Necessary Conditions for Isolation of Special Classes of Bilinear Autoregressive Moving Average Vector (BARMAV) Models

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Abstract: Bilinear Autoregressive Moving Average Vector (BARMAV) Models are models aggregated with the linear and non-linear vector components of autoregressive and moving average processes. The linear part is the sum of the two vector processes, while the non-linear part is the product of the processes. From the general BARMAV models, Bilinear Autoregressive Vector (BARV) Models and Bilinear Moving Average Vector (BMAV) Models have been isolated. Under certain conditions, the models are proved to exist. Empirically, Nigerian consumer price index and inflation rate are used to test the fitness of the bilinear models. Data for the analysis are from Central Bank of Nigeria Statistical Bulletin, collected from January 2009 to December 2016 with November 2009 as the base year for each of the series. The bilinear autoregressive moving average vector models are fitted to the data. Parameters are tested and found to be significant. The adequacy of each estimated model is confirmed with ACF, PACF and descriptive statistics adopted in the paper. The plots of the actual and fitted CPI and IR have shown that models are adequate as estimates compete favourably with the actual values. The models are useful in modelling some economic and financial data that exhibit some characteristics of non-linearity.

Keywords: AR Process, MA Process, Linear and Bilinear Models

1. Introduction

When dealing with classical time series models, the two popular processes that explain the behaviour of empirical data in a stationary time series are autoregressive and moving average processes. These processes are described on the basis of autocorrelation and partial autocorrelation functions of empirical data. The popular Autoregressive Moving Average (ARMA) model in time series is a model of linear relationship between a time series process \(X_t\) and the lag variables of both the process and error term. The General ARMA \((p,q)\) model is expressed in a linear form as,

\[
X_t = \mu_t + \sum_{i=1}^{p} \phi_i X_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t
\]

(1)

Where, \(X_t\) is the process, \(\phi_i\) and \(\theta_j\) are parameters of autoregressive and moving average processes respectively, \(\epsilon_t\) is the error term.

The above model is a univariate linear time series model for the two processes from which AR or MA model can be isolated on condition that \(j = 0\) or \(i = 0\) respectively, [2, 5, 8, 9].

The interest in this paper is to identify special classes of bilinear autoregressive moving average vector models under certain conditions. The fact is that in time series modelling, a process may be described by either AR, MA or both. In as much as AR and MA exist independently each as a univariate linear, multivariate linear, univariate bilinear; it follows that there exists multivariate bilinear model for each of the AR and MA processes under certain conditions.

1.1. Multivariate ARMA Models

The multivariate ARMA model is presented in the matrix form as
The above model “3” is a multivariate linear autoregressive model with each of the processes isolated; such as parametric autoregressive and moving average processes respectively, \( \alpha_{a,fr}(a = 1, ..., p; f = 1, ..., r; h = 1, ..., c) \) are rxc matrices of autoregressive parameters, and \( \beta_{b,fr}(b = 1, ..., q; f = 1, ..., r; h = 1, ..., c) \) are rxc matrices of moving average parameters. The above model “3” is a multivariate non-linear Autoregressive Moving Average Model. On multivariate time series, cited in this paper includes [17], [6], [11]. From “3” above, if certain condition(s) are introduced to the parameters, each of the processes is isolated; such as \( b = 0 \) changes “3” to Vector Autoregressive (VAR) model in the form,

\[
X_{it} = \sum_{a=1}^{p} \sum_{f=1}^{r} \sum_{h=1}^{c} \alpha_{a,fr}X_{j,t-a} + \epsilon_{kt} + \sum_{b=1}^{q} \sum_{f=1}^{r} \sum_{h=1}^{c} \sum_{k=1}^{w} \beta_{b,fr} \epsilon_{kt-b} 
\]

Model (2) can be written as

\[
X_{it} = \sum_{a=1}^{p} \sum_{f=1}^{r} \sum_{h=1}^{c} \alpha_{a,fr}X_{j,t-a} + \epsilon_{kt} + \sum_{b=1}^{q} \sum_{f=1}^{r} \sum_{h=1}^{c} \sum_{k=1}^{w} \beta_{b,fr} \epsilon_{kt-b} 
\]

