Backlund transformations for the Boussinesq equation and merging solitons

Alexander G Rasin and Jeremy Schiff

1 Department of Mathematics, Ariel University, Ariel 40700, Israel
2 Department of Mathematics, Bar-Ilan University, Ramat Gan, 52900, Israel

E-mail: rasin@ariel.ac.il and schiff@math.biu.ac.il

Received 23 January 2017, revised 15 June 2017
Accepted for publication 22 June 2017
Published 12 July 2017

Abstract
The Backlund transformation (BT) for the ‘good’ Boussinesq equation and its superposition principles are presented and applied. Unlike other standard integrable equations, the Boussinesq equation does not have a strictly algebraic superposition principle for 2 BTs, but it does for 3. We present this and discuss associated lattice systems. Applying the BT to the trivial solution generates both standard solitons and what we call ‘merging solitons’—solutions in which two solitary waves (with related speeds) merge into a single one. We use the superposition principles to generate a variety of interesting solutions, including superpositions of a merging soliton with 1 or 2 regular solitons, and solutions that develop a singularity in finite time which then disappears at a later finite time. We prove a Wronskian formula for the solutions obtained by applying a general sequence of BTs on the trivial solution. Finally, we obtain the standard conserved quantities of the Boussinesq equation from the BT, and show how the hierarchy of local symmetries follows in a simple manner from the superposition principle for 3 BTs.

Keywords: Backlund transformation, Boussinesq, merging, solitons, superposition, lattice, symmetries

(Some figures may appear in colour only in the online journal)
1. Introduction

In this paper we explore the Bäcklund transformation (BT) of the Boussinesq equation (BEq)

\[ U_{tt} - 4\beta U_{xx} + \frac{1}{3} U_{xxxx} - 2(U^2)_{xx} = 0 \]  

(1)

where \( \beta \) is a positive constant. The BEq is one of the oldest of the classical integrable nonlinear partial differential equations (PDE) \([9, 10]\), and its BT was given in bilinear form by Hirota and Satsuma \([23]\) and in standard form by Chen \([11]\), who also gave a superposition principle (see also \([24, 44, 48]\)). However, certain aspects seem not to have been discussed. There is a second superposition principle, and using this it is possible to give a superposition principle for 3 BTs that is algebraic (as opposed to the superposition principle of [11] that involves derivatives). In addition, there does not seem to be a systematic study of solutions generated by the BT, and this involves several surprises, as we shall shortly explain.

Our original motivation for looking at the BT of the BEq was connected with lattice versions of the equation. In recent years there has been substantial interest in integrable lattice equations. One of the origins of these is from superposition principles of BTs of integrable PDE (for example, the Q4 equation in the ABS classification \([3]\) was originally discovered by Adler as the superposition principle for the Krichever-Novikov equation \([2]\)). Discrete versions of the BEq have been given by Nijhoff et al \([34]\) as a scalar equation on a large stencil (see also \([33]\)), and by Tongas and Nijhoff \([41]\) as a system of equations for 3 fields on a rectangular plaquette. These, along with related ‘modified’ and ‘Schwarzian’ systems, have attracted much attention recently \([5, 6, 17–20, 32, 45–47]\). We investigate what superposition principles exist for the continuum BEq and show there are two associated lattice systems. One is a system of 2 equations for 2 fields on a rectangular plaquette (like the discrete modified and Schwarzian BEqs, as introduced in \([5, 33]\)). The other is a system of 2 equations for a single field on a cube.

However, it seems there is much to be learnt from simply applying the BT and its superposition principles, and this is the main focus of this paper. Note that in (1) we have written the ‘good’ version of the BEq, in which the signs of the \( U_{tt} \) and \( U_{xxxx} \) terms are the same, and there is no long wave instability (we also restrict in the current paper to the case \( \beta > 0 \) for which there is no short wave instability). For the ‘bad’ version, in which the signs of the \( U_{tt} \) and \( U_{xxxx} \) terms are opposite, the \( N \)-soliton solutions of the BEq equation were given by Hirota \([21]\), using his eponymous method. For the good BEq there is a subtlety in applying the Hirota method, and there are a variety of interesting, non-standard, soliton-type solutions discovered (apparently independently) by Tajiri and Nishitani \([40]\), Lambert et al \([26]\), Manoranjan et al \([30, 31]\) and Bogdanov and Zakharov \([7]\), citing unpublished work of Orlov. (Similar phenomena were observed by Hietarinta and Zhang \([19]\) in their study of solitons in a modified discrete BEq.) We show that applying the BT to the trivial solution can generate both standard solitons, and also what we call ‘merging solitons’—solutions in which two solitary waves (with related speeds) merge into a single one (Lambert et al \([26]\) use the term ‘soliton resonances’, Wang et al \([43]\) use the term ‘soliton fusion’). The superposition principle enables us to superpose a merging soliton with a standard 1-soliton or a standard 2-soliton. We have not succeeded so far to obtain a nonsingular solution involving the superposition of 2 or more merging solitons, but from the superposition of 3 merging solitons we find a solution which initially describes 6 solitary waves, becomes singular in finite time, but then becomes regular again, leaving 3 solitary waves. The possibility of finite time singularities in the BEq is well-known, originating, we believe, in \([25]\).
We also use the BT to prove a Wronskian formula for the general soliton solution of the B Eq, a generalization of the formula given in [35] for the bad BEq.

Finally, we show how to use the BT of the B Eq to generate its conservation laws and symmetries. The idea of using a BT to generate conservation laws of an integrable PDE is very old, see for example [42]. In [38] we showed how the superposition principles of BTs of integrable PDEs can be used to generate their symmetries. This works for the B Eq, but it is necessary to use the superposition principle for 3 BTs. This is a consequence of the fact that the B Eq is associated with the Lie group $SL(3)$, while equations such as Korteweg-de Vries, Sine-Gordon and Camassa-Holm are associated with $SL(2)$. We show how to use the 3 BT superposition principle to obtain the local symmetries of the B Eq, and also obtain the recursion operator and some nonlocal symmetries.

This paper is structured as follows: In section 2 we give the BT of the B Eq and its standard superposition principles. In section 3 we discuss associated lattice equations and obtain the algebraic superposition principle for 3 BTs. In section 4 we describe solutions of the B Eq generated by the BT and the superposition principles. In section 5 we use the BT to generate the symmetries and conservation of the B Eq. In section 6 we conclude and indicate areas for further study.

