Quantum Diffusion with Drift and the Einstein Relation II

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Abstract: This paper is a companion to [3]. Its purpose is to describe and prove a certain number of technical results used in [3], but not proven there. Both papers concern long-time properties (diffusion, drift) of the motion of a driven quantum particle coupled to an array of thermal reservoirs. The main technical results derived in the present paper are (1) an asymptotic perturbation theory applicable for small driving, and, (2) the construction of time-dependent correlation functions of particle observables.

KEY WORDS: diffusion, kinetic limit, quantum brownian motion

1. Introduction

In [3] and in the present paper, we study a model of a quantum tracer particle hopping on the lattice $\mathbb{Z}^d$ ($d \geq 1$) and interacting with an array of thermal reservoirs placed at the sites of $\mathbb{Z}^d$, which are described quantum-mechanically: When the particle visits a site $x \in \mathbb{Z}^d$ it can emit or absorb quanta of the thermal reservoir at site $x$, thus changing its own momentum. Those quanta correspond to non-interacting, massless modes – phonons or photons – in a state of thermal equilibrium at a positive temperature $T = \beta^{-1}$, the same for all the reservoirs. Reservoirs at different sites are independent of one another. As a consequence, memory effects only arise when the particle returns to sites it has visited previously. At positive temperature and assuming a certain analyticity property of the particle-reservoir coupling, such memory effects turn out to decay exponentially fast in time. There is a constant external force acting on the tracer particle. The particular model described here has first been studied in [8], for a vanishing external force, and it has been proven there that the particle exhibits “quantum Brownian motion”; (see also [2, 5] for related results). The purpose of the analysis presented in [3] and in this paper is to determine asymptotic properties of the motion of the particle in the presence of a non-vanishing external force, as time $t$ tends to $\infty$. As announced in [3], one expects that, for a sufficiently weak external force and weak particle-reservoir interactions, the state of the quantum particle (that is, a state on the algebra generated by functions of its momentum) approaches a “non-equilibrium steady state” (NESS). This state describes a uniform motion of the tracer particle with a mean drift velocity $v$ that depends on the force pushing it, the strength of interaction of the particle with the reservoirs and the temperature of the reservoirs. Furthermore,

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the particle exhibits diffusion—“quantum Brownian motion”—around its mean motion. The diffusion constant at zero force is related to the derivative, at zero external force, of the drift velocity with respect to the force by the Einstein relation.

One major difficulty encountered in our analysis is that, for a non-zero external force, the state of the particle (a density matrix on the particle Hilbert space obtained by tracing out the degrees of freedom of all the reservoirs) appears to show phase coherence over fairly long distance scales. In contrast, when the external force vanishes, the state of the particle exhibits exponential decoherence in particle position space. Because of the lack of decoherence in the presence of an external force our result concerning the approach of the particle state to a NESS only holds in an ergodic mean, and our formula for the diffusion constant involves an abelian limit.

While the material in [3] is quite elegant and of some general conceptual interest, the present paper is primarily technical. Most technicalities to be coped with in the following are related to the thermodynamic limit and to the need to control large-time asymptotics, in particular cluster properties of connected correlation functions of particle observables, as time differences tend to \( \infty \). The need to introduce finite-volume approximations of the system studied in our papers comes from the structure of our proof of the Einstein relation. Technical difficulties arise because the components of the particle position are unbounded operators and the potential of a constant force grows linearly in the particle position.

Background from physics and an account of the difficulties encountered in the analysis of this and of more realistic models of quantum transport have been presented in [3] and will not be repeated here. Suffice it to say that we expect that one would face major problems if one attempted to extend the results of [3] and of the present paper to more realistic models of particle transport—in particular, continuum models—or if one tried to prove stronger convergence results for the model studied here.

The organization of our paper is as follows:

In Section 2, we recall some notation and the definition of the model studied in [3] and in this paper. We then define the effective quantum dynamics of the particle, after tracing out the degrees of freedom of the thermal reservoirs, and we introduce correlation (Green) functions of operators representing properties of the particle (particle observables) in various states of the system. We define the time-reversal operator and recall the KMS condition characterizing a thermal equilibrium state. Equilibrium states at vanishing external force appear in our proof of the Einstein relations.

In Section 3, the main assumptions concerning our model are summarized, and the main results proven in [3] and in this paper are stated. All results are only proven for a weak external force and at weak coupling of the tracer particle to the reservoirs, (where “weak” depends on the temperature of the reservoirs and the kinetic energy operator of the particle). In Lemmas 3.1 and 3.2 we state our results on the existence and some properties of the thermodynamic limit. Theorem 3.3 describes “return to equilibrium” at vanishing external force. Theorem 3.4 concerns the approach, in an ergodic mean, of the state of the tracer particle to a “non-equilibrium stationary (or steady) state” (NESS). In Theorem 3.5, the convergence of the mean velocity of the particle to a non-zero drift velocity, as time \( t \) tends to \( \infty \), is asserted, and a formula for the diffusion constant involving an abelian limit is presented. The Einstein relation forms the content of Theorem 3.6.

In Section 4, the Dyson expansion of the propagator describing the time-evolution of general mixed states of the system in powers of the operator describing the interactions between the particle and the reservoirs is derived. This expansion also yields an expansion of the effective particle dynamics and of correlation functions of particle observables when the degrees of freedom of the reservoirs are traced out; (Subsections 4.2 through 4.4). In Subsection 4.5, the Dyson expansions are cast in the form of “polymer expansions” for dilute gases of extended particles with hard-core interactions contained in a one-dimensional “space”, with “space” corresponding to the time axis of the original system.

In Section 5, the expansions derived in Section 4 are further studied and used to prove existence and properties of the thermodynamic limit of various quantities; see Subsection 5.1. (These results can be used to prove Lemmas 3.1 and 3.2.) In Subsection 5.2, bounds on the effective dynamics of the tracer particle, after tracing out the reservoir degrees of freedom, are proven. In Subsection 5.3, the Laplace transform in the time variable of the effective particle dynamics is introduced, yielding an object resembling a resolvent of an effective Hamiltonian that depends on a spectral parameter with the interpretation of an energy. This “pseudo-resolvent” plays a fundamental role in the proofs of the main results stated in Section 3. In Subsection 5.4, a direct-integral (fiber-momentum) decomposition of translation-invariant operators acting on the particle state and in particular of the effective particle dynamics is recalled. In Subsection 5.5, the main contributions, in fibers of fixed momentum, to the pseudo-resolvents introduced in Subsection 5.3 are identified, and some key spectral properties of these operators are described; (but see also [3]).
In Section 6, the behavior of the “pseudo-resolvents” near the origin of the complex spectral parameter plane (“zero energy”) is analyzed, which yields information on large-time asymptotics of the effective dynamics.

In Section 7, the effective dynamics and the correlation functions are studied for a particle in a vanishing external force. In this situation, the state of the system (restricted to continuous functions of the particle momentum operator) is shown to approach an equilibrium state at the temperature of the reservoirs. This yields a proof of Theorem 3.3.

In Section 8, the Einstein relation (Theorem 3.6) is proven.

All other results, in particular Theorems 3.4 and 3.5, have already been proven in the companion paper [3], using the results established in Sects. 5 and 6 of this paper. In order to render the present paper comprehensible and more or less self-contained, it is unavoidable to repeat a certain amount of material concerning definitions and notation from [3]. The reader is recommended to consult the companion paper [3] for background from physics, motivation and discussion of the main results.

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2. Definition of the model

In this section, we recall the definition of the model studied in [3] and in this paper in a finite volume approximation. Among other things, we also repeat some important notations and conventions, the definition of the reservoir equilibrium states and the definition of the effective particle dynamics.

2.1. Notations and conventions.

2.1.1. Banach spaces. Given a Hilbert space $E$, we use the standard notation $B_p(E) := \left\{ A \in B(E) : \text{Tr} \left( (A^*A)^{p/2} \right) < \infty \right\}$, $1 \leq p \leq \infty$, with $B_\infty(E) \equiv B(E)$ the bounded operators on $E$, and

$$\|A\|_p := \left( \text{Tr} \left( (A^*A)^{p/2} \right) \right)^{1/p}, \quad \|A\| := \|A\|_\infty.$$  

For operators acting on $B_p(E)$, e.g., elements of $B(B_p(E))$, we often use the calligraphic font: $V, W$ etc.. An operator $A \in B(E)$ determines bounded operators $\text{Ad}(A), \text{ad}(A), A_l, A_r$ on $B_p(E)$ by

$$\text{Ad}(A)B := ABA^*, \quad \text{ad}(A)B := [A,B] = AB - BA$$

and

$$A_lB := AB, \quad A_rB := BA^*, \quad B \in B_p(E).$$  \hfill (2.1)

Note that $(A_1)_l(A_2)_l = (A_2)_l(A_1)_l$, as operators on $B_p(E)$, $A_1, A_2 \in B(E)$, i.e., the left- and right multiplications commute. The norm of operators in $B(B_p(E))$ is defined by

$$\|W\| := \sup_{A \in B_p(E)} \frac{\|W(A)\|_p}{\|A\|_p}.$$  

In the following, we usually set $p = 1$ or 2.

2.1.2. Scalar products. For vectors $\kappa \in \mathbb{C}^d$, we let $\text{Re} \kappa$ denote the vector $(\text{Re} \kappa^1, \ldots, \text{Re} \kappa^d)$, where Re denotes the real part. Similar notation is used for the imaginary part, Im. The scalar product on $\mathbb{C}^d$ is written as $\langle \kappa_1, \kappa_2 \rangle$ or $\kappa_1 \cdot \kappa_2$ and the norm as $|\kappa| := \sqrt{\langle \kappa, \kappa \rangle}$. The scalar product on an infinite-dimensional Hilbert space $E$ is written as $\langle \cdot, \cdot \rangle$, or, occasionally, as $\langle \cdot, \cdot \rangle_E$. All scalar products are defined to be linear in the second argument and anti-linear in the first one.
2.1.3. Kernels. For $\mathcal{E} = \ell^2(\mathbb{Z}^d)$, we can represent $A \in \mathcal{B}(\mathcal{E})$ by its kernel $A(x, y)$, i.e., $(Af)(x) = \sum_y A(x, y)f(y)$, $f \in \mathcal{E}$. Similarly, an operator, $\mathcal{A}$, acting on $\mathcal{B}(\mathcal{E})$ can be represented by its kernel $\mathcal{A}(x, y, x', y')$ satisfying $(\mathcal{A} \rho)(x, y) = \sum_{x', y'} \mathcal{A}(x, y, x', y') \rho(x', y')$, $\rho \in \mathcal{B}(\mathcal{E})$. Occasionally, we use the notation $|x\rangle$ for $\delta_x \in \mathcal{E}$, defined by $\delta_x(x') = \delta_{x, x'}$, and $\langle x|$ for $\langle \delta_x, \cdot \rangle$. In this notation $|x\rangle\langle y|$ stands for the rank-one operator $\delta_x \delta_y \langle \cdot, \cdot \rangle$. Similarly, for the choice $\mathcal{E} = L^2(\mathbb{T}^d)$, we often use the notation $|f\rangle$ for $f \in L^2(\mathbb{T}^d)$ and $\langle g|$ for $\langle g, \cdot \rangle$, $g \in L^2(\mathbb{T}^d)$. In this ‘Dirac notation’, $|f\rangle\langle g|$ stands for the rank-one operator $f \delta_{g, \cdot} \langle \cdot, \cdot \rangle$ on $L^2(\mathbb{T}^d)$.

2.2. The particle. Consider the hypercube $\Lambda = \Lambda_L = \mathbb{Z}^d \cap [-L/2, L/2]^d$, for some $L \in 2\mathbb{N}$. The particle Hilbert space is chosen as $\mathcal{H}_S = \ell^2(\Lambda)$ where the subscript $S$ refers to ‘system’.

To describe the hopping term (kinetic energy), we choose a real function $\varepsilon : \mathbb{T}^d \to \mathbb{R}$ and we consider the self-adjoint operator $T \equiv T^\Lambda$ on $\ell^2(\Lambda)$ with symmetric kernel

$$T(x, x') = \varepsilon(x - x'),$$

with $\varepsilon$ the Fourier transform of $\varepsilon$. Since we will assume $\varepsilon$ to be analytic, the hopping is short range.

A natural choice for the dispersion law is $\varepsilon(k) = \sum_j 2(1 - \cos k^j)$, corresponding to $T = -\Delta$, with $\Delta$ the lattice Laplacian on $\ell^2(\Lambda)$ with Dirichlet boundary conditions. This choice satisfies all our assumptions, to be stated in Section 2.3.1.

We define the particle Hamiltonian as

$$H_S := T - F \cdot X,$$

where $F \in \mathbb{R}^d$ is an external force field, e.g., an electric field, and $X \equiv X^\Lambda$ denotes the position operator on $\mathcal{H}_S$, defined by $X f(x) = x f(x)$. In what follows we will write $F = \lambda^2 \chi$, with $\chi$ a rescaled field, (a notation to be motivated later).

2.3. The reservoir.

2.3.1. Dynamics. For each $x \in \mathbb{Z}^d$, we define a reservoir Hilbert space at site $x$ by

$$\mathcal{H}_R_x := \Gamma_x(\ell^2(\bar{\Lambda})),$$

where $\bar{\Lambda} = \bar{\Lambda}_L = \mathbb{R}^d \cap [-L/2, L/2]^d$ and $\Gamma_x(\mathcal{E})$ is the symmetric (bosonic) Fock space over the Hilbert space $\mathcal{E}$. We assume that the reader is familiar with basic concepts of second quantization, such as Fock space and creation/annihilation operators; (we refer to, e.g., [3] for definitions and background). The total reservoir Hilbert space is defined by

$$\mathcal{H}_R := \bigotimes_{x \in \bar{\Lambda}} \mathcal{H}_R_x.$$

Note that for all $x$, the spaces $\mathcal{H}_R_x$ are isomorphic to each other. We remark that there is no compelling reason to restrict the one-site reservoirs to the same region, $[-L/2, L/2]^d$, as the particle system, but this simplifies our notation. The reservoir Hamiltonian is defined as

$$H_R := \sum_{x \in \bar{\Lambda}} \sum_{q \in \bar{\Lambda}^*} \omega(q) a_{x, q}^* a_{x, q},$$

where $\bar{\Lambda}^* = \frac{2\pi}{L} \mathbb{Z}^d$ is the set of quasi-momenta for the reservoir at site $x$, and the operators $a_{x, q}^*$ are the canonical creation/annihilation operators satisfying the commutation relations

$$[a_{x, q}, a_{x', q'}^*] = \delta_{x, x'} \delta_{q, q'}, \quad [a_{x, q}, a_{x', q'}] = [a_{x, q}^*, a_{x', q'}^*] = 0,$$

and we choose the dispersion law $\omega(q) = |q| + \delta_{q, 0}$. Note that this dispersion law corresponds to photons or phonons, except for $q = 0$, where we have modified this dispersion law at $q = 0$ by adding an infrared regularization that does not affect any of our results; e.g., if we replace $\delta_{q, 0}$ by $K \delta_{q, 0}$, with $K > 0$, then all infinite-volume objects studied in this paper are independent of $K$.

1 Later, we will consider $T^\Lambda$ as an operator on $\ell^2(\mathbb{Z}^d)$ by the natural embedding of $\ell^2(\Lambda)$ into $\ell^2(\mathbb{Z}^d)$. As such, it has the kernel

$$T^\Lambda(x, x') = \begin{cases} \varepsilon(x - x'), & \text{if } x, x' \in \Lambda, \\ 0, & \text{else} \end{cases}$$

i.e., we impose Dirichlet boundary conditions.
2.3.2. *Equilibrium state.* Next, we introduce the *Gibbs state* of the reservoir at inverse temperature $\beta$, $0 < \beta < \infty$. It is given by the density matrix

$$\rho_{R,\beta} := \frac{1}{Z_{R,\beta}} e^{-\beta H_R}, \quad \text{where} \quad Z_{R,\beta} = \text{Tr}_R[e^{-\beta H_R}], \quad (2.4)$$

where $\text{Tr}_R$ denotes the trace over $\mathcal{H}_R$.

An alternative way to describe this density matrix is to specify the expectation values of arbitrary observables, which we denote by $\langle O \rangle_{\rho_{R,\beta}} := \text{Tr}[O \rho_{R,\beta}]$. For $\varphi \in L^2(\bar{\Lambda}^*)$, we write $a_x(\varphi) = \sum_{q \in \Lambda^*} \varphi(q) a_{x,q}$, and we choose observables, $O$, to be polynomials in the operators $a_x(\varphi)$. One then finds that, for any $x, x'$ and $\varphi, \varphi' \in L^2(\bar{\Lambda}^*)$:

i. **Gauge-invariance:**

$$\langle a^*_x(\varphi) \rangle_{\rho_{R,\beta}} = \langle a_x(\varphi) \rangle_{\rho_{R,\beta}} = 0; \quad (2.5)$$

ii. **Two-point correlations:** Let $g_\beta := (e^{\beta \omega} - 1)^{-1}$, with the one-particle dispersion law $\omega(q) = |q| + \delta_{q,0}$, be the Bose-Einstein density (operator). Then

$$\left( \begin{array}{cc} \langle a^*_x(\varphi) a_{x'}(\varphi') \rangle_{\rho_{R,\beta}} & \langle a^*_x(\varphi) a^*_{x'}(\varphi') \rangle_{\rho_{R,\beta}} \\ \langle a_x(\varphi) a_{x'}(\varphi') \rangle_{\rho_{R,\beta}} & \langle a_x(\varphi) a^*_{x'}(\varphi') \rangle_{\rho_{R,\beta}} \end{array} \right) = \delta_{x,x'} \left( \begin{array}{cc} 0 & 0 \\ 0 & \langle \varphi, (1 + g_\beta) \varphi' \rangle \end{array} \right);$$

iii. **Wick’s theorem:**

$$a^{x_{2n}}(\varphi_{2n}) \ldots a^{x_1}(\varphi_1)_{\rho_{R,\beta}} = \sum_{\pi \in \text{Pair}(n)} \prod_{(r,s) \in \pi} \langle a^{x_r}(\varphi_s) a^*_{x_s}(\varphi_r) \rangle_{\rho_{R,\beta}}, \quad (2.6)$$

$$a^{x_{2n+1}}(\varphi_{2n+1}) \ldots a^{x_1}(\varphi_1)_{\rho_{R,\beta}} = 0, \quad (2.7)$$

where $\text{Pair}(n)$ denotes the set of partitions of $\{1, \ldots, 2n\}$ into $n$ pairs and the product is over these pairs $(r, s)$, with the convention that $r < s$. Here, $\#$ stands either for $*$ or nothing.

2.4. **The interaction.** We define the Hilbert space of state vectors of the coupled system (particle and reservoirs) by

$$\mathcal{H} := \mathcal{H}_S \otimes \mathcal{H}_R .$$

We pick a smooth ‘structure factor’ $\phi \in L^2(\mathbb{R}^d)$ and we define its finite volume version $\phi^\Lambda \in L^2(\bar{\Lambda}^*)$ by $\phi^\Lambda(q) = (2\pi/L)^{d/2} \phi(q)$, with the normalization chosen such that $\|\phi\|_{L^2(\mathbb{R}^d)} = \lim_{L \to \infty} \|\phi^\Lambda\|_{L^2(\bar{\Lambda}^*)}$. We will drop the superscript $\Lambda$. The interaction between the particle and the reservoir at site $x$ is given by

$$\mathbb{1}_x \otimes \Psi_x(\phi), \quad \text{where} \quad \Psi_x(\phi) = a_x(\phi) + a_x^*(\phi)$$

is the field operator, and $\mathbb{1}_x = |x\rangle \langle x|$ denotes the projection onto the lattice site $x$. The interaction Hamiltonian is taken to be

$$H_{SR} := \sum_{x \in \Lambda} \mathbb{1}_x \otimes \Psi_x(\phi) \quad \text{on} \quad \mathcal{H}_S \otimes \mathcal{H}_R .$$

The total Hamiltonian of the interacting system on $\mathcal{H}$ is then given by

$$H := T \otimes 1 - \lambda^2 X \otimes 1 + 1 \otimes H_R + \lambda H_{SR} , \quad (2.8)$$

where $\lambda \in \mathbb{R}$ is a coupling constant. The interaction term $H_{SR}$ is relatively bounded w.r.t. $H_S + H_R$ with arbitrarily small relative bound. It follows that $H$ is essentially selfadjoint on the domain $\mathcal{H}_S \otimes \text{Dom}(H_R)$, (where $\text{Dom}(H_R)$ denotes the domain of $H_R$).
2.5. Effective dynamics and correlation functions. The time-evolution in the Schrödinger picture is given by
\[ \rho_t = e^{-itH} \rho e^{itH}, \quad \rho \in \mathcal{B}(\mathcal{H}). \]

We will first choose an initial state \( \rho \) of the form \( \rho = \rho_S \otimes \rho_{R,\beta} \), with \( \rho_{R,\beta} \) as defined above. Of course, \( \rho_t \), with \( t > 0 \), will in general not be a simple tensor product, but we can always take the partial trace, \( \text{Tr}_R[\cdot] \), over \( \mathcal{H}_R \) to obtain the ‘reduced density matrix’ \( \rho_{S,t} \) of the system;
\[ \rho_{S,t} = \text{Tr}_R \left[ e^{-itH} (\rho_S \otimes \rho_{R,\beta}) e^{itH} \right] =: \mathcal{Z}_{[0,t]} \rho_S, \]
and we call \( \mathcal{Z}_{[0,t]} : \mathcal{B}(\mathcal{H}_S) \to \mathcal{B}(\mathcal{H}_S) : \rho_S \mapsto \rho_{S,t} \) the reduced or effective dynamics. It is a trace-preserving and completely positive map.

