ON EMBEDDINGS OF THE GRASSMANNIAN $Gr(2, m)$ INTO THE GRASSMANNIAN $Gr(2, n)$

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ABSTRACT. In this paper, we consider holomorphic embeddings of $Gr(2, m)$ into $Gr(2, n)$. We can study such embeddings by finding all possible total Chern classes of the pull-back of the universal bundles under these embeddings. To do this, we use the relations between Chern classes of the universal bundles and Schubert cycles together with properties of complex vector bundles of rank 2 on Grassmannians. Consequently, we find a condition on $m$ and $n$ for which any holomorphic embedding of $Gr(2, m)$ into $Gr(2, n)$ is linear.

1. Introduction

For any $n \geq m \geq d$, there is a natural holomorphic embedding of $Gr(d, m)$ into $Gr(d, n)$; let $f : \mathbb{C}^m \hookrightarrow \mathbb{C}^n$ be an injective linear map, then $f$ induces the embedding $\tilde{f} : Gr(d, m) \hookrightarrow Gr(d, n)$ which maps a $d$-dimensional subspace $L$ of $\mathbb{C}^m$ to the $d$-dimensional subspace $f(L)$ of $\mathbb{C}^n$. We call such an embedding to be linear. It is interesting to find an answer for the following question on the linearity of embeddings:

**Question.** Under what conditions on $m,n$ and $d$, is a holomorphic embedding $Gr(d, m) \hookrightarrow Gr(d, n)$ always linear?

In this paper, we consider this question for the case when $d = 2$ and show the following Main Theorem:

**Main Theorem.** Assume that the pair $(m,n)$ of integers satisfies one of the conditions

- $5 \leq m = n - 1 \leq 8$;
- $9 \leq m \leq n \leq \frac{3m-6}{2}$.

Then any holomorphic embedding $\varphi : Gr(2, m) \hookrightarrow Gr(2, n)$ is linear. Moreover, any holomorphic embedding $\varphi : Gr(2, 4) \hookrightarrow Gr(2, 5)$ is either linear or $\varphi_0 \circ \phi$ where $\varphi_0 : Gr(2, 4) \hookrightarrow Gr(2, 5)$ is a linear embedding and $\phi : Gr(2, 4) \rightarrow Gr(2, 4)$ is the dual map which is defined by

$\phi(x) :=$ orthogonal complement of $L_x$.

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This theorem is follows from Theorem 5.1 and Proposition 6.3. The condition of Main Theorem may not be effective, that is, there may be a more general condition on \( m \) and \( n \) for the linearity of embeddings.

There have been some approaches for the above question. S. Feder answered this question for the case when \( d = 1 \). In [6], he showed that if \( 2m > n \), then any holomorphic embedding

\[
\varphi : \mathbb{P}^m \hookrightarrow \mathbb{P}^n
\]

is linear (or equivalently, of degree 1), but there is an embedding \( \mathbb{P}^m \hookrightarrow \mathbb{P}^{2m} \) of degree 2 for all \( m \geq 1 \). H. Tango answered for a slightly modified question. In [12], he considered holomorphic embeddings

\[
\varphi : Gr(1, n - 1) = \mathbb{P}^{n-2} \hookrightarrow Gr(2, n)
\]

and classified all of them. In particular, he showed that every such embedding is of degree 1 or 2 (See Example 3.2 for the case when \( n = 4 \)).

Although S. Feder and H. Tango dealt different cases, they used similar numerical techniques. To describe these techniques generally, denote the embedding (1.1) or (1.2) by

\[
\varphi : Gr(d_1, m) \hookrightarrow Gr(d_2, n).
\]

Let \( E(d_2, n) \) be the universal bundle on \( Gr(d_2, n) \) and \( E \) the pull-back of the dual bundle of \( E(d_2, n) \) under the embedding \( \varphi \). Then the total Chern class of \( E \) can be written uniquely as a linear sum of Schubert cycles on \( Gr(d_1, m) \) with coefficients \( a, b, \cdots, c \) in \( \mathbb{Z} \) (Equation (3.2)) and from these coefficients, we characterize the linearity of \( \varphi \).

Let \( N \) be the pull-back of the normal bundle of \( \varphi(Gr(d_1, m)) \) in \( Gr(d_2, n) \) under \( \varphi \). Then we can construct an equation of the total Chern class of \( N \) in terms of Chern classes of \( E, E(d_1, m) \) and their dual bundles, tensor bundles. So we obtain Diophantine equations on \( a, b, \cdots, c \) (Lemma 4.1 and Proposition 4.2). In this way, we can think of the problem on the classification of embeddings between two Grassmannians as the problem on solving the obtained Diophantine equations.

In general, it is hard to solve these kinds of Diophantine equations without additional informations. To overcome this difficulty for our case when \( d_1 = d_2 = 2 \), we get inequalities on \( a, b, \cdots, c \) (Proposition 5.3 and 5.10). In particular, Proposition 5.3 (b) makes us to apply W. Barth and A. van de Ven’s results (Proposition 2.8 and 2.9) on the complex vector bundle \( E \) on \( Gr(2, m) \) to be decomposable or \( E(2, m) \otimes L \) for some line bundle \( L \) on \( Gr(2, m) \). With help of these results, we can find all possible \( a, b, \cdots, c \) satisfying the Diophantine equations and conclude that any embedding \( \varphi : Gr(2, m) \hookrightarrow Gr(2, n) \) is linear if either \( 5 \leq m = n - 1 \leq 8 \) or \( 9 \leq m \leq n \leq \frac{3m-6}{2} \).

Throughout this paper, we assume that every map is holomorphic.
2. Preliminaries

In this section, we introduce the results on Schubert cycles, a universal (quotient) bundle on $Gr(d, m)$ and complex vector bundles of rank 2 on $\mathbb{P}^k$ or on $Gr(d, m)$. For more details on Schubert cycles, see [2], §1.5 Grassmannians of [3], for more details on a universal (quotient) bundle, see [12] and for more details on complex vector bundles of rank 2, see [3], [4].

2.1. Schubert cycles on $Gr(d, m)$. For an $m$-dimensional complex vector space $V$ and $d \leq m$, let $Gr(d, V)$ be the set of all $d$-dimensional subspaces of $V$ and call it a Grassmannian. When $V = \mathbb{C}^m$, we denote $Gr(d, \mathbb{C}^m)$ simply by $Gr(d, m)$. For a partial flag $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_d$ of $\mathbb{C}^m$, let $\omega(A_1, A_2, \cdots, A_d)$ be the subvariety of $Gr(d, m)$ given by

$$\{x \in Gr(d, m) \mid \text{dim}(L_x \cap A_i) \geq i \text{ for all } i\}$$

where $L_x$ is the $d$-dimensional subspace of $\mathbb{C}^m$ corresponding to $x$. We call such an $\omega(A_1, A_2, \cdots, A_d)$ a Schubert variety of type $(a_1, a_2, \cdots, a_d)$ where $a_i := m - d + i - \text{dim}(A_i)$ for $i = 1, 2, \cdots, d$. The (complex) codimension of $\omega(A_1, A_2, \cdots, A_d)$ in $Gr(d, m)$ is $\sum_{i=1}^d a_i$.

Example 2.1. There are some familiar Schubert varieties on $Gr(d, m)$.

(a) Let $\omega(A_1, \cdots, A_{d-1}, A_d)$ be a Schubert variety of type $(m-d, \cdots, m-d, 0)$. Since $\text{dim}(A_i) = i$ for all $1 \leq i \leq d - 1$ and $\text{dim}(A_d) = m$,

$$A_i = \text{span}\{v_1, \cdots, v_i\} \quad \text{for all } 1 \leq i \leq d - 1 \quad \text{and}$$

$$A_d = \mathbb{C}^m$$

for some linearly independent vectors $v_1, \cdots, v_{d-1} \in \mathbb{C}^m$. So we have

$$\omega(A_1, \cdots, A_{d-1}, A_d) = \{x \in Gr(d, m) \mid A_{d-1} \subseteq L_x\}$$

$$\simeq \mathbb{P}(\mathbb{C}^m / A_{d-1}) \simeq \mathbb{P}^{m-d}.$$  

It is an $(m - d)$-dimensional projective space and is maximal in $Gr(d, m)$, that is, there is not a projective space in $Gr(d, m)$ containing $\omega(A_1, \cdots, A_{d-1}, A_d)$ properly.

(b) Let $\omega(A_1, \cdots, A_d)$ be a Schubert variety of type $(k, \cdots, k)$. Since $\text{dim}(A_i) = m - d + i - k$ for all $1 \leq i \leq d$,

$$\omega(A_1, \cdots, A_d) = \{x \in Gr(d, m) \mid L_x \supseteq A_d\}$$

$$= Gr(d, A_d) \simeq Gr(d, m - k).$$

If an embedding $\varphi : Gr(d, m - k) \hookrightarrow Gr(d, m)$ is semi-linear (See Definition 3.1 (a).), then its image is of this form.

Two Schubert varieties of type $(a_1, \cdots, a_d)$ and $(b_1, \cdots, b_d)$ correspond to the same homology class if and only if $(a_1, \cdots, a_d) = (b_1, \cdots, b_d)$. We denote the Poincaré dual to a Schubert variety of type $(a_1, \cdots, a_d)$ by $\omega_{a_1, \cdots, a_d}$.
and call it the Schubert cycle of type \((a_1, \cdots, a_d)\). Since the codimension of a Schubert variety of type \((a_1, \cdots, a_d)\) is \(\sum_{i=1}^d a_i\),
\[
\omega_{a_1, \cdots, a_d} \in H^2(\sum_{i=1}^d a_i)(Gr(d, m), \mathbb{Z})
\]
and the set of Schubert cycles
\[
\left\{ \omega_{a_1, \cdots, a_d} \mid m - d \geq a_1 \geq \cdots \geq a_d \geq 0 \text{ and } \sum_{i=1}^d a_i = k \right\}
\]
form a basis of the \(\mathbb{Z}\)-module \(H^{2k}(Gr(d, m), \mathbb{Z})\). In particular, when \(k = \dim(Gr(d, m)) = d(m - d)\), \(H^{2d(m-d)}(Gr(d, m), \mathbb{Z}) \simeq \mathbb{Z}\) is generated by \(\omega_{m-d, \cdots, m-d} = \omega_{1, \cdots, 1}^{m-d}\). So every \(\gamma \in H^{2d(m-d)}(Gr(d, m), \mathbb{Z})\) is of the form \(c_\gamma \omega_{1, \cdots, 1}^{m-d}\) for some integer \(c_\gamma\), thus we may identify \(\gamma\) with \(c_\gamma \in \mathbb{Z}\). The multiplications of Schubert cycles are commutative and satisfy the following rule, named Pieri’s formula.

**Lemma 2.2** (Pieri’s formula). In \(Gr(d, m)\), for \(m - d \geq a_1 \geq a_2 \geq \cdots \geq a_d \geq 0\) and \(m - d \geq h \geq 0\),
\[
\omega_{a_1, a_2, \cdots, a_d} \omega_{b_0, \cdots, b_d} = \sum_{(b_1, b_2, \cdots, b_d) \in I} \omega_{b_1, b_2, \cdots, b_d}
\]
where \(I\) is the set of all pairs \((b_1, b_2, \cdots, b_d) \in \mathbb{Z}^d\) satisfying
\[
m - d \geq b_1 \geq a_1 \geq b_2 \geq a_2 \geq \cdots \geq b_d \geq a_d \geq 0 \text{ and } \left(\sum_{i=1}^d a_i\right) + h = \sum_{i=1}^d b_i.
\]

For the proof of Lemma 2.2, see the page 203 of [8]. In particular, when \(d = 2\), we have the refined rules for the multiplications of Schubert cycles.

**Convention 2.3.** In \(Gr(2, m)\), let \(\omega_{k,l} = 0\) unless \(m - 2 \geq k \geq l \geq 0\).

**Corollary 2.4.** Under **Convention 2.3**, Schubert cycles on \(Gr(2, m)\) satisfy the following relations:

(a) (Lemma 4.2 (i) of [12]) \(\omega_{i,j} \omega_{1,1} = \omega_{i+1,j+1}\).

