GLOBAL WELL-POSEDNESS AND EXISTENCE OF THE
GLOBAL ATTRACTOR FOR THE KADOMTSEV-PETVIASHVILI
II EQUATION IN THE ANISOTROPIC SOBOLEV SPACE

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Abstract. The global well-posedness for the KP-II equation is established in
the anisotropic Sobolev space $H^{s,0}$ for $s > -\frac{1}{8}$. Even though conservation
laws are invalid in the Sobolev space with negative index, we explore the asymp-
totic behavior of the solution by the aid of the $I$-method in which Colliander,
Keel, Staffilani, Takaoka, and Tao introduced a series of modified energy terms.
Moreover, a-priori estimate of the solution leads to the existence of global at-
tractor for the weakly damped, forced KP-II equation in the weak topology of
the Sobolev space when $s > -\frac{1}{8}$.

1. Introduction. In this article, we consider the initial value problem for the
Kadomtsev-Petviashvili II (KP-II) equation

$$\begin{cases}
\partial_t u + (\partial_x^3 + \partial_x^{-1}\partial_y^2)u + \frac{1}{2}\partial_x u^2 = 0, & (x, y) \in \mathbb{R}^2, \ t \geq 0 \\
u(x, y, 0) = u_0(x, y) \in H^{s,0} (\mathbb{R}^2),
\end{cases}$$

and the global attractor for the weakly damped, forced Kadomtsev-Petviashvili II
dfKP-II) equation

$$\begin{cases}
\partial_t u + (\partial_x^3 + \partial_x^{-1}\partial_y^2)u + \frac{1}{2}\partial_x u^2 + \gamma u = f, & (x, y) \in \mathbb{R}^2, \ t \geq 0 \\
u(x, y, 0) = u_0(x, y) \in H^{s,0} (\mathbb{R}^2),
\end{cases}$$

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where \( u = u(x, y, t) \) is a real-valued function, the damping parameter \( \gamma \) is positive and the external forcing term \( f \in L^2(\mathbb{R}^2) \) is independent of \( t \).

The KP-II equation was initially derived by B.B. Kadomtsev and V.I. Petviashvili [8] to describe the propagation of weakly transverse water waves in the long wave regime with small surface tension. It can be seen as the two-dimensional generalization of the Korteweg-de Vries (KdV) equation. The KP-II equation is completely integrable. In fact, there exists an infinite sequence of conserved quantities [17], such as

\[
E(u)(t) = \int_{\mathbb{R}^2} u^2(x, y, t) \, dx \, dy \quad (3)
\]

and

\[
E_p(u)(t) = \int_{\mathbb{R}^2} \frac{1}{2} (\partial_x u)^2 - (\partial_x^{-1} \partial_y u)^2 - \frac{1}{3} u^3 \, dx \, dy. \quad (4)
\]

However, due to the appearance of \( \partial_x^{-1} \), these conserved quantities besides \( E(u) \) seem to be useless for proofs of well-posedness.

In terms of the Cauchy problem for the KP-II equation, J. Bourgain [2] firstly proved global well-posedness in \( L^2(T^2) \) and \( L^2(\mathbb{R}^2) \) by creatively using the Fourier restriction norm method. H. Takaoka [20] obtained local well-posedness in the anisotropic Sobolev spaces \( H^{-\frac{1}{2}+\epsilon,0} \) and in the anisotropic homogeneous Sobolev space \( H^{-\frac{1}{2}+\epsilon,0} \cap H^{-\frac{1}{2}+\epsilon,0} \). In [4], M. Hadac not only proved local well-posedness for the KP-II equation in the inhomogeneous Sobolev space \( H^{s,0} \) in full sub-critical range \( s > -\frac{1}{2} \) without any additional low frequency assumption, but also considered well-posedness for the generalized KP-II equation in the anisotropic Sobolev space \( H^{s_1,s_2} \). Local well-posedness for the critical level of regularity \( s = -\frac{1}{2} \) in both homogeneous and inhomogeneous cases are solved by Hadac, Herr and Koch via using the atomic space. In [5], they acquired small data global well-posedness and scattering result as well. With regard to global well-posedness for large data below \( L^2 \), Isaza and Mejía showed global result in \( H^{s,0} \) for \( s > -\frac{1}{11} \) by means of the high-low frequency technique and the almost conservation law (see [6, 7]).

Despite that KP-II equation possesses remarkable rich structure and advantage, one cannot neglect energy dissipation mechanism and external excitation in reality. Therefore, sometimes we need add a weak dissipation and an external forcing term to the original KP-II equation. That is the dKP-II equation (2) we consider here. Global attractor is an invariant compact subset which attracts all trajectories when \( t \) approaches to \( +\infty \). Tsugawa [22] proved the existence of global attractor for the weakly damped, forced KdV equation on the Sobolev space of negative index by applying the I-method. This idea is also effective for the mKdV equation [15] and the Zakharov-Kuznetsov equation [18, 19]. But, not like the torus case, we only showed weak global attractor for the damped, forced Zakharov-Kuznetsov equation because the bounded set in \( \mathbb{R}^2 \) is just a weakly compact set. What’s more, for some kinds of reaction diffusion equations, the weighted \( L^2 \) space and the uniform Hölder space are used to recover the precompactness of orbits (see [1] and [14]) . Besides, localization estimate plays an important role in the proofs of precompactness in [23] and [10].

In this article, we refine the global well-posedness for the KP-II equation in \( H^{s,0} \) by pushing \( s \) from \( -\frac{1}{14} \) to \( -\frac{3}{8} \) through utilizing the atomic space and the I-method. The advantage of \( U^2, V^2 \) spaces is to obtain sharp estimates in the time variable, which provide us with better decay in high frequency. Connecting this merit of the
atomic space to $I$-method, one can explore the global behavior of energy. Then
the increment of modified energy allows us to prove a a-priori estimate which helps
us better understand the global dynamic of the solution below $L^2$ (see Proposition
12). Due to a lack of compactness in $\mathbb{R}^2$, we make an attempt to obtain global
attractor in the sense of a stronger topology by using localization estimate as we
carried it out for the damped, forced Zakharov-Kuznetsov equation in [10]. But the
situation for the KP-II equation is very different from the ZK equation. Because
the anti-derivative operator $\partial_x^{-1}$ will cause increment instead of decay for the low
frequency part. What’s more, we have to work in negative regularities which seems
to bring another difficulty at the technical level.

We now state the main results of this paper.

**Theorem 1.1.** The initial value problem (1) is globally well-posed in $H^{s,0}(\mathbb{R}^2)$ for $-\frac{3}{8} < s < 0$ .

**Theorem 1.2.** Let $-\frac{1}{8} < s < 0$. Then, there exists a semi-group $A(t)$ and maps $M_1$ and $M_2$ such that $A(t)u_0$ is the unique solution of (2) satisfying

\[
A(t)u_0 = M_1(t)u_0 + M_2(t)u_0
\]

and for $t > T_1$

\[
\sup_{t > T_1} \|M_1(t)u_0\|_{L^2} < K
\]

\[
\|M_2(t)u_0\|_{H^{s,0}} < Ke^{-\gamma(t-T_1)}
\]

where $T_1$ depends on $\|u_0\|_{H^{s,0}}$, $\|f\|_{L^2}$ and $\gamma$, the constant $K$ depends only on $\|f\|_{L^2}$ and $\gamma$.

According to this global dynamic result, we gain the weak global attractor in $H^{s,0}$ from Theorem 1.1 of [21].

**Corollary 1.** The global attractor for (2) exists in the sense of weak topology in $H^{s,0}(\mathbb{R}^2)$ for $-\frac{1}{8} < s < 0$ .

**Organization of the paper.** In Section 2, we shortly list some propositions of $U^p$ and $V^p$. In Section 3, we prove global well-posedness by using the $I$-method on atomic space. Section 4 is devoted to a priori estimate and the proof of Theorem 1.2. Finally, in Section 5 we prove well-posedness, the weakly continuous and Corollary 1.

Now we give the notations used throughout this paper. Let $c < 1$, $C > 3$, the notation $c+\epsilon$ stands for $c + \epsilon$ for some $0 < \epsilon \ll 1$. We always use a fixed smooth cut-off function $\chi \in C_0^{\infty}([-2,2])$ which satisfies that $\chi$ is even, nonnegative, and $\chi = 1$ on $[-1,1]$. We denote spatial variables by $x, y$ and their dual Fourier variables by $\xi, \eta$. While, $\tau$ is the dual variable of the time $t$. Let $\hat{f}$ denote the Fourier transform of $f$ in both time and spatial variables and $\hat{f}$ denote its Fourier transform only in space or in time. We write $\zeta = (\xi, \eta)$, $\lambda = (\xi, \eta, \tau)$ and $\mu = \tau - \xi^3 + \xi^{-1}\eta^2$ for brevity. The capital letters $N, M, N_1, N_2$ and $N_3$ denote dyadic numbers and we write $\sum_{N \geq 1} a_N = \sum_{n \in \mathbb{N}} a_{2^n}$, $\sum_{N \geq M} a_N = \sum_{n \in \mathbb{N} \cap 2^n \geq M} a_{2^n}$ for dyadic summations. The non-isotropic Sobolev space $H^{s_1, s_2}(\mathbb{R}^2)$ is a space of complex valued temperate distributions with the norm

\[
\|u\|_{H^{s_1, s_2}} = \left( \int_{\mathbb{R}^2} \langle \xi \rangle^{2s_1} \langle \eta \rangle^{2s_2} |\hat{u}(\xi, \eta)|^2 d\xi d\eta \right)^{\frac{1}{2}}.
\]
2. Function spaces. $U^p$ and $V^p$ spaces (see [5, 11, 12, 13]) are powerful tools to handle low regularity well-posedness for dispersive equations.

Let $1 \leq p < q < \infty$ and $\mathcal{Z}$ be the set of finite partitions $-\infty = t_0 < t_1 < \ldots < t_{K-1} < t_K = \infty$. For any $\{t_k\}_{k=0}^K \subset \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \in L^2$ with $\sum_{k=0}^{K-1} \|\phi_k\|_2^p = 1$, $\phi_0 = 0$, $U^p$-atom is given by

$$a = \sum_{k=1}^K 1_{[t_{k-1}, t_k)} \phi_{k-1}$$

and the atomic space is

$$U^p = \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j : a_j \text{ $U^p$-atom}, \lambda_j \in \mathbb{C}, \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}.$$

The norm of $U^p$ is defined by

$$\|u\|_{U^p} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j, a_j \text{ $U^p$-atom}, \lambda_j \in \mathbb{C} \right\}.$$

$V^p$ is the normed space of all functions $v : \mathbb{R} \to L^2$ such that $\lim_{t \to \pm \infty} v(t)$ exist and for which the norm

$$\|v\|_{V^p} = \sup_{\{t_k\}_{k=0}^K \subset \mathcal{Z}} \left( \sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_{L^2}^p \right)^{1/p}$$

is finite. We say $v \in V^p$ if $v(-\cdot) \in V^p$ and use $V^p_{rc}$ ($V^p_{-rc}$) to denote the closed subspace of all right continuous functions in $V^p$ ($V^p$).

There is a more intimate connection between $U^p$ and $V^p$. (see Section 2 of [5])

**Proposition 1.** Let $1 < p < q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. We have

(i) $U^p$, $V^p$, $V^p_{rc}$, $V^p_{-rc}$ and $V^p_{rc}$ are Banach spaces,

(ii) $U^p \subset V^p_{-rc} \subset U^q$,

(iii) $\|u\|_{U^p} = \sup_{\|v\|_{V^p} = 1} |\int \langle u'(t), v(t) \rangle \, dt|$ if $u \in V^p_{-rc}$ is absolutely continuous on compact interval.

