LOCALLY NILPOTENT LINEAR GROUPS

A. S. DETINKO AND D. L. FLANNERY

Abstract. We survey aspects of locally nilpotent linear groups. Then we obtain a new classification; namely, we classify the irreducible maximal locally nilpotent subgroups of $\text{GL}(q,F)$ for prime $q$ and any field $F$.

1. Why locally nilpotent linear groups?

Linear (matrix) groups are commonly used to represent abstract groups. Early work on linear groups was undertaken in the second half of the nineteenth century. Linear group theory is now a rich subject deeply connected to other areas of mathematics. Over the past few decades, revived interest in matrix groups has been driven partly by the development of computational group theory.

Recall that a group is said to be locally nilpotent if every finite subset generates a nilpotent subgroup. Locally nilpotent groups therefore generalize nilpotent groups. Many structural and classification results for locally nilpotent linear groups are known. Further progress can be made via computational techniques.

Group-theoretic algorithms accept a finite generating set as input. The celebrated ‘Tits alternative’ states that a finitely generated matrix group either is solvable-by-finite (i.e., it has a normal solvable subgroup of finite index), or it contains a nonabelian free subgroup. For groups of the latter kind, basic computational problems, such as membership testing and the conjugacy problem, are undecidable in general. On the other hand, locally nilpotent linear groups, being solvable and nilpotent-by-finite, are more tractable for computation. (Note that a solvable linear group is nilpotent-by-abelian-by-finite.) This point is accentuated by Gromov’s result [4], which implies that a finitely generated group has polynomial growth if and only if it is nilpotent-by-finite. Hence, as explained in [1], certain algorithmic efficiency problems can be overcome for locally nilpotent linear groups.

Extra motivation for continued study of locally nilpotent linear groups arises from their applications. Here we mention almost crystallographic groups, which appear as nilpotent-by-finite linear groups over $\mathbb{Q}$ [5] [§8.2.3]. Moreover, a finitely generated nilpotent group is polycyclic and so isomorphic to a subgroup of $\text{GL}(n,\mathbb{Z})$ for some $n$. Algorithms for nilpotent subgroups of $\text{GL}(n,\mathbb{Q})$ thereby serve as a key step toward algorithms for abstract finitely generated nilpotent groups.
2. THE STRUCTURE OF LOCALLY NILPOTENT LINEAR GROUPS

A systematic investigation of locally nilpotent linear groups was carried out by D. A. Suprunenko, starting in the late 1940s [13]. He classified the maximal locally nilpotent subgroups of $GL(n, F)$ for algebraically closed $F$. Several authors extended Suprunenko’s results to other $F$. In particular, criteria for the number of $GL(n, F)$-conjugacy classes of maximal locally nilpotent subgroups to be finite, and classification in partial cases, were enabled by a detailed description of locally nilpotent linear groups over an arbitrary field $F$ (see, e.g., [8]).

In this section, we outline some of the most important structural results about locally nilpotent linear groups. Much research has focused on the maximals, due to the fact that each locally nilpotent subgroup of $GL(n, F)$ is contained in a maximal locally nilpotent subgroup (by contrast, a nilpotent subgroup of $GL(n, F)$ might not be contained in a maximal nilpotent subgroup). We follow a standard reduction scheme: reducible $\rightarrow$ completely reducible $\rightarrow$ irreducible $\rightarrow$ absolutely irreducible $\rightarrow$ primitive.

2.1. Reducible locally nilpotent linear groups. We use terminology for linear groups as in [14, 15]. A reducible subgroup $G$ of $GL(n, F)$ is conjugate to a group of block upper triangular matrices, where the main diagonal blocks (irreducible parts) give irreducible representations of $G$ over $F$ in smaller degree. Suppose that $G$ is locally nilpotent and indecomposable. Then the irreducible parts of $G$ are pairwise equivalent [14, p. 223, Theorem 2], so that $G$ may be conjugated to a group of block upper triangular matrices

$$
\begin{bmatrix}
  a(g) & a_{12}(g) & \cdots & a_{1k}(g) \\
  0 & a(g) & \cdots & a_{2k}(g) \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & a(g)
\end{bmatrix}, \quad g \in G
$$

where $a_{ij}(g) \in \text{Mat}(n/k, F)$ and $a(G) = \{a(g) \mid g \in G\} \leq GL(n/k, F)$ is irreducible (and locally nilpotent). If $F$ is perfect then $a_{ij}(g) = c_{ij}(g)a(g)$ where $c_{ij}(g)$ centralizes $a(G)$; thus $G$ is contained in the direct product of a unitriangular group over a division algebra and a completely reducible group over $F$ with equivalent irreducible parts. This reduces study of locally nilpotent linear groups to the irreducible case.

Another passage to the completely reducible case uses the Jordan decomposition. For each $g \in GL(n, F)$ there are a unique unipotent matrix $g_u \in GL(n, \overline{F})$ and a unique diagonalizable matrix $g_d \in GL(n, \overline{F})$ such that $g = g_d g_u = g_u g_d$ ($\overline{F}$ denoting the algebraic closure of $F$); see [15] 7.2. If $F$ is perfect then $g_u, g_d \in GL(n, F)$. 

Theorem 2.1 ([14, p. 240, Theorem 6] and [15, 7.11]). Let $G \leq \text{GL}(n, \mathbb{F})$ be locally nilpotent. Define $G_u = \{ g \in G \mid g \text{ unipotent} \}$, $G_d = \{ g \in G \mid g \text{ diagonalizable} \}$. Then $G_u, G_d \triangleleft G$ and $\langle G_u, G_d \rangle = G_u \times G_d$.

By Theorem 2.1, if $G$ is completely reducible locally nilpotent then every subgroup of $G$ is completely reducible; so the elements of $G$ are diagonalizable (see [14, p. 239, Theorem 5] and [15, 7.12]).

