The 3-Stage Optimization of K-Out-Of-N Redundant IRM With Multiple Constraints

KEYWORDS
IRM, k-out-of-n systems, heuristic, Dynamic, Langrangean and Sensitivity Analysis.

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ABSTRACT
In the present scenario Reliability plays a key role for solving complex executive problems. This paper addresses a redundant Integrate Reliability Model (IRM) optimization for the k-out-of-n configuration system with multiple constraints. Generally the reliability of a system is treated as function of cost, but in many real life situations other considerations apart from conventional cost constraint like weight, volume, size, space etc., play vital role in optimizing the system reliability. Quite a few IRM’s are reported with cost constraint only in optimizing the system reliability. As the literature informs that few authors mentioned IRM’S with Redundancy and this paper focuses a novel method of optimizing a Redundant IRM with multiple constraints to encounter the hidden impact of additional constraints apart from the cost constraint while the system is optimized by considering the k-out-of-n configuration system.

1. INTRODUCTION
The paper is focused to design and to optimize an Integrated Reliability Model for Redundant system with multiple constraints for the k-out-of-n configuration system as a beginning in the mentioned area of the research and initiated optimizing the system reliability. Integrated Reliability Model (IRM) refers to the determination of the number of components \(x_j\), component reliabilities \(r_j\), Stage reliabilities \(R_j\) and the system reliability \(R_s\) where in the problem considers both the unknowns that is the components reliabilities and the number of components in each stage for the given cost constraint to maximize the system reliability. So far in literature the integrated reliability models are optimized using cost constraint alone where there is an established truth between cost and reliability.

This prompted the author to present a piece of novel aspect of Reliability Optimization through modeling by considering an Integrated Reliability Model for a Redundant System by treating Weight and Volume as additional constraints apart from the conventional Cost constraint to optimize the System Reliability, to negotiate the hidden impact of the additional constraints like Weight and Volume for the k-out-of-n configuration reliability model.

3. MATHEMATICAL MODEL
The objective function and the constraints of the model

\[ R_s = \prod_{j=1}^{n} R_j \]

maximize \( R_s = \sum_{k=1}^{x_k} \left( r_j \right)^{k} (1 - r_j)^{x_k - k} \)  
subject to the constraints

\[ \sum_{j=1}^{n} c_j x_j \leq C_c \]

\[ \sum_{j=1}^{n} w_j x_j \leq W_w \]

\[ \sum_{j=1}^{n} v_j x_j \leq V_v \]

non-negative restriction that \( x_j \) is an integer and \( r_j, R_j > 0 \)

4. MATHEMATICAL FUNCTION
To establish the mathematical model, the most commonly used function is considered for the purpose of reliability design and analysis. The proposed mathematical function

\[ c_j = e^{\left(\frac{f_j - r_{j\text{min}}}{r_{j\text{max}} - r_j}\right)} \]

Where \( a_j, b_j \) are constants.

System reliability for the given function

\[ R_s = \prod_{j=1}^{n} R_j \]

The number of components at each stage \( X_j \) is given through the relation

\[ X_j = \frac{\ln(R_j)}{\ln(r_j)} \]

The problem under consideration is

maximize \( R_s = \prod_{j=1}^{n} \left( 1 - (1 - r_j)^{X_j} \right) \)

subject to the constraints

\[ \sum_{j=1}^{n} e^{f_j} (r_{j\text{min}} - r_j) x_j - C_0 \leq 0 \]

\[ \sum_{j=1}^{n} e^{g_j} (r_{j\text{min}} - r_j) x_j - W_0 \leq 0 \]

\[ \sum_{j=1}^{n} e^{h_j} (r_{j\text{max}} - r_j) x_j - V_0 \leq 0 \]

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5. THE LAGRANGIAN METHOD:
Solving the proposed formulation using the Lagrangean method

