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Intertwining operator for $AG_2$ Calogero–Moser–Sutherland system

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Abstract

We consider generalised Calogero–Moser–Sutherland quantum Hamiltonian $H$ associated with a configuration of vectors $AG_2$ on the plane which is a union of $A_2$ and $G_2$ root systems. The Hamiltonian $H$ depends on one parameter. We find an intertwining operator between $H$ and the Calogero–Moser–Sutherland Hamiltonian for the root system $G_2$. This gives a quantum integral for $H$ of order 6 in an explicit form thus establishing integrability of $H$.

1 Introduction

The study of Calogero–Moser–Sutherland (CMS) integrable systems goes back to the works [1]–[3]. Olshanetsky and Perelomov introduced generalised CMS systems related to root systems of Weyl groups [4], which includes the non-reduced root system $BC_n$. The corresponding Hamiltonians are closely related to radial parts of Laplace-Beltrami operators on symmetric spaces [5], [6]. In the case of root system $G_2$ the rational version of the corresponding CMS system was considered earlier by Wolfes [7]. A uniform proof of integrability for all root systems via trigonometric version of Dunkl operators was given by Heckman in [8]. Another more involved proof was provided earlier by Opdam in [9]. In the case of integer values of coupling parameters these CMS systems admit additional quantum integrals and they are algebraically integrable as it was established by Chalykh, Styrkas and Veselov in [10] (see also [11]).

It was found by Chalykh, Veselov and one of the authors in [12], [13] that there are integrable generalisations of CMS type quantum systems which correspond to special configurations of vectors generalising root systems. Examples of such configurations include deformations of the root systems $A_n$ and $C_n$. These examples are related to symmetric superspaces, [14]–[17], and to special representations of Cherednik algebras [18]. The corresponding configurations of vectors have to satisfy so-called locus conditions [19]. It is expected that there are very few such configurations, but they are not classified yet. We refer to [20] for a survey of results on locus configurations and integrability of rational, trigonometric and elliptic generalised CMS systems (see also [21] and reference therein for the elliptic case).

The work [22] of Fairley and one of the authors deals with a class of trigonometric locus configurations on the plane. In the process of classification of such configurations a new locus configuration $AG_2$ was found in [22] (see also [23] where this configuration appears as well in different but related context of WDVV equations). This configuration of vectors with multiplicities depends on one integer parameter $m$. Being a locus configuration, it follows from the general results of Chalykh [20] that the corresponding generalised CMS operator $H$ has an intertwining
relation with the Laplacian. This implies integrability and, moreover, algebraic integrability of the Hamiltonian $H$ with $m \in \mathbb{Z}$. The latter means existence of some additional quantum integrals; see [11], [10] for more details on algebraic integrability including precise definition and examples of intertwining operators.

Let us describe the configuration of vectors $AG_2$ in detail, following [22]. This is a non-reduced configuration consisting of the union of root systems $G_2$ and $A_2$. A positive half of this configuration is shown on Figure 1.

The short vectors from the root system $G_2$ are denoted as $\pm \beta_1, \pm \beta_2, \pm \beta_3$, and the long vectors from the root system $G_2$ are denoted as $\pm \alpha_1, \pm \alpha_2, \pm \alpha_3$. Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean inner product on the plane. Then the ratio $\frac{\langle \alpha_i, \alpha_i \rangle}{\langle \beta_i, \beta_i \rangle} = 3$ for any $i$. The vectors from the additional root system $A_2$ are $\pm 2\beta_1, \pm 2\beta_2, \pm 2\beta_3$. Configuration $AG_2$ consists of 18 vectors $\pm \beta_i, \pm 2\beta_i, \pm \alpha_i, i = 1, 2, 3$. Note that the numbering of $\alpha$’s is chosen in such a way that $\langle \alpha_i, \beta_i \rangle = 0$ for all $i = 1, 2, 3$.

Note that
\[ 2\beta_2 - \alpha_1 = \alpha_1 - 2\beta_3 = \beta_1, \quad 2\beta_3 - \alpha_2 = \alpha_2 + 2\beta_1 = \beta_2, \quad 2\beta_2 - \alpha_3 = \alpha_3 - 2\beta_1 = \beta_3, \]
and
\[ \alpha_3 - \beta_2 = \beta_3 - \alpha_2 = \beta_2 - \beta_3 = \beta_1, \text{ and } \alpha_1 - \beta_2 = \beta_3. \]

The configuration $AG_2$ is invariant under the $G_2$ Weyl group action and it belongs to a two-dimensional lattice spanned by $\beta_1$ and $\alpha_2$. However, it is not a root system. In order to see this let us consider, for instance, vectors $\beta_1, \beta_2$ which are symmetric about the line orthogonal to vector $2\beta_3$. We have $\beta_2 - \beta_1 = k \cdot 2\beta_3$, where $k = \frac{1}{2}$ is not an integer so the crystallographic condition fails.

Let $H_0$ be the CMS Hamiltonian for the root system $G_2$ with multiplicities for the long and short roots $m$ and $3m$, respectively. And let $H$ be the Hamiltonian of the generalised CMS system under study, associated to the above collection of vectors $AG_2$ where $\alpha_i, \beta_i$ and $2\beta_i$ have multiplicities $m, 3m$ and $1$, respectively, where we use conventions for non-reduced systems coming from theory of symmetric spaces (see e.g. [16]). More precisely,

\[
H_0 = -\Delta + \sum_{i=1}^{3} \left( v_i(x) + u_i(x) \right),
\]
\[
H = -\Delta + \sum_{i=1}^{3} \left( v_i(x) + \tilde{u}_i(x) \right),
\]

(1.1)
where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian, and for $x = (x_1, x_2) \in \mathbb{C}^2$ we have

$$v_i(x) = \frac{m(m+1)\langle \alpha_i, \alpha_i \rangle}{\sinh^2\langle \alpha_i, x \rangle},$$

$$u_i(x) = \frac{3m(3m+1)\langle \beta_i, \beta_i \rangle}{\sinh^2\langle \beta_i, x \rangle},$$

and

$$\tilde{u}_i(x) = \frac{9m(m+1)\langle \beta_i, \beta_i \rangle}{\sinh^2\langle \beta_i, x \rangle} + \frac{2(2\beta_i, 2\beta_i)}{\sinh^2(2\beta_i, x)}$$

$$= \frac{(3m+1)(3m+2)\langle \beta_i, \beta_i \rangle}{\sinh^2\langle \beta_i, x \rangle} - \frac{2\langle \beta_i, \beta_i \rangle}{\cosh^2\langle \beta_i, x \rangle}.$$

In addition, we introduce the following notation for the difference $\hat{u}_i(x) - u_i(x)$:

$$\hat{u}_i(x) := \tilde{u}_i(x) - u_i(x) = \frac{2(3m+1)\langle \beta_i, \beta_i \rangle}{\sinh^2\langle \beta_i, x \rangle} - \frac{2\langle \beta_i, \beta_i \rangle}{\cosh^2\langle \beta_i, x \rangle}. \quad (1.3)$$

Let $\partial_i$ denote the partial derivative $\frac{\partial}{\partial x_i}$. For any vector (or a vector field) $\gamma = (\gamma^{(1)}, \gamma^{(2)}) \in \mathbb{C}^2$, we will write $\partial_i$ for the directional derivative operator $\gamma^{(1)}\partial_1 + \gamma^{(2)}\partial_2$. In particular, if $\phi$ is a scalar field on the plane and $\nabla(\phi) = (\partial_1(\phi), \partial_2(\phi))$ is its gradient, then by $\partial_\nabla(\phi)$ we will mean $\partial_1(\phi)\partial_1 + \partial_2(\phi)\partial_2$.

In this paper we establish an intertwining relation between the Hamiltonian $H$ and the integrable Hamiltonian $H_0$ of the CMS system associated with the root system $G_2$. This relation is valid for any value of the parameter $m$ which is allowed to be non-integer. This leads to integrability of $H$ for any $m$ thus generalizing integrability for integer $m$ known from [22], [20]. We also find the intertwining operator $\mathcal{D}$ of order 3 in an explicit form. This, in turn, gives quantum integral of $H$ of order 6. We note that direct application of results of [20] in the case of integer $m$ leads to a higher order intertwiner and a higher order integral of $H$. The degree 6 for the integral of $H$ is expected to be minimal possible. Indeed, it follows from [24] that for generic $m$ an independent integral for the rational version of $H$ with constant highest term has to be of degree at least 6 since such highest term should be $G_2$-invariant.

The intertwining operator $\mathcal{D}$ has the form

$$\mathcal{D} = \partial_{\beta_1}\partial_{\beta_2}\partial_{\beta_3} + \sum_{\sigma \in A_3} f_{\sigma(1)}\partial_{\beta_{\sigma(2)}}\partial_{\beta_{\sigma(3)}} + \sum_{i=1}^3 g_i\partial_{\beta_i} + h, \quad (1.4)$$

where $A_3 = \{id, (1,2,3), (1,3,2)\}$ is the alternating group on 3 elements, and $f_i, g_i$ ($i = 1, 2, 3$) and $h$ are some functions which we specify explicitly. We will use the notation $\sum_{\sigma}$ throughout the paper as a shorthand for the cyclic sum $\sum_{\sigma \in A_3}$. To be more precise, we will prove the following main theorem.

**Theorem 1.** There exists a third-order differential operator $\mathcal{D}$ of the form (1.4) such that

$$HD = \mathcal{D}H_0. \quad (1.5)$$

We obtain quantum integrability of $H$ and a quantum integral of motion as a direct corollary by making use of a general statement from [25]. Let us recall the notion of the formal adjoint differential operator $D^*$ for a differential operator $D$. It can be defined by the relations $\partial_i^* = -\partial_i$, $f^* = f$ for any function $f$, and $(AB)^* = B^*A^*$ for any differential operators $A, B$. 

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Theorem 2. Let $\mathcal{D}$ satisfy (1.5) and let $\mathcal{D}^*$ be the formal adjoint of $\mathcal{D}$. Let $I$ be any differential operator such that the commutator $[I,H_0] = 0$. Then $\mathcal{D}I\mathcal{D}^*$ commutes with $H$. In particular, $[\mathcal{D}^*H,H] = 0$.

Indeed, taking the formal adjoint of the relation (1.5) gives $\mathcal{D}^*H = H_0\mathcal{D}^*$. Hence

$$HD\mathcal{D}^* = \mathcal{D}H_0\mathcal{D}^* = D IH_0\mathcal{D}^* = \mathcal{D}^*H.$$  

Note that for integer $m$ the operator $H_0$ is algebraically integrable as the commutative ring of quantum integrals is larger than the ring of $G_2$-invariants [11], [10]. Therefore this gives a way to see algebraic integrability of the operator $H$ for integer $m$ (see also [20]). We also note that in the rational limit the operator $\mathcal{D}\mathcal{D}^*$ reduces to a quantum integral for the rational CMS system associated with the root system $G_2$ with multiplicities $m$ and $3m+1$ for the long and short roots, respectively.