Corollary 2.2 ([15, 7.13]). If $G \leq \text{GL}(n, \mathbb{F})$ is locally nilpotent and splittable (meaning that $g_u, g_d \in G$ for each $g \in G$), then $G = G_u \times G_d$.

For example, if $\mathbb{F}$ is finite and $G$ is nilpotent then $G$ is splittable. More generally:

Theorem 2.3 ([11, p. 136, Proposition 3]). Let $G$ be a nilpotent subgroup of $\text{GL}(n, \mathbb{F})$, where $\mathbb{F}$ is a perfect field. Then $\widehat{G}_u := \{ g_u \mid g \in G \}$, $\widehat{G}_d := \{ g_d \mid g \in G \}$ are subgroups of $\text{GL}(n, \mathbb{F})$, $\widehat{G}_d$ is completely reducible, and $G \leq \widehat{G}_u \times \widehat{G}_d$.

2.2. Irreducible locally nilpotent linear groups. An irreducible maximal locally nilpotent subgroup of $\text{GL}(n, \mathbb{F})$ can be thought of as an absolutely irreducible maximal locally nilpotent subgroup of $\text{GL}(m, \mathbb{E})$ for some $m$ dividing $n$ and field $\mathbb{E} \supset \mathbb{F}$ (see [14, p. 217, Theorem 4]). This affords a reduction to the absolutely irreducible case, particularly in the classification of irreducible maximal locally nilpotent subgroups of $\text{GL}(n, \mathbb{F})$. Subsequent reduction is possible in two directions, which are not mutually exclusive: to $p$-subgroups of $\text{PGL}(n, \mathbb{F})$, and to primitive groups. The first of these possibilities is based on the next result ($1_m$ denotes the $m \times m$ identity matrix).

Theorem 2.4 ([14 pp. 220–221]). Let $n = p_1^{a_1} \cdots p_k^{a_k}$ where the $p_i$ are distinct primes.

(i) If $G$ is a maximal absolutely irreducible locally nilpotent subgroup of $\text{GL}(n, \mathbb{F})$, then $G = G_1 \otimes \cdots \otimes G_k$ where $G_i \leq \text{GL}(p_i^{a_i}, \mathbb{F})$ for $1 \leq i \leq k$ is maximal absolutely irreducible locally nilpotent.

(ii) Let $k = 1$, and suppose that $G$ is an absolutely irreducible subgroup of $\text{GL}(n, \mathbb{F})$ containing $1 \times \mathbb{F}^n$. Then $G$ is a maximal absolutely irreducible locally nilpotent subgroup of $\text{GL}(n, \mathbb{F})$ if and only if $G/\mathbb{F}^n$ is a Sylow $p$-subgroup of $\text{PGL}(n, \mathbb{F})$.

An irreducible locally nilpotent linear group is center-by-periodic; indeed, its central quotient is a direct product of $p$-groups (see [12, Corollary 3.2.4]).

Except when $\mathbb{F}$ is finite or algebraically closed, the description of Sylow $p$-subgroups of $\text{PGL}(n, \mathbb{F})$ is quite different from the description of Sylow $p$-subgroups of $\text{GL}(n, \mathbb{F})$. The former were considered in [6], mainly for $p > 2$. Classifying the Sylow 2-subgroups of $\text{PGL}(n, \mathbb{F})$ is difficult. In [8], $p$-subgroups of $\text{PGL}(n, \mathbb{F})$ are handled by the same techniques as those used for locally nilpotent linear groups.
The reduction to primitives is not as straightforward for locally nilpotent linear groups as it is for other types of linear groups. To appreciate this disparity, note that an irreducible imprimitive solvable subgroup of $GL(n, F)$ is conjugate to a subgroup of $G \wr T$ where $G \leq GL(m, F)$ is primitive solvable and $T$ is a transitive solvable permutation group of degree $n/m$ [14, p. 129, Theorem 5]; however, the wreath product of a locally nilpotent linear group and a nilpotent permutation group need not even be (locally) nilpotent.

As in [8, §2], nilpotent primitive groups $G$ may be treated using the series $G \geq H \geq K \geq 1$, where $K = [G, G]$ and $H = C_G(K)$. Some basic information follows.

**Theorem 2.5** ([8 Theorem 2]). Let $n$ be a power of a prime $p \neq \text{char } F$, and let $G$ be a primitive absolutely irreducible locally nilpotent subgroup of $GL(n, F)$. Then $K$ is an abelian $p$-group, $\Sigma = \langle K \rangle_F$ is a field, $G/H \cong \text{Gal}(\Sigma/F)$, $[H, H] \leq F^\times 1_n$, and $H$ is a primitive absolutely irreducible class 2 nilpotent subgroup of $GL(m, \Sigma)$, $m = n/|\Sigma : F|$.

We remark that the primitive nilpotent linear groups over finite fields are classified in [2].

The paper [8] also gives methods to classify maximal locally nilpotent subgroups of $GL(n, F)$ for arbitrary $F$. Special attention is paid to the problem of determining when the number of $GL(n, F)$-conjugacy classes of these subgroups is finite. This depends on finiteness of the groups $F^\times/(F^\times)^m$ for $m$ dividing $n$.

Groups over an algebraically closed field have been the most intensively examined.

**Theorem 2.6** ([13, Chapter III], [14, Chapter VII]). Let $F$ be algebraically closed.

(i) Irreducible locally nilpotent subgroups of $GL(n, F)$ exist if and only if $\text{char } F$ does not divide $n$, in which case there exists an irreducible nilpotent subgroup of $GL(n, F)$ of every nilpotency class.

(ii) Irreducible maximal locally nilpotent subgroups of $GL(n, F)$ are monomial and pairwise conjugate.

