Application of the Veneziano Model in Charmonium Dalitz Plot Analysis

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We adapt the Veneziano model to the analysis of vector charmonium decays. Starting from a set of covariant Veneziano terms we show how to construct partial waves amplitudes that receive contributions from selected Regge trajectories. The amplitudes, nevertheless retain the proper asymptotic limit. This arises from duality between directly produced resonances and cross-channel Reggeon and in practical applications helps remove uncertainties in the parametrization of backgrounds.

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I. INTRODUCTION

We consider the generalized Veneziano amplitude \cite{1} and its application in analyses of decays of heavy quarkonia. Specifically we focus on the decays of vector charmonia, e.g. $J/\psi$ and $\psi'$ to three pions.

Decays of charmonia have been investigated by MARKII, CLEO, BaBar, BES and more recently by BE- SIII \cite{2–3}. One of the original motivations was to verify perturbative QCD \cite{10}. The QCD calculations are based on the assumption that the initial quarkonium wave function and the wave functions of the light hadrons in the final state factorize. The latter is supposed to be produced through a calculable short distance process that follows annihilation of the $c\bar{c}$ pair. The predicted ratio of branching ratios, $BR(\psi' \to \rho\pi)/BR(J/\psi \to \rho\pi) \sim 12\%$ appears, however, to be significantly above the measurements, which determine this ratio to be of the order of 1\%. The di-pion spectrum is dominated by the $\rho(770)$ resonance and this so-called $\rho - \pi$ puzzle still remains largely unresolved \cite{11–13}. To better understand its origin may require gaining further information about $c\bar{c}$ wave functions and/or light quark production dynamics. These can be determined, at least indirectly, by comparing microscopic model predictions with measurements of charmonium couplings to light quark resonances other then the $\rho(770)$ meson. In this paper we discuss methods for determining these couplings.

To be able to determine which resonances are produced in a given reaction it is necessary to perform a partial wave analysis (PWA). While the full reaction amplitude is a function of the energies, momenta and helicities of external particles, partial waves emerge after the amplitude is projected onto waves with well defined angular momenta. These are associated with resonances appearing in the intermediate states. In data analysis, partial waves are often used from the start without reference to the underlying reaction amplitude. A finite sum of partial waves, however, cannot reproduce singularities of the full amplitude and analyses based solely on a model for partial waves are insensitive to a large set of dynamical constraints. Without prior knowledge, using only the energy dependence of a single partial wave, it is difficult to determine resonance parameters unambiguously. On the other hand the full amplitude does in principle contain information about all resonances. In particular the asymptotic behavior in the cross-channel energy variable is given by the leading Regge pole of the direct channel partial waves. Partial waves possess a rich analytical structure which extends beyond the energy dependence at fixed angular momentum. They are analytical functions of complex angular momentum and the motion of singularities in the angular momentum plane as a function of energy gives a connection between resonances of varying spins and masses. This connection is specific to the underlying dynamics responsible for resonance formation. For example, a linear rise of a Regge pole trajectory is a manifestation of confinement and is related to the existence of an infinite number of bound states.

While it is unknown how to construct reaction amplitudes that take full advantage of S-matrix constraints, when applicable, the Veneziano model and its various extensions are a good starting point in developing amplitude models. In this paper, using the Veneziano model, we want to show how important it is to go beyond individual partial waves and to illustrate the benefits of considering the full amplitude by applying the model to the specific case of charmonium decays.

The paper is organized as follows. In Section II we give a brief description of the Veneziano model. There is an extensive literature on the subject and for more details we refer the reader to \cite{14} and references therein. In Section III we discuss the procedure used to isolate selected poles. The amplitudes obtained in this section are used in Section IV to analyze di-pion mass distributions from charmonium decays. Summary and outlook are given in Section V.