(b) (Restate of Lemma 2.2) \(\omega_{i,0} \omega_{j,0} = \omega_{i+j,0} + \omega_{i+j-1,1} + \cdots + \omega_{i+1,j-1} + \omega_{i,j}\).

Using Corollary 2.4 and the commutativity, we can compute the multiplications of Schubert cycles on \(Gr(2, m)\) easily. For example,
\[
\omega_{8,5} \omega_{7,3} = (\omega_{3,0}^{8} \omega_{1,1}^{5}) (\omega_{4,0}^{3} \omega_{1,1}^{3}) = \omega_{4,0}^{8} \omega_{3,0}^{3} \omega_{1,1}^{8}
\]
\[
= (\omega_{7,0} + \omega_{6,1} + \omega_{5,2} + \omega_{4,3}) \omega_{1,1}^{8}
\]
\[
= \omega_{15,8} + \omega_{14,9} + \omega_{13,10} + \omega_{12,11}
\]
(Some terms can be omitted if \(m < 17\)).
2.2. Universal bundle and universal quotient bundle on $Gr(d, m)$. Let $E(d, m)$ be the universal bundle on $Gr(d, m)$ whose total space is

$$\{(x, v) \in Gr(d, m) \times \mathbb{C}^m \mid v \in L_x\}$$

(We denote the universal bundle on $Gr(d, V)$ by $E(d, V)$.) We have the following canonical short exact sequence:

$$0 \to E(d, m) \to \bigoplus_{i=0}^m O_{Gr(d, m)} \to Q(d, m) \to 0$$

where $Q(d, m) := \bigoplus_{i=0}^m O_{Gr(d, m)} / E(d, m)$, which is called the universal quotient bundle on $Gr(d, m)$.

**Proposition 2.5** (Lemma 1.3 and 1.4 of [12]). In $Gr(d, m)$, the total Chern classes of $E(d, m)$ and $Q(d, m)$ are as follows:

(a) $c(E(d, m)) = 1 - \omega_{1,0,\ldots,0} + \omega_{1,1,0,\ldots,0} - \cdots + (-1)^d \omega_{1,1,1,\ldots,1}$.

(b) $c(Q(d, m)) = 1 + \omega_{1,0,\ldots,0} + \omega_{2,0,\ldots,0} + \cdots + \omega_{m-2,0,\ldots,0}$.

**Definition 2.6.** Let $Y$ be a non-singular variety. A cohomology class $\gamma \in H^{2k}(Y, \mathbb{Z})$ is numerically non-negative if the intersection numbers $\gamma \cdot Z$ are non-negative for all subvarieties $Z$ of $Y$ of dimension $k$.

In our case when $Y = Gr(2, m)$, a cohomology class $\gamma \in H^{2k}(Gr(2, m), \mathbb{Z})$ is numerically non-negative means that when we write $\gamma$ as the linear sum with respect to the basis

$$\{\omega_{k-i,i} \mid 0 \leq i \leq k - i \leq m - 2\},$$

every coefficient in $\gamma$ is non-negative. The following proposition tells us a sufficient condition for the numerical non-negativities of all Chern classes of complex vector bundles $\mathcal{E}$.

**Proposition 2.7** (Proposition 2.1 (i) of [12]). Let $Z$ be a non-singular variety and let $\mathcal{E}$ be a complex vector bundle of arbitrary rank on $Z$ which is generated by global sections. Then each Chern class $c_i(\mathcal{E})$ of $\mathcal{E}$ is numerically non-negative for all $i = 1, 2, \cdots \dim(Z)$.

### 2.3. Vector bundles of rank 2 on $\mathbb{P}^k$ or on $Gr(d, m)$.

Let $\mathcal{E}$ be a complex vector bundle on $\mathbb{P}^k$ of rank 2. For a projective line $\ell$ in $\mathbb{P}^k$, $\mathcal{E}|_{\ell}$ is decomposable by Grothendieck theorem (Theorem 2.1.1 of [11]), that is,

$$\mathcal{E}|_{\ell} = O_\ell(a_1) \oplus O_\ell(a_2)$$

for some integers $a_1, a_2$ unique up to the permutation. For such $a_1$ and $a_2$, define $b(\mathcal{E}|_{\ell})$ by the integer $\left\lfloor \frac{a_1 - a_2}{2} \right\rfloor$ where $\lfloor \bullet \rfloor$ is the maximal integer which does not exceed $\bullet$, and using this, let

$$B(\mathcal{E}) := \max \left\{ b(\mathcal{E}|_{\ell}) \mid \mathbb{P}^1 \simeq \ell \subset \mathbb{P}^k \right\}.$$

The following proposition tells us a sufficient condition for the decomposability of complex vector bundles $\mathcal{E}$ on $\mathbb{P}^k$ of rank 2.
Proposition 2.8 (Theorem 5.1 of [4]). Let $E$ be a complex vector bundle on $\mathbb{P}^k$ of rank 2 satisfying $B(E) < \frac{k-2}{4}$. Then $E$ is decomposable.

Also, there is a sufficient condition for complex vector bundles on $Gr(d, m)$ of rank 2 to be decomposable or some special form.

Proposition 2.9 (Theorem 4.1 of [4]). Let $E$ be a complex vector bundle on $Gr(d, m)$ of rank 2 with $m - d \geq 2$ satisfying that the restrictions $E|_Y$ are decomposable for all Schubert varieties $Y \subset Gr(d, m)$ of type $(m-d, \cdots, m-d, 0)$. Then either $E$ is decomposable, or $d = 2$ and $E \simeq E(2, m) \otimes L$ for some line bundle $L$ on $Gr(2, m)$.

Proposition 2.8 and 2.9 will play an important role in proving Theorem 5.1.

3. Embeddings of $Gr(2, m)$ into $Gr(2, n)$

In this section, we will consider the linearity of embeddings of $Gr(2, m)$ into $Gr(2, n)$.

Definition 3.1. (a) An embedding $\varphi : Gr(d, m) \hookrightarrow Gr(d, n)$ is semi-linear if there is an $m$-dimensional subspace $H_\varphi$ of $\mathbb{C}^n$ satisfying that the image of $\varphi$ is $Gr(d, H_\varphi)$.

(b) An embedding $\varphi : Gr(d, m) \hookrightarrow Gr(d, n)$ is linear if $\varphi$ is induced by an injective linear map $f : \mathbb{C}^m \hookrightarrow \mathbb{C}^n$, that is,

$$\varphi(x) = d\text{-dimensional subspace } f(L_x) \text{ of } \mathbb{C}^n$$

for all $x \in Gr(d, m)$.

(c) When $m = 2d$, an embedding $\varphi : Gr(d, 2d) \hookrightarrow Gr(d, n)$ is twisted linear if $\varphi$ is of the form $\varphi_0 \circ \phi$ where $\varphi_0 : Gr(d, 2d) \hookrightarrow Gr(d, n)$ is a linear embedding and $\phi : Gr(d, 2d) \to Gr(d, 2d)$ is the dual map which is defined by

$$\phi(x) := \text{orthogonal complement of } L_x.$$

Example 3.2. Consider an embedding $\varphi : Gr(2, 3) \hookrightarrow Gr(2, 4)$. Since $Gr(2, 3) \simeq \mathbb{P}^2$, we may assume that $\varphi$ is an embedding of $\mathbb{P}^2$ into $Gr(2, 4)$. Write an element in $Gr(2, 4)$ as an equivalence class of a $2 \times 4$ matrix:

$$\left[ \begin{array}{cccc} * & * & * & * \\ * & * & * & * \end{array} \right] .$$

By Theorem 5.1 (i) of [12], there are following 4 types of $X = \varphi(\mathbb{P}^2)$ up to linear transformations, namely

$$X^0_{3,1} := \left\{ \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & x_0 & x_1 \\ 0 & x_0 & x_2 \end{array} \right] \mid [x_0 : x_1 : x_2] \in \mathbb{P}^2 \right\} ,$$

$$X^1_{3,1} := \left\{ \left[ \begin{array}{ccc} x_0 & x_1 & 0 \\ 0 & x_0 & x_2 \\ 0 & x_0 & x_1 \end{array} \right] \mid [x_0 : x_1 : x_2] \in \mathbb{P}^2 \right\} ,$$

$$\hat{X}^0_{3,1} := \phi(X^0_{3,1}) \text{ and}$$

$$\hat{X}^1_{3,1} := \phi(X^1_{3,1})$$
where \( \phi : Gr(2, 4) \to Gr(2, 4) \) is the dual map given as in (3.1). If \( X = X_{3,1}^0 \) or \( \tilde{X}_{3,1}^0 \), then the degree of \( \varphi \) is 1. In particular, \( X_{3,0}^0 \simeq \mathbb{P}^2 \) is a Schubert variety of type \((2, 0)\) and \( \tilde{X}_{3,0}^0 \) is a Schubert variety of type \((1, 1)\). Since Schubert varieties of type \((2, 0)\) and \((1, 1)\) are sub-Grassmannians in \( Gr(2, 4) \), \( \varphi \) is linear. On the other hands, if \( X = X_{3,1}^1 \) or \( \tilde{X}_{3,1}^1 \), then the degree of \( \varphi \) is 2, so \( \varphi \) is not linear. In particular, when \( X = X_{3,1}^1 \), the composite of the Plücker embedding \( Gr(2, 4) \hookrightarrow \mathbb{P}(\wedge^2 \mathbb{C}^4) \) with the corresponding embedding \( \varphi \) is given by

\[
[ x_0 : x_1 : x_2 ] \mapsto \begin{bmatrix} x_0 & x_1 & x_2 & 0 \\ 0 & x_0 & x_1 & x_2 \end{bmatrix}
\mapsto [ x_0^2 : x_0 x_1 : x_0 x_2 : x_1^2 - x_0 x_2 : x_1 x_2 : x_2^2 ].
\]

Hence, we can show that \( \varphi \) is not linear directly.

Let \( \varphi : Gr(2, m) \hookrightarrow Gr(2, n) \) be an embedding with the image \( X \) and \( E := \varphi^*(\tilde{E}(2, n)) \) where \( \tilde{E}(2, n) \) is the dual bundle of \( E(2, n) \). To distinguish Schubert cycles on two Grassmannians, denote Schubert cycles on \( Gr(2, n) \) (resp. \( Gr(2, m) \)) by \( \tilde{\omega}_{i,j} \) (resp. \( \omega_{k,l} \)) (Of course, all properties in Section 2 hold for \( Gr(2, n) \) and \( \tilde{\omega}_{i,j} \)). By Proposition 2.5 (a),

\[
c(\tilde{E}(2, n)) = 1 + \tilde{\omega}_{1,0} + \tilde{\omega}_{1,1}
\]

and when \( m \geq 4 \),

\[
\begin{align*}
(3.2) \quad c_1(E) &= \varphi^*(\tilde{\omega}_{1,0}) = a \omega_{1,0}, \\
 & \quad c_2(E) = \varphi^*(\tilde{\omega}_{1,1}) = b \omega_{1,0}^2 + c \omega_{1,1} = b \omega_{2,0} + (b + c) \omega_{1,1} 
\end{align*}
\]

for some \( a, b, c \in \mathbb{Z} \).

**Lemma 3.3.** Let \( n \geq m \geq 4 \). Then the Poincaré dual to the homology class of \( X \) is

\[
\sum_{i=0}^{n-m} (X \cdot \tilde{\omega}_{n-2-i, 2m-n-2+i}) \tilde{\omega}_{2n-2m-i, i}
\]

where \( X \cdot \tilde{\omega}_{n-2-i, 2m-n-2+i} \) is the intersection number in \( Gr(2, n) \).

