Wave Properties of Plasma Surrounding the Event Horizon of a Non-Rotating Black Hole

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Abstract

We have studied the wave properties of the cold and isothermal plasma in the vicinity of the Schwarzschild black hole event horizon. The Fourier analyzed perturbed 3+1 GRMHD equations are taken on the basis of which the complex dispersion relations are obtained for non-rotating, rotating non-magnetized and rotating magnetized backgrounds. The propagation and attenuation vectors along with the refractive index are obtained (shown in graphs) to study the dispersive properties of the medium near the event horizon. The results show that no information can be obtained from the Schwarzschild magnetosphere. Further, the pressure ceases the existence of normal dispersion of waves.

Keywords: 3+1 formalism, GRMHD equations, dispersion relations.
PACS numbers: 95.30.Sf, 95.30.Qd, 04.30.Nk

1 Introduction

Black holes are solutions of the Einstein field equations (EFEs) and Schwarzschild solution is the simplest black hole solution. The most physical black hole solution is the Kerr metric which is axisymmetric and rotating. This reduces

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to the Schwarzschild black hole when its angular momentum decreases to zero by energy extraction [1]. Around the Schwarzschild event horizon, the powerful gravitational force pulls the magnetized plasma of the surrounding space towards the event horizon in the form of an accretion cloud. The accreting plasma creates a magnetic field. The dynamical effects of accreting magnetospheric plasmas near a black hole’s event horizon enforce us to use the theory of general relativistic magnetohydrodynamics (GRMHD) to study them.

The gravitomagnetic waves and gravitational perturbations in the black hole regime have always been of interest. Many people [2]–[5] studied perturbations in the Schwarzschild regime. To make the results of GR accessible for astrophysicists, Arnowitt, Deser and Misner [6] developed a formulation (ADM 3+1 formalism) which splits the four-dimensional spacetime into three-dimensional hypersurfaces labeled by time. This makes the results of GR comparable with those of Newtonian physics. Several authors [7]–[12] applied this formalism to judge different aspects in GR.

Thorne and Macdonald [13]–[14] extended the formulation to electromagnetic fields of black hole theory. Holcomb and Tajima [15], Holcomb [16] and Dettmann et al. [17] investigated the wave propagation in Friedmann universe. Buzzi et al. [18] discussed one-dimensional radial propagation of transverse and longitudinal waves, in two component plasma, close to the Schwarzschild event horizon. The stationary symmetric GRMHD theory of black holes was developed by Zhang [19]. He also investigated the behavior of perturbations of cold plasma in the ergosphere of Kerr black hole [20]. Recently, Sharif and Umber [21]–[25] have found some interesting wave properties of cold and isothermal plasmas (with constant rest-mass density) in the vicinity of Schwarzschild black hole event horizon. They have evaluated real and complex wave numbers using 3+1 GRMHD equations.

In this paper, we shall consider variable rest-mass density to study the wave properties of cold and isothermal plasmas in Schwarzschild planar analogue by using complex wave vector components. The paper has been organized as follows: Section 1 contains the description of planar analogue with respective background, perturbation and Fourier analysis assumptions. Sections 2, 3 and 4 contain the dispersion relations obtained for cold plasma living in non-rotating, rotating non-magnetized and rotating magnetized backgrounds whereas sections 5, 6 and 7 constitute the dispersion relations for isothermal plasma living in the same backgrounds. All this has been done for variable mass density and pressure. The results will be discussed in the
last section.

2 Schwarzschild Planar Analogue and Relative Assumptions

The Schwarzschild planar analogue can be described as [20]

\[ ds^2 = -\alpha^2(z)dt^2 + dx^2 + dy^2 + dz^2. \] (2.1)

The directions \( z, x \) and \( y \) are analogous to Schwarzschild’s \( r, \phi \) and \( \theta \) respectively. In this planar analogue, we assume the existence of cold and isothermal plasmas with the respective equations of state [20]

\[ \mu = \frac{\rho}{\rho_0}, \quad \mu = \frac{\rho + p}{\rho_0}, \] (2.2)

where \( \rho_0, \rho \) and \( p \) denote the rest, moving mass densities and pressure respectively.

In non-rotating background, the perturbed flow of fluid is only along \( z \)-axis. The FIDO measured magnetic field and velocity are given by

\[ B = B e_z, \quad V = u(z)e_z. \] (2.3)

In rotating background, FIDO measured velocity and magnetic field are two-dimensional. In planar analogue, these quantities can be expressed in \( xz \)-plane by the following expressions

\[ B = B[\lambda(z)e_x + e_z], \quad V = V(z)e_x + u(z)e_z. \] (2.4)

Here \( \lambda, u \) and \( V \) are related to each other by the following equation [21]

\[ V = \frac{V_F}{\alpha} + \lambda u, \] (2.5)

where \( V^F \) is an integration constant.

We assume linear (first order) perturbations in flow variables (mass density \( \rho \), pressure \( p \), velocity \( V \) and magnetic field \( B \)) of the fluid.

\[ \rho = \rho^0 + \delta \rho = \rho^0 + \rho \tilde{\rho}, \quad p = p^0 + \delta p = p^0 + p \tilde{p}, \]

\[ V = V^0 + \delta V = V^0 + v, \quad B = B^0 + \delta B = B^0 + B b, \] (2.6)
where \( \rho^0, \ p, \ V^0 \) and \( B^0 \) are unperturbed quantities, \( \delta \rho, \ \delta p, \ \delta V \) and \( \delta B \) represent perturbed quantities. The dimensionless perturbed quantities \( \tilde{\rho}, \ \tilde{p}, \ \tilde{V} \) and \( \tilde{b} \) can be written as follows

\[
\tilde{\rho} = \tilde{\rho}(t, z), \quad \tilde{p} = \tilde{p}(t, z), \\
\tilde{v} = \delta V = v_x(t, z)e_x + v_z(t, z)e_z, \\
\tilde{b} = \frac{\delta B}{B} = b_x(t, z)e_x + b_z(t, z)e_z. 
\]  

(2.7)

For Fourier analysis, we assume the harmonic space and time dependence of perturbations

\[
\tilde{\rho}(t, z) = c_1e^{-i(\omega t - k z)}, \quad \tilde{p}(t, z) = c_2e^{-i(\omega t - k z)}, \\
v_x(t, z) = c_3e^{-i(\omega t - k z)}, \quad v_z(t, z) = c_4e^{-i(\omega t - k z)}, \\
b_x(t, z) = c_5e^{-i(\omega t - k z)}, \quad b_z(t, z) = c_6e^{-i(\omega t - k z)}. 
\]  

(2.8)

3  3+1 Perfect GRMHD Equations in Planar Analogue

For the Schwarzschild planar analogue, the perfect GRMHD equations can be written as [21, 26]

\[
\frac{\partial B}{\partial t} = \nabla \times (\alpha V \times B), 
\]  

