STABILITY ESTIMATES FOR SINGULAR SDES AND APPLICATIONS

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Abstract. We consider multidimensional SDEs with singular drift \( b \) and Sobolev diffusion coefficients \( \sigma \), satisfying Krylov–Röckner type assumptions. We prove several stability estimates, comparing solutions driven by different \((b', \sigma')\), both for Itô and Stratonovich SDEs, possibly depending on negative Sobolev norms of the difference \( b^1 - b^2 \). We then discuss several applications of these results to McKean–Vlasov SDEs, criteria for strong compactness of solutions and Wong–Zakai type theorems.

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1. Introduction

In this paper we consider multidimensional singular SDEs of the form

\[
\begin{align*}
\mathrm{d}X_t = b_t(X_t) \, \mathrm{d}t + \sigma_t(X_t) \, \mathrm{d}W_t, \quad X_{t=0} = X_0
\end{align*}
\]

where \( b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, \sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d} \) and \( W \) is a \( d \)-dimensional Brownian motion. The interval \([0, T]\) is finite, although arbitrarily large, and the initial condition \( X_0 \) is possibly random and independent of \( W \). The drift \( b \) is integrable, possibly unbounded; \( b \) and \( \sigma \) will be always assumed to satisfy relatively standard assumptions under which strong well-posedness is known to hold. To properly formulate them, let us first define for \( \beta \in [0, 1] \) the sets

\[
\mathcal{J}_\beta := \left\{ (p, q) : p, q \in \left( \frac{2}{2-\beta}, \infty \right), \frac{2}{q} + \frac{d}{p} < 2 - \beta \right\};
\]

we will be mostly interested in considering \( \mathcal{J}_0 \) and \( \mathcal{J}_1 \).

Assumption 1.1. The following hold.

\((\mathcal{H}^b)\) There exist \((p_1, q_1) \in \mathcal{J}_1\) such that \( b \in L_{t}^{p_1} L_{x}^{q_1} \).

\((\mathcal{H}_1^\sigma)\) There exists a constant \( K \) such that \( \sigma \) is uniformly bounded and nondegenerate, namely

\[
K^{-1} |\xi|^2 \leq |\sigma^*(t, x) \xi|^2 \leq K|\xi|^2 \quad \forall \xi \in \mathbb{R}^d, (t, x) \in [0, T] \times \mathbb{R}^d.
\]

\((\mathcal{H}_2^\sigma)\) \( \sigma \) is uniformly continuous in space, uniformly in time, with modulus of continuity \( h \), in the sense that

\[
|\sigma(t, x) - \sigma(t, y)| \leq h(|x - y|) \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}^{2d}.
\]
Let schemes for singular SDEs are considered; although technically it doesn’t contain stability this work is hugely inspired by the previous work \[19\] by one of the authors, where numerical integrability coefficients are generalization of all of the aforementioned results, allowing for any value \(m\) estimates can be used to solve McKean–Vlasov SDEs. Theorem 1.2 can be considered a great fBm; however in the Brownian setting knowledge, the rst paper to consider stability estimates in negative Sobolev norms (namely for suitable SDE is also allowed to have a multiplicative Lévy noise; however, the condition general stability estimate, true for any \(\sigma\) in Assumption 1.1 as well as parameters (\(b\), \(\sigma\)) in \(\mathcal{F}_1\), but again the resulting estimate is only in function of \(\sigma\) \(\sigma\) needed to show well-posedness of the corresponding Kolmogorov equation, and the focus on \(m = 2\). Later, again for \(\sigma \neq \sigma\), the stability result was generalized in \[20\], for any values \((p, q) \in \mathcal{F}_1\), but again the resulting estimate is only in function of \(\sigma\) \(\sigma\) holds for \(m = 2\). A quite general stability estimate, true for any \(m \in [1, \infty)\), is presented in \[30\], where the SDE is also allowed to have a multiplicative Lévy noise; however, the condition \(\nabla b^1 \in L^{q_1}_x L^p_x\) for suitable \(q', p' \in (1, \infty)\) must be assumed to obtain the stability result. To the best of our knowledge, the rst paper to consider stability estimates in negative Sobolev norms (namely in the style of \(1.3\)) is \[8\], which considers the more general framework of SDEs driven by fBm; however in the Brownian setting \(H = 1/2\), it only allows for additive noise \(\sigma = 1\) and drifts \(b\) of spatial regularity at least \(C^\alpha\) for some \(\alpha > 0\). The same work also shows how such estimates can be used to solve McKean–Vlasov SDEs. Theorem 1.2 can be considered a great generalization of all of the aforementioned results, allowing for any value \(m \in [1, \infty)\), different integrability coefficients \((p_1, q_1) \neq (p_2, q_2)\) for \(b\) and \(\sigma\) in Assumption 1.1 as well as parameters \((\tilde{p}_1, \tilde{q}_1), (\tilde{p}_2, \tilde{q}_2)\) different from them in the estimates \(1.2\) and \(1.3\). Let us finally mention that this work is hugely inspired by the previous work \[19\] by one of the authors, where numerical schemes for singular SDEs are considered; although technically it doesn’t contain stability

\[(\mathcal{H}_3^p)\]  
There exist \((p_2, q_2) \in \mathcal{F}_1\) such that \(\nabla \sigma \in L^{q_2}_x L^p_x\).

It is well established that under Assumption 1.1, the SDE \(1.1\) is strongly well-posed for any initial \(X_0\) independent of \(W\), see e.g. \[29\] and the basic recap in Section 2.1. In this work we are instead interested in deriving stability estimates, in the sense of comparing two distinct solutions \(X^1, X^2\) driven by the same \(W\) but with respect to different initial conditions, drifts and diffusions. The next statement exemplifies some of our main ndings in this regard.

**Theorem 1.2.** Let \((b^1, \sigma^1), (b^2, \sigma^2)\) satisfy Assumption 1.1 for the same parameters \((p_1, q_1), (p_2, q_2)\), \(K, h\) and denote by \(X^1, X^2\) the associated solutions to \(1.1\). Then for any \(m \in [1, \infty), T > 0\) and any \((\tilde{p}_1, \tilde{q}_1) \in \mathcal{F}_0, (\tilde{p}_2, \tilde{q}_2) \in \mathcal{F}_1\) there exists a constant \(C\) (depending on all the aforementioned parameters as well as \(d, \|b^1||_L^{q_1}L^{p_1}\) and \(\|\nabla \sigma^1||_L^{q_2}L^{p_2}\)) such that

\[
\sup_{t \in [0, T]} \left\| X^1_t - X^2_t \right\|_{L^m_x} \leq C \left[ \|X^1_0 - X^2_0\|_{L^m_x} + \|b^1 - b^2\|_{L^\tilde{q}_1X^1X^2} + \|\sigma^1 - \sigma^2\|_{L^{q_2}_x L^p_x} \right]. \tag{1.2}
\]

If moreover \(q_2 = \infty\) and \(4/q_1 + d/p_1 < 1\), we have

\[
\sup_{t \in [0, T]} \left\| X^1_t - X^2_t \right\|_{L^m_x} \leq \tilde{C} \left[ \|X^1_0 - X^2_0\|_{L^m_x} + \|b^1 - b^2\|_{L^{q_1}W^{1-1/p_1}_x} + \|\sigma^1 - \sigma^2\|_{L^{q_2}L^p_x} \right] \tag{1.3}
\]

where \(\tilde{C}\) is another positive constant with similar dependence on the parameters.
estimates, it is clear that the quantity \( \varphi_n \) appearing in [19, Theorem 1.1] is closely related to the error committed by comparing two SDEs driven respectively by drift \( b^n \) and \( b \) respectively, in the case of common diffusion \( \sigma \). We refer to Section 3.4.1 for a deeper discussion on this point.

The proof of Theorem 1.2 will be presented in Section 2.3 and is in fact consequence of a more abstract and much more general stability result, see Proposition 2.11; the same strategy of proof allows for numerous variants, see Remark 2.13, as well as similar results for Stratonovich SDEs, cf. Corollary 2.12.

In order to motivate the importance of results in the style of Theorem 1.2, we will present in Section 3 several applications, ranging from existence and uniqueness results for distribution dependent (or McKean–Vlasov) SDEs, criteria for strong compactness of solutions to singular SDEs and Wong–Zakai type theorems. We will also shortly discuss in Section 3.4 other applications of such results which have already appeared in the literature, although without it being stated explicitly; indeed, we believe one of nice contributions of this work is also to highlight the importance of stability estimates for SDEs in a wide range of problems, which have so far being treated without recognizing their key role in them.

Remark 1.3. For simplicity in this paper we only considered drifts with global bounds in \( L^q_t L^p_x \). Up to technical details, we expect the same strategy to work in many other cases, including:

a) \( b, \nabla \sigma \in L^q_t L^p_x \) where \( L^p_x \) is the so called localized \( L^p_x \) considered in [29]; one of the advantages of this case is that bounded \( b \) is covered as well.

b) The case of \( b \) of the form \( b = b^1 + \ldots + b^n \) with \( b^j \in L^q_t L^p_x \) with \( (q_i, \rho_i) \in J_i \).

c) The presence of an additional Lipschitz term, i.e. \( b = b^1 + b^2 \) with \( b^1 \in L^q_t L^p_x \) and \( b^2 \in L^\infty_t C^1 \) as in [31].

d) Coefficients belonging to mixed-normed spaces, i.e. \( b \in L^q_t L^p_{x,1} \ldots L^p_{x,d} \) with \( 2/q + \sum_i 1/p_i < 1 \), as considered in [21].

Structure of the paper. We start by recalling in Section 2.1 some basic facts related to the SDE (1.1) and deriving some key lemmas needed for later; we then pass to prove several stability results, including Theorem 1.2, in Section 2.2 and Section 2.3. We then present several applications in Section 3: well-posedness for some classes of McKean-Vlasov SDEs in Section 3.1; strong compactness of solutions to singular SDEs Section 3.2; finally, a Wong–Zakai theorem for singular SDEs in Section 3.3. We conclude with Appendix A, containing more details on the proof of the Wong–Zakai theorem under classical regularity conditions, i.e. \( b \in C^1 \) and \( \sigma \in C^2 \).

Notations and conventions. We always work on a finite time interval \([0, T]\). We write \( a \leq b \) to mean that there exists a positive constant \( C \) such that \( a \leq Cb \); we use the index \( a \leq b \) to highlight the dependence \( C = C(\lambda) \). For vectors \( x, y \in \mathbb{R}^d \), we write \( x \cdot y := \sum_{i=1}^n x_i y_i \) and \( |x| = \sqrt{x \cdot x} \); for matrices \( A, B \in \mathbb{R}^{d \times d} \), we use the convention \( A : B := \sum_{1 \leq i, j \leq n} A_{ij} B_{ij} \). Given a real number \( z \), we denote by \( \lfloor z \rfloor \) and \( \{ z \} \) respectively its integer and fractional parts.

For any \( N \in \mathbb{N} \) and \( p \in [1, \infty] \), we denote by \( L^p(\mathbb{R}^d; \mathbb{R}^N) \) the standard Lebesgue space; when there is no risk of confusion in the parameter \( N \), we will simply write \( L^p_x \) for short and denote by \( \| \cdot \|_{L^p_x} \) the corresponding norm. Similarly for the Bessel potential spaces \( W^{\beta,p}_x = W^{\beta,p}(\mathbb{R}^d; \mathbb{R}^N) \),
which are defined for $\beta \in \mathbb{R}$, with corresponding norm

$$
\|f\|_{W^{\beta,p}} := \|(\mathbf{I} - \Delta)^{\beta/2} f\|_{L^p_x};
$$

For $\alpha \in [0, +\infty)$, $C^\alpha = C^\alpha(\mathbb{R}^d; \mathbb{R}^N)$ stands for the usual Hölder continuous function space, made of continuous bounded functions with with continuous and bounded derivatives up to order $[\alpha] \in \mathbb{N}$ and with globally $(\alpha)$-Hölder continuous derivatives of order $[\alpha]$.

We denote by $C_T = C([0, T]; \mathbb{R}^d)$ the path space of continuous functions on $[0, T]$, endowed with the supremum norm $\|\varphi\|_{C_T} = \sup_{t \in [0, T]} |\varphi_t|$.

Given a Banach space $E$ and a parameter $q \in [1, \infty]$, we denote by $L^q_{t,x} = L^q(0, T; E)$ the space of measurable functions $f : [0, T] \to E$ such that

$$
\|f\|_{L^q_{t,x}} := \left( \int_0^T \|f_t\|_E^q \, dt \right)^{\frac{1}{q}} < \infty
$$

with the usual convention of the essential supremum norm in the case $q = \infty$. Similarly, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $m \in [1, \infty)$, we denote by $L^m_{\omega} = L^m(\Omega, \mathcal{F}, \mathbb{P}; E)$ the space of $E$-valued $\mathcal{F}$-measurable random variables $X$ such that

$$
\|X\|_{L^m_{\omega}} := \mathbb{E}\left[ \|X\|^m \right]^{\frac{1}{m}} < \infty
$$

where $\mathbb{E}$ denotes expectation w.r.t. $\mathbb{P}$. The above definitions can be concatenated by choosing at each step a different $E$, so that one can define $L^m_{\omega}C_T, L^q_{t,x}L^p_{t,x}$ and so on. Whenever $q = p$, we might write for simplicity $L^p_{t,x}$ in place of $L^p_{t,x}L^p_{t,x}$.

2. Stability estimates

2.1. Preliminaries. We shortly recall here several facts related to the SDE (1.1) which will be needed in the sequel, including Girsanov theorem, Zvonkin transform and Khasminskii type estimates. They are by now very classical tools in the study of singular SDEs, see for instance [29] for a nice self-contained presentation (in a slightly different setting).

We start by recalling that any solution $X$ can be regarded after a change of measure as a solution to a driftless SDE, thanks to Girsanov theorem.

Lemma 2.1. Let $(b, \sigma)$ satisfy Assumption 1.1 and let $X$ be the unique solution to (1.1). Then there exists another measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, and a $\mathbb{Q}$-Brownian motion $\tilde{W}$ such that $X$ solves the SDE $dX_t = \sigma_t(X_t) \, d\tilde{W}_t$. Furthermore it holds

$$
\mathbb{E}_\mathbb{P}\left[ \left( \frac{d\mathbb{Q}}{d\mathbb{P}} \right)^n \right] < \infty \quad \forall \, n \in \mathbb{Z}.
$$

The next classical ingredient we need are suitable estimates for parabolic PDEs of the form (2.1), strictly related to the so called Zvonkin transform. In the next statement, we are adopting the notations $a = \sigma\sigma^*$ and $a : D^2f = \sum_{i,j} a_{ij} \partial^2_{ij} f, b \cdot \nabla f = \sum_{i} b_i \partial_i f$. 

Lemma 2.2. Let $(b, \sigma)$ satisfy Assumption 1.1. Then there exists $\lambda_0 \geq 1$, depending on $T, d, K, h, p_1, q_1$ and $\|b\|_{L^q_{t,x} L^p_{t,x}}$, such that for all $\lambda \geq \lambda_0$ there is a unique solution $u \in L^q_{t,x} W^{2,p_1}_x \cap W^{1,q_1}_{t,x}$ to the equation

$$
\partial_t u + \frac{1}{2} a : D^2 u + b \cdot \nabla u - \lambda u = -b, \quad u|_{t=0} = 0.
$$

(2.1)

Furthermore there exists $\epsilon > 0$, depending on $d, p_1, q_1$, such that

$$
\lambda^\epsilon \|u\|_{L^\infty_t C^2_x} + \|\partial_t u\|_{L^q_{t,x} L^p_{t,x}} + \|u\|_{L^q_{t,x} W^{2,p_1}_x} \leq \|b\|_{L^q_{t,x} L^p_{t,x}}.
$$

(2.2)

Proof. The result is a special case of [29, Theorem 3.2]; indeed, by time reversal we can reduce ourselves to the case of a forward parabolic equation with $u|_{t=0} = 0$ as therein. Let $\epsilon > 0$ be a sufficiently small parameter satisfying $2/q_1 + d/p_1 > 1 - 4\epsilon$ and choose $p' \geq p$ large enough so that $d/p' = 2\epsilon$; setting $q' = \infty$ and $\alpha = 1 + 4\epsilon$, we can apply the result from [29] to get an estimate for $\lambda^\epsilon \|u\|_{L^\infty_t C^2_x}$.

