INVARIANT SMOOTH QUARTIC SURFACES BY ALL FINITE PRIMITIVE GROUPS OF
PGL$_4$(C)

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ABSTRACT. For each finite primitive subgroup $G$ of PGL$_4$(C), we find all the smooth $G$-invariant quartic surfaces. We also find all the faithful representations in PGL$_4$(C) of the smooth quartic $G$-invariant surfaces by the groups: $\mathfrak{A}_5$, $\mathfrak{S}_5$, PSL$_2(F)$, $\mathfrak{A}_6$, $\mathfrak{Z}_3 \times \mathfrak{Z}_5$ and $\mathfrak{Z}_3 \times D_{10}$. The primitive representation of these groups are precisely the subgroups of PGL$_4$(C) for which $\mathbb{P}^3$ is not $G$-super rigid. As a byproduct, we show that the smooth quartic surface with the biggest group of projective automorphism is given by

\[
\{x_0^4 + x_1^4 + x_2^4 + x_3^4 + 12x_0x_1x_2x_3 = 0\}
\]

(unique up to projective equivalence).

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1. INTRODUCTION

It is well known that the simplest algebraic varieties are the hypersurfaces. However, it is a good place to test and find examples of general theories. In the history of algebraic geometry, there has been a lot of research around hypersurfaces, since the classical works of Cayley in the 19th century, to the more recent works about rigidity presented by J. Kollár [Kol19b], where the author synthesized a century of mathematical effort which start at times of Fano, and show that any smooth hypersurfaces of dimension $n$ with degree $d = n + 1$ are super rigid. Concerning automorphisms of hypersurfaces, a remarkable result due to Matsumura and Monsky, see [MM64, Theo. 2], states that for hypersurfaces of degree $d$ and dimension $n$, except for $n = 2$ and $d = 4$, the group of projective automorphisms of a hypersurface $X$, denoted by PAut$(X)$, and the automorphisms of the hypersurface $X$, denoted by Aut$(X)$, are the same group. More recently, V. González-Aguilera, A. Liendo, and P. Montero showed that for $d \geq 3$, $n \geq 1$ and $(n, d) \neq (2, 3), (3, 4)$, the automorphism group of every smooth hypersurface of dimension $n - 1$ and degree $d$ in $\mathbb{P}^n$ is liftable, (i.e., we can realize Aut$(X)$ as a subgroup of GL$(V)$) if and only if the number of variables and the degree are relative prime, i.e., $\gcd(n + 1, d) = 1$, see, e.g., [GALM20]. On the other hand, we have that even for the most basic K3 surfaces $S$, the quartic surfaces, the groups PAut$(S)$, and Aut$(S)$ are different, see, for instance, [MM64, GALM20]. Furthermore, to study the group PAut$(S)$ it is necessary to go through the groups $G$ which are subgroups of PGL$_4$(C) and not just subgroups of GL$_4$(C).

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Recall that, a variety $X \subseteq \mathbb{P}^n$ is $G$-invariant for a subgroup $G \leq \text{PGL}_n(\mathbb{C})$ if $G \subseteq \text{PAut}(X)$. To find $G$-invariant smooth quartic surfaces in $\mathbb{P}^3$, we need to have a list of possible subgroups of $\text{PGL}_4(\mathbb{C})$. The finite subgroups of $\text{PGL}_4(\mathbb{C})$ were classified by H. Blichfeld in his keynote book [Bli18, Chap. VII]. In [CS19, Appendix A], the authors’ present diagrams of all finite primitive groups of $\text{PGL}_4(\mathbb{C})$ including their inclusions, see the Figures 1, 2, 3, and 4.

The main goal of our work is to find all the smooth quartic surfaces in $\mathbb{P}^3$ invariant by the finite primitive subgroups $G$ of $\text{Aut}(\mathbb{P}^3)$. Also, we found all the smooth quartic surfaces invariant by the finite primitive subgroups of $\text{PGL}_4(\mathbb{C})$ for which $\mathbb{P}^3$ is not $G$-super rigid. These groups are isomorphic to $\mathfrak{A}_5$, $\mathfrak{S}_5$, $\text{PSL}_2(\mathbb{F}_7)$, $\mathfrak{A}_6$, $\mathbb{Z}_2^4 \times \mathbb{Z}_5$, and $\mathbb{Z}_2^4 \times D_{10}$, see [CS19, Theo.1.3].

Recall that $\mathbb{P}^3$ is said to be $G$-birational super-rigid, if:

1. There are no other $G$-Fano varieties $G$-equivariantly birational to $\mathbb{P}^3$,
2. There are no $G$-equivariant birational map from $\mathbb{P}^3$ to a variety $X$ such that there is a (non-birational) $G$-equivariant epimorphism $\pi : X \to Z$ where $\dim(X) > \dim(Z) \neq 0$ and whose general fiber is an irreducible rationally connected variety, and
3. $\text{Bir}^G(\mathbb{P}^3) = \text{Aut}^G(\mathbb{P}^3)$, where $\text{Bir}^G(\mathbb{P}^3)$ is the normalizer of the group $G$ in $\text{Bir}(\mathbb{P}^3)$. Similarly, $\text{Aut}^G(\mathbb{P}^3)$. For a detailed explanation, see [CS15, Def. 3.1.1].

The classification of groups of automorphisms of hypersurfaces is at an early state. Some results are known; for instance, in [DD19], the authors classify all possible automorphism groups of smooth cubic surfaces over an algebraically closed field of arbitrary characteristics. For non-singular cubic hypersurfaces in $\mathbb{P}^4$ over algebraically closed fields of characteristic 0, the work is completely done, see [WY20]. A partial classification of automorphisms of cubic fourfolds is given in [LZ22]. Oguiso and Yu in [OY19] studied the automorphisms of the smooth quintic threefold.

In the case of quartic surfaces, there is no complete description. However, there are several works in that direction. The quartic surfaces $G$-invariants by the alternating group $\mathfrak{A}_5$, icosahedron group, were found in [FMPK16] using projective representation theory. By different approaches, such surfaces were also found by I. Dolgachev in [Dol18]. The quartic surfaces $G$-invariant by $G \in \{\mathfrak{A}_6, \mathfrak{S}_5\}$, were studied in [FMPK16]. In particular, the authors show that there are no $\mathfrak{A}_6$-invariants quartic surfaces.

In [MPK19], the authors determine for every $p \geq 5$, all $\mathbb{Z}_p$-invariant non-singular quartic surfaces in $\mathbb{P}^3$ over an algebraically closed field of characteristic zero. The invariant quartic surfaces by the Heisenberg group, $H \cong \mathbb{Z}_2^4$, were studied by D. Eklund in [Ek118]. In that work, the author finds all the $H$-invariant quartic surfaces and shows that for a generic of surfaces, its Picard number is 16.

Here we tackle the problem using the techniques presented in [FMPK16, MPK19]. We develop, use, and implement a computational method, see Appendix A. Using that method, we retrieve the results presented in [FMPK16] and in [Dol18]. Beyond the previous results, we find the smooth invariant quartic surfaces for each one of the primitive subgroups of $\text{Aut}(\mathbb{P}^3) \cong \text{PGL}_4(\mathbb{C})$.

The computational method presented in Appendix A can be used to resolve the general version of the problem of finding the basis of the space of invariant polynomials. Nevertheless, the big obstacle is to understanding the finite subgroups $G$ of $\text{PGL}_n(\mathbb{C})$, for $n \geq 4$, and to describe any projective faithful representation for each group. In particular, for Calabi-Yau hypersurfaces, it is more complicated because, in these cases, the degree and number of variables are not relatively primes. For that reason, the results presented by V. González-Aguilera, A. Liendo, and P. Montero [GALM20, Theo. 3.5] can not be applied.

The main result obtained in this work is the description of all the smooth quartic surfaces invariant for primitive groups of $\text{PGL}_4(\mathbb{C})$. Yet another important motivation for studying quartic surfaces is a longstanding conjecture by Burnside. Burnside established in a conjectural way that every smooth quartic surface has a group of projective automorphisms of order bounded above by $2^4 \cdot 120$, and that if a projective surface has a group of projective automorphisms of order $1920 = 2^4 \cdot 120$, it must be projective equivalent to $\{x_0^4 + x_1^4 + x_2^3 + x_3^3 + 12x_0x_1x_2x_3 = 0\}$, see [Bur55, Sect. 272, Ex. 6]. Kondō in [Kon99] showed that the maximum order group of automorphisms of a K3 surface is $3840$. The maximum order is achieved only for the Kummer surface $\text{Kum}(E_i \times E_i)$, where $E_i$ is the elliptic curve $\mathbb{C}/(1, i)$, and the group acting is a $\mathbb{Z}_4$ extension of the Mathieu Group $M_{20} \cong \mathbb{Z}_2^4 \times \mathfrak{A}_5$. Recently, Bonnafé and Sarti
proved in [BS21] that there are only two K3 surfaces admitting an action of a group extension $G$ of $M_{20}$ with $|G| = 1920$. Finally, by [BS21, Remark 3.3], the Kondō Kummer is not a quartic surface. On the other hand, in the same work is proved that their examples (unique) are the smooth quartic surfaces \( \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 + 12x_0x_1x_2x_3 = 0 \} \) and the Kummer surface $\text{Kum}(E_7 \times E_{27})$, where $\tau = \frac{-1 + \sqrt{7}}{2}$ which cannot be a quartic surface.

