Crossover from adiabatic to sudden interaction quench in a Luttinger liquid

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Motivated by recent experiments on interacting cold atoms, we analyze interaction quenches in Luttinger liquids (LL), where the interaction is ramped from zero to a finite value within a finite time. The fermionic single particle density matrix reveals several regions of spatial and temporal coordinates relative to the quench time, termed as Fermi liquid, sudden quench LL, adiabatic LL regimes, and a LL regime with time dependent exponent. The various regimes can also be observed in the momentum distribution of the fermions, directly accessible through time of flight experiments. Most of our results apply to arbitrary quench protocols.

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Non-equilibrium dynamics and strong-correlation phenomena in quantum many body systems are topics at the forefront of contemporary physics. When these two fields are combined, namely when strongly correlated systems are driven out-of-equilibrium, we face a real challenge. Experimental advances on ultracold atoms1 have made many-body systems possible, and in particular, quantum quenches (SQ) of the interaction in LLs have become of particular relevance for cold atomic systems. Sudden quenches (SQ) of the interaction in LLs have been considered recently by several authors14–16, and the idea of the other extreme limit of adiabatic parameter ramps is often invoked, too. However, experimental ramps cannot take infinite time, and are not instantaneous, either. Here we study, how a nonzero quench time, τ ≠ 0, influences the final state of the system after a quantum quench. As we show, a finite τ leads to ‘heating’ effects, and generates excitations in the final state. Moreover, it amounts in the appearance of additional energy (∼ 1/τ) and corresponding length scales: while in certain space-time regions the system reveals universal near-equilibrium (adiabatic) correlations3, in other regimes renormalized Fermi liquid (FL) or sudden quench (SQ) behavior is found.

Thus motivated, let us study the LL Hamiltonian10

\[ H = \sum_{q \neq 0} \omega(q) b_q^\dagger b_q + \frac{g(q,t)}{2} [b_q b_{-q} + b_q^\dagger b_{-q}^\dagger], \]  

with \( \omega(q) = v|q| \) (v being the bare “sound velocity”), and \( b_q^\dagger \) the creation operator of a bosonic density wave.

The interaction \( g \) is changed from zero to a nonzero value within a quench time \( \tau \), \( g(q,t) = g_2(q)|Q(t)| \), with \( Q(t) \) encoding the explicit quench protocol, and satisfying \( Q(t > \tau) = 1 \) and \( Q(t < 0) = 0 \). In particular, for a linear quench, \( Q(t) = t\Theta(t(\tau-t))/\tau + \Theta(t-\tau) \) with \( \Theta(t) \) the Heaviside functions.

We describe time-evolution using the Heisenberg equation of motion, leading to

\[ i\partial_t b_q = [b_q, H] = \omega(q) b_q + g(q,t) b_{-q}^\dagger, \]  

and similarly, \( i\partial_t b_{-q}^\dagger = -\omega(q) b_{-q}^\dagger - g(q,t)b_q \). Solutions of these are of the form

\[ b_q(t) = u(q,t) b_q(0) + v^*(q,t) b_{-q}^\dagger(0), \]  

where all the time dependence is carried by the prefactors, \( u(q,t) \) and \( v(q,t) \), and the operators on the r.h.s. refer to non-interacting bosons before the quench. All expectation values are thus taken in terms of the initial density matrix of the latter (or vacuum at \( T = 0 \)). The bosonic nature of the quasiparticles requires
\[ |u(q,t)|^2 - |v(q,t)|^2 = 1. \] From Eqs. 2-3, we obtain
\[ i\partial_t \begin{bmatrix} u(q,t) \\ v(q,t) \end{bmatrix} = \begin{bmatrix} \omega(q) & g(q,t) \\ -g(q,t) & -\omega(q) \end{bmatrix} \begin{bmatrix} u(q,t) \\ v(q,t) \end{bmatrix}, \tag{4} \]
with the initial condition \( u(q,0) = 1, \ v(q,0) = 0 \). Since both \( \omega(q) \) and \( g(q,t) \) are even functions of \( q \), \( u(q,t) \) and \( v(q,t) \) must be so, too. By Eq. 4, all time dependence has been transferred to the Bogoliubov coefficients, and therefore expectation values of the time dependent bosonic modes and non-equilibrium dynamics are calculable using standard techniques developed for equilibrium 14, once the solutions of Eq. 4 are known.

