Minlos—Faddeev Regularization of Zero-Range Interactions in the Three-Body Problem

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To regularize the three-body problem, Minlos and Faddeev suggested a modification of zero-range model, which diminishes interaction at the triple-collision point. The analysis reveals that this regularization results in four alternatives depending on the regularization parameter \( \sigma \). Explicitly, Efimov or Thomas effects remain for \( \sigma < \sigma_c \), the additional boundary conditions of two types should be imposed at the triple-collision point for \( \sigma_c \leq \sigma < \sigma_r \), and the problem is regularized for \( \sigma \geq \sigma_r \). Critical values \( \sigma_c < \sigma_r \) separating different alternatives are determined both for a two-component three-body system and for three identical bosons.

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In studies of the universal low-energy dynamics, e.g., in ultracold quantum gases, it is natural to use a zero-range model for short-range two-body interactions. Nevertheless, it is not a trivial task to introduce this in the few-body problem, which requires additional efforts. A basic origin of complication is connected with overlap of at least two zero-range interactions, which takes place near the triple-collision point [1].

Minlos and Faddeev proposed a specific modification of zero-range model in the three-body problem [1]. Twenty years later an equivalent form written as a boundary condition in the configuration space was given in [2]. Both papers showed that an additional three-body force of the strength \( \sigma \) is able to preclude the Efimov and Thomas effects for sufficiently large \( \sigma \) exceeding the explicitly given critical value \( \sigma_c \). Moreover, the condition \( \sigma_c \leq \sigma \) was assumed to provide an unambiguous description of the three-body problem. Recently, a proof of this conjecture was presented for three identical bosons [3] and for \( N \) identical bosons interacting with a distinct particle [4].

The Minlos—Faddeev regularization of zero-range model is studied in this note both for the two-component system consisting of two identical bosons interacting with a distinct particle and for the system containing three identical bosons. The analysis is based on reducing the problem to consideration of an ordinary differential equation with singular coefficients. It is shown that the proposed regularization leads to different results in four intervals of the parameter \( 0 < \sigma < \infty \) separated by three critical values \( \sigma_c, \sigma_r, \) and \( \sigma_r \).

More exactly, in the interval \( \sigma_c \leq \sigma < \sigma_r \), it is necessary to set an additional boundary condition in the triple-collision point, which depends on an arbitrary real-valued parameter \( b \). Moreover, in the interval \( \sigma_c \leq \sigma < \sigma_r \), this boundary condition should be written in a special form by taking account of the lower order terms in expansion near the triple-collision point. To elucidate consequences of the described regularization, dependence of the bound-state energy of three identical bosons on \( b \) and \( \sigma \) is determined for \( \sigma_c \leq \sigma < \sigma_r \).

Consider the system consisting of a distinct particle 1 of mass \( m_1 \) and two identical bosons 2 and 3 of masses \( m_2 = m_3 = m \). In the center-of-mass frame, define the scaled Jacobi variables as \( x = \sqrt{2\mu} (r_2 - r_1) \) and \( y = \sqrt{2\mu} \left( r_1 - \frac{m_1 r_2 + m r_3}{m_1 + m} \right) \), where \( r_i \) is the position vector of \( i \)th particle and the reduced masses are denoted by \( \mu = \frac{m m_2}{m_1 + m} \) and \( \bar{\mu} = \frac{m(m + m_1)}{m_1 + 2m} \). In the zero-range model, the two-body interaction is determined by a single parameter, the scattering length \( a \). The units are chosen as \( \hbar = |\sigma| = 2\mu = 1 \), which gives the unit two-body binding energy \( (\varepsilon_{2b} = 1) \). The mass ratio \( m/m_1 \) remains a single essential parameter that for convenience can be interchangeably replaced by the kinematic angle \( \omega \) defined by \( \sin \omega = 1/(1 + m_1/m) \).

