Gradient Estimates on Dirichlet Eigenfunctions

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Abstract

By methods of stochastic analysis on Riemannian manifolds, we derive explicit constants \(c_1(D)\) and \(c_2(D)\) for a \(d\)-dimensional compact Riemannian manifold \(D\) with boundary such that

\[
c_1(D)\sqrt{\lambda}\|\phi\|_{\infty} \leq \|\nabla \phi\|_{\infty} \leq c_2(D)\sqrt{\lambda}\|\phi\|_{\infty}
\]

holds for any Dirichlet eigenfunction \(\phi\) of \(-\Delta\) with eigenvalue \(\lambda\). In particular, when \(D\) is convex with nonnegative Ricci curvature, this estimate holds for

\[
c_1(D) = \frac{1}{de}, \quad c_2(D) = \sqrt{e} + \frac{e\sqrt{2}}{\sqrt{\pi}}.
\]

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1 Introduction

Let $D$ be a $d$-dimensional compact Riemannian manifold with boundary $\partial D$. We write $(\phi, \lambda) \in \text{Eig}(\Delta)$ if $\phi$ is a Dirichlet eigenfunction of $-\Delta$ in $D$ with eigenvalue $\lambda > 0$. According to [6], there exist two constants $c_1(D), c_2(D) > 0$ such that

$$(1.1) \quad c_1(D) \sqrt{\lambda} \|\phi\|_{\infty} \leq \|\nabla \phi\|_{\infty} \leq c_2(D) \sqrt{\lambda} \|\phi\|_{\infty}, \quad (\phi, \lambda) \in \text{Eig}(\Delta).$$

In this paper, by using stochastic analysis of the Brownian motion on $D$, we present explicit expressions of these two constants in terms of the lower bounds of $\text{Ric}_D$ and $\mathbb{I}_{\partial D}$ where $\text{Ric}_D$ is the Ricci curvature on $D$ and $\mathbb{I}_{\partial D}$ the second fundamental form of $\partial D$.

**Theorem 1.1.** Let $K, \theta \geq 0$ be two constants such that

$$\text{Ric}_D \geq -K, \quad \mathbb{I}_{\partial D} \geq -\theta.$$ 

Let

$$\alpha_0 = \frac{1}{2} ((d-1)\theta + \sqrt{(d-1)K}).$$

Then, for any nontrivial $(\phi, \lambda) \in \text{Eig}(\Delta),$

$$\frac{\lambda}{\sqrt{d e(\lambda + K)}} \leq \frac{\|\nabla \phi\|_{\infty}}{\|\phi\|_{\infty}} \leq \sqrt{\lambda}\left(\sqrt{e} + \frac{e\sqrt{2}}{\sqrt{\pi}} + \alpha_0 \wedge \frac{\alpha_0^2}{2\pi\lambda}\right).$$

In particular, when $\text{Ric}_D, \mathbb{I}_{\partial D} \geq 0$,

$$\frac{\sqrt{\lambda}}{\sqrt{d e}} \leq \frac{\|\nabla \phi\|_{\infty}}{\|\phi\|_{\infty}} \leq \sqrt{\lambda}\left(\sqrt{e} + \frac{e\sqrt{2}}{\sqrt{\pi}}\right), \quad (\phi, \lambda) \in \text{Eig}(\Delta).$$

**Proof.** This result follows from Theorem 2.1 and Theorem 3.1 below for the special case $V = 0$. In this case, $\text{Ric}_V = \text{Ric}_D \geq -K$ is equivalent to (2.1) with $n = d$. \hfill \Box

**Remark 1.1.** Various other estimates can be obtained from our method. For instance, let $\alpha \in \mathbb{R}$ be such that $\frac{1}{2} \Delta \rho_{\partial D} \leq \alpha$ outside the focal set, where $\rho_{\partial D}$ denotes the distance to boundary $\partial D$. Then for $\lambda < \alpha^2/4$,

$$(1.3) \quad \frac{\|\nabla \phi\|_{\infty}}{\|\phi\|_{\infty}} \leq \sqrt{e(\lambda + K)} + e\left(2 \max(\alpha, 0) + \frac{\alpha^2 e^2}{4 \sqrt{2\pi}} \right) e^{-\frac{\alpha^2}{2\pi\lambda}}).$$

This relies on Remark 3.1 where another estimate of the right hand side of (3.13) is given. It improves the estimate in Theorem 1.1 in the case when $\alpha < 0$ and $|\alpha|$ is large. See Theorem 3.5 below for case that $k = 0$ and $\theta < 0$.

By (1.2), when $D$ is convex with nonnegative Ricci curvature, (1.1) holds with

$$c_1(D) = \frac{1}{\sqrt{d e}}, \quad c_2(D) = \sqrt{e} + \frac{e\sqrt{2}}{\sqrt{\pi}}.$$
To estimate $c_1(D)$ and $c_2(D)$ for positive $K$ or $\theta$, let $\lambda_1 > 0$ be the first Dirichlet eigenvalue of $-\Delta$ on $D$. Then Theorem 1.1 implies that the inequalities (1.1) hold for

$$c_1(D) = \sqrt{\frac{\lambda_1}{\text{det}(\lambda_1 + K)}},$$
$$c_2(D) = \sqrt{\frac{e(\lambda_1 + K)}{\lambda_1}} + e \left( \frac{(d-1)\theta + \sqrt{K(d-1)}}{2\sqrt{\lambda_1}} + \sqrt{\frac{2}{\pi}} + \frac{(d-1)\theta + \sqrt{K(d-1)}}{4\lambda_1\sqrt{2\pi}} \right).$$

This is due to the fact that the first expression is an increasing function of $\lambda$ and the second one is a decreasing function of $\lambda$. Since there exist explicit lower bound estimates on $\lambda_1$ (see [8] and references within), this gives explicit lower bounds of $c_1(D)$ and upper bounds of $c_2(D)$.

