Distortion elements in group of diffeomorphisms of the 2-sphere

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Abstract

We prove that every distortion element in the group of diffeomorphisms of the 2-sphere which has some recurrent point that is not fixed is an irrational pseudo-rotation. Moreover we prove that the differential of a distortion element in the group of diffeomorphisms of the 2-sphere having at least three fixed points at a fixed point has a unique eigenvalue which is 1.

1 Introduction

We recall the concept of distortion in a group. We will say that $f$ in a group $G$ is a distortion element in $G$ if it has infinite order and there exists a finite subset $S$ of $G$ such that:

(i) $f$ belongs to the group generated by $S$,

(ii) If $|f^n|$ is the word length of $f^n$ in the generators of $S$, then

$$\lim_{n \to \infty} \frac{|f^n|}{n} = 0.$$ 

This limit exists because the sequence $(|f^n|)_{n \in \mathbb{N}}$ is sub-additive. The simplest examples of groups which contain a distortion element of infinite order are the solvable Baumslag-Solitar groups:

$$BS(1; p) := < f, g : gfg^{-1} = f^p >,$$ where $p \geq 2$.

Indeed, for the group relation we have for every integer $n \geq 1$: $g^nfg^{-n} = f^{pn}$. Taking as generators the set $S = \{f, g\}$, we have $|f^{pn}| \leq 2n + 1$ and the element $f$ is distorted in the group $BS(1; p)$, with $p \geq 2$. Another example is the discrete Heisenberg group $H_3$. It is the group of integer matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

There are generators $g$ and $h$ of $H_3$ such that their commutator $f = [g, h]$ has infinite order and generates the center of $H_3$. This implies that $f^{n^2} = [g^n, h^n]$ and hence that $f$ is a distortion element in $H_3$. 

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Before stating our main result we recall some results of distortion elements in subgroups of the group of homeomorphisms on manifolds. Given a manifold $M$, we will denote by $\text{Diff}^r_0(M)$ (resp. $\text{Homeo}_0(M)$) the connected component of the group of $C^r$-diffeomorphisms (resp. homeomorphisms) of $M$ containing the identity. The following result due to Calegari and Freedman (see [3]) shows that every homeomorphism of the $d$-dimensional sphere is distorted.

**Theorem 1.1** (Calegari and Freedman, [3]). Let $d \geq 1$. Then every element in the group $\text{Homeo}_0(S^d)$ is distorted.

In this paper we will consider the group of diffeomorphisms of the 2-sphere. As it is cleverly noticed in [3], we have the following restrictions for a distortion element $f$ in $\text{Diff}^1_0(M)$.

$(P_1)$ If $x$ is a fixed point of $f$, then eigenvalues of $Df(x)$ have absolute value 1, and

$(P_2)$ $f$ has zero topological entropy

The authors of [3] use Oseledec’s theorem and Pesin-Ruelle inequality to proving property 2. We note that recently Navas gives a most elementary proof of property 2 which is valid for bi-Lipschitz homeomorphisms (see [14]). The following result improves the above second property $(P_2)$.

**Theorem 1.2** (Franks and Handel, [6]). Let $M$ be a closed oriented surface of genus at least two. Then there are no distortion element in $\text{Diff}^1_0(M, \mu)$ the subgroup of the group $\text{Diff}^1_0(M)$ which preserve a Borel probability measure on $M$ with total support.

Also we can find infinitely many distortion elements in $\text{Diff}^1_0(M)$ for any manifold $M$. More precisely we can see the solvable Baumslag-Solitar group (defined above) $BS(1; p)$, with $p \geq 2$ as a subgroup of $\text{Diff}^1_0(M)$.

**Example** (Solvable Baumslag-Solitar group embeds in $\text{Diff}^1_0(M)$). Choose a ball $B$ in a manifold $M$. Let us consider a $C^1$-action of $BS(1; p)$ on $[0, 1]$ such that the generators $f, g$ satisfy $f'(0) = g'(0) = 1$ and $f'(1) = g'(1) = 1$ (see [2]). Extend it to an action on $B$ which fixes an interior point and the boundary $\partial B$ so that it can be extend as the identity outside $B$.