**Proof.** Since the codimension of \( X \simeq Gr(2, m) \) in \( Gr(2, n) \) is \( 2n - 2m \), the Poincaré dual to the homology class of \( X \) is

\[
\sum_{i=0}^{n-m} d_i \tilde{\omega}_{2n-2m-i, i}
\]

for some integers \( d_i \). By Lemma 4.2 (ii) of [12],

\[
X \cdot \tilde{\omega}_{n-2-j, 2m-n-2+j} = \sum_{i=0}^{n-m} d_i (\tilde{\omega}_{2n-2m-i, i} \cdot \tilde{\omega}_{n-2-j, 2m-n-2+j}) = d_j
\]

as desired. \( \square \)
Lemma 3.4 (Theorem 1.1 of [5]). The automorphism group of $Gr(d, m)$ is

\[ \text{Aut}(Gr(d, m)) = \begin{cases} PGL(d, m), & \text{if } m \neq 2d \\ PGL(d, m) \cup (\phi \circ PGL(d, C)), & \text{if } m = 2d \end{cases} \]

where $\phi : Gr(d, 2d) \to Gr(d, 2d)$ is the dual map given as in [3.1].

Proposition 3.5. For $n \geq m \geq 4$, let $\varphi : Gr(2, m) \hookrightarrow Gr(2, n)$ be an embedding and the integers $a, b, c$ as in (3.2).

(a) If either $\varphi$ is linear, or $m = 4$ and $\varphi$ is twisted linear, then $\varphi$ is semi-linear.

(b) Assume that $\varphi$ is semi-linear. If $m \geq 5$, then $\varphi$ is linear, and if $m = 4$, then $\varphi$ is either linear or twisted linear.

(c) An embedding $\varphi$ is linear if and only if $(a, b, c) = (1, 0, 1)$. Moreover, when $m = 4$, $\varphi$ is twisted linear if and only if $(a, b, c) = (1, 1, -1)$.

Proof. (a) If $\varphi$ is linear, then $\varphi$ is induced by an injective linear map $f : C^m \hookrightarrow C^n$. So $X = \varphi(Gr(2, m)) = Gr(2, f(C^m))$, thus $\varphi$ is semi-linear. Moreover, if $m = 4$ and $\varphi$ is twisted linear, then $\varphi = \varphi_0 \circ \phi$ for some linear embedding $\varphi_0 : Gr(2, 4) \hookrightarrow Gr(2, n)$. So we have

\[ X = \varphi(Gr(2, 4)) = \varphi_0(\phi(Gr(2, 4))) = \varphi_0(Gr(2, 4)), \]

thus $\varphi = \varphi_0 \circ \phi$ is semi-linear.

(b) Since $\varphi$ is semi-linear, $X = Gr(2, H_\varphi)$ for some $m$-dimensional subspace $H_\varphi$ of $C^m$. For such an $H_\varphi$, fix an isomorphism $\psi_\varphi : Gr(2, H_\varphi) \to Gr(2, m)$ which is induced by a linear isomorphism $H_\varphi \to C^m$. Then $\psi_\varphi \circ \varphi$ is an automorphism on $Gr(2, m)$ as follows:

\[ \begin{CD} \psi_\varphi \circ \varphi @>\cong>>& Gr(2, m) \\
@. @| \psi_\varphi \\
Gr(2, m) @>\cong>>& Gr(2, H_\varphi) \subset Gr(2, n). \end{CD} \]

If $m \geq 5$, then $\psi_\varphi \circ \varphi \in PGL(d, C)$ by Lemma 3.4 thus $\varphi$ is linear. If $m = 4$, then $\psi_\varphi \circ \varphi \in PGL(d, C) \cup (PGL(d, C) \circ \phi)$ by Lemma 3.4 thus $\varphi$ is either linear or twisted linear.

(c) First, assume that either $\varphi$ is linear, or $m = 4$ and $\varphi$ is twisted linear. Then $X = Gr(2, H_\varphi)$ for some $m$-dimensional subspace $H_\varphi$ of $C^n$ by (a). The total space of $E(2, n)|_X$ is

\[ \{(x, v) \in X \times C^n \mid v \in L_x \subset C^n\} = \{(x, v) \in Gr(2, H_\varphi) \times H_\varphi \mid v \in L_x \subset H_\varphi\} \]
which is exactly equal to the total space of $E(2, H_\varphi)$. So $E(2, n)|_X = E(2, H_\varphi)$ and we have

$$E = \varphi^*(\tilde{E}(2, n)) = \varphi^*(\tilde{E}(2, n)|_X) = \varphi^*(\tilde{E}(2, H_\varphi)).$$

If $\varphi$ is linear, then $\varphi^*(\tilde{E}(2, H_\varphi)) \simeq \tilde{E}(2, m)$. So $c(E) = 1 + \omega_{1,0} + \omega_{1,1}$, thus $(a, b, c) = (1, 0, 1)$. If $m = 4$ and $\varphi$ is twisted linear, then $\varphi = \varphi_0 \circ \phi$ for some linear embedding $\varphi_0 : Gr(2, m) \hookrightarrow Gr(2, n)$. So we have

$$\varphi^*(\tilde{E}(2, H_\varphi)) = \phi^*(\varphi_0^*(\tilde{E}(2, H_\varphi))) \simeq \phi^*(\tilde{E}(2, m)) = Q(2, m),$$

thus $c(E) = 1 + \omega_{1,0} + \omega_{2,0} = 1 + \omega_{1,0} + \omega_{1,0}^2 - \omega_{1,1}$, that is, $(a, b, c) = (1, 1, -1)$.

Assume that $(a, b, c) = (1, 0, 1)$. Then $\varphi^*(\tilde{w}_{1,0}) = \omega_{1,0}$ and $\varphi^*(\tilde{w}_{1,1}) = \omega_{1,1}$. Note that $H^\bullet(Gr(2, n), \mathbb{Z})$ (resp. $H^\bullet(Gr(2, m), \mathbb{Z})$) is generated by $\tilde{w}_{1,0}$ and $\tilde{w}_{1,1}$ (resp. $\omega_{1,0}$ and $\omega_{1,1}$) as a graded ring. Moreover, the multiplicative structures of $H^\bullet(Gr(2, n), \mathbb{Z})$ and $H^\bullet(Gr(2, m), \mathbb{Z})$ are exactly same when we adopt Convention 2.3. So we have

$$\varphi^*(\tilde{w}_{i,j}) = \omega_{i,j}$$

for all $n - 2 \geq i \geq j \geq 0$, thus the Poincaré dual to the homology class of $X$ is

$$\sum_{i=0}^{n-m} (X : \tilde{w}_{n-2-i,2m-n-2+i}) \tilde{w}_{2n-2m-i,i} = \sum_{i=0}^{n-m} (\varphi^*(\tilde{w}_{n-2-i,2m-n-2+i})) \tilde{w}_{2n-2m-i,i} = (\omega_{m-2,m-2}) \tilde{w}_{n-m,n-m} = \tilde{w}_{n-m,n-m}$$

by Lemma 3.3.

Assume that $m = 4$ and $(a, b, c) = (1, 1, -1)$. Then $\varphi^*(\tilde{w}_{1,0}) = \omega_{1,0}$ and $\varphi^*(\tilde{w}_{1,1}) = \omega_{2,0}^2 - \omega_{1,1} = \omega_{2,0}$. Note that

$$\varphi^*(\tilde{w}_{1,0}) = \varphi^*(\tilde{w}_{1,0}^4 - 3 \tilde{w}_{1,0}^2 \tilde{w}_{1,1} + \tilde{w}_{1,1}^2) = \omega_{1,0}^4 - 3 \omega_{1,0}^2 \omega_{2,0} + \omega_{2,0}^2 = 0,$$

(3.3) $$\varphi^*(\tilde{w}_{1,1}) = \varphi^*(\tilde{w}_{1,0}^2 \tilde{w}_{1,1} - \tilde{w}_{1,1}^2) = \omega_{1,0}^2 \omega_{2,0} - \omega_{1,1}^2 = 0$$

and

$$\varphi^*(\tilde{w}_{2,0}) = \omega_{2,0}^2 = 1$$
because \( \omega_{1,0}^4 = 2 \) and \( \omega_{1,0}^2 \omega_{2,0} = 1 = \omega_{2,0}^2 \) in \( H^8(Gr(2,4), \mathbb{Z}) \simeq \mathbb{Z} \). So the Poincaré dual to the homology class of \( X \)

\[
\sum_{i=0}^{n-4} (X \cdot \tilde{\omega}_{n-2-i,6-n+i}) \tilde{\omega}_{2n-8-i,i} = \sum_{i=\max\{0,n-6\}}^{n-4} (X \cdot \tilde{\omega}_{n-2-i,6-n+i}) \tilde{\omega}_{2n-8-i,i}
\]

(\because \text{ If } i < n - 6, \text{ then } \tilde{\omega}_{2n-8-i,i} = 0.)

\[
= \sum_{i=\max\{0,n-6\}}^{n-4} (\varphi^*(\tilde{\omega}_{n-2-i,6-n+i})) \tilde{\omega}_{2n-8-i,i}
\]

\[
= (\varphi^*(\tilde{\omega}_{4,0})) \tilde{\omega}_{n-2,n-6} + (\varphi^*(\tilde{\omega}_{3,1})) \tilde{\omega}_{n-3,n-5} + (\varphi^*(\tilde{\omega}_{2,2})) \tilde{\omega}_{n-4,n-4}
\]

\[
= \tilde{\omega}_{n-4,n-4} \quad (\because \text{ Equation (3.3)})
\]

by Lemma 3.3.

Hence, if either \((a, b, c) = (1, 0, 1)\), or \(m = 4\) and \((a, b, c) = (1, 1, -1)\), then the Poincaré dual to the homology class of \( X \) is \( \tilde{\omega}_{n-m,n-m} \). Any subvariety of \( Gr(2, n) \) which is corresponding to \( \tilde{\omega}_{n-m,n-m} \) is of the form \( Gr(2, H) \) where \( H \) is an \( m \)-dimensional subspace of \( \mathbb{C}^n \) (See Example 2.1 (b).), so \( \varphi \) is semi-linear. Applying (b) and the converse results, we obtain the desired results. \( \square \)

4. Numerical conditions for an embedding \( \varphi \)

Let \( \varphi : Gr(2, m) \rightarrow Gr(2, n) \) be an embedding with the image \( X \) and for \( E := \varphi^*(\mathcal{E}(2, n)) \), the integers \( a, b \) and \( c \) be given as in (3.2). Let \( N \) be the pull-back of the normal bundle of \( X \) in \( Gr(2, n) \) under \( \varphi \). In this section, we will find Diophantine equations (Lemma 4.1 and Proposition 4.2) and inequalities on \( a, b, c \) (Proposition 4.5, 4.6 and 4.10) by computing the total Chern class of \( N \). For more details on Chern classes, see [10].

4.1. Total Chern class of \( N \).

**Lemma 4.1.** Let \( n \geq m \geq 4 \). Then the total Chern class \( c(N) = c_0 + c_1 + c_2 + \cdots + c_{2n-2m} \) of \( N \) satisfies the following equation:

\[
(1 + (4b - a^2) \omega_{1,0}^2 + 4c \omega_{1,1}^2)(1 + \omega_{1,0} + \omega_{1,1})^m
\]

\[
= (1 + a \omega_{1,0} + b \omega_{1,0}^2 + c \omega_{1,1}^2)^n(1 - \omega_{1,0}^2 + 4 \omega_{1,1}).
\]

Moreover, the first and second Chern classes of \( N \) are

\[
c_1 = (an - m) \omega_{1,0} \quad \text{and} \quad c_2 = \left\{ \left( \frac{n}{2} \right) a^2 - amn + m^2 - \left( \frac{m}{2} \right) + a^2 - 1 + b(n - 4) \right\} \omega_{1,0}^2 + \left\{ c(n - 4) - m + 4 \right\} \omega_{1,1}.
\]
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Proof. Taking the tensor product of (2.1) with $\tilde{E}(2, m)$, we obtain

\begin{equation}
0 \rightarrow E(2, m) \otimes \tilde{E}(2, m) \rightarrow \bigoplus_{i=1}^{m} \tilde{E}(2, m) \rightarrow Q(2, m) \otimes \tilde{E}(2, m) \rightarrow 0,
\end{equation}

and after replacing $m$ with $n$, we also obtain

\begin{equation}
0 \rightarrow E(2, n) \otimes \tilde{E}(2, n) \rightarrow \bigoplus_{i=1}^{n} \tilde{E}(2, n) \rightarrow Q(2, n) \otimes \tilde{E}(2, n) \rightarrow 0.
\end{equation}

