One Loop Field Strengths of Charges and Dipoles on a Locally de Sitter Background

H. Degueldre†

*Department of Nonlinear Dynamics*
Max-Planck-Institute for Dynamics and Self-Organization
37077 Goettingen, GERMANY

and

R. P. Woodard‡

*Department of Physics*
*University of Florida*
*Gainesville, FL 32611 USA*

**ABSTRACT**

We use the one loop vacuum polarization induced by scalar quantum electrodynamics to compute the electric and magnetic fields of point charges and magnetic dipoles on a locally de Sitter background. Our results are consistent with the physical picture of an inflating universe filling with a vast sea of charged particles as more and more virtual infrared scalar are ripped out of the vacuum. One consequence is that vacuum polarization quickly becomes nonperturbatively strong. Our computation employs the Schwinger-Keldysh effective field equations and is done in flat, conformal coordinates. Results are also obtained for static coordinates.

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† e-mail: hdeguel@nld.ds.mpg.de
‡ e-mail: woodard@phys.ufl.edu
1 Introduction

The phenomenon of vacuum polarization is a triumph of flat space Quantum Electrodynamics (QED). The enhanced high energy coupling strength it predicts has not only been verified experimentally, it also provided the crucial paradigm for understanding renormalization group flows. In spite of these successes, the enhancement is only about 2% at the highest energy accelerators we have so far been able to build [1].

Loop effects such as vacuum polarization derive ultimately from the response to virtual particles (in this case, electrons and positrons) which are the quantum field theoretic manifestation of 0-point motion. It has long been realized that these effects must be strengthened in an expanding universe, essentially because the expansion of spacetime tends to hold virtual quanta apart [2]. Leonard Parker made the first quantitative computations in the late 60’s [3]. He found that the effect is largest for massless particles that are not conformally invariant [4], which includes massless, minimally coupled scalars and gravitons [5]. At a fixed expansion rate the effect increases with the cosmological acceleration. During primordial inflation massless virtual quanta are actually ripped out of the vacuum, which is thought to be the origin of primordial scalar and tensor perturbations [6].

From this discussion one can infer that the largest possible vacuum polarization occurs during de Sitter inflation (which has the highest acceleration consistent with stability) and derives from massless charged particles that are not conformally invariant. These conditions are realistic. The measured value of the scalar power spectrum, and the current upper bound on the tensor-to-scalar ratio [7, 8] suggest that the deceleration parameter (which is minus the acceleration) of primordial inflation was less than -.993 [9], which amply justifies taking the de Sitter limit of -1. Primordial inflation is thought to have occurred at such an enormous scale that all known charged particles would have been effectively massless. However, massless fermions are conformally invariant on the classical level, which means they can only experience the expansion of spacetime through the conformal anomaly. This is responsible for the vacuum polarization of ordinary QED being only slightly enhanced during primordial inflation [10].

Much larger effects can come from a charged scalar (such as the components of the Higgs which become the longitudinal parts of the $W^\pm$ in low energy physics) provided it is massless and not conformally invariant. Davis, Dimopoulos, Prokopec and Törnkvist made the remarkable proposal that
such a particle might even endow the photon with mass during primordial inflation [11, 12]. Their idea was confirmed with a dimensionally regulated and fully renormalized one loop computation of the vacuum polarization from Scalar Quantum Electrodynamics (SQED) on de Sitter background [13]. Although the one loop effect grows without bound [14], a nonperturbative resummation of the leading secular terms reveals that the photon mass approaches a constant value of about 1.8163 times the Hubble constant [15].

The quantum-corrected, linearized Maxwell equations in an arbitrary metric \( g_{\mu\nu} \) read,

\[
\partial_\nu \left( \sqrt{-g} \ g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}(x) \right) + \int d^4x' \left[ \mu_{\Pi} \right](x; x') A_\mu(x') = J^\mu(x) . \tag{1}
\]

Here \( A_\mu(x) \) stands for the vector potential, \( [\mu_{\Pi}](x; x') \) for the vacuum polarization and \( J^\mu(x) \) is the current density. It is immediately obvious that the same vacuum polarization that reveals the effective mass of dynamical photons can also be used to compute the electrodynamic response to point charges and current dipoles. That is the purpose of this paper.

Throughout this paper we employ a spacelike metric. Section 2 derives corrections to the field strengths of point charges and dipoles from the one loop vacuum polarization of flat space SQED. That serves as a useful correspondence limit and also illustrates the basics of the more challenging de Sitter computation. In section 3 we review the de Sitter geometry. Section 4 presents the actual de Sitter computation, with the messy details consigned to two appendices. Our conclusions comprise section 5.

### 2 Vacuum Polarization in Flat Space

The purpose of this section is to work out the effect at one loop from a massless, charged scalar in flat space. Our analysis parallels recent computations, made using the Schwinger-Keldysh formalism [16, 17], of the one loop effect of gravitons on electromagnetism [18] and of a massless, minimally coupled scalar on linearized gravity in flat space background [19]. We begin with some considerations following from the general tensor structure of the vacuum polarization and from the fact that we only have the result to some finite order in the loop expansion. Then the one loop correction is derived for a point charge and for a point magnetic dipole.
2.1 General considerations

Poincaré invariance and transversality constrain the vacuum polarization at any order to take the form,

$$[\mu \Pi^\nu](x; x') = [\partial \cdot \partial \eta^{\mu\nu} - \partial^\mu \partial^\nu] \Pi(x - x') = - [\partial \cdot \partial' \eta^{\mu\nu} - \partial'^\mu \partial'^\nu] \Pi(x - x').$$  (2)

This means we can integrate by parts to express the effective field equation (1) as,

$$\partial_\nu F^\nu_{\mu}(x) - \int d^4x' \Pi(x - x') \partial'_{\nu} F^\nu_{\mu}(x') + \text{(Surface Terms)} = J^\mu(x).$$  (3)

The causality of the Schwinger-Keldysh formalism precludes there being any spatial surface terms, or any surface terms at the upper limit of the temporal integration [20]. There can be surface terms at the lower temporal limit, which corresponds to the time when the state was released [21]. We shall assume that these are completely absorbed by perturbative corrections to the initial state wave functional [22]. If we agree to exclude these state corrections from \(\Pi(x - x')\), the linearized effective field equations become just,

$$\partial_{\nu} F^{\nu\mu}(x) - \int d^4x' \Pi(x - x') \partial_{\nu}' F^{\nu\mu}(x') = J^\mu(x).$$  (4)

The one loop result for \(\Pi(x - x')\) is [13],

$$\Pi^{(1)}(x - x') = -\frac{\alpha \partial^4}{96\pi^2} \left\{ \theta(\Delta t - \Delta x) \left\{ \ln \left[ \mu^2 (\Delta t^2 - \Delta x^2) \right] - 1 \right\} \right\}.$$  (5)

Here \(\partial^2 \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu = -\partial_t^2 + \nabla^2\), \(\Delta t \equiv t - t'\), \(\Delta x \equiv \| \vec{x} - \vec{x}' \|\), and \(\alpha \equiv e^2/4\pi\) is the loop-counting parameter of SQED. Because we do not possess the higher order corrections, equation (4) can only be solved perturbatively. That is, one expands the field strength in powers of \(\alpha\),

$$F_{\mu\nu} = F^{(0)}_{\mu\nu} + F^{(1)}_{\mu\nu} + F^{(2)}_{\mu\nu} + \ldots,$$  (6)

and then distills (4) into terms of the same order. Because the current density is zeroth order we have,

$$\partial^\nu F^{(0)}_{\nu\mu}(x) = J_\mu(x),$$  (7)

$$\partial^\nu F^{(1)}_{\nu\mu}(x) = \int d^4x' \Pi^{(1)}(x - x') \partial^\nu F^{(0)}_{\nu\mu}(x'),$$  (8)
and so on. We can therefore define the source of the one loop field strength to be the “one loop current density,”

\[ J_\mu^{(1)}(x) \equiv \int d^4x' \Pi^{(1)}(x-x')J_\mu(x'). \tag{9} \]

It remains to explain that we will always solve for the field strength tensor directly, rather than going through the intermediate step of finding the vector potential. Note that contracting the Levi-Civita density into the gradient of \( \partial^\alpha F_{\alpha\nu} \) gives,

\[ \epsilon^{\rho\sigma\mu\nu} \partial_\mu \partial_\nu F_{\rho\sigma} = \frac{1}{2} \partial^2 \epsilon^{\rho\sigma\mu\nu} F_{\rho\sigma}. \tag{10} \]

Substituting (9) into (8) and then using (10) implies,

\[ \partial^2 F_\mu^{(1)} = \partial_\mu J_\nu^{(1)} - \partial_\nu J_\mu^{(1)}. \tag{11} \]

This form is advantageous in view of expression (5) for \( \Pi^{(1)}(x-x') \). Up to a homogeneous solution, we can write the one loop field strength tensor as

\[ F_\mu^{(1)} = \partial_\mu J_\nu^{(1)} - \partial_\nu J_\mu^{(1)}, \]

where \( J_\mu^{(1)} \) is obtained by simply removing a factor of \( \partial^2 \) from \( J_\mu^{(1)} \),

\[ J_\mu^{(1)}(x) = -\alpha q \delta^0_\mu \delta^3(\vec{x}) \int_0^\infty \frac{dt}{t_I} \left\{ \ln[\mu^2(t^2 - t_I)] - 1 \right\} J_\mu(x'). \tag{12} \]

The ambiguity regarding homogeneous solutions can be settled by appealing to initial conditions.

