Relativistic Constraints on the Structure
of Fundamental Forces

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PACS No: 03.30.+p, 03.50.De

Abstract:

It is proved that Special Relativity imposes constraints on the structure of fundamental forces. The orthogonality of the 4-force exerted on an elementary particle and its 4-velocity is discussed. The significance of the energy-momentum tensor associated with the field is analyzed. Relying on these issues, it is proved that the Lorentz force is consistent with all constraints whereas a force derived from a scalar potential does not satisfy all requirements. This analysis explains a general discussion of Goldstein, Poole and Safko.
1. Introduction

It is well known that the theory of Special Relativity (SR) changed dramatically old concepts concerning the structure of the physical world. The following lines mention several examples of this kind. Thus, probably the most well known relativistic formula $E = mc^2$ unifies mass and energy conservation laws. (In this formula $m$ denotes the mass measured in the laboratory frame. In all other cases, $m$ is a scalar denoting the self mass of a particle.) Another notion is the absolute property ascribed to length and time intervals. This idea has been forsaken and a single absolute quantity $(ds)^2 = (dt)^2 - (dr)^2$ is defined. Velocity takes a new form. In SR, velocity of a massive particle is always smaller than the speed of light $v < c$. In the case of massless particles, like the photon, the theory claims that it travels in the speed of light $c$ in all inertial frames and that $c$ is independent of the velocity of the source. The introduction of SR into quantum mechanics enabled Dirac to construct his celebrated equation. Solutions of this equation yield very good values for the atomic hydrogen energy levels, for the spin of the electron and for its g-factor. Another result of this equation is the prediction of the positron - an antiparticle of the electron.

SR shows that one does not need to postulate the existence of ether in order to explain wave properties of electromagnetic radiation. Thus, the following very well known textbooks on electrodynamics [1-3] do not have the words ”ether” or ”aether” in the subject index. The very detailed book of Jackson (see [4], pp. 503-515) discusses ether and other ideas which are inconsistent with SR and describes relevant experiments aiming to test these ideas. All results are consistent with SR.

Another difficulty of the ether theory emerges from quantum mechani-
cal effects. Experiments prove and this theory explains the particle/wave duality of other elementary particles, like the electron as well as that of nonelementary objects like the proton and the neutron. Thus, if one postulates that for every wave there should be a material medium (analogous to the case of acoustic waves) then a need for other kinds of ether (or new properties of the same ether) becomes a necessity. This argument explains the above mentioned status of ether in the current scientific literature. In spite of that, the notion of ether still can be found in modern scientific and nonscientific texts. Thus, a search of the Internet by means of Google shows about 1000 cases for the strings "ether waves", "aether waves". This figure should be compared with the 495,000 cases found for "electromagnetic waves".

SR changed the notions of probability (or charge) density and of energy density. Thus, in Newtonian mechanics volume is invariant and so is time. It follows that, at any instant, the amount of physical objects (like charge or mass) enclosed inside a given volume, takes the same value in all inertial frames. For this reason, density is an invariant in Newtonian mechanics. This property does not hold in SR because volumes undergo a Lorentz contraction and simultaneity of events is not conserved. Thus, it is proved that probability density (and charge density) become 0-component of the current 4-vector (see [1], pp. 69-71). This 4-vector is parallel to the local 4-velocity.

In the case of energy density, one finds that it is represented by the \(T^{00}\) component of the energy-momentum tensor \(T^{\mu\nu}\). Thus, energy density, momentum density, energy current and the \(3 \times 3\) tensor (like the Maxwell stress tensor of electrodynamics) are entries of one tensor \(T^{\mu\nu}\) (see [1], pp. 77-83). A Lorentz transformation of this tensor indicates the close relations between these notions.
SR explains a tremendous amount of data. Perhaps the most striking case is found in the colliding electron/positron beams of the LEP accelerator at CERN. Here electrons and positrons acquire a kinetic energy larger than 100GeV. It means that $E_K > 200,000 \, m_e c^2$, where $m_e$ denotes the rest mass of the electron (and of the positron). Now, in spite of this gigantic kinetic energy, the velocity of the particles does not exceed the speed of light $c$. Moreover, for these beams, an electron-positron interaction yields plenty of massive particles, demonstrating the interrelation between mass and energy, which is expressed by the formula $E = mc^2$. All kinds of processes like this abide by the law where an elementary particle and its antiparticle are produced in pairs and the overall energy and momentum (calculated by the laws of SR) are conserved.

