SOFT INTERACTION OF HEAVY FERMIONS

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Abstract

We discuss a formula for the sum of tree diagrams of \( n \)-bosons scattering from a heavy fermion. This formula can be considered as a generalization of the eikonal formula to include non-abelian vertices. It will be used to demonstrate the consistency of multi-meson-fermion scattering in the large-\( N_c \) limit.

1 Introduction

This work is carried out in collaboration with Keh-Fei Liu of the University of Kentucky\(^1\). The central piece is a tree-level formula for the sum of the \( n! \) diagrams describing the scattering of \( n \) bosons from a heavy fermion. This formula is kinematical and combinatorial, valid for any coupling and internal quantum numbers. It relies on the presence of a heavy fermion, just like the heavy-quark effective theory. It will be used to demonstrate the consistency of meson-baryon scattering in large-\( N_c \) QCD\(^2\) for an arbitrary number of mesons, under certain criteria. We are in the process of extending it to study the consistency of large-\( N_c \) QCD in loops, baryon-baryon scattering, and heavy quarks.

This formula bears some superficial similarity to formulas for soft-pion productions\(^3\), in that multiple commutators appear in both. However, the present formula does not rely on chiral symmetry, nor current algebra. As stressed before, it is kinematical and combinatorial, it is generally valid under no particular dynamical assumptions. Its validity depends on the smallness of all energy scales compared to the heavy-fermion mass \( M \), and not some
\[ q_n = p' \quad V_n \quad q_{n-1} \quad V_{n-1} \quad \ldots \quad k_n \quad k_{n-1} \quad k_3 \quad k_2 \quad k_1 \]

Figure 1: A tree diagram. Solid and wavy lines are respectively fermions and bosons.

dynamical quantity like \( f_\pi \). If we need an analogy with known formulas, then it is much more appropriate to think of it as a non-abelian generalization of the eikonal formula\(^4\).

2 Sum of Feynman Diagrams

Fig. 1 depicts one of the \( n! \) tree diagrams for the process

\[ F'(p') \rightarrow F(p) + B_1(k_1) + B_2(k_2) + \cdots + B_n(k_n), \quad (1) \]

in which an initial fermion \( F' \) emits \( n \) bosons \( B_i \) to become a final fermion \( F \). The momenta of the particles are enclosed in parentheses. If a boson in (1) is moved from right to left, we will get an inelastic boson-fermion scattering process. Both possibilities are considered using the notation of (1). The initial and the final fermions are assumed to have a heavy mass \( M \):

\[ p^2 = (p')^2 = M^2. \]

We shall use the notation \( T_a = [t_1 t_2 t_3 \cdots t_n] \) to denote a tree graph obtained from Fig. 1 by permuting the \( n \) boson lines. The index \( a \) runs from 1
to \( n! \), and each \( t_i \) takes on a different value between 1 and \( n \). The numbers between the square brackets indicate the order of the boson lines from final to initial states. In this language Fig. 1 is denoted by \( T_1 = [123 \cdots n] \).

One encounters these diagrams in calculating the emission of soft photons from a (comparatively) heavy fermion. The result, summed over the \( n! \) diagrams, is described by the familiar 'eikonal formula'\(^4\), according to which the \( n \) photons may be treated as being emitted independently from the heavy source. Note that such factorization does not occur in each Feynman diagram so the sum is vastly simpler than the individual diagrams. Since we need to obtain a non-abelian generalization of the eikonal formula, it is useful first to recall how the abelian case is obtained.

The \( i \)th propagator of Fig. 1 is given by

\[
S_i = \frac{M_i + \gamma q_i}{M_i^2 - q_i^2 - i\epsilon}, \quad q_i = p + \sum_{j=1}^{i} k_j.
\]  

(2)

For other permuted diagrams \( k_j \) should be replaced by \( k_{t_j} \), but in every case we have \( q_0 = p \) and \( q_n = p' \). Let \( \omega = p \cdot k_i / M \) be the energy of the \( i \)th boson in the rest frame of the final fermion. Our central assumption is \( \omega_i \ll M \), i.e., that the interaction is soft compared to the mass of the fermion. Then the numerator of propagator (2) may be approximated by \( M + \gamma p \) provided the intermediate masses \( M_i \) are not too different from \( M \): \( \Delta M_i \equiv (M_i^2 - M^2) / 2M \ll M \). In that case (2) can be approximated by

\[
S_i = \mathcal{P} D_i,
\]  

