Minimal completely asymmetric \((4, n)\)-regular matchstick graphs

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Abstract

A matchstick graph is a graph drawn with straight edges in the plane such that the edges have unit length, and non-adjacent edges do not intersect. We call a matchstick graph \((m, n)\)-regular if every vertex has only degree \(m\) or \(n\). In this article we present the latest known \((4, n)\)-regular matchstick graphs for \(4 \leq n \leq 11\) with a minimum number of vertices and a completely asymmetric structure.

We call a matchstick graph \textit{completely asymmetric}, if the following conditions are complied.

- The graph is rigid.
- The graph has no point, rotational or mirror symmetry.
- The graph has an asymmetric outer shape.
- The graph can not be decomposed into rigid subgraphs and rearrange to a similar graph which contradicts to any of the other conditions.

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1 Introduction

A matchstick graph is a graph drawn with straight edges in the plane such that the edges have unit length, and non-adjacent edges do not intersect. We call a matchstick graph \((m, n)\)-regular if every vertex has only degree \(m\) or \(n\).

For \(m \leq n\) minimal \((4, n)\)-regular matchstick graphs with a minimum number of vertices only exist for \(4 \leq n \leq 11\). The smallest known \((4, n)\)-regular matchstick graph for \(n = 4\), also named 4-regular, is the so called Harborth graph consisting of 52 vertices and 104 edges. Except for \(n = 10\) all currently smallest known \((4, n)\)-regular matchstick graphs for \(4 \leq n \leq 11\) are symmetric. They consist of 104, 115, 117, 159, 126, 273, 231 and 811 edges and were presented in an earlier paper by the authors [7].

It is an open problem how many different \((4, n)\)-regular matchstick graphs with a minimum number of vertices for \(4 \leq n \leq 11\) exist and which is the least minimal number. "Our knowledge on matchstick graphs is still very limited. It seems to be hard to obtain rigid mathematical results about them. Matchstick problems constructing the minimal example can be quite challenging. But the really hard task is to rigidly prove that no smaller example can exist." [2]

In this article we present the \((4, n)\)-regular matchstick graphs for \(4 \leq n \leq 11\) with the smallest currently known number of vertices and a completely asymmetric structure. The definition of a completely asymmetric matchstick graph is given in chapter 3. The graphs were discovered in the days from March 17 - August 12, 2016 and were presented for the first time in the German mathematics internet forum Matroids Matheplanet [3].

The rigidity of the graphs has been verified by Stefan Vogel. Details can be found in the thread of the graph theory forum [3]. The method which was used for the calculations he describes in a separate German article [6]. The rigidity of the graphs is also a proof for their geometry and their existenz.
2 Rigid subgraphs

The geometry of the graphs in this paper is based on two types of rigid subgraphs we call the kite (Fig. 1 a) and the triplet kite (Fig. 1 b). The kite is a \((4, 2)\)-regular matchstick graph consisting of 12 vertices and 21 edges and has a vertical symmetry. The triplet kite is a \((4, 3, 2)\)-regular matchstick graph consisting of 22 vertices and 41 edges and has a vertical symmetry.

![Kite and Triplet Kite](image)

(a) kite  
(b) triplet kite

(c) double kite  
(d) reverse double kite

Figure 1: Rigid subgraphs

Two kites can be connected to each other in two useful ways. We call these subgraphs the double kite (Fig. 1 c) and the reverse double kite (Fig. 1 d), both consisting of 22 vertices and 42 edges. What makes the subgraphs (c) and (d) so useful is the fact that they have only two vertices of degree 2. Two of these subgraphs can be used to connect two vertices of degree 2 at different distances by using them like clasps. This property has been used for \(n = 8, \ldots, 11\) (Fig. 9 - 12).

The geometry of the graphs for \(n = 4, \ldots, 7\) is based on the triplet kite. The geometry of the graphs for \(n = 8, \ldots, 10\) is based on the kite. The geometry of the graph for \(n = 11\) is a special one, but also this graph contains kites.
3 Completely asymmetric matchstick graphs

We call a matchstick graph completely asymmetric, if the following conditions are complied.

- The graph is rigid.
- The graph has no point, rotational or mirror symmetry.
- The graph has an asymmetric outer shape.
- The graph can not be decomposed into rigid subgraphs and rearrange to a similar graph which contradicts to any of the other conditions.

For example the next two matchstick graphs are asymmetric, but not completely asymmetric, because only three of the four conditions are complied.

Figure 2: 4-regular matchstick graphs with 63 vertices and 126 edges. Figure 2a shows an asymmetric graph consisting only of one type of rigid subgraph - the kite. And six kites can be rearrange to a symmetric graph with a rotational symmetry of order 3 (Fig. 2b). Therefore the graph in Figure 2a is asymmetric, but not completely asymmetric.

Figure 3: Figure 3a shows a (4,5)-regular matchstick graph with 60 vertices and 121 edges. This asymmetric graph has a point symmetric outer shape (Fig. 3b). Therefore the graph in Figure 3a is asymmetric, but not completely asymmetric.
4 The smallest known completely asymmetric \((4, n)\)-regular matchstick graphs for \(4 \leq n \leq 11\)

Figure 4: 4-regular matchstick graph with 66 vertices and 132 edges.

Figure 5: \((4, 5)\)-regular matchstick graph with 62 vertices and 125 edges.
Figure 6: (4,6)-regular matchstick graph with 63 vertices and 128 edges - v1.