Given $U^p_S = e^{S}U^p$ with norm $\|u\|_{U^p_S} = \|e^{-S}u\|_{U^p}$ and $V^p_S = e^{-S}V^p$ with norm $\|v\|_{V^p_S} = \|e^{-S}v\|_{V^p}$, where $S = -\partial_x^2 - \partial_x^{-1}\partial_y^2$.

Define the Littlewood-Paley multipliers by

$$\widehat{P_0u} = \chi(2\xi)\hat{u},$$

$$\widehat{P_Nu} = \psi_N(|\xi|)\hat{u},$$

$$\mathcal{F}(Q_Mu)(\xi, \eta, \tau) = \psi_M(\tau)\mathcal{F}u(\xi, \eta, \tau),$$

$$\mathcal{F}(Q^S_Mu)(\xi, \eta, \tau) = \psi_M(\tau - \xi^3 + \xi^{-1}\eta^2)\mathcal{F}u(\xi, \eta, \tau),$$

as well as $Q^S_M = \sum_{N \geq M} Q^S_N$ and $Q^S_{<M} = I - Q^S_M$, where $\psi(x) = \chi(x) - \chi(2x)$ and $\psi_N = \psi(N^{-1}\cdot)$. Note that $Q^S_M = e^{S}Q_M e^{-S}$.
Lemma 2.1. Let $N_2 \leq N_1$, $2 < q \leq \infty$ and $2 < r < \infty$. Assume that $(q, r)$ satisfies $rac{1}{q} + \frac{1}{r} = \frac{1}{2}$, then we have

\[\|Q^S_{\geq M}u\|_{L^2(\mathbb{R}^3)} \lesssim M^{-\frac{1}{2}}\|u\|_{V^3_{\infty}}.\]  

(8)

\[\|Q^S_{\geq M}u\|_{L^p_{x,y}(\mathbb{R}^3)} \lesssim \|u\|_{L^p_{x,y}(\mathbb{R}^3)} \]

(9)

\[\|Q^S_{\geq M}u\|_{V^p_{x,y}} \lesssim \|u\|_{V^p_{x,y}}.\]  

(10)

\[\|u\|_{L^1_tL^1_{x,y}(\mathbb{R}^3)} \lesssim \|u\|_{L^3_{x,y}(\mathbb{R}^3)}.\]  

(11)

\[\|u_{N_1}v_{N_2}\|_{L^2(\mathbb{R}^3)} \lesssim \left(\frac{N_2}{N_1}\right)^{\frac{1}{2}}\|u_{N_1}\|_{L^2_{x,y}(\mathbb{R}^3)}\|v_{N_2}\|_{L^2_{x,y}(\mathbb{R}^3)}.\]  

(12)

\[\|u_{N_1}v_{N_2}\|_{L^2(\mathbb{R}^3)} \lesssim \left(\frac{N_2}{N_1}\right)^{\frac{1}{2}}\|u_{N_1}\|_{L^2_{x,y}(\mathbb{R}^3)}\|v_{N_2}\|_{L^2_{x,y}(\mathbb{R}^3)}.\]  

(13)

Proof. (8) - (10) are from Corollary 2.18 in [5]. Proposition 2.3 of [16] shows that the estimate (11) holds true for free solution. Thus, the claim (11) can be showed by proposition 2.19 of [5]. (12) and (13) follow from (27) and (28) of Corollary 2.21 in [5].

\[\square\]

3. Global well-posedness. Given $M : \mathbb{R}^k \to \mathbb{C}$, $M$ is said to be symmetric if

\[M(\xi_1, \cdots, \xi_k) = M(\sigma(\xi_1, \cdots, \xi_k))\]

for all $\sigma \in S_k$, where $S_k$ is the permutation group for $k$ elements. We define a $k$-linear functional acting on functions $u_1, \cdots, u_k$ for each $m$

\[\Lambda_k(M; u_1, \cdots, u_k) = \int_{\xi_1 + \cdots + \xi_k = 0} M(\xi_1, \cdots, \xi_k) \hat{u}_1(\xi_1) \cdots \hat{u}_k(\xi_k).\]

Actually, one can define $k$-linear functional as above if $M$ is a symbol with respect to $(\zeta_1, \cdots, \zeta_k)$. We write $\Lambda_k(M)$ instead of $\Lambda_k(M; u_1, \cdots, u)$ for convenience.

$m(\xi)$ is a smooth, radially symmetric, non-increasing function satisfying

\[m(\xi) = \begin{cases} \left(\frac{\xi}{N}\right)^s & |\xi| \leq N \\ 0 & |\xi| \geq 2N \end{cases},\]

where $s < 0$, $N \gg 1$. The corresponding Fourier multiplier operator is

\[\hat{\tilde{M}}(\xi) = m(\xi) \hat{f}(\xi).\]

Let $\lambda > 0$ and $N' = \frac{N}{\lambda}$, we also define

\[m'(\xi) = \begin{cases} \left(\frac{\xi}{N'}\right)^s & |\xi| \leq N' \\ 0 & |\xi| \geq 2N' \end{cases},\]

and rescaled operator

\[\hat{\tilde{M}}(\xi) = m'(\xi) \hat{f}(\xi).\]

We start from well-posedness for the KP-II equation. Acting multiplier operator $I$ on both sides of (1), we obtain

\[\partial_t u + (\partial_x^3 + \partial_x^{-1} \partial_y^2) u + \frac{1}{2} \partial_x u^2 = 0.\]  

(14)

One can write this as an integral equation

\[Iu = e^{iS} Iu_0 - \frac{1}{2} \int_0^t e^{(t-t')S} \partial_x I u^2(t') dt'.\]
In the next place, we will estimate the Duhamel term following the idea of Proposition 3.1 in [5]. The work space is denoted by \( Y^s \) which can be defined via the norm
\[
\|u\|_{Y^s} = \left( \sum_{N} N^{2s} \|P_N u\|_{L^2}^2 \right)^{1/2}.
\]
It’s easy to see that \( Y^s \subset L^\infty H^{s,0} \).

Lemma 3.1. Let \( N, N_1, N_2 \) and \( N_3 \) be dyadic numbers. Then, it holds
(i) if \( N_1 \sim N_2 \gg N_2 \),
\[
|\int \chi(t) I(u_{N_1}, v_{N_2}) \partial_x w_{N_2} dx dy dt| \leq N^s N_2^{-\frac{1}{2} + s} \|Iu_{N_1}\|_{L^2} \|Iv_{N_2}\|_{L^2} \|w_{N_2}\|_{V^2}, \tag{15}
\]
(ii) if \( N_1 \sim N_2 \gg N_3 \),
\[
|\int \chi(t) I(u_{N_1}, v_{N_2}) \partial_x w_{N_2} dx dy dt| \leq N_1^{-1} N_3^2 C(N_1, N_3) \|Iu_{N_1}\|_{L^2} \|Iv_{N_2}\|_{L^2} \|w_{N_2}\|_{V^2}, \tag{16}
\]
where \( C(N_1, N_3) = \begin{cases} 1 & N \gg N_1 \sim N_2 \gg N_3 \\ \frac{N_1}{N} & N_1 \sim N_2 \gg N \gg N_3 \\ N^s N_1^{-2s} N_3^2 & N_1 \sim N_2 \gg N_3 \gg N \end{cases} \).

Proof. Set \( w_{N_3} = \chi(t) w_{N_3} \). From the fact \( \|w_{N_3}\|_{V^2} \lesssim \|w_{N_3}\|_{V^2} \) and Parseval’s formula, it suffices to control
\[
\int \sum_{j=1}^3 \lambda_j \frac{m(\xi_1 + \xi_2)}{m(\xi_1)m(\xi_2)} \hat{u}_{N_1}(\xi_1) \hat{v}_{N_2}(\xi_2) \hat{w}_{N_3}(\lambda_j)
\]
by \( \|u_{N_1}\|_{L^2}, \|v_{N_2}\|_{L^2} \) and \( \|w_{N_3}\|_{V^2} \).

In order to prove (15), we decompose \( Id = Q_{<M}^S + Q_{\geq M}^S \) and divide the integral into eight pieces.

Case 1. \( Q_j^S = Q_{<M}^S \) for \( j = 1, 2, 3 \).

We claim that
\[
\int \xi_3 \frac{m(\xi_1 + \xi_2)}{m(\xi_1)m(\xi_2)} \mathcal{F}(Q_{<M}^S u_{N_1}) \mathcal{F}(Q_{<M}^S v_{N_2}) \mathcal{F}(Q_{<M}^S w_{N_3}^*) = 0.
\]
In the low frequency situation, one has \( |\lambda_j| < M \) and \( |\xi_j| \geq N_j/2 \) due to the cut off operators \( Q_{<M}^S \) and \( P_{N_j} \). By resonance identity
\[
\lambda_1 + \lambda_2 + \lambda_3 = -3\xi_1 \xi_2 \xi_3 - \left( \frac{\xi_1 \eta_2 - \xi_2 \eta_1}{\xi_1 \xi_2 \xi_3} \right)^2,
\]
we know
\[
\frac{1}{8} N_1 N_2 N_3 \leq |\xi_1 \xi_2 | \xi_3 | \leq max \{|\lambda_1|, |\lambda_2|, |\lambda_3|\} < N.
\]
Hence, if we choose \( M = \frac{1}{10} N_1 N_2 N_3 \), the integral vanishes.

Case 2. At least one of \( Q_j^S \) is high frequency, for instance \( Q_1^S = Q_{\geq M}^S \).
Using the $L^4$ Strichartz estimate (11), we have
\[
\left| \int \xi_3 \frac{m(\xi_1 + \xi_2)}{m(\xi_1) m(\xi_2)} \mathcal{F}(Q_{\geq M}^S u_{N_1}) \mathcal{F}(Q_{\geq M}^S v_{N_2}) \mathcal{F}(Q_{\leq M}^S w_{N_3}) \right|
\leq \frac{N_3}{m(N_2)} \|Q_{\geq M}^S u_{N_1}\|_{L^2(\mathbb{R}^3)} \|Q_{\geq M}^S v_{N_2}\|_{L^4(\mathbb{R}^3)} \|Q_{\leq M}^S w_{N_3}\|_{L^4(\mathbb{R}^3)}
\leq N_3 \|Q_{\geq M}^S u_{N_1}\|_{L^2(\mathbb{R}^3)} \|Q_{\geq M}^S v_{N_2}\|_{L^2(\mathbb{R}^3)} \|Q_{\leq M}^S w_{N_3}\|_{L^2(\mathbb{R}^3)}
\leq N_3^{1 - \frac{3}{2}} \|Q_{\geq M}^S u_{N_1}\|_{L^2(\mathbb{R}^3)} \|Q_{\geq M}^S v_{N_2}\|_{L^2(\mathbb{R}^3)} \|Q_{\leq M}^S w_{N_3}\|_{L^2(\mathbb{R}^3)}
\leq N_3^{1 - \frac{3}{2}} C(N_1, N_3) \|u_{N_1}\|_{L^2(\mathbb{R}^3)} \|v_{N_2}\|_{L^2(\mathbb{R}^3)} \|w_{N_3}\|_{L^2(\mathbb{R}^3)}.
\]

The other cases can be dealt with in exactly the same way.

