BOUNDARY VALUE PROBLEMS FOR THE 2nd-ORDER SEIBERG-WITTEN EQUATIONS

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Abstract. It is shown that the non-homogeneous Dirichlet and Neuman problems for the 2nd-order Seiberg-Witten equation admit a regular solution once the $H$-condition is satisfied. The approach consist in applying the elliptic techniques to the variational setting of the Seiberg-Witten equation.

1. Introduction

Let $X$ be a compact smooth 4-manifold with non-empty boundary. In our context, the Seiberg-Witten equations are the 2nd-order Euler-Lagrange equation of the functional defined in 2.2.1. When the boundary is empty, their variational aspects were first studied in [9] and the topological ones in [2]. Thus, the main aim is to obtain the existence of a solution to the non-homogeneous equations whenever $\partial X \neq \emptyset$. The non-emptyness of the boundary inflicts boundary conditions on the problem. Classically, this sort of problem is classified according with its boundary conditions in Dirichlet Problem (D) or Neumann Problem (N).

1.1. Spin$^c$ Structure. The space of Spin$^c$ structures on $X$ is identified with

$$Spin^c(X) = \{ \alpha + \beta \in H^2(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z}_2) \mid w_2(X) = \alpha(\text{mod } 2) \}.$$  

For each $\alpha \in Spin^c(X)$, there is a representation $\rho_\alpha: SO_4 \to Cl_4$, induced by a Spin$^c$ representation, and a pair of vector bundles $(\mathcal{S}_\alpha^+, \mathcal{L}_\alpha)$ over $X$ (see [11]). Let $P_{SO_4}$ be the frame bundle of $X$, so

- $\mathcal{S}_\alpha = P_{SO_4} \times_{\rho_\alpha} V = S_\alpha^+ \oplus S_\alpha^-$. The bundle $S_\alpha^+$ is the positive complex spinors bundle (fibers are $Spin_4^c$ - modules isomorphic to $\mathbb{C}^2$).
- $\mathcal{L}_\alpha = P_{SO_4} \times_{\det(\alpha)} \mathbb{C}$. It is called the determinant line bundle associated to the Spin$^c$-structure $\alpha$. ($c_1(\mathcal{L}_\alpha) = \alpha$)

Thus, for each $\alpha \in Spin^c(X)$ we associate a pair of bundles

$$\alpha \in Spin^c(X) \rightsquigarrow (\mathcal{L}_\alpha, \mathcal{S}_\alpha^+).$$

From now on, we considered on $X$ a Riemannian metric $g$ and on $\mathcal{S}_\alpha$ an hermitian structure $h$.

Let $P_\alpha$ be the $U_1$-principal bundle over $X$ obtained as the frame bundle of $\mathcal{L}_\alpha$ ($c_1(P_\alpha) = \alpha$). Also, we consider the adjoint bundles

Key words and phrases. connections, gauge fields, 4-manifolds
MSC 58J05, 58E50.
where $Ad(U_1)$ is a fiber bundle with fiber $U_1$, and $ad(u_1)$ is a vector bundle with fiber isomorphic to the Lie Algebra $u_1$.

1.2. The Main Theorem. Let $\mathcal{A}_\alpha$ be (formally) the space of connections (covariant derivative) on $L_\alpha$, $\Gamma(S^+_{\alpha})$ is the space of sections of $S^+_{\alpha}$ and $\mathcal{G}_\alpha = \Gamma(Ad(U_1))$ is the gauge group acting on $\mathcal{A}_\alpha \times \Gamma(S^+_{\alpha})$ as follows:

(1.1) \[ g.(A, \phi) = (A + g^{-1}dg, g^{-1}\phi). \]

$\mathcal{A}_\alpha$ is an affine space which vector space structure, after fixing an origin, is isomorphic to the space $\Omega^2(ad(u_1))$ of $ad(u_1)$-valued 1-forms. Once a connection $\nabla^0 \in \mathcal{A}_\alpha$ is fixed, a bijection $\mathcal{A}_\alpha \leftrightarrow \Omega^2(ad(u_1))$ is explicited by $\nabla^A \leftrightarrow A$, where $\nabla^A = \nabla^0 + A$. $\mathcal{G}_\alpha = Map(X, U_1)$, since $Ad(U_1) \simeq X \times U_1$. The curvature of a 1-connection form $A \in \Omega^2(ad(u_1))$ is the 2-form $F_A = dA \in \Omega^2(ad(u_1))$.

**Definition 1.2.1.**

1. (1) the configuration space of the $D$-problem is

\[ C^D = \{(A, \phi) \in \mathcal{A}_\alpha \times \Gamma(S^+_{\alpha}) \mid (A, \phi) \mid_{\partial X} \sim (A_0, \phi_0)\}, \]

2. (2) the configuration space of the $N$-problem is

\[ C^N = \mathcal{A}_\alpha \times \Gamma(S^+_{\alpha}) \]

Although each boundary problem requires its own configuration space, the superscripts $D$ and $N$ will be used whenever the distinction is necessary, since most arguments work for both sort of problems.

The Gauge Group $\mathcal{G}_\alpha$ action on each of the configuration space is given by $\nabla^0$.

The Dirichlet ($D$) and Neumann ($N$) boundary value problems associated to the $SW_{\alpha}$-equations are the following: Let’s consider $\Theta, \sigma \in \Omega^2(ad(u_1)) \oplus \Gamma(S^+_{\alpha})$ and $(A_0, \phi_0)$ defined on the manifold $\partial X$ ($A_0$ is a connection on $L_\alpha$; $\phi_0$ is a section of $\Gamma(S^+_{\alpha})|_{\partial X}$). In this way, find $(A, \phi) \in C^D$ satisfying $D$ and $(A, \phi) \in C^N$ satisfying $N$, where

\[
\mathcal{D} = \begin{cases} 
  d^*F_A + 4\Phi^*(\nabla^A)\phi = \Theta, \\
  \Delta_A\phi + \frac{(|\phi|^2 + k_\alpha)}{4}\phi = \sigma, \\
  (A, \phi)|_{\partial X} \sim (A_0, \phi_0),
\end{cases} \quad \mathcal{N} = \begin{cases} 
  d^*F_A + 4\Phi^*(\nabla^A)\phi = \Theta, \\
  \Delta_A\phi + \frac{(|\phi|^2 + k_\alpha)}{4}\phi = \sigma, \\
  i^*(\ast F_A) = 0, \nabla^A_\phi \phi = 0,
\end{cases}
\]

and

1. the operator $\Phi^* : \Omega^1(S^+_{\alpha}) \to \Omega^1(u_1)$ is locally given by

\[ \Phi^*(\nabla^A)\phi = \frac{1}{2}\nabla^A (|\phi|^2) = \sum_i < \nabla_i^A \phi, \phi > \eta_i, \]

and $\eta = \{\eta_i\}$ is an orthonormal frame in $\Omega^1(ad(u_1))$.

