A mollifier approach to regularize a Cauchy problem for the inhomogeneous Helmholtz equation

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Abstract

The Cauchy problem for the inhomogeneous Helmholtz equation with non-uniform refraction index is considered. The ill-posedness of this problem is tackled by means of the variational form of mollification. This approach is proved to be consistent, and the proposed numerical simulations are quite promising.

1 Introduction

Let $V$ be a $C^{3,1}$ bounded domain of $\mathbb{R}^3$ with boundary $\partial V$. For $x' \in \partial V$, we denote by $\nu(x')$ the unit normal vector to $\partial V$ pointing outward $V$. Let $\Gamma$ be a nonempty open subset of $\partial V$. We consider the Cauchy problem for the inhomogeneous Helmholtz equation

$$\Delta u(x) + k^2 \eta(x) u(x) = S(x), \quad x \in V, \quad (1)$$
$$\partial_n u(x') = f(x'), \quad x' \in \Gamma, \quad (2)$$
$$u(x') = g(x'), \quad x' \in \Gamma. \quad (3)$$

Here, $u = u(x)$ is the unknown amplitude of the incident field, $\eta \in L^\infty(\Omega)$ is the refraction index, $k$ is a positive wave number, $S \in L^2(\Omega)$ is the source function, and $f \in L^2(\Gamma)$ and $g \in L^2(\Gamma)$. This work is supported in part by the grant ANR-17-CE40-0029 of the French National Research Agency ANR (project MultiOnde).
\( g \in L^2(\Gamma) \) are empirically known boundary conditions.

The Helmholtz equation arises in a large range of applications related to the propagation of acoustic and electromagnetic waves in the time-harmonic regime. In this paper, we consider the inverse problem of reconstructing an acoustic or electromagnetic field from partial data given on an open part of the boundary of a given domain. This problem called the Cauchy problem for the Helmholtz equation is known to be ill-posed if \( \Gamma \) does not occupy the whole boundary \( \partial V \). In [17], the above system was considered in the particular case where the refraction index is constant. However, in practice, the emitted wave travels through an environment in which the refraction index fails to be constant, and we need to investigate the corresponding problem.

We are facing a linear inverse problem. Our aim is to derive a stable approximation method for this problem, which yields stable and amenable computational scheme. Our main focus will be on mollification, in the variational sense of the term, which turns out to be both flexible and numerically efficient.

Mollifiers were introduced in partial differential equations by K.O. Friedrichs [27, 11]. The term mollification has been used in the field of inverse problems since the eighties. In the original works on the subject, mollifiers were used to smooth the data prior to inversion. In his book, D.A. Murio [23] provides an overview of this approach and its application to some classical inverse problems. Let us also mention the paper by D.N. Hao [14], which provides a wide framework for the mollification approach in this initial meaning. In [21], A.K. Louis and P. Maass proposed another approach, based on inner product duality. This approach has been subsequently referred to as the method of approximate inverses [24]. The approximate inverses are particularly well adapted to problems in which the adjoint equation has explicit solutions. A third approach, based on a variational formulation, also appeared in the same period of time. In [20], A. Lannes et al. gave such a formulation while studying the problems of Fourier extrapolation and deconvolution. This variational formulation was not studied further until the papers by N. Alibaud et al. [2] and by X. Bonnefond and P. Maréchal [7], where convergence properties of the variational formulation was considered.

A definite advantage of the variational approach to mollification lies in the fact that it offers a quite flexible framework, just like the Tikhonov regularization, while being more respectful of the initial model equation than the latter.

The paper is organized as follows. In Section 2, we introduce the linear operator associated to the Cauchy problem for the inhomogeneous Helmholtz equation. We propose a regularized variational formulation of the ill-posed problem based on mollification. Under an additional smoothness assumption on the targeted solution we show in Theorem 7 that the unique minimier converges strongly to the minimum-norm least square solution. Section 3 is devoted to numerical experiments. We consider two numerical examples in order to illustrate the efficiency of our regularization approach.
2 Functional setting and regularization

We shall work in a functional space which enables us to interpret the ideal (exact) data as the image of $u$ by a bounded linear operator. We observe that:

(I) the Laplacian $\Delta$ is a bounded operator from $H^2(V)$ to $L^2(V)$, so that, since $\eta \in L^\infty(\Omega)$, the operator $T_1u = (\Delta + k^2\eta)u$ is also bounded from $H^2(V)$ to $L^2(V)$ [12];

(II) $\nabla u \in (H^1(V))^3$, so that $u \mapsto \partial_v u \in H^{1/2}(\Gamma)$ is a continuous linear operator, which implies in turn that the operator $T_2u = \partial_v u|_{\Gamma}$ is compact from $H^2(V)$ to $L^2(\Gamma)$;

(III) the trace operator $u \mapsto u|_{\partial V}$ maps $H^2(V)$ to $H^{3/2}(\Gamma)$ continuously, so that the operator $T_3u = u|_{\Gamma}$ is compact from $H^2(V)$ to $L^2(\Gamma)$.

Therefore, a natural choice for our workspace is $H^2(V)$. We can then write our system in the form

$$Tu = v,$$

in which

$$T: H^2(V) \rightarrow L^2(V) \times L^2(\Gamma) \times L^2(\Gamma)$$

$$u \mapsto (\Delta u + k^2\eta u, \partial_v u, u)$$

and

$$v = (S, f, g) \in G := L^2(V) \times L^2(\Gamma) \times L^2(\Gamma).$$

We first show that $T$ is injective.

**Proposition 1.** The linear bounded map $T$ defined by (4) is injective.

**Proof.** For a proof of this classical result, we refer to the Fritz John’s book [19]. More recent proofs based on Carleman estimates can be found in [18, 9, 1, 26]. The principal idea is to show that a part of $\Gamma$ is non-characteristic with respect to the Helmholtz operator, which in turns leads to the existence of a small neighborhood of that part of the boundary in $V$ where the solution is identically zero. Since $\eta \in L^\infty(\Omega)$, the Helmholtz operator $T_1$ possesses the unique continuation property in $V$, and hence the solution is identically zero in the whole domain $V$ which completes the proof. $\square$

We are now going to set up our approach to the regularization of the problem. We consider the mollifier

$$\varphi_\alpha(x) = \frac{1}{\alpha^3} \varphi \left( \frac{x}{\alpha} \right), \quad \alpha \in (0, 1], \quad x \in \mathbb{R}^3,$$

in which $\varphi$ is an integrable function such that

1. $\text{supp} \varphi \subset B_r := \{x \in \mathbb{R}^3 | \|x\| \leq r\}$ for some positive $r$,

2. $\int \varphi(x) \, dx = 1$. 

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Desirable additional properties of $\varphi$ are, as usual, nonnegativity, isotropy, smoothness, radial decrease.

We denote by $C_{\alpha}$ the convolution operator by $\varphi_{\alpha}$: for every $u \in L^2(\mathbb{R}^3)$, $C_{\alpha}u := \varphi_{\alpha} * u$. Our regularization principle will control smoothness by means of $C_{\alpha}$, and $\alpha$ will play the role of the regularization parameter. One difficulty lies in the fact that convolving $u$ by $\varphi_{\alpha}$ entails extrapolating $u$ from $V$ to the larger set $V + \alpha B_r$. Zero padding is obviously forbidden here since we wish to preserve $H^2$ regularity. It is then necessary to introduce an extension operator that preserves the properties of the solution on $V$.