(3.1)

\[ \nabla \cdot B = 0, \]  

(3.2)

\[
\frac{\partial \rho}{\partial t} + (\alpha V) \cdot \nabla \rho + \rho \gamma^2 V \cdot \frac{\partial V}{\partial t} + \rho \gamma^2 V \cdot (\alpha V \cdot \nabla) V \\
+ \rho \nabla \cdot (\alpha V) = 0, 
\]  

(3.3)

\[
\begin{align*}
\{ (\rho \gamma^2 + \frac{B^2}{4\pi}) \delta_{ij} + \rho \gamma^4 V_i V_j - \frac{1}{4\pi} B_i B_j \} \left( \frac{1}{\alpha} \frac{\partial}{\partial t} + V \cdot \nabla \right) V^j \\
- \left( \frac{B^2}{4\pi} \delta_{ij} - \frac{1}{4\pi} B_i B_j \right) V^j V^k + \rho_0 \gamma^2 V_i \left( \frac{1}{\alpha} \frac{\partial}{\partial t} + (V \cdot \nabla) \mu \right) \\
= -\rho \gamma^2 a_i - p_{,i} + \frac{1}{4\pi} (V \times B)_i \nabla \cdot (V \times B) - \frac{1}{8\pi \alpha^2 (\alpha B)^2, i} \\
+ \frac{1}{4\pi \alpha} (\alpha B)_i j B^j - \frac{1}{4\pi \alpha} [B \times \{ V \times (\nabla \times (\alpha V \times B)) \}]_i, \\
\end{align*}
\]  

(3.4)

\[
\gamma^2 \left( \frac{1}{\alpha} \frac{\partial}{\partial t} + V \cdot \nabla \right) (\mu p_0) - \frac{1}{\alpha} \frac{\partial p}{\partial t} + 2\rho_0 \mu \gamma^4 V \cdot \left( \frac{1}{\alpha} \frac{\partial}{\partial t} + V \cdot \nabla \right) V 
\]  

(3.5)
\[ +2\rho_0\mu\gamma^2(V \cdot a) + \rho_0\mu\gamma^2(\nabla \cdot V) + \frac{1}{4\pi\alpha}[\nabla \times (V \times \alpha B)] \]
\[ + (V \times B) \cdot \frac{\partial}{\partial t} (V \times B) \]. \tag{3.5} \]

The cold plasma model can be described by Eqs. (3.1)-(3.4) whereas isothermal plasma needs Eq. (3.5) for a better interpretation.

In a stationary symmetric background, the observers move along symmetric directions and thus do not see any change in flow around themselves. For such magnetized fluids, Phinney proposed the stream functions \(h\), \(l\) and \(e\) \[19\], i.e.,
\[ h = -\frac{4\pi u}{B}, \quad l = \gamma V - \alpha\lambda B/h, \quad e = \gamma\alpha, \tag{3.6} \]
where the rest-mass conservation law in three dimensions is \(\alpha\rho_0\gamma u = -1\) with specific enthalpy \(\mu = 1\) and the constant \(V_F = 0\).

## 4 Non-Rotating Background with Cold Plasma

The Fourier analyzed perturbed GRMHD equations in non-rotating background (Eqs.(3.15)-(3.18) of [21]) are
\[ -\frac{i\omega}{\alpha} c_5 = 0, \tag{4.1} \]
\[ ik c_5 = 0, \tag{4.2} \]
\[ c_1 \left( -\frac{i\omega}{\alpha} + ik \right) + c_3 \left\{ (1 + \gamma^2 u^2)ik - (1 + \gamma^2 u^2)(1 - 2\gamma^2 u^2)u' \right\} \]
\[ -\frac{i\omega}{\alpha} \gamma^2 u = 0, \tag{4.3} \]
\[ c_1 \gamma^2 \{ a_z + uu'(1 + \gamma^2 u^2) \} + c_3 \left[ \gamma^2 (1 + \gamma^2 u^2) \left( -\frac{i\omega}{\alpha} + iuk \right) \right. \]
\[ + \left. \gamma^2 \{ u'(1 + \gamma^2 u^2)(1 + 4\gamma^2 u^2) + 2\gamma^2 u a_z \} \right] = 0. \tag{4.4} \]

Equations (4.1) and (4.2) show that \(c_5 = 0\) which means that there are no perturbations in magnetic field. Thus the non-magnetized as well as magnetized backgrounds admit the same perturbed Fourier analyzed GRMHD equations.
4.1 Numerical Solutions

In order to find the numerical solutions, we assume the time lapse \( \alpha = \frac{1}{10} \tanh(10z) \). For stationary flow, we take \( \alpha \gamma = 1 \) which implies that \( \gamma = 1/\alpha \). For the inflow of fluid into the black hole event horizon we take \( u = -\sqrt{1-\alpha^2} \). Using these assumptions, the mass conservation law in three dimensions gives \( \rho = -\frac{1}{u} \). These quantities satisfy the GRMHD equations for the region \( 0.5 \leq z \leq 10 \). For these assumptions, \( e = 1 \).

The dispersion relation from Eqs. (4.3) and (4.4) can be obtained (using Mathematica) by equating determinant of the coefficients of constants \( c_1 \) and \( c_3 \) to zero \(^{[27]} \). This determinant leads to a complex dispersion relation of the form

\[
A_1(z, \omega)k^2 + A_2(z, \omega)k + A_3(z, \omega) + i\{A_4(z, \omega)k + A_5(z, \omega)\} = 0 \quad (4.5)
\]

which gives two complex values of \( k \). Since \( k \) is the \( z \)-component of the wave vector, it gives quantities corresponding to \( z \)-direction. The real and imaginary parts of \( k \) yield the propagation and attenuation vectors respectively. The propagation vector gives refractive index on the basis of which the mode of dispersion can be found. The sinusoidal expression then takes the form

\[
e^{-i(\omega t - k_1 z - i k_2 z)} = e^{-i(\omega t - k_1 z) - k_2 z}, \quad \text{where} \quad k_1 = \text{Re}(k) \quad \text{and} \quad k_2 = \text{Im}(k).
\]

The two values of \( k \) are shown in Figure 1 and 2. Figure 1 indicates that the propagation vector decreases with the increase in \( z \) and \( \omega \). It means that the waves propagate slowly as they move away from the event horizon and as their angular frequency increase. The attenuation vector admits positive and negative values randomly which shows random damping and growth of waves. The refractive index is greater than one and its variation with respect to angular frequency is positive in the region \( 1.9 \leq z \leq 10, \quad 1 \leq \omega \leq 10 \) which gives that the region admits normal dispersion of waves \(^{[28]} \). Figure 2 shows that the attenuation vector takes positive values at random points. The propagation vector decreases with the increase in angular frequency. The attenuation vector takes random values which means that the waves damp and grow randomly with the increase in angular frequency. The refractive index is greater than one but \( \frac{dn}{d\omega} < 0 \) in most of the region giving anomalous dispersion of waves \(^{[28]} \).