We use the same decomposition as above to prove (16). Here \( \frac{m(\xi_1 + \xi_2)}{m(\xi_1) m(\xi_2)} \) can be bounded by \( C(N_1, N_3) \).

When \( Q_{\leq M}^S \) or \( Q_{\geq M}^S \) is high frequency, by using estimate (8) and bilinear Strichartz estimate (13), one has
\[
\left| \int \xi_3 \frac{m(\xi_1 + \xi_2)}{m(\xi_1) m(\xi_2)} \mathcal{F}(Q_{\geq M}^S u_{N_1}) \mathcal{F}(Q_{\geq M}^S v_{N_2}) \mathcal{F}(Q_{\leq M}^S w_{N_3}) \right|
\leq \frac{N_3 m(N_3)}{m^2(N_1)} \|Q_{\geq M}^S u_{N_1}\|_{L^2(\mathbb{R}^3)} \|Q_{\geq M}^S v_{N_2}\|_{L^2(\mathbb{R}^3)} \|Q_{\leq M}^S w_{N_3}\|_{L^2(\mathbb{R}^3)}
\leq \frac{N_3 m(N_3)}{m^2(N_1)} (N_1 N_2 N_3)^{-\frac{1}{2}} \|Q_{\geq M}^S u_{N_1}\|_{L^2(\mathbb{R}^3)} \|Q_{\geq M}^S v_{N_2}\|_{L^2(\mathbb{R}^3)} \|Q_{\leq M}^S w_{N_3}\|_{L^2(\mathbb{R}^3)}
\leq N_1^{-\frac{3}{2}} N_3^2 C(N_1, N_3) \|u_{N_1}\|_{L^2(\mathbb{R}^3)} \|v_{N_2}\|_{L^2(\mathbb{R}^3)} \|w_{N_3}\|_{L^2(\mathbb{R}^3)}.
\]

When \( Q_{\leq M}^S = Q_{\geq M}^S \) is high frequency, it holds
\[
\left| \int \xi_3 \frac{m(\xi_1 + \xi_2)}{m(\xi_1) m(\xi_2)} \mathcal{F}(Q_{\leq M}^S u_{N_1}) \mathcal{F}(Q_{\leq M}^S v_{N_2}) \mathcal{F}(Q_{\leq M}^S w_{N_3}) \right|
\leq N_3 C(N_1, N_3) \|Q_{\leq M}^S u_{N_1}\|_{L^2(\mathbb{R}^3)} \|Q_{\leq M}^S v_{N_2}\|_{L^2(\mathbb{R}^3)} \|Q_{\leq M}^S w_{N_3}\|_{L^2(\mathbb{R}^3)}
\leq N_3 C(N_1, N_3) (N_1 N_2 N_3)^{-\frac{1}{2}} \|Q_{\leq M}^S u_{N_1}\|_{L^2(\mathbb{R}^3)} \|Q_{\leq M}^S v_{N_2}\|_{L^2(\mathbb{R}^3)} \|Q_{\leq M}^S w_{N_3}\|_{L^2(\mathbb{R}^3)}
\leq N_1^{-1} N_3^2 C(N_1, N_3) \|u_{N_1}\|_{L^2(\mathbb{R}^3)} \|v_{N_2}\|_{L^2(\mathbb{R}^3)} \|w_{N_3}\|_{L^2(\mathbb{R}^3)}.
\]

Hence, the proof is completed.

From the proof for Lemma (3.1) above, \( \frac{m(\xi_1 + \xi_2)}{m(\xi_1) m(\xi_2)} \) are controlled respectively by \( N^s N_2^{-s} \) and \( C(N_1, N_3) \) under the conditions \( N_1 \sim N_3 \gg N_2 \) and \( N_1 \sim N_2 \gg N_3 \). The technique is the same for \( \frac{m(\xi_1 + \xi_2)}{m(\xi_1) m(\xi_2)} = 1 \). Therefore, it’s easy to see that the following trilinear estimates hold true.

**Lemma 3.2.** Let \( N_1, N_2, N_3 \) be dyadic numbers. Then, we have
(i) if \( N_1 \sim N_3 \gg N_2 \),
\[
\left| \int \chi(t)(u_{N_1} v_{N_2}) \partial_x w_{N_3} \, dx \, dy \, dt \right|
\leq N_2^{-\frac{1}{2}} \|u_{N_1}\|_{L^2(\mathbb{R}^3)} \|v_{N_2}\|_{L^2(\mathbb{R}^3)} \|w_{N_3}\|_{L^2(\mathbb{R}^3)},
\]
(ii) if \( N_1 \sim N_2 \gg N_3 \),
\[
\left| \int \chi(t)(u_{N_1} v_{N_2}) \partial_x w_{N_3} \, dx \, dy \, dt \right|
\]
Combining Proposition 1 (iii) with (15), we obtain

\[ \lesssim N_1^{-\frac{1}{2}} N_3^\frac{1}{2} \| u_{N_1} \|_{U_3^1} \| v_{N_2} \|_{U_3^1} w_{N_3} \|_{Y_3^1}, \]  

(18)

\textbf{Proposition 2.} Let \(-\frac{3}{2} < s < 0\). We have

\[ \| \int_0^t e^{(t-t')S} \chi(t) \partial_x I(u)(t') dt' \|_{Y^0} \lesssim \| u \|_{Y^0} \| I u \|_{Y^0}. \]  

(19)

\textbf{Proof.} By definition of \(Y^0\) and symmetry, it suffices to consider the following two terms

\[ J_1 = \sum_{N_3} \| \sum_{N_1 \gg N_2} \int_0^t e^{(t-t')S} \chi(t) \partial_x P_{N_3} I(u_{N_1} v_{N_2})(t') dt' \|_{U_3^1}^2 \]

and

\[ J_2 = \sum_{N_3} \| \sum_{N_1 \sim N_2} \int_0^t e^{(t-t')S} \chi(t) \partial_x P_{N_3} I(u_{N_1} v_{N_2})(t') dt' \|_{U_3^1}^2. \]

Combining Proposition 1 (iii) with (15), we obtain

\[ J_1 \lesssim \sum_{N_3} \left( \sum_{N_1 \gg N_2} \sup_{w_{N_3} \|_{L_3^\infty} \leq 1} \left| \int_{\mathbb{R}^3} \chi(t) I(u_{N_1} v_{N_2}) \partial_x w_{N_3} dx dy dt \right| \right)^2 \]

\[ \lesssim \sum_{N_3} \left( \sum_{N_1 \gg N_2} N^n N_2^{-\frac{1}{2} - s} \| u_{N_1} \|_{U_3^1} \| v_{N_2} \|_{U_3^1} \right)^2 \]

\[ \lesssim \sum_{N_3} \| u_{N_1} \|_{U_3^1}^2 \sum_{N_2} \| v_{N_2} \|_{U_3^1}^2 \sum_{N_2} N_s^\frac{1}{2} \sum_{N_2} N_2^{-(1+2s)} \]

\[ \lesssim \sum_{N_3} \| u_{N_1} \|_{U_3^1}^2 \sum_{N_2} \| v_{N_2} \|_{U_3^1}^2 N_s \sum_{N_2} N_2^{-(1+2s)} \]

\[ \lesssim \| u \|_{Y^0}^2 \| I u \|_{Y^0}^2. \]

The second term can be controlled via (16) and Minkowski’s inequality,

\[ J_2 \lesssim \sum_{N_3} \left( \sum_{N_1 \sim N_2} \sup_{w_{N_3} \|_{L_3^\infty} \leq 1} \left| \int_{\mathbb{R}^3} \chi(t) I(u_{N_1} v_{N_2}) \partial_x w_{N_3} dx dy dt \right| \right)^2 \]

\[ \lesssim \sum_{N_3} \left( \sum_{N_1 \gg N_2} N_1^{-1} N_3^\frac{1}{2} \left( \| u_{N_1} \|_{U_3^1} \| v_{N_2} \|_{U_3^1} \right)^2 \right. \]

\[ + \sum_{N_3 \gg N_1} \left( \sum_{N_1 \ll N_3} N_1^{-1} N_3^\frac{1}{2} \left( \frac{N_1}{N} \right)^{-2s} \| u_{N_1} \|_{U_3^1} \| v_{N_1} \|_{U_3^1} \right)^2 \]

\[ + \sum_{N_3 \gg N_2} \left( \sum_{N_2 \ll N_3} N_2^{-1} N_3^\frac{1}{2} N_2^{-(1)} N_2^{-2s} \| u_{N_1} \|_{U_3^1} \| v_{N_1} \|_{U_3^1} \right)^2 \]

\[ + \sum_{N_3 \sim N_1} \left( \sum_{N_1} N_1^{-1} N_3^\frac{1}{2} N_1^{-(1-s)} \| u_{N_1} \|_{U_3^1} \| v_{N_1} \|_{U_3^1} \right)^2 \]

\[ \lesssim \sum_{N_1} N_1^{-1} \left( \sum_{N_3 \gg N_1} \right)^\frac{1}{2} \| u_{N_1} \|_{U_3^1} \| v_{N_1} \|_{U_3^1} \right)^2 \]

\[ + \sum_{N_3 \ll N_1} N_3^{\frac{1}{2}} N_3^{-\frac{1}{2} - s} \left( \sum_{N_1} \| u_{N_1} \|_{U_3^1} \| v_{N_1} \|_{U_3^1} \right)^2 \]

\[ + \sum_{N_3 \gg N_1} N_3^{\frac{1}{2}} N_3^{-\frac{1}{2} - s} \left( \sum_{N_1} \| u_{N_1} \|_{U_3^1} \| v_{N_1} \|_{U_3^1} \right)^2 \]
\[ + \sum_{N_1} N_1^{-1-2s}\|Iu_{N_1}\|_{U_2}^2 \|Iv_{N_1}\|_{U_2}^2 \]
\[ \lesssim \|Iu\|_{Y_0}^2 \|Iv\|_{Y_0}^2. \]

The proof is completed. \hfill \Box

Analogously, one can estimate the nonlinear term without operator \( I \) as Proposition 3 with the help of Lemma 3.2.

**Proposition 3.** Let \(-\frac{3}{8} < s < 0\). We have
\[
\| \int_0^t e^{(t-t')S} \chi(t) \partial_x (uv)(t') dt' \|_{Y^s} \lesssim \|u\|_{Y^s} \|v\|_{Y^s}.
\]

**Proposition 4.** Let \(-\frac{3}{8} < s < 0\). Assume \( u_0 \) satisfies \( E(Iu_0) < \epsilon_0 \ll 1 \). Then there exists a unique solution \( u \) to (1) on \([0,1]\), such that
\[
\|Iu\|_{Y_0} \lesssim \epsilon_0.
\]

**Proof.** From Duhamel’s principle, we get
\[
Iu = \chi(t) e^{tS} Iu_0 - \frac{1}{2} \int_0^t e^{(t-t')S} \chi(t) \partial_x I(u^2)(t') dt'.
\]
Applying Proposition 3, it gives
\[
\|Iu\|_{Y_0} \leq \|\chi(t) e^{tS} Iu_0\|_{Y_0} + \| \int_0^t e^{(t-t')S} \chi(t) \partial_x I(u^2)(t') dt' \|_{Y_0}
\[
\leq \|Iu_0\|_{L^2(\mathbb{R}^2)} + C \|Iu\|_{Y_0}^2,
\]
and
\[
\|Iu - Iv\|_{Y_0} \leq C(\|Iu\|_{Y_0} + \|Iv\|_{Y_0}) \|Iu - Iv\|_{Y_0}.
\]
Then, We get the existence of local solution by the contraction mapping principle. A bootstrap argument yields \( \|Iu\|_{Y_0} \lesssim \epsilon_0 \).