2. $i^*(\ast F_A) = F_4$, where

$F_4 = (F_{14}, F_{24}, F_{34}, 0)$ is the local representation of the $4^{th}$-component (normal to $\partial X$) of the 2-form of curvature in the local chart $(x, U)$ of $X$;

$x(U) = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4; \|x\| < \epsilon, x_4 \geq 0\}$, and
x(U ∩ ∂X) ⊂ \{x ∈ x(U) \mid x_4 = 0\}. Let \{e_1, e_2, e_3, e_4\} be the canonical base of \(\mathbb{R}^4\), so \(\nu = -e_4\) is the normal vector field along \(\partial X\).

**Main Theorem 1.2.2.** If the pair \((\Theta, \sigma) \in L^{k,2} \oplus (L^{k,2} \cap L^{\infty})\) satisfies the \(H\)-condition, then the problems \(D\) and \(N\) admit a \(C^r\)-regular solution \((A, \phi)\), whenever \(2 < k\) and \(r < k\).

2. Basic Set Up

2.1. Sobolev Spaces. As a vector bundle \(E\) over \((X, g)\) is endowed with a metric and a covariant derivative \(\nabla\), we define the Sobolev norm of a section \(\phi \in \Omega^0(E)\) as

\[
|| \phi ||_{L^{k,p}} = \sum_{|i| = 0}^k (\int_X |\nabla^i \phi|^p)^{\frac{1}{p}}.
\]

In this way, the \(L^{k,p}\)-Sobolev Spaces of sections of \(E\) is defined as

\[
L^{k,p}(E) = \{\phi \in \Omega^0(E) \mid || \phi ||_{L^{k,p}} < \infty\}.
\]

In our context, in which we fixed a connection \(\nabla^0\) on \(L_\alpha\), a metric \(g\) on \(X\) and an hermitian structure on \(S_\alpha\), the Sobolev Spaces on which the basic setting is made are the following:

- \(A_\alpha = L^{1,2}(\Omega^1(\text{ad}(u_1)))\);
- \(\Gamma(S_\alpha^+) = L^{1,2}(\Omega^0(X, S_\alpha^+))\);
- \(C_\alpha = A_\alpha \times \Gamma(S_\alpha^+)\);
- \(\mathcal{G}_\alpha = L^{2,2}(X, U_1) = L^{2,2}(\text{Map}(X, U_1))\).

\(\mathcal{G}_\alpha\) is an \(\infty\)-dimensional Lie Group which Lie algebra is \(g = L^{1,2}(X, u_1)\).

The Sobolev spaces above induce a Sobolev structure on \(C^D_\alpha\) and on \(C^N_\alpha\). From now on, the configuration spaces will be denoted by \(C_\alpha\) by ignoring the superscripts, unless if it needed be.

The most basic analytical results needed to achieve the main result is the Gauge Fixing Lemma (Uhlenbeck - [15]) and the estimate [241] both extended by Marini, A. [12] to manifolds with boundary:

**Lemma 2.1.1.** (Gauge Fixing Lemma) - Every connection \(\hat{A} \in A_\alpha\) is gauge equivalent, by a gauge transformation \(g \in \mathcal{G}_\alpha\) named Coulomb (C) gauge, to a connection \(A \in A_\alpha\) satisfying

1. \(d^\tau A_\nu = 0\) on \(\partial X\),
2. \(d^\ast A = 0\) on \(X\),
3. In the \(N\)-problem, the connection \(A\) satisfies \(A_\nu = 0\) (\(\nu \perp \partial X\)).

**Corollary 2.1.2.** Under the hypothesis of [241], there exists a constant \(K > 0\) such that the connection \(A\), gauge equivalent to \(\hat{A}\) by the Coulomb gauge, satisfies the following estimates:

\[
|| A ||_{L^{1,p}} \leq K, \quad || F_A ||_{L^p}
\]

**notation:** \(*_f\) is the Hodge operator in the flat metric and the index \(\tau\) denotes tangencial components.
2.2. Variational Formulation. A global formulation for problems $D$ and $N$ is made using the Seiberg-Witten functional;

**Definition 2.2.1.** Let $\alpha \in \text{Spin}^c(X)$. The Seiberg-Witten functional $SW_\alpha : \mathcal{C}_\alpha \to \mathbb{R}$ is defined as

\[
SW_\alpha(A, \phi) = \int_X \left( \frac{1}{4} |F_A|^2 + |\nabla^A \phi|^2 + \frac{1}{8} |\phi|^4 + \frac{k_g}{4} |\phi|^2 \right) dv_g + \pi^2 \alpha^2.
\]

where $k_g = \text{scalar curvature of } (X, g)$.

**Remark 2.2.2.** The $G_\alpha$-action on $\mathcal{C}_\alpha$ has the following properties;

1. the $SW_\alpha$-functional is $G_\alpha$-invariant.
2. the $G_\alpha$-action on $\mathcal{C}_\alpha$ induces on $TC_\alpha$ a $G_\alpha$-action as follows:
   let $(\Lambda, V) \in T_{(A, \phi)}\mathcal{C}_\alpha$ and $g \in G_\alpha$,
   \[
g.(\Lambda, V) = (\Lambda, g^{-1}V) \in T_g(\mathcal{C}_\alpha).
\]
   Consequently, $d(SW_\alpha)_{g.(A, \phi)}(g.(\Lambda, V)) = d(SW_\alpha)_{(A, \phi)}(\Lambda, V)$.

The tangent bundle $TC_\alpha$ decomposes as

\[TC_\alpha = \Omega^1(ad(u_1)) \oplus \Gamma(S^+_\alpha) .\]

In this way, the 1-form $dSW_\alpha \in \Omega^1(\mathcal{C}_\alpha)$ admits a decomposition $dSW_\alpha = d_1SW_\alpha + d_2SW_\alpha$, where

\[d_1(SW_\alpha)_{(A, \phi)} : \Omega^1(ad(u_1)) \to \mathbb{R}, \quad d_1(SW_\alpha)_{(A, \phi)} \cdot \Lambda = d(SW_\alpha)_{(A, \phi)}(\Lambda, 0)\]

\[d_2(SW_\alpha)_{(A, \phi)} : \Gamma(S^+_\alpha) \to \mathbb{R}, \quad d_2(SW_\alpha)_{(A, \phi)} \cdot V = d(SW_\alpha)_{(A, \phi)}(0, V) .\]

By performing the computations, we get

1. for every $\Lambda \in \mathcal{A}_\alpha$,

\[
d_1(SW_\alpha)_{(A, \phi)} \cdot \Lambda = \frac{1}{4} \int_X \text{Re} \{ <F_A, d_A \Lambda > + 4 <\nabla^A(\phi), \Phi(\Lambda) > \} \, dx ,
\]

where $\Phi : \Omega^1(u_1) \to \Omega^1(S^+_\alpha)$ is the linear operator $\Phi(\Lambda) = \Lambda(\phi)$, which is dual defined in 1.5

2. for every $V \in \Gamma(S^+_\alpha)$,

\[
d_2(SW_\alpha)_{(A, \phi)} \cdot V = \int_X \text{Re} \{ <\nabla^A \phi , \nabla^A V > + <\frac{1}{4} |\phi|^2 + k_g \phi , V > \} \, dx.
\]

Therefore, by taking $\text{supp}(\Lambda) \subset \text{int}(X)$ and $\text{supp}(V) \subset \text{int}(X)$, we restrict to the interior of $X$, and so, the gradient of the $SW_\alpha$-functional at $(A, \phi) \in \mathcal{C}_\alpha$ is

\[
\text{grad}(SW_\alpha)(A, \phi) = (d^1_{\omega}F_A + 4\Phi^*(\nabla^A \phi), \triangle_A \phi + \frac{1}{4} |\phi|^2 + k_g \phi).
\]
It follows from the $G_\alpha$-action on $T\mathcal{C}_\alpha$ that

\begin{equation}
\text{grad}(\text{SW}_\alpha)(g.(A, \phi)) = \left( d^*_A F_A + 4\Phi^* (\nabla^A \phi), g^{-1}.(\triangle_A \phi + \frac{1}{4} \phi |^2 + k_\alpha \phi) \right).
\end{equation}

An important analytical aspect of the $\text{SW}_\alpha$-functional is the Coercivity Lemma proved in [9].

