Bases and metric dimension of composition product of some graph families

P.V. Shamsudheen¹* and A.T. Shahida²

Abstract
A subset of vertices $S$ resolves a graph $G$ if every vertex of $G$ is uniquely determined by its vector of distances to the vertices in $S$. A resolving set of minimum cardinality for a graph $G$ is called a minimum resolving set. A minimum resolving set is usually called a basis for $G$ and the cardinality of basis is called the metric dimension of $G$, denoted by $\dim(G)$. For the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ its composition product is denoted by $G_1[G_2]$ is the graph and two vertices $(u,v)$ and $(x,y)$ are adjacent in $G_1[G_2]$ whenever $ux \in E_1$ or, $u=x$ and $vy \in E_2$. In this paper, metric dimension of composition product of $S_n[O_n], S_n[S_n], S_n[S_m], S_n[O_m]$ are obtained. Also discusses about general properties of the bases of graphs.

Keywords
Metric Dimension, Resolving sets, Composition Product, Bases.

AMS Subject Classification
Primary; 05C12, Secondary; 05E30.

Contents
1 Introduction ................................................. 587
2 Preliminaries ................................................ 587
3 Main Results ............................................... 588
4 Conclusion ............................................... 589
References ..................................................... 589

1. Introduction
The concept metric dimension of connected graphs and its related properties are first introduced by by PJ Slater [5] in 1975, independently by Harary and Melter [1] in 1976. The concept of minimum resolving set has a significant role in various context such as organic chemistry [4], Robotic navigation [3], coin weighing problem [9], drug discovery [9], network discovery and verification [2]. The motivation behind the results on this work is due to the large range of application of resolving sets in various fields as mentioned as network discovery and verification. In the field of chemistry, to provide mathematical representations for a set of chemical compounds in a way that gives distinct representations to distinct compounds. Also provides the chemical structure for the compounds. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of $V(G)$. The representation of a vertex $v$ of $G$ with respect to $W$ is defined as the $k$-vector $r(v/W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$. The set $W$ is called a resolving set of $G$ if every two distinct vertices $x, y \in V(G)$ satisfy $r(x/W) \neq r(y/W)$. A basis of $G$ is a resolving set of $G$ with the minimum cardinality, and the metric dimension of $G$ refers to its cardinality and is denoted by $\dim(G)$. In this paper, mainly discussed about the metric dimension and related basis properties of composition products of graphs.

2. Preliminaries
In this section we includes the basic definitions and results required for this paper. The following definitions are from [6], [9], [1], [7] and [8]

Definition 2.1. Let $G=(V,E)$ be a graph. For an ordered subset $W = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$ and for any vertex $v \in V$, the metric representation of $v$ with respect to $W$ is the $k$-vector which is denoted and defined as $r(v/W) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$.

Definition 2.2. The composition product of two graphs $G_1(V_1,E_1)$ and $G_2(V_2, E_2)$ is the graph $G$, denoted by $G_1[G_2]$, whose vertex set is $V_1 \times V_2$ and two vertices $(u_i, v_j)$ and $(u_j, v_m)$ are adjacent in $G$ whenever $u_iu_j \in E_1$ or $u_i = u_j$ and $v_jv_m \in E_2$. 
The number of vertices in above composition graph is $V_1 \times V_2$. Number of edges is $(V_1 \times E_2) + (E_1 \times V_2^2)$

**Definition 2.3.** The set $W$ is called a resolving set for $G$ if $r(V_1/W) = r(V_2/W)$ implies that $V_1 = V_2$ for all $V_1, V_2 \in V(G)$.

**Definition 2.4.** A vertex $x \in V(G)$ is said to resolve a pair of vertices $u, v$ in $G$ if $d(u, x) \neq d(v, x)$.

**Definition 2.5.** The minimum cardinality of a resolving set of $G$ is called the metric dimension of $G$ and is denoted by $\dim(G)$. A resolving set of minimum cardinality for a graph $G$ is called a minimum resolving set. A minimum resolving set is usually called a basis for $G$.

**Theorem 2.6.** [9] The metric dimension of the star graph $S_n = n - 2$ for $n > 1$.

**Theorem 2.7.** [9] A connected graph $G$ of order $n > 2$ has dimension $n - 1$ if and only if $G = K_n$.

### 3. Main Results

In this section includes some fundamental results about the metric dimension in the context of composition product of graph families.

**Theorem 3.1.** The metric dimension of the composition product of star graph $S_n$ and empty graph $O_m$ is

$$\dim(S_n[O_m]) = \begin{cases} 1 & \text{if } n = 2, m = 1 \\ nm - 2 & \text{if } n \geq 2, m \geq 1, n \neq 2, m \neq 1 \end{cases}$$

Proof. **case.1:** for $n = 1$ and for any value of $m$, composition product $S_n[O_m]$ is a disconnected graph. Since metric dimension is defined only for connected graphs, there is no need to check the dimension of the composition product for $n = 1$.