Now we turn to the growth of \( E(Iu)(t) \).

**Proposition 5.** Assume \( u \) satisfies (1) and \( M \) is a symmetric function. Then
\[
\frac{d\Lambda_k(M)}{dt} = \Lambda_k(M h_k) - \frac{ik}{2} \Lambda_{k+1}(M (\xi_1, ..., \xi_k-1, \xi_k + \xi_{k+1}))(\xi_k + \xi_{k+1}),
\]
where \( h_k = i \sum_{j=1}^k (\xi_j^3 - \xi_j^{-1} \eta_j^2) \).

**Proof.** See (2.3) in [3]. \hfill \Box

Parseval’s formula provides
\[
E(Iu) = \Lambda_2(m(\xi_1)m(\xi_2)).
\]
It follows from Proposition 5 that
\[
\frac{dE(Iu)(t)}{dt} = \Lambda_2(m(\xi_1)m(\xi_2)h_2) + i \Lambda_3(\sum_{j=1}^3 \xi_j m^2(\xi_j)).
\]
The first term vanishes since \( \xi_1^3 + \xi_2^3 - \xi_1^{-1} \eta_1^2 - \xi_2^{-1} \eta_2^2 = 0 \) for \( \xi_1 + \xi_2 = 0 \) and \( \eta_1 + \eta_2 = 0 \).
Denote
\[ M_3(\xi_1, \xi_2, \xi_3) = i \sum_{j=1}^{3} \xi_j m^2(\xi_j). \]
We further give a new modified energy
\[ E^1(Iu) = E(Iu) + \Lambda_3(\sigma_3) \]
where \( \sigma_3 \) will be set to make a cancelation.

Using Proposition 5 again, one has
\[ \frac{dE^1(Iu)(t)}{dt} = \Lambda_3(M_3) + \Lambda_3(\sigma_3 h_3) + \Lambda_4(M_4), \]
where
\[ M_4(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{3}{2} \left[ \sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4) + \sigma_3(\xi_1, \xi_2 + \xi_4, \xi_3)(\xi_2 + \xi_4) + \sigma_3(\xi_1 + \xi_4, \xi_2, \xi_3)(\xi_1 + \xi_4) \right]. \]

Then the two trilinear terms in (22) cancel by taking
\[ \sigma_3 = -\frac{M_3}{h_3} = -\frac{M_3}{3\xi_1\xi_2\xi_3 + \frac{(\xi_1 n_2 - \xi_2 n_1)^2}{\xi_1 \xi_2 \xi_3}}, \]
and we have
\[ \frac{dE^1(Iu)(t)}{dt} = \Lambda_4(M_4). \]

Here is the estimate for \( M_3 \) (see Proposition 5.16 in [24]).

**Proposition 6.** Let \( N_1, N_2 \) and \( N_3 \) be dyadic numbers. Then, in the hyperplane \( \{ \xi_1 + \xi_2 + \xi_3 = 0, |\xi_j| \sim N_j \} \), it holds
\[ |M_3(\xi_1, \xi_2, \xi_3)| \lesssim \max \{ m^2(\xi_1), m^2(\xi_2), m^2(\xi_3) \} \min \{ N_1, N_2, N_3 \}. \]  
\[ (23) \]

**Proposition 7.** Let \(-\frac{3}{2} < s < 0\). We have
\[ \| \chi(t)\Lambda_3(\sigma_3) \|_{L^\infty} \lesssim N^{-2s} \| Iu \|_{Y_0}^3. \]  
\[ (24) \]

**Proof.** The fact
\[ \sum_{N_1} N_1^0 \| P_{N_1} Iu \|_{U_3^2} \leq (\sum_{N_1} N_1^0)^{\frac{3}{2}} \| Iu \|_{Y^0} \]
indicates that it suffices to prove
\[ \| \int \sum_{j=1}^{3} \frac{M_3(\xi_1, \xi_2, \xi_3)}{m(\xi_1)m(\xi_2)m(\xi_3)h_3} \prod_{j=1}^{3} \partial_j(\xi_j) \|_{L^2} \]
\[ \lesssim N^{-2s} (N_1 N_2 N_3)^{\frac{3}{2}} \prod_{j=1}^{3} \| u_j \|_{U_3^2} \]
\[ (25) \]

for any function \( u_j \) \((j = 1, 2, 3)\) with frequency supported on \( |\xi_j| \sim N_j \). One can assume \( N_1 \sim N_2 \gtrsim N_3 \) and \( N_1 \gtrsim N \), otherwise the symbol \( M_3 \) vanishes. Notice that \( |h_3| > 3|\xi_1 \xi_2 \xi_3| \). Moreover, from the definition of atom, we know that if \( a \) is a \( U^p \)-atom then so is \( \mathcal{F}_{xy}^{-1} a \). Therefore, we can also assume that the Fourier transform of \( u \) with respect to spatial variables is nonnegative due to
\[ \| \mathcal{F}_{xy}^{-1} a \|_{U^p} \lesssim \| u \|_{U^p}. \]
Case 1. \(N_1 \sim N_2 \sim N_3 \sim N\)
By (23) and (11), we obtain
\[
\left\| \sum_{j=0}^{N^2} a_j(\zeta_j) \right\|_{L^2_t} \approx \frac{1}{m(N_1)} \prod_{j=1}^{3} \left\| u_j \right\|_{L^1_tL^2_y} \leq N^{-2^+} (N_1 N_2 N_3)^{0-} \prod_{j=1}^{3} \left\| u_j \right\|_{U^2_y}.
\]

Case 2. \(N_1 \sim N_2 \gg N_3, N_1 \sim N\)
(23), (11) and bilinear Strichartz estimate (12) imply
\[
\left\| \sum_{j=0}^{N^2} a_j(\zeta_j) \right\|_{L^2_t} \approx \frac{1}{m(N_1)} \prod_{j=1}^{3} \left\| u_j \right\|_{L^1_tL^2_y} \leq N^{-2^+} (N_1 N_2 N_3)^{0-} \prod_{j=1}^{3} \left\| u_j \right\|_{U^2_y}.
\]

This completes the proof of (25), and hence (24). \(\square\)

Proposition 8. Let \(-\frac{3}{8} < s < 0\). We have
\[
\left\| \int_0^1 \Lambda_4(M_4) dt \right\| \lesssim N^{-1^+} \| Iu \|_{Y^0}^4.
\]

Proof. Even though \(\sigma_3\) is not symmetric with respect to \(\xi_1, \xi_2\), and \(\xi_3\), we control \(\sigma_3\) by \(\frac{M_4(\xi_1, \xi_2, \xi_3)}{\xi_1 \xi_2 \xi_3}\) all over the proof. Therefore, as previous discussion, it suffices to show
\[
\left\| \int_0^1 \Lambda_4(M_4) dt \right\| \lesssim N^{-1^+} \prod_{j=1}^{4} \left\| u_j(\lambda_j) \right\| \lesssim N^{-1^+} \prod_{j=1}^{4} N_3^{0^+} \left\| u_j \right\|_{U^2_y}.
\]

We may assume \(N_1 \geq N_2, N_3 \geq N_4\) by symmetry.

Case 1. \(N_1 \sim N_2 \sim N_3 \gg N\)
This gives \(N_3 \sim N_4 \gg N_1 \geq N_2, N_3 \sim N_4 \gg N\). By using (23) and bilinear Strichartz estimate (12), we have
\[
\left\| \sum_{j=1}^{N_4} a_j(\zeta_j) \right\|_{L^2_t} \approx \frac{1}{m(N_1)} \prod_{j=1}^{3} \left\| u_j \right\|_{L^1_tL^2_y} \leq N^{-2^+} (N_1 N_2 N_3)^{0-} \prod_{j=1}^{3} \left\| u_j \right\|_{U^2_y}.
\]
We complete the proof of (27).

Let

\[ \text{Proposition 9.} \]

\[ \text{Case 3.} \]

\[ \text{where } N_{12} \sim |\xi_1 + \xi_2|. \]

\[ \text{Case 2.} \]

\[ \text{We have } N_1 \sim N_3 \gg N \text{ in this case, therefore} \]

\[ \text{Proposition 9. Let } -\frac{3}{8} < s < 1, N \gg 1. \text{ Assume } E(Iu_0) < \epsilon_0^2 \ll 1, \text{ then there exists a unique solution} \]

\[ u(x, y, t) \in C([0, 1], H^{s,0}(\mathbb{R}^2)) \]

\[ \text{of (1) satisfying} \]

\[ E(Iu)(1) = E(Iu)(0) + O(N^{-1+\epsilon_0^3}). \]

\[ \text{(28)} \]

\[ \text{Proof. From Proposition 4, there exists a unique solution } u \text{ to (1) on } [0, 1] \text{ satisfying} \]

\[ ||Iu||_{Y_0} \ll 1. \]
Combining Proposition 7 and Proposition 8, we get

\[
|E(Iu)(1) - E(Iu)(0)| = |\Lambda_3(\sigma_3)(0) - \Lambda_3(\sigma_3)(1) + \int_0^1 \Lambda_4(M_4)dt| \\
\lesssim N^{-2+}||Iu||_{Y^0}^3 + N^{-1+}||Iu||_{Y^0}^4 \\
\lesssim N^{-1+} \epsilon_0^3.
\]

The rescaling solution of (1) is

\[ u_\lambda(x, y, t) = \lambda^{-2}u(\lambda^{-1}x, \lambda^{-2}y, \lambda^{-3}t) \]

with initial date \( u_{0,\lambda}(x, y) = \lambda^{-2}u_0(\lambda^{-1}x, \lambda^{-2}y) \), where \( \lambda > 0 \). For any \( T > 0 \), the solution \( u \) exists on \([0, T]\) is equivalent to \( u_\lambda \) exists on \([0, \lambda^3T]\).

By simple calculation, we know that

\[ E(Iu_{0,\lambda}) \lesssim \lambda^{-3-2s}N^{-2s}||u_0||_{H^{s,0}}^2. \]

Taking \( \lambda \sim N^{-\frac{3}{8s+}} \) \((N \gg 1)\) such that

\[ E(Iu_{0,\lambda}) \lesssim N^{0-}||u_0||_{H^{s,0}}^2 < \epsilon_0^2 \ll 1, \]

then we can use Proposition 9 to extend the solution \( u_\lambda \) from \([0, 1]\) to \([1, 2]\). Iterating this procedure \( N^{1-} \) steps, we achieve the solution \( u_\lambda \) on \([0, N^{1-}]\). Choosing \( N \) sufficiently large such that

\[ N^{1-} > \lambda^3T \sim N^{-\frac{3}{8s}+}T. \]

This is possible as \(-\frac{3}{8} < s\). It means that the solution \( u_\lambda \) is extended to \([0, \lambda^3T]\).

Hence, we show the global well-posedness for the KP-II equation when \(-\frac{3}{8} < s < 0\).

4. **A-priori estimates and global dynamic.** We follow Tsugawa’s idea that he used to obtain the global attractor for the KdV equation on the Sobolev space of negative index (see [22]). The rescaled equation associated to (2) is

\[
\begin{align*}
\partial_t v + (\partial_x^3 + \partial_x^{-1}\partial_y^2)v + \frac{1}{2}\partial_x v^2 + \gamma \lambda^{-3}v &= \lambda^{-3}g, \\
v(x, y, 0) &= v_0(x, y) \in H^s(\mathbb{R}^2),
\end{align*}
\] (29)

where \( v(x, y, t) = \lambda^{-2}u(\lambda^{-1}x, \lambda^{-2}y, \lambda^{-3}t) \), \( v_0(x, y) = \lambda^{-2}u_0(\lambda^{-1}x, \lambda^{-2}y) \) and \( g(x, y) = \lambda^{-2}f(\lambda^{-1}x, \lambda^{-2}y) \).

