Contrasting formulations of cosmological perturbations in a magnetic FLRW cosmology

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Abstract
In this paper we contrasted two cosmological perturbation theory formalisms, the $1 + 3$ covariant gauge invariant and the gauge invariant, by comparing the gauge invariant variables associated with the magnetic field defined in each approach. In the first part we give an introduction to each formalism assuming the presence of a magnetic field. We found that gauge invariant quantities defined by the $1 + 3$ covariant approach are related to spatial variations of the magnetic field (defined in the gauge invariant formalism) between two closed fundamental observers. This relation was computed by choosing the comoving gauge in the gauge invariant approach in a magnetized universe. Furthermore, we have derived the gauge transformations for electromagnetic potentials in the gauge invariant approach, and the Maxwell equations have been written in terms of these potentials.

Keywords: cosmological perturbation theory, magnetic fields, cosmology

1. Introduction

Cosmological perturbation theory has become a standard tool in modern cosmology to understand the formation of the large-scale structure in the universe, and also to calculate the fluctuations in the cosmic microwave background (CMB) \cite{1}. The first treatment of perturbation theory within general relativity was developed by Lifshitz \cite{2}, where the evolution of
structures in a perturbed Friedmann–Lemaître–Robertson–Walker universe (FLRW) under synchronous gauge was addressed. Later, the covariant approach of perturbation theory was formulated by Hawking [3] followed by Olson [4], where curvature perturbation was worked on rather than metric perturbation. Then, based on early works by Gerlach and Sengupta [5], Bardeen [6] introduced a full gauge invariant approach to first order in cosmological perturbation theory. In his work, he built a set of gauge invariant quantities related to density perturbations commonly known as Bardeen potentials (see also the extensive review by Kodama and Sasaki [7]).

However, alternative representations of previous formalisms were appearing due to the gauge problem [8]. This issue arises in cosmological perturbation theory due to the fact that splitting all metric and matter variables into a homogeneous and isotropic space-time plus small deviations of the background is not unique. Basically, perturbations in any quantity are defined by choosing a correspondence between a fiducial background space-time and the physical universe. However, given the general covariance in perturbation theory, which states that there is no preferred correspondence between these space-times, a freedom in the method of identifying points between two manifolds appears [9]. This arbitrariness generates a residual degree of freedom, which would imply that variables might not have a physical interpretation.

Following the research mentioned above, two main formalisms have been developed for studying the evolution of matter variables and to deal with the gauge problem, that will be reviewed in this paper. The first is known as the $1 + 3$ covariant gauge invariant formalism and was presented by Ellis and Bruni [10]. This approach is based on earlier works of Hawking and of Stewart and Walker [11]. The idea is to define variables covariantly such that they vanish in the background; therefore, they can be considered as gauge invariant under gauge transformation according to the Stewart–Walker lemma [12]. In the $1 + 3$ covariant gauge invariant, gauge invariant variables manage the gauge ambiguities and acquire a physical interpretation. Since the covariant variables do not assume linearization, exact equations are found for their evolution. The second approach considers arbitrary order perturbations in a geometrical perspective; it has been deeply discussed by Bardeen [6], Kodama and Sasaki [7], Mukhanov, Feldman and Brandenberger [13] and Bruni [14], and it is known as the gauge invariant approach. Here, perturbations are decomposed into the so-called scalar, vector and tensor parts and the gauge invariant ones are found with the gauge transformations and using the Stewart–Walker lemma. The gauge transformations are generated by arbitrary vector fields, defined on the background space-time and associated with a one-parameter family of diffeomorphisms. This approach allows us to find the conditions for the gauge invariance of any tensor field, although at high order it sometimes appears unclear. As an alternative description of the latter approach, it is important to comment on the work done by Nakamura [15], where he splits the metric perturbations into a gauge invariant and a gauge variant part, and thus evolution equations are written in terms of gauge invariant quantities.

Given the importance and advantage of these two approaches is necessary to find equivalences between them. Some authors have compared different formalisms, for example [16] discussed the invariant quantities found by Bardeen with the ones built on the $1 + 3$ covariant gauge invariant in a specific coordinate system; moreover, the authors of [17] found a way to reformulate the Bardeen approach in a covariant scenario and the authors of [18] contrasted the nonlinear approach described by Malik et al [19] with the Nakamura approach.

The purpose of this paper is to present a way to contrast the approaches mentioned above. With this aim, we follow the methodology used by [16] and [20], where a comparison of

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4 The only restriction is that perturbation be small with respect to its value in the background; even so, this does not help to specify the map in a unique way.
gauge invariant quantities built in each approach is made. However, we address the treatment in the cosmological magnetic field context, where cosmological perturbation theory has played an important role for explaining the origin of magnetic fields in galaxies and clusters from a weak cosmological magnetic field originated before the recombination era. This means that magnetic fields can leave imprints of their influence on the evolution of the universe, whether in nucleosynthesis or CMB anisotropies [21–23]. In fact, the study of primordial magnetic fields will offer a qualitative window on the very early universe [24]. Cosmological perturbation models permeated by a large-scale primordial magnetic field have been widely worked on by Tsagas [25–27] and Ellis [28], who found the complete equation system which shows a direct coupling between the Maxwell and the Einstein fields, and also gauge invariance for magnetic fields was built into the frame of the 1 + 3 covariant approach. Furthermore, in previous works, we have obtained a set of equations which describe the evolution of cosmological magnetic fields up to second order in the gauge invariant approach, with the respective gauge transformations for the fields, important for building the gauge invariant magnetic variables [29]. Therefore, studying in detail the magnetic gauge invariant quantities in each of the formalisms, we can find equivalences between them. In addition, we have built the invariant gauge for the electromagnetic four-potentials and the Maxwell equations are written in terms of these potentials.

The outline of the paper is as follows. In sections 2 and 3, the 1 + 3 covariant and gauge invariant formalisms are reviewed and the key gauge invariant variables are defined. In section 4, we introduce the electromagnetic four-potentials in perturbation theory using the gauge invariant formalism; also the gauge transformations are deduced and the Maxwell equations are written here in terms of the potentials. Section 5 shows the equivalence between the 1 + 3 covariant and gauge invariant formalisms, studying in detail the invariant gauge quantities and discussing the physical meaning of these variables. The last section is devoted to a discussion of the main results.

We use Greek indices $\mu, \nu, ...$ for space-time coordinates and Roman indices $i, j, ...$ for purely spatial coordinates. We also adopt units where the speed of light $c = 1$ and a metric signature $(-, +, +, +)$.