Before discussing a continuous quench, let's see how limiting cases are recovered from Eq. 4. The adiabatic limit follows from replacing \( g(q,t) \) with its time independent final value, and looking for the stationary solutions of Eq. 4 at a given energy while ignoring the initial conditions. The SQ limit requires only the replacement of Eq. (4) at a given energy while looking for the stationary solutions for \( q > 0 \) with the initial condition \( u(q,0) = 1, \ v(q,0) = 0 \) for all \( t \) and \( q \), and solving Eq. (4) perturbatively in the interaction. To lowest order in \( g_2(q) \), we obtain \( u(q,t) \approx \exp(-i\omega(q)t) \) and
\[ v(q,t > 0) \approx i \int_0^t dt' g(q,t') \exp(i\omega(q)(t-2t')). \tag{5} \]
Higher order corrections to \( u(q,t) \) and \( v(q,t) \) are of order \( g_2^2(q) \) and \( g_2^2(q) \), respectively. We have also checked numerically that Eq. 5 is indeed applicable for any \( t \) and \( \tau \), as long as \( g_2(q) \ll v \) \[24\]. In the SQ \( (\tau \to 0) \) and adiabatic \( (\tau \to \infty) \) limits we obtain
\[ v(q,t > \tau) \approx \frac{g_2(q)}{2v} \times \left\{ \begin{array}{ll} 2i \sin(\omega(q)t) & \text{for } \tau \to 0, \\ -\exp(-i\omega(q)t) & \text{for } \tau \to \infty, \end{array} \right. \tag{6} \]
reproducing to lowest order in \( g_2(q) \) the SQ results 14, 16 and the equilibrium Bogoliubov transformation 10, 21, respectively.

We are now in position to obtain information about physical observables. We start with the evolution of the total energy of the system. We take the energy of our initial vacuum state to be zero. In the fermionic setting, this corresponds to measuring the energy with respect to the energy of the non-interacting Fermi sea. The expectation value of Eq. 1 is then evaluated in the Heisenberg picture, where the expectation value is taken with the non-interacting ground state, and \( u(q,t) \) and \( u(q,t) \) as obtained from Eq. 3 keeping track of the time evolution of the system. We thus obtain for \( \langle H(t) \rangle \) after the quench \( (t > 0) \)
\[ \langle H \rangle = \sum_{q \neq 0} \omega(q)n_B(q) + (2n_B(q) + 1)\text{Im}[v^*(q,t)\partial_t v(q,t)], \]
with \( n_B(q) = 1/(\exp(\omega(q)/T) - 1) \) the Bose function. The expression above is time independent for \( t > \tau \), as expected. At \( T = 0 \) and \( t > \tau \), and an interaction of finite range, \( g_2(q) = g_2 \exp(-R_0|q|/2) \), we obtain
\[ \langle H \rangle = E_{gs} \left[ 1 - \left( \frac{\tau_0}{\tau} \right)^2 \ln \left( 1 + \left( \frac{\tau}{\tau_0} \right)^2 \right) \right] \tag{7} \]
for a linear quench. Here we introduced the microscopic time scale, \( \tau_0 \equiv R_0/2v \), and \( E_{gs} = -Lg_2^2/4v^2R_0^2 \) is the adiabatic ground state energy shift to lowest order in \( g_2 \), with \( L \) the system size. The second term corresponds to quasiparticle excitations resulting from the finite quench speed. In the SQ limit, \( \tau \ll \tau_0 \), the energy of the system is only slightly shifted \[22\], \( \langle H \rangle = E_{gs}(\tau/\tau_0)^2/2 \). This holds true for a general quench, i.e. \( \langle H \rangle \sim (\tau/\tau_0)^2 \) when \( \tau \to 0 \) with a quench dependent coefficient. In the adiabatic limit, \( \tau \gg \tau_0 \), on the other hand, the excess energy (or “heating”) vanishes as \(-2E_{gs}\ln(\tau/\tau_0)^2 \tau^2/2 \) in accord with the so-called analytic response of Ref. \[23\]. This remains valid for general smooth quenches displaying kink(s) (discontinuity in the derivative) and bounded \( \partial_t Q(t) \). Smooth quenches without kinks but with bounded \( \partial_t Q(t) \) produce also a universal decay as \( \sim 1/\tau^2 \), while the \( \tau \) dependence of the heating becomes non-universal for protocols with a diverging \( \partial_t Q(t) \). \[24\]. The crossover between the SQ and adiabatic limits occurs when \( \tau \sim \tau_0 \), which typically translates to \( \tau \sim 1/J \) in an optical lattice, with \( J \) the hopping integral in the underlying microscopic lattice Hamiltonian.