2. Bäcklund transformation and superposition

We work with the potential B Eq in the form

$$f_t = (f_x - f^2 - 2h)_x,$$  \hfill (2)

$$h_t = \left(\frac{2}{3}f_{xx} - h_x + \frac{2}{3}f^3 - 2ff_x\right)_x + 2fh_x.$$  \hfill (3)

This arises from the consistency of the Lax pair $Y_x = AY$, $Y_t = BY$ where

$$A = \begin{pmatrix} f & 1 & 0 \\ f_x - f^2 - h & 0 & 1 \\ \lambda + f_{xx} + f^3 - 3ff_x - 2hh_x + 2fh & h & -f \end{pmatrix},$$

$$B = \begin{pmatrix} -h & f & 1 \\ f_x - f^2 & 0 & 1 \\ B_{31} & \lambda + f_{xx} + f^3 - 3ff_x - hh_x + fh & h + f^2 - f_x \end{pmatrix},$$

$$B_{31} = -\frac{1}{2}f_{xxx} + f_x^2 + fxx - f^2f_x - h^2 + h(f_x - f^2).$$

The reason to take the equation in this apparently complicated form is that it simplifies the action of the BT, as we shall see shortly. It is easy to check that (2)–(3) imply

$$f_{tt} = -\frac{1}{3}f_{xxxx} + 4f_{x}f_{xx}$$  \hfill (4)

which is the standard form of the potential B Eq. If we define $w = h + \frac{1}{2}f^2 - f_x$ then the above system simplifies to

$$f_t = (-2w - f_x)_x,$$

$$w_t = w_{xx} + \frac{2}{3}f_{xxx} - f_x^2.$$
So if \( u = f_x, \ v = w_x \) then
\[
\begin{align*}
\dot{u}_t &= (-2v - u_x)_x, \\
\dot{v}_t &= \left(v_x + \frac{2}{3}u_{xx} - u^2\right)_x.
\end{align*}
\] (5)
\( (6) \)

This is the two component form of the BEq that we use (it is maybe more standard to replace the field \( v \) by \( \tilde{v} = -2v - u_x \) to simplify the first equation, but we find our form marginally more convenient). By eliminating \( v \) we obtain the scalar form of the BEq:
\[
\dot{u}_{tt} = -\frac{1}{3}u_{xxxx} + \left(2u^2\right)_{xx}.
\] (7)

Finally, if we write \( u = U + \beta \) we recover the familiar form (1). However, we will work with the form (7), and just remember, when looking at explicit solutions, that we are interested in solutions with \( u \to \beta \) at spatial infinity. We will focus on the case \( \beta > 0 \), in which case (1) is a linearly stable perturbed wave equation.

It is straightforward to verify that the potential BEq in the \( f, h \) form (2)–(3) has a BT
\[
\begin{align*}
f &\to f_{\text{new}} = f - s, \\
h &\to h_{\text{new}} = h - f_x + fs
\end{align*}
\] (8)

where \( s \) satisfies the equations
\[
\begin{align*}
s_{xx} &= \theta - 3ss_x - s^3 + 3fs_x + 3fs_x - 3fx - 3hx, \\
s_t &= \theta - ss_x - s^3 + 3fs_s + f_{xx} - 3ff_x - 3hx.
\end{align*}
\] (9)
(10)

or, equivalently,
\[
\begin{align*}
s_{xx} &= \theta - 3ss_x - s^3 + 3us - 3v, \\
s_t &= \theta - ss_x - s^3 + 3us - 2u_x - 3v.
\end{align*}
\] (11)
(12)

This is a BT in the sense that if \( f, h \) satisfy the potential BEq system (2)–(3), then so do \( f_{\text{new}}, h_{\text{new}} \) given by (8). Furthermore, the equations for \( s \), (11)–(12), are consistent if and only if \( u, v \) satisfy the BEq system (5)–(6). The BT (8) was originally given by Chen [11].

Denote by \( f_1, h_1 \) \( (f_2, h_2) \) the solution obtained from \( f, h \) using a BT with parameter \( \theta_1 \) \( (\theta_2) \), and by \( f_{12}, h_{12} \) \( (f_{21}, h_{21}) \) the solution obtained from \( f_1, h_1 \) \( (f_2, h_2) \) using a BT with parameter \( \theta_2 \) \( (\theta_1) \). Assuming commutativity of BTs gives
\[
\begin{align*}
f_{12} &= f_{21}, \\
h_{12} &= h_{21}.
\end{align*}
\]

Eliminating \( s \) from 4 copies of (8) we have
\[
\begin{align*}
h_1 &= h - f_x + f(f - f_1), \\
h_2 &= h - f_x + f(f - f_2), \\
h_{12} &= h_1 - f_{xx} + f_1(f_1 - f_{12}), \\
h_{21} &= h_2 - f_{xx} + f_2(f_2 - f_{21}).
\end{align*}
\]

Using commutativity and taking the obvious linear combination of these equations to eliminate all \( h \) fields, we arrive at the superposition principle, as given by Chen [11]
\[
\begin{align*}
f_{2x} - f_{xx} + (f + f_{12} - f_1 - f_2)(f_2 - f_1) = 0,
\end{align*}
\] (13)

which can be solved for \( f_{12} \):
\[ f_{12} = f_1 + f_2 - f - \frac{f_{2x} - f_{1x}}{f_2 - f_1}. \] (14)

In place of giving a proof for commutativity, it is possible to directly verify that the new solution given by (14) is a solution of the potential BEq (4).

But in fact there is also a second superposition formula. Assuming the commutativity of 2 BTs, we have four versions of equation (9):

\[
\begin{align*}
(f - f_1)_{xx} &= \theta_1 - 3(f - f_1)(f - f_1) - (f - f_1)^3 + 3f_x(f - f_1) + 3f_{xx} - 3f_{1x} - 3h_x, \\
(f - f_2)_{xx} &= \theta_2 - 3(f - f_2)(f - f_2) - (f - f_2)^3 + 3f_x(f - f_2) + 3f_{xx} - 3f_{1x} - 3h_x, \\
(f_1 - f_{12})_{xx} &= \theta_1 - 3(f_1 - f_{12})(f_1 - f_{12}) - (f_1 - f_{12})^3 + 3f_x(f_1 - f_{12}) + 3f_{xx} - 3f_{11x} - 3h_{1x}, \\
(f_2 - f_{12})_{xx} &= \theta_1 - 3(f_2 - f_{12})(f_2 - f_{12}) - (f_2 - f_{12})^3 + 3f_x(f_2 - f_{12}) + 3f_{xx} - 3f_{22x} - 3h_{2x},
\end{align*}
\]

where \( h_1 = h - f_x + f(f - f_1), h_2 = h - f_x + f(f - f_2). \) Taking a suitable linear combination of these equations eliminates the second derivatives of \( f, f_1, f_2, f_{12} \) and the function \( h \), giving the result

\[
0 = (f_1 - f_2) (f_{12x} + f_x - 2f_2 - f_1^2 - f_{12}^2 + 2f_1 f_2 - f_{12} f_1 + f_1 f_2) + \theta_2 - \theta_1 + (f_1 - f) f_{1x} - (f_2 - f) f_{2x}.
\]

Finally, adding \( f \) times equation (13) gives

\[
0 = (f_1 - f_2) (f_{12} + f_{12x} - f^2 - f_1^2 - f_{12}^2 + 2f_1 f_2 + f_{12} f_1 + f_{12} f_2) + \theta_2 - \theta_1 + f_1 f_{1x} - f_2 f_{2x}.
\] (15)

This can be written in the form

\[
f_{12x} = f_1^2 + f_{12}^2 + f_{12} - (f + f_{12})(f_1 + f_2) - f_{12} f_2 + \frac{\theta_1 - \theta_2 + f_1 f_{1x} - f f_{1x}}{f_1 - f_2} - f_x.
\] (16)

Equation (16) does not follow directly from (14). Differentiating the right hand side of (14) will include second derivatives of \( f_1, f_2 \). However note that if these are eliminated by using the first two of the 4 versions of (9) above, then (16) can be proved directly, without needing to assume commutativity.

