4D, $\mathcal{N} = 1$ supersymmetry genomics (II)

S. James Gates Jr.,$^a$ Jared Hallett,$^b$ James Parker,$^a$ Vincent G.J. Rodgers$^c$ and Kory Stiffler$^a$

$^a$Center for String and Particle Theory, Department of Physics, University of Maryland, College Park, MD 20742-4111 U.S.A.
$^b$Department of Mathematics, Williams College, Williamstown, MA 01267 U.S.A.
$^c$Department of Physics and Astronomy, University of Iowa, Iowa City, IA 52242 U.S.A.

E-mail: gatess@wam.umd.edu, jdh4@williams.edu, jp@jamesparker.me, vrodgers@newton.physics.uiowa.edu, kstiffle@gmail.com

ABSTRACT: We continue the development of a theory of off-shell supersymmetric representations analogous to that of compact Lie algebras such as SU(3). For off-shell 4D, $\mathcal{N} = 1$ systems, quark-like representations have been identified [1] in terms of cis-Adinkras and trans-Adinkras and it has been conjectured that arbitrary representations are composites of $n_c$-cis and $n_t$-trans representations. Analyzing the real scalar and complex linear superfield multiplets, these “chemical enantiomer” numbers are found to be $n_c = n_t = 1$ and $n_c = 1, n_t = 2$, respectively.

KEYWORDS: Extended Supersymmetry, Superspaces

ArXiv ePrint: 1112.2147
1 Introduction

This paper is part of continuing efforts to create a comprehensive representation theory of off-shell supersymmetry (SUSY), a project we refer to as supersymmetric ‘genomics’ [1].
From our perspective, building the SUSY representation theory is an important undertaking. Representation theory for compact Lie algebras gives us a clear mathematical framework with which to separate matter from force carriers in the standard model: matter transforms in the fundamental representation of the gauge group, force carriers in the adjoint. Perhaps an analogous relationship exists among yet to be discovered SUSY particles. If so, an analogous representation theory of graded Lie algebras and their associated SUSY systems would prove quite useful. In addition, SUSY representation theory would be useful in building gauge/gravity correspondences and finding relationships amongst them.

There is also the matter of the most general four dimensional superstring or heterotic string theory. Although ‘compactification’ of higher dimensional superstring or heterotic string theories have long been pursued, there is no ‘science’ to the notion of compactification. One piece of evidence for this is the fact that episodically ‘new’ methods of compactification appear in the physics literature. The first widely studied method along these lines was ‘Calabi-Yau compactification’. But more recently newer methods called ‘G-2 structures’ and ‘non-geometrical compactifications’ have appeared. We believe it is a fair question to ask, “Is there a way to categorize and know all possible methods of compactification?” Whatever the future brings along these lines, at their core lies four dimensional \( \mathcal{N} = 1 \) supersymmetric models and their representation theory. In short, any theory which incorporates supersymmetry could benefit from the existence of a SUSY representation theory.

Typically, supersymmetric theories are better understood on-shell than they are off-shell. Today, one case of popular interest is in superstring theory: the AdS/CFT correspondence. The low energy effective field theory for Superstring/M-Theory is well known to be an on-shell representation of either a 10 or 11 dimensional supergravity \([2, 3]\). In the case of the AdS/CFT correspondence, an application manifests itself in the duality between the type IIB supergravity theory and another theory which is best understood on-shell: that of 4D, \( \mathcal{N} = 4 \) Super Yang-Mills theory \([4, 5]\).

The issue of on-shell SUSY being better understood than off-shell SUSY has been long standing. Since 1981 \([6]\) it has remained a question as to whether or not an infinite number of auxiliary fields are needed to close the 4D, \( \mathcal{N} = 4 \) Super Yang-Mills off-shell algebra,

\[
\{ Q^I_{\alpha}, Q^J_{\beta} \} = 2\delta^{IJ} P_{\alpha\beta},
\]

or if indeed this can be accomplished with a finite set. To resolve this issue it has been quite evident for some time that new tools are needed. Recently, an in depth investigation of the central charges and internal symmetries of the 4D, \( \mathcal{N} = 4 \) Super Yang-Mills algebra was undertaken \([7]\). In 1995, evidence was shown in \([8]\) that the minimal size of the off-shell representation with vanishing central charges for 4D, \( \mathcal{N} = 4 \) supersymmetry is 128 bosonic and 128 fermionic degrees of freedom. This is related to evidence that the representation theories of all superalgebras in any dimension are fully encoded in the representation theory of a corresponding one dimensional theory. It was, of course, satisfying to see that these numbers correspond exactly to the numbers in 4D, \( \mathcal{N} = 4 \) superconformal supergravity \([9]\).

Related to this, the graph theoretic tool of Adinkras were introduced in 2004 \([10]\). Adinkras are graphic representations of the 1D ‘shadows’ of supersymmetric representations. It has been conjectured that these are ubiquitous in higher dimensions, an example
of which is summarized in figure 1. They are similar to Feynman diagrams and Dynkin diagrams in that they convey a large amount of mathematical information in a visual form. Paying homage to the phrase, ‘a picture is worth a thousand words’, it will be seen more literal in this paper that in the case of Adinkras, a picture can be worth many, many more equations.

Since their inception, Adinkras have been proposed as a useful way to attack the problems of supersymmetric field theory that nonetheless does not actually rely on field theory. In a sense, the field theory problems have been mapped (we believe with complete fidelity) into a realm of graph theory, coding theory, and other mathematical formalism that permit a rigorous study of all such problems. In fact, recently there has been proposed a highly formalized mathematical framework [11] that may allow the investigation (and hopefully resolution) of long unsolved problems in this field.

One place where this tool has been used is in the problem of the over-abundance of lower dimensional SUSY systems versus higher dimensional systems. For example, not all one dimensional SUSY systems are dimensional reductions of higher dimensional systems. It is straightforward to take the higher dimensional theory and reduce it to one dimension; going the other way is not so intuitive. Recently, such a program of dimensionally enhancing Adinkras has been undertaken by Faux, Iga, and Landweber [12, 13]. Other Adinkra applications have been in building supersymmetric actions [14], study of error correcting codes [15], and the main topic of this paper, SUSY representation theory [1]. Also along these lines [16], the existence of a stringent criterion that bisects one dimensional representations into two classes (one class only exists for one dimensional representations and the other can be either one dimensional or two dimensional representations) has been identified.

In the first part [1], it was shown how valise Adinkras (Adinkras which are reduced to one fermion row and one boson row as in figure 1) can be used as an organizational tool for off-shell supersymmetric systems. For instance, two distinct, well known 4D off-shell supersymmetric systems, the chiral multiplet (figure 1(a)) and the vector multiplet (figure 1(b)), were shown to have two distinct valise Adinkras: there are no re-definitions solely on bosons or alternately solely on fermions which map one of these to the other. Notice in figure 1 that a ‘parity reflection about the orange axis’ does relate the two diagrams. More generally, any odd number of such color parity reflections will also relate the two. This is not true of an even number of such reflections. However, it should be noted that color parity reflections correspond to redefinitions of supersymmetry generators, not field component redefinitions. Such ‘color parity reflections’ are analogous to spatial reflections of chemical enantiomers, figure 1(c). Paying homage to this analogy, we define the ‘SUSY enantiomer numbers’ $n_c$ and $n_f$ as in figure 1.

There is one additional subtlety. The matter of which of the cis-Adinkra or trans-Adinkra is associated with the chiral or vector multiplets is strictly a matter of a choice of conventions. As noted in the work of [12], if one begins with both valise Adinkras, it is equally valid to assign one to correspond to the chiral multiplet and then necessarily the other to the vector multiplet and then to implement a dimensional extension algorithm that will be consistent.

\footnote{This statement works equally well with the tensor multiplet replacing the vector multiplet.}
(a) The cis-Adinkra \((n_c = 1, n_t = 0)\). This is the valise Adinkra for the 4D, \(\mathcal{N} = 1\) Chiral multiplet.

(b) The trans-Adinkra \((n_c = 0, n_t = 1)\). This is the valise Adinkra for the 4D, \(\mathcal{N} = 1\) vector and tensor multiplets.

(c) One example of chemical enantiomers: the two chiral forms of bromo-chloro-fluoro-methane (CHBrClF).

**Figure 1.** Our conventions for the (a) cis- and (b) trans-Adinkra. They are ‘color parity reflections’ of each other about the ‘orange axis’, (c) analogous to chemical enantiomers which are mirror reflections of each other.

It was conjectured that all 4D, \(\mathcal{N} = 1\) off-shell component descriptions of supermultiplets are associated with the number of cis-valise \((n_c)\) and trans-valise \((n_t)\) Adinkras in the representation. In this paper, we supply evidence to this conjecture. The 4D, \(\mathcal{N} = 1\) real scalar superfield off-shell multiplet is found to have the SUSY enantiomer numbers \(n_c = n_t = 1\), and the 4D, \(\mathcal{N} = 1\) complex linear superfield off-shell multiplet is found to have SUSY enantiomer numbers \(n_c = 1\) and \(n_t = 2\). The discovery of these SUSY enantiomer numbers is also analogous to other results much more familiar to particle physicists. It is well known that for SU(3), the dimension \(d_{SU(3)}\) of all arbitrary representation is specified by two integers (we can denote by \(p\) and \(q\)). For SU(3) the integers \(p\) and \(q\) also are related to graphical images called ‘Young tableaux’ which take the generic form shown in figure 2 where it can be seen that \(p\) corresponds to the number of height-1 boxes and \(q\) the number of height-2 boxes in the SU(3) Tableaux. One goal of this SUSY genomics project is to build the representation table for SUSY systems in analogy to Lie algebras, as shown in table 1.

This paper is structured as follows. Section 2 reviews parts of genomics part I [1] pertaining to the current work of part II. Specifically this includes reviews of the 4D, \(\mathcal{N} = 1\) off-shell chiral and vector multiplets. New in this work is the specific form of the Lagrangians of these multiplets as well as their zero-brane reduction. The Adinkras for these multiplets can be found in the supporting extended review in appendix A. New in these Adinkras are the explicit nodal field content.
The bulk of the new results of this paper are found in section 3 which introduces the real scalar superfield multiplet and section 4 which introduces the complex linear superfield multiplet. Both these sections follow the same structure. First are shown the SUSY transformations laws and the Lagrangians of which they are a invariance. Following this, their zero-brane reductions are explicitly shown, and their Adinkras are drawn with explicit nodal field content. In both cases, it is shown how the SUSY enantiomer numbers can be read directly from these Adinkras. Also, traces referred to in part I [1] as chromocharacters are derived for each of these multiplets. Spring boarding off the trace results of part I, it is shown how the SUSY enantiomer numbers of these two newly investigated multiplets are encoded within these traces. Additionally in section 3 the real pseudoscalar supermultiplet and its relation to the real scalar supermultiplet are briefly discussed.

Finally, in section 5, we define the characteristic polynomial for Adinkras. We show how this polynomial encodes the precise numerical discrepancy between bosonic and fermionic nodes in the Adinkra. All conventions, including those for the gamma matrices, are as in part I [1], unless otherwise specified.