In the fermionic context, the structure of the non-equilibrium dynamics can be well demonstrated by means of the fermionic one-particle density matrix. Since the fermion field decomposes to right-going and a left-going part \( \Psi(x) = e^{-ikr_F x} \Psi_r(x) + e^{-ikr_F x} \Psi_l(x) \), it is enough to concentrate on the right-going part of the density matrix,
\[ G_r(x,t) \equiv \langle \Psi^*_r(x,t)\Psi_r(0,t) \rangle, \]
describing excitations around the right Fermi momentum, \( k \approx k_F \). The right-going field, \( \Psi_r(x) \), can be expressed in terms of the LL bosons as \[10\]
\[ \Psi_r(x) = \frac{\eta_r}{\sqrt{2\pi}a} \exp(i\phi_r(x)), \tag{8} \]
where \( \eta_r \) denotes the Klein factor, and \( \phi_r(x) = \sum_{q>0} \sqrt{2\pi/|q|} Le^{iqx-a|q|/2}b_q + h.c. \). Following standard
one-particle density matrix exhibits universal properties, \( Q \), that, independently of the quench protocol, \( \gg \tau \). The prefactor note the perturbative sudden quench and adiabatic extensions of energy \( \gamma \) for \( \tau > \tau_0 \), while the short distance behavior is described by the adiabatic exponent. It is remarkable that for slow quenches, \( \tau \gg \tau_0 \), the quench time manifests itself explicitly through an adiabatically enhanced prefactor \( A \sim (\tau/\tau_0)^\gamma_{ad} \) of the asymptotic tail as also shown in Fig. 1. Thus while the spatial decay of Eq. (10) contains the SQ exponent, its dependence on the quench protocol[25] as \( n(k, t) \) exhibits a jump of size \( \sim Z(t) \) at \( k = k_F \), while it approximately scales for \( |k| \gg 1/2v\tau \) as

\[
n(k) - \frac{1}{2} \sim -\text{sign}(\tilde{k}) \times \left\{ \begin{array}{ll}
A(\tau/\tau_0) |\tilde{k} R_0|^{\gamma_{SQ}}, & |\tilde{k}| \ll \frac{1}{2v\tau} \\
|\tilde{k} R_0|^{\gamma_{ad}}, & |\tilde{k}| \gg \frac{1}{2v\tau},
\end{array} \right.
\]

(12)

for \( \tilde{k} \equiv k - k_F, |\tilde{k}| \ll k_F \), and \( t \gg \tau \). Thus the time scale of the quench is also imprinted in the momentum distribution, which also shows a cross-over behavior between the SQ and the adiabatic limits. For adiabatic quenches, \( \tau \to \infty \), we recover the equilibrium LL exponent, while close to \( k_F \), the momentum distribution is enhanced by a factor \( A(\tau/\tau_0) \) compared to the SQ behavior[14, 16].

The above analysis can be extended to the short time region, \( t \ll \tau \), where the behavior found depends explicitly on the quench protocol[25] as

\[
G_r(x, t) \sim G^0_r(x) \left( \frac{R_0}{\min\{|x|, 2v\tau\}} \right)^{\gamma(t)},
\]

(13)

where \( \gamma(t) = g_2^2 Q^2/2v^2 + \ldots \). For short distances, \( |x| \ll 2v\tau \), the spatial correlations decay with a time-dependent exponent, and this region can thus be characterized as a weakly interacting LL (t-LL). For \( |x| \gg 2v\tau \), on the other hand, similar to \( t \gg \tau \), correlations remain almost unaffected by interaction, and a Fermi liquid regime is found. For \( t \ll \tau_0 \), \( Z(t) \approx 1 \) as in the initial
Fermi gas, but for \( t \gg \tau_0 \) we recover a Fermi liquid behavior with a reduced quasiparticle weight as

\[
Z(\tau_0 \ll t \ll \tau) \sim \left( \frac{\tau_0}{\tau} \right)^{\gamma(t)} \tag{14}
\]

