A braided monoidal category for symplectic fermions

A. Davydov\textsuperscript{1}, I. Runkel\textsuperscript{2}  \\
\textsuperscript{1} Dept. Math., Ohio Univ., Athens, Ohio 45701, USA  
\textsuperscript{2} Dept. Math., Hamburg Univ., Bundesstr. 55, 20146 Hamburg, Germany

Abstract

We describe a class of examples of braided monoidal categories which are built from Hopf algebras in symmetric categories. The construction is motivated by a calculation in two-dimensional conformal field theory and is tailored to contain the braided monoidal categories occurring in the study of the Ising model, their generalisation to Tambara-Yamagami categories, and categories occurring for symplectic fermions.

1 Introduction

In this short note we summarise some of the results in [1, 2], where also more extensive references can be found.

We are interested in a particular type of $\mathbb{Z}/2\mathbb{Z}$-graded braided monoidal categories. The grade 0 component is the monoidal category $\text{Rep}_S(H)$ of modules over a Hopf algebra $H$ in a symmetric monoidal category $S$, and the grade 1 component is the category $S$ itself. We will write $\mathcal{C} = \mathcal{C}_0 + \mathcal{C}_1$ with $\mathcal{C}_0 = \text{Rep}_S(H)$ and $\mathcal{C}_1 = S$. The tensor product functor $*$ on the various components is defined as:

*IR thanks the organisers of the XXIX Internat. Colloq. on Group-Theoretical Methods in Physics (August 20–26, 2012, Chern Inst., Tianjin) for the opportunity to speak.
A B A * B comments
\begin{array}{llll}
\mathcal{C}_0 & \mathcal{C}_0 & A \otimes B & \in \mathcal{C}_0 \quad \text{the } H\text{-action is via the coproduct of } H \\
\mathcal{C}_0 & \mathcal{C}_1 & F(A) \otimes B & \in \mathcal{C}_1 \quad F : \text{Rep}_S(H) \to S \text{ is the forgetful functor} \\
\mathcal{C}_1 & \mathcal{C}_0 & A \otimes F(B) & \in \mathcal{C}_1 \\
\mathcal{C}_1 & \mathcal{C}_1 & H \otimes A \otimes B & \in \mathcal{C}_0 \quad \text{the } H\text{-action is by multiplication}
\end{array}

This somewhat ad-hoc looking definition of the tensor product is actually quite natural. The mixed tensor products are the natural left and right action of the monoidal category \text{Rep}_S(H) on \mathcal{S}. To obtain the last line in the table, assume that \mathcal{C} can be made rigid. Writing \mathcal{T} for the tensor unit of \mathcal{S} considered as an object in \mathcal{C}_1, we have for all \text{H-modules } M

\begin{equation}
\text{Hom}_{\mathcal{C}_0}(\mathcal{T}^* \otimes \mathcal{T}, M) \cong \text{Hom}_{\mathcal{C}_1}(\mathcal{T}, \mathcal{T}^* \otimes M) \cong \text{Hom}_S(1, F(M)). \quad (1)
\end{equation}

This means that \mathcal{T}^* \otimes \mathcal{T} is a representing object for the functor \( M \mapsto \text{Hom}_S(1, F(M)) \), and so \( \mathcal{T}^* \otimes \mathcal{T} \cong H \) as an \text{H-module}. If we in addition demand that \( \mathcal{T}^* \cong \mathcal{T} \), the last line in the above table follows.

Given the above form of the tensor product functor \( \ast \) on \mathcal{C}, one can ask if it is possible to describe associativity and braiding isomorphisms for \( \ast \) in terms of Hopf algebraic data on \( H \). Our results for this question are given in Section 4. But before getting there, in Sections 2 and 3 we would like to give the two examples of such \( \mathbb{Z}/2\mathbb{Z} \)-graded braided monoidal categories which were our main motivation when setting up the formalism.

2 Tambara-Yamagami categories

For simplicity, we will work over the field \( \mathbb{C} \). Consider a fusion category \( \mathcal{C} \) over \( \mathbb{C} \) whose simple objects are labelled by \( G \cup \{m\} \) where \( G \) is a finite group and \( m \) is an extra label. Suppose that the tensor product \( \ast \) is of the form, for \( a, b \in G \),

\begin{equation}
a \ast b \cong ab \ , \quad m \ast a \cong m \cong a \ast m \ , \quad m \ast m \cong \bigoplus_{g \in G} g . \quad (2)
\end{equation}