The lower bound estimate of $\|\nabla \phi\|_\infty$ will be derived by using Itô’s formula for $|\nabla \phi|^2(X_t)$ where $X_t$ is a Brownian motion (with drift) on $D$, see Section 2 for details. A powerful probabilistic tool for establishing upper bound gradient estimates is the use of Bismut type formulas for the Dirichlet semigroup $P^D_t$ on $D$, which gives

$$|\nabla P^D_t f(x)| \leq \frac{c(t)}{\rho_{\partial D}} \|f\|_\infty, \quad t > 0, \ f \in \mathcal{B}_b(D),$$

where $\rho_{\partial D}$ is the Riemannian distance to $\partial D$ and $c(t)$ an explicit quantity depending on the geometry of $D$, see [7] for details. However, as this estimate blows up at the boundary $\partial D$, it does not give the wanted upper bound estimate of $\|\nabla \phi\|_\infty$ near the boundary. To achieve the goal of a uniform upper bound on $D$, we will construct some martingales to reduce $\|\nabla \phi\|_\infty$ to $\|\nabla \phi\|_{\partial D, \infty} := \sup_{\partial D} |\nabla \phi|$, and to estimate the latter using $\|\phi\|_\infty$, see Section 3 for details.

In general, we will consider Dirichlet eigenfunctions for the symmetric operator $L := \Delta + \nabla V$ on $D$ where $V \in C^2(D)$. We denote by $\text{Eig}(L)$ the set of pairs $(\phi, \lambda)$ where $\phi$ is a Dirichlet eigenfunction of $-L$ on $D$ with eigenvalue $\lambda$.

## 2 Lower bound estimate

In this Section we will estimate $\|\nabla \phi\|_\infty$ from below using the following Bakry-Émery curvature-dimension condition:

$$(2.1) \quad \frac{1}{2} L|\nabla f|^2 - \langle \nabla Lf, \nabla f \rangle \geq -K|\nabla f|^2 + \frac{(Lf)^2}{n}, \quad f \in C^\infty(D),$$

where $K \in \mathbb{R}$, $n \geq d$ are two constants. When $V = 0$, this condition with $n = d$ is equivalent to $\text{Ric}_D \geq -K$.

**Theorem 2.1** (Lower bound estimate). Assume that (2.1) holds. Then

$$(2.2) \quad \|\nabla \phi\|_\infty^2 \geq \|\phi\|_\infty^2 \sup_{t > 0} \frac{\lambda^2(e^{Kt} - 1)}{nK e^{(\lambda+K)t}}, \quad (\phi, \lambda) \in \text{Eig}(L).$$

Consequently, for $K^+ := \max\{0, K\}$ there holds

$$(2.3) \quad \|\nabla \phi\|_\infty^2 \geq \frac{\lambda^2}{n e(\lambda + K^+)} \|\phi\|_\infty^2, \quad (\phi, \lambda) \in \text{Eig}(L).$$
Proof. Let $X_t$ be the diffusion process generated by $\frac{1}{2} L$ in $D$, and let 

$$\tau_D := \inf\{ t \geq 0 : X_t \in \partial D \}.$$ 

By Itô’s formula, we have

$$d|\nabla \phi|^2(X_t) = \frac{1}{2} L|\nabla \phi|^2(X_t) \, dt + dM_t, \quad t \leq \tau_D,$$

for some martingale $M_t$. By the curvature dimension condition (2.1) and $L \phi = -\lambda \phi$, we obtain

$$\frac{1}{2} L|\nabla \phi|^2 = \frac{1}{2} L|\nabla \phi|^2 - \langle \nabla L \phi, \nabla \phi \rangle - \lambda |\nabla \phi|^2 \geq -(K + \lambda)|\nabla \phi|^2 + \frac{\lambda^2}{n}\phi^2.$$ 

Therefore, (2.4) gives

$$d|\nabla \phi|^2(X_t) \geq \left( \frac{\lambda^2}{n}\phi^2 - (K + \lambda)|\nabla \phi|^2 \right)(X_t) \, dt + dM_t, \quad t \leq \tau_D.$$

Hence, for any $t > 0$,

$$e^{(K+\lambda)t} \| \nabla \phi \|_\infty^2 \geq \mathbb{E} \left[ |\nabla \phi|^2(X_{t\wedge \tau_D})e^{(K+\lambda)(t\wedge \tau_D)} \right]$$

$$\geq \frac{\lambda^2}{n} \mathbb{E} \left[ \int_0^{t\wedge \tau_D} e^{(K+\lambda)s} \phi(X_s)^2 \, ds \right]$$

$$= \frac{\lambda^2}{n} \mathbb{E} \left[ \int_0^t 1_{\{s < \tau_D\}} e^{(K+\lambda)s} \phi(X_s)^2 \, ds \right].$$

Since $\phi|_{\partial D} = 0$ and $L \phi = -\lambda \phi$, by Jensen’s inequality we have

$$\mathbb{E} \left[ 1_{\{s < \tau_D\}} \phi(X_s)^2 \right] \geq (\mathbb{E}[\phi(X_{s\wedge \tau_D})])^2 = e^{-\lambda \phi(x)^2},$$

where $x = X_0 \in D$ is the starting point of $X_t$. Then, by taking $x$ such that $\phi(x)^2 = \|\phi\|_\infty^2$, we arrive at

$$e^{(K+\lambda)t} \| \nabla \phi \|_\infty^2 \geq \frac{\lambda^2}{n} \int_0^t e^{(K+\lambda)s} e^{-\lambda \phi(x)^2} \, ds$$

$$= \frac{\lambda^2}{n} \|\phi\|_\infty^2 \int_0^t e^{Ks} \, ds = \lambda^2 (e^{Kt} - 1) \frac{1}{nK} \|\phi\|_\infty^2.$$ 

This completes the proof of (2.2).

