LARGE DEVIATIONS OF SUBGRAPH COUNTS
FOR SPARSE ERDŐS–RÉNYI GRAPHS

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Abstract. For each fixed integer \( \ell \geq 3 \) and \( u > 0 \) we establish the leading order of the exponential rate function for the probability that the number of cycles of length \( \ell \) in the Erdős–Rényi graph \( G(N, p) \) exceeds its expectation by a factor \( 1 + u \), assuming \( N^{-1/2} \ll p \ll 1 \) (up to log corrections) when \( \ell \geq 4 \), and \( N^{-1/3} \ll p \ll 1 \) in the case of triangles. We additionally obtain the sharp upper tail for Schatten norms of the adjacency matrix, and for general subgraph counts in a narrower range of \( p \), as well as the sharp lower tail for counts of graphs for which Sidorenko’s conjecture holds. As in recent works [CD16, Eld] on the emerging theory of nonlinear large deviations, our general approach applies to “low complexity” functions on product spaces, though the notion of complexity used here is somewhat different. For the application to subgraph counts, our argument yields a quantitative refinement of Szemerédi’s regularity lemma for random graphs in the large deviations regime.

1. Introduction
1.1. The “infamous” upper tail for triangle counts. For a graph \( H = ([n], E) \), the associated homomorphism counting function on graphs \( G \) over the vertex set \( [N] \) is given by

\[
\text{hom}_H(G) := \sum_{i_1, \ldots, i_n \in [N]} \prod_{k \in E} A_G(i_k, i_l),
\]

(1.1)

where \( A_G \) is the \( N \times N \) adjacency matrix for \( G \). That is, \( \text{hom}_H(G) \) counts the number of edge-preserving maps from \( [n] \) to \( [N] \). When \( H = C_\ell \), the cycle on \( \ell \geq 3 \) vertices, we have

\[
\text{hom}_{C_\ell}(G) = \text{Tr} A_G^\ell.
\]

For now we focus on the triangle homomorphism counting function \( \text{hom}_{C_3}() \). For \( N \) large and \( p \in (0, 1) \) possibly depending on \( N \), let \( G \sim G(N, p) \) be the Erdős–Rényi random graph on vertex set \( [N] \). One easily computes

\[
\mathbb{E} \text{hom}_{C_3}(G) = N(N-1)(N-2)p^3 = (1 + o(1))N^3 p^3.
\]

(Unless stated otherwise, all asymptotic notation is with respect to the limit \( N \to \infty \); see Section 1.7 for our notational conventions.) The “infamous upper tail problem” [JR02] is to determine the exponential rate function for the probability that \( \text{hom}_{C_3}(G) \) exceeds its expectation by a constant factor, that is to estimate

\[
\text{UT}(C_3, N, p, u) := -\log \mathbb{P}(\text{hom}_{C_3}(G) \geq (1 + u)N^3 p^3), \quad u > 0.
\]

(1.2)
A moment’s thought yields upper bounds on \( UT(C_3, \cdot) \) which turn out to be asymptotically tight (at least for some range of \( p \)). Indeed, one way to create on the order of \( N^3p^3 \) extra triangles is via the event

\[
\text{Clique}(a): \quad \text{Vertices } 1, \ldots, [aNp] \text{ form a clique,} \quad (1.3)
\]

for fixed \( a > 0 \). The probability of this event is

\[
\mathbb{P}(\text{Clique}(a)) = p^{\binom{[aNp]}{2}} \geq \exp\left( -\frac{1}{2}a^2N^2p^2 \log(1/p) \right).
\]

On this event the clique contributes \( (aNp)^3 \) extra triangle homomorphisms (assuming \( Np \to \infty \) and \( p = o(1) \)). Thus, taking \( a = u^{1/3} \), and intersecting with the high-probability (and independent) event that the complement of the clique contains \( (1 + o(1))N^3p^3 \) triangles, we have

\[
UT(C_3, N, p, u) \leq (1 + o(1)) \frac{u^{2/3}}{2} N^2p^2 \log(1/p).
\]

We get another upper bound on \( UT(C_3, N, p, u) \) by considering the event

\[
\text{Hub}(b): \quad \text{Vertices } 1, \ldots, [bNp^2] \text{ are connected to all other vertices} \quad (1.4)
\]

for fixed \( b > 0 \) (assuming \( p \gg N^{-1/2} \)). On this event, every edge in the complement of the hub \([bNp^2]\) forms a triangle with every vertex in the hub, giving \( \sim 3bN^3p^3 \) extra triangle homomorphisms (if \( p = o(1) \)). Taking \( b = u/3 \), we obtain \( UT(C_3, N, p, u) \leq (1 + o(1)) \frac{u}{3} N^2p^2 \log(1/p) \).

Thus we have

\[
UT(C_3, N, p, u) \leq (1 + o(1)) \min\left\{ \frac{u^{2/3}}{2}, \frac{u}{3} \right\} N^2p^2 \log(1/p). \quad (1.5)
\]

There is a third natural event to consider, that \( G \) has on the order of \( N^2p \) extra edges distributed uniformly across the graph. Indeed, this event turns out dominate the tail event for triangle counts in much of the dense regime (with \( p \) fixed) \cite{CV11, LZ15}. However, a short computation reveals that in the sparse regime \( p \to 0 \) this event can not compete with the events Clique and Hub (though, as seen in Section 1.5, it does give the leading order contribution for the lower tails for certain functions).

Lower bounds on \( UT(C_3, N, p, u) \) (that is, upper bounds on upper tail for triangle counts in \( G \)) have a long history in the literature. Using the machinery of polynomial concentration, Kim and Vu showed \cite{KV04}

\[
UT(C_3, N, p, u) \gtrsim_u N^2p^2
\]

for all \( p \geq N^{-1} \log N \) and fixed \( u > 0 \), which matches the upper bound (1.5) up to the factor \( \log(1/p) \). The missing logarithm was obtained in work of Chatterjee \cite{Cha12} and DeMarco and Kahn \cite{DK12b}, who showed

\[
UT(C_3, N, p, u) \lesssim_u N^2p^2 \log(1/p)
\]

for all \( p \geq N^{-1} \log N \) and fixed \( u > 0 \). The focus then shifted to the asymptotic dependence of \( UT(C_3, \cdot) \) on \( u \), i.e. to find a formula for \( c_3(u) \) such that

\[
UT(C_3, N, p, u) \sim c_3(u) N^2p^2 \log(1/p).
\]

In the dense regime where \( p \in (0, 1) \) is fixed, Chatterjee and Varadhan used the theory of graphons to express \( \lim_{N \to \infty} \frac{1}{N} \mathbb{E} UT(C_3, N, p, u) \) as the solution to a variational problem \cite{CV11}, which was subsequently analyzed by Lubetzky and Zhao in \cite{LZ15}.

The sparse case with \( p = N^{-c} \) for some constant \( c \in (0, 1) \) falls outside the purview of graphon theory. A breakthrough was made in \cite{CD16}, which introduced a general nonlinear
large deviations framework, and as an application showed that for \( N^{-\kappa_3} \log N \leq p \ll 1 \), \( \kappa_3 = 1/42 \), the value of \( \text{UT}(C_3, N, p, u) \) is asymptotically given by the solution of a variational problem (stated in (1.12) below). This variational problem was solved by Lubetzky and Zhao in [LZ17]. For more on these developments we refer to Chatterjee’s survey [Cha16]. The sparsity range was improved in the subsequent work [Eld] of Eldan to have \( \kappa_3 = 1/18 \) as a consequence of general advances in the theory of nonlinear large deviations. Together with the analysis of [LZ17], one thus has for \( p_\ast(N) = N^{-1/18} \log N \) and any fixed \( u \geq 0 \),

\[
\text{UT}(C_3, N, p, u) = (1 + o(1)) \min \left\{ \frac{u^{2/3}}{2}, \frac{u}{3} \right\} N^2 p^2 \log(1/p) \quad p_\ast(N) \ll p \ll 1. \tag{1.6}
\]

The above is expected to hold for \( p_\ast(N) = N^{-1/2} \); see Open Problem 4 in [Cha16, Section 11]. In Corollary 1.6 below, we obtain the optimal range \( p_\ast(N) = N^{-1/2} \) (up to logarithmic corrections), for counts of cycles of length \( \ell \geq 4 \), matching the full range of validity of a formula established by Bhattacharya et al. [BGLZ17] for the solution to the associated variational problem (see Theorem 1.5). At the same time we improve (1.6) up to \( p_\ast(N) = N^{-1/3} \). For homomorphism counts of general connected graphs \( H \) of maximum degree \( \Delta \geq 2 \) we cover the range up to \( p_\ast(N) = N^{-1/(3\Delta - 2)} \) (see Theorem 1.7, with a wider range for irregular graphs). The exponent \( (3\Delta - 2)^{-1} \) is within a constant factor of the optimal exponent of \( \Delta^{-1} \) for the formula in Theorem 1.5. Our general approach also gives bounds for the lower tail for counts of cycles and of graphs having the “Sidorenko” property, as well as the Schatten norms of the random adjacency matrix \( A_G \) (see (1.33)). As a byproduct of our approach we obtain quantitative versions of the classic regularity and counting lemmas, optimized for sparse random graphs, which may be of independent interest (see Subsection 2.3); in particular, we address Open Problem 5 in [Cha16, Section 11], which asks for a version of the regularity lemma suitable for the study of large deviations for sparse random graphs.

1.2. Results for homomorphism counts. Given a graph \( H = (V,E) \), the homomorphism counting function of (1.1) extends to symmetric \( N \times N \) matrices \( X \) as follows:

\[
\text{hom}_H(X) := \sum_{\varphi: V \to [N]} \prod_{e=uv \in E} X_{\varphi(u)\varphi(v)}. \tag{1.7}
\]

(When \( E = \emptyset \) we take the empty product to be 1, so that \( \text{hom}_H(X) = N^{|V|} \) in this case.) We denote by \( X_N \) the set of all symmetric \( N \times N \) matrices with entries in \([0,1]\) and zeros on the diagonal, and by \( A_N \subset X_N \) the set of adjacency matrices for graphs on \([N]\) vertices. For \( d \in \mathbb{N}, x \in [0,1]^d \) and \( p \in [0,1] \) denote

\[
I_p(x) := \sum_{i=1}^d x_i \log \frac{x_i}{p} + (1 - x_i) \log \frac{1 - x_i}{1 - p}, \tag{1.8}
\]

which is the Kullback–Leibler divergence \( D_{KL}(\mu_x||\mu_p) \) between the (product) Bernoulli measures with centers of mass \( x = (x_i) \) and \( p \) (we make the convention \( 0 \log 0 := 0 \)). Then, for \( h : [0,1]^d \to \mathbb{R}, \; p \in [0,1] \) and \( t \in \mathbb{R} \) write

\[
\phi_p(h, t) := \inf \left\{ I_p(x) : x \in [0,1]^d, \; h(x) \geq t \right\}, \tag{1.9}
\]

\[
\psi_p(h, t) := \inf \left\{ I_p(x) : x \in [0,1]^d, \; h(x) \leq t \right\}. \tag{1.10}
\]
We will often have \( d = \binom{N}{2} \) with \([0, 1]^d \cong \mathcal{X}_N\) and \( h = N^{-|V|}p^{-|E|} \hom_H(\cdot)\) for some graph \( H\), in which case we denote
\[
I_p(X) = \sum_{1 \leq i < j \leq N} I_p(x_{ij}), \quad X = (x_{ij}) \in \mathcal{X}_N, \tag{1.11}
\]
and
\[
\Phi_{N,p}(H,t) := \inf \left\{ I_p(X) : X \in \mathcal{X}_N, \hom_H(X) \geq tN^{|V|}p^{|E|} \right\}, \tag{1.12}
\]
\[
\Psi_{N,p}(H,t) := \inf \left\{ I_p(X) : X \in \mathcal{X}_N, \hom_H(X) \leq tN^{|V|}p^{|E|} \right\}. \tag{1.13}
\]

Our results here and in Subsections 1.4–1.5 establish the above quantities as the large deviation rate for the upper and lower tails, respectively, of the associated homomorphism counts in \( G \), with the corresponding expressions for Schatten norms of the adjacency matrix. In terms of the assumed range of sparsity parameter \( p \) our strongest results for the upper tail are for cycles \( C_\ell \) (cf. Theorem 4.1 for a more quantitative version).

**Theorem 1.1** (Large deviations for cycles counts, qualitative version). Fix an integer \( \ell > 2 \). If \( 0 < p \ll 1 \) satisfies
\[
p \gg \max \left( N^{\frac{3}{2} - 1}, \frac{(\log N)^{\frac{\ell - 4}{2\ell - 2}}}{\sqrt{N}} \right), \tag{1.14}
\]
then for any fixed \( t > 1 \),
\[
\mathbb{P} \left( \hom_{C_\ell}(G) \geq tN^\ell p^\ell \right) \leq \exp \left( -\Phi_{N,p}(C_\ell, t - o(1)) + o(N^2 p^2 \log(1/p)) \right). \tag{1.15}
\]
If
\[
\left( \frac{\log N}{N} \right)^{\frac{\ell - 2}{2\ell - 2}} \ll p \ll 1, \tag{1.16}
\]
then for any fixed \( 0 < t < 1 \),
\[
\mathbb{P} \left( \hom_{C_\ell}(G) \leq tN^\ell p^\ell \right) \leq \exp \left( -\Psi_{N,p}(C_\ell, t + o(1)) + o(N^2 p) \right). \tag{1.17}
\]

**Remark 1.2.** Ignoring the log factors, the exponent \( \frac{\ell - 2}{2\ell - 2} = \frac{1}{2} - \frac{1}{2\ell - 2} \) of \( N \) in (1.16) asymptotically matches the exponent \( 1/2 \) in (1.14) as \( \ell \to \infty \). For the case of even \( \ell \), Theorem 1.21 extends (1.17) to hold for all \( p = p(N) \in (0,1) \). For the case \( \ell = 3 \), whereas (1.16) enforces \( p \gg (\log N)/N)^{1/4} \), recent independent work of Kozma and Samotij [KS] establishes (1.17) for \( p \gg N^{-1/2} \).

**Remark 1.3.** In the independent work [Aug], posted to arXiv shortly after this paper, Auger obtains (1.15) for all \( \ell \geq 3 \) and \( p \gg (\log N)^2/\sqrt{N} \). This improvement for the case \( \ell = 3 \) is an outcome of a general advance on large deviations for nonlinear functions on product spaces having the *low-complexity gradient* condition used in [CD16,Eld], whereas in the present work we avoid this condition. Actually, [Aug] eliminates the first term in the maximum in (1.14) by relying on strong concentration of the empirical spectral measure of \( A \) around its even, semi-circle, limit. Indeed, upon replacing our use of the Schatten norm \( \|A\|_{S_i} \) to control the bulk contribution by the concentration results from [Aug, Proof of Lemma 4.1], one recovers her improvement for \( \ell = 3 \), without appealing to a low-complexity gradient.

**Remark 1.4.** As the quantitative version Theorem 4.1 shows, we can allow \( \ell = \ell(N) \) to grow at rate \( (\log N)^{o(1)} \), and we can further allow \( \ell \sim (\log N)^C \) for fixed \( C < \infty \) at the expense of increasing the power of the logarithmic corrections in the lower bounds on \( p \) by \( O(C) \) (as can be seen from (4.7), taking \( W(N) \) to grow poly-logarithmically).
The following result of [BGLZ17] (extending the earlier work [LZ17] for the case of cliques), solves the upper tail variational problem (1.12) in a wide range of values of $p$.

**Theorem 1.5** ([BGLZ17]). Fixing a connected graph $H = (V, E)$, let $H^* = H[V^*]$ denote the induced subgraph on the subset $V^* \subseteq V$ of vertices of maximal degree $\Delta \geq 2$ (so $H^* = H$ when $H$ is regular). For $u > 0$ let $\theta_H(u)$ be the unique $\theta > 0$ satisfying $P_{H^*}(\theta) = 1 + u$, for the independence polynomial $P_{H^*}(x) = 1 + \sum_{k=1}^{\ell} a_k x^k$, where $a_k$ counts the independent sets of size $k$ in $H^*$. For $N^{-1/\Delta} \ll p \ll 1$ and any fixed $u > 0$, 

$$\phi_{N,p}(H, 1 + u) = (c_H(u) + o(1)) N^{2p^{2/\Delta} \log(1/p)}$$

where

$$c_H(u) := \begin{cases} \min\{\theta_H(u), \frac{1}{2} u^{2/|V|}\} & \text{if } H \text{ is regular}, \\ \theta_H(u) & \text{if } H \text{ is irregular}. \end{cases} \quad (1.18)$$

For the case of cycles we have recursively that 

$$P_{C_{\ell}}(x) = 1 + 2x, \quad P_{C_{\ell+1}}(x) = 1 + 3x, \quad P_{C_{\ell+2}}(x) = P_{C_{\ell+1}}(x) + x P_{C_{\ell}}(x), \quad \ell \geq 4. \quad (1.19)$$

Writing $c_\ell(u) := c_{C_{\ell}}(u)$, we have for instance that 

$$c_3(u) = \begin{cases} \frac{1}{2} u & u \leq 27/8, \\ \frac{1}{2} u^{2/3} & u > 27/8, \end{cases} \quad c_4(u) = \begin{cases} -1 + \sqrt{1 + \frac{1}{2} u} & u \leq 16, \\ \frac{1}{2} \sqrt{u} & u \geq 16. \end{cases}$$

Extending the notation (1.2) to general graphs $H = (V, E)$ as 

$$\text{UT}(H, N, p, u) := - \log \mathbb{P} \left( \text{hom}_H(G) \geq (1 + u) N^{|V|} p^{|E|} \right), \quad (1.20)$$

yields the following corollary upon combining Theorem 1.1 and Theorem 1.5.

**Corollary 1.6** (Upper tail for cycle counts). Fix an integer $\ell \geq 3$ and let $0 < p \ll 1$ be as in (1.14). For any fixed $u > 0$, 

$$\text{UT}(C_{\ell}, N, p, u) = (c_\ell(u) + o(1)) N^{2p^{2/\Delta} \log(1/p)}, \quad (1.21)$$

where 

$$c_\ell(u) = \min \left\{ \theta_\ell(u), \frac{1}{2} u^{2/\ell} \right\} \quad (1.22)$$

and $\theta_\ell(u)$ is the unique $\theta > 0$ such that $P_{C_{\ell}}(\theta) = 1 + u$ for $P_{C_{\ell}}(\cdot)$ of (1.19).

**Proof.** The lower bound on $\text{UT}(\cdot)$ in (1.21) is an immediate consequence of Theorems 1.1 and 1.5. The matching upper bound is established similarly to (1.5) by consideration of the events $\text{Clique}(a)$ and $\text{Hub}(b)$ for appropriate $a = a'(\ell) u^{1/\ell}$ and $b = b'(\ell) u$, essentially following the lines of the proof of [BGLZ17, Proposition 2.4].