3. Lattice equations and triple superposition

Writing \( u = f_x \) in (13) and (15) we obtain the pair of quad-graph equations

\[
0 = u_{12} - u_1 + (f + f_{12} - f_1 - f_2)(f_2 - f_1),
\] (17)

\[
0 = (f_1 - f_2) (u + u_{12} - f^2 - f_1^2 - f_{12}^2 + 2f_1 f_2 + f_{12} f_1 + f_{12} f_2) + \theta_2 - \theta_1 + f_1 u_1 - f_2 u_2.
\] (18)

(Here we are thinking of \( f, f_1, f_2, f_{12} \) as 4 values of the field \( f \) around the vertices of a rectangle. Other notations common in the literature are \( f, \tilde{f}, \tilde{f}, \tilde{f} \) and \( f_{nm}, f_{n+1,m}, f_{n,m+1}, f_{n+1,m+1} \).) These equations are simplified by introducing the field \( g = u - f^2 \):

\[
0 = g_2 - g_1 + (f + f_{12})(f_2 - f_1).
\] (19)
\[ 0 = (f_1 - f_2)(g + g_{12} + (f + f_{12})(f_1 + f_2) + f f_{12}) + \theta_2 - \theta_1 + f_1 g_1 - f_2 g_2. \quad (20) \]

It is straightforward to check that these equations have the consistency around the cube (CAC) property [3], and also arise as the consistency conditions for the following Lax pair:

\[
Y_1 = \begin{pmatrix}
 f_1 & -1 & 0 \\
-(f f_1 + g_1) & f & 1 \\
\theta_1 + f^2 f_1 + f g_1 - f_1 g & g + g_1 - f^2 & -(f + f_1)
\end{pmatrix} Y,
\]

\[
Y_2 = \begin{pmatrix}
 f_2 & -1 & 0 \\
-(f f_2 + g_2) & f & 1 \\
\theta_2 + f^2 f_2 + f g_2 - f_2 g & g + g_2 - f^2 & -(f + f_2)
\end{pmatrix} Y.
\]

Thus the system (19)–(20) is presumably an integrable lattice system, which we call the lattice potential Boussinesq system.

The system (19)–(20) should be compared with the lattice Boussinesq system of Tongas and Nijhoff [41]. Their system involves 3 fields \(u, v, w\), satisfying 5 equations on an elementary plaquette, 4 of which are the “same” equation on the 4 sides of the plaquette. Using \(u, v, w\) for the fields, as in [41] (and not as in the rest of this paper), the equations are

\[
w_1 = uu_1 - v, \quad w_2 = uu_2 - v, \quad w_{12} = u_2 u_{12} - v_2, \quad w_{12} = u_1 u_{12} - v_1, \quad w = uu_{12} - v_{12} + \frac{\theta_2 - \theta_1}{u_2 - u_1}. \]

We would argue that since the Tongas-Nijhoff system involves 5 relations between 12 quantities on an elementary plaquette (3 fields at each of 4 vertices), whereas our system involves 2 relations between 8 quantities, there is a fundamental difference. However, we suspect there may be relations between solutions of the two systems. Likewise, we suspect there are relations with the lattice modified Boussinesq system introduced in [33] and the lattice Schwarzian Boussinesq system that appears in [5], both of which are systems of 2 equations for 2 fields on an elementary plaquette.

In checking the CAC property for (19)–(20) it emerges that it is possible to eliminate the field \(g\) when considering the equations on a cube. So we introduce a third BT, with parameter \(\theta_3\), and denote by \(f_3\) the solution obtained from \(f\) via this BT, and consider the set of 8 solutions \(f, f_1, f_2, f_3, f_{12}, f_{13}, f_{23}, f_{123}\) associated with vertices of a cube, as indicated in figure 1. These solutions satisfy the equations

\[
f_{12}(f_2 - f_1) + f_{23}(f_3 - f_2) + f_{13}(f_1 - f_3) = 0 \quad (21)
\]

and

\[
f_{123} = f + \frac{(\theta_3 - \theta_2)f_1 + (\theta_1 - \theta_3)f_2 + (\theta_2 - \theta_1)f_3}{(f_2 - f_3)f_2 f_{23} + (f_3 - f_1)f_2 f_{13} + (f_1 - f_2)f_3 f_{12}}. \quad (22)
\]

Once again, we have 2 relations between 8 quantities. The first of these equations has a superficial similarity to the Hirota DAGTE equation [22], which was used as a starting point to find discrete Boussinesq systems in [20]. We strongly suspect that the system (21)–(22) can be considered as an integrable system on a three dimensional lattice. However, (22) is important.
in its own right, as it is an algebraic superposition principle for constructing new solutions of the BEq, which we will use in section 4.3.

4. Solitons of the Boussinesq equation

4.1. 1 BT solutions

As explained in section 2, we wish to look at solutions of the BEq with (7) with $u \to \beta$ at spatial infinity, with $\beta > 0$. Equivalently, we want solutions of the potential BEq (4) for which

$$f \sim \beta x + \gamma_+ \text{ as } x \to \infty \text{ and } f \sim \beta x + \gamma_- \text{ as } x \to -\infty,$$

where $\gamma_\pm$ are constants. We obtain such solutions by applying the BT to the starting solution $f = \beta x$.

Applying the BT once gives new solutions

$$f_1 = \beta x - \frac{\lambda_1 y}{2} \quad \text{where} \quad y = C_1 e^{\lambda_1 x + \lambda_1 t} + C_2 e^{\lambda_2 x + \lambda_2 t} + C_3 e^{\lambda_3 x + \lambda_3 t}. \quad (23)$$

Here $\lambda_1, \lambda_2, \lambda_3$ are the three roots of the cubic equation $\lambda^3 = 3\beta\lambda + \theta$, and $C_1, C_2, C_3$ are constants, not all zero, which can be jointly rescaled without changing the solution. There are two main situations to look at: the case $\theta^2 < 4\beta^3$ when $\lambda_1, \lambda_2, \lambda_3$ are all real and distinct, and the case $\theta^2 > 4\beta^3$ when one is real and the other two are a complex conjugate pair. (Note that the first situation can only happen if $\beta > 0$). As we wish to focus on soliton-type solutions, we look only at the first case, when there are 3 real, distinct roots.

