Cosmological expansion on a dilatonic brane-world

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In this paper we study brane-world scenarios with a bulk scalar field, using a covariant formalism to obtain a 4D Einstein equation via projection onto the brane. We discuss, in detail, the effects of the bulk on the brane and how the scalar field contribute to the gravitational effects. We also discuss choice of conformal frame and show that the frame selected by the induced metric provides a natural choice. We demonstrate our formalism by applying it to cosmological scenarios of Randall-Sundrum and Hořava-Witten type models. Finally we consider the cosmology of models where the scalar field couples non-minimally to the matter on the brane. This gives rise to a novel scenario where the universe expands from a finite scale factor with an initial period of accelerated expansion, thus avoiding the singularity and flatness problem of the standard big bang model.

I. INTRODUCTION

The standard Big-Bang model based on the FRW spacetime now has many observational successes, notably the observed expansion of distant galaxies, the synthesis of light elements during Big Bang Nucleosynthesis (BBN) and the isotropy of the Cosmic Microwave Background (CMB). These successes mainly probe what, from the point of view of this paper, we shall describe as the late universe, that is, after neutrino decoupling — which is just before BBN. Our knowledge of the universe before BBN is very limited and hence it is possible for substantial modifications to our understanding of the expansion of the universe during such an epoch.

Motivated by attempts to solve the hierarchy problem of particle physics and to reconcile cosmology with string theory, much interest has been generated by the idea that this Big-Bang model is embedded inside some higher dimensional spacetime: the 4D universe that we see being thought as a 3-brane or brane-world. In such models the matter fields of the standard model are localized by some mechanism to only have support on the 3-brane, with gravity and other undetected particles, such as scalar fields, propagating in the directions perpendicular to the brane or ‘bulk’ as they have become known. In the most of popular of these models, the Randall-Sundrum (RS) scenario, the brane is a hypersurface (or co-dimension 1 brane) in a 5D spacetime with $Z_2$ symmetry along the extra dimension.

In fact, RS made two simple proposals (see also ref. for related earlier work): the first has two branes of opposite tension in an anti-de-Sitter (adS) background spacetime with $Z_2$ symmetry. In such a model the global spacetime structure is warped, that is, the spatial 3-metric contains an exponential or warp factor which is a function of the coordinate along the extra-dimension. They suggested that this could possibly solve the hierarchy problem of particle physics by relating the Planck scale on our brane (the one with negative tension) to that on the other via an exponential function of the distance between them. Hence, a large hierarchy ($\sim 10^{16}$) can be generated from a relatively small dimensionless number.

In the second such model, there is only a single positive tension brane: it being possible to consider this as a special case of RS1 with physical interpretation of the branes reversed and the negative tension brane taken off to infinity. In this scenario, which has an effectively infinite extra-dimension cut-off by the adS horizon, they showed that there exists a normalizable gravitational zero mode coupled to the brane which implies that to some degree the gravitational effects on the brane are close to those predicted by Newtonian theory and Einstein’s 4D General Relativity (GR), even though the spacetime has an extra dimension. More precisely they showed that the gravitational theory experienced by observers on the brane is a tensor theory, being mediated by a massless spin 2 particle, with a tower of massive Kaluza-Klein type states due to the extra dimension. Although there is no natural solution to the hierarchy problem in this model, much interest has been generated by the possibility of creating such a model within a modern fundamental theory based on string or M-theory.

Interestingly, a very similar proposal already existed in the literature: the Hořava and Witten (HW) theory. In this model, which is derived from the strong coupling behaviour of $E_8 \times E_8$ string theories in 10D, the $Z_2$ symmetry is natural; it being a consequence of the orbifold structure of the related 11D M-Theory. Orbifolds, manifolds which are smooth except for a finite number of points, occur naturally in realistic compactifications of string theories since compactification on smooth manifolds cannot generate certain aspects of the Standard Model, for example, the chiral properties of neutrinos. Furthermore, the fixed points of the orbifold give natural places for the branes to live. In the HW case the orbifold is $S^1/Z_2$ so there are two fixed points. This is similar to the RS1 model except that vacuum
energy contributions are generated by the effects of a scalar field with bulk support. Simple cosmological solutions were studied, in refs. [23,24], although very little was learned about the detailed dynamics of the expansion.

The existence of the gravitational zero mode coupled to the brane suggests that the gravitational effects are close to those of 4D GR, but there are subtle differences, particularly on small and large scales. For example, using a linearized analysis, it was shown in ref. [10] that a localized mass distribution on the positive tension brane of RS1 will effect the gravity on the negative tension brane in such a way as to make it appear to be like 4D Brans-Dicke (BD) gravity. The fact that any brane-world model must give rise to a universe which expands in an essentially similar way to the standard FRW universe, at least from the epoch of BBN, implies that one can probe the gravitational dynamics of such model at large distances using cosmology. Probing the viability of brane-world models in which there is an extra scalar field with bulk support using cosmology is the main motivation of this paper, along with the possibility that such scenarios might alleviate the many naturalness problems, such as the cosmological constant problem, prevalent in 4D FRW models.

A large number of papers have attempted to understand the cosmological expansion rate in RS type brane-world models, this being the first thing that one must get right before making any more exotic predictions. Here, we give a brief, and no doubt incomplete, history of the main recent developments. One of the remarkable aspects of this area is that it has been shown that there exist spatially homogeneous and isotropic brane solutions whose expansion can be understood almost totally without knowledge of the detailed 5D solution, with only small corrections to the expansion rate due to the effects of the bulk. Initially, it was suggested [11,12] that a simple model for a 3-brane in a flat 5D background would lead to a Hubble expansion rate $H \propto \rho$, where $\rho$ is the density of the matter on the brane, in complete conflict with the observed expansion rate ($H^2 \propto \rho$ in a 4D FRW universe) and hence with BBN. However, this ignored the lesson of the RS1 and RS2 models. In the simplest versions of these two models vacuum energy, or cosmological constant (CC), contributions were placed on the two branes along with a negative CC in the bulk to give asymptotic adS geometry. Decomposing the matter on the brane into a CC plus other matter and including a negative CC in the bulk rectified the problem assuming the standard RS relation between the CCs [13–17]. This modification leads to the rather perplexing result that on the negative tension brane $H^2 \propto -\rho$. It was suggested in [18,19], and more recently in [20,21] that an averaging procedure a la Kaluza-Klein could rectify this situation if one introduced a bulk scalar field to stabilize the extra dimension via, for example, the Goldberger-Wise mechanism [22]. Another interesting consequence of these models is a radiation type contribution to the expansion rate due to the effects of the bulk.

Cosmological solutions in brane-world models with a bulk scalar field (see refs. [25–27] for previous work on this subject) are necessarily more complicated due to the extra field. However, they are probably more interesting since it would seem unlikely that only gravity would propagate in the bulk. In this paper we attempt to make some progress in understanding how the inclusion of such a field will effect the cosmological evolution, and in particular whether the standard FRW expansion is possible at late times. In contrast to the case of an RS type brane-world model, we find that there is an obstacle to a closed form solution in these models since the evolution on the brane can be affected in an almost arbitrary way by the dynamics of the scalar field in the bulk. In particular, it is not possible to derive an effective equation for the evolution of the scalar field on the brane, in terms of quantities which are only defined on the brane. In order to make any statements about the late-time behaviour of the expansion, therefore we are forced to make the reasonable assumption that the value of the scalar field on the brane is either stabilized or slow varying. We will see that the expansion rate is related to value of the scalar field, and hence if it is varying, one would have a model with a variable effective gravitational constant, which is constrained by observations to be at least slowly varying after BBN. An interesting consequence of this approximation is that the cosmological expansion of stabilized or slowly varying dilatonic brane worlds is historically that of an appropriately tuned RS type model.

Our approach to this problem will be to use the Gauss-Codacci formalism which gives a coordinate independent description of the dynamics on the brane in terms of effective 4D Einstein equations [17]. In section II we extend the analysis of ref. [17] to the situation where there is a scalar field in the bulk. Note that these methods are very different to those used in most of the other literature [11,12,23,24] probing cosmology on RS type brane-worlds where specific form of the metric is used. We show how the results of these two different approaches can be reconciled easily and, as a consequence, gain an improved understanding of the effects of the bulk. As we are dealing with a theory with a scalar field, the possibility of choosing a different conformal frame arises, as in 4D scalar-tensor gravity theories. We discuss the possibility of conformal scalings of the metric in section III. We show, firstly, that our frame is the preferable one on the grounds that it is the only frame in which the energy-momentum contribution due to the

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1 We shall refer to a RS type model as one with just constant vacuum energy terms, as opposed to those which are generated by the existence of a scalar field.
scalar field does not contain second derivatives of the scalar field and, secondly, that the conformal scaling, although allowing the sign of the Hubble parameter to change, does not affect the relationship between the signs of the brane tension and the effective Newton constant.

In section III we apply these methods, first, to the simplest RS type model obtaining the results of ref. [17], before applying them to the more complicated models which include a stabilizing scalar field potential for the size of the extra dimension. We show that once the scalar field becomes stabilized the 4D Einstein equations become exactly those of the RS model and hence a negative tension brane leads to $H^2 \propto -\rho$ as in the case of constant potentials. We discuss in detail the conflict between the results obtained by this approach and those of other analyses, which have concluded that the standard FRW result should apply. This raises the important question of what to interpret as the effective 4D theory.

We then study a simple HW type model with exponential potentials in the bulk and on the brane. Assuming that the matter on the brane is not coupled to the scalar field, and the scalar field on the brane is varying slowly, we show that the RS type cosmological evolution is still valid, albeit now with an effective gravitational constant which is varying with time. It is interesting to note that in such models energy conservation on the brane is modified by the variation of the scalar field giving rise to a novel scenario for the variation of the coupling constants. If one allows the matter on the brane to be coupled to the scalar field as well, then it is possible to achieve accelerated expansion on the brane leading to an interesting bulk driven inflation model even when the scalar field is stabilized.