The quasiparticle weight thus slowly decreases during the quench, and excitations remain similar to those in the initial Fermi gas with a reduced weight \((Z < 1)\) for \( t < \tau \). After the quench, \( t > \tau \), the quasiparticle weight continues to decrease as a power-law, and it resembles the equilibrium and SQ LL exponents. A finite quasiparticle residue is retained during the quench, reflecting the Fermi gas nature of the initial state, getting suppressed gradually after the quench. The variety of quench in-ternal and external parameters include the momentum distribution, \( \langle |p| \rangle \), the Fermi energy, \( E_F \), and the quench time, \( \tau_0 \). The resulting dynamics is largely influenced by the finite quench time for fermions, and in particular, the momentum distribution exhibits a crossover from the adiabatic LL to that of the SQ with the extra adiabatic enhancement factor \((\tau/\tau_0)^{\alpha}\), revealing both the equilibrium and SQ LL exponents. A finite quasiparticle residue is retained during the quench, reflecting the Fermi gas nature of the initial state, getting suppressed gradually after the quench. The variety of quench induced phases offers a unique opportunity to design low dimensional correlated states on demand.

In summary, we have studied continuous interaction quenches in LL, bridging smoothly between the SQ and adiabatic limits. The resulting dynamics is largely influenced by the finite quench time for fermions, and in particular, the momentum distribution exhibits a crossover from the adiabatic LL to that of the SQ with the extra adiabatic enhancement factor \((\tau/\tau_0)^{\alpha}\), revealing both the equilibrium and SQ LL exponents. A finite quasiparticle residue is retained during the quench, reflecting the Fermi gas nature of the initial state, getting suppressed gradually after the quench. The variety of quench induced phases offers a unique opportunity to design low dimensional correlated states on demand.

We thank A. Polkovnikov for stimulating comments. B. D. thanks for the hospitality of MPIPKS in Dresden. This research has been supported by the Hungarian Scientific Research Funds Nos. K72613, K73361, and the momentum distribution has been measured in time of flight (ToF) experiments in 2D and 3D Fermi gases. Therefore, by applying ToF imaging or momentum resolved rf spectroscopy, the observation of the momentum distribution of Eq. (12) is within reach for 1D fermions. Furthermore, the specific momentum distribution of a LL has already been observed in the Tonks-Girardeau limit of 1D Bose systems, which exhibit fermionic properties in this strongly interacting regime, and reveal after an interaction quench features similar to the ones found for fermions.

In summary, we have studied continuous interaction quenches in LL, bridging smoothly between the SQ and adiabatic limits. The resulting dynamics is largely influenced by the finite quench time for fermions, and in particular, the momentum distribution exhibits a crossover from the adiabatic LL to that of the SQ with the extra adiabatic enhancement factor \((\tau/\tau_0)^{\alpha}\), revealing both the equilibrium and SQ LL exponents. A finite quasiparticle residue is retained during the quench, reflecting the Fermi gas nature of the initial state, getting suppressed gradually after the quench. The variety of quench induced phases offers a unique opportunity to design low dimensional correlated states on demand.

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[1] I. Bloch, et al., Rev. Mod. Phys. 80, 885 (2008).
[2] P. Calabrese, and J. Cardy, Phys. Rev. Lett. 96, 136801 (2006).
[3] M. Rigol, et al., Nature 452, 854 (2008).
[4] J. Dziarmaga, Adv. Phys. 59, 1063 (2010).
[5] A. Polkovnikov, et al., arXiv:1007.5331.
[6] S. Hofferberth, et al., Nature 449, 324 (2007).
[7] E. Haller, et al., Nature 466, 597 (2010).
[8] T. Kinoshita, et al., Nature 440, 900 (2006).
[9] B. Paredes, et al., Nature 429, 277 (2004).
[10] T. Giamarchi, Quantum Physics in One Dimension (Oxford University Press, Oxford, 2004).
[11] A. O. Gogolin, et al., Bosonization and Strongly Correlated Systems (Cambridge University Press, Cambridge, 1998).
[12] B. D. Gutman, et al., Phys. Rev. B 81, 085436 (2010).
[13] E. Perfetto, et al., Phys. Rev. Lett. 105, 156802 (2010).
[14] M. A. Cazalilla, Phys. Rev. Lett. 97, 156403 (2006).
[15] G. S. Uhrig, Phys. Rev. A 80, 061602(R) (2009).
[16] A. Iucci, and M. A. Cazalilla, Phys. Rev. A 80, 063619
To avoid instabilities, \( v > |g_2(q)| \) is assumed.