We note that for $N^{-1} \ll p \ll N^{-1/2}$ the upper tail no longer has the form of the right hand side of (1.21) – see [BGLZ17, Section 1.3] for further discussion of this. (In particular, the event $\text{Hub}(b)$ is no longer viable in this regime of sparsity.) The variational problem (1.12) was solved in the regime $N^{-2/\Delta} \ll p \ll N^{-1/\Delta}$ for the case of regular graphs in [LZ17, BGLZ17]. For $p \ll N^{-1/\Delta}$ and general $H$, even the order of $\text{UT}(H, N, p, u)$ up to constants depending only on $u$ has not been completely settled. Indeed, the conjectured dependence on $H, N, p$ from [DK12a] has recently been refuted in certain cases, see [SW] and the references therein on the rich history of this problem.
For upper tails of homomorphism counts our sharpest results are for cycles, but as stated next, we also improve on all previous works for the general case. To this end, let

$$\Delta_s(H):=\frac{1}{2}\max_{\{v_1,v_2\}\in E}\{\deg_H(v_1)+\deg_H(v_2)\} \geq 1.$$  \hfill(1.23)

**Theorem 1.7** (Upper tail for homomorphism counts). Let \(H=(V,E)\) be a connected graph of maximal degree \(\Delta \geq 2\) and with \(\Delta_s = \Delta_s(H)\) as in (1.23). Then, for any fixed \(u > 0\) and \(N^{-1/((\Delta+2\Delta_s)-2)}(\log N)^5|V|/\Delta \ll p \ll 1\),

with \(c_H(u)\) as in (1.18), we have

\[ UT(H,N,p,u) = (c_H(u) + o(1))N^2p^{\Delta} \log(1/p). \]

Note that \(\Delta + 2\Delta_s - 2 \in [2\Delta - 1, 3\Delta - 2]\), with the upper bound holding when \(H\) is regular, and the lower bound holding when \(H\) is a star. We note in particular that in Theorem 1.7, the reciprocal of the exponent of \(N\) in the lower bound on \(p\) scales linearly with the maximum degree of \(H\), rather than with the number of edges in \(H\) as in the previous works [CD16, Eld].

### 1.3. Relation to previous works and new ideas

Before stating our results for Schatten norms and for lower tails of homomorphism counts for Sidorenko graphs (see Subsections 1.4–1.5), we pause to discuss the ideas leading to our main results.

Previous work on nonlinear large deviations focused on approximating the partition function \(Z\) for Gibbs measures on the Hamming cube. Specifically, given a Hamiltonian \(f: \{0,1\}^d \to \mathbb{R}\) with associated Gibbs measure \(\mu\) of density \(Z^{-1}e^{f(x)}\) on \(\{0,1\}^d\) the aim is to approximate

\[ Z = \sum_{x \in \{0,1\}^d} \exp(f(x)). \]

This generalizes the problem of determining the large deviations of a function \(h\) of a vector \(x \in \{0,1\}^d\) with i.i.d. Bernoulli\((p)\) components, i.e. of approximating

\[ \log \mathbb{P}(h(x) \geq t \mathbb{E} h(x)), \]

which corresponds to \(\log Z\) for

\[ f_h(x) := g(h(x)) + d \log(1-p) + \sum_{i=1}^d x_i \log \frac{p}{1-p}, \]

where \(g(s) \equiv 0\) for \(s > t \mathbb{E} h(x)\) and \(g(s) \equiv -\infty\) for \(s < t \mathbb{E} h(x)\).

The Gibbs variational principle frames the log-partition function (or Helmholtz free energy) as the solution to a variational problem:

\[ \log Z = \sup_{\nu \in \mathcal{M}_1((0,1)^d)} \int f(x) d\nu(x) - \text{D}_{\text{KL}}(\nu\|\mu), \]

where the supremum ranges over all probability measures on the cube, and \(\text{D}_{\text{KL}}(\cdot\|\cdot)\) is the Kullback–Leibler divergence. The feasible region for optimization has dimension exponential in \(d\); to reduce dimensionality it is common practice in physics to invoke the naive mean field approximation, which is to restrict \(\nu\) to range over product measures. When the Hamiltonian has the separable form \(f(x) = f_1(x_1) + \cdots + f_d(x_d)\), so that \(\mu\) is itself a product measure, the naive mean field approximation is an exact identity.

The main idea introduced in [CD16] (see also the survey [Cha16]) is that the naive mean field approximation can be rigorously justified when \(f\) has low-complexity gradient, meaning that the image of \(\nabla f\) can be efficiently approximated using a net (in particular when \(f\) is
affine, so that $\mu$ is a product measure, the image of $\nabla f$ is a single point). By “efficient” we mean that the metric entropy of the image of the gradient is small in comparison with the free energy $\log Z$. This idea was further developed by Eldan in [Eld] where the complexity of the gradient is quantified in terms of the Gaussian width of its image rather than covering numbers. In addition, he showed a low complexity gradient yields an approximation of the Gibbs measure $\mu$ by a mixture of tilted measures, each of which is close to a product measure (see also more recent works [EG18, Aus]). However, when $h(x)$ stands for subgraph counts, the leading term $UT(\cdot)$ decreases as $p = p(N) \to 0$ and while this approach is relatively general, between the required smooth approximation of $g(\cdot)$ we must employ in (1.25), and the move from $\nabla f_h$ to $f_h$, the ability to recover the optimal range of $p(N)$ is completely lost.

To overcome this deficiency we take here a different approach, better tuned to yield sharper results in specific applications. As in [CD16, Eld], our approach involves a notion of low complexity, now working directly with (1.24) using nets to approximate the values of the function $h$ rather than $\nabla f_h$. Specifically, we construct efficient coverings of the cube $\{0, 1\}^N$ (identified with the space of $N \times N$ adjacency matrices in the natural way) by convex bodies $B_t$ on which the function $h$ is nearly constant. This can be regarded as a quantitative version of the approach from [CV11], which relied on the compactness of the spaces of graphons; the coverings we construct in Sections 4–5 using spectral arguments quantify the compactness of the space of adjacency matrices (see Section 2 for further discussion of these ideas).

Turning to state explicitly how such coverings yield tail bounds, fix $h : [0, 1]^d \to \mathbb{R}$ and consider for $t \in \mathbb{R}$ the corresponding super- and sub-level sets

$$L_\geq(h, t) := \{y \in [0, 1]^d : h(y) \geq t\}, \quad L_\leq(h, t) := \{y \in [0, 1]^d : h(y) \leq t\}. \quad (1.26)$$

Setting for any $\mathcal{T} \subseteq \mathbb{R}^d$,

$$I_p(\mathcal{T}) := \inf \{I_p(x) : x \in \mathcal{T} \cap [0, 1]^d\}, \quad (1.27)$$

we have $\phi_p(h, t) = I_p(L_\geq(h, t))$ and $\psi_p(h, t) = I_p(L_\leq(h, t))$ in (1.9) and (1.10). Our aim is to show that for the product Bernoulli($p$) measure $\mu_p$ on $\{0, 1\}^d$,

$$\mu_p(L_\geq(h, t)) \leq \exp (-\phi_p(h, t) - \text{Error}), \quad (1.28)$$

where Error is of lower order than the main term $\phi_p(h, t)$, and similarly for the lower tail (namely, for $\mu_p(L_\leq(h, t))$). It is well known that (1.28) holds with Error $= 0$ whenever $h$ is an affine function, namely, for half-spaces $L_\geq(h, t)$ (hence the exactness of the naive mean field approximation for the associated tilted measure). From this fact, together with the convexity of the function $I_p(\cdot)$, such a zero-error, non-asymptotic bound applies for any closed convex set:

**Proposition 1.8** ([DZ02, Eq. (4.5.6) and Exer. 2.2.23(b)]). For $p \in [0, 1]$ and closed convex $\mathcal{K} \subseteq \mathbb{R}^d$,

$$\mu_p(\mathcal{K}) \leq \exp (-I_p(\mathcal{K})).$$

The above does not at first appear to be useful for proving Theorems 1.1 and 1.7, since super-level sets for homomorphism counting functions are non-convex (except for the trivial edge-counting function). For example, in the case $h = \text{hom}_{\ell}$ for $\ell$ even, $L_\geq(h, t)$ is the complement of a convex set. However, our key observation is that such super-level sets can be efficiently covered by convex sets on which $h$ has small fluctuations, thereby utilizing the following easy consequence of Proposition 1.8 and the union bound.
Corollary 1.9. Let $h : [0, 1]^d \to \mathbb{R}$. Suppose there is a finite family $\{B_i\}_{i \in I}$ of closed convex sets in $\mathbb{R}^d$, an “exceptional” set $\mathcal{E} \subset \{0, 1\}^d$, and $\delta > 0$ such that

$$\{0, 1\}^d \setminus \mathcal{E} \subseteq \bigcup_{i \in I} B_i$$

(1.29)

and

$$\forall i \in I, \forall x, y \in B_i, \quad h(y) - h(x) \leq \delta.$$  

(1.30)

Then, for any $p \in [0, 1]$ and $t \in \mathbb{R}$,

$$\mu_p(\mathcal{L}_\geq(h, t)) \leq |I| \exp(-\phi_p(h, t - \delta)) + \mu_p(\mathcal{E}),$$  

(1.31)

$$\mu_p(\mathcal{L}_\leq(h, t)) \leq |I| \exp(-\psi_p(h, t + \delta)) + \mu_p(\mathcal{E}).$$  

(1.32)

Proof. Denoting by $I' \subset I$ the set of $i$ for which $(B_i \setminus \mathcal{E}) \cap \mathcal{L}_\geq(h, t) \neq \emptyset$, we have by the union bound, followed by Proposition 1.8 that

$$\mu_p(\mathcal{L}_\geq(h, t)) \leq \mu_p(\mathcal{E}) + \sum_{i \in I'} \mu_p(B_i) \leq \mu_p(\mathcal{E}) + \sum_{i \in I'} e^{-I_p(B_i)}$$

$$\leq |I'| \exp\left\{ - \min_{i \in I'} I_p(B_i) \right\} + \mu_p(\mathcal{E}).$$

From (1.30) it follows that $B_i \cap [0, 1]^d \subseteq \mathcal{L}_\geq(h, t - \delta)$ for any $i \in I'$. Hence,

$$\min_{i \in I'} I_p(B_i) \geq I_p(\mathcal{L}_\geq(h, t - \delta)) = \phi_p(h, t - \delta)$$

and (1.31) follows. The same line of reasoning yields also (1.32).

□

Remark 1.10. Our proof shows that it suffices for (1.31) to have (1.30) for $x \in B_i$ and $y \in B_i \setminus \mathcal{E}$, whereas for (1.32) it suffices to have (1.30) for $y \in B_i$ and $x \in B_i \setminus \mathcal{E}$.

Proposition 1.8 and Corollary 1.9 yield only upper bounds on tail probabilities. In some cases tilting arguments can show that these bounds are sharp. For our main application even this is unnecessary since the variational problem (1.12) was solved in [LZ17, BGLZ17], with sharpness directly verified by considering the events Clique and Hub from (1.3), (1.4), respectively.

Whereas Corollary 1.9 is rather elementary, the real technical challenge is in the design of coverings $\{B_i\}_{i \in I}$ for the space of $N \times N$ adjacency matrices (up to well-chosen exceptional sets) that are efficient enough to allow the sparsity parameter $p = p(N)$ to decay as quickly as in our stated results. For this task we employ techniques of high-dimensional geometry and spectral analysis. A key intermediate step is to obtain upper tail bounds for “outlier” eigenvalues of $A$, i.e. eigenvalues of size $\sqrt{Np} \ll |\lambda_j(A)| \lesssim Np$, which might be of independent interest. For a slowly growing parameter $R(N)$ this allows us to approximate $A$ by its rank-$R$ projection, which in turn can be approximated by a point $Y \in X_N$ in a net of size $O(RN \log N)$. This provides an efficient covering by operator-norm balls (up to an exceptional event containing matrices with many large outlier eigenvalues), and the proof of Theorem 1.7 then reduces to a careful control of the modulus of continuity of the functionals $\text{hom}_{H^*}(\cdot)$ with respect to the operator norm. For the case that $H$ is a cycle we can take a more refined approach. In particular, we take advantage of the approximate orthogonality of the images of the rank-$R$ approximation $Y$ for $A$ and of the residual matrix $A - Y$ to get improved control on the fluctuation of $\text{hom}_{C^*_Y}(\cdot)$ on a convex body $B_Y$ that is specially designed to exploit this orthogonality. See Subsection 2.2 for further discussion of these ideas. In particular, the probabilistic parts of our arguments are confined to Proposition 2.4(a) and Lemma 3.3, which are used to control the exceptional set $\mathcal{E}$ in our applications of Corollary 1.9 (and to
Lemmas 6.3 and 7.1 for converses of Proposition 1.8 that we utilize for matching lower bounds in Proposition 1.13, and in Theorems 1.16, 1.21, respectively).

As detailed in Subsection 2.3, our approach is a quantitative strengthening of a well-known spectral proof of Szemerédi’s regularity lemma [FK99,Sze11] (see also [Tao12]), for the setting of random graphs in the large deviations regime for subgraph counts. In the language of modern graph limit theory (as covered in [Lov12]), the regularity method rests on two facts about the cut-norm topology: the compactness of the space of graphons (the regularity lemma) and the continuity of the functions hom\(_H(\cdot,\cdot)\) (the counting lemma). It is well known that the regularity method, which underlies the work in [CV11] for large deviations in the dense regime, breaks down for sparse graphs (having edge density \(\sim N^{-c}\) for fixed \(c > 0\)). One notes, however, that for the problem of large deviations for hom\(_H(G)\) for a single fixed graph \(H\), the regularity method gives us more than we need: it gives coverings by neighborhoods on which all counts hom\(_F(\cdot,\cdot)\) (with \(|E(F)|\) below a given threshold) have controlled fluctuations. In the present work we are able to address sparser graphs by designing coverings more specifically tailored to the graph \(H\) under consideration.

1.4. Results for Schatten (and operator) norms. Denote by \(A = AG \in \mathcal{A}_N\) the (random) adjacency matrix for \(G \sim G(N,p)\) and recall the Schatten norms

\[
\|X\|_{S_\alpha} = \left( \sum_{j=1}^{N} |\lambda_j(X)|^\alpha \right)^{1/\alpha}, \quad \alpha \in [1, \infty],
\]

(1.33)
defined in terms of the eigenvalues of \(X\). Clearly, \(\text{hom}_{C_2}(X) = \|X\|_{S_2}^2\), so Theorem 1.1 gives large deviations bounds for the Schatten norms \(\|A\|_{S_\alpha}\) of even order \(\ell \geq 4\). An inspection of the proof of Theorem 1.1 reveals that with slight modifications our argument applies also to Schatten norms of any order above two, yielding our next result.

Proposition 1.11 (Large deviations, Schatten norms). The conclusion of Theorem 1.1 holds with \(\ell \in \mathbb{N}\) changed to \(\alpha \in (2, \infty)\), with \(\text{hom}_{C_\ell}(G)\), \(\phi_{N,p}(C_\ell, s)\) and \(\psi_{N,p}(C_\ell, s)\) replaced by \(\|A\|_{S_\alpha}, \phi_p(\| \cdot \|_{S_\alpha}, Nq)\) and \(\psi_p(\| \cdot \|_{S_\alpha}, Nq)\), respectively, with \(s = q/p\) fixed.

Remark 1.12. Theorem 1.21 below dramatically improves the range of \(p\) for the lower tail in Proposition 1.11. As for the upper tail, while \(\|A\|_{S_\alpha}\) reduces to tail estimates for the binomial distribution, note that \(E\|A\|_{S_2}^2 \propto \sqrt{E\|A\|_{S_2}^4} = \sqrt{N(N-1)p} \gg Np\) and the upper tail exponential decay rate is then \(N^2p\) (unlike for \(\alpha > 2\)). It is also easy to check that \(E\|A\|_{S_\alpha} \gtrsim N^{1/\alpha}\sqrt{Np} \gg Np\) whenever \(N^{-1} \ll p \ll N^{2/\alpha - 1}\), with the upper tail large deviations of \(\|A\|_{S_\alpha}\) exhibiting a qualitative transition as \(p\) crosses \(N^{2/\alpha - 1 + o(1)}\).

In Corollary 1.6 the matching lower bound for the upper tail of cycle counts is due to the asymptotic solution of the variational problem \(\phi_{N,p}(C_{\ell}, 1+u)\) provided by Theorem 1.5. Whereas the analogous result for \(\phi_p(\| \cdot \|_{S_\alpha}, Nq)\) is lacking, we do get such matching bounds for \(\alpha = \infty\), namely for the upper tail of the Perron–Frobenius eigenvalue \(\lambda_1(A) = \|A\|_{op}\) and further bound the upper tail decay for \(\lambda_2(A)\) – the eigenvalue of second-largest modulus.

Proposition 1.13. For \(N^{-1/2} \ll p \leq 1/2\) and fixed \(s = q/p > 1\),

\[
- \log P(\|A\|_{op} \geq Nq) = (1 + o(1)) \phi_p(\| \cdot \|_{op}, (1 + o(1))Nq).
\]

(1.34)

Moreover, for \(N^{-1}\log N \ll p \leq 1/2\) and any \(t \gg \sqrt{N}\),

\[
- \log P(\lambda_2(A) \geq t) \geq - \log P(\|A - p11^T\|_{op} \geq t)
= (1 + o(1)) \phi_p(\| \cdot - p11^T\|_{op}, t + o(t)).
\]

(1.35)
Remark 1.14. The upper bounds on the LHS of (1.34) and (1.35), hold up to \( p \geq N^{-1} \log N \) and \( t \geq C \sqrt{Np} \), respectively. By eigenvalue interlacing \( \| A - p 1 1^T \|_{op} \geq \lambda_2(A) \), trivially yielding the inequality in (1.35), where one may further replace \( p 1 1^T \) by \( E A = p(1 1^T - I) \).

Remark 1.15. In [GH], Guionnet and Husson establish a large deviations principle for the largest eigenvalue of \( N \)-dimensional Wigner matrices, rescaled by \( \sqrt{N} \), whose independent, standardized entries have uniformly sub-Gaussian mgf-s (allowing for Rademacher entries). However, such uniform sub-Gaussian domination does not apply to \( A - E A \) when \( p = o(1) \). Indeed, [GH] concerns deviations of the largest eigenvalue at the scale \( \sqrt{N} \) of the bulk spectral distribution, whereas (1.35) is about larger deviations (we expect (1.35) to fail for \( t \propto \sqrt{Np} \)).

1.5. Sharp lower tails for homomorphism counts and Schatten norms. In this subsection we consider two families of functions of \( G \) for which we can obtain the sharp lower tail for a wide range of \( p \). Moreover, we provide an explicit formula for the tail, which in both cases is asymptotically given by \( \binom{N}{2} I_p(q) \), the relative entropy of the distribution \( G(N,q) \) with respect to \( G(N,p) \), for an appropriate \( q < p \). In particular, the rate matches the log-probability that the edge density is uniformly lowered from \( p \) to \( q \). This contrasts with our results for the upper tail, where the rate asymptotically matches the log-probability for a small planted structure (see (1.3) and (1.4)).

