COHOMOLOGY WITH LOCAL COEFFICIENTS OF SOLVMANIFOLDS AND MORSE-NOVIKOV THEORY

DMITRI V. MILLIONSCHIKOV

Abstract. We study the cohomology $H^\lambda_\omega(G/\Gamma, \mathbb{C})$ of the deRham complex $\Lambda^*(G/\Gamma) \otimes \mathbb{C}$ of a compact solvmanifold $G/\Gamma$ with a deformed differential $d_\lambda\omega = d + \lambda\omega$, where $\omega$ is a closed 1-form. This cohomology naturally arises in the Morse-Novikov theory. We show that for a solvable Lie group $G$ with a completely solvable Lie algebra $g$ and a cocompact lattice $\Gamma \subset G$ the cohomology $H^\lambda_\omega(G/\Gamma, \mathbb{C})$ coincides with the cohomology $H^\lambda_\omega(g)$ of the Lie algebra associated with the one-dimensional representation $\rho_{\lambda\omega} : g \to \mathbb{R}, \rho_{\lambda\omega}(\xi) = \lambda\omega(\xi)$. Moreover $H^\lambda_\omega(G/\Gamma, \mathbb{C})$ is non-trivial if and only if $-\lambda [\omega]$ belongs to the finite subset $\{0\} \cup \tilde{\Omega}_g$ in $H^1(G/\Gamma, \mathbb{C})$ well defined in terms of $g$.

Introduction

In the beginning of the 80-th S.P. Novikov constructed ([N1], [N2]) an analogue of the Morse theory for smooth closed 1-forms on a compact smooth manifold $M$. In particular he introduced the Morse-type inequalities (Novikov’s inequalities) for numbers $m_p(\omega)$ of zeros of index $p$ of a closed 1-form $\omega$ on $M$. A lot of papers was devoted to this problem in the following years (see [N3] for references). In [Pa] a method of obtaining the torsion-free Novikov inequalities in terms of the deRham complex of manifold was proposed. This method was based on Witten’s approach [W] to the Morse theory. A. Pazhitnov obtained some important results in this direction in [Pa]. The cohomology of the deRham complex $\Lambda^*(M)$ with the deformed differential $d + \lambda\omega$ coincides with the cohomology $H^\lambda_{\rho_\omega}(M, \mathbb{C})$ with coefficients in the local system $\rho_{\lambda\omega}$ of groups $\mathbb{C}$, $\rho_{\lambda\omega}(\gamma) = \exp \int_{\gamma} \lambda\omega$ and for sufficiently large real numbers $\lambda$ we have the following estimate (see [Pa]):

$$m_p(\omega) \geq \dim H^p_{\rho_{\lambda\omega}}(M, \mathbb{C}), \forall p.$$  

L. Alania in [Al] studied $H^\lambda_{\rho_{\omega}}(M_n, \mathbb{C})$ of a class of nilmanifolds $M_n$. He proved that $H^\lambda_{\rho_{\omega}}(M_n, \mathbb{C})$ is trivial if $\lambda\omega \neq 0$. The partial answer for the case $\lambda\omega = 0$ was obtained by the present author in [Mill]. In both cases the proof was based on the Nomizu theorem [N2] and the computations were made in terms of the corresponding nilpotent Lie algebra $\mathbb{N}_n$. The starting point of this article was the intention to improve the results of [Al] in more general situation considering solvmanifolds and to find examples of manifolds $M$ with non-trivial $H^p_{\rho_{\omega}}(M, \mathbb{C}), \lambda\omega \neq 0$. One of the first observations that was made in this direction: for a nilmanifold $G/\Gamma$ the cohomology $H^\lambda_{\rho_{\omega}}(G/\Gamma, \mathbb{C})$ coincides with the cohomology $H^\lambda_{\rho_{\omega}}(g)$ associated with the

1991 Mathematics Subject Classification. 58A12, 17B30, 17B56 (Primary) 57T15 (Secondary).

Key words and phrases. Solvmanifolds, nilmanifolds, cohomology, local system, Morse-Novikov theory, solvable Lie algebras.

Partially supported by the Russian Foundation for Fundamental Research, grant no. 99-01-00090 and PAI-RUSSIE, dossier no. 04495UL.

1
one-dimensional representation of the Lie algebra \(\rho_{\lambda\omega} : g \to \mathbb{C}, \rho_{\lambda\omega}(\xi) = \lambda\omega(\xi)\) and hence \(H^*_{\lambda\omega}(g) = 0\) by Dixmier’s theorem [3] (Corollary 2.3).

Applying Hattori’s theorem [H] one can observe that the isomorphism \(H^*_{\lambda\omega}(G/\Gamma, \mathbb{C}) \cong H^*_{\lambda\omega}(g)\) still holds on for compact solvmanifolds \(G/\Gamma\) with completely solvable Lie group \(G\). A kind of minimal model of a solvable Lie algebra \(g\), the free \(d\)-algebra \((\Lambda^*(\omega_1, \ldots, \omega_n), d)\) with differential \(d\)

\[
d\omega_i = 0, \ i = 1, \ldots, k; \quad d\omega_j = \alpha_j \wedge \omega_j + P_j(\omega_1, \ldots, \omega_{j-1}), \ j = k+1, \ldots, n.
\]

is considered (Lemma 3.2). By means of \((\Lambda^*(\omega_1, \ldots, \omega_n), d)\) a spectral sequence \(E_r\) that converges to the \(H^*_{\lambda\omega}(g)\) is constructed. \(E_r\) degenerates at the first term \(E_1\) if \(-\lambda\omega\) doesn’t belong to the finite set \(\Omega_g \subset H^1(g)\). \(\Omega_g\) is defined by the collection \(\{\alpha_{k+1}, \ldots, \alpha_n\}\) of the closed 1-forms that have invariant sense as the weights of completely reducible representation associated to the restriction \(\text{ad}|_{[g, g]}\) of adjoint representation \(\text{ad} : g \to g\) (Theorem 3.3).