Where, \( X_{it} \) \( (i = 1,2, ..., n) \) are time series variables, \( \epsilon_{kt} \) are error variables associated with \( X_{it} \) with each \( i \)th time process corresponding to each \( k \)th error term \( i = k \), \( X_{j,t-a} \) and \( \epsilon_{kt-b} \) are autoregressive and moving average processes respectively, \( \alpha_{a,fr}(a = 1, ..., p; f = 1, ..., r; h = 1, ..., c) \) are rxc matrices of autoregressive parameters, \( \beta_{b,fr}(b = 1, ..., q; f = 1, ..., r; h = 1, ..., c) \) are rxc matrices of moving average parameters. The above model “4” is a multivariate linear Autoregressive Moving Average Model. On multivariate time series, cited in this paper includes [17], [6], [11]. From “3” above, if certain condition(s) are introduced to the parameters, each of the processes is isolated; such as \( b = 0 \) changes “3” to Vector Autoregressive (VAR) model in the form,

\[
X_{it} = \sum_{a=1}^{p} \sum_{f=1}^{r} \sum_{h=1}^{c} \alpha_{a,fr}X_{j,t-a} + \epsilon_{kt} + \sum_{b=1}^{q} \sum_{f=1}^{r} \sum_{h=1}^{c} \sum_{k=1}^{w} \beta_{b,fr} \epsilon_{kt-b} 
\]

Similar to “4” are Dufour [3], Usoro and Omekara [13]. Also if \( a = 0 \), a pure moving average vector model is isolated, and it becomes,

\[
X_{it} = \sum_{b=1}^{q} \sum_{f=1}^{r} \sum_{h=1}^{c} \beta_{b,fr} \epsilon_{kt-b} + \epsilon_{kt} 
\]

Where, the parameters are as described above.

1.2. Non-Linear ARMA Models

The non-linear Autoregressive Moving Average models are presented in the form,

\[
\begin{bmatrix}
X_{1t} \\
X_{2t} \\
\vdots \\
X_{nt}
\end{bmatrix} = \begin{bmatrix}
\varphi_{10.11} & \varphi_{10.12} & \ldots & \varphi_{10.1c} \\
\varphi_{10.21} & \varphi_{10.22} & \ldots & \varphi_{10.2c} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_{10.r1} & \varphi_{10.r2} & \ldots & \varphi_{10.rc}
\end{bmatrix} \begin{bmatrix}
X_{1t-1} \\
X_{2t-1} \\
\vdots \\
X_{nt-1}
\end{bmatrix} + \begin{bmatrix}
\varphi_{p1.11} & \varphi_{p1.12} & \ldots & \varphi_{p1.1c} \\
\varphi_{p1.21} & \varphi_{p1.22} & \ldots & \varphi_{p1.2c} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_{p1.r1} & \varphi_{p1.r2} & \ldots & \varphi_{p1.rc}
\end{bmatrix} \begin{bmatrix}
\epsilon_{1t} \\
\epsilon_{2t} \\
\vdots \\
\epsilon_{nt}
\end{bmatrix}
\]

The expansion of the above matrices gives non-linear ARMA models for \( X_{1t}, X_{2t}, ..., X_{nt} \). The models are reduced to

\[
X_{it} = \sum_{a=1}^{p} \sum_{f=1}^{r} \sum_{h=1}^{c} \sum_{k=1}^{w} \beta_{ab,fr} X_{j,t-a} \epsilon_{kt-b} + \epsilon_{kt}
\]

Model (6) can be written as

\[
X_{it} = \sum_{a=1}^{p} \sum_{b=1}^{q} \sum_{f=1}^{r} \sum_{h=1}^{c} \sum_{k=1}^{w} \varphi_{ab,fr} X_{j,t-a} \epsilon_{kt-b} + \epsilon_{kt}
\]

Model “7” is a multivariate non-linear Autoregressive Moving Average Model. Combining “5” and “7”, we have Multivariate Bilinear Autoregressive Moving Average Models also known as BARMA models as
2. Isolation of Special Models

In this section, conditions for isolation of special classes of bilinear autoregressive moving average vector models are considered. Usoro [15] identified special classes of bilinear time series models. Under certain conditions BAR and BMA were identified from the mixed BARMA model. Here, we consider the multivariate case of Usoro [15,16].