(3)

where

\[
D_i = \frac{1}{\Delta M_i - W_i - i\epsilon},
\]  

(4)

\( W_i = \sum_{j=1}^{i} \omega_j \) is the total boson energy preceding that propagator, and \( \mathcal{P} = \frac{1}{2}(1 + \gamma p / M) \) is a projection operator. With this approximation, the vertex factor \( \gamma^\mu \) can be replaced by the velocity \( v^\mu = p^\mu / M \) of the fermion, because

\[
\mathcal{P} \gamma^\mu \mathcal{P} = v^\mu \mathcal{P}.
\]  

(5)

For the abelian eikonal formula, a single fermion without resonance is considered, so \( \Delta M_i = 0 \). The off-shell (for line \( p' \)) photon-emission amplitude
from Fig. 1 is then

$$A^*_{0}[T_1] = \prod_{i=1}^{n} (2 ev \cdot \epsilon(k_i) D_{i0}) ,$$

(6)

where the subscript 0 is there to remind us of the condition \(\Delta M_i = 0\), and the asterisk denotes ‘off-shell’. Note that \(W_i\) in \(D_i\) contains a sum of meson energies so there is no factorization here.

With a certain amount of combinatorial algebra, it can be shown\(^4\) that the total off-shell emission amplitude is

$$A^*_{0} = \sum_{a=1}^{n!} A^*[T_a] = \prod_{i=1}^{n} \left( -\frac{ev \cdot \epsilon(k_i)}{p \cdot k_i} \right).$$

(7)

This eventual factorization of the boson variables then allows the conclusion that the \(n\) photons are emitted independently.

For \textit{on-shell} amplitudes, the condition \(M^2 - (p')^2 = 0\) is equivalent to the energy-conservation condition

$$\frac{1}{M} p \cdot \sum_{i=1}^{n} k_i = \sum_{i=1}^{n} \omega_i = 0$$

(8)

in the approximation \(\omega_i \ll M\). The on-shell amplitude \(A_0\) is equal to the off-shell amplitude \(A^*_{0}\) with the \(n\)th propagator removed, so it is

$$A_0 \sim (M^2 - p'^2) A^*_{0} = 0$$

(9)

A more careful calculation shows that \(A_0\) is actually proportional to the products of \(\delta(\omega_i)^4\). Formulas (7) and (9) will be referred to as the \textit{abelian eikonal formula}.

We shall now proceed to consider the non-abelian generalization of this formula. In that case, the on-shell amplitude from Fig. 1 is

$$A[T_1] = \langle p \lambda | V_1 S_1 V_2 S_2 V_3 \cdots V_{n-1} S_{n-1} V_{n} | p' \lambda' \rangle$$

(10)

where \(V_i\) is the \(i\)th vertex attached to the boson \(B_i\), \(S_i\) is the propagator in (2), and \(\lambda, \lambda'\) represent helicities and other internal quantum numbers. We allow in this non-abelian version multi-channel configurations where both \(V_i\) and \(\Delta M_i\) should be regarded as matrices connecting fermions of various
helicities and internal quantum numbers. These matrices do not commute and it is in this sense that the amplitude is non-abelian. Within the same heavy-fermion approximation, the numerator $M + \gamma q_i$ in (2) may be replaced by its on-shell value and written as $\sum_{\lambda_i} u_{\lambda_i} (q_i) \bar{u}_{\lambda_i} (q_i)$. In this way (10) can be written as

$$A[T_1] = V_1 D_1 V_2 D_2 V_3 \cdots V_{n-1} D_{n-1} V_n$$

provided we take

$$V_i = \langle q_i \lambda_i | V_i | q_{i-1} \lambda_{i-1} \rangle$$

and $D_i$ defined in (4). For the permuted diagrams, $V_i$ and $\Delta M_i$ are to be replaced by $V_{ti}$ and $\Delta M_{ti}$.

Eq. (11) will be our starting point for further analyses. As discussed above, it is valid to the leading order of $\omega_i/M$. However, if we interpret $A[T_1]$ as the pole-part of the amplitude, then (11) is exact provided the bosons are massless and the boson momenta are all parallel to one another (‘forward scattering’). This is so because under that condition $k_i \cdot k_j = 0$ so the expression $D_i$ in (3) and (4) is exact. So is eq. (8). Moreover, by taking the pole part of the amplitude, the numerator of (2) should be taken on shell (the residue of the poles) so the vertices can be rigorously replaced by their helicity matrix elements as in (12).

