BELLMAN FUNCTION AND LINEAR DIMENSION-FREE
ESTIMATES IN A THEOREM OF D. BAKRY

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Abstract. By using an explicit Bellman function, we prove a bilinear embedding theorem for the Laplacian associated with a weighted Riemannian manifold \((M, \mu_\varphi)\) having the Bakry-Emery curvature bounded from below. The embedding, acting on the cartesian product of \(L^p(M, \mu_\varphi)\) and \(L^q(T^*M, \mu_\varphi)\), \(1/p + 1/q = 1\), involves estimates which are independent of the dimension of the manifold and linear in \(p\). As a consequence we obtain linear dimension-free estimates of the \(L^p\) norms of the corresponding shifted Riesz transform. All our proofs are analytic.

1. Introduction

Consider a complete Riemannian manifold \((M, g, \mu_0)\) with Riemannian metric \(g\) and Riemannian measure \(\mu_0\). Let \(d, \nabla, \text{Grad}\) and \(\Delta\) denote, respectively, the exterior and the covariant derivative, the gradient, and the nonnegative Laplace-Beltrami operator on \(M\). Given \(\varphi \in C^\infty(M)\), consider the weighted measure on \(M\) defined by
\[
d\mu_\varphi(x) = e^{-\varphi(x)} d\mu_0(x),
\]
and denote by \(\mathcal{L}_\varphi\) the nonnegative weighted Laplacian defined on \(C^\infty_c(M)\) by
\[
\mathcal{L}_\varphi f = \Delta f + df(\text{Grad}(\varphi)).
\]
It was proved in [1, 14] that \(\mathcal{L}_\varphi\) is essentially self-adjoint on \(L^2(M, \mu_\varphi)\), and with an abuse of notation we still denote by \(\mathcal{L}_\varphi\) its unique self-adjoint extension. The Bakry-Emery curvature tensor associated with \(\mathcal{L}_\varphi\) is defined by
\[
\text{Ric}_\varphi = \text{Ric} + \nabla^2 \varphi,
\]
where \(\text{Ric}\) denotes the Ricci curvature tensor on \(M\). For every \(a \in \mathbb{R}\), consider the shifted Riesz transform defined by
\[
\mathcal{R}_a = d(a^2 I + \mathcal{L}_\varphi)^{-1/2}.
\]
The following celebrated result was first proven by D. Bakry [1].

Theorem 1. Suppose that \(\text{Ric}_\varphi \geq -a^2 g\). Then, for every \(p\) in \((1, \infty)\), there is \(C(p) > 0\) such that
\[
\|\mathcal{R}_a f\|_{L^p(T^*M, \mu_\varphi)} \leq C(p) \|f\|_{L^p(M, \mu_\varphi)},
\]
for all \(f \in R(a^2 I + \mathcal{L}_\varphi) \cap L^p(M, \mu_\varphi)\).

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This result has been recently improved by Li in [9, Theorem 1.4], where the author obtained an explicit upper estimate, namely, \( C(p) = 2(p^* - 1)(1 + 4\|τ\|_p) \). Here \( p^* = \max\{p, q\}, 1/p + 1/q = 1 \), and \( τ \) is the exit time of the standard 3-dimensional Brownian motion from the unit ball in \( \mathbb{R}^3 \). One can determine the asymptotic behaviour of \( \|τ\|_p \) by means of the distribution function of \( τ \), which has been calculated by Ciesielski and Taylor [4]. As a result one quickly computes that \( \|τ\|_p \sim p \) as \( p \to \infty \). Thus the estimate in [9] is quadratic in \( p \) for \( a > 0 \), except in the case \( a = 0 \), when the author showed that it suffices to take \( C(p) = 2(p^* - 1) \). Apparently, a further improvement was made by the same author in [10, Theorems 1.5, 1.6] by demonstrating that, if \( a > 0 \), one can take \( C(p) = 2(p^* - 1)^{3/2} \).

The same papers by Li [9, 10] also contain a thorough review of numerous earlier results about Riesz transforms on various classes of Riemannian manifolds, as well as several applications that further motivate the pursuit of the dimension-free boundedness of Riesz transforms in such generality.

The proofs in [1, 9, 10] are probabilistic, and in [9, p. 269] the author specifically raises the question of finding an analytic proof of Theorem 1. The main objective of this paper is to give a short analytic proof of Bakry’s result with explicit linear estimates in \( p \) (see Corollary 4). We accomplish this by employing the technique of Bellman functions. It was brought into harmonic analysis by Nazarov, Treil and Volberg in the 1994 preprint version of their paper [12]. Here we will follow the scheme laid out in papers [5, 6, 7]. Accordingly, the result in question will be a corollary of the so-called bilinear embedding theorem for the weighted Laplacian \( \Delta_\phi \) on \( M \) (see Theorem 3). We are able to make the passage from the embedding theorem to the Riesz transforms without using spectral multipliers. This is in contrast with [5] and [6], although one of the results there (dimension-free estimates of Riesz transforms associated with the Ornstein-Uhlenbeck operator, see [5]) is a particular case of Corollary 4. In this light our method can be viewed as an improvement over [5].

As remarked above, unlike in Li [9, 10] our proofs are purely analytic. Moreover, the estimate we obtain in Corollary 4 is linear in \( p \) and thus better than those obtained by Li in [9, 10], except when \( a = 0 \). In the latter case both estimates are linear, yet the one due to Li exhibits a smaller numerical constant. A better numerical constant in our theorem could be obtained by using a “sharper” Bellman function. Such a function does exist, see [5] and [13], but since it is not explicit, it seems much more difficult to work with. The advantage of the Bellman function we utilise in this paper (and which originated in the work of Nazarov and Treil [11]) is its simplicity and the fact that it admits satisfactory estimates of its partial derivatives (see Theorem 7).

