^2_u u(t, t)\|_{L^2}\|u(t, t)\|_{L^\infty} + \|\nabla u(t, t)\|_{L^2}^2 \right] \]
\[ + M \left[ \|D^2_u B(t, t)\|_{L^2}\|B(t, t)\|_{L^2} + \|\nabla B(t, t)\|_{L^\infty} \right] \]
\[ \leq MX(t)^\alpha. \] (2.15)
Applying (2.15) on (2.13) and (2.14) and adding it to (2.11),
\[ \frac{d}{dt} X(t) + \left[ \|\nabla u(t, t)\|_{L^2}^2 + \|u(t, t)\|_{L^2}^2 \right] \leq MX(t)^\alpha, \]
and hence (2.7) follows.

**proof of Theorem 1.1.** Using Galerkin method, we construct an approximated solution \( u \) to the momentum equation (2.1), and then substitute \( u \) into the magnetic field equation (2.2) to obtain approximated solution \( B \). Let \( \{u^{(m)}, B^{(m)}\} \) be a sequence of approximated solution to the system (2.4) - (2.8). By (2.4), (2.7) and (2.8),
\[ \frac{d}{dt} \left[ \|B^{(m)}(t, t)\|_{H^2}^2 + \|u^{(m)}(t, t)\|_{H^2}^2 + |B(t, t)|_{L^2}^2 + |u(t, t)|_{L^2}^2 \right] \]
\[ + \left[ \|\nabla u^{(m)}(t, t)\|_{L^2}^2 + \|u^{(m)}(t, t)\|_{L^2}^2 \right] \]
\[ \leq \left[ \|B^{(m)}(t, t)\|_{H^2}^2 + \|u^{(m)}(t, t)\|_{H^2}^2 + |B(t, t)|_{L^2}^2 + |u(t, t)|_{L^2}^2 \right]. \]
Therefore, there exists $T > 0$ which is independent of $m$ and depends only on $||u_0||^2_{H^2}$ and $||H_0 - \tilde{H}||^2_{H^2}$, such that we can find a constant $\tilde{M} = \tilde{M}(T)$ satisfying
\[
\sup_{0 \leq t \leq T} \left[ ||B^{(m)}(\cdot, t)||^2_{H^2} + ||u^{(m)}(\cdot, t)||^2_{H^2} + ||B_t(\cdot, t)||^2_{L^2} + ||u_t^{(m)}(\cdot, t)||^2_{L^2} \right] \\
+ \int_0^T \left[ ||\nabla u^{(m)}(\cdot, t)||^2_{L^2} + ||u_t^{(m)}(\cdot, t)||^2_{L^2} \right] dt \leq \tilde{M}. \tag{2.16}
\]
Taking $m \to \infty$ on (2.16), we obtain $(u, B)$ which is a solution to (2.1)-(2.3) on $[0, T) \times \mathbb{R}^3$ satisfying (1.9)-(1.10). \qed

3. Global Existence: Proof of Theorem 1.2

In this section we show the global existence of solution to the system (2.1)-(2.3) and hence prove Theorem 1.2. The most subtle part of our analysis is to extract dissipative structure of the system, which can be partially accomplished by introducing auxiliary variable functions $w$ and $v$ in Lemma 3.1. The methods we use here are reminiscent of those given by Lei-Liu-Zhou [9] for incompressible viscoelastic fluids, and we refer the reader to [9] for further details.

Specifically, we introduce an auxiliary variable $w^j$ as follows. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis of $\mathbb{R}^3$, and for $i = 1, 2, 3$, we define $w^j$ as in
\[
w^j = \Delta u + 3\lambda^{-1}(\nabla \times B) \times e^j. \tag{3.1}
\]
We are ready to obtain higher-order estimates for $u$ and $B$ with the help of $w^j$:

**Lemma 3.1** Assume that the hypotheses and notations of Theorem 1.2 are in force. Then there exist $M' > 0$ depends only on $\lambda$ such that
\[
\frac{d}{dt} \left[ ||u(\cdot, t)||^2_{H^2} + ||B(\cdot, t)||^2_{H^2} + \frac{1}{2} ||w(\cdot, t)||^2_{L^2} \right] + ||D_x^3 u(\cdot, t)||^2_{L^2} \\
+ \frac{\lambda}{2} \sum_j ||\nabla w^j(\cdot, t)||^2_{L^2} + \frac{\lambda}{2} \sum_{i \neq j} ||\nabla (w^i + w^j)(\cdot, t)||^2_{L^2} \\
\leq M' \left[ ||u(\cdot, t)||_{H^2} + ||B(\cdot, t)||_{H^2} \right] \\
\times \left[ ||\nabla u(\cdot, t)||^2_{L^2} + ||D_x^3 u(\cdot, t)||^2_{L^2} + \sum_j ||\nabla w^j(\cdot, t)||^2_{L^2} \right] \\
+ M' ||\nabla u(\cdot, t)||^2_{L^2}, \tag{3.2}
\]
where $w^j$ is defined in (3.1) and $w = \sum_j w^j$.