The next lemma, which allows to control integrals of the form $\int_0^T f_r(X_r) \, dr$ for $f$ merely satisfying $L^q_t L^p_x$-integrability, is also classical; its proof comes from a combination of so called Krylov’s estimates and Khasminskii’s lemma.

Lemma 2.3. Let $(b, \sigma)$ satisfy Assumption 1.1, $X$ be a solution to (1.1) and $(\tilde{p}, \tilde{q}) \in \mathcal{F}_0$. Then there exists a constant $C > 0$ (depending on $T, d, K, h, p_1, q_1, \|b\|_{L^q_{t,x} L^p_{t,x}}, \tilde{p}, \tilde{q}$) such that

$$
\mathbb{E} \left[ \exp \left( \lambda \int_0^T f_r(X_r) \, dr \right) \right] \leq C \exp \left( C \lambda^{\tilde{q}} \|f\|_{L^\tilde{q}_{t,x} L^{\tilde{p}_{t,x}}}^{\tilde{q}} \right) \quad \forall \lambda > 0, \quad f \in L^\tilde{q}_t L^{\tilde{p}_x}.
$$

(2.3)

Proof. By [29, Theorem 1.1 (A)], for any $s < t$ it hold

$$
\mathbb{E} \left[ \int_s^t f_r(X_r) \, dr \right]_{\mathcal{F}_s} \leq \|f\|_{L^{\tilde{q}}(s,t; L^{\tilde{p}_x})} =: w(s, t)^{1/\tilde{q}},
$$

where $w$ is a continuous control; estimate (2.3) then follows from an application of the quantitative version of Khasminskii’s lemma from [19, Lemma 3.5], for the choice $\gamma = 1/\tilde{q}$. \hfill \Box

Our next ingredient will be an alternative estimate for additive functional $\int_0^T f_r(X_r) \, dr$ in the case where $f$ only enjoys negative regularity; this result is less classical and improves on the one from [19, Proposition 6.6].

Lemma 2.4. Let $b$, $\sigma$ satisfy Assumption 1.1 with $q_2 = \infty$, $X$ be a solution to (1.1) and $(\tilde{p}, \tilde{q}) \in \mathcal{F}_1$. Then there exists a constant $C > 0$ (depending on $T, d, K, h, p_1, q_1, q_2, \|b\|_{L^{q_1}_{t,x} L^{p_1}_{t,x}}, \|\nabla \sigma\|_{L^\infty_{t,x} L^\infty_{t,x}}, \tilde{p}, \tilde{q}$) such that for all smooth function $f$ it holds

$$
\mathbb{E} \left[ \exp \left( \lambda \sup_{t \in [0,T]} \left| \int_0^t f_r(X_r) \, dr \right| \right) \right] \leq C \exp \left( C \lambda^{\tilde{q}} \|f\|_{L^\tilde{q}_{t,x} W^{-1,\tilde{p}}_x}^{\tilde{q}} \right) \quad \forall \lambda > 0.
$$

(2.4)

By linearity and density, we can extend the definition of $\int_0^T f_r(X_r) \, dr$ as a continuous process to any $f \in L^\tilde{q}_t W^{-1,\tilde{p}}_x$, in which case estimate (2.4) still holds.
Proof: First observe that, thanks to Lemma 2.1, we can assume without loss of generality $b \equiv 0$; we can also assume $f$ to be $\mathbb{R}$-valued, otherwise we can argue componentwise, and by homogeneity we can take $\lambda = 1$. Given a smooth $f$, denote by $v$ the solution to the parabolic PDE

$$\partial_t v + \frac{1}{2} a : D^2 v = f, \quad v|_{t=T} = 0.$$  

Since $dX_t = \sigma_t(X_t) \, dW_t$, by Itô’s formula we get

$$\int_0^t f_r(X_r) \, dr = v_t(X_t) - v_0(x) - \int_0^t (\sigma^* \nabla v)_r(X_r) \cdot \, dW_r.$$  

On the other hand, thanks to our assumptions and [19, Theorem A.4] (together with some standard interpolation estimates), it holds

$$\|v\|_{L_T^q W_s^{1,p}} + \|\partial_t v\|_{L_T^q W_s^{1,p}} + \|v\|_{L_T^\infty w_s^{1,p}} \leq \|f\|_{L_T^q W_s^{1,p}} \tag{2.5}$$  

Therefore we have the estimate

$$\mathbb{E} \left[ \exp \left( \sup_{t \in [0,T]} \left| \int_0^t f_r(X_r) \, dr \right| \right) \right] \leq e^{2\|\sigma\|_{L_T^\infty}} \mathbb{E} \left[ \exp \left( \sup_{t \in [0,T]} \left| \int_0^t (\sigma^* \nabla v)_r(X_r) \cdot \, dW_r \right| \right) \right].$$

using Doob’s inequality for the submartingale $M_t := \exp \left( \lambda \int_0^t (\sigma^* \nabla v)_r(X_r) \cdot \, dW_r \right)$, as well as the basic inequality $e^{|x|} \leq e^x + e^{-x}$, we can find a constant $\kappa > 0$ such that

$$\mathbb{E} \left[ \sup_{t \in [0,T]} |M_t| \right] \leq I^+ + I^-, \quad I^\pm := \mathbb{E} \left[ \exp \left( \pm \kappa \int_0^T (\sigma^* \nabla v)_r(X_r) \cdot \, dW_r \right) \right].$$

Let us show how to estimate $I^+$, the other term being treated similarly. By estimate (2.5), $\nabla v \in L_T^{\tilde{q}/2} L_T^\tilde{p}/2$, thus $|\sigma^* \nabla v|^2 \in L_T^{\tilde{q}/2} L_T^\tilde{p}/2$ where $(\tilde{q}/2, \tilde{p}/2) \in \mathcal{F}_0$. In particular Lemma 2.3 applies to $f = |\sigma^* \nabla v|^2$; but then by Girsanov’s theorem and Novikov’s condition, this implies that

$$N_t := \exp \left( 2\kappa \int_0^t (\sigma^* \nabla v)_r(X_r) \cdot \, dW_r - 2\kappa^2 \int_0^t |(\sigma^* \nabla v)_r(X_r)|^2 \, dt \right)$$

is a martingale, which together with Hölder’s inequality yields

$$I^+ \leq \mathbb{E} \left[ N_T \right]^{1/2} \mathbb{E} \left[ \exp \left( 2\kappa^2 \int_0^T |(\sigma^* \nabla v)_r(X_r)|^2 \, dr \right) \right]^{1/2} \leq C \exp \left( C \|\sigma^* \nabla v\|_{L_T^{\tilde{q}/2} L_T^\tilde{p}/2} \right).$$

Combining all of the above with the estimate

$$\|\sigma^* \nabla v\|_{L_T^{\tilde{q}/2} L_T^\tilde{p}/2} \leq \|\nabla v\|_{L_T^{\tilde{q}/2} L_T^\tilde{p}/2} \leq \|f\|_{L_T^q W_s^{1,p}},$$

where the second passage comes (2.5), and relabelling the constant $C$ to include $\kappa$, we finally obtain the desired (2.4).  \qed
Remark 2.5. The realisation that suitable diffusion processes can be integrated along distributions goes back at least to the works of Bass and Chen [1], where the case of measures in a suitable Kato class is considered. Among more recent developments let us mention [6] and [33, Theorem 5.1], where autonomous $\hat{f} \in W^{-\alpha,p}$ are allowed for $\alpha \leq 1/2$, $p > d/(1 - \alpha)$. As already mentioned, Lemma 2.4 is an improvement over [19], as it allows $v = 1$ without further assumptions of $f$ and reaches exponential integrability.

Among other applications of Lemma 2.4 of independent interest, let us mention that for the choice $f = \text{div}b \in L_t^{\tilde{q}_1}W_x^{-1,\tilde{p}_1}$ it yields uniform $L_t^m$ estimates for the Jacobian $J_{\phi_t}(x) = \det D_x\phi_t(x)$ of the flow $\phi_t$ associated to the SDE (1.1), which is formally given by the formula

$$J_{\phi_t}(x) = \exp \left( \int_0^t \text{div}b_r(X_r^t) \, dr \right).$$

Compare this fact with [5, Section 3], the additional assumption $\text{div}b \in L^p$, $p > 2$ is imposed in order to analyse such term and derive regularity estimates.

Remark 2.6. Interpolating between $L_t^{\tilde{q}_1}L_x^{\tilde{p}_1}$ and $L_t^{\tilde{q}_1}W_x^{-1,\tilde{p}_1}$ for $(\tilde{q}_0, \tilde{p}_0) \in \mathcal{J}_0$, $(\tilde{q}_1, \tilde{p}_1) \in \mathcal{J}_1$ and applying Lemmas 2.3-2.4, one can in fact prove a more general statement: for any $\beta \in [0,1]$, $(\tilde{q}_\beta, \tilde{p}_\beta) \in \mathcal{J}_\beta$ and any $f \in L_t^{\tilde{q}_\beta}W_x^{-\beta,\tilde{p}_\beta}$, the process $\int_0^t f_r(X_r) \, dr$ is well defined and

$$\mathbb{E} \left[ \exp \left( \lambda \sup_{t \in [0,T]} \left| \int_0^t f_r(X_r) \, dr \right| \right) \right] \leq C \exp \left( C\lambda \tilde{q}_\beta \| f \|^{\tilde{q}_\beta}_{L_t^{\tilde{q}_\beta}W_x^{-\beta,\tilde{p}_\beta}} \right) \quad \forall \lambda > 0.$$

Remark 2.7. Estimates (2.3)-(2.4) immediately imply moment bounds of the form

$$\mathbb{E} \left[ \left( \int_0^T f_r(X_r) \, dr \right)^m \right] \leq \| f \|^m_{L_t^{\tilde{q}_1}L_x^{\tilde{p}_1}}, \quad \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t f_r(X_r) \, dr \right|^m \right] \leq \| f \|^m_{L_t^{\tilde{q}_1}W_x^{-1,\tilde{p}_1}}$$

for all $m \in [1,\infty)$. For instance, to deduce the first estimate in (2.6), it suffices to apply (2.3) for the choice $\tilde{f} = f / \| f \|_{L_t^{\tilde{q}_1}L_x^{\tilde{p}_1}}$, so that

$$\| f \|_{L_t^{\tilde{q}_1}L_x^{\tilde{p}_1}}^{-m} \mathbb{E} \left[ \left( \int_0^T f_r(X_r) \, dr \right)^m \right] \leq \mathbb{E} \left[ \exp \left( \int_0^T \tilde{f}_r(X_r) \, dr \right) \right] \leq 1.$$

As a final ingredient, we need to recall an inequality involving maximal functions; the standard version, sometimes referred to as Hajlasz inequality, is classical [27], but the one-sided one presented here seems to be of much recent derivation, cf. [2].

Lemma 2.8. Let $f$ be a measurable function such that $\nabla f = \psi^1 + \psi^2$, where $\psi^1 \in L_t^{\tilde{q}_1}L_x^{\tilde{p}_1}$ for some parameters $(\tilde{q}_1, \tilde{p}_1)$ satisfying $\tilde{q}_1 \in [1,\infty)$ and $\tilde{p}_1 \in (d,\infty)$; then there exists a negligible Lebesgue set $N \subset [0,T] \times \mathbb{R}^d$ and another measurable function $g$ such that

$$|f_t(x) - f_t(y)| \leq [g^1_t(x) + g^2_t(x)]|x - y| \quad \forall y \in \mathbb{R}^d, \forall (t, x) \in [0,T] \times \mathbb{R}^d \setminus N; \quad (2.7)$$

moreover there exists a constant $C = C(d, \tilde{p}_1, \tilde{p}_2)$ such that $g$ can be chosen of the form $g = g^1 + g^2$ satisfying $\| g^i \|_{L_t^{\tilde{q}_1}L_x^{\tilde{p}_1}} \leq C\| \psi^i \|_{L_t^{\tilde{q}_1}L_x^{\tilde{p}_1}}$ for $i = 1,2$. 

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Proof. In the case of autonomous functions \( f \), the statement is a simple consequence of [2, Lemma 5.1]; more precisely, by fixing another \( p' \in (d, \hat{p}_1 \wedge \hat{p}_2) \) and applying the result therein, inequality (2.7) holds for \( g(x) \sim [\mathcal{M}(|\nabla f|^{p'})(x)]^{1/p'} \), where \( \mathcal{M} \) denotes the Hardy–Littlewood maximal function operator. By standard properties of \( \mathcal{M} \), \( g \lesssim g^1 + g^2 \) for \( g'(x) := \mathcal{M}(|\psi|^{p'})(x) \); by construction, \( |\psi_i|^{p'} \in L^{\hat{p}_i/p'}_L \) with \( \hat{p}_i/p' > 1 \), therefore again by properties of \( \mathcal{M} \) it holds \( \|\mathcal{M}(|\psi|^{p'})\|_{L^{p'/p'}} \lesssim \|\psi_i\|_{L^{\hat{p}_i}}^{p'} \), from which we can deduce that \( \|g_i\|_{L^{\hat{p}_i}} \lesssim \|\psi_i\|_{L^{\hat{p}_i}}^{p'} \).

In the case of time-dependent functions \( f \), one can either: i) observe that for a.e. \( t \in [0, T] \), \( \nabla f_i \in L^{\hat{p}_i}_L + L^{\hat{p}_2}_L \) and apply the result at such fixed \( t \); or ii) consider a sequence \( f^n \) of smooth functions such that \( f^n \to f \) and they admits decompositions \( \nabla f^n = \psi^{n,1} + \psi^{n,2} \) with \( \psi^{n,i} \to \psi^i \) in \( L^{\hat{p}^i}_L L^{\hat{p}^2}_X \); then to any \( f^n \) there is an associated \( g^n = g^{n,1} + g^{n,2} = \mathcal{M}(|\psi^{n,1}|^{p'})^{1/p'} + \mathcal{M}(|\psi^{n,2}|^{p'})^{1/p'} \) such that (2.7) holds and clearly \( g^{n,i} \to g^i \) in \( L^{\hat{p}^i}_L L^{\hat{p}^2}_X \), so that inequality (2.7) still holds after the limit (up to a Lebesgue negligible set). \( \square \)

2.2. A general abstract estimate. We present here a fundamental estimate which is at the heart of the proof of Theorem 1.2, as well as of several applications. It focuses on comparing the solution \( \tilde{X} \) of the SDE (1.1) associated to \( (X_0, b, \sigma) \) w.r.t. to any semimartingale perturbation of it; more precisely, we will consider a process \( Y \) solving

\[
Y_t = Y_0 + \int_0^t [b_s(Y_s) + R^1_Y + \sigma_s(Y_s) + R^2_Y] \, dW_s
\]

where \( R^1, R^2 \) are predictable processes, taking values respectively in \( \mathbb{R}^d \) and \( \mathbb{R}^{d \times d} \). In the following, we will enforce the following conditions on \( Y \).