The main result of this article (Theorem 4.13): the cohomology with local coefficients \(H^*_{\lambda\omega}(G/\Gamma, \mathbb{C})\) of a compact solvmanifold \(G/\Gamma\), where \(G\) is completely solvable Lie group is non-trivial if and only if \(-\lambda[\omega] \in \tilde{\Omega}_g\), where \(\tilde{\Omega}_g\) is the finite subset in \(H^1(G/\Gamma, \mathbb{C})\) well defined in terms of \(g\).

The author is grateful to L. Alania for helpful discussions and attention to this work.

1. Deformed deRham complex and Morse-Novikov theory

Let us consider a closed compact \(C^\infty\)-manifold \(M\) and its deRham complex \((\Lambda^*(M), d)\) of differential forms. Let \(\omega\) be a closed 1-form on \(M\) and \(\lambda \in \mathbb{R}\). Now one can define a new algebraic complex \((\Lambda^*(M), d_{\lambda\omega})\) with a deformed differential

\[
d_{\lambda\omega} = d + \lambda\omega : \Lambda^*(M) \to \Lambda^*(M)
\]
i.e. for any form \(a \in \Lambda^*(M)\):

\[
d_{\lambda\omega}(a) = da + \lambda\omega \wedge a.
\]

Now taking \(\lambda \in \mathbb{C}\) and considering the complexification \(\Lambda^*(M) \otimes \mathbb{C}\) we come to the following important

**Lemma 1.1** (N3, [Pa]). 1) For a closed 1-form \(d\omega = 0\) the cohomology \(H^*_{\lambda\omega}(M, \mathbb{C})\) of the algebraic complex \((\Lambda^*(M) \otimes \mathbb{C}, d_{\lambda\omega})\) coincides with the cohomology \(H^*_{\rho_{\lambda\omega}}(M, \mathbb{C})\) with coefficients in local system of groups \(\mathbb{C}\) defined by the representation \(\rho_{\lambda\omega} : \pi_1(M) \to \mathbb{C}^*\) of fundamental group defined by the formula

\[
\rho_{\lambda\omega}(\gamma) = \exp \int_\gamma \lambda\omega, \quad \gamma \in \pi_1(M),
\]

2) For any pair \(\omega, \omega'\) of 1-forms such that \(\omega - \omega' = d\phi, \phi \in C^\infty(M)\) the cohomology \(H^*_{\lambda\omega}(M, \mathbb{C})\) and \(H^*_{\lambda\omega'}(M, \mathbb{C})\) are isomorphic to each other. This isomorphism can be given by the gauge transformation

\[
a \to e^{\lambda\phi}a; \quad d \to e^{\lambda\phi}de^{-\lambda\phi} = d + \lambda d\phi \wedge
\]

We denote corresponding Betti numbers by \(b_p(\lambda, \omega)\), where \(b_p(\lambda, \omega) = \dim H^*_{\rho_{\lambda\omega}}(M, \mathbb{C})\).

**Remark.** The representation \(\rho_{\lambda\omega} : \pi_1(M) \to \mathbb{C}^*\) defines a local system of groups \(\mathbb{C}\) on the manifold \(M\) in the sense of Steenrod (see [R] for details).
The cohomology $H^*(M, \mathbb{C})$ naturally arises in the Morse-Novikov theory: we assume now that $\omega$ is a Morse 1-form, i.e., in a neighbourhood of any point $\omega = df$, where $f$ is a Morse function. The zeros of $\omega$ are isolated, and one can define the index of each zero. The number of zeros of $\omega$ of index $p$ is denoted by $m_p(\omega)$.

**Theorem 1.2** (A. Pazhitnov, [Pa]). For sufficiently large real numbers $\lambda$, \[ m_p(\omega) \geq b_p(\lambda, \omega) \forall p. \]

**Theorem 1.3** (A. Pazhitnov, [Pa]). Assume that all the periods of the form $\omega$ are commensurable. If $Re\lambda$ is sufficiently large, then $b_p(\lambda, \omega) = b_p(\omega)$, where $b_p(\omega)$ is a Novikov number.

2. Dixmier’s exact sequence of Lie algebra cohomology

Let $\mathfrak{g}$ be a $n$-dimensional Lie algebra. The dual of the Lie bracket $[,] : \Lambda^2(\mathfrak{g}) \to \mathfrak{g}$ gives a linear mapping $d_1 : \mathfrak{g}^* \to \Lambda^2(\mathfrak{g}^*)$ which extends in a standard way to a differential $d$ of a cochain complex of the Lie algebra $\mathfrak{g}$:

$$
\mathbb{K} \xrightarrow{d_0 = 0} \mathfrak{g}^* \xrightarrow{d_1} \Lambda^2(\mathfrak{g}^*) \xrightarrow{d_2} \Lambda^3(\mathfrak{g}^*) \xrightarrow{d_3} \ldots
$$

$$
d(\rho \wedge \eta) = d\rho \wedge \eta + (-1)^{deg \rho} \rho \wedge d\eta, \forall \rho, \eta \in \Lambda^*(\mathfrak{g}^*).
$$

Vanishing of the $d^2$ corresponds to the Jacobi identity.