The same Nazarov-Treil Bellman function was recently used by A. Volberg and the second author [6, 7] in order to obtain similar results for (generalised) Schrödinger operators with nonnegative potentials, again yielding dimension-free estimates with sharp (linear) estimates in \( p \) involving explicit constants.

Since the introduction of the Bellman function method in harmonic analysis by Nazarov, Treil and Volberg in mid-90’s, there has been a whole series of (sharp) inequalities treated with great success by this method. Yet to the best of our knowledge this paper is the very first case of applying Bellman functions on general manifolds rather than on Euclidean spaces. As such it may open a path for a wide range of similar applications in the future. For example, we are currently studying \( L^p \) spectral multipliers on weighted Riemannian manifolds by using Bellman functions techniques. This will be contained in a forthcoming paper.
2. Preliminaries

For each \( x \) in \( M \), we denote the tangent and the cotangent space at \( x \) respectively by \( T_x M \) and \( T^*_x M \). For every \( j, k \in \mathbb{N} \), we set

\[
T^{j,k}_x M = T^*_x M \otimes \cdots \otimes T^*_x M \otimes T_x M \otimes \cdots \otimes T_x M,
\]

and we denote by \( T^{j,k}_x M \) the fiber bundle over \( M \) whose fibre at \( x \) is \( T^{j,k}_x M \). A tensor of type \((j, k)\) is just a section of \( T^{j,k}_x M \). We denote the space of smooth tensors of type \((j, k)\) by \( C^\infty(T^{j,k}_x M) \) and identify functions on \( M \) with tensors of type \((0, 0)\).

For each \( k = 0, \ldots, \dim M \), let \( \wedge^k T^*_x M \) denote the bundle of alternating tensors of type \((0, k)\), also referred to as \( k\)-forms. Recall that for every \( j, k \in \mathbb{N} \) and \( x \in M \) the Riemannian scalar product on \( T_x M \) induces a scalar product \( \langle \cdot, \cdot \rangle \) on \( T^{j,k}_x M \); this clearly induces a scalar product on \( \wedge^k T^*_x M \), for all \( k = 0, \ldots, \dim M \). We set \( \| \cdot \|_{T^{j,k}_x M} = \langle \cdot, \cdot \rangle_{T^{j,k}_x M} \). For each \( p \in [1, \infty] \) and \( j, k \in \mathbb{N} \), let \( L^p(T^{j,k}_x M, \mu_\varphi) \) be the Banach space of all measurable tensors \( u \) of type \((j, k)\)

\[
\|u\|_{L^p(T^{j,k}_x M, \mu_\varphi)} = \left( \int_M |u(x)|^p_{T^{j,k}_x M} \, d\mu_\varphi(x) \right)^{1/p} < \infty, \quad \text{if } p \in [1, \infty);
\]
\[
\text{ess sup}_{x \in M} |u(x)|_{T^{j,k}_x M} < \infty, \quad \text{if } p = \infty.
\]

When there will be no ambiguity, we shall denote \( \| \cdot \|_{T^{j,k}_x M} \) simply by \( \| \cdot \| \), and \( L^p(T^{j,k}_x M, \mu_\varphi) \) by \( L^p(\mu_\varphi) \). If \( A \) is an operator on \( L^2(\mu_\varphi) \), we denote respectively by \( R(A) \) and \( N(A) \) its range and null-space.

Furthermore, let

\[
d : C^\infty(\wedge^k T^*_x M) \to C^\infty(\wedge^{k+1} T^*_x M) \quad \text{and} \quad \nabla : C^\infty(T^{j,k}_x M) \to C^\infty(T^{j,k+1}_x M)
\]

be the exterior and the total covariant derivative, respectively, and \( d^*_\varphi \) and \( \nabla^*_\varphi \) their adjoints on \( L^2(\mu_\varphi) \). Recall that on functions \( d \) and \( \nabla \) coincide with the differential, \( d^2 = 0 \), and, for every \( u \in T^{j,k}_x M, \eta_i \in C^\infty(T^*_x M) \) and \( X, Y \in C^\infty(TM) \), \( \nabla u(\eta_1, \ldots, \eta_r, X, Y_1, \ldots, Y_s) = \nabla_X u(\eta_1, \ldots, \eta_r, Y_1, \ldots, Y_s) \), where \( \nabla_X u \in T^{j,k}_x M \) denotes the covariant derivative of \( u \) with respect to \( X \). Given a system of local coordinates \((x^1, \ldots, x^n)\), we set \( \nabla_i = \nabla_{\partial_i} \), \( i = 1, \ldots, n \).

An easy computation gives

\[
d^*_\varphi = d_0^* + i_{\text{Grad}(\varphi)},
\]

where \( i_{\text{Grad}(\varphi)} \) denotes the inner multiplication by \( \text{Grad}(\varphi) \) on \( \wedge^{k+1} T^*_x M \). The (non-negative) weighted Hodge-De Rham Laplacian acting on \( k \)-forms is defined by

\[
\square_{k,\varphi} = d d^*_\varphi + d^*_\varphi d.
\]

