Additivity for transpose depolarizing channels

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Additivity of the minimal output entropy for the family of transpose depolarizing channels introduced by Fannes et al. [4] is considered. It is shown that using the method of our previous paper [3] allows us to prove the additivity for the range of the parameter values for which the problem was left open in [4]. Together with the result of [4], this covers the whole family of transpose depolarizing channels.

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INTRODUCTION

In a recent paper [4] Fannes et al. considered the one parameter family of transpose depolarizing channels

$$\Phi(\mu) = t \mu^T + (1-t)\text{Tr} \mu \frac{I_d}{d},$$

(1)

where

$$-\frac{1}{d-1} \leq t \leq \frac{1}{d+1}.$$  

(2)

Here $\mu$ is an arbitrary complex $d \times d$ matrix, $\mu^T$ denotes its transpose, and $I$ is the $d \times d$ unit matrix. The channel $\Phi$ is irreducibly covariant since for any arbitrary unitary transformation $U$

$$\Phi(U\mu U^*) = \bar{U} \Phi(\mu) \bar{U}^*,$$

(3)

where $\bar{U}$ is the complex–conjugate of $U$ in a fixed basis. Note that $\Phi(\mu)$ can be written as

$$\Phi(\mu) = c \left( t + \frac{1}{d-1} \right) \Phi_+(\mu) - c \left( t - \frac{1}{d+1} \right) \Phi_-(\mu),$$

(4)

where $c = (d^2 - 1)/2d$ and

$$\Phi_{\pm}(\mu) := \frac{1}{d \pm 1} \left( \text{ITr} \mu \pm \mu^T \right).$$

(5)

The channels $\Phi_{\pm}(\mu)$ admit the following Kraus decompositions

$$\Phi_{\pm}(\mu) = \frac{1}{2(d \pm 1)} \sum_{i,j=1}^d \left( |i\rangle \langle j| \pm |j\rangle \langle i| \right) \mu \left( |i\rangle \langle j| \pm |j\rangle \langle i| \right)^*.$$

(6)

From [4] it follows that the channel $\Phi$ interpolates between the channels $\Phi_+$ and $\Phi_-$, where $\Phi_-$ is the Werner–Holevo channel introduced in [9] and studied extensively (see e.g. [1, 3, 7]).

Fannes et al. proved additivity of the minimal output entropy of the channels [4] for

$$-\frac{2}{d^2 - 2} \leq t \leq \frac{1}{d+1}.$$  

(7)
The values of the parameter \( t \) given by (7) does not however cover the full range of values (2). The aim of this paper is to extend the validity of the additivity relation for the whole range of values of \( t \) given by (2). More precisely, we prove additivity of the minimum output entropy for

\[-\frac{1}{d-1} \leq t \leq 0.\]

The minimum output entropy of a channel \( \Phi \) is

\[h(\Phi) := \min_\rho S(\Phi(\rho)),\]

where the minimization is over all possible input states \( \rho \) (i.e., density matrices) of the channel. Here and below \( S(\sigma) := -\text{Tr} \sigma \log \sigma \) denotes the von Neumann entropy of the density matrix \( \sigma \). Fannes et al. proved the additivity relation

\[h(\Phi \otimes \Phi) = 2h(\Phi)\]

for the values of \( t \) given by (7). For simplicity of exposition we also consider the case \( d_1 = d_2 = d \) although the proof can be easily extended to the case \( d_1 \neq d_2 \), i.e.

\[h(\Phi_1 \otimes \Phi_2) = h(\Phi_1) + h(\Phi_2).\]

The proof employs the method developed in [3].

Consider the Schmidt decomposition

\[| \psi_{12} \rangle = \sum_{\alpha=1}^{d} \sqrt{\lambda_{\alpha}} |\alpha; 1\rangle \otimes |\alpha; 2\rangle.\]

Here \( \{ |\alpha; j\rangle \} \) is an orthonormal basis in \( \mathcal{H}_j \), \( j = 1, 2 \), and \( \lambda = (\lambda_1, \ldots, \lambda_d) \) is the vector of the Schmidt coefficients. The state \( |\psi_{12}\rangle \langle \psi_{12}| \) can then be expressed as

\[|\psi_{12}\rangle \langle \psi_{12}| = \sum_{\alpha,\beta=1}^{d} \sqrt{\lambda_{\alpha}\lambda_{\beta}} |\alpha; 1\rangle \langle \beta| \otimes |\alpha; 2\rangle \langle \beta; 2|.\]