2 Review of genomics(I)

In this section, we review the results from genomics(I) [1] and pertinent to our current work herein contained. In [1], we studied the well known $4D, \mathcal{N} = 1$ chiral multiplet and the $4D, \mathcal{N} = 1$ vector multiplet, both in a Majorana spinor representation. The supersymmetric transformations, algebra, and reductions to the zero-brane were investigated for these representations. The zero-brane reduction of the supersymmetric transformations

Figure 2. The integers $p$ and $q$ in SU(3) Young tableaux.
led us to the Adinkra representations of these two systems. In the following two sections, we review the Lagrangians for each system, the zero-brane reduction of this Lagrangian, and the resulting Adinkras which encode the supersymmetric transformation laws for the zero-brane reduced Lagrangians.

2.1 The 4D $\mathcal{N} = 1$ chiral multiplet

The 4D, $\mathcal{N} = 1$ chiral multiplet is very well known to consist of a scalar $A$, a pseudoscalar $B$, a Majorana fermion $\psi_a$, a scalar auxiliary field $F$, and a pseudoscalar auxiliary field $G$. The Lagrangian for this system which is supersymmetric with respect to the transformation laws investigated in [1] is:

$$L_{CM} = -\frac{1}{2}(\partial_\mu A)(\partial^\mu A) - \frac{1}{2}(\partial_\mu B)(\partial^\mu B) + i\frac{1}{2}(\gamma^\mu)^{ab}\psi_a\gamma_\mu\psi_b + \frac{1}{2}F^2 + \frac{1}{2}G^2.$$  \hspace{1cm} (2.1)

Its zero-brane reduction is acquired by assuming only time dependence of the fields. This Lagrangian with time derivatives denoted by a prime ($'$) is:

$$L^{(0)}_{CM} = \frac{1}{2}(A'_1)^2 + \frac{1}{2}(B'_1)^2 + i\frac{1}{2}(\gamma^\mu)^{ab}\psi_a\psi'_b + \frac{1}{2}F^2 + \frac{1}{2}G^2.$$  \hspace{1cm} (2.2)

The zero-brane reduced SUSY transformation laws which are a symmetry of this Lagrangian were shown in [1] to be succinctly written as the valise cis-Adinkra in figure 1(a). This is reviewed in appendix A.

2.2 The 4D $\mathcal{N} = 1$ vector multiplet

The 4D, $\mathcal{N} = 1$ vector multiplet off-shell is described by a vector $A_\mu$, a Majorana fermion $\lambda_a$, and a pseudoscalar auxiliary field $d$. The Lagrangian for this system which is supersymmetric with respect to the transformation laws investigated in [1] is:

$$L_{VM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}i(\gamma^\mu)^{ab}\lambda_a\partial_\mu\lambda_b + \frac{1}{2}d^2.$$  \hspace{1cm} (2.3)

Its zero-brane reduction is acquired by assuming only time dependence of the fields. This Lagrangian with time derivatives denoted by a prime ($'$) is:

$$L^{(0)}_{VM} = \frac{1}{2}((A'_1)^2 + (A'_2)^2 + (A'_3)^2) + \frac{1}{2}i\delta^{ab}\lambda_a\lambda'_b + \frac{1}{2}d^2.$$  \hspace{1cm} (2.4)

The zero-brane reduced SUSY transformation laws which are a symmetry of this Lagrangian were shown in [1] to be succinctly written as the valise trans-Adinkra in figure 1(b). This is reviewed in appendix A.

3 The 4D, $\mathcal{N} = 1$ real scalar superfield multiplet

The 4D, $\mathcal{N} = 1$ real scalar superfield is a multiplet that is very well known. It consists of a scalar $K$, a Majorana fermion $\zeta$, a scalar field $M$, a pseudoscalar field $N$, and axial vector field $U$, a Majorana fermion field $\Lambda$, and another scalar field $d$. All together, their are eight bosonic and eight fermionic degrees.
3.1 Supersymmetry transformation laws

We use the following set of transformation laws of the supercovariant derivative, $D_a$, acting on each component of the real scalar superfield multiplet:

\[
\begin{align*}
D_a K &= \zeta_a \\
D_a M &= \frac{1}{2} \Lambda_a - \frac{1}{2} (\gamma^\nu)_a \partial_\nu \zeta_d \\
D_a N &= -i \frac{1}{2} (\gamma^5)_a \Lambda_d + i \frac{1}{2} (\gamma^5 \gamma^\nu)_a \partial_\nu \zeta_d \\
D_a U_\mu &= i \frac{1}{2} (\gamma^5 \gamma_\mu)_a \Lambda_d - i \frac{1}{2} (\gamma^5 \gamma^\nu \gamma_\mu)_a \partial_\nu \zeta_d \\
D_a d &= -i \frac{1}{2} (\gamma^\nu)_a \partial_\nu \Lambda_d \\
D_a \zeta_b &= i (\gamma^\mu)_{ab} \partial_\mu K + (\gamma^5 \gamma^\mu)_{ab} U_\mu + i C_{ab} M + (\gamma^5)_{ab} N \\
D_a \Lambda_b &= i (\gamma^\mu)_{ab} \partial_\mu M + (\gamma^5 \gamma^\mu)_{ab} \partial_\mu N + (\gamma^5 \gamma^\mu \gamma^\nu)_{ab} \partial_\mu U_\nu + i C_{ab} d.
\end{align*}
\] (3.1)

A direct calculation shows that

\[\{D_a, D_b\} = 2i (\gamma^\mu)_{ab} \partial_\mu\] (3.2)

is satisfied for all fields in the multiplet. Furthermore, the following Lagrangian is invariant, up to total derivatives, with respect to the $D_a$ transformations in eq. (3.1):

\[L = -\frac{1}{2} M^2 - \frac{1}{2} N^2 + \frac{1}{2} U_\mu U^\mu - \frac{1}{2} K d + i \frac{1}{2} \zeta_a C_{ab} \Lambda_b\] (3.3)

It is clear this Lagrangian implies that all fields in this multiplet are non-propagating.

The real scalar supermultiplet can be used to derive the structure of the real pseudo-scalar supermultiplet by making the substitutions

\[
\begin{align*}
K &\rightarrow L \\
\zeta_a &\rightarrow i (\gamma^5)_a \rho_d \\
M &\rightarrow \hat{N} \\
N &\rightarrow -\hat{M} \\
U_\mu &\rightarrow V_\mu \\
\Lambda_a &\rightarrow -i (\gamma^5)_a \hat{\Lambda}_d \\
d &\rightarrow \hat{d}
\end{align*}
\] (3.4)

and this yields the pseudo-scalar supermultiplet, which satisfies

\[
\begin{align*}
D_a L &= i (\gamma^5)_a \rho_d \\
D_a \rho_b &= - (\gamma^5 \gamma^\mu)_{ab} \partial_\mu L + i (\gamma^\mu)_{ab} V_\mu + (\gamma^5)_{ab} \hat{N} + i C_{ab} \hat{M} \\
D_a \hat{N} &= -i \frac{1}{2} (\gamma^5)_a \hat{\Lambda}_d + i \frac{1}{2} (\gamma^5 \gamma^\mu)_a \partial_\mu \rho_d \\
D_a \hat{M} &= \frac{1}{2} \hat{\Lambda}_a - \frac{1}{2} (\gamma^\mu)_a \partial_\mu \rho_d \\
D_a V_\mu &= -\frac{1}{2} (\gamma_\mu)_a \hat{\Lambda}_d + \frac{1}{2} (\gamma^\nu \gamma_\mu)_a \partial_\nu \rho_d \\
D_a \hat{d} &= -i (\gamma^5 \gamma^\mu)_a \partial_\mu \hat{\Lambda}_d \\
D_a \hat{\Lambda}_b &= (\gamma^5 \gamma^\mu)_{ab} \partial_\mu \hat{N} + i (\gamma^\mu)_{ab} \partial_\mu \hat{M} + i (\gamma^5 \gamma^\mu \gamma^\nu)_{ab} \partial_\mu V_\nu - (\gamma^5)_{ab} \hat{d}
\end{align*}
\] (3.5)
3.2 One dimensional reduction

On our way to the Adinkra picture of the real scalar superfield multiplet, we here first reduce the transformation laws, eq. (3.1), to one dimension by considering the fields to have only time dependence. In the following, we list these time-only dependent transformation laws, with time derivatives denoted by a prime ('). The transformation laws on the bosons are

\[
\begin{align*}
D_1 K &= \zeta_1 & D_2 K &= \zeta_2 & D_3 K &= \zeta_3 & D_4 K &= \zeta_4 \tag{3.6}
\end{align*}
\]

\[
\begin{align*}
D_1 M &= \frac{1}{2} \Lambda_1 - \frac{1}{2} \zeta_2' & D_2 M &= \frac{1}{2} \Lambda_2 + \frac{1}{2} \zeta_1' \\
D_3 M &= \frac{1}{2} \Lambda_3 + \frac{1}{2} \zeta_4' & D_4 M &= \frac{1}{2} \Lambda_4 - \frac{1}{2} \zeta_3'
\end{align*}
\]

\[
\begin{align*}
D_1 N &= \frac{1}{2} \Lambda_1 - \frac{1}{2} \zeta_3' & D_2 N &= -\frac{1}{2} \Lambda_2 - \frac{1}{2} \zeta_4' \\
D_3 N &= \frac{1}{2} \Lambda_2 + \frac{1}{2} \zeta_1' & D_4 N &= -\frac{1}{2} \Lambda_1 + \frac{1}{2} \zeta_2'
\end{align*}
\]

\[
\begin{align*}
D_1 d &= -\Lambda'_2 & D_2 d &= \Lambda'_1 & D_3 d &= \Lambda'_4 & D_4 d &= -\Lambda'_3 \tag{3.9}
\end{align*}
\]

\[
\begin{align*}
D_1 U_0 &= \frac{1}{2} \Lambda_3 + \frac{1}{2} \zeta_4' & D_1 U_1 &= \frac{1}{2} \zeta_4' - \frac{1}{2} \Lambda_3 \\
D_1 U_2 &= \frac{1}{2} \Lambda_1 + \frac{1}{2} \zeta_2' & D_1 U_3 &= \frac{1}{2} \Lambda_4 + \frac{1}{2} \zeta_3' \tag{3.10}
\end{align*}
\]

\[
\begin{align*}
D_2 U_0 &= \frac{1}{2} \Lambda_4 - \frac{1}{2} \zeta_3' & D_2 U_1 &= \frac{1}{4} \Lambda_4 + \frac{1}{2} \zeta_1' \\
D_2 U_2 &= \frac{1}{2} \Lambda_2 - \frac{1}{2} \zeta_4' & D_2 U_3 &= \frac{1}{2} \Lambda_3 - \frac{1}{2} \zeta_2'
\end{align*}
\]

\[
\begin{align*}
D_3 U_0 &= \frac{1}{2} \zeta_4' - \frac{1}{2} \Lambda_1 & D_3 U_1 &= -\frac{1}{2} \Lambda_1 - \frac{1}{2} \zeta_2' \\
D_3 U_2 &= \frac{1}{2} \zeta_2' - \frac{1}{2} \Lambda_3 & D_3 U_3 &= \frac{1}{4} \Lambda_2 - \frac{1}{2} \zeta_1' 	ag{3.12}
\end{align*}
\]

\[
\begin{align*}
D_4 U_0 &= -\frac{1}{2} \Lambda_2 - \frac{1}{2} \zeta_1' & D_4 U_1 &= \frac{1}{4} \Lambda_2 - \frac{1}{2} \zeta_4' \\
D_4 U_2 &= -\frac{1}{2} \Lambda_4 - \frac{1}{2} \zeta_3' & D_4 U_3 &= \frac{1}{4} \Lambda_1 + \frac{1}{2} \zeta_2' \tag{3.13}
\end{align*}
\]