In the present paper, we will mainly consider observables of the form \( O \otimes 1 \) with \( O \in \mathcal{B}(\mathcal{H}_S) \), in which case we can also write
\[ \langle O(t) \rangle_{\rho_S \otimes \rho_{R,\beta}} := \text{Tr} \left[ O(t) \rho_S \otimes \rho_{R,\beta} \right] = \mathcal{Z}_S \langle O_{S,t} \rangle, \quad (2.9) \]
where the trace \( \text{Tr}[\cdot] \) is over the Hilbert space \( \mathcal{H} \), the trace \( \text{Tr}_S[\cdot] \) is over the particle Hilbert space \( \mathcal{H}_S \) and \( O(t) \) is the Heisenberg picture time evolution of the observable \( O \otimes 1 \), i.e.,
\[ O(t) := e^{itH} (O \otimes 1) e^{-itH}. \quad (2.10) \]
Note that \( O(t) \) is, in general, not of the product form \( O' \otimes 1 \), for some \( O' \).

Next, consider several observables \( O_1, \ldots, O_m \in \mathcal{B}(\mathcal{H}) \) and a set of times \( t_1, \ldots, t_m \in \mathbb{R} \). For \( \rho_S \in \mathcal{B}(\mathcal{H}_S) \) we define correlation functions by the formula
\[ \langle O_{m}(t_m) \ldots O_1(t_1) \rangle_{\rho_S \otimes \rho_{R,\beta}} := \text{Tr} \langle O_{m}(t_m) \ldots O_1(t_1) \rangle_{\rho_{S,t} \otimes \rho_{R,\beta}} \],
the trace being over the Hilbert space \( \mathcal{H} \).

2.5.1. Equilibrium states. Apart from an initial state (density matrix) of the product form \( \rho_S \otimes \rho_{R,\beta} \), we also consider the Gibbs state of the coupled system when the external force field vanishes, \( \chi = 0 \). In finite volume, it is defined by
\[ \rho_{\beta} := \frac{1}{Z_{\beta}} e^{-\beta H_{x=0}}, \quad Z_{\beta} = \text{Tr} e^{-\beta H_{x=0}}, \quad H_{x=0} = T \otimes 1 \otimes H_{R} + \lambda H_{SR}, \]
and one easily checks that \( \rho_{\beta} \in \mathcal{B}(\mathcal{H}) \). The correlation functions determined by \( \rho_{\beta} \) are written as
\[ \langle O_{m}(t_m) \ldots O_1(t_1) \rangle_{\rho_{\beta}} := \text{Tr} \langle O_{m}(t_m) \ldots O_1(t_1) \rangle_{\rho_{\beta}}, \]
with \( O_1, \ldots, O_m \) and \( t_1, \ldots, t_m \) as in (2.11).

2.5.2. Time-reversal. We define an anti-linear time-reversal operator \( \Theta = \Theta_S \otimes \Theta_R \), where \( \Theta_S \) is given by
\[ \Theta_S f(x) = f(-x), \quad f \in \ell^2(\Lambda), \]
and \( \Theta_R \) by \( \Theta_R := \Gamma_x(\theta_R) \), with the one-particle operator \( \theta_R \) given by
\[ \theta_R \varphi_x(q) = \varphi_x(-q), \quad \varphi_x \in \ell^2(\Lambda^*), \quad x \in \Lambda. \]
If the dispersion law \( \varepsilon \) of the particle and the form factor \( \phi \) are invariant under time-reversal, i.e., \( \varepsilon(k) = \varepsilon(-k) \), \( \phi(q) = \phi(-q) \) (as will be assumed) then we have that
\[ \Theta H_{x=0} \Theta = H_{x=0}, \quad \Theta \rho_{\beta} \Theta = \rho_{\beta}, \]
expressing time-reversal invariance of the model.
2.5.3. KMS condition. The KMS condition characterizing the Gibbs state $\rho_\beta$ can be expressed as follows: Denote by $\mathbb{H}_\beta$ the strip

$$\mathbb{H}_\beta := \{ z \in \mathbb{C} : 0 \leq \text{Im} z \leq \beta \}. \quad (2.12)$$

Then, for $O_1, O_2 \in \mathcal{B}(\mathcal{H})$, the correlation function $z \mapsto \langle O_2^{\delta=0}(z)O_1 \rangle_{\rho_\beta}$ is analytic in the interior of the strip $\mathbb{H}_\beta$, bounded and continuous on $\mathbb{H}_\beta$, and satisfies the KMS (boundary) condition

$$\langle O_2^{\delta=0}(t)O_1 \rangle_{\rho_{\beta,n}} = \langle O_1 O_2^{\delta=0}(t+i\beta) \rangle_{\rho_{\beta,n}}, \quad t \in \mathbb{R}. \quad (2.13)$$

This follows from the cyclicity of the trace. Note that we write $O^{\delta=0}(t)$ to indicate that, here, the time evolution is generated by $H^{\delta=0}$.

3. Assumptions and Results

3.1. Assumptions. The model introduced in the last section is parametrized by two functions: the dispersion law $\varepsilon : \mathbb{T}^d \to \mathbb{R}$, and the form factor $\phi : \mathbb{R}^d \to \mathbb{C}$. In this subsection, we formulate our assumptions on these two functions. The (multi-) strip $\mathcal{V}_\delta$ is defined by

$$\mathcal{V}_\delta := \{ z \in (\mathbb{T} + iT)^d : |\text{Im} z| \leq \delta \}. \quad (3.1)$$

**Assumption A.** [Particle dispersion] The function $\varepsilon$ extends to an analytic function in a region containing a strip $\mathcal{V}_\delta, \delta > 0$. In particular, the norm

$$\| \varepsilon \|_{\infty, \delta} := \sup_{p \in \mathcal{V}_\delta} |\varepsilon(p)|$$

is finite, for some $\delta > 0$. Furthermore, there does not exist any $v \in \mathbb{R}^d$ such that the function

$$\mathbb{T}^d \ni k \mapsto (v, \nabla \varepsilon(k))$$

vanishes identically.

This assumption allows us to estimate the free particle propagator $e^{-itH_\beta}$ on the particle Hilbert space $\mathcal{H}_\beta = \ell^2(\Lambda_L)$ as follows:

$$|(e^{-itH_\beta})(x,x')| \leq Ce^{-\nu|x-x'|}e^{\ell\|\varepsilon\|_{\infty, \nu}}. \quad (3.2)$$

For $L = \infty$, the bound $|3.2$ is the Combes-Thomas bound; for finite $L$, it can be established in an analogous way. If we replace $\mathbb{T}^d$ by $\mathbb{R}^d$, any physically acceptable dispersion law $\varepsilon$ is unbounded, and there is no exponential decay in $|x-x'|$. This is the main reason why the system studied in this paper is defined on a lattice.

The next assumption deals with the ‘time-dependent’ correlation function defined (in finite-volume) as

$$\hat{\psi}^\Lambda(t) := \sum_{q \in \Lambda} |\phi(q)|^2 \left( \frac{e^{-it\omega(q)}}{e^{\beta\omega(q)} - 1} + \frac{e^{it\omega(q)}}{1 - e^{-\beta\omega(q)}} \right), \quad (3.3)$$

and in the thermodynamic limit as

$$\hat{\psi}(t) := \int dq |\phi(q)|^2 \left( \frac{e^{-it|q|}}{e^{\beta|q|} - 1} + \frac{e^{it|q|}}{1 - e^{-\beta|q|}} \right). \quad (3.4)$$

Since the correlation function $\hat{\psi}$ is determined by the form factor $\phi$, the following assumption is in fact a constraint on the choice of $\phi$.

**Assumption B.** [Decay of reservoir correlation function] The form factor $\phi$ is a spherically symmetric function, i.e., $\phi(q) \equiv \phi(|q|)$. The correlation functions $\hat{\psi}^\Lambda(z), \hat{\psi}(z)$ are uniformly bounded in $\Lambda$ and $z \in \mathbb{H}_\beta$, (see $2.12$), and

$$\lim_{\Lambda} \hat{\psi}^\Lambda(z) = \hat{\psi}(z)$$
holds uniformly on compacts in $\mathbb{H}_\beta$, where $\lim_\Lambda$ stands for $\lim_{L \to \infty}$ (recall that $\Lambda \equiv \Lambda_L$). Furthermore, the number

$$\sum_{q \in \Lambda^*} \omega(q)^{-1} |\phi^\Lambda(q)|^2$$

is bounded uniformly in $\Lambda$. Most importantly, $\hat{\psi}(z)$ is continuous on $\mathbb{H}_\beta$ and

$$|\hat{\psi}(z)| \leq C e^{-\gamma n|z|}, \quad z \in \mathbb{H}_\beta.$$

This assumption entails that the reservoirs exhibit exponential loss of memory. This is a key ingredient for our analysis.

Often, one also considers the ‘spectral density’

$$\psi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \hat{\psi}(t) e^{it\omega}.$$  \hspace{1cm} (3.6)

It satisfies the so-called ‘detailed balance’ property $e^{i\omega}\psi(\omega) = \psi(-\omega)$, which expresses, physically, that the reservoir is in thermal equilibrium at inverse temperature $\beta$.

Assumptions [A] and [B] are henceforth required and will not be repeated.

3.2. Thermodynamic limit. Up to this point, we have considered a system in a finite volume (cube), $\Lambda$ or $\hat{\Lambda}$, characterized by its linear size $L$. However, if we wish to study dissipative effects, we must, of course, pass to the thermodynamic limit, in order to eliminate finite-volume effects such as Poincaré recurrence. This amounts to taking $\Lambda = \mathbb{Z}^d, \hat{\Lambda} = \mathbb{R}^d$ and is accomplished below.

In this section, we will explicitly put a label $\Lambda$ on all quantities referring to a system in a finite volume.

3.2.1. Observables of the system. We begin by defining some classes of infinite-volume system observables, (i.e., certain types of bounded operators on $\mathcal{H}_s$). We say that an operator $O \in \mathcal{B}(\mathcal{H}_s)$ is exponentially localized whenever

$$|O(x, x')| \leq C e^{-\nu(|x| + |x'|)}, \quad \text{for some } \nu > 0.$$  \hspace{1cm} (3.7)

An important rôle is played by the so-called quasi-diagonal operators. These are operators $O \in \mathcal{B}(\mathcal{H}_s)$ with the property that

$$|O(x, x')| \leq C e^{-\nu(x-x')} , \quad \text{for some } \nu > 0.$$  \hspace{1cm} (3.8)

We denote by $\mathfrak{O}$ the class of quasi-diagonal operators and by $\mathfrak{A}$ its norm-closure.

An observable $O \in \mathcal{B}(\mathcal{H}_s)$ is said to be translation-invariant whenever $T_y O = O$, for arbitrary $y \in \mathbb{Z}^d$, where $T_y O(x) := O(x + y, x' + y)$. Translation-invariant operators on $\mathcal{H}_s$ form a commutative $C^*$-algebra denoted by $\mathcal{C}_{ti}$. We also introduce the algebras

$$\mathfrak{O}_{ti} := \mathfrak{C}_{ti} \cap \mathfrak{O}, \quad \mathfrak{A}_{ti} := \mathfrak{C}_{ti} \cap \mathfrak{A}.$$  

An operator $O \in \mathcal{C}_{ti} / \mathfrak{O}_{ti} / \mathfrak{A}_{ti}$ can be identified with a multiplication operator, $M_f$, on the Hilbert space $L^2(\mathbb{T}^d)$, i.e., $M_f g = f g$, $g \in L^2(\mathbb{T}^d)$, with $f : \mathbb{T}^d \to \mathbb{C}$ a bounded and measurable/real-analytic/continuous function. Physically, the variable in $\mathbb{T}^d$ is the momentum of the particle.

These classes of operators are introduced because certain expansions used in our analysis will apply to quasi-diagonal operators or translation-invariant quasi-diagonal operators, and they can be extended to the closures of these algebras by density.

In analyzing diffusion and in the proof of the Einstein relation we also need to consider certain observables that are unbounded operators: We introduce the $*$-algebra $\mathfrak{X}$ that consists of polynomials in the components, $X^i, i = 1, \ldots, d$, of the particle-position operator $X$.

Given an infinite-volume observable $O \in \mathcal{B}(\mathcal{H}_s), \mathcal{H}_s = \ell^2(\mathbb{Z}^d)$, or $O \in \mathfrak{X}$, we associate an observable $O^\Lambda = 1_\Lambda O 1_\Lambda$ on $\mathcal{H}_s^\Lambda = \ell^2(\Lambda)$ with it, where $1_\Lambda$ is the orthogonal projection $\ell^2(\mathbb{Z}^d) \to \ell^2(\Lambda)$. 

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3.2.2. Dynamics. We choose not to construct directly the time-evolution of infinite-volume observables and infinite-volume states, although this could be done by using the Araki-Woods representation of the system in the thermodynamic limit. Instead, we will analyze the infinite-volume dynamics of ‘reduced’ states, i.e., of states restricted to particle observables and correlation (Green) functions of particle-observables by constructing these objects as thermodynamic limits of finite-volume expressions.

An infinite-volume density matrix of the particle system \( \rho_S \in \mathcal{B}_1(\mathcal{H}_S) \) is called exponentially localized if

\[ |\rho_S(x, x')| \leq Ce^{-\nu(|x|+|x'|)}, \quad \text{for some } \nu > 0. \]

(3.7)

Given such an infinite-volume density matrix \( \rho_S \), we associate finite-volume density matrices

\[ \rho^\Lambda_S := \frac{1}{Z^\Lambda_S} \mathbb{1}_\Lambda \rho_S \mathbb{1}_\Lambda \in \mathcal{B}_1(\mathcal{H}^\Lambda_S), \quad Z^\Lambda_S := \text{Tr}_S[\mathbb{1}_\Lambda \rho_S \mathbb{1}_\Lambda], \]

(3.8)

with it. Note that, due to the normalization by \( Z^\Lambda_S \), \( \rho^\Lambda_S \) is a density matrix on \( \mathcal{H}^\Lambda_S \).

Recall the definition of the reduced dynamics, \( Z^\Lambda_{[0,\ell];} \), introduced in Section 2.5.

We set

\[ Z_{[0,\ell];} := \lim_{\Lambda \to \infty} Z^\Lambda_{[0,\ell];}. \]

(3.9)

The next lemma asserts that the thermodynamic limit (as \( \Lambda \) and \( \Lambda \) increase to \( \mathbb{Z}^d \), \( \mathbb{R}^d \), respectively) in (3.9) exists, and that the resulting reduced dynamics \( Z_{[0,\ell];} \) is translation-invariant.

**Lemma 3.1.** The limit on the right side of Equation (3.9) exists in \( \mathcal{B}_1(\mathcal{H}_S) \), and this defines the map \( Z_{[0,\ell];} : \mathcal{B}_1(\mathcal{H}_S) \to \mathcal{B}_1(\mathcal{H}_S) \). The map \( Z_{[0,\ell];} \) preserves the trace, i.e., \( \text{Tr}_S[\rho_S \rho_S] = \text{Tr}_S[\rho_S] \), positivity and exponential localization of the state of the particle, i.e., if \( \rho_S \) has any of these properties, then so does \( Z_{[0,\ell];} \rho_S \). Moreover, \( Z_{[0,\ell];} \) is translation-invariant; \( T_y Z_{[0,\ell];} T_y = Z_{[0,\ell];} \) for \( y \in \mathbb{Z}^d \) with \( T_y \) as in Subsection 3.2.1. As a consequence of the above, for \( O \in \mathfrak{A} \) or \( \mathfrak{X} \), and for an exponentially localized state \( \rho_S \), we can define

\[ \langle O(t) \rangle_{\rho_S^{\Lambda}} := \text{Tr}_S[O Z_{[0,\ell];} \rho_S]. \]

3.2.3. Correlation functions. Next, we define the infinite-volume analogues of the finite-volume correlation functions

\[ \langle O^\Lambda_m(t_m) \cdots O^\Lambda_1(t_1) \rangle_{\rho^\Lambda_S} \quad \text{and} \quad \langle O_m^\Lambda(t_m) \cdots O_1^\Lambda(t_1) \rangle_{\rho^\Lambda_S}, \]

that have been introduced in Section 2.5.

Consider observables \( O_1, \ldots, O_m \) in \( \mathfrak{A} \) or \( \mathfrak{X} \), times \( 0 \leq t_1 < \ldots < t_m \), and an exponentially localized density matrix \( \rho_S \). We then define

\[ \langle O_m(t_m) \cdots O_1(t_1) \rangle_{\rho_S^{\Lambda}} := \lim_{\Lambda \to \infty} \langle O^\Lambda_m(t_m) \cdots O^\Lambda_1(t_1) \rangle_{\rho^\Lambda_S}. \]

(3.10)

Similarly, for observables \( O_1, \ldots, O_m \in \mathfrak{A} \) and times \( 0 \leq t_1 < \ldots < t_m \), we define

\[ \langle O_m(t_m) \cdots O_1(t_1) \rangle_{\rho_S} := \lim_{\Lambda \to \infty} \langle O^\Lambda_m(t_m) \cdots O^\Lambda_1(t_1) \rangle_{\rho^\Lambda_S}. \]

(3.11)

Note that we construct the thermodynamic limit of equilibrium correlation functions only for translation-invariant observables, since, pictorially, the particle is uniformly distributed in space and hence the expectation values of localized observables vanish. Also note that, in Equation (3.11), we do not constrain the time-evolution to be the one generated by the Hamiltonian with \( \chi = 0 \). But, of course, we have to do so if we want the correlation functions to be stationary in time, as in the following lemma.

**Lemma 3.2.** The limits on the right hand sides of Equations (3.10) and (3.11) exist. For \( m = 1, 2 \), they exist for \( t_1, t_2 \in \mathbb{R} \) (i.e., not necessarily positive or ordered). For the dynamics with \( \chi = 0 \), and for arbitrary observables \( O_1, O_2 \in \mathfrak{A} \), writing \( O(t) \), instead of \( O^{x=0}(t) \), and \( O_\Theta \), instead of \( \Theta_S O \Theta_S \), the following properties hold:

i. Stationarity: \( \langle O_2(t_2) O_1(t_1) \rangle_{\rho_S} = \langle O_2(t_2 + t) O_1(t_1 + t) \rangle_{\rho_S}, \) for any \( t \in \mathbb{R}; \)

ii. Time-reversal invariance: \( \langle O_2,\Theta(-t_2) O_1,\Theta(-t_1) \rangle_{\rho_S} = \langle O_2(t_2) O_1(t_1) \rangle_{\rho_S}; \)
exists a real-analytic function \( f_{O_1, O_2}(z) \), analytic in the interior of the strip \( \mathbb{H}_\beta \), bounded and continuous on \( \mathbb{H}_\beta \), that satisfies the KMS (boundary) condition

\[
f_{O_1, O_2}(t) := \langle O_2 O_1(t) \rangle_{\rho^s}, \quad f_{O_1, O_2}(t + i\beta) = \langle O_1(t) O_2 \rangle_{\rho^s}, \quad t \in \mathbb{R}.
\]

We remark that there is no particular reason to limit our construction of general correlation functions to one- and two-point functions, \( m = 1, 2 \). However, focusing on these special cases will enable us to keep our notation manageable in the technical sections. Lemmas 5.1 and 5.2 are proven in Section 5 using rather straightforward estimates. These are the only ones among our results that do not require exponential decay of the reservoir correlation function \( \hat{\psi} \) (cf. Assumption [B]), nor small coupling, \( \lambda \), or weak external field, \( \chi \).

### 3.3. Results

Next, we summarize our main results. Throughout this section, it is understood that we consider the infinite-volume system; i.e., \( \Lambda = \mathbb{Z}^d \), \( \hat{\Lambda} = \mathbb{R}^d \).

Theorem 3.3 concerns the system in equilibrium, i.e., in a vanishing external force field, and asserts that, in this case, the system has the property of ‘return to equilibrium’. Theorem 3.4 states the corresponding result for an off-equilibrium system: It claims that the state of the system approaches, for small external force fields \( \chi \), a ‘Non-Equilibrium Stationary State’ (NESS) in the limit of large times. Theorem 3.5 asserts that the motion of the particle is diffusive at large times. For \( \chi \neq 0 \), this diffusive motion is around an average uniform motion (i.e., a drift at a constant velocity). Our last result, Theorem 3.6, confirms the fluctuation-dissipation formula for the particle’s motion to the field \( \chi \) through the ‘Einstein relation’.

Theorems 3.3 and 3.4 have already appeared in [3], (see Theorems 3.2 and 3.3). We restate them here for completeness. The main purpose of the present paper is to prove Theorems 3.3 and 3.6 besides developing analytical techniques and establishing technical results that have been used to prove various results in [3].

Our first result (partially contained in [8]) concerns the model without external force field, i.e., \( \chi = 0 \).

**Theorem 3.3.** [Return to equilibrium] Let \( \chi = 0 \). Then there exist a constant \( k_\lambda > 0 \) and a decay rate \( g > 0 \) such that, for \( 0 < |\lambda| < k_\lambda \), the following holds. For an arbitrary exponentially localized density matrix \( \rho_s \), arbitrary observables \( O_1, \ldots, O_m \in \mathfrak{A}_{ni} \) and times \( 0 \leq t_1 < \ldots < t_m \),

\[
\langle O_m(t_m) \cdots O_1(t_1) \rangle_{\rho_s \otimes \rho_{ni}} = \langle O_m(t_m) \cdots O_1(t_1) \rangle_{\rho_s} + O(e^{-\lambda^2 g t_1}), \quad \text{as } t_1 \to \infty,
\]

and the correlation functions exhibit the following ‘exponential cluster property’:

\[
\langle O_2(t_2) O_1(t_1) \rangle_{\rho_s} = \langle O_1 \rangle_{\rho_s} \langle O_2 \rangle_{\rho_s} + O(e^{-\lambda^2 g |t_2 - t_1|}), \quad \text{as } t_2 - t_1 \to \infty.
\]

(In these equations, \( O(t) \) stands for \( O^{\chi=0}(t) \).)

