AN ELEMENTARY VIEW OF FAMILIAL PSEUDOFUNCTORS

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Abstract. A classical result due to Diers shows that a presheaf $F: A \to \text{Set}$ on a category $A$ is a coproduct of representables precisely when each connected component of $F$’s category of elements has an initial object. Most often, this condition is imposed on a presheaf of the form $B(X, L(-))$ for a functor $L: A \to B$, in which case this property says that $L$ admits generic factorisations at $X$, or equivalently that $L$ has a left multiadjoint at $X$.

Here we generalize these results to the two dimensional setting, replacing $A$ with an arbitrary bicategory $\mathcal{A}$, and $\text{Set}$ with $\text{Cat}$. In this two dimensional setting, simply asking that a bi-presheaf $F: \mathcal{A} \to \text{Cat}$ be a coproduct of representables is often too strong of a condition. Instead, we will only ask that $F$ be a lax conical colimit of representables. This is turn allows for the weaker notion of lax generic factorisations (and lax multiadjoints) for pseudofunctors of bicategories $L: \mathcal{A} \to \mathcal{B}$.

We also compare our lax multiadjoints to Weber’s familial 2-functors, finding our description is more general (not requiring a terminal object in $\mathcal{A}$), though essentially equivalent when a terminal object does exist. Moreover, our description of lax generics allows for an equivalence between lax generic factorisations and famility.

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1. Introduction

Given a category $\mathcal{A}$ and presheaf $F: \mathcal{A} \to \textbf{Set}$, it is often a natural question to ask whether this presheaf is a coproduct of representable presheaves; meaning

$$F \cong \sum_{m \in \mathcal{M}} \mathcal{A}(P_m, -)$$

for some set $\mathcal{M}$ and function $P(-): \mathcal{M} \to \mathcal{A}$. Such presheaves have a straightforward characterization: a presheaf $F$ is a coproduct of representables precisely when each connected component of $\text{el} \ F$ has an initial object. Said more explicitly, this means that for any $(D, w)$ in $\text{el} \ F$ there exists an $(A, x)$ and morphism $k: (A, x) \to (D, w)$ where $(A, x)$ satisfies the following property: for any diagram in $\text{el} \ F$ as below

$$
\begin{array}{ccc}
(C, z) & \xrightarrow{h} & (B, y) \\
\downarrow{g} & & \downarrow{f} \\
(A, x) & \xrightarrow{j} & (D, w)
\end{array}
$$

there exists a $h: (A, x) \to (C, z)$ such that the diagram commutes, and moreover $h$ is the unique morphism $(A, x) \to (C, z)$.

Of particular interest is the case where $F$ is of the form $B(X, L-)$ for a functor $L: \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$. Here, asking that each connected component of $\text{el} \ B(X, L-)$ has an initial object amounts to asking that for any $w: X \to LD$ there exists an $x: X \to LA$ and $k: A \to D$ such that $w = Lk \cdot x$, and $x$ is “generic” meaning that it satisfies the following property: given any commuting square as on the left below

$$
\begin{array}{ccc}
X & \xrightarrow{z} & LB \\
\downarrow{x} & & \downarrow{Lg} \\
LA & \xrightarrow{Lj} & LC
\end{array}
$$

there exists a unique $h: A \to B$ such that $Lh \cdot x = z$ (note that $g \cdot h = f$ can be shown as a consequence). In this case we say $L$ admits generic factorisations, and call $x: X \to LA$ a generic morphism.

The reader will notice that the above condition on $L$ makes no mention of terminal objects, and indeed there are natural examples of generic factorisations without terminal objects, such as composition of spans in a category $\mathcal{E}$ with pullbacks

$$\text{Span} \ (\mathcal{E}) \ (Y, Z) \times \text{Span} \ (\mathcal{E}) \ (X, Y) \to \text{Span} \ (\mathcal{E}) \ (X, Z).$$

Thus higher analogues of generic factorisations should also not require the existence of terminal objects.

It is the purpose of this paper to generalize these notions of famility to the two dimensional setting, replacing the category $\mathcal{A}$ with a bicategory $\mathcal{A}$, and replacing $\textbf{Set}$ with $\mathbf{Cat}$. However, this is not a straightforward generalization, as asking that a bi-presheaf $F: \mathcal{A} \to \mathbf{Cat}$ be a coproduct of representables is often too strong of a condition. To see why, consider the case where a pseudofunctor $L: \mathcal{A} \to \mathcal{B}$ is such that each $\mathcal{B}(X, L-)$ is a coproduct of representables, meaning we have an
equivalence
\[ \mathcal{B}(X,L-) \simeq \sum_{m \in \mathcal{M}} \mathcal{A}(P_m,-) \]
for some set \( \mathcal{M} \). With such an equivalence, we would then have for each 2-cell \( \alpha \) as on the left below

\[ X \xrightarrow{f} LA \quad \Rightarrow \quad m, \quad P_m \xrightarrow{\exists \pi} A \]

assigned to an \( \exists \pi : f \Rightarrow g \) as on the right above, that \( f \simeq Lf \cdot \delta \) and \( g \simeq Lg \cdot \delta \) for the same generic \( \delta : X \to LP_m \) corresponding to the identity at \( P_m \). This is an unreasonably strong condition: we should not expect two 1-cells to factor through the same generic \( \delta \) just because there is a comparison map between them. In general, this should only be expected when the comparison map is invertible.

To address this problem, we weaken the condition on \( \mathcal{B}(X,L-) \), now only asking that it be a lax conical colimit of representables. In this paper we will give a characterization of when a pseudofunctor \( F : \mathcal{A} \to \text{Cat} \) is a lax conical colimit of representables (also giving appropriate notions of generic object and morphism in this setting), and then go on to specialize this characterization to the case where \( F \) is of the form \( \mathcal{B}(X,L-) \). We will see that in this setting, the generics are morphisms \( x : X \to LA \) for which we have universal factorisations of any 2-cell \( \alpha \) as on the left below

\[ X \xrightarrow{z} LB \quad \xrightarrow{\exists \pi} LA \]

into a diagram as on the right above. The factorization being universal means it must satisfy a number of axioms detailed later in Definition 37.

To see why admitting lax-generic factorisations is a natural condition on a pseudofunctor \( L : \mathcal{A} \to \mathcal{B} \), consider the problem of calculating a left extension as below

\[ \mathbf{[A^{op}, \text{Cat}]} \xrightarrow{\text{lan}_L} \mathbf{[B^{op}, \text{Cat}]} \]

for a given pseudofunctor \( L \) (where \( \mathcal{A} \) and \( \mathcal{B} \) are small). In general this left extension should not be expected to have a nice form. However, if \( L \) is a pseudofunctor which admits lax-generic factorisations, so that each \( \mathcal{B}(X,L-) \) is a lax conical colimit of representables, then this left extension will have a simple description. An important example of this situation is given by taking \( L \) as the canonical inclusion
of a small category $\mathcal{E}$ into its bicategory of spans $\text{Span}(\mathcal{E})$

and forming the left extension $\text{lan}_L$ as above, with right adjoint $\text{res}_L$ given by restricting along $L$. Now, recognizing $[\text{Span}(\mathcal{E})^{\text{op}}, \text{Cat}]$ as the 2-category of fibrations with sums (by the universal property of spans) [2], and noting that the extension-restriction adjunction is pseudomonadic (a consequence of $L$ being bijective on objects) [5], the reader will recognize this left extension as the free functor for the pseudomonad $\Sigma_\mathcal{E}$ for fibrations over $\mathcal{E}$ with sums. In this way one can derive the pseudomonad for fibrations with sums, and understand why this pseudomonad has a simple description. Note the same can be done for fibrations with products, replacing $\text{Span}(\mathcal{E})$ with $\text{Span}(\mathcal{E})^{\text{co}}$.

2. Background

In this section we will recall the necessary background for this paper. We first recall the basic theory of generic factorisations in the one-dimensional case, and then go on to recall the basics of lax conical colimits and the Grothendieck construction, which will replace the category of elements in the two dimensional setting.

2.1. Generic factorisations in one dimension. In the simple one dimensional case, the study of familial representability and generic factorisations stems from the following.

**Problem 1.** When is a presheaf $F: \mathcal{A} \to \text{Set}$ a coproduct of representables, meaning it is equivalent to the colimit of

$$\mathcal{M}^{\text{op}} \xrightarrow{P^{\text{op}}} \mathcal{A}^{\text{op}} \xrightarrow{\mathcal{A}} [\mathcal{A}, \text{Set}]$$

for some $\mathcal{M} \in \text{Set}$ and functor $P(-): \mathcal{M} \to \mathcal{A}$? In particular, when is a functor $L: \mathcal{A} \to \mathcal{B}$ such that

$$\mathcal{B}(X, L-) : \mathcal{A} \to \text{Set}$$

is a coproduct of representables for all $X \in \mathcal{B}$?

The classical answer to these questions is given by Diers [3, 4] (also see [8] for a more recent account), which we will recall after a couple of definitions.

**Definition 2.** Given a presheaf $F: \mathcal{A} \to \text{Set}$, define the category of elements of $F$ as the category with objects given by pairs $(A \in \mathcal{A}, x \in FA)$ and morphisms $(A, x) \to (B, y)$ given by maps $f: A \to B$ such that $Ff(x) = y$. We denote this category $\text{el} F$. 

Definition 3. Given a presheaf $F : \mathcal{A} \to \text{Set}$, we say an object $(A, x) \in \text{el } F$ is generic if for any given objects $(B, y), (C, z)$ and morphisms $f$ and $g$ as below

\[
\begin{array}{ccc}
(C, z) & \xrightarrow{h} & (B, y) \\
\downarrow & & \downarrow \\
(A, x) & \xrightarrow{f} & \xrightarrow{g} (B, y)
\end{array}
\]

there exists a morphism $h : (A, x) \to (C, z)$ such that the diagram commutes. Moreover, we ask that $h$ is the only morphism $(A, x) \to (C, z)$.

Remark 4. The above may be simply stated by asking $(A, x)$ is initial within its connected component.

Remark 5. The reader will note that this is stronger than asking for the existence of a unique lifting $h$. In fact, asking that $h$ be the unique morphism (and not just the unique lifting), is a condition which will turn out to often be too strong in dimension two.

The answer to the first part of Problem 1 is then the following.

Proposition 6 (Diers). Given a presheaf $F : \mathcal{A} \to \text{Set}$, the following are equivalent:

1. $F : \mathcal{A} \to \text{Set}$ is a coproduct of representables;
2. each connected component of $\text{el } F$ has an initial object;
3. for any $(B, y) \in \text{el } F$ there exists a generic object $(A, x)$ and morphism $f : (A, x) \to (B, y)$.

Remark 7. Of course (3) above is simply expanding (2) into more detail. This detailed version will be more analogous to the characterizations we give in the higher dimensional case.

We now consider the second part of Problem 1 concerning functors $L : \mathcal{A} \to \mathcal{B}$, first recalling the notion of “generic morphism” (also known as “diagonally universal morphism” in the work of Diers).

Definition 8. Given a functor $L : \mathcal{A} \to \mathcal{B}$ we say that a morphism $x : X \to LA$ for some $X \in \mathcal{B}$ and $A \in \mathcal{A}$ is generic if for any commuting square as on the left below

\[
\begin{array}{ccc}
X & \xrightarrow{z} & LB \\
\downarrow & & \downarrow \\
LA & \xrightarrow{L_f} & LC
\end{array}
\quad
\begin{array}{ccc}
X & \xrightarrow{z} & LB \\
\downarrow & & \downarrow \\
LA & \xrightarrow{L_f} & LC
\end{array}
\]

there exists a unique $h : A \to B$ such that $Lh \cdot x = z$. That $f = g \cdot h$ follows as a consequence of this property.

The following characterization generalizes $L$ having a left adjoint.