and on the fermions are

\[
\begin{align*}
D_1 \zeta_1 &= iK' & D_1 \zeta_2 &= iU_2 - iM \\
D_1 \zeta_3 &= iU_3 - iN & D_1 \zeta_4 &= iU_0 + iU_1 \tag{3.14}
\end{align*}
\]

\[
\begin{align*}
D_2 \zeta_1 &= iM - iU_2 & D_2 \zeta_2 &= iK' \\
D_2 \zeta_3 &= iU_1 - iU_0 & D_2 \zeta_4 &= -iN - iU_3 \tag{3.15}
\end{align*}
\]

\[
\begin{align*}
D_3 \zeta_1 &= iN - iU_3 & D_3 \zeta_2 &= iU_0 - iU_1 \\
D_3 \zeta_3 &= iK' & D_3 \zeta_4 &= iM + iU_2 \tag{3.16}
\end{align*}
\]

\[
\begin{align*}
D_4 \zeta_1 &= -iU_0 - iU_1 & D_4 \zeta_2 &= iN + iU_3 \\
D_4 \zeta_3 &= -iM - iU_2 & D_4 \zeta_4 &= iK' \tag{3.17}
\end{align*}
\]

\[
\begin{align*}
D_1 \Lambda_1 &= iU_2' + iM' & D_1 \Lambda_2 &= -id \\
D_1 \Lambda_3 &= iU_0' - iU_1' & D_1 \Lambda_4 &= iN' + iU_3' \tag{3.18}
\end{align*}
\]

\[
\begin{align*}
D_2 \Lambda_1 &= id & D_2 \Lambda_2 &= iU_0' + iM' \\
D_2 \Lambda_3 &= iU_3' - iN' & D_2 \Lambda_4 &= iU_0' + iU_1' \tag{3.19}
\end{align*}
\]
\[ D_3 A_1 = -iU_0' - iU_1' \quad D_3 A_2 = iN' + iU_3' \]
\[ D_3 A_3 = iM' - iU_2' \quad D_3 A_4 = i \text{id} \]  \hspace{1cm} (3.20)
\[ D_4 A_1 = iU_3' - iN' \quad D_4 A_2 = iU_1' - iU_0' \]
\[ D_4 A_3 = -i \text{id} \quad D_4 A_4 = iM' - iU_2' \]  \hspace{1cm} (3.21)

These transformation laws can be depicted as the Adinkra in figure 3. The rules for drawing Adinkras are reviewed in appendix A.1.

### 3.3 Canonical fields, \( \Phi \) and \( \Psi \)

Defining the nodes canonically from the Adinkra shown in figure 3 as

\[
\Phi_i = \begin{pmatrix}
\int d \, dt \\
M + U_2 \\
U_0 + U_1 \\
N + U_3 \\
-N + U_3 \\
U_0 - U_1 \\
-M + U_2 \\
K'
\end{pmatrix}, \quad i\hat{\Psi}_i = \begin{pmatrix}
\Lambda_1 \\
\Lambda_2 \\
\Lambda_3 \\
\Lambda_4 \\
\zeta_1' \\
\zeta_2' \\
\zeta_3' \\
\zeta_4'
\end{pmatrix}
\]

we can write the transformation laws succinctly as

\[ D_I \Phi_i = i(L_I)_{ij} \Psi_j, \quad D_I \Psi_j = (R_I)_{ji} \Phi'_i \]  \hspace{1cm} (3.22)

where the Adinkra matrices

\[
L_1 = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

\[ , \hspace{1cm} (3.23) \]

\[
L_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

\[ , \hspace{1cm} (3.24) \]
Figure 3. An Adinkra for the real scalar superfield multiplet. Color convention: \( D_1, D_2, D_3, D_4 \).

\[
L_3 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad (3.25)
\]

\[
L_4 = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad (3.26)
\]
satisfy the orthogonal relationship

\[ \mathbf{R}_I = \mathbf{L}_I^t = \mathbf{L}_I^{-1}, \]  

(3.27)

and satisfy the garden algebra

\[
\begin{align*}
\mathbf{L}_I \mathbf{R}_J + \mathbf{L}_J \mathbf{R}_I &= 2 \delta_{IJ} \mathbf{I}_8 \\
\mathbf{R}_I \mathbf{L}_J + \mathbf{R}_J \mathbf{L}_I &= 2 \delta_{IJ} \mathbf{I}_8
\end{align*}
\]  

(3.28)

with \( \mathbf{I}_n \) the \( n \times n \) identity matrix.

### 3.4 Finding \( n_c \) and \( n_t \) via Adinkras

Applying the field redefinitions

\[
\begin{align*}
\hat{\mathbf{\Phi}} &= \mathbf{X} \mathbf{\Phi} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\int \frac{d}{dt} - K' \\
2M \\
2N \\
2U_0 \\
-2U_2 \\
2U_1 \\
-2U_3
\end{pmatrix},
\hat{\mathbf{\Psi}} &= \mathbf{Y} \mathbf{\Psi} = i \frac{1}{\sqrt{2}} \begin{pmatrix}
\zeta'_1 + \Lambda_2 \\
\zeta'_2 - \Lambda_1 \\
\zeta'_3 - \Lambda_4 \\
-\zeta'_4 - \Lambda_3 \\
-\zeta'_1 + \Lambda_1 \\
-\zeta'_4 + \Lambda_3 \\
\zeta'_3 + \Lambda_4
\end{pmatrix}
\end{align*}
\]  

(3.29)

where the matrices are elements of the \( O(8) \) group and given by

\[
\mathbf{X} = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 0
\end{pmatrix},
\]  

(3.30)

\[
\mathbf{Y} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{pmatrix},
\]  

(3.31)

take us to another Adinkraic basis via

\[
\hat{\mathbf{L}}_I = \mathbf{X} \mathbf{L}_I \mathbf{Y}, \quad \hat{\mathbf{R}}_I = \mathbf{Y}^t \mathbf{R}_I \mathbf{X}^t
\]  

(3.32)
where the Adinkra matrices now are block diagonal:

\[
\hat{L}_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

(3.33)

\[
\hat{L}_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

(3.34)

\[
\hat{L}_3 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix},
\]

(3.35)

\[
\hat{L}_4 = \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

(3.36)

\[
\hat{R}_I = \hat{L}^t_I = \hat{L}_I^{-1}.
\]

(3.37)

These can be written in terms of the SO(4) generators

\[
i\alpha^1 = i\sigma^2 \otimes \sigma^1, \quad i\alpha^2 = i\sigma^1 \otimes \sigma^2, \quad i\alpha^3 = i\sigma^2 \otimes \sigma^3, \\
i\beta^1 = i\sigma^1 \otimes \sigma^2, \quad i\beta^2 = i\sigma^2 \otimes I_2, \quad i\beta^3 = i\sigma^3 \otimes \sigma^2
\]

(3.38)
Figure 4. The 8 x 8 valise Adinkra for the real scalar superfield multiplet with \( n_c = n_t = 1 \) (compare with figure 1). All fermionic nodes have the same engineering dimension and all bosonic nodes have the same engineering dimension.

As

\[
\begin{align*}
\hat{L}_1 &= I_8, \\
\hat{L}_2 &= iI_2 \otimes \beta^3, \\
\hat{L}_3 &= i\sigma^3 \otimes \beta^2, \\
\hat{L}_4 &= -iI_2 \otimes \beta^1
\end{align*}
\]  

(3.39)

When comparing with eqs. (A.9) and (A.10), it is clear the block diagonal representation (3.39) of the real scalar superfield Adinkra matrices are composed of the cis-Adinkra matrices in the upper block and the trans-Adinkra matrices in the lower block.

Furthermore, the Adinkra for these matrices is as in figure 4. This Adinkra is easily seen, upon comparison with figure 1, to split into one cis- and one trans-Adinkra. Clearly then the real scalar superfield multiplet has the SUSY enantiomer numbers \( n_c = n_t = 1 \). This will be shown another way in the next section, with traces of the Adinkra matrices which is independent of node definition of the Adinkras.

### 3.5 Traces

Because the \( L \) and \( \hat{L} \) matrices are related by orthogonal transformations, the following traces

\[
Tr[L_i L_d^j] = 8 \delta_{ij} \tag{3.40}
\]

\[
Tr[L_i L_j^k L_k^l L_d^j] = 8 \left( \delta_{ij} \delta_{KL} - \delta_{IK} \delta_{HL} + \delta_{IL} \delta_{JK} \right) \tag{3.41}
\]
are identical for both sets. We see here that the real scalar superfield multiplet fits into the conjectured trace formulas from part I \[1\]

\[
\begin{align*}
\text{Tr}[L^i I^j] &= 4 \left( n_c + n_t \right) \delta_{ij} \\
\text{Tr}[L^i L^j L^K L^L] &= 4 \left( n_c + n_t \right) \left( \delta_{ij} \delta_{KL} - \delta_{iK} \delta_{jL} + \delta_{iL} \delta_{jK} \right) \\
&+ 4 \left( n_c - n_t \right) \epsilon_{ijkl}
\end{align*}
\]  

(3.42)

where \(n_c\) and \(n_t\) are respectively, the number of cis-valise and trans-valise Adinkras contained in an arbitrary multiplet. We see that \(n_c = n_t\) and as \(n_c\) and \(n_t\) are constrained by the size of the multiplet \((2d = 16)\) to satisfy

\[
4(n_c + n_t) = d
\]  

(3.43)

we find as in figure 4 that

\[
n_c = n_t = 1.
\]  

(3.44)

4 The 4D, \(\mathcal{N} = 1\) complex linear superfield multiplet and Adinkras

The 4D, \(\mathcal{N} = 1\) complex linear superfield multiplet consists of a scalar \(K\), a pseudoscalar \(L\), a Majorana fermion \(\zeta\), a Majorana fermion auxiliary field \(\rho\), a scalar auxiliary field \(M\), a pseudoscalar auxiliary field \(N\), a vector auxiliary field \(V\), an axial vector auxiliary field \(U\), and a Majorana fermion auxiliary field \(\beta\).