The case when two of the constants $C_1, C_2, C_3$ are zero is trivial. If one is zero, say $C_3$, then the new solution is

$$f_1 = \beta x + \frac{c}{2} - p \tanh (p(x - ct) + \alpha)$$

or

$$f_1 = \beta x + \frac{c}{2} - p \coth (p(x - ct) + \alpha)$$

where $c = -(\lambda_1 + \lambda_2) = \lambda_3$, $p = \frac{1}{2}(\lambda_1 - \lambda_2)$ and $\alpha$ is an arbitrary constant. The corresponding solutions of the BEq are

$$u_1 = \beta - p^2 \text{sech}^2 (p(x - ct) + \alpha), \quad u_1 = \beta + p^2 \text{csch}^2 (p(x - ct) + \alpha).$$

These are the standard soliton and singular soliton solutions. A direct calculation confirms that these are solutions provided

$$\frac{c^2}{4} + \frac{p^2}{3} = \beta. \quad (24)$$
From this we again deduce the need for $\beta$ to be positive, and obtain bounds on both the velocity $c$ and the amplitude parameter $p$ for fixed $\beta$. Note that there are solutions with both positive and negative velocity, but that the solutions do not depend on the sign of $p$. Note also that for solitons $u < \beta$ and for singular solitons $u > \beta$.

Proceeding to the case where all three constants $C_1, C_2, C_3$ are nonzero, we need to distinguish between the case that all have the same sign, in which case the solution will be nonsingular, and the case that there are differing signs, in which case there is singularity. We start with the former. Looking at the expression for $y$ in (23), at a given time $t$ and position $x$ we will ‘see’ a soliton if two of the terms balance and are much bigger than the third term. So for example, we will see a soliton determined by the first two terms at position

$$x \approx \frac{1}{\lambda_1 - \lambda_2} \log \left( \frac{C_2}{C_1} \right) - (\lambda_1 + \lambda_2)t$$

(this is obtained from balance between the first two terms), provided

$$t(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) < K$$

where $K$ is some constant (this being the condition that at the given $x$, the first two terms are much bigger than the third one). Clearly a critical role is played by the sign of the product $(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$. If this is positive we ‘see’ a soliton determined by the first two terms for large negative times, if it is negative we will see the soliton for large positive times. It is straightforward to check that this product is positive for two of the three possible pairs of terms in $y$ and negative for the other pair. Thus the solutions describe the merger of two solitary waves into a single one. Furthermore, if we choose, without loss of generality, $\lambda_1 < \lambda_2 < \lambda_3$, then the incoming solitary waves are those of velocity $\lambda_1$ and $\lambda_3$, one of which is negative and

Figure 2. The merging soliton. Parameter values $\beta = 5$ and $\theta = -10$, so $\lambda_1, \lambda_2, \lambda_3 \approx -4.17, 0.69, 3.48$. The constants $C_1, C_2, C_3$ all taken to be 1. Plots of $u = f_t$ (with $f$ given by (23)), displayed for times $t = -0.7, -0.4, -0.1, 0.2, 1.5, 2.8$. Note smaller solitons are faster (see equation (24)).
one positive, and the outgoing one has velocity $\lambda_2$. Since $\lambda_1 + \lambda_2 + \lambda_3 = 0$ there is a ‘law of conservation of speed’. See figure 2 (compare with figures 3.1 and 5.1 in [30] and figure 11 in [7]). In this plot, as in all subsequent plots in this section, $u$ is plotted as a function of $x$. We call this solution a ‘merging soliton’.

Similar considerations can be applied in the case that all three constants $C_1, C_2, C_3$ in (23) are nonzero, but their signs differ. There will be one pair with the same sign and two pairs with opposite signs. The solutions can describe the absorption of a standard soliton by a singular soliton, see figure 3, or the merger of two singular solitons to a standard soliton, see figures 4 and 5 (compare figure 12 in [7]).

In summary, we have obtained 5 types of solution by a single application of the BT to the starting solution $f = \beta x$: standard solitons, singular solitons, a merging soliton, a singular soliton absorbing a soliton, and the merger of a pair of singular solitons to a single soliton. We refer to these solutions collectively as ‘1 BT solutions’.

4.2. 2 BT solutions

We now consider superpositions of two 1 BT solutions of the form (23) using the superposition principle (14). The solution takes the simple form

$$f_{12} = \beta x - \frac{y_1 y_{2xx} - y_2 y_{1xx}}{y_1 y_{2x} - y_2 y_{1x}}. \quad (25)$$

We have not succeeded to give a complete (analytic) classification of these solutions, but we report cases in which we have found superpositions without singularities:

- For certain parameter values, a pair of standard soliton solutions can be superposed to give a colliding 2-soliton solution. The 2 solitons should be taken with velocities of dif-
fering signs; this is a necessary, but not sufficient, condition for such a superposition to be possible.

- For certain parameter values, a standard soliton and a singular soliton can be superposed to give a 2-soliton solution. The resulting solutions include both colliding pairs and pairs moving in the same direction.
- For certain parameter values, a standard soliton solution can be superposed with one of the two types of singular solution describing a merger, to give a solution with three solitary waves merging to two. See figure 6, in which the soliton with parameters $\theta = 8, C_1 = 1, C_2 = 1, C_3 = 0$ is superposed with the 1 BT solution with $\theta = -12, C_1 = -1, C_2 = 1, C_3 = 1$ (for $\beta = 5$). In the cases of this that we have found, the initial configuration always has two solitary waves moving in one direction, and the other in the opposite direction, but the final configuration can have two moving in one direction, or one in each direction. We have not found cases of mergers of 3 moving in the same direction to 2, but we cannot currently exclude this possibility.
So far we have not found any cases of the merger of 4 solitary waves to 2, though we cannot currently exclude this possibility. The superpositions of a pair of solutions describing a merger always seem to be singular, describing, for example, the merger of 4 solitary waves to 2 singular solitons, or the absorption of two singular solitons by two standard solitary waves.

4.3. 3 BT solutions

The superposition of 3 1 BT solutions of the form (23), using equations (22) and (14), takes the form

\[ f_{123} = \beta x - \frac{\theta_1 y_1 (y_2 y_3 x - y_3 y_2 x) + \theta_2 (y_3 y_1 x - y_1 y_3 x) + \theta_3 (y_1 y_2 x - y_2 y_1 x)}{y_1 (y_2 x y_3 x - y_3 x y_2 x) + y_2 (y_3 x y_1 x - y_1 x y_3 x) + y_3 (y_1 x y_2 x - y_2 x y_1 x)}. \]  

\[ (26) \]

Once again, we do not have a full analytic classification of solutions, but numerical experiments indicate that this is simpler than for superpositions of 2 solutions. 3-soliton solutions are obtained from (certain) superpositions of 2 standard solitons and 1 singular soliton. Solutions

![Figure 6. 3 solitary waves merge to 2. Superposition of two solutions of type (23) with \( \theta = 8, C_1 = 1, C_2 = 1, C_3 = 0 \) and \( \theta = -12, C_1 = -1, C_2 = 1, C_3 = 1 \) for \( \beta = 5 \). Plots of \( u \) against \( x \) for times \( t = -1.8, -1.0, -0.6, -0.16, 0.0, 0.16, 0.6, 1.0, 1.8 \).]
describing the merger of 4 waves to 3 are obtained from (certain) superpositions of a standard soliton, a singular soliton, and a merging soliton. See figure 7 for an example.