For $d : \Lambda^q(\mathfrak{g}^*) \to \Lambda^{q+1}(\mathfrak{g}^*)$ and $f \in \Lambda^q(\mathfrak{g}^*)$ the following formula holds on:

$$
df(X_1, \ldots, X_{q+1}) = \sum_{1 \leq i < j \leq q+1} (-1)^{i+j-1} f([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{q+1}).
$$

A cohomology of this complex is called the cohomology (with trivial coefficients) of the Lie algebra $\mathfrak{g}$ and is denoted by $H^*(\mathfrak{g})$.

From the definition it follows that $H^1(\mathfrak{g})$ is the dual space to $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ and so

1) $b^1(\mathfrak{g}) = \dim H^1(\mathfrak{g}) \geq 2$ for a nilpotent Lie algebra $\mathfrak{g}$,

2) $b^1(\mathfrak{g}) \geq 1$ for a solvable Lie algebra $\mathfrak{g}$,

3) $b^1(\mathfrak{g}) = 0$ for a semi-simple Lie algebra $\mathfrak{g}$.

Now we take a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ with a non-trivial $H^1(\mathfrak{g})$. Let $\omega \in \mathfrak{g}^*, \omega \neq 0, d\omega = 0$ and $\lambda \in \mathbb{K}$. One can define

1) a new deformed differential $d_{\lambda \omega}$ in $\Lambda^*(\mathfrak{g}^*)$ by the formula

$$
d_{\lambda \omega}(a) = da + \lambda \omega \wedge a.
$$

2) an one-dimensional representation

$$
\rho_{\lambda \omega} : \mathfrak{g} \to \mathbb{K}, \rho_{\lambda \omega}(\xi) = \lambda \omega(\xi), \xi \in \mathfrak{g}.
$$

Now we recall the definition of the Lie algebra cohomology associated with a representation. Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ its linear representation. We denote by $C^0(\mathfrak{g}, V)$ the space of $\mathfrak{g}$-linear alternating mappings of $\mathfrak{g}$ into $V$. Then one can consider an algebraic complex:

$$
V = C^0(\mathfrak{g}, V) \xrightarrow{d_0} C^1(\mathfrak{g}, V) \xrightarrow{d_1} C^2(\mathfrak{g}, V) \xrightarrow{d_2} \ldots
$$

where the differential $d_q$ is defined by:


\[ (d_q f)(X_1, \ldots, X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} \rho(X_i) f(X_1, \ldots, \hat{X}_i, \ldots, X_{q+1}) + \sum_{1 \leq i < j \leq q+1} (-1)^{i+j-1} f([X_i, X_j], X_1, \ldots, \hat{X}_i, \hat{X}_j, \ldots, X_{q+1}). \]

The cohomology of the complex \((C^*(g, V), d)\) is called the cohomology of the Lie algebra \(g\) associated to the representation \(\rho : g \to V\).

**Proposition 2.1.** Let \(g\) be a Lie algebra and \(\omega\) is a closed 1-form. Then the complex \((\Lambda^*(g^*), d_{\lambda\omega})\) coincides with the cochain complex of the Lie algebra \(g\) associated with one-dimensional representation \(\rho_{\lambda\omega} : g \to \mathbb{K}\), where \(\rho_{\lambda\omega}(\xi) = \lambda\omega(\xi), \xi \in g\).

The proof follows from

\[ (\lambda \omega \wedge a)(X_1, \ldots, X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} \lambda\omega(X_i)(a(X_1, \ldots, \hat{X}_i, \ldots, X_{q+1})). \]

One can deduce that \(H_{\lambda\omega}^0(g) = 0\) for a non-trivial \(\lambda\omega\), as well as \(H_{\lambda\omega}^0(g) = 0\) for an unimodular \(n\)-dimensional Lie algebra \(g\).

**Remark.** The cohomology \(H_{\lambda\omega}^*\) coincides with the Lie algebra cohomology \(H^*(g)\) with trivial coefficients if \(\lambda = 0\). If \(\lambda \neq 0\) the deformed differential \(d_{\lambda\omega}\) is not compatible with the exterior product \(\wedge\) in \(\Lambda^*(g)\)

\[ d_{\lambda\omega}(a \wedge b) = d(a \wedge b) + \lambda\omega \wedge a \wedge b \neq d_{\lambda\omega}(a) \wedge b + (-1)^{\text{deg}a} a \wedge d_{\lambda\omega}(b) \]

and the cohomology \(H_{\lambda\omega}^*(g)\) has no natural multiplicative structure and therefore no Poincare duality in the case of unimodular Lie algebra \(g\). The corresponding Euler characteristic \(\chi_{\lambda\omega}(g)\) is still equal to zero.

Let \(\omega \in g^*, \omega \neq 0, d\omega = 0\). Then \(b_\omega = \{x \in g, \omega(x) = 0\}\) is an ideal of codimension 1 in \(g\). One can choose an element \(X \in g, \omega(X) = 1\).

**Theorem 2.2** (Dixmier [3]). There exists a long exact sequence of Lie algebra cohomology:

\[ \cdots \to adX_i^* + \lambda Id \to H_{\lambda\omega}^i(b_\omega) \xrightarrow{\omega \wedge} H_{\lambda\omega}^{i+1}(g) \xrightarrow{r_i} H^i(b_\omega) \xrightarrow{dX_i^* + \lambda Id} H^{i+1}(b_\omega) \to \cdots \]

where

1) the homomorphism \(r_i : H^i(g) \to H^i(b)\) is the restriction homomorphism;
2) \(\omega \wedge : H^{*+1}(b_\omega) \to H^{*}(g)\) is induced by the multiplication \(\omega \wedge : \Lambda^{*+1}(b_\omega) \to \Lambda^*(g^*)\);
3) the homomorphisms \(adX_i^* : H^i(b_\omega) \to H^i(b_\omega)\) are induced by the derivation \(adX_i^*\) of degree zero of \(\Lambda^*(b_\omega^*)\) \((adX^*a \wedge b + a \wedge adX^*b, \forall a, b \in \Lambda^*(b_\omega^*))\) that continues a dual mapping \(adX^* : b_\omega^* \to b_\omega^*\) to the \(adX\) : \(b_\omega \to b_\omega\) operator. The derivation \(adX^*\) commutes with \(d\) and corresponding mapping in \(H^*(b_\omega)\) we denote by the same symbol \(adX^*\). \(Id\) is the identity operator.

