How hard are verifiable delay functions?

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Abstract. Verifiable delay functions (VDF) are functions that take a specified number of sequential steps to be evaluated but can be verified efficiently. In this paper, we introduce a new complexity class that contains all the VDFs. We show that this new class VDF is a subclass of CLS (continuous local search) and RELAXED-SINK-OF-VERIFIABLE-LINE is a complete problem for the class VDF.

Keywords: Verifiable delay functions, Sequentiality, Turing machine, Space-time hierarchy

1 Introduction

In 1992, Dwork and Naor introduced the very first notion of VDF under a different nomenclature “pricing function” [9]. It is a computationally hard puzzle that needs to be solved to send a mail, whereas the solution of the puzzle can be verified efficiently. Later, the concept of verifiable delay functions was formalized in [5].

Given the security parameter $\lambda$ and delay parameter $T$, the prover needs to evaluate the VDF in time $T$. The verifier verifies the output in $\text{poly}(\lambda, \log T)$-time using some proofs produced by the prover. A crucial property of VDFs, namely sequentiality, ensures that the output can not be computed in time much less than $T$ even in the presence of $\text{poly}(\lambda, T)$-parallelism. VDFs have several applications ranging from non-interactive time-stamping to resource-efficient blockchains, however, are really rare in practice because of the criteria sequentiality. In order to design new VDFs we must find problems that offer sequentiality. To the best of our knowledge so far, all the practical VDFs are based on two inherently sequential algebraic problems – modular exponentiation in groups of unknown order [21,24] (fundamentally known as the time-lock puzzle [22]) and isogenies over super-singular curves [11]. The security proofs of these VDFs are essentially polynomial-time reductions from one of these assumptions to the corresponding VDFs. Thus, from the perspective of the designers, the first hurdle is to find inherently sequential problems.

The main motivation behind this study has been where should we search for such inherently sequential problems which are also efficiently verifiable? In this paper, we show that the class of all VDFs, namely VDF, is a subclass of the class CLS (continuous local search). In particular, we prove that RELAXED-SINK-OF-VERIFIABLE-LINE [6] is a complete problem for this new class VDF.
1.1 Proof Sketch

The key challenges with this aim are,

**VDF-hardness** reducing any arbitrary VDF into a hard distribution of rSVL (abbreviation for RELAXED-SINK-OF-VERIFIABLE LINE) instances.

**VDF-membership** producing VDFs from a family of subexponentially-hard rSVL instances.

First, we define VDF as a language that helps us, in turn, to define the class VDF as a special case of interactive proofs. The main idea is to detach the Fiat–Shamir transformation from the traditional definition of VDF in order to find its hardness irrespective of any random oracle.

For the first task, Choudhuri et al. suggest a method in [7] for the VDFs that need proof. We show that it is possible even with the VDFs that have no proofs. The trick is to define the Eval function of a VDF in terms of an iterated sequential function $f$. Use this $f$ in order to design the successor circuit $S$ of rSVL instances. As a much easier case, we also show that the permutation VDFs can be reduced to such hard rSVL instances even without using any $f$.

We accomplish the second task by deriving a permutation VDF from a subexponentially hard family of rSVL instances. The derived VDF is proven to be secure. Although it suffices for the membership in the class VDF, we also deduce a generic VDF from the same rSVL instances but using an one-way injective function.

2 Related Work

In this section, first we describe some well-known schemes that are qualified as VDFs.

**Squaring over $\mathbb{Z}_p$** The pricing function by Dwork–Naor scheme [9] asks a prover, given a prime $p = 3 \pmod{4}$ and a quadratic residue $x$ modulo $p$, to find a $y$ such that $y^2 = x \pmod{p}$. The prover has no other choice other than using the identity $y = x^{(p+1)/4} \pmod{p}$, but the verifier verifies the correctness using $y^2 = x \pmod{p}$. The drawback of this design is that the delay parameter $T = \mathcal{O}(\log p)$. Thus the difference between the evaluation and the verification may be made up by a prover with $\text{poly}(T)$-processors by parallelizing the field multiplications. Moreover, it is difficult to generate the public parameters of this VDF for sufficiently large $T$ as $\text{Setup}$ needs to sample a prime $p > 2^{\mathcal{O}(T)}$.

**Injective Rational Maps** In 2018, Dan et al. [5] propose a VDF based on injective rational maps of degree $T$, where the fastest possible inversion is to compute the polynomial GCD of degree-$T$ polynomials. They conjecture that it achieves $(T^2, o(T))$ sequentiality using permutation polynomials as the candidate map. However, it is a weak VDF as it needs $\mathcal{O}(T)$ processors to evaluate the output in time $T$. 
RSW Puzzle: Rivest, Shamir, and Wagner [22] introduced the time-lock puzzle stating that it needs at least \( T \) number of sequential squaring to compute \( y = g^{2^T} \mod \Delta \) when the factorization of \( \Delta \) is unknown. Therefore they proposed this encryption that can be decrypted only sequentially. Starting with \( \Delta = pq \) such that \( p, q \) are large primes, the key \( y \) is enumerated as \( y = g^{2^T} \mod \Delta \). Then the verifier, uses the value of \( \phi(\Delta) \) to reduce the exponent to \( e = 2^T \mod \phi(\Delta) \) and finds out \( y = g^e \mod \Delta \). On the contrary, without the knowledge of \( \phi(\Delta) \), the only option available to the prover is to raise \( g \) to the power \( 2^T \) sequentially. As the verification stands upon a secret, the knowledge of \( \phi(\Delta) \), it is not a VDF as verification should depend only on public parameters.

Pietrzak [21] and Wesolowski [24] circumvent this issue independently. We describe both the VDFs in the generic group \( G \) as these schemes can be instantiated over two different groups – the RSA group \( (\mathbb{Z}/\Delta\mathbb{Z})^\times \) and the class group of imaginary quadratic number field. Both the protocols use a common random oracle \( H_G : \{0,1\}^* \rightarrow G \) to map the input statement \( x \) to the generic group \( G \). We assume \( g := H_G(x) \).

Pietrzak’s VDF: It exploits the identity \( z^r \cdot y = (g^r z)^{2^{T/2}} \) where \( y = g^{2^T} \), \( z = g^{2^{T/2}} \) and \( r \in \mathbb{Z}_{2^\lambda} \) is chosen at random. So the prover is asked to compute the output \( y = g^{2^T} \) and the proof \( \pi = \{ u_1, u_2, \ldots, u_{\log T} \} \) such that \( u_{i+1} = u_i^{r_i+2^{T/2}}, \quad r_i = H(u_i, T/2^{i-1}, u_i, u_i^{2^{T/2i-1}}) \) and \( v_i = u_i^{r_i 2^{T/2i} + 2^{T/2}} \). The verifier computes the \( v_i = u_i^{r_i 2^{T/2i} + 2^{T/2}} \) and checks if \( v_{\log T} = u_{\log T}^2 \). So the verifier performs \( \sum_{i=1}^{\log T} \log r_i \) number of sequential squaring. As \( H \) samples \( r_i \) uniformly from its range \( \mathbb{Z}_{2^\lambda} \), we have \( \sum_{i=1}^{\log T} \log r_i = \mathcal{O}(\lambda \log T) \). The effort to generate the proof \( \pi \) is in \( \mathcal{O}(\sqrt{T \log T}) \).