2. The 1 + 3 covariant approach: preliminaries

We first briefly review the covariant formalism of Ellis and Bruni [10] and its extension with magnetic field described by Tsagas and Barrow [26, 30]. The average motion of matter in the universe defines a future-directed timelike four-velocity $u^\alpha$, corresponding to a fundamental observer ($u_\mu u^\mu = -1$), and generates a unique splitting of space-time into the tangent three-spaces orthogonal to $u_\alpha$. The second order rank symmetric tensor $h_{\alpha\beta}$ written as

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$$

is the projector tensor which defines the spatial part of the local rest frame of the fundamental observers ($h_\alpha^\beta u_\beta = 0$). The proper time derivative along the fluid-flow lines and spatial derivative in the local rest frame for any tensorial quantity $T^{\alpha\beta...}_{\gamma\delta...}$ are given by

$$\dot{T}^{\alpha\beta...}_{\gamma\delta...} = u^\lambda \nabla_\lambda T^{\alpha\beta...}_{\gamma\delta...} \quad \text{and} \quad D_\lambda T^{\alpha\beta...}_{\gamma\delta...} = h_\lambda^\kappa h_\mu^\nu h_\rho^\sigma \nabla_\lambda T^{\alpha\beta\kappa\mu\rho\sigma...}$$

respectively. The operator $D_\lambda$ is the covariant derivative operator orthogonal to $u_\alpha$. The kinematic variables are introduced by splitting the covariant derivative of $u_\alpha$ into its spatial and temporal parts, thus
\[ \nabla_\alpha u_\beta = \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{\Theta}{3} h_{\alpha\beta} - a_\beta u_\alpha, \]  

where the variable \( a_\alpha = u^\beta \nabla_\beta u_\alpha \) is the acceleration (\( a_\alpha u^\alpha = 0 \)), \( \Theta = \nabla_\alpha u^\alpha \) is the volume expansion, \( \sigma_{\alpha\beta} = D_{(\alpha\beta)} - \frac{\Theta}{3} h_{\alpha\beta} \) is the shear (\( \sigma_{\alpha\beta} u^\alpha = 0 \), \( \sigma_{\alpha\beta} = \frac{1}{\Theta} h_{\alpha\beta} \)) and \( \omega_{\alpha\beta} = D_{(\alpha\beta)} \) is the vorticity (\( \omega_{\alpha\beta} u^\alpha = 0 \), \( \omega_{\alpha\beta} = \frac{1}{\theta} \)). Also, on using the totally antisymmetric Levi–Civita tensor, one defines the vorticity vector \( \omega^\alpha = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} h_{\beta\gamma} \). A length scale factor \( a \) is introduced along the fluid flow of \( u_\alpha \) by means of \( H = \frac{a}{a} = \frac{\Theta}{\theta} \), with \( H \) the local Hubble parameter. Now, we summarize some of the results of the covariant studies of electromagnetic fields. The Maxwell equations in their standard tensor form are written as

\[ \nabla_\alpha F_{\beta\gamma} = 0 \quad \text{and} \quad \nabla^\beta F_{\alpha\beta} = j_\alpha. \]  

These equations are covariantly characterized by the antisymmetric electromagnetic tensor \( F_{\alpha\beta} \), where \( j_\alpha \) is the four-current that sources the electromagnetic field \( [31] \). Using the four-velocity, the electromagnetic fields can be expressed as a four-vector electric field \( E_\alpha \) and magnetic field \( B_\alpha \) as

\[ E_\alpha = F_{\alpha\beta} u_\beta \quad \text{and} \quad B_\alpha = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F_{\beta\gamma} u^\delta. \]  

By definition, the electromagnetic four-vectors must be purely spatial and orthogonal to the four-velocity (\( E_\alpha u^\alpha = B_\alpha u^\alpha = 0 \)). We can write the electromagnetic tensor in terms of the electric and magnetic fields

\[ F_{\alpha\beta} = E_\alpha E_\beta - E_\beta E_\alpha + B_\gamma \epsilon_{\alpha\beta\gamma\delta} u^\delta. \]  

The electromagnetic tensor determines the energy–momentum tensor of the field, which is given by

\[ T^{(EM)}_{\alpha\beta} = -F_{\alpha\gamma} F^\gamma_\beta - \frac{1}{4} \epsilon_{\alpha\beta\gamma\delta} F_{\gamma\delta} F^{\gamma\delta}. \]  

Using the four-vector \( u_\alpha \) and the projection tensor \( h_{\alpha\beta} \), one can decompose the Maxwell equations (4) into a timelike and a spacelike component, obtaining the following set of equations [30]:

\[ h_{\alpha\gamma} E^\beta = \left( \sigma_{\alpha\beta} + \omega_{\alpha\beta} - \frac{2}{3} \Theta \delta_{\alpha\beta} \right) E^\beta + \epsilon^{\alpha\beta\gamma\delta} B_\delta u_\gamma + \text{curl } B^\alpha - J^\alpha, \]  

\[ h_{\alpha\gamma} B^\beta = \left( \sigma_{\alpha\beta} + \omega_{\alpha\beta} - \frac{2}{3} \Theta \delta_{\alpha\beta} \right) B^\beta - \epsilon^{\alpha\beta\gamma\delta} E_\delta u_\gamma - \text{curl } E^\alpha, \]  

\[ D^\alpha E_\alpha = \rho - 2 \omega^\alpha B_\alpha, \]  

\[ D^\alpha B_\alpha = 2 \omega^\alpha E_\alpha. \]

When the curl operator is defined as \( \text{curl } E^\alpha = \epsilon^{\beta\gamma\delta} u_\beta \nabla_\gamma E_\delta \) and the four-current \( J_\alpha \) splits along and orthogonal to \( u^\alpha \) [26], then

\[ \rho = -J_\alpha u^\alpha \quad \text{and} \quad J_\beta = h_{\alpha\beta} J_\alpha \quad \text{with} \quad J_\alpha u^\alpha = 0. \]  

Finally, using the antisymmetric electromagnetic tensor together with Maxwell’s equations (4), one arrives at a covariant form of the charge density conservation law

\[ \dot{\rho} = -\Theta \rho - D^\alpha J_\alpha - \omega^\alpha J_\alpha. \]  

In this approach, Ellis and Bruni [10] built gauge invariant quantities associated with the orthogonal spatial gradients of the energy density \( \mu \), pressure \( P \) and fluid expansion \( \Theta \).
Assuming that the unperturbed background universe is represented by an FLRW metric, the following basic variables are considered:

\[ X_\alpha = \kappa h^\beta_\alpha \nabla_\beta \mu, \quad Y_\alpha = \kappa h^\beta_\alpha \nabla_\beta P \quad \text{and} \quad Z_\alpha = \kappa h^\beta_\alpha \nabla_\beta \Theta, \]

where \( \kappa = 8\pi G \). In fact, the variables such as pressure or energy density are usually nonzero in the FLRW background and so are not gauge invariant. However, the spatial projection of these variables defined in equation (11) vanishes in the background, and so they are gauge invariant and covariantly defined in the physical universe. It is also important to define quantities which are easier to measure, thus we define the fractional density gradient

\[ \lambda_\alpha = \frac{X_\alpha}{\kappa \mu} \quad \text{and} \quad \gamma_\alpha = \frac{Y_\alpha}{\kappa P}. \]

In the same way, one can define the gauge invariant for magnetic fields \( B_\alpha \) in a magnetized universe [32]. For instance, the comoving fractional magnetic energy density distributions and the magnetic field vector can be defined as follows:

\[ B_\alpha = D_\alpha B^2, \quad (13a) \]
\[ B = \frac{a^2}{B^2} D^\alpha B_\alpha, \quad (13b) \]
\[ M_{\alpha \beta} = a D_\alpha B_\beta. \]

with \( B^2 \) the local density of the magnetic field. As has been argued by Tsagas et al [26], they describe the spatial variation in the magnetic energy density and the spatial inhomogeneities in the distribution of the vector field \( B_\alpha \), as measured by a pair of neighboring fundamental observers (which represent the motion of typical observers in the Universe with the four-velocity its vector tangent) in a gauge invariant way. A further discussion of fundamental observers and the meaning of this gauge invariance with respect to these observers is given in section 6.3.1 of [27].

