SCHWARZ MAPS ASSOCIATED WITH THE TRIANGLE GROUPS (2, 4, 4) AND (2, 3, 6)

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Abstract. We study the Schwarz maps with monodromy groups isomorphic to the triangle groups (2, 4, 4) and (2, 3, 6) and their inverses. We apply our formulas to the study of mean iterations.

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1. Introduction

The Gauss hypergeometric function 
\[ \mathcal{F}(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha, n)(\beta, n)}{(\gamma, n)(1, n)} z^n, \]
where \(z\) is the main variable in the unit disk \(D = \{ z \in \mathbb{C} \mid |z| < 1 \}\), \(\alpha, \beta, \gamma\) are parameters with \(\gamma \neq 0, -1, -2, \ldots\), and \((\alpha, n) = \alpha(\alpha + 1) \cdots (\alpha + n - 1)\). This function admits an integral representation
\[ \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_{1}^{\infty} t^{\beta - \gamma}(t - z)^{-\beta}(t - 1)^{\gamma - \alpha - 1} dt, \]
and satisfies the hypergeometric differential equation
\[ \mathcal{F}(\alpha, \beta, \gamma) : z(1 - z) f''(z) + \{\gamma - (\alpha + \beta + 1)z\} f'(z) - \alpha \beta f(z) = 0, \]
which has only singular points of regular type at \(z = 0, 1, \infty\). The Schwarz map is defined by the continuation to \(X = \mathbb{C} \setminus \{0, 1\}\) of the ratio of two linearly independent solutions to \(\mathcal{F}(\alpha, \beta, \gamma)\) in a small simply connected domain in \(X\).

In this paper, we study the Schwarz maps for two sets of the parameters
\[ (\alpha, \beta, \gamma) = \left(\frac{1}{4}, 0, \frac{1}{2}\right), \left(\frac{1}{3}, 0, \frac{1}{2}\right). \]
The monodromy groups of \(\mathcal{F}(\alpha, \beta, \gamma)\) for these sets of parameters are reducible and isomorphic to the triangle groups \((2, 4, 4)\) and \((2, 3, 6)\), respectively. We give circuit matrices generating these groups in Corollary 1. The images of the Schwarz maps...
are the quotient of the complex torus $E_i = \mathbb{C}/(i\mathbb{Z} + \mathbb{Z})$ by the the multiplicative group $(i) = \{±1, ±i\}$ for $(α, β, γ) = \left(\frac{1}{4}, 0, \frac{1}{2}\right)$, and that of $E_ζ = \mathbb{C}/(ζ\mathbb{Z} + \mathbb{Z})$ by $\langle ζ \rangle = \{±1, ±ζ, ±ζ^2\}$ for $(α, β, γ) = \left(\frac{1}{3}, 0, \frac{1}{2}\right)$, where $i = \sqrt{-1}$ and $ζ = \frac{1 + \sqrt{3}i}{2}$.

We consider elliptic curves

$$C_i : u^4 = t^2(t - 1), \quad C_ζ : u^6 = t^3(t - 1),$$

and relate these Schwarz maps and the Abel-Jacobi maps

$$ζ_i : C_i \to E_i, \quad ζ_ζ : C_ζ \to E_ζ$$

defined by incomplete elliptic integrals on $C_i$ and on $C_ζ$. We express the inverses of these Schwarz maps in terms of the theta function $ϑ(ζ)$, with characteristics $a, b$; see Theorem 1 and Theorem 3. As corollaries, some $ϑ_{a,b}(0,i)$ and $ϑ_{a,b}(0,ζ)$ are evaluated in terms of the Gamma function. We study the pull-back of the $(1 + i)$-multiple on $E_i$ and that of the $(1 + ζ)$-multiple $E_ζ$ under the corresponding Abel-Jacobi maps. We apply these results to the study of mean iterations. We show that the formula in Theorem 2 yields a limit formula of a mean iteration

$$\lim_{n \to \infty} \frac{m \circ \cdots \circ m(a, b)}{n} = \frac{a}{F(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; 1 - \frac{a^2}{4})},$$

where $a > b > 0$ and

$$m : (a, b) \mapsto \left(\frac{a + b}{2}, \sqrt{\frac{a(a + b)}{2}}\right).$$

We have a similar result from the $(1 + ζ)$-multiple formula on the elliptic curve $E_ζ$ in Theorem 4. It is studied in [HKM] that these limit formulas for mean iterations can be obtained from transformation formulas for the hypergeometric function in [G]. We elucidate a geometric background of these limit formulas as multiplications on the complex tori $E_i$ and $E_ζ$.

2. The Schwarz map

2.1. Fundamental system of solutions to $F(a, b, c)$. We define the Schwarz map as the ratio of solutions to $F(α, β, γ)$ given by the Euler type integral representations

$$f_1(x) = \int_1^x t^{β-γ}(t - x)^{-β} (t - 1)^{γ-α} \frac{dt}{t - 1}, \quad f_2(x) = \int_1^∞ t^{β-γ}(t - x)^{-β} (t - 1)^{γ-α} \frac{dt}{t - 1},$$

where

$$0 < \text{Re}(α) < \text{Re}(γ), \quad \text{Re}(β) < 1.$$  

For an element $x$ in $U = \{x ∈ X \mid \max(|x|, |1 - x|) < 1, \text{Im}(x) > 0\}$, they can be expressed by the hypergeometric series. By (1.1),

$$f_2(x) = B(γ - α, α) \cdot F(α, β, γ; x),$$

where $B(*, *)$ denotes the beta function. By the variable change

$$s = \frac{x - 1}{t - 1}, \quad i.e. \quad t = \frac{s + x - 1}{s}, \quad dt = \frac{(x - 1)ds}{s^2},$$

for the integral representation of $f_1(x)$ and (1.1), we have

$$f_1(x) = e^{πi(γ-α)} B(γ - α, 1 - β)(1 - x)^{γ-α-β} \cdot F(γ - α, γ - β, γ - α - β + 1; 1 - x),$$
where \( \theta_1 = \arg x \) and \( \theta_2 = \arg(1 - x) \) belong to the open interval \((-\pi/2, \pi/2)\), and the arguments of \( t, t - x, t - 1 \) on the open segments \((1, x)\) and \((1, \infty)\) take values in the intervals in Table 1. Here pay your attention to the argument of \( t - 1 \) and the orientation of the path integral.

\[
\begin{array}{|c|c|c|}
\hline
\text{arg}(t) & t \in (x, 1) & t \in (1, \infty) \\
\text{arg}(t - x) & \min(0, \theta_1), \max(0, \theta_1) & 0 \\
\text{arg}(t - 1) & \theta_2 & \min(0, \theta_2), \max(0, \theta_2) \\
\hline
\end{array}
\]

Table 1. Arguments of \( t, t - x \) and \( t - 1 \)

**Remark 1.** When \( \beta = 0 \), the solution \( f_1(x) \) is expressed as

\[
f_1(x) = e^{\pi i (\gamma - \alpha)} \cdot (1 - x)^{\gamma - \alpha} \cdot F(\gamma - \alpha, \gamma, \gamma - \alpha + 1; 1 - x)
\]

for \(|x - 1| < 1\), and the solution \( f_2(x) \) reduces to a constant

\[
B(\gamma - \alpha, \alpha) = \frac{\Gamma(\gamma - \alpha) \Gamma(\alpha)}{\Gamma(\gamma)}.
\]

**2.2. Monodromy representation of \( \mathcal{F}(a, b, c) \).** We take a base point \( \hat{x} \) in \( U \). Let \( \mathcal{M} \) be the monodromy representation of \( \mathcal{F}(\alpha, \beta, \gamma) \) with respect to the base point \( \hat{x} \). It is the homomorphism from the fundamental group \( \pi_1(X, \hat{x}) \) to the general linear group of the local solution space to \( \mathcal{F}(\alpha, \beta, \gamma) \) on \( U \) arising from the analytic continuation along a loop with terminal \( \hat{x} \). We denote the image of \( \ell \in \pi_1(X, \hat{x}) \) by \( \mathcal{M}_\ell \). Let \( \ell_0 \) and \( \ell_1 \) be a loop starting from \( \hat{x} \) turning positively around the point \( x = 0 \) and that around the point \( x = 1 \), respectively. Since \( \pi_1(X, \hat{x}) \) is generated by \( \ell_0 \) and \( \ell_1 \), \( \mathcal{M} \) is determined by \( \mathcal{M}_0 = \mathcal{M}_{\ell_0} \) and \( \mathcal{M}_1 = \mathcal{M}_{\ell_1} \). By the basis \( \Gamma(f_1(x), f_2(x)) \), the transformations \( \mathcal{M}_0 \) and \( \mathcal{M}_1 \) are represented by matrices \( \mathcal{M}_0 \) and \( \mathcal{M}_1 \). That is, the basis \( \Gamma(f_1(x), f_2(x)) \) is transformed into

\[
\mathcal{M}_\ell \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}
\]

by the analytic continuation along the loop \( \ell_\ell \). They are expressed by the intersection matrix

\[
H = \begin{pmatrix}
\frac{e(\gamma - \alpha) - e(\beta)}{e(\gamma - \alpha) - 1} & -\frac{e(\gamma - \alpha)}{e(\gamma - \alpha) - 1} \\
\frac{-e(\beta) + 1}{e(\gamma - \alpha) - 1} & \frac{-e(\gamma) + 1}{(e(\gamma - \alpha) - 1)(e(\alpha) - 1)}
\end{pmatrix}
\]

as in [Ma].

**Proposition 1.** Suppose that

\[
\alpha, \alpha - \gamma, \beta - \gamma \notin \mathbb{Z}.
\]

Then we have

\[
\mathcal{M}_0 = \lambda_0 I_2 - \frac{\lambda_0 - 1}{e_2 H e_2^*} H e_2^* e_2 = \begin{pmatrix} e(-\gamma) & 1 - e(-\alpha) \\ 0 & 1 \end{pmatrix},
\]

\[
\mathcal{M}_1 = I_2 - \frac{1 - \lambda_1}{e_1 H e_1^*} H e_1^* e_1 = \begin{pmatrix} e(\gamma - \alpha - \beta) & 0 \\ -1 + e(-\beta) & 1 \end{pmatrix}.
\]
where \( \lambda_0 = e^{-\gamma} \), \( \lambda_1 = e(\gamma - \alpha - \beta) \),
\[
I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = (1, 0), \quad e_2 = (0, 1), \quad e_1^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

We use this proposition for \( \beta \in \mathbb{Z} \) with a base change
\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 - e(\alpha) \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}.
\]

**Corollary 1.** In this case, \( M_0 \) and \( M_1 \) are transformed into
\[
N_0 = \begin{pmatrix} e(\gamma) & -e(\alpha) \\ 0 & 1 \end{pmatrix}, \quad N_1 = \begin{pmatrix} e(\gamma - \alpha) & 0 \\ 0 & 1 \end{pmatrix},
\]
respectively. When \( (\alpha, \beta, \gamma) = \left( \frac{1}{2}, 0, 0 \right) \), \( N_0, N_1, (N_0N_1)^{-1} \) are
\[
\begin{pmatrix} -1 & i \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} i & 1 \\ 0 & 1 \end{pmatrix}.
\]

The group generated by these matrices is isomorphic to the triangle group \( (2, 4, 4) \),
and to the semi-direct product \( (i) \rtimes \mathbb{Z}[i] \). When \( (\alpha, \beta, \gamma) = \left( \frac{1}{3}, 0, \frac{1}{2} \right) \), \( N_0, N_1, (N_0N_1)^{-1} \) are
\[
\begin{pmatrix} -1 & \zeta \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \zeta^2 & 1 \\ 0 & 1 \end{pmatrix}, \quad \zeta = \frac{1 + \sqrt{3}i}{2}.
\]

The group generated by these matrices is isomorphic to the triangle group \( (2, 3, 6) \),
and to the semi-direct product \( (\zeta) \rtimes \mathbb{Z}[\zeta] \).

### 3. Theta Functions

#### 3.1. Basic properties of \( \vartheta_{a,b} \).

The theta function with characteristics is defined by
\[
\vartheta_{a,b}(z, \tau) = \sum_{n \in \mathbb{Z}} \exp(i(n + a)^2\tau + 2\pi i(z + b)),
\]
where \( z \in \mathbb{C} \) and \( \tau \in \mathbb{H} \) are main variables, and \( a, b \) are rational parameters. For a fixed \( \tau \), we denote \( \vartheta_{a,0}(z, \tau) \) by \( \vartheta_{a,0}(z) \). In this subsection, we collect useful formulas for \( \vartheta_{a,b}(z, \tau) \) in our study from [I] and [Mu].