Wesolowski’s VDF: It asks the prover to compute an output \( y = g^{2^T} \) and a proof \( \pi = g^{2^T/\ell} \), where \( \ell = H_{\text{prime}}(\text{bin}(g)||\text{bin}(y)) \) is a 2\( \lambda \)-bit prime. It needs \( \mathcal{O}(T/\log T) \) time to do the same. The verifier checks if \( y = \pi^\ell \cdot g^{2^T \mod \ell} \). Hence the verification needs at most \( 2 \log \ell = 4 \lambda \) squaring.

Isogenies over Super-Singular Curves: Feo et al. [11] presents two VDFs based on isogenies over super-singular elliptic curves. They start with five groups \( \langle G_1, G_2, G_3, G_4, G_5 \rangle \) of prime order \( T \) with two non-degenerate bilinear pairing maps \( e_{12} : G_1 \times G_2 \rightarrow G_5 \) and \( e_{34} : G_3 \times G_4 \rightarrow G_5 \). Also there are two group isomorphisms \( \phi : G_1 \rightarrow G_3 \) and \( \phi : G_4 \rightarrow G_2 \). Given all the above descriptions as the public parameters along with a generator \( P \in G_1 \), the prover needs to find \( \overline{Q} \), where \( Q \in G_4 \), using \( T \) sequential steps. The verifier checks if \( e_{12}(P, \overline{Q}) = e_{34}(\phi(P), Q) \) in \( \text{poly}(\log T) \) time. It runs on super-singular curves over \( \mathbb{F}_p \) and \( \mathbb{F}_{p^2} \) as two candidate groups. While being inherently non-interactive, there are two drawbacks as mentioned by the authors themselves. First, it requires a trusted setup, and second, the setup phase may turn out to be slower than the evaluation.

Mahmoody et al. recently rule out the possibility of having a VDF out of random oracles only [18].
Table 1. Comparison among the existing VDFs. $T$ is the targeted time bound, $\lambda$ is the security parameter, $\Delta$ is the number of processors. All the quantities may be subjected to $O$-notation, if needed.

| VDF (by authors) | Eval Sequential | Eval Parallel | Verify | Setup | Proof size |
|------------------|-----------------|--------------|--------|-------|------------|
| Dwork and Naor [9] | $T$ | $T^{2/3}$ | $T^{2/3}$ | $T$ | $-$ |
| Dan et al. [5] | $T^2$ | $T > T - o(T)$ | $\log T$ | $\log T$ | $-$ |
| Wesolowski [24] | $(1 + \frac{1}{\log T})T$ | $(1 + \frac{1}{\Delta \log T})T$ | $\lambda^4$ | $\lambda^4$ | $\lambda^4$ |
| Pietrzak [21] | $(1 + \frac{2}{\log T})T$ | $(1 + \frac{2}{\sqrt{T}})T$ | $\log T$ | $\lambda^3$ | $\log T$ |
| Feo et al. [11] | $T$ | $T$ | $\lambda^3 T \log \lambda$ | $-$ | $-$ |

PPAD-hardness of Cryptographic Protocols

Now, we briefly mention few works in order to show the significance of the class PPAD in the context of cryptography.

Abbot, Kane and Valiant were the first to show that virtual black-box obfuscation [1] can be used to generate hard instances of End-of-Line (EOL). Since, virtual black-box obfuscation is known only for certain functions, Bitansky et al. consider indistinguishability obfuscation (iO) to show that quasi-polynomially hard iO and subexponentially hard one-way function reduce to EOL via a new problem Sink-of-Verifiable-Line [4]. Building up further, Garg et al. derived PPAD-hardness from polynomially-hard iO or compact public-key functional encryption and one-way permutation [13]. Relying on sub-exponentially hard injective one-way functions, Komargodski and Segev show that quasi-polynomially hard private-key functional encryption implies PPAD-hardness.

Hubáček and Yogev introduced a new total search problem End-of-Metered-Line (EOML) proving that it is hard to search local optima even over continuous domains. They also show that EOML belongs to a subclass, namely continuous local search (CLS), of PPAD.

A striking result by Choudhuri et al. shows that relative to a random oracle (used in Fiat–Shamir transformation), hardness in the class $\#P$ implies hardness in CLS [6]. In particular, they derive a new verifiable procedure applying the Fiat–Shamir transformation on the sumcheck protocol for $\#SAT$ and reduce it to a new problem Relaxed-Sink-of-Verifiable-Line (rSVL) in CLS.

A similar work by the same group of authors inspires our work [7]. It reduces the problem of finding $g^x \mod N$ relative to a random oracle (used in Fiat–Shamir transformation) to RSVL. A typical property in Pietrzak’s VDF [21] called ”proof-merging” is at the core of this reduction. Suppose, $\pi_{g \rightarrow y}^T = \{u_1, \ldots, u_{\log T}\}$ denotes the proof for $h = g^x$ in Pietrzak’s VDF. The property proof-merging is the observation that given two proof $\pi_{g \rightarrow h}^T$ and $\pi_{h \rightarrow y}^T$, finding the proof $\pi_{g \rightarrow y}^T$ reduces to finding a proof $\pi_{u \rightarrow v}^T$ such that, $u := g^r \cdot h$, $v := h^r \cdot y$ and $r := H(u_1, g, y, 2T)$. It works because the element $h$ must be present in the proof $\pi_{g \rightarrow y}^T = \{u_1, \ldots, u_{\log T+1}\}$ as $u_1$. Therefore, the merged proof $\pi_{u \rightarrow v}^T = \{u_1', \ldots, u_{\log T}'\}$ is equivalent to the proof $\pi_{g \rightarrow y}^T = \{u_1, u_1', \ldots, u_{\log T}\}$.
3 Preliminaries

We start with the notations.

3.1 Notations

We denote the security parameter with $\lambda \in \mathbb{Z}^+$. The term $\text{poly}(\lambda)$ refers to some polynomial of $\lambda$, and $\text{negl}(\lambda)$ represents some function $\lambda^{-\omega(1)}$. If any randomized algorithm $A$ outputs $y$ on an input $x$, we write $y \leftarrow A(x)$. By $x \leftarrow \mathcal{X}$, we mean that $x$ is sampled uniformly at random from $\mathcal{X}$. For a string $x$, $|x|$ denotes the bit-length of $x$, whereas for any set $\mathcal{X}$, $|\mathcal{X}|$ denotes the cardinality of the set $\mathcal{X}$. If $x$ is a string then $x[i \ldots j]$ denotes the substring starting from the literal $x[i]$ ending at the literal $x[j]$. We consider an algorithm $A$ as efficient if it runs in probabilistic polynomial time (PPT).

3.2 Verifiable Delay Function

We borrow this formalization from [5].

Definition 1. (Verifiable Delay Function). A verifiable delay function from domain $\mathcal{X}$ to range $\mathcal{Y}$ is a tuple of algorithms $(\text{Setup}, \text{Eval}, \text{Verify})$ defined as follows,

- $\text{Setup}(1^\lambda, T) \rightarrow pp$ is a randomized algorithm that takes as input a security parameter $\lambda$ and a targeted time bound $T$, and produces the public parameters $pp$. We require $\text{Setup}$ to run in $\text{poly}(\lambda, \log T)$ time.
- $\text{Eval}(pp, x) \rightarrow (y, \pi)$ takes an input $x \in \mathcal{X}$, and produces an output $y \in \mathcal{Y}$ and a (possibly empty) proof $\pi$. $\text{Eval}$ may use random bits to generate the proof $\pi$. For all $pp$ generated by $\text{Setup}(\lambda, T)$ and all $x \in \mathcal{X}$, the algorithm $\text{Eval}(pp, x)$ must run in time $T$.
- $\text{Verify}(pp, x, y, \pi) \rightarrow \{0, 1\}$ is a deterministic algorithm that takes an input $x \in \mathcal{X}$, an output $y \in \mathcal{Y}$, and a proof $\pi$ (if any), and either accepts (1) or rejects (0). The algorithm must run in $\text{poly}(\lambda, \log T)$ time.