This tensor product is a special case of the one in the above table: the underlying symmetric category \( \mathcal{S} \) is \( \text{vect}(\mathbb{C}) \), the category of finite dimensional \( \mathbb{C} \) vector spaces. The component \( \mathcal{C}_0 \) is spanned by the simple objects \( g \in G \); the component \( \mathcal{C}_1 \) is spanned by \( m \) alone, so that \( \mathcal{C}_1 \cong \text{vect}(k) \). The Hopf algebra \( H \in \mathcal{S} \) is the function algebra \( \text{Fun}(G, \mathbb{C}) \).
For any such fusion category $C$, the group $G$ is necessarily abelian and $C$ is monoidally equivalent to $C(\chi, \tau)$, which is defined as follows [3, Thm. 3.2]. $C(\chi, \tau)$ has simple objects and fusion rules as in (2), and its associator is determined by a symmetric non-degenerate bicharacter $\chi : G \times G \to \mathbb{C}^\times$ and a choice of $\tau \in \mathbb{C}^\times$ such that $\tau^2 = |G|^{-1}$. The associator is a bit lengthy and we refer to [3].

The category $C(\chi, \tau)$ allows for a braiding if and only if $G$ is an elementary 2-group (i.e. $gg = e$ for all $g \in G$) [4]. The braiding isomorphisms are determined by a quadratic form $\sigma$ associated to the bicharacter $\chi$ and a number $\beta$ such that $\beta \neq 0$ and $\beta^2 = \tau \sum_{a \in G} \sigma(a)$. Explicitly, under the identifications (1) the braiding is

$$c_{a,b} = \chi(a,b) id_{ab}, c_{a,m} = \sigma(a) id_m = c_{m,a}, c_{m,m} = \bigoplus_{g \in G} \beta \sigma(g)^{-1} id_g,$$

(3)

An important example of a braided monoidal category of the above type is provided by the two-dimensional critical Ising model. There, one considers the three irreducible representations $\hat{1}$, $\hat{\varepsilon}$, $\hat{\sigma}$ of the Virasoro algebra which have central charge $c = \frac{1}{2}$ and lowest $L_0$-weights $h_1 = 0$, $h_\varepsilon = \frac{1}{2}$ and $h_\sigma = \frac{1}{16}$. The fusion rules are of the form (1) where $\hat{1}, \hat{\varepsilon}$ generate the group $G = \mathbb{Z}/2\mathbb{Z}$ and $\hat{m} = \hat{\sigma}$ has fusion rule $\hat{\sigma} \ast \hat{\sigma} \cong \hat{1} \oplus \hat{\varepsilon}$. The braiding isomorphism $c_{r,s}$ projected to the simple object $t \in r \ast s$ is multiplication by $\exp(\pi i (h_r + h_s - h_t))$. Comparing to (3) shows that the braided monoidal structure is determined by $\sigma(\hat{\varepsilon}) = \exp(\pi i/2)$, $\beta = \exp(\pi i/8)$ and thus $\chi(\hat{\varepsilon}, \hat{\varepsilon}) = -1, \tau = 1/\sqrt{2}$.

3 Symplectic fermions

Continuing with examples from two-dimensional conformal field theory, we now consider symplectic fermions [5]. The mode algebra of $n$ pairs of symplectic fermions is determined by a $2n$-dimensional symplectic vector space $\mathfrak{h}$. It is convenient to think of $\mathfrak{h}$ as a purely odd abelian Lie super-algebra with non-degenerate super-symmetric pairing $(-,-)$; we will use this language in the following. The symplectic fermion mode algebra is the affinisation $\hat{\mathfrak{h}}$ of $\mathfrak{h}$ with central element $K$ and graded bracket $[a_m, b_n] = m(a, b) \delta_{m+n,0} K$, where $m, n \in \mathbb{Z}$ for untwisted (Neveu-Schwarz) representations and $m, n \in \mathbb{Z} + \frac{1}{2}$ for twisted (Ramond) representations. 

$\dagger$ This means that $\sigma : G \to \mathbb{C}^\times$ satisfies $\sigma(a) = \sigma(a^{-1})$, $\sigma(e) = 1$, and that $\chi(a, b) \sigma(a) \sigma(b) = \sigma(ab)$ for all $a, b \in G$. 

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Denote by $S(h)$ the symmetric algebra of $h$ in $svect(C)$, the category of finite-dimensional complex super-vector spaces. Note that as a vector space, $S(h)$ is simply the exterior algebra of the vector space underlying $h$; in particular, $S(h)$ is finite-dimensional. The categories of untwisted and twisted representations of $\hat{h}$ (of a certain type) are equivalent to [2, Thms. 2.4 & 2.8]:

\[
\begin{align*}
(\text{untwisted}) \quad C_0 & := \text{Rep}_{svect} S(h) \\
(\text{twisted}) \quad C_1 & := svect(C).
\end{align*}
\]

We would like to stress that for $\dim h = 2n > 0$, $C_0$ is not semi-simple.

A conformal field theory calculation endows the category $C = C_0 + C_1$ with a $\mathbb{Z}/2\mathbb{Z}$-graded tensor product [2, Thm. 3.13]. This tensor product is of the form stated in Section 1 with symmetric category $S = svect(C)$ and Hopf algebra $H = S(h)$.