The structure of the paper is as follows. We collect some preliminary trigonometric identities associated with vectors from the configuration $AG_2$ in Section 2. We introduce all the coefficients of the intertwining operator (1.4) in Section 3, where we also establish some preliminary results on these coefficients. In Section 4 we prove the main Theorem 1 on the intertwining relation. We present results on the rational limit in Section 5. We outline some future directions in Section 6.

2 Preliminary trigonometric identities

In this section we collect some trigonometric identities involving vectors from the configuration $AG_2$. We will use these identities later in the proof of the intertwining relation (1.5) in Section 4.

One can choose a coordinate system where the vectors take the form $\beta_1 = \omega(\sqrt{2},0)$, $\beta_2 = \omega(\frac{\sqrt{2}}{2},\frac{3\sqrt{2}}{2})$, $\beta_3 = \omega(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $\alpha_1 = \omega(0, \sqrt{6})$, $\alpha_2 = \omega(-\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $\alpha_3 = \omega(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ for some non-zero $\omega \in \mathbb{C}$. We will use inner products between the vectors but not the specific coordinates of the vectors.

Remark 1. In most cases we state only one particular form of each identity, but other variants can be obtained by rotating or scaling the vectors. More precisely, the relevant transformations will be the replacement of $\beta_i$ with $2\beta_i$, and the two rotations by $\frac{\pi}{2}$, clockwise and anti-clockwise. These rotations can alternatively be defined by the following replacement rules: $\beta_1 \rightarrow -\beta_3$, $\beta_2 \rightarrow \beta_1$, $\beta_3 \rightarrow \beta_2$, $\alpha_1 \rightarrow \alpha_3$, $\alpha_2 \rightarrow \alpha_1$, $\alpha_3 \rightarrow -\alpha_2$, and $\beta_1 \rightarrow \beta_2$, $\beta_2 \rightarrow \beta_3$, $\beta_3 \rightarrow -\beta_1$, $\alpha_1 \rightarrow \alpha_2$, $\alpha_2 \rightarrow -\alpha_3$, $\alpha_3 \rightarrow \alpha_1$, respectively.

The vectors $\alpha_i$ and $\beta_i$ in the configuration $AG_2$ satisfy the following trigonometric identities, where we omit the argument $x = (x_1, x_2)$. Thus we write $\coth \beta_i$ for $\coth(\beta_i, x)$, etc.

Lemma 2.1. We have

$$\sum_{1 \leq j < k \leq 3} \langle \beta_j, \beta_k \rangle \coth \beta_j \coth \beta_k = \omega^2. \quad (2.1)$$

Proof. By a difference of cotangents formula and the fact that $\beta_2 - \beta_3 = \beta_1$, two terms in the sum become

$$\omega^2 \coth \beta_1 (\coth \beta_2 - \coth \beta_3) = -\frac{\omega^2 \cosh \beta_1}{\sinh \beta_2 \sinh \beta_3}. \quad (2.2)$$

By expanding $\cosh \beta_1$ in terms of $\beta_2$ and $\beta_3$, we can rearrange the right-hand side of (2.2) further as

$$-\frac{\omega^2 \cosh \beta_2 \cosh \beta_3 - \omega^2 \sinh \beta_2 \sinh \beta_3}{\sinh \beta_2 \sinh \beta_3} = -\omega^2 \coth \beta_2 \coth \beta_3 + \omega^2,$$

as required. ∎
It will be convenient to use the following notation throughout the paper:

\[ X = \omega^2 (\sinh \beta_1 \sinh \beta_2 \sinh \beta_3)^{-1} , \quad (2.3) \]
\[ Y = \omega^2 (\sinh 2\beta_1 \sinh 2\beta_2 \sinh 2\beta_3)^{-1} . \quad (2.4) \]

Here are some identities involving these functions.

**Lemma 2.2.** We have

\[ \sum_{\sigma} \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \coth \beta_{\sigma(1)}}{\sinh^2 \beta_{\sigma(2)} \sinh^2 \beta_{\sigma(3)}} = -2X . \quad (2.5) \]

**Proof.** Multiplying equality (2.1) by \( 2\omega^{-2} \) and regrouping terms as in the proof of Lemma 2.1, we get

\[ 2 = \coth \beta_1 (\coth \beta_2 - \coth \beta_3) + \coth \beta_2 (\coth \beta_1 + \coth \beta_3) + \coth \beta_3 (\coth \beta_2 - \coth \beta_1) \]
\[ = -\frac{\cosh \beta_1}{\sinh \beta_2 \sinh \beta_3} + \frac{\cosh \beta_2}{\sinh \beta_1 \sinh \beta_3} - \frac{\cosh \beta_3}{\sinh \beta_1 \sinh \beta_2} . \quad (2.6) \]

The statement follows by dividing (2.6) by \( \sinh \beta_1 \sinh \beta_2 \sinh \beta_3 \). \( \square \)

There is also the following version of Lemma 2.2 involving the function \( \tanh \) rather than \( \coth \).

**Lemma 2.3.** We have

\[ -\frac{1}{2} \sum_{\sigma} \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \tanh \beta_{\sigma(1)}}{\sinh^2 \beta_{\sigma(2)} \sinh^2 \beta_{\sigma(3)}} = X + 4Y . \quad (2.7) \]

**Proof.** Note the relation \( \tanh z = \coth z - (\sinh z \cosh z)^{-1} \) valid for all \( z \in \mathbb{C} \). Hence by Lemma 2.2 we get

\[ \sum_{\sigma} \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \tanh \beta_{\sigma(1)}}{\sinh^2 \beta_{\sigma(2)} \sinh^2 \beta_{\sigma(3)}} = \sum_{\sigma} \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \coth \beta_{\sigma(1)}}{\sinh^2 \beta_{\sigma(2)} \sinh^2 \beta_{\sigma(3)}} - \sum_{\sigma} \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle}{\sinh \beta_{\sigma(1)} \cosh \beta_{\sigma(1)} \sinh^2 \beta_{\sigma(2)} \sinh^2 \beta_{\sigma(3)}} \]
\[ = -2X - \sum_{1 \leq j < k \leq 3} \frac{\langle \beta_j, \beta_k \rangle \coth \beta_j \coth \beta_k}{\sinh \beta_1 \cosh \beta_1 \sinh \beta_2 \cosh \beta_2 \sinh \beta_3 \cosh \beta_3} . \]

The result follows by applying Lemma 2.1. \( \square \)

**Lemma 2.4.** We have

\[ -X \sum_{i=1}^{3} \coth \beta_i \partial_{\beta_i} = \sum_{\sigma} \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle}{\sinh^2 \beta_{\sigma(2)} \sinh^2 \beta_{\sigma(3)}} \partial_{\beta_{\sigma(1)}} . \quad (2.8) \]

**Proof.** Let us replace \( \partial_{\beta_1} = \partial_{\beta_2} - \partial_{\beta_3} \) in (2.8). Then the coefficient of \( \partial_{\beta_2} \) in the left-hand side equals

\[ -X (\coth \beta_1 + \coth \beta_2) = -\frac{\omega^2 \sinh(\beta_1 + \beta_2)}{\sinh^2 \beta_1 \sinh^2 \beta_2 \sinh \beta_3} . \]

Then on the right-hand side of equality (2.8) the coefficient at \( \partial_{\beta_2} \) is

\[ \frac{\omega^2}{\sinh^2 \beta_2 \sinh^2 \beta_3} - \frac{\omega^2}{\sinh^2 \beta_1 \sinh^2 \beta_3} = -\frac{\omega^2 (\sinh^2 \beta_2 - \sinh^2 \beta_1)}{\sinh^2 \beta_1 \sinh^2 \beta_2 \sinh^2 \beta_3} = -\frac{\omega^2 \sinh(\beta_1 + \beta_2)}{\sinh^2 \beta_1 \sinh^2 \beta_2 \sinh \beta_3} \]
as \( \sinh^2 \beta_2 - \sinh^2 \beta_1 = \sinh(\beta_1 + \beta_2) \sinh(\beta_2 - \beta_1) \). Similarly, the coefficient at \( \partial_{\beta_3} \) matches too. \( \square \)
As an immediate corollary of Lemma 2.4 we get the following statement.

**Corollary 2.5.** We have

\[
\partial \nabla (X) = \sum_\sigma \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle}{\sinh^2 \beta_{\sigma(2)} \sinh^2 \beta_{\sigma(3)}} \partial \beta_{\sigma(1)},
\]

and

\[
\partial \nabla (Y) = \sum_\sigma \frac{2 \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle}{\sinh^2 2 \beta_{\sigma(2)} \sinh^2 2 \beta_{\sigma(3)}} \partial \beta_{\sigma(1)}.
\]

The following lemma can be proven by a straightforward calculation with the help of Lemma 2.1.

**Lemma 2.6.** The following two equalities hold:

\[
\Delta(X) = 4\omega^2 \left( 2 + \sum_{j=1}^3 \frac{1}{\sinh^2 \beta_j} \right) X,
\]

(2.9)

\[
\Delta(Y) = 16\omega^2 \left( 2 + \sum_{j=1}^3 \frac{1}{\sinh^2 2 \beta_j} \right) Y.
\]

(2.10)

**Lemma 2.7.** The following identities hold:

\[
-\left( \frac{1}{\cosh^2 \beta_2} + \frac{1}{\cosh^2 \beta_3} \right) \frac{1}{\sinh^2 \alpha_1} + 2(\tanh \beta_2 + \tanh \beta_3) \frac{\coth \alpha_1}{\sinh^2 \alpha_1} = \frac{1}{\cosh^2 \beta_2 \cosh^2 \beta_3},
\]

(2.11)

\[
\left( \frac{1}{\sinh^2 \beta_2} + \frac{1}{\sinh^2 \beta_3} \right) \frac{1}{\sinh^2 \alpha_1} + 2(\coth \beta_2 + \coth \beta_3) \frac{\coth \alpha_1}{\sinh^2 \alpha_1} = \frac{1}{\sinh^2 \beta_2 \sinh^2 \beta_3},
\]

(2.12)

\[
-\left( \frac{1}{\cosh^2 \beta_2} + \frac{1}{\cosh^2 \beta_3} \right) \frac{1}{\sinh^2 \beta_1} + 2(\tanh \beta_2 - \tanh \beta_3) \frac{\coth \beta_1}{\sinh^2 \beta_1} = \frac{1}{\cosh^2 \beta_2 \cosh^2 \beta_3},
\]

(2.13)

\[
\left( \frac{1}{\sinh^2 \beta_2} + \frac{1}{\sinh^2 \beta_3} \right) \frac{1}{\sinh^2 \beta_1} + 2(\coth \beta_2 - \coth \beta_3) \frac{\coth \beta_1}{\sinh^2 \beta_1} = \frac{1}{\sinh^2 \beta_2 \sinh^2 \beta_3}.
\]

(2.14)

**Proof.** Since \(\beta_2 + \beta_3 = \alpha_1\), we have

\[
\tanh \beta_2 + \tanh \beta_3 = \tanh \alpha_1 (1 + \tanh \beta_2 \tanh \beta_3).
\]

Therefore the left-hand side of the relation (2.11) multiplied by \(\sinh^2 \alpha_1\) takes the form

\[
- \frac{1}{\cosh^2 \beta_2} - \frac{1}{\cosh^2 \beta_3} + 2(1 + \tanh \beta_2 \tanh \beta_3) = (\tanh \beta_2 + \tanh \beta_3)^2 = \frac{\sinh^2 (\beta_2 + \beta_3)}{\cosh^2 \beta_2 \cosh^2 \beta_3},
\]

which implies the relation (2.11). The equalities (2.12) – (2.14) can be proved by following a similar sequence of steps, using in addition that \(\coth x + \coth y = \tanh(x + y)(1 + \coth x \coth y)\) for \(x, y \in \mathbb{C}\).