**Theorem A.** The smooth quartic surfaces invariant by at least one finite primitive group of $\text{PGL}_4(\mathbb{C})$ are listed as follows:

1. The invariant quartic smooth surface under the primitive action of the groups $\mathbb{Z}_2^4 : S_5$, $\mathbb{Z}_2^4 : A_5$, $\mathbb{Z}_2^4 : (\mathbb{Z}_5 \times \mathbb{Z}_4)$, $\mathbb{Z}_2^2 \times D_{10}$, and $\mathbb{Z}_2^4 \times \mathbb{Z}_5$ is
   \[
   \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 + 6 \left( x_0^2 x_1^2 - x_0^2 x_1^2 + x_0^2 x_1^2 + x_1^2 x_2^2 - x_0^2 x_2^2 + x_0^2 x_3^2 \right) = 0 \}.
   \]
   Moreover, the surface presented above is projectively equivalent to the Burnside quartic surface
   \[
   \{ x_0^4 + x_1^4 + x_2^4 + x_3^4 + 12x_0x_1x_2x_3 = 0 \}.
   \]

2. All the quartic surfaces invariant by the primitive representations in $\text{PGL}_4(\mathbb{C})$ of $\mathfrak{A}_5$ are in the pencils of quartics
   (a) The Pencil
   \[
   \left\{ \lambda_0 \left( \frac{x_0^4}{2\sqrt{3}} + \frac{x_1^4}{3} + x_2^4 + x_3^4 + x_0^2 x_1^2 + \frac{1}{3} x_1^3 x_0 + \frac{1}{3} \sqrt{2} x_2^3 x_0 + \sqrt{\frac{2}{3}} x_3^3 x_0 \right)
   + \sqrt{\frac{2}{3}} x_1 x_2 - \frac{1}{3} \sqrt{2} x_1 x_2 + x_2 x_3 x_3 - \frac{x_0^2}{2\sqrt{3}} \right) + \lambda_1 \left( \frac{x_1^4}{6} + \frac{x_1 x_2}{\sqrt{3}} - \frac{1}{2} x_1^2 x_0 + \sqrt{3} x_2 x_3 x_0 + \frac{1}{3} \sqrt{2} x_3^3 x_0 + x_1 x_2 x_3 x_0
   + \frac{2}{3} \sqrt{2} x_1 x_2 - \sqrt{\frac{2}{3}} x_1 x_3 - \frac{1}{3} x_2 x_3 - \frac{x_0^2}{3} \right) = 0 \},
   \]
   where the singular fibers are finitely many and at least must include the fibers over the following points:
   \[
   [ -2\sqrt{2} : \sqrt{i \sqrt{15} - 1} ] , [ -2\sqrt{2} : \sqrt{i \sqrt{15} - 1} ] , [ 47 : 8 \left( 1 - 4\sqrt{3} \right) ]
   \]
   \[
   [ 47 : -8 \left( 1 + 4\sqrt{3} \right) ] , [ 4 : \sqrt{3} + i \sqrt{5} ] , [ 4 : \sqrt{3} - i \sqrt{5} ] , \text{ and } [ 1 : -\sqrt{3} ] .
   \]

(b) The pencil
   \[
   \left\{ \lambda_0 \left( \frac{x_0^4}{\sqrt{15}} - \left( \frac{x_2^4 x_0^2}{2\sqrt{15}} - \frac{x_1^3 x_0}{3} - \frac{1}{3} x_2 x_0 - \frac{1}{3} x_3^3 x_0 + x_1 x_2 x_3 x_0 \right)
   \right) + \frac{1}{6} \sqrt{\frac{5}{3}} x_1 - \frac{1}{3} \sqrt{\frac{5}{3}} x_2 x_3^2 + \frac{1}{3} \sqrt{\frac{5}{3}} x_1 x_3^2 + \frac{1}{2} \sqrt{\frac{5}{3}} x_2 x_3^2 \right) + \lambda_1 \left( \frac{x_0^4}{4} + \frac{1}{2} x_1^2 x_2 + x_2 x_3 x_0^2 + \frac{x_0^4}{4} + x_2 x_3^2 + x_2 x_3^2 \right) = 0 \},
   \]
   where the singular ones are the fibers on the five points of $\mathbb{P}^1_{\lambda_0, \lambda_1}$
   \[
   [0 : 1], [1 : 0], [\sqrt{15} : -4], [2\sqrt{15} : -5], \text{ and } [6\sqrt{15} : 1] .
   \]
(3) The smooth quartic surfaces invariant by the primitive representations in $\text{PGL}_4(\mathbb{C})$ of $\mathfrak{S}_5$ are in the pencil of quartics (b), plus the surfaces
\[
\begin{align*}
&\{-\frac{x_0^4}{12} + \frac{x_1^2 x_0^3}{2\sqrt{3}} - \frac{x_2^3 x_0}{6} - \frac{x_1^3 x_0}{\sqrt{2}} + \frac{x_3^3 x_0}{\sqrt{6}}\}, \\
&+ \frac{x_1 x_0^3}{3\sqrt{2}} - \frac{x_2^2 x_0^2}{2} - \frac{x_1^2 x_3 x_0}{2} - \frac{x_3^3 x_0}{\sqrt{6}} - \frac{x_1^3 x_0}{\sqrt{2}} - \frac{x_3^3 x_0}{\sqrt{6}}
\end{align*}
\]
and
\[
\begin{align*}
&\left\{\frac{x_0^4}{2\sqrt{3}} + \frac{1}{2} x_1 x_0^3 + x_2^2 x_3 x_0^2 + \frac{1}{3} x_1^3 x_0 + \frac{1}{3} \sqrt{2} x_2 x_0^3 + \sqrt{\frac{2}{3}} x_3 x_0
\right. \\
&+ \sqrt{\frac{2}{3}} x_1 x_0^3 + \frac{1}{3} \sqrt{2} x_1 x_3 x_0^2 + x_1^2 x_2 x_0 - \frac{x_1^4}{2\sqrt{3}} = 0\}
\end{align*}
\]

(4) The unique smooth quartic surface invariant by the primitive representations in $\text{PGL}_4(\mathbb{C})$ of $\text{PSL}_2(\mathbb{F}_7)$ is $\{2x_0^4 + 6x_1^2 x_2 x_3 x_0 + x_1 x_3^3 + x_1^2 x_2 + x_2^2 x_3 = 0\}$. 

**Corollary B.** The smooth quartic surface invariant by the biggest group of projective automorphisms being finite primitive in $\text{PGL}_4(\mathbb{C})$ is $\{x_0^4 + x_1^4 + x_2^4 + x_3^4 + 12x_0 x_1 x_2 x_3 = 0\}$ (unique up to projective equivalence). Furthermore, its group of projective automorphisms is isomorphic to $\mathbb{Z}_5^4 \rtimes \mathfrak{S}_5$ (with order 1920).

**Remark 1.1.** In [LZ22], the authors affirm that the Kondô surface $X_{K0} = \text{Kum}(E_i \times E_i)$ is the quartic $\{x_0^4 + x_1^4 + x_2^4 + x_3^4 + 12x_0 x_1 x_2 x_3 = 0\}$, which seen to be an inaccuracy because of the work of Bonnafé and Sarti, [BS21, Remark 3.3].

**Outline of the article.** The paper is organized as follows. In Section 2 gathers notation, global conventions, and known results that will be used throughout the paper. In Section 3 is presented the mathematics behind the computational algorithms. Section 4 contains relabeling of the diagrams presented in [CS19, Appendix A]. These diagrams show some inclusions between all finite primitive subgroups of $\text{PGL}_4(\mathbb{C})$, up to conjugates. The description of all these groups can be seen in [Bli18, Chapter VII]. In Section 5, we hand over explicitly all the projective primitive representations of the groups studied. In Section 6, for the sake of completeness, we present all the Non-Primitive projective representations of the groups $\mathfrak{A}_5, \mathfrak{S}_5, \mathfrak{A}_6, Z_2^4 \rtimes Z_5, Z_2^4 \rtimes D_{10}$ and $\text{PSL}_2(\mathbb{F}_7)$. In Section 7 we find for each group $G$ in the previous two sections the collection of all (maximal) $G$-invariant subspace of quartics. In Section 8 we list all the smooth quartic surfaces by the finite primitive groups of $\text{PGL}_4(\mathbb{C})$. Finally, in Appendix A, we display the pseudocode $G$-invariant forms of degree $d$ implemented in the Program Mathematica [Inc], used to compute the invariant smooth quartic surfaces.

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## 2. Notations and Preliminaries

For the reader’s convenience and to set up the notation, we now briefly recall the needed notions.

**Some Constants**

- $\omega$: Primitive 3rd root of unity, $\omega^3 = 1$.
- $i$: Primitive 4-th root of unity, $i^4 = -1$.
- $\mu$: Primitive 5-th root of unity, $\mu^5 = 1$. 


\( \zeta \): Primitive 7-th root of unity, \( \zeta^7 = 1 \).

\( \psi \): Primitive 8-th root of unity, \( \psi^2 = i \).

\( s_\pm := \zeta^2 \pm \zeta^5 \).

\( t_\pm := \zeta^4 \pm \zeta^3 \).

\( u_\pm := \zeta \pm \zeta^6 \).

\( \lambda_\pm := (-1 \pm i \sqrt{15})/4 \).

\( \nu_\pm := (\sqrt{3} \pm i \sqrt{5})/2 \sqrt{2} \).

**Conventions and Notations**

- \( n \) always denote an integer \( \geq 2 \) and \( d \) always denote an integer \( \geq 3 \).
- For us \( \mathbb{Z}_n \) denotes the cyclic group of order \( n \).
- \( \mathbb{F}_q \) is the finite field of \( q = p^a \) elements.
- The alternating and symmetric groups in \( n \) letter are denoted by \( \mathfrak{A}_n \) and \( \mathfrak{S}_n \) respectively.
- A dihedral group of order \( n \) is denoted by \( D_n \). In particular, \( D_{10} \) is the dihedral group of 10 elements.
- Given a group \( G \), here we denote \( G \times \mathbb{Z}_2 \) the group \((G \times G) \rtimes \mathbb{Z}_2 \), where the action is given by exchanging factors. Similarly, the groups \((\mathfrak{A}_4 \times \mathfrak{A}_4) \rtimes \mathbb{Z}_2 \) in Diagram 5 denote an extension of these groups by an element of order two which exchange the factors of \( \mathfrak{A}_4 \times \mathfrak{A}_4 \).
- Here \( H.G \) denoted a nonsplit extension of \( G \) by \( H \), i.e., there is a short exact sequence of groups \( 1 \to H \to H.G \to G \to 1 \) and \( H.G \) is not isomorphic to \( H \times G \).
- The groups \( \text{GL}_n(\mathbb{C}) \) and \( \text{PGL}_n(\mathbb{C}) \) are the general linear group of degree \( n \) over the complex number \( \mathbb{C} \) and its projectivization.
- The group \( \text{SL}_n(\mathbb{C}) \) is the special linear group of degree \( n \) over complex numbers.
- \( E_n \) is the identity matrix of size \( n \times n \).