Before we proceed to the security of VDFs we need the precise model of parallel adversaries [5].

Definition 2. (Parallel Adversary) A parallel adversary $A = (A_0, A_1)$ is a pair of non-uniform randomized algorithms $A_0$ with total running time $\text{poly}(\lambda, T)$, and $A_1$ which runs in parallel time $\sigma(T) < T - o(T)$ on at most $\text{poly}(\lambda, T)$ number of processors.

Here, $A_0$ is a preprocessing algorithm that precomputes some state based only on the public parameters, and $A_1$ exploits this additional knowledge to solve in parallel running time $\sigma$ on $\text{poly}(\lambda, T)$ processors.

The three desirable properties of a VDF are now introduced.
Definition 3. (Correctness) A VDF is correct with some error probability \( \varepsilon \), if for all \( \lambda, T \), parameters \( pp \), and \( x \in X \), we have

\[
\Pr \left[ \text{Verify}(pp, x, y, \pi) = 1 \mid pp \leftarrow \text{Setup}(1^{\lambda}, T), x \leftarrow X, (y, \pi) \leftarrow \text{Eval}(pp, x) \right] \geq 1 - \text{negl}(\lambda).
\]

Definition 4. (Soundness) A VDF is computationally sound if for all non-uniform algorithms \( A \) that run in time \( \text{poly}(T, \lambda) \), we have

\[
\Pr \left[ y \neq \text{Eval}(pp, x) \mid \text{Verify}(pp, x, y, \pi) = 1 \right] \leq \text{negl}(\lambda).
\]

Further, a VDF is called statistically sound when all adversaries (even computationally unbounded) have at most \( \text{negl}(\lambda) \) advantage. Even further, it is called perfectly sound if we want this probability to be 0 against all adversaries. Hence, perfect soundness implies statistical soundness which implies computational soundness but not the reverse.

Definition 5. (Sequentiality) A VDF is \((\Delta, \sigma)-\text{sequential}\) if there exists no pair of randomized algorithms \( A_0 \) with total running time \( \text{poly}(T, \lambda) \) and \( A_1 \) which runs in parallel time \( \sigma \) on at most \( \Delta \) processors, such that

\[
\Pr \left[ y = \text{Eval}(pp, x) \mid pp \leftarrow \text{Setup}(1^{\lambda}, T), \text{state} \leftarrow A_0(1^{\lambda}, T, pp), x \leftarrow X \right] \leq \text{negl}(\lambda).
\]

Definition 6. (Permutation VDF). Permutation VDFs are the VDFs with \( X = Y \) where \( X \) and \( Y \) denote the input and output domains respectively.

We reiterate an important remark from [5] but as a lemma.

Lemma 1. \((T \in \text{SUBEXP}(\lambda)). \text{ If } T > 2^{o(\lambda)} \text{ then there exists an adversary that breaks the sequentiality of the VDF with non-negligible advantage.}\)

Proof. \( A \) observes that the algorithm \( \text{Verify} \) is efficient. So given a statement \( x \in X \), \( A \) chooses an arbitrary \( y \in Y \) as the output without running \( \text{Eval}(x, pp, T) \). Now, \( A \) finds the proof \( \pi \) by a brute-force search in the entire solution space with its \( \text{poly}(T) \) number of processors. In each of its processors, \( A \) checks if \( \text{Verify}(x, pp, T, y, \pi_i) = 1 \) with different \( \pi_i \). The advantage of \( A \) is \( \text{poly}(T)/2^{O(\lambda)} \geq \text{negl}(\lambda) \) as \( T > 2^{o(\lambda)} \).

So we need \( T \leq 2^{o(\lambda)} \) to restrict the advantage of \( A \) upto \( 2^{o(\lambda)}/2^{O(\lambda)} = 2^{-\Omega(\lambda)} \).
3.3 Search Problems

In this section, we review the basics of search problems from [6, 20, 19].

Suppose, \( R \subseteq \{0, 1\}^* \times \{0, 1\}^* \) is a relation such that, for all \((x, y) \in R\),

i) \( R \) is polynomially-balanced i.e., \(|y| \leq \text{poly}(|x|)\).

ii) \( R \) is efficiently-recognizable.

A search problem \((\mathcal{L}, R)\) is defined by a set of instances \( \mathcal{L} \subseteq \{0, 1\}^* \) and a relation \( R \). In particular, given a \( x \in \mathcal{L} \), the search problem \((\mathcal{L}, R)\) is to find an \( y \) if there exists an \((x, y) \in R\), otherwise say "no". The set of all search problems is called as functional-\( \text{NP} \) or \( \text{FNP} \). For example, the search version of \( 3\text{-SAT} \) belongs to \( \text{FNP} \).

The relation \( R \) is called total if, for every \( x \), there is always a \( y \) such that \((x, y) \in R\). A search problem \((\mathcal{L}, R)\) is called total when \( R \) is total. It means total search problems always have solutions e.g., FACTORING. The set of all total search problems is called as total-\( \text{FNP} \) or \( \text{TFNP} \).

A notable subclass of \( \text{TFNP} \) is called as \( \text{PPAD} \) which stands for "polynomial parity argument in a directed graph". It is defined as the set of all problems that are polynomial-time reducible in \( \text{End-of-Line} \) problem \([20]\).

**Definition 7. (End-of-Line problem EOL).** An \( \text{End-of-Line} \) instance \((S, P)\) consists of a pair of circuits \( S, P : \{0, 1\}^n \rightarrow \{0, 1\}^n \) such that \( P(0^n) = 0^n \) and \( S(0^n) \neq 0^n \). The goal is to find a vertex \( v \in \{0, 1\}^n \) such that \( P(S(v)) \neq v \) or \( S(P(v)) \neq v \neq 0^n \).

Thus, EOL deals with a directed graph over the vertices \( \{0, 1\}^n \) and the edges of the form \((u, v)\) if and only if \( S(u) = v \) and \( P(v) = u \). Here, \( S \) and \( P \) represent the successor and predecessor functions for this directed graph. The in-degree and out-degree of every vertex in this graph is at most one except that in-degree of \( 0^n \) is 0, the goal is to find a vertex \( v \), other than \( 0^n \), which is either a source (in-degree is 0) or a sink (out-degree is 0). Such a vertex always exists by the parity argument for a graph - the number of odd degree vertex in a graph is even. Therefore, EOL is in \( \text{TFNP} \) and by definition in \( \text{PPAD} \).

A subclass of \( \text{PPAD} \), namely continuous local search \( \text{CLS} \) is the class of problems that are polynomial-time reducible to the problem \( \text{Continuous-Local-Optimum} \) \([8]\). An interesting but not known to be complete problem in \( \text{CLS} \) is \( \text{End-of-Metered-Line} \) \([17]\).

