Representations of Lie superalgebras in prime characteristic I

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Abstract

We initiate the representation theory of restricted Lie superalgebras over an algebraically closed field of characteristic $p > 2$. A superalgebra generalization of the celebrated Kac–Weisfeiler conjecture is formulated, which exhibits a mixture of $p$-power and 2-power divisibilities of dimensions of modules. We establish the conjecture for basic classical Lie superalgebras.

1. Introduction

1.1. The finite-dimensional complex simple Lie superalgebras were classified by Kac [10] in the 1970s. A main subclass in this classification, independently obtained by Scheunert, Nahm, and Rittenberg [14, 15] (with contributions from Kaplansky and others), is called basic classical Lie superalgebras, which by definition admits an even non-degenerate supersymmetric bilinear form and whose even subalgebras are reductive. It consists of several infinite series and three exceptional ones.

The modular representations of restricted Lie algebras in prime characteristic have been developed over the years with intimate connections to algebraic groups (see [6, 12, 20]; cf. Jantzen [8] for a review). In [12] Premet developed new ideas to establish a long-standing conjecture of Kac and Weisfeiler [20] for Lie algebras of reductive groups. Among other things Premet made a crucial use of the powerful machinery of support varieties for Lie algebras developed by Friedlander–Parshall [5], Jantzen and others.

1.2. In this article and its sequels we will initiate and develop systematically the modular representations of Lie superalgebras over an algebraically closed field $K$ of characteristic $p > 2$. To our best knowledge, there has not been any serious study in this direction, perhaps because the representation theory of simple Lie superalgebras over $\mathbb{C}$ (for example, the irreducible character problem) is already very difficult and remains to be better understood. It turns out that the modular super representation theory is a very promising direction full of non-trivial yet accessible problems and conjectures, and it exhibits a novel phenomenon of non-defining characteristics with primes $p$ and 2.

There has been increasing interest in modular representation theory of algebraic supergroups in connection with other areas in recent years, thanks to the work of Brundan, Kleshchev, Kujawa, and others (see [17] for references and historical remarks). As usual, when $\mathfrak{g}$ is the Lie (super)algebra of an algebraic supergroup $G$, the study of $\mathfrak{g}$-modules with zero $p$-character is closely related to the study of rational modules of $G$ or of its Frobenius kernel. The current work sets the study of restricted modules of Lie superalgebras in a wider context. It is possible that some basic duality between categories of modular representations of Lie superalgebras and Lie algebras will emerge when the super theory is more adequately developed.
1.3. For a (finite-dimensional) restricted Lie superalgebra $g = g_0 \oplus g_1$, one defines a $p$-character $\chi \in g_0^*$ and the associated reduced enveloping superalgebra $U_\chi(g)$. Let $g_\chi = g_{\chi,0} \oplus g_{\chi,1}$ be the centralizer of $\chi$ in $g$ of codimension $d_0/d_1$. It is well known that $d_0$ is even, but $d_1$ can possibly be odd. Let $[a]$ denote the least integer upper bound of $a$. We formulate at the end of Section 2 the Super KW Conjecture or Super KW Property which asserts that every $U_\chi(g)$-module has dimension divisible by $p^{d_0/2}2^{[d_1/2]}$. The celebrated Kac–Weisfeiler conjecture (Premet’s theorem) states that the KW Property above holds for the Lie algebra $g$ of a reductive algebraic group (with $g_1 = 0$ and $d_1 = 0$ above) under mild assumptions on $p$.

The main result of this article is that the Super KW Property as formulated above holds for basic classical Lie superalgebras (the general linear Lie superalgebras, though not simple, are also included). In this article we shall exclude the usual simple Lie algebras from the basic classical Lie superalgebras, even though all the proofs make sense for them as well.

For restricted Lie superalgebras, the theory of support varieties has yet to be developed (see, however, an interesting construction over $\mathbb{C}$ of Boe, Kuwada, and Nakano [1]). Our approach will take full advantage of a combination of techniques developed in Premet [12, 13] and in Skryabin [18], which allow us to establish the Super KW Property for basic Lie superalgebras with nilpotent $p$-characters bypassing completely the support variety machinery. In addition, we establish a Morita equivalence to reduce general $p$-characters to nilpotent ones, adapting Friedlander–Parshall [6] (also see [20]) to the superalgebra set-up. At several places we have to find ways to overcome new implications and difficulties that are not presented in the usual Lie algebra set-up. Let us explain in some detail.

1.4. In Section 3 we construct a natural $\mathbb{Z}$-grading on an arbitrary basic classical Lie superalgebra $g$ associated to a given nilpotent $p$-character $\chi$. For the Lie superalgebras of type $\mathfrak{sl}, \mathfrak{gl},$ and $\mathfrak{osp}$, our explicit construction, which works for every prime $p > 2$, is built on the one in Jantzen [9] for the classical Lie algebras. For the exceptional Lie superalgebras, we impose somewhat stronger conditions (which can presumably be relaxed) on the prime $p$.

In Section 4, we construct a $p$-subalgebra $m$ of $g$ from such a $\mathbb{Z}$-grading associated to a nilpotent $p$-character $\chi$, following Premet [12] (the cases with $d_1$ odd offer some new perspectives). We then use the ingenious and elementary method in Skryabin [18] to prove that every simple $U_\chi(g)$-module is free over the algebra $U_\chi(m)$ of dimension $[a] := \frac{d_0}{2}2^{[d_1/2]}$, where $[a]$ denotes the largest integer lower bound of $a$. We are done if $d_1$ is even; for $d_1$ odd, $[\frac{d_1}{2}] - [\frac{d_1}{2}] = 1$, and an extra factor 2 required in the super KW conjecture is then supplied by a $2$-dimensional endomorphism algebra of simple modules ‘of type $Q’$, which is a pure super phenomenon.

For a basic classical Lie superalgebra $g$, we further construct a $K$-superalgebra (called a finite $W$-superalgebra) $W_\chi(g) = \text{End}_{U_\chi(g)}(U_\chi(g) \otimes U_\chi(m) K_\chi)$ and show that $U_\chi(g)$ is isomorphic to the matrix algebra $M_d(W_\chi(g)^{op})$, following Premet’s argument [13] with a modification to avoid completely the use of support variety machinery. Again the type $Q$ phenomenon is implicit behind the scene here. This provides a conceptual explanation for the Super KW Property of $g$. The structures and representations of the superalgebra $W_\chi(g)$ and its complex counterpart deserves to be studied separately.

One can verify by inspection that the centralizer of a semisimple (even) element in $g$ is always a Levi subalgebra of $g$, and the case-by-case verification is elementary yet tedious due to the existence of non-conjugate Borel subalgebras. In Section 5, we establish a Morita equivalence theorem, which relates the reduced enveloping algebra $U_\chi(g)$ with an arbitrary $p$-character $\chi$.
of \( \mathfrak{g} \) to that of a Levi subalgebra of \( \mathfrak{g} \) with a nilpotent \( p \)-character. This is a superalgebra generalization of the classical results [6, 20], and our proof follows largely the general strategy in [6], with a few modifications to deal with the new super features (for example, the existence of non-conjugate Borel subalgebras and three types of roots that give rise to three rank-one Lie superalgebras). This Morita equivalence together with the results in Section 4 completes the proof of the Super KW Conjecture for \( \mathfrak{g} \).

Finally in Section 6, we work out by hand completely the representation theory of the simple Lie superalgebra \( \mathfrak{osp}(1|2) \) for all \( p \)-characters, which is very similar to the \( \mathfrak{sl}(2) \) case with some additional interesting super type \( Q \) phenomenon. In particular, we show that there is no projective simple \( U_\chi(\mathfrak{osp}(1|2)) \)-module when \( \chi \) is zero or regular nilpotent, in contrast to the \( \mathfrak{sl}(2) \) case.

1.5. Throughout we work with an algebraically closed field \( K \) with characteristic \( p > 2 \) as the ground field (unless specified otherwise for the exceptional Lie superalgebras). We exclude \( p = 2 \) since in that case Lie superalgebras coincide with Lie algebras.

A superspace is a \( \mathbb{Z}_2 \)-graded vector space \( V = V_0 \oplus V_1 \), in which we call elements in \( V_0 \) and \( V_1 \) even and odd, respectively. Write \( |v| \in \mathbb{Z}_2 \) for the parity (or degree) of \( v \in V \), which is implicitly assumed to be \( \mathbb{Z}_2 \)-homogeneous. A bilinear form \( f \) on \( V \) is supersymmetric if \( f(u, v) = (-1)^{|u||v|} f(v, u) \) for all homogeneous \( u, v \in V \). We will use the notation\[
\dim V = \dim V_0 | \dim V_1; \quad \dim V = \dim V_0 + \dim V_1.\]

All Lie superalgebras \( \mathfrak{g} \) will be assumed to be finite dimensional. We will use \( U(\mathfrak{g}) \) to denote its universal enveloping superalgebra.

According to Walls [19], the finite-dimensional simple associative superalgebras over \( K \) are classified into two types: besides the usual matrix superalgebra (called type \( M \)) there are in addition simple superalgebras of type \( Q \). Alternatively, a superalgebra analog of Schur’s Lemma states that the endomorphism ring of an irreducible module of a superalgebra is either 1-dimensional or 2-dimensional (in the latter case it is isomorphic to a Clifford algebra), cf., for example, Kleshchev [11, Chapter 12]. An irreducible module is of type \( M \) if its endomorphism ring is 1-dimensional and it is of type \( Q \) otherwise.

By vector spaces, derivations, subalgebras, ideals, modules, and submodules etc. we mean in the super sense unless otherwise specified.

2. Basic results for restricted Lie superalgebras

The materials in this section (except Subsection 2.7) are standard generalizations from Lie algebras and should not be surprising to experts, but we find it convenient to formulate them precisely for the latter use.

2.1. Restricted Lie Superalgebras

The notion of restricted Lie superalgebras can be easily formulated as follows (cf., for example, Farnsteiner [4]).

**Definition 2.1.** A Lie superalgebra \( \mathfrak{g} = \mathfrak{g}_\mathbb{0} \oplus \mathfrak{g}_\mathbb{1} \) is called a restricted Lie superalgebra, if there is a \( p \)th power map \( \mathfrak{g}_\mathbb{0} \to \mathfrak{g}_\mathbb{0} \), denoted as \( [^p] \), satisfying

(a) \( (kx)^[^p] = kp^x[^p] \) for all \( k \in K \) and \( x \in \mathfrak{g}_\mathbb{0} \),

(b) \( [x[^p], y] = (ad x)^p(y) \) for all \( x \in \mathfrak{g}_\mathbb{0} \) and \( y \in \mathfrak{g} \).
(c) \((x + y)^[p] = x^[p] + y^[p] + \sum_{i=1}^{p-1} s_i(x, y)\) for all \(x, y \in \mathfrak{g}_0\) where \(s_i\) is the coefficient of \(\lambda^{i-1}\) in \((ad(\lambda x + y))^{p-1}(x)\).

In short, a restricted Lie superalgebra is a Lie superalgebra whose even subalgebra is a restricted Lie algebra and the odd part is a restricted module by the adjoint action of the even subalgebra.

For example, the Lie algebra of an algebraic supergroup is restricted (see [17]).

All the Lie (super)algebras in this article will be assumed to be restricted.

2.2. Basic classical Lie superalgebras

Following [10, 14, 15], we recall the list of basic classical Lie superalgebras. In loc. cit., the ground field is \(\mathbb{C}\). We observe that these Lie superalgebras (whose even subalgebras are Lie algebras of reductive algebraic groups) are well defined over \(K\) and remain to be simple over \(K\) of characteristic \(p > 2\) (and \(p > 3\) in case of \(D(2, 1; \alpha)\) and \(G(3)\)), and they admit a non-degenerate even supersymmetric bilinear form. This is clear for the infinite series, and for the exceptional Lie superalgebras it follows from the construction of [14, 15] (it should also be possible via the method of contragredient Lie superalgebras of Kac, cf. [10] and the references therein).