For our first such result we recall some notation from graph limit theory. Consider the space \( W \) of all bounded symmetric measurable functions \( f : [0,1]^2 \to \mathbb{R} \), and for a simple graph \( H = (V, E) \) define the associated homomorphism density functional

\[
t_H : W \to \mathbb{R}, \quad t_H(f) := \int_{[0,1]^V} \prod_{k \in E} f(x_k, x_l) \prod_{k \in V} dx_k.
\]

This extends to \( W \) the homomorphism counting functionals (1.7). Indeed, associating to each \( X \in X_N \) the element \( f_X \in W \) with

\[
f_X(x, y) = X_{\lfloor Nx \rfloor, \lfloor Ny \rfloor}
\]

it follows that \( t_H(f_X) = N^{-|V|} \text{hom}_H(X) \). A simple graph \( H = (V, E) \) is Sidorenko if

\[
0 \leq f \in W \quad \implies \quad t_H(f) \geq t_{K_2}(f)^{|E|}.
\]

It was conjectured by Erdős and Simonovits [Sim84] and Sidorenko [Sid93] that all bipartite graphs are Sidorenko. While the conjecture remains open as of this writing, (1.38) has been established for complete bipartite graphs, trees and even cycles [Sid93], hypercubes [Hat10] and bipartite graphs with a vertex complete to the other side [CFS10], among others; see the recent works [Sze,CKLL] and references therein for further results. In the following theorem we provide a lower bound for the lower tail of \( \text{hom}_H(G) \), valid for any simple graph \( H \), and show that this bound is tight if \( H \) is Sidorenko. In particular, conditional on Sidorenko’s conjecture, (1.39) and (1.40) provide the sharp lower tail for homomorphism counts of any bipartite graph.

Theorem 1.16 (Lower tail, Sidorenko graphs). Let \( H = (V, E) \) be a finite, simple, graph with \( n \geq 1 \) vertices and \( m \geq 1 \) edges. If \( N^{-1/2(D^-)^{-1}} \ll p \leq 1/2 \) (for \( D^- \) as in Theorem 1.7), then fixing \( q/p \in (0,1) \) and setting \( \hat{q} := q - q/N \), we have when \( N \to \infty \),

\[
\mathbb{P} \left( \text{hom}_H(G) \leq \hat{q}^m N^n \right) \geq e^{-(1+o(1)) \binom{N}{2} I_p(q)}.
\]

Moreover, if \( H \) is Sidorenko, then for any \( 0 < q < p < 1 \) and \( N \in \mathbb{N} \),

\[
\mathbb{P} \left( \text{hom}_H(G) \leq \hat{q}^m N^n \right) \leq e^{-\binom{N}{2} I_p(q)}.
\]
Remark 1.17. We stress that the upper bound (1.40) is non-asymptotic, applying for any fixed $N$ and $0 < q < p < 1$; thus, if $N$ is an asymptotic parameter then $p$ and $q$ can depend in an arbitrary way on $N$. The same goes for (1.42) below.

Remark 1.18. Such bounds for Sidorenko graphs $H$ are derived for the regime of fixed $0 < q < p < 1$ in [LZ15], and in [Zha17] for general $H$, when $N^{-a_H} \leq p \ll 1$ and $\bar{s}_H < q/p < 1$ for some $\bar{s}_H \in (0, 1)$ and an extremely small $a_H > 0$. Moreover, [LZ15, Prop. 3.5] shows that conditional on the event $\{\text{hom}_H(G) \leq q^m N^m\}$, the corresponding graphon $f_A$ is close in cut-norm to the constant $q \in W$.

Remark 1.19. Previous works considered the lower tail for subgraph counts $\text{sub}_H(G)$. For $p \gg N^{-1/\Delta(H)}$, with high probability $\text{sub}_H(G)$ and $\text{hom}_H(G)$ differ by a non-random, fixed factor ($\sim \text{aut}(H)$, the number of graph automorphisms of $H$). In contrast, $\text{sub}_H(G)$ and $\text{hom}_H(G)$ have substantially different behavior for smaller $p$. For general $H$, [JW16] obtains upper and lower bounds for the lower tail $\log P(\text{sub}_H(G) \leq (1 - \varepsilon) \mathbb{E}\text{sub}_H(G))$ matching up to a constant factor, whereas Theorem 1.16 obtains the sharp lower tail (with asymptotically matching upper and lower bounds) for Sidorenko graphs. Such a sharp lower tail is obtained in [JW16, Theorem 3] for a wide class of graphs $H$ including $2$-balanced graphs, but only in a regime of sufficiently small $p = p(N)$ that does not overlap with Theorem 1.16.

For convex functions of $A$, such as the Schatten norms of Proposition 1.11, we can obtain strong results for the lower tail via the following special case of Proposition 1.8.

**Proposition 1.20.** Fix $N \in \mathbb{N}$, $h : \mathcal{X}_N \to \mathbb{R}$ and $p \in (0, 1)$. If $t \in \mathbb{R}$ is such that the sub-level set $\{X \in \mathcal{X}_N : h(X) \leq t\}$ is convex, then

$$\mathbb{P}(h(A) \leq t) \leq \exp(-\psi_p(h, t)). \quad (1.41)$$

Proposition 1.20 applies to any semi-norm of $A$. Here, we consider the lower tail for Schatten norms $\|A\|_{S_{\alpha}}$, showing in particular that the leading order is the same for all $\alpha \in (2, \infty]$ and $N^{\frac{\alpha}{\alpha - 1}} \ll p \leq 1/2$ (for smaller $p$ there may be slack in (1.42)).

**Theorem 1.21** (Lower tail, Schatten norms). For $0 < q < p < 1$, $\alpha \in [1, \infty]$ and $N \in \mathbb{N}$,

$$\mathbb{P}(\|A\|_{S_{\alpha}} \leq q(N - 1)) \leq e^{-\psi_p(\|\cdot\|_{S_{\alpha}} q(N - 1))} \leq e^{-(\frac{N}{2})I_p(q)}. \quad (1.42)$$

Moreover, if $\alpha \in (2, \infty]$ and $p = p(N)$ satisfies $1/2 \geq p \gg N^{\frac{\alpha}{\alpha - 1}}$ as $N \to \infty$ (taking $p(N) \gg \log N/N$ for $\alpha = \infty$), then for fixed $s := q/p \in (0, 1)$, we have

$$\mathbb{P}(\|A\|_{S_{\alpha}} \leq q(N - 1)) \geq e^{-(1 + o(1))(\frac{N}{2})I_p(q)}. \quad (1.43)$$

**Remark 1.22.** Whereas even-length cycles are Sidorenko [Sid93], taking $\alpha = 2\ell \in 2\mathbb{N}$ in Theorem 1.21 improves upon the range $p \gg N^{-1/3}$ required for $H = C_{2\ell}$ in Theorem 1.16.

### 1.6. Organization of the paper.

In Section 2 we motivate our spectral approach to the covering and continuity arguments mentioned in Subsection 1.3, and state our versions of the regularity and counting lemma for sparse random graphs, which will be applied in Section 5 to prove Theorem 1.7. In Section 3 we establish some preliminary control on the spectrum of $A$, towards bounding the fluctuation of homomorphism counts on certain small sets. In Section 4 we prove Theorem 1.1 as a direct consequence of the non-asymptotic version, Theorem 4.1; the necessary modifications to obtain Proposition 1.11 are given in Section 4.5. In Sections 5 and 6 we prove Theorem 1.7 and Proposition 1.13, respectively. Lastly, in Section 7 we establish Theorems 1.16 and 1.21.
1.7. Notation and conventions.

Asymptotic notation. Unless otherwise stated, $C, C', C_o, c$, etc.
denote universal constants; if they depend on parameters we indicate this by writing e.g. $C_\kappa, C(H)$. The notations $f = O(g)$, $f \lesssim g$ and $g \gtrsim f$ are synonymous to having $|f| \leq Cg$ for some universal constant $C$, while $f = \Theta(g)$ and $f \asymp g$ mean $f \lesssim g \lesssim f$. We indicate dependence of the implied constant on parameters (such as $H$ or $u$) with subscripts, e.g. $f \lesssim_H g$. The statements $f = o(g)$, $g = \omega(f)$, $f \ll g$, $g \gg f$ are synonymous to having $f/g \to 0$ as $N \to \infty$, where the rate of convergence may depend on fixed parameters such as $H$ and $u$ without being indicated explicitly. While our results use the qualitative $o(\cdot)$ notation, in the proofs we often give quantitative estimates with more explicit dependence on fixed parameters for the sake of clarity. We assume throughout that $N \geq 2$ (so that $\log N \gtrsim 1$).

Matrices and normed spaces. We endow $\mathbb{R}^N$ with the $\ell_r$ norms $\| \cdot \|_r, r \in [1, \infty]$ and Euclidean inner product $\langle \cdot, \cdot \rangle$, and denote by $S^{d-1}$ the unit Euclidean sphere in $\mathbb{R}^d$. We write $1 = 1_N \in \mathbb{R}^N$ for the all-ones vector and $I = I_N$ for the $N \times N$ identity matrix. For a set $\Omega$ we write $\text{Sym}_{\Omega}(\Omega)$ for the set of symmetric $N \times N$ matrices with entries in $\Omega$, and $\text{Sym}_0^0(\Omega) \subseteq \text{Sym}_N(\Omega)$ for the subset of symmetric matrices with zeros along the diagonal. For $1 \leq R \leq N$ we write $\text{Sym}_{N,R}(\Omega) \subseteq \text{Sym}_N(\Omega)$ for the subset of elements of rank at most $R$. We abbreviate

$$X_N := \text{Sym}_N^0([0, 1]), \quad A_N := \text{Sym}_N^0([0, 1])$$

as these sets will appear frequently. When invoking Corollary 1.9 we implicitly identify the above sets with $[0, 1]^{N\choose 2}$ and $[0, 1]^{N^2}$, respectively. Note that $A_N$ is the set of adjacency matrices for simple (and undirected) graphs on $N$ vertices. Throughout we let $A \in A_N$ denote the adjacency matrix of $G \sim G(N, p)$, with $\mu_p(\cdot) = \mathbb{P}(A \in \cdot)$ the corresponding product Bernoulli measure on $A_N$. We denote the adjacency matrix for the complete graph on $N$ vertices by

$$J = J_N := 1_N 1_N^T - I_N \in A_N.$$  

We label the eigenvalues of an element $X \in \text{Sym}_N(\mathbb{R})$ in non-increasing order of modulus:

$$|\lambda_1(X)| \geq |\lambda_2(X)| \geq \cdots \geq |\lambda_N(X)|$$

and recall the Schatten norms on $\text{Sym}_N(\mathbb{R})$ as in (1.33). In particular, $\| X \|_s = |\lambda_1(X)|$ equals the $\ell^2_N \to \ell^2_N$ operator norm

$$\| X \|_{op} = \sup_{u \in S^{N-1}} \| Xu \|_2 = \sup_{u \in S^{N-1}} \langle u, Xu \rangle.$$  

Moreover, $\| X \|_{s_2}$ equals the Hilbert–Schmidt norm for the inner product

$$\langle X, Y \rangle_{HS} = \text{Tr}(XY), \quad \| X \|_{HS} = (\text{Tr}X^2)^{1/2},$$

with the closed Hilbert–Schmidt ball in $\text{Sym}_N(\mathbb{R})$ of radius $t$ denoted by $B_{HS}(t)$. By the non-commutative Hölder inequality, whenever $1/\alpha + 1/\beta = 1/\gamma$,

$$\| XY \|_{s_\gamma} \leq \| X \|_{s_\alpha} \| Y \|_{s_\beta}$$

(see [Sim05, Theorem 2.8]), and in particular

$$\| XY \|_{s_\alpha} \leq \| X \|_{op} \| Y \|_{s_\alpha}.$$  

For $X \in \text{Sym}_N(\mathbb{R})$ having spectral decomposition

$$X = \sum_{j=1}^N \lambda_j u_j u_j^T,$$

we implicitly identify the $1.33$ (see [N with the closed Hilbert–Schmidt ball in $\text{Sym}_N(\mathbb{R})$ of radius $t$ denoted by $B_{HS}(t)$. By the non-commutative Hölder inequality, whenever $1/\alpha + 1/\beta = 1/\gamma$,

$$\| XY \|_{s_\gamma} \leq \| X \|_{s_\alpha} \| Y \|_{s_\beta}$$

(see [Sim05, Theorem 2.8]), and in particular

$$\| XY \|_{s_\alpha} \leq \| X \|_{op} \| Y \|_{s_\alpha}.$$  

For $X \in \text{Sym}_N(\mathbb{R})$ having spectral decomposition

$$X = \sum_{j=1}^N \lambda_j u_j u_j^T,$$
with eigenvalues arranged as in (1.46), and for any $1 \leq R \leq N$, we further have that

$$X = X_{\leq R} + X_{> R}, \quad X_{\leq R} := \sum_{j \leq R} \lambda_j u_j u_j^T, \quad X_{> R} := \sum_{j > R} \lambda_j u_j u_j^T. \quad (1.49)$$

**Graph theory.** All graphs are assumed to be simple (without self-loops or multiple edges) unless stated otherwise. For a graph $H = (V, E)$ we write $V(H) = V$, $E(H) = E$, $\nu(H) = |V|$, and $\delta(H) = |E|$. We say that a graph $H$ is nonempty if $E(H) \neq \emptyset$. For $v \in V(H)$, $\deg_H(v)$ denotes the degree of $v$, and $\Delta(H) := \max_{v \in V(H)} \{\deg_H(v)\}$ denotes the maximum degree of $H$. We often take $V = [n]$. We use $F \leq H$ to mean that $F$ is a subgraph of $H$ (obtained by removing some of the vertices and/or edges of $H$). We further write

$$F \preceq H, \quad F \prec H \quad (1.50)$$

when $F$ is an induced subgraph of $H$ (i.e. $F = H[V']$ for some $V' \subseteq V(H)$), or a strictly induced subgraph of $H$, respectively.

### 2. Spectral regularity method for random graphs

To establish Theorems 1.1 and 1.7 via Corollary 1.9, we need to find a covering of “most” of $\mathcal{A}_N$ by convex sets on which the functions $\text{hom}_d(\cdot)$ have small fluctuations. In effect, our approach is a quantitative refinement of the argument in [CV11], which uses the topological space of graphons with the cut metric to obtain such coverings in the dense setting. In this section, we first motivate our spectral approach to covering constructions, and how it can be optimized towards Theorem 1.1 for cycle counts. Then, in Subsection 2.3, we make the connection with graphon methods more precise by stating quantitative versions of the regularity and counting lemmas tailored for applications to sparse random graphs; along with Corollary 1.9, these are the key ingredients for establishing Theorem 1.7.

#### 2.1. A simple argument for triangle counts

We begin with a short, crude version of our argument for the normalized homomorphism counting function

$$h_3(X) := \frac{1}{N^3 p^3} \text{hom}_{C_3}(X).$$

It yields the upper tail (1.6) for $p_3(N) = ((\log N)/N)^{1/8}$ and motivates the derivation of refined estimates on the spectrum of $A$ in Section 3. Specifically, observe that with eigenvalues as in (1.46) and $1 \leq k \leq N$,

$$\|X_{\leq k}\|_{HS}^2 := \sum_{j=1}^k \lambda_j^2(X) \geq k \lambda_k^2(X). \quad (2.1)$$

Thus, we have for the projection of $X \in \mathbb{B}_{HS}(N)$ to $X_{\leq R} \in \text{Sym}_{N,R}(\mathbb{R})$, as in (1.49), that

$$\|X - X_{\leq R}\|_{op} = \|X_{> R}\|_{op} = |\lambda_{R+1}(X)| \leq \frac{N}{\sqrt{R+1}}. \quad (2.2)$$

By a standard argument, the set $\text{Sym}_{N,R}(\mathbb{R}) \cap \mathbb{B}_{HS}(N)$ can be covered by $O(NR \log \frac{1}{\epsilon N})$ operator-norm balls of radius $\epsilon N$. Hence, from (2.2) and the triangle inequality, for some $1 \leq R \leq N$ to be chosen later we have a set $\mathcal{N} \subset \mathcal{X}_N$ of size $O(RN \log N)$ consisting of matrices of rank at most $R$ such that for any $X \in \mathcal{X}_N$ there exists $Y \in \mathcal{N}$ with

$$\|X - Y\|_{op} \leq \epsilon N, \quad \text{where} \quad \epsilon \lesssim R^{-1/2}. \quad (2.3)$$

This is the key fact is behind the quantitative covering of [CD16]; incidentally, it also underlies a well-known spectral proof of the regularity lemma [FK99, Sze11, Tao12]. Note that whereas
in [CD16] such a net is used to approximate the gradient of the functions $\text{hom}_H(\cdot)$, here we use nets to approximate the values of the functions themselves.

To each $Y \in \mathcal{N}$ we associate the closed, convex set $B_Y = \{X \in \mathcal{X}_N : \|X - Y\|_{\text{op}} \leq \varepsilon N\}$. By Weyl's inequality, upon ordering the eigenvalues of $M_1, M_2 \in \text{Sym}_N(\mathbb{R})$ on $\mathbb{R}$ (instead of by modulus), we have that

$$|\lambda_j(M_1) - \lambda_j(M_2)| \leq \|M_1 - M_2\|_{\text{op}} \quad \forall 1 \leq j \leq N.$$  \hfill (2.4)

Since $|a^\ell - b^\ell| \leq \ell |a - b|(|a|^{\ell - 1} + |b|^{\ell - 1})$ for any $a, b \in \mathbb{R}$, $\ell \in \mathbb{N}$, it follows that

$$|\text{Tr } M_1^\ell - \text{Tr } M_2^\ell| \leq \sum_{j=1}^N |\lambda_j(M_1)^\ell - \lambda_j(M_2)^\ell|$$

$$\leq \ell \|M_1 - M_2\|_{\text{op}} (\|M_1\|_{S_{\ell - 1}}^{\ell - 1} + \|M_2\|_{S_{\ell - 1}}^{\ell - 1}).$$  \hfill (2.5)

Considering (2.5) for $\ell = 3$ and matrices $Y, X \in B_Y \subseteq \mathcal{X}_N \subset \mathbb{B}_{\text{HS}}(N)$, we have by (2.3) that

$$|\text{Tr } X^3 - \text{Tr } Y^3| \leq 3\|X - Y\|_{\text{op}} (\|X\|_{\text{HS}}^2 + \|Y\|_{\text{HS}}^2) \leq 6\varepsilon N^3.$$  \hfill (2.6)

Consequently, when $\varepsilon = o(p^3)$, for which it suffices to take $R \gg p^6$ (see (2.3)), get by the triangle inequality, that uniformly over $Y \in \mathcal{N}$ and $X_1, X_2 \in B_Y$,

$$|h_3(X_1) - h_3(X_2)| = \frac{1}{N^3p^3} |\text{Tr } X_1^3 - \text{Tr } X_2^3| = o(1).$$  \hfill (2.7)

Hence, by Corollary covering with $\{B_i\}_{i \in I} = \{B_Y\}_{Y \in \mathcal{N}}$ and $\mathcal{E} = \emptyset$, we deduce that

$$\mathbb{P}(h_3(A) \geq t) \leq |\mathcal{N}| \exp (-\phi_p(h_3, t - o(1)))$$

$$= \exp (-\phi_{N,p}(C_3, t - o(1)) + O(RN \log N)) .$$  \hfill (2.8)

The main term in (2.8) dominates the error term when $RN \log N \ll N^2p^2$. We can satisfy this and our requirement that $R \gg p^{-6}$, provided $p \gg ((\log N)/N)^{1/8}$.