By (ii), a completely reducible locally nilpotent linear group over an algebraically closed field is monomial. The matrix form of the groups in (ii) is discussed in [13, Chapter III, §7].

To sum up: although locally nilpotent linear groups are well-studied, significant gaps yet remain. Most results are concerned with absolutely irreducible maximal locally nilpotent groups, and do not readily yield analogues for groups that are not maximal or not absolutely irreducible (cf. [2]). Complete classifications of locally nilpotent subgroups of $GL(n, F)$ are achievable only by placing restrictions on $F$ or $n$. In the sequel, we allow arbitrary fields but restrict the degree.
3. PRIME DEGREE LOCALLY NILPOTENT LINEAR GROUPS

In this section, we illustrate how established theory of locally nilpotent linear groups may be applied to obtain a full classification in that theory. Specifically, we classify the irreducible maximal locally nilpotent subgroups of \( \text{GL}(q, F) \), where \( q \) is prime and \( F \) is any field. This classification is in the form of a list of \( \text{GL}(q, F) \)-conjugacy class representatives of the groups, with each listed group defined by a generating set of matrices. We provide criteria to decide conjugacy between listed groups.

Restricting to prime degree \( q \) offers various advantages. An irreducible nonabelian subgroup \( G \) of \( \text{GL}(q, F) \) is absolutely irreducible, and is either primitive or monomial. If \( G \) is locally nilpotent then \( GF^\times 1_q/F^\times 1_q \) lies in a Sylow \( q \)-subgroup of \( \text{PGL}(q, F) \), whose structure is simpler than that of a Sylow subgroup of \( \text{PGL} (n, F) \) for composite degree \( n \).

A partial classification of the irreducible maximal locally nilpotent subgroups of \( \text{GL}(q, F) \) can be derived from a description of the absolutely irreducible maximal locally nilpotent subgroups of \( \text{GL}(q^a, F) \). However, here we propose novel methods, and give a complete, self-contained result, which may be extended to a complete classification in degrees \( q^a \). In particular, we give an exact description of the Sylow 2-subgroups of \( \text{PGL}(2, F) \), omitted by other authors. This is of signal importance because classifying the Sylow 2-subgroups of \( \text{PGL} (n, F) \) in arbitrary degree \( n \) depends on having solved the problem for \( n = 2 \) (cf. the case of 2-subgroups in [7, 10]).

We first consider absolutely irreducible groups; abelian groups will be dealt with at the end. Note that the methods of this section were originally developed in [3], for the purpose of classifying maximal irreducible periodic subgroups of \( \text{PGL}(q, F) \).

By [14, p. 217, Theorem 6], \( \text{GL}(q, F) \) has absolutely irreducible locally nilpotent subgroups if and only if there exists an element of order \( q \) in \( F^\times \). Let \( D \) be the set \( \{ \text{diag}(\beta_1, \ldots, \beta_q) \mid \beta_i \in \text{Syl}_q(F^\times), \beta \in F^\times \} \) of diagonal matrices, where \( \text{Syl}_q \) denotes Sylow \( q \)-subgroup. For \( \alpha \in F^\times \), define

\[
I_\alpha = \begin{bmatrix} 0 & 1_{q-1} \\ \alpha & 0 \end{bmatrix} \in \text{GL}(q, F)
\]

and \( I := I_1 \). If \( H \leq \text{GL}(q, F) \) then we write \( \text{Det}(H) \) for \( \{ \det(h) \mid h \in H \} \).

Assuming that \( F^\times \) has an element of order \( q \), let \( H_\alpha = \langle D, I_\alpha \rangle \). The subgroup \( H_\alpha \) of \( \text{GL}(q, F) \) is monomial and absolutely irreducible. Since \( H_\alpha/F^\times 1_q \) is a \( q \)-subgroup of \( \text{PGL}(q, F) \), \( H_\alpha \) is locally nilpotent. When \( \text{Syl}_q(F^\times) \) is finite, \( H_\alpha \) is nilpotent with nilpotency class \( 1 + (q - 1) \log_q |\text{Syl}_q(F^\times)| \).

Denote by \( \pi \) the natural epimorphism from the group of all monomial matrices in \( \text{GL}(q, F) \) onto the group \( \text{Sym}(q) \) of \( q \times q \) permutation matrices; \( \ker \pi \) is the group \( \text{D}(q, F) \) of all diagonal matrices in \( \text{GL}(q, F) \).
Lemma 3.1 (Cf. [3] Lemma 22). Let \( a, b \in D(q, F) \). The following are equivalent:

(i) \( Ia \) is \( GL(q, F) \)-conjugate to \( Ib \);
(ii) \( Ia \) is \( D(q, F) \)-conjugate to \( Ib \);
(iii) \( \det(a) = \det(b) \).

Lemma 3.2. Let \( H \) be an irreducible monomial locally nilpotent subgroup of \( GL(q, F) \). Then \( H \) is conjugate in \( GL(q, F) \) to a subgroup of some \( H_\alpha \).

Proof. If \( H \) is abelian then \( \pi(H) \leq Sym(q) \) is transitive abelian, i.e., cyclic of order \( q \); thus \( H \cap D(q, F) \leq F^x 1_q \). If \( H \) is absolutely irreducible then \( H^{F^x} / F^x 1_q \) is a \( q \)-group, so \( H \cap D(q, F) \leq D \), and \( |\pi(H)| = q \) again. Hence, up to conjugacy, \( H \leq \langle D, Ia \rangle \) for some \( a \in D(q, F) \). By Lemma 3.1 \( H \) is then conjugate to a subgroup of \( H_{\det(a)} \). \( \Box \)

Denote \( \text{Det}(D) = \text{Syl}_q(F^x)(F^x)^g \) by \( S \).