II. FORMALISM

A. Geometrical description of the spacetime

The basic method is to foliate the 5D space-time in the direction of the extra-dimension as done in ref. [17]. This makes a 4+1 split of the Einstein equations in a similar way to the standard 3+1 split used in treatments of the Cauchy initial value problem in GR (see, for example, ref. [28]). This geometrical split allows us to understand the precise gravitational dynamics on the brane without any possible problems with unphysical gauge degrees of freedom.

More precisely, we wish to study space-times including a smooth 4D time-like hypersurface (or co-dimension 1 brane) which we can express, in terms of the coordinates $x^a$ with $(a = 0, 1, 2, 3, 4)$, by $f(x^a) = 0$, where $f$ is a real-valued function. Hence, we can define a space-like normal vector $n^a$ and a unit vector $n^a$ parallel to it by

$$n_a = \partial_a f / (\partial_b f \partial^b f)^{1/2}. \quad (1)$$

One would like to extend $n_a$ to follow a geodesic congruence, that is, so that it satisfies the geodesic equation, $n^b \nabla_b n^a = 0$. Near the brane, this will define a smooth foliation, but it may not be possible to foliate the entire bulk in this way. Moreover, in cases where there is more than one brane, a geodesic congruence which is normal to one brane may not be normal to another. For example, a coordinate system which is Gaussian-normal with respect to one of the branes, that is, $n_a = (0, 0, 0, 0, 1)$ on that brane, is not necessarily Gaussian-normal with respect to another [30]. (See also [31] for a discussion of the regions covered by Gaussian normal coordinates in Schwarzschild-adS space.)

One can define the metric, or (first) fundamental tensor, of the foliation, $h_{ab}$, in terms of the normal vector $n_a$ to be its orthogonal complement,

$$h^a_b = \delta^a_b - n^a n_b. \quad (2)$$

As well as being the higher-dimensional manifestation of the induced metric on the brane, this is a projection tensor which, along with the it orthogonal complement, allows one to project other tensors tangential on to the brane. As an example consider any vector, $V^a$, which can be written as

$$V^a = h^a_b V^b + (n_b V^b) n_a. \quad (3)$$

Clearly, the first part is tangential to the brane and the other is perpendicular. We shall see in subsequent sections that this projection property is an important concept when to attempting to understand brane-world since it allows one to split the dynamics up into those tangential to the brane and those perpendicular.

\footnote{Note that, throughout, we will use the same sign conventions as [28,20]: in particular, we use a $(-+++)$ signature for the 5D space-time metric.}
The extrinsic curvature, or second fundamental tensor, is defined to be

\[ K_{ab} = \nabla_a n_b = h_a^c h_b^d \nabla_c n_d, \tag{4} \]

which is symmetric by Frobenius’ theorem and tangent to the hypersurface. It can also be written in terms of Lie derivatives of the metric with respect to the normal

\[ 2K_{ab} = \mathcal{L}_n g_{ab} = \mathcal{L}_n h_{ab}, \tag{5} \]

and its Lie derivative is given by

\[ \mathcal{L}_n K_{ab} = K_{ac} K_b^c - R_{abcd} n^c n^d, \tag{6} \]

where \( R_{abcd} \) is the Riemann curvature tensor of the background space-time. In a Gaussian-normal coordinate system, the extrinsic curvature is half of the derivative of the hypersurface metric, \( h_{ab} \), with respect to the coordinate normal to the hypersurface.

Within this setup, we can use \(^7\) the Gauss-Codacci formalism to compute effective 4D Einstein equations for the gravitational dynamics on the brane. We will denote all curvature tensors pertaining to the hypersurface metric, \( h_{ab} \), with a bar over the letter, whereas those pertaining to \( g_{ab} \) will have no bar. Similarly, \( \nabla \) is the covariant derivative that preserves \( h_{ab} \) whereas \( \nabla \) is the covariant derivative that preserves \( g_{ab} \). The Gauss and Codacci equations are then

\[
\bar{R}_{abcd} = h^a_i h^b_j h^c_k h^d_l R_{ijkl} + 2K_{a[c} K_{d]b}, \tag{7}
\]

\[
\nabla_b K^a_c - \nabla_a K = n^c h^b_a R_{bc} = n^c h^b_a G_{bc}. \tag{8}
\]

Following the approach of ref. \(^7\), we use the decomposition of the 5D Riemann tensor \(^2\)

\[
R_{abcd} = \frac{2}{3} \left( g_{a[c} R_{d]b} - g_{b[c} R_{d]a} \right) - \frac{1}{6} R g_{a[c} g_{d]b} + C_{abcd}, \tag{9}
\]

which we substitute into \(^7\). Contracting appropriately gives the following expression for the 4D effective Einstein tensor,

\[
\bar{G}_{ab} = \frac{2}{3} \left\{ G_{cd} h^c_a h^d_b + \left( G_{cd} n^c n^d - \frac{1}{4} G \right) h_{ab} \right\} + KK_{ab} - K_a^c K_b^c - \frac{1}{2} \left( K^2 - K^{cd} K_{cd} \right) h_{ab} - E_{ab}, \tag{10}
\]

where \( E_{ab} \) is the electric part of the Weyl tensor with respect to \( n_a \), given by

\[
E_{ab} = C_{achd} n^c n^d. \tag{11}
\]

We should note that at this stage everything we have derived is a geometric identity and hence true on all of the 4D hyperspaces in the foliation (assumed to be well-defined). In subsequent sections we will use junction conditions to compute \( K_{ab} \) close to the brane and use this expression for the Einstein tensor to make a comparison with the predictions for cosmological expansion of the brane and those of 4D GR and other gravity theories.

B. The action and variation

In order to impose some dynamics on the geometrical identities of the previous section we will derive equations of motion from a variational principle which includes the standard Einstein-Hilbert term albeit in 5D, a scalar field \( \phi \) which has support in the bulk as well as on the brane, and a matter Lagrangian density, \( \mathcal{L}^{(i)}(\phi) \) on each of the branes,

\[
S = \frac{1}{2\kappa_5^2} \int_M d^5x \sqrt{-g} \left\{ R - \frac{1}{2} \left( \partial \phi \right)^2 - \kappa_5^2 V(\phi) \right\} - \frac{1}{2\kappa_5^2} \sum_i \int_{M_i} d^4x \sqrt{-h^{(i)}} \left\{ 2 \left[ K^{(i)} \right]_+^+ + \kappa_5^2 U^{(i)}(\phi) + \kappa_5^2 \mathcal{L}^{(i)}(\phi) \right\}, \tag{12}
\]

where \( M \) is the 5D manifold with metric \( g_{ab} \), and the 4D hypersurfaces \( M_i \) are the branes each having a projected metric \( h_{ab}^{(i)} \). The constant \( \kappa_5^2 \), which is effectively the 5D gravitational constant, has mass dimensions of \( \mathcal{O}(m^{-3}) \) (\( m \) is a unit of mass), \( \phi \) is dimensionless, the bulk potential \( V(\phi) \) has dimensions \( \mathcal{O}(m^5) \) and the brane potentials \( U^{(i)}(\phi) \)
have dimensions $\mathcal{O}(m^4)$. The $2 [K^{(i)}]^\perp$ terms added to action on the brane are the appropriate Gibbons-Hawking
terms, required for a well-formulated variational problem.

Varying this action with respect to the metric gives

$$
\delta_g S = \frac{1}{2\kappa_5^2} \int_M d^5x \sqrt{-g} \left\{ G_{ab} - \frac{1}{2} \partial_a \phi \partial_b \phi + \frac{1}{2} \delta g_{ab} \left( \frac{1}{2} (\partial \phi)^2 + \kappa_5^2 V(\phi) \right) \right\} \delta g^{ab}
+ \frac{1}{2\kappa_5^2} \sum_i \int_{M_i} d^4x \sqrt{-h} \left\{ [Kh_{ab} - K_{ab}]^+ + \frac{\kappa_5^2}{2} \left( U(\phi) h_{ab} + L(\phi) h_{ab} - 2 \frac{\delta L}{\delta g^{ab}} \right) \right\} \delta g^{ab},
$$

where we have now dropped the $i$ labels on the terms in the brane action for convenience; suffice to say, all the
subsequent expressions apply on each of the branes. The $[Kh_{ab} - K_{ab}]^+$ term is a combination of the boundary terms
from the variation of the bulk Einstein-Hilbert term and that of the Gibbons-Hawking term (see ref. [22] for more
details as to its derivation). By demanding these expressions be true for all variations, one can read off the bulk
Einstein equations and the Darmois-Israel junction conditions, to be

$$
G_{ab} = \frac{1}{2} \partial_a \phi \partial_b \phi - \frac{1}{2} \left( \frac{1}{2} (\partial \phi)^2 + \kappa_5^2 V(\phi) \right) g_{ab},
$$

$$
[Kh_{ab} - K_{ab}]^+ = \kappa_5^3 \left( \tau_{ab} - \frac{1}{2} U(\phi) h_{ab} \right),
$$

where we have defined $\tau_{ab}$, which we will interpret as the energy-momentum of ‘ordinary’ brane matter, by

$$
\tau_{ab} = \frac{\delta L}{\delta h_{ab}} - \frac{1}{2} U(\phi) h_{ab} = (\sqrt{-h})^{-1} \frac{\delta}{\delta h^{ab}} \left( \sqrt{-h} L \right).
$$

Varying (12) with respect to $\phi$ gives

$$
\delta_\phi S = \frac{1}{2\kappa_5^2} \int_M d^5x \sqrt{-g} \left\{ \nabla^2 \phi - \kappa_5^2 \frac{dV}{d\phi} \right\} \delta \phi + \frac{1}{2\kappa_5^2} \sum_i \int_{M_i} d^4x \sqrt{-h} \left\{ [n.\partial \phi]^+ - \frac{\kappa_5^2}{2} \frac{dU}{d\phi} - \frac{\kappa_5^2}{2} \frac{\delta L}{\delta \phi} \right\} \delta \phi,
$$

where $\nabla^2$ is the covariant 5D wave operator $g^{ab} \nabla_a \nabla_b$. Again, the $i$ labels on brane terms have been dropped for
convenience. Reading off the $\phi$ equation of motion in the bulk and the jump condition at each brane gives

$$
\nabla^2 \phi = \kappa_5^2 \frac{dV}{d\phi}, \quad [n.\partial \phi]^+ = \kappa_5^3 \left( \frac{dU}{d\phi} + \frac{\delta L}{\delta \phi} \right).
$$

By using (3) and an appropriate generalization, one can write $\nabla^2 \phi$ in terms of the 4D covariant wave operator, $\nabla^2$
acting on $\phi$ and normal derivatives of $\phi$ as follows

$$
\nabla^2 \phi = \nabla^2 \phi + K n.\partial \phi + n.\partial (n.\partial \phi) = \kappa_5^3 \frac{dV}{d\phi}.
$$

This is expression is the result of the equivalent procedure used to derive (10) for the Einstein tensor. However, as we
shall discuss in the subsequent sections, it is impossible to write this equation in terms of local quantities only defined
on the branes due to our lack of knowledge of $n.\partial (n.\partial \phi)$ on the brane. This is the main obstacle to the derivation of
a closed form solution when there is a bulk scalar field in an equivalent way to the RS type models.