Given that, it is more satisfying to regard (11) in this way, as an exact expression for the pole part of the forward massless scattering amplitude, so that an exact non-abelian formula for the sum can be obtained. At the end, we might want to put in boson masses, consider non-forward scattering, and the non-pole part of the tree amplitude. These lead to corrections of order $\omega_i/M$, so the formula would still be valid to the zeroth order of this small parameter.

To proceed further, let us first continue to ignore the possibility of resonances and put all $\Delta M_i = 0$. Eq. (11) then becomes

$$A_0[T_1] = V[T_1] D_0[T_1] ,$$

where

$$V[T_1] = V_1 V_2 V_3 \cdots V_{n-1} V_n ,$$

and

$$D_0[T_1] = \prod_{i=1}^{n-1} D_{i0} = \prod_{i=1}^{n-1} \frac{1}{W_i - i\epsilon} .$$
Figure 2: Same as Fig. 1 but with spurious bosons (dashed lines) put in to take care of resonance-mass corrections.

The sum over all diagrams $A_0 = \sum_{a=1}^{n!} A_0[T_a]$ can then be obtained by a combinatorial argument to be

$$A_0 = \sum_{b=1}^{(n-1)!} V^{MC}[T'_b n] D_0[T'_b n] ,$$

(16)

with the following notations. $[T'_b n] = [t'_{b_1} t'_{b_2} \cdots t'_{b_{n-1}} n]$ is the subset of trees $T_a$ with the boson $B_n$ fixed at the end, and the sum in (16) is over the $(n-1)!$ permutations of the remaining boson lines. $V^{MC}$ stands for ‘multiple commutator’

$$V^{MC}[T'_b n] = [V_{b_1} [V_{b_2} [V_{b_3} \cdots [V_{b_{n-1}}, V_n] \cdots]]] .$$

(17)

Line $n$ is singled out in (16) but we could have singled out any other line instead. Note that the derivation of (16) is not trivial, as can be seen from the fact that each multiple commutator contains $2^{n-1}$ terms so (16) contains a total of $(n-1)!2^{n-1}$ terms, whereas (15) where this comes from contains a total of only $n!$ terms.

Finally let us see how the formula changes when resonances are included to make $\Delta M_i \neq 0$. In that case, instead of $D_{i0}$, we should use the full $D_i$ in
By using the expansion
\[ D_i = D_{i0} \sum_{\ell_i=0}^{\infty} (-\Delta M_i D_{i0})^{\ell_i}, \tag{18} \]
we can cast this amplitude into sums of amplitudes without resonances by introducing spurious bosons. Specifically, for a given \( \ell = \sum_{i=1}^{n-1} \ell_i \), the expanded amplitude can be represented by Fig. 2, which differs from Fig. 1 by having \( \ell \) spurious bosons (indicated by dotted lines) added. These dotted lines must appear between the first and the last wavey lines, but otherwise they are to be inserted in all possible ways. The dotted lines carry zero energy and a vertex factor
\[ U_{t_i} = -\Delta M_{t_i}, \]
where \( t_i \) is the original (wavey line) boson immediately to the right of the dotted line concerned. All dotted lines between two consecutive wavey lines carry the same zero energy and possess the same vertex, so they are to be thought of as identical particles. In the presence of resonances, the sum over all diagrams are again given by (16), but now modified to describe Fig. 2 instead of Fig. 1, and a sum over all \( \ell \) must also be taken. The permutation in (16) is similarly modified to include all permutation of non-identical bosons in Fig. 2. In spite of the many sums present, the important thing is that it remains to be given by multiple commutators of the \( V_i \)'s and the \( U_i \)'s.

3 Large-\( N_c \) QCD

We can use (16) to demonstrate the consistency of inelastic meson-baryon scattering amplitude in large-\( N_c \) QCD. The requirement of consistency for \( n = 2 \) and 3 led to very interesting physical predictions\(^2\). For an arbitrary \( n \), to our knowledge the consistency has not been demonstrated and we shall use (16) to show it.