The present work is devoted to the analysis of relativistic constraints imposed on the structure of fundamental forces. A brief discussion of the special case of the Lorentz force and of 2 approaches to the problem can be found in a recent edition of a well known textbook (see [5], pp. 297-300). This discussion indicates that the status of the problem is still indecisive.

In the notation used here Greek indices run from 0 to 3. The Lorentz metric $g_{\mu\nu}$ is diagonal and its entries are (1,-1,-1,-1). $\tau$ denotes the proper time. In the system of units used here $\hbar = c = 1$ and the dimensions of every quantities is a power of the length $L$. Thus, mass, energy and momentum take the dimension $[L^{-1}]$ whereas length and time have the dimension $[L]$.

2. The Notion of Force

People are aware of several kinds of forces encountered in everyday life,
like the force exerted by an extended (or contracted) spring or other objects obeying Hooke’s law; the force exerted by the pressure of gas on the surface of a piston; several kinds of friction forces; forces associated with biological activity of muscles etc. These forces are treated in textbooks where appropriate formulas are used. However, these kinds of forces do not have a fundamental nature and their description applies phenomenological formulas. The phenomenological nature of these forces is explained in the following lines.

In theoretical physics a genuine elementary particle is pointlike. This property holds in classical physics (see [1], pp. 43-44). Here the authors use several arguments that rely on SR. One of their arguments assumes that an elementary particle having a nonzero volume exists. As an elementary particle, no relative motion between its parts can take place. Now assume that at a certain instant $t$ a force is exerted on one of its sides. By the elementary nature of the particle, all its parts must move at the same speed. Hence, at the volume of this particle, interaction propagates at an infinite speed. This result contradicts SR.

Similarly, the pointlike nature of elementary particles is obtained in quantum mechanics and in quantum field theory, where a genuine elementary particle is described by a wave function (field function) $\psi(x^\mu)$. This function depends on a single set of space-time coordinates $x^\mu$. Hence, $\psi(x^\mu)$ describes a pointlike particle.

The pointlike nature of genuine elementary particles, like electrons, muons and quarks is consistent with experimental data which indicate that their size is smaller than $10^{-16}$ cm. This limit depends on the energy used in the experiments. Today it is believed that genuinely elementary particles like the electron, the muon and the quarks are pointlike. (The discussion carried
out below uses both pointlike expressions like $ev^\mu$ and density expressions like the 4-current $j^\mu$ (see [1], pp. 69-71). Using the Dirac $\delta$-function and performing an integration, one derives the first kind of expression from the second one.)

Evidently, pointlike particles cannot collide. Hence, a fundamental force cannot stem from a contact interaction between 2 particles. It follows that a mediating field $F$ is required for explaining acceleration of a particle as well as energy and momentum exchanged between interacting particles. Thus, the rest of this work is devoted to the analysis of the structure of this field $F$ and to its interrelations with the force exerted on particles. Here the tensorial nature of $F$ is still undefined. A discussion of this aspect of the problem is presented in the rest of this work. (Hereafter, the word ‘collision’ refers to cases where, at the interaction instant, the distance between the particles is very short.)