Definition 9. We say a functor $L : \mathcal{A} \to \mathcal{B}$ has a left multiadjoint if for every $X \in \mathcal{B}$ the presheaf $\mathcal{B}(X, L-) : \mathcal{A} \to \text{Set}$ is a coproduct of representables.

Applying Proposition 6 to presheaves of the form $\mathcal{B}(X, L-)$ for a given functor $L : \mathcal{A} \to \mathcal{B}$, we recover the following.

Proposition 10 (Diers). Given a functor $L : \mathcal{A} \to \mathcal{B}$ the following are equivalent:
(1) the functor $L$ has a left multiadjoint;
(2) for every morphism $f : X \to LT$ there exists a generic morphism $\delta : X \to LA$ and morphism $\bar{f} : A \to T$ such that $f = L\bar{f} \cdot \delta$.

Remark 11. Condition (2) is usually stated by saying “$L$ admits generic factorisations”.

2.2. Lax conical colimits and the Grothendieck construction. Here we give the required background on lax conical colimits and the Grothendieck construction.

Definition 12 (lax conical colimits). Given a category $A$, a bicategory $\mathcal{K}$, and pseudofunctor $F : A \to \mathcal{K}$, the lax colimit of $F$ consists of an object $T \in \mathcal{K}$, along with for every $A \in A$ a map $\varphi_A : FA \to T$ and for every morphism $f : A \to B$ in $A$ a 2-cell

$$\varphi_A \downarrow \varphi_B$$

$$FA \xrightarrow{Ff} FB$$

compatible with the binary and nullary constraints of $F$. This data, which may be seen as a lax natural transformation $\varphi : \Delta^1 \Rightarrow \mathcal{K}(F-,T) : A^{op} \to \mathcal{K}$, is required to be universal in that

$$\mathcal{K}(T,S) \to \{A^{op}, \text{Cat}\}(\Delta^1, \mathcal{K}(F-, S))$$

$\alpha \mapsto \mathcal{K}(F-, \alpha) \cdot \varphi$

defines an equivalence (where $[A^{op}, \text{Cat}]$ is the 2-category of pseudofunctors, lax natural transformations, and modifications).

Remark 13. It is worth noting that the above definition can be used when $F : A \to \mathcal{K}$ is only required to be a lax functor. Also, one may note that lax conical colimits can be seen as an instance of weighted bi-colimits (though we will not use this).

When $\mathcal{K} = \text{Cat}$, such a lax colimit can easily be evaluated by the so called Grothendieck construction. We describe this construction below (though we will be more general by replacing the category $A$ with a bicategory $\mathcal{A}$).

Definition 14 (Grothendieck construction). Given a bicategory $\mathcal{A}$ and pseudofunctor $F : \mathcal{A} \to \text{Cat}$, the category of elements of $F$, denoted by $\text{el}\, F$ or by $\hat{\mathcal{A}} \in A_{\lax} F$ is the bicategory with:

- **Objects:** An object is a pair of the form $(A \in \mathcal{A}, x \in FA)$;
- **Morphisms:** A morphism $(A , x) \to (B , y)$ is a morphism $f : A \to B$ in $\mathcal{A}$ and a morphism $\alpha : Ff(x) \to y$ in $FB$;
- **2-cells:** A 2-cell $(f, \alpha) \Rightarrow (g, \beta) : (A , x) \to (B , y)$ is a 2-cell $\nu : f \Rightarrow g$ in $\mathcal{A}$ such that

$$\begin{array}{ccc}
Ff(x) & \xrightarrow{(F\nu)} & Fg(x) \\
\alpha & \Rightarrow & \beta \\
\end{array}$$

is equal to $\alpha$.

The bicategory $\int_{A \in \mathcal{A}} F$ with its canonical projection to $\mathcal{A}$ is called the Grothendieck construction of $F$, especially in the case where $\mathcal{A}$ is a 1-category.
Remark 15. When $\mathcal{A}$ is a category, the notation $\int^{A \in \mathcal{A}}_{\text{lax}} FA$ is justified as the category of elements can be written as a lax colimit as in Definition 12. In the case where $\mathcal{A}$ is a bicategory, el $F$ is an appropriate tri-colimit of $F$, and the notation is still justified (though in a more technical sense that we will not burden this paper with; see [1]).

Taking $[\mathcal{A}, \text{Cat}]$ as the 2-category of pseudofunctors $\mathcal{A} \to \text{Cat}$, pseudonatural transformations, and modifications, we are now ready to state the main goal of this paper, which is to answer the following:

**Problem 16.** When is a bi-presheaf $F: \mathcal{A} \to \text{Cat}$ a lax conical colimit of representables, meaning it is equivalent to the lax colimit of

$$\mathcal{M}^{\text{op}} \xrightarrow{P_{(-)}} \mathcal{A}^{\text{op}} \xrightarrow{M} [\mathcal{A}, \text{Cat}]$$

for some $\mathcal{M} \in \text{Cat}$ and pseudofunctor $P_{(-)}: \mathcal{M} \to \mathcal{A}$? In particular, when is a pseudofunctor $L: \mathcal{A} \to \mathcal{B}$ such that

$$\mathcal{B}(X, L(-)): \mathcal{A} \to \text{Cat}$$

is a lax conical colimit of representables for all $X \in \mathcal{B}$ (such that the construction of these lax colimits is natural in $X$ in an appropriate sense)?

Note that given an $F$ arising as in the first part of this problem, we may write

$$F \simeq \int_{\mathcal{M}}^{m \in \mathcal{M}} \mathcal{A}(P_m, -)$$

as the analogue of the usual notation $F \simeq \sum_{m \in \mathcal{M}} A(P_m, -)$ in one dimension. Moreover, it is easy to see $\int_{\mathcal{M}}^{m \in \mathcal{M}} \mathcal{A}(P_m, -)$ is evaluated as the pseudofunctor $\mathcal{A} \to \text{Cat}$ sending each $T \in \mathcal{A}$ to the category with objects given by pairs $(m \in \mathcal{M}, f: P_m \to T)$ and morphisms given by morphisms $\lambda$ in $\mathcal{M}$ and 2-cells $\alpha$ in $\mathcal{A}$ as below

\[
\begin{array}{ccc}
(m \in \mathcal{M}, f: P_m \to T) & \xrightarrow{\lambda \alpha} & (n \in \mathcal{M}, g: P_n \to T)
\end{array}
\]

In the next section we will characterize when $F: \mathcal{A} \to \text{Cat}$ is a lax conical colimit of representables in terms of properties satisfied by el $F$, using the fact that for such an $F$ we know el $F$ has the form

$$\int_{\mathcal{M}}^{m \in \mathcal{M}} \mathcal{A}(P_m, A) \simeq \int_{\mathcal{M}}^{m \in \mathcal{M}} \mathcal{A}(P_m, A).$$

Finally, we recall the notion of a fibration, which characterizes functors $p: F \to \mathcal{E}$ (with $\mathcal{E}$ a 1-category) which arise from a pseudofunctor $F: \mathcal{E}^{\text{op}} \to \text{Cat}$ via the Grothendieck construction (here we mean the dual version of Definition 14 using oplax colimits in place of lax colimits).

---

1 An extra condition ensuring naturality in $X$ is not required in the simpler dimension one case.
Definition 17. A fibration is a functor $p: \mathcal{F} \to \mathcal{E}$ such that for any morphism $f: X \to pB$ in $\mathcal{E}$ there exists a morphism $\phi: f^*B \to B$ in $\mathcal{F}$ such that $p(\phi) = f$ and for any $\psi: A \to B$ and $r: pA \to X$ rendering commutative the right diagram below

there exists a unique $\tau: A \to f^*B$ such that $p(\tau) = r$ and the left diagram commutes. Moreover, we say a morphism $\phi: f^*B \to B$ in $\mathcal{F}$ is cartesian if the above property is satisfied when $f = p(\phi)$.

Remark 18. Dually, we have an equivalence between pseudofunctors $F: \mathcal{E} \to \text{Cat}$ and opfibrations over $\mathcal{E}$, with the equivalence given by Definition 14. It is worth noting that for such a pseudofunctor $F: \mathcal{E} \to \text{Cat}$, the morphisms of the form $(f, \alpha): (A, x) \to (B, y)$ with $\alpha$ invertible are the opcartesian arrows of $\text{el } F$ with respect to the corresponding opfibration $\text{el } F \to \mathcal{E}$.

3. Lax generics in bicategories of elements

Before we can describe lax-generic objects and morphisms in bicategories of elements, we will have to introduce the language needed to describe them. In particular, we define “mixed left liftings” which are similar to left liftings, except that the induced arrow’s direction is reversed. Note that basic properties for left liftings, such as the pasting lemma, or the lifting through an identity being itself, do not hold in general for mixed left liftings.

Definition 19 (mixed left lifting property). Let $\mathcal{C}$ be a bicategory. We say a diagram as on the left below

exhibits $(h, \nu)$ as the mixed left lifting of $f$ through $g$ if for any diagram as on the right above, there exists a unique 2-cell $\lambda: k \Rightarrow h$ such that

Moreover, we say such a lifting $(h, \nu)$ is strong if $h$ is sub-terminal in $\mathcal{C} (A, C)$.

Remark 20. It is clear that strong mixed liftings are unique up to unique isomorphism. Indeed, it is this stronger notion that will be used though this section.

The following lemma shows that an arrow $h$ which arises as a strong mixed lifting has the property that the strong mixed lifting of $h$ through the identity is itself.
Lemma 21. Suppose the left diagram below

\[\begin{array}{ccc}
A & \xrightarrow{f} & B \\
h \downarrow & & \downarrow g \\
C & \xrightarrow{\ell} & D
\end{array}\]

exhibits \((h, \nu)\) as the strong mixed lifting of \(f\) through \(g\). Then the right diagram above exhibits \((h, \text{id})\) as the strong mixed lifting of \(h\) through \(1_C\).

Proof. Given any \(k: A \to C\) and \(\zeta: h \Rightarrow k\) we have by universality of \((h, \nu)\) an induced \(\lambda: k \Rightarrow h\) such that

\[\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow g \\
C & \xrightarrow{\ell} & D
\end{array}\]

that is, since \(h\) is subterminal, a unique induced \(\lambda: k \Rightarrow h\) such that \(\lambda \zeta\) is the identity. This proves the result. \(\square\)

We now have the required theory to define notions of lax-generic object and lax-generic morphism in bicategories of elements.

Definition 22 (lax-generic objects). Let \(\mathcal{A}\) be a bicategory and \(F: \mathcal{A} \to \text{Cat}\) be a pseudofunctor. We say that an object \((A, x)\) in \(F\) is lax-generic if:

1. for any \((B, y), (C, z), (f, \alpha)\) and \((g, \beta)\) as below with \(\beta\) invertible

\[\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow g \\
C & \xrightarrow{\ell} & D
\end{array}\]

there exists a strong mixed left lifting \((h, \gamma) : (A, x) \to (C, z)\) exhibited by a 2-cell \(\nu: f \Rightarrow gh\);

2. if \(\alpha\) is invertible above, then both \(\gamma\) and \(\nu\) are also invertible.

Remark 23. If we replace the isomorphism \(\beta\) with an identity above the definition remains equivalent.

Definition 24 (generic morphisms). Let \(\mathcal{A}\) be a bicategory and \(F: \mathcal{A} \to \text{Cat}\) be a pseudofunctor, and suppose that \((A, x)\) is a lax-generic object in \(F\). We say that a morphism \((\ell, \phi) : (A, x) \to (D, w)\) out of \((A, x)\) in \(F\) is generic if the diagram below

\[\begin{array}{ccc}
A & \xrightarrow{\ell} & D \\
\downarrow & & \downarrow 1_D \\
(D, w)
\end{array}\]

exhibits \((\ell, \phi)\) as the strong mixed left lifting of \((\ell, \phi)\) through \((1_D, \text{id})\).
Remark 25. It is an easy consequence of the universal property that every 2-cell out of \((\ell, \phi)\) is a section (in a unique way); and consequently that any 2-cell between generic 1-cells is invertible. Moreover, as \((\ell, \phi)\) is sub-terminal within its hom-category it follows that any isomorphism between generic 1-cells is unique. It follows that if \((A, x)\) and \((B, y)\) are generic objects, then the category of generic morphisms \((A, x) \to (B, y)\) is equivalent to a discrete category (a set).

