Modification Cross’ Theorem on Triangle with Congruence

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Abstract: Cross’ theorem states if any triangle $ABC$, on each side constructed a square and vertices of the square are connected, it will form another triangle which has an equal area to triangle $ABC$. In this paper we will discuss the modification Cross’ theorem on triangle, which is to construct a square outcase direction on each side triangle produced by Cross’ theorem. The purpose of this paper is to modify Cross’ theorem and give simple proof even junior or senior high school can answer. The process of proof is done in simple way, that is using a congruence approach. The result obtained are square vertices that are connected is form a trapezoid and have area 5 times the area of triangle $ABC$.

Keywords: Triangle, Square, Congruence, Cross’ Theorem

1. Introduction

Geometry has been studied starting from elementary to college, one of the material is about plane. On plane we will know such as triangles, rectangles, squares or trapeziums. At the elementary school level students are just beginning to recognize the nature, elements and determine area, while at the middle and high school levels students have begun to understand concepts and theorems in mathematics.

One of the many theorems in geometry that discusses about triangles is the Cross’ theorem. Cross’ theorem states if it is any triangle $ABC$ a square constructed on each side of triangle and a squares vertices are connected so it will form another triangle which has an area equal to the triangle $ABC$.

Cross’ theorem discovered and named by 14 years old schoolboy David Cross, which was posed by Faux (2004). Cross’ theorem can be proved by using congruence [7]. Congruence essentially means that two figures or objects are of the same shape and size which is has been studied at junior high school.

In general, this Cross’ theorem only applies to triangles, but several authors have developed this Cross’ theorem like Cross’ theorem on the triangle using a rectangle and the Cross’ Theorem on quadrilateral [4, 11]. The results reveal that there are the relationship between the area area formed from the new triangles constructed and the initial triangle.
Lots of software that can be used in learning mathematics especially in geometry. One software which can be used is Geogebra. Geogebra application is dynamic mathematical software that can be used as a tool in mathematics learning. In this paper using software Geogebra which is very helpful in constructing points and lines. Geogebra is a versatile software for learning mathematics in schools and colleges, Geogebra is used as a media for demonstration and visualization, tools for finding mathematical concepts and preparing materials for teaching.

Based on the description above, the author discusses proof of modification Cross’ theorem using easy material and can be discusses by middle or high school students.

2. Cross’ Theorem

Cross’ theorem was first put forward by Faux (2004) for reader of the Mathematics Teaching Journal. Faux states that a triangle $ABC$ on each sides triangle a squares are constructed, then a near square vertices connected it will form a new triangles and has an equal area to triangle $ABC$ [2].

This Cross’ theorem was discovered by David Cross. In general Cross’ theorem applies to triangles, but some authors have developed Cross’ theorem on triangles using rectangles and Cross’ theorem on quadrilateral [4, 11]. Cross’ theorem and some of the proofs has been discussed that triangles are formed has the same area from initial triangle [1, 3], as explained in Theorem 2.1.

Theorem 2.1. Let $ABC$ denote any triangle, and construct squares on each side of the triangle outward which is $ABIH$, $BCGF$ and $ACDE$. If line $EH$, $IF$ and $DG$ constructed, then will form $\triangle AEH$, $\triangle BIF$ and $\triangle CDG$ which has area same to $\triangle ABC$, shown in Figure 2.

\begin{proof}
Suppose that $AB = c$, $BC = a$ and $AC = b$, using trigonometric formula to prove area triangle $ABC$, we have

$$L \triangle ABC = \frac{1}{2} ac \sin \angle ABC = \frac{1}{2} ab \sin \angle ACB = \frac{1}{2} bc \sin \angle BAC.$$ 

Consider $\triangle CDG$, since $\angle DCG = 180 - \angle ACB$, $CD = b$, $CG = a$, hence area $\triangle CDG$

$$L \triangle CDG = \frac{1}{2} ab \sin \angle DCG$$

$$L \triangle CDG = \frac{1}{2} ab \sin (180 - \angle ACB)$$

$$L \triangle CDG = L \triangle ABC$$

Similar way for $\triangle AEH$ and $\triangle BIF$, then

$$L \triangle CDG = L \triangle AEH = L \triangle BIF = L \triangle ABC$$

This completes the proof.

Furthermore Cross’ theorem using rectangles and not square on each side of the triangle it has been discussed in reference [4]. Because of many of rectangles that can be constructed, so conclude the side belongs to the rectangle must have the same proportion [4]. The result obtain is only new triangle has the same area, shown in Figure 3 and explained in Theorem 2.2.