### 3. Gauge invariant approach

Let us begin by reviewing some general ideas about the gauge invariant approach. Following [14, 16], consider two Lorentzian manifolds \((M, g)\) and \((M_0, g_0)\), that represent the physical and the background space-times respectively. The perturbation of a tensor field \( T \) is defined as the difference between the values that the quantity takes in \( M \) and \( M_0 \), evaluated at points corresponding to the same physical event. To compare any quantity in the two space-times, a diffeomorphism \( \phi : M \rightarrow M_0 \) is defined which enables the identification of points between \( M \) and \( M_0 \). However, this identification map is completely arbitrary; this freedom arises in the cosmological perturbation theory and one may refer to it as gauge freedom of the second kind in order to distinguish it from the usual gauge freedom of general relativity [8]. Once an identification map \( \phi \) has been assigned, perturbations (living on \( M_0 \)) can be defined as

\[ \Delta^0 T \bigg|_{M_0} = \phi^* T - T_0, \]

with \( T_0 \) the background tensor field corresponding to \( T \) and \( \phi^* T \) the pullback which gives the representation of \( T \) over \( M_0 \). To define the perturbation to a given order, the fields are expanded in a Taylor power series and the above mentioned iteration scheme is then used. For this, consider a family of four-submanifold \( M_\lambda \) with \( \lambda \in \mathbb{R} \) embedded in a five-manifold \( \mathcal{N} = \mathcal{M} \times \mathbb{R} \) [9, 14]. Each submanifold in the family represents a perturbed space-time and
the background is represented when \( \lambda = 0 \) (namely \( \mathcal{M}_0 \)). In each submanifold, the Einstein and Maxwell equations must be fulfilled:

\[
E_{\left[ g_0, T_\lambda \right]} = 0 \quad \text{and} \quad M_{\left[ F_\lambda, \tilde{A}_\lambda \right]} = 0.
\]

To generalize the definition of perturbation given in equation (14), a one-parametric group of diffeomorphisms \( \mathcal{X}_\lambda \) is introduced in order to identify points of the background with the physical space-time labeled with \( \lambda \). Therefore, one obtains a way to define the perturbation for any tensor field:

\[
\Delta^\phi T \big|_{\mathcal{M}_0} = \mathcal{X}_\lambda^* T - T_0.
\]

The first term of equation (16) which lives on \( \mathcal{M}_0 \) admits an expansion around \( \lambda = 0 \) given by

\[
\mathcal{X}_\lambda^* T = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta_\lambda^{(k)} T = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathcal{L}_X T \big|_{\mathcal{M}_0} = \exp(\lambda \mathcal{L}_X) T \big|_{\mathcal{M}_0},
\]

where \( \mathcal{L}_X T \) is the Lie derivative of \( T \) along to the vector field \( X \) that generates the flow \( \mathcal{X} \), \( k \) refers to the expansion order and \( \delta_\lambda^{(k)} T \) represents the \( k \)th order perturbation of \( T \). If we choose another vector field (gauge choice) \( \mathcal{Y} \), the expansion of \( T \) is written as

\[
\mathcal{Y}_\lambda^* T = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \delta_\lambda^{(k)} T = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathcal{L}_Y T \big|_{\mathcal{M}_0}.
\]

At this point, it is useful to define fields on \( \mathcal{M} \) that are intrinsically gauge independent. We say that a quantity is gauge invariant if its value at any point of \( \mathcal{M} \) does not depend on the gauge choice, namely \( \mathcal{Y}_\lambda^* T = \mathcal{X}_\lambda^* T \). An alternative way to define a gauge invariant quantity at order \( n \geq 1 \) (see proposition 1 in [14]) is iff

\[
\mathcal{L}_\xi \delta_\lambda^{(k)} T = 0
\]

is satisfied. Here \( \xi \) is any vector field on \( \mathcal{M} \) and \( \forall k \leq n \). At first order (\( k = 1 \)) the Stewart–Walker lemma is found [12]. In cases where the tensor field is gauge dependent, it is useful to represent this tensor from a particular gauge \( \mathcal{X} \) in other \( \mathcal{Y} \). For this, the identification map \( \Phi \) on \( \mathcal{M}_0 \), \( \Phi_\lambda : \mathcal{M}_0 \to \mathcal{M}_0 \) is defined by

\[
\Phi_\lambda = \mathcal{X}_\lambda \circ \mathcal{Y}_\lambda \quad \text{that implies} \quad \mathcal{Y}_\lambda^* T = \Phi_\lambda^* \mathcal{X}_\lambda^* T.
\]

Therefore, \( \Phi \) induces a pullback which changes the representation \( X \) of \( T \) to the representation \( Y \) of \( T \). Now, to generalize equation (17) and using the Baker–Campbell–Haussdorf formula [38], the gauge transformation on \( \mathcal{M}_0 \) of \( T \) is

\[
\Phi_\lambda^* \mathcal{X}_\lambda^* T = \exp \left( \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \mathcal{L}_{\xi_1} \right) \mathcal{X}_\lambda^* T,
\]

with \( \xi_1 \) a vector field on \( \mathcal{M}_0 \). The relations to first and second order perturbations of \( T \) in two different gauge choices are found by substituting equations (17, 18) in equation (21), obtaining

\[
\delta_\mathcal{Y}^{(1)} T - \delta_\mathcal{X}^{(1)} T = \mathcal{L}_{\xi_1} T \big|_{\mathcal{M}_0},
\]

and

\[
\delta_\mathcal{Y}^{(2)} T - \delta_\mathcal{X}^{(2)} T = 2 \mathcal{L}_{\xi_1} \delta_\mathcal{X}^{(1)} T \big|_{\mathcal{M}_0} + \left( \mathcal{L}_2 \xi_1 + \mathcal{L}_{\xi_1} \right) T \big|_{\mathcal{M}_0}.
\]
where the generators of the gauge transformation $\Phi$ are
\[ \xi_1 = Y - X \quad \text{and} \quad \xi_2 = [X, Y]. \] (23)
This vector field can be split in the time and space parts
\[ \xi^{(k)}_{\mu} = \left( \alpha^{(k)}, \partial_t \beta^{(k)} + d_t^{(k)} \right), \] (24)
where $\alpha^{(k)}$ and $\beta^{(k)}$ are arbitrary scalar functions, and $\partial_t d_t^{(k)} = 0$. The function $\alpha^{(k)}$ determines the choice of constant time hypersurfaces, while $\beta^{(k)}$ and $d_t^{(k)}$ fix the spatial coordinates within the hypersurface.