The notion of force holds in classical physics where measurements of position, velocity and acceleration are assumed to yield results having adequate accuracy. Let us examine the motion of a particle $P$ from point $A$ to point $B$ along a curve $C$ (see fig. 1). The particle $P$ accelerates and, in the laboratory frame, its energy at $B$ is higher that that of $A$. Let $\Delta E$ denote the amount of energy acquired by $P$. Due to the laws of SR, energy cannot travel faster than light. Hence, when the particle $P$ was at $A$, the energy $\Delta E$ was inside the spherical shell $S$. But $S$ contains nothing except the pointlike particle $P$ and the field of force $F$. This argument proves that energy density must be associated with the field of force. The physical expression for energy density and related quantities is the energy-momentum tensor $T^{\mu\nu}$ (see [1], pp. 77-80). Here the entry $T^{00}$ represents energy density. This tensor is a symmetric second rank tensor.
A simple dimensional analysis yields the relations between the field of force and its associated energy-momentum tensor. Using Newton’s law $f = ma$, one finds that, in the system of units used here, the dimension of force is $L^{-2}$. Now, energy has the dimension $L^{-1}$. Therefore, energy density has the dimension $L^{-4}$. These values restrict the relations between the field $F$ and $T^\mu{}_{\nu}$.

Another restriction imposed on the system is that, in the vacuum, the energy-momentum tensor must be divergenceless, $T^\mu{}_{\nu} = 0$. This restriction relies on the fact that the vacuum can be neither a source nor a sink of energy and momentum.

Before proceeding further, let us examine the case of the electromagnetic force. This case provides an important illustration of the problem and is used in the analysis carried out later.

3. The Electromagnetic Force

The electromagnetic force (called the Lorentz force) can be derived from the action principle. Here we have a charged particle and an electromagnetic field. The particle’s part of the Lagrangian is (see [1], pp. 45-49)

$$L = -m(1 - v^2)^{1/2} - e(\Phi - A \cdot \mathbf{v}),$$

(1)

where $m$ and $e$ denote the particle’s self mass and charge, respectively. Applying the Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}} = \nabla L$$

(2)

to (1), one isolates the time derivative of the mechanical momentum and
obtains the Lorentz force

\[ f = \frac{dp}{dt} = e(E + v \times B) \]  

(3)

This expression can be written in a covariant form

\[ f^\mu = ma^\mu = eF^{\mu\nu}v_\nu, \]  

(4)

where \( v^\mu \) and \( a^\mu = dv^\mu/d\tau \) are the particle’s 4-velocity and 4-acceleration, respectively. Now, since \( v^\mu \) is dimensionless, one finds that the electromagnetic fields and the Lorentz force \( f^\mu \) of (4) have the same dimensions \([L^{-2}]\).

The energy-momentum tensor of the electromagnetic fields is (see [1], p. 81 or [4], p. 605))

\[ T^{\mu\nu}_F = \frac{1}{4\pi} (F^{\mu\alpha} F^\beta_{\alpha\nu} g_{\beta\nu} + \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g^{\mu\nu}) \]  

(5)

Here we see a realization of the discussion presented in the previous Section. In (4) we see the relation between the mediating electromagnetic field \( F^{\mu\nu} \) and the force exerted on a charged particle. In (5) one finds the relation between this field and the energy-momentum tensor. Thus, the field’s part of the energy-momentum tensor is a homogeneous quadratic function of the fields tensor \( F^{\mu\nu} \) and it is divergenceless at the vacuum (see [1], p. 78)

\[ T^{\mu\nu}_{\text{vac}} = 0. \]  

(6)

Moreover, at the position of a charge, it satisfies (see [1], pp. 82, 83)

\[ T^{\mu\nu}_{\text{charge}} = -F^{\mu\nu} j_{\nu}. \]  

(7)

This relation proves energy-momentum conservation where the amount lost
The form of the Lorentz force (3) can be used as a guide for the definition of force. As explained above, we treat here elementary pointlike particles. Thus, the force is taken as the time derivative of the mechanical momentum

\[ f = \frac{dp}{dt} \]  

(see a discussion in [5], pp. 297-300). The covariant form of the force is given in (4). Relation (4) yields the following constraint on a relativistic force

\[ v^\mu v_\mu = 1 \rightarrow a^\mu v_\mu = 0 \rightarrow f^\mu v_\mu = 0. \]  

Relation (9) proves that a relativistic force is spacelike. Indeed, let us examine (9) in the rest frame of the particle where \( v^\mu = (1, 0, 0, 0) \). It follows that, in this frame, \( f^\mu = (0, f) \) which is a spacelike vector.