The Schmidt coefficients form a probability distribution:

\[\lambda_{\alpha} \geq 0 ; \quad \sum_{\alpha=1}^{d} \lambda_{\alpha} = 1;\]

thus the vector \( \lambda \) varies in the \((d-1)\)–dimensional simplex \( \Sigma_d \), defined by these constraints. The extreme points (vertices) of \( \Sigma_d \) correspond precisely to unentangled vectors \(|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \). Then the additivity follows if for every choice of the bases \( \{ |\alpha; 1\rangle \} \) and \( \{ |\alpha; 2\rangle \} \), the function

\[\lambda \rightarrow S(\sigma_{12}(\lambda)),\]

where

\[\sigma_{12}(\lambda) := (\Phi \otimes \Phi) (|\psi_{12}\rangle \langle \psi_{12}|) = \sum_{\alpha,\beta=1}^{d} \sqrt{\lambda_{\alpha}\lambda_{\beta}} \Phi(|\alpha; 1\rangle \langle \beta| \otimes \Phi(|\alpha; 2\rangle \langle \beta; 2|).\]
is the channel output state, attains its minimum at the vertices of $\Sigma_d$. Owing to \( \Sigma_d \), we can choose for \( \{\alpha; i\} \) the canonical basis of real vectors \( \{|\alpha\rangle\} \) in $\mathcal{H}_i \simeq \mathbb{C}^{d_i}$; $i = 1, 2$. Moreover from the definition \( \Pi \) of the channel $\Phi$ it follows that

$$
\Phi (|\alpha\rangle \langle \beta|) = (1 - t)\delta_{\alpha\beta} \frac{I}{d} + t|\beta\rangle \langle \alpha|,
$$

since $|\alpha\rangle$ and $|\beta\rangle$ are real. Hence,

$$
\sigma_{12}(\lambda) = \sum_{\alpha,\beta=1}^{d} \sqrt{\lambda_\alpha \lambda_\beta} \Phi(|\alpha\rangle \langle \beta|) \otimes \Phi(|\alpha\rangle \langle \beta|)
$$

$$
= \sum_{\alpha,\beta=1}^{d} |\alpha\beta\rangle \langle \alpha\beta| \left[ \frac{(1-t)^2}{d^2} + \frac{t(1-t)}{d} (\lambda_\alpha + \lambda_\beta) \right] + \sum_{\alpha,\beta=1}^{d} t^2 \sqrt{\lambda_\alpha \lambda_\beta} |\alpha\alpha\rangle \langle \beta\beta|. \tag{17}
$$

Here we have used the constraint \( \Pi \) and the fact that $I = \sum_{\alpha=1}^{d} |\alpha\rangle \langle \alpha|$.

To find the minimum output entropy of the product channel $\Phi \otimes \Phi$, we first evaluate the eigenvalues of $\sigma_{12}(\lambda)$. For this purpose it is useful to express $\sigma_{12}(\lambda)$ in the form of a $d^2 \times d^2$ matrix $A$ with elements

$$
A_{ij} = (\mu_i + \eta_i)\delta_{ij} + \sqrt{\eta_i\eta_j}(1 - \delta_{ij}), \tag{18}
$$

where we identify $i$ or $j$ with a pair $(\alpha, \beta)$ and define

$$
\mu_i \equiv \mu_{\alpha\beta} = \frac{(1-t)^2}{d^2} + \frac{t(1-t)}{d} (\lambda_\alpha + \lambda_\beta) \ ; \ \eta_j \equiv \eta_{\alpha\beta} = \lambda_\alpha t^2 \delta_{\alpha\beta}, \ \alpha, \beta = 1, \ldots, d. \tag{19}
$$

As shown in \( \Sigma_d \), the characteristic equation $\det(A - \gamma I) = 0$ can be written as

$$
\prod_{1 \leq \alpha, \beta \leq d} (\mu_{\alpha\beta} - \gamma) \left[ \prod_{\alpha' = 1}^{d} (\mu_{\alpha'\alpha'} - \gamma) \left\{ 1 + \sum_{\alpha'' = 1}^{d} \frac{t^2 \lambda_{\alpha''}}{(\mu_{\alpha'\alpha''} - \gamma)} \right\} \right] = 0.
$$

This implies that $\sigma_{12}(\lambda)$ has the following sets of eigenvalues:

1. $d(d - 1)$ eigenvalues of the form

$$
\gamma_{\alpha\beta} = \mu_{\alpha\beta} = \frac{(1-t)^2}{d^2} + \frac{t(1-t)}{d} (\lambda_\alpha + \lambda_\beta), \ \alpha \neq \beta, \alpha, \beta = 1, \ldots, d. \tag{20}
$$

2. $d$ eigenvalues \( \{g_\alpha, \alpha = 1, \ldots, d\} \), given by the roots of the equation

$$
\prod_{\alpha = 1}^{d} (\mu_{\alpha\alpha} - g) \left\{ 1 + \sum_{\alpha' = 1}^{d} \frac{t^2 \lambda_{\alpha'}}{(\mu_{\alpha'\alpha'} - g)} \right\} = 0. \tag{21}
$$

This equation can be written as

$$
\prod_{\alpha = 1}^{d} (c_1 + c_2 \lambda_\alpha - g) \left\{ 1 + \sum_{\alpha' = 1}^{d} \frac{t^2 \lambda_{\alpha'}}{(c_1 + c_2 \lambda_{\alpha'} - g)} \right\} = 0. \tag{22}
$$

Here we have defined

$$
c_1 = \frac{(1-t)^2}{d^2} \ ; \ c_2 = \frac{2t(1-t)}{d}. \tag{23}
$$
Since $t$ is in the range $[8]$

$$c_2 \leq 0, \quad -2 \leq c_2/c_1 = \frac{2td}{1-t} \leq 0.$$  

(24)

The von Neumann entropy of the output of the product channel can be expressed as a sum

$$S(\sigma_{12}(\lambda)) = S_1(\lambda) + S_2(\lambda)$$  

(25)

where

$$S_1(\lambda) := -\sum_{1 \leq \alpha, \beta \leq d, \beta \neq \alpha} \gamma_{\alpha \beta} \log \gamma_{\alpha \beta}, \quad S_2(\lambda) := -\sum_{\alpha=1}^d g_{\alpha} \log g_{\alpha}.  $$  

(26)

Note that

$$\sum_{1 \leq \alpha, \beta \leq d, \beta \neq \alpha} \gamma_{\alpha \beta} = \frac{d-1}{d}(1-t)^2 := c.$$  

(27)

Moreover, using the fact that the eigenvalues of $\sigma_{12}(\lambda)$ sum to 1 we get

$$\sum_{\alpha=1}^d g_{\alpha} = 1 - c.$$  

(28)

Using the above relations we can define sets of non–negative variables

$$\widetilde{\gamma}_{\alpha \beta} := \frac{1}{c} \gamma_{\alpha \beta}, \quad \alpha, \beta = 1, \ldots, d.$$  

(29)

and

$$\widetilde{g}_{\alpha} := \frac{1}{1-c} g_{\alpha}, \quad \alpha = 1, 2, \ldots, d,$$

(30)

each of which sum to unity, i.e.,

$$\sum_{1 \leq \alpha, \beta \leq d, \beta \neq \alpha} \widetilde{\gamma}_{\alpha \beta} = 1 \quad ; \quad \sum_{\alpha=1}^d \widetilde{g}_{\alpha} = 1,$$

and hence define probability distributions. In terms of these variables we have

$$S_1(\lambda) = cH(\{\widetilde{\gamma}_{\alpha \beta}\}) + \text{const}  $$  

(31)

$$S_2(\lambda) = (1-c)H(\{\widetilde{g}_{\alpha}\}) + \text{const}  $$  

(32)

Here $H(\{x_i\})$ denotes the Shannon entropy of a probability distribution $\{x_1, \ldots, x_n : x_i \geq 0, \sum_{i=1}^n x_i = 1\}$. Since $H(\{x_i\})$ is a concave function of the variables $x_i$, i = 1, ..., n, it follows from (31) that $S_1(\lambda)$ is a symmetric concave function of the variables $\widetilde{\gamma}_{\alpha \beta}$. These variables (defined by (29) and (20)) are affine functions of the Schmidt coefficients $\lambda_1, \ldots, \lambda_d$. Hence, $S_1$ is a concave function of $\lambda$, and attains its global minimum at the vertices of the simplex $\Sigma_d$, defined by the constraints (14).

Let us now analyze $S_2$. We wish to prove the following:

**Theorem.** The function $S_2$ is Schur-concave in $\lambda \in \Sigma_d$ i.e., $\lambda \prec \lambda' \implies S_2(\lambda) \geq S_2(\lambda')$, where $\prec$ denotes the stochastic majorization (see [3]).