For $s \in [0, 1)$, we denote by $H^{2+s}(\mathbb{R}^3)$ the set of functions in $H^{2+s}(\mathbb{R}^3)$ having a compact support. Our objective is to derive an extension operator $E: H^{2+s}(V) \to H^{2+s}(\mathbb{R}^3)$ that in addition of satisfying $Eu|_V = u$ for all $u \in H^{2+s}(V)$, is bounded and invertible. Note that there are many extension operators satisfying these properties. We next provide a complete characterization of a useful extension operator suited for $C^{3,1}$ smooth bounded domains. For other extension operators on Sobolev spaces under weaker regularity assumptions on the domain $V$, see, for example, [10, 22, 8, 25].

For $\varepsilon \in (0, 1)$ small enough define the tubular domains

$$V_{\varepsilon}^\pm := \{x' \pm t\nu(x') : (x', t) \in \partial V \times (0, \varepsilon)\}, \quad and \quad V_{\varepsilon} := V \cup \partial V \cup V_{\varepsilon}^+.$$  \hspace{1cm} (5)

We first notice that $V_{\varepsilon}^- \subset V \subset V_{\varepsilon}$ for all $\varepsilon \in (0, 1)$. Due to the regularity of $\partial V$, the function defined by

$$\phi(x', t) = x' - t\nu(x'),$$  \hspace{1cm} (6)

is a $C^{2,1}$-diffeomorphism from $\partial V \times (0, \varepsilon)$ onto $V_{\varepsilon}^-$ for $\varepsilon$ small enough.

Let $u$ be fixed in $H^2(V)$. The first step is to construct an extension $u_{\varepsilon}$ of $u$ to $H^2(V_{\varepsilon}/2)$. For $\alpha, \beta \in \mathbb{R}$, set

$$\tilde{u}(x' + t\nu(x')) = \alpha u(x' - t\nu(x')) + \beta u(x' - 2t\nu(x')), \quad for \ (x', t) \in \partial V \times (0, \varepsilon/2).$$  \hspace{1cm} (7)

Since $\nu$ is a $C^2$ vector field, $\tilde{u}$ lies in $H^2(V_{\varepsilon}^+)$, and verifies

$$\tilde{u}(x') = (\alpha + \beta)u(x'), \quad and \quad \partial_t \tilde{u}(x') = (-\alpha - 2\beta)\partial_t u(x'), \quad x' \in \partial V.$$  \hspace{1cm} (8)

Let $u_{\varepsilon}$ be defined by

$$u_{\varepsilon}(x) = \begin{cases} \tilde{u}(x), & x \in V_{\varepsilon}^+; \\ u(x), & x \in V. \end{cases}$$  \hspace{1cm} (9)

By construction, we have $u_{\varepsilon} \in H^2(V) \cup H^2(V_{\varepsilon}^+)$. Considering the traces (8) and taking $\alpha = 3$ and $\beta = -2$, implies that $u_{\varepsilon}$ and its first derivatives have no jumps across $\partial V$, and thus $u_{\varepsilon} \in H^2(V_{\varepsilon}/2)$.

Let $\chi_{\varepsilon} \in C_0^\infty(\mathbb{R}^3)$ be a cut off function satisfying

$$\chi_{\varepsilon} = 1, \ on \ V_{\varepsilon/8}, \ and \ \chi_{\varepsilon} = 0, \ on \ \mathbb{R}^3 \setminus \overline{V_{3\varepsilon/8}}.$$  \hspace{1cm} (10)


Now, we are ready to introduce the operator $E$. For $u \in H^2(V)$, define

$$Eu = \chi_\varepsilon u_\varepsilon,$$

where $u_\varepsilon$ and $\chi_\varepsilon$ are respectively given in (9) and (10).

**Proposition 2.** Let $s \in [0, 1)$ be fixed. The extension operator $E : H^{2+s}(V) \to H_0^{2+s}(\mathbb{R}^3)$ is bounded, invertible, and satisfies

$$\|u\|_{H^{2+s}(V)} \leq \|Eu\|_{H^{2+s}(\mathbb{R}^3)} \leq C_s \|u\|_{H^{2+s}(V)},$$

where $C_s > 1$ is a constant that only depends on $s$, $\varepsilon$ and $V$. In addition, $\text{Supp}(Eu) \subset V_\varepsilon$.

**Proof.** The left side inequality is straightforward. The functions $\psi_j : V_{\varepsilon/j}^+ \to V_{\varepsilon/j}^-$, $j = 1, 2$, defined by

$$\psi_j(x' + t\nu(x')) = x' - jt\nu(x'), \quad (x', t) \in \partial V \times (0, \varepsilon), \ j = 1, 2,$$

are $C^{2,1}$-diffeomorphisms. Forward calculations give

$$\int_{\mathbb{R}^3 \setminus V} |Eu|^2 + |\nabla Eu|^2 + |\nabla^2 Eu|^2 \, dx \leq \|\chi_\varepsilon\|_{C^{2,1}(\mathbb{R}^3)}^2 \sum_{j=1}^2 \int_{V_{\varepsilon/2}^+} |\tilde{u}|^2 + |\tilde{\nabla}\tilde{u}|^2 + |\tilde{\nabla}^2\tilde{u}|^2 \, dx,$$

with $D\psi_j^{-1}$ is the gradient of the vector field $\psi_j^{-1}$, and $\kappa > 0$ is a universal constant. Therefore

$$\|Eu\|_{H^2(\mathbb{R}^3)} \leq C \|u\|_{H^2(V)},$$

where

$$C^2 = \kappa\|\chi_\varepsilon\|_{C^{2,1}(\mathbb{R}^3)}^2 \sum_{j=1}^2 \left( \|\psi_j\|_{C^{2,1}(V_{\varepsilon/j}^+)}^2 + 1 \right) \|D\psi_j^{-1}\|_{C^1(V_{\varepsilon/j}^-)}^2.$$

Similarly, tedious calculations lead to the following estimate of the seminorm

$$|D_\beta Eu|_{s,\mathbb{R}^3}^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|D_\beta Eu(x) - D_\beta Eu(y)|^2}{|x - y|^{3+2s}} \, dx \, dy \leq \tilde{C}_s^2 (|D_\beta u|_{s,V}^2 + \|u\|_{H^2(V)}^2),$$

for all $\beta \in \mathbb{N}^3$, satisfying $|\beta| = 2$, where $\tilde{C}_s > 0$, depends on $s, \varepsilon$ and $V$. By taking $C_s = \max(1, C, \tilde{C}_s)$, we complete the proof of the proposition. \qed
The defined extension operator $E$ opens the way to the following variational formulation of mollification:

$$ (\mathcal{P}) \quad \begin{aligned} \text{Minimize} \quad & J_\alpha(u, v, T) := \|v - Tu\|^2 + \|(I - C_\alpha) Eu\|^2_{H^2(\mathbb{R}^3)} \\ \text{subject to} \quad & u \in H^2(V). \end{aligned} $$

Our aim is now to prove

1. the well-posedness of the above variational problem, that is, that the solution $u_\alpha$ depends continuously on the data $v$;

2. the consistency of the regularization, that is, that $u_\alpha$ converges to $T^*u$ in some sense as $\alpha \downarrow 0$, where $T^*$ is the pseudo-inverse of $T$.