**Lemma 2.2.3. Coercivity** - For each $(A, \phi) \in \mathcal{C}_\alpha$, there exists $g \in G_\alpha$ and a constant $K^{(A, \phi)}_C > 0$, where $K^{(A, \phi)}_C$ depends on $(X, g)$ and $\text{SW}_\alpha(A, \phi)$, such that

\[ || g.(A, \phi) ||_{L^1,2} < K^{(A, \phi)}_C. \]

**Proof.** Lemma 2.3 in [9]. The gauge transform is the Coulomb one given in the Gauge Fixing Lemma 2.1.1. □

Considering the gauge invariance of the $\text{SW}_\alpha$-theory, and the fact that the gauge group $G_\alpha$ is an infinite dimensional Lie Group, we can’t hope to handle the problem in the general. So forth, we need to restrict the problem to the space

\begin{equation}
\mathcal{C}^\xi_\alpha = \{(A, \phi) \in \mathcal{C}_\alpha; || (A, \phi) ||_{L^1,2} < K^{(A, \phi)}_C\},
\end{equation}

The superscript $D$ and $\mathcal{N}$ are being ignored for simplicity, although each one should be taken in account according with the problem. These choice of spaces is a a property of the $G_\alpha$ action on $\mathcal{C}_\alpha$, it is suggested by the Gauge Fixing Lemma and the Coercivity Lemma; this sort of property is not shared by most actions.

### 3. Existence of a Solution

#### 3.1. Non Homogeneous Palais-Smale Condition - $\mathcal{H}$.

In the variational formulation, the problems $D$ and $\mathcal{N}$ (1.4) are written as

\begin{equation}
(D) = \begin{cases}
\text{grad}(\text{SW}_\alpha)(A, \phi) = (\Theta, \sigma), \\
(A, \phi) |_{\partial X} \underset{\text{gauge}}{\sim} (A_0, \phi_0).
\end{cases} \quad (N) = \begin{cases}
\text{grad}(\text{SW}_\alpha)(A, \phi) = (\Theta, \sigma), \\
\iota^*(\ast F_A) = 0, \nabla^A_A \phi = 0.
\end{cases}
\end{equation}

The equations in (1.4) may not admit a solution for any pair $(\Theta, \sigma) \in \Omega^1(ad(u_1)) \oplus \Gamma(S^+_\alpha)$. In finite dimension, if we consider a function $f : X \rightarrow \mathbb{R}$, the analogous question would be to find a point $p \in X$ such that, for a fixed vector $u$, $\text{grad}(f)(p) = u$. This question is more subtle if $f$ is invariant by a Lie group action on $X$. Therefore, we need a premiss on the pair $(\Theta, \sigma) \in \Omega^1(ad(u_1)) \oplus \Gamma(S^+_\alpha)$.

**Condition 3.1.1. ($\mathcal{H}$)** - Let $(\Theta, \sigma) \in \Omega^{1,2}(\Omega^1(ad(u_1))) \oplus (L^{1,2}(\Gamma(S^+_\alpha)) \cap L^\infty(\Gamma(S^+_\alpha)))$ be a pair such that there exists a sequence $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}} \subset \mathcal{C}^\xi_\alpha$ (2.7) with the following properties;

1. $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}} \subset L^{1,2}(A_\alpha) \times (L^{1,2}(\Gamma(S^+_\alpha)) \cup L^\infty(\Gamma(S^+_\alpha)))$ and there exists a constant $c_\infty > 0$ such that, for all $n \in \mathbb{Z}$, $|| \phi_n ||_\infty < c_\infty$.
2. There exists $c \in \mathbb{R}$ such that, for all $n \in \mathbb{Z}$, $\text{SW}_\alpha(A_n, \phi_n) < c$. 


the sequence \( \{d(SW_n)_{(A_n,\phi_n)}\}_{n \in \mathbb{Z}} \subset (L^{1,2}(\Omega^1(ad(u_1))) \oplus L^{1,2}(\Gamma(S_\alpha^+)))^* \), of linear functionals, converges weakly to

\[
L_\Theta + L_\sigma : TC_\alpha \rightarrow \mathbb{R},
\]

where

\[
L_\Theta(\Lambda) = \int_X < \Theta, \Lambda >, \quad L_\sigma(V) = \int_X < \sigma, V >.
\]

## 3.2. Strong Converge of \( \{(A_n, \phi_n)\}_{n \in \mathbb{Z}} \) in \( L^{1,2} \).

As an immediate consequency of (2.2.3), the sequence \( \{(A_n, \phi_n)\}_{n \in \mathbb{Z}} \) given by the \( H \)-condition converges to a pair \( (A, \phi) \):

1. weakly in \( C_\alpha \),
2. weakly in \( L^1(A_\alpha \times \Gamma(S_\alpha^+)) \),
3. strongly in \( L^p(A_\alpha \times \Gamma(S_\alpha^+)) \), for any \( p < 4 \).

### Remark 3.2.1.
Let \( \{A_n\}_{n \in \mathbb{N}} \subset L^2 \) be a converging sequence in \( L^2 \) satisfying \( d^*A_n = 0 \), for all \( n \in \mathbb{N} \), and let \( A = \lim_{n \to \infty} A_n \in L^2 \). So, \( d^*A = 0 \), once

\[
|< d^*A, \rho >| \leq |A - A_n||_{L^2} \cdot |d\rho||_{L^2},
\]

for all \( \rho \in \Omega^0(ad(u_1)) \).

### Theorem 3.2.2. A
The limit \( (A, \phi) \in L^2(A_\alpha \times \Gamma(S_\alpha^+)) \), obtained as a limit of the sequence \( \{(A_n, \phi_n)\}_{n \in \mathbb{Z}} \), is a weak solution of (1.4).