Remark 26. It is worth noting that for any generic object \((A, x)\) and strong mixed lifting as below

\[
\begin{array}{ccc}
(C, z) & \xrightarrow{(h, \gamma)} & (B, y) \\
\downarrow{(g, \beta)} & & \downarrow{(f, \alpha)} \\
(A, x) & \xrightarrow{(s, \phi)} & (B, y)
\end{array}
\]

with \(\beta\) invertible, the induced morphism \((h, \gamma)\) is a generic morphism as a consequence of Lemma 21.

The following proposition is a step towards characterizing when an \(F: \mathcal{A} \to \text{Cat}\) is a lax conical colimit of representables.

**Proposition 27.** Let \(\mathcal{A}\) be a bicategory and \(F: \mathcal{A} \to \text{Cat}\) be a pseudofunctor. Suppose that generic morphisms between generic objects compose to generic morphisms. Define \(\mathcal{A}_g^F\) as the locally full sub-bicategory of \(\text{el} F\) consisting of lax-generic objects and 1-cells. Define \(\mathcal{M}\) as the category consisting of lax-generic objects in \(\text{el} F\) and representatives of isomorphism classes of generic 1-cells between them. Observe \(\mathcal{A}_g^F \simeq \mathcal{M}\). Take \(P(-): \mathcal{M} \to \mathcal{A}\) as the assignment taking a generic object \((A, x)\) to \(A\) and a representative generic morphism between generic objects \((s, \phi): (A, x) \to (B, y)\) to \(s: A \to B\). Then \(P(-): \mathcal{M} \to \mathcal{A}\) defines a pseudofunctor, and for every \(T \in \mathcal{A}\) there exists fully faithful functors

\[\Lambda_T: \int_{\text{lax}}^m \mathcal{M} (P_m, T) \to FT\]

pseudo-natural in \(T \in \mathcal{A}\).

**Proof.** Firstly note that \(P(-): \mathcal{M} \to \mathcal{A}\) defines a pseudofunctor since it may be written as the composite \(\mathcal{M} \to \mathcal{A}_g^F \to \text{el} F \to \mathcal{A}\). We may then define \(\Lambda_T\) on objects by the assignment \((A, x, f) \mapsto Ff(x)\), and on morphisms by the assignment (suppressing the pseudofunctoriality constraints of \(F\))

\[
\begin{array}{ccc}
(A, x, f: A \to T) & \xrightarrow{\gamma} & A \\
\downarrow{(h, \gamma, \nu)} & & \downarrow{h} \\
(B, y, g: B \to T) & \xrightarrow{\nu} & B
\end{array}
\]

\[
\begin{array}{ccc}
(A, f) & \xrightarrow{Ff(x)} & (B, g) \\
\downarrow{(F\nu)_x} & & \downarrow{Fg(y)} \\
(FgFh(x) & \xrightarrow{Fg(\gamma)} & Fg(y)
\end{array}
\]

Observe that we have the following conditions satisfied.
**Functoriality.** Given another

\[(B, y, g : B \to T) \quad \xrightarrow{(k, \zeta, \mu)} \quad (C, z, q : A \to T)\]

the commutativity of

\[
\begin{array}{cccccc}
Ff(x) & \xrightarrow{(F\nu)} & FgFh(x) & \xrightarrow{Fg(y)} & Fg(y) & \xrightarrow{(F\mu)} \\
& & & \downarrow_{(F\mu)_{Fh(x)}} & & \\
& & & FqFkFh(x) & & \\
\end{array}
\]

by naturality of \(F\mu\) exhibits binary functoriality. It is trivial that identities are preserved.

**Fullness.** Given any \((A, x, f : A \to T)\) and \((B, y, g : B \to T)\) with a \(\phi : Ff(x) \to Fg(y)\), we may construct the universal diagram

\[
\begin{array}{cc}
(A, x) & \xrightarrow{(h, \gamma)} \\
\downarrow_{(f, \phi)} & \downarrow_{(g, \text{id})} \\
(B, g) & \xrightarrow{(g, \text{id})} \\
\end{array}
\]

using lax-genericity of \((A, x)\). Now \((h, \gamma)\) is generic by Lemma 21, and without loss of generality we can assume it is a representative generic. Then \((h, \gamma, \nu)\) is assigned to \(\phi\).

**Faithfulness.** Given another triple \((k, \psi, \omega)\) which also maps to \(\phi\), we have the diagram

\[
\begin{array}{cc}
(A, x) & \xrightarrow{(k, \psi)} \\
\downarrow_{(f, \phi)} & \downarrow_{(g, \text{id})} \\
(B, g(y)) & \xrightarrow{(g, \text{id})} \\
\end{array}
\]

But as \((k, \psi)\) and \((h, \gamma)\) are both generics, the induced \((k, \psi) \Rightarrow (h, \gamma)\) arising from universality of \((h, \gamma)\) must be invertible. Also, as they are both representative, they must be equal. As the identity must then be the induced morphism we conclude \(k = h, \psi = \gamma\) and \(\omega = \nu\).

**Pseudo-naturality.** Clearly given any 1-cell \(\alpha : T \to S\) in \(\mathcal{A}\) the squares

\[
\begin{array}{ccc}
(A, x, f : A \to T) & \xrightarrow{\alpha(-)} & (A, x, \alpha f : A \to S) \\
\downarrow_{\Lambda_T} & & \downarrow_{\Lambda_S} \\
Ff(x) & \xrightarrow{F\alpha(-)} & F(\alpha f)(x) \\
\end{array}
\]
commute up to pseudo-functoriality constraints of $F$, and the above squares satisfy the required naturality, nullary and binary coherence conditions as a consequence of the corresponding pseudo-functoriality coherence conditions.

**Remark 28.** Given any $(h, \gamma, \nu)$ as in (3.1) we also have

\[
\begin{array}{ccc}
(A, x, f : A \to T) & A & A \\
\downarrow (id, id, \nu) & \downarrow id & \downarrow id \\
(A, x, gh : A \to T) & A & B \\
\end{array}
\]

\[
\begin{array}{ccc}
 & f & \uparrow \theta \\
\downarrow & & \\
\downarrow & Ff(x) & \leftarrow Fgh(x) \\
\end{array}
\]

\[
\begin{array}{ccc}
 & Fgh(id) & \uparrow \phi \\
\downarrow & & \\
\downarrow & Fgh(x) & \rightarrow T \\
\end{array}
\]

**Remark 29.** Each $\Lambda_T$ is well defined, but not necessarily fully faithful, taking $\mathfrak{M}$ as the category given by el $F$ with no 2-cells (after replacing the bicategory el $F$ with an equivalent 2-category).

We can now characterize precisely when a bi-presheaf $F : \mathcal{A} \to \text{Cat}$ is a lax conical colimit of representables.

**Theorem 30.** Let $\mathcal{A}$ be a bicategory and $F : \mathcal{A} \to \text{Cat}$ be a pseudofunctor. Then the following are equivalent:

1. the pseudofunctor $F : \mathcal{A} \to \text{Cat}$ is a lax conical colimit of representables;
2. the following conditions hold:
   a. for every object $(B, y)$ in el $F$ there exists a lax-generic object $(A, x)$ and morphism $(f, \alpha) : (A, x) \to (B, y)$ with $\alpha$ invertible;
   b. generic morphisms between lax-generic objects compose to generic morphisms.

**Proof.** The direction (2) $\Rightarrow$ (1) is clear from Proposition 27 as condition (a) means that for any $B \in \mathcal{A}$ and $y \in FB$ we have a lax generic $(A, x)$ and morphism $(f, \alpha) : (A, x) \to (B, y)$ in el $F$ with $\alpha$ invertible, so that

\[\Lambda_B(A, x, f : A \to B) = Ff(x) \xrightarrow{\alpha} y\]

which witnesses the essential surjectivity of $\Lambda_B$ at $y \in FB$.

For (1) $\Rightarrow$ (2), suppose we are given a category $\mathfrak{M}$ and pseudofunctor $P(\cdot) : \mathfrak{M} \to \mathcal{A}$ (assuming without loss of generality that $P(\cdot)$ strictly preserves identities) such that $F \simeq \int^{m \in \mathfrak{M}} \mathcal{A}(P_m, -)$, and consequently

\[\text{el } F \simeq \int^\text{lax} \mathcal{A}(P_m, -).
\]

This exhibits el $F$ as the bicategory with:

**Objects:** An object is a triple of the form $(m \in \mathfrak{M}, A \in \mathcal{A}, x : P_m \to A)$;

**Morphisms:** The morphisms $(m, A, x) \to (n, B, y)$ are triples comprising a morphism $u : m \to n$ in $\mathfrak{M}$, a morphism $f : A \to B$ in $\mathcal{A}$ and a 2-cell

\[
\begin{array}{ccc}
P_m & x & A \\
\downarrow & \downarrow u & \\
P_n & y & B \\
\end{array}
\]

\[
\begin{array}{ccc}
P_m & x & \downarrow f \uparrow \phi \\
\downarrow & \downarrow & \\
P_n & y & B \\
\end{array}
\]
in \( \mathcal{A} \);

2-cell: A 2-cell \( \lambda : (u, f, \theta) \Rightarrow (u, g, \phi) : (m, A, x) \rightarrow (n, B, y) \) is a 2-cell \( \lambda : f \Rightarrow g \) in \( \mathcal{A} \) such that

\[
\begin{array}{c}
P_m \xrightarrow{x} A \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
P_u \| \theta \| \| f \| \| g \| \| \phi \| \\
P_n \xrightarrow{y} B
\end{array}
\]

\[
\begin{array}{c}
P_m \xrightarrow{x} A \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
P_u \| \phi \| \| g \| \| f \| \\
P_n \xrightarrow{y} B.
\end{array}
\]

Existence of expected lax-generics. We first show that each

\[
(m \in M, P_m \in \mathcal{A}, \text{id} : P_m \rightarrow P_m)
\]

in \( F \) is lax-generic. Consider a diagram

\[
\begin{array}{c}
\begin{array}{c}
(m, P_m, \text{id}) \rightarrow (n, B, y) \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
(u, f, \alpha) \rightarrow (u, h, \gamma)
\end{array}
\end{array}
\]

where \((u, f, \alpha)\) and \((\text{id}, g, \text{id})\) are respectively

\[
\begin{array}{c}
P_m \xrightarrow{\text{id}} P_m \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
P_u \| \alpha \| \| f \| \| \beta \| \\
P_n \xrightarrow{y} B
\end{array}
\]

\[
\begin{array}{c}
P_n \xrightarrow{z} C \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
P_m \| \beta \| \| g \| \| \phi \| \\
P_n \xrightarrow{y} B
\end{array}
\]

then we recover a canonical \((u, h, \gamma)\) as

\[
(3.2)
\]

with the 2-cell \( \nu : f \Rightarrow gh = gzP_u = yP_u \) given as \( \alpha \). Now, for universality, suppose we have a \((u, k, \phi)\) given as

\[
\begin{array}{c}
P_m \xrightarrow{\text{id}} P_m \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
P_u \| \phi \| \| k \| \\
P_n \xrightarrow{z} C
\end{array}
\]

with a 2-cell \( \psi : f \Rightarrow gk \) such that

\[
(3.3)
\]
Then we can take our induced map \( \lambda : k \Rightarrow h \) as \( \phi : k \Rightarrow z \cdot P_u \). It is trivial that

\[
\begin{align*}
\lambda & \quad \Rightarrow \\
P_m & \xrightarrow{id} P_m \\
P_n & \xrightarrow{z} C
\end{align*}
\]

so that \( \lambda \) is a 2-cell \((u, k, \phi) \Rightarrow (u, h, \gamma)\). Also, from (3.4) it is clear that \( \lambda = \phi \) is the only 2-cell \((u, k, \phi) \Rightarrow (u, h, \gamma)\), meaning \((u, h, \gamma)\) is sub-terminal within its hom-category. Moreover, (3.3) shows \( \psi \) pasted with \( \lambda = \phi \) is \( \alpha = \nu \).