In Figures 1 and 2, the propagation vector takes negative values which shows that the waves move towards the black hole event horizon. In addition, a small region near the event horizon admits decrease in refractive index with the increase in \( z \).
Figure 1: The waves are directed towards the event horizon. Dispersion is normal in most of the region.

Figure 2: The waves move towards the event horizon. Most of the region admits anomalous dispersion of waves.
5 Rotating, Non-magnetized Background with Cold Plasma

For the rotating non-magnetized background, the Fourier analyzed perturbed perfect GRMHD equations are given by Eqs.(4.13)-(4.15) of [21].

\begin{align}
\frac{c_1}{\alpha} \left( -\frac{\omega}{\alpha} + iuk \right) + c_3 \left[ -\frac{\omega}{\alpha} \gamma^2 u + (1 + \gamma^2 u^2)\frac{u'}{u} - (1 - 2\gamma^2 u^2) \right] \\
\times (1 + \gamma^2 u^2) \frac{u'}{u} + 2\gamma^4 u^2 V V' \right] + c_4 \gamma^2 \left[ \left( -\frac{\omega}{\alpha} + iku \right) V + \gamma^2 u \right] \\
\times \left\{ (1 + 2\gamma^2 V^2) V' + 2\gamma^2 u V u' \right\} = 0, \\
\end{align}

(5.1)

\begin{align}
& c_1 \gamma^2 u \{ (1 + \gamma^2 V^2) V' + \gamma^2 u V u' \} + c_3 \gamma^2 \left[ \left( -\frac{\omega}{\alpha} + iku \right) \gamma^2 u V \right] \\
& + \{(1 + 2\gamma^2 u^2)(1 + 2\gamma^2 V^2) - \gamma^2 V^2 \} V' + 2\gamma^2 (1 + 2\gamma^2 u^2) u V u' \right] \\
& + c_4 \left[ \left( -\frac{\omega}{\alpha} + iku \right) \gamma^2 (1 + \gamma^2 V^2) + \gamma^4 u \{ (1 + 4\gamma^2 V^2) u u' \right. \\
& + 4V V'(1 + \gamma^2 V^2) \} \right] = 0, \\
\end{align}

(5.2)

\begin{align}
& c_1 \gamma^2 \{ a_z + (1 + \gamma^2 u^2) u u' + \gamma^2 u^2 V V' \} + c_3 \gamma^2 \left[ (1 + \gamma^2 u^2) \left( -\frac{\omega}{\alpha} \right. \right. \\
& \left. + iuk \right) + \gamma^2 [ u'(1 + \gamma^2 u^2)(1 + 4\gamma^2 u^2) + 2u \gamma^2 \{ a_z + (1 + 2\gamma^2 u^2) V V' \} ] \right] \\
& + c_4 \gamma^4 \left[ \left( -\frac{\omega}{\alpha} + iku \right) V u + u^2 V' (1 + 4\gamma^2 V^2) \right] \\
& + 2V (a_z + (1 + 2\gamma^2 u^2) u u') \right] = 0. \\
\end{align}

(5.3)

5.1 Numerical Solutions

We assume the same time lapse which we have assumed for non-rotating background. The stationary flow assumption is \( \alpha \gamma = 1 \) with \( V = u \) leads to \( u = -\sqrt{\frac{\omega}{2u^2}} \). Thus the mass conservation law in three dimensions gives \( \rho = -\frac{1}{u} \). These values satisfy the perfect GRMHD equations for the range \( 0.75 \leq z \leq 10 \). In rotating non-magnetized background, the constant values of \( u = V = -0.703562 \) make the flow constants \( l = -7.03562 \) and \( e = 1 \). Consequently, we obtain the following form of complex dispersion relation
Figure 3: The waves move towards the event horizon. Most of of the region shows normal dispersion of waves cubic in $k$.

\[
A_1(z)k^2 + A_2(z, \omega)k + A_3(z, \omega) + \iota\{A_4(z)k^3 + A_5(z, \omega)k^2 + A_6(z, \omega)k + A_7(z, \omega)\} = 0
\] (5.4)

This dispersion relation gives three complex values of $k$ with corresponding graphs shown in Figures 3-5.

In Figure 3, the propagation vector decreases with the increase in $z$. The attenuation vector decreases with the increase in $z$ in a small region near the event horizon. Thus, in the small region near the event horizon, the waves damp as they move towards the event horizon. The refractive index is greater than one and $\frac{dn}{d\omega} > 0$ except for the region $2.8 \leq z \leq 10$, $1 \leq \omega \leq 2.1$ which shows normal dispersion of waves. Figure 4 shows that the attenuation vector takes random negative values with increase in angular frequency. In a small region near the event horizon, the attenuation vector increases with increase in $z$. This indicates that the waves grow as they move towards the event horizon. The refractive index is greater than one and its variation with respect to angular frequency is negative which shows that the whole region admits anomalous dispersion of waves. Figure 5 indicates that
Figure 4: The waves move towards the event horizon. The region shows anomalous dispersion.

Figure 5: The waves are directed towards the event horizon. A small region near the event horizon admits normal dispersion of waves.
the attenuation vector takes negative values. Near the event horizon, the attenuation vector decreases with the increase in $z$. It means that waves grow near the event horizon. The refractive index is greater than one and its variation with respect to angular frequency is greater than zero in the region $0.75 \leq z \leq 1.1$, $1.5 \times 10^{-5} \leq \omega \leq 10$ which admits normal dispersion of waves. In rest of the region, dispersion is anomalous and normal randomly due to variation in the refractive index with respect to angular frequency which attain positive and negative values randomly respectively.

In Figures 3-5, the propagation is negative in the whole region. This quantity decreases with the increase in angular frequency. In Figures 4 and 5, a small region near the event horizon shows that the refractive index decreases with the increase in $z$.