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{figure3.png}
  \caption{Illustration Cross’ theorem using rectangles.}
  \label{fig:cross_theorem}
\end{figure}

\begin{proof}
Let $AB = c$, $BC = a$ and $AC = b$, suppose each rectangle have same $r$ proportion, from Theorem 2.1 its easy to show

$$L \triangle CDG = L \triangle AEH = L \triangle BIF = L \triangle ABC \times r^2$$

This completes the proof.

Furthermore, Cross’ theorem on a quadrilateral has been discussed in reference [11], the theorem on a quadrilateral shows that sum of two pairs triangle at the opposite angle will be the same as the initial quadrilateral area, as explained in Theorem 3.3.

Theorem 2.2. Let $ABC$ denote any triangle, and construct rectangles on each side of the triangle outward $ABIH$, $BCGF$ and $ACDE$ and has a width $\frac{1}{2}$ sides triangle. If line $EH$, $IF$ and $DG$ constructed, then will form $\triangle AEH$, $\triangle BIF$ and $\triangle CDG$ which has area same each other.

\begin{proof}
Let $AB = c$, $BC = a$ and $AC = b$, suppose each rectangle have same $r$ proportion, from Theorem 2.1 its easy to show

$$L \triangle CDG = L \triangle AEH = L \triangle BIF = L \triangle ABC \times r^2$$

This completes the proof.

Furthermore, Cross’ theorem on a quadrilateral has been discussed in reference [11], the theorem on a quadrilateral shows that sum of two pairs triangle at the opposite angle will be the same as the initial quadrilateral area, as explained in Theorem 3.3.

Theorem 3.3 Let $ABCD$ denote any quadrilateral, square construction on each side of the quadrilateral outward $ADGH$, $ABFE$, $BCLK$ and $CDJI$. If line $EH$, $FK$, $LI$ and $JG$ constructed, then will form $\triangle AEH$, $\triangle BFK$, $\triangle CLI$ and $\triangle JDG$ which has area $L \triangle AEH + L \triangle BIF + L \triangle CDG = L \triangle ABCD$, shown in Figure 4.

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{figure4.png}
  \caption{Illustration Cross’ theorem using rectangles.}
  \label{fig:cross_theorem}
\end{figure}
Many ideas of congruence concepts are discussed [5-7]. Then in this paper prove the modification Cross’ theorem with concepts understood by junior and senior high school students the concept of congruence. The proof pattern in this paper is widely used [8-10]. Based on the Cross’ theorem is constructing a square on the sides of a triangle so author is interested in doing modification by constructing a square on a triangle formed from the Cross’ theorem.

3. Modification Cross’ Theorem

Modification Cross’ theorem is based on triangles formed from the Cross’ theorem, then construction a square outward direction on each sides of the triangle, if the square vertex is connected it will form new shape, explained in Theorem 3.1.

Theorem 3.1 Let $AEH$, $BIF$ and $CDG$ are triangles formed from Cross’ theorem triangle $ABC$. Furthermore, $EHRQ$, $IFPO$ and $DGMN$ is constructed on each side triangle Cross’ theorem, if vertices a squares are connected then it will form a trapezoid and have area $5 \text{L} \triangle ABC$, shown in Figure 5.

Proof. Will be shown that $FGMP$, $HIOR$ and $DEQN$ are trapezoid and has an area $5 \text{L} \triangle ABC$. To show $FGMP$ trapezoid it will proved $FG//PM$. Suppose $A'$, $D'$, $I'$, $P'$ and $M'$ respectively is projection of point $A$ to $BC$, $D$ to $GC$, $I$ to $FG$ and $M$ to $FG$ so that several triangle are formed its $\triangleABA'$, $\triangleCDD'$, $\triangleBII'$, $\triangleFPP'$ and $\triangleGMM'$, as in Figure 6.

