AN ELEMENTARY PROOF OF FEDIĬ’S THEOREM AND EXTENSIONS

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Abstract. We present an elementary, $L^2$, proof of Fediĭ’s theorem on arbitrary (e.g., infinite order) degeneracy and extensions. In particular, the proof allows and shows $C^\infty$, Gevrey, and real analytic hypoellipticity, and allows the coefficients to depend on the remaining variable as well.

1. Introduction

In 1971, V.S. FediĬ [Fed71] proved local hypoellipticity for the operator

$D_x^2 + a^2(x)D_t^2$

where $a(x) \geq 0$, and $a(x) \neq 0$ for $x \neq 0$. Related and more recent results include those of Kusuoko and Strook [KuStr85], Morimoto [Mori87], Christ [Christ95] and Bell and Mohammed [BellMo95]. Here, thanks in part to helpful conversations with A. Bove, we will give a flexible and utterly elementary proof of FediĬ’s result which proves hypoellipticity in the smooth, Gevrey, and real analytic categories rapidly, when appropriate.

Theorem 1.1. Let $a(x)$ have the above properties and $b(t)$ be a smooth (resp. real analytic) non-zero function of $t$ near $t_0$. Then the operator

$P = D_x^2 + a^2(x)b^2(t)D_t^2 = X^2 + Y^2$

is hypoelliptic at $(0, t_0)$ in the $C^\infty$, Gevrey, and real analytic categories, assuming, of course, that the coefficients belong to that class.

2. Proof of the Theorem

We make a few preliminary observations.
First, for \( x \neq 0 \), the operator is elliptic, where the results are known. Thus our localization will be assumed to be in a neighborhood of \( x = 0 \) and the associated localizing function(s) may be taken to depend on \( t \) alone, since using a product of a cut-off in \( x \) as well would only clutter up the notation, and whenever such a function received a derivative, we would be thrown into the elliptic region.

Second, we will estimate derivatives of a solution \( u \) in \( L^2 \) norm, using the Sobolev embedding theorem.

Third, using the pseudodifferential calculus and microlocalizing in the standard ways, we shall demonstrate only that derivatives in the variable \( t \) grow as desired. The restrictions of this microlocalization are that if \( a(x) \) belongs to a given differentiability class then we will be able to prove hypoellipticity in that class (in \( x \)) but, as we will see below, the regularity in \( t \) will be limited only by that of the coefficient \( b(t) \).

Fourth, taking all inner products in \( L^2 \), and using the identity 1 = \( D_x x \) we have, for smooth \( v \) supported near \( x = 0 \),

\[
\|v\|^2_{L^2} = |((D_x x)v, v)| \leq |(xD_x v, v)| + |(D_x v, xv)|
\]

\[
\leq \frac{1}{2}\|v\|^2_{L^2} + C\|D_x v\|^2_{L^2} \leq \frac{1}{2}\|v\|^2_{L^2} + C\|D_x v\|^2_{L^2} + C\|abD_t v\|^2_{L^2}
\]

\[
\leq \frac{3}{4}\|v\|^2_{L^2} + C'(\|Pv, v\|)
\]

so that we have the following \( a \) priori inequality (in \( L^2 \) norms) for \( v \) of small \( x \)– support:

\[
\|v\|^2 + \|D_x v\|^2 + \|abD_t v\|^2 = \|v\|^2 + \|Xv\|^2 + \|Yv\|^2 \lesssim \|Pv, v\|.
\]

It is important to note that the estimate is not subelliptic in the usual sense (which would require \( \|v\|^2 \) on the left), and of course this corresponds to the fact that for general \( a(x) \), which may degenerate to infinite order at \( x = 0 \), Hörmander’s bracket condition may be violated.

We will concentrate on the analytic hypoellipticity of \( P \), assuming the solution is already smooth; showing that a distribution solution is smooth can be accomplished by introducing a cutoff function and a mollifier and observing that any brackets with \( P \) are rapidly handled by using a weighted Schwarz inequality and maximality of the estimate.
We shall see more of this below as we handle a solution \( u \) known to be smooth.