It holds \( ||Iv||_{L^2} = \lambda^{-\frac{3}{2}}||Iu||_{L^2} \) and \( ||I'g||_{L^2} = \lambda^{-\frac{3}{2}}||If||_{L^2} \) from the definition of rescaled operator \( I' \).

We first give local well-posedness result for the weakly damped forced KP-II equation.

**Proposition 10.** Let \(-\frac{1}{2} < s < 0\). Assume \( I'v_0 \in L^2(\mathbb{R}^2) \) and \( I'g \in L^2(\mathbb{R}^2) \), then there is a constant \( \delta = \delta(||I'v_0||_{L^2(\mathbb{R}^2)}, \lambda^{-3}||I'g||_{L^2(\mathbb{R}^2)}, \gamma \lambda^{-3}) > 0 \) so that there exists a unique solution \( v(x, y, t) \in C([0, \delta], H^{s,0}(\mathbb{R}^2)) \)

of (29) satisfying

\[ ||I'v||_{Y^0} \lesssim ||I'v_0||_{L^2(\mathbb{R}^2)} + \lambda^{-3}||I'g||_{L^2(\mathbb{R}^2)}, \]

and

\[ \sup_{t \in [0, \delta]} ||I'v(t)||_{L^2(\mathbb{R}^2)} \lesssim ||I'v_0||_{L^2(\mathbb{R}^2)} + \lambda^{-3}||I'g||_{L^2(\mathbb{R}^2)}. \]
Proof. Acting $I'$ on (29)

$$\partial_t I'v + (\partial_x^2 + \partial_x^{-1} \partial_y^2)I'v + \frac{1}{2} \partial_x I'v^2 + \gamma \lambda^{-3} I'v = \lambda^{-3} I'g,$$

we rewrite (30) as an integral equation

$$I'v = \mathcal{T}_I'v,$$

where

$$\mathcal{T}_I'v = \chi(t)e^{tS}I'v_0 - \int_0^t e^{(t-t')S} \chi(t)(\frac{1}{2} \partial_x I'v^2 + \gamma \lambda^{-3} I'v - \lambda^{-3} I'g)dt'.$$

From the definition of $Y^0$ and the duality of $U^p$, we know that

$$\| \int_0^t e^{(t-t')S} \chi(t)(\gamma \lambda^{-3} I'v - \lambda^{-3} I'g)dt'\|_{Y^0} \lesssim (\sup_{N_1} \| w \|_{L^2_x})^2 \lesssim (\| \gamma \lambda^{-3} P_N, I'v - \lambda^{-3} P_N, I'g \|_{L^2_x} \| \chi(t)w \|_{L^1_x} \|_{L^2_y})^2$$

$$\lesssim (\| \gamma \lambda^{-3} P_N, I'v \|_{L^2_y} + \lambda^{-3} \| P_N, I'g \|_{L^2_y})^2$$

$$\lesssim \gamma \lambda^{-3} \| I'v \|_{Y^0} + \lambda^{-3} \| I'g \|_{L^2}.$$  (31)

Set

$$B = \{ I'v \in Y^0 \mid \| I'v \|_{Y^0} < C_0(\| I'v_0 \|_{L^2} + \lambda^{-3} \| I'g \|_{L^2}) \},$$

by using Proposition 3 and (31), we get

$$\| \mathcal{T}_I'v \|_{Y^0} \lesssim \| I'v_0 \|_{L^2} + \| I'v \|_{Y^0} + \gamma \lambda^{-3} \| I'v \|_{Y^0} + \lambda^{-3} \| I'g \|_{L^2}$$

$$\lesssim (\| I'v_0 \|_{L^2} + \lambda^{-3} \| I'g \|_{L^2})^2 (1 + C_0 \gamma \lambda^{-3} (\| I'v_0 \|_{L^2} + \lambda^{-3} \| I'g \|_{L^2})$$

and

$$\| \mathcal{T}_I'v_1 - \mathcal{T}_I'v_2 \|_{Y^0} \lesssim (\gamma \lambda^{-3} + \| I'v_1 \|_{Y^0} + \| I'v_2 \|_{Y^0}) \| I'v_1 - I'v_2 \|_{Y^0}$$

$$\lesssim [\gamma \lambda^{-3} + C_0(\| I'v_0 \|_{L^2} + \lambda^{-3} \| I'g \|_{L^2})] \| I'v_1 - I'v_2 \|_{Y^0}.$$  

Then, it is easy to see that

$$\mathcal{T} : B \to B$$

is a strict contraction mapping as we assume

$$\lambda^{-3} \gamma < 1, \quad \| I'v_0 \|_{L^2} < 1, \quad \lambda^{-3} \| I'g \|_{L^2} < 1.$$  (32)

We consider $\sigma$—scaling of $v$

$$w(x, y, t) = \sigma^{-2} v(\sigma^{-1} x, \sigma^{-2} y, \sigma^{-3} t).$$

If $\sigma$ is chosen to be sufficiently large, it holds

$$(\sigma \lambda)^{-3} \gamma < 1,$$

$$\| I''w_0 \|_{L^2} \lesssim \sigma^{-\frac{1}{2}} \| I'v_0 \|_{L^2} < 1,$$

$$(\sigma \lambda)^{-3} \sigma^{-\frac{1}{2}} \| I'g \|_{L^2} < 1.$$  

This verifies (32), so there exists a unique solution $w$ on $[0, 1]$. Hence, (30) is locally well-posed on $[0, \delta]$ by taking $\delta = \sigma^{-3}$.  

$\square$
Next, we explore the global dynamic of $I'v$.

From (30), we obtain

\[
\frac{dE(I'v)(t)}{dt} = -2 \int_{\mathbb{R}^2} [(\partial_x^3 + \partial_x^{-1}\partial_y^2)I'v + \frac{1}{2} \partial_x I'v^2 + \gamma \lambda^{-3} I'v - \lambda^{-3} I'g]I'v \, dx \, dy
\]

\[
= -2 \gamma \lambda^{-3} E(I'v) + 2 \int_{\mathbb{R}^2} \lambda^{-3} I'gI'v \, dx \, dy + \Lambda_3(M_3; v)
\]

and

\[
\frac{dE^1(I'v)(t)}{dt} = -2 \gamma \lambda^{-3} E(I'v) + 2 \int_{\mathbb{R}^2} \lambda^{-3} I'gI'v \, dx \, dy
\]

\[
+ \Lambda_4(M_4; v) - 3 \gamma \lambda^{-3} \Lambda_3(\sigma_3; v) + 2 \lambda^{-3} \Lambda_3^2(\sigma_3)
\]

\[
= -2 \gamma \lambda^{-3} E^1(I'v) + 2 \int_{\mathbb{R}^2} \lambda^{-3} I'gI'v \, dx \, dy
\]

\[
+ \Lambda_4(M_4; v) - \gamma \lambda^{-3} \Lambda_3(\sigma_3; v) + \lambda^{-3} \Lambda_3^2(\sigma_3),
\]

where $M_3, M_4$ are given in Section 3 and

\[
\Lambda_3^2(\sigma_3) = \Lambda_3(\sigma_3; g, v, v) + \Lambda_3(\sigma_3; v, g, v) + \Lambda_3(\sigma_3; v, v, g).
\]

(34) implies

\[
\frac{d}{dt} E^1(I'v)(t)e^{2\gamma \lambda^{-3} t} = 2 \int_{\mathbb{R}^2} \lambda^{-3} I'gI'v e^{2\gamma \lambda^{-3} t} \, dx \, dy - \gamma \lambda^{-3} \Lambda_3(\sigma_3; v)e^{2\gamma \lambda^{-3} t}
\]

\[
+ \Lambda_4(M_4; v)e^{2\gamma \lambda^{-3} t} + \lambda^{-3} \Lambda_3^2(\sigma_3)e^{2\gamma \lambda^{-3} t}.
\]

Integrating (35) over $[0, T']$, one gets

\[
E^1(I'v)(T')e^{2\gamma \lambda^{-3} T'} - E^1(I'v)(0)
\]

\[
= 2 \int_0^{T'} \int_{\mathbb{R}^2} \lambda^{-3} I'gI'v e^{2\gamma \lambda^{-3} t} \, dx \, dy \, dt - \gamma \lambda^{-3} \int_0^{T'} \Lambda_3(\sigma_3; v)e^{2\gamma \lambda^{-3} t} \, dt
\]

\[
+ \int_0^{T'} \Lambda_4(M_4; v)e^{2\gamma \lambda^{-3} t} \, dt + \lambda^{-3} \int_0^{T'} \Lambda_3^2(\sigma_3)e^{2\gamma \lambda^{-3} t} \, dt.
\]

**Lemma 4.1.** Assume that $v$ is a solution of (29) on $[0, T']$. Then, there exists $C_1 > 0$ such that

\[
\sup_{t \in [0, T']} ||I'v(t)||_{L^2}^2 e^{2\gamma \lambda^{-3} t}
\]

\[
\leq C_1(||I'v_0||_{L^2}^2 + \frac{e^{2\gamma \lambda^{-3} T'}}{\gamma^2} ||I'g||_{L^2}^2 + \sup_{t \in [0, T']} |\Lambda_3(\sigma_3; v)| e^{2\gamma \lambda^{-3} t}
\]

\[
+ \int_0^{T'} \Lambda_4(\sigma_3; v)e^{2\gamma \lambda^{-3} t} \, dt + \lambda^{-3} \int_0^{T'} \Lambda_3^2(\sigma_3)e^{2\gamma \lambda^{-3} t} \, dt
\]

\[
+ \int_0^{T'} \Lambda_4(M_4; v)e^{2\gamma \lambda^{-3} t} \, dt).
\]
Proof. From (36) and $E^1(I'v) = E(I'v) + \Lambda_3(\sigma_3; v)$, we know

$$E(I'v)(T')e^{2\gamma\lambda^{-3}T'}$$

$$\lesssim E(I'v)(0) + |\Lambda_3(\sigma_3; v)(0)| + |\Lambda_3(\sigma_3; v)(T')|e^{2\gamma\lambda^{-3}T'}$$

$$+ \frac{e^{\gamma\lambda^{-3}T'}}{\gamma} \left\| I'g \right\|_{L^2} \sup_{t \in [0, T']} \left\| I'v(t) \right\|_{L^2} e^{2\gamma\lambda^{-3}t} + \left| \int_0^{T'} \Lambda_3(\sigma_3; v)e^{2\gamma\lambda^{-3}t} dt \right|$$

$$+ \lambda^{-3} \left| \int_0^{T'} \Lambda_3^2(\sigma_3)e^{2\gamma\lambda^{-3}t} dt \right| + \left| \int_0^{T'} \Lambda_4(M_4; v)e^{2\gamma\lambda^{-3}t} dt \right|.$$