This example being an action of the line has no recurrence. Milotg gives another class of distorted element in $\text{Diff}^\infty_0(M)$. He uses Avila’s techniques (see [1]) and a local perfection result (see [9]), he obtains that every recurrent element in $\text{Diff}^\infty_0(M)$, i.e. for which arbitrarily large iterates are arbitrarily close to the identity in the $C^\infty$-topology, is distorted.

**Theorem 1.3** (Milotg, [13]). Let $M$ be a compact manifold. Then every recurrent element in the group $\text{Diff}^\infty_0(M)$ is distorted.

For instance, irrational rotations of the 2-sphere are distorted (previously showed by Calegari and Freedman (see [3])). More generally, using the Anosov-Katok method (see [10] and [5]), we can build recurrent elements in the case of the 2-sphere which are not conjugate to a rotation (such an element are called irrational pseudo-rotation). We start by introducing some definitions. Let $f$ be a homeomorphism of $S^2$. We say that $z \in S^2$ is a (positively) recurrent point that is not fixed for $f$ if there are arbitrarily large (positively) iterates $f^n(z)$ which are arbitrarily close
to $z$ and $f(z) \neq z$. We say that a homeomorphism $f$ of $\mathbb{S}^2$ is an irrational pseudo-rotation if:

(i) $f$ has exactly two periodic points, $z_0$ and $z_1$, which are both fixed;

(ii) there exists a point $z$ whose $\alpha$-limit set or $\omega$-limit set is not included in $\{z_0, z_1\}$; and

(iii) if $\tilde{f}$ is a lift of $f|_{\mathbb{S}^2 \setminus \{z_0, z_1\}}$ to the universal covering space of $\mathbb{S}^2 \setminus \{z_0, z_1\}$, its unique rotation number is an irrational number.

Our main result is the following.

**Theorem A.** Let $f$ be a distortion element in $\text{Diff}^1_0(\mathbb{S}^2)$ such that has a (positively) recurrent point that is not fixed. Then $f$ is an irrational pseudo-rotation.

Clearly, by Theorem 1.1 the hypothesis of differentiability is necessary. Moreover the solvable Baumslag-Solitar group embeds in $\text{Diff}^1_0(\mathbb{S}^2)$ (above example) and Heisenberg group $H_3$ acts smoothly on $\mathbb{S}^2$ by restricting and projectivizing the standard action of $GL(3, \mathbb{R})$ on $\mathbb{R}^3$. These examples show that hypothesis of the existence of recurrence in Theorem A is also necessary.

What about above property 1 ($P_1$)? A priori this property can be improve with an additional very strong hypothesis. If a distortion element $f$ belongs to a finitely generated group which acts with a global fixed point $z$. Then the differential at $z$ induces an action of the group on $GL(2, \mathbb{R})$, and as it is noticed in [3] the eigenvalues of $Df(z)$ are roots of unity.

The property “eigenvalues are roots of unity” is again true in some cases for our examples of groups having distortion elements without any additional hypothesis.

Case of solvable Baumslag-Solitar group. Here Guéman and Lioussé showed the existence of a $g$-minimal set $\Lambda$ contained in $\text{Fix}(f)$ (see [7]). Using the fact that every point in $\Lambda$ is recurrent, the authors of [7] proved, in [8], that for every $z \in \Lambda$ the eigenvalues of $Df(z)$ are roots of unity.

Case of Heisenberg group. If $H_3$ acts on $\mathbb{S}^2$ by diffeomorphisms, then Ribón insures the existence of an orbit of cardinality at most 2 (see [13]). Considering power of the generating elements (if necessary) we can suppose that $H_3$ acts with a global fixed point $z$. Then Franks and Handel (see [6]) proved that $Df(z) = \pm Id$.

Our second result shows that for distortion elements in $\text{Diff}^1_0(\mathbb{S}^2)$ any additional hypothesis is necessary.

**Theorem B.** Let $f$ be a distortion element in $\text{Diff}^1_0(\mathbb{S}^2)$ which has at least three fixed points. If $z$ is a fixed point of $f$, then $Df(z)$ has a unique eigenvalue which is 1.

