Eight-dimensional Octonion-like but Associative Normed Division Algebra

Joy Christian
Einstein Centre for Local-Realistic Physics, Oxford OX2 6LB, United Kingdom

ABSTRACT:
We present an eight-dimensional even sub-algebra of the $2^4 = 16$-dimensional associative Clifford algebra $\text{Cl}_{4,0}$ and show that its eight-dimensional elements denoted as $X$ and $Y$ respect the norm relation $||XY|| = ||X|| \cdot ||Y||$, thus forming an octonion-like but associative normed division algebra, where the norms are calculated using the fundamental geometric product instead of the usual scalar product so that the underlying coefficient algebra resembles that of split complex numbers instead of reals. The corresponding 7-sphere has a topology that differs from that of octonionic 7-sphere.

(Note added to proof: The results of this paper are published in Section 2.8 of Ref. [16] listed in the bibliography.)

Consider the following eight-dimensional vector space with graded Clifford-algebraic basis and orientation $\lambda = \pm 1$:

$$\text{Cl}^\lambda_{3,0} = \text{span}\{ 1, \lambda e_x, \lambda e_y, \lambda e_z, \lambda e_x e_y, \lambda e_y e_z, \lambda e_z e_x \}.$$  \hspace{1cm} (1)

As we shall see, the choice of orientation, $\lambda = +1$ or $\lambda = -1$ does not affect our argument. In what follows we will use the language of Geometric Algebra, as used, for example, in Refs. [1] and [2]. Accordingly, in the above definition \{ $e_x, e_y, e_z$ \} is a set of anti-commuting orthonormal vectors in $\mathbb{R}^3$ such that $e_i e_j = -e_j e_i$ for any $i \neq j = x, y, or z$. In general the vectors $e_i$ satisfy the following geometric product in this associative but non-commutative algebra [1, 2]:

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j,$$  \hspace{1cm} (2)

with

$$e_i \cdot e_j := \frac{1}{2} \{ e_i e_j + e_j e_i \}$$  \hspace{1cm} (3)

being the symmetric inner product and

$$e_i \wedge e_j := \frac{1}{2} \{ e_i e_j - e_j e_i \}$$  \hspace{1cm} (4)

being the anti-symmetric outer product, giving $(e_i \wedge e_j)^2 = -1$. There are thus basis elements of four different grades in this algebra: An identity element $e_i^2 = 1$ of grade-0, three orthonormal vectors $e_i$ of grade-1, three orthonormal bivectors $e_j e_k$ of grade-2, and a trivector $I_3 = e_x e_y e_z$ of grade-3 representing a volume element in $\mathbb{R}^3$. Since in $\mathbb{R}^3$ there are $2^3 = 8$ ways to combine the vectors $e_i$ using the geometric product (2) such that no two products are linearly dependent, the resulting algebra, $\text{Cl}^\lambda_{3,0}$, is a linear vector space of eight dimensions, spanned by these graded bases.

In this paper we are interested in a certain conformal completion\footnote{The conformal space we are considering is an inhomogeneous version of the space usually studied in Conformal Geometric Algebra [2]. It can be viewed as an 8-dimensional subspace of the 32-dimensional representation space postulated in Conformal Geometric Algebra. The larger representation space results from a homogeneous freedom of the origin within $\mathbb{E}^3$, which does not concern us in this paper.} of this algebra, originally presented in Ref. [3]. This is accomplished by using an additional vector, $e_\infty$, to close the lines and volumes of the Euclidean space, giving

$$\mathcal{K}^\lambda = \text{span}\{ 1, \lambda e_x e_y, \lambda e_x e_z, \lambda e_y e_z, \lambda e_x e_\infty, \lambda e_y e_\infty, \lambda e_z e_\infty, \lambda I_3 e_\infty \}.$$  \hspace{1cm} (5)

*Electronic address: jjc@bu.edu
importantly, we shall soon see that for vectors $K$ multiplication. Suppose $X$ and $e_\infty$ have some remarkable properties \[3\]. To begin with, it is a powerful property. More

\[
\begin{array}{|c|cccccccc|}
\hline
* & 1 & \lambda e_x e_y & \lambda e_x e_z & \lambda e_y e_z & \lambda e_x e_\infty & \lambda e_y e_\infty & \lambda e_z e_\infty & \lambda I_3 e_\infty \\
\hline
1 & 1 & \lambda e_x e_y & \lambda e_x e_z & \lambda e_y e_z & \lambda e_x e_\infty & \lambda e_y e_\infty & \lambda e_z e_\infty & \lambda I_3 e_\infty \\
\lambda e_x e_y & \lambda e_x e_y & -1 & e_y e_x & -e_x e_y & -e_y e_\infty & e_x e_\infty & -e_z e_\infty & -I_3 e_\infty \\
\lambda e_x e_z & \lambda e_x e_z & -e_y e_z & -1 & e_x e_y & e_z e_\infty & -I_3 e_\infty & -e_x e_\infty & \lambda I_3 e_\infty \\
\lambda e_y e_z & \lambda e_y e_z & e_x e_z & -e_y e_x & -1 & I_3 e_\infty & \lambda I_3 e_\infty & -e_x e_\infty & \lambda I_3 e_\infty \\
\lambda e_x e_\infty & \lambda e_x e_\infty & e_y e_\infty & -e_x e_\infty & -I_3 e_\infty & e_x e_\infty & \lambda I_3 e_\infty & -e_y e_\infty & -e_x e_\infty \\
\lambda e_y e_\infty & \lambda e_y e_\infty & -e_x e_\infty & -I_3 e_\infty & e_x e_\infty & e_y e_\infty & -e_x e_\infty & -e_y e_\infty & e_y e_\infty \\
\lambda e_z e_\infty & \lambda e_z e_\infty & -e_y e_\infty & I_3 e_\infty & -e_x e_\infty & -e_y e_\infty & e_z e_\infty & -e_z e_\infty & -e_z e_\infty \\
\lambda I_3 e_\infty & \lambda I_3 e_\infty & -e_y e_\infty & \lambda I_3 e_\infty & -e_x e_\infty & -e_y e_\infty & -e_z e_\infty & -e_z e_\infty & \lambda I_3 e_\infty \\
\hline
\end{array}
\]

TABLE I: Multiplication Table for a “Conformal Geometric Algebra” $K^\lambda$ of $E^3$. Here $I_3 = e_x e_y e_z$, $e_\infty^2 = +1$, and $\lambda = \pm 1$.

With unit vector $e_\infty$, this is an eight-dimensional even sub-algebra of the $2^4 = 16$-dimensional Clifford algebra $Cl_{4,0}$. Unlike the seven imaginaries of octonions, there are only six basis elements of $K^\lambda$ that are imaginary. The seventh, $\lambda I_3 e_\infty$, squares to $+1$. This is evident from the multiplication table\[1\]. We therefore call it an “octonian-like” algebra. As an eight-dimensional linear vector space, $K^\lambda$ has some remarkable properties \[3\]. To begin with, it is closed under multiplication. Suppose $X$ and $Y$ are two vectors in $K^\lambda$. Then $X$ and $Y$ can be expanded in the graded basis of $K^\lambda$:

\[
X = X_0 + X_1 \lambda e_x e_y + X_2 \lambda e_x e_z + X_3 \lambda e_y e_z + X_4 \lambda e_x e_\infty + X_5 \lambda e_y e_\infty + X_6 \lambda e_z e_\infty + X_7 \lambda I_3 e_\infty
\]  

(6)

and

\[
Y = Y_0 + Y_1 \lambda e_x e_y + Y_2 \lambda e_x e_z + Y_3 \lambda e_y e_z + Y_4 \lambda e_x e_\infty + Y_5 \lambda e_y e_\infty + Y_6 \lambda e_z e_\infty + Y_7 \lambda I_3 e_\infty
\]  

(7)

And using the definition $||X||^2 := X \cdot X^\dagger$ for the quadratic form $Q(X)$ (where $^\dagger$ represents the reverse operation \[1\] and $X \cdot X^\dagger$ represents the scalar part of the geometric product $XX^\dagger$) the multivectors $X$ and $Y$ can be normalized as

\[
||X||^2 = \sum_{\mu=0}^{7} X_\mu^2 = 1 \quad \text{and} \quad ||Y||^2 = \sum_{\nu=0}^{7} Y_\nu^2 = 1.
\]  

(8)

Now it is evident from the multiplication table above (Table\[1\]) that if $X, Y \in K^\lambda$, then so is their product $Z = XY$:

\[
Z = Z_0 + Z_1 \lambda e_x e_y + Z_2 \lambda e_x e_z + Z_3 \lambda e_y e_z + Z_4 \lambda e_x e_\infty + Z_5 \lambda e_y e_\infty + Z_6 \lambda e_z e_\infty + Z_7 \lambda I_3 e_\infty = XY.
\]  

(9)

Thus $K^\lambda$ remains closed under arbitrary number of multiplications of its elements. This is a powerful property. More importantly, we shall soon see that for vectors $X$ and $Y$ in $K^\lambda$ (not necessarily unit) the following norm relation holds:

\[
||XY|| = ||X|| \cdot ||Y||,
\]  

(10)
provided the norms are calculated employing the fundamental geometric product instead of the usual scalar product. In particular, this means that for any two unit vectors \( \mathbf{X} \) and \( \mathbf{Y} \) in \( \mathcal{K}^\lambda \) with the geometric product \( \mathbf{Z} = \mathbf{X} \mathbf{Y} \) we have

\[
\| \mathbf{Z} \|^2 = \sum_{\rho = 0}^{7} Z^2_{\rho} = 1.
\]  

(11)

Now, in order to prove the norm relation (10), it is convenient to express the elements of \( \mathcal{K}^\lambda \) as “dual” quaternions. The idea of dual numbers, \( z \), analogous to complex numbers, was introduced by Clifford in his seminal work as follows:

\[
z = r + d \varepsilon,
\]

(12)

where \( \varepsilon \) is the dual operator, \( r \) is the real part, and \( d \) is the dual part. Similar to how the “imaginary” operator \( i \) is introduced in the complex number theory to distinguish the “real” and “imaginary” parts of a complex number, Clifford introduced the dual operator \( \varepsilon \) to distinguish the “real” and “dual” parts of a dual number. The dual number theory can be extended to numbers of higher grades, including to numbers of composite grades, such as quaternions.

In analogy with dual numbers, but with \( \varepsilon^2 = +1 \), it is convenient for our purposes to write the elements of \( \mathcal{K}^\lambda \) as

\[
Q_z = q_r + q_d \varepsilon,
\]

(13)

where \( q_r \) and \( q_d \) are quaternions and \( Q_z \) may now be viewed as a “dual”-quaternion (or in Clifford’s terminology, as a bi-quaternion). Next, recall that the set of unit quaternions is a 3-sphere, which can be normalized to a radius \( \varrho_r \) and written as the set

\[
S^3 = \left\{ q_r := q_0 + q_1 \lambda \mathbf{e}_x \mathbf{e}_y + q_2 \lambda \mathbf{e}_z \mathbf{e}_x + q_3 \lambda \mathbf{e}_y \mathbf{e}_z \left| \|q_r\| = \sqrt{q_r q_r^\dagger} = \varrho_r \right. \right\}.
\]

(14)

Consider now a second, dual copy of the set of quaternions within \( \mathcal{K}^\lambda \), corresponding to the fixed orientation \( \lambda = +1 \):

\[
S^3 = \left\{ q_d := -q_7 + q_6 \mathbf{e}_x \mathbf{e}_y + q_5 \mathbf{e}_z \mathbf{e}_x + q_4 \mathbf{e}_y \mathbf{e}_z \left| \|q_d\| = \sqrt{q_d q_d^\dagger} = \varrho_d \right. \right\}.
\]

(15)

If we now identify \( \lambda I_3 \mathbf{e}_\infty \) appearing in (15) as the duality operator \( -\varepsilon \), then (in the reverse additive order) we obtain

\[
\varepsilon \equiv -\lambda I_3 \mathbf{e}_\infty, \quad \text{with} \quad \varepsilon^\dagger = \varepsilon \quad \text{and} \quad \varepsilon^2 = +1 \quad \text{(since} \mathbf{e}_\infty \text{is a unit vector in} \mathcal{K}^\lambda),
\]

(16)
\[ q_d \epsilon = -q_d \lambda I_3 e_\infty = q_4 \lambda e_4 e_\infty + q_5 \lambda e_5 e_\infty + q_6 \lambda e_6 e_\infty + q_7 \lambda I_3 e_\infty, \]  

which is a multi-vector “dual” to the quaternion \( q_d \). Note that we write \( \epsilon \) as if it were a scalar because it commutes with all element of \( \mathcal{K}^\lambda \) in [5]. Comparing (14) and (17) with (5) we now wish to write \( \mathcal{K}^\lambda \) as a set of paired quaternions,

\[ \mathcal{K}^\lambda = \left\{ Q_z := q_r + q_d \epsilon \mid ||Q_z|| = \sqrt{Q_z Q_z^\dagger} = \sqrt{\epsilon^2 + \epsilon_d^2}, \, 0 < \epsilon_r < \infty, \, 0 < \epsilon_d < \infty \right\}, \]

in analogy with (14) or (15), with \( Q_z Q_z^\dagger \) being the geometric product between \( Q_z \) and \( Q_z^\dagger \) (instead of the inner product \( Q_z \cdot Q_z^\dagger \) used in (5) to calculate the value of \( ||Q_z|| \)). But this definition \( ||Q_z|| = \sqrt{Q_z Q_z^\dagger} = \sqrt{\epsilon^2 + \epsilon_d^2} \) for the norm is possible only if we require \( q_r q_d^\dagger + q_d q_r^\dagger = 0 \), rendering every \( q_r \) orthogonal to its dual \( q_d \) (cf. Fig. 1). In other words,

\[ ||Q_z|| = \sqrt{Q_z Q_z^\dagger} = \sqrt{\epsilon^2 + \epsilon_d^2} \iff q_r q_d^\dagger + q_d q_r^\dagger = 0, \]

or equivalently, \( (q_r q_d^\dagger)_s = 0 \); i.e., \( q_r q_d^\dagger \) must be a pure quaternion (for a pedagogical discussion of (19) see section 7.1 of Ref. [4]). We can see this by working out the geometric product of \( Q_z \) with \( Q_z^\dagger \) while using \( \epsilon^2 = +1 \), which gives

\[ Q_z Q_z^\dagger = \left( q_r q_r^\dagger + q_d q_d^\dagger \right) + \left( q_r q_d^\dagger + q_d q_r^\dagger \right) \epsilon. \]

Now, using definitions (14) and (15), it is easy to see that \( q_r q_r^\dagger = \epsilon_r^2 \) and \( q_d q_d^\dagger = \epsilon_d^2 \), reducing the above product to

\[ Q_z Q_z^\dagger = \epsilon_r^2 + \epsilon_d^2 + \left( q_r q_d^\dagger + q_d q_r^\dagger \right) \epsilon. \]

In terms of the coefficients of \( Q_z \) the quantity \( q_r q_d^\dagger + q_d q_r^\dagger \) can be worked out and it turns out to be a scalar as well:

\[ q_r q_d^\dagger + q_d q_r^\dagger = -2 q_0 q_7 + 2 \lambda q_1 q_6 + 2 \lambda q_2 q_5 + 2 \lambda q_3 q_4. \]

Consequently, since \( \epsilon \) appearing in (21) is a pseudoscalar, the product \( Q_z Q_z^\dagger \) between \( Q_z \) and \( Q_z^\dagger \) is always of the form

\[ Q_z Q_z^\dagger = (\text{a scalar}) + (\text{a scalar}) \times \epsilon \]

\[ = (\text{a scalar}) + (\text{a pseudoscalar}). \]

It is thus clear that for \( Q_z Q_z^\dagger \) to be a scalar, \( q_r q_d^\dagger + q_d q_r^\dagger \) must vanish, or equivalently \( q_r \) must be orthogonal to \( q_d \).

But there is more to the normalization condition \( q_r q_d^\dagger + q_d q_r^\dagger = 0 \) than meets the eye. It also leads to the crucial norm relation (10), which is at the heart of the only known four normed division algebras \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) and \( \mathbb{O} \) associated with the four parallelizable spheres \( S^0, S^1, S^3 \) and \( S^7 \), with octonions forming a non-associative algebra in addition to forming a non-commutative algebra \( \mathbb{R}, \mathbb{C}, \mathbb{H} \). However, before we prove the norm relation (10), let us take a closer look at definition (19) within Geometric Algebra. In (5) we used the following definition of norm for a general multivector:

\[ ||X|| = \sqrt{X \cdot X^\dagger}. \]

The left-hand side of this equation — by definition — is a scalar number; namely, \( ||X|| \). But what is important to recognize for our purpose of proving (10) is that there are two equivalent ways of working out this scalar number:

(a) \( ||X|| = \) square root of the scalar part \( X \cdot X^\dagger \) of the geometric product \( XX^\dagger \) between \( X \) and \( X^\dagger \), or

(b) \( ||X|| = \) square root of the geometric product \( XX^\dagger \) with the non-scalar part of \( XX^\dagger \) set to zero.
The above two definitions of the norm $||X||$ are entirely equivalent. They give one and the same scalar value for the norm. Moreover, in general, given a product denoted by $\ast$, the quantity $X^\dagger$ is said to be the conjugate of $X$ if $X \ast X^\dagger$ happens to be equal to unity, $X \ast X^\dagger = 1$, as in the case of quaternionic products in (14) and (15). On the other hand, in Geometric Algebra the fundamental product between any two multivectors $X$ and $Y$ is the geometric product, $XY$, not the scalar product $X \cdot Y$ (or the wedge product $X \wedge Y$ for that matter). Therefore the product that must be used in computing the norm $||X||$ that preserves the Clifford algebraic structure of $\mathcal{K}^\lambda$ is the geometric product $XX^\dagger$, not the scalar product $X \cdot X^\dagger$. To be sure, in practice, if one is interested only in working out the value of the norm $||X||$, then it is often convenient to use the definition (a) above. However, our primary purpose here in working out the norms of $X$ and $Y$ is to preserve the algebraic structure of the space $\mathcal{K}^\lambda$ in the fundamental relation (10), and therefore the definition of the norm we must use is necessarily the second definition stated above; i.e., definition (b).

With the above comments in mind, we are now ready to prove the norm relation (10). To this end, suppose the multivectors $X$ and $Y$ belonging to $\mathcal{K}^\lambda$ as spelled out in (6) and (7) are normalized using the definition (b) as follows:

$$||X|| = \sqrt{XX^\dagger} = \sqrt{(q_2^2 + 0 \times \varepsilon)} = \varrho_X := \sqrt{\varrho_{r1}^2 + \varrho_{d1}^2}$$

and

$$||Y|| = \sqrt{YY^\dagger} = \sqrt{(q_2^2 + 0 \times \varepsilon)} = \varrho_Y := \sqrt{\varrho_{r2}^2 + \varrho_{d2}^2},$$

where $\varrho_X$ and $\varrho_Y$ are fixed scalars. Then, according to (11), their product $XY$ in $\mathcal{K}^\lambda$ is another multivector, giving

$$||XY|| = \sqrt{(XY)(XY)^\dagger}$$

$$= \sqrt{XYY^\dagger X^\dagger}$$

$$= \sqrt{X (q_2^2 + 0 \times \varepsilon) X^\dagger}$$

$$= \sqrt{(XX^\dagger) \varrho_Y}$$

$$= \sqrt{(q_2^2 + 0 \times \varepsilon)} \varrho_Y$$

$$= \varrho_X \varrho_Y$$

$$= ||X|| \cdot ||Y||.$$

Thus, at first sight, the norm relation (10) appears to be trivially true. However, the above simple proof is not quite satisfactory because we have assumed that $X$ and $Y$ are normalized using the definition (b), which requires us to set the non-scalar parts of the geometric products $XX^\dagger$ and $YY^\dagger$ equal to zero. That is not difficult to do for both $X$ and $Y$, but what is involved in the above proof is the geometric product $XY$ and its conjugate $(XY)^\dagger$, which makes the proof less convincing. It is therefore important to spell out the proof in full detail by assuming only that the non-scalar parts of the geometric products $XX^\dagger$ and $YY^\dagger$ are zero but without assuming a priori that the non-scalar part of the geometric product $(XY)(XY)^\dagger$ is zero; i.e., we would like to derive the latter by assuming only the former.

To that end, we first work out the right-hand side of the norm relation (10) in the notations of the condition (19):

$$||Q_{z1}|| \cdot ||Q_{z2}|| = \left( \sqrt{\varrho_{r1}^2 + \varrho_{d1}^2} \right) \left( \sqrt{\varrho_{r2}^2 + \varrho_{d2}^2} \right) = \sqrt{\varrho_{r1}^2 \varrho_{r2}^2 + \varrho_{r1}^2 \varrho_{d2}^2 + \varrho_{d1}^2 \varrho_{r2}^2 + \varrho_{d1}^2 \varrho_{d2}^2}. \quad (34)$$

Now, to verify the left-hand side of the norm relation (10), consider a product of two distinct members of the set $\mathcal{K}^\lambda$,

$$Q_{z1} Q_{z2} = (q_{r1} q_{r2} + q_{d1} q_{d2}) + (q_{r1} q_{d2} + q_{d1} q_{r2}) \varepsilon,$$

(35)

together with their individual definitions

$$Q_{z1} = q_{r1} + q_{d1} \varepsilon \quad \text{and} \quad Q_{z2} = q_{r2} + q_{d2} \varepsilon.$$

(36)
If we now use the fact that \( \varepsilon \), along with \( \varepsilon^\dagger = \varepsilon \) and \( \varepsilon^2 = 1 \), commutes with every element of \( K^\lambda \) defined in (12) and consequently with all \( q_r, q_1^\dagger, q_d \) and \( q_d^\dagger \), and work out \( Q_{1z_1}, Q_{1z_2}^\dagger \) and the products \( Q_{z_1} Q_{z_2}^\dagger \) and \( (Q_{z_1} Q_{z_2})^\dagger \) as

\[
Q_{1z_1} = q_1^\dagger + q_d^\dagger \varepsilon, \\
Q_{1z_2} = q_1^\dagger + q_d^\dagger \varepsilon, \\
Q_{z_1} Q_{z_2} = (q_1 q_1^\dagger + q_d q_d^\dagger) \varepsilon, \\
Q_{z_2} Q_{z_1} = (q_2 q_2^\dagger + q_d q_d^\dagger) \varepsilon, \\
\text{and } (Q_{z_1} Q_{z_2})^\dagger = (q_1 q_1^\dagger + q_d q_d^\dagger) \varepsilon,
\]

then, using the same normalization condition \( q_r, q_1^\dagger + q_d q_d^\dagger = 0 \) of (19), the norm relation (10) is not difficult to verify.

To that end, we first work out the geometric product \( (Q_{z_1} Q_{z_2})(Q_{z_1} Q_{z_2})^\dagger \) using expressions (39) and (40), which gives

\[
(Q_{z_1} Q_{z_2})(Q_{z_1} Q_{z_2})^\dagger = \left\{ (q_1 q_2 + q_d q_d) (q_1 q_2^\dagger + q_d q_d^\dagger) + (q_1 q_2^\dagger + q_d q_d^\dagger) (q_1 q_2 + q_d q_d^\dagger) \right\} \varepsilon
\]

Now the “real” part of the above product simplifies to (41) as follows:

\[
\left\{ (Q_{z_1} Q_{z_2})(Q_{z_1} Q_{z_2})^\dagger \right\}_{\text{real}} = q_1 q_2 q_1 q_2^\dagger q_d q_d^\dagger + q_d q_d + q_1 q_2 q_d q_d^\dagger q_1 q_2^\dagger + q_1 q_2 q_1 q_2^\dagger q_d q_d^\dagger q_1 q_2^\dagger + q_1 q_2 q_1 q_2^\dagger q_d q_d^\dagger q_d q_d^\dagger q_1 q_2^\dagger + q_1 q_2 q_1 q_2^\dagger q_d q_d^\dagger q_d q_d^\dagger q_1 q_2^\dagger
\]

Similarly, the “dual” part of the product (42) simplifies to

\[
\left\{ (Q_{z_1} Q_{z_2})(Q_{z_1} Q_{z_2})^\dagger \right\}_{\text{dual}} = q_1 q_2 q_d q_d^\dagger q_d q_d^\dagger q_1 q_2^\dagger + q_1 q_2 q_d q_d^\dagger q_d q_d^\dagger q_1 q_2^\dagger + q_1 q_2 q_d q_d^\dagger q_d q_d^\dagger q_1 q_2^\dagger + q_1 q_2 q_d q_d^\dagger q_d q_d^\dagger q_1 q_2^\dagger + q_1 q_2 q_d q_d^\dagger q_d q_d^\dagger q_1 q_2^\dagger + q_1 q_2 q_d q_d^\dagger q_d q_d^\dagger q_1 q_2^\dagger + q_1 q_2 q_d q_d^\dagger q_d q_d^\dagger q_1 q_2^\dagger
\]

We can see this again by inserting into (43) the normalization condition (19) in the form \( q_r q_1^\dagger = -q_d q_d^\dagger \) and the normalization conditions \( ||q||^2 = q q^\dagger = \varepsilon^2 \) for the real and dual quaternions specified in (14) and (15), for each of the four terms of (43).

Here (44) follows from (43) upon inserting the normalization condition (19) in the form \( q_r q_1^\dagger = -q_d q_d^\dagger \) into the second and third terms of (43), which then cancel out with the sixth and seventh terms of (43), respectively; and (45) follows from (43) upon inserting the normalization conditions \( ||q||^2 = q q^\dagger = \varepsilon^2 \) for the quaternions in (14) and (15), which cancels out the first four terms of (43) with the last four. Consequently, combining the results of (43) and (44), for the left-hand side of (40) we have

\[
||Q_{z_1} Q_{z_2}|| = \sqrt{\varepsilon^2 r_1^2 r_2^2 + \varepsilon^2 d_1^2 d_2^2 + \varepsilon^2 d_1^2 r_2^2 + \varepsilon^2 d_2^2 r_2^2}.
\]

Thus, comparing the results in (38) and (32), we finally arrive at the relation

\[
||Q_{z_1} Q_{z_2}|| = ||Q_{z_1}|| ||Q_{z_2}||,
\]

which is evidently the same as the norm relation (10) in every respect apart from the appropriate change in notation. This result is facilitated by the definition (b) of the norm [or of the quadratic form \( Q(X) \)] explained below Eq. (24). We have thus proved that the finite-dimensional algebra \( K^\lambda \) over the reals can be equipped with a positive definite quadratic form \( Q \) (the square of the norm) such that \( Q(XY) = Q(X) Q(Y) \) for all \( X \) and \( Y \) in \( K^\lambda \). Consequently, a product \( XY \) would vanish if and only if \( X \) or \( Y \) vanishes. In other words, \( K^\lambda \), equipped with \( Q \), is a division algebra.
In Appendix B we prove the composition law \(|Q_{z_1}Q_{z_2}|^2 = |Q_{z_1}|^2|Q_{z_2}|^2\) in full generality without assuming (19), and in Appendix C we prove that the orthogonality of the quaternions \(q_r\) and \(q_d\) is preserved under multiplication.

Without loss of generality we can now restrict \(K^\lambda\) in (18) to a unit 7-sphere by setting the radii \(g_r\) and \(g_d\) to \(\frac{1}{\sqrt{2}}\):

\[
K^\lambda \supset S^7 := \left\{ Q_z := q_r + q_d \varepsilon \mid \|Q_z\| = 1 \text{ and } q_r q_d^\dagger + q_d q_r^\dagger = 0 \right\},
\]

(50)

where \(\varepsilon = -\lambda I_3 e_\infty\), \(e^\dagger = \varepsilon\), \(e^2 = e_\infty^2 = +1\),

\[
q_r = q_0 + q_1 \lambda e_x e_y + q_2 \lambda e_x e_z + q_3 \lambda e_y e_z, \quad \text{and} \quad q_d = -q_7 + q_0 e_x e_y + q_5 e_z e_x + q_4 e_y e_z,
\]

so that

\[
Q_z = q_0 + q_1 \lambda e_x e_y + q_2 \lambda e_x e_z + q_3 \lambda e_y e_z + q_4 \lambda e_x e_\infty + q_5 \lambda e_y e_\infty + q_6 \lambda e_z e_\infty + q_7 \lambda I_3 e_\infty.
\]

(52)

Needless to say, since all Clifford algebras are associative algebras by definition, unlike the non-associative octonionic algebra the 7-sphere we have constructed here corresponds to an associative (but non-commutative) division algebra.

Note that in terms of the components of \(q_r\) and \(q_d\) the condition \(q_r q_d^\dagger + q_d q_r^\dagger = 0\) is equivalent to the constraint

\[
f_c = -q_0 q_7 + \lambda q_1 q_6 + \lambda q_2 q_5 + \lambda q_3 q_4 = 0.
\]

(53)

This constraint reduces the space \(K^\lambda\) to the sphere \(S^7\), thereby reducing the 8 dimensions of \(K^\lambda\) to the 7 dimensions of \(S^7\) defined in (50). But the 7-sphere thus constructed has a topology that is different from that of the octonionic 7-sphere, and the difference between the two is captured by the difference in the corresponding normalizing constraints

\[
f_0 (q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7) = 0.
\]

(54)

More precisely, the two normalizing constraints giving rise to the two topologically distinct 7-spheres of radius \(\rho\) are:

\[
f_0 = q_0^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 + q_7^2 - \rho^2 = 0,
\]

(55)

which reduces the set \(\emptyset\) of unit octonions to the sphere \(S^7\) made up of eight-dimensional vectors of fixed length \(\rho\), and

\[
f_c = -q_0 q_7 + \lambda q_1 q_6 + \lambda q_2 q_5 + \lambda q_3 q_4 = 0,
\]

(56)

which reduces the set \(K^\lambda\) to the sphere \(S^7\) made up of a different collection of eight-dimensional vectors of fixed length \(\rho = \sqrt{g_r^2 + g_d^2}\). Both constraints, (53) and (56), involve the same eight variables of the embedding space \(\mathbb{R}^8\), namely, \(q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7\), giving the same dimensions for the sphere \(S^7\) of radius \(\rho\), albeit respecting different topologies. This difference arises because we have used the geometric product \(XX^\dagger\) rather than the scalar product \(X \cdot X^\dagger\) to derive the constraint \(f_c = 0\). But both definitions of the norm \(||X||\) give identical results, as explained above.

Given the quadratic form \(Q(X)\) and the norm relation (19), we may now view the four associative normed division algebras in the only possible dimensions 1, 2, 4, and 8, respectively \(\mathbb{R}, \mathbb{C}, \mathbb{H}\), as even sub-algebras of the Clifford algebras

\[
Cl^\lambda_{1,0} = \text{span}\{1, \lambda e_x\},
\]

(56)

\[
Cl^\lambda_{2,0} = \text{span}\{1, \lambda e_x, \lambda e_y, \lambda e_z e_y\},
\]

(57)

\[
Cl^\lambda_{3,0} = \text{span}\{1, \lambda e_x, e_y, \lambda e_z, \lambda e_x e_y, \lambda e_z e_y, \lambda e_x e_z\},
\]

(58)

and

\[
Cl^\lambda_{4,0} = \text{span}\{1, \lambda e_x, \lambda e_y, \lambda e_z, \lambda e_x e_y, \lambda e_z e_x, \lambda e_y e_z, \lambda e_z e_\infty, \lambda e_x e_\infty, \lambda e_y e_\infty, \lambda e_z e_\infty, \lambda e_x e_y e_\infty, \lambda e_z e_x e_\infty, \lambda e_y e_z e_\infty\}.
\]

(59)

It is easy to verify that the even subalgebras of \(Cl^\lambda_{1,0}, Cl^\lambda_{2,0}\) and \(Cl^\lambda_{3,0}\) are indeed isomorphic to \(\mathbb{R}, \mathbb{C}, \mathbb{H}\), respectively.

In practice, the above eight-dimensional algebra sometimes appears in the guise of a ‘1d up’ approach to Conformal Geometric Algebra in the engineering and computer vision applications \(\mathbb{R}, \mathbb{C}, \mathbb{H}\). Such physical applications would
benefit from explicitly using the quadratic form $Q(X)$ and the corresponding division algebra we have presented in this paper. For instance, it may help in removing the “singularities” or non-zero zero divisors from occurring in such applications. An illustration of how that may work can be found in Ref. [3] where we have applied the quadratic form $Q(X)$ and the corresponding division algebra to understand the geometrical origins of quantum correlations within the 7-sphere constructed in this paper. In the broader context of relativistic quantum theory, it is well known that between 1932 and 1952 Jordan attempted to use an alternative ring of octonions with non-associative multiplication rules to transfer the probabilistic interpretation of quantum theory to what is now known as exceptional Jordan algebra [11]. But as Dirac has noted [12], Jordan’s attempt to obtain a generalized quantum theory in this manner was not successful, because the non-associative multiplication rules are not compatible with any physically meaningful group of transformations such as the Poincaré group. However, the octonion-like algebra $\mathcal{K}^\lambda$ with six rather than seven imaginaries we have presented in this paper is *associative* by construction, and therefore it will be amenable to Jordan type application to quantum theory. Apart from these applications, in Section 5 of Ref. [7] Baez has discussed more mathematically oriented applications of the norm division algebras in four dimensions. These application can now be extended to eight dimensions, thanks to the associativity of $\mathcal{K}^\lambda$. The Clifford-algebraic investigations by Lounesto in normed division algebras and octonions may also benefit from the associativity of $\mathcal{K}^\lambda$ [13]. To facilitate these applications, in Appendix A we illustrate how non-zero zero divisors are precluded from the $\mathcal{K}^\lambda$ equipped with $Q(X)$.

**Appendix A: Illustration of How the Definition (b) of the Norm Precludes Zero Divisors**

According to Frobenius theorem [14] — which uses scalar products (instead of geometric products we have used) as an essential ingredient in its proof, a finite-dimensional associative division algebra over the reals is necessarily isomorphic to either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ in the 1, 2, and 4 dimensions, respectively. Since Clifford algebras are finite-dimensional associative algebras, Frobenius theorem suggests that those Clifford algebras that are not isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ may contain non-zero zero divisors or idempotent elements. It is therefore important to understand how the definition (b) of the norm leading to the quadratic form $Q(X)$ prevents non-zero zero divisors from occurring in the algebra $\mathcal{K}^\lambda$.

To that end, recall that the elements of $\mathcal{K}^\lambda$ are of the following general form in terms of quaternions $q_r$ and $q_d$:

$$Q_z = q_r + q_d \varepsilon,$$  \hspace{1cm} (A1)

where $\varepsilon^\dagger = \varepsilon$ and $\varepsilon^2 = +1$ as in [10] and the normalization of $Q_z$ requires that $q_r$ and $q_d$ must satisfy the condition

$$q_rq_d^\dagger + q_dq_r^\dagger = 0 \iff q_rq_d = -q_dq_r^\dagger.$$  \hspace{1cm} (A2)

This condition follows from the definition (b) of the norm $||Q_z||$ discussed below Eq. (24). It respects the fundamental geometric product $Q_zQ_z^\dagger$ and gives the same scalar value for the norm $||Q_z||$ as that calculated using definition (a).

Now, for the sake of argument, consider the following idempotent quantities as candidate non-zero zero divisors:

$$Z_\pm = \frac{1}{2}(1 \pm \varepsilon).$$  \hspace{1cm} (A3)

We call the quantities $Z_\pm$ idempotent quantities because they square to themselves, which can be easily verified:

$$Z_\pm^2 = Z_+ Z_+^\dagger = Z_+ \hspace{1cm} \text{and} \hspace{1cm} Z_\pm^2 = Z_- Z_+^\dagger = Z_-.$$  \hspace{1cm} (A4)

But $Z_+$ and $Z_-$ are also orthogonal to each other because their products vanish, which can also be easily verified:

$$Z_+Z_- = Z_-Z_+ = 0.$$  \hspace{1cm} (A5)

Now consider two multivectors, $X$ and $Y$, confined to a two-dimensional subspace of $\mathcal{K}^\lambda$, by setting $X_0 = Y_0 = 1/\sqrt{2}$, $Y_7 = -X_7 = 1/\sqrt{2}$, and the remaining twelve coefficients equal to zero in the Eqs. (6) and (7), along with $\varepsilon \equiv -\lambda I_3 e_\infty$:

$$X = \frac{1}{\sqrt{2}}(1 + \varepsilon) = \sqrt{2}Z_+ \hspace{1cm} \text{and} \hspace{1cm} Y = \frac{1}{\sqrt{2}}(1 - \varepsilon) = \sqrt{2}Z_-.$$  \hspace{1cm} (A6)
Then (A5) implies that \( \|XY\| = 0 \), which can be verified by substituting for the multivectors \( X \) and \( Y \) from (A6):

\[
\|XY\| = \left\| \frac{1}{\sqrt{2}}(1 + \varepsilon) \right\| \left\| \frac{1}{\sqrt{2}}(1 - \varepsilon) \right\|
\]

\[
= \left\| \frac{1}{2} (1 - \varepsilon^2) \right\|
\]

\[
= \|0\|
\]

\[
= 0,
\]

where \( \varepsilon^2 = 1 \) is used. Next, using \( \varepsilon^\dagger = \varepsilon \) and \( \varepsilon^2 = 1 \), we evaluate the right-hand side of the norm relation (10), giving

\[
\|X\| ||Y|| = \left\| \frac{1}{\sqrt{2}}(1 + \varepsilon) \right\| \left\| \frac{1}{\sqrt{2}}(1 - \varepsilon) \right\|
\]

\[
= \left( \sqrt{\frac{1}{2}(1 + \varepsilon)} \right) \left( \sqrt{\frac{1}{2}(1 - \varepsilon)} \right)
\]

\[
= \sqrt{(1 + \varepsilon)(1 - \varepsilon)}
\]

\[
= \sqrt{1 - \varepsilon^2}
\]

\[
= \sqrt{0}
\]

\[
= 0,
\]

where we have used geometric products to evaluate the norms. Comparing (A10) and (A18) we see that the norm relation \( \|XY\| = \|X\| ||Y|| \) is satisfied for the multivectors in (A6), despite (A4) and (A5). On the other hand, if we insist on using scalar products for evaluating the norms, then (A10) remains the same but instead of (A18) we obtain

\[
\|X\| ||Y|| = \left\| \frac{1}{\sqrt{2}}(1 + \varepsilon) \right\| \left\| \frac{1}{\sqrt{2}}(1 - \varepsilon) \right\|
\]

\[
= \left( \sqrt{\frac{1}{2}(1 + \varepsilon)} \right) \left( \sqrt{\frac{1}{2}(1 - \varepsilon)} \right)
\]

\[
= \sqrt{(1 + \varepsilon)(1 - \varepsilon)}
\]

\[
= \sqrt{1 - \varepsilon^2}
\]

\[
= \sqrt{0}
\]

\[
= 1,
\]

which seems to imply that \( \|XY\| \neq \|X\| ||Y|| \). This is because the norms are evaluated inconsistently in arriving at the contradictory results (A10) and (A24). While the product between \( X \) and \( Y \) for (A10) is evaluated using the geometric product giving \( XY = 0 \) so that \( \|XY\| = 0 \), the norms \( \|X\| \) and \( ||Y|| \) are evaluated for (A24) using the scalar products giving \( \|X\| = 1 \) and \( ||Y|| = 1 \). But using scalar products to evaluate norms is inconsistent with the choices made in (A6) for the coefficients of \( X \) and \( Y \). To appreciate this, recall again that the fundamental product in Geometric Algebra is the geometric product, not the scalar product, and the geometric product such as \( X \dagger Y \) is worked out in Eq. (21) above, from which it is clear that the norm \( \|X\| = \sqrt{XX\dagger} \) can reduce to a scalar quantity if and only if \( a_r q_r^\dagger + a_d q_d^\dagger = 0 \). And, as we saw in (22), in terms of the coefficients of \( X \) this condition is equivalent to

\[
-X_0 X_7 + \lambda X_1 X_6 + \lambda X_2 X_5 + \lambda X_3 X_4 = 0.
\]

\[
(A25)
\]
It is now easy to see that the choices of the coefficients in \(X\) and \(Y\) are incompatible with this condition. The choices of coefficients made in (A8) are \(X_0 = \frac{1}{\sqrt{2}}\) and \(X_7 = -\frac{1}{\sqrt{2}}\) for \(X\) and \(Y_0 = \frac{1}{\sqrt{2}}\) and \(Y_7 = \frac{1}{\sqrt{2}}\) for \(Y\), with all other coefficients set to zero. Substituting these values in (A2), we immediately arrive at the contradictions \(\pm \frac{1}{2} = 0\), proving that the \(ad\ hoc\) coefficients chosen to define \(X\) and \(Y\) are not compatible with the use of scalar products to evaluate the norms. In other words, the \(ad\ hoc\) coefficients chosen in (A8) to define \(X\) and \(Y\) are not compatible with the definition (b) for the norms. On the other hand, if the norms are evaluated consistently on both sides of Eq. (10), as for (A10) and (A15), then the norm relation \(\|XY\| = \||X||\|Y\|\) holds for any elements \(X\) and \(Y\) in \(K^\lambda\). This completes the illustration of how using the definition (b) for the norms precludes non-zero zero divisors from \(K^\lambda\), and leads us to the following general proof of the composition law for \(K^\lambda\).

\textbf{Appendix B: Proof of the Composition Law for }K^\lambda\textbf{ without Assuming }q_r q_d^1 + q_d q_r^1 = 0

In the proof of the norm relations (10) we assumed the normalization condition \(q_r q_d^1 + q_d q_r^1 = 0\). It turns out, however, that the composition law holds for the algebra \(K^\lambda\) even without assuming this condition. In this appendix we first prove the composition law explicitly and then obtain the norm relation (10) as its special case. To this end, recall from Eq. (23) that the product \(Q_z Q_{z'}^\dagger\) between the general element \(Q_z\) in \(K^\lambda\) and its conjugate \(Q_{z'}^\dagger\) is of the form

\begin{equation}
Q_z Q_{z'}^\dagger = (\text{a scalar}) + (\text{a scalar}) \times \varepsilon,
\end{equation}

with \(\varepsilon^2 = +1\). Thus, the square of the norm resembles a split complex number \(i\) rather than a real scalar. However, the composition law still holds. To prove this, we begin by evaluating the right-hand side of (10) using (39) and (10):

\begin{equation}
\|Q_{z_1}\|^2 \|Q_{z_2}\|^2 = \left(Q_{z_1} Q_{z_1}^\dagger\right) \left(Q_{z_2} Q_{z_2}^\dagger\right)
\end{equation}

where we have used \(q_r q_1^1 = \varepsilon q_2^1, q_d q_1^1 = \varepsilon q_2^1, etc\). Recalling from (22) that the quantities such as \(q_r q_1^1 + q_d q_1^1\) are scalar quantities, we see that the above product also resembles a split complex number similar to \(Q_z Q_{z'}^\dagger\) in (11).

Next, using (35), (41), and their product evaluated in (42), we can evaluate the left-hand side of (10) as follows:

\begin{equation}
\|Q_{z_1} Q_{z_2}\|^2 = (Q_{z_1} Q_{z_2}) (Q_{z_1} Q_{z_2})^\dagger
\end{equation}

where

\begin{equation}
\text{B7)}
\end{equation}
Thus, the algebra we set \( q \) split complex number similar to \( Q \).

More importantly, the right-hand sides of (B6) and (B13) are identical. We have thus proved that, although norms

\[ \frac{1}{2} q_{11}^2 + \frac{1}{2} q_{12}^2 + \frac{1}{2} q_{13}^2 + \frac{1}{2} q_{14}^2 = 0 \text{ and } q_{21} q_{22} + q_{23} q_{24} + q_{24} q_{23} = 0 \]  

Again, since the quantities such as \( q_{11}^2 + q_{21}^2 \) are scalar quantities, we see that the above product also resembles a split complex number similar to \( Q \) in \( \mathbb{H}^1 \). In other words, it is a sum of a scalar and a pseudoscalar, as in \( \mathbb{K}^2 \).

We have thus proved that, although norms in \( \mathbb{K}^2 \) resemble split complex numbers rather than scalars, the composition law continues to hold for the algebra \( \mathbb{K}^2 \):

\[ ||Q_z, Q_z||^2 = ||Q_z||^2 \text{ and } ||Q_z||^2 = 1. \]  

Thus, the algebra \( \mathbb{K}^2 \) is a normed division algebra even without assuming \( q_r q_d^\dagger + q_d q_r^\dagger = 0 \). On the other hand, if we set \( q_{11}^2 + q_{21} = 0 \) and \( q_{12} q_{12}^\dagger + q_{22} q_{22}^\dagger = 0 \) in (B6) and (B13) as a special case [as we did in proving the relation (A9)], then the above composition law reduces to the norm relation (19), confirming our main thesis above:

\[ ||Q_z, Q_z|| = \sqrt{\frac{1}{2} q_{11}^2 + \frac{1}{2} q_{12}^2 + \frac{1}{2} q_{13}^2 + \frac{1}{2} q_{14}^2} = ||Q_z||, \]  

The advantage of the special case is that it reduces the norms from a split complex form to purely scalar quantities.

Appendix C: Orthogonality of the Quaternions \( q_r \) and \( q_d \) is Preserved under Multiplication

In this appendix we prove that the 7-sphere we have constructed in this paper and defined in Eq. (50), namely

\[ \mathbb{K}^2 \subset S^7 := \left\{ Q_z := q_r + q_d \varepsilon \left| ||Q_z|| = 1 \text{ and } q_r q_d^\dagger + q_d q_r^\dagger = 0 \right. \right\}, \]  

remains closed under multiplication. This may not be obvious because of the orthogonality condition \( q_r q_d^\dagger + q_d q_r^\dagger = 0 \) we have imposed for normalizing the elements \( Q_z \) of the algebra \( \mathbb{K}^2 \). But the orthogonality of the quaternions \( q_r \) and \( q_d \) in \( Q_z \) is preserved under multiplication of the elements of \( \mathbb{K}^2 \). To prove this, consider two distinct elements of \( \mathbb{K}^2 \),

\[ Q_{z1} = q_{r1} + q_{d1} \varepsilon \quad \text{and} \quad Q_{z2} = q_{r2} + q_{d2} \varepsilon, \]

satisfying the orthogonality conditions \( q_{r1} q_{d2}^\dagger + q_{d1} q_{r2}^\dagger = 0 \) and \( q_{r2} q_{d2}^\dagger + q_{d2} q_{r2}^\dagger = 0 \), together with their product

\[ Q_{z3} := Q_{z1} Q_{z2} = (q_{r1} q_{r2} + q_{d1} q_{d2}) + (q_{r1} q_{d2} + q_{d1} q_{r2}) \varepsilon, \]

[...the rest of the text continues as...]

as considered in Eq. (C5). If we now define the quaternions
\[ q_{r3} := (q_{r1} q_{r2} + q_{d1} q_{d2}) \quad \text{and} \quad q_{d3} := (q_{r1} q_{d2} + q_{d1} q_{r2}), \]
then the product (C3) can be expressed as the following third element of \( K^\lambda:\)
\[ Q_{r3} = q_{r3} + q_{d3} \varepsilon. \]
It is now easy to prove that the quaternions \( q_{r3} \) and \( q_{d3} \) are also orthogonal, or equivalently, \( q_{r3} q_{d3} + q_{d3} q_{r3} = 0:\)
\[ q_{r3} q_{d3} + q_{d3} q_{r3} = (q_{r1} q_{r2} + q_{d1} q_{d2}) (q_{r1} q_{d2} + q_{d1} q_{r2}) + (q_{r1} q_{d2} + q_{d1} q_{r2}) (q_{r1} q_{r2} + q_{d1} q_{d2}) = 0. \]
Here Eq. (C10) follows from Eq. (C9) because, as shown in Eq. (22), the quantities such as \( q_i \) are scalar quantities, and therefore we can use \( q_{r1} q_{r1} = q_{d1} q_{d1} \), etc., as before, to reduce Eq. (C9) to Eq. (C10). And the orthogonality conditions \( q_{r1} q_{d1} + q_{d1} q_{r1} = 0 \) and \( q_{r2} q_{d2} + q_{d2} q_{r2} = 0 \) then reduces the RHS of Eq. (C10) to zero. Consequently, the 7-sphere defined in (50) remains closed under multiplication, analogously to the octonionic 7-sphere.

Acknowledgements:
I thank Tevian Dray for his comments on the previous version of this preprint, which led to the proof in Appendix B.

References:
[1] C. Doran and A. Lasenby, Geometric Algebra for Physicists (Cambridge University Press, Cambridge, 2003).
[2] L. Dorst, D. Fontaine, and S. Mann, Geometric Algebra for Computer Science (Elsevier, Amsterdam, 2007).
[3] J. Christian, Quantum correlations are woven by the spinors of the Euclidean primitives, R. Soc. Open Sci., 5, 180526 (2018). https://doi.org/10.1098/rsos.180526 See also https://arxiv.org/abs/1806.02392 (2018).
[4] B. Kenwright, A beginners guide to dual-quaternions: what they are, how they work, and how to use them for 3D character hierarchies, in Proceedings of the 20th International Conference on Computer Graphics, Visualization and Computer Vision, 1–10 (2012).
[5] A. Hurwitz, Über die Composition der quadratischen Formen von beliebig vielen Variablen, Nachr. Ges. Wiss. Göttingen, 1898, 309–316 (1898).
[6] A. Hurwitz, Über die Komposition der quadratischen Formen, Math. Ann., 88 (1–2), 1–25 (1923).
[7] J. C. Baez, The octonsions, Bull. Am. Math. Soc., 39, 145–205 (2002).
[8] J. W. Milnor, Topology from the Differentiable Viewpoint (Princeton University Press, Princeton, New Jersey, 1997).
[9] A. Lasenby, Recent applications of conformal geometric algebra, in Computer Algebra and Geometric Algebra with Applications, 298–328 (Springer, New York, 2004).
[10] A. Lasenby, Rigid body dynamics in a constant curvature space and the ‘3D up’ approach to conformal geometric algebra, in Guide to Geometric Algebra in Practice, 371–389 (Springer, New York, 2011).
[11] P. Jordan, Über die Multiplikation quantenmechanischer Größen, Zeitschrift für Physik, 80, 285–291 (1933).
[12] P. A. M. Dirac, The relation between mathematics and physics, Proceedings of the Royal Society (Edinburgh), 59, Part II, 122–129 (1939).
[13] P. Lourenço, Octonions and triality, Advances in Applied Clifford Algebras, 11, 191 (2001).
[14] F. G. Frobenius, Über lineare Substitutionen und bilineare Formen, Journal für die reine und angewandte Mathematik, 84, 1–63 (1878).
[15] T. Dray and C. A. Manogue, The Geometry of the Octonions (World Scientific, Singapore, 2015), Chapter 5.
[16] J. Christian, Response to ‘Comment on “Quantum correlations are woven by the spinors of the Euclidean primitives”’, R. Soc. Open Sci., 9, 220147 (2022).