Therefore,

$$\| I'v(T') \|^2_{L^2} e^{2\gamma\lambda^{-3}T'}$$

$$\lesssim \| I'v(0) \|^2_{L^2} + \sup_{t \in [0, T']} |\Lambda_3(\sigma_3; v)| e^{2\gamma\lambda^{-3}t} + \left| \int_0^{T'} \Lambda_3(\sigma_3; v)e^{2\gamma\lambda^{-3}t} dt \right|$$

$$+ \lambda^{-3} \left| \int_0^{T'} \Lambda_3^2(\sigma_3)e^{2\gamma\lambda^{-3}t} dt \right| + \left| \int_0^{T'} \Lambda_4(M_4; v)e^{2\gamma\lambda^{-3}t} dt \right|$$

$$+ \frac{C_3}{\gamma^2} e^{2\gamma\lambda^{-3}T'} \| I'g \|^2_{L^2} + \varepsilon \sup_{t \in [0, T']} \| I'v(t) \|^2_{L^2} e^{2\gamma\lambda^{-3}t}. \quad (38)$$

Taking $\varepsilon$ sufficiently small, the last term of (38) will be absorbed by the left side, hence we obtain (37). \hfill $\Box$

**Lemma 4.2.** Let $l \in \mathbb{N}^+$ and $\delta > 0$. Then, we have

$$\sup_{t \in [0, (l+1)\delta]} |\Lambda_3(\sigma_3; v)| e^{2\gamma\lambda^{-3}t} \lesssim (l\delta)^{\frac{1}{2}} (N')^{-2+} \sum_{m=0}^{l} \| I'v \|^3_{Y^0_{[m\delta, (m+1)\delta]}} e^{2\gamma\lambda^{-3}m\delta} \quad (39)$$

$$\left| \int_0^{(l+1)\delta} \Lambda_3(\sigma_3; v)e^{2\gamma\lambda^{-3}t} dt \right| \lesssim (l\delta)^{\frac{1}{2}} (N')^{-2+} \sum_{m=0}^{l} \| I'v \|^3_{Y^0_{[m\delta, (m+1)\delta]}} e^{2\gamma\lambda^{-3}m\delta} \quad (40)$$

$$\left| \int_0^{(l+1)\delta} \Lambda_3^2(\sigma_3)e^{2\gamma\lambda^{-3}t} dt \right| \lesssim (l\delta)^{\frac{1}{2}} (N')^{-2+} \| I'g \|_{L^2} \sum_{m=0}^{l} \| I'v \|^2_{Y^0_{[m\delta, (m+1)\delta]}} e^{2\gamma\lambda^{-3}m\delta} \quad (41)$$

**Proof.** On one hand, from Proposition 7, one has

$$\left\| \chi \left( \frac{t}{(l+1)\delta} \right) \Lambda_3(\sigma_3; v)e^{2\gamma\lambda^{-3}t} \right\|_{L^\infty_t}$$

$$\lesssim (l\delta)^{\frac{1}{2}} \| \Lambda_3(\sigma_3; v e^{\frac{2}{\gamma}\lambda^{-3}t}) \|_{L^2_{[0, (l+1)\delta]}}$$

$$\lesssim (l\delta)^{\frac{1}{2}} \sum_{m=0}^{l} \| \Lambda_3(\sigma_3; v e^{\frac{2}{\gamma}\lambda^{-3}t}) \|_{L^2_{[m\delta, (m+1)\delta]}}$$

$$\lesssim (l\delta)^{\frac{1}{2}} (N')^{-2+} \sum_{m=0}^{l} \| I'v \|^3_{Y^0_{[m\delta, (m+1)\delta]}} e^{2\gamma\lambda^{-3}m\delta}. \quad \Box$$
Besides,
\[
| \int_0^{(l+1)\delta} A_3(\sigma_3; v) e^{2\gamma_3 t^3} dt | \\
\lesssim | \sum_{m=0}^{l} \int_{m\delta}^{(m+1)\delta} A_3(\sigma_3; ve^{2\gamma_3 t^3}) dt | \\
\lesssim \delta^\frac{1}{2} \sum_{m=0}^{l} || A_3(\sigma_3; ve^{2\gamma_3 t^3}) ||_{L^2([m\delta,(m+1)\delta])} \\
\lesssim \delta^\frac{1}{2} (N')^{-2+} \sum_{m=0}^{l} || I' v ||_{V_0}^{\frac{1}{2}} || I' v ||_{V_0}^{\frac{1}{2}} e^{2\gamma_3 t^3} m\delta.
\]

On the other hand, in order to prove (41), we need to show
\[
\| A_3(\sigma_3; g, v) \|_{L^2_t[0,\delta]} \\
\lesssim N'^{-2+} \| I' g \|_{L^2(R^2)} || I' v ||_{V_0}^{\frac{1}{2}}.
\]
Therefore, it suffices to show
\[
\| \int \sum_{j=1}^3 \phi_j \| \int_0^3 \frac{M_3(\xi_1, \xi_2, \xi_3)}{m'(\xi_1) m'(\xi_2) m'(\xi_3)} \hat{g}(\xi_1) \hat{u}_2(\xi_2) \hat{u}_3(\xi_3) \|_{L^2}
\lesssim N'^{-2+}(N_1 N_2 N_3)^0 \| g \|_{L^2(R^2)} || u_2 ||_{V_0} || u_3 ||_{V_0} (42)
\]
for function with frequency supported on \(|\xi_j| \sim N_j\). One can assume \(N_2 \geq N_3\) and \(N_2 \geq N\).

**Case 1.** \(N_2 \sim N_3 \sim N\)
\[
\| \int \sum_{j=1}^3 \phi_j \| \int_0^3 \frac{M_3(\xi_1, \xi_2, \xi_3)}{m'(\xi_1) m'(\xi_2) m'(\xi_3)} \hat{g}(\xi_1) \hat{u}_2(\xi_2) \hat{u}_3(\xi_3) \|_{L^2}
\lesssim \frac{N_1 m_2^2(N_1)}{N_1 N_2 N_3 m'(N_1) m'(N_2) m'(N_3)} \| g \|_{L^2(R^2)} || u_2 ||_{L^4(R^3)} || u_3 ||_{L^4(R^3)} \\
\lesssim N'^{-2+}(N_1 N_2 N_3)^0 \| g \|_{L^2(R^2)} || u_2 ||_{V_0} || u_3 ||_{V_0}.
\]

**Case 2.** \(N_1 \sim N_2 \gg N_3\)
\[
\| \int \sum_{j=1}^3 \phi_j \| \int_0^3 \frac{M_3(\xi_1, \xi_2, \xi_3)}{m'(\xi_1) m'(\xi_2) m'(\xi_3)} \hat{g}(\xi_1) \hat{u}_2(\xi_2) \hat{u}_3(\xi_3) \|_{L^2}
\lesssim \frac{N_1 m^2(N_3)}{N_1 N_2 N_3 m'(N_1) m'(N_2) m'(N_3)} \| g \|_{L^2(R^2)} || u_2 ||_{L^2(R^2)} \\
\lesssim \frac{N_3 m(N_3)}{N_1 N_2 N_3} \| g \|_{L^2(R^2)} || u_2 ||_{V_0} || u_3 ||_{V_0} \\
\lesssim N'^{-2+}(N_1 N_2 N_3)^0 \| g \|_{L^2(R^2)} || u_2 ||_{V_0} || u_3 ||_{V_0}.
\]
Hence, the proof is completed.

We give an impactful a-priori estimate which will be used to control \(\| u \|_{H^{r,s}}\).

**Proposition 11.** Let \(C_2 \ll 1\), \(C_3 > 1\), \(C_4 \gg 1\) and \(T' > 0\). Assume \(v\) is a solution of (29) on \([0, T']\). If \(\lambda^3 \geq \gamma\), \((N')^{-1} \geq C_4 T'\) and
\[
|| I' v ||_{L^2}^2 + \frac{1}{\gamma^2} || I' g ||_{L^2}^2 e^{2\gamma_3 t^3} \leq C_2,
\]
then

\[ \| I' (T')\|_{L^2}^2 e^{2\gamma \lambda^{-3} T'} \leq C_3 (\| I' v_0\|_{L^2}^2 + \frac{1}{\gamma^2} \| I' g\|_{L^2}^2 e^{2\gamma \lambda^{-3} T'}). \]

**Proof.** The local existence time for the solution of (29) is

\[ \delta = \delta (\| I' v_0\|_{L^2}, \lambda^{-3} \| I' g\|_{L^2}, \gamma \lambda^{-3}). \]

Set \( j \in \mathbb{N} \) satisfying \( \delta j = T' \). For \( 0 \leq k \leq j, \ k \in \mathbb{N} \), we will prove

\[ \| I' (k\delta)\|_{L^2}^2 e^{2\gamma \lambda^{-3} k\delta} \leq 4C_1 (\| I' v_0\|_{L^2}^2 + \frac{1}{\gamma^2} \| I' g\|_{L^2}^2 e^{2\gamma \lambda^{-3} k\delta}) \] (43)

by induction.

For \( k = 0 \), (43) holds true trivially. We assume (43) holds for \( k = l \) where \( 0 \leq l \leq j - 1 \).

From Lemma 4.1, one has

\[ \| I' v((l+1)\delta)\|_{L^2}^2 e^{2\gamma \lambda^{-3} (l+1)\delta} \leq C_1 (\| I' v_0\|_{L^2}^2 + \frac{e^{2\gamma \lambda^{-3} (l+1)\delta}}{\gamma^2} \| I' g\|_{L^2}^2 + \sup_{t \in [0,(l+1)\delta]} |(\Lambda_3 (\sigma_3; v))| e^{2\gamma \lambda^{-3} t} \]

\[ + \int_0^{(l+1)\delta} \Lambda_3 (\sigma_3; v) e^{2\gamma \lambda^{-3} t} dt | + \lambda^{-3} | \int_0^{(l+1)\delta} \Lambda_3^2 (\sigma_3) e^{2\gamma \lambda^{-3} t} dt | \]

\[ + \int_0^{(l+1)\delta} \Lambda_4 (M_4; v) e^{2\gamma \lambda^{-3} t} dt |). \]

Therefore, it remains to prove

\[ \sup_{t \in [0,(l+1)\delta]} |(\Lambda_3 (\sigma_3; v))| e^{2\gamma \lambda^{-3} t} \leq \| I' v_0\|_{L^2}^2 + \frac{1}{\gamma^2} \| I' g\|_{L^2}^2 e^{2\gamma \lambda^{-3} (l+1)\delta}, \] (44)

\[ \int_0^{(l+1)\delta} \Lambda_3 (\sigma_3; v) e^{2\gamma \lambda^{-3} t} dt \leq \| I' v_0\|_{L^2}^2 + \frac{1}{\gamma^2} \| I' g\|_{L^2}^2 e^{2\gamma \lambda^{-3} (l+1)\delta}, \] (45)

\[ \lambda^{-3} \int_0^{(l+1)\delta} \Lambda_3^2 (\sigma_3) e^{2\gamma \lambda^{-3} t} dt \leq \| I' v_0\|_{L^2}^2 + \frac{1}{\gamma^2} \| I' g\|_{L^2}^2 e^{2\gamma \lambda^{-3} (l+1)\delta}, \] (46)

and

\[ \int_0^{(l+1)\delta} \Lambda_4 (M_4; v) e^{2\gamma \lambda^{-3} t} dt | v \leq \| I' v_0\|_{L^2}^2 + \frac{1}{\gamma^2} \| I' g\|_{L^2}^2 e^{2\gamma \lambda^{-3} (l+1)\delta}. \] (47)

By Lemma 4.2, Proposition 10 and (43), we know that

\[ \sup_{t \in [0,(l+1)\delta]} |(\Lambda_3 (\sigma_3; v))| e^{2\gamma \lambda^{-3} t} \]

\[ \lesssim (l\delta)^{\frac{1}{2}} (N')^{-2+} \sum_{m=0}^{l} \| I' v\|_{L^2}^2 e^{2\gamma \lambda^{-3} (m+1)\delta} \]

\[ \lesssim (l\delta)^{\frac{1}{2}} (N')^{-2+} \sum_{m=0}^{l} (C_1 C_2)^{\frac{1}{2}} C_1 (\| I' v_0\|_{L^2}^2 + \frac{1}{\gamma^2} \| I' g\|_{L^2}^2 e^{2\gamma \lambda^{-3} (m+1)\delta}) \]
which is possible because \( C_2 \ll 1 \) and \( C_4 \gg 1 \). Furthermore, we can get (45) and (46) in a similar way.