Each form \(f \in \Lambda^*(g)\) can be decomposed as \(f = \omega \wedge f' + f''\), where \(f' \in \Lambda^{*+1}(b_\omega^*)\) and \(f'' \in \Lambda^*(b_\omega^*)\). And one can write out a short exact sequence of algebraic complexes

\[ 0 \to \Lambda^{*+1}(b_\omega^*) \xrightarrow{\omega \wedge} \Lambda^*(g^*) \to \Lambda^*(b_\omega^*) \to 0 \]
where $\Lambda^*(b_\omega^*)$ has the standard differential $d$, $\Lambda^*(g^*)$ has the deformed differential $d_{\omega\wedge}$ and $\Lambda^{*−1}(b_\omega^*)$ is taken with the differential $−d$ as $d(\omega\wedge c) = d_{\omega\wedge}(\omega\wedge c) = −\omega\wedge dc$.

The short exact sequence of algebraic complexes gives us the long exact sequence of cohomology. Everything is clear with the homomorphisms $d_*$. The following formula holds:

(4) $(df)_X(X_1, \ldots, X_{q+1}) = \sum_{1 \leq i \leq q+1} (-1)^i f(adX(X_j), X_1, \ldots, \hat{X}_j, \ldots, X_{q+1}) + \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} f([X_i, X_j], X_1, \ldots, \hat{X}_j, \ldots, X_{q+1}) = (adX^*_g(f) + d(fX))(X_1, \ldots, X_{q+1})$.

Hence the homomorphism $H^q(b_\omega) \to H^q(b_\omega)$, $[f] \to [f' + \lambda f]$ of long exact sequence in cohomology coincides with homorphism induced by $adX^* + \lambda Id$.

**Corollary 2.3.** Let $g$ be an $n$-dimensional Lie algebra and $\omega \in g^*$, $\omega \neq 0$, $d\omega = 0$ and $Spec^k(\omega)$ be the set of eigenvalues of operator $adX^*_k : H^k(b_\omega) \to H^k(b_\omega)$, then

1) the cohomology $H^*_\lambda(\omega)(g)$ is non-trivial if and only if

$−\lambda \in \cup_{k=1}^n Spec^k(\omega)$;

2) the $i$-th Betti number $b_i^\lambda(\omega)(g) = dimH^i(\lambda_\omega)(g)$ can be expressed in a following way:

(5) $b_i^\lambda(\omega)(g) = k_i^\lambda + k_i^{\lambda−1}$,

where by $k_i^\lambda$, we denote the dimension of the kernel of $adX^*_i + \lambda Id : H^i(b_\omega) \to H^i(b_\omega)$.

**Example 2.4.** Let $g$ be a Lie algebra defined by the basis $X, e_1, e_2, \ldots, e_n$ and commutating relations (trivial ones are omitted):

$[X, e_1] = e_1, [X, e_2] = e_2, \ldots, [X, e_n] = e_n$.

Thus $g$ is a semidirect sum of $K$ and abelian $K^n$ defined by the operator $adX$ with the identity matrix $E$ in the basis $e_1, e_2, \ldots, e_n$ of $K^n$. We take $\omega, \omega_1, \ldots, \omega_n$ as the corresponding dual basis in $g$. In particular

$\omega(X) = 1, \omega(e_i) = 0, i = 1, \ldots, n.$

d$\omega = 0$ and $adX^*(\omega_1 \wedge \cdots \wedge \omega_p) = \omega_1^p \wedge \cdots \wedge \omega_p^p$. Hence $Spec^p(\omega) = \{p\}$ and for $\lambda = p$ we have only two non-trivial Betti numbers:

$b_p^{p−1}(g) = b_p^p(\omega)(g) = \binom{n}{p}$.

**Corollary 2.5** (Dixmier, [D]). Let $g$ be a nilpotent Lie algebra. The cohomology $H^*_\lambda(\omega)(g)$ is trivial for all non-trivial $\lambda\omega$ and coincides with the Lie algebra cohomology $H^*(g)$ in trivial $\lambda\omega = 0$ case.
The operator \( adX \) is nilpotent and therefore the same is \( adX^* : H^i(b_\omega) \to H^i(b_\omega). \) Hence \( adX^* + \lambda I \) is non-degenerate operator for all \( \lambda \neq 0. \) We obtain the proof of Theorem 1 from [A] as the corollary of Dixmier’s theorem [B] for cohomology of nilpotent Lie algebras.

**Remark.** We represented in this article only a version of Dixmier’s exact sequence adapted to our special case of 1-dimensional Lie algebra representation (see [D] for all details), the last thing that we want to recall is Dixmier’s estimate for Betti numbers \( \dim H^p(g) \) of a nilpotent Lie algebra \( g. \)

**Corollary 2.6 ([B]).** Let \( g \) be a nilpotent Lie algebra. Then
\[
\dim H^p(g) \geq 2, \ p = 1, \ldots, n-1.
\]

It follows from the Corollary 2.3, we have \( \lambda \omega = 0 \) and \( \dim H^p(g) = k^p + k^{p-1}, \) where \( k^p \geq 1, k^{p-1} \geq 1 \) are the dimensions of the kernels of nilpotent operators \( adX_0^*, \ adX_1^* \) in the spaces \( H^p(b_\omega), \ H^{p-1}(b_\omega) \) with \( \dim H^p(b_\omega) \geq 2, \dim H^{p-1}(b_\omega) \geq 2 \) by inductive assumption.

