LINEAR KOSZUL DUALITY AND FOURIER TRANSFORM 
FOR CONVOLUTION ALGEBRAS

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ABSTRACT. In this paper we prove that the linear Koszul duality isomorphism for convolution algebras in $K$-homology of [MR3] and the Fourier transform isomorphism for convolution algebras in Borel–Moore homology of [EM] are related by the Chern character. So, Koszul duality appears as a categorical upgrade of Fourier transform of constructible sheaves. This result explains the connection between the categorification of the Iwahori–Matsumoto involution for graded affine Hecke algebras in [EM] and for usual affine Hecke algebras in [MR3].

INTRODUCTION

0.1. This article is a sequel to [MR1, MR2, MR3]. It links two kinds of “Fourier” transforms prominent in mathematics, the Fourier transform for constructible sheaves and the Koszul duality. This is done in a particular situation which is of interest in representation theory.

0.2. Chern character map. Our setting consists of two vector subbundles $F_1, F_2$ of a trivial vector bundle $X \times V$ over a (smooth and proper) algebraic variety $X$. We consider the fiber product $F_1 \times_V F_2$ as well as the dual object — the fiber product $F_1^\perp \times_{V^*} F_2^\perp$ of orthogonal complements of $F_1$ and $F_2$ inside the dual vector bundle $X \times V^*$. The linear Koszul duality mechanism from [MR1, MR2, MR3] is a geometric version of the standard Koszul duality between graded modules over the symmetric algebra of a vector space and graded modules over the exterior algebra of the dual vector space. Here, this formalism provides an equivalence of categories of equivariant coherent sheaves on the derived fiber products $F_1 \times_V^R F_2$ and $F_1^\perp \times_{V^*}^R F_2^\perp$ (in the sense of dg-schemes). In particular we get an isomorphism of equivariant $K$-homology groups of algebraic varieties $F_1 \times_V F_2$ and $F_1^\perp \times_{V^*} F_2^\perp$.\footnote{Note that $K$-homology does not distinguish the derived fiber product from the usual fiber product of varieties, see [MR3].} On the other hand, the Fourier transform for constructible sheaves provides an isomorphism of equivariant Borel–Moore homologies of fiber products $F_1 \times_V F_2$ and $F_1^\perp \times_{V^*} F_2^\perp$, see [EM].\footnote{For simplicity we work under a technical assumption on $F_i$’s which is satisfied in all known applications.}

Our main result shows that the maps in $K$-homology and in Borel–Moore homology are related by the Chern character map (the “Riemann-Roch map”) from equivariant $K$-homology to (completed) equivariant Borel–Moore homology.\footnote{In this way, linear Koszul duality appears as a categorical upgrade of the topological Fourier transform.}
0.3. Convolution algebras. In representation theory the above setting provides a geometric construction of algebras. Indeed, when $F_1 = F_2 = F$ then the equivariant $K$-homology and Borel–Moore homology of $F \times_V F$ have structures of convolution algebras; denote these $A_K(F)$ and $A_{BM}(F)$. The Chern character provides a map of algebras $A_K(F) \to \hat{A}_{BM}(F)$ from the $K$-homology algebra to a completion of the Borel–Moore homology algebra $\hat{C}G, \hat{K}at, \hat{L}1$. This gives a strong relation between their representation theories: one obtains results on the representation theory of the (more interesting) algebra $A_K(F)$ through the relation to the representation theory of the algebra $A_{BM}(F)$ which is more accessible.\footnote{The reason is the powerful machinery of perverse sheaves that one can use in the topological setting, see \cite{CG}.}

In this setting, the maps $i_K : A_K(F) \xrightarrow{\sim} A_K(F^\perp)$, $i_{BM} : A_{BM}(F) \xrightarrow{\sim} A_{BM}(F^\perp)$ induced respectively by linear Koszul duality and by Fourier transform are isomorphisms of algebras.

The original example of this mechanism appeared in the study of affine Hecke algebras, see \cite{KL, CG}. The Steinberg variety $Z$ of a complex connected reductive algebraic group $G$ (with simply connected derived subgroup) is of the above form $F \times_V F$ where the space $X$ is the flag variety $B$ of $G$, the vector space $V$ is the dual $g^*$ of the Lie algebra $g$ of $G$, and $F$ is the cotangent bundle $T^*B$. The $G \times G_m$-equivariant $K$-homology and Borel–Moore homology of the Steinberg variety $Z$ are then known to be realizations of the affine Hecke algebra $\mathcal{H}_{aff}$ of the dual reductive group $\hat{G}$ and of the graded affine Hecke algebra $\mathcal{H}_{gr}^{aff}$. In this case the dual version $F^\perp \times_V F^\perp$ turns out to be another – homotopically equivalent – version of the Steinberg variety $Z$. Therefore, $i_K$ and $i_{BM}$ are automorphisms of $\mathcal{H}_{aff}$ and $\mathcal{H}_{gr}^{aff}$. In fact these are (up to an inversion) geometric realizations of the Iwahori–Matsumoto involution on $\mathcal{H}_{aff}^{gr}$ (see \cite{EM}) and $\mathcal{H}_{aff}$ (see \cite{MR3}). So, in this situation, Theorem \ref{thm:main} explains the relation between results of \cite{MR3} and \cite{EM}.

0.4. Character cycles and characteristic cycles. In \cite{Kas}, Kashiwara introduced for a group $G$ acting on a space $X$ an invariant of a $G$-equivariant constructible sheaf $F$ on $X$. This is an element $\text{ch}_G(F)$ of the Borel–Moore homology of the stabilizer space $G_X := \{ (g,x) \in G \times X \mid gx = x \}$. He linearized this to an element $\text{ch}_g(F)$ of the Borel–Moore homology of the analogous stabilizer space $gX$ for the Lie algebra $g$ of the group $G$. Under some assumptions (that put one in the above geometric setting) he proved that the characteristic cycle of $F$ is the image of $\text{ch}_g(F)$ under a Fourier transform map in Borel–Moore homology (see \cite{Kas, \S 1.9}). This work is the origin of papers on Iwahori–Matsumoto involution \cite{EM} and linear Koszul duality \cite{MR1}. From this point of view, the present paper is a part of the effort to categorify Kashiwara’s character cycles.

0.5. Organization of the paper. In Section \ref{sec:main} we define precisely all our morphisms, and state our main result (Theorem \ref{thm:main}). In Sections \ref{sec:compatibility} and \ref{sec:proof} we prove some compatibility statements for all our constructions, and we apply these results in Section \ref{sec:proof} to the proof of Theorem \ref{thm:main}. Finally, Appendix \ref{sec:proof} contains the proofs of some technical lemmas needed in other sections.
LINEAR KOSZUL DUALITY AND FOURIER TRANSFORM

1. Definitions and statement

1.1. Equivariant homology and cohomology. If $A$ is a complex algebraic group acting on a complex algebraic variety $Y$, we denote by $\mathcal{D}_\text{const}^A(Y)$ the equivariant derived category of constructible complexes on $Y$ with complex coefficients, see [BL]. Let $\underline{\mathcal{O}}_Y$, respectively $\underline{\Omega}_Y$, be the constant, respectively dualizing, sheaf on $Y$. These are objects of $\mathcal{D}^\text{const}_A(Y)$. We also denote by $\mathcal{D}_Y : \mathcal{D}^\text{const}_A(Y) \xrightarrow{\sim} \mathcal{D}^\text{const}_A(Y)^{\text{op}}$ the Grothendieck–Verdier duality functor.

If $M$ is in $\mathcal{D}^\text{const}_A(Y)$, the $i$-th equivariant cohomology of $Y$ with coefficients in $M$ is by definition

$$H_i^A(X, M) := \text{Ext}^i_{\mathcal{D}^\text{const}_A(Y)}(\underline{\mathcal{O}}_Y, M).$$

In particular, the equivariant cohomology and Borel–Moore homology of $Y$ are defined by

$$H_i^A(Y) := H_i^A(Y, \underline{\mathcal{O}}_Y), \quad H_i^A(Y) := H_i^A(Y, \underline{\Omega}_Y).$$

We will also use the notation

$$H^*_A(Y) := \bigoplus_{i \in \mathbb{Z}} H_i^A(Y), \quad \hat{H}^*_A(Y) := \prod_{i \in \mathbb{Z}} H_i^A(Y),$$

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Note that with our conventions, one can have $H_i^A(Y) \neq 0$ for $i < 0$. We will use the general convention that we denote by the same symbol an homogeneous morphism between vector spaces of the form $H^*_A(\cdot)$ and the induced morphism between the associated vector spaces $\hat{H}^*_A(\cdot)$.

There exists a natural (right) action of the algebra $H^*_A(Y)$ on $H^*_A(Y)$ induced by composition of morphisms in $\mathcal{D}^\text{const}_A(Y)$; it extends to an action of the algebra $\hat{H}^*_A(Y)$ on $\hat{H}^*_A(Y)$.

The basic constructions in Borel–Moore homology (see [CG] §2.6]) can be easily generalized to the equivariant setting. As a reference for the constructions related to the equivariant Riemann–Roch theorem we will use [BZ]. (This topic is also treated with more details in [EG], but in the context of equivariant Chow groups rather than equivariant Borel–Moore homology.) In particular, assume that $A$ is connected and reductive; then with the same notation as above we have a “Riemann–Roch” morphism

$$\tau^A : K^A(Y) \to \hat{H}^*_A(Y)$$

defined in terms of the Chern character and which is compatible with proper pushforwards (see [BZ §5] and [EG] §3). If $Y$ is smooth, we denote by $\text{Td}^A_Y$ the equivariant Todd class (see [BZ] §3) of the tangent bundle of $Y$. It is an invertible element of the algebra $\hat{H}^*_A(Y)$.

We will also frequently use the following constructions. If $Z \subset Y$ is an $A$-stable closed subvariety then there exist natural pushforward morphisms

$$K^A(Z) \to K^A(Y), \quad \text{resp.} \quad H^*_A(Z) \to H^*_A(Y),$$

or...
see [CG] §5.2.13, resp. [CG] §2.6.8. On the other hand, if \( Y \) is smooth, \( Y' \subset Y \) is an \( A \)-stable smooth closed subvariety, and \( Z \subset Y \) is a not necessarily smooth closed subvariety, then we have “restriction with supports” morphisms

\[
K^A(Z) \to K^A(Z \cap Y'), \quad \text{resp.} \quad H^*_n(Z) \to H^*_n(2\dim(Y)+2\dim(Y')(Z \cap Y'))
\]

associated with the inclusion \( Y' \hookrightarrow Y \), see [CG] p. 246, resp. [CG] §2.6.21. (The definition of the second morphism is recalled in [A.3].) Finally, if \( E \to Y \) is a vector bundle, then we have the Thom isomorphism

\[
H^*_n(E) \cong H^*_n(2\rk(E)).
\]

1.2. Fourier–Sato transform. Let \( G \) be a complex connected reductive algebraic group, and let \( X \) be a \( G \)-variety. If \( r : E \to X \) is a \( G \)-equivariant (complex) vector bundle, we equip it with a \( G \times \mathbb{G}_m \)-action where \( t \in \mathbb{G}_m \) acts by multiplication by \( t^{-2} \) along the fibers of \( r \). We denote by \( E^\circ \) the \( G \times \mathbb{G}_m \)-equivariant dual vector bundle (so that \( t \in \mathbb{G}_m \) acts by multiplication by \( t^2 \) along the fibers), and by \( E^* \) the dual \( G \)-equivariant vector bundle, which we equip with a \( \mathbb{G}_m \)-action where \( t \in \mathbb{G}_m \) acts by multiplication by \( t^{-2} \) along the fibers. We denote by \( \hat{r} : E^* \to X \) the projection.

The Fourier–Sato transform defines an equivalence of categories

\[
\Phi_E : \mathcal{D}^G_{\text{const}}(E) \sim \mathcal{D}^G_{\text{const}}(E^\circ).
\]

This equivalence is constructed as follows (see [KS] §3.7; see also [AHJR] §2.7 for a reminder of the main properties of this construction). Let \( Q := \{(x, y) \in E \times X \mid \Re((x, y)) \leq 0\} \), and let \( q : Q \to E, \bar{q} : Q \to E^\circ \) be the projections. Then we have

\[
\Phi_E := \bar{q}q^*.
\]

(This equivalence is denoted \((\cdot)^\wedge\) in [KS]; it differs by a shift of the equivalence \( T_E \) of [AHJR].)

Inverse image under the automorphism of \( G \times \mathbb{G}_m \) which sends \((g, t)\) to \((g, t^{-1})\) establishes an equivalence of categories

\[
\mathcal{D}^G_{\text{const}}(E^\circ) \sim \mathcal{D}^G_{\text{const}}(E^*).
\]

We will denote by

\[
\mathcal{F}_E : \mathcal{D}^G_{\text{const}}(E) \sim \mathcal{D}^G_{\text{const}}(E^*)
\]

the composition of (1.2.1) and (1.2.2).

