S-, P- and R-striations as attractors for electron phase trajectories in spatially periodic resonance fields

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Abstract. A new point of view on the appearance of S-, P- and R-striations in a positive column of inert gases is proposed, based on a dynamic analysis of the resonance properties of electron phase trajectories in spatially periodic fields. The positive column may be considered as a resonator containing a set of resonant modes. Like a tuning fork, being disturbed, it responds with one of the modes, in particular with of S-, P-, or R-modes or striations, depending on the discharge conditions. The dynamic approach eliminates the difficulties of the kinetic theory associated with the long length of the solution of Boltzmann equation, which is much greater than the length of the positive column.

1. Introduction
Until now striations (S-, P-, R-types) have been investigated using either hydrodynamic [1] or kinetic approaches [2] [3] [4]. Full review can be found in Ref. [5].

In this work, a dynamic approach to the analysis of electron motion is proposed. The approach is based on the analysis of resonance properties of the electron phase trajectories in spatially periodic fields [6]. It turned out, that case, ignored in kinetics approach, with total energy losses $\Delta \varepsilon_{ex} = 0$ is the most important one [7]. The case allows to clearly demonstrate the existence of resonant fields with the spatial periods $L_0 = \varepsilon_{ex}/(eE_0)$ and $L_0/p$ with integer $p$, corresponding to the fundamental mode or S-striation, and higher harmonics or the so-called integer resonances, respectively. Here $L_0$ is the resonant length of electric field, $\varepsilon_{ex}$ is the first excitation level of atoms, $e$ is the elementary electric charge, $E_0$ is the electric field amplitude. In particular, the length with $p = 2$ describes the P-striation. Along with integer resonances, there are noninteger resonances $qL_0/p$, where $q/p$ is a rational fraction. For example, the length with $q = 2$ and $p = 3$ corresponds to the R-striation.

2. Electron movement with full energy conserved
At very low pressures, it can be expected that electrons, being accelerated in a constant electric field $E_0$, will reach the excitation threshold $\varepsilon_{ex}$ without energy losses due to elastic collisions. At low currents, electron-electron collisions have no noticeable effect on the motion of electrons and their acceleration in the field. In this idealized case, electrons move with conservation of total energy $\varepsilon$ up to the excitation threshold, experience an inelastic impact, lose a quantum of energy $\varepsilon_{ex}$, accelerated again in the kinetic energy range $0 < w < \varepsilon_{ex}$. The process is periodically
repeated. This case corresponds to the experiments of Frank and Hertz \[8\] in a constant field, in which inelastic impacts were demonstrated for the first time.

From the gas discharge physics point of view, it is of interest to analyze the motion of electrons in spatially periodic electric fields. The analysis allows to elucidate the resonant behavior of the plasma electron component and to make comparison with the experimentally observed striations. The first considered case corresponds to the condition \( \lambda > L_0 \), where \( \lambda \) is the electron mean free path with respect to elastic collisions, \( L_0 = \frac{\varepsilon_{ex}}{\varepsilon E_0} \) is the length at which the electrons gain energy equal to the excitation threshold. Thus, a spatial scale \( L_0 \) appears, which determines the resonance properties of the behavior of electrons.

Let us consider electron movement in modulated fields \( E(x) \) with corresponding potential energy of electrons in this field \( V(x) \) of the following form

\[
E(x) = E_0 \left( 1 + \alpha \cos \frac{2\pi x}{L} \right),
\]

\[
V(x) = -\int_0^x eE(x)dx = -eE_0x - \frac{\varepsilon E_0L}{2\pi} \sin \frac{2\pi x}{L},
\]

where \( \alpha \) is the modulation depth, \( L \) - period of the field. The equation of the electron kinetic energy \( w(x) \) has the following form and boundary conditions:

\[
\frac{dw}{dx} = eE(x) - \varepsilon_{ex} \sum_n \delta(x - x_n), \; n \in \mathbb{N}
\]

\[
w(x=0) = w_0, \; w(x_n-0) = \varepsilon_{ex}, \; w(x_n+0) = 0, \; n \in \mathbb{N}.
\]

The last term in the eq. 3 describes inelastic collisions with energy loss \( \varepsilon_{ex} \). The points \( x_n \) is determined by the condition \( w(x_n) = \varepsilon_{ex} \). \( \delta(x) \) is the Dirac delta function.

The solution of eq. 3 in the field \[1\] has the form

\[
w(x) = w_0 + eE_0 \left( x + \frac{\alpha L}{2\pi} \sin \frac{2\pi x}{L} \right) - \varepsilon_{ex} \sum_n \Theta(x - x_n), \; n \in \mathbb{N},
\]

where \( \Theta(x) \) is the Heaviside step function.

In a phase space \( (\varepsilon = w(x) + V(x), x) \) “full energy – coordinate”, electron trajectories described by eq. 3 are lines which are parallel to the \( x \)-axis, with domains defined by \( x_n < x < x_{n+1} \) (see fig. 1).

