The Affine Hidden Symmetry and Integrability of Type IIB Superstring in $AdS_5 \times S^5$

Bo-Yu Hou$^a$, Dan-Tao Peng$^{ab}$, Chuan-Hua Xiong$^a$, Rui-Hong Yue$^a$

$^a$Institute of Modern Physics, Northwest University, Xi’an, 710069, P. R. China
$^b$Interdisciplinary Center for Theoretical Study, University of Science and Technology of China, Hefei, Anhui, 230026, P. R. China

Abstract

In this paper, we motivate how the Hodge dual related with S-duality gives the hidden symmetry in the moduli space of IIB string. Utilizing the static $\kappa$-symmetric Killing gauge, if we take the Hodge dual of the vierbeins keeping the connection invariant, the duality of Maure-Cartan equations and the equations of motion becomes manifest. Thus by twistly transforming the vierbein, we can express the BPR currents as the Lax connections by a unique spectral parameter. Then we construct the generators of the infinitesimal dressing symmetry, the related symmetric algebra becomes the affine $gl(2,2|4)^{(1)}$, which can be used to find the classical $r$ matrix.

1 Introduction

Maldacena [1] proposed the AdS/CFT correspondence between the classical theory of SUGRA on $AdS_5 \otimes S^5$ in the bulk and the quantum conformal SUSY Yang-Mills theory...
on the boundary.

In the last years, the existence of integrability structures on both sides of the correspondence have been discussed in papers [2–14]. Bena, Polchinski and Roiban [15] found an infinite set currents of classically conserved current for the Green-Schwarz [16] string on $AdS_5 \times S^5$, such that it may be exact solvable. Dolan, Nappi and Witten [17] have described the equivalences between this integrable structure and the Yangian symmetry of the nonlocal currents as Bernards’ paper [18].

For this infinite dimensional symmetry, the quantization procedure will preserve it. But, to quantize such system, one has to handle infinite number of the second class constraints given by the Dirac quantization method. The ghost will appear, BRST method is needed and the symmetry e.g. $\kappa$ symmetry will not be manifest [19].

We try to disclose the complete symmetries of this classical system and to construct the Lax-connection correspondently. Then the fundamental Poisson bracket [20] will give out the classical $r$ matrix [21]. Further, the quantization of $r$ matrix gives the quantum double $R$ matrix which determines the scattering matrix. The amplitudes will be given in terms of Bethe Ansatz method.

In this paper, we investigate the duality between the Maurer-Cartan equations (MCE) and the equations of motion (EOM) and obtain the Lax-matrix by using the twisted dual transformation which represents a dressing symmetry for GS string embedding into $AdS_5 \otimes S^5$. Physically, the invariance of re-parametrization admits such twist transformation. However, since on the MT action has imposed the conformal gauge, the action is not invariant under re-parametrization. If we take the improved stress-energy tensor the conformal symmetry will recovered as conformal affine symmetry. The full symmetric algebra will be enlarged from $PSU(2,2|4)$ into $gl(2,2|4)^{(1)}$.

### 2 The Metsaev-Tseytlin action of Green-Shwarz superstring in $AdS^5 \otimes S^5$ [22]

The $AdS_5 \otimes S^5$ is a coset space $\frac{SO(4,2)}{SO(4,1)} \otimes \frac{SO(6)}{SO(5)}$. It also preserves the full supersymmetry of the SUGRA and corresponds to the maximally supersymmetric background vacuum of IIB SUGRA. Combining the bosonic $SO(4,2) \otimes SO(6)$ isometry symmetry with the full supersymmetry, the symmetry turns to be the $PSU(2,2|4)$ acting on the super
coset space $\frac{PSU(2,2|4)}{SO(4,1) \otimes SO(5)}$. In what follows, we adapt the conventions introduced by [22]:

\[a, b, c = 0, 1, \cdots, 4 \text{ so}(4,1) \text{ vector indices (} AdS_5 \text{ tangent space)}\]
\[a', b', c' = 5, \cdots, 9 \text{ so}(5) \text{ vector indices (} S^5 \text{ tangent space)}\]
\[\alpha, \beta, \gamma, \delta = 1, \cdots, 4 \text{ so}(4,1) \text{ spinor indices (} AdS_5 \text{)}\]
\[\alpha', \beta', \gamma', \delta' = 1, \cdots, 4 \text{ so}(5) \text{ spinor indices (} S^5 \text{)}\]
\[\alpha, \hat{\beta}, \hat{\gamma} = 1, \cdots, 32 \text{ } D = 10 \text{ Majorana-Weyl spinor indices}\]
\[I, J, K, L = 1, 2 \text{ } SO(2) \text{ labels of the } N = 2 \text{ two sets of spinors}\]