In (3.1.3), we consider only two observables; but an analogous statement holds for \( n > 2 \) observables.

As already remarked, \( \mathfrak{A}_{ni} \) is commutative. Hence the positive and normalized functional \( O \mapsto \langle O \rangle_{\beta} \) on \( \mathfrak{A}_{ni} \) can be expressed in terms of a probability measure. Recall that \( O \in \mathfrak{A}_{ni} \) is of the form \( M_f \) for \( f \in C(\mathbb{T}^d) \). Anticipating the notation of Theorem 3.3, we can write

\[
\langle M_f \rangle_{\rho_s} = \langle f, \zeta^{0, \lambda} \rangle_{L^2(\mathbb{T}^d)}.
\]

Next, we state results that hold off equilibrium: In the theorems below, we use the notation \( O(t) \) for \( O^{\chi}(t) \), even if \( \chi \neq 0 \).

Our next result describes the approach of the state of the system to a ‘Non-Equilibrium Stationary State’ (NESS), in the limit of large times. However, it is slightly weaker, because we are forced to consider ergodic averages, since the external force field attenuates dissipative effects of the reservoirs; for a more extended discussion we refer to [3]. In fact, in statement (3.14) of Theorem 3.5 we cannot even control the ergodic average, but only the abelian average.

**Theorem 3.4.** [Approach to NESS] There are constants \( k_0, k_\chi \), such that, for \( 0 < |\lambda| < k_0 \), \( |\chi| < k_\chi \), there exists a real-analytic function \( \zeta \equiv \zeta^{0, \lambda} \) on \( \mathbb{T}^d \), satisfying \( \zeta \geq 0 \) and \( \int_{\mathbb{T}^d} dk \zeta(k) = 1 \), i.e., \( \zeta \) is a probability density, such that the following statements hold for any exponentially localized density matrix, \( \rho_s \), and continuous function \( f : \mathbb{T}^d \to \mathbb{R} \):
i. For $\chi \neq 0$,
\[
\frac{1}{T} \int_0^T dt \langle M_f(t) \rangle_{\rho_\beta \otimes \rho_m, \beta} = \langle f, \zeta^{(\Lambda)} \chi \rangle_{L^2(T^* \mathcal{V})} + \mathcal{O}(1/T), \quad \text{as } T \to \infty.
\]

ii. For $\chi = 0$,
\[
\langle M_f(t) \rangle_{\rho_\beta \otimes \rho_m, \beta} = \langle f, \zeta^{0,\Lambda} \rangle_{L^2(T^* \mathcal{V})} + \mathcal{O}(e^{-\lambda^2 g t}), \quad \text{as } t \to \infty,
\]
where $g > 0$ is the decay constant appearing in (3.12) and (3.13). Moreover $\zeta^{0,\Lambda}$ satisfies ‘time reversal invariance’; $\zeta^{0,\Lambda}(k) = \zeta^{0,\Lambda}(-k)$.

Our next result asserts that the motion of the particle is diffusive around an average uniform motion.

**Theorem 3.5. [Diffusion]** Under the same assumptions as in Theorem 3.4
\[
\lim_{t \to \infty} \frac{1}{t} \langle X(t) \rangle_{\rho_\beta \otimes \rho_m, \beta} = v(\chi),
\]
where $v(\chi)$ is the ‘asymptotic velocity’ of the particle and is given by $v(\chi) = \langle \nabla \varepsilon, \zeta^{(\Lambda)} \chi \rangle$. For $\chi \neq 0$, we have $v(\chi) \neq 0$. The dynamics of the particle is diffusive, in the sense that the limits
\[
D^{ij}(\chi) := \lim_{T \to \infty} \frac{1}{T^2} \int_0^T \int_0^T dt e^{-\frac{t}{T}} \langle (X^i(t) - v(\chi)t)(X^j(t) - v(\chi)t) \rangle_{\rho_\beta \otimes \rho_m, \beta}
\]
exist, where the ‘diffusion tensor’ $D(\chi)$ is positive-definite, with $D(\chi) = \mathcal{O}(\lambda^{-2})$, as $\lambda \to 0$.

Note that the claim about the asymptotic velocity follows formally from Theorem 3.4 by defining the velocity operator as
\[
V^i := i[H, X^i] = i[T, X^i] = M_{V^i, \chi},
\]
and writing $X(t) = X(0) + \int_0^t ds V(s)$. Although it is quite easy to make this reasoning precise, we warn the reader that, at this point, it is formal, because the Heisenberg-picture observables $X(t)$ and $V(t)$ have not been constructed as operators in the thermodynamic limit. They are formal objects appearing in correlation functions that are constructed as thermodynamic limits of finite-volume correlation functions.

Our next result states that the equilibrium diffusion matrix $D(\chi = 0)$ (which is in fact a multiple of the identity matrix) is related to the response of the particle’s motion to the field $\chi$. The corresponding identity is known as the ‘Einstein relation’:

**Theorem 3.6. [Einstein relation]** Under the same assumptions as in Theorem 3.4
\[
\frac{\partial}{\partial \chi} \bigg|_{\chi = 0} v^j(\chi) = \lambda^2 \beta D^{ij}(\chi = 0) = 0,
\]
where $D(\chi = 0)$ is defined in Equation (3.14) and it equals
\[
D^{ij}(\chi = 0) = \frac{1}{2} \int \langle V^i(t)V^j(t) \rangle_{\rho_\beta}.
\]

Note that, by the positivity and isotropy of the diffusion matrix, this theorem also shows that, for small but non-zero $\chi$, $v(\chi)$ does not vanish. The origin of the unfamiliar factor $\lambda^2$ on the right side of (3.16) is found in the fact that the driving force field in the Hamiltonian is $\lambda^2 \chi$, rather than $\chi$.

### 4. Dyson expansion: The formalism

In this section and the next one, we expand the effective dynamic $\mathcal{Z}_{[0,t]}^{\Lambda}$ (see Section 2.5) and correlation functions in absolutely convergent series. This task is carried out in two steps: First, we derive the expansions without worrying about careful estimates. To avoid ambiguities concerning the definition of operators, we do this in finite volume. This is done in the present section, which is therefore essentially an algebraic exercise, because the convergence of the expansions is a trivial matter. In a second step, which is postponed to Section 5, we derive bounds on these expansions and prove their convergence uniformly in $\Lambda$.

In the present section, we set $\mathcal{H}_{\Lambda}^\beta = \ell^2(L^2(\Lambda)), \mathcal{H}_{\Lambda}^{\text{R}} = \Gamma_*\ell^2(\Lambda)$, etc., as in Section 2.3. To keep notations simple, we temporarily drop the superscript $\Lambda$ everywhere. We start by defining ‘Green functions’. 
4.1. Green functions. The (interacting) Green functions are defined as follows: Recall the equilibrium density matrix of the reservoirs $p_{r_\beta} = Z_{r_\beta}^{-1} e^{-β H_{r_\beta}}$, $Z_{r_\beta} = \text{Tr} e^{-β H_{r_\beta}}$, and let $\rho$ be a density matrix on $\mathcal{H}_S$. We define a map $Q : \mathcal{B}_1(\mathcal{H}_R) \to \mathcal{B}_1(\mathcal{H}) : \rho \mapsto \rho \otimes p_{r_\beta}$. Let $I \subset \mathbb{R}_+$ be a finite time interval. Subsequently, we abbreviate $\inf I$ and $\sup I$ by $t_-(I)$ and $t_+(I)$, respectively. The length of an interval $I$ is denoted by $|I| := t_+(I) - t_-(I)$. Let $S_1, \ldots, S_m$ be operators acting on $\mathcal{B}_2(\mathcal{H}_S)$. The most relevant choice will be $S_j$ equal to $(O_i)_t$ or $(O_i)_r$ for some ‘observables’ $O_i \in \mathcal{B}(\mathcal{H}_S)$, $i = 1, \ldots, m$, (we use here the left- and right-multiplications that were defined in (2.1)). The (interacting) Green function on $I$ is defined as the map

$$G_I(S_1^{s_1}, \ldots, S_m^{s_m}) : \mathcal{B}_2(\mathcal{H}_S) \longrightarrow \mathcal{B}_2(\mathcal{H}_S)$$

by

$$G_I(S_1^{s_1}, \ldots, S_m^{s_m})(\cdot) := \text{Tr}_R \left[ e^{-i(t_+ - t_-) S} e^{-i(t_m - t_{m-1}) S} \cdots e^{-i(t_1 - t_0) S} \mathcal{Q} (\cdot) \right],$$

(4.1)

where $t_-(I) \leq s_1 < s_2 < \ldots < s_m \leq t_+(I)$, and the trace is over the reservoir Hilbert space $\mathcal{H}_R$. Here, $L := \text{ad}(H)[H, \cdot]$ denotes the Liouvillian associated to $H$, an essentially self-adjoint operator on $\mathcal{B}_1(\mathcal{H}_S)$. The notation $S_j^{s_j}$, $j = 1, \ldots, m$, merely indicates where the operator $S_j$ should be placed on the right side of (4.1). In particular, $S_j^\alpha$ is not the operator $S_j$ time-evolved to $\alpha$. Since the operators carry a time label $s_j$, their order in the bracket on the left side of (4.1) is irrelevant. We remark that we have defined the Green functions as operators on $\mathcal{B}_2(\mathcal{H}_S)$, i.e., we view density matrices as Hilbert-Schmidt operators through the embedding $\mathcal{B}_1(\mathcal{H}_S) \subset \mathcal{B}_2(\mathcal{H}_S)$ since it is more convenient to work with the Hilbert space $\mathcal{B}_2(\mathcal{H}_S)$.

A special case of interest is $m = 0$, i.e., when no $S_j$’s are present in the Green function. We set

$$Z_I(\cdot) := G_I(\emptyset)(\cdot) = \text{Tr}_R \left[ e^{-i|I| L} \mathcal{Q} (\cdot) \right].$$

For $I = [0, t]$ this notation agrees with the notation for the effective dynamics in Section 2.5.

For later purposes, we define a special class of operators on $\mathcal{B}_2(\mathcal{H}_S)$: We say $S \in \mathcal{B}(\mathcal{B}_2(\mathcal{H}_S))$ is quasi-diagonal, whenever its kernel satisfies

$$|S(x_1, x_r; x_1', x_r')| \leq C e^{-\nu|x_1 - x_1'| - |x_r - x_r'|}, \quad \text{for some } \nu > 0, \quad (x_1, x_r, x_1', x_r') \in \mathbb{Z}^d.$$  

(4.2)

Note the analogy with quasi-diagonal observables $O \in \mathcal{B}(\mathcal{H}_S)$ defined in Section 3.2 in particular $S = (O)_{t \in [1, r]}$ is quasi-diagonal if and only if $O$ is quasi-diagonal.

4.2. An expansion for $Z_I$. In this subsection, we derive an expansion for $Z_I$, with $I \subset \mathbb{R}_+$ a finite interval. We define the free particle dynamics, $U_t$, on $\mathcal{B}_2(\mathcal{H}_S)$ by

$$U_{[t_1, t_2]} := e^{-i(t_2 - t_1) \text{ad}(H_S)}, \quad t_1, t_2 \in \mathbb{R}_+, \quad H_S = T - λ^2 \chi \cdot X,$$

(4.3)

and the particle-reservoir interaction, $H_{SR}(t)$, in the interaction picture, which we decompose in spatially localized terms

$$H_{SR}(t) := e^{itH_S} H_{SR} e^{-itH_S} = \sum_x \Psi_x(t), \quad \Psi_x(t) := e^{itH_S} (a_x^+(\phi) + a_x(\phi)) e^{-itH_S}.$$

Iterating Duhamel’s formula

$$e^{it\text{ad}(H_S)} e^{-it\text{ad}(H)} = U_{[0, t]} - iλ \int_0^t ds U_{[s, t]} \text{ad}(H_{SR}(s)) e^{is\text{ad}(H_S)} e^{-is\text{ad}(H)},$$

we find the Lie-Schwinger- or Dyson series for $Z_I$:

$$Z_I(\cdot) = \sum_{n \geq 0} (-iλ)^n \int_{t_-(I) \leq t < t_1 < \ldots < t_n < t_+(I)} dt_1 \cdots dt_n \text{Tr}_R \left[ U_{[t_1, t_2]}(t_1) \cdots \text{ad}(H_{SR}(t_n)) \cdots \text{ad}(H_{SR}(t_1)) U_{[t_n, t_-(I)]}(\cdot) \mathcal{Q}(\cdot) \right].$$

(4.4)

where the $n = 0$ term on the right side is understood as $U_I$. We refrain from giving a proof of the (norm)-convergence of this series, since we establish similar, but more involved, bounds in Section 5. We will use the shorthand notations

$$1_{x, c} := (1_x)_c, \quad \Psi_{x,c}(t) := (-i\Psi_x(t))_c.$$
for \(x \in \Lambda, \varsigma \in \{1, r\}\) (the left- and right multiplications \((\cdot)\) were introduced in (2.1)). In this notation, the formal Lie-Schweringer series for \(Z_I\) can be rewritten as:

\[
Z_I(\cdot) = \sum_{n \geq 0} (-\lambda)^n \sum_{x \in \Lambda^n} \sum_{\varsigma \subseteq \{1, r\}^n} \int_{\Delta^n_I} \text{Tr}_R \left[ \Psi_{x_n, \varsigma_n}(t_n) \cdots \Psi_{x_1, \varsigma_1}(t_1) \rho_{R, \beta} \right] \\
\times U_{(t_n, t_{+}(I))} \mathbb{1}_{x_n, \varsigma_n} U_{(t_{n-1}, t_n)} \cdots \mathbb{1}_{x_1, \varsigma_1} U_{(t_-(I), t_1)}(\cdot),
\]

where we use the shorthand

\[
\int_{\Delta^n_I} dt := \delta_{n, 0} + \int_{t_-(I) < t_1 < \ldots < t_n < t_+} dt_1 \cdots dt_n.
\]

In a next step, we evaluate the trace over the reservoir Hilbert space in Equation (4.4) using the quasi-free property of \(\rho_{R, \beta}\). To do so, we introduce some more notation also applicable to the slightly more complicated expansions of general Green functions.

### 4.3. Free Green functions, reservoir correlators and the path expansion of \(Z_I\).

Let \(S_1, \ldots, S_m \) be operators acting on \(\mathcal{B}_2(\mathcal{H}_S)\). Let \(I \subset \mathbb{R}_+\) be an interval, and choose a set of times \(t_- (I) \leq s_1 < s_2 < \ldots < s_m \leq t_+(I)\). We define **free Green functions** by

\[
G_I^0(\mathcal{S}_0^1, \ldots, \mathcal{S}_m^m) := U_{(s_m, t_+(I))} \mathcal{S}_m \cdots \mathcal{U}_{(s_1, s_2)} \mathcal{S}_1 \mathcal{U}_{(t_-(I), s_1)}.
\]

For \(m = 0\), we set \(G_I^0(\emptyset) = U_I\). As in (4.1), the time labels \(s_i\) just indicate where the operators should be placed.

Since the operators \(\mathbb{1}_{x, \varsigma}\) often show up in combination with the free time evolution \(U_I\) on \(\mathcal{B}_2(\mathcal{H}_S)\), we introduce the following shorthand notation: A **path**, \(\varpi\), over a (closed) interval \(I \subset \mathbb{R}_+\) is a finite collection of triples

\[
(x_i, \varsigma_i, t_i), \quad i = 1, 2, \ldots,
\]

where \(x_i \in \Lambda, \varsigma_i \in \{1, r\}\) and \(t_i \in I\). The number of triples in a path \(\varpi\) is denoted by \(|\varpi|\). The set of all paths over an interval \(I\), referred to as ‘path space’, is denoted by \(\mathcal{P}_I\). The free Green function associated to a path, \(G_I^0(\varpi)\), is defined as

\[
G_I^0(\varpi) := U_{(t_n, t_+(I))} \mathbb{1}_{x_n, \varsigma_n} U_{(t_{n-1}, t_n)} \cdots \mathbb{1}_{x_1, \varsigma_1} U_{(t_-(I), t_1)},
\]

where \(\varpi = ((x_1, \varsigma_1, t_1), \ldots, (x_n, \varsigma_n, t_n))\), with \(t_- (I) < t_1 < t_2 < \ldots < t_n \leq t_+(I), I \subset \mathbb{R}_+\) an interval.

We can evaluate the term containing the partial trace in (4.4) using the quasi-free property, or Wick rule, of the reservoir state: If \(n\) is odd, this term vanishes. Hence, we replace \(n\) by \(2n\) subsequently. Denote by \(\text{Pair}(n)\) be the set of partitions \(\pi\) of the integers \(1, \ldots, 2n\) into \(n\) pairs. We write \((r, s) \in \pi\) if \((r, s)\) is one of these pairs, with the convention that \(r < s\). Wick’s rule states that

\[
\lambda^{2n} \text{Tr}_R \left[ \Psi_{x_{2n}, \varsigma_{2n}}(t_{2n}) \cdots \Psi_{x_1, \varsigma_1}(t_1) \rho_{R, \beta} \right] = \sum_{\pi \in \text{Pair}(2n)} \zeta(\varpi, \pi),
\]

where \(\varpi = ((x_1, \varsigma_1, t_1), \ldots, (x_{2n}, \varsigma_{2n}, t_{2n}))\), \(\text{Pair}(\varpi) \equiv \text{Pair}(n)\), \(|\varpi| = 2n\), and

\[
\zeta(\varpi, \pi) := \prod_{(r, s) \in \pi} \lambda^2 h_{r, \varsigma, s, \varsigma'}(r, s, \varsigma, \varsigma') \delta_{x_r, x_s},
\]

where we have set, for \(u, v \in \mathbb{R}_+\),

\[
h(u, v, \varsigma, \varsigma') :=
\begin{cases}
-\hat{\psi}(u - v), & \text{if } \varsigma = 1, \varsigma' = 1, \\
-\hat{\psi}(v - u), & \text{if } \varsigma = r, \varsigma' = r, \\
\hat{\psi}(v - u), & \text{if } \varsigma = r, \varsigma' = 1, \\
\hat{\psi}(u - v), & \text{if } \varsigma = 1, \varsigma' = r,
\end{cases}
\]

with \(\hat{\psi}(t) \equiv \hat{\psi}^A(t)\) as defined in (3.3).
If \( n = 0 \), i.e., \( \varpi = \emptyset \), we set the right side of Equation (4.4) equal to one. Integration over path space \( P_I, I \subset \mathbb{R}_+ \) an interval, is denoted by the shorthand

\[
\int_{P_I} d\varpi F(\varpi) := F_0(\emptyset) + \sum_{n \geq 1} \sum_{r,s} \int \Delta_{2^n} \, d\xi F_n(\varpi),
\]

(4.10)

for \( F = (F_n)_{n \in \mathbb{N}} \), where now \( \xi \in \Lambda^{2n}, \xi \in \{1, r\}^{2n} \). We will treat \( d\varpi \) merely as a shorthand notation, though it is straightforward to check that \( d\varpi \) indeed defines a measure on an appropriate measure space. In this notation, the expansion for \( Z_I \) in Equation (4.4) takes the compact form

\[
Z_I = \int_{P_I} d\varpi \sum_{\pi \in \text{Pair}(\varpi)} \zeta(\varpi, \pi) G^0_I(\varpi),
\]

(4.11)

which we call the ‘path expansion’ of \( Z_I \).

### 4.4. Path expansion of the correlation functions.

With the formalism introduced in the previous subsections, it is straightforward to derive the path expansion for the interacting Green function \( G_I(S_1, \ldots, S_m), I \subset \mathbb{R}_+ \) an interval, where \( S_1, \ldots, S_m \) are operators acting on \( \mathcal{H}_2(\mathbb{R}) \): We expand each propagator \( e^{-i(s_j-s_{j-1})\xi} \) in Equation (1.1) in its Lie-Schwinger series and proceed as previously. As a result we obtain the path expansion for the Green function:

\[
G_I(S_1, \ldots, S_m) = \int_{P_I} d\varpi \sum_{\pi \in \text{Pair}(\varpi)} \zeta(\varpi, \pi) G^0_I(S_1, \ldots, S_m | \varpi),
\]

where we use the shorthand notation

\[
G^0_I(S_1, \ldots, S_m | \varpi) := G^0_I(S_1^{\varpi}, \ldots, S_m^{\varpi}, \|_{x_1, \xi_1}, \ldots, \|_{x_2n, \xi_{2n}}),
\]

with \( \varpi = ((x_1, \xi_1, t_1), \ldots, (x_{2n}, \xi_{2n}, t_{2n})) \).

### 4.5. Polymer expansions.

In this subsection, we rearrange our expansions in a ‘polymer’ form. This will enable us to explore the exponential decay of the reservoir correlation function \( \hat{\psi} \) in the next sections. Given that paths \( \varpi \) are collections of triples, we can define the ‘union’ path \( \varpi = \varpi_1 \cup \varpi_2 \). Let us write \( \min(t(\varpi)) \), \( \max(t(\varpi)) \) for the smallest and largest time in the path \( \varpi \), respectively. If \( \max(t(\varpi)) < \min(t(\varpi_2)) \), then we write \( \varpi_1 < \varpi_2 \).