### 2.2. Refined approach.

The element $Y = Y(A) \in \mathcal{N}$ was obtained by approximating the rank $R$ projection $A_{\leq R}$ of each adjacency matrix $A \in \mathcal{A}_N$. In doing so, we can even take $\delta = N^{-3\delta}$ and the net $\mathcal{N}$ fine enough to ensure $\|A_{\leq R} - Y(A)\|_{\text{HS}} \leq 3\delta N$ while still having $\log |\mathcal{N}| \lesssim R \log N$ (cf. Lemma 4.2). Thus, $Y$ is essentially the rank $R$ projection of $A$. In particular, the images of $Y$ and $A - Y$ are nearly orthogonal sub-spaces. This property roughly carries over to any matrix $X$ in the convex hull $B'_Y$ of all $A \in \mathcal{A}_N$ with $\|A_{\leq R} - Y\|_{\text{HS}} \leq 3\delta N$ (see (4.21)). Consequently,

$$|\text{Tr } X^3 - \text{Tr } Y^3| \approx |\text{Tr } (X - Y)^3| \leq \|X - Y\|_{S_{\delta}}^3 \approx \|X_{\geq R}\|_{S_{\delta}}^3 \quad \forall X \in B'_Y ,$$  \hfill (2.9)

thereby reducing the task of controlling the fluctuation of $h_3(X)$ on sets $B'_Y$ to that of bounding the tail of the (absolute) third moment of the spectrum. Such approximate orthogonality applies to any spectral function of $A$ that is dominated by the large eigenvalues (among $\text{hom}_H(\cdot)$ these are precisely $\text{hom}_{O_k}(\cdot)$, but Schatten norms also have this property). The bound $N^3/\sqrt{R}$ of (2.6) is the best we can achieve in (2.9) with the bound (2.2) on $\{|\lambda_j|, j > R\}$. While it is essentially sharp for general elements of $\mathcal{A}_N$, for random elements of $\mathcal{A}_N$ (under the Erdős–Rényi measure $\mu_p$) we can do much better. Indeed, with probability $1 - o(1)$ we have $\lambda_1(A) \sim Np$ and $|\lambda_2(A)| = O(\sqrt{Np})$. In fact, reordering the eigenvalues as $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_N(A)$, we have that $\lambda_2(A)/\sqrt{Np}$ and $\lambda_N(A)/\sqrt{Np}$ “stick” to the edges $\pm 2$ of the support of Wigner’s semicircle distribution (see Lemma 3.5). However, we are limited to exploiting properties of random elements holding with probability $1 - \exp(-\omega(N^2p^2 \log(1/p)))$, which do not include the event that $|\lambda_2(A)| = O(\sqrt{Np})$. Indeed, on
the events Clique and Hub from (1.3), (1.4) the Perron-Frobenius eigenvalue \( \lambda_1(A) \) is joined by a second “outlier” eigenvalue at scale \( Np \) (on Clique it is a positive outlier, while on Hub it is negative). Additional outlier eigenvalues correspond to having a large-scale pattern for the edge distribution which is of rank \( \geq 3 \) (see [Tao12] for one formalization of this heuristic).

Fortunately, for \( C_\ell \)-counts we only need

\[
\|A_{>R}\|_{S_\ell} = o(Np) \tag{2.10}
\]

with \( R = R(N) \) growing poly-logarithmically, in order to allow \( p \) of size \( N^{-1/2}(\log N)^C \). Using the appropriate exceptional sets, we accomplish this by utilizing Proposition 3.2 (for triangle counts we must further assume \( p \gg N^{-1/3} \)).

For the lower tail bound in Theorem 1.1 we can only exclude events of probability \( 1 - \exp(-\omega(N^2p)) \), hence the somewhat larger lower limit on \( p \) in (1.16), but as seen in Section 1.5, for even \( \ell \) we have no such restrictions (by the convexity of sub-level sets).

A key feature of cycle homomorphism counts is that they can be expressed as functions of the spectrum of \( A \) alone, which lets us get sharp control on the fluctuations of these functions on the sets \( B_\ell^i \) from (2.9) via (2.10). For general \( H \) as in Theorem 1.7 we lack a spectral representation of hom\(_H\)(\( \cdot \)), so instead of the sets \( B_\ell^i \) we use a covering by operator-norm balls. In particular, we cannot exploit orthogonality of the images of \( Y \) and the residual \( A - Y \) as we do for cycles to obtain sharp control on the fluctuations of hom\(_H\)(\( \cdot \)). Nevertheless, after removing improbable events involving extremely large values of hom\(_H\)(\( A \)) for subgraphs \( F \) of \( H \), we get strong control on fluctuations of hom\(_H\)(\( \cdot \)) by an iterative “pruning” procedure on \( H \), resulting in the “counting lemma” of Proposition 2.4 below.

2.3. Regularity and counting lemmas for random graphs. We first recall the definitions of the space \( W_0 \) of graphons and the cut metric. Denote by \( W \) the space of bounded, symmetric, Lebesgue-measurable functions \( f : [0,1]^2 \to \mathbb{R} \) (as in Section 1.5), equipped with the cut-norm

\[
\|f\|_\Box = \sup_{S,T \subseteq [0,1]} |\int_{S \times T} f(x,y)dxdy|,
\]

where the supremum is taken over measurable subsets of \([0,1]\). The cut-metric on \( W \) is then

\[
\delta_\Box(f,g) := \inf_{\sigma \in \Sigma} \{ \|f - g^\sigma\|_\Box \},
\]

where \( g^\sigma(x,y) := g(\sigma(x),\sigma(y)) \) and the infimum is taken over all measure-preserving bijections \( \sigma : [0,1] \to [0,1] \). On \( W \) we have the equivalence relation \( f \sim g \) if and only if \( f = g^\sigma \) for some \( \sigma \in \Sigma \), and denote by \( \overline{g} \) the \( \delta_\Box \)-closure of the corresponding orbit \( \{g^\sigma : \sigma \in \Sigma\} \) of \( g \in W \). Setting as \( W_0 \) the collection of elements \( f \in W \) with \( 0 \leq f \leq 1 \), the associated quotient spaces \( \overline{W} = \{\overline{g} : g \in W_0\} \) are thus \( \delta_\Box \)-measurable.

Graphons provide a topological reformulation of the regularity method from extremal graph theory, which rests on two key facts: Szemerédi’s regularity lemma, and the counting lemma. These can be formulated for graphons as follows (cf. [Lov12]):

**Lemma 2.1** (Weak regularity lemma for graphons). For every \( f \in W_0 \) and \( k \geq 1 \) there exists a step function \( g \in W_0 \) with \( k \) steps (i.e. a partition \( \mathcal{P} \) of \([0,1]\) into \( k \) measurable sets, such that \( g \) is constant on \( S \times T \) for all \( S,T \in \mathcal{P} \)) such that

\[
\|f - g\|_\Box \leq \frac{2}{\sqrt{\log k}}.
\]

**Lemma 2.2** (Counting lemma for graphons). For every simple graph \( H \) and every \( f,g \in W_0 \),

\[
|t_H(f) - t_H(g)| \leq e(H)\delta_\Box(f,g).
\]
(Recall the homomorphism density functionals $t_H(\cdot)$ from (1.36).)

The weak regularity lemma is closely related to the fact that $(\tilde{W}_0, \delta_\varepsilon)$ is a compact metric space, while the counting lemma says that the functionals $t_H(\cdot)$ are continuous with respect to the cut-metric. Taken together, they allow one to cover large deviation events for dense Erdős–Rényi graphs by a bounded collection of graphon neighborhoods on which the functionals $\text{hom}_H(\cdot)$ are essentially constant, which was the approach taken in [CV11]. Towards proving Theorem 1.7 we obtain the following quantitative analogues of the regularity and counting lemmas for the probability space $(\mathcal{A}_N, \mu_p)$, where a notable feature is to replace the cut-metric by the operator norm.

**Proposition 2.3** (Regularity lemma for random graphs). For some $c > 0$, $C_* < \infty$, any $N \in \mathbb{N}$, $K \geq 1$, $p \in (0, 1)$ such that $Np \geq \log N$, and all $1 \leq R \leq Np$ there exists a partition $\mathcal{A}_N = \bigsqcup_{j=0}^J \mathcal{E}_j$ having the following properties:

(a) $\log J \leq RN \log(3 + \frac{R}{Np})$;
(b) $\mu_p(\mathcal{E}_0) \leq \exp\left(-cK^2N^2p^2\right)$;
(c) for each $1 \leq j \leq J$, there exists $Y_j \in \text{Sym}_{N,R}(\mathbb{R}) \cap \mathcal{B}_{\text{HS}}(N)$ such that

$$\max_{A \in \mathcal{E}_j} \{\|A - Y_j\|_{\text{op}}\} \leq \frac{C_*KNp}{\sqrt{R}}. \quad (2.11)$$

A key feature making the above useful for sparse graphs is that the right hand side of (2.11) saves a factor $p$ over the bound (2.3). (For the application to Theorem 1.7 we only have to take $K$ of size $\log N$ to ensure the event $\mathcal{E}_0$ is of negligible size.)

**Proposition 2.4** (Operator-norm counting lemma). Let $H = (V, E)$ be a finite, simple graph of maximal degree $\Delta \geq 1$ and let $\Delta_* = \Delta_*(H) \geq 1$ be as in (1.23). Let $N \in \mathbb{N}$, $p \in (0, 1)$. For $K \geq 1$ define the exceptional set

$$\mathcal{E}_H(K) = \left\{ X \in \mathcal{X}_N : \exists F < H \text{ with } \text{hom}_F(X) > KN^{\nu(F)}p^{e(F)} \right\}. \quad (2.12)$$

(a) If $N^{-1/\Delta} < p < 1$, then for some finite $C(H), c(H) > 0$ and all $K \geq C(H)$,

$$\mu_p(\mathcal{E}_H(K)) \lesssim_H \exp\left(-c(H)K^{1/\nu(H)}N^2p^\Delta\right). \quad (2.13)$$

(b) There exists an increasing function $f : \mathbb{N} \to \mathbb{N}$ depending only on $\Delta_*$ such that if

$$\sup_{X,Y \in B} \|X - Y\|_{\text{op}} \leq \varepsilon_0Np^{\Delta_*} \quad (2.14)$$

for some $p \in (0, 1)$, $\varepsilon_0 \in [0, 1]$, $K \geq 1$ and a convex set $B \subseteq \mathcal{X}_N$ for which

$$B \cap \mathcal{E}_H(K)^c \neq \emptyset, \quad (2.15)$$

then for all $F \leq H$,

$$\text{Fluct}(F; B) := \sup_{X,Y \in B} \left| \text{hom}_F(X) - \text{hom}_F(Y) \right| \leq \varepsilon_0 f(e(F))KN^{\nu(F)}p^{e(F)}. \quad (2.16)$$

**Remark 2.5.** In part (a) the bound (2.13) holds even if in (2.12) we replace $F < H$ by $F \leq H$. Further, in part (b) we retain (2.16) when relaxing (2.15) to the weaker assumption that $\inf_{X \in B} \{\text{hom}_F(X)\} \leq KN^{\nu(F)}p^{e(F)}$ per fixed $F < H$. Finally, if we replace $F < H$ with $F \prec H$ in (2.12), then (2.16) still holds for all $F \preceq H$. 

Whereas Proposition 2.3 is essentially optimal for our purposes, we believe there is room for improvement in Proposition 2.4. In particular, replacing the factor $p^\Delta$, with $p$ on the right hand side of the condition (2.14) would relax the sparsity assumption in Theorem 1.7 to the optimal range $N^{-1/\Delta} \ll p \ll 1$, up to poly-logarithmic corrections.

3. Preliminary control on the spectrum

We consider for $1 \leq k \leq N$, the norms
\[ \|A_{\leq k}\|_{\text{HS}} = \sup_{W: \dim W = k} \|\Pi_W(A)\|_{\text{HS}}, \] (3.1)
where $\Pi_W$ denotes the operator for projection to the subspace $W$, and link the growth of $k \mapsto \|X_{\leq k}\|_{\text{HS}}$ to the decay of $R \mapsto \|X_{\geq R}\|_{S_\alpha}$ (when $\alpha > 2$).

**Lemma 3.1.** Fixing finite $L, D \geq 0$, let
\[ G(L, D) := \{ X \in \text{Sym}_N(\mathbb{R}) : \|X_{\leq k}\|_{\text{HS}} \leq L + \sqrt{k} D, \quad \forall 1 \leq k \leq N \}. \] (3.2)
Then, for $\kappa_\alpha := (\frac{2}{\alpha^2 - 2})^{1/\alpha}$, any $L, D$, $\alpha \in (2, \infty]$, $1 \leq R \leq N$ and $X \in G(L, D)$,
\[ \|X_{> R}\|_{S_\alpha} \leq (N - R)^{1/\alpha} D + \kappa_\alpha L R^{1/\alpha - 1/2}. \] (3.3)

**Proof.** Recall from (2.1), that if $X \in G(L, D)$, then for any $k \in [N]$,
\[ |\lambda_k(X)| = \|X_{\geq k}\|_{\text{op}} \leq k^{-1/2}\|X_{\leq k}\|_{\text{HS}} \leq D + Lk^{-1/2}, \quad \forall 1 \leq k \leq N. \] (3.4)
That is, (3.3) holds at $\alpha = \infty$ (with $\kappa_\infty = 1$). Having (3.4) at all $k \in (R, N]$, it follows by the triangle inequality, that for any finite $\alpha > 2$,
\[ \|X_{> R}\|_{S_\alpha} \leq (N - R)^{1/\alpha} D + L\|((k^{-1/2} 1_{k>R})\|_{\alpha}. \]
Further, bounding the latter $\ell_\alpha$-norm on $\mathbb{R}^N$, we get that for any $\alpha > 2$ and $R \geq 1$,
\[ \|((k^{-1/2} 1_{k>R})\|_{\alpha} \leq \left( \int_R^\infty u^{-\alpha/2} du \right)^{1/\alpha} = \kappa_\alpha R^{1/\alpha - 1/2}, \]
thereby establishing (3.3). \hfill \Box

The main result of this section, used for controlling the exceptional set $\mathcal{E}$ in Corollary 1.9, is as follows.

**Proposition 3.2.** For some $C, C', c > 0$, any $K \geq 2$ and $Np \geq \log N$,
\[ \mathbb{P}\left( A \notin G(KNp, C' \sqrt{Np}) \right) \leq C \exp\left( -cK^2N^2p^2 \right) =: P_{\text{excep}}(K). \] (3.5)
Hence, up to probability $P_{\text{excep}}(K)$, the matrix $A$ satisfies (3.3) with $L = KNp$, $D = C' \sqrt{Np}$, any $\alpha \in (2, \infty)$ and all $1 \leq R \leq N$.

Via a union bound over $k \leq N$, our next lemma, with $t = (K - 1)Np$, yields Proposition 3.2 for $C' = C_1$.

**Lemma 3.3.** For some $c > 0$, finite $C_\kappa$, any $\kappa > 0$, $t \geq 0$, $Np \geq \kappa \log N$ and $1 \leq k \leq N$,
\[ \mathbb{P}\left( \|A_{\leq k}\|_{\text{HS}} \geq t + Np + C_\kappa \sqrt{Np} \right) \leq 4e^{-t^2/16}. \]
In proving Lemma 3.3 we employ the following well-known concentration inequality.
Theorem 3.4 (cf. [Tal96, Theorem 6.6]). Suppose $F : [-1, 1]^d \to \mathbb{R}$ is convex and $L$-Lipschitz with respect to the Euclidean metric for some $L < \infty$ and the random vector $\xi \in [-1, 1]^d$ has independent components. Then, for any median $m$ of $F(\xi)$ and $t \geq 0$,

$$\mathbb{P}(|F(\xi) - m| \geq t) \leq 4 \exp\left(-\frac{t^2}{16L^2}\right).$$

We further need some control on the spectral gap of $A$, as in the following result about the operator norm of sparse Wigner matrices (whose root goes back [FK81]).

Lemma 3.5 (cf. [BGBK, Theorem 3.2], [LHY, Example 4.10]). Let $J$ be as in (1.45). For any $\kappa > 0$, there exists $C_\kappa < \infty$ such that $C_\kappa \to 4$ as $\kappa \to \infty$ and if $\kappa \log N \leq Np \leq N/2$, then

$$\mathbb{E}\|A - pJ\|_{op} \leq \frac{C_\kappa}{2}\sqrt{Np}.$$

Proof of Lemma 3.3. The mapping $A \mapsto \|A_{\leq k}\|_{HS}$ is convex and 1-Lipschitz with respect to $\|\cdot\|_{HS}$. Hence, in view of Theorem 3.4 it suffices to show that

$$\mathbb{P}\left(\|A_{\leq k}\|_{HS} \leq Np + C_\kappa \sqrt{kNp}\right) \geq \frac{1}{2}. \quad (3.6)$$

Turning to establish (3.6), since $\sqrt{Np} \gg p$ we can replace $J$ by $11^T = J + I$ in Lemma 3.5, and have by Markov’s inequality,

$$\mathbb{P}\left(\|A - p11^T\|_{op} \leq C_\kappa \sqrt{Np}\right) \geq \frac{1}{2}. \quad (3.7)$$

Further, with $\|p11^T\|_{op} = Np$, applying the triangle inequality for $\|\cdot\|_{op}$ we deduce that with probability at least $1/2$,

$$|\lambda_2(A)| \leq \|A - p11^T\|_{op} \leq C_\kappa \sqrt{Np} \quad \text{and} \quad |\lambda_1(A)| \leq Np + C_\kappa \sqrt{Np} \quad (3.8)$$

(as $A$ is a rank-1 perturbation of $A - p11^T$, the left-most inequality in (3.8) follows by the eigenvalue interlacing property). Recalling our ordering (1.46) of eigenvalues, given (3.8),

$$\|A_{\leq k}\|_{HS} \leq (Np)^2 + 2NpC_\kappa \sqrt{kNp} + kC_\kappa^2 \sqrt{kNp} \leq (Np + C_\kappa \sqrt{kNp})^2.$$

This establishes (3.6) and thereby concludes the proof. \qed

4. PROOF OF THEOREM 1.1: UPPER AND LOWER TAILS FOR CYCLE COUNTS

As we show next, Theorem 1.1 is a straightforward consequence of the following non-asymptotic tail bounds.