It is well-known that \( \square_{k,\varphi} \), initially defined on smooth \( k \)-forms with compact support, is essentially selfadjoint on \( L^2(\wedge^k T^*_x M, \mu_\varphi) \) (see \([13]\)). Note that \( \square_{0,\varphi} = \mathcal{L}_\varphi \), and by the Bochner-Weitzenböck formula we have

\[
\square_{1,\varphi}(\omega) = \nabla^2 \omega = \nabla_0 \nabla \omega + \nabla_{\text{Grad}(\varphi)} \omega + \text{Ric}_{\varphi}(\cdot, \cdot) \omega,
\]

where \( \sharp : T^*_x M \to T_x M \) is the duality defined by \( \omega(X) = \langle \sharp \omega, X \rangle_T M \) for all \( \omega \in T^*_x M \) and \( X \in T_x M \) \([13, 8]\).
We set \( \mathcal{L}_\varphi = \Box_{1, \varphi} \), \( P^a_t = \exp\left(-t(\sigma^2 I + \mathcal{L}_\varphi)\right) \) and \( P^a_t = \exp\left(-t(\sigma^2 I + \mathcal{L}_\varphi)\right) \). Note that \( L^2(M, \mu_{\varphi}) = R(a^2 I + \mathcal{L}_\varphi) \oplus N(a^2 I + \mathcal{L}_\varphi) \) where the sum is orthogonal. The Riesz transform \( \mathcal{R}_\alpha \) initially defined on \( R(a^2 I + \mathcal{L}_\varphi) \) extends to an isometry

\[
\mathcal{R}_\alpha : R(a^2 I + \mathcal{L}_\varphi) \to L^2(T^* M).
\]

Note that if \( \alpha > 0 \) then \( R(a^2 I + \mathcal{L}_\varphi) = L^2(M, \mu_{\varphi}) \). Moreover, \( N(\mathcal{L}_\varphi) \neq \{0\} \) if and only if \( \mu_{\varphi}(M) < \infty \); in this case \( N(\mathcal{L}_\varphi) = \{ \text{constant functions on } M \} \). When \( \alpha > 0 \), \( \mathcal{R}_\alpha \) is often called local Riesz transform.

**Lemma 2.** For every \( f \in C^\infty_c(M) \), \( \omega \in C^\infty_c(T^* M) \), \( r \geq 1 \) and \( a \geq 0 \),

(a) \( \Delta \mathcal{L}_\varphi f = \mathcal{L}_\varphi d f \) and \( d^* \mathcal{L}_\varphi \omega = \mathcal{L}_\varphi d^* \omega \).

(b) \( \Delta P^a_t f = \mathcal{L}_\varphi P^a_t f \) and \( d^* \mathcal{L}_\varphi \mathcal{R}_\alpha \omega = P^a_t d^* \mathcal{R}_\alpha \omega \).

(c) \( |P^a_t f(x)|^r \leq P^a_t |f|^r(x) \).

If also \( \text{Ric}_{\varphi} \geq -a^2 g \), then

(d) \( |e^{-t(\sigma^2 I + \mathcal{L}_\varphi)} \omega(x)|_{T^*_x M} \leq e^{ta^2} |e^{-t(\sigma^2 I + \mathcal{L}_\varphi)} \omega(x)|_{T^*_x M} \).

(e) \( |P^a_t \omega(x)|_{T^*_x M} \leq P^a_t \|\omega\|_{T^*_x M} \).

**Proof.** The first three items in the lemma have been proved in [1, Prop. 1.7]. Since \( \mathcal{L}_\varphi \) generates a Markovian semigroup on \( (M, \mu_{\varphi}) \) [2], we quickly get

\[
|e^{-t(\sigma^2 I + \mathcal{L}_\varphi)} f(x)|^r \leq e^{-t(\sigma^2 I + \mathcal{L}_\varphi)} |f|^r(x).
\]

Set \( dm(s) = (\pi s)^{-1/2} e^{-s} \, ds \). One readily see that

\[
P^a_t = \int_0^\infty e^{-t(\sigma^2 I + \mathcal{L}_\varphi)} dm(s).
\]

Hence [1] also holds with \( P^a_t \) in place of \( e^{-t(\sigma^2 I + \mathcal{L}_\varphi)} \), and [3] is proved. Similarly, [3] follows from a combination of the item [4] and the subordination formula [2]. \( \square \)

3. Bilinear embedding theorem and Riesz transforms

We now state the bilinear embedding theorem which is the principal result of the paper. The proof will be given in Section 5. Denote by \( \nabla \) the total covariant derivative on \( M \times \mathbb{R}_+ \). Then, for every \( \eta \in C^\infty(T^* R^k(M \times \mathbb{R}_+)) \), \( |\nabla \eta| = \sqrt{|\nabla \eta|^2 + |\nabla \eta|^2} \).

**Theorem 3.** Suppose that \( M \) is a complete Riemannian manifold with \( \text{Ric}_{\varphi} \geq -a^2 g \). Then for all \( p \) in \( (1, \infty) \), \( f \in C^\infty_c(M) \) and \( \omega \in C^\infty_c(T^* M) \),

\[
\int_0^\infty \int_M |\nabla P^\#_t f(x)| |\nabla P^\#_t \omega(x)| t \, d \mu_{\varphi}(x) \, dt \leq 3(p^* - 1) \|f\| L^p(M, \mu_{\varphi}) \|\omega\| L^p(T^* M, \mu_{\varphi}).
\]

The bilinear embedding theorem implies a dimension free estimate for the \( L^p \) norms of the Riesz transform.