II. GENERALIZED VENEZIANO AMPLITUDES

Properties of the Veneziano model will be illustrated by considering the decays of vector charmonia, $J/\psi$ and $\psi'$ to three pions, $\pi^+\pi^-\pi^0$. For simplicity, we neglect
the pion mass since \( M_\pi^2 \gg m_\pi^2 \): in units of GeV, \( M_\pi^2 = O(10) \) is the mass squared of the decaying particle. The generalization to other reactions that involve four external particles is in principle straightforward and a number of 2-to-2 and 1-to-3 process have been considered this way in the past. It is worth noting that the original paper by Veneziano \( [1] \) deals with another vector-to-three pion decay, namely \( \omega \to 3\pi \). The main difference between \( \omega \) and charmonium decays is that in the latter the 3\( \pi \) phase space is significantly larger and direct production of several di-pion resonances above the \( \rho(770) \) is possible.

In the Veneziano model the amplitude \( A \), describing a decay of a vector meson with momentum \( p \) and helicity \( \lambda \) to three pions, \( V(p, \lambda) \to \pi^+(p_1)\pi^0(p_2)\pi^-(p_3) \), after the isospin tensor \( \epsilon_{ijk} \) has been factored out, is given by

\[
A(s, t, u) = K[A_{n,m}(s, t) + A_{n,m}(s, u) + A_{n,m}(t, u)], \tag{1}
\]

Here \( s, t \) and \( u \) are the standard Mandelstam variables, \( s + t + u = M_\pi^2 \) and the scalar functions \( A_{n,m} \) are given by

\[
A_{n,m}(s, t) = \frac{\Gamma(n - \alpha_0)\Gamma(n - \alpha_1)}{\Gamma(n + m - \alpha_0 - \alpha_1)}, \tag{2}
\]

with \( n, m \) being positive integers and \( 1 \leq m \leq n \). The lower limit on \( m \) guarantees that \( A(s, t, u) \) has the expected high-energy behavior (see below) and the upper limit eliminates double poles in overlapping channels. The leading, linear Regge trajectory \( \alpha(s) = \alpha_0 + \alpha's \) is denoted by \( \alpha_s \) for short. Finally \( K \) is a kinematical factor,

\[
K = \epsilon_{\mu\nu\alpha\beta} \epsilon_{\rho\sigma}(p, \lambda) p_\mu^\rho p_\nu^\sigma p_\alpha^\lambda p_\beta^\lambda \tag{3}
\]

originating from presence of an odd-number of pions (unnatural parity) in the final state.

The Veneziano formula exhibits the expected behavior of the amplitude in the large-\( N_c \) limit, with the QCD boson spectrum saturated by narrow resonances and confinement resulting in linear Regge trajectories. This spectrum is manifested in the singularities of the amplitudes \( A_{n,m}(s, t) \), which have simple poles. For given \( n \), there is an infinity of \( s \)-channel poles labeled by a nonnegative integer \( k \) that are located at \( s = s_{n+k} \), satisfying

\[
\alpha(s_{n+k}) = n + k. \tag{4}
\]

In the vicinity of the pole the amplitude \( A_{n,m}(s, t) \) is given by

\[
A(s \sim s_{n+k}) = \frac{\beta_{n,m,k}(t)}{s_{n+k} - s} \tag{5}
\]

where the residue,

\[
\beta_{n,m,k}(t) = \frac{(-1)^k}{\alpha' k!} \frac{\Gamma(n - \alpha_t)}{\Gamma(n - m - k - \alpha_t)} \tag{6}
\]

is a polynomial in \( t \) of the order \( L_{\text{max}} \equiv k + n - m \geq 0 \). We thus conclude that for each \( k \) the full amplitude of Eq. (1), in each \( s \)-channel \( (s, t, u) \), describes a finite number of degenerate, narrow (zero) width resonances that have spins in the range, \( 1 \leq \alpha \leq \alpha_{\text{max}} = L_{\text{max}} + 1 \). The additional unit of angular momentum arises from the angular dependence of the kinematic factor \( K \). The integers \( n \) and \( m \) determine which resonances contribute (poles) to the amplitude. It follows from Eqs. (4,6) that amplitudes with \( m = 1 \), i.e. \( A_{n,1} \), have poles whose location is determined by the leading trajectory and from all subsequent daughter trajectories. The amplitudes \( A_{n,2} \) have poles originating from the \( 1^{\text{st}} \) daughter and subsequent daughters, \( A_{n,3} \) from the \( 2^{\text{nd}} \) and all subsequent daughters, etc. The daughter trajectories are defined by,