Several other identities can be derived from Lemma 2.7, which we put in Lemmas 2.8 and 2.9 below.
Lemma 2.8. The following relation is satisfied:

\[
\left( \frac{\coth \beta_3 - \coth \beta_2}{\sinh^2 \beta_3 - \sinh^2 \beta_2} \right) \frac{1}{\sinh^2 \alpha_1} + \left( \frac{1}{\sinh^2 \beta_3 - \sinh^2 \beta_2} \right) \frac{1}{
\frac{\sinh^2 \beta_2}{\sinh^2 \beta_3} - \frac{1}{\sinh^2 \beta_2} \right) \frac{1}{\sinh^2 \alpha_1} = \frac{\coth \beta_3 - \coth \beta_2}{\sinh^2 \beta_2 \sinh^2 \beta_3}.
\]  

(2.15)

Proof. By multiplying the relation (2.12) by \( \coth \beta_3 - \coth \beta_2 \), and then using \( \coth \beta_2 - \coth \beta_2 = \sinh^{-2} \beta_3 - \sinh^{-2} \beta_2 \), we get

\[
\left( \frac{\coth \beta_3 - \coth \beta_2}{\sinh^2 \beta_2} \right) + \frac{\coth \beta_3 - \coth \beta_2}{\sinh^2 \beta_3} \frac{1}{\sinh^2 \alpha_1} + 2 \left( \frac{1}{\sinh^2 \beta_3 - \sinh^2 \beta_2} \right) \frac{1}{\sinh^2 \alpha_1} = \frac{\coth \beta_3 - \coth \beta_2}{\sinh^2 \beta_2 \sinh^2 \beta_3}.
\]

(2.16)

Comparing relations (2.16) and (2.15), it remains to prove that

\[
\left( \frac{\coth \beta_3 - \coth \beta_2}{\sinh^2 \beta_3} \right) - \frac{1}{\sinh^2 \alpha_1} = \frac{1}{\sinh^2 \beta_3 - \sinh^2 \beta_2} \frac{1}{\sinh^2 \alpha_1} = 0.
\]

(2.17)

Note that since \( \alpha_1 = \beta_2 + \beta_3 \) we get

\[
\frac{\coth \alpha_1 - \coth \beta_2}{\sinh^2 \beta_3} = \frac{\coth \alpha_1 - \coth \beta_3}{\sinh^2 \beta_2} = \frac{\sinh \beta_3}{\sinh^2 \beta_3 \sinh \alpha_1 \sinh \beta_2} + \frac{\sinh \beta_2}{\sinh \beta_3 \sinh \alpha_1 \sinh^2 \beta_2} = 0,
\]

which implies that relation (2.17) holds as required.

Lemma 2.9. The following identity holds:

\[
\sum_\sigma \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \left( \frac{1}{\sinh^2 \beta_{\sigma(2)}} + \frac{1}{\sinh^2 \beta_{\sigma(3)}} \right) \frac{\coth \beta_{\sigma(1)}}{\sinh^2 \beta_{\sigma(1)}} = 2 \left( 2 + \sum_{j=1}^3 \frac{1}{\sinh^2 \beta_j} \right) X.
\]

(2.18)

Proof. Let us multiply the identity (2.14) in Lemma 2.7 by \( \coth \beta_1 \). It follows that

\[
\left( \frac{1}{\sinh^2 \beta_2} + \frac{1}{\sinh^2 \beta_3} \right) \coth \beta_1 = \frac{\coth \beta_1}{\sinh^2 \beta_2 \sinh^2 \beta_3} - 2(\coth \beta_2 - \coth \beta_3) \frac{\coth \beta_1}{\sinh^2 \beta_1} = \frac{-\coth \beta_1}{\sinh^2 \beta_2 \sinh^2 \beta_3} + \frac{2 \coth \beta_1}{\sinh \beta_1 \sinh \beta_2 \sinh \beta_3} = \frac{\coth \beta_1}{\sinh^2 \beta_2 \sinh^2 \beta_3} + 2 e^{-2} \left( 1 + \frac{1}{\sinh^2 \beta_1} \right) X.
\]

(2.19)

We obtain two more variants of the relation (2.19) by applying \( \pm \frac{\pi}{2} \) rotations and interchanging the \( \beta \)'s accordingly (see Remark 1). By adding together the resulting three equalities, we get, with use of Lemma 2.2, that the left-hand side of the identity (2.18) equals

\[
\sum_\sigma \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \coth \beta_{\sigma(1)} = 2 \left( 3 + \sum_{j=1}^3 \frac{1}{\sinh^2 \beta_j} \right) X = 4X + \left( \sum_{j=1}^3 \frac{2}{\sinh^2 \beta_j} \right) X.
\]

\]

Lemma 2.10. The following identity holds:

\[
\sum_\sigma \langle \alpha_{\sigma(2)}, \alpha_{\sigma(3)} \rangle \left( \frac{1}{\sinh^2 \alpha_{\sigma(2)}} - \frac{1}{\sinh^2 \alpha_{\sigma(3)}} \right) \frac{\coth \alpha_{\sigma(1)}}{\sinh^2 \alpha_{\sigma(1)}} = 0.
\]

(2.20)

Proof. Let us multiply both sides of (2.20) by \( -\frac{1}{2} \sinh^2 \alpha_1 \sinh^2 \alpha_2 \sinh^2 \alpha_3 \). We need to prove that
Lemma 3.1. For any single-variable function $F$ and vectors $\alpha, \beta, \gamma$ such that $\langle \gamma, \gamma \rangle \neq 0$ we have

$$\partial_\alpha \partial_\beta (F((\gamma, x))) = \frac{\langle \gamma, \alpha \rangle \langle \gamma, \beta \rangle}{\langle \gamma, \gamma \rangle} \Delta (F((\gamma, x))).$$

Proof. By the chain rule of differentiation, $\Delta (F((\gamma, x))) = \langle \gamma, \gamma \rangle F''((\gamma, x))$, while $\partial_\alpha \partial_\beta (F((\gamma, x))) = \langle \gamma, \alpha \rangle \langle \gamma, \beta \rangle F''((\gamma, x))$, where $F''$ denotes the second derivative of the function $F$. 

In the expression (1.4) for the operator $D$, let

$$f_j = -(3m + 1) \langle \beta_j, \beta_j \rangle \coth \beta_j - \langle \beta_j, \beta_j \rangle \tanh \beta_j,$$

where $j = 1, 2, 3$.

In the next lemma we calculate the gradient and Laplacian of the functions $f_j$.

Lemma 3.2. The functions $f_j$ defined by expression (3.1), $j = 1, 2, 3$, satisfy the following relations:

1. $\nabla (f_j) = \frac{1}{2} \hat{u}_j \beta_j$, (or, equivalently, $\partial_\nabla (f_j) = \frac{1}{2} \hat{u}_j \partial_\beta_j$),

2. $-\Delta (f_j) + \hat{u}_j f_j = \partial_\beta_j (u_j)$.

Proof. Part (1) follows from the equality

$$\partial_i (f_j) = \left( \frac{(3m + 1) \langle \beta_j, \beta_j \rangle}{\sinh^2 \beta_j} - \frac{\langle \beta_j, \beta_j \rangle}{\cosh^2 \beta_j} \right) \beta_j^{(i)} = \frac{1}{2} \hat{u}_j \beta_j^{(i)},$$

$i = 1, 2$, where $\beta_j = (\beta_j^{(1)}, \beta_j^{(2)})$ and we used the definition (1.3).

To establish property (2) we note that

$$\Delta (f_j) = - \frac{2(3m + 1) \langle \beta_j, \beta_j \rangle^2 \coth \beta_j}{\sinh^2 \beta_j} + \frac{2 \langle \beta_j, \beta_j \rangle^2 \tanh \beta_j}{\cosh^2 \beta_j}.$$ 

Expanding and simplifying the product $\hat{u}_j f_j$ yields

$$\hat{u}_j f_j = - \frac{2(3m + 1)^2 \langle \beta_j, \beta_j \rangle^2 \coth \beta_j}{\sinh^2 \beta_j} + \frac{2 \langle \beta_j, \beta_j \rangle^2 \tanh \beta_j}{\cosh^2 \beta_j}.$$
Therefore,
\[-\Delta(f_j) + \tilde{u}_j f_j = -\frac{6m(3m + 1)(\beta_j, \beta_j)^2 \coth \beta_j}{\sinh^2 \beta_j} = \partial_{\beta_j}(u_j),\]
by relation (1.2), as required.

For each \( j = 1, 2, 3 \), let \( g_j \) in the operator (1.4) be defined by

\[ g_j = g_j^{(I)} + g_j^{(II)} + g_j^{(III)}, \tag{3.2} \]

where

\[ g_j^{(I)} = \prod_{k \neq j} f_k, \tag{3.3} \]
\[ g_j^{(II)} = -\frac{\prod_{k \neq j} (\alpha_j, \beta_k)}{\langle \alpha_j, \alpha_j \rangle} v_j, \tag{3.4} \]
\[ g_j^{(III)} = -\frac{\prod_{k \neq j} (\beta_j, \beta_k)}{\langle \beta_j, \beta_j \rangle} u_j, \tag{3.5} \]
or, more explicitly,

\[ g_1 = f_2 f_3 - \frac{9m(m + 1)\omega^4}{\sinh^2 \alpha_1} + \frac{3m(3m + 1)\omega^4}{\sinh^2 \beta_1}, \]
\[ g_2 = f_1 f_3 - \frac{9m(m + 1)\omega^4}{\sinh^2 \alpha_2} - \frac{3m(3m + 1)\omega^4}{\sinh^2 \beta_2}, \]
\[ g_3 = f_1 f_2 - \frac{9m(m + 1)\omega^4}{\sinh^2 \alpha_3} + \frac{3m(3m + 1)\omega^4}{\sinh^2 \beta_3}. \]

In the next lemma we find gradients of the functions \( g_j^{(II)}, g_j^{(III)} \).

