Loss of the sensitivity of a gravitational wave detector due to temperature

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Abstract

Using an explicit solution for the Schrödinger equation describing a model of the gravitational wave detector (LIGO-project), we prove that the SQL (standard quantum limit) for dimensionless amplitude for gravitational perturbations of the metric exceeds $10^{-19}$ at temperature 100K.

The first estimate of the sensitivity of a gravitational wave detector was obtained by V. B. Braginsky [1]. Our estimate of the SQL bases on the model proposed in [2, 3]. Actually, we use the Hamiltonian $\hat{H}$ [3], and we find that the unitary group generated by $\hat{H}$ slightly differs from [3] (see [4, 5]). The main observation of this paper is that the temperature of the initial state of the oscillator decreases the sensitivity of the measuring device.

1. Let the space of the product system be $l_2 \otimes l_2$, where the first Hilbert space is the state space of the electromagnetic field and the second factor corresponds to the state space of the quantum oscillator. On this product space, consider the Hamiltonian:

$$H = \frac{1}{\hbar}(H_0 + H_{int}) = \omega a^\dagger a \otimes I + I \otimes \omega_0 b^\dagger b - (\kappa a^\dagger a + f(t)) \otimes (b + b^\dagger),$$ (1)

where $f(t) = \frac{F(t)}{\sqrt{2m\omega_0\hbar}}, F(t)$ is a classical force generated by a gravitational wave, $\kappa = \frac{\omega_0}{L}\sqrt{\frac{2\hbar}{m\omega_0}}$ is a coupling constant, $\omega$, $L$, $\omega_0$, $m$ are the optical frequency, the length of the cavity, the eigenfrequency and the mass of the oscillator correspondingly.

In paper [4], we proved that an explicit solution of the Schrödinger evolution equation

$$\frac{d}{dt}U_t = iH U_t, \quad U_t|_{t=0} = I \otimes I$$

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with the Hamiltonian (1) reads as follows:

\[ U_t = e^{iH_0 t} e^{-C(t)} e^{-ib\hat{\beta}(t)} e^{-ib\beta(t)}, \tag{2} \]

where

\[ \beta(t) = \int_0^t d\tau (\kappa a^\dagger a + f(\tau)) e^{i\omega_0 \tau}, \]

\[ C(t) = \int_0^t d\tau (\kappa a^\dagger a + f(\tau)) e^{i\omega_0 \tau}. \]

As an observable, we choose a quantized output of the classical two-beam interference

\[ A(t) = \frac{\hat{I}_N}{2i} (e^{i\frac{\omega x}{c}} - e^{-i\frac{\omega x}{c}}), \]

where \( x_t \) is the displacement of the oscillator caused by the gravitational force:

\[ \hat{A} = \frac{\hat{I}_N}{2i} (e^{ig(b^\dagger+b)} - e^{-ig(b^\dagger+b)}) = \frac{\hat{I}_N}{2i} e^{-\frac{g^2}{2}} (e^{igb^\dagger} e^{igb} - e^{-igb^\dagger} e^{-igb}), \tag{3} \]

\[ \langle \hat{A}^2 \rangle = (\hat{I}_N)^2 \left\{ \frac{1}{2} - \frac{e^{-2g^2}}{4} \left( e^{-2igb^\dagger} e^{-2igb} + e^{-2igb^\dagger} e^{-2igb} \right) \right\}, \]

\[ \theta_g = \frac{\omega}{\omega_0} \int_0^t \frac{F_g(\tau)}{mc} \sin \omega_0 (t - \tau) d\tau, \quad F_g(\tau) = Lm h_0 \omega_0^2 \cos(\omega_g \tau) \tag{6} \]
corresponds to the gravitational source, $\omega_g$ is the frequency of the gravitational wave and $h_0$ is the dimensionless amplitude of gravitational perturbations of the metric tensor, and

$$\theta_1 = 2\kappa g (1 - \cos(\omega_0 t))$$ \hspace{1cm} (7)

describes the light pressure of the laser beam on the oscillator.

In order to estimate the sensitivity of the detector, we use the following condition on the mean value of the output signal (4) and its dispersion (5):

$$I(t) > \sqrt{D(t)}.$$ \hspace{1cm} (8)

In resonant case $\omega_0 = \omega_g$, this inequality gives the standard estimate of the SQL [1] corresponding to the ground state of the oscillator (for detailed discussion see [4]):

$$h_0 > h_{SQL} = \frac{1}{L t \omega_0} \sqrt{\frac{\hbar}{m \omega_0}} = 5 \cdot 10^{-24}.$$ \hspace{1cm} (9)

All numerical values are obtained by using realistic parameters of LIGO II project:

$$\omega_g = 30 \text{ sec}^{-1}, \quad L = 4 \cdot 10^3 \text{ m}, \quad m = 10 \text{ kg}, \quad \omega = 1.8 \cdot 10^{15} \text{ sec}^{-1}.$$ \hspace{1cm} (10)

Estimate (9) coincides with the famous V. B. Braginsky formula [1] for SQL. This solvable model allows us to obtain the SQL for a thermal equilibrium initial state of the oscillator.