**Proof.** First recall from Lemma 2.2 that
\[
\frac{1}{2} \frac{d}{dt} ||D_x^2 B(\cdot, t)||^2_{L^2} \leq M ||B(\cdot, t)||_{H^2} \left[ ||D_x^2 B(\cdot, t)||^2_{L^2} + ||D_x^3 u(\cdot, t)||^2_{L^2} + ||\nabla u(\cdot, t)||^2_{L^2} \right]. \tag{3.3}
\]
Next, we differentiate (2.1) twice with respect to \(x\), multiply it by \(D_x^2 u\) and integrate to obtain

\[
\frac{d}{dt} ||D_x^2 u(\cdot, t)||_{L^2}^2 + \lambda ||D_x^3 u(\cdot, t)||_{L^2}^2 
\leq M \left[ ||D_x^2 u(\cdot, t)||_{L^2}^2 ||\nabla u(\cdot, t)||_{L^\infty} + ||D_x^2 B(\cdot, t)||_{L^2} ||D_x^3 u(\cdot, t)||_{L^2} \right. 
\leq M \left[ ||u(\cdot, t)||_{H^2} + ||B(\cdot, t)||_{H^2} \right] \left[ ||D_x^2 B(\cdot, t)||_{L^2}^2 + ||D_x^3 u(\cdot, t)||_{L^2}^2 + ||\nabla u(\cdot, t)||_{L^2}^2 \right].
\]  

(3.4)

Adding (3.3) and (3.4),

\[
\frac{d}{dt} ||D_x^2 u(\cdot, t)||_{L^2}^2 + ||D_x^2 B(\cdot, t)||_{L^2}^2 + ||D_x^3 u(\cdot, t)||_{L^2}^2 
\leq M \left[ ||u(\cdot, t)||_{H^2} + ||B(\cdot, t)||_{H^2} \right] \left[ ||D_x^2 B(\cdot, t)||_{L^2}^2 + ||D_x^3 u(\cdot, t)||_{L^2}^2 + ||\nabla u(\cdot, t)||_{L^2}^2 \right].
\]  

(3.5)

Recall the definition of \(w^j\) that

\[ w^j = \Delta u + 3\lambda^{-1}(\nabla \times B) \times e^j. \]

Then using (2.1) and (2.2), for any pair \((i, j)\), we have

\[
\int_{\mathbb{R}^3} w^i \cdot w^j \, dx - \lambda \int_{\mathbb{R}^3} \Delta w^i \cdot w^j \, dx 
= \int_{\mathbb{R}^3} \Delta [(u \times B) \times (\tilde{H} - 3e_i)] \cdot w^j \, dx - \int_{\mathbb{R}^3} \Delta \nabla p \cdot w^j \, dx 
+ \lambda^{-1} \int_{\mathbb{R}^3} [\nabla \times ((B \cdot \nabla) u) \times 3e_i] \cdot w^j \, dx - \lambda^{-1} \int_{\mathbb{R}^3} [\nabla \times ((u \cdot \nabla) B) \times 3e_i] \cdot w^j \, dx 
+ \lambda^{-1} \int_{\mathbb{R}^3} [\nabla \times (\tilde{H} \cdot \nabla u) \times 3e_i] \cdot w^j \, dx
\]  

(3.6)