Let \( F \subset E \) be a \( G \)-stable subbundle, and denote by \( F^\perp \subset E^* \) the orthogonal to \( F \). Then one can consider the constant sheaf \( \underline{\mathbb{C}}_F \) as an object of \( \mathcal{D}^G_{\text{const}}(E) \). (Here and below, we omit direct images under closed inclusions when no confusion is likely.) Similarly, we have the object \( \underline{\mathbb{C}}_{F^\perp} \) of \( \mathcal{D}^G_{\text{const}}(E^*) \). The following result is well known; we reproduce the proof for future reference.

**Lemma 1.2.3.** There exists a canonical isomorphism

\[
\mathcal{F}_E(\underline{\mathbb{C}}_F) \cong \underline{\mathbb{C}}_{F^\perp}[-2\rk(F)].
\]

**Proof.** It is equivalent to prove a similar isomorphism for \( \Phi_E \). For simplicity we denote \( F^\perp \) by the same symbol when it is considered as a subbundle of \( E^\circ \).
By definition of \( \mathfrak{F}_E \) we have a canonical isomorphism
\[
\mathfrak{F}_E(\mathbb{C}_F) \cong \tilde{q} F! \mathbb{C}_{Q_F},
\]
where \( Q_F := q^{-1}(F) \subset Q \) and \( \tilde{q} \) is the composition of \( \tilde{q} \) with the inclusion \( Q_F \hookrightarrow Q \). There is a natural closed embedding \( i_F : F \times_X F^\perp \hookrightarrow Q_F \); we denote by \( U_F \) the complement and by \( j_F : U_F \hookrightarrow Q_F \) the inclusion. The natural exact triangle \( j_F ! \mathbb{C}_{U_F} \to \mathbb{C}_{Q_F} \to i_{F!*} \mathbb{C}_{F \times_X F^\perp} \to +1 \) provides an exact triangle
\[
q F! j_F ! \mathbb{C}_{U_F} \to q F! \mathbb{C}_{Q_F} \to q F! i_{F!*} \mathbb{C}_{F \times_X F^\perp} \to +1.
\]
One can easily check that \( q F! j_F ! \mathbb{C}_{U_F} = 0 \), so that the second map in this triangle is an isomorphism. Finally, \( q_F \circ i_F : F \times_X F^\perp \to E^\circ \) identifies with the composition of the projection \( F \times_X F^\perp \to F^\perp \) with the embedding \( F^\perp \hookrightarrow E^\circ \). We deduce a canonical isomorphism
\[
q F! \mathbb{C}_{Q_F} \cong \mathbb{C}_{F^\perp}[-2 \text{rk}(F)],
\]
which finishes the proof. \( \square \)

We will mainly use these constructions in the following situation. Let \( V \) be a \( G \)-module (which we will consider as a \( G \)-equivariant vector bundle over the variety \( pt := \text{Spec}(\mathbb{C}) \)), and let \( E := V \times X \), a \( G \)-equivariant vector bundle over \( X \). We denote by \( p : E \to V \), \( \tilde{p} : E^* \to V^* \) the projections. As above, let \( F \subset E \) be a \( G \)-stable subbundle.

**Corollary 1.2.4.** There exists a canonical isomorphism
\[
F_V(\mathbb{C}_F) \cong \tilde{p}_! \mathbb{C}_{F^\perp}[-2 \text{rk}(F)].
\]

\textbf{Proof.} By [KS] Proposition 3.7.13 (see also [ALJR] §A.4) we have a canonical isomorphism of functors
\[
F_V \circ p! \cong \tilde{p}_! \circ F_E.
\]
In particular we deduce an isomorphism \( F_V(\mathbb{C}_F) \cong \tilde{p}_! F_E(\mathbb{C}_F) \). Then the result follows from Lemma 1.2.3. \( \square \)

### 1.3. Equivariant homology as an Ext-algebra.

From now on we let \( G \) be a complex connected reductive algebraic group, \( X \) be a smooth and proper complex algebraic variety, and \( V \) be a finite dimensional \( G \)-module. Let \( E := V \times X \), considered as a \( G \times \mathbb{G}_m \)-equivariant vector bundle as in \ref{1} and let \( F_1, F_2 \) be \( G \)-stable subbundles of the vector bundle \( E \) over \( X \). As in \ref{1} we denote by \( p : E \to V \) the projection, and by \( F_1^\perp, F_2^\perp \subset E^* \) the orthogonals to \( F_1 \) and \( F_2 \). Then there exists a canonical isomorphism
\[
H^\bullet_{G \times \mathbb{G}_m}(F_1 \times_X F_2) \cong \text{Ext}_{\mathcal{O}^\bullet_{G \times \mathbb{G}_m}(V)}^{2 \dim(F_2) - \bullet}(p_{\ast} \mathbb{C}_{F_1}, p_{\ast} \mathbb{C}_{F_2}).
\]

Let us explain (for future reference) how this isomorphism can be constructed, following [CG] L3. Consider the cartesian diagram
\[
\begin{array}{ccc}
E \times_X E & \xrightarrow{\mu} & E \times E \\
\downarrow p \times p & & \downarrow p \times p \\
V \times V & \xrightarrow{\Delta} & V \times V
\end{array}
\]
where Δ is the diagonal embedding. Then in [CG, Equation (8.6.4)] (see also [EM, §1.15 and §2.4]) the authors construct a canonical and bifunctorial isomorphism

$$\mu_*j^!_*(\mathbb{D}_E(A_1) \boxtimes A_2) \cong R\mathcal{H}om_C(p_!A_1, p_!A_2)$$

for $A_1, A_2$ in $\mathcal{D}^{G \times G_m}_{\text{const}}(E)$. Applying equivariant cohomology, we obtain an isomorphism

$$(1.3.2) \quad \text{Ext}^\bullet_{\mathcal{D}^{G \times G_m}_{\text{const}}(V)}(p_!A_1, p_!A_2) \cong \text{H}^\bullet_{G \times G_m}(E \times V, j^!_*(\mathbb{D}_E(A_1) \boxtimes A_2)).$$

Setting $A_1 = \mathbb{C}_{F_1}$, $A_2 = \mathbb{C}_{F_2}$ we obtain an isomorphism

$$\text{Ext}^\bullet_{\mathcal{D}^{G \times G_m}_{\text{const}}(V)}(p_!\mathbb{C}_{F_1}, p_!\mathbb{C}_{F_2}) \cong \text{H}^\bullet_{G \times G_m}(E \times V, j^!_*(\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2})).$$

Let $a : F_1 \times F_2 \hookrightarrow E \times E$ be the inclusion, and consider the cartesian diagram

$$\begin{array}{ccc}
F_1 \times V & \xrightarrow{b} & E \times V \\
\downarrow & & \downarrow \ j \\
F_1 \times F_2 & \xrightarrow{a} & E \times E.
\end{array}$$

Then using the base change isomorphism we obtain

$$\text{H}^\bullet_{G \times G_m}(E \times V, j^!_*(\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2})) \cong \text{H}^\bullet_{G \times G_m}(E \times V, j^!_*(\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2}))$$

$$\cong \text{H}^\bullet_{G \times G_m}(E \times V, b_*k^!((\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2})) \cong \text{H}^\bullet_{G \times G_m}(F_1 \times V, F_2, k^!((\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2})).$$

Now we use the canonical isomorphisms $\mathbb{C}_{F_2} \cong \mathbb{D}_{F_2}[\pm 2\dim(F_2)]$ (since $F_2$ is smooth) and $k^!((\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2}) \cong k^!((\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2}) \cong \mathbb{D}_{F_1 \times V} F_2$ to deduce $(1.3.1)$.

1.4. The Fourier isomorphism. We continue with the setting of $(1.3)$ and denote by $\bar{p} : E^* \to V^*$ the projection. Using $(1.3.1)$ there exist canonical isomorphisms

$$\text{H}^\bullet_{G \times G_m}(F_1 \times V, F_2) \cong \text{Ext}^{2\dim(F_2)}_{\mathcal{D}^{G \times G_m}_{\text{const}}(V)}(p_!\mathbb{C}_{F_1}, p_!\mathbb{C}_{F_2}).$$

$$\text{H}^\bullet_{G \times G_m}(F_1 \times V, F_2) \cong \text{Ext}^{2\dim(F_2)}_{\mathcal{D}^{G \times G_m}_{\text{const}}(V^*)}(\bar{p}_!\mathbb{C}_{F_1}, \bar{p}_!\mathbb{C}_{F_2}).$$

On the other hand, through the canonical isomorphisms $\mathcal{F}_V(p_!\mathbb{C}_{F_i}) \cong \bar{p}_!\mathbb{C}_{F_i}[\pm 2\text{rk}(F_i)]$ for $i = 1, 2$ (see $(1.2.3)$), the functor $\mathcal{F}_V$ induces an isomorphism

$$\text{Ext}^\bullet_{\mathcal{D}^{G \times G_m}_{\text{const}}(V)}(p_!\mathbb{C}_{F_1}, p_!\mathbb{C}_{F_2}) \Rightarrow \text{Ext}^{-2\text{rk}(F_2)+2\text{rk}(F_1)}_{\mathcal{D}^{G \times G_m}_{\text{const}}(V^*)}(\bar{p}_!\mathbb{C}_{F_1}, \bar{p}_!\mathbb{C}_{F_2}).$$

We denote by

$$\text{Fourier}_{F_1, F_2} : \text{H}^\bullet_{G \times G_m}(F_1 \times V, F_2) \to \text{H}^\bullet_{G \times G_m}(F_1 \times F_2, F_2)$$

the resulting isomorphism. This isomorphism, considered in particular in [EM], was the starting point of our work on linear Koszul duality.
1.5. **Linear Koszul duality.** Let us recall the definition and main properties of linear Koszul duality, following [MR1] [MR2] [MR3]. In this paper we will only consider the geometric situation relevant for convolution algebras, as considered in [MR3] §4. However we will allow using two different vector bundles $F_1$ and $F_2$; the setting of [MR3] §4 corresponds to the choice $F_1 = F_2$.

We continue with the setting of §4.3. We denote by $\Delta V \subset V \times V$ the diagonal copy of $V$. We will consider the derived category

$$\mathcal{D}^c_{G \times G_m}((\Delta V \times X \times X))^R_{\mathcal{E} \times E} (F_1 \times F_2)$$

as defined in [MR3] §3.1. It is a subcategory of the derived category of $G \times G_m$-equivariant quasi-coherent dg-modules over a certain sheaf of $\mathcal{O}_{X \times X}$-dg-algebras on $X \times X$. Note that the derived intersection

$$(\Delta V \times X \times X))^R_{\mathcal{E} \times E} (F_1 \times F_2)$$

is quasi-isomorphic to the derived fiber product $F_1 \times_V F_2$ in the sense of [BR] §3.7.

Similarly we have a derived category

$$\mathcal{D}^c_{G \times G_m}((\Delta V^* \times X \times X))^R_{\mathcal{E}^* \times E^*} (F_1^\perp \times F_2^\perp)$$

We denote by $\omega_X$ the canonical line bundle on $X$. Then by [MR3] Theorem 3.1 there exists a natural equivalence of triangulated categories

$$\mathcal{R}_{F_1,F_2} : \mathcal{D}^c_{G \times G_m}((\Delta V \times X \times X))^R_{\mathcal{E} \times E} (F_1 \times F_2) \sim \mathcal{D}^c_{G \times G_m}((\Delta V^* \times X \times X))^R_{\mathcal{E}^* \times E^*} (F_1^\perp \times F_2^\perp)^{\text{op}}.$$

More precisely, [MR3] Theorem 3.1 provides an equivalence of categories

$$\kappa_{F_1,F_2} : \mathcal{D}^c_{G \times G_m}((\Delta V \times X \times X))^R_{\mathcal{E} \times E}(F_1 \times F_2) \sim \mathcal{D}^c_{G \times G_m}((\Delta V^* \times X \times X))^R_{\mathcal{E}^* \times E^*}(F_1^\perp \times F_2^\perp)^{\text{op}}$$

where $\overline{\Delta V^*} \subset V^* \times V^*$ is the antidiagonal copy of $V^*$. (The construction of [MR3] depends on the choice of an object $\mathcal{E}$ in $D^b \mathcal{Coh}^G(X \times X)$ whose image in $D^b \mathcal{Coh}(X \times X)$ is a dualizing object; here we take $\mathcal{E} = \mathcal{O}_X \boxtimes_{\omega_X} [\text{dim}(X)]$.) Then $\mathcal{R}_{F_1,F_2}$ is the composition of $\kappa_{F_1,F_2}$ with the natural equivalence

$$\mathcal{D}^c_{G \times G_m}((\Delta V^* \times X \times X))^R_{\mathcal{E}^* \times E^*}(F_1^\perp \times F_2^\perp) \sim \mathcal{D}^c_{G \times G_m}((\Delta V^* \times X \times X))^R_{\mathcal{E}^* \times E^*}(F_1^\perp \times F_2^\perp)^{\text{op}}$$

(see [MR3] §4.3) and the natural equivalence

$$\mathcal{D}^c_{G \times G_m}((\Delta V \times X \times X))^R_{\mathcal{E} \times E}(F_1 \times F_2^\perp) \sim \mathcal{D}^c_{G \times G_m}((\Delta V \times X \times X))^R_{\mathcal{E} \times E}(F_1 \times F_2^\perp)^{\text{op}}$$

induced by the automorphism of $G_m$ sending $t$ to $t^{-1}$.