In Table 1, we list all the basic classical Lie superalgebras over \(K\) with restrictions on \(p\) (the general linear Lie superalgebra, though not simple, is also included). We impose somewhat stronger restrictions on the prime \(p\) (which can probably be relaxed) for the three exceptional Lie superalgebras for our latter purpose of constructing suitable \(\mathbb{Z}\)-gradings.

In this table, Lie superalgebra \(D(2, 1; \alpha)\), \(\alpha \in K^* \setminus \{0, -1\}\), is 17-dimensional for which \(D(2, 1; \alpha)_0\) is a Lie algebra of type \(A_1 \oplus A_1 \oplus A_1\) and its adjoint representation on \(D(2, 1; \alpha)_1\) is \(V \otimes V \otimes V\), where \(V\) is the natural representation of \(A_1\).

The Lie superalgebra \(F(4)\) is 40-dimensional for which \(F(4)_0\) is a Lie algebra of type \(B_3 \oplus A_1\) and its adjoint representation on \(F(4)_1\) is \(U \otimes V\), where \(U\) is the 8-dimension spin representation of \(B_3\).

The Lie superalgebra \(G(3)\) is 31-dimensional for which \(G(3)_0\) is a Lie algebra of type \(G_2 \oplus A_1\), and the adjoint \(G(3)_0\)-module \(G(3)_1\) is the tensor product of the 7-dimensional simple \(G_2\)-module with \(V\). (We thank Zongzhu Lin for helpful clarification on restriction of prime \(p\) for representations of Lie algebra of type \(G_2\).)

2.3. Reduced Enveloping Superalgebras

Let \(\mathfrak{g}\) be a restricted Lie superalgebra. For each \(x \in \mathfrak{g}_0\), the element \(x^p - x^[p] \in U(\mathfrak{g})\) is central by Definition 2.1. We refer to \(\mathbb{Z}_p(\mathfrak{g}) = K(x^p - x^[p] \mid x \in \mathfrak{g}_0)\) as the \(p\)-center of \(U(\mathfrak{g})\). Let \(x_1, \ldots, x_s\) and \(y_1, \ldots, y_t\) be the basis of \(\mathfrak{g}_0\) and \(\mathfrak{g}_1\), respectively. The following proposition is a consequence of the PBW theorem for \(U(\mathfrak{g})\).

| Lie superalgebra | Characteristic of \(K\) |
|-----------------|--------------------------|
| \(\mathfrak{gl}(m|n)\) | \(p > 2\) |
| \(\mathfrak{sl}(m|n)\) | \(p > 2, p \nmid (m - n)\) |
| \(B(m, n), C(n), D(m, n)\) | \(p > 2\) |
| \(D(2, 1; \alpha)\) | \(p > 3\) |
| \(F(4)\) | \(p > 15\) |
| \(G(3)\) | \(p > 15\) |
Proposition 2.2. Let $\mathfrak{g}$ be a restricted Lie superalgebra. Then $Z_p(\mathfrak{g})$ is a polynomial algebra isomorphic to $K[x_i^p - x_i^{[p]} | i = 1, \ldots, s]$, and the enveloping superalgebra $U(\mathfrak{g})$ is free over $Z_p(\mathfrak{g})$ with basis
\[
\{x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_t^{b_t} | 0 \leq a_i < p; b_j = 0, 1 \text{ for all } i, j\}.
\]

Proposition 2.3. Let $\mathfrak{g}$ be a restricted Lie superalgebra. Then all irreducible $U(\mathfrak{g})$-modules are finite-dimensional, and their dimensions are bounded by a constant $M(\mathfrak{g})$ which depends only on $\mathfrak{g}$.

Proof. The proof (using Proposition 2.2) is the same as in the non-super case [3], and will be skipped.

Let $V$ be a simple $U(\mathfrak{g})$-module and $x \in \mathfrak{g}_0$. By Schur’s lemma, the central element $x^p - x^{[p]}$ acts by a scalar $\zeta(x)$, which can be written as $\chi_V(x)^p$ for some $\chi_V \in \mathfrak{g}_0^*$. We call $\chi_V$ the $p$-character of the module $V$.

Fix $\chi \in \mathfrak{g}_0^*$. Let $I_\chi$ be the ideal of $U(\mathfrak{g})$ generated by the even central elements $x^p - x^{[p]} - \chi(x)^p$. The quotient algebra $U_\chi(\mathfrak{g}) := U(\mathfrak{g})/I_\chi$ is called the reduced enveloping superalgebra with $p$-character $\chi$. We often regard $\chi \in \mathfrak{g}^*$ by letting $\chi(\mathfrak{g}_1) = 0$. If $\chi = 0$, then $U_0(\mathfrak{g})$ is called the restricted enveloping superalgebra.

The following proposition is an easy consequence of Proposition 2.2.

Proposition 2.4. The superalgebra $U_\chi(\mathfrak{g})$ has a basis
\[
\{x_1^{a_1} \cdots x_s^{a_s} y_1^{b_1} \cdots y_t^{b_t} | 0 \leq a_i < p; b_j = 0, 1 \text{ for all } i, j\}.
\]
In particular, $\dim U_\chi(\mathfrak{g}) = p^{\dim \mathfrak{g}_0} 2^{\dim \mathfrak{g}_1}$.

Remark 2.5. The adjoint algebraic group $G_0$ of $\mathfrak{g}_0$ acts on $\mathfrak{g}$ by adjoint action since $\mathfrak{g}_1$ is a rational $G_0$-module. The representation theory of the superalgebra $U_\chi(\mathfrak{g})$ depends only on the orbit of $\chi$ under the coadjoint action of $G_0$, since $U_\chi(\mathfrak{g}) \simeq U_{\chi'}(\mathfrak{g})$ for $\chi$ and $\chi'$ in the same orbit.

2.4. The baby Verma modules

Let $\mathfrak{g}$ be one of the Lie superalgebras in Section 2.2. Fix a triangular decomposition
\[ \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \]
where $\mathfrak{n}^+ = \mathfrak{n}_0^+ \oplus \mathfrak{n}_1^+$ and $\mathfrak{n}^- = \mathfrak{n}_0^- \oplus \mathfrak{n}_1^-$ are the Lie subalgebras of positive and negative root vectors, respectively, and $\mathfrak{h}$ is a Cartan (even) subalgebra of $\mathfrak{g}_0$ of rank $l$.

By Remark 2.5 we may choose $\chi \in \mathfrak{g}_0^*$ with $\chi(\mathfrak{n}_0^+) = 0$ without loss of generality. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Each $\lambda \in \mathfrak{h}^*$ defines a 1-dimensional $\mathfrak{h}$-module $K_\lambda$, where each $h \in \mathfrak{h}$ acts as multiplication by $\lambda(h)$. The module $K_\lambda$ is a $U_\chi(\mathfrak{h})$-module if and only if $\lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p$ for all $h \in \mathfrak{h}$. Set
\[ \Lambda_\chi = \{ \lambda \in \mathfrak{h}^* | \lambda(h)^p - \lambda(h^{[p]}) = \chi(h)^p \text{ for all } h \in \mathfrak{h} \}. \]
Note that $|\Lambda_\chi| = p^{\dim b}$. Now for $\lambda \in \Lambda_\chi$, $K_\lambda$ is regarded as a $b$-module with $n^+$ acting trivially, and the baby Verma module is defined to be the induced module $Z_\chi(\lambda) := U_\chi(g) \otimes_{U_\chi(b)} K_\lambda$.

2.5. $p$-Nilpotent Lie superalgebras

A restricted Lie superalgebra $g$ is called $p$-nilpotent if, for each $x \in g_\bar{0}$, there exists $r > 0$ such that $x^{[p]^r} = 0$.

Proposition 2.6. If $g$ is a $p$-nilpotent Lie superalgebra, then each $U_\chi(g)$ has a unique simple module (up to isomorphism). Moreover, if $\chi = 0$, then the trivial module is the only simple module.

Proof. The proof is essentially the same as for $p$-nilpotent Lie algebras (see, for example, proofs of Proposition 3.2 and Theorem 3.3 in [8]), and will be skipped here.

2.6. $U_\chi(g)$ as Frobenius and symmetric algebras

Recall that the supertrace of an endomorphism $X$ on a vector space $V_\bar{0} \oplus V_\bar{1}$ is defined to be $\text{str}(X) = \text{tr}(X|_{V_\bar{0}}) - \text{tr}(X|_{V_\bar{1}})$. An associative superalgebra $A$ with a supersymmetric nondegenerate bilinear form will be referred to as a symmetric (super)algebra. The standard properties for a symmetric algebra remain valid for symmetric superalgebras. The following is a generalization of a result of Friedlander and Parshall [6] for restricted Lie algebras, and it can be proved in the same way.

Proposition 2.7. For each $\chi \in g_\bar{0}$, the reduced enveloping superalgebra $U_\chi(g)$ is a Frobenius algebra (with a non-degenerate bilinear form denoted by $\tilde{\mu}$). Moreover, $\tilde{\mu}$ is supersymmetric if and only if $\text{str}(ad x) = 0$ for all $x \in g_\bar{0}$. In particular, if $g$ is simple, then $U_\chi(g)$ is a symmetric superalgebra.

2.7. The Super KW Property

Let $\chi \in g_\bar{0}$ and we always regard $\chi \in g^*$ by setting $\chi(g_\bar{1}) = 0$. Denote the centralizer of $\chi$ in $g$ by $g_\chi = g_{\chi,0} + g_{\chi,1}$, where $g_{\chi,i} = \{y \in g_i | \chi([y,g]) = 0\}$ for $i \in \mathbb{Z}_2$. Set $d_i = \dim g_i - \dim g_{\chi,i}$. Recall that $\lfloor a \rfloor$ denotes the least integer upper bound of $a$.

We formulate the following superalgebra generalization of the Kac–Weisfeiler conjecture.

Super KW Property. The dimension of every $U_\chi(g)$-module is divisible by $p^{d_0/2} \lfloor d_1/2 \rfloor$.

It is well known that $d_0$ is an even integer and $d_1$ can possibly be odd. When $g$ is the Lie algebra of a reductive algebraic group (that is, $g_\bar{1} = 0$), the celebrated Kac–Weisfeiler (KW) conjecture [20] states that the (Super) KW Property holds (where no $2$-power is involved). The original KW conjecture has been completed by Premet [12].

It is known that the KW Property holds for many interesting examples beyond the set-up of the original KW conjecture, and nevertheless, it does not hold for all Lie algebras (and so superalgebras).

Question. For which Lie superalgebras does the Super KW Property hold?

The main goal of this article is to establish the Super KW Conjecture for all the basic classical Lie superalgebras with assumptions on $p$ as in Table 1. In the remainder of this article, $g$ is assumed to be one of the basic classical Lie superalgebras with such a restriction on $p$. 
3. The \( \mathbb{Z} \)-gradings of Lie superalgebras

3.1. The \( \mathbb{Z} \)-gradings with favorable properties

Let \( \mathfrak{g} \) be one of the basic classical Lie superalgebras in Section 2.2. The Lie superalgebra \( \mathfrak{g} \) admits a non-degenerate invariant even bilinear form \( \langle \cdot, \cdot \rangle \), whose restriction on \( \mathfrak{g}_0 \) gives an isomorphism \( \mathfrak{g}_0 \cong \mathfrak{g}_0^\ast \). Let \( \chi \in \mathfrak{g}_0^\ast \) be a nilpotent character, that is, it is the image of some nilpotent element \( X \) in \( \mathfrak{g}_0 \) under the above isomorphism. Then \( \mathfrak{g}_X \) is equal to the usual centralizer \( \mathfrak{g}_X = \{ y \in \mathfrak{g} \mid [X, y] = 0 \} \).