\[
\alpha^{(m)}(s) \equiv \alpha(s) - (m - 1). \tag{7}
\]

So that the leading trajectory \( \alpha(s) \) corresponds to \( \alpha^{(1)}(s) \), \( \alpha^{(2)} \) is the \( 1^{\text{st}} \) daughter and so on. The trajectories and the spectrum are illustrated in Fig. [1].

For fixed-\( t \), the asymptotic behavior of \( A_{n,m}(s, t) \) at large-\( s \) reflects the presence of an infinite number of resonances in the \( t \)-channel. Using Stirling's formula one finds,

\[
A_{n,m}(s \to \infty, t) \propto \frac{1}{s} \frac{\Gamma(n - \alpha_t)}{\Gamma(n - m - k - \alpha_t)} (s)^{\alpha^{(m)}(k)}. \tag{8}
\]

For large-\( s \) the kinematical factor in Eq. (1) contributes an additional power of \( s \) so that, the full amplitude has
the expected Regge limit,
\[ A(s, t, u) \propto (-s)^\alpha(t) \] (9)
that arises from the leading, \( m = 1 \) trajectory.

### III. REMOVAL OF POLES

As described in the preceding section, for given \( n \) and \( m \) the amplitude \( A_{n,m} \) contains an infinite number of poles. Since production of resonances is reaction dependent, it is necessary to find a mechanism that allows for the residues to be process dependent and in particular for the possibility that some of them vanish if a resonance formation is forbidden, e.g. by a conservation law. One possibility is to consider linear combinations

\[ A_{n,m}(s, t) \rightarrow \mathcal{A}(s, t) = \sum_{n \geq 1, n \leq m \leq 1} c_{n,m} A_{n,m}(s, t). \] (10)

The coefficients \( c_{n,m} \) need to be chosen in such a way that \( \mathcal{A} \)'s only couples to resonances that contribute to the process in question. In the case considered here, of an isoscalar boson decaying to three pions, isospin conservation demands each pair of pions to be produced in the isospin-1 state, which together with Bose statistics forbids production of spin-even resonances in \( s, t \) and \( u \) channels.

One way to proceed is to construct combinations of \( A_{n,m} \)'s that result in \( \mathcal{A} \) containing only a finite number of Regge poles. As will be shown below, this requires an infinite number of terms in Eq. (10). Alternatively one can attempt data analysis with a finite number of linear combinations of the \( A_{n,m} \)'s and let the fit determine the coefficients \( c_{n,m} \) [17–19]. We find the first approach more appealing particularly in the context of resonance production. Resonance properties are constrained by unitarity. This forces Regge trajectories to be non-linear, but the Veneziano model relies on trajectories that are real and linear. Even though there are extensions of the Veneziano model that introduce non-linear trajectories [20] [21], it is far simpler to implement unitarity at the level of an isolated Regge pole [22] [24]. An amplitude that contains only a finite number of resonance poles, however, does not reproduce the Regge limit in the crossed channel. And it is important to preserve the asymptotic behavior since it helps constraining the background under directly produced resonances. Thus to take the full advantage of the Veneziano model, we will, at the end, need to consider an infinite number of poles.