6 Rotating Magnetized Background with Cold Plasma

In this background, the Fourier analyzed form of the perturbed GRMHD equations (Eqs.(5.15)-(5.20) of [21]) is given as follows

$$
c_4(\alpha' + i k \alpha) - c_3(\alpha \lambda' + i k \alpha \lambda) + c_5(\alpha V)' - c_6(\alpha u)' - \omega + i k u a = 0, \tag{6.1}
$$

$$
c_5\left(\frac{-i \omega}{\alpha} + i k u\right) = 0, \tag{6.2}
$$

$$
c_5 i k = 0, \tag{6.3}
$$

$$
c_1\left(\frac{-i \omega}{\alpha} + i u k\right) + c_3\left[\frac{-i \omega}{\alpha} \gamma^2 u + (1 + \gamma^2 u^2) i k - (1 - 2 \gamma^2 u^2)\right]
\times (1 + \gamma^2 u^2) \frac{u'}{u} + 2 \gamma^4 u^2 V' V + c_4 \gamma^2 \left[\frac{-i \omega}{\alpha} V + i k u V\right]
+ u \{(1 + 2 \gamma^2 V^2) V' + 2 \gamma^2 u V u'\}] = 0, \tag{6.4}
$$

$$
c_1 \gamma^2 \rho u \{(1 + \gamma^2 V^2) V' + \gamma^2 u V u'\} - \frac{B^2}{4 \pi} c_6 \{(1 - u^2) i k + \frac{\alpha'}{\alpha}(1 - u^2) - u u'\} + c_3[-(\rho \gamma^4 u V - \frac{\lambda B^2}{4 \pi} \frac{i \omega}{\alpha} + (\rho \gamma^4 u V + \frac{\lambda B^2}{4 \pi}) i k u
+ \rho \gamma^2 \{(1 + 2 \gamma^2 u^2)(1 + 2 \gamma^2 V^2) - \gamma^2 V^2\} V' + 2 \rho \gamma^4 (1 + 2 \gamma^2 u^2) u V u'
+ \frac{B^2 u}{4 \pi \alpha}(\alpha \lambda')\}
+ c_4[-\{\rho \gamma^2 (1 + \gamma^2 V^2) + \frac{B^2}{4 \pi} \frac{i \omega}{\alpha}\} + \{\rho \gamma^2 (1 + \gamma^2 V^2)\}
+ \frac{B^2}{4 \pi \alpha}(\alpha \lambda')] + c_5\left[\frac{-i \omega}{\alpha} \gamma^2 u + (1 + \gamma^2 u^2) i k - (1 - 2 \gamma^2 u^2)\right]
\times (1 + \gamma^2 u^2) \frac{u'}{u} + 2 \gamma^4 u^2 V' V + c_4 \gamma^2 \left[\frac{-i \omega}{\alpha} V + i k u V\right]
+ u \{(1 + 2 \gamma^2 V^2) V' + 2 \gamma^2 u V u'\}] = 0, \tag{6.4}
$$

$$
c_1 \gamma^2 \rho u \{(1 + \gamma^2 V^2) V' + \gamma^2 u V u'\} - \frac{B^2}{4 \pi} c_6 \{(1 - u^2) i k + \frac{\alpha'}{\alpha}(1 - u^2) - u u'\} + c_3[-(\rho \gamma^4 u V - \frac{\lambda B^2}{4 \pi} \frac{i \omega}{\alpha} + (\rho \gamma^4 u V + \frac{\lambda B^2}{4 \pi}) i k u
+ \rho \gamma^2 \{(1 + 2 \gamma^2 u^2)(1 + 2 \gamma^2 V^2) - \gamma^2 V^2\} V' + 2 \rho \gamma^4 (1 + 2 \gamma^2 u^2) u V u'
+ \frac{B^2 u}{4 \pi \alpha}(\alpha \lambda')\}
+ c_4[-\{\rho \gamma^2 (1 + \gamma^2 V^2) + \frac{B^2}{4 \pi} \frac{i \omega}{\alpha}\} + \{\rho \gamma^2 (1 + \gamma^2 V^2)\}
+ \frac{B^2}{4 \pi \alpha}(\alpha \lambda')] + c_5\left[\frac{-i \omega}{\alpha} \gamma^2 u + (1 + \gamma^2 u^2) i k - (1 - 2 \gamma^2 u^2)\right]
\times (1 + \gamma^2 u^2) \frac{u'}{u} + 2 \gamma^4 u^2 V' V + c_4 \gamma^2 \left[\frac{-i \omega}{\alpha} V + i k u V\right]
+ u \{(1 + 2 \gamma^2 V^2) V' + 2 \gamma^2 u V u'\}] = 0, \tag{6.4}
$$
\begin{align*}
- \frac{B^2}{4\pi} iku + \rho \gamma^4 u \{(1 + 4 \gamma^2 V^2) uu' + 4 VV'(1 + \gamma^2 V^2)\} \\
- \frac{B^2 u \alpha'}{4\pi \alpha} = 0, \quad (6.5)
\end{align*}

\begin{align*}
&c_1 \rho \gamma^2 \{a_z + (1 + \gamma^2 u^2) uu' + \gamma^2 u^2 V V'\} + c_3 [-\rho \gamma^2 (1 + \gamma^2 u^2) \\
&+ \frac{\lambda^2 B^2}{4\pi} \frac{i \omega}{\alpha} + \rho \gamma^2 (1 + \gamma^2 u^2) - \frac{\lambda^2 B^2}{4\pi} i ku + \rho \gamma^2 \{u'(1 + \gamma^2 u^2) \\
&\times (1 + 4 \gamma^2 u^2) + 2 u \gamma^2 (a_z + (1 + 2 \gamma^2 u^2) V V')\} - \frac{\lambda B^2 u}{4\pi \alpha} (\alpha \lambda')\} \\
&+ c_4 [-(\rho \gamma^4 u V - \frac{\lambda B^2}{4\pi} \frac{i \omega}{\alpha} + (\rho \gamma^4 u V + \frac{\lambda B^2}{4\pi} i ku \\
&\{\rho \gamma^4 \{u^2 V(1 + 4 \gamma^2 V^2) + 2 V(a_z + (1 + 2 \gamma^2 u^2) uu')\} + \frac{\lambda B^2 \alpha' u}{4\pi \alpha}\} \\
&\frac{B^2}{4\pi} c_6 [\lambda (1 - u^2) i k + \frac{\alpha'}{\alpha} (1 - u^2) - \lambda uu' + \frac{(\alpha \lambda')'}{\alpha} = 0. \quad (6.6)
\end{align*}

Equations (6.2) and (6.3) give \( c_5 = 0 \) which shows that there are no perturbations in \( z \)-component of magnetic field.

### 6.1 Numerical Solutions

We assume the same values of time lapse, density, \( x \) and \( z \)-component of velocity which we have mentioned in the previous section. Substituting \( u = V \) in Eq. (2.5) with the assumption that \( V^F = 0 \), it gives that \( \lambda = 1 \). Also we take \( B = \sqrt{\frac{176}{7}} \). These values satisfy the GRMHD equations for the region \( 1 \leq z \leq 10 \). For this region, the numerical values of the flow constants (due to constant \( u = V = -0.703562, B \) and \( \lambda \)) are \( e = 1, l = -6.83562 \) and \( h = -2.50663 \).