3.1. Perturbations on a magnetized FLRW background

At zero order (background), the universe is well described by a spatially flat FLRW
\[ ds^2 = a^2(\tau) \left( -d\tau^2 + \delta_{ij} dx^i dx^j \right), \] (25)
with $a(\tau)$ the scale factor and $\tau$ the conformal time. The Einstein tensor components in this background are given by
\[ G^0_0 = -\frac{3}{a^2} H^2, \quad G^i_j = -\frac{1}{a^2} \left( 2 \frac{a''}{a} - H^2 \right) \delta^i_j, \] (26)
with $H = \frac{\dot{a}}{a}$ the Hubble parameter, and the prime denotes the derivative with respect to $\tau$. We consider the background filled with a single barotropic fluid where the energy–momentum tensor is
\[ T^\mu_{\nu(0)} = \left( \mu(0) + P(0) \right) u^\mu_{(0)} u^\nu_{(0)} + P(0) \delta^\mu_{\nu}, \] (27)
with $\mu(0)$ the energy density and $P(0)$ the pressure. The comoving observers are defined by the four-velocity $u^\mu = (a^{-1}, 0, 0, 0)$ with $u^\mu u_\mu = -1$ and the conservation law for the fluid yields
\[ \mu^\mu_{(0)} + 3H \left( \mu_{(0)} + P_{(0)} \right) = 0. \] (28)
We also allow the presence of a weak and spatially homogeneous large-scale magnetic field in our FLRW background with the property $B^{(0)}_{00} \ll \mu_{(0)}$. This field must be sufficiently random to satisfy $\langle B^{(0)}_0 \rangle = 0$ and $\langle B^{(2)}_{00} \rangle = 0$ to ensure that symmetries and the evolution of the background remain unaffected. Working under the MHD approximation in large scales, the plasma is globally neutral; this means that the charge density is neglected and the electric field with the current should be zero, thus the only nonzero magnetic variable in the background is $B^{(2)}_{00}$. The evolution of the energy density magnetic field is given by
\[ B^{(2)}_{00} = -4HB^{(0)}_{00}, \] (29)
showing $B^2 \sim a^{-4}$ in the background. Fixing the background, we consider the perturbations up to second order about this FLRW magnetized universe, so that the metric tensor is given by
\[ g_{00} = -a^2(\tau) \left( 1 + 2 \psi^{(1)} + \psi^{(2)} \right), \] (30)
\[ g_{0i} = a^2(\tau) \left( \omega^{(1)}_i + \frac{1}{2} \omega^{(2)}_i \right), \] (31)
\[ g_{ij} = a^2(\tau) \left[ (1 - 2\psi^{(1)} - \phi^{(2)})\delta_{ij} + \chi^{(1)}_{ij} + \frac{1}{2} \chi^{(2)}_{ij} \right] \]  

(32)

The perturbations are split into a scalar, a transverse vector part and a transverse trace-free tensor

\[ \omega^{(k)}_i = \partial_i \omega^{(k)} + \omega^{(k)\perp}_i, \]

with \( \partial_i \omega^{(k)\perp}_i = 0 \), and \( k = 1, 2 \) \cite{14}. Similarly we can split \( \chi^{(k)}_{ij} \) as

\[ \chi^{(k)}_{ij} = D_{ij} \chi^{(k)\parallel} + \partial_i \chi^{(k)\perp} + \partial_j \chi^{(k)\perp} + \chi^{(k)\top}, \]

(34)

for any tensor quantity. Keeping in mind that to zero order the variables depend only on \( \tau \), we expand the scalar variables such as the energy density of the matter and the magnetic field as

\[ \mu = \mu_0 + \mu_1 + \frac{1}{2} \mu_2, \]

(35)

\[ B^2 = B^2_0 + B^2_1 + \frac{1}{2} B^2_2, \]

(36)

and the vector variables such as magnetic and electric field and four-velocity among others as

\[ B^i = \frac{1}{a^2(\tau)} \left( B^i_0 + \frac{1}{2} B^i_1 \right), \]

(37)

\[ E^i = \frac{1}{a^2(\tau)} \left( E^i_0 + \frac{1}{2} E^i_1 \right), \]

(38)

\[ u^\mu = \frac{1}{a(\tau)} \left( \sigma^\mu_0 + \psi^\mu_1 + \frac{1}{2} \psi^\mu_2 \right). \]

(39)

Again, the four-velocity \( u^\mu \) is subject to the normalization condition \( u^\mu u_\mu = -1 \), and in any gauge it can be expressed as

\[ u^\mu = a \left[ -1 - \psi^{(1)} + \frac{1}{2} \psi^{(2)} + \frac{1}{2} \psi^{(1)} \psi^{(1)} - \psi^{(1)} v^{(1)}_i v^{(1)}_i, \right. \]

\[ \omega^{(1)}_i + v^{(1)}_i + \frac{1}{2} \left( \omega^{(2)}_i + v^{(2)}_i - \omega^{(1)}_i \psi^{(1)} + \psi^{(1)} \chi^{(1)}_i - 2 v^{(1)} \phi^{(1)} \right) \]

(40)

\[ u^\mu = \frac{1}{a} \left[ 1 - \psi^{(1)} + \frac{1}{2} \left( 3 \psi^{(1)} - \psi^{(2)} + \psi^{(1)} v^{(1)}_i + 2 \omega^{(1)}_i v^{(1)}_i, \right) v^{(1)}_i + \frac{1}{2} v^{(2)}_i \right]. \]

(41)

With the four-velocity one can also define the acceleration as

\[ a_{ij} = u^\nu \nabla_\nu u_\mu. \]

(42)

Using equation (22a), we can find the transformation of the metric and matter variables at first order,

\[ \gamma^{(1)} = \psi^{(1)} + \frac{1}{a} \left( a \alpha^{(1)} \right)^T, \]

(43a)

5 With \( \partial \chi^{(k)\top} = \partial \chi^{(k)\perp} = 0 \), \( \chi^{(k)\perp} = 0 \) and \( D_i \equiv \partial_i \partial_j - \frac{1}{3} \delta^{ij} \partial_k \partial^k \).
\[
\varphi^{(1)} = \phi^{(1)} - H\alpha^{(1)} - \frac{1}{3} \nabla^2 \beta^{(1)},
\]
and with these latter equations we can build the gauge invariant variables. One way of getting
the gauge invariant is to fix the vector field \(\xi\) at a particular gauge, for example the
longitudinal gauge (set the scalar perturbations \(\omega\) and \(\chi\) being zero). Therefore, the scalar
gauge invariant variables at first order are given by
\[
\Psi^{(1)} \equiv \psi^{(1)} + \frac{1}{a} \left( S_{(1)}^{(1)} a \right), \quad \text{and} \quad \Phi^{(1)} \equiv \phi^{(1)} + \frac{1}{6} \nabla^2 \chi^{(1)} - H S_{(1)}^{(1)},
\]
with \(S_{(1)}^{(1)} \equiv \left( \omega^{(1)} - \frac{1}{2} \rho^{(1)} \right)\) the scalar contribution of the shear. These are commonly called
the Bardeen potentials, and were interpreted by Bardeen as the spatial dependence of the
proper time intervals between two nearby observers and curvature perturbations respectively
[6]. Other scalar invariants are
\[
\Delta^{(1)} \equiv \mu^{(1)} + \left( \rho^{(0)} \right) S_{(1)}^{(1)}, \quad \text{and} \quad \Delta_{(1)}^{(1)} \equiv P^{(1)} + \left( P^{(0)} \right) S_{(1)}^{(1)},
\]
which describe the energy density and pressure of the matter. The vector modes yield
\[
\omega_{(1)}^{(1)} \equiv \omega_{i}^{(1)} - \left( \chi_{i}^{(1)} \right) \rho, \quad \text{and} \quad \psi_{(1)}^{(1)} \equiv \omega_{i}^{(1)} + \dot{\psi}_{i}^{(1)},
\]
related to the vorticity of the fluid. There are other gauge invariant variables at first order,
such as the three-current, the charge density and the electric and magnetic fields, because they
vanish in the background. Tensor quantities are also gauge invariant because they are null in
the background [12]. In order to study the evolution of magnetic field at large scales we must
rewrite Maxwell’s equation (4) in this formalism. The deduction of the following equations is
shown in [29]. At first order the Maxwell equations are expressed as
\[
\partial_{i} E_{(1)}^{i} = a \varphi^{(1)},
\]
\[
\partial_{i} B_{(1)}^{i} = 0,
\]
\[
\epsilon^{ijk} \partial_{j} B_{k}^{(1)} = \left( E_{(1)}^{i} \right)^{\rho} + 2HE_{(1)}^{i} + a \mathcal{H}^{(1)},
\]
\[
\left( B_{(1)}^{i} \right)^{\rho} + 2HR_{(1)}^{i} = -\epsilon^{ijk} \partial_{j} E_{k}^{(1)},
\]
these equations represent the evolution of fields in a totally invariant way. Furthermore, the
energy density of the magnetic field is the unique variable; it is gauge dependent and evolves
under the MHD approximation as \(\sim a^{-4}\) and transforms to first order as
\[
\bar{B}_{(1)}^{2} = B_{(0)}^{2} + \left( B_{(0)}^{2} \right) \rho.
\]
To second order, the Maxwell equations are given by [29]