Since (2.1) holds for $K^+$ replacing $K$, we may and do assume that $K \geq 0$. By taking $t = \frac{1}{\lambda + K}$ in (2.2), we obtain

$$\| \nabla \phi \|_\infty^2 \geq \frac{\lambda^2}{nK e} \|\phi\|_\infty^2 \geq \frac{\lambda^2}{ne(\lambda + K)} \|\phi\|_\infty^2.$$ 

Hence (2.3) holds. \qed
3 Upper bound estimate

Let $\text{Ric}^V_D = \text{Ric}_D - \text{Hess}_V$.

**Theorem 3.1 (Upper bound estimate).** Let $K_V, K_0, \theta \geq 0$ be constants such that

$$\text{Ric}^V_D \geq -K_V, \quad \text{Ric}_D \geq -K_0, \quad \|\partial_D \| \geq -\theta.$$

Let

$$\alpha = \frac{1}{2} \left( (d-1)\theta + \sqrt{(d-1)K_0} + \|\nabla V\|_\infty \right).$$

Then, for any $(\phi, \lambda) \in \text{Eig}(L)$,

$$\|\nabla \phi\|_\infty \leq \|\phi\|_\infty \left\{ \sqrt{e(\lambda + K_V)} + e\left( \alpha + \frac{\sqrt{2}\lambda}{\sqrt{\pi}} + \alpha \wedge \frac{\alpha^2}{\sqrt{2}\pi} \right) \right\}.$$

To prove this result, we first estimate $\|\nabla \phi\|_\infty$ in terms of $\|\phi\|_\infty$ and $\|\nabla \phi\|_{\partial_D, \infty}$ where $\|f\|_{\partial_D, \infty} := \|1_{\partial_D} f\|_\infty$ for a function $f$ on $D$.

**Lemma 3.2.** Assume $\text{Ric}^V_D \geq -K_V$ for some constant $K_V \in \mathbb{R}$. Then, for any $(\phi, \lambda) \in \text{Eig}(L)$,

$$\|\nabla \phi\|_\infty \leq e^{\frac{(\lambda + K_V)}{2} t} \|\nabla \phi\|_{\partial_D, \infty} + \|\phi\|_\infty e^{\frac{1}{2}t} \left( \frac{K_V}{1 - e^{-K_V t}} \right)^{1/2}, \quad t > 0.$$

Consequently,

$$\|\nabla \phi\|_\infty \leq e^{1/2} \left( \|\nabla \phi\|_{\partial_D, \infty} + \sqrt{\lambda + K_V^+} \|\phi\|_\infty \right), \quad (\phi, \lambda) \in \text{Eig}(L).$$

**Proof.** We first recall some facts concerning the diffusion process generated by $\frac{1}{2}L$, see for instance [3]. For any $x \in D$, the diffusion $X_t$ solves the SDE

$$\text{d}X_t = \frac{1}{2}\nabla V(X_t) \text{d}t + u_t \circ \text{d}B_t, \quad X_0 = x, \quad t \leq \tau_D,$$

where $B_t$ is a $d$-dimensional Brownian motion, $u_t$ is the horizontal lift of $X_t$ onto the orthonormal frame bundle $O(D)$ with initial value $u_0 \in O_x(D)$, and

$$\tau_D := \inf\{ t > 0 : X_t \in \partial D \}$$

is the hitting time of $X_t$ to the boundary $\partial D$. Setting $Z := \nabla V$, we have

$$\text{d}u_t = \frac{1}{2}Z^*(u_t) \text{d}t + \sum_{i=1}^d H_i(u_t) \circ \text{d}B^i_t$$

where $Z^*(u) := h_u(Z_{\pi(u)})$ and $H_i(u) := h_u(u e_i)$ are defined by means of the horizontal lift $h_u: T_{\pi(u)}D \to T_u O(D)$ at $u \in O(D)$. Note that formally $h_u(u_t \circ \text{d}B_t) = \sum_i h_u(u_t e_i) \circ \text{d}B^i_t = \sum_i H_i(u_t) \circ \text{d}B^i_t$. 

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For $f \in C^\infty(D)$, let $a := df \in \Gamma(T^*D)$. Setting $m_t := u_t^{-1}a(X_t)$, we see by Itô’s formula that

$$dm_t = \frac{1}{2}u_t^{-1} (\Box a + \nabla_Z a)(X_t) \, dt$$

where $\Box a = \text{tr} \nabla^2 a$ denotes the so-called connection (or rough) Laplacian on 1-forms and $\equiv$ equality modulo the differential of a local martingale.