Note that the Grothendieck group of the triangulated category $\mathcal{D}^c_{G \times G_m}((\Delta V \times X \times X))^R_{\mathcal{E} \times E}(F_1 \times F_2)$ is naturally isomorphic to $K^{G \times G_m}(F_1 \times V F_2)$, see [MR3] Lemma 5.1. We have a similar isomorphism for $F_1^\perp$ and $F_2^\perp$. Hence the equivalence $\mathcal{R}_{F_1,F_2}$ induces an isomorphism

$$\text{Koszul}_{F_1,F_2} : \mathcal{K}^{G \times G_m}(F_1 \times V F_2) \sim \mathcal{K}^{G \times G_m}(F_1^\perp \times V F_2^\perp).$$
1.6. Grothendieck–Serre duality. Consider the “duality” equivalence
\[ D^{G \times G_m}_{F_1 \times F_2} : \mathcal{D}^b \mathcal{Coh}^{G \times G_m}_{F_1 \times \times F_2}(F_1 \times F_2^\perp) \to \mathcal{D}^b \mathcal{Coh}^{G \times G_m}_{F_1 \times \times F_2}(F_1 \times F_2^\perp)^{\text{op}} \]
associated with the dualizing complex \( \mathcal{O}_{F_1 \times \times F_2} \boxtimes \omega_{F_2}(\dim(F_2^\perp)) \), which sends \( \mathcal{G} \) to
\[ R\text{Hom}_{\mathcal{O}_{F_1 \times F_2}^\perp}(\mathcal{G}, \mathcal{O}_{F_1 \times \times F_2} \boxtimes \omega_{F_2}(\dim(F_2^\perp))) \]
(see e.g. [MR3, §2.1] and references therein). (Here, \( \omega_{F_2} \) is the canonical line bundle on \( F_2^\perp \), endowed with its natural \( G \times G_m \)-equivariant structure.) This equivalence induces an isomorphism in K-theory which we denote by
\[ (1.6.1) \quad D^{G \times G_m}_{F_1 \times F_2} : K^{G \times G_m}(F_1 \times_{V^*} F_2^\perp) \xrightarrow{\sim} K^{G \times G_m}(F_1 \times_{V^*} F_2^\perp). \]

1.7. Riemann–Roch maps. Following [CG §5.11], we consider the “bivariant Riemann–Roch maps”
\[ \text{RR}_{F_1, F_2} : K^{G \times G_m}(F_1 \times V \times F_2) \to \hat{H}^{G \times G_m}(F_1 \times V \times F_2), \]
\[ \text{RR}_{F_1^\perp, F_2^\perp} : K^{G \times G_m}(F_1^\perp \times_{V^*} F_2^\perp) \to \hat{H}^{G \times G_m}(F_1^\perp \times_{V^*} F_2^\perp) \]
defined by
\[ \text{RR}_{F_1, F_2}(c) = (1 \boxtimes (\text{Td}_{F_2}^{G \times G_m}))^{-1} \cdot \tau^{G \times G_m}(c), \]
\[ \text{RR}_{F_1^\perp, F_2^\perp}(d) = ((\text{Td}_{F_1^\perp}^{G \times G_m})^{-1} \cdot \text{Td}_X^{G \times G_m} \boxtimes (\text{Td}_X^{G \times G_m})^{-1}) \cdot \tau^{G \times G_m}(d). \]
In the expression for \( \text{RR}_{F_1, F_2} \), \( 1 \boxtimes (\text{Td}_{F_2}^{G \times G_m})^{-1} \) is considered as an element of \( \hat{H}^{G \times G_m}(F_1 \times V \times F_2) \) through the composition
\[ \hat{H}^*(G \times G_m)^2(F_1 \times F_2) \to \hat{H}^{G \times G_m}(F_1 \times F_2) \to \hat{H}^{G \times G_m}(F_1 \times V \times F_2) \]
where the first morphism is the restriction of scalars for the diagonal embedding of \( G \times G_m \), and the second morphism is the pullback in equivariant cohomology. In the expression for \( \text{RR}_{F_1^\perp, F_2^\perp} \), first we consider \( \text{Td}_X^{G \times G_m} \) as an element of \( \hat{H}^{G \times G_m}(E^*) \) using the Thom isomorphism \( \hat{H}^{G \times G_m}(E^*) \xrightarrow{\sim} \hat{H}^{G \times G_m}(X) \); then the same conventions as above allow to consider \( (\text{Td}_{F_1^\perp}^{G \times G_m})^{-1} \cdot \text{Td}_X^{G \times G_m} \boxtimes (\text{Td}_X^{G \times G_m})^{-1} \) as an element in \( \hat{H}^{G \times G_m}(F_1^\perp \times_{V^*} F_2^\perp) \).

1.8. Statement. The main result of this paper is the following.

Theorem 1.8.1. Assume that the pushforward morphism
\[ (1.8.2) \quad h^{G \times G_m}(F_1^\perp \times_{V^*} F_2^\perp) \to h^{G \times G_m}(F_1^\perp \times_{V^*} E^*) \]
induced by the inclusion \( F_2^\perp \hookrightarrow E^* \) is injective. Then the following diagram commutes:
\[ \begin{array}{ccc}
K^{G \times G_m}(F_1 \times V \times F_2) & \xrightarrow{\text{Koszul}_{F_1, F_2}} & K^{G \times G_m}(F_1^\perp \times_{V^*} F_2^\perp) \\
\text{RR}_{F_1, F_2} & & \text{RR}_{F_1^\perp, F_2^\perp}
\end{array} \]
\[ \begin{array}{ccc}
\hat{H}^{G \times G_m}(F_1 \times V \times F_2) & \xrightarrow{\text{Fourier}_{F_1, F_2}} & \hat{H}^{G \times G_m}(F_1^\perp \times_{V^*} F_2^\perp)
\end{array} \]
The proof of Theorem 1.8.1 is given in §1.3. It is based on compatibility (or functoriality) results for all the maps considered in the diagram, which are stated in Sections 2 and 3; some of these results might be of independent interest. Let us point out that our assumption is probably not needed for the result to hold.

Remark 1.8.3. The fiber product $F_1 \times_{V^*} E^*$ is isomorphic to $F_1 \times X$, hence is a vector bundle over $X^2$. In particular, by the Thom isomorphism we have

\[(1.8.4) \quad H^G_{\ast \times G_m}(F_1 \times_{V^*} E^*) \cong H^G_{\ast \times G_m}(X \times X).\]

Moreover, by [CG Lemma 5.4.35] the following diagram commutes:

\[(1.8.5) \quad \begin{array}{ccc}
H^G_{\ast \times G_m}(F_1 \times_{V^*} F_2) & \to & \text{Hom}_{H^G_{\ast \times G_m}(pt)}(H^G_{\ast \times G_m}(F_2), H^G_{\ast \times G_m}(F_1)) \\
\downarrow & & \downarrow \\
H^G_{\ast \times G_m}(X \times X) & \to & \text{Hom}_{H^G_{\ast \times G_m}(pt)}(H^G_{\ast \times G_m}(X), H^G_{\ast \times G_m}(X)).
\end{array}\]

Here the horizontal arrows are induced by convolution, the left vertical arrow is induced (via isomorphism (1.8.4)) by (1.8.2), and the right vertical arrow is induced by the respective Thom isomorphisms. Assume now that $H^G_{\ast \times G_m}(X) = 0$ (e.g. that $X$ is paved by affine spaces). Then one can easily check that the lower horizontal arrow in diagram (1.8.5) is an isomorphism. Hence in this case our assumption is equivalent to injectivity of the upper horizontal arrow. If moreover $F_1 = F_2 = F$, then $H^G_{\ast \times G_m}(F_1 \times_{V^*} F_1)$ is an algebra and $H^G_{\ast \times G_m}(F_1)$ is a module over this algebra. In this case our assumption means that the action on this module is faithful.

1.9. An injectivity criterion for (1.8.2). The following result gives an easy criterion which ensures that the assumption of Theorem 1.8.1 is satisfied.

**Proposition 1.9.1.** Assume that $H^G_{\ast \times G_m}(F_1 \times_{V^*} F_2) = 0$. Then the pushforward morphism

$$(1.9.1) \quad H^G_{\ast \times G_m}(F_1 \times_{V^*} F_2) \to H^G_{\ast \times G_m}(F_1 \times_{V^*} E^*)$$

induced by the inclusion $F_2 \hookrightarrow E^*$ is injective.

**Proof.** Let $T$ be a maximal torus of $G$. Then we have a commutative diagram

\[
\begin{array}{ccc}
H^G_{\ast \times G_m}(F_1 \times_{V^*} F_2) & \to & H^G_{\ast \times G_m}(F_1 \times_{V^*} E^*) \\
\downarrow & & \downarrow \\
H^T_{\ast \times G_m}(F_1 \times_{V^*} F_2) & \to & H^T_{\ast \times G_m}(F_1 \times_{V^*} E^*)
\end{array}
\]

where horizontal arrows are pushforward morphisms, and vertical arrows are forgetful maps. The left vertical arrow is injective: indeed, by our assumption and [L2 Proposition 7.2], there exist (non-canonical) isomorphisms

\[
(1.9.2) \quad H^G_{\ast \times G_m}(F_1 \times_{V^*} F_2) \cong H^G_{\ast \times G_m}(pt) \otimes_C H^G_{\ast \times G_m}(F_1 \times_{V^*} F_2),
\]

\[
(1.9.3) \quad H^T_{\ast \times G_m}(F_1 \times_{V^*} F_2) \cong H^T_{\ast \times G_m}(pt) \otimes_C H^T_{\ast \times G_m}(F_1 \times_{V^*} F_2)
\]
such that our forgetful morphism is induced by the natural morphism $H^*_G \times Gm (pt) \to H^*_T \times Gm (pt)$, which is well known to be injective. Hence, to prove that the upper horizontal arrow is injective it is sufficient to prove that the lower horizontal arrow is injective.

If $Q$ denotes the fraction field of $H := H^*_T \times Gm (pt)$, then using again isomorphism \[1.93\], the natural morphism

$$H^*_T \times Gm(F^\perp_1 \times V^* F^\perp_2) \to Q \otimes_H H^*_T \times Gm(F^\perp_1 \times V^* F^\perp_2)$$

is injective. We deduce that to prove the proposition it suffices to prove that the induced morphism

$$Q \otimes_H H^*_T \times Gm(F^\perp_1 \times V^* F^\perp_2) \to Q \otimes_H H^*_T \times Gm(F^\perp_1 \times V^* E^*)$$

is injective. Let $Y := (X \times X)^T$ denote the $T$-invariants in $X \times X$. Then we have

$$Y = (F^\perp_1 \times V^* F^\perp_2)^T \times Gm = (F^\perp_1 \times V^* E^*)^T \times Gm.$$

Consider the commutative diagram

\[
\begin{array}{ccc}
H^*_T \times Gm(F^\perp_1 \times V^* F^\perp_2) & \xrightarrow{\alpha} & H^*_T \times Gm(F^\perp_1 \times V^* E^*) \\
\downarrow{\beta} & & \downarrow{\gamma} \\
H^*_T \times Gm(Y) & & 
\end{array}
\]

where all morphisms are pushforward in homology. Then by the localization theorem (see \[L3\], Proposition 4.4 or \[EM\], Theorem B.2) both $\beta$ and $\gamma$ become isomorphisms after applying $Q \otimes_H (\cdot)$. Hence the same is true for $\alpha$; in particular $Q \otimes_H \alpha$ is injective, which finishes the proof. $\square$

**Remark 1.9.4.** Using a non-equivariant variant of isomorphism Fourier$_{F_1,F_2}$, one can check that the condition $H^c\text{odd}(F^\perp_1 \times V^* F^\perp_2) = 0$ is equivalent to the condition $H^c\text{odd}(F_1 \times V F_2) = 0$.

### 1.10. The case of affine Hecke algebras

Our main motivation for studying the commutativity of the diagram in Theorem \[1.8.1\] comes from the following geometric situation. Assume that $G$ is semisimple and simply connected, and let $g$ be its Lie algebra. Choose a Borel subgroup $B \subset G$ with Lie algebra $\mathfrak{b}$, and let $B := G/B$ be the flag variety of $G$. One can apply the constructions of Theorem \[1.8.1\] in the case $X = B$, $V = g^*$, and when $F$ is the subbundle

$$\tilde{N} := \{(\xi, gB) \in g^* \times B \mid \xi_{[g, b]} = 0\}.$$

(This variety is isomorphic to the Springer resolution of the nilpotent cone.) The Killing form defines a $G$-equivariant isomorphism $(g^*)^* \cong g^*$, hence a $G \times Gm$-equivariant isomorphism $E \cong E^*$. Via this isomorphism, the orthogonal $F^\perp$ identifies with

$$\tilde{g} := \{(\xi, gB) \in g^* \times B \mid \xi_{[g, b]} = 0\}.$$

(This variety is isomorphic to the Grothendieck simultaneous resolution.) Consider the varieties

$$Z := \tilde{N} \times g^* \tilde{N}, \quad Z := \tilde{g} \times g^* \tilde{g}.$$

Then the $G \times Gm$-equivariant K-theory groups $K^{G \times Gm}(Z)$ and $K^{G \times Gm}(Z)$ are both endowed with a convolution product, and are both isomorphic to the affine Hecke algebra $\mathcal{H}_{aff}$ associated with $G$ through Kazhdan–Lusztig–Ginzburg’s isomorphism; see \[MR3\] §5.2 for details and references. On the other hand, the Borel–Moore homology groups $H^*_G \times Gm(Z)$ and $H^*_G \times Gm(Z)$ can also be...
endowed with a convolution product, and are isomorphic to Lusztig’s graded affine Hecke algebra $\mathcal{H}_{\text{aff}}$ associated with $G$, see [L2, L3]. Moreover, the Riemann–Roch morphisms can be identified with some version of the morphism from $\mathcal{H}_{\text{aff}}$ to a completion of $\mathcal{H}_{\text{aff}}^\text{op}$ defined in [L1].