3. **COHOMOLOGY OF SOLVABLE LIE ALGEBRAS**

**Definition 3.1.** A real solvable Lie algebra \( g \) is called completely solvable if \( ad(X) : g \to g \) has only real eigenvalues \( \forall X \in g. \)

**Lemma 3.2.** Let \( g \) be a \( n \)-dimensional solvable over \( \mathbb{C} \) (or real completely solvable Lie algebra) and \( b^1(g) = \dim H^1(g) = k. \) Then exists a basis \( \omega_1, \ldots, \omega_n \) in \( g^\ast \) such that
\[
\begin{align*}
d\omega_1 &= \cdots = d\omega_k = 0, \\
d\omega_{k+1} &= \alpha_{k+1} \wedge \omega_{k+1} + P_{k+1}(\omega_1, \ldots, \omega_k), \\
&\vdots \\
d\omega_n &= \alpha_n \wedge \omega_n + P_n(\omega_1, \ldots, \omega_{n-1}),
\end{align*}
\]
where \( \alpha_{k+1}, \ldots, \alpha_n \) are closed 1-forms, that are the weights of completely reducible representation associated to \( ad_{\mathfrak{g}||\mathfrak{g}} \) and \( P_1(\omega_1, \ldots, \omega_{i-1}) \in \Lambda^2(\omega_1, \ldots, \omega_{i-1}). \)

For the proof we apply Lie’s theorem to the adjoint representation \( ad \) restricted to the commutant \( [\mathfrak{g}, \mathfrak{g}]. \) Namely we can choose a basis \( e_{k+1}, \ldots, e_n \) such that the subspaces \( \text{Span}(e_i, \ldots, e_n), i = k+1, \ldots, n \) are invariant with respect to the representation \( ad_{\mathfrak{g}||\mathfrak{g}}. \) Then we add \( e_1, \ldots, e_k \) in a way that \( e_1, \ldots, e_n \) form the basis of \( g. \) For the dual forms \( \omega_1, \ldots, \omega_n \) in \( g^\ast \) we have formulas (B).

**Remark.** One can consider the free \( d \)-algebra \( (\Lambda^\ast(\omega_1, \ldots, \omega_n), d) \) as a kind of minimal model of the cochain complex \( \Lambda^\ast(\mathfrak{g}^\ast) \) because the mapping \( \Lambda^\ast(\omega_1, \ldots, \omega_n) \to \Lambda^\ast(\mathfrak{g}^\ast) \) induces the isomorphism in cohomology.

Now we start with a solvable Lie algebra over \( \mathbb{C} \) (or real completely solvable) and take a basis \( \omega_1, \ldots, \omega_n \) constructed in Lemma 3.2. Let us denote by \( \Lambda^\ast \) the exterior subalgebra in \( \Lambda^\ast(\omega_1, \ldots, \omega_n) \) generated by \( \omega_1, \ldots, \omega_k. \) One can define a filtration \( F \) of the cochain complex \( \Lambda^\ast(\mathfrak{g}^\ast): \)
\[
0 \subset \Lambda^\ast \subset F^{k+1} \subset F^{k+2} \subset F^{k+1,k+2} \subset F^{k+3} \subset F^{k+1,k+3} \subset \cdots \subset F^{k,\ldots,n} = \Lambda^\ast(\mathfrak{g}^\ast)
\]
where the system of subspaces \( \{ F^{j_1,\ldots,j_p}, k < j_1 < \cdots < j_p \leq n \} \) is defined by the following conditions:
A subspace $F^{j_1,\ldots,j_r}$ is spanned by monomials $a = \omega_{i_1} \land \cdots \land \omega_{i_q}$ with $l_1 < \cdots < l_q$ such that
- $l_q < j_p$; or $l_q = j_p, l_q-1 < j_p-1$;
- or $l_q = j_p, l_q-1 = j_p-1, l_q-2 < j_p-2$; or $\ldots$;
- or $l_q = j_p, \ldots, l_q-p+1 = j_1, l_q-p \leq k$.

Thus for example:

\begin{align*}
F^{k+1} & = \Lambda^* \oplus \Lambda^* \land \omega_{k+1}, & F^{k+2} & = \Lambda^* \oplus \Lambda^* \land \omega_{k+1} \oplus \Lambda^* \land \omega_{k+2}, \\
F^{k+1,k+2} & = \Lambda^* \oplus \Lambda^* \land \omega_{k+1} \oplus \Lambda^* \land \omega_{k+2} \oplus \Lambda^* \land \omega_{k+1} \land \omega_{k+2}, & F^{k+3} & = \Lambda^* \oplus \Lambda^* \land \omega_{k+1} \oplus \Lambda^* \land \omega_{k+2} \oplus \Lambda^* \land \omega_{k+1} \land \omega_{k+2} \oplus \Lambda^* \land \omega_{k+3}. \\
\end{align*}

The filtration $F$ is compatible with differential $d + \lambda \omega$ and one can consider the corresponding spectral sequence $E_r$. To obtain its first term $E_1$ one have to calculate the cohomology of complexes $(\Lambda^* \land \omega_{j_1} \land \cdots \land \omega_{j_p}, d_0)$:

$$d_0(\tilde{a} \land \omega_{j_1} \land \cdots \land \omega_{j_p}) = (\alpha_{j_1} + \cdots + \alpha_{j_p} + \lambda \omega) \tilde{a} \land \omega_{j_1} \land \cdots \land \omega_{j_p}.$$ 

