Schwarzschild black hole levitating in the hyperextreme Kerr field

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Abstract

The equilibrium configurations between a Schwarzschild black hole and a hyperextreme Kerr object are shown to be described by a three–parameter subfamily of the extended double–Kerr solution. For this subfamily, its Ernst potential and corresponding metric functions, we provide a physical representation which employs as arbitrary parameters the individual Komar masses and relative coordinate distance between the sources. The calculation of horizon’s local angular velocity induced in the Schwarzschild black hole by the Kerr constituent yields a simple expression inversely proportional to the square of the distance parameter.

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I. INTRODUCTION

The general solution of the extended double–Kerr equilibrium problem is represented by the Ernst complex potential of the form \[1\]

\[ \mathcal{E} = \frac{(\Lambda + \Gamma)}{(\Lambda - \Gamma)}, \quad \Lambda = \sum_{1 \leq i < j \leq 4} \lambda_{ij} r_i r_j, \quad \Gamma = \sum_{i=1}^{4} \gamma_i r_i, \]

\( \lambda_{ij} = (-1)^{i+j} (\alpha_i - \alpha_j)(\alpha_{i'} - \alpha_{j'}) X_i X_j, \quad (i', j' \neq i, j; \ i < j'), \)

\( \gamma_i = (-1)^i (\alpha_{i'} - \alpha_{j})(\alpha_{i'} - \alpha_{k'}) (\alpha_{j'} - \alpha_{k'}) X_i, \quad (i', j', k' \neq i; \ i' < j' < k'), \)

\( r_i = \sqrt{\rho^2 + (z - \alpha_i)^2}, \)

where the constants \( \alpha_i, \ i = 1, 2, 3, 4, \) can take on arbitrary real values or occur in complex conjugate pairs, and the quantities \( X_i \) are defined as follows:

\[ X_1 = \frac{v_1 - \phi}{\phi^{-1} - v_1}, \quad X_2 = \frac{1 - \phi v_1}{\phi^{-1} v_1 - 1}, \quad X_3 = \frac{1 + i \phi v_4}{1 - i \phi^{-1} v_4}, \quad X_4 = \frac{-\phi + i v_4}{\phi^{-1} + i v_4}, \]

\[ v_1 = \epsilon_1 \left[ \frac{(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4)}{(\alpha_2 - \alpha_3)(\alpha_2 - \alpha_4)} \right]^{1/2}, \quad v_4 = \epsilon_4 \left[ \frac{(\alpha_1 - \alpha_4)(\alpha_2 - \alpha_4)}{(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3)} \right]^{1/2}, \]

\( \phi \) being an arbitrary complex constant of modulus one \( (\phi \bar{\phi} = 1); \ \epsilon_1 = \pm 1 \) and \( \epsilon_4 = \pm 1. \)

The above potential \( \mathcal{E} \) satisfies the Ernst equation \[2\] and describes the equilibrium configurations of two arbitrary aligned Kerr sources which can be black holes, hyperextreme objects or their combinations. Because the four parameters \( \alpha_i \) determine location of the sources on the symmetry axis, they can always be parametrized by three arbitrary real constants, so that formulas \([1]\) and \([2]\) involve, accounting for \( \phi, \) four real independent parameters which can be related to the masses and angular momenta of the Kerr constituents. In the paper \[3\] the Komar quantities \[4\] were calculated for each component of an equilibrium configuration and the following general equilibrium law for two Kerr constituents was established:

\[ J + s \left( \frac{j_1}{m_1} + \frac{j_2}{m_2} \right) + \delta \epsilon (M + s)^2 = 0, \]

\[ M = m_1 + m_2, \quad J = j_1 + j_2, \quad \delta = \pm 1, \quad \epsilon = \pm 1, \]

which indicates at which separation distance \( s \) the equilibrium of spinning sources occurs for some given Komar masses \( m_l, \ l = 1, 2, \) and Komar angular momenta \( j_l. \)

The above equilibrium law raises an interesting question of whether balance is still possible when the angular momentum of one of the constituents, say \( j_1, \) is equal to zero? On the one
hand, one is tempted to say that equilibrium in this case is impossible because the spin–spin repulsive force which ensures balance with the gravitational attractive force and emerges due to the interaction of sources’ angular momenta must have zero value. On the other hand, as was already observed in [5], in the non–equilibrium configurations composed of a rotating and a non–rotating black holes kept apart by a strut (this type of stationary subextreme systems is covered by the usual double–Kerr solution of Kramer and Neugebauer [6]) the spinning black hole necessarily involves in rotation the horizon of the non–rotating black hole, so that the spin–spin repulsive force in such configurations is still present. But whether this ‘secondary’ spin–spin interaction is sufficient for removing a strut?