A hypersurface of the complex projective space \( \mathbb{P}^n \) is defined as the zero locus of a degree \( d \) form \( f \) in the polynomial ring \( \mathbb{C}[x_0, \ldots, x_n] \). A good reference to introduce into the study of hypersurfaces is presented by J. Kollár in [Kol19a] or by O. Debarre in [Deb17]. Also, I. Dolgachev presented a concise introduction to automorphisms of algebraic varieties in his notes written for the occasion of the “Fourth Latin American School on Algebraic Geometry and its Applications” ELGA, in Talca, Chile, [Dol19].

We denote by \( \text{Form}_{n,d} \) to the subset of \( \mathbb{C}[x_0, \ldots, x_n] \) of all homogeneous polynomials of degree \( d \) in the variables \( x_0, \ldots, x_n \), and the zero polynomial. It is clear \( \text{Form}_{n,d} \) is a linear subspace of \( \mathbb{C}^{n+1} \) of finite dimension given by \( (\binom{\binom{n+d}{n}}{n}) \).

For \( f, g \) \in \( \mathbb{C}[x_0, \ldots, x_n] \) we write \( f \sim g \) if \( f = \lambda g \) for some \( \lambda \) \in \( \mathbb{C}^* \).

For \( f \in \mathbb{C}[x_0, \ldots, x_n] \) and \( A \in \text{GL}_{n+1}(\mathbb{C}) \) we define \( Af \) as the polynomial in \( \mathbb{C}[x_0, \ldots, x_n] \) given by

\[
Af(x) = f(Ax),
\]

where

\[
Ax = A \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}.
\]

For \( V \) a subspace of \( \mathbb{C}^{n+1} \), and matrix \( (A) \in \text{PGL}_{n+1}(\mathbb{C}) \) we write \( (A)V = \{Av : v \in V\} \).

**Definition 2.1.** Recall that a hypersurface of \( \mathbb{P}^n \) is the zero locus of a single irreducible polynomial \( f \in \mathbb{C}[x_0, \ldots, x_n] \), i.e., \( X = \{f = 0\} \). Let \( X \) and \( Y \) be hypersurfaces in \( \mathbb{P}^n \). We say \( X \) and \( Y \) are projectively equivalent if and only if \( X = (A)Y \) for some \( (A) \in \text{PGL}_{n+1}(\mathbb{C}) \).

**Definition 2.2.** Let \( X \) be a hypersurface in \( \mathbb{P}^n \) and \( f \in \mathbb{C}[x_0, \ldots, x_n] \) homogeneous non-constant polynomial. Define the following subgroups of \( \text{PGL}_{n+1}(\mathbb{C}) \):

\[
\text{PAut}(X) = \{(A) \in \text{PGL}_{n+1}(\mathbb{C}) : (A)X = X\},
\]

\[
\text{PAut}(f) = \{(A) \in \text{PGL}_{n+1}(\mathbb{C}) : Af \sim f\}.
\]
Example 2.3. Let $$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \in \GL_3(\mathbb{C})$$, and consider the forms of degree two in $$\mathbb{C}[x, y, z]$$:

$$f(x, y, z) = x^2 + y^2 - z^2, \quad g(x, y, z) = xz + yz + xy \quad \text{and} \quad h(x, y, z) = x^2 + yz.$$  

We see that $$\{f = 0\} = \{g = 0\}$$ if and only if $$f \sim Af$$, then $$Af(x, y, z) = a^2x^2 + b^2y^2 - c^2z^2$$, so, the entries of the matrix $$A$$ must satisfy the relations $$a^2 = b^2 = -c^2$$, therefore $$A = \lambda D$$, where $$\lambda \in \mathbb{C}^\times$$ and $$D$$ is a diagonal matrix with entries 1 or $$\pm i$$. Similarly, if we do an analysis for the degree two form $$g(x, y, z) = xz + yz + xy$$, we get that $$\{g = 0\}$$ if and only if $$Ag(x, y, z) = (ac)xz + (bc)yz + (ab)xy$$, and so the equality $$g = \lambda \cdot Ag$$ implies that $$a = b = c$$, therefore $$A = \lambda E_3$$. Finally, we leave it to the reader to verify that the conditions which must satisfy the entries of the matrix $$A$$ to get that $$h \sim Ah$$ are given by $$\det \begin{bmatrix} a & b \\ c & a \end{bmatrix} = 0$$.

Definition 2.4. Let $$X = \{f = 0\}$$ be a hypersurface in $$\mathbb{P}^n$$, and let $$g \in \mathbb{C}[x_0, \ldots, x_n]$$ be a homogeneous non-constant polynomial, and $$G$$ be a subgroup of $$\PGL_{n+1}(\mathbb{C})$$. We say $$X$$ is $$G$$-invariant if and only if $$(A)X = X$$ for all $$A \in G$$, i.e., $$G \subseteq \PAut(X)$$. Likewise, we say $$f$$ is $$G$$-invariant if and only if $$Af \sim f$$ for all $$A \in G$$, i.e., $$G \subseteq \PAut(f)$$ for all $$f \in V$$.

Definition 2.5. Let $$V$$ be a subspace of $$\Form_{n,d}$$ and $$G$$ be a subgroup of $$\PGL_{n+1}(\mathbb{C})$$. We say $$V$$ is a $$G$$-invariant subspace of $$\Form_{n,d}$$ if and only if $$Af \sim f$$ for all $$A \in V$$ and $$A \in G$$, i.e., $$G \subseteq \PAut(f)$$ for all $$f \in V$$.

Definition 2.6. Given a group $$G$$ of $$\PGL_{n+1}(\mathbb{C})$$, a $$G$$-decomposition of $$\mathbb{C}^{n+1}$$ is a set $$\Gamma$$ of non-trivial subspaces of $$\mathbb{C}^{n+1}$$ such that

1) $$|\Gamma| \geq 2$$.
2) $$\mathbb{C}^{n+1} = \bigoplus_{V \in \Gamma} V$$.
3) For each $$\langle A \rangle \in G$$ and $$V \in \Gamma$$ we have $$\langle A \rangle V = W$$ for some $$W \in \Gamma$$.

Definition 2.7. A subgroup $$G$$ of $$\PGL_{n+1}(\mathbb{C})$$ is called primitive if and only if there is no $$G$$-decomposition of $$\mathbb{C}^{n+1}$$.

Definition 2.8. Let $$G$$ be a non-primitive subgroup of $$\PGL_{n+1}(\mathbb{C})$$. Then $$G$$ is called imprimitive if and only if for each $$\Gamma$$ a $$G$$-decomposition of $$\mathbb{C}^{n+1}$$ and $$V, W \in \Gamma$$ there exists $$A \in G$$ such that $$\langle A \rangle V = W$$.

Definition 2.9. Let $$G$$ be a subgroup of $$\PGL_{n+1}(\mathbb{C})$$. We say $$G$$ is intransitive if and only if $$G$$ is neither primitive nor imprimitive.

Example 2.10. (1) For $$n \geq 2$$ let $$G = \langle (A) \rangle$$ be the subgroup of $$\PGL_{n}(\mathbb{C})$$ generated by $$A \in \GL_{n}(\mathbb{C})$$. We have $$G$$ is intransitive since every invertible matrix is diagonalizable. In particular, for $$n = 4$$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and $$G = \langle (A) \rangle \subset \PGL_{4}(\mathbb{C})$$, we have the $$G$$-decomposition of $$\mathbb{C}^4$$: $$\Gamma = \{V_1, V_2\}$$, where $$V_1$$ is generated by $$e_1, e_2$$ and $$V_2$$ is generated by $$e_3, e_4$$, as usual $$e_j$$ denotes the $$j$$-th column of identity matrix $$E_4$$.

(2) On the other hand, consider the representation of $$\PSL_2(\mathbb{F}_7)$$ given in Remark 6.1, denoted by $$\overline{G}_{168}$$ in the notation used in [CS19]. It is clear that $$\overline{G}_{168}$$ cannot be intransitive since $$\overline{G}_{168}$$ does not fix any point on $$\mathbb{P}^2$$. If $$\overline{G}_{168}$$ is imprimitive, then we find a non-trivial homomorphism of groups $$\varphi : \overline{G}_{168} \to \mathfrak{S}_3$$, and since $$\overline{G}_{168}$$ is simple, then $$\varphi$$ is injective and we get a contradiction by the orders of $$\overline{G}_{168}$$ and $$\mathfrak{S}_3$$. Therefore, $$\overline{G}_{168}$$ is primitive.
3. THEORETICAL INVARIANT SUBSPACES

Let $G$ be a subgroup of $\operatorname{PGL}_{n+1}(\mathbb{C})$. By the preliminaries, the problem of finding the smooth hypersurfaces of degree $d$ invariant by $G$ is equivalent to finding the homogeneous non-singular forms of degree $d$ invariant by $G$. This task can be done in two parts:

1) To find all homogeneous forms $f \in \operatorname{Form}_{n,d}$ such that $f \sim Af$ for all $(A) \in G$, and
2) determine which of such $f$ are non-singular.

Here we will present a solution to the first problem when $G$ is finite. Moreover, the solution is computable. In fact, we implemented an algorithm in Mathematica [Inc] to solve the item 1) for a given finite subgroup of $\operatorname{PGL}_d(\mathbb{C})$, see Appendix A.

With respect to item 1), we relate the concepts of $G$-invariant subspace of $\operatorname{Form}_{n,d}$, and maximal $G$-invariant subspace of $\operatorname{Form}_{n,d}$. It turns out to be a finite number of such maximal $G$-invariant subspace of $\operatorname{Form}_{n,d}$, see Corollary 3.4, and the description of these subspaces is straightforward given a set of generators of $G$, see Theorem 3.9. Therefore, Corollary 3.5 concludes the item 1).

Concerning item 2), we recall basic results of the theory of resultants and elimination. In particular the discriminant of $f$ solves the question of determining if $f$ is neither singular nor non-singular. However, the computation of the discriminant could be an exaggeration to solve this question in many cases, so we will state the Lemma 3.10 which is much easier to test on any homogeneous form, although this result is not an equivalent criterion of non-singularity.

3.1. Invariant Subspaces.

**Lemma 3.1.** Let $V$ be a $G$-invariant non-trivial subspace of $\operatorname{Form}_{n,d}$, $(A) \in G$ and $p \in V$ non-zero. If we have that $Ap = \lambda p$ for some unique $\lambda \in \mathbb{C}^*$, then $Af = \lambda f$ for all $f \in V$.