The cohomology $H^*(\Lambda^* \land \omega_{j_1} \land \cdots \land \omega_{j_p}, d_0)$ coincides with the cohomology of $(\Lambda^*, \tilde{d})$ where differential $\tilde{d}$ acts as exterior multiplication by 1-form $\alpha_{j_1} + \cdots + \alpha_{j_p} + \lambda \omega$. Hence $H^*(\Lambda^* \land \omega_{j_1} \land \cdots \land \omega_{j_p}, d_0)$ is trivial if $\alpha_{j_1} + \cdots + \alpha_{j_p} + \lambda \omega = 0$. So taking $\lambda \omega$ such that

$$\alpha_{j_1} + \cdots + \alpha_{j_p} + \lambda \omega \neq 0 \forall j_1 < \cdots < j_p, p = k+1, \ldots, n$$

one can see that the spectral sequence $E_r$ degenerates at the first term $E_1$. Taking the complexification of real solvable Lie algebra $g$ we come to the following

**Theorem 3.3.** Let $g$ be a solvable Lie algebra, $\dim g = n$ and $\{\alpha_{k+1}, \ldots, \alpha_n\}$ is the collection of the weights of completely reducible representation associated to $\text{ad}[g,g]$. Let $\Omega_g$ denote the set of all $p$-sums $\alpha_{i_1} + \cdots + \alpha_{i_p}$, $k+1 \leq i_1 < \cdots < i_p \leq n$, $p = 1, \ldots, n$. Then exists a spectral sequence $E_r$ that converges to the cohomology $H_{\lambda \omega}^*(g)$ and its first term $E_1$ degenerates if $-\lambda \omega \notin \Omega_g$.

The set $\Omega_g$ is defined by $\lambda \omega$ such that the term $E_1$ is non-trivial, but generally $E_1$ doesn’t coincide with $E_\infty$. So we have to introduce $\Omega_g \subset \Omega_g$ such that $E_\infty \neq 0$ if and only if $-\lambda \omega \in \{0\} \cup \Omega_g$.

**Corollary 3.4.** Let $g$ be a solvable Lie algebra. Then $H_{\lambda \omega}^*(g)$ is non-trivial if and only if $-\lambda \omega \in \{0\} \cup \Omega_g$ – the finite subset in $H^1(g)$.

4. **Cohomology of Solvmanifolds.**

**Definition 4.1.** A solvmanifold (nilmanifold) $M$ is a compact homogeneous space of the form $G/\Gamma$, where $G$ is a simply connected solvable (nilpotent) Lie group and $\Gamma$ is a lattice in $G$.

Let $g$ denote a Lie algebra of $G$. Recall that $G$ is solvable if and only if $g$ is solvable Lie algebra, the last condition is equivalent to nilpotency of derived algebra $[g,g]$.

We start with examples of nilmanifolds.

**Example 4.2.** A $n$-dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. 

Example 4.3. The Heisenberg manifold $M_3 = \mathcal{H}_3/\Gamma_3$, where $\mathcal{H}_3$ is the group of all matrices of the form
\[
\begin{pmatrix}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{pmatrix}, \ x, y, z \in \mathbb{R},
\]
and a lattice $\Gamma_3$ is a subgroup of matrices with $x, y, z \in \mathbb{Z}$.

Theorem 4.4 (A.I. Malcev [Mal]). Let $G$ be a simply connected nilpotent Lie group with a tangent Lie algebra $\mathfrak{g}$. Then $G$ has a co-compact lattice $\Gamma$ (i.e. $G/\Gamma$ is a compact space) if and only if there exists a basis $e_1, e_2, \ldots, e_n$ in $\mathfrak{g}$ such that the constants $\{c^k_{ij}\}$ of Lie structure $[e_i, e_j] = c^k_{ij}e_k$ are all rational numbers.

This theorem gives us a practical tool for construction of nilmanifolds: let $\mathfrak{g}$ be a nilpotent Lie algebra defined by its basis $e_1, e_2, \ldots, e_n$ and commutating relations $[e_i, e_j] = c^k_{ij}e_k$, where all numbers $c^k_{ij} \in \mathbb{Q}$. Now one can define a group structure $\ast$ in the vector space $\mathfrak{g}$ using the Campbell-Hausdorff formula. The nilpotent group $G = (\mathfrak{g}, \ast)$ has a co-compact lattice $\Gamma$ (a subgroup generated by basic elements $e_1, e_2, \ldots, e_n$) and one can consider corresponding nilmanifold $G/\Gamma$.

Example 4.5. Let $V_n$ be a nilpotent Lie algebra with a basis $e_1, e_2, \ldots, e_n$ and a Lie bracket:
\[
[e_i, e_j] = \begin{cases} (j - i)e_{i+j}, & i + j \leq n; \\ 0, & i + j > n. \end{cases}
\]
Cohomology of the corresponding family of nilmanifolds $M_n$ was studied in [Al] and [Mill].

The situation with non-nilpotent solvable Lie groups is much more difficult: the crucial point is the problem of existence of cocompact lattice (see [R] for details). For example, if a solvable Lie group $G$ admits a cocompact lattice $\Gamma$ then the corresponding Lie algebra $\mathfrak{g}$ is unimodular, hence $\alpha_{k+1} + \cdots + \alpha_n = 0$. The condition of unimodularity of $\mathfrak{g}$ is not sufficient. See [R] for general information in this domain.

