Estimates on the Lower Bound of the First Gap *

Jun LING

Abstract

We give a new lower bound for the first gap \( \lambda_2 - \lambda_1 \) of the Dirichlet eigenvalues of the Schrödinger operator on a bounded convex domain \( \Omega \) in \( \mathbb{R}^n \) or \( S^n \) and greatly sharpens the previous estimates. The new bound is explicit and computable.

1 Introduction

In this paper, we give a new estimate on the lower bound of the gap of the first two Dirichlet eigenvalues of the Schrödinger operator on a bounded strictly convex domain \( \Omega \) in \( \mathbb{R}^n \) or \( S^n \). Let \( \lambda_1 \) and \( \lambda_2 \) be the first two Dirichlet eigenvalues of the Schrödinger operator \(-\Delta + V\) to the eigenvalue problem

\[
-\Delta u + Vu = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]

where \( \Delta \) is the Laplacian on \( \Omega \), \( V : \Omega \rightarrow \mathbb{R}^1 \) a nonnegative convex smooth function, \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) or \( S^n \) whose second fundamental form of the boundary with respect to the outward normal to the boundary is positive definite. It is an interesting and important problem to find a lower bound for the first gap \( \lambda_2 - \lambda_1 > 0 \). There has been a lot of work on this problem. See [1] and [2] for the references on the problem. In 1983, M. van den Berg [7] conjectured that the lower bound is \( 3\pi^2/d^2 \), where \( d \) is the diameter of the domain. See also S. T. Yau’s Problem Section in [21] and M. Ashbaugh [1] about the conjecture. In recent years, M. Ashbaugh and R. Benguria [2,3] and R. Bañuelos and P.J. Méndez-Hernández [6], R. Bañuelos and P. Kroger [5], and B. Davis [11] proved the conjecture for some special potential \( V \) and for some special class of symmetric domains in \( \mathbb{R}^2 \). For a general bounded convex domain in \( \mathbb{R}^n \), Singer, Wong, Yau and Yau [23] showed that \( \lambda_2 - \lambda_1 \geq \pi^2/(4d^2) \). Yu and Zhong [26] later removed the

*2000 Mathematics Subject Classification Primary 35J10; Secondary 35P15, 53C21
factor 4, using the interior log-convexity of a positive first eigenfunction. Lee and Wang [14] showed that one still has interior log-convexity if the domain is in $S^n$ and therefore the above estimate remains true for the Laplacian on a bounded convex domain in $S^n$. The author [18] proved that global log-convexity holds if the domain in $R^n$ or $S^n$ is strictly convex and therefore one has the strict lower bound $\lambda_2 - \lambda_1 > \pi^2/d^2$. R. G. Smits [22] gave an alternative derivation of the last inequality. In this paper, we give a new general bound for a general bounded convex domain. The new bound is explicit and computable.

In [25], S. T. Yau gave an estimate $\lambda_2 - \lambda_1 \geq \theta \pi^2/d^2 + 2 \cos^2(\sqrt{\theta} \pi) \alpha$, where $\theta$ is any constant with $0 \leq \theta \leq 1/4$, and $\alpha > 0$ is a quantity determined by the global log-convexity and is defined in (2). Let us first recall the notion of "log-convexity". Let $f$ be a positive first eigenfunction. Let $\alpha(x) = \inf_{\tau \in T_x \Omega, |\tau| = 1} \left[ \nabla^2 (\ln f) \right] (\tau, \tau) (x)$. It is known from the work of Brascamp and Lieb [8], Caffarelli and Friedman [9], Korevaar [12], Korevaar and Lewis [13], and Lee and Wong [14] that the interior log-convexity $\alpha(x) > 0$ for $x$ in $\Omega$ holds. Let $\alpha$ be the constant

$$\alpha = \inf_{x \in \Omega} \alpha(x).$$

The author [18] showed that the global log-convexity $\alpha > 0$ holds. Yau [25] gave an interesting estimate on the lower bound $\alpha$ in terms of the potential $V$.

Our main result is the following Theorem [11]

**Theorem 1.** If $\Delta$ is the Laplacian in $R^n$ or $S^n$ and if $\lambda_1$ and $\lambda_2$ are the first two Dirichlet eigenvalues of the Schrödinger operator $-\Delta + V$ with a nonnegative convex potential $V$ on a bounded strictly convex domain $\Omega$ in $R^n$ or $S^n$, then we have the following estimate

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2} + \frac{31}{50} \alpha,$$

where $\alpha$ is the quantity in (2) and $d$ is the diameter of $\Omega$.

**Remark 1.** If the domain $\Omega$ has a certain symmetric property that the "midrange" of the ratio of eigenfunctions is zero, then Theorem [11] shows

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2} + \alpha.$$

**Remark 2.** The above result holds when $\Omega$ is in a manifold with non-negative Ricci curvature and positive $\alpha$. 

2
In order to improve the known results on the gap estimate, we need to construct suitable test functions where detailed technical work is essential. In the last section we construct the test function \( \xi \). We explore the properties of the function \( \xi \), the Zhong-Yang function \( \eta \) and the ratio \( \xi / \eta \). Those properties are essential to the construction of the suitable test functions. Because those functions are complicated combinations of trigonometric and rational functions, the needed properties such as monotonic and convex properties are hard to prove. In the past, though we know that many nice properties might be true, only a few of them could be proven strictly in mathematics by the canonical calculus method. We are able to prove those properties effectively now by studying the differential equations those functions satisfied and using the Maximum Principle. Since the constructions and proofs in that part are quite technical by nature, we put them in the last section. Readers may refer to that section when in need. The functions \( \xi \) and \( \eta \) and their properties have also other important applications. In our recent estimates on the first non-zero eigenvalues of compact manifolds, function \( \xi \) plays major role. We prove the main result in Section 2. Last section is for deriving some preliminary estimates and the conditions for test functions and for proving the properties of the functions used in the proof of the Theorem 1.

2 Proof of the Main Result

Let \( f_2 \) be the second eigenfunction and \( f \) a positive first eigenfunction of Problem \( \Pi \). It is known \( f_2 / f \) changes its signs in \( \Omega \) (see [10]) and is smooth up to the boundary \( \partial \Omega \) (see [23]) and can be normalized so that

\[
\sup_{\Omega} f_2 / f = 1, \quad \inf_{\Omega} f_2 / f = -k, \quad \text{and} \quad 0 < k \leq 1.
\]

Let \( \lambda = \lambda_2 - \lambda_1 \),

\[
(3) \quad v = \left[ f_2 / f - (1 - k)/2 \right] / [(1 + k)/2].
\]

Then the function \( v \) satisfies the following

\[
(4) \quad \Delta v = -\lambda(v + a) - 2\nabla v \nabla (\ln f) \quad \text{in} \ \Omega,
\]

\[
(5) \quad \frac{\partial v}{\partial N} = 0 \quad \text{on} \ \partial \Omega,
\]
\[ (6) \quad \sup_{\Omega} v = 1 \quad \text{and} \quad \inf_{\Omega} v = -1 \]

where \( N \) is the outward normal of \( \partial \Omega \), and

\[ (7) \quad a = (1 - k)/(1 + k) \]

is the "midrange" of the ratio \( f_2/f \). Note that \( 0 \leq a < 1 \).

We set in this paper, unless otherwise stated,

\[ (8) \quad \lambda = \lambda_2 - \lambda_1 \quad \text{and} \quad \delta = \alpha/\lambda. \]

and let

\[ Z(t) = \max_{x \in \overline{\Omega}, t = \sin^{-1}(v(x)/b)} \frac{\| \nabla v \|^2}{b^2 - v^2}/\lambda. \]

for \( t \in [-\sin^{-1}(1/b), \sin^{-1}(1/b)] \).