Assumption 2.9. The following hold for any \( T > 0 \):

i) \( \int_0^T \left[ |b_s(Y_s)| + |R^1_s| + |\sigma_s(Y_s)|^2 + |R^2_s|^2 \right] \, ds < \infty \) \( \mathbb{P} \)-a.s.

ii) For any \( f \in L_{L^\infty}^{\hat{q}^1}L_X^{\hat{p}^2} \) with \( (\hat{q}, \hat{p}) \in J_0 \), it holds \( \int_0^T |f_s(Y_r)| \, dr < \infty \) \( \mathbb{P} \)-a.s. Moreover for any sequence of smooth, bounded functions \( f^\epsilon \) such that \( f^\epsilon \to f \) in \( L^{\hat{q}^1}L^{\hat{p}^2}_X \) as \( \epsilon \to 0 \), it holds

\[
\lim_{\epsilon \to 0} \int_0^T |f^\epsilon(Y_s) - f(Y_s)| \, ds = 0 \quad \mathbb{P} \text{-a.s.}
\]

iii) The processes \( Y, R^1, R^2 \) are adapted to the filtration generated by \( (Y_0, W, \tilde{W}) \), where \( \tilde{W} \) is another \( \mathbb{R}^m \)-valued process, for some \( m \in \mathbb{N} \), and \( Y_0, W, \tilde{W} \) are independent.

Remark 2.10. While Condition i) is natural in order to give meaning to (2.8), the others are less classical. Point ii) will be needed in order to make sense of an Itô-type formula for functions with only integrable second order derivatives. Instead iii) simplifies the proof of Proposition 2.11 below by allowing to disintegrate w.r.t. the initial conditions \( (X_0, Y_0) \); depending on the specific structure of the processes \( R^1, R^2 \) it might not be even needed. In the simplest cases, one can just take \( \tilde{W} = 0 \), but there are also applications of interest where a nontrivial choice of \( \tilde{W} \) must be considered, as we will see in Section 3.
Before presenting the statement, we need a few preparations. Given \((b, \sigma)\) satisfying Assumption 1.1, by Lemma 2.2 we can choose \(\lambda\) large enough so that the solution \(u\) to \((2.1)\) has Lipschitz constant smaller than \(1/2\) for all \(t \in [0, T]\); from now on, we will always work with such a fixed \(\lambda\) and denote by \(u\) the associated solution, so that the map \(\phi_t(x) := x + u_t(x)\) is 2-Lipschitz with 2-Lipschitz inverse for all \(t \in [0, T]\). For such \(u\), we define an associated process

\[
V_t := \int_0^t \left( \frac{1}{2} [(R^2 + \sigma)(R^2 + \sigma)^\ast - \sigma^\ast] : D^2u + R^1 \cdot (I + \nabla u) \right)_r(Y_r) \, dr + \int_0^t [R^2(I + \nabla u)]_r(Y_r) \, dW_r.
\]

which is well defined thanks to Assumption 2.9.

The proof of the next result shares some clear analogies with those of stochastic Grönwall lemmas, cf. [19, Lemma 3.8], as well as [9] for a general overview; however, in order not to impose any restriction on the \(m\)-th moment we want to estimate, we will exploit crucially the structure of the SDE which allows us to “recenter” it at each step, a property which doesn’t hold in the more general setting of those results.

**Proposition 2.11.** Let \(X\) be a solution to \((1.1)\) associated to \((X_0, b, \sigma)\) satisfying Assumption 1.1 and let \(Y\) be a solution to \((2.8)\) satisfying Assumption 2.9. Then for any \(m \in [1, \infty)\), any \(\gamma > 1\) and any \(T > 0\) there exists a constant \(C\) (depending on \(m, \gamma, T, d, (q_i, p_i), K, h, \|b\|_{L^q_tL^p_x}, \|\nabla \sigma\|_{L^q_tL^p_x}\) such that

\[
\left\| \sup_{t \in [0, T]} |X_t - Y_t| \right\|_{L^m} \leq C \left( \left\| X_0 - Y_0 \right\|_{L^m} + \left\| \sup_{t \in [0, T]} |V_t| \right\|_{L^m} \right)
\]

where the process \(V\) is defined as in \((2.9)\).

**Proof:** It suffices to consider the case where the initial data \(X_0\) and \(Y_0\) are deterministic; indeed in the general case, as they are random variables independent of the driving \((W, \bar{W})\), we can first condition on the knowledge of \((X_0, Y_0)\), obtain the estimate, and then take expectation again to conclude. From now on we will fix \(m \in [1, \infty)\).

We start by applying Zvonkin transform to both \(X\) and \(Y\); by classical computations, setting \(\hat{\sigma} := \sigma(I + \nabla u)\) it holds

\[
d(\phi_t(X_t)) = \lambda u_t(X_t) \, dt + \tilde{\sigma}_t(X_t) \, dW_t;
\]

the computation for \(Y\) is slightly more involved, but it is not difficult to check that, thanks to our definition of \(V\), in the end one arrives at

\[
d(\phi_t(Y_t)) = \lambda u_t(Y_t) \, dt + \tilde{\sigma}_t(Y_t) \, dW_t + \, dV_t.
\]

Indeed, the computation is rigorously justified thanks to Assumption 2.9, as we can first apply Itô formula to \(\phi^\varepsilon\), associated to some smooth approximation \(b^\varepsilon\) of \(b\), and then pass to the limit as \(\varepsilon \to 0\). As a consequence, if we integrate both equations over any interval \([s, t]\) and take the
modulus, we find the estimate

\[ |\phi_t(X_t) - \phi_t(Y_t)| \leq |\phi_0(X_0) - \phi_0(Y_0)| + \lambda \int_0^t |u_r(X_r) - u_r(Y_r)| \, dr \]

\[ + \left| \int_0^t [\tilde{\sigma}_r^1(X_r) - \tilde{\sigma}_r^1(Y_r)] \, dW_r \right| + \tilde{V} \]

where we set \( \tilde{V} := 2 \sup_{t \in [0,T]} V_t \); using the properties of \( \phi, u \), we arrive at

\[ \sup_{t \in [s,s+h]} |X_t - Y_t| \leq |X_0 - Y_0| + \lambda h \sup_{t \in [s,s+h]} |X_t - Y_t| + \sup_{t \in [s,s+h]} \int_0^t [\tilde{\sigma}_r^1(X_r) - \tilde{\sigma}_r^1(Y_r)] \, dW_r \]

(2.11)

Observe that this is a pathwise inequality that holds for any choice of \( s, h \), which we may and will take to be random as well.

Moreover, since \( \nabla \sigma \in L_{t}^{q_i} L_{x}^{p_i} \) and \( u \in L_{t}^{\infty} C_{x}^{1} \cap L_{t}^{q_i} W_{x}^{2,p_i} \), the new diffusion term satisfies \( \nabla \tilde{\sigma} \in L_{t}^{q_i} L_{x}^{p_i} + L_{t}^{q_i} L_{x}^{p_i} \) as well; therefore we can apply Lemma 2.8 to find functions \( g' \in L_{t}^{q_i} L_{x}^{p_i} \), \( i = 1, 2 \), for which (2.7) holds for \( f = \tilde{\sigma} \). Let us define the process

\[ A_t := \int_0^t \left[ \lambda^2 T + |g_r^1|^2(X_r) + |g_r^2|^2(X_r) \right] \, dr; \]

for a constant \( \tilde{K} > 0 \) to be chosen later; we also set an increasing sequence of stopping times \( \tau^0 = 0 \) and

\[ \tau^{n+1} = \inf \{ t \geq \tau^n : t \leq T, A_t - A_{\tau^n} \geq (4\tilde{K})^{-2/m} \}, \]

with the convention that \( \tau_{n+1} = T \) if there is no \( t \in [\tau_n, T] \) such that \( A_t - A_{\tau^n} \geq (4\tilde{K})^{-2/m} \). Now let us take \( s = \tau^n \) in (2.11) and elevate both sides to the \( m \)-th power, so to obtain

\[ \sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m \leq \sup_{t \in [\tau^n, \tau^{n+1}]} \left[ (\tau^n)^m + 2 \sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m + \tilde{V}^m \right] \]

where we set \( Z := X - Y \) and

\[ M^n_t := \int_{\tau^n}^{\tau^{n+1}} [\tilde{\sigma}_r(X_r) - \tilde{\sigma}_r(Y_r)] \, dW_r. \]

By Doob’s optimal stopping theorem \( M^n \) is a continuous martingale w.r.t. to the filtration \( \mathcal{F}_t = \mathcal{F}_{\tau^{n+1}} \); by the pathwise BDG inequality from [26, Theorem 3], there exists another continuous \( \mathcal{F}_t \)-martingale \( \tilde{M}^n \) such that

\[ \sup_{t \in [0,T]} |M^n_t|^m \leq [M^n]_{h}^{m/2} + \tilde{M}^n_h \]

where \( [M^n] \) denotes the quadratic variation of \( M^n \); by Lemma 2.8 it holds

\[ [M^n]_t = \int_{\tau^n}^{\tau^{n+1}} [\tilde{\sigma}_r(X_r) - \tilde{\sigma}_r(Y_r)]^2 \, dr \leq \int_{\tau^n}^{\tau^{n+1}} \left[ |g_r^1|^2(X_r) + |g_r^2|^2(X_r) \right] |Z_r|^2 \, dr. \]
Combining everything and taking $h = \tau_{n+1} - \tau_n$, we arrive at an inequality of the form

$$\sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m \leq C \left[ |Z_{\tau^n}|^m + \tilde{M}^n_{\tau^{n+1} - \tau^n} + |\tilde{V}|^m \right]$$

$$+ \sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m \left[ \lambda^m T^{m/2} (\tau_{n+1} - \tau^n)^{m/2} + \left( \int_{\tau^n}^{\tau_{n+1}} (|g_1|^2 + |g_2|^2) \mu_r \ dr \right)^{m/2} \right]$$

$$\leq C \left[ |Z_{\tau^n}|^m + \tilde{M}^n_{\tau^{n+1} - \tau^n} + |\tilde{V}|^m \right] + 2C \left( A_{\tau^{n+1}} - A_{\tau^n} \right)^{m/2} \sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m$$

Taking $C = \tilde{K}$ as the desired constant in the definition of $\tau^n$, we find

$$\sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m \leq 2\tilde{K} \left[ |Z_{\tau^n}|^m + \tilde{M}^n_{\tau^{n+1} - \tau^n} + |\tilde{V}|^m \right];$$

using the fact that $\tilde{M}^n$ is a $\mathcal{G}_t$-martingale, taking expectation we are left with

$$E \left[ \sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m \right] \leq 2\tilde{K} E \left[ \sup_{t \in [\tau^{n-1}, \tau^n]} |Z_t|^m \right] + 2\tilde{K} E[|\tilde{V}|^m],$$

which iteratively yields (for an appropriately chosen constant $\kappa > 0$ in function of $\tilde{K}$)

$$E \left[ e^{-\kappa n} \sup_{t \in [\tau^n, \tau^{n+1}]} |Z_t|^m \right] \leq |x_0 - y_0|^m + E[|\tilde{V}|^m]. \quad (2.12)$$

Now set $\Delta := (4\tilde{K})^{-2/m}$, $C := 2\kappa / \Delta$, then it holds

$$E \left[ e^{-CA_T} \sup_{t \in [0, T]} |Z_t|^m \right] = \sum_{n \in \mathbb{N}} E \left[ e^{-CA_T} \sup_{t \in [0, T]} |Z_t|^m 1_{A_T \in [\Delta n, \Delta(n+1) ]} \right]$$

$$\leq \sum_{n \in \mathbb{N}} E \left[ e^{-2\kappa n} \sup_{j=1, \ldots, n+1} \sup_{t \in [\tau_{j-1}, \tau_j]} |Z_t|^m 1_{A_T \in [\Delta n, \Delta(n+1) ]} \right]$$

$$\leq \sum_{n \in \mathbb{N}} e^{-\kappa n} \sum_{j=1}^{n+1} E \left[ e^{-\kappa n} \sup_{t \in [\tau_{j-1}, \tau_j]} |Z_t|^m \right]$$

$$\leq |x_0 - y_0|^m + E[|\tilde{V}|^m]$$

where in the last passage we applied (2.12). Observe that since $g^i \in L^{q_i}_t L^{p_i}_x$, $|g|^2 \in L^{4q_i}_t L^{4p_i}_x$ with $(q_i/2, p_i/2) \in \mathcal{J}_0$, thus by Lemma 2.3 the process $A$ satisfies $E[\exp(\lambda A)] < \infty$ for all $\lambda \in \mathbb{R}$; therefore for any $\delta \in (0, 1)$ it holds

$$E \left[ \sup_{t \in [0, T]} |Z_t|^\delta \right] \leq E \left[ e^{-CA_T} \sup_{t \in [0, T]} |Z_t|^m \right]^\delta E \left[ e^{C_{\tau^n} A_T} \right]^{1-\delta}$$

$$\leq \delta |x_0 - y_0|^\delta + E[|\tilde{V}|^m]^{\delta}.$$

Taking the $(\delta m)^{-1}$-power on both sides and relabelling $\tilde{m} = m\delta$, $\gamma \tilde{m} = m$, overall yields the desired result. □
2.3. **Comparison of SDEs with different coefficients.** With Proposition 2.11 at hand, we are now ready to give the

**Proof of Theorem 1.2.** Let us set $X^1 = X$, $X^2 = Y$, $b^1 = b$, $\sigma^1 = \sigma$, $R^1_t = (b^2 - b^1)_t(X^2_t)$ and $R^2_t = (\sigma^2 - \sigma^1)_t(X^2_t)$ in the setting of Section 2.2; it is then clear by Section 2.1 that Assumption 2.9 holds and so by Proposition 2.11 our task is reduced to provide estimates for $\|\tilde{V}\|_{L^{mV}}$; The process $V$, as defined in (2.9), in this case can be decomposed as $V = I^1 + I^2 + I^3$ for

\[
I^1_t := \frac{1}{2} \int_0^t [(a^2 - a^1) : D^2 u]_r(X^2_r) \, dr, \quad I^2_t := \int_0^t [(\sigma^2 - \sigma^1)(I + \nabla u)]_r(X^2_r) \, dW_r, \\
I^3_t := \int_0^t [(b^2 - b^1) \cdot (I + \nabla u)]_r(X^2_r) \, dr
\]

where $a^i = \sigma^i(\sigma^i)^*$ for $i = 1, 2$; we treat each term separately. From now on for simplicity we will write $m$ in place of $mV$, as it doesn’t affect the computation.