We have not succeeded in obtaining nonsingular solutions from a superposition using more than one merger-type solution. However there are some remarkable singular solutions. In figure 8 we present plots of the superposition of three merger-type solutions, describing the evolution of 6 solitary waves into 3, via a brief, finite time duration singularity. The singularity forms after $t = -0.1$ and disappears before $t = 0.3$.

4.4. Wronskian solutions

The forms (23), (25) and (26) for the solutions obtain by 1, 2 or 3 applications of the BT suggest that in general the form of the general solution obtained by $n$ applications of the BT to the starting solution $f = \beta x$ should be
where $W$ is the Wronskian

\[
W = \det \begin{pmatrix}
y_1 & y_2 & \cdots & y_n \\
y_1' & y_2' & \cdots & y_n' \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{pmatrix}
\]  

and each of the functions $y_i$ are of the form in (23), i.e. $y_i$ is a general solution of the differential equation $y_i'' = 3\beta y_i' + \theta y_i$. (In this paragraph we use primes to denote differentiation with respect to $x$.) We prove this as follows. Assuming

\[
f = \beta x - \frac{w'}{w}, f_1 = \beta x - \frac{w_1'}{w_1}, f_2 = \beta x - \frac{w_2'}{w_2}, f_{12} = \beta x - \frac{w_{12}'}{w_{12}}
\]

substituting in (14), simplifying and integrating once gives the requirement

\[
WW_{12} = K(W_1W_2^2 - W_2W_1^2)
\]

where $K$ is an arbitrary constant (note that each of the $W$’s is only defined up to an overall constant). Now if $W, W_1, W_2, W_{12}$ all have the form of Wronskians, of dimensions $n - 2, n - 1, n - 1, n$ respectively, with $W_{12}$ being exactly the determinant of the matrix in (28), then

- $W$ is the determinant of the same matrix with the $(n - 1)^\prime$th and $n$’th rows and columns deleted,
- $W_1$ is the determinant of the same matrix with the $n$’th row and $n$’th column deleted,
- $W_2$ is the determinant of the same matrix with the $n$’th row and $(n - 1)^\prime$th column deleted,
- $W_1'$ is the determinant of the same matrix with the $(n - 1)^\prime$th row and $n$’th column deleted,
- $W_2'$ is the determinant of the same matrix with the $(n - 1)^\prime$th row and $(n - 1)^\prime$th column deleted.

The desired identity therefore follows from a case of Sylvester’s theorem for determinants [16], that if $A$ is an arbitrary $n \times n$ matrix, $C$ is the same matrix with the $(n - 1)^\prime$th and $n$’th rows and columns deleted, $B_1$ is the same matrix with the $n$’th row and $n$’th column deleted, $B_2$ is the same matrix with the $(n - 1)^\prime$th row and $n$’th column deleted, $B_3$ is the same matrix with the $n$’th row and $(n - 1)^\prime$th column deleted and $B_4$ is the same matrix with the $(n - 1)^\prime$th row and $(n - 1)^\prime$th column deleted, then

\[
\det C \det A = \det B_1 \det B_2 - \det B_3 \det B_4.
\]

The general result on the form of the solution follows by induction. A similar result appeared for the bad BEq in [35], however taking each $y_i$ to be the sum of only two exponentials.

### 4.5. Comparison with the Hirota method

In [21], Hirota gave the general solution of the bad BEq using the ‘Hirota method’ and it is interesting now to see how this works for the good BEq. For this paragraph we work directly with the BEq in the form (1), with $\beta > 0$. Writing $U = -(\log \tau)_{xx}$, the equation becomes

\[
\tau \tau_{ff} - \tau_f^2 - 4\beta (\tau \tau_{xx} - \tau_x^2) + \frac{1}{3} (\tau \tau_{xxxx} - 4\tau_x \tau_{xxx} + 3\tau_{xxx}^2) = 0
\]

which can be written in ‘Hirota bilinear form’
\[(D_t^2 - 4\beta D_x^2 + \frac{1}{3} D_x^4)\tau \cdot \tau = 0,\]

where $D_t, D_x$ are the usual Hirota derivatives. This has ‘multisoliton’ solutions in the usual form.
\[ \tau = 1 + \sum_i c_i e^{\eta_i} + \sum_{i<j} c_i c_j \phi_{ij} e^{\eta_i + \eta_j} + \sum_{i<j<k} c_i c_j c_k \phi_{ijk} e^{\eta_i + \eta_j + \eta_k} + \ldots \]

where \( \eta_i = a_i(x + b_i t), a_i, b_i, c_i \) constants, with \( a_i^2 + 3b_i^2 = 12 \beta \), and

\[ \phi_{ij} = \frac{(a_i - a_j)^4 - 12\beta(a_i - a_j)^2 + 3(a_i b_j - a_j b_i)^2}{(a_i + a_j)^4 - 12\beta(a_i + a_j)^2 + 3(a_i b_j + a_j b_i)^2}. \]

To guarantee that all these solutions are nonsingular requires \( \phi_{ij} > 0 \) for all choices of the constants \( a_i, b_i, c_i \) (in addition to choosing the constants \( c_i > 0 \)) and that is not the case here. Furthermore, it is possible to choose \( a_i, a_j, b_i, b_j \) such that \( \phi_{ij} = 0 \). A straightforward calculation shows that if this happens then \( b_i^2 + b_i b_j + b_j^2 = 3\beta \), implying that there is some constant \( \theta \) for which \( b_i, b_j \) are distinct solutions of the cubic equation \( b^3 - 3\beta b + \theta \). This is the origin of the merging soliton solutions in the the Hirota framework. A more extensive discussion appears in [26], and the significance of the criterion \( \phi_{ij} = 0 \) is emphasized in [43].

5. Conservation laws and symmetries

The remarkably simple method for finding conservation laws from a BT is very old, see for example [42]. For the BEq, we simply need to observe that (11)–(12) implies

\[ s_i + (2u - s_x - s^2)_x = 0. \]

Thus \( s \), which depends on \( \theta \), provides a generating function for (densities of) conservation laws. To obtain the standard conservation laws, observe that the solution \( s \) to (11) can be written as an asymptotic series in \( \theta \) in the form

\[ s \sim \sum_{i=1}^{\infty} \theta^{-i/3} s_i. \]

Each of the coefficients \( s_i \) is the density for a conservation law. The first few coefficients are given as follows:

\[ s_0^3 = 1, \quad s_0 = 0, \quad s_1 = \frac{u}{s_{-1}}, \quad s_2 = \frac{v + u_s}{s_{-1}^2}. \]