We have the following estimates (9) and (10), Lemma 1 and corollaries 1 and 2. The proofs are in the last section.

\[ (9) \quad 0 < \delta \leq \frac{1}{2}. \]

\[ (10) \quad Z(t) \leq 1 + a \quad t \in [-\sin^{-1}(1/b), \sin^{-1}(1/b)]. \]

**Lemma 1.** If the function \( z : [-\sin^{-1}(1/b), \sin^{-1}(1/b)] \mapsto \mathbb{R}^1 \) satisfies the following

1. \( z(t) \geq Z(t) \quad t \in [-\sin^{-1}(1/b), \sin^{-1}(1/b)] \),
2. there exists some \( x_0 \in \Omega \) such that at point \( t_0 = \sin^{-1}(v(x_0)/b) \)
   \[ z(t_0) = Z(t_0), \]
3. \( z(t_0) > 0 \),

then we have the following

\[ 0 \leq \frac{1}{2} z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 - z(t_0) + 1 + c \sin t_0 - 2\delta \cos^2 t_0 \]

\[ - \frac{z'(t_0)}{4z(t_0)} \cos t_0 [z'(t_0) \cos t_0 - 2z(t_0) \sin t_0 + 2 \sin t_0 + 2c]. \]
Corollary 1. If in addition to the above conditions 1-3 in Lemma 1, $z'(t_0) \geq 0$ and $1 - c \leq z(t_0) \leq 1 + a$, then we have the following

$$0 \leq \frac{1}{2}z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 - z(t_0) + 1 + c \sin t_0 - 2\delta \cos^2 t_0.$$ 

Corollary 2. If $a = 0$, which is defined in (7), and if in addition to the above conditions 1-3 in Lemma 1, $z'(t_0) \sin t_0 \geq 0$ and $z(t_0) \leq 1$, then we have the following

$$0 \leq \frac{1}{2}z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 - z(t_0) + 1 + 2\delta \cos^2 t_0.$$ 

We now prove our result.

Theorem 2. If $a > 0$ and $\mu \delta \leq \frac{4}{\pi^2}a$ for a constant $\mu \in (0, 1)$, then

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2} + \mu \alpha$$

Proof. Let $\mu_\epsilon = \mu - \epsilon > 0$ for small positive constant $\epsilon$. Take $b > 1$ close to 1 such that $\mu_\epsilon \delta < \frac{4}{\pi^2}c$. Let

$$z(t) = 1 + c\eta(t) + \mu_\epsilon \delta \xi(t),$$

where $\xi$ and $\eta$ are the functions defined by (30) and (38), respectively. Let $\tilde{I} = [-\sin^{-1}(1/b), \sin^{-1}(1/b)]$. We claim that

$$Z(t) \leq z(t) \quad \text{for } t \in \tilde{I}.$$ 

By Lemma 3 and Lemma 4 we have

$$\frac{1}{2}z'' \cos^2 t - z' \cos t \sin t - z = -1 - c \sin t + 2\mu_\epsilon \delta \cos^2 t,$$

(15) $$z'(t) > 0$$

(16) $$0 < 1 - \frac{a}{b} = z(-\frac{\pi}{2}) \leq z(t) \leq z(\frac{\pi}{2}) = 1 + \frac{a}{b} \leq 1 + a,$$

where (15) is due to the following.

$$z'(t) = c\eta'(t) + \mu_\epsilon \delta \xi'(t) = \mu_\epsilon \delta \eta'(t) \left( \frac{c}{\mu_\epsilon \delta} + \frac{\xi'(t)}{\eta'(t)} \right)$$

$$\geq \mu_\epsilon \delta \eta'(t) (\frac{c}{\mu_\epsilon \delta} - \frac{\pi^2}{4}) > 0.$$
Let $P \in \mathbb{R}^1$ and $t_0 \in [-\sin^{-1}(1/b), \sin^{-1}(1/b)]$ such that
\[ P = \max_{t \in I} (Z(t) - z(t)) = Z(t_0) - z(t_0). \]

Thus
\[ (17) \quad Z(t) \leq z(t) + P \quad \text{for} \quad t \in \bar{I} \quad \text{and} \quad Z(t_0) = z(t_0) + P. \]

Suppose that $P > 0$ Then $z + P$ satisfies the inequality in Corollary 1 of Lemma 1. Then
\[
\begin{align*}
z(t_0) + P &= Z(t_0) \\
&\leq \frac{1}{2} (z + P)'(t_0) \cos^2 t_0 - (z + P)'(t_0) \cos t_0 \sin t_0 + 1 + c \sin t_0 - 2\delta \cos^2 t_0 \\
&= \frac{1}{2} z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 + 1 + c \sin t_0 - 2\delta \cos^2 t_0 \\
&\leq \frac{1}{2} z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 + 1 + c \sin t_0 - 2\mu \delta \cos^2 t_0 \\
&= z(t_0).
\end{align*}
\]

This contradicts the assumption $P > 0$. Thus $P \leq 0$ and (13) must hold.

Now we have
\[ |\nabla t|^2 \leq \lambda z(t) \quad \text{for} \quad t \in \bar{I}, \]
that is
\[ (18) \quad \sqrt{\lambda} \geq \frac{|\nabla t|}{\sqrt{z(t)}}. \]

Let $q_1$ and $q_2$ be two points in $\bar{\Omega}$ such that $v(q_1) = -1$ and $v(q_2) = 1$ and let $L$ be the minimum geodesic segment between $q_1$ and $q_2$. $L$ lies on $\bar{\Omega}$ completely, since $\bar{\Omega}$ is convex. We integrate both sides of (18) along $L$ and change variable and let $b \to 1$. Then
\[ (19) \quad \sqrt{\lambda} d \geq \int_L \frac{|\nabla t|}{\sqrt{z(t)}} dt = \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{z(t)}} dt \geq \frac{\left( \int_{-\pi/2}^{\pi/2} dt \right)^{3/2}}{\left( \int_{-\pi/2}^{\pi/2} z(t) dt \right)^{1/2}} \geq \left( \frac{\pi^3}{\int_{-\pi/2}^{\pi/2} z(t) dt} \right)^{1/2}. \]

Square the two sides. Then
\[ \lambda \geq \frac{\pi^3}{d^2 \int_{-\pi/2}^{\pi/2} z(t) dt}. \]
Now
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} z(t) \, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1 + a\eta(t) + \mu_\varepsilon \delta \xi(t)] \, dt = (1 - \mu_\varepsilon \delta)\pi, \]
where we used the facts that \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \eta(t) \, dt = 0 \) since \( \eta \) is an even function, and that \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \xi(t) \, dt = -\pi \) (by (33) in the Lemma 3). Therefore
\[ \lambda \geq \frac{\pi^2}{(1 - \mu_\varepsilon \delta) d^2} \quad \text{and} \quad \lambda \geq \frac{\pi^2}{d^2} + \mu_\varepsilon \alpha. \]
Letting \( \varepsilon \to 0 \), we get
\[ \lambda \geq \frac{\pi^2}{(1 - \mu \delta) d^2} \quad \text{and} \quad \lambda \geq \frac{\pi^2}{d^2} + \mu_\varepsilon \alpha. \]

\[ \square \]

**Theorem 3.** If the "midrange" \( a = 0 \), then
\[ (20) \quad \lambda_2 - \lambda_1 \geq \frac{\pi^2}{d^2} + \alpha. \]

**Proof.** Let
\[ y(t) = 1 + \delta \xi. \]
By Lemma 3 for \( -\frac{\pi}{2} < t < \frac{\pi}{2} \), we have
\[ (21) \quad \frac{1}{2} y'' \cos^2 t - y' \cos t \sin t - y = -1 + 2\delta \cos^2 t, \]
\[ (22) \quad y'(t) \sin t \geq 0, \quad \text{and} \]
\[ (23) \quad y(\pm \frac{\pi}{2}) = 1 \quad \text{and} \quad 0 < y(t) < 1. \]

We need only show that \( Z(t) \leq y(t) \) on \( [-\pi/2, \pi/2] \). If it is not true, then there is \( t_0 \) and a number \( P > 0 \) such that \( P = Z(t_0) - y(t_0) = \max Z(t) - y(t) \).

Note that \( y(t) + P \geq 1 - \frac{1}{2}(\frac{\pi^2}{d^2} - 1) + P > 0 \). So \( y + P \) satisfies the inequality in the Corollary 2 in Lemma 1.

Therefore
\[ y(t_0) + P = Z(t_0) \]
\[ \leq \frac{1}{2} (y + P)''(t_0) \cos^2 t_0 - (y + P)'(t_0) \cos t_0 \sin t_0 + 1 - 2\delta \cos^2 t_0 \]
\[ = \frac{1}{2} y''(t_0) \cos^2 t_0 - y'(t_0) \cos t_0 \sin t_0 + 1 - 2\delta \cos^2 t_0 \]
\[ = y(t_0). \]

This contradicts the assumption \( P > 0 \). The rest of the proof is similar to that of Theorem 2 just noticing that \( \delta \leq \frac{1}{2} < \frac{4}{\pi^2 - 1}. \)

\[ \square \]
Proof of Theorem 1 Since $0 \leq a < 1$, either $a = 0$ or $0 < a < 1$.