Another result that follows (9) is that the force \( f^\mu \) exerted on a particle, must depend on its 4-velocity \( v^\mu \). Indeed, there is only one force which is independent of \( v^\mu \) and satisfies (9). This is the null force \( f^\mu = 0 \).

Requirement (9) is satisfied by the Lorentz force (4)

\[ f^\mu v_\mu = eF^{\mu\nu}v_\nu v_\mu = 0, \]  

where the final result is obtained from the antisymmetry of the tensor \( F^{\mu\nu} \) and the symmetry of the product \( v_\nu v_\mu \).

The dependence of the 4-force \( f^\mu \) on the particle’s 4-velocity \( v^\mu \) means that it is not identical to the mediating field \( F \) introduced in the second Section. Thus, in the case of electrodynamics, the Lorentz force (4) illustrates
the field takes the form of an antisymmetric tensor \( F^{\mu\nu} \) whereas the associated force is (obviously) a 4-vector obtained from the tensorial contraction of \( F^{\mu\nu} \) and the 4-velocity \( v_{\nu} \).

The discussion carried out above proves 3 requirements that should be satisfied by a relativistic force:

A. The 4-force \( f^{\mu} \) exerted on a particle must be orthogonal to its 4-velocity \( v^{\mu} \).

B. The mediating field \( F \) must yield a symmetric energy-momentum tensor \( T^{\mu\nu} \). In the vacuum, relation (6) \( T^{\mu\nu}_{\,\,\nu} = 0 \) must hold.

C. At the space-time point where a particle is located, energy-momentum balance yields the following relation \( T^{\mu\nu}_{\,\,\nu} + f^{\mu} = 0 \) (see (7) for the case of electrodynamics).

These requirements are used in the following discussion. They are denoted by the letters A, B and C, respectively. Requirements B and C look like very serious constraints imposed on an arbitrary formula of force. As a matter of fact, they have a standard solution for cases where the dynamics of the system is derived from a Lagrangian density of the fields (see [1], pp. 77-80, 270-273). This aspect provides another argument for the usage of a Lagrangian density and of the variational principle as a basis of any field theory.

5. The Scalar Potential

The student of Newtonian mechanics regards a scalar potential defined in the 3-dimensional space, as a self-evident and a very useful expression
of the theory. This is not the case discussed here because the potential is regarded as a scalar in Minkowski space. The realization of a relativistic scalar potential is the Yukawa potential (see [6], p. 211; [7], p. 122)

\[ \phi = -g^2 e^{-\mu r} \]

(11)

where \( \phi \) satisfies the Klein-Gordon (KG) equation (see [8], p. 26)

\[ (\Box + \mu^2)\phi = 0. \]

(12)

Here \( g \) is a dimensionless coupling constant and \( \mu \) denotes the mass of the particle represented by the field of force \( \phi \). This kind of particle is associated with the field of force and is certainly distinguished from the particle upon which the force is exerted.

In a free space, namely at points which are free of particles, we have only the scalar potential \( \phi \) and all tensorial expressions can be obtained from an application of the 4-derivative operator \( \partial_\mu \). Now we can analyze the Yukawa interaction.