To explore high derivatives, we start with powers of \( D_t \), localized by a function \( \varphi(t) \) (see above). We have, in \( L^2 \) norms and inner product, since \( \varphi_x = 0 \) near the point in question,

\[
(*_{D_t^r}) : \quad \|\varphi D_t^r u\|^2 + \|D_x \varphi D_t^r u\|^2 + \|abD_t \varphi D_t^r u\|^2 \leq \|P \varphi D_t^r u, \varphi D_t^r u\|
\]

\[
\leq |\langle \varphi D_t^r P u, \varphi D_t^r u \rangle| + |\langle P, \varphi D_t^r u \rangle, \varphi D_t^r u \rangle|
\]

\[
\leq |\langle \varphi D_t^r P u, \varphi D_t^r u \rangle| + |\langle Y^2, \varphi D_t^r u \rangle, \varphi D_t^r u \rangle|
\]

\[
\leq C_\varepsilon \|\varphi D_t^r P u\|^2 + 2\|\varphi D_t^r u\|^2 + 2\|\langle Y, \varphi D_t^r u \rangle, \varphi D_t^r u \rangle, \varphi D_t^r u \rangle + |\langle Y, [Y, \varphi D_t^r \rangle, \varphi D_t^r u \rangle, \varphi D_t^r u \rangle|.
\]

Now \( \|Y^* \varphi D_t^r u\|^2 \) may be added to the left side of the inequality for \( |x| \) small, since \( Y^* = -Y - ab' \) and \( ab' \) will be small for \( |x| \) small, and

\[
|\langle Y, \varphi D_t^r \rangle, \varphi D_t^r u \rangle, \varphi D_t^r u \rangle < |\langle abD_t \varphi D_t^{r-1} u, \varphi D_t^r u \rangle| + |\langle ab' D_t \varphi D_t^{r-1} u, \varphi D_t^r u \rangle| + \ldots
\]

\[
< \frac{1}{2}(\varphi_{D_t^r}) + C_\varepsilon(\varphi_{D_t^{r-1}}) + r^2(\varphi_{D_t^{r-1}}) + \ldots
\]

and

\[
|\langle Y, [Y, \varphi D_t^r \rangle, \varphi D_t^r u \rangle, \varphi D_t^r u \rangle < |\langle abab D_t \varphi D_t^{r-2} u, \varphi D_t^r u \rangle| + |\langle abab' D_t \varphi D_t^{r-2} u, \varphi D_t^r u \rangle|
\]

\[
+ \frac{1}{2}(\varphi_{D_t^r}) + C_\varepsilon(\varphi_{D_t^{r-2}}) + C_\varepsilon r^2(\varphi_{D_t^{r-2}}) + C_\varepsilon r^4(\varphi_{D_t^{r-2}}) + \ldots
\]

or, in all,

\[
(\varphi_{D_t^r}) < (\varphi_{D_t^{r-1}}) + (\varphi_{D_t^{r-2}}) + r^2(\varphi_{D_t^{r-2}}) + r^4(\varphi_{D_t^{r-2}}) + \ldots
\]

where under . . . we include terms where we must move one \( D_t \) across a \( \varphi \), thus increasing the number of derivatives on \( \varphi \) by one but decreasing \( r \) by one.
All of this may be iterated until we have \( C^r \) terms each with \( r \) reduced to zero and at most \( r \) derivatives on the localizing function \( \varphi(t) \). The result is hypoellipticity in \((x,t)\) in the appropriate spaces.

**Remark 1.** We have not emphasized the \( C^\infty \) hypoellipticity of \( P \). In the case of \( b(t) \equiv 1 \), as in the paper of Kohn \[Koh05\], one may introduce a pseudodifferential cut-off in the variable \( \tau \) dual to \( t \) which is equal to one for \( |\tau| \leq N \) and then smoothly to zero by the time \( |\tau| \geq 2N \), and, since the resulting function is smooth in \( t \), apply the a priori estimates and derivatives, then let \( N \to \infty \) to see that the corresponding norms are finite. When the coefficient \( b(t) \) is not constant, one must introduce a mollifier in the variable \( t \), treat the brackets of functions with the mollifier as in the classical works of Friedrichs, Hörmander and others, and then let the mollifier approach the identity. Note that it is important here that \( b(t) \) is never zero.

**Remark 2.** When one works in the real analytic category, the localizing function \( \varphi(t) \), must be taken to belong to the Ehrenpreis class: \( \varphi(t) \) is the convolution of \( N \) identical bump functions with derivative proportional to \( N \) with the characteristic function of an intermediate set. Such a function will depend on \( N \) but have the property that, with \( C \) independent of \( N \), \( \varphi = \varphi_\varphi \equiv 1 \) on \( I_0 \), \( \varphi \in C^\infty_0(I_2) \), and

\[
|D^k \varphi| \leq C^{k+1}N^k, \quad k \leq N.
\]

This is enough to prove analyticity (when the coefficients are analytic).

**References**

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