Example 4.6. Let us consider a semidirect product $G_0 = \mathbb{R} \rtimes \mathbb{R}^2$ where $\mathbb{R}$ acts on $\mathbb{R}^2$ via
\[
t \rightarrow \phi(t) = \begin{pmatrix} a^t & 0 \\ 0 & a^{-t} \end{pmatrix},
\]
where $a + a^{-1} = n \in \mathbb{N}, a \neq 1, a > 0$. Then
\[
\phi(1) = C^{-1} \begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix} C,
\]
for some matrix $C \in GL(2, \mathbb{R})$. Then exists a lattice $L \subset \mathbb{R}^2$ invariant with respect to $\phi(1)$. $\mathbb{Z}$ acts on $L$ via $\phi(1)^m$ and we define a lattice $\Gamma \subset G$ as $\mathbb{Z} \rtimes_{\phi(1)^m} L$. The lattices corresponding to different values of $n$ are, generally speaking, non-isomorphic. So corresponding solvmanifolds are non-diffeomorphic to. But we study the cohomology $H_\ast^p(G/\Gamma, \mathbb{C})$ over $\mathbb{C}$ and as we will see it doesn’t depend on the choice of $\Gamma \subset G$.

The solvmanifold from the previous example is a fibre bundle over $S^1$ with $T^2$ as fibre. It can be generalized by the following

Theorem 4.7 (G.D. Mostow [Mos1]). Any compact solvmanifold is a bundle with toroid as base space and nilmanifold as fibre.
Definition 4.8. A solvable Lie group $G$ is called completely solvable if its tangent Lie algebra $\mathfrak{g}$ is completely solvable.

One can identify deRham complex $\Lambda^*(G/\Gamma)$ with subcomplex $\Lambda^{*}_{\Gamma}(G) \subset \Lambda^*(G)$ of left-invariant forms on $G$ with respect to the action of the lattice $\Gamma$. $\Lambda^{*}_{\Gamma}(G)$ contains the subcomplex $\Lambda^*_{\mathfrak{g}}(G)$ of left-invariant forms with respect to the whole action of $G$. $\Lambda^*_{\mathfrak{g}}(G)$ is naturally isomorphic to the Lie algebra cochain complex $\Lambda^*(\mathfrak{g})$. Let us consider the corresponding inclusion

$$\psi : \Lambda^*(\mathfrak{g}) \rightarrow \Lambda^*(G/\Gamma).$$

Theorem 4.9 (Hattori [3]). Let $G/\Gamma$ be a compact solvmanifold, where $G$ is a completely solvable Lie group, then the inclusion $\psi : \Lambda^*(\mathfrak{g}) \rightarrow \Lambda^*(G/\Gamma)$ induces the isomorphism $\psi^* : H^*(\mathfrak{g}) \rightarrow H^*(G/\Gamma, \mathbb{R})$ in cohomology.

Remark. In fact Hattori’s theorem is the generalization of the theorem proved by Nomizu [2] for nilmanifolds. For an arbitrary solvmanifold $G/\Gamma$ the mapping $\psi^*$ is not isomorphism but it is an inclusion (see [3]).

So every class $[\omega] \in H^1(G/\Gamma, \mathbb{R})$ can be represented by the left-invariant (with respect to the action of $G$) 1-form $\omega$. By means of $\omega$ one can define a one-dimensional representation $\rho_{\lambda\omega} : G \rightarrow \mathbb{C}^*$:

$$\rho_{\lambda\omega}(g) = \exp \int_{\gamma(e,g)} \lambda \omega,$$

where $\gamma(e,g)$ is a path connecting the identity $e$ with $g \in G$ (let us recall that $G$ is a simply-connected). As $\omega$ is the left invariant 1-form then

$$\int_{\gamma(e,g_1,g_2)} \lambda \omega = \int_{\gamma(e,g_1)} \lambda \omega + \int_{\gamma(g_1,g_1,g_2)} \lambda \omega = \int_{\gamma(e,g_1)} \lambda \omega + \int_{g_1^{-1}\gamma(e,g_2)} \lambda \omega$$

holds on and $\rho_{\lambda\omega}(g_1g_2) = \rho_{\lambda\omega}(g_1)\rho_{\lambda\omega}(g_2)$. $\rho_{\lambda\omega}$ induces the representation of corresponding Lie algebra $\mathfrak{g}$ (we denote it by the same symbol): $\rho_{\lambda\omega}(X) = \lambda \omega(X)$.

In this situation it’s possible to make some generalizations using Mostow’s theorem. Namely following [3] we give

Definition 4.10. Let $G$ be a simply-connected Lie group and $\Gamma \subset G$ a lattice. Let $\rho$ be a finite dimensional representation of $G$ on a complex vector space $F$. Let $Ad$ denote the adjoint representation of $G$ on its Lie algebra $\mathfrak{g}$ as well as the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$. We will say that the representation $\rho$ is $\Gamma$-supported if $\rho(\Gamma)$ and $\rho(G)$ have the same Zariski closure in $\text{Aut}_\mathbb{C}(F)$. The representation $\rho$ is $\Gamma$-admissible if $\rho \oplus Ad$ (on $F \oplus \mathfrak{g}_\mathbb{C}$) is $\Gamma$-supported.

Theorem 4.11 (Mostow, Theorem 7.26 in [3]). Let $G$ be a simply-connected solvable Lie group and $\Gamma \subset G$ a lattice. Let $\rho$ be a finite dimensional $\Gamma$-admissible representation in a complex vector space $F$.

Then the inclusion $\psi : \Lambda^*(\mathfrak{g}, F) \rightarrow \Lambda^*(G/\Gamma, F)$ of complexes of differential forms with values in $F$ induces the isomorphism $\psi^* : H^*_\rho(\mathfrak{g}, F) \rightarrow H^*_\rho(G/\Gamma, F)$ in cohomology where $\rho$ is used also to denote the representation of the Lie algebra $\rho : \mathfrak{g} \rightarrow F$ induced by $\rho : G \rightarrow \text{Aut}_\mathbb{C}(F)$.

