Measurable centres in convolution semigroups

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Abstract
In a convolution semigroup over a locally compact group, measurability of the translation by a fixed element implies continuity. In other words, the measurable centre coincides with the topological centre.

1 Overview
For a topological group $G$, let $rG$ denote $G$ with its right uniformity, and $\text{LUC}(G)$ the space of bounded uniformly continuous functions on $rG$. The norm dual $\text{LUC}(G)^*$ of $\text{LUC}(G)$ with convolution $\ast$ is a Banach algebra of some importance in harmonic analysis. A useful tool for investigating the structure of $\text{LUC}(G)^*$ is its topological centre

$$\Lambda(\text{LUC}(G)^*) = \{ m \in \text{LUC}(G)^* \mid \text{the mapping } n \mapsto m \ast n \text{ is weak}^* \text{ continuous on } \text{LUC}(G)^* \}.$$

The space $\mathcal{M}_\tau(G)$ of (finite signed) Radon measures on $G$ naturally embeds in $\text{LUC}(G)^*$: A measure $\mu \in \mathcal{M}_\tau(G)$ maps to the functional $f \mapsto \int f \, d\mu$, $f \in \text{LUC}(G)$. By the theorem of Lau [12], $\Lambda(\text{LUC}(G)^*) = \mathcal{M}_\tau(G)$ for every locally compact group $G$.

In this paper I prove a stronger version of Lau’s result, in which weak$^*$ continuity in the definition of $\Lambda(\text{LUC}(G)^*)$ is replaced by measurability. I also prove similar characterizations for such generalized (measurable) centres of subsemigroups of $\text{LUC}(G)^*$. In particular, the result applies to the semigroup $\beta G$ for any discrete group $G$, thus extending a recent result of Glasner [10, Th.2.1].

2 Preliminaries
All topological spaces and groups considered in this paper are assumed to be Hausdorff, and all linear spaces to be over the field $\mathbb{R}$ of reals. Functions (including linear functionals) are real-valued. It is a simple exercise to extend the results that follow to linear spaces over the complex field and complex-valued functions.

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A pseudometric $\Delta$ on a group $G$ is right-invariant iff $\Delta(x, y) = \Delta(xz, yz)$ for all $x, y$ and $z$ in $G$. The right uniformity on a topological group $G$ is induced by the set of all right-invariant continuous pseudometrics, denoted by $\text{RP}(G)$. A bounded pseudometric $\Delta$ on $G$ is uniformly continuous in $rG$ if and only if there exists $\Delta' \in \text{RP}(G)$ such that $\Delta \leq \Delta'$.

When $\Delta$ is a pseudometric on $G$, write

$$\text{BLip}_b(\Delta) = \{f : G \to \mathbb{R} \mid -1 \leq f(x) \leq 1 \text{ and } |f(x) - f(y)| \leq \Delta(x, y) \text{ for all } x, y \in G\}.$$ Then $\text{BLip}_b(\Delta)$ is a compact subset of the product space $\mathbb{R}^G$; in the sequel $\text{BLip}_b(\Delta)$ is always considered with this compact topology.

Let $G$ be a group, $f$ a real-valued function on $G$ and $x \in G$. Define $\rho^f(z)$ (the right translation of $f$ by $x$) to be the function $z \mapsto f(zx)$, $z \in G$. The set $\text{orb}(f) := \{\rho^f(x) \mid x \in G\}$ is the (right) orbit of $f$. The closure of $\text{orb}(f)$ in the product space $\mathbb{R}^G$ is denoted $\overline{\text{orb}}(f)$. For every $f \in \text{LUC}(G)$ the set $\overline{\text{orb}}(f)$ is norm bounded and uniformly equicontinuous on $rG$, and thus $\overline{\text{orb}}(f)$ is a $G$-pointwise compact subset of $\text{LUC}(G)$.

**Fact 2.1 (Cor. 15 in [15])** Let $G$ be a locally compact group that is not compact. For every $\Delta \in \text{RP}(G)$ there is $f \in \text{LUC}(G)$ such that $\text{BLip}_b(\Delta) \subseteq \overline{\text{orb}}(f)$.