Theorem 4.1 (Quantitative large deviations for cycle counts). There are constants $c > 0$ and $C' < \infty$ such that for any integer $t \geq 3$, $N^{-1/2} \leq p \leq 1/2$, $K \geq 2$, $1 \leq R \leq N$, we have for any $t > 1$,

$$\mathbb{P}\left(\text{hom}_{C_t}(G) \geq tN^t p^t\right) \leq \exp\left(-\Phi_{Np}(C_t, t - \varepsilon_{\text{fluct}}) + E_{\text{complexity}}\right) + P_{\text{except}}, \quad (4.1)$$

and for any $0 \leq t \leq 1$,

$$\mathbb{P}\left(\text{hom}_{C_t}(G) \leq tN^t p^t\right) \leq \exp\left(-\Psi_{Np}(C_t, t + \varepsilon_{\text{fluct}}) + E_{\text{complexity}}\right) + P_{\text{except}}, \quad (4.2)$$

where the fluctuation term is $\varepsilon_{\text{fluct}} = 3\varepsilon^t$ with

$$\varepsilon(K, R) := \frac{C'}{N^{1/2 - 1/\ell} p^{1/2}} + \frac{\kappa \ell K}{R^{1/2 - 1/\ell}}$$

(4.3)
and \( \kappa_\ell \) as in Lemma 3.1, the complexity term is
\[
E_{\text{complexity}}(R) = O(\ell RN \log N),
\]
and \( P_{\text{excep}} = P_{\text{excep}}(K) \) is the exceptional probability from (3.5).

The bounds (4.1)–(4.2) are the result of applying Corollary 1.9 with a covering \( \{ B_i \} \in \mathcal{I} \) of \( \mathcal{A}_N \cap \mathcal{G}(KNp, C' \sqrt{\mathcal{N}p}) \), throughout which the corresponding bound (3.3) holds. Thanks to Proposition 3.2, the \( \mu_p \)-probability of its complement, exceptional set \( \mathcal{E} \), is at most \( P_{\text{excep}} \). The error term \( E_{\text{complexity}} \) is log \( |\mathcal{I}| \), which in our case is basically the metric entropy of \( \text{Sym}_{N,R}([0,1]) \).

**Proof of Theorem 1.1.** Starting with (1.15), the first term in the definition (4.3) of \( \varepsilon(K, R) \) is \( o(1) \) as long as \( p \gg N^{2/\ell - 1} \). Fixing an arbitrarily slowly growing function \( W = W(N) \), we take
\[
K = (W^2 \log(1/p))^{1/2}, \quad R = (W^4 \log N)^{\ell/(\ell - 2)}.
\]
Since \( Np \geq 1 \), with these choices we have that
\[
\frac{K}{R^{1 - 1/\ell}} = \left(\frac{W^2 \log(1/p)}{W^4 \log N}\right)^{1/2} \leq W^{-1} = o(1),
\]
hence also \( \varepsilon_{\text{fluct}} = o(1) \). Furthermore, for such \( K \),
\[
P_{\text{excep}} = C \exp(-cK^2 N^2 p^2) = \exp(-\omega(N^2 p^2 \log(1/p))).
\]
For
\[
\left(\frac{W^6 \log N}{\sqrt{N}}\right)^{\ell - 1} \leq p \leq N^{-1/10}
\]
we have
\[
\frac{\ell RN \log N}{N^2 p^2 \log(1/p)} \asymp \frac{R}{N^2 p^2} \leq W^{-2} = o(1),
\]
whereas for \( N^{-1/10} \leq p \ll 1 \),
\[
\frac{\ell RN \log N}{N^2 p^2 \log(1/p)} \leq \frac{R \log N}{N p^2} \leq \frac{R \log N}{N^{0.8}} = o(1).
\]
To conclude the proof of (1.15) it remains to dominate the error (4.6) by the first term on the RHS of (4.1), for which it suffices to show the analogue of (1.5), namely
\[
\Phi_{N,p}(C_\ell, t - \varepsilon_{\text{fluct}}) \precsim N^2 p^2 \log(1/p) \quad (4.8)
\]
for any fixed \( t \geq 1 \). While we could appeal to Theorem 1.5, it is easy to verify (4.8) directly. That is, for the projection \( I_{[N_0]} \) to the first \( N_0 \) coordinates, consider the matrix
\[
X_* = p(1_1 1^T - 1) + (1 - p)(1_{[N_0]} 1_{[N_0]}^T - I_{[N_0]}) \in X_N.
\]
As \( I_p(X_*) = \binom{N_0}{2} \log(1/p) \), taking \( N_0 = [ap] \) for fixed \( a = a(t) > 0 \) to be chosen gives
\[
I_p(X_*) \precsim_t N^2 p^2 \log(1/p).
\]
Moreover, for any fixed \( \ell \in \mathbb{N} \),
\[
\text{Tr} X_\ell^\ell \precsim \text{Tr}(I_{[N_0]} 1_{[N_0]}^T - I_{[N_0]})^\ell = \frac{N_0!}{(N_0 - \ell)!} = (aNp - O(1))^\ell.
\]
With \( p \ll 1 \), we can take \( a = (2t)^{1/\ell} \), yielding that
\[
\Phi_{N,p}(C_\ell, t - \varepsilon_{\text{fluct}}) \precsim \Phi_{N,p}(C_\ell, t) \leq I_p(X) \precsim_t N^2 p^2 \log(1/p),
\]
as claimed in (4.8). Turning to prove (1.17), let
\[ K = W^p^{-1/2}, \quad R = (W^4/p)^{\ell/(\ell-2)}. \] (4.10)
By (1.16) we have that \( p \gg N^{2/\ell-1} \) (since \( \frac{\ell-2}{2\ell-2} \leq 1 - \frac{2}{\ell} \)). Hence, from (4.3) and (4.10),
\[ \varepsilon(K, R) = \frac{1}{N^{1/2-1/\ell}p^{1/2}} + \frac{1}{W} = o(1), \]
yielding that \( \varepsilon_{\text{fluct}} = o(1) \). Further, from (3.5) we now have that
\[ P_{\text{except}} = C \exp(-cK^2N^2p^2) = \exp(-\omega(N^2p)). \]
Next, assuming
\[ p \geq \left( \frac{\log N}{N} \right)^{\frac{\ell-2}{\ell}} W^{3\ell/(\ell-1)}, \]
it follows that
\[ RN \log N \leq \frac{(\log N) W^{4\ell/(\ell-2)}}{N p^{(2\ell-2)/(\ell-2)}} \leq W^{4\ell} \frac{\ell}{\ell-2} \leq W^{-2} = o(1) \]
and so it only remains to show that
\[ \psi_{N,p}(C_{\ell,t} + \varepsilon_{\text{fluct}}) \lesssim_{t} N^2p. \] (4.11)
For this consider the matrix \( X_0 = bpJ_N \in \mathcal{X}_N \) for some fixed \( b = b(t) \in [0, 1] \). Clearly
\[ I_p(X_o) = (\frac{N}{2})I_p(bp), \]
whereas since \( p = o(1) \),
\[ I_p(bp) \sim p(b \log b - b + 1). \]
Thus \( I_p(X_o) \lesssim_{t} N^2p \). Moreover,
\[ \text{Tr} X_0^\ell = (bp)^\ell \frac{N!}{(N-\ell)!}, \]
which for \( b = t^{1/\ell} \) yields that
\[ \psi_{N,p}(C_{\ell,t} + \varepsilon_{\text{fluct}}) \lesssim_{t} \psi_{N,p}(C_{\ell,t}) \lesssim_{t} I_p(X_o) \lesssim_{t} N^2p \]
as claimed in (4.11).

\[ \square \]

4.1. Constructing a net. For \( R \in \mathbb{N} \) let
\[ \Lambda_R = \{ \lambda = (\lambda_1, \ldots, \lambda_R) \in \mathbb{R}^R : |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_R| \} \]
(4.12)
and for \( L < \infty \) write
\[ \Lambda_R(L) = \{ \lambda \in \Lambda_R : \|\lambda\|_2 \leq L \}. \] (4.13)
For \( 1 \leq R \leq N \) we denote the Stiefel manifold \( \text{St}(N, R) \) of ordered orthonormal bases for sub-spaces of \( \mathbb{R}^N \) of dimension \( R \) by
\[ \text{St}(N, R) = \{ \mathbf{u} = (u_1, \ldots, u_R) \text{ orthonormal in } \mathbb{R}^N \}. \]
We denote a mapping
\[ M : \Lambda_R \times \text{St}(N, R) \to \text{Sym}_{N,R}(\mathbb{R}), \quad M(\lambda, \mathbf{u}) = \sum_{j \leq R} \lambda_j u_j u_j^\top, \] (4.14)
which is a surjection by the spectral theorem (recall from Section 1.7 that \( \text{Sym}_{N,R}(\mathbb{R}) \) is the set of symmetric \( N \times N \) matrices of rank at most \( R \)). We equip \( \Lambda_R \) and \( \text{St}(N, R) \) with the Euclidean metrics (where elements are naturally associated to points in \( \mathbb{R}^R \) and \( \mathbb{R}^{N^2} \), respectively).

Recall that for a subset \( E \) of a metric space \( (X, d) \) and \( \delta > 0 \), a \( \delta \)-net for \( E \) is a set \( \Sigma \subset E \) such that for every \( x \in E \) there exists \( y \in \Sigma \) with \( d(x, y) \leq \delta \).
Lemma 4.2. Fixing \( c > 0 \), let \( 1 \leq R \leq N \) and \( 3\delta \leq R^{-c} \). There exist \( \delta \)-nets \( \Sigma \subset \Lambda_R(N) \), \( V \subset \text{St}(N,R) \) (with respect to the Euclidean metrics) of size

\[ |\Sigma| \leq \exp(R(2\log(2N) + |\log \delta|)), \quad |V| \leq \exp(O(NR|\log \delta|)). \]

Furthermore, if \( X = M(\lambda, u) \in \text{Sym}_{N,R}^{N,R}(\mathbb{R}) \cap B_{\text{HS}}(N) \) and \( Y = M(\mu, v) \in \Sigma \times V \) is such that \( \|\lambda - \mu\|_2, \|u - v\|_{\text{HS}} \leq \delta \), then

\[ \|X - Y\|_{\text{HS}} \leq 3N\delta. \]  

(4.15)

Remark 4.3. From (4.15) we have that \( M(\Sigma \times V) \) is a \( 3N\delta \)-net for \( \text{Sym}_{N,R}^{N,R}(\mathbb{R}) \cap B_{\text{HS}}(N) \) in the Hilbert–Schmidt metric. In the proof of Theorem 4.1 it will be convenient to separately approximate the spectrum and the eigenbasis of rank \( R \) projections of matrices \( A \in \mathcal{A}_N \), which is why we have defined the net in terms of \( \Sigma \) and \( V \).

Proof. For \( \Sigma \) we intersect \( \Lambda_R(N) \) with the \( R \)-th Cartesian power of an \( \delta/(2N) \)-mesh of the interval \([-N,N]\). Recalling that \( \text{St}(N,R) \) is a compact sub-manifold of the Euclidean sphere of radius \( \sqrt{R} \) in \( \mathbb{R}^{N \times R} \), let \( V \subset \text{St}(N,R) \) be any \( \delta \)-separated set which is maximal under set inclusion. Clearly, such \( V \) is a \( \delta \)-net for \( \text{St}(N,R) \) (otherwise, there exists \( u \in \text{St}(N,R) \) not within \( \delta \) of any element of \( V \), in contradiction with the maximality of \( V \)). Let \( B_{\text{HS}}(1)^\circ \) denote the open unit Euclidean (Hilbert–Schmidt) ball in \( \mathbb{R}^{N \times R} \). Since \( V \) is \( \delta \)-separated, the set \( V + \frac{\delta}{2} \cdot B_{\text{HS}}(1)^\circ \) is a union of disjoint Euclidean balls of radius \( \delta/2 \), which by the triangle inequality is contained in \( (\sqrt{R} + \delta/2) B_{\text{HS}}(1)^\circ \). Thus, by the monotonicity of the volume measure on \( \mathbb{R}^{N \times R} \),

\[ \text{vol}(B_{\text{HS}}(1))(\delta/2)^{NR}|V| = \text{vol}(V + (\delta/2) \cdot B_{\text{HS}}(1)) \leq (\sqrt{R} + \delta/2)^{NR} \text{vol}(B_{\text{HS}}(1)) \]

yielding for \( R \geq 1 \geq \delta \) the bound

\[ |V| \leq \left(1 + \frac{2\sqrt{R}}{\delta}\right)^{NR} \leq e^{NR\log(3R/\delta)}. \]  

(4.16)

Substituting \( \log R \leq \frac{1}{c}\log 1/(3\delta) \), we arrive at the stated bound on \( |V| \).

Turning to show (4.15), by the triangle inequality and Cauchy–Schwarz,

\[ \|X - Y\|_{\text{HS}} \leq \sum_{j \leq R} \|\lambda_j u_j u_j^T - \mu_j v_j v_j^T\|_{\text{HS}} \]

\[ \leq \sum_{j \leq R} |\lambda_j - \mu_j| \|u_j u_j^T\|_{\text{HS}} + \sum_{j \leq R} |\mu_j| \|u_j u_j^T - v_j v_j^T\|_{\text{HS}} \]

\[ \leq \sqrt{R} \|\lambda - \mu\|_2 + \|\mu\|_2 \left(\sum_{j \leq R} \|u_j u_j^T - v_j v_j^T\|_{\text{HS}}^2\right)^{1/2} \]

\[ \leq \sqrt{R} \delta + N \left(\sum_{j \leq R} \|u_j u_j^T - v_j v_j^T\|_{\text{HS}}^2\right)^{1/2}. \]

Next note that for any \( u, w \in \mathbb{R}^N \),

\[ 2\|u - w\|_2^2 - \|uu^T - ww^T\|_{\text{HS}}^2 = 2(\langle u, w \rangle - 1)^2 - (\|u\|_2^2 - 1)^2 - (\|w\|_2^2 - 1)^2, \]  

(4.17)

which is non-negative for pairs of unit vectors such as \( u_j, v_j \). Summing over \( 1 \leq j \leq R \) gives

\[ \sum_{j \leq R} \|u_j u_j^T - v_j v_j^T\|_{\text{HS}}^2 \leq 2 \sum_{j \leq R} \|u_j - v_j\|_2^2 = 2\|u - v\|_{\text{HS}}^2 \leq 2\delta^2. \]

Consequently, \( \|X - Y\|_{\text{HS}} \leq (1 + \sqrt{2})N\delta \leq 3N\delta \), as claimed. \( \square \)
4.2. Proof of Theorem 4.1. Fix $\ell > 2$ and $R \in [N]$. For $X \in \text{Sym}_N(\mathbb{R})$ recall the decomposition $X = X_{\leq R} + X_{>R}$ of (1.49), omitting hereafter the subscript $R$, with the induced parameters

$$u_{\leq}(X) = (u_1, \ldots, u_R) \in \text{St}(N, R), \quad \lambda_{\leq}(X) = (\lambda_1, \ldots, \lambda_R) \in \Lambda_R.$$  (4.18)

In order to apply Corollary 1.9 for

$$h_\ell(X) = \frac{1}{N^{\ell-\mu}} \text{Tr} X^\ell,$$

we specify for $\varepsilon = \varepsilon(K, R)$ of (4.3), the “exceptional” set

$$\mathcal{E}(\varepsilon) := \{ X \in \mathbb{B}_{\text{HS}}(N) : \| X \|_{S_\ell} > \varepsilon Np \}.  \quad (4.19)$$

Then, for the covering by closed convex sets, let $\Sigma$ and $\mathcal{V}$ be as in Lemma 4.2 and for each $X \in \mathbb{B}_{\text{HS}}(N)$ choose any $y(X) = (\mu(X), v(X)) \in \Sigma \times \mathcal{V}$ such that

$$\| \lambda_{\leq}(X) - \mu(X) \|_2 \leq \delta \quad \text{and} \quad \| u_{\leq}(X) - v(X) \|_{\text{HS}} \leq \delta.  \quad (4.20)$$

Setting $\delta' = 5\delta\sqrt{N}$, for each $y = (\mu, v) \in \Sigma \times \mathcal{V}$ consider the convex set

$$\mathcal{B}_y(\varepsilon) := \{ X \in \mathbb{B}_{\text{HS}}(N) : \exists Z \in \text{Sym}_N(\mathbb{R}), \\ \text{Im}(Z) \subseteq \ker(M(y)), \\ \| Z \|_{S_\ell} \leq \varepsilon Np, \| X - M(y) - Z \|_{\text{HS}} \leq \delta' N \}.  \quad (4.21)$$

We have the following two claims:

Claim 4.4. For any $\ell > 2$, $\varepsilon > 0$, $\delta > 0$ and $X \in \mathbb{B}_{\text{HS}}(N) \cap \mathcal{E}(\varepsilon)^c$ we have $X \in \mathcal{B}_y(X)(\varepsilon)$.