Further terms can be computed using the recurrence relation

\[ s_{k+2} = \frac{1}{3s_{-1}} \left( 3us_k - \sum_{j=-1}^{k+1} \sum_{i=\max(-1,-j-1)}^{\min(k+1,2-j+1)} s_{k-i-j}s_j \right) + \frac{1}{3} \sum_{i=-1}^{k+1} s_i s_{(k-i)x} - s_{kxx} \], \quad k \geq 1. \]

So for example

\[ s_3 = \frac{2}{3} u_{sx} + v_s, \]

\[ s_4 = \frac{1}{3s_{-1}} (3uv - u_{xxt} - 2v_{xx}), \]

\[ s_5 = \frac{1}{9s_{-1}^2} (u_{xxxt} + 3v_{xxx} - 3u^3 + 3uu_{xx} - 9uv_x - 9v^2 - 18u_x v). \]
For each $i = 1, 2, \ldots, s$, $s_i$ is the density $F$ of a conservation law $F_i + G_s = 0$. For $i = 3$ the conservation law is evidently trivial ($F = H_3$, $G = -H_1$ for some $H$). Indeed we will shortly show that all the conservation laws for $i = 3, 6, 9, \ldots$ are trivial. The associated flux $G$ is the coefficient of $\theta^{-i/3}$ in $2u - s - s'$. Thus for $i = 1, 2, 4, 5$ we have fluxes

\begin{align*}
G_1 &= \frac{1}{s-1}(u_s + 2v), \\
G_2 &= -\frac{1}{s-1}\left(u^2 + v_x + \frac{1}{3}u_{xx}\right), \\
G_4 &= \frac{1}{s-1}\left(\frac{2}{3}u^3 - uv_x - 2u^2u_x - u_x^2 + u_xv + v^2 + \frac{1}{9}u_{xxx}\right), \\
G_5 &= \frac{1}{9s-1}\left(18uv^2 - 9u^2u_x - 3u^2u_{xxx} - 21uv_{xx} - 36u_{xx} - 30u_{xxx}v - 36v_{xx} - 9u_xu_{xx} - 45u_xv_x \\
&\qquad\qquad+ 3v_{xxxx} + u_{xxxx}\right).
\end{align*}

We note there are 3 possible series for $s_i$ corresponding to the 3 possible choices of $s_{i-1}$. The dependence of $s_{1}, s_{2}, \ldots$ on the choice of $s_{i-1}$ is clear, and can be verified to be consistent with the recursion relation. We denote the three solutions of (11) with these three asymptotic series by $s^{(1)}, s^{(2)}, s^{(3)}$. If we define $\sigma = s^{(1)} + s^{(2)} + s^{(3)}$ then $\sigma$ has asymptotic series $\sum_{i=1}^{\infty} \sigma_s \theta^{-i}$. However, if we define

\[ A = (s^{(2)} - s^{(3)})s^{(1)} + (s^{(3)} - s^{(1)})s^{(2)} + (s^{(1)} - s^{(2)})s^{(3)} - (s^{(1)} - s^{(2)})s^{(2)} - s^{(3)}(s^{(3)} - s^{(1)}), \]

then it can be verified (using (11)) for each of the functions $s^{(1)}, s^{(2)}, s^{(3)}$ that

\[ (\log A)_1 = s^{(1)} + s^{(2)} + s^{(3)} = -(\log A)_x. \]

It follows that $s_{3,i}$ is a total $x$ derivative for all $i$, and the associated conservation laws are trivial.

The use of a BT to generate symmetries is rather newer [38]. The critical observation made in [38] for the KdV, Sin Gordon and Camassa Holm equations, was that while individual BTs are not ‘small’ transformations (and thus not directly related to symmetries, which are transformations of solutions that are infinitesimally close to the identity), the composition of two BTs can be small in this sense. Equivalently, the inverse of a BT is not a (single) BT, but rather the composition of two BTs. These facts have their origins in the fact that the Lax pair for the BEq is a $3 \times 3$ matrix Lax pair.

Equation (22) for a triple BT can be written, using (13), in the form

\[ f_{123} = f + \frac{(\theta_3 - \theta_1)f_1 + (\theta_1 - \theta_2)f_2 + (\theta_2 - \theta_3)f_3}{(f_1^2 - f_2^2 - f_{s_2} + f_{s_3})f_1 + (f_2^2 - f_3^2 - f_{s_3} + f_{s_4})f_2 + (f_3^2 - f_{s_4}^2 - f_{s_3}^2 + f_{s_4}^2)f_3} \]

\[ = f - \frac{(\theta_3 - \theta_2)s_1 + (\theta_1 - \theta_2)s_2 + (\theta_2 - \theta_1)s_3}{(s_2 - s_1)(s_{s_2} + (s_3 - s_1)s_{s_2} + (s_3 - s_2)s_{s_3} - (s_1 - s_2)(s_2 - s_3)(s_3 - s_1)^3).} \]

The critical observation is that as $\theta_2, \theta_3$ tend to $\theta_1$, the numerator of the second term becomes small, but the denominator can remain large by taking $s_1, s_2, s_3$ to be distinct solutions of (11)–(12). Thus, writing $\theta_1 = \theta, \theta_2 = \theta + \epsilon$, $\theta_3 = \theta + \epsilon \alpha$ and taking the limit $\epsilon \to 0$, we obtain the following generator for infinitesimal symmetries acting on $f$ via $f \to f + c\theta f$: \[ \frac{\partial}{\partial \theta} \].
\[ Q_i(\theta) = \frac{a(s^{(1)} - s^{(2)}) + b(s^{(3)} - s^{(1)})}{(s^{(2)} - s^{(1)}) \frac{\partial}{\partial s^1} + (s^{(3)} - s^{(1)}) \frac{\partial}{\partial s^2} + (s^{(1)} - s^{(2)}) \frac{\partial}{\partial s^3} - (s^{(1)} - s^{(2)}) (s^{(2)} - s^{(3)}) (s^{(3)} - s^{(1)})}. \]

(32)

Here \( s^{(1)}, s^{(2)}, s^{(3)} \) are distinct solutions of (11)–(12). The generator for the field \( h \) can be written down but is long and complicated. The generator for \( w \) (see section 2) takes a simpler form:

\[ Q_w(\theta) = \frac{aw^{(2)}(s^{(1)} - s^{(3)})}{(s^{(2)} - s^{(1)}) \frac{\partial}{\partial s^1} + (s^{(3)} - s^{(1)}) \frac{\partial}{\partial s^2} + (s^{(1)} - s^{(2)}) \frac{\partial}{\partial s^3} - (s^{(1)} - s^{(2)}) (s^{(2)} - s^{(3)}) (s^{(3)} - s^{(1)})}. \]

(33)