**Corollary 4.** Under the above conditions,

\[
\|\mathcal{R}_\alpha f\| L^p(T^* M, \mu_{\varphi}) \leq 12(p^* - 1) \|f\| L^p(M, \mu_{\varphi}),
\]

for all \( f \in \mathcal{R}(a^2 I + \mathcal{L}_\varphi) \cap L^p(M, \mu_{\varphi}) \).
Proof. We claim that for every \( f \in C_c^\infty(M) \cap \text{R}(a^2I + \mathcal{L}_\varphi) \) and \( \omega \in C_c^\infty(T^*M) \) we have that
\[
\int_M \langle R_a f(x), \omega(x) \rangle \, d\mu_\varphi(x) = 4 \int_0^\infty \int_M \left( \frac{dP_t^a f(x)}{dt} \right) \omega(x) \, d\mu_\varphi(x) \, t \, dt.
\]
Assuming the claim \((3)\), Corollary 4 follows immediately from Theorem 3 and the Cauchy-Schwarz inequality. To prove the claim \((3)\), consider the function
\[
\varphi(t) = \langle P_t^a R_a f, P_t^a \omega \rangle_{L^2(\mu_\varphi)}.
\]
Since \( \langle R_a f, \omega \rangle_{L^2(\mu_\varphi)} = \varphi(0) \), it suffices to show that
\[
\varphi(0) = \int_0^\infty \varphi''(t) \, dt = 4 \int_0^\infty \left( \frac{dP_t^a f}{dt} \right) \omega(t) \, dt.
\]
In order to prove the first equality it is enough to show that both \( \varphi(t) \) and \( t\varphi'(t) \) tend to zero as \( t \to \infty \). First note that, by Lemma \(2\), \( P_t^a R_a f = R_a P_t^a f \). Therefore, by the \( L^2 \) contractivity of both \( R_a \) and \( P_t^a \), \( \|\varphi(t)\| \leq \|P_t^a f\|_{L^2(\mu_\varphi)} \|\omega\|_{L^2(\mu_\varphi)} \). Since \( f \in \text{R}(a^2I + \mathcal{L}_\varphi) \), the spectral theorem gives that \( P_t^a f \to 0 \) in \( L^2(\mu_\varphi) \) as \( t \to \infty \).

Similarly, Lemma \(2\) gives
\[
\varphi'(t) = 2\langle (a^2I + \mathcal{L}_\varphi) P_t^a f, (a^2I + \mathcal{L}_\varphi)^{-1/2} f \rangle_{L^2(\mu_\varphi)} = 2\langle P_t^a f, P_t^a d^* \omega \rangle_{L^2(\mu_\varphi)},
\]
therefore \( \lim_{t \to \infty} t|\varphi'(t)| = 0 \) as before. The second equality in \((4)\) can be verified by a straightforward calculation, again with the help of Lemma \(2\). \( \square \)

4. Bellman function

As announced above, the main tool in the proof of the bilinear embedding Theorem 3 will be a particular Bellman function. Throughout this section we assume that \( p \geq 2 \), \( q = p/(p-1) \) and \( \delta = q(q-1)/8 \) are fixed. Observe that \( \delta \sim (p-1)^{-1} \).

Fix \( n \in \mathbb{N} \) and define the Bellman function \( Q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+ \) by setting
\[
Q(\zeta, \eta) = \frac{1}{2} \beta(|\zeta|, |\eta|),
\]
where
\[
\beta(u, v) = u^p + v^q + \delta \begin{cases} u^2v^{2-q} & ; \ u^p \leq v^q \\ \frac{2}{p} u^p + \left( \frac{2}{q} - 1 \right) v^q & ; \ u^p \geq v^q \end{cases}
\]
for any \( u, v \geq 0 \). For every \( (\zeta, \eta) \in \mathbb{R} \times \mathbb{R}^n \) set \( U(\zeta, \eta) = |\zeta|, |\eta| \). The function \( Q \) belongs to \( C^1(\mathbb{R} \times \mathbb{R}^n) \), and it is of order \( C^2 \) everywhere except on the set \( U^{-1}(\Upsilon_0) \), where
\[
\Upsilon_0 = \{(u, v) \in \mathbb{R}^+ \times \mathbb{R}^+ ; (v = 0) \lor (u^p = v^q)\}.
\]

Remark 5. The origins of this function lie in the paper of Nazarov and Treil [11]. A modification of their function was later applied in [5, 6]. Here we use a simplified variant which comprises only two variables. It was introduced in [7]. The function \( Q \) above is the same as in [7], except that it differs by a sign; thus it is nonnegative while the function in [7] was nonpositive.

Remark 6. In contrast to [3, 5, 7], to keep our notation reasonable and to gain some transparency and simplicity in the proofs, we use a Bellman function involving only real variables. This allows us to prove Theorem 3 and Corollary 4 just for real-valued functions and differential forms; the corresponding estimates for complex-valued functions...
and differential forms easily follow by estimating separately the real and imaginary part. Note that the argument above gives an appropriately bigger constant. In order to preserve the same constants one could readily instead use a “complex” Bellman function as in [3, 6, 7] and prove Theorem 3 and Corollary 4 for complex-valued functions and differential forms.

Throughout the rest of the paper we shall use the following notation: if \( m \in \mathbb{N}, \ \Omega \subset \mathbb{R}^m \) is open, \( \Phi \in C^\infty(\Omega), \ \omega \in \Omega \) and \( x \in \mathbb{R}^m \), then we set

\[
H_{\Phi}(\omega; x) = (\text{Hess}(\Phi)_{\omega}, x)_{\mathbb{R}^m},
\]

where \( \text{Hess}(\Phi)_{\omega} \) is the Hessian matrix of \( \Phi \) at \( \omega \), i.e. \( [\partial_{x,x_j} \Phi(\omega)]_{i,j=1}^m \).

The following result, essentially proved in [6], summarizes the properties of \( Q \).

**Theorem 7.** For every \( (u, v) \in \mathbb{R}_+ \times \mathbb{R}_+ \)

(i) \( 0 \leq \beta(u, v) \leq (1 + \delta)(u^p + v^q) \).

If \( \xi = (\zeta, \eta) \in (\mathbb{R} \times \mathbb{R}^n) \setminus \operatorname{U}^{-1}(Y_0) \), then there exists \( \tau = \tau(|\zeta|, |\eta|) > 0 \) such that

(ii) \( H_Q(\xi; w) \geq \delta(\tau|w_1|^2 + \tau^{-1}|w_2|^2) \), for all \( w = (w_1, w_2) \in \mathbb{R} \times \mathbb{R}^n \).