Claim 4.5. For any $\ell > 2$, $\varepsilon > 0$, $\delta' \leq N^{-2\ell}$, $y \in \Sigma \times \mathcal{V}$ and $X \in \mathcal{B}_y(\varepsilon)$,

$$|h_\ell(X) - h_\ell(M(y))| \leq \varepsilon \ell + O(N^{-\ell}).$$

We defer the proofs of these claims to subsequent subsections and conclude the proof of Theorem 4.1. From Claim 4.4 we have that for any $\varepsilon > 0$,

$$\mathbb{B}_{\text{HS}}(N) \setminus \mathcal{E}(\varepsilon) \subseteq \bigcup_{y \in \Sigma \times \mathcal{V}} \mathcal{B}_y(\varepsilon).$$

From Claim 4.5 and the triangle inequality we have for any $y \in \Sigma \times \mathcal{V}$, $\varepsilon > 0$ and $X, X' \in \mathcal{B}_y(\varepsilon)$,

$$|h_\ell(X) - h_\ell(X')| \leq 2\varepsilon \ell + O(N^{-\ell}).  \quad (4.22)$$

It is easy to check that for $L = KNp$, $D = C\sqrt{Np}$ and $\varepsilon = \varepsilon(K, R)$,

$$N^{1/\ell} D + \kappa_\ell L R^{1/\ell - 1/2} = \varepsilon Np.$$  

Hence, by (3.3) we have that $\| A \|_{S_\ell} \leq \varepsilon Np$ on the event that $A \in \mathcal{G}(L, D)$. From Proposition 3.2, the latter holds up to $\mu_p$-probability $P_{\text{except}}(K)$ of (3.5). In particular,

$$\mu_p(\mathcal{E}(\varepsilon)) \leq P_{\text{except}}(K).$$

Further, for such $\varepsilon$ the RHS of (4.22) is controlled by $\varepsilon_{\text{fluct}} = 3\varepsilon \ell$. Thus, (4.1) and (4.2) follow by applying Corollary 1.9 with $\mathcal{E} = A_N \cap \mathcal{E}(\varepsilon)$ and $\{ \mathcal{B}_y \}_{i \in \mathcal{I}} = \{ \mathcal{B}_y(\varepsilon) \}_{y \in \Sigma \times \mathcal{V}}$. \qed
4.3. **Proof of Claim 4.4.** Fix $X \in B_{HS}(N) \cap \mathcal{E}(\varepsilon)^c$ with spectral decomposition

$$X = \sum_{j=1}^{N} \lambda_j u_j u_j^T$$

and write $Y := M(\mu, v)$ for $(\mu, v) = y(X)$ (with notation as in (4.14)). Consider the matrix $V$ with columns $v_1, \ldots, v_R$ and the corresponding projection matrix $\Pi = I - VV^T$ onto $\text{span}(v_1, \ldots, v_R)^\perp$. Evidently, $\text{Im}(Z) \subseteq \text{span}(v_1, \ldots, v_R)^\perp \subseteq \ker(Y)$ for $Z := \Pi X \Pi$. Proceeding to establish (4.21) for $X, Y$ and $Z$, upon applying (1.48), our assumption that $X \notin \mathcal{E}(\varepsilon)$ yields that

$$\|Z\|_i \leq \|\Pi\|_\text{op}^2 \|X\|_i \leq \|X\|_i \leq \varepsilon Np.$$ 

Further, setting $w_j = \Pi u_j$, we have by the triangle inequality and Cauchy–Schwarz, that

$$\|X - Z\|_{HS} = \|\sum_{j > R} \lambda_j (u_j u_j^T - w_j w_j^T)\|_{HS} \leq \sum_{j > R} |\lambda_j| \|u_j u_j^T - w_j w_j^T\|_{HS} \leq \|X\|_{HS} \left( \sum_{j > R} \|u_j u_j^T - w_j w_j^T\|_{HS} \right)^{1/2}. \tag{4.23}$$

Recall (4.17), that $\|u_j u_j^T - w_j w_j^T\|_{HS} \leq 2\|u - w\|_2$ whenever $\|u\|_2^2 = \langle u, u \rangle$ and $\|u\|_2 = 1$. With $\Pi$ a projection matrix, this applies for $w_j = \Pi u_j$ and since $\|V\|_{\text{op}} = 1$, yields the bound

$$\|u_j u_j^T - w_j w_j^T\|_{HS} \leq 2\|(I - \Pi) u_j\|_2^2 = 2\|VV^T u_j\|_2^2 \leq 2\|V^T u_j\|_2^2. \tag{4.24}$$

Further, denoting by $U$ the matrix of columns $u_1, \ldots, u_R$, as $\{u_j\}$ are orthonormal, $U^T u_j = 0$ for any $j > R$ and from (4.20) we deduce that

$$\|V^T u_j\|_2 = \|(V - U)^T u_j\|_2 \leq \|V - U\|_{HS} \leq \delta. \tag{4.25}$$

Combining (4.23)–(4.25), and recalling that $\|X\|_{HS} \leq \|X\|_{HS} \leq N$ for $X \in B_{HS}(N)$, yields

$$\|X - Z\|_{HS} \leq \|X\|_{HS} \sqrt{2N\delta} \leq 2N^{3/2}\delta. \tag{4.26}$$

Finally, by the triangle inequality and (4.15) we have that

$$\|X - Y - Z\|_{HS} \leq \|X - Y\|_{HS} + \|X - Z\|_{HS} \leq 3N\delta + \|X - Z\|_{HS}.$$ 

In view of (4.26), we see that $\|X - Y - Z\|_{HS} \leq 5N^{3/2}\delta$ as desired for (4.21). \hfill $\square$

4.4. **Proof of Claim 4.5.** For $Y = M(y)$ and $X \in B_{y}(\varepsilon)$, let $Z$ be as in (4.21). Considering (2.5) for matrices $X, Y, Z$ and $Z$, we get by the monotonicity of $\ell \mapsto \| \cdot \|_{S_{\ell-1}}$ that

$$|\text{Tr} X^\ell - \text{Tr}(Y + Z)^\ell| \leq \ell \|X - Y - Z\|_{\text{op}} \|X\|_{S_{\ell-1}} \|Y + Z\|_{S_{\ell-1}} \lesssim \ell \|X - Y - Z\|_{HS} \|X\|_{HS} \|Y + Z\|_{HS} . \tag{4.27}$$

Further, $\|X - Y - Z\|_{HS} \leq \delta' N$ by (4.21), and $\|X\|_{HS} \leq N$. Thus, with $\delta' \leq 1$,

$$\|Y + Z\|_{HS} \leq \|X\|_{HS} + \|X - Y - Z\|_{HS} \leq N + \delta' N \leq 2N.$$ 

Along with (4.27), the preceding yields

$$|\text{Tr} X^\ell - \text{Tr}(Y + Z)^\ell| \leq \ell \delta' N (N^{\ell-1} + (2N)^{\ell-1}) \leq \delta' (4N)\ell. \tag{4.28}$$

Since $\text{Im}(Z) \subseteq \ker(Y)$, with $Y, Z \in \text{Sym}_N(\mathbb{R})$, the eigenvalues of $Y + Z$ are the $R' \leq R$ non-zero eigenvalues of $Y$ and the $N - R'$ eigenvalues of the restriction of $Z$ to ker$(Y)$. Hence,

$$\text{Tr}(Y + Z)^\ell = \text{Tr} Y^\ell + \text{Tr} Z^\ell. \tag{4.29}$$
Since \( \delta' \leq N^{-2\ell} \), we see that
\[
|\text{Tr} X^\ell - \text{Tr} Y^\ell| \leq |\text{Tr} Z^\ell| + O(N^{-\ell}) \leq \|Z\|_S^\ell + O(N^{-\ell}) \leq (\varepsilon Np)^\ell + O(N^{-\ell}),
\]
and the claim follows from dividing through by \( N^\ell p^\ell \geq 1 \). This concludes the proof of Claim 4.5 and hence of Theorem 4.1.

4.5. Proof of Proposition 1.11. Fix \( \alpha \in (2, \infty) \) and for \( X \in \operatorname{Sym}_N(\mathbb{R}) \) denote
\[
g_\alpha(X) = \frac{1}{Np} \|X\|_{S_\alpha}
\]
Setting \( \ell = \alpha \in (2, \infty] \) possibly non-integer, \( t = q/p, \varepsilon_{\text{fluc}} = 3\varepsilon \), while replacing \( h_\ell(\cdot) \) by \( g_\alpha(\cdot) \), only three items of the proof of Theorem 1.1 require modification. First, since \( \|X\|_{S_\alpha} \geq \frac{1}{\alpha} \|X\|_{\text{op}} \geq N_0^{-1} 1^T_{[N_0]} X 1_{[N_0]} \) for any \( X \in \operatorname{Sym}_N(\mathbb{R}) \), verifying that \( 1^T_{[N_0]} X_* 1_{[N_0]} = N_0(N_0 - 1) \) for \( X_* \) of (4.9), yields the analog of (4.8). Similarly, having \( \|J_N\|_{S_\alpha} = N(1 + o(1)) \), yields the analog of (4.11). Lastly, replacing Claim 4.5 with the following substitute eliminates the factor \( \ell \) of (4.7), thereby handling also \( \alpha = \infty \).

Claim 4.6. For any \( \alpha \in (2, \infty], \delta' \leq N^{-2}, \varepsilon > 0, y \in \Sigma \times V \) and \( X \in B_y(\varepsilon) \),
\[
\left|g_\alpha(X) - g_\alpha(M(y))\right| \leq \varepsilon + N^{-1}. \tag{4.30}
\]

Proof. For \( Y = M(y) \) and \( Z \) as in (4.21) and \( \alpha \in (2, \infty] \), by the triangle inequality
\[
\|X\|_{S_\alpha} - \|Y + Z\|_{S_\alpha} \leq \|X - Y - Z\|_{S_\alpha} \leq \|X - Y\|_{\text{HS}} \leq \delta' N \leq N^{-1}.
\]
By the same reasoning,
\[
\|Y + Z\|_{S_\alpha} - \|Y\|_{S_\alpha} \leq \|Z\|_{S_\alpha} \leq \varepsilon Np.
\]
Adding the preceding inequalities and dividing by \( Np \geq 1 \), yields the bound (4.30). \( \square \)

Remark 4.7. As with Theorem 1.1, our argument yields a quantitative version of Proposition 1.11, which is the same as Theorem 4.1 but with the integer \( \ell \geq 3 \) and \( \operatorname{hom}_{C_\ell}(G) \) replaced by \( \alpha \in (2, \infty] \) and \( \|A\|_{S_\alpha} \), respectively, where \( \phi_{N,p}(C_\ell, t), \psi_{N,p}(C_\ell, t) \), are correspondingly replaced with (1.9) and (1.10) for \( h = (Np)^{-1} \|\cdot\|_{S_\alpha} \).

5. Proof of Theorem 1.7: Upper tail for general homomorphism counts

We first prove our regularity and counting lemmas for random graphs, namely Propositions 2.3 and 2.4. We then combine these with Corollary 1.9 to establish Theorem 1.7.

5.1. Proof of Proposition 2.3. We may and will assume wlog that \( K \geq 2 \). From Proposition 3.2 we have property (b) for \( \mathcal{E}_0 := \mathcal{A}_N \setminus \mathcal{G}(KNp, C'\sqrt{Np}) \), while from Lemma 3.1 (at \( \alpha = \infty \)), we further have that for any \( 1 \leq R \leq Np \),
\[
\|A > R\|_{\text{op}} \leq C' \sqrt{Np} + \frac{KNp}{\sqrt{R}} \leq (C' + 1) \frac{KNp}{\sqrt{R}} \quad \forall A \in \mathcal{A}_N \setminus \mathcal{E}_0. \tag{5.1}
\]
Setting \( \delta_* = 3\delta = (KP \wedge 1)/\sqrt{R} \), from Lemma 4.2 there is an \( N\delta_* \)-net \( \mathcal{N} = M(\mu, v) \) for \( \operatorname{Sym}_{N,R}(\mathbb{R}) \cap B_{\text{HS}}(N) \) under the Hilbert–Schmidt metric, with
\[
\log |\mathcal{N}| \leq \log |\Sigma| + \log |\mathcal{V}| \lesssim R \log \left(3 + \frac{R}{KP}\right). \tag{5.2}
\]
In particular, for any \( A \in \mathcal{A}_N \) there exists \( Y \in \mathcal{N} \) such that
\[
\|A_{\leq R} - Y\|_{\text{op}} \leq \|A_{\leq R} - Y\|_{\text{HS}} \leq \frac{NKp}{\sqrt{R}},
\]
which upon setting \(C_* = C' + 2\) and
\[
    \mathcal{B}_Y := \left\{ X \in \mathcal{X}_N : \|X - Y\|_{op} \leq \frac{C_* K N p}{\sqrt{R}} \right\}, \quad Y \in \mathcal{N},
\]
implies, in conjunction with (5.1) and the triangle inequality, that
\[
    \mathcal{A}_N \setminus \mathcal{E}_0 \subset \bigcup_{Y \in \mathcal{N}} \mathcal{B}_Y.
\]
Fixing an enumeration \(\{Y_j\}_{j=1}^J\) of those \(Y \in \mathcal{N}\) for which \(\mathcal{B}_Y\) intersects \(\mathcal{A}_N\), we have property (a) thanks to (5.2) and can cover \(\mathcal{A}_N \setminus \mathcal{E}_0\) by pairwise disjoint sets \(\mathcal{E}_j \subseteq \mathcal{A}_N \cap \mathcal{B}_{Y_j}\), whereby property (c) follows from the definition of \(\mathcal{B}_{Y_j}\). \(\square\)

5.2. **Proof of Proposition 2.4(a)**. We will use a standard relation between homomorphism counts and *injective* homomorphism counts; for additional background we refer to [Lov12, Chapter 5]. For a graph \(H = ([n], E)\) we denote the *injective homomorphism counting function* on symmetric \(N \times N\) matrices \(X\) by
\[
    \text{inj}_H(X) = \sum_{\varphi: [n] \to [N]} \prod_{e=kt \in E} X_{\varphi(k)\varphi(l)}, \quad (5.3)
\]
where the sum ranges over injective mappings from \([n]\) to \([N]\). For adjacency matrices \(A\) we have the identity
\[
    \text{hom}_H(A) = \sum_P \text{inj}_{H/P}(A), \quad (5.4)
\]
where the sum ranges over partitions \(P\) of \(V(H) = [n]\), and \(H/P\) is the quotient graph obtained by identifying vertices within parts of the partition and deleting multiple edges (but leaving self-loops – in particular all parts of the partition must be independent sets in \(H\) to give a nonzero contribution to the sum, since \(A\) is the adjacency matrix for a graph without self-loops). For future reference, note that if \(F = H/P\) is a simple graph, then
\[
    e(H) - e(F) \leq (\nu(H) - \nu(F))\Delta(H), \quad \Delta(F) \leq \Delta(H), \quad (5.5)
\]
Indeed, for the first inequality, we obtain \(F\) by \(k = \nu(H) - \nu(F)\) operations, where in each step the size of one part in the partition increases by one and the number of parts decreases by one. Denoting by \(H = H_0, H_1, \ldots, H_k = F\) the corresponding shrinking graphs, we have
\[
    e(H) - e(F) = \sum_{j=0}^{k-1} e(H_j) - e(H_{j+1}) \leq k\Delta(H)
\]
(since with each step the number of edges in \(H_j\) can go down by at most \(\Delta(H_j) \leq \Delta(H)\)).

The main ingredient for the proof of Proposition 2.4(a) is the following result of Janson, Oleszkiewicz and Ruciński [JOR04].

**Theorem 5.1.** For any graph \(F\), some \(c_0(F) > 0\), any \(N^{-1/\Delta(F)} < p < 1\) and \(K \geq 2\),
\[
    \mathbb{P}(\text{inj}_F(A) \geq K \mathbb{E}\text{inj}_F(A)) \leq \exp \left( -c_0(F) K^{1/\nu(F)} N^2 p^{\Delta(F)} \right). \quad (5.6)
\]

**Proof.** Combining [JOR04, Theorems 1.2, 1.5], we have that for all \(K > 1\) there exists \(c(K, F) > 0\) such that for all \(N \geq \nu(F)\) and all \(N^{-1/\Delta(F)} < p < 1\),
\[
    \mathbb{P}(\text{inj}_F(A) \geq K \mathbb{E}\text{inj}_F(A)) \leq \exp \left( -c(K, F) N^2 p^{\Delta(F)} \right).
\]
(Note that while [JOR04] considers *subgraph counts* rather than injective homomorphism counts, the two differ by a fixed combinatorial factor which cancels from both sides of the
LHS inequality.) From [JOR04, Remark 8.2], one can take \( c(K, F) \geq \frac{K^{1/C(F)}}{\max_{F' \leq F} \alpha_{F'}^*} \), where the maximum ranges over subgraphs \( F' \) of \( F \) and \( \alpha^*_F \) is the fractional independence number of \( F' \) (c.f. [JOR04, Appendix A]). The claim then follows from bounding \( \alpha^*_F \leq v(F') \leq v(F) \) for all \( F' \leq F \).

Returning to Proposition 2.4(a), for any \( G \leq H \) we have from the identity (5.4), linearity of the expectation, and the union bound, that

\[
\mathbb{P} (\hom_G(A) \geq K \mathbb{E} \hom_G(A)) \leq \sum_F \mathbb{P} (\text{inj}_F(A) \geq K \mathbb{E} \text{inj}_F(A))
\]

(5.7)

where the sum ranges over the \( C_o(G) \) possible simple quotient graphs \( F = G/P \). Let \( c'(G) = \min \{ c_o(F)/C_o(G)^{1/v(F)} \text{ over such } F \} > 0 \). Since \( p > N^{-1/\Delta} \geq N^{-1/\Delta(G)} \) we have from (5.5) that

\[
p^{e(G) - e(F)} \geq p^{(v(G) - v(F))\Delta(G)} \geq N^{v(F) - v(G)},
\]

and hence

\[
\mathbb{E} \hom_G(A) = \sum_F \mathbb{E} \text{inj}_F(A) \leq \sum_F N^{v(F)} p^{e(F)} \leq C_o(G) N^{v(G)} p^{e(G)}.
\]

Consequently, we deduce from (5.6) and (5.7) that for any \( K \geq 2C_o(G) \)

\[
\mathbb{P} \left( \hom_G(A) \geq K N^{v(G)} p^{e(G)} \right) \leq \sum_F \exp \left( -c_o(F)(K/C_o(G))^{1/v(F)} N^2 p^{\Delta(F)} \right)
\]

\[
\leq C_o(G) \exp \left( -c'(G) K^{1/v(G)} N^2 p^{\Delta(G)} \right),
\]

(5.8)

(using the RHS of (5.5) in the last inequality). Next, from (5.8) and the union bound we have that for \( K \geq C(H) := 2 \max_{G \leq H} \{ C_o(G) \} \),

\[
\mu_p(\mathcal{E}_H(K)) \leq \sum_{G \leq H} C_o(G) \exp \left( -c'(G) K^{1/v(G)} N^2 p^{\Delta(G)} \right) \lesssim_H \exp \left( -c(H) K^{1/v(H)} N^2 p^\Delta \right),
\]

which is precisely the stated bound (2.13) of Proposition 2.4(a).