Before we impose the asymptotic behavior on the forms given by Eq. (10), it is nevertheless useful to ask what choice of \( c_{n,m} \)'s produces an \( \mathcal{A} \) that contains only a finite number of resonance poles. Since the \( A_{n,m} \) amplitudes contain an infinite number of poles, in order to cancel all, but a finite number of them, an infinite number of \( c_{n,m} \)'s in Eq. (10) must be non-vanishing. It is not difficult to find a relation between the coefficients that decouples all, but a finite number of poles. Consider, for example, keeping only the pole at \( \alpha(s) = 1 \), i.e. at \( s = s_1 \). This pole is only present in the amplitude \( A_{1,1} \) since amplitudes with \( n > 1 \) have poles at \( s_n \geq s_2 \) (cf. Fig. [1]). There is only one amplitude \( A_{1,m} = A_{1,1} \) so a single coefficient \( c_{1,1} \) determines the residue and ultimately coupling to the pole at \( s = s_1 \). The amplitude \( A_{1,1} \), however, also has poles at higher masses located at \( \alpha_s = 2, 3, \cdots \) with residues that are polynomials in \( t \) of the order of \( O(1), O(2), \cdots \), respectively. If we only want to keep the pole at \( \alpha(s) = 1 \), these high mass poles of \( A_{1,1} \) must be canceled by the same poles in amplitudes \( A_{n,m} \) with \( n \geq 2 \). Specifically, the pole in \( A_{1,1} \) at \( \alpha_s = 2 \) can only be canceled by the same pole in the two amplitudes: \( A_{2,m}, m = 1, 2 \), since for \( n > 2 \) no other \( A_{n,m} \) contains this pole. The amplitudes \( A_{2,1} \) and \( A_{2,2} \) are polynomials in \( t \) of the order of \( O(1) \) and \( O(0) \), respectively. We can therefore uniquely determine the two coefficients, \( c_{2,1} \) and \( c_{2,2} \) in terms of \( c_{1,1} \) so that the first order polynomial in \( t \) at the \( s = s_2 \) pole of \( A_{1,1} \) is identical to the first order polynomial \( t \) at the pole of \( A_{2,1} \) and \( A_{2,2} \). This way we can make the sum of residues between the two amplitudes, \( A_{1,1} \), \( A_{2,1} \) and \( A_{2,2} \) at the pole \( \alpha(s) = 2 \) vanish identically. Similarly, at the \( \alpha_s = 3 \) pole of \( A_{1,1} \), the residue is an \( O(2) \) polynomial in \( t \). This pole is also present in \( A_{2,1} \) and \( A_{2,2} \) with residues order, \( O(2) \) and \( O(1) \) polynomials, respectively, and it is also present in \( A_{3,1} \), \( A_{3,2} \) and \( A_{3,3} \) with residues of the order of \( O(2) \), \( O(1) \) and \( O(0) \), respectively. With \( c_{2,1} \) and \( c_{2,2} \) already fixed, \( c_{3,1}, c_{3,2} \) and \( c_{3,3} \) are now uniquely determined in terms of \( c_{1,1} \) by the requirement that the total residue of the \( \alpha_s = 3 \) pole, which is an \( O(2) \) polynomial in \( t \), vanishes. Continuing in this way all poles in \( s \) satisfying \( \alpha(s) > 1 \) can be eliminated. It is easy to check that this is achieved by setting, for \( n \geq 2 \)
\[ c_{n,1} = \frac{c_{1,1}}{\Gamma(n)}, \quad c_{n,2} = -\frac{c_{1,1}}{\Gamma(n-1)}, \quad c_{n,m} = 0 \text{ for } m > 3. \] (11)

The resulting amplitude \( \mathcal{A} \) is then given by
\[ \mathcal{A}(s, t) = c_{1,1} \frac{2 - \alpha_s - \alpha_t}{(1 - \alpha_s)(1 - \alpha_t)}. \] (12)
where the subscript indicates the location of the pole. This simple result could have been anticipated. The combination of the \( \Gamma \) functions in \( A_{n,m}(s, t) \) can be written as an infinite sums of simple poles in \( s \). The amplitude is symmetric in \( s \) and \( t \), therefore if all poles but the one at \( \alpha_s = 1 \) are left, by \( s \leftrightarrow t \) symmetry \( \mathcal{A} \) must also contain a pole in \( t \) but not a double pole. This leaves Eq. (12) as the only possibility. As expected once the infinite number of poles has been eliminated, \( \mathcal{A} \) no longer exhibits the Regge limit. We will return to this point in the following subsection.