**Proof.** The proof goes along the same lines as in the 2nd-step in the proof of the Main Theorem in [9].

1. for every \( \Lambda \in A_\alpha \),

\[
(3.2) \quad d_1(SW_\alpha)_{(A_n,\phi_n),\Lambda} = \frac{1}{4} \int_X \text{Re}\{< F_{A_n}, d_{A_n} \Lambda > + 4 < \nabla A_n(\phi_n), \Phi(\Lambda) >\}dx
\]

\[
(3.3) \quad + \int_{\partial X} \text{Re}\{\Lambda \wedge *F_{A_n}\}
\]

where

(a) \( \Phi : \Omega^1(u_1) \rightarrow \Omega^1(S_\alpha^+) \) is the linear operator \( \Phi(\Lambda) = \Lambda(\phi) \); it’s dual is defined in [5].

Assuming \( \phi \in L^\infty \), it follows that

\[
\lim_{n \to \infty} d_1(SW_\alpha)_{(A_n,\phi_n),\Lambda} = d_1(SW_\alpha)_{(A,\phi),\Lambda}.
\]

Therefore, \( d_1(SW_\alpha)_{(A,\phi),\Lambda} = \int_X < \Theta, \Lambda >. \)

(b) \( \Lambda \wedge *F_A = - < \Lambda, F_4 > dx_1 \wedge dx_2 \wedge dx_3. \)

Since the equation above is true for all \( \Lambda \), let \( \text{supp}(\Lambda) \subset \partial X \), so \( F_4 = 0 \)
\( (\Rightarrow i*(F_A) = 0) \).

2. for every \( V \in \Gamma(S_\alpha^+) \),
(3.4)  
\[ d_2(SW_\alpha)(A_n, \phi_n).V = \int_X \text{Re}\{<\nabla A_n \phi_n,\nabla A_n V>\} + <\frac{|\phi_n|^2 + k_g \phi_n, V>\} dx \]

(3.5)  
\[ + \int_{\partial X} \text{Re}\{<\nabla^\nu A_n \phi_n, V>\}. \]

Analogously, it follows that \((A, \phi)\) is a weak solution of the equation

\[ d_2(SW_\alpha)(A_n, \phi_n).V = \int_X <\sigma, V> . \]

So, in the \(N\)-problem, \(\nabla^\nu \phi = 0\).

In order to pursue the strong \(L^{1,2}\)-convergence for the sequence \(\{(A_n, \phi_n)\}_{n \in \mathbb{Z}}\), next we obtain an upper bound for \(||\phi||_{L^\infty}\), whenever \((A, \phi)\) is a weak solution.

**Lemma 3.2.3.** Let \((A, \phi)\) be a solution of either \(\mathcal{D}\) or \(\mathcal{N}\) in \([1,2]\), so

1. If \(\sigma = 0\), then there exists a constant \(k_{X,g}\), depending on the Riemannian metric on \(X\), such that

\[ ||\phi||_{L^\infty} < k_{X,g} \text{vol}(X). \]

2. If \(\sigma \neq 0\), then there exist constants \(c_1 = c_1(X, g)\) and \(c_2 = c_2(X, g)\) such that

\[ ||\phi||_{L^p} < c_1 + c_2 \|\sigma\|_{L^{3p}}^2. \]

In particular, if \(\sigma \in L^\infty\) then \(\phi \in L^\infty\).

**Proof.** Fix \(r \in \mathbb{R}\) and suppose that there is a ball \(B_r^{-1}(x_0)\), around the point \(x_0 \in X\), such that

\[ |\phi(x)| > r, \quad \forall x \in B_r^{-1}(x_0). \]

Define

\[ \eta = \begin{cases} \left(1 - \frac{r}{|\phi|}\right) \phi, & \text{if } x \in B_r^{-1}(x_0), \\ 0, & \text{if } x \in X - B_r^{-1}(x_0) \end{cases} \]

So,

\[ |\eta| \leq |\phi| \]

\[ \nabla \eta = r <\phi, \nabla \phi> \frac{1}{|\phi|^3} \phi + (1 - \frac{r}{|\phi|}) \nabla \phi \]

\[ \Rightarrow |\nabla \eta|^2 = r^2 <\phi, \nabla \phi>^2 \frac{1}{|\phi|^4} + 2r(1 - \frac{r}{|\phi|}) <\phi, \nabla \phi>^2 \frac{1}{|\phi|^3} + (1 - \frac{r}{|\phi|})^2 |\nabla \phi|^2 \]

\[ \Rightarrow |\nabla \eta|^2 < r^2 \frac{|\nabla \phi|^2}{|\phi|^2} + 2r(1 - \frac{r}{|\phi|}) \frac{|\nabla \phi|^2}{|\phi|^3} + (1 - \frac{r}{|\phi|})^2 |\nabla \phi|^2. \]
Since $r < |\phi|$, 

(3.9) \[ |\nabla \eta|^2 < 4 |\nabla \phi|^2. \]

Hence, by 3.8 and 3.9, $\eta \in L^{1,2}$.

The directional derivative of $SW_\alpha$ at direction $\eta$ is given by

$$d(SW_\alpha)_{(A,\phi)}(0, \eta) = \int_X \left[ < \nabla^A \phi, \nabla^A \eta > + \frac{|\phi|^2 + k_g}{4} |\phi| (|\phi| - r) \right].$$

By 2.4,

$$\int_X \left[ < \nabla^A \phi, \nabla^A \eta > + \frac{|\phi|^2 + k_g}{4} |\phi| (|\phi| - r) \right] = \int_X < \sigma, (1 - \frac{r}{|\phi|}) \phi >.$$

However,

$$\int_X < \nabla^A \phi, \nabla^A \eta > = \int_X \left[ r < \phi, \nabla^A \phi >^2 + (1 - \frac{r}{|\phi|}) |\nabla \phi|^2 \right] > 0.$$

So,

$$\int_X \frac{|\phi|^2 + k_g}{4} |\phi| (|\phi| - r) < \int_X < \sigma, (1 - \frac{r}{|\phi|}) \phi > < \int_X |\sigma| (|\phi| - r).$$

Hence,

$$\int_X (|\phi| - r) \left( \frac{|\phi|^2 + k_g}{4} |\phi| - |\sigma| \right) < 0.$$

Since $r < |\phi(x)|$, whenever $x \in B_{r-1}(x_0)$, it follows that

(3.10) \[ (|\phi|^2 + k_g) |\phi| < 4 |\sigma|, \quad \text{almost everywhere in } B_{r-1}(x_0). \]

There are two cases to be analysed independently:

1. $\sigma = 0$.

   In this case, we get

   (3.11) \[ (|\phi|^2 + k_g) |\phi| < 0, \quad \text{almost everywhere.} \]

   The scalar curvature plays a central role here: if $k_g \geq 0$ then $\phi = 0$; otherwise,

   $$|\phi| \leq \max \{0, (-k_g)^{1/2} \}.$$

   Since $X$ is compact, we let $k_{X,g} = \max_{x \in X} \{0, [-k_g(x)]^{1/2} \}$, and so,

   $$|| \phi ||_{\infty} < k_{X,g} vol(X).$$

2. Let $\sigma \neq 0$.

   The inequality 3.10 implies that

   $$|\phi|^3 + k_g |\phi| - 4 |\sigma| < 0 \quad \text{a.e.}$$

   Consider the polynomial
\[
Q_{\sigma(x)}(w) = w^3 + k_g w - 4 |\sigma(x)|.
\]

A estimate for \( |\phi| \) is obtained by estimating the largest real number \( w \) satisfying \( Q_{\sigma(x)}(w) < 0 \). \( Q_{\sigma(x)} \) being monic implies that \( \lim_{w \to \infty} Q_{\sigma(x)}(w) = +\infty \). So, either \( Q_{\sigma(x)} > 0 \), whenever \( w > 0 \), or there exist a root \( \rho \in (0, \infty) \).