The third and the fourth term on the right side of (3.6) can be estimated as follows:

\[
\left| \lambda \int_{\mathbb{R}^3} [\nabla \times ((B \cdot \nabla) u) \times 3e_i] \cdot w^j \, dx - \lambda \int_{\mathbb{R}^3} [\nabla \times ((u \cdot \nabla) B) \times 3e_i] \cdot w^j \, dx \right| 
\leq M \left[ ||\nabla u(\cdot, t)||_{L^\infty}^2 + ||D_x^2 u(\cdot, t)||_{L^2}^2 ||u(\cdot, t)||_{L^\infty} \right] ||\nabla w^j(\cdot, t)||_{L^2} 
+ M \left[ ||D_x^2 u(\cdot, t)||_{L^2} ||B(\cdot, t)||_{L^\infty} \right] ||\nabla w^j(\cdot, t)||_{L^2} 
\leq M \left[ ||u(\cdot, t)||_{H^2} + ||B(\cdot, t)||_{H^2} \right] ||\nabla w^j(\cdot, t)||_{L^2} 
\times \left[ ||\nabla w^j(\cdot, t)||_{L^2}^2 + ||\nabla u(\cdot, t)||_{L^2}^2 + ||D_x^2 B(\cdot, t)||_{L^2}^2 + ||D_x^3 u(\cdot, t)||_{L^2}^2 \right].
\]  

(3.7)

The last term on the right side of (3.6) is bounded by

\[
\left| \lambda \int_{\mathbb{R}^3} [\nabla \times (\tilde{H} \cdot \nabla u) \times 3e_i] \cdot w^j \, dx \right| \leq M ||\nabla w(\cdot, t)||_{L^2} ||\nabla u(\cdot, t)||_{L^2}.
\]  

(3.8)

For the term \(- \int_{\mathbb{R}^3} \Delta \nabla p \cdot w^j \, dx\) as appeared on the right side of (3.3), we apply the divergence operator \(\text{div}(\cdot)\) on (2.1) to get

\[ \Delta p = \text{div}(u \cdot \nabla u) + \text{div}((\nabla \times B) \times B) + \text{div}((\nabla \times B) \times \tilde{H}), \]
and therefore
\[- \int_{\mathbb{R}^3} \Delta \nabla p \cdot w^j dx = \int_{\mathbb{R}^3} (\text{div}(u \cdot \nabla)u + \text{div}(\nabla \times B) \times B) \text{div}(w^j) dx
\]
\[+ \int_{\mathbb{R}^3} \text{div}(\nabla \times B) \times \tilde{H} \text{div}(w^j) dx. \quad (3.9)\]

The first 2 terms on the right side of (3.9) can be bounded in a similar way as (3.7), while for the term \(\int_{\mathbb{R}^3} \text{div}(\nabla \times B) \times \tilde{H} \text{div}(w^j) dx\), using the fact that, for \(\text{div} B = 0\),
\[\Delta B^j = -\text{div}[\nabla \times B] e_j\]
for each \(j\), so we have
\[\int_{\mathbb{R}^3} \text{div}(\nabla \times B) \times \tilde{H} \text{div}(w^j) dx = \sum_k \int_{\mathbb{R}^3} \text{div}[\nabla \times B] \times e_k \text{div}(w^j) dx\]
\[= - \int_{\mathbb{R}^3} (\sum_k \Delta B^k) \text{div}(w^j) dx\]
\[= \frac{\lambda}{3} \int_{\mathbb{R}^3} \text{div}(w) \text{div}(w^j) dx,\]

where the last equality of the above follows from the definition of \(w\). By direct computation
\[\sum_{i,j} \int_{\mathbb{R}^3} \text{div}(w) \text{div}(w^j) dx = \frac{1}{2} \sum_{i \neq j} \int_{\mathbb{R}^3} \text{div}(w) \text{div}(w^i + w^j) dx,\]
and hence we have
\[\sum_{i,j} \int_{\mathbb{R}^3} \text{div}(\nabla \times B) \times \tilde{H} \text{div}(w^i + w^j) dx = \frac{\lambda}{6} \sum_{i \neq j} \int_{\mathbb{R}^3} \text{div}(w) \text{div}(w^i + w^j) dx\]
\[\leq \frac{\lambda}{6} \left( \sum_j |w^j|^2 dx \right)^{\frac{1}{2}} \times \left( \sum_{i \neq j} \int_{\mathbb{R}^3} |\nabla (w^i + w^j)|^2 dx \right)^{\frac{1}{2}}. \quad (3.10)\]

Summing over \(i\) and \(j\) on (3.6), applying (3.7), (3.8), (3.9), (3.10) on it and absorbing appropriate terms, the result (3.2) follows immediately. \(\square\)

**proof of Theorem 1.2.** By Theorem 1.1, there is positive constant \(\varepsilon > 0\) and a positive time \(T_\varepsilon > 0\) such that, if the initial data is given satisfying
\[\|u_0\|_{H^2} + \|H_0 - \tilde{H}\|_{H^2} < \varepsilon\]
then there is a solution \((u, B)\) defined on \(\mathbb{R}^3 \times [0, T_\varepsilon)\) satisfying
\[u \in C([0, T_\varepsilon); H^2) \cap L^\infty([0, T_\varepsilon); H^2) \cap L^2([0, T_\varepsilon); H^3),\]
\[H - \tilde{H} \in C([0, T_\varepsilon); H^2) \cap L^2([0, T_\varepsilon); H^3),\]