The associativity isomorphism is determined by the copairing $C \in h \otimes h$ dual to the super-symmetric pairing $(-, -)$ on $h$, and by a top-form $\lambda$ on $S(h)$ such that $(\lambda \otimes \lambda)(e^{-C}) = 1$ [2, Thm. 6.2]. To be more specific, pick a basis $\{e_i\}_{i=1, \ldots, 2n}$ of $h$ such that the pairing takes the standard form $(e_{2k-1}, e_{2k}) = 1 = -(e_{2k}, e_{2k-1})$ for $k = 1, \ldots, n$. Then $C = \sum_{k=1}^{n}(e_{2k} \otimes e_{2k-1} - e_{2k-1} \otimes e_{2k})$ and, if we set $\hat{C} = -2 \sum_{k=1}^{n} e_{2k} \otimes e_{2k-1}$, the top-form $\lambda$ is determined by $\lambda(\hat{C}^n) = n!(-2i)^n$. The explicit form of the associativity isomorphisms will be given as a special case of Theorem 1 below.

To describe the braiding, denote by $\omega_V$ the parity involution on a super-vector space $V$, and by $s_{V,W} : V \otimes W \rightarrow W \otimes V$ the symmetric structure on $svect(C)$. Then [2, Thm. 6.4]:

\[
\begin{align*}
& A \quad B \quad c_{A,B} \quad : \quad A \ast B \rightarrow B \ast A \\
& C_0 \quad C_0 \quad s_{A,B} \circ \exp(-C) \\
& C_0 \quad C_1 \quad s_{A,B} \circ (\exp(\frac{1}{2}\hat{C}) \otimes id_B) \\
& C_0 \quad C_0 \quad s_{A,B} \circ (id_A \otimes \exp(\frac{1}{2}\hat{C})) \circ (id_A \otimes \omega_B) \\
& C_1 \quad C_1 \quad e^{-i\pi \frac{n}{4}} \cdot (id_{S(h)} \otimes s_{A,B}) \circ (\exp(-\frac{1}{2}\hat{C}) \otimes id_A \otimes \omega_B)
\end{align*}
\]

In the last line, note that for $A, B \in C_1$ we have $A \ast B = S(h) \otimes A \ast B$.

4 A unified framework

The braiding isomorphisms \( \text{(3)} \) and \( \text{(5)} \) in the two examples just discussed may look quite different at first glance, but—just as was the case for the tensor product $\ast$ itself—they are actually two instances of the same structure.
Figure 1: Associativity isomorphism $\alpha_{A,B,C} : A*(B*C) \to (A*B)*C$. The label $abc$ means that $A \in C_a$, $B \in C_b$, $C \in C_c$. In the three non-listed cases 000, 001, 100, $\alpha_{A,B,C}$ is the identity (or rather the associator of the underlying category $S$). The diagrams are read from bottom to top, the empty and solid dot denote $S$ and $S^{-1}$, respectively, and the three-valent vertices are the product and coproduct. The arrowhead depicts the action of $H$ on a module.

Namely, let $S$ be a pivotal symmetric monoidal category (i.e. a ribbon category with symmetric braiding $s_{A,B} : A \otimes B \to B \otimes A$) and let $H$ be a Hopf algebra in $S$ with invertible antipode. Denote by $\text{Rep}_S(H)$ the monoidal category of left $H$-modules in $S$ and set

$$C = C_0 + C_1 \quad \text{with} \quad C_0 = \text{Rep}_S(H), \quad C_1 = S.$$ \hfill (6)

On $C$ we fix the $\mathbb{Z}/2\mathbb{Z}$-graded tensor product functor $*$ from Section[1]. Denote by $\mu$ the multiplication of $H$, by $\Delta$ the coproduct, and by $S$ the antipode. For a morphisms $x : 1 \to H$ in $S$, write $xM$ and $M_x$ for the endomorphism of $H$ given by left- and right-multiplication with $x$, respectively, and $\text{Ad}_x$ for the endomorphism given by $x(-)x^{-1}$. Associativity isomorphisms for $*$ can be obtained from Hopf-algebraic data as follows [1, Cor. 3.17]:

**Theorem 1.** Let $\gamma : 1 \to H \otimes H$ and $\lambda : H \to 1$ be two morphisms in $S$ such that

1. $\gamma$ is a non-degenerate Hopf-copairing,
2. $\lambda$ is a right cointegral for $H$, such that there exits $g : 1 \to H$ with $(id \otimes \lambda) \circ \Delta = g \circ \lambda$, and such that $(\lambda \otimes \lambda) \circ (id \otimes S) \circ \gamma = id_1$,
3. $\gamma$ satisfies the symmetry condition $s_{H,H} \circ \gamma = (id \otimes (S^2 \circ \text{Ad}_g^{-1})) \circ \gamma$.

