Fast Algorithms for Hop-Constrained Flows and Moving Cuts*

Bernhard Haeupler
Carnegie Mellon University &
ETH Zürich
haeupler@cs.cmu.edu

D Ellis Hershkowitz
Carnegie Mellon University &
ETH Zürich
dhershko@cs.cmu.edu

Thatchaphol Saranurak
University of Michigan
thsa@umich.edu

Abstract

Hop-constrained flows and their duals, moving cuts, are two fundamental quantities in network optimization. Up to poly-logarithmic factors, they characterize how quickly a network can accomplish numerous distributed primitives. In this work, we give the first efficient algorithms for computing $(1 \pm \epsilon)$-optimal $h$-hop-constrained flows and moving cuts with high probability. Our algorithms take $\tilde{O}(m \cdot \text{poly}(h))$ sequential time, $\tilde{O}(\text{poly}(h))$ parallel time and $\tilde{O}(\text{poly}(h))$ distributed CONGEST time. We use these algorithms to efficiently compute hop-constrained cutmatches, an object at the heart of recent advances in expander decompositions.

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1 Introduction

Throughput and latency are two of the most fundamental quantities in a communication network. Given nodes $s$ and $t$, throughput measures the rate at which bits can be delivered from $s$ to $t$ while the worst-case latency measures the maximum time it takes for a bit sent by $s$ to arrive at $t$. If we imagine that our network is modeled as a weighted graph $G$ where edge weights give connection bandwidth then the minimum $s$-$t$ cut in $G$ exactly gives the maximum throughput attainable in sending from $s$ to $t$; likewise, its dual, the maximum flow, gives a collection of paths along which bits can be sent to achieve this throughput.

However, the paths given by the maximum flow provide no latency guarantees. Indeed the paths given by the maximum flow can consist of arbitrarily-many edges; if we imagine that each edge incurs some latency, then, the worst-case latency induced by the maximum flow is unboundedly bad. Thus, a natural question in network optimization is:

*How can we achieve high throughput while maintaining a low latency?*

The optimal way to achieve a high throughout subject to a latency constraint is exactly captured by hop-constrained flows and their duals, moving cuts. Given some hop constraint $h \geq 1$, a hop-constrained flow is defined in the same way as a classic max $s$ to $t$ flow but where the flow paths are restricted to have at most $h$ edges. The dual of a hop-constrained flow is called a moving cut. A moving cut is a minimum weight assignment of lengths to edges which guarantees that any $s$ to $t$ path consisting of at most $h$ edges has weight at least 1.

The centrality of the maximum hop-constrained flow (or equivalently, the minimum moving cut) in network optimization is evidenced by recent work. Haeupler et al. [9] showed that, up to poly-log factors, the maximum hop-constrained flow gives the minimum makespan of multiple unicasts in a distributed network, even when (network) coding is allowed. Similarly, the “shortcut quality” of a graph gives, up to poly-log factors, the optimal running time in the CONGEST model of distributed computation for numerous distributed optimization problems such as minimum spanning tree (MST), approximate min-cut and approximate shortest paths [3–8, 11]. The shortcut quality of a graph can be characterized as the largest moving cut in a graph, up to poly-logarithmic factors [10].

1.1 Our Contributions

In this work, we give the first efficient algorithms for computing these primitives in several well-studied models of computation. In particular, given a graph with $n$ nodes, $m$ edges and a hop constraint $h \geq 1$, we show how to compute maximum $h$-hop-constrained flows and minimum moving cuts that are $(1 \pm \varepsilon)$-optimal with high probability in $\tilde{O}(m \cdot \text{poly}(h))$ sequential time, $\tilde{O}(\text{poly}(h))$ parallel time and $\tilde{O}(\text{poly}(h))$ distributed CONGEST time. Building on these algorithms, we give algorithms with the same running times that compute hop-constrained cutmatches, a basic primitive at the heart of recent advances in expander decompositions.

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1We use “with high probability” to mean with probability at least $1 - \frac{1}{\text{poly}(n)}$ and we use $\tilde{O}$ notation to suppress dependence on $\text{poly}(\log n)$ factors.
2 Notation and Conventions

Before moving on to a formal statement of hop-constrained flows, moving cuts and our results we introduce some notation and conventions. Suppose we are given a graph $G = (V, E)$.

**Graph Notation** In general we will treat a path a series of consecutive edges in $E$. If $G$ is edge-weighted by $l : E \rightarrow \mathbb{R}_{>0}$ then we let $l(P) := \sum_{e \in P} l_e$ stand for the weight of a path where we use $l_e$ as shorthand for $l(e)$. We denote with $d_l(s, t)$ (or just $d(s, t)$ when $G$ is clear from context) the minimum weight of a path from $s$ to $t$. We use $\delta(v)$ to denote all edges incident to $v$.

**Hop-Constrained Notation** For $s, t \in V$, we let $P_h(s, t)$ be all paths between $s$ and $t$ which consist of at most $h$ edges. If $G$ is edge-weighted by $l$ then we let $d_l^{(h)}(s, t)$ or just $d^{(h)}(s, t)$ give $\min_{P \in P_h(s, t)} l(P)$. We will say that a path is $h$-hop if it consists of at most $h$ edges.

**CONGEST.** The classic CONGEST model of distributed communication is defined as follows [14]. In this model, the network is modeled as a graph $G = (V, E)$ with $n = |V|$ nodes and $m = |E|$ edges. Communication is conducted over discrete, synchronous rounds. During each round each node can send an $O(\log n)$-bit message along each of its incident edges.

3 Hop-Constrained Flows, Moving Cuts and Main Results

We proceed to more formally define a hop-constrained flow, moving cuts and our results. Suppose we are given a graph $G = (V, E)$.

A maximum $s$ to $t$ flow in $G$ in the classic sense can be defined as a collection of paths between $s$ and $t$ where each path receives some value and the total value incident to an edge does not exceed its capacity. This definition, in contrast to the standard max-flow LP, naturally extends to the hop-constrained setting where we define a hop-constrained flow as a collection of $s$ to $t$ paths consisting of at most $h$ edges where each such path $P$ receives some value $f_P$ and these values respect the capacities of edges. More precisely, we have the following LP with a variable $f_P$ for each path $P \in P_h(s, t)$. In this work we will assume all edges have unit capacity.

$$\max \sum_{P \in P_h(s, t)} f_P \quad \text{s.t.} \quad \sum_{P, e \in P} f_P \leq 1 \quad \forall e \in E$$

$$0 \leq f_P \leq 1 \quad \forall P \in P_h(s, t)$$

An $h$-hop-constrained flow, then, is simply a feasible solution to this LP.