The generators of the so(4,1) and so(5) Clifford algebras are $4 \times 4$ matrices $\gamma^a$ and $\gamma^{a'}$

\[\gamma^{(a}\gamma^{b)} = \eta^{ab} = (-++++) \quad \gamma^{(a'}\gamma^{b')} = \eta^{a'b'} = (++++)\]

\[\hat{\gamma}^a \equiv \gamma^a, \quad \hat{\gamma}^{a'} \equiv i\gamma^{a'}\] (1)

satisfying $(\gamma^a)^\dagger = \gamma^0 \gamma^a \gamma^0$, $(\gamma^{a'})^\dagger = \gamma^{a'}$ and the Majorana condition is diagonal with respect to the two supercharges

\[\bar{Q}_{aa'I} \equiv (Q_1^{\beta\beta'})^\dagger (\gamma^0)_{\alpha}^{\beta} \delta_{\alpha'}^{\beta'} = -Q_1^{\beta\beta'} C_{\beta\alpha} C_{\beta'\alpha'} \] (2)

Here $C = (C_{\alpha\beta})$ and $C' = (C_{\alpha'\beta'})$ are the charge conjugation matrices of the so(4,1) and so(5) Clifford algebras $Q_{aa'I} \equiv Q_1^{\beta\beta'} C_{\beta\alpha} C_{\beta'\alpha'}$. The bosonic generators are antihermitean: $P_a^{\dagger} = -P_a$, $P_{a'}^{\dagger} = -P_{a'}$, $J_{ab}^{\dagger} = -J_{ab}$, $J_{a'b'}^{\dagger} = -J_{a'b'}$. The SO(2) $2 \times 2$ matrices are $\epsilon^{IJ} = -\epsilon^{JI}, \epsilon^{12} = 1$, and $s^{IJ} \equiv \text{diag}(1, -1)$.

The 10-dimensional $32 \times 32$ Dirac matrices $\Gamma^a$ of SO(9,1) ($\Gamma^{(a} \Gamma^{b)} = \eta^{ab}$) and the corresponding charge conjugation matrix $C$ can be represented as

\[\Gamma^a = \gamma^a \otimes I \otimes \sigma_1, \quad \Gamma^{a'} = I \otimes \gamma^{a'} \otimes \sigma_2, \quad C = C \otimes C' \otimes i\sigma_2, \] (3)

where $I$ is the $4 \times 4$ unit matrix and $\sigma_i$ are the Pauli matrices. The chirality is the eigenvalue of $\sigma_3$ in the last factor.

The generators of superalgebra $g \text{ su}(2,2|4)$ are divided into: a. the even generators $B$ which includes two pairs of translations and rotations $- (P_a, J_{ab}) \equiv (h, k)$ for $AdS_5$ and $(P_{a'}, J_{a'b'}) \equiv (h', k')$ for $S^5$ respectively; b. the odd generators $F$ are the two $D = 10$ Majorano-Weyl spinors $Q_{I}^{a'} \equiv F_I$. 

3
The commutation relations for the generators $T_A = (P_a, P_{a'}, J_{ab}, J_{a'b'})|Q_{\alpha\alpha'}I \equiv (\mathfrak{h}, \mathfrak{h'}, \mathfrak{e}, \mathfrak{e'}|\mathcal{F}_I) \equiv (\hat{\mathfrak{h}}; \hat{\mathfrak{e}}|\mathcal{F})$ are

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + 3 \text{ terms} , \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} ,$$

(4)

$$[J_{a'b'}, J_{c'd'}] = \eta_{b'c'}J_{a'd'} + 3 \text{ terms} , \quad [\mathfrak{h'}', \mathfrak{h'}'] \subset \mathfrak{h'}' ,$$

(5)

$$[P_a, P_b] = J_{ab} , \quad [\mathfrak{e}, \mathfrak{e}] \subset \mathfrak{h}$$

(6)

$$[P_{a'}, P_{b'}] = -J_{a'b'} , \quad [\mathfrak{e'}, \mathfrak{e'}'] \subset \mathfrak{h'}'$$

(7)

$$[P_a, J_{bc}] = \eta_{ab}P_c - \eta_{ac}P_b , \quad [\mathfrak{h}, \mathfrak{e}] \subset \mathfrak{h} ,$$