These results are part of a bigger plan which seeks to characterize dynamics of distortion elements of the 2-sphere.

We will proved Theorem A using the new criterion for the existence of topological horseshoes for surface homeomorphisms which are isotopic to the identity recently find for Le Calvez and Tal (see [11]).
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2 Preliminary results

In this section we recall some results of distortion elements in the group of diffeomorphisms of the 2-sphere and homeomorphisms of the 2-sphere with no topological horseshoes that we will use in the rest of the article.

2.1 Distortion elements in the group of diffeomorphisms of the 2-sphere

The following result is Theorem 1.2 for the case of the 2-sphere.

**Theorem 2.1** (Theorem 1.3, [6]). Let $f$ be a distortion element in $\text{Diff}^1_0(S^2)$ and let $\mu$ be an $f$-invariant Borel probability measure. If $f^n$ has at least three fixed points for some smaller integer $n$, then $\text{Supp}(\mu) \subset \text{Fix}(f^n)$.

We state the following corollary which will be used in the rest of the article.

**Corollary 2.2.** Let $f$ be a distortion element in $\text{Diff}^1_0(S^2)$. Suppose that $f$ has at least three fixed points, then

(i) the set of periodic points of $f$ coincides with that of its fixed points;
(ii) if $X$ is a closed $f$-invariant set, then $X$ contains a fixed point; and
(iii) if $h(f)$ denotes the topological entropy of $f$, then $h(f) = 0$.

2.2 Homeomorphisms of the 2-sphere with no topological horseshoes

In this section we state some results which were proved in [11], where the authors study some properties of homeomorphisms of the 2-sphere with no topological horseshoes. A compact subset $Y$ of a manifold $M$ is a topological horseshoe if it is invariant by an iterated $f^q$ of $f$, $f^q|_Y$ is semi-conjugated to $g : Z \to Z$ on a Hausdorff compact space, so that the fibers by the factor map are all finite with uniform bound $m$ in their cardinality, and $g$ is semi-conjugated to the Bernoulli shift $\sigma : \{1, \cdots, N\} \to \{1, \cdots, N\}$, where $N \geq 2$ is such that the preimage of every $s$-periodic sequence of $\{1, \cdots, N\}$ by the factor map contains a $s$-periodic point of $g$.

**Remark 1.** Note that, if $h(f)$ denotes the topological entropy of $f$, then

$$qh(f) = h(f^q) \geq h(f^q|_Y) = h(g) \geq h(\sigma) = \log(N),$$

and that $f^{jn}$ has at least $q^n/m$ fixed points for every $n \geq 1$. 

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The first result is about rotation numbers for annular homeomorphisms with no topological horseshoes. We will write $\mathbb{A} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ the open annulus and denote by $\pi : \mathbb{R}^2 \to \mathbb{A}$ the universal covering projection of $\mathbb{A}$ and by $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ the projection in the first coordinate.

**Theorem 2.3 (Theorem A, [11]).** Let $f$ be a homeomorphism of $\mathbb{A}$ which is isotopic to the identity and let $\tilde{f}$ be a lift of $f$ to $\mathbb{R}^2$. We suppose that $f$ has no topological horseshoe. Then each point $z$ such that its $\omega$-limit set is non-empty has a well defined rotation number $\text{Rot}_{\tilde{f}}(z)$, i.e. for every compact set $K \subset \mathbb{A}$ and every increasing sequence of integers $(n_k)_{k \in \mathbb{N}}$ such that $z$ and $f^{n_k}(z)$ belong to $K$, we have

$$\lim_{k \to \infty} \frac{1}{n_k} (\pi_1(\tilde{f}^{n_k}(\tilde{z}) - \tilde{z})) = \text{Rot}_{\tilde{f}}(z),$$

where $\tilde{z}$ is a lift of $z$.

**Theorem 2.4 (Theorem F, [11]).** Let $f$ be a homeomorphism of $\mathbb{A}$ which is isotopic to the identity and let $\tilde{f}$ be a lift of $f$ to $\mathbb{R}^2$. We suppose that $\tilde{f}$ is fixed point free and that there exists a positively recurrent point $z$ such that $\text{Rot}_{\tilde{f}}(z)$ is well defined and equal to $0$. Then $f$ has a topological horseshoe.