**Definition 3.1 (h-Hop-Constrained Flow).** Given graph $G = (V, E)$ and vertices $s, t \in V$, an $h$-hop-constrained $s$ to $t$ flow is any feasible solution to Hop-Constrained Flow LP.

With the above definition of hop-constrained flows we can now define moving cuts as the dual of hop-constrained flows. In particular, taking the dual of the above LP we get the moving cut LP with a variable $l_e$ for each $e \in E$. 

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min \sum_{e \in E} l_e \quad \text{s.t.} \quad \sum_{e \in P} l_e \geq 1 \quad \forall P \in \mathcal{P}_h(s, t) \\
0 \leq l_e \leq 1 \quad \forall e \in E

(Moving Cut LP)

Similarly, an \(h\)-hop moving cut is simply a feasible solution to this LP.

**Definition 3.2 (h-Hop Moving Cut).** Given graph \(G = (V, E)\) and vertices \(s, t \in V\), an \(h\)-hop moving cut is any feasible solution to Moving Cut LP.

We will use \(f\) and \(l\) to stand for solutions to Hop-Constrained Flow LP and Moving Cut LP respectively. We say that \((f, l)\) is a feasible pair if both \(f\) and \(l\) are feasible for their respective LPs and that \((f, l)\) is \((1 \pm \epsilon)\)-optimal for \(\epsilon > 0\) if the moving cut certifies the value of the hop-constrained flow up to a \((1 - \epsilon)\); i.e. if \((1 - \epsilon) \sum_e l_e \leq \sum_P f_P\).

With the above notions, we can now state our main results which say that one can efficiently compute a feasible pair \((f, l)\) sequentially, in parallel and distributively. In the following we say \(f\) is integral if it assigns either value 1 or 0 to all paths in \(\mathcal{P}_h(s, t)\).

**Theorem 3.1.** One can with high probability compute a feasible hop-constrained flow, moving cut pair \((f, l)\) that is \((1 \pm \epsilon)\)-optimal where \(\delta = \Theta(\epsilon^2)\), 
\(k = \tilde{O} \left( \frac{h}{\epsilon^4} \right)\), 
\(f = \delta \cdot \sum_{j=1}^{k} f_j\) and \(f_j\) is an integral \(h\)-hop-constrained flow in:

1. Sequential time \(\tilde{O} (m \cdot \text{poly} (h, \frac{1}{\epsilon}))\);
2. Parallel time \(\tilde{O} (\text{poly} (h, \frac{1}{\epsilon}))\) with work \(\tilde{O} (m \cdot \text{poly} (h, \frac{1}{\epsilon}))\);
3. CONGEST time \(\tilde{O} (\text{poly} (h, \frac{1}{\epsilon}))\).

We note that \(k\) in the above result is optimal up to \(\text{poly}(\log n, \frac{1}{\epsilon})\) factors. Andrews et al. [1] showed that finding an integral solution to hop-constrained flow (i.e. a maximum collection of edge-disjoint \(s\) to \(t\) paths) is \(\Omega(h)\)-hard-to-approximate under standard complexity assumptions and any \(o(h)\) value for \(k\) would, by averaging, immediately give an an integral \(o(h)\)-approximation to the problem of finding a maximum collection of edge-disjoint \(s\) to \(t\) paths.

Lastly, the above result serves as the basis for our algorithms to compute hop-constrained cutmatches.

**4 Overview of Approach**

Before moving on to the details of our approach, we give an overview of our strategy for computing hop-constrained flows and moving cuts.

Computing a hop-constrained flow, moving cut pair is naturally suggestive of the following multiplicative-weights-type approach. We initialize \(l_e\) to some very small value. Then, we find a shortest path from \(s\) to \(t\) according to \(l\), send some small \((\approx \epsilon)\) amount of flow along this path and multiplicatively increase \(l\) of all edges in this path by \(\approx (1 + \epsilon)\). We repeat this until \(s\) and \(t\) are at least 1 apart according to \(d_1^{(h)}\).
The principle shortcoming of such an algorithm is that it is easy to construct examples where there are polynomially-many edge-disjoint $h$-hop paths between $s$ and $t$ and so we would clearly have to repeat the above process at least polynomially-many times until $s$ and $t$ are at least 1 apart according to $d_1^{(h)}$. This is not consistent with our stated time complexities.

To solve this issue, we use an algorithm similar to the above but instead of sending flow along a single path at a time, we send it along a large batch of paths at once. We would like to be able to send flow along these paths and make the above updates to lengths in parallel so we use an edge-disjoint collection of paths.

What can we hope to say about how long such an algorithm takes to make $s$ and $t$ at least 1 apart according to $d_1^{(h)}$? If each collection of paths that we send flow along was guaranteed to be incident to every $h$-hop path between $s$ and $t$ of length $d_1^{(h)}(s,t)$ then after each batch we would know that we increased $d_1^{(h)}$ by some non-zero amount. However, there is no way to lower bound this amount; in principle we might only increase $d_1^{(h)}$ by some tiny $\epsilon' > 0$. To solve this issue we find a collection of edge-disjoint paths which have length essentially $d_1^{(h)}(s,t)$ but which share an edge with every $h$-hop path with length at most $(1 + \epsilon) \cdot d_1^{(h)}(s,t)$. This allows us to lower bound the rate at which $d_1^{(h)}(s,t)$ increases, thereby allowing us to argue that this algorithm completes quickly.

Thus, in summary we repeatedly find an edge-disjoint collection of $h$-hop paths between $s$ and $t$ which have length about $d_1^{(h)}(s,t)$; these paths satisfy the property that every $h$-hop path from $s$ to $t$ with length at most $(1 + \epsilon) \cdot d_1^{(h)}(s,t)$ shares an edge with at least one of these paths; we call such a collection an $h$-hop $(1 + \epsilon)$-shortest path blocker. We then send a small amount of flow along these paths and multiplicatively increase the length of all incident edges, appreciably increasing $d_1^{(h)}(s,t)$. We repeat this until our lengths form a feasible moving cut.