Given two pairings \( \pi_1 \in \text{Pair}(\varpi_1), \pi_2 \in \text{Pair}(\varpi_2) \), with \( \varpi_1 < \varpi_2 \), we define \( \pi = \pi_1 \cup \pi_2 \in \text{Pair}(\varpi_1 \cup \varpi_2) \) in the obvious way, namely \( (r, s) \in \pi \) if, either \( (r, s) \in \pi_1 \), or \( (|\varpi_1| + r, |\varpi_1| + s) \in \pi_2 \). In an analogous way, we define unions of a finite ordered set of paths, i.e., \( \varpi_1 < \varpi_2 < \ldots < \varpi_I \) and pairings over them. Note the factorization property of the weights \( \zeta \) defined in Equation (4.8):

\[
\zeta(\varpi_1 \cup \varpi_2 \cup \ldots \cup \varpi_I, \pi_1 \cup \pi_2 \cup \ldots \cup \pi_I) = \zeta(\varpi_1, \pi_1) \zeta(\varpi_2, \pi_2) \ldots \zeta(\varpi_I, \pi_I).
\]

(4.12)

We call a pairing \( \pi \in \text{Pair}(\varpi) \) irreducible if it can be written as a union \( \pi = \pi_1 \cup \pi_2 \) with \( \pi_j \in \varpi_j \) with \( \varpi = \varpi_1 \cup \varpi_2 \) and \( \varpi_1 < \varpi_2 \).

We call \( D := (\varpi, \pi), \varpi \in P_I, \pi \in \text{Pair}(\varpi) \), a diagram. A diagram \( D = (\varpi, \pi) \) is irreducible over an interval \( I \) if \( \pi \) is irreducible and \( t_-(I) = \min(t(\varpi)) \), \( t_+(I) = \max(t(\varpi)) \). We define the domain of a diagram \( D \) as

\[
\text{Dom}(D) := \bigcup_{(r, s) \in \pi} [t_r, t_s].
\]

Note that a diagram \( D = (\varpi, \pi) \) is irreducible over \( I \) if and only if \( \text{Dom}(D) = I \). If a diagram is not irreducible it can be decomposed uniquely into irreducible diagrams with pairwise disjoint domains:

\[
\text{Dom}(D) = \bigcup_{j=1}^{l} \text{Dom}(D_j), \quad D_j \text{ irreducible over } I_j \subseteq I,
\]

(4.13)

for some \( l \geq 1 \), with \( D_j = (\pi_j, \varpi_j) \), such that \( \pi = \cup_j \pi_j, \varpi = \cup_j \varpi_j \) and \( \varpi_1 < \varpi_2 < \ldots < \varpi_l \). It follows that the diagrams \( D_j \) are ordered in the sense that \( \sup \text{Dom}(D_j) < \inf \text{Dom}(D_{j+1}), j = 1, \ldots, l - 1 \) and that \( I_j \) are intervals.
Let $I \subset \mathbb{R}_+$ be an interval. We define $\mathcal{V}_I \in \mathcal{B}(\mathcal{B}_2(\mathcal{H}_S))$ as the sum over the ‘amplitudes’ of all irreducible diagrams over $I$: Given a path $\varpi \in \mathcal{P}_I$, we use the ‘δ-function’
\[
\delta(\partial I - \partial \varpi) := \delta(\text{min}(t(\varpi)) - t_{-}(I)) \delta(\text{max}(t(\varpi)) - t_{+}(I)),
\]
(4.14)
to restrict the integration over path space $\mathcal{P}_I$ to paths having an element (i.e., a triple) at the initial time $t_{-}(I)$ and an element at the final time $t_{+}(I)$ of the interval $I$. Hence, $\mathcal{V}_I$ is given by
\[
\mathcal{V}_I := \int_{\mathcal{P}_I} d\varpi \, \delta(\partial I - \partial \varpi) \sum_{\pi \in \text{Pair}(\varpi) \atop \pi \text{ irreducible}} \zeta(\varpi, \pi) \mathcal{G}^0_I(\varpi),
\]
(4.15)
where we sum only over irreducible $\pi$ (alternatively, such $\pi$ that render the diagram $D = (\varpi, \pi)$ irreducible over $I$). As in (4.10), the $d\varpi$-integral and the ‘δ-function’ in (4.15) are a shorthand notation for certain sums and integrals; the δ-function in fact indicates that the integral over the time-coordinates is over $\Delta^{\mathcal{H}_S}_{\mathcal{V}^2}$ instead of $\Delta^{\mathcal{H}_S}_{\mathcal{V}^2}$. Using the factorization property of $\zeta$ and the definition of $\mathcal{V}$ we rewrite the path expansion of $\mathcal{Z}_I$ in Equation (4.11) in terms of irreducible diagrams:
\[
\mathcal{Z}_I = \sum_{l=0} \int_{\Delta^{\mathcal{H}_S}_{\mathcal{V}^2}} 1 dt \, \mathcal{G}_I(1)_{t_{[2l, t_{+}(I)]}} \mathcal{Z}_{[t_{2l}, t_{2l+1}]}(1) \mathcal{Z}_{[t_{2l-2}, t_{2l-1}]}(1) \cdots \mathcal{V}_{[t_{2}, t_{2l}]}(1) \mathcal{Z}_{[t_{-}(I), t_{2l}]}(1).
\]
This expansion can be viewed as a one-dimensional ‘polymer’ expansion. The polymers correspond to connected subsets of the interval $I$ with weights given by $\mathcal{V}$. Two polymers ‘interact’ via hard core exclusion taken into account in the integration domain $\Delta^{\mathcal{H}_S}_{\mathcal{V}^2}$. In Equation (4.15), we may consider diagrams with $|\varpi| > 2$ as ‘excitations’. Diagrams without ‘excitations’, i.e., diagrams whose irreducible decomposition contains only paths $\varpi$ with $|\varpi| = 2$, are called ladder diagrams. The origin of the nomenclature becomes clear when one only retains diagrams with $|\varpi| = 2$ in the expansion (4.16). In the following sections, we will argue that the leading contributions to $\mathcal{Z}_I$ arise from ladder diagrams.

The formalism developed above can also be applied to Green functions: We extend the definition (4.15) by setting
\[
\mathcal{V}_I(S_1^{s_1}, \ldots, S_m^{s_m}) := \int_{\mathcal{P}_I} d\varpi \, \delta(\partial I - \partial \varpi) \sum_{\pi \in \text{Pair}(\varpi) \atop \pi \text{ irreducible}} \zeta(\varpi, \pi) \mathcal{G}_I^0(S_1^{s_1}, \ldots, S_m^{s_m} | \varpi),
\]
(4.17)
for $S_1^{s_1}, \ldots, S_m^{s_m} \in \mathcal{B}(\mathcal{B}_2(\mathcal{H}_S))$, $t_{-}(I) \leq s_1 < \ldots < s_m \leq t_{+}(I)$. We refer to $\mathcal{V}_I$ as the ‘dressing operator’ in the following. For simplicity, we restrict our discussion to Green functions involving two ‘observables’ $S_1, S_2 \in \mathcal{B}(\mathcal{B}_2(\mathcal{H}_S))$. Given two times $s_1, s_2 \in \mathbb{R}_+$, we use the shorthand notation (1), (2), for $S_1^{s_1}, S_2^{s_2}$, respectively. Using the factorization property of $\zeta$ and the definition of the dressing operator $\mathcal{V}$ we find
\[
\mathcal{G}_I((1), (2)) = \mathcal{Z}_{[s_2, t_{+}(I)]} \mathcal{Z}_{[s_1, s_2]} \mathcal{S}_1 \mathcal{Z}_{[t_{-}(I), s_1]} + \int_{s_1}^{s_2} dt_2 \int_{t_{-}(I)}^{s_1} dt_1 \mathcal{Z}_{[s_2, t_{+}(I)]} \mathcal{S}_2 \mathcal{Z}_{[t_{2}, t_{2}]}(1) \mathcal{Z}_{[t_{-}(I), t_{1}]}(1)
\]
\[
+ \int_{s_2}^{t_{+}(I)} dt_2 \int_{s_1}^{s_2} dt_1 \mathcal{Z}_{[t_{2}, t_{+}(I)]} \mathcal{S}_1 \mathcal{Z}_{[t_{1}, t_{2}]}(2) \mathcal{Z}_{[t_{1}, t_{1}]} \mathcal{S}_1 \mathcal{Z}_{[t_{-}(I), s_1]}(1)
\]
\[
+ \int_{s_2}^{t_{+}(I)} dt_2 \int_{t_{-}(I)}^{s_1} dt_1 \mathcal{Z}_{[t_{2}, t_{+}(I)]} \mathcal{S}_1 \mathcal{Z}_{[t_{1}, t_{2}]}(1) \mathcal{Z}_{[t_{-}(I), t_{1}]}(1).
\]
(4.18)
and
\[
\mathcal{G}_I((1)) = \mathcal{Z}_{[s_1, t_{+}(I)]} \mathcal{S}_1 \mathcal{Z}_{[t_{-}(I), s_1]} + \int_{s_1}^{t_{+}(I)} dt_2 \int_{t_{-}(I)}^{s_1} dt_1 \mathcal{Z}_{[t_{2}, t_{+}(I)]} \mathcal{S}_1 \mathcal{Z}_{[t_{1}, t_{2}]}(1) \mathcal{Z}_{[t_{-}(I), t_{1}]}(1).
\]
(4.19)
4.6. Generalized expansions. The expansion presented in the previous subsections were appropriate for initial states of the form \( \rho_S \otimes \rho_{R,\beta} \). In the present section, we replace the factorized initial states by the coupled ‘Gibbs’ state of the interacting system at vanishing external force \( \chi \):

\[
\rho_{\beta} = \frac{1}{Z_{\beta}} e^{-\beta H_{x=0}}, \quad Z_{\beta} = \text{Tr}[e^{-\beta H_{x=0}}], \quad H_{x=0} = T + H_{R} + \lambda H_{SR}.
\]

To extend our formalism to this particular initial state, we define operators

\[ D := e^{-\beta H/2} e^{\beta H_{x=0}/2}, \quad D(\mathcal{O}) := \text{Ad}(D)\mathcal{O} = D\mathcal{O}^*, \]

for observables \( \mathcal{O} \) in some subspace of \( \mathcal{B}_2(\mathcal{H}) \). Note that \( D \) and \( \mathcal{D} \) are unbounded operators, even in finite volume \( \Lambda \). Their use lies in a non-commutative ‘Radon-Nykodim’ identity:

\[
\rho_{\beta} = \frac{Z_{\beta,R}}{Z_{\beta}} D(\mathbb{1} \otimes \rho_R).
\]

Since we are in finite volume and we do not have periodic boundary conditions, the operators \( D \) and \( D \) are not translation-invariant. If they were translation-invariant and \( \mathcal{O} \) were translation-invariant observable, then we could write

\[
\text{Tr}[\mathcal{O}\rho_{\beta}] = \frac{Z_{\beta,R}}{Z_{\beta}} \text{Tr}[D(\mathbb{1} \otimes \rho_R)] = \frac{Z_{\beta,R} |\Lambda|}{Z_{\beta}} \text{Tr}[D(\mathbb{1}_{x=0} \otimes \rho_{R,\beta})] = \text{Tr}[D(\eta_{\beta} \otimes \rho_{R,\beta})],
\]

with \( \eta_{\beta} := \frac{Z_{\beta,R} |\Lambda|}{Z_{\beta}} \mathbb{1}_0 \) and the assumed translation-invariance is used in the second equality. The rank-one operator \( \eta_{\beta} \) is positive but not normalized, i.e., \( \text{Tr}_S[\eta_{\beta}] \neq 1 \), and it has a well-defined thermodynamic limit. Since translation invariance is broken only at the boundary of \( \Lambda \), Equality (4.22) is correct up to an error that vanishes in the thermodynamic limit; (recall the definition of quasi-diagonal operators in (4.22)).

**Lemma 4.1.** Assume that the operators \( S_1, \ldots, S_m \) on \( \mathcal{B}_2(\mathcal{H}_S) \) are quasi-diagonal and translation-invariant in the sense that

\[
S_i(x_1, x_2, x_3) = S_i(x_1 + y, x_2 + y, x_3 + y),
\]

whenever all variables are in \( \Lambda \). Then, for any \( 0 \leq s_1 < s_2 < \cdots < s_m \),

\[
\text{Tr}\left[ S_m \ldots S_2 \eta_{s_2}^i \rho_{\beta} \right] = \text{Tr}\left[ S_m \ldots S_2 e^{i(x_2-s_1)\mathcal{L}_S} e^{i x_1 \mathcal{L}_S} \rho_{\beta} \right] = \mathcal{O}\left( \frac{|\partial \Lambda|}{|\Lambda|} \right),
\]

where the error term \( \mathcal{O}(\frac{|\partial \Lambda|}{|\Lambda|}) \) is understood as \( \mathcal{O}(L^{-1}) \), as \( L \to \infty \). Furthermore, the limit \( \lim_{\Lambda} \frac{Z_{\beta,R} |\Lambda|}{Z_{\beta}} \) exists and is finite (note that the number \( \frac{Z_{\beta,R} |\Lambda|}{Z_{\beta}} \) appears in the operator \( \eta_{\beta} \) in (4.22)).

Note that (4.20) is (4.22) with \( m = 1, S_1 = (O) \) and \( s_1 = 0 \). The proof of Lemma 4.1 is postponed to Section 5.1.

The representation of equilibrium expectations (and correlations) on the right side of (4.22) is useful for us, because it allows us to treat equilibrium correlations on par with correlations in a state where the particle is initially localized on the lattice. Indeed, for, e.g., \( m = 2 \), the expression on the right side of (4.22) differs from the previously considered expressions \( \text{Tr}_S[G_l(S_{2x}^{S} S_{1x}^{S})(\rho_S)] \) only through the particular choice \( \rho_S = \eta_{\beta} \) (keeping in mind that \( \eta_{\beta} \) is not normalized) and the presence of the operator \( D \). Our strategy will be to expand \( D \) in its Lie-Schwinger series (treating again \( \lambda H_{SR} \) as a perturbation) and to merge this expansion with the one developed in the previous sections. Things are set up so that the only change to be made in the framework developed above is that the interval \( I \) is now a subset of \( \mathbb{R}_{\beta} := [-\beta/2, \infty) \), and the objects \( U_I \) and \( h(s, s', \xi, \xi') \) need to be redefined whenever \( I \) or \( \{ s, s' \} \) have a non-zero intersection with \( [-\beta/2, 0) \). The necessary generalizations are:

i. Free particle propagation \( U_I \): For any interval \( I \subset \mathbb{R}_{\beta} \), we set \( U_I := U_{I_2} U_{I_1} \), with \( I_1 \cup I_2 = I \) and \( I_1 \subset [-\beta/2, 0], I_2 \subset \mathbb{R}_+ \), and we define

\[
U_I(\cdot) := \begin{cases} e^{-i[I_1 H_S](\cdot)} e^{i[I_2 H_S]}, & \text{if } I \subset \mathbb{R}_+, \\ e^{-i[I_2 H_T](\cdot)} e^{-i[I_1 H_T]}, & \text{if } I \subset [-\beta/2, 0]. \end{cases}
\]
ii. Correlation function: It is convenient to introduce the maps $m_{\pm} : \mathbb{R}_\beta \to \mathbb{C}$ defined by $m_{\pm}(s) := s$, for $s \geq 0$, and $m_{\pm}(s) := \pm is$, for $s \in [-\beta/2, 0]$. Then we set

$$h(s, s', \varsigma, \varsigma') := \sigma(s, \varsigma)\sigma(s', \varsigma') \begin{cases} \hat{\upsilon}(m_{-}(s) - m_{-}(s')), & \text{if } \varsigma = 1, \varsigma' = 1, \\ \hat{\upsilon}(m_{+}(s') - m_{+}(s)), & \text{if } \varsigma = r, \varsigma' = r, \\ \hat{\upsilon}(m_{-}(s') - m_{+}(s)), & \text{if } \varsigma = r, \varsigma' = 1, \\ \hat{\upsilon}(m_{-}(s) - m_{+}(s')), & \text{if } \varsigma = 1, \varsigma' = r, \end{cases} \quad (4.24)$$

where $\sigma(s \leq 0, \varsigma) = 1$ and $\sigma(s > 0, \varsigma) = -i, i$, for $\varsigma = 1, r$, respectively.

With these modifications, we can extend the definitions of $V_l, Z_f, G_I$ and all relations between them; in particular (4.17), (4.18) and (4.19), remain valid. For example, we have from (4.22)

$$\text{Tr}[O(s)\rho_\beta] = \text{Tr}_{\varsigma}[O(s) Z_{[-\beta/2, 0]}] + \mathcal{O}\left(\frac{\partial|A|}{|A|}\right), \quad (4.25)$$

where $\varsigma = 1$ or $\varsigma = r$; and $Z_{[-\beta/2, 0]}$ may be decomposed as

$$Z_{[-\beta/2, 0]} = Z_{[0, 0]}Z_{[-\beta/2, 0]} + \int_{-\beta/2}^0 \, du \int_0^t \, dv \, Z_{[u, v]} V_{[u, v]} Z_{[-\beta/2, 0]} \cdot (4.26)$$

This concludes our discussion on finite-volume expansions. In the next section, we move on to discussing the thermodynamic limit and convergence of the series expansions introduced above.

5. Dyson series: Analysis and bounds

In this section, we analyze the Dyson series. The first part, Section 5.1, though lengthy, contains only soft estimates that require neither the full power of Assumption 13 nor the crucial fact that we consider separate reservoirs at each lattice point. This part could have been avoided by defining the model in the thermodynamic limit from the start. In contrast, Section 5.2 contains the crucial estimates that are specific to our model.

5.1. Thermodynamic limit. In the previous section, we have derived the expansion

$$Z_I = \int_{P_I} \, d\varpi \sum_{\pi \in \Pi_{\mu}^{\pi}(\varpi)} \varsigma(\varpi, \pi) G_0^\pi(\varpi), \quad (5.1)$$

where $I \subset [-\beta/2, \infty)$, with all objects in finite volume $\Lambda = \Lambda_L$. Next, we propose to pass to the thermodynamic limit. First, we note that the operators $\varsigma(\varpi, \pi) G_0^\pi(\varpi)$ on the right side of (5.1) are well-defined as operators on $B_2(\mathcal{H}_0)$, for $\Lambda = \mathbb{Z}^d$, $\Lambda = \mathbb{R}^d$, respectively. Indeed, the correlation functions $\varsigma$ are products of the functions $\hat{\upsilon}$, which were defined for $\Lambda = \mathbb{Z}^d$ in (3.4), and the operators $U_I$ are well-defined on $B_2(\ell^2(\mathbb{Z}^d))$ because $H_\mathcal{S} = T - \lambda^2 X \cdot X$ is a self-adjoint operator. Finally, the shorthand $\int d\varpi$ contains sums over $xi \in \Lambda$, which have to be interpreted now as sums over $\mathbb{Z}^d$. This gives meaning to $Z_I$ as a series of operators on $B_2(\mathcal{H}_0)$. Likewise, the series for the correlation functions $V_I$ and $G_I$ are well-defined term by term, respectively, for $S_I \in B(B_2(\mathcal{H}_0))$.

Below, we prove convergence of these series and we establish that they are indeed the limits of their natural finite-volume counterparts. The main ingredient here is the thermodynamic limit of correlation functions in Assumption 13.

We introduce some notation that will also be used in the subsequent analysis. Let $\Lambda = \Lambda_L$ or $\mathbb{Z}^d$. We write $w$ to denote walks in $\Lambda \times \Lambda$, i.e., sequences $w = (w_0, w_1, \ldots, w_n)$, where each $w$ is of the form $(w_1, w_2)$, with $w_1, w_2 \in \Lambda$, and we write $l(w) := n + 1$ for the length of the walk. (The walks $w$ should not be confused with the paths of triples $\varpi$.) On $\Lambda \times \Lambda$ we use the distance

$$|w|_{\Lambda \times \Lambda} = |(w_1, w_2)|_{\Lambda \times \Lambda} := |w_1| + |w_2|,$$
where \(| \cdot |\) denotes the Euclidean distance on \(\Lambda\). Moreover, we set \(|w| := \sum_{j=1}^{n} |w_{j} - w_{j-1}|_{\Lambda \times \Lambda}\). For simplicity, we abbreviate \(|w| := |w|_{\Lambda \times \Lambda}\) hereafter. Finally, we write \(w : w \to w'\) whenever \(w = w_{n}, w' = w_{n}, i.e., w\) is a walk starting at \(w\) and ending at \(w'\).

Subsequently, we use the above notation of walks for kernels of operator acting on \(\mathcal{B}(\ell^{2}(\mathbb{Z}^{d}))\): Instead of writing \(S(w_{1}, w_{2}, w_{3}', w_{4}')\), we write \(S(w, w')\), with \(w = (w_{1}, w_{2})\), \(w' = (w_{3}', w_{4}')\), for \(S \in \mathcal{B}(\ell^{2}(\mathbb{Z}^{d}))\). For example, the definition of quasi-diagonal operators in \([12]\), reads \(|S(w', w)| \leq C e^{-c|w' - w|}\), for some \(c > 0\), in this notation.

Before we are able to analyze the thermodynamic limit, we still have to define what is the ‘natural finite volume analogue’ of \(\mathcal{V}_{I}(S_{1}^{s_{1}}, \ldots, S_{m}^{s_{m}}), G_{I}(S_{1}^{s_{1}}, \ldots, S_{m}^{s_{m}})\) when starting from infinite volume \(S_{I}\)-operators. We set

\[ S_{I}^{A} := \text{Ad}(1_{\Lambda}) S_{I} \text{Ad}(1_{\Lambda}), \]

where \(1_{\Lambda}\) is the orthogonal projection \(\ell^{2}(\mathbb{Z}^{d}) \to \ell^{2}(\Lambda)\).