**Definition 8. (End-of-Metered-Line problem EOML).** An \( \text{End-of-Metered-Line} \) instance \((S, P, M)\) consists of circuits \( S, P : \{0, 1\}^n \rightarrow \{0, 1\}^n \) and \( M : \{0, 1\}^n \rightarrow \{0, \ldots, 2^n - 1\} \) such that \( P(0^n) = 0^n \neq S(0^n) \) and \( M(0^n) = 1 \). The goal is to find a vertex \( v \in \{0, 1\}^n \) satisfying one of the following,

i) \text{End of Line:} \, \text{either} \, P(S(v)) \neq v \text{ or } S(P(v)) \neq v \neq 0^n .

ii) \text{False source:} \, v \neq 0^n \text{ and } M(v) = 1 .

iii) \text{Miscount:} \, \text{either} \, M(v) > 0 \text{ and } M(S(v)) = M(v) \neq 1 \text{ or } M(v) > 1 \text{ and } M(v) - M(P(v)) \neq 1 .
Clearly, EOML reduces to EOL, but the "odometer" circuit $M$ makes EOML easier than EOL. The circuit $M$ outputs the number of steps required to reach $v$ from the source. Observe that any vertex, for which $M$ contradicts its correct behaviour, solves the problem. Thus, there exists a solution for every EOML instance. Hence, EOML $\in$ TFNP.

The problem Sink-of-Verifiable-Line, introduced by Valiant et. al and further, developed in [4], is defined as follows,

**Definition 9. (Sink-of-Verifiable-Line problem SVL).** An Sink-of-Verifiable-Line instance $(S, V, T, v_0)$ consists of $T \in \{1, \ldots, 2^n\}$, $v_0 \in \{0, 1\}^n$, and two circuits $S : \{0, 1\}^n \rightarrow \{0, 1\}^n$ and $V : \{0, 1\}^n \times \{1, \ldots, T\} \rightarrow \{0, 1\}$ with the guarantee that for every $v \in \{0, 1\}^n$ and $i \in \{1, \ldots, T\}$, it holds that $V(v, i) = 1$ if and only if $v = S^i(v_0)$. The goal is to find a vertex $v \in \{0, 1\}^n$ such that $V(v, T) = 1$ (i.e., the sink).

Similar to, EOL and EOML the circuit $S$ implements a successor function for the directed graph. However, in SVL the graph is a single line with the source $v_0$. The circuit $V$ allows to test if a vertex $v$ is at a distance of $i$ on the line from $v_0$. The goal is to find the vertex at distance $T$ from $v_0$. Given an arbitrary instance $(S, V, T, v_0)$, we do not know how to efficiently check if $V$ behaves correctly. Therefore, every instance of SVL may not be valid and may not have solutions. So SVL $\notin$ TFNP.

Although, in [4], SVL was defined with the fixed source $v_0 = 0^n$, it is equivalent to SVL with arbitrary source. The instance $(S, V, T, 0^n)$ reduces to the instance $(S, V, T, v_0)$ considering $v_0 = 0^n$. On the other hand, the instance $(S, V, T, v_0)$ reduces to $(S', V', T, 0^n)$ when we define $S'(v) := S(v \oplus v_0)$ and $V'(v, i) := V(v \oplus v_0, i)$. Most importantly, this reduction works for any search problem in the context of TFNP that considers fixed vertex in its input (like EOL with $0^n$).

In order to solve the issue SVL $\notin$ TFNP, Choudhuri et al. introduced a relaxed version of it, namely Relaxed-Sink-of-Verifiable-Line. In this version, the circuit $V$ allows a few vertices off the main line starting at $v_0$. So, these off-the-line vertices also act as solutions. Hence,

**Definition 10. (Relaxed-Sink-of-Verifiable-Line problem rSVL).** An Relaxed-Sink-of-Verifiable-Line instance $(S, V, T, v_0)$ consists of $T \in \{1, \ldots, 2^n\}$, $v_0 \in \{0, 1\}^n$, and two circuits $S : \{0, 1\}^n \rightarrow \{0, 1\}^n$ and $V : \{0, 1\}^n \times \{1, \ldots, T\} \rightarrow \{0, 1\}$ with the guarantee that for every $v \in \{0, 1\}^n$ and $i \in \{1, \ldots, T\}$, it holds that $V(v, i) = 1$ if and only if $v = S^i(v_0)$. The goal is to find:

1. The sink: a vertex $v \in \{0, 1\}^n$ such that $V(v, T) = 1$
2. False positive: a pair $(v, i) \in \{0, 1\}^n \times \{1, \ldots, 2^n\}$ such that $v \neq S^i(v_0)$ and $V(v, i) = 1$.

Lemma 10 in cf. [6] shows that Relaxed-Sink-of-Verifiable-Line is many-one reducible to End-of-Metered-Line. Since, EOML $\in$ CLS, thus rSVL $\in$ CLS $\subset$ PPAD. The off the line vertices guarantee the existence of the solutions for any arbitrary instances.
3.4 Interactive Proof System

Goldwasser et al. were the first to show that the interactions between the prover and randomized verifier recognizes class of languages larger than $\text{NP}$ [15]. They named the class as $\text{IP}$ and the model of interactions as the interactive proof system. Babai and Moran introduced the same notion of interactions in the name of Arthur-Merlin games however with a restriction on the verifiers' side [2]. Later, Goldwasser and Sipser proved that both the models are equivalent [16]. Two important works in this context that motivate our present study are by the Shamir showing that $\text{IP} = \text{PSPACE}$ [23] and by the Goldwasser et al. proving that $\text{PSPACE} = \text{ZK}$, the set of all zero-knowledge protocols. We summarize the interactive proof system from [3].

An interactive proof system $(\mathcal{P} \leftrightarrow \mathcal{V})$ consists of a pair of Turing machines $(\mathcal{T} \mathcal{M})$, $\mathcal{P}$ and $\mathcal{V}$, with common alphabet $\Sigma = \{0, 1\}$. $\mathcal{P}$ and $\mathcal{V}$ each have distinguished initial and quiescent states. $\mathcal{V}$ has distinguished halting states out of which there is no transitions. $\mathcal{P}$ and $\mathcal{V}$ operates on various one-way infinite tapes,

i. $\mathcal{P}$ and $\mathcal{V}$ have a common read-only input tape.
ii. $\mathcal{P}$ and $\mathcal{V}$ each have a private random tape and a private work tape.
iii. $\mathcal{P}$ and $\mathcal{V}$ have a common communication tape.
iv. $\mathcal{V}$ is polynomially time-bounded. This means $\mathcal{V}$ halts on input $x$ in time $\text{poly}(|x|)$. $\mathcal{V}$ is in quiescent state when $\mathcal{P}$ is running.
v. $\mathcal{P}$ is computationally unbounded but runs in finite time. This means $\mathcal{P}$ may compute any arbitrary function $\{0, 1\}^* \rightarrow \{0, 1\}^*$ on input $x$ in time $f(|x|)$.

Feldman proved that “the optimum prover lives in $\text{PSPACE}$” [4].

vi. The length of the messages written by $\mathcal{P}$ into the common communication tape is bounded by $\text{poly}(|x|)$. Since $\mathcal{V}$ runs in $\text{poly}(|x|)$ time, it can not write messages longer than $\text{poly}(|x|)$.

Execution begins with $\mathcal{P}$ in its quiescent state and $\mathcal{V}$ in its start state. $\mathcal{V}$’s entering its quiescent state arouses $\mathcal{P}$, causing it to transition to its start state. Likewise, $\mathcal{P}$’s entering its quiescent state causes $\mathcal{V}$ to transition to its start state. Execution terminate when $\mathcal{V}$ enters in its halting states. Thus $(\mathcal{P} \leftrightarrow \mathcal{V})(x) = 1$ denotes $\mathcal{V}$ accepts $x$ and $(\mathcal{P} \leftrightarrow \mathcal{V})(x) = 0$ denotes $\mathcal{V}$ rejects $x$.