To lower bound the quality of our solution we observe that, after scaling by $d_1^{(h)}(s,t)$ our lengths always induce a feasible dual solution. Since such a solution always costs more than the optimal dual solution, this then allows us to lower bound the amount of flow we send in terms of the optimal dual solution, ultimately showing that our total flow is within a multiplicative $\epsilon$ of the optimal dual solution.

### 5 $h$-Hop $(1 + \epsilon)$-Shortest Path Blockers

In this section we show how to efficiently accomplish our main subroutine, the computation of what we call $h$-hop $(1 + \epsilon)$-shortest near-shortest path blockers.

**Definition 5.1** ($h$-Hop $(1 + \epsilon)$-Shortest Path Blockers). Let $G = (V, E, l)$ be a weighted graph with $l_e > 0$ for all $e \in E$. Fix $\epsilon > 0$, $h \geq 1$ and $s, t \in V$. Let $\mathcal{P} \subseteq \mathcal{P}_h(s,t)$ be a collection of edge-disjoint $s$ to $t$ paths. $\mathcal{P}$ is an $h$-hop near-shortest path blockers blocker if:

1. **Near-Shortest**: $P \in \mathcal{P}$ has length at most $(1 + 2\epsilon) \cdot d^{(h)}(s,t)$;

2. **Near-Shortest Path Edge-Blocking**: If $P' \in \mathcal{P}_h(s,t)$ has length at most $(1 + \epsilon) \cdot d^{(h)}(s,t)$ then there is some $P \in \mathcal{P}$ such that $P \cap P' \neq \emptyset$.

The main theorem of this section we show is how to compute our shortest path blockers efficiently.
Theorem 5.1. One can with high probability compute an $h$-hop $(1 + \epsilon)$-shortest path blocker in

1. Sequential time $\tilde{O}(m \cdot \text{poly}(h))$;
2. Parallel time $\tilde{O}(\text{poly}(h))$ with work $\tilde{O}(m \cdot \text{poly}(h))$;
3. CONGEST time $\tilde{O}(\text{poly}(h))$.

The main idea for computing this collection of paths is to reduce finding them to finding a series of exact versions of them. In particular, by rounding edge weights we can, at a small penalty in the quality of our paths, reduce finding $h$-hop paths of length at most $(1 + 2\epsilon) \cdot d^h(s, t)$ which “hit” all $h$-hop paths of length at most $(1 + \epsilon) \cdot d^h(s, t)$ to simply finding $h$-hop paths of length $d^h(s, t)$ which hit all $h$-hop paths with length $d^h(s, t)$.

5.1 Computing $h$-Hop Shortest Path Blockers

More formally, the exact version of these objects we consider are what we call $h$-hop shortest path blockers and are defined as follows.

Definition 5.2 ($h$-Hop Shortest Path Blockers). Let $G = (V, E, I)$ be a weighted graph, let $s, t \in V$ and let $\mathcal{P} \subseteq \mathcal{P}_h(s, t)$ be a collection of $h$-hop shortest $s$ to $t$ paths. Then we say that $\mathcal{P}$ is an $h$-hop shortest path blocker if:

1. Shortest: $P \in \mathcal{P}$ has length (exactly) $d^h(s, t)$;
2. Shortest Path Edge-Blocking: If $P'$ is a path with at most $h$ edges with length (exactly) $d^h(s, t)$ then there is some $P \in \mathcal{P}$ such that $P \cap P' \neq \emptyset$.

We show how to efficiently compute these objects with the following theorem. The basic idea is to simulate an algorithm similar to Luby’s algorithm for maximum independent set (MIS) [13] but where we are computing an MIS in the graph where each vertex corresponds to an $h$-hop shortest path between $s$ and $t$ and an edge in this graph is between two vertices if the corresponding $h$-hop paths share an edge. This graph has up to $\Theta(n^h)$ vertices so we cannot afford to explicitly write down the graph while being consistent with the desired runtimes. Rather, we only “simulate” the algorithm of Luby. This general idea is an adaptation of an algorithm of Lotker et al. [12] for the hop-unconstrained setting to the hop-constrained setting.

Theorem 5.3. Given a weighted graph with edge weights greater than 0 and $s, t \in V$, there is an algorithm that computes an $h$-hop shortest path blocker with high probability:

1. Sequentially in time $\tilde{O}(m \cdot \text{poly}(h))$;
2. In parallel time $\tilde{O}(\text{poly}(h))$ with work $\tilde{O}(m \cdot \text{poly}(h))$;
3. In CONGEST in $\tilde{O}(\text{poly}(h))$ rounds.

Proof. We initialize our solution $\mathcal{P}$ to $\emptyset$. We will work in phase $i = 1, 2, \ldots h$ where we guarantee that at the end of the $i$th phase every path consisting of at most $i$ edges between $s$ and $t$ which has length at most $d^h(s, t)$ intersects the edges of at least one path in $\mathcal{P}$ and all paths in $\mathcal{P}$ are edge-disjoint. It follows that at the end of the $h$th phase our solution is an $h$-hop shortest path blocker with high probability.
We proceed to describe the $i$th phase. First, some notation. Let $\mathcal{P}^*(s, t, i)$ be all paths with length $d^{(i)}(s, t)$ consisting of exactly $i$ edges between $s$ and $t$. For $e = \{v, u\}$, we let $n_{vu}(t, j)$ be the number of paths from $v$ to $t$ which use $e$, have length $d^{(j)}(v, t)$ and exactly $j \geq 1$ edges. Similarly, we let $n_v(t, j)$ be the number of paths from $v$ to $t$ with length $d^{(j)}(v, t)$ which use exactly $j \geq 0$ edges. Notice that we can equivalently define these quantities through the mutual recursions:

$$n_{vu}(t, j) := n_u(t, j - 1) \cdot \mathbb{I}[d^{(j)}(v, t) = d^{(j-1)}(u, t) + l_e]$$

and

$$n_v(t, j) := \begin{cases} \mathbb{I}[v = t] & \text{if } j = 0 \\ \sum_{e \in \delta(v)} n_{vu}(t, j) & \text{otherwise.} \end{cases}$$

We say that a path in $\mathcal{P}^*(s, t, i)$ is covered if some path in $\mathcal{P}$ shares an edge with it and we let $\mathcal{U}$ be all uncovered paths in $\mathcal{P}^*(s, t, i)$. For a path $P \in \mathcal{U}$, we let $N(P)$ be all paths in $\mathcal{U}$ which share an edge with $P$. Notice that $|N(P)| \leq n^i$.