**Theorem 3.1.** Let \( \mathfrak{g} \) be one of the basic classical Lie superalgebras in Section 2.2. Then there exists a \( \mathbb{Z} \)-grading \( \mathfrak{g} = \oplus_{k \in \mathbb{Z}} \mathfrak{g}(k) \) satisfying:

\[
X \in \mathfrak{g}(2);
\]

\[
(\mathfrak{g}(k), \mathfrak{g}(l)) = 0, \quad \text{if } k + l \neq 0;
\]

\[
\mathfrak{g}_X = \oplus_{k \in \mathbb{Z}} \mathfrak{g}_X(k) \quad \text{where } \mathfrak{g}_X(k) = \mathfrak{g}_X \cap \mathfrak{g}(k);
\]

\[
\mathfrak{g}_X(s) = 0 \quad \forall s < 0;
\]

\[
\dim \mathfrak{g}_X = \dim \mathfrak{g}(0) + \dim \mathfrak{g}(1).
\]

This grading is compatible with the \( \mathbb{Z}_2 \)-grading, that is, \( \mathfrak{g}(k) = \mathfrak{g}(k)_0 \oplus \mathfrak{g}(k)_1 \) where \( \mathfrak{g}(k)_i = \mathfrak{g}(k) \cap \mathfrak{g}_i \) for \( i \in \mathbb{Z}_2, k \in \mathbb{Z} \).

We shall construct the \( \mathbb{Z} \)-gradings in the following subsections according to the types of the Lie superalgebras. The constructions for the Lie superalgebras of \( \mathfrak{gl} \) and \( \mathfrak{osp} \) types are natural generalizations of those for classical Lie algebras, cf. [9, Chapters 1–5].

3.2. The \( \mathbb{Z} \)-gradings for \( \mathfrak{gl} \)

Let \( V = V_0 \oplus V_1 \) be a vector space with \( \dim V = m \mid n \). Identify \( \mathfrak{gl}(m \mid n) \) with \( \mathfrak{gl}(V) \), whose even part is \( \mathfrak{gl}(V)_0 = \mathfrak{gl}(V_0) \oplus \mathfrak{gl}(V_1) \). To a nilpotent element \( X \in \mathfrak{gl}(V)_0 \), we associate with a pair of partitions \( (\pi_0, \pi_1) \), where \( \pi_i = (\lambda_i^1, \ldots, \lambda_i^r) \) of length \( r_i \) is the shape of the Jordan canonical form of the summand of \( X \) in \( \mathfrak{gl}(V_i) \), respectively.

There exist \( v_1, \ldots, v_{r_0} \in V_0 \) and \( u_1, \ldots, u_{r_1} \in V_1 \) such that all \( X^j v_i \) and \( X^j u_i \) with \( 1 \leq i \leq r_0 \) (and \( r_1 \)); \( 0 \leq j < \lambda_i^0 \) and \( 0 \leq j < \lambda_i^1 \) are the basis for \( V_0 \) and \( V_1 \) and such that \( X^{\lambda_i^0} v_i = 0 \) and \( X^{\lambda_i^1} u_i = 0 \).

Each homogeneous \( Z \in \mathfrak{gl}(V)_X \) is determined by \( Z(v_i) \) and \( Z(u_j) \) with \( 1 \leq i \leq r_0, 1 \leq j \leq r_1 \) because \( Z(X^k v_i) = X^k Z(v_i) \) and \( Z(X^k u_j) = X^k Z(u_j) \) for all \( i, j \), and \( k \). Furthermore, \( X^{\lambda_i^0} Z(v_i) = 0 \) and \( X^{\lambda_i^1} Z(u_i) = 0 \) for all \( i \) and \( j \). Using this, one checks that if \( Z \) is even, then \( Z(v_i) \) and \( Z(u_j) \) have the forms

\[
Z(v_i) = \sum_{l=1}^{r_0} \sum_{k=\max(0, \lambda_i^0 - \lambda_i^0)}^{\lambda_i^0 - 1} a_{k,l;i} X^k v_i,
\]

\[
Z(u_j) = \sum_{l=1}^{r_1} \sum_{k=\max(0, \lambda_i^1 - \lambda_i^1)}^{\lambda_i^1 - 1} b_{k,l;j} X^k u_i.
\]
If \( Z \) is odd, then
\[
Z(v_i) = \sum_{l=1}^{r_1} \sum_{k=\max\{0,\lambda^1_i-\lambda^0_j\}}^{\lambda^1_i-1} c_{k,l;i} X^k v_i,
\]
(3.8)
\[
Z(u_j) = \sum_{l=1}^{r_0} \sum_{k=\max\{0,\lambda^0_j-\lambda^1_i\}}^{\lambda^0_j-1} d_{k,l;j} X^k v_l.
\]
(3.9)
The coefficients \( a_{k,l;i}, b_{k,l;j}, c_{k,l;i}, \) and \( d_{k,l;j} \) above can be chosen arbitrarily in \( K \). We compute that
\[
\dim \mathfrak{gl}(V)_{X,\bar{0}} = \sum_{i,j=1}^{r_0} \min(\lambda^0_i, \lambda^0_j) + \sum_{i,j=1}^{r_1} \min(\lambda^1_i, \lambda^1_j),
\]
\[
\dim \mathfrak{gl}(V)_{X,\bar{1}} = 2 \sum_{i,j=1}^{r_0} \sum_{i,j=1}^{r_1} \min(\lambda^0_i, \lambda^1_j).
\]
Define a \( \mathbb{Z} \)-grading of the vector space \( V \) (which is compatible with the \( \mathbb{Z}_2 \)-grading) as follows. Set \( V(k)_{\bar{0}} \) and \( V(k)_{\bar{1}} \) equal to the span of \( X^j v_i \) with \( k = 2j + 1 - \lambda^0_i \) and \( X^j u_i \) with \( k = 2j + 1 - \lambda^1_i \), respectively. Note that
\[
\dim V(l) = \dim V(-l),
\]
(3.10)
\[
XV(l) \subseteq V(l + 2).
\]
(3.11)

Remark 3.2. The grading defined above can be thought of coming from the \( \mathfrak{sl}(2) \)-theory in characteristic zero. Indeed, let \( e, f, \) and \( h \) be the standard basis of \( \mathfrak{sl}(2) \). We can make \( V \) into an \( \mathfrak{sl}(2) \)-module such that \( e \) acts as \( X \), \( h \) acts as on each \( V(m) \) as multiplication with \( m \), and \( f \) annihilates all \( v_i \) and \( u_j \), and maps each \( X^k v_i \) and \( X^k u_j \) to suitable multiples of \( X^{k-1} v_i \) and \( X^{k-1} u_j \), respectively.

The grading on \( V \) induces a grading \( \mathfrak{gl}(V) = \bigoplus_{k \in \mathbb{Z}} \mathfrak{gl}(V)(k) \) with
\[
\mathfrak{gl}(V)(k) = \{ f \in \mathfrak{gl}(V) | f(V(l)) \subseteq V(l + k) \text{ for all } l \in \mathbb{Z} \}.
\]
(3.12)
This gives a \( \mathbb{Z} \times \mathbb{Z}_2 \)-grading on the Lie superalgebra, that is,
\[
[\mathfrak{gl}(V)(k), \mathfrak{gl}(V)(l)] \subseteq \mathfrak{gl}(V)(k + l),
\]
(3.13)
and
\[
\mathfrak{gl}(V)(k) = \mathfrak{gl}(V)(k)_{\bar{0}} \oplus \mathfrak{gl}(V)(k)_{\bar{1}},
\]
(3.14)
where \( \mathfrak{gl}(V)(k)_i = \mathfrak{gl}(V)_i \cap \mathfrak{gl}(V)(k) \).
We have
\[
X \in \mathfrak{gl}(V)(2)
\]
by (3.11). This implies easily that
\[
\mathfrak{gl}(V)_X = \bigoplus_{k \in \mathbb{Z}} \mathfrak{gl}(V)_X(k) \text{ where } \mathfrak{gl}(V)_X(k) = \mathfrak{gl}(V)_X \cap \mathfrak{gl}(V)(k).
\]

Example 3.3. Let \( V \) be a superspace of \( \dim V = 3|2 \). Therefore, \( \mathfrak{gl}(V) \) is isomorphic to the Lie superalgebra \( \mathfrak{gl}(3|2) \) consisting of all \( (3 + 2) \times (3 + 2) \) square matrices. Let \( X \) be a nilpotent element in \( \mathfrak{gl}(V)_{\bar{0}} \) corresponding to the pair of partitions \( (3; 2) \). Then there exist \( v_1 \in V_{\bar{0}} \) and
$u_1 \in \mathcal{V}$ such that $\{X^2v_1, Xv_1, v_1\}$ and $\{Xu_1, u_1\}$ form bases of $\mathcal{V}_0$ and $\mathcal{V}_1$ respectively. Under this basis of $\mathcal{V}$, the element $X$ has matrix form

$$
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}.
$$

The grading $\mathcal{V} = \mathcal{V}(-2) \oplus \mathcal{V}(-1) \oplus \mathcal{V}(0) \oplus \mathcal{V}(1) \oplus \mathcal{V}(2)$ is specified as follows:

- $\mathcal{V}(2) = \mathcal{X}X^2v_1$, $\mathcal{V}(1) = \mathcal{K}Xu_1$, $\mathcal{V}(0) = \mathcal{K}Xv_1$,
- $\mathcal{V}(-1) = \mathcal{K}u_1$, $\mathcal{V}(-2) = \mathcal{K}v_1$, $\mathcal{V}(l) = 0$ for $|l| > 2$.

Any element $Z \in \mathfrak{gl}(\mathcal{V})_X$ is of the matrix form

$$
Z = \begin{bmatrix}
x_0 & x_2 & x_4 & y_1 & y_3 \\
0 & -x_0 & x_2 & 0 & 1 \\
0 & -4 & 0 & 0 & 0 \\
0 & -1 & z_1 & z_3 & w_0 \\
0 & -3 & 0 & 0 & w_0
\end{bmatrix},
$$

where $x_i, y_i, z_i$, and $w_i$ are arbitrary scalars in $K$, $0_i = 0$, and the index $i$ indicates the $Z$-gradings of the corresponding matrix entries. Hence, $\dim \mathfrak{gl}(\mathcal{V})_X = 5|4$.

Note that the centralizer $\mathfrak{gl}(\mathcal{V})_X$ is concentrated on the non-negative degrees.

3.3. The $Z$-gradings for $\mathfrak{osp}$

Let $\phi$ be a non-degenerate even supersymmetric bilinear form on $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$, that is, $\mathcal{V}_0$ and $\mathcal{V}_1$ are orthogonal, the restriction $\phi_0$ of $\phi$ to $\mathcal{V}_0$ is symmetric, and the restriction $\phi_1$ of $\phi$ to $\mathcal{V}_1$ is skew-symmetric. Then the ortho-symplectic Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(\mathcal{V})$ with $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is defined by

$$
\mathfrak{g}_i = \{ f \in \mathfrak{gl}(\mathcal{V})_i \mid \phi(f(x), y) = -(-1)^{|x||y|}\phi(x, f(y)) \forall x, y \in \mathcal{V} \}, \quad i \in \mathbb{Z}_2.
$$

According to [10], the ortho-symplectic Lie superalgebras $\mathfrak{osp}(\mathcal{V})$ are further divided into infinite series of type $B(m,n), C(n), \text{ and } D(m,n)$, depending on whether or not $\dim \mathcal{V}$ is $(2m + 1)(2n, 2n)$, and $2m|2n$, respectively.