**Lemma 3.** Let $K, s > 0$. Let $\varphi \in L^1(\mathbb{R}^3)$ be such that

$$ \hat{\varphi}(0) = 1 \quad \text{and} \quad 1 - \hat{\varphi}(\xi) \sim K \|\xi\|^s \quad \text{as} \quad \xi \to 0. $$

Assume in addition that $\hat{\varphi}(\xi) \neq 1$ for every $\xi \in \mathbb{R}^3 \setminus \{0\}$. Define

$$ m_\alpha = \min_{\|\xi\|=1} |1 - \hat{\varphi}(\alpha \xi)|^2 \quad \text{and} \quad M_\alpha = \max_{\|\xi\|=1} |1 - \hat{\varphi}(\alpha \xi)|^2. $$

Then,

(i) for every $\alpha > 0$, $0 < m_\alpha \leq M_\alpha \leq (1 + \|\varphi\|_1)^2$;

(ii) $\sup_{\alpha > 0} M_\alpha / m_\alpha < \infty$ and $M_\alpha \to 0$ as $\alpha \downarrow 0$;

(iii) there exists $\nu_0, A_0 > 0$ such that, for every $\alpha \in (0, 1]$, for every $\xi \in \mathbb{R}^3 \setminus \{0\}$,

$$ \nu_0 \left( \|\xi\|^{2s} \mathbb{1}_{B_{1/\alpha}}(\xi) + \frac{1}{M_\alpha} \mathbb{1}_{B_{1/\alpha}^c}(\xi) \right) \leq \frac{|1 - \hat{\varphi}(\alpha \xi)|^2}{|1 - \hat{\varphi}(\alpha \xi/ \|\xi\|)|^2} \leq \mu_0 \|\xi\|^{2s}. $$

**Proof.** See [2, Lemma 12]. \(\blacksquare\)

**Corollary 4.** Let $\varphi, m_\alpha, M_\alpha, \nu_0, \mu_0$ be as in Lemma 3, and let $C_\alpha$ be the operator of convolution with $\varphi_\alpha$. For every $u \in L^2(\mathbb{R}^3)$,

$$ \|(I - C_\alpha) u\|^2_{L^2(\mathbb{R}^3)} \geq \nu_0 m_\alpha \int_{\mathbb{R}^3} \left( \|\xi\|^{2s} \mathbb{1}_{B_{1/\alpha}}(\xi) + \frac{1}{M_\alpha} \mathbb{1}_{B_{1/\alpha}^c}(\xi) \right) |\hat{u}(\xi)|^2 \, d\xi. \quad (13) $$

**Proof.** From Lemma 3, we have:

$$ \|(I - C_\alpha) u\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |1 - \hat{\varphi}(\alpha \xi)|^2 |\hat{u}(\xi)|^2 \, d\xi $$

$$ = \int_{\mathbb{R}^3} |1 - \hat{\varphi}(\alpha \xi/ \|\xi\|)|^2 \|\xi\|^{2s} \frac{|1 - \hat{\varphi}(\alpha \xi/ \|\xi\|)|^2}{|1 - \hat{\varphi}(\alpha \xi/ \|\xi\|)|^2} |\hat{u}(\xi)|^2 \, d\xi $$

$$ \geq m_\alpha \nu_0 \int_{\mathbb{R}^3} \left( \|\xi\|^{2s} \mathbb{1}_{B_{1/\alpha}}(\xi) + \frac{1}{M_\alpha} \mathbb{1}_{B_{1/\alpha}^c}(\xi) \right) |\hat{u}(\xi)|^2 \, d\xi. $$
**Lemma 5.** Let $C_\alpha$ be the operator of convolution with $\varphi_\alpha$, where $\varphi$ is as in Lemma 3. There exists a positive constant $A_\alpha$, depending on $V$ only, such that for every $s \geq 0$ and every $u \in H^s(V)$,

$$A_\alpha m_\alpha \|u\|_{H^s(\mathbb{R}^3)}^2 \leq \|(I - C_\alpha)u\|_{H^s(\mathbb{R}^3)}^2. \quad (14)$$

**Proof.** We start with the case $s = 0$. From (13), we have:

$$\|(I - C_\alpha)u\|_{L^2(\mathbb{R}^3)}^2 \geq m_\alpha \nu_\alpha \int_{\mathbb{R}^3} \left( \|\xi\|^{2s} 1_{B_1/\alpha} \left( \xi + \frac{1}{M_\alpha} 1_{B_1/\alpha} (\xi) \right) \right) |\hat{u}(\xi)|^2 \, d\xi$$

$$\geq m_\alpha \nu_\alpha (1 + \|\varphi\|_1)^{-2} \int_{\mathbb{R}^3} 1_{B_1^c}(\xi) |\hat{u}(\xi)|^2 \, d\xi$$

$$\geq m_\alpha \nu_\alpha (1 + \|\varphi\|_1)^{-2} \left\| T_{B_1^c}^{-1} \right\|_2^2 \|u\|_{L^2(\mathbb{R}^3)}^2,$$

in which $T_{B_1^c} := 1_{B_1^c} F$, the operator of Fourier truncation to $B_1^c$. Thus

$$A_\alpha = \nu_\alpha (1 + \|\varphi\|_1)^{-2} \left\| T_{B_1^c}^{-1} \right\|_2^2$$

is suitable. Notice that, for $s = 0$, Parseval’s identity enables to rewrite (14) in the form

$$A_\alpha m_\alpha \|\hat{u}\|_{L^2(\mathbb{R}^3)}^2 \leq \|(1 - \hat{\varphi}(\alpha \cdot) \hat{u}\|_{L^2(\mathbb{R}^3)}^2, \quad (15)$$

in which $F$ denotes the Fourier-Plancherel operator. Now, let $s > 0$ and assume that $u \in H^s(V)$. We readily see that $F^{-1}((1 + \|\xi\|^2)^{s/2} \hat{u}(\xi))$ belongs to $L^2(\mathbb{R}^3)$. Applying (15) to the latter function yields

$$A_\alpha m_\alpha \|(1 + \|\cdot\|^{2s/2} \hat{u}\|_{L^2(\mathbb{R}^3)}^2 \leq \|(1 - \hat{\varphi}(\alpha \cdot))(1 + \|\cdot\|^{2s/2} \hat{u}\|_{L^2(\mathbb{R}^3)}^2,$$

and (14) follows. \(\blacksquare\)

**Lemma 6.** Let $C_\alpha$ be as in the previous lemma, $s > 0$, and $s_\circ \geq 0$. If $u \in H^{s_\circ + s}(\mathbb{R}^3)$, then

$$\|(I - C_\alpha)u\|_{H^{s_\circ}(\mathbb{R}^3)}^2 \leq \mu_\circ M_\alpha \|u\|_{H^{s_\circ + s}(\mathbb{R}^3)}^2,$$

with $\mu_\circ$ is the positive constant provided by Lemma 3.

**Proof.** We have:

$$\|(I - C_\alpha)u\|_{H^{s_\circ}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} (1 - \hat{\varphi}(\alpha \xi/\|\xi\|)^2 \left| \frac{1 - \hat{\varphi}(\alpha \xi/\|\xi\|)}{1 - \hat{\varphi}(\alpha \xi/\|\xi\|)} \right|^2 (1 + \|\xi\|^{2s_\circ} |\hat{u}(\xi)|^2 \, d\xi$$

$$\leq \mu_\circ M_\alpha \int_{\mathbb{R}^3} \|\xi\|^{2s_\circ} (1 + \|\xi\|^2) |\hat{u}(\xi)|^2 \, d\xi$$

$$\leq \mu_\circ M_\alpha \int_{\mathbb{R}^3} (1 + \|\xi\|^{2s_\circ + s} |\hat{u}(\xi)|^2 \, d\xi$$

$$= \mu_\circ M_\alpha \|u\|_{H^{s_\circ + s}(\mathbb{R}^3)}^2,$$

in which the first inequality stems from Lemma 3. \(\blacksquare\)
Theorem 7. Assume \( u^\dagger \in H^{2+s}(\mathbb{R}^3) \), with \( s \in (0, 1) \), and \( v = Tu_{|V}^\dagger \), so that \( u_{|V}^\dagger = T^\dagger v \). Let \( u_\alpha \) be the solution to Problem (\( \mathcal{P} \)). Then \( u_\alpha \to T^\dagger v \) in \( H^2(V) \) as \( \alpha \downarrow 0 \).