(3.11) (3.12)
\[ \partial_t u \in L^\infty([0, T_\varepsilon); L^2) \cap L^2([0, T_\varepsilon); H^1), \tag{3.13} \]

\[ \partial_t H \in L^\infty([0, T_\varepsilon); L^2), \tag{3.14} \]

\[ ||u(\cdot, t)||_H^2 + ||H(\cdot, t) - \tilde{H}||_H^2 \leq \frac{\lambda(M')^{-1}}{4}, \tag{3.15} \]

where \( M' \) is defined as in Lemma 3.1 which we now fix. Using (3.13) on (3.12), integrating with respect to \( t \) and absorbing terms, we get

\[ \sup_{0 \leq s \leq T_\varepsilon} ||u(\cdot, s)||_H^2 + ||H - \tilde{H}(\cdot, s)||_H^2 + \int_0^{T_\varepsilon} ||\nabla u(\cdot, s)||_H^2 ds \]
\[ \leq \left( \frac{\lambda}{4} + M' \right) \int_0^\infty ||\nabla u(\cdot, s)||_L^2 ds \]
\[ \leq \left( \frac{1}{8} + \frac{M'\lambda^{-1}}{2} \right) \left[ ||u_0||_L^2 + ||H_0 - \tilde{H}||_L^2 \right], \]

where the last inequality follows from Lemma 2.1. We choose \( a > 0 \) such that

\[ a < \min \left\{ \varepsilon, \sqrt{\frac{\lambda(M')^{-1}}{4} \left( \frac{1}{8} + \frac{M'\lambda^{-1}}{2} \right)^{-1}} \right\}, \]

then if \( ||u_0||_H^2 + ||H_0 - \tilde{H}||_H^2 < a \), we have a solution \((u, B)\) defined on \( \mathbb{R}^3 \times [0, T_\varepsilon) \) satisfying (3.11)-(3.14) and

\[ ||u(\cdot, t)||_H^2 + ||H - \tilde{H}(\cdot, t)||_H^2 < \frac{\lambda(M')^{-1}}{4} \]

and hence we can extend \((u, B)\) continuous to a solution on \( \mathbb{R}^3 \times [0, T_\varepsilon) \). By taking \( t = T_\varepsilon \) as the new initial time and using , we obtain a solution on \( \mathbb{R}^3 \times [0, 2T_\varepsilon) \) with

\[ \sup_{0 \leq s \leq 2T_\varepsilon} ||u(\cdot, s)||_H^2 + ||H - \tilde{H}(\cdot, s)||_H^2 + \int_0^{2T_\varepsilon} ||\nabla u(\cdot, s)||_H^2 ds \]
\[ \leq \frac{1}{2} \left[ ||u_0||_L^2 + ||H_0 - \tilde{H}||_L^2 \right], \]

We repeat the above argument and finally obtain a global solution \((u, B)\) defined on \( \mathbb{R}^3 \times [0, \infty) \) satisfying

\[ \sup_{t} ||u(\cdot, t)||_H^2 + ||H - \tilde{H}(\cdot, t)||_H^2 + \int_0^\infty ||\nabla u(\cdot, s)||_H^2 ds \]
\[ \leq \frac{1}{2} \left[ ||u_0||_L^2 + ||H_0 - \tilde{H}||_L^2 \right] \]

which finishes the proof. \( \square \)
4. Incompressible limits: Proof of Theorem 1.3

In this section we consider the approximate compressible MHD system and prove that solutions for \((1.1)-(1.4)\) can be obtained as the incompressible limit of the corresponding compressible system when the Mach number tends to zero. The results we obtained here are parallel to those in [9] for viscoelastic flow. Here we also recall a result [7] obtained by Jiang-Ju-Li recently about incompressible limit of the compressible non-isentropic MHD equations with zero magnetic diffusivity and general initial data in both \(\mathbb{R}^2\) and \(\mathbb{R}^3\).