Recall that, under Assumption 1.1, by Lemma Lemma 2.2 it holds $u \in L^q_t W_x^{2,p_1} \cap L^\infty_t C^1_x$; moreover $(p_1, q_1)$, $(\tilde{p}_2, \tilde{q}_2) \in \mathcal{F}_1$, thus if we define $1/p' = 1/p + 1/\tilde{p}_2$ and $1/q' = 1/q_1 + 1/\tilde{q}_2$, it holds $(p', q') \in \mathcal{F}_0$. By Hölder’s inequality and estimate (2.6), we can then control $I^1$ by

\[
\left\| \sup_{t \in [0,T]} |I^1_t| \right\|_{L^{m}_t} \lesssim \left\| (a^2 - a^1) : D^2 u \right\|_{L^{p'}_t L^p_\delta}
\lesssim \|a^2 - a^1\|_{L^{\tilde{p}_2}_t L^{\tilde{q}_2}_x} \|D^2 u\|_{L^{q_1}_t L^{p_1}_x} \lesssim \|\sigma^2 - \sigma^1\|_{L^{q_2}_t L^{p_2}_x}
\tag{2.13}
\]

where in the final passage we also exploited the bound $\|\sigma^1\|_{L^{\infty}_t L^\infty_x} \leq K$. A similar idea applies to $I^2$, combined with BDG’s inequality:

\[
\left\| \sup_{t \in [0,T]} |I^2_t| \right\|_{L^{m}_t} \lesssim \left\| \int_0^T \left| (\sigma^2 - \sigma^1)(I + \nabla u_r) \right|^2 (X^2_r) \, dr \right|^{1/2}_{m/2}
\lesssim \left\| (\sigma^2 - \sigma^1)^2 \right\|^1_{L^{\tilde{q}_2/2}_t L^{\tilde{p}_2/2}_x} \|I + \nabla u\|_{L^{\infty}_t L^\infty_x} \lesssim \|\sigma^1 - \sigma^2\|_{L^{\tilde{q}_2}_t L^{\tilde{p}_2}_x},
\tag{2.14}
\]

where this time we applied (2.6) for $(\tilde{p}_2/2, \tilde{q}_2/2) \in \mathcal{F}_0$. Similarly, since $(\tilde{p}_1, \tilde{q}_1) \in \mathcal{F}_0$, it holds

\[
\left\| \sup_{t \in [0,T]} |I^3_t| \right\|_{L^{m}_t} \lesssim \left\| (b^2 - b^1) \cdot (I + \nabla u) \right\|_{L^{\tilde{p}_1}_t L^{\tilde{q}_1}_x}
\lesssim \|b^2 - b^1\|_{L^{\tilde{p}_1}_t L^{\tilde{q}_1}_x} \|(I + \nabla u)\|_{L^{\infty}_t L^\infty_x} \lesssim \|b^2 - b^1\|_{L^{\tilde{p}_1}_t L^{\tilde{q}_1}_x}.
\tag{2.15}
\]

In the case $q_2 = \infty$ and $4/q_1 + d/p_1 < 1$, we can give an alternative estimate for $I^3$; to this end, let us recall that by the product rules for distributions (see e.g. [19, Lemma A.2 (ii)]), for $p \geq d \vee 2$ it holds

\[
\|fg\|_{W_x^{-1,p}} \leq \|f\|_{W_x^{-1,p}} \|g\|_{W_x^{1,p}} \quad \forall f \in W_x^{-1,p}, g \in W_x^{1,p}.
\tag{2.16}
\]
By Hölder’s inequality and the second estimate in (2.6) (for \( \tilde{q} = q_1/2, \tilde{p} = p_1 \), which under our assumptions satisfy \((\tilde{q}, \tilde{p}) \in \mathcal{J}_0\)), we then find
\[
\left\| \sup_{t \in [0,T]} |L^2_t|^m \right\|_{L^m} \lesssim \|(b^2 - b^1) \cdot (I + \nabla u)\|_{L^{q_1/2,1,p_1}} \lesssim \|b^2 - b^1\|_{L^{q_1,1,p_1}} \|u\|_{L^{q_1,1,p_1}}. \tag{2.17}
\]

Overall, estimates (2.13), (2.14), combined with (2.15) (respectively (2.17)) and Proposition 2.11 yield (1.2) (respectively (1.3)).

We can also obtain stability estimates for Stratonovich SDEs.

**Corollary 2.12.** Let Assumption 1.1 hold and consider \( X^i \) solutions to the Stratonovich SDEs
\[
dX^i_t = b^i_t(X^i_t) \, dt + \sigma^i_t(X^i_t) \circ dW_t, \quad X^i|_{t=0} = X^i_0.
\]
Then for any \( m \in [1, \infty) \) and any \((\tilde{p}_1, \tilde{q}_1), (\tilde{p}_2, \tilde{q}_2) \in \mathcal{J}_0 \) and \((\tilde{p}_3, \tilde{q}_3) \in \mathcal{J}_1 \), it holds
\[
\left\| \sup_{t \in [0,T]} |X^1_t - X^2_t|^m \right\|_{L^m} \lesssim \|X^1_0 - X^2_0\|_{L^m} + \|b^1 - b^2\|_{L^{q_1,1,p_1}} + \|\nabla (\sigma^1 - \sigma^2)\|_{L^{q_2,1,p_2}} + \|\sigma^1 - \sigma^2\|_{L^{q_1,1,p_1}}. \tag{2.18}
\]
If additionally \( q_2 = \infty \) and \( 4/q_1 + d/p_1 < 1 \), we have
\[
\left\| \sup_{t \in [0,T]} |X^1_t - X^2_t|^m \right\|_{L^m} \lesssim \|X^1_0 - X^2_0\|_{L^m} + \|b^1 - b^2\|_{L^{q_1,1,p_1}} + \|\sigma^1 - \sigma^2\|_{L^{q_1,1,p_1}}. \tag{2.19}
\]

**Proof.** The proof is very similar to that of Theorem 1.2, so we mostly sketch it. We can rewrite the Stratonovich SDEs into the corresponding Itô ones, with corresponding new drift terms \( \tilde{b}^i \) in place of \( b^i \) given by
\[
\tilde{b}^i_t := b^i_t + \frac{1}{2} (\sigma^i \cdot \nabla) \sigma^i, \quad \text{where } [(\sigma \cdot \nabla) \sigma]_j = \sum_{k,l} \partial_i \sigma_{jk} \sigma_{lk}.
\]
Estimates (2.18)-(2.19) then follow respectively from (1.2)-(1.3), once we estimate the difference \( \tilde{b}^1 - \tilde{b}^2 \) in appropriate norms. In order to obtain (2.18), we can use
\[
(\sigma^1 \cdot \nabla) (\sigma^1 - \sigma^2) = (\sigma^1 \cdot \nabla) (\sigma^1 - \sigma^2) + [(\sigma^1 - \sigma^2) \cdot \nabla] \sigma^2
\]
and then
\[
\|(\sigma^1 \cdot \nabla) (\sigma^1 - \sigma^2)\|_{L^{q_1,1,p_2}} \lesssim \|\sigma^1\|_{L^{q_1,1,p_2}} \|\nabla \sigma^1 - \nabla \sigma^2\|_{L^{q_1,1,p_2}}
\]
\[
\|[(\sigma^1 - \sigma^2) \cdot \nabla] \sigma^2\|_{L^{q_1,1,p_2}} \lesssim \|\sigma^1 - \sigma^2\|_{L^{q_1,1,p_2}} \|\nabla \sigma^2\|_{L^{q_1,1,p_2}}
\]
where we set \( 1/p' = 1/\tilde{p}_3 + 1/p_2, 1/q' = 1/\tilde{q}_3 + 1/q_2 \) so that \((p', q') \in \mathcal{J}_0 \). Instead for obtaining (2.19), in the case \( q_2 = \infty \) we can estimate the first term differently. To this end, let us recall that by \([19, \text{Lemma A.2 (iii)}] \), whenever \( p > d \) and \( 1/p + 1/p < 1 \), it holds
\[
\|f g\|_{W^{1,p}_x} \lesssim (\|f\|_{L^{\infty}_x} + \|\nabla g\|_{L^{p'_x}}) \|f\|_{W^{1,p}_x} \quad \forall f \in W^{1,p}_x, \, g \in W^{1,p}_x. \tag{2.20}
\]
Applying estimate (2.20) for the choice \( p = p_2 \) and \( \tilde{p} = \tilde{p}_3 \), we obtain
\[
\| \sigma^1 \cdot \nabla (\sigma^1 - \sigma^2) \|_{L^q_t W_x^{-\tilde{p}_1,\tilde{p}_3}} \leq \left[ \| \sigma^1 \|_{L^\infty_t} + \| \nabla \sigma^1 \|_{L^\infty_t} \right] \| \nabla (\sigma^1 - \sigma^2) \|_{L^q_t W_x^{-\tilde{p}_1,\tilde{p}_3}} \\
\leq \| \sigma^1 - \sigma^2 \|_{L^q_t W_x^{-\tilde{p}_1,\tilde{p}_3}}.
\]

Combined with Proposition 2.11 and estimates (2.13), (2.14), (2.15) and (2.17), this yields the conclusion. \( \square \)

**Remark 2.13.** Theorem 1.2 and Corollary 2.12 admit several other variants, in the sense of employing other quantities of the form \( \| b^1 - b^2 \|_{L^q_t W_x^{-\tilde{p}_1,\tilde{p}_3}} \) in the r.h.s. of (1.3) and (2.19). Indeed going through the proof, the only relevant difference comes from estimating the term \( I^3 \) related to \( \int_0^1 [(b^2 - b^1) \cdot (I + \nabla u)]_\tau (X^2) \, d\tau \), which can be accomplished as in Remark 2.6 under the assumption \( q_2 = \infty \). There are two different regimes, related to the parameters \( (\beta, q_1, p_1) \):

1. \( \beta < 1 - 2/q_1 \). In this case, since \( (p_1, q_1) \in \mathcal{F}_\beta \), we can assume \( \beta \) to satisfy \( \beta > d/p_1 \) as well (this proves a stronger estimate, as for \( \tilde{\beta} \leq \beta \) one can use \( W_x^{-\tilde{\beta},\tilde{p}_3} \leftrightarrow W_x^{-\tilde{\beta},\tilde{p}_3} \)). By Lemma 2.2 and standard interpolation techniques, it is easy to show that \( \nabla u \in L^\infty_t W_x^{\tilde{\beta},\tilde{p}_3} \) and one can then apply \([19, \text{Lemma A.2 (ii)}]\) to obtain
\[
\| (b^2 - b^1) \cdot (I + \nabla u) \|_{L^q_t W_x^{-\tilde{p}_1,\tilde{p}_3}} \leq \| b^1 - b^2 \|_{L^q_t W_x^{-\tilde{p}_1,\tilde{p}_3}} (1 + \| \nabla u \|_{L^\infty_t W_x^{\tilde{\beta},\tilde{p}_3}})
\]
eventually leading to
\[
\left\| \sup_{t \in [0,T]} |X^1_t - X^2_t| \right\|_{L^m_w} \leq \| X^1_0 - X^2_0 \|_{L^m_w} + \| b^1 - b^2 \|_{L^q_t W_x^{-\tilde{p}_1,\tilde{p}_3}} + \| \sigma^1 - \sigma^2 \|_{L^q_t L^\infty_x} \quad (2.21)
\]
for any choice of parameters \( (\tilde{p}_1, \tilde{q}_1) \in \mathcal{F}_\beta \). This is in agreement with (1.2), which is recovered for \( \beta = 0 \).

2. \( \beta > 1 - 2/q_1 \). In this case we only know that \( \nabla u \in L^\tilde{q}_t W^{\tilde{p}_3,\tilde{p}_1} \) for \( 1/\tilde{q} = 1/q - (1 - \beta)/2 \); combined again with \([19, \text{Lemma A.2 (ii)}]\), this implies \( (b^2 - b^1) \cdot (I + \nabla u) \in L^{\tilde{q}_t} W_x^{-\tilde{p}_1,\tilde{p}_3} \) for \( 1/q' = 1/\tilde{q} + 1/\tilde{q}_1 \) and the relevant condition becomes \( (\tilde{p}_1, \tilde{q}_1) \in \mathcal{F}_\beta \). Overall we still obtain a stability estimate where \( \| b^1 - b^2 \|_{L^q_t W_x^{-\tilde{p}_1,\tilde{p}_3}} \) appears, but now only under the more restrictive condition
\[
\frac{2}{q_1} + \frac{2}{\tilde{q}_1} + \frac{d}{\tilde{p}_1} < 3 - 2\beta, \quad \frac{d}{\tilde{p}_1} < \beta. \quad (2.22)
\]
Condition (2.22) agrees with \( 4/q_1 + d/p_1 < 1 \) needed to derive (1.3), which corresponds to \( \beta = 1, \tilde{p}_1 = p_1 \) and \( \tilde{q}_1 = q_1 \).

### 3. Applications

#### 3.1. McKean–Vlasov equations

Throughout this section, for any \( t \in [0, T] \) and any \( X \), we will denote by \( X_t^s \) the stopped process \( X_{t^\land s}^s := X_{t^\land s} \), which we ask the reader not to confuse with
the pointwise evaluation $X_t$. We consider McKean–Vlasov (or distribution dependent) SDEs of the form
\[
dX_t = B_t(X_t, \mu_t) \, dt + \Sigma_t(X_t, \mu_t) \, dW_t, \quad \mu_t = \mathcal{L}(X_t)
\] (3.1)
for suitable coefficients $B, \Sigma$, see Assumption 3.1 below.

In the case of regular (e.g. Lipschitz) continuous functions $B, \Sigma$, equation (3.1) can be solved classically; however in the presence of a singular drift $B$, several important results in the literature guaranteeing existence and uniqueness for (3.1) require the diffusion term $\Sigma$ not to depend on $\mu$, see e.g. [22, Section 3.2], or anyway more restrictive assumption on $\Sigma$, cf. [24, Section 4]; the reason lies in the use of transport-cost inequalities and Girsanov transform, see also [18]. On the other hand there is another class of results, including [13, 14], possibly not fitting in this dichotomy, see for instance [12, 23, 16]; finally, let us mention the remarkable work [34] for existence and uniqueness results, in the presence of $\mu$-dependent $\Sigma$, under yet another set of assumptions.

Here we do not follow any of the aforementioned approaches, instead we derive existence and uniqueness results for (3.1) by readapting the strategy from [8], based on the use of stability estimates in negative Sobolev spaces and Wasserstein distances. Our assumptions cover the basic convolutional case $\Sigma = \delta$ (cf. Corollary 3.5), but also allow for $\Sigma$ to depend on $\mu$, as well as a more general abstract class of coefficients $B$ and $\Sigma$ with nonlinear dependence on $\mu$ (cf. Assumption 3.1).

In the following, given a separable Banach space $E$, we denote by $\mathcal{P}(E)$ the set of probability measures on $E$ and by $W_m$ the associated $m$-Wasserstein distance, namely
\[
W_m(\mu, \nu) = \inf \left\{ \left( \int \|x - y\|_E^m \, \pi(dx, dy) \right)^{1/m} : \pi \in \Pi(\mu, \nu) \right\}
\]
where $\Pi(\mu, \nu) \subset \mathcal{P}(E \times E)$ denotes the set of couplings of $(\mu, \nu)$. We refer to [28] for a complete survey on the properties of $W_m$; let us only recall that $W_m(\mu, \nu) \leq W_\tilde{m}(\mu, \nu)$ whenever $m \leq \tilde{m}$ and that for any $m \in [1, \infty)$, $\mu, \nu \in \mathcal{P}(E)$ there exists an optimal coupling for $W_m(\mu, \nu)$, which can be realised as a pair of $E$-valued random variables $(Y, Z)$ such that $\mathcal{L}(Y) = \mu, \mathcal{L}(Z) = \nu$ and $W_m(\mu, \nu) = \mathbb{E}[[Y - Z]_E^m]^{1/m}$. We will mainly consider $E$ as the path space $C_T := C([0, T]; \mathbb{R}^d)$ or $E = \mathbb{R}^d$.

We are now ready to introduce our assumptions on the coefficients $B, \Sigma$.

**Assumption 3.1.** There exist $p_1, p_2 \in (d, \infty), m \in [1, \infty)$ and $C > 0$ such that the maps $B : [0, T] \times \mathbb{R}^d \times \mathcal{P}(C_T) \to \mathbb{R}^d, \Sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}(C_T) \to \mathbb{R}^{d \times d}$ satisfy the following:

i) $\|B_t(\cdot, \nu)\|_{L_{p_1}^E} \leq C, \|B_t(\cdot, \mu) - B_t(\cdot, \nu)\|_{W_2^{-1, p_1}} \leq C W_m(\mu, \nu)$ for all $t \in [0, T], \mu, \nu \in \mathcal{P}(C_T)$.

ii) $\|\Sigma_t(\cdot, \mu)\|_{L_{p_2}^E} \leq C, \|\Sigma_t(\cdot, \mu) - \Sigma_t(\cdot, \nu)\|_{L_{p_2}^E} \leq C W_m(\mu, \nu)$ for all $t \in [0, T], \mu, \nu \in \mathcal{P}(C_T)$. 