**Proof.** We walk in the steps of the proof of Theorem 11 in [2], which we adapt to the present context. The main differences lie in that the regularization term uses a Sobolev norm and in that we make use of the extension operator \( \bar{E} \) in order to cope with boundary constraints. In Step 1, we show that the family \((u_\alpha)\) is bounded in \( H^2(V) \), thus weakly compact; in Step 2, we establish the weak convergence of \( u_\alpha \) to \( T^\dagger v \), and finally in Step 3, we use a compactness argument to show that the convergence is, in fact, strong.

**Step 1.** By construction, we have:

\[
\|(I - C_\alpha)Eu_\alpha\|_{H^2(\mathbb{R}^3)}^2 \leq J_\alpha(u_\alpha; v, T) \leq J_\alpha(u_{|V}^\dagger; v, T) \leq \left\|(I - C_\alpha)Eu_{|V}^\dagger\right\|^2_{H^2(\mathbb{R}^3)}. \tag{16}
\]

Using Lemma 5 and Lemma 6, we obtain:

\[
\|u_\alpha\|_{H^2(V)}^2 \leq \|Eu_\alpha\|_{H^2(\mathbb{R}^3)}^2 \leq \frac{\mu_0 M_\alpha}{A_\alpha m_\alpha} \left\|Eu_{|V}^\dagger\right\|^2_{H^{2+s}(\mathbb{R}^3)},
\]

and Lemma 3(ii) then shows that the set \((u_\alpha)_{\alpha \in (0,1]}\) is bounded in \( H^2(V) \), therefore is weakly compact.

**Step 2.** Denote \( \|\cdot\|_G \) the natural norm in the Hilbert space \( G \). Now, let \((\alpha_n)\) be a sequence converging to 0. There then exists a subsequence \((u_{\alpha_{n_k}})\) which converges weakly in \( H^2(V) \). Let \( \tilde{u} \) be the weak limit of this subsequence. We then have:

\[
\|v - Tu_{\alpha_{n_k}}\|_G^2 \leq J_{\alpha_{n_k}}(u_{\alpha_{n_k}}; v, T) \leq J_{\alpha_{n_k}}(u_{|V}^\dagger; v, T)
\]

\[
= \left\|(I - C_{\alpha_{n_k}})u_{|V}^\dagger\right\|^2_{H^2(\mathbb{R}^3)}
\]

\[
\leq \frac{\mu_0 M_{\alpha_{n_k}}}{A_{\alpha_{n_k}} m_{\alpha_{n_k}}} \left\|u_{|V}^\dagger\right\|^2_{H^{2+s}(\mathbb{R}^3)}.
\]

Since \( M_{\alpha_{n_k}} \) goes to zero as \( k \to \infty \), so does \( \|v - Tu_{\alpha_{n_k}}\|_G^2 \). By weak lower semicontinuity of the norm on the Hilbert space \( G \), we see that the weak limit \( \tilde{u} \) satisfies:

\[
\|v - T\tilde{u}\|_G^2 \leq \liminf_{k \to \infty} \|v - Tu_{\alpha_{n_k}}\|_G^2 = \lim_{k \to \infty} \|v - Tu_{\alpha_{n_k}}\|_G^2 = 0.
\]

Therefore, \( T\tilde{u} = v \) and the injectivity of \( T \) (Proposition 1) implies that \( \tilde{u} = u^\dagger \).

**Step 3.** We will show that, for every multi-index \( \beta \in \mathbb{N}^3 \) such that \( |\beta| \leq 2 \),

\[
D\beta Eu_\alpha \to D\beta Eu_{|V}^\dagger \quad \text{in} \quad L^2(\mathbb{R}^3) \quad \text{as} \quad \alpha \downarrow 0,
\tag{17}
\]

which will imply the announced strong convergence. Observe first that, by the previous step and the continuity of \( E \),

\[
Eu_\alpha \to Eu_{|V}^\dagger \quad \text{in} \quad H^2(\mathbb{R}^3) \quad \text{as} \quad \alpha \downarrow 0,
\]
so that, for every multi-index $\beta \in \mathbb{N}^3$ such that $|\beta| \leq 2$,

$$D_{\beta}E u_{\alpha} \to D_{\beta}E u_{\alpha}^\dagger \text{ in } L^2(\mathbb{R}^3) \text{ as } \alpha \downarrow 0,$$

(18)

Fix $\beta \in \{\beta' \in \mathbb{N}^3 \mid |\beta'| \leq 2\}$ and let $(\alpha_n)_n$ be a sequence converging to 0, as in the previous step. For convenience, let $u_n := u_{\alpha_n}$, $C_n := C_{\alpha_n}$, $M_n := M_{\alpha_n}$ and $m_n := m_{\alpha_n}$. Since $E u_n$ has compact support, it is obvious that

$$\lim_{n \to \infty} \sup_{\|x\| \geq n} |D_{\beta}E u_n(x)|^2 \, dx = 0.$$  

(19)

Now, for every $h \in \mathbb{R}^3$ and every function $u$, let $\mathcal{T}_h u$ denote the translated function $x \mapsto u(x - h)$. We proceed to show that

$$\sup_n \|\mathcal{T}_h D_{\beta}E u_n - D_{\beta}E u_n\|_{L^2(\mathbb{R}^3)}^2 \to 0 \text{ as } \|h\| \to 0.$$  

(20)

Together with (18) and (19), this will establish (17) via the Fréchet-Kolmogorov Theorem (see e.g. [16, Theorem 3.8 page 175]). We have:

$$\|\mathcal{T}_h D_{\beta}E u_n - D_{\beta}E u_n\|_{L^2(\mathbb{R}^3)}^2 = \|F(\mathcal{T}_h D_{\beta}E u_n - D_{\beta}E u_n)\|_{L^2(\mathbb{R}^3)}^2$$

$$= \int |e^{-2i\pi \langle h, \xi \rangle} - 1|^2 |F D_{\beta}E u_n(\xi)|^2 \, d\xi$$

$$= I_1 + I_2,$$

in which

$$I_1 := \int_{\|\xi\| \leq 1/\alpha_n} |e^{-2i\pi \langle h, \xi \rangle} - 1|^2 |F D_{\beta}E u_n(\xi)|^2 \, d\xi,$$

$$I_2 := \int_{\|\xi\| > 1/\alpha_n} |e^{-2i\pi \langle h, \xi \rangle} - 1|^2 |F D_{\beta}E u_n(\xi)|^2 \, d\xi.$$

We now bound $I_1$ and $I_2$. On the one hand,

$$I_1 = \int_{0 \leq \|\xi\| \leq 1/\alpha_n} \frac{|e^{-2i\pi \langle h, \xi \rangle} - 1|^2}{\|\xi\|^{2s}} |\|\xi\|^{2s} |F D_{\beta}E u_n(\xi)|^2 \, d\xi$$

$$\leq \sup_{\xi \neq 0} \frac{|e^{-2i\pi \langle h, \xi \rangle} - 1|^2}{\|\xi\|^{2s}} \int_{\|\xi\| \leq 1/\alpha_n} \|\xi\|^{2s} |F D_{\beta}E u_n(\xi)|^2 \, d\xi.$$