2. Suppose that the oscillator is at thermal equilibrium with the environment at temperature $\theta = k T$, where $k$ is Boltzmann constant and $T$ is Kelvin temperature. Then the initial state of the oscillator is described by the density matrix with Gibbs weight:

$$\rho_0 = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-\frac{\hbar \omega_n}{\theta}} |n\rangle\langle n|,$$

where $Z = \frac{1}{1 - e^{-\frac{\hbar \omega}{\theta}}}$ is the statistical sum. Let us estimate minimal detectable perturbations of the metric, when the oscillator has non-zero temperature before measurements.
For observable (3) under evolution (2), we obtain the following partially averaged output signal

\[ \hat{I}(t) = \hat{I}_N \text{Tr}_{\text{osc}} \{ e^{ig b^\dagger e^{ig b} U_t \rho_0 U_t^\dagger} \} = \]

\[ = \frac{1}{Z} \hat{I}_N e^{-g^2 \frac{t}{2}} e^{i \theta_g + in \theta_l} \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)! k!} g^{2k} e^{-\frac{n \hbar \omega \theta}{2}}, \tag{11} \]

where \( \theta_g \) and \( \theta_l \) are given by (6) and (7). The sum in (11) can be calculated explicitly. The Laguerre polynomial equals

\[ L_n(z) = L_n^{(0)}(z) = 1 - \left( \frac{n}{1!} \right) z \frac{z^2}{2!} - \cdots + (-1)^n \left( \frac{n}{n} \right) \frac{z^n}{n!} \]

and their generating function is given by

\[ e^{-\frac{zt}{1-t}} = \sum_{n=0}^{\infty} L_n(z) t^n \quad (|t| < 1). \]

In our case, \( t = e^{-\frac{\hbar \omega \theta}{g}} < 1 \) and for \( \langle \hat{A} \rangle_{\text{osc}} = \Re \hat{m} \hat{I}(t) \), we obtain

\[ \langle \hat{A} \rangle = \alpha \hat{I}_N e^{-g^2 \frac{t}{2}} \sin(\theta_g + a^\dagger a \theta_l), \tag{12} \]

where

\[ \alpha = \exp \left\{ -\frac{g^2}{e^{\hbar \omega / \theta} - 1} \right\}. \tag{13} \]

Considering the exponential distribution of photons in the laser beam at large \( N \sim 10^{17} \) and at small \( |\theta_l| = 1.2 \cdot 10^{-19} \), the mean value and the dispersion of the signal reads as follows:

\[ I = \alpha I_N e^{g^2 \frac{t}{2}} \sin(\theta_g + N \theta_l), \tag{14} \]

\[ \langle \hat{A}^2 \rangle = \frac{I_N^2}{2} \left( 1 - \alpha^4 e^{-2g^2 - 2N \theta_l^2} \cos(2\theta_g + 2N \theta_l) \right) \]

\[ D = \langle \hat{A}^2 \rangle - I^2. \tag{15} \]

For temperatures from \( 10^{-9}\text{K} \) to \( 10^{10}\text{K} \), the coefficient \( \alpha \) (13) equals approximately \( \alpha \approx 1 - \frac{g^2 \theta}{\hbar \omega} \). The condition for detection of a gravitational wave (8)
\[ I(t) > \sqrt{D(t)} \text{ can be estimated by linearizing expressions (14) and (15) since } g \ll 1 \]

\[ |\theta_g + N\theta| > \sqrt{g^2 + g^2 \frac{\theta}{h \omega_0} + N\theta^2}. \]

If we compare the signal from the gravitational wave \( \theta_g \) with its dispersion in semi-classical case, when number of photons does not fluctuate (as was done in [4]), we obtain the SQL for perturbations of the metric (see (9))

\[ h_{SQL}(T) = h_{SQL} \sqrt{1 + \frac{kT}{h \omega_0}} \sim 10^{-19} \quad (16) \]

at temperature \( T \sim 100K \) and for realistic values of the parameters of LIGO project [10]. It is clear that thermal noises have a great significance for the sensitivity of a measuring device. The corresponding SQL (16) is much higher (about 10^5) than the SQL in vacuum [9].

References

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