C. Effective 4D equations

Having derived the appropriate bulk Einstein equations and equations motion for $\phi$ along with their appropriate
junction conditions, we are now in a position to derive effective 4D equations by applying the geometrical projection
technology of the section II A. The first step in this direction is to impose $Z_2$ symmetry which relates quantities at each side of the brane and allows us to evaluate $K_{ab}$ and $n.\partial \phi$ close to, but not on, the brane using the junction conditions,

$$K_{ab} = -\frac{1}{2}\kappa_5^2 \left( \tau_{ab} - \frac{1}{3} h_{ab} \tau + \frac{1}{6} U h_{ab} \right), \quad n.\partial \phi = \frac{1}{2}\kappa_5^2 \left( \frac{dU}{d\phi} + \frac{\delta L}{\delta \phi} \right). \tag{20}$$

Substituting the expression for $K_{ab}$, and the bulk Einstein equation (14) into (10) gives the effective 4D Einstein equation,

$$\overline{G}_{ab} = -\Lambda(\phi) h_{ab} + \frac{\kappa_4^4}{12} U(\phi) \tau_{ab} + \frac{\kappa_4^4}{16} U'(\phi) \frac{\delta L}{\delta \phi} h_{ab} + \frac{\kappa_4^4}{8} \left( \frac{\delta L}{\delta \phi} \right)^2 h_{ab} + \frac{1}{3} \nabla_a \phi \nabla_b \phi - \frac{5}{24} (\nabla \phi)^2 h_{ab} - E_{ab}, \tag{21}$$

where, for convenience, we have written

$$\Lambda(\phi) = \frac{\kappa_4^4}{48} U(\phi)^2 + \frac{\kappa_2^2}{4} V(\phi) - \frac{\kappa_4^4}{32} U'(\phi)^2, \tag{22}$$

$$\pi_{ab} = -\frac{1}{4} \tau_a c \tau_{bc} + \frac{1}{12} \tau_{ab} + \frac{1}{8} \tau^{cd} \tau_{cd} h_{ab} - \frac{1}{24} \tau^2 h_{ab}. \tag{23}$$

$\Lambda(\phi)$ is an effective cosmological constant type term, $\pi_{ab}$ is the quadratic correction to the Einstein equations first deduced, but interpreted somewhat differently, in ref. [11], and $E_{ab}$ is the only non-local term, it being a projection of the bulk Weyl tensor onto the brane. In the next section we shall discuss how its effects can be understood simply in terms of quantities which are local to the brane in the case where the metric on the brane is spatially isotropic and homogeneous.

By taking the trace of (21) we can obtain an expression for the 4D Ricci scalar,

$$\overline{R} = 4\overline{\Lambda}(\phi) - \frac{\kappa_4^4}{12} U(\phi) \tau - \frac{\kappa_4^4}{4} U'(\phi) \frac{\delta L}{\delta \phi} + \frac{\kappa_4^4}{12} \left( \tau^2 - 3 \tau^{cd} \tau_{cd} \right) - \frac{\kappa_4^4}{8} \left( \frac{\delta L}{\delta \phi} \right)^2 + \frac{1}{2} (\nabla \phi)^2, \tag{24}$$

which is independent of the contribution from the bulk Weyl tensor $E_{ab}$ and will be useful when considering the expansion rate.

Similarly, substituting (21) into the Codacci equation (8) on the brane gives us the following expression for the divergence of the brane energy-momentum tensor

$$\nabla^b \tau_{ab} = -\frac{1}{2} \nabla_a \phi \frac{\delta L}{\delta \phi}, \tag{25}$$

which is an effective energy momentum conservation equation for the matter on the brane. Interestingly, the brane energy-momentum is only conserved either if $L$ is independent of $\phi$, or if $\phi$ is constant on the brane.

One can also substitute (21) into the the equation of motion for $\phi$ on the brane which gives,

$$\nabla^b \phi + n.\partial (n.\partial \phi) = \kappa_5^2 V'(\phi) - \frac{\kappa_4^4}{12} (\tau - 2 U(\phi)) \left( U'(\phi) + \frac{\delta L}{\delta \phi} \right). \tag{26}$$

We have written this in terms of a d’Alembertian wave operator on the brane, local source terms, and the second derivative of $\phi$ in the direction of normal to the brane. This term, which is analogous to $E_{ab}$ in the case of the effective Einstein tensor, is the only term which is dependent on the bulk. Unfortunately, in contrast to $E_{ab}$, it is not possible to understand it simply in terms of quantities local to the brane and hence in our subsequent discussion we will be forced to make some assumptions as to the behaviour of $\phi$ on the brane. In previous discussion we have pointed out that RS showed that the theory of gravity on the brane was a tensor theory, and we see that in the effective 4D Einstein equations are only slightly modified by the bulk in the weak-field limit, but there is clearly no way of thinking of the bulk scalar field in the same way; it being a truly 5D quantity, and must be treated as such.

Now, motivated by the isotropy of the CMB and homogeneity of the observed galaxy distribution, we choose the metric on the brane to be an embedding of the 4D FRW universe, that is,

$$h_{ab} dx^a dx^b = -dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j, \tag{27}$$

where $\gamma_{ij}$ is the metric of the three-spaces of constant curvature. In this case, standard results give
where dots denote differentiation with respect to cosmic time, $a$ is the scale factor and $H = \dot{a}/a$ is the Hubble parameter. Therefore, in order to derive equations for $H$, and hence an analogy to the Friedmann equation, we have two alternatives: either compute $E_{00}$ on the brane and hence $G_{00}$ as advocated in ref. [17], or integrate the equation for $H$ in terms of $\mathcal{R}$ which implies the existence of an integration constant. This exactly the equivalent approach to those taken in refs. [11][13]. In the subsequent sections on applications we shall perform these calculations explicitly for a number of cases, to show that they are equivalent. For the rest of this section we shall attempt to compare our model of the universe. Note that in making this definition we have justified the assumption of ignoring the quadratic gravitational constant given by

$$L = \frac{1}{3} \mathcal{R}_{00} = H^2 + \frac{k}{a^2},$$

$$\mathcal{R} = 6 \left( \dot{H} + 2H^2 + \frac{k}{a^2} \right),$$

where $\kappa_4^2 = \frac{8\pi G}{k}$.

If we now assume that $\mathcal{L}$ is independent of $\phi$ and that the quadratic corrections are small, then

$$\mathcal{R} = 4\mathcal{X}(\phi) - \frac{\kappa_4^2}{12} U(\phi) \tau + \frac{1}{2} (\nabla \phi)^2.$$

The equivalent expression for Einstein gravity with a cosmological constant is

$$\mathcal{G}_{ab} = 8\pi G \mathcal{N}_{ab} - \mathcal{N} h_{ab}, \quad \mathcal{R} = 4\mathcal{X} - 8\pi G \mathcal{N}.$$

Therefore, we see the directly the interpretation of $\mathcal{X}$ as a cosmological constant and if one defines the an effective gravitational constant given by

$$\frac{\kappa_4^2}{12} U(\phi) = 8\pi G \mathcal{N}(\phi),$$

then one we will get the same expansion rate as Einstein gravity in this limit. As we have already discussed, from the point of view of the brane we have no control over the dynamics of the scalar field, but the fact that it is related to the effective gravitational constant means that it must not have changed much since the time of BBN in any realistic model of the universe. Note that in making this definition we have justified the assumption of ignoring the quadratic terms at late since they will be suppressed by a factor of $\tau/U$.

Clearly, by making these identifications we can make some interesting general statements. First, we should note that the expression for the Ricci scalar is independent of $E_{ab}$ and hence, if $\phi$ is constant on the brane, it is completely insensitive to anything which is not on the brane. This is not to say that the gravitational dynamics are in general independent of the bulk, just that in this specialized case the only effect on the effective Friedmann equation is via an integration constant. Secondly, we see that $U(\phi)$ must be a strictly positive number, at least after the epoch of BBN, in order to recover the desired expansion rate. We will discuss this in detail in the context of the Goldberger-Wise mechanism for stabilizing the extra dimension; suffice to say if the effective tension on the brane is negative then the corresponding expansion rate will also be negative. This is effectively rules out the RS1 model for solving the hierarchy problem, even when there exists a scalar field to stabilize the extra dimension. Finally, in such models there is a simple and at first sights uncomplicated way to ensure a zero effective cosmological constant on the brane, by setting

$$U'(\phi)^2 - \frac{2}{3} U(\phi)^2 = \frac{8}{\kappa_5^2} V(\phi),$$

an equation which can be solved analytically in the simple case where $V(\phi) = 0$ and $V(\phi)$ is a constant. This self-tuning mechanism, first suggested in refs. [37][38], has been the subject of a great deal of debate in the context of supergravity models where $U$ and $V$ are derived from the same super-potential (see also [39][40]). Unfortunately, it appears [41] that no such supersymmetric model exists without the a naked singularity somewhere in the bulk.