**Classification of lax-generics.** We now show that an object \((m \in \mathbf{M}, A \in \mathcal{A}, x : P_m \to A)\) in el \( F \) is lax-generic if and only if \( x \) is an equivalence. It is clear the above argument generalizes if one replaces \((m, P_m, \text{id})\) with \((m, A, x)\) where \( x \) is an equivalence.

Conversely, if \((m, A, x)\) is a generic object then we may construct the universal diagram

\[
\begin{array}{ccc}
(m, P_m, \text{id}) & \xrightarrow{(1, x^* \gamma)} & (m, A, x) \\
\downarrow & \downarrow & \downarrow \\
(m, A, x) & \xrightarrow{(1, 1, \text{id})} & (m, A, x)
\end{array}
\]

noting that \( \nu \) and \( \gamma \) are both invertible. In fact, this gives an adjoint equivalence. That \( \nu \) is a 2-cell says

\[
\begin{align*}
P_m & \xrightarrow{x} A \\
id & \xrightarrow{id} \xrightarrow{id} \xrightarrow{id} \\
P_m & \xrightarrow{x} A
\end{align*}
\]

which gives one triangle identity. For the other identity, note that 2-cells \( \xi : (1, x^* xx^*, \gamma \gamma) \Rightarrow (1, x^*, \gamma) \), meaning 2-cells \( \xi \) such that

\[
\begin{align*}
P_m & \xrightarrow{x} A \\
id & \xrightarrow{id} \xrightarrow{id} \xrightarrow{id} \\
P_m & \xrightarrow{x} A
\end{align*}
\]

are unique, as \((1, x^*, \gamma)\) is sub-terminal within its hom-category. But we may take \( \xi \) to be

\[
\gamma x^* : (1, x^* xx^*, \gamma \gamma) \Rightarrow (1, x^*, \gamma)
\]

or

\[
x^* \nu^{-1} : (1, x^* xx^*, \gamma \gamma) \Rightarrow (1, x^*, \gamma)
\]

which both satisfy (3.5). Thus \( \gamma x^* = x^* \nu^{-1} \) and so \( \gamma x^* \cdot x^* \nu = \text{id} \) giving the other triangle identity.
Existence of lax-generic factorisations. Suppose we are given a \((n, B, y: P_n \to B)\) in \(el F\). We have the map \((n, P_n, \text{id}: P_n \to P_n) \to (n, B, y: P_n \to B)\) given as

\[
\begin{array}{ccc}
P_n & \xrightarrow{\text{id}} & P_n \\
P_n & \downarrow{\text{id}} & \downarrow{y} \\
P_n & \xrightarrow{y} & B
\end{array}
\]

which is of the required form since the 2-cell involved is invertible.

Generic morphisms form a category. Before showing that generic morphisms form a category, we will need a characterization of them. Now, specializing the earlier argument of “existence of expected lax-generics” to the case when \(g\) is the identity (though generalizing the identity on \(P_m\) to an equivalence \(x: P_m \to A\)) we see that if \((m, A, x)\) is generic (i.e. \(x\) is an equivalence)

\[
\begin{array}{c}
(m, A, x) \xrightarrow{(u, f, \alpha)} \xrightarrow{(g, \beta)} (n, B, y)
\end{array}
\]

the lifting \((u, h, \gamma)\) above, constructed as in (3.2), has \(\gamma\) invertible. It is also clear that if \((u, f, \alpha)\) is such that \(\alpha\) is invertible, then the lifting \((u, h, \gamma)\) through \((\text{id}, \text{id}, \text{id})\) constructed as in (3.2) is given by \((u, f, \alpha)\).

This shows that the generic morphisms between generic objects are diagrams of the form

\[
\begin{array}{ccc}
P_m & \xrightarrow{x} & A \\
P_n & \downarrow{\Phi} & \downarrow{f} \\
P_n & \xrightarrow{y} & B
\end{array}
\]

with \(\alpha\) invertible, and it is clear that these are closed under composition and that identities are such diagrams. □

Remark 31. When \(F: \mathcal{A} \to \textbf{Cat}\) is a lax conical colimit of representables, and from a generic object \((A, x)\) we construct the universal diagram

\[
\begin{array}{c}
(A, x) \xrightarrow{(f, \alpha)} (B, y)
\end{array}
\]

the 2-cell \(\nu\) is the unique 2-cell \((f, \alpha) \Rightarrow (g, \beta) \cdot (h, \gamma)\). This is since for such an \(F\), generic morphisms compose and any map \((g, \beta)\) with \(\beta\) invertible is generic. Sub-terminality of \((g, \beta) \cdot (h, \gamma)\) then gives uniqueness.

Remark 32. When \(F: \mathcal{A} \to \textbf{Cat}\) is a lax conical colimit of representables, written \(F \simeq \int_{\mathcal{A}}^{\text{strict}} F\), then \(\mathcal{M}\) is equivalent to the category of strict\(^2\) lax-generic objects \((A, x)\) and representative generic morphisms in \(el F\). This is a consequence of the characterization of lax-generic objects and morphisms given in the above proof of

\(^2\)Strict here means if both \(\alpha\) and \(\beta\) are identities, then both \(\nu\) and \(\gamma\) are identities.
Theorem 30. Moreover, as Theorem 30 constructs \( M \) as the the category of lax-generic objects and morphisms, we conclude this non-strict choice of \( M \) is also equivalent.

It is a natural question to ask if Theorem 30 has a variant which does not require generic morphisms to compose; and it turns out that this is the case. Given a bi-presheaf \( F : \mathcal{A} \to \mathbf{Cat} \) one can again define \( M \) as the category containing generic objects \((A, x) \in \mathrm{el} F\) and representative generic morphisms between them, but now defining the composite of two generic morphisms

\[
\begin{aligned}
(A, x) &\xrightarrow{(h, \gamma)} (B, y) &\xrightarrow{(k, \zeta)} (C, z) \\
(A, x) &\xrightarrow{(h, \gamma)} (B, y) &\xrightarrow{(k, \zeta)} (C, z)
\end{aligned}
\]

to be the mixed lifting through the identity as below.

Now, it is not hard to verify that this situation of generics not directly composing corresponds to the following weaker notion of family.

**Definition 33.** A pseudofunctor \( F : \mathcal{A} \to \mathbf{Cat} \) is a weak lax conical colimit of representables if there exists a category \( M \) and normal\(^3\) lax functor \( P(-) : M \to \mathcal{A} \) such that

\[
F \simeq \int_{lax}^m \mathcal{A}(P_m, -)
\]

Meaning that we find the following variant of Theorem 30.

**Theorem 34.** Let \( \mathcal{A} \) be a bicategory and \( F : \mathcal{A} \to \mathbf{Cat} \) be a pseudofunctor. Then the following are equivalent:

1. the pseudofunctor \( F : \mathcal{A} \to \mathbf{Cat} \) is a weak lax conical colimit of representables;
2. for every object \((B, y)\) in \( \mathrm{el} F \) there exists a lax-generic object \((A, x)\) and morphism \((f, \alpha) : (A, x) \to (B, y)\) with \( \alpha \) invertible.

Note however that in practice, we will usually want the reindexing \( P(-) : M \to \mathcal{A} \) to be a pseudofunctor. Indeed, \( P(-) \) is to be a pseudofunctor in all of the examples of Section 7.

4. Lax generic factorisations and lax multiadjoints

Here we specialize the results of the previous section to the case when \( F : \mathcal{A} \to \mathbf{Cat} \) is of the form \( \mathcal{B} (X, L- \) for a pseudofunctor \( L : \mathcal{A} \to \mathcal{B} \). The following is a generalization of “left multiadjoint” in Definition 9 to the case of a pseudofunctor \( L : \mathcal{A} \to \mathcal{B} \).

**Definition 35.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be bicategories and let \( L : \mathcal{A} \to \mathcal{B} \) be a pseudofunctor. We say that \( L \) has a left lax multiadjoint if there exists a pseudofunctor \( \mathcal{M}_{(-)} : \mathcal{B}^{\text{op}} \to \mathbf{Cat} \) and a pseudofunctor \( P : \int_{lax}^m \mathcal{B}_X \to \mathcal{A} \) such that

\[
\mathcal{B} (X, L-) \simeq \int_{lax}^m \mathcal{B}_X \to \mathcal{A} (P_m, -)
\]

\(^3\text{By normal we mean the unit constraints are required to be invertible.}\)
for all $X \in \mathcal{B}$, where each $P^X_\mathcal{M}: \mathcal{M}_X \to \mathcal{A}$ is obtained from $\mathcal{P}$ by including $\mathcal{M}_X \to j_{lax}^{X \in \mathcal{B}} \mathcal{M}_X$.

**Remark 36.** One might wonder why we did not simply define $L$ to have a left lax multiadjoint when every

$\mathcal{B}(X, L^{-}): \mathcal{A} \to \text{Cat}$

is a lax conical colimit of representables. The reason is that this condition would only be sufficient to force $\mathcal{P}$ (which may be constructed from this condition) to be a normal lax functor.

Before applying Theorem 30 to bi-presheaves of the form $\mathcal{B}(X, L^{-})$, we will need the appropriate notions of genericity with respect to a pseudofunctor $L: \mathcal{A} \to \mathcal{B}$.

**Definition 37.** Let $\mathcal{A}$ and $\mathcal{B}$ be bicategories and let $L: \mathcal{A} \to \mathcal{B}$ be a pseudofunctor. Then a 1-cell $\delta: X \to LA$ is lax-generic if for any diagram and 2-cell $\alpha$ as on the left below

there exists a diagram and 2-cells $\nu$ and $\gamma$ as on the right above (suppressing the constraint $Lg \cdot Lh \cong Lgh$) which is equal to $\alpha$, such that:

1. the top triangle is “sub-terminal” meaning that given any 2-cells $\omega, \tau: k \Rightarrow h$ as below

we have $\omega = \tau$;
2. given any other diagram

equal to $\alpha$, there exists a (necessarily unique) 2-cell $\psi: k \Rightarrow h$ such that
and
\[
\begin{array}{c}
\begin{array}{c}
A \xymatrix{ \ar[r]^{f} & C }
\end{array}
\end{array}
\xymatrix{ \ar[r]^{g} & B }
\xymatrix{ \ar[r]^{h} & C }
\]

(3) if \( \alpha \) is invertible, then both \( \gamma \) and \( \nu \) are invertible.

We call a factorization
\[
\begin{array}{c}
\begin{array}{c}
X \xymatrix{ \ar[r]^{z} & LB } \ar@{=>}[d]_{\delta} \end{array}
\end{array}
\xymatrix{ \ar[r]^{Lg} & LC } \xymatrix{ \ar[r]^{h} & LA }
\]

the \textit{universal factorization} of \( \alpha \) if both (1) and (2) are satisfied above.

Earlier in Definition 24 we defined a 1-cell to be generic when it satisfied a certain strong mixed lifting property. Translating this definition into the context of a pseudofunctor \( L: \mathcal{A} \to \mathcal{B} \) results in the below definition.