Let $\chi \in \mathfrak{g}_0^*$ be a nilpotent $p$-character, which can be regarded as $\chi \in \mathfrak{g}^*$ by declaring that $\chi(\mathfrak{g}_1) = 0$. Let $X \in \mathfrak{g}_0$ be the nilpotent element corresponding to $\chi$ via the form $(\cdot, \cdot)$ on $\mathfrak{g}$, and write it as $X = X_0 + X_1$ where $X_i \in \mathfrak{gl}(\mathcal{V}_i) \cap \mathfrak{g}_0$. The $X_i$ determines the partition $\pi_i = (\lambda_1^i, \ldots, \lambda_{r_i}^i)$, and $\pi_i$ is the shape of the Jordan canonical form of $X_i$. According to [9, Chapter 1], there exist bases $v_1, \ldots, v_{r_0} \in \mathcal{V}_0$ and $u_1, \ldots, u_{r_1} \in \mathcal{V}_1$ satisfying properties in [9, 1.11, Theorems 1 and 2], respectively.

We retain the $Z$-grading on $\mathcal{V}$ as defined in Section 3.2 in the case of $\mathfrak{gl}(\mathcal{V})$, so that (3.10) and (3.11) still hold. Using the arguments in [9, 3.4], one can show that the grading on $\mathcal{V}$ is compatible with the bilinear form $\phi$ in the following sense:

$$
\phi(\mathcal{V}(k), \mathcal{V}(l)) = 0 \text{ unless } k + l = 0.
$$

We claim that

$$
\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}(k) \text{ where } \mathfrak{g}(k) := \mathfrak{g} \cap \mathfrak{gl}(\mathcal{V})(k).
$$

Indeed, let us take $Y \in \mathfrak{g}_1$ (the argument for even elements is similar and thus skipped) and decompose $Y = \sum_k Y_k$ with $Y_k \in (\mathfrak{gl}(\mathcal{V})(k))_1$. We have to show that $Y_k \in \mathfrak{g}$, that is, $\phi(Y_k(v), w) = -(-1)^{|v||w|}\phi(v, Y_k(w))$ for $v \in \mathcal{V}(s)$ and $w \in \mathcal{V}(t)$. Note that $Y_k(v) \in \mathcal{V}(k + s)$ and
implies that \((3.16)\) implies that

\[
\phi(Y_k(v), w) = 0 = \phi(v, Y_k(w)) \quad \text{if } s + k + t \neq 0.
\]

On the other hand, if \(s + k + t = 0\), then \((3.16)\) implies that

\[
\phi(Y_k(v), w) = \sum_l \phi(Y_l(v), w) = \phi(Y(v), w) = -(-1)^{|v|} \phi(v, Y(w))
\]

\[
= -(-1)^{|v|} \sum_l \phi(v, Y_l(w)) = -(-1)^{|v|} \phi(v, Y_k(w)).
\]

This proves \((3.17)\).

By definition, \(X \in g(2)\), that is, \((3.1)\) holds. This implies, as in Section 3.2, that

\[
g_X = \bigoplus_{k \in \mathbb{Z}} g_X(k),
\]

where \(g_X(k) = g_X \cap g(k)\). This verifies \((3.3)\).}

**Remark 3.4.** Define a homomorphism of algebraic groups \(\tau = \tau_0 \times \tau_1\) from the multiplicative group \(K^\times\) to \(GL(V) = GL(V_0) \times GL(V_1)\) as follows. For each \(t \in K^\times\), let \(\tau(t) = \tau_0(t) \times \tau_1(t)\) be the linear map with \(\tau_0(t)(v) = t^k v\) for all \(v \in V(k)_i\) and all \(k\). Now the equality \((3.10)\) shows that \(\tau(t)\) takes values in \(SL(V_0) \times SL(V_1)\). In the set-up of Section 3.3, one checks easily using \((3.16)\) that \(\tau(K^\times)\) is contained in \(SO(V_0) \times Sp(V_1)\). Now the \(\mathbb{Z}\)-gradings defined in Sections 3.2 and 3.3 can be described via

\[
gl(V)(k) = \{ Z \in gl(V) \mid Ad(\tau(t))(Z) = t^k Z \text{ for all } t \in K^\times \},
\]

\[
osp(V)(k) = \{ Z \in osp(V) \mid Ad(\tau(t))(Z) = t^k Z \text{ for all } t \in K^\times \}.
\]

### 3.4. Completing the proof of Theorem 3.1 for \(gl\), \(sl\), and \(osp\)

Throughout this subsection, it is assumed that \(g = gl(V)\) or \(osp(V)\) as in subsection 3.2 or 3.3.

The invariance of the even non-degenerate bilinear form \((\cdot, \cdot)\) under all \(\tau(t)\) with \(t \in K^\times\) implies that \((g(k), g(l)) = 0\) if \(k + l \neq 0\), while \((\cdot, \cdot)\) induces a perfect pairing between \(g(k)_i\) and \(g(-k)_i\) for \(i \in \mathbb{Z}_2\). Thus \((3.2)\) has been verified.

We now prove \((3.4)\). Let \(0 \neq Z \in g_X(s)_i\) (the argument for \(Z\) odd is similar) and write \(Z(v_i)\) and \(Z(u_j)\) as in \((3.6)\) and \((3.7)\), respectively. We have \(Z(v_i) \in V(s + 1 - \lambda^0_i)_i\) and \(Z(u_j) \in V(s + 1 - \lambda^1_j)_j\). Therefore, the coefficients \(a_{k,l;i}\) in \((3.6)\) and \(b_{k,l;i,j}\) in \((3.7)\) are 0 unless \(s + 1 - \lambda^0_i = 2k + 1 - \lambda^0_i\) and \(s + 1 - \lambda^1_j = 2k + 1 - \lambda^1_j\), respectively. On the other hand, we have \(k \geq \max(0, \lambda^0_i - \lambda^0_i)\) and \(k \geq \max(0, \lambda^1_j - \lambda^1_j)\) by \((3.6)\) and \((3.7)\). Therefore, \(a_{k,l;i} \neq 0\) and \(b_{k,l;i,j} \neq 0\) imply that

\[
s = 2k + \lambda^0_i - \lambda^0_i \geq k \geq 0 \quad \text{and} \quad s = 2k + \lambda^1_j - \lambda^1_j \geq k \geq 0,
\]

respectively. In either case, we conclude that \(s \geq 0\), whence \((3.4)\).

To complete the proof of Theorem 3.1 for \(g\), it remains to verify the identity \((3.5)\). Using the argument of \([9, \text{Lemma 5.7}]\), one can show that

\[
[g(k-2)_i, X] = g(k)_i \iff g_X \cap g(-k) = 0
\]

\((3.18)\)

for each \(k \in \mathbb{Z}\) and \(i \in \mathbb{Z}_2\). Now by \((3.4)\) and \((3.18)\), the map \(ad X : g(k)_i \rightarrow g(k+2)_i\) is surjective with kernel the subspace \(g(k)_i \cap g_X_i\) for \(k \geq 0\) and \(i \in \mathbb{Z}_2\). The highest degree component in the grading is contained in \(g_X\); counting the dimension of \(g_X\) backwards through the grading, we prove \((3.5)\).

This completes the proof of Theorem 3.1 for \(g = gl(V)\) or \(osp(V)\).
Remark 3.5. For $\mathfrak{g} = sl(m|n)$ with $p \mid m - n$ and $p > 2$, the arguments in Subsections 3.2, 3.3 and the above carry over readily, and so Theorem 3.1 holds for $sl(m|n)$ as well.

3.5. The $\mathbb{Z}$-gradings for exceptional Lie superalgebras

Now suppose that $\mathfrak{g}$ is one of the exceptional Lie superalgebras of type $D(2,1;\alpha), F(4),$ or $G(3)$ as in Table 1, which admit a non-degenerate even supersymmetric bilinear form and whose even subalgebras are Lie algebras of reductive algebraic groups. In each case, the lower bound for the characteristic of $K$ is indeed $\geq 3(h - 1)$, where $h$ is the maximum of the Coxeter numbers for the irreducible summands of the even subalgebra $\mathfrak{g}_0$. As before let $X \in \mathfrak{g}$ denote the nilpotent element corresponding to a nilpotent $p$-character $\chi \in \mathfrak{g}_0^\ast$. With this restriction on $p$, by Carter [2] (also cf. [9, Section 5.5]), there exists an $sl(2)$-triple $\{X,H,Y\}$ so that $KH$ lifts to a one-dimensional torus $\gamma : K^\times \rightarrow G_0$ in the simply connected algebraic group associated to $\mathfrak{g}_0$ which induces a $\mathbb{Z}$-grading

$$\mathfrak{g} = \oplus_{k \in \mathbb{Z}} \mathfrak{g}(k),$$

where $\mathfrak{g}(k) = \{ z \in \mathfrak{g} \mid \text{Ad}(\gamma(t))(z) = t^k z \text{ for all } t \in K^\times \}.$

Moreover, $\mathfrak{g}_0$ is semisimple under the adjoint $sl(2)$-action. By the explicit information on $\mathfrak{g}_1$ given in Subsection 2.2, we further observe that $\mathfrak{g}_1$ is also semisimple under the adjoint $sl(2)$-action. Now it follows that this grading satisfies all the properties (3.1)–(3.5) in Theorem 3.1 just as for semisimple Lie algebras in characteristic zero (compare with [9]).

4. Proof of Super KW Conjecture with nilpotent $p$-characters

4.1. The subalgebra $\mathfrak{m}$

Let $\mathfrak{g}$ be one of the basic classical Lie superalgebras in Section 2.2 with a non-degenerate supersymmetric even bilinear form denoted by $(\cdot, \cdot)$. Indeed, take a non-zero $x \in \mathfrak{g}(-1)_i$ for $i \in \mathbb{Z}_2$. Then it follows by (3.4) that $0 \neq [X,x] \in \mathfrak{g}(1)$. By the non-degeneracy of the pairing between $\mathfrak{g}(1)_i$ and $\mathfrak{g}(-1)_i$, there exists $y \in \mathfrak{g}(-1)_i$ with $0 \neq ([X,x],y) = ([X,x],y) = \langle x,y \rangle$.

Note that $\dim \mathfrak{g}(-1)_0$ is even. Take $\mathfrak{g}(-1)_0' \subset \mathfrak{g}(-1)_0$ to be a maximal isotropic subspace with respect to $\langle \cdot, \cdot \rangle$. It satisfies $\dim \mathfrak{g}(-1)_0' = \dim \mathfrak{g}(-1)_0/2$.

Denote $r = \dim \mathfrak{g}(-1)_1$. There is a basis $v_1, \ldots, v_r$ of $\mathfrak{g}(-1)_1$ under which the symmetric form $\langle \cdot, \cdot \rangle$ has matrix form

$$\begin{bmatrix}
0 & \cdots & 1 \\
1 & \ddots & \\
& & 0
\end{bmatrix}.$$

If $r$ is even, then take $\mathfrak{g}(-1)_1' \subset \mathfrak{g}(-1)_1$ to be the subspace spanned by $v_1, \ldots, v_{r/2}$. If $r$ is odd, then take $\mathfrak{g}(-1)_1' \subset \mathfrak{g}(-1)_1$ to be the subspace spanned by $v_1, \ldots, v_{(r-1)/2}$. Set $\mathfrak{g}(-1)' = \mathfrak{g}(-1)_0' \oplus \mathfrak{g}(-1)_1'$ and introduce the subalgebras

$$\mathfrak{m} = \bigoplus_{k \neq 2} \mathfrak{g}(k) \oplus \mathfrak{g}(-1)',$$

$$\mathfrak{m}' = \begin{cases} 
\mathfrak{m} \oplus K^\times v_{r+1}, & \text{for } r \text{ odd} \\
\mathfrak{m}, & \text{for } r \text{ even}.
\end{cases}$$
The subalgebra $m$ is $p$-nilpotent, and the linear function $\chi$ vanishes on the $p$-closure of $[m, m]$. It follows by Proposition 2.6 that $U_\chi(m)$ has the trivial module as its only irreducible module and $U_\chi(m)/N_m = K$, where $N_m$ is the Jacobson radical of $U_\chi(m)$.