\[ \partial_i E_i^{(2)} = -4E_i^{(1)} \partial_t (\psi^{(1)} - 3\phi^{(1)}) + a\varrho^{(2)}, \]  

(49a)

\[ (\nabla \times B^{(2)})_i = 2E_i^{(1)} \left( 2\left( \psi^{(1)} \right)' - 6\left( \phi^{(1)} \right)' \right) + \left( E_i^{(1)} \right)' + 2HE_i^{(2)} + 
\]

\[ + 2 \left( \nabla \left( 2\psi^{(1)} - 6\phi^{(1)} \right) \times B_i^{(1)} \right)' + aJ_i^{(2)}, \]  

(49b)

\[ \frac{1}{a^2} \left( a^2 B_k^{(2)} \right)' + \left( \nabla \times E_j^{(2)} \right)_k = 0, \]  

(49c)

\[ \partial_i B_i^{(2)} = 0, \]  

(49d)

dependent on gauge choice. The magnetic gauge dependent variables transform as

\[ \tilde{E}_i^{(2)} = E_i^{(2)} + 2 \left[ \left( \frac{a^2 E_i^{(1)} \alpha^{(1)}}{a^2} \right)' + \left( \xi_i^{(1)} \times B_i^{(1)} \right)' + \xi_i^{(1)} \partial_t E_i^{(1)} + E_i^{(1)} \partial_t \xi_i^{(1)} \right], \]  

(50)

\[ \tilde{B}_i^{(2)} = B_i^{(2)} + 2 \left[ \frac{\alpha^{(1)}}{a^2} \left( a^2 B_i^{(1)} \right)' + \left( \nabla \times \left( B_i^{(1)} \times \xi^{(1)} \right) + E_i^{(1)} \times \nabla \alpha^{(1)} \right) \right], \]  

(51)

where \( \varrho^{(2)} \) and \( J_i^{(2)} \) transform according to equations (80) and (81) in [29]. The energy density at second order evolves as equation (117) in [29] and it transforms

\[ \tilde{E}_i^{(2)} = E_i^{(2)} + 2 \left[ \left( \frac{a^2 E_i^{(1)} \alpha^{(1)}}{a^2} \right)' + \left( \xi_i^{(1)} \times B_i^{(1)} \right)' + \xi_i^{(1)} \partial_t E_i^{(1)} + E_i^{(1)} \partial_t \xi_i^{(1)} \right]. \]  

(50)

\[ \tilde{B}_i^{(2)} = B_i^{(2)} + 2 \left[ \frac{\alpha^{(1)}}{a^2} \left( a^2 B_i^{(1)} \right)' + \left( \nabla \times \left( B_i^{(1)} \times \xi^{(1)} \right) + E_i^{(1)} \times \nabla \alpha^{(1)} \right) \right], \]  

(51)

where \( \alpha^{(2)} \) and \( J_i^{(2)} \) transform according to equations (80) and (81) in [29]. The energy density at second order evolves as equation (117) in [29] and it transforms

\[ \tilde{E}_i^{(2)} = E_i^{(2)} + 2 \left[ \left( \frac{a^2 E_i^{(1)} \alpha^{(1)}}{a^2} \right)' + \left( \xi_i^{(1)} \times B_i^{(1)} \right)' + \xi_i^{(1)} \partial_t E_i^{(1)} + E_i^{(1)} \partial_t \xi_i^{(1)} \right]. \]  

(50)

\[ \tilde{B}_i^{(2)} = B_i^{(2)} + 2 \left[ \frac{\alpha^{(1)}}{a^2} \left( a^2 B_i^{(1)} \right)' + \left( \nabla \times \left( B_i^{(1)} \times \xi^{(1)} \right) + E_i^{(1)} \times \nabla \alpha^{(1)} \right) \right], \]  

(51)

where \( \varrho^{(2)} \) and \( J_i^{(2)} \) transform according to equations (80) and (81) in [29]. The energy density at second order evolves as equation (117) in [29] and it transforms

\[ \tilde{E}_i^{(2)} = E_i^{(2)} + 2 \left[ \left( \frac{a^2 E_i^{(1)} \alpha^{(1)}}{a^2} \right)' + \left( \xi_i^{(1)} \times B_i^{(1)} \right)' + \xi_i^{(1)} \partial_t E_i^{(1)} + E_i^{(1)} \partial_t \xi_i^{(1)} \right]. \]  

(50)

\[ \tilde{B}_i^{(2)} = B_i^{(2)} + 2 \left[ \frac{\alpha^{(1)}}{a^2} \left( a^2 B_i^{(1)} \right)' + \left( \nabla \times \left( B_i^{(1)} \times \xi^{(1)} \right) + E_i^{(1)} \times \nabla \alpha^{(1)} \right) \right], \]  

(51)

where \( \alpha^{(2)} \) and \( J_i^{(2)} \) transform according to equations (80) and (81) in [29]. The energy density at second order evolves as equation (117) in [29] and it transforms

\[ \tilde{E}_i^{(2)} = E_i^{(2)} + 2 \left[ \left( \frac{a^2 E_i^{(1)} \alpha^{(1)}}{a^2} \right)' + \left( \xi_i^{(1)} \times B_i^{(1)} \right)' + \xi_i^{(1)} \partial_t E_i^{(1)} + E_i^{(1)} \partial_t \xi_i^{(1)} \right]. \]  

(50)

\[ \tilde{B}_i^{(2)} = B_i^{(2)} + 2 \left[ \frac{\alpha^{(1)}}{a^2} \left( a^2 B_i^{(1)} \right)' + \left( \nabla \times \left( B_i^{(1)} \times \xi^{(1)} \right) + E_i^{(1)} \times \nabla \alpha^{(1)} \right) \right], \]  

(51)

where \( \alpha^{(2)} \) and \( J_i^{(2)} \) transform according to equations (80) and (81) in [29].