Since every $\lambda \in \Sigma_d$ is majorized by the vertices of $\Sigma_d$, this will imply that $S_2(\lambda)$ also attains its minimum at the vertices. Thus $S(\lambda) = S_1(\lambda) + S_2(\lambda)$ is minimized at the vertices, which correspond to unentangled states. As was observed, this implies the additivity (10).
PROOF OF THE THEOREM

In [8] it was proved that the Shannon entropy \( H(x) = (x_1, \ldots, x_d) \in \Sigma_d \), is a monotonically non-decreasing function of the elementary symmetric polynomials \( s_q(x_1, x_2, \ldots, x_d) \) (see e.g. [2]) in the variables \( x_1, x_2, \ldots, x_d, q = 2, \ldots, d \). This implies that \( S_2 \) is a monotonically non-decreasing function of the symmetric polynomials

\[
\sum_{\nu} (\nu_1 + \nu_2 + \cdots + \nu_d) \leq \sum_{\nu} (\nu_1 + \nu_2 + \cdots + \nu_d - 1)
\]

Therefore, to prove the Theorem it is sufficient to prove that the functions \( \tilde{s}_q(\lambda) \) are Schur concave in \( \lambda \in \Sigma_d \) for \( q = 2, \ldots, d \).

Let us define the variables

\[
\nu_\alpha := 1 + \frac{c_2}{c_1} \lambda_\alpha, \quad \alpha = 1, 2, \ldots, d.
\]

This together with (34) implies that

\[
1 + c_2/c_1 \leq \nu_\alpha \leq 1, \quad \sum_{\alpha=1}^{d} \nu_\alpha = d + c_2/c_1.
\]

Defining \( \gamma = g/c_1 \), (32) can be expressed in terms of the variables \( \nu_\alpha \) as follows

\[
\prod_{\alpha=1}^{d} (\nu_\alpha - \gamma) \left( 1 + \sum_{\alpha'=1}^{d} (\nu_{\alpha'} - 1) c_2^{\nu_{\alpha'}} \right) = 0.
\]

Denote \( \gamma_i := g_i/c_1 \) for \( i = 1, \ldots, n \), where \( g_1, g_2, \ldots, g_n \) are the roots of eq. (32). Therefore \( \gamma_1, \ldots, \gamma_d \) are the zeroes of the product \( (\gamma_1 - \gamma)(\gamma_2 - \gamma) \ldots (\gamma_d - \gamma) \) and equation (36) can be expressed in terms of these roots as follows:

\[
\sum_{k=0}^{d} \gamma^k (-1)^k s_{d-k}(\gamma_1, \gamma_2, \ldots, \gamma_d) = 0.
\]

In terms of the elementary symmetric polynomials \( s_l \) of the variables \( \nu_1, \nu_2, \ldots, \nu_d \), (37) can be rewritten as

\[
\sum_{k=0}^{d} \gamma^k (-1)^k s_{d-k}(\nu_1, \nu_2, \ldots, \nu_d) + \sum_{k=0}^{d-1} \gamma^k (-1)^k \sum_{l=1}^{d} s_{d-1-k}(\nu_1, \ldots, \nu_l, \ldots, \nu_d) \frac{(\nu_l - 1)c_2}{c_2} = 0,
\]

where the symbol \( \cdot \) means that the variable \( \nu_l \) has been omitted from the arguments of the corresponding polynomial. Equating the LHS of (37) with the LHS of (36) yields, for each \( 0 \leq k \leq d - 1 \):

\[
s_{d-k}(\gamma_1, \gamma_2, \ldots, \gamma_d) = s_{d-k}(\nu_1, \nu_2, \ldots, \nu_d) + \sum_{l=1}^{d} s_{d-1-k}(\nu_1, \ldots, \nu_l, \ldots, \nu_d) \frac{(\nu_l - 1)c_2}{c_2}.
\]

Note that in (39), values \( s_{d-k}(\gamma_1, \gamma_2, \ldots, \gamma_d) \) are expressed in terms of values of elementary symmetric polynomials in the variables \( \nu_1, \nu_2, \ldots, \nu_d \) (which are themselves linear functions of the Schmidt coefficients \( \lambda_1, \ldots, \lambda_d \)).