For a linear quench, we can integrate Eq. (5) in the main text to obtain
\[
v(q,t) = \frac{g_2(q)|q|}{2\omega^2(q)\tau} \left[ \sin(\omega(q)t) - \omega(q)t \exp(-i\omega(q)t) \right]
\]
for \( 0 < t < \tau \), and
\[
v(q,t) = \frac{ig_2(q)|q|}{4\omega^2(q)\tau} \left[ \exp(i\omega(q)(t-2\tau)) - \exp(i\omega(q)t) + 2i\omega(q)\tau \exp(-i\omega(q)t) \right]
\]
for \( t > \tau \).

The exponent in the one-particle density matrix in Eq. (9) in the main text is then evaluated in closed form for \( t > \tau, T = 0 \) and \( L \to \infty \) as
\[
- \sum_{q > 0} \left( \frac{2\pi}{qL} \right) 4 \sin^2 \left( \frac{qx}{2} \right) |v(q,t)|^2 = -\frac{g_2^2}{v^2}(I_1(\tau, x, R_0) + I_2(t, \tau, x, R_0)),
\]
where
\[
I_1(\tau, x, R_0) = \int_0^\infty dq \frac{\exp(-R_0q)}{q^3\omega^2(q)\tau^2} \sin^2 \left( \frac{qx}{2} \right) \left[ \sin^2(\omega(q)\tau) + (\omega(q)\tau)^2 \right],
\]
\[
I_2(t, \tau, x, R_0) = \int_0^\infty dq \frac{\exp(-R_0q)}{q^2\omega^2(q)\tau^2} \sin^2 \left( \frac{qx}{2} \right) \left[ -\sin(2\omega(q)t) + \sin(2\omega(q)(t-\tau)) \right],
\]
which are evaluated as
\[
I_1(\tau, x, R_0) = \frac{1}{32(\nu\tau)^2} \left[ \sum_{r, s = \pm 1} \left\{ \ln(2iv\nu\tau + R_0 + isx)(sx - iR_0 + r2v\tau)^2 \right\} + 
+ 2 \sum_{s = \pm 1} \left\{ \ln(R_0 + isx)(R_0 + isx)^2 + (2\nu\tau)^2 \right\} + \ln(2isv\nu\tau + R_0)(R_0 + 2isv\nu\tau)^2 \right] - 4 \ln(R_0)[R_0^2 + (2\nu\tau)^2]
\]
and
\[
I_2(t, \tau, x, R_0) = \frac{1}{2} \ln \left( R_0^2 + 4\nu^2(\tau - t)^2 \right) - \sum_{s = \pm 1} \frac{1}{4} \ln \left( R_0^2 + (2\nu\tau - 2vt - sx)^2 \right) + 
+ \frac{i}{8\nu\tau} \sum_{s = \pm 1} \left[ 2(2iv\nu\tau - sR_0) \ln \left( \frac{sR_0 + 2iv(\tau - t)}{sR_0 - 2ivt} \right) + \sum_{r = \pm 1} (R_0 - 2ivt + isx) \ln \left( \frac{R_0 + i(2\nu\tau - 2vt + sx)}{R_0 - i(2\nu\tau - 2vt + sx)} \right) \right].
\]
The one-particle fermionic density matrix of Eq. (9) in the main text at dependence is used to generate Fig. 1. in the main text.

For arbitrary quench protocols lead to the same behaviour. For arbitrary quench protocol \( Q \), the asymptotic expansion of the exact results for a linear quench.

In this section, we demonstrate that the asymptotic behavior what we obtained for a linear quench is universal, and arbitrary quench protocols lead to the same behaviour. For arbitrary quench protocol \( Q(t) \), the exponent in the one-particle fermionic density matrix of Eq. (9) in the main text at \( T = 0 \) and \( L \to \infty \) can be rewritten as

\[
I = -\sum_{q > 0} \left( \frac{2\pi}{qL} \right) 4\sin^2 \left( \frac{qx}{2} \right) |v(q, t)|^2 = -\left( g_2 \frac{qL}{v^2} \right)^2 (I_1(t, x, R_0) + I_2(t, t, x, R_0)).
\]  

(22)

Expanding these in various limits are used to obtain the results cited in the paper, and their general \( (t, \tau, x, R_0) \) dependence is used to generate Fig. 1. in the main text.