We have already noted that it was deduced in ref. [17] that there is effective BD gravity on the negative tension brane in the RS1 model due to a localized mass distribution on the positive tension brane, and this may at first sight appear to in conflict with our results. However, here we are only considering the effect on expansion in the spatially homogeneous and isotropic model, and hence their assumptions do not apply. In fact, a localized mass distribution off the brane cannot contribute the overall expansion rate of the brane since its only gravitational effect must be via $E_{ab}$ which is traceless and hence does not contribute to Ricci scalar.

Even though the analysis of ref. [17] does not apply here, it is sensible to compare our model to alternative theories of gravity such as BD, rather than just Einstein gravity since it can help us to constrain further the parameters of our model. The field equations for BD gravity in 4D without a cosmological constant are [29]
Here, we have kept the bars and $h_{ab}$ as the 4D metric to avoid ambiguity. Taking the trace of (34) and substituting (35) gives an expression for the Ricci scalar

$$R = -\frac{8\pi}{\varphi} \left( \frac{2\omega}{3+2\omega} \right) \tau + \frac{\omega}{\varphi^2} (\nabla \varphi)^2,$$

and hence we make the following identification

$$\frac{\kappa^4}{12} U(\phi) = \frac{8\pi}{\varphi} \left( \frac{2\omega}{3+2\omega} \right).$$

Thus, if $U(\phi) < 0$ and $\varphi > 0$, the Brans-Dicke parameter, $\omega$, must be in the range $-3/2 < \omega < 0$. Note that the same bounds on $w$ were found in [11] when considering localized mass distributions on the negative tension brane. Such values of $\omega$ can be ruled out by experiment [14], reiterating that the RS1 model, and associated models with a stabilized scalar field, are ruled out by experiment.

### D. Conformal Transformations

There has been considerable debate in the literature of scalar-tensor gravity theories (such as Brans-Dicke theory) as to which frame represents the physical one. This issue is also pertinent in the case of dilatonic brane-worlds due to the existence of a scalar field. Scalar-tensor theories can be related by conformal transformations of the metric, $g_{ab} \rightarrow \tilde{g}_{ab} = \Omega(\phi)^2 g_{ab}$ and it is well understood that changing to a different frame in a cosmological solution can change the expansion rate (in particular, it could change the results of the following section. We have adopted the view of [18] that the frame in the formulation of the action (12) is the physically significant one in the sense that it is with respect to this metric that free particles will follow geodesics. As this viewpoint is not universal, even in the 4D case (see [19] for a summary of different opinions) it is interesting to study how our effective Einstein equations behave under conformal transformations of the metric. In particular, we can demonstrate that, unlike with the Jordan frame of Brans-Dicke theory [48,49], the frame chosen by the induced metric is the one in which the energy-momentum of the scalar field does not contain second derivatives of $\phi$. This means that any other frame would be philosophically problematic in that second derivative terms necessarily give an energy-momentum which is not positive-definite. (Again, there are differences of opinion: not everyone considers violations of positive-definiteness in the gravitational scalar sector to be problematic [18]!)

If the 4D induced metric transforms as $h_{ab} \rightarrow \hat{h}_{ab} = \Omega(\phi)^2 h_{ab}$ then the covariant derivatives which preserve these metrics are related by, for example,

$$\tilde{\nabla}_a v_b = \nabla_a v_b + \frac{\Omega'}{\Omega} \left( h_{cd} \partial_{(a} \phi \right) h_{b) d} + \frac{2}{\Omega} \left( 2\Omega^2 \right) \nabla_a \phi \nabla_b \phi,$$

and the Einstein tensor transforms to (see [28])

$$\tilde{G}_{ab} = \frac{\Lambda}{\Omega^2} \tilde{h}_{ab} + \frac{\kappa^4 U'}{16\Omega^2} \nabla_a \nabla_b \phi + \frac{2}{\Omega} \left( 2\Omega^2 - \Omega'' \right) \nabla_a \phi \nabla_b \phi + \frac{2}{\Omega} \left( \Omega \nabla_c \nabla_d \phi + \Omega' \nabla_c \phi \nabla_d \phi \right) h_{ab} h^{cd}.$$

Note that $\tilde{n}_a = \Omega n_a$ and $\tilde{n}^a = \Omega^{-1} n^b$ and hence $E_{ab}$ must be conformally invariant as both $C^a_{\ bcd}$ and $n^a n_b$ are. Likewise, $\nabla^a n_b$ and $\nabla^a X_b$ are conformally invariant. Substituting for $\tilde{G}_{ab}$ from (34) and expressing everything in terms of derivatives relating to $h_{ab}$, one can deduce the effective Einstein equation in the new frame, which is

$$\tilde{G}_{ab} = \frac{\Lambda}{\Omega^2} \tilde{h}_{ab} + \frac{\kappa^4 U'}{16\Omega^2} \nabla_a \nabla_b \phi + \frac{2}{\Omega} \left( 2\Omega^2 - \Omega'' \right) \nabla_a \phi \nabla_b \phi + \left( \frac{1}{3} + \frac{2}{\Omega^2} + \frac{2}{\Omega''} \right) \nabla_a \phi \nabla_b \phi \left( \frac{\tilde{E}}{\Omega^2} \right)^2 + \frac{2}{\Omega} \nabla^2 \phi + \left( \frac{2}{\Omega''} \right) \nabla^2 \phi + \left( \frac{\tilde{E}}{\Omega^2} - \frac{\tilde{E}}{\Omega''} - \frac{1}{24} \right) \left( \nabla \phi \right)^2 \tilde{h}_{ab}.$$
Note that we can substitute for $\nabla^2 \phi$ using the conformal transform of \([20]\) and that some of the terms can be set to zero by suitable choice of $\Omega$. However, the $\nabla_a \nabla_b \phi$ term will always be present unless $\Omega$ is constant, which, of course, corresponds merely to a rescaling of units. Thus the original frame is the only one in which the terms corresponding to the effective 4D energy-momentum of the scalar field can be positive definite. Also note that, in all frames, the coefficient of $\tau_{ab}$ has the same sign as the vacuum energy, as in Brans-Dicke theory. Frames other than the one we have considered have an additional term proportional to $\tau_{ab}$; the coefficient of this term is constrained by gravitational tests such as those carried out on solar system measurements.

In order to consider the expansion rate in the new frame can also calculate the equation of motion for the transformed Ricci scalar, which is

$$
\bar{R} = 4 \bar{\Lambda} + \frac{\kappa^4 U}{12\Omega^2} \bar{\tau} + \frac{\kappa^3 U' \delta}{4\Omega^2} \left( \frac{\bar{L}}{\Omega^2} \right) + \frac{\kappa^4 U'^2}{8\Omega^2} \left( \frac{\delta}{\delta \phi} \left( \frac{\bar{L}}{\Omega^2} \right) \right)^2 + 6 \frac{\Omega'}{\Omega} \bar{\tau}^2 + \left( 6 \frac{\Omega''}{\Omega} + 14 \frac{\Omega^2}{\Omega^2} - \frac{1}{2} \right) (\bar{\nabla}^2 \phi)^2. \tag{41}
$$

As mentioned above, it is possible to substitute for $\nabla^2 \phi$ using \([20]\), which gives

$$
\nabla^2 \phi = 2 \frac{\Omega'}{\Omega} (\nabla \phi)^2 - n. \partial (n. \partial \phi) + \kappa^4 U' (\phi) - \frac{\kappa^4}{12} (\bar{\tau} - 2U(\phi)) \left( U'(\phi) + \frac{\delta}{\delta \phi} \left( \frac{\bar{L}}{\Omega^2} \right) \right). \tag{42}
$$

Thus we can remove the second brane derivatives in $\phi$ at the expense of introducing the second derivative normal to the brane, $n. \partial (n. \partial \phi)$. This gives

$$
\bar{R} = 4 \bar{\Lambda} + \left( \frac{\kappa^4 U}{12\Omega^2} + \frac{\kappa^3 U' \delta}{2\Omega} \right) \bar{\tau} + \frac{\kappa^4 U'}{4\Omega^2} \frac{\delta}{\delta \phi} \left( \frac{\bar{L}}{\Omega^2} \right) + \frac{\kappa^3 U''}{2\Omega^2} \left( \frac{\delta}{\delta \phi} \left( \frac{\bar{L}}{\Omega^2} \right) \right)^2 + \left( 6 \frac{\Omega'}{\Omega} + 2 \frac{\Omega^2}{\Omega^2} - \frac{1}{2} \right) (\bar{\nabla}^2 \phi)^2 - 6 \frac{\Omega'}{\Omega} n. \partial (n. \partial \phi), \tag{43}
$$

where we have defined

$$
\bar{\Lambda} = \frac{\Lambda}{\Omega^2} + \frac{\Omega'}{4\Omega} (6\kappa^2 U' + \kappa^4 U U'). \tag{44}
$$

It can be seen that the coefficient of $\bar{\tau}$ in this frame is different from the coefficient of $\tau$ in the original frame. In particular, it is possible to choose $\Omega$ so that the expansion rate is positive even when the brane has negative tension. However, the effect of the $n. \partial (n. \partial \phi)$ term is impossible to quantify. From the expression for the Einstein tensor, it is possible to see that, although one can change the sign of the dependence of the Hubble parameter on the matter content by changing frame, one cannot change the sign of the effective Newton constant.

Our original choice of frame for the 5D action was the 5D Einstein frame and one might want to argue the case for other 5D frames. The result should be recoverable from our conformally transformed equation \([40]\). This has been studied in \([51]\) starting from a 5D action in a general frame:

$$
S = \frac{1}{2\kappa^5} \int_M d^5x \sqrt{-g} \left\{ F(\phi) R + \cdots \right\} - \frac{1}{2\kappa^5} \sum_i \int_{M_i} d^4x \sqrt{-h^{(i)}} \left\{ 2F(\phi) \left( K^{(i)} \right)^+ + \cdots \right\}. \tag{45}
$$

We refer the reader to \([41]\) for a discussion of the results and merely quote that the effective Einstein equation derived from such an action has the form

$$
\bar{G}_{ab} = 8\pi G_N \bar{\tau}_{ab} + 8\pi G_A \tau \bar{h}_{ab} + \cdots, \tag{46}
$$

where $G_N$ is in agreement with our conformally transformed equation \([40]\) above as expected. In particular, it should be noted that $G_N$ always has the same sign as the vacuum energy of the brane.