Then the natural isomorphisms in Figure[1] define associativity isomorphisms for $*$. 

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Figure 2: Braiding isomorphisms $c_{A,B} : A * B \rightarrow B * A$. The notation is as in Figure [1]

Note that it is not claimed that the above description gives all associativity isomorphisms for $*$; outside of $\mathcal{S} = \text{vect}(k)$ this would require extra assumptions. In [1], the above theorem is actually proved in the more general setting of $\mathcal{S}$ being ribbon but not necessarily symmetric. The relevant Hopf algebra notions are reviewed in [1, Sec. 2].

Example 2. 1. The Tambara-Yamagami categories are recovered for $\mathcal{S} = \text{vect}(\mathbb{C})$ and $H = \text{Fun}(G, \mathbb{C})$. The cointegral and copairing are $\lambda = \tau \sum_{a \in G} \delta_a$ and $\gamma = \sum_{a,b \in G} \chi(a, b) \delta_a \otimes \delta_b$.
2. For symplectic fermions take $\mathcal{S} = \text{svect}(\mathbb{C})$ and $H = S(\mathfrak{h})$. The copairing is $\gamma = e^C$ and the cointegral $\lambda$ is as given in Section 3.
3. Another example for $\mathcal{S} = \text{vect}(\mathbb{C})$ is provided by Sweedler’s four-dimensional Hopf algebra, which is not semi-simple [1, Sec. 3.8.3]. This illustrates that Theorem 1 is more general than Tambara-Yamagami categories even in the vector space case.

For the braiding isomorphisms we need to fix an involutive monoidal automorphism $\omega$ of the identity functor on $\mathcal{S}$. For $\mathcal{S} = \text{vect}(\mathbb{C})$, $\omega$ is necessarily the identity, but for $\mathcal{S} = \text{svect}(\mathbb{C})$ there are already two choices: the identity and parity involution. We have [1, Thm. 1.2 & Rem. 4.11]:

Theorem 3. Let $H$, $\gamma$, $\lambda$ and $g$ be as in Theorem 1 and let $\sigma : 1 \rightarrow H$ and $\beta : 1 \rightarrow 1$ be invertible. Suppose that

1. $\gamma$ is determined through $\sigma$ by $\gamma = (\sigma^{-1} M \otimes M_{\sigma^{-1}}) \circ \Delta \circ \sigma$.
2. $\lambda$ satisfies $\lambda \circ S = \lambda \circ \text{Ad}_\sigma$ and $\lambda \circ \sigma = \beta \circ \beta$. 

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3. \( \text{Ad}_\sigma \) is a Hopf-algebra isomorphism \( H \to H_{\text{cop}} \) (the opposite coalgebra).

4. \( S \circ \sigma = gM \circ \sigma = M_{g^{-1}} \circ \sigma \).

5. \( \omega \) evaluated on \( H \) satisfies \( (\text{id} \otimes \omega_H) \circ \gamma = (\text{Ad}_\sigma \otimes (\text{Ad}^{-1}_\sigma \circ S)) \circ \gamma \).

Then the natural isomorphisms in Figure 2 define a braiding on \( \mathcal{C} \).

If \( \sigma^2 \) is central in \( H \), then \( \mathcal{C} \) can be made into a ribbon category with twist isomorphisms \( \theta_A = \sigma^{-2}(\cdot) \) for \( A \in \mathcal{C}_0 \) (i.e. the left action of \( \sigma^{-2} \) on the \( H \)-module \( A \)), and \( \theta_A = \beta^{-1}\omega_A \) for \( A \in \mathcal{C}_1 \), see [1, Prop. 4.18].

Example 4. 1. In the Tambara-Yamagami case, \( \sigma \) and \( \beta \) are as in Section 2. Comparing (3) and Figure 2 shows that the braiding isomorphisms match \( (S = \text{id}_H \text{ for an elementary 2-group, and Ad}_\sigma = \text{id}_H \text{ since } H \text{ is commutative}). \)

2. For symplectic fermions choose \( \sigma = \exp(\frac{1}{2} \hat{C}) \) and \( \beta = e^{-\pi in/4} \). Then (5) agrees with Figure 2.

3. Sweedler’s Hopf algebra is quasi-triangular, but the resulting braiding on \( \mathcal{C}_0 \) does not extend to all of \( \mathcal{C} \) (at least via the above construction). However, one can find a 16-dimensional semi-simple Hopf algebra in \( \text{vect}(\mathbb{C}) \) which is neither commutative nor co-commutative for which Theorems 1 and 3 apply [1, Sec. 4.7.4]. This is another instance where our setting is more general than the Tambara-Yamagami case even for \( \mathcal{S} = \text{vect}(\mathbb{C}) \).

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