*Proof.* Clearly, $Af = \lambda f$ if $f = 0$. Suppose then that $f \neq 0$, we have $f, f + p \in V$, then $Af = af$ and $A(f + p) = b(f + p)$ for some $a, b \in \mathbb{C}^*$. Thus $bf + bp = b(f + p) = A(f + p) = Af + Ap = af + \lambda p$ and so $(b - a)f = (\lambda - b)p$. If $b - a = 0$, then $(\lambda - b)p = 0$ and $\lambda = b = a$. On the other hand, if $b - a \neq 0$, then $f = \tau p$ for some $\tau \in \mathbb{C}^*$ and $af = Af = A(\tau p) = \tau(Ap) = \tau \lambda p = \lambda f$, thus $a = \lambda$. In any case we get that $a = \lambda$. \hfill $\Box$

**Theorem 3.2.** Let $V, W$ be two $G$-invariant subspaces of $\operatorname{Form}_{n,d}$. Then $V \cap W \neq 0$ implies $V + W$ is a $G$-invariant subspace of $\operatorname{Form}_{n,d}$.

*Proof.* There is nothing to prove if $V$ or $W$ is trivial. Therefore, we can suppose that $V$ and $W$ are non-trivial, and take $p \in V \cap W$ non-zero polynomial. For $(A) \in G$ we have that $Ap = \lambda p$ for some $\lambda \in \mathbb{C}^*$. By the previous Lemma 3.1, we see that $Af = \lambda f$, and $Ag = \lambda g$ for all $f \in V$ and $g \in W$.

If $f + g \in V + W$, $f \in V$ and $g \in W$, then $A(f + g) = Af + Ag = \lambda f + \lambda g = \lambda(f + g)$ and so $A(f + g) \sim f + g$. \hfill $\Box$

**Corollary 3.3.** If $f \in \operatorname{Form}_{n,d}$ is a non-zero polynomial $G$-invariant, then there exists a unique $V$ maximal $G$-invariant subspace of $\operatorname{Form}_{n,d}$ such that $f \in V$.

**Corollary 3.4.** There is a finite number of maximal $G$-invariant subspaces of $\operatorname{Form}_{n,d}$.

By the preliminaries results we obtain the correspondence between Smooth surfaces of degree $d$ contained in $\mathbb{P}^n$ and the non-singular forms contained in the collection of all the maximal $G$-invariant subspaces of $\operatorname{Form}_{n,d}$. In sum, we have Corollary 3.5.

**Corollary 3.5.** Let $\Gamma$ be the finite collection of all maximal $G$-invariant subspaces of $\operatorname{Form}_{n,d}$. Then we have a correspondence between

$$\{\text{Smooth hypersurfaces of degree } d \text{ in } \mathbb{P}^n\} \leftrightarrow \{\text{Non-singular forms in } \bigcup \Gamma\}.$$

In the rest of this section we suppose that $G$ is a finite group. We are going to construct all maximal $G$-invariant subspaces of $\operatorname{Form}_{n,d}$. In first place, let’s take $\{A_i\}_{i \leq m} \subset \operatorname{SL}_{n+1}(\mathbb{C})$ such that $G$ is generated by $\{(A_i)\}_{i \leq m}$. Let $\sigma_1$ be the order of $(A_i)$, and $\kappa \geq 1$ such that $\kappa d$ is the least common
multiple of $n+1$ and $d$, i.e., $\text{lcm}(n+1,d) = kd$. Thus, $A_i^{\alpha_i} = \tau_i E_{n+1}$ for some $\tau_i \in \mathbb{C}^*$, since $\det(A_i) = 1$, then $\tau_i^{n+1} = \det(\tau_i E_{n+1}) = \det(A_i^{\alpha_i}) = 1$, $\tau_i^{n+1} = 1$. Now let's define

$$K = \{k \in \mathbb{C}^m \mid k_i \alpha_i = 1, \text{ for each } 1 \leq i \leq m\}.$$ 

For each $1 \leq j \leq m$, and for any $k \in K$ denote by $B_{k,j}$ the linear transformation over $\text{Form}_{n,d}$ given by the formula

$$B_{k,j} f = A_j f - k_j f.$$ 

For each $k \in K$, we define $T_k: \text{Form}_{n,d} \rightarrow \text{Form}_{n,d}$ by $(T_k g)_j = B_{k,j} g$, for $1 \leq j \leq m$, and $g \in \text{Form}_{n,d}$. It is easy to see that $T_k$ is a linear map since each $B_{j,k}$ is a linear transformation. Let

$$\Gamma = \{\ker(T_k) : k \in K, \ker(T_k) \neq 0\}$$ 

be a collection, possibly empty, of non-trivial subspaces of $\text{Form}_{n,d}$.

**Lemma 3.6.** If $f \in \text{Form}_{n,d}$ is $G$-invariant, then $f \in \ker(T_k)$ for some $k \in K$.

**Proof.** If $f = 0$, then $f \in \ker(T_k)$ for all $k \in K$. On the other hand, if $f \neq 0$, then $A_i f = \lambda_i f$ for some $\lambda_i \in \mathbb{C}^*$. Thus, $\lambda_i^\alpha_i f = A_i^{\alpha_i} f = (\tau_i E_{n+1}) f = \tau_i^d f$, $\lambda_i^\alpha_i = \tau_i^d$, $\lambda_i^\alpha_i = \tau_i^d = 1$, $\lambda_i \neq \tau_i$. Therefore, $\lambda = (\lambda_i) \in K$ and $f \in \ker(T_k)$. □

**Lemma 3.7.** The subspace $\ker(T_k)$ is a $G$-invariant subspace of $\text{Form}_{n,d}$ for all $k \in K$.

**Proof.** If $f \in \ker(T_k)$, then $A_i f = k_i f$ for all $i$, so $\{(A_i)\}_{1 \leq i \leq m} \subset \text{PAut}(f)$. Since $G$ is generate by $\{(A_i)\}_{1 \leq i \leq m}$, then $G \subset \text{PAut}(f)$, so $\ker(T_k)$ is a $G$-invariant subspace of $\text{Form}_{n,d}$. □

**Lemma 3.8.** If $\ker(T_k) \cap \ker(T_\lambda) = 0$, then $k = \lambda$ and $\ker(T_k) = \ker(T_\lambda)$.

**Proof.** Suppose $p \in \ker(T_k) \cap \ker(T_\lambda)$ is a non-zero element, then $T_k(p) = 0$ and $T_\lambda(p) = 0$, this implies $A_i p = k_i p$ and $A_j p = \lambda_j p$ for all $i$, so $k_i p = \lambda_j p$, $k_i = \lambda_j$, $k = \lambda$ and $\ker(T_k) = \ker(T_\lambda)$. □

**Theorem 3.9.** Let $\Gamma$ as in Corollary 3.5. If $\Gamma \neq \emptyset$, then $\Gamma$ is the set of all maximal $G$-invariant subspaces of $\text{Form}_{n,d}$. Otherwise, if $\Gamma = \emptyset$, then the trivial space is the unique $G$-invariant subspace of $\text{Form}_{n,d}$.

**Proof.** Suppose $\Gamma = \emptyset$, then we want to prove there is not a non-zero $G$-invariant polynomial in $\text{Form}_{n,d}$. If $f \in \text{Form}_{n,d}$ is a $G$-invariant non-zero polynomial, then by Lemma 3.6 we have $f \in \ker(T_k)$ for some $k \in K$, so $\ker(T_k) = 0$ and $\ker(T_k) \notin \Gamma$, which gives a contradiction.

Now we want to prove that $V$ is a non-trivial $G$-invariant subspace of $\text{Form}_{n,d}$ if and only if $V \in \Gamma$. Let $V$ be a maximal $G$-invariant non-trivial subspace of $\text{Form}_{n,d}$, and let $p \in V$ be non-zero element. Then by Lemma 3.6 we have that $p \in \ker(T_k)$ for some $k \in K$, then $A_i p = k_i p$ for all $i$. If $f \in V$ by Lemma 3.1 we have that $A_i f = k_i f$ for all $i$, thus, $f \in \ker(T_k)$ and $V \subset \ker(T_k)$. However, $\ker(T_k)$ is a $G$-invariant subspace of $\text{Form}_{n,d}$ because of Lemma 3.7, thus $V = \ker(T_k)$ and $V \in \Gamma$. Then $\ker(T_k) \in \Gamma$ is contained in some $\ker(T_\lambda) \in \Gamma$ maximal $G$-invariant subspace of $\text{Form}_{n,d}$. By Lemma 3.8, we have that $k = \lambda$, and $\ker(T_k)$ is a maximal $G$-invariant subspace of $\text{Form}_{n,d}$. □

The relevance of Theorem 3.9 is that it provides an algorithm to find all maximal $G$-invariant subspace of $\text{Form}_{n,d}$. Indeed, as the set of generators of $G$ is finite since $G$ is finite, then $K$ is finite too, and the linear maps $T_k$ have a finite size matrix representations. With such matrices we find the kernels of $T_k$ by software computations. Under these considerations we implemented the algorithm Invariant-Forms[$G,d$] in Mathematica [Inc], see Appendix A.

### 3.2. Non-Singularity

In this section we recall some well-known facts in theory of resultants and elimination: the following references give a basic background in this matter [CLO05, GKZ94, vdW49]. Furthermore, we present a result by K. Oguiso, and X. Yu which says that under certain conditions a homogeneous polynomial is singular, see Lemma 3.10.