This elimination procedure can be generalized to produce amplitudes with isolated poles at any higher, integer value of \( \alpha_s \). For example, to construct an amplitude with a single pole in \( s \) at \( \alpha(s) = 3 \), one starts with the three amplitudes \( A_{3,m}, m = 1, 2, 3 \) and determines the coefficients \( c_{n,m} \) for \( n \geq 4 \) in terms of \( c_{3,1}, c_{3,2} \) and \( c_{3,3} \) that
remove all poles at \( \alpha(s) > 3 \). The most general structure of the amplitude with the pole at \( \alpha(s) = 3 \) only is therefore given by

\[
A_3(s, t) = \frac{(6 - \alpha_s - \alpha_t)(3 - \alpha_s)(3 - \alpha_t)}{(3 - \alpha_s)(3 - \alpha_t)} \sum_{i=1}^{3} a_{3,i}(-\alpha_s - \alpha_t)^{i-1}.
\]

The first factor in the numerator guarantees that \( A_3 \) does not have the double pole at \( \alpha_s = \alpha_t = 3 \). It is followed by a product of two monomials in \( \alpha_s + \alpha_t \) that generate \( O(2) \) polynomial in \( s \) or \( t \) at the pole located at \( \alpha_t = 3 \) or \( \alpha_s = 3 \), respectively. Having three parameters \( a_{3,m} \), \( m = 1, 2, 3 \) determining the amplitude \( A_3(s, t) \) enables to decompose the residue in terms of an arbitrary linear combination of partial waves with \( l = 0, 1, 2 \). We note, however, that once the kinematic factor \( K \) is taken into account, cf. Eq. (1), the \( \alpha(s) = 3 \) pole actually represents (narrow) resonances with spins \( l = 1, 2, 3 \). The coefficients \( a_{3,m} \), \( m = 1, 2, 3 \) can therefore be chosen to decouple the \( l = 2 \) isobar. In general we find that an amplitude \( A_n \), which has a single pole at \( \alpha_s = n \) or \( \alpha_t = n \) is given by

\[
A_n(s, t) = \frac{(2n - \alpha_s - \alpha_t)(n - \alpha_s)(n - \alpha_t)}{(n - \alpha_s)(n - \alpha_t)} \sum_{i=1}^{n} a_{n,i}(-\alpha_s - \alpha_t)^{i-1}.
\]

### A. Regge asymptotics

The large-\( s \) behavior of the amplitude \( A_n \), is given by \( s^{n-1} \). The expected, Regge asymptotic behavior, however, should be \( s^{\alpha(t)-1} \), cf. Eqs. (3-4). The Regge behavior can only emerge from an infinite number of poles, therefore we need to modify the procedure outlined above and allow for an infinite number of poles to be present in \( A \). If we include an infinite number of poles located at say, \( n > N \) where \( N \) is chosen such that \( N >> \alpha'M_\psi^2 \), \( (\alpha' \sim 0.9 \text{ GeV}^{-2} \) is the Regge trajectory slope), these poles will contribute a smooth background in the decay region.