In the system of units used here the action \( \int L d^3x dt \) is dimensionless. Hence, all terms of the Lagrangian density \( L \) must have the dimension \([L^{-4}]\). Now, the Lagrangian density of a KG particle has a term \( m^2 \phi^2 \) (see [8], p. 26). It follows that the dimension of a KG wave function \( \phi \) is \([L^{-1}]\). This argument can be used in an examination of the 4-vector obtained from a differentiation of the Yukawa potential (11). This is a radial force

\[ f_{Yukawa}^\mu = -g^2 (\mu r + 1) e^{-\mu r} \]

(13)

which has the dimension \([L^{-2}]\). (Note that in this expression \( \mu \) denotes mass and is not an index.)

Here we see that the 4-vector \( f_{Yukawa}^\mu \) of (13) has the dimensions of force. However, unlike the Lorentz force (4), \( f_{Yukawa}^\mu \) is independent of the 4-velocity.
of the particle upon which it is exerted. Hence, in spite of the fact that
the Yukawa force is derived from the relativistic KG equation (12), \( f_{\text{Yukawa}}^\mu \) violates requirement \( A \) presented at the end of Section 4. Therefore, it is
relativistically unacceptable.

This problem holds also for the general case of a scalar potential having
a dimension which is different from \([L^{-1}]\). Indeed, in order to have a 4-force
which is orthogonal to the 4-velocity, one requires a field tensor which is
\textit{antisymmetric} in 2 indices (like that of the electromagnetic fields). However,
such a tensor cannot be obtained from a scalar function because the curl of
a gradient vanishes.

6. General Aspects of the Problem

It is explained above how the pointlike nature of a relativistic element-
ary particle entails the requirement for a mediating field associated with the
force \textit{and} with the energy-momentum exchanged between interacting parti-
cles. A simple relativistic analysis (9) proves that the 4-force exerted on
an elementary particle must be orthogonal to its 4-velocity. Furthermore, a
pointlike particle may be regarded as an integral of density, provided appro-
priate Dirac \( \delta \)-functions are used. Therefore, as explained in the intro-
ductive Section, probability density is a 0-component of a 4-vector which is parallel
to the local 4-velocity. This is a very good reason for a derivation of the
4-force as a certain mathematical function of the mediating field \( F \) \textit{and} the
particle’s 4-velocity.

Another issue is the connection between the mediating field \( F \) and a
well defined energy-momentum tensor \( T^{\mu\nu} \). In the vacuum (namely, in the
entire space, except the points where particles are located) this tensor can
depend only on \( F \) and it must satisfy \( T_{\mu\nu}^{\mu\nu} = 0 \) (see B at the end of Section 4).
Requirement B is a very stringent mathematical restriction on the form of the
mediating field. As is well known (see [1], pp. 77-80, 270-273) the standard
method of constructing the energy-momentum tensor is by an application
of the Lagrangian density \( \mathcal{L} \). For this reason, let us examine \( \mathcal{L} \) and find
restrictions imposed by it.

In the system of units used here, the action

\[
S = \int \mathcal{L} d^3x \, dt
\]  

(14)
is dimensionless. Hence, the dimension of every term of \( \mathcal{L} \) must be \( [L^{-4}] \).
Moreover, since the action \( S \) is a Lorentz scalar, and so is the product \( d^3x \, dt \),
one finds that every term of \( \mathcal{L} \) must be a Lorentz scalar.

Another issue is the dimension of the mediating field \( F \). A general phys-
ical argument states that \( F \) must vanish at infinity. Hence, the dimension of
\( F \) must be \( [L^{-n}] \), where \( n \) is a positive integer. Now the interaction term of
the Lagrangian density must be a Lorentz scalar obtained from a tensorial
contraction of the particle’s density (represented by the particle’s 4-current
\( j^\mu \)), whose dimension is \( [L^{-3}] \), and another 4-vector, \( A^\mu \), associated with the
mediating field \( F \). In order to comply with the dimension of the Lagrangian
density \( [L^{-4}] \), \( A^\mu \) must have the dimension \( [L^{-1}] \).