Assume that $r$ is odd. The induced $U_\chi(m')$-module $V = U_\chi(m') \otimes_{U_\chi(m)} K$ is 2-dimensional, irreducible, and admits an odd automorphism of order 2 induced from $v_{r+1/2}$. By Frobenius reciprocity, it is the only irreducible $U_\chi(m')$-module. Denote by $q_d(K)$ the simple superalgebra of type $Q$ consisting of all $2d \times 2d$ matrices of the form

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

with $A$ and $B$ arbitrary $d \times d$ matrices. We summarize the above as follows.

**Proposition 4.1.** Assume that $\dim g(-1)_1$ is odd. Then $U_\chi(m')$ has a unique simple module; it is isomorphic to $V$, which is 2-dimensional and of type $Q$. Moreover, $U_\chi(m')/N_{m'}$ is isomorphic to the simple superalgebra $q_1(K)$.

We have the following commutative diagram:

$$U_\chi(m) \supset U_\chi(m') \cup N_m = N_{m'}$$

4.2. **Freeness over $U_\chi(m)$**

For a $U_\chi(g)$-module $M$ set

$$M^m = \{ v \in M | N_m \cdot v = 0 \} = \{ v \in M | N_{m'} \cdot v = 0 \}. \tag{4.2}$$

**Proposition 4.2.** Let $g$ be one of the basic classical Lie superalgebras. Then every $U_\chi(g)$-module is $U_\chi(m)$-free.

**Proof.** Since our proof is a straightforward superalgebra generalization of the proof of [18, Theorem 1.3], we only formulate the main steps below and refer to loc. cit. for details.

Recall the $Z$-grading on $g = \oplus_{k \in \mathbb{Z}} g(k)$ from Section 3. Define a decreasing filtration $\{g^k\}$ of $g$ by setting $g^k = \sum_{l \geq k} g(-l)$ with induced $\mathbb{Z}_2$-grading $g^k = g^k_0 \oplus g^k_1$ for $k \in \mathbb{Z}$. It clearly satisfies the conditions in [18, (b1)-(b6), p. 568].

First of all, we have $[g^2, N_m] \subset N_m$, and so $M^m$ is stable under $g^2$ for every $U_\chi(g)$-module $M$.

Let $y_{a,1,\ldots,y_{0,r_0}}$ and $y_{1,1,\ldots,y_{0,r_1}}$ be the respective bases for $m_0$ and $m_1$ compatible with the $Z$-grading on $g$. Define $d_{0,s} > 0$ and $d_{1,s} > 0$ by the condition $y_{0,s} \in g_{d_{0,s}}^0 \setminus g_{d_{0,s}+1}$ and $y_{1,s} \in g_{d_{1,s}}^1 \setminus g_{d_{1,s}+1}^1$, respectively. Let $\Xi = \{ a = (a_1, \ldots, a_{r_0}; \epsilon_1, \ldots, \epsilon_{r_1}) \mid 0 \leq a_s < p, \epsilon_t \in \{0, 1\} \}$. For $a \in \Xi$ set

$$|a| = \sum a_s + \sum \epsilon_t,$$

$$\text{wt } a = \sum d_{0,s} a_s + \sum d_{1,t} \epsilon_t,$$

$$y_a = (y_{0,1} - \eta(y_{0,1}))^{a_1} \ldots (y_{0,r_0} - \eta(y_{0,r_0}))^{a_{r_0}} y_{1,1}^{t_1} \ldots y_{1,r_1}^{t_1} \in U_\chi(m),$$

where $\eta \in m_0^*$ is the linear function defining the 1-dimensional representation on $m$ with $p$-character $\chi_m$.

For each pair $i, j$ denote $\Lambda(i, j) = \{ b \in \Xi \mid \text{wt } b = i \text{ and } |b| = j \}$, and consider any linear ordering $\preceq$ on $\Xi$ subject to the following condition: $a < b$ whenever either $\text{wt } a < \text{wt } b$ or
wt $a = wt b$, $|a| > |b|$. Let $M$ be a $U_\chi(g)$-module. Following [18, proof of Theorem 1.3], for each $v \in M^m$ and $b \in \Lambda(i, j)$, there exists $v_b \in M$ such that

$$y_av_a = v \quad \text{and} \quad y_av_b = 0 \quad \text{when} \quad a > b.$$  

(4.3)

Let $m$ operate in $E = \text{Hom}_K(U_\chi(m), M^m)$ by $(yf)(u) = (-1)^{|y||f|+|u|}f(uy)$ for homogeneous $y \in m$, $f \in E$, and $u \in U_\chi(m)$. Take any even linear map $\pi: M \to M^m$ such that $\pi|_{M^m} = \text{id}$ and define $\varphi: M \to E$ setting $\varphi(w)(u) = (-1)^{|u||w|}w(aw)$ for $w \in M$ and $u \in U_\chi(m)$. Clearly $\varphi$ is a homomorphism of $U_\chi(m)$-modules. By (4.3) $\varphi(v_a)(y_a) = v$ and $\varphi(v_b)(y_a) = 0$ when $a > b$ for any $v \in M^m$. Given $\psi \in E$, it can be shown by downward induction on $b$ that there exists $w \in M$ such that $\varphi(w)(y_a) = \psi(y_a)$ for all $a \geq b$. Since the elements $y_a$ form a basis for $U_\chi(m)$, the map $\varphi$ is surjective. Furthermore, $\ker(\varphi)$ is an $m$-submodule of $M$ that has zero intersection with $M^m$ because $\varphi(v)(1) = v$ for every $v \in M^m$. Then $\ker \varphi = 0$. Thus $\varphi$ is an isomorphism of $U_\chi(m)$-modules. Since $U_\chi(m)$ is a Frobenius algebra by Proposition 2.7, the $U_\chi(m)$-module $E \cong U_\chi(m)^* \otimes M^m$ is free. Hence $M$ is $U_\chi(m)$-free.

4.3. The Super KW Property with nilpotent $p$-characters

THEOREM 4.3. Let $g$ be one of the basic classical Lie superalgebras as in Section 2.2, and let $\chi \in \mathfrak{g}_0^\ast$ be nilpotent. Let $d_i = \dim g_i - \dim g_{X,i}$, $i \in \mathbb{Z}_2$. Then the dimension of every $U_\chi(g)$-module $M$ is divisible by $p^{d_0/2}d_1^{1/2}$. 

Note that $p^{d_0/2}d_1^{1/2} = \dim U_\chi(m') (\neq \dim U_\chi(m) \text{ in general}).$

Proof. By (3.5) and since $\dim g(k) = \dim g(-k)$, we have

$$\dim g - \dim g_\chi = \sum_{k \geq 2} 2\dim g(-k) + \dim g(-1).$$  

(4.4)

In particular, $\dim g(-1) \overline{1}$ and $d_1$ have the same parity. It follows now from the definition of $m$ that either (1) $\frac{d_0}{2}\mid \frac{d_1}{2} = \dim m$ when $\dim g(-1) \overline{1}$ and $d_1$ are even, or (2) $\frac{d_0}{2}\mid \frac{d_1-1}{2} = \dim m$ when $\dim g(-1) \overline{1}$ and $d_1$ are odd.

In case (1), the theorem follows immediately from Proposition 4.2.

In case (2), for each $U_\chi(g)$-module $M$, $M^m$ is a module over the superalgebra $U_\chi(m')/N_m \cong q_1(K)$, by Proposition 4.1. Since the (unique) simple module of $q_1(K)$ is 2-dimensional, $M^m$ has dimension divisible by 2. Now the isomorphism $M \cong U_\chi(m)^* \otimes M^m$ (cf. Proposition 4.2 and its proof) implies the desired divisibility.

Theorem 4.3 can be somewhat strengthened in the following form.

THEOREM 4.4. Set $\delta = \dim U_\chi(m)$ and denote by $Q_m$ the induced $U_\chi(g)$-module $U_\chi(g) \otimes_{U_\chi(m)} K_\chi$. Then $Q_m$ is a projective $U_\chi(g)$-module and

$$U_\chi(g) \cong M_\delta(W_\chi(g)^{op}), \quad \text{where} \quad W_\chi(g) = \text{End}_{U_\chi(g)}(Q_m).$$

Proof. Let $V_1, \ldots, V_k$ and $W_1, \ldots, W_l$ be all inequivalent simple $U_\chi(g)$-modules of type $M$ and of type $Q$, respectively. Let $P_i$ and $Q_j$ denote the projective cover of $V_i$ and $W_j$, respectively. By Proposition 4.2, $V_i$ and $W_j$ are free over $U_\chi(m)$. It follows by Frobenius reciprocity that

$$\dim \text{Hom}_g(Q_m, V_i) = \dim \text{Hom}_m(K_\chi, V_i) =: a_i.$$  

By Frobenius reciprocity and Proposition 2.6,

$$\dim \text{Hom}_g(Q_m, W_j) = \dim \text{Hom}_m(K_\chi, W_j) = \dim \text{Hom}_m(U_\chi(m), W_j)$$
which has to be an even number, say $2b_j$, since as a type $Q$ module $W_j$ admits an odd involution commuting with $m$. It follows that the ranks of the free $U_\chi(m)$-modules $V_i$ and $W_j$ are $a_i$ and $2b_j$, respectively. Assign

$$P = \bigoplus_{i=1}^s P_i^{a_i} \bigoplus_{j=1}^t Q_j^{b_j}.$$  

Then $P$ is projective and has the same head as $Q_m$. Therefore, there is a surjective homomorphism $\psi : P \to Q_m$.

Since $\dim V_i = \delta a_i$ and $\dim W_j = 2\delta b_j$, by Wedderburn theorem for superalgebras (cf. Kleshchev [11, Theorem 12.2.9]) the left regular $U_\chi(g)$-module is isomorphic to $P^{\delta}$. The equality of dimensions $\dim P = \dim U_\chi(g)/\delta = \dim Q_m$ implies that $\psi$ is an isomorphism. Finally,

$$U_\chi(g) \cong \text{End}_{U_\chi(g)}(U_\chi(g))^{\text{op}} \cong \text{End}_{U_\chi(g)}(P^{\delta})^{\text{op}} \cong (M_\delta(\text{End}_{U_\chi(g)}(P)))^{\text{op}} \cong M_\delta(W_\chi(g)^{\text{op}}).$$

This completes the proof of the theorem. \qed

**Remark 4.5.** In the case $g$ is a Lie algebra (that is, $g_0 = 0$), we recover a theorem of Premet [13, Theorem 2.3(i),(ii)] with somewhat modified arguments that avoid the use of support variety machinery.

The algebra $W_\chi(g)$ has a counterpart over $\mathbb{C}$, which is usually referred to as a finite $W$-superalgebra in the math physics literature.

**Remark 4.6.** Theorems 4.3 and 4.4 remain valid when $g$ is a direct sum of basic classical Lie superalgebras.

5. A reduction from general to nilpotent $p$-characters

In this section, we will establish a Morita equivalence which reduces the case of a general $p$-character to a nilpotent one, completing the proof of the Super KW Conjecture for $g$.

5.1. An equivalence of categories

Let $g$ be a basic classical Lie superalgebra as in Section 2.2.

Let $\chi = \chi_s + \chi_n$ be the Jordan decomposition of $\chi \in g_0^*$ (we regard $\chi \in g^*$ by letting $\chi(g_1) = 0$). Under the isomorphism $g_0^* \cong g_0$ induced by the non-degenerate bilinear form $(\cdot, \cdot)$ on $g_0$, this can be identified with the usual Jordan decomposition $s + n$ on $g_0$. Take a Cartan subalgebra $h$ of $g$ which contains $s$, and recall that $g$ admits a root space decomposition (cf. [10])

$$g = h \bigoplus_{\alpha \in \Phi} g_\alpha.$$  

Then it follows that $g_s =: l = l_0 \oplus l_1$ also has a root space decomposition

$$l = h \bigoplus_{\alpha \in \Phi(l)} g_\alpha,$$

where $\Phi(l) = \{\alpha \mid \alpha(s) = 0\}$. 
It is a well-known super phenomenon that all the systems of simple roots, the systems of positive roots, or Borel subalgebras of \( \mathfrak{g} \) are not equivalent under the Weyl group \( W \). An explicit list of non-\( W \)-equivalent systems of positive roots can be found in Kac [10, pp. 51–53].