From Proposition 8, it holds

\[
\left| \int_0^{(t+1)\delta} \Lambda_4(M_4;v) e^{2\gamma \lambda^3 t} dt \right| \\
\lesssim (N')^{-1} \sum_{m=0}^t \| I'v \|_{\gamma v_{\max, (m+1)\delta}}^4 e^{2\gamma \lambda^3 (m+1)\delta} \\
\lesssim (N')^{-1} \sum_{m=0}^t C_1 C_2 (\| I'v_0 \|_{L^2}^2 + \frac{1}{\gamma^2} \| I'g \|_{L^2}^2 e^{2\gamma \lambda^3 (t+1)\delta}) \\
\lesssim C_1 C_2 C_4^{-1} \delta^{-1} (\| I'v_0 \|_{L^2}^2 + \frac{1}{\gamma^2} \| I'g \|_{L^2}^2 e^{2\gamma \lambda^3 (t+1)\delta}),
\]

which gives (47) by taking \( C_4 \) sufficiently large and \( C_2 \) sufficiently small. \( \square \)

**Proposition 12.** Let \( C_2 \ll 1, C_3 > 1, C_4 \gg 1 \) and \( T > 0 \) be given. Assume \( u \) is a solution of (2) on \([0,T]\). If \( N^{1-} \geq \gamma, N^{0+} \geq C_4 T \), and

\[
\| Iu_0 \|_{L^2}^2 + \frac{1}{\gamma^2} \| If \|_{L^2}^2 e^{2\gamma T} \leq C_2 N^{1-}.
\]

then we have

\[
\| Iu(T) \|_{L^2}^2 e^{2\gamma T} \leq C_3 (\| Iu_0 \|_{L^2}^2 + \frac{1}{\gamma^2} \| If \|_{L^2}^2 e^{2\gamma T}).
\]

**Proof.** Due to \( \| I'v \|_{L^2}^2 = \lambda^{-1} \| Iu \|_{L^2}^2 \) and \( \| I'g \|_{L^2}^2 = \lambda^{-1} \| If \|_{L^2}^2 \), Proposition 12 can be acquired easily via taking \( \lambda = N^{1-}, N' = \frac{N}{\lambda} \) and \( T' = \lambda^{3} T \) in Proposition 11. \( \square \)

Next, we show the global dynamic of the solution.

**Proof of Theorem 1.2.** Fixing \( 0 < \epsilon < 1 + 8s \), we choose \( T_1 > 0 \) so that

\[
e^{2\gamma T_1} > \gamma^2 \| f \|_{L^2}^2 \quad \max \left\{ \gamma \frac{8s}{1+\epsilon}, (C_4 T_1)^{-\frac{4s}{1+\epsilon}}, \left( \frac{C_2}{2} \| u_0 \|_{H^{s,0}} - \frac{4s}{1+\epsilon} \right), \left( 2C_2^{-1} \gamma^{-2} \| f \|_{L^2}^2 e^{2\gamma T_1} \right)^{-\frac{8s}{1+\epsilon}} \right\} \leq \left( \frac{C_2}{2} \| u_0 \|_{H^{s,0}} \right)^{-\frac{4s}{1+\epsilon}},
\]

which is possible because \( \frac{8s}{1+\epsilon} < 1 \). \( T_1 \) depends only on \( \| u_0 \|_{H^{s,0}}, \| f \|_{L^2} \) and \( \gamma \).

Set

\[
N = \max \left\{ \gamma \left( \frac{1}{8s} \right), (C_4 T_1)^{-\frac{4s}{1+\epsilon}}, \left( \frac{C_2}{2} \| u_0 \|_{H^{s,0}} \right)^{-\frac{4s}{1+\epsilon}}, \left( 2C_2^{-1} \gamma^{-2} \| f \|_{L^2}^2 e^{2\gamma T_1} \right)^{-\frac{8s}{1+\epsilon}} \right\},
\]

then we have

\[
N^{\frac{3(1-s)}{1+\epsilon}} \geq \gamma, \quad N' \geq C_4 T_1,
\]
and
\[ \|Iu_0\|_{L^2}^2 \leq N^{-2s}\|u_0\|_{H^{s,0}}^2 \leq \frac{C_2}{2} N^{\frac{1-s}{4}}, \]
\[ \gamma^{-2}\|If\|_{L^2}^2 e^{2\gamma T_1} \leq \frac{C_2}{2} N^{\frac{1-s}{4}}. \]

Hence, from Proposition 12, one gains
\[ \|u(T_1)\|_{H^{s,0}}^2 \leq \|Iu(T_1)\|_{L^2}^2 \]
\[ \leq C_3(\|u_0\|_{L^2}^2 e^{-2\gamma T_1} + \frac{1}{\gamma^2} \|If\|_{L^2}^2) \]
\[ \leq C_3(N^{-2s}\|u_0\|_{H^{s,0}}^2 e^{-2\gamma T_1} + \frac{1}{\gamma^2} \|f\|_{L^2}^2). \]

From (48) and (49), we know
\[ N^{-2s}e^{-2\gamma T_1}\|u_0\|_{H^{s,0}}^2 < \frac{1}{\gamma^2} \|f\|_{L^2}^2. \]

This gives the bound
\[ \|u(T_1)\|_{H^{s,0}}^2 \leq \frac{2C_2}{\gamma^2} \|f\|_{L^2}^2 < K_1, \]
where \( K_1 \) depends only on \( \|f\|_{L^2} \) and \( \gamma \).

Then, one can fix \( T_2 > 0 \) and solve (2) on time interval \([T_1, T_1 + T_2]\) with initial data \( u(T_1) \). Let \( K_2 > 0 \) be sufficiently large such that
\[ K_2 e^{2\gamma t} > \max \left\{ \gamma^\frac{8s}{(\gamma^{4s} - 8s)}, (C_4 t)^{\frac{2s}{2s}}, (2C_2^{-1} K_1)^{\frac{8s}{2s + 2}}, \frac{2C_2^{-1} \gamma^{-2} \|f\|_{L^2}^2 e^{2\gamma t}}{\gamma^\frac{8s}{(\gamma^{4s} - 8s)}} \right\}, \]
(50)
for any \( t > 0 \), where \( K_2 \) depends only on \( \|f\|_{L^2} \) and \( \gamma \). Set \( N^{-2s} = K_2 e^{2\gamma T_2} \), then the assumptions in Proposition 12 are verified by (50). Therefore, we obtain
\[ \|Iu(T_1 + T_2)\|_{L^2}^2 \leq C_3(\|Iu(T_1)\|_{L^2}^2 e^{-2\gamma T_2} + \frac{1}{\gamma^2} \|If\|_{L^2}^2) \]
\[ \leq C_3(N^{-2s}\|u(T_1)\|_{H^{s,0}}^2 e^{-2\gamma T_2} + \frac{1}{\gamma^2} \|f\|_{L^2}^2) \]
\[ \leq C_3(K_1 K_2 + \frac{1}{\gamma^2} \|f\|_{L^2}^2) \]
\[ < K_3, \]
where \( K_3 \) depends only on \( \|f\|_{L^2} \) and \( \gamma \).

Define the maps \( M_1(t) \) and \( M_2(t) \) for \( t > T_1 \) as
\[ M_1(t)u_0 = \hat{A}(t)u_0|_{|\xi|<N}, \quad M_2(t)u_0 = \hat{A}(t)u_0|_{|\xi|>N}, \]
where \( A(t)u_0 = u(t) \) and \( N = (K_2 e^{2\gamma(t-T_1)})^{-\frac{1}{2}} \), then we have
\[ \|M_1(t)u_0\|_{L^2}^2 \leq \|Iu(t)\|_{L^2}^2 < K_3, \]
\[ \|M_2(t)u_0\|_{H^{s,0}}^2 \leq N^{2s}\|Iu(t)\|_{L^2}^2 < K_2^{-1} K_3 e^{-2\gamma(t-T_1)}. \]

Hence, taking \( K = \max\{K_2^{\frac{1}{2}}, K_2^{-\frac{1}{2}}, K_3^{\frac{1}{2}}\} \), one can obtain Theorem 1.2. \( \square \)
5. **Global attractor in weak topology.** In order to define an infinite-dimensional dynamical system from the evolution equation (2), first of all we should make sure that the corresponding initial value problem is well-posed in $H^{s,0}$.

**Proposition 13.** Let $-\frac{3}{8} < s < 1$. Assume $u_0 \in H^{s,0}$, then there exists $T = T(\|u_0\|_{H^{s,0}}) > 0$ and
\[
u(x,y,t) \in C([0,T],H^{s,0}(\mathbb{R}^2))
\]
such that $u(x,y,t)$ is the unique solution of (2).

**Proof.** By Duhamel’s formulation
\[
u = \chi(t)\eta^{(s)}u_0 - \int_0^t \eta^{(t-t')}\chi(t)(\frac{1}{2}\partial_x u^2 + \gamma u - f)dt'.
\]
From the definition of $Y^s$ and the duality of $U^p$, one knows that
\[
\|\int_0^t \eta^{(t-t')}\chi(t)(\gamma u - f)dt'\|_{Y^s} \\ \lesssim \left(\sum_{N_1} N_1^{2s} \sup_{\|w\|_Y \leq 1} \|\int_{\mathbb{R}^3} \chi(t)(\gamma P_{N_1} u - P_{N_1} f)wdxv dt\|_x^2\right)^{\frac{1}{2}} \\ \lesssim \left(\sum_{N_1} N_1^{2s} \sup_{\|w\|_Y \leq 1} \|\gamma P_{N_1} u - \chi(t)P_{N_1} f\|_{L_x^2 L_y^2} \|\chi(t)w\|_{L_t^1 L_x^1 L_y^1}\right)^{\frac{1}{2}} \\ \lesssim \left(\sum_{N_1} N_1^{2s} \|\gamma P_{N_1} u\|_{L_x^2}^2 + \|P_{N_1} f\|_{L_x^2}^2\right)^{\frac{1}{2}} \\ \approx \gamma \|u\|_{Y^s} + \|f\|_{L_x^2}. \tag{51}
\]
Set
\[
B = \{u \in Y^s \mid \|u\|_{Y^s} < C_0(\|u_0\|_{L_x^2} + \|f\|_{L_x^2})\},
\]
(20) and (51) give
\[
\|u\|_{Y^s} \lesssim \|u_0\|_{L_x^2} + \|u\|_{Y^s}^2 + \gamma \|u\|_{Y^s} + \|f\|_{L_x^2} \\ \lesssim C_0^2(\|u_0\|_{L_x^2} + \|f\|_{L_x^2})^2 + (1 + C_0\gamma)\|u_0\|_{L_x^2} + \|f\|_{L_x^2}, \tag{52}
\]
and
\[
\|u_1 - u_2\|_{Y^s} \lesssim (\gamma + \|u_1\|_{Y^s} + \|u_2\|_{Y^s})\|u_1 - u_2\|_{Y^s} \lesssim (\gamma + C_0(\|u_0\|_{L_x^2} + \|f\|_{L_x^2}))\|u_1 - u_2\|_{Y^s}. \tag{53}
\]
We rescale equation (2) to get (29) for constructing a strict contraction mapping on $\bar{B}$, where
\[
\bar{B} = \{v \in Y^s \mid \|v\|_{Y^s} < C_0(\|v_0\|_{L_x^2} + \lambda^{-3}\|g\|_{L_x^2})\}.
\]
Similarly to (52) and (53),
\[
\|v\|_{Y^s} \lesssim \|v_0\|_{L_x^2} + \|v\|_{Y^s}^2 + \gamma \lambda^{-3}\|v\|_{Y^s} + \lambda^{-3}\|g\|_{L_x^2} \\ \lesssim C_0^2(\|v_0\|_{L_x^2} + \lambda^{-3}\|g\|_{L_x^2})^2 + (1 + C_0\gamma \lambda^{-3})\|v_0\|_{L_x^2} + \lambda^{-3}\|g\|_{L_x^2}, \tag{54}
\]
and
\[
\|v_1 - v_2\|_{Y^s} \lesssim (\gamma \lambda^{-3} + \|v_1\|_{Y^s} + \|v_2\|_{Y^s})\|v_1 - v_2\|_{Y^s} \lesssim (\gamma \lambda^{-3} + C_0(\|v_0\|_{L_x^2} + \lambda^{-3}\|g\|_{L_x^2}))\|v_1 - v_2\|_{Y^s}. \tag{55}
\]
Choosing $\lambda$ sufficiently large, we have
\[
\gamma \lambda^{-3} \ll 1,
\]
Hence there exists a unique solution \( v \) on \([0, 1]\) from the fixed point argument. Thus, (2) is locally well-posed on \([0, T]\) by taking \( T = \lambda^{-3} \).