### 4. Electromagnetic potentials

In order to study the behavior of electromagnetic fields in scenarios such as inflation, vector–tensor theories [33, 34] or quantization of gauge theories in nontrivial space-times [35], it is more convenient to write the Maxwell equations in terms of a four-potential. Therefore, in this section we will apply the gauge invariant approach to scenarios where the presence of the electromagnetic four-potential becomes relevant. The covariant form of the Maxwell equations (see homogeneous equation (4)) reflects the existence of a four-potential [26]. This means that we can define the four-potential as \( A_\mu (x_\nu) \) with the antisymmetric condition given by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). To first order, the four-potential is gauge invariant (because they are null at the background). Using the homogeneous Maxwell equations, we can define the fields in terms of four-vector potentials:

\[ \alpha' + 3 H \varrho + \nabla \cdot J = 0. \]  

(53)

Here, to a first order approximation, the equation is completely invariant, but at second order the involved variables are gauge dependent and transform according to (80) and (81) in [29].
Therefore the inhomogeneous Maxwell equations could be reduced to two invariant equations

\[
\nabla^2 \varphi^{(1)} + \frac{1}{a^2} \frac{\partial}{\partial t} \left( \nabla \cdot \left( a^2 A^{(1)} \right) \right) = -a \varphi_{(1)}
\]

\[
\nabla^2 A^{(1)}_i - \frac{1}{a^2} \frac{\partial^2}{\partial t^2} \left( a^2 A^{(1)}_i \right) - \frac{1}{a^2} \frac{\partial}{\partial t} \left( \nabla \cdot A^{(1)} \right) + \frac{1}{a^2} \frac{\partial}{\partial t} \left( a^2 \varphi^{(1)} \right) = -a J^{(1)}_i.
\]

The latter equations, although they are written in terms of gauge invariant quantities, have an arbitrariness in the potentials known in electrodynamics given by the transformations

\[
A_i^{(1)} \rightarrow A_i^{(1)} + \partial_i \Lambda
\]

and

\[
\varphi^{(1)} \rightarrow \varphi^{(1)} - \frac{1}{a^2} \frac{\partial}{\partial t} (a^2 \Lambda),
\]

with \(\Lambda\) some scalar function of the same order as the potentials and where the fields are left unchanged under this transformation. As is commonly known in the literature, the freedom given by this transformation implies that we can choose the set of potentials satisfying the Lorenz conditions, which in this case is

\[
\nabla \cdot A^{(1)} + \frac{1}{a^2} \frac{\partial}{\partial t} \left( a^2 \varphi^{(1)} \right) = 0.
\]

Therefore, we can arrive at an uncoupled set of equations for the potentials, which are equivalent to the Maxwell equations

\[
\nabla^2 \varphi^{(1)} - \frac{1}{a^2} \frac{\partial^2}{\partial t^2} \left( a^2 \varphi^{(1)} \right) = -a \varphi_{(1)}
\]

\[
\nabla^2 A^{(1)}_i - \frac{1}{a^2} \frac{\partial^2}{\partial t^2} \left( a^2 A^{(1)}_i \right) = -a J^{(1)}_i.
\]

At second order the procedure is more complex given the gauge dependence of the potentials. Using the antisymmetrization and the gauge transformation equation (22b), we have found that the four-potential transforms as

\[
\tilde{\varphi}^{(2)} = \varphi^{(2)} + 2 \frac{\alpha_{(1)}}{a^2} \left( a^2 \varphi^{(1)} \right)' + \xi_{(1)} \partial_t \varphi^{(1)} + \alpha_{(1)} \varphi_{(1)} - \xi_{(1)}' A_{(1)}
\]

\[
\tilde{A}_i^{(2)} = A_i^{(2)} + 2 \left[ \frac{\alpha_{(1)}}{a^2} \left( a^2 A_i^{(1)} \right)' + \partial_t A_i^{(1)} \xi_{(1)} - \varphi_{(1)} \partial_\alpha^{(1)} + A_{(1)} \partial \xi_{(1)} \right].
\]

Applying the curl operator to vector potential \(A_i^{(2)}\) and after some long but otherwise straightforward algebra, we obtain the transformation of magnetic field given by equation (51) and the vector potential can expressed as

\[
B_i^{(2)} = \left( \nabla \times \tilde{A}_i^{(2)} \right);
\]

which is an original result of this paper. Similarly, we can use the induction equation (49c) found in the previous section, and with some algebra we find that the scalar potential is described in terms of electric field equation (50) via

\[
\partial_t \tilde{\varphi}^{(2)} = -\tilde{E}_i^{(2)} - \frac{1}{a^2} \left( a^2 \tilde{A}_i^{(2)} \right)';
\]

again the four-potential at this order has a freedom mediated by some scalar function \(\Lambda\) with the same order, and under transformations similar to those shown at first order the fields \(E_i^{(2)}\) and \(B_i^{(2)}\) are left unchanged. Let us continue with the Maxwell equation at second order written in terms of the four-potential. For this purpose, we substitute equations (62) and (63) in the inhomogeneous Maxwell equations (49a) and (49c), obtaining a coupling set of
equations given by

\[ \nabla^2 \varphi^{(2)} + \frac{1}{a^2} \frac{\partial}{\partial t} \left( \nabla \cdot (a^2 A^{(2)}) \right) - 4 \left( \frac{1}{a^2} (a^2 A_{(1)}^{(1)})' \right) + \partial_i \varphi^{(1)} \times \partial^i \left( \psi^{(1)} - 3 \phi^{(1)} \right) = -a \varphi^{(2)}, \]  

(64)

\[ \nabla^2 A_{(2)}^{(2)} - \frac{1}{a^2} \frac{\partial^2}{\partial t^2} \left( a^2 A^{(2)} \right) - \partial_i \left( \nabla \cdot A^{(2)} + \frac{1}{a^2} \frac{\partial}{\partial t} (a^2 \varphi^{(2)}) \right) \]

\[ - 4 \left( \frac{1}{a^2} (a^2 A_{(1)}^{(1)})' \right) \left( \psi^{(1)} - 3 \phi^{(1)} \right) + 4 \left( \nabla^2 A_{(1)}^{(1)} - \partial_i (\nabla \cdot A^{(1)}) \right) \times \left( \psi^{(1)} - 3 \phi^{(1)} \right) = -a J_{i}^{(2)}, \]  

(65)
in a dependent gauge way. The gravitational potentials \( \psi \) and \( \phi \) transform via equations (43a) and (43b). With these equations we can see a strong dependence between the electromagnetic fields and the gravitational effects with first order couplings between these variables. The Maxwell equations found above are still gauge dependent due to the fact that electromagnetic and gravitational potentials have a freedom in the choice of \( \xi^\mu \), the gauge vector. Thus, fixing the value of \( \xi^\mu \), the variables might take on their given meaning. For example, assuming that

\[ \psi^{(1)} - 3 \phi^{(1)} = 0, \]  

(66)
in order to have the same expression as obtained in the first order case, and using equations (43a) and (43b), an important constraint for the vector part of the gauge dependence is found:

\[ - \nabla^2 \beta^{(1)} = \psi^{(1)} - 3 \phi^{(1)} + 4 H \alpha^{(1)} + \alpha_{(1)}', \]  

(67)

With this choice, the conservation equation given by expression (53) reads as

\[ \Delta \psi^{(2)} + 3 H \Delta (\psi^{(2)} + \partial_i J_{(2)}^{(1)} + 2 \phi_{(1)} (\psi^{(1)} - 3 \phi^{(1)}) + 2 J_{(1)} \partial_i (\psi^{(1)} - 3 \phi^{(1)}) = 0, \]  

(68)

which is gauge invariant and equivalent to equation (B2) in [29]. We can also use the Lorenz condition by fixing the freedom of the fields,

\[ \nabla \cdot A^{(2)} + \frac{1}{a^2} \frac{\partial}{\partial t} (a^2 \varphi^{(2)}) = 0, \]  

(69)

obtaining the Maxwell equation in terms of the potential and written in a covariant way.