Since $T_{Gr(2, n)} \simeq Q(2, n) \otimes \tilde{E}(2, n)$ and $T_{Gr(2, m)} \simeq Q(2, m) \otimes \tilde{E}(2, m)$,

\begin{equation}
c(\varphi^*(T_{Gr(2, n)})) = \frac{\varphi^*(c(\tilde{E}(2, n)))^n}{c(\varphi^*(E(2, n) \otimes E(2, n)))} = \frac{c(E)^n}{c(\tilde{E} \otimes E)} \quad \text{and}
\end{equation}

\begin{equation}
c(T_{Gr(2, m)}) = \frac{c(\tilde{E}(2, m))^m}{c(E(2, m) \otimes \tilde{E}(2, m))}
\end{equation}

by (4.2) and (4.3). So we have the following equation:

\begin{equation}
c(N) = \frac{c(\varphi^*(T_{Gr(2, n)}))}{c(T_{Gr(2, m)})} = \frac{c(E)^n/c(\tilde{E} \otimes E)}{c(E(2, m))^m/c(E(2, m) \otimes \tilde{E}(2, m))},
\end{equation}

that is,

\begin{equation}
(4.4) \quad c(N)c(\tilde{E} \otimes E)c(\tilde{E}(2, m))^m = c(E)^n c(E(2, m) \otimes \tilde{E}(2, m)).
\end{equation}

Note that

\begin{equation}
c(E) = 1 + a \omega_{1,0} + b \omega_{1,0}^2 + c \omega_{1,1},
\end{equation}

\begin{equation}
c(\tilde{E} \otimes E) = 1 + (4b - a^2) \omega_{1,0}^2 + 4c \omega_{1,1},
\end{equation}

\begin{equation}
c(\tilde{E}(2, m)) = 1 + \omega_{1,0} + \omega_{1,1} \quad \text{and}
\end{equation}

\begin{equation}
c(E(2, m) \otimes \tilde{E}(2, m)) = 1 - \omega_{1,0}^2 + 4 \omega_{1,1}.
\end{equation}

Putting (4.5) into (4.4), we obtain the equation (4.1).

Moreover, comparing the cohomology classes of degree 2 in both sides of (4.1), we have

\begin{equation}
c_1 + m \omega_{1,0} = an \omega_{1,0},
\end{equation}

so we obtain

\begin{equation}
(4.6) \quad c_1 = (an - m) \omega_{1,0}.
\end{equation}

Comparing the cohomology classes of degree 4 in both sides of (4.1), we have

\begin{equation}
(4.7) \quad c_2 + c_1 \cdot m \omega_{1,0} + (4b - a^2) \omega_{1,0}^2 + 4c \omega_{1,1} + \binom{m}{2} \omega_{1,0}^2 + m \omega_{1,1}
\end{equation}

\begin{equation}
= \binom{n}{2} a^2 \omega_{1,0}^2 + bn \omega_{1,0}^2 + cn \omega_{1,1} - \omega_{1,0}^2 + 4 \omega_{1,1}.
\end{equation}
Putting (4.6) into (4.7), we obtain
\[
c_2 = \left\{ \left( \frac{n}{2} \right) a^2 - amn + m^2 - \left( \frac{m}{2} \right) + a^2 - 1 + b(n - 4) \right\} \omega_{1,0}^2 + \{c(n - 4) - m + 4\} \omega_{1,1}
\]
as desired. □

For \(0 \leq k \leq m - 2\), we can choose the basis
\[
\{ \omega_{1,0}^{k-2i} \omega_{1,1}^i \mid 0 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor \}
\]
of \(H^{2k}(\text{Gr}(2,m),\mathbb{Z})\) instead of \(\{\omega_{k-i,i} \mid 0 \leq i \leq k\}\). Using the equation (4.11) and this new basis (4.8), we obtain the following Proposition:

**Proposition 4.2.** Let \(4 \leq m \leq n \leq \frac{2m - 2}{2}\). For \(0 \leq k \leq 2n - 2m\), write the \(k\)-th Chern class \(c_k\) of \(N\) as

\[
c_k = \begin{cases} 
\alpha_0 = 1 = \beta_0, & \text{if } k = 0 \\
\alpha_1 \omega_{1,0} = \beta_1 \omega_{1,0}, & \text{if } k = 1 \\
\alpha_k \omega_{1,0}^k + \cdots + \beta_k \omega_{1,1}^{k/2}, & \text{if } k \geq 2 \text{ is even} \\
\alpha_k \omega_{1,0}^k + \cdots + \beta_k \omega_{1,1}^{(k-1)/2}, & \text{if } k \geq 3 \text{ is odd}
\end{cases}
\]

with respect to the basis (4.8). Then we have two modular equations

\[
\left( \sum_{k=0}^{2n-2m} \alpha_k \omega_{1,0}^k \right) (1 + (4b - a^2) \omega_{1,0}^2)(1 + \omega_{1,0})^{m-1} = (1 + a \omega_{1,0} + b \omega_{1,0}^2)^n(1 - \omega_{1,0}) \pmod{\omega_{1,0}^{m-1}}
\]

and

\[
\left( \sum_{k=0}^{n-m} \beta_2 k \omega_{1,1}^k \right) (1 + 4c \omega_{1,1})(1 + \omega_{1,1})^m = (1 + c \omega_{1,1})^n(1 + 4 \omega_{1,1}) \pmod{\omega_{1,1}^{(m-2)/2} + 1}
\]

where \(f(x) \equiv g(x) \pmod{x^k}\) if and only if \(f(x) - g(x) = \sum_{i=k}^{\infty} a_i x^i\) for some integers \(a_i\).

**Proof.** First, note that it is possible to express each \(c_k\) uniquely as in (4.9) because \(2n - 2m \leq m - 2\). Take the image under the canonical projection \(H^\bullet(\text{Gr}(2,m),\mathbb{Z}) \to H^\bullet(\text{Gr}(2,m),\mathbb{Z})/\left(\oplus_{k=m-1}^{2m-4} H^{2k}(\text{Gr}(2,m),\mathbb{Z})\right)\) to both sides of (4.11). Since \(H^\bullet(\text{Gr}(2,m),\mathbb{Z})/\left(\oplus_{k=m-1}^{2m-4} H^{2k}(\text{Gr}(2,m),\mathbb{Z})\right) \simeq \oplus_{k=0}^{m-2} H^{2k}(\text{Gr}(2,m),\mathbb{Z})\) as a \(\mathbb{Z}\)-module, we can expand each term with respect to the basis (4.8).

When we only consider the terms of the form \(\omega_{1,0}^k\) with \(k \leq m - 2\), we have

\[
\left( \sum_{k=0}^{2n-2m} \alpha_k \omega_{1,0}^k \right) (1 + (4b - a^2) \omega_{1,0}^2)(1 + \omega_{1,0})^m = (1 + a \omega_{1,0} + b \omega_{1,0}^2)^n(1 - \omega_{1,0}^2) \pmod{\omega_{1,0}^{m-1}},
\]

where \(f(x) \equiv g(x) \pmod{x^k}\) if and only if \(f(x) - g(x) = \sum_{i=k}^{\infty} a_i x^i\) for some integers \(a_i\).
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and after dividing $1 + \omega_{1,0}$ into the both sides, we can obtain the equation (4.10). The equation (4.11) can be obtained similarly by considering only the terms of the form $\omega_{1,1}^k$ with $2k \leq m - 2$. □

Next, we derive some inequalities on the integers $a, b, c$ and $\alpha_k$ which are helpful to solve the Diophantine equations (4.10) and (4.11).

**Lemma 4.3.** Let $\gamma \in H^{2k}(Gr(2, m), \mathbb{Z})$ with $0 \leq k \leq m - 2$. Then the coefficient of $\omega_{1,0}^k$ in $\gamma$ with respect to the basis (4.8) coincides with that of $\omega_{k,0}$ in $\gamma$ with respect to (2.2).

**Proof.** Note that each basis element in (4.8) is expressed as follows:

\[
\begin{align*}
\omega_{1,0}^k &= \omega_{k,0} + a_{0,1} \omega_{k-1,1} + \cdots + a_{0,2k} \omega_{k-2k,2k}, \\
\omega_{1,0}^{k-2} \omega_{1,1} &= \omega_{k-1,1} + a_{1,2} \omega_{k-2,2} + \cdots + a_{1,2k} \omega_{k-2k,2k}, \\
&\vdots \\
\omega_{1,0}^{k-2h+2} \omega_{1,1}^{h-1} &= \omega_{k-h+1,h-1} + a_{h-1,h} \omega_{k-h,h} \\
\omega_{1,0}^{k-2h} \omega_{1,1}^h &= \omega_{k-h,h}
\end{align*}
\]

for some non-negative integers $a_{i,j}$ and $h := \lfloor \frac{k}{2} \rfloor$ by Corollary 2.4. Since $\omega_{k,0}, \omega_{k-1,1}, \cdots, \omega_{k-h+1,h-1}, \omega_{k-h,h}$ are all non-zero, the coefficient of $\omega_{1,0}^k$ in $\gamma$ with respect to the basis (4.8) is equal to that of $\omega_{k,0}$ in $\gamma$ with respect to (2.2). □

**Lemma 4.4.** Let $\varphi : Gr(2, m) \hookrightarrow Gr(2, n)$ be an embedding and $E(2, n)$, $Q(2, n)$ be complex vector bundles given as in Subsection 2.2 and the introductory part of Section 4. Then each Chern class of $E = \varphi^*(E(2, n))$, $\varphi^*(Q(2, n))$ and $N$ is numerically non-negative.

**Proof.** Note that a vector bundle $E$ on $Z$ is generated by global sections if and only if there is a surjective bundle morphism from a trivial bundle on $Z$ (of any rank) to $E$. By the short exact sequence (2.1) for $n$ and its dual, $E$ and $\varphi^*(Q(2, n))$ are generated by global sections. Moreover, by the short exact sequence (4.3), $T_{Gr(2, n)} \simeq Q(2, n) \otimes E(2, n)$ is generated by global sections and from this, we can conclude that $N = \varphi^*(T_{Gr(2, n)})/T_{Gr(2, m)}$ is generated by global sections. Hence, for $E = E$, $\varphi^*(Q(2, n))$ and $N$, $c_0(E) = 1 \geq 0$ and $c_1(E), \cdots, c_{2m-4}(E)$ are all numerically non-negative by Proposition 2.7. □

**Proposition 4.5.** For $4 \leq m \leq n \leq \frac{3m-2}{2}$, let $a, b, c$ and $\alpha_k$ be the integers given as in (3.2) and (1.9).

(a) For $0 \leq k \leq 2n - 2m$, $\alpha_k \geq 0$.
(b) $a \geq 1$, $b \geq 0$ and $b + c \geq 0$ with $a^2 \geq b$.
(c) If $m \geq 5$, then $a^2 \geq 2b$.
(d) If $m \geq 6$, then $a^2 > 2b$. 

Proof. (a) By Lemma \[\text{4.3}\] the coefficient of $\omega_{k,0}$ in $c_k$ with respect to the basis $\{\omega_1, \omega_2\}$ is equal to $\alpha_k$. Then by Lemma \[\text{4.4}\] each $c_k = c_k(N)$ is numerically non-negative, thus $\alpha_k$ is non-negative for all $0 \leq k \leq 2n - 2m$.

(b) By Lemma \[\text{4.4}\] each Chern class of $E$ and $N$ is numerically non-negative. By \[\text{(3.2)}\],

$$c(E) = 1 + a\omega_{1,0} + b\omega_{2,0} + (b + c)\omega_{1,1},$$

so $a, b + c \geq 0$, and by Lemma \[\text{4.1}\]

$$c_1 = (an - m)\omega_{1,0},$$

so $a \geq 1$.

By Lemma \[\text{4.4}\] again,

$$c_2(\varphi^*(Q(2, n))) = \varphi^*(\tilde{\omega}_{2,0}) \quad (\because \text{Proposition \[\text{2.5}\] (b)})$$

$$= \varphi^*(\tilde{\omega}_{2,0}^2 - \tilde{\omega}_{1,1}) \quad (\because \text{Corollary \[\text{2.4}\]})$$

$$= (a\omega_{1,0})^2 - (b\omega_{1,0}^2 + c\omega_{1,1})$$

$$= (a^2 - b)\omega_{1,0}^2 - c\omega_{1,1}$$

$$= (a^2 - b)\omega_{2,0} + (a^2 - b - c)\omega_{1,1}$$

is numerically non-negative. Since $m \geq 4$, $\omega_{2,0}$ is not zero, so we have $a^2 \geq b$.