**Lemma 5.1.** Let \(A\) be one of the following (infinite volume) operators

\[ Z_{I}, \quad G_{I}(S_{1}^{s_{1}}, \ldots, S_{m}^{s_{m}}), \quad \mathcal{V}_{I}(S_{1}^{s_{1}}, \ldots, S_{m}^{s_{m}}), \]

with \(I \subset [-\beta/2, \infty)\), and \(S_{1}, \ldots, S_{m}\) quasi-diagonal. Denote by \(A^{\Lambda}\) their finite-volume analogues as discussed above, in particular including the replacement of \(S_{I}\) by \(S_{I}^{A}\). Then, for \(\Lambda\) finite and for \(\Lambda = \mathbb{Z}^{d}\), the series defining these operators converge absolutely in norm (as operators on \(\mathcal{B}(\ell^{2}(\mathbb{Z}^{d}))\)). Moreover,

\[ |A^{\Lambda}(w, w')| \leq C e^{-c|w' - w|}, \quad |A(w, w')| \leq C e^{-c|w' - w|}, \]

where the constant \(C_{I}\) can be chosen uniformly in \(\Lambda\) (for \(|\Lambda|\) large enough), including \(\Lambda = \mathbb{Z}^{d}\), independent of the times \(s_{1}, \ldots, s_{m}\) and uniform on compacts in \(t_{-}(I), t_{+}(I)\). The exponent \(c > 0\) can be chosen to depend only on the observables \(S_{1}, \ldots, S_{m}\). Finally, for any \(w, w'\),

\[ \lim_{\Lambda} A^{\Lambda}(w, w') = A(w, w'), \]

uniformly on compacts in the variables \(s_{1}, \ldots, s_{m}\) and \(t_{-}(I), t_{+}(I)\).

**Proof.** The following bounds apply alike in finite and infinite volume. Denoting by \(\sum_{w}^{\mathbb{Z}^{d}}\) the sum over the \(x\)- and \(\varsigma\)-coordinates in the path \(\mathbb{w} = ((x_{1}, \varsigma_{1}, t_{1}), \ldots, (x_{2n}, \varsigma_{2n}, t_{2n}))\), we obtain, for \(\nu > 0\) sufficiently small, and uniformly in \(\mathbb{L}\),

\[ \sum_{\mathbb{w}} |g_{I}(\mathbb{w})(w, w')| \leq \|e^{\nu I}\|_{\text{Imc}} \sum_{\mathbb{w} : w \to w'} |\mathbb{w}| = |\mathbb{w}| + 2 \prod_{j=1}^{|\mathbb{w}|+1} \|U_{(t_{j-1}, t_{j})}(w_{j-1}, w_{j})\| \]

\[ \leq C^{1+|\mathbb{w}|} e^{\nu |\mathbb{w}|} \sum_{\mathbb{w} : w \to w'} \prod_{j=1}^{|\mathbb{w}|+1} e^{-2\nu |w_{j} - w_{j-1}|} \]

\[ \leq C e^{C |I|^{1+|\mathbb{w}|}} e^{-c|w' - w'|} (C e^{2d} |\mathbb{w}|), \quad \text{(5.3)} \]

where we introduced the dummy variables \(t_{0} = t_{-}(I), t_{|\mathbb{w}|+1} = t_{+}(I)\) in the first line. To get the second line, we used the propagation bound \((3.3)\) and its imaginary time version

\[ \|e^{\tau T}(x, x')\| \leq C e^{C |\tau|^{1+|\mathbb{w}|}} e^{-2\nu |x - x'|}, \]

for \(\tau \leq \beta/2\). The term between brackets on the last line of \((5.3)\) originates from the sums \(\sum_{w} e^{-\nu |w|}\).

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From (5.1) and (5.3) we get

\[
|Z_t(w, w')| \leq \sum_{n \geq 0} \sum_{\text{Pair}(n)} \int dt_1 \cdots dt_{2n} \sum_{\Delta} |\zeta(\varpi, \pi)| |G^0_{\varpi}(w, w')|
\]

\[
\leq Ce^{-\nu|w-w'|} |\text{Pair}(n)| C^1|\nu| \int dt_1 \cdots dt_{2n} C^n
\]

\[
\leq Ce^{-\nu|w-w'|} |\text{Pair}(n)| C^1|\nu| \sum_{n \geq 0} (2n - 1)! \frac{C^n |I|^{2n}}{(2n)!}
\]

\[
\leq Ce^{-\nu|w-w'|} \sum_{n \geq 0} (2n - 1)! \frac{C^n |I|^{2n}}{(2n)!},
\]  

(5.4)

where \((2n - 1)! \equiv (2n - 1)(2n - 3) \cdots 1\) is the number of pairings of \(2n\) elements, i.e., \(|\text{Pair}(n)|\), and the \(n = 0\) term is understood as \(|U_t(w, w')|\). Note that we have used Assumption B to get \(|\zeta(\varpi, \pi)| \leq C^n\). A similar estimate holds for \(G_I(\cdot), V_I(\cdot)\), since ultimately we only used that the operators \(U_t\) satisfy the propagation estimate \(|U_t(w, w')| \leq Cte^{-\nu|w-w'|}\); see (5.2).

We now turn to convergence of kernels. By the uniform bounds above, the convergence of kernels follows once we have proved that \(U_t^\beta(w, w') \rightarrow U_t(w, w')\) and \(\zeta^\beta(\varpi, \pi) \rightarrow \zeta(\varpi, \pi)\) for any \(w, w', \pi, \varpi\), and uniformly on compacts in the time-arguments. The first claim is obvious because \(H_S^\beta \rightarrow H_S\) strongly, moreover \(H_S^\beta\) is bounded for finite \(\Lambda\). Thus, functions of \(H_S^\beta\) converge strongly to functions of \(H_S\). The second claim follows from Assumption B since \(\zeta(\varpi, \pi)\) is a product of the correlation functions \(\tilde{\psi}(t), t \in H_\beta\).

We now move towards the proof of Lemma 4.1. We recall that both the particle and the reservoirs are restricted to finite volumes \(\Lambda, \bar{\Lambda}\) that are related by \(\Lambda = \bar{\Lambda} \cap \mathbb{Z}^d\). However, since each lattice point is connected to a separate reservoir, there is no compelling reason for these volumes to be related and this is exploited in the present proof: We first perform the thermodynamic limit for the particle \((\Lambda \to \mathbb{Z}^d)\) but not for the reservoirs \((\bar{\Lambda} \to \mathbb{Z}^d)\). To that end, we introduce operators \(\tilde{A}\) on \(\mathcal{B}(\ell^2(\mathbb{Z}^d))\), with \(A = Z_I, V_I, G_I(S^1, \ldots, S^m)\), for \(\tilde{S}_j\) translation-invariant and quasi-diagonal, and interval \(I \subset [-\beta/2, \infty]\). These ‘tilde operators’ are obtained from the ones without tildes by choosing the correlation functions \(\zeta\) to be \(\zeta^\beta\), i.e., in finite volume, but choosing the operators \(U_t\) and \(\tilde{S}_j\) in infinite volume. Therefore, the operators \(A\) are translation-invariant. Note that given a collection of translation-invariant and quasi-diagonal operators \(\tilde{S}_j\), we now have three types of objects, namely \(A^\beta, \tilde{A}, A\). The latter of the three does not play a rôle in the proof of Lemma 4.1.

**Lemma 5.2.** Let \(A^\beta, \tilde{A}\) be as described above. Then

\[
|A^\beta(w, w') - \tilde{A}(w, w')| \leq C_1 e^{-c'|w-w'|},
\]

(5.5)

where \(d(A, B) = \inf_{x \in A, y \in B} |x - y|\) for \(A, B \subset \Lambda\), and \(\Lambda_c = \mathbb{Z}^d \setminus \Lambda\). The constants \(c, c'\) can be chosen uniformly in \(I, s_1, \ldots, s_m\) and \(S_1, \ldots, S_m\).

**Proof.** We decompose

\[
H_S = H_S^\Lambda + T^{0\Lambda} + H_S^{\Lambda_c},
\]

(5.6)

where the operator on the left side corresponds to infinite volume, \(H_S^{\Lambda_c}\) acts on \(\ell^2(\Lambda_c)\) and

\[
T^{0\Lambda}(x, x') = \begin{cases} \tilde{c}(x' - x) & x \in \Lambda, x' \in \Lambda_c \text{ or } x' \in \Lambda, x \in \Lambda_c, \\ 0 & \text{otherwise} \end{cases}
\]

(5.7)

Duhamel’s principle yields

\[
\mathbb{I}_\Lambda \left( e^{-itH_S} - e^{-itH_S^\Lambda} \right) \mathbb{I}_\Lambda = \int_0^t ds \mathbb{I}_\Lambda e^{-i(t-s)H_S} T^{0\Lambda} e^{-isH_S^\Lambda} \mathbb{I}_\Lambda.
\]

(5.8)

Applying the propagation bound for both \(e^{-itH_S^\Lambda}\) and \(e^{-itH_S}\) and the exponential decay of the function \(\tilde{c}(\cdot)\), we conclude that

\[
\left| (e^{-itH_S} - e^{-itH_S^\Lambda})(x, x') \right| \leq Ce^{Ct} e^{-c|x-x'|} e^{-c' \max(d(x, \Lambda_c), d(x', \Lambda_c))}, \quad x, x' \in \Lambda,
\]

(5.9)
and hence we also have
\[ |U^1_{\beta}(w, w') - U^1(\bar{w}, \bar{w}')| \leq C_{1} e^{-c' \text{dist}(w_1, \bar{w}_1, \Delta^2)} e^{-c|w - w'|}. \]  
(5.10)

Obviously, we can equally well choose \( \{w'_j, w''_j\} \) instead of \( \{w_j, w'_j\} \) on the right side of (5.10). Furthermore, we can also replace the difference \( U^1_{\beta} - U^1 \) on the left side by \( S^\Lambda - S \) for quasi-diagonal \( S \), and we can allow \( I \subseteq [-\beta/2, \infty) \).

To address \( A^\Lambda - \bar{A} \) as required, we recall that the correlation functions \( \zeta \) in both operators are the same (i.e. the finite-volume ones) so that the only differences originate in the difference on the left side of (5.10) (or the generalizations just mentioned). By repeatedly applying (5.10) and using the same strategy as in Lemma 5.1 to sum/integrate over \( \bar{\omega} \), we get the claim of the Lemma.

**Proof of Lemma 4.1.** We abbreviate \( g(\Lambda) := \frac{Z_{\beta,\Lambda}}{Z_{\beta}} \). Recall that
\[ \rho_{\beta} = g(\Lambda) \frac{1}{|\Lambda|} \mathcal{D}(\mathbb{1} \otimes \rho_{R}) = g(\Lambda) \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \mathcal{D}(\mathbb{1}_x \otimes \rho_{R}). \]  
(5.11)

Therefore, the left side and right side of (5.12) may be written as
\[ g(\Lambda) \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \text{Tr} [A^\Lambda(\mathbb{1}_x)] , \quad g(\Lambda) \text{Tr} [A^\Lambda(\mathbb{1}_0)] , \]  
(5.12)
respectively, with \( A^\Lambda = g^\Lambda_{[-\beta/2, \beta]}(S^\Lambda_{(m)}, \ldots, S^\Lambda_{(n)}) \). Using kernels, we recast
\[ \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \text{Tr}[A^\Lambda(\mathbb{1}_x)] - \text{Tr}[A^\Lambda(\mathbb{1}_0)] = \frac{1}{|\Lambda|} \sum_{x, y \in \Lambda} A^\Lambda(y, y, x, x) - \sum_{y \in \Lambda} A^\Lambda(y, y, 0, 0). \]  
(5.13)

Since the \( (S_j) \) were assumed to be translation-invariant on \( \Lambda \) and quasi-diagonal, they are finite-volume restrictions of truly translation-invariant and quasi-diagonal operators. Therefore, Lemma 5.2 applies to the operator \( A^\Lambda \) above and the associated \( \bar{A} \) are translation-invariant. Let us split \( A^\Lambda = \bar{A} + \kappa^\Lambda \) in (5.13) with \( \kappa^\Lambda := A^\Lambda - \bar{A} \).

By translation invariance, we can drop the terms containing \( \bar{A} \), so (5.13) equals
\[ \frac{1}{|\Lambda|} \sum_{x, y \in \Lambda} \kappa^\Lambda(y, y, x, x) - \sum_{y \in \Lambda} \kappa^\Lambda(y, y, 0, 0). \]  
(5.14)

By the bounds of Lemma 5.2 this difference is bounded by \( C_1^{\beta/|\Lambda|} \) and hence we obtain
\[ \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \text{Tr}[A^\Lambda(\mathbb{1}_x)] - \text{Tr}[A^\Lambda(\mathbb{1}_0)] \leq \frac{1}{|\Lambda|} \sum_{x \in \Lambda} C e^{-c|x, \Lambda^c|} \leq C_1^{\beta/|\Lambda|}. \]  
(5.15)

Recall that we need to prove that the difference between the two expressions in (5.12) is \( O((\beta |\Lambda|)/|\Lambda|) \). This follows from (5.15) provided that \( g(\Lambda) \) remains bounded as \( \Lambda \to \mathbb{Z}^d \), as we show now: By an application of the Golden-Thompson inequality and exploiting the fact that \( c \beta \leq e^{\beta \lambda_2} \leq C \beta \), for \( 0 < c < C < \infty \), uniformly in \( \Lambda \), it suffices to check that
\[ 0 < c' \leq \frac{\text{Tr} e^{-\beta(H_{\Lambda} + \lambda H_{\Lambda})}}{|\Lambda| \text{Tr} e^{-\beta H_{\Lambda}}} \leq C' < \infty, \]  
(5.16)
uniformly in \( \Lambda \). The operators in the exponent can be explicitly diagonalized (they are polynomials of order 2 in creation and annihilation operators) and (5.10) follows then after a straightforward calculation by the bound on (3.33). Note for further reference that this also shows that \( 1/g(\Lambda) \) is uniformly bounded as \( \Lambda \to \mathbb{Z}^d \).

It remains to show that \( \lim_{\Lambda} g(\Lambda) \) exists. Since \( \text{Tr} [\rho_{\beta}] = 1 \), we have
\[ \frac{1}{g(\Lambda)} = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \text{Tr} [\mathcal{D}(\mathbb{1}_x \otimes \rho_{R})], \]  
(5.17)
and by (5.15), taking now $\mathcal{A}^\Lambda = Z_{[-\beta/2, 0]}^\Lambda$, we get
\[
\frac{1}{g(\Lambda)} = \text{Tr}[Z_{[-\beta/2, 0]}^\Lambda (I)] + \mathcal{O} \left( \frac{|\partial \Lambda|}{|\Lambda|} \right).
\]
(5.18)

By Lemma 5.1, the limit $\lim_{\Lambda} \text{Tr}[Z_{[-\beta/2, 0]}^\Lambda (I)]$ on the right side exists and is finite. Hence, either $g(\Lambda)$ diverges or $\lim_{\Lambda} g(\Lambda)$ exists and is finite. But the first possibility was excluded above, hence the proof is complete.

To construct the correlation functions $\mathbf{3.10}$ with $\mathbf{3.11}$ with $\mathbf{O} \in \mathfrak{X}$, (X the $*$-algebra generated by the position operator $(X^j)$), we want to consider $S_j = f_j(X^j)\chi$, $\zeta = 1, r$, for some polynomials $f_j$. In this case, one can no longer expect $V_\zeta(\cdot)$ or $\mathbf{G}_\zeta(\cdot)$ to be a bounded operators; but their kernels are obviously well-defined, and we can still follow the proof of Lemma 5.1 bounding $|S_j(w', w)| \leq C\delta_{w, w'}|w|^{N_j}$, where $N_j$ is the degree of the polynomial $f_j$. We obtain
\[
|\mathbf{A}(w, w')| = C|w|^{N} e^{-v|w - w'|},
\]
uniformly in $\Lambda$, for some $N$ that is determined by the $S_1, \ldots, S_m$. This polynomial growth in $|w|$ is compensated by the exponential decay, so that, for exponentially localized $\rho_S$, (see $\mathbf{5.7}$), expressions such as $\mathbf{G}_\ell(\cdot)\rho_S, \mathbf{G}_\ell(\cdot)\rho_S$, etc. are again exponentially localized operators. For example, the following identity (trivial in finite volume) holds for exponentially localized $\rho_S$ and interval $I \subset \mathbb{R}_+$:
\[
\partial_\ell \mathbf{G}_\ell(S_1^a, \ldots, S_m^a)\rho_S = \lambda^2 \sum_{\ell=1, r} \left( \delta_{\ell, 1} - \delta_{\ell, r} \right) \int_I ds \mathbf{G}_\ell((X^s)\rho_S, (X^s)\rho_S)\rho_S.
\]
(5.19)

Another useful identity is obtained by applying the relation $X(t) - X(0) = \int_0^t ds V(s)$, with $V$ the velocity operator, in correlation functions, e.g.,
\[
\langle (X(t) - X(0))^2 \rangle_{\rho_S \otimes \rho_{s, \beta}} = \int_0^t ds_2 \int_0^t ds_1 (V(s_2)(V(s_1))_{\rho_S \otimes \rho_{s, \beta}},
\]
(5.20)
\[
\langle (X(t) - X(0)) \rangle_{\rho_S \otimes \rho_{s, \beta}} = \int_0^t ds_1 (V(s_1))_{\rho_S \otimes \rho_{s, \beta}},
\]
(5.21)

for exponentially localized $\rho_S$. Here, the left sides should be interpreted as linear combinations of correlations functions, e.g.,
\[
\langle (X(t) - X(0))^2 \rangle_{\rho_S \otimes \rho_{s, \beta}} = \langle (X(t))^2 \rangle_{\rho_S \otimes \rho_{s, \beta}} - \langle X(t)X(0) \rangle_{\rho_S \otimes \rho_{s, \beta}} - \langle X(0)X(t) \rangle_{\rho_S \otimes \rho_{s, \beta}} + \langle (X(0))^2 \rangle_{\rho_S \otimes \rho_{s, \beta}}.
\]

**Proof of Lemmas 5.7 and 5.9** The thermodynamic limit of $\mathbf{Z}_\zeta$ is immediate from Lemma 5.1 by the convergence of kernels and the exponential bounds. To deal with correlation functions, we note that, for finite $\Lambda$,
\[
\text{Tr} \mathbf{G}_0(0)(S_1^a, \ldots, S_m^a)(\rho_S) = \langle O_m(s_m) \ldots O_1(s_1) \rangle_{\rho_S \otimes \rho_{s, \beta}},
\]
(5.22)

with $S_j = (O_j)_{1 \leq s_1 < \ldots < s_m \leq t}$. Using the existence of the thermodynamic limit for $\mathbf{G}_\ell(\cdot)$ and the exponential bounds, we show the convergence for exponentially localized $\rho_S$ and $O_j \in \mathfrak{A}$ (i.e., $O_j$ quasi-diagonal) or $O_j \in \mathfrak{X}$, thus defining the right side of 5.22 for $\Lambda = \mathbb{R}^d$. The extension to $\mathfrak{A}$ is by density. (For $m = 2$, the ordering of the times can be relaxed on the right side of 5.22 by setting $S_1 = (O_1)_t$, which leads to an exchange of $O_1(s_1)$ and $O_2(s_2)$ on the right side.)

For equilibrium correlation functions, we first recall that the rank-one operator $\eta_\beta$ is well-defined in the thermodynamic limit by Lemma 4.1. Then, the argument is analogous to the one above, but replacing 5.22 by
\[
\text{Tr} \mathbf{G}_\ell(-\beta, \ell)(S_1^a, \ldots, S_m^a)\eta_\beta = \langle O_m(s_m) \ldots O_1(s_1) \rangle_{\rho_\beta}.
\]
(5.23)

We set $\chi = 0$, i.e., $O_j(s)$ replaced by $O_j^{\chi = 0}(s)$ and we consider 5.23 with $m = 2$. The time-reversal invariance and stationarity of (5.25) follow from the finite volume system, where they are explicit. The only thing left to prove is the infinite-volume KMS condition: we note that the construction of 5.25 can be carried out when $s_1 - s_2$ is in the strip $\mathbb{H}_\beta$ and the conclusions of Lemmas 4.1 and 5.1 remain valid. This is checked straightforwardly by using that the operators $e^{-i\mathfrak{H}_\beta}$ remain quasi-diagonal for $s \in \mathbb{H}_\beta$ (in fact, for any $s \in C$) and the correlation functions $\zeta(\varpi, \pi)$ remain well-defined, because, upon taking $0 \leq \text{Im}(s_1 - s_2) \leq \beta$, all arguments of the function $\psi$. 21
in (4.24) remain in the strip $\mathbb{H}_\beta$, as one verifies by inspection. Hence, the thermodynamic limit is still valid for $s_1 - s_2 \in \mathbb{H}_\beta$ in the sense that the correlation functions $\langle O_s^A (s_2) O_s^A (s_1) \rangle_{\nu, \lambda}$ are bounded uniformly in $\Lambda$ and converge uniformly on compact sets in $s_1 - s_2 \in \mathbb{H}_\beta$. Therefore, the limit of the finite-volume correlation function is analytic in the interior of the strip and continuous on the boundary. Thus, the KMS condition in infinite-volume follows from the one in finite volume.

5.2. Bounds on the effective dynamics. Up to now, we have established bounds on the free correlation functions $G^0_1$, from which we could derive crude bounds on the interacting correlation functions. This is sufficient to prove the existence of the thermodynamic limit. In what follows, we prove sharper bounds on the interacting correlation functions in infinite volume, using the decay properties of the reservoir correlation function. From now on, all quantities refer to infinite volume, unless mentioned otherwise.