The first case would imply that

\[
Q_{\sigma(x)}(|\phi(x)|) > 0, \text{ a.e.,}
\]

contradicting 3.10. By the same argument, there exists a root \( \rho \in (0, \infty) \) such that \( Q_{\sigma(x)}(w) \) changes its sign in a neighborhood of \( \rho \). Let \( \rho \) be the largest root in \( (0, \infty) \) with this property. By the Corollary A.0.11 there exist constants \( c_1 = c_1(X, g) \) and \( c_2 \) such that

\[
|\rho| < c_1 + c_2 |\sigma(x)|^3.
\]

Consequently,

(3.12) \( |\phi(x)| < c_1 + c_2 |\sigma(x)|^3 \), a.e. in \( B_{r-1}(x_0) \)

and

(3.13) \( \| \phi \|_{L^p} < C_1 + C_2 \| \sigma \|_{L^3}^3 \), restricted to \( B_{r-1}(x_0) \)

where \( C_1, C_2 \) are constants depending on \( \text{vol}(B_{r-1}(x_0)) \).

The inequality 3.13 can be extended over \( X \) by using a \( C^\infty \) partition of unity. Moreover, if \( \sigma \in L^\infty \), then

(3.14) \( \| \phi \|_\infty < C_1 + C_2 \| \sigma \|_{\infty}^3 \),

where \( C_1, C_2 \) are constants depending on \( \text{vol}(X) \).

\[\square\]

In [9], it was proved a sort of concentration lemma, which is extended as follows;

Lemma 3.2.4. Let \( \{(A_n, \phi_n)\}_{n \in \mathbb{Z}} \) be the sequence given by the \( \mathcal{H} \)-condition 3.1.1. So,

\[
\lim_{n \to \infty} \int_X < \Phi^*(\nabla A_n \phi_n), A_n - A > = 0.
\]

Proof. By equation 1.65

\[
\lim_{n \to \infty} \int_X < \Phi^*(\nabla A_n \phi_n), A_n - A > = \lim_{n \to \infty} \int_X < \nabla_i A_n \phi_n, \phi_n > \cdot < \eta_i, A_n - A >
\]
Theorem 3.2.5. B - Let \((\Theta, \sigma)\) be a pair satisfying the H- condition. So, the sequence \(\{(A_n, \phi_n)\}_{n \in \mathbb{Z}}\), given by (3.2.4), converges strongly to \((A, \phi) \in \mathcal{C}_\alpha\).

Proof. From (3.2.2) \(\{(A_n, \phi_n)\}_{n \in \mathbb{Z}}\) converges weakly in \(L^{1,2}\) to \((A, \phi) \in \mathcal{C}_\alpha\). The prove is splitted into 2 parts:

1. \(\lim_{n \to \infty} \|A_n - A\|_{L^{1,2}} = 0\). Let \(d^* : \Omega^1(ad(u_1)) \to \Omega^0(ad(u_1))\). The operator \(d : \ker(d^*) \to \Omega^2(ad(u_1))\) being elliptic implies, by the fundamental-elliptic estimate, that

\[
\|A_n - A\|_{L^{1,2}} \leq c \|d(A_n - A)\|_{L^2} + \|A_n - A\|_{L^2}.
\]

The first term in the right-hand-side is estimate as follows;

\[
\|dA_n - dA\|_{L^2}^2 = \int_X <d(A_n - A), d(A_n - A)> = \\
= \int_X <dA_n, d(A_n - A)> - \int_X <dA, d(A_n - A)> = \\
= \int_X <d^* F_{A_n}, A_n - A> - \int_X <d^* F_A, A_n - A> = \\
= d(S\mathcal{W}_\alpha)(A_n, \phi_n)(A_n - A) - 4 \int_X <\Phi^*(\nabla A_n \phi_n), A_n - A> - \\
\quad d(S\mathcal{W}_\alpha)(A, \phi)(A_n - A) - 4 \int_X <\Phi^*(\nabla A \phi), A_n - A> + o(1) = \\
= -4 \left\{ \int_X <\Phi^*(\nabla A_n \phi_n), A_n - A> + \int_X <\Phi^*(\nabla A \phi), A_n - A> \right\} \\
+ o(1), \quad \lim_{n \to \infty} o(1) = 0.
\]

So, it follows from (3.2.4) that \(\lim_{n \to \infty} \|A_n - A\|_{L^{1,2}} = 0\), and consequently, \(A_n \to A\) strongly in \(L^2\).

2. \(\lim_{n \to \infty} \|\phi_n - \phi\|_{L^{1,2}} = 0\).
\[ \| \nabla^0 \phi_n - \nabla^0 \phi \|_{L^2}^2 = \int_X \left( \nabla^0 \phi_n, \nabla^0 (\phi_n - \phi) \right) - \int_X \left( \nabla^0 \phi, \nabla^0 (\phi_n - \phi) \right) \]

The term (1) leads to

\[ \int_X \left( \nabla^0 \phi_n, \nabla^0 (\phi_n - \phi) \right) = \int_X \left( \nabla^A_n \phi_n, (\nabla^A_n - A_n) (\phi_n - \phi) \right) - \int_X \left( \nabla^A_n \phi_n, A_n (\phi_n - \phi) \right) - \int_X \left( A_n \phi_n, \nabla^A_n (\phi_n - \phi) \right) + \int_X \left( A_n \phi_n, A_n (\phi_n - \phi) \right) = \]

\[ d(SW_n)_{(A_n, \phi_n)} (\phi_n - \phi) - \int_X \left( \phi_n \right)^2 + k_g < \phi, \phi_n - \phi > - \]

\[ \int_X \left( \nabla^A_n \phi_n, A_n (\phi_n - \phi) \right) - \int_X \left( A_n \phi_n, \nabla^A_n (\phi_n - \phi) \right) + \int_X < A_n \phi_n, A_n (\phi_n - \phi) > . \]

The term (2) in (3.15) leads to similar terms named (21), (22), (23) and (24).

Let’s analyze each one of the overbraced terms obtained above:

(a) terms (11) and (21).

\[ d(SW_n)_{(A_n, \phi_n)} (\phi_n - \phi) - \int_X \left( \phi_n \right)^2 + k_g < \phi, \phi_n - \phi > + o(1) = \]

\[ < \sigma, \phi_n - \phi > - \int_X \left( \phi_n \right)^2 + k_g < \phi, \phi_n - \phi > - \int_X \left( \phi_n \right)^2 + k_g < \phi, \phi_n - \phi > + o(1) \leq < \sigma, \phi_n - \phi > - \int_X \left( \phi_n \right)^2 + k_g < \phi, \phi_n - \phi > + o(1) \]

\[ \leq \| \sigma \|_{L^2}^2 \cdot \| \phi_n - \phi \|_{L^2}^2 + \| \phi_n \|_{L^2}^2 \cdot \| \phi \|_{\infty} \cdot \| \phi_n - \phi \|_{L^2}^2 + o(1), \]

where \( \lim_{n \to \infty} o(1) = 0 \). By the similarity among (11) and (21), we conclude the boundeness of term (22).