A Lagrangian function is formulated as

\[ F = R_S + \lambda \left[ \sum_{j=1}^{n} \left( \frac{e^{1-f_j}(r_j-r_j_{\text{min}})}{r_j_{\text{max}}-r_j_{\text{min}}} - 1 \right) \ln(1-R_j) \right] - C_0 \]

\[ + \frac{1}{2} \sum_{z=1}^{3} \left( \sum_{j=1}^{n} e^{1-f_j}(r_j-r_j_{\text{min}})/(r_j_{\text{max}}-r_j_{\text{min}}) \right) \right]^{1/2} \ln(1-R_j) \right) - \frac{1}{2} \sum_{z=1}^{3} \left( \sum_{j=1}^{n} e^{1-f_j}(r_j-r_j_{\text{min}})/(r_j_{\text{max}}-r_j_{\text{min}}) \right) \right]^{1/2} \ln(1-R_j) \right) = 0 \]

\[ \frac{\partial F}{\partial \lambda} = \sum_{j=1}^{n} \left[ e^{1-f_j}(r_j-r_j_{\text{min}})/(r_j_{\text{max}}-r_j_{\text{min}}) \right] - C_0 = 0 \]

\[ \frac{\partial F}{\partial \lambda} = \sum_{j=1}^{n} \left[ e^{1-f_j}(r_j-r_j_{\text{min}})/(r_j_{\text{max}}-r_j_{\text{min}}) \right] - W_0 = 0 \]

Where \( \lambda_1, \lambda_2, \lambda_3 \) are Lagrangean multipliers.

CASE PROBLEM:
To derive the optimum component reliability(\( r_j \)), stage reliability(\( R_j \)), number of components in each stage(\( x_j \)) and the system reliability(\( R_S \)) not to exceed system cost Rs.1000, weight of the system 1500 kg and volume of the system 2000 cm\(^3\).

CONSTANTS:

| Stage | \( f_j \) | \( g_j \) | \( h_j \) | \( r_{j_{\text{min}}} \) | \( r_{j_{\text{max}}} \) |
|-------|---------|---------|---------|----------------|----------------|
| 1     | 0.9     | 0.5     | 0.2     | 0.5            | 0.99           |

Reliability design relating to Cost, weight and volume – Without \( X \) Rounding Off:

i) RELIABILITY DESIGN RELATING TO COST

| Stage | \( r_j \) | \( R_j \) | \( x_j \) | \( C_{x_{100}} \) | \( c_j \) | \( x \) |
|-------|---------|---------|---------|----------------|------|------|
| 01    | 0.701   | 0.9399  | 2.33    | 110.27         | 256.93 |
| 02    | 0.70    | 0.9518  | 2.52    | 110.00         | 277.2 |
| 03    | 0.80    | 0.9886  | 2.78    | 110.51         | 307.21 |

Total Cost

841.34

ii) RELIABILITY DESIGN RELATING TO WEIGHT

| Stage | \( r_j \) | \( R_j \) | \( x_j \) | \( W_{x_{100}} \) | \( W \) | \( X \) |
|-------|---------|---------|---------|----------------|------|------|
| 01    | 0.701   | 0.9399  | 2.33    | 163.08         | 379.97 |
| 02    | 0.70    | 0.9518  | 2.52    | 161.6          | 407.23 |
| 03    | 0.80    | 0.9886  | 2.78    | 164.8          | 458.14 |

Total Weight

1245.34

iii) RELIABILITY DESIGN RELATING TO VOLUME

| Stage | \( r_j \) | \( R_j \) | \( x_j \) | \( V_{x_{100}} \) | \( V \) | \( X \) |
|-------|---------|---------|---------|----------------|------|------|
| 01    | 0.701   | 0.9399  | 2.33    | 218.7          | 509.57 |
| 02    | 0.70    | 0.9518  | 2.52    | 215.5          | 543.06 |
| 03    | 0.80    | 0.9886  | 2.78    | 222.6          | 618.83 |

Total Volume

1671.46

SYSTEM RELIABILITY = \( R_S = 0.8844 \)

RELIABILITY DESIGN RELATING TO COST, WEIGHT AND VOLUME – WITH \( X \) Rounding Off:

i) RELIABILITY DESIGN RELATING TO COST

| Stage | \( r_j \) | \( R_j \) | \( x_j \) | \( c_{x_{100}} \) | \( c_j \) | \( x \) |
|-------|---------|---------|---------|----------------|------|------|
| 01    | 0.701   | 0.9732  | 3       | 110.27         | 330.31 |
| 02    | 0.70    | 0.9730  | 3       | 110.00         | 330.00 |
| 03    | 0.80    | 0.9920  | 3       | 110.51         | 331.53 |