We obtain a complex dispersion relation quartic in \( k \)

\begin{align*}
A_1(z)k^4 + A_2(z, \omega)k^3 + A_3(z, \omega)k^2 + A_4(z, \omega)k + A_5(z, \omega) \\
+i \{A_6(z)k^3 + A_7(z, \omega)k^2 + A_8(z, \omega)k + A_9(z, \omega)\} = 0. \quad (6.7)
\end{align*}

The corresponding graphs are shown in Figures 6-9.

Figure 6 shows that the propagation vector decreases with the increase in \( \omega \) and \( z \). The attenuation vector attains random values in the region \( 2.1 \leq z \leq 10, \ 0 \leq \omega \leq 10 \) showing that waves damp and grow randomly. In the region \( 1 \leq z < 2.1 \), the attenuation vector increases with the increase in \( z \).
Figure 6: The waves move towards the event horizon. Most of the region admits the properties of anomalous dispersion.

The variation of refractive index with respect to $\omega$ is negative in most of the region which shows anomalous dispersion of waves. There are some points in the region where $\frac{dn}{d\omega} > 0$ and dispersion is normal. Figure 7 indicates that the attenuation vector takes negative values. The propagation vector decreases with the increase in angular frequency and $z$. The attenuation vector takes random values with the increase in angular frequency. The refractive index is greater than one and its variation with respect to angular frequency admit positive values at some points which indicates normal dispersion of waves. Figure 8 shows that the attenuation vector is positive throughout the region. The propagation vector decreases with the increase in angular frequency and $z$. The attenuation vector randomly increases and decreases with the increase in angular frequency. It shows that the waves randomly damp and grow with the increase in angular frequency. The refractive index is greater than one and $\frac{dn}{d\omega} > 0$ in most of the region which results that the waves disperse normally. Figure 9 shows that the attenuation vector is positive for the region $1 \leq z \leq 5, 10^{-5} \leq \omega \leq 10$. In a small region near the event horizon, the attenuation vector decreases with the increase in $z$ which indicates that the waves damp as they move towards the event horizon. The refractive index
Figure 7: The waves are directed towards the event horizon. The dispersion is found to be normal and anomalous randomly.

Figure 8: The waves move towards the event horizon. Most of the region admits normal dispersion.
Figure 9: The waves are directed towards the event horizon. Some of the region admits normal dispersion is greater than one and its variation with respect to angular frequency is greater than zero at random points which admit normal dispersion of waves.

In Figures 6-9, the propagation vector is negative in the whole region. Moreover, Figures 6 and 9 admit a small region near the event horizon where the refractive index increases with the decrease in $z$.

7 Non-Rotating Background with Isothermal Plasma

For the isothermal plasma model, the Fourier analyzed perturbed GRMHD equations (3.3.1)-(3.3.5) of [26] are given as follows

\[-\frac{i\omega}{c_5} = 0, \tag{7.1}\]
\[\kappa c_5 = 0, \tag{7.2}\]
\[c_1 \{-\rho\omega + ik\rho u - (u_\alpha p)' - \alpha u^2 \gamma^2 pu'\} + c_2 \{-pu_\omega + ikp\rho u + (u_\alpha p)' + \alpha u^2 \gamma^2 pu'\} + c_3 (\rho + p) \{\alpha (1 + \gamma^2 u^2) \kappa - \alpha (1 - 2\gamma^2 u^2)\} - \alpha (1 - 2\gamma^2 u^2)\]
\( x(1 + \gamma^2 u^2) \frac{u'}{u} - i \omega \gamma^2 u = 0, \) \hspace{1cm} (7.3)

\( c_1 \rho \gamma^2 \{ a_z + uu'(1 + \gamma^2 u^2) \} + c_2 \{ p \gamma^2 \{ a_z + uu'(1 + \gamma^2 u^2) \} \}
+i k p + p' + c_3 (\rho + p) \gamma^2 [(1 + \gamma^2 u^2)(-\frac{i \omega}{\alpha} + i ku) + \{ u'(1 + \gamma^2 u^2) \}
\times (1 + 4 \gamma^2 u^2) + 2 \gamma^2 u a_z \} = 0, \) \hspace{1cm} (7.4)

\( c_1 \rho \gamma^2 \left\{ -\frac{i \omega}{\alpha} + u (a_z + \gamma^2 uu') \right\} + c_2 p \left\{ -\frac{i \omega}{\alpha} (\gamma^2 - 1) + \gamma^2 u (a_z + \gamma^2 uu') \right\}
+ c_3 (\rho + p) \gamma^2 \left\{ -\frac{2 i \omega}{\alpha} \gamma^2 u + \gamma^2 u^2 i k + (2 \gamma^2 uu' + a_z) \right\}
\times (1 + 2 \gamma^2 u^2) \} = 0. \) \hspace{1cm} (7.5)

Equations (7.1) and (7.2) show that \( c_5 = 0 \) which means that there are no perturbations in magnetic field.

7.1 Numerical Solutions

In order to find the numerical solutions we assume the same time lapse and \( z \)-component of fluid velocity as in Section 4.1. When we substitute these values with the assumption of stiff fluid, i.e., \( \rho = p \), the mass conservation law in three dimensions gives \( \rho = -\frac{1}{2u} = p \). The complex dispersion relation is of the form

\[ A_1(z) k^2 + A_2(z, \omega) k + A_3(z, \omega) + i \left\{ A_4(z) k^3 + A_5(z, \omega) k^2 \right\} + A_6(z, \omega) k + A_7(z, \omega) \} = 0. \] \hspace{1cm} (7.6)

This gives three complex values of \( k \) shown in Figures 10-12.

Figure 10 indicates that the propagation decreases while the attenuation vector increases with the increase in angular frequency. The refractive index is greater than one and its variation with respect to \( \omega \) is greater than zero in the region \( 0.5 \leq z \leq 2 \). Rest of the region admits points of normal as well as anomalous dispersion. Figure 11 shows that the propagation vector decreases with the increase in angular frequency. The attenuation vector is also negative and it decreases with the increase in angular frequency and \( z \). It shows that the waves grow with the increase in angular frequency and \( z \). The refractive index is greater than one and its variation with respect to \( \omega \) is negative which indicates that the region admits anomalous dispersion. Figure 12 shows that the attenuation vector admits negative values. The
Figure 10: The waves move towards the event horizon. A small region near the event horizon admits normal dispersion.

Figure 11: The waves are directed towards the event horizon. Dispersion is found to be anomalous.
Figure 12: The waves move toward the event horizon. The dispersion is normal in a small region near the event horizon. The propagation vector decreases while the attenuation vector shows random increase and decrease with the increase in angular frequency. The attenuation vector decreases near the event horizon which indicates that the waves grow near the event horizon. The refractive index is greater than one. The change in refractive index with respect to angular frequency is greater than zero in the region $0.5 \leq z \leq 3, \quad 0 \leq \omega \leq 10$ which indicates normal dispersion.