Since $|e^{-2i\pi \langle h, \xi \rangle} - 1| = O(\|\xi\|)$ and $s \leq 1$, the above supremum is finite. Let $\gamma_0$ denote its value. Therefore

$$I_1 \leq \gamma_0 \|h\|^{2s} \left( \int_{\|\xi\| \leq 1/\alpha_n} |F D_{\beta}E u_n(\xi)|^2 \, d\xi + \int_{\|\xi\| \leq 1/\alpha_n} \|\xi\|^{2s} |F D_{\beta}E u_n(\xi)|^2 \, d\xi \right)$$

$$\leq \gamma_0 \|h\|^{2s} \left( \|F D_{\beta}E u_n\|_{L^2(\mathbb{R}^3)}^2 + \int_{\|\xi\| \leq 1/\alpha_n} \|\xi\|^{2s} |F D_{\beta}E u_n(\xi)|^2 \, d\xi \right).$$
Since \((u_n)\) is bounded in \(H^2(V)\) independently of \(n\), so is \(\|FD_\beta Eu_n\|_{L^2(\mathbb{R}^3)}\). Moreover, by using Corollary 4 with \(D_\beta Eu_n\) in place of \(u\), we get

\[
\int_{\|\xi\| \leq 1/\alpha_n} \|\xi\|^{2s} |FD_\beta Lu_n(\xi)|^2 d\xi \leq \frac{1}{\nu_0 m_n} \|(I - C_n)D_\beta Eu_n\|_{L^2(\mathbb{R}^3)}^2
\]

\[
= \frac{1}{\nu_0 m_n} \|D_\beta (I - C_n)Eu_n\|_{L^2(\mathbb{R}^3)}^2
\]

\[
\leq \frac{1}{\nu_0 m_n} \|(I - C_n)Eu_n\|_{H^2(\mathbb{R}^3)}^2\]

\[
\leq \frac{1}{\nu_0 m_n} \|Eu_\dagger_{V}\|_{H^2(\mathbb{R}^3)}^2 \leq \frac{\mu_0 M_n}{n_0 m_n} \|Eu_\dagger_{V}\|_{H^{2+s}(\mathbb{R}^3)}^2,
\]

in which the last two inequalities are respectively due to the inequality (16) and Lemma 6 (with \(s_0 = 2\)). It follows that \(I_1 = \mathcal{O}(\|h\|^{2s})\). On the other hand, using again Corollary 4 with \(D_\beta Eu_n\) in place of \(u\), we have:

\[
I_2 \leq 4 \int_{\|\xi\| > 1/\alpha_n} |FD_\beta Eu_n(\xi)|^2 d\xi
\]

\[
\leq \frac{4 M_n}{\nu_0 m_n} \|(I - C_n)D_\beta Eu_n\|_{L^2(\mathbb{R}^3)}^2
\]

\[
= \frac{4 M_n}{\nu_0 m_n} \|D_\beta (I - C_n)Eu_n\|_{L^2(\mathbb{R}^3)}^2
\]

\[
\leq \frac{4 M_n}{\nu_0 m_n} \|(I - C_n)Eu_n\|_{H^2(\mathbb{R}^3)}^2
\]

\[
\leq \frac{4 M_n}{\nu_0 m_n} \|Eu_\dagger_{V}\|_{H^2(\mathbb{R}^3)}^2 \leq \frac{\mu_0 M_n}{\nu_0 m_n} M_{n} \|Eu_\dagger_{V}\|_{H^{2+s}(\mathbb{R}^3)}^2
\]

in which the last two inequalities are respectively due to the inequality (16) and Lemma 6 (with \(s_0 = 2\)). It follows that \(I_2 = \mathcal{O}(M_n)\). Gathering the obtained bounds on \(I_1\) and \(I_2\), we see that there exists a positive constant \(K\) such that

\[
\|\mathcal{T}_n D_\beta Eu_n - D_\beta Eu_n\|_{L^2(\mathbb{R}^3)}^2 \leq K \left( \|h\|^{2s} + M_n \right).
\]

(21)

Now, fix \(\varepsilon > 0\). There exists \(n_\varepsilon \in \mathbb{N}^*\) such that for every \(n \geq n_\varepsilon\), \(M_n \leq \varepsilon\). From (21), we see that

\[
\sup_n \|\mathcal{T}_n D_\beta Eu_n - D_\beta Eu_n\|_{L^2(\mathbb{R}^3)}^2 \leq \max \left\{ \max_{1 \leq m \leq n_\varepsilon} \|\mathcal{T}_m D_\beta Eu_n - D_\beta Eu_n\|_{L^2(\mathbb{R}^3)}^2, K \left( \|h\|^{2s} + \varepsilon \right) \right\}
\]

By the \(L^2\)-continuity of translation, we have:

\[
\forall n \in \mathbb{N}^*, \quad \|\mathcal{T}_n D_\beta Eu_n - D_\beta Eu_n\|_{L^2(\mathbb{R}^3)}^2 \to 0 \quad \text{as} \quad \|h\| \to 0.
\]
Consequently,
\[
\max_{1 \leq n \leq n^*} \| \mathcal{T}_n D_\beta E u_n - D_\beta E u_n \|_{L^2(\mathbb{R}^3)}^2 \to 0 \quad \text{as} \quad \| h \| \to 0,
\]
so that
\[
\limsup_{h \to 0} \sup_n \| \mathcal{T}_n D_\beta E u_n - D_\beta E u_n \|_{L^2(\mathbb{R}^3)}^2 \leq K \varepsilon.
\]
Since \( \varepsilon > 0 \) was arbitrary, (20) is established, which achieves the proof. \( \blacksquare \)

**Remark 8.** Notice that the real \( s > 0 \) in Theorem 7 can be taken arbitrary large. The proof is similar, we only need to consider an extension operator \( E \) that is bounded from \( H^{2+s}(V) \) into \( H_0^{2+s}(\mathbb{R}^3) \).

### 3 Numerical experiments

In this section, we consider two numerical examples in order to illustrate the accuracy and the efficiency of our regularization approach in the resolution of the inhomogeneous Helmholtz equation with non-constant refraction index.

In order to reduce the computational complexity, we consider the system (1)-(2)-(3) in two dimensions (as in [15]) on a rectangular domain as follows:

\[
\begin{align*}
    u_{xx}(x, y) + u_{yy}(x, y) + k^2 \eta(x, y) u(x, y) &= S(x, y), & \quad (x, y) \in [a, b] \times [0, 1], \\
    u_y(x, 0) &= f(x), & \quad x \in [a, b], \\
    u(x, 0) &= g(x), & \quad x \in [a, b].
\end{align*}
\]

(22)

Given the boundary data \( f \) and \( g \) at \( y = 0 \), we aim at approximating the solution \( u(\cdot, y) \) for \( y \in (0, 1] \).