It is easy to see that this function satisfies
\[
\vartheta_{a,b}(z, \tau) = e^{\frac{a^2 \tau}{2} + a(z + b)} \vartheta_{0,0}(z + a \tau + b, \tau), \\
\vartheta_{-a,-b}(z, \tau) = \vartheta_{a,b}(-z, \tau), \\
\vartheta_{a,b}(z + p \tau + q, \tau) = e(aq - \frac{p^2 \tau}{2} - p z - b p) \vartheta_{a,b}(z, \tau), \\
\vartheta_{a+p,b+q}(z, \tau) = e(aq) \vartheta_{a,b}(z, \tau), \\
\vartheta_{a,b}(z + c \tau + d, \tau) = e(c(b' - b)) \frac{\vartheta_{a+c,b+d}(z, \tau)}{\vartheta_{a'+c,b'+d}(z, \tau)},
\]
where \( p, q \in \mathbb{Z} \) and \( a', b' \in \mathbb{Q} \).

It is known that \( \vartheta_{a,b}(z) = 0 \) if and only if
\[
(-a + p + \frac{1}{2}) \tau + (-b + q + \frac{1}{2}) \quad (p, q \in \mathbb{Z}),
\]
and they are simple zeroes. If \( (a_1, b_1), \ldots, (a_r, b_r) \) and \( (a'_1, b'_1), \ldots, (a'_r, b'_r) \) satisfy
\[
\sum_{i=1}^{r} (a_i, b_i) = \sum_{i=1}^{r} (a'_i, b'_i) \mod \mathbb{Z}^2
\]
then the product

$$F(z) = \prod_{i=1}^{r} \frac{\vartheta_{a_i,b_i}(z)}{\vartheta_{a_i,b_i}(0)}$$

becomes an elliptic function with respect to the lattice $L_\tau = \mathbb{Z} \tau + \mathbb{Z}$, i.e., it is meromorphic on $\mathbb{C}$ and satisfies

$$F(z) = F(z + 1) = F(z + \tau).$$

**Fact 1** (Jacobi’s derivative formula).

$$\frac{\partial}{\partial z} \vartheta_{\frac{1}{z},\frac{1}{\tau}}(z, \tau) \bigg|_{z=0} = -\pi \vartheta_{0,0}(0, \tau) \vartheta_{0,\frac{1}{2}}(0, \tau) \vartheta_{\frac{1}{2},0}(0, \tau).$$

**Fact 2** (Transformation formulas).

$$\vartheta_{a,b}(z, \tau + 1) = e \left( \frac{a(1-a)}{2} \right) \vartheta_{a+a+b-\frac{1}{2}}(z, \tau),$$

$$\vartheta_{a,b}\left(\frac{z}{\tau}, \frac{-1}{\tau}\right) = e(ab) \sqrt{\frac{\tau}{4}} \vartheta_{b,-a}(z, \tau),$$

where $\sqrt{\tau/4}$ is positive when $\tau$ is purely imaginary.

**Fact 3** (Addition formulas, Jacobi’s identity).

$$\vartheta_{0,0}(z_1 + z_2)\vartheta_{0,0}(z_1 - z_2)\vartheta_{0,0}(0)^2 = \vartheta_{0,0}(z_1)^2\vartheta_{0,0}(z_2)^2 + \vartheta_{\frac{1}{2},\frac{1}{2}}(z_1)^2\vartheta_{\frac{1}{2},\frac{1}{2}}(z_2)^2,$$

$$\vartheta_{0,\frac{1}{2}}(z_1 + z_2)\vartheta_{0,\frac{1}{2}}(z_1 - z_2)\vartheta_{0,\frac{1}{2}}(0)^2 = \vartheta_{0,0}(z_1)^2\vartheta_{0,0}(z_2)^2 - \vartheta_{\frac{1}{2},\frac{1}{2}}(z_1)^2\vartheta_{\frac{1}{2},\frac{1}{2}}(z_2)^2,$$

$$\vartheta_{\frac{1}{2},0}(z_1 + z_2)\vartheta_{\frac{1}{2},0}(z_1 - z_2)\vartheta_{\frac{1}{2},0}(0)^2 = \vartheta_{0,0}(z_1)^2\vartheta_{0,0}(z_2)^2 - \vartheta_{\frac{1}{2},\frac{1}{2}}(z_1)^2\vartheta_{\frac{1}{2},\frac{1}{2}}(z_2)^2,$$

$$\vartheta_{\frac{1}{2},\frac{1}{2}}(z_1 + z_2)\vartheta_{\frac{1}{2},\frac{1}{2}}(z_1 - z_2)\vartheta_{\frac{1}{2},\frac{1}{2}}(0)^2 = \vartheta_{\frac{1}{2},\frac{1}{2}}(z_1)^2\vartheta_{\frac{1}{2},\frac{1}{2}}(z_2)^2 - \vartheta_{0,0}(z_1)^2\vartheta_{0,0}(z_2)^2.$$

3.2. **Formulas for $\tau = i$**. In this subsection, we obtain several formulas for $\vartheta_{a,b}(z, i)$ in the case of $\tau = i$.

**Lemma 1.** We have

$$\vartheta_{a,b}(iz, i) = e(ab) \exp(\pi z^2) \vartheta_{-b,a}(z, i),$$

$$\vartheta_{0,0}(iz, i) = \exp(\pi z^2) \vartheta_{0,0}(z, i), \quad \vartheta_{0,\frac{1}{2}}(iz, i) = \exp(\pi z^2) \vartheta_{\frac{1}{2},0}(z, i),$$

$$\vartheta_{\frac{1}{2},0}(iz, i) = \exp(\pi z^2) \vartheta_{\frac{1}{2},0}(z, i), \quad \vartheta_{\frac{1}{2},\frac{1}{2}}(iz, i) = i \exp(\pi z^2) \vartheta_{\frac{1}{2},\frac{1}{2}}(iz, i),$$

$$\vartheta_{0,\frac{1}{2}}(0, i) = \vartheta_{\frac{1}{2},0}(0, i) = \frac{\vartheta_{0,0}(0, i)}{\sqrt{2}}.$$

**Proof.** For the $i$-multiple formulas, we have only to substitute $\tau = i$ into the second formula for $\vartheta_{-a,-b}$. We have $\vartheta_{0,\frac{1}{2}}(0) = \vartheta_{\frac{1}{2},0}(0)$ by substituting $z = 0$ into the identity between $\vartheta_{0,\frac{1}{2}}(iz)$ and $\vartheta_{\frac{1}{2},0}(z)$. By Jacobi’s identity, we have $\vartheta_{0,0}(0)^4 = 2\vartheta_{0,\frac{1}{2}}(0)^4$. Note that $\vartheta_{0,0}(0)$ and $\vartheta_{0,\frac{1}{2}}(0)$ take positive real values. □
Lemma 2. We have
\[ \vartheta_{0,0}(1+i)z,i) = \frac{\vartheta_{0,0}(0,i)\vartheta_{0,0}(z,i)\vartheta_{\frac{1}{2},0}(z,i)}{\exp(\pi i(1+i)z^2)\vartheta_{\frac{1}{2},0}(0,i)}; \]
\[ \vartheta_{\frac{1}{2},0}(1+i)z,i) = e(\frac{1}{8}) \frac{\vartheta_{0,0}(0,i)\vartheta_{0,0}(z,i)\vartheta_{\frac{1}{2},0}(z,i)}{\exp(\pi i(1+i)z^2)\vartheta_{\frac{1}{2},0}(0,i)}; \]
\[ \vartheta_{\frac{1}{2},0}(1+i)z,i) = \frac{\vartheta_{0,0}(z,i)^4 - \vartheta_{0,0}(z,i)^2\vartheta_{\frac{1}{2},0}(z,i)^2}{\exp(2\pi i(1+i)z^2)\vartheta_{\frac{1}{2},0}(0,i)\vartheta_{\frac{1}{2},0}(0,i)}. \]

Proof. We set
\[ \eta(z) = \exp(\pi i(1+i)z^2)\vartheta_{0,0}(1+i)z,i). \]
Since \( \vartheta_{0,0}(z) \) has simple zero at \( z = \frac{i+1}{2} \), the function \( \eta(z) \) has simple zero at \( z = \frac{1+i}{2} \). By using the quasi periodicity of \( \vartheta_{0,0}(z) \), we can show that
\[ \eta(z+1) = -\eta(z), \quad \eta(z+i) = -\exp(-2\pi i(1+2z))\eta(z). \]
Thus the function
\[ \frac{\eta(z)}{\vartheta_{\frac{1}{2},0}(z)\vartheta_{\frac{1}{2},0}(z)} \]
is a holomorphic elliptic function with respect to \( L_1 \); it is a constant. We can determine this constant by putting \( z = 0 \). The second formula is obtained by the substitution \( z + \frac{1}{2} \) into \( z \) for the first formula. We show the third formula. By Fact 3 for \( z_1 = z \) and \( z_2 = iz \), we have
\[ \vartheta_{\frac{1}{2},0}(z+iz)\vartheta_{\frac{1}{2},0}(z-iz)\vartheta_{0,0}(0)^2 = \vartheta_{0,0}(z)^2\vartheta_{0,0}(iz)^2 - \vartheta_{\frac{1}{2},0}(z)^2\vartheta_{\frac{1}{2},0}(iz)^2. \]
This identity together with Lemma 1 leads the third formula. \( \square \)

3.3. Formulas for \( \tau = \zeta \). In this subsection, we obtain several formulas for \( \vartheta_{a,b}(z,\zeta) \) in the case of \( \tau = \zeta = \frac{1+\sqrt{3}i}{2} \).

Lemma 3. We have
\[ \vartheta_{a,b}(\omega z,\zeta) = e(\frac{a^2}{2} + ab - \frac{1}{24})e(\frac{z^2}{2\zeta})\vartheta_{-a-b-\frac{1}{2},a}(z,\zeta), \]
\[ \vartheta_{a,b}(\omega^2 z,\zeta) = e(ab + \frac{b^2 + b}{2} + \frac{1}{24})e(\frac{z^2}{2\zeta})\vartheta_{b, a-b-\frac{1}{2}}(z,\zeta), \]
\[ \vartheta_{0,0}(\omega z,\zeta) = e(\frac{-1}{24})e(\frac{z^2}{2\zeta})\vartheta_{\frac{1}{2},0}(z,\zeta), \quad \vartheta_{0,0}(\omega^2 z,\zeta) = e(\frac{1}{24})e(\frac{z^2}{2\zeta})\vartheta_{0,0}(z,\zeta), \]
\[ \vartheta_{\frac{1}{2},0}(\omega z,\zeta) = e(\frac{-1}{24})e(\frac{z^2}{2\zeta})\vartheta_{\frac{1}{2},0}(z,\zeta), \quad \vartheta_{\frac{1}{2},0}(\omega^2 z,\zeta) = e(\frac{1}{24})e(\frac{z^2}{2\zeta})\vartheta_{\frac{1}{2},0}(z,\zeta), \]
\[ \vartheta_{\frac{1}{2},0}(\omega^2 z,\zeta) = \omega^2 e(\frac{z^2}{2\zeta})\vartheta_{\frac{1}{2},0}(z,\zeta), \quad \vartheta_{\frac{1}{2},0}(\omega^2 z,\zeta) = \omega^2 e(\frac{z^2}{2\zeta})\vartheta_{\frac{1}{2},0}(z,\zeta), \]
where \( \omega = \zeta^2 = \frac{-1 + \sqrt{3}i}{2} \).

Proof. Fact 2 yields that
\[ \vartheta_{a,b}(\zeta,\frac{-1}{\zeta}) = e(ab)e(\frac{-1}{24})e(\frac{z^2}{2\zeta})\vartheta_{b,-a}(z,\zeta), \]
\[ = \vartheta_{a,b}(-\omega z,\zeta - 1) = e(\frac{z(a-1)}{2})\vartheta_{a,-a+b}(-\omega z,\zeta) = e(\frac{z(a-1)}{2})\vartheta_{-a,a-b-\frac{1}{2}}(\omega z,\zeta). \]
By rewriting \((a', b') = (-a, a - b - \frac{1}{2})\) i.e., \((a, b) = (-a', -a' - b' - \frac{1}{2})\) for the identity
\[
ee(ab)e\left(-\frac{1}{24}\right)e\left(\frac{x^2}{2\zeta}\right)g_{b, -a}(z, \zeta) = e\left(\frac{a(a - 1)}{2}\right)g_{-a, a - b - \frac{1}{4}}(\omega z, \zeta),
\]
we have the first formula. To get the second formula, substitute \(z = \omega^2 z\) into the first formula. These formulas yield the others.