The Hamiltonian is formally defined as the six-dimensional Laplace operator supplemented by the boundary conditions imposed at zero distance.
between the interacting particles. In [1] it was suggested to introduce in the momentum-space equation a term containing the convolution-type operator \( K(k - k') \) depending on the relative momentum \( k \) between the interacting pair’s center-of-mass and the third particle, whose asymptotic form \( K(\xi) \rightarrow \frac{\sigma}{\xi^2} \) for \( \xi \rightarrow \infty \). The equivalent form of this regularization in the configuration-space representation was proposed in [2] and considered recently in [3, 4], where an additional term was introduced to modify the boundary conditions at zero distance between the interacting particles,

\[
\lim_{\xi \rightarrow \infty} \left[ \frac{\partial \log(x|\Psi)}{\partial x} - \frac{\sigma}{\cos \omega} \theta(y) \right] = -\text{sgn}(a). \tag{1}
\]

Here, \( \sigma \) is a regularization parameter, which controls the wavefunction near the triple-collision point, the factor \( \frac{1}{\cos \omega} \) is introduced for convenience, and \( \theta(y) \) is an arbitrary bounded function normalized by \( \theta(0) = 1 \)

and \( \frac{d\theta}{dy} \big|_{y=0} \neq 0 \). Only one boundary condition (1) should be imposed if the symmetry under permutations of identical particles is taken into account.

Clearly, the described procedure adds a kind of the three-body interaction, thus providing the regularization of Hamiltonian in the triple-collision point. As is well established, the zero total angular momentum and positive parity \( \mathcal{P} (L^p = 0^+) \) are those quantum numbers, for which the regularization is certainly required, therefore, namely this case will be considered in this note. One should mention that the regularization is also needed for even \( \mathcal{P} \) and positive parity of the three-body problem and its regularization. The eigenvalue Eqs. (5) and (6), whose different branches determine an infinite set of eigenvalues \( \gamma_n^2(\rho) \), are of the well-known form [5, 8, 11], in which the regularization gives rise to an additional term proportional to \( \sigma \theta(\rho) \).

By means of the expansion (2), the original problem is formulated as a set of coupled hyper-radial equations [8–10] for the channel functions \( f_n(\rho) \),

\[
\left[ \frac{d^2}{d\rho^2} - \frac{\gamma_n^2 - 1/4}{\rho^2} + P_{nn} + E \right] f_n(\rho) - \sum_{m\neq n} \left[ P_{nm} - Q_{nm} \frac{d}{d\rho} - \frac{d}{d\rho} Q_{nm} \right] f_m(\rho) = 0, \tag{7}
\]

where matrix elements \( Q_{nm}(\rho) = \int \Phi_{n} \frac{\partial \Phi_{m}}{\partial \rho} d\Omega \) and \( P_{nn}(\rho) = \int \frac{\partial \Phi_{n}}{\partial \rho} \frac{\partial \Phi_{n}}{\partial \rho} d\Omega \) can be given in the analytical form via \( \gamma_n(\rho) \) and their derivatives [9, 11, 12].

The diagonal terms in the hyper-radial equations (7), which are singular near the triple-collision point \( (\rho \rightarrow 0) \), are of principal importance for analysis of the three-body problem and its regularization. The matrix elements \( Q_{nm}(\rho) \) and \( P_{nm}(\rho) \), as follows from their analytical expressions [9, 11, 12], are finite and differentiable functions for any \( \rho \) and do not influence essential properties of the solution near \( \rho = 0 \). Explicitly, the principal singularity comes from the branch

\[
The functions \( \Phi_n(\alpha, \hat{x}, \hat{y}; \rho) \) for \( L^p = 0^+ \) satisfy the equation

\[
\left[ \frac{\partial^2}{\partial \rho^2} + 4 \tan 2\alpha \frac{\partial}{\partial \alpha} + \gamma^2(\rho) - 4 \right] \Phi(\alpha, \hat{x}, \hat{y}; \rho) = 0, \tag{3}
\]

the boundary condition

\[
\lim_{\alpha \rightarrow 0} \frac{\partial \log[\alpha \Phi(\alpha, \hat{x}, \hat{y}; \rho)]}{\partial \alpha} = -\rho \text{sgn}(a) + \frac{\sigma \theta(\rho)}{\cos \omega} \tag{4}
\]