(8)

$$[P_{a'}, J_{b'c'}] = \eta_{a'b'}P_{c'} - \eta_{a'c'}P_{b'} , \quad [\mathfrak{h'}', \mathfrak{e'}] \subset \mathfrak{h'}'$$(9)

$$[Q_I, P_a] = -\frac{i}{2} \epsilon_{IJ}Q_j\gamma_a , \quad [\mathcal{F}_I, \mathfrak{e}] \subset \epsilon_{IJ}\mathcal{F}_J ,$$

(10)

$$[Q_I, P_{a'}] = \frac{1}{2} \epsilon_{IJ}Q_j\gamma_{a'} , \quad [\mathcal{F}_I', \mathfrak{e}'] \subset \epsilon_{IJ}\mathcal{F}_J ,$$

(11)

$$[Q_I, J_{ab}] = -\frac{1}{2} Q_I\gamma_{ab} , \quad [\mathcal{F}_I, \mathfrak{h}] \subset \mathcal{F}_I ,$$

(12)

$$[Q_I, J_{a'b'}] = -\frac{1}{2} Q_I\gamma_{a'b'} , \quad [\mathcal{F}_I, \mathfrak{h}'] \subset \mathcal{F}_I .$$

(13)

$$\{Q_{\alpha\alpha'I}, Q_{\beta\beta'I}J\} = \delta_{IJ} \left[ -2i C_{C}^{alpha'beta'}(C\gamma^a)_{alpha}P_a + 2 C_{C}^{alpha'beta'}(C'\gamma^a)_{alpha'}P_{a'} \right]$$

$$+ \epsilon_{IJ} \left[ C_{C}^{alpha'beta'}(C\gamma^a)_{alpha}J_{ab} - C_{C}^{alpha'beta'}(C'\gamma^a)_{alpha'}J_{a'b'} \right] , \quad [\mathcal{F}; \mathcal{F}] \subset \mathcal{B} .$$

(14)

The left-invariant Cartan 1-forms

$$L^A = dX^M L^A_M , \quad X^M = (x, \theta)$$

(15)
are given by

\[ G^{-1}dG = L^A T_A = L^a P_a + L^a' P_{a'} + \frac{1}{2} L_{ab} J_{ab} + \frac{1}{2} L_{a'b'} J_{a'b'} + L^{\alpha\alpha'} Q_{\alpha\alpha'} , \]  

(16)

where \( G = G(x, \theta) \) is a coset representative in \( PSU(2,2|4) \), e.g. by using the S-gauge given by [23] or the KRP gauge [24, 25] used by Roiban and Siegel [27].

We call \( \hat{H}(H, H') \), \( \hat{K}(K, K') \), \( K_F \) forms respectively for the Cartan connections \( L_{ab} \) and \( L_{a'b'} \), the super-beins \( \hat{L}^\alpha \) including funf-beins \( L^a \) and \( L^{a'} \) and the 2-spinor 16-beins \( L^{\alpha\alpha'} \). They satisfy the Maurer-Cartan (MC) equation, i.e. the structure equation of basic one forms on the superspace \( SU(2|4) \otimes SO(4,1) \)

\[ d(G^{-1}dG) + (G^{-1}dG) \wedge (G^{-1}dG) = 0. \]  

(17)

Then the super Gauss equations of the induced curvatures \( F_{ab} \) and \( F_{a'b'} \) defined by \( F = dH + H \wedge H \) are

\[ F_{ab} \equiv dL_{ab} + L_{ac} \wedge L^{cb} = -L^a \wedge L^b + \epsilon^{IJ} L^I \gamma_{ab} \wedge L^J , \]  

(18)

\[ F_{a'b'} \equiv dL_{a'b'} + L_{a'c} \wedge L^{c'b'} = L^{a'} \wedge L^{b'} - \epsilon^{IJ} L^I \gamma_{a'b'} \wedge L^J . \]  

(19)

The super Coddazi equation for the funf-beins are

\[ dL^a + L^b \wedge L^{ba} = -iL^J \gamma^a \wedge L^J , \quad dL^{a'} + L^{b'} \wedge L^{ba'} = L^J \gamma^{a'} \wedge L^J , \]  

(20)

and the super Coddazi equation for the spinor 16-beins are

\[ dL^I - \frac{1}{2} \gamma^a L^I \wedge L^a - \frac{1}{4} \gamma_{a'b'} L^I \wedge L^{a'b'} = -L^a \epsilon^{IJ} L^J \wedge L^a + \frac{1}{2} \epsilon^{IJ} L^J \gamma^{a'} L^a \wedge L^{a'} . \]  

(21)

In the super Gauss equations and the Coddazi equations, the terms on the left hand side are the usual gauge covariant exterior derivative \( d + H \wedge \), while the right hand side include the contributions of curvature and torsion by the fermions.