Lemma 4. For \(\tau = \zeta\), we have
\[
\begin{align*}
\vartheta_{0, 0}((1 + \zeta)z) &= \frac{e\left(\frac{1}{24}\right)e((\omega^2 + \frac{1}{2})z^2)}{\vartheta_{0, 0}(0)^2}\vartheta_{\frac{1}{2}, 0}(z)\vartheta_{0, 0}(z)^2 - i\vartheta_{\frac{1}{2}, 0}(z)^2, \\
\vartheta_{0, \frac{1}{2}}((1 + \zeta)z) &= \frac{e\left(\frac{1}{24}\right)e((\omega^2 + \frac{1}{2})z^2)}{\vartheta_{\frac{1}{2}, 0}(0)^2}\vartheta_{0, \frac{1}{2}}(z)\vartheta_{0, 0}(z)^2 - \vartheta_{\frac{1}{2}, 0}(z)^2, \\
\vartheta_{\frac{1}{2}, 0}((1 + \zeta)z) &= \frac{e\left(\frac{1}{24}\right)e\left((\omega^2 + \frac{1}{2})z^2\right)}{\vartheta_{\frac{1}{2}, 0}(0)^2}\vartheta_{\frac{1}{2}, 0}(z)\vartheta_{0, 0}(z)^2 + i\vartheta_{\frac{1}{2}, 0}(z)^2, \\
\vartheta_{\frac{1}{2}, \frac{1}{2}}((1 + \zeta)z) &= \frac{e\left(\frac{1}{24}\right)e\left((\omega^2 + \frac{1}{2})z^2\right)}{\vartheta_{\frac{1}{2}, 0}(0)^2}\vartheta_{\frac{1}{2}, \frac{1}{2}}(z)\vartheta_{0, 0}(z)^2 + i\vartheta_{\frac{1}{2}, \frac{1}{2}}(z)^2.
\end{align*}
\]

Proof. We apply addition formulas in Fact 3 to \(z_1 = z\) and \(z_2 = \zeta z\), and use Lemma 3. For example, we have
\[
\begin{align*}
\vartheta_{0, 0}((1 + \zeta)z)\vartheta_{0, 0}((1 - \zeta)z)\vartheta_{0, 0}(0)^2 &= \vartheta_{0, \frac{1}{2}}(z)^2\vartheta_{0, \frac{1}{2}}(z)\vartheta_{0, 0}(z)^2 + \vartheta_{\frac{1}{2}, 0}(z)^2\vartheta_{\frac{1}{2}, 0}(z)^2, \\
\vartheta_{0, 0}(1 - \zeta) &= \vartheta_{0, 0}(\omega z) = \vartheta_{0, 0}(\bar{\omega} z) = e\left(-\frac{1}{24}\right)e\left(\frac{x^2}{2\zeta}\right)\vartheta_{0, 0}(z), \\
\vartheta_{0, \frac{1}{2}}(z)^2 &= \vartheta_{0, \frac{1}{2}}(\omega z) = \vartheta_{0, \frac{1}{2}}(\bar{\omega} z) = e\left(-\frac{1}{6}\right)e\left(\frac{x^2}{\omega}\right)\vartheta_{\frac{1}{2}, 0}(z)^2, \\
\vartheta_{\frac{1}{2}, 0}(z)^2 &= \vartheta_{\frac{1}{2}, 0}(\omega z)^2 = \vartheta_{\frac{1}{2}, 0}(\bar{\omega} z)^2 = e\left(\frac{1}{12}\right)e\left(\frac{x^2}{\omega}\right)\vartheta_{\frac{1}{2}, 0}(z)^2,
\end{align*}
\]
which yield the first formula.

Lemma 5. Some theta constants \(\vartheta_{a, b}(0, \zeta)\) are related as follows:
\[
\begin{align*}
\vartheta_{0, \frac{1}{2}}(0, \zeta) &= e\left(-\frac{1}{24}\right)\vartheta_{0, 0}(0, \zeta), & \vartheta_{\frac{1}{2}, 0}(0, \zeta) &= e\left(-\frac{1}{24}\right)\vartheta_{0, 0}(0, \zeta), \\
\vartheta_{\frac{1}{2}, \frac{1}{2}}(0, \zeta) &= e\left(-\frac{1}{8}\right)\vartheta_{\frac{1}{2}, \frac{1}{2}}(0, \zeta), & \vartheta_{\frac{1}{2}, \frac{1}{2}}(0, \zeta) &= e\left(-\frac{17}{24}\right)\vartheta_{\frac{1}{2}, \frac{1}{2}}(0, \zeta). \\
\vartheta_{\frac{1}{2}, \frac{1}{2}}(0, \zeta) &= e\left(\frac{1}{18}\right)\frac{1}{\sqrt{2}}\vartheta_{0, 0}(0, \zeta), & \vartheta_{\frac{1}{2}, 0}(0, \zeta) &= e\left(\frac{1}{18}\right)\frac{\sqrt{3}}{\sqrt{2}}\vartheta_{0, 0}(0, \zeta).
\end{align*}
\]

Proof. By substituting \(z = 0\) and \(z = (\zeta + 1)/3\) into formulas in Lemma 3, we have the formulas in the first and second lines in this lemma. We show the formulas in the third line. Substitute \(z = (\zeta + 1)/3\) and \(z = (\zeta + 1)/6\) into the first formula in Lemma 4. Then we have
\[
\begin{align*}
\vartheta_{0, 0}(\zeta) &= \frac{e\left(\frac{1}{24}\right)e\left((\omega^2 + \frac{1}{2})(\zeta + 1)^2\right)}{\vartheta_{0, 0}(0)^2}\vartheta_{\frac{1}{2}, 0}(0, \zeta)\left\{ \vartheta_{0, 0}(\frac{\zeta + 1}{3})^2 - i\vartheta_{\frac{1}{2}, 0}(\frac{\zeta + 1}{3})^2 \right\}, \\
\vartheta_{0, 0}(\frac{\zeta}{2}) &= \frac{e\left(\frac{1}{24}\right)e\left((\omega^2 + \frac{1}{2})(\zeta + 1)^2\right)}{\vartheta_{0, 0}(0)^2}\vartheta_{\frac{1}{2}, 0}(\frac{\zeta}{3})\left\{ \vartheta_{0, 0}(\frac{\zeta + 1}{6})^2 - i\vartheta_{\frac{1}{2}, 0}(\frac{\zeta + 1}{6})^2 \right\}.
\end{align*}
\]
By using shown formulas in the lemma, we can transform these identities into

\[
\vartheta_{0,0}(0, \zeta)^3 = \frac{2}{\zeta} \vartheta_{1,1/2}(0, \zeta)^3, \\
\vartheta_{0,0}(0, \zeta)^3 = \vartheta_{1,1/2}(0, \zeta) (\vartheta_{1,1/2}(0, \zeta)^2 - \zeta \vartheta_{1,1/2}(0, \zeta)^2). 
\]

Note that the last identity is equivalent

\[
\vartheta_{1,1/2}(0, \zeta)^2 = \frac{\vartheta_{0,0}(0, \zeta)^3 + \zeta \vartheta_{1,1/2}(0, \zeta)^3}{\vartheta_{0,0}(0, \zeta)} = \frac{\zeta + 1}{2} \vartheta_{0,0}(0, \zeta)^2.
\]

By numerical computations, we can see that the identity

\[
\vartheta_{1,1/2}(0, \zeta) = e(\frac{1}{18}) \frac{1}{\sqrt{2}} \vartheta_{0,0}(0, \zeta)
\]

holds. This identity yields that

\[
\vartheta_{1,1/2}(0, \zeta)^2 = e(\frac{1}{36}) \frac{\sqrt{3}}{\sqrt{4}} \vartheta_{0,0}(0, \zeta)^2.
\]

By numerical computations, we can select a square root of \( e(1/36) \) so that identity between \( \vartheta_{1,1/2}(0, \zeta) \) and \( \vartheta_{0,0}(0, \zeta) \) holds. \( \square \)

4. The Schwarz map for \((\alpha, \beta, \gamma) = (1/4, 0, 1/2)\)

We study the Schwarz map for \((\alpha, \beta, \gamma) = (1/4, 0, 1/2)\) and its inverse by using an elliptic curve with \(i\)-action and \(\vartheta_{a,b}(z,i)\).

4.1. Abel-Jacobi map for \(C_i\). Let \(C_i\) be an algebraic curve in \(\mathbb{P}^2\) defined by

\[
C_i : s_i^2 = s_0 s_i^3 (s_1 - s_0).
\]

By affine coordinates \((t, u) = (s_1/s_0, s_2/s_0)\), \(C_i\) is expressed by

\[
u^4 = t^2(t - 1).
\]

Note that the point \((t, u) = (0, 0)\) in \(C_i\) is a node. We use the same symbol \(C_i\) for a non-singular model of \(C_i\). By a projection \(pr\) from the non-singular model \(C_i\) to the complex projective line \(\mathbb{P}^1\) arising from

\[
C_i \ni (t, u) \mapsto t \in \mathbb{C},
\]

we regard \(C_i\) as a branched covering \(\mathbb{P}^1\) with a covering transformation \(\rho_i\) arising from a map

\[
\rho_i : C_i \ni (t, u) \mapsto (t, iu) \in C_i.
\]

The branch points of \(pr\) are \(t = 0, 1, \infty\). Each preimage of \(pr^{-1}(1)\) and \(pr^{-1}(\infty)\) consists of a point; \(P_1 = pr^{-1}(1)\) and \(P_\infty = pr^{-1}(\infty)\) are expressed as \((t, u) = (1, 0)\) and \([s_0, s_1, s_2] = [0, 1, 0]\), respectively. On the other hand, the preimage \(pr^{-1}(0)\) consists of two points, which are denoted by \(P_{0,1}\) and \(P_{0,2}\). The point \(P_{0,1}\) corresponds to

\[
\lim_{x \to 0^+} (x, \sqrt[4]{x^2(x - 1)}), \quad \arg x^2(x - 1) = \pi
\]

for \(x\) in the open interval \((0, 1)\), and \(P_{0,2}\) is given by \(\rho_i(P_{0,1})\). By the Hurwitz formula, \(C_i\) is an elliptic curve.

Let \(I_\infty\) be an oriented path in \(C_i\) given by

\[
(x, \sqrt[4]{x^2(x - 1)}) \in C_i, \quad x \in [1, \infty],
\]

where \(\sqrt[4]{x^2(x - 1)}\) takes real values for \(x \in [1, \infty)\) and the interval \([1, \infty]\) is naturally oriented. We define a cycle \(B\) by \(I_\infty - \rho_i^{-1}I_\infty\) and a cycle \(A\) by \(\rho_i^{-1}B\). Since

\[
B \cdot A = 1,
\]
A and B form a basis of \( H_1(C_i, \mathbb{Z}) \).

The space of holomorphic 1-forms on \( C_i \) is one dimensional and it is spanned by a form expressed by

\[
\varphi = \frac{udt}{t(t-1)} = \frac{dt}{\sqrt{t^2(t-1)^3}}
\]

The period integral \( \int_B \varphi \) is evaluated as

\[
(1-i) \int_1^\infty \frac{dt}{\sqrt{t^2(t-1)^3}} = (1-i)B\left(\frac{1}{4}, \frac{1}{4}\right).
\]

On the other hand, we have

\[
\int_A \varphi = \int_{\rho_i(B)} \varphi = \int_B \rho_i^*(\varphi) = i \int_B \varphi.
\]

We normalize \( \varphi \) to \( \varphi_1 \) as

\[
\varphi_1 = \frac{1}{(1-i)B\left(\frac{1}{4}, \frac{1}{4}\right)} \varphi.
\]

Then we have

\[
\int_B \varphi_1 = 1, \quad \int_A \varphi_1 = i
\]

and the Abel-Jacobi map

\[
j_i : C_i \ni P = (x, \sqrt{x^2(x-1)}) \mapsto z = \int_{P_i} \varphi_1 \in E_i = \mathbb{C}/L_i,
\]

where \( L_i = Zi + \mathbb{Z} \subseteq \mathbb{C} \). The map \( j_i \) is an isomorphism between \( C_i \) and \( E_i \).