(c) By Lemma \[\text{4.4}\]

$$c_3(\varphi^*(Q(2, n))) = \varphi^*(\tilde{\omega}_{3,0}) \quad (\because \text{Proposition \[\text{2.5}\] (b)})$$

$$= \varphi^*(\tilde{\omega}_{3,0}^2 - 2\omega_{1,0}^2\omega_{1,1}) \quad (\because \text{Corollary \[\text{2.4}\]})$$

$$= (a\omega_{1,0})^3 - 2a\omega_{1,0}(b\omega_{1,0}^2 + c\omega_{1,1})$$

$$= a(a^2 - 2b)\omega_{1,0}^3 - 2ac\omega_{1,0}\omega_{1,1}$$

$$= a(a^2 - 2b)\omega_{3,0} + 2a(a^2 - 2b - c)\omega_{2,1}$$

is numerically non-negative. Since $m \geq 5$, $\omega_{3,0}$ is not zero, so we have $a(a^2 - 2b) \geq 0$. By (b), $a \geq 1$, thus $a^2 \geq 2b$.

(d) By Lemma \[\text{4.4}\]

$$c_4(\varphi^*(Q(2, n))) = \varphi^*(\tilde{\omega}_{4,0}) \quad (\because \text{Proposition \[\text{2.5}\] (b)})$$

$$= \varphi^*(\tilde{\omega}_{4,0} - 3\omega_{1,0}^2\tilde{\omega}_{1,1} + \tilde{\omega}_{1,1}^2) \quad (\because \text{Corollary \[\text{2.4}\]})$$

$$= (a\omega_{1,0})^4 - 3(a\omega_{1,0})^2(b\omega_{1,0}^2 + c\omega_{1,1}) + (b\omega_{1,0}^2 + c\omega_{1,1})^2$$

$$= (a^4 - 3a^2b + b^2)\omega_{1,0}^4 + (-3a^2c + 2bc)\omega_{1,0}^2\omega_{1,1} + c^2\omega_{1,1}^2$$

$$= (a^4 - 3a^2b + b^2)\omega_{4,0} + a\omega_{3,1} + \beta\omega_{2,2} \quad (\because \text{Lemma \[\text{4.3}\]})$$

is numerically non-negative where $\alpha, \beta$ are some equations in $a, b$ and $c$. Since $m \geq 6$, $\omega_{4,0}$ is not zero, so we have $a^4 - 3a^2b + b^2 \geq 0$, that is, either $a^2 \geq \left(\frac{3 + \sqrt{5}}{2}\right)b$ or $a^2 \leq \left(\frac{3 - \sqrt{5}}{2}\right)b$. But $a^2 \leq \left(\frac{3 - \sqrt{5}}{2}\right)b$ is impossible by (b), thus we have

$$a^2 \geq \left(\frac{3 + \sqrt{5}}{2}\right)b > 2b.$$
Proposition 4.6. Let $6 \leq m \leq n \leq \frac{3m-6}{2}$ and the integer $c$ be given as in (3.2). Then $c \geq 1$.

Proof. We will complete the proof by showing that the following two cases are impossible:

Case 1. $c \leq -1$ and Case 2. $c = 0$.

Case 1. Suppose that $c \leq -1$. By (4.11), we have

$$
\sum_{k=0}^{n-m} \beta_{2k} \omega_{1,1}^k (1 + 4c \omega_{1,1}) \equiv (1 + c \omega_{1,1})^n (1 + \omega_{1,1})^{-m} (1 + 4 \omega_{1,1}) \pmod{\omega_{1,1}^{\lfloor (m-2)/2 \rfloor + 1}}.
$$

Since $2(n-m+2) \leq m-2$, we can compare the coefficient of $\omega_{1,1}^{n-m+2}$ in both sides of (4.12). Then we have

$$
0 = \sum_{k=0}^{n-m+2} \binom{n}{k} \binom{m + (n - m + 2 - k) - 1}{n - m + 2 - k} c^k (-1)^{n-m+2-k}
$$

$$
+ 4 \sum_{k=0}^{n-m+1} \binom{n}{k} \binom{m + (n - m + 1 - k) - 1}{n - m + 1 - k} c^k (-1)^{n-m+1-k}
$$

$$
= \sum_{k=0}^{n-m+2} \binom{n}{k} \binom{n + 1 - k}{m-1} (-c)^k (-1)^{n-m+2}
$$

$$
+ 4 \sum_{k=0}^{n-m+1} \binom{n}{k} \binom{n - k}{m-1} (-c)^k (-1)^{n-m+1}.
$$

After dividing $(-1)^{n-m+1}$ into both sides of (4.13),

$$
4 \sum_{k=0}^{n-m+1} \binom{n}{k} \binom{n - k}{m-1} (-c)^k = \sum_{k=0}^{n-m+2} \binom{n}{k} \binom{n + 1 - k}{m-1} (-c)^k.
$$

Since

$$
\binom{n}{k} \binom{n - k}{m-1} = \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{(m-1)!(n-m + 1 - k)!}
$$

$$
= \frac{n!}{(m-1)!(n-m + 1)!} \cdot \frac{(n-m + 1)!}{k!(n-m + 1 - k)!}
$$

$$
= \binom{n}{m-1} \binom{n-m + 1}{k},
$$

we have

$$
4 \sum_{k=0}^{n-m+1} \binom{n}{k} \binom{n - k}{m-1} (-c)^k = \sum_{k=0}^{n-m+2} \binom{n}{k} \binom{n + 1 - k}{m-1} (-c)^k.
$$
the left hand side of (4.14) is equal to

\[(4.15) \quad \text{(LHS)} = 4 \sum_{k=0}^{n-m+1} \binom{n}{m-1} \binom{n-m+1}{k} (-c)^k \]

\[= 4 \binom{n}{m-1} (1-c)^{n-m+1}. \]

Similarly, since

\[\binom{n}{k} \left( \begin{array}{c} n+1-k \ \\ m-1 \end{array} \right) = \frac{n!}{k!(n-k)!} \cdot \frac{(n+1-k)!}{(m-1)!(n-m+2-k)!} \]

\[= \frac{(n+1)!}{(m-1)!(n-m+2)!} \frac{(n+1-k)}{k!(n-m+2-k)!} \]

\[= \frac{(n+1)!}{m-1} \binom{n-m+2}{k} \frac{n+1-k}{n+1} \]

\[\geq \binom{n}{k} \left( \begin{array}{c} n+1 \ \\ m-2 \end{array} \right) \frac{m-1}{n+1} \binom{n-m+2}{k} \]

for \(0 \leq k \leq n-m+2\), the right hand side of (4.14) satisfies

\[(4.16) \quad \text{(RHS)} \geq \binom{n}{m-2} (1-c)^{n-m-2}. \]

Applying (4.15) and (4.16) to (4.14), since \(1-c \neq 0\), we have

\[ (1-c) \binom{n}{m-2} \leq 4 \binom{n}{m-1}. \]

So we have \((1-c) (m-1) \leq 4 (n-m+2)\), that is,

\[4n \geq (5-c) m + c - 9. \]

But since \(4n \leq 6m - 12\), we have

\[(4.18) \quad (c+1) m \geq c + 3 \]

by (4.17).

If \(c = -1\), then (4.18) is impossible clearly. If \(c \leq -2\), then \(c+1 \leq -1\), so we have

\[m \leq \frac{c+3}{c+1} = 1 + \frac{2}{c+1} < 1 \]

by (4.18), which implies a contradiction.

Case 2. Suppose that \(c = 0\). Putting \(c = 0\) into (4.11), we have

\[(4.19) \quad \sum_{k=0}^{n-m} \beta_{2k} \omega_{1,1}^k \equiv (1 + \omega_{1,1})^{-m} (1 + 4 \omega_{1,1}) \left( \mod \omega_{1,1}^{[(m-2)/2]+1} \right). \]
Comparing the coefficient of \( \omega_{1,i}^{n-m+i} \) in both sides of (4.19) for \( i = 1 \) and 2, we have

\[
0 = \left( m + (n - m + i) - 1 \right) \left( \frac{1}{n - m + i} \right) (-1)^{n-m+i} \\
+ 4 \left( m + (n - m + i - 1) - 1 \right) \left( \frac{1}{n - m + i - 1} \right) (-1)^{n-m+i-1}
\]

\[
= (-1)^{n-m+i} \left\{ \left( \frac{n + i - 1}{m - 1} \right) - 4 \left( \frac{n + i - 2}{m - 1} \right) \right\}
\]

\[
= (-1)^{n-m+i} \left( \frac{n + i - 1}{m - 1} \right) \left( 1 - 4 \cdot \frac{n - m + i}{n + i - 1} \right).
\]

So we obtain

\[
3n = 4m - 3i - 1
\]

for \( i = 1 \) and 2, which implies a contradiction.

Hence, we conclude that \( c \geq 1 \). \( \square \)

4.2. Euler class of \( N_R \).

**Proposition 4.7.** For \( n \geq m \geq 4 \), let \( \varphi : \text{Gr}(2,m) \hookrightarrow \text{Gr}(2,n) \) be an embedding with the image \( X \), and \( N_R \) the real vector bundle which is corresponding to \( N \). Then the Euler class of \( N_R \) is

\[
e(N_R) = \sum_{i=0}^{n-m} (X \cdot \tilde{\omega}_{n-2-i,2m-n-2+i}) \varphi^* (\tilde{\omega}_{2n-2m-i,i}).
\]

In particular, when \( n = m + 1 \),

\[
e(N_R) = (X \cdot \tilde{\omega}_{m-1,m-3}) \varphi^* (\tilde{\omega}_{2,0}) + (X \cdot \tilde{\omega}_{m-2,m-2}) \varphi^* (\tilde{\omega}_{1,1}).
\]

**Proof.** By Theorem 1.3 of [6],

\[
e(N_R) = \varphi^* (\varphi_*(1))
\]

where 1 is the cohomology class in \( H^*(\text{Gr}(2,m), \mathbb{Z}) \) which is corresponding to \( \text{Gr}(2,m) \) itself. Since \( \varphi_*(1) \) is the cohomology class in \( H^*(\text{Gr}(2,n), \mathbb{Z}) \) which is corresponding to \( X \),

\[
e(N_R) = \varphi^* \left( \sum_{i=0}^{n-m} (X \cdot \tilde{\omega}_{n-2-i,2m-n-2+i}) \tilde{\omega}_{2n-2m-i,i} \right)
\]

\[
= \sum_{i=0}^{n-m} (X \cdot \tilde{\omega}_{n-2-i,2m-n-2+i}) \varphi^* (\tilde{\omega}_{2n-2m-i,i})
\]

by Lemma 3.3. \( \square \)

**Lemma 4.8** (Example 14.7.11 of [7] or page 364 of [9]). In \( \text{Gr}(2,m) \),

\[
\omega_{i,j}^{2m-4-i-j} = \frac{(2m - 4 - j)! (i - j + 1)!}{(m - 2 - i)! (m - j)!}
\]

for \( m - 2 \geq i \geq j \geq 0 \).
Lemma 4.9. Let $6 \leq m \leq n \leq \frac{3m-6}{2}$ and the integers $\alpha_{2n-2m}$ be given as in \eqref{eq:alpha}. Then 
\[
\alpha_{2n-2m} \geq \frac{(2m-4)!}{(m-2)!(m-1)!} \cdot b^{n-2}.
\]

Proof. Note that the top Chern class $c_{2n-2m}$ of $N$ is equal to the Euler class $e(N_R)$ where $N_R$ is the real vector bundle which is corresponding to $N$. So $\alpha_{2n-2m}$ is equal to the coefficient of $\omega_{1,0}^{2n-2m}$ in 
\[
e(N_R) = \sum_{i=0}^{n-m} d_i \varphi^*(\omega_{2n-2m-i,i})
\]
where $d_i := X \cdot \tilde{\omega}_{2n-2i,2m-n-2+i}$, by Proposition 4.4. Here, $d_i \geq 0$ for all $0 \leq i \leq n - m$ because each $d_i$ is the intersection number of two subvarieties of $Gr(2,n)$.