**Lemma 3.3.** The functions \( g_j^{(II)}, g_j^{(III)} \) defined by formulas (3.4) and (3.5) satisfy the following relations for all \( \sigma \in A_3 \):

1. \( 2 \partial_{\nabla(g_{j(\sigma)^{(II)}})} = -\partial_{\beta_\sigma(2)}(v_{\sigma(1)}) \partial_{\beta_\sigma(3)} - \partial_{\beta_\sigma(3)}(v_{\sigma(1)}) \partial_{\beta_\sigma(2)}, \)
2. \( 2 \partial_{\nabla(g_{j(\sigma)^{(III)}})} = -\partial_{\beta_\sigma(2)}(u_{\sigma(1)}) \partial_{\beta_\sigma(3)} - \partial_{\beta_\sigma(3)}(u_{\sigma(1)}) \partial_{\beta_\sigma(2)}. \)

**Proof.** We give proof for \( \sigma = id \), the other cases are analogous. In the right-hand side of part (1) we have

\[-\partial_{\beta_2}(v_1) \partial_{\beta_3} - \partial_{\beta_3}(v_1) \partial_{\beta_2} = -\langle \alpha_1, \beta_2 \rangle v'_1 \partial_{\beta_3} - \langle \alpha_1, \beta_3 \rangle v'_1 \partial_{\beta_2} = -3v_1 \omega^2 (\partial_{\beta_3} + \partial_{\beta_2}) = -3v'_1 \omega^2 \partial_{\alpha_1}, \]

where \( v'_1(x) = \frac{dV}{dz} |_{z=\langle \alpha_1, x \rangle} \), \( V(z) = m(m + 1)(\alpha_1, \alpha_1) \sinh^{-2} z \). And in the left-hand side of relation (1) we get

\[ 2 \partial_{\nabla(g_{j(\sigma)^{(II)}})} = -2 \frac{\langle \alpha_1, \beta_2 \rangle}{\langle \alpha_1, \alpha_1 \rangle} \partial_{\nabla(v_1)} = -2 \frac{\langle \alpha_1, \beta_2 \rangle}{\langle \alpha_1, \alpha_1 \rangle} v'_1 \partial_{\alpha_1} = -3\omega^2 v'_1 \partial_{\alpha_1}, \]

so the two sides are equal. The proof of part (2) is similar.

It will be useful to combine gradients of functions \( g_j^{(I)}, g_j^{(II)}, g_j^{(III)} \) as in the following lemma.
Lemma 3.4. Functions $g_j^{(I)}$, $g_j^{(II)}$, $g_j^{(III)}$ defined by formulas (3.3) – (3.5) satisfy the following relations:

(1) \[ 2 \sum_{i=1}^{3} \partial_{\nabla(g_i^{(I)})} \partial_{\beta_i} = \sum_{\sigma} \left( \sum_{j \neq \sigma(1)} \hat{u}_j \right) f_{\sigma(1)} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}}, \]

(2) \[ 2 \sum_{i=1}^{3} \partial_{\nabla(g_i^{(II)})} \partial_{\beta_i} = - \sum_{\sigma} \left( \partial_{\beta_{\sigma(2)}} \left( v_{\sigma(1)} \partial_{\beta_{\sigma(3)}} + \partial_{\beta_{\sigma(3)}} \left( u_{\sigma(1)} \partial_{\beta_{\sigma(2)}} \right) \right) \right) \partial_{\beta_{\sigma(1)}} = - \sum_{\sigma} \partial_{\beta_{\sigma(1)}} \left( \sum_{j=1}^{3} v_j \right) \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}}, \]

(3) \[ 2 \sum_{i=1}^{3} \partial_{\nabla(g_i^{(III)})} \partial_{\beta_i} = - \sum_{\sigma} \left( \partial_{\beta_{\sigma(2)}} \left( u_{\sigma(1)} \partial_{\beta_{\sigma(3)}} + \partial_{\beta_{\sigma(3)}} \left( u_{\sigma(1)} \partial_{\beta_{\sigma(2)}} \right) \right) \right) \partial_{\beta_{\sigma(1)}} = - \sum_{\sigma} \partial_{\beta_{\sigma(1)}} \left( \sum_{j \neq \sigma(1)} u_j \right) \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}}. \]

Proof. In order to prove part (1), we note that by the definition (3.3) we have

\[ 2 \sum_{i=1}^{3} \partial_{\nabla(g_i^{(I)})} \partial_{\beta_i} = 2 \sum_{\sigma} \partial_{\nabla(f_{\sigma(2)}, f_{\sigma(3)})} \partial_{\beta_{\sigma(1)}} =\]

\[ = 2 \sum_{\sigma} f_{\sigma(3)} \partial_{\nabla(f_{\sigma(2)})} \partial_{\beta_{\sigma(1)}} + 2 \sum_{\sigma} f_{\sigma(2)} \partial_{\nabla(f_{\sigma(3)})} \partial_{\beta_{\sigma(1)}}, \]

Substituting the result of Lemma 3.2 part (1) for $\partial_{\nabla(f)}$ into the expression (3.6) we obtain

\[ \sum_{\sigma} f_{\sigma(3)} \hat{u}_{\sigma(2)} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(1)}} + \sum_{\sigma} f_{\sigma(2)} \hat{u}_{\sigma(3)} \partial_{\beta_{\sigma(3)}} \partial_{\beta_{\sigma(1)}}, \]

which equals the right-hand side of (1).

The equalities (2) and (3) follow from Lemma 3.3. \[ \square \]

In the next lemma we deal with combining gradients of functions $f_j$ and $g_j^{(I)}$, $g_j^{(II)}$, $g_j^{(III)}$.

Lemma 3.5. The functions $g_j^{(I)}$, $g_j^{(II)}$, $g_j^{(III)}$ defined by formulas (3.3) – (3.5) satisfy also the following relations:

(1) \[ \sum_{i=1}^{3} \langle \nabla(f_i), \nabla(g_i^{(I)}) \rangle = \frac{1}{2} \sum_{\sigma} \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \hat{u}_{\sigma(2)} \hat{u}_{\sigma(3)} f_{\sigma(1)}, \]

(2) \[ \sum_{i=1}^{3} \langle \nabla(f_i), \nabla(g_i^{(II)}) \rangle = 0, \]

(3) \[ \sum_{i=1}^{3} \langle \nabla(f_i), \nabla(g_i^{(III)}) \rangle = \sum_{i=1}^{3} \hat{u}_i \partial_{\beta_i} (g_i^{(III)}). \]

Proof. The left-hand side of (1) can be expanded using the product rule as

\[ \sum_{i=1}^{3} \langle \nabla(f_i), \nabla(g_i^{(I)}) \rangle = \sum_{\sigma} \langle \nabla f_{\sigma(1)}, \nabla (f_{\sigma(2)} f_{\sigma(3)}) \rangle \]

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Lemma 3.6. The functions $g_i^{(II)}$ and $g_i^{(III)}$ defined by formulas (3.4) and (3.5) satisfy the following relations for all $i = 1, 2, 3$:

1. $\Delta (g_i^{(II)}) = - \prod_{k \neq i} \partial_{\beta_k} v_i = - \prod_{k \neq i} \partial_{\beta_k} \sum_{j=1}^{3} v_j$,

2. $\Delta (g_i^{(III)}) = - \prod_{k \neq i} \partial_{\beta_k} u_i$.

Proof. Statement (1) follows by formula (3.4) and Lemma 3.1. Similarly, property (2) follows directly from Lemma 3.2 and formula (3.5).

Lemma 3.7. Functions $g_i^{(I)}$, $g_j^{(II)}$, $g_j^{(III)}$ defined by formulas (3.3) – (3.5) satisfy the following relations:

1. $\sum_{i=1}^{3} \Delta (g_i^{(I)}) \partial_{\beta_i} = \sum_{i=1}^{3} \left( \sum_{j \neq i} \hat{u}_j \right) g_i^{(I)} \partial_{\beta_i} + \frac{1}{2} \sum_{\sigma} \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \hat{u}_{\sigma(2)} \hat{u}_{\sigma(3)} \partial_{\sigma(1)}$

$$- \sum_{\sigma} f_{\sigma(1)} \left( \partial_{\beta_{\sigma(2)}} (u_{\sigma(2)}) \partial_{\beta_{\sigma(3)}} + \partial_{\beta_{\sigma(3)}} (u_{\sigma(3)}) \partial_{\beta_{\sigma(2)}} \right) ,$$

2. $\sum_{i=1}^{3} \Delta (g_i^{(III)}) \partial_{\beta_i} = - \sum_{\sigma} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}} \left( \sum_{j=1}^{3} v_j \right) \partial_{\sigma(1)} ,$

3. $\sum_{i=1}^{3} \Delta (g_i^{(III)}) \partial_{\beta_i} = - \sum_{\sigma} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}} (u_{\sigma(1)}) \partial_{\sigma(1)} .$

Proof. Let us first consider $\Delta (g_1^{(I)})$. By Lemma 3.2, we have

$$\Delta (g_1^{(I)}) = \Delta (f_2 f_3) = \Delta (f_2) f_3 + 2 \langle \nabla (f_2), \nabla (f_3) \rangle + \Delta (f_3) f_2$$

$$= \left( \hat{u}_2 f_2 - \partial_{\beta_2} (u_2) \right) f_3 + \frac{1}{2} \hat{u}_2 \hat{u}_3 \langle \beta_2, \beta_3 \rangle + \left( \hat{u}_3 f_3 - \partial_{\beta_3} (u_3) \right) f_2$$

$$= \left( \sum_{j \neq 1} \hat{u}_j \right) g_1^{(I)} + \frac{1}{2} \hat{u}_2 \hat{u}_3 \langle \beta_2, \beta_3 \rangle - \left( \partial_{\beta_2} (u_2) f_3 + \partial_{\beta_3} (u_3) f_2 \right) .$$ (3.7)
By multiplying (3.7) by $\partial_{\beta_1}$, and adding it with similar expressions for $\Delta(g_2^{(1)})\partial_{\beta_2}$ and $\Delta(g_3^{(1)})\partial_{\beta_3}$, we obtain property (1).

Properties (2) and (3) follow from Lemma 3.6 parts (1) and (2), respectively, by multiplying these equalities by $\partial_{\beta_i}$ and summing them up over $i = 1, 2, 3$.