Denote by $Q_t: T_xD \to T_xD$ the solution, along the paths of $X_t$, to the covariant ordinary differential equation

$$DQ_t = -\frac{1}{2} (\text{Ric}_D^V)^\sharp Q_t \, dt, \quad Q_0 = \text{id}_{T_xD}, \quad t \leq \tau_D,$$

where $D := u_t du_t^{-1}$ and where by definition

$$(\text{Ric}_D^V)^\sharp v = \text{Ric}_D^V(\cdot, v)^\sharp, \quad v \in T_xD.$$

Thus, condition $\text{Ric}_D^V \geq -K_V$ implies

$$|Q_t v| \leq e^{K_V t/2} |v|, \quad t \leq \tau_D.$$

Finally, note that for any smooth function $f$ on $D$, we have by the Weitzenböck formula:

$$d(\Delta + Z) f = d(-d^*df + (df)Z)$$

$$= \Delta^{(1)} df + \nabla_Z df + \langle \nabla, Z, \nabla f \rangle$$

$$= (\Box + \nabla_Z)(df) - \text{Ric}_D^V(\cdot, \nabla f)$$

$$= (\Box - \text{Ric}_D^V + \nabla_Z)(df)$$

where $\Delta^{(1)}$ denotes the Hodge-deRham Laplacian on 1-forms.

Now let $(\phi, \lambda) \in \text{Eig}(L)$, i.e. $L\phi = -\lambda \phi$, where $L = \Delta + Z$. For $v \in T_xD$, consider the process

$$n_t(v) := (d\phi)(Q_t v).$$

Then

$$n_t(v) = \langle \nabla \phi(X_t), Q_t v \rangle = \langle u_t^{-1}(\nabla \phi)(X_t), u_t^{-1}Q_t v \rangle.$$

Using (3.6), we see by Itô’s formula and formula (3.8) that

$$dn_t(v) = \frac{1}{2} (\Box d\phi + \nabla_Z d\phi)(X_t) IS_0 dt + d\phi(X_t)(DQ_t v) dt = -\frac{\lambda}{2} n_t(v) dt.$$

It follows that

$$e^{\lambda t/2} n_t(v) = e^{\lambda t/2} \langle \nabla \phi(X_t), Q_t v \rangle, \quad t \leq \tau_D,$$

is a martingale, and consequently, for any function $h \in C^1([0, \infty); \mathbb{R})$,

$$h_t e^{\lambda t/2} \langle \nabla \phi(X_t), Q_t v \rangle - \int_0^t h_s e^{\lambda s/2} \langle \nabla \phi(X_s), Q_s v \rangle ds, \quad t \leq \tau_D,$$
is a martingale as well. By the formula
\[ e^{\lambda t/2} \phi(X_t) = \phi(X_0) + \int_0^t e^{\lambda s/2} \langle \nabla \phi(X_s), u_s dB_s \rangle \]
we see then that
\[ N_t(v) := h_t e^{\lambda t/2} \langle \nabla \phi(X_t), Q_tv \rangle - e^{\lambda t/2} \phi(X_t) \int_0^t \langle \dot{h}_s Q_s v, u_s dB_s \rangle, \quad t \leq \tau_D, \]
is a martingale.

Now, for fixed \( t > 0 \), we take \( h \in C^1([0,t]; [0,1]) \) such that \( h_0 = 1 \) and \( h_t = 0 \). Then, by the martingale property of \( \{N_s \wedge \tau_D(v)\}_{s \in [0,t]} \) we obtain
\[ |\nabla \phi(x)| = |N_0(v)| = |E N_{t \wedge \tau_D}(v)| \]
This together with (3.7) yields
\[ |\nabla \phi(x)| \leq e^{(\lambda + K_V)t/2} \|\nabla \phi\|_{\partial D, \infty} + e^{\lambda t/2} \|\phi\|_{\infty} \left( \int_0^t (\dot{h}_s)^2 e^{K_V s} ds \right)^{1/2}. \]
Taking \( h_s = \frac{e^{-K_V s} - e^{-K_V t}}{e^{-K_V t} - 1}, \quad s \in [0, t] \), we obtain (3.2). Finally, noting that
\[ \frac{K_V}{1 - e^{-K_V t}} \leq \frac{K_V^+}{1 - e^{-K_V^+ t}} \leq t^{-1} e^{K_V^+ t}, \]
and taking \( t = (K_V^+ + \lambda)^{-1} \) in (3.2), we prove (3.3).

To estimate the term \( \|\nabla \phi\|_{\partial D, \infty} \), we shall compare \( \phi(x) \) and
\[ \psi(t, x) := \mathbb{P}(\tau_D^x > t), \quad t > 0, \]
for small \( \rho_{\partial D}(x) := \text{dist}(x, \partial D) \). Let \( P_t^D \) be the Dirichlet semigroup generated by \( \frac{1}{2} L \). Then \( \psi(t, x) = P_t^D 1_D(x) \), so that
\[ \partial_t \psi(t, x) = \frac{1}{2} L \psi(t, x) (x), \quad t > 0. \]

**Lemma 3.3.** For any \((\phi, \lambda) \in \text{Eig}(L)\),
\[ \|\nabla \phi\|_{\partial D, \infty} \leq \|\phi\|_{\infty} \inf_{t > 0} e^{\lambda t/2} \|\nabla \psi(t, \cdot)\|_{\partial D, \infty}. \]
Proof. To prove (3.10), we fix \( x \in \partial D \). For small \( \varepsilon > 0 \), let \( x^\varepsilon = \exp_\varepsilon(\varepsilon N) \), where \( N \) is the inward unit normal vector field of \( \partial D \). Since \( \phi|_{\partial D} = 0 \) and \( \psi(t, \cdot)|_{\partial D} = 0 \), we have

\[
(3.11) \quad |\nabla \phi(x)| = |N \phi(x)| = \lim_{\varepsilon \to 0} \frac{|\phi(x^\varepsilon)|}{\varepsilon}, \quad |\nabla \psi(t, \cdot)(x)| = \lim_{\varepsilon \to 0} \frac{|\psi(t, x^\varepsilon)|}{\varepsilon}.
\]