If $a = 0$, then we apply Theorem 3 to get the bound with $\mu = 1$,

$$\lambda \geq \frac{\pi^2}{d^2} + \alpha.$$

If $0 < a < 1$, then there are several cases altogether.

- (I): $a \geq \frac{\pi^2}{4} \delta$.
- (II): $a < \frac{\pi^2}{4} \delta$.
  - (II-a): $a \geq 0.765$.
  - (II-b): $0 < a < 0.765$.
    * (II-b-1): $a \geq 1.53 \delta$.
    * (II-b-2): $a < 1.53 \delta$.

For Case (I): $0 < a < 1$ and $a \geq \frac{\pi^2}{4} \delta$, we apply Theorem 3 for $\mu = 1$ to get the following lower bound

$$\frac{\pi^2}{d^2} + \alpha$$

For Case (II-a): $0.765 \leq a < \frac{\pi^2}{4} \delta$, we apply Theorem 2 with $\mu = \frac{4a}{\pi^2 \delta}$ since $(\frac{4a}{\pi^2 \delta}) \delta \leq \frac{1}{\pi} a$ and $0 < \frac{4a}{\pi^2 \delta} < 1$. Then

$$\lambda \geq \frac{\pi^2}{d^2} + \frac{4a}{\pi^2 \delta} \alpha = \frac{\pi^2}{d^2} + \frac{4a}{\pi^2} \lambda$$

Thus

$$\lambda \geq \frac{1}{1 - \frac{4a}{\pi^2} \frac{\pi^2}{d^2}}.$$

On the other hand we have bound (9),

$$\lambda \geq 2\alpha.$$

The above two estimates give

$$\lambda \geq \frac{\pi^2}{d^2} + \frac{4a}{\pi^2} 2\alpha \geq \frac{\pi^2}{d^2} + \frac{8(0.765)}{\pi^2} \alpha > \frac{\pi^2}{d^2} + \frac{31}{50} \alpha.$$

The theorem is proved in this case.
For Case (II-b-1): \(0 < a < 0.765\), \(a < \frac{\pi^2}{4}\delta\) and \(a \geq 1.53\delta\), we apply Theorem 2 with \(\mu = \frac{4}{\pi^2} a\) since \((\frac{4}{\pi^2} a) \leq \frac{4}{\pi^2} a\) and \(0 < \frac{4}{\pi^2} a < 1\). Then
\[
\lambda \geq \frac{\pi^2}{d^2} + \frac{4}{\pi^2} a \alpha \geq \frac{\pi^2}{d^2} + \frac{4}{\pi^2} 153 \alpha \quad \alpha > \frac{\pi^2}{d^2} + \frac{31}{50} \alpha,
\]
which is what we want to prove.

For the remaining Case (II-b-2): \(0 < a < 0.765\), \(a < \frac{\pi^2}{4}\delta\) and \(a < 1.53\delta\), we define a function \(z\) by
\[
z(t) = 1 + c\eta(t) + (\delta - \sigma c^2)\xi(t) \quad \text{on } [-\sin^{-1} \frac{1}{b}, \sin^{-1} \frac{1}{b}],
\]
where
\[
(24) \quad \sigma = \frac{\tau}{\left(\frac{3}{2} - \pi^2 \frac{2}{3} - \frac{\pi^2}{32} - \frac{1}{6} \frac{153}{100} - \frac{\pi^2}{32} \frac{153}{100} \frac{1}{[-1+(12-\pi^2)]^{1/2}}\right) c}
\]
and
\[
(25) \quad \tau = \frac{2}{3\pi^2} \left(\frac{4}{3(4-\pi)} + \frac{3(4-\pi)}{4} - 2\right).
\]

Let \(\bar{I} = [-\sin^{-1} \frac{1}{b}, \sin^{-1} \frac{1}{b}]\). We now show that
\[
(26) \quad Z(t) \leq z(t) \quad \text{on } \bar{I}.
\]

If \(26\) is not true, then there exists a constant \(P > 0\) and \(t_0\) such that
\[
P c^2 = \frac{Z(t_0) - z(t_0)}{-\xi(t_0)} = \max_{t \in [-\sin^{-1} \frac{1}{b}, \sin^{-1} \frac{1}{b}]} \frac{Z(t) - z(t)}{-\xi(t)}.
\]
Let \(w(t) = z(t) - P c^2 \xi(t) = 1 + c\eta(t) + m\xi(t)\), where \(m = \delta - \sigma c^2 - P c^2\). Then
\[
Z(t) \leq w(t) \quad \text{on } \bar{I} \quad \text{and} \quad Z(t_0) = w(t_0).
\]

By Lemma 2, \(w(t_0) > 0\). So \(w\) satisfies \(11\) in Lemma 1
\[
0 \leq -2(\sigma + P)c^2 \cos^2 t_0 - \frac{w'(t_0)}{4w(t_0)} \cos t_0 \left(\frac{8c}{\pi} \cos t + 4mt \cos t\right).
\]

We used \(31\), \(32\), \(39\) and \(11\) to get the above inequality. Thus
\[
(27) \quad \sigma + P \leq -\frac{w'(t_0)}{2c^2 w(t_0)} \left(\frac{2c}{\pi} + mt\right) = -\frac{\eta'(t_0)}{\pi w(t_0)} \left(1 + \frac{m \xi'(t_0)}{c \eta'(t_0)}\right) \left(1 + \frac{\pi m}{2c} t_0\right).
\]
The righthand side is not positive for \( t_0 \geq 0 \), by Lemmas 3 and 4. Thus \( t_0 < 0 \), and

\[
\begin{align*}
&= - \left(1 + \frac{m \xi'(t_0)}{c \eta'(t_0)} \right) \left(1 + \frac{\pi m}{2c} t_0 \right) \\
&= \frac{2 \xi'(t_0)}{\pi t_0 \eta'(t_0)} \left( \frac{\pi t_0 \eta'(t_0)}{2 \xi'(t_0)} + \frac{\pi m}{2c} t_0 \right) \left( -1 - \frac{\pi m}{2c} t_0 \right) \\
&\leq \frac{1}{4} \left( \frac{2 \xi'(t_0)}{\pi t_0 \eta'(t_0)} \right)^2 \left( \frac{\pi t_0 \eta'(t_0)}{2 \xi'(t_0)} - 1 \right)^2 \\
&= \frac{3(12 - \pi^2)}{8} \left( \frac{2 \xi'(t_0)}{\pi t_0 \eta'(t_0)} \right) \leq \frac{4}{3(4 - \pi)}.
\end{align*}
\]

By Lemmas 3 and 4 we have \( 2(3 - \frac{\pi^2}{8}) \leq \frac{\xi'(t)}{\tau} \leq \frac{4}{3} \) and \( 2(\frac{4}{\pi} - 1) \leq \eta'(t) \leq \frac{8}{3\pi} \). So

\[
\frac{3(12 - \pi^2)}{8} \leq \frac{2 \xi'(t_0)}{\pi t_0 \eta'(t_0)} \leq \frac{4}{3(4 - \pi)}.
\]

Note that the function \( f(t) = t + \frac{1}{t} - 2 \) achieves its maximum on \([A, B]\) not containing 0 at an endpoint. Therefore

\[
\left| - \left(1 + \frac{m \xi'(t_0)}{c \eta'(t_0)} \right) \left(1 + \frac{\pi m}{2c} t_0 \right) \right| \leq \frac{1}{4} \left( \frac{4}{3(4 - \pi)} + \frac{3}{3(4 - \pi)} - 2 \right).
\]

Now (27) becomes

\[
\sigma + P \leq \frac{\tau}{w(t_0)}.
\]

On the other hand, by Lemma 2

\[
\begin{align*}
\sigma + P &\geq \left(3 - \frac{\pi^2}{8} - \left(\frac{\pi^2}{32} \right) \frac{153}{100} \frac{200}{153} - \left(\frac{12 - \pi^2}{153} \right) \frac{(8}{3\pi} - \frac{\pi}{3})^2 \right) \frac{\tau}{\sigma} > 0.
\end{align*}
\]

Since \(-P \xi(t_0) \geq 0\), we have \( w(t_0) \geq z(t_0) \). This fact, (28) and (29) imply that for \( P > 0 \)

\[
\sigma + P < \sigma,
\]

which is impossible.