Our aim is to prove that \( \tilde{s}_q(\lambda) \) is Schur concave in the Schmidt coefficients \( \lambda_1, \ldots, \lambda_d \), for \( q = 2, \ldots, d \). Eq. (32) implies that this amounts to proving Schur concavity of \( s_{d-k}(\gamma_1, \gamma_2, \ldots, \gamma_d) \) as a function of \( \lambda_1, \ldots, \lambda_d \), for all \( 0 \leq k \leq d - 2 \). The functions

\[
\Phi_k(\nu_1, \ldots, \nu_d) := s_{d-k}(\nu_1, \ldots, \nu_d) + \sum_{l=1}^{d} s_{d-1-k}(\nu_1, \ldots, \nu_l, \ldots, \nu_d) \frac{(\nu_l - 1)c_2}{c_2} \equiv \text{RHS of } (39)
\]

(40)
are symmetric in the variables \( \nu_1, \nu_2, \ldots, \nu_d \), and hence in the variables \( \lambda_1, \ldots, \lambda_d \). By the necessary and sufficient condition for Schur concavity [2] it is enough to prove

\[
(\lambda_i - \lambda_j) \left( \frac{\partial \Phi_k}{\partial \lambda_i} - \frac{\partial \Phi_k}{\partial \lambda_j} \right) = (\nu_i - \nu_j) \left( \frac{\partial \Phi_k}{\partial \nu_i} - \frac{\partial \Phi_k}{\partial \nu_j} \right) \leq 0, \quad \forall \ 1 \leq i, j \leq d. \tag{41}
\]

By the rule of differentiation of the elementary symmetric polynomials, see e.g. [2], we have

\[
\frac{\partial}{\partial \nu_i} \Phi_k(\nu_1, \ldots, \nu_d) = \frac{\partial}{\partial \nu_i} s_{d-k}(\nu_1, \ldots, \nu_d) + \frac{\partial}{\partial \nu_i} \sum_{l=1}^{d} s_{d-1-k}(\nu_1, \ldots, \mu_l, \ldots, \nu_d) \frac{(\nu_l - 1)t^2}{c_2} = \frac{t^2}{2} s_{d-1-k}(\nu_1, \ldots, \nu_d). \tag{42}
\]

Therefore,

\[
\left( \frac{\partial \Phi_k}{\partial \nu_i} - \frac{\partial \Phi_k}{\partial \nu_j} \right) (\nu_1, \ldots, \nu_d) = s_{d-1-k}(\nu_1, \ldots, \nu_d) - s_{d-1-k}(\nu_1, \ldots, \nu_d) + \sum_{1 \leq i < j \leq d} s_{d-1-k}(\nu_1, \ldots, \mu_i, \ldots, \nu_d) \frac{(\nu_i - 1)t^2}{c_2} - \sum_{1 \leq i < j \leq d} s_{d-1-k}(\nu_1, \ldots, \mu_j, \ldots, \nu_d) \frac{(\nu_j - 1)t^2}{c_2} + \frac{t^2}{c_2} [s_{d-1-k}(\nu_1, \ldots, \nu_d) - s_{d-1-k}(\nu_1, \ldots, \nu_d)]. \tag{43}
\]

Using a transformation rule for the elementary symmetric polynomials, see e.g. [2], we get

\[
\left( \frac{\partial \Phi_k}{\partial \nu_i} - \frac{\partial \Phi_k}{\partial \nu_j} \right) (\nu_1, \ldots, \nu_d) = (\nu_j - \nu_i) s_{d-k-2}(\nu_1, \ldots, \mu_i, \mu_j, \ldots, \nu_d) + \frac{2t^2}{c_2} (\nu_j - \nu_i) s_{d-k-2}(\nu_1, \ldots, \mu_i, \mu_j, \ldots, \nu_d) + \sum_{1 \leq i < j \leq d} \frac{(\nu_i - 1)t^2}{c_2} [s_{d-k-2}(\nu_1, \ldots, \mu_i, \ldots, \nu_d) - s_{d-k-2}(\nu_1, \ldots, \mu_j, \ldots, \nu_d)] + 1 + \frac{2t^2}{c_2} (\nu_j - \nu_i) s_{d-k-2}(\nu_1, \ldots, \mu_i, \mu_j, \ldots, \nu_d). \tag{44}
\]