(b) terms (12) and (22).
i. (12)
\[
\int_X < \nabla A_n \phi_n, A_n (\phi_n - \phi) > =
\int_X < \nabla A_n \phi_n, (A_n - A)(\phi_n - \phi) > + \int_X < \nabla A_n \phi_n, A(\phi_n - \phi) >
\leq \int_X | \nabla A_n \phi_n |^2 \cdot \int_X | A_n - A |^4 \cdot \int_X | \phi_n - \phi |^4 +
\int | \nabla A_n \phi_n |^2 \cdot \int | A(\phi_n - \phi) |^2
\]

ii. (21)
\[
\int_X < \nabla A \phi, A(\phi_n - \phi) > \leq \int_X | \nabla A \phi |^2 \cdot \int_X | A(\phi_n - \phi) |^2
\]

The term \( \int_X | \nabla A \phi |^2 \) is bounded by (10.4) and \( A \in C^0 \) by (10.8).

(c) term \{ (13) - (23) \}.
\[
\int_X < A_n \phi_n, \nabla A_n (\phi_n - \phi) > - \int_X < A \phi, \nabla A (\phi_n - \phi) > =
\int_X < (A_n - A) \phi_n, \nabla A_n (\phi_n - \phi) > - \int_X < A \phi_n, \nabla A_n (\phi_n - \phi) >
\]

In each of the last two lines above, the first terms are bounded by \( \| A_n - A \|_{L^4} \), while the term \{ (i) - (ii) \} can be written as
\[
\int_X < (A_n - A) \phi_n, \nabla A_n (\phi_n - \phi) > + \int_X < A_n (\phi_n - \phi), \nabla A_n (\phi_n - \phi) > +
\int_X < A_n \phi_n, (\nabla A_n - \nabla A)(\phi_n - \phi) >
\]
So, it is also bounded by \( \| A_n - A \|_{L^4} \).

(d) term \{ (14) - (24) \}.
\[
\int_X < A_n \phi_n, A_n (\phi_n - \phi) > - \int_X < A \phi, A(\phi_n - \phi) > =
\int_X < A_n \phi_n, (A_n - A)(\phi_n - \phi) > + \int_X < (A_n - A) \phi_n, A((\phi_n - \phi) > +
\int | A(\phi_n - \phi) |^2
\]

Since \( A \in C^0 \), it follows that \( \lim_{n \to \infty} \| A(\phi_n - \phi) \|^2 = 0 \).
4. Regularity of the Solution \((A, \phi)\)

Let \(\beta = \{e_i; 1 \leq i \leq 4\}\) be an orthonormal frame fixed on \(TX\) with the following properties; for all \(i, j \in \{1, 2, 3, 4\}\)

1. (commuting) \([e_i, e_j] = 0\),
2. \(\nabla_{e_i} e_j = 0\) (\(\nabla = \) Levi-Civita connection on \(X\)).

Let \(\beta^* = \{dx_1, \ldots, dx_n\}\) be the dual frame induced on \(\mathcal{S}_\alpha^*\). From the 2\textsuperscript{nd} property of the frame \(\beta\), it follows that \(\nabla_{e_i} dx^j = 0\) for all \(i, j \in \{1, 2, 3, 4\}\). For the sake of simplicity, let \(\nabla^A_i = \nabla_i^A\). Therefore, \(\nabla^A : \Omega^0(\text{ad}(u_1)) \to \Omega^1(\text{ad}(u_1))\) is given by

\[
\nabla^A \phi = \sum_l (\nabla^A_l \phi) dx_l \quad \Rightarrow \quad |\nabla^A \phi|^2 = \sum_l |\nabla^A_l \phi|^2,
\]

and

\[
(\nabla^A)^2 = \sum_{k,l} (\nabla^A_k \nabla^A_l \phi) dx_l \wedge dx_k \quad \Rightarrow \quad |(\nabla^A)^2|^2 = \sum_{k,l} |\nabla^A_k \nabla^A_l \phi|^2.
\]

In this setting, the 2-form of curvature of the connection \(A\) is given by

\[
(F_A)_{kl} = F_{kl} = \nabla^A_k \nabla^A_l - \nabla^A_l \nabla^A_k.
\]

In order to compute the operator \(\Delta_A = (\nabla^A)^* \nabla^A : \Omega^0(\mathcal{S}_\alpha^+) \to \Omega^0(\mathcal{S}_\alpha^+)\), let \(* : \Omega^l(\mathcal{S}_\alpha) \to \Omega^{4-l}(\mathcal{S}_\alpha)\) be the Hodge operator and consider the identity

\[
(\nabla^A)^* = -* \nabla^A : \Omega^1(\mathcal{S}_\alpha^+) \to \Omega^0(\mathcal{S}_\alpha^+).
\]

Hence,

\[
\Delta_A \phi = - \sum_k \nabla^A_k \nabla^A_k \phi.
\]

In this way,

\[
|\Delta_A \phi|^2 = \sum_{k,l} <\nabla^A_k \nabla^A_l \phi, \nabla^A_l \nabla^A_k \phi> =
\]

\[
= \sum_{k,l} \left[\nabla^A_k (<\nabla^A_k \phi, \nabla^A_l \nabla^A_k \phi>) - <\nabla^A_k \phi, \nabla^A_k \nabla^A_l \phi>\right] =
\]

\[
= \sum_{k,l} \left[\nabla^A_k (<\nabla^A_k \phi, \nabla^A_l \nabla^A_k \phi>) - <\nabla^A_k \phi, \nabla^A_k \nabla^A_l \phi> - <\nabla^A_k \phi, F_{lk} \nabla^A_l \phi>\right] =
\]

\[
= \sum_{k,l} \left[\nabla^A_k (<\nabla^A_k \phi, \nabla^A_l \nabla^A_k \phi>) - \nabla^A_k (<\nabla^A_k \phi, \nabla^A_k \nabla^A_l \phi>) + <\nabla^A_k \phi, F_{lk} \nabla^A_l \phi>\right] +
\]

\[
+ \sum_{k,l} \left[<\nabla^A_k \nabla^A_k \phi, \nabla^A_l \nabla^A_k \phi> + <\nabla^A_k \phi, F_{lk} \nabla^A_k \phi>\right] =
\]

\[
(4.1)
\]
(4.2)
\[
= \sum_{k,l} [\nabla_k^A (\langle \nabla_k^A \phi, \nabla_l^A \phi \rangle) - \nabla_i^A (\langle \nabla_k^A \phi, \nabla_i^A \phi \rangle)] + \sum_{k,l} | \nabla_k^A \nabla_i^A \phi |^2 + \\
+ \sum_{k,l} [\langle F_{kl} \phi, \nabla_k^A \nabla_i^A \phi \rangle + \langle \nabla_k^A \phi, F_{kl} \nabla_i^A \phi \rangle]
\]

and so,
\[
| (\nabla^A)^2 \phi |^2 \leq | \Delta_A \phi |^2 + \sum_{k,l} \{ | \nabla_k^A (\langle \nabla_k^A \phi, \nabla_i^A \nabla_i^A \phi \rangle) | \} + \\
\sum_{k,l} \{ | \nabla_i^A (\langle \nabla_k^A \phi, \nabla_k^A \nabla_i^A \phi \rangle) | \} + \sum_{k,l} \{ | \langle F_{kl} \phi, \nabla_k^A \nabla_i^A \phi \rangle | \} + \\
\sum_{k,l} \{ | \langle \nabla_k^A \phi, F_{kl} \nabla_i^A \phi \rangle | \}
\]