We guess $\max_{P \in \mathcal{U}} |N(P)|$ as $\tilde{\Delta} = n^i, n^i/2, n^i/4, \ldots, 1$. For each of these guesses we repeat the following process $\Theta(\log n)$ times. We imagine that $s$ begins with some number of balls which it then tosses to neighbors who then toss it to their neighbors and so on. The paths traced out by these balls will give us a collection of paths $\mathcal{S}$. We let $\mathcal{S}'$ be all paths $P \in \mathcal{S}$ which are edge-disjoint from all paths in $\mathcal{S} \setminus \{P\}$. We then add $\mathcal{S}'$ to $\mathcal{P}$ and delete all edges in $\mathcal{S}'$ from our graph at which point we recompute $n_{vu}(t, j)$ and $n_v(t, j)$ for every edge $j \leq i$ and $v, u \in V$.

To compute $\mathcal{S}$ we do the following. If $d^{(i)}(s, t) \neq d^{(h)}(s, t)$ then we set $\mathcal{S}$ to be $\emptyset$. Otherwise, we do the following. For each edge $e = (s, v)$ we have $s$ initially toss a random $B\left(n_{vu}(s, t), \frac{1}{2\tilde{\Delta}}\right)$-many balls to $v$ where $B(n, p)$ is a binomial with $n$ trials each with probability of success $p$. Then, for $j = i - 1, i - 2, \ldots, 0$ rounds each vertex $v$ throws each of its balls independently to a neighbor $u$ along edge $e = \{v, u\}$ with probability $n_{vu}(t, j)/n_v(t, j)$. It is easy to see by our assumption on non-zero edge weights that a given path in $\mathcal{U}$ is added to $\mathcal{S}$ by this process independently with probability $\frac{1}{2\Delta}$.

We now argue that every path in $\mathcal{P}^*(s, t, i)$ is covered with high probability by the end of the $i$th phase. To do so, it suffices to argue that just before we halve $\tilde{\Delta}$ it holds that $|N(P)| \leq \tilde{\Delta}/2$ for every $P \in \mathcal{U}$ with high probability. By induction, to show this it suffices to argue that for $P \in \mathcal{U}$ where $|N(P)| \geq \frac{\tilde{\Delta}}{2}$ and $\max_{P \in \mathcal{U}} |N(P')| \leq \tilde{\Delta}$ then each repetition of the above process covers $P$ with constant probability; this is because we repeat the above process $\Theta(\log n)$ times for each guess $\tilde{\Delta}$ and so after $\Theta(\log n)$ repetitions such a $P$ will be covered with high probability or have $|N(P)|$ reduced below $\frac{\tilde{\Delta}}{2}$; by a union bound this will hold for all such $P$ meaning we reduced $|N(P)|$ by half for every $P \in \mathcal{U}$.

Thus, consider such a $P$. We claim that each repetition of the above process covers $P$ with probability at least $\Omega(1)$. To see this, notice that $P$ is certainly covered if there is a $P' \in N(P)$ which is in $\mathcal{S}$ and no distinct $P''$ is in both $\mathcal{S}$ and $N(P')$. Since we know $|N(P)| \geq \frac{\tilde{\Delta}}{2}$ by assumption, we have that the probability that some such $P'$ is sampled is at least

$$\Pr(\exists P' \in \mathcal{S} \cap N(P)) \geq 1 - \left(1 - \frac{1}{2\tilde{\Delta}}\right)^{\frac{\tilde{\Delta}}{2}} \geq 1 - \exp\left(-\frac{1}{4}\right)$$
≥ .1.

On the other hand, conditioning on this event we have that for a fixed $P'$ the probability that no such $P''$ is sampled by a union bound is at least

$$\Pr(\nexists P'' \in S \cap N(P') \setminus \{P'\} \mid P' \in S \cap N(P)) \geq 1 - \tilde{\Delta} \cdot \frac{1}{2\Delta} \geq .1.$$ 

Thus, combining these bounds we have that the probability that $P$ is covered in one of the above repetitions is at least $\Omega(1)$ as desired. Concluding with a union bound it follows that $\tilde{\Delta}$ is always an upper bound on $\max_{P \in U} |N(P)|$ with high probability and so by the end of the $i$th phase we have covered all paths in $\mathcal{P}^*(s, t, i)$.

To see the runtime of our algorithm it suffices to argue that we can implement each repetition in each phase in the stated complexities since we have $O(h \log n)$ repetitions. We can first compute $d^{(j)}(v, t)$ for every $v \in V$ and $j \in [h]$ by, e.g. $h$ rounds of Bellman-Ford in the stated complexities. This, in turn, allows us to compute $n_{vu}(t, j)$ and $n_v(t, j)$ using the above mutual recursions in the stated complexities. Lastly, passing the balls, computing $S'$ from $S$ and deleting $S'$ from our graph is trivial to do in the stated complexities.

5.2 Computing $h$-Hop $(1 + \epsilon)$-Shortest Path Blockers

Having shown how to compute an exact version of $h$-hop shortest path blockers in the preceding section, we now use this to construct our $h$-hop $(1 + \epsilon)$-shortest path blockers by appropriately rounding edge weights and then taking a series of $h$-hop shortest path blockers. Our algorithm for doing so on weighted graph $G = (V, E, l)$ is given below.