2.1. Bilinear Autoregressive Vector (BARV) Model

BARV model is given as

$$X_{it} = \sum_{a=1}^{p} \sum_{f,h,j=1}^{rcn} \alpha_{a,fh} X_{jt-a} + \sum_{b=1}^{q} \sum_{f,h,k=1}^{rcw} \beta_{b,fh} \epsilon_{kt-b} + \sum_{b=1}^{p} \sum_{f,h,k=1}^{rcw} \varphi_{ab,fh} X_{jt-a} \epsilon_{kt-b} + \epsilon_{kt}$$

Proof:

From “8”,

Let $X_{it} = A_t + B_t + C_t + \epsilon_{kt}$

Where,

$$A_t = \sum_{a=1}^{p} \sum_{f,h,j=1}^{rcn} \alpha_{a,fh} X_{jt-a}$$

and

$$C_t = \sum_{a=1}^{p} \sum_{b=1}^{q} \sum_{f,h,k=1}^{rcw} \varphi_{ab,fh} X_{jt-a} \epsilon_{kt-b}$$

Special Condition: if $b = 0 \Rightarrow q = 0, \beta_{a,fh} (f = 1, ..., r; h = 1, ..., c) = 0$, the moving average component of the model is uncorrelated at any lag, $\Rightarrow B_t = 0$, as $\epsilon_{kt} \sim i.i.d(0, \sigma^2_{\epsilon})$.

then

$$A_t = \sum_{a=1}^{p} \sum_{f,h,j=1}^{rcn} \alpha_{a,fh} X_{jt-a} \ \text{and} \ C_t = \sum_{a=1}^{p} \sum_{f,h,j=1}^{rcn} \varphi_{ab,fh} X_{jt-a} \epsilon_{kt}, \ \text{such that}$$

$$X_{it} = A_t + C_t + \epsilon_{kt} = \sum_{a=1}^{p} \sum_{f,h,j=1}^{rcn} \alpha_{a,fh} X_{jt-a} + \sum_{a=1}^{p} \sum_{f,h,j=1}^{rcn} \varphi_{ab,fh} X_{jt-a} \epsilon_{kt} + \epsilon_{kt}$$

Hence, “9” is BARV model with $\alpha_{a,fh}, 1 \leq a \leq p, 1 \leq f, h, j \leq r, c, n; \varphi_{ab,fh}, 1 \leq a \leq p, 1 \leq f, h, j, k \leq r, c, n, w$. This completes the proof.

2.2. Bilinear Moving Average Vector (BMAV) Model

BMAV model is given as

$$X_{it} = \sum_{b=1}^{Q} \sum_{f,h,k=1}^{rcw} \beta_{b,fh} \epsilon_{kt-b} + \sum_{b=1}^{Q} \sum_{f,h,k=1}^{rcw} \varphi_{ab,fh} X_{jt} \epsilon_{kt-b} + \epsilon_{kt}$$

Corollary

From “8”

Let $X_{it} = D_t + E_t + F_t + \epsilon_{kt}$

Where,

$$D_t = \sum_{a=1}^{p} \sum_{f,h,j=1}^{rcn} \alpha_{a,fh} X_{jt-a} E_t = \sum_{b=1}^{q} \sum_{f,h,k=1}^{rcw} \beta_{b,fh} \epsilon_{kt-b}$$
\[
F_t = \sum_{a=1}^{P} \sum_{b=1}^{Q} \sum_{f,h,j,k=1}^{rcnw} \phi_{ab fh} X_{jt-a} \varepsilon_{kt-b}
\]

Special Condition: if \( a = 0 \Rightarrow p = 0 \), \( \alpha_{b fh}(f = 1, \ldots, r; h = 1, \ldots, c) = 0 \), the autoregressive component of the model is uncorrelated at any lag, \( => D_t = 0 \), as \( \varepsilon_{kt} \simiid (0, \sigma^2) \), then

\[
E_t = \sum_{b=1}^{q} \sum_{f,h,j=1}^{rcw} \beta_{b fh} \varepsilon_{kt-b} \quad \text{and} \quad F_t = \sum_{b=1}^{Q} \sum_{f,h,j,k=1}^{rcnw} \phi_{ob fh} X_{jt} \varepsilon_{kt-b}, \text{such that}
\]

\[
X_{it} = E_t + F_t + \varepsilon_{kt} = \sum_{b=1}^{q} \sum_{f,h,j=1}^{rcw} \beta_{b fh} \varepsilon_{kt-b} + \sum_{b=1}^{Q} \sum_{f,h,j,k=1}^{rcnw} \phi_{ob fh} X_{jt} \varepsilon_{kt-b} + \varepsilon_{kt}
\]

Hence, “10” is BMAV model with \( \beta_{b fh}, 1 \leq b \leq q, 1 \leq f, h, j \leq r, c, w; \phi_{ob fh}, 1 \leq b \leq Q, 1 \leq f, h, j, k \leq r, c, n, w \). This completes the proof.