In summary, although there are differences in opinion in the literature of scalar-tensor gravity theories, the frame we have used is in good agreement with the philosophy of both \([43]\) and \([49]\) in the first case because we have interpreted the metric of the original frame as that governing the geodesics and in the second because ours is the only frame in which the scalar field has positive definite energy-momentum.
E. Understanding the effects of the bulk

Our 4D effective equations for the gravitational sector \(^{[21]}\) and the scalar field \(^{[28]}\) both contain terms which specify how effects from the bulk can manifest themselves on the brane. In the case of the scalar field it is impossible to derive a simple expression in terms of just local quantities forcing us to make sensible assumptions as to the behaviour of \(\phi\) on the brane. However, it is possible to do so for the in the gravitational sector.

One can deduce an equation for \(E_{ab}\) on the brane by realizing that the 4D Einstein tensor must satisfy the 4D Bianchi identity \(\nabla^a E_{ab} = 0\). From \(^{[21]}\) one can deduce that

\[
\nabla^a E_{ab} = -\mathcal{K}(\phi)\nabla_b \phi + 8\pi G_N(\phi)\tau_{ab} \nabla^a \phi - 4\pi G_N(\phi) \frac{\delta L}{\delta \phi} \nabla_b \phi + \kappa_5^4 \nabla^a \pi_{ab} + \kappa_5^4 \frac{U''(\phi)}{16} \left( \frac{\delta L}{\delta \phi} \nabla_b \phi \right) - \frac{\kappa_5^4}{16} \nabla_b \phi \nabla_b \nabla_b \phi - \frac{1}{24} \nabla_b (\nabla_b \phi)^2 .
\]

(47)

The tensor \(E_{ab}\) is traceless, tangential to the brane and satisfies the above equation. Therefore, from the point of view of the brane it includes 5 independent quantities, 3 of which are related to spatial isotropy of the brane via \(E_{0i}\) and the other 2 are tensor degrees of freedom due to gravitational waves in the bulk.

If we assume that the scalar field and the scalar field on the brane are spatially homogeneous and there are no bulk gravitational waves then \(E_{0i} = 0\) since \(\tau_{0i} = 0\), and \(E_{ij} = f(t)\tau_{ij}\) for some function \(f(t)\). Therefore, \(E_{ab}\) is entirely specified by the \(b = 0\) component of (47),

\[
E^0_0 + 4\frac{\dot{a}}{a} E_{00} = \dot{\phi} \left\{ \nabla^a (\phi) + 8\pi G_N(\phi)\tau_{00} + 4\pi G_N(\phi) \frac{\delta L}{\delta \phi} - \kappa_5^4 \frac{U''(\phi)}{16} \frac{\delta L}{\delta \phi} - \frac{1}{4} \right\} - \kappa_5^4 \nabla^0 \pi_{a0} \quad (48)
\]

\[
E^i_0 + 4\frac{\dot{a}}{a} E_{i0} = \dot{\phi} \left\{ \nabla^a (\phi) + 8\pi G_N(\phi)\tau_{i0} + 4\pi G_N(\phi) \frac{\delta L}{\delta \phi} - \kappa_5^4 \frac{U''(\phi)}{16} \frac{\delta L}{\delta \phi} - \frac{1}{4} \right\} .
\]

(49)

If \(L\) is independent of \(\phi\) and we consider a perfect fluid, we can write \(\dot{\tau}_{ab} = \text{diag}(-\rho, p, p, p)\) and hence we find that \(\nabla^a \pi_{a0} = 0\). Thus

\[
E_{00} + 4\frac{\dot{a}}{a} E_{00} = \dot{\phi} \left\{ \nabla^a (\phi) + 8\pi G_N(\phi)\rho \right\} ,
\]

(50)

and so, if \(\phi\) is constant,

\[
\frac{1}{a^2} \frac{d}{dt} \left(a^4 E_{00}\right) = 0 .
\]

(51)

which integrates to give \(E_{00} = Ma^{-4}\) where the integration constant, \(M\), has dimensions \(O(m^2)\). This is the exact same integration constant as discussed in terms of integrating the equation for the Ricci scalar in the previous section. Clearly, if \(\phi\) is slowly varying then modifications to this result can be derived in a perturbative expansion.

We have shown, therefore, that the effect of the bulk on a spatially isotropic and homogeneous brane is to contribute an effective radiation type term to the Friedmann equation \(^{[12,14,23,24]}\). The physical interpretation of the integration constant is that it quantifies the mass outside the brane, which in the case of RS2 is zero and in RS1 is just that of the other brane. This is almost equivalent to Gauss’ law in electrostatics and is the analogue of Birkhoff’s theorem for spherically symmetric space-times (see \(^{[14]}\) for a discussion of Birkhoff’s theorem applied to domain walls in \(\text{ads}\) space). In fact this result is much deeper. In ref. \(^{[12]}\) it was shown that the only global spacetime compatible with spatial isotropy on the brane is Schwarzschild-anti-de-Sitter and this integration constant is the equivalent of the Schwarzschild mass term in usual treatment of vacuum spacetimes in 4D. Therefore, the precise magnitude this contribution to the expansion rate not only depends on the mass of other branes, but that of any black holes nucleated by thermal effects in the bulk \(^{[10,17]}\). Since it behaves like radiation on the brane, the magnitude of such a contribution to the expansion rate can be constrained using BBN.

A corollary of this, reiterating our earlier point, is that when \(\dot{\phi} = 0\), there is no possibility of changing the sign of the effective gravitational constant due to effects from the bulk. Hence when \(\phi\) is stabilized a negative tension brane will have a negative expansion rate. We should note that this does not preclude such effects when the scalar field is varying, nor does it prevent effects being transmitted between branes by perturbations from spatial isotropy and homogeneity, for example, the localized mass distribution of ref. \(^{[10]}\). Such effects would, however, be at perturbative order in the cosmological context.
III. APPLICATIONS

A. Randall-Sundrum type models

The basic RS scenarios are a very special case of the general class of models we have discussed where \( \phi \) is a constant at all times. The potentials will then take constant values, related to the CCs by

\[
V(\phi) = 2\Lambda, \quad U_i(\phi) = 2\lambda_i, \tag{52}
\]

where \( \Lambda \) is the bulk CC and \( \lambda_i \) are the brane CCs with \( i = 1, 2 \) and \( \lambda_1 = -\lambda_2 \) in RS1. Substituting these potentials into (21) gives the result obtained in ref. [17], that

\[
G_{ab} = \left( \frac{\kappa^4 5^2 3^2}{12} + \frac{\kappa^2 5^2}{2} \right) h_{ab} + \frac{\kappa^4 5^2}{6} \tau_{ab} + \kappa^4 5^2 \pi_{ab} - E_{ab}, \tag{53}
\]

and the standard energy conservation equation \( \nabla^a \tau_{ab} = 0 \). Note that this result holds for both the RS scenarios: there will be an equation for each brane in the first scenario, but that the \( i \) labels have been dropped for convenience.

From this we can deduce the the only way in which information about the bulk can be communicated to the brane is by the \( E_{ab} \) term. The first term, which is the effective 4D cosmological constant, is usually set to zero by tuning \( \Lambda \), but we shall not assume this to be so here. Note, once again, that recovering a positive expansion rate requires that \( \lambda > 0 \), as found in ref. [17], and on our brane, we identify

\[
G_N = \frac{\kappa^4 5^4}{48\pi}. \tag{54}
\]

We can recover the Friedmann equations for these solutions using our formalism in two ways as described in the previous section. Using (28) and assuming that the matter on the brane to be a comoving fluid with energy-momentum tensor given by \( \tau_{ab} = \text{diag}(-\rho, p, p, p) \), one can deduce that

\[
H^2 = \left( \frac{\kappa^4 5^2 3^2}{36} + \frac{\kappa^2 5^2}{6} \right) + \frac{8\pi G_N}{3} \rho + \frac{\kappa^4 5^2}{36} \rho^2 - \frac{1}{3} E_{00} - \frac{k}{a^2}. \tag{55}
\]

This is the now well known Friedmann equation for a RS type model and includes a term analogous to that found in 4D, a quadratic order correction and the often ignored unknown \( E_{00} \). When the brane metric is spatially isotropic and homogeneous, we can determine \( E_{00} \) from (51) which give

\[
H^2 = \left( \frac{\kappa^4 5^2 3^2}{36} + \frac{\kappa^2 5^2}{6} \right) + \frac{8\pi G_N}{3} \rho + \frac{\kappa^4 5^2}{36} \rho^2 + \frac{M}{a^4} - \frac{k}{a^2}. \tag{56}
\]

The other way to recover this solution is to use (29), which gives

\[
\dot{H} + 2H^2 = \left( \frac{\kappa^4 5^2 18}{18} + \frac{\kappa^2 5^2}{3} \right) + \frac{4\pi G_N}{3} (\rho - 3p) - \frac{\kappa^4 5^2}{36} (\rho^2 + 3\rho p) - \frac{k}{a^2}. \tag{57}
\]

in combination with the conservation of energy-momentum on the brane, \( \dot{\rho} + 3H (\rho + p) = 0 \), which can be used to remove \( p \) in favour of \( \rho \) and \( \dot{\rho} \). If the dependent variable is changed from \( t \) to \( a \), the differential equation for \( H^2 \) derived in ref. [15] is recovered and the solution (56) is once again found. In fact, this analysis is slightly more general than that of ref. [15] as it only requires homogeneity and isotropy on the brane, not necessarily in the bulk.