**Definition 38.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be bicategories and let \( L: \mathcal{A} \to \mathcal{B} \) be a pseudofunctor. Let \( \delta: X \to LA \) be a generic 1-cell. Then a pair \((h, \gamma)\) of the form
\[
\begin{array}{c}
\begin{array}{c}
X \xymatrix{ \ar[r]^{z} & LB } \ar@{=>}[d]_{\delta} \end{array}
\end{array}
\xymatrix{ \ar[r]^{Lg} & LC } \xymatrix{ \ar[r]^{h} & LA }
\]

is \textit{generic} if:

1. the diagram is “sub-terminal” meaning that given any 2-cells \( \omega, \tau: k \Rightarrow h \) as below
\[
\begin{array}{c}
\begin{array}{c}
X \xymatrix{ \ar[r]^{z} & LB } \ar@{=>}[d]_{\delta} \end{array}
\end{array}
\xymatrix{ \ar[r]^{Lg} & LC } \xymatrix{ \ar[r]^{h} & LA }
\]

we have \( \omega = \tau \);
2. given any other diagram
\[
\begin{array}{c}
\begin{array}{c}
X \xymatrix{ \ar[r]^{z} & LB } \ar@{=>}[d]_{\delta} \end{array}
\end{array}
\xymatrix{ \ar[r]^{Lg} & LC } \xymatrix{ \ar[r]^{h} & LA }
\]

and \( \lambda: h \Rightarrow k \) such that
\[
\begin{array}{c}
\begin{array}{c}
X \xymatrix{ \ar[r]^{z} & LB } \ar@{=>}[d]_{\delta} \end{array}
\end{array}
\xymatrix{ \ar[r]^{Lg} & LC } \xymatrix{ \ar[r]^{h} & LA }
\]
there exists a (necessarily unique) $\lambda^*: k \Rightarrow h$ such that

$$X \xrightarrow{\delta} LA \xrightarrow{\lambda} Lk \xleftarrow{\phi} LB$$

and $\lambda^* \lambda = \text{id}_h$.

From this definition, the following is clear.

**Corollary 39.** For any universal factorization

$$X \xrightarrow{\delta} LB \xrightarrow{Lg} LC$$

it follows that $(h, \gamma)$ is a generic 2-cell.

Before proving the main theorem of this section, it is worth defining the spectrum of a pseudofunctor. This is to be the two dimensional analogue of Diers’ definition of spectrum of a functor [4, Definition 3].

**Definition 40.** Let $\mathcal{A}$ and $\mathcal{B}$ be bicategories and let $L: \mathcal{A} \to \mathcal{B}$ be a pseudofunctor such that $\mathcal{B}(X, L-) \equiv \text{lax conical colimit of representables for every } X \in \mathcal{B}$. For each $X \in \mathcal{B}$, define $\mathcal{M}_X$ as the category with objects given by lax-generic morphisms out of $X$ and morphisms given by representative generic cells between them. We define the spectrum of $L$ to be the pseudofunctor

$$\text{Spec}_L: \mathcal{B}^{\text{op}} \to \text{Cat}$$

assigning an object $X \in \mathcal{B}^{\text{op}}$ to $\mathcal{M}_X$ and a morphism $f: Y \to X$ in $\mathcal{B}$ to the functor $\mathcal{M}_f: \mathcal{M}_X \to \mathcal{M}_Y$ which takes a generic morphism $\delta: X \to LA$ to $\delta': X \to LP$ where $\delta \cdot f \cong Lu \cdot \delta'$ is a chosen generic factorization of $\delta \cdot f$, and takes a generic 2-cell $\gamma: Lh \cdot \delta \Rightarrow \sigma$ as on the left below to the 2-cell $\gamma': Lh \cdot \delta' \Rightarrow \sigma'$ as on the right below

constructed as the universal factorization of the left pasting above.

**Remark 41.** When $\mathcal{A}$ has a terminal object the spectrum has an especially simple form, namely as the functor $\mathcal{B}(-, L1): \mathcal{B}^{\text{op}} \to \text{Cat}$.

We can now apply Theorem 30 to the case where $F: \mathcal{A} \to \text{Cat}$ is of the form $\mathcal{B}(X, L-)$ for a pseudofunctor $L: \mathcal{A} \to \mathcal{B}$ to help prove the following theorem.

**Theorem 42.** Let $\mathcal{A}$ and $\mathcal{B}$ be bicategories and let $L: \mathcal{A} \to \mathcal{B}$ be a pseudofunctor. Then the following are equivalent:

1. the pseudofunctor $L: \mathcal{A} \to \mathcal{B}$ has a left lax multiadjoint;
2. the following conditions hold:
(a) for every object $X \in \mathcal{A}$ and 1-cell $y : X \to LC$ in $\mathcal{B}$, there exists a lax-generic morphism $\delta : X \to LA$ and 1-cell $f : A \to C$ such that $Lf \cdot \delta \cong y$.

(b) for any triple of lax-generic morphisms $\delta$, $\sigma$ and $\omega$, and pair of generic cells $(h, \theta)$ and $(k, \phi)$ as below

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow^{\delta} & \searrow^{\theta} & \downarrow^{\phi} \\
LA & \xrightarrow{Lh} & LB \xrightarrow{Lk} LC
\end{array}
\]

the above pasting $(kh, \phi f \cdot \theta)$ is a generic cell\(^4\).

Proof. (1) $\Rightarrow$ (2) : Supposing that $L$ has a left lax multiadjoint, it follows that each $\mathcal{B} (X, L-)$ is a lax conical colimit of representables. By Theorem 30, we have (2)(a), as well as 2(b) when $f$ and $g$ are both the identity at $X$. To get the full version of (2)(b) we use that

\[
P : \int_{\text{lax}}^{X \in \mathcal{B}} \mathcal{M}_X \to \mathcal{A}
\]

is a pseudofunctor, where we have assumed without loss of generality that each $\mathcal{M}_X$ is the category of generic morphisms out of $X$ and representative cells, using Remark 32. Indeed, $\int_{\text{lax}}^{X \in \mathcal{B}} \mathcal{M}_X$ is the bicategory with objects pairs $(X, \delta : X \to LA)$ and morphisms $(X, \delta : X \to LA) \rightarrow (Y, \sigma : Y \to LB)$ given by triples $(f, h, \theta)$ as below

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow^{\delta} & \searrow^{\theta} & \downarrow^{\phi} \\
LA & \xrightarrow{Lh} & LB
\end{array}
\]

such that $(h, \theta)$ is a generic cell. As the lax functoriality constraints of $P$ are given by factoring diagrams such as (4.1) though a generic, the invertibility of these lax constraints of $P$ forces (2)(b).

(2) $\Rightarrow$ (1) : Applying Theorem 30 to the conditions 2(a) and 2(b) (only needing the case when $f$ and $g$ are identities at $X$), it follows that we may write

\[
\mathcal{B} (X, L-) \simeq \int_{\text{lax}}^{m \in \mathcal{M}_X} \mathcal{A} (P_m X, -)
\]

where $\mathcal{M}_X$ is the category of generic morphisms out of $X$ and representative generic cells between them. From this, we recover the spectrum $\text{Spec}_L : \mathcal{B}^{\text{op}} \rightarrow \text{Cat}$ taking each $X$ to $\mathcal{M}_X$. Also, we again we have the canonical normal lax functor

\[
P : \int_{\text{lax}}^{X \in \mathcal{B}} \mathcal{M}_X \to \mathcal{A}
\]

defined as in the reverse implication. The full version of (2)(b) forces this to be a pseudofunctor as required. \hfill \square

\(^4\)Suppressing pseudofunctoriality constraints of $L$. 

Remark 43. The reader will notice that condition (2)(b) where $f$ and $g$ are identities at $X$ is what is required to ensure that $P^X: \mathcal{M}_X \to \mathcal{A}$ is a pseudofunctor, whilst the full version of 2(b) is what is required to ensure

$$\mathbf{P}: \int^{X \in \mathcal{B}} \mathcal{M}_X \to \mathcal{A}$$

is a pseudofunctor.

Under the conditions of this theorem, we also have a notion of generic factorisations on 2-cells, in a sense we now describe.

Remark 44. Suppose $L$ has a left lax multiadjoint, $\delta$ and $\sigma$ are generic objects, and consider a 2-cell $\alpha: Lf \cdot \delta \Rightarrow Lg \cdot \sigma$. Then $\alpha$ has a $L$-generic factorization

$$\begin{array}{ccc}
X & \xrightarrow{\psi} & LC \\
\sigma & \xrightarrow{\lambda} & \delta \\
\downarrow & & \downarrow \\
LB & \xleftarrow{\nu} & LG
\end{array} = \begin{array}{ccc}
X & \xrightarrow{\psi} & LC \\
\sigma & \xrightarrow{\lambda} & \delta \\
\downarrow & & \downarrow \\
LB & \xleftarrow{\nu} & LG
\end{array}$$

Also note that any map $k: X \to LC$ can be factored as $L\kappa \cdot \xi$ for some generic $\xi$ and morphism $L\kappa$, and so when $L$ is surjective on objects we have a $L$-generic factorization of every 1-cell and 2-cell in the bicategory $\mathcal{B}$.

5. An alternative characterization

In Section 3 we gave a characterization of when a pseudofunctor $F: \mathcal{A} \to \mathbf{Cat}$ is a lax conical colimit of representables in terms of lax-generic objects and morphisms. However, it is natural to ask if we can also give a characterization in terms of what we will call “pseudo-generic” factorisations. Here we address this problem in the case where $\mathcal{A}$ is a 1-category $\mathcal{E}$, giving a simple description of when a pseudofunctor $F: \mathcal{E} \to \mathbf{Cat}$ is a lax conical colimit of representables.

These pseudo-generics are to be defined in terms of a pseudo-lifting property which we now recall.

**Definition 45 (pseudo-lifting property).** Let $\mathcal{C}$ be a bicategory. We say a diagram as on the left below

$$\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow & & \downarrow \\
B & \xleftarrow{g} & f
\end{array}$$

with $\nu$ invertible exhibits $(h, \nu)$ as the pseudo lifting of $f$ through $g$ if for any diagram as on the right above with $\psi$ invertible, there exists a unique invertible 2-cell $\lambda: k \Rightarrow h$ such that

$$\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow & \xrightarrow{\lambda} & \downarrow \\
B & \xleftarrow{g} & f
\end{array} = \begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow & \xrightarrow{\psi} & \downarrow \\
B & \xleftarrow{g} & f
\end{array}$$

Moreover, we say such a lifting $(h, \nu)$ is strong if $h$ is sub-terminal in $\mathcal{C} (A, C)$. 


Remark 46. Note that when $\mathcal{A}$ is a 1-category $\mathcal{E}$, the category of elements $\text{el } F$ is a 1-category, and so the mixed and pseudo lifting properties both become the usual one-dimensional lifting properties.

**Definition 47** (pseudo-generic objects). Let $\mathcal{A}$ be a bicategory and $F: \mathcal{A} \to \text{Cat}$ be a pseudofunctor. We say that an object $(A, x)$ in $\text{el } F$ is *pseudo-generic* if:

1. for any $(B, y), (C, z), (f, \alpha)$ and $(g, \beta)$ as below with both $\alpha$ and $\beta$ invertible

\[
\begin{array}{ccc}
(A, x) & \xrightarrow{(f, \alpha)} & (B, y) \\
\downarrow & & \downarrow \\
(C, z) & \xleftarrow{(g, \beta)} & (A, x) \\
\end{array}
\]

there exists a strong pseudo lifting $(h, \gamma) : (A, x) \to (C, z)$ exhibited by an invertible 2-cell $\nu : f \Rightarrow gh$;

2. every pseudo-lifting $(h, \gamma)$ as above has $\gamma$ invertible.