To embed the IIB superstring into the super coset space \( \mathcal{M} \), we should pull back the Cartan form down to the world sheet \( \Sigma(\sigma, \tau) \) as

\[ L^A = L^A_M dx^M = L^A_M \partial_i x^M d\sigma^i = L^A_i d\sigma^i \equiv L^A_1 d\tau + L^A_2 d\sigma . \]  

(22)

Then the MC 1-form becomes

\[ G^{-1} \partial_i G = L^A_i P_A = L^a_i P_a + L^{a'}_i P_{a'} + \frac{1}{2} (L^a_{iJ} J_{ab} + L^{a'b'}_{iJ} J_{a'b'}) + L^{\alpha\alpha'}_i Q_{\alpha\alpha'} . \]  

(23)
and e.g. the super Coddazi equations for the vector 5-beins \([20]\) become

\[
\begin{align*}
\epsilon^{ij}(\partial_i L^a_j + L^{ab}_i L^b_j) + i\epsilon^{ij} \bar{L}^I_i \gamma^a L^I_j &= 0 , \\
\epsilon^{ij}(\partial_i L^{a'}_j + L^{a'b'}_i L^{b'}_j) - \epsilon^{ij} \bar{L}^I_i \gamma^{a'} L^I_j &= 0 .
\end{align*}
\]

(24)

(25)

The MC equations for the vierbeins describes the geometric behavior for the embedding of the type IIB string worldsheet into the target space \(AdS_5 \times S^5\).

Now turn to the string dynamics. The \(AdS_5 \otimes S^5\) GS superstring action is given as \(SU(2,2|4) / SO(4,1) \otimes SO(5)\) superspace sigma model \([22,23]\).

\[
I = -\frac{1}{2} \int_{\partial M_3} d^2 \sigma \sqrt{g} g^{ij}(L^a_i L^a_j + L^{a'b'}_i L^{b'}_j) + i \int_{M_3} s^{IJ}(L^a \wedge \bar{L}^I_i \gamma^a \wedge L^J + iL^{a'} \wedge \bar{L}^I_i \gamma^{a'} \wedge L^J).
\]

(26)

This action is invariant with respect to the local \(\kappa\)-transformations in terms of \(\delta x^a = \delta X^M L^a_M, \delta x^{a'} = \delta X^M L^{a'}_M, \delta \theta^I = \delta X^M L^I_M\)

\[
\begin{align*}
\delta \kappa^{\bar{I}} = 2(\bar{L}^I_i \gamma^a - i\bar{L}^I_i \gamma^{a'}) \kappa^i I \\
\delta (\sqrt{g} g^{ij}) = -16i\sqrt{g}(P^{jk}_- L^I_k \kappa^{i1} + P^{jk}_+ L^I_k \kappa^{i2}) .
\end{align*}
\]

(27)

(28)

Here \(P^{ij}_\pm \equiv \frac{1}{2}(g^{ij} \pm \frac{1}{\sqrt{g}} \epsilon^{ij})\) are the projection operators, and 16-component spinor \(\kappa^{iI}\) (the corresponding 32-component spinor has opposite chirality to that of \(\theta\)) satisfy the (anti) self duality constraints

\[
P^{ij}_- \kappa^I_j = \kappa^{i1}, \quad P^{ij}_+ \kappa^I_j = \kappa^{i2},
\]

(29)

which can be written as \(\frac{1}{\sqrt{g}} \epsilon^{ij} \kappa^I_j = -\kappa^{i1}, \frac{1}{\sqrt{g}} \epsilon^{ij} \kappa^I_j = \kappa^{i2}\), i.e. \(\frac{1}{\sqrt{g}} \epsilon^{ij} \kappa^I_j = -S^{IJ} \kappa^{iJ}\).