We first begin with the compressible MHD system which takes the following form:

\[
\rho_t^\varepsilon + \text{div}(\rho^\varepsilon u^\varepsilon) = 0, \tag{4.1}
\]

\[
(\rho^\varepsilon u^\varepsilon)_t + \text{div}(\rho^\varepsilon u^\varepsilon u^\varepsilon) + \varepsilon^{-2}P(\rho^\varepsilon)_x_j + (\frac{1}{2} |H^\varepsilon|^2)_x_j - \text{div}(H^\varepsilon J^\varepsilon) = \mu \Delta u^\varepsilon + \lambda \text{div} u^\varepsilon, \tag{4.2}
\]

\[
H^\varepsilon_t + \text{div}(H^\varepsilon u^\varepsilon - u^\varepsilon J^\varepsilon) = 0, \tag{4.3}
\]

\[
\text{div} H^\varepsilon = 0. \tag{4.4}
\]

where \(\rho^\varepsilon = \rho^\varepsilon(x, t)\) is the density function, \(P = P(\rho^\varepsilon)\) is a given function of pressure having the form

\[
P(\rho^\varepsilon) = K(\rho^\varepsilon)^\gamma, \tag{4.5}
\]

with \(K > 0\) and \(\gamma \geq 1\) being independent of \(\varepsilon\), and \(\varepsilon\) is the Mach number. The system is solved subjected to initial data

\[
(\rho^\varepsilon(\cdot, 0), u^\varepsilon(\cdot, 0), H^\varepsilon(\cdot, 0)) = (\rho_0^\varepsilon, u_0^\varepsilon, H_0^\varepsilon) \tag{4.6}
\]

and \((\rho_0^\varepsilon(x), u_0^\varepsilon(x), H_0^\varepsilon(x))\) satisfies

\[
\rho^\varepsilon(x, 0) = \rho_0^\varepsilon(x) = \hat{\rho} + \tilde{\rho}_0, \tag{4.7}
\]

\[
u^\varepsilon(x, 0) = u_0^\varepsilon(x) = u_0(x) + \tilde{u}_0^\varepsilon, \tag{4.8}
\]

\[
H^\varepsilon(x, 0) = H_0^\varepsilon(x) = H_0(x) + \tilde{H}_0^\varepsilon, \tag{4.9}
\]

where \((u_0, H_0)\) satisfy (4.1.5) and there are constants \(\tilde{H} \in \mathbb{R}^3\), \(C > 0\) and \(\tilde{\rho}, \tilde{\rho} > 0\) independent of \(\varepsilon\) such that \(\rho^\varepsilon(x), u^\varepsilon(x), H^\varepsilon(x)\) are assumed to satisfy

\[
\tilde{\rho} > \tilde{\rho}_0 > 0, \tag{4.10}
\]

\[
\|\tilde{\rho}_0\|_{L^3} \leq C\varepsilon^2, \tag{4.11}
\]

\[
\|\tilde{u}_0\|_{H^4} \leq C\varepsilon, \tag{4.12}
\]

\[
\|\tilde{H}_0\|_{H^3} \leq C\varepsilon. \tag{4.13}
\]

The proof of Theorem 1.3 relies on the following lemma which can be proved in a similar way as given in Suen [13]. It can be stated as follows:

**Lemma 4.1** Assume that \(P\) satisfies (4.1.5) and let \(\tilde{\rho}, \mu, \lambda > 0\) and \(\tilde{H} \in \mathbb{R}^3\) be given. Then given \(\tilde{\rho} > 0\) and \(\varepsilon > 0\), there is a positive time \(T\) depending on \(\tilde{\rho}, \varepsilon\) and on the system parameters \(\mu, \lambda, P\) such that if initial data \((\rho_0^\varepsilon, u_0^\varepsilon, H_0^\varepsilon)\) is given satisfying (4.1.7)-(4.1.9) and (4.1.10)-(4.1.12), then there is a classical solution \((\rho^\varepsilon, u^\varepsilon, H^\varepsilon)\) to (4.1)-(4.4). The solution satisfies the following:

\[
\rho^\varepsilon - \tilde{\rho}, H^\varepsilon - \tilde{H} \in C([0, T]; H^3(\mathbb{R}^3)) \cap C^1([0, T]; H^2(\mathbb{R}^3))
\]
where $C_{\varepsilon,0} = ||\varepsilon(\rho^\varepsilon - \tilde{\rho})||_{H^3} + ||u_0||_{H^3} + ||H_0 - \tilde{H}||_{H^3}$.

**proof of Theorem 1.3.** It follows by a similar compactness argument as given by Lei-Zhou [11] with the use of the estimates (4.14)-(4.15) in Lemma 4.1. We omit the details here. \qed

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