Consider $\triangle IBF$ on Figure 6, we have

\[
\angle IBF + \angle ABI + \angle ABC + \angle CBF = 360 \\
\angle IBF + 90 + \angle ABC + 90 = 360 \\
\angle IBF + \angle ABC + 180 = 360 \\
\angle IBF = 360 - 180 - \angle ABC \\
\angle IBF = 180 - \angle ABC
\]

Thus on $\triangle IBF$ we have

\[
\angle IBI' + \angle IBF = 180 \\
\angle IBI' + (180 - \angle ABC) = 180 \\
\angle IBI' = 180 - (180 - \angle ABC) \\
\angle IBI' = \angle ABC
\]

Hence, for $\triangle BAA'$ and $\triangle BII'$, since $\angle IBI' = \angle ABA'$, $\angle BA'A = \angle BII' = 90^\circ$ and $BA = BI$ then $\triangle BAA' \cong \triangle BII'$, so that

\[
\text{AA'} = \text{II'} \quad (1)
\]

For $\triangle CD'D$ and $\triangle CAA'$, with similar way obtained $\angle DCG = 180 - \angle ACA'$ and $\angle DCD' = \angle ACA'$, since $\angle DCD' = \angle ACA'$, $\angle CD'D = \angle CA'A = 90^\circ$ and $CD = CA$ then $\triangle CDD' \cong \triangle CAA'$, so that

\[
\text{DD'} = \text{AA'} \quad (2)
\]

Next, on $\triangle GMM'$ and $\triangle GDD'$, with similar way obtained $\angle FG'M = 180 - \angle CGD$ and $\angle MGM' = \angle CGD$, since $\angle MGM' = \angle CGD$, $\angle GM'M = \angle GD'D = 90^\circ$ and $GM = GD$ then $\triangle GMM' \cong \triangle GDD'$, so that

\[
\text{MM'} = \text{DD'} \quad (3)
\]

And then for $\triangle FPP'$ and $\triangle FII'$, with similar way obtained $\angle GFP = 180 - \angle IFI'$ and $\angle FPP' = \angle IFI'$, since $\angle GFP = \angle IFI'$, $\angle FPP' = \angle FII' = 90^\circ$ and $FP = FI$ then $\triangle FPP' \cong \triangle FII'$, so that

Figure 4. Illustration Cross’ theorem on quadrilateral.

Figure 5. Illustration modification Cross’ theorem.

Figure 6. Illustration projection point in modification Cross’ theorem.
Based on (1), (2), (3) and (4) obtained
\[ MM' = DD' = AA' = II' = PP' \] (5)

Because \( MM' = PP' \) then \( PM/PM' \), so that \( FG/PM \).
Moreover, to calculate area \( FGMP \), consider trapezoid \( P'M'MP \) on Figure 6, we have equation obtained
\[ PM = P'M' = FG + FP' + GM' \] (6)

Consider line \( FP' \), since \( FPP' = FI' = FB + BI' = FB + BA' \)
\[ FP' = FI' = FB + BI' = FB + BA' \] (7)

Same for line \( GM' \), since \( GMM' = GD' = GC + CD = GC + CA' \)
\[ GM' = GD' = GC + CD = GC + CA' \] (8)

Substitution (7) and (8) to (6), hence
\[ PM = 4a. \]

Next, suppose \( L, K \) and \( S \) each point that divides line \( PM \) to be 4 parts and each one have distance \( a \), shown in Figure 7.

This completes the proof.
Next, will discuss about the extension of line on the sides of expansion trapezoid. In general, if the sides of the triangle produced by the Cross’ theorem is extended so that it intersects, there is no relation between the new triangle formed with the initial triangle. Unique to this modification Cross’ theorem, if the outer edges of the trapezoid is extended will form a new triangle that is uniform with the initial triangle, as explained in Theorem 3.2.

Theorem 3.2 Let \( FGMP \), \( HIOR \) and \( DEQN \) is a trapezoid from modification Cross’ theorem on triangle \( ABC \), then suppose \( E_n, E_b \) and \( E_s \) are intersects each line \( RO \) to \( NQ \), \( RO \) to \( PM \) and \( NQ \) to \( PM \), hence will form \( \Delta E_nE_bE_s \) which is uniform to \( \Delta ABC \), shown in Figure 10.
Proof. To proof $\triangle E_aE_bE_c \approx \triangle ABC$ will show that each angle on triangle have same size. Therefore from proof Theorem 3.1 we have $FG//PM$ so that $BC//PM$ then on other side $AB//RO$ and $AC//NQ$. Because each line are parallel to each side then obviously the extension will also form a same angle which is $\angle E_aE_bE_c = \angle ABC$, $\angle E_bE_cE_a = \angle BCA$ and $\angle E_aE_cE_b = \angle CAB$. Thus $\triangle E_aE_bE_c \approx \triangle ABC$.

4. Conclusion

In this paper the authors only discuss the shape formed in modification Cross’ theorem on triangle and his area. In addition there is relationship between trapezoid and initial triangle at modification Cross’ theorem on triangle. Therefore, further discussing could be focused on sum of area modification Cross’ theorem on triangle include the square, and more expansion on the side of trapezoid.

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