Now, by applying the inequalities
\[
\left( \sum_{i} a_i \right)^p \leq K_p \sum_{i} | a_i |^p, \quad \sqrt{\sum_{i=1}^{n} a_i} \leq \sum_{i=1}^{n} \sqrt{a_i}
\]
to (4.2), we get
\[
| (\nabla^A)^2 \phi |^p \leq K_p | \Delta_A \phi |^p + K_p \sum_{k,l} \{ | \nabla_k^A (\langle \nabla_k^A \phi, \nabla_i^A \nabla_i^A \phi \rangle) | \} + \\
K_p \sum_{k,l} \{ | \nabla_i^A (\langle \nabla_k^A \phi, \nabla_k^A \nabla_i^A \phi \rangle) | \} + \sum_{k,l} \{ | \langle F_{kl} \phi, \nabla_k^A \nabla_i^A \phi \rangle | \} + \\
\sum_{k,l} \{ | \langle \nabla_k^A \phi, F_{kl} \nabla_i^A \phi \rangle | \};
\]

After integrating, it follows that
\[
k_1. \| (\nabla^A)^2 \phi \|_{L^p} \leq \| \Delta_A \phi \|_{L^p} + k_2. \| \nabla^A \phi \|_{L^p}^2 + k_3, \| F_A(\phi) \|_{L^p} + \\
+ k_4. \| F_A(\nabla^A \phi) \|_{L^p} + k_5. \sum_{k,l} \left\{ \| \nabla_k^A (\langle \nabla_k^A \phi, \nabla_i^A \nabla_i^A \phi \rangle) \|_{L^p} \right\} + \\
+ k_6. \sum_{k,l} \int_X \left\{ | \nabla_i^A (\langle \nabla_k^A \phi, \nabla_i^A \nabla_i^A \phi \rangle) \|_{L^p} \right\}
\]

The boundness of the right hand side of (4.4) results from the analysis of each term.

**Proposition 4.0.6.** Let \((A, \phi) \in C_\alpha\) be a solution of equations in \((1.4)\). If \(\sigma \in L^{\infty}\), then

1. \(\nabla^A \phi \in L^2\),
2. \(\Delta_A \phi \in L^2\).
Proof. (1) $\nabla^A \phi \in L^2$

$$< \Delta A \phi, \phi > + \left( \frac{\left| \phi \right|^2 + k_g}{4} \right) \left| \phi \right|^2 = < \sigma, \phi >$$

$$\Rightarrow \left| \nabla^A \phi \right|^2 + \left( \frac{\left| \phi \right|^2 + k_g}{4} \right) \left| \phi \right|^2 = < \sigma, \phi > \leq$$

$$\leq \frac{1}{\epsilon^2} \left| \sigma \right|^2 + \epsilon^2 \left| \phi \right|^2$$

Therefore,

$$\left| \nabla^A \phi \right|^2 < \frac{1}{\epsilon^2} \left| \sigma \right|^2 + (\epsilon^2 - k_g) \left| \phi \right|^2$$

From (4.3), there exists a polynomial $p$, which coefficients depend on $(X, g)$ and $\epsilon$, such that

(4.5) $\| \nabla^A \phi \|^2_{L^2} < p(\| \sigma \|_{\infty})$

So, $\nabla^A \phi \in L^2$.

(2) $\Delta A \phi \in L^2$.

$$< \Delta A \phi, \Delta A \phi > + \frac{\left| \phi \right|^2 + k_g}{4} < \phi, \Delta A \phi > = < \sigma, \Delta A \phi >$$

let $0 < \epsilon < 1$,

$$\left| \Delta A \phi \right|^2 + \frac{\left| \phi \right|^2 + k_g}{4} \left| \nabla^A \phi \right|^2 = < \sigma, \Delta A \phi > <$$

$$< \frac{1}{\epsilon^2} \left| \sigma \right|^2 + \epsilon^2 \left| \Delta A \phi \right|^2$$

(4.6) $(1 - \epsilon^2) \left| \Delta A \phi \right|^2 + \frac{\left| \phi \right|^2 + k_g}{4} \left| \nabla^A \phi \right|^2 < \frac{1}{\epsilon^2} \left| \sigma \right|^2$

By the boundness of the term

(4.7) $\int_X \left| \phi \right|^2 \cdot \left| \nabla^A \phi \right|^2 < \| \phi \|_{L^\infty} \cdot \| \nabla^A \phi \|^2_{L^2}$,

it follows the existence of a polynomial $q$, which coefficients depending on $\epsilon$ and $(X, g)$, such that

(4.8) $\| \Delta A \phi \|_{L^2} < q(\| \sigma \|_{\infty})$.

□

Proposition 4.0.7. Let $(A, \phi)$ be solutions of the $SW_\alpha$-equations where $(\Theta, \sigma) \in L^{1.2} \times (L^{1.2} \cap L^\infty)$, then $F_A \in L^q$, for all $q < \infty$. 

Proof. By \( \Phi^* (\nabla^A \phi) = \frac{1}{2} \nabla^A (| \phi |^2) \), and so,

\[
d^* F_A + 4 \Phi^* (\nabla^A \phi) = \Theta \quad \Rightarrow \quad d^* F_A \|_{L^2} \leq \| \phi \|_{L^{1,2}}^2 + \| \Theta \|_{L^2}
\]

There are two cases to be analysed:

1. \( F_A \) is harmonic.

Since the Laplacian defined on \( u_1 \)-forms is an elliptic operator, the fundamental inequality for elliptic operators claims that there exists a constant \( C_k \) such that

\[
\| F_A \|_{L^{k+2,2}} \leq \| \Delta A F_A \|_{L^{k,2}} + C_k \| F_A \|_{L^2}.
\]

Consequently, \( F_A \) being harmonic implies, for all \( k \in \mathbb{N} \), that

\[
\| F_A \|_{L^{k+2,2}} \leq C_k \| F_A \|_{L^2} \quad \Rightarrow \quad F_A \in C^\infty.
\]

2. \( F_A \) is not harmonic.

In this case, since \( \Theta \in L^{1,2}, \phi \in L^\infty \) and

\[
\Delta A F_A = d(< \phi, \nabla^A \phi >) + d\Theta = < \phi, F_A(\phi) > + d\Theta,
\]

it follows that \( F_A \in L^{2,2} \).