From the variation of action \([20]\), the equations of motion (EOM) are obtained \([22]\)

\[
\begin{align*}
\sqrt{g} g^{ij}(\nabla_i L^a_j + L^{ab}_i L^b_j) + i\epsilon^{ij} s^{IJ} \bar{L}^I_i \gamma^a L^J_j &= 0 , \\
\sqrt{g} g^{ij}(\nabla_i L^{a'}_j + L^{a'b'}_i L^{b'}_j) - \epsilon^{ij} s^{IJ} \bar{L}^I_i \gamma^{a'} L^J_j &= 0 , \\
(\gamma^a L^a_i + i\gamma^{a'} L^{a'}_i)(\sqrt{g} g^{ij} \delta^{IJ} - \epsilon^{ij} s^{IJ}) L^J_j &= 0 ,
\end{align*}
\]

(30)

(31)

(32)

where \(\nabla_i\) is the \(g_{ij}\)-covariant derivative on the worldsheet \(\Sigma(\sigma, \tau)\).
The Hodge dual and duality between MCE and EOM

The equations of motion (30) and (31) can be rewritten as
\[ g^{ij}(\partial_i(\sqrt{g}L^a_j) + L^a_{ib}L^b_j + i\epsilon^{ij}S^{IJ}L_i^I\gamma^aL_j^J = 0 , \tag{33} \]
\[ g^{ij}(\partial_i(\sqrt{g}L^a_{ij}) + L^a_{ij'}L^b_j - i\epsilon^{ij}S^{IJ}L_i^I\gamma^aL_j^J = 0 , \tag{34} \]
where we have used
\[ \nabla_i L^a_i = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g}L^a_i) . \tag{35} \]

In order to disclose the duality between the MCE and the EOM, we first describes the Hodge dual of bosonic and fermionic forms.

As usual, the Hodge dual of the coordinates of world-sheet is given by
\[ *(dz^i) = -\frac{1}{\sqrt{g}} \epsilon^{ij}dz_j , \quad (dz^1) = d\tau , \quad (dz^2) = d\sigma , \]
\[ \epsilon_{12} = -\epsilon_{21} = \epsilon_{21} = -\epsilon_{12} = 1 . \tag{36} \]

So the even beins simply become
\[ L^i \leftrightarrow *L^i \equiv ( *L^i )^i = -\frac{\epsilon^{ij}}{\sqrt{g}}L^j_i , \quad L^i \leftrightarrow *L^i \equiv \epsilon^{ij}\sqrt{g}L^j_i . \tag{37} \]

As for the odd part, i.e. the spinors in the static \( \kappa \) symmetric Killing gauge [24,25,28], we have
\[ P_{ij}^L J^1 = L^i_j , \quad P_{ij}^L J^2 = L^2_i , \]
\[ P_{ij}^\kappa J^1 = \kappa^i_j , \quad P_{ij}^\kappa J^2 = \kappa^2_j , \quad *\theta^I \equiv S^{IJ}\theta^J , \tag{38} \]

here \( \kappa (x) , \theta (x) \) are ± eigenvector of local \( \Gamma^{11} (x) \) which satisfy Killing equation [29]. This implies that all the local tangent vectors and eigen spinors are covariant in the same local gauge, which is covariantly moving on \( AdS_5 \). And this covariance is enabled by the pure geometrical behavior of \( AdS^5 \otimes S^5 \) in SUGRA [30]. Then the MCE (24) and (25) can be rewritten as
\[ \partial_i(\sqrt{g} *L^a_i) + L^a_{ib}\sqrt{g} *L^b_i - i\epsilon^{ij}L^I_i\gamma^aL^J_j = 0 , \tag{39} \]
\[ \partial_i(\sqrt{g} *L^a_{ij}) + L^a_{ij'}\sqrt{g} *L^b_j + i\epsilon^{ij}L^I_i\gamma^aL^J_j = 0 . \tag{40} \]
Applying the above transformations, it is easy to find that the EOM (30) and (31) are the dual of the 5-bein super Codazzi equations (39) and (40) respectively.

It is clear that the GS string action is invariant under the above dual transformation. There exists no dual between MC eq. (21) and EOM eq. (32), because the \( L^I \) only appears in the Wess-Zumino-Witten term and has no dynamical contribution to the action. Under the dual transformation, the 3rd EOM (32) changes into

\[
(\gamma^a \ast L^a_i + i\gamma^a \ast L^{a'}_i)(\sqrt{g}g^{ij} \delta^{Ij} \epsilon^{ij}s^{IJ}L^I_j) = 0 . \tag{41}
\]

Namely, only the first factor takes the dual form.

For the \( L^{ab} \), it does not change under duality since it is not dynamical and does not appear in the GS string action.