\( \mathcal{L}(C_T) \)

Under Assumption 3.1, one can give meaning to equation (3.1) as follows: given a continuous adapted process \( \sigma \), we can define \( \mu_t = \mathcal{L}(Y^t) \) and then

\[
\sigma^Y_t(x) := \Sigma_t(x, \mu_t); \quad \sigma^\mu_t(x) := \Sigma_t(x, \mu_t);
\]

it can be readily checked that \((b^Y, \sigma^Y)\) satisfy Assumption 1.1 \(^1\), thus the associated SDE is wellposed. We then define \( X \) to be a solution to (3.1) if and only if it is a solution to the SDE associated to \((b^X, \sigma^X)\); the concepts of strong existence, pathwise uniqueness and uniqueness in law for (3.1) can then be readapted similarly. Let us finally recall that, as usual for McKean–Vlasov SDEs, we will consider the initial datum \( X_0 \) to be random and independent of \( W \); depending on how one wants to formulate it, the data of the problem can then be considered equivalently \((X_0, B, \Sigma)\) or \((\mu_0, B, \Sigma)\), where \( \mu_0 = \mathcal{L}(X_0) \).

**Theorem 3.2.** Let \( B, \Sigma \) satisfy Assumption 3.1 for some \( m \in [1, \infty) \). Then for any \( \mu_0 \in \mathcal{P}_m(\mathbb{R}^d) \) strong existence, pathwise uniqueness and uniqueness in law holds for the SDE (3.1). A similar statement holds for Stratonovich version of (3.1), i.e. with \( \Sigma_t(X_t, \mu_t) \) \( dW_t \) replaced by \( \Sigma_t(X_t, \mu_t) \circ dW_t \).

**Proof.** We give the proof in the Itô case, the Stratonovich one being identical (up to applying Corollary 2.12 in place of Theorem 1.2). Standard arguments involving the structure of eq. (3.1) (see for instance [7, Propositions 3.5-3.7]) allow to show that any weak solution is a strong one and that pathwise uniqueness implies uniqueness in law; thus our task is reduced to show existence and uniqueness among strong solutions. Consider the filtration \( \mathcal{F}_t \) generated by \((X_0, W)\) and let \( E \) be the space of continuous, \( \mathcal{F}_t \)-adapted paths \( Y \in L^m_{\omega}C_T \), endowed with the metric

\[
d_E(Y, Z) := \sup_{t \in [0, T]} \mathbb{E} \left[ \sup_{s \leq t} |Y_s - Z_s|^m \right]^{1/m} = \sup_{t \in [0, T]} e^{-\lambda t} \|Y^t - Z^t\|_{L^m_{\omega}C_T};
\]

for a suitable \( \lambda > 0 \) to be chosen later; define a map \( \mathcal{I} \) from \( E \) to itself, which to a given process \( Y \) associates the unique strong solution \( \mathcal{I}(Y) \) to the SDE associated to \((X_0, b^Y, \sigma^Y)\). It is then clear that \( X \) is a solution to (3.1) if and only if \( X = \mathcal{I}(X) \) and that our task is reduced to showing that \( \mathcal{I} \) is a contraction on \((E, d)\), for suitable choice of \( \lambda \). To this end, let us fix a parameter \( q \in (2, \infty) \) sufficiently large so that \( 4/q + d/p_1 < 1 \), \( 2/q + d/p_2 < 1 \); then by estimate (1.3) for the choice \((\tilde{p}_2, \tilde{q}_2) = (p_2, q)\) and Assumption 3.1, for any \( t \in [0, T] \) it holds

\[
\|\mathcal{I}(Y)^t - \mathcal{I}(Z)^t\|_{L^m_{\omega}C_T} \leq \left\| b^Y - b^Z \right\|_{L^q_{\omega}W^{1,p_1}} + \left\| \sigma^Y - \sigma^Z \right\|_{L^q_{\omega}T^{p_2}} \\
\lesssim \int_0^t \mathbb{W}_m(\mathcal{L}(Y^s), \mathcal{L}(Z^s)) ds \\
\lesssim \int_0^t \mathbb{E}^{\lambda q^2} d_E(Y, Z)^q ds \lesssim \lambda^{-1} e^{\lambda q^2} d_E(Y, Z)^q;
\]

\(^1\)Since \( \nabla \sigma^Y \in L^p \) for some \( p > d \), Morrey’s inequality implies that it is \( \alpha \)-Hölder in space for \( \alpha = 1 - d/p \), uniformly in time; thus conditions concerning the modulus of continuity of \( \sigma^X \) are automatically satisfied.
overall, this implies the existence of another constant $\tilde{C} > 0$ such that
\[
d_E(I(Y), I(Z)) \leq \tilde{C}\lambda^{-1/q}d_E(Y, Z)
\]
which implies contractivity of $I$ once we choose $\lambda$ large enough.

We now pass to provide sufficient conditions for Assumption 3.1 in a practical case of interest, given by convolutional drifts; in particular here we take $\mu_t = \mathcal{L}(X_t) \in \mathcal{P}(\mathbb{R}^d)$ and consider maps $B, \Sigma$ with linear dependence on $\mu$ by means of a convolution.

**Lemma 3.3.** For any $p \in [1, \infty]$, $f \in L^p_x$ and $g$ such that $\nabla g \in L^p_x$, it holds
\[
\|f * (\mu - \nu)\|_{W^{-1,p}_x} \leq \|f\|_{L^p_x} \mathcal{W}_1(\mu, \nu), \quad \|g * (\mu - \nu)\|_{L^p_x} \leq \|\nabla g\|_{L^p_x} \mathcal{W}_1(\mu, \nu) \quad \forall \mu, \nu \in \mathcal{P}(\mathbb{R}^d).
\]

**Proof.** Let us first prove the claim for $g$. Recall the basic fact that, for any $h_t \in \mathbb{R}^d$, it holds
\[
\|g(\cdot + h) - g(\cdot + h')\|_{L^p_x} \leq \|\nabla g\|_{L^p_x} |h - h'|.
\]
Let $(Y, Z)$ be an optimal coupling for $(\mu, \nu)$, then by definition of convolution it holds
\[
\|g * (\mu - \nu)\|_{L^p_x} = \|\mathbb{E}[g(\cdot - Y) - g(\cdot - Z)]\|_{L^p_x} \leq \mathbb{E}\|g(\cdot - Y) - g(\cdot - Z)\|_{L^p_x}
\]
\[
\leq \|\nabla g\|_{L^p_x} \mathbb{E}[|Y - Z|] = \|\nabla g\|_{L^p_x} \mathcal{W}_1(\mu, \nu).
\]
The case of $f$ then follows from a duality argument. Observe that $\langle f * (\mu - \nu), g \rangle = \langle f, g * (\tilde{\mu} - \tilde{\nu}) \rangle$, where $\tilde{\mu}(A) = \mu(-A)$ is the reflection of $\mu$; it is clear that $\mathcal{W}_1(\tilde{\mu}, \tilde{\nu}) = \mathcal{W}_1(\mu, \nu)$, therefore
\[
\|f * (\mu - \nu), g\| \leq \|f\|_{L^p_x} \|g * (\tilde{\mu} - \tilde{\nu})\|_{L^{p'}} \leq \|f\|_{L^p_x} \|g\|_{W^{-1,p'}_x} \mathcal{W}_1(\mu, \nu)
\]
which yields the conclusion by taking the supremum over all $g \in W^{1,p'}_x$ with $\|g\|_{W^{1,p'}_x} = 1$.

**Lemma 3.4.** Let $\sigma$ satisfy conditions (H1$^q$)-(H2$^q$) from Assumption 1.1 with $q_2 = \infty$, $p_2 > d$ and constant $K$ and let $\pi : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ satisfy
\[
\nabla \pi \in L_{t,x}^\infty L^{p_2}_x, \quad \sup_{t,x} |\pi_t(x)| \leq (1 - \delta)K^{-1/2}
\]
for some $\delta \in (0, 1)$. Then $\Sigma_t(x, \mu) := \sigma_t(x) - (\pi_t * \mu)(x)$ satisfies conditions ii)-iii) from Assumption 3.1.

**Proof.** By standard properties of convolution, for any $\mu \in \mathcal{P}(\mathbb{R}^d)$ it holds
\[
\|\nabla (\pi_t * \mu)\|_{L^p_x} \leq \|\nabla \pi_t\|_{L^{p_2}_x} \leq \|\nabla \pi\|_{L^{\infty}_tL^{p_2}_x}
\]
which immediately implies a uniform bound on $\|\nabla \Sigma_t(\cdot, \mu)\|_{L^{p_2}}$; a similar argument shows uniform boundedness of $\Sigma_t(x, \mu)$. By our assumptions and triangular inequality, it holds
\[
|\Sigma_t(x, \mu)\xi| \geq |\sigma_t(x)\xi| - \|\pi * \mu\|_{L^{\infty}_t}|\xi| \geq |\sigma_t(x)\xi| - \|\pi\|_{L^{\infty}_t}|\xi| \geq \delta K^{-1/2}|\xi|
\]
which shows uniform ellipticity.

Combining Lemma 3.3 and Lemma 3.4 immediately implies the following...
Then strong wellposedness holds for the associated McKean–Vlasov SDE (3.1) and its dependence on a modulus of continuity $h$.

Remark 3.6. It is clear that other variants of Theorem 3.2 are allowed; for instance in Assumption 3.1-i) can be modified by requiring the existence of parameters $(q_1, p_1)$ satisfying $4/q_1 + d/p_1 < 1$ and a function $t \in L^{p_1}$ such that

$$\|B_t(\cdot, \mu)\|_{L^{p_1}} t_t \leq C, \quad \|B_t(\cdot, \mu) - B_t(\cdot, \nu)\|_{W^{1,p_1}} \leq t_t \mathbf{W}_m(\mu, \nu).$$

In the style of Remark 2.13, one could also formulate a condition on $B$ involving $W^{-\nu,p_1}$-norms for other values $\nu \in [0, 1]$, although the choice $\nu = 1$ is natural in view of Lemma 3.3.

Remark 3.7. Unfortunately, our result in general does not cover the choice $\Sigma_t(x, \mu) := (\sigma_t \mu)(x)$; the problem is that, even if $\sigma$ is uniformly elliptic, the same doesn’t need to hold for $\sigma \mu$, as cancellations might happen inside the convolution. This difficulty can be overcome under more information on $\sigma$; for instance if $\sigma_t(x) = \gamma_t(x) A$, where $\gamma : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ and $A$ is a fixed, uniformly elliptic $\mathbb{R}^{d \times d}$ matrix, then the continuity and uniform ellipticity requirements on $\sigma$ imply that $\gamma_t(x) \geq \delta > 0$ (or $\gamma_t(x) \leq -\delta < 0$, which doesn’t change the analysis). Due to the positivity of $\mu$, we then find

$$|\langle \sigma_t \mu \rangle^*(x) \xi| = |\gamma_t \mu(x)| |A^* \xi| \geq \delta |A^* \xi|$$

showing again uniform ellipticity of $\Sigma_t(x, \mu) = (\sigma_t \mu)(x)$.

Remark 3.8. Other examples of $B$ and $\Sigma$ satisfying conditions similar to Assumption 3.1, are discussed in [8, Section 2.1]. Among them let us mention the case of dependence on statistics of $\mu$, e.g. $B_t(x, \mu) = b_t(x - \langle \phi, \mu \rangle)$ where $\langle \phi, \mu \rangle := \int \phi(\omega) \mu(d\omega)$ and $\phi : C_T \to \mathbb{R}^d$ is a uniformly Lipschitz map. Indeed, by properties of translations in Sobolev spaces and Kantorovich-Rubinstein duality, it holds

$$\|b_t(\cdot - \langle \phi, \mu \rangle) - b_t(\cdot - \langle \phi, \nu \rangle)\|_{W^{1,p}} \leq \|b_t\|_{L^p} |\langle \phi, \mu - \nu \rangle| \leq \|b_t\|_{L^p} \|\phi\|_{L^p} \mathbf{W}_1(\mu, \nu).$$

3.2. Strong compactness of solutions to singular SDEs. We show here that, under quite minimal assumptions, the solutions to SDEs of the form (1.1), driven by the same reference Brownian motion $W$, form a compact set in the strong topology of $L^p_{x,a} C_T$. This is much stronger than standard tightness results for $L(X)$.

Throughout this section we will assume for simplicity $X_0 = x \in \mathbb{R}^d$ to be deterministic and fixed, so the dependence of the solution on it will not appear in the sequel. Given a pair $(b, \sigma)$ satisfying Assumption 1.1, we denote by $X^{b,\sigma}$ the associated solution to the Itô SDE (1.1) with initial condition $x$; similarly, $\tilde{X}^{b,\sigma}$ denotes the solution to the corresponding Stratonovich SDE.

We start by dealing with autonomous drift and diffusion; thus fix some parameters $p_1, p_2 \in (d, +\infty)$ corresponding to $b \in L^{p_1}_{x,a}, \nabla \sigma \in L^{p_2}_{x,a}$.\footnote{As in Section 3.1, $\nabla \sigma \in L^{p_2}_{x,a}$ implies $\sigma$ being globally Hölder, so that we can drop assumption $|\mathcal{H}|$ and the dependence on a modulus of continuity $h$.}
Theorem 3.9. Consider a family of coefficients \( \mathcal{A} \subset \{(b, \sigma) : b \in L^p_t, \sigma \in L^\infty, \nabla \sigma \in L^p_x\} \) satisfying Assumption 1.1 in a uniform way; namely, there exist constants \( M, K > 0 \) such that for all \( (b, \sigma) \in \mathcal{A} \) it holds \( \|b\|_{L^p_t} + \|\sigma\|_{L^\infty} + \|\nabla \sigma\|_{L^p_x} \leq M \) and \( |\sigma^* \xi| \geq K^{-1}|\xi|^2 \) for all \( \xi \in \mathbb{R}^d \). Then the families \( \{X^{b,\sigma} : (b, \sigma) \in \mathcal{A}\} \) and \( \{\bar{X}^{b,\sigma} : (b, \sigma) \in \mathcal{A}\} \) are precompact compact in \( L^m_0 C_T \), for all \( m \in [1, \infty) \).

Proof. We focus on the Itô case, the Stratonovich one being almost identical up to applying Corollary 2.12 instead of Theorem 1.2. By well-known results concerning the SDE (1.1) and our assumptions, the solutions \( X^{b,\sigma} \) satisfy uniform moment estimates, namely for all \( m \in [1, \infty) \) it holds \( \sup_{(b,\sigma) \in \mathcal{A}} \|X^{b,\sigma}\|_{L^m_0 C_T} < \infty \); thus it suffices to show precompactness w.r.t. convergence in probability.

Given any sequence \( (b^n, \sigma^n)_n \subset \mathcal{A} \), by the assumptions and weak compactness we can extract a (not relabelled) subsequence such that \( b^n \) converge weakly in \( L^p_t \) to some \( b \) and \( \sigma^n \) converge weakly-* in \( L^\infty_x \) to some \( \sigma \), with the additional property that \( \nabla \sigma^n \) converge weakly to \( \nabla \sigma \) in \( L^p_x \). We claim that \( X^{b^n,\sigma^n} \) converge in probability to \( X^{b,\sigma} \), which yields the conclusion in the Itô case.