Moreover, there is a certain absolute \( C = C(p) > 0 \) such that for every \( u, v > 0 \),

(iii) \( 0 \leq \partial_u \beta(u, v) \leq C \max\{u^{p-1}, v\} \) and \( 0 \leq \partial_v \beta(u, v) \leq C v^{q-1} \).

As noted earlier, while \( Q \) is of class \( C^1 \), it is not globally \( C^2 \). One can fix this in a standard fashion by taking convolutions with mollifiers. More precisely, denote by \( B^{n+1} \) the open unit ball in \( \mathbb{R}^{n+1} \) and define

\[
\psi(x) = c_{n+1} e^{-\frac{1}{1-|x|^2}} \chi_{B^{n+1}}(x),
\]

where \( c_{n+1} \) is chosen so that the integral of \( \psi \) over \( \mathbb{R}^{n+1} \) is equal to one. For any \( \kappa > 0 \) and \( x \in \mathbb{R}^{n+1} \) set

\[
\psi_\kappa(x) = \frac{1}{\kappa^{n+1}} \psi\left(\frac{x}{\kappa}\right).
\]

The (regular) Bellman function \( Q_\kappa \) is defined on \( \mathbb{R} \times \mathbb{R}^n \) by

\[
Q_\kappa = \psi_\kappa * Q,
\]

where \( * \) denotes the convolution in \( \mathbb{R}^{n+1} \). Since both \( Q \) and \( \psi_\kappa \) are biradial, there exists \( \beta_\kappa : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) such that

\[
Q_\kappa(\zeta, \eta) = \frac{1}{2} \beta_\kappa(|\zeta|, |\eta|),
\]

for all \( (\zeta, \eta) \in \mathbb{R} \times \mathbb{R}^n \).

**Theorem 8.** Let \( \kappa \in (0, 1) \). Then \( Q_\kappa \in C^\infty(\mathbb{R}^{n+1}) \) and, for any \( (u, v) \in \mathbb{R}_+ \times \mathbb{R}_+ \),

(i) \( 0 \leq \beta_\kappa(u, v) \leq (1 + \delta)[(u + \kappa)^p + (v + \kappa)^q] \).

For any \( \xi = (\zeta, \eta) \in \mathbb{R} \times \mathbb{R}^n \), there exists \( \tau_\kappa = \tau_\kappa(|\zeta|, |\eta|) > 0 \) such that

(ii) \( H_{Q_\kappa}(\xi; w) \geq \delta(\tau_\kappa|w_1|^2 + \tau^{-1}_\kappa|w_2|^2) \), for all \( w = (w_1, w_2) \in \mathbb{R} \times \mathbb{R}^n \).

Moreover, there is a certain absolute \( C = C(p) > 0 \) such that for every \( u, v > 0 \),

(iii) \( 0 \leq \partial_u \beta_\kappa(u, v) \leq C \max\{u^{p-1}, v + \kappa\} \) and \( 0 \leq \partial_v \beta_\kappa(u, v) \leq C(v + \kappa)^{q-1} \).
Theorem 3. Consider the operators \( T \) where 
\[
\begin{align*}
\tau
\end{align*}
\] 
It follows that 
\[
\begin{align*}
|T(x, η)| &\leq \frac{1}{2} β_κ(|ξ|, |η|), 
|T(x, η)| &\leq \frac{1}{2} β_κ(|ξ|, |η|), 
\end{align*}
\] 
Then we have 
\[
\begin{align*}
\left|\sum_{i=1}^{n+1} H_{Q_κ}(S_x(v); S_x(∇_i v)) - \frac{∂_i β_κ(|ξ|, |η|)}{2|ξ|} ξ \mathcal{L}_φ \zeta + \frac{∂_i β_κ(|ξ|, |η|)}{2|η|} \left( Ric(ζ, η) - \langle \mathcal{L}_φ η, η \rangle \right) \right| &< \epsilon 
\end{align*}
\] 
Lemma 9. Let \( ζ \in C^∞(M × R^+), \ η \in C^∞ E \), and set \( v(x,t) = (ζ(x,t), η(x,t)) \). Then, for every \( (x,t) \in M × R^+ \), in exponential local coordinates centered at \( x \) we have 
\[
\begin{align*}
-\mathcal{L}_φ Q_κ &\ni \sum_{i=1}^{n+1} H_{Q_κ}(S_x(v); S_x(∇_i v)) - \frac{∂_i β_κ(|ξ|, |η|)}{2|ξ|} ξ \mathcal{L}_φ \zeta + \frac{∂_i β_κ(|ξ|, |η|)}{2|η|} \left( Ric(ζ, η) - \langle \mathcal{L}_φ η, η \rangle \right) \end{align*}
\]
where \( \nabla_{n+1} = \nabla_t, \zeta = \zeta(x, t), \eta = \eta(x, t) \) and \( \tilde{S}_x \) is defined by \( \tilde{S}_x(\zeta, \eta) = (\zeta, S_x(\eta)) \).

**Proof.** The lemma follows (by direct computation in exponential local coordinates) from the very definition of \( \tilde{Q}_\kappa \) and the Bochner formula [11] eq. (0.3)] which says that

\[
\frac{1}{2} \mathcal{L}_\varphi(\omega^2) = |\nabla \omega|^2 - \langle \mathcal{L}_{\varphi, \omega}, \omega \rangle + \text{Ric}_\omega(\omega, \omega)
\]

for all \( \omega \in C^\infty(T^* M) \).