**Example 1:** For the first example, we set \( [a, b] = [-1, 1] \), \( k = 3 \) and define the refraction index \( \eta \) and the exact solution \( u \) as:

\[
\eta(x, y) = \begin{cases} 
2 - \left( x^2 + \frac{(2y-1)^2}{0.8^2} \right)^{1/2} & \text{if } x^2 + \frac{(2y-1)^2}{0.8^2} \leq 1, \\
1 & \text{otherwise}
\end{cases}
\]

and

\[
u(x, y) = (x - 2y + 1) \sin \left( \frac{k}{\sqrt{2}} (x + 2y - 1) \right).
\]

The source term \( S \) and the boundary data \( f \) and \( g \) are defined accordingly:

\[
\begin{cases}
    S(x, y) &= -\frac{5}{2} k^2 (x - 2y + 1) \sin \left( \frac{k}{\sqrt{2}} (x + 2y - 1) \right) - 3k\sqrt{2} \cos \left( \frac{k}{\sqrt{2}} (x + 2y - 1) \right) \\
    &\quad + k^2 \eta(x, y) u(x, y), \\
    f(x) &= 2k \sqrt{2} (x + 1) \cos \left( \frac{k}{\sqrt{2}} (x - 1) \right) - 2 \sin \left( \frac{k}{\sqrt{2}} (x - 1) \right), \\
    g(x) &= (x + 1) \sin \left( \frac{k}{\sqrt{2}} (x - 1) \right).
\end{cases}
\]

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Example 2: For the second example, we consider a simpler setting where 
\[ a, b = [-1.5, 1.5] \]
\[ k = 1 \]
and define the refraction index \( \eta \) (depending only on \( y \)) and the exact solution \( u \) as:

\[
\eta(x, y) = 1 + y^2,
\]
and

\[
u(x, y) = \frac{4(1 + y)}{\sqrt{2\pi}} e^{-8x^2}.
\]

The source term \( S \) and the boundary data \( f \) and \( g \) are defined accordingly:

\[
\begin{cases}
S(x, y) = \frac{4(1+y)}{\sqrt{2\pi}} e^{-8x^2} (256x^2 - 15 + y^2), \\
f(x) = g(x) = \frac{4}{\sqrt{2\pi}} e^{-8x^2}.
\end{cases}
\]

In both cases, we consider a Gaussian convolution kernel \( \varphi_\alpha \) i.e.

\[
\varphi_\alpha(x, y) = \frac{1}{\alpha^2 2\pi} e^{-\frac{x^2+y^2}{2\alpha^2}},
\]

which satisfies the Levy-kernel condition of Lemma 3 with \( s = 2 \).

Discretization setting

For the discretization of the system (22)-(23)-(24), we use a finite difference method of order 2 described as follows.

We first define the uniform grid \( \Gamma \) on the bounded domain \([a, b] \times [0, 1]\):

\[
\Gamma^n = (x_j, y_n) \quad \text{with} \quad \begin{cases} x_j = a + (j - 1) \Delta_x, & j = 1, \ldots, n_x \\ y_n = (n - 1) \Delta_y, & n = 1, \ldots, n_y, \end{cases}
\]

where \( \Delta_x \) and \( \Delta_y \) are the discretization steps given by

\[
\Delta_x = (b - a)/(n_x - 1), \quad \Delta_y = 1/(n_y - 1).
\]

We then approximate the second derivatives \( u_{xx} \) and \( u_{yy} \) by means of the five-point stencil finite-difference scheme and \( u_y \) using a central finite difference and derive the discrete system:

\[
\begin{align*}
\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^{n-1}}{\Delta_y^2} + \frac{u_{j+1}^n - 2u_{j}^{n} + u_{j-1}^{n-1}}{\Delta_x^2} + k^2 \eta_j^n u_j^n &= S_j^n \\
u_j^1 - u_j^0 &= f_j \\
u_j^1 &= g_j,
\end{align*}
\]

where

\[
u_j^n \approx u(x_j, y_n), \quad \eta_j^n := \eta(x_j, y_n), \quad S_j^n := S(x_j, y_n), \quad f_j := f(x_j), \quad \text{and} \quad g_j := g(x_j).
\]
In summary, we obtain the following iterative scheme:

\[
\begin{align*}
&u_{xx}(x_1, y_n) \approx (2u_1^n - 5u_1^n + 4u_3^n - u_2^n)/\Delta_x^2, \\
&u_{xx}(x_{n_x}, y_n) \approx (2u_{n_x}^n - 5u_{n_x-1}^n + 4u_{n_x-2}^n - u_{n_x-3}^n)/\Delta_x^2.
\end{align*}
\]  

(29)

Hence at the boundary nodes \( j = 1 \) and \( j = n_x \), \( u_{xx}(x_j, y_n) \) (i.e. \( j = 1 \) and \( j = n_x \)), we can rewrite the discrete system (32)-(36) in the matrix form:

\[
\begin{pmatrix}
\frac{u_1^{n+1} - 2u_1^n + u_1^{n-1}}{\Delta y^2} + 2u_1^n - 5u_1^n + 4u_3^n - u_2^n + k^2\eta_1^n u_1^n & = & S_1^n, \\
\frac{u_{n_x}^{n+1} - 2u_{n_x}^n + u_{n_x}^{n-1}}{\Delta y^2} + 2u_{n_x}^n - 5u_{n_x-1}^n + 4u_{n_x-2}^n - u_{n_x-3}^n + k^2\eta_{n_x}^n u_{n_x}^n & = & S_{n_x}^n.
\end{pmatrix}
\]  

(30), (31)

In summary, we obtain the following iterative scheme:

\[
\begin{align*}
u_j^n &= g_j \\
u_j^{n+1} - (\Lambda_j^n - 4\gamma)u_j^n + \gamma(-u_4^n + 4u_3^n - 5u_2^n + u_1^{n-1}) &= \Delta_y^2S_1^n, \\
u_j^{n+1} - \Lambda_j^n u_j^n + u_j^{n-1} + \gamma(u_{j+1}^n + u_{j-1}^n) &= \Delta_y^2S_j^n, \\
u_{n_x}^{n+1} - (\Lambda_{n_x}^n - 4\gamma)u_{n_x}^n + \gamma(-u_{n_x-3} + 4u_{n_x-2}^n - 5u_{n_x-1}^n) + u_{n_x}^{n-1} &= \Delta_y^2S_{n_x}^n
\end{align*}
\]  

(34), (35), (36)

where

\[
\Lambda_j^n = 2 + 2\gamma - k^2\Delta_y^2\eta_j^n \quad \text{and} \quad \gamma = \Delta_y^2/\Delta_x^2.
\]

In (34), (35), (36), the index \( n \) runs from 1 to \( n_y - 1 \). In (35), the index \( j \) runs from 2 to \( n_x - 1 \).

By defining the column vector

\[
U^n = (u_1^n, u_2^n, ..., u_{n_x}^n)^\top, \quad n = 0, 1, ..., n_y.
\]

we can rewrite the discrete system (32)-(36) in the matrix form:

\[
\begin{align*}
U^1 &= G \\
U^2 - U^0 &= 2\Delta_x F \\
U^n - A_1U^{n-1} + U^{n-1} &= \Delta_y^2S^n, \\
U^{n+1} - A_nU^n + U^{n-1} &= \Delta_y^2S^n, \quad n = 2, ..., n_y - 1.
\end{align*}
\]  

(37), (38), (39), (40)

where

\[
G := \begin{pmatrix}
g(x_1) \\
g(x_2) \\
\vdots \\
g(x_{n_x})
\end{pmatrix}, \quad F := \begin{pmatrix}
f(x_1) \\
f(x_2) \\
\vdots \\
f(x_{n_x})
\end{pmatrix}, \quad S^n := \begin{pmatrix}
S(x_1, y_n) \\
S(x_2, y_n) \\
\vdots \\
S(x_{n_x}, y_n)
\end{pmatrix},
\]  

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and $A_n$ is the nearly tridiagonal matrix defined by

$$A_n = \begin{pmatrix} -\Lambda_1^n + 4\gamma & -5\gamma & 4\gamma & -\gamma & 0 & \cdots & 0 \\ \gamma & -\Lambda_2^n & \gamma & 0 & 0 & \cdots & : \\ 0 & \gamma & -\Lambda_3^n & \gamma & 0 & \cdots & : \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & 0 & 0 & \gamma & -\Lambda_{n_z-1}^n & \gamma \\ 0 & \cdots & 0 & -\gamma & 4\gamma & -5\gamma & -\Lambda_{n_z}^n + 4\gamma \end{pmatrix}.$$  \hspace{1cm} (41)

From (38) and (39), we can get rid of the additional unknown vector $U^0$ and get the system

$$\begin{align*} U^1 &= G \\ 2U^2 + A_1U^1 &= \Delta_2^y S^1 + 2\Delta_y F, \\ U^{n+1} + A_n U^n + U^{n-1} &= \Delta_2^y S^n, \quad n = 2, \ldots, n_y - 1. \end{align*}$$  \hspace{1cm} (42)

In order to model the noise in the measured data $f$ and $g$, we consider the noisy versions $G_\epsilon$ and $F_\epsilon$ of the vectors $G$ and $F$ defined by

$$G_\epsilon = G + \epsilon \vartheta, \quad \text{and} \quad F_\epsilon = F + \epsilon \vartheta,$$  \hspace{1cm} (43)

where $\vartheta$ is a $n_z$-column vector of zero mean drawn using the normal distribution.