### 2.2 Response to a point charge

The current density of a stationary charge \( q \) at the origin is,

\[ J_\mu(t, \vec{x}) = q \delta^0_\mu \delta^3(\vec{x}) \quad \leftrightarrow \quad J_\mu(t, \vec{x}) = -q \delta^0_\mu \delta^3(\vec{x}). \tag{13} \]

Taking the initial time to be \( t_I \) and inserting (13) in expression (12) gives,

\[ J_\mu^{(1)}(t, \vec{x}) = \frac{\alpha q \delta^0_\mu}{96\pi^2} \int_{t_I}^{t-x} dt' \left\{ \ln[\mu(t-t'-x)] + \ln[\mu(t-t'+x)] - 1 \right\} J_\mu(x'). \tag{14} \]

\[ = \frac{\alpha q \delta^0_\mu}{96\pi^2} \partial^2 \left\{ -2x \ln(2\mu x) + 3(x-t+t_I) + (t-t_I-x) \ln[\mu(t-t-I-x)] + (t-t_I+x) \ln[\mu(t-t_I+x)] \right\}. \tag{15} \]
Before acting the d’Alembertian it is useful to specialize it to functions of \( t \) and \( x \),
\[
\partial^2 \longrightarrow -\partial_t^2 + \frac{1}{x} \partial_x^2 x = \frac{1}{2x} (\partial_t - \partial_x)(\partial_t + \partial_x)(t-x) - \frac{1}{2x} (\partial_t - \partial_x)(\partial_t + \partial_x)(t+x) . \tag{16}
\]
We now use (16) to act the d’Alembertian in (15),
\[
\mathcal{J}_\mu^{(1)}(t, \vec{x}) = \frac{\alpha q \delta_\mu^0}{96\pi^2} \left\{ -\frac{4}{x} \ln(2\mu x) + \frac{2}{x} \ln\left[\frac{t-t_I+x}{t-t_I-x}\right] \right\} . \tag{17}
\]
The homogeneous terms would vanish if acted upon by another d’Alembertian, but they also drop out when \( t_I \) is taken to the infinite past,
\[
\lim_{t_I \to -\infty} \mathcal{J}_\mu^{(1)}(t, \vec{x}) = -\frac{\alpha q \delta_\mu^0 \ln(2\mu x)}{24\pi^2} . \tag{18}
\]
The one loop field strengths follow,
\[
\begin{align*}
F_{0i}^{(1)}(t, \vec{x}) & = -\frac{\alpha q}{24\pi^2} \frac{x^i}{x^3} \left[ \ln(2\mu x) - 1 \right] = -\frac{\alpha}{6\pi} \left[ \ln(2\mu x) - 1 \right] \times F_{0i}^{(0)}(t, \vec{x}) , \tag{19} \\
F_{ij}^{(1)}(t, \vec{x}) & = 0 . \tag{20}
\end{align*}
\]
From expression (19) we see that the regularization scale \( \mu \) gives rise to a characteristic length, \( L \equiv e^1/2\mu \). Outside this length the charge \( q \) is screened by quantum corrections, whereas it is enhanced for \( x < L \). Why this happens becomes clearer from examining the one loop current density,
\[
J_\mu^{(1)}(t, \vec{x}) = \partial^2 \mathcal{J}_\mu^{(1)}(t, \vec{x}) = \frac{\alpha q \delta_\mu^0}{24\pi^2} \left\{ -\frac{1}{x^3} + \infty \times \delta^3(\vec{x}) \right\} . \tag{21}
\]
Of course it is the positive contribution at the origin which enhances the electrostatic force for \( x < L \), and the negative cloud of charge density screens it for \( x > L \).

Vacuum polarization can be understood physically from the response of virtual charged particles (in this case massless scalars) to the classical source. The energy-time uncertainty principle allows a pair of virtual particles with mass \( m \) and wave number \( k \) to exist for a time \( \Delta t \sim 1/\sqrt{m^2 + k^2} \). During this time the partner whose charge is opposite to \( q \) will be pulled into the source, and the partner with the same charge as \( q \) will be pushed away. By itself, that would lead to a negative induced charge at the origin, which is
exactly opposite to (21). However, two features complicate the physics of vacuum polarization in this model: renormalization and the masslessness of our scalars. The first feature means that what we call the one loop current density $J^{(1)}_\mu(t, \vec{x})$ actually includes an infinite constant times the classical current density. Had the mass been nonzero, this infinite constant would have been chosen to completely null the one loop correction to the fields at spatial infinity. However, the masslessness of our charged scalars means that there continues to be an effect out to arbitrarily large distances. In that case, one chooses the renormalization constants to make the one loop field strengths vanish at some fixed distance, $L = e^1/2\mu$.

### 2.3 Response to a point magnetic dipole

Shrinking a current loop with magnetic dipole moment $\vec{m}$ down to the origin gives the following current density,

$$J^\mu(t, \vec{x}) = -\epsilon^{0\mu\rho\sigma} m_\rho \partial_\sigma \delta^3(\vec{x}) \quad \Longrightarrow \quad J^0 = 0 \quad , \quad \vec{J} = -\vec{m} \times \vec{\nabla} \delta^3(\vec{x}) .$$

This differs from the current density (13) of a point charge only by the replacement,

$$q\delta^\mu_0 \rightarrow -\epsilon^{0\mu\rho\sigma} m_\rho \partial_\sigma .$$

(23)

Because the derivative must be spatial it can be partially integrated, then converted from $\vec{\nabla}^\prime$ to $-\vec{\nabla}$, so the result for $J^{(1)}_\mu(t, \vec{x})$ follows by making the same replacement (23) in expression (18),

$$\lim_{t_1 \rightarrow -\infty} J^{(1)}_\mu(t, \vec{x}) = -\frac{\alpha}{24\pi^2} \epsilon^{0\mu\rho\sigma} m_\rho \partial_\sigma \left[ \frac{\ln(2\mu x)}{x} \right] .$$

(24)

It is useful to $3 + 1$ decompose (24),

$$J^{(1)}_0(t, \vec{x}) = 0 ,$$

(25)

$$J^{(1)}_j(t, \vec{x}) = \frac{\alpha}{6\pi} \left[ \ln(2\mu x) - 1 \right] \times \frac{\epsilon^{jk\ell} m_k x^\ell}{4\pi x^3} .$$

(26)

The resulting field strengths are,

$$F_{0i}(t, \vec{x}) = 0 ,$$

(27)

$$F_{ij}(t, \vec{x}) = \frac{\alpha \epsilon^{ijk}}{24\pi^2} \left\{ \frac{(3\tilde{x}^k \hat{x} \cdot \vec{m} - m_k)}{x^3} \left[ \ln(2\mu x) - 1 \right] + \frac{(m_k - \tilde{x}^k \hat{x} \cdot \vec{m})}{x^3} \right\} .$$

(28)
In view of the relation $F_{ij} = -\epsilon^{ijk} B^k$ we can read off the one loop correction to the magnetic field from (28),

$$\vec{B}^{(1)}(t, \vec{x}) = -\frac{\alpha}{6\pi} \ln(2\mu x) - 1 \times \vec{B}^{(0)}(t, \vec{x}) - \frac{\alpha}{6\pi} \times \frac{\vec{m} - \vec{x} \cdot \vec{m}}{4\pi x^3},$$

where (away from $\vec{x} = 0$) the classical result is,

$$\vec{B}^{(0)}(t, \vec{x}) = \frac{(3\vec{x} \cdot \vec{m} - \vec{m})}{4\pi x^3}. \tag{30}$$

Expression (29) represents the same sort of screening we found in (19) for a point charge. The second term in (29) is just the residue necessary to keep $\vec{B}$ transverse with the coordinate-dependent screening factor. Except for the usual divergence at the origin, the one loop current density which supports this field is,

$$\vec{J}^{(1)}(t, \vec{x}) = -\frac{\alpha}{8\pi^2} \frac{\vec{m} \times \vec{x}}{x^4} \quad (\vec{x} \neq 0). \tag{31}$$

The current rotates clockwise with respect to $\vec{m}$.

## 3 de Sitter Geometry

The de Sitter geometry is the unique, maximally symmetric solution of Einstein’s equation with a positive cosmological constant. Any coordinatization $x^\mu$ of de Sitter can be described by a mapping $x^\mu \rightarrow X^A(x)$ to the 4-dimensional submanifold of 5-dimensional Minkowski space such that,

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 \equiv \eta_{AB} X^A X^B = \frac{1}{H^2}. \tag{32}$$

Here $H$ is the Hubble constant, which is related to the cosmological constant by $\Lambda = 3H^2$. The de Sitter metric in $x^\mu$ coordinates is given by,

$$g_{\mu\nu}(x) = \frac{\partial X^A}{\partial x^\mu} \frac{\partial X^B}{\partial x^\nu} \eta_{AB}. \tag{33}$$

The most convenient coordinates for our work are open conformal coordinates, $x^\mu = (\eta, \vec{x})$. Whereas each of the spatial coordinates $x^i$ runs from
$-\infty$ to $+\infty$, the conformal time runs from $\eta \to -\infty$ (the infinite past) to $\eta \to 0$ (the infinite future). The 5-dimensional embedding is,

$$X^0 = \frac{a}{2H} \left[ 1 + H^2(x^2 - \eta^2) \right],$$  \hspace{1cm} (34)

$$X^i = ax^i,$$  \hspace{1cm} (35)

$$X^4 = \frac{a}{2H} \left[ 1 - H^2(x^2 - \eta^2) \right],$$  \hspace{1cm} (36)

where the scale factor is $a \equiv -1/H\eta$. The inverse transformation is,

$$\eta = -\sqrt{\frac{\eta_{AB}X^A X^B}{H(X^0 + X^4)}}, \quad x^i = \frac{X^i}{H(X^0 + X^4)}.$$  \hspace{1cm} (37)

From expression (33) we see that the invariant element is,

$$ds^2 = a^2(\eta) \left[ -d\eta^2 + d\vec{x} \cdot d\vec{x} \right], \quad a(\eta) = -\frac{1}{H\eta}.$$  \hspace{1cm} (38)

Hence $g_{\mu\nu} = a^2 \eta_{\mu\nu}$.

Because 4-dimensional electrodynamics is conformally invariant we can reduce many aspects of the de Sitter computation (in conformal coordinates) to familiar manipulations in flat space. One feature of conformal coordinates that sometimes disturbs the mathematically inclined is that they do not cover the full de Sitter manifold. One can see this by adding (34) and (36) to get,

$$X^0 + X^4 = \frac{a}{H} > 0.$$  \hspace{1cm} (39)

This is of no import because $\eta = \text{const}$ defines a Cauchy surface, so information from any part of the full de Sitter manifold must enter the conformal coordinate submanifold as an initial condition. The full de Sitter manifold is irrelevant if one regards de Sitter as a paradigm for the geometry of primordial inflation. From that perspective the conformal coordinate submanifold is just a special case of the larger class of spatially flat, Friedman-Robertson-Walker (FRW) geometries which are relevant to inflationary cosmology.

We shall also be interested in static coordinates, $x^u = (t, \vec{r})$. Although the time coordinates runs from $t \to -\infty$ to $t \to +\infty$, the spatial radius obeys $0 \leq r < 1/H$. The 5-dimensional embedding is,

$$X^0 = \frac{1}{H} \sinh(HT) \sqrt{1 - H^2 r^2},$$  \hspace{1cm} (40)

$$X^i = r^i,$$  \hspace{1cm} (41)

$$X^4 = \frac{1}{H} \cosh(HT) \sqrt{1 - H^2 r^2}.$$  \hspace{1cm} (42)

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The inverse is,

\[ t = \frac{1}{H} \tanh^{-1}\left( \frac{X^0}{X^4} \right), \quad r^i = \frac{X^i}{H \sqrt{\eta_{AB} X^A X^B}}. \]  

(43)

Combining with (33) we find the invariant element,

\[ ds^2 = -\left[ 1 - H^2 r^2 \right] d\tau^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 d\Omega^2. \]  

(44)

Of course this is why the system is called “static coordinates.”

As we have mentioned, mathematically oriented physicists sometimes imagine that subtle errors must result from failing to formulate quantum field theory on the full de Sitter manifold. Static coordinates pose an equal peril for confusing those who seek to deny the possibility of secular effects associated with the continuous creation of massless, minimally coupled scalars and gravitons during inflation. Quantum field theory cannot be formulated in static coordinates because \( t = \text{const} \) does not constitute a Cauchy surface. The static coordinate \( t \) is the proper time of an observer in free fall at the origin. Although no point outside the static coordinate system can causally influence this observer, that is only true for \( \vec{r} = 0 \). All other points in the static coordinate system are in causal contact with points from outside.