The following proposition is proved by case-by-case calculations, which is completely elementary yet tedious and thus omitted. From the detailed calculation we find that the following system of simple roots for \( F(4) \) up to \( W \)-equivalence

\[
\left\{-\delta, \frac{1}{2}(\delta - \varepsilon_1 - \varepsilon_2 + \varepsilon_3), \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\right\}
\]

is missing from Kac’s list. (We learned that this was also noticed by Serganova.)

**Proposition 5.1.** Let \( l = \mathfrak{g}_s \) with \( s \) in a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \). There exists a system \( \Pi \) of simple roots of \( \mathfrak{g} \) such that \( \Pi \cap \Phi(l) \) is a system of simple roots for \( \Phi(l) \). In particular \( l \) is always a direct sum of basic classical Lie superalgebras.

Let \( \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \) be the Borel subalgebra associated to \( \Pi \). Then we can define a parabolic subalgebra \( \mathfrak{p} = l + \mathfrak{b} = l \oplus \mathfrak{u} \), where \( \mathfrak{u} \) denotes the nilradical of \( \mathfrak{p} \).

Note that \( \chi(u) = 0 \) since \( s + n \in \mathfrak{g}_x \subset \mathfrak{g}_x = l \), and \( \chi(u) = (s + n, u) \subset (l, u) = 0 \). Also note that \( \chi_s = 0 \) and hence \( \chi = \chi_{n|} \) is nilpotent. Thus any \( U_\chi(l) \)-module can be regarded as a \( U_\chi(p) \)-module with a trivial action of \( u \). Here and below, by abuse of notation, we will use the same letter \( \chi \) for its restrictions on \( p, l, \) or \( u \).

Given an associative \( k \)-superalgebra \( A \), \( A\text{-mod} \) denotes the category of finite-dimensional \( A \)-(super)modules with even morphisms, which turns out to be an abelian category. One can easily switch back and forth by a parity functor between this category and the category of finite-dimensional \( A \)-(super)modules with arbitrary morphisms (cf., for example, [1]).

**Theorem 5.2.** Let \( \mathfrak{g} \) be a basic classical Lie superalgebra as in Section 2.2. Let \( \chi = \chi_s + \chi_n \in \mathfrak{g}_0 \) be a Jordan decomposition, let \( l = \mathfrak{g}_x \), and let \( \mathfrak{p} = l \oplus \mathfrak{u} \) as above. Then \( -u : U_\chi(\mathfrak{g})\text{-mod} \to U_\chi(l)\text{-mod} \) is an equivalence of categories, and its inverse is given by \( U_\chi(l)\text{-mod} \to U_\chi(\mathfrak{g})\text{-mod} \). Moreover, \( U_\chi(\mathfrak{g}) \) and \( U_\chi(l) \) are Morita equivalent.

The above theorem is a super analog of a theorem of Friedlander and Parshall [6, Theorem 3.2] which was in turn built on the earlier work of Kac and Weisfeiler [20]. By the same argument as in [6, pp. 1068], we reduce the proof of Theorem 5.2 to the following theorem, which is a super analogue of the main theorem of [20] and [6, Theorem 8.5].

**Theorem 5.3.** Retain the above notation. Then for any irreducible \( U_\chi(\mathfrak{g}) \)-module \( M \), \( M^u \) is an irreducible \( U_\chi(p) \)-module and the natural map

\[
U_\chi(\mathfrak{g}) \otimes_{U_\chi(p)} M^u \longrightarrow M
\]

is an isomorphism. Also, \( M \) is a projective \( U_\chi(u) \)-module.

5.2. **Proof of Theorem 5.3**

The proof of Theorem 5.3 follows the same strategy as in the Lie algebra case given in [6, Section 8], with a few modifications. In the following, we only formulate and establish the parts that differ more substantially, while omitting the parts that are completely analogous to the Lie algebra case and referring to loc. cit. for details.
The proof in [6, Section 8] is based on four lemmas (Lemmas 8.1, 8.2, 8.3, and 8.4 therein), and it can be literally copied to the super set-up once the super analogues of the four lemmas are formulated and established. The super analogs of Lemmas 8.1 and 8.3 are obtained in a straightforward manner and thus skipped. On the other hand, the super analogs of the remaining lemmas need some extra care.

The complication in the following super analog of [6, Lemma 8.2] arises from the fact that there are three types of roots in \( g \) as follows:

1. \( \Phi \cap \mathbb{Q}\delta = \{ \pm \delta \} \) with \( \delta \) even;
2. \( \Phi \cap \mathbb{Q}\delta = \{ \pm \delta \} \) with \( \delta \) odd; and
3. \( \Phi \cap \mathbb{Q}\delta = \{ \pm \delta, \pm 2\delta \} \) with \( \delta \) odd and \( 2\delta \) even.

They correspond to three rank-one Lie superalgebras \( \mathfrak{sl}(2), \mathfrak{sl}(1|1), \) and \( \mathfrak{osp}(1|2) \), respectively. For latter purpose, for such a \( \delta \), we shall denote

\[
\delta^* = \begin{cases} 
\delta, & \text{in case (i) and (ii)} \\
\{ \delta, 2\delta \}, & \text{in case (iii)}.
\end{cases}
\]

**Lemma 5.4.** Let \( \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} \) be a parabolic subalgebra of a basic classical Lie superalgebra \( \mathfrak{g} \) containing a Borel subalgebra \( \mathfrak{b} \). Assume that the commutator subalgebra \( \mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}] \) of \( \mathfrak{l} \) is isomorphic to one of the following:

1. \( \mathfrak{sl}(2) \) with standard basis \( \{ e, f, h \} \) such that \( \chi(e) = 0 = \chi(f) \) and \( \chi(h) \neq 0 \);
2. \( \mathfrak{sl}(1|1) \) with basis

\[
\begin{align*}
X &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
y &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\
h &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{align*}
\]

such that \( \chi(h) \neq 0 \);
3. \( \mathfrak{osp}(1|2) \) with standard basis \( \{ e, f, h; E, F \} \) (see Section 6) such that \( \chi(e) = 0 = \chi(f) \) and \( \chi(h) \neq 0 \).

Let \( V \) be a finite dimensional \( U_{\chi}(\mathfrak{b}) \)-module upon which \( e \) and \( E \), \( X \) act trivially. Denote by \( I = U_{\chi}(\mathfrak{p}) \otimes_{U_{\chi}(\mathfrak{b})} - : U_{\chi}(\mathfrak{b}) \text{-mod} \to U_{\chi}(\mathfrak{p}) \text{-mod} \) the induction functor. Then \( I(V)^{I_{\chi}} = V \) and \( I(V)^{E} = V, \ I(V)^{X} = V \).

**Proof.** The key here as in the proof of [6, Lemma 8.2] is to show that \( U_{\chi}(\mathfrak{l}') \) is a semisimple superalgebra. This is well known for the \( \mathfrak{sl}(2) \) case, and we will prove it for the \( \mathfrak{osp}(1|2) \) case in Section 6 (see Proposition 6.1 for a more precise statement).

We now consider the case when \( \mathfrak{l}' \) is isomorphic to \( \mathfrak{sl}(1|1) \) with \( \chi(h) \neq 0 \). For \( \lambda \) satisfying \( \lambda^p - \lambda - \chi(h)^p = 0 \), the baby Verma module

\[
Z_{\chi}(\lambda) = U_{\chi}(\mathfrak{sl}(1|1)) \otimes_{U_{\chi}(KX+Kh)} K_{\lambda}
\]

is 2-dimensional with a basis \( \{ v_{\lambda} = 1 \otimes 1, Yv_{\lambda} \} \), where \( K_{\lambda} = K\mathfrak{l}_{\lambda} \) denotes the 1-dimensional representation of \( KX + Kh \) with \( X.1_{\lambda} = 0, h.1_{\lambda} = \lambda 1_{\lambda} \). The action of \( \mathfrak{sl}(1|1) \) is given by

\[
Xv_{\lambda} = 0, \quad Yv_{\lambda} = \lambda v_{\lambda}, \quad hv_{\lambda} = \lambda v_{\lambda}, \quad hYv_{\lambda} = \lambda Yv_{\lambda}, \quad YYv_{\lambda} = 0.
\]

Observe that each \( Z_{\chi}(\lambda) \) is irreducible of type \( M \), and \( Z_{\chi}(\lambda) \) are pairwise non-isomorphic (there are \( p \) of them in total). Since \( \dim U_{\chi}(\mathfrak{sl}(1|1)) = 4p \), this forces each \( Z_{\chi}(\lambda) \) to be projective and \( U_{\chi}(\mathfrak{sl}(1|1)) \) to be semisimple.

Thus each \( U_{\chi}(\mathfrak{l}') \)-module is projective and hence projective as a module over its subalgebra generated by \( X \) and by \( E \) in the \( \mathfrak{osp}(1|2) \) case. Hence \( I(V)^{X} \) and \( I(V)^{E} \) have dimension equal to \( \dim I(V)/2 \) and \( \dim I(V)/2p \), which coincide with \( \dim V \).
We now go back to the notation for \( \chi, \mathfrak{b}, \) and \( \mathfrak{p} \), as in Section 5.1. Let \( \mathfrak{p} = l \oplus \mathfrak{u} \) be the parabolic subalgebra opposite to \( \mathfrak{p} \). Denote

\[
\Phi^+_s := \{ \text{roots whose root vectors lie in } \mathfrak{n} \cap l \}, \quad \Phi^-_s = -\Phi^+_s;
\]
\[
\Phi_u := \{ \text{roots whose root vectors lie in } \mathfrak{u} \}, \quad \Phi^-_u = -\Phi^-_u.
\]

The following is a super analog of [6, Lemma 8.4] with a different proof. The complication in the superalgebra set-up arises from the fact that there are odd roots and the longest element in the Weyl group \( W \) does not send a system of positive roots to its opposite. We shall need the notion of odd reflections (which has been used by Serganova and others in various situations; cf. [16, 17] for references).

**Lemma 5.5.** We can enumerate \( \Phi_u = \{ \delta_1^+, \ldots, \delta_t^+ \} \) as a sequence of singletons or pairs of roots so that for each \( i \),

\[
\Phi^+_i := \Phi^+_s \cup \{ -\delta^+_1, \ldots, -\delta^+_{i-1}, \delta^+_i, \ldots, \delta^+_t \}
\]

is a system of positive roots for \( \Phi \) in which \( \delta_i \) is a simple root. Moreover, for each \( i, \Psi_i := \{ -\delta^+_1, \ldots, -\delta^+_i \} \) is a closed subsystem of \( \Phi \) normalized by \( \Phi^+_s \).

**Proof.** For a root \( \delta \) in a set of simple roots \( \tilde{\Pi} \) associated to the system of positive roots \( \tilde{\Phi}^+ \), let \( r_\delta : \Phi \to \Phi \) be the (even or odd) reflection associated to \( \alpha \). (see [16] for the basic properties of odd reflections). If \( 2\delta \) is a root, then \( r_\delta \) is by definition the even reflection \( r_{2\delta} \). It is known that \( r_\delta \tilde{\Phi}^+ \) is a system of positive roots, \( -\delta^* \in r_\delta \tilde{\Phi}^+, \) and \( r_\delta \tilde{\Phi}^+ \cap \tilde{\Phi}^+ = \tilde{\Phi}^+ \backslash \delta^* \).