**Fact 2.2 ([1] and [5])** Let $G$ be a locally compact group and $m \in \text{LUC}(G)^*$. If the restriction of $m$ to the set $\text{BLip}_b(\Delta)$ is continuous for every $\Delta \in \text{RP}(G)$ then $m \in \text{M}_1(G)$.

By combining the first two facts we obtain a characterization of finite Radon measures on locally compact groups: A functional $m \in \text{LUC}(G)^*$ is in $\text{M}_1(G)$ if and only if for every $f \in \text{LUC}(G)$ the restriction of $m$ to $\overline{\text{orb}}(f)$ is $G$-pointwise continuous.

Let $X$ be a (Hausdorff as always) topological space and $A \subseteq X$. Say that $A$ is a CBP set iff for every continuous mapping $\varphi : K \to X$ from a compact space $K$ the set $\varphi^{-1}(A)$ has the Baire property in $K$. Say that a real-valued function $f$ on $X$ is CBP measurable iff $f^{-1}(U)$ is a CBP set in $X$ for every open subset $U$ of $\mathbb{R}$.

When $X$ is compact, CBP subsets of $X$ are exactly the universally Baire-property sets in the terminology of Fremlin [9]. Evidently the CBP subsets of $X$ form a $\sigma$-algebra, every Borel set is CBP, and every Borel measurable mapping is CBP measurable. If a mapping $f : X \to Y$ is CBP measurable then so is its restriction to any subspace of $X$.

Denote by $\mathcal{P}I$ the set of all subsets of a set $I$, and identify $\mathcal{P}I$ with the compact set $2^I$.

**Fact 2.3 (Lemma 2.1 in [3])** Let $I$ be an infinite set, and let $\mu : \mathcal{P}I \to \mathbb{R}$ be finitely additive. If $\mu$ is CBP measurable then it is a measure on $\mathcal{P}I$ and $\mu(A) = \sum_{i \in A} \mu(\{i\})$ for every $A \subseteq I$.

The next theorem and its proof are due to Fremlin. A slightly different version appears in [9, 1E].

**Theorem 2.4** Let $K_0$ be a compact space and $\varphi_0 : K_0 \to X$ a continuous surjective mapping onto a compact space $X$. Let $A \subseteq X$ be such that $\varphi_0^{-1}(A)$ is a CBP set in $K_0$. Then $A$ is a CBP set in $X$.

**Proof.** Take any continuous mapping $\varphi_1 : K_1 \to X$ from a compact space $K_1$. For $i = 0, 1$ let $\pi_i : K_0 \times K_1 \to K_i$ be the canonical projections. Then

$$K := \{(x_0, x_1) \in K_0 \times K_1 \mid \varphi_0(x_0) = \varphi_1(x_1)\}$$
Lemma 2.6 For every locally compact group $G$ the uniform space $rG$ has the $(\ell_1)$ property.

For metrizable locally compact groups this is Lemma 4.1 in [3]. Essentially the same proof works for the general case, and I do not repeat it here.

Lemma 2.7 Let $G$ be a topological group and let $m \in \text{LUC}(G)^*$ be such that for every $\Delta \in \text{RP}(G)$ the restriction of $m$ to BLip$_b(\Delta)$ is CBP measurable. Let $I$ be a non-empty index set and $\varphi: rG \to \ell_1(I)$ a uniformly continuous mapping from $rG$ to $\ell_1(I)$ with the $\|\cdot\|_1$ norm, and such that $\|\varphi(x)\|_1 \leq 1$ for every $x \in G$. Then

$$\sum_{i \in I} |m(\varphi_i)| < \infty \quad \text{and} \quad \sum_{i \in A} m(\varphi_i) = m\left(\sum_{i \in A} \varphi_i\right) \quad \text{for} \quad A \subseteq I$$

where $\varphi_i(x) := \varphi(x)(i)$ for $i \in I$, $x \in X$. 

Remark 2.8 The following theorem is an essential step in the proof of the main result in the next section. As before, BLip$_b(\Delta)$ is considered with the $G$-pointwise topology.