The morphism induced by linear Koszul duality in equivariant K-theory (as in §1.3) in this geometric context is studied in [MR3], and shown to coincide with (a modification of) the Iwahori–Matsumoto involution of $\mathcal{H}_{\text{aff}}$. On the other hand, the Fourier isomorphism of [L4] is studied in this context in [EM], and shown to coincide with (a modification of) the Iwahori–Matsumoto involution of $\mathcal{H}_{\text{aff}}^\text{op}$. In this situation Proposition [L9.1] ensures that the assumption of Theorem [L8.1] is satisfied, since $Z$ is paved by affine spaces. Hence, in this particular case, Theorem [L8.1] explains the relation between these “categorification” results of [MR3] and [EM].

2. Compatibility of the Fourier isomorphism with inclusions

In this section and the next one we will consider compatibility properties of our morphisms in two geometric situations. We use the same setting and notation as in §§1.3, 1.8.

2.1. Further notation. First we will consider a situation which we will refer to as Setting (A): here we are given an additional subbundle $F'_2 \subset E$ containing $F_2$ and such that $F_2, F'_2$ and $E$ can be locally simultaneously trivialized. In this case we will consider the morphisms

\begin{equation}
K^G_{\times \mathbb{G}_m}(F_1 \times V F'_2) \to K^G_{\times \mathbb{G}_m}(F'_1 \times V F_2)
\end{equation}

(restriction with supports in K-theory induced by the inclusion $F_2 \hookrightarrow F'_2$),

\begin{equation}
H^G_{\times \mathbb{G}_m}(F_1 \times V F_2) \to H^G_{\times \mathbb{G}_m}(F'_1 \times V F_2)
\end{equation}

(restriction with supports in Borel–Moore homology induced by the inclusion $F_2 \hookrightarrow F'_2$),

\begin{equation}
K^G_{\times \mathbb{G}_m}(F'_1 \times V^* F'_2) \to K^G_{\times \mathbb{G}_m}(F'_1 \times V^* F_2)
\end{equation}

(pushforward in K-theory induced by the inclusion $(F'_2)^\perp \hookrightarrow F'_2$) and

\begin{equation}
H^G_{\times \mathbb{G}_m}(F'_1 \times V^* F'_2) \to H^G_{\times \mathbb{G}_m}(F'_1 \times V^* F_2)
\end{equation}

(pushforward in Borel–Moore homology induced by the inclusion $(F'_2)^\perp \hookrightarrow F'_2$).

We will also consider a situation which we will refer to as Setting (B): here we are given an additional subbundle $F'_1 \subset E$ containing $F_1$ and such that $F_1, F'_1$ and $E$ can be locally simultaneously trivialized. In this case we will consider the morphisms

\begin{equation}
K^G_{\times \mathbb{G}_m}(F_1 \times V F_2) \to K^G_{\times \mathbb{G}_m}(F'_1 \times V F_2)
\end{equation}

(pushforward in K-theory induced by the inclusion $F_1 \hookrightarrow F'_1$),

\begin{equation}
H^G_{\times \mathbb{G}_m}(F_1 \times V F_2) \to H^G_{\times \mathbb{G}_m}(F'_1 \times V F_2)
\end{equation}

(pushforward in Borel–Moore homology induced by the inclusion $F_1 \hookrightarrow F'_1$),

\begin{equation}
K^G_{\times \mathbb{G}_m}(F'_1 \times V^* F'_2) \to K^G_{\times \mathbb{G}_m}(F'_1 \times V^* F_2)
\end{equation}

(restriction with supports in K-theory induced by the inclusion $(F'_1)^\perp \hookrightarrow F'_1$) and

\begin{equation}
H^G_{\times \mathbb{G}_m}(F'_1 \times V^* F'_2) \to H^G_{\times \mathbb{G}_m}(F'_1 \times V^* F_2)
\end{equation}

(restriction with supports in Borel–Moore homology induced by the inclusion $(F'_1)^\perp \hookrightarrow F'_1$).
2.2. Convolution algebras and inclusion of subbundles. Consider Setting (A) of §2.1. Then we have natural morphisms induced by adjunction

$$\text{adj}_{F_2,F_2}^*: \mathbb{C}^*_{F_2^*} \to \mathbb{C}^*_{F_2} \quad \text{and} \quad \text{adj}_{(F_2)^*,F_2}^*: \mathbb{C}^*_{(F_2)^*} \to \mathbb{C}^*_{F_2^*} [2\text{rk}(F_2^*) - 2\text{rk}((F_2)^*)].$$

The proof of the following result being rather technical (and the details not needed), it is postponed to the appendix (see §A.6–A.7).

**Proposition 2.2.1.**

1. The following diagram commutes:

$$
\begin{array}{ccc}
\text{H}^\bullet_{\mathbb{C}^*}(F_1 \times V, F_2) & \xrightarrow{\text{(1.3.1)}} & \text{Ext}^{2\dim(F_2^*)}_{\mathbb{C}^*}(V) (p \mathbb{C}^*_{F_1}, p \mathbb{C}^*_{F_2}) \\
\text{H}^\bullet_{\mathbb{C}^*}(F_1 \times V, F_2) & \xrightarrow{\text{(1.3.1)}} & \text{Ext}^{2\dim(F_2^*)}_{\mathbb{C}^*}(V) (p \mathbb{C}^*_{F_1}, p \mathbb{C}^*_{F_2}).
\end{array}
$$

2. The following diagram commutes:

$$
\begin{array}{ccc}
\text{H}^\bullet_{\mathbb{C}^*}(F_1^\perp \times V^*, (F_2')^\perp) & \xrightarrow{\text{(1.3.1)}} & \text{Ext}^{2\dim((F_2')^\perp)}_{\mathbb{C}^*}(V^*) (p \mathbb{C}^*_{F_1}, p \mathbb{C}^*_{(F_2')^\perp}) \\
\text{H}^\bullet_{\mathbb{C}^*}(F_1^\perp \times V^*, (F_2')^\perp) & \xrightarrow{\text{(1.3.1)}} & \text{Ext}^{2\dim((F_2')^\perp)}_{\mathbb{C}^*}(V^*) (p \mathbb{C}^*_{F_1}, p \mathbb{C}^*_{(F_2')^\perp}).
\end{array}
$$

Consider now Setting (B) of §2.1. We have natural morphisms induced by adjunction

$$\text{adj}_{F_1,F_1}^*: \mathbb{C}^*_{F_1^*} \to \mathbb{C}^*_{F_1} \quad \text{and} \quad \text{adj}_{(F_1)^*,F_1}^*: \mathbb{C}^*_{(F_1)^*} \to \mathbb{C}^*_{F_1^*} [2\text{rk}(F_1^*) - 2\text{rk}((F_1)^*)].$$

The proof of the following proposition is similar to that of Proposition 2.2.1 and is therefore omitted.

**Proposition 2.2.2.**

1. The following diagram commutes:

$$
\begin{array}{ccc}
\text{H}^\bullet_{\mathbb{C}^*}(F_1 \times V, F_2) & \xrightarrow{\text{(1.3.1)}} & \text{Ext}^{2\dim(F_2^*)}_{\mathbb{C}^*}(V) (p \mathbb{C}^*_{F_1}, p \mathbb{C}^*_{F_2}) \\
\text{H}^\bullet_{\mathbb{C}^*}(F_1 \times V, F_2) & \xrightarrow{\text{(1.3.1)}} & \text{Ext}^{2\dim(F_2^*)}_{\mathbb{C}^*}(V) (p \mathbb{C}^*_{F_1}, p \mathbb{C}^*_{F_2}).
\end{array}
$$

2. The following diagram commutes:

$$
\begin{array}{ccc}
\text{H}^\bullet_{\mathbb{C}^*}(F_1^\perp \times V^*, F_2^\perp) & \xrightarrow{\text{(1.3.1)}} & \text{Ext}^{2\dim((F_2)^\perp)}_{\mathbb{C}^*}(V^*) (p \mathbb{C}^*_{F_1}, p \mathbb{C}^*_{(F_2)^\perp}) \\
\text{H}^\bullet_{\mathbb{C}^*}(F_1^\perp \times V^*, F_2^\perp) & \xrightarrow{\text{(1.3.1)}} & \text{Ext}^{2\dim((F_2)^\perp)}_{\mathbb{C}^*}(V^*) (p \mathbb{C}^*_{F_1}, p \mathbb{C}^*_{(F_2)^\perp}).
\end{array}
$$
2.3. Fourier transform and inclusion of subbundles. In the next lemma, $E$ can be an arbitrary $G$-equivariant vector bundle on an arbitrary smooth $G$-variety $X$. We consider subbundles $F \subset F' \subset E$ which can be locally simultaneously trivialized. (In practice, $E$ and $X$ will be as above, and we will take $F = F_i$, $F' = F'_i$ for $i \in \{1, 2\}$.) Adjunction induces morphisms

$$\text{adj}^i_{\mathcal{E}} : \mathcal{E}_{\mathcal{E}} \rightarrow \mathcal{F}_{\mathcal{F}} \quad \text{and} \quad \text{adj}^i_{\mathcal{E}} : \mathcal{E}_{\mathcal{E}} \rightarrow \mathcal{F}_{\mathcal{F}} \mathcal{H}_{\mathcal{H}} \{2\text{rk}(F') - 2\text{rk}(F')\}.$$

**Lemma 2.3.1.** The following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{E}_{\mathcal{E}} \rightarrow & \mathcal{F}_{\mathcal{F}} & \rightarrow \mathcal{F}_{\mathcal{F}} \\
\downarrow & \downarrow & \downarrow \\
\mathcal{E}_{\mathcal{E}} \mathcal{H}_{\mathcal{H}} \{2\text{rk}(F')\} & \mathcal{E}_{\mathcal{E}} \mathcal{H}_{\mathcal{H}} \{2\text{rk}(F)\} & \\
\end{array}
$$

where vertical isomorphisms are provided by Lemma 1.2.3.

**Proof.** It is equivalent to prove a similar isomorphism for $\mathcal{E}_E$; for simplicity we still denote by $F'$, $(F')$ the orthogonals viewed in $E^\circ$, and by $\mathcal{E}_E \rightarrow X$ the projection. By the construction of Lemma 1.2.3 we have natural isomorphisms

$$\mathcal{E}_E(\mathcal{E}_{\mathcal{E}}) \cong \mathcal{E}_E(\mathcal{E}_{\mathcal{E}}) \quad \text{and} \quad \mathcal{E}_E(\mathcal{E}_{\mathcal{E}}) \cong \mathcal{E}_E(\mathcal{E}_{\mathcal{E}}),$$

where $Q_{F'} := q^{-1}(F')$, $Q_{F} := q^{-1}(F)$. It follows from the definitions that the morphism $\mathcal{E}_E(\mathcal{E}_{\mathcal{E}})$ is the image under $q$ of the morphism $\mathcal{E}_E(\mathcal{E}_{\mathcal{E}}) \rightarrow \mathcal{E}_E(\mathcal{E}_{\mathcal{E}})$ induced by adjunction (for the inclusion $Q_{F} \hookrightarrow Q_{F'}$). Hence we want to describe the morphism $\varphi$ in the following diagram, where the upper arrow is induced by adjunction as above, and the vertical isomorphisms are as in the proof of Lemma 1.2.3.

$$
\begin{array}{ccc}
\mathcal{E}_E(\mathcal{E}_{\mathcal{E}}) & \mathcal{E}_E(\mathcal{E}_{\mathcal{E}}) & \\
\downarrow & \downarrow & \downarrow \\
\mathcal{E}_E(\mathcal{E}_{\mathcal{E}}) & \mathcal{E}_E(\mathcal{E}_{\mathcal{E}}) & \\
\end{array}
$$

Now we have canonical isomorphisms

$$\mathcal{E}_E(\mathcal{E}_{\mathcal{E}}) \cong \mathcal{E}_E(\mathcal{E}_{\mathcal{E}}), \quad \mathcal{E}_E(\mathcal{E}_{\mathcal{E}}) \cong \mathcal{E}_E(\mathcal{E}_{\mathcal{E}}),$$

and one can check that the functor $\mathcal{E}_E$ induces an isomorphism

$$\text{Hom}_{\text{const}}(E) \mathcal{E}_E(\mathcal{E}_{\mathcal{E}}) \cong \mathcal{E}_E(\mathcal{E}_{\mathcal{E}}), \quad \mathcal{E}_E(\mathcal{E}_{\mathcal{E}}) \cong \mathcal{E}_E(\mathcal{E}_{\mathcal{E}}),$$

sending $\text{adj}^i_{\mathcal{E}}$ to the identity morphism of $\mathcal{E}_E(\mathcal{E}_{\mathcal{E}})$. Hence it is enough to prove that $\varphi$ is the identity morphism of $\mathcal{E}_E(\mathcal{E}_{\mathcal{E}})$. (This is allowed by combining [BL] Proposition 2.5.3)
By local triviality, one can then assume that \( X = \text{pt} \) (i.e. that \( E \) is a vector space and that \( F, F' \subset E \) are subspaces).