Let \( X_t^\varepsilon \) be the \( L \)-diffusion starting at \( x^\varepsilon \) and \( \tau_D^\varepsilon \) its first hitting time of \( \partial D \). Note that \( N_t := \phi(X_t^\varepsilon) e^{\lambda(t \wedge \tau_D^\varepsilon)/2 \varepsilon} \), \( t \geq 0 \), is a martingale. Thus, for each fixed \( t > 0 \), we can estimate as follows:

\[
|\nabla \phi(x)| = \lim_{\varepsilon \to 0} \frac{|\phi(x^\varepsilon)|}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\mathbb{E}[\psi(t, x^\varepsilon)]}{\varepsilon} = \|\phi\|_\infty e^{\lambda t/2} \lim_{\varepsilon \to 0} \frac{\mathbb{E}[\mathbf{1}_{t < \tau_D^\varepsilon}]}{\varepsilon} \leq \|\phi\|_\infty e^{\lambda t/2} \lim_{\varepsilon \to 0} \frac{\psi(t, x^\varepsilon)}{\varepsilon} = \|\phi\|_\infty e^{\lambda t/2} |\nabla \psi(t, \cdot)(x)|.
\]

Taking the infimum over \( t \) gives the claim.

We now estimate \( \|\nabla \psi(t, \cdot)\|_\infty \). Let \( \text{cut}(D) \) be the cut-locus of \( \partial D \), which is a zero-volume closed subset of \( D \) such that \( \rho_{\partial D} := \text{dist}(\cdot, \partial D) \) is smooth in \( D \setminus \text{cut}(D) \).

**Proposition 3.4.** Let \( \alpha \in \mathbb{R} \) be such that

\[
(3.12) \quad \frac{1}{2} L \rho_{\partial D}(x) \leq \alpha, \quad x \in D \setminus \text{cut}(D).
\]

Then

\[
(3.13) \quad \|\nabla \psi(t, \cdot)\|_{\partial D, \infty} \leq \alpha + \sqrt{\frac{2}{\pi t}} + \int_0^t \frac{1 - e^{-s^2}}{\sqrt{2\pi s^3}} \, ds \leq \sqrt{\frac{2}{\pi t}} + \min \left\{ 2\alpha^+, \, \alpha + \frac{\alpha^2 \sqrt{t}}{2\sqrt{\pi}} \right\}.
\]

**Proof.** Let \( x \in D \) and let \( X_t \) solve SDE (3.4). As shown in [5], \( (\rho_{\partial D}(X_t))_{t \leq \tau_D} \) is a semi-martingale satisfying

\[
(3.14) \quad \rho_{\partial D}(X_t) = \rho_{\partial D}(x) + b_t + \frac{1}{2} \int_0^t L \rho_{\partial D}(X_s) \, ds - l_t, \quad t \leq \tau_D,
\]

where \( b_t \) is a real-valued Brownian motion starting at 0, and \( l_t \) a non-decreasing process which increases only when \( X_t^\varepsilon \in \text{cut}(D) \). Hence, setting \( \varepsilon = \rho_{\partial D}(x) \), we deduce from (3.12) and (3.14) that

\[
(3.15) \quad \rho_{\partial D}(X_t(x)) \leq Y_t^\alpha(\varepsilon) := \varepsilon + b_t + \alpha t, \quad t \leq \tau_D.
\]

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Consequently, letting $T^\alpha(\epsilon)$ be the first hitting time of 0 by $Y^\alpha_t(\epsilon)$, we obtain

$$
(3.16) \quad \psi(t, x) \leq P(t < T^\alpha(\epsilon)).
$$

On the other hand, since $\psi(t, \cdot)$ vanishes on the boundary and is positive in $D$, we have for all $y \in \partial D$

$$
(3.17) \quad |\nabla \psi(t, y)| = \lim_{x \in D, x \to y} \frac{\psi(t, x)}{\rho^\partial_D(x)}.
$$

Hence, by (3.16), to prove the first inequality in (3.13) it is enough to establish that

$$
(3.18) \quad \limsup_{\epsilon \downarrow 0} \frac{P(t < T^\alpha(\epsilon))}{\epsilon} \leq \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - e^{-\frac{\alpha^2 s}{2}}}{\sqrt{2\pi s^3}} ds.
$$