1. $\mathcal{P} \leftarrow \emptyset$

2. Let $j_e$ be the minimum $j$ such that $l_e \geq j \cdot \frac{\epsilon}{h} \cdot d_1^{(h)}(s, t)$ and let $\tilde{G}$ be the weighted graph where edge $e$ is replaced by an edge of weight $\tilde{l}_e := j_e \cdot \frac{\epsilon}{h} \cdot d_1^{(h)}(s, t)$

3. For $i = 0, 1, 2, \ldots, 2h$

   (a) $\mathcal{P}_i = \emptyset$

   (b) If $d_i^{(h)}(s, t) = d_1^{(h)}(s, t) \cdot (1 + i \cdot \frac{\epsilon}{h})$ then let $\mathcal{P}_i$ be an $h$-hop shortest path blocker in $\tilde{G}$ (compute using Theorem 5.3) and delete from $\tilde{G}$ all edges in $\mathcal{P}_i$.

4. Return $\mathcal{P} := \bigcup_i \mathcal{P}_i$.

We conclude with our proof that we can efficiently compute our $h$-hop $(1 + \epsilon)$-shortest path blockers.

**Theorem 5.1.** One can with high probability compute an $h$-hop $(1 + \epsilon)$-shortest path blocker in

1. Sequential time $\tilde{O}(m \cdot \text{poly}(h))$;

2. Parallel time $\tilde{O} \left( \text{poly}(h) \right)$ with work $\tilde{O}(m \cdot \text{poly}(h))$. 

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3. **CONGEST** time $\tilde{O}(\text{poly}(h))$.

Proof. We use the above algorithm.

We first verify that the result of this algorithm is indeed an $h$-hop $(1+\epsilon)$-shortest shortest path blocker (Definition 5.1). Each of the paths we return is between $s$ and $t$ and consists of at most $h$ edges since each $P_i$ is an $h$-hop shortest path blocker in $\tilde{G}$. Moreover, the paths we return are edge-disjoint since each $P_i$ is an $h$-hop shortest path blocker and we delete all edges in $P_i$ from $\tilde{G}$ before computing $\tilde{P}_j$ for $j > i$. Thus, it remains to argue that each $P \in \mathcal{P}$ has length at most $(1+2\epsilon) \cdot d^{(h)}(s,t)$ and that any $P' \in \mathcal{P}_h(s,t)$ with length at most $(1+\epsilon) \cdot d^{(h)}_1(s,t)$ in $G$ intersects at least one edge of a path in $\mathcal{P}$.

Consider an edge $e$ with weight $l_e$ in $G$ and weight $\tilde{l}_e$ in $\tilde{G}$. Observe that since we are rounding edge weights up we have $l_e \leq \tilde{l}_e$. Combining this with the fact that we are rounding to multiples of $\frac{1}{h} \cdot d^{(h)}_1(s,t)$ we have that

$$l_e \leq \tilde{l}_e \leq l_e + \frac{\epsilon}{h} \cdot d^{(h)}_1(s,t)$$

We now argue that each path we return has length at most $(1+2\epsilon) \cdot d^{(h)}(s,t)$ in $G$. By construction any path $P$ that we return has length at most $d^{(h)}_1(s,t) + h \cdot \tilde{\epsilon} \cdot d^{(h)}_1(s,t) = (1+2\epsilon) \cdot d^{(h)}_1(s,t)$ in $\tilde{G}$. Moreover, since lengths in $G$ are only less than those in $\tilde{G}$ in $G$ this path has length at most $(1+2\epsilon) \cdot d^{(h)}_1(s,t)$ in $G$ as desired.

We continue by arguing that if $P'$ has length at most $(1+\epsilon) \cdot d^{(h)}_1(s,t)$ and at most $h$ edges then there is some path in $\mathcal{P}$ which shares an edge with $P'$. Notice that since $P'$ consists of at most $h$ edges then by Equation (1) its weight in $\tilde{G}$ is at most $(1+\epsilon) \cdot d^{(h)}_1(s,t) + h \cdot \tilde{\epsilon} \cdot d^{(h)}_1(s,t) = (1+2\epsilon) \cdot d^{(h)}_1(s,t)$. Thus, there must be some $i_{P''} \in [2h]$ for which $P''$ has length $d^{(h)}_1(s,t) \cdot (1+i_{P''} \cdot \frac{\epsilon}{h})$ in $\tilde{G}$. Additionally, by definition of an $h$-hop shortest path blocker and our rounding, when our algorithm considers $i$ we know that $d^{(h)}_i(s,t) \geq d^{(h)}_1(s,t) \cdot (1+i \cdot \frac{\epsilon}{h})$. So, when $i_{P''}$ is considered for $i$ then either $d^{(h)}_i(s,t) > d^{(h)}_1(s,t) \cdot (1+i_{P''} \cdot \frac{\epsilon}{h})$ in which case we must have already removed an edge of $P'$ from $\tilde{G}$ (and so $\mathcal{P}$ already includes an edge of $P'$) or by definition of an $h$-hop shortest path blocker we include at least one edge of $P'$ in $\mathcal{P}_i$.

Next, we consider the computational complexity of implementing the above algorithm. $j_e$ can easily be computed in the stated complexities since $d^{(h)}_i(s,t)$ and $d^{(h)}_1(s,t)$ can be computed by, e.g. running $h$ rounds of Bellman-Ford. Lastly, we must compute at most $O(h)$ $h$-hop shortest path blockers which by Theorem 5.3 can be done in the stated complexities.

\[ \square \]

### 6 Computing Hop-Constrained Flows and Moving Cuts

Having shown how to compute an $h$-hop $(1+\epsilon)$-shortest path blocker, we now use a series of these as batches to which we apply multiplicative-weights-type updates. The result is our algorithm which returns both a hop-constrained flow and a (near) certifying moving cut.

Formally, our algorithm is as follows.

1. Let $\epsilon_0 = \frac{\epsilon}{2-\epsilon}$ and let $\delta = \frac{\epsilon_0^2}{(1+\epsilon_0) \ln m}$

2. $l_e \leftarrow \left( \frac{1}{m} \right)^{1/\epsilon_0}$ for all $e \in E$
3. \( f_P \leftarrow 0 \) for all \( P \in \mathcal{P}_h(s,t) \)

4. While \( d_1^{(h)}(s,t) < 1 \):
   
   (a) Compute an \( h \)-hop \((1 + \epsilon_0)\)-shortest path blocker \( \mathcal{P} \) (using Theorem 5.1)
   
   (b) For \( P \in \mathcal{P} \):
      
      i. **Hop-Constrained Flow (Primal Update)**: \( f_P \leftarrow f_P + \delta \)
      
      ii. **Moving Cut (Dual Update)**: \( l_e \leftarrow (1 + \epsilon_0) \cdot l_e \) for every \( e \in P \)

5. Return \((f, l)\)

We first prove the feasibility of our solution.