**Proposition 2.** The Abel-Jacobi map \( j_i \) sends points \( P_1, \ P_\infty, \ P_{0,1} \) and \( P_{0,2} \) to

\[
j_i(P_1) = 0, \quad j_i(P_\infty) = \frac{i+1}{2}, \quad j_i(P_{0,1}) = \frac{i}{2}, \quad j_i(P_{0,2}) = \frac{1}{2}
\]

as elements of \( E_i \).

**Proof.** It is clear that \( j_i(P_1) = 0 \) and \( j_i(P_\infty) = \frac{i+1}{2} \). We have

\[
j_i(P_{0,1}) = \frac{1}{(1-i)B\left(\frac{1}{4}, \frac{1}{4}\right)} \int_{s_0}^0 \exp(\pi i/4) \frac{\sqrt{s^2(1-s)}}{s(s-1)} = \frac{i}{\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)^2 \Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}\right)^2 \Gamma\left(\frac{1}{4}\right)}
\]

\[
= \frac{i}{\sqrt{2}} \frac{\pi}{\pi/\sin \frac{\pi}{4}} = \frac{i}{2}.
\]

Since \( P_{0,2} = \rho_1(P_{0,1}) \), \( j_i(P_{0,2}) \) is equal to \( ij_i(P_{0,1}) = -\frac{1}{2} \equiv \frac{1}{2} \mod L_i \).

We consider the relation between the Abel-Jacobi map \( j_i \) and the Schwarz map

\[
x \mapsto \frac{f_1(x)}{(1-i)f_2(x)} = \frac{2\sqrt{2i}}{B\left(\frac{1}{4}, \frac{1}{4}\right)} \sqrt{1-xF\left(\frac{1}{4}, \frac{5}{4}, \frac{1}{4}, \frac{1}{4}\right)}
\]

for \( \mathcal{F}\left(\frac{1}{4}, 0, \frac{1}{4}\right) \). By Corollary 1, its monodromy group is generated by the three transformations

\[
N_0 : z \mapsto -z + i, \quad N_1 : z \mapsto iz, \quad (N_0N_1)^{-1} : z \mapsto iz + 1,
\]

and this group is isomorphic to the semi-direct product \( \langle i \rangle \ltimes \mathbb{Z}\langle i \rangle \). Note that the information of a branch of \( \sqrt{x^2(x-1)} \) is lost in the Schwarz map. Thus we can regard the Schwarz map as the Abel-Jacobi map \( j_i \) modulo the actions of \( \rho_i \) and \( i \); that is

\[
C_i/\langle \rho_i \rangle \ni x \mapsto \int_1^x \varphi_1 \in E_i/\langle i \rangle,
\]

where \( \langle \rho_i \rangle \) and \( \langle i \rangle \) are the groups generated by \( \rho_i \) and \( i \), respectively.
4.2. The inverse of \( j_i \). In this subsection, we express the inverse of the Abel-Jacobi map \( j_i \) in terms of \( \vartheta_{a,b}(z,\tau) \). We fix the variable \( \tau \) to \( i \) and denote \( \vartheta_{a,b}(z, i) \) by \( \vartheta_{a,b}(z) \) in short. Since the pull-backs \( j_i^{-1}(t) \) and \( j_i^{-1}(u) \) are elliptic functions with respect to the lattice \( L_i \), they can be expressed as
\[
j_i^{-1}(t) = \theta_{a}(z), \quad j_i^{-1}(u) = \theta_{a}(z)
\]
in terms of \( \vartheta_{a,b}(z) \). It turns out that the map
\[
E_i \ni z \mapsto (\theta_{a}(z), \theta_{a}(z)) \in C_i
\]
is the inverse of \( j_i \).

**Theorem 1.** The inverse of \( j_i : C_i \ni (t, u) \mapsto z \in E_i \) is given by
\[
t = 2 \frac{\vartheta_{0,1}(z, i)^2 \vartheta_{0,0}(z, i)^2}{\vartheta_{0,0}(z, i)^4} = 1 - \frac{\vartheta_{1,1}(z, i)^4}{\vartheta_{0,0}(z, i)^4},
\]
where
\[
u = -(1 - i) \frac{\vartheta_{0,1}(z, i) \vartheta_{0,0}(z, i) \vartheta_{1,1}(z, i) }{\vartheta_{0,0}(z, i)^4}.
\]
The holomorphic 1-form \( \varphi = \frac{udt}{t(t-1)} \) on \( C_i \) corresponds to
\[
2(1 - i)\pi \vartheta_{0,0}(0, i)^2 dz = (1 - i)B(\frac{1}{4}, \frac{1}{4}) dz
\]
by the Abel-Jacobi map \( j_i \).

**Proof.** We regard the coordinate \( t \) of \( C_i \) as a meromorphic function on \( C_i \). Its divisor is
\[
2P_{0,1} + 2P_{0,2} - 4P_{\infty}.
\]
We construct an elliptic function for \( L_i \) with zero of order 2 at \( z = \frac{i}{2} \) and pole of order 4 at \( z = \frac{i + 1}{2} \). Since
\[
2 \cdot \left(0, \frac{1}{2} \right) + 2 \cdot \left(\frac{1}{2}, 0 \right) \equiv 4 \cdot \left(\frac{1}{2}, \frac{1}{2} \right) \mod \mathbb{Z}^2,
\]
the function
\[
\frac{\vartheta_{0,1}(z)^2 \vartheta_{0,0}(z)^2}{\vartheta_{0,0}(z)^4}
\]
becomes an elliptic function for \( L_i \). Moreover, it has zero of order 2 at \( z = \frac{i}{2}, \frac{1}{2} \), and pole of order 4 at \( z = \frac{i + 1}{2} \), since \( \vartheta_{a,b}(z) = 0 \) if and only if \( z \equiv (-a+\frac{1}{2})i+(-b+\frac{1}{2}) \mod \mathbb{Z}^2 \). Thus the pull-back \( F(P) \) of this function under the map \( j_i \) is a constant multiple of \( t \) by Proposition 2. Let us determine this constant. Lemma 1 yields that
\[
\frac{\vartheta_{0,1}(0)^2 \vartheta_{0,0}(0)^2}{\vartheta_{0,0}(0)^4} = \frac{\vartheta_{0,1}(0)^4}{\vartheta_{0,0}(0)^4} = \frac{1}{2}.
\]
Thus \( 2F(P) \) is equal to \( t \).

Similarly we regard \( t - 1 \) as a meromorphic function on \( C_i \) whose divisor is
\[
4P_1 - 4P_\infty.
\]
The function
\[
\frac{\vartheta_{1,1}(z)^4}{\vartheta_{0,0}(z)^4}
\]
becomes an elliptic function for $L_i$ with zero of order 4 at $z = 0$ and pole of order 4 at $z = \frac{i + 1}{2}$. The pull-back of this function under the map $j_i$ is a constant multiple of $t - 1$. By substituting $P_{0,1}$ into this pull-back, we can determine the constant. We have

$$t - 1 = -\frac{\partial_{+ \frac{1}{2}}(z)^4}{\partial_{0,0}(z)^4}.$$ 

By regarding the coordinate $u$ of $C_i$ as a meromorphic function on $C_i$, we see that its divisor is

$$P_{0,1} + P_{0,2} + P_1 - 3P_{\infty}.$$ 

Thus it is the pull-back of

$$c \cdot \frac{\partial_{0,\frac{1}{2}}(z)\partial_{\frac{1}{2}}(z)\partial_{2\frac{1}{2}}(z)}{\partial_{0,0}(z)^3}$$

under $j_i$, where $c$ is a constant. Let us determine $c$. By $u^4 = t^2(t - 1)$, we have

$$c^4 \cdot \frac{\partial_{0,\frac{1}{2}}(z)^4\partial_{\frac{1}{2}}(z)^4\partial_{2\frac{1}{2}}(z)^4}{\partial_{0,0}(z)^{12}} = \frac{4\partial_{0,\frac{1}{2}}(z)^4\partial_{\frac{1}{2}}(z)^4}{\partial_{0,0}(z)^8} \cdot -\frac{\partial_{2\frac{1}{2}}(z)^4}{\partial_{0,0}(z)^4},$$

which yields that $c^4 = -4$, i.e., $c = i^k \cdot (1 + i)$ for some $k \in \{0, 1, 2, 3\}$.

By the expressions $t$, $t - 1$ and $u$ in terms of $\vartheta_{a,b}(z)$, it turns out that the holomorphic 1-from $\varphi = \frac{udt}{t(t - 1)}$ corresponds to

$$i^k(1 + i) \cdot \frac{\partial_{0,\frac{1}{2}}(z)\partial_{\frac{1}{2}}(z)\partial_{2\frac{1}{2}}(z)}{\partial_{0,0}(z)^3} \cdot \frac{\partial_{0,0}(z)^4}{2\partial_{0,\frac{1}{2}}(z)^2\partial_{\frac{1}{2}}(z)^2} \cdot -\frac{\partial_{2\frac{1}{2}}(z)^4}{\partial_{0,0}(z)^4} \cdot$$

$$\frac{4\{\partial_{0,0}(z)^4\partial_{0,0}(z)^4\partial_{2\frac{1}{2}}(z)^4 - \partial_{\frac{1}{2}}(z)^3\partial_{2\frac{1}{2}}(z)^4\partial_{0,0}(z)(z)\partial_{0,0}(z)\}}{\partial_{0,\frac{1}{2}}(z)\partial_{\frac{1}{2}}(z)}dz,$$

which should be a constant multiple of $dz$. By putting $z = 0$ and using Fact 1, we have

$$\varphi = -2i^k(1 + i)\pi \vartheta_{0,0}(0)^2 j_i^*(dz).$$

Since $\vartheta_{0,0}(0)^2$ and

$$B\left(\frac{1}{4}, \frac{1}{4}\right) = \int_{1}^{\infty} \varphi = \int_{j_i(P_1)}^{j_i(P_{\infty})} -2i^k(1 + i)\pi \vartheta_{0,0}(0)^2dz = -2i^k(1 + i)\pi \vartheta_{0,0}(0)^2 \frac{1 + i}{2}$$

are positive real, $k$ is equal to 1. Hence we have the expressions of $u$ and $\varphi$. □

**Corollary 2.** Let $z \in E_j$ be the image of $(t, u) \in C_i$ under the Abel-Jacobi map $j_i$.

Then we have

$$\frac{iu^2}{t} = \frac{\vartheta_{+ \frac{1}{2}}(z)^2}{\vartheta_{0,0}(z)^2}, \quad 1 + \frac{u^2}{t} = \sqrt{2} \frac{\vartheta_{0,\frac{1}{2}}(z)^2}{\vartheta_{0,0}(z)^2}, \quad 1 - \frac{u^2}{t} = \sqrt{2} \frac{\vartheta_{\frac{1}{2}}(z)^2}{\vartheta_{0,0}(z)^2}.$$ 

Moreover, $\vartheta_{a,b}(z)$’s satisfy relations

$$\sqrt{2}\vartheta_{0,\frac{1}{2}}(z)^2 = \vartheta_{0,0}(z)^2 + \vartheta_{\frac{1}{2}}(z)^2, \quad \sqrt{2}\vartheta_{\frac{1}{2},0}(z)^2 = \vartheta_{0,0}(z)^2 - \vartheta_{+ \frac{1}{2}}(z)^2.$$

**Proof.** The first identity is a direct consequence of Theorem 1. The right hand side of the second identity is an elliptic function with respect to $L_i$. It has zero of order 2 at $j_i(P_{0,1})$ and pole of order 2 at $j_i(P_{\infty})$. Since $P_{0,1}$ corresponds to the limit as $t \to 0$ given by the branch of $u$ with $\arg(u) = \frac{\pi}{4}$ on the interval $(0, 1)$,

$$\lim_{t \to 0} \frac{i u^2}{t} = -1.$$ 

By comparing the zero and pole of both functions, $1 + i\frac{u^2}{t}$ is a
constant multiple of the pull-back of $\frac{\vartheta_{0,\frac{1}{2}}(z)^2}{\vartheta_{0,0}(z)^2}$ under $j_t$. We can determine this constant by the substitution $z = 0$. The third identity is obtained by the action of $p_t$ on the second identity. By eliminating $i\frac{u^2}{t}$ from these identities, we have the relations among $\vartheta_{a,b}(z)$'s. 