Let $h$ in the operator (1.4) be defined by

$$h = h^{(1)} + h^{(II)} + h^{(III)} + h^{(IV)},$$

where

$$h^{(1)} = f_1f_2f_3,$$  

$$h^{(II)} = \sum_{i=1}^{3} f_i(g_i^{(II)} + g_i^{(III)}),$$  

$$h^{(III)} = \sum_{i=1}^{3} \partial_{\beta_i}(g_i^{(III)}) = -\sum_{i=1}^{3} \prod_{k \neq i}(\beta_i, \beta_k) \partial_{\beta_i}(u_i)$$

$$= -\frac{12m(3m+1)\omega^6}{\sinh^2 \beta_1} \coth \beta_1 + \frac{12m(3m+1)\omega^6}{\sinh^2 \beta_2} \coth \beta_2 - \frac{12m(3m+1)\omega^6}{\sinh^2 \beta_3} \coth \beta_3,$$

$$h^{(IV)} = -3m(3m+1)\omega^{-2} \prod_{i=1}^{3}(\beta_i, \beta_i) X - 4(3m+1)\omega^{-2} \prod_{i=1}^{3}(\beta_i, \beta_i) Y.$$

In the next Lemmas 3.8, 3.9 we calculate gradients and Laplacians of the functions $h^{(1)}$, $h^{(II)}$, $h^{(III)}$.

**Lemma 3.8.** The functions $h^{(1)}$, $h^{(II)}$, $h^{(III)}$ defined by formulas (3.9) – (3.12) satisfy the following relations:

1. $2\partial_{\nabla(h^{(1)} + h^{(II)})} = \sum_{i=1}^{3} \hat{u}_i g_i \partial_{\beta_i} - \sum_{\sigma} f_{\sigma(1)} \left( \partial_{\beta_{\sigma(2)}}(v_{\sigma(1)} + u_{\sigma(1)}) \partial_{\beta_{\sigma(3)}}(v_{\sigma(1)} + u_{\sigma(1)}) \partial_{\beta_{\sigma(2)}} \right)$,

2. $2\partial_{\nabla(h^{(III)})} = -\sum_{\sigma} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}} \left( \sum_{j \neq \sigma(1)} u_j \right) \partial_{\beta_{\sigma(1)}}$.

**Proof.** We have that

$$\partial_j(h^{(1)}) = \partial_j(f_1)f_2f_3 + \partial_j(f_2)f_1f_3 + \partial_j(f_3)f_1f_2 = \sum_{i=1}^{3} \partial_j(f_i)g_i^{(1)},$$

therefore by Lemma 3.2 part (1),

$$2\partial_{\nabla(h^{(1)})} = \sum_{i=1}^{3} 2g_i^{(1)} \partial_{\nabla(f_i)} = \sum_{i=1}^{3} \hat{u}_i g_i^{(1)} \partial_{\beta_i}. \quad (3.13)$$

On the other hand,

$$2\partial_{\nabla(h^{(II)})} = 2 \sum_{i=1}^{3} \partial_{\nabla(f_i)(g_i^{(III)} + g_i^{(IV)})} = 2 \sum_{i=1}^{3} (g_i^{(III)} + g_i^{(IV)}) \partial_{\nabla(f_i)} + 2 \sum_{i=1}^{3} f_i \left( \partial_{\nabla(g_i^{(III)})} + \partial_{\nabla(g_i^{(IV)})} \right). \quad (3.14)$$
By Lemma 3.2 part (1) and Lemma 3.3 we can rearrange the expression (3.14) as
\[
\sum_{i=1}^{3} \hat{u}_i(g_i^{(II)} + g_i^{(III)}) \partial_{\beta_i} - \sum_{\sigma} f_{\sigma(1)} \left( \partial_{\beta_{\sigma(2)}} (v_{\sigma(1)} + u_{\sigma(1)}) \partial_{\beta_{\sigma(3)}} + \partial_{\beta_{\sigma(3)}} (v_{\sigma(1)} + u_{\sigma(1)}) \partial_{\beta_{\sigma(2)}} \right).
\] (3.15)

The statement (1) follows by adding up equalities (3.13) and (3.15).

In the right-hand side of statement (2), the coefficient at \( \partial_{\beta_1} \) is equal to
\[
-\partial_{\beta_2} \partial_{\beta_3} (u_2 + u_3) = -24m(3m + 1)\omega^6 \left( \frac{2 \coth^2 \beta_2}{\sinh^2 \beta_2} + \frac{1}{\sinh^4 \beta_2} + \frac{2 \coth^2 \beta_3}{\sinh^2 \beta_3} + \frac{1}{\sinh^4 \beta_3} \right).
\] (3.16)

In the left-hand side of statement (2) one can check that \( 2\partial_{\hat{v}_{(h^{(III))}}} \) is equal to
\[
\frac{24m(3m + 1)\omega^6 \times}{2} \left( \frac{2 \coth^2 \beta_1}{\sinh^2 \beta_1} + \frac{1}{\sinh^4 \beta_1} \right) \partial_{\beta_1} - \left( \frac{2 \coth^2 \beta_2}{\sinh^2 \beta_2} + \frac{1}{\sinh^4 \beta_2} \right) \partial_{\beta_2} + \left( \frac{2 \coth^2 \beta_3}{\sinh^2 \beta_3} + \frac{1}{\sinh^4 \beta_3} \right) \partial_{\beta_3}.
\] (3.17)

Let us substitute in expression (3.17) \( \partial_{\beta_1} = \partial_{\beta_2} - \partial_{\beta_3} \), \( \partial_{\beta_2} = \partial_{\beta_1} + \partial_{\beta_3} \), and \( \partial_{\beta_3} = \partial_{\beta_2} - \partial_{\beta_1} \). Then one can see that the coefficient at \( \partial_{\beta_1} \) equals expression (3.16). Similarly, the coefficients at \( \partial_{\beta_2} \) and \( \partial_{\beta_3} \) also match on both sides of equality (2).

**Lemma 3.9.** Functions \( h^{(I)} \), \( h^{(II)} \), \( h^{(III)} \) given by formulas (3.9) – (3.11) satisfy the following relations:

1. \( \Delta(h^{(I)} + h^{(III)}) = \sum_{i=1}^{3} \hat{u}_i f_i g_i - \sum_{i=1}^{3} \partial_{\beta_i} (u_i) g_i + \frac{1}{2} \sum_{\sigma} \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \hat{u}_{\sigma(2)} \hat{u}_{\sigma(3)} f_{\sigma(1)} \)
   \[+ \sum_{i=1}^{3} \hat{u}_i \partial_{\beta_i} (g_i^{(III)}) - \sum_{\sigma} f_{\sigma(1)} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}} (v_{\sigma(1)} + u_{\sigma(1)}), \]

2. \( \Delta(h^{(III)}) = -\partial_{\beta_1} \partial_{\beta_2} \partial_{\beta_3} \left( \sum_{j=1}^{3} u_j \right). \)

**Proof.** Firstly, by Lemma 3.2 part (2) and Lemma 3.5 part (1) we have
\[
\Delta(h^{(I)}) = \Delta(f_1 f_2 f_3) = \sum_{i=1}^{3} \Delta(f_i) g_i^{(I)} + \sum_{i=1}^{3} \langle \nabla(f_i), \nabla(g_i^{(I)}) \rangle
\]
\[= \sum_{i=1}^{3} \hat{u}_i f_i g_i^{(I)} - \sum_{i=1}^{3} \partial_{\beta_i} (u_i) g_i^{(I)} + \frac{1}{2} \sum_{\sigma} \hat{u}_{\sigma(2)} \hat{u}_{\sigma(3)} \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle f_{\sigma(1)}.
\] (3.18)
Secondly, by Lemma 3.2 part (2), by Lemma 3.5 parts (2) and (3), and Lemma 3.6 we have

\[
\Delta(h^{(II)}) = \Delta \left( \sum_{i=1}^{3} f_i (g_i^{(II)} + g_i^{(III)}) \right)
= \sum_{i=1}^{3} \Delta(f_i) (g_i^{(II)} + g_i^{(III)}) + 2 \sum_{i=1}^{3} \langle \nabla(f_i), \nabla(g_i^{(II)} + g_i^{(III)}) \rangle + \sum_{i=1}^{3} f_i \Delta(g_i^{(II)} + g_i^{(III)})
= \sum_{i=1}^{3} \hat{u}_i f_i (g_i^{(II)} + g_i^{(III)}) - \sum_{i=1}^{3} \partial_{\beta_i} (u_i) (g_i^{(II)} + g_i^{(III)})
+ \sum_{i=1}^{3} \hat{u}_i \partial_{\beta_i} (g_i^{(III)}) - \sum_{\sigma} f_{\sigma(1)} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}} (v_{\sigma(1)} + u_{\sigma(1)}).
\]

The statement (1) follows by adding the equalities (3.18) and (3.19). By Lemma 3.1 for any \( j \) we have

\[
\Delta \left( - \frac{\prod_{k \neq j} \langle \beta_j, \beta_k \rangle}{\langle \beta_j, \beta_j \rangle} \partial_{\beta_j} (u_j) \right) = - \left( \prod_{k \neq j} \partial_{\beta_k} \right) \partial_{\beta_j} (u_j) = - \partial_{\beta_1} \partial_{\beta_2} \partial_{\beta_3} (u_j). \tag{3.20}
\]

We get result (2) by summing equalities (3.20) over \( j = 1, 2, 3 \).

\[\square\]

### 4 Proof of the intertwining relation

Let \( A = A(x, \partial_1, \partial_2) \) be a differential operator of order \( N \). Then \( A \) can be represented as

\[
A = \sum_{k=0}^{N} A^{(k)}, \quad \text{with} \quad A^{(k)} = \sum_{i+j=k} a_{ij}(x) \partial_i^j
\]

for some functions \( a_{ij}(x) \) so \( A^{(k)} \) denotes the \( k \)-th order part of \( A \). That is \( A^{(k)} \) is the sum of all terms in \( A \) that contain exactly \( k \) derivatives when all the derivatives are put on the right.

Both operators \( HD \) and \( DH_0 \) have order 5. It is easy to see that the respective terms of orders 5 and 4 in both operators are the same. We are going to show that this is also true for lower orders.