We can now give a simple characterization of when a pseudofunctor $F: \mathcal{E} \to \text{Cat}$ is a lax conical colimit of representables.

Remark 48. For proving the below theorem, simplified descriptions of pseudo-genericity would suffice as it concerns 1-categories $\mathcal{E}$ (for example every morphism becomes sub-terminal within its hom-category in this case). However, we will leave the descriptions in full generality above in case it is possible to generalize the below theorem to the bicategorical case.

**Theorem 49.** Let $\mathcal{E}$ be a category and $F: \mathcal{E} \to \text{Cat}$ be a pseudofunctor. Then the following are equivalent:

1. the pseudofunctor $F: \mathcal{E} \to \text{Cat}$ is a lax conical colimit of representables;
2. for every object $(B, y)$ in $\text{el } F$ there exists a lax-generic object $(A, x)$ and morphism $(f, \alpha) : (A, x) \to (B, y)$ with $\alpha$ invertible;
3. the following conditions hold:
   
   a. for every object $(B, y)$ in $\text{el } F$ there exists a pseudo-generic object $(A, x)$ and morphism $(f, \alpha) : (A, x) \to (B, y)$ with $\alpha$ invertible;
   
   b. for every morphism $f: X \to Y$ in $\mathcal{E}$ the functor $Ff: FX \to FY$ is a fibration.

Moreover, if any of the above equivalent conditions hold we then have

\[
F \simeq \int_{\text{lax}}^{m \in \mathfrak{M}} \mathcal{E}(P_m, -)
\]

where $P(-): \mathfrak{M} \to \mathcal{E}$ is the canonical projection of the category $\mathfrak{M}$ with:

**Objects:** An object is a pseudo-generic $(A, x)$ in $\text{el } F$;

**Morphisms:** A morphism $(A, x) \to (B, y)$ is a morphism $f : A \to B$ in $\mathcal{E}$ equipped with a morphism $\alpha : Ff(x) \to y$ in $FB$.

---

\(^5\)One could omit this condition and still prove Theorem 49, however, we give it here as it forces the lax-generic objects and pseudo-generic objects to coincide when $F: \mathcal{E} \to \text{Cat}$ is a lax conical colimit of representables.
Proof: Firstly note (1) ⇔ (2) by Theorem 30. For (1, 2) ⇒ (3), suppose that $F$ is a lax conical colimit of representables, i.e. that there exists a category $\mathfrak{M}$ and pseudofunctor $P(-) : \mathfrak{M} \to \mathcal{E}$ and equivalences

$$FT \simeq \int_{\text{lax}}^{m \in \mathfrak{M}} \mathcal{E}(P_m, T)$$

pseudonatural in $T \in \mathcal{E}$. Then as every lax-generic object $(A, x)$ is also pseudo-generic, we have the pseudo-generic factorisations of condition (a). Now consider a morphism $f : X \to Y$ in $\mathcal{E}$ and the functor $Ff : FX \to FY$. We know that $Ff$ is equivalent to (via an appropriate pseudo-naturality square) the functor

$$f \circ (-) : \int_{\text{lax}}^{m \in \mathfrak{M}} \mathcal{E}(P_m, X) \to \int_{\text{lax}}^{m \in \mathfrak{M}} \mathcal{E}(P_m, Y)$$

and this functor is a fibration since for any $\lambda : (m, u) \Rightarrow (n, v)$ as on the right below

\begin{align*}
\begin{array}{ccc}
m & \downarrow P_m & n \\
\lambda & \downarrow v \cdot P_\lambda & \lambda \\
n & \downarrow P_n & n \\
\end{array}
\quad \quad \\
\begin{array}{ccc}
m & \downarrow P_m & m \\
\lambda & \downarrow v \cdot P_\lambda & \lambda \\
n & \downarrow P_n & n \\
\end{array}
\end{align*}

we recover the $f \circ (-)$-cartesian lift on the left above. To see this lift is cartesian, and in fact that every morphism in $\int_{\text{lax}}^{m \in \mathfrak{M}} \mathcal{E}(P_m, X)$ is $f \circ (-)$-cartesian, note that given any $\lambda : (m, u) \Rightarrow (n, v)$ as on the left below and $\xi : (r, fw) \Rightarrow (m, fu)$ as on the top right below

\begin{align*}
\begin{array}{ccc}
r & \downarrow P_r & r \\
\xi & \downarrow v \cdot P_\xi & \xi \\
m & \downarrow v \cdot P_m & m \\
\lambda & \downarrow v \cdot P_\lambda & \lambda \\
n & \downarrow v \cdot P_n & n \\
\end{array}
\quad \quad \\
\begin{array}{ccc}
r & \downarrow P_r & r \\
\xi & \downarrow v \cdot P_\xi & \xi \\
m & \downarrow v \cdot P_m & m \\
\lambda & \downarrow v \cdot P_\lambda & \lambda \\
n & \downarrow v \cdot P_n & n \\
\end{array}
\end{align*}

for which the right of (5.1) can be seen as the result of some assignment

\begin{align*}
\begin{array}{ccc}
r & \downarrow P_r & m \\
\phi & \downarrow v \cdot P_\phi & \phi \\
n & \downarrow P_n & n \\
\end{array}
\quad \quad \\
\begin{array}{ccc}
r & \downarrow P_r & m \\
\phi & \downarrow v \cdot P_\phi & \phi \\
n & \downarrow P_n & n \\
\end{array}
\end{align*}

the induced unique lift $\xi : (r, w) \Rightarrow (m, u)$ given on the left in (5.1) is well defined since $u \cdot P_\xi = v \cdot P_\xi \cdot P_\lambda = v \cdot P_\phi = w$.

(3) ⇒ (1): Define $\mathfrak{M}$ as above, i.e. the full sub-category of el $F$ on the pseudo-generic objects. Now, $\int_{\text{lax}}^{m \in \mathfrak{M}} \mathcal{E}(P_m, T)$ is the category consisting of:

**Objects:** An object is a pair of the form $(A \in \mathcal{E}, x \in FA, f : A \to T)$
Morphisms: A morphism \((A, x, f: A \to T) \rightarrow (B, y, g: B \to T)\) is a morphism \(\alpha: A \to B\) in \(E\) rendering commutative

\[
\begin{array}{ccc}
A & \xrightarrow{f} & T \\
\downarrow{\alpha} & & \\
B & \xrightarrow{g} & \\
\end{array}
\]

equipped with a morphism \(\xi: F\alpha(x) \to y\) in \(FB\).

It suffices to check that the functor \(\lim_{m \in M} E(P_m, Y) \to FT\) defined by the assignation \((A, x, f) \mapsto Ff(x)\) on objects, and by

\[
\begin{array}{ccc}
(A, x, f) & \xrightarrow{(\alpha, \xi)} & (F\alpha(x), Ff(x)) \\
\downarrow{(\alpha, \xi)} & & \downarrow{\xi} \\
(B, y, g) & & (y, Fg(y))
\end{array}
\]

on morphisms (suppressing pseudo-functoriality constraints) is an equivalence. Functoriality is clear, and so it suffices to check the following.

Essentially Surjective. For any \(t \in FT\) we have \((T, t) \in el F\), and thus by (a) a pseudo-generic \((A, x)\) and morphism \((k, \phi): (A, x) \to (Y, t)\) with \(\phi\) invertible. Now note that \((A, x, k) \mapsto Fk(x) \cong t\) as required.

Full. Suppose we are given a morphism \(\zeta: Ff(x) \to Fg(y)\) in \(FT\). We may then take the \(Fg\)-cartesian lift \(\zeta^*y \to y\) and construct the universal diagram

\[
\begin{array}{ccc}
(B, \zeta^*y) & \xrightarrow{(h, \gamma)} & (g, id) \\
\downarrow{\nu} & & \downarrow{\eta} \\
(A, x) & \xrightarrow{(f, id)} & (T, Ff(x))
\end{array}
\]

with \(\gamma\) invertible. Note that \(\nu\) is necessarily an identity and so \(Fg(\gamma)\) is the identity (suppressing pseudo-functoriality constraints). It then suffices to observe that we have the assignation

\[
\begin{array}{ccc}
A & \xrightarrow{Fh(x)} & Ff(x) \\
\downarrow{h} & & \downarrow{\zeta} \\
B & \xrightarrow{\zeta^*y} & Fg(y)
\end{array}
\]

Faithful. Now, given another

\[
\begin{array}{ccc}
A & \xrightarrow{Fk(x)} & Ff(x) \\
\downarrow{k} & & \downarrow{\zeta} \\
B & \xrightarrow{y} & Fg(y)
\end{array}
\]
mapping to $\zeta$, we have $Fg(\phi) = \zeta$ and thus a factorization of $\phi$ through the cartesian lift

$$
\begin{array}{c}
Fk(x) \\
\downarrow \phi \\
\zeta^* y \\
\downarrow \phi \\
y
\end{array}
$$

with $Fg(\lambda)$ the identity. Thus we have a diagram

$$
\begin{array}{c}
(B, \zeta^* y) \\
\downarrow \text{id} \\
(A, x) \\
\downarrow (f, \text{id}) \\
(T, Ff(x)) \\
\end{array}
$$

and so $(k, \lambda) = (h, \gamma)$ by uniqueness. Hence $(k, \phi)$ is equal to $(h, \zeta \gamma)$ from earlier.

6. Comparing to Weber’s familial 2-functors

The purpose of this section is to compare our definition of a familial 2-functor $L: \mathcal{A} \to \mathcal{B}$ between 2-categories (assuming $\mathcal{A}$ has a terminal object) with Weber’s definition. It turns out that these two definitions are essentially equivalent. Note also that Weber’s definition assumes some “strictness conditions” (such as identity 2-cells factoring into identity 2-cells) which are natural conditions on 2-functors, but arguably less natural in the case of pseudofunctors.

We first recall the notion of generic morphism corresponding to what Weber refers to as the “naive” 2-categorical analogue of parametric right adjoints [9].

Definition 50. Suppose $\mathcal{A}$ and $\mathcal{B}$ are 2-categories. Given a 2-functor $L: \mathcal{A} \to \mathcal{B}$ we say a morphism $x: X \to LA$ is naive-generic if:

1. for any commuting square as on the left below

$$
\begin{array}{c}
X \rightarrow \rightarrow LB \\
x \downarrow \downarrow Lg \\
LA \rightarrow \rightarrow LC \\
x \downarrow \downarrow Lf \\
\end{array}
$$

there exists a unique $h: A \to B$ such that $Lh \cdot x = z$ and $f = gh$;

2. for two commuting diagrams

$$
\begin{array}{c}
X \rightarrow \rightarrow LB \\
x \downarrow \downarrow Lh \\
LA \rightarrow \rightarrow LC \\
x \downarrow \downarrow Lf \\
\end{array}
\quad \quad \quad
\begin{array}{c}
X \rightarrow \rightarrow LB \\
x \downarrow \downarrow Lg \\
LA \rightarrow \rightarrow LC \\
x \downarrow \downarrow Lf \\
\end{array}
$$

the 2-cells $\theta: z_1 \Rightarrow z_2$ such that $Lg \cdot \theta = \text{id}$ bijectively correspond to 2-cells $\theta': h_1 \Rightarrow h_2$ such that $L(\theta') \cdot x = \theta$ and $g \cdot \theta' = \text{id}$.

Definition 51. Suppose $\mathcal{A}$ and $\mathcal{B}$ are 2-categories, and that $\mathcal{A}$ has a terminal object. We say a 2-functor $L: \mathcal{A} \to \mathcal{B}$ is a naive parametric right adjoint if every $f: X \to LA$ factors as $Lf \cdot x$ for a naive-generic morphism $x$. 
Weber’s definition of famility requires certain maps in a 2-category to be fibrations. Thus we will need to recall the definition of fibration in a 2-category $\mathcal{B}$. Note that when $\mathcal{B}$ is finitely complete there are other equivalent characterizations of fibrations [6].