5.2.1. Bounds on diagrams. In analogy to the operator $\mathcal{V}_I$ defined in (4.17), we set, for $n \in \mathbb{N}$,

$$\hat{\mathcal{V}}^{(n)}_I (S_1^{s_1}, \ldots, S_m^{s_m}) := \int_{|\varpi| \geq 2n} d\varpi \delta(\partial I - \partial \varpi) \sum_{\pi \in \text{Pair}(\varpi)} \zeta(\varpi, \pi) G^0_1 (S_1^{s_1}, \ldots, S_m^{s_m} | \varpi), \quad (5.24)$$

with $\hat{\mathcal{V}}^{(2)}_I = \mathcal{V}_I$, because the smallest irreducible diagrams have $|\varpi| = 2$. In Lemma 5.3 we provide a bound on the right side. To save writing, we introduce

$$\tilde{h}(t) := \sup_{s-s' \in \mathbb{H}_\beta, \varsigma_1, \varsigma_2 \in \{1, 2\}} |h(s, s', \varsigma_1, \varsigma_2)|. \quad (5.25)$$

From Assumption [B] we get

$$|\tilde{h}(t)| \leq C e^{-g_0 |t|}.$$  

Lemma 5.3. For sufficiently small $\lambda, \nu > 0$, for any $n \geq 1$, any interval $I \subset [-\beta/2, \infty)$ and an arbitrary collection (possibly empty), $S_1, \ldots, S_m$, of observables with associated times $s_1, \ldots, s_m \in I$,  

$$\left| \hat{\mathcal{V}}^{(n)}_I (S_1^{s_1}, \ldots, S_m^{s_m}) (w, w') \right| \leq C^{m+\nu} \lambda^{2n} \max(1, |I|^{n-1}) e^{-g_0 |I|} \times \prod_{j=1}^{m} \left| \mathcal{S}_I (w_{2j-1}, w_{2j}) \right| e^{\nu |w_{2j-1} - w_{2j}|}, \quad (5.26)$$

where the constant $C$ depends only on $\tilde{h}$, the particle dispersion relation $\varepsilon$, and the spatial dimension $d$.

Proof. Recalling the definition of the integration measure $d\varpi$ on $\mathcal{P}_I$ in (4.10), we perform the integration on the right-hand side of (5.24), by first fixing the number of triples in $\varpi$, $(|\varpi| = 2p, p \geq n)$, and the time coordinates in $\varpi$, $(t_1, \ldots, t_{2p})$, while summing over the spatial- and $\varsigma$-coordinates. The times $s_1, \ldots, s_m$ induce a partition of the time interval $I$ into $m + 1$ intervals $I^{(j)}$, $j = 0, \ldots, m$, and, almost surely with respect to the fixed times $t_1, \ldots, t_{2p}$, also a partition of the path $\varpi \in \mathcal{P}_I$ into subpaths $\varpi^{(j)}$ (they can be empty, i.e., $\varpi^{(j)} = \emptyset$). We denote by $\sum_{\varpi^{(j)}}$ the sum over the $\varpi$- and $\varsigma$-coordinates of $\varpi^{(j)}$. For any one of those $I^{(j)}$, $\varpi^{(j)}$, we use (5.3) with $\nu$ replaced by $\nu$, to obtain, for any $0 \leq \nu \leq \tilde{\nu}$,
where we set $p := \sum_{\Delta} |G^0_0(S^{m}_i) | \omega_i(w, w') |$

\[
\leq \sum_{w', w \to w'} \sum_{l(w) = 2m + 2} |G^0_{(0)}(\omega^{(0)})(w_0, w_1)| \left( \prod_{j=1}^{m} |S^j(w_{2j-1}, w_{2j})| \sum_{\Delta} |G^0_{(i)}(\omega^{(i)})(w_{2j}, w_{2j+1})| \right) 
\]

\[
\leq \sum_{w', w \to w'} \left( \prod_{j=0}^{m} C(C_\nu^{-2d}) \omega^{(j)} |e^{-|w|}(w_{2j+1} - w_{2j})| \prod_{j=1}^{m} |S^j(w_{2j-1}, w_{2j})| \right) 
\]

\[
\leq C^{m+1}(C_\nu^{-2d}) |\omega| e^{C_\nu |I|} \sum_{w', w \to w'} e^{-|w|} \left( \prod_{j=1}^{m} |S^j(w_{2j-1}, w_{2j})| |e^{(w_{2j} - w_{2j-1})} \right). 
\quad (5.27)
\]

This way, we have bounded the $x$- and $z$-sums in \([5.24]\), so that we are left with the $I$-integral, $\pi$-sum, and the sum over $p \geq n$. More precisely, to pass from \([5.24]\) to \([5.26]\), it suffices to show

\[
e^{C_\nu |I|} \sum_{p \geq n} (C_\nu^{-2d})^{2p} \int_{\Delta^p_{\pi}} \frac{dt}{\prod_{(r, s) \in \pi} \sup_{\lambda \text{ irreducible}} |\omega_{(r, \pi)}|} \leq (C_\nu^{2n} \max(1, |I|^{n-1}) e^{2n |I|}, 
\quad (5.28)
\]

for some sufficiently small $\tilde{\nu}$, and all sufficiently small $\lambda$, depending on $\nu$. The supremum in the left hand expression is over all $x$-$z$-coordinates of $\omega$. Consider any pairing $\pi$ in the sum on the left hand side; by irreducibility, we know that $\sum_{(r, s) \in \pi} |t_r - t_s| \geq |I|$ and that $t_1 = t_{-}(I), t_2 = t_{+}(I)$, so we can bound the left hand side by

\[
\leq \frac{2n |I|}{4} \sum_{p \geq n} \sum_{\pi \text{ irreducible}} \int_{\Delta^{p-2}} \frac{dt}{\prod_{(r, s) \in \pi} |k(t_r - t_s)|} \left. \left. \prod_{t_1 = t_{-}(I), t_2 = t_{+}(I)} \right. \right. 
\quad (5.29)
\]

where we set

\[
k(t) := \lambda^2 (C_\nu^{-4d}) e^{t(C_\nu^{-2d})} h(t). 
\quad (5.30)
\]

To deal with \([5.29]\), we first develop a combinatorial estimate:

**Lemma 5.4.** Let $e^x_n := \sum_{j \geq n} x^j / j!$ and $e^x_n := e^x$, for $n \in \mathbb{N}$. Then, for $k \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$,

\[
\sum_{p \geq n} \sum_{\pi \text{ irreducible}} \int_{\Delta^{p-2}} \frac{dt}{\prod_{(r, s) \in \pi} |k(t_r - t_s)|} \left. \left. \prod_{t_1 = t_{-}(I), t_2 = t_{+}(I)} \right. \right. 
\quad (5.31)
\]

\[
\sum_{p \geq n} \sum_{\pi \text{ irreducible}} \int_{\Delta^{p-2}} \frac{dt}{\prod_{(r, s) \in \pi} |k(t_r - t_s)|} \left. \left. \prod_{t_1 = t_{-}(I), t_2 = t_{+}(I)} \right. \right. 
\quad (5.32)
\]

where the $p = 0$ term on the left hand side of \([5.31]\) is understood to equal $1$. The bounds \([5.29], [5.29]\) hold for $n \geq 0$, $n \geq 1$, respectively.

**Proof.** For any pairing $(r, s) \in \pi$ and the corresponding time coordinates $t_r, t_s$ we set $u_i := t_r$ and $v_i := t_s$, where the indices $i = 1, \ldots, n$ are chosen such that $t_{-}(I) \leq u_1 < u_2 < \ldots < u_n \leq t_{+}(I)$. Note that, by our definition of a pairing, $u_i < v_i$. By using the change of variables $\Delta^p \to (u, \omega)$, we rewrite the left side of \([5.31]\) as

\[
Z_t(n) := \sum_{p \geq n} \int_{\Delta^p_{\Delta^p_{\Delta^p}}} 
\quad (5.33)
\]

with the $p = 0$ term being $1$, and we set $Z_t(n < 0) := Z_t(0)$. Each $v_j$-integral is bounded by $|k||I|$; the integral over $\Delta^p_{\Delta^p}$ gives $|I|^n / n!$ and \([5.31]\) follows. To derive \([5.32]\), we split the expression according to whether the
pairing \( \pi \) contains the pair \((1,2n)\) or not. In the first case, we note that the time-coordinates of all pairs other than \((1,2n)\) are not constrained and hence we get the estimate
\[
|k(t_+(I) - t_-(I))| |Z_I(n-1)|, \tag{5.34}
\]
where the first factor originates from the pair \((1,2n)\). In the second case, there are pairs \((1,j),(j',2n)\) with \( j \neq 2n, j' \neq 1 \). The time coordinates of all other pairs are again unconstrained, so we get the estimate
\[
\int dv |k(v - t_-(I))| \int du |k(t_+(I) - u)| |Z_I(n-2)|, \tag{5.35}
\]
where the first and second factor originate from the pairs \((1,j),(j',2n)\), respectively. Equation (5.32) then follows from adding (5.34) and (5.35), and using (5.31) to evaluate \( Z_I \).

Finally, we use \( e^{\frac{2g}{R}} \) and hence, by Lemma 5.3, for \( \lambda, \nu > 0 \) sufficiently small, with \( \tilde{\nu} \) small enough such that \( \frac{2g}{R} - C\tilde{\nu} > 0 \) and hence \( k \) as defined in (5.36) belongs to \( L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \). Then, by (5.32), we bound (5.29) by
\[
e^{-\frac{2g}{R} |I|} \left( \|k\|_\infty e^{\frac{|I|}{k_1}} + \|k\|_1^n \right) / (\tilde{\nu} \lambda)^n. \tag{5.36}
\]
Finally, we use \( e^{\frac{2g}{R}} \leq x^ne^{\nu}/n! \) for \( x \geq 0 \), and we choose \( |\lambda| \) small enough compared to \( (\tilde{\nu})^{-2d} \), so that \( \|k\|_1 \leq gR/3 \) and (5.36) is bounded by the right side of (5.25).

5.2.2. Bounds on correlations functions with unbounded observables. As argued previously, we cannot bound the correlation functions \( G_I(\cdot), V_I(\cdot) \) in norm when \( S_I \) is unbounded, for some \( j \), but we can bound their kernels. For future use in Section 6 let us bound \( \partial_\chi \mathcal{V}_I^{(n)} \rho_S \) for an exponentially localized \( \rho_S \) and interval \( I \subset \mathbb{R}_+ \). This quantity is given by (see (5.19))
\[
\partial_\chi \mathcal{V}_I^{(n)} \rho_S = \lambda^2 \sum_{c=1,r} (\delta_{c,1} - \delta_{c,r}) \int_I ds \mathcal{V}_I^{(n)}((X_s)^c) \rho_S, \tag{5.37}
\]
and hence, by Lemma 5.3 for \( \lambda, \nu > 0 \) sufficiently small,
\[
\left| \left( \partial_\chi \mathcal{V}_I^{(n)} \rho_S \right)(w) \right| \leq \max(1, |I|^n)(C|\lambda|)^{2n+2} e^{-\frac{2g}{R} |I|} \sum_{w_0,w_1} |w_1| e^{-\nu(|w-w_1|+|w_1-w_0|+|w_0|)} \tag{5.38}
\]
where \( \nu \) is chosen so small that \( \rho_S(w_0) \leq Ce^{-\nu|w_0|} \). Higher derivatives lead to an obvious generalization of (5.38); the \( k \)th derivative will produce the factor \( \max(1, |I|^{n+k}) (C\lambda)^{2n+k} \) on the right side (because the \( k \)th derivative corresponds to \( k \) time-integrations over \( s_1, \ldots, s_k \) in the generalization of (5.37)) and \( k \) factors \( |w_i|, i = 1, \ldots, k \), but these can still be controlled by the exponential decay in \( |w_0| \) and \( |w_{i+1} - w_i| \). An obvious consequence of (5.38) is that, for exponentially localized \( \rho_S \), the function
\[
\chi \mapsto \mathcal{V}_I^{(n)} \rho_S \tag{5.39}
\]
is \( C^\infty \) and that all derivatives are exponentially localized operators, too.

5.2.3. Bounds on the effective dynamics \( Z_I \). Next, we show how the bounds in Lemma 5.3 help to control the reduced evolution \( Z_I \) and the correlation functions \( G_I(S_1^{n_1}, \ldots, S_m^{n_m}) \). If we demand that \( S_1, \ldots, S_m \) are quasi-diagonal, then Lemma 5.3 immediately yields
\[
\mathcal{V}_I^{(n)}(S_1^{n_1}, \ldots, S_m^{n_m}) (w, w) \leq C'(C|\lambda|)^{2n} \max(1, |I|^{n-1}) e^{-\frac{2g}{R} |I|} e^{-\nu |w-w'|}, \tag{5.40}
\]
for \( \lambda, \nu > 0 \) sufficiently small, with \( C' \) depending on the \( S_j \)’s. To get a bound on the operator norm, we note that
\[
\|S\| \leq \sup_w \sum_{w'} |S(w', w)|, \tag{5.41}
\]
where the supremum and the sum are over \( \mathbb{Z}^d \times \mathbb{Z}^d \).
Next, we define operators \( J_{\theta} \), with \( \theta = (\theta_1, \theta_2) \in \mathbb{C}^d \times \mathbb{C}^d \), by
\[
J_{\theta}O := e^{-i(\theta_1,X)}O e^{-i(\theta_2,X)}, \quad O \in \mathcal{B}(\mathcal{H}_S).
\]
(5.41)

Note that \( J_{\theta} \) is unbounded if \( \theta \) has an imaginary part. Also note that an operator \( O \in \mathcal{B}_2(\mathcal{H}_S) \), is exponentially localized iff \( \| J_{\theta}O \|_2 < \infty \), for \( \theta = (\theta_1, \theta_2) \) in some complex neighborhood of \((0,0)\).

From (5.39), we get for \( 0 < \lambda \) and \( \theta \in \mathbb{C}^{2d} \) sufficiently small, in particular \( |\theta| \leq \nu \),
\[
\| J_{\theta}V_1(S_1^{(1)}, \ldots, S_m^{(m)})_J_{-\theta} \| \leq C'(|\lambda|)^{2n} \max(1, |I|^{n-1})e^{-\frac{2n}{\lambda}d|I|}.
\]
(5.42)

This implies that \( V_1^{(n)}(S_1^{(1)}, \ldots, S_m^{(m)}) \) preserves the subspace of exponentially localized density operators.

Using the above bounds on \( V_1^{(n)}(\cdot) \), with \( m = 0 \), and propagation bounds on \( U_t \), namely
\[
\| J_{\theta}U_t J_{-\theta} \| \leq Ce^{\lambda^2|I|O(|\theta|)} \quad \text{for} \quad I \subset \mathbb{R}_+ , \quad \| J_{\theta}U_t J_{-\theta} \| \leq C , \quad \text{for} \quad I \subset [-\beta/2, 0],
\]
we can bound the series in (4.16) and (4.20) by
\[
\| J_{\theta}Z J_{-\theta} \| \leq Ce^{2|I|O(|\theta|)} \sum_{i=0}^{\infty} \int_{\Delta t^2} \text{d} u (C|\lambda|)^{2i} \leq Ce^{2|I|O(|\theta|)} , \quad I \subset [-\beta/2, \infty).
\]

5.3. Laplace transform of Green functions. As already mentioned in the introduction, it is more convenient to conduct our analysis of the long-time behaviour of \( Z_t \) in the energy-domain, instead of the time-domain. For sufficiently large \( \text{Re} \ z \), we set
\[
R(z) := \int_0^{\infty} \text{d} t e^{-zt} Z_{[0,t]} , \quad R_{\beta}(z) := \int_0^{\infty} \text{d} t e^{-zt} Z_{[-\beta/2,t]} ,
\]
(5.43)
\[
M(z) := \int_0^{\infty} \text{d} t e^{-zt} (V_{[0,t]} - V_{[0,t]}^{(2)}), \quad R_{\text{ex}}(z) := \int_0^{\infty} \text{d} t e^{-zt} V_{[0,t]}^{(2)},
\]
(5.44)

and (as elucidated below),
\[
Y(z) := Z_{[-\beta/2,0]} + \int_0^{\infty} \text{d} v e^{-zv} \int_{-\beta}^{v} \text{d} u V_{[u,v]} Z_{[-\beta/2,u]}.
\]

Note that \( M \) is the sum/integral of the lowest order diagrams. When writing \( \| A \| \), where \( A \) is an operator acting on a subspace of \( \mathcal{B}_2(\mathcal{H}_S) \), we understand \( \| \cdot \| \) to be the standard operator norm on \( \mathcal{B}(\mathcal{B}_2(\mathcal{H}_S)) \).

Recall the definition of the operators \( J_{\theta} \), with \( \theta \in \mathbb{C}^{2d} \), in (5.11).

Lemma 5.5. The operator-valued function \((z, \theta) \mapsto J_{\theta}A(z)J_{-\theta}, \text{ with } A = M, R_{\text{ex}}, Y \), is analytic in the region \(|\theta| < \theta_0, \text{Re } z > -k_z, \text{ for some } k_z, k_\theta > 0, \text{ and satisfies the bounds} \quad (as \lambda \to 0) \)
\[
\sup_{|\theta| < \theta_0, \text{Re } z > -k_\theta} \left\{ \begin{array}{l}
\| J_{\theta}M(z)J_{-\theta} \| = O(\lambda^2), \\
\| J_{\theta}R_{\text{ex}}(z)J_{-\theta} \| = O(\lambda^2), \\
\| J_{\theta}Y(z)J_{-\theta} \| = O(\lambda^0).
\end{array} \right.
\]
(5.45)

Moreover, for \( \text{Re } z > 0 \),
\[
R_{\beta}(z) = R(z)Y(z)
\]
(5.46)
and
\[
R(z) = (z - L_S - M(z) - R_{\text{ex}}(z))^{-1},
\]
(5.47)

where \( L_S = \text{ad}(H_S) \) is the Liouvillian of the particle system.
Proof. The bounds on $R_{cx}(z), M(z)$ are (the Laplace transform of) the bound in (5.42) for $m = 0$ and $n = 2, n = 1$, respectively, with $k_0 < \nu$, and $k_z = gR/4$. For the bound on $\mathcal{Y}(z)$, we also use that $\|Z_I\| \leq C$, for intervals $I \subset [-\beta/2, 0]$, as established above. To get (5.47), let us abbreviate

$$R_{\text{irr}}(z) := M(z) + R_{cx}(z), \quad R_S(z) := (z - L_S)^{-1}.$$  

Since $L_S$ is selfadjoint (as an operator on $\mathcal{B}_2(\mathcal{H}_S)$), we have $\|R_S(z)\| \leq |\text{Re} z|^{-1}$. We choose $\lambda$ sufficiently small and Re $z$ sufficiently large such that $\|R_{\text{irr}}(z)R_S(z)\| \leq |\text{Re} z|^{-1}\|R_{\text{irr}}(z)\| < 1$. Starting from the ‘polymer expansion’ of $Z_{[0,\ell]}$ in Equation (4.16) and taking the Laplace transform, we find that

$$\int_0^\infty dt \, e^{-z t} Z_{[0,\ell]} = \sum_{n=0}^\infty R_S(z)^n R_{\text{irr}}(z) R_S(z) = R_S(z)(1 - R_{\text{irr}}(z)R_S(z))^{-1},$$

which is (5.47), for $\text{Re} z$ large enough. Since $\|Z_I\| \leq C$, the left side of (5.47) is an analytic function in the region $\{z \in \mathbb{C} : \text{Re} z > 0\}$ and we can extend (5.47) to that region by analytic continuation. Finally, the relation $R_S(z) = R(z)Y(z)$ follows (first for $\text{Re} z$ large enough) by taking the thermodynamic limit and the Laplace transform of (4.26) (and then by continuing analytically in $z$).

5.4. Fiber decomposition. We interrupt our analysis of Green functions in order to recall the fiber decomposition.

To start with, we note that $\mathcal{B}_1(\mathcal{H}_S) \subset \mathcal{B}_2(\mathcal{H}_S), \mathcal{H}_S = l^2(\mathbb{Z}^d)$. Hence, we may view density matrices on $\mathcal{H}_S$ as elements of the space of Hilbert-Schmidt operators, $\mathcal{B}_2(\mathcal{H}_S) \simeq L^2(\mathbb{T}^d \times \mathbb{T}^d, dk_3dk_r)$. We define

$$\hat{O}(k_1, k_r) := \frac{1}{(2\pi)^d} \sum_{x_1, x_r \in \mathbb{Z}^d} O(x_1, x_r) e^{-i k_1 x_1 + i k_r x_r}, \quad O \in \mathcal{B}_2(l^2(\mathbb{Z}^d)).$$

In what follows, we will write $O$ for $\hat{O}$. To conveniently cope with the translation invariance of our model, we make the following change of variables

$$k := \frac{k_1 + k_r}{2}, \quad p := k_1 - k_r,$$

and, for a.a. $p \in \mathbb{T}^d$, we obtain a well-defined function $O_p \in L^2(\mathbb{T}^d)$ by putting

$$(O_p)(k) := O \left( k + \frac{p}{2}, k - \frac{p}{2} \right). \quad (5.48)$$

This follows from the fact that the Hilbert space $\mathcal{B}_2(\mathcal{H}_S) \simeq L^2(\mathbb{T}^d \times \mathbb{T}^d, dk_3dk_r)$ can be represented as a direct integral

$$\mathcal{B}_2(\mathcal{H}_S) \simeq \int_{\mathbb{T}^d} dp \, \mathcal{H}_p, \quad O = \int_{\mathbb{T}^d} dp \, O_p, \quad (5.49)$$

where each ‘fiber space’ $\mathcal{H}_p$ can be identified with $L^2(\mathbb{T}^d)$.