Denote by \( \Pi_0 \) the set of simple roots associated to the system of positive roots \( \Phi^+_0 = \Phi^-_s \cup \Phi_u \). Pick \( \delta_1 \in \Pi_0 \cap \Phi_u \). We proceed inductively. Assume that we have defined \( \Pi_0, \ldots, \Pi_{i-1} \) and have chosen \( \delta_1 \in \Pi_0 \cap \Phi_u, \ldots, \delta_i \in \Pi_{i-1} \cap \Phi_u \). Set \( \Pi_i = r_{\delta_i}(\Pi_{i-1}) \) and define \( \Phi^+_i \) to be the positive system determined by \( \Pi_i \). Then we have

\[
\Phi^+_i = \Phi^+_s \cup \{ -\delta^+_1, \ldots, -\delta^+_{i-1}, \delta^+_i \}
\]

It follows that

\[
\Phi^+_i \cap \Phi^-_0 = (\Phi^+_i \cap \Phi^-_s) \bigcup \{ \delta_i^+ \},
\]

and thus \( |\Phi^+_i \cap \Phi^-_s| \) is 1 or 2 less than \( |\Phi^+_i \cap \Phi^-_s| \). Repeating this process, we obtain \( \Phi^+_i \) and \( \Psi_i := \Phi^+_i \cap \Phi^-_0 \) in (1). Until \( \Phi^+_t \cap \Phi^-_0 = \emptyset \).

It follows from \( \Psi_i = \Phi^+_i \cap \Phi^-_u \) that \( \Psi \) is a closed subsystem of \( \Phi \). Given \( \alpha \in \Phi^+_s \), so that \( \alpha - \delta_j \) (or \( \alpha - 2\delta_j \)) is a root, then \( \alpha - \delta_j \) (or \( \alpha - 2\delta_j \)) lies in \( \Phi^+_r \); moreover it lies in \( \Psi_i = \Phi^+_r \cap \Phi^-_0 \) since \( \Phi^-_u \) is normalized by \( \Phi^+_r \).

5.3. **Proof of the Super KW Property for \( \mathfrak{g} \)**

Now we are in a position to prove the Super KW Property with arbitrary \( p \)-characters.

**Theorem 5.6** (Super Kac–Weisfeiler Property). *Let \( \mathfrak{g} \) be a basic classical Lie superalgebra as in Section 2.2, and let \( \chi \in \mathfrak{g}^*_0 \). Let \( d_i = \dim \mathfrak{g}_i - \dim \mathfrak{g}_{\chi,i}, i \in \mathbb{Z}_2. \) Then the dimension of every \( U_\chi(\mathfrak{g}) \)-module \( M \) is divisible by \( p^{d_0/2}(d_1/2) \).

**Proof.** Observe that

\[
\dim \mathfrak{g} - \dim \mathfrak{g}_\chi = \dim \mathfrak{g} - \dim \mathfrak{l}_\chi \leq 2 \dim \mathfrak{u} + (\dim (1 - \dim \mathfrak{l}_\chi)).
\]
The theorem is now an easy consequence of Remarks 2.5 and 4.6, and Theorems 4.3 and 5.3.

**Corollary 5.7.** Assume that $\chi$ is regular semisimple (that is, $g_\chi$ is a Cartan subalgebra $\mathfrak{h}$ of $g$). Then $U_\chi(g)$ is a semisimple superalgebra. Furthermore, the baby Verma modules $Z_\chi(\lambda)$ with $\lambda$ such that $\lambda(h)p - \lambda(h) = \chi(h)p$ for $h \in \mathfrak{h}$ form a complete list of simple $U_\chi(g)$-modules.

**Proof.** By Theorem 5.2, $U_\chi(g)$ is Morita equivalent to $U_\chi(\mathfrak{h})$ which is semisimple by the assumption of regular semisimplicity on $\chi$. Hence $U_\chi(g)$ is a semisimple superalgebra. Note that $\dim g - \dim g_\chi = 2 \dim \mathfrak{n}_0|2 \dim \mathfrak{n}_1$. By Theorem 5.6, the $g$-module $Z_\chi(\lambda)$ is simple, since the dimension of $Z_\chi(\lambda)$ is $p^{\dim \mathfrak{n}_0}2^{\dim \mathfrak{n}_1}$. For different $\lambda$, $Z_\chi(\lambda)$ are non-isomorphic by high weight consideration.

**Corollary 5.8.** Let $g = \mathfrak{osp}(1|2n)$ be the Lie superalgebra of type $B(0, n)$. Let $\chi$ be regular nilpotent (that is, the corresponding $X \in g_\bar{0}$ is a regular nilpotent element). Then, the baby Verma modules of $U_\chi(g)$ are simple.

**Proof.** It is well known that $\dim g_\bar{0} = n$, which is the rank of $g_\bar{0} = \mathfrak{sp}(2n)$. We claim that $\dim g_{\bar{0}, \bar{1}} = 1$. Indeed, $g_{\bar{1}}$ is the natural $\mathfrak{sp}(2n)$-module, and $X$ can be regarded as a matrix of corank 1. Therefore, we have $\dim m' = \dim m^-$ (recall the subalgebra $m'$ from Section 4.1). Now having dimension equal to $\dim U_\chi(m')$, the baby Verma modules of $U_\chi(g)$ must be simple by Theorem 4.3.

6. The modular representations of $\mathfrak{osp}(1|2)$

In this section, we give a complete description of the modular representation theory of $g = \mathfrak{osp}(1|2)$, with many similarities to the well-known $\mathfrak{sl}(2)$ case [6] (also cf. [8, Section 5]). It turns out that there are no projective simple $U_\chi(g)$-modules in contrast to the $\mathfrak{sl}(2)$-case.

6.1. Lie superalgebra $\mathfrak{osp}(1|2)$

Recall that $g = \mathfrak{osp}(1|2)$ consists of $3 \times 3$ matrices in the following $(1|2)$-block form

\[
\begin{bmatrix}
0 & v & u \\
u & a & b \\
-v & c & -a
\end{bmatrix}
\]

with $a, b, c, u, v \in K$. The even subalgebra is generated by

\[
e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

A basis of $g_{\bar{1}}$ is given by

\[
E = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.
\]

The adjoint $g_{\bar{0}}(\cong \mathfrak{sl}(2))$-module $g_{\bar{1}}$ is the 2-dimensional natural module.
We collect the commutation relations of these basis elements below:

\[ [h, E] = E, \quad [h, F] = -F \]
\[ [e, E] = 0, \quad [e, F] = -E \]
\[ [f, E] = -F, \quad [f, F] = 0 \]
\[ [E, E] = 2e, \quad [E, F] = h, \quad [F, F] = -2f. \]

It is easy to check that the relations \( e^{[p]} = 0 = f^{[p]} \) and \( h^{[p]} = h \) provide a restricted structure on the Lie superalgebra \( g \).

Since \( g_0 \cong sl(2) \), there are three coadjoint orbits of \( g_0^* \) with the following representatives:

(i) regular nilpotent: \( \chi(e) = \chi(h) = 0 \) and \( \chi(f) = 1 \);
(ii) regular semisimple: \( \chi(e) = 0 = \chi(f) \) and \( \chi(h) = a^p \) for some \( a \in K^* \);
(iii) restricted: \( \chi(e) = \chi(f) = \chi(h) = 0 \).

6.2. The baby Verma modules of \( \mathfrak{osp}(1|2) \)

Fix \( \chi \in g_0^* \) such that \( \chi(e) = 0 \) and denote the Borel subalgebra \( b := K e + K h + K E \) with Cartan subalgebra \( h = K h \). Recall from Section 2.4 the baby Verma module \( Z_\chi(\lambda) = U_\chi(g) \otimes_{U_\chi(b)} K_\lambda \) with \( \lambda^p - \lambda = \chi(h)^p \). The set \( \{ v_i := F^i \otimes 1 | 0 \leq i < 2p \} \) is a basis for \( Z_\chi(\lambda) \) with \( g \)-action given by

\[
\begin{align*}
hv_i &= (\lambda - i)v_i, \\
fv_i &= \begin{cases} 
- v_{i+2} & 0 \leq i < 2p - 2, \\
\chi(f)^p v_0 & i = 2p - 2, \\
\chi(f)^p v_1 & i = 2p - 1.
\end{cases} \\
ev_i &= \begin{cases} 
-i 2 (\lambda + 1 - \frac{i}{2}) v_{i-2} & \text{if } i \text{ is even}, \\
-i 2 (\lambda - \frac{i}{2}) v_{i-2} & \text{if } i \text{ is odd}.
\end{cases} \\
Fv_i &= \begin{cases} 
v_{i+1} & 0 \leq i < 2p - 1, \\
-\chi(f)^p v_0 & i = 2p - 1.
\end{cases} \\
E v_i &= \begin{cases} 
-\frac{i}{2} v_{i-1} & \text{if } i \text{ is even}, \\
(\lambda - \frac{i}{2}) v_{i-1} & \text{if } i \text{ is odd}.
\end{cases}
\end{align*}
\]

(6.1) (6.2) (6.3) (6.4) (6.5)

Here we use the convention that \( v_i = 0 \) when \( i < 0 \).

Let \( M \) be an irreducible \( U_\chi(g) \)-module. Note that \( \chi(e) = 0 \) and \( E^2 = e \) so \( E^{2p} = e^p = 0 \). Thus the set \( \{ m \in M | E \cdot m = 0 = e \cdot m \} \), which is \( K h \)-stable, is non-zero. Hence there exists \( 0 \neq m_0 \in M \) such that \( e \cdot m_0 = 0 = E \cdot m_0 \) and \( h \cdot m_0 = \lambda m_0 \) for some \( \lambda \in K \), which satisfies \( \lambda^p - \lambda = \chi(h)^p \). Then by the Frobenius reciprocity there exists an epimorphism of \( g \)-modules: \( Z_\chi(\lambda) \rightarrow M \). Thus by Remark 2.5, every simple module appears as a homomorphic image of some baby Verma module \( Z_\chi(\lambda) \).

6.3. The regular semisimple case

In this case \( \chi(h) \neq 0 \), so \( \lambda \notin F_p \) for those satisfying \( \lambda^p - \lambda = \chi(h)^p \), where \( F_p \) denotes the finite field of \( p \) elements. It follows from (6.1–6.5) that the only vectors annihilated by \( E \) (and \( e \)) are scalar multiples of \( v_0 \). Since each non-zero submodule (either graded or non-graded) of \( Z_\chi(\lambda) \) contains a non-zero vector killed by \( E \), \( Z_\chi(\lambda) \) is irreducible and of type \( M \). By Section 6.2 the baby Verma modules \( Z_\chi(\lambda) \) with \( \lambda^p - \lambda = \chi(h)^p \) are all the irreducibles and pairwise non-isomorphic. Now each \( Z_\chi(\lambda) \) is of dimension \( 2p \) and \( \dim U_\chi(g) = 4p^3 \), the algebra \( U_\chi(g) \) has
to be isomorphic to the semisimple superalgebra \( \oplus_1 M_{2p}(K) \) by dimension counting, where \( M_d(K) \) denotes the simple algebra of all \( d \times d \) matrices over \( K \). Summarizing, we have the following proposition.

**Proposition 6.1.** Let \( \mathfrak{g} = \mathfrak{osp}(1|2) \), and let \( \chi \in \mathfrak{g}_0^* \) be regular semisimple with \( \chi(e) = \chi(f) = 0 \). Then

(i) The algebra \( U_\chi(\mathfrak{g}) \) is semisimple and isomorphic to the algebra \( \oplus_1 M_{2p}(K) \).

(ii) The algebra \( U_\chi(\mathfrak{g}) \) has \( p \) distinct isomorphism classes of irreducible modules, each represented by some \( Z_\chi(\lambda) \) with \( \lambda^p - \lambda = \chi(h)^p \). The modules \( Z_\chi(\lambda) \) are of type \( M \).