From (42), we can rewrite our discrete system into a single matrix equation:

$$AU = B_\epsilon,$$

where $U$, $B_\epsilon$ are $n_xn_y$-column vectors and $A$ is the $n_xn_y \times n_xn_y$ block-triangular matrix respectively defined by

$$U = \begin{pmatrix} U^1 \\ U^2 \\ U^3 \\ \vdots \\ \vdots \\ U^{n_y-2} \\ U^{n_y-1} \\ U^{n_y} \end{pmatrix}, \quad B_\epsilon = \begin{pmatrix} G_\epsilon \\ \Delta_2^y S^1 + 2\Delta_y F_\epsilon \\ \Delta_2^y S^2 \\ \Delta_2^y S^3 \\ \vdots \\ \vdots \\ \Delta_2^y S_{n_y-2} \\ \Delta_2^y S_{n_y-1} \end{pmatrix},$$
and

$$A = \begin{pmatrix}
I_{n_x} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
A_1 & 2I_{n_x} & \vdots & & & & & & \\
I_{n_x} & A_2 & I_{n_x} & \vdots & & & & & \\
0 & I_{n_x} & A_3 & I_{n_x} & \vdots & & & & \\
& & & & \ddots & & & & \\
& & & & & \ddots & & & \\
& & & & & & \ddots & & \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & I_{n_x} & A_{n_y-2} & I_{n_x} \\
& & & & & & & I_{n_x} & A_n \\
\end{pmatrix},$$

where $I_{n_x}$ is the square identity matrix of size $n_x$ and the matrices $A_n$ are the sub-matrices defined in (41).

The regularized solution $U^\epsilon_\alpha$ is defined as the solution of the minimization problem

$$U^\epsilon_\alpha = \arg\min_{U \in \mathbb{R}^{n_xn_y}} \left( \|AU - B_\epsilon\|^2 + \|(I - C_\alpha)EU\|^2 + \|D_x(I - C_\alpha)EU\|^2 \\
+ \|D_y(I - C_\alpha)EU\|^2 + \|D_{xx}(I - C_\alpha)EU\|^2 + \|D_{yy}(I - C_\alpha)EU\|^2 + 2\|D_{xy}(I - C_\alpha)EU\|^2 \right),$$

where $D_x, D_y, D_{xx}, D_{yy}, D_{xy}$ are discrete versions of the partial differential operators $\partial_x, \partial_y, \partial_{xx}, \partial_{yy}, \partial_{xy}, C_\alpha$ is the matrix approximating the convolution with the function $\varphi_\alpha$ defined in (25) and $E$ is the matrix modeling the extension operator. From (44), we compute $U^\epsilon_\alpha$ as the solution of the matrix equation

$$[A^\top A + E^\top (I_{n_xn_y} - C_\alpha)^\top D(I_{n_xn_y} - C_\alpha)E] U^\epsilon_\alpha = A^\top B_\epsilon,$$

where $\alpha$ is the regularization parameter, $I_{n_xn_y}$ is the square identity matrix of size $n_xn_y$ and $D$ is the matrix defined by

$$D = I_{n_xn_y} + D_x^\top D_x + D_y^\top D_y + D_{xx}^\top D_{xx} + D_{yy}^\top D_{yy} + 2D_{xy}^\top D_{xy}.$$

**Selection of the regularization parameter**

The choice of the regularization parameter $\alpha$ is a crucial step of the regularization. Indeed, the reconstruction error $U - U^\epsilon_\alpha$ has two components: the regularization error $U - U_\alpha$ (corresponding to exact data) and the data error propagation $U_\alpha - U^\epsilon_\alpha$. The former error is generally monotonically increasing with respect to $\alpha$ and attains its minimum at $\alpha = 0$ while the latter error blows up as $\alpha$ goes to 0 and decreases when $\alpha$ gets larger. Consequently, the reconstruction error norm $\|U - U^\epsilon_\alpha\|$ is minimal in some located region (depending on the noise level $\epsilon$ in the data) where both error terms have approximately the same magnitude. Outside that region, the reconstruction error is dominated by one of the two error terms which leads to an undesirable approximate solution $U^\epsilon_\alpha$. 

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In the following, we consider the heuristic selection rule (46)-(47) which has a similitude with the discrete quasi-optimality rule [3, 4, 5, 6] except for the denominator which in our case is not equal to one.

Let \((\alpha_n)_n\) be a sample of the regularization parameter \(\alpha\) on a discrete grid defined as
\[
\alpha_n := \alpha_0 q^n, \quad \alpha_0 \in (0, \|T\|^2], \quad 0 < q < 1, \quad n = 1, ..., N_0,
\] (46)
we consider the parameter \(\alpha(\epsilon)^*\) defined by
\[
\alpha(\epsilon)^* = \alpha_{n_*}, \quad \text{with} \quad n_* = \arg\min_{n \in \{1, ..., N_0\}} \frac{\|U^\epsilon_{\alpha_n} - U^\epsilon_{\alpha_{n+1}}\|}{\alpha_n - \alpha_{n+1}}.
\] (47)

The heuristic behind the rule (47) is the following:

Indeed, we aim at approximating the best regularization parameter \(\alpha(\epsilon)\) (over the chosen grid) which minimizes the reconstruction error norm \(\|U - U^\epsilon_{\alpha}\|\) over the grid \((\alpha_n)_n\), i.e.
\[
\alpha(\epsilon) = \alpha_{n(\text{opt})} \quad \text{with} \quad n(\text{opt}) = \arg\min_n K(\alpha_n) := \|U - U^\epsilon_{\alpha_n}\|.
\]

Given that minimizers of a differentiable function are critical points of that function, provided the function \(K\) is differentiable, \(\alpha_{n(\text{opt})}\) can be characterized as a minimizer of the absolute value of the derivative of function \(K\), that is
\[
n(\text{opt}) \approx \arg\min_n |K'(\alpha_n)|.
\]

By approximating the derivative \(K'(\alpha_n)\) of the function \(K\) at \(\alpha_n\) by its growth rates over the grid \((\alpha_n)_n\), we get that
\[
n(\text{opt}) \approx \arg\min_n \left| \frac{K(\alpha_{n+1}) - K(\alpha_n)}{\alpha_{n+1} - \alpha_n} \right|.
\]

However, since the exact solution \(u\) is unknown, we cannot evaluate the function \(K\). In such a setting, we search a tight upper bound of the function \(K\) and aim at minimizing that upper bound. Using the triangle inequality, we have
\[
|K(\alpha_{n+1}) - K(\alpha_n)| = \|U - U^\epsilon_{\alpha_{n+1}}\| - \|U - U^\epsilon_{\alpha_n}\| \leq \|U^\epsilon_{\alpha_{n+1}} - U^\epsilon_{\alpha_n}\| \quad (48)
\]

Hence from (48), we get an upper bound of the unknown term \(|K(\alpha_{n+1}) - K(\alpha_{n+1})|\) which is actually computable.

