Aggregation of the fuzzy logic sets in terms of the functions of the triangular norm and triangular co-norm

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Abstract. The purpose of this article is to explore the group of the operators, which can be used for aggregation of the fuzzy sets. There were scrutinized operators to be used for the intersection and union such as the triangular norm and triangular co-norm in the article. All of those operators are defined as the binary operations, where there was indicated that the norms and co-norms are two-valued functions of [0,1]. Furthermore, there were discussed properties of norm and co-norm functions such as symmetry, associativity, neutral entry and null entry properties, monotonicity properties. Moreover, the consideration of the properties of norm and co-norm functions have routed investigation of the norm and co-norm operators to go further to prove theorem, which converts the union into intersection and vice versa in terms to all possible norm and co-norm functions based on the fuzzy logic sets.

1. Introduction

The main principles of the Fuzzy Set Theory have been introduced by Zadeh in [1, 2, 3, 4, 5, 6]. Let’s describe by $U$ the universe of the sets and some subset $F$ of $U$ can be described as the set of ordered pairs:

$$F = (x, \vartheta(x)),$$

or alternatively, in terms of pairs of ordinate value and abscissa, as

$$F = \left\{ \frac{\vartheta(x)}{x} \mid x \in F \right\},$$  \hspace{1cm} (1)

where $F$ is defined as the Fuzzy Set [6,7].

$\vartheta(x)$ is a characteristic function or a membership function, which assumes solely values of 0 and 1, where measure for the membership, such as the value of 1 designates a membership, while 0 value designates a non-membership:

$$\vartheta(x) = \begin{cases} 1, & \text{entry } x \in U_n \\ 0, & \text{entry } x \notin U_n. \end{cases}$$

Alternatively, we may explain $\vartheta(x)$ as it follows:

Let $\vartheta(x)$ be a function from $X$ into $Y$. Next, let $A \subset X$ and $B \subset Y$. If $(x, y) \in \vartheta(x)$, then $y$ is called direct image of $x$ under $\vartheta$, where $x \in A$, $y \in B \subset [0;1]$. 

The membership function $\vartheta(x) \in [0; 1]$, while $F$ is the fuzzy set (1) which assigns each element from $A$ to a strictly closed interval $B \in [0; 1]$ of real numbers. If an interval is strictly closed, then such fuzzy set of $F$ is the so-called segment valued fuzzy set.

Furthermore, the membership function $\vartheta(x)$ can be described as it is:

$$\vartheta(x): A \to B \subset [0; 1].$$

The closed interval $[0; 1]$ represents the set of all possible closed sub-intervals, between 0 and 1.

This type of membership function is understood as the interval-real valued closed fuzzy set.

A type of a fuzzy set with the precisely defined membership, which is one-to-one function from $A$ into $B \subset [0; 1]$ always implies that if $x = x'$ for $x, x' \in X$, then

$$\vartheta(x) = \vartheta(x').$$

A type of the fuzzy set, which involved with one-to one membership function defines an ordinary fuzzy set.

If $\vartheta(x)$ is a one-to one membership function from $A$ into $[0; 1]$, then there exists an inverse function to a membership function, which we can define as $\vartheta^{-1}$ from range onto $A$ by the property of the inverse functions as it is:

$$([0; 1], x) \in \vartheta^{-1} \text{ if } (x, [0; 1]) \in \vartheta.$$ 

In a case, where a membership function is an ordinary fuzzy set, we may further consider a composition of the membership functions as it follows:

$$(\vartheta \circ \vartheta_1) = \vartheta(\vartheta_1) \text{ for each } x \in A.$$ 

In this case, where membership function is a function of another fuzzy set, we deal with the fuzzy set called the fuzzy set of type 2:

$$\vartheta: A \to \vartheta([0,1]).$$

The Ultra-fuzzy set of $F_{\text{ultra}}$ is the fuzzy set, when the membership function is the fuzzy set of type 2.

2. Operators for the intersection and the union of the fuzzy sets

The intersection and the union of the fuzzy sets can be utilized in terms of the aggregation of the fuzzy sets [7]. Such aggregation(s) can be introduced through the operators such as triangular $\Delta$– norm for the intersection and triangular $\Delta_\ast$– co-norm for the union of the fuzzy sets.

Definition: The triangular $\Delta$– norm is the binary operation: $\Delta: \vartheta(x): [0,1] \times [0,1] \to [0,1]$.