4.1 Supersymmetry transformation laws

The supersymmetry variations of the components of the complex linear superfield multiplet can be cast in the forms

\[
\begin{align*}
D_a K &= \rho_a - \zeta_a \\
D_a M &= \beta_a - \frac{1}{2} (\gamma^\nu)_a d \partial_\nu \rho_d \\
D_a N &= -i (\gamma^5)_a d \beta_d + \frac{1}{2} (\gamma^5 \gamma^\nu)_a d \partial_\nu \rho_d \\
D_a L &= i (\gamma^5)_a d (\rho_d + \zeta_d) \\
D_a U_\mu &= i (\gamma^5 \gamma_\mu)_a d \beta_d - i (\gamma^5)_a d \partial_\mu (\rho_d + 2 \zeta_d) - \frac{i}{4} (\gamma^5 \gamma_\mu)_a d \partial_\nu (\rho_d + 2 \zeta_d) \\
D_a V_\mu &= - (\gamma_\mu)_a d \beta_d + \partial_\mu (\rho_a - 2 \zeta_a) + \frac{1}{2} (\gamma^\nu \gamma_\mu)_a d \partial_\nu \rho_d (\rho_d + 2 \zeta_d) \\
D_a \zeta_b &= -i (\gamma^\mu)_{ab} \partial_\mu K - (\gamma^5 \gamma^\mu)_{ab} \partial_\mu L - \frac{1}{2} (\gamma^5 \gamma_\mu)_{ab} U_\mu + i \frac{1}{2} (\gamma^\mu)_{ab} V_\mu \\
D_a \rho_b &= i C_{ab} M + (\gamma^5)_{ab} N + \frac{1}{2} (\gamma^5 \gamma^\mu)_{ab} U_\mu + i \frac{1}{2} (\gamma^\mu)_{ab} V_\mu \\
D_a \beta_b &= \frac{1}{2} (\gamma^\mu)_{ab} \partial_\mu M + \frac{1}{2} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu L + \frac{1}{2} (\gamma^5 \gamma^\nu)_{ab} \partial_\mu U_\nu + \frac{1}{2} (\gamma^\mu \gamma^\nu)_{ab} \partial_\mu V_\nu + \frac{i}{2} (\gamma^5 \gamma^\mu)_{ab} \partial_\mu V_\nu + \eta^\mu \nu \partial_\mu \partial_\nu (-i C_{ab} K + (\gamma^5)_{ab} L).
\end{align*}
\]  

(4.1)
As in the case of the real scalar superfield multiplet, the commutator algebra for the D-operator calculated from (4.1) takes the form

$$\{D_a, D_b\} = 2i(\gamma^\mu)_{ab} \partial_\mu$$

(4.2)

for all fields in the complex linear superfield multiplet. The D-operator (4.1) is an invariance, up to total derivatives, of the Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial_\mu K \partial^\mu K - \frac{1}{2} \partial_\mu L \partial^\mu L - \frac{1}{2} M^2 - \frac{1}{2} N^2 + \frac{1}{4} U_\mu U^\mu + \frac{1}{4} V_\mu V^\mu$$

$$+ \frac{1}{2} i (\gamma^\mu)_{ab} \partial_\mu \zeta_a \zeta_b + i \rho_a C^{ab} \beta_b$$

(4.3)

As a bridge to section 4.2, we show the zero-brane reduction of this Lagrangian, where all fields are assumed to have only time dependence, and time derivatives are denoted by a prime ('):

$$\mathcal{L}^{(0)}_0 = \frac{1}{2} K^2 + \frac{1}{2} L^2 - \frac{1}{2} M^2 - \frac{1}{2} N^2 + \frac{1}{4} U_\mu U^\mu + \frac{1}{4} V_\mu V^\mu$$

$$+ i \frac{1}{2} (\zeta_1 \zeta'_1 + \zeta_2 \zeta'_2 + \zeta_3 \zeta'_3 + \zeta_4 \zeta'_4) + i (\rho_2 \beta_1 - \rho_1 \beta_2 + \rho_3 \beta_4 - \rho_4 \beta_3)$$

(4.4)

This Lagrangian can be acquired by the following calculation, performed on the zero-brane:

$$D_1 D_2 D_3 D_4 \mathcal{L}^{(0)}_{\text{Superspace}} = \mathcal{L}^{(0)} + \text{total derivatives}$$

(4.5)

where $\mathcal{L}^{(0)}_{\text{Superspace}} = -\frac{1}{8} (K^2 + L^2)$, the Lagrangian for the complex linear superfield multiplet in superspace. This calculation is shown explicitly in appendix B.2.

### 4.2 Zero brane reduction

Before creating the Adinkra for the complex linear superfield multiplet, we must first reduce the transformation laws, eq. (4.1), to one dimension by considering the fields to have only time dependence. The resulting SUSY transformation laws are, for the bosons:

$$D_1 K = \rho_1 - \zeta_1 \quad D_2 K = \rho_2 - \zeta_2 \quad D_3 K = \rho_3 - \zeta_3 \quad D_4 K = \rho_4 - \zeta_4$$

(4.6)

$$D_1 M = -\frac{1}{2} \rho'_2 + \beta_1 \quad D_2 M = \frac{\rho'_2}{2} + \beta_2$$

$$D_3 M = \frac{\rho'_3}{2} + \beta_3 \quad D_4 M = -\frac{\rho'_3 + \beta_3}{2}$$

(4.7)

$$D_1 N = -\frac{\beta'_4}{2} + \beta_4 \quad D_2 N = -\frac{1}{2} \rho'_4 - \beta_3$$

$$D_3 N = \frac{\beta'_4}{2} + \beta_2 \quad D_4 N = \frac{\beta'_4}{2} - \beta_1$$

(4.8)

$$D_1 L = -\rho_4 - \zeta_4 \quad D_2 L = \rho_3 + \zeta_3 \quad D_3 L = -\rho_2 - \zeta_2 \quad D_4 L = \rho_1 + \zeta_1$$

(4.9)

$$D_1 U_0 = \beta_3 + \zeta'_4 + \frac{3 \beta'_4}{2} \quad D_1 U_1 = -\beta_3 - \zeta'_4 + \frac{\beta'_4}{2}$$

$$D_1 U_2 = \beta_1 - \zeta'_2 + \frac{\beta'_2}{2} \quad D_1 U_3 = \beta_4 - \zeta'_3 + \frac{\beta'_3}{2}$$

(4.10)

$$D_2 U_0 = \beta_4 - \zeta'_4 - \frac{3 \beta'_4}{2} \quad D_2 U_1 = \beta_4 - \zeta'_3 + \frac{\beta'_3}{2}$$

$$D_2 U_2 = \beta_2 + \zeta'_1 - \frac{\beta'_1}{2} \quad D_2 U_3 = \beta_3 + \zeta'_4 - \frac{\beta'_4}{2}$$

(4.11)
\[
\begin{align*}
D_3 U_0 &= -\beta_1 + \zeta'_1 + \frac{3\beta'_2}{2} \\
D_3 U_2 &= -\beta_3 - \zeta'_1 + \frac{3\beta'_2}{2} \\
D_4 U_0 &= -\beta_2 - \zeta'_1 + \frac{3\beta'_2}{2} \\
D_4 U_2 &= -\beta_4 + \zeta'_3 - \frac{\beta'_2}{2} \\
D_1 V_0 &= \beta_2 - \zeta'_1 + \frac{3\beta'_2}{2} \\
D_1 V_2 &= \beta_4 + \zeta'_3 - \frac{\beta'_2}{2} \\
D_2 V_0 &= -\beta_1 - \zeta'_2 + \frac{\beta'_2}{2} \\
D_2 V_2 &= -\beta_3 + \zeta'_4 + \frac{\beta'_2}{2} \\
D_3 V_0 &= -\beta_4 - \zeta'_3 + \frac{3\beta'_2}{2} \\
D_3 V_2 &= -\beta_2 + \zeta'_4 + \frac{\beta'_2}{2} \\
D_4 V_0 &= \beta_3 - \zeta'_4 + \frac{3\beta'_2}{2} \\
D_4 V_2 &= \beta_1 + \zeta'_2 + \frac{\beta'_2}{2}
\end{align*}
\]

and for the fermions:

\[
\begin{align*}
D_1 \zeta_1 &= \frac{\nu_0}{2} + \frac{\nu_1}{2} - iK' \\
D_1 \zeta_3 &= \frac{\nu_0}{2} - \frac{\nu_1}{2} \\
D_2 \zeta_1 &= \frac{\nu_0}{2} - \frac{\nu_1}{2} \\
D_2 \zeta_3 &= \frac{\nu_0}{2} + \frac{\nu_1}{2} + iL' \\
D_3 \zeta_1 &= \frac{\nu_0}{2} + \nu'_1 \\
D_3 \zeta_3 &= \frac{\nu_0}{2} - \nu'_1 - iK' \\
D_4 \zeta_1 &= \nu'_2 + \frac{\nu_1}{2} + iL' \\
D_4 \zeta_3 &= \nu'_2 + \frac{\nu_1}{2} \\
D_1 \rho_1 &= \frac{\nu_0}{2} + \nu'_1 \\
D_1 \rho_3 &= -iN + \frac{\nu_0}{2} + \frac{\nu_2}{2} \\
D_2 \rho_1 &= iM - \frac{\nu_0}{2} - \frac{\nu_2}{2} \\
D_2 \rho_3 &= \frac{\nu_0}{2} - \nu'_2 \\
D_3 \rho_1 &= iN - \frac{\nu_0}{2} + \frac{\nu_2}{2} \\
D_3 \rho_3 &= \frac{\nu_0}{2} - \frac{\nu_1}{2} \\
D_4 \rho_1 &= -\frac{\nu_0}{2} - \frac{\nu_1}{2} \\
D_4 \rho_3 &= -iM - \frac{\nu_0}{2} + \frac{\nu_2}{2}
\end{align*}
\]
With the rules for Adinkras reviewed in appendix A.1, it can be seen that these transformation laws can be succinctly displayed as the Adinkra in figure 5.

Next, we define fields from the nodes of figure 5

\[
\Phi = \begin{pmatrix}
-M \\
K' - V_0 \\
-L' - U_0 \\
N \\
U_1 \\
U_2 \\
V_0 - 2K' \\
-U_1 \\
U_3 \\
-V_3 \\
V_1 \\
-2L' - U_0 \\
V_2
\end{pmatrix}
\]

\[
i\Psi = \begin{pmatrix}
\frac{\rho}{4} - \beta_1 \\
-\beta_2 - \frac{\rho}{4} \\
-\beta_3 - \frac{\rho}{4} \\
\beta_4 - \frac{\rho}{4} \\
\beta_1 - \zeta_2 + \frac{\rho}{4} \\
\beta_2 + \zeta_1 - \frac{\rho}{4} \\
\beta_3 + \zeta_4 - \frac{\rho}{4} \\
\beta_4 - \zeta_3 + \frac{\rho}{4} \\
\beta_1 + \zeta_2 + \frac{\rho}{4} \\
-\beta_2 + \zeta_1 + \frac{\rho}{4} \\
-\beta_3 + \zeta_4 + \frac{\rho}{4} \\
\beta_4 + \zeta_3 + \frac{\rho}{4}
\end{pmatrix}
\]

such that

\[
D_1\Phi_i = i (L_1)_{ij} \Psi_j, \quad D_1\psi_i = (R_1)_{ki} \Phi^i.
\]

where the \( L \) and \( R \) Adinkra matrices are

\[
L_1 = I_3 \otimes I_4, \quad L_2 = i I_3 \otimes \beta_3,
\]

\[
L_3 = i \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix} \otimes \beta_2, \quad L_4 = -i I_3 \otimes \beta_1
\]

and

\[
R_1 = L_1^t = L_1^{-1}.
\]
Figure 5. The $12 \times 12$ valise Adinkra for the complex linear superfield multiplet with $n_c = 1$ and $n_t = 2$ (compare with figure 1). All fermionic nodes have the same engineering dimension and all bosonic nodes have the same engineering dimension.