Remark: this duality is the generalization of the usual bosonic Hodge dual between the dynamical first fundamental form (metric 1-form) of the pseudosphere and its geometric 2nd fundamental form. For the pseudo-sphere with negative constant curvature

\[
d\omega_{12} = -k\omega_1 \wedge \omega_2 , \tag{42}
\]

where \( \omega_{12} \) is connection 1-form and the constant curvature \( k = -1/\sqrt{g} \). The metric 1-form is \( ds^2 = d\omega_1^2 + d\omega_2^2 \). Under the same moving frame i.e. the same gauge, one has

\[
\omega^i = \epsilon^{ij} * d\omega_{3j} . \tag{43}
\]

With the help of eq. (43), the eq. (42) changes into

\[
d\omega_{12} = -\omega_{13} \wedge \omega_{23} , \tag{44}
\]

which is the 3rd component of the Maure-Cardan equation. Since the connection form \( \omega_{12} \) which is shared by both sides of duality, will be changed in the same way under same gauge [31, 32]. So while we take the duality for other two components (Coddazi eq.) to interchange the MCE into the EOM, the \( \omega_{12} \) will be the same. The geodesic motion on pseudo-sphere gives dynamical Sine-Gordon equation of the angle between asymptotic lines. The image of the normal line of pseudo-sphere gives the non-linear \( \sigma \) model on the sphere. For the non-linear \( \sigma \) model on \( AdS_5 \), the conformal metric is the Poincaré metric \( \sum_{i=1}^{3} \frac{dx_i^2 + dr^2}{r^2} \) as the higher dimensional generalization for the metric of pseudo-sphere. The dual of (30) and (31) to eq. (39) and (40) can be considered as the generalization of such duality in \( AdS_5 \otimes S^5 \) with the \( \kappa \) symmetric gauge.
4 The twisted dual and integrability

Now we introduce the twisted dual transformation of vierbein as follows. The duality discussed in previous section will be included as a special case of it. On the world sheet $\Sigma(\sigma, \tau)$, it is the re-parametrization transformations along the two directions of the positive and negative light-cone $\tau \pm \sigma$ with the scale factors $\lambda = e^{2\phi}$ and $\lambda^{-1}$ correspondently, the even vierbein forms $L^a$ will be Lorentz rotate by $\pm 2\phi$ oppositely, and the odd vierbein forms $L^I$ will rotate oppositely by $\pm \varphi$ together with $\theta^I$ and $\kappa^I$. All vierbeins are rotate around same axis, the normal line of $AdS_5$ surface in the same gauge especially in Killing gauge.

For the even funf-bein form part, we have

$$L^{\hat{a}}(\lambda) \equiv \exp^{2\phi} P^{ij} L^a_j + \exp^{-2\phi} P^{\hat{i}\hat{j}} L^{\hat{a}}_j = \frac{1}{2}(\lambda + \lambda^{-1}) L^{\hat{a}}_j + \frac{1}{2}(\lambda - \lambda^{-1}) L^{\hat{a}}_j, \quad (45)$$

here $\lambda = \exp^{2\phi}$.

The transformations of odd vierbein are

$$L^{i1}(\lambda) \equiv \lambda^{-\frac{1}{2}} L^1_j = e^{\varphi} P^{ij} L^1_j, \quad (46)$$

$$L^{i2}(\lambda) \equiv \lambda^{\frac{1}{2}} L^2_j = e^{-\varphi} P^{ij} L^2_j, \quad (47)$$

here the Killing gauge are used. Combined (46) and (47), it yields

$$L^{iI}(\lambda) = \exp^{\varphi} P^{i j} L^I_j + \exp^{-\varphi} P^{i j} L^I_j, \quad (48)$$

$$\tilde{L}^{iI}(\lambda) = \exp^{\varphi} P^{i j} \tilde{L}^I_j + \exp^{-\varphi} P^{i j} \tilde{L}^I_j. \quad (49)$$

Notice that the re-parametrization invariance of action implies the loop group symmetry, while the WZW term as a 2-cocycle, further gives the central extension. Here the $GL(1) \otimes GL(1)$ supplies an axial symmetry $A$ of the extended $N = 2$ SUSY, missing in the $PSU(2, 2|4)$ GS string action in Ref. [22], and the $U(1)$ S-duality of $\frac{SL(2, R)}{U(1)} \sim \frac{SU(1, 1)}{U(1)}$ in super-gravity. The dual twist breaks the invariance of the action in conformal gauge, $T^a_a \neq 0$ superficially. But, one can recover it by using the improved stress-energy tensor with contributions from the $U(1) \times U(1)$ fields which correspond to the central and grade derivative operators in affine algebra [26]. This is not the symmetry of MT action, actually it is the hidden symmetry in the moduli space, which is described by the continuous spectral parameter $\lambda$. 