**Lemma 6.1.** The pair \((f, l)\) returned by the above algorithm are feasible for **Hop-Constrained Flow LP** and **Moving Cut LP** respectively.

**Proof.** First, observe that by definition of a \((1 + \epsilon_0)\)-shortest path blocker, each iteration of the while loop increases \( d_1^{(h)}(s,t) \) and so our algorithm terminates. It follows that \( l \) is feasible by construction since our algorithm only returns \( l \) once \( d_1(s,t) \geq 1 \).

To see that \( f \) is feasible it suffices to argue that for each edge \( e \) the number of times a path containing \( e \) has its primal value increased is at most \( \frac{1}{\delta} \). Notice that each time we increase the primal value on a path containing edge \( e \) by \( \delta \) we increase the dual value of this edge by a multiplicative \((1 + \epsilon_0)\); this is because the paths in our \( h \)-hop \((1 + \epsilon_0)\)-shortest path blockers are disjoint. Since the length of our edges according to \( l \) start at \((\frac{1}{m})^{1/\epsilon_0}\), it follows that if we increase the primal value of \( k \) paths incident to edge \( e \) then \( l_e = (1 + \epsilon_0)^k \cdot (\frac{1}{m})^{1/\epsilon_0} \). On the other hand, by assumption when we increase the dual value of an edge \( e \) it must be the case that \( l_e < 1 \) since otherwise \( d_1(s,t) \geq 1 \), contradicting the condition of our while loop. It follows that \((1 + \epsilon_0)^k \cdot (\frac{1}{m})^{1/\epsilon_0} \leq 1 \) and so applying the fact that \( \ln(1 + \epsilon_0) \geq \frac{\epsilon_0}{1 + \epsilon_0} \) for \( \epsilon_0 > -1 \) we get

\[
  k \leq \frac{1}{\epsilon_0 \cdot \ln(1 + \epsilon_0)} \log m \\
  \leq \frac{1 + \epsilon_0}{\epsilon_0^2} \ln m \\
  = \frac{1}{\delta}
\]

as desired. \( \Box \)

We next prove the near-optimality of our solution.

**Lemma 6.2.** The pair \((l, f)\) returned by the above algorithm satisfies \((1 - \epsilon) \sum_e l_e \leq \sum_P f_P \)

**Proof.** Let \( k_i \) be the number of paths in \( \mathcal{P} \) in the \( i \)th iteration of our while loop, let \( \alpha_i \) be \( d_1^{(h)}(s,t) \) at the start of this iteration and let \( D_i := \sum_e l_e \) be our total dual value at the start of this iteration. Notice that \( \frac{1}{\alpha_i} \cdot 1 \) is dual feasible and has cost \( \frac{D_i}{\alpha_i} \). If \( \beta \) is the optimal dual value then by optimality it follows that \( \beta \leq \frac{D_i}{\alpha_i} \), giving us the upper bound on \( \alpha_i \) of \( \frac{D_i}{\beta} \). Since in the \( i \)th iteration we increase
our dual value by $k_i \alpha_i \epsilon_0$ we get the following recurrence to which we apply our bound on $\alpha_i$ as follows:

$$D_{i+1} \leq D_i + k_i \alpha_i \epsilon_0$$

$$\leq D_i \left(1 + \frac{k_i \epsilon_0}{\beta}\right)$$

$$\leq D_i \cdot \exp\left(\frac{k_i \epsilon_0}{\beta}\right)$$

Let $T - 1$ be the index of the last iteration of our algorithm; notice that $D_T$ is the value of $l$ in our returned solution. Let $K := \sum_i k_i$. Then, repeatedly applying this recurrence gives us

$$D_T \leq D_0 \cdot \exp\left(\frac{K \epsilon_0}{\beta}\right)$$

$$= \left(\frac{1}{m}\right)^{1/\epsilon_0 - 1} \exp\left(\frac{K \epsilon_0}{\beta}\right)$$

On the other hand, we know that since $l$ is dual feasible when we return it, so it must be the case that $D_T \geq 1$. Combining this with the above upper bound on $D_T$ gives us $1 \leq \left(\frac{1}{m}\right)^{1/\epsilon_0 - 1} \exp\left(\frac{K \epsilon_0}{\beta}\right)$. Solving for $K$ gives us

$$\left(\frac{\beta}{\epsilon_0}\right) \left(\frac{1}{\epsilon_0} - 1\right) \ln m \leq K.$$ 

However, notice that $K \delta$ is the primal value of our solution so rewriting this inequality in terms of $K \delta$ gives us

$$\beta \frac{1 - \epsilon_0}{1 + \epsilon_0} \leq K \delta.$$ 

Moreover, by our choice of $\epsilon_0$ we have that $\frac{1 - \epsilon_0}{1 + \epsilon_0} = 1 - \epsilon$ and so we conclude

$$\beta \cdot (1 - \epsilon) \leq K \delta$$

as desired. \qed

We conclude with our main theorem by proving that we need only run the while loop of our algorithm $\tilde{O}\left(\frac{1}{\epsilon^3}\right)$ times.

**Theorem 3.1.** One can with high probability compute a feasible hop-constrained flow, moving cut pair $(f, l)$ that is $(1 \pm \epsilon)$-optimal where $\delta = \Theta(\epsilon^2)$, $k = \tilde{O}\left(\frac{h}{\epsilon^2}\right)$, $f = \delta \cdot \sum_{j=1}^{k} f_j$ and $f_j$ is an integral $h$-hop-constrained flow in:

1. **Sequential time** $\tilde{O}\left(m \cdot \text{poly}\left(h, \frac{1}{\epsilon}\right)\right)$;
2. **Parallel time** $\tilde{O}\left(\text{poly}\left(h, \frac{1}{\epsilon}\right)\right)$ with work $\tilde{O}\left(m \cdot \text{poly}\left(h, \frac{1}{\epsilon}\right)\right)$;
3. **CONGEST time** $\tilde{O}\left(\text{poly}\left(h, \frac{1}{\epsilon}\right)\right)$. 