The generators for the fields \( u \) and \( v \) are \( x \)-derivatives of the generators for \( f \) and \( w \), respectively. In computing these derivatives, it is useful to notice that the quantity in the denominator \( Q_\phi(\theta) \) and \( Q_w(\theta) \) is the quantity \( \Lambda \) introduced above in the discussion of conservation laws, see (29), which satisfies \( \Lambda = -A(s^{(1)} + s^{(2)} + s^{(3)}) \). Using the asymptotic expansions for \( s^{(1)}, s^{(2)}, s^{(3)} \) obtained in the discussion of conservation laws we obtain the first few local symmetries:

\[ X_1 = \frac{\partial}{\partial w}, \]
\[ X_2 = \frac{\partial}{\partial f}, \]
\[ X_3 = u \frac{\partial}{\partial f} + v \frac{\partial}{\partial w}, \]
\[ X_5 = (-2v - u_x) \frac{\partial}{\partial f} + \left( v_x + \frac{2}{3} u_{xx} - u_t \right) \frac{\partial}{\partial w} \quad \left( = f_t \frac{\partial}{\partial f} + w_t \frac{\partial}{\partial w} \right), \]
\[ X_7 = 3(6uv_x + 12uv - u_{xxx} - 2v_{xx}) \frac{\partial}{\partial f} \]
\[ + \left( 12u^3 - 18u_{tx}u - 18u_{vx} - 9u_x^2 + 18v^2 + 2u_{xxxx} + 3v_{xxx} \right) \frac{\partial}{\partial w}, \]
\[ X_8 = (15u^3 - 15u_{tx}u + 45u_x v + 45v^2 + u_{xxxx}) \frac{\partial}{\partial f} \]
\[ + \left( 18u_x u + 45v u^2 - 6u_{xxx}u - 12u_{tx}u - 6u_{xx}u_x - 12u_{tx}v - 9u_x v_x - 18v v_x \right) \frac{\partial}{\partial w}. \]

Here we have taken, without loss of generality, \( a = 1, b = 0 \). The index \( i \) on the vector field \( X_i \) indicates that it is obtained from the coefficient of \( \theta^{-i/3} \) in the expansions of the generators. Note that the vector fields \( X_1, X_2, \ldots \) vanish, in analogy of the situation for conservation laws.

The local symmetries listed above are generated from the (32)–(33) by expansion in powers of \( \theta \). Using the identity (30) the full symmetry can be written (for the case \( a = 1, b = 0 \)) in the form

\[ X = (s^{(1)} - s^{(2)}) e^{\int f^{(i)} + f^{(2)} + f^{(3)} dt} \left( \frac{\partial}{\partial f} - s^{(3)} \frac{\partial}{\partial w} \right). \]

This is a symmetry provided \( s^{(1)}, s^{(2)}, s^{(3)} \) are solutions of (11)–(12). Taking, for example, \( s^{(3)} = s^{(1)} \) we obtain the nonlocal symmetry

\[ X = (s^{(1)} - s^{(2)}) e^{\int 2u^{(i)} + f^{(3)} dt} \left( \frac{\partial}{\partial f} - s^{(1)} \frac{\partial}{\partial w} \right). \]
There are 6 distinct versions of this symmetry arising from permutations of $s^{(1)}, s^{(2)}, s^{(3)}$. Nonlocal symmetries are useful as it is possible to construct invariant solutions with respect to nonlocal, as well as local, symmetries [28].

Returning to local symmetries, it is straightforward to verify, using just (11), that the symmetry generators

$$Q_f(\theta) = (s^{(1)} - s^{(2)}) e^{-s^{(3)} + s^{(2)} + s^{(1)} dx}, \quad Q_w(\theta) = -s^{(3)} Q_f(\theta)$$

satisfy the linear differential equations

$$\begin{pmatrix} -\frac{2}{3} D^3 + Du + uD \\ \frac{1}{2} D^4 - uD^2 + 2vD + Dv \end{pmatrix} \frac{2}{3} D^5 - \frac{2}{3} (uD^3 + D^3 u) + u^2 D + Du^2 - (v_1 D + D v_1) \begin{pmatrix} Q_w \\ Q_f \end{pmatrix} = \theta \begin{pmatrix} 0 & -D \\ D & 0 \end{pmatrix} \begin{pmatrix} Q_w \\ Q_f \end{pmatrix}. $$

(Here $D$ denotes differentiation with respect to $x$). Denoting the matrix differential operator on the LHS of this equation as $P_2$, and the one on the RHS as $P_1$, we have

$$P_1^{-1} P_2 \begin{pmatrix} Q_w \\ Q_f \end{pmatrix} = \theta \begin{pmatrix} Q_w \\ Q_f \end{pmatrix}$$

implying that the operator $P_1^{-1} P_2$ can be identified as the recursion operator [36] for the potential BEq. Since $Q_w(\theta) = Q_f(\theta)$, and $Q_w(\theta) = Q_w(\theta)$, the operator $P_2 P_1^{-1}$ can be identified as the recursion operator of the BEq. (Note that the recursion operator differs from the standard one for the BEq, as given, for example, in [37], as our form of the BEq (5)–(6) is slightly different.)

6. Conclusion

The theme of this paper has been how the Bäcklund transformation, and particularly its superposition principles, can give so much insight into the properties of the Boussinesq equation. Specifically, we have obtained two systems of lattice equations associated with the superposition principles, we have used the superposition principle to study the soliton solutions of the equation, which have a rich structure that has not yet been fully explored, and we have given a concise and complete account of the theory of conservation laws and symmetries of the equation, using a generating function for symmetries derived immediately from the superposition principle of 3 BTs.

The novelty in this work in the context of the theory of Bäcklund transformations, in comparison, say, to our recent work on the BT for the Camassa-Holm equation [39], is in the need to look at the superposition principle for 3 BTs. For the BEq, the superposition principle of 2 BTs is not purely algebraic, whereas for 3 BTs it is. We expect this structure to be shared by the many interesting equations associated with the Lie algebra $SL(3)$ (i.e. with $3 \times 3$ matrix Lax pairs).

A number of open questions have emerged in the course of this paper. In our work on lattice systems, we arrived at the system of equations (21)–(22) on a cube, and it would be interesting to have a characterization of the integrability of this system. In our work on soliton solutions, we have seen that although we have a formula for the general multisoliton solution, we still lack much in the physical understanding of these solutions. In particular, the question of whether there exists a nonsingular solution describing the merger of 4 solitons to 2 is open,
and there is much work to be done understanding the changes in speeds (and also phase shifts) between initial and final solitons in the merger solutions.

Another open direction is to understand the action of the BT and application of the superposition principles to rational solutions [1, 4, 12–14] and symmetry reductions of the BEq [8, 15, 27, 29], and in particular to investigate the possible reductions associated with the nonlocal symmetries given in section 5.

Acknowledgments

We thank Ralph Willox for references [26, 40] and Danny Hershkowitz for information about Sylvester’s theorem for determinants.