9
Now we can construct the Lax connection $A_i(\lambda)$ with the spectral parameter $\lambda$ as

\[
A_i(\lambda) = H + K(\lambda) + F(\lambda)
\]

\[
= \frac{1}{2} L_i^a J_{a\bar{b}} + L_i^a(\lambda) P_a + L_i^{\alpha\alpha'}(\lambda) Q_{\alpha\alpha'}
\]

\[
= \frac{1}{2} (L_i^{ab} J_{ab} + L_i^{a'b'} J_{a'b'}) + \frac{1}{2} (\lambda + \lambda^{-1})(L_i^a P_a + L_i^{a'} P_{a'})
\]

\[
+ \frac{1}{2} (\lambda - \lambda^{-1}) \left[ * (L^a) P_a + *(L^{a'}) P_{a'} \right]
\]

\[
+ \lambda^{-\frac{1}{2}} L_i^{\alpha\alpha'} Q_{\alpha\alpha'}^{1} + \lambda^{\frac{1}{2}} L_i^{\alpha\alpha'} Q_{\alpha\alpha'}^{2}
\]

\[
(50)
\]

which looks like the original Cartan form with beins replaced by $L(\lambda)$. Such an O(2) transformation, should be defined in the same covariantly shifted moving frame, (the same gauge) with covariant constant $N(x)$. Thus the $H$ including in the covariant derivative will not be twisted. Obviously if $\lambda = 1, \phi = 0$, it is the original Cartan form \[10\]. On the "wick rotated" worldsheet we may take

\[
\lambda = \exp^{2\phi} = i.
\]

Then

\[
L^{i\bar{a}} = i \epsilon^{ij} L_{j}^{\bar{a}},
\]

\[
(52)
\]

Similarly

\[
L^{\bar{i}} = (-i)^{\frac{1}{2}} L^{\bar{i}}, \quad \bar{L}^{\bar{i}} = i^{\frac{1}{2}} L^{\bar{i}},
\]

\[
(53)
\]

\[
L^{i\bar{i}} = (-i)^{\frac{1}{2}} \bar{L}^{\bar{i}}, \quad \bar{L}^{i\bar{i}} = i^{\frac{1}{2}} \bar{L}^{\bar{i}},
\]

\[
(54)
\]

\[
i.e. L^{\bar{a}} = *L^{\bar{a}} \text{ and } L^{J} = S^{IJ} *L^{J},
\]

\[
(55)
\]

here i appears, from the difference of sign of Hodge star in $M_2$ and in $E_2$. Thus the vierbein $L(i)$ becomes simply the Hodge dual of original vierbein on Euclidean world sheet. And it implies the dual symmetry of the MCE and the EOM. It is obvious that the Lax connections $A_i(\lambda)$ satisfy the zero curvature (flat connection) condition:

\[
\partial_i A_j(\lambda) - \partial_j A_i(\lambda) + [A_i(\lambda), A_j(\lambda)] = 0,
\]

\[
(56)
\]

as the linear combination of MCE and EOM i.e. the system is integrable, and we may introduce the transfer matrices $U(\lambda, \sigma)$

\[
\partial_i U(\lambda, \sigma) = A_i(\lambda, \sigma) U(\lambda, \sigma).
\]

\[
(57)
\]
5 Infinite set of conserved currents

The EOM \((30,31)\) is the conservation law of the Noether currents with respect to the local transitive operator \(T^a\). In the KPR gauge, let the normal line \(N(x), N^2(x) = 1\) (the normalized \(Y(x)\) in Ref. [27]) be \(e_6\). These generators \(T^a \sim T^{6a} \in SO(4,2)\) in the right “body axis” becomes the translation generated by Killing vectors along the same direction, here \(e_a (a = 1 \cdots 5)\) are the tangent vectors on \(AdS_5\).

The existence of transfer matrix \(U(\lambda, x)\) implies that Noether currents generated by \(U^{-1}(\lambda, x)T^a U(\lambda, x)\) is conserved. We omit the detail of the proof. It’s essentially the generalization of Ref. [31, 37, 38]. Here besides

\[
D_i (\lambda) U (\lambda) = 0, \tag{58}
\]

\[
D_i (\lambda) = \partial_i + A_i (\lambda), \tag{59}
\]

the covariant constancy of

\[
N (\lambda; x) = U^{-1}(\lambda) N (1; x) U (\lambda), \tag{60}
\]

\[
D_i (\lambda) N (\lambda; x) = 0, \tag{61}
\]

here \(D_i (\lambda) = (\partial_i + [A_i (\lambda), .])\) in adjoint representation is used.