The discriminant of a homogeneous form of degree $d$, $f \in \text{Form}_{n,d}$, which we denote by $\text{Disc}_d(f)$ or $\text{Disc}(f)$, is a homogeneous polynomial in the coefficients $f$ of degree $(n+1)(d-1)^n$ such that $\text{Disc}(f) \neq 0$ if and only if $f$ is non-singular.
For example, if \( n = 1, d = 2, \) and \( f = ax_0^2 + bx_0x_1 + c \in \mathbb{C}[x_0, x_1], \) then \( \text{Disc}(f) = 4ac - b^2 \) and we have \( f \) is non-singular if and only if \( 4ac - b^2 \neq 0. \)

It is well-known that for fixed \( n \) and \( d, \) the discriminant \( \text{Disc}(\cdot) \) exists as a polynomial function defined in \( \text{Form}_{n,d}. \) The problem is how to compute it. From theory of Resultants we know \( \text{Disc}(f) \) is the resultant of all partial derivatives of \( f. \) We follow the Macaulay’s construction of this resultant to find the discriminant \( \text{Disc}(f). \)

The Macaulay’s method: Consider two square matrices \( Q(f) \) and \( Q'(f), \) where the latter is a submatrix of the former, such that \( \det(Q(f)) = \text{Disc}(f) \cdot \det(Q'(f)). \) Thus, if \( \det(Q'(f)) \neq 0, \) we have the expression

\[
\text{Disc}(f) = \frac{\det(Q(f))}{\det(Q'(f))}
\]

To overcome the difficulty occasioned by the possibility of vanishing of \( \det(Q'(f)), \) we can take an element \( \varphi \in \text{Form}_{n,d} \) such that \( \det(Q'(\varphi)) \neq 0. \) e.g., for the Fermat polynomial of degree \( d, \) \( \varphi = d^{-1}(x_0^d + x_1^d + \cdots + x_n^d), \) we have that \( Q(\varphi) \) and \( Q'(\varphi) \) are identity matrices. Now, if \( \lambda \) and \( \mu \) are indeterminates we have that

\[
\det(Q(\lambda f + \mu \varphi)) = \text{Disc}(\lambda f + \mu \varphi) \cdot \det(Q'(\lambda f + \mu \varphi)).
\]

For fixed \( f, \) this expression is an identity of polynomials in the variables \( \lambda \) and \( \mu, \) and \( \det(Q'(\lambda f + \mu \varphi)) \) is not the zero polynomial since for \( \lambda = 0 \) and \( \mu = 1 \) we have \( \det(Q'(\lambda f + \mu \varphi)) = \det(Q'(\varphi)) = 1. \)

Therefore \( \det(Q'(\lambda f + \mu \varphi)) \) divides \( \det(Q(\lambda f + \mu \varphi)) \) as polynomials in the variables \( \lambda \) and \( \mu, \) and we get the expression

\[
\text{Disc}(f) = \frac{\det(Q(\lambda f + \mu \varphi))}{\det(Q'(\lambda f + \mu \varphi))}_{(\lambda, \mu)=(1,0)}
\]

of discriminant of \( f, \) which it is valid in general.

Certainly, in many cases it could be an exaggeration to compute the discriminant of \( f \) to determine if \( f \) is non-singular. The following lemma gives us some sufficient conditions for singularity of a homogeneous form, which it is proved in [OY19, Section 3, Lemma 3.2, Proposition 3.3].

**Lemma 3.10.** Let \( \{ f = 0 \} \) a hypersurface of degree \( d \) given by the homogeneous form \( f \in \text{Form}_{n,d}. \) Let \( a, b \) two nonnegative integers such that \( 2a + b \leq n. \) If there are \( a + b \) distinct variables \( x_{i_1}, \ldots, x_{i_a+b} \) such that \( f \in (x_{i_1}, \ldots, x_{i_a}) + (x_{i_{a+1}}, \ldots, x_{i_{a+b}})^2, \) then \( \{ f = 0 \} \) is singular. In particular, if \( \{ f = 0 \} \) is a smooth hypersurface, then for every \( i \in \{ 1, \ldots, n + 1 \}, \) the monomial \( x_i^{d-1} \) must appear with non-zero coefficient in the form \( f \) for some \( j. \)

The last statement is useful because it can be easily tested for any \( f \in \text{Form}_{n,d}. \)

4. **Diagrams**

Here, we present a relabeling of the diagrams presented in [CS19, Appendix A]. These diagrams show some inclusions between all finite primitive subgroups of \( \text{PGL}_4(\mathbb{C}) \), up to conjugates. The description of all these groups can be viewed in [Bli18, Chapter VII]. In the next section, we will present a set of generators for some of these groups.
The following diagrams point out the isomorphic class to which each one of these groups belongs.
5. **Finite Primitive Groups**

5.1. **Diagram: 1° - 12°.** For Diagram 1 it is enough to describe the group (1°).

\[
E_2 \star S = \sqrt{2} \begin{bmatrix}
i & i & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & i & i \\
0 & 0 & 1 & -1
\end{bmatrix},
E_2 \star W_1 = \begin{bmatrix}
i & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & -i
\end{bmatrix},
\]

\[
S \star E_2 = \sqrt{2} \begin{bmatrix}
i & 0 & i & 0 \\
0 & i & 0 & i \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1
\end{bmatrix},
W_1 \star E_2 = \begin{bmatrix}
i & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & -i & 0 \\
0 & 0 & 0 & -i
\end{bmatrix}, (1°)
\]

5.2. **Diagram: 13° - 21°.**
For Diagram 2 it is enough to describe the groups $13^\circ$, $17^\circ$ and $19^\circ$. Moreover, we have that $(13^\circ)$ is isomorphic to $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$, $(14^\circ)$ is isomorphic to $\mathbb{Z}_2^4 \rtimes D_{10}$ and $(19^\circ)$ is isomorphic to $\mathbb{Z}_2^4 \rtimes S_5$, see [CS19].

\[ S_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad T = \frac{\psi}{\sqrt{2}} \begin{bmatrix} -i & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -i & i \end{bmatrix}. \quad (13^\circ)

\[ (13^\circ) \text{ and } R^2 = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (14^\circ)

\[ (13^\circ) \text{ and } R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & -i \end{bmatrix}. \quad (15^\circ)

\[ (13^\circ) \text{ and } SB = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}. \quad (16^\circ)

\[ (13^\circ) \text{ and } BR = \frac{\psi}{\sqrt{2}} \begin{bmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 1 & i \end{bmatrix}. \quad (17^\circ)

\[ (13^\circ) \text{ and } B = \psi \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (19^\circ)

5.3. Diagram: (A), (C) - (F), (G), and (K). For Diagram 3 it is enough to describe the groups (A), (G), (C) y (E). In addiction, we have that (A) is isomorphic to $\mathfrak{A}_5$, (G) is isomorphic to $\mathfrak{S}_5$, (C) is isomorphic to $\mathfrak{A}_6$, and (E) is isomorphic to $\text{PSL}_2(\mathbb{F}_7)$, see [Bli18, Chapter VII].

\[ F_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}, \quad F_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & 0 & -1 \end{bmatrix}, \quad F_3 = \frac{1}{2} \begin{bmatrix} \sqrt{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad (A)

\[ (A) \text{ and } F'' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \quad (G)
it is enough to describe both groups, $(A)$ and $(H)$. In particular, we see that there are no non-primitive representations for the groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Apart from the primitive representations of the groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we have that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Likewise, we have that $(B)$ is isomorphic to $\mathbb{S}_5$ and $(H)$ it is isomorphic to $\mathbb{S}_5$, see [Bli18, Chapter VII].

$$F_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}, \quad F_2' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \quad F_3' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (B)$$

$$(B) \quad \text{and} \quad F' = \psi \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (H)$$

6. Some Non-Primitive Finite Groups

Apart from the primitive representations of the groups $\mathbb{S}_5, \mathbb{S}_6, \mathbb{S}_6, \mathbb{A}_6, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times D_{10}$ and $\text{PSL}_2(\mathbb{F}_7)$

given in the previous section, we present in this section the remaining faithful representations of these groups. In particular, we see that there are no non-primitive representations for the groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and $\mathbb{Z}_2 \times D_{10}$, moreover, this is the case for all groups in the Diagram 6.

$$Q_{11} = F_1, \quad Q_{12} = F_2' \quad \text{and} \quad Q_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_+ \\ 0 & 0 & \lambda_- & 0 \end{bmatrix} \quad (Q_1)$$

$$Q_{41} = \begin{bmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^2 \end{bmatrix}, \quad Q_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \end{bmatrix}, \quad Q_{43} = \begin{bmatrix} 0 & 0 & 0 & \nu_+ \\ 0 & 0 & \nu_- & 0 \\ \nu_- & 0 & 0 & 0 \end{bmatrix} \quad (Q_4)$$

$$Q_{51} = Q_{41}, \quad Q_{52} = Q_{42} \quad \text{and} \quad Q_{53} = \begin{bmatrix} 0 & 0 & 0 & \nu_+ \\ 0 & 0 & \nu_- & 0 \\ \nu_+ & 0 & 0 & 0 \\ \nu_- & 0 & 0 & 0 \end{bmatrix} \quad (Q_5)$$

$$(Q_5) \quad \text{and} \quad R_{54} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad (R_3)$$
Remark

By the genus formula, the Klein curve one finite primitive group in the Blichfeldt’s classification of order 160. This argument works for all $g^A$ are all faithful representations, up to conjugates, of $S$. By the work of H. Maschke in 1898 (Mas98), we have that the groups $(A), (B), (Q_1), (Q_4),$ and $(Q_5)$ are all faithful representations, up to conjugates, of $\mathfrak{A}_5$ in $\text{PGL}_4(C)$. The groups $(G), (H),$ and $(R)$ are all faithful representations, up to conjugates, of $\mathfrak{S}_5$ in $\text{PGL}_4(C)$. Finally, the groups $(C),$ and $(Q_7)$ are all faithful representations, up to conjugates, of $\mathfrak{A}_6$ in $\text{PGL}_4(C)$.

The projective special linear group $\text{PSL}_2(F_7)$ is the second-smallest non-cyclic simple group. For geometric reasons, it is an important group. For instance, it is the group of automorphisms of the Klein quartic curve, $K = \{x^3 + y^3 + 3z^2 = 0\} \subseteq C^2$. By a famous theorem of Hurwitz, see [Mir95, Chap. III, Theo. 3.9], we know that for smooth curves $C$ of genus $g \geq 2$, the order of its group of automorphism is bounded in terms of the genus $g$. More precisely, the bound is $|\text{Aut}(C)| \leq 84(g - 1)$. By the genus formula, the Klein curve $K$ has genus $g = 3$, and it is well known that the unique curve of genus $g = 3$ attains the Hurwitz bound. In fact, the order of $\text{PSL}_2(F_7)$ is $168 = 84(3 - 1)$.

By [MPK19, Proposition 4.5, (4)] we have that $(C)$ and $(P)$ are all faithful representations, up to conjugates, of $\text{PSL}_2(F_7)$ in $\text{PGL}_4(C)$.