**Corollary 3.** We have

$$\vartheta_{0,0}(0,i) = \frac{\Gamma(1)}{\sqrt[4]{\pi}} = \sqrt[4]{\pi}, \quad \vartheta_{0,\frac{1}{2}}(0,i) = \vartheta_{\frac{1}{2},0}(0,i) = \frac{\Gamma(1)}{\sqrt{(2\pi)^4}} = \sqrt[4]{2\pi}. $$

**Proof.** By Theorem 1, we have

$$2\pi \vartheta_{0,0}(0)^2 = B\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{\Gamma(1)}{\sqrt[4]{\pi}}. $$

Note that $\vartheta_{0,0}(0)$ and $\Gamma(1)$ are positive. To show the rest, use the inversion formula for the $\Gamma$-function and Lemma 1. 

**Corollary 4.** The inverse of the Schwarz map (4.1) for $F(\frac{1}{4}, 0, \frac{1}{2})$ is given by

$$x = 2\frac{\vartheta_{0,\frac{1}{2}}(z)^2 \vartheta_{\frac{1}{2},0}(z)^2}{\vartheta_{0,0}(z)^4} = 1 - \frac{\vartheta_{\frac{1}{2},\frac{1}{2}}(z)^4}{\vartheta_{0,0}(z)^4}. $$

**Proof.** It is clear by Theorem 1. We can check this map is invariant under the action of $(i)$ by Lemma 1. 

**Corollary 5.** For any point $z$ around 0, we have

$$- \frac{2\sqrt[4]{2\pi}}{\Gamma(1)} \cdot \frac{\vartheta_{\frac{1}{2},\frac{1}{2}}(z,i)}{\vartheta_{0,0}(z,i)} \cdot F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; \frac{\vartheta_{\frac{1}{2},\frac{1}{2}}(z,i)^4}{\vartheta_{0,0}(z,i)^4}\right) = z. $$

**Proof.** By Corollary 4, we have

$$\frac{2\sqrt[4]{2\pi}}{\Gamma(1)} \cdot \frac{\vartheta_{\frac{1}{2},\frac{1}{2}}(z,i)}{\vartheta_{0,0}(z,i)} \cdot F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; \frac{\vartheta_{\frac{1}{2},\frac{1}{2}}(z,i)^4}{\vartheta_{0,0}(z,i)^4}\right) \equiv z $$

modulo the monodromy group of $F(\frac{1}{4}, 0, \frac{1}{2})$. Since the both sides of the above become 0 for $z = 0$, their difference is represented as the group $(i)$. Consider the limit of the both sides as $z \to i\frac{1}{2}$ along the imaginary axis. Use

$$\frac{\vartheta_{\frac{1}{2},\frac{1}{2}}(\frac{1}{2}, i)}{\vartheta_{0,0}(\frac{1}{2}, i)} = e^{\frac{1}{2}} \cdot \frac{\vartheta_{0,\frac{1}{2}}(0,i)}{\vartheta_{\frac{1}{2},0}(0,i)} = -i, $$

and the Gauss-Kummer formula

$$F(\alpha, \beta, \gamma; 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)}$$

for $\text{Re}(\gamma - \alpha - \beta) > 0$. 

**4.3. $(1 + i)$-multiplication.**

**Theorem 2.** Let $z \in E_i$ be the image of $(t, u) \in C_i$ under the Abel-Jacobi map $j_t$. Then we have

$$(4.2) \quad j_t^{-1}(1 + i)z = \left(\frac{t - 2}{t}, (1 + i)\frac{u(2 - t)}{t^2}\right). $$
Proof. We set
\[(t', u') = \frac{1}{j_i^{-1}}((1 + i)z)\].

By Theorem 1, we have
\[
t' = 2 - \frac{\vartheta_{0, 0}(0)\vartheta_{\frac{1}{2}, 0}((1 + i)z)^2}{\vartheta_{\frac{1}{2}, 0}(0)(1 + i)z)^4}
\]
\[
u' = -(1 - i)\frac{\vartheta_{0, 0}(0)\vartheta_{\frac{1}{2}, 0}((1 + i)z)^2}{\vartheta_{\frac{1}{2}, 0}(0)(1 + i)z)^4}.
\]

We transform them as
\[
t' = \frac{\vartheta_{0, 0}(0)\vartheta_{\frac{1}{2}, 0}(0)}{\vartheta_{\frac{1}{2}, 0}(0)^2} \cdot \frac{\vartheta_{0, 0}(z)^4 - \vartheta_{\frac{1}{2}, 0}(z)^2\vartheta_{\frac{1}{2}, 0}(0)^2}{\vartheta_{\frac{1}{2}, 0}(z)^4\vartheta_{\frac{1}{2}, 0}(0)^2} = \left(\frac{2}{t} - 1\right)^2,
\]
\[
u' = -(1 - i) \cdot \frac{\vartheta_{0, 0}(0)\vartheta_{0, 0}(z)^4 - \vartheta_{0, 0}(z)^2\vartheta_{0, 0}(0)^2}{\vartheta_{0, 0}(0)^2} \cdot \frac{\vartheta_{0, 0}(0)\vartheta_{\frac{1}{2}, 0}(0)}{\vartheta_{\frac{1}{2}, 0}(0)^3},
\]
\[
= \sqrt{2} \cdot \frac{\vartheta_{0, 0}(0)\vartheta_{\frac{1}{2}, 0}(0)}{\vartheta_{\frac{1}{2}, 0}(0)^2} \cdot \frac{\vartheta_{0, 0}(z)^4 - \vartheta_{\frac{1}{2}, 0}(z)^2\vartheta_{\frac{1}{2}, 0}(0)^2}{\vartheta_{\frac{1}{2}, 0}(z)^4\vartheta_{\frac{1}{2}, 0}(0)^2} = \left(\frac{2}{t} - 1\right)^2
\]
\[
= \frac{4}{(1 - i)^2} - \frac{2}{(1 - i)^2} = (1 + i)\frac{u(2 - t)}{t^2},
\]
by Lemma 2 and Theorem 1. □

5. The Schwarz Map for \((\alpha, \beta, \gamma) = (1/3, 0, 1/2)\)

In this section, we study the Schwarz map for \((\alpha, \beta, \gamma) = (1/3, 0, 1/2)\) and its inverse by using an elliptic curve with \(\zeta\)-action and \(\vartheta_{a,b}(z, \zeta)\), where \(\zeta = \frac{1 + \sqrt{3}i}{2}\).

5.1. The Abel-Jacobi Map for \(C_\zeta\).

Let \(C_\zeta\) be an algebraic curve in \(\mathbb{P}^2\) defined by
\[
C_\zeta : s_3^2 = s_0^3s_1^2(s_1 - s_0).
\]

By affine coordinates \((t, u) = (s_1/s_0, s_2/s_0)\), \(C_\zeta\) is expressed as
\[
u^6 = t^4(t - 1).
\]

Note that \((t, u) = (0, 0)\) is a triple node and \([s_0, s_1, s_2] = [0, 1, 0]\) is a node. We use the same symbol \(C_\zeta\) for a non-singular model of \(C_\zeta\). We regard \(C_\zeta\) as a cyclic 6-fold covering of the \(t\)-space with covering transformation
\[
\rho_\zeta : (t, u) \mapsto (t, \zeta u), \quad \zeta = \frac{1 + \sqrt{3}i}{2}.
\]

The branching information of this covering is as in Table 2. Here we set some points in the non-singular model \(C_\zeta\) as follows:
\[
P_{0, 1} = \lim_{t \to 0} (t, t^{1/2}(t - 1)^{1/6}), \quad P_{0, 2} = \rho_\zeta(P_{0, 1}), \quad P_{0, 3} = \rho_\zeta^2(P_{0, 1}),
\]
The preimage of \( I \) under the natural projection consists of six copies \( \rho^i \cdot I \) \((i = 0, 1, \ldots, 5)\). Since the terminal points of \( \rho^2 \cdot I \equiv I_{\infty} \) coincide with that of \( I_{\infty} \), the formal difference \( B = \rho^1 \cdot I_{\infty} - \rho^2 \cdot I_{\infty} = (1 - \rho^2) \cdot I_{\infty} \) is a cycle of \( C_\zeta \). Let \( A \) be the cycle \( \rho \cdot B \). Then the intersection number \( A \cdot B \) of the cycles \( A \) and \( B \) is 1. Thus the cycles \( A \) and \( B \) form a basis of the first homology group \( H_1(C_\zeta, \mathbb{Z}) \) of \( C_\zeta \).

A non-zero holomorphic 1-form \( \psi \) on \( C_\zeta \) is given by

\[
\psi = \frac{t^2 dt}{u^5} = \frac{udt}{t(t-1)} = \frac{t^{1/2}(t-1)^{1/6} dt}{t(t-1)}.
\]

It is easy to see that

\[
\rho^*(\psi) = \zeta \psi.
\]
Note that
\[
\int_{I_{\infty}} \psi = \int_{1}^{\infty} t^{1/2 - 1}(t - 1)^{1/6 - 1} dt = \int_{0}^{1} s^{1/3 - 1}(1 - s)^{1/6 - 1} ds = B\left(\frac{1}{3}, \frac{1}{6}\right),
\]
\[
\int_{A} \psi = \zeta(1 - \zeta^2)B\left(\frac{1}{3}, \frac{1}{6}\right), \quad \int_{B} \psi = (1 - \zeta^2)B\left(\frac{1}{3}, \frac{1}{6}\right).
\]
We normalize \( \psi \) to \( \psi_1 \) as
\[
\psi_1 = \frac{1}{(1 - \zeta^2)B\left(\frac{1}{3}, \frac{1}{6}\right)} \psi,
\]
then we have
\[
\int_{A} \psi_1 = \zeta, \quad \int_{B} \psi_1 = 1.
\]
The Abel-Jacobi map is defined by
\[
\psi_{\zeta} : C_{\zeta} \ni P \mapsto \int_{P} \psi_1 \in E_\zeta = \mathbb{C}/L_\zeta,
\]
where \( L_\zeta = \mathbb{Z}\zeta + \mathbb{Z} \subset \mathbb{C} \). The map \( \psi_{\zeta} \) is an isomorphism between \( C_{\zeta} \) and \( E_\zeta \).

**Proposition 3.** We have
\[
\psi_{\zeta}(P_1) = 0, \quad \psi_{\zeta}(P_{\infty, 1}) = \frac{\zeta + 1}{3}, \quad \psi_{\zeta}(P_{\infty, 2}) = \frac{2\zeta + 2}{3},
\]
\[
\psi_{\zeta}(P_{0, 1}) = \frac{\zeta}{2}, \quad \psi_{\zeta}(P_{0, 2}) = \frac{\zeta + 1}{2}, \quad \psi_{\zeta}(P_{0, 3}) = \frac{1}{2}
\]
as elements of \( E_\zeta \).