One clear problem of these simple RS type models is the fine-tuning of the CCs when one tunes \( \Lambda \) to make the effective 4D cosmological constant zero to get a Ricci-flat brane in the absence of matter. It has been suggested that this unwanted aspect of the models can be removed by using a scalar field to stabilize the extra dimension in an RS1 type model, and our approach has been set up to deal with this kind of scenario. Such a mechanism was proposed by Goldberger and Wise [22] with

\[
V(\phi) = \frac{\mu^2}{\kappa^4} \phi^2 + 2\Lambda, \quad U_i(\phi) = \frac{\mu_i}{\kappa^4} (\phi^2 - \nu_i^2)^2 + 2\lambda_i, \tag{58}
\]
where the $\mu$ and $\mu_i$ have dimensions $O(m)$, while the $\nu_i$ are dimensionless. If the field $\phi$ is varying, the equation for $E_{ab}$ can no longer be integrated exactly, nor can the analogue of (57), both due to the rather complicated dependence on $\phi$. The expansion rate will in general be very different and hence the scalar field must stabilize before BBN. Not surprisingly, however, if the field is constant then the whole situation just returns to that of a simple RS model and the integration can once again be done exactly. If the field is slowly varying then, at zeroth order, the solution will be that for $\phi$ constant with small perturbative corrections. The model stabilizes when the $\phi$ takes the values $\pm \nu_i$ on the branes. Thus, after stabilization, the coefficient of $\rho$ in the expression for $H^2$ is the same as (58). Hence, if $\lambda_i$ is negative then negative expansion is expected at late times as in the simple RS model.

What we have deduced in a rather roundabout way — adding the scalar field and then making constant after stabilization — is that the scalar field can have no effect on the sign of the expansion rate once it is stabilized; the reason being that, as (57) comes from the trace of (21) and $E_{ab}$ is trace-free, there is no contribution from the bulk Weyl tensor in (57). Thus the only contribution to $H^2$ that can possibly arise from the bulk Weyl tensor is that term arising from the integration constant when solving (57). This is the argument discussed in the previous section on the general case applied to the specific case under consideration here. In particular, we note that recovering the standard Hubble expansion when $\rho$ is small requires that $\lambda$ be positive $\lambda$. The same result was discovered in (57), where an explicit 5D metric was used. From section 111 we see that it is possible to recover a positive expansion rate for cosmological solutions in the case of a negative tension brane, although this is unsatisfactory as it does not give the usual Einstein equation and leads to an indefinite contribution to the energy-momentum from the scalar field.

This result is almost the total opposite of that obtained in (18–21). In these papers, the standard Friedmann equations are recovered from the RS1 scenario by averaging over the bulk to obtain an effective 4D theory in a similar way to the dimensional reduction performed in Kaluza-Klein theories. In simple terms, these authors make a specific coordinate choice and integrate over one of the coordinates. To compare this to our covariant formalism of projection onto the brane, we now attempt to formulate such an averaging for a general manifold with co-dimension one branes without reference to any particular coordinate system.

The covariant analogue of integrating over the bulk coordinate at each point on the brane is to integrate along a congruence of curves. Note that to integrate tensors we must use a pull-back to bring them into the tangent space at the point on the brane. This averaging procedure will depend on the congruence chosen and, in the case where there is more than one brane, on the brane chosen. For any brane, the natural choice for the congruence is clearly given by the geodesics tangent to the normal on the brane. This gives rise to the same foliation discussed in section 1, which is well-defined near the brane but may not defined throughout the bulk. The most serious objection to this in models with two branes is that the geodesics may not be normal to the second brane and so a different result would be obtained working from the other brane. Note that the method of projecting onto the brane only requires the foliation to be well-defined in a neighbourhood of the brane.

This brane-world averaging should be contrasted with the usual Kaluza-Klein dimensional reduction where there are no branes and all matter experiences all of the dimensions. (Of course, our projection method is meaningless in theories such as these where there is no preferred hypersurface.) In such theories, everything is considered to be relatively homogeneous across the extra dimensions, i.e., there are no preferred points. In contrast, with brane worlds the matter has support only on the branes and, in addition, the background is usually highly warped. The effect of integrating over the bulk with the exponential weight factor is to give 4D quantities which are largely determined by the matter on or close to the shadow brane. It is questionable whether it is sensible to interpret the average as physically meaningful, especially as our experiences relate to a very atypical point in the spacetime.

As the Goldberger-Wise method for stabilising the radius involves introducing a scalar field, it is possible to reinterpret the results by a conformal rescaling, as discussed in section II. Some authors, such as 20, use a conformal scaling chosen by integrating over the bulk. For the cosmological solutions they consider, this gives a positive expansion rate, even for negative tension branes: an effect which can be seen by considering the term proportional to $\tau_i$ in (43). However, as we have already observed from (11) this does not recover a positive Newton constant in the full Einstein equations.

The difference in the results obtained by averaging and projecting raises the philosophical issue of which should be interpreted as the correct 4D effective theory. This difference does not seems to have been discussed in the literature.

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4We should note that our definitions are such that the dimensions and the particular forms differ from those used in ref. 22. However, the physics are exactly the same.

5One could recover an expanding universe with negative $\lambda$ during the radiation era by choosing $M$ sufficiently large to dominate the ordinary radiation, but one would never get matter domination, since as soon as the matter terms begin to dominate the negative expansion terms would kick in and contraction would ensue. The process of expansion and contraction would take place indefinitely in such a model.
where some authors, e.g., [18–21], have adopted the averaging method whereas others, e.g., [11,15,51], have obtained different results by hand-on versions of our method where a specific form of the 5D metric is used. In particular, [51] concludes, as we do, that the brane tension must be positive to get the correct cosmological expansion even in presence of a stabilized scalar field, whereas [21] conclude the opposite by an averaging analysis. This discrepancy for cosmological models could be resolved by reinterpreting the metric via a conformal scaling, but this is problematic for reasons we have already discussed. One technical advantage of the projection procedure is that it can be defined on a larger class of manifolds. On a more philosophical level, the defining property of brane-world models is that we are confined to the brane and so the only matter we observe via non-gravitational interactions is that on our brane and the curvature quantities we measure are those of the metric restricted to the 4D submanifold, so the projection method seems a more natural interpretation.

B. Hořava-Witten model

The Hořava-Witten model is a realistic scenario for a brane-world based on the a compactification of 11D description of $E_8 \times E_8$ heterotic string theory, often called heterotic M-theory [3]. The general model contains many fields, most of which can be set to zero self-consistently. However, one, a scalar field, cannot since it corresponds to the deformation properties of the compactified Calabi-Yau manifold which is used to reduce the model from 11D to 5D. In this section we shall explore the implications of similar models with a scalar field. As a result of our analysis in section [11], we will work in the 4D conformal frame selected by the induced metric.

Within the general framework in which we have been working, we shall consider two branes and the potentials

$$V(\phi) = \kappa_5^{-10/3}C e^{-2\beta \phi}, \quad U_i(\phi) = \kappa_5^{-8/3}C_i e^{-\beta \phi},$$

(59)

where the powers of $\kappa_5$ have been chosen to make the $C$ and $C_i$ dimensionless. We expect these potentials to correspond to ‘roll-slow’ behaviour of $\phi$. Spherically symmetric cosmological solutions for dilaton domain walls with these potentials have been found in ref. [2] using Birkhoff’s theorem. The effective Einstein equation (21) now has

$$\Xi(\phi) = \frac{\kappa_5^{-4/3}}{96} e^{-2\beta \phi} (24C + (2 - 3\beta^2)C_i^2), \quad 8\pi G_N(\phi) = \frac{\kappa_5^{4/3}}{12} C_i e^{-\beta \phi},$$

(60)

which leads to a variable cosmological constant term, similar to quintessence, and also a variable effective gravitational constant. From ref. [8] we see that, for the simplest HW theory, $\beta = 1$ and $24C = C_i^2$; thus $\Xi = 0$ in this model. We should note, however, that the mechanism by which the flat brane ($\Xi = 0$) is achieved is subtly different from the RS type models since the potential in the bulk is positive; the cancellation being due to spatial gradients of the scalar field. We also have $C_1 = -C_2$ so, from the previous analysis, we must live on the brane which has a positive value of $C_i$ to recover Einstein gravity.

We study the cosmology of this model using the same methods we employed to the RS type models. The equation for $H$ (28) gives us the following differential equation:

$$\dot{H} + 2H^2 = \frac{\kappa_5^{4/3}}{72} C_i e^{-\beta \phi}(\rho - 3p) - \frac{\kappa_5^4}{36}(\rho^2 + 3p^2) - \frac{1}{12}\beta^2 - \frac{k}{a^2},$$

(61)

where we have assumed, for simplicity, that $C$ is tuned in terms of $C_i$ and $\beta$ so that the cosmological term vanishes identically. The equation for $E_{00}$ (29) becomes

$$\dot{E}_{00} + \frac{4}{a} E_{00} = \frac{1}{12} \phi \left( 3\phi - \kappa_5^{4/3} \beta C_i e^{-\beta \phi} \rho \right),$$

(62)

and we have the standard energy-momentum conservation condition $\dot{\rho} + 3H(\rho + p) = 0$. As we have already noted the effective gravitational constant varies, but in the way which is different from that usually assumed in 4D. The standard way to allow $G_N$ to vary in 4D GR is to make $\nabla^a (G_N T_{ab}) = 0$ as a consequence of the Bianchi identities, but this is no longer true in this case and $G_N$ can vary without a modification to energy-momentum conservation. Hence this HW type model provides a novel mechanism for the variation of $G_N$.

At late times, we will require $\phi$ to be almost constant. The solution of zeroth order in $\phi$ is effectively the RS solution, but in a specialized case we can attempt recover the next-to-leading order correction in $\phi$. If we ignore the $\phi^2$ term, the equation (11) still cannot be integrated in general due to the $\phi$ dependence in the first term. However, if the matter satisfies the equation of state $\rho = 3p$, that is, if the universe is radiation dominated, then the first term will be identically zero. Integrating the energy conservation equation, gives the usual solution for radiation, $\rho = \rho_0 a^{-4}$,
where \( \rho_0 \) is the density of radiation today, if we assume \( a(t_0) = 1 \) and \( t_0 \) is the time of the present day. Thus we can write (61) as

\[
a \frac{dH^2}{da} + 4H^2 = -\frac{\kappa_0^2 \rho_0^2}{9a^8} - \frac{2k}{a^2},
\]

which can be solved to give

\[
H^2 = \frac{C}{a^8} + \frac{\kappa_0^2 \rho_0^2}{36a^8} - \frac{k}{a^2}.
\]

(64)

where \( C \) is a constant of integration. This integration constant corresponds to the combination of the bulk effect and the effect of the radiation localized to the brane.