From now on we will adopt the shorter notation \( X^n = X^{b^n,\sigma^n}, X = X^{b,\sigma} \). To show the claim, let us fix some \( \delta > 0 \) and choose \( R > 0 \) large enough so that \( \mathbb{P}(\|X^n\|_{C_T} > R) < \delta \), \( \mathbb{P}(\|X\|_{C_T} > R) < \delta \), which is possible by the uniform moment estimates; next, let us choose \( \psi \in C^\infty_c \) such that \( \psi \equiv 1 \) on \( B_R \) and \( \psi \equiv 0 \) on \( B_{R+1} \), where \( B_R := \{x \in \mathbb{R}^d : |x| \leq R\} \). Denote by \( \bar{X}^n \) the solutions associated to \( (\psi b^n, \psi \sigma^n) \), similarly \( \bar{X} \). Clearly \( \psi b^n \) converge weakly to \( \psi b \), which together with the compact embedding \( L^p_t \hookrightarrow W^{-1,p}_x \) and the presence of the cutoff \( \psi \) implies \( \|\psi b^n - \psi b\|_{W^{-1,p}_x} \to 0 \) as \( n \to \infty \); similarly \( \|\psi \sigma^n - \psi \sigma\|_{L^p_x} \to 0 \) and so by Theorem 1.2 we deduce that \( \bar{X}^n \to \bar{X} \) in \( L^m_0 C_T \). On the other hand, by construction \( \bar{X}^n \) and \( X^n \) coincide as long as \( X^n \) do not get outside \( B_R \), similarly for \( X \), thus we obtain

\[
\lim_{n \to \infty} \mathbb{P}(\|X^n - X\| > \varepsilon) \leq \lim_{n \to \infty} \mathbb{P}(\|X^n\| > R) + \mathbb{P}(\|X\| > R) + \mathbb{P}(\|\bar{X} - \bar{X}\| > \varepsilon)
\leq 2\delta + \lim_{n \to \infty} \mathbb{P}(\|\bar{X} - \bar{X}\| > \varepsilon) = 2\delta.
\]

By the arbitrariness of \( \delta > 0 \), the conclusion follows. \( \square \)

In the case of time-dependent coefficients, we cannot go through the same proof, since local compactness in weak topologies is not obvious anymore; however, under mild additional time regularity assumptions, we can invoke the partial compactness results from [25, Section 9]. In the next statement, we fix \( (p_1, q_1) \in \mathcal{J}_1 \) and consider \( q_2 = \infty, p_2 \in (d, +\infty) \).

Corollary 3.10. Consider a family \( \mathcal{A} \subset \{(b, \sigma) : b \in L^p_t L^q_x, \sigma \in L^\infty_x, \nabla \sigma \in L^p_x L^q_x \} \) with the following property: there exist some parameters \( s > 0 \) and \( N \in \mathbb{N} \), as well as constants \( M, K > 0 \)
such that for all \((b, \sigma) \in \mathcal{A}\) it holds\(^3\)

\[
\|b\|_{L^p_t L^p} + \|b\|_{W^{1, N}_t W^{1, p}_x} + \|\sigma\|_{L^p_t} + \|\nabla \sigma\|_{L^p_t L^p} + \|b\|_{W^{1, N}_t W^{1, p}_2} \leq M
\]

and \(|\sigma^* \xi| \geq K^{-1} |\xi|^2\) for all \(\xi \in \mathbb{R}^d\). Then the families \(\{X^{b, \sigma} : (b, \sigma) \in \mathcal{A}\}\) and \(\{\tilde{X}^{b, \sigma} : (b, \sigma) \in \mathcal{A}\}\) are precompact compact in \(L^m_0 C_T\), for all \(m \in [1, \infty)\).

**Proof.** Introducing the cutoff \(\psi\) as in the proof of Theorem 3.9, we can always assume \((b, \sigma)\) to be compactly supported on \(B_R\); thus the only relevant issue is to show the existence of a subsequence \((b^n, \sigma^n)\) converging to \((b, \sigma)\) in a suitable weak topology, which still has to be strong enough in order for our stability estimates to apply. For any \(\beta > 0\), we can apply [25, Corollary 7] for the choice \(X = L^p_t, B = W_x^{-\beta, p_2}\) and \(Y = W_x^{-\beta - N, p_2}\) to deduce the existence of a subsequence such that \(b^n \to b\) in \(L^{\hat{q}_1}_t W_x^{\beta, p_1}\) for any \(\hat{q}_1 < q_1\), where \(b \in L^{q_1}_t L^{p_1}_t\); similarly \(\sigma^n \to \sigma\) in \(L^{\hat{q}_2}_t L^{p_2}_x\) for any \(\hat{q}_2 < +\infty\), where \(\sigma\) still satisfies the same bounds as the elements from \(\mathcal{A}\). In particular, the SDE associated to \((b, \sigma)\) is well defined, with solution \(X^{b, \sigma}\). Since \((p_1, q_1) \in \mathcal{J}_\beta\) and \(p_2 > d\), we can always find \(\tilde{q}_1, \tilde{q}_2\) and \(\beta\) satisfying the above and such that additionally \(\beta < 1 - 2/q_1\), \((\tilde{q}_1, p_1) \in \mathcal{J}_{\beta}\) and \((\tilde{q}_2, p_2) \in \mathcal{J}_1\), which by the stability estimate \((2.21)\) implies \(X^{b_n, \sigma_n} \to X^{b, \sigma}\) and thus the conclusion in the Itô case.

The reasoning for Stratonovich SDEs is almost identical, but we give some additional details since we haven’t precisely stated a version of \((2.21)\) in this case. By rewriting the SDE in the corresponding Itô form, it suffices to verify that \(\sigma_n \cdot \nabla \sigma^n\) converge strongly to \(\sigma \cdot \nabla \sigma\) in suitable topologies; by [25, Corollary 7], we can show \(\sigma^n \to \sigma\) in \(L^{\tilde{q}_1}_t W_x^{\beta, p_2}\) for any \(\beta < 1\), thus \(\sigma^n \to \sigma\) in \(L^{\tilde{q}_2}_t W_x^{\beta - 1, p_2}\) and by \((2.16)\) we can deduce that \(\sigma^n \cdot \nabla \sigma^n \to \sigma \cdot \nabla \sigma\) in \(L^{\tilde{q}_1}_t W_x^{\beta - 1, p_2}\). As the argument holds for any \(\tilde{q}_2 < \infty\) and any \(\beta < 1\) and by assumption \(p_2 > d\), we can always choose such parameters in a way that \((\tilde{q}_2/2, p_2) \in \mathcal{J}_{1 - \beta}\), yielding the conclusion again by an application of \((2.21)\).

**Remark 3.11.** We conjecture that the time regularity condition, in the form of a uniform bound in \(W^{1, N}_t W_x^{1, p}\), is not needed in order to derive the strong compactness result of Corollary 3.10. In order to avoid it, one would need a stability estimate where the difference of different coefficients is measured in a negative norm of both time and space, e.g. \(\|b^1 - b^2\|_{W^{1, \gamma}_t W^{-\gamma, p}_x}\). In this direction, let us mention that it is possible to obtain estimates for integrals of the form \(\int f(X_t) \, dt\), by means of nonlinear Young integrals (cf. [3, 11]), for instance when \(f \in C^{\gamma}_{t, x}\) for \(\gamma < 1/2\); combining this result with Lemma 2.4, giving instead estimates when \(f\) has positive regularity in time but negative in space, and interpolation techniques, there is some hope to achieve estimates for \(f\) enjoying (possibly very small) negative regularity in both variables. We leave this problem for future investigations.

\(^3\)Here we follow the notation from [25]; for a Banach space \(E\), the space \(W^{1, 1}_x E = W^{1, 1}(0, T; E)\) is defined as the set of integrable functions \(f : [0, T] \to E\) such that

\[
\|f\|_{W^{1, 1}_x E} := \|f\|_{L^1_t E} + \|f\|_{W^{1, 1}_t E} < \infty, \quad \text{where} \quad \|f\|_{W^{1, 1}_t E} := \int_{[0,T]^2} \frac{\|f_r - f_u\|_E}{|r - u|^{1+\gamma}} \, dr \, du.
\]
3.3. **Improved Wong-Zakai Theorem.** In this section we apply our stability results in the context of Wong–Zakai approximation of singular SDEs. The main difficulty lies in the fact that, due to the singularity of the drift $b$, the SDE associated to a smooth approximation $W^n$ in place of $W$ does not need to be wellposed, making even the correct formulation of the result unclear. To overcome this difficulty, we follow the strategy introduced in [20], based on first replacing the coefficients $(b, \sigma)$ by some smooth approximations of them and then applying the Wong–Zakai result therein; one thus needs to control the overall error committed in this two-step approximation. Compared to [20], we improve the result from by not only allowing $b$ to be singular, but also considering $\sigma$ only Sobolev differentiable and not $C^2_x$.

We start by introducing some basic notations and conventions for this section. For simplicity, here we will only consider autonomous coefficients $(b, \sigma)$ satisfying **Assumption 1.1**, as well as deterministic initial data $x_0 \in \mathbb{R}^d$. We always work on the Wiener space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$, that is to say, $\Omega := \{\omega \in C([0, \infty), \mathbb{R}^d) : \omega(0) = 0\}$, $\mathbb{P}$ is the Wiener measure defined on the Borel $\sigma$-algebra $\mathcal{F}$ of $\Omega$, $W_t(\omega) := \omega(t)$, $t \geq 0$ is a Wiener process and $(\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by $(W_t)_{t \geq 0}$.

We shall consider the following class of approximations:

**Definition 3.12.** [15, VI. Definition 7.1] By an approximation of the Wiener process $W_t$, we mean a family $\{W^n\}_{n \geq 0}$ of $d$-dimensional continuous processes defined on the Wiener space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ such that

1. for every $\omega \in \Omega$, $t \mapsto W^n_t(\omega)$ is continuous and piecewise continuously differentiable;
2. $W^n_0(\omega)$ is $\mathcal{F}_{1/n}$-measurable and $\mathbb{E}W^n_0 = 0$;
3. $W^n_{t+n/(k+1)}(\omega) = W^n_t(\theta_{k/n}\omega) + \omega(k/n)$, for every $k \in \mathbb{N}$, $t \geq 0$ and $\omega \in \Omega$, where $\theta_t$ is the shift operator defined by $(\theta_t\omega)(s) = \omega(t+s) - \omega(t)$;
4. there exists a positive constant $C$ such that
   \[
   \mathbb{E}[|W^n_0|^6] \leq Cn^{-3}, \quad \mathbb{E}\left(\int_0^{1/n} |\dot{W}^{n_4}_s|^2 \, ds\right) \leq Cn^{-3}.
   \]

From [15, VI. Section 7] we know that for such a approximation sequence $W^n$ we have

$$
\lim_{n \to \infty} \mathbb{E} \sup_{0 \leq t \leq T} |W^n_t - W^n_t|^2 = 0
$$

for every $T > 0$. We further introduce the following notations for $t > 0$ and $i, j \in \{1, \ldots, d\}$:

\[
s^n_{ij}(t) := s_{ij}(t, n) := \frac{1}{2t} \mathbb{E} \left[ \int_0^t \left( W^n_{s,i,j} \dot{W}^{n}_{s,i,j} - W^n_{s,i} \dot{W}^{n}_{s,j} \right) \, ds \right]
\]

and

\[
c^n_{ij}(t) := c_{ij}(t, n) := \frac{1}{t} \mathbb{E} \left[ \int_0^t \dot{W}^{n}_{s,i,j} \left( W^n_{s,i,j} - W^n_{s,i} \right) \, ds \right].
\]

\[\dot{W}^{n}_s := \frac{dW^n_s}{ds}\]
Notice that $s(t, n)$ is a skew-symmetric $d \times d$-matrix for each $t$ and $n$, i.e. $s_{ij}(t, n) = -s_{ji}(t, n)$. In order to control the convergence rate of the solutions $X^n$ driven by $W^n$ to $X$ driven by $W$, we will impose the following:

**Assumption 3.13.** There exists a skew-symmetric $d \times d$-matrix $s$ and a rate function $f_n$ such that

$$|s_{ij}(n^{-1}) - s_{ij}| \leq f_n, \quad \lim_{n \to \infty} f_n = 0, \quad i, j = 1, \ldots, d.$$  

Under **Assumption 3.13**, we define

$$c_{ij} := s_{ij} + \frac{1}{2} \delta_{ij}, \quad i, j = 1, \ldots, d. \quad (3.5)$$

Correspondingly, for any pair of functions $\varphi, \psi : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ sufficiently regular, we define a vector field $c : \varphi \cdot \nabla \psi : \mathbb{R}^d \to \mathbb{R}^d$ componentwise by

$$(c : \varphi \cdot \nabla \psi)_k(x) := \sum_{i,j=1}^d c_{ij} \varphi_{ij}(x) \partial_i \psi_{jk}(x) \quad \forall k = 1, \ldots, d.$$  

The next result gives us the dependence between the convergence rate of $X^n$ to $X$ and the coefficients $b, \sigma$; it is just a useful rewriting of [15, IV. Theorem 7.2], making explicitly the dependence on the coefficients $(b, \sigma)$ and the rate of convergence. For more details on its derivation, we refer the interested reader to Appendix A.

**Theorem 3.14** (Wong–Zakai). Let $b \in C^1_x$, $\sigma \in C^2_x$ and $(W^n_t)_{n \geq 1}$ be an approximation of $W_t$, in the sense of **Definition 3.12**, satisfying **Assumption 3.13**. Consider the SDEs

$$X_t = x_0 + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s + \int_0^t (c : \sigma \cdot \nabla \sigma)(X_s) \, ds,$$

and

$$X^n_t = x_0 + \int_0^t b(X^n_s) \, ds + \int_0^t \sigma(X^n_s) \, dW^n_s.$$  

Then for each $T > 0$ there exists a constant $C$ such that

$$\mathbb{E} \left[ \sup_{t \in [0, T]} |X_t - X^n_t|^2 \right] \leq \exp \left( C(1 + \|b\|_{C^1_x}^2 + \|\sigma\|_{C^2_x}^4 + \|\sigma\|_{C^2_x}^2 \|\sigma\|_{C^2_x}^2) \right) \left( f_n^2 + n^{-\frac{1}{2}} \right), \quad (3.6)$$

where $f_n$ is from **Assumption 3.13** and $C$ depends on $T$ and $\max_{i,j} |c_{ij}|$, but not on $b, \sigma$ and $n$. \(^5\)

In order to control the error we commit by applying **Theorem 3.14** at the level of smooth approximations of our coefficients, we introduce the following classes for $b$ and $\sigma$.

**Definition 3.15.** Let $b \in L^p_x$ for some $p_1 \in (d, +\infty)$, $p_1 \geq 2$, and $r \in [0, \infty) \to [0, \infty)$ be an increasing function. For a sequence $(b^n)_{n \geq 1}$ of continuous functions $b^n : \mathbb{R}^d \to \mathbb{R}^d$, we write $(b^n)_{n \geq 1} \in \mathcal{A}(b, p_1, r)$ if:

\(^5\)It is claimed in [20, Theorem 3.9] that the resulting error in (3.6) is $f_n^2 + n^{-1+\epsilon}$, where $\epsilon$ can be chosen arbitrarily small; unfortunately this is not the case, as there are other error terms of the form $n^{-\epsilon}$ which become too slow if $\epsilon$ is chosen too small. For more details, see Appendix A.
We are now ready to state our result.