The triangular $\Delta$– norm as the function is symmetric, associative, monotonic.

For $k, l, m, n, s \in [0,1]$ there are the following properties applied:

- Symmetric: $\Delta(k,l) = \Delta(l,k)$;
- Associative: $\Delta(k,\Delta(l,m)) = \Delta(\Delta(k,l),m)$;
- $\Delta(k,1) = k$ and $\Delta(1,k) = k$;
- $\Delta(k,0) = 0$ ;
- Monotonicity: if $k \leq l$ and $m \leq n$, then $\Delta(k,m) \leq \Delta(l,m)$.

Thereafter, the intersection of two fuzzy sets $F_1 \cap_\Delta F_2$ may be are – introduced through the $\Delta$ – norm for fuzzy sets: $Z := F_1 \cap_\Delta F_2$, where membership function

$$\vartheta_Z(x) := \Delta(\vartheta_{F_1}(x), \vartheta_{F_2}(x)) \text{ for all } x \in X.$$ 

Since the symmetry and associativity properties of the intersection upheld, the following properties uphold for fuzzy sets, too:

$$F_1 \cap_\Delta F_2 = F_2 \cap_\Delta F_1,$$
$$F_1 \cap_\Delta (F_2 \cap_\Delta F_3) = (F_1 \cap_\Delta F_2) \cap_\Delta F_3,$$
Now let's define the convexity of fuzzy sets as it follows:

For a strict inequality, function is called a strictly convex.

Theorem: There is existing functional relation between \( K \) and \( O \in (K \cup O) \).

We have to note that all above intersections are derived from \( \Delta - \) norms. As the consequence of those intersections through the \( \Delta - \) norm following intersections hold:

\[ F_1 \cap \Delta F_2 \subseteq F_1 \text{ and } F_1 \cap \Delta F_2 \subseteq F_2. \]

Moreover, \( F_1 \cap \Delta F_2 \subseteq F_1 \cap F_2. \)

The dual intersection based on \( \Delta - \) norm can be presented for possible intersections:

\[ F_1 \cap \Delta F_2 = (F_1^2 \cap \Delta F_2^2). \]

**Theorem:** If \( F_1 \) and \( F_2 \) are subsets of the universe \( U_n \) and \( F'_1 = U_n \setminus F_1 \), \( F'_2 = U_n \setminus F_2 \), where \( F'_1 \) and \( F'_2 \) are the compliments of \( F_1 \) and \( F_2 \) in \( U_n \) then \( (F_1 \cup \Delta F_2)' = F'_1 \cap \Delta F'_2 \).

**Proof:** Let \( t \in (F_1 \cup \Delta F_2)' \). If so, then \( t \notin F_1 \cup \Delta F_2 \). Hence, \( t \notin F_1 \) and \( t \notin F_2 \).

From here we may conclude that \( t \in F'_1 \), which means that \( t \in F'_1 \cap \Delta F'_2 \). Secondly, let \( t \in F'_1 \cap \Delta F'_2 \). If so, then \( t \in F'_1 \) and \( t \in F'_2 \). Hence \( t \notin F_1 \) and \( t \notin F_2 \). From here it follows that \( t \notin (F_1 \cup \Delta F_2) \) and, thereafter \( t \in (F_1 \cup \Delta F_2)' \).

Therefore, \( F'_1 \cap \Delta F'_2 \subseteq (F_1 \cup \Delta F_2)' \), which accomplished the proof.

The other class of aggregation operators is the \( \Delta - \) co-norm, which is denoted as \( \Delta_c \). These group of operators are utilized to form the union of the fuzzy sets.

**Definition:** \( \Delta_c \) is a map, which performs binary operation in \([0;1]\): a mapping \( \Delta_c: [0,1] \times [0,1] \rightarrow [0,1] \).

These co-norm operators have the following features:

- Symmetry: \( \Delta_c(p,q) = \Delta_c(q,p) \);
- Associativity: \( \Delta_c(p,\Delta_c(r,t)) = \Delta_c(\Delta_c(p,r),t) \);
- Null element and neutral element properties: \( \Delta_c(p,0) = \Delta_c(0,p) = p, \Delta_c(p,1) = 1 \);
- Monotonicity property: If \( p \leq q \) and \( r \leq s \), then \( \Delta_c(p,r) \leq \Delta_c(q,s) \).