Next, we consider the weakly continuous dependence of solution on initial data in \( H^{s, 0} \) for the Cauchy problem of (2). Without loss of generality, we may assume \( \gamma = 0 \) and \( f = 0 \).

We put \( D_x u = F^{-1} [\xi \hat{u}(\xi)] \). The derivative operators \( D_y \) and \( D_t \) are similarly defined. We begin with the following smoothing estimate (see [9, Lemma 3.2 b) on page 1126]).

**Lemma 5.1.** Let \( 0 \leq a \leq 1 \).

\[
\left\| (D_x)^a (D_x^{-1} D_y)^{(1-a)} e^{\tau S} u_0 \right\|_{L_T^\infty L_x^2} \lesssim \left\| u_0 \right\|_{L_x^2_y}.
\]

Lemma 5.1 with \( a = 1/3 \) and \( a = 1/2 \) and the interpolation yield the following lemma.

**Lemma 5.2.** (i) Let \( 0 < \theta < 1 \) and \( 1/p = (1 - \theta)/2 \).

\[
\left\| (D_x^{-1} D_y^2)^{(\theta/3)} f \right\|_{L_T^\infty L_x^2} \lesssim \left\| f \right\|_{X^{0, \theta/2+}}.
\]

(ii) Let \( 0 < \theta < 1/2 \) and \( 1/p = (1 - 2\theta)/2 \).

\[
\left\| D_y^\theta f \right\|_{L_T^\infty L_x^2} \lesssim \left\| f \right\|_{X^{0, +}}.
\]

We now explain how to show that if a sequence of initial data \( \{u_{0n}\} \) converges to \( u_0 \) weakly in \( H^{s, 0} \) and a sequence of solutions \( \{u_n\} \) with \( u_0(0) = u_{0n} \) converges to a solution \( u \) with \( u(0) = u_0 \) weakly in \( X_T^{s, 0, 1/2+} \) for \( s > -1/2 \), then \( u_n(t) \) converges to \( u(t) \) weakly in \( H^{s, 0} \) for each \( t \in [0, T] \). Here we use \( X_T^{s, 0, 1/2+} \) instead of \( Y^s \) to avoid being too lengthy and tedious. The prerequisite “if” part can be proved through the standard local existence theorem of solution (see Remark 1 below).

Let \( \psi \in C_0^\infty (\mathbb{R}^2) \) and let \( T' \) be a fixed positive constant with \( 0 < T' < T \). The Duhamel formula yields

\[
(e^{-TS}(u_n(T') - u(T')), \psi)
= (u_{0n} - u_0, \psi) + 2 \int_0^{T'} (e^{-tS}((u_n - u)(u_n + u))(t), \partial_x \psi) \, dt
= (u_{0n} - u_0, \psi) + 2(\langle u_n - u \rangle (u_n + u), f),
\]

where we write \( u \) and \( u_n \) for \( \chi_{\{0 \leq t \leq T'\}} u \) and \( \chi_{\{0 \leq t \leq T'\}} u_n \), moreover we put \( f = \chi_{\{0 \leq t \leq T'\}} e^{tS} \partial_x \psi \). It suffices to prove that the the last term on the right hand side of (56) converges to zero as \( n \to \infty \), since \( \{u_n\} \) is bounded in \( L^\infty(0, T; H^{s, 0}) \).

We put \( \sigma = \tau - S(\xi, \eta) \). By \( P_N \), we denote the projection : \( u \in L^2(\mathbb{R}_x) \mapsto F^{-1} [\chi_{\{\xi \leq N\}}(\xi) \hat{u}(\xi)] \). We set \( u_N = P_N u \) and \( u_{\geq N} = (I - P_N) u \). We define a cut-off function \( \varphi(x, y, t) \in C_0^\infty(\mathbb{R}^3) \) as follows.

\[
\varphi(x) = \begin{cases} 
1, & x^2 + y^2 + t^2 \leq 1, \\
0, & x^2 + y^2 + t^2 \geq 4.
\end{cases}
\]
For $R > 1$, we put $\varphi_R(x, y, t) = \varphi(x/R, y/R, t/R)$. We first note that
\[
(\tau - \hat{S}(\xi, \eta)) + (\tau_1 - \hat{S}(\xi_1, \eta_1)) + (\tau_2 - \hat{S}(\xi_2, \eta_2)) = -\xi_1 \xi_2 \left(3 + \frac{(|\eta_1/\xi_1 - \eta_2/\xi_2|^2)}{\xi_2^2}\right),
\]
where $\tau = \tau_1 + \tau_2$ and $\xi = \xi_1 + \xi_2$. The quadratic nonlinear interaction of $u$ and $v$ is decomposed into the following four terms.
\[
uv = u_{>N}v_{>N} + u_{N}v_{>N} + u_{>N}v_{N} + u_{N}v_{N}.
\]
It follows from (57) that the first three terms on the right hand side can be made arbitrarily small as $N$ gets larger and larger. So, we have only to consider the term $u_{N}v_{N}$. The contribution of the term $u_{N}v_{N}$ can be divided into the following two terms.
\[
\langle u_{N}v_{N}, f \rangle = \langle u_{N}v_{N}, (1 - \varphi_R)f \rangle + \langle u_{N}v_{N}, \varphi_R f \rangle =: I_1 + I_2.
\]
For $I_1$, we have
\[
|I_1| \leq C(R), \quad R > 1,
\]
where $C(R) \to 0$ as $R \to \infty$. We have by H"older’s inequality
\[
|I_2| \leq \|u_N\|_{L^2_{x,y,t}(|z|<2R)} \|v_N\|_{L^2_{x,y,t}(|z|<2R)} \|f\|_{L^\infty_{x,y,t}},
\]
where $z = (x, y, t) \in \mathbb{R}^3$.

We now show that the mapping $u \mapsto u_N$ is compact from $X^{0,1/4+}$ to $L^2_{x,y,t}(|z|<2R)$ for each $R > 1$. We first note that
\[
|\tau| \leq |\sigma| + |\xi|^3 + |\xi|^{-1}|\eta|^2.
\]
Hence, for $\varepsilon > 0$, we have
\[
\|D^\varepsilon u\|_{L^2(\mathbb{R}^3)} \lesssim \|u\|_{X^{\alpha, \varepsilon}} + \|D^\varepsilon u\|_{L^2(\mathbb{R}^3)} + \|(D^\varepsilon_1 D^\varepsilon_2)^\varepsilon u\|_{L^2(\mathbb{R}^3)}.
\]
Let $\mathcal{R}$ be a mapping which restricts function $f$ on $\mathbb{R}^3$ to $f_1(|z|<2R)$. Since we have the compact embedding
\[
\mathcal{R}(1 + D_x + D_y + D_t)^{-\varepsilon} L^2_{x,y,t}(\mathbb{R}^3) \subset L^2_{x,y,t}(|z|<2R), \quad \varepsilon > 0,
\]
we can conclude by Lemma 5.2 (i) and (ii) that the mapping $u \mapsto u_N$ is compact from $X^{0,1/4+}$ to $L^2_{x,y,t}(|z|<2R)$ for small $\varepsilon > 0$. Therefore, the weakly continuous dependence in $H^{s,0}$, $s > -1/2$ of solution on initial data follows from the above argument and (56).

Remark 1. Let $T$ be a positive constant which denotes the existence time of solution to (2). For the weakly continuous dependence, we consider the integral equation associated with the Cauchy problem (2).
\[
u(t) = \alpha(t)e^{tS}u_0 - \alpha(t)\int_0^t e^{(t-t')S} \partial_x(\alpha_T(t')u(t'))^2 dt', \quad t \in \mathbb{R},
\]
where $\alpha$ is a cut-off function in $C_0^\infty(\mathbb{R})$ such that $\alpha(t) = 1$ ($|t| < 1$) and $\alpha(t) = 0$ ($|t| > 2$) and $\alpha_T(t) = \alpha(t/T)$. All what we need to do is to prove that if a sequence of initial data $\left\{u_{0n}\right\}$ converges to $u_0$ weakly in $H^{s,0}$ for $s > -1/2$, a solution $u$ of (60) is given by a weak limit of the sequence of solutions $u_n$ of (60) with $u_0$.
replaced by $u_{0n}$. In fact, we can extract a subsequence $\{u_{n'}\}$ converges to some $u$ weakly in $H^{s, 0}$, since a sequence of solutions $\{u_n\}$ of (60) with $u_0$ replaced by $u_{0n}$ is bounded in $H^{s, 0}$ for $s > -1/2$ (see [4, Theorem 1.1 on page 6556]). Furthermore, by the duality argument similar to above, we can easily see that $u$ is a solution of (60), which is unique in $H^{s, 0}$ for $s > -1/2$ (see [4, Theorem 1.1 on page 6556]). Therefore, the whole sequence of $\{u_n\}$ converges to $u$ weakly in $H^{s, 0}$.

**Proof of Corollary 1.** By Proposition 13 the initial value problem is well-posed in $H^{s, 0}$, hence one can define an infinite-dimensional dynamical system from the evolution equation (2). Moreover, we get the weakly continuous in $H^{s, 0}$ by Remark 1. From Theorem 1.2, we know that $M_1(t)$ is a bounded mapping and $M_2(t)$ converges uniformly to 0 in $H^{s, 0}$. It means that the semi-group $A(t)$ is asymptotically compact in the sense of weak topology. Therefore we gain the existence of the global attractor in $H^{s, 0}$ by Remark 1.4 and Theorem 1.1 in [21].

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