**Definition 11. (Interactive Proof System)** $(\mathcal{P} \leftrightarrow \mathcal{V})$. $(\mathcal{P} \leftrightarrow \mathcal{V})$ is an interactive proof system for the language $\mathcal{L} \subseteq \{0, 1\}^*$ if

(Correctness). $(x \in \mathcal{L}) \implies \Pr[(\mathcal{P} \leftrightarrow \mathcal{V})(x) = 1] \geq 1 - \text{negl}(|x|)$.

(Soundness). $(x \notin \mathcal{L}) \implies \forall \mathcal{P}', \Pr[(\mathcal{P}' \leftrightarrow \mathcal{V})(x) = 1] < \text{negl}(|x|)$.

The class of polynomial-time interactive proofs $\text{IP}$ is defined as the class of the languages that have $(\mathcal{P} \leftrightarrow \mathcal{V})$ such that $\mathcal{P}$ and $\mathcal{V}$ talk for at most $\text{poly}(n)$-rounds. Thus

$$\text{IP} = \{ \mathcal{L} \mid \mathcal{L} \text{ has a poly}(n)\text{-round } (\mathcal{P} \leftrightarrow \mathcal{V}) \}.$$  

Alternatively and more specifically,

\footnote{We could not find a valid citation.}
Definition 12. (The Class IP).

$$\text{IP} = \bigcup_{k \in \text{poly}(n)} \text{IP}[k].$$

For every $k$, $\text{IP}[k]$ is the set of languages $L$ such that there exist a probabilistic polynomial time TM $V$ that can have a $k$-round interaction with a prover $P : \{0,1\}^* \rightarrow \{0,1\}^*$ having these two following properties

(Correctness). $(x \in L) \implies \Pr[(P \leftrightarrow V)(x)] = 1 \geq 1 - \text{negl}(|x|)$.

(Soundness). $(x \notin L) \implies \forall P', \Pr[(P' \leftrightarrow V)(x)] = 1 < \text{negl}(|x|)$.

4 Fiat–Shamir Transformation

Any interactive protocol $(P \leftrightarrow V)$ can be transformed into a non-interactive protocol if the messages from the verifier $V$ are replaced with the response of a random oracle $H$. This is known as Fiat–Shamir transformation (FS) [12]. In particular, the $i$-th message from $V$ is computed as $y_i := H(x, x_1, y_1, \ldots, x_i, y_{i-1})$ where $x_i$ denotes the $i$-th response of $P$. When $H$ is specified in the public parameters of a $k$-round protocol, the transcript $x, x_1, y_1, \ldots, x_k, y_{k-1}$ can be verified publicly. Thus, relative to a random oracle $H$, a $k$-round interactive proof protocol $(P \leftrightarrow V)$ can be transformed into a two-round non-interactive argument $(P_{FS} \leftrightarrow V_{FS})$ where $P_{FS}$ sends the entire transcript $x, x_1, y_1, \ldots, x_k, y_k$ to $V_{FS}$ in a single round. Under the assumption that $H$ is one-way and collision-resistant, $V_{FS}$ accepts $x \in L$ in the next round if and only if $V$ accepts. Here we summarize two claims on Fiat–Shamir transformation stated in [10].

Lemma 2. If there exists an adversary $A$ who breaks the soundness of the non-interactive protocol $(P_{FS} \leftrightarrow V_{FS})$ with the probability $p$ using $q$ queries to a random oracle then there exists another adversary $A'$ who breaks the soundness of the $k$-round interactive protocol $(P \leftrightarrow V)$ with the probability $p/q^k$.

Proof. See [14] for details.

Lemma 3. Against all non-uniform probabilistic polynomial-time adversaries, if a $k$-round interactive protocol $(P \leftrightarrow V)$ achieves $\text{negl}(|x|^k)$-soundness then the non-interactive protocol $(P_{FS} \leftrightarrow V_{FS})$ has $\text{negl}(|x|)$-soundness.

Proof. Since, all the adversaries run in probabilistic polynomial time, the number of queries $q$ to the random oracle must be upper-bounded by $\text{poly}(|x|)$. Putting $q = |x|^c$ for any $c \in \mathbb{Z}^+$ in lemma 2 it follows the claim.

5 VDF Characterization

In this section, we investigate the possibility to model VDFs as a language in order to define its hardness. It seems that there are two hurdles,
VDF ⊆ CLS

Eliminating Fiat–Shamir The prover \( P \) in Def. 16 generates the proof \( \pi := f(x, y, T, H(x, y, T)) \) using Fiat–Shamir transformation where \( y := \text{Eval}(x, pp, T) \). Unless Fiat–Shamir is eliminated from VDF, its hardness remains relative to the random oracle \( H \). Sect. 5.1 resolves this issue.

Modelling Parallel Adversary How to model the parallel adversary \( A \) (Def. 2) in terms of computational complexity theory? We model \( A \) as a special variant of Turing machines described in Def. 14.

We address the first issue now.

5.1 Interactive VDFs

We introduce the interactive VDFs in order to eliminate the Fiat–Shamir. In the interactive version of a VDF, the \( V \) replaces the randomness of Fiat–Shamir heuristic. In particular, a non-interactive VDF with the Fiat–Shamir transcript \( \langle x, x_1, y_1, \ldots, x_k, y_k \rangle \) can be translated into an equivalent \( k \)-round interactive VDF allowing \( V \) to choose \( y_i \)s in each round.

Definition 13. (Interactive Verifiable Delay Function). An interactive verifiable delay function is a tuple \( \text{Setup}, \text{Eval}, \text{Open}, \text{Verify} \) that implements a function \( X \rightarrow Y \) as follows,

- \textbf{Setup}(1^\lambda, T) → pp is a randomized algorithm that takes as input a security parameter \( \lambda \) and a delay parameter \( T \), and produces the public parameters \( pp \) in \( \text{poly}(\lambda, \log T) \) time.
- \textbf{Eval}(pp, x) → y takes an input \( x \in X \), and produces an output \( y \in Y \). For all \( pp \) generated by \textbf{Setup}(\lambda, T) and all \( x \in X \), the algorithm \textbf{Eval}(pp, x) must run in time \( T \).
- \textbf{Open}(x, y, pp, T, t) → \pi takes the challenge \( t \) chosen by \( V \) and recursively computes a proof \( \pi \) in \( k \in \text{poly}(\lambda, \log T) \)-rounds of interaction with \( V \). In general, for some \( k \in \text{poly}(\lambda, \log T) \), \( \pi = \{\pi_1, \ldots, \pi_k\} \) can be computed as \( \pi_{i+1} := \text{Open}(x_i, y_i, pp, T, t_i) \) where \( x_i \) and \( y_i \) depend on \( \pi_i \). Observing \( (x_i, y_i, \pi_i) \) in the \( i \)-th round, \( V \) chooses the challenge \( t_i \) for the \( (i+1) \)-th round. Hence, \( V \) can efficiently collect all the proofs \( \pi = \{\pi_1, \ldots, \pi_k\} \) in \( \text{poly}(\lambda, \log T) \)-rounds of interactions. \textbf{Open} does not exist for the VDFs that need no proof (e.g., [11]).
- \textbf{Verify}(pp, x, y, \pi) → \{0, 1\} is a deterministic algorithm that takes an input \( x \in X \), an output \( y \in Y \), and the proof vector \( \pi \) (if any), and either accepts (1) or rejects (0). The algorithm must run in \( \text{poly}(\lambda, \log T) \) time.