The preceding considerations preclude computing the vacuum polarization in static coordinates, but they in no way prevent us from transforming results from conformal coordinates into static coordinates. If the source is at the static coordinate origin then an observer at fixed \( \vec{r} \) would appear, in conformal coordinates, to be moving towards the origin with precisely the velocity needed to counteract the expansion of spacetime and maintain a constant physical distance from the source. Such an observer will experience effects associated with boosting the conformal field strength, and other effects associated with shrinking the conformal coordinate separation to zero.

By substituting (40-42) in (37) we see that the coordinate transformation is,

\[ \eta = \frac{-e^{-Ht}}{H \sqrt{1 - H^2 r^2}}, \quad x^i = \frac{r^i e^{-Ht}}{\sqrt{1 - H^2 r^2}}. \]  

(45)

The various components of the Jacobian are,

\[ \frac{\partial \eta}{\partial t} = \frac{e^{-Ht}}{\sqrt{1 - H^2 r^2}}, \quad \frac{\partial x^i}{\partial t} = -\frac{H r^i e^{-Ht}}{\sqrt{1 - H^2 r^2}}, \]  

(46)

\[ \frac{\partial \eta}{\partial r^j} = -\frac{H r^j e^{-Ht}}{\left[ 1 - H^2 r^2 \right]^{3/2}}, \quad \frac{\partial x^i}{\partial r^j} = \frac{\delta^{ij} e^{-Ht}}{\sqrt{1 - H^2 r^2}} + \frac{H^2 r^i r^j e^{-Ht}}{\left[ 1 - H^2 r^2 \right]^{3/2}}. \]  

(47)
We denote the static coordinate field strengths with a tilde,
\[ \tilde{F}_{0i} = (e^{-2Ht}/1-H^2r^2) \left\{ F_{0i} + Hr^jF_{ij} \right\}, \]
\[ \tilde{F}_{ij} = (e^{-Ht}/1-H^2r^2)^2 \left\{ -Hr^iF_{0j} + Hr^jF_{0i} + (1-H^2r^2)F_{ij} \right\} + H^2r^i r^k F_{kj} - H^2r^j r^k F_{ki}. \]

The factors of $e^{-2Ht}$ seem to drive $\tilde{F}_{\mu\nu}$ to zero but one has to keep in mind that a factor of $1/x^2$ grows at fixed $r$,
\[ \frac{1}{x^2} = \frac{e^{2Ht}}{r^2} \left[ 1-H^2r^2 \right]. \]

Note also that the scale factor is,
\[ a = e^{Ht} \sqrt{1-H^2r^2}. \]

### 4 Vacuum Polarization in de Sitter Space

The plan of this section is the same as that of section 2. We begin with some general considerations in which the one loop result for the $[\mu\Pi^\nu](x;x')$ is presented in open conformal coordinates, and the one loop currents it engenders are evaluated as much as possible for a general source. We then specialize the classical source to a point charge and a point magnetic dipole. In each case the one loop current density and the one loop field strengths are first derived in open conformal coordinates, and then transformed to static coordinates.

#### 4.1 General Considerations

An important simplification associated with conformal coordinates in $D = 4$ spacetime dimensions is that all the scale factors in the classical Maxwell equation cancel out,
\[ \partial_\nu \left[ \sqrt{-g} g^{\nu\rho} g^{\mu\sigma} F_{\rho\sigma} \right] = \partial_\nu \left[ \eta^{\nu\rho} \eta^{\mu\sigma} F_{\rho\sigma} \right]. \]
This means much of the flat space formalism of section 2 still applies. In particular, the equation for the one loop correction to the field strength is,

$$\partial^2 F^{(1)}_{\mu\nu} = \partial_\mu J^{(1)}_\nu - \partial_\nu J^{(1)}_\mu,$$

where $$\partial^2 \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu = -\partial^2_\eta + \nabla^2$$ is the flat space d’Alembertian and the one loop current is,

$$J^{(1)}_\mu(x) \equiv \eta_{\mu\rho} \int d^4 x' \left[ \rho \Pi^{\nu}_{(1)}(x; x') A^{(0)}_\nu(x') \right].$$

The retarded solution for (53) is,

$$F^{(1)}_{\mu\nu}(x) = -\int d^4 x' \frac{\delta(\Delta \eta - \Delta x)}{4\pi \Delta x} \left[ \partial'_\mu J^{(1)}_\nu(x') - \partial'_\nu J^{(1)}_\mu(x') \right],$$

where the conformal coordinate separations are $$\Delta \eta \equiv \eta - \eta'$$ and $$\Delta x \equiv ||\vec{x} - \vec{x}'||$$.

Because the massless, minimally coupled scalar is not de Sitter invariant, the vacuum polarization it engenders contains two distinct tensor structures. One of these is proportional to the covariant transverse projection operator, whereas the other one is proportional to the purely spatial transverse projection operator constructed from

$$[\mu \Pi^{\nu}](x; x') = \left[ \eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma} \right] \partial'_\mu \partial'_\sigma \left\{ \Pi_F(x; x') + \Pi_C(x; x') + \Pi_K(x; x') \right\}$$

$$+ \left[ \tilde{\eta}^{\mu\nu} \tilde{\eta}^{\rho\sigma} - \tilde{\eta}^{\mu\rho} \tilde{\eta}^{\nu\sigma} \right] \partial'_\mu \partial'_\sigma \Pi_K(x; x').$$

It is convenient to break the covariant structure function up into the old, flat space contribution $$\Pi_F$$, a term $$\Pi_C$$ like the conformal anomaly, and a nonlocal de Sitter contribution $$\Pi_G$$. At one loop order one has [13, 14],

$$\Pi^{(1)}_F(x; x') = -\frac{\alpha}{96\pi^2} \partial^4 \left[ \theta(\Delta \eta - \Delta x) \left\{ \ln \left[ \mu^2 (\Delta \eta^2 - \Delta x^2) \right] - 1 \right\} \right],$$

$$\Pi^{(1)}_C(x; x') = -\frac{\alpha}{6\pi} \ln(a) \delta^4(x-x'),$$

$$\Pi^{(1)}_G(x; x') = -\frac{\alpha H^2 a}{8\pi^2} \partial^2 \left[ a' \theta(\Delta \eta - \Delta x) \left\{ \ln \left[ H^2 (\Delta \eta^2 - \Delta x^2) \right] + 1 \right\} \right].$$

1One might wonder if there is any advantage to employing a covariant representation based on covariant derivatives and de Sitter invariant basis tensors. However, a systematic investigation of this formalism reveals that it is cumbersome and that it obscures the essential physics [23]. The procedure for converting our noncovariant — but simple — representation (56) to the complicated and counter-intuitive covariant representation can be found in [24].
The one loop noncovariant structure function is \[13, 14\],

\[
\Pi_K^{(1)}(x; x') = \frac{\alpha H^4 a^2 a'^2}{4\pi^2} \theta(\Delta \eta - \Delta x) \left\{ \ln[H'^2(\Delta \eta^2 - \Delta x^2)] + 2 \right\}. \tag{60}
\]

Here \(H' \equiv e^{\gamma-1}H\), where \(\gamma \approx 0.57721\) is Euler’s constant. Although the spatial dependence of the various structure functions is limited to the coordinate separation, \(\vec{x} - \vec{x}'\), factors of \(a\) and \(a'\) complicate the temporal dependence.

The various one loop current densities can be expressed in terms of the classical current density \(J_{\mu}\) and the classical field strength \(F_{\mu\nu}^{(0)}\),

\[
J_{\mu}^{1F}(x) = -\alpha \frac{\partial^4}{96\pi} \int d^4x' \theta(\Delta \eta - \Delta x) \left\{ \ln[H'^2(\Delta \eta^2 - \Delta x^2)] - 1 \right\} J_{\mu}(x'), \tag{61}
\]

\[
J_{\mu}^{1G}(x) = -\frac{\alpha}{6\pi} \left[ -\ln(a) F_{\nu\mu}^{(0)}(x) \right] = \frac{\alpha}{6\pi} \left[ -\ln(a) J_{\mu}(x) + H a F_{0\mu}^{(0)}(x) \right], \tag{62}
\]

\[
J_{\mu}^{1K}(x) = -\frac{\alpha H^2}{8\pi^2} \partial^\nu \left[ a \partial^2 \int d^4x' a' \theta(\Delta \eta - \Delta x) \right.
\]

\[
\times \left\{ \ln[H'^2(\Delta \eta^2 - \Delta x^2)] + 1 \right\} F_{\nu\mu}^{(0)}(x') \right]. \tag{63}
\]

\[
J_{\mu}^{1G}(x) = \frac{\alpha H^4 a^2}{4\pi^2} \int d^4x' a'^2 \theta(\Delta \eta - \Delta x) \left\{ \ln[H'^2(\Delta \eta^2 - \Delta x^2)] + 1 \right\} \partial^\nu F_{\nu\mu}^{(0)}(x'), \tag{64}
\]

Of course \(J_{\mu}^{1F}(x)\) is the same as the flat space result of section 2. Note that \(J_0^{1K}(x) = 0\) because the index \(\bar{\mu}\) is purely spatial. It is useful to 3 + 1 decompose \(J_{\mu}^{1G}(x)\),

\[
J_0^{1G}(x) = -\frac{\alpha H^2 a}{8\pi^2} \partial^2 \int d^4x' a' \theta(\Delta \eta - \Delta x) \left\{ \ln[H'^2(\Delta \eta^2 - \Delta x^2)] + 1 \right\} J_0(x'), \tag{65}
\]

\[
J_i^{1G}(x) = -\frac{\alpha H^2 a}{8\pi^2} \partial^2 \int d^4x' a' \theta(\Delta \eta - \Delta x) \left\{ \ln[H'^2(\Delta \eta^2 - \Delta x^2)] + 1 \right\} J_i(x'), \tag{66}
\]

\[
+ \frac{\alpha H^2 a}{8\pi^2} \partial^2 \int d^4x' a'^2 \theta(\Delta \eta - \Delta x) \left\{ \ln[H'^2(\Delta \eta^2 - \Delta x^2)] + 1 \right\} F_{0i}^{(0)}(x'),
\]

\[
+ \frac{\alpha H^2 a}{8\pi^2} \partial^2 \int d^4x' a'^2 \theta(\Delta \eta - \Delta x) \left\{ \ln[H'^2(\Delta \eta^2 - \Delta x^2)] + 1 \right\} F_{0i}^{(0)}(x'). \tag{66}
\]
We should also comment on the range of coordinates. Unlike the flat space case, it is not possible to release this system in free vacuum infinitely far back in the past because perturbation theory breaks down after a finite interval of conformal time \[15\]. The one loop results \([57-60]\) were computed with initial time \(\eta_I = -1/H\), corresponding to \(a_I = 1\). For a source at the origin, the factors of \(\theta(\Delta \eta - \Delta x)\) in \([57-60]\) imply that there is no one loop effect for radii \(x\) outside the range,

\[
\eta_I < \eta - x \quad \implies \quad Hx < 1 - \frac{1}{a}.
\]