**Theorem 2.5 (Theorem G, [11]).** Let $f$ be an orientation-preserving homeomorphism of $S^2$ that has no topological horseshoe. Then every non-wandering point $z$ whose $\alpha$-limit set or $\omega$-limit set in not included in the set of fixed points of $f$ lies in an open annulus $A$ which is a fixed point free and $f|_A$ is isotopic to the identity.

### 3 Main Proposition

In this section, we will prove the following proposition which is key in the demonstrations of Theorems A and B.

**Proposition 3.1.** Let $f$ be a distortion element in $\text{Diff}_0^1(S^2)$ and let $z_0$ and $z_1$ be two distinct fixed points of $f$. Let $f_{\text{ann}}$ be the homeomorphism of the closed annulus $\overline{A}$ obtained from $S^2$ by blowing up both points $z_0$ and $z_1$. If $\tilde{f}_{\text{ann}}$ is a lift of $f_{\text{ann}}$ to the universal covering space of $\overline{A}$, then it has a unique rotation number, i.e. there exists $\rho$ such that

$$\lim_{n \to \infty} \frac{1}{n} (\pi_1(\tilde{f}_{\text{ann}}^n(\tilde{z}) - \tilde{z})) = \rho,$$

where $\tilde{z}$ is a lift of $z \in \overline{A}$.

To prove this proposition we need the concept of spread introduced by Franks and Handel in [6].

**Spread.** Let $\gamma$ be a smooth curve with endpoints $z_0$ and $z_1$ (smooth at the endpoints) and $\beta$ be a simple nullhomotopic closed curve on $S^1$ such that the point $z_1$ is contained in the disk bounded by $\beta$. For any curve $\alpha$, the spread $L_{\alpha, \beta}(\alpha)$ is going to measure how many times $\alpha$ rotates around $\beta$ with respect to $\gamma$. More formally the definition of spread is the following: let $\overline{A}$ be the compact annulus obtained from $S^2$ by blowing up
both points $z_0$ and $z_1$. In this case the universal covering space of $\mathcal{A}$ can be identified with $\mathbb{R} \times [0,1]$ and the covering translation can be identified by $T(x, y) = (x + 1, y)$. For each lift $\hat{\alpha}$ of $\alpha$ and $\hat{\gamma}$ of $\gamma$ to $\mathbb{R} \times [0,1]$, there exist integers $a < b$ such that $\hat{\alpha} \cap T^i(\hat{\gamma}) \neq \emptyset$ if and only if $a < i < b$. Define $L_{\hat{\beta}, \hat{\gamma}}(\hat{\alpha}) = \max\{0, b - a - 2\}$.

Define $\hat{\alpha} = \max\{L_{\hat{\beta}, \hat{\gamma}}(\hat{\alpha})\}.$

Define the spread of $\alpha$ with respect to $f$, $\beta$ and $\gamma$ as

\[ \sigma_{f, \beta, \gamma}(\alpha) := \lim_{n \to \infty} L_{\beta, \gamma}(f^n(\alpha)). \]

In [6] the authors give some criteria for undistortion, one of them is positive spread.

**Lemma 3.2.** Let $g_i, 1 \leq i \leq k$ be a finite set of elements of $\text{Diff}^1_0(S^2)$. There exists a constant $C > 0$ such that the following property holds: If $f$ belongs to the group generated by the $g_i$, and if $|f^n|$ is the word length of $f^n$ in the generators $g_i$, then for all curves $\alpha$, $\beta$, $\gamma$ and all integer $n > 0$,

\[ L_{\beta, \gamma}(f^n(\alpha)) \leq L_{\beta, \gamma}(\alpha) + C|f^n|. \]

It follows the following proposition.