5.3. Proof of Proposition 2.4(b). We begin with a crude bound on the directional derivatives of \( \hom_H(\cdot) \).

**Lemma 5.2** (Derivatives of homomorphism counts). For \( W, Z \in \text{Sym}_N(\mathbb{R}) \) and a simple graph \( H = (V, E) \), the directional derivative of \( \hom_H \) at \( W \) in the direction \( Z \) is

\[
\mathcal{D}_H(W, Z) := \langle Z, \nabla \hom_H(W) \rangle_{\text{HS}} = \sum_{1 \leq i < j \leq N} Z_{ij} \partial_{W_{ij}} \{ \hom_H(W) \}.
\]

(5.9)

Fixing a non-empty simple graph \( H = ([n], E) \), for \( v \in V \) let \( H_{(v)} \) denote the induced subgraph of \( H \) on the vertices \( V \setminus \{ v \} \). Then, if \( W \in \mathcal{X}_N \),

\[
|\mathcal{D}_H(W, Z)| \leq \|Z\|_{\text{op}} \sum_{\{v_1, v_2\} \in E} \sqrt{\hom_{H_{(v_1)}}(W) \hom_{H_{(v_2)}}(W)}.
\]

(5.10)

**Proof.** For \( i = (i_1, \ldots, i_n) \in [N]^n \), \( W \in \text{Sym}_N(\mathbb{R}) \), and \( E' \subseteq E \), we denote

\[
W_{E'}(i) := \prod_{e' = k' \ell' \in E'} W_{i_{k'}, i_{\ell'}}^{'}, \quad W_\emptyset(i) := 1,
\]

so that

\[
\hom_H(W) = \sum_{i \in [N]^n} W_E(i).
\]
All directional derivatives are zero when $E = \emptyset$. Thus, assuming wlog that $n \geq 2$ and $m = |E| \geq 1$, from (5.9) we can express the directional derivative as a sum over "labelled" homomorphism counts in which all but one of the edges are labelled by entries of $W$, with the remaining edge labelled by an entry of $Z$:

$$D_H(W, Z) = \sum_{e \in E} \text{hom}_H(L^e), \quad \text{hom}_H(L^{\{k,l\}}) := \sum_{i \in [N]^n} Z_{i_k,i_l} W_{E \setminus \{k,l\}}(i).$$

Hence, it suffices to show that for any $W \in \mathcal{X}_N$ and $e = \{v_1, v_2\} \in E$,

$$|\text{hom}_H(L^e)| \leq \|Z\|_{\text{op}} \sqrt{\text{hom}_{H(v_1)}(W) \text{hom}_{H(v_2)}(W)}. \quad (5.11)$$

To this end, wlog take $e = \{1, 2\}$ and partition $E$ to $\{e\}, E_1, E_2, E_3$, where for $j = 1, 2$, we denote by $E_j$ the set of edges incident to vertex $j$ in the graph $H$, with the exception of $e$. With $W_{E_3}(i)$ independent of $i_1, i_2$, we have that

$$\text{hom}_H(L^{\{1,2\}}) = \sum_{i_3, \ldots, i_n \in [N]} W_{E_3}(i) \sum_{i_1, i_2 \in [N]} W_{E_1}(i) Z_{i_1,i_2} W_{E_2}(i).$$

Further, for any fixed $i_3, \ldots, i_n$, the value of $W_{E_1}(i)$ depends only on $i_1$, with $W_{E_2}(i)$ depending only on $i_2$. The inner sum is thus a quadratic form in $Z$, yielding that

$$\left| \sum_{i_1, i_2 \in [N]} W_{E_1}(i) Z_{i_1,i_2} W_{E_2}(i) \right| \leq \|Z\|_{\text{op}} \left( \sum_{i_1 \in [N]} W_{E_1}(i)^2 \right)^{1/2} \left( \sum_{i_2 \in [N]} W_{E_2}(i)^2 \right)^{1/2}
\leq \|Z\|_{\text{op}} \left( \sum_{i_1 \in [N]} W_{E_1}(i) \right)^{1/2} \left( \sum_{i_2 \in [N]} W_{E_2}(i) \right)^{1/2},$$

where in the last inequality we used the fact that $W_{E'_i}(i) \in [0,1]$ for any $i$, $E'$ and $W \in \mathcal{X}_N$. Consequently, by the above bound and Cauchy–Schwarz,

$$|\text{hom}_H(L^{\{1,2\}})| \leq \sum_{i_3, \ldots, i_n \in [N]} W_{E_3}(i) \left| \sum_{i_1, i_2 \in [N]} W_{E_1}(i) Z_{i_1,i_2} W_{E_2}(i) \right|
\leq \|Z\|_{\text{op}} \left( \sum_{i_1, i_3, \ldots, i_n \in [N]} W_{E_3}(i) W_{E_1}(i) \right)^{1/2} \left( \sum_{i_2, i_3, \ldots, i_n \in [N]} W_{E_3}(i) W_{E_2}(i) \right)^{1/2}
= \|Z\|_{\text{op}} \left( \text{hom}_{H^{(2)}}(W) \right)^{1/2} \left( \text{hom}_{H^{(1)}}(W) \right)^{1/2}.$$

The same holds for any $e \in E$, resulting with (5.11) and thereby with (5.10). \hfill \square

For any set $\mathcal{B} \subseteq \mathcal{X}_N$ and any graph $F$ (including when $e(F) = 0$), we trivially have that

$$\text{Max}(F; \mathcal{B}) := \sup_{X \in \mathcal{B}} \{\text{hom}_F(X)\} \leq \text{Max}(F; \mathcal{X}_N) \leq N^{v(F)}. \quad (5.12)$$

We also have the following immediate consequence of Lemma 5.2.

**Lemma 5.3.** For any non-empty simple graph $F$ and convex set $\mathcal{B} \subseteq \mathcal{X}_N$,

$$\text{Fluct}(F; \mathcal{B}) \leq \sup_{X,Y \in \mathcal{B}} \{\|X - Y\|_{\text{op}}\} \sum_{\{v_1, v_2\} \in E(F)} \sqrt{\text{Max}(F(v_1); \mathcal{B}) \text{Max}(F(v_2); \mathcal{B})}. \quad (5.13)$$

**Proof.** Fixing $X, Y \in \mathcal{B}$, for $t \in [0,1]$ let $W_t = (1 - t)Y + tX$. Note that

$$\text{hom}_F(X) - \text{hom}_F(Y) = \int_0^1 \frac{d}{dt}\{\text{hom}_F(W_t)\} dt = \int_0^1 D_F(W_t, X - Y) dt.$$
Applying the bound (5.10) on the expression on the RHS,
\[ |\hom_F(X) - \hom_F(Y)| \leq \|X - Y\|_{op} \int_0^1 \sum_{\{v_1, v_2\} \in E(F)} \sqrt{\hom_{F(v_1)}(W_t) \hom_{F(v_2)}(W_t)} \, dt. \]
Since \( B \) is convex, \( W_t \in B \) for all \( t \in [0, 1] \). Hence
\[ \hom_{F(v)}(W_t) \leq \Max(F(v); B) \]
and (5.13) follows by combining the previous two displays. \( \square \)

We proceed to establish Proposition 2.4(b) by iterating the preceding lemma (thereby sharpening the argument from [CD16, Lemma 5.4]).

Proof of Proposition 2.4(b). Set \( f_1(\cdot) = f(\cdot) + 1, f(k) = k \) for \( k \leq \Delta_\ast \) and thereafter set \( f(k) = kf_1(k-1) \) recursively, to guarantee that for any subgraph \( F \) of \( H \) with \( e(F) > \Delta_\ast \)
\[ \sum_{\{v_1, v_2\} \in E(F)} \sqrt{f_1(e(F(v_1)))f_1(e(F(v_2)))} \leq f(e(F)). \] (5.14)
By (1.23) we have for any \( e = \{v_1, v_2\} \in E(F) \) and \( F \leq H \),
\[ \Delta_\ast + e(F(v_1))/2 + e(F(v_2))/2 \geq e(F). \] (5.15)
We establish (2.16) by induction on \( e(F) \). To this end, note that from Lemma 5.3 together with (2.14) and (5.12), we have for any nonempty graph \( F \),
\[ \text{Fluct}(F; B) \leq \varepsilon_0 Np^{\Delta_\ast} \sum_{e \in E(F)} N^{\nu(e(F)) - 1} = \varepsilon_0 e(F)N^{\nu(F)}p^{\Delta_\ast}. \]
This also holds trivially in the case \( E(F) = \emptyset \) for which \( \text{Fluct}(F; B) = 0 \), thereby establishing (2.16) for any \( F < H \) having \( e(F) \leq \Delta_\ast \). Next, let \( k \in \{\Delta_\ast + 1, \ldots, e(H)\} \) and assume inductively that (2.16) holds whenever \( F < H \) has \( e(F) < k \). For such \( F \) we then have from (2.15) and the triangle inequality that
\[ \Max(F; B) \leq \inf_{X \in B} \{\hom_F(X)\} + \text{Fluct}(F; B) \]
\[ \leq KN^{\nu(F)}p^{e(F)} + \text{Fluct}(F; B) \leq f_1(e(F))K N^{\nu(F)}p^{e(F)}. \]
Considering \( F \leq H \) with \( e(F) = k \), the preceding applies to all \( \{F(v), v \in V(F)\} \). Hence, by Lemma 5.3,
\[ \text{Fluct}(F; B) \leq \varepsilon_0 Np^{\Delta_\ast} \sum_{\{v_1, v_2\} \in E(F)} \sqrt{f_1(e(F(v_1)))f_1(e(F(v_2)))} K N^{\nu(F) - 1}p^{e(F(v_1))/2}p^{e(F(v_2))/2} \]
\[ \leq \varepsilon_0 f(e(F))K N^{\nu(F)}p^{e(F)}, \]
as claimed, where in the second inequality we have used (5.14) and (5.15). \( \square \)

5.4. Proof of Theorem 1.7. Fix a simple, connected graph \( H = ([n], E) \) of \( m = |E| \) edges and maximal degree \( \Delta \geq 2 \). Similarly to Corollary 1.6, the lower bound can be established by considering the events Clique and Hub from (1.3), (1.4) with appropriate choices of \( a, b \), following the lines of the proof of [BGLZ17, Proposition 2.4]. Turning to the upper bound, towards an application of Corollary 1.9 we first specify the exceptional event as
\[ \mathcal{E} = \mathcal{E}(H, R, K_0, K_1) := \mathcal{E}_0(R, K_0) \cup \mathcal{E}_H(K_1) \] (5.16)
for 1 ≤ R ≤ Np and K_0, K_1 ≥ 1, to be chosen later, where \( \mathcal{E}_0(R, K_0) \) denotes the part \( \mathcal{E}_0 \) of the partition \( \mathcal{A}_N = \bigcup_{j=0}^J \mathcal{E}_j \) from Proposition 2.3, with \( K_0 \) in place of \( K \), and \( \mathcal{E}_H(K_1) \) is as in (2.12). Denoting by \( \mathcal{C}_j \subseteq \mathcal{X}_N, j \geq 1 \), the closed convex hull of \( \mathcal{E}_j \) and taking
\[
\mathcal{I} = \{ j \in [J] : \mathcal{C}_j \cap (\mathcal{A}_N \setminus \mathcal{E}) \neq \emptyset \},
\]
yields the following analogue of Claim 4.4:
\[
\mathcal{A}_N \setminus \mathcal{E} \subseteq \bigcup_{j \in \mathcal{I}} \mathcal{C}_j.
\]
(5.18)

We further have the following analogue of Claim 4.5 for general \( H \).

**Claim 5.4.** If \( \varepsilon_0 := 2C_* K_0 R^{-1/2} p^{1-\Delta_*} \leq 1 \) for \( \Delta_* \) of (1.23), then for all \( j \in \mathcal{I} \),
\[
\text{Fluct}(H; \mathcal{C}_j) \lesssim_m \varepsilon_0 K_1 N^n p^m.
\]

**Proof.** Fixing \( j \in \mathcal{I} \), in view of (5.16) and (5.17), the condition (2.15) of Proposition 2.4 holds with \( K = K_1 \). Furthermore, by the triangle inequality, the convexity of \( X \mapsto \|X\|_{op} \), property (2.11) and our choice of \( \varepsilon_0 \),
\[
\max_{X,Y \in \mathcal{C}_j} \|X - Y\|_{op} \leq 2 \max_{X \in \mathcal{C}_j} \|X - Y_j\|_{op} = 2 \max_{X \in \mathcal{C}_j} \|X - Y_j\|_{op} \leq \frac{2C_* K_0 N p}{\sqrt{R}} = \varepsilon_0 N p^{\Delta_*}.
\]
Thus, the condition (2.14) is also met and applying Proposition 2.4 yields our claim. \( \square \)

Fix \( t > 1 \) and denote
\[
h(X) = \frac{1}{N^n p^m} \text{hom}_H(X).
\]

We take
\[
K_0 = \log N, \quad K_1 = (\log N)^{2n}, \quad R = \frac{K_0^2 K_1^2 \log N}{p^{2\Delta_* - 2}}
\]
(5.19)

By our assumption that \( p \geq N^{-1/(\Delta+2\Delta_*-2)} \) and the fact that \( \Delta \geq 1 \), for all \( N \) sufficiently large the choice (5.19) results with \( R \leq N p \) and \( \varepsilon_0 = 2C_* / (K_1 \sqrt{\log N}) \leq 1 \) in Claim 5.4. Thus, by (5.18) and Claim 5.4, we can apply Corollary 1.9 with \( \mathcal{E} \cap \mathcal{A}_N \) for \( \mathcal{E} \) and \( \mathcal{I} \) of (5.16) and (5.17), respectively, and conclude in view of Proposition 2.3(a), that
\[
\mathbb{P}(h(A) \geq t) \leq \mathbb{P}(A \in \mathcal{E}) + \left| \mathcal{I} \right| \exp \left( -\phi_{N,p}(H,t - \varepsilon_{\text{fluct}}) \right) \leq \exp \left( -\phi_{N,p}(H,t - \varepsilon_{\text{fluct}}) + O(RN \log N) \right) + P_{\text{excep}}
\]
with
\[
\varepsilon_{\text{fluct}} \lesssim_m \varepsilon_0 K_1 = \frac{2C_*}{\sqrt{\log N}} = o(1),
\]
(5.20)

while, with \( K_1 \geq C(H) \), by (2.13) and Proposition 2.3(b),
\[
P_{\text{excep}} = O_H \left( \exp \left( -c(H)K_1^{1/n} N^2 p^\Delta \right) \right) + O \left( \exp \left( -cK_0^2 N^2 p^2 \right) \right) = e^{-\omega(N^2 p^\Delta \log N)}.
\]

Finally,
\[
\frac{RN \log N}{N^2 p^\Delta \log(1/p)} \leq \frac{(\log N)^{4n+3}}{Np^{\Delta+2\Delta_*-2}} = o(1)
\]
as long as
\[
\frac{(\log N)^{(4n+3)/\Delta}}{N^{1/(\Delta+2\Delta_*-2)}} \ll p \leq \frac{1}{e},
\]
which holds since, by assumption, \( n \geq 1 + \Delta \geq 3 \). Theorem 1.7 follows from the preceding three estimates and Theorem 1.5.
6. The upper tail for largest eigenvalues

Proposition 1.13 is a direct consequence of the following more general, quantitative bounds.

**Theorem 6.1.** For \( B \in \text{Sym}_N(\mathbb{R}) \) (non-random), let
\[
g_B : \mathcal{X}_N \to \mathbb{R}_+, \quad g_B(X) = \|X + B\|_{\text{op}}.\]
Then, for any such \( B \) and all \( N \in \mathbb{N} \), \( p \in (0, 1) \), \( \delta \in (0, \frac{1}{3}) \) and \( t \geq 0 \),
\[
\phi_p(g_B, (1 - 3\delta)t) - N \log(9/\delta) \leq -\log \mathbb{P}(g_B(A) \geq t) \leq \phi_p(g_B, t + 2) + \log 2. \tag{6.1}
\]

**Remark 6.2.** Slight modifications in the proof of Theorem 6.1 yield the same bounds on the right-most eigenvalue, namely for \( g_B^+(X) := \sup_{u \in \mathbb{S}^{N-1}} \langle u, (X + B)u \rangle \).

**Proof of Proposition 1.13.** We start with (1.34). Fix \( s = q/p > 1 \). With \( Np \geq \kappa \log N \), for all \( N \) large enough, \( t = \frac{s}{2} Np \geq C \kappa \sqrt{Np} \), so from Lemma 3.3 (noting that \( \|X_{\leq 1}\|_{\text{HS}} = \|X\|_{\text{op}} \)) we deduce that
\[
-\log \mathbb{P}(\|A\|_{\text{op}} \geq Nq) \gtrsim_s (Np)^2. \tag{6.2}
\]
Combined with the RHS of (6.1) for \( B = 0 \), this implies that for \( N^{-1} \log N \lesssim p \leq 1/2 \),
\[
\phi_p(\| \cdot \|_{\text{op}}, Nq) \gtrsim_s (Np)^2. \tag{6.3}
\]
In particular, the upper bound in (1.34) on the LHS of (6.2) holds for any such \( p \). In case \( p \gg N^{-1/2} \), we have by (6.3) that the leading term on the LHS of (6.1) (at \( B = 0 \)) is at least \( (Np)^2 \gg N \). We can then set \( \delta(N) \to 0 \) sufficiently slowly for it to dominate the error term \( N \log(9/\delta) \), yielding the matching lower bound in (1.34).

Turning to (1.35), taking \( Np \geq \kappa \log N \) and \( t \geq C \kappa \sqrt{Np} \), we re-run the proof of Lemma 3.3, now for the function \( A \mapsto \|A - p \mathbf{1} \mathbf{1}^T\|_{\text{op}} \) with (3.7) instead of (3.6), to deduce that
\[
-\log \mathbb{P}(\|A - p \mathbf{1} \mathbf{1}^T\|_{\text{op}} \geq t) \gtrsim t^2. \tag{6.4}
\]
Setting hereafter \( B = -p \mathbf{1} \mathbf{1}^T \), combined with the RHS of (6.1) this yields that
\[
\phi_p(\| \cdot - p \mathbf{1} \mathbf{1}^T \|_{\text{op}}, t) = \phi_p(g_B, t) \gtrsim t^2. \tag{6.5}
\]
Thus, the upper bound in (1.35) on the LHS of (6.4) holds for any such \( t(N) \) and \( p(N) \). Similarly to our proof of (1.34), when \( t \gg \sqrt{N} \) the leading term in the LHS of (6.1) is much larger than \( N \), so taking \( \delta(N) \to 0 \) sufficiently slowly yields the matching lower bound. \( \square \)

To establish Theorem 6.1 we will use the following standard converse to Proposition 1.8 for the case that \( \mathcal{K} \) is a closed half-space.

**Lemma 6.3.** For \( s \in \mathbb{R} \) and non-zero \( v \in \mathbb{R}^d \), let \( \mathcal{H}_v(s) = \{ x \in \mathbb{R}^d : \langle v, x \rangle \geq s \} \). Then,
\[
\mu_p(\mathcal{H}_v(s)) \geq \frac{1}{2} \exp \left( -I_p(\mathcal{H}_v(s + \sqrt{2}\|v\|_2)) \right). \tag{6.5}
\]

**Proof.** Let \( x \in \{0, 1\}^d \) have distribution \( \mu_p \) and \( \Lambda(\beta) := \log \mathbb{E} e^{\beta T} \), the CGF of \( T := \langle v, x \rangle \). Recall [DZ02, Exer. 2.2.23(b)] that for any \( \beta \geq 0 \) and \( y \in [0, 1]^d \),
\[
I_p(y) = \sum_{i=1}^d I_p(y_i) \geq \sum_{i=1}^d \{ \beta v_i y_i - \log \mathbb{E} e^{\beta v_i x_i} \} = \beta \langle v, y \rangle - \Lambda(\beta).
\]
Consequently,
\[
I_p(\mathcal{H}_v(t)) = \inf_{\{y : \langle v, y \rangle \geq t\}} I_p(y) \geq \sup_{\beta \geq 0} \{ \beta t - \Lambda(\beta) \}. \tag{6.6}
\]
Next, with $\mathbb{E}_\beta$ denoting expectation under the tilted product measure $\mu_{p, \beta}$ such that

$$\frac{d\mu_{p, \beta}}{d\mu_p} = e^{\beta T - \Lambda(\beta)},$$

recall that $m_\beta := \mathbb{E}_\beta T = \Lambda'(\beta)$ is an increasing function, with $\Lambda'(\beta) \uparrow m_\infty < \infty$ as $\beta \to \infty$.

In particular, setting $w = 2^{-1/2}||v||_2$, we deduce from (6.6) that whenever $s + w \geq m_\infty$ we have $I_p(H_v(s + 2w)) = \infty$ and (6.5) trivially holds. Further, $\text{Var}_\beta(T) = \Lambda'(\beta) \leq \frac{1}{4}||v||_2^2$ for any $\beta$. Hence, for $J_\beta := [m_\beta - w, m_\beta + w]$ we have from Chebychev’s inequality that

$$\mathbb{P}_\beta(T \notin J_\beta) \leq w^{-2} \text{Var}_\beta(T) \leq \frac{1}{2}. \quad (6.7)$$

This yields (6.5) when $s + w \leq m_0$, since

$$\mu_p(H_v(s)) = 1 - \mathbb{P}_0(T < s) \geq 1 - \mathbb{P}_0(T \notin J_0) \geq \frac{1}{2}.$$ 

If $s + w \in (m_0, m_\infty)$, then $s + w = m_\beta$ for some $\beta > 0$ with $J_\beta \subseteq [s, s + 2w]$. Hence,

$$\mu_p(H_v(s)) \geq \mathbb{P}(T \in J_\beta) = e^{\Lambda(\beta)} \mathbb{E}_\beta[e^{-\beta T} \mathbb{1}(T \in J_\beta)] \geq e^{\Lambda(\beta) - \beta(s + 2w)} \mathbb{P}_\beta(T \in J_\beta). \quad (6.8)$$

Combining (6.6) at $t = s + 2w$ with (6.7) and (6.8), we again get (6.5).