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### 5. Proof of Theorem 3

We first prove the theorem for \( p \geq 2 \). Let \( f \in C^\infty_c(M) \) and \( \omega \in C^\infty_c(T^* M) \). In view of Remark 8 we can assume \( f, \omega \) to be real-valued. Fix \( a \in M \) and \( \varepsilon > 0 \). For every \( s, l > 0 \), define \( K_{s,l} = B(a, 2l) \times [1/s, s] \) and

\[
\kappa_{s,l} = \varepsilon \inf_{(x,t) \in K_{s,l}} \min \{ P_t^a |f|(x), P_t^a |\omega|(x) \}.
\]

Since \( P_t^a \) is an integral operator with positive kernel, and \( (x, t) \rightarrow P_t^a u(x) \) is continuous for all nice \( u \), it follows that \( \kappa_{s,l} > 0 \). Next define the function \( b_{s,l} \) by setting

\[
b_{s,l}(x, t) = \tilde{Q}_{\kappa_{s,l}}(P_t^a f(x), \tilde{P}_t^a \omega(x)),
\]

for all \( (x, t) \in M \times \mathbb{R}_+ \).

Similar to the Euclidean case [6] the bulk of the proof of Theorem 3 will consist of estimating an integral involving \( \mathcal{L}_\varphi b_{s,l} \) from below and above. This will be the content of Propositions [11] and [12] respectively.

**Proposition 10.** Suppose that \( \text{Ric}_\varphi \geq -a^2 g \). Then, for all \( (x, t) \in M \times \mathbb{R}_+ \),

\[
-\mathcal{L}_\varphi b_{s,l}(x, t) \geq 2\delta |\nabla P_t^a f(x)||\nabla \tilde{P}_t^a \omega(x)|.
\]

**Proof.** We apply Lemma 9 with \( \zeta = P_t^a f, \eta = \tilde{P}_t^a \omega, \nu(x,t) = (P_t^a f(x), \tilde{P}_t^a \omega(x)) \) and \( \kappa = \kappa_{s,l} \). Since \( \mathcal{L}_\varphi P_t^a f = -a^2 P_t^a f, \mathcal{L}_\varphi \tilde{P}_t^a \omega = -a^2 \tilde{P}_t^a \omega, \text{Ric}_\varphi(\tilde{P}_t^a \omega, \tilde{P}_t^a \omega) \geq -a^2 |\tilde{P}_t^a \omega|^2 \) and, by Theorem 3 [11]), the partial derivatives of \( \beta_{s,l} \) are nonnegative, in exponential local coordinates centered at \( x \) we have that

\[
-\mathcal{L}_\varphi b_{s,l}(x, t) \geq \sum_{i=1}^{n+1} H_{Q_{\kappa_{s,l}}}(\tilde{S}_x(\nu(x,t)); \tilde{S}_x(\nabla_i \nu(x,t))).
\]

Therefore, Theorem 3 [11] and the identity \([S_x(\eta)]_{\mathbb{R}^n} = |\eta|_{T^* M}, \eta \in T^*_x M\), imply

\[
-\mathcal{L}_\varphi b_{s,l}(x, t) \geq 2\delta \left( \sum_{i=1}^{n+1} |\nabla_i P_t^a f(x)|^2 \right) \left( \sum_{i=1}^{n+1} |\nabla_i \tilde{P}_t^a \omega(x)|^2 \right),
\]

which is the statement from the proposition.

In order to estimate \( -\mathcal{L}_\varphi b_{s,l} \) from above we need a preliminary result.

**Lemma 11.** Suppose that \( \text{Ric}_\varphi \geq -a^2 g \). Then, for every \( (x, t) \in K_{s,l} \),

\[
b_{s,l}(x, t) \leq \frac{1 + \delta}{2} (1 + \varepsilon)^p (P_t^a |f|(x) + P_t^a |\omega|(x)).
\]

Moreover, there exists \( C = C(\varepsilon, p) \) such that, for every \( (x, t) \in K_{s,l} \),

\[
|\partial_t b_{s,l}(x, t)| \leq C \left( \max\{P_t^a |f|(x)\}^{p-1}, P_t^a |\omega|(x) \right) \left( \partial_t P_t^a f(x) + (P_t^a |\omega|(x))^{p-1} |\partial_t \tilde{P}_t^a \omega(x)| \right).
\]
Proof. By combining Theorem (6) with Lemma (3) and (8), we get
\[ b_{s,t}(x,t) \leq \frac{1 + \delta}{2} \left[ (P_{s}^nf(x) + \kappa_{s,t})^p + (P_{s}^\omega(x) + \kappa_{s,t})^q \right]. \]

The first part of the lemma now follows from the definition of \( \kappa_{s,t} \) and Lemma (3), (8).

Observe that
\[ 2|\partial_{t}b_{s,t}(x,t)| \leq \partial_{u}\beta_{\kappa_{s,t}}(|P_{s}^nf(x)|, |P_{s}^\omega(x)|)|\partial_{t}P_{s}^nf(x)| \\
+ \partial_{x}\beta_{\kappa_{s,t}}(|P_{s}^nf(x)|, |P_{s}^\omega(x)|)|\partial_{t}P_{s}^\omega(x)|. \]

The second part of the lemma follows from the definition of \( \kappa_{s,t} \), by combining the above inequality with Theorem (6) and Lemma (3), (9), (11).