**Proof.** It is obvious that \( \psi_{\zeta}(P_1) = 0 \). It is easy to see that
\[
\psi_{\zeta}(P_{\infty, 1}) = \int_{I_{\infty}} \psi_1 = \frac{1}{1 - \zeta^2} = \frac{\zeta + 1}{3},
\]
\[
\psi_{\zeta}(P_{\infty, 2}) = \int_{P_{\infty}} \psi_1 = \int_{I_{\infty}} \rho_\zeta'(\psi_1) = \zeta \psi_{\zeta}(P_{\infty, 1}) = \frac{\zeta^2 + \zeta}{3} = \frac{2\zeta + 2}{3} \mod L_\zeta.
\]
Note that
\[
\psi_{\zeta}(P_{0, 1}) = \int_{I_0} \psi_1 = \frac{1}{(1 - \zeta^2)B\left(\frac{1}{3}, \frac{1}{6}\right)} \int_{1}^{0} t^{1/2}(t - 1)^{1/6} \frac{dt}{t(t - 1)},
\]
\[
\int_{1}^{0} t^{1/2}(t - 1)^{1/6} \frac{dt}{t(t - 1)} = e\left(\frac{1}{12}\right) \int_{1}^{1} t^{1/2}(1 - t)^{1/6} \frac{dt}{t(1 - t)} = e\left(\frac{1}{12}\right) B\left(\frac{1}{2}, \frac{1}{6}\right).
\]
Thus we have
\[
\psi_{\zeta}(P_{0, 1}) = \frac{e\left(\frac{1}{12}\right) B\left(\frac{1}{2}, \frac{1}{6}\right)}{1 - \zeta^2} = \frac{(\zeta + 1)e\left(\frac{1}{12}\right)}{3} \frac{\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{3}\right)} = \frac{3\sqrt{3}e\left(\frac{1}{6}\right)}{2} \frac{\sqrt{3}}{2} = \frac{\zeta}{2}.
\]
The rests are obtained as
\[
\psi_{\zeta}(P_{0, 2}) = \psi_{\zeta}(P_{0, 1}) \equiv \frac{\zeta + 1}{2} \mod L_\zeta, \quad \psi_{\zeta}(P_{0, 3}) = \zeta^2 \psi_{\zeta}(P_{0, 1}) \equiv \frac{1}{2} \mod L_\zeta,
\]
since \( P_{0, 2} = \rho_\zeta \cdot P_{0, 1} \) and \( P_{0, 3} = \rho_\zeta^2 \cdot P_{0, 1} \).

We consider the relation between the Abel-Jacobi map \( \psi_{\zeta} \) and the Schwarz map
\[
(5.1) \quad x \mapsto \frac{f_1(x)}{(1 - \zeta^2) f_2(x)} = \frac{2\sqrt{3} \zeta}{B\left(\frac{1}{3}, \frac{1}{6}\right)} \psi_1(x) F\left(\frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - x\right)
\]
for \( F\left(\frac{1}{3}, 0, \frac{1}{2}\right) \). By Corollary 1, its monodromy group is generated by the three transformations
\[
N_0 : z \mapsto -z + \zeta, \quad N_1 : z \mapsto \zeta z, \quad (N_0 N_1)^{-1} : z \mapsto \zeta^2 z + 1,
\]
and this group is isomorphic to the semi-direct product \(\langle \zeta \rangle \ltimes \mathbb{Z}[\zeta] \). Note that the information of a branch of \( u = \sqrt[3]{x^3(x-1)} \) is lost in the Schwarz map. Thus we can regard the Schwarz map as the Abel-Jacobi map \( j_\zeta \) modulo the actions of \( \rho_\zeta \) and \( \zeta \); that is

\[
C_\zeta / (\rho_\zeta) \ni x \mapsto \int_1^x \psi_1 \in E_\zeta / \langle \zeta \rangle.
\]

5.2. The inverse of \( j_\zeta \). We express the inverse of the Abel-Jacobi map \( j_\zeta \). We regard the coordinates \( t \) and \( u \) as meromorphic functions on \( C_\zeta \). The pull-backs \( j_{\zeta}^{-1}(t) \) and \( j_{\zeta}^{-1}(u) \) are elliptic functions with respect to the lattice \( L_\zeta \), they can be expressed as

\[
j_{\zeta}^{-1}(t) = \theta_t(z), \quad j_{\zeta}^{-1}(u) = \theta_u(z)
\]

in terms of theta functions with characteristics. It turns out that the map

\[
E_\zeta \ni z \mapsto (\theta_t(z), \theta_u(z)) \in C_\zeta
\]

is the inverse of \( j_\zeta \).

Lemma 6. Let \( z \) be the image of \((t, u) \in C_\zeta \) under the Abel-Jacobi map. Then we have

\[
1 + \frac{t}{u^2} = \sqrt{3} \vartheta_{0,0}(z, \zeta)^2, \quad 1 + \frac{\zeta^2 t}{u^2} = -\sqrt{3} \vartheta_{1,2}(z, \zeta)^2, \quad 1 + \frac{\zeta^4 t}{u^2} = \sqrt{3} \vartheta_{1,1}(z, \zeta)^2.
\]

Proof. By Table 3, we have

\[
1 + \frac{t}{u^2} = c \cdot \vartheta_{0,0}(z)^2,
\]

where \( c \) is a constant. We substitute \( P_{0,1} \) into the above, we have

\[
1 - \omega = c \cdot \frac{\vartheta_{0,0}(\zeta/2)^2}{\vartheta_{1,1}(\zeta/2)^2} = c \cdot \left( - \frac{\vartheta_{1,2}(0)^2}{\vartheta_{0,0}(0)^2} \right) = -c \cdot \left( \frac{1}{6} \right),
\]

which yields \( c = \sqrt{3} \). The rest can be shown similarly.

Lemma 7. The functions \( \vartheta_{1,1}(z, \zeta)^2 \) and \( \vartheta_{1,0}(z, \zeta)^2 \) are expressed as linear combinations of \( \vartheta_{0,0}(z, \zeta)^2 \) and \( \vartheta_{1,0}(z, \zeta)^2 \):

\[
\vartheta_{1,1}(z, \zeta)^2 = e \left( \frac{1}{12} \right) \left( \vartheta_{0,0}(z, \zeta)^2 - \omega^2 \vartheta_{1,1}(z, \zeta)^2 \right),
\]

\[
\vartheta_{1,0}(z, \zeta)^2 = e \left( \frac{1}{12} \right) \left( \vartheta_{0,0}(z, \zeta)^2 + \omega \vartheta_{1,0}(z, \zeta)^2 \right).
\]

Proof. By Lemma 6, we have

\[
-\sqrt{3} \frac{\vartheta_{1,0}(z)^2}{\vartheta_{1,1}(z)^2} - 1 = \omega \frac{t}{u^2} = \omega \left( \sqrt{3} \frac{\vartheta_{0,0}(z)^2}{\vartheta_{1,1}(z)^2} - 1 \right),
\]

\[
\sqrt{3} \frac{\vartheta_{1,0}(z)^2}{\vartheta_{1,1}(z)^2} - 1 = \omega^2 \frac{t}{u^2} = \omega^2 \left( \sqrt{3} \frac{\vartheta_{0,0}(z)^2}{\vartheta_{1,1}(z)^2} - 1 \right),
\]

which yield this lemma.

Lemma 8. Let \( z \) be the image of \((t, u) \in C_\zeta \) under the Abel-Jacobi map. Then we have

\[
\frac{u^3}{t(t-1)} = e \left( \frac{1}{8} \right) \sqrt{27} \frac{\vartheta_{0,0}(z, \zeta) \vartheta_{0,2}(z, \zeta) \vartheta_{1,0}(z, \zeta)}{\vartheta_{1,1}(z, \zeta)^3}.
\]
Proof. By Table 3, we have

$$\frac{u^3}{t(t - 1)} = c' \frac{\vartheta_{0,0}(z) \vartheta_{0,1}(z) \vartheta_{0,0}(z)}{\vartheta_{1/2,1}(z)^3},$$

where $c'$ is a constant. We consider the limit as $t \to \infty$ with $t \in (1, \infty)$, $u \in (0, \infty)$. The left hand side of the above converges to 1. On the other hand, the right hand side of the above converges to

$$c' \frac{(\frac{1}{3} - \frac{1}{8}) \vartheta_{0,0}(0) \vartheta_{0,1}(0) \vartheta_{0,0}(0)}{\vartheta_{1/2,1}(0)^3} = c' \frac{(\frac{1}{3} - \frac{1}{8}) \vartheta_{1/2,1}(0)^3}{\vartheta_{1/2,1}(0)^3} = c' \left(1 + \frac{1}{\sqrt{27}}\right) \vartheta_{1/2,1}(0)^3$$

by Lemma 5. Hence we have $c' = e(\frac{1}{\sqrt{27}}) \vartheta_{1/2,1}(0)^3$. □

**Theorem 3.** The inverse of $\mathcal{J}_\zeta : C_\zeta \ni (t, u) \mapsto z \in E_\zeta$ is given by

$$t = \frac{-3\sqrt{3}i\vartheta_{0,0}(z, \zeta) \vartheta_{1/2,1}(z, \zeta)^2 \vartheta_{1,0}(z, \zeta)^2}{(\vartheta_{1/2,1}(z, \zeta)^2 - \vartheta_{1/2,1}(z, \zeta)^3)^2},$$

$$u = e\left(\frac{1}{8}\right) \vartheta_{0,0}(z, \zeta) \vartheta_{1/2,1}(z, \zeta) \vartheta_{1,0}(z, \zeta) \vartheta_{2,0}(z, \zeta)$$

$$\cdot \frac{(\vartheta_{1/2,1}(z, \zeta)^2 - \vartheta_{1/2,1}(z, \zeta)^3)^2}{(\vartheta_{1/2,1}(z, \zeta)^2 - 3\sqrt{3}i\vartheta_{0,0}(z, \zeta)^2 - \vartheta_{1/2,1}(z, \zeta)^6)}.$$

Proof. Note that

$$(1 + \frac{t}{u^2})(1 + \frac{c't}{u^2})(1 + \frac{c't^2}{u^2}) = 1 + \frac{t^3}{u^6} = 1 + \frac{1}{t-1}.$$

By Lemma 6, we have

$$1 + \frac{1}{t-1} = -3\sqrt{3}i \frac{\vartheta_{0,0}(z)^2 \vartheta_{1/2,1}(z)^2 \vartheta_{1,0}(z)^2}{\vartheta_{1/2,1}(z)^6},$$

which yields

$$t = \frac{3\sqrt{3}i\vartheta_{0,0}(z)^2 \vartheta_{1/2,1}(z)^2 \vartheta_{1,0}(z)^2}{3\sqrt{3}i\vartheta_{0,0}(z)^2 \vartheta_{1/2,1}(z)^2 \vartheta_{1,0}(z)^2 + \vartheta_{1/2,1}(z)^6}.$$

Rewrite $\vartheta_{0,0}(z)^2$ and $\vartheta_{1/2,1}(z)^2$ in the denominator of this expression by $\vartheta_{0,0}(z)^2$ and $\vartheta_{1/2,1}(z)^2$ by Lemma 7. Then it can be factorized as

$$3\sqrt{3}i\vartheta_{0,0}(z)^2 \vartheta_{1/2,1}(z)^2 \vartheta_{1,0}(z)^2 + \vartheta_{1/2,1}(z)^6 = -3\sqrt{3}i\vartheta_{0,0}(z)^2 - \vartheta_{1/2,1}(z)^6.$$

Hence we have the expression of $t$.

By Lemmas 6 and 8, the functions $1 + \frac{t}{u^2}$ and $\frac{u^3}{t(t - 1)}$ are expressed in terms $\vartheta_{a,b}(z, \zeta)$. We have

$$u = \frac{u^3}{t(t - 1)} \cdot \left(1 + \frac{t}{u^2}\right)^{-1} \cdot (t - 1)$$

$$= e\left(\frac{-1}{8}\right) \vartheta_{0,0}(z) \vartheta_{1/2,1}(z) \vartheta_{1,0}(z) \vartheta_{2,0}(z)$$

$$\cdot \frac{(\vartheta_{1/2,1}(z)^2 - \vartheta_{1/2,1}(z)^3)^2}{\vartheta_{1/2,1}(z)^3} - \vartheta_{1/2,1}(z)^6$$

$$\cdot \frac{3\sqrt{3}i\vartheta_{0,0}(z)^2 \vartheta_{1/2,1}(z)^2 \vartheta_{1,0}(z)^2 + \vartheta_{1/2,1}(z)^6}{3\sqrt{3}i\vartheta_{0,0}(z)^2 \vartheta_{1/2,1}(z)^2 \vartheta_{1,0}(z)^2 - \vartheta_{1/2,1}(z)^6}.$$
Note that the denominator of the last term is \((\sqrt{3}i\vartheta_{0,0}(z)^2 - \vartheta_{\frac{1}{2},\frac{1}{2}}(z)^2)^3\). Hence we have the expression of \(u\).