Therefore we have the estimate (26). Now we proceed as in the proof of Theorem 2. We get the following

\[
\lambda d^2 \geq \frac{\pi^3}{\pi [1 - (\delta - \sigma c^2)]}.
\]

10
Since $\delta - \sigma c^2 > 0.625\delta$ by Lemma 2, we have
\[
\lambda \geq \frac{1}{|1 - (\delta - \sigma c^2)|} \frac{\pi^2}{d^2} > \frac{1}{|1 - 0.625\delta|} \frac{\pi^2}{d^2}
\]
and
\[
\lambda \geq \frac{\pi^2}{d^2} + 0.625\alpha > \frac{\pi^2}{d^2} + \frac{31}{50}.
\]

We now present a Lemma that is used in the proof of the Theorem 1.

Lemma 2. If $a < 1.53\delta$ and $0 < a < 0.765$ then
\[
z(t) = 1 + c\eta(t) + \delta\xi(t) \geq \left(\frac{3}{2} - \frac{\pi^2}{8} - \frac{\pi^2}{32} + \frac{153}{600} \frac{1}{153} - \frac{\pi^2}{16} - \frac{\pi^2}{16}ight) c > 0,
\]
for $t \in [-\pi/2, \pi/2]$ and
\[
\delta - \sigma c^2 \approx 0.62516283437 > 0.625\delta,
\]
where $c = a/b$ and $b > 1$ is any constant and $\sigma$ is the constant in (24).

Proof. By Lemmas 5, Lemma 3 and 4, the function $z$ on $[-\pi/2, \pi/2]$ has a unique critical point $t_1 \in (-\pi/2, 0)$ if $0 < a < \frac{\pi^2}{4}\delta$ and $z$ is decreasing on $[-\pi/2, t_1]$ and increasing on $[t_1, \pi/2]$. Therefore
\[
\min_{[-\pi/2, \pi/2]} z = \min_{[-\pi/2, 0]} z = z(t_1).
\]
So we need only consider the restricted function $z|_{[-\pi/2, 0]}$ for the minimum.

Now first consider the Taylor expansion of $\xi$ at $0$ for $t \in [-\pi/2, 0]$. By Lemma 3, $\xi(0) = -\frac{\pi^2}{4} + 1$, $\xi'(0) = 0$ and $\xi''(0) = 2(3 - \frac{\pi^2}{4})$ and $\xi'''(t) < 0$ on $(-\pi/2, 0)$.

Thus
\[
\xi(t) = \xi(0) + \xi'(0) + \frac{\xi''(0)}{2!} t^2 + \frac{\xi'''(t_2)}{2!} t^3 \\
\geq \xi(0) + \xi'(0) + \frac{\xi''(0)}{2!} t^2 \\
= -(\frac{\pi^2}{4} - 1) + (3 - \frac{\pi^2}{4}) t^2,
\]
where $t_2$ is a constant in $(t, 0)$. Similarly, using the data $\eta(-\pi/2) = -1$, $\eta'(-\pi/2) = \frac{8}{3\pi}$ and $\eta''(t) > 0$ on $(-\pi/2, 0)$ (actually on $[-\pi/2, \pi/2]$), and the Taylor expansion of $\eta$ at $-\pi/2$, we have for $t \in [-\pi/2, 0]$,

\[
\eta(t) = \eta(-\pi/2) + \eta'(-\pi/2)(t + \pi/2) + \frac{\eta''(-\pi/2)}{2!}(t + \pi/2)^2 + \frac{\eta'''(t_3)}{3!}(t + \pi/2)^3
\]

\[
\geq \eta(-\pi/2) + \eta'(-\pi/2)(t + \pi/2) + \frac{\eta''(-\pi/2)}{2!}(t + \pi/2)^2
\]

\[
= -1 + \frac{8}{3\pi}(t + \pi/2) - \frac{1}{4}(t + \pi/2)^2
\]

\[
= -\left(\frac{\pi^2}{16} - \frac{1}{3}\right) + \frac{8}{3\pi} - \frac{\pi}{4}t - \frac{1}{4}t^2,
\]

where $t_3$ is some constant in $(-\pi/2, t)$. Therefore on $[-\pi/2, 0]$,

\[
z(t) = 1 + c\eta(t) + \delta \xi(t)
\]

\[
\geq 1 - \left(\frac{\pi^2}{16} - \frac{1}{3}\right)c - (\frac{\pi^2}{4} - 1)\delta + \left(\frac{8}{3\pi} - \frac{\pi}{4}\right)ct + \left[-\frac{1}{4}c + (3 - \frac{\pi^2}{4})\delta\right]t^2
\]

Let $\nu = 1.53$ and $a_0 = 0.765$. That $a \leq \nu \delta$ implies $c = a/b < \nu \delta$, where $b > 1$ is a constant. Using conditions \([2]\) \(\delta \leq \frac{n-1}{2n} < \frac{1}{2}\) and $a \leq a_0$, we get

\[
1 - \left(\frac{\pi^2}{16} - \frac{1}{3}\right)c - (\frac{\pi^2}{4} - 1)\delta
\]

\[
\geq 1 - \left(\frac{\pi^2}{16} - \frac{1}{3}\right)\nu \delta - (\frac{\pi^2}{4} - 1)\delta
\]

\[
\geq \frac{3}{2} - \frac{\pi^2}{8} - \left(\frac{\pi^2}{32} - \frac{1}{6}\right)\nu
\]

\[
> \left(\frac{3}{2} - \frac{\pi^2}{8} - \left(\frac{\pi^2}{32} - \frac{1}{6}\right)\nu\right)\frac{1}{a_0}c
\]

and

\[
1 + c\eta(t) + \delta \xi(t)
\]

\[
\geq \left(\frac{3}{2} - \frac{\pi^2}{8} - \left(\frac{\pi^2}{32} - \frac{1}{6}\right)\nu\right)\frac{1}{a_0}c + \left(\frac{8}{3\pi} - \frac{\pi}{4}\right)ct + \left[-\frac{1}{4}c + (3 - \frac{\pi^2}{4})\frac{1}{\nu}\right]t^2
\]

\[
= \left(\frac{3}{2} - \frac{\pi^2}{8} - \left(\frac{\pi^2}{32} - \frac{1}{6}\right)\nu\right)\frac{1}{a_0}c + \left(\frac{8}{3\pi} - \frac{\pi}{4}\right)t + \left[-\frac{1}{4} + (3 - \frac{\pi^2}{4})\frac{1}{\nu}\right]t^2
\]
\[
\geq \left( \frac{3}{2} - \frac{\pi^2}{8} - \left( \frac{\pi^2}{32} - \frac{1}{6} \right) \nu \right) \frac{1}{a_0} - \frac{\left( \frac{8}{3\pi} - \frac{\pi}{4} \right)^2}{4 \left[ -1 + \left( 12 - \pi^2 \right) \frac{1}{\nu} \right]} \right) c
\]
\[
\geq \left( \frac{3}{2} - \frac{\pi^2}{8} - \left( \frac{\pi^2}{32} - \frac{1}{6} \right) \nu \right) \frac{1}{a_0} - \frac{\left( \frac{8}{3\pi} - \frac{\pi}{4} \right)^2}{\left[ -1 + \left( 12 - \pi^2 \right) \frac{1}{\nu} \right]} \right) c > 0.5433 > 0.
\]

Let \( \tau \) be the constant in (25). Then
\[
\sigma c^2 = \frac{\tau c}{\left( \frac{3}{2} - \frac{\pi^2}{8} - \left( \frac{\pi^2}{32} - \frac{1}{6} \right) \nu \right) \frac{1}{a_0} - \frac{\left( \frac{8}{3\pi} - \frac{\pi}{4} \right)^2}{\left[ -1 + \left( 12 - \pi^2 \right) \frac{1}{\nu} \right]} c}
\]
\[
\leq \frac{\tau \nu \delta}{\left( \frac{3}{2} - \frac{\pi^2}{8} - \left( \frac{\pi^2}{32} - \frac{1}{6} \right) \nu \right) \frac{1}{a_0} - \frac{\left( \frac{8}{3\pi} - \frac{\pi}{4} \right)^2}{\left[ -1 + \left( 12 - \pi^2 \right) \frac{1}{\nu} \right]} c} \approx 0.374837516563 \delta
\]
and
\[
\delta - \sigma c^2 > 0.625 \delta.
\]

\[\square\]