Recall the definition of the operators $\mathcal{J}_\theta$, with $\theta \in \mathbb{C}^{2d}$, in (5.41). The following lemma captures some identities used later on.

Lemma 5.6. Let $O \in \mathcal{B}_1(\mathcal{H}_S)$, then

$$\text{Tr}_S[Oe^{ip \cdot x}] = \langle 1, O_p \rangle_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} dk \, O_p(k), \quad p \in \mathbb{T}^d. \quad (5.50)$$

If there is a $\delta > 0$ such that $\|\mathcal{J}_\theta O\|_2 < \infty$, for $|\theta| \leq \delta$, then $p \mapsto O_p$ is analytic in the interior of the strip $\mathcal{S}_\delta$. (In the discussion above, the fiber operator $O_p$ is defined for a.a. $p$, for $O \in \mathcal{B}_2(\mathcal{H}_S)$, but in the context of Lemma 5.6, $O_p$ can be defined for arbitrary $p$. The first statement of the lemma follows from the singular-value decomposition for trace-class operators and standard properties of the Fourier transform. The second statement of Lemma 5.6 is the Paley-Wiener theorem, i.e., the relation between exponential decay of functions and analyticity of their Fourier transforms; see [7].)
The fiber decomposition in Equation (5.49) is useful when one deals with operators $A$ acting on $\mathcal{H}_2(\mathbb{H}_S)$ that are translation-invariant (TI), i.e., $T_{\theta}A T_{-\theta} = A$, with $T_{\theta}$ defined as in Section 3.2. An important example of a TI operator $A$ is the reduced time-evolution $Z_{[0,t]}$, see Lemma 3.1. For TI operators $A$, we find that $(AO)_p$ depends on $O_p$ only, and hence it makes sense to write

\[(AO)_p = A_p O_p, \quad A = \int_{\mathbb{T}^d} dp A_p.\]  

(5.51)

Similarly to Lemma 5.6 above, we find that, if $J_{\theta/2}AJ_{-\theta/2}$ is bounded for all $\theta = (\theta_1, \theta_2)$, with $|\theta_1| \leq \delta$, then the map $p \mapsto A_p$ is analytic in a strip $\mathcal{V}_\delta$.

5.5. Identifying the lowest order contributions: $L_S + \mathcal{M}(z)$. We return to our the analysis of Green functions. Identifying the fiber spaces $\mathcal{H}_p$ with $L^2(\mathbb{T}^d)$, we interpret $(L_S)_p + \mathcal{M}(z)_p$ as an operator acting on $L^2(\mathbb{T}^d)$.

First, we observe that

\[\lambda^{-2}(L_S)\lambda^2 \nu = i \nabla \epsilon - \chi \cdot \nabla + O(\lambda^2 \kappa),\]  

(5.52)

in the limit $\kappa \to 0$, $\lambda \to 0$.

Second, displaying the $\chi$-dependence in $\mathcal{M}(z)$ explicitly, a straightforward calculation yields, (see Section 6.2 in [3]),

\[\lambda^{-2}(\mathcal{M}(z = 0, \chi = 0)\lambda^2 \nu f)(k) = (Gf)(k) + (Lf)(k), \quad f \in L^2(\mathbb{T}^d),\]  

(5.53)

where

\[(Gf)(k) := \int_{\mathbb{T}^d} dk' r(k', k)f(k'), \quad (Lf)(k) := -\int_{\mathbb{T}^d} dk' r(k', k)f(k),\]  

(5.54)

with $r(\cdot, \cdot) : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{C}$, the ‘rate function’

\[r(k, k') = \psi|\epsilon(k') - \epsilon(k)|,\]  

(5.55)

where $\psi$ is the ‘spectral reservoir density’ defined in [3.6]. Starting from Assumptions A and B, it is straightforward to verify that $r(\cdot, \cdot)$ is a real-analytic function in both variables, which is strictly positive almost everywhere for real arguments.

Hence, taking into account the contributions in (5.52), we are led to consider the operator

\[M^{\kappa, \chi} := i \nabla \epsilon - \chi \cdot \nabla + G + L,\]  

(5.56)

which is densely defined on $L^2(\mathbb{T}^d)$, with core $C^\infty(\mathbb{T}^d)$.

The operator $M^{\kappa, \chi}$ has the physical interpretation of a generator of a one-parameter (strongly continuous) semigroup on $L^2(\mathbb{T}^d)$, often referred to as linear Boltzmann evolution. A detailed analysis of the spectrum of $M^{\kappa, \chi}$ and of the associated evolution equation has been carried out in [3]. The next lemma captures the main results of Section 6 of [3].

Lemma 5.7. There exist constants $k_\kappa, k_\chi > 0$, such that for $|\kappa| \leq k_\kappa$ and $|\chi| \leq k_\chi$ the following holds.

i. The spectrum of the operator $M^{\kappa = 0, \chi} = |\chi| \leq k_\chi$, satisfies

\[\sigma(M^{\kappa = 0, \chi}) \subset \{0\} \cup \{ z \in \mathbb{C} : \Re z \leq -g_M(\chi) \},\]  

(5.57)

for some $g_M(\chi) > 0$. Moreover, 0 is a simple eigenvalue. The spectral projection associated with the eigenvalue 0 is of the form $|\zeta_M(\chi)\rangle\langle 1|$, where $\zeta_M \in L^2(\mathbb{T}^d)$ is a strictly positive, smooth function on $\mathbb{T}^d$, normalized such that $\langle 1, \zeta_M \rangle_{L^2(\mathbb{T}^d)} = 1$.

ii. The spectrum of the operator $M^{\kappa, \chi}$, $|\kappa| \leq k_\kappa$, $|\chi| \leq k_\chi$, satisfies

\[\sigma(M^{\kappa, \chi}) \subset \{ u_M(\kappa, \chi) \} \cup \{ z \in \mathbb{C} : \Re z \leq -g_M(\kappa, \chi) \},\]  

(5.58)

where $u_M(\kappa, \chi) = O(\kappa)$ and $g_M(\kappa, \chi) = g_M(\chi) + O(\kappa) > 0$. Moreover, $u_M(\kappa, \chi)$ is a simple (isolated) eigenvalue, whose associated spectral projection, $P_M^{\kappa, \chi}$, can be written as $|\zeta_M^{\kappa, \chi}\rangle\langle \zeta_M^{\kappa, \chi}|$, with $\zeta_M^{\kappa, \chi}, \zeta_M^{\kappa, \chi} \in L^2(\mathbb{T}^d)$ two smooth functions on $\mathbb{T}^d$, normalized such that $\langle \zeta_M^{\kappa, \chi}, \zeta_M^{\kappa, \chi} \rangle = 1$. 

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In the following we will refer to \( \zeta_M^{\kappa,\chi} \) as the invariant state of \( M^{\kappa,\chi} \).

We refer to Section 6 of [3], for a detailed proof of this lemma. One key ingredient is that the spectrum of the multiplication operator \( L \) has a gap, as follows from the strict positivity of the rate function \( r(\cdot,\cdot) \). Since \( \chi \cdot \nabla \) is anti-self adjoint, the operator \( -\chi \cdot \nabla + L \) has the same gap. Next, since the rate function is analytic in both variables, \( G \) is a compact operator. Then Weyl’s theorem on the stability of the essential spectrum and a Perron-Frobenius type argument, using again the positivity of the rate function, yield part i. Part ii follows from analytic perturbation theory since \( i\kappa \cdot (\nabla \epsilon) \) is a bounded operator.

Next, recalling the definition of \( \mathcal{M} \), we obtain from (5.52) and (5.53), for \( \chi = 0 \),

\[
(\mathcal{L}_S + \mathcal{M}(z = 0, \chi = 0))\lambda_{\kappa} = \lambda^2 M^{\kappa,\chi=0} + \mathcal{O}(\lambda^4|\kappa|),
\]

as \( \lambda \to 0, \kappa \to 0 \). For \( \chi \neq 0 \), the situations is more subtle: In (5.53), we have set \( \chi = 0 \). However, as it turns out, \( \mathcal{M}(z,\chi)_p \) is not an analytic perturbation of \( \mathcal{M}(z,\chi = 0)_p \). To overcome this technical difficulty, we define an operator \( \tilde{M}^{\lambda,\kappa,\chi} \) on \( L^2(\mathbb{T}^d) \) by

\[
\tilde{M}^{\lambda,\kappa,\chi} := i\kappa \cdot (\nabla \epsilon) - \chi \cdot \nabla + \lambda^{-2}(\mathcal{M}(z = 0, \chi))\lambda_{\kappa},
\]

with core \( \mathcal{D} = C^\infty(\mathbb{T}^d) \), such that

\[
(\mathcal{L}_S + \mathcal{M}(z = 0, \chi))\lambda_{\kappa} = \lambda^2 \tilde{M}^{\lambda,\kappa,\chi} + \mathcal{O}(\lambda^4|\kappa|),
\]

as \( \lambda \to 0, \kappa \to 0 \), holds.

In Section 6.2 of [3], we have proven the following lemma, relating the spectrum of \( \tilde{M} \equiv \tilde{M}^{\lambda,\kappa,\chi} \) to the spectrum of \( M \equiv M^{\kappa,\chi} \) in a small neighborhood of zero. Recall the definition of the gap \( g_M(\chi) \) in Lemma 5.7 and define \( B_r \) to be the disk \( B_r := \{ z \in \mathbb{C} : |z| \leq r \} \).

**Lemma 5.8.** There is a constant \( r > 0, r \propto g_M(0) \), such that, inside the ball \( B_r \), the operators \( M \) and \( \tilde{M} \) have unique simple eigenvalues \( u_M \equiv u_M(\kappa,\chi) \) and \( u_{\tilde{M}} \equiv u_{\tilde{M}}(\lambda,\kappa,\chi) \), respectively, with \( |u_{\tilde{M}} - u_M| = \mathcal{O}(\lambda^2) \).

Moreover, for \( z \in B_r \),

\[
\frac{1}{z - M} = \frac{1}{z - u_{\tilde{M}}} P_{\tilde{M}} + \mathcal{O}(z^0).
\]

For a proof we refer to Lemma 6.3 in [3].

This concludes our discussion on the lowest order contributions. In the following section, we explain how the higher order contribution can be controlled.

### 6. Analysis of \( \mathcal{R}(z) \) around \( z = 0 \)

In this section, we show that the map \( z \mapsto \mathcal{R}(z) \), a priori defined for \( \text{Re} \ z > 0 \), can be analytically extended into the region \( \{ z \in \mathbb{C} : |z| < \lambda^2 r \} \), for some \( r > 0 \) and \( \lambda > 0 \) sufficiently small. This is accomplished by applying perturbation theory to the (fibers of the) operators \( \mathcal{R}(z) \). The guiding idea is that \( (\mathcal{R}(z))\lambda^2 \kappa \) is a small perturbation of \( (z - \lambda^2 \tilde{M}^{\lambda,\kappa,\chi})^{-1} \). The small parameters will be the coupling constant \( \lambda \), the (rescaled) fiber momentum \( \kappa \) and the field \( \chi \).

All of these three parameters are assumed to be sufficiently small throughout, and we do not repeat this at every step.

In Lemma 7.4 of [3] we have shown that the map \( z \mapsto (\mathcal{R}(z))\lambda^2 \kappa \) has a unique simple pole in a neighborhood of \( z = 0 \), whose residue, \( P \equiv P^{\lambda,\kappa,\chi} \) is a rank-one operator with the property that, in the fiber indexed by \( \kappa = 0 \),

\[
P^{\lambda,\kappa=0,\chi} = |\zeta\rangle\langle 1|, \quad \text{with} \quad \| \zeta - \zeta_M \|_{L^2(\mathbb{T}^d)} = \mathcal{O}(\lambda^2),
\]

where \( \zeta_M \equiv \zeta_M^{\kappa,\chi} \) is the invariant state of the generator \( M \equiv M^{\kappa,\chi} \); see Lemma 5.4. In Lemma 6.1 we establish that \( P \) and \( \zeta \) are regular function of \( \chi \).

To start with, we define an operator acting on \( L^2(\mathbb{T}^d) \):

\[
S \equiv S(z,\chi,\lambda,\kappa) := (\mathcal{L}_S + \mathcal{M}(z) + \mathcal{R}_{\text{ex}}(z))\lambda^2 \kappa,
\]

(6.2)
such that \((R(z))_{\lambda \gamma \kappa} = (z - S)^{-1}\) whenever the left side is well-defined. Note that \(S\) is a closed operator on \(L^2(\mathbb{T}^d)\): It is bounded except for the term \(\chi \cdot \nabla\) that comes from \(L_S\).

For simplicity, we often abbreviate \(S(z, \lambda, \kappa, \chi)\) by, e.g., \(S(z)\), when considering the function \(z \mapsto S(z)\), with the other variables kept fixed. We use similar shorthand notations for \(u_M \equiv u_M(\kappa, \chi)\), \(P \equiv P^{\lambda, \kappa, \chi}\), etc. in this and the remaining sections.

Recalling (5.60), we write
\[
S(z) = \lambda^2 \tilde{M} + (M(z, \chi) - M(0, \chi))_{\lambda \gamma \kappa} + (R(z))_{\lambda \gamma \kappa}.
\]
From the definition of \(M(z, \chi)\) in (5.44), we infer, using (5.42), that
\[
\|(M(z, \chi) - M(0, \chi))_{\lambda \gamma \kappa}\| \leq C \lambda^2 |z|,
\]
as an operator on \(L^2(\mathbb{T}^d)\). Moreover, from Lemma 5.5, \(\|(R_{ex})_{\lambda \gamma \kappa}\| \leq C \lambda^4\). Thus for \(z \in B_{\lambda \gamma r}, r > 0\), sufficiently small, we have
\[
S(z) = \lambda^2 \tilde{M} + O(\lambda^4 (1 + |\kappa|)).
\]

Recall the constants \(k_z\) and \(k_\theta\) from Lemma 5.5.

**Lemma 6.1.** Let \(D \subset L^2(\mathbb{T}^d)\) be the dense subspace of real-analytic functions on \(\mathbb{T}^d\).

i. \(D\) is a core for \(S\) and \(SD \subset D\). For all \(z \in \mathbb{C}\) satisfying \(\text{Re} z \geq -k_z\) and such that \((z - S(z))^{-1}\) exists (i.e., as a bounded operator), we have \((z - S(z))^{-1}D \subset D\). Moreover, the differences \(S(z) - S(z = 0)\) and \(S(\kappa) - S(\kappa = 0)\) are bounded operators, and they are analytic in the variables \(\kappa, z\) in the region \(\text{Re} z > -k_z\) and \(|\kappa| < k_\theta\).

ii. For \(f \in D\), the function \(\chi \mapsto S(\chi)f\) is \(C^\infty\), and all derivatives are bounded in the variables \(z, \lambda, \kappa, \chi\), uniformly on compacts. Moreover, all derivatives of \(\chi \mapsto S(\chi)f, f \in D,\) are in \(D\).

iii. Fix \(r > 0\) sufficiently small, \(e.g., r = g_M(0)/4\). Then there is a unique \(z = z^* (\lambda, \kappa, \chi)\) in \(B_{\lambda \gamma r}\) such that \(z - S(z)\) is not invertible, i.e., such that \(z \in \sigma(S(z))\). Denoting this unique \(z^*\) by \(u^{\lambda, \kappa, \chi}\), we have that \(u^{\lambda, \kappa, \chi}\) is an isolated simple eigenvalue of \(S(u^{\lambda, \kappa, \chi})\), and when considered as pole of the map \(z \mapsto S(z)\), the residue of \(\chi \mapsto S(\chi)f, f \in D\), is a rank-one operator.

iv. For \(r > 0\) sufficiently small,
\[
\frac{1}{z - S(z)} = \frac{1}{z - u^{\lambda, \kappa, \chi}} P^{\lambda, \kappa, \chi} + R^{\lambda, \kappa, \chi}(z),
\]
for \(z \in B_{\lambda \gamma r}\), where \(z \mapsto R^{\lambda, \kappa, \chi}(z)\) is bounded analytic in \(B_{\lambda \gamma r}\).

v. The pole \(u \equiv u^{\lambda, \kappa, \chi}\) and the operators \(P^\kappa \equiv P^{\lambda, \kappa, \chi}\), \(R^\kappa(z) \equiv R^{\lambda, \kappa, \chi}(z)\) are analytic in \(\kappa\) and \(\gamma\). The pole \(u\) is a \(C^\infty\)-function in \(\chi\), and, for \(B = P^\kappa\), \(R^\kappa(z)\), the function \(\chi \mapsto B(\chi)f\) is \(C^\infty\), for any \(f \in D\).

**Proof.** Statements i, iii, iv and the statements about analyticity in \(\kappa\) in item iv have been proven in Section 7 of [3]. Here, we only give the proofs of statement ii and of the claim on regularity in \(\chi\) in v.

Item ii is a consequence of the discussion in Subsection 5.3.2. In particular, the first derivative is constructed by restricting \([5.38]\) to a fiber (take \(n = 2\) to get \(\mathcal{R}_{ex}\) and \(n = 1\) to get \(\mathcal{M}\), for \(L_S\), the differentiability is explicit).
Higher derivatives are dealt with analogously; see the remark after (5.38).

To prove the regularity claim in iv, we first note that residue and pole of the map \(z \mapsto (z - S(z))^{-1}\), can be expressed as contour integrals,
\[
P = \frac{1}{2\pi i} \int_C dz \frac{1}{z - S(z)}, \quad uP = \frac{1}{2\pi i} \int_C dz \frac{z}{z - S(z)},
\]
with the (positively oriented) contour \(C := \{z \in \mathbb{C} : |z| = \lambda^2 r/2\}\). We claim that
\[
\left(\frac{1}{z - S(z, \chi')} - \frac{1}{z - S(z, \chi)}\right) f = \frac{1}{z - S(z, \chi')} (S(z, \chi') - S(z, \chi)) \frac{1}{z - S(z, \chi)} f, \quad f \in D.
\]
By statement i, both sides are well defined for $z \in \mathcal{C}$ and the equality is checked by multiplying both sides with the invertible operator $z - S(z, \chi')$. Since $(z - S(z, \chi'))^{-1}$ is uniformly bounded for all $z \in \mathcal{C}$, and since $(z - S(z, \chi'))^{-1}f$ is in $\mathcal{D}$, it follows from i that the right side of (6.8) converges to 0, as $\chi' \to \chi$, for $f \in \mathcal{D}$ (and by density this holds for all $f \in L^2(\mathbb{T}^d)$). Thus, forming the difference quotient, we obtain

$$\frac{\partial}{\partial \chi} \frac{1}{z - S(z, \chi)} f = \frac{1}{z - S(z, \chi)} \left( \frac{\partial}{\partial \chi} S(z, \chi) \right) \frac{1}{z - S(z, \chi)} f, \quad f \in \mathcal{D},$$

(6.9)

where we used that $S(z, \chi)f, f \in \mathcal{D}$, is differentiable in $\chi$, with derivative uniformly bounded for $z \in \mathcal{C}$; see item ii. We also claim that the right hand side defines a function in $\mathcal{D}$. This follows from $(z - S(z, \chi))^{-1} \mathcal{D} \subset \mathcal{D}$, see i, and $\frac{\partial}{\partial \chi} \mathcal{D} \subset \mathcal{D}$, see ii. Thus the right side of (6.8) is a function in $\mathcal{D}$, which is uniformly bounded on compacts in the variables $\lambda, \kappa, \chi$ and uniformly bounded for all $z \in \mathcal{C}$. Thus the above procedure can be iterated, and we infer that $(z - S(z, \chi))^{-1}f, f \in \mathcal{D}$, is a $C^\infty$-function in the variable $\chi$, whose derivatives are uniformly bounded on compacts in the variables $\lambda, \kappa, \chi$ and for all $z \in \mathcal{C}$.

The identities in (6.9) then immediately lead to the conclusion that $u, Pf, R(z)f, f \in \mathcal{D}$, are $C^\infty$-functions of $\chi$. We refer to [6] for a more detailed treatment of asymptotic perturbation theory.

Replacing $S(z)$ by the Boltzmann generator $M^{\kappa, \chi}$, the same proof also shows that $u_M^{\kappa, \chi}$, $\xi_M^{\kappa, \chi}$, in Lemma 5.7 are $C^\infty$-functions of $\chi$.

Next, we recall Lemma 5.8 to compare the eigenvalue $u^{\lambda, \kappa, \chi}$, the rank-one operator $P^{\lambda, \kappa, \chi}$, with the corresponding quantities of the operator $M^{\kappa, \chi}$, from Section 5.5. Combination of Lemma 5.8 with Lemma 6.1 yields:

**Lemma 6.2.** For $\kappa = 0$, the residue at $z = 0$, $P^{\kappa=0}$, can be written as $P^{\kappa=0} = |\zeta^{\lambda, \kappa=0, \chi} \rangle \langle 1|$, with $\zeta \equiv \zeta^{\lambda, \kappa, \chi}$ a real-analytic function on $\mathbb{T}^d$ satisfying

$$\| \zeta - \zeta_M \| = O(\lambda^2),$$

where $\zeta_M \equiv \zeta_M^{\kappa, \chi}$ is the invariant state of $M \equiv M^{\kappa, \chi}$. For $\kappa = 0$, $\zeta$ is a probability density on $\mathbb{T}^d$. The function $u = u(\lambda, \kappa, \chi) \in \mathcal{C}$ satisfies

$$u(\kappa) = u(-\kappa), \quad |u - \lambda^2 u_M| = O(\lambda^2).$$

Moreover, we have that

$$u(\kappa = 0) = 0, \quad P^{\kappa=0} R^{\kappa=0}(z) = 0.$$  

(6.10)

7. The equilibrium regime $\chi = 0$

In this section, we discuss properties of the equilibrium correlation functions. We will often need restrictions of operators acting on $B_2(\mathcal{H}_0)$ to the fiber space indexed by $\kappa = 0$. We indicate these restrictions by writing $(A)_0$, for $A \in B_2(\mathcal{H}_0)$, below; in particular, $(A)_0$ acts on $L^2(\mathbb{T}^d) \cong \mathcal{H}_{\kappa=0}$. Whenever we use such fiber restrictions of operators pointwisely, i.e., for a given fiber indexed by $\kappa = 0$, this can be justified by Lemma 5.7 because all operators are quasi-diagonal on $B_2(\mathcal{H}_0)$, and we will omit these justifications.