All the three security properties remain same for the interactive VDF. Sequentiality is preserved by the fact that \textbf{Open} runs after the computation of \( y := \text{Eval}(x, pp, T) \). For soundness, we rely on lemma. 8. The correctness of interactive VDFs implies the correctness of the non-interactive version as the randomness that determines the proof is not in the control of \( P \). Therefore, an honest prover always convinces \( V \).
Although the interactive VDFs do not make much sense as publicly verifiable proofs in decentralized distributed networks, it allow us to analyze its hardness irrespective of any random oracle.

In order to model parallel adversary, we consider a well-known variant of Turing machine that suits the context of parallelism. We describe the variant namely parallel Turing machine as briefly as possible from (Sect. 2 in cf. [25])

5.2 Parallel Turing Machine

Intuitively, a parallel Turing machine has multiple control units (CU) (working collaboratively) with a single head associated with each of them working on a common read-only input tape and a common read-write work tape.

Definition 14. (Parallel Turing Machine). A parallel Turing machine is a tuple \( PTM = (Q, \Gamma, \Sigma, q_0, F, \delta) \) where

1. \( Q \) is the finite and nonempty set of states.
2. \( \Gamma \) is the finite and non-empty set of tape alphabet symbols including the input alphabet \( \Sigma \).
3. \( q_0 \in Q \) is the initial state.
4. \( F \subseteq Q \) is the set of halting states.
5. \( \delta : 2^Q \times \Gamma \rightarrow 2^{Q \times D} \times \Gamma \) where \( D = \{-1, 0, +1\} \) is the set of directions along the tape.

A configuration of a PTM is a pair \( c = (p, b) \) of mappings \( p : \mathbb{Z}^+ \rightarrow 2^Q \) and \( b : \mathbb{Z}^+ \rightarrow \Gamma \). The mapping \( p(i) \) denotes the set of states of the CUs currently pointing to the \( i \)-th cell in the input tape and \( b(i) \) is the symbol written on it. So it is impossible for two different CUs pointing to the same cell \( i \) while staying at the same state simultaneously. During transitions \( c' = (M'_i, b'(i)) = \delta(c) = \delta(p(i), b(i)), \) the set of CUs may be replaced by a new set of CUs \( M'_i \subseteq Q \times D \). The \( p'(i) \) in the configuration \( c' \) is defined as \( p'(i) = \{ q \mid (q, +1) \in M'_{i-1} \cup (q, 0) \in M'_{i} \cup (q, -1) \in M'_{i+1} \} \).

Without loss of generality, the cell 1 is observed in order to find the halting condition of PTM. We say that a PTM halts on a string if and only if \( p(1) \subseteq F \) after some finite time. The notion of decidability by a PTM is exactly same as in TM. We denote \( PTM(s, t, h) \) as the family of all languages for which there is a PTM recognizing them using space \( s \), time \( t \) and \( h \) processors. Thus languages decidable by a TM is basically decidable by a PTM\( (s, t, 1) \). Assuming \( TM(s, t) \) is the set of languages recognized by a TM in space \( s \) and time \( t \), we mention Theorem 15 from (cf. [25]) without the proof.

We observe that the parallel adversary \( A \) defined in Def. 2 is essentially a PTM having \( poly(\lambda, T) \) processors running on \( poly(\lambda, T) \) space in time \( \sigma(T) \). We will refer such a PTM with \( poly(\lambda, T) \)-PTM (w.l.o.g.) in our subsequent discussions.
5.3 VDF As A Language

Now we characterize VDFs in terms of computational complexity theory. We observe that, much like \((P \leftrightarrow V)\), VDFs are also proof system for the languages,

\[
\mathcal{L} = \left\{(x, y, T) \mid pp \leftarrow \text{Setup}(1^\lambda, T), \ x \in \{0,1\}^\lambda, \ y \leftarrow \text{Eval}(pp, x)\right\}.
\]

\(P\) tries to convince \(V\) that the tuple \((x, y, T) \in \mathcal{L}\) in polynomially many rounds of interactions. In fact, Pietrzak represents his VDF using such a language (Sect. 4.2 in cf. [21]) where it needs \(\log T\) (i.e., \(\text{poly}(\lambda)\)) rounds of interaction. However, by design, the VDF is non-interactive. It uses Fiat–Shamir transformation.

Thus, a VDF closely resembles an \((P \leftrightarrow V)\) except on the fact that it stands sequential (see Def. 5) even against an adversary (including \(P\)) possessing subexponential parallelism. Notice that a \(\text{poly}(\lambda, T)\)-PTM (see Def. 14) precisely models the parallel adversary described in Def. 2. In case of interactive proof systems, we never talk about the running time of \(P\) except its finiteness. On the contrary, \(P\) of a VDF must run for at least \(T\) time in order to satisfy its sequentiality. Hence, we define VDF as follows,

**Definition 15. (Verifiable Delay Function \((P \leftrightarrow V)\)).** For every \(\lambda \in \mathbb{Z}^+, \ T \in 2^{o(\lambda)}\) and for all \(s = (x, y, T) \in \{0,1\}^{2\lambda + [\log T]}\), \((P \leftrightarrow V)\) is a verifiable delay function for a language \(L \subseteq \{0,1\}^*\) if

1. (Correctness). \((s \in L) \implies \Pr[\langle P \leftrightarrow V \rangle(s) = 1] \geq 1 - \text{negl}(\lambda)\).
2. (Soundness). \((s \notin L) \implies \forall A, \Pr[\langle A \leftrightarrow V \rangle(s) = 1] \leq \text{negl}(\lambda)\).
3. (Sequentiality). \((s \in L) \implies \forall B, \Pr[\langle B \leftrightarrow V \rangle(s) = 1] \leq \text{negl}(\lambda)\).

where,

i. \(P: \{0,1\}^* \rightarrow \{0,1\}^*\) is a TM that runs in time \(\geq T\),
ii. \(A: \{0,1\}^* \rightarrow \{0,1\}^*\) is a TM that runs in time \(\text{poly}(\lambda, T)\),
iii. \(B\) is a \(\text{poly}(\lambda, T)\)-PTM (see Def. [14]) that runs in time \(\leq T\).

Further, we define the class of all verifiable delay functions as,

**Definition 16. (The Class VDF).**

\[
\text{VDF} = \bigcup_{k \in \text{poly}(\lambda)} \text{VDF}[k].
\]

For every \(k \in \mathbb{Z}^+, \text{VDF}[k]\) is the set of languages \(\mathcal{L}\) such that there exists a probabilistic polynomial-time TM \(V\) that can have a \(k\)-round interaction with

i. \(P: \{0,1\}^* \rightarrow \{0,1\}^*\) is a TM that runs in time \(\geq T\),
ii. \(A: \{0,1\}^* \rightarrow \{0,1\}^*\) is a TM that runs in time \(\text{poly}(\lambda, T)\),
iii. \(B\) is a \(\text{poly}(\lambda, T)\)-PTM (see Def. [14]) that runs in time \(\leq T\).

satisfying these three following properties,

1. (Correctness). \((s \in \mathcal{L}) \implies \Pr[\langle P \leftrightarrow V \rangle(s) = 1] \geq 1 - \text{negl}(\lambda)\).
2. (Soundness). \((s \notin \mathcal{L}) \implies \forall A, \Pr[\langle A \leftrightarrow V \rangle(s) = 1] \leq \text{negl}(\lambda)\).
3. (Sequentiality). \((s \in \mathcal{L}) \implies \forall B, \Pr[\langle B \leftrightarrow V \rangle(s) = 1] \leq \text{negl}(\lambda)\).
6 VDF-completeness of rSVL

In this section, we show that rSVL is a complete problem the class VDF.