By approximating \(|K(\alpha_{n+1}) - K(\alpha_{n+1})|\) by its upper bound in (48), we get
\[
n(\text{opt}) \approx \arg\min_n \left| \frac{\|U^\epsilon_{\alpha_{n+1}} - U^\epsilon_{\alpha_n}\|}{\alpha_{n+1} - \alpha_n} \right| = \left| \frac{U^\epsilon_{\alpha_{n+1}} - U^\epsilon_{\alpha_n}}{\alpha_n - \alpha_{n+1}} \right|.
\]

which is precisely the definition of our heuristic selection rule (47).

To illustrate the efficiency of the selection rule (47), on Figure 1 (resp. Figure 2), we exhibit the curve of the reconstruction error along with the selected parameter \(\alpha(\epsilon)^*\) for each noise level for Example 1 (resp. Example 2).
Relative noise level $\sim 10^{-2}$

$\|U - U_{\alpha(\epsilon)^*}\|_2 / \|U\|_2 \approx 10^{-3}$

Relative noise level $\sim 10^{-4}$

$\|U - U_{\alpha(\epsilon)^*}\|_2 / \|U\|_2 \approx 10^{-3}$

Results and comments

In the simulations, we consider three noise levels $\epsilon_i$ ($i = 2, 3, 4$) such that the relative error in the data ($Red$) satisfies

$$Red = \frac{\|F - F_{\epsilon_i}\|_2}{\|F\|_2} \approx \frac{\|G - G_{\epsilon_i}\|_2}{\|G\|_2} \approx 10^{-i}.$$  

We choose Matlab as the coding environment and we solve equation (45) using a generalised minimal residual method (GMRES) with ortho-normalization based on Householder reflection. We choose as initial guess the solution from the Matlab direct solver $\text{ldivide}$.

Figures 3 (resp. 6) compares the exact solution $u$ to the reconstruction $u_{\alpha(\epsilon)^*}$ for each noise level for Example 1 (resp. Example 2). From these Figures, we observe that the reconstruction gets better as the noise level decreases.

On Figures 4 and 5 (resp. 7 and 8), we compare the exact function $u$ and the regularized solution $u_{\alpha(\epsilon)^*}$ at $y = 0.75$ and $y = 1$ for Example 1 (resp. Example 2) for each noise level.

Table 1 (resp. 2) presents the numerical values of the relative errors

$$\frac{\|u(\cdot, y) - u_{\alpha(\epsilon)^*}(\cdot, y)\|_2}{\|u(\cdot, y)\|_2}$$
for \( y = 0, 0.25, 0.5, 0.75 \) and \( y = 1 \) for Example 1 (resp. Example 2) for each noise level. From these Tables, we observe that the reconstruction error get smaller as \( y \) approaches 0 and as the noise level decreases.

From Figures 3 to 8 and Tables 1 and 2, we can see that our mollifier regularization approach yields quite good results. Moreover, as predictable, the reconstruction gets better when the noise level decreases and when we get closer to the boundary side where boundary data are given.

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![Figure 3](image-url): Comparison of the exact solution \( u \) of Example 1 and the regularized solution \( u^{\alpha(\epsilon)}_{\ast} \) for each noise level.

![Figure 4](image-url): Comparison of the exact solution \( u(\cdot, y) \) (black curve) of Example 1 and the regularized solution \( u^{\alpha(\epsilon)}_{\ast}(\cdot, y) \) at \( y = 0.75 \) (blue curve) for each noise level.
Figure 5: Comparison of the exact solution $u(\cdot, y)$ (black curve) of Example 1 and the regularized solution $u_{\alpha(\epsilon)}^\epsilon(\cdot, y)$ (blue curve) at $y = 1$ for each noise level.

$||u(\cdot, y) - u_{\alpha(\epsilon)}^\epsilon(\cdot, y)||^2_2/||u(\cdot, y)||^2_2$

| y   | $\text{Red} \sim 10^{-4}$ | $\text{Red} \sim 10^{-3}$ | $\text{Red} \sim 10^{-2}$ |
|-----|---------------------------|---------------------------|---------------------------|
| 0   | $8.2467 \times 10^{-5}$   | $7.4086 \times 10^{-4}$   | $6.8610 \times 10^{-3}$   |
| 0.25| $2.0430 \times 10^{-3}$   | $4.1855 \times 10^{-3}$   | $5.1843 \times 10^{-2}$   |
| 0.5 | $6.7946 \times 10^{-3}$   | $1.5606 \times 10^{-2}$   | $2.2334 \times 10^{-1}$   |
| 0.75| $5.5590 \times 10^{-2}$   | $8.6677 \times 10^{-2}$   | $5.0774 \times 10^{-1}$   |
| 1   | $1.4908 \times 10^{-1}$   | $1.8441 \times 10^{-1}$   | $4.4632 \times 10^{-1}$   |

Table 1: Relative $L^2$ error between the exact solution $u(\cdot, y)$ of Example 1 and the regularized solution $u_{\alpha(\epsilon)}^\epsilon(\cdot, y)$ for $y = 0, 0.25, 0.5, 0.75, 1$.

Figure 6: Comparison of the exact solution $u$ of Example 2 and the regularized solution $u_{\alpha(\epsilon)}^\epsilon$ for each noise level.
Relative noise level $\sim 10^{-2}$

Relative noise level $\sim 10^{-3}$

Relative noise level $\sim 10^{-4}$

Figure 7: Comparison of the exact solution $u(\cdot, y)$ (black curve) of Example 2 and the regularized solution $u_{\alpha(\epsilon)}^\epsilon(\cdot, y)$ at $y = 0.75$ (blue curve) for each noise level.

Relative noise level $\sim 10^{-2}$

Relative noise level $\sim 10^{-3}$

Relative noise level $\sim 10^{-4}$

Figure 8: Comparison of the exact solution $u(\cdot, y)$ (black curve) of Example 2 and the regularized solution $u_{\alpha(\epsilon)}^\epsilon(\cdot, y)$ (blue curve) at $y = 1$ for each noise level.

| $y$     | $\|u(\cdot, y) - u_{\alpha(\epsilon)}^\epsilon(\cdot, y)\|_2/\|u(\cdot, y)\|_2$ |
|---------|----------------------------------------------------------------------------------|
| $0$     | $4.6134 \times 10^{-5}$ $4.7109 \times 10^{-4}$ $1.8718 \times 10^{-3}$ |
| $0.25$  | $3.8348 \times 10^{-4}$ $1.6691 \times 10^{-3}$ $2.9628 \times 10^{-3}$ |
| $0.5$   | $2.7372 \times 10^{-3}$ $8.3065 \times 10^{-3}$ $2.9628 \times 10^{-2}$ |
| $0.75$  | $1.7155 \times 10^{-2}$ $4.0734 \times 10^{-2}$ $4.6936 \times 10^{-2}$ |
| $1$     | $1.1859 \times 10^{-1}$ $2.1162 \times 10^{-1}$ $2.3045 \times 10^{-1}$ |

Table 2: Relative $L^2$ error between the exact solution $u(\cdot, y)$ of Example 2 and the regularized solution $u_{\alpha(\epsilon)}^\epsilon(\cdot, y)$ for $y = 0, 0.25, 0.5, 0.75, 1$. 
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