In this case the claim boils down to the fact that the dotted arrow in the following diagram is the identity:

\[
\begin{array}{ccc}
\mathsf{H}^2_{\text{c}}(E'; (F')^\perp) & \xrightarrow{\sim} & \mathsf{H}^2_{\text{c}}(Q) \\
\downarrow & & \downarrow \\
\mathsf{H}^2_{\text{c}}(E; (F \times F')^\perp) & \xrightarrow{\sim} & \mathsf{H}^2_{\text{c}}(Q) \\
\end{array}
\]

To prove this fact we regard \( E \times E^* \) as a real vector space, endowed with the non-degenerate quadratic form given by \( q(x, \xi) := \text{Re}(\langle \xi, x \rangle) \). The orthogonal group \( H \) of this form stabilizes \( Q \), hence acts on \( \mathsf{H}^2_{\text{c}}(E; (F \times F')^\perp) \), and this action factors through the group of components \( H/H^\circ \). Now \( F \times F^\perp \) and \( F' \times (F')^\perp \) are conjugate under the action of \( H^\circ \), with finishes the proof. \( \square \)

In the following proposition we get back to the assumption that \( E = V \times X \), and we let \( p : E \to V \) be the projection. The following result is an immediate consequence of Lemma 2.3.1 and the isomorphism of functors \( F_V \circ p_! \cong \check{p} \circ F_E \), see the proof of Corollary 1.2.4.

**Proposition 2.3.2.** The following diagram is commutative:

\[
\begin{array}{ccc}
F_V(p_! F) & \xrightarrow{F_V(p_!(\text{adj}_{F,F'}^*))} & F_V(p_! \mathbb{C}_F) \\
\downarrow^{1.2.5} & & \downarrow^{1.4.5} \\
\check{p}_!(-2\text{rk}(F')) & \xrightarrow{\check{p}_!(\text{adj}_{F,F'}^*)_{\perp,F'}} & \check{p}_! \mathbb{C}_F[-2\text{rk}(F)].
\end{array}
\]

2.4. The Fourier isomorphism and inclusion of subbundles. We come back to Setting (A) of 2.3.

**Proposition 2.4.1.** The following diagram commutes:

\[
\begin{array}{ccc}
\mathsf{H}^\bullet_{G \times \mathbb{G}_m}(F_1 \times V F_2') & \xrightarrow{\text{Fourier}_{F_1,F_2'}} & \mathsf{H}^\bullet_{G \times \mathbb{G}_m}(F_1 \times V +2\text{dim}((F_2')^\perp)-2\text{dim}(F_1) (F_1^\perp \times V^* (F_2')^\perp) \\
\downarrow^{2.4.2} & & \downarrow^{2.4.4} \\
\mathsf{H}^\bullet_{G \times \mathbb{G}_m}(F_1 \times V F_2) & \xrightarrow{\text{Fourier}_{F_1,F_2}} & \mathsf{H}^\bullet_{G \times \mathbb{G}_m}(F_1 \times V +2\text{dim}((F_2')^\perp)-2\text{dim}(F_1) (F_1^\perp \times V^* F_2^\perp). \\
\end{array}
\]

**Proof.** First, by Proposition 2.2.1(1) the following diagram commutes:

\[
\begin{array}{ccc}
\mathsf{H}^\bullet_{G \times \mathbb{G}_m}(F_1 \times V F_2') & \xrightarrow{1.3.1} & \text{Ext}^{2\text{dim}(F_2')\bullet}_{\mathcal{D}_{\text{const}}(V)}(p_! \mathbb{C}_{F_2}, p_! \mathbb{C}_{F_2'}) \\
\downarrow^{2.1.2} & & \downarrow^{(p_!(\text{adj}_{F_2,F_2'})^*)(\cdot)} \\
\mathsf{H}^\bullet_{G \times \mathbb{G}_m}(F_1 \times V F_2) & \xrightarrow{1.3.1} & \text{Ext}^{2\text{dim}(F_2')\bullet}_{\mathcal{D}_{\text{const}}(V)}(p_! \mathbb{C}_{F_2}, p_! \mathbb{C}_{F_2}).
\end{array}
\]
Then, by functoriality the following diagram commutes, where horizontal maps are induced by the functor $F$:

\[
\begin{array}{ccc}
\text{Ext}_{D^{G \times \mathbb{G}_m}}^{2 \dim(F_2) - \bullet}(p_1 \mathbb{G}_m F_1, p_2 \mathbb{G}_m F_2) & \overset{\sim}{\longrightarrow} & \text{Ext}_{D^{G \times \mathbb{G}_m}}^{2 \dim(F_2) - \bullet}(\mathbb{G}_m F_1, \mathbb{G}_m F_2) \\
(p_! \text{adj}^*_F F_2) \circ (\cdot) & & (p_! \text{adj}^*_F F_2) \circ (\cdot)
\end{array}
\]

By Proposition 2.3.2 the following diagram commutes, where vertical maps are induced by the isomorphisms $F_V(\mathbb{G}_m F_1) \cong \mathbb{G}_m F_1[-2 \text{rk}(F)]$ for $F_1, F_2$ or $F_2'$ (see (1.2.5)):

\[
\begin{array}{ccc}
\text{Ext}_{D^{G \times \mathbb{G}_m}}^{2 \dim(F_2) - \bullet}(\mathbb{G}_m F_1, \mathbb{G}_m F_2) & \overset{\sim}{\longrightarrow} & \text{Ext}_{D^{G \times \mathbb{G}_m}}^{2 \dim(F_2) - \bullet}(\mathbb{G}_m F_1, \mathbb{G}_m F_2) \\
F_V(p_! \text{adj}^*_F F_2) \circ (\cdot) & & (p_! \text{adj}^*_F F_2) \circ (\cdot)
\end{array}
\]

Finally, by Proposition 2.2.1(2) the following diagram commutes:

\[
\begin{array}{ccc}
\text{Ext}_{D^{G \times \mathbb{G}_m}}^{2 \dim(F_1) - \bullet}(\mathbb{G}_m F_1, \mathbb{G}_m F_2) & \overset{\sim}{\longrightarrow} & \text{Ext}_{D^{G \times \mathbb{G}_m}}^{2 \dim(F_1) - \bullet}(\mathbb{G}_m F_1, \mathbb{G}_m F_2) \\
(p_! \text{adj}^*_F F_2) \circ (\cdot) & & (p_! \text{adj}^*_F F_2) \circ (\cdot)
\end{array}
\]

Pasting all these diagrams provides the result. \qed

Now we consider Setting (B) of \[2.1\]. The proof of the following proposition is similar to that of Proposition 2.4.1 (replacing Proposition 2.2.1 by Proposition 2.2.2), and is therefore omitted.

**Proposition 2.4.2.** The following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{H}^{G \times \mathbb{G}_m}(F_1 \times V F_2) & \overset{\sim}{\longrightarrow} & \mathbb{H}^{G \times \mathbb{G}_m}(F_1 \times V F_2) \\
\overset{\text{Fourier}_{F_1 F_2}}{\longrightarrow} & & \overset{\text{Fourier}_{F_1 F_2}}{\longrightarrow} \\
\mathbb{H}^{G \times \mathbb{G}_m}(F_1' \times V F_2) & \overset{\sim}{\longrightarrow} & \mathbb{H}^{G \times \mathbb{G}_m}(F_1' \times V F_2)
\end{array}
\]

3. Compatibility of the remaining constructions with inclusions

### 3.1. Compatibility for linear Koszul duality.

Consider Setting (A) of \[2.1\]. Then we have equivalences of triangulated categories $\mathbb{R}_{F_1 F_2}$ and $\mathbb{R}_{F_1' F_2}$ constructed as in \[1.5\]. We also have
Proposition 3.1.2. The following diagram commutes:

\[
\begin{array}{ccc}
K^{G \times \mathbb{G}_m}(F_1 \times F_2) & \xrightarrow{\text{Koszul}_{F_1,F_2}} & K^{G \times \mathbb{G}_m}(F_1^\perp \times V^*, (F_2')^\perp) \\
\downarrow \text{(2.1.1)} & & \downarrow \text{(2.1.3)} \\
K^{G \times \mathbb{G}_m}(F_1 \times V \times F_2) & \xrightarrow{\text{Koszul}_{F_1,F_2}} & K^{G \times \mathbb{G}_m}(F_1^\perp \times V^*, F_2^\perp).
\end{array}
\]

Now, consider Setting (B) of §2.1. The same considerations as above allow to prove the following result.

Proposition 3.1.1. The following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{D}^c_{G \times \mathbb{G}_m}((\Delta V \times X \times X)^R_{E \times E}(F_1 \times F_2)) & \xrightarrow{\text{Koszul}_{F_1,F_2}} & \mathcal{D}^c_{G \times \mathbb{G}_m}((\Delta V \times X \times X)^R_{E \times E}(F_1 \times F_2)) \\
\downarrow \text{(2.1.3)} & & \downarrow \text{(2.1.7)} \\
\mathcal{D}^c_{G \times \mathbb{G}_m}((\Delta V \times X \times X)^R_{E \times E}(F_1 \times F_2)) & \xrightarrow{\text{Koszul}_{F_1,F_2}} & \mathcal{D}^c_{G \times \mathbb{G}_m}((\Delta V \times X \times X)^R_{E \times E}(F_1 \times F_2)).
\end{array}
\]

3.2. Compatibility for duality. Consider Setting (A) of §2.1

Proposition 3.2.1. The following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{D}^c_{G \times \mathbb{G}_m}((\Delta V \times X \times X)^R_{E \times E}(F_1 \times F_2)) & \xrightarrow{\text{Koszul}_{F_1,F_2}} & \mathcal{D}^c_{G \times \mathbb{G}_m}((\Delta V \times X \times X)^R_{E \times E}(F_1 \times F_2)) \\
\downarrow \text{(2.1.3)} & & \downarrow \text{(2.1.7)} \\
\mathcal{D}^c_{G \times \mathbb{G}_m}((\Delta V \times X \times X)^R_{E \times E}(F_1 \times F_2)) & \xrightarrow{\text{Koszul}_{F_1,F_2}} & \mathcal{D}^c_{G \times \mathbb{G}_m}((\Delta V \times X \times X)^R_{E \times E}(F_1 \times F_2)).
\end{array}
\]
Proposition 3.2.2. \[\text{The following diagram commutes:}\]

![Diagram](image)

Consider now Setting (B) of §2.1.

Proposition 3.3.1. \[\text{Assume that the pushforward morphism}\]

\[H_\bullet^{G\times \mathbb{G}_m}(F_1 \times V \cdot F_2) \to H_\bullet^{G\times \mathbb{G}_m}(F_1 \times F_2)\]

is injective. Then the following diagram commutes:

![Diagram](image)
Proof. Consider the following cube:

\[
\begin{array}{ccc}
K^{G \times \mathbb{G}_m}(F_1 \times V F_2') & \xrightarrow{\mathbb{R}R_{F_1,F_2'}} & \hat{H}^{G \times \mathbb{G}_m}(F_1 \times V F_2') \\
\downarrow & & \downarrow \\
K^{G \times \mathbb{G}_m}(F_1 \times V F_2) & \xrightarrow{(\cdot)^*} & \hat{H}^{G \times \mathbb{G}_m}(F_1 \times V F_2) \\
\downarrow & & \downarrow \\
K^{G \times \mathbb{G}_m}(F_1 \times F_2) & \xrightarrow{(\cdot)^*} & \hat{H}^{G \times \mathbb{G}_m}(F_1 \times F_2) \\
\end{array}
\]

Here the labels \((\cdot)^*\), resp.\(n (\cdot)^*\), indicate restriction with supports (always with respect to the morphism induced by \(F_2 \hookrightarrow F_2'\)), resp. pushforward, the arrow labelled by \((\cdot)\) is given by \((1 \boxtimes (Td_{G \times \mathbb{G}_m}^{-1})) \cdot \tau^{G \times \mathbb{G}_m}_{F_1 \times F_2}\), and the arrow labelled by \((\cdot)^*\) by \((1 \boxtimes (Td_{G \times \mathbb{G}_m}^{-1})) \cdot \tau^{G \times \mathbb{G}_m}_{F_1 \times F_2}\). The upper and lower faces of this cube commute by the equivariant Riemann–Roch theorem (see [BZ, Theorem 5.1]) and the projection formula. The left face commutes by definition, and the right one by Lemma [A.5.3]. The front face commutes by [EG, Theorem 3.1.(d)], using the equality

\[
(Td_{G \times \mathbb{G}_m}^{-1}) = \text{td}(N) \cdot ((Td_{G \times \mathbb{G}_m}^{-1})|_{F_2})
\]

(\(N\) is the normal bundle to \(F_2\) inside \(F_2'\)). Using our assumption, we deduce the commutativity of the back face, which finishes the proof. \(\square\)

Now, consider Setting (B) of §2.1

**Proposition 3.3.2.** The following diagram commutes:

\[
\begin{array}{ccc}
K^{G \times \mathbb{G}_m}(F_1 \times V F_2) & \xrightarrow{\mathbb{R}R_{F_1,F_2}} & \hat{H}^{G \times \mathbb{G}_m}(F_1 \times V F_2) \\
\downarrow & & \downarrow \\
K^{G \times \mathbb{G}_m}(F_1' \times V F_2') & \xrightarrow{\mathbb{R}R_{F_1',F_2'}} & \hat{H}^{G \times \mathbb{G}_m}(F_1' \times V F_2').
\end{array}
\]

**Proof.** This follows from the equivariant Riemann–Roch theorem, see [BZ Theorem 5.1]. \(\square\)

### 3.4. Compatibility for \(\mathbb{R}R\)

The proofs in this subsection are analogous to those of the corresponding statements in §3.3; they are therefore omitted.