Recall the results of Theorem 5.1. Statement i, for $\chi \neq 0$, is proven in [3]; statement ii, for $\chi = 0$, has been proven in [2]. Note that the statement for $\chi = 0$ is stronger. In the notation of Section 5.3, this follows from the observation that $\tilde{M} = M$, for $\chi = 0$. As argued in [4][5], the function $z \mapsto \langle R(z) \rangle_{\lambda \kappa}$ consequently has only one pole, namely $u(\lambda, \kappa, \chi = 0)$, in the region $\text{Re} z > -\lambda^2 g_M(\kappa, \chi = 0) + O(\lambda^4)$. Then, the pole $u(\lambda, \kappa, \chi = 0)$ determines the long-time properties. By the inverse Laplace transform, one then proves the following theorem.

**Theorem 7.1.** [Equilibrium asymptotics] Take $\chi = 0$. Then, for $0 < \lambda$ and $\kappa$ sufficiently small, there is $g > 0$ such that

$$\left\| (Z_{[0,t]})_{\lambda \kappa} - e^{u(\kappa)t} P^{\kappa} \right\| = O(e^{-g \lambda^2 t}), \quad t \to \infty,$$

as operators on $L^2(\mathbb{T}^d)$.

Note that $g$ can be chosen to be given by $g = g_M(0)/5$. Also recall that $u(\kappa = 0) = 0$, by Lemma 6.2. For the proof, we refer to Theorem 4.5 of [3].
Lemma 7.2. The topic of the next Lemma.

Proof. In finite volume, cyclicity of the trace implies
\[ \langle O_2(s_2)O_1(s_1) \rangle_{\varrho_0} = \text{Tr} \left[ S_2 \mathcal{G}_{[0,s_2]}(S_1^*) \varrho_0 \right], \]
\[ \langle O_2(s_2)O_1(s_1) \rangle_{\rho} = \text{Tr} \left[ S_2 \mathcal{G}_{[-\beta/2,s_2]}(S_1^*) \eta_\beta \right], \]
for \( s_2 > s_1 \geq 0, \mathcal{S} = (O_i) \) and \( \rho_\beta \) an exponentially localized density matrix. We note, however, that the representation of correlation functions in (7.1) is not unique, since the value of
\[ \text{Tr}_S \mathcal{G}_{[0,\beta]}(S_1^*,S_2^*) \varrho_0 \]
and \( \text{Tr}_S \mathcal{G}_{[-\beta/2,\beta]}(S_1^*,S_2^*) \eta_\beta \),
with \( t \geq s_2 > s_1 \geq 0 \), is independent of \( t \), as long as \( t \geq s_2 \), and we may equally well write
\[ \langle O_2(s_2)O_1(s_1) \rangle_{\varrho_0} = \text{Tr} \left[ \mathcal{G}_{[0,\beta]}(S_1^*,S_2^*) \varrho_0 \right], \]
\[ \langle O_2(s_2)O_1(s_1) \rangle_{\rho} = \text{Tr} \left[ \mathcal{G}_{[-\beta/2,\beta]}(S_1^*,S_2^*) \eta_\beta \right]. \]
This freedom is not apparent in our expansions for \( \mathcal{G}_{[0,\beta]}(S_1^*,S_2^*) \) (or \( \mathcal{G}_{[-\beta/2,\beta]}(S_1^*,S_2^*) \)) because these expansions contain diagrams ‘crossing’ the time \( s_2 \). This suggests that there is a ‘sum rule’ in our expansions, and this is the topic of the next Lemma.

Lemma 7.2. Let \( P^0 = [\zeta^0]\{1\} \) be the spectral projection as given in Lemma 7.1 (for \( \kappa = 0 \)). Then, for all \( s \geq s_1 \geq 0 \),
\[ \int_{s_1}^{s} dt_2 \int_{s_1}^{s_1} dt_1 \, P^0 \left( V_{[t_1,t_2]}(S_1) Z_{[\tau,t_1]} \right) = 0, \]
where \( u = 0 \) or \( u = -\beta/2 \).

Proof. In finite volume, cyclicity of the trace implies
\[ \text{Tr} [S_1 e^{-iS_1^* \mathcal{L} \varrho_0 \otimes \rho_{R,\beta}}] = \text{Tr} \left[ e^{i(s-s_1)\mathcal{L}} \left( S_1 e^{-iS_1^* \mathcal{L} \varrho_0 \otimes \rho_{R,\beta}} \right) \right]. \]
Upon inserting the expansions as above and passing to the thermodynamic limit, we find that
\[ P^0 \left( S_1 Z_{[0,s_1]} \right) = P^0 \left( Z_{[s_1,s]} S_1 Z_{[0,s_1]} \right) + P^0 \int_{s_1}^{s} dt_2 \int_{0}^{s_1} dt_1 \left( Z_{[t_2,t]} \right) V_{[t_1,t_2]}(S_1) Z_{[\tau,t_1]} \right) = 0. \]
The operator \( P^0 \) in front of all terms in (7.6) corresponds to the traces in Equation (7.5). Observe that \( P^0 Z_I = P^0 \) for \( I \subset [0,\infty), \) since \( Z_I \) is trace preserving. Using this in the two terms of the right side of (7.6) yields (7.4) for \( u = 0 \). The proof for \( u = -\beta/2 \) is similar.

We are now prepared to prove Theorem 3.3. Since the technical input—the exponential decay in Theorem 7.1—has been prepared, all that remains are some straightforward algebraic manipulations.

Proof of Theorem 3.3. The case \( m = 1 \): By Theorem 7.1 we know that, for \( f \) a continuous function on \( \mathbb{T}^d \), (hence \( M_f \in \mathfrak{A}_i \)),
\[ \text{Tr}_S [M_f Z_{[0,t]} \rho_0] = \langle f, \zeta^0 \rangle + \mathcal{O}(e^{-\beta \Lambda t^2}), \]
so the only thing in need of a proof is that \( \langle f, \zeta^0 \rangle = \langle M_f \rangle_{\rho_0} \). Recall that, by the construction of the equilibrium dynamics, we have the stationarity property \( \langle Ox=0(t) \rangle_{\rho_0} = \langle O \rangle_{\rho_0} \), \( O \in \mathfrak{A}_i \), i.e., \( \langle Z_{[-\beta/2,\beta]}(t) \eta_\beta \rangle_0 = \langle Z_{[-\beta/2,\beta]}(t) \eta_\beta \rangle_0, \) \( t \geq 0 \), where \( \eta_\beta = \lim_{\Lambda} \mathcal{Z}_\beta \frac{\pi \eta \mathcal{L}}{Z_\beta} \mathcal{L} \); see Lemma 4.1. Thus
\[ \langle M_f \rangle_{\rho_0} = \int_0^\infty dt e^{-\beta t} \langle f, (Z_{[-\beta/2,\beta]} \eta_\beta) \rangle_0 = \langle f, (\mathcal{R}(z) \eta_\beta) \rangle_0 = z \langle f, (\mathcal{R}(z) \eta_\beta) \rangle_0 = z \langle f, \mathcal{R}(\mathcal{Y}(z) \eta_\beta) \rangle_0 = z \langle f, \mathcal{P}(\mathcal{Y}(z) \eta_\beta) \rangle_0 + z \langle f, \mathcal{R}(\mathcal{Y}(z) \eta_\beta) \rangle_0. \]
The second term on the right side of the second line vanishes at $z = 0$ by the analyticity of $R(z), \mathcal{Y}(z)$. Choosing $f \equiv 1$, we obtain

$$1 = \langle 1 \rangle_{\rho_S} = \langle 1, \rho^0 (\mathcal{Y}(z = 0) \eta_\beta)_0 \rangle = \langle 1, (\mathcal{Y}(z = 0) \eta_\beta)_0 \rangle,$$

(7.8)

and hence, setting $z = 0$ in (7.7), $\langle M_f \rangle_{\rho_S} = \langle 1, \zeta^0 \rangle$, in particular, $\langle Z_{(-\beta/2,1)} \eta_\beta \rangle_0 = \langle Z_{(-\beta/2,0)} \eta_\beta \rangle_0 = \zeta^0$.

The case $m = 2$: First, we prove (3.12). We start by considering the correlation functions $\text{Tr}_S [S^*_2 G_{[0,s_2]}(S^*_1) \rho_S]$ and $\text{Tr}_S [S^*_2 G_{[-\beta/2,s_2]}(S^*_1) \eta_\beta]$. Because we take the trace, and $S_1, S_2$ are translation-invariant (hence fiber-preserving), it is sufficient to consider $(G_{[-s_2]}(1))_0$ with (1) standing for $(S^*_1)$. We obtain that

$$G_{[0,s_2]}(1) = Z_{[s_1,s_2]} S_1 Z_{[0,s_1]} + \int_{s_1}^{s_2} ds_2 \int_{s_1}^{s_2} dt_1 Z_{[t_2,s_2]} V_{[t_1,t_2]}(1) Z_{[0,t_1]}$$

(7.9)

and

$$G_{[-\beta/2,s_2]}(1) = Z_{[s_1,s_2]} S_1 Z_{[-\beta/2,s_1]} + \int_{s_1}^{s_2} ds_2 \int_{-\beta/2}^{s_2} dt_1 Z_{[t_2,s_2]} V_{[t_1,t_2]}(1) Z_{[-\beta/2,t_1]}.$$  

(7.10)

It then follows from Theorem 7.1, the bounds (5.40) and $\| (Z_\rho) \| \leq C$ that

$$G_{[0,s_2]}(1)_0 = \langle Z_{[s_1,s_2]} S_1 \rangle_0 P^0 + \int_{s_1}^{s_2} ds_2 \int_{s_1}^{s_2} dt_1 \langle Z_{[t_2,s_2]} V_{[t_1,t_2]}(1) \rangle_0 P^0 + O(e^{-g\lambda^2 \sigma_1}).$$

(7.11)

If we replace $G_{[0,s_2]}(1)$ by $G_{[-\beta/2,s_2]}(1)$, we get a similar identity, except that $P^0$ is replaced by $P^0 \mathcal{Y}(z = 0)$. However, once (7.11) is applied to $(\eta_\beta)_0$, we can use (7.3), to conclude

$$G_{[0,s_2]}(1) = G_{[-\beta/2,s_2]}(S^*_1) \eta_\beta + O(e^{-g\lambda} \sigma_1).$$

This proves (3.12) for $m = 2$.

In order to prove the ‘cluster property’ of the correlation function, i.e., (3.13), we consider the limit $s_2 - s_1 \to \infty$ in (7.10):

$$G_{[-\beta/2,s_2]}(1)_0 = P^0 \{ S_1 Z_{[-\beta/2,s_1]} \} + \int_{s_1}^{s_2} ds_2 \int_{s_1}^{s_2} dt_1 P^0 \{ V_{[t_1,t_2]}(1) Z_{[-\beta/2,t_1]} \} + O(e^{-g\lambda \sigma_1(s_1 - s_2)}).$$

where we have used the ‘sum rule’ (7.4) in the second line. Applying the above equation to $(\eta_\beta)_0$ yields

$$\text{Tr}_S [S^*_2 G_{[-\beta/2,s_2]}(S^*_1) \eta_\beta] = (1, (S_2) \eta_\beta^0)(1, (S_1 Z_{[-\beta/2,s_1]} \eta_\beta)_0) + O(e^{-g\lambda \sigma_1(s_1 - s_2)}),$$

$$= (1, (S_2) \eta_\beta^0)(1, (S_1 \zeta^0) + O(e^{-g\lambda \sigma_1(s_1 - s_2)}),$$

where we used the stationarity to get the second line. This proves the desired cluster property for $m = 2$.

The cases $m > 2$: Straightforward generalization of the above arguments.

\[\square\]

8. PROOF OF THEOREM 3.6

We proceed to proving the Einstein relation. Recall the definition of the velocity operator

$$V^J := i[H, X^J] = i[T, X^J].$$

(8.1)

In a finite volume $\Lambda$, the derivative with respect to $\chi$ of $(V^\Lambda(t))_{\rho_\beta}$ can be computed using Duhamel’s principle:

$$\frac{\partial}{\partial \chi^I} \langle V^\Lambda(t) \rangle_{\rho_\beta} = -i\sqrt{\lambda} \int_0^t ds \langle [X^{\Lambda, I}(t-s), V^\Lambda(t)] \rangle_{\rho_\beta}. $$

(8.2)
Here, it is well-understood that time-evolution on the right side is at \( \chi = 0 \), whereas on the left side the force field is set to 0 only after the differentiation. For simplicity, we drop the spatial indices \( i, j \) in the following.

Using the time-translation invariance of the state \( \rho_{\beta}^{2} \), the KMS condition \((2.13)\) and the time-reversal invariance of the model at vanishing \( \chi \), one finds that

\[
\frac{\partial}{\partial X} \bigg|_{X = 0} \langle V^X(t) \rangle_{\rho_{\beta}^{2}} = \frac{-i\lambda^{2} \beta}{2} \int_{-t}^{t} ds \langle V^X V^\Lambda(s) \rangle_{\rho_{\beta}^{2}} + Q^X(t), \tag{8.3}
\]

where

\[
Q^X(t) := \frac{1}{2} \int_{0}^{\beta} \int_{u}^{\beta} ds \langle V^X V^\Lambda(is + t) \rangle_{\rho_{\beta}^{2}} - \frac{1}{2} \int_{0}^{\beta} \int_{0}^{u} ds \langle V^X V^\Lambda(is - t) \rangle_{\rho_{\beta}^{2}}.
\]

Again, it is understood that time-evolution on the right side of \((8.3)\) is taken for a vanishing external field. For a detailed derivation of Equation \((8.3)\), we refer to Section 4.2 in \[4\]. Note that the correlation functions on the right side involve imaginary times. Finally, we observe that, by the discussion following Lemma \ref{lemma8.1} \((8.6)\) holds in infinite volume as well.

To complete the proof of Theorem \ref{theorem3.6} we show that, in the thermodynamic limit, \( Q(t) \to 0 \), as \( t \to \infty \).

We first establish a lemma concerning the behaviour of correlation functions continued to imaginary times.

**Lemma 8.1.** The equilibrium correlation functions satisfy the bound

\[
\|\langle O_{2}(t_{2})O_{1}(t_{1}) \rangle_{\rho_{\beta}}\| \leq C\|O_{2}\|\|O_{1}\|\|e^{-\lambda^{2}g}\Re(t_{2} - t_{1})\|, \tag{8.4}
\]

for \( t_{2} - t_{1} \in \mathbb{H}_{\beta} \) and \( O_{1}, O_{2} \in \mathfrak{A}_{\beta} \).

**Proof.** Recall the finite volume approximations \( \langle O_{2}(t_{2})O_{1}(t_{1}) \rangle_{\rho_{\beta}} \). By Proposition 5.3.7. in \[1\],

\[
\|\langle O_{2}(t_{2})O_{1}(t_{1}) \rangle_{\rho_{\beta}}\| \leq \|O_{2}\|\|O_{1}\|, \quad t_{1} - t_{2} \in \mathbb{H}_{\beta} . \tag{8.5}
\]

Since \( \|O^{X}\| \leq \|O\| \), we conclude that the correlation functions in infinite volume satisfy \((8.5)\) too. We set \( t_{2} = 0 \) and define

\[
f_{a}(t) := e^{\lambda^{2}st - at^{2}} \langle O_{2}(0)O_{1}(t) \rangle_{\rho_{\beta}}, \quad t \in \mathbb{H}_{\beta},
\]

with the decay rate \( \lambda^{2}g > 0 \) as in Theorem \ref{theorem7.1} and \( a > 0 \). From Theorem \ref{theorem7.1} and the KMS condition we infer that

\[
\sup_{a > 0} \sup_{t \in \partial H_{\beta}} |f_{a}(t)| < \infty .
\]

Furthermore, by the infinite-volume version of \((8.5)\), \( f_{a} \) is bounded on the whole strip \( \mathbb{H}_{\beta} \), and the KMS condition implies that it is continuous on \( \mathbb{H}_{\beta} \) and analytic in the interior. Therefore, the maximum principle for infinite domains (the Phragmen-Lindelöf theorem) yields

\[
\sup_{a > 0} \sup_{t \in \partial H_{\beta}} |f_{a}(t)| \leq \sup_{a > 0} \sup_{t \in \partial H_{\beta}} |f_{a}(t)| < \infty .
\]

By varying \( a \), we deduce that \( \langle O_{2}(0)O_{1}(t) \rangle_{\rho_{\beta}} \leq Ce^{-\lambda^{2}g} \Re(t) \). By time-translation invariance, the claim of the lemma follows. \( \square \)

**Proof of Theorem 3.6** Recall the definition of the operator \( S \) in \((6.2)\). Starting from the bounds in Subsection 5.2.2, Lemma \ref{lemma6.1} says that \( \chi \mapsto S(z, \chi) f \) is \( C^{\infty} \), for \( f \in \mathcal{D} \), (\( \mathcal{D} \) the set of real-analytic functions on \( T^{d} \)). As pointed out in Lemma \ref{lemma6.1} this implies that the function

\[
(\mathcal{R}(z))_{\lambda^{2}e} f = \frac{1}{z - S(z, \lambda, \kappa, \chi)} f, \quad f \in \mathcal{D},
\]

is also \( C^{\infty} \) in \( \chi \). Inspecting the reasoning in Subsection 5.2.2 leading to \((5.39)\) and restricting to fibers, it is clear that \( \chi \mapsto (\mathcal{Q}(z))_{\lambda^{2}e} f \), for \( (z, \chi) \) in a neighborhood of \((0, 0)\), is \( C^{\infty} \), with all derivatives uniformly bounded on compacts. Hence, by \((5.40)\), we get smoothness of \( \chi \mapsto (\mathcal{R}_{\beta}(z))_{\lambda^{2}e} f \). Therefore, for exponentially localized \( \rho_{S} \) (in particular \( \eta_{\beta} \)), the function

\[
(z, \chi) \mapsto z\langle \nabla \epsilon, (\mathcal{R}_{\beta}(z)\rho_{S})_{0} \rangle,
\]

is also \( C^{\infty} \) in \( \chi \).
is analytic in $z$ and $C^\infty$ in $\chi$ for $(z, \chi)$ in a neighborhood of $(0, 0)$, with the corresponding derivatives uniformly bounded on compacts.

Next, starting from the identity $v(\chi) = \langle \nabla \epsilon, \zeta^0 \rangle = \lim_{z \to 0} z \langle \nabla \epsilon, (R_\beta(z))_0(\eta_\beta)_0 \rangle$, we obtain, using the above regularity properties, that

$$\frac{\partial}{\partial \chi} \bigg|_{\chi=0} v(\chi) = \lim_{z \to 0} z \frac{\partial}{\partial \chi} \bigg|_{\chi=0} \langle \nabla \epsilon, (R_\beta(z))_0(\eta_\beta)_0 \rangle .$$

Hence, setting $z = 1/T$, using the definition of $R_\beta$ and the bounds $|\langle V^X(t) \rangle_{\rho_\beta}| \leq C$ and $|\partial_\chi \langle V^X(t) \rangle_{\rho_\beta}| \leq \lambda^2 C t$, for $\chi$ sufficiently small, as follows from (5.19), we get

$$\begin{align*}
\frac{\partial}{\partial \chi} \bigg|_{\chi=0} v(\chi) &= \lim_{T \to \infty} \frac{1}{T} \int_0^\infty dt \, e^{-\frac{t}{T}} \frac{\partial}{\partial \chi} \bigg|_{\chi=0} \langle V^X(t) \rangle_{\rho_\beta} \\
&= \frac{\beta \lambda^2}{2} \lim_{T \to \infty} \frac{1}{T} \int_0^\infty dt \, e^{-\frac{t}{T}} \left( \int_{-t}^t ds \, \langle VV(s) \rangle_{\rho_\beta} + Q(t) \right) \\
&= \frac{\beta \lambda^2}{2} \int_\mathbb{R} ds \, \langle VV(s) \rangle_{\rho_\beta} .
\end{align*}$$

The second equality is (8.3) (in the thermodynamic limit), and the third equality follows because, by Lemma 8.1 $Q(t) \to 0$, as $t \to \infty$, and $\langle VV(s) \rangle_{\rho_\beta} = O(e^{-\lambda^2 gs})$, by Theorem 5.3.

To complete the derivation of the Einstein relation we have to argue that the equilibrium auto-correlation function $\int_\mathbb{R} ds \, \langle VV(s) \rangle_{\rho_\beta}$ is indeed equal to the diffusion constant $D$. This is checked by using the exponential decay of the correlation function and the identity

$$\langle X^i(t)X^j(t) \rangle_{\rho_\beta} = \int_0^t ds_1 \int_0^t ds_2 \langle V^i(s_1)V^j(s_2) \rangle_{\rho_\beta} ,$$

cf., (5.20).

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