**Theorem 2.5** Let $G$ be a locally compact group and $m \in \text{LUC}(G)^*$, and assume that for every $\Delta \in \text{RP}(G)$ the restriction of $m$ to BLip$_b(\Delta)$ is CBP measurable. Then for every $\Delta \in \text{RP}(G)$ the restriction of $m$ to BLip$_b(\Delta)$ is continuous.
**Proof.** Since \( \varphi \) is uniformly continuous in the \( \| \cdot \|_1 \) norm, there is \( \Delta \in \text{RP}(G) \) such that 
\[
\| \varphi(x) - \varphi(y) \|_1 \leq \Delta(x, y)
\]
for all \( x, y \in G \). The expression
\[
\psi(A) := \sum_{i \in A} \varphi_i(x), \quad A \subseteq I,
\]
defines a continuous finitely additive mapping \( \psi : \mathcal{P} I \to \text{BLip}_b(\Delta) \). Hence the function \( m \circ \psi \) is finitely additive and CBP measurable on \( \mathcal{P} I \). Apply Fact 2.3.

**Proof of Theorem 2.5** Take any \( \Delta \in \text{RP}(G) \) and any net \( \{ f_\gamma \} \) of functions \( f_\gamma \in \text{BLip}_b(\Delta) \) such that \( \lim_{\gamma} f_\gamma = 0 \) for all \( x \in G \). Fix an arbitrary \( \varepsilon > 0 \). By Lemma 2.6 there is a partition of unity \( p \) on \( G \) that is subordinated to \( \Delta/\varepsilon \) and uniformly continuous from \( r G \) to \( \ell_1(I) \) with the \( \| \cdot \|_1 \) norm.

For each \( i \in I \) choose a point \( x_i \in G \) such that \( \Delta(x, x_i) \leq \varepsilon \) whenever \( p_i(x) > 0 \). Then
\[
\left| f_\gamma(x) - \sum_{i \in I} f_\gamma(x_i) \cdot p_i(x) \right| \leq \sum_{i \in I} p_i(x) \cdot |f_\gamma(x) - f_\gamma(x_i)| \leq \sum_{i \in I} p_i(x) \cdot \Delta(x, x_i) \leq \varepsilon
\]
for all \( \gamma \) and all \( x \in G \).

For a fixed \( \gamma \), define \( \varphi : G \to \ell_1(I) \) by \( \varphi(x)(i) := f_\gamma(x_i) \cdot p_i(x), \ x \in X, \ i \in I, \) and apply Lemma 2.7 to get
\[
\sum_{i \in I} f_\gamma(x_i)m(p_i) = m\left( \sum_{i \in I} f_\gamma(x_i)p_i \right).
\]

By Lemma 2.7 there is a finite set \( D \subseteq I \) such that \( \sum_{i \in I \setminus D} |m(p_i)| < \varepsilon \). For almost all \( \gamma \) we have \( |f_\gamma(x_i)| < \varepsilon \) when \( i \in D \), and
\[
|m(f_\gamma)| \leq \left| m\left( f_\gamma - \sum_{i \in I} f_\gamma(x_i) \cdot p_i \right) \right| + \left| \sum_{i \in I} f_\gamma(x_i)m(p_i) \right|
\]
\[
\leq \|m\|\varepsilon + \sum_{i \in D} |f_\gamma(x_i)| \cdot |m(p_i)| + \sum_{i \in I \setminus D} |f_\gamma(x_i)| \cdot |m(p_i)|
\]
\[
\leq \|m\|\varepsilon + 2\|m\|\varepsilon + \varepsilon = (3\|m\| + 1)\varepsilon.
\]

As this holds for every \( \varepsilon > 0 \), \( m \) is continuous on \( \text{BLip}_b(\Delta) \). \( \Box \)