For the union on \( \Delta_c \) such as \( F_1 \cup \Delta_c F_2 \), we may define the intersection:

\[ T = F_1 \cap \Delta_c F_2, \text{ where } \theta_T(x) = \Delta_c \left( \theta_{F_1}(x), \theta_{F_2}(x) \right). \]

Moreover, for all possible union between fuzzy there are the following properties hold:

\[ F_1 \cup \Delta_c F_2 = F_2 \cup \Delta_c F_1; \]

\[ F_1 \cup \Delta_c (F_2 \cup \Delta_c F_3) = (F_1 \cup \Delta_c F_2) \cup \Delta_c F_3; \]

\[ F_1 \cup \Delta_c \emptyset = F_1, F_1 \cup \Delta_c X = X. \]

There is existing functional relation between \( \Delta - \) norm and \( \Delta_c - \) conorm:

\[ \Delta_c(p,q) = 1 - \Delta(1 - p, 1 - q), \text{ and } \Delta(s,t) = 1 - \Delta_c(1 - s, 1 - t). (2) \]

Now let's define the convexity of fuzzy sets as it follows:

**Definition:** The fuzzy set \( F \) is said to be a convex, if for

\[ x_1, x_2 \in A, 0 \leq \varepsilon \leq 1, \theta(\varepsilon x_1 + (1 - \varepsilon)x_2) \leq \varepsilon \theta(x_1) + (1 - \varepsilon)\theta(x_2). \]

For a strict inequality, function is called a strictly convex.

If we reverse the direction of the inequality to greater (\( \geq \)), then function is to be concave.

In terms of the singleton pairs \((x, \theta(x))\) we define a concept of a support plane of a membership function as it is:

**Definition:** A support plane of the function \( \theta \):

\[ \theta: X \rightarrow \theta(x)[0;1] \] is a plane, which touches the graph at \((x, \theta(x))\). If the support plane is unique, the it is called a tangent plane or a tangent.
Thereafter, if $\theta$ is a convex, then existence of the singleton of $\theta(x)$ is equivalent to the differentiability.

If $\theta$ is a convex at $A \subset X$, then $\theta$ has a right derivative and left derivative:

$$\theta'_+ = \lim_{h \to 0^+} \frac{\theta(x+h) - \theta(x)}{h},$$

$$\theta'_- = \lim_{h \to 0^-} \frac{\theta(x) - \theta(x-h)}{h}.$$

3. **Analytical representation of the fuzzy sets with further graphical representation of the fuzzy sets**

The fuzzy sets can be described in various ways such as modeling with type 1 to type 3:

(1) **Modeling with Type 1**: Here in this type the singletons are in the form of discrete pairs $\theta_\alpha(x_n)/x_n$.

(2) **Type 2**: The triangular representation of Type 3 analytically presented as the piece-wise function(s):

$$\theta_\alpha(x) = 0 \text{ for } x < x_1, \theta_\alpha(x) = (x - x_1)/(x_2 - x_1) \text{ for } x_1 \leq x \leq x_2;$$

$$\theta_\alpha(x) = \frac{(x_3 - x)}{(x_3 - x_2)} \text{ for } x_2 \leq x \leq x_3;$$

$$\theta_\alpha(x) = 1 \text{ for } x_n < x.$$  

(3) **Type 3**: $\Gamma(\gamma)$ – function type of the membership function:

$$\theta_\alpha(x) = 0 \text{ for } x < x_1,$$

$$\theta_\alpha(x) = \frac{x - x_1}{x_2 - x_1} \text{ for } x_1 \leq x \leq x_2,$$

$$\theta_\alpha(x) = 1 \text{ for } x_n < x.$$  

(4) **Type 4**: Smoothed alternative $\Gamma - \gamma$ function:

$$\theta_\alpha(x) = 0 \text{ for } 0 \leq x \leq x_1,$$

$$\theta_\alpha(x) = 1 - e^{-k(x-x_1)^2} \text{ for } x_1 > x, k > 0.$$  

(5) **Type 5**: Fuzzy set representation with the so-called S-function by Zadeh:

$$\theta_\alpha(x) = 0 \text{ for } x < x_1;$$

$$\theta_\alpha(x) = 2 \left( \frac{x - x_1}{x_2 - x_1} \right)^2, \text{ for } x_1 \leq x \leq x_2;$$

$$\theta_\alpha(x) = 1 - 2((x - x_3)/(x_3 - x_1)^2, \text{ for } x_2 \leq x \leq x_3;$$

$$\theta_\alpha(x) = 1, \text{ for } x_2 = \frac{x_1 + x_3}{2} \text{ for } x > x_4.$$  

