The SD Oscillator and Its Attractors

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Abstract: We propose a new archetypal oscillator for smooth and discontinuous systems (SD oscillator). This oscillator behaves both smooth and discontinuous system depending on the value of the smoothness parameter. New dynamic behaviour is presented for the transitions from the smooth to discontinuous regime.

1. Introduction

This study is motivated by a growing interest in the non-smooth dynamics where various physical systems have been studied. Examples include problems from mechanical and civil engineering [1], electronics [2], control [3], biology [4] and others. Although some theoretical foundations have been laid in the work by Filippov [5] and others, there is a large disproportion between development and understanding of smooth and discontinuous (non-smooth) systems.

We study the nonlinear dynamics of an archetypal oscillator whose nonlinearity can be smooth or discontinuous depending on the value of the smoothness parameter $\alpha$, which was first proposed in our previous works, see Cao and \textit{et. al.} [6, 7]. As the considered oscillator has properties of both smooth and discontinuous systems (at the limit), potentially a wealth of knowledge can be drawn from the well developed theory of continuous dynamics. Physically as shown in Fig. 1a, this oscillator is similar to a snap-through truss system. It comprises a mass, $m$, linked by a pair of inclined elastic springs which are capable of resisting both tension and compression; each spring of stiffness $k$ is pinned to a rigid support. This model is inspired on the elastic arch described by Thompson and Hunt in [8] (see Fig. 1b). Although the springs themselves provide linear restoring resistance, the resulting vertical force on the mass is strongly nonlinear because of changes to the geometric configuration. The equation of motion can be written as,

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\[ X'' + 2kX \left(1 - \frac{L}{\sqrt{X^2 + l^2}}\right) = 0, \]  

(1)

where \( L \) is the equilibrium length, \( X \) is the mass displacement and \( l \) is the half distance between the rigid supports. Now suppose system (1) is perturbed by a viscous damping and an external harmonic excitation of amplitude \( F_0 \) and frequency \( \Omega \). This leads to the following system,

\[ X'' + \delta X' + 2kX \left(1 - \frac{L}{\sqrt{X^2 + l^2}}\right) = F_0 \cos \Omega t. \]  

(2)

System (1) can be made dimensionless by letting \( \omega_0^2 = \frac{2k}{m} \), \( x = \frac{X}{L} \) and \( \alpha = \frac{l}{L} \neq 0 \),

\[ \ddot{x} + \omega_0^2 x \left(1 - \frac{1}{\sqrt{x^2 + \alpha^2}}\right) = 0. \]  

(3)

The smoothness parameter \( \alpha \) defines not only the geometry of the oscillator (Fig.1a) but also has physical meaning. For \( \alpha > 1 \) the system represents a pre-tensioned discrete elastic string. For \( \alpha = 0 \), the model corresponds to an oscillating mass supported by two parallel vertical springs.

Again system (2) can be written in a dimensionless form by letting \( \tau = \omega_0 t \), \( f_0 = \frac{F_0}{2kL} \),

\[ \ddot{\xi} + \frac{\delta}{2m \omega_0} \dot{\xi} + \left(1 - \frac{1}{\sqrt{\xi^2 + \alpha^2}}\right) = f_0 \cos \omega \tau. \]  

(4)

**Figure 1.** a) The dynamical model in a the form of a nonlinear oscillator, where a mass is supported by a pair of springs pinned to rigid supports and (b) a simple elastic arch.
The nonlinear restoring force \( F(x) = -\omega_0^2 x \left( 1 - \frac{1}{\sqrt{x^2 + \alpha^2}} \right) \) is plotted for \( \omega_0 = 1 \) in Fig. 2a for different values of parameter \( \alpha \). The solid line represents the discontinuous case \( \alpha = 0 \), the dotted, the dashed dotted and the dashed lines mark the smooth cases, for \( \alpha = 0.1, \alpha = 0.5 \), and \( \alpha = 0.75 \) respectively.

2. Unperturbed Oscillator

When \( \alpha = 0 \) system (3) can be written in the following form,

\[
\dot{x} + \omega_0^2 (x - \text{sign}(x)) = 0. \tag{5}
\]

It is worth reiterating here that the discontinuous dynamics is obtained by decreasing of the smoothness parameter \( \alpha \) to 0.

To examine the influence of parameter \( \alpha \) on the dynamics of (3) we construct the supercritical pitchfork bifurcation at \( \alpha = 1 \) where the stable branch \( x = 1 \) bifurcates into two stable branches with \( x = \pm \sqrt{1 - \alpha^2} \). The stationary \( x = 0 \) solution is now unstable, exhibiting the standard hyperbolic structure.

![Fig. 2](image)

**Fig. 2** (a) Nonlinear restoring force, \( F(x) \); solid line marks the discontinuous case for \( \alpha = 0 \), dotted, dash-dotted and dashed curves are the smooth cases for for \( \alpha = 0.1, \alpha = 0.5 \), and \( \alpha = 0.75 \) respectively; (b) smooth case for \( \alpha = 0.5 \).

The Hamiltonian for system (3) can be written as

\[
H(x, y) = \frac{1}{2} y^2 + \frac{1}{2} x^2 - \omega_0^2 \sqrt{x^2 + \alpha^2} + \omega_0 \alpha, \tag{6}
\]

where \( y = x' \).