3 Some Estimates and Lemmas

Lemma 3. Let
\[
(30) \quad \xi(t) = \frac{\cos^2 t + 2t \sin t \cos t + t^2 - \pi^2}{\cos^2 t} \quad \text{on} \quad \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right].
\]
Then the function $\xi$ satisfies the following

\begin{align}
(31) \quad & \frac{1}{2} \xi'' \cos^2 t - \xi' \cos t \sin t - \xi = 2 \cos^2 t \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \\
(32) \quad & \xi' \cos t - 2 \xi \sin t = 4 \cos t \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \\
(33) \quad & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \xi(t) \, dt = -\frac{\pi}{2},
\end{align}

$1 - \frac{\pi^2}{4} = \xi(0) < \xi(\frac{\pi}{2}) = 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$,

$\xi'$ is increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\xi'(\frac{\pi}{2}) = \pm \frac{2\pi}{3}$,

$\xi'(t) < 0$ on $(-\frac{\pi}{2}, 0)$ and $\xi'(t) > 0$ on $(0, \frac{\pi}{2})$,

$\xi''(\frac{\pi}{2}) = 2$, $\xi''(0) = 2(3 - \frac{\pi^2}{2})$ and $\xi''(t) > 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$,

$\left(\frac{\xi'(t)}{t}\right)' > 0$ on $(0, \pi/2)$ and $2(3 - \frac{\pi^2}{2}) \leq \frac{\xi'(t)}{t} < \frac{4}{3}$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$,

$\xi''(\frac{\pi}{2}) = \frac{8\pi}{15}$, $\xi''(t) < 0$ on $(-\frac{\pi}{2}, 0)$ and $\xi''(t) > 0$ on $(0, \frac{\pi}{2})$.

**Proof.** For convenience, let $q(t) = \xi'(t)$, i.e.,

\begin{align}
(34) \quad & q(t) = \xi'(t) = \frac{2(2t \cos t + t^2 \sin t + \cos^2 t \sin t - \frac{\pi^2}{4} \sin t)}{\cos^3 t}.
\end{align}

Equation (31) and the values $\xi(\pm \frac{\pi}{2}) = 0$, $\xi(0) = 1 - \frac{\pi^2}{4}$ and $\xi'(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3}$ can be verified directly from (30) and (31). The values of $\xi''$ at 0 and $\pm \frac{\pi}{2}$ can be computed via (31). By (32), $(\xi(t) \cos^2 t)' = 4t \cos^2 t$. Therefore

\begin{align}
\xi(t) \cos^2 t &= \int_{\frac{t}{2}}^{\frac{\pi}{2}} 4s \cos^2 s \, ds,
\end{align}

\begin{align}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} & \xi(t) \, dt = 2 \int_{0}^{\frac{\pi}{2}} \xi(t) \, dt = -8 \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{1}{\cos^2 t} \int_{t}^{\frac{\pi}{2}} s \cos^2 s \, ds \right) \, dt \\
& = -8 \int_{0}^{\frac{\pi}{2}} \left( \int_{0}^{s} \frac{1}{\cos^2(t)} \, dt \right) s \cos^2 s \, ds = -8 \int_{0}^{\frac{\pi}{2}} s \cos s \sin s \, ds = -\pi.
\end{align}

It is easy to see that $q$ and $q'$ satisfy the following equations

\begin{align}
(35) \quad & \frac{1}{2} q'' \cos t - 2q' \sin t - 2q \cos t = -4 \sin t,
\end{align}

14
Let \( p(t) = \eta(t) \), i.e.,

\[
p(t) = \eta(t) = \frac{2(\frac{4}{\pi} \cos t + \frac{4}{t} \sin t - \sin^2 t - 1)}{\cos^3 t}.
\]

Lemma 4. Let

\[
\eta(t) = \frac{4t + \frac{4}{\pi} \cos t \sin t - 2 \sin t}{\cos^2 t}
\]
on \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

Then the function \( \eta \) satisfies the following

\[
\frac{1}{2} \eta'' \cos^2 t - \eta' \cos t \sin t - \eta = -\sin t \quad in \quad (-\frac{\pi}{2}, \frac{\pi}{2}),
\]

\[
\eta' \cos t - 2\eta \sin t = \frac{8}{\pi} \cos t - 2 \quad in \quad (-\frac{\pi}{2}, \frac{\pi}{2}),
\]

\[-1 = \eta(-\frac{\pi}{2}) \leq \eta(t) \leq \eta(\frac{\pi}{2}) = 1 \quad on \quad [-\frac{\pi}{2}, \frac{\pi}{2}],
\]

\[0 < 2(\frac{4}{\pi} - 1) = \eta'(0) \leq \eta'(\pm \frac{\pi}{2}) = \frac{8}{3\pi} \quad on \quad [-\frac{\pi}{2}, \frac{\pi}{2}],
\]

\[-1/2 = \eta''(\frac{\pi}{2}) \leq \eta''(\pm \frac{\pi}{2}) \leq \eta''(\frac{\pi}{2}) = 1/2 \quad on \quad [-\frac{\pi}{2}, \frac{\pi}{2}],
\]

\[
\eta'''(t) > 0 \quad on \quad [-\frac{\pi}{2}, \frac{\pi}{2}] \quad and \quad \eta'''(\pm \frac{\pi}{2}) = \frac{32}{15\pi}.
\]

Proof. Let \( p(t) = \eta'(t) \), i.e.,

\[
p(t) = \eta'(t) = \frac{2(\frac{4}{\pi} \cos t + \frac{4}{t} \sin t - \sin^2 t - 1)}{\cos^3 t}.
\]
Equation \(39\), \(\eta(\pm \frac{\pi}{2}) = \pm 1, \eta'(0) = 2(\frac{4}{\pi} - 1)\) and \(\eta'(\pm \frac{\pi}{2}) = \frac{8}{3\pi}\) can be verified directly. We get \(\eta''(\pm \frac{\pi}{2}) = \pm 1/2\) from the above values and equation \(39\). By \(39\), \(q = \eta', q' = \eta''\) and \(p'' = \eta''\) satisfy the following equations in \((-\frac{\pi}{2}, \frac{\pi}{2})\)

(42) \[\frac{1}{2}p'' \cos t - 2p' \sin t - 2p \cos t = -1,\]

and

\[
\frac{\cos^2 t}{2(1 + \cos^2 t)} p'' - \frac{2 \cos t \sin t}{1 + \cos^2 t} p'' - 2p' = -\frac{\sin t}{1 + \cos^2 t},
\]

Therefore

\[
\frac{\cos^2 t}{2(1 + \cos^2 t)}(p'')'' - \frac{\cos t \sin t(3 + 2 \cos^2 t)}{(1 + \cos^2 t)^2} (p'')' - \frac{2(5 \cos^2 t + \cos^4 t)}{(1 + \cos^2 t)^2} (p'') = -\frac{\cos t (2 + \sin t)}{(1 + \cos^2 t)^2}.
\]

(43)

The coefficient of \((p'')\) in \(13\) is obviously negative in \((-\frac{\pi}{2}, \frac{\pi}{2})\) and the right-hand side of \(13\) is also negative. So \(p''\) cannot achieve its non-positive local minimum at a point in \((-\frac{\pi}{2}, \frac{\pi}{2})\). On the other hand, \(p''(\frac{\pi}{2}) = \frac{32}{\sqrt{3\pi}} > 0\) (see the proof below), \(p''(t) > 0\) on \([-\frac{\pi}{2}, \frac{\pi}{2}]\). Therefore \(p'\) is increasing and \(-1/2 = p'(-\frac{\pi}{2}) \leq p'(t) \leq p'\left(\frac{\pi}{2}\right) = 1/2\). Note that \(p'(0) = 0\) (\(p'\) is an odd function). So \(p'(t) > 0\) on \((0, \frac{\pi}{2})\) and \(p\) is increasing on \([0, \frac{\pi}{2}]\). Therefore \(2(4/\pi - 1) = p(0) \leq p(t) = \eta'(t) \leq p\left(\frac{\pi}{2}\right) = \frac{8}{3\pi}\) on \([0, \frac{\pi}{2}]\), and on \([-\frac{\pi}{2}, \frac{\pi}{2}]\) since \(p\) is an even function. We now show that \(p\left(\frac{\pi}{2}\right) = \frac{8}{3\sqrt{\pi}}, p'\left(\frac{\pi}{2}\right) = 1/2\) and \(p''\left(\frac{\pi}{2}\right) = \frac{32}{\sqrt{3\pi}}\). The first is from a direct computation by using \(11\). By \(39\),

\[
\frac{1}{2}p'(\frac{\pi}{2}) = \frac{1}{2} \eta''(\frac{\pi}{2}) = \lim_{t \to \frac{\pi}{2}^-} \frac{\eta'(t) \cos t \sin t + \eta(t) - \sin t}{\cos^2 t} = -\frac{1}{2}[\eta''(\frac{\pi}{2}) - 1].
\]

So \(p'(\frac{\pi}{2}) = 1/2\). Similarly, by \(52\),

\[
\frac{1}{2}p''\left(\frac{\pi}{2}\right) = \lim_{t \to \frac{\pi}{2}^-} \frac{2p'(t) \sin t - 1}{\cos t} + 2p\left(\frac{\pi}{2}\right) = -2p''\left(\frac{\pi}{2}\right) + \frac{16}{3\pi}
\]

Thus \(p''\left(\frac{\pi}{2}\right) = \frac{32}{3\sqrt{\pi}}\).