**case.2:** if $n = 2$ and $m = 1$, the composition product is a star graph with 2 vertices. Therefore $d(S_2[O_1]) = d(S_2) = 1$.

![Figure 1](image1.png)

**case.3:** if $n \geq 2, m \geq 1, (n \neq 2, m \neq 1)$, in this case the vertices of the composition product of $S_n[O_m]$ as shown in figure 2, this graph has $n$ vertices with degree $(n - 1)m$, therefore the resolving set contains $n - 1$ vertices and also $n(m - 1) - 1$ vertices. Since in this graph $n(m - 1)$ vertices has the degree $m$ therefore the minimum cardinality of the resolving set is $(n - 1) + n(m - 1) - 1 = nm - 2$.

**Corollary 3.2.** From the above theorem, the metric dimension of the composition product of $S_n[O_m]$ with $n = m$ is $\dim(S_n[O_n]) = n^2 - 2$ for all $n \geq 2$.

**Corollary 3.3.** The lower bound of the metric dimension of the composition product of the graph $S_n[O_m]$ is the metric dimension of $S_n$. That is $\dim(S_n) \leq \dim(S_n[O_m])$. In this case, there does not exist independent bases. All the bases are connected.

**Theorem 3.4.** The metric dimension of the composition product of two star graphs of order $n$ and $m$ is

$$\dim(S_n[S_m]) = \begin{cases} 1 & \text{if } n = 2, 3, m = 1 \\ 3 & \text{if } n = 2, 3, m = 2 \\ m - 2 & \text{if } n = 1, m \geq 3 \\ 2m - 3 & \text{if } n = 2, m \geq 3 \\ 3m - 5 & \text{if } n = 3, m \geq 3 \end{cases}$$

Proof. The composition product of $S_n[S_m]$ as shown in the Figure 3.

**case.1:** If $2 \leq n \leq 3, m = 1$, the composition product must be $S_2[S_1]$ and $S_3[S_1]$. Where $S_2[S_1] = P_2$ and $S_3[S_1] = P_3$. Therefore $\dim(S_2[S_1]) = \dim(P_2) = 1$ and $\dim(S_3[S_1]) = \dim(P_3) = 1$.

**case.2:** For $2 \leq n \leq 3, m = 2$. Since the composition product of $S_2[S_2] = K_4$. So $\dim(S_2[S_2]) = 3$.

**case.3:** If $n = 1$, and for any $m \geq 3$, the composition product $S_1[S_m]$ is a star graph with $m$ vertices. Therefore $\dim(S_1[S_m]) = m - 2$.

**case.4:** If $n = 2, m \geq 3$. In this case every resolving set having at least $2m - 3$ vertices is a basis. Therefore $\dim(S_2[S_m])$ for any $m \geq 3$ is $2m - 3$.

**case.5:** If $n = 3, m \geq 3$. In this case every resolving set having at least $3m - 5$ vertices is a basis. Therefore $\dim(S_3[S_m])$ for any $m \geq 3$ is $3m - 5$. 

![Figure 2](image2.png)
4. Conclusion

In this paper we tried to find out the metric dimension of composition product of star graphs, star graphs and empty graphs. Also investigate the properties of resolving sets and bounds of metric dimension of the composition product of the graphs.

Acknowledgment

The work of first author is supported by the UGC-Ministry of Human Resource Development. Under the grant number F.No.16-9(June 2018)/2019(NET/CSIR), UGC-Ref.No. :994/(CSIR-UGC NET JUNE 2018) Dated 16 APRIL 2019.

References

[1] F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Combinatoria*, 2(2)(1976), 191-195.
[2] Z. Beerliova, F. Eberhard, T. Erlebach, A. Hall, M. Homann, M. Mihalak, and L. Ram, Network Discovery and Verification, *IEEE J. on Selected Areas in Communications*, 24 (2006), 2168-2181.
[3] S. Kuller, B. Raghavachari and A.A.Rosenfeld, *Localization in Graphs*, Technical Report CS-TR-3326, University of Maryland at College Park, 1994.
[4] M. A. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, *J. Biopharm. Statist.*, 3(1993), 203-236.
[5] P. J. Slater, Leaves of trees, *Proc. 6th Southeastern Conference on Combinatorics, Graph Theory and Computing, Congr.*, 14(37)(1975), 549-559.
[6] Carmen Hernando, Mer’ce Mora, Ignacio M. Pelayo, Carlos Seara, On the metric dimension of some families of graphs, *Electronic Notes in Discrete Mathematics*, 22(2005), 129-133.
[7] V. Saenpholphat, P. Zhang, Conditional resolvability in graphs, *International Journal of Mathematics and Mathematical Sciences*, 38(2004), 1997-2017.
[8] F.Harary, *Graph Theory*, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1