Proposition 12. Suppose that \( \text{Ric} \geq -a^2 g \). Then
\[ \limsup_{s \to \infty} \sup_{l \to \infty} \int_{1/s}^{s} \int_{B(o,l)} -\mathcal{L}_{\varphi}b_{s,t}(x,t) \, d\mu_{\varphi}(x) \, t \, dt \leq \frac{1 + \delta}{2} (1 + \varepsilon)^p (\|f\|_p^p + \|\omega\|_q^q). \]

Proof. Recall that \( o \in M \) was fixed at the beginning of this section. Set \( r(x) = \rho(x,o) \), where \( \rho \) denotes the geodesic distance on \( M \). Thus \( B(o, \delta) = \{ x \in M : r(x) < \delta \} \). Since \( \text{Ric} \geq -a^2 g \), by (16) Theorem 3.1 (see also (13) Theorem 2.4) for the unweighted case, we have that
\[ -r\mathcal{L}_{\varphi}r \leq Cr^2 \quad \text{for} \quad r \geq 1 \]
weakly, and for almost all \( x \in M \). Take a nonincreasing function \( \Lambda \in C_c^\infty([0, \infty)) \) such that \( 0 \leq \Lambda \leq 1 \), \( \Lambda = 1 \) in \([0, 1]\) and \( \Lambda = 0 \) in \([2, \infty)\). For \( l > 0 \) and \( x \in M \) define
\[ F_l(x) = \Lambda \left( \frac{r(x)^2}{l^2} \right). \]

Observe that \( (\text{supp } F_l) \times [1/s, s] \subset K_{s,l} \). By Proposition (11) \( \mathcal{L}_{\varphi}b_{s,l} \geq 0 \), so that
\[ \int_{1/s}^{s} \int_{B(o,l)} -\mathcal{L}_{\varphi}b_{s,l}(x,t) \, d\mu_{\varphi}(x) \, t \, dt \leq \int_{1/s}^{s} \int_{M} -\mathcal{L}_{\varphi}b_{s,l}(x,t) F_l(x) \, d\mu_{\varphi}(x) \, t \, dt \\
= \int_{1/s}^{s} \int_{M} (\partial_{t} - \mathcal{L}_{\varphi})b_{s,l}(x,t) F_l(x) \, d\mu_{\varphi}(x) \, t \, dt \cdot \]
Therefore, to complete the proof it suffices to show that
\[ \limsup_{s \to \infty} \sup_{l \to \infty} \int_{1/s}^{s} \int_{M} \partial_{t}b_{s,l}(x,t) F_l(x) \, d\mu_{\varphi}(x) \, t \, dt \leq \frac{1 + \delta}{2} (1 + \varepsilon)^p (\|f\|_p^p + \|\omega\|_q^q) \]
and
\[ \lim_{l \to \infty} \int_{1/s}^{s} \int_{M} \mathcal{L}_{\varphi}b_{s,l}(x,t) F_l(x) \, d\mu_{\varphi}(x) \, t \, dt = 0 \]
for all \( s > 0 \).

We first prove (9). An integration by parts in the variable \( t \) gives
\[ \int_{1/s}^{s} \partial_{t}b_{s,l}(x,t) \, t \, dt = s\partial_{t}b_{s,l}(x,s) - s^{-1} \partial_{t}b_{s,l}(x,s,1/s) + b_{s,l}(x,1/s) - b_{s,l}(x,s). \]
Theorem (6) and Lemma (11) imply, for all \((x,t) \in K_{s,l}\),
\[ b_{s,l}(x,1/s) - b_{s,l}(x,s) \leq b_{s,l}(x,1/s) \leq \frac{1 + \delta}{2} (1 + \varepsilon)^p \left( P_{1/s}^nf(x)^p + F_{1/s}^\omega(x)^q \right). \]
It follows that
\[
\int_{1/s}^{s} \int_{M} \frac{\partial^2 b_{s,l}(x,t)}{t} F_l(x) \, d\mu_\varphi(x) \, t \, dt \\
\leq \frac{1+\delta}{2} \left( (1+\varepsilon)^p \left( \|f\|_p^p + \|\omega\|_q^q \right) + \|s\partial_\varphi b_{s,l}(x,s) + s^{-1}\partial_\varphi b_{s,l}(x,1/s)\|_{L^1(\mu_\varphi)} \right),
\]
where in the last inequality we used the fact that for every \( r \in [1,\infty] \) the semigroup \( P_t^a \) is contractive on \( L^r \). Therefore, in order to prove (19) it is enough to show that
\[
\lim_{s \to \infty} \|s\partial_\varphi b_{s,l}(x,s) + s^{-1}\partial_\varphi b_{s,l}(x,1/s)\|_{L^1(\mu_\varphi)} = 0. \tag{11}
\]
Since the semigroup \( P_t^a \) is contractive in \( L^r \) for all \( r \in [1,\infty] \), there exists \( h = h(p) > 2 \) such that
\[
\|(P_t^a |f|^p) + P_t^0 |\omega| + (P_t^0 |\omega|)^{q-1}\|_{L^h(\mu_\varphi)} \leq C(f, \omega, p)
\]
uniformly in \( t > 0 \). Hence, by Lemma \( \ref{lem:holder} \) and Hölder’s inequality, to prove (11) it suffices to show that
\[
\lim_{t \to \infty} \|t\partial_\varphi P_t^a f\| + |t\partial_\varphi P_t^a \omega|_{L^{h}(\mu_\varphi)} = 0, \tag{12}
\]
where \( h' \) is the conjugate exponent of \( h \). To prove (12), simply observe that by the spectral theorem \( t\partial_\varphi P_t^a f \) and \( t\partial_\varphi P_t^a \omega \) converge to 0 in \( L^2 \) as \( t \to 0, \infty \), and that \( \|t\partial_\varphi P_t^a f\|_{L^1} + |t\partial_\varphi P_t^a \omega|_{L^p} \) is uniformly bounded in \( t \) for all \( r \in (1,\infty) \[8, Thm 4.6 (c)\] \), because \( \text{Ric}_\varphi \geq -a^2 \text{g} \) implies that the semigroups \( P_t^a \) and \( P_t^0 \) are both analytic on \( L^\infty(\mu_\varphi) \), for all \( r \in (1,\infty) \).