5. Equivalence between the two approaches

In this section we present a method to find the equivalence between the two approaches mentioned above. To do this, we compare the gauge invariant quantities built into each approach, similar to that used by [16] and [20]. The comoving gauge is defined by choosing spatial coordinates such that the three-velocity of the fluid vanishes, \( \vec{u}^i = 0 \), and the four-velocity is orthogonal to the hypersurface of constant time [19]. From equation (40) we have \( \bar{\psi}^{(1)} + \bar{\psi}^{(2)} = 0 \), and using equations (43c) and (43d) we fix the values for the gauge transformation generator vector field \( \xi^\mu \) given by
\[ \begin{align*}
\ddot{\omega}^{(1)} + \dot{\omega}^{(1)} &= 0 \rightarrow \alpha^{(1)} = v^{\parallel} + \omega^{\parallel}, \\
\ddot{\psi}^{(1)} &= 0 \rightarrow \beta^{(1)} = \int v^{\parallel} d\tau + C^{(1)}(x^{i}), \\
\ddot{v}^{(1)} &= 0 \rightarrow d^{(1)} = \int v^{\perp} d\tau + C^{(1)}(x^{i}),
\end{align*} \tag{70}\]

with \( C(x^{i}) \) a residual gauge freedom. Therefore, by using this constraint for \( \xi^{\mu} \) (see equation (24)), we can define a gauge invariant quantity related to the energy density of the magnetic field in the gauge invariant approach by substituting the value of \( \xi^{\mu} \) from (70) into equation (48), obtaining

\[ \Delta_{\text{mag}}^{(1)} := \tilde{B}^{2\perp}_{(1)} = B^{2\perp}_{(1)} + (B^{2\parallel}_{(0)}) \left( \frac{\partial^{(1)}}{\partial (1)} + \omega^{\parallel}\right), \rightarrow \text{comoving gauge.} \tag{71} \]

Now, we start expanding equation (13a), where we use the projector defined in equation (2) and the four-velocity given by equation (40); to first order we obtain

\[ B_{0} = D_{0}B_{2\perp}^{2} = 0, \tag{72} \]

for the temporal part. For the spatial part we obtain

\[ B_{i} = D_{i}B_{2\perp}^{2} = \partial_{i} \left( B^{2\perp}_{(1)} + (B^{2\parallel}_{(0)}) \left( \frac{\partial^{(1)}}{\partial (1)} + \omega^{\parallel}\right) \right). \tag{73} \]

where both equations correspond to the gauge invariant in the 1 + 3 covariant approach. If we compare the latter equation with the gauge invariant quantity corresponding to the energy density of the magnetic field (see equation (71)), we have finally

\[ B_{i} = D_{i}B_{2\perp}^{2} \equiv \partial_{i}\Delta_{\text{mag}}^{(1)} \tag{74} \]

The authors of [20] found similar results for the matter density case. To describe the equivalence to second order, we will make use of \( \tilde{u}_{i} = 0 \) again (comoving condition); thus checking equation (40) we find that

\[ \frac{1}{2} \left( \ddot{\omega}^{(2)} + \ddot{\psi}^{(2)} \right) - \ddot{\psi}^{(1)} \psi^{(1)} - 2\dot{\psi}^{(1)} \dot{\phi}^{(1)} + \dot{\psi}^{(1)} \chi^{(i)} = 0. \tag{75} \]

Substituting equations (43a)–(43e) and values for \( \ddot{\psi}^{(2)}, \psi^{(2)} \) and \( \chi^{(i)} \) in the last equation, we obtain the temporal gauge dependence \( \alpha^{(2)} \) written in the comoving gauge given by

\[ \begin{align*}
\partial_{i}\alpha^{(2)} &= \omega^{(2)} + \psi^{(2)} - 4\psi^{(1)} \omega^{(2)} + 2\dot{\psi}^{(1)} \left( \dot{\phi}^{(1)} - 2\phi^{(1)} \right) \\
&\quad + \left( \omega^{(1)} + \psi^{(1)} \right) \left( \omega^{(1)} + \psi^{(1)} \right) - \left( \omega^{(1)} + \psi^{(1)} \right) \left( \dot{\omega}^{(1)} + \dot{\psi}^{(1)} \right) \\
&\quad + \partial_{i}\chi^{(i)} \left( \omega^{(1)} + \psi^{(1)} \right) + 2\chi^{(i)} \dot{v}^{i} + \xi^{(i)} \partial_{i} \left( \omega^{(1)} + \psi^{(1)} \right);
\end{align*} \tag{76} \]

the deduction of this equation is given in appendix. We can also define a gauge invariant quantity related to the energy density of the magnetic field in the gauge invariant approach at second order, fixing the value of \( \alpha^{(2)} \) from (76) and \( \xi^{(1)} \) from (70) in equation (52), which yields

\[ \Delta_{\text{mag}}^{(2)} := \tilde{B}^{2\perp}_{(2)}, \rightarrow \text{comoving gauge.} \tag{77} \]

On the other hand, expanding equation (13a) to second order (which comes from the 1 + 3 covariant approach), the temporal part corresponds to

\[ B_{0} = D_{0}B_{2\perp}^{2} = -v^{(1)}B^{2\perp}_{(0)} \left( \frac{\partial^{(1)}}{\partial (1)} + \omega^{\parallel}\right) - v^{(1)} \partial_{0}B_{2\perp}^{2}, \tag{78} \]
which is the same result as found in (73) multiplied by \( v_{(i)} \), therefore the temporal part is zero and gives us an important constraint for our work. For the spatial part we found the following:

\[
B_i = D_i B^2 = \frac{1}{2} \partial_i B_i^2 + \left( \omega_i^{(1)} + v_i^{(1)} \right) B_i^{(2)} + B_{(0)} \left( \frac{1}{2} \left( \omega_i^{(2)} + v_i^{(2)} \right) - 2 \omega_i^{(1)} v_i^{(1)} \right) \tag{79}
\]

Now, applying the gradient operator \( \partial_i \) to \( \Delta_{\text{mag}}^{(2)} \) shown in equation (77), which is an invariant quantity associated with energy density to second order, we obtain

\[
\begin{align*}
\partial_i \Delta_{\text{mag}}^{(2)} &= \frac{1}{2} \partial_i B_i^2 + \partial_i \alpha_i^{(2)} B_i^{(2)} + 2 \alpha_i^{(1)} \partial_i \alpha_i^{(1)} B_i^{(2)} + B_{(0)} \left( \alpha_i^{(1)} \partial_i \alpha_i^{(1)} - \omega_i^{(1)} v_i^{(1)} \right) + 2 B_{(1)} \partial_i B_i^{(2)} + 2 \alpha_i^{(1)} \partial_i B_i^{(1)} + \partial_i \xi_i^{(1)} \partial_i \alpha_i^{(1)} B_i^{(2)} + \xi_i^{(1)} \partial_i \partial_i \alpha_i^{(1)} B_i^{(2)} + 2 \xi_i^{(1)} \partial_i \alpha_i^{(1)} B_i^{(2)} + 2 \partial_i \xi_i^{(1)} \partial_i \alpha_i^{(1)} B_i^{(2)} + 2 \partial_i \xi_i^{(1)} \partial_i B_i^{(1)} + 2 \partial_i \alpha_i^{(1)} \partial_i B_i^{(1)} \tag{80}
\end{align*}
\]