**Proposition 4.1.** The third order terms in the intertwining relation (1.5) satisfy

\[
(HD)^{(3)} = (DH_0)^{(3)}.
\]

**Proof.** We have

\[
(HD - DH_0)^{(3)} = -2 \sum_{\sigma} \partial_{\nabla(f_{\sigma(1)})} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}} + \left( \sum_{j=1}^{3} \hat{u}_j \right) \partial_{\beta_1} \partial_{\beta_2} \partial_{\beta_3}.
\]

By Lemma 3.2 part (1) we get

\[
\sum_{\sigma} \partial_{\nabla(f_{\sigma(1)})} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}} = \frac{1}{2} \sum_{\sigma} \hat{u}_{\sigma(1)} \partial_{\beta_{\sigma(1)}} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}} = \frac{1}{2} \left( \sum_{j=1}^{3} \hat{u}_j \right) \partial_{\beta_1} \partial_{\beta_2} \partial_{\beta_3},
\]

and the statement follows. \[\square\]
Lemma 3.4. which is zero by applying Lemma 3.2 part (2) and adding equalities from all three parts of

Note that the sum of expressions (4.2), (4.3) divided by

Similarly, by the identities (2.13) and (2.14) in Lemma 2.7 we get

Firstly we let

For any

Lemma 4.3. For any \( \sigma \in A_3 \),

\[
\frac{1}{2} \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \tilde{u}_{\sigma(2)} \tilde{u}_{\sigma(3)} + \left( \sum_{j \neq \sigma(1)} \tilde{u}_j \right) \left( g_{\sigma(1)}^{(II)} + g_{\sigma(1)}^{(III)} \right) - \left( f_{\sigma(2)} \partial_{\beta_{\sigma(3)}} + f_{\sigma(3)} \partial_{\beta_{\sigma(2)}} \right) \left( v_{\sigma(1)} + u_{\sigma(1)} \right)
\]

Proof. Firstly we let \( \sigma = \text{id} \). By the identities (2.11) and (2.12) in Lemma 2.7 we get

\[
\left( \sum_{j \neq 1} \tilde{u}_j \right) g_1^{(II)} - \left( f_2 \partial_{\beta_3} + f_3 \partial_{\beta_2} \right) (v_1) = -36m(m+1)\omega^6 \left( \frac{3m+1}{\sinh^2 \beta_2 \sinh^2 \beta_3} + \frac{1}{\cosh^2 \beta_2 \cosh^2 \beta_3} \right).
\]

Similarly, by the identities (2.13) and (2.14) in Lemma 2.7 we get

\[
\left( \sum_{j \neq 1} \tilde{u}_j \right) g_1^{(III)} - \left( f_2 \partial_{\beta_3} + f_3 \partial_{\beta_2} \right) (u_1) = 12m(3m+1)\omega^6 \left( \frac{3m+1}{\sinh^2 \beta_2 \sinh^2 \beta_3} + \frac{1}{\cosh^2 \beta_2 \cosh^2 \beta_3} \right).
\]

Note that the sum of expressions (4.2), (4.3) divided by \( \omega^6 \) together with the term \(-\frac{1}{2} \tilde{u}_2 \tilde{u}_3\) divided by \( \omega^4 \) equals

\[
- \frac{48m(3m+1)}{\sinh^2 \beta_2 \sinh^2 \beta_3} - 8(3m+1) \left( \frac{1}{\sinh^2 \beta_2} - \frac{1}{\cosh^2 \beta_2} \right) \left( \frac{1}{\sinh^2 \beta_3} - \frac{1}{\cosh^2 \beta_3} \right)
\]

\[
= - \frac{48m(3m+1)}{\sinh^2 \beta_2 \sinh^2 \beta_3} - \frac{128(3m+1)}{\sinh^2 2\beta_2 \sinh^2 2\beta_3},
\]

which is the right-hand side of the equality (4.1) divided by \( \omega^6 \) as required. The cases \( \sigma \neq \text{id} \) follow from versions of (2.11) – (2.14) obtained by rotating vectors (see Remark 1).

Proposition 4.4. The first order terms in the intertwining relation (1.5) satisfy

\[
(HD)^{(1)} = (DH_0)^{(1)}.
\]
Proof. We have that

\[(HD - DH_0)^{(1)} = - \sum_{i=1}^{3} \Delta(g_i) \partial_{\beta_i} - 2 \partial \nabla(h) + \sum_{i=1}^{3} \left( \sum_{j=1}^{3} \hat{u}_j \right) g_i \partial_{\beta_i} - \sum_{\sigma} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}} \left( \sum_{j=1}^{3} (v_j + u_j) \right) \partial_{\beta_{\sigma(1)}} \]

\[- \sum_{\sigma} f_{\sigma(1)} \left( \partial_{\beta_{\sigma(2)}} \left( \sum_{j=1}^{3} (v_j + u_j) \right) \right) \partial_{\beta_{\sigma(3)}} \partial_{\beta_{\sigma(3)}} \left( \sum_{j=1}^{3} (v_j + u_j) \right) \partial_{\beta_{\sigma(2)}} \right). \tag{4.4}\]

We substitute the expression for \(\sum_{i=1}^{3} \Delta(g_i) \partial_{\beta_i}\) from Lemma 3.7 and the expression for \(\partial \nabla(h^{(i)}+h^{(ii)}+h^{(iii)})\) from Lemma 3.8 into the formula (4.4). Then the expression (4.4) can be rearranged as

\[- \frac{1}{2} \sum_{\sigma} \beta_{\sigma(2)} \beta_{\sigma(3)} \hat{u}_{\sigma(2)} \hat{u}_{\sigma(3)} \partial_{\beta_{\sigma(1)}} \sum_{\sigma} \left( \sum_{j=1}^{3} \hat{u}_j \right) \left( g_{\sigma(1)} + g_{\sigma(1)} \right) \partial_{\beta_{\sigma(1)}} \]

\[- \sum_{\sigma} \left( f_{\sigma(2)} \partial_{\beta_{\sigma(3)}} + f_{\sigma(3)} \partial_{\beta_{\sigma(2)}} \right) (v_{\sigma(1)} + u_{\sigma(1)}) \partial_{\beta_{\sigma(1)}} - 2 \partial \nabla(h^{(iv)}) , \tag{4.5}\]

which is zero by Lemma 4.3 and Corollary 2.5.

The following lemma is needed in order to consider the zero order terms in the intertwining relation.

**Lemma 4.5.** The zero order terms satisfy

\[(HD - DH_0)^{(0)} = A + B + C + D\]

where

\[A = \sum_{i=1}^{3} \left( \sum_{j \neq i} \hat{u}_j \right) \partial_{\beta_i} \left( g_i^{(iii)} \right) - \sum_{\sigma} f_{\sigma(1)} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}} \left( \sum_{j \neq \sigma(1)} u_j \right) , \tag{4.6}\]

\[B = - \frac{1}{2} \sum_{\sigma} \beta_{\sigma(2)} \beta_{\sigma(3)} \hat{u}_{\sigma(2)} \hat{u}_{\sigma(3)} \partial_{\beta_{\sigma(1)}} \sum_{\sigma} \left( \sum_{j \neq \sigma(1)} \hat{u}_j \right) \left( g_{\sigma(1)} + g_{\sigma(1)} \right) \partial_{\beta_{\sigma(1)}} , \tag{4.7}\]

\[C = \left( \sum_{j=1}^{3} \hat{u}_j \right) h^{(iv)} - \Delta(h^{(iv)}) , \tag{4.8}\]

\[D = - \sum_{i=1}^{3} g_i \partial_{\beta_i} \left( \sum_{j \neq i} (v_j + u_j) \right) . \tag{4.9}\]

Moreover, the term \(D\) can be rearranged as \(D = D_1 + D_2\), where

\[D_1 = - \sum_{i=1}^{3} \left( g_i^{(ii)} + g_i^{(iii)} \right) \partial_{\beta_i} \left( \sum_{j \neq i} (v_j + u_j) \right) , \tag{4.10}\]

\[D_2 = - \sum_{\sigma} f_{\sigma(1)} \left( f_{\sigma(2)} \partial_{\beta_{\sigma(3)}} + f_{\sigma(3)} \partial_{\beta_{\sigma(2)}} \right) (v_{\sigma(1)} + u_{\sigma(1)}) . \tag{4.11}\]
Proof. We have

\[(H \circ D)^{(0)} - (D \circ H_0)^{(0)} = -\Delta(h) + \left(\sum_{j=1}^{3} \hat{u}_j\right) h - \partial_{\beta_1} \partial_{\beta_2} \partial_{\beta_3} \left(\sum_{j=1}^{3} u_j\right) \]

\[- \sum_{\sigma} f_{\sigma(1)} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}} \left(v_{\sigma(1)} + \sum_{j=1}^{3} u_j\right) \]  \hspace{1cm} (4.12)

\[- \sum_{i=1}^{3} g_i \partial_{\beta_1} \left(\sum_{j \neq i} v_j + \sum_{j=1}^{3} u_j\right) . \]

By putting in the results of Lemma 3.9, the expression (4.12) takes the required form \(A + B + C + D\). By expanding \(g_i = g_i^{(II)} + g_i^{(III)} + g_i^{(I)}\), we also have that

\[D = D_1 - \sum_{\sigma} f_{\sigma(2)} f_{\sigma(3)} \partial_{\beta_{\sigma(1)}} \left(\sum_{j \neq \sigma(1)} (v_j + u_j)\right) = D_1 + D_2\]
as required.

In the next Lemmas 4.6 – 4.9 we rearrange the expressions for the zero order terms \(A, B, C, D\). Namely, we rewrite these terms explicitly as functions of \(\beta_j\).

**Lemma 4.6.** The function \(A\) given by expression (4.6) can be rearranged as follows:

\[\frac{A}{48 \omega^6} = 2m(3m + 1)^2 X - m(3m + 1) \sum_{\sigma} \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \coth \beta_{\sigma(1)}}{\cosh^2 \beta_{\sigma(2)} \cosh^2 \beta_{\sigma(3)}} \]

\[+ 3m^2(3m + 1) \left(\sum_{j=1}^{3} \frac{1}{\sinh^2 \beta_j}\right) X + 8m(3m + 1) \left(\sum_{j=1}^{3} \frac{1}{\sinh^2 \beta_j}\right) Y \]

\[+ 24m(3m + 1) Y . \]  \hspace{1cm} (4.13)

**Proof.** Consider the term involving \(g_i^{(III)}\) in \(\sum_{i=1}^{3} \left(\sum_{j \neq i} \hat{u}_j\right) \partial_{\beta_1} \left(g_i^{(III)}\right)\). It gives

\[\frac{1}{48 \omega^6} \left(\sum_{j \neq 1} \hat{u}_j\right) \partial_{\beta_1} \left(g_i^{(III)}\right) = -m(3m + 1)^2 \left(\frac{1}{\sinh^2 \beta_2} + \frac{1}{\sinh^2 \beta_3}\right) \coth \beta_1 \]

\[+ m(3m + 1) \left(\frac{1}{\cosh^2 \beta_2} + \frac{1}{\cosh^2 \beta_3}\right) \coth \beta_1 . \]  \hspace{1cm} (4.14)

Now consider the terms involving \(u_1\) in \(- \sum_{\sigma} f_{\sigma(1)} \partial_{\beta_{\sigma(2)}} \partial_{\beta_{\sigma(3)}} \left(\sum_{j \neq \sigma(1)} u_j\right)\). They produce

\[- \frac{1}{48 \omega^6} \left(f_2 \partial_{\beta_3} \partial_{\beta_1} (u_1) + f_3 \partial_{\beta_1} \partial_{\beta_2} (u_1)\right) = \]

\[= -2m(3m + 1)^2 \left(\coth \beta_2 - \coth \beta_3\right) \coth^2 \beta_1 - 2m(3m + 1) \left(\tanh \beta_2 - \tanh \beta_3\right) \frac{\coth^2 \beta_1}{\sinh^2 \beta_1} \]

\[+ m(3m + 1) \left((3m + 1) \left(\coth \beta_3 - \coth \beta_2\right) + \left(\tanh \beta_3 - \tanh \beta_2\right)\right) \frac{1}{\sinh^4 \beta_1} . \]  \hspace{1cm} (4.15)
It follows from Lemma 2.7 (namely, equalities (2.13) and (2.14) multiplied by $\coth \beta_1$) that the sum of the right-hand side of equality (4.14) with the first two terms in the right-hand side of equality (4.15) equals

$$-m(3m+1)^2 \frac{\coth \beta_1}{\sinh^2 \beta_2 \sinh^2 \beta_3} - m(3m+1) \frac{\coth \beta_1}{\cosh^2 \beta_2 \cosh^2 \beta_3},$$  

(4.16)

Adding expression (4.16) with analogous ones coming from the terms $g_2^{(III)}$, $u_2$, and $g_3^{(III)}$, $u_3$ in the left-hand side of equality (4.13) we get the 1st line of the right-hand side of equality (4.13) by Lemma 2.2.