### 3 Generalized centres

For any topological group \( G \), the convolution in \( \text{LUC}(G)^* \) may be written as
\[
m \ast n(f) = m(\int_x n(\int_y f(xy)))
\]
for \( m, n \in \text{LUC}(G)^* \) and \( f \in \text{LUC}(G) \). Here \( \int_x f(\ldots) \) means “\( f(\ldots) \) as a function of \( x \)”. This formula applies not only in \( \text{LUC}(G)^* \) for a topological group \( G \) but also in analogous spaces over more general semiuniform semigroups \([14]\).
LUC\( (G)^* \) with convolution is a Banach algebra. Here we mostly treat it as a semigroup with the \(*\) operation. The group \( G \) naturally embeds in LUC\( (G)^* \): An element \( x \in G \) maps to the functional \( f \mapsto f(x), f \in \text{LUC}(G) \). The embedding is a homeomorphism of \( G \) onto its image in LUC\( (G)^* \) with the weak* topology. The embedding also preserves the algebraic structure, so that \( G \) may be identified with a subgroup of LUC\( (G)^* \).

The weak* closure of \( G \) in LUC\( (G)^* \), denoted here \( G^{\text{LUC}} \), is a weak* compact subsemigroup of LUC\( (G)^* \). It is known as the canonical LC-compactification \([2]\), universal enveloping semigroup \([4]\), LUC-compactification \([13]\), or greatest ambit \([16]\) of \( G \); or, in the language of uniform spaces, a uniform (or Samuel) compactification of \( rG \). When \( G \) is discrete, \( G^{\text{LUC}} \) is its Čech–Stone compactification \( \beta G \). When \( G \) is locally compact, \( G^{\text{LUC}} \cap M_t(G) = G \).

For any topological group \( G \) and any \( S \subseteq \text{LUC}(G)^* \) define

\[
\Lambda(S) := \{ m \in S \mid \forall f \in \text{LUC}(G) \text{ the function } n \mapsto m \ast n(f) \text{ is weak* continuous on } S \} \\
\Lambda^{\text{CBP}}(S) := \{ m \in S \mid \forall f \in \text{LUC}(G) \text{ the function } n \mapsto m \ast n(f) \text{ is weak* CBP measurable on } S \}
\]

(the \textit{topological centre} and the \textit{weak* CBP measurable centre} of \( S \)).

It is well known and easy to prove that \( S \cap M_t(G) \subseteq \Lambda(S) \subseteq \Lambda^{\text{CBP}}(S) \subseteq S \). If \( G \) is compact then LUC\( (G)^* = M_t(G) \) and therefore \( \Lambda(S) = \Lambda^{\text{CBP}}(S) = S = S \cap M_t(G) \) for every \( S \subseteq \text{LUC}(G)^* \).

Now we come to the main result of this paper. The proof strategy is the same as in section 5 of \([15]\).

**Theorem 3.1** Let \( G \) be a locally compact group and \( G^{\text{LUC}} \subseteq S \subseteq \text{LUC}(G)^* \). Then

\[
\Lambda(S) = \Lambda^{\text{CBP}}(S) = S \cap M_t(G).
\]

**Proof.** In view of the preceding discussion, it is enough to prove that \( \Lambda^{\text{CBP}}(S) \subseteq S \cap M_t(G) \) when \( G \) is not compact.

For \( f \in \text{LUC}(G) \) define the mapping \( \varphi_f : \text{LUC}(G)^* \rightarrow \text{LUC}(G) \) by

\[
\varphi_f(n) := \{ x \mapsto n(xy), n \in \text{LUC}(G)^* \}.
\]

Then for every \( m \in \text{LUC}(G)^* \) the mapping \( n \mapsto m \ast n(f) \) from \( \text{LUC}(G)^* \) to \( \mathbb{R} \) is the composition \( m \circ \varphi_f \). By \([15] \text{ L.19}\), \( \varphi_f \) is continuous from \( G^{\text{LUC}} \) to \( \text{LUC}(G) \) with the \( G \)-pointwise topology, and \( \varphi_f(G^{\text{LUC}}) = \text{orb}(f) \).