Remark 6.1. In [Kle78], the author classified the unique faithful representation $\rho$ of $\text{PSL}_2(F_7)$ in $\text{PGL}_3(C)$. The subgroup $\rho(\text{PSL}_2(F_7)) < \text{PGL}_3(C)$ is generated by the matrices

$$
\begin{bmatrix}
\zeta^4 & 0 & 0 \\
0 & \zeta^2 & 0 \\
0 & 0 & \zeta
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
u & 0 & 0 \\
0 & \sqrt{7} & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

Note that the projective representation $(P)$ of $\text{PSL}_2(F_7)$ is a trivial extension of the last representation, found by Klein.

By [CS19, Corollary 5.2.], we have that $(13^6)$ is the unique faithful representation, up to conjugates, of $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$ in $\text{PGL}_4(C)$. The proof of this corollary shows that there is no non-primitive subgroup of $\text{PGL}_4(C)$ isomorphic to $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$. Now we are in conditions to show that $(14^2)$ is the unique faithful representation, up to conjugates, of the group $\mathbb{Z}_2^4 \rtimes D_{10}$ in $\text{PGL}_4(C)$. Suppose that $\overline{G}_{160}$ is a subgroup of $\text{PGL}_4(C)$ isomorphic to $\mathbb{Z}_2^4 \rtimes D_{10}$, as $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$ is a subgroup of $\mathbb{Z}_2^4 \rtimes D_{10}$, then we have that $\overline{G}_{80}$ is a subgroup of $\overline{G}_{160}$ and $\text{PGL}_4(C)$ isomorphic to $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$. By the previous considerations $\overline{G}_{80}$ is a primitive subgroup of $\text{PGL}_4(C)$, thus $\overline{G}_{160}$ is also a primitive subgroup of $\text{PGL}_4(C)$. By Blichfeld’s classification of all finite primitive subgroups of $\text{PGL}_4(C)$, we get that $\overline{G}_{160}$ is conjugate to $(14^2)$, note that there is an only one finite primitive group in the Blichfeld’s classification of order 160. This argument works for all groups in the diagram 6. In conclusion, from the discussion presented in the last paragraph, we obtain Corollary 6.2.

Corollary 6.2. Let $G$ be subgroup of $\text{PGL}_4(C)$ isomorphic to some group of the Diagram 6. Then $G$ is conjugate to some group of the Diagram 2.

7. Invariant Subspaces of Quartics

In this section, we find for each group $G$ in the previous two sections the collection of all (maximal) $G$-invariant subspace of quartics. Since all these linear subspaces are finite-dimensional, then it is enough
to show a finite basis by each of such subspaces. For each one of these basis \( \{ p_1, p_2, \ldots, p_r \} \) we show a criterion over the scalar \( \lambda_1, \lambda_2, \ldots, \lambda_r \) to determinate which linear combinations \( \lambda_1 p_1 + \lambda_2 + \cdots + \lambda_r p_r \) are non-singular. In particular, if the basis consists of only one polynomial \( p \), we will tag this polynomial with (*) to mean \( p \) is singular, by the contrary, if \( p \) is non-singular, then we will tag \( p \) with (**). For example, the polynomial \( x_1^2 x_2^2 + x_2^3 x_3^3 - 2x_0 x_1 x_2 x_3 \) in 7.1.1 is singular, but the polynomial \( 2x_1^4 + 6x_1 x_2 x_3 x_0 + x_1 x_3^3 + x_1^2 x_2 + x_2^3 x_3 \) in 7.1.1 is non-singular.

**Respect to Diagram 1.**

**7.1. Invariant subspaces of quartics by (1\(^{\circ}\)).**

**7.1.1. First subspace \( I_1 \).**

\[ x_1^2 x_2^2 + x_0^2 x_3^2 - 2x_0 x_1 x_2 x_3. \] (*)

**7.1.2. Second subspace \( I_2 \).**

\[ x_1^2 + x_1^4 + x_2^4 + 8x_0 x_1 x_2 x_3 - 2 (x_1^2 x_2^2 + x_0 x_3^2) + 2i\sqrt{3} (x_0^2 x_1^2 + x_2^2 x_3^2 + x_0 x_2 x_3^2). \] (*)

**7.1.3. Third subspace \( I_3 \).**

\[ x_1^4 + x_1^2 + x_2^4 + 8x_0 x_1 x_2 x_3 + 2 (x_1^2 x_2^2 + x_0 x_3^2) + 2i\sqrt{3} (x_0^2 x_1^2 - x_3^2 x_1^2 - x_0 x_2 x_3^2) \] (*)

**7.1.4. Fourth subspace \( I_4 \).**

\[ x_1^4 + x_1^2 + x_2^4 + 8x_0 x_1 x_2 x_0 + 2 (x_1^2 x_2^2 + x_0 x_3^2) - 2i\sqrt{3} (x_0^2 x_1^2 - x_3^2 x_1^2 - x_0 x_2 x_3^2) \] (*)

**7.1.5. Fifth subspace \( I_5 \).**

\[ x_1^4 + x_1^2 + x_2^4 + 8x_1 x_2 x_3 x_0 - 2 (x_1^2 x_2^2 + x_0 x_3^2) - 2i\sqrt{3} (x_0^2 x_1^2 + x_3^2 x_1^2 + x_0 x_2 x_3^2) \] (*)

For the sake of clarity, we want to mention that it is unnecessary to compute invariant quartics for the remaining groups on Diagram 1. Any other group in Diagram 1 contains (1\(^{\circ}\)), and the invariant quartics by (1\(^{\circ}\)) are singular (*). Therefore, the other groups in this diagram are done.

**Respect to Diagram 2.**

**7.2. Invariant subspaces of quartics by (13\(^{\circ}\)).**

**7.2.1. First subspace \( \text{XIII}_1 \).**

\[ k = x_0^4 + x_1^4 + x_2^4 + x_3^4 + 6 \left(x_0^2 x_1^2 - x_2^2 x_1^2 + x_3^2 x_1^2 + x_0 x_2^2 - x_0 x_3^2 + x_2 x_3^2\right). \] (**)

**7.2.2. Second subspace \( \text{XIII}_2 \).**

\[ -\frac{1 + 4\mu - \mu^2 - 2\mu^3 + 2\mu^4}{\mu^2 - 1} \frac{6\mu^2 - \mu^3 - \mu^4}{1 + \mu} (x_0^2 x_2^2 + x_1^2 x_3^2) - 6\frac{\mu^2 - 1}{\mu^2 - 1} \frac{1 + \mu^2}{(x_0 x_2^3 + x_1 x_2^3)} + 24x_0 x_1 x_2 x_3. \] (*)

**7.2.3. Third subspace \( \text{XIII}_3 \).**

\[ (1 + 2\mu + 4\mu^2 + 2\mu^3 + \mu^4)(x_0^4 + x_1^4 + x_2^4 + x_3^4) + 6(1 + \mu^4)(x_0^2 x_1^2 + x_2 x_3^2) \]
\[ - 6(1 - 2\mu^3 - \mu^4)(x_0^2 x_3^2 + x_1^2 x_3^2) + 6(4 \mu^2 - 2\mu - 1)(x_0 x_2 x_3 + x_1 x_2 x_3) + 24(\mu^4 - 1) x_0 x_1 x_2 x_3. \] (*)

**7.2.4. Fourth subspace \( \text{XIII}_4 \).**

\[ -(1 + 2\mu + 4\mu^2 + 2\mu^3 + \mu^4)(x_0^4 + x_1^4 + x_2^4 + x_3^4) + 6(1 + \mu^4)(x_0^2 x_2^2 + x_1^2 x_2^2) \]
\[ - 6(1 - 2\mu^3 - \mu^4)(x_0^2 x_3^2 + x_1^2 x_3^2) + 6(2\mu - 1 - \mu^4)(x_0 x_2 x_3 + x_1 x_2 x_3) + 24(-1 + \mu^4) x_0 x_1 x_2 x_3. \] (*)

**7.2.5. Fifth subspace \( \text{XIII}_5 \).**

\[ -(1 + 4\mu - \mu^2 - 2\mu^3 + 2\mu^4)(x_0^4 + x_1^4 + x_2^4 + x_3^4) - 6(1 + \mu^2)(x_0^2 x_2^2 + x_1^2 x_2^2) \]
\[ - 6(1 - \mu^2 + 2\mu^4)(x_0^2 x_3^2 + x_1^2 x_3^2) + 6(1 - \mu^2 + 2\mu^4)(x_0^2 x_3^2 + x_1^2 x_2^2) + 24(-1 + \mu^2) x_0 x_1 x_2 x_3. \] (*)
7.3. Invariant subspaces of quartics by (14°).

7.3.1. XIV. It is the subspace 7.2.1.

7.4. Invariant subspaces of quartics by (15°).

7.4.1. XV. It is the subspace 7.2.1.

7.5. Invariant subspaces of quartics by (16°). There is no non-trivial subspace of invariant quartics by this group. The last statement implies that there is also no non-trivial subspace of invariant quartics by groups (18°), (20°), and (21°).

7.6. Invariant subspaces of quartics by (17°).

7.6.1. XVII. It is the subspace 7.2.1.

7.7. Invariant subspaces of quartics by (19°).

7.7.1. XIX. It is the subspace 7.2.1.

Respect to Diagram 3.