Substituting (44) in (41), using (23) and rearranging factors, we obtain that the Schur concavity holds if and only if

\[
\sum_{1 \leq i < j \leq d} (1 - \nu_i) s_{d-k-3}(\nu_1, \ldots, \mu_i, \mu_j, \ldots, \nu_d) - \frac{2(1 + t(d-1))}{td} s_{d-k-2}(\nu_1, \ldots, \mu_i, \mu_j, \ldots, \nu_d) \geq 0, \tag{45}
\]

for all \( 1 \leq i, j \leq d \) and \( 0 \leq k \leq d - 2 \). The variables \( \nu_i \) and \( \nu_j \) do not appear in (45). Owing to symmetry, without loss of generality, we can choose \( i = d - 1 \) and \( j = d \). Then omitting \( \nu_{d-1} \) and \( \nu_d \) and setting \( n = d - 2 \),
we obtain that the functions $\Phi_k$ defined in (40) are Schur concave in the Schmidt coefficients $\lambda_1, \ldots, \lambda_d$ if and only if
\[
\sum_{l=1}^{n} (1 - \nu_l) s_{n-k-1}(\nu_1, \ldots, \nu_n) - \frac{2(1 + t(d-1))}{td} s_{n-k}(\nu_1, \ldots, \nu_n) \geq 0,
\]
for $0 \leq k \leq n - 1$. Here the variables $\nu_l$, $1 \leq l \leq n$, satisfy the constraints
\[
\nu_l \leq 1; \quad \sum_{l=1}^{n} \nu_l \geq n + c_2/c_1 = n + \frac{2td}{1-t} \geq n - 2,
\]
following from (44), the relations: $\lambda_l \geq 0$ for all $l$, and $\sum_{l=1}^{n} \lambda_l = \sum_{l=1}^{d-2} \lambda_l \leq 1$, and (24).

The above constraint implies that $1 - \nu_l \leq 0$ for $1 \leq l \leq n$. Thus if all $\nu_1, \ldots, \nu_n \geq 0$, (46) obviously holds. The constraint (47) also implies that at most one of the variables $\nu_1, \ldots, \nu_n$ can be negative. Hence, we need to prove (46) only in the case in which one and only one of the variables $\nu_1, \ldots, \nu_n$ is negative.

We now proceed to prove (46). We first notice that for all values of $0 \leq k \leq n$ nonnegativity of the first term in the LHS of (46) under the constraints
\[
\nu_l \leq 1; \quad \sum_{l=1}^{n} \nu_l \geq n - 2,
\]
which are weaker than (47), and coincide with them for $t = -\frac{1}{d-1}$, was proven in (8). Next we prove that the second term on the LHS of (46) is positive for $k = 1, 2, \ldots, n$. These two facts together prove (46) for all $k = 1, 2, \ldots, n$. For $k = 0$ the second term is not positive. In this case we prove (46) by considering the sum of the two terms on the LHS of (46).

Let us now analyze the second term on the LHS of (46) for $1 \leq k \leq n$. In the range (8) we have
\[
-\frac{(1 + t(d-1))}{td} \geq 0.
\]

Also
\[
s_{n-k}(\nu_1, \ldots, \nu_n) = \sum_{2 \leq i_1 < i_2 < \ldots < i_{n-k} \leq n} \nu_{i_1}, \ldots, \nu_{i_{n-k}} + \nu_1 \sum_{2 \leq i_1 < i_2 < \ldots < i_{n-k-1} \leq n} \nu_{i_1}, \ldots, \nu_{i_{n-k-1}}
\]
\[
= \frac{1}{(n-k-1)!} \sum_{i_1, i_2, \ldots, i_{n-k-1}=2}^{n} \nu_{i_1}, \ldots, \nu_{i_{n-k-1}} (\nu_1 + \sum_{r \neq i_1, \ldots, i_{n-k-1}}^{n} \nu_r)
\]
\[
= \frac{1}{(n-k-1)!} \sum_{i_1, i_2, \ldots, i_{n-k-1}=2}^{n} \nu_{i_1}, \ldots, \nu_{i_{n-k-1}} (\nu_1 + \sum_{r=1}^{n} \nu_r - \nu_1 - (\nu_{i_1} + \ldots + \nu_{i_{n-k-1}}))
\]
\[
\geq \frac{1}{(n-k-1)!} \sum_{i_1, i_2, \ldots, i_{n-k-1}=2}^{n} \nu_{i_1}, \ldots, \nu_{i_{n-k-1}} \times ((n-2) - (n-k-1))
\]
\[
= \frac{1}{(n-k-1)!} \sum_{i_1, i_2, \ldots, i_{n-k-1}=2}^{n} \nu_{i_1}, \ldots, \nu_{i_{n-k-1}} \times (k-1) \geq 0,
\]
(49)
since \( k \geq 1 \). Hence,
\[
\text{[2nd term on LHS of (46)]} \geq 0 \quad \text{for } 1 \leq k \leq n. \tag{50}
\]
In the second last line of eq. (49) we have used the constraint (47). The negativity of the first term on the LHS of (46) (as proved in [3]) together with (50) implies that the inequality (46) holds for all \( k = 1, 2, \ldots, n \).