These are some of the questions that will be answered in the present paper. We will demonstrate that a Schwarzschild black hole hole can freely levitate above the super–spinning Kerr object, and we will obtain the general exact solution describing this specific equilibrium model. In Sec. II we reparametrize the ‘Schwarzschild–Kerr’ equilibrium problem by using the Komar masses of the constituents and the separation distance as arbitrary real parameters, and derive a representation of the Ernst complex potential and of all metric functions in terms of these constant quantities. In Sec. III the results of Sec. II are applied to the analysis of the physical properties of the levitating Schwarzschild black hole. Concluding remarks are given in Sec. IV.

II. THE SOLUTION AND METRIC FUNCTIONS

The derivation of the solution describing equilibrium configurations of our interest partially simplifies if we take into consideration that

(i) both Komar masses of our solution must have positive values: \( m_1 > 0, m_2 > 0; \)

(ii) it is well known that in the double–Kerr solution the equilibrium states between two subextreme constituents with positive Komar masses do not exist [1, 7], which means that the spinning partner of the Schwarzschild black hole in the binary system can only be a hyperextreme Kerr constituent whose location is defined by a complex conjugate pair \( \alpha_4 = \bar{\alpha}_3 \) (see Fig. 1a);

(iii) from (3) it follows that, when \( j_1 = 0 \), the equilibrium condition can be solved most easily with respect to the remaining angular momentum \( j_2 \), hence the set of physical parameters of our solution is likely to be comprised by the individual Komar masses \( m_1 \) and
$m_2$ jointly with the relative coordinate distance $s$ (three parameters in total).

With these remarks in mind, we now solve the equilibrium condition (3) for $j_2$:

$$j_1 = 0 \implies j_2 = \epsilon \frac{m_2(s + m_1 + m_2)^2}{s + m_2}, \quad \epsilon = \pm 1,$$

where we have taken into account that, according to the paper [3], we must choose $\delta = -1$ in (3) for our concrete ‘subextreme–hyperextreme’ configuration, the factor $\epsilon$ now defining the orientation of rotation.

The next step is to express the quantities $\alpha_i$ and $X_i$ in terms of $m_1$, $m_2$ and $s$. This can be done with the aid of the formulas for the Komar masses and angular momenta elaborated in the paper [3]. Then, after a very tedious but straightforward algebra we finally get the following concise expressions for $\alpha_i$ and $X_i$:

$$\alpha_1 = s + m_1, \quad \alpha_2 = s - m_1, \quad \alpha_3 = -i\sigma, \quad \alpha_4 = +i\sigma,$$

$$X_1 = i\epsilon, \quad X_2 = \frac{s + m_2 + im_1}{m_1 + i\epsilon(s + m_2)},$$

$$X_3 = -\frac{s + m_1 + m_2(1 - i\epsilon) - \sigma}{s + m_1 + m_2(1 + i\epsilon) + \sigma}, \quad X_4 = -\frac{s + m_1 + m_2(1 - i\epsilon) + \sigma}{s + m_1 + m_2(1 + i\epsilon) - \sigma},$$

$$\sigma = \sqrt{s^2 - m_1^2 + 2s[m_1(1 + \mu) + m_2]}, \quad \mu := \frac{m_1}{s + m_2}.$$  

Note that for the derivation of (5) we have found it advantageous to place the hyperextreme Kerr constituent at the origin of coordinates, the Schwarzschild black hole locating above it (see Fig. 1b).

The substitution of (5) into (11) yields the desired representation of the Ernst potential determining the ‘Schwarzschild–Kerr’ equilibrium configurations:

$$\mathcal{E} = \frac{(A - B)}{(A + B)},$$

$$A = m_2 \sigma \mu [(R_+ + R_-)(r_+ - r_-) - 2(R_+R_- - r_+r_-)] + i\sigma^2 (R_+ - R_-)(r_+ + r_-) - \epsilon s(1 + \mu)$$

$$\times [m_2 \mu (R_+ - R_-)(r_+ - r_-) + i\epsilon m_2 (R_+ - R_-)(r_+ + r_-) + i\sigma (R_+ + R_-)(r_+ + r_-)],$$