Lemma 3.3. If \( \alpha \in S \) then \( H_\alpha \) is \( D(q, F) \)-conjugate to \( H_1 \). Let \( \alpha_1, \alpha_2 \not\in S \); then \( H_{\alpha_1} \) and \( H_{\alpha_2} \) are \( GL(q, F) \)-conjugate if and only if \( \text{Det}(H_{\alpha_1}) = \text{Det}(H_{\alpha_2}) \), i.e., \( (\alpha_1 S) \) and \( \langle \alpha_2 S \rangle \) are identical subgroups of \( F^x / S \) of order \( q \).

Proof. Suppose that \( \alpha = \beta_1 \beta'^q \) for some \( \beta_1 \in \text{Syl}_q(F^x) \) and \( \beta \in F^x \). Then \( \text{Det}(I_\alpha) = \text{Det}(I_b) \) where \( b = \text{diag}(\beta_1 \beta, \beta, \ldots, \beta) \in D \). Thus \( H_\alpha \) is \( D(q, F) \)-conjugate to \( \langle Ib, D \rangle = H_1 \) by Lemma 3.1

Suppose that \( \alpha_1, \alpha_2 \not\in S \) and \( \alpha_1 \in \langle \alpha_2, S \rangle \). Then \( \text{Det}(I_{\alpha_1}) = \text{Det}(I_{\alpha_2} c) \) for some \( c \in D \) and \( 1 \leq r \leq q - 1 \). Also, there exists \( x \in \text{Sym}(q) \) such that \( x I_{\alpha_2} c x^{-1} = Ib \) for some \( b \in D(q, F) \). Hence, by Lemma 3.1 once more, \( H_{\alpha_1} \) and \( H_{\alpha_2} \) are conjugate (this time by a monomial matrix). \( \Box \)

Corollary 3.4. Define \( \mathcal{H} = \{ H_\alpha \mid \alpha \in F^x \setminus S \} \). The \( GL(q, F) \)-conjugacy classes of the groups in \( \mathcal{H} \) are in one-to-one correspondence with the distinct subgroups of \( F^x / S \) of order \( q \). Therefore the number of such classes is finite if and only if \( F^x / S \) is finite.

Remark 3.5. If \( F \) is algebraically closed or finite then \( \mathcal{H} = \emptyset \): a maximal absolutely irreducible monomial locally nilpotent subgroup of \( GL(q, F) \) is conjugate to \( H_1 \).

We turn next to primitive groups.

Lemma 3.6. Let \( H \) be a primitive locally nilpotent subgroup of \( GL(q, F) \). Then \( H \) has an irreducible abelian normal subgroup.

Proof. Since \( H \) is a locally nilpotent linear group, it is solvable. Thus \( H \) has an abelian normal subgroup \( A \) of finite index [14, p. 135, Theorem 6]. If \( A \leq Z(H) \) then \( H/Z(H) \) is finite and so \( H \) is nilpotent. Recall that a nonabelian nilpotent group contains a noncentral abelian normal subgroup. Any such subgroup \( B \) of \( H \) must be irreducible.
For if it were reducible then $B$ would be diagonalizable with inequivalent irreducible parts, contradicting primitivity of $H$. \qed

By Lemma 3.6 a primitive locally nilpotent subgroup of $GL(q, \mathbb{F})$ is contained in the $GL(q, \mathbb{F})$-normalizer of the multiplicative group of a field extension $\Delta$ of $\mathbb{F}1_q$ of degree $q$. Since this degree is prime, $\Delta$ is a cyclic extension of $\mathbb{F}$, with Galois group of order $q$. If $\mathbb{F}^\times$ has an element $\xi$ of order $q$, then $\Delta = \langle h \rangle_{\mathbb{F}}$ for some $h \in GL(q, \mathbb{F})$ such that $h^q = \beta 1_q \in \mathbb{F}^\times 1_q$ (see [9, p. 289, Theorem 6.2]). Since $\beta \in (\mathbb{F}^\times)^q$ implies that $h$ is scalar, $\beta = \alpha^r \gamma^q$ for some $\alpha, \gamma \in \mathbb{F}^\times$ and $1 \leq r \leq q - 1$. Then $\gamma^{-1}h$ and $I^r_\alpha$ have the same characteristic (minimal) polynomial $X^q - \alpha^r 1_q$, which is $\mathbb{F}$-irreducible; so $\gamma^{-1}h$ and $I^r_\alpha$ are conjugate. Hence $\Delta$ is conjugate to $\Delta_\alpha := \langle I_\alpha \rangle_{\mathbb{F}}, \alpha \notin (\mathbb{F}^\times)^q$. We see that $N_{GL(q, \mathbb{F})}(\Delta^r_\alpha) = \langle \Delta^r_\alpha, d \rangle$ where $d = \text{diag}(1, \xi, \ldots, \xi^{q-1})$. Denote by $G(\alpha, b)$ the subgroup $\langle A_\alpha, db \rangle$ of $\langle \Delta^r_\alpha, d \rangle$, where $A_\alpha/\mathbb{F}^\times 1_q$ is the Sylow $q$-subgroup of $\Delta^r_\alpha/\mathbb{F}^\times 1_q$ and $b \in \Delta^r_\alpha$. Since $A_\alpha$ is a noncentral irreducible subgroup, $G(\alpha, b)$ is absolutely irreducible.

**Lemma 3.7.** An absolutely irreducible primitive locally nilpotent group $H \leq GL(q, \mathbb{F})$ is conjugate to a subgroup of some $G(\alpha, b)$.