We can also attempt to derive this solution by computing \( E_{00} \). With \( \rho = \rho_0 a^{-4} \), (62) can be solved to first order in derivatives of \( \phi \);

\[
\frac{d}{dt}(a^4 E_{00}) = \frac{d}{dt}\left(\frac{\kappa_0^4}{12} C_i e^{-\beta \phi} \rho_0 \right) \Rightarrow E_{00} = \frac{1}{a^4} \left( \frac{\kappa_0^4}{12} C_i e^{-\beta \phi} - 3C \right),
\]

(65)

where we have chosen \(-3C\) as the constant of integration for consistency with (64). Equation (28) gives us

\[
H^2 = \frac{\kappa_0^4 \rho_0^2}{36a^8} C_i e^{-\beta \phi} + \frac{\kappa_0^2 \rho_0^2}{36a^8} - \frac{k}{a^2} - \frac{1}{3} E_{00}.
\]

(66)

We note that, when we substitute the solution for \( E_{00} \), there is a cancellation between the terms dependent on \( \phi \) and the result is consistent with (64). Thus we see that there are no first-order corrections in derivatives of \( \phi \) to the Friedmann equation in the radiation dominated era. There is, however, a \( \phi \) dependent correction to \( E_{00} \) which cancels the term proportional to \( \rho \) in (66). We conclude, therefore, that the RS type solution is valid if one just ignores term which are second-order in derivatives.

This solution is valid during the radiation era and when \( \phi \) is rolling slowly enough for any second-order terms in derivatives of \( \phi \) to be neglected. By the time the universe has reached the matter domination, one might hope that the evolution of \( \phi \) is now sufficiently slow that \( \phi \) will be almost constant, so that all derivatives of \( \phi \) may be neglected and hence the RS solution can recovered in the matter era as well. Thus one might imagine that the cosmological solutions during matter-domination and during radiation-domination, except possibly at very early times when \( \phi \) is not changing slowly, are those of the RS model. Given the very different nature of the HW and RS models, this is somewhat remarkable.

C. Cosmology with matter coupled to \( \phi \)

Our setup allows us to study the case where the matter on the brane is coupled to the scalar field \( \phi \), i.e., when \( \delta L/\delta \phi \neq 0 \). We now study such a scenario in a cosmological context. Since (24) has \( \delta L/\delta \phi \) terms, we need an explicit form of the Lagrangian. Since we want a cosmological solution, we want the energy-momentum tensor on the brane to be \( \tau_{ab} = F(\phi) (\rho h_{ab} + (\rho + p) U_a U_b) \): \( F(\phi) \) is an arbitrary function, \( \rho \) is the energy density, \( p \) is the pressure and \( U \) is the velocity of the flow. This suggests that we try \( F(\phi) \) times the Lagrangian for a perfect fluid so we consider the variation where the matter part of the action is given by

\[
S_{\text{matter}} = -2 \int \sqrt{-g} d^4 x F(\phi) \left[ p(\varepsilon, s) - \frac{n}{2\varepsilon} (h_{ab} \Omega_a \Omega_b + \varepsilon^2) \right].
\]

(67)

Here, \( s \) is the entropy, \( \varepsilon \) is the enthalpy, \( n \) is a Lagrange multiplier and \( \Omega \) is a 4-vector field, which can be written as \( \Omega_a = \nabla_a \chi + \alpha \nabla_a \beta + \beta \nabla_a s \) for some scalar fields \( \chi, \alpha, \beta \) and \( \theta \); respectively the three Clebsch potentials and the thermasy. Varying (67) with respect to \( n \) gives the constraint \( h^{ab} \Omega_a \Omega_b + \varepsilon^2 = 0 \) so we define the velocity \( U = \Omega / \varepsilon \). Varying with respect to the other parameters \( \varepsilon, \chi, \alpha, \beta, \theta \) and \( s \) gives

\[\text{See (24) for details of how this is constructed.}\]
\[ \frac{\partial \rho}{\partial \varepsilon} = n, \quad \nabla_a [nF(\phi)U^a] = 0, \quad U^a \nabla_a \beta = U^a \nabla_a \alpha = U^a \nabla_a s = 0, \quad \frac{\partial p}{\partial s} + nU^a \nabla_a \theta = 0. \tag{68} \]

Varying with respect to the metric gives, as required,

\[ \tau_{ab} = F(\phi) (ph_{ab} + nzU_a U_b) = F(\phi) (ph_{ab} + (\rho + p)U_a U_b). \tag{69} \]

Thus, on-shell where \( U^a U_a = -1 \), the Lagrangian and its variation with respect to \( \phi \)

\[ \mathcal{L}(\phi) = -2F(\phi)p(\varepsilon, s), \quad \frac{\delta \mathcal{L}}{\delta \phi} = -2F'(\phi)p(\varepsilon, s). \tag{70} \]

From \([21]\), we see that the effect of coupling the matter can be thought of as changing the relation between \( \rho \) and \( p \). Couplings of scalar fields to photons and baryons are tightly constrained; coupling to dark matter is less constrained (see \([53]\) and refs. therein for more details). As before we have chosen to work in the frame determined by the metric induced on the brane. As discussed in section II D, there is an issue of which frame to choose. An alternative to the above approach would be to transform the action into the Jordan frame before varying the action and then transform the equations of motion back to the Einstein frame. The variation of the action in arbitrary frames is given in \([50]\).

Now that we have the expression for the Lagrangian that we need, let us turn our attention to the cosmology of such models. If we assume that \( V(\phi) \) has been tuned so that \( \Lambda = 0 \) and that \( \phi \) is spatially homogeneous on the brane (i.e., \( \phi \) on the brane depends only on \( t \) ) then \([24]\) and \([29]\) give us

\[ \dot{H} + 2H^2 = \frac{k^4}{72} U(\phi)F(\phi)(\rho - 3p) + \frac{k^4}{12} U'(\phi)F'(\phi)p - \frac{k^4}{36} F(\phi)^2(\rho^2 + 3pp) - \frac{k^4}{12} F'(\phi)^2p^2 - \frac{1}{12} \dot{\phi}^2 - \frac{k}{a^2}, \tag{71} \]

and \([20]\) gives us

\[ \dot{\rho} + 3H(\rho + p) = -\frac{F'(\phi)}{F(\phi)} \dot{\phi}(\rho + p). \tag{72} \]

For an equation of state \( p = \omega \rho \), \([22]\) has solution \( \rho = \rho_0 \left( a^3 F \right)^{-(1+\omega)} \). For dust \((p = 0)\), substituting this solution into \((71)\) gives

\[ \dot{H} + 2H^2 = \frac{k^4}{72} U(\phi) \rho_0 - \frac{k^4}{36a^6} \rho_0^2 - \frac{1}{12} \dot{\phi}^2 - \frac{k}{a^2}, \tag{73} \]

which is identical to the non-coupled case. This is not surprising as, the energy-momentum tensor is \( \tau^a_b = \text{diag}(\rho a^{-3}, 0, 0, 0) \) so we have the same matter content as for non-coupled dust. It is also a promising start for this model as the standard Big-Bang model has been very successful as a description of the matter-dominated universe and so we want our model to look like the standard cosmology during the matter era.

Now consider radiation matter in the specific case where \( \phi \) is constant. For convenience, we will assume that \( \rho^2 \) and \( \dot{\phi} \) terms can be neglected. Then \((71)\) becomes

\[ \frac{d}{dt}(a^4 H^2) = A \frac{\dot{a}}{a} - 2ka\dot{a}, \tag{74} \]

where \( A \) is a collection of constants. We get the solution

\[ H^2 = \frac{A \log a}{a^4} + \frac{C}{a^4} - \frac{k}{a^2}. \tag{75} \]

For the non-coupled radiation matter, \( A \) in \([23]\) is always zero but the coupling allows a non-zero, positive value. As \( a \) grows, this new term will dominate the \( a^{-4} \) term. Also, as \( a \to 0 \), the first term will dominate and will be negative. Thus there will be a value of \( a \), dependent on the values of \( \rho_0, C \) and \( k \), for which \( H^2 = 0 \).

The term proportional to \( a^{-4} \) contains a contribution from the bulk and one from radiation matter on the brane. To calculate these individual contributions, we can use \([28]\) which, ignoring quadratic effects, gives

\[ H^2 = \frac{k^4 \rho_0}{48 a^4} \left( 4 U F^{-1/3} + U' F F^{-4/3} \right) - \frac{1}{3} E_{00} + \frac{k}{a^2}. \tag{76} \]

We can calculate \( E_{00} \) from \([49]\) which reduces to
$$\frac{d}{dt}(a^4E_{00}) = \frac{\kappa^2}{8F_{4/3}^4} \frac{d}{dt}(\log a) ,$$  \hspace{1cm} (77)

where quadratic matter terms have again been neglected. This generates the $a^{-4}\log a$ term and the other part of the $a^{-4}$ term in (73).

We have a Friedmann equation which is like that of the standard cosmology but with the extra $a^{-4}\log a$ term. Motivated by this, we now study the cosmology generated by a Friedmann equation with this new term in addition to the usual radiation, dust and curvature terms. Thus we consider

$$H^2 = \frac{A\log a}{a^4} + \frac{B}{a^4} + \frac{C}{a^3} - \frac{k}{a^2},$$  \hspace{1cm} (78)

where $A$, $B$ and $C$ are positive constants. We will consider this very much as a dynamical system, although we will make some physically motivated assumptions about the relative magnitudes of the constants.