Thus, the resonant values of the field periods is expected to be multiples of \( L_0 \). In general, the analysis can be carried out as follows. After substituting the expression for the potential \[2\] into eq. 1 a transcendental equation is obtained, determining the points \( x_n \) as a function of the period length \( L \) and the initial kinetic energy \( w_0 \) in accordance with

\[
w_0 - (n + 1)\varepsilon_{ex} + eE_0 \left( x_n - \frac{\alpha L}{2\pi} \sin \frac{2\pi x_n}{L} \right) = 0.
\]

Let us introduce angle variable \( \theta_n = \frac{2\pi x_n}{L} \) and initial phase \( \phi_0 = \frac{2\pi w_0}{eE_0L} \). Using these variables, eq. 5 can be rewritten as

\[
\alpha \sin \theta_n = \theta_n + \phi_0 - 2\pi(n + 1)\frac{L_0}{L}.
\]

In a simple case when \( w_0 = 0 \Rightarrow \phi_0 = 0 \), it becomes obvious, that eq. 6 has solution if

\[
2\pi(n + 1)\frac{L_0}{L} = 2\pi N, \; n \in \mathbb{N}.
\]
Figure 1. Phase trajectory of electron with $w_0 = 0$ under assumption of full energy conservation (without energy losses due to elastic collisions). Experiences a loss of an energy quantum $\varepsilon_{ex}$ due to inelastic collisions are shown as vertical arrows.

The equality (7) defines resonant spatial periods of the electric field. If $L = qL_0/p$, the eq. (8) transforms to

$$(n + 1)\frac{p}{q} = N, \text{ or } (n + 1)p = Nq.$$  

If eq. (8) holds, the right-hand side of eq. (6) changes by a value proportional to $2\pi$, and the solution is reproduced periodically. In addition, the condition $w_n(x) = w_{n+1}(x)$ is satisfied. However, if the ratio $q/p$ is irrational, the coordinates of inelastic collisions occurrence $x_n$ are not periodic.

Resonances with $q = 1$ and $p \in \mathbb{N}$ can be called integer resonances. There is a fundamental mode $q = 1; p = 1, 2, \ldots$ with length $L = L_0$ and higher harmonics $p = 2, 3, \ldots$ Along with integer resonances, non-integer resonances take place if the fraction $q/p$ is a rational number. In the next sections, resonances with lengths $L_0, L_0/2, L_0/3, 2L_0/3$ is analysed. These resonances correspond to S-, P-, Q- and R-striations respectively, observed in the experiments. The striations lengths, measured in the experiment, to some extent coincide with those, obtained in the dynamic approach.

The experimental potential drop over the length of the S-striation slightly exceeds the excitation threshold $\varepsilon_{ex}$. For the P- and R-striations, the potential drops are also a bit more than $\varepsilon_{ex}/2$, and $2\varepsilon_{ex}/3$, respectively. These slight inequalities are of very fundamental importance, that is described in the next section.

In order to clarify the sharpness of the obtained resonances, the number of inelastic collisions $k$, which are required to reproduce the solution to eq. (5) was calculated in the case of spatial periods with weak detuning $\Delta L = \pm 0.01L_0$. Fig. 2 shows the results of calculations for the S-resonance. The ordinate represents the value $(x_{k+1} - x_k)/L_0$ i.e. normalized length between two collisions, depending on $k$. For such a weak detuning, the value of $k$ should be large enough (in our case $k \sim 300$). In the case of exact resonance, $x_{k+1} - x_k = L_0$ (dashed line in fig. 2).

The number of required intervals which are necessary to reproduce the solution of eq. (5) in dependence on the spatial period length are shown in fig. 3. The domain $0.2 < L/L_0 < 1.1$ covers the S-, P-, Q- and R-resonances. The calculations were performed with a step of $0.05L_0$ with refining to $0.001L_0$ in the peak’s vicinities.

Fig. 3 shows that there are well pronounced integer and rational resonances. Resonances of $L_0/p$ type (S-, P-, and Q-striations) have maximal amplitudes. The resonance at length $2L_0/3$ (R-striation) has maximal amplitude among the non-integer resonances. The performed analysis
suggests that it is the S-, P-, and R-striations that are most likely to be realized in gas discharge. This conclusion, based on the resonance properties of the electron component, does not require full-scale modeling.

It also can be shown that the same resonance effects take place for electrons having nonzero initial kinetic energy.