6.4. The regular nilpotent case

In this case, \( \chi(h) = 0 \) and \( \chi(f) = 1 \). The \( \lambda \) satisfying the equation \( \lambda^p - \lambda = 0 \) lies in \( \mathbb{F}_p = \{0, 1, \ldots, p - 1\} \). We observe that

\[
\{ v \in Z_\chi(\lambda)|E \cdot v = 0 \} = K v_0 \oplus K v_{2\lambda+1}.
\]

Assume \( \lambda \neq (p - 1)/2 \) first. Note in this case \( h \) acts on \( v_0 \) and \( v_{2\lambda+1} \) with different eigenvalues. Any (graded or non-graded) submodule of \( Z_\chi(\lambda) \) contains a \( Kh \)-stable vector killed by \( E \), and hence either \( v_0 \) or \( v_{2\lambda+1} \). However, since \( v_0 = -K^{2p-2\lambda-1}v_{2\lambda+1} \), the submodule must be equal to \( Z_\chi(\lambda) \). Therefore, \( Z_\chi(\lambda) \) is irreducible and of type \( M \).

Now suppose that \( \lambda = (p - 1)/2 \), and so \( 2\lambda + 1 = p \). In this case, \( v_0 \) and \( v_{2\lambda+1} \) have opposite \( Z_2 \)-parities but identical \( h \)-eigenvalues. It follows that \( Z_\chi(\lambda) \) is irreducible of type \( Q \). Note that \( Z_\chi(\lambda) \) contains two \( p \)-dimensional non-graded simple submodules \( Z_\chi((p - 1)/2)^+ = K\{v_i + \sqrt{-1}v_{p+i}|0 \leq i < p\} \) and \( Z_\chi((p - 1)/2)^- = K\{v_i - \sqrt{-1}v_{p+i}|0 \leq i < p\} \). There is an odd \( \mathfrak{g} \)-module involution of \( Z_\chi(\lambda) \) which exchanges \( v_i \) and \( v_{i+p} \) for \( 0 \leq i \leq p - 1 \).

A second look at the space of vectors annihilated by \( E \) shows that \( Z_\chi(\mu) \) is isomorphic to \( Z_\chi(\lambda) \) if and only if \( \mu = \lambda \) or \( \mu = \lambda^* \), where we denote \( \lambda^* = p - \lambda - 1 \). Therefore, a complete list of simple \( \mathfrak{g} \)-modules consists of \( Z_\chi(\lambda) \), where \( 0 \leq \lambda \leq (p - 1)/2 \).

From now on, let \( 0 \leq \lambda, \mu \leq (p - 1)/2 \). By the exactness of the functor \( U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b})} - \), the number of composition factors isomorphic to \( Z_\chi(\lambda) \) of the left regular \( U_\chi(\mathfrak{g}) \)-module equals the number of composition factors isomorphic to \( K_\lambda \) or \( K_{\lambda^*} \) in the left regular \( U_\chi(\mathfrak{b}) \)-module. This number is easily seen to equal \( 2p \) if \( \lambda = (p - 1)/2 \) (and so \( \lambda^* = \lambda \)) and \( 4p \) otherwise.

Denote by \( P_\chi(\lambda) \) the projective cover of \( Z_\chi(\lambda) \). We claim that the module \( Z_\chi(\lambda) \) for each \( \lambda \) is not projective. Otherwise, \( Z_\chi(\lambda) \) cannot appear as a section of any \( P_\chi(\mu) \) with \( \mu \neq \lambda \) and it appears once as a section of \( P_\chi(\lambda) = Z_\chi(\lambda) \). For \( \lambda \neq (p - 1)/2 \), this would imply that the number of composition factors isomorphic to \( Z_\chi(\lambda) \) (which is of type \( M \)) in \( U_\chi(\mathfrak{g}) \) equals \( \dim Z_\chi(\lambda) = 2p \). This contradicts \( 4p \) as claimed in case for \( \lambda \neq (p - 1)/2 \) in the preceding paragraph, and thus the module \( Z_\chi(\lambda) \) for \( \lambda \neq (p - 1)/2 \) is not projective. Now if \( Z_\chi((p - 1)/2) \) (which is of type \( Q \)) were projective, then the number of composition factors isomorphic to \( Z_\chi((p - 1)/2) \) in \( U_\chi(\mathfrak{g}) \) equals \( \frac{1}{2} \dim Z_\chi((p - 1)/2) = p \), by the Wedderburn theorem for superalgebras (cf. [11, Theorem 12.2.9]), which contradicts \( 2p \) as claimed in the preceding paragraph.

Note that \( U_\chi(\mathfrak{g}) \) is a symmetric algebra by Proposition 2.7. Then the head and socle of \( P_\chi(\lambda) \) for each \( \lambda \) must be two distinct copies of \( Z_\chi(\lambda) \), and hence \( \dim P_\chi(\lambda) \geq 4p \). Now \( \dim P_\chi(\lambda) = 4p \) for each \( \lambda \) thanks to the following calculation:

\[
4p^3 = \dim U_\chi(\mathfrak{g}) = \sum_{0 \leq \lambda < \frac{p - 1}{2}} 2p \cdot \dim P_\chi(\lambda) + p \cdot \dim P_\chi\left(\frac{p - 1}{2}\right) \geq (p - 1)/2 \cdot 2p \cdot 4p + p \cdot 4p = 4p^3.
\]
For \( \lambda \neq (p - 1)/2 \), the endomorphism algebra of the module \( P_\lambda(\lambda) \) is a 2-dimensional local algebra with basis \( \{Id, \pi\} \), where \( \pi \) is the projection of \( P_\lambda(\lambda) \) to its socle, that is,

\[
\text{End}_{U_\chi(\mathfrak{g})}(P_\lambda(\lambda)) \cong K[x]/(x^2).
\]

For \( \lambda = (p - 1)/2 \), the endomorphism algebra \( \text{End}_{U_\chi(\mathfrak{g})}(P_\lambda((p - 1)/2)) \) is \((2,2)\)-dimensional with basis \( \{Id, \pi \text{ (even)}; J', \pi \circ J' \text{ (odd)}\} \), where \( \pi \) is the projection of \( P_\lambda((p - 1)/2) \) to its socle and \( J' \) the lift of the odd automorphism of \( Z\chi((p - 1)/2) \). We have the following isomorphisms of algebras:

\[
\text{End}_{U_\chi(\mathfrak{g})}(P_\lambda \left( \frac{p - 1}{2} \right)) \cong K[x]/(x^2) \otimes q_1(K) \cong q_1(K[x]/(x^2)).
\]

Finally set \( T = \oplus_{0 \leq \lambda < (p - 1)/2} P_\lambda((p - 1)/2) \). Then the left regular module \( U_\chi(\mathfrak{g}) \) is isomorphic to \( T^p \) and

\[
U_\chi(\mathfrak{g}) \cong \text{End}_{U_\chi(\mathfrak{g})}(U_\chi(\mathfrak{g}))^{op} \cong \text{End}_{U_\chi(\mathfrak{g})}(T^p)^{op} \cong (M_\mathfrak{p}(\text{End}_{U_\chi(\mathfrak{g})}(T)))^{op}
\]

\[
\cong \left( \oplus_{0 \leq \lambda < (p - 1)/2} M_2p(K[x]/(x^2)) \oplus M_\mathfrak{p}(q_1(K[x]/(x^2))) \right)^{op}
\]

\[
\cong \left( \oplus_{0 \leq \lambda < (p - 1)/2} M_2p(K[x]/(x^2)) \oplus q_\mathfrak{p}(K[x]/(x^2)) \right)^{op}.
\]

Summarizing, we have proved the following proposition.

**Proposition 6.2.** Let \( \mathfrak{g} = \mathfrak{osp}(1|2) \), and let \( \chi \in \mathfrak{g}_0^+ \) be regular nilpotent with \( \chi(e) = \chi(h) = 0 \). Then

(i) The superalgebra \( U_\chi(\mathfrak{g}) \) has \((p + 1)/2\) isomorphism classes of irreducible modules, that is, \( Z_\chi(\lambda) \) for \( \lambda = 0, 1, \ldots, (p - 1)/2 \).

(ii) For \( \lambda \in \mathbb{F}_p \), the module \( Z_\chi(\lambda) \) is isomorphic to \( Z_\chi(p - \lambda - 1) \), and there is no other isomorphism among the baby Verma modules.

(iii) The module \( Z_\chi(\lambda) \) is of type \( M \) for \( \lambda \neq (p - 1)/2 \); and \( Z_\chi((p - 1)/2) \) is of type \( Q \).

(iv) Each projective cover \( P_\lambda(\lambda) \) is a self-extension of \( Z_\chi(\lambda) \).

(v) As algebras, \( U_\chi(\mathfrak{g})^{op} \cong \oplus_{0 \leq \lambda < (p - 1)/2} M_2p(K[x]/(x^2)) \oplus q_\mathfrak{p}(K[x]/(x^2)) \).

6.5. **The restricted case**

In this case \( \chi = 0 \), and \( U_\chi(\mathfrak{g}) = U_0(\mathfrak{g}) \) is the restricted enveloping superalgebra. For \( \lambda \in \mathbb{F}_p = \{0, \ldots, p - 1\} \), let \( L(\lambda) \) be the \( \mathfrak{g} \)-module with basis \( v_0, \ldots, v_{2\lambda} \) and with the action given by formulas (6.1), (6.3), (6.5) and

\[
f v_i = -v_{i+2}, \quad F v_i = v_{i+1}.
\]

Each module \( L(\lambda) \) is irreducible and of type \( M \). Dropping the subscript \( \chi = 0 \), we shall denote by \( Z(\lambda) \) and \( P(\lambda) \) the baby Verma module and projective cover of \( L(\lambda) \), respectively, for each \( \lambda \in \mathbb{F}_p \).

It is straightforward to verify that the baby Verma module \( Z(\lambda) \) has two composition factors \( L(\lambda) \) and \( L(p - \lambda - 1) \). Indeed, \( Z(\lambda) \) has a unique proper submodule generated by \( v_{2\lambda + 1} \), where \( E.v_{2\lambda + 1} = 0 \) by (6.5). This submodule is simple and is isomorphic to \( L(p - \lambda - 1) \).

The general results by Holmes and Nakano [7] apply in our set-up, since all the simple modules \( L(\lambda) \) are of type \( M \). In particular, by [7, Theorems. 4.5 and 5.1] the projective cover \( P(\lambda) \) of \( L(\lambda) \) has a baby Verma filtration, and for any \( \lambda, \mu \in \mathbb{F}_p \) one has the Brauer type reciprocity

\[
(P(\lambda) : Z(\mu)) = [Z(\mu) : L(\lambda)],
\]

where \( (P(\lambda) : Z(\mu)) \) is the multiplicity of \( Z(\mu) \) appearing in the baby Verma filtration of \( P(\lambda) \), and \( [Z(\mu) : L(\lambda)] \) is the multiplicity of \( L(\lambda) \) in a composition series of \( Z(\mu) \). It follows by the
discussion on the composition factors of $Z(\mu)$ that $(P(\lambda) : Z(\mu)) = 1$ for $\mu = \lambda$ or $p - \lambda - 1$, and is 0 otherwise.

In summary, we have proved the following theorem.

**Proposition 6.3.** Let $\mathfrak{g} = \mathfrak{osp}(1|2)$. Then

(i) The algebra $U_0(\mathfrak{g})$ has $p$ isomorphism classes of irreducible modules $L(\lambda)$ for $\lambda = 0, \ldots, p - 1$. Moreover, $L(\lambda)$ has dimension $2\lambda + 1$.

(ii) For each $\lambda$, $Z(\lambda)$ has two composition factors: $L(\lambda)$ and $L(p - \lambda - 1)$.

(iii) For each $\lambda$, the projective cover $P(\lambda)$ has a baby Verma filtration with $Z(p - \lambda - 1)$ and $Z(\lambda)$ as subquotients.

Clearly, Propositions 6.1, 6.2, and 6.3 fit well with the Super KW Property established in Theorem 5.6.

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