Let $\sigma$ satisfy condition $(\mathcal{H}_1^\sigma)$ from Assumption 1.1 with constant $K$, $\nabla \sigma \in L_\infty^{P_1}$; let $r \in [0, \infty) \to [0, \infty)$ be an increasing function. For a sequence $(\sigma^n)_{n \geq 1}$ of regular functions $\sigma^n : \mathbb{R}^{d \times d} \to \mathbb{R}^d$, we write $(\sigma^n)_{n \geq 1} \in \mathcal{A}(\sigma, p_2, r)$ if:

(i) $\sigma^n$ satisfy condition $(\mathcal{H}_1^\sigma)$ with the same constant $K$, uniformly in $n$,
(ii) $\sigma^n \in C^2_x$,
(iii) $\|\sigma - \sigma^n\|_{L_\infty^{P_1}} \to 0$ as $n \to \infty$,
(iv) $\sup_{r \in \mathbb{N}} \|\nabla \sigma^n\|_{L_\infty^{P_2}} < \infty$ and $\|\sigma^n\|_{C^2_x} + K\|\sigma^n\|_{C^2_x} \leq r(n)\|\sigma\|_{L_\infty^{P_2}}$.

We are now ready to state our result.

**Theorem 3.17.** Let $(p_1, p_2) \in (d, +\infty)^2$, $p_1 \geq 2$, and let $b \in L_\infty^{P_1}$, $\sigma$ satisfy condition $(\mathcal{H}_1^\sigma)$ in Assumption 1.1 and $\nabla \sigma \in L_\infty^{P_1}$. Assume further that $(W^n)_{n \geq 1}$ is an approximation of $W$, in the sense of Definition 3.12, satisfying Assumption 3.13. Let $X$ be the solution to

$$X_t = x_0 + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s + \int_0^t (c : \sigma \cdot \nabla \sigma)(X_s) \, ds$$  \hspace{1cm} (3.7)

for $c$ as defined in (3.5). Let $(b^n)_{n \geq 1} \in \mathcal{A}(b, p_1, r)$, $(\sigma^n)_{n \geq 1} \in \mathcal{A}(\sigma, p_2, r)$ be approximation sequences for $b$ and $\sigma$ such that

$$\lim_{n \to \infty} \exp \left( Cr(n)^2 (\|b\|_{L_\infty^{P_1}}^2 + \|\sigma\|_{L_\infty^{P_1}}^2) \right) (f_n^2 + n^{-1/5}) = 0$$  \hspace{1cm} (3.8)

where $C$ is the constant appearing in (3.6) and $f_n$ is given in Assumption 3.13. Then we have

$$\lim_{n \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t - X^n_t|^2 \right] = 0,$$  \hspace{1cm} (3.9)

where $(X^n_t)_{t \geq 0}$ is the solution to the equation

$$X^n_t = x_0 + \int_0^t b^n(X^n_s) \, ds + \int_0^t \sigma^n(X^n_s) \, dW^n_s.$$

**Proof.** Observe that, under our assumptions, $c : \sigma \cdot \nabla \sigma \in L_\infty^{P_2}$, therefore the SDE (3.7) is well-posed (either by [29] or the facts recalled in Section 2.1). Let $X_{m,t}$ be the solution to

$$X_{m,t} = x_0 + \int_0^t b^m(X_{m,s}) \, ds + \int_0^t \sigma^m(X_{m,s}) \, dW_s + \int_0^t (c : \sigma^m \cdot \nabla \sigma^m)(X_{m,s}) \, ds.$$

By Theorem 3.14, we know that for any $m \geq 1$ it holds

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_{m,t} - X^n_{m,t}|^2 \right] \leq \exp \left( C (\|b^m\|_{C^1_x}^2 + \|\sigma^m\|_{C^2_x}^2) + (f_n^2 + n^{-1/5}) \right) \left( f_n^2 + n^{-1/5} \right).$$
where $C$ is independent of $m$ and $n$ and $X_{m,t}^n$ is the solution to
\[ X_{m,t}^n = x_0 + \int_0^t b^m(X_{m,s}^n) \, ds + \int_0^t \sigma^m(X_{m,s}^n) \, \xi^m_s \, ds. \]
By choosing $m = n$ and using the condition (3.8), we obtain
\[ \lim_{n \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_{n,t}^n - X_{n,t}^n|^2 \right] = 0. \]
On the other hand, arguing similarly to the proof of Corollary 2.12, using the estimates
\[ \|c : (\sigma^m - \sigma) \cdot \nabla \sigma^m\|_{L^p_x} \lesssim \|\sigma^m - \sigma\|_{L^2} \|\sigma^m\|_{L^p_x} \lesssim \|\sigma^m - \sigma\|_{L^2}, \]
\[ \|c : \sigma \cdot \nabla (\sigma^m - \sigma)\|_{W^{1,p}_x} \lesssim \|\nabla (\sigma^m - \sigma)\|_{W^{1,p}_x} (\|\sigma\|_{L^2} + \|\nabla \sigma\|_{L^p_x}) \lesssim \|\sigma^m - \sigma\|_{L^2}, \]
it is not difficult to show that
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t - X_{n,t}^n|^2 \right] \lesssim \|b - b^n\|_{L^p_x}^2 + \|\sigma - \sigma^n\|_{L^p_x}^2. \]
Overall we obtain
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t - X_{n,t}^n|^2 \right] \lesssim \|b - b^n\|_{L^p_x}^2 + \|\sigma - \sigma^n\|_{L^p_x}^2 + \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_{n,t} - X_{n,t}^n|^2 \right] \]
which implies (3.9) by letting $n \to \infty$. \hfill \Box

3.4. **Other applications of stability estimates in the literature.** We briefly discuss here two further examples of implicit applications of the stability estimates from Proposition 2.11 in the literature. In this case we do not provide any new results, yet we believe that highlighting this step makes the arguments from the papers [19] and [10] clearer.

3.4.1. **Euler-Maruyama schemes for singular SDEs.** Similarly to Section 3.3, when dealing with singular drifts one cannot try to simulate the solution to the SDE directly, as the unboundedness of $b$ can be a major source of errors. Again, one can circumvent this problem by introducing a two step approximation, based on first mollifying $b$, and then applying an Euler–Maruyama scheme to the SDE with approximate drift $b^n$. This idea has seen precursors in [17], in the study of weak error rates for integrable drifts, and in [4], in the one-dimensional setting with $b$ only enjoying some negative Hölder regularity. Here we follow the recent work [19], which studied the strong error in the general multidimensional setting, with $(b, \sigma)$ satisfying Assumption 1.1; therein this two-step approximation was directly unified in a unique way and thus stability estimates were only used in a very implicit way.

Let $(b, \sigma)$ satisfy Assumption 1.1 and $X$ be the solution to (1.1); there are two different (yet similar) ways one can proceed.

On one hand, given a sequence of smooth bounded approximations $(b^n)_n$ of $b$, one can introduce the solution $\hat{X}^n$ associated to $(b^n, \sigma)$ and then apply the Euler–Maruyama scheme of step $n$ to $(b^n, \sigma)$, yielding a numerical approximation $\hat{X}^{n,n}$; the overall error splits into $\epsilon = \epsilon^1 + \epsilon^2$, associated respectively to $X^n - \hat{X}^n$ and $\hat{X}^n - \hat{X}^{n,n}$. The first error $\epsilon^1$ can be estimated by means of either Theorem 1.2 or the more refined estimates from Section 2.3 and yields a quantity in
the style of \( \omega_n \) as defined in [19]. The error \( e_2 \) instead is the purely numerical one, which has to be treated by different techniques and will result in a polynomial decay rate \( n^{-\alpha} \). Alternatively, one can proceed more directly as in [19] and, given the approximations \( (b^n)_n \), compare \( X \) to the solutions \( X^n \) to

\[
\frac{dX_t^n}{dt} = b^n_t(X^n_{k_n(t)}) \, dt + \sigma_t(X^n_{k_n(t)}) \, dW_t, \quad X^n|_{t=0} = X_0^n,
\]

where \( k_n(t) = \lfloor tn \rfloor /n \), \( \lfloor \cdot \rfloor \) denoting the integer part. For fixed \( n \in \mathbb{N} \), we are in the setting of Proposition 2.11 by considering \( Y_t := X_t^n \), corresponding to (2.8) for the choice

\[
R_1^t := b^n_t(X^n_{k_n(t)}) - b_t(X^n_t), \quad R_2^t := \sigma_t(X^n_{k_n(t)}) - \sigma_t(X^n_t).
\]

Assuming for the moment that \( (Y, R^1, R^2) \) satisfy Assumption 2.9, an application of Proposition 2.11 then yields

\[
\sup_{t \in [0,T]} |X_t - X^n_t| \lesssim \|X_0 - X^n_0\|_{L^m_{\omega}} + \sup_{t \in [0,T]} \|V_t\|_{L^m_{\omega}}
\]

for \( V \) as defined in (2.9). By our assumptions, the properties of \( u \) and martingale inequalities, it is easy to deduce that

\[
\sup_{t \in [0,T]} \|V_t\|_{L^m_{\omega}} \leq \left\| \int_0^T (b^n_t - b_t)(X^n_t) \, dt \right\|_{L^m_{\omega}} + \left\| \int_0^T (b^n_t(X^n_t) - b^n_t(X^n_{k_n(t)})) \, dt \right\|_{L^m_{\omega}}
\]

\[
+ \left\| \int_0^T (\sigma_t(X^n_t) - \sigma_t(X^n_{k_n(t)})) |D^2 u_t(X^n_t)| \, dt \right\|_{L^m_{\omega}}
\]

\[
+ \left\| \left( \int_0^T (\sigma_t(X^n_t) - \sigma_t(X^n_{k_n(t)}))^2 \, dt \right)^{1/2} \right\|_{L^m_{\omega}}.
\]

(3.11)

Again, the first term gives rise to the approximation error \( e_1 \) related to \( \omega_n \), while all the others are numerical errors producing a decay \( n^\alpha \). This is the point where the fine analysis on the properties of the process \( X^n \) conducted in [19, Section 5], also based on novel tools like stochastic sewing techniques, must kick in. The same kind of results in fact verify that \( (Y, R^1, R^2) \) verify Assumption 2.9, see for instance Theorem 5.1 and Lemma 5.14 from [19].

Nonetheless, we find it important to stress how reaching an estimate of the form (3.11) can be performed in the framework of well established SDE tools and do not need any substantially new techniques.

3.4.2. Interacting particle systems. We now set ourselves in the setting of [10], which studied the propagation of chaos for interacting particle systems with singular kernel of Krylov-Röckner type. Therein again Proposition 2.11 was applied, although implicitly; we make this point clear as follows.
For any $N \in \mathbb{N}$, consider the interacting particle system
\begin{equation}
\frac{dX_t^{N,i}}{dt} = B_t(X_t^{N,i}, \mu_t^N) \, dt + \sigma_t(X_t^{N,i}) \, dW_t^i, \quad \mu_t^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{N,i}}, \quad \forall \, i = 1, \ldots, N,
\end{equation}
where $X_0^{N,i} = \xi^i$ for some i.i.d. $\{\xi^i\}_{i \in \mathbb{N}}$, $\{W_i\}_{i \in \mathbb{N}}$ are independent Brownian motions, $\sigma$ satisfies properties $(\mathcal{H}_\sigma^t)$–$(\mathcal{H}_\sigma^\infty)$ from Assumption 1.1 and $B$ is of the form considered in [10]. As $N \to \infty$, each particle $X_t^{i,N}$ of system (3.12) is expected to converge to an independent copy of the associated McKean–Vlasov SDE
\begin{equation}
\frac{dX_t^i}{dt} = B_t(X_t^i, \mu_t) \, dt + \sigma_t(X_t^i) \, dW_t^i, \quad \mu_t = L(X_t^i),
\end{equation}
a property which is usually referred to as propagation of chaos; in [10] the authors show that such convergence holds in a strong sense, for suitable singular $B$, see Assumption (H)$^b$ therein.

To explain how their techniques are related to ours, let us first recall some basic facts. Under (H)$^b$, equation (3.13) is wellposed and the associated drift $b_t^\mu(x) := B_t(x, \mu_t)$ satisfies an $L_1^\mu$-$L_2^\mu$ integrability condition. Moreover by exchangeability of (3.12), it suffices to show convergence for $i = 1$. Finally, by setting $X_t^{N,1} = Y$ and
\begin{align*}
R_t^1 := B_t(X_t^{N,1}, \mu_t^N) - b_t^{\mu}(X_t^{N,1}), \quad R_t^2 := 0,
\end{align*}
we are in the setting of Proposition 2.11. The triple $(Y, R_t^1, R_t^2)$ satisfies 2.9 thanks to an application of the partial Girsanov transform, cf. [10, Lemma 5.3 (ii)]; observe that $X_t^{1,N}$ is measurable w.r.t. to the filtration generated by $\{\xi^i, W_i\}_{i \in \mathbb{N}}$, which is covered by Assumption 2.9-iii) by taking $Z = (\xi^i, W_i)_{i=2}^N$. Therefore by Proposition 2.11 and the usual properties of $u$ we deduce that
\begin{equation*}
\left\| \sup_{t \in [0,T]} |X_t^{N,1} - X_t^1| \right\|_{L^m} \leq \left\| \int_0^T \left| B_t(X_t^{N,1}, \mu_t^N) - b_t^{\mu}(X_t^{N,1}) \right| \, dt \right\|_{L^m}.
\end{equation*}
The convergence of the quantity on the r.h.s. to 0 as $N \to \infty$ is established in [10, Lemma 6.1] for $m \gamma = 2$ (but the other cases follow similarly), which yields the desired propagation of chaos property.

Let us point out two interesting facts coming from our analysis:

i) In order to apply our stability results, although some key information on the processes $X_t^{N,i}$ is needed, the main requirement is the regularity of $b^\mu$. This can be better than the one of the original $B$, especially in the case of convolutional drifts $B_t(x, \mu) = (\phi_t * \mu)(x)$, thus there is hope to establish such results also in cases where $\phi$ does not satisfy the Krylov–Röckner condition, but $b^\mu$ does.

ii) We took $\sigma$ measure independent as in [10], but this doesn’t really play a role in our analysis; thus it might be possible to extend several result to the $\mu$-dependent diffusion as well. The key problem would then be to establish some analogue of [10, Lemma 6.1] in this setting.

\footnote{Actually, the authors in [10] use the local spaces $L_t^q$-$L_x^p$, but at the level of exposition this doesn’t change much.}
Appendix A. More details on the proof of Theorem 3.14

In this appendix, we follow carefully the proof of [15, IV. Theorem 7.2] in order to indicate the precise dependence on the regularity of the coefficients \((b, \sigma)\) and the resulting convergence as stated in Theorem 3.14.

It will be convenient in several passages to keep the same notations as in [15], based on the use of a parameter \(\delta \ll 1\) and a function \(N(\delta) : (0, 1) \to \mathbb{N}\) satisfying the hypothesis of [15, IV. Lemma 7.1], so let us explain shortly how it translates into our framework. The parameter \(\delta\) corresponds to \(n^{-1}\), while the function \(N(\delta)\) is given by \(N(\delta) := [\delta^{-r/4}]\) for a suitable parameter \(r \in (0, 1)\) to be chosen later on. Observe that by construction, for any \(\varepsilon \in (0, 1)\) it holds

\[
\lim_{\delta \to 0^+} N(\delta)^{4\delta} = 0, \quad \lim_{\delta \to 0} N(\delta) = +\infty. \tag{A.1}
\]

For such choice of \(N(\delta)\), we set \(\tilde{\delta} := N(\delta)\delta\) and

\[
[s]^+(\tilde{\delta}) = (k + 1)\delta, \quad [s]^{-}(\tilde{\delta}) = k\tilde{\delta} \quad \text{if} \quad k\tilde{\delta} \leq s < (k + 1)\tilde{\delta}.
\]

Several terms, previously indexed over \(n\) in Section 3.3, will now be indexed over \(\delta\), e.g. we will write \(X^\delta, W^\delta, f^\delta_s\) and \(s(t, \delta)\) in place of \(X^n, W^n, f^n_s\) and \(s(t, n)\). Finally, throughout this appendix we will adopt the convention that, whenever the symbol \(\lesssim\) appears in the estimates, the hidden constant may depend on \(T, \max_{ij} |c_{ij}|\) or \(d\), but no other parameters.