and inherit the permutation symmetry of the total wavefunction. Solving the boundary value problem specified by Eqs. (3) and (4), one obtains the equation

\[
\rho \text{sgn}(a) - \frac{2\sigma \theta(\rho)}{\sqrt{3}} \sin \frac{\pi}{2} = \gamma \cos \frac{\pi}{2} - 2 \frac{\sin \frac{\pi}{2}}{\sin 2\omega} \tag{5}
\]

for the two-component system. Similarly, for three identical bosons, taking into account that \( \omega = \pi/6 \) and all three particles interact, one finds

\[
\rho \text{sgn}(a) - \frac{2\sigma \theta(\rho)}{\sqrt{3}} \sin \frac{\pi}{2} = \gamma \cos \frac{\pi}{2} - 8 \frac{\sin \frac{\pi}{6}}{\sqrt{3}} \tag{6}
\]

The eigenvalue Eqs. (5) and (6), whose different branches determine an infinite set of eigenvalues \( \gamma_n(\rho) \), are of the well-known form [5, 8, 11], in which the regularization gives rise to an additional term proportional to \( \sigma \theta(\rho) \).

In study of the proposed regularization, a key point is to determine its impact on the wavefunction in the triple-collision point. To proceed, one introduces the hyper-radius and hyper-angular variables on a hypersphere \( \Omega = \{ \alpha, \hat{x}, \hat{y} \} \), which are defined by \( x = \rho \cos \alpha \), \( y = \rho \sin \alpha \) and \( \hat{x} = x/x \), and \( \hat{y} = y/y \).

As in [5, 8–10], the total wavefunction is expanded in a set of functions \( \Phi_n(\alpha, \hat{x}, \hat{y}; \rho) \), which are the solutions of an auxiliary boundary value problem on a hypersphere (for fixed \( \rho \)),

\[
\Psi(x, y) = \rho^{5/2} \sum_{n=1}^{\infty} f_n(\rho) \Phi_n(\alpha, \hat{x}, \hat{y}; \rho). \tag{2}
\]
corresponding to the smallest $\gamma^2(\rho)$, whose singular part is
\[ V_{\text{sing}}(\rho) = \frac{\rho^2 - 1/4}{\rho^2} + \frac{q}{\rho} \quad (8) \]
for $\rho \to 0$. Here the notations $\gamma \equiv \gamma_i(0)$ and $q \equiv \frac{d\gamma^2}{d\rho}|_{\rho=0}$ are introduced for brevity. A number of aspects concerning the Schrödinger equation with inverse square singularity, as in $V_{\text{sing}}(\rho)$, were multiply discussed in literature, e.g., in [8, 13–15]. Following [8, 10], the essential features of the quantum problem for the singular potential of the form $V_{\text{sing}}(\rho)$ (8) are briefly summarized below.

There are four variants for unambiguous description of the problem, which correspond to four intervals of the real-valued $\gamma^2$, as is understood by considering two solutions $f_\pm(\rho)$ of the Schrödinger equation with the potential $V_{\text{sing}}(\rho)$, whose leading-order terms for $\rho \to 0$ are given by $f_\pm(\rho) \sim \rho^{1/2\gamma}$. Consider first the case $\gamma^2 \geq 1$, for which one of the solutions, $f_+(\rho)$, is not square integrable and should be ruled out, thus, the problem is defined by the condition of square integrability or simply by $f(\rho) \to 0$. On the other hand, for $\gamma^2 < 1$ both solutions $f_\pm(\rho)$ are square integrable for $\rho \to 0$ and any linear combination of them is acceptable. To get rid of this ambiguity, it is necessary to choose a specific linear combination by introducing one additional real-valued parameter. In particular, for $\gamma^2 < 0$, i.e., pure imaginary $\gamma$, both solutions $f_\pm(\rho) \sim \rho^{1/2\gamma\pm i|\gamma|}$ oscillate for $\rho \to 0$. One possibility to define the unambiguous problem in this case is to fix a relative phase $\delta$ of two oscillating functions, i.e., by the requirement $f(\rho) \to 0$, $f_\pm(\rho) \to \rho^{1/2\gamma\pm i\delta} (\log(\rho) + \delta)$ [13, 14]. As an important consequence, one obtains the asymptotic energy spectrum, which depends exponentially on the level’s number, $E_n \sim e^{-2\gamma n/\gamma}$. In fact, these considerations explain the Efimov effect in the three-body problem [5, 16, 17]. One should also notice that the asymptotic energy spectrum of this form was derived already in [1].