**Corollary 6.** The pull back of the holomorphic 1-from \(\psi = \frac{udt}{t(t-1)}\) under the map \(\mathcal{X}_\zeta^{-1}\) is

\[
e^{\left(-\frac{1}{8}\right)2\pi\sqrt{27}\vartheta_{0,0}(0,\zeta)^2dz}.
\]

The theta constant \(\vartheta_{0,0}(0,\zeta)\) is evaluated as

\[
\vartheta_{0,0}(0,\zeta) = e^{\left(-\frac{1}{48}\right)}\frac{\sqrt{3}}{\sqrt{4\pi}}\Gamma\left(\frac{1}{3}\right)^{3/2}.
\]

The other theta constants \(\vartheta_{a,b}(0,\zeta)\) are

\[
\vartheta_{0,\frac{1}{2}}(0,\zeta) = e^{\left(-\frac{1}{48}\right)}\frac{\sqrt{3}}{\sqrt{4\pi}}\Gamma\left(\frac{1}{3}\right)^{3/2},
\]

\[
\vartheta_{\frac{1}{2},0}(0,\zeta) = e^{\left(-\frac{11}{144}\right)}\frac{\sqrt{27}}{2\pi}\Gamma\left(\frac{1}{3}\right)^{3/2},
\]

\[
\vartheta_{\frac{1}{2},\frac{1}{2}}(0,\zeta) = e^{\left(-\frac{7}{144}\right)}\frac{\sqrt{3}}{\sqrt{2\pi}}\Gamma\left(\frac{1}{3}\right)^{3/2}.
\]

**Proof.** Recall that

\[
\frac{t}{t-1} = -3\sqrt{3i}\frac{\vartheta_{0,0}(z)^2\vartheta_{0,\frac{1}{2}}(z)^2\vartheta_{\frac{1}{2},0}(z)^2}{\vartheta_{\frac{1}{2},\frac{1}{2}}(z)^6}.
\]

Thus we have

\[
\frac{dt}{t^2} = d\left(1 - \frac{1}{t}\right) = d\left(\frac{t-1}{t}\right) = \frac{dz}{-3\sqrt{3i}}\left[\frac{6\vartheta_{\frac{1}{2},\frac{1}{2}}(z)^2\vartheta_{0,\frac{1}{2}}(z)\vartheta_{0,\frac{1}{2}}(z)\vartheta_{\frac{1}{2},0}(z)^2\vartheta_{0,0}(z)\vartheta_{0,\frac{1}{2}}(z)^2\vartheta_{\frac{1}{2},\frac{1}{2}}(z)^2\vartheta_{\frac{1}{2},0}(z)^2 - \vartheta_{\frac{1}{2},\frac{1}{2}}(z)^6\cdot(\vartheta_{0,0}(z)^2\vartheta_{0,\frac{1}{2}}(z)^2\vartheta_{\frac{1}{2},0}(z)^2)\vartheta_{0,0}(z)}{\vartheta_{0,0}(z)^2\vartheta_{0,\frac{1}{2}}(z)^2\vartheta_{\frac{1}{2},0}(z)^2}\right],
\]

where \(f(z)' = \frac{df(z)}{dz}\). Since \(\psi = u \cdot \frac{t}{t-1} \cdot \frac{dt}{t^2}\), the pull-back of \(\psi\) under the map \(\mathcal{X}_\zeta^{-1}\) is \(e^{\left(-\frac{1}{8}\right)2\pi\sqrt{27}\vartheta_{0,0}(0,\zeta)^2}\) times

\[
\frac{6\vartheta_{0,0}(z)^2\vartheta_{0,\frac{1}{2}}(z)^2\vartheta_{\frac{1}{2},0}(z)^2\vartheta_{\frac{1}{2},\frac{1}{2}}(z)^2\vartheta_{0,0}(z)\vartheta_{0,\frac{1}{2}}(z)^2\vartheta_{\frac{1}{2},\frac{1}{2}}(z)^2\vartheta_{\frac{1}{2},0}(z)^2 - \vartheta_{\frac{1}{2},\frac{1}{2}}(z)^6\cdot(\vartheta_{0,0}(z)^2\vartheta_{0,\frac{1}{2}}(z)^2\vartheta_{\frac{1}{2},0}(z)^2)\vartheta_{0,0}(z)}{\vartheta_{0,0}(z)^2\vartheta_{0,\frac{1}{2}}(z)^2\vartheta_{\frac{1}{2},0}(z)^2}\) \(dz\).

It should be a constant times \(dz\). We determine this constant by substituting \(z = 0\) into the above. By Fact 1 and Lemma 5, we have

\[
\mathcal{X}_\zeta^{-1}\psi = e^{\left(-\frac{1}{8}\right)2\pi\sqrt{27}\vartheta_{0,0}(0,\zeta)^2}.
\]

Note that

\[
B\left(\frac{1}{3},\frac{1}{6}\right) = \int_1^\infty \psi = \int_1^{\mathcal{X}_\zeta(P_{\infty,1})} \mathcal{X}_\zeta^{-1}\psi = e^{\left(-\frac{1}{8}\right)2\pi\sqrt{27}\vartheta_{0,0}(0,\zeta)^2} \cdot \left(\frac{\zeta+1}{3} - 0\right)
\]

by Proposition 3. The well-known formula

\[
\Gamma\left(\frac{1}{6}\right) = \frac{1}{\sqrt[3]{2\sqrt{\pi}}} \Gamma\left(\frac{1}{3}\right)^2
\]
yields that
\[ B(\frac{1}{3}, \frac{1}{6}) = \frac{\Gamma(\frac{1}{3}) \Gamma(\frac{1}{6})}{\Gamma(\frac{1}{2})} = \frac{\sqrt{3}}{\sqrt[3]{2\pi}} \Gamma(\frac{1}{3}). \]

Hence we evaluate the theta constant as
\[ \vartheta_{0,0}(0,\zeta)^2 = e^{\left(\frac{1}{24}\right)} \frac{\sqrt{3}}{\sqrt[3]{16\pi^2}} \Gamma(\frac{1}{3}). \]

We can determine the sign of \( \vartheta_{0,0}(0,\zeta) \) by a numerical computation. The rests can be obtained by Lemma 5.

**Corollary 7.** The inverse of the Schwarz map (5.1) for \( F(\frac{1}{3}, 0, \frac{1}{4}) \) is given by
\[ x = \frac{-3\sqrt{3}i \vartheta_{0,0}(z, \zeta)^2 \vartheta_{0,\frac{1}{4}}(z, \zeta)^2 \vartheta_{\frac{1}{4},0}(z, \zeta)^2}{(\sqrt{3}i \vartheta_{0,0}(z, \zeta)^2 - \vartheta_{\frac{1}{4},\frac{1}{4}}(z, \zeta)^2)^3}. \]

**Proof.** It is clear by Theorem 3. We can check this map is invariant under the action of \( (\zeta) \) by Lemma 3.

**Corollary 8.** For any point \( z \) around 0, we have
\[ \frac{\sqrt{16\pi\zeta^2}}{\Gamma(\frac{1}{3})^3}, \frac{1}{1-x} = \frac{F\left(\frac{1}{6}, \frac{1}{2}; \frac{7}{6}; \frac{1}{\vartheta_{\frac{1}{4},\frac{1}{4}}(z, \zeta)^6} \left(\vartheta_{\frac{1}{4},\frac{1}{4}}(z, \zeta)^2 - \sqrt{3}i \vartheta_{0,0}(z, \zeta)^2\right)^3\right)}{\vartheta_{\frac{1}{4},\frac{1}{4}}(z, \zeta)^6}, \]
where the branch of the square root is selected as \( \sqrt{\zeta^2} = \zeta \) for \( z = \frac{\zeta}{2} \).

**Proof.** Let \( z \) be the image of the Schwarz map (5.1). We have seen in Proof of Theorem 3 that
\[ \frac{1}{1-x} = \frac{3\sqrt{3}i \vartheta_{0,0}(z)^2 \vartheta_{0,\frac{1}{4}}(z)^2 \vartheta_{\frac{1}{4},0}(z)^6 + \vartheta_{\frac{1}{4},\frac{1}{4}}(z)^6}{\vartheta_{\frac{1}{4},\frac{1}{4}}(z)^6}. \]

Thus we have the desired identity modulo the monodromy group of \( F(\frac{1}{3}, 0, \frac{1}{4}) \). Since the both sides of the above become 0 for \( z = 0 \), their difference is represented as the group \( (\zeta) \). Consider the limit of the both sides as \( z \to \frac{\zeta}{2} \) along the segment connecting 0 and \( \frac{\zeta}{2} \). Since 1/(1-x) converges to 1 by this limit, it turns out that \( x \) converges to 0.

Use
\[ 1 - \sqrt{3}i \vartheta_{0,0}(\zeta, \zeta)^2 \vartheta_{0,\frac{1}{4}}(\zeta, \zeta)^2 \vartheta_{\frac{1}{4},0}(\zeta, \zeta)^2 = 1 + \sqrt{3}i \vartheta_{0,0}(0, \zeta)^2 \vartheta_{0,\frac{1}{4}}(0, \zeta)^2 \vartheta_{\frac{1}{4},0}(0, \zeta)^2 = 1 + \sqrt{3}i \zeta = \zeta^2 \]
and the Gauss-Kummer formula.

5.3. \((1+\zeta)-\text{multiplication.}\)

**Theorem 4.** Let \( z \in E_\zeta \) be the image of \( (t, u) \in C_\zeta \) under the Abel-Jacobi map \( \zeta \).

Then we have
\[ \zeta^{-1}(1+\zeta)z = \left(\frac{t(9-8t)^2}{(4t-3)^3}, e^{\frac{1}{12}} \sqrt{3}u, \frac{9-8t}{(4t-3)^2}\right). \]

**Proof.** We set \((t', u') = \zeta^{-1}(1+\zeta)z\). Then \( t' \) is given by the substitution \( z \to (z+1)z \) into the expression of \( t \) in Theorem 3. Rewrite \( \vartheta_{a,b}(1+\zeta)z \) in terms of \( \vartheta_{a,b}(z) \) by Lemma 4. Its numerator \( N(t') \) is
\[ N(t') = 3\sqrt{3}i \vartheta_{0,0}(z)^2 \vartheta_{0,\frac{1}{4}}(z)^2 \vartheta_{\frac{1}{4},0}(z)^2 \times (\vartheta_{0,0}(z)^2 - i \vartheta_{0,\frac{1}{4}}(z)^2)^2 (\vartheta_{0,\frac{1}{4}}(z)^2 + i \vartheta_{\frac{1}{4},0}(z)^2)^2 (\vartheta_{\frac{1}{4},\frac{1}{4}}(z)^2 - \vartheta_{\frac{1}{4},0}(z)^2)^2, \]
and its denominator $D(t')$ is
\[
D(t') = (-\sqrt[3]{3} \vartheta_{0,0}(z)^2 + \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2(\vartheta_{0,0}(z)^4 - \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^4)) \\
+ 2i(\sqrt[3]{3} \vartheta_{0,0}(z)^2 - \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2)\vartheta_{0,0}(z)^2 \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2)^3.
\]

In this computation, the theta constants $\vartheta_{0,0}(0)$, $\vartheta_{\frac{1}{3}, \frac{2}{3}}(0)$, $\vartheta_{\frac{1}{3}, \frac{2}{3}}(0)$ are canceled by Lemma 5. Divide them by $\vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^{18}$ and rewrite
\[
\frac{\vartheta_{0,0}(z)^2}{\vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2} = \frac{1 + \frac{t}{y^2}}{\sqrt[3]{3i}}, \quad \frac{\vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2}{\vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2} = \frac{1 + \omega t}{\sqrt[3]{3}}, \quad \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2 = \frac{1 + \omega^2 t}{\sqrt[3]{3}}.
\]

Then we have
\[
t' = \left( \frac{-(t^3 + u^6)(t^3 - 8u^6)^2}{27u^{18}} \right) \left/ \frac{-(t^3 + 4u^6)^3}{27u^{18}} \right. = \frac{t(9 - 8t)^2}{(4t - 3)^3},
\]

where we use the relation $u^6 = t^3(t - 1)$.