Let $\gamma_i$ be the coefficient of $\omega_{1,0}^{2n-2m-2i}$ in $\varphi^*(\tilde{\omega}_{2n-2m-2i,0})$ with respect to the basis \eqref{eq:omega}. Then the coefficient of $\omega_{1,0}^{2n-2m}$ in 
\[
\varphi^*(\tilde{\omega}_{2n-2m-i,i}) = \varphi^*(\tilde{\omega}_{2n-2m-2i,0}) \varphi^*(\omega_{1,1}^i)
= \varphi^*(\tilde{\omega}_{2n-2m-2i,0}) (b\omega_{1,0}^2 + c\omega_{1,1}^i)
\]
with respect to \eqref{eq:omega} is equal to $\gamma_i b^i$. By Lemma 4.3 and 4.4, $\gamma_i$ is equal to the coefficient of $\omega_{2n-2m-2i,0}$ in $\varphi^*(\tilde{\omega}_{2n-2m-2i,0})$ with respect to \eqref{eq:omega} and is non-negative for all $0 \leq i \leq n - m$. So we have 
\[
\alpha_{2n-2m} = \sum_{i=0}^{n-m} d_i \gamma_i b^i \geq d_{n-m} \gamma_{n-m} b^{n-m} \quad (\because b \geq 0 \text{ by Proposition 4.5 (b)})
= d_{n-m} b^{n-m} \quad (\because \text{Since } \varphi^*(\omega_{0,0}) = \omega_{0,0}, \gamma_{n-m} = 1.).
\]

Applying
\[
d_{n-m} = \varphi^*(\tilde{\omega}_{1,1}^{m-2}) = (b\omega_{1,0}^2 + c\omega_{1,1})^{m-2}
= \sum_{i=0}^{m-2} \left( \begin{array}{c} m-2 \\ i \\ \end{array} \right) b^{m-2-i} c^i \omega_{1,0}^{2m-4-2i} \omega_{1,1}^{i}
\geq b^{m-2} \omega_{1,0}^{2m-4} \quad (b, c, \omega_{1,0}^{2m-4} \omega_{1,1}^{i} \geq 0)
\]
by Proposition 4.5 (b), 4.6 and Lemma 4.8
\[
= \frac{(2m-4)!}{(m-2)!(m-1)!} \cdot b^{m-2} \quad (\because \text{Lemma 4.8})
\]
to \eqref{eq:alpha}, we obtain the desired inequality. \hfill \Box

Now, we ready to prove an inequality on $a, b$ better than that of Proposition 4.5 (d).

Proposition 4.10. For $6 \leq m \leq n \leq \frac{3m-6}{2}$, let $a$ and $b$ be the integers given as in \eqref{eq:4.2}. Then $a^2 > 4b$. 

**Proof.** Suppose that \(a^2 \leq 4b\). By Proposition 4.5 (b), \(a \geq 1\), so \(b \geq 1\). Then by (4.10), we have

\[
\sum_{k=0}^{2n-2m} \alpha_k \omega_{1,0}^k \leq \mod (1 + a \omega_{1,0} + b \omega_{1,0}^2)^n \leq \mod (1 + \sqrt{b} \omega_{1,0})^{2n} \mod \omega_{1,0}^{m-1}
\]

where \(f(x) \leq \mod g(x) \mod x^k\) if and only if the coefficient of \(x^i\) in \(g(x) - f(x)\) is non-negative for all \(0 \leq i < k\). Comparing the coefficient of \(\omega_{1,0}^{2n-2m}\) in both sides of (4.21),

\[
\alpha_{2n-2m} \leq \left( \frac{2n}{2n-2m} \right) b^{n-m} = \left( \frac{2n}{2m} \right) b^{n-m}.
\]

By Lemma 4.9 and (4.22), we have

\[
\frac{(2m-4)!}{(m-2)!(m-1)!} \cdot b^{n-2} \leq \left( \frac{2n}{2m} \right) b^{n-m},
\]

so we have

\[
b^{m-2} \leq \frac{2n}{2m} \cdot \frac{(m-2)!(m-1)!}{(2m-4)!} \leq \frac{(3m-6)(3m-7)\cdots(2m+1)}{(2m-4)(2m-5)\cdots(m+3)} \cdot \frac{(m-2)(m-3)(m-4)}{(m+2)(m+1)m} \cdot (m-5)
\]

\[
\leq 2^{m-6} \cdot (m-5).
\]

Hence, \(b \leq 2\).

By Proposition 4.5 (d), \(2b < a^2 \leq 4b\), so the only possible pair \((a,b)\) is \((2,1)\). Putting \((a,b) = (2,1)\) into (4.10), we have

\[
\sum_{k=0}^{2n-2m} \alpha_k \omega_{1,0}^k \equiv (1 + \omega_{1,0})^{2n-m+1}(1 - \omega_{1,0}) \mod \omega_{1,0}^{m-1}).
\]

Since \(2n - 2m + 1 \leq m - 2\), we can compare the coefficient of \(\omega_{1,0}^{2n-2m+1}\) in both sides of (4.24). Then we have

\[
0 = \left( \frac{2n-m+1}{2n-2m+1} \right) - \left( \frac{2n-m+1}{2n-2m} \right) = \left( \frac{2n-m+1}{2n-2m+1} \right) \left( 1 - \frac{2n-2m+1}{m+1} \right).
\]

So \(\frac{2n-2m+1}{m+1} = 1\), that is, \(3m = 2n \leq 3m - 6\) which implies a contradiction. Hence, \(a^2 > 4b\). \(\square\)
5. Proof of Main Theorem: general case

In this section, we prove Main Theorem for \(9 \leq m \leq n \leq \frac{3m-6}{2}\).

**Theorem 5.1.** Assume that the pair \((m, n)\) of integers satisfies \(9 \leq m \leq n \leq \frac{3m-6}{2}\). Then any embedding \(\varphi : Gr(2, m) \hookrightarrow Gr(2, n)\) is linear.

Let \(a, b\) and \(c\) be the integers given as in (3.2). To prove Theorem 5.1 with helps of Proposition 2.8 and 2.9, we need an upper bound of \(a\).

**Lemma 5.2.** If \(m \geq 7\), then 12 divides \(ab(a^2 - b + 3)\).

**Proof.** By Lemma 4.10 of [12], if \(k \geq 5\) and \(E\) is a complex vector bundle on \(P^k\) of rank 2 with 
\[
c(E) = 1 + \alpha H + \beta H^2
\]
where \(H\) is a hyperplane of \(P^k\), then \(\alpha \beta (a^2 - \beta + 3)\) is divisible by 12. In our case, \(Gr(2, m)\) contains a maximal projective space \(Y \cong P^{m-2}\) and the total Chern class of the restriction of \(E\) to \(Y\) is
\[
c(E|_Y) = 1 + a H + b H^2.
\]
Since \(m - 2 \geq 5\), \(ab(a^2 - b + 3)\) is divisible by 12. \[\square\]

**Proposition 5.3.** Assume that the pair \((m, n)\) of integers satisfies the condition of Theorem 5.1. Then we have the following inequalities:

(a) \(\sqrt{a^2 - 4b} < \frac{m-4}{3}\).

(b) \(a < \frac{m-4}{2}\).

**Proof.** (a) First, by Proposition 4.10, the expression \(\sqrt{a^2 - 4b}\) is well defined. By (4.10), we have
\[
\left(\sum_{k=0}^{2n-2m} \alpha_k \omega_{1,0}^k\right) \left(1 + \omega_{1,0}\right)^{m-1} \left(1 + (4b - a^2) \omega_{1,0}^2\right)
\equiv (1 + a \omega_{1,0} + b \omega_{1,0}^2)^n (1 - \omega_{1,0}) \pmod{\omega_{1,0}^{m-1}}.
\]
For convenience, let
\[
\sum_{k=0}^{2m-4} A_k \omega_{1,0}^k := \left(\sum_{i=0}^{2n-2m} \alpha_i \omega_{1,0}^i\right) \left(1 + \omega_{1,0}\right)^{m-1}
\quad \text{and}
\sum_{k=0}^{2m-4} B_k \omega_{1,0}^k := (1 + a \omega_{1,0} + b \omega_{1,0}^2)^n.
\]
Then we have 
\[A_{k+2} + (4b - a^2)A_k = B_{k+2} - B_{k+1}\]
for all \(0 \leq k \leq m - 4\), and in particular, when \(k = 2n - 2m + 2 \leq m - 4\),
\[A_{2n-2m+4} + (4b - a^2)A_{2n-2m+2} = B_{2n-2m+4} - B_{2n-2m+3}.
\]
Suppose that \(\sqrt{a^2 - 4b} \geq \frac{m-4}{3}\). We will derive a contradiction by comparing the signs of both sides of (5.1).
• Since \( A_k = \sum_{i=0}^{2n-2m} \alpha_i \binom{m-1}{k-i} \) for all \( k \geq 2n - 2m \), we have

\[
\text{(LHS)} = A_{2n-2m+4} + (4b - a^2)A_{2n-2m+2}
\]
\[
= \sum_{i=0}^{2n-2m} \alpha_i \left\{ \left( \frac{m - 1}{2n - 2m + 4 - i} \right) - (a^2 - 4b) \left( \frac{m - 1}{2n - 2m + 2 - i} \right) \right\}
\]
\[
= \sum_{i=0}^{2n-2m} \alpha_i \left( \frac{m - 1}{2n - 2m + 2 - i} \right) \cdot C_i
\]

where \( C_i := \frac{(-2n+3m-3+i)(-2n+3m-4+i)}{(2n-2m+3+i)(2n-2m+3-i)} - (a^2 - 4b) \). Since

\[
C_i \leq \frac{(m - 3)(m - 4)}{4 \cdot 3} - (a^2 - 4b)
\]
\[
\leq \frac{(m - 4)(m - 7)}{36} < 0 \quad (\because m \geq 9)
\]

for all \( 0 \leq i \leq 2n - 2m \), the left hand side of (5.1) is negative.

• Since \( B_k = \sum_{i=0}^{\frac{n-m}{2}} \binom{n}{k-i} a^{k-2i} b^i \), we have

\[
\text{(RHS)} = B_{2n-2m+4} - B_{2n-2m+3}
\]
\[
\geq \sum_{i=0}^{n-m+1} \frac{n}{i} a^{2n-2m+3-2i} b^i \left\{ \left( \frac{n-i}{2n-2m+4-2i} \right) a - \left( \frac{n-i}{2n-2m+3-2i} \right) \right\}
\]
\[
= \sum_{i=0}^{n-m+1} \left( \frac{n}{i} \right) a^{2n-2m+3-2i} b^i \left( \frac{n-i}{2n-2m+4-2i} \right) \cdot D_i
\]

where \( D_i := a - \frac{2n-2m+4-2i}{n+m-2m-3+i} \). Since

\[
D_i \geq \sqrt{a^2 - 4b} - \frac{2n - 2m + 4}{-n + 2m - 3}
\]
\[
\geq \frac{m - 4}{3} - \frac{(3m - 6) - 2m + 4}{\frac{3m-6}{2} + 2m - 3} \quad (\because n \leq \frac{3m - 6}{2})
\]
\[
= \frac{m - 4}{3} - \frac{2m - 4}{m}
\]
\[
= \frac{(m - 5)^2 - 13}{3m} > 0 \quad (\because m \geq 9)
\]

for all \( 0 \leq i \leq n - m + 1 \), the right hand side of (5.1) is positive.

So the equality (5.1) does not hold, thus this implies a contradiction. Hence, \( \sqrt{a^2 - 4b} < \frac{m-4}{2} \).