Thus, substituting equations (78) and (70) in the last equation, we obtain

\[
\begin{align*}
\partial_i \Delta_{\text{mag}}^{(2)} &= \frac{1}{2} \partial_i B_i^2 + \left( \omega_i^{(1)} + v_i^{(1)} \right) B_i^{(2)} + B_{(0)} \left( \frac{1}{2} \left( \omega_i^{(2)} + v_i^{(2)} \right) - 2 \omega_i^{(1)} v_i^{(1)} \right) \tag{81}
\end{align*}
\]

which is the expression found in equation (79). Therefore we have obtained the desired result, an equivalence between the invariants of the two approaches up to second order:

\[
B_i = D_i B^2 = \partial_i \Delta_{\text{mag}}^{(2)} \tag{82}
\]

For the gauge invariant vector field defined in equation (13c) we have

\[
\begin{align*}
\mathcal{M}_{0,0} &= 0. \tag{83a} \\
\mathcal{M}_{0,i} &= \left( aB_{(2)} \right)^i + a v_i B_{(1)}^{(1)}. \tag{83b} \\
\mathcal{M}_{i,j} &= a \left( \partial_i B_j^{(2)} + B_{(1)}^{(1)} \nu_j \right). \tag{83c} \\
\mathcal{M}_{i}^j &= a \left( \partial_i B_{(2)} \right)^j - \frac{1}{a} B_{(1)}^{(1)} \left( a \nu_i \right)^j - 3 B_{(1)}^{(1)} \partial_i \phi \tag{83d}
\end{align*}
\]

If we consider neither the magnetic field nor the vorticity in linear perturbation theory in equation (83d), we obtain the usual equation of divergence of the magnetic field (which confirms a claim in [36]). Making the antisymmetric product between the four-acceleration equation (42) and the magnetic field gives an equation of the type

\[
\begin{align*}
a_{\mu}^{(1)} B_{j}^{(1)} &= B_{\mu}^{(1)} v_j^{(1)} + B_{\mu}^{(1)} \partial_j v_j^{(1)} + H B_{\mu}^{(1)} v_j^{(1)}, \tag{84}
\end{align*}
\]

where we use the four-velocity expressed in equation (40) from section 3.1 and where the temporal part is zero. If we contract the indices in equation (84) and use equation (66), we obtain a consistency condition with equation (83d) under a null electric field condition. Therefore a magnetic field with no accompanying electric field or current provides the relation

\[
\begin{align*}
a_{\alpha}^{(1)} B_{\beta}^{(1)} &= M_{\alpha, \beta}. \tag{85}
\end{align*}
\]

establishing an important relation between the gradient of the magnetic field and a kinematic quantity as has been argued by [26]. Taking the curl of equation (51) and using Maxwell’s equation (49c), we find that
\[
\left( \nabla \times B^{(2)} \right)_i \equiv \left( \nabla \times B^{(2)} \right)_i = a \left( \nabla \times B^{(2)} \right)_i, \quad (86)
\]

where the electric field and vorticity (this assumption will be reflected as \( \epsilon^{ij} \partial_j \xi_5^{(1)} = 0 \)) have been ignored. Here \( \tilde{H}^{(2)}_i \) is the gauge invariant quantity related to the magnetic field vector in the gauge invariant approach. Therefore, equations (83c) and (86) allow us to find the vector equivalence up to second order given as

\[
\epsilon^{ij} M_{[i, j]} = \left( \nabla \times \tilde{H} \right)^k, \quad (87)
\]

which can be described as the variations of the magnetic field vector. In short, assuming a magnetized universe we have verified the equivalence of the two approaches by finding connections between their gauge invariant quantities via equations (74) and (82) for the scalar and (87) for the tensor case.

6. Discussion

Relativistic perturbation theory has been an important tool in theoretical cosmology to link scenarios of the early universe with cosmological data such as CMB fluctuations. However, there is an issue in the treatment of this theory, which is called the gauge problem. Due to the general covariance, a gauge degree of freedom arises in cosmological perturbation theory. If the correspondence between a real and a background space-time is not completely specified, the evolution of the variables will have unphysical modes. Different approaches have been developed to overcome this problem, among them the \( 1+3 \) covariant gauge invariant and gauge invariant approaches, which have been studied in the present paper. Following some results shown in [16, 37, 38] and [20], we have contrasted these formalisms, comparing their gauge invariant variables defined in each case. Using a magnetic scenario, we have shown a strong relation between the two formalisms; indeed, we found that the gauge invariant defined by the \( 1+3 \) covariant approach is related to spatial variations of the magnetic field energy density (a variable defined in the invariant gauge formalism) between two closed fundamental observers as noticed in equations (74), (82) and (87). Moreover, we have also derived the gauge transformations for electromagnetic potentials, equations (60) and (61), which are relevant in the study of evolution of primordial magnetic fields in scenarios such as inflation or later phase transitions. With the description of the electromagnetic potentials, we have expressed the Maxwell equations in terms of these ones, finding again an important coupling with the gravitational potentials.

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Appendix. Spatial part of the gauge transformation generator

In order to obtain equation (76), we use the expression (22b) to find the way that \( \psi^{(2)}_i \) and \( \omega^{(2)}_i \) transform to second order. The expression was obtained first by [14] and becomes
By substituting equations for the three-velocity. The couplings between the shear part and for the three-velocity. The couplings between first order terms are then given as

\[ \tilde{\omega}_i^{(2)} = \omega_i^{(2)} - \partial_i \alpha^{(1)} + \xi_i^{(2)'} + \xi_i^{(1)'} \left( 2 \alpha^{(1)} \omega_i^{(1)} - \partial_i \alpha^{(1)} + \partial_i \xi_i^{(1)} \right) \]

\[ + \alpha^{(1)} \left[ 2 \left( \omega_i^{(1)} + 2 H \omega_i^{(1)} \right) - \partial_i \alpha^{(1)} + \xi_i^{(1)''} - 4 H \left( \partial_i \alpha^{(1)} - \xi_i^{(1)} \right) \right] \]

\[ + \alpha_i^{(1)} \left( 2 \omega_i^{(1)} - 3 \partial_i \alpha^{(1)} + \xi_i^{(1)} \right) + \xi_i^{(1)'} \left( -4 \phi^{(1)} \delta_{ij} + 2 \chi_i^{(1)} + 2 \xi_i^{(1)} + \xi_i^{(1)} \right) \]

\[ + \xi_i^{(1)'} \left( 2 \omega_i^{(1)} - \partial_i \alpha^{(1)} \right) - 4 \psi^{(1)} \partial_i \alpha^{(1)} \]

(A.1)

for the shear part and

\[ \tilde{v}_i^{(2)} = v_i^{(2)} - \xi_i^{(2)'} + \alpha_i^{(1)} \left[ 2 \left( \psi_i^{(1)'} - H \psi_i^{(1)} \right) - \left( \xi_i^{(1)''} - 2 H \xi_i^{(1)} \right) \right] \]

\[ + \xi_i^{(1)'} \partial_i \left( 2 \nu_i^{(1)} - \xi_i^{(1)} \right) - \partial_i \xi_i^{(1)} \left( 2 \nu_i^{(1)} - \xi_i^{(1)} \right) + \xi_i^{(1)'} \left( 2 \psi_i^{(1)} + \alpha_i^{(1)} \right) \]

(A.2)

for the three-velocity. As an alternative method, we can use equation (A12) from [29] and transform it from the Poisson to the comoving gauge.

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