$$B = 2\epsilon s \mu [(m_2(s + m_1 + m_2)(1 - i\epsilon \mu) - i\epsilon \sigma^2)(R_- - R_+) - i\sigma [s + m_1 + (1 + i\epsilon)m_2]$$

$$\times (R_+ + R_-)] - 2i\epsilon m_2 \sigma [(1 + \mu)^2 r_+ + (1 - i\epsilon \mu)^2 r_-],$$

where we have introduced new notations for the functions $r_i$:

$$R_\pm = \sqrt{\rho^2 + (z \pm i\sigma)^2}, \quad r_\pm = \sqrt{\rho^2 + (z - s \pm m_1)^2}. $$
It is easy to check that if \( m_2 = 0 \), the potential \( z > s + m_1 \)
reduces to that defining the Schwarzschild solution with the additional shift \( s \) along the symmetry axis, and if \( m_1 = 0 \), it describes the hyperextreme Kerr solution \( j_2^2 > m_2^2 \). The axis value \( e(z) \) of the
potential obtained has the form
\[
e(z) = \frac{z^2 - (1 + i\epsilon)(s + m_1 + m_2)z - (s + m_1)^2 + (1 + i\epsilon)(\sigma^2 - m_1m_2 - m_2s)}{z^2 + (1 - i\epsilon)(m_1 + m_2 - i\epsilon)sz + i\epsilon(s - m_1)(s + m_1 + (1 + i\epsilon)m_2)}.
\]  
\( (8)\)

The calculation of the metrical fields \( f, \gamma \) and \( \omega \) entering the stationary axisymmetric
line element
\[
ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2d\varphi^2] - f(dt - \omega d\varphi)^2
\]
can be performed in our case with the aid of the general formulas of the paper \( 9 \), yielding
the following final result:
\[
f = \frac{A\bar{A} - B\bar{B}}{(A + B)(\bar{A} + \bar{B})}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B}}{K_0R_+R_-r_+r_-} \quad \omega = \omega_0 - \frac{2\text{Im}[G(\bar{A} + \bar{B})]}{AA - BB},
\]
\[
G = -zB + 4m_2\epsilon\sigma r_+r_- + s(1 + \mu)^2r_-\{\sigma[m_2 + (1 - i\epsilon)s](R_+ + R_-) - [m_2(\epsilon s + im_1) + (\epsilon - i)s(s + m_1 + m_2)](R_+ - R_-)\} + i\epsilon(1 + i\epsilon\mu)r_+
\]
\[
\times \{[s(1 + \mu) + m_2 - i\epsilon\sigma](s + m_1 - i\epsilon\sigma)R_+ - [s(1 + \mu) + m_2 + i\epsilon\sigma](s + m_1 + i\epsilon\sigma)R_-\}
\]
\[
- 2i\epsilon m_2\epsilon\sigma[(s + m_1)(1 + \mu)^2r_+ + (s - m_1)(1 - i\epsilon\mu)^2r_-]
\]
\[
- 2\epsilon\sigma\{[m_2(s + m_1 + m_2)(1 - i\epsilon\mu) - i\epsilon\sigma^2][2s(R_+ - R_-) + i\sigma(R_+ + R_-)]
\]
\[
+ i\sigma[s + m_1 + (1 + i\epsilon)m_2][2s(R_+ + R_-) + i\sigma(R_+ - R_-)]\},
\]
\[
K_0 = 16s^2\sigma^2(1 + \mu)^2, \quad \omega_0 = 2\epsilon(s + m_1 + m_2).
\]  
\( (10)\)

Formulas \( 10 \) define an asymptotically flat metric regular on all parts of the symmetry
axis outside the location of sources, i.e., on its upper part \( \rho = 0, z > s + m_1 \), its lower
part \( \rho = 0, z < 0 \), and on the segment separating the sources \( \rho = 0, 0 < z < s - m_1 \),
the metric functions \( \gamma \) and \( \omega \) on these parts of the axis taking zero values. Therefore, the
Schwarzschild black hole and hyperextreme Kerr constituent indeed form an equilibrium
configuration due to a specific realization of the spin–spin interaction mechanism which will
be discussed in the next section.

III. PHYSICAL PROPERTIES OF A LEVITATING BLACK HOLE

We first note that the application of the solution derived in the previous section to
concrete equilibrium configurations of a Schwarzschild black hole and a Kerr hyperextreme
object is very simple: one only needs to choose some particular values of the masses \( m_1 \) and \( m_2 \), together with the value of the distance \( s \) \((s > m_1)\) at which equilibrium must occur, and then find from formula (4) the corresponding value of the angular momentum \( j_2 \) ensuring equilibrium. The space–time geometry of that particular configuration is determined by formulas (6), (7) and (10).