(b) Suppose that \( a \geq \frac{m-4}{2} \). Then by (a),

\[
a^2 - 4b < \left( \frac{m - 4}{9} \right)^2 \leq \left( \frac{2a}{3} \right)^2.
\]
By (4.10) and (5.2),

\[
\sum_{k=0}^{2n-2m} \alpha_k \omega_{1,0}^k \leq_{\text{mod}} (1 + a \omega_{1,0} + b \omega_{1,0}^2)^n \left\{ \sum_{i=0}^{2m-4} \left( \sqrt{a^2 - 4b \omega_{1,0}} \right)^i \right\}
\]

(5.3) \quad \leq_{\text{mod}} \left( 1 + \frac{a}{2} \omega_{1,0} \right)^{2n} \left\{ \sum_{i=0}^{2m-4} \left( \frac{2a}{3} \omega_{1,0} \right)^i \right\} \pmod{\omega_{1,0}^{n-1}}

where \( f(x) \leq_{\text{mod}} g(x) \pmod{x^k} \) if and only if the coefficient of \( x^i \) in \( g(x) - f(x) \) is non-negative for all \( 0 \leq i < k \). Comparing the coefficient of \( \omega_{1,0}^{2n-2m} \) in both sides of (5.3),

\[
\alpha_{2n-2m} \leq \sum_{i=0}^{2n-2m} \binom{2n}{i} \left( \frac{a}{2} \right)^i \left( \frac{2a}{3} \right)^{2n-2m-i}
\]

\[
= \left\{ \sum_{i=0}^{2n-2m} \binom{2n}{i} \left( \frac{1}{2} \right)^i \left( \frac{2}{3} \right)^{2n-i} \right\} \cdot \left( \frac{3}{2} \right)^{2m} a^{2n-2m}
\]

\[
\leq \left\{ \frac{1}{2} \sum_{i=0}^{2n} \binom{2n}{i} \left( \frac{1}{2} \right)^i \left( \frac{2}{3} \right)^{2n-i} \right\} \cdot \left( \frac{3}{2} \right)^{2m} a^{2n-2m} \quad (\because \: 2n - 2m \leq n)
\]

\[
= \frac{81}{32} \left( \frac{1}{2} + \frac{2}{3} \right)^{2n} \cdot \left( \frac{3}{2} \right)^{2m-4} a^{2n-2m}
\]

\[
\leq \frac{81}{32} \left( \frac{7}{6} \right)^{3m-6} \cdot \left( \frac{3}{2} \right)^{2m-4} a^{2n-2m} \quad (\because \: 2n \leq 3m - 6)
\]

\[
= \frac{81}{32} \left( \frac{7^3}{96} \right)^{m-2} a^{2n-2m}.
\]

By Lemma 4.9, we have

\[
(5.4) \quad \frac{(2m-4)!}{(m-2)!(m-1)!} \cdot b^{n-2} \leq \frac{81}{32} \left( \frac{7^3}{96} \right)^{m-2} a^{2n-2m}.
\]
By (5.2), \( b > \frac{5}{36} a^2 \), so the left hand side of (5.4) satisfies

\[
\frac{(2m - 4)!}{(m - 2)!(m - 1)!} \cdot b^{n-2} = \frac{(2m - 4)(2m - 5) \cdots m}{(m - 2)(m - 3) \cdots 2} \cdot b^{n-2}
\]

\[
\geq 2^{m-3} \cdot \left( \frac{5}{36} \right)^{n-2} a^{2n-4}
\]

\[
\geq 2^{m-3} \cdot \left( \frac{5}{36} \right)^{(3m-10)/2} a^{2n-4} \quad (\because n \leq \frac{3m - 6}{2})
\]

\[
\geq \frac{1}{2} \left( \frac{36}{5} \right)^{2} \cdot 2^{m-2} \cdot \left( \frac{5}{36} \right)^{(3m-6)/2} a^{2n-4}
\]

\[
> 24 \left( \frac{5 \sqrt{5}}{3 \cdot 6^2} \right)^{m-2} a^{2n-4},
\]

thus,

\[
24 \left( \frac{5 \sqrt{5} a^2}{3 \cdot 6^2} \right)^{m-2} < \frac{81}{32} \left( \frac{\tau}{96} \right)^{m-2}.
\]

So \( \frac{5 \sqrt{5} a^2}{3 \cdot 6^2} \leq \frac{\tau}{96} \), that is, \( a^2 \leq \frac{9 \cdot \tau}{40 \sqrt{5}} \simeq 34.5137 \), thus \( a \leq 5 \).

Since \( 4b < a^2 < \frac{36}{3} b \), the only possible pairs \((a,b)\) are \((3,2), (4,3), (5,4), (5,5)\) and \((5,6)\).

Case 1. \((a,b) = (3,2), (4,3)\) or \((5,4)\) : In this case, \( a = b + 1 \) with \( b = 2, 3 \) or 4. Then \( 1 + a \omega_{1,0} + b \omega_{1,0}^2 = (1 + \omega_{1,0})(1 + b \omega_{1,0}) \) and \( a^2 - 4b = (b-1)^2 \), so we have

\[
(5.5) \quad \left( \sum_{k=0}^{2n-2m} \alpha_k \omega_{1,0}^k \right) (1 - (b - 1)^2 \omega_{1,0}^2)
\]

\[
\equiv (1 + \omega_{1,0})^{n-m+1}(1 + b \omega_{1,0})^n(1 - \omega_{1,0}) \pmod{\omega_{1,0}^{m-1}}
\]

by (4.10). Comparing the coefficient of \( \omega_{1,0}^{2n-2m+2} \) in both sides of (5.5),

\[
0 \geq -(b - 1)^2 \alpha_{2n-2m}
\]

\[
= \sum_{k=0}^{n-m+1} \binom{n - m + 1}{k} \left\{ \binom{n}{2n - 2m + 2 - k} b^{2n-2m+2-k} - \binom{n}{2n - 2m + 1 - k} b^{2n-2m+1-k} \right\}
\]

\[
(5.6)
\]

\[
= \sum_{k=0}^{n-m+1} \binom{n - m + 1}{k} \binom{n}{2n - 2m + 2 - k} b^{2n-2m+2-k} A_k
\]
where $A_k := b - \frac{2n - 2m + 2}{n + 2m - 1}$. Since

\[
A_k \geq b - \frac{2n - 2m + 2}{n + 2m - 1} \\
\geq b - \frac{(3m - 6) - 2m + 2}{(3m - 6) + 2m - 1} \quad (\because n \leq \frac{3m - 6}{2}) \\
= b - \frac{2m - 8}{m + 4} \\
> b - 2 \\
\geq 0 \quad (\because b = 2, 3 \text{ or } 4)
\]

for $0 \leq k \leq n - m + 1$, the right hand side of (5.6) is positive which implies a contradiction.

Case 2. $(a, b) = (5, 5) : ab(a^2 - b + 3) = 25 \cdot (25 - 5 + 3) = 25 \cdot 23$ is not divisible by 12, so this case is impossible by Lemma 5.2.

Case 3. $(a, b) = (5, 6) :$ Since $a^2 - 4b = 1$, we have by (4.10),

\[
\sum_{k=0}^{2n-2m} \alpha_k \omega_{1,0}^k \leq_{mod} (1 + 5 \omega_{1,0} + 6 \omega_{1,0}^2)^n \\
\leq_{mod} \left(1 + \frac{5}{2} \omega_{1,0}\right)^{2n} \left(\mod \omega_{1,0}^{m-1}\right)
\]

where $f(x) \leq_{mod} g(x) \mod x^k$ if and only if the coefficient of $x^i$ in $g(x) - f(x)$ is non-negative for all $0 \leq i < k$. Comparing the coefficients of $\omega_{1,0}^{2n-2m},$

\[
\alpha_{2n-2m} \leq \left(\frac{2n}{2n - 2m}\right) \left(\frac{5}{2}\right)^{2n-2m} \\
= \left(\frac{2n}{2m}\right) \left(\frac{5}{2}\right)^{2n-4} \left(\frac{2}{5}\right)^{2m-4}
\]

Applying Lemma 4.9 to (5.8),

\[
\left(\frac{24}{25}\right)^{n-2} \leq \left(\frac{2n}{2m}\right) \cdot (m - 2)! \cdot (m - 1)! \left(\frac{2}{5}\right)^{2m-4} \\
\leq 2^{n-6} \cdot (m - 5) \left(\frac{2}{5}\right)^{2m-4} \\
\quad (\because \text{By the same argument with (4.23)}) \\
= \frac{m - 5}{16} \left(\frac{8}{25}\right)^{m-2}.
\]
But since \( n \leq \frac{3m-6}{2} \),

\[
\left(\frac{24}{25}\right)^{n-2} \geq \left(\frac{25}{24}\right)^2 \left\{ \left(\frac{24}{25}\right)^{3/2} \right\}^{m-2},
\]

so we have \( \frac{48\sqrt{6}}{125} = \left(\frac{24}{25}\right)^{3/2} \leq \frac{8}{25} \) which implies a contradiction.

Hence, \( a < \frac{m-4}{2} \) as desired. \( \square \)

**Proof of Theorem 5.1.** Let \( Y \cong \mathbb{F}^{m-2} \) be a Schubert variety of \( \text{Gr}(2, m) \) of type \((m - 2, 0)\). Then the total Chern class of the restriction of \( E \) to \( Y \) is

\[
c(E|_Y) = 1 + aH + bH^2
\]

where \( H \) is a hyperplane of \( Y \). For any projective line \( \ell \) in \( Y \),

\[
(E|_Y)|_\ell = E|_\ell \cong \mathcal{O}_\ell(a_1) \oplus \mathcal{O}_\ell(a_2)
\]

for some integers \( a_1, a_2 \) with \( a_1 + a_2 = a \). Since \( (E|_Y)|_\ell \) is generated by global sections by Lemma 4.4, \( a_1 \) and \( a_2 \) are non-negative. So we have

\[
B(E|_Y) \leq \frac{a - 0}{2} < \frac{m - 4}{4}
\]

by Proposition 5.3 (b) (For the definition of \( B(E|_Y) \), see (2.3)). Hence, \( E|_Y \) is decomposable by Proposition 2.8. Since \( Y \cong \mathbb{F}^{m-2} \) is arbitrary, \( E \) is either decomposable or \( E(2, m) \otimes L \) for some line bundle \( L \) on \( \text{Gr}(2, m) \) by Proposition 2.9.

**Case 1.** \( E \cong E(2, m) \otimes L \): Let \( c(L) = 1 + r \omega_{1,0} \) with \( r \in \mathbb{Z} \). Then

\[
c(E(2, m) \otimes L) = 1 + (2r + 1) \omega_{1,0} + r(r + 1) \omega^2_{1,0} + \omega_{1,1},
\]

that is,

\[
a = 2r + 1, \quad b = r(r + 1) \quad \text{and} \quad c = 1
\]

with \( r \geq 0 \) by Proposition 4.5 (b). Since \( 4b - a^2 = -1 \), putting (5.9) into (4.10), we have

\[
\sum_{k=0}^{2n-2m} a_k \omega^k_{1,0} \equiv (1 + r \omega_{1,0})^n (1 + (r + 1) \omega_{1,0})^n \pmod{\omega_{1,0}^{m-1}}.
\]

Comparing the coefficient of \( \omega_{1,0}^{2n-2m} \) in both sides of (5.10),

\[
\alpha_{2n-2m} \leq \frac{2n}{2n - 2m} (r + 1)^{2n-2m} = \frac{2n}{2m} (r + 1)^{2n-2m}.
\]
By Lemma 4.9, we have
\[(r(r + 1))^{n-2} \leq \binom{2n}{2m} \cdot \frac{(m-2)!(m-1)!}{(2m-4)!} \cdot (r + 1)^{2n-2m} \]
(5.12)
\[\leq 2^{m-6} \cdot (m-5)(r + 1)^{2n-2m} \]
\[(\therefore \text{ By the same argument with (4.23)).} \]

After dividing \((r + 1)^{2n-2m}\) into both sides of (5.12),
\[(5.13)\]
\[r^{n-2}(r + 1)^{-n+2m-2} \leq \frac{m-5}{16} \cdot 2^{m-2}. \]

Since \(-n + 2m - 2 \geq -\left(\frac{3m-6}{2}\right) + 2m - 2 = \frac{m+2}{2} > 0\), we have
(5.14)
\[r^{2m-4} \leq r^{n-2}(r + 1)^{-n+2m-2}. \]

Combining (5.13) and (5.14), \(r^2 \leq 2\), thus, \(r = 0\) or 1.