With the \( c \)'s given by Eq. (11) and the sum over \( n \) in Eq. (10) truncated at \( n = N \) we find that instead of a single pole at \( \alpha = 1 \) we obtain

\[
A_1(s, t) \rightarrow A_1(s, t; N) = a_{1,1} \frac{2 - \alpha_s - \alpha_t}{(1 - \alpha_s)(1 - \alpha_t)} \times \frac{\Gamma(N + 1 - \alpha_s)\Gamma(N + 1 - \alpha_t)}{\Gamma(N)\Gamma(N + 2 - \alpha_s - \alpha_t)}.
\]

For \( s >> N \) this amplitude has the desired Regge behavior \( \propto s^{\alpha(t)-1} \). As expected it is free from poles in the range \( 2 \leq \alpha(s) \leq N \) and of course the same holds in the \( t \)-channel. For \( N \) large enough i.e. \( N >> \alpha'M_\psi^2 \) the contribution to the decay region of the undesired, high-energy poles located at \( \alpha \geq N + 1 \) is power suppressed

\[
A_1({s, t} < M_\psi^2; N) = a_{1,1} \frac{2 - \alpha_s - \alpha_t}{(1 - \alpha_s)(1 - \alpha_t)} \times \left[ 1 + O \left( \frac{\alpha'M_\psi^2}{N} \right) \right]
\]

thus as mentioned above, can be interpreted as background. The generalization of Eq. (15) to an amplitude, which in the decay region has a pole at \( \alpha = n \), i.e. generalization of Eq. (14) to an amplitude with proper Regge asymptotics is given by

\[
A_n(s, t; N) = \frac{2n - \alpha_s - \alpha_t}{(n - \alpha_s)(n - \alpha_t)} \sum_{i=1}^{n} a_{n,i}(-\alpha_s - \alpha_t)^{i-1} \times \frac{\Gamma(N + 1 - \alpha_s)\Gamma(N + 1 - \alpha_t)}{\Gamma(N + 1 - n)\Gamma(N + n + 1 - \alpha_s - \alpha_t)}.
\]

In the following we use these amplitudes to describe \( J/\psi \) and \( \psi' \), three pion decays.

### IV. APPLICATION TO VECTOR CHARMONIUM DECAYS

Both, the \( J/\psi \) and \( \psi' \) decays show a clear signal of \( \rho(770) \) production. In additional there is indication of higher mass resonance production in \( \psi' \) decays. This is not necessarily the case in \( J/\psi \) decays, nevertheless the single \( \rho(770) \) does not saturate the spectrum either. In the past we attempted to describe the \( J/\psi \) decay distribution with additional partial waves. We found that interference effects are strong and even after adding \( \pi\pi \) interactions up to \( \sim 1.6 \text{ GeV} \) the description remained quite poor. Continuing to expand the partial wave basis to cover even higher mass region would lead to a quite unconstrained analysis. On the other hand with the amplitudes developed in this paper, all partial waves are related to the same Regge trajectory and that gives a very strong constraint on amplitude analysis.

We will thus attempt to fit the di-pion mass distribution using a combination of amplitudes given by Eq. (17) truncated up to some maximal value of \( n = n_{\text{max}} \). For di-pion mass up to \( \sim 3.5 \text{ GeV} \) which is accessible in \( \psi' \) decay, resonances with masses corresponding to \( n \) up to \( \sim 12 \) can be directly produced. We have found however that when using only the di-pion mass projection the fit is quite insensitive to amplitudes with \( n \) larger than \( \sim 5-6 \). In the following we will therefore truncate the sum over \( n \) at \( n_{\text{max}} = 6 \). As long as \( N >> \alpha'M_\psi^2 \) we find little sensitivity to \( N \), so we take \( N = 20 \), which is comfortably above the boundary of available phase space in both \( J/\psi \) and \( \psi' \) decays. In terms of the \( s \)-channel partial waves, \( f_i(s) \), the scalar amplitude \( F \equiv A/K \) in Eq. (1)

\[
F(s, t, u) = \sum_{n=1}^{n_{\text{max}}} \left[ A_n(s, t; N) + A_n(s, u; N) + A_n(t, u; N) \right]
\]