The case \( k = 0 \):
In this case we have
\[
\text{LHS of (46) = } e_n \left[ \sum_{l=1}^{n} \frac{(1 - \nu_l)}{\nu_l} - \frac{2(1 + t(d - 1))}{td} \right] \tag{51}
\]
where \( e_n := \nu_1, \ldots, \nu_n < 0 \), since one and only one of the variables \( \nu_1, \ldots, \nu_n \) is negative. Hence in this case the inequality (46) reduces to
\[
\sum_{l=1}^{n} \frac{(1 - \nu_l)}{\nu_l} \leq \frac{2(1 + t(d - 1))}{td} \leq 0. \tag{52}
\]
Without loss of generality we can choose \( \nu_1 < 0 \) and \( \nu_l > 0 \) for all \( l = 2, 3, \ldots, n \). The function
\[
f(\nu) = \frac{1 - \nu}{\nu} = \frac{1}{\nu} - 1 \tag{53}
\]
is nonincreasing for all \( \nu \), convex for \( \nu > 0 \) and \( f(1) = 0 \). Denote
\[
g(\nu_2, \ldots, \nu_n) = \sum_{l=2}^{n} f(\nu_l), \tag{54}
\]
then \( g \) is convex on the simplex
\[
\nu_2 + \cdots + \nu_n \geq n + c_2/c_1 - \nu_1, \quad (0 \leq) \nu_l \leq 1, \ l = 2, \ldots, n, \tag{55}
\]
where \( \nu_1 \) is fixed, and hence attains its maximum on its extreme points. These are
\[
(2 + c_2/c_1 - \nu_1, 1, \ldots, 1) \tag{56}
\]
and its permutations, and \((1, 1, \ldots, 1)\). In the first case
\[
\nu_1 + \nu_2 = 2 + c_2/c_1 = \frac{2(1 + t(d - 1))}{1 - t}, \tag{57}
\]
and we have to show that
\[
\frac{1}{\nu_1} + \frac{1}{\nu_2} - 2 \leq \frac{2(1 + t(d - 1))}{td}. \tag{58}
\]
The second case reduces to this because it corresponds to \( \nu_2 = 1 \) (and \( \nu_l = 1, l > 2 \)), and the LHS of (58) is then maximal for the minimal possible value \( \nu_1 = 1 + c_2/c_1 \) (see (55)), for which the condition (57) is satisfied.
To prove (58) we take into account that \( \nu_1 < 0, \nu_2 > 0 \). Then it reduces to
\[
\frac{2(1 + t(d - 1))}{1 - t} = \nu_1 + \nu_2 \geq 2\nu_1\nu_2 \left[ \frac{1 + t(d - 1)}{td} + 1 \right]. \tag{59}
\]
The product

$$\nu_1\nu_2 = \nu_1 \left[ \frac{2(1 + t(d - 1))}{1 - t} - \nu_1 \right]$$

is nonpositive and monotonically increases from the value

$$\nu_1^0 = \frac{2(1 + t(d - 1))}{1 - t} - 1 < 0$$

to zero. Since the LHS of (59) is nonnegative, it is sufficient, whatever the sign of the last factor on the right hand side of (59) is, to check it only for \( \nu_1 = \nu_1^0 \). Substituting this value and making common denominator, we get, taking into account that \( t < 0 \),

$$(1 + t(d - 1))td \leq 2(1 + t(d - 1))^2 + (1 + t(d - 1))(2td - 1 + t) - (1 - t)td$$

or

$$0 \leq 2(1 + t(d - 1))^2 + (td)^2 - (1 - t)^2 - (1 - t)(td)$$

or

$$0 \leq 3(td)^2 + 3(1 - t)(td) + (1 - t)^2,$$

which is indeed true.

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