Let me review briefly the problem encountered in the limit of large-\( N_c \).\(^2\) The baryon mass \( M \) in that limit is of order \( N_c \), so from (2)–(4) the baryon propagator is of order 1. However, the meson-baryon Yukawa coupling constant grows like \( \sqrt{N_c} \), so a diagram like Fig. 1 will be of order \( N_c^{n/2} \). On the other hand, the complete \( n \)-meson baryon amplitude is known to go down with \( n \) like \( N_c^{1-n/2} \). There is then a discrepancy of \( n - 1 \) powers of \( N_c \) between an individual diagram and the complete amplitude. How can one effect such
a huge cancellation between the individual diagrams in the sum? This is the consistency problem mentioned above. The problem of course gets worse as $n$ increases.

For $n = 2$ and $3$, a multiple-commutator formula similar to (16) had been used to demonstrate the consistency$^2$. It proceeds by showing that though individual vertices $V_i$ are of order $\sqrt{N_c}$ so the product of $n$ vertices behave like $N_c^{n/2}$, the multiple commutators of these vertices in fact behaves like $N_c^{1-n/2}$, as desired. The same proof works for an arbitrary $n$ as long as a formula like (16) is available to convert the sum of diagrams into multiple commutators.

To understand why multiple commutators lead to such an enormous cancellation, we have to go back to study how a large-$N_c$ baryon is made of and how it couples. A baryon in this framework is a color-singlet object composed of $N_c$ quarks in the S state. Different baryons arise from different spin and flavor excitations, but the wave function must be symmetric in the spin and flavor variables of the $N_c$ quarks. Spin is described by the rotational group $SU(2)_J$ so the spin of a baryon is represented by the Young tableau in Fig. 3(a). Each box in the tableau represents a quark. Those columns with two boxes are spin singlets, and those columns with one box are spin doublets symmetrical under permutation of these columns, so the spin of the baryon is $J$ if there are $2J$ single-box columns. Fig. 3(b) is for the flavor group. The tableau is identical to the one in 3(a) to enforce the symmetry of the wave function. This time each box represents the flavor state of a quark. Under isotopic-spin symmetry the flavor group is $SU(2)_F$. The total isospin of the baryon is $I = J$ because there are $2J = 2I$ single-box columns in 3(b). Putting these together we conclude that the allowed baryon resonances have $I = J = \frac{1}{2}, \frac{3}{2}, \cdots, \frac{N_c}{2}$.

On the other hand, if we assume a flavor symmetry of $SU(3)_F$, then the two-box columns in 3(b) give rise to a $\overline{3}$ representation while the single-box columns give rise to a $3$ representation. If there are $p$ single-box columns and $q$ two-box columns in 3(b), so that $N_c = p + 2q$, then the $SU(3)_F$ representation of the baryon is $(p,q)$ (where $(1,1)$ is the octet, $(3,0)$ is the decuplet, $(2,2)$ is the 27-plet, etc). For large $N_c$, one or both of $(p,q)$ must be large, and we are then talking about $SU(3)_F$ representaions too large to appear in Nature. Consequently it is useless to talk about large $N_c$ as
an approximation to Nature if we insist on $SU(3)_F$ symmetry as well as
the identical-particle symmetry of the baryonic wave function. In that sense
large $N_c$ hates $SU(3)_F$, which makes the successful broken-$SU(3)_F$ predictions
from the large-$N_c$ analyses\(^2\) even more intriguing. To consider large $N_c$ as
a useful tool we must go back to $SU(2)_F$ symmetry. We may still include
strange and other quarks, but in no way they should be regarded as identical
to the up and down quarks. New quarks of each species in the baryon are to
be represented by a new tableau in Fig. 3(a), with $2K$ single-box columns
if that is the number of new quarks of that species. They carry a total spin
of $K$. There will be no corresponding tableau for these new quarks in 3(b)
because they are strong-isospin singlets. Consequently the total spin carried
by the up and down quarks is still $I$, so the allowed spin $J$ of the baryon
is obtained by adding $I$ and $K$ vectorially, \textit{i.e.}, it ranges from $|I + K|$ to
$|I - K|$. We are now ready to describe how the baryon matrix elements for the
vertices are computed. Let $q^\dagger, q$ be free-quark creation and annihilation op-
erators. Let us abbreviate a one-body (free) quark operator as
\begin{equation}
q^\dagger \Gamma q \equiv \{ \Gamma \} ,
\end{equation}
where $\Gamma$ is either the (spin) Pauli matrices $\sigma^i$, the flavor matrices $t^a$, their
products, or $1$. For two flavors, $t^a = \tau^a$ are the Pauli matrices, and for
three flavors, they are the Gell-Mann matrices $t^a = \lambda^a$. The use of $SU(3)_F$
quark operators and matrices does not imply that the baryon wavefunction
is $SU(3)_F$ symmetric.
Any operator $O$ can always be written as sums of products of operators of the form (17), the only constraint being that they must have the same quantum numbers as $O$. A product of $m$ one-body operators will be referred to as an $m$-body operator, and it can be shown$^2$ that an explicit factor $F_m = N_c^{1-m}$ must accompany the presence of an $m$-body operator. We need to find out the $N_c$-dependence of the baryon matrix elements of $\{\Gamma\}$ and its products. We are interested only in baryons of low $J$ and $I$ which exist in Nature, hence most of the columns (of order $N_c$) in Fig. 3 are two-box columns. They constitute singlet spins and singlet isospins and contribute zero to matrix elements of $\Gamma = \sigma^i$ and $\tau^a$. For these $\Gamma$, the baryon matrix element