Total Cost

991.84

VARIATION IN TOTAL COST = 18.82%

ii) RELIABILITY DESIGN RELATING TO WEIGHT

| Stage | \( r_j \) | \( R_j \) | \( x_j \) | \( V_{x_{100}} \) | \( V \) | \( X \) |
|-------|---------|---------|---------|----------------|------|------|
| 01    | 0.701   | 0.9732  | 3       | 218.7          | 489.24 |
| 02    | 0.70    | 0.9730  | 3       | 216.1          | 484.8 |
| 03    | 0.80    | 0.9920  | 3       | 216.8          | 494.4 |

Total Weight

1468.44

VARIATION IN TOTAL WEIGHT = 18.83%

iii) RELIABILITY DESIGN RELATING TO VOLUME

| Stage | \( r_j \) | \( R_j \) | \( x_j \) | \( V_{x_{100}} \) | \( V \) | \( X \) |
|-------|---------|---------|---------|----------------|------|------|
| 01    | 0.701   | 0.9732  | 3       | 222.6          | 667.8 |
| 02    | 0.70    | 0.9730  | 3       | 225.5          | 646.5 |
| 03    | 0.80    | 0.9920  | 3       | 222.6          | 667.8 |

Total Volume

1970.4

VARIATION IN TOTAL VOLUME = 18.79%

SYSTEM RELIABILITY = 0.9394

VARIATION IN SYSTEM RELIABILITY = 0.622%

6. HEURISTIC METHOD
The Lagrangean multipliers method gives a solution to arrive at an optimal design quickly rather than sophisticated algorithms. This is of course done at the cost of treating the number of components in each stage(\( x_j \)) as real. This disad-
vantage can be overcome, by the heuristic approach. Heur-
istic methods, in most cases employ experimentation and
trial-and-error techniques. A heuristic method is particu-
larily used to come rapidly to a solution that is reasonably close to
the best possible answer, or ‘optimal solution’.

i) RELIABILITY DESIGN RELATING TO COST

| Stage | \(r_j\) | \(R_j\) | \(x_j\) | \(c_{\text{pf}}\) | \(c_{\text{x}}\) |
|-------|-------|-------|-------|-------|-------|
| 01    | 0.701 | 0.9732 | 3     | 110.27 | 330.31 |
| 02    | 0.70  | 0.9730 | 3     | 110.00 | 330.00 |
| 03    | 0.80  | 0.9600 | 2     | 110.51 | 221.02 |
| Total Cost | | | | | 881.33 |

VARIATION IN TOTAL COST = 11.86%

ii) RELIABILITY DESIGN RELATING TO WEIGHT

| Stage | \(r_j\) | \(R_j\) | \(x_j\) | \(w_{\text{pf}}\) | \(w_{\text{x}}\) |
|-------|-------|-------|-------|-------|-------|
| 01    | 0.701 | 0.9732 | 3     | 163.08 | 489.24 |
| 02    | 0.70  | 0.9730 | 3     | 161.6  | 484.8 |
| 03    | 0.80  | 0.9600 | 2     | 164.8  | 329.60 |
| Total Weight | | | | | 1303.64 |

VARIATION IN TOTAL WEIGHT = 13.09%

iii) RELIABILITY DESIGN RELATING TO VOLUME

| Stage | \(r_j\) | \(R_j\) | \(x_j\) | \(v_{\text{pf}}\) | \(v_{\text{x}}\) |
|-------|-------|-------|-------|-------|-------|
| 01    | 0.701 | 0.9732 | 3     | 218.7  | 656.1 |
| 02    | 0.70  | 0.9730 | 3     | 215.5  | 646.5 |
| 03    | 0.80  | 0.9600 | 2     | 226.6  | 445.2 |
| Total Volume | | | | | 1747.8 |

SYSTEM RELIABILITY (RS) = 0.9091
VARIATION IN TOTAL VOLUME = 12.61%
VARIATION IN SYSTEM RELIABILITY = 0.279%

7.2 SENSITIVITY ANALYSIS

It is observed that when the input data of constraints is
increased by 10% there is only a 4.09% increase in system reli-
ability. When the input data is decreased 10%, there is only an
8.3% decrease in system reliability. When one factor is varied, keeping all the other factors unchanged, the variation in the
system reliability is as shown in the following Table.

| Variation in factors | System Reliability |
|----------------------|--------------------|
| Cost                 | System Reliability |
| 10% decrease         | No change          |
| 10% Increase         | No change          |
| Weight               | System Reliability |
| 10% decrease         | No change          |
| 10% Increase         | No change          |
| Volume               | System Reliability |
| 10% decrease         | 8.37% decreases    |
| 10% Increase         | 4.09% increase     |

The analysis confirms that the volume factor is more sensitive to input data than are cost and weight.