It is well known that the (sub-probability) density $f_{\alpha, \epsilon}$ of $T^\alpha(\epsilon)$ is

$$
(3.19) \quad f_{\alpha, \epsilon}(s) = \frac{\epsilon \exp\left(-\frac{(\epsilon + \alpha s)^2}{2s}\right)}{\sqrt{2\pi s^3}},
$$

which can be obtained by the reflection principle for $\alpha = 0$ and the Girsanov transform for $\alpha \in \mathbb{R}$. Thus

$$
(3.20) \quad P(t \geq T^\alpha(\epsilon)) = \epsilon \int_0^t \frac{\exp\left(-\frac{(\epsilon + \alpha s)^2}{2s}\right)}{\sqrt{2\pi s^3}} ds
$$

$$
= \epsilon \exp(-\alpha \epsilon) \int_0^t \frac{e^{-\alpha^2 r}}{\sqrt{2\pi r^3}} \exp\left(-\frac{\epsilon^2}{2s}\right) ds
$$

$$
= \exp(-\alpha \epsilon) \int_0^{2t/\epsilon^2} \frac{e^{-1/r}}{\sqrt{\pi r^3}} \exp\left(-\frac{\alpha^2 \epsilon^2 r}{4}\right) dr,
$$

where we have made the change of variable $r = 2s/\epsilon^2$. With the change of variable $v = 1/r$ we easily check that

$$
(3.21) \quad \int_0^{\infty} r^{-3/2} e^{-1/r} dr = \Gamma(1/2) = \sqrt{\pi},
$$

and this allows to write

$$
(3.22) \quad P(t \geq T^\alpha(\epsilon)) = \exp(-\alpha \epsilon) \left(1 - \int_{2t/\epsilon^2}^{\infty} \frac{e^{-1/r}}{\sqrt{\pi r^3}} dr - \int_0^{2t/\epsilon^2} \frac{e^{-1/r}}{\sqrt{\pi r^3}} \left(1 - e^{-\frac{\alpha^2 \epsilon^2 r}{4}}\right) dr\right).
$$

As $\epsilon \to 0$,

$$
\int_{2t/\epsilon^2}^{\infty} \frac{e^{-1/r}}{\sqrt{\pi r^3}} dr = \int_{2t/\epsilon^2}^{\infty} \frac{1}{\sqrt{\pi r^3}} dr + o(\epsilon) = \frac{\epsilon \sqrt{2}}{\sqrt{t}} + o(\epsilon),
$$

and

$$
\int_0^{2t/\epsilon^2} \frac{e^{-1/r}}{\sqrt{\pi r^3}} \left(1 - e^{-\frac{\alpha^2 \epsilon^2 r}{4}}\right) dr = \int_0^{2t/\epsilon^2} \frac{1}{\sqrt{\pi r^3}} dr + o(\epsilon) = \frac{\epsilon \sqrt{2}}{\sqrt{t}} + o(\epsilon),
$$

where we have used the fact that $\int_0^{\infty} \frac{1}{\sqrt{\pi r^3}} dr = \frac{\sqrt{\pi}}{\sqrt{t}}$. Thus
and with change of variable $s = \frac{1}{2} \varepsilon^2 r$

$$
\int_0^{2t/\varepsilon^2} \frac{e^{-1/r}}{\sqrt{\pi r^3}} \left( 1 - e^{-\frac{\alpha^2 s^2}{2r}} \right) \, dr = \varepsilon \int_0^t \frac{e^{-\frac{s^2}{2r}}}{\sqrt{2\pi s^3}} \left( 1 - e^{-\frac{\alpha^2 s^2}{2r}} \right) \, ds
\begin{align*}
&= \varepsilon \int_0^t \frac{1 - e^{-\frac{\alpha^2 s^2}{2r}}}{\sqrt{2\pi s^3}} \, ds + o(\varepsilon)
\end{align*}
$$

by monotone convergence. Combining these with $e^{-\alpha \varepsilon} = 1 - \alpha \varepsilon + o(\varepsilon)$, we deduce from (3.22) that

$$
\mathbb{P}(t \geq T^\alpha(\varepsilon)) = 1 - \varepsilon \left( \alpha + \sqrt{\frac{2}{\pi t}} + \int_0^t \frac{1 - e^{-\frac{\alpha^2 s^2}{2r}}}{\sqrt{2\pi s^3}} \, ds \right) + o(\varepsilon)
$$

which yields (3.18).

Obviously, the inequality $1 - e^{-s} \leq s$ for $s \geq 0$ implies

$$
\int_0^t \frac{1 - e^{-\frac{\alpha^2 s^2}{2r}}}{\sqrt{2\pi s^3}} \, ds \leq \int_0^t \frac{\alpha^2}{2\sqrt{2\pi s}} \, ds = \frac{\alpha^2 \sqrt{t}}{\sqrt{2\pi}}.
$$

Moreover, we will show that

$$
\int_0^t \frac{1 - e^{-\frac{\alpha^2 s^2}{2r}}}{\sqrt{2\pi s^3}} \, ds = |\alpha| - \frac{\sqrt{2}}{\sqrt{\pi t}} \int_0^{|\alpha|\sqrt{t}} \, dr \int_r^\infty e^{-\frac{s^2}{2}} \, ds \leq |\alpha|,
$$

which then together with (3.24) gives the second inequality in (3.13).