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Proof. By Lemma 6.1 and Lemma 6.2 we know that our solution is feasible and \((1 + \epsilon)\)-optimal so it only remains to argue the runtime of our algorithm and that the returned flow decomposes in the stated way.

We first argue that the while loop of the above algorithm must run at most \(O\left(\frac{h \log^2 n}{\epsilon^2}\right)\) times. Consider the \(i\)th iteration, let \(\alpha_i = d_1^{(h)}(s, t)\) at the beginning of this iteration and let \(\alpha'_i\) be the value of \(d_1^{(h)}(s, t)\) after an additional \(\Theta\left(\frac{h \log^2 n_{i+\epsilon_0}}{\epsilon_0}\right) = \Theta\left(\frac{h \log n}{\epsilon_0}\right)\) iterations of the while loop. Also, let \(P_j\) be our shortest path blocker in the \(j\)th iteration.

We claim that \(\alpha'_i \geq (1 + \epsilon_0) \cdot \alpha_i\). Assume for the sake of contradiction that \(\alpha'_i < (1 + \epsilon_0) \cdot \alpha_i\). It follows that there is some path \(P \in P_h(s, t)\) with length at most \(\alpha_i \cdot (1 + \epsilon_0)\) after \(i + \Theta\left(\frac{h \log^2 n_{i+\epsilon_0}}{\epsilon_0}\right)\) many iterations. However, notice that by definition of an \(h\)-hop \((1 + \epsilon_0)\)-shortest path blocker (Definition 5.1), we know that \(P_j\) contains an edge of \(P\) for every \(j \in [i, i + \Theta\left(\frac{h \log^2 n_{i+\epsilon_0}}{\epsilon_0}\right)]\). By averaging, it follows that there is some edge \(e \in P\) which is in at least \(\Theta\left(\frac{h \log^2 n_{i+\epsilon_0}}{\epsilon_0}\right)\) of these \(P_j\) for \(j \in [i, i + \Theta\left(\frac{h \log^2 n_{i+\epsilon_0}}{\epsilon_0}\right)]\). Since every edge starts with dual value \(\left(\frac{1}{m}\right)^{1/\epsilon_0}\) and multiplicatively increases by \((1 + \epsilon_0)\) each time it is in a \(P_j\), such an edge after \(i + \Theta\left(\frac{h \log^2 n_{i+\epsilon_0}}{\epsilon_0}\right)\) many iterations must have \(l_e\) value at least \(\left(\frac{1}{m}\right)^{1/\epsilon_0} \cdot (1 + \epsilon_0)\) \(\Theta\left(\frac{h \log^2 n_{i+\epsilon_0}}{\epsilon_0}\right) \geq n^{1/\epsilon_0}\). However, by assumption the length of \(P\) is at most \((1 + \epsilon_0) \cdot \alpha_i\) after \(i + \Theta\left(\frac{h \log^2 n_{i+\epsilon_0}}{\epsilon_0}\right)\) iterations and this is at most 2 since \(\alpha_i < 1\) since otherwise our algorithm would have halted. But \(2 < n^{1/\epsilon_0}\) and so we have arrived at a contradiction.

Since \(\alpha_i\) increases by a multiplicative \((1 + \epsilon_0)\) after every \(\Theta\left(\frac{h \log n}{\epsilon_0}\right)\) iterations and starts at least \(\left(\frac{1}{m}\right)^{1/\epsilon_0}\), it follows that after \(y \cdot \Theta\left(\frac{h \log n}{\epsilon_0}\right)\) iterations the \(h\)-hop distance between \(s\) and \(t\) is at least \((1 + \epsilon_0)^y \cdot \left(\frac{1}{m}\right)^{1/\epsilon_0}y\). Thus, for \(y \geq \Omega\left(\frac{\ln n_{i+\epsilon_0}}{\epsilon_0}\right)\) we have that \(s\) and \(t\) are at least 1 apart in \(h\)-hop distance. Consequently, our algorithm must run for at most \(O\left(\frac{h \log^2 n}{\epsilon^2}\right) = O\left(\frac{h \log^2 n}{\epsilon^2}\right)\) many iterations of the while loop.

The running time of our algorithm is immediate from the fact that \(d_1^{(h)}(s, t)\) can be computed in the stated complexities (by e.g. \(h\) rounds of Bellman-Ford), the bound of \(O\left(\frac{h \log^2 n}{\epsilon^2}\right)\) on the number of iterations of the while loop and the running times given in Theorem 5.1 for computing our \(h\)-hop \((1 + \epsilon_0)\)-shortest path blocker.

Lastly, the flow decomposes in the stated way because we have at most \(O\left(\frac{h \log^2 n}{\epsilon^2}\right)\) iterations and each \(P_j\) consists of edge-disjoint \(h\)-hop paths from \(s\) to \(t\). Thus, each \(P_j\) corresponds to an integral hop-constrained flow \(f_j\) where our final solution is \(\delta \cdot \sum_{j=1}^{k} f_j\) and \(k = \tilde{O}\left(\frac{1}{\epsilon}\right)\). \(\square\)

7 Hop-Constrained Cutmatches

As it captures low-latency communication subject to bandwidth constraints, the problem of computing low-congestion \(h\)-hop paths between two set of nodes \(S\) and \(T\) occurs often in network optimization. We say that such a collection of paths is a matching if each node in \(S\) and \(T\) occurs as the endpoint of at most one path and a perfect matching if each node in \(S\) and \(T\) occurs as the
endpoint of exactly one path.

In this section we give algorithms that either find a perfect low-congestion matching between two sets of nodes or, if this is not possible, finds as large of a matching as possible together with a moving cut that (approximately) certifies that there is no low-congestion way of extending the current matching. This type of strong matching primitive is used, for example, in recent expander decomposition algorithms; see, e.g., [2].