First, consider Setting (A) of §2.1
Proposition 3.4.1. The following diagram commutes:

\[
\begin{array}{ccc}
K^{G \times \mathbb{G}_{m}}(F_1^\perp \times_{V^*} (F_2^\perp)) & \xrightarrow{RR\left(F_1^\perp, (F_2^\perp)^\perp\right)} & \hat{H}^{G \times \mathbb{G}_{m}}(F_1^\perp \times_{V^*} (F_2^\perp)) \\
\downarrow^{(2.1.3)} & & \downarrow^{(4.1.8)} \\
K^{G \times \mathbb{G}_{m}}(F_1^\perp \times_{V^*} F_2^\perp) & \xrightarrow{RR\left(F_1^\perp, F_2^\perp\right)} & \hat{H}^{G \times \mathbb{G}_{m}}(F_1^\perp \times_{V^*} F_2^\perp).
\end{array}
\]

Now, consider Setting (B) of §2.1

Proposition 3.4.2. Assume that the pushforward morphism

\[
H^{G \times \mathbb{G}_{m}}((F_1^\perp) \times_{V^*} F_2^\perp) \rightarrow H^{G \times \mathbb{G}_{m}}((F_1^\perp) \times F_2^\perp)
\]

is injective. Then the following diagram commutes:

\[
\begin{array}{ccc}
K^{G \times \mathbb{G}_{m}}(F_1^\perp \times_{V^*} F_2^\perp) & \xrightarrow{RR\left(F_1^\perp, F_2^\perp\right)} & \hat{H}^{G \times \mathbb{G}_{m}}(F_1^\perp \times_{V^*} F_2^\perp) \\
\downarrow^{(2.1.7)} & & \downarrow^{(4.1.8)} \\
K^{G \times \mathbb{G}_{m}}((F_1^\perp) \times_{V^*} F_2^\perp) & \xrightarrow{RR\left(F_1^\perp, (F_2^\perp)^\perp\right)} & \hat{H}^{G \times \mathbb{G}_{m}}((F_1^\perp) \times_{V^*} F_2^\perp).
\end{array}
\]

4. Proof of Theorem 1.8.1

4.1. A particular case. In this subsection we study the case when $F_1 = E$ and $F_2 = X$ (considered as the zero-section of $E$) so that $F_1^\perp = X$, $F_2^\perp = E^*$. In this case, the assumption of Theorem 1.8.1 is trivially satisfied.

Lemma 4.1.1. Under the identifications $E \times V^* X = X \times X = X \times V^* E^*$, the isomorphism

\[
\text{Fourier}_{E, X} : H^{G \times \mathbb{G}_{m}}(E \times V^* X) \xrightarrow{\sim} H^{G \times \mathbb{G}_{m}}(X \times V^* E^*)
\]

is the identity morphism of $H^{G \times \mathbb{G}_{m}}(X \times X)$.

Proof. Using isomorphism (1.3.1) in the case $V = \{0\}$, $F_1 = F_2 = X$ we obtain an isomorphism

\[
\alpha : H^{G \times \mathbb{G}_{m}}(X \times X) \xrightarrow{\sim} \text{Ext}^{2\dim(X)-\bullet}_{P^{G \times \mathbb{G}_{m}}(pt)}((p_0)_!\underline{C}_X, (p_0)_!\underline{C}_X),
\]

where $p_0 : X \rightarrow pt$ is the projection. Then the composition

\[
H^{G \times \mathbb{G}_{m}}(X \times X) = H^{G \times \mathbb{G}_{m}}(E \times V^* X) \cong \text{Ext}^{2\dim(X)-\bullet}_{P^{G \times \mathbb{G}_{m}}(V)}(p_!\underline{C}_E, p_!\underline{C}_V)
\]

sends each $c \in H^{G \times \mathbb{G}_{m}}(X \times X)$ to the morphism

\[
p_!\underline{C}_E = \underline{C}_V \boxtimes (p_0)_!\underline{C}_X \xrightarrow{\varphi \boxtimes (c)(\cdot)} \underline{C}_V \boxtimes (p_0)_!\underline{C}_X [2\dim(X) - i] = p_!\underline{C}_X [2\dim(X) - i]
\]

where $\varphi : \underline{C}_V \rightarrow \underline{C}_V(0)$ is the $(*, *)$-adjunction morphism for the inclusion $\{0\} \hookrightarrow V$, and we use the identification $\bar{V} = V \times pt$. Similarly, the composition

\[
H^{G \times \mathbb{G}_{m}}(X \times E^*) = H^{G \times \mathbb{G}_{m}}(X \times V^* E^*) \cong \text{Ext}^{2\dim(E^*+\bullet)-\bullet}_{P^{G \times \mathbb{G}_{m}}(V^*)}(p_!\underline{C}_E, p_!\underline{C}_E^*)
\]

\[
\text{Ext}^{2\dim(X)-\bullet}_{P^{G \times \mathbb{G}_{m}}(V)}(p_!\underline{C}_E, p_!\underline{C}_V) \cong \text{Ext}^{2\dim(X)+\bullet}_{P^{G \times \mathbb{G}_{m}}(V^*)}(p_!\underline{C}_E, p_!\underline{C}_E^*)
\]

\[
\varphi \boxtimes (c)(\cdot)
\]

\[
\underline{C}_V \boxtimes (p_0)_!\underline{C}_X [2\dim(X) - i]
\]

\[
p_!\underline{C}_X [2\dim(X) - i]
\]

\[
\underline{C}_X \boxtimes (p_0)_!\underline{C}_V [2\dim(X) - i]
\]

\[
\underline{C}_V \boxtimes (p_0)_!\underline{C}_X [2\dim(X) - i]
\]
sends each \(c \in H^*_G(X \times X)\) to the morphism
\[
\hat{p}^n \hat{\mathcal{L}}_X = \mathcal{L}(0) \boxtimes (p_0)! \hat{\mathcal{L}}_X \xrightarrow{\psi \otimes \alpha(c)} \mathcal{L}_{V^*} \boxtimes (p_0)! \hat{\mathcal{L}}_X[2 \dim(E^*) - i]
\]
where \(\psi : \mathcal{L}(0) \to \mathcal{L}_{V^*}[2 \dim(V^*)]\) is the \((1,1)\)-adjunction morphism for the inclusion \(\{0\} \hookrightarrow V^*\), and we use the identification \(V^* = V \times \text{pt}\). Now using Lemma \[\ref{lem:2.3.1}\] we obtain that \(F_V\) sends \(\varphi \boxtimes \alpha(c)\) to \(\psi \boxtimes \alpha(c)\), and the lemma follows. \(\square\)

With this result in hand we can prove Theorem \[\ref{thm:1.8.1}\] in our particular case.

**Lemma 4.1.2.** Theorem \[\ref{thm:1.8.1}\] holds in the case \(F_1 = E, F_2 = X\).

**Proof.** In our case we have \(F_1 \times V F_2 = X \times X\), and also \(F_1^\perp \times V F_2^\perp = X \times X\). There exists a natural morphism of dg-schemes
\[
(DV \times X \times X)_{\mathcal{E}}^R(E \times X) \to (X \times X)_{\mathcal{E}}^R X \times X
\]
associated with the morphism of vector bundles \(p \times p : E \times E \to X \times X\), see \[MR3, \S 3.2\]. In our case it is easily checked that this morphism is a quasi-isomorphism, hence it induces an equivalence of categories
\[
L\Phi^* : D^c_{G \times Gm}((X \times X)_{\mathcal{E}}^R X \times X) \simot D^c_{G \times Gm}((DV \times X \times X)_{\mathcal{E}}^R E \times E).
\]
Similarly, the morphism dual to \(p \times p\) induces a quasi-isomorphism
\[
(X \times X)_{\mathcal{E}}^R X \times X \to (DV^* \times X \times X)_{\mathcal{E}}^R E^* \times E^*
\]
hence an equivalence of categories
\[
R\Psi^* : D^c_{G \times Gm}((X \times X)_{\mathcal{E}}^R X \times X) \simot D^c_{G \times Gm}((DV^* \times X \times X)_{\mathcal{E}}^R E^* \times E^*).
\]
Moreover, if \(\mathcal{R}_{X,X}\) denotes the linear Koszul duality equivalence defined as \[\ref{def:3.3}\] (in the case \(V = \{0\}, F_1 = F_2 = E = X\), by \[MR3, \text{Proposition 3.4}\] there exists an isomorphism
\[
\mathcal{R}_{X,X} \circ L\Phi^* \cong R\Psi^* \circ \mathcal{R}_{E,E}.
\]
Now by definition the equivalence \(\mathcal{R}_{X,X}\) coincides with the equivalence
\[
\begin{cases}
  \mathcal{D}^b\text{Coh}^G_{X \times X} & \to \mathcal{D}^b\text{Coh}^G_{X \times X} \\
  G & \mapsto R\text{Hom}_{\mathcal{O}_{X \times X}}(G, \mathcal{O}_{X \times X} \boxtimes \omega_X)[\dim(X)]
\end{cases}
\]
Using this one can easily check that the composition
\[
\mathcal{R}R_{X,E^*} \circ \mathcal{D}_{X,E^*} \circ \text{Koszul}_{E,X}
\]
coincides with the morphism
\[
(4.1.3) \quad \begin{cases}
  K^G_{X \times X} & \to \hat{H}^*_{G \times Gm}(X \times X) \\
  c & \mapsto (1 \boxtimes (Td^G_{X \times X})^{-1}) \cdot \tau^G_{X \times X}(c)
\end{cases}
\]
Using Lemma \[\ref{lem:1.1.1}\] we observe that the composition \(\text{Fourier}_{X,E} \circ \mathcal{R}R_{X,E}\) also coincides with morphism \(\eqref{eq:1.1.3}\), which finishes the proof of the Lemma. \(\square\)
4.2. Compatibility with inclusion. Consider first Setting (A) of §2.1

Proposition 4.2.1. (1) The following diagram commutes:

\[
\begin{array}{ccc}
K^{G \times G_m}(F_1 \times V F_2') & \xrightarrow{\mathbb{R} \circ \mathbb{D} \circ \text{Koszul}_{F_1, F_2}} & \mathbb{H}^{G \times G_m}(F_1^{\perp \times V} (F_2')^\perp) \\
\downarrow & & \downarrow \\
K^{G \times G_m}(F_1 \times V F_2) & \xrightarrow{\text{Fourier}_{F_1, F_2} \circ \mathbb{R} \circ \mathbb{D}} & \mathbb{H}^{G \times G_m}(F_1^{\perp \times V} F_2^\perp) \\
\end{array}
\]

(2) Assume that the pushforward morphism

\[\mathbb{H}^{G \times G_m}(F_1 \times V F_2) \to \mathbb{H}^{G \times G_m}(F_1 \times F_2)\]

is injective. Then the following diagram commutes:

\[
\begin{array}{ccc}
K^{G \times G_m}(F_1 \times V F_2) & \xrightarrow{\text{Fourier}_{F_1, F_2} \circ \mathbb{R} \circ \mathbb{D}} & \mathbb{H}^{G \times G_m}(F_1^{\perp \times V} (F_2')^\perp) \\
\downarrow & & \downarrow \\
K^{G \times G_m}(F_1 \times V F_2) & \xrightarrow{\text{Fourier}_{F_1, F_2} \circ \mathbb{R} \circ \mathbb{D}} & \mathbb{H}^{G \times G_m}(F_1^{\perp \times V} F_2^\perp) \\
\end{array}
\]

Proof. (1) follows from Propositions 3.1.1, 3.2.1, and 3.4.1. (2) follows from Propositions 3.3.1 and 2.4.1.