3. Electron movement with weak energy losses

In the previous section, it was assumed that the energy is accumulated by electrons over the striation length without elastic collisions up to the excitation threshold, i.e. the condition $\lambda > L_0$ holds. Now, the more important case of electron motion in the presence of weak energy losses due to elastic collisions is analyzed. This case is very important for understanding the mechanism of stratification with S-, P-, Q-, and R-striations formation. The range of applicability corresponds to the condition $\lambda < L_0 < \lambda_\varepsilon$, where $\lambda_\varepsilon \sim \sqrt{(m/M)\lambda_\varepsilon}$ is the energy relaxation length, $m$ and $M$ are electron and atom masses, respectively.

In inert gases, the energy relaxation length is approximately two orders of magnitude greater than the mean free path, which is a consequence of the difference in the momentum and energy relaxation times of electrons during elastic collisions.

In the equation of motion (3), an additional term appears which describes the energy loss in elastic collisions:

$$\frac{dw}{dx} = eE(x) - \frac{w}{\lambda_\varepsilon} - \varepsilon_{\text{ext}} \sum_n \delta(x - x_n), \; n \in \mathbb{N}$$

$$w|_{x=0} = w_0, \; w|_{x_n-0} = \varepsilon_{\text{ext}}, \; w|_{x_n+0} = 0, \; n \in \mathbb{N}. \tag{9}$$

The energy relaxation length depends on the kinetic energy. For simplicity, the case $\lambda_\varepsilon \sim \text{const}$ is considered. A similar behaviour is approximately fulfilled for neon. Generalization

Figure 2. The dependence of $(x_{k+1} - x_k)/L_0$ on the number of inelastic collisions $k$. (a) for positive detuning and (b) for negative one. S-striation.

Figure 3. The resonance spectrum: the dependence of the number of inelastic collisions $k$, which is required to reproduce the solution of eq. 5, on the length of the spatial period $L$. 
to other cases can be easily made. The solution of eq. 9 has the form
\[
w(x) = w_0 e^{-\frac{x}{\lambda e}} + e^{-\frac{x'}{\lambda e}} \int_0^x eE(x') e^{-\frac{x'}{\lambda e}} dx' - \varepsilon_{ex} \sum_n \Theta(x - x_n) e^{-\frac{x - x_n}{\lambda e}}.
\]

Eq. 10 includes solution 4 as limiting case for \(\lambda e \to \infty\).

Weak energy losses due to elastic collisions appear to be a new channel of energy dissipation, which leads to an increase in the distance at which the electron reaches the excitation threshold. In order to calculate the new resonance length \(L_S\), the motion of electrons in a constant field is considered in the domain \(x_0 = 0 < x < x_1 = L_S\) with the conditions \(w(L_S) = \varepsilon_{ex}\) and \(w(x_0) = 0\). Expanding the exponent in eq. 10 in a series using parameter \(L_S/\lambda e \ll 1\), the equation for determining the length \(L_S\) is obtained:
\[
eE_0 \lambda e \left(1 - \exp\left(-\frac{L_S}{\lambda e}\right)\right) = \varepsilon_{ex}.
\]

Expanding the exponent in eq. 11 till the quadratic term, one can obtain the resonant length in the form
\[
L_S = \lambda e \left(1 \pm \sqrt{1 - \frac{2L_0}{\lambda e}}\right) \approx L_0 + \frac{L_0^2}{2\lambda e}.
\]

Thus, the energy losses in elastic collisions increase the length of the resonant field for the S-striation by \(\Delta L = L_0^2/(2\lambda e)\), as follows from eq. 12. Fig. 4 compares the phase trajectories of electrons under the conditions of total energy conservation (dashed lines) and low energy losses due to elastic collisions (solid lines).

It can be seen from the fig. 4 that elastic collisions increase the energy gain length by \(\Delta L\) and, accordingly, increase energy losses by \(\Delta \varepsilon = \Delta L \times E_0\), as follows from eq. 11, 12.

Fig. 5 shows the phase trajectories of electrons with \(0 < w_0 < \varepsilon_{ex}\) in cases of S-, P-, R-striations in the phase plane (\(\varepsilon, x\)).

It can be seen from the fig. 5 that the trajectories of electrons with different \(w_0\) are converging to some stable periodical trajectories after several periods. Similar patterns are observed in the

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**Figure 4.** Phase trajectory of an electron with \(w_0 = 0\) in a constant field. Dotted curves describes motion with conserved full energy. Solid curves is for motion with low energy losses due to elastic collision. \(\Delta L\) and \(\Delta \varepsilon\) are the increments for resonance length and energy losses due to elastic collisions, respectively.
Figure 5. Electron phase trajectories in phase space \((\varepsilon, x)\) converged from initial energies \(\{w_0\}\) to the established periodic state for (a) S-striation, (b) P-striation, and (c) R-striation.

P- and R-striations (fig. 5 b, c). The difference is that for P-striation (fig. 5 b) there are two attractors, since \(L_S = 2L_P\). For the R-striation (fig. 5 c), there are also two attractors, since \(2L_S = 3L_R\).

A comparison of the dynamic and kinetic approaches will be carried out in the next work.

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