Notice: Three different covariant derivatives

1. covariant derivative \(\nabla_i\) on the world sheet (eq.\((30,32)\))

2. gauge covariant exterior derivative \(d + H \Lambda\) on \(S^5\) and (or) on \(AdS_5\) surface.

3. covariant derivative \(D_i \) \((59)\) \((61)\) in the flat “zero curvature” 6-dim space.

6 Outlook

Soon later, after the Yale preprint [37] Ueno [39] pointed out that \(U^{-1}TU, T \in g\) is the infinitesimal dressing transformation. It is the generator of dressing transformation [40] which is related with finite Riemann Hilbret problem \(U^{-1}gU, g \in G\). The Riemann Hilbret problem with poles and zeros will determine the left and right part of Leznov, Sovileevs soliton solution. So Riemann Hilbret factorization gives the holomorphic and antiholomorphic of left and right moving part of strings and branes.
They combine into amplitude as that in CFT how correlation function is obtained from conformal block. Furthermore, the **Poisson Lie structure** [21] of this affine dressing group is given. The soliton generation operator will be constructed by exponentiate the $r$ matrix of the classical double, it is just the $\pm$ frequency part of the vertex operator of principle realization of affine Kac Moody algebra [41].

We will find classical solitonic solution of IIB string and brane. The **loop parameter** describes their moduli space. The loop parameter $\lambda$ also characterized the constant change of phase or length of the complexified $O(2)$ rotation around the normal line $N$, which is the well know covariant constant field in SUGRA. This $O(2)$ is the S duality $U(1)$ in $SU(1,1)$. This can be generalize into the maximally symmetry $AdS_7 \times S^4$, $AdS_4 \times S^7$ of M theory, with pure geometrical behavior. And then transform to other dimensions by T duality.

The Hodge twist transform is the case of unequal fermion parameter $\alpha \neq \bar{\alpha}$ in Witten’s topological nonlinear $\sigma$ model, thus $\lambda$ gives different moduli of IIA, IIB [42]. At last $\lambda$ will describe the affinization of the R symmetry and dilation of the dimension.

Studying the **Poisson structure**, Dolan find the so called Kac Moody alg. of principle chiral model. She points out that it is the loop parameter expansion of the Poisson structure of the transformation in Ref. [31]. In fact, it’s the fundamental Poisson bracket given by Faddeev

$$\{U(\lambda), \otimes U(\mu)\} = [r(\lambda - \mu), U(\lambda) \otimes U(\mu)], \quad (62)$$

and the Poisson Lie bracket

$$\{L(\varphi), \otimes L(\psi)\} = [r(\varphi - \psi), L(\varphi) \otimes 1 + 1 \otimes L(\psi)], \quad (63)$$

here $L$ is the $A_0$ component in (57).

In fact, the $r$-matrix may be derived from the symplectic form given by the action with WZW term [43]. It is just the classical algebra. The quantum version of (63) is given by Drinfeld, Jimbo, Faddeev, Reshetikin and Takhtajan [34–36],

$$R_{12}(\phi - \psi)L_1(\phi)L_2(\psi) = L_2(\psi)L_1(\phi)R_{12}(\phi - \psi). \quad (64)$$

In a further paper [hep-th/0406250] we will give the gauged form of Roiban-Siegel action [27], corresponding to the embedding of IIB string in $AdS_5 \times S^5$. Its bosonic part is the the gauged WZW action for affine Toda theory [33]. Thus, the the dressing transformation gives all the soliton solution, i.e. the moduli space of the string background
states. After the quantization, it will become quantum $R$ and $L$ matrices [44] for the quantum Affine Toda. The scattering amplitude can also be obtained as Ref. [45]. The correlation function of vertex operators is given by the double scaling limit of the quantum Virasoso and W algebra [46], which is equivalent to the q-deformation of Yangian double with centre [47] at the critical level [48]. Thus the dressing symmetry revealed in this paper discloses that both the classical and quantum integrable structure of IIB GS string on $AdS_5 \times S^5$ is given by the Affine Toda system. e.g. why the Seiberg-Witten curve is given by the spectral determinant of affine Toda.

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