2.3. Test for Linearity and Bilinearity

The above test involves the parameters of both the linear and non-linear components of bilinear time series models.

(i) Linear Parameter: A linear parameter is the unknown coefficient of either AR, MA or Mixed ARMA process.
(ii) Non-Linear Parameter: This is the unknown coefficient of the non-linear component(s) of the bilinear model.

**Test Statistic:**
The test statistic is given as

\[
t = \frac{\hat{\alpha}_{afh}}{\hat{S}_{afh}} \quad \text{for the linear AR component,} \quad \frac{\hat{\beta}_{bfh}}{\hat{S}_{bfh}} \quad \text{for the linear MA component,} \quad \frac{\hat{\phi}_{ab fh}}{\hat{S}_{ab fh}} \quad \text{for the non-linear ARMA component.}
\]

**Hypotheses:**
\[H_0: \hat{\alpha}_{afh} = \hat{\beta}_{bfh} = \hat{\phi}_{ab fh} = 0\]
\[H_1: \hat{\alpha}_{afh} \neq 0, \hat{\beta}_{bfh} \neq 0, \hat{\phi}_{ab fh} \neq 0\]

From the above hypotheses;
(i) if \( H_0 \) is rejected for at least one of \( \hat{\alpha}_{afh} \) and \( \hat{\beta}_{bfh} \), but accepted for \( \hat{\phi}_{ab fh} \), there is no effect of bilinearity in the series. It is characterised by a pure linear process.
(ii) if \( H_0 \) is accepted for both \( \hat{\alpha}_{afh} \) and \( \hat{\beta}_{bfh} \), but rejected for only \( \hat{\phi}_{ab fh} \), the series is characterised by a complete non-linear process.
(iii) if \( H_0 \) is rejected for at least one of \( \hat{\alpha}_{afh} \) and \( \hat{\beta}_{bfh} \), and rejected for \( \hat{\phi}_{ab fh} \), there is effect of bilinearity in the series. A bilinear process is a process that has effects of both linear and non-linear components in the series.

3. Model Fitting to Empirical Data

In this section, we consider fitting the bilinear models to the empirical data. For illustration, Nigeria Consumer Price Index (CPI) and Inflation rates (IR) are fitted with the bilinear models. The procedures of fitting bilinear models to time series data are not different from the ordinary ARMA models.

3.1. Time Plots of the CPI and IR

![Time Series Plot of Consumer Price Index](image)

**Figure 1. Time Series Plot of Consumer Price Index.**
“Figures 1” and “Figure 2” are the time series plots of consumer price index and inflation rates from January 2009 to December 2016, with November 2009 as the base year. The consumer price index exhibits long term increase over time, while interest rate shows some level of randomness, which explains non-linearity characteristics as indicated from January 2009 with a drop 2013, and sharp increase in 2016.

3.2. Autocorrelation and Partial Autocorrelation Functions of CPI and IR

The above ACF’s and PACF’s for the CPI and IR suggest BARMA(1,1,1,1) for each CPI and IR.

3.3. Estimation of Parameters and Interpretation of Results

| Predictor | Coeff. | SE. Coeff | t-statistics | P  |
|-----------|--------|-----------|--------------|----|
| Constant  | 0.9864 | 0.7978    | 1.24         | 0.220 |
| X_{t-1}   | 0.9972 | 0.0042    | 240.12       | 0.000 |
| X_{t-1}   | 0.01896 | 0.0328 | 0.58 | 0.564 |
| ε_{t-1}   | -0.8520 | 0.3779 | 2.25 | 0.027 |
| ε_{t-1}   | 1.2686 | 0.6063 | 2.09 | 0.039 |
| X_{t-1}ε_{t-1} | 0.0093 | 0.0024 | 3.92 | 0.000 |
| X_{t-1}ε_{t-1} | -0.1412 | 0.0533 | 2.65 | 0.009 |