For the interval $1/4 \leq \gamma^2 < 1$, it is necessary to take into account the less singular term $q/\rho$ in $V_{\text{sing}}(\rho)$ (8) because the next-to-leading order term in $f_\pm(\rho) \sim \rho^{1/2-\gamma} [1 + q \rho/(1 - 2\gamma)]$ is of principal importance. Thus, to define the problem for $0 \leq \gamma^2 < 1$ one should introduce an additional real-valued parameter $b$ by imposing the boundary condition, e.g., of the form proposed in [8, 10]
\[ f(\rho) \to \rho^{1/4} - \text{sgn}(b) |b|^{1/2} \rho^{1/4} \left(1 + \frac{q \rho}{1 - 2\gamma}\right), \quad (9) \]
where the $q$-dependent term can be omitted for $0 \leq \gamma^2 < 1/4$.

Special treatment is needed for two critical values $\gamma^2 = 0$ and $1/4$ separating different types of the problem definition. In particular, for $\gamma^2 = 0$ the boundary condition could be written in the form
\[ f(\rho) \to \rho^{1/2} \log \rho \quad (10) \]
where only positive values of the parameter are admitted ($b > 0$). For $\gamma^2 = 1/4$, one of the solutions is $f_- \sim 1 + q \rho \log \rho$, for $\rho \to 0$ and the $q$-dependent boundary condition can be written in the form
\[ f(\rho) \to \rho - b(1 + q \rho \log \rho) \quad (11) \]

The main result of this letter is a complete analysis of the Minlos–Faddeev regularization by using the explicitly given dependences $\gamma$ and $q$ on $\sigma$. Taking $\rho = 0$ in Eqs. (5) and (6), one finds
\[ \sigma = \frac{\sin \gamma \omega \pi}{\sin \omega} - \frac{\gamma \cos \omega \cos \gamma \pi}{2} \sin \frac{\gamma \pi}{2} \quad (12) \]
for two identical bosons and a distinct particles and
\[ \sigma = \frac{4 \gamma \pi - \sqrt{\gamma}}{6} \gamma \cos \gamma \pi \frac{\gamma \pi}{2} \sin \frac{\gamma \pi}{2} \quad (13) \]
for three identical bosons. Both Eqs. (12) and (13) implicitly determine monotonically increasing functions $\gamma(\sigma)$, which are shown in Fig. 1 both for the two-component system in the case $m/m_i = 1$ and for three identical bosons. Without regularization, i.e., if $\sigma = 0$, the well-known values of pure imaginary $\gamma = i 0.4136973$ for the two-component system with $m/m_i = 1$ and $\gamma = i 1.0062378$ for three identical bosons are obtained. Another parameter entering in the boundary condition for $\rho \to 0$ is expressed via $\gamma$ by using its definition and Eqs. (5) and (6), which gives for two identical bosons and a distinct particles
\[ q = \frac{4 \text{sgn}(a) \gamma \sin \frac{2\gamma \pi}{2}}{\sin \gamma \pi - \gamma \pi} \quad (14) \]
and for three identical bosons

\[ q = \frac{4 \text{sgn}(\gamma) \sin^2 \gamma \pi}{\sin \gamma \pi - \gamma \pi - \frac{32 \pi}{3 \sqrt{3}} \sin \gamma \pi \sin^2 \gamma \pi} \]  

(15)

The dependences \( q(\sigma) \) for the two-component system in the case \( m/m_1 = 1 \) and for three identical bosons are depicted in Fig. 1.