Thus, to finish the proof, it remains to establish (3.25). Indeed, noting that

$$
\sqrt{\frac{2}{\pi t}} \int_0^{|\alpha|\sqrt{t}} \left( \int_0^r e^{-s^2/2} \, ds \right) \, dr = \int_0^t \frac{1 - e^{-\frac{\alpha^2 s^2}{2r}}}{\sqrt{2\pi s^3}} \, ds.
$$

Two changes of variables give

$$
\sqrt{\frac{2}{\pi t}} \int_0^{|\alpha|\sqrt{t}} \left( \int_0^r e^{-s^2/2} \, ds \right) \, dr = \frac{\alpha^2}{2\sqrt{2\pi t}} \int_0^t \frac{1}{\sqrt{u}} \left( \int_0^u \frac{1}{\sqrt{v}} e^{-\frac{\alpha^2 v}{2}} \, dv \right) \, du.
$$

A first integration by parts choosing $2(\sqrt{u} - \sqrt{t})$ as primitive of $1/\sqrt{u}$ yields

$$
\frac{\alpha^2}{2\sqrt{2\pi t}} \int_0^t \frac{1}{\sqrt{u}} \left( \int_0^u \frac{1}{\sqrt{v}} e^{-\frac{\alpha^2 v}{2}} \, dv \right) \, du = \frac{\alpha^2}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{u}} e^{-\frac{\alpha^2 u}{2}} \, du - \frac{\alpha^2}{\sqrt{2\pi t}} \int_0^t e^{-\frac{\alpha^2 u}{2}} \, du
\begin{align*}
&= \frac{\alpha^2}{\sqrt{2\pi}} \int_0^t \frac{1}{\sqrt{u}} e^{-\frac{\alpha^2 u}{2}} \, du - \frac{\sqrt{2}}{\sqrt{\pi t}} \left( 1 - e^{-\frac{\alpha^2 u}{2}} \right) + o(\varepsilon)
\end{align*}
$$

A second integration by parts shows that the right hand side is equal to $\int_0^t \frac{1 - e^{-\frac{\alpha^2 u}{2}}}{\sqrt{2\pi s^3}} \, ds$. Therefore, (3.26) holds and hence (3.25) as well. \qed
Remark 3.1. We proved that
\[
\| \nabla \psi(t, \cdot) \|_{\partial D, \infty} \leq \alpha + \frac{\sqrt{2}}{\sqrt{\pi t}} + \int_0^t \frac{1 - e^{-\frac{s^2}{2t}}}{\sqrt{2\pi s^3}} \, ds
\]
(3.27)
\[
= 2 \max(0, \alpha) + \sqrt{\frac{2}{\pi t}} \int_0^\infty \left( \int_r^\infty e^{-s^2/2} \, ds \right) \, dr
\]
where the last equality follows from (3.25) and the observation that \( \int_0^\infty \int_0^\infty e^{-s^2/2} \, ds \, dr = 1 \).

This implies that for all \( a > 1 \)
\[
\| \nabla \psi(t, \cdot) \|_{\partial D, \infty} \leq 2 \max(0, \alpha) + \sqrt{\frac{2}{\pi t}} e^{-\frac{a^2 t}{2t}} \int_0^\infty \left( \int_0^\infty e^{-\frac{s^2 (a-1)}{4a}} \, ds \right) \, dr
\]
\[
\leq 2 \max(0, \alpha) + \sqrt{\frac{2}{\pi t}} \frac{a}{a - 1} e^{-\frac{a^2 t}{2t}}.
\]

In the case when \( 4 < t \alpha^2 \), taking \( a = \frac{\alpha^2 t}{\alpha^2 t - 4} \) yields
\[
\| \nabla \psi(t, \cdot) \|_{\partial D, \infty} \leq 2 \max(0, \alpha) + \frac{e^2 \alpha^2 t}{4} \sqrt{\frac{2}{\pi t}} e^{-\frac{\alpha^2 t}{2t}}
\]
and in particular
\[
\| \nabla \psi(1/\lambda, \cdot) \|_{\partial D, \infty} \leq 2 \max(0, \alpha) + \frac{e^2 \alpha^2}{4\lambda} \sqrt{2\pi} e^{-\frac{\alpha^2}{2\lambda}}.
\]

Combined with the Lemmas 3.2 and 3.3, this gives estimate (1.3) of Remark 1.1.

Finally, to estimate the constant \( \alpha \) in (3.12), we shall use the Laplacian comparison theorem to bound \( \Delta \rho_{\partial D} \) from above. See [4, 9] for the corresponding lower bound estimate.

Theorem 3.5. Let \( \theta, k \in \mathbb{R} \) be such that \( I_{\partial D} \geq -\theta \) and \( \text{Ric}_D \geq -(d - 1)k \). For \( t \geq 0 \) let
\[
h(t) = \begin{cases} 
\cos(\sqrt{-k} t) + \frac{\theta}{\sqrt{-k}} \sin(\sqrt{-k} t), & \text{if } k < 0, \\
1 + \theta t, & \text{if } k = 0, \\
\cosh(\sqrt{k} t) + \frac{\theta}{\sqrt{k}} \sinh(\sqrt{k} t), & \text{if } k > 0.
\end{cases}
\]

Let \( h^{-1}(0) \) be the first zero of \( h \) (where \( h^{-1}(0) := \infty \) if \( h(t) > 0 \) for all \( t \geq 0 \)). Then for any \( x \in D \setminus \text{cut}(D) \) such that \( \rho_{\partial D}(x) < h^{-1}(0) \), there holds
\[
\Delta \rho_{\partial D}(x) \leq (d - 1) \frac{h'}{h}(\rho_{\partial D}(x)).
\]
(3.28)