(18)
is given by [25] 

$$F(s, t, u) = \sum f_l(s)P^l(z)$$  \hspace{1cm} (19)$$

where, ignoring the pion mass, $z = (t - u)/(M^2_{\psi} - s)$ is the cosine of $s$-channel scattering angle and $P_l(z)$ are the Legendre polynomials. The $s$-channel pole of $A$ located at $\alpha(s) = n$ contains partial waves with $l = 0, \cdots n$. At the pole located at $\alpha(s) = n$ the partial waves $f_l(s)$ are given by $f_l(s) = g_{n,l}/(n - \alpha(s))$ with

$$g_{n,l} = \int_{-1}^{1} \frac{dz}{2} [P_{l-1}(z) - P_{l+1}(z)] \text{Res}A_n(n, t; N) + (t \to u)$$  \hspace{1cm} (20)$$

where

$$\text{Res}A_n(n, t; N) = \sum_{i=1}^{n} a_{n,i}(-n - \alpha_i)^{-1} \equiv \sum_{i=1}^{n} a'_{n,i}i^{-1}.$$  \hspace{1cm} (21)$$

The residue $g_{n,l}$ is the product of two couplings. One of them is the coupling of the charmonium to a di-pion resonance of spin $l$ and mass $m_r$, given by $\alpha(m_r^2) = n$ and the other is the coupling of this resonance to the di-pion decay channel.

Decoupling of the spin-even resonances implies that $\text{Res}A_n(n, t; N)$ should be an even function of $t$. For $n = 1$ $\text{Res}A_n(n, t; N) = a'_{1,1}$ and from Eq (20) we can determine charmonium coupling to the $\rho(770)\pi$ intermediate state. For $n = 1$, we need to set $a'_{2,2} = 0$ to decouple the $l = 2$ wave. The $\alpha(m_r^2) = 2$ pole would then correspond to the first excitation of the $\rho$-meson, i.e. the $\rho(1450)$. For $n = 3$ we set $a'_{3,2} = 0$ and the pole at $\alpha(m_r^2) = 3$ describes the $l = 1$, second excitation of the $\rho$, i.e. the $\rho(1570)$ and the $l = 3$, $\rho_3(1690)$ resonance. For $n = 4$, $a'_{4,2} = a'_{4,4} = 0$ and we find two degenerate resonances with masses given by $\alpha(m_r^2) = 4$, and spins $l = 1, 3, 5$ that may be associated with $\rho(1900)$ and $\rho_3(1900)$, respectively. Similarly the $\alpha(m_r^2) = 5$ pole, with $a'_{5,2} = a'_{5,4} = 0$ describes resonances with $l = 1, 3, 5$, which can correspond to $\rho(2150)$, $\rho_3(2250)$ and $\rho_5(2350)$, respectively. No higher mass $\rho$’s are known [26]. The pole at $\alpha(m_r^2) = 6$ produces additional three resonances with spins $l = 1, 3, 5$ and if the fit was robust, it would constitute a discovery of these states.

For each channel, $J/\psi$ and $\psi'$ we thus fit twelve real parameters, $a'_{n,i}$ for $n = 1 \cdots 6$ and $i = 1, \cdots n$, $i$ = even which via Eq. (20) determine production times decay coupling of the twelve di-pion resonances discussed above. In addition we allow the trajectories to be imaginary when appearing in the denominators of the $A_n$’s in order to be able to account for the finite width of the resonances as required by unitarity. The $\rho$ trajectory is expected to be approximately equal to

$$\alpha(s) = 1 + \alpha'(s - m^2_{\rho}) + i\alpha' m_{\rho} \Gamma_{\rho}$$

$$\sim 0.47 + 0.9s + 0.1i\sqrt{s - 0.07}$$  \hspace{1cm} (22)$$

where we also included the phase space factor, $\sqrt{s - 4m^2_{\pi}}$ in the imaginary part.