As follows from the above results, the regularization suggested by Minlos and Faddeev gives rise to four separate outcomes depending on the value of \( \sigma \), which is a consequence of the correspondence with \( \gamma^2 \). Different types of the regularized problem are separated by the critical values \( \sigma_r \), \( \sigma_c \), and \( \sigma_e \), which correspond to \( \gamma^2 = 0 \), \( 1/4 \), and 1, respectively. Explicitly, the three-body problem becomes regularized for \( \sigma \geq \sigma_r \), and one real-valued parameter should be introduced to impose the boundary condition (9) in the triple-collision point for \( \sigma_r > \sigma \geq \sigma_c \). Furthermore, the condition should be \( q \)-dependent for \( \sigma_r > \sigma > \sigma_c \), whereas \( q \)-dependence can be safely omitted for \( \sigma_r > \sigma \geq \sigma_c \). At last, the Efimov or Thomas effect takes place for \( \sigma_c > \sigma \) and the famous exponential dependence on the level’s number, \( E_n \sim e^{-2\pi n/|\gamma|} \) could be obtained by introducing one-parameter boundary condition in the triple-collision point. At two critical points \( \sigma = \sigma_c \) and \( \sigma = \sigma_r \), one should use the specific boundary conditions (10) and (11), respectively.

As follows from Eq. (12), the critical values for the two-component system are given by

\[ \sigma_c = \frac{2}{\pi} \left( \frac{\omega}{\sin \omega} - \cos \omega \right), \]  

(16)

and \( \sigma_e = 1 \). The dependences \( \sigma_c(m/m_1) \) and \( \sigma_e(m/m_1) \) are depicted in Fig. 2. For the case of equal masses \( (m/m_1 = 1) \), from Eq. (12) one finds \( \sigma_r = 1 \), \( \sigma_c = 3\sqrt{3}/4 - 1 \approx 0.29904 \), and \( \sigma_e = 2/3 - \sqrt{3}/\pi \approx 0.11534 \). This value of \( \sigma_c \) was also given in [4]. Similarly, from Eq. (13) one finds the critical values \( \sigma_r = 2 \), \( \sigma_c = 7\sqrt{3}/4 - 2 \approx 1.03109 \), and \( \sigma_e = 4/3 - \sqrt{3}/\pi \approx 0.78200 \) for three identical bosons. Exactly this value of \( \sigma_e \) was also given in [1–3].

It is worthwhile to present three values of \( q \) corresponding to \( \sigma_r, \sigma_c, \) and \( \sigma_e \), namely, \( |q(\sigma_r)| = 486/\left( \pi (81 + 8\sqrt{3}) \right) \approx 1.24225 \), \( |q(\sigma_c)| = 18/(8\sqrt{3} - 2) \).
18 \pi - 3 \pi \approx 1.11757, \text{ and } |q(\sigma_r)| = \frac{12}{(5\pi)} \approx 0.76394 \text{ from Eq. (14) for the two-component system in the case } m/m_t = 1 \text{ and } |q(\sigma_r)| = \frac{486}{[\pi(81 + 16\sqrt{3}\pi)]} \approx 0.920483, \text{ and } |q(\sigma_r)| = \frac{12}{(7\pi)} \approx 0.545674 \text{ from Eq. (15) for three identical bosons.}

To elucidate the above results, the bound-state energies of three identical bosons are determined as a function of \( \sigma \) in the interval \( \sigma \leq \sigma < \sigma_c \). The calculated energy dependence is presented in Fig. 3 for the following values of \( b \) (top to bottom): in the left part of panel (a) \( \{−0.08, −0.16, −0.32, −0.64, −2.0, ∞, 2.0, 0.64, 0.32\} \), in the right part of panel (a) \( \{0.64, 0.32, 0.16, 0.08\} \), in panel (b) \( \{0, −0.01, −0.02, −0.04, −0.08, −0.16, −0.32, −0.64, −2.0, ∞\} \), in the left part of panel (c) \( \{0.32, 0.16, 0.08, 0.04\} \), and in the right part of panel (c) \( \{−0.16, −0.32, −0.64, −2.0, ∞, 2.0, 0.64, 0.32, 0.16, 0.08\} \).