Now assume that \( m \in \Lambda^{\text{CBP}}(S) \), which means that for every \( f \in \text{LUC}(G) \) the mapping \( m \circ \varphi_f \) is CBP measurable on \( S \), and therefore also on \( G^{\text{LUC}} \subseteq S \). By Theorem \([24]\) \( m \) is CBP measurable.
on $\text{orb}(f)$. By Fact 2.1, $m$ is CBP measurable on $\text{BLip}_b(\Delta)$ for every right-invariant continuous pseudometric $\Delta$ on $G$, and therefore also continuous on $\text{BLip}_b(\Delta)$ by Theorem 2.5. Hence $m \in M_t(G)$ by Fact 2.2. □

By choosing $S = \text{LUC}(G)^*$ and $S = G^{\text{LUC}}$ we obtain two corollaries. The first one is the promised strengthening of Lau’s theorem [12].

**Corollary 3.2** $\Lambda(\text{LUC}(G)^*) = \Lambda^{\text{CBP}}(\text{LUC}(G)^*) = M_t(G)$ for every locally compact group $G$.

The second corollary is a common generalization of the theorems of Lau and Pym [13] and Glasner [10].

**Corollary 3.3** $\Lambda(G^{\text{LUC}}) = \Lambda^{\text{CBP}}(G^{\text{LUC}}) = G$ for every locally compact group $G$.

Note that Theorem 3.1 applies also to many other sets between $G^{\text{LUC}}$ and $\text{LUC}(G)^*$ — for example, the set of positive elements in $\text{LUC}(G)^*$, or the set of means on $\text{LUC}(G)^*$.

4 Variations and open problems

One may ask to what extent Theorem 3.1 and its corollaries depend on the group $G$ being locally compact. The results in [6] and [15] suggest that the space $M_u(rG)$ of uniform measures should take the place of $M_t(G)$ in describing the centres in convolution semigroups as we move beyond locally compact groups ($M_u(rG)$ and $M_t(G)$ coincide when $G$ is locally compact). This leads to the question whether $\Lambda^{\text{CBP}}(\text{LUC}(G)^*) = M_u(rG)$ for every topological group $G$, or at least for some interesting class of non-locally-compact groups.

With the same approach as in the proof of Theorem 3.1, we get that $\Lambda^{\text{CBP}}(S) = S \cap M_u(rG)$ for $G^{\text{LUC}} \subseteq S \subseteq \text{LUC}(G)^*$ whenever $G$ is an ambitable topological group [15] for which $rG$ has the $(\ell_1)$ property. However, infinite-dimensional normed spaces do not have the $(\ell_1)$ property by the theorem of Zahradn’ık [20].

One may also try to weaken the measurability condition in the definition of $\Lambda^{\text{CBP}}(S)$. In one direction, Schachermayer’s example [18] marks a limit of such generalizations: For the additive group $c_0$ and the metric $\Delta$ of the sup norm on $c_0$, there is a bounded linear functional $m$ on $\text{LUC}(c_0)$ whose restriction to $\text{BLip}_b(\Delta)$ is Baire-property measurable and yet $m$ is not in $M_t(c_0)$.

In another direction, for $S \subseteq \text{LUC}(G)^*$ define

$$\Lambda^{\text{URM}}(S) = \{m \in S \mid \forall f \in \text{LUC}(G) \text{ the function } n \mapsto m \star n(f) \text{ is weak* universally Radon-measurable on } S\}.$$ 

The characterization of $\Lambda^{\text{URM}}(S)$ is not as straightforward as that of $\Lambda^{\text{CBP}}(S)$, even for the group $\mathbb{Z}$ of integers with the discrete topology. On one hand, Glasner’s proof of Theorem 2.1 in [10] demonstrates that if $G$ is a countable discrete group then $\Lambda^{\text{URM}}(\beta G) = G$, which improves (for such groups) Corollary 3.3. On the other hand, the statement $\Lambda^{\text{URM}}(\text{LUC}(\mathbb{Z})^*) = M_t(\mathbb{Z})$, which may be written simply as $\Lambda^{\text{URM}}(\ell_\infty^*) = \ell_1$, is neither provable nor disprovable in the ZFC set theory. That follows from old results about medial limits, covered by Fremlin [8, 538Q], along with a recent result of Larson [11].
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