Now we rearrange the other terms in the right-hand side of equality (4.15). We have that

$$m(3m+1) \left( (3m+1) \left( \coth \beta_3 - \coth \beta_2 \right) + \left( \tanh \beta_3 - \tanh \beta_2 \right) \right) \frac{1}{\sinh^4 \beta_1}$$

$$= m(3m+1) \left( \frac{3m \sinh \beta_1}{\sinh \beta_2 \sinh \beta_3} + \frac{\sinh \beta_1}{\sinh \beta_2 \sinh \beta_3} - \frac{\sinh \beta_1}{\cosh \beta_2 \cosh \beta_3} \right) \frac{1}{\sinh^4 \beta_1}$$

$$= \frac{3m^2 (3m+1) \omega^{-2}}{\sinh^2 \beta_1} X + \frac{m(3m+1) \cosh \beta_1}{\sinh^3 \beta_1 \sinh \beta_2 \sinh \beta_3 \cosh \beta_2 \cosh \beta_3}$$

$$= \frac{3m^2 (3m+1) \omega^{-2}}{\sinh^2 \beta_1} X + \frac{m(3m+1)(1+\sinh^2 \beta_1)}{\sinh^3 \beta_1 \sinh \beta_2 \sinh \beta_3 \cosh \beta_1 \cosh \beta_2 \cosh \beta_3}$$

$$= \frac{3m^2 (3m+1) \omega^{-2}}{\sinh^2 \beta_1} X + \frac{8m(3m+1) \omega^{-2}}{\sinh^2 \beta_1} Y + \frac{8m(3m+1) \omega^{-2}}{\sinh^2 \beta_1} Y.$$

Similarly, terms from the right-hand side of a version of equality (4.15) for $g_2^{(III)}$, $u_2$, and $g_3^{(III)}$, $u_3$, add up to

$$3m^2 (3m+1) \omega^{-2} \left( \frac{1}{\sinh^2 \beta_2} + \frac{1}{\sinh^2 \beta_3} \right) X + 8m(3m+1) \omega^{-2} \left( \frac{1}{\sinh^2 \beta_2} + \frac{1}{\sinh^2 \beta_3} \right) Y + 16m(3m+1) \omega^{-2} Y.$$  

(4.18)

The sum of terms in the last line of (4.17) together with (4.18) equals the terms in the 2nd and 3rd line in the right-hand side of equality (4.13) \(\Box\).

**Lemma 4.7.** Consider the functions $B$ and $D_2$ given by (4.7) and (4.11). We have

$$\frac{B + D_2}{64 \omega^6} = -3m(3m+1)(3m+2)X - 12m(3m+1)Y - 16(3m+1)Y$$

$$+ \sum_{\sigma} \frac{12m(3m+1)\langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \coth \beta_{\sigma(1)}}{\sinh^2 2\beta_{\sigma(2)} \sinh^2 2\beta_{\sigma(3)}}.$$  

(4.19)

**Proof.** As a consequence of Lemma 4.3 we have

$$\frac{B + D_2}{64 \omega^6} = - \sum_{\sigma} \frac{3m(3m+1) \omega^{-2} \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle f_{\sigma(1)}}{4 \sinh^2 \beta_{\sigma(2)} \sinh^2 \beta_{\sigma(3)}} - \sum_{\sigma} \frac{2(3m+1) \omega^{-2} \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle f_{\sigma(1)}}{\sinh^2 2\beta_{\sigma(2)} \sinh^2 2\beta_{\sigma(3)}}.$$  

(4.20)

We substitute $f_j$ given by (3.1) into the first sum in the relation (4.20), and we substitute

$$f_j = -2\omega^2 (3m \coth \beta_j + 2 \coth 2 \beta_j),$$

$j = 1, 2, 3$, in the second sum in the relation (4.20). By Lemma 2.2, as well as its version with $\beta_j$ replaced with $2 \beta_j$, and by Lemma 2.3, we can rearrange the right-hand side of (4.20) into the required form. \(\Box\)
Lemma 4.8. The function $C$ given by (4.8) can be rearranged as follows:

\[
\frac{C}{32\omega^6} = -9m^2(3m + 1) \left( \sum_{j=1}^{3} \frac{1}{\sinh^2 \beta_j} \right) X + 3m(3m + 1) \left( 2 + \sum_{j=1}^{3} \frac{1}{\cosh^2 \beta_j} \right) X
\]

\[-12m(3m + 1) \left( \sum_{j=1}^{3} \frac{1}{\sinh^2 \beta_j} \right) Y + 32(3m + 1)Y .
\]

(4.21)

Proof. By Lemma 2.6, we have that

\[
-\Delta \left( \frac{h^{(IV)}}{32\omega^6} \right) = 3m(3m + 1) \left( 2 + \sum_{j=1}^{3} \frac{1}{\sinh^2 \beta_j} \right) X + 16(3m + 1) \left( 2 + \sum_{j=1}^{3} \frac{1}{\sinh^2 2\beta_j} \right) Y .
\]

(4.22)

The product \( \frac{1}{32\omega^6} (\sum_{j=1}^{3} \hat{u}_j) h^{(IV)} \) can be rearranged as

\[
- \left( 3m \sum_{j=1}^{3} \frac{1}{\sinh^2 \beta_j} + \sum_{j=1}^{3} \frac{1}{\sinh^2 \beta_j} - \sum_{j=1}^{3} \frac{1}{\cosh^2 \beta_j} \right) 3m(3m + 1)X
\]

\[- \left( 3m \sum_{j=1}^{3} \frac{1}{\sinh^2 \beta_j} + 4 \sum_{j=1}^{3} \frac{1}{\sinh^2 2\beta_j} \right) 4(3m + 1)Y .
\]

(4.23)

The equality (4.21) follows by multiplying (4.23) out and combining it with (4.22).

Lemma 4.9. The function $D_1$ given by (4.10) can be rearranged as

\[
D_1 = 144m^2(3m + 1)\omega^6 \left( 2 + \sum_{j=1}^{3} \frac{1}{\sinh^2 \beta_j} \right) X .
\]

(4.24)

Proof. Note that Lemma 2.10 can be restated in the following form:

\[
\sum_{i=1}^{3} g^{(II)}_i \partial_{\beta_i} \left( \sum_{j \neq i} v_j \right) = 0 .
\]

Consider the terms in \( -\sum_{i=1}^{3} g^{(III)}_i \partial_{\beta_i} \left( \sum_{j \neq i} v_j \right) \) that involve $v_1$, that is $j = 1$. They equal

\[
108m^2(m + 1)(3m + 1)\omega^8 \left( \frac{1}{\sinh^2 \beta_3} - \frac{1}{\sinh^2 \beta_2} \right) \frac{\coth \alpha_1}{\sinh^2 \alpha_1} .
\]

(4.25)

Now let us look at terms in \( -\sum_{i=1}^{3} g^{(III)}_i \partial_{\beta_i} \left( \sum_{j \neq i} u_j \right) \) that involve $g^{(II)}_1$. These terms are equal to

\[
108m^2(m + 1)(3m + 1)\omega^8 \left( \frac{\coth \beta_3}{\sinh^2 \beta_3} - \frac{\coth \beta_2}{\sinh^2 \beta_2} \right) \frac{1}{\sinh^2 \alpha_1} .
\]

(4.26)

By Lemma 2.8 the sum of expressions (4.25) and (4.26) equals

\[
108m^2(m + 1)(3m + 1)\omega^8 \frac{\coth \beta_3 - \coth \beta_2}{\sinh^2 \beta_2 \sinh^2 \beta_3} .
\]

(4.27)
We also have that \(- \sum_{i=1}^{3} g_i^{(III)} \partial_{\beta_i} \left( \sum_{j \neq i} u_j \right)\) is equal to
\[
36m^2(3m + 1)^2 \omega^8 \left( \frac{\coth \beta_2 - \coth \beta_3}{\sinh^2 \beta_2 \sinh^2 \beta_3} - \frac{\coth \beta_1 + \coth \beta_3}{\sinh^2 \beta_1 \sinh^2 \beta_3} + \frac{\coth \beta_2 - \coth \beta_1}{\sinh^2 \beta_1 \sinh^2 \beta_2} \right). \tag{4.28}
\]
By adding expression (4.27) and the first term of expression (4.28) we get
\[
72m^2(3m + 1) \omega^8 \frac{\coth \beta_3 - \coth \beta_2}{\sinh^2 \beta_2 \sinh^2 \beta_3}. \tag{4.29}
\]
By grouping similarly the remaining terms in the left-hand side of identity (4.24) and by using variants of Lemma 2.8 obtained by rotating \(\beta\)'s by \(\pm \frac{\pi}{3}\) (see Remark 1), we get that the left-hand side of (4.24) can be rearranged as
\[
72m^2(3m + 1) \omega^6 \sum_{\sigma} \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \left( \frac{1}{\sinh^2 \beta_{\sigma(2)}} + \frac{1}{\sinh^2 \beta_{\sigma(3)}} \right) \coth \beta_{\sigma(1)} \sinh \beta_{\sigma(1)}. \tag{4.30}
\]
The result follows by Lemma 2.9.

\[\square\]