It turns out that at most one bound state exists and the bound-state energy monotonically increases with increasing \( b \) at fixed \( \sigma \). If \( \sigma > \sigma_c \) only a bound state exists for \( a < 0 \) and its energy monotonically increases from \( E_\text{c} \approx −1.0670 \) to the threshold \( E_\text{th} = −1 \) with increasing \( \sigma \) within the small interval \( \sigma_c \leq \sigma < \sigma_0 \approx 0.7825 \). If \( b \to \infty \), there is a bound state, whose energy for \( a > 0 \) decreases from \( E_\text{c} \) to \( −\infty \) with \( \sigma \) increasing from \( \sigma_0 \) to \( \sigma_\text{cr} \), and for \( a < 0 \) increases from \( −\infty \) to \( E_\text{c} \approx −0.38792 \) with \( \sigma \) increasing from \( \sigma_0 \) to \( \sigma_\text{cr} \). Only positive \( b \) are admitted at the critical value \( \sigma_\text{cr} \), therefore, all the energies for \( b < 0 \) tend to same value \( E_\text{c} \) for \( \sigma \to \sigma_\text{cr} \). At another boundary of the interval, all the existing energies tend to \( E_\text{c} \) for \( a < 0 \) and disappear for \( a > 0 \). Recall that there are no bound states for \( \sigma \geq \sigma_\text{cr} \).

In the limit \( |a| \to \infty \), the hyper–radial Eqs. (7) become decoupled and one bound state exists for \( b > 0 \), whose energy is \( E = −\frac{4}{b^2}\left[\frac{\Gamma(\gamma)}{\Gamma(−\gamma)}\right]^{1/\gamma} \) and the channel function is \( f(\rho) = \rho^{1/2}\mathcal{K}_{\gamma}(\sqrt{E}\rho) \), where \( \tilde{\gamma} \) is related to \( \sigma \) by Eq. (13) and \( \mathcal{K}_\gamma(x) \) is a modified Bessel function.

Starting from the first suggestion to modify the two-body zero-range interaction proposed by Minlos and Faddeev [1] it was declared [2–4] that the three-body problem becomes regularized, if the regularization parameter \( \sigma \) is sufficiently large to suppress the Efimov or Thomas effects, i.e., if \( \sigma \) exceeds the critical value \( \sigma_\text{c} \) defined in Eq. (16).
In this work it was shown that the proposed regularization leads to different results in four intervals of the non-negative parameter $\sigma$, in particular, another and stricter condition $\sigma > \sigma_\text{c} > \sigma_\text{r}$ is necessary for unambiguous description of the three-body problem. Concerning the interval $\sigma_\text{c} \leq \sigma < \sigma_\text{r}$, it is necessary to set a boundary condition in the triple-collision point depending on one real-valued parameter $b$. Among different possibilities, the boundary condition depending on the additional parameter $b$ is chosen in this work as (9) for the interval $\sigma_\text{c} < \sigma < \sigma_\text{r}$ and (10) or (11) for two specific values of $\sigma$. It is essential that for the smaller interval $\sigma < \sigma_\text{c}$ the boundary condition contains the parameter $q$, which is exactly determined by Eqs. (14) and (15). At last, the Efimov or Thomas effects are present for $\sigma < \sigma_\text{c}$ and the exponential asymptotic tail of the energy spectrum takes place after introducing the boundary condition in the triple-collision point.

To exemplify in details the main conclusions, three critical values $\sigma_\text{c}$, $\sigma_\text{r}$, and $\sigma_\text{e}$ are determined both for the two-component system consisting of two identical bosons and a distinct particle and for the system consisting of three identical bosons. The effect of regularization is additionally demonstrated by the calculation of the bound-state energy of three identical bosons as a function of $\sigma$ and $b$.

It is worthwhile to mention that the described scenario is quite general. In fact, it could be anticipated for any problem, whose essential properties are determined by the effective potential with the singular part $\sim \rho^{-2}$, which strength goes through the critical values. Besides the two-component system consisting of two identical particles (either fermions or bosons) and a distinct one, which was described in \cite{8, 10}, this scenario could be of importance also for the three-body problem in the mixed dimensions \cite{18–20} or in the presence of spin-orbit interaction \cite{21–23}.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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