**Definition 52.** We say a morphism $p: E \to B$ in a 2-category $\mathcal{B}$ is a *fibration* if:

1. for every $X \in \mathcal{B}$, the functor $\mathcal{B}(X, p): \mathcal{B}(X, E) \to \mathcal{B}(X, B)$ is a fibration;
2. for every $f: X \to Y$ in $\mathcal{B}$, the functor $\mathcal{B}(f, E): \mathcal{B}(Y, E) \to \mathcal{B}(X, E)$ preserves cartesian morphisms.

If we have a choice of cartesian lifts which strictly respects composition and identities we say the fibration splits.

We now have the required background to define famility in the sense of Weber.

**Definition 53.** Suppose $\mathcal{A}$ and $\mathcal{B}$ are 2-categories and that $\mathcal{A}$ has a terminal object. We say a 2-functor $L: \mathcal{A} \to \mathcal{B}$ is *Weber-familial* if

1. $L$ is a naive parametric right adjoint;
2. for every $A \in \mathcal{A}$, and unique $t_A: A \to 1$ in $\mathcal{A}$, the morphism $Lt_A: LA \to L1$ is a split fibration in $\mathcal{B}$.

The following is Weber’s analogue of lax-generic morphisms.

**Definition 54.** Suppose $\mathcal{A}$ and $\mathcal{B}$ are 2-categories. Given a 2-functor $L: \mathcal{A} \to \mathcal{B}$ for which each $Lt_A: LA \to L1$ is a split fibration, we say a morphism $x: X \to LA$ is *Weber-lax-generic* if for any 2-cell $\alpha$ as on the left below,

\[
\begin{array}{ccc}
X & \xrightarrow{\delta} & LB \\
\downarrow x & & \downarrow Lg \\
LA & \xrightarrow{Lh} & LC
\end{array}
\]

there exists a unique factorization $(h, \gamma, \nu)$ as above such that $(h, \gamma)$ is chosen $Lt_B: LB \to L1$ cartesian.\(^6\)

The following lemma shows that for Weber-familial 2-functors $L$, the lax-generics of both our sense and Weber’s coincide, and our generic 2-cells can equivalently be characterized as certain cartesian morphisms.

**Lemma 55.** Suppose $\mathcal{A}$ and $\mathcal{B}$ are 2-categories and that $\mathcal{A}$ has a terminal object. Let $L: \mathcal{A} \to \mathcal{B}$ be a Weber-familial 2-functor. Define $\mathcal{M}$ as the category with objects given by chosen naive-generics $\delta: X \to LA$ (meaning to be identified with another naive-generic $\sigma: X \to LB$ if there exists a pair $(h, \gamma)$ as below with $h$ invertible and $\gamma$ an identity), and morphisms given by pairs $(h, \gamma)$

\[
\begin{array}{ccc}
X & \xrightarrow{\delta} & LA \\
\downarrow \sigma & & \downarrow Lh \\
LB & \xrightarrow{Lg} & L1
\end{array}
\]

where $\gamma$ is chosen $Lt_B: LB \to L1$ cartesian. Then:

\(^6\)This definition of lax-generics has the downside that it assumes some famility conditions, thus not allowing for a theorem describing an equivalence between famility and lax-generic factorisations.
(1) for every $X \in \mathcal{B}$ we have isomorphisms
\[ \mathcal{B}(X, L-) \cong \int^{m \in \mathcal{M}} \mathcal{A}(P_m, -) ; \]

(2) a map $\delta : X \to LA$ in $\mathcal{B}$ is naive-generic if and only if it is strict lax-generic;

(3) a 2-cell in $\mathcal{B}$ as below
\[
\begin{array}{c}
X \downarrow \delta \\
\downarrow z \\
LB
\end{array}
\]

is generic if and only if it is $L\tau_B : LB \to L1$ cartesian.

**Proof.** (1) : It suffices to check that the functors
\[ \int^{m \in \mathcal{M}} \mathcal{A}(P_m, T) \to \mathcal{B}(X, LT) \]
are isomorphisms. That this assignment is bijective on objects is a consequence of the well known one-dimensional case (for instance, see [7, Prop. 7]). That the assignment on morphisms
\[
\begin{array}{ccc}
LP_m & \xrightarrow{\delta_m} & LA \\
\downarrow h & & \downarrow \delta_m \\
LP_{m'} & \xrightarrow{\delta_{m'}} & LA
\end{array}
\]
is bijective follows from the fact each naive-generic is Weber-lax generic [9, Lemma 5.8]. Naturality is also an easy consequence of this fact.

(2) : If $\delta$ is naive-generic, and thus isomorphic to a representative naive-generic, then $\delta$ is lax-generic by (1). If $\delta$ is strict lax-generic, then from a $\theta : z_1 \Rightarrow z_2$ we have a universal factorization
\[
\begin{array}{ccc}
X \xrightarrow{x} LA & \xrightarrow{\theta} & \xrightarrow{Lh} LB \\
\downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \\
LA & \xrightarrow{Lh_1} & LB
\end{array}
\]
where we have used that $Lg \cdot \theta$ is an identity to see the top right triangle above can be taken as an identity. In this way, we recover the bijection required of a naive-generic.

---

7By strict we mean identity 2-cells universally factor into identity 2-cells.
(3) Consider a 2-cell

\[
\begin{array}{c}
X \\
\delta \\
\gamma \\
\zeta \\
\zeta \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
LA \\
Lh \\
LB \\
\end{array}
\]

If this 2-cell is generic, then we have a factorization

\[
\begin{array}{c}
X \\
\delta \\
\gamma \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
LA \\
Lh \\
LB \\
\end{array}
= 
\begin{array}{c}
X \\
\delta \\
\gamma \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
LA \\
Lh \\
LB \\
\end{array}
\]

where \( \phi \) is chosen cartesian. By genericity of \( \gamma \), we have an \( \lambda^* : k \Rightarrow h \) such that

\[
\begin{array}{c}
X \\
\delta \\
\gamma \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
LA \\
Lh \\
LB \\
\end{array}
= 
\begin{array}{c}
X \\
\delta \\
\gamma \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
LA \\
Lh \\
LB \\
\end{array}
\]

and \( \lambda^* \lambda = \text{id}_k \). Substituting \( 6.1 \) into \( 6.2 \) and using that \( \delta \) is Weber-lax-generic gives \( \lambda \lambda^* = \text{id}_k \). Conversely, if this 2-cell is cartesian we then have a factorization

\[
\begin{array}{c}
X \\
\delta \\
\gamma \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
LA \\
Lh \\
LB \\
\end{array}
= 
\begin{array}{c}
X \\
\delta \\
\gamma \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
LA \\
Lh \\
LB \\
\end{array}
\]

where \( (k, \phi) \) is a generic 2-cell (which must also be cartesian by the above argument). Since \( \phi \) and \( \gamma \) are cartesian, and thus isomorphic to chosen cartesian morphisms, it follows that \( \lambda \) is invertible (by uniqueness of chosen cartesian factorisations).

Finally, we give the main result of this section, showing that for 2-functors \( L : \mathcal{A} \to \mathcal{B} \) our lax-multiadjoint condition is essentially equivalent to Weber’s familiarity condition.

**Theorem 56.** Suppose \( \mathcal{A} \) and \( \mathcal{B} \) are 2-categories and that \( \mathcal{A} \) has a terminal object. Then for a 2-functor \( L : \mathcal{A} \to \mathcal{B} \) the following are equivalent:

1. \( L \) is Weber-familial;
2. \( L \) has a strict\(^8\) left lax multiadjoint.

**Proof.** (1) \( \Rightarrow \) (2) : Supposing \( L : \mathcal{A} \to \mathcal{B} \) is Weber-familial, we have that each \( \mathcal{B}(X, L-) \) is a lax conical colimit of representables by Lemma 55 part (1). Also, as the generic 2-cells may be identified with the cartesian 2-cells, we know since the fibration \( L|_B : LB \to L1 \) respects precomposition we have the following property:

---

\( ^8 \)By strict we mean isomorphic to a lax conical colimit of representables in place of equivalent, and that the reindexings \( P^\Lambda_{(-)} \) are 2-functors instead of pseudofunctors.
for any generic 2-cell out of an $X \in \mathcal{B}$ as on the left below

\[(6.3)\]

and map $k : Y \to X$ in $\mathcal{B}$, the right diagram is a generic 2-cell. It is this property (along with closure of generic cells under composition) which gives (2)(b) of Theorem 42.

(2) ⇒ (1) : Suppose $L : \mathcal{A} \to \mathcal{B}$ is a strict left lax multiadjoint. Then $L$ is a naive parametric right adjoint since $L$ has strict lax generic factorisations, and lax-generic implies naive generic (shown in the proof of Lemma 55).

It remains to check that each $L_{tA} : LA \to L1$ is a split fibration. To see this, note that for each $X \in \mathcal{B}$ the functor $\mathcal{B}(X, LA) \to \mathcal{B}(X, L1)$ may be written as the functor

\[
\int_{\text{lax}} \mathcal{A}(P_m, A) \to \int_{\text{lax}} \mathcal{A}(P_m, 1) \cong \mathfrak{M}
\]

defined by the assignment

\[
m \quad P_m \quad f \quad m' \quad P_{m'}
\]

\[
\lambda \quad P_{\lambda} \quad \beta \quad A \quad \lambda
\]

\[
m' \quad P_{m'} \quad g \quad m'
\]

It is straightforward to verify that for each $(m', g : P_m \to A)$ and $\lambda : m \to m'$ we recover a cartesian lift

\[
m \quad P_m \quad g \cdot P_{\lambda} \quad m
\]

\[
\lambda \quad P_{\lambda} \quad \beta \quad A \quad \lambda
\]

\[
m' \quad P_{m'} \quad g \quad m'
\]

and it is clear the canonical choice of cartesian lifts given above splits. The cartesian morphisms are diagrams as above (with the identity 2-cell possibly replaced by an isomorphism), and these correspond to generic cells in $\mathcal{B}(X, LA)$. That for each $k : Y \to X$ the functor $\mathcal{B}(k, LA) : \mathcal{B}(Y, LA) \to \mathcal{B}(X, LA)$ preserves cartesian morphisms then follows from condition (2)(b) of Theorem 42.

\[\square\]

7. Examples

We will first consider some simple examples of lax multiadjoints which concern pseudofunctors $L : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ is a 1-category. Our first and simplest examples of such pseudofunctors $L : \mathcal{A} \to \mathcal{B}$ concern the universal embeddings into bicategories of spans and polynomials.
The reader will also recall that in this setting where $\mathcal{A}$ is a 1-category, $el\ F = el\ B(X,L-)$ is a 1-category for each $X \in B$, and so the mixed lifting properties become the usual lifting properties. Indeed, it is clear that in such cases every pair $(h,\gamma)$ out of a generic 1-cell is a generic 2-cell.