**Lemma 5.** The function \(r(t) = \xi'(t)/\eta'(t)\) is an increasing function on \([-\frac{\pi}{2}, \frac{\pi}{2}\]), i.e., \(r'(t) > 0\), and \(|r(t)| \leq \frac{2}{\sqrt{\pi}}\) holds on \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

**Proof.** Let \(p(t) = \eta'(t)\) as in \(11\) and \(q(t) = \xi'(t)\). Then \(r(t) = q(t)/p(t)\). It is easy to verify that \(r(\pm \frac{\pi}{2}) = \pm \frac{\pi^2}{4}\). By \(42\) and \(52\),

\[(1/2)p(t)r'' \cos t + (p'(t) \cos t - 2p(t) \sin t)r' - r = -4 \sin t.
\]

16
Differentiating the last equation, we get
\[
\frac{1}{2}p(t) \cos t (r')'' + [\frac{3}{2}p'(t) \cos t - \frac{5}{4}p(t) \sin t](r')' + [p''(t) \cos t - 3p'(t) \sin t - 2p(t) \cos t - 1](r') = -4 \cos t.
\]

Using (42), the above equation becomes
\[
\frac{1}{2}p(t) \cos t (r')'' + [\frac{3}{2}p'(t) \cos t - \frac{5}{4}p(t) \sin t](r')' + [p''(t) \cos t - 3p'(t) \sin t - 2p(t) \cos t - 1](r') = -4 \cos t.
\]

The coefficient of \((r')\) in (44) is negative, for
\[
p'(t) \sin t + 2p(t) \cos t - 3 < \frac{1}{4} + \frac{16}{3\pi} - 3 < 0.
\]
This fact and the negativity of the righthand side of (44) in \((-\frac{\pi}{2}, \frac{\pi}{2})\) imply that \(r'\) cannot achieve its non-positive minimum on \([-\frac{\pi}{2}, \frac{\pi}{2}]\) at a point in \((-\frac{\pi}{2}, \frac{\pi}{2})\). Now
\[
\lim_{t \to \frac{\pi}{2}} r'(t) = \lim_{t \to \frac{\pi}{2}} s(t) \cos^2 t/(\frac{4}{\pi} \cos t + \frac{4}{\pi} t \sin t - \sin^2 t - 1)^2
\]
\[
= \lim_{t \to \frac{\pi}{2}} [s(t)/\cos^4 t]/[(\frac{4}{\pi} \cos t + \frac{4}{\pi} t \sin t - \sin^2 t - 1)/\cos^3 t]_t^2
\]
\[
= \lim_{t \to \frac{\pi}{2}} [s(t)/\cos^4 t]/[\frac{1}{2} y'(t)]_t^2
\]
\[
= \left(\frac{4}{3\pi} - \frac{\pi}{12}\right)/\left(\frac{4}{3\pi}\right)_t^2
\]
\[
> 0,
\]
where
\[
s(t) = \frac{4}{\pi} t^2 - t^2 \cos t + \frac{12}{\pi} \cos^2 t + \frac{8}{\pi} t \sin t \cos t - \cos t \sin^2 t + (\frac{\pi^2}{4} - 3) \cos t - \pi + 4t \sin t.
\]

Therefore \(r'(t) > 0\) and \(r\) is an increasing function on \([-\frac{\pi}{2}, \frac{\pi}{2}]\). 

\textit{Proof of the estimate (8)}. We estimate the maximum of the function
\[
(45) \quad P(x) = |\nabla v|^2 + Av^2,
\]
where \(v\) is the function in (3), and where \(A \geq 0\) is a constant.
Let \( A = 0 \) in (45). Function \( P \) must achieve its maximum at some point \( x_0 \in \bar{\Omega} \). Suppose that \( x_0 \in \partial \Omega \). Choose an orthornormal frame \( \{ e_1, \ldots, e_n \} \) about \( x_0 \) such that \( e_n \) is a outward normal to \( \partial \Omega \). By (5), \( v_n = \partial v / \partial N = 0 \).

Thus at \( x_0 \)

\[
v_{in} = e_i e_n v - (\nabla e_i e_n) v
\]

\[
= -(\nabla e_i e_n) v
\]

\[
= - \sum_{j=1}^{n-1} h_{ij} v_j
\]

and

\[
P_n = 2 \sum_{j=1}^{n} v_j v_{jn} + 2 A v v_n = \sum_{j=1}^{n-1} v_j v_{jn}
\]

\[
= -2 \sum_{i,j=1}^{n-1} h_{ij} v_i v_j
\]

\[
\leq 0 \quad \text{by the convexity of } \partial \Omega.
\]

On the other hand, that \( P \) attains the maximum at the boundary point \( x_0 \) implies that

\[
P_n \geq 0.
\]

Thus at \( x_0 \), \(-2 \sum_{i,j=1}^{n-1} h_{ij} v_i v_j = P_n = 0 \). By the strict convexity, \( v_1 = \cdots = v_{n-1} = 0 \) and \( \nabla v = 0 \) at \( x_0 \). Therefore \( v \) is a constant. This is impossible, so \( x_0 \in \Omega \). \( \nabla v(x_0) \neq 0 \) (otherwise \( v \) is a constant). At \( x_0 \),

\[
\nabla P(x_0) = 0 \quad \text{and} \quad \Delta P(x_0) \leq 0.
\]

Take a local frame so that

\[
v_1(x_0) = \nabla v(x_0) \quad \text{and} \quad v_i(x_0) = 0, \quad i \geq 2.
\]

Thus at \( t_0 \) we have

\[
0 = \frac{1}{2} \nabla P_i = v_j v_{ji} + A v v_i,
\]

(46) \( v_{11} = -Av \quad \text{and} \quad v_{1i} = 0 \quad i \geq 2 \),
and

\[ 0 \geq \frac{1}{2} \Delta P(x_0) = v_{ji}v_{ji} + v_jv_{jii} + Av_iv_i + Avv_{ii} \]

\[ = v_{ji}^2 + v_1(\Delta v)_1 + R_{ji}v_iv_i + A|\nabla v|^2 + Av\Delta v \]

\[ \geq v_{11}^2 + v_1(\Delta v)_1 + A|\nabla v|^2 + Av\Delta v \]

\[ = (-Av)^2 - \lambda|\nabla v|^2 - 2v_1(\nabla v\nabla \ln f)_1 + A|\nabla v|^2 \]

\[ - \lambda Av(v + a) - 2Av\nabla v\nabla \ln f \]

\[ = -(\lambda - A)|\nabla v|^2 - Av^2(\lambda - A) - a\lambda Av \]

\[ - 2v_1^2(\ln f)_{11} - 2v_1(\ln f)_1(v_{11} + Av), \]

where we have used (46) and (8). Therefore at \( x_0 \),

\[ (47) \quad 0 \geq -(\lambda - 2\alpha - A)|\nabla v|^2 - A(\lambda - A)v^2 - a\lambda Av. \]

Using the fact that \( A = 0 \) in the above inequality, we get the (9). □

**Proof of the estimate (17).** We first prove the following

\[ (48) \quad \frac{|\nabla v|^2}{b^2 - v^2} \leq \lambda(1 + a), \]

where \( b > 1 \) is an arbitrary constant.