Before going to the actual proof of Theorem 3.14, we need to collect a few basic observations coming from that of [15, IV. Lemma 7.1]. Therein it is shown that, for any \(k \in \mathbb{N}\), it holds

\[
k\delta s_{ij}(k\delta, \delta) = k\delta s_{ij}(\tilde{\delta}, \tilde{\delta}),(\delta, \delta)
\]

and moreover that

\[
s_{ij}(\delta, \delta) - c_{ij}(k\delta, \delta) + \frac{1}{2}\delta_{ij} = \frac{1}{k\delta} \mathbb{E}\left[W^\delta_{0,i}(0) W^\delta_{k\delta,j} - W^\delta_{0,i} W^\delta_{0,j}\right]. \tag{A.2}
\]

for the choice \(k = N(\delta)\), this readily implies the estimate

\[
|s_{ij}(\delta, \delta) - c_{ij}(\tilde{\delta}, \tilde{\delta}) + \frac{1}{2}\delta_{ij}| \leq \tilde{\delta}^{-1}[\tilde{\delta}^{3/2}\delta^{1/2} + \delta] \lesssim N(\delta)^{-1/2}
\]

which combined with Assumption 3.13 implies

\[
|c_{ij}(\tilde{\delta}, \tilde{\delta}) - c_{ij}|^2 \leq f^2_\delta + N(\delta)^{-1}. \tag{A.3}
\]

We can now proceed with the proof of Theorem 3.14. First by [15, IV. 7 (7.47)], for every \(s \leq t\) we have

\[
|X^\delta_t - X^\delta_s| \leq \|b\|_\infty (t - s) + \|

σ\|_\infty \int_s^t |W^\delta_s| \, ds. \tag{A.4}
\]

As in [15, IV. 7 (7.48)], we decompose \(X^n - X\) into the following four terms:

\[
X^\delta_t - X_t := H_1(t) + H_2(t) + H_3(t) + H_4(t) \tag{A.5}
\]
where

\[ H_1(t) = \int_{[t]} (\sigma(X_s^\delta) \, dW_s^\delta - \int_{[t]} (c : \sigma \cdot \nabla \sigma)(X_s) \, ds, \]

\[ H_2(t) = \int_{\delta} (\sigma(X_s^\delta) \, dW_s^\delta - \int_{\delta} (c : \sigma \cdot \nabla \sigma)(X_s) \, ds, \]

\[ H_3(t) = \int_0^\delta (\sigma(X_s^\delta) \, dW_s^\delta - \int_0^\delta (c : \sigma \cdot \nabla \sigma)(X_s) \, ds \]

and

\[ H_4(t) = \int_0^t b(X_s^\delta) \, ds - \int_0^t b(X_s) \, ds. \]

Easily observe that

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |H_4(t)|^2 \right] \leq \|\nabla b\|_\infty^2 \int_0^T \mathbb{E}[|X_s^\delta - X_s|^2] \, ds. \quad (A.6) \]

We rewrite

\[ H_1(t) = H_{11}(t) - H_{12}(t) - H_{13}(t). \]

Following from [15, IV. 7 (7.54)] we get

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |H_{13}(t)|^2 \right] \leq \|\sigma \cdot \nabla \sigma\|_\infty^2 N(\delta)^2 \delta^2, \]

and [15, IV. 7 (7.55)] gives us

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |H_{11}(t)|^2 \right] \leq \|\sigma\|_\infty^2 N(\delta)^{3/2} \delta^{5/2}. \]

Note that

\[ H_{12} = \sigma(X_{[t]} - (\delta))(W_t - W_{[t]} - (\delta) + \int_{[t]} (\sigma(X_s) - \sigma(X_{[t]} - (\delta))) \, dW_s =: H_{121}(t) + H_{122}(t). \]

Then [15, IV. 7 (7.56)] yields

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |H_{121}(t)|^2 \right] \leq \|\sigma\|_\infty^2 N(\delta)^{3/2} \delta^{1/2} \]

and [15, IV. 7 (7.57)] implies

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |H_{122}(t)|^2 \right] \leq \|\nabla \sigma\|_\infty^2 (\|\sigma\|_\infty^2 + \|b\|_\infty^2) \delta, \]

where inside these passages we used the inequality

\[ \mathbb{E}[|X_s(x) - x|^2] \leq \|\sigma\|_\infty^2 s + \|b\|_\infty^2 s^2. \]
Putting all these estimates together we get
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |H_1(t)|^2 \right] \lesssim (1 + \|\sigma\|_{C^2}^2 + \|b\|_{\infty}^4) N(\delta)^{3/2} \delta^{1/2}.
\] (A.7)

Since \( |H_3| \leq \sup_{0 \leq t \leq T} |H_1(t)| \),
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |H_3(t)|^2 \right] \leq \mathbb{E}\left[ \sup_{0 \leq t \leq T} |H_1(t)|^2 \right] \lesssim (1 + \|\sigma\|_{C^1}^4 + \|b\|_{\infty}^4) N(\delta)^{3/2} \delta^{1/2}.
\] (A.8)

We are only left with estimating \( H_2 \). By integration by parts, we have
\[
\int_{k\delta}^{(k+1)\delta} \sigma(X_s^\delta) dW_s^\delta = \sigma(X_{k\delta}^\delta) [W_{(k+1)\delta}^\delta - W_{k\delta}^\delta]
\]
\[
+ \int_{k\delta}^{(k+1)\delta} \nabla \sigma(X_s^\delta) [\sigma(X_s^\delta) W_s^\delta + b(X_s^\delta)] \cdot [W_{(k+1)\delta}^\delta - W_{k\delta}^\delta] \, ds
\]
\[=: J_1(k) + J_2(k). \]

Also
\[
J_1(k) = \sigma(X_{k\delta-\delta}^\delta)(W_{(k+1)\delta}^\delta - W_{k\delta}^\delta) + [\sigma(X_{k\delta}^\delta) - \sigma(X_{k\delta-\delta}^\delta)] [W_{(k+1)\delta}^\delta - W_{k\delta}^\delta]
\]
\[+ \sigma(X_{k\delta-\delta}^\delta) (W_{(k+1)\delta}^\delta - W_{(k+1)\delta-\delta}) + \sigma(X_{k\delta-\delta}^\delta) (W_{k\delta}^\delta - W_{(k+1)\delta-\delta})
\]
\[=: J_{11}(k) + J_{12}(k) + J_{13}(k) + J_{14}(k). \]

Write
\[
H_2(t) = I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t)
\]
where
\[
I_1(t) = \sum_{k=1}^{m(t)-1} J_{11}(k) + \int_{\delta}^{[t/\delta]} \sigma(X_s^\delta) \, dW_s^\delta,
\]
with \( m(t) :=[t/\delta] \), and
\[
I_i(t) = \sum_{k=1}^{m(t)-1} J_{1i}(k), \quad i = 2, 3, 4,
\]
\[
I_5(t) = \sum_{k=1}^{m(t)-1} J_2(k) - \int_{\delta}^{[t/\delta]} (c : \nabla \sigma)(X_s^\delta) \, ds.
\]

From [15, IV. 7 (7.63)] we have
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |I_1(t)|^2 \right] \leq \|\nabla \sigma\|_{L^2}^2 \int_0^T \mathbb{E}[|X_s^\delta - X_{(k+1)\delta}^\delta|^2] \, ds + (\|\sigma\|_{L^2}^2 + \|b\|_{L^2}^2) N(\delta)^2 \delta^{1/2}.
\]
and \([15, IV. 7 (7.64)]\) shows
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |I_2(t)|^2 \right] \leq \|\nabla \sigma\|_{\infty}^2 N(\delta)^{-1}.
\]
\([15, IV. 7 (7.65), (7.66)]\) give us
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |I_3(t)|^2 \right] + \mathbb{E} \left[ \sup_{0 \leq t \leq T} |I_4(t)|^2 \right] \leq \|\sigma\|_{\infty}^2 N(\delta)^{-1}
\]
For \(I_5\) we rewrite
\[
I_5(t) = I_{51}(t) + I_{52}(t) + I_{53}(t) + I_{54}(t) + I_{55}(t)
\]
with
\[
I_{51}^{(h)}(t) = \sum_{k=1}^{m(t)-1} \int_{(k+1)\delta}^{k\delta} \sum_{i,l,j=1}^{d} \left[ (\sigma_{il} \partial \sigma_{jk})(X^\delta_i) - (\sigma_{il} \partial \sigma_{jk})(X^\delta_{k\delta}) \right] \dot{W}^{\delta}_{s,j} \left[ W^{\delta}_{(k+1)\delta,j} - W^{\delta}_{s,j} \right] \, ds,
\]
\[
I_{52}^{(h)}(t) = \sum_{k=1}^{m(t)-1} \int_{(k+1)\delta}^{k\delta} \sum_{i,l,j=1}^{d} \left[ (\sigma_{il} \partial \sigma_{jk})(X^\delta_i) \right] \left[ W^{\delta}_{s,j} \left( W^{\delta}_{(k+1)\delta,j} - W^{\delta}_{s,j} \right) - c_{ij}(\delta, \delta) \right] \, ds,
\]
\[
I_{53}^{(h)}(t) = \sum_{k=1}^{m(t)-1} \int_{(k+1)\delta}^{k\delta} \sum_{i,l,j=1}^{d} c_{ij} \left[ (\sigma_{il} \partial \sigma_{jk})(X^\delta_i) - (\sigma_{il} \partial \sigma_{jk})(X^\delta_{k\delta}) \right] \, ds,
\]
\[
I_{54}^{(h)}(t) = \sum_{k=1}^{m(t)-1} \int_{(k+1)\delta}^{k\delta} \sum_{i,l,j=1}^{d} c_{ij} \left[ (\sigma_{il} \partial \sigma_{jk})(X^\delta_i) \right] \left[ W^{\delta}_{s,j} \left( W^{\delta}_{(k+1)\delta,j} - W^{\delta}_{s,j} \right) - c_{ij}(\delta, \delta) \right] \, ds,
\]
\[
I_{55}^{(h)}(t) = \sum_{k=1}^{m(t)-1} \sum_{i,l,j=1}^{d} \left[ \delta(\sigma_{il} \partial \sigma_{jk})(X^\delta_{k\delta}) - c_{ij} \right].
\]
First we get from \([15, IV. 7 (7.67)]\)
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |I_{55}(t)|^2 \right] \leq \sum_{i,j=1}^{d} \|\sigma \cdot \nabla \sigma\|_{\infty}^2 (c_{ij}(\delta, \delta) - c_{ij})^2
\]
and by \([15, IV. 7 (7.68)]\)
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |I_{54}(t)|^2 \right] \leq \|\nabla (\sigma \cdot \nabla \sigma)\|_{\infty}^2 \left[ \int_{0}^{T} \mathbb{E}[|X^\delta_s - X^\delta_t|^2] \, ds + (\|\sigma\|_{\infty}^2 + \|b\|_{\infty}^2) N(\delta)^2 \delta \right].
\]
Furthermore by \([15, IV. 7 (7.69)]\)
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |I_{52}(t)|^2 \right] \leq \|b \cdot \nabla \sigma\|_{\infty}^2 N(\delta)^2 \delta
\]
and for \(I_{51}\) \([15, IV. 7 (7.70)]\) yields
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |I_{51}(t)|^2 \right] \leq (\|\sigma\|_{\infty}^2 + \|b\|_{\infty}^2) N(\delta)^4 \delta.
\]
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Applying identity (A.2) for the choices $k = N(\delta), k = N(\delta) - 1$, one can see that

\[
|N(\delta)^{-1} c_{ij}^*(\bar{\delta}, \delta)| = |N(\delta)^{-1} \left[ s_{ij}(\delta, \delta) + \frac{1}{2} \delta_{ij} \right] - (\bar{\delta})^{-1} \mathbb{E} \left[ W_{0,i}^\delta \left( W_{N(\delta)\delta,j} - W_{(N(\delta) - 1)\delta,j} \right) \right] |
\]

\[
\lesssim N(\delta)^{-1} + (\bar{\delta})^{-1} \delta^1 \delta^2 = N(\delta)^{-1}.
\]

Therefore

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |I_5(t)|^2 \right] \lesssim \|\sigma\|^2_{\infty} (N(\delta)^3 \delta + N(\delta)^{-1}),
\]

Collecting all the terms related to $I_5$ and applying (A.3), we get the following:

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |I_5(t)|^2 \right] \lesssim \|\nabla (\sigma \cdot \nabla \sigma)\|^2_{\infty} \int_0^T \mathbb{E} \left[ |X^\delta_{\epsilon} - X_{\epsilon}|^2 \right] \, ds
\]

\[
+ \left( 1 + \|\sigma\|_{C_2^4}^g + \|b\|_{\infty}^4 \right) \left( f^2_\delta + N(\delta)^4 \delta + N(\delta)^{-1} \right).
\]

Therefore

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |H_5(t)|^2 \right] \lesssim \left( \|\nabla \sigma\|^2_{\infty} + \|\nabla (\sigma \cdot \nabla \sigma)\|^2_{\infty} \right) \int_0^T \mathbb{E} \left[ |X^\delta_{\epsilon} - X_{\epsilon}|^2 \right] \, ds
\]

\[
+ \left( 1 + \|\sigma\|_{C_2^4}^g + \|b\|_{\infty}^4 \right) \left( f^2_\delta + N(\delta)^4 \delta + N(\delta)^{-1} \right). \tag{A.9}
\]

By combining (A.5), (A.6), (A.7), (A.8) and (A.9) we can conclude that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t - X^\delta_t|^2 \right] \lesssim \left( 1 + \|b\|^2_{C_1^2} + \|\sigma\|^4_{C_2^4} + \|\sigma\|_{\infty} \|\sigma\|^2_{C_2^2} \right) \int_0^T \mathbb{E} \left[ |X^\delta_{\epsilon} - X_{\epsilon}|^2 \right] \, ds
\]

\[
+ \left( 1 + \|\sigma\|_{C_2^4}^g + \|b\|_{\infty}^4 \right) \left( f^2_\delta + N(\delta)^4 \delta + N(\delta)^{-1} \right).
\]

Grönwall’s inequality then implies the estimate

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t - X^\delta_t|^2 \right] \lesssim \exp \left( C' \left( 1 + \|b\|^2_{C_1^2} + \|\sigma\|^4_{C_2^4} + \|\sigma\|_{\infty} \|\sigma\|^2_{C_2^2} \right) \right) \times
\]

\[
\times \left( 1 + \|\sigma\|_{C_2^4}^g + \|b\|_{\infty}^4 \right) \left( f^2_\delta + N(\delta)^4 \delta + N(\delta)^{-1} \right). \tag{A.10}
\]

Observe that, up to relabelling the constant $C'$, the polynomial term $1 + \|\sigma\|^g_{C_2^4} + \|b\|^4_{\infty}$ in (A.10) can be reabsorbed in the exponential. Moreover recall that $\delta = n^{-1}$ and $N(\delta) \sim n^{t/4}$, so that

\[
N(\delta)^4 \delta + N(\delta)^{-1} \sim n^{\frac{t}{4} + \frac{1}{2}} + n^{-1} + n^{-\frac{1}{2}};
\]

\[31\]
optimizing it w.r.t. $\varepsilon$ yields the choice $\varepsilon = 4/5$ and a corresponding decay rate $n^{-1/5}$. Combining these two last observations with (A.10) yields the desired estimate (3.6). □

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