For future reference we define the scale factors \(a_{\pm}\) evaluated at conformal times \(\eta \pm x\),

\[
a_{\pm} \equiv -\frac{1}{H(\eta \pm x)} = \frac{1}{a^{-1} \pm Hx} = e^{Ht} \sqrt{\frac{1 \pm Hr}{1 \mp Hr}}.
\]

The late time behavior of these scale factors depends critically upon whether or not one fixes the conformal radius \(x\) or static radius \(r\),

\[
(a_{\pm})_{x \text{ fixed}} \rightarrow \mp \frac{1}{Hx}, \quad (a_{\pm})_{r \text{ fixed}} \rightarrow e^{Ht} \sqrt{\frac{1 \pm Hr}{1 \mp Hr}}.
\]

### 4.2 Response to a point charge

The electromagnetic coupling to a point charge \(q\) with worldline \(\chi^\mu(\tau)\) is,

\[
S_q = q \int d\tau \dot{\chi}^\mu A_\mu(\chi).
\]

Because \(S_q\) is independent of the metric, the current density of a point charge is unchanged from \([13]\), nor are the 0th order field strengths changed from section 2.1,

\[
J^\mu(\eta, \vec{x}) = q \delta^\mu_0 \delta^3(\vec{x}) \quad , \quad F^{(0)}_{0i}(\eta, \vec{x}) = \frac{q}{4\pi} \frac{x^i}{x^3} \quad , \quad F^{(0)}_{ij}(\eta, \vec{x}) = 0.
\]

One consequence is that \(J^{1F}_\mu\) must agree with expression \([21]\), and the associated one loop field strengths must agree with \([19-20]\),

\[
J^{1F}_\mu(\eta, \vec{x}) = \frac{\alpha q}{24\pi^2} \frac{\delta^\mu_0}{x^3} (\vec{x} \neq 0) \quad , \quad F^{1F}_{\mu \nu}(\eta, \vec{x}) = -\frac{\alpha}{6\pi} \left[ \ln(2\mu x) - 1 \right] \times F^{(0)}_{\mu \nu}(\eta, \vec{x}).
\]
Substituting (71) into (62) gives,

\[ J_0^{1C}(\eta, \vec{x}) = \frac{\alpha q}{6\pi} \ln(a) \delta^3(\vec{x}) \quad , \quad J_i^{1C}(\eta, \vec{x}) = \frac{\alpha q H a x^i}{24\pi^2 x^3} = -\partial_i \left\{ \frac{\alpha q H a}{24\pi^2 x} \right\}. \]

Hence the C-type charge density represents a fractional screening of the classical charge density by \(-\frac{\alpha}{6\pi} \ln(a)\), with the current density \(J_i^{1C}\) carrying off the positive charge to infinity. We can easily solve for the C-type electric field by noting,

\[ \partial^2 F_{0i}^{1C}(\eta, \vec{x}) = \partial_0 J_i^{1C}(\eta, \vec{x}) - \partial_i J_0^{1C}(\eta, \vec{x}) = \partial^2 \partial_i \left\{ \frac{\alpha q \ln(a)}{24\pi^2 x} \right\}. \]

Therefore, the C-type field strengths are,

\[ F_{\mu\nu}^{1C}(\eta, \vec{x}) = -\frac{\alpha}{6\pi} \ln(a) \times F_{\mu\nu}^{(0)}(\eta, \vec{x}). \]  

Substituting (71) into (64) implies that all components of the K-type current density vanish. It is better to approach the G-type current density and its associated field strength indirectly. Suppose the temporal and spatial components of the current density take the form,

\[ J_0^{1G}(\eta, \vec{x}) = f(\eta, x) \quad , \quad J_i^{1G}(\eta, \vec{x}) = \partial_i g(\eta, x). \]

Current conservation implies,

\[ 0 = -\partial_\eta f(\eta, x) + \nabla^2 g(\eta, x) \implies g(\eta, x) = \int_0^x dx' \left( 1 - \frac{x'}{x} \right) x' \partial_\eta f(\eta, x'). \]

The magnetic field is obviously zero and we can find the electric field by noting,

\[ \partial^2 F_{0i}^{1G}(\eta, \vec{x}) = \partial_\eta J_i^{1G}(\eta, \vec{x}) - \partial_i J_0^{1G}(\eta, \vec{x}) = \partial_\eta \left\{ \nabla^2 \frac{1}{\nabla^2 f - f} \right\}, \]

\[ = -\partial^2 \partial_i \int_0^x dx' \left( 1 - \frac{x'}{x} \right) x' f(\eta, x'). \]

Hence everything follows from the zero component,

\[ J_i^{1G}(\eta, \vec{x}) = \frac{x^i}{x^3} \int_0^x dx' x'^2 \partial_\eta J_0^{1G}(\eta, x'), \]

\[ F_{0i}^{1G}(\eta, \vec{x}) = -\frac{x^i}{x^3} \int_0^x dx' x'^2 J_0^{1G}(\eta, x'), \]

\[ F_{ij}^{1G}(\eta, \vec{x}) = 0. \]
Expression (82) is recognizable as the integral form of Gauss’s law, and expression (81) is the Maxwell displacement current. It remains to evaluate

\[ J_1 G_0 (\eta, x) = \frac{\alpha q H^2 a}{8\pi} \partial^2 \int_{\eta_i}^{\eta} d\eta' d' \left\{ \ln[H'(\Delta \eta - x)] + \ln[H'(\Delta \eta + x)] + 1 \right\}. \] (84)

Expression (84) is sufficiently intricate that the best strategy is to treat each of the three integrands separately, and to make an additional distinction between the two derivative operators which result when the d’Alembertian is specialized to functions of just \( \eta \) and \( x \) as in (16),

\[ \partial^2 \rightarrow \frac{1}{2x} (\partial_\eta - \partial_x)(\partial_\eta + \partial_x)(\eta - x) - \frac{1}{2x} (\partial_\eta - \partial_x)(\partial_\eta + \partial_x)(\eta + x). \] (85)

The first of the integrands in (84) depends only upon \( \eta - x \) so it vanishes when acted upon by the first operator of (85),

\[ \frac{1}{2x} (\partial_\eta - \partial_x)(\partial_\eta + \partial_x)(\eta - x) \int_{\eta_i}^{\eta} d\eta' d' \ln[H'(\Delta \eta - x)] = 0. \] (86)

A nonzero result emerges when this same first integrand is acted on by the second operator in (85),

\[ -\frac{1}{2x} (\partial_\eta - \partial_x)(\partial_\eta + \partial_x)(\eta + x) \int_{\eta_i}^{\eta} d\eta' d' \ln[H'(\Delta \eta - x)] = -\frac{1}{x} (\partial_\eta - \partial_x) \int_{\eta_i}^{\eta} d\eta' d' \ln[H'(\Delta \eta - x)], \] (87)
\[ = -\frac{2a_-}{x} \left\{ \gamma - 1 - \ln(a_-) + \ln(1 - \frac{1}{a_-}) \right\}. \] (88)

(We remind the reader of the scale factors \( a_\pm \) defined in (68).) Acting the first part of the d’Alembertian (85) on the second integrand in (84) gives,

\[ \frac{1}{2x} (\partial_\eta - \partial_x)(\partial_\eta + \partial_x)(\eta - x) \int_{\eta_i}^{\eta} d\eta' d' \ln[H'(\Delta \eta + x)] \]
\[ = \frac{1}{x} (\partial_\eta - \partial_x)(\eta - x) \int_{\eta_i}^{\eta} d\eta' d' \frac{1}{\Delta \eta + x}, \] (89)
\[ = \frac{a_+}{x} (\partial_\eta - \partial_x)(\eta - x) \left\{ -\ln(a_-) - \ln(2Hx) + \ln(1 - \frac{1}{a_+}) \right\}, \] (90)
\[ = \frac{2a_+}{x} \left\{ -\ln(a_-) - \ln(2Hx) + \ln(1 - \frac{1}{a_+}) + \frac{1}{2} - \frac{1}{2aHx} \right\}. \] (91)
The second part of the d’Alembertian (85) acts on this second integrand to produce,
\[-\frac{1}{2x}(\partial_\eta - \partial_x)(\partial_\eta + \partial_x)(\eta + x) \int_{\eta}^{\eta-x} d\eta' d\eta'' \ln[H'(\Delta \eta + x)] \]
\[= -\frac{a_-}{x}(\partial_\eta + \partial_x)(\eta + x) \ln(2Hx) \]
\[= -\frac{2a_-}{x} \left\{ \ln(2Hx) + \frac{1}{2} - \frac{1}{2aHx} \right\} , \quad (92)\]

And the final term of the integrand in (84) is simple enough that the d’Alembertian can be kept together,
\[\partial^2 \int_{\eta}^{\eta-x} d\eta' d\eta'' = \partial^2 \left\{ \frac{\ln(a_-)}{H} \right\} = -\frac{2a_-}{x} . \quad (94)\]

Summing expressions (86), (88), (91), (93) and (94), and multiplying by the prefactor of (84) gives the $G$-type charge density,
\[J_{1G}^0(\eta, x) = \alpha q H^2 a^3 \frac{\ln(a_-) - \ln(2Hx) - \ln(1 - \frac{1}{a_-}) - \gamma - \frac{1}{2} + \frac{1}{2aHx}}{4\pi^2 x} \]
\[+ \frac{a_+}{x} \left\{ -\ln(a_-) - \ln(2Hx) + \ln(1 - \frac{1}{a_+}) + \frac{1}{2} - \frac{1}{2aHx} \right\} . \quad (95)\]

Expression (95) is complicated but its import becomes clear at late times, either at fixed $x$ or at fixed $r = x/a$,
\[\left( J_{1G}^0 \right)_{\eta \gg \eta_I \ x \ fixed} = \frac{\alpha q H a}{4\pi^2 x^2} \left\{ 2 \ln \left( \frac{1}{Hx} \right) - \ln \left( 1 - H^2 x^2 \right) + O(1) \right\} , \quad (96)\]
\[\left( J_{1G}^0 \right)_{\eta \gg \eta_I \ r = x/a} = \frac{\alpha q H^2 a^3}{4\pi^2 r} \left\{ 2 \ln(a) - \frac{2Hr \ln(1 + Hr)}{1 - H^2 r^2} - \frac{2 \ln(2Hr)}{1 - H^2 r^2} + O(1) \right\} . \quad (97)\]

In each case we see that the one loop charge density (which is minus $J_{1G}^0$) is opposite to $q$, implying screening. We also see that the effect from $J_{1G}^0$ is much stronger than that of $J_{1C}^0$ (powers of $a$ versus $\ln(a)$), which is itself stronger than that of $J_{1F}^0$.