**Proof.** Up to conjugacy, $H = \langle H \cap \Delta^r_\alpha, db \rangle$ for some $\alpha \in \mathbb{F}^\times \setminus (\mathbb{F}^\times)^q$ and $b \in \Delta^r_\alpha$. Then $H \cap \Delta^r_\alpha \leq A_\alpha$ by Theorem 2.4. \qed

Let $\varepsilon_k$ be an element of multiplicative order $2^k$ in $\mathbb{F}$. We drop the subscript when $k = 2$, i.e., $\varepsilon$ is a square root of $-1$.

**Lemma 3.8.** Suppose that $\text{char } \mathbb{F} \neq 2$ and $\varepsilon \notin \mathbb{F}$. Let $\mathbb{E} = \mathbb{F}(\varepsilon)$, and let $\sigma$ be the $\mathbb{F}$-involution of $\mathbb{E}$. If $\text{Syl}_2(\mathbb{E}^\times) = \langle \varepsilon_m \rangle$ is cyclic then $\text{Syl}_2(\mathbb{E}^\times/\mathbb{F}^\times)$ is cyclic. Explicitly, one of the following must be true:

(i) $\sigma(\varepsilon_m) = -\varepsilon_m^{-1}$ and $\text{Syl}_2(\mathbb{E}^\times/\mathbb{F}^\times) = \langle \varepsilon_m \mathbb{F}^\times \rangle$ has order $2^{m-1}$;

(ii) $\sigma(\varepsilon_m) = \varepsilon_m^{-1}$ and $\text{Syl}_2(\mathbb{E}^\times/\mathbb{F}^\times) = \langle (1 + \varepsilon_m) \mathbb{F}^\times \rangle$ has order $2^m$.

**Proof.** We make some preliminary comments. Either $\sigma(\varepsilon_m) = \varepsilon_m^{-1}$ or $\sigma(\varepsilon_m) = -\varepsilon_m^{-1}$. Suppose that $\sigma(\varepsilon_k) = \varepsilon_k^{-1}$. Then

$$\sigma((1 + \varepsilon_k)^{2^k}) = \sigma(1 + \varepsilon_k)^{2^k} = (1 + \varepsilon_k^{-1})^{2^k} = \left(\frac{1 + \varepsilon_k}{\varepsilon_k}\right)^{2^k} = (1 + \varepsilon_k)^{2^k}.$$ 

Thus $(1 + \varepsilon_k)^{2^k} \in \mathbb{F}$ and

(1) \hspace{1cm} $(1 + \varepsilon_k)^{\mathbb{F}^\times} \in \text{Syl}_2(\mathbb{E}^\times/\mathbb{F}^\times)$

if $k \geq 2$. Also, if $k > 2$ then $\varepsilon_k^{-1}(\varepsilon_{k-1} + 1) = \varepsilon_k^{-1}(\varepsilon_k^2 + 1) = \varepsilon_k + \varepsilon_k^{-1} \in \mathbb{F}^\times$, and hence

(2) \hspace{1cm} $1 + \varepsilon_{k-1} \in \langle \varepsilon_k \rangle \mathbb{F}^\times$. 


Let $x \in \mathbb{F}^\times$ be a nontrivial element of $\text{Syl}_2(\mathbb{E}^\times/\mathbb{F}^\times)$ of order $2^i$; so $x^{2^i} \in \mathbb{F}^\times \setminus (\mathbb{F}^\times)^2$.

Suppose that $l = 1$. Write $x = a + \varepsilon b$ for $a, b \in \mathbb{F}$. Then $2ab \varepsilon \in \mathbb{F}$ implies that $a = 0$, i.e.,

$$x \in \langle \varepsilon \mathbb{F}^\times \rangle \leq \langle \varepsilon_m \mathbb{F}^\times \rangle.$$  

Suppose next that $l \geq 2$. We have $\sigma(x) = xy$ for some $y \in \text{Syl}_2(\mathbb{E}^\times)$, $y^{2^l} = 1$. Now $y \neq -1$, because if $y = -1$ then $|x \mathbb{F}^\times| = 2 < 2^l$. Further, $x = \sigma^2(x) = \sigma(xy) = \sigma(y)yx$ and so

$$\sigma(y) = y^{-1}.$$  

Since $\text{tr}(x) = (1 + y)x \in \mathbb{F}$,

$$x \in (1 + y)^{-1} \mathbb{F}^\times = (1 + \sigma(y)) \mathbb{F}^\times = (1 + y^{-1}) \mathbb{F}^\times.$$  

We are ready to complete the proof that (i) or (ii) must be true. Let $\sigma(\varepsilon_m) = -\varepsilon_m^{-1}$, so $m > 2$. By (3), we may take $l \geq 2$, in which event $4 < |y| \leq 2^{m-1}$ by (1). Then $(1 + y^{-1}) \mathbb{F}^\times \leq \langle \varepsilon_m \mathbb{F}^\times \rangle$ by (2), and by (5), this yields (i).

Let $\sigma(\varepsilon_m) = \varepsilon_m^{-1}$. By (1), $(1 + \varepsilon_m) \mathbb{F}^\times \leq \text{Syl}_2(\mathbb{E}^\times/\mathbb{F}^\times)$. As $\sigma$ fixes $\varepsilon_m^{-1}(1 + \varepsilon_m)^2$, additionally $\varepsilon_m \mathbb{F}^\times \in ((1 + \varepsilon_m) \mathbb{F}^\times)^2$. Part (ii) follows from (2), (3), and (5).  

Corollary 3.9. If $\text{Syl}_2(\mathbb{E}^\times)$ is quasicyclic then $\text{Syl}_2(\mathbb{E}^\times/\mathbb{F}^\times)$ is quasicyclic, and equal to $\{\langle \varepsilon_k \mathbb{F}^\times \rangle \mid \varepsilon_k \in \text{Syl}_2(\mathbb{E}^\times)\}$.