As is well known, electrodynamics satisfies all the requirements mentioned
above. The electromagnetic 4-potential \( A^\mu \) has the dimension \( [L^{-1}] \); The
interaction term of the Lagrangian density is proportional to the Lorentz
scalar \( j^\mu A_\mu \); the electromagnetic field is the antisymmetric tensor \( F^{\mu\nu} \) which
is the 4-curl of \( A_\mu \); the Lorentz force (4) is orthogonal to the 4-velocity.

The Yukawa interaction does not satisfy all the requirements stated above.
Indeed, the Yukawa interaction term of the Lagrangian density is the follow-
ing Lorentz scalar (see [7], p. 79)

$$\mathcal{L}_{\text{int}} = -g\phi\bar{\psi}\psi.$$  \hspace{1cm} (15)

This expression is very strange, because it relies not on the particle’s density in the laboratory frame $\psi^\dagger\psi$ but on its scalar density $\bar{\psi}\psi$. This is a deviation from the quantum mechanical rule where expectation values are calculated by means of the ordinary density $\psi^\dagger\psi$ and $\psi$ belongs to an orthonormal basis of the Hilbert space. In particular, consider a moving Dirac particle. Here, due to the Lorentz contraction, one finds that in the laboratory frame $\int \bar{\psi}\psi d^3x < 1$. This result is inconsistent with the orthonormality of wave functions belonging to the basis of the Hilbert space. Therefore, it casts doubts on the consistency of the Yukawa scalar interaction. Indeed, as explained at the end of Section 5, a scalar potential cannot yield a relativistically acceptable force.

Another issue is the problem of a non-linear force like

$$f_{NL}^\mu = G_{\mu\nu\lambda}^\nu v_\lambda$$  \hspace{1cm} (16)

where $G_{\mu\nu\lambda}$ is antisymmetric in $\mu\nu$. It can be shown that such a force is inconsistent with the Lagrangian density approach where density of dynamical quantities is used. Thus, one replaces the 4-velocities of (16) with 4-currents and finds that the dimension of the product $j_\nu j_\lambda$ is $[L^{-6}]$. Now, the dimension of force is $[L^{-2}]$ and that of force density is $[L^{-5}]$. It follows that the dimension of $G_{\mu\nu\lambda}$ is $[L]$. Therefore, $G_{\mu\nu\lambda}$ does not vanish at infinity. This property proves that this $G_{\mu\nu\lambda}$ is unphysical. The situation is worse for a $G$ whose tensorial rank is higher than 3.

The discussion carried out in this work relies on SR. Thus, the Lorentz metric is uniform and it may be regarded as an inert background. Hence, if the space-like 4-acceleration $a^\mu = dv^\mu/d\tau$ does not vanish in a certain frame
then it cannot vanish in any frame. This property certainly does not hold for a gravitational field. In this case, the metric acquires dynamical properties and the equation of motion is (see [1] p. 245)

\[
\frac{dv^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} v^\alpha v^\beta = 0.
\] (17)

As is well known, the Christoffel symbol \(\Gamma^\mu_{\alpha\beta}\) vanishes in frames satisfying certain conditions (see [1], p. 239). It follows that in these frames gravitational acceleration vanishes and particles move inertially (like astronauts inside a spaceship). Hence, the case of a gravitational field is certainly outside the framework of the discussion carried out above.

It is proved above that if one adopts the requirement that the theory can be derived from a Lagrangian density then the only self-consistent fundamental force takes the form of the Lorentz force. Thus, if another kind of force exists within the framework of SR then it cannot be derived from a Lagrangian density. However, such a force should satisfy the three very stringent requirements stated at the end of Section 4. As reported recently (see [5], pp. 297-300), efforts to find such a force ended in vain.
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Figure Captions

Fig. 1:

A particle $P$ moves from point $A$ to point $B$ along a curve $C$. $\Delta t$ is the time duration of this motion. The circle denotes a spherical shell $S$ (whose scale differs from that of $C$). The distance between any point on $S$ and a point on $C$ is larger than $c \Delta t$. 