The $G$-type contribution to the one loop electric field follows from substituting each of the ten terms of expression (95) in (82). The integrals are not
illuminating and have been consigned to expressions (150-159) of Appendix A. After considerable rearrangement the final result is,

\[
F^{1G}_{0i}(\eta, \vec{x}) = -\frac{\alpha}{\pi} \times F^{(0)}_{0i}(\eta, \vec{x}) \times \left\{ aHx \left[ (1-\gamma) \left[ 1 - \ln \left( \frac{1}{a} + Hx \right) \right] ight] 
+ 2 - \left( \frac{a-1}{aHx} \right) \ln \left( \frac{1 + \frac{Hx}{a-1}}{1 - \frac{Hx}{a-1}} \right) - \ln \left[ (1 - \frac{1}{a})^2 - H^2 x^2 \right] 
- \ln \left[ 1 + aHx \right] \left[ 2 \ln(a) - \frac{1}{2} \ln \left[ 1 + aHx \right] + 1 - \gamma - \ln(2) \right] - \frac{1}{2} \text{Li}_2 \left[ 1 - a^2 H^2 x^2 \right] 
+ \text{Li}_2 \left[ \frac{1}{2} \frac{aHx}{2} \right] - \frac{\pi^2}{12} + \frac{1}{2} \ln^2(2) + \text{Li}_2 \left[ \frac{1}{a} - Hx \right] - \text{Li}_2 \left[ \frac{1}{a} + Hx \right] \right\}. \tag{98}
\]

The symbol \( \text{Li}_2(z) \) denotes the dilogarithm function,

\[
\text{Li}_2(z) \equiv \int_0^z \frac{\ln(1-t)}{t} dt = \sum_{k=1}^\infty \frac{z^k}{k^2} = -\frac{1}{2} \ln^2(-z) - \frac{\pi^2}{6} - \sum_{k=1}^\infty \frac{1}{k^2 z^k}. \tag{99}
\]

Note that we have employed the identity,

\[
\text{Li}_2(-x) - \text{Li}_2(1-x) = - \ln(x) \ln(1+x) - \frac{\pi^2}{6} - \frac{1}{2} \text{Li}_2(1-x^2). \tag{100}
\]

Expression (98) is unwieldy. One can understand it better by taking the limit of late time at fixed \( x \),

\[
\left( F^{1G}_{0i} \right)_{\text{fixed}} \longrightarrow -\frac{\alpha}{\pi} \times \frac{q}{4\pi} \frac{x^i}{x^3} \times aHx \times F(Hx). \tag{101}
\]

The proportionality function is positive definite,

\[
F(Hx) = (1-\gamma) \left[ 1 - \ln(Hx) \right] + 2 - \frac{1}{Hx} \ln \left[ \frac{1+Hx}{1-Hx} \right] - \ln(1 - H^2 x^2). \tag{102}
\]

(Recall that \( \gamma \approx 0.57721 \) is Euler’s constant, so \( 1 - \gamma \approx 0.42278 \).) For small \( Hx \) it has the expansion,

\[
F(Hx) = (1-\gamma) \ln \left[ \frac{1}{Hx} \right] + (1-\gamma) + \frac{4}{3} (Hx)^2 + \frac{3}{10} (Hx)^4 + \ldots \tag{103}
\]

The largest \( Hx \) can get is one, at which point,

\[
F(1) = 3 - \gamma - 2 \ln(2) \approx 1.03649. \tag{104}
\]
Combining expressions (72), (75) and (101) gives the total one loop correction to the electric field at late times for fixed co-moving position $x$,

$$
\left(F_{0i}^{(1)}\right)_{x_{\text{fixed}}} \rightarrow -\frac{\alpha}{\pi} \times F_{0i}^{(0)} \left\{ \frac{1}{6} \left[ \ln(2\mu x) - 1 \right] + \frac{1}{6} \ln(a) + aHx \times F(Hx) \right\}. \tag{105}
$$

It will be seen that the $1G$ contribution totally dominates the $1C$ and $1F$ contributions. The one loop correction we have found is consistent with screening from the vacuum polarization induced by the vast ensemble of charged scalars produced during inflation. The presence of a scale factor means that the one loop correction cancels the tree order effect after only about $\ln(a) \sim \ln(\pi/\alpha) \approx 6$ e-foldings! Of course this does not mean fixed $x$ observers see the field of a negative charge after that time; what happens instead is that they see effectively zero field strength. Perturbation theory breaks down because screening has become nonperturbatively strong.

The result is curiously different at fixed $r$ in static coordinates. To see why, let us first evaluate the $1G$ contribution (98) at late times holding $r = aHx$ fixed,

$$
\left(F_{0i}^{1G}\right)_{r_{\text{fixed}}} \rightarrow -\frac{\alpha}{\pi} \times \frac{q}{4\pi} \frac{a^2 r^4}{r^3} \times \ln(a) \left[ -2 \ln(1+Hr) + (1-\gamma)Hr \right]. \tag{106}
$$

This is negative definite for all $0 < Hr < 1$, which is consistent with anti-screening. Of course that is what happens in flat space as well, when one gets very close to the source. Note that the factor of $\ln(a)$ cancels between the $1F$ and $1C$ contributions at fixed $r$,

$$
\frac{1}{6} \left[ \ln(2\mu x) - 1 \right] + \frac{1}{6} \ln(a) = \frac{1}{6} \left[ \ln(2\mu r) - 1 \right]. \tag{107}
$$

Hence the $1G$ contribution is still dominant.

To derive the electric field strength in static coordinates we must still transform the vector indices. Substituting (106) into expression (48) gives,

$$
\left(\bar{F}_{0i}^{(1)}\right)_{r_{\text{fixed}}} \rightarrow -\frac{\alpha}{\pi} \times \frac{q}{4\pi} \frac{r^4}{r^3} \times Ht \left[ -2 \ln(1+Hr) + (1-\gamma)Hr \right], \tag{108}
$$

$$
= \bar{F}_{0i}^{(0)} \times \frac{\alpha}{\pi} \times Ht \left[ 2 \ln(1+Hr) - (1-\gamma)Hr \right]. \tag{109}
$$

We see that the one loop correction at fixed static coordinate $r$ (109) has both the opposite sign and a much slower growth than the result in conformal coordinates (105). The physical interpretation seems to be that the
static coordinate observer experiences the logarithmic running of the electro-
magnetic coupling which is built into the Bunch-Davies vacuum, no matt er
what coordinate system one employs. The effect becomes nonpert urbatively
strong after about $Ht \sim \pi/\alpha \approx 430$ e-foldings.

4.3 Response to a point magnetic dipole

In conformal coordinates the current density and classical field strengths of
a point dipole are unchanged from flat space,

$$J^0(\eta, \vec{x}) = 0 \quad , \quad \vec{J}(\eta, \vec{x}) = -\vec{m} \times \vec{\nabla} \delta^3(\vec{x}) \quad , \quad (110)$$
$$F^{(0)}_{0i}(\eta, \vec{x}) = 0 \quad , \quad F^{(0)}_{ij}(\eta, \vec{x}) = \frac{\epsilon^{ijk}}{4\pi} \left[ m^k \nabla^2 - \vec{m} \cdot \vec{\nabla} \delta_k \right] \left\{ -\frac{1}{x} \right\} . \quad (111)$$

It follows that the $1F$ contributions to the one loop induced current density
and field strengths are the same as we found in section 2.3,

$$J^{1F}_0(\eta, \vec{x}) = 0 \quad , \quad \vec{J}^{1F}(\eta, \vec{x}) = -\frac{\alpha}{8\pi^2} \frac{\vec{m} \times \hat{x}}{x^4} \quad , \quad (112)$$
$$F^{1F}_{0i}(\eta, \vec{x}) = 0 \quad , \quad F^{1F}_{ij}(\eta, \vec{x}) = \frac{\alpha\epsilon^{ijk}}{24\pi^2} \left[ m^k \nabla^2 - \vec{m} \cdot \vec{\nabla} \partial_k \right] \left\{ -\frac{\ln(2\mu x)}{x} \right\} . \quad (113)$$

Note that we have neglected delta function terms in (112).

Substituting (110)-(111) in expression (62) gives the $1C$ contribution to the
one loop induced current density,

$$J^{1C}_0(\eta, \vec{x}) = 0 \quad , \quad J^{1C}_i(\eta, \vec{x}) = \frac{\alpha}{6\pi} \frac{\ln(a)}{x} \epsilon^{ijk} m^i \partial_k \delta^3(\vec{x}) . \quad (114)$$

Using these currents, with a few partial integrations, in relation (55) produces
the associated field strengths,

$$F^{1C}_{0i}(\eta, \vec{x}) = \frac{\alpha\epsilon^{ijk}}{24\pi^2} m^i \partial_j \partial_0 \left\{ \frac{\ln(Hx + \frac{1}{a})}{x} \right\} , \quad (115)$$
$$F^{1C}_{ij}(\eta, \vec{x}) = \frac{\alpha\epsilon^{ijk}}{24\pi^2} \left[ m^k \nabla^2 - \vec{m} \cdot \vec{\nabla} \delta_k \right] \left\{ \frac{\ln(Hx + \frac{1}{a})}{x} \right\} . \quad (116)$$

Recall that we are assuming $0 < Hx < 1 - 1/a$.

It is best to add the $G$ and $K$ currents together before inferring the field
strengths they induce, so we derive them together. From expression (65) we
see that \( J^G_0(\eta, \vec{x}) \) vanishes because the classical charge density is zero for a point magnetic dipole. Substituting the other classical values \( (110, 111) \) into expression \( (66) \) and performing some partial integrations reduces the induced \( G \)-type current density to a single integral over conformal time,

\[
J^G_i(\eta, \vec{x}) = \frac{\alpha H^2 a}{8\pi^2} \partial^2 \epsilon^{ijk} m^j \partial_k \int_{\eta_i}^{\eta} d\eta' a' \left\{ \ln[H^2(\Delta \eta^2 - \Delta x^2)] + 1 \right\} . \tag{117}
\]

Of course the temporal component of \( J^K(\eta, \vec{x}) \) always vanishes. From expression \( (64) \) and the classical relation \( \partial_j F_{ji}^{(0)}(\eta, \vec{x}) = -\epsilon^{ijk} m^j \partial_k \delta^3(\vec{x}) \) we infer,

\[
J^K_i(\eta, \vec{x}) = -\frac{\alpha H^4 a^2}{4\pi^2} \epsilon^{ijk} m^j \partial_k \int_{\eta}^{\eta-x} d\eta' a'^2 \left\{ \ln[H^2(\Delta \eta^2 - x^2)] + 2 \right\} . \tag{118}
\]

The next step is to perform the conformal time integrations in expressions \( (117) \) and \( (118) \). For the \( G \)-type current density comparison with \( (84) \) reveals that we can read off the result by making the replacement \( q \rightarrow \epsilon^{ijk} m^j \partial_k \) in \( (95) \),

\[
J^G_i(\eta, \vec{x}) = \frac{\alpha H^2 a}{4\pi^2} \epsilon^{ijk} m^j \partial_k \left\{ \frac{a_+}{x} \left[ \ln\left( \frac{a_-}{2Hx} \right) - \ln\left( 1 - \frac{1}{a_-} \right) - \frac{\gamma}{2} + \frac{1}{2aHx} \right] + \frac{a_-}{x} \left[ -\ln(a_-) - \ln(2Hx) + \ln\left( 1 - \frac{1}{a_+} \right) + \frac{1}{2} - \frac{1}{2aHx} \right] \right\} . \tag{119}
\]