In particular, if \( \theta, k \geq 0 \) we have
\[
\Delta \rho_{\partial D}(x) \leq (d - 1)(\theta + \sqrt{k}), \quad x \in D \setminus \text{cut}(D).
\]
(3.29)
Proof. The proof of (3.28) is adapted from [10, Theorem 1.2.2] where the corresponding Hessian upper bound is presented. For fixed \( x \in D \setminus \text{cut}(D) \), let \( p \) be the orthogonal projection of \( x \) on \( \partial M \), which is the unique point on \( \partial D \) such that \( \text{dist}(x, p) = \rho := \rho_{\partial D}(x) \). Then

\[
\gamma(s) := \exp_s(pN), \quad s \in [0, \rho],
\]
is the minimal geodesic in \( D \) linking \( p \) and \( x \). Let \( X_0(0) = N(p) \), and \( \{X_i(0)\}_{1 \leq i \leq d-1} \) be an orthonormal basis of \( T_p \partial D \). For \( 0 \leq i \leq d-1 \), let

\[
X_i(s) = /_{p \to \gamma(s)} X_i(0), \quad s \in [0, \rho],
\]
be the parallel transport of \( X_i(0) \) along the geodesic \( \gamma \). Moreover, for any \( 1 \leq i \leq d-1 \), let \( \{J_i(s)\}_{s \in [0, \rho]} \) be the Jacobi field along \( \gamma \) such that \( J(\rho) = X_i(\rho) \) and

\[
\langle J_i(0), U \rangle = - \|D(J_i(0), U)\|, \quad U \in T_p \partial D.
\]

By the second variational formula (see e.g. page 321 in [2]), we have

\[
\Delta \rho_{\partial D}(x) = \sum_{i=1}^{d-1} \text{Hess}_{\rho_{\partial D}}(X_i, X_i)(\rho_{\partial D}(x)) = - \sum_{i=1}^{d-1} \|D(J_i(0), J_i(0))\|
\]

\[
+ \sum_{i=0}^{d-1} \int_0^{\rho_{\partial D}(x)} \left( |\dot{J}_i(s)|^2 - \langle \mathcal{R}(X_0(s), J_i(s))X_0(s), J_i(s) \rangle \right) ds
\]

where \( \mathcal{R} \) is the curvature tensor. Define

\[
\ddot{J}_i(s) = \frac{h(s)}{h(\rho_{\partial D}(x))} X_i(s), \quad s \in [0, \rho_{\partial D}(x)], \quad 0 \leq i \leq d - 1.
\]

Then \( \ddot{J}_i(\rho_{\partial D}(x)) = J_i(\rho_{\partial D}(x)) = X_i \), and by \( \|D \| \geq -\theta \),

\[
\langle \dot{J}_i(0), \ddot{J}_i(0) \rangle = \frac{\theta}{h(\rho_{\partial D}(x))^2} \geq - \|D(\ddot{J}_i(0), \ddot{J}_i(0))\|, \quad 1 \leq i \leq d - 1.
\]

Hence, by the index lemma (see the first displayed formula on page 322 in [2]), and using the lower bound conditions on \( \|D \| \) and \( \text{Ric}_D \), we deduce from (3.30) that

\[
\Delta \rho_{\partial D}(x) \leq - \sum_{i=1}^{d-1} \|D(\ddot{J}_i(0), \ddot{J}_i(0))\|
\]

\[
+ \sum_{i=0}^{d-1} \int_0^{\rho_{\partial D}(x)} \left( |\dot{J}_i(s)|^2 - \langle \mathcal{R}(X_0(s), \ddot{J}_i(s))X_0(s), \ddot{J}_i(s) \rangle \right) ds
\]

\[
\leq \frac{1}{h(\rho_{\partial D}(x))^2} \left( - (\text{tr} \|D\|)(p) + \int_0^{\rho_{\partial D}(x)} \left\{ h(s)^2 - h(s) \text{Ric}(X_0(s), X_0(s)) \right\} ds \right)
\]

\[
\leq \frac{d-1}{h(\rho_{\partial D}(x))^2} \left( - \theta + \int_0^{\rho_{\partial D}(x)} \left\{ h'(s)^2 + kh(s)^2 \right\} ds \right)
\]

\[
= (d-1)\frac{h'}{h}(\rho_{\partial D}(x)),
\]
where in the last step we used the facts that \((hh')(0) = \theta\) and \(h'' = kh\), and the latter implies
\[(h')^2 + kh^2 = (hh')' - hh'' + kh^2 = (hh')'.\]
Thus (3.28) holds. When \(\theta, k \geq 0\), we have \(h^{-1}(0) = \infty\) and
\[
\frac{h'(t)}{h(t)} = \frac{\sqrt{k \sinh(\sqrt{kt}) + \theta \cosh(\sqrt{kt})}}{\cosh(\sqrt{kt}) + \frac{\theta}{\sqrt{k}} \sinh(\sqrt{kt})} \leq \frac{\sqrt{k \cosh(\sqrt{kt}) + \theta \cosh(\sqrt{kt})}}{\cosh(\sqrt{kt})} = \sqrt{k} + \theta.
\]
Then (3.29) follows from (3.28).

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Theorem 3.5 with \(k = \frac{K\lambda}{d-1}\), condition (3.12) holds for \(\alpha\) as given in (3.1). Applying Lemmas 3.2, 3.3 and Proposition 3.4 with \(t = s = \frac{1}{\lambda}\), we obtain
\[
\|\nabla \phi\|_\infty \leq e^{1/2} \left( \sqrt{\lambda + K\lambda} \|\phi\|_\infty + \|\nabla \phi\|_{\partial D,\infty} \right) \\
\leq \|\phi\|_\infty \left\{ \sqrt{e(\lambda + K\lambda)} + e \left( \alpha + \frac{\sqrt{2\lambda}}{\sqrt{\pi}} + \alpha \wedge \frac{\alpha^2}{\sqrt{2\pi\lambda}} \right) \right\}.
\]
The result follows by substitution.

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