To determine whether this models suffers from a flatness problem, we define the curvature density in the usual way

$$\Omega_k = -\frac{k}{a^2H^2},$$

hence we derive the relation,

$$\dot{\Omega}_k = -\frac{2}{H}(\dot{H} + H^2)\Omega_k .$$  \hspace{1cm} (79)

This tells us that $\Omega_k = 0$ is a stable fixed point if $\dot{H} + H^2 > 0$. From (78), we get

$$\dot{H} + H^2 = \left(\frac{A}{2} - B\right) \frac{1}{a^4} - \frac{A\log a}{a^4} - \frac{3C}{2a^3} .$$  \hspace{1cm} (80)

So there will be value of $a$, dependent on $A$, $B$ and $C$, below which $\Omega_k = 0$ will be an attractor. This stability condition can be written,

$$A - 2B - 2A\log a - 3Ca > 0 .$$  \hspace{1cm} (81)

The r.h.s. of (78) can be negative for some values of $a$. Clearly we must have $H^2 \geq 0$, so we also require that $a$ only takes values for which the r.h.s. of (78) is non-negative. We will neglect the last term both because its relative effect is only significant for large $a$ and because we hope to ensure that the universe is driven towards flatness. Thus we must also satisfy the condition

$$A\log a + B + Ca > 0 .$$  \hspace{1cm} (82)

If we can satisfy both (81) and (82) we will get a period of accelerated expansion which drive the universe towards flatness. Adding twice the second inequality to the first gives us $A > \frac{C}{\alpha}$. Substituting $a = A/C$ into the second inequality gives

$$\log \left(\frac{C}{A}\right) < 1 + \frac{B}{A} .$$  \hspace{1cm} (83)

If this is not satisfied, then (82) cannot hold for any value of $a < A/C$. Thus, the coefficients $A$, $B$ and $C$ must satisfy (83) for us to get a period in the history of the universe where $\Omega_k = 0$ is an attractor. Note that this is a necessary but not sufficient condition. We now consider whether we can achieve a period of accelerated expansion for some reasonable values of the parameters $A$, $B$ and $C$. Define $B = B/A$ and $C = C/A$. Then conditions (81) and (82) become

$$f_1(a) \equiv 1 - 2B - 2\log a - 3Ca > 0 ,$$  \hspace{1cm} (84)

$$f_2(a) \equiv \log a + B + Ca > 0 .$$  \hspace{1cm} (85)

Now $f_1$ decreases monotonically from $f_1(0) = \infty$ whereas $f_2$ increases monotonically from $f_2(0) = -\infty$. Thus $\exists \alpha > 0$ such that $f_1(\alpha) = f_2(\alpha)$. If $f_1(\alpha) = f_2(\alpha) > 0$ then there is an interval $I \ni \alpha$ such that, $\forall \alpha \in I$, $f_1(\alpha) > 0$ and $f_2(\alpha) > 0$.

The ratio $C/B = C/B$ is well known and, if we set $a = 1$ at present times, it is $10^4$ to good approximation [54]. By plotting $f_1$ and $f_2$ against $a$ for different values of $B$ we can determine that, for $B > 11$, there will be a period of accelerated expansion. Choosing larger values of $B$ makes this period occur for smaller values of $a$. In the standard cosmology, BBN occurs when $a \sim 10^{-11}$. So we expect the period of accelerated expansion to finish before this.
Choosing $B = 30$ gives a period of acceleration $10^{-13} \lesssim a \lesssim 1.5 \times 10^{-13}$, which would be acceptable. In practice, $B$ is probably much larger than this as no effect like the $a^{-4} \log a$ term has been observed.

As discussed above, there is a finite value of $a$ for which $H = 0$. Clearly, it is important that this is an unstable fixed point so that the universe expands from this initial value. Hence we now analyse the stability of the fixed point. Changing (88) to conformal time, we find

$$\left(\frac{da}{d\tau}\right)^2 = A\log a + B + Ca - ka^2. \quad (86)$$

Scaling $\tau \to \beta \tau$ and $a \to \gamma a$ gives

$$\frac{\gamma^2}{\beta^2} \left(\frac{da}{d\tau}\right)^2 = A\log a + (B + A\log \gamma) + C\gamma a - k\gamma^2 a^2, \quad (87)$$

so we can choose $\beta$ and $\gamma$ to give

$$a'^2 = \left(\frac{da}{d\tau}\right)^2 = \log a + \hat{A}a - \hat{B}a^2, \quad (88)$$

where $\hat{A}$ and $\hat{B}$ are redefined constants with $\hat{A}$ being positive and $\hat{B}$ having the same sign as $k$. Primes denote differentiation with respect to the rescaled conformal time. If $k \leq 0$ then $a' = 0$ has exactly one root, whereas, if $k > 0$, the number of roots will be two, one or zero depending on the values of $\hat{A}$ and $\hat{B}$. The ratio of $\hat{A}$ to $\hat{B}$ is

$$\hat{A} / \hat{B} = \frac{C}{k} e^{B/A}. \quad (89)$$

Since we expect $B \gg A$ for a reasonable cosmology, this ratio will be large, so there will be two roots. The case $k > 0$ with fewer than two roots does not allow $a'^2 > 0$, so does not produce any sort of cosmology.

To analyse the stability of this fixed point (the one with the smaller value of $a$ in the case where there are two) we calculate $a''$ by differentiating (88). We find

$$2a'' = a^{-1} + \hat{A} - 2\hat{B}a. \quad (90)$$

If $k \leq 0$, $a'' > 0 \forall a$. If $k > 0$ then $\exists \xi$ such that $a > \xi \Rightarrow a'' < 0$ and $a < \xi \Rightarrow a'' > 0$. However, $\hat{A}/\hat{B} \gg 1$ so $\xi \gg 1$ and so we have $a'' > 0$ at the fixed point with the smaller value of $a$. Thus, in all cases, we have a cosmology where the universe expands from a finite value of $a$. In the case $k \leq 0$, the universe will expand indefinitely whereas, if $k > 0$, the universe will expand to the other fixed point. To determine the stability of the other fixed point in the latter case, note that

$$a'' = \frac{1}{a} \left(\frac{1}{2} + \frac{\hat{A}}{2} - \hat{B}a^2\right) \lesssim \log a + \hat{A} - \hat{B}a^2 \quad \text{if} \quad a > e^{1/2}. \quad (91)$$

Thus $a'' < 0$ at the second fixed point if $a > e^{1/2}$, which is likely as $\hat{A} \gg \hat{B}$, so the universe will recollapse.

The upshot of all this analysis is that we have models where the universe expands from a finite value of $a$ with an early period of accelerated expansion (i.e. where $\Omega_k = 0$ is an attractor). These cosmological scenarios are interesting as they avoid an initial singularity and alleviate the flatness problem of the standard cosmology. In the above discussion, we have ignored the terms quadratic in the matter density. Thus the accelerated expansion must occur at low enough energy scales for the quadratic effects to be negligible but at high enough scales not to affect BBN.

**IV. CONCLUSIONS**

Using the Gauss-Codacci formalism we have derived effective 4D Einstein equations for a brane-world scenario with a bulk scalar field. By studying the trace of these equations we have been able to circumvent our ignorance of the contribution from the bulk in the gravitational sector, but not for the scalar field. This allows us to determine necessary properties for recovering the standard cosmological expansion in the late time limit. We have shown that the two-brane RS scenario cannot be made consistent with the desired expansion rate from standard 4D Einstein
gravity nor with observational constraints on Brans-Dicke gravity on the negative tension brane. This is true even if there is a stabilization mechanism, that is, constant or very slowly changing scalar field. A necessary condition for the recovery of a positive expansion rate is that the vacuum energy of the brane must be positive when the scalar field has been stabilized.

The expansion rate of a cosmological solution can be changed in sign by introducing a conformal scaling of the metric. This is a highly contentious issue in the literature of scalar-tensor gravity theories. We have adopted the view that the induced metric must be that which governs the geometry of the spacetime (and hence the Hubble parameter of a cosmological solution). In section II D, we showed that, in any other frame, there would be second derivatives of the scalar field into the Einstein equation, resulting in an energy-momentum for the scalar field which violates positive definiteness. In addition, we see that, as with 4D Brans-Dicke gravity, the term in the Einstein equations proportional to the energy-momentum tensor still has a negative coefficient in the case of a negative tension brane, the possible change in sign of the Hubble parameter being due to the additional trace term. Dilatonic brane-world models are fundamentally different from 4D scalar-tensor theories in that the choice of frame is really an issue for the 5D theory, which then determines the frame of the 4D effective theory. In our calculation, we have started from a 5D Einstein frame and shown that the induced metric determines a frame which is preferable according to both of the criteria normally advanced in the study of 4D scalar-tensor theories, namely that matter follows geodesics of the metric and that the contribution to the Einstein equations from the scalar field is positively definite.

Differences with results obtained by other authors who perform a dimensional reduction of the 5D spacetime by integrating over the bulk lead us to examine the difference between interpreting averaged or projected quantities as the relevant physical 4D entities. This is potentially an important philosophical issue in the study of higher dimensional spaces with warped geometries. We have argued that the averaging is difficult to define in a unique, coordinate-free way, whereas projection suffers from no such problem. In particular, it must be correct in the case that we have studied where the matter on the brane is localized by a $\delta$-function, it may not necessarily be the case when the matter is defined in terms of excitations of higher dimensional gravity as in Kaluza-Klein models, for example. Clearly, this issue requires some understanding of how the matter is localized onto the brane from a fundamental point of view before it can be answered with any certainty.

For a model based on a HW type dilaton, we have been able to obtain a next-to-leading order correction in $\dot{\phi}$ to the RS solution in the case of a universe dominated by radiation. The Friedmann equation is the same as for the RS model due to a cancellation of $\phi$ dependent terms in the bulk contribution. Given that the HW type models have a rather different mechanism for achieving a cancellation of the effective cosmological constant, and that the asymptotic bulk geometry is likely to be somewhat different, it might seem surprising that the they reduce to the same effective Friedmann equation when the scalar field is assumed to be slowly varying. This suggests that form of the RS Friedmann equation is in fact universal in a brane-world which has any chance of being compatible with the standard Big-Bang model.

It is natural to consider the case where the brane matter is non-minimally coupled to the scalar field. We discover that this does not change the cosmology when in a matter dominated era, but in the case of relativistic matter there is an additional term in the Friedmann equation. This gives rise to a cosmology where the universe expands from a finite value of the scale factor, with a period of accelerated expansion in the early universe. Although a toy model, it has the advantages of avoiding the initial singularity and flatness problem of the standard big bang model. Clearly, more should be directed in toward understanding this novel scenario.

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