For the \( K \)-type current density it is best to change variables from \( \eta' \) to \( a' = -1/H\eta' \) in expression \( (118) \),

\[
J^K_i(\eta, \vec{x}) = -\frac{\alpha H^3 a^2}{4\pi^2} \epsilon^{ijk} m^j \partial_k \int_1^{a} da' \left\{ \ln\left[ \frac{a'}{a} \right] + \ln\left[ \frac{1}{a'} - \frac{1}{a} \right] + 2\gamma \right\} , \tag{120}
\]

\[
= \frac{\alpha H^3 a^2}{4\pi^2} \epsilon^{ijk} m^j \partial_k \left\{ (a_+ - a_-) \ln(a_+ - a_-) - (a_- - 1) \ln(a_- - 1) - (a_+ - 1) \ln(a_+ - 1) - (a_- - 1) \ln(a_- - 1) - 2\gamma a_- \right\} . \tag{121}
\]

The final step before combining the \( G \)-type and \( K \)-type currents is to put them in a common form by isolating distinct logarithms and making judicious use of the identities,

\[
2a_+ a_- = a(a_+ + a_-) = \frac{(a_+ - a_-)}{Hx} \quad , \quad aHxa_\pm = \pm a_\pm \mp a . \tag{122}
\]
The results are,

\[
J^i_{\ell} (\eta, \vec{x}) = \frac{\alpha H^2}{4\pi^2} \varepsilon^{ijk} m^j \partial_k \left\{ \frac{a}{x} \left[ - (a_+ + a_-) \ln(2Hx) - (a_+ - a_-) \ln(a_-) \right. \right.
\]
\[
\left. \left. - a_- \ln\left(1 - \frac{1}{a_-}\right) + a_+ \ln\left(1 - \frac{1}{a_+}\right) - (\gamma + 1)a_- \right] \right\}, \quad \text{(123)}
\]

\[
J^{\ell K} (\eta, \vec{x}) = \frac{\alpha H^2}{4\pi^2} \varepsilon^{ijk} m^j \partial_k \left\{ \frac{a}{x} \left[ (a_+ + a_- - 2a) \ln(2Hx) \right.ight.
\]
\[
\left. \left. + (a_+ + a_- + 2aHx - 2a) \ln(a_-) + (a_+ + aHx - a) \ln\left(1 - \frac{1}{a_-}\right) \right. \right.
\]
\[
\left. \left. - (a_- - aHx - a) \ln\left(1 - \frac{1}{a_+}\right) + 2\gamma(a_- - a) \right] \right\}. \quad \text{(124)}
\]

Adding expressions (123) and (124) gives a current density which is much simpler than either of them separately,

\[
J^{\ell K} (\eta, \vec{x}) = \frac{\alpha H^2}{4\pi^2} \varepsilon^{ijk} m^j \partial_k \left\{ \frac{a}{x} \left[ -2a \ln(2Hx) + 2(a_- + aHx - a) \ln(a_-) \right. \right.
\]
\[
\left. \left. \left. - a(1 - Hx) \ln\left(1 - \frac{1}{a_-}\right) + a(1 + Hx) \ln\left(1 - \frac{1}{a_+}\right) - a_- + \gamma(a_- - 2a) \right] \right\}. \quad \text{(125)}
\]

Finally, it is useful to introduce the symbol \( J^{\ell K} (\eta, x) \) for \( H^2 \) times the part of (125) within the curly brackets,

\[
J^{\ell K}_i (\eta, \vec{x}) \equiv \frac{\alpha}{4\pi^2} \varepsilon^{ijk} m^j \partial_k J^{\ell K} (\eta, x). \quad \text{(126)}
\]

From (53) we see that the one loop field strength induced by (125) obeys the equation,

\[
\partial^2 F^{\ell K}_{\mu\nu} (\eta, \vec{x}) = \partial_\mu J^{\ell K}_{\nu} (\eta, \vec{x}) - \partial_\nu J^{\ell K}_\mu (\eta, \vec{x}). \quad \text{(127)}
\]

The retarded solution (55) takes the form,

\[
F^{\ell K}_{\mu\nu} (\eta, \vec{x}) = \partial_\mu J^{\ell K}_{\nu} (\eta, \vec{x}) - \partial_\nu J^{\ell K}_\mu (\eta, \vec{x}), \quad \text{(128)}
\]

where,

\[
J^{\ell K}_\mu (\eta, \vec{x}) = - \int d^4 \vec{x}' \frac{\delta (\Delta \eta - \Delta x)}{4\pi \Delta x} J^{\ell K}_\mu (\eta', \vec{x}'). \quad \text{(129)}
\]
We can commute the differential operator in (126) through the Greens function to define a scalar function $J_{x, \bar{x}}^{GK}(\eta, x)$,

\begin{equation}
J_{x, \bar{x}}^{GK}(\eta, x) = \frac{\alpha}{4\pi^2} \epsilon_{ijk} \frac{m^i}{m^j} \partial_k J_{x, \bar{x}}^{GK}(\eta, x), \tag{130}
\end{equation}

\begin{equation}
\equiv -\frac{\alpha}{4\pi^2} \epsilon_{ijk} \frac{m^i}{m^j} \int d^4x' \frac{\delta(\Delta\eta - \Delta x)}{4\pi \Delta x} J^{GK}(\eta', x'). \tag{131}
\end{equation}

In terms of $J^{GK}(\eta, x)$ the electric and magnetic field strengths are,

\begin{equation}
F_{0i}^{GK}(\eta, \bar{x}) = \frac{\alpha \epsilon_{ijk}}{4\pi^2} \frac{m^j}{m^k} \partial_0 J_{x, \bar{x}}^{GK}(\eta, x), \tag{132}
\end{equation}

\begin{equation}
F_{ij}^{GK}(\eta, \bar{x}) = \frac{\alpha \epsilon_{ijk}}{4\pi^2} \left[ m^k \nabla^2 - \bar{m} \cdot \vec{\nabla} \partial^k \right] J_{x, \bar{x}}^{GK}(\eta, x). \tag{133}
\end{equation}

It remains to evaluate the function $J^{GK}(\eta, x)$. Note that for a function $f(\eta, x)$ of just $\eta$ and $x \equiv \|\bar{x}\|$, the integral against the retarded Green’s function gives,

\begin{equation}
-\int d^4x' \frac{\delta(\Delta\eta - \Delta x)}{4\pi \Delta x} \times f(\eta', x') = -\frac{1}{2x} \int_{\eta}^{\eta + \Delta\eta} \int_{|x' - \Delta\eta|}^{x + \Delta\eta} dx' f(\eta', x'). \tag{134}
\end{equation}

In our case the function $f(\eta, x)$ is $J^{GK}(\eta, x)$ and we must remember that causality makes it vanish for $x > \eta - \eta_I$. We can therefore express $J^{GK}(\eta, x)$ as the sum of three integrals,

\begin{equation}
J^{GK}(\eta, x) = -\frac{1}{2x} \left\{ \int_{\eta - x}^{\eta} \int_{\eta - x}^{\eta - \eta_I} dx' \right. \left. + \int_{\eta - x}^{\eta} \int_{\eta - \eta_I}^{\eta - \eta_I} dx' \right. \left. + \int_{\eta}^{\eta} \int_{\eta - \eta_I}^{\eta - \eta_I} dx' \right. \left. + \int_{\eta + \eta_I}^{\eta + \eta_I} \int_{x - \eta_I}^{x + \eta - \eta_I} dx' \right. \left. \right\} x' J^{GK}(\eta', x'). \tag{135}
\end{equation}

These integrals are evaluated in Appendix B. Here we present just the leading results for large $a$ with $x$ fixed, and with $r = x/a$ fixed,

\begin{equation}
J_{x, \bar{x}}^{GK} \rightarrow \frac{\ln(a)}{x} \times \mathcal{G}(Hx), \quad J_{r, \bar{r}}^{GK} \rightarrow \frac{\ln(a)}{x} \times \mathcal{K}(Hr), \tag{136}
\end{equation}

where the functions $\mathcal{G}(z)$ and $\mathcal{K}(z)$ are,

\begin{equation}
\mathcal{G}(z) = 2\gamma + 2\ln(2) - \ln \left( \frac{1 + z}{1 - z} \right) - z \ln(1 - z^2), \tag{137}
\end{equation}

\begin{equation}
\mathcal{K}(z) = \frac{\pi^2}{12} + 2\ln(1 + z) + \text{Li}_2 \left( \frac{z - 1}{z + 1} \right). \tag{138}
\end{equation}
The function $G(z)$ is positive at $z = 0$ so the $GK$ terms dominate over the $F$ and $C$ terms at late times for fixed $x$,

$$
(F^{(1)}_{0i})_{\text{fixed } x} \to \frac{\alpha e^{ijk}}{4\pi^2} m^j \partial_k \partial_0 \left\{ \frac{\ln(a)}{x} \times G(Hx) \right\},
$$

(139)

$$
(F^{(1)}_{ij})_{\text{fixed } x} \to \frac{\alpha e^{ijk}}{4\pi^2} \left[ m^k \nabla^2 - \bar{m} \cdot \nabla \partial_k \right] \left\{ \frac{\ln(a)}{x} \times G(Hx) \right\}.
$$

(140)

Near $x = 0$ the term in curly brackets is the positive constant $2\gamma + 2\ln(2)$ times $\ln(a)/x$. This screens the classical result of $-1/x$, although only logarithmically.