**Theorem 1. (rSVL is VDF-complete).** rSVL is a complete problem for the class VDF.

**Proof.** By theorem [2] rSVL belongs to the class VDF. By theorem [3] rSVL is a hard problem for the class VDF. Hence, rSVL is a complete problem for the class VDF.

**Theorem 2. (rSVL ∈ VDF).** For the parameters \( \lambda \in \mathbb{Z}^+ \) and \( T = T(\lambda) \in 2^{o(\lambda)} \), let \( S : \{0,1\}^\lambda \rightarrow \{0,1\}^\lambda, V : \{0,1\}^\lambda \times \{1, \ldots, T\} \rightarrow \{0,1\} \) and \( x \in \{0,1\}^\lambda \).

If there exists a family of rSVL instances \( \{S,V,v,T\}_{v \in \{0,1\}^\lambda} \) such that each instance allows at most polynomially many (i.e., \( \text{poly}(\lambda) \)) false positive vertices then there exists a permutation VDFs with \( X = Y = \{0,1\}^\lambda \) and the delay parameter \( T \).

**Proof.** Given any \( \lambda \in \mathbb{Z}^+ \) and any \( T \in 2^{o(\lambda)} \), we derive a permutation VDF from a sub-family of rSVL, \( \{S,V,v,T\}_{v \in \{0,1\}^\lambda} \) as follows,

- **Setup\((1^\lambda, T) \rightarrow pp\)** It samples a rSVL sub-family \( \{S,V,v,T\}_{v \in \{0,1\}^\lambda} \) as the public parameter \( pp \) from the family \( \{S,V,v,T\}_{v,\lambda} \), uniformly at random.
- **Eval\((pp, x) \rightarrow (y, \perp)\)** It takes an input \( x \in X = \{0,1\}^\lambda \), and produces an output \( y := S^T(x) \). There is no proof, so \( \pi = \perp \).
- **Verify\((pp, x, y, \perp) \rightarrow \{0,1\}\)** It returns \( V(y, T) \). Note that the input \( x \) is implicit to the circuit \( V \). Thus, Verify is not independent of \( x \).

We prove the correctness, computational soundness and sequentiality of this VDF in Theorem [4][5] and [6]

**Theorem 3. (Correctness).** The derived VDF, in Theorem [4] is correct.

**Proof.** For the rSVL instance \( (S,V,v,T), V(u,T) = 1 \) if and only if either \( u = S^T(v) \) or \( u \) is a false positive vertex i.e., \( V(u,T) = 1 \) but \( u \neq S^T(v) \). In this VDF, \( \text{Eval}(x) = S^T(x) \) and \( \text{Verify}(x,y,T) = V(y,T) \). Therefore, \( \text{Verify}(x,y,T) = 1 \) if and only if either \( y = \text{Eval}(x,T) \) or \( y \) is false positive vertex off the main line in the rSVL instance \( (S,V,x,T) \). Since, the number of false positive vertices is in \( \text{poly}(\lambda) \), the probability that a random vertex \( y \) is false positive is at most \( \text{poly}(\lambda)/2^\lambda = \text{negl}(\lambda) \). Therefore,

\[
\Pr[V(y,T) = 1 \land y = S^T(x)] \geq 1 - \text{negl}(\lambda).
\]

**Theorem 4. (Soundness).** For the parameters \( \lambda \in \mathbb{Z}^+ \) and \( T = T(\lambda) \in 2^{o(\lambda)} \), let \( S : \{0,1\}^\lambda \rightarrow \{0,1\}^\lambda, V : \{0,1\}^\lambda \times \{1, \ldots, T\} \rightarrow \{0,1\} \) and \( v \in \{0,1\}^\lambda \).

If there exists an adversary \( A \) that breaks the soundness of this permutation VDFs with \( X = Y = \{0,1\}^\lambda \) and the delay parameter \( T \) with a non-negligible probability \( \epsilon = \epsilon(\lambda) \), then there exists another adversary \( B \) who finds false positive vertices for the instances from the rSVL family \( \{S,V,v,T\}_{v \in \{0,1\}^\lambda} \) with the probability \( \epsilon \).
Proof. For the rSVL instance $(S, V, v, T)$, $V(u, T) = 1$ if and only if either $u = S^T(v)$ or $u$ is a false positive vertex i.e., $V(u, T) = 1$ but $u \neq S^T(v)$. In this VDF, \( \text{Eval}(x) = S^T(x) \) and \( \text{Verify}(x, y, T) = V(y, T) \). Therefore, \( \text{Verify}(x, y, T) = 1 \) if and only if either $y = \text{Eval}(x, T)$ or $y$ is false positive vertex off the main line in the rSVL instance $(S, V, x, T)$.

As the adversary $A$ breaks the soundness of this VDF, he must find a $y \neq \text{Eval}(x, T)$ but $\text{Verify}(x, y, T) = 1$. Therefore, in order to find a false positive vertex $y$ in the rSVL instance $(S, V, x, T)$, $B$ runs $A$ on input $(x, T)$. When $A$ outputs $y$, $B$ returns $y$ as the false positive vertex. Therefore,

\[
\Pr[B \text{ wins }] = \Pr[A \text{ wins }] = \epsilon(\lambda).
\]

Since, the number of false positive vertices is in $\text{poly}(\lambda)$, the probability that a random vertex $y$ is false positive is at most $\text{poly}(\lambda)/2^\lambda = \text{negl}(\lambda)$. Therefore, $A$ has negligible advantage.

**Theorem 5. (Sequentiality).** For the parameters $\lambda \in \mathbb{Z}^+$ and $T = T(\lambda) \in 2^{o(\lambda)}$, let $S : \{0, 1\}^\lambda \rightarrow \{0, 1\}^\lambda$, $V : \{0, 1\}^\lambda \times \{1, \ldots, T\} \rightarrow \{0, 1\}$ and $v \in \{0, 1\}^\lambda$. If there exists an adversary $A$ that breaks the sequentiality of this permutation VDF’s with $X = Y = \{0, 1\}^\lambda$ and the delay parameter $T$ in time $T_A = T(\lambda) < T$ with the non-negligible probability $\epsilon = \epsilon(\lambda)$ then there exists another adversary $B$ who solves the instances from the rSVL family $\{S, V, v, T\}_{v \in \{0, 1\}^\lambda}$ in time $T_A + O(1) < T$ with the probability $\epsilon$.

Proof. For the rSVL instance $(S, V, v, T)$, $V(u, T) = 1$ if and only if either $u = S^T(v)$ or $u$ is a false positive vertex i.e., $V(u, T) = 1$ but $u \neq S^T(v)$. In this VDF, \( \text{Eval}(x) = S^T(x) \) and \( \text{Verify}(x, y, T) = V(y, T) \). Therefore, \( \text{Verify}(x, y, T) = 1 \) if and only if either $y = \text{Eval}(x, T)$ or $y$ is false positive vertex off the main line in the rSVL instance $(S, V, x, T)$.

As the adversary $A$ breaks the sequentiality of this VDF, he must find the $y = \text{Eval}(x, T)$ or a $y' \neq \text{Eval}(x, T)$ but $\text{Verify}(x, y', T) = 1$, in time $T_A < T$. Therefore, in order to solve the rSVL instance $(S, V, x, T)$, $B$ runs $A$ on input $(x, T)$. When $A$ outputs $y$, $B$ returns $y$ as the solution in time $T_A + O(1)$. Therefore,

\[
\Pr[B \text{ wins }] = \Pr[A \text{ wins }] = \epsilon(\lambda).
\]

Theorem 2 gives rise to permutation VDF. Although, it suffices to prove that rSVL $\in$ VDF, deriving a VDF with $X \neq Y$, from rSVL needs a family of injective one-way function $H = \{H : X \rightarrow Y\}$.