7.8. Invariant subspaces of quartics by (A).

7.8.1. A₁.

\[
\begin{align*}
    h_0 &= \frac{x_0^4}{2\sqrt{3}} + \frac{1}{3}x_1x_0^3 + x_2x_3x_0^2 + \frac{1}{3}x_1^3x_0 + \frac{1}{3}\sqrt{2}x_2^3x_0 + \frac{1}{3}\sqrt{3}x_3^3x_0 \\
    &+ \frac{1}{2}\sqrt{2}x_1x_2^3 - \frac{1}{3}\sqrt{2}x_1x_3^3 + x_1^2x_2x_3 - x_1^4/\sqrt{3}, \\
    h_1 &= \frac{x_0^4}{6} + \frac{1}{3}x_1x_0^3 - \frac{1}{2}x_1^2x_0^2 + \sqrt{3}x_2x_3x_0^2 + \frac{1}{3}\sqrt{2}x_3^3x_0 + x_1x_2x_3x_0 \\
    &+ \frac{2}{3}\sqrt{2}x_1x_2^3 - \frac{1}{2}x_1^2x_3^3 - \frac{1}{3}x_1^3. \\
\end{align*}
\]

Denote by \( D(\lambda_0, \lambda_1) \) the discriminant of the linear combination \( \lambda_0h_0 + \lambda_1h_1 \), it is a homogeneous polynomial in the parameters \( \lambda_0, \lambda_1 \) of degree \((3 + 1) \cdot (4 - 1)^3 = 108\). Since the zero locus of \( D(\lambda_0, \lambda_1) \) lies in \( \mathbb{P}^1 \), then there are finite many non-trivial solutions of \( D(\lambda_0, \lambda_1) = 0 \). Moreover, the number of solutions is bounded by 108, the degree of \( D(\lambda_0, \lambda_1) \). Some of the points \([\lambda_0 : \lambda_1] \in \mathbb{P}^1\) such that \( \{\lambda_0h_0 + \lambda_1h_1 = 0\} \) is a singular invariant surface by (A), these points \([\lambda_0 : \lambda_1]\) are the following:

\[
\begin{bmatrix}
    -2\sqrt{2} : \sqrt{i\sqrt{15} - 1}, \\
    -2\sqrt{2} : \sqrt{-i\sqrt{15} - 1}, \\
    1 : -\sqrt{3}, \\
    47 : 8 \left(1 - 4\sqrt{3}\right), \\
    47 : -8 \left(1 + 4\sqrt{3}\right), \\
    4 : -\sqrt{3} + i\sqrt{3}, \\
    \end{bmatrix}
\]

We found these points numerically with the help of the software Mathematica. Unfortunately, to find all zeros of \( D(\lambda_0, \lambda_1) \) numerically, or to compute the discriminant \( D(\lambda_0, \lambda_1) \) by Macaulay’s method is computationally heavy.

7.9. Invariant subspaces of quartics by (G).

7.9.1. G₁.

\[
\begin{align*}
    f_0 &= -\frac{x_0^4}{12} + \frac{x_1x_0^3}{2\sqrt{3}} - \frac{1}{2}x_1^2x_0^2 + \frac{1}{3}\sqrt{3}x_2x_3x_0^2 + x_1x_2x_3x_0 - \frac{x_3^3x_0}{3\sqrt{2}} - \frac{x_1x_0^3}{2\sqrt{3}} - \frac{x_3^3x_0}{\sqrt{6}} \\
    &+ \frac{x_1x_2^3}{3\sqrt{2}} - \frac{x_2^3x_3}{2\sqrt{3}} - \frac{x_1^3x_2}{\sqrt{3}} - \frac{x_1^3}{12} - \frac{x_1^3x_3}{\sqrt{6}}. \\
\end{align*}
\]
7.9.2. $G_2$. 

$$f_1 = \frac{x_0^4}{2\sqrt{3}} + \frac{1}{3} x_1 x_3^3 + x_2 x_3^2 x_0 + \frac{1}{3} x_1^2 x_0 + \frac{1}{3} \sqrt{2} x_2 x_0 + \sqrt{\frac{2}{3}} x_3^2 x_0$$

$$+ \sqrt{\frac{2}{3}} x_1 x_2^3 - \frac{1}{3} \sqrt{2} x_1 x_3^3 + x_1^2 x_2 x_3 - \frac{x_1^4}{2\sqrt{3}}.$$

7.10. **Invariant subspaces of quartics by (C).** There is no a non-trivial subspace of invariant quartics by this group.

7.11. **Invariant subspaces of quartics by (E).**

7.11.1. $E_1$. 

$$q = 2x_0^4 + 6x_1 x_2 x_3 x_0 + x_1 x_3^3 + x_1^2 x_2 + x_3 x_3. \quad (*)$$

Respect to Diagram 4.

7.12. **Invariant subspaces of quartics by (B).**

7.12.1. $B_1$. 

$$g_0 = \frac{x_0^4}{\sqrt{15}} - \frac{x_2 x_3 x_0^2}{\sqrt{15}} - \frac{x_1^2 x_0^2}{2\sqrt{15}} - \frac{1}{3} x_1 x_0^3 - \frac{1}{3} x_3 x_0 - \frac{1}{3} x_3^3 x_0 + x_1 x_2 x_3 x_0$$

$$+ \frac{1}{6} \frac{\sqrt{5}}{3} x_4^1 - \frac{1}{3} \frac{\sqrt{5}}{3} x_1 x_0^2 - \frac{1}{3} \frac{\sqrt{5}}{3} x_3 x_3^3 + \frac{1}{2} \frac{\sqrt{5}}{3} x_2^2 x_3^2,$$

$$g_1 = \frac{x_0^4}{4} + \frac{1}{2} x_1 x_0^2 + x_2 x_3 x_0^2 + \frac{x_4^1}{4} + x_2 x_3^2 + x_1 x_2 x_3.$$

The linear combination $\lambda_0 g_0 + \lambda_1 g_1$ is non-singular if and only if 

$$-20\lambda_1 \lambda_0^3 + 107 \sqrt{15\lambda_0^2 \lambda_0^3} + 1140 \lambda_0^3 \lambda_0^3 + 180 \sqrt{15\lambda_1^2 \lambda_0} \neq 0$$

7.13. **Invariant subspaces of quartics by (H).**

7.13.1. $H_1$. It is the subspace 7.12.1.

The non-primitive groups.

7.14. **Invariant subspaces of quartics by $(Q_1)$**.

7.14.1. $Q_{11}$. 

$$x_1^4 + 4x_2 x_3 x_0^2 + 4x_2^2 x_3^2, x_1^2 x_0^2 + 2x_2 x_3 x_0^2,$$

$$x_0^4. \quad (*)$$

Let’s note that all linear combinations of these three polynomials are singular, since each one of these has degree less than 2 in the variable $x_2$. Then each polynomial in this subspace has degree less than 2 in the variable $x_2$ and by Lemma 3.10 we get that all polynomials in this subspace are singulars.

7.15. **Invariant subspaces of quartics by $(Q_4)$**.

7.15.1. $Q_{41}$. 

$$-\frac{1}{2} x_0^2 x_2^2 + x_0 x_1 x_3 x_2 - \frac{1}{2} x_1^2 x_3^2. \quad (*)$$

7.16. **Invariant subspaces of quartics by $(Q_5)$**. There is no non-trivial subspace of invariant quartics by this group.

7.17. **Invariant subspaces of quartics by $(R_3)$**. There is no non-trivial subspace of invariant quartics by this group.
7.18. Invariant subspaces of quartics by \((Q_7)\). There is no non-trivial subspace of invariant quartics by this group.

7.19. Invariant subspaces of quartics by \((P)\).

7.19.1. \(P_1\).

\[
\begin{align*}
p_0 &= x_0^4, \\
p_1 &= x_2x_1^3 + x_3^3x_1 + x_2^3x_3.
\end{align*}
\]

The linear combination \(\lambda_0p_0 + \lambda_1p_1\) is non-singular if and only if \(\lambda_0\lambda_1 \neq 0\). Certainly, by Lemma 3.10 it is necessary that \(\lambda_1\) and \(\lambda_2\) are non-zero. Now, if \(\bar{p} \in \mathbb{P}^2\) is a singular point of \(\lambda_0p_0 + \lambda_1p_1\), then we find a singular point \(\bar{b} \in \mathbb{P}^2\) of \(p_1\), but \(\{p_1 = 0\}\) is the smooth Klein quartic curve, contradiction. Therefore, \(\lambda_0\) and \(\lambda_1\) are non-zero is a criterion of non-singularity of \(\lambda_0p_0 + \lambda_1p_1\), this criterion is equivalent to \(\lambda_0\lambda_1 \neq 0\).

Remark 7.1. All computations have been made with the help of the software Mathematica.

8. Results

Here we present the results obtained in the last two sections. It means, we lists all the smooth quartic surfaces by the finite primitive groups of \(\text{PGL}_4(\mathbb{C})\).

Theorem 8.1 (Diagram 1). There are no smooth quartic surfaces invariant by at least one primitive subgroup listed in Diagram 1.

Proof. See Section 7, subsection 7.1.1. \(\square\)

Theorem 8.2 (Diagram 2). The unique smooth quartic surface invariant by at least one primitive group on Diagram 2 is given by the polynomial

\[
k = x_0^4 + x_1^4 + x_2^4 + x_3^4 + 6 \left( x_0^2x_1^2 - x_2^2x_1^2 + x_3^2x_1^2 + x_0^2x_2^2 - x_0^2x_3^2 + x_2^2x_3^2 \right).
\]

Moreover, such surface is invariant by the group \((19^0)\), which is isomorphic to \(\mathbb{Z}_3\times S_5\), and by its subgroups \((13^0), (14^0), (15^0), (17^0)\), and \((19^0)\). On the other hand, the primitive groups \((16^0), (18^0), (20^0)\), and \((21^0)\) do not fix any smooth quartic surface.

Proof. By Section 7 it is enough to prove that none of groups \((16^0), (18^0), (20^0)\), or \((21^0)\) fixes any smooth surface of degree 4. If \(X = \{f = 0\}\) is a smooth quartic surface invariant by some of these groups, then by Diagram 2 \(X\) must be invariant by \((16^0)\), thus \(f\) is a non-singular quartic form invariant by \((16^0)\), contradiction with 7.5. \(\square\)

Theorem 8.3 (Diagram 3). The smooth quartic surfaces fixed by primitives groups on Diagram 3 are the following:

(A) All the \((A)\)-invariant quartic surfaces are in the pencil of quartics

\[
\begin{align*}
\left\{ \lambda_0 \left( \frac{x_0^4}{2\sqrt{3}} + \frac{1}{3}x_1x_0^3 + x_2x_3x_0^2 + \frac{1}{3}x_1^3x_0 + \frac{1}{3}\sqrt{2}x_2x_0 + \sqrt{\frac{2}{3}}x_3x_0 \\
+ \sqrt{\frac{2}{3}}x_1x_0^3 - \frac{1}{3}\sqrt{2}x_1x_3^2 + x_2^2x_2x_3 - \frac{x_1^4}{2\sqrt{3}} \right) + \\
\lambda_1 \left( \frac{x_0^4}{6} + \frac{x_1x_0^3}{\sqrt{3}} - \frac{1}{2}x_2^2x_0 + \sqrt{3}x_2x_3x_0^2 + \frac{1}{3}\sqrt{2}x_3x_0 + x_1x_2x_3x_0 \\
+ \frac{2}{3}\sqrt{2}x_1x_0^3 - \frac{1}{3}\sqrt{2}x_2x_3^2 - \frac{x_1^4}{3} \right) = 0 \right\}.
\end{align*}
\]
where the singular fibers are finitely many and at least must include the fibers over the following points:
\[
\begin{align*}
[-2\sqrt{2} : \sqrt{3}\sqrt{15} - 1], 
[-2\sqrt{2} : -\sqrt{3}\sqrt{15} - 1], 
[1 : -\sqrt{3}], 
[47 : 8 \left(1 - 4\sqrt{3}\right)], 
[47 : -8 \left(1 + 4\sqrt{3}\right)], 
[4 : -\sqrt{3} + i\sqrt{5}], 
\text{and} 
[4 : -\sqrt{3} - i\sqrt{5}].
\end{align*}
\]