**Proposition 4.10.** The zero order terms satisfy
\[(HD)^{(0)} = (DH_0)^{(0)}.\]

**Proof.** By Lemmas 4.5 – 4.9 we have
\[
\frac{(HD - DH_0)^{(0)}}{48m(3m + 1) \omega^6} = 2 \left( \sum_{j=1}^{3} \frac{1}{\cosh^2 \beta_j} \right) X - 2X + 8Y - \sum_{\sigma} \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \coth \beta_{\sigma(1)} \cosh^2 \beta_{\sigma(2)} \cosh^2 \beta_{\sigma(3)} \tag{4.30}
\]
\[
+ \sum_{\sigma} \frac{16 \langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \coth \beta_{\sigma(1)}}{\sinh^2 2 \beta_{\sigma(2)} \sinh^2 2 \beta_{\sigma(3)}}. \]
Let us replace \(\sinh^{-2} 2 \beta_{\sigma(2)} \sinh^{-2} 2 \beta_{\sigma(3)}\) in the last sum in (4.30) with
\[
\frac{1}{16} \left( \frac{1}{\sinh^2 \beta_{\sigma(2)}} - \frac{1}{\cosh^2 \beta_{\sigma(2)}} \right) \left( \frac{1}{\sinh^2 \beta_{\sigma(3)}} - \frac{1}{\cosh^2 \beta_{\sigma(3)}} \right).
\]
By using Lemma 2.2 the right-hand side of (4.30) can be rewritten as \(E + F\), where
\[E = -4X + 8Y \tag{4.31}\]
and
\[F = 2 \left( \sum_{j=1}^{3} \frac{1}{\cosh^2 \beta_j} \right) X - \sum_{\sigma \in S_3} \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle \coth \beta_{\sigma(1)}}{\sinh^2 \beta_{\sigma(2)} \cosh^2 \beta_{\sigma(3)}}, \tag{4.32}\]
where the summation is over the symmetric group. Let us collect terms with \(\cosh^{-2} \beta_1\) in \(F\). We have
\[
\left( 2X + \frac{\omega^2 \coth \beta_2}{\sinh^2 \beta_3} - \frac{\omega^2 \coth \beta_3}{\sinh^2 \beta_2} \right) \frac{1}{\cosh^2 \beta_1}
\]
\[
\begin{align*}
&= \left( \frac{\sinh(\beta_2 - \beta_1) + \sinh \beta_1 \cosh \beta_2}{\sinh \beta_3} + \frac{\sinh(\beta_1 + \beta_3) - \sinh \beta_1 \cosh \beta_3}{\sinh \beta_2} \right) \frac{X}{\cosh^2 \beta_1} \\
&= \left( \frac{\sinh \beta_2}{\sinh \beta_3} + \frac{\sinh \beta_3}{\sinh \beta_2} \right) \frac{X}{\cosh \beta_1} = \omega^2 \left( \frac{1}{\sinh^2 \beta_2} + \frac{1}{\sinh^2 \beta_3} \right) \frac{\tan \beta_1}{\sinh^2 \beta_1}.
\end{align*}
\] (4.33)

Note that by multiplying the relation (2.14) in Lemma 2.7 through by $\tan \beta_1$ we rearrange (4.33) as
\[
\left( \frac{1}{\sinh^2 \beta_2} + \frac{1}{\sinh^2 \beta_3} \right) \frac{\tan \beta_1}{\sinh \beta_1} = \frac{\tan \beta_1}{\sinh^2 \beta_2 \sinh^2 \beta_3} = \frac{\tan \beta_1}{\sinh^2 \beta_2 \sinh^2 \beta_3} + 2 \omega^{-2} X.
\] (4.34)

Similarly, we collect and rearrange terms in $F$ with $\cosh^{-2} \beta_2$ and $\cosh^{-2} \beta_3$. Then by using variants of the identity (4.34) obtained by rotating $\beta$’s (see Remark 1) we get
\[
F = \sum_{\sigma} \frac{\langle \beta_{\sigma(2)}, \beta_{\sigma(3)} \rangle}{\sinh^2 \beta_{\sigma(2)} \sinh^2 \beta_{\sigma(3)}} \frac{\tan \beta_{\sigma(1)}}{\sinh \beta_{\sigma(1)}} + 6X = 4X - 8Y
\]
by Lemma 2.3. Hence $E + F = 0$ as required.

\section{5 Rational limit}

In the rational limit $\omega \to 0$ the operator $H_0$ takes the form
\[
H_0^r = -\Delta + \sum_{i=1}^{3} \left( \frac{m(m+1)\langle \tilde{\alpha}_i, \tilde{\alpha}_i \rangle}{\langle \tilde{\alpha}_i, x \rangle^2} + \frac{3m(3m+1)\langle \tilde{\beta}_i, \tilde{\beta}_i \rangle}{\langle \tilde{\beta}_i, x \rangle^2} \right),
\] (5.1)

where vectors $\tilde{\alpha}_i$, $\tilde{\beta}_i$ can be taken as the original vectors $\alpha_i$, $\beta_i$ with any fixed non-zero value of $\omega$. The Hamiltonian $H$ in the rational limit becomes
\[
H^r = -\Delta + \sum_{i=1}^{3} \left( \frac{m(m+1)\langle \tilde{\alpha}_i, \tilde{\alpha}_i \rangle}{\langle \tilde{\alpha}_i, x \rangle^2} + \frac{(3m+1)(3m+2)\langle \tilde{\beta}_i, \tilde{\beta}_i \rangle}{\langle \tilde{\beta}_i, x \rangle^2} \right).
\] (5.2)

And the explicit form of the intertwining operator $D$ in the rational limit is the operator $D^r = \lim_{\omega \to 0} \omega^{-3} D$ which takes the form
\[
D^r = \partial_{\tilde{\alpha}_1} \partial_{\tilde{\alpha}_2} \partial_{\tilde{\alpha}_3} - \sum_{\sigma} \frac{(3m+1)^2 \langle \tilde{\beta}_{\sigma(1),}, \tilde{\beta}_{\sigma(1)} \rangle}{\langle \tilde{\beta}_{\sigma(1)}, x \rangle^2} \partial_{\tilde{\beta}_{\sigma(2)}} \partial_{\tilde{\beta}_{\sigma(3)}} + \sum_{\sigma} \frac{(3m+1)^2 \langle \tilde{\beta}_{\sigma(1)}, \tilde{\beta}_{\sigma(1)} \rangle^2}{\langle \tilde{\beta}_{\sigma(2)}, x \rangle \langle \tilde{\beta}_{\sigma(3)}, x \rangle} \partial_{\tilde{\beta}_{\sigma(1)}}
\]
\[
- \sum_{i=1}^{3} \left( \frac{m(m+1)\prod_{k \neq i} \langle \tilde{\alpha}_i, \tilde{\beta}_k \rangle}{\langle \tilde{\beta}_i, x \rangle^3} + \frac{3m(3m+1)\prod_{k \neq i} \langle \tilde{\beta}_i, \tilde{\beta}_k \rangle}{\langle \tilde{\beta}_i, x \rangle^3} \right) \partial_{\tilde{\beta}_i}
\]
\[
+ \sum_{i=1}^{3} \frac{9m(m+1)(3m+1)\prod_{k=1}^{3} \langle \tilde{\beta}_i, \tilde{\beta}_k \rangle}{\langle \tilde{\beta}_i, x \rangle^3} + \sum_{i=1}^{3} \frac{m(m+1)(3m+1)\prod_{k \neq i} \langle \tilde{\alpha}_i, \tilde{\beta}_i \rangle}{\langle \tilde{\beta}_i, x \rangle^3} \langle \tilde{\beta}_i, x \rangle^2
\]
\[
- \frac{3(3m+1)(6m^2 + 6m + 1)\prod_{i=1}^{3} \langle \tilde{\beta}_i, \tilde{\beta}_i \rangle}{2\langle \tilde{\beta}_1, x \rangle \langle \tilde{\beta}_2, x \rangle \langle \tilde{\beta}_3, x \rangle}.
\] (5.3)

\textbf{Theorem 3.} The operators defined by formulas (5.1), (5.2) and (5.3) satisfy the intertwining relation
\[
H^r D^r = D^r H_0^r.
\] (5.4)
Similarly to the trigonometric case we derive quantum integrals in the factorised form, following [25]:

\[
[H^r, D^r D^{r*}] = 0, \quad [H^r_0, D^{r*} D^r] = 0.
\]

**Remark 2.** The operator \(H^r\) is the ordinary \(G_2\) Calogero–Moser operator with multiplicities \(m, 3m + 1\). Therefore the operator \(D^r\) is the rational version of the corresponding Opdam’s shift operator in a suitable gauge for the \(G_2\)-orbit containing \(\tilde{\beta}_i\) [26]. Hence the operator \(D^r\) can also be constructed via the product of the corresponding (rational) Dunkl operators \(\nabla_{\tilde{\beta}_1} \nabla_{\tilde{\beta}_2} \nabla_{\tilde{\beta}_3}\) as it was demonstrated by Heckman for any root system in [27].

### 6 Concluding remarks

We established integrability of the CMS system associated with the collection of vectors \(AG_2\) and an arbitrary value of the parameter \(m\). This configuration of vectors is interesting as it is an example of a slightly weakened notion of a root system. Indeed, the configuration is invariant under the Weyl group \(G_2\) and the root vectors belong to the invariant lattice but the crystallographic condition between the root vectors is no longer satisfied. This makes it harder to study the corresponding CMS system as, for instance, we could not define (trigonometric) Dunkl operators with good properties for the model \(AG_2\). Nonetheless integrability property appears to be present.

There are a number of further questions about this system. Firstly, it is natural to consider elliptic version and investigate its integrability. Secondly, it would be interesting to clarify whether the classical analogue of the system is integrable. In the case of root system \(G_2\) Lax pairs for the corresponding CMS model were constructed in [28], [29] (see also [30]), which may be a starting point for approaching classical \(AG_2\) CMS system. Another approach could be to investigate classical version of the quantum integral \(DD^{*}\). On the other hand let us consider the operator \(\hbar^2 H\) and take the limit \(\hbar \to 0, m \to \infty\) such that \(\hbar m \to const\). It is easy to see that the resulting classical Hamiltonian is the ordinary \(G_2\) Hamiltonian. This suggests that the classical analogue of \(H\) where potential is the same as in the quantum case may be non-integrable.

Thirdly, it would be interesting to investigate bispectrality of the considered Hamiltonian \(H\). More specifically, existence of the intertwining operator \(D\) implies that for integer \(m\) the Hamiltonian \(H\) has Baker–Akhiezer eigenfunction

\[
\psi(k, x) = D\phi(k, x), \quad H\psi(k, x) = (k_1^2 + k_2^2)\psi(k, x),
\]

where \(\phi(k, x)\) is the Baker–Akhiezer function for \(G_2\) CMS system [11, 10], and \(k = (k_1, k_2)\) is the spectral parameter. Bispectral dual Hamiltonian, if exists, would be an operator of Ruijsenaars–Macdonald type acting in \(k\)-variables of \(\psi(k, x)\) so that \(\psi(k, x)\) is its eigenfunction. In the root system case and for type \(A\) deformed CMS system such type of bispectrality is established in [31] (see also [32] for other examples).

We hope to return to some of these questions soon.

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