**Proof of Theorem 6.1.** Starting with the lower bound in (6.1), let $V \subset S^{N-1}$ be a Euclidean $\delta$-net of size at most $(3/\delta)^N$ (for example take $R = 1$ in (4.16)). Note that for all $X \in \text{Sym}_N(\mathbb{R})$,

$$g_B(X) = \sup_{u,v \in S^{N-1}} \langle u, (X + B)v \rangle \geq \max_{u,v \in V} \langle u, (X + B)v \rangle \geq (1 - 3\delta)g_B(X). \quad (6.9)$$

Indeed, for (6.10), supposing that $u_* = u_*(X), v_* = v_*(X)$ attain the supremum in (6.9), there exist $\tilde{u}, \tilde{v} \in V$ with $||\tilde{u} - u_*||_2, ||\tilde{v} - v_*||_2 \leq \delta$, whence

$$\langle \tilde{u}, (X + B)\tilde{v} \rangle \geq \langle u_*, (X + B)v_* \rangle - (2\delta + \delta^2)\|X + B\|_{op} \geq (1 - 3\delta)g_B(X).$$

Further, from (6.9), each super-level set $L_{\geq}(g_B, s), s \geq 0$, is the union of the closed half-spaces $H_{u,v}(s) := \{X \in \text{Sym}_N(\mathbb{R}) : \langle u, (X + B)v \rangle \geq s\}$ over $u,v \in S^{N-1}$. Consequently,

$$\phi_p(g_B, s) = I_p(L_{\geq}(g_B, s)) = \inf_{u,v \in S^{N-1}} I_p(H_{u,v}(s)). \quad (6.11)$$

Thus, with $s = (1 - 3\delta)t$, applying (6.10), the union bound and Proposition 1.8 yields

$$\mathbb{P}(g_B(A) \geq t) \leq \sum_{u,v \in V} \mu_p(H_{u,v}(s)) \leq |V|^2 \max_{u,v \in V} \{e^{-I_p(H_{u,v}(s))}\} \leq |V|^2 e^{-\phi_p(g_B, s)}.$$ 

The lower bound in (6.1) follows from substituting the bound on $|V|$ and taking logarithms.

Viewing $\text{Sym}_N^d(\mathbb{R}) \cong \mathbb{R}^d$ for $d = \binom{N}{2}$, we see that $H_{u,v}(t) = H_{y}(t - \langle u, Bv \rangle)$ for $H_y(\cdot)$ of Lemma 6.3, where $y \neq 0$ is the upper-triangular part of $uv^T + vu^T$. It is easy to check that

$$\|y\|_2^2 \leq \|u\|_2^2 \|v\|_2^2 + \langle u, v \rangle^2 \leq 2,$$

hence from Lemma 6.3, we have that

$$\mu_p(H_{u,v}(t)) \geq \frac{1}{2} \exp \left(-I_p(H_{u,v}(t + 2))\right). \quad (6.12)$$

Now from the identities (6.9), (6.11) and the bound (6.12) we have

$$\mathbb{P}(g_B(A) \geq t) \geq \sup_{u,v \in S^{N-1}} \mu_p(H_{u,v}(t)) \geq \frac{1}{2} \sup_{u,v \in S^{N-1}} \{e^{-I_p(H_{u,v}(t + 2))}\} = \frac{1}{2} e^{-\phi_p(g_B, t + 2)},$$

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and the upper bound in (6.1) follows.

7. Lower tails: proofs of Theorems 1.16 and 1.21

In proving Theorems 1.16 and 1.21 we set for $r \in (0, 1), \alpha \in [2, \infty]$ and $\varepsilon > 0,$

$$B_r(\varepsilon, \alpha) := \{X \in \mathcal{X}_N : \|X - r J_N \|_{S_\alpha} \leq \varepsilon r N\} \quad (7.1)$$

and use the following lemma (which is established by a tilting argument).

**Lemma 7.1.** Let $\alpha \in [2, \infty]$ and $N^{-1} \log N \leq r < p \leq 1/2.$ If

$$\varepsilon \geq \frac{C_1 N^{1/\alpha}}{\sqrt{Nr}} \quad (7.2)$$

for $C_1$ of Lemma 3.5, then

$$\mu_p(B_r(\varepsilon, \alpha)) \geq \frac{1}{2} e^{-(N/2) I_p(r) - 2\varepsilon^2 N^2}. \quad (7.3)$$

**Proof.** Since $\| \cdot \|_{S_\alpha} \leq N^{1/\alpha} \| \cdot \|_{op},$ thanks to condition (7.2),

$$B_r := \{X \in \mathcal{X}_N : \|X - r J_N \|_{op} \leq C_1 \sqrt{Nr}\} \subseteq B_r(\varepsilon, \alpha),$$

so (7.3) is an immediate consequence of

$$\mu_p(B_r) \geq \frac{1}{2} e^{-(N/2) I_p(r) - C_1 \sqrt{2pN}}. \quad (7.4)$$

Turning to prove (7.4), since

$$\frac{\mu_p(B_r)}{\mu_r(B_r)} = \frac{1}{\mu_r(B_r)} \int_{B_r} \exp \left(- \log \frac{d\mu_r}{d\mu_p}\right) d\mu_r,$$

applying Jensen’s inequality, we find that

$$\mu_p(B_r) \geq \mu_r(B_r) \exp \left(- \frac{1}{\mu_r(B_r)} \int_{B_r} \log \frac{d\mu_r}{d\mu_p} d\mu_r\right). \quad (7.5)$$

Furthermore, by Lemma 3.5 and Markov’s inequality, for $A_r \sim \mu_r$ and $r \geq N^{-1} \log N,$

$$\mu_r(B_r^c) = \mathbb{P}(\|A_r - r J_N \|_{op} > C_1 \sqrt{Nr}) \leq \frac{1}{2}. \quad (7.6)$$

Turning to the integrand in (7.5), note that for any $X \in \mathcal{X}_N$

$$\log \frac{d\mu_r}{d\mu_p}(X) = \left(\frac{N}{2}\right) I_p(r) \sum_{1 \leq i < j \leq N} (r - X_{ij}),$$

with the non-negative

$$\kappa(r, p) := \log \frac{1 - r}{1 - \frac{r}{p}} + \log \frac{p}{r} \leq \log \left(\frac{2p}{r}\right)$$

(as $p \leq 1/2$). We further estimate for any $X \in \mathcal{B}_r$

$$\sum_{1 \leq i < j \leq N} (r - X_{ij}) = \frac{1}{2} 1^T(r J_N - X) 1 \leq \frac{N}{2} \|X - r J_N \|_{op} \leq \frac{C_1}{2} N \sqrt{Nr}.$$

Combining the previous three displays (with $x \log(1/x) \leq 1$), we have for any $X \in \mathcal{B}_r$

$$\log \frac{d\mu_r}{d\mu_p}(X) - \left(\frac{N}{2}\right) I_p(r) \leq C_1 N \sqrt{Nr} \frac{1}{2} \log \left(\frac{2p}{r}\right) \leq C_1 N \sqrt{2pN}.$$

Substituting the latter bound and (7.6) into (7.5) yields our claim (7.4). \qed
Proof of Theorem 1.16. We first establish (1.40). The event on the LHS of (1.40) is $t_H(f_A) \leq \frac{\bar{q}|E|}{N}$, which by the Sidorenko property (1.38) is contained in the event $t_{K_2}(f_A) \leq \bar{q}$. The latter is the restriction

$$\sum_{1 \leq i<j \leq N} A_{ij} \leq \left( \frac{N}{2} \right) q.$$ 

Since the LHS has the $\text{Bin}(\frac{N}{2}, p)$ distribution, the claim follows from a classical result for tails of the binomial distribution (or one can apply Proposition 1.20 and follow the lines after (7.8) in the proof of Theorem 1.21 below).

Turning to the lower bound (1.39), recall that $\text{hom}_F(r J_N) \leq N^{\gamma(F)} r^\delta(F)$ for any subgraph $F$ and $r \in (0, 1)$. Hence, Proposition 2.4(b) applies with $K = 1$ and $r$ in place of $p$, for $B = B_r(\varepsilon, \infty)$ and any $\varepsilon = \frac{1}{2} r^{\Delta_* - 1}$, $\varepsilon_0 \leq 1$. Thus, for some $C = C(m)$ and any such $\varepsilon$,

$$\sup_{X \in B_r(\varepsilon, \infty)} \left| \text{hom}_H(X) - \text{hom}_H(r J_N) \right| \leq \varepsilon_0 C r^m N^n,$$

implying by the triangle inequality that for all $X \in B_r(\varepsilon, \infty)$,

$$\text{hom}_H(X) \leq \text{hom}_H(r J_N) + \varepsilon_0 C r^m N^n \leq (1 + \varepsilon_0 C) r^m N^n.$$

For $r = \frac{\bar{q}}{1 + \varepsilon_0 C} 1^{1/m}$ the RHS is at most $\bar{q}^m N^n$, hence

$$\{ X \in \mathcal{X}_N : \text{hom}_H(X) \leq \bar{q}^m N^n \} \supseteq B_r(\varepsilon, \infty).$$

Thanks to our assumption that $p^{2s_* - 1} \gg 1/\sqrt{Np}$, taking $q = sp$ for fixed $s \in (0, 1)$, Lemma 7.1 applies for $\alpha = \infty$ and some $\varepsilon_0(N) \to 0$ (such that $\varepsilon \geq C_1/\sqrt{Nr}$), giving

$$\mathbb{P}(\text{hom}_H(A) \leq \bar{q}^m N^n) \geq \mu_p(B_r(\varepsilon, \infty)) \geq \frac{1}{2} e^{-\left( \frac{N}{2} \right) I_p(r) - 2 \varepsilon p N^2}.$$

This completes the proof, since $I_p(r)/I_p(sp) \to 1$ and $p^{-1} I_p(sp)$ is bounded away from zero for such $p = p(N)$ and $r = r(N)$. 

Proof of Theorem 1.21. We first prove (1.42). The first inequality is a direct consequence of Proposition 1.20. For the second inequality in (1.42) it suffices to show that

$$\inf \{ I_p(X) : X \in \mathcal{X}_N, \|X\|_{S_\alpha} \leq (N - 1)q \} \geq \left( \frac{N}{2} \right) I_p(q).$$

If $X \in \mathcal{X}_N$ is such that $\|X\|_{S_\alpha} \leq (N - 1)q$, then by the monotonicity of $\beta \mapsto \| \cdot \|_{S_\beta}$

$$\frac{1}{\binom{N}{2}} \sum_{1 \leq i<j \leq N} X_{ij} = \frac{1}{N(N-1)} \mathbf{1}^T X \mathbf{1} \leq \frac{1}{N-1} \|X\|_{\text{op}} \leq \frac{1}{N-1} \|X\|_{S_\alpha} \leq q,$$  

(7.8)

Since $I_p(\cdot)$ is convex on $[0, 1]$ and decreasing on $[0, p]$, it follows from (7.8) that

$$\frac{1}{\binom{N}{2}} I_p(X) = \frac{1}{\binom{N}{2}} \sum_{1 \leq i<j \leq N} I_p(X_{ij}) \geq I_p\left( \frac{1}{\binom{N}{2}} \sum_{1 \leq i<j \leq N} X_{ij} \right) \geq I_p(q),$$

for all $X \in \mathcal{X}_N$ such that $\|X\|_{S_\alpha} \leq (N - 1)q$. This yields (7.7) and thereby (1.42).

Turning to the lower bound (1.43), by the triangle inequality and monotonicity of $\beta \mapsto \| \cdot \|_{S_\beta}$, we have that for any $r \in (0, 1)$, $\alpha \in [2, \infty]$ and $X \in B_r(\varepsilon, \alpha)$,

$$\|X\|_{S_\alpha} \leq r \|J_N\|_{S_\alpha} + \|X - r J_N\|_{S_\alpha} \leq r \|J_N\|_{\text{HS}} + \varepsilon r N \leq (1 + \varepsilon) r N.$$

For $r = q/(1 + \varepsilon)^2$ and $\varepsilon \geq 1/(N - 1)$ the RHS is at most $q(N - 1)$, hence

$$\{ X \in \mathcal{X}_N : \|X\|_{S_\alpha} \leq q(N - 1) \} \supseteq B_r(\varepsilon, \alpha).$$
Taking \( q = sp \) for fixed \( s \in (0, 1) \), thanks to our assumption that \( \sqrt{Np} \gg N^{1/\alpha} \) (or \( Np \gg \log N \) in case \( \alpha = \infty \)), Lemma 7.1 applies for some \( \varepsilon = \varepsilon(N) \to 0 \). The proof then concludes exactly as in the proof of Theorem 1.16. \( \square \)

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**References**

[Aug] F. Augeri. Nonlinear large deviation bounds with applications to traces of Wigner matrices and cycles counts in Erdős-Rényi graphs. Preprint, arXiv:1810.01558.

[Aus] T. Austin. The structure of low-complexity Gibbs measures on product spaces. Preprint, arXiv:1810.07278.

[BGBK] F. Benaych-Georges, C. Bordenave, and A. Knowles. Spectral radii of sparse random matrices. Preprint, arXiv:1704.02945.

[BGLZ17] B. B. Bhattacharya, S. Ganguly, E. Lubetzky, and Y. Zhao. Upper tails and independence polynomials in random graphs. *Adv. Math.*, 319:313–347, 2017.

[CD16] S. Chatterjee and A. Dembo. Nonlinear large deviations. *Advances in mathematics*, 299:396–450, 2016.

[CFS10] D. Conlon, J. Fox, and B. Sudakov. An approximate version of Sidorenko’s conjecture. *Geom. Funct. Anal.*, 20(6):1354–1366, 2010.

[Cha12] S. Chatterjee. The missing log in large deviations for triangle counts. *Random Structures Algorithms*, 40(4):437–451, 2012.

[Cha16] S. Chatterjee. An introduction to large deviations for random graphs. *Bull. Amer. Math. Soc. (N.S.)*, 53(4):617–642, 2016.

[CKLL] D. Conlon, J. H. Kim, C. Lee, and J. Lee. Some advances on Sidorenko’s conjecture. To appear in *J. London Math. Soc.*

[CV11] S. Chatterjee and S. R. S. Varadhan. The large deviation principle for the Erdős–Rényi random graph. *Combinatorica*, 1(3):233–241, 1981.

[DK12a] B. DeMarco and J. Kahn. Tight upper tail bounds for cliques. *Random Structures Algorithms*, 41(4):469–487, 2012.

[DK12b] B. DeMarco and J. Kahn. Upper tails for triangles. *Random Structures Algorithms*, 40(4):452–459, 2012.

[DZ02] A. Dembo and O. Zeitouni. Large deviations and applications. In *Handbook of stochastic analysis and applications*, volume 163 of *Statist. Textbooks Monogr.*, pages 361–416. Dekker, New York, 2002.

[EG18] R. Eldan and R. Gross. Decomposition of mean-field Gibbs distributions into product measures. *Electron. J. Probab.*, 23:Paper No. 35, 24, 2018.

[Eld] R. Eldan. Gaussian-width gradient complexity, reverse log-Sobolev inequalities and nonlinear large deviations. *Geom. Funct. Anal.*, to appear. Preprint at arXiv:1612.04346.

[FK81] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981.

[FK99] A. Frieze and R. Kannan. A simple algorithm for constructing Szemerédi’s regularity partition. *Electron. J. Combin.*, 6:Research Paper 17, 7, 1999.

[GH] A. Guionnet and J. Husson. Large deviations for the largest eigenvalue of rademacher matrices. Preprint. arXiv:1810.01188.

[Hat10] H. Hatami. Graph norms and Sidorenko’s conjecture. *Israel J. Math.*, 175:125–150, 2010.

[JOR04] S. Janson, K. Oleszkiewicz, and A. Ruciński. Upper tails for subgraph counts in random graphs. *Israel J. Math.*, 142:61–92, 2004.

[JR02] S. Janson and A. Ruciński. The infamous upper tail. *Random Structures Algorithms*, 20(3):317–342, 2002. Probabilistic methods in combinatorial optimization.

[JW16] S. Janson and L. Warnke. The lower tail: Poisson approximation revisited. *Random Structures Algorithms*, 48(2):219–246, 2016.

[KS] G. Kozma and W. Samotij. Private communication.

[KV04] J. H. Kim and V. H. Vu. Divide and conquer martingales and the number of triangles in a random graph. *Random Structures Algorithms*, 24(2):166–174, 2004.
LARGE DEVIATIONS OF SUBGRAPH COUNTS FOR SPARSE RANDOM GRAPHS

[1] R. Latała, R. V. Handel, and P. Youssef. The dimension-free structure of nonhomogeneous random matrices. Inventiones Math., to appear. (arXiv:1711.00807).

[2] L. Lovász. Large networks and graph limits, volume 60 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2012.

[3] E. Lubetzky and Y. Zhao. On replica symmetry of large deviations in random graphs. Random Structures Algorithms, 47(1):109–146, 2015.

[4] E. Lubetzky and Y. Zhao. On the variational problem for upper tails in sparse random graphs. Random Structures Algorithms, 50(3):420–436, 2017.

[5] A. Sidorenko. A correlation inequality for bipartite graphs. Graphs Combin., 9(2):201–204, 1993.

[6] M. Simonovits. Extremal graph problems, degenerate extremal problems, and supersaturated graphs. In Progress in graph theory (Waterloo, Ont., 1982), pages 419–437. Academic Press, Toronto, ON, 1984.

[7] B. Simon. Trace ideals and their applications, volume 120 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, second edition, 2005.

[8] M. Šileikis and L. Warnke. A counterexample to the DeMarco–Kahn upper tail conjecture. Preprint, arXiv:1809.09995.

[9] B. Szegedy. An information theoretic approach to Sidorenko’s conjecture. Preprint, arXiv:1406.6738.

[10] B. Szegedy. Limits of kernel operators and the spectral regularity lemma. European Journal of Combinatorics, 32(7):1156 – 1167, 2011. Homomorphisms and Limits.

[11] M. Talagrand. A new look at independence. Ann. Probab., 24(1):1–34, 1996.

[12] T. Tao. The spectral proof of the Szemeredi regularity lemma. URL: https://terrytao.wordpress.com/2012/12/03/the-spectral-proof-of-the-szemeredi-regularity-lemma/, December 2012.

[13] Y. Zhao. On the lower tail variational problem for random graphs. Combin. Probab. Comput., 26(2):301–320, 2017.

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