But we’ll not discuss the details of Mostow’s theorem and possible generalizations we restrict ourselves to the case of completely solvable Lie group $G$. Namely we’ll prove by means of Hattori’s theorem the following important
Corollary 4.12. Let $G/\Gamma$ be a compact solvmanifold, $G$ has a completely solvable Lie group and $\omega$ is a closed 1-form on $G/\Gamma$. The cohomology $H^*_{\omega}(G/\Gamma, \mathbb{C})$ is isomorphic to the Lie algebra cohomology $H^*_{\omega'}(\mathfrak{g})$ where $\omega' \in \mathfrak{g}^*$ is the left-invariant 1-form that represents the class $[\omega] \in H^1(G/\Gamma, \mathbb{R})$.

The Corollary 4.12 together with Corollary 3.4 gives us

Theorem 4.13. Let $G/\Gamma$ be a compact solvmanifold, $G$ is a completely solvable Lie group and $\omega$ is a closed 1-form on $G/\Gamma$. The cohomology $H^*_{\omega}(G/\Gamma, \mathbb{C})$ is non-trivial if and only if $-\lambda[\omega] \in \{0\} \cup \Omega_0$ — the finite subset in $H^1(G/\Gamma, \mathbb{R})$ well-defined in terms of the corresponding Lie algebra $\mathfrak{g}$.

Corollary 4.14 (Al). Let $G/\Gamma$ be a compact nilmanifold. The cohomology $H^*_{\omega}(G/\Gamma, \mathbb{C}) = 0$ if and only if $\lambda \omega \neq 0$.

Let us consider a 3-dimensional solvmanifold $M = G_0/\Gamma$ defined in the Example 4.6. The corresponding Lie algebra $\mathfrak{g}_0$ is isomorphic to the Lie algebra defined by its basis $e_1, e_2, e_3$ and following non-trivial brackets:

$[e_1, e_2] = e_2, [e_1, e_3] = -e_3.$

We take a dual basis $\omega_1, \omega_2, \omega_3$ in $\mathfrak{g}^*$. Then

$d\omega_1 = 0, d\omega_2 = \omega_1 \wedge \omega_2, d\omega_3 = -\omega_1 \wedge \omega_3.$

So $\alpha_2 = \omega_1$ and $\alpha_3 = -\omega_1$. It is easy to see that $H^*_{\omega}(\mathfrak{g}_0)$ is non-trivial if and only if $\lambda \omega = 0, \pm \omega_1$. For corresponding Betti numbers $b^p_{\lambda \omega}(M) = \dim H^p_{\lambda \omega}(G_0/\Gamma) = \dim H^p_{\lambda \omega}(\mathfrak{g}_0)$ of the solvmanifold $M = G_0/\Gamma$ we have:

$b^0_{\pm \omega_1} (M) = 0, b^1_{\pm \omega_1} (M) = b^2_{\pm \omega_1} (M) = 1, b^3_{\pm \omega_1} (M) = 0;

b^0 (M) = b^1 (M) = b^2 (M) = b^3 (M) = 1.$

REFERENCES

[Al] L. Alania, Cohomology with local system of certain nilmanifolds, Russian Math. Surveys 54:5 (1999), 1019–1020.

[D] J. Dixmier, Cohomologie des algebres de Lie nilpotentes, Acta Sci. Math. Szeged 16 (1955), 246–250.

[H] A. Hattori, Spectral sequence in the de Rham cohomology of fibre bundles, J. Fac. Sci. Univ. Tokyo, Sect. 1, 8:4, (1960), pp. 289–331.

[Ma] A. Malcev, On a class of homogeneous spaces, Amer. Math. Soc. Transl. (1) 9 (1962), 276–307.

[Mill] D.V. Millionschikov, Cohomology of nilmanifolds and Gontcharova’s theorem, in “Global Differential geometry: The Mathematical Legacy of Alfred Gray”, M. Fernandez and J.Wolf ed., AMS CONM 288 (2001), 381–385.

[Mos1] G.D. Mostow, Factor spaces of soluble groups, Ann. of Math. 60 (1954), 1–27.

[Mos2] G.D. Mostow, Cohomology of topological groups and solvmanifolds, Ann. of Math. 73 (1961), 20–48.

[Nz] K. Nomizu, On the cohomology of homogeneous spaces of nilpotent Lie groups, Ann. of Math. 59 (1954), 531–538.

[N1] S. P. Novikov, Multivalued functions and functionals. An analogue of the Morse theory, Soviet Math. Dokl. 24 (1981), 222–226.

[N2] S. P. Novikov, The hamiltonian formalism and a many-valued analogue of Morse theory, Russian Math. Surveys 37:5 (1982), 1–56.

[N3] S. P. Novikov, Bloch homology. Critical points of functions and closed 1-forms, Soviet Math. Dokl. 33:5 (1986), 551-555.
[N4] S. P. Novikov, *On the exotic De-Rham cohomology. Perturbation theory as a spectral sequence*, arXiv:math-ph/0201019.

[Pa] A.V. Pazhitnov, *An analytic proof of the real part of Novikov’s inequalities*, Soviet Math. Dokl. 35 (1987), 1–2.

[R] M.S. Raghunathan, *Discrete Subgroups of Lie Groups*, Springer-Verlag, 1972.

[W] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. 17 (1982), 661–692.

Department of Mathematics and Mechanics, Moscow State University, 119899 Moscow, Russia

Current address: Université Louis Pasteur, UFR de Mathématique et d’Informatique, 7 rue René Descartes - 67084 Strasbourg Cedex (France)

E-mail address: million@mech.math.msu.su