We now prove (19). We first show that
\[
\liminf_{l \to \infty} \int_{1/s}^{s} \int_{M} -\mathcal{L}_\varphi b_{s,l}(x,t) F_l(x) \, d\mu_\varphi(x) \, t \, dt \geq 0. \tag{13}
\]
A simple computation based on \[8\] gives
\[
-\mathcal{L}_\varphi F_l = \frac{-2r \mathcal{L}_\varphi r}{t} \Lambda' r^2/t^2 + \frac{4r^2}{t} \left| \frac{dr}{t} \right| \Lambda'' r^2/t^2.
\]
Since \( \|dr\|_\infty \leq 1, \Lambda' \leq 0 \), and \( \Lambda' = 0 \) on \([0,1] \cup [2,\infty) \), by (17) there exists \( C > 0 \) such that
\[
-\mathcal{L}_\varphi F_l \geq -C \left( \|\Lambda'\|_\infty + \|\Lambda''\|_\infty \right) \chi_{B(o,2l) \setminus B(o,l)}, \tag{14}
\]
for all \( l \geq 1 \). Integrating by parts in (13), and combining (13) with Lemma \( \ref{lem:holder} \) we obtain
\[
\int_{M} -\mathcal{L}_\varphi b_{s,l}(x,t) F_l(x) \, d\mu_\varphi(x) \geq -C \int_{B(o,2l) \setminus B(o,l)} (P_t^a |f|^p + P_t^0 |\omega|^{q}) \, d\mu_\varphi.
\]
Denote the integral on the right hand side by \( \Psi_l(t) \). Since \( \lim_{t \to \infty} \Psi_l(t) = 0 \) pointwise on \( \mathbb{R}_+ \) and \( 0 \leq \Psi_l(t) \leq |f|_p^p + |\omega|_q^q \), the Lebesgue dominated convergence theorem implies (13).

It remains to prove that
\[
\limsup_{l \to \infty} \int_{1/s}^{s} \int_{M} -\mathcal{L}_\varphi b_{s,l}(x,t) F_l(x) \, d\mu_\varphi(x) \, t \, dt \leq 0. \tag{15}
\]
Consider the function \( R = (1/2)(1 + \delta)(1 + \epsilon)^p(P_t^p |f|^p + P_t^0 |\omega|^q) \). By Lemma 11, \( b_{s,l} - R \leq 0 \) on \( K_{s,l} \), and an argument similar to the one we used to prove (13) shows that
\[
\limsup_{l \to \infty} \int_{1/s}^{s} \int_{M} -\mathcal{L}_\varphi(b_{s,l}(x,t) - R(x,t))F_l(x) \, d\mu_\varphi(x) \, t \, dt \leq 0.
\]
We now prove that
\[
\limsup_{l \to \infty} \int_{1/s}^{s} \int_{M} \mathcal{L}_\varphi R(x,t)F_l(x) \, d\mu_\varphi(x) \, t \, dt = 0. \tag{16}
\]
Since \( \|dr\|_\infty \leq 1 \) and \( F_l = 0 \) for \( r \geq 2l \), we have that
\[
\|dF_l\|_\infty \leq 4\|\Lambda'\|_\infty l.
\]
Therefore, by integrating by parts we get
\[
\left| \int_{1/s}^{s} \int_{M} \mathcal{L}_\varphi R(x,t)F_l(x) \, d\mu_\varphi(x) \, t \, dt \right| \leq 4\|\Lambda'\|_\infty \int_{1/s}^{s} \int_{M} |dR(x,t)| \, d\mu_\varphi(x) \, t \, dt.
\]
By Lemma 2,
\[
dR(x,t) = C(\|dP_t^a f\|^p + \|dP_t^0 \omega\|^q) = C(P_t^a \, d|f|^p + P_t^0 \, d|\omega|^q),
\]
where the right-hand side is in \( L^1(M \times [1/s,s], d\mu_\varphi \, t \, dt) \) because \( f \) and \( \omega \) are regular, compactly supported and \( |P_t^0 \omega| \leq e^{t\alpha^2} P_t^0 |\omega| \). This implies (10) and concludes the proof of the proposition.

**Proof of Theorem 3.** Suppose that \( p \geq 2 \). By combining Propositions 10 and 12 using the Fatou lemma and passing to the limit as \( \epsilon \to 0 \), we get
\[
2\delta \int_{0}^{\infty} \int_{M} |\nabla P_t^a f| |\nabla P_t^0 \omega| \, d\mu_\varphi \, t \, dt \leq \frac{1 + \delta}{2} (\|f\|^p_p + \|\omega\|^q_q) .
\]
Now apply the above inequality to \( \lambda f \) and \( \lambda^{-1} \omega \) instead of \( f, \omega \), respectively, and minimize in \( \lambda \geq 0 \). The result is
\[
\int_{0}^{\infty} \int_{M} |\nabla P_t^a f| |\nabla P_t^0 \omega| \, d\mu_\varphi \, t \, dt \leq C_q(p - 1)\|f\|_p \|\omega\|_q ,
\]
where
\[
C_q = \frac{8 + q(q - 1)}{4} (q - 1)^{1/q - 1} .
\]
The substitution \( s = q - 1 \) returns
\[
\sup_{q \in (1,2]} C_q = \frac{1}{4} \sup_{s \in (0,1)} (s^2 + s + 8)s^{-s+1} < 2.8 < 3 .
\]
When \( 1 < p < 2 \), interchange \( P_t^a f \) and \( P_t^0 \omega \) in the definition of \( b_{s,l} \), and proceed as before. \( \square \)
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