8. DYNAMIC PROGRAMMING

The heuristic approach commonly provides a workable solu-
tion which is approximate one. To validate the established
redundant reliability system and to obtain the much needed integer solution the Dynamic Programming method is ap-
plied. The Lagrangean Method can be used as the input for
the Dynamic Programming Approach, in order to determine
the stage Reliabilities, System Reliabilities, Stage Cost and
the System Cost. The Dynamic Programming Approach pro-
vides flexibility in determining the number of components in
each stage; Stage Reliabilities and the System Reliability for
the given System Cost. As per the procedure the parameter
values derived from the Lagrangean are given as inputs for
the Dynamic Programming Approach to obtain the integer
solution.

i) RELIABILITY DESIGN RELATING TO COST CONSTRAINT

| Stage | \(r_j\) | \(R_j\) | \(x_j\) | \(c_{\text{pf}}\) | \(c_{\text{x}}\) |
|-------|-------|-------|-------|-------|-------|
| 01    | 0.9075 | 0.9794 | 4     | 1920   | 1920  |
| 02    | 0.9277 | 0.9811 | 5     | 1590   | 1590  |
| 03    | 0.9278 | 0.9416 | 1     | 878    | 878   |
| Total Cost | | | | | 4388 |

VARIATION IN TOTAL COST = 12.24%

ii) RELIABILITY DESIGN RELATING TO WEIGHT CONSTRAINT

| Stage | \(r_j\) | \(R_j\) | \(x_j\) | \(w_{\text{pf}}\) | \(w_{\text{x}}\) |
|-------|-------|-------|-------|-------|-------|
| 01    | 0.9075 | 0.9794 | 4     | 2560   | 2560  |
| 02    | 0.9277 | 0.9811 | 5     | 2385   | 2385  |
| 03    | 0.9278 | 0.9416 | 1     | 1676   | 1676  |
| Total weight | | | | | 6621 |

VARIATION IN TOTAL WEIGHT = 11.72%

iii) RELIABILITY DESIGN RELATING TO VOLUME:

| Stage | \(r_j\) | \(R_j\) | \(x_j\) | \(v_{\text{pf}}\) | \(v_{\text{x}}\) |
|-------|-------|-------|-------|-------|-------|
| 01    | 0.9075 | 0.9794 | 4     | 1920   | 1920  |
| 02    | 0.9277 | 0.9811 | 5     | 1590   | 1590  |
| 03    | 0.9278 | 0.9416 | 1     | 878    | 878   |
| Total volume | | | | | 4388 |

SYSTEM RELIABILITY (RS) = 0.9047
VARIATION IN TOTAL VOLUME = 12.24%
VARIATION IN SYSTEM RELIABILITY = 19.42%

9. CONCLUSIONS

The integrated reliability models for redundant systems with
multiple constraints for the k-out-of-n configuration system is
established for the commonly used mathematical function
using Lagrangean method approach where component reli-
abilities (r) and the number of components (x) in each stage
are treated as unknowns. The system reliability (R) is maxi-
mized for the given cost, weight and volume by determining
the component reliabilities (r) and the number of components
required for each stage (x). The Lagrangean Multiplier Meth-
od provide a real valued solution, the Heuristic approach is
considered for analysis purpose which provided a near opti-
mum solution wherein the values of component reliabilities
(r) are taken as input to carry out heuristic analysis. The analy-
ysis of Heuristic approach results in gaining a solution which
ought to be an approximate one even after its validation and
to derive the much needed scientific integer solutions for the
defined problem, the Dynamic Programming approach is
applied. The advantage of Dynamic Programming is that
the number of components required for each stage (x) directly
gives an integer value along with the other values of the pa-
rameters, which is very convenient for practical implementa-
tion for the real life problems.