(6) **Type 6**: Fuzzy sets represented by a generalized trapezoidal function to represent modeling of the fuzzy functions:

$$\theta_\alpha(x) = 0 \text{ for } x < x_1;$$

$$\theta_\alpha(x) = \frac{b_2(x - x_1)}{x_2 - x_1} \text{ for } x_1 \leq x \leq x_2;$$

$$\theta_\alpha(x) = \frac{(b_3 - b_2)(x - x_2)}{x_3 - x_2} + b_2, \text{ for } x_2 \leq x \leq x_3;$$

$$\theta_\alpha(x) = 1 \text{ or } b_3 - b_4 \text{ for } x_3 \leq x \leq x_4;$$
\[ \vartheta_A(x) = \frac{(b_2 - b_3)(x_5 - x)}{x_5 - x_4} + b_5 \text{ for } x_4 \leq x \leq x_5; \]
\[ \vartheta_A(x) = \frac{b_5(x_6 - x)}{x_6 - x_5} \text{ for } x_5 \leq x \leq x_6; \]
\[ \vartheta_A(x) = 0 \text{ for } x_6 < x. \]

The construction of the membership functions by utilizing modeling types 1 to type 6 is extremely delicate process. The process of the utilization of types 1 – 6 allow to apply different modeling techniques in the theory and applications of the various sciences, e.g. such as control systems. Utilization of the modeling types 1 – 6 is applicable to the modeling scenarios where the parameters of the prospective constructed membership function can be approximated commensurate to the parameters described with modeling of membership functions from type 1to type 6.

4. Conclusion
The Fuzzy Logic models allow to present the fuzzy logic elements by the characteristic functions and their graphs. The fuzzy set modeling are so-called linguistic methods with further representation of the linguistic elements in the form of fuzzy sets, which enable to represent the fuzzy sets graphically as the convex or concave membership functions.

The main concept of the modeling with fuzzy sets are embedded with linguistic methods, where the fuzzy sets represent the linguistic elements. The fuzzy logic models are utilizing the fuzzy logic characteristic function with domain and co-domain based on the fuzzy logic sets. Furthermore, there were considered the operations involving standard fuzzy sets in the article here. There was shown that for the union, intersection and complement operators with fuzzy sets there are existing broad range of the operators, designated as the \( \Delta \)-norms and \( \Delta_c \)-co-norms, which represent the binary operations from \([0; 1]\) to \([0; 1]\).

There was presented the theorem, which connects the norm and co-norm operations in terms of the union and intersection of the fuzzy set.

Moreover, the definition of the norm and co-norms were coherent to the properties of the binary operations with norms and co-norms.

Fuzzy sets modeling provides ample information to visualize the fuzzy sets by the characteristic functions and corresponding fuzzy curves.

It is important when the fuzzy sets and corresponding membership function have been analyzed within the concepts of the modern Analysis. Moreover, it is imperative to introduce the main properties of the fuzzy sets in terms of the modern Analysis. The utilization of the properties of the union and intersection of the fuzzy sets with further application of the \( \Delta \)-norm and \( \Delta_c \)-co-norms operators lead to establish the commutativity, associativity and monotonicity properties, which hold for the fuzzy set operators.

Furthermore, fuzzy logic characteristic or membership functions and fuzzy set singleton pairs (fuzzy logic set pair) can be used for modeling purposes to model human phenomena activities, e.g. fuzzy logic pairs are used to represent temperature, volume, frequency, age, degree, or pressure, or other forms of the natural or human factor phenomena problems involved.

References
[1] Zadeh L 1971 *J Information Sciences* 3 pp 117-200
[2] Zadeh L 1978 *J Fuzzy Sets and Systems* 1 pp 3-22
[3] Zadeh L 1965 *J Information and Controls* 8 pp 338-353
[4] Zadeh L 1973 *J IEEE Trans Systems, Man, and Cybernetics* 3 pp 28-144
[5] Zadeh L 1971 *J Information Sciences* 3 pp 177-200
[6] Aliev R, Fazlollahi B, Aliev R 2004 *Soft Computing and Its Applications in Business and Economics* (Springer-Verlag-Berlin Heidelberg)
[7] Gadjiev D 2019 *The soft Programming and Fuzzy Logic Mathematics Modeling concepts with*
the application of the Mathematical Analysis (Chebyshevskii sbornik)