Proof. Since $\sigma(\varepsilon_k) = \varepsilon_k^{-1}$ for each $\varepsilon_k \in \text{Syl}_2(\mathbb{E}^\times)$, we cannot have $\sigma(\varepsilon_k) = -\varepsilon_k^{-1}$. The corollary is then a consequence of Lemma 3.3 (ii) and (2).  

Lemma 3.10. Let $|E : F| = q$ and $E = F(a)$, where $a^{d} \in F$. Suppose that $\mathbb{F}^\times$ has an element of order $q$, and $E \neq F(\varepsilon)$ if $q = 2$. Then $\text{Syl}_q(\mathbb{E}^\times/\mathbb{F}^\times) = \langle a \mathbb{F}^\times \rangle$.

Proof. If $q > 2$, or $q = 2$ and $\varepsilon \in \mathbb{F}$, then the result follows from [8, Lemma 2].

Let $q = 2$. Select $x \mathbb{F}^\times \in \mathbb{E}^\times/\mathbb{F}^\times$ of order $2^m \geq 2$; say $x^{2^m} = \alpha \in \mathbb{F}^\times \setminus (\mathbb{F}^\times)^2$. If $\alpha = -4 \gamma^4$ for $\gamma \in \mathbb{F}^\times$, then $\gamma^{2^{m-1}}/2\gamma^2$ is a square root of $-1$, contradicting $E \neq F(\varepsilon)$.

Suppose that $\alpha \not= -4(\mathbb{F}^\times)^4$. The polynomial $X^4 - \alpha$ is $F$-irreducible, so that if $m \geq 2$ then $|E : F| \geq 4$. Hence $m = 1$, $a = \sqrt{\beta}$ for some $\beta \in \mathbb{F}^\times$, and $x = \sqrt{\alpha}$. Now $\sqrt{\alpha} = x_1 + x_2 \sqrt{\beta}$ for some $x_1, x_2 \in \mathbb{F}$. Then $\alpha = x_1^2 + \beta x_2^2 + 2x_1 x_2 \sqrt{\beta}$ implies that $x_1 = 0$ or $x_2 = 0$. As the latter is impossible, $x \in \langle a \mathbb{F}^\times 1_2 \rangle$.  

Lemma 3.11. If $q > 2$ or $\alpha \not= -(\mathbb{F}^\times)^2$ then $A_\alpha$ is the monomial group $\langle I_\alpha, \mathbb{F}^\times 1_q \rangle$. Otherwise, $A_\alpha$ is primitive.

Proof. In Lemma 3.10 put $E = \Delta_\alpha$ and $a = I_\alpha$. Then $A_\alpha = \langle I_\alpha, \mathbb{F}^\times 1_q \rangle$ unless $q = 2$ and $\Delta_\alpha \cong F(\varepsilon)$, i.e., $\alpha \not= -(\mathbb{F}^\times)^2$. If $A_{-\gamma^2} \leq \text{GL}(2, \mathbb{F})$ were monomial then its square would be scalar; however, $1_2 + \gamma^{-1} I_{-\gamma^2} \in A_{-\gamma^2}$ has projective order 4.
We refer to the set of hypotheses \( q = 2 \) and \( \alpha \in -((\mathbb{F}^\times)^2) \) as case (*). Lemma 3.8 and Corollary 3.9 describe the \( A_\alpha \) in case (*). Actually, a group \( G(-\gamma^2, b) \) in this case is conjugate to some \( G(-1, b') \), since \( I_{-\gamma^2} = D(q, \mathbb{F}) \)-conjugate to \( \gamma I_{-1} \) by Lemma 3.11

In all but case (*), \( |G(\alpha, b)/\mathbb{F}^\times 1_q| = q^2 \) (because \( (db)^q = det(b)1_q \) and \( [I_\alpha, d] \) is scalar), so \( G(\alpha, b) \) is class 2 nilpotent. The locally nilpotent group \( G(-1, b) \) is nilpotent only if \( Syl_2(\Delta_\alpha) \) is cyclic, when \( G(-1, b)/\mathbb{F}^\times 1_q \) is a dihedral 2-group and \( G(-1, b) \) has nilpotency class \( \log_2 |Syl_2(\Delta_\alpha)|= 2 \).

**Lemma 3.12.** In case (*), \( G(\alpha, b) \) is primitive. In all other cases, \( G(\alpha, b) \) is primitive if and only if \( det(b) \not\in (-1)^{q-1} \alpha, (\mathbb{F}^\times)^q \) = Det(\( A_\alpha \)).

*Proof.* By Lemma 3.11, assume that we are not in case (*). According to [3] Lemma 1, \( G(\alpha, b) \) is primitive if and only if its elements of order \( q \) are all scalar. Suppose that \( det(b) \not\in Det(\alpha) \) and let \( h \in G(\alpha, b) \), \( |h| = q \). If \( h \not\in \alpha \), i.e., \( h = dbb_1 \) for some \( b_1 \in A_{\alpha} \), then \( h^q = det(bb_1b)^q \) implies that \( det(b) \in Det(\alpha) \). Thus \( h \in \alpha \), and \( h \) is scalar by Lemma 3.11. Conversely, if \( det(b) \in Det(\alpha) \) then \( dbx \) for some \( x \in A_\alpha \) is a nonscalar element of \( G(\alpha, b) \) of order \( q \). \( \square \)

**Remark 3.13.** Except in case (*), if \( \mathbb{F} \) is finite then \( G(\alpha, b) \) is monomial.

**Lemma 3.14.** For \( i = 1, 2 \), let \( g_i = db_i \) where \( b_i \in \Delta_\alpha \). The following are equivalent:

(i) \( g_1 \) is \( GL(q, \mathbb{F}) \)-conjugate to \( g_2 \);

(ii) \( g_1 \) is \( \Delta_\alpha \)-conjugate to \( g_2 \);

(iii) \( det(b_1) = det(b_2) \).