**EVALUATION OF THE ASYMPTOTICS OF BOSONIC CORRELATOR OF EQ. (9) IN THE MAIN TEXT FOR GENERAL QUENCH PROTOCOL**

In this section, we demonstrate that the asymptotic behavior what we obtained for a linear quench is universal, and arbitrary quench protocols lead to the same behaviour. For arbitrary quench protocol \( Q(t) \), the exponent in the one-particle fermionic density matrix of Eq. (9) in the main text at \( T = 0 \) and \( L \to \infty \) can be rewritten as

\[
I = -\sum_{q > 0} \left( \frac{2\pi}{qL} \right) 4\sin^2 \left( \frac{qx}{2} \right) |v(q, t)|^2 = -\left( g_2 \frac{qL}{v^2} \right)^2 \int_0^t dt_1 \int_0^t dt_2 Q(t_1)Q(t_2)\partial_t \partial_t f(t_1 - t_2).
\]  

(23)

where

\[
f(t) = \ln \left( 1 + \frac{x^2}{(R_0 - 2ivt)^2} \right)
\]  

(24)

After partial integrations, it takes the form

\[
I = -\frac{g_2^2}{4v^2} \left( Q^2(t)f(0) - 2Q(t)\int_0^t dt_1 Q'(t_1)\text{Re} f(t_1 - t) + \int_0^t dt_1 Q'(t_1) \int_0^t dt_2 Q'(t_2)f(t_1 - t_2) \right).
\]  

(25)

Here, by using \( Q(t > \tau) = 1 \), the upper limit of integration would reduce to \( \min\{t, \tau\} \). However, our considerations remain valid for smooth quench functions as well, reaching 1 only asymptotically, i.e \( Q(t \gg \tau) \to 1 \). Let us first consider the properties of the steady state in the limit of \( t \gg (\tau, x/v) \), when the middle term in Eq. does not contribute, and \( Q(t \gg \tau) \approx 1 \). For \( t \gg |x/v| \gg \tau \), the first term yields \( 2\ln(|x|/R_0) \), while the last integral produces similar spatial decay and the adiabatic enhancement as \( 2\ln(|x|/2v\tau) \). The exponent is

\[
I = -\frac{g_2^2}{v^2} \ln \left| \frac{x}{\sqrt{R_0}2v\tau} \right| \text{ for } t \gg |x/v| \gg \tau.
\]  

(26)

In the \( t \gg \tau \gg |x/v| \) limit, the last term also vanishes, and we are left with the adiabatic exponent

\[
I = -\frac{g_2^2}{v^2} \ln \left| \frac{x}{R_0} \right| \text{ for } t \gg \tau \gg |x/v|.
\]  

(27)

The other limit of interest is \( |x| \gg (v\tau, vt) \), when the exponent simplifies in Eq. with \( f(t) \) replaced by \( f_1(t) = -2\ln(R_0 - 2ivt) \). The first term always only contributes with a constant, \( -2\ln(R_0) \). In the limit of \( |x| \gg vt \gg v\tau \), \( Q(t) = 1 \), the third term becomes independent of both \( x \) and \( t \) and gives rise to the adiabatic enhancement factor as \( -2\ln(\tau) \), and only the second term determines the temporal decay. The exponent is obtained as

\[
I = -\frac{g_2^2}{v^2} \ln \left( \frac{t}{\sqrt{\tau}v} \right) \text{ for } |x| \gg vt \gg v\tau.
\]  

(28)

In the limit of \( |x| \gg v\tau \gg vt \), i.e. during the quench, the time dependence of \( Q(t) \) is essential. The second term yields \( 4Q^2(t)\ln(t) \) to leading order, while the third term gives \(-1/2 \) times the second term. Altogether, the exponent reads as

\[
I = -\frac{g_2^2}{v^2} Q^2(t) \ln \left( \frac{t}{\tau_0} \right) \text{ for } |x| \gg v\tau \gg vt.
\]  

(29)

These agree with the asymptotic expansion of the exact results for a linear quench.