The propagation vector takes negative values for Figures 10-12. In a small region near the event horizon, the refractive index increases with the increase in $z$ for Figures 11-12.

8 Rotating Non-magnetized Background with Isothermal Plasma

The Fourier analyzed GRHD equations for isothermal plasma are given by Eqs.(3.4.1)-(3.4.4) of [26],

$$c_1\{(-i\omega + i\alpha ku)\rho - \alpha' up - \alpha u' p - \alpha up' - \alpha \gamma^2 up(VV' + uu')\}$$
\[ + c_2 \{ - \omega + \iota \alpha k u \} p + \alpha' u p + \alpha u' p + \alpha \gamma^2 u p (VV' + uu') \} \]
\[ + c_3 (\rho + p) \{ - \omega \gamma^2 u + i k \alpha (1 + \gamma^2 u^2) - \alpha \{ (1 - 2 \gamma^2 u^2) (1 + \gamma^2 u^2) \} \frac{u'}{u} \]
\[ - 2 \gamma^4 u^2 V V' \} \] + \[ c_4 (\rho + p) \{ \gamma^2 V ( - \omega + i k \alpha u) + \alpha \gamma^2 u \{ (1 + 2 \gamma^2 V^2) V' \]
\[ + 2 \gamma^2 u V u' \} \} = 0. \] (8.1)
\[ c_1 \rho \gamma^2 u \{(1 + \gamma^2 V^2) V' + \gamma^2 u V u' \} + c_2 p r \gamma^2 u \{(1 + \gamma^2 V^2) V' + \gamma^2 u V u' \} \]
\[ + c_3 (\rho + p) \{ \gamma^2 u V \left( \frac{- i \omega}{\alpha} + i k \alpha \right) \} + \{ (1 + 2 \gamma^2 V^2) \} (1 + 2 \gamma^2 u^2) - \gamma^2 V^2 \} V' \]
\[ + 2 \gamma^2 u V u' \{(1 + 2 \gamma^2 u^2) \} + \[ c_4 (\rho + p) \gamma^2 \left( (1 + \gamma^2 u^2) \right) \left( \frac{- i \omega}{\alpha} + i k \alpha \right) \] \]
\[ + \gamma^2 u \{(1 + 4 \gamma^2 u^2) u u' + 4 (1 + \gamma^2 V^2) V V' \} \} = 0, \] (8.2)
\[ c_1 \rho \gamma^2 \{ a_z + u u' \{(1 + \gamma^2 u^2) + \gamma^2 u^2 V V' \} \} + c_2 p r \gamma^2 \{ a_z + u u' \{(1 + \gamma^2 u^2) \} \]
\[ + \gamma^2 u^2 V V' \} + p' + i k p \} + \[ c_3 (\rho + p) \gamma^2 \left( (1 + \gamma^2 u^2) \right) \left( \frac{- i \omega}{\alpha} + i k \alpha \right) \] \]
\[ + u' \{(1 + \gamma^2 u^2) \} \left( 1 + 4 \gamma^2 u^2 \right) + 2 \gamma^2 \{(1 + 2 \gamma^2 u^2) V V' + a_z \} \] + \[ c_4 (\rho + p) \gamma^4 \left[ u V \left( \frac{- i \omega}{\alpha} + i k \alpha \right) \right] + u^2 V' \{(1 + 4 \gamma^2 u^2) \}
\[ + 2 V \{(1 + 2 \gamma^2 u^2) u u' + a_z \} \} = 0, \] (8.3)
\[ c_1 \rho \gamma^2 \left[ \frac{- i \omega}{\alpha} + u \{ a_z + \gamma^2 (V V' + uu') \} \} \right] + \[ c_2 p \left[ \frac{- i \omega}{\alpha} \{(\gamma^2 - 1) + u \gamma^2 \{ a_z 
\[ + \gamma^2 (V V' + uu') \} \} \} \right] + c_3 (\rho + p) \gamma^2 \left\{ \gamma^2 u \left( \frac{- 2 i \omega}{\alpha} + i k \alpha \right) \right\} + \left\{ a_z + \gamma^2 u u' \right\} \]
\[ (1 + 2 \gamma^2 u^2) + \gamma^2 V V' \{(1 + 4 \gamma^2 u^2) \} + c_4 (\rho + p) \gamma^2 \left\{ \gamma^2 V \left( \frac{- 2 i \omega}{\alpha} + i k \alpha \right) \right\} \]
\[ + \gamma^2 u V' \{(1 + 4 \gamma^2 V^2) + 2 \gamma^2 u V \{ a_z + 2 \gamma^2 u u' \} \} \} = 0. \] (8.4)

### 8.1 Numerical Solutions

Using the same assumptions as given in Section 5.1, we obtain a complex dispersion relation of the form

\[ A_1(z) k^4 + A_2(z, \omega) k^3 + A_3(z, \omega) k^2 + A_4(z, \omega) k + A_5(z, \omega) \]
\[ + i \{ A_6(z) k^3 + A_7(z, \omega) k^2 + A_8(z, \omega) k + A_9(z, \omega) \} = 0. \] (8.5)

The four complex values of \( k \) are represented in Figures 13-16.
Figure 13: The waves move towards the event horizon. A small region near the event horizon admits anomalous dispersion.

Figure 13 shows that the propagation vector takes negative values. The propagation vector decreases with the increase in angular frequency and $z$. The attenuation vector increases with the increase in angular frequency. The refractive index is greater than one and its variation with respect to $\omega$ is less than zero in the region $0.75 \leq z \leq 4.5$, $0 \leq \omega \leq 10$ which leads to anomalous dispersion. Random points of normal and anomalous dispersion are found otherwise. Figure 14 indicates that the propagation and attenuation vectors decrease with the increase in $z$ and $\omega$. It shows that the waves grow with the increase in angular frequency and $z$. The refractive index is greater than one and its variation with respect to $\omega$ is positive in the region $0.75 \leq z \leq 4$, $0 \leq \omega \leq 10$ which indicates normal dispersion of waves. Random points of normal and anomalous dispersion lies in rest of the region. Figure 15 shows that the propagation vector decreases with the increase in angular frequency and $z$. The attenuation vector decreases as $z$ decreases which indicates that the waves grow as they move towards the event horizon. The refractive index is greater than one. The quantity $\frac{dn}{d\omega} > 0$ at random points which leads to normal dispersion of waves at those points. In Figure 16, the attenuation vector decreases as $z$ decreases in the region except for $\omega = 0$ which shows the
Figure 14: The waves are directed towards the event horizon. Normal dispersion of waves is found near the event horizon.