**Proposition 3.3.** Let $f$ be a distortion element in $\text{Diff}^1_0(S^2)$. Then for all curves $\alpha$, $\beta$, $\gamma$ we have

\[ \sigma_{f, \beta, \gamma}(\alpha) = 0. \]

**Proof.** Since $f$ is a distortion element in $\text{Diff}^1_0(S^2)$, there exists $g_i$, $1 \leq i \leq k$ a finite set of elements of $\text{Diff}^1_0(S^2)$ such that $f$ belongs to the group generated by the $g_i$. Moreover if $|f^n|$ is the word length of $f^n$ in the generators $g_i$,

\[ \lim_{n \to \infty} \frac{|f^n|}{n} = 0. \]

According to the definition of spread and above lemma we have

\[ \sigma_{f, \beta, \gamma}(\alpha) = \lim_{n \to \infty} \frac{L_{\beta, \gamma}(f^n(\alpha))}{n} \leq \lim_{n \to \infty} \frac{L_{\beta, \gamma}(\alpha) + C|f^n|}{n} = 0. \]

The following result is about rotation numbers for homeomorphisms of the closed annulus with no topological horseshoe.

**Lemma 3.4.** Let $f$ be a homeomorphism of $(\mathbb{R}/\mathbb{Z}) \times [0,1]$ which is isotopic to the identity and let $\hat{f}$ be a lift of $f$ to $\mathbb{R} \times [0,1]$. We suppose that $f$ has no topological horseshoe. Then each point $z \in (\mathbb{R}/\mathbb{Z}) \times [0,1]$ has a well defined rotation number

\[ \text{Rot}_f(z) = \lim_{n \to \infty} \frac{\pi_1(\hat{f}^n(\hat{z})) - \pi_1(\hat{z})}{n}, \]

where $\hat{z}$ is a lift to $z$.  

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Proof. Write \( f(x, y) = (f_1(x, y), f_2(x, y)) \), we can extend \( f \) to a homeomorphism \( f^* \) of the open annulus \( \mathcal{A} \) such that \( f^*(x, y) = (f_1(x, 1), y) \) if \( y \geq 1 \) and \( f^*(x, y) = (f_1(x, 0), y) \) if \( y \leq 0 \). Note that \( f^* \) has no horseshoe, because \( f \) has no. As \( (\mathbb{R}/\mathbb{Z}) \times [0, 1] \) is included in a compact set of \( \mathcal{A} \), the result follows from Theorem 2.3.  

### 3.1 Proof of Proposition 3.1

In this section, we prove Proposition 3.1. Suppose by contradiction that there exist \( z \) and \( z' \) (in the interior of \( \mathcal{A} \)) such that \( \text{Rot}_{\tilde{f}_{\text{ann}}}^{\alpha}(z) = \rho \) and \( \text{Rot}_{\tilde{f}_{\text{ann}}}^{\alpha}(z') = \rho' \), with \( \rho < \rho' \). Conjugation we can suppose that \( \tilde{f}_{\text{ann}} \) is a homeomorphism of \( \mathbb{R} \times [0, 1] \) and a lift of \( \gamma \) to \( \mathbb{R} \times [0, 1] \) is \( \{(0, z)\} \times [0, 1] \).

Let \( \alpha \) be a smooth curve joining \( z \) and \( z' \) and let \( \tilde{\alpha} \) be a lift of \( \alpha \) with endpoints \( \tilde{z} \) and \( \tilde{z}' \). For \( \epsilon < (\rho' - \rho)/2 \) and for every \( n \) large enough we have

\[
\pi_1(\tilde{f}_{\text{ann}}^n(\tilde{z})) < n\epsilon + n\rho + \pi_1(\tilde{z})
\]

and

\[
-n\epsilon + n\rho' + \pi_1(\tilde{z}') < \pi_1(\tilde{f}_{\text{ann}}^n(\tilde{z}'))
\]

Therefore we have

\[
-2\epsilon n + (\rho' - \rho)n < \pi_1(\tilde{f}_{\text{ann}}^n(\tilde{z}')) - \pi_1(\tilde{f}_{\text{ann}}^n(\tilde{z})) 
\leq L_{\beta, \gamma}(f^n(\alpha)) + 2.
\]

And so

\[
0 < \sigma_{f, \beta, \gamma}(\alpha).
\]

This contradicts Proposition 3.3.

Suppose now that there exists \( z \) (in the interior of \( \overline{\mathcal{A}} \)) such that \( \text{Rot}_{f_{\text{ann}}}^{\alpha}(z) = \rho \) and that the rotation number of the boundary component associated to \( z_0 \) is \( \rho_0 \), with \( \rho < \rho_0 \). We have two cases.