| Predictor | Coeff. | SE. Coeff | t-statistics | P  |
|-----------|--------|-----------|--------------|----|
| Constant  | 1.1815 | 0.8569    | 1.38         | 0.171 |
| X_{t-1}   | -0.0058 | 0.0045 | -1.30 | 0.196 |
| X_{t-1}   | 0.9529 | 0.0352 | 27.08 | 0.000 |
| ε_{t-1}   | -1.0104 | 0.4058 | -2.49 | 0.015 |
| ε_{t-1}   | 0.1491 | 0.6512 | 0.23 | 0.819 |
| X_{t-1}ε_{t-1} | 0.0082 | 0.0026 | 3.20 | 0.002 |
| X_{t-1}ε_{t-1} | -0.0308 | 0.0573 | -0.54 | 0.592 |

The models with parameter estimates are presented as follows.
\[
\begin{bmatrix}
X_{1t} \\
X_{2t}
\end{bmatrix} = \begin{bmatrix}
1.00183 & 0.04905 \\
-0.000286 & 0.98898 \\
-0.5241 & 1.3913 \\
-0.6176 & 0.2961 \\
\end{bmatrix} \begin{bmatrix}
X_{1t-1} \\
X_{2t-1} \\
\epsilon_{1t-1} \\
\epsilon_{2t-1}
\end{bmatrix} + \begin{bmatrix}
0.007128 & -0.15476 \\
0.005565 & -0.04692
\end{bmatrix} \begin{bmatrix}
X_{1t-1} & \epsilon_{1t-1} \\
X_{2t-1} & \epsilon_{2t-1}
\end{bmatrix}
\]

(11)

The above results have it that parameters of the bilinear model fitted to CPI are all significant, except for \(X_{2t-1}\). For IR, evidence has it that some parameters of the linear components are significant with one of the non-linear components. This is a true indication of model fitness to the data. Further evidence is shown in the ACF and PACF of the model residual.

**Figure 7.** ACF of the Model Residual.

**Figure 8.** PACF of the Model Residual.

“Figure 7” and “Figure 8” are evident that the residual is a pure white noise process, which is identically and independently distributed with zero mean and constant variance as shown in table 3.

| Variable  | N  | N* | Mean | SE Mean | STD  | Min  | Q1   | Median | Q3   | Max  |
|-----------|----|----|------|---------|------|------|------|-------|------|------|
| Residual  | 95 | 1  | 0.000| 0.073   | 0.708| -1.3 | -0.3 | -0.11 | 0.23 | 3.0  |

**Table 3.** Descriptive Statistics: Residual.
4. Conclusion

There is no gainsaying the fact that each of the AR and MA processes are combined to form mixed ARMA process. This is explained by the behaviour of the empirical data as always shown in the distribution of the autocorrelation and partial autocorrelation functions. A process that is only described by either AR or MA remains a singular process, except characterised by both. The idea about this paper is that if there exist condition(s) for isolation of AR or MA from ARMA, isolation of BAR or BMA from BARMA models, therefore, BARV and BMAV are conditionally isolated from BARMAV model. For a pure bilinear autoregressive vector model, the non-linear component of the model is the product of lagged $X_{it}$ and non-lagged $\epsilon_{it}$. That is the multiplication of $X_{it-1}, X_{it-2}, \ldots, X_{it-p}$ by $\epsilon_{it}$. That means each of the non-zero lags of $X_{it}$ is multiplied by zero lag of $\epsilon_{it}$ to form the non-linear part of the model. Similarly, for a pure bilinear moving average vector model, the non-linear component of the model is the product of lagged $\epsilon_{it}$ and non-lagged $X_{it}$. That is the multiplication of $\epsilon_{it-1}, \ldots, \epsilon_{it-q}$ by $X_{it}$. Here, each of the non-zero lags of $\epsilon_{it}$ is multiplied by zero lag of $X_{it}$ to form the non-linear part of the BMAV model. Empirically, see Usoro and Omekara (2008) and Usoro (2018). Empirically, the monthly consumer price index and inflation rate used in this paper are characterised by both AR and MA process. This called for adoption of “8” in the analysis. The plots of the actual and fitted CPI and IR data in figures “9” and “10” have shown that estimates compete favourably with the actual. Hence, the models are suitable in modelling time series data that exhibit some form of non-linearity characteristics.
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