*Proof.* See [3] Lemma 23]. \( \square \)

**Corollary 3.15.** Primitive groups \( G(\alpha, b_1), G(\alpha, b_2) \) not in case (*) are conjugate if and only if \( Det(G(\alpha, b_1)) = Det(G(\alpha, b_2)) \) and \( det(b_1) = det(b_2c) \) for some \( c \in A_\alpha \).

*Proof.* Suppose that \( tG(\alpha, b_1)t^{-1} = G(\alpha, b_2) \). Since \( t \) normalizes \( A_\alpha \) (Lemma 3.12), we have \( t \in \langle d, \Delta_\alpha \rangle \). Then it may be checked that \( tdb_1t^{-1} \in db_2A_\alpha \). The other direction is clear by Lemma 3.14. \( \square \)

Denote by \( \mathcal{G} \) the set of all primitive \( G(\alpha, b) \) such that the only case (*) groups in \( \mathcal{G} \) are the \( G(-1, b) \).

**Remark 3.16.** \( \mathcal{G} = \emptyset \) if \( \mathbb{F} \) is algebraically closed, for then \( G(\alpha, b) \) is not defined. When \( \mathbb{F} \) is finite, \( \mathcal{G} \neq \emptyset \) if and only if \( q = 2 \) and \( |\mathbb{F}| \equiv 3 \) (mod 4).

**Lemma 3.17.** The subset \( \tilde{\mathcal{G}} \) of \( \mathcal{G} \) consisting of all groups not in case (*) splits up into finitely many \( GL(q, \mathbb{F}) \)-conjugacy classes if and only if \( \mathbb{F}^\times/(\mathbb{F}^\times)^q \) is finite.
Proof. If the number of conjugacy classes in $\tilde{G}$ is finite, then $\mathbb{F}^\times/(\mathbb{F}^\times)^q$ is finitely generated and hence finite. Conversely, if there are only finitely many candidates for $\text{Det}(G(\alpha, b))$, then the number of conjugacy classes in $\tilde{G}$ is finite by Corollary 3.15. □

We now present the main classification.

**Theorem 3.18.** Suppose that $\mathbb{F}^\times$ has an element of order $q$. A subgroup of $\text{GL}(q, \mathbb{F})$ is an absolutely irreducible maximal locally nilpotent subgroup of $\text{GL}(q, \mathbb{F})$ if and only if it is conjugate to a group in $\mathcal{N} = \{H_1 \cup H \cup \mathcal{G},$ with the following exceptions when $q = 2$ and $\varepsilon \not\in \mathbb{F}$:

(i) $H_1$ is a proper subgroup of $G(-1, 1) \in \mathcal{G}$;

(ii) if $\alpha \not\in -(\mathbb{F}^\times)^2$ and either $\det(b) \in -(\mathbb{F}^\times)^2$ or $\det(b) \in \alpha(\mathbb{F}^\times)^2$, then $G(\alpha, b)$ is conjugate to a proper subgroup of $G(-1, c)$ where $\det(c) = \alpha$.

Proof. We observed previously that all groups in $\mathcal{N}$ are absolutely irreducible locally nilpotent. By Lemmas 3.2, 3.3, 3.7 and remarks after Lemma 3.11 an absolutely irreducible locally nilpotent subgroup of $\text{GL}(q, \mathbb{F})$ is conjugate to a subgroup of a group in $\mathcal{N}$. It remains to show that the $H_\alpha$ and $G(\alpha, b) \in \mathcal{G}$ are maximal locally nilpotent, with the stated exceptions.

Let $G$ be a maximal locally nilpotent subgroup of $\text{GL}(q, \mathbb{F})$ containing $H_\alpha$. If $G$ is monomial then $tGt^{-1} = H_\beta$ for some $t$, $H_\beta$. If $tDt^{-1} \neq D$ then $tDt^{-1} \cap D$ is scalar of index $q$ in $D$, so $|H_\beta/\mathbb{F}^\times 1_q| = q^2$; but $H_\beta/\mathbb{F}^\times 1_q$ has cardinality at least $q^{q+1}$. Thus $tDt^{-1} = D$, and then $q = |H_\beta : D| \geq |tH_\alpha t^{-1} : D| = |H_\alpha : D| = q$. Therefore $tH_\alpha t^{-1} = H_\beta$, i.e., $H_\alpha = G$. Next suppose that $G$ is primitive, hence conjugate to some $G(\alpha_1, b)$. In every case other than $(s)$, $|G/\mathbb{F}^\times 1_q| = q^2$ is less than the cardinality of $H_\alpha/\mathbb{F}^\times 1_q$. Hence $q = 2$, $\varepsilon \not\in \mathbb{F}^\times$, $G$ is conjugate to $G(-1, b)$, and $H_\alpha = \langle d, I_\alpha, \mathbb{F}^\times 1_2 \rangle$. Either $I_\alpha$ or $I_{-\alpha} = dI_\alpha$ is conjugate to $h \in A_{-1}$ such that $h^2 = \mathbb{F}^\times 1_2$. Now $h$ has the form $\eta I_{-1}$, $\eta \in \mathbb{F}^\times$, and by comparing determinants we get $\alpha = \pm \eta^2$. Thus if $H_\alpha \in \mathcal{H}$ then $H_\alpha$ is maximal. However, $H_1 = \langle d, I_{-1}, \mathbb{F}^\times 1_2 \rangle$ is a proper subgroup of $G(-1, 1)$.