Case 1. The local rotation set at \( z_0 \) is empty. In this case following [12], we have one of the next three possibilities

(i) \( z_0 \) is a global attractor,

(ii) \( z_0 \) is a global repeller,

(iii) \( f \) is locally conjugate to the application \( z \mapsto e^{2\pi ip/q} z(1 + z^{qr}) \), for some integers \( p, q \) and \( r \).

If possibility (i) or (iii) is valid, then there exists a point \( z' \) (in the interior of \( \overline{\mathcal{A}} \)) whose \( \omega \)-limit set is contained in the boundary component of \( \overline{\mathcal{A}} \) associated to \( z_0 \) and so \( \text{Rot}_{f_{\text{ann}}}^{\alpha}(z') = \rho_0 \). This contradicts the first part. If possibility (ii) is valid for both boundary components of \( \overline{\mathcal{A}} \) the result follows by the first part.

Case 2. The local rotation set at \( z_0 \) is not empty. In this case following [4], we know that the rotation set of the open annulus is an interval. Take a rational number written in an irreducible way \( p/q \) such that \( \rho < p/q < \rho_0 \). We have again three possibilities (see [4]). Again, there exists a point \( z' \) (in the interior of \( \overline{\mathcal{A}} \)) such that \( \rho < p/q \leq \text{Rot}_{f_{\text{ann}}}^{\alpha}(z') \). This is again a contradiction with the first part.

This proves the result.
4 Proof of main results

4.1 Proof of Theorem A

In this section, we prove Theorem A. Let $f$ be a distortion element in $\text{Diff}_0^1(S^2)$. We will consider two cases:

Case 1: $f$ has at least three fixed points.
Let $z$ be a (positively) recurrent point of $f$ which is not fixed. By Theorem 2.5 there exists an open annulus $A$ which is maximal fixed point free and that contains $z$. As $f|_A$ is isotopic to the identity, by Corollary 2.2 we can fix two fixed points $z_0$ and $z_1$ one in each connected component of the complement of $A$. Let $z_2$ be a third fixed point and let $\tilde{f}$ be a lift of $f|_{S^2 \setminus \{z_0, z_1\}}$ to the universal covering space of $S^2 \setminus \{z_0, z_1\}$ that fixes a lift of $z_2$. Then by Proposition 3.1 it has a unique rotation number which must be $\{0\}$. In this case the positively recurrent point $z$ in $A$ has a well defined rotation number $\text{Rot}_{\tilde{f}}(z)$ and it must be equal to 0. From Theorem 2.4 $f|_A$ has a topological horseshoe. This contradicts Corollary 2.2 (see Remark 1).

Case 2: $f$ has exactly two fixed points.
Let $z_0$ and $z_1$ be the two fixed points of $f$ and let $\tilde{f}$ be a lift of $f|_{S^2 \setminus \{z_0, z_1\}}$ to the universal covering space of $S^2 \setminus \{z_0, z_1\}$. Then as above it has a unique rotation number. If this is rational as above we can show that $f$ has a topological horseshoe which contradicts Corollary 2.2 (see Remark 1). Hence the unique rotation number of $\tilde{f}$ is irrational and so $f$ is an irrational pseudo-rotation.

4.2 Proof of Theorem B

In this section, we prove Theorem B. Let $f$ be a distortion element in $\text{Diff}_0^1(S^2)$. Suppose that $f$ has at least three fixed points. Fix us three distinct fixed points $z_0$, $z_1$ and $z$ of $f$. Let $f_{\text{ann}}$ be the homeomorphism of the closed annulus $\overline{A}$ obtained from $S^2$ by blowing up the fixed points $z_0$ and $z$. If $f_{\text{ann}}$ is a lift of $f_{\text{ann}}$ to the universal covering space of $\overline{A}$ which fixes a lift of $z_2$, then by Proposition 3.1 it has a unique rotation number which must be $\{0\}$. Therefore the rotation number of both boundary components of $\overline{A}$ is zero. Hence $Df(z)$ has a unique eigenvalue which is 1.

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