References

[1] Ablowitz M J and Satsuma J 1978 Solitons and rational solutions of nonlinear evolution equations J. Math. Phys. 19 2180–6
[2] Adler V E 1998 Bäcklund transformation for the Krichever–Novikov equation Int. Math. Res. Not. 1 1–4
[3] Adler V E, Bobenko A I and Suris Y B 2003 Classification of integrable equations on quad-graphs. The consistency approach Commun. Math. Phys. 233 513–43
[4] Ankiewicz A, Bassom A P, Clarkson P A and Dowie E 2017 Conservation laws and integral relations for the Boussinesq equation Studies Appl. Math. (at press) (https://doi.org/10.1111/sapm.12174)
[5] Atkinson J 2008 Bäcklund transformations for integrable lattice equations J. Phys. A: Math. Theor. 41 135202
[6] Atkinson J, Lobb S B and Nijhoff F W 2012 An integrable multicomponent quad-equation and its Lagrangian formulation Theor. Math. Phys. 173 1644–53
Atkinson J, Lobb S B and Nijhoff F W 2012 Teor. Mat. Fiz. 173 363–74 (Russian version appears)
[7] Bogdanov L V and Zakharov V E 2002 The Boussinesq equation revisited Physica D 165 137–62
[8] Boiti M and Pempinelli F 1980 Similarity solutions and Bäcklund transformations of the Boussinesq equation Nuovo Cimento B 56 148–56
[9] Boussinesq J 1871 Théorie de lintumescence liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire C. R. Acad. Sci. 72 755–9
[10] Boussinesq J 1872 Théorie des ondes et des remous qui se propagent le long d’un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond J. Math. Pures Appl. 17 55–108
[11] Chen H H 1976 Relation between Bäcklund transformations and inverse scattering problems Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications (Lecture Notes in Mathematics vol 515) (Berlin: Springer) pp 241–52 (Workshop Contact Transformations, Vanderbilt Univ., Nashville, Tenn., 1974)
[12] Clarkson P A 2008 Rational solutions of the Boussinesq equation Anal. Appl. 6 349–69 (Singapore)
[13] Clarkson P A 2009 Rational solutions of the classical Boussinesq system Nonlinear Anal. Real World Appl. 10 3360–71
[14] Clarkson P A and Dowie E 2016 Rational solutions of the Boussinesq equation and applications to rogue waves (arXiv:1609.00503)
[15] Clarkson P A and Kruskal M D 1989 New similarity reductions of the Boussinesq equation J. Math. Phys. 30 2201–13
[16] Gantmacher F R 1959 The Theory of Matrices vol 1, 2 (New York: Chelsea Publishing Co.) (translated by K A Hirsch)
[17] Hietarinta J 2011 Boussinesq-like multi-component lattice equations and multi-dimensional consistency J. Phys. A: Math. Theor. 44 165204
[18] Hietarinta J and Zhang D-J 2010 Multisoliton solutions to the lattice Boussinesq equation J. Math. Phys. 51 033505
[19] Hietarinta J and Zhang D-J 2011 Soliton taxonomy for a modification of the lattice Boussinesq equation SIGMA 7 14
[20] Hietarinta J and Zhang D-J 2013 Hirota’s method and the search for integrable partial difference equations. 1. Equations on a 3 × 3 stencil J. Differ. Equ. Appl. 19 1292–316
[21] Hirota R 1973 Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices J. Math. Phys. 14 810–4
[22] Hirota R 1981 Discrete analogue of a generalized Toda equation J. Phys. Soc. Japan 50 3785–91
[23] Hirota R and Satsuma J 1977 Nonlinear evolution equations generated from the Bäcklund transformation for the Boussinesq equation Prog. Theor. Phys. 57 797–807
[24] Huang X C 1982 A two-parameter Bäcklund transformation for the Boussinesq equation J. Phys. A: Math. Gen. 37 3609–28
[25] Kalantarov V K and Ladyženskaja O A 1977 Formation of collapses in quasilinear equations of parabolic and hyperbolic types Zapiski Nauchn. Seminarov Leningr. Otd. Mat. Inst. Steklov. 69 77–102, 274 (Boundary value problems of mathematical physics and related questions in the theory of functions, 10)
[26] Lambert E, Musette M and Kesteloot E 1987 Soliton resonances for the good Boussinesq equation J. Math. Phys. 38 597–621
[27] Levi D and Winternitz P 1989 Nonclassical symmetry reduction: example of the Boussinesq equation J. Phys. A: Math. Gen. 22 937–49
[28] Lou S, Hu X and Chen Y 2012 Nonlocal symmetries related to Boussinesq equation J. Math. Phys. 53 153503
[29] Lou S Y 1990 A note on the new similarity reductions of the Boussinesq equation J. Phys. A: Math. Gen. 23 2431–8
[30] Manoranjan V S, Mitchell A R and Morris J L 1984 Numerical solutions of the good Boussinesq equation SIAM J. Sci. Stat. Comput. 5 946–57
[31] Manoranjan V S, Ortega T and Sanz-Serna J M 1988 Soliton and antisoliton interactions in the ‘good’ Boussinesq equation J. Math. Phys. 29 1964–8
[32] Maruno K I and Kajiwara K 2010 The discrete potential Boussinesq equation and its multisoliton solutions J. Phys. Soc. Japan 79 593–609
[33] Nimmo J J C and Freeman N C 1983 A method of obtaining the N-soliton solution of the Boussinesq equation in terms of a Wronskian J. Nonlinear Math. Phys. 13 372–81
[34] Olver P J 1977 Evolution equations possessing infinitely many symmetries J. Math. Phys. 18 1212–5
[35] Olver P J 1986 Applications of Lie Groups to Differential Equations (Graduate Texts in Mathematics vol 107) (New York: Springer)
[36] Rasin A G and Schiff J 2013 The Gardner method for symmetries J. Phys. A: Math. Theor. 46 155202
[37] Rasin A G and Schiff J 2016 Bäcklund transformations for the Camassa–Holm equation J. Nonlinear Sci. 27 45–69
[38] Tajiri M and Nishitani T 1982 Two-soliton resonant interactions in one spatial dimension: solutions of Boussinesq type equation J. Phys. Soc. Japan 51 3720–3
[39] Tongas A and Nijhoff F 2005 The Boussinesq integrable system: compatible lattice and continuum structures Glasgow Math. J. 47 205–19
[40] Wadati M, Sanuki H and Konno K 1975 Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws Prog. Theor. Phys. 53 419–36
[41] Wang S, Tang X Y and Lou S Y 2004 Soliton fission and fusion: Burgers equation and Sharma–Tasso–Olver equation Chaos Solitons Fractals 21 231–9
[42] Weiss J 1985 The Painlevé property and Bäcklund transformations for the sequence of Boussinesq equations J. Math. Phys. 26 258–69
[43] Xenitidis P and Nijhoff F 2012a Lattice Schwarzian Boussinesq equation and two-component systems (arXiv: 1202.5767)
[46] Xenitidis P and Nijhoff F 2012b Symmetries and conservation laws of lattice Boussinesq equations
Phys. Lett. A 376 2394–401
[47] Zhang Y, Chang X, Hu J, Hu X and Tam H W 2015 Integrable discretization of soliton equations via
bilinear method and Bäcklund transformation Sci. China Math. 58 279–96
[48] Zhang Y and Chen D Y 2005 A modified Bäcklund transformation and multi-soliton solution for
the Boussinesq equation Chaos Solitons Fractals 23 175–81