The matrices in eq. (4.32) are written in terms of the SO(4) generators in eq. (3.38) and comparing with eqs. (A.9) and (A.10), it is clear that the block diagonal representation (4.32) of the complex linear superfield Adinkra matrices are composed of the cis-Adinkra matrices in the upper block and trans-Adinkra matrices in each of the lower two blocks. It is then clear from eq. (4.32) that the complex linear superfield has the SUSY enantiomer numbers $n_c = 1, n_t = 2$. This same conclusion can be easily arrived at from comparing figure 5 to the definitions of the cis- and trans-valise Adinkras in figure 1. We will confirm the SUSY enantiomer numbers through basis independent traces in section 4.3. Furthermore, making the substitution $\beta_2 \rightarrow -\beta_2$ would transform the Adinkra matrices in eq. (4.32) into the $n_c = 2, n_t = 1$ SUSY representation.
We can write the matrices in eq. (4.32) as

\[
\begin{align*}
L_1 &= I_3 \otimes I_4, & L_2 &= iI_3 \otimes \beta_3, \\
L_3 &= i(h_4 - \frac{1}{3}I_3 \otimes \frac{1}{3}h_5) \otimes \beta_2, & L_4 &= -iI_3 \otimes \beta_1,
\end{align*}
\] (4.34)

where they are written in terms of the in the following \(12 \times 12\) basis

\[
M_{12 \times 12} = a_0 I_3 \otimes I_4 + \sum_I a_I a_I \otimes I_4 + \sum \Delta h_\Delta \otimes I_4 \\
+ \sum_I c_I I_3 \otimes \alpha_I + \sum_{I, K} d_{I, K} a_I \otimes \alpha_K + \sum \Delta \epsilon I h_\Delta \otimes \alpha_I \\
+ \sum_I \tilde{c}_I I_3 \otimes \beta_I + \sum_{I, K} \tilde{d}_I I_3 \otimes \beta_K + \sum \Delta \tilde{\epsilon}_I h_\Delta \otimes \beta_I \\
+ \sum_{I, J} f_{I, J} I_3 \otimes \alpha_I \beta_J + \sum_{I, J, K} g_{I, J, K} a_I \otimes \alpha_J \beta_K \\
+ \sum \Delta \tilde{\epsilon}_{I, J} h_I h_J \otimes \alpha_I \beta_J,
\] (4.35)

where \(a_I\) and \(h_\Delta\) are the SL(3) generators

\[
\begin{align*}
a_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, & a_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, & a_3 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{align*}
\] (4.36)

and

\[
\begin{align*}
h_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & h_2 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & h_3 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{align*}
\] (4.37)

\[
\begin{align*}
h_4 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & h_5 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{align*}
\]

**4.3 Traces**

Calculating the traces

\[
Tr[L_I(L_J)_t] = 12 \delta_{IJ}
\] (4.38)

and

\[
Tr[L_I L_I^t L_K L_K^t] = 12(\delta_{IJ} \delta_{KL} - \delta_{IK} \delta_{JL} + \delta_{IL} \delta_{JK}) - 4 \epsilon_{IJKL}
\] (4.39)

and comparing with the conjecture in eq. (3.42), for the complex linear superfield multiplet we find

\[
n_c = 1, \quad n_t = 2.
\] (4.40)

This is a basis independent confirmation of our findings from section 4.2, i.e., orthogonal transformations on the fields like those performed on the real scalar superfield multiplet in section 3.4 won’t change this result.
5 Characteristic polynomials for Adinkras

In this final section, we investigate yet another way to distinguish SUSY representations, the characteristic polynomial

$$P_I(\mathcal{J}) \equiv \det \begin{pmatrix} \mathcal{J}I_n & -L_I \\ -R_I & \mathcal{J}I_m \end{pmatrix}, \quad \text{no } I \text{ sum}$$

(5.1)

for $n \times m$ Adinkra matrices $L_I$ and $m \times n$ Adinkra matrices $R_I$ with $\mathcal{J}$ an arbitrary constant. Results for several 4D, $\mathcal{N} = 1$ systems are listed in table 2, which are all color independent.

The off-shell cases I, III, V, the real scalar, and complex linear superfield multiplets clearly follow a similar pattern here. This stems from the fact that each of their Adinkras have equal numbers of bosonic and fermionic nodes and as a consequence have square $L_I$ and $R_I$ matrices. For mutually commuting $d \times d$ matrices $A$, $B$, $C$, and $D$ we have the mathematical identity [18]

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det (AD - BC).$$

(5.2)

This leads us to the following result for the square Adinkra matrices

$$P_I(\mathcal{J}) \equiv \det \begin{pmatrix} \mathcal{J}I_d & -L_I \\ -R_I & \mathcal{J}I_d \end{pmatrix}, \quad \text{no } I \text{ sum}$$

(5.3)

$$= \det (\mathcal{J}^2 I_d - L_I R_I)$$

which, owing to the orthogonality of these off-shell $d \times d$ Adinkra matrices $R_I = L_I^{-1} = L_I^t$, leads us to the formula

$$P_I(\mathcal{J}) = (\mathcal{J}^2 - 1)^d, \quad d_L = d_R = d.$$ 

(5.4)

Furthermore, this result is independent of the Adinkraic basis, as $O(d)$ transformations, e.g. eqs. (3.30) and (3.31), will clearly not change the result. Cases II, IV, and VI clearly do not fit so nicely into such a succinct formula, as $d_L \neq d_R$ in each of these cases. It is interesting to note that each of the characteristic polynomials in table 2 do follow the pattern

$$P_I(\mathcal{J}) = \mathcal{J}^{d_L - d_R} \times (\text{Polynomial in } \mathcal{J}).$$

(5.5)

This leads us to the conjecture:

**Conjecture 1** The number of factors of $\mathcal{J}$ that factor out of the characteristic polynomial is equal to the absolute value of the difference in bosonic and fermionic nodes.

It is not clear at present what information the left over polynomial multiplying the $|d_L - d_R|$ factors of $\mathcal{J}$ conveys.

Finally, we point out that case IV is unique. As shown in part I [1], its algebra does not close and is an on-shell representation with no known off-shell representation. Along
Table 2. Characteristic polynomials for Adinkras of several 4D, $\mathcal{N} = 1$ systems with $d_L$ ($d_R$) bosonic (fermionic) nodes. The Adinkra matrices for cases I through VI are as in part I [1].

| SUSY rep.                              | $P_1(J)$                                      | $d_L$ | $d_R$ |
|----------------------------------------|-----------------------------------------------|-------|-------|
| off-shell chiral multiplet (case I)     | $(J^2 - 1)^4$                                 | 4     | 4     |
| on-shell chiral multiplet (case II)     | $J^2(J^2 - 1)^2$                              | 2     | 4     |
| off-shell tensor multiplet (case III)   | $(J^2 - 1)^4$                                 | 4     | 4     |
| double tensor multiplet (case IV)       | $J^2(J^4 - 3J^2 + 2)^2$                       | 6     | 4     |
| off-shell vector multiplet (case V)     | $(J^2 - 1)^4$                                 | 4     | 4     |
| on-shell vector multiplet (case VI)     | $J(J^2 - 1)^3$                                | 3     | 4     |
| real scalar superfield multiplet        | $(J^2 - 1)^8$                                 | 8     | 8     |
| complex linear superfield multiplet     | $(J^2 - 1)^{12}$                              | 12    | 12    |

this line of thought, we point out similarities between cases IV and II. These have the same value of $|d_L - d_R|$ and so their characteristic polynomials follow the same pattern: $P_1(J) = J^2 \times (\text{Polynomial in } J)^2$. This is not too surprising as in [1] transformations relating these two multiplets were pointed out, though an exact mapping was not found. In fact these systems have been known to be related for quite some time as the double tensor multiplet was spawned in [19] over the relationship between antisymmetric tensor fields and scalar particles of supergravity.

6 Conclusion

Through the genomics project, we seek to find a categorization system for SUSY representations in analogy to the representation theory of Lie algebras. In part I [1], we found two distinct Adinkras, cis- and trans-, that we gave in the present work the SUSY enantiomer numbers $n_c = 1$, $n_t = 0$ and $n_c = 0$, $n_t = 1$, respectively. In part I [1], it was conjectured that these cis- and trans-Adinkras were a basis from which all off-shell 4D, $\mathcal{N} = 1$ SUSY representations could be described. In our work here in part II, we found this to be the case for two more off-shell 4D, $\mathcal{N} = 1$ SUSY representations. These two systems are the real scalar ($n_c = n_t = 1$) and complex linear ($n_c = 1$, $n_t = 2$) superfield multiplets.

We showed two methods of finding these SUSY enantiomer numbers, one graphical, another computational. The graphical way is of course to decompose the Adinkra into its cis- and trans- components. The computational method was using traces of the Adinkra matrices, termed chromocharacters in the previous work [1]. Both methods arrived at the same numbers, as expected.

Finally, in our continuing quest to find various mathematical and graphical objects with which to classify SUSY representations, we introduced the SUSY characteristic polynomial for Adinkras. In the eight representations investigated, we saw the consistent result that the polynomial encodes the precise numerical discrepancy in numbers of bosonic and fermionic nodes in the Adinkra. Whether this trend is a feature of all Adinkras or not, and what other information the polynomial holds, both remain to be seen. We will continue investigations...
of polynomials such as these in future works, as well as derive the Adinkras and SUSY enantiomer numbers of other systems to continue building the SUSY representation table.

“Details create the big picture.” - Sanford I. Weill

Acknowledgments

This research was supported in part by the endowment of the John S. Toll Professorship, the University of Maryland Center for String & Particle Theory, National Science Foundation Grant PHY-0354401. SJG and KS offer additional gratitude to the M.L.K. Visiting Professorship and to the M.I.T. Center for Theoretical Physics for support and hospitality facilitating part of this work. We thank Leo Rodriguez, Abdul Khan, and the many other students who toiled over various aspects of the elusive complex linear superfield multiplet Adinkra. Many of these students were a part of various SSTPRS summer school sessions held either at the University of Maryland or the University of Iowa, to which we also extend gratitude. We thank Tristan H¨ubsch for recently renewing our interest in this difficult problem. Most Adinkras were drawn with the aid of Adinkramat © 2008 by G. Landweber.

A Adinkra review of genomics I

In this appendix we review the construction of the Adinkras for the 4D $\mathcal{N} = 1$ chiral and vector multiplets found in [1], where the SUSY transformation laws for these systems were studied in depth. Here, we review their zero-brane reduction and how to construct the Adinkras from these transformation laws. Also, it is reviewed that the chiral multiplet and vector multiplet have SUSY enantiomer numbers $(n_c = 1, n_t = 0)$ and $(n_c = 0, n_t = 1)$. The chiral multiplet valise Adinkra is then precisely the cis-Adinkra and the vector multiplet is precisely the trans-Adinkra. These two Adinkras are shown in the body of this paper to be the building blocks of both the real scalar and complex linear superfield multiplet’s valise Adinkras.

A.1 The 4D $\mathcal{N} = 1$ chiral multiplet

The zero-brane reduced SUSY transformation laws for the chiral multiplet, which are a symmetry of the Lagrangian (2.2), are:

\[
\begin{align*}
D_1A &= \psi_1 \\
D_1B &= -\psi_4 \\
D_1F &= \psi'_2 \\
D_1G &= -\psi'_3 \\
D_2A &= \psi_2 \\
D_2B &= \psi_3 \\
D_2F &= -\psi'_1 \\
D_2G &= -\psi'_4 \\
D_3A &= \psi_3 \\
D_3B &= -\psi_2 \\
D_3F &= -\psi'_4 \\
D_3G &= \psi'_1 \\
D_4A &= \psi_4 \\
D_4B &= \psi_1 \\
D_4F &= \psi'_3 \\
D_4G &= \psi'_2.
\end{align*}
\]

and

\[
\begin{align*}
D_1\psi_1 &= iA' \\
D_1\psi_2 &= iF \\
D_1\psi_3 &= -iG \\
D_1\psi_4 &= -iB' \\
D_2\psi_1 &= -iF \\
D_2\psi_2 &= iA' \\
D_2\psi_3 &= B' \\
D_2\psi_4 &= -iG \\
D_3\psi_1 &= iG \\
D_3\psi_2 &= -iB' \\
D_3\psi_3 &= iA' \\
D_3\psi_4 &= -iF \\
D_4\psi_1 &= iB' \\
D_4\psi_2 &= iG \\
D_4\psi_3 &= iF \\
D_4\psi_4 &= iA'.
\end{align*}
\]

where a prime (’) denotes a time derivative.
Figure 6. How to read SUSY transformation laws from an Adinkra.