For small $z$ the function $K(z)$ has the expansion,

$$
K(z) = 2 \ln(2) z - 2z^2 + O(z^3).
$$

(141)

The linear term drops out from the one loop field strengths, so the $GK$ terms are negligible compared with the $F$ and $C$ terms for large $a$ at fixed $r$. The resulting field strengths are,

$$
(F^{(1)}_{0i})_{\text{fixed } r} \to \frac{\alpha e^{ijk}}{24\pi^2} m^j \partial_k \partial_0 \left\{ \frac{-\ln(2\mu x) + \ln(Hx + \frac{1}{2})}{x} \right\},
$$

(142)

$$
= \frac{\alpha e^{ijk}}{24\pi^2} m^j \hat{r}^k \times H a^3 \times \left\{ \frac{1}{r^2} - \frac{H^2}{(1 + H r)^2} \right\},
$$

(143)

$$
(F^{(1)}_{ij})_{\text{fixed } r} \to \frac{\alpha e^{ijk}}{24\pi^2} \left[ m^k \nabla^2 - \bar{m} \cdot \nabla \partial_k \right] \left\{ \frac{-\ln(2\mu x) + \ln(Hx + \frac{1}{2})}{x} \right\},
$$

(144)

$$
= \frac{\alpha e^{ijk}}{24\pi^2} \times a^3 \times \left\{ \frac{1}{2\mu r} \left[ \frac{m^k - 3\bar{m} \cdot \hat{r} \hat{r}^k}{r^3} \right] + \frac{2}{r^3} \left[ m^k - 2\bar{m} \cdot \hat{r} \hat{r}^k \right] - \frac{H}{r^2} \left[ \frac{m^k - \bar{m} \cdot \hat{r} \hat{r}^k}{1 + H r} \right] - \frac{H^2}{r (1 + H r)^2} \right\}.
$$

(145)

We must employ expressions (48) and (49) to convert (143) and (145) into the static field strengths. The final result for the (late time form of the) static electric field is relatively simple,

$$
\tilde{F}_{0i}^{(1)} \to \frac{\alpha e^{ijk}}{24\pi^2} m^j r^k \times He^{H t} \sqrt{1 - H^2 r^2} \left\{ - \frac{\ln(1 + H r)}{2\mu r} + \frac{1}{r^2} + \frac{H}{r (1 + H r)} \right\}.
$$

(146)

The factor of $e^{H t}$ makes this seem like a substantial modification but one should keep in mind that the tree order result carries the same factor in
static coordinates,
\[
\tilde{F}^{(0)}_{\mu\nu} = \frac{\epsilon_{ijk}}{4\pi} m^i \tilde{r}^j \times -\frac{He^{Ht} \sqrt{1-H^2 r^2}}{r^2}.
\]  \hfill (147)

It is therefore better to view the one loop correction as a time independent
screening of the classical result by the \( r \)-dependent enhancement (for small \( r \)) of about \( \alpha/6\pi \times \ln(1/2\mu r) \). This is exactly what one finds in flat space \(^{29}\) and is a manifestation of the running of the electrodynamic coupling.

The tensor structure of the one loop static magnetic field is complicated but
its leading term for small \( r \) bears the same relation to the tree order result,
\[
\tilde{F}^{(0)}_{ij} = \frac{\epsilon_{ijk}}{4\pi} e^{Ht} \sqrt{1-H^2 r^2} \left\{ \frac{m^k-3\vec{m}\cdot\vec{r}^k}{r^3} \right\}, \hfill (148)
\]
\[
\tilde{F}^{(1)}_{ij} \rightarrow \frac{\alpha \epsilon_{ijk}}{24\pi^2} e^{Ht} \sqrt{1-H^2 r^2} \left\{ \ln \left( \frac{1+H r}{2\mu r} \right) \left[ \frac{m^k-3\vec{m}\cdot\vec{r}^k}{r^3} \right] + \ldots \right\}. \hfill (149)
\]

5 Discussion

In this study we have explored the effect that the inflationary production
of charged, massless and minimally coupled scalars have on the electric and
magnetic fields induced by a point charge and by a point magnetic dipole.
The sources were held fixed at the spatial origin. We derived results for the
fields perceived by two observers, one stationary in conformal coordinates
and the other stationary in static coordinates.

The conformal observer sees the one loop correction \(^{105}\) to the elec-
tric field of a point charge screen the classical result exponentially rapidly in
physical time. That is to be expected because the physical distance between
this observer and the source grows exponentially, and the intervening space
is filled with a nearly constant density of charged particles. In contrast, the
static observer perceives the one loop correction \(^{109}\) to slowly enhance the
classical result. It is more difficult to understand this result. A possible
interpretation is that it derives from the running of the electrodynamic cou-
ing which is built into the scalar vacuum. However, it should be noted
that the enhancement does not come from merely probing the one loop cor-
rection in flat space \(^{19}\) at the ever-smaller coordinate separations needed
to keep the physical distance to the source constant. That effect is entirely
canceled by the conformal anomaly term \(^{175}\); the enhancement originates
in the intrinsically de Sitter contribution \(^{98}\).
Our results for a current dipole are interestingly different. The con-
formal observer sees much weaker screening (140) which grows only linearly in
physical time. The static observer sees an enhancement (149) which is con-
stant in time but grows as one approaches the source. Because the static
result derives from the flat space and conformal anomaly terms it is the most
simply understood in terms of the flat space result (29), and is clearly a
manifestation of the running of the electrodynamic coupling.

As far as we know this work is unique in having solved the effective field
equations for the constrained part of a force field on de Sitter background.
There have been many previous studies of the propagation of dynamical
quanta on de Sitter, both from the production of scalars [25, 13, 14, 26, 27]
and from the production of gravitons [28, 29, 30]. There have also been
many flat space background studies of how massless scalars affect the force
of gravity [31, 19], and of how gravitons affect electric and magnetic forces
[32, 18]. But no one has previously combined de Sitter and force laws to
reveal the surprising dichotomy between screening in conformal coordinates
to anti-screening in static coordinates.

Although this seems to be the first time one loop corrections to force
laws have been studied on de Sitter, it will not be the last. Computations
have already been made of the one loop graviton self-energy from inflationary
gravitons [33] and from inflationary scalar [34], and the vacuum polarization
from inflationary gravitons will soon be completed [35]. These results can
be used to quantum-correct the effective Maxwell and Einstein equations,
and then proceed to a study of the same type we have just completed. We
anticipate that the current work will play an important role in organizing
and understanding these studies. The project to quantum-correct Maxwell’s
equations to account for inflationary gravitons is especially noteworthy be-
cause it is a straightforward approach to check the puzzling screening effect
that Kitamoto and Kitazawa have inferred from graviton loop corrections to
the off-shell effective field equations at fixed sub-horizon scales [36].

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comments.
6 Appendix A: Integrals from Subsection 4.2

The following integrals (times $-\alpha/\pi \times F_{0i}^{(0)}$) sum to give our result (98) for $F_{0i}^{(G)}$ in section 4.2,

\[
aH^2 \int_0^x \! dx' \, a'_- \ln(a'_-) = \frac{1}{2} \ln^2(a_-) - \frac{1}{2} \ln^2(a_-) - \frac{a}{a_-} \ln(a_-) - \ln(a_-) + aHx , \quad (150)
\]

\[
-aH^2 \int_0^x \! dx' \, a'_- \ln(2Hx') = \ln(1+aHx) \ln(2Hx) + \text{Li}_2(-aHx) ,
\]

\[
-aH^2 \int_0^x \! dx' \, a'_- \ln(1 - \frac{1}{a'_-}) = (a - \frac{a}{a_-}) \ln(1 - \frac{1}{a_-}) - (a - 1) \ln(1 - \frac{1}{a_-}) + \text{Li}_2\left(\frac{1}{a}\right) - \text{Li}_2\left(\frac{1}{a_-}\right) + aHx , \quad (152)
\]

\[
-(\gamma + \frac{1}{2})aH^2 \int_0^x \! dx' \, a'_- = \left(\gamma + \frac{1}{2}\right)\left[\ln(a) - \frac{a}{a_-} \ln(a_-) - aHx\right] , \quad (153)
\]

\[
aH^2 \int_0^x \! dx' \, \frac{a'_-}{2aH} = \frac{1}{2} \ln(a) - \frac{1}{2} \ln(a_-) , \quad (154)
\]

\[
-aH^2 \int_0^x \! dx' \, a'_+ \ln(a'_+) = \ln\left(\frac{a}{2}\right) \ln(1-aHx) + \frac{a}{a_-} \ln(a_-) - \ln(a) + aHx ,
\]

\[
-\text{Li}_2\left(\frac{1}{2}\right) + \text{Li}_2\left(\frac{1}{2} - \frac{1}{2}aHx\right) , \quad (155)
\]

\[
-aH^2 \int_0^x \! dx' \, a'_+ \ln(2Hx') = - \ln\left(\frac{a}{2}\right) \ln(1-aHx) + aHx \ln(2Hx) - aHx + \text{Li}_2(1) - \text{Li}_2(1-aHx) , \quad (156)
\]

\[
aH^2 \int_0^x \! dx' \, a'_+ \ln\left(1 - \frac{1}{a'_+}\right) = -(a - \frac{a}{a_+}) \ln\left(1 - \frac{1}{a_+}\right) + (a - 1) \ln\left(1 - \frac{1}{a}\right)
\]

\[
-\text{Li}_2\left(\frac{1}{a}\right) + \text{Li}_2\left(\frac{1}{a_+}\right) + aHx , \quad (157)
\]

\[
\frac{1}{2}aH^2 \int_0^x \! dx' \, a'_+ = - \frac{1}{2} \ln(1-aHx) - \frac{1}{2}aHx , \quad (158)
\]

\[
-aH^2 \int_0^x \! dx' \, \frac{a'_+}{2aH} = \frac{1}{2} \ln(1-aHx) . \quad (159)
\]

The integrands of (155-156) and (158-159) diverge at $x' = -\eta$, which results in the ill-defined factors of $\pm \ln(1-aHx)$. However, these divergences cancel between (155-156) and between (158-159), as do the problematic logarithms.
Appendix B: Integrals from Subsection 4.3

We begin with some general comments. The expression we must evaluate is a double integration over three regions which can be termed “I”, “II” and “III”,

$$J^{GK}(\eta, x) = -\frac{1}{2x}\left[\int d\eta'dx' + \int d\eta'dx' + \int d\eta'dx'\right] x' J^{GK}(\eta', x') .$$

Each of these double integrals takes the form,

$$\int_A^B d\eta' \int_C^{D(\eta')} dx' ,$$

where the various limits are,

$$A_I = \eta - x , \quad A_{II} = \frac{1}{2}(\eta + \eta + x) , \quad A_{III} = \eta ,$$

$$B_I = \frac{1}{2}(\eta + \eta - x) , \quad B_{II} = A_I , \quad B_{III} = A_{II} ,$$

$$C_I = \eta' - \eta , \quad C_{II} = C_I , \quad C_{III} = \eta + x - \eta' ,$$

$$D_I = \eta - x - \eta' , \quad D_{II} = -D_I , \quad D_{III} = -D_I .$$

Of course the integrand is of crucial importance. From expressions (125) and (126) we see that $x J^{GK}(\eta, x)$ can be broken up into seven components,

$$I_1(\eta, x) = -2H^2 a^2 \ln(2Hx) ,$$

$$I_2(\eta, x) = +2H^2 a a_- \ln(a_-) ,$$

$$I_{34}(\eta, x) = +2H^2 a^2 (1-Hx) \ln\left[1 + H(\eta - x)\right] ,$$

$$I_{56}(\eta, x) = -a^2 (1-Hx) \ln\left[1 + H(\eta - x)\right] ,$$

$$I_{78}(\eta, x) = +a^2 (1+Hx) \ln\left[1 + H(\eta + x)\right] ,$$

$$I_9(\eta, x) = (\gamma - 1)a a_- ,$$

$$I_{10}(\eta, x) = -2a^2 .$$

In each case the $x'$ integration can be expressed in terms of elementary functions. For $I_{10}$ the $\eta'$ integration also results in elementary functions. $I_2$ and $I_9$ give polylogarithms, about which more later. The remaining integrands — $I_1, I_{34}, I_{56}$ and $I_{78}$ — all take the form of $a^2$ times the $x'$ derivative of
an elementary function of $\eta'$ and $x'$. For these integrands the best strategy is partially integrate on $\eta'$ after having performed the $x'$ integration,

$$H\int_B^A d\eta' a^2 \int_{D(\eta')}^{C(\eta')} \frac{\partial F(\eta', x')}{\partial x'} = H\int_B^A d\eta' a^2 \left\{ F(\eta', C(\eta')) - F(\eta', D(\eta')) \right\} . (173)$$

$$= a' \left\{ F(\eta', C) - F(\eta', D) \right\} \bigg|_B^A - \int_B^A d\eta' a' \frac{\partial}{\partial \eta'} \left\{ F(\eta', C) - F(\eta', D) \right\} . (174)$$