\(\text{(G)}\) There are only two smooth quartic surfaces invariant by \((G)\), they are:
\[
\begin{align*}
&\{ -\frac{x_4^4}{12} + \frac{x_1^3 x_0}{2\sqrt{3}} - \frac{1}{2} x_1^2 x_0 + \frac{1}{2} \sqrt{3} x_1^2 x_0 x_2^3 + x_1 x_2 x_3 x_0 - \frac{x_3^3 x_0}{3\sqrt{2}} - \frac{x_1^3 x_0}{2\sqrt{3}} - \frac{x_3^3 x_0}{\sqrt{6}} \\
&\quad + \frac{x_1 x_0^3}{3\sqrt{2}} - \frac{1}{2} x_2^3 x_0 - \frac{1}{2} \sqrt{3} x_2 x_3 x_0 - \frac{x_4^4}{12} - \frac{x_1 x_0^3}{\sqrt{6}} = 0 \},
\end{align*}
\]

and
\[
\begin{align*}
&\left\{ \frac{x_3^3}{2\sqrt{3}} + \frac{1}{3} x_1 x_0 + x_2 x_3 x_0 + \frac{1}{3} x_3 x_0 + \frac{1}{3} \sqrt{2} x_3 x_0 + \frac{1}{3} \sqrt{2} x_3 x_0 \\
&\quad + \frac{1}{3} \sqrt{2} x_3 x_0 + \frac{1}{3} \sqrt{2} x_3 x_0 + \frac{1}{3} \sqrt{2} x_3 x_0 - \frac{x_4^4}{12} = 0 \right\}.
\end{align*}
\]

\(\text{(E)}\) The unique smooth quartic invariant surface by the group \((E)\) is:
\[
\{ 2x_4^4 + 6x_1 x_2 x_3 x_0 + x_1 x_3^2 + x_2^2 x_2 + x_3^2 x_3 = 0 \}.
\]

In addition, the primitive groups \((C), (D), (K), \text{and} (F)\) do not fix any smooth quartic surface.

\textbf{Proof.} It follows immediately by Diagram 3 and Section 7. \(\square\)

\textbf{Theorem 8.4 (Diagram 4).} The quartic surfaces invariant by primitives groups \((B), \text{and} (H)\) are given by the pencil of quartic surfaces
\[
\begin{align*}
&\left\{ \lambda_0 \left(\frac{x_0^4}{\sqrt{15}} - \frac{x_2 x_3 x_0^2}{\sqrt{15}} - \frac{x_2 x_0^2}{2\sqrt{15}} - \frac{1}{3} x_3^3 x_0 - \frac{1}{3} x_3^3 x_0 - \frac{1}{3} x_3^3 x_0 + x_1 x_2 x_3 x_0 \\
&\quad + \frac{1}{6} \sqrt{5} x_1^4 - \frac{1}{3} \sqrt{5} x_1^3 x_2 + \frac{1}{3} \sqrt{5} x_1 x_3 + \frac{1}{3} \sqrt{5} x_2 x_3 + \frac{1}{2} \sqrt{5} x_2^2 x_3 \right) \\
&\quad + \lambda_1 \left(\frac{x_0^4}{4} + \frac{1}{2} x_1 x_0^2 + x_2 x_3 x_0^2 + \frac{x_3^4}{4} + x_2^2 x_0^2 + x_1^2 x_2 x_3 = 0 \right) \right\},
\end{align*}
\]

where the singular ones are the fibers on the five points of \(\mathbb{P}^1_{\lambda_0, \lambda_1} \)
\[
[0 : 1], [1 : 0], [\sqrt{15} : -4], [2\sqrt{15} : -5], \text{and} (6\sqrt{15} : 1).
\]

\textbf{Proof.} It is a straightforward consequence by the results presented in Section 7. \(\square\)

\textbf{Corollary 8.5.} The projective automorphism group of the smooth quartic surface \(\{ k = 0 \}\) is the group \((19^\circ)\). Moreover, \(\text{PAut}(k = 0) \cong \mathbb{Z}_2^4 \cdot \mathfrak{S}_5\), and \(\{ k = 0 \}\) is projectively equivalent to
\[
\{ h_{12} = x_0^4 + x_1^4 + x_2^4 + x_3^4 + 12x_0 x_1 x_2 x_3 = 0 \}.
\]

Note that, \(h_{12}\) is the quartic form which Burnside conjectured to be the most symmetric smooth quartic surface.

\textbf{Proof.} By 7.7, we have that \(\text{PAut}(k = 0) = \text{PAut}(k) \supset (19^\circ)\). This implies \(\text{PAut}(k)\) is a primitive group of \(\text{PGL}_4(\mathbb{C})\), which it is finite by [MM64, Theorem 1]. By Blichfeldt’s classification of finite primitive groups of \(\text{PGL}_4(\mathbb{C})\) we have \(\text{PAut}(k)\) is conjugate to some unique group in the diagrams of Section 4, namely \(G\). The order of \(G\) is \(|\text{PAut}(k)| \geq |(19^\circ)| = 1920\) and by Section 7 we have \((19^\circ)\) is the group of maximum order between the groups of diagrams in Section 4 which fix some smooth
quartic surface, thus $|\text{PAut}(k)| = |G| = 1920$. Therefore $\text{PAut}(k) \supset (19^\circ)$ have the same order and they must to be equal. Therefore, $\text{PAut}\{k = 0\} = \text{PAut}(k) = (19^\circ)$.

Now we are going to prove that $\{k = 0\}$ and $\{h_{12} = 0\}$ are projectively equivalent. By [FMPK16, Section 5] we have $h_{12}$ is non-singular, $|\text{PAut}\{h_{12} = 0\}| = |\text{PAut}(h_{12})| = 1920$ and $\text{PAut}(h_{12})$ has a subgroup conjugate to the primitive group $(A)$. Thus, $\text{PAut}(h_{12})$ is a primitive finite group of $\text{PGL}_4(\mathbb{C})$ which is conjugate of $(19^\circ) = \text{PAut}(k)$ by Blichfeldt’s classification. Let’s take $(T) \in \text{PGL}_4(\mathbb{C})$ such that $\text{PAut}(k)(T) = (T) \text{PAut}(h_{12})$. It is clear $(T)\{h_{12} = 0\}$ is invariant by $\text{PAut}(k) = (19^\circ)$, and by 7.7 must be the case $(T)\{h_{12} = 0\}$ and $\{k = 0\}$ are equals. Therefore, $\{h_{12} = 0\}$ and $\{k = 0\}$ are projective equivalent. □

By the previous result, we retrieve a noticeable outcome. In fact, we get that the surface $\{h_{12} = 0\}$ is the most symmetric quartic smooth surface, up to projective equivalence. The previous proof also proves Corollary B in the introduction. For completeness we present the smooth quartic invariant surfaces by the non-primitive representations of the groups in Diagram 3.

**Theorem 8.6** (Non-Primitive Groups in Diagram 3). The group $\text{PSL}_2(\mathbb{F}_7)$ seen in $\text{PGL}_4(\mathbb{C})$ as the non-primitive group $(P)$ is the only that fix smooth quartic surfaces. Moreover, these surfaces are in the pencil of quartics

$$\{\lambda_1(x_0^4) + \lambda_1(x_2x_3^3 + x_3^3x_1 + x_2^3x_3) = 0\},$$

with singular fibers at the point $[1 : 0], [0 : 1] \in \mathbb{P}^4_{\lambda_0, \lambda_1}$.

**Proof.** See Section 7, Subsection 7.19.1. □
Algorithm 1: $G$-invariant forms of degree $d$

**Input:** $n$, $d$, $m$, $(\sigma_i)_{1 \leq i \leq m}$, $(A_i)_{1 \leq i \leq m}$ as appears in Section 3.1

**Output:** Basis for each $ker T_k$, $k \in K$

```
begin
C = Array(c, δ)
X = Array(x, n + 1)
E = Exponents(d, n + 1)
δ = Binomial(n + d, n)

Def GeneralPolynomial(Y, X):
    return $\sum_{1 \leq i \leq \delta} Y_i \cdot \text{Monomial}(X, E_i)$

Def LinearB(U, r, Y, X):
    return GeneralPolynomial(Y, U \cdot X) − r \cdot GeneralPolynomial(Y, X)

Def MatrixRepresentationB(U, r):
    M = ConstantArray(0, \{δ, δ\})
    p = LinearB(U, r, A, X)
    for 1 \leq i \leq δ do
        b = Coefficient(p, x_i)
        for 1 \leq j \leq δ do
            M_{i,j} = Coefficient(b, c_j)
    return M

Def MatrixRepresentationT(U, m):
    P = ConstantArray(0, \{mδ, δ\})
    for 1 \leq k \leq m do
        M = MatrixRepresentationB(U_k, r_k)
        for 1 \leq i \leq δ do
            for 1 \leq j \leq δ do
                P_{i+(k-1)\delta,j} = M_{i,j}
        return P

Def BasisT(U, m):
    P = MatrixRepresentationT(U, m)
    Q = NullSpace(P)
    q = Length(Q)
    R = ConstantArray(0, q)
    for 1 \leq i \leq q do
        R_i = GeneralPolynomial(Q_i, X)
    return R

Define $K$ as appears in Section 3.1.

l = Length(K)

G-InvariantForms = ConstantArray(0, l)
for 1 \leq i \leq l do
    G-InvariantForms_i = BasisT((A_j)_{1 \leq j \leq m}, (K_{i,j})_{1 \leq j \leq m}, m)
return G-InvariantForms
```