**Example 57.** The canonical pseudofunctor $L: \mathcal{E} \to \text{Span}(\mathcal{E})$ has a left lax multi-adjoint. To see this, first observe that a span $X \rightrightarrows LA$ is generic if it is isomorphic to the form

\[
\begin{array}{c}
X \xrightarrow{(s,1)} LA \\
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow \\
LA \xrightarrow{Lp} LA
\end{array}
\]

This is since for a general span $(s,t)$ genericity would imply we can factor the diagram on the left below

\[
\begin{array}{c}
X \xrightarrow{(s,1)} LM \\
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow \\
LA \xrightarrow{Lp} LA
\end{array} = \begin{array}{c}
X \xrightarrow{(s,1)} LM \\
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow \\
LA \xrightarrow{Lp} LA
\end{array}
\]

as on the right above, where $\nu$ is necessarily an identity and $\gamma$ invertible. Hence $tu = id$ and $ut$ is invertible, showing that $t$ is invertible. Conversely, to see such a $(s,1)$ is generic, note that any diagram as on the left below

\[
\begin{array}{c}
X \xrightarrow{(u,v)} LM \\
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow \\
LA \xrightarrow{Lp} LA
\end{array} \quad \overset{\alpha}{=} \quad \begin{array}{c}
X \xrightarrow{(u,v)} LM \\
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow \\
LA \xrightarrow{Lp} LA
\end{array}
\]

universally factors as on the right above, where $\alpha$ and $\gamma$ are the respective morphisms of spans

\[
\begin{array}{c}
\alpha: \\
\begin{array}{c}
X \xrightarrow{\nu} LA \\
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow \\
LA \xrightarrow{\nu} LA
\end{array}
\end{array}
\quad \overset{\gamma}{=} \quad \begin{array}{c}
X \xrightarrow{\nu} LA \\
\downarrow \quad \downarrow \\
\downarrow \quad \downarrow \\
LA \xrightarrow{\nu} LA
\end{array}
\]

As all cells between generic morphisms are generic, it follows that the category $\mathcal{M}_X$ of generics out of $X$ is the slice $\mathcal{E}/X$, and so for any $X \in \mathcal{E}$ we may take $P(-)$ as the functor $\text{dom}: \mathcal{E}/X \to \mathcal{E}$, giving

\[
\text{Span}(\mathcal{E})(X,L-) \cong \int_{\text{lax}}^{m \in \mathcal{E}/X} \mathcal{E}(P_m,-)
\]

Dual to the above, we see that $L: \mathcal{E} \to \text{Span}(\mathcal{E})^\text{co}$ admits oplax-generic factorisations; indeed we may write

\[
\text{Span}(\mathcal{E})^\text{co}(X,L-) \cong \int_{\text{oplax}}^{m \in \mathcal{E}/X} \mathcal{E}(P_m,-)
\]
Moreover, the pseudofunctor $L : \mathcal{E} \to \text{Span}_{\text{iso}}(\mathcal{E})$ admits both lax and oplax generic factorisations, as we may write

$$\text{Span}_{\text{iso}}(\mathcal{E})(X, L--) \cong \int_{\text{lax}}^{m \in (\mathcal{E}/X)_{\text{iso}}} \mathcal{E}(P_m, -) \cong \int_{\text{oplax}}^{m \in (\mathcal{E}/X)_{\text{iso}}} \mathcal{E}(P_m, -)$$

where $(\mathcal{E}/X)_{\text{iso}}$ contains the objects of $\mathcal{E}/X$ and only those morphisms which are invertible. The reader will also note that we do not have

$$\text{Span}_{\text{iso}}(\mathcal{E})(X, L--) \simeq \sum_{\text{ob } \mathcal{E}/X} \mathcal{E}(P_m, -)$$

As for each $T \in \mathcal{E}$, the right above is a discrete category, but isomorphisms of spans are not unique (and so the canonical assignment is not fully faithful).

In the following examples we will omit the verification that the generic morphisms are classified correctly.

**Example 58.** The canonical pseudofunctor $L : \mathcal{E} \to \text{Poly}(\mathcal{E})$ has a left lax multi-adjoint. Indeed a polynomial $X \rightarrow LA$ is generic precisely when it is isomorphic to the form

$$X \xrightarrow{s} LM \xrightarrow{p} LA \xrightarrow{id} LA$$

and all cells between are generic. Consequently, we may take $P(-)$ as the functor $p : \Pi \mathcal{E}(\mathcal{E}/X) \to \mathcal{E}$ where $\Pi \mathcal{E}(\mathcal{E}/X)$ is the category with objects given by spans

$$X \xrightarrow{f} T \xrightarrow{g} U$$

out of $X$, and morphisms of spans from $(f, g) \rightarrow (f', g')$ given by a pair $\alpha : W \rightarrow T$ and $\beta : U \rightarrow U'$ rendering commutative the diagram

$$X \xrightarrow{f} T \xrightarrow{g} U \xrightarrow{\beta} U'$$

such that the pullback is chosen. As a consequence we have

$$\text{Poly}(\mathcal{E})(X, L--) \cong \int_{\text{lax}}^{m \in \Pi \mathcal{E}(\mathcal{E}/X)} \mathcal{E}(P_m, -)$$

for all $X \in \text{Poly}(\mathcal{E})$.

**Remark 59.** By the above, the usual inclusion $\text{Span}(\mathcal{E}) \rightarrow \text{Poly}(\mathcal{E})$ can be seen as coming from the unit components $u_{\mathcal{E}/X} : \mathcal{E}/X \rightarrow \Pi \mathcal{E}(\mathcal{E}/X)$ of the pseudomonad $\Pi \mathcal{E}$ for fibrations with products. Indeed, the family of functors $\text{Span}(\mathcal{E})(X, Y) \rightarrow \text{Poly}(\mathcal{E})(X, Y)$ may be written as the resulting functors

$$\int_{\text{lax}}^{m \in \mathcal{E}/X} \mathcal{E}(P_m, Y) \rightarrow \int_{\text{lax}}^{m \in \Pi \mathcal{E}(\mathcal{E}/X)} \mathcal{E}(P_m, Y)$$
We now give a more complicated example, where $\mathcal{A}$ is not a 1-category. In this situation the mixed lifting properties are necessary (unlike the earlier examples where usual liftings would suffice), and so it is no longer the case that every $(h, \gamma)$ out of a generic morphism is a generic 2-cell.

**Example 60.** The canonical pseudofunctor $L: \text{Span}(\mathcal{E})^{co} \to \text{Poly}(\mathcal{E})$ is such that $L^{op}$ has a left lax multiadjoint. Here a polynomial $LA \triangleright\leftarrow X$ is opgeneric (meaning the opposite polynomial is generic) if it is isomorphic to the form

\[
\begin{array}{ccc}
\text{id} & LA & \text{id} \\
LA & \downarrow & LA \\
& f & X
\end{array}
\]

and a pair $((s, t), \gamma)$ out of an opgeneric as below

\[
\begin{array}{ccc}
(1,1,f) & \gamma & (s,t) = (s,t,1) \\
LA & \downarrow & LB \\
\text{opgeneric} & \downarrow & \text{poly} \\
(v,u,g) & \gamma & \downarrow \gamma
\end{array}
\]

is generic when $\gamma: (s, t, f) \Rightarrow (v, u, g)$ is a cartesian morphism of polynomials. We note also that cartesian morphisms of polynomials are closed under vertical composition as well as precomposition by another polynomial.

Given a general morphism of polynomials $\phi: (s, t, f) \Rightarrow (v, u, g)$ as given by the diagram

\[
\begin{array}{ccc}
LA & \downarrow & X \\
P & \downarrow & M \\
\text{id} & LA & \text{id} \\
& \downarrow & f \\
& \downarrow & X
\end{array}
\]

the op-generic factorization of $\phi$ is given by

\[
\begin{array}{ccc}
LA & \downarrow & X \\
& \downarrow & LA \\
& \downarrow & LA \\
& \downarrow & f \\
& \downarrow & X
\end{array}
\]

where $\nu$ is the reversed morphism of spans on the left below

\[
\begin{array}{ccc}
LA & \downarrow & M & \downarrow \nu & X \\
\text{op generic} & \downarrow & \text{poly} & \downarrow & \text{poly} \\
P & \downarrow & N & \downarrow \nu & X \\
\text{id} & LA & \text{id} & \downarrow & f \\
& \downarrow & f & \downarrow & X
\end{array}
\]

and $\gamma$ is the cartesian morphism of polynomials on the right above. It follows that for any $X \in \mathcal{E}$ we may take $P_{(-)}$ as the functor

$$\mathcal{E}/X \xrightarrow{\text{dom}} \mathcal{E} \xrightarrow{\iota} \text{Span}(\mathcal{E})^{\text{coop}}$$
where \( \iota \) assigns each morphism \( h: A \to B \) to \( (h, 1_A) \in \text{Span}(E)_{\text{coop}} \), and get

\[
\text{Poly}(E)^{\text{op}}(X, L-) \cong \int_{\text{lax}} m \in E/X \text{Span}(E)_{\text{coop}}(P_m, -).
\]

We now give a natural example which does not come from a pseudo functor of bicategories \( L: \mathcal{A} \to \mathcal{B} \). Indeed, the following may be seen as the main motivating example for this paper.

**Example 61.** Consider the bi-presheaf \( \text{Fam}: \text{CAT} \to \text{CAT} \) sending a category \( C \) to the category \( \text{Fam}(C) \) with objects given by families of objects of \( C \) denoted \( (A_i \in C: i \in I) \), and morphisms \( (A_i \in C: i \in I) \to (B_j \in C: j \in J) \) given by a reindexing \( \varphi: I \to J \) along with comparison maps \( A_i \to B_{\varphi(i)} \) for each \( i \in I \).

Now, the generic objects of \( \text{el Fam} \) are those elements of the form \( (I, (A_i \in C: i \in I)) \) for a set \( I \). And it is clear that for any general element \( (C, (B_j : j \in J)) \) of \( \text{el Fam} \) that we have the “generic factorization” (that is an opcartesian map from a generic)

\[
(J, (j : j \in J)) \xrightarrow{(H_{(-), \text{id}})} (C, (B_j : j \in J))
\]

Also, a general morphism out of a generic object

\[
(I, (i : i \in I)) \xrightarrow{(H_{(-), (\varphi, \gamma)})} (C, (B_j : j \in J))
\]

consists of a functor \( H_{(-)}: I \to C \), a function \( \varphi: I \to J \), and morphisms \( \gamma_i: H_i \to B_{\varphi(i)} \) indexed over \( i \in I \). Such a morphism is generic precisely when every \( \gamma_i \) is invertible.

It is then clear that the category of generic objects and generic morphisms between them (note \( H_{(-)} \) is uniquely determined by \( \varphi \) in this case) is isomorphic to \( \text{Set} \). It follows that the \( \text{Fam} \) construction is given by

\[
\text{Fam}(C) = \int_{\text{lax}} X \in \text{Set} C^X, \quad C \in \text{CAT}
\]

It is worth noting that restricting to the category of finite sets \( \text{Set}_{\text{fin}} \), yields the finite families construction \( \text{Fam}_{\text{fin}} \), and restricting further the category of finite sets and bijections \( \mathbb{P} \) yields the free symmetric (strict) monoidal category construction.

The above shows that \( \text{Fam} \) is familial in the sense that it is a lax conical colimit of representables, however \( \text{Fam} \) is also familial in another sense: it has a left lax multiadjoint.

**Example 62.** The pseudofunctor \( \text{Fam}: \text{CAT} \to \text{CAT} \) has a left lax multiadjoint. Here the generic morphisms are those functors of the form

\[
\delta_F: C \to \text{Fam}(\text{el } F): X \to ((X, x) \in \text{el } F: x \in FX)
\]

for a presheaf \( F: C \to \text{Set} \) (Weber refers to these as “functors endowing \( C \) with elements” [9, Definition 5.10]). A cell out of such a generic morphism

\[
\begin{array}{ccc}
\text{Fam}(\text{el } F) & \xrightarrow{\delta} & \text{Fam}(H) \\
\downarrow \gamma & & \downarrow \text{Fam}(H) \\
\text{Fam}(B) & \xrightarrow{z} & \text{Fam}(B)
\end{array}
\]
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is generic when the comparison maps (not necessarily the reindexing maps) comprising each $\gamma_X$ for $X \in \mathcal{C}$ are required invertible. It follows that this lax multiadjoint is exhibited by the formula

$$\text{CAT} \left( \mathcal{C}, \text{Fam} \left( - \right) \right) \cong \int_{\text{lax}} \text{CAT} \left( \text{el} F, - \right)$$

for each $\mathcal{C} \in \text{CAT}$.

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