Let \( A = \lambda(1 + a) + \epsilon \) in (15) for small \( \epsilon > 0 \). \( P \) achieves its maximum at some \( x_0 \in \bar{\Omega} \). If \( \nabla v(x_0) \neq 0 \) and \( x_0 \in \Omega \), then (47) holds at \( x_0 \) with \( A = \lambda(1 + a) + \epsilon \). Thus

\[ |\nabla v(x_0)|^2 + \lambda(1 + a)v(x_0)^2 \leq \frac{a\lambda v}{a\lambda + \epsilon} [\lambda(1 + a) + \epsilon] \leq [\lambda(1 + a) + \epsilon]. \]

This estimate holds if \( x_0 \in \bar{\Omega} \) with \( \nabla v(x_0) = 0 \). If \( x_0 \in \partial \Omega \), then the convexity of \( \Omega \) and previous argument in the proof of (19) imply that the above estimate holds. So we have the estimate (18). By the definition of \( Z \), we have (19). □

**Proof of Lemma 7** Define

\[ J(x) = \left\{ \frac{|\nabla v|^2}{b^2 - v^2} - \lambda z \right\} \cos^2 t, \]

where \( t = \sin^{-1}(v(x)/b) \). Then

\[ J(x) \leq 0 \quad \text{for } x \in \bar{\Omega} \quad \text{and} \quad J(x_0) = 0. \]
If $\nabla v(x_0) = 0$ then

$$0 = J(x_0) = -\lambda z \cos^2 t.$$ 

This contradicts the condition 3 in the theorem. Therefore

$$\nabla v(x_0) \neq 0.$$ 

If $x_0 \in \partial \Omega$, then by an argument in the proof of (9), the convexity of $\Omega$ and that $J(x_0)$ is the maximum would imply that $\nabla v(x_0) = 0$. Thus $x_0 \in \Omega = \bar{\Omega} \setminus \partial \Omega$. The Maximum Principle implies that

$$\nabla J(x_0) = 0 \quad \text{and} \quad \Delta J(x_0) \leq 0.$$ 

$J(x)$ can be rewritten as

$$J(x) = \frac{1}{b^2} |\nabla v|^2 - \lambda z \cos^2 t.$$ 

Thus (49) is equivalent to

$$20
\begin{align*}
2 \sum_i v_i v_{ij} \bigg|_{x_0} &= \lambda \cos t \left[z' \cos t - 2z \sin t \right]_{x_0} \\
2 \frac{\nabla^2}{b^2} + 2 \sum_i v_i v_{ijj} - \lambda \left( z'' |\nabla t|^2 + z' \Delta t \right) \cos^2 t \\
&\quad + 4 \lambda z' \cos t \sin t |\nabla t|^2 - \lambda z \Delta \cos^2 t \bigg|_{x_0}.
\end{align*}
$$

Choose a normal coordinate around $x_0$ such that $v_1(x_0) \neq 0$ and $v_i(x_0) = 0$ for $i \geq 2$. Then (50) implies

$$v_{11} \bigg|_{x_0} = \frac{\lambda b}{2} \left(z' \cos t - 2z \sin t \right) \bigg|_{x_0} \quad \text{and} \quad v_{1i} \bigg|_{x_0} = 0 \text{ for } i \geq 2.$$ 

20
Now we have

\[
|∇v|^2_{x_0} = \lambda b^2 z \cos^2 t |_{x_0},
\]

\[
|∇t|^2_{x_0} = |∇v|^2_{b^2 - v^2} = \lambda z |_{x_0},
\]

\[
\frac{Δv}{b}_{x_0} = Δ \sin t = \cos t Δt - \sin t |∇t|^2_{x_0},
\]

\[
\Delta t_{x_0} = \frac{1}{\cos t} (|∇t|^2 + \frac{Δv}{b})
\]

\[
= \frac{1}{\cos t} [\lambda z \sin t - \frac{\lambda}{b} (v + a) - \frac{2}{b} v_1 (\ln f)_1] |_{x_0}, \quad \text{and}
\]

\[
Δ \cos^2 t_{x_0} = Δ \left(1 - \frac{v^2}{b^2}\right) = \frac{2}{b^2} |∇v|^2 - \frac{2}{b^2} v Δv
\]

\[
= -2\lambda z \cos t + \frac{2}{b^2} \lambda v (v + a) + \frac{4}{b^2} v_1 (\ln f)_1 |_{x_0}.
\]

Therefore,

\[
\frac{2}{b^2} \sum_{i,j} v_{ij}^2 |_{x_0} \geq \frac{2}{b^2} v_{11}^2
\]

\[
= \frac{\lambda^2}{2} (z')^2 \cos^2 t - 2\lambda^2 zz' \cos t \sin t + 2\lambda^2 z^2 \sin^2 t |_{x_0},
\]

\[
\frac{2}{b^2} \sum_{i,j} v_i v_{ijj} |_{x_0} = \frac{2}{b^2} (\nabla v \nabla (Δv) + R(∇v, ∇v)) \geq \frac{2}{b^2} \nabla v \nabla (Δv)
\]

\[
= -2\lambda^2 z \cos^2 t - \frac{4}{b^2} v_1 v_{11} (\ln f)_1 - \frac{4}{b^2} v_1^2 (\ln f)_{11} |_{x_0},
\]

\[
-λ(z'')|∇t|^2 + z' Δt \cos^2 t |_{x_0}
\]

\[
= -\lambda^2 zz'' \cos^2 t - \lambda^2 zz' \cos t \sin t
\]

\[
+ \frac{1}{b} \lambda^2 z' (v + a) \cos t + \frac{2}{b} \lambda z' v_1 (\ln f)_1 \cos t |_{x_0},
\]

and

\[
4λz' \cos t \sin t |∇t|^2 - \lambda z Δ \cos^2 t |_{x_0}
\]

\[
= 4\lambda^2 zz' \cos t \sin t + 2\lambda^2 z^2 \cos^2 t - \frac{2}{b} \lambda^2 \sin t (v + a) - \frac{4}{b} λz \sin t v_1 (\ln f)_1 |_{x_0}.
\]
Putting these results into (51) we get

\[
0 \geq -\lambda^2 z z'' \cos^2 t + \frac{\lambda^2}{2} (z')^2 \cos^2 t \\
+ \lambda^2 z' \cos t (z \sin t + c + \sin t) \\
+ 2\lambda^2 z^2 - 2\lambda^2 z \\
- 2\lambda^2 cz \sin t - 4\lambda z^2 t (\ln f)_{11} \\
- \frac{4}{b^2} \left[ v_{11} - \frac{\lambda b}{2} (z' \cos t - 2z \sin t) \right] v_1 (\ln f)_1 \bigg|_{x_0}.
\]

The last term in (53) is 0 due to (52). Now (54)

\[
z(t_0) > 0,
\]

by the condition 3 in the theorem, and

(55)

\[
- \frac{(\ln f)_{11}}{\lambda} \geq \delta,
\]

by the definition of \(\delta\). Dividing two sides of (53) by \(2\lambda^2 z \bigg|_{x_0}\) and taking (55) into account, we have

\[
0 \geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + \frac{1}{2} z'(t_0) \cos t_0 \left( \sin t_0 + \frac{c + \sin t_0}{z(t_0)} \right) + z(t_0) \\
- 1 - c \sin t_0 + 2\delta \cos^2 t_0 \\
+ \frac{1}{4z(t_0)} (z'(t_0))^2 \cos^2 t_0.
\]

Therefore,

\[
0 \geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + z'(t_0) \cos t_0 \sin t_0 + z(t_0) - 1 - c \sin t_0 + 2\delta \cos^2 t_0 \\
+ \frac{z'(t_0)}{4z(t_0)} \cos t_0 [z'(t_0) \cos t_0 - 2z(t_0) \sin t_0 + 2 \sin t_0 + 2c].
\]

Proof of Corollary 4. By Condition 2 in the theorem, (10), \(|\sin t_0| = |v(t_0)/b| \leq 1/b\) and \(1 - c \leq z(t_0) \leq 1 + a\). Thus for \(t_0 \geq 0\),

\[-z(t_0) \sin t_0 + \sin t_0 + c \geq - \sin t_0 - a \sin t_0 + \sin t_0 + c \geq a (\frac{1}{b} - \sin t_0) \geq 0,\]

and for \(t_0 < 0\),

\[-z(t_0) \sin t_0 + \sin t_0 + c \geq - \sin t_0 + c \sin t_0 + \sin t_0 + c \geq c(1 + \sin t_0) \geq 0.\]

In any case the last term in the (11) is non-negative. \(\square\)
Proof of Corollary 2. The last term in the (11) is nonnegative.  