By the same way, we can express $u'$ in terms of $\vartheta_{a,b}(z)$’s, whose numerator $N(u')$ and denominator $D(u')$ are
\[
N(u') = e\left( \frac{1}{8} \sqrt[3]{3} \vartheta_{0,0}(z)\vartheta_{\frac{1}{3}, \frac{2}{3}}(z)\vartheta_{\frac{1}{3}, \frac{2}{3}}(z)\vartheta_{\frac{1}{3}, \frac{2}{3}}(z) - \vartheta_{0,0}(z)^2 \right) \\
\times (\vartheta_{0,0}(z)^4 + \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^4)(\vartheta_{0,0}(z)^2 + i\vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2),
\]
\[
D(u') = \left\{ (\sqrt[3]{3} \vartheta_{0,0}(z)^2 + \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2)(\vartheta_{0,0}(z)^4 - \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^4) \\
- 2i(\sqrt[3]{3} \vartheta_{0,0}(z)^2 - \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2)\vartheta_{0,0}(z)^2 \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2 \right\}^2.
\]

Divide them by $\vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^{8}(\sqrt[3]{3i}\vartheta_{0,0}(z)^2 - \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2)^2$. We factor out $u$ from the numerator as
\[
\frac{N(u')}{\vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^8}(\sqrt[3]{3i}\vartheta_{0,0}(z)^2 - \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2)^2 = iu \frac{(\vartheta_{0,0}(z)^2 - \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2)(\vartheta_{0,0}(z)^4 + \vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^4)(\vartheta_{0,0}(z)^2 + i\vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^2)}{\vartheta_{\frac{1}{3}, \frac{2}{3}}(z)^8}.
\]

Since the rest terms are expressed in terms of $\vartheta_{a,b}(z)^2$, we can compute them quite similarly to the case of $t'$. Hence we have
\[
u' = iu \left( \frac{(3i - \sqrt[3]{3})(-t^3 + 8u^6)t}{18u^8} \right) \left/ \frac{t^3 + 4u^6)^3}{9t^2u^8} \right. = \frac{3 - \sqrt[3]{3}}{2} u \cdot \frac{t(9 - 8t)^2}{(4t - 3)^3}.
\]

It is easy to see that $(t', u')$ satisfies $u'^6 = u'^3(t' - 1)$. \hfill \Box

6. LIMITS OF MEAN ITERATIONS

6.1. LIMIT FORMULA BY $F(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}; x)$. Theorem 2 is interpreted as follows.

**Theorem 5.** Let $P_x = (x, \sqrt[3]{x^2(x - 1)})$ be a point of the curve $C$. We set
\[
P_{x'} = \frac{(2 - x^2)^2}{x^2}, \quad \frac{(1 + i)(2 - x) \sqrt[3]{x^2(x - 1)}}{x^2} \in C.
\]

Then we have
\[
\int_{P_{x'}} \varphi \equiv (1 + i) \int_{P_{x}} \varphi \mod (i) \leq Z[i].
\]
Corollary 9. The following identity holds around $x = 1$:

$$
\frac{1}{\sqrt{x}} F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; 1 - \frac{(2 - x)^2}{x^2}\right) = F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; 1 - x\right).
$$

Proof. Theorem 5 implies that

$$
\lim_{n \to \infty} \frac{\text{hypergeometric series}}{\text{limit formula for the sequences defined by the mean iteration (6.1)}}.
$$

Theorem 5 implies that

$$
\text{Proof.}
$$

We can show that the sequences

$$
\text{Proof.}
$$

and determine the action of

$$
\text{desired identity.}
$$

Note that

$$
\text{Lemma 1 in [HKM]}. \text{Substitute}
$$

and

$$
\text{desired identity.}
$$

for $0 < x < 1$ and $\arg(1 - (2 - x)^2/x^2) = \pi$. We can cancel the factor $\sqrt{1-x}$ and determine the action of $\langle i \rangle \cong \mathbb{Z}[i]$ by the substitution $x = 1$. Thus we have the desired identity. \hfill \Box

Let $a = a_1$ and $b = b_1$ be positive real numbers. We define a pair $\{a_n, b_n\}_{n \in \mathbb{N}}$ of sequences by the recursive relations

$$
(6.1) \quad a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{\frac{a_n(a_n + b_n)}{2}}.
$$

Corollary 10 (A formula in Theorem 2 in [HKM]). We have

$$
\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \frac{a}{F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; 1 - \frac{b^2}{a^2}\right)^2}.
$$

Proof. We can show that the sequences $\{a_n\}$ and $\{b_n\}$ converge and $\lim a_n = \lim b_n$ by Lemma 1 in [HKM]. Substitute $x = 2a/(a+b)$ into the identity between hypergeometric series in Corollary 9. Since

$$
\frac{2 - x}{x} = \frac{b}{a},
$$

we have

$$
\frac{\sqrt{a+b}}{\sqrt{2a}} F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; 1 - \frac{b^2}{a^2}\right) = F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; 1 - \frac{2a}{a+b}\right).
$$

$$
\text{Proof.}
$$

$$
\text{Proof.}
$$

The last term is equal to $\lim_{n \to \infty} a_n$ since $\lim_{n \to \infty} \frac{b_n^2}{a_n^2} = 1$ and $F\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}; 0\right) = 1$. \hfill \Box

Hence we see that the $(1+i)$-multiple formula (4.2) in Theorem 2 implies this limit formula for the sequences defined by the mean iteration (6.1).
6.2. Limit formula by $F(\frac{1}{6}, \frac{1}{6}, \frac{2}{6}; x)$. Theorem 4 is interpreted as follows.

**Theorem 6.** Let $P_x = (x, \sqrt[3]{x^3(x - 1)})$ be a point of the curve $C_\zeta$. We set

$$P_{x'} = \left( \frac{x(9 - 8x)^2}{(4x - 3)^3}, e^{\frac{1}{12}}\sqrt{3} \sqrt[3]{x^3(x - 1)} \frac{9 - 8x}{(4x - 3)^2} \right) \in C_\zeta.$$ 

Then we have

$$\int_{P_1}^{P_{x'}} \psi \equiv (1 + \zeta) \int_{P_1}^{P_x} \psi \mod (\zeta) \times \mathbb{Z}[[\omega]].$$

**Corollary 11.** The following identity holds around $x = 1$:

$$F\left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - x \right) = \frac{1}{\sqrt{4x - 3}} F\left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - \frac{x(9 - 8x)^2}{(4x - 3)^3} \right),$$

where $\sqrt{4x - 3} = 1$ for $x = 1$.

**Proof.** Theorem 6 implies that

$$\int_1^{x'} \sqrt{F(t - 1) dt} \equiv (1 + \zeta) \int_1^x \sqrt{F(t - 1) dt} \mod (\zeta) \times \mathbb{Z}[[\omega]],$$

for $x' = \frac{x(9 - 8x)^2}{(4x - 3)^3}$. By this relation, there exists $k \in \mathbb{N}$ such that

$$\zeta^k \sqrt[3]{27(x - 1)} F\left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - \frac{x(9 - 8x)^2}{(4x - 3)^3} \right) = (1 + \zeta) \sqrt[3]{1 - x} F\left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - x \right).$$

We cancel $(1 + \zeta) \sqrt[3]{1 - x}$ and $\sqrt[3]{27(x - 1)}$, and choose $k = 0$ so that the identity holds for $x = 1$. \hfill \Box

By Corollary 11, we define two means as follows. We solve the cubic equation

$$\frac{x(9 - 8x)^2}{(4x - 3)^3} = \frac{b^2}{a^2}$$

of the variable $x$, where we assume $0 < a < b$. A real solution $x_0$ of this equation is

$$3 \left[ \frac{\sqrt[3]{a^2}}{\sqrt{b^2 - a^2}} \left( \sqrt{b + \sqrt{b^2 - a^2}} - \sqrt{b - \sqrt{b^2 - a^2}} \right) + 2 \right].$$

We set

$$\eta_1 = b + \sqrt{b^2 - a^2}, \quad \eta_2 = b - \sqrt{b^2 - a^2}.$$  

Note that

$$\eta_1 \eta_2 = a^2, \quad \frac{\eta_1 + \eta_2}{2} = b, \quad \frac{\eta_1 - \eta_2}{2} = \sqrt{b^2 - a^2}.$$  

We express $x_0$ and $4x_0 - 3$ in terms of $\eta_1$ and $\eta_2$ as

$$x_0 = \frac{3}{8} \left[ \frac{\sqrt[3]{\eta_1 \eta_2}}{(\eta_1 - \eta_2)/2} \left( \sqrt[3]{\eta_1} - \sqrt[3]{\eta_2} \right) + 2 \right] = \frac{3}{4} \left[ \frac{\sqrt[3]{\eta_1 \eta_2}}{\sqrt[3]{\eta_1} + \sqrt[3]{\eta_1 \eta_2} + \sqrt[3]{\eta_2}} + 1 \right],$$

$$4x_0 - 3 = \frac{3}{2} \left[ \frac{\sqrt[3]{\eta_1 \eta_2}}{(\eta_1 - \eta_2)/2} \left( \sqrt[3]{\eta_1} - \sqrt[3]{\eta_2} \right) + 2 \right] - 3 = \frac{3\sqrt[3]{\eta_1 \eta_2}}{\sqrt[3]{\eta_1} + \sqrt[3]{\eta_1 \eta_2} + \sqrt[3]{\eta_2}}.$$
Thus the identity in Corollary 11 is transformed into

\[
F \left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - \frac{\sqrt{\eta_1^2 + \eta_2^2}}{2} \right) / \left( \frac{\eta_1^{2/3} + \eta_2^{1/3} \eta_1^{1/3} + \eta_2^{2/3}}{3} \right)
\]

\[
= \frac{1}{\sqrt{a}} \sqrt{\eta_1^{2/3} + \eta_2^{1/3} \eta_1^{1/3} + \eta_2^{2/3}} F \left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - \frac{b^2}{a^2} \right)
\]

This formula is equivalent to

\[
(6.3) \quad \frac{a}{F \left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - \frac{b^2}{a^2} \right)} = \frac{m_1(a, b)}{F \left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - \frac{m_2(a, b)^2}{m_1(a, b)} \right)}
\]

if we define two means \(m_1\) and \(m_2\) of positive real numbers \(a\) and \(b\) by

\[
m_1(a, b) = \frac{a^{2/3} \sqrt{\eta_1^{2/3} + \eta_1^{1/3} \eta_2^{1/3} + \eta_2^{2/3}}}{\sqrt{3}}, \quad m_2(a, b) = \frac{a^{2/3} (\eta_1^{1/3} + \eta_2^{1/3})}{2},
\]

where \(\eta_1\) and \(\eta_2\) are given in (6.6) with conditions

\[-\frac{\pi}{6} < \arg(\eta_i^{1/3}) < \frac{\pi}{6}, \quad \eta_1^{1/3} \eta_2^{1/3} = a^{2/3}.\]

Let \(a_1 = a\) and \(b_1 = b\) be positive real numbers. We give a pair of sequences \(\{a_n, b_n\}_{n \in \mathbb{N}}\) with initial terms \(a_1 = a\), \(b_1 = b\) by the recursive relations

\[
a_{n+1} = m_1(a_n, b_n), \quad b_{n+1} = m_2(a_n, b_n).
\]

**Corollary 12** (A formula in Theorem 3 in [HKM]). We have

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \frac{a}{F \left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - \frac{b^2}{a^2} \right)}.
\]

**Proof.** It is shown in §5 of [HKM] that the sequences \(\{a_n\}\) and \(\{b_n\}\) converge and satisfy \(\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n\). By (6.3), we have

\[
\frac{a}{F \left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - \frac{b_2^2}{a_2^2} \right)} = \frac{a_2}{F \left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - \frac{b_3^2}{a_3^2} \right)} = \cdots = \frac{a_n}{F \left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 1 - \frac{b_n^2}{a_n^2} \right)} = \cdots = \lim_{n \to \infty} a_n,
\]

since \(\lim_{n \to \infty} \frac{b_n^2}{a_n^2} = 1\) and \(F \left( \frac{1}{6}, \frac{1}{2}, \frac{7}{6}; 0 \right) = 1\).

Hence we see that the \((1 + \zeta)\)-multiple formula (5.2) in Theorem 4 implies this limit formula for the sequences defined by the mean iteration (6.4).

**References**

[1] J. Igusa, Theta functions, Springer-Verlag, Berlin, 1972.
[2] [HKM] R. Hattori, T. Kato and K. Matsumoto, Mean iterations derived from transformation formulas for the hypergeometric function, *Hokkaido Math. J.*, 38 (2009), 563–586.
[3] K. Matsumoto, Monodromy representation of hypergeometric system \(F_D\) even in the reducible case, in preparation.
[4] D. Mumford, *Tata lectures on theta I*, Birkhäuser, Boston, 1983.
[5] E.M. Goursat, Sur l’Équation Différentielle Linéaire qui Admet pour Intégrale la Série Hypergéométrique, *Ann. Sci. l’Ecole Normale Sup.* (2) 10 (1881), 3–112.
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