While the preceding parts of this paper have used a moving cut between pairs of vertices, moving cuts naturally extend to pairs of sets of nodes. In particular, given a pair of sets of vertices \(S\) and \(T\), let \(P(S,T)\) be all \(h\)-hop paths between vertices in \(S\) and \(T\). Then a moving cut between \(S\) and \(T\) is any feasible solution to the LP which is identical to Moving Cut LP but which replaces \(P(s,t)\) with \(P(S,T)\). A hop-constrained flow between \(S\) and \(T\), likewise, is the dual of this LP. The following definition summarizes the object we compute in this section. It is easy to see by, e.g., adding a node connected to all of \(S\) and another node connected to all of \(T\), that our algorithms from the preceding section can be used to compute moving cuts between pairs of sets of vertices.

**Definition 7.1** (h-Hop Cutmatch). Given graph \(G = (V,E)\), an \(h\)-hop \(φ\)-sparse cutmatch of congestion \(γ\) between two node sets \(S,T \subseteq V\) with \(|S| \leq |T|\) consists of:

- A partition of \(S\) into \(S_M\), \(S_U \subseteq S\);
- A partition of \(T\) into \(T_M\), \(T_U \subseteq T\) with \(|T_M| = |S_M|\);
- A set of \(h\)-hop paths \(P\) matching \(S_M\) and \(T_M\) using any edge in \(G\) at most \(γ\) often;
- A moving cut \(l\) of \(S_U\) and \(T_U\) of value \(\sum_e |l_e| \leq φ|S_U|\).

We proceed to show how to efficiently compute a cutmatch using our previous algorithm.

**Theorem 7.2.** There is an algorithm that, given two node sets \(S,T \subseteq V\) with \(|S| \leq |T|\) and two integer parameters \(h \geq 1\) and \(φ \geq 1\) outputs an \(h\)-hop \(φ\)-sparse cutmatch of congestion \(γ\) between \(S\) and \(T\), where \(γ = \Theta\left(\frac{1}{h}\right)\) with high probability. This algorithm runs in

1. Sequential time \(\tilde{O}(γm \cdot \text{poly}(h))\);
2. Parallel time \(\tilde{O}(γ \cdot \text{poly}(h))\) with work \(\tilde{O}(γm \cdot \text{poly}(h))\);
3. CONGEST time \(\tilde{O}(γ \cdot \text{poly}(h))\).

**Proof.** Start with \(S_0 = S\), \(T_0 = T\), \(P_0 = \emptyset\), and \(C = \emptyset\). The algorithm runs for at most \(O(γ)\) iterations for a small hidden constant. In each iteration \(i \in [1, O(γ)]\) we use Theorem 3.1 with \(ε = 2\) (any constant would suffice) to find a hop-constrained flow, moving cut pair, \((f, l)\) where \(\delta = \Theta(1)\), \(k = \tilde{O}(h)\), \(f = \delta \cdot \sum_{j=1}^{k} f_j\) and \(f_j\) is an integral \(h\)-hop-constrained flow from \(S\) to \(T\).

Consider the \(h\)-hop paths induced by all \(f_j\). It is easy to see that we may assume that these paths are indeed a matching at the cost of only a multiplicative constant in the total number of paths. In particular, a priori nothing guarantees that there are not vertices in \(S\) and \(T\) which are the endpoints for multiple such paths. To fix this, we do the following. For each \(v \in S\) and \(u \in T\) we add a node \(v'\) connected only to \(v\) and a node \(u'\) connected only to \(u\). Then, we let \(S'\) be all such \(v'\) and \(T'\) be all such \(u'\). Next, we apply the above algorithm to sets \(S'\) and \(T'\) (increasing the hop-constraint to \(h+2\)). By our \(δ\) value, each node in \(S'\) and \(T'\) are endpoints for a constant number
of the computed paths. If each node, therefore, samples one of its paths uniformly at random then a path has a constant probability of being sampled by both of its endpoints. By standard union bound arguments, repeating this $\Theta(\log n)$ times guarantees with high probability that we find one such collection of paths whose cardinality is within a constant of that of all paths induced by $f_i$. Thus, we let $P_i'$ be the disjoint $h$-hop paths induced by all $f_i$ which have both of their endpoints sampled in this manner.

We let $S_{i-1}$ and $T_{i-1}$ be the nodes in $G$ matched by this set $P_i'$. Let $S_i'$ be the nodes in $S_{i-1}$ matched by $P_i'$ and let $T_i'$ be the nodes in $T_{i-1}$ matched by $P_i'$. If $|P_i'| > \Omega(\frac{\log n \cdot |S_{i-1}|}{\gamma})$ for a sufficiently large hidden constant we set $P_i = P_{i-1} \cup P_i'$, $S_i = S_{i-1} \cap S_i'$, $T_i = T_{i-1} \cap T_i'$ and we continue with iteration $i + 1$.

In each iteration $i$ in which $|P_i'| > \Omega(\frac{\log n \cdot |S_{i-1}|}{\gamma})$, we have that $S_i$ is of size less than $(1 - \frac{2\log n}{\gamma})|S_{i-1}|$. Since $\left(1 - \frac{2\log n}{\gamma}\right)^\gamma < \frac{1}{|S|}$ such a shrinkage of the $S$ set cannot happen for more than $\gamma$ iterations. Thus, we eventually have an iteration $i \leq \gamma$ where $|P_i'| \leq \frac{2\log n \cdot |S_{i-1}|}{\gamma}$. In this case, we take as our moving cut for the cutmatch the $1$ computed in this iteration and we let $P = P_{i-1}$, $S_U = S_{i-1}$, $T_U = T_{i-1}$, $S_M = S \setminus S_U$, $T_M = T \setminus T_U$, and then terminate.

The running time is exactly that of running at most $\gamma$ invocations of Theorem 3.1 and the above sampling of paths procedure to guarantee a matching. By construction the set $P$ consists of $h$-hop paths which match $S_M$ with $T_M$. Since the paths added in each iteration have $O(1)$ edge-congestion and since we run for at most $O(\gamma)$ iterations for a small hidden constant the congestion of the paths in $P$ is at most $\gamma$. The $\phi$-sparsity of our cutmatch follows from the fact that in our last iteration we have $|P_i'| \leq \frac{2\log n \cdot |S_{i-1}|}{\gamma}$ and so the moving cut we compute must have value $\tilde{O}(\frac{|S_U|}{\gamma})$. □

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