Figure 15: The waves are directed towards the event horizon. Dispersion is found to be normal at random points.
Figure 16: The waves move towards the event horizon. Dispersion is found to be normal at random points.

growth of waves as they move towards the event horizon. The refractive index is greater than one and its variation with respect to $\omega$ is greater than zero at random points which shows that the waves disperse normally at random points.

In Figures 13-16, the propagation vector admits negative values which indicates that the waves move towards the event horizon. The refractive index increases in a small region near the event horizon for all these figures.

9 Rotating Magnetized Background with Isothermal Plasma

This is the most general and physical case considered to investigate the wave properties, i.e., we take rotating isothermal plasma which is obviously highly magnetized. The respective Fourier analyzed perturbed GRMHD equations are given by Eqs.(3.5.1)-(3.5.7) of [26]

\[
c_4(\alpha' + \imath k \alpha) - c_3(\alpha \lambda') + \imath k \alpha \lambda + c_5(\alpha V)' - c_6(\alpha u)' - \omega
\]
\[+iku\alpha\} = 0, \tag{9.1}\]
\[c_5\left(\frac{-i\omega}{\alpha} + ik\alpha\right) = 0, \tag{9.2}\]
\[c_5ik = 0, \tag{9.3}\]
\[c_1\{(\omega + i\omega)\rho - \alpha'up - \alpha'p - \alpha u'p - \alpha\gamma^2up(VV'' + uu')\} \]
\[+ c_2\{(\omega + i\omega)p + \alpha'up + \alpha'u\rho + \alpha u'p + \alpha\gamma^2up(VV' + uu')\} \]
\[+ c_3(\rho + p)(-\omega\gamma^2u + i\omega(1 + \gamma^2u^2) - \alpha\{(1 - 2\gamma^2u^2)(1 + \gamma^2u^2)\}u' \]
\[+ \alpha\gamma^2u((1 + 2\gamma^2V^2)V' + 2\gamma^2uV'u')\} = 0, \tag{9.4}\]
\[c_4(\rho + p)[\gamma^2V(-\omega + ik\omega)\}
\[+ \alpha\gamma^2u((1 + 2\gamma^2V^2)V' + 2\gamma^2uV'u')\} = 0, \tag{9.4}\]
\[c_1\gamma^2\rho u\{(1 + \gamma^2V^2)V' + 2\gamma^2uV'u'\} + c_2\gamma^2\rho u\{(1 + \gamma^2V^2)V' + 2\gamma^2uV'u'\}
\[+ c_3[-\{(\rho + p)\gamma^2uV - \frac{\lambda B^2}{4\pi}\}u\omega + \{(\rho + p)\gamma^2uV + \frac{\lambda B^2}{4\pi}\}iku \]
\[+(\rho + p)\gamma^2\{(1 + 2\gamma^2u^2)(1 + 2\gamma^2V^2) - \gamma^2V^2\}V' + 2(\rho + p)\gamma^4(1 \]
\[+ 2\gamma^2u^2)uV'u' + B^2\omega(\alpha\lambda')\} + c_4[-\{(\rho + p)\gamma^2(1 + \gamma^2V^2) + \frac{B^2}{4\pi}\}u\omega \]
\[+ \{(\rho + p)\gamma^2(1 + 2\gamma^2V^2) - \frac{B^2}{4\pi}\}iku + (\rho + p)\gamma^4u((1 + 4\gamma^2V^2)uu' \]
\[+ 4VV'(1 + \gamma^2V^2) - \frac{B^2}{4\pi}\} - \frac{B^2}{4\pi}\]c_6\{(1 - u^2)ik + \frac{\alpha'}{\alpha}(1 - u^2)
\[= 0, \tag{9.5}\]
\[c_1(\omega + 1 + \gamma^2u^2)u\omega + 2\gamma^2u^2V'V') + c_2[p\gamma^2\{(a_z + (1 + 2\gamma^2u^2)uu' \]
\[+ \gamma^2V' + 1 + kp + p'] + c_3[-\{(\rho + p)\gamma^2(1 + 2\gamma^2V^2) + \frac{\lambda B^2}{4\pi}\}u\omega \]
\[+(\rho + p)\gamma^2{\{(1 + 2\gamma^2u^2) - \frac{\lambda B^2}{4\pi}\}iku + \{(\rho + p)\gamma^2\{u'(1 + 2\gamma^2u^2) \]
\[(1 + 4\gamma^2V^2) + 2\gamma(V^2(a_z + (1 + 2\gamma^2u^2)V') - \frac{\lambda B^2u}{4\pi\alpha}(\alpha\lambda')\} \]
\[+ c_4[-\{(\rho + p)\gamma^2uV - \frac{\lambda B^2}{4\pi}\}u\omega + \{(\rho + p)\gamma^2uV + \frac{\lambda B^2}{4\pi}\}iku \]
\[+ \{(\rho + p)\gamma^2{\{u'(1 + 2\gamma^2u^2)uu'\}} + \frac{\lambda B^2}{4\pi\alpha}\}\}
\[+ \frac{B^2}{4\pi}\]c_6\{\lambda(1 - u^2)ik + \frac{\alpha'}{\alpha}(1 - u^2) - \lambda uu' + \frac{(\alpha\lambda')}{\alpha} = 0, \tag{9.6}\]
\[c_1\rho\gamma^2[-\frac{i\omega}{\alpha} + u\{a_z + \gamma^2(VV' + uu')\}] + c_3[-\frac{2i\omega}{\alpha}\{(\rho + p)\gamma^4u \]

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\[ \frac{B^2 \lambda}{4 \pi} (u \lambda - V) + ik \{ (\rho + p) \gamma^4 u^2 + \frac{\lambda^2 B^2}{4 \pi} \} + (\rho + p) \gamma^2 \{ (a_z \\
+ 2 \gamma^2 uu')(1 + 2 \gamma^2 u^2) + \gamma^2 V V' (1 + 4 \gamma^2 u^2) \} + \frac{\lambda B^2}{4 \pi} (3 \lambda a_z + \lambda') \]
\[ + c_6 \frac{B^2}{4 \pi} \left\{ \frac{i \omega}{\alpha} \{ \lambda + 2u(u \lambda - V) \} + ik u \lambda + 3a_z (2 \lambda u - V) + 2u' \lambda \\
+ u' \lambda - V' \right\} + c_2 p \left\{ \frac{i \omega}{\alpha} (\gamma^2 - 1) + u \gamma^2 \{ a_z + \gamma^2 (V V' + uu') \} \right\} = 0. \tag{9.7} \]

Equations (9.2) and (9.3) give \( c_5 = 0 \) which shows that the \( z \)-component of magnetic field is not affected by gravity.