With the help of the Hamiltonian function (6), the trajectories can be classified and analyzed. For both the continuous and discontinuous cases, the phase portraits of systems (3) and (5) are plotted for different values of the Hamiltonian \( H(x, y) = E \). For instance, for the smooth nonlinearity, \( \alpha = 0.5 \), the dynamic behaviour of double-well is similar to that of the Duffing oscillator, see [9], shown in Fig. 2b. For \( \alpha = 0 \), the behaviour is singular, as shown in Fig. 3a and 3b (local structure): the orbits for
$E > 0$ are comprised of two large segments of circles with their centres located at $(-1,0)$ and $(1,0)$ connected at $x = 0$. The case of $E < 0$ is represented by two families of circles.

**Fig. 3** Phase portraits: (a) discontinuous case for $\alpha = 0$, (b) local amplified saddle-like singular structure.

It is the most interesting that $E = 0$ for the discontinuous system (5) is made up of two circles centred at $(\pm 1,0)$ connecting at the singular point $(0,0)$, which form special singular homoclinic-like orbits. The structure around the point indicates a saddle-like behavior, therefore $(0,0)$ is named the saddle-like singular. The hyperbolicity at origin $(0,0)$ is lost due to the tangency of the stable and unstable eigen directions. This isolated singularity has neither eigenvalue nor eigenvector. The pair of circles excluding the point $(0,0)$ is not the manifolds of the singularity, but the flow along these circles approaches the point as $x \to 0$, and it will be trapped by the singularity. The solution of the special homoclinic-like orbits can be formulated as

$$\Gamma = \left\{ (x(t), y(t)), t \in \left[ -\frac{\pi}{\omega_0}, \frac{\pi}{\omega_0} \right] \right\} \cup \{(0,0)\},$$

where $x(t), y(t) = (\pm 1 \pm \cos \omega t, \mp \sin \omega t)$, which is named the homoclinic-like orbit.

### 3. Perturbed Attractors: Transition from Smooth to Discontinuous Dynamics

Numerical simulations have been carried out for system (4), assuming $f_0 = 0.83, \xi = 0.04$, and $\omega = 1.0606$. Figure 4 shows bifurcation diagrams constructed for $\gamma$ sampled stroboscopically at phase zero versus control parameter $\alpha$ as it decreases from 1 to 0. Periodic windows and chaotic regions are found.

Figure 5 shows the transient of the phase plots as the parameter $\alpha$ changes. In the large chaotic region, the right side in Fig. 5, the largest Lyapunov exponent of the attractor will decrease, as shown in Fig. 5a and 5b. When $\alpha$ is small enough, the system exhibits chaotic transients throughout. This behaviour can be characterized by a chaotic saddle, see [10]. The transient and the final periodic attractors are shown in Fig. 5c for $\alpha = 0.01$. As $\alpha$ decreases to 0, the volume of the chaotic saddle increases to chaotic attractor when $\alpha = 0$, as shown in Fig. 4 and Fig. 5d.

The Poincaré maps shown in Fig. 5 for $\alpha = 0.2, 0.01$ and 0 show the topological similarity now associated with the discontinuity at the origin. The co-existing period 2 solutions persist for $\alpha = 0$ and the chaotic saddle becomes a chaotic attractor for $\alpha = 0$ as shown in Fig. 5d. For both $\alpha = 0$ the
computation is inferred via a semi-analytical method, see [11]. The largest Lyapunov exponents for all the chaotic attractors presented in this paper have been calculated using the chaos synchronization method, see [12] for instance, as shown in the captions for the corresponding figures.

Fig. 4 Bifurcation diagrams for $y$ versus $\alpha$ constructed for decreasing from $\alpha = 1$ to 0 with the same initial condition (1,0).

Fig. 5. Poincare sections: (a) Chaotic attractor for $\alpha = 0.2$ with the largest Lyapunov exponent 0.0909, (b) Chaotic attractor for $\alpha = 0.05$ with the largest Lyapunov exponent 0.0572, (c) chaotic saddle leading to period 2 solution for $\alpha = 0.01$ (d) chaos for $\alpha = 0$ with the largest Lyapunov exponent 0.0268.
4. Closing Remarks
We proposed an archetypal system named SD oscillator and discussed its dynamics. This system behaves both as smooth and discontinuous depending on the value of the smoothness parameter. New dynamics have been presented for the transition regimes. For the smooth regime, the oscillator exhibits standard dynamics of double well for $\alpha > 0$. For the discontinuous regime, when $\alpha = 0$, the oscillator shows both standard double-well and nonstandard dynamics, and homoclinic-like dynamic structures near the saddle-like singular $(0,0)$.

The presented oscillator is being actively studied by the authors in several main directions. Firstly, the peculiar properties at the limit of $\alpha = 0$ are being analysed in more details to better understand the bifurcation structures under varying damping and external forcing. This research relates to the studies of preloaded oscillators, e.g. [1]; The second pursued direction is to develop analytical measures (e.g. construction of Melnikovian) to predict the border of chaos. This research bears a significant analogy to the predictions made for the Duffing oscillator, see e.g [9]; The most motivated direction is to investigate the application for this special oscillator in other physical systems, taken biological system as example, see [12].

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