The following Lemma is due to the author [18]. We enclose it here for the completeness.

**Lemma 6.** Let $f$ be the first eigenfunction of (1) with $f > 0$ in $\Omega$. Then there exists an $\epsilon > 0$ such that the function $-\ln f$ is strictly convex in the $\epsilon$-neighborhood of $\partial \Omega$.

**Proof.** Choose a normal coordinate about $x_0$ such that $\partial/\partial x_1$ is the outward unit normal vector field of $\partial \Omega$ near $x_0$. Take a point $\bar{x}$ with small distance $d$ to $\partial \Omega$ and $d = dist(\bar{x}, x_0) = dist(\bar{x}, \partial \Omega)$. Then by the Strong Maximum Principle $f_1|_{x_0} < 0$, and $f_i|_{x_0} = 0$, for $i \geq 2$. Therefore

$$f_1|_{\bar{x}} \sim c_1 d, \quad c_1 = -f(x_0) > 0$$

and

$$f_i|_{\bar{x}} \sim O(d) \quad \text{for} \quad i \geq 2.$$ 

Here "$\sim A/d^\alpha$" means "$(A + o(1))/d^\alpha$" with $o(1) \to 0$ as $d \to 0$. Let $w = \ln f$. Then

$$w_{ij}\big|_{\bar{x}} = (f_{ij}/f - f_i f_j/f^2 - \Gamma_{ij}^k f_k/f)\big|_{\bar{x}}.$$ 

Thus we obtain

$$w_{11}\big|_{\bar{x}} \sim O(1/d) - f_1^2/d^2$$

and

$$w_{1i}\big|_{\bar{x}} \sim O(1/d), \quad i \geq 2.$$ 

For $i, j \geq 2$,

$$f_{ij}\big|_{\bar{x}} = \nabla_{\partial/\partial x_i} \nabla_{\partial/\partial x_j} f \big|_{\bar{x}} - \nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} f \big|_{\bar{x}} \sim O(d) - \nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} f \big|_{x_0}.$$ 

Now

$$\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} f \big|_{x_0} = (\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}) \frac{\partial}{\partial x_k} f \big|_{x_0} = (\nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_1} f \big|_{x_0},$$

$$(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j}) = 0, \quad j \geq 2.$$
\[
0 = \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j} \right) = (\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j}) = h_{ij} + (\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_1}),
\]

where \((h_{ij})_{n-1,n-1}\) is the second fundamental form of \(\partial \Omega\) to \(\partial/\partial x_1\). Therefore
\[
f_{ij} \big|_{\tilde{x}} \sim O(d) + h_{ij} f_1 \big|_{x_0} \quad \text{and} \quad w_{ij} \big|_{\tilde{x}} \sim h_{ij} f_1 \big|_{x_0}, \quad i, j \geq 2,
\]
and
\[
(-w_{ij})_{n,n} \big|_{\tilde{x}} \sim \left( \begin{array}{cc} \frac{f_1^2}{d^2} & O(1/d) \\ O(1/d) & O(1/d)(-f_1 h_{ij})_{n-1,n-1} \end{array} \right) \bigg|_{x_0}
\]

Since \((h_{ij})\) is positive definite, so is \((-w_{ij})\) for \(\tilde{x}\) near the boundary \(\partial \Omega\). \(\Box\)

References

[1] M. Ashbaugh Some eigenvalue comparison results for domains in \(S^n\) and for annular domains in \(R^{n+1}\), The Erwin Schrödinger International Institute for Mathematical Physics - Workshop on Geometrical Aspects of Spectral Theory, Editors Leonid Friedlander and Thomas Hoffmann-Ostenhof, \texttt{ftp://ftp.esi.ac.at/pub/Preprints/esi768.pdf}, East Tyola, 1999.

[2] M. Ashbaugh and R. Benguria, Optimal lower bounds for the eigenvalue gaps for Schrödinger operators with symmetric single-well potentials and related results, Maximum Principles and Eigenvalue Problems in Partial Differential Equations, Longman, White Plains New York 1988.

[3] M. Ashbaugh and R. Benguria, Optimal lower bounds for the gap between the first two eigenvalues of one dimensional Schrödinger operators with symmetric single-well potentials, Proc. Amer. Math. Soc., 105(1989), 419–424.

[4] M. Ashbaugh and R. Benguria, A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions, Ann. of Math, 135(1992), 601–628.

[5] R. Bañuelos and P. Kroger, Gradient estimates for ground state Schrodinger operators and applications, Comm. Math. Phy., 224(20002), 545–550.
[6] R. Bañuelos and P.J. Méndez-Hernández, Sharp inequalities for ratios of heat kernels of Schrödinger operators and applications to spectral gaps, *Journal Funct. Anal.*, **176**(2000), 368–399.

[7] M. van den Berg, On condensation in the free-boson gas and the spectrum of Laplacian, *J. Statist. Phys.*, **31**(1983), 623–637.

[8] H. Brascamp and E. Lieb, On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, *J. Func. Anal.*, **22**(1976), 366–389.

[9] L. A. Caffarelli and A. Friedman, Convexity of solutions of semilinear elliptic equations, *Duke Mathematical Journal*, **52**(1985).

[10] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, Orlando, Fla, 1984.

[11] B. Davis, On the spectral gap for fixed membranes, *Ark. Mat.*, **39**(2001), 65–74.

[12] N. J. Korevaar, Convexity properties of solutions to elliptic pdes, *Duke Mathematical Journal*, **52**(1985), 431–456.

[13] N. J. Korevaar and J. L. Lewis. Convexity solutions of certain elliptic equations have constant rank Hessians, *Arch. Rational Mech. Anal.*, **97**(1987), 19–32, 1987.

[14] Y. I. Lee and A. N. Wang, Estimate of $\lambda_2 - \lambda_1$ on spheres, *Chinese J. Math.*, **15**(1987), 95–97.

[15] P. Li, *Lecture Notes on Geometric Analysis, Lecture Notes Series*, #6, Seoul National University, Seoul, Korea.

[16] P. Li and S. T. Yau., On the Schrödinger equation and the eigenvalue problem, *Comm. Math. Phys.*, **88**(1983), 309–318.

[17] P. Li and S. T. Yau, Estimates of eigenvalues of a compact Riemannian manifold, *AMS Proc. Symp. Pure Math.*, **36**(1980), 205–239.

[18] J. Ling, A lower bound for the gap between the first two eigenvalues of Schrödinger operators on convex domains in $S^n$ or $R^n$, *Michigan Math. J.*, **40**(1983), 259–270.
[19] J. Ling, A Bound for the First Fundamental Gap, Ph.D. Dissertation, State University of New York at Buffalo.

[20] L. Payne, G. Pólya, and H. Weinberger, On the ratio of consecutive eigenvalues, J. Math. Phys., 35(1956), 289–298.

[21] R. Schoen, and S. T. Yau, Lecture Notes on Differential Geometry, Conference Proceedings and Lecture Notes in Geometry and Topology, Vol 1, International Press, 1994.

[22] R. G. Smits, Spectral gaps and rates to equilibrium for diffusions in convex domains, Michigan Math. J., 43(1996), no. 1, 141-157.

[23] I. M. Singer, B. Wong, S. T. Yau, and S. S. T. Yau, An estimate of the gap of the first two eigenvalues, Ann. Scuola Norm. Sup. Pisa, CI. Sci., 12(1985), 319–333.

[24] S. T. Yau, editor., Problem Section, Seminar on Diff. Geom., Princeton University Press, Princeton, 1982.

[25] S. T. Yau, An estimate of the gap of the first two eigenvalues in the Schrödinger operator, In SunYung Alice Chang, Chang-Shou Lin and Horng-Tzer Yau, editors, Lectures on Partial Differential Equations: Proceedings in honor of Louis Nirenberg’s 75th Birthday, pages 223–235, International Press, 2003.

[26] Q. H. Yu and J.-Q. Zhong, Lower bounds of the gap between the first and second eigenvalues in the Schrödinger operators, Trans. Amer. Math. Sco., 294(1986), 341–349.

Department of mathematics, Utah Valley State College, Orem, Utah 84058
E-mail address: lingju@uvsc.edu