Consider now Setting (B) of §2.1

Proposition 4.2.2. (1) Assume that the pushforward morphism

\[\mathbb{H}^{G \times G_m}((F_1')^{\perp \times V} F_2^\perp) \to \mathbb{H}^{G \times G_m}((F_1')^{\perp \times F_2^\perp})\]

is injective. Then the following diagram commutes:

\[
\begin{array}{ccc}
K^{G \times G_m}(F_1 \times V F_2') & \xrightarrow{\mathbb{R} \circ \mathbb{D} \circ \text{Koszul}_{F_1, F_2}} & \mathbb{H}^{G \times G_m}(F_1^{\perp \times V} F_2^\perp) \\
\downarrow & & \downarrow \\
K^{G \times G_m}(F_1' \times V F_2) & \xrightarrow{\mathbb{R} \circ \mathbb{D} \circ \text{Koszul}_{F_1', F_2}} & \mathbb{H}^{G \times G_m}((F_1')^{\perp \times V} F_2^\perp) \\
\end{array}
\]

(2) The following diagram commutes:

\[
\begin{array}{ccc}
K^{G \times G_m}(F_1 \times V F_2') & \xrightarrow{\text{Fourier}_{F_1, F_2} \circ \mathbb{R} \circ \mathbb{D}} & \mathbb{H}^{G \times G_m}(F_1^{\perp \times V} (F_2')^\perp) \\
\downarrow & & \downarrow \\
K^{G \times G_m}(F_1' \times V F_2) & \xrightarrow{\text{Fourier}_{F_1', F_2} \circ \mathbb{R} \circ \mathbb{D}} & \mathbb{H}^{G \times G_m}((F_1')^{\perp \times V} F_2^\perp) \\
\end{array}
\]

Proof. (1) follows from Propositions 3.1.2, 3.2.2, and 3.4.2. (2) follows from Propositions 3.3.2 and 2.4.2.
4.3. Proof of Theorem 1.8.1. By assumption, the pushforward morphism
\[ H^*_{G \times G_m}(F_1^\perp \times_{V^*} F_2^\perp) \to H^*_{G \times G_m}(F_1^\perp \times_{V^*} E^*) \]
is injective. Hence the same is true for the induced morphism
\[ \hat{H}^*_{G \times G_m}(F_1^\perp \times_{V^*} F_2^\perp) \to \hat{H}^*_{G \times G_m}(F_1^\perp \times_{V^*} E^*). \]

By Proposition 4.2.1 applied to the inclusion \( X \subset F_2 \), we deduce that it suffices to prove the theorem in the case \( F_2 = X \). (Note that the inclusion \( F_1 \times_{V^*} X \to E_1 \times_{V^*} X \) is the inclusion of the zero section in the vector bundle \( F_1 \times X \) over \( X \times X \). Hence the injectivity assumption in Proposition 4.2.1(2) holds by Lemma A.8.2.)

Now consider the inclusion of vector subbundles \( F_1 \subset E \) (again with \( F_2 = X \)). In this case, the restriction with supports morphism
\[ H^*_{G \times G_m}(E^*) \to H^*_{G \times G_m}(X \times E^*) = H^*_{G \times G_m}(X \times X) \]
(see (2.1.8)) is the Thom isomorphism for the vector bundle \( F_1^\perp \times_{V^*} E^* \cong F_1^\perp \times_{V^*} X \) over \( X \times X \); in particular it is injective. Using Proposition 4.2.2 we deduce that it suffices to prove the theorem in the case \( F_1 = E, F_2 = X \). (Note that in our situation the inclusion \( E_1^\perp \times_{V^*} X \subset E_1^\perp \times_{V^*} X \) is the inclusion of the zero section in the vector bundle \( X \times E^* \) over \( X \times X \), so that the injectivity assumption in Proposition 4.2.2(1) holds by Lemma A.8.2.) In this case the theorem holds by Lemma 4.1.2, hence our proof is complete.

Appendix A. Proofs of some technical results

A.1. Conventions. In §§A.2–A.4 we work in the \( A \)-equivariant constructible derived category of some complex algebraic \( A \)-varieties (for some arbitrary complex algebraic group \( A \)). If \( X, Y, Z \) are \( A \)-varieties and \( f : X \to Y, g : Y \to Z \) are \( A \)-equivariant morphisms, then there exist canonical “composition” isomorphisms
\[ g_* f_* \cong (g \circ f)_*, \quad g! f! \cong (g \circ f)!, \quad f^* g^* \cong (g \circ f)^*, \quad f^! g^! \cong (g \circ f)^!, \]
which we will all indicate by (Comp). Similarly, given a cartesian square

\[
\begin{array}{ccc}
Y' & \xrightarrow{f'} & Z' \\
g' \downarrow & & \downarrow g \\
Y & \xrightarrow{f} & Z
\end{array}
\]
of \( A \)-equivariant morphisms, there exist canonical “base change” isomorphisms
\[ f^* g^! \cong (g')_!(f')^!, \quad f^! g_* \cong (g')_*(f')^!, \]
which we will indicate by (BC).
A.2. Some commutative diagrams. Consider a commutative diagram of $A$-varieties and $A$-equivariant morphisms

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
| & \downarrow{a} & | \\
Y' & \xrightarrow{g'} & Z'
\end{array}
\]

where all squares are cartesian. The following lemma is a restatement of [AHR, Lemma B.7(d)].

**Lemma A.2.1.** The following diagram of isomorphisms of functors commutes:

\[
\begin{array}{c}
(g'')_!f_* \xrightarrow{(\text{Comp})} (g'')_!c_*d_* \xrightarrow{(\text{BC})} b_!(g')!d_* \\
\downarrow{\sim} & \downarrow{\sim} & \downarrow{\sim} \\
(BC) \downarrow{i} & (\text{Comp}) \downarrow{i} & (BC) \downarrow{i} \\
c_*g' \xrightarrow{(\text{Comp})} b_!a_*g'.
\end{array}
\]

Now, consider $A$-equivariant morphisms

\[
W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z.
\]

The following lemma is a restatement of [AHR, Lemma B.4(a) and Lemma B.4(d)].

**Lemma A.2.2.** The following diagrams of isomorphisms of functors commute:

\[
\begin{array}{c}
h_*g_*f_* \xrightarrow{(\text{Comp})} h_!(g \circ f)_* \xrightarrow{(\text{Comp})} f'!(h \circ g)' \\
(\text{Comp}) \downarrow{i} & (\text{Comp}) \downarrow{i} & (\text{Comp}) \downarrow{i} \\
(h \circ g)_*f_* \xrightarrow{(\text{Comp})} (h \circ g \circ f)_*.
\end{array}
\]

A.3. Base change and adjunction. Consider a cartesian diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{f'} & Z' \\
\downarrow{g'} & \square & \downarrow{g} \\
Y & \xrightarrow{f} & Z.
\end{array}
\]

Then there exists a canonical morphism of functors

\[
(f')_!(g')! \rightarrow g'f!
\]

which can be defined equivalently as the composition

\[
(f')_!(g')! \rightarrow (f')_!(g')_!f'_! \xrightarrow{(\text{Comp})} (f'\circ g')_!f'_! \xrightarrow{(\text{Comp})} (f')_!(f')_!g'_!f'_! \rightarrow g'f!
\]

or as the composition

\[
(f')_!(g')! \rightarrow g'f!(f')_!(g')_! \xrightarrow{(\text{Comp})} g'_!(f \circ f')_!(g')_! \xrightarrow{(\text{Comp})} g'_!f!(g')_! \rightarrow g'f!
\]

where the unlabelled arrows are induced by the appropriate adjunction morphisms. (We leave it to the reader to check that these compositions coincide.)
As stated in [KS, Exercise III.9], the following diagram is commutative, where vertical arrows are induced by the canonical morphisms $f_! \to f_*$ and $(f')_! \to (f')_*$:

\[
\begin{array}{ccc}
(f')_!(g')! & \xrightarrow{(\text{A.3.2})} & g^! f_! \\
\downarrow & & \downarrow \\
(f')_*(g')! & \xrightarrow{(\text{BC})} & g^! f_*
\end{array}
\]

We deduce the following.

**Lemma A.3.3.** If $f$ (hence also $f'$) is proper, then the base change isomorphism $(f')_*(g')! \cong g^! f_*$ coincides, under the natural identifications $f_! = f_*$ and $(f')_! = (f')_*$, with morphism (A.3.2).

**A.4. Some consequences.** Consider again a cartesian diagram (A.3.1), and assume that $f$ (hence also $f'$) is proper.

First, one can consider the diagram of morphisms of functors

\[
\begin{array}{ccc}
(f')_*(g')! f^! & \xrightarrow{(\text{BC})} & g^! f_* f^! \\
\downarrow (\text{Comp}) & & \downarrow \\
(f')_*(f \circ g')! & \xrightarrow{(\text{Comp})} & g^!
\end{array}
\]

(A.4.1)

where the right vertical arrow is induced by the adjunction morphism $f_* f^! = f_! f^! \to \text{id}$ and the lower horizontal arrow is induced by the adjunction morphism $(f')_*(f')^! = (f')_! (f')^! \to \text{id}$.

**Lemma A.4.2.** Diagram (A.4.1) is commutative.

**Proof.** The claim follows from Lemma A.3.3 (using the first description of morphism (A.3.2)) and the fact that the composition of adjunction morphisms $f^! \to f^! f_! f^! \to f^!$ is the identity. \qed

One can also consider the diagram of morphisms of functors

\[
\begin{array}{ccc}
g_! (f')_*(g')! & \xrightarrow{(\text{BC})} & gg^! f_* \\
\downarrow (\text{Comp}) & & \downarrow \\
(g \circ f')_!(g')! & \xrightarrow{(\text{Comp})} & f_*
\end{array}
\]

(A.4.3)
where unlabelled arrows are induced by adjunction, and in the left-hand side we use the identifications $f_1 = f_*$ and $(f')_! = (f')_*$.

**Lemma A.4.4.** Diagram [A.4.3] is commutative.

**Proof.** The claim follows from Lemma [A.3.3] (using the second description of morphism [A.3.2]) and the fact that the composition of adjunction morphisms

$$g! \to g! g_! \to g!$$

is the identity. □

### A.5. Restriction with supports in homology.

As in §1.1, let $A$ be a complex algebraic group, let $Y$ be a smooth complex $A$-variety, and let $Y' \subset Y$ be a smooth $A$-stable closed subvariety. Consider another $A$-stable closed subvariety $Z \subset Y$, not necessarily smooth, and set $Z' := Z \cap Y'$. Then we have a cartesian diagram of inclusions

$$
\begin{array}{ccc}
Z' & \rightarrow & Y' \\
\downarrow{g} & & \downarrow{f} \\
Z & \rightarrow & Y
\end{array}
$$

Set $N := 2 \dim(Y) - 2 \dim(Y')$. The “restriction with supports” morphism

(A.5.1)

$$H^*_A(Z) \to H^*_{-N}(Z')$$

associated with the inclusion $Y' \hookrightarrow Y$ is defined as follows. Consider the composition

$$i_!^! \to i_!^! f_* f^* \overset{(BC)}{\sim} g_! (i'_!)^! f^*$$

where the first morphism is induced by the adjunction morphism $\text{id} \to f_* f^*$. Then applying this composition to $D_Y$ and using the isomorphisms

$$i_!^! D_Y \cong D_Z, \quad f_* D_Y \cong f_* \mathbb{C}_Y[2 \dim(Y)] \cong \mathbb{C}_{Y'}[2 \dim(Y)] \cong D_{Y'}[N], \quad \text{and} \quad (i'_!)^! D_Y \cong D_{Z'}.$$

we obtain a morphism

$$D_Z \to g_* D_{Z'}[N].$$

Then taking (equivariant) cohomology provides our morphism [A.5.1].

We will need a compatibility property for this construction. Consider the following diagram:

(A.5.2)

$$
\begin{array}{ccc}
H^*_A(Z) & \overset{(\text{A.5.1})}{\longrightarrow} & H^*_{-N}(Z') \\
\downarrow{\cdot} & & \downarrow{\cdot} \\
H^*_A(Y) & \longrightarrow & H^*_{-N}(Y')
\end{array}
$$

where the vertical morphism are pushforward morphisms for the inclusions $Z \hookrightarrow Y$ and $Z' \hookrightarrow Y'$, and the lower horizontal morphism is the morphism given by the construction above in the case $Z = Y$ (i.e. this morphism is induced by the morphism $D_Y \to f_* D_Y \cong f_* D_{Y'}[N]$).