Now we show how to build an Adinkra from these transformation laws. Adinkras are defined by fermionic and bosonic nodes and colored connections as in figure 6. The rules are as follows:

1. The color encodes the identity of the transformation law. Here and throughout this paper, we hold to the conventions of [1] and identify the colors as \( D_1, D_2, D_3, \) and \( D_4 \).
2. Applying the \( D \)-operator to the lower node yields the upper node.
3. Applying the \( D \)-operator to the upper node yields \( \partial_t \) acting on the lower node.
4. A dashed line encodes an overall additional minus sign in both associated transformations.
5. Transformations from fermion to boson yield an additional factor of \( i \) multiplying the boson.
6. Field nodes on the same horizontal level have the same engineering dimension. Engineering dimensions increase by one half when ascending to the row directly above.

With these simple rules, we can easily see that the transformation laws (A.1) and (A.2) encode the cis-Adinkra in figure 7, there written in valise form of one row of fermions over one row of bosons. The cis-Adinkra has SUSY enantiomer numbers \( n_c = 1 \) and \( n_t = 0 \). Identifying the nodes in figure 7 as

\[
\begin{align*}
  i\Psi_1 &= \psi_1, & i\Psi_2 &= \psi_2, & \Psi_3 &= \psi_3, & \Psi_4 &= -\psi_4 \\
  \Phi_1 &= A & \Phi_2 &= \int dtF & \Phi_3 &= -\int dtG & \Phi_4 &= B
\end{align*}
\]

we can succinctly write the zero-brane transformation laws of the chiral multiplet as:

\[
D_i\Phi_i = i(L_i)_{ij}\Psi_j, \quad D_i\Psi_j = (R_i)_{ji}\Phi'_i
\]
where the Adinkra matrices are the cis-Adinkra matrices in eq. (A.9), and
\[ R_I = L_I^t = L_I^{-1}. \] (A.4)

A.2 The 4D \( \mathcal{N} = 1 \) vector multiplet

The zero-brane reduced SUSY transformation laws for the vector multiplet, which are a symmetry of the Lagrangian (2.4), are:

\[
\begin{align*}
D_1A_1 &= \lambda_2 & D_2A_1 &= \lambda_1 & D_3A_1 &= \lambda_4 & D_4A_1 &= \lambda_3 \\
D_1A_2 &= -\lambda_4 & D_2A_2 &= \lambda_3 & D_3A_2 &= \lambda_2 & D_4A_2 &= -\lambda_1 \\
D_1A_3 &= \lambda_1 & D_2A_3 &= -\lambda_2 & D_3A_3 &= \lambda_3 & D_4A_3 &= -\lambda_4 \\
D_1d &= -\lambda_3' & D_2d &= -\lambda_4' & D_3d &= \lambda_1' & D_4d &= \lambda_2' \quad (A.5)
\end{align*}
\]

and

\[
\begin{align*}
D_1\lambda_1 &= iA_3' & D_2\lambda_1 &= iA_1' & D_3\lambda_1 &= i\lambda_3 & D_4\lambda_1 &= -iA_2' \\
D_1\lambda_2 &= iA_1' & D_2\lambda_2 &= -iA_3' & D_3\lambda_2 &= iA_2' & D_4\lambda_2 &= iA_1' \\
D_1\lambda_3 &= -i\lambda_4 & D_2\lambda_3 &= iA_2' & D_3\lambda_3 &= iA_3' & D_4\lambda_3 &= iA_1' \\
D_1\lambda_4 &= -iA_2' & D_2\lambda_4 &= -i\lambda_3 & D_3\lambda_4 &= iA_1' & D_4\lambda_4 &= -iA_3' \quad (A.6)
\end{align*}
\]

following the Adinkra construction rules as in figure 6, we arrive at the valise form for the vector multiplet Adinkra in figure 8, which is seen to be the trans-Adinkra from [1].

Identifying the nodes in figure 8 as

\[
\begin{align*}
i\Psi_1 &= \lambda_1, & i\Psi_2 &= -\lambda_2, & \Psi_3 &= -\lambda_3, & \Psi_4 &= \lambda_4 \\
\Phi_1 &= A_3, & \Phi_2 &= -A_1, & \Phi_3 &= \int dt & \Phi_4 &= -A_2 \quad (A.7)
\end{align*}
\]

we can succinctly write the zero-brane transformation laws of the vector multiplet as:

\[ D_1\Phi_i = i(L_i)_{ij}\Psi_j, \quad D_1\Psi_j = (R_1)_{ji}\Phi_i' \quad (3.22) \]
where the Adinkra matrices are the trans-Adinkra matrices in eq. (A.10), and
\[ R_I = L_I = L_I^{-1}. \]  
(A.8)

### A.3 The cis- and trans-Adinkras: SUSY enantiomers

The cis- and trans-Adinkra matrices can be succinctly written as

\[
\begin{align*}
L_1 &= I_4, & L_2 &= i\beta^3, & L_3 &= i\beta^2, & L_4 &= -i\beta^1 \quad \text{cis} \quad (A.9) \\
L_1 &= I_4, & L_2 &= i\beta^3, & L_3 &= -i\beta^2, & L_4 &= -i\beta^1 \quad \text{trans}. \quad (A.10)
\end{align*}
\]

where \( \beta^i \) are the SO(4) generators defined in eq. (3.38). Here we see clearly the origination of the the names cis- and trans:- the matrices are the same, modulo a ‘reflection’ about the ‘orange axis’ \( L_3 \), in an analogous fashion to how chemical enantiomers are identical modulo a reflection about a spatial axis as in figure 1(c).

### B More of the complex linear superfield

In this appendix, we explicitly show calculations pertinent to results for the complex linear multiplet reported in section 4. In appendix B.1, we show explicitly the SUSY transformations of a certain set of linear combinations of the complex linear multiplet. This set of transformation laws will prove useful in appendix B.2 where we show as a consistency check that, on the zero-brane, the following identity holds

\[ D_1 D_2 D_3 D_4 \mathcal{L}_{\text{Superspace}}^{(0)} = \mathcal{L}^{(0)} + \text{total derivatives} \]  
(B.1)

where \( \mathcal{L}_{\text{Superspace}}^{(0)} = -\frac{1}{8}(K^2 + L^2) \), the Lagrangian for the complex linear multiplet in superspace, and \( \mathcal{L}^{(0)} \) is the Lagrangian (4.4) for the CLM in bosonic space-time.
B.1 Linear combinations of the 4D $\mathcal{N} = 1$ complex linear multiplet

From section 4.2, we have following transformation laws:

\[
\begin{align*}
D_1 K &= \rho_1 - \zeta_1, & D_2 K &= \rho_2 - \zeta_2, \\
D_3 K &= \rho_3 - \zeta_3, & D_4 K &= \rho_4 - \zeta_4 \\
D_1 L &= -\rho_4 - \zeta_4, & D_2 L &= \rho_3 + \zeta_3 \\
D_3 L &= -\rho_2 - \zeta_2, & D_4 L &= \rho_1 + \zeta_1
\end{align*}
\]

We calculate the transformation laws for the linear combinations of $\rho$ and $\zeta$ found on the right hand side:

\[
\begin{align*}
D_1 (\rho_1 - \zeta_1) &= iK' \\
D_1 (\rho_3 - \zeta_3) &= -i (N - U_3) \\
D_2 (\rho_1 - \zeta_1) &= i (M - U_2) \\
D_2 (\rho_3 - \zeta_3) &= -i (U_0 - U_1 + L') \\
D_3 (\rho_1 - \zeta_1) &= i (N - U_3) \\
D_3 (\rho_3 - \zeta_3) &= iK' \\
D_4 (\rho_1 - \zeta_1) &= -i (U_0 + U_1 + L') \\
D_4 (\rho_3 - \zeta_3) &= -i (M + U_2) \\
D_1 (\rho_1 + \zeta_1) &= i (V_0 + V_1 - K') \\
D_1 (\rho_3 + \zeta_3) &= -i (N - V_2) \\
D_2 (\rho_1 + \zeta_1) &= i (M - V_3) \\
D_2 (\rho_3 + \zeta_3) &= iL' \\
D_3 (\rho_1 + \zeta_1) &= i (N + V_2) \\
D_3 (\rho_3 + \zeta_3) &= i (V_0 - V_1 - K') \\
D_4 (\rho_1 + \zeta_1) &= iL' \\
D_4 (\rho_3 + \zeta_3) &= -i (M - V_3)
\end{align*}
\]

The twelve linear combinations on the right hand sides of these have the transformation laws:

\[
\begin{align*}
D_1 (M - U_2) &= \zeta_2' - \rho_2' \\
D_3 (M - U_2) &= 2\beta_3' + \zeta_4' \\
D_1 (M + U_2) &= 2\beta_1 - \zeta_2' \\
D_3 (M + U_2) &= \rho_4' - \zeta_4' \\
D_1 (N - U_3) &= \zeta_4 - \rho_4' \\
D_3 (N - U_3) &= \rho_1' - \zeta_1'
\end{align*}
\]
This is a total of 48 equations. Half of these uncover eight new linear combinations of fermions, these equations are:

\[
\begin{align*}
D_1 (U_3 + N) &= 2\beta_4 - \zeta'_3 \\
D_3 (U_3 + N) &= 2\beta_2 + \zeta'_1 \\
D_1 (M + V_3) &= -\zeta'_2 - \rho'_2 \\
D_3 (M + V_3) &= \zeta'_4 + \rho'_4 \\
D_1 (M - V_3) &= 2\beta_1 + \zeta'_2 \\
D_3 (M - V_3) &= 2\beta_3 - \zeta'_4 \\
D_1 (V_2 + N) &= 2\beta_4 + \zeta'_3 \\
D_3 (V_2 + N) &= \zeta'_1 + \rho'_1 \\
D_1 (N - V_2) &= -\zeta'_3 - \rho'_3 \\
D_3 (N - V_2) &= 2\beta_2 - \zeta'_1 \\
D_1 (-K' + V_0 + V_1) &= \zeta'_1 + \rho'_1 \\
D_3 (-K' + V_0 + V_1) &= -2\beta_4 - \zeta'_3 \\
D_1 (-K' + V_0 - V_1) &= 2\beta_2 - \zeta'_1 \\
D_3 (-K' + V_0 - V_1) &= \zeta'_4 + \rho'_4 \\
D_1 (L' + U_0 + U_1) &= \rho'_4 - \zeta'_4 \\
D_3 (L' + U_0 + U_1) &= \zeta'_2 - 2\beta_1 \\
D_1 (L' + U_0 - U_1) &= 2\beta_3 + \zeta'_4 \\
D_3 (L' + U_0 - U_1) &= \rho'_2 - \zeta'_2 \\
D_2 (-K' + V_0 + V_1) &= -2\beta_1 - \zeta'_2 \\
D_4 (-K' + V_0 + V_1) &= \zeta'_4 + \rho'_4 \\
D_2 (-K' + V_0 - V_1) &= \zeta'_2 + \rho'_2 \\
D_4 (-K' + V_0 - V_1) &= 2\beta_3 - \zeta'_4 \\
D_2 (L' + U_0 + U_1) &= \zeta'_3 + 2\beta_4 - \zeta'_3 \\
D_4 (L' + U_0 + U_1) &= \zeta'_1 - \rho'_1 \\
D_1 (L' + U_0 - U_1) &= 2\beta_3 - \zeta'_4 \\
D_4 (L' + U_0 - U_1) &= -2\beta_2 - \zeta'_1 \\
D_1 (L + V_3 - U_3) &= -2\beta_3 - \zeta'_1 \\
D_2 (L + V_3 - U_3) &= \zeta'_2 - 2\beta_1 \\
D_3 (L + V_3 - U_3) &= \zeta'_4 + \rho'_4 \\
D_4 (L + V_3 - U_3) &= 2\beta_3 - \zeta'_4 \\
\end{align*}
\]
The $D_a$ transformations of these eight new fermionic fields are:

\[
\begin{align*}
D_1 (2\beta_1 + \zeta'_2) &= i M' - i V'_2 \\
D_3 (2\beta_1 + \zeta'_2) &= -3i L'' - 2i U'_0 \\
D_1 (2\beta_1 - \zeta'_2) &= i M' + i U'_2 \\
D_3 (2\beta_1 - \zeta'_2) &= -i L'' - i U'_0 - i U'_1 \\
D_1 (2\beta_2 + \zeta'_1) &= 2i V'_0 - 3i K'' \\
D_3 (2\beta_2 + \zeta'_1) &= i U'_3 + i N' \\
D_1 (2\beta_2 - \zeta'_1) &= -i K'' + i V'_0 - i V'_1 \\
D_3 (2\beta_2 - \zeta'_1) &= i N' - i V'_2 \\
D_1 (2\beta_3 + \zeta'_4) &= i L'' + i U'_0 - i U'_1 \\
D_3 (2\beta_3 + \zeta'_4) &= i M' - i U'_2 \\
D_1 (2\beta_3 - \zeta'_4) &= 3i L'' + 2i U'_0 \\
D_3 (2\beta_3 - \zeta'_4) &= i M' - i V'_3 \\
D_1 (2\beta_4 + \zeta'_3) &= i V'_2 + i N' \\
D_3 (2\beta_4 + \zeta'_3) &= i K'' - i V'_0 - i V'_1 \\
D_1 (2\beta_4 - \zeta'_3) &= i U'_3 + i N' \\
D_3 (2\beta_4 - \zeta'_3) &= 3i K'' - 2i U'_0 \\
\end{align*}
\]

Only eight of these 24 equations contain on their right hand side one of the two new linear field combinations. These eight equations are:

\[
\begin{align*}
D_3 (2\beta_1 + \zeta'_2) &= -3i L'' - 2i U'_0 \\
D_1 (2\beta_2 + \zeta'_1) &= 2i V'_0 - 3i K'' \\
D_4 (2\beta_2 + \zeta'_1) &= -3i L'' - 2i U'_0 \\
D_2 (2\beta_4 + \zeta'_3) &= 2i V'_0 - 3i K'' \\
D_1 (2\beta_4 - \zeta'_3) &= 3i K'' - 2i U'_0 \\
\end{align*}
\]

\[\text{B.2 From the CLM superspace Lagrangian to the CLM bosonic space-time Lagrangian}\]

In this section, we show that the zero-brane-reduced bosonic space-time Lagrangian

\[
\mathcal{L}^{(0)} = \frac{1}{2} K^2 + \frac{1}{2} L^2 - \frac{1}{2} M^2 - \frac{1}{2} N^2 + \frac{1}{2} U^{\mu} U^{\mu} + \frac{1}{2} V^{\mu} V^{\mu} + i\left(\xi_1 \zeta_1' + \zeta_2 \zeta_2' + \zeta_3 \zeta_3' + \zeta_4 \zeta_4'\right) + i(\rho_2 \beta_1 - \rho_1 \beta_2 + \rho_3 \beta_4 - \rho_4 \beta_3)
\]

(4.4)
is recovered via proper application of four zero-brane reduced supercovariant derivatives on the full superspace Lagrangian:

\[ D_1 D_2 D_3 D_4 \mathcal{L}_{\text{Superspace}} = D_1 D_2 D_3 D_4 \mathcal{L}_{\text{Superspace}} \]
\[ = - \frac{1}{8} D_1 D_2 D_3 D_4 (K^2 + L^2) \]  
\[ = \mathcal{L}^{(0)} + \text{total derivatives}. \]  

We calculate:

\[ -4D_1 D_2 D_3 D_4 \mathcal{L}_{\text{Superspace}} = (D_2 D_3 K)(D_1 D_4 K) - (D_1 D_3 K)(D_2 D_4 K) + \]
\[ + (D_1 D_2 K)(D_3 D_4 K) + KD_1 D_2 D_3 D_4 K + \]
\[ + (D_2 D_3 L)(D_1 D_4 L) - (D_1 D_3 L)(D_2 D_4 L) + \]
\[ + (D_1 D_2 L)(D_3 D_4 L) + LD_1 D_2 D_3 D_4 L + \]
\[ + (D_1 D_2 D_3 K)D_4 K + (D_3 K)(D_1 D_2 D_4 K) + \]
\[ - (D_2 K)(D_1 D_3 D_4 K) + (D_1 K)(D_2 D_3 D_4 K) + \]
\[ + (D_1 D_2 D_3 L)D_4 L + (D_3 L)(D_1 D_2 D_4 L) + \]
\[ - (D_2 L)(D_1 D_3 D_4 L) + (D_1 L)(D_2 D_3 D_4 L) \]  

We break the work up by first calculating the first eight bosonic terms of eq. (B.3). We have for the bosonic \( K \) terms

\[ (D_2 D_3 K)(D_1 D_4 K) = D_2 (\rho_3 - \zeta_3) D_1 (\rho_4 - \zeta_4) \]
\[ = -U_1^2 + U_0^2 + L^2 + 2U_0 L' \]  
\[ (D_1 D_3 K)(D_2 D_4 K) = D_1 (\rho_3 - \zeta_3) D_2 (\rho_4 - \zeta_4) \]
\[ = U_3^2 - N^2 \]  
\[ (D_1 D_2 K)(D_3 D_4 K) = D_1 (\rho_2 - \zeta_2) D_3 (\rho_4 - \zeta_4) \]
\[ = M^2 - U_2^2 \]  
\[ KD_1 D_2 D_3 D_4 K = KD_1 D_2 D_3 (\rho_4 - \zeta_4) \]
\[ = iKD_1 D_2 (M + U_2) \]
\[ = iKD_1 (2\beta_2 + \zeta_1) \]
\[ = K(-2V_0' + 3K'') \]
\[ = -3K'^2 + 2K'V_0 + \text{total derivatives} \]
In the last line here, we have used integration by parts. We have for the bosonic $L$ terms

\[
(D_2 D_3 L)(D_1 D_4 L) = D_2 (-\rho_2 - \zeta_2) D_1 (\rho_1 + \zeta_1) \\
= V_0^2 - 2 V_0 K' + K'^2 - V_1^2 \tag{B.8}
\]

\[
(D_1 D_3 L)(D_2 D_4 L) = -D_1 (\rho_2 + \zeta_2) D_2 (\rho_1 + \zeta_1) \\
= -M^2 + V_3^2 \tag{B.9}
\]

\[
(D_1 D_2 L)(D_3 D_4 L) = D_1 (\rho_3 + \zeta_3) D_3 (\rho_1 + \zeta_1) \\
= N^2 - V_2^2 \tag{B.10}
\]

\[
LD_1 D_2 D_3 D_4 L = LD_1 D_2 D_3 (\rho_1 + \zeta_1) \\
= i LD_1 D_2 (N + V_2) \\
= i LD_1 (-2 \beta_3 + \zeta_4') \\
= 2 L U_0' + 3 L L'' + \text{total derivatives} \tag{B.11}
\]

In the last line we have again used integration by parts. Putting it all together we have

\[
D_1 D_2 D_3 D_4 \mathcal{L}^{\text{bosons}}_{\text{Superspace}} = -\frac{1}{4} \left( -U_1^2 + U_0^2 + L^2 + 2 U_0 L' - U_3^2 + N^2 + \\
+ M^2 - U_2^2 - 3 K'^2 + 2 K' V_0 + V_0^2 - 2 V_0 K' + \\
+ K'^2 - V_1^2 + M^2 - V_3^2 + N^2 - V_2^2 - 2 L' U_0 - 3 L'^2 \right) \tag{B.12}
\]

+ total derivatives

\[
= \frac{1}{2} L^2 + \frac{1}{2} K'^2 - \frac{1}{2} M^2 - \frac{1}{2} N^2 + \frac{1}{4} U_{\mu} U_{\mu} + \frac{1}{4} V_{\mu} V_{\mu} + \\
+ \text{total derivatives}
\]

which is the bosonic part of the Lagrangian (4.4).

Next, we compute the eight fermionic parts of eq. (B.3)

\[
(D_1 D_2 D_3 K) D_4 K = (D_1 D_2 (\rho_3 - \zeta_3)) (\rho_4 - \zeta_4) \\
= -i (D_1 (U_0 - U_1 + L')) (\rho_4 - \zeta_4) \\
= i (2 \beta_3 + \zeta_4') (-\rho_4 + \zeta_4) \\
= i (2 \beta_4 - 2 \zeta_3 \beta_3 + \rho_4 \zeta_4' - \zeta_4 \zeta_4') \tag{B.13}
\]

\[
(D_3 K) (D_1 D_2 D_4 K) = (\rho_3 - \zeta_3) D_1 D_2 (\rho_4 - \zeta_4) \\
= -i (\rho_3 - \zeta_3) D_1 (N + U_3) \\
= i (\rho_3 - \zeta_3) (-2 \beta_4 + \zeta_4') \\
= i (-2 \rho_3 \beta_4 + 2 \zeta_3 \beta_4 + \rho_3 \zeta_4' - \zeta_3 \zeta_4') \tag{B.14}
\]
\[(D_2 K)(D_1 D_3 D_4 K) = (\rho_2 - \zeta_2)(D_1 D_3 (\rho_1 - \zeta_1))
= i(\rho_2 - \zeta_2)D_1(M + U_2)
= i(\rho_2 - \zeta_2)(2\beta_1 - \zeta_2')
= i(2\rho_2\beta_1 - 2\zeta_2\beta_1 - \rho_2\zeta_2' + \zeta_2\zeta_2') \quad \text{(B.15)}\]

\[(D_1 K)(D_2 D_3 D_4 K) = (\rho_1 - \zeta_1)D_2 D_3 (\rho_1 - \zeta_4)
= i(\rho_1 - \zeta_1)D_2(M + U_2)
= i(\rho_1 - \zeta_1)(2\beta_2 + \zeta_1')
= i(2\rho_1\beta_2 - 2\zeta_1\beta_2 + \rho_1\zeta_1' - \zeta_1\zeta_1') \quad \text{(B.16)}\]

\[(D_1 D_2 D_3 L)(D_4 L) = -(D_1 D_2 (\rho_2 + \zeta_2))(\rho_1 + \zeta_1)
= i(D_1(V_0 - V_1 - K'))(-\rho_1 - \zeta_1)
= i(2\beta_2 - \zeta_1')(-\rho_1 - \zeta_1)
= i(2\rho_1\beta_2 - 2\zeta_1\beta_2 - \rho_1\zeta_1' - \zeta_1\zeta_1') \quad \text{(B.17)}\]

\[(D_3 L)(D_1 D_2 D_4 L) = (\rho_2 - \zeta_2)(D_1 D_2 (\rho_1 + \zeta_1))
= i(-\rho_2 - \zeta_2)D_1(M - V_3)
= i(-\rho_2 - \zeta_2)(2\beta_1 + \zeta_2')
= i(-2\rho_2\beta_1 - 2\zeta_2\beta_1 - \rho_2\zeta_2' - \zeta_2\zeta_2') \quad \text{(B.18)}\]

\[(D_2 L)(D_1 D_3 D_4 L) = (\rho_3 + \zeta_3)D_1 D_3 (\rho_1 + \zeta_1)
= i(\rho_3 + \zeta_3)D_1(N + V_2)
= i(\rho_3 + \zeta_3)(2\beta_4 + \zeta_3')
= i(2\rho_3\beta_4 + 2\zeta_3\beta_4 + \rho_3\zeta_3' + \zeta_3\zeta_3') \quad \text{(B.19)}\]

\[(D_1 L)(D_2 D_3 D_4 L) = (-\rho_4 - \zeta_4)D_2 D_3 (\rho_1 + \zeta_1)
= i(-\rho_4 - \zeta_4)D_2(N + V_2)
= i(\rho_4 + \zeta_4)(2\beta_3 - \zeta_4')
= i(2\rho_4\beta_3 + 2\zeta_4\beta_3 - \rho_4\zeta_4' - \zeta_4\zeta_4') \quad \text{(B.20)}\]