**Theorem 6. (rSVL $\in$ VDF, in general).** For the parameters $\lambda \in \mathbb{Z}^+$ and $T = T(\lambda) \in 2^{o(\lambda)}$, let $S : \{0, 1\}^\lambda \rightarrow \{0, 1\}^\lambda$, $V : \{0, 1\}^\lambda \times \{1, \ldots, T\} \rightarrow \{0, 1\}$ and $v \in \{0, 1\}^\lambda$. If there exists an adversary $A$ that $x \in \{0, 1\}^\lambda$. If there exists a family of rSVL instances $\{S, V, v, T\}_{v \in \{0, 1\}^\lambda}$ such that each instance allows at most polynomially many (i.e., $\text{poly}(\lambda)$) false positive vertices then there exists a permutation VDF’s with $X \in \{0, 1\}^*$, $Y = \{0, 1\}^\lambda$ and the delay parameter $T$, assuming a family of injective one-way function $H = \{H : X \rightarrow Y\}$.
Proof. Given any \( \lambda \in \mathbb{Z}^+ \) and any \( T \in 2^{o(\lambda)} \), we derive a permutation VDF from a sub-family of rSVL, \{S, V, v, T\}_{v \in \{0,1\}^\lambda} as follows,

- **Setup** \((1^\lambda, T) \rightarrow pp\) It samples a rSVL sub-family \{S, V, v, T\}_{v \in \{0,1\}^\lambda} as the public parameter \( pp \) from the family \{S, V, v, T\}_{v \in \{0,1\}^\lambda}, uniformly at random. Apart from the rSVL instance, it also chooses an \( H \in \mathcal{H} \), uniformly at random.

- **Eval** \((pp, x) \rightarrow (y, \perp)\) It takes an input \( x \in \{0,1\}^* \), and produces an output \( y := S^T(H(x)) \). There is no proof, so \( \pi = \perp \).

- **Verify** \((pp, x, y, \perp) \rightarrow \{0,1\}\) It returns \( V(y, T) \). Note that the input \( H(x) \) is implicit to the circuit \( V \). Thus, Verify is not independent of \( x \).

The proofs for the correctness, computational soundness and sequentiality of this VDF are same as Theorem 3, 4 and 5.

As before, first we show that every permutation VDF reduces to rSVL.

**Theorem 7. (Reduction from Permutation VDF to rSVL).** For the parameters \( \lambda \in \mathbb{Z}^+ \) and \( T = T(\lambda) \in 2^{o(\lambda)} \), let \((\text{Setup}, \text{Eval}, \text{Open}, \text{Verify})\) be an interactive permutation VDF on the domain \{0,1\}^\lambda. Then there exists a hard distribution of rSVL instances \{S, V, v, T\}_{v \in \{0,1\}^\lambda} that have at most polynomially many (i.e., \( \text{poly}(\lambda) \)) false positive vertices, such that \( S : \{0,1\}^\lambda \rightarrow \{0,1\}^\lambda \), \( V : \{0,1\}^\lambda \times \{1, \ldots, T\} \rightarrow \{0,1\} \) and \( x \in \{0,1\}^\lambda \).

**Proof.** Given any \( \lambda \in \mathbb{Z}^+ \) and any \( T \in 2^{o(\lambda)} \), we derive a hard distribution of rSVL instances, \{S, V, v, T\}_{v \in \{0,1\}^\lambda} from a permutation VDF, as follows,

**The \( S \) circuit** In order to design \( S \), we observe that, for every permutation VDF, \( \text{Eval}(x, T) = \text{Eval}(\text{Eval}(x, T - 1)) \) as \( \mathcal{X} = \mathcal{Y} \). This vertex \( u \) is the source in the rSVL instances. Therefore, we keep \( S(u) = \text{Eval}(pp, u, T = 1) \).

In particular,

**Algorithm 1** \( S(u) \) from Eval

1: \( y := \text{Eval}(pp, u, 1) \)
2: \( \text{return } y \).

More generally, \( S^i(v) = \text{Eval}(pp, v, i) \) for all \( i \leq T \).

**The \( V \) circuit** We take the advantage of the interactive VDFs in order to design \( V \). In particular,

**Algorithm 2** \( V(v, T) \) from Eval

1: \( \pi := \text{Open}(pp, v, y, T) \)
2: \( w := \text{Verify}(pp, v, y, T, \pi) \).
3: \( \text{return } w \).
Since, $\text{Open}$ takes at most $\text{poly}(\lambda, \log T)$-rounds, $V$ is efficient.

Now, the challenge is to reduce any arbitrary VDF into an hard rSVL instances. We can not follow the approach used in [7]. They label the rSVL graph with the proofs in the Pietrzak’s VDF using the ”proof-merging” technique described in Sect. 2. This approach does not work for the VDFs that needs no proof e.g., the isogenie-based VDF [11]. Labelling the $i$-th node in the rSVL graph with $\text{Eval}(pp, x, i)$ for $i \leq T$, needs $T$ different computation of $\text{Eval}$ for each $i$ as $X \neq Y$. We tackle this problem in the following theorem using a special function $f$.

**Theorem 8. (rSVL is VDF-hard).** For the parameters $\lambda \in \mathbb{Z}^+$ and $T = T(\lambda) \in 2^{o(\lambda)}$, let $(\text{Setup}, \text{Eval}, \text{Open}, \text{Verify})$ be an interactive VDF on the domain $X = \{0, 1\}^*$ and the range $Y = \{0, 1\}^\lambda$. Then there exists a hard distribution of rSVL instances $\{S, V, v, T\}_{v \in \{0, 1\}^\lambda}$ that have at most polynomially many (i.e., $\text{poly}(\lambda)$) false positive vertices, such that $S : \{0, 1\}^\lambda \rightarrow \{0, 1\}^\lambda$, $V : \{0, 1\}^\lambda \times \{1, \ldots, T\} \rightarrow \{0, 1\}$ and $x \in \{0, 1\}^\lambda$.

**Proof.** Given any $\lambda \in \mathbb{Z}^+$ and any $T \in 2^{o(\lambda)}$, we derive a hard distribution of rSVL instances, $\{S, V, v, T\}_{v \in \{0, 1\}^\lambda}$ from a VDF, as follows,

First, we define a function $f$ as,

$$f(\text{Eval}(pp, x, T = 0), i) = \text{Eval}(pp, x, i) \quad \forall i \leq T.$$  

The function $\text{Eval}(x, T = 0)$ maps $x$ into the range $Y$.

**The $S$ circuit** We observe that, for every VDF, $\text{Eval}(pp, v, T) = f(v, T) = f(f(v, T - 1))$ with the base case $v = \text{Eval}(pp, x, 0)$. This vertex $v$ is the source in the rSVL instances. Therefore, we keep $S(u) = f(pp, u, T = 1)$. In particular,

**Algorithm 3 $S(u)$ from Eval**

1: $y := f(pp, u, 1)$
2: return $y$.

More generally, $S^i(v) = f(pp, v, i)$ for all $i \leq T$.

**The $V$ circuit** This circuit is same as in the previous theorem. In particular,

**Algorithm 4 $V(v, T)$ from Eval**

1: $\pi := \text{Open}(pp, v, y, T)$
2: $w := \text{Verify}(pp, v, y, T, \pi)$.
3: return $w$.

Since, $\text{Open}$ takes at most $\text{poly}(\lambda, \log T)$-rounds, $V$ is efficient.
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