### 9.1 Numerical Solutions

We assume the same values of time lapse, \( x \) and \( z \)-components of the fluid velocity, \( \lambda \) and \( B \) as given in section 6.1. Using these values with the assumption \( \rho = p \), the mass conservation law in three dimensions gives \( \rho = -\frac{1}{2u} = p \).

We obtain a complex dispersion relation quintic in \( k \), i.e., of the form

\[ A_1(z)k^4 + A_2(z, \omega)k^3 + A_3(z, \omega)k^2 + A_4(z, \omega)k + A_5(z, \omega) \\
+ i \{ A_6(z)k^5 + A_7(z, \omega)k^4 + A_8(z, \omega)k^3 + A_9(z, \omega)k^2 + A_{10}(z, \omega)k \\
+ A_{11}(z, \omega) \} = 0, \tag{9.1} \]

which can not be solved to get the exact solutions. We have found the numerical solutions with the help of software Mathematica. We obtain five complex values of \( k \) shown in Figures 17-21.

Figure 17 shows that the propagation vector decreases with the increase in angular frequency. In a small region near the event horizon, the attenuation vector increases as \( z \) decreases. The refractive index is greater than one and its variation with respect to \( \omega \) is positive at random points which shows normal dispersion of the waves. Figure 18 shows that the propagation vector decreases with the increase in angular frequency. The attenuation vector randomly increases and decreases in the region \( 1 \leq z \leq 2.2, \ 0 \leq \omega \leq 10 \).

The refractive index is greater than one and its variation with respect to \( \omega \) is positive at random points which shows that the waves disperse normally at
Figure 17: The waves are directed towards the event horizon. Normal dispersion is found at random points.

Figure 18: The waves move towards the event horizon. The waves disperse normally at random points.
Figure 19: The waves move towards the event horizon. Dispersion is found to be normal at random points.

Figure 20: The waves move towards the event horizon. Normal dispersion is found in a small region near the event horizon.
Figure 21: The waves are directed towards the event horizon. Normal dispersion is found at random points those points. Figure 19 indicates that both the propagation and attenuation vectors are negative and decrease with the increase in angular frequency. The refractive index is greater than one and its variation with respect to $\omega$ is positive at random points which shows that the waves disperse normally at those points. Figure 20 indicates that both the propagation and the attenuation vectors decrease with the increase in angular frequency. The refractive index is greater than one and its variation with respect to $\omega$ is greater than zero in the region $1 \leq z \leq 1.2$, $0 \leq \omega \leq 10$ which shows normal dispersion. Random points of normal and anomalous dispersion are found otherwise. Figure 21 shows that the propagation vector decreases with the increase in angular frequency. The attenuation vector increases and decreases randomly in the region $1 \leq z \leq 2.2$, $0 \leq \omega \leq 10$. The refractive index is greater than one and its variation with respect to $\omega$ is positive near the event horizon which indicates normal dispersion of waves.

In Figures 17-21, the propagation vector admits negative values indicating the direction of waves towards the event horizon. Furthermore, in all these figures, a small region near the event horizon admits the increase in refractive index with the decrease in $z$. 

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10 Outlook

We have discussed the cold and isothermal plasma wave properties of the Schwarzschild black hole magnetosphere in 3+1 formalism.

For this purpose, we have used the planar analogue (given by Eq.(2.1)) due to the following reasons:

- It is difficult to study the MHD stream equation and the behavior of a perturbed magnetosphere in the Schwarzschild spacetime. We have chosen a different way and altered the spacetime (i.e., we have taken the planar analogue) which simplifies the analysis of magnetosphere. It has been done while preserving the following key features of the Schwarzschild metric:

  1. The behavior of the Schwarzschild time coordinate as \( r \to 2M \) is analogous to the behavior of Schwarzschild planar analogue time coordinate as \( z \to 0 \).

  2. The planar spacetime and its stationary MHD magnetosphere have nice computational features that all aspects of the magnetosphere should become asymptotically \( z \) independent at large \( z \).

- The planar analogue is not empty which is not important for our analysis because it serves merely as a testbed for studying various aspects of the interactions of gravity with plasma [20]. Therefore, we are free to assume that there is no direct non-gravitational interaction between the plasma and materials whose stress-energy produce the spacetime curvature.

The plasma is assumed to be living in this planar analogue for which the dispersion relations are obtained from the Fourier analyzed perturbed 3+1 GRMHD equations for the non-rotating, rotating non-magnetized and rotating magnetized backgrounds. These relations give the value of \( z \)-component of the wave vector which yields the propagation and attenuation vectors, the refractive index and its change with respect to angular frequency. The graphs are obtained to have a clear viewpoint. A summary of the results is given below.

For the cold plasma living in the vicinity of Schwarzschild regime, normal dispersion of waves is found in the Figures 1, 3 and 8 which indicate
that the waves can pass through the medium. Figures 2, 4, 6 and 9 represent that the waves disperse anomalously in most of the region. Random dispersion of waves is found in the Figures 5 and 7. Figure 3 shows that the waves damp in a small region near the event horizon. In Figures 4 and 5, the waves grow near the event horizon in a small region. In Figure 9, the attenuation vector increases with the decrease in $z$ in a small region near the event horizon. In Figures 1-9, the waves damp and grow randomly with the increase in angular frequency.

For the isothermal plasma surrounding the Schwarzschild black hole, Figure 14 admits normal dispersion near the event horizon. The waves disperse anomalously in Figure 11. In Figure 10, the attenuation vector increases whereas it decreases for Figure 11 with the increase in angular frequency. It shows that the waves damp in Figure 10 and grow for Figure 11 with the increase in angular frequency. Figure 12 shows that the waves grow and damp randomly with the increase in angular frequency. In Figures 15 and 16, the waves grow in a small region near the event horizon. In Figure 17, the waves damp whereas they grow in Figures 18 and 21 with the decrease in $z$ in small regions near the event horizon. In Figures 19 and 20, the attenuation vector decreases with the increase in angular frequency.

For the cold plasma living in the non-rotating Schwarzschild background, there exists a case where the waves disperse normally (Figure 1) whereas for the isothermal plasma, normal dispersion lies at random points. In the rotating non-magnetized background, the cold plasma shows normal dispersion in Figure 3 whereas the isothermal plasma admits normal dispersion in a small region near the event horizon in Figure 14. In the rotating magnetized background, Figure 8 indicates that the cold plasma admits normal dispersion in most of the region whereas the isothermal plasma admits normal dispersion at random points. Thus we can conclude that the pressure ceases the normal dispersion of waves.

In most of the cases, the refractive index increases as the waves move towards the event horizon in a small region near the event horizon. This shows that the refraction of waves increases as they move towards the event horizon. For all the backgrounds of cold and isothermal plasma, the propagation vector takes negative values which shows that the waves move towards the event horizon.

**Acknowledgment:** We would like to thank Miss Umber Skeikh for the fruitful discussions during this work.
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