It turns out that the surface terms in (174) cancel between the three regions,

$$a' \left\{ F(\eta', C_I) - F(\eta', D_I) \right\} \bigg|_{B_I}^{A_I} = a_{A_I} \left\{ F(A_I, \eta - \eta_I) - F(A_I, 0) \right\} - 0 , (175)$$

$$a' \left\{ F(\eta', C_{II}) - F(\eta', D_{II}) \right\} \bigg|_{B_{II}}^{A_{II}} = a_{A_{II}} \left\{ F(A_{II}, \frac{1}{2}(\eta + x - \eta_I)) - F(A_{II}, \frac{1}{2}(\eta_I - \eta + 3x)) \right\} - a_{A_I} \left\{ F(A_I, \eta - \eta_I) - F(A_I, 0) \right\} , (176)$$

$$a' \left\{ F(\eta', C_{III}) - F(\eta', D_{III}) \right\} \bigg|_{B_{III}}^{A_{III}} = 0 - a_{A_{III}} \left\{ F(A_{III}, \frac{1}{2}(\eta + x - \eta_I)) - F(A_{III}, \frac{1}{2}(\eta_I - \eta + 3x)) \right\} . (177)$$

Integrands $I_2$ and $I_9$, and the volume terms (174) from $I_1$, $I_{34}$, $I_{56}$ and $I_{78}$, all give rise to $\eta'$ integrations of the form $1/\eta'$ times logarithms. Integrations of this form result in polylogarithms, of which the two we require are,

$$\text{Li}_2(z) \equiv - \int_0^z dt \frac{\ln(1-t)}{t} , \quad \text{Li}_3(z) \equiv \int_0^z dt \frac{\text{Li}_2(t)}{t} . (179)$$

Both are real for $z \leq 1$. Their expansions for small $z$ and for large $-z$ are,

$$\text{Li}_2(z) \to z + O(z^2) , \quad \text{Li}_2(z) \to -\frac{1}{2} \ln^2(-z) - \frac{\pi^2}{6} + O\left(\frac{1}{z}\right) , (180)$$

$$\text{Li}_3(z) \to z + O(z^2) , \quad \text{Li}_3(z) \to -\frac{1}{6} \ln^3(-z) - \frac{\pi^2}{6} \ln(-z) + O\left(\frac{1}{z}\right) . (181)$$

Care must be taken to arrange things so that the arguments of $\text{Li}_2(z)$ and $\text{Li}_3(z)$ lie in the range $z \leq 1$ for which the function is real. This is simple to
accomplish for the dilogarithm $\text{Li}_2(z)$ but it requires some effort for $\text{Li}_3(z)$. $\text{Li}_3(z)$ derives exclusively from $\mathcal{I}_2$, when changing to dimensionless variables converts the $\eta'$ integration to either $\int dt/t \times \ln(1-t)$ or $\int dt/t \times \ln(t) \ln(1-t)$. The final integral only makes sense for $0 \leq t \leq 1$ so it involves no choices,

$$\int dt \frac{\ln(t) \ln(1-t)}{t} = \text{Li}_3(t) - \ln(t)\text{Li}_2(t). \quad (182)$$

If the dimensionless parameter $t$ lies in the range $0 \leq t \leq 1$ we write the first integral as,

$$\int dt \frac{\ln^2(1-t)}{t} = -2\text{Li}_3(1-t) + 2 \ln(1-t)\text{Li}_2(1-t) + \ln^2(1-t)\ln(t). \quad (183)$$

However, if $t < 0$ we must use Landen’s identity to re-express it as,

$$\int dt \frac{\ln^2(1-t)}{t} = -2\text{Li}_3(t) - 2\text{Li}_3\left(\frac{-t}{1-t}\right) - 4\text{Li}_3\left(\frac{1}{1-t}\right) + 2 \ln\left(\frac{1}{1-t}\right)\text{Li}_2\left(\frac{-t}{1-t}\right)$$

$$+ 4 \ln\left(\frac{1}{1-t}\right)\text{Li}_2\left(\frac{1}{1-t}\right) + 2 \ln^2\left(\frac{1}{1-t}\right)\ln\left(\frac{-t}{1-t}\right). \quad (184)$$

It can happen that the appropriate choice to make between (183) and (184) depends upon whether $x$ or $r = ax$ is held fixed for large $a$, in which case we report the choice appropriate for holding $x$ fixed at large $a$.

Our results for the seven integrals are,

$$-2x J_{GK}^1 = 2\text{Li}_2\left(1-aHx\right) - 2\text{Li}_2\left(\frac{1}{1+aHx}\right) + 2\text{Li}_2\left[\frac{1}{2}\left(1 + \frac{1}{a} - Hx\right)\right]$$

$$- 2\text{Li}_2\left[\frac{1}{2}\left(1 + \frac{1}{a} + Hx\right)\right] + 2\text{Li}_2\left[2\left(1 + aHx\right)\right] - 2\text{Li}_2\left[2\left(\frac{1}{a+1} + aHx\right)\right]$$

$$+ \ln^2\left(\frac{a}{2}\right) - \ln\left(1 + aHx\right)\ln\left[2\left(\frac{1}{a} + Hx\right)\right] + \ln^2\left[2\left(1 + \frac{1}{a} + Hx\right)\right]$$

$$- \ln^2\left[2\left(1 + \frac{1}{a} - Hx\right)\right], \quad (185)$$

$$-2x J_{GK}^2 = 2\text{Li}_3\left(-\frac{a}{aHx + 1}\right) - 2\text{Li}_3\left(\frac{a}{aHx - 1}\right) + 2\text{Li}_3\left(\frac{aHx}{aHx - 1}\right) - 2\text{Li}_3\left(-1\right)$$

$$+ 2 \ln\left(\frac{1}{a} + Hx\right)\left[\text{Li}_2(-1) - \text{Li}_2\left(\frac{aHx}{aHx - 1}\right)\right] + \ln^2\left(\frac{1}{a} + Hx\right)\ln\left(1 + aHx\right), \quad (186)$$

$$-2x J_{GK}^3 = \left[4 + \frac{2}{a} + 2Hx\right] \left\{-\text{Li}_2\left(\frac{1-aHx}{2}\right) + \text{Li}_2\left(\frac{1}{2}\right) - \text{Li}_2\left(\frac{1+aHx}{a+1+aHx}\right)\right\}$$
\[ \frac{1}{2} \ln^2\left(\frac{1+aHx}{2a}\right) - 2 \ln\left(\frac{1}{a} + Hx\right) - 2 \left(1 - \frac{1}{a} - Hx\right) \ln\left(\frac{1+aHx}{a+1-aHx}\right) - 2 \ln\left(\frac{1+aHx}{a+1-aHx}\right) - 2 \frac{1}{2} \ln^2\left(\frac{1}{a} - Hx\right) \]

- \[ -2x J_5^{GK} = \left[1 + \frac{1}{a} + Hx\right] \left\{ \text{Li}_2\left(\frac{2+2aHx}{a1+aHx}\right) - \text{Li}_2(1) + \ln\frac{a+1+aHx}{2+2aHx} \right\} \]

+ \ln\left(1 + \frac{1}{a} + Hx\right) \ln\left(\frac{a+1+aHx}{2+2aHx}\right) - \left[1 + \frac{1}{a} - Hx\right] \left\{ \text{Li}_2\left(\frac{2}{a1-aHx}\right) - \text{Li}_2\left(\frac{2+2aHx}{a1-aHx}\right) \right\} \]

- \ln\left(\frac{a+1+aHx}{2+2aHx}\right) + \frac{2}{a} \ln\left(\frac{a+1-aHx}{2}\right) - 2 \frac{1}{2} \ln\left(1 - \frac{1}{a} - Hx\right) \ln\left(\frac{a+1-aHx}{2+2aHx}\right) \]

\[ -2x J_7^{GK} = \left(1 - \gamma\right) \left\{ \text{Li}_2\left(\frac{1-aHx}{2}\right) - \text{Li}_2\left(\frac{1}{2}\right) + \text{Li}_2\left(\frac{1+aHx}{a+1+aHx}\right) \right\} \]

- \left(1 - \gamma\right) \left\{ \text{Li}_2\left(\frac{1-aHx}{2}\right) + \ln\left(1 + aHx\right) \right\} + \frac{1}{2} \ln^2\left(1 + aHx\right) \]

\[ -2x J_9^{GK} = 4\gamma \left\{ - \ln\left(1 + aHx\right) + \ln\left(\frac{a+1+aHx}{a+1-aHx}\right) \right\} \]

It remains just to give the expansions for large \( a \) at fixed \( x \), and at fixed \( r = ax \). In both cases the leading term goes like \( \ln(a) \). For fixed \( x \) we find,

\[ -2x J_1^{GK} = -4\ln(2Hx) \ln(a) + O(1) \]
\[ -2x \mathcal{J}^{GK}_2 = 0 \times \ln(a) + O(1) , \]
\[ -2x \mathcal{J}^{GK}_{34} = 4 \ln(Hx) \ln(a) + O(1) , \]
\[ -2x \mathcal{J}^{GK}_{56} = [2Hx - 2(1-Hx) \ln(1-Hx)] \ln(a) + O(1) , \]
\[ -2x \mathcal{J}^{GK}_{78} = [2Hx + 2(1+Hx) \ln(1+Hx)] \ln(a) + O(1) , \]
\[ -2x \mathcal{J}^{GK}_9 = 0 \times \ln(a) + O(1) , \]
\[ -2x \mathcal{J}^{GK}_{10} = -4\gamma \ln(a) + O(1) . \]

At fixed \( r \) the results are,
\[ -2x \mathcal{J}^{GK}_1 = 0 \times \ln(a) + O(1) , \]
\[ -2x \mathcal{J}^{GK}_2 = -\left[ \frac{\pi^2}{6} + 2 \text{Li}_2 \left( \frac{Hr-1}{Hr+1} \right) \right] \ln(a) + O(1) , \]
\[ -2x \mathcal{J}^{GK}_{34} = -4 \ln(1+Hr) \ln(a) + O(1) , \]
\[ -2x \mathcal{J}^{GK}_{56} = 0 \times \ln(a) + O(1) , \]
\[ -2x \mathcal{J}^{GK}_{78} = 0 \times \ln(a) + O(1) , \]
\[ -2x \mathcal{J}^{GK}_9 = 0 \times \ln(a) + O(1) , \]
\[ -2x \mathcal{J}^{GK}_{10} = 0 \times \ln(a) + O(1) . \]

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