$$\langle \Gamma \rangle = \langle q_i \lambda_i \{\Gamma\} | q_{i-1} \lambda_{i-1} \rangle$$

comes only from the single-box columns so it is of order 1. For $\Gamma = 1$, or $\sigma^i \tau^a$, or if it involves matrices connecting the up/down quarks to other quarks, the two-box columns do contribute so the matrix element $\langle \Gamma \rangle$ is generally of order $N_c$, the order of the number of columns. Similar arguments show that the matrix element $\langle \{\Gamma\}^m \rangle$ of an $m$-body operator is at most of order $N_c^m$, so with the factor $F_m$ in front, the matrix element of every term of $O$ is at most of order $N_c$ (assuming that there are no explicit factors of $N_c$ accompanying the definition of $O$).

Though $\langle \{\Gamma\}^m \rangle$ is generally of order $N_c$, it might be as suppressed as $N_c^{1-m}$ if all the $\Gamma$’s are made up of $\sigma^i$’s and $\tau^a$’s. When this occurs important phenomenological consequences often follow$^2$. For example, if all the $m > 1$-body operators are suppressed, then the dominant contribution comes from the one-body operator, but that is just treating the quarks inside the baryon as if they were free, so this gives rise to a large-$N_c$ justification of the quark model. Another example has to do with the mass of a baryon. It can be computed from the matrix elements of the operators $\{1\}, \{\vec{J}\} \cdot \{\vec{J}\}/N_c$, etc., so it is of the form $M_i = aN_c + bJ(J+1)/N_c$. The mass split $\Delta M_i$ is not only small compared to $M$ as demanded by the approximations needed to reach (16), it is actually of order $N_c^{-1}$ and not just order 1. Like the Skyrme model, or the strong-coupling model, the baryonic resonances are rotational levels very closely packed.

It is now easy to see why (16) leads to the enormous cancellation between individual Feynman diagrams needed for the consistency. From free-quark
commutation relation, the commutator of two 1-body operators is again a 1-body operator, and the commutator of an \( m \)-body operator with an \( m' \)-body operator is an \( (m + m' - 1) \)-body operator. As far as matrix elements are concerned, this means that each time we commute we lose one power of \( N_c \), so all together \( (n - 1) \) powers of \( N_c \) are lost by the multiple commutator in (16), and this is precisely the amount needed to match the discrepancy between individual and the sum of meson-baryon scattering diagrams.

If baryon resonances are included, (16) can still be used with modifications, as explained. The argument above valid for \( \Delta M_i = 0 \) will still be valid if the additional vertices \( U_i = -\Delta M_i \) are of order 1. As we saw above, this is actually of order \( N_c^{-1} \) so there is no problem when baryon resonances are present either.

In this way we prove the consistency of the large-\( N_c \) amplitudes if (16) is used, with or without baryonic resonances. Eq. (16) is exact for the pole part of the massless forward scattering amplitude. When boson masses are taken into account and the forward scattering restriction is relaxed, eq. (16) is only approximate. As such cancellations arising from the multiple commutators are still present, but (16) alone is not sufficient to demonstrate the complete cancellation of the \( (n - 1) \) powers of \( N_c \). We shall not discuss this more difficult problem here.

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5 References

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