**Lemma A.5.3.** Diagram [A.5.2] is commutative.
Proof. Consider the following diagram:

\[
\begin{array}{ccc}
& i_! & i_! f_* f^* \\
\downarrow & & \downarrow \\
\downarrow & f_*(i'_!) & f^*(i'_!) \\
& id & f_* f^*.
\end{array}
\]

Here the unlabelled arrows are induced by the appropriate adjunction morphisms, and the arrow labelled with (\(\downarrow\)) is induced by the composition of natural isomorphisms

\[
i_! g_* \cong i_! g \xrightarrow{(\text{Comp})} \sim (i \circ g)! \xrightarrow{(\text{Comp})} f_!(i'_!)! \cong f_*(i'_!).
\]

The left part of the diagram is clearly commutative, and the right part is commutative by Lemma A.4.4. Hence the diagram as a whole is commutative. Now, when applied to \(\mathcal{D}Y\) and after taking equivariant cohomology, this diagram induces diagram (A.5.2), hence these remarks prove the lemma. (In this argument we also use the left diagram in Lemma A.2.2, which allows to forget about the “(Comp)” isomorphisms in the right-hand side of the diagram once equivariant cohomology is taken.) \(\square\)

A.6. Proof of Proposition 2.2.1(1). By functoriality of isomorphism (1.3.2) the following diagram commutes:

\[
\begin{array}{ccc}
\text{Ext}^\bullet_{G \times \mathbb{G}_m(V)}(p_* \mathbb{C}_{F_1}, p_* \mathbb{C}_{F_2}) & \xrightarrow{1.3.2} & H^\bullet_{G \times \mathbb{G}_m}(j^!(\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2})) \\
(p_* \text{adj}^*_{F_2, F_2}) \circ (\cdot) & \downarrow & \downarrow \\
\text{Ext}^\bullet_{G \times \mathbb{G}_m(V)}(p_* \mathbb{C}_{F_1}, p_* \mathbb{C}_{F_2}) & \xrightarrow{1.3.2} & H^\bullet_{G \times \mathbb{G}_m}(j^!(\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2}))
\end{array}
\]

where the right vertical morphism is induced by \(\text{adj}^*_{F_2, F_2}\).

Now, consider the following diagram, where all squares are cartesian and all morphisms are closed inclusions:

\[
\begin{array}{ccc}
F_1 \times_V F_2 & \xrightarrow{c} & F_1 \times V F'_2 & \xrightarrow{b' \circ} & E \times E \\
\downarrow k & & \downarrow k' & & \downarrow j \\
F_1 \times F_2 & \xrightarrow{d} & F_1 \times F'_2 & \xrightarrow{a' \circ} & E \times E.
\end{array}
\]

Then under the natural identifications \(H^\bullet_{G \times \mathbb{G}_m}(j^!(\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2})) \cong H^\bullet_{G \times \mathbb{G}_m}(j^! a_*(\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2}))\), \(H^\bullet_{G \times \mathbb{G}_m}(j^!(\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2})) \cong H^\bullet_{G \times \mathbb{G}_m}(j^!(a'\star(\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2})))\), the right vertical morphism in (A.6.1) identifies with the morphism

\[
\begin{array}{c}
H^\bullet_{G \times \mathbb{G}_m}(j^!(a'\star(\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2}))) \to H^\bullet_{G \times \mathbb{G}_m}(J^! a_*(\mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2}))
\end{array}
\]
induced by the adjunction morphism \( \text{id} \to d_\ast d^\ast \) (through the “composition” isomorphism \((a')_s d_\ast \cong a_\ast\)).

Consider the following diagram of morphisms of functors:

\[
\begin{array}{ccc}
  j^!(a')_s & \sim & (b')_s(k')^! \\
  \downarrow \text{(BC)} & & \downarrow \text{(BC)} \\
  j^!(a')_s d_\ast d^\ast & \sim & (b')_s(k')^! d_\ast d^\ast \\
  \downarrow \text{(Comp)} & & \downarrow \text{(Comp)} \\
  j^! a_\ast d^\ast & \sim & (b')_s c_\ast (k')^! d^\ast.
\end{array}
\]

Here the upper vertical arrows are induced by the adjunction morphism \( \text{id} \to d_\ast d^\ast \), and other arrows are either base change or composition isomorphisms as indicated. The upper square is clearly commutative, and the lower square is commutative by Lemma A.2.1. Hence the whole diagram is commutative, which allows to define the dotted arrows uniquely. The arrow labelled with \((\ast)\) is the morphism which defines (A.6.2), and the arrow labelled with \((\dagger)\) is the morphism used in the definition of restriction with supports (2.1.2), see A.5. Applying this diagram to \( \mathbb{D}_{F_1} \boxtimes \mathbb{C}_{F_2} \) and taking equivariant cohomology allows to finish the proof of Proposition 2.2.1(1). (In this argument we also use the left diagram in Lemma A.2.2, which allows e.g. to forget about the “(Comp)” isomorphism on the lower line once equivariant cohomology is taken.)

A.7. Proof of Proposition 2.2.1(2). Consider the following diagram, where all squares are cartesian and all morphisms are closed inclusions:

\[
\begin{array}{ccc}
  F_1^\perp \times V^\ast & \xrightarrow{\tilde{\epsilon}} & F_1^\perp \times V^\ast \\
  \downarrow \tilde{k} & & \downarrow \tilde{k}' \\
  F_1^\perp \times (F_2')^\perp & \xrightarrow{j} & F_1^\perp \times F_2^\perp \\
  \downarrow \tilde{a} & & \downarrow \tilde{a}' \\
  F_1^\perp \times (F_2')^\perp & \xrightarrow{\tilde{\alpha}} & F_1^\perp \times F_2^\perp \xrightarrow{\tilde{a}} E^\ast \times E^\ast.
\end{array}
\]

Then by functoriality of isomorphism (1.3.2) we have a commutative diagram

\[
\begin{align*}
\text{Ext}^\ast_{\mathcal{D}_G \times \mathbb{G}_m(V^\ast)}(\tilde{\alpha}_s \mathbb{C}_{F_1^\perp}, \tilde{\alpha}_s \mathbb{C}_{(F_2')^\perp}) & \sim H^\ast_{G \times \mathbb{G}_m}(j^! \tilde{a}_s(\mathbb{D}_{F_1^\perp} \boxtimes \mathbb{C}_{(F_2')^\perp})) \\
(\tilde{\alpha}_s \text{adj}_{(F_2')^\perp}, (F_2')^\perp)^!(\cdot) & \Downarrow \sim \Downarrow \\
\text{Ext}^\ast_{\mathcal{D}_G \times \mathbb{G}_m(V^\ast)}(\tilde{\alpha}_s \mathbb{C}_{F_1^\perp}, \tilde{\alpha}_s \mathbb{C}_{(F_2')^\perp}) & \sim H^\ast_{G \times \mathbb{G}_m}(j^! \tilde{a}_s(\mathbb{D}_{F_1^\perp} \boxtimes \mathbb{C}_{(F_2')^\perp}))
\end{align*}
\]

where horizontal arrows are induced by isomorphism (1.3.2) and the right vertical morphism is induced by the adjunction morphism \( \tilde{d} \tilde{d}' \to \text{id} \) (through the isomorphisms \((\tilde{a}')_s \tilde{d}_\ast \cong (\tilde{a}')_s \tilde{d}_\ast \cong \tilde{a}_\ast \) and \( \mathbb{C}_{(F_2')^\perp} \cong \mathbb{D}_{(F_2')^\perp}[-2 \dim((F_2')^\perp)] \), \( \mathbb{D}_{F_2^\perp} \cong \mathbb{D}_{F_2^\perp}[-2 \dim(F_2^\perp)] \)).
Consider the following diagram of morphisms of functors:

\[
\begin{array}{cccc}
\tilde{g}^!(\bar{a}'_s\bar{d}^l) & \xrightarrow{(BC)} & \tilde{b}'_s\tilde{k}'^!\bar{d}^l \\
\text{(Comp)} & \tilde{i} & \text{(Comp)} & \tilde{i} \\
(\#) & \tilde{g}^!\tilde{a}'_s\tilde{d}^l & \xrightarrow{(BC)} & (\tilde{b}')_s\tilde{c}'_s\tilde{k}'^!\bar{d}^l \\
\text{(Comp)} & \tilde{l} & \text{(Comp)} & \tilde{l} \\
\tilde{g}^!(\tilde{a}')_s\tilde{d}^l & \xrightarrow{(BC)} & (\tilde{b}')_s(\tilde{k}')^!\tilde{d}^l \\
\end{array}
\]

Here all the unlabelled arrows are induced by the appropriate adjunction morphisms. The upper square is commutative by Lemma [A.2.4], the lower square is obviously commutative, and the right square is commutative by Lemma [A.4.2]. Hence the diagram as a whole is commutative, which allows to define the dotted arrows uniquely. The arrow labelled with (\#) is the one which induces the right arrow in diagram [A.7.1] (when applied to \(D_1\)) and the right square is commutative by Lemma A.4.2. Hence the diagram as a whole is commutative, and the lower square is obviously commutative, and the upper square is commutative by Lemma A.2.1, the lower square is obviously commutative, and the upper square is commutative by Lemma A.2.1, hence it is enough to prove that Eu(\(F\)) is not a zero-divisor in \(H^r_{G\times G_m}(Y)\). Note that, as \(G_m\) acts trivially on \(Y\), there exists a canonical isomorphism

\[(A.8.1) \quad H^*_{G\times G_m}(Y) \cong H^*_{G}(Y) \otimes \mathbb{C} H^*_G(pt).\]

**Lemma A.8.2.** The pushforward morphism

\[H^*_{G\times G_m}(Y) \to H^*_{G\times G_m}(F)\]

in equivariant Borel–Moore homology is injective.

**Proof.** It is well known that the composition of our morphism with the Thom isomorphism \(H^*_{G\times G_m}(F) \cong H^*_{G\times G_m}(Y)\) identifies with the action of the equivariant Euler class \(\text{Eu}(F) \in H^*_{G\times G_m}(Y)\) of \(F\), see e.g. [L3, §1.19]. By our assumption on \(Y\), the equivariant homology \(H^*_{G\times G_m}(Y)\) is a free module of rank one over \(H^*_G(Y)\), hence it is enough to prove that Eu(\(F\)) is not a zero-divisor in \(H^*_G(Y)\). However one can check that (due to our choice of \(G_m\)-action) this Euler class can be written, using isomorphism (A.8.1), as

\[
\text{Eu}(F) = 1 \otimes (-2u)^r + x
\]

where \(1 \in H^0_G(Y)\) is the unit, \(u \in H^2_G(pt)\) is the canonical generator and \(x \in \bigoplus_{i \geq 2} H^i_G(Y) \otimes H^r_{G_m}(pt)\). It follows that this element is indeed not a zero-divisor. \(\square\)
References

[AHR] P. Achar, A. Henderson, S. Riche, Geometric Satake, Springer correspondence, and small representations II, preprint arXiv:1205.5089.

[AHRJR] P. Achar, A. Henderson, D. Juteau, S. Riche, Weyl group actions on the Springer sheaf, preprint arXiv:1304.2642 to appear in Proc. Lond. Math. Soc.

[BL] J. Bernstein, V. Lunts, Equivariant sheaves and functors, Lecture Notes in Math. 1578, Springer, 1994.

[BR] R. Bezrukavnikov, S. Riche, Affine braid group actions on Springer resolutions, Ann. Sci. École Norm. Sup. 45 (2012), 535–599.

[BZ] J.-L. Brylinski, B. Zhang, Equivariant Todd classes for toric varieties, preprint arXiv:0311318.

[CG] N. Chriss, V. Ginzburg, Representation theory and complex geometry, Birkhäuser, 1997.

[EG] D. Edidin, W. Graham, Riemann–Roch for equivariant Chow groups, Duke Math. J. 102 (2000), 567–594.

[EM] S. Evens, I. Mirković, Fourier transform and the Iwahori–Matsumoto involution, Duke Math. J. 86 (1997), 435–464.

[Ha] R. Hartshorne, Residues and duality, Lecture Notes in Math. 20, Springer, 1966.

[Kas] M. Kashiwara, Character, character cycle, fixed point theorem and group representations, RIMS-569 preprint (1987), available on the author’s web page.

[KS] M. Kashiwara, P. Schapira, Sheaves on manifolds, Springer, 1990.

[Kat] S. Kato, On the combinatorics of unramified admissible modules, Publ. Res. Inst. Math. Sci. 42 (2006), no. 2, 589–603.

[KL] D. Kazhdan, G. Lusztig, Proof of the Deligne–Langlands conjecture for Hecke algebras, Invent. Math. 87 (1987), 153–215.

[Le] E. Letellier, Fourier transforms of invariant functions on finite reductive Lie algebras, Lecture Notes in Math. 1859, Springer, 2005.

[L1] G. Lusztig, Affine Hecke algebras and their graded version, J. Amer. Math. Soc. 2 (1989), 599–635.

[L2] G. Lusztig, Cuspidal local systems and graded Hecke algebras. I. Inst. Hautes Études Sci. Publ. Math. No. 67 (1988), 145–202.

[L3] G. Lusztig, Cuspidal local systems and graded Hecke algebras. II. CMS Conf. Proc. 16, Representations of groups (Banff, AB, 1994), 217–275, Amer. Math. Soc., 1995.

[MR1] I. Mirković, S. Riche, Linear Koszul duality, Compos. Math. 146 (2010), 233–258.

[MR2] I. Mirković, S. Riche, Linear Koszul duality II – Coherent sheaves on perfect sheaves, preprint arXiv:1301.3924.

[MR3] I. Mirković, S. Riche, Iwahori–Matsumoto involution and linear Koszul duality, preprint arXiv:1301.4008 to appear in Int. Math. Res. Not.

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