Let $G$ be a maximal locally nilpotent subgroup of $\text{GL}(q, \mathbb{F})$ containing $G(\alpha, b) \in \mathcal{G}$. Then $tGt^{-1} = G(\alpha_1, b_1) \in \mathcal{G}$ for some $t$. Apart from when $q = 2$, $\varepsilon \not\in \mathbb{F}^\times$, and $\alpha_1 = -1$, we have $|G(\alpha, b)/\mathbb{F}^\times 1_q| = |G(\alpha_1, b_1)/\mathbb{F}^\times 1_q| = |G/\mathbb{F}^\times 1_q| = q^2$, and thus $G(\alpha, b) = G$. Suppose that $q = 2$, $\varepsilon \not\in \mathbb{F}$, and $\alpha_1 = -1$. If $\alpha \not\in -(\mathbb{F}^\times)^2$ then $G(\alpha, b)$ is conjugate to some $G(-1, b_2)$, so that $G(\alpha, b) = G$. Therefore, if $G(\alpha, b)$ is not maximal then $\alpha \not\in -(\mathbb{F}^\times)^2$. For the rest of the proof, $\alpha \not\in -(\mathbb{F}^\times)^2$, which means that $G(\alpha, b) = \langle I_\alpha, db, \mathbb{F}^\times 1_2 \rangle$. One of the following occurs: $tI_\alpha t^{-1} \not\in A_{-1}$ and $tdbt^{-1} \in A_{-1}$, or $tI_\alpha t^{-1} \not\in A_{-1}$ and $tdbt^{-1} \not\in A_{-1}$. In the first situation, $\det(b) \in -(\mathbb{F}^\times)^2$; in the second, $[I_\alpha, db]$ scalar forces $tdbt^{-1} \in I_{-1}dI_\alpha t^{-1}\mathbb{F}^\times$, so $\det(b) \in \alpha(\mathbb{F}^\times)^2$. Suppose that $\det(b) \in -(\mathbb{F}^\times)^2$. Then $\det(db) = \det(\eta I_{-1})$ for some $\eta \in \mathbb{F}^\times$. Since $db$ and $\eta I_{-1}$ have zero trace, these elements are conjugate, say $sdb^{-1} = \eta I_{-1}$. Also, $sI_\alpha s^{-1}$ normalizes
$\Delta^*_1$, because $I_\alpha dB I_\alpha^{-1} = -db$. Thus $sI_\alpha s^{-1} = dc$, where $c \in \Delta^*_1$ and $\det(c) = \alpha$. It follows that $G(\alpha, b)$ is conjugate to a proper subgroup of $G(-1, c)$. Entirely similar reasoning leads to this same conclusion when $\det(b) \in \alpha(F^\times)^2$.

The next lemma completes our classification.

**Lemma 3.19.** Suppose that $GL(q, F)$ has irreducible abelian subgroups, and let $H$ be an irreducible maximal abelian subgroup of $GL(q, F)$.

(i) If $F^\times$ has no element of order $q$, then $H$ is a maximal locally nilpotent subgroup of $GL(q, F)$, and any maximal locally nilpotent subgroup of $GL(q, F)$ is abelian.

(ii) Suppose that $F^\times$ has an element of order $q$. Then $H$ is a maximal locally nilpotent subgroup of $GL(q, F)$ unless $q = 2$ and $\varepsilon \notin F$; in that case, $H$ is a maximal locally nilpotent subgroup if and only if $H/F^\times 1_2$ is not a 2-group.

**Proof.** We prove (ii). Clearly $H$ is not monomial. By Lemma 3.11 if $H$ is not maximal locally nilpotent then $q = 2$, $\varepsilon \notin F$, and $H = A_\alpha \leq G(\alpha, \beta) \in \mathcal{G}$. □

**References**

1. R. Beals, *Algorithms for matrix groups and the Tits alternative*, J. Comput. System Sci. **58** (1999), 260–279.
2. A. S. Detinko and D. L. Flannery, *Classification of nilpotent primitive linear groups over finite fields*, Glasg. Math. J. **46** (2004), no. 3, 585–594.
3. A. S. Detinko and D. L. Flannery, *Periodic subgroups of projective linear groups in positive characteristic*, Cent. Eur. J. Math. **6** (2008), no. 3, 384–392.
4. M. L. Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. No. 53 (1981), 53–73.
5. D. F. Holt, B. Eick, and E. A. O’Brien, *Handbook of computational group theory*, Chapman & Hall/CRC, Boca Raton, 2005.
6. V. S. Konyukh, *Sylow p-subgroups of a projective linear group*, Vestsı Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk (1985), no. 6, 23–29, 124–125.
7. V. S. Konyukh, *On linear p-groups*, Vestsı Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk (1987), no. 1, 3–8, 124.
8. V. S. Konyukh, *Irreducible locally nilpotent linear groups*, Fundam. Prikl. Mat. **4** (1998), 1345–1364.
9. S. Lang, *Algebra*, Graduate Texts in Mathematics **211**, Springer-Verlag, New York, 2002.
10. C. R. Leedham-Green and W. Plesken, *Some remarks on Sylow subgroups of general linear groups*, Math. Z. **191** (1986), 529–535.
11. D. Segal, *Polycyclic groups*, Cambridge University Press, Cambridge, 1983.
12. M. Shirvani and B. A. F. Wehrfritz, *Skew linear groups*, London Mathematical Society Lecture Note Series, vol. 118, Cambridge University Press, Cambridge, 1986.
13. D. A. Suprunenko, *Soluble and nilpotent linear groups*, Transl. Math. Monogr., vol. 9, American Mathematical Society, Providence, RI, 1963.
14. D. A. Suprunenko, *Matrix groups*, Transl. Math. Monogr., vol. 45, American Mathematical Society, Providence, RI, 1976.
15. B. A. F. Wehrfritz, *Infinite linear groups*, Springer-Verlag, New York, 1973.