The data and results of the fit are shown in Figs. 2, 3. The data is taken from for the resent measurement by the BESIII collaboration [9]. Unfortunately having no access to the Dalitz plot distribution we were only able to analyze the di-pion mass projection. Fitting mass projection carries larger systematic uncertainty compared to an event-by-event fit. We therefore also allow for the parameters of the trajectory, the intercept, slope and the magnitude of the imaginary part to float. The trajectory parameters,
\[ \alpha(s) = (0.61 \pm 0.04) + (0.68 \pm 0.08)s \\
+ (0.11 \pm 0.02)i \sqrt{s - 0.07} \]  
\[ \text{(23)} \]

obtained from the fit to the $J/\psi$ mass distribution are in good agreement with Eq. (22). As an estimate for systematic uncertainty we fit the $\rho(770)$ mass region with the $n = 1$ amplitude alone. In this case we find

\[ \alpha(s) = (0.57 \pm 0.08) + (0.73 \pm 0.12)s \\
+ (0.12 \pm 0.03)i \sqrt{s - 0.07}. \]  
\[ \text{(24)} \]

The $J/\psi$ mass distribution is dominated by the $\rho(770)$ and by fitting the mass projection, in general we find weak sensitivity to the higher mass resonances. This is reflected in large uncertainties we obtain for the fit parameters $\alpha_n$, and for this reason we do not attempt to determine resonance couplings. Nevertheless, examining Fig. 2 it is clear that the $\rho$ determine resonance couplings. Nevertheless, examining Fig. 2 it is clear that the $\rho(770)$ production alone is not capable of reproducing the data. Fitting data on an event-by-event basis might be possible to obtain a more reliable estimate of higher mass resonance production.

The $\rho(770)$ is much less pronounced in the decay of the $\psi'$. If we try to determine $\rho(770)$ production alone by restricting the fit to the $n = 1$ amplitude in the $M_{4\pi} < 1$ GeV mass region we find

\[ \alpha(s) = (0.45 \pm 0.51) + (1.0 \pm 0.9)s \\
+ (-0.08 \pm 0.02)i \sqrt{s - 0.07} \]  
\[ \text{(25)} \]

which is consistent with $J/\psi$ fit results but carries large statistical uncertainty. The fit with all six amplitudes gives

\[ \alpha(s) = (0.55 \pm 0.2) + (0.65 \pm 0.1)s \\
+ (0.26 \pm 0.01)i \sqrt{s - 0.07} \]  
\[ \text{(26)} \]

which well reproduces the real part but leads to a $\rho(770)$ width that is twice as large as observed. This is clearly seen in Fig. 3. We find that the $\psi'$ seems to be dominated by resonances in the $\alpha = 5$ mass region, i.e. $\rho(2150)$, $\rho_3(2250)$ and $\rho_5(2350)$ seen as the large bump in Fig. 3.

We expect that an event-by-event, likelihood fit would remove the large uncertainties we find in the couplings of these resonances [27].

V. SUMMARY AND OUTLOOK

Based on the Veneziano model we constructed a set of amplitudes which i) isolate contributions from individual Regge trajectories, including the daughters, ii) preserve the asymptotic behavior emerging from cross-channel Reggeons. The first property enables introduction of finite resonance widths, while the second helps avoid uncertainties in background parametrization common to analyses based on truncated partial wave expansion. Given the limited sensitivity of a fit to mass projections of the Dalitz plot, we retained the simple, linear parametrization of the real part of the trajectory, even though it introduces a high level of degeneracy between resonances. The partial wave amplitudes incorporate Regge poles. This eliminates a freedom in choosing which resonances to include in the analysis and gives a link between the measured spectrum and the underlying dynamics, i.e. in the large-$N_c$ limit the QCD spectrum is expected to match that of the Veneziano model. For precision studies based on event-by-event analysis, the approximations in Regge trajectories can be easily eliminated and tailored to reproduce the data. Since Regge poles factorize, self-consistency can be tested by comparing resonance couplings between various decay modes containing the same set of resonance, e.g. $K\bar{K}\pi^0$. Extensions of the Veneziano approach beyond four-particles are known [25] and the approach discussed here can therefore be generalized to higher multiplicities.

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