Suppose that \(r = 1\). Then by (5.10), we have
\[(5.15)\]
\[\sum_{k=0}^{2n-2m} \alpha_k \omega_{1,0}^k \equiv (1 + \omega_{1,0})^{n-m}(1 + 2 \omega_{1,0})^n \pmod{\omega_{1,0}^{m-1}}. \]

Comparing the coefficient of \(\omega_{1,0}^{2n-2m+1}\) in both sides of (5.15),
\[0 = \sum_{i=0}^{n-m} \binom{n-m}{i} \binom{n}{2n-2m+1-i} \cdot 2^{2n-2m+1-i} \]
which implies a contradiction. Hence, \(r = 0\).

Case 2. \(E \simeq L_1 \oplus L_2\) : Let \(c(L_1) = 1 + r_1 \omega_{1,0}\) and \(c(L_2) = 1 + r_2 \omega_{1,0}\) with \(r_1, r_2 \in \mathbb{Z}\). Then
\[c(L_1 \oplus L_2) = 1 + (r_1 + r_2) \omega_{1,0} + r_1 r_2 \omega_{1,1}, \]
that is,
\[a = r_1 + r_2, \quad b = r_1 r_2 \quad \text{and} \quad c = 0. \]

But by Proposition 5.6, \(c = 0\) cannot be happened.

As a result, \(E \simeq E(2,m) \otimes L\) where \(L\) is the trivial line bundle on \(Gr(2,m)\) and thus \((a,b,c) = (1,0,1)\). Hence, \(\varphi : Gr(2,m) \hookrightarrow Gr(2,n)\) is linear by Proposition 3.5 (c).

6. Proof of Main Theorem : special case

In this section, we prove Main Theorem for \(4 \leq m = n - 1 \leq 8\).

Theorem 6.1. For \(5 \leq m \leq 8\), any embedding \(\varphi : Gr(2,m) \hookrightarrow Gr(2,m+1)\) is linear.

Proposition 6.2. For \(m \geq 4\), let \(a, b\) and \(c\) be the integers given as in (3.2).
(a) The following two equations hold:

\[(m+1)\binom{m+1}{2} (a-1)^2 + a^2 - 1 + b (m - 3) = (a^2 - b) d_0 + b d_1\]

and

\[c (m - 3) - m + 4 = c (d_1 - d_0)\]

where \(d_0 := \varphi^* (\tilde{\omega}_{m-1,m-3})\) and \(d_1 := \varphi^* (\tilde{\omega}_{m-2,m-2})\).  

(b) If \(m \geq 5\), then \(c\) divides \(m - 4\). Moreover, in this case, \(2b + 2c - a^2 > 0\) and \(c \geq 1\).

Proof. (a) Since \(c_2 = c_2(N) = e(N_{\mathbb{R}})\), we can compare the coefficients of \(\omega_{2,0}\) and \(\omega_{1,1}\) in \(c_2\) (Lemma 4.1) with \(e(N_{\mathbb{R}})\) (Proposition 4.7). Then we obtain the desired equations (6.1) and (6.2).

(b) If \(m \geq 5\), then \(c\) divides \(m - 4\) by (6.2) and in particular, \(c \neq 0\). In this case, after dividing \(c\) into both sides of (6.2),

\[m - 3 - \frac{m - 4}{c} = d_1 - d_0\]

\[= \{(2b - a^2) \omega_{1,0}^2 + 2c \omega_{1,1}\} (b \omega_{1,0}^2 + c \omega_{1,1})^{m-3}\]

\[(6.3) = \{(2b - a^2) \omega_{2,0}^2 + (2b + 2c - a^2) \omega_{1,1}\} (b \omega_{2,0} + (b + c) \omega_{1,1})^{m-3}\]

\[= \sum_{i=0}^{m-3} \binom{m - 3}{i} b^i (b + c)^{m-3-i} .\]

Since \(m - 3 - \frac{m - 4}{c} \geq m - 3 - \frac{m - 4}{m - 3} = 1\), the right hand side of (6.3) is positive. Furthermore, since \(b, b + c, a^2 - 2b \geq 0\) by Proposition 4.5 (b), (c) and \(\omega_{2,0}^2 \omega_{1,1}^{m-3-i}, \omega_{2,0}^2 \omega_{1,1}^{m-2-i} \geq 0\), we have

\[2b + 2c - a^2 > 0.\]

Furthermore, \(c > \frac{1}{2} (a^2 - 2b) \geq 0\) by Proposition 4.5 (c) again. \(\square\)

**Proposition 6.3.** An embedding \(\varphi : Gr(2, 4) \hookrightarrow Gr(2, 5)\) is either linear or twisted linear.

**Proof.** Note that since \(\omega_{1,0}^4 = 2\) and \(\omega_{1,0}^2 \omega_{1,1} = 1 = \omega_{1,1}^2\) in \(H^8(Gr(2, 4), \mathbb{Z}) \cong \mathbb{Z}\),

\[(6.4) \quad d_0 = b (a^2 - b) + (b + c) (a^2 - b - c) \quad \text{and} \quad d_1 = b^2 + (b + c)^2.\]

By (6.1) and (6.2), we have

\[(6.5) 10 (a - 1)^2 + a^2 - 1 + b = (a^2 - b) d_0 + b d_1\]

and

\[(6.6) c = c (d_1 - d_0).\]
Case 1. \( c \neq 0 \): By (6.4) and (6.6),

\[
1 = d_1 - d_0 = b(2b - a^2) + (b + c)(2b + 2c - a^2).
\]

Applying (6.7) to (6.5), we have

\[
11a^2 - 20a + 9 + b = a^2d_0 + b(d_1 - d_0)
\]

(6.8)

\[
= a^2d_0 + b.
\]

So \( a \) divides 9 and \( a^2 \) divides \(-20a + 9\), thus, \( a = 1 \). Furthermore, in this case, \( d_0 = -b^2 + 2b + c - (b + c)^2 \)

by (6.4) and (6.8), that is,

\[
b(b - 1) = (b + c)\{1 - (b + c)\}.
\]

Since the right hand side of (6.9) is less than or equal to \( \frac{1}{4} \), \( b = 0 \) or 1. The only possible pairs \((b, c)\) satisfying (6.7) and (6.9) are \((b, c) = (0, 1)\) or \((1, -1)\). Hence, \((a, b, c) = (1, 0, 1)\) or \((1, 1, -1)\).

Case 2. \( c = 0 \): Applying (6.4) to (6.5),

\[
10(a - 1)^2 + a^2 - 1 + b = 2b\{(a^2 - b)^2 + b^2\}.
\]

Suppose that \( b > 0 \). After dividing \( b \) into both sides of (6.10),

\[
2b^2\left\{\left(\frac{a^2}{b} - 1\right)^2 + 1\right\} < 11\cdot\frac{a^2}{b} + 1,
\]

that is,

\[
2b^2 t^2 - (4b^2 + 11)t + 4b^2 - 1 < 0
\]

where \( t := \frac{a^2}{b} \). The discriminant of (6.11) is

\[
D = (4b^2 + 11)^2 - 4 \cdot 2b^2 \cdot (4b^2 - 1)
\]

\[
= -16b^4 + 96b^2 + 121
\]

\[
= -16(b^2 - 3)^2 + 265.
\]

If \( b \geq 3 \), then \( D < 0 \), so (6.11) is impossible. For \( b = 1 \) or 2, we can show directly that (6.10) does not have any integer solution \( a \). Hence, \( b = 0 (= c) \), so \( a = 1 \) by (6.10).

By Proposition 3.5 (c), if \((a, b, c) = (1, 0, 1)\), then \( \varphi \) is linear and if \((a, b, c) = (1, 1, -1)\), then \( \varphi \) is twisted linear.

To complete the proof, it suffices to show that the pair \((a, b, c) = (1, 0, 0)\) is impossible. In this case, by Lemma 4.1, we have

\[
c(N) = 1 + \omega_{1,0} = c(E).
\]

After dividing (6.12) into both sides of (4.11), we have

\[
(1 - \omega_{1,0}^2)(1 + \omega_{1,0} + \omega_{1,1})^4 = (1 + \omega_{1,0})^4(1 - \omega_{1,0}^2 + 4\omega_{1,1}).
\]

(6.13)
Comparing the cohomology classes of degree 6 in (6.13),
\[
\binom{4}{3} \omega^3_{1,0} + \binom{4}{2} \cdot 2 \omega_{1,0} \omega_{1,1} - \binom{4}{1} \omega^3_{1,0} = \binom{4}{3} \omega^3_{1,0} + \binom{4}{1} \omega_{1,0} (-\omega^2_{1,0} + 4 \omega_{1,1})
\]
and from this, we have
\[
(4 \cdot 2 + 12 - 4 \cdot 2) \omega_{2,1} = \{4 \cdot 2 + 4 \cdot (-2 + 4)\} \omega_{2,1} \quad (\because \omega^3_{1,0} = 2 \omega_{2,1})
\]
which implies a contradiction.

Proof of Theorem 6.1. By Proposition 4.5 (c), \(a^2 - b \geq b\), so we have
\[
\binom{m+1}{2} (a-1)^2 + a^2 - 1 + b(m-3) \geq b(d_0 + d_1)
\]
(6.14)
\[
= a^2 b \omega^2_{1,0} (b \omega^2_{1,0} + c \omega_{1,1})^{m-3} \\
\geq a^2 b^{m-2} \omega^2_{1,0} \omega^2_{1,0} (\because b, c, \omega^2_{1,1} - \omega_{1,1} - 2 \omega_{1,1}) \geq 0
\]
by Proposition 4.5 (b), 6.2 (b) and Lemma 4.8
\[
= \frac{(2m-4)!}{(m-2)!(m-1)!} \cdot a^2 b^{m-2} \quad (\because \text{Lemma 4.8})
\]
by (6.1). Since the left hand side of (6.14) is less than
\[
a^2 \left\{ \binom{m+1}{2} + m - 2 \right\}
\]
because \(a^2 \geq b\) by Proposition 4.5 (b), after dividing \(a^2\) into both sides of (6.14),
\[
\binom{m+1}{2} + m - 2 \geq \frac{(2m-4)!}{(m-2)!(m-1)!} \cdot b^{m-2}.
\]
Hence, \(b = \begin{cases} 0 \text{ or } 1, & \text{for } m = 5, 6 \\ 0, & \text{for } m = 7, 8 \end{cases}\).

Assume that \(b = 1\) with \(m = 5\) or 6. By Proposition 4.5 (c) and Proposition 6.2 (b),
\[
2 = 2b \leq a^2 < 2b + 2c = 2 + 2c
\]
and \(c \geq 1\) divides \(m - 4\). The only possible pair \((m, a, c)\) satisfying these properties is \((6, 2, 2)\). Apply \((m, a, b, c) = (6, 2, 1, 2)\) to (6.14), then
\[
27 = \binom{7}{2} + 6 \geq \frac{8!}{4! \cdot 5!} \cdot 4 = 56
\]
which implies a contradiction.
Assume that $b = 0$ with $5 \leq m \leq 8$. Then by (6.2),
\[
c(m - 3) - m + 4 = c(-a^2 \omega_{1,0}^2 + 2c \omega_{1,1})(c \omega_{1,1})^{m-3}
\]
\[
= (2c - a^2)c^{m-2}
\]
\[
\geq c^{m-2} \quad (\because \text{Proposition 6.2 (b)}),
\]
thus $c = 1$. By Proposition [6.2] (b), $2 - a^2 > 0$, so $a = 1$.

Hence, $(a, b, c) = (1, 0, 1)$ for $5 \leq m \leq 8$, thus any embedding $\varphi : Gr(2, m) \hookrightarrow Gr(2, m + 1)$ is linear by Proposition 3.5 (c).

\[\square\]

**Proof of Main Theorem.** The proof of the case when $9 \leq m \leq n \leq \frac{3m-6}{2}$ is followed from Theorem 5.1, and the proof of the case when $5 \leq m = n - 1 \leq 8$ is followed from Theorem 6.1. Moreover, the proof of the case when $m = n - 1 = 4$ is followed from Proposition 6.3. \[\square\]

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