All together now we have for the fermions:

\[
D_1 D_2 D_3 D_4 L_{\text{Superspace}}^{\text{fermions}} = -i\frac{1}{2}(2\rho_1\beta_3 - 2\zeta_4\beta_3 + \rho_4\zeta_4' - \zeta_4\zeta_4' - 2\rho_3\beta_4 + 2\zeta_4\beta_4 + \rho_4\zeta_4' - \zeta_4\zeta_4' + \\
- 2\rho_2\beta_1 + 2\rho_2\beta_1 + 2\rho_2\beta_1 + 2\rho_2\beta_1 - 2\zeta_2\beta_2 + 2\rho_1\beta_2 - 2\zeta_1\beta_2 + \rho_1\zeta_1' - \zeta_1\zeta_1' + \\
+ 2\rho_1\beta_2 + 2\zeta_1\beta_2 - \rho_1\zeta_1' - \zeta_1\zeta_1' - 2\rho_2\beta_1 - 2\zeta_2\beta_1 - \rho_2\zeta_2' - \zeta_2\zeta_2' + \\
- 2\rho_3\beta_4 - 2\zeta_4\beta_4 - \rho_3\zeta_4' - \zeta_4\zeta_4' + 2\rho_4\beta_3 + 2\zeta_4\beta_3 - \rho_4\zeta_4' - \zeta_4\zeta_4')
= -i\frac{1}{2}(-\zeta_1\zeta_1' - \zeta_2\zeta_2' - \zeta_3\zeta_3' - \zeta_4\zeta_4' + \\
- i(\rho_1\beta_2 - \rho_2\beta_1 + \rho_1\beta_3 - \rho_3\beta_4) \\
i\frac{1}{2}(-\zeta_1\zeta_1' + \zeta_2\zeta_2' + \zeta_3\zeta_3' + \zeta_4\zeta_4') + \\
i(\rho_2\beta_1 - \rho_1\beta_2 + \rho_3\beta_4 - \rho_4\beta_3) \quad \text{(B.21)}
\]
which is indeed the fermionic part of the Lagrangian (4.4). Putting the bosons and fermions together, we conclude that
\[ D_1D_2D_3D_4 \mathcal{L}_{\text{Superspace}} = \mathcal{L}^{(0)} + \text{total derivatives.} \quad (B.22) \]

\section{C Multiplication rules for the $a_I$ and $h_\Delta$ matrices}

The matrices denoted by $a_I$ and $h_\Delta$ defined eqs. (4.36) and (4.37) form a representation of the $\text{sl}(3,\mathbb{R})$ algebra (i.e. traceless linear three by three matrices with real entries). With our definitions, the multiplication of these is given by

\begin{align*}
a_1a_1 &= -\frac{2}{3}I_3 + \frac{1}{2}h_4 + \frac{1}{6}h_5, & a_1a_2 &= \frac{1}{2}a_3 + \frac{1}{2}h_3, \\
a_2a_2 &= -\frac{2}{3}I_3 - \frac{1}{2}h_4 + \frac{1}{6}h_5, & a_2a_3 &= \frac{1}{2}a_1 + \frac{1}{2}h_1, \\
a_3a_3 &= \frac{2}{3}I_3 - \frac{1}{2}h_5, & a_3a_1 &= \frac{1}{2}a_2 + \frac{1}{2}h_2, \\
h_1h_1 &= \frac{2}{3}I_3 - \frac{1}{2}h_4 - \frac{1}{6}h_5, & h_1h_2 &= \frac{1}{2}a_3 + \frac{1}{2}h_3, \\
h_1h_3 &= -\frac{1}{2}a_2 + \frac{1}{2}h_2, & h_1h_4 &= -\frac{1}{2}a_1 - \frac{1}{2}h_1, \\
h_1h_5 &= \frac{3}{2}a_1 - \frac{1}{2}h_1, \\
h_2h_2 &= \frac{2}{3}I_3 + \frac{1}{2}h_4 - \frac{1}{6}h_5, & h_2h_3 &= \frac{1}{2}a_1 + h_1, \\
h_2h_4 &= -\frac{1}{2}a_2 + \frac{1}{2}h_2, & h_2h_5 &= -\frac{3}{2}a_2 - \frac{1}{2}h_2, \\
h_3h_3 &= \frac{2}{3}I_3 + \frac{1}{3}h_5, & h_3h_1 &= a_3, \\
h_3h_5 &= h_3, & h_3h_4 &= \frac{2}{3}I_3 + \frac{1}{3}h_5, \\
h_4h_5 &= h_4, & h_5h_5 &= 2I_3 - h_5. \quad (C.2)
\end{align*}

\begin{align*}
a_1h_1 &= \frac{1}{2}h_4 - \frac{1}{2}h_5, & a_1h_3 &= -\frac{1}{2}a_2 + \frac{1}{2}h_2, \\
a_2h_1 &= -\frac{2}{3}a_3 + \frac{1}{2}h_3, & a_2h_3 &= -\frac{1}{2}a_1 - \frac{1}{2}h_1, \\
a_3h_1 &= -\frac{1}{2}a_2 - \frac{1}{2}h_2, & a_3h_3 &= -h_4, \\
a_1h_2 &= -\frac{1}{2}a_3 - \frac{1}{2}h_3, & a_1h_4 &= -\frac{1}{2}a_1 - \frac{1}{2}h_1, \\
a_2h_2 &= \frac{1}{2}h_4 + \frac{1}{2}h_5, & a_2h_4 &= \frac{1}{2}a_2 - \frac{1}{2}h_2, \\
a_3h_2 &= -\frac{1}{2}a_1 + \frac{1}{2}h_1, & a_3h_4 &= h_3, \\
a_1h_5 &= -\frac{1}{2}a_1 + \frac{3}{2}h_1, & a_2h_5 &= -\frac{1}{2}a_2 - \frac{3}{2}h_2, \\
a_3h_5 &= a_3. \quad (C.3)
\end{align*}

Using the facts that under the matrix transposition operator we find
\[ (a_I)^t = -a_I, \quad (h_\Delta)^t = +h_\Delta \quad (C.4) \]
the equations in (C.1)–(C.3) can easily be used to derive the commutator algebra of all these matrices. Some of the simplest examples of this process are illustrated below.

\begin{align*}
a_3h_3 &= -h_4 \quad \rightarrow \quad (a_3h_3)^t = -(h_3)^t \quad \rightarrow \quad (h_3)^t(a_3)^t = -h_4 \quad \rightarrow \quad h_3a_3 = h_4 \quad \rightarrow \quad [a_3, h_3] = -2h_4 \quad (C.5)
\end{align*}
\[ a_3 h_4 = h_3 \rightarrow (a_3 h_4)^t = (h_3)^t \rightarrow \]
\[ (h_4)^t (a_3)^t = h_3 \rightarrow -h_4 a_3 = h_3 \rightarrow \]
\[ [a_3, h_4] = 2h_3 \quad \text{(C.6)} \]

References

[1] S.J. Gates Jr. et al., 4D, \( N = 1 \) Supersymmetry Genomics (I), JHEP 12 (2009) 008 [arXiv:0902.3830] [nSPIRE].

[2] M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory. Vol. 2, Cambridge University Press, Cambridge U.K. (1987).

[3] J. Polchinski, String Theory. Vol. II: Superstring Theory and Beyond, Cambridge University Press, Cambridge U.K. (1998).

[4] J.M. Maldacena, The large-\( N \) limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1133] [hep-th/9711200] [nSPIRE].

[5] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, Large-\( N \) field theories, string theory and gravity, Phys. Rept. 323 (2000) 183 [hep-th/9905111] [nSPIRE].

[6] W. Siegel and M. Roček, On off-shell supermultiplets, Phys. Lett. B 105 (1981) 275 [nSPIRE].

[7] S.J. Gates Jr. et al., A detailed Investigation of First and Second Order Supersymmetries for Off-Shell \( N = 2 \) and \( N = 4 \) Supermultiplets, arXiv:1106.5475 [nSPIRE].

[8] S.J. Gates Jr. and L. Rana, A theory of spinning particles for large-\( N \) extended supersymmetry. 2., Phys. Lett. B 369 (1996) 262 [hep-th/9510151] [nSPIRE].

[9] E. Bergshoeff, M. de Roo and B. de Wit, Extended conformal supergravity, Nucl. Phys. B 182 (1981) 173 [nSPIRE].

[10] M. Faux and S.J. Gates Jr., Adinkras: A graphical technology for supersymmetric representation theory, Phys. Rev. D 71 (2005) 065002 [hep-th/0408004] [nSPIRE].

[11] Y.X. Zhang, Adinkras for mathematicians, arXiv:1111.6055 [nSPIRE].

[12] M. Faux, K. Iga and G. Landweber, Dimensional enhancement via supersymmetry, Adv. Math. Phys. 2011 (2011) 259089 [arXiv:0907.3605] [nSPIRE].

[13] M.G. Faux and G.D. Landweber, Spin holography via dimensional enhancement, Phys. Lett. B 681 (2009) 161 [arXiv:0907.4543] [nSPIRE].

[14] C. Doran et al., Adinkras and the Dynamics of Superspace Prepotentials, hep-th/0605269 [nSPIRE].

[15] C. Doran et al., Relating Doubly-Even Error-Correcting Codes, Graphs and Irreducible Representations of \( N \)-Extended Supersymmetry, arXiv:0806.0051 [nSPIRE].

[16] S.J. Gates Jr. and T. Hubsch, On Dimensional Extension of Supersymmetry: From Worldlines to Worldsheets, arXiv:1104.0722 [nSPIRE].

[17] H. Georgi, Lie Algebras in Particle Physics, second edition, Perseus Books, New York U.S.A. (1999).

[18] J.R. Silvester, Determinants of Block Matrices, Math. Gaz. 84 (2000) 460.

[19] D.Z. Freedman, Gauge Theories of Antisymmetric Tensor Fields, Caltech. preprint calt-68-624 (1977), unpublished.