Therefore, by the Sobolev embedding theorem \( F_A \in L^q \), for all \( q < \infty \).

\[ \square \]

**Proposition 4.0.8.** Let \((A, \phi)\) be solutions of the \( \mathcal{SW}_\alpha \)-equations where \((\Theta, \sigma) \in L^{1,2} \times (L^{1,2} \cap L^\infty)\), then \((\nabla^A)^2 \phi \in L^p\), for all \(1 < p < 2\).

**Proof.** In \(4.4\) we must take care of the last terms;

1. \( F(\nabla^A \phi) \in L^p\), for all \(1 < p < 2\). By Young's inequality,

\[
\| F(\nabla^A \phi) \|_{L^p} \leq \| F_A \|_{L^\frac{2p}{2-p}} \cdot \| \nabla^A \phi \|_{L^2}.
\]

2. There is no contribution from the divergent terms, since

\[
\int_X \left\{ | \nabla^A_k (< \nabla^A_k \phi, \nabla^A_l \nabla^A \phi >) |^{\frac{2}{p}} \right\} \leq [\text{vol}(X)]^{\frac{2-p}{2p}} \int_X \left\{ | \nabla^A_k (< \nabla^A_k \phi, \nabla^A_l \nabla^A \phi >) | \right\}.
\]

In the same way,

\[
\sum_{k,l} \int_X \left\{ | \nabla^A_k (< \nabla^A_k \phi, \nabla^A_l \nabla^A \phi >) |^{\frac{2}{p}} \right\} = 0
\]

\[
\sum_{k,l} \int_X \left\{ | \nabla^A_l (< \nabla^A_k \phi, \nabla^A_l \nabla^A \phi >) |^{\frac{2}{p}} \right\} = 0.
\]

The estimates above applied to \(4.4\) implies that

\[
\| (\nabla^A)^2 \phi \|_{L^p} \leq k_1 \| \Delta A \phi \|_{L^p} + k_2 \| \nabla^A \phi \|_{L^p}^p + k_3 \| \nabla^A \phi \|_{L^p}^p + k_4 \| F_A(\phi) \|_{L^p} + k_5 \| F_A \|_{L^\frac{2p}{2-p}} \cdot \| \nabla^A \phi \|_{L^p}^p,
\]

\[ \square \]
Thus, \( \phi \in L^{2,p} \), for all \( 1 < p < 2 \). Considering that \( \sigma \in L^{1,2} \), the bootstrap argument applied on \( L^2 \) implies that \( \phi \in L^{3,p} \), for every \( k \geq 2 \) and \( 1 < p < 2 \). Hence, by Sobolev embedding theorem, \( \phi \in C^0 \).

**Theorem 4.0.9.** Let \((A, \phi)\) be a solution of the SW\(_\alpha\)-equations where \((\Theta, \sigma) \in L^{k,2}(\Omega^1(\text{ad}(u_1))) \oplus (L^{k,2}(\Gamma(S^+_\alpha))) \cap L^\infty(\Gamma(S^+_\alpha)))\), then \((A, \phi) \in L^{k+2,p} \times (L^{k+2,2} \cap L^\infty)\), for all \( 1 < p < 2 \). Moreover, if \( k > 2 \), then \((A, \phi) \in C^r \times C^r\), for all \( r < k \).

**Proof.**

(1) If \( \Theta \in L^{k,2} \), then by 4.0.7 \( F_A \in L^{k+1,2} \). Consequently, by 2.1.2 \( A \in L^{k+2,2} \).

(2) The Sobolev class of \( \phi \) is obtained by the bootstrap argument.

\[ \square \]

**Appendix A. Estimates for solutions of 3\(^rd\)-degree equation**

Let \( p, q \in \mathbb{R} \) and consider the equation

\[ x^3 + px + q = 0 \]

**Proposition A.0.10.** The solutions of (A.1) are given in \( \mathbb{R} \) by

\[ x_1 = z_1 + z_2, \quad x_2 = z_1 + \lambda z_2 \quad \text{and} \quad y_3 = z_1 + \lambda^2 z_2, \]

where

\[ z_1 = 3\sqrt{-\frac{q}{2} + \sqrt{D}}, \quad z_2 = 3\sqrt{-\frac{q}{2} - \sqrt{D}}, \]

\[ D = \frac{p^3}{27} + \frac{q^2}{4}, \]

and \( \lambda \in \mathbb{C} \) satisfies \( \lambda^3 = 1 \).

**Corollary A.0.11.** Let \( q, p \in \mathbb{R} \) such that \( q < 0 \) and \( p < 0 \). So, the solutions of equation (A.1) are estimates according with the following cases;

1. \( D \geq 0 \)

\[ |x_i| \leq \frac{8}{3} + \frac{1}{3} |q| + \frac{1}{12} q^2 + \frac{1}{81} p^3 \]

2. \( D < 0 \)

\[ |x_i| \leq 3 + \frac{1}{6} q^2 + \frac{1}{81} |p|^3 \]

**Proof.** Since

\[ |x_i| \leq |z_1| + |z_2| \]

it is enough to estimate \( |z_1| \) and \( |z_2| \). The basics identity needed are the following: suppose \( x \geq 0 \), so

\[ \sqrt{x} \leq 1 + \frac{1}{2} x \]

\[ \sqrt{x} \leq 1 + \frac{1}{3} x \]
In this case, \( z_1, z_2 \in \mathbb{R} \) and

\[
|z_1| = \sqrt[3]{\frac{-q}{2} + \sqrt[3]{D}} \leq 1 + \frac{1}{3} \left| \frac{-q}{2} + \sqrt[3]{D} \right| \leq 1 + \frac{1}{6} \left| q \right| + \frac{1}{6} D
\]

So,

\[
|z_1| \leq \frac{4}{3} + \frac{1}{6} \left| q \right| + \frac{1}{24} q^2 + \frac{1}{162} p^3
\]

The same estimate can be obtained for \( |z_2| \). Hence,

\[
|x_i| \leq \frac{8}{3} + \frac{1}{3} \left| q \right| + \frac{1}{12} q^2 + \frac{1}{81} p^3
\]

In this case, \( z_1, z_2 \in \mathbb{C} - \mathbb{R} \). Since \( D \in \mathbb{R} \), we can write \( \sqrt[3]{D} = i \sqrt[3]{|D|} \) and

\[
z_1 = \sqrt[3]{-\frac{1}{2}q + i \sqrt[3]{D}}, \quad z_2 = \sqrt[3]{-\frac{1}{2}q - i \sqrt[3]{D}}
\]

Therefore,

\[
|z_1|^2 = \sqrt[3]{\frac{q^2}{4} + |D|} < 1 + \frac{1}{12} q^2 + \frac{1}{3} \left| D \right| \leq 1 + \frac{1}{6} q^2 + \frac{1}{81} \left| p \right|^3
\]

and

\[
|z_1| < \frac{3}{2} + \frac{1}{12} q^2 + \frac{1}{162} \left| p \right|^3
\]

Hence,

\[
|x_i| < 3 + \frac{1}{6} q^2 + \frac{1}{81} \left| p \right|^3
\]

\[\square\]

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