Figure 7: (4,6)-regular matchstick graph with 63 vertices and 128 edges - v2.
Figure 8: $(4,7)$-regular matchstick graph with 91 vertices and 185 edges.

Figure 9: $(4,8)$-regular matchstick graph with 87 vertices and 176 edges.
Figure 10: \((4,9)\)-regular matchstick graph with 137 vertices and 277 edges.

Figure 11: \((4,10)\)-regular matchstick graph with 114 vertices and 231 edges.
Figure 12: (4,11)-regular matchstick graph with 405 vertices and 817 edges.
The (4,11)-regular matchstick graph is the graph with the most complicated geometry. The interesting part lies in the neighborhood of the vertices of degree 11, as the next detail image shows. The whole graph is rigid, but this part of the graph is flexible. There exists a very small rhombus. The long outside edges measure exactly two unit lengths.

![Figure 13: Detail around the top vertex of degree 11.](image)

The degrees of the angles between the edges around the centered vertex. Counterclockwise beginning with the angle between the red edges.

\[
\begin{align*}
32.882386539359686, & \quad 39.600593591737876, \quad 25.654399576214260, \\
35.256568295080810, & \quad 31.867413841561630, \quad 34.483762832351330, \\
29.304619620735510, & \quad 36.230077308279220, \quad 29.505850754987293, \\
35.136791905120454, & \quad 30.077535734571963.
\end{align*}
\]
5 Remarks on the graphs

Figur 4: This 4-regular matchstick graph with 132 edges was discovered on June 16, 2016 by Mike Winkler. The graph has a triplet-kite-based geometry. There exist two other versions of this graph with a slightly different internal geometry, both with 134 edges.

Figur 5: This (4, 5)-regular matchstick graph with 125 edges was discovered on June 25, 2016 by Mike Winkler. The graph has a triplet-kite-based geometry and contains two triplet-kites. There exist seven other versions of this graph with a slightly different internal geometry, two with 126 edges, two with 127 edges, two with 128 edges, and one with 129 edges.

Figur 6 and 7: These (4, 6)-regular matchstick graphs with 128 edges were discovered on June 25, 2016 by Mike Winkler. Each graph has a triplet-kite-based geometry and contains two triplet-kites. Each version has a slightly different internal geometry.

Figur 8: This (4, 7)-regular matchstick graph with 185 edges was discovered on June 16, 2016 by Mike Winkler. The graph has a triplet-kite-based geometry and arises from a fusion of the graphs for $n = 4$ and $n = 5$.

Figur 9: This (4, 8)-regular matchstick graph with 176 edges was discovered on August 12, 2016 by Peter Dinkelacker. The graph has a kite-based geometry and contains two double-kites, two kites and two slightly modified kites.

Figur 10: This (4, 9)-regular matchstick graph with 277 edges was discovered on August 11, 2016 by Peter Dinkelacker. The graph has a kite-based geometry and contains four double-kites and four slightly modified kites. This graph is based on the currently smallest known symmetric equivalent with 273 edges.

Figur 11: This (4, 10)-regular matchstick graph with 231 edges was discovered on March 17, 2016 by Peter Dinkelacker. The graph has a kite-based geometry and consists only of kites. Three double kites, two reverse double kites and one kite. This version is also the currently smallest known one. It remains an interesting question whether a symmetric (4, 10)-regular matchstick graph with 231 edges or less exists.

Figur 12 and 13: This (4, 11)-regular matchstick graph with 817 edges was discovered on June 5, 2016 by Mike Winkler and Stefan Vogel. Idea and design by Winkler, calculation of the exact angles between the edges around the vertices of degree 11 by Vogel. This graph contains five double kites and five reverse double kites. There exists a few asymmetric variations of this graph with 817 edges, because the clasps can be varied.
6 References

1. Richard K. Guy and Robert E. Woodrow (eds.), The Lighter Side of Mathematics: Proceedings of the Eugène Strens Memorial Conference on Recreational Mathematics and its History, Calgary, Canada, July 27 - August 2, 1986, Spectrum Series. Washington, WA: Mathematical Association of America. viii, 367 p. MAA mem., 1994.

2. Sascha Kurz and Giuseppe Mazzuoccolo, 3-regular matchstick graphs with given girth. (weblink)
(http://arxiv.org/pdf/1401.4360v1.pdf)

3. Matroids Matheplanet, Thread in the graph theory forum. (weblink)
(http://www.matheplanet.de/matheplanet/nuke/html/viewtopic.php?topic=216644&start=0)
(Nicknames used in the forum: haribo = Peter Dinkelacker, Slash = Mike Winkler)

4. Siemens PLM Software, Solid Edge 2D-Drafting ST8. (weblink)
(http://www.plm.automation.siemens.com/de_de/)

5. Wikipedia, Matchstick graph. (weblink)
(https://en.wikipedia.org/wiki/Matchstick_graph)

6. Stefan Vogel, Beweglichkeit eines Streichholzgraphen bestimmen, July 2016. (weblink)
(http://www.matheplanet.de/matheplanet/nuke/html/article.php?sid=1757&mode=&order=0)

7. Mike Winkler and Peter Dinkelacker, New minimal (4,n)-regular matchstick graphs, April 2016, Update in September 2016, arXiv:1604.07134[math.MG]. (weblink)
(http://arxiv.org/pdf/1604.07134v1.pdf)

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