Turning now to the general properties of our solution, it should be emphasized that the levitating Schwarzschild black hole does not contribute to the general angular momentum of the system, and this fact can be readily verified, e.g., with the help of Tomimatsu’s angular–momentum formula [10]. At the same time, it is not difficult to calculate the horizon’s local angular velocity \( \Omega \) [11] induced by the Kerr constituent in the Schwarzschild black hole because this quantity is equal to the inverse value of the metric function \( \omega \) evaluated on the horizon. From (6), (7) and (10) we get the following simple formula:

\[
\Omega = \frac{em_2}{2(s + m_2)(s + m_1 + m_2)},
\]

which means that \( \Omega \) is inversely proportional to the square of the distance between the constituents and has the same sign as the angular momentum \( j_2 \). The corotating Kerr hyperextreme object and the Schwarzschild horizon give rise to the spin–spin repulsive force compensating the gravitational attraction of the constituents, which is in agreement with the general observations made in the aforementioned paper [5] by Varzugin. At the same time, an estimation made in [5] concerning \( \Omega \) resulted in the induced angular velocity proportional to \( s^{-3} \); however, such a more rapid, compared to our formula (11), decreasing of \( \Omega \) with distance could be explained by the presence in [5] of a strut attached to the horizon and slowing down its velocity.

For the calculation of the horizon area one can use the formula 

\[4\pi m_1[-\omega_H^2 \exp(2\gamma_H)]^{1/2},\]

where \( \omega_H \) and \( \gamma_H \) are values of the functions \( \omega \) and \( \gamma \) on the horizon; after its application we get

\[
A_H = 16\pi m_1^2\left(1 + \frac{m_2}{s}\right),
\]

and the well–known expression for the area of the isolated Schwarzschild horizon is recovered from (12) in the limit \( s \to \infty \), or in the absence of the second body \((m_2 = 0)\). Apparently, the entropy of the Schwarzschild black hole increases as a result of the interaction with the Kerr constituent.
It is worth noting that an induced angular velocity of the horizon is not the only peculiar feature the non–rotating black hole acquires due to the spin–spin interaction with a Kerr source. Thanks to the latter interaction, the levitating black hole also develops a stationary limit surface touching the event horizon at the points $\rho = 0, z = s \pm m_1$. One may speculate in this respect that the usual Penrose process of the energy extraction from a rotating black hole \cite{12} could work in the case of our black–hole constituent too, most probably supplying it with a non–zero Komar angular momentum.

\textbf{IV. CONCLUSIONS}

In the present paper we have demonstrated a remarkable property of the Schwarzschild black hole to form equilibrium configurations with a hyperextreme Kerr source. Our analysis is essentially based on the three–parameter subfamily of the extended double–Kerr solution for which we have been able to work out a simple parametrization involving Komar masses and separation distance as arbitrary parameters. Although the Komar angular momentum of the Schwarzschild black–hole constituent remains equal to zero all the time, the black hole horizon becomes involved in rotation due to the stationarity of the spacetime, and the resulting spin–spin interaction with the Kerr source turns out sufficient for counteracting the gravitational attractive force and attaining equilibrium of the two constituents.

In view of the presence in our solution of a naked singularity which is a well–known characteristic of the hyperextreme Kerr object, it would be interesting to see whether the five–dimensional black rings \cite{13, 14} could replace the hyperextreme Kerr constituent in the 4+1–analogs of our equilibrium configurations as the black rings can have arbitrarily large angular momenta for fixed masses but at the same time they do not develop naked singularities. It seems that having a slight generalization of the black Saturn solution \cite{15} in which the black hole and the black ring could be located in two different planes would probably be enough for reproducing our main results obtained for the levitating Schwarzschild black hole. The construction of such generalization looks feasible if one takes into account a formal similarity of the solution–generating techniques which are being currently used in the four– and five–dimensional gravities \cite{16}. 
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FIG. 1: Location of the Schwarzschild black hole (a bar on the symmetry axis) and Kerr hyperextreme object (a cut perpendicular to the axis) parametrized by two different parameter sets: (a) using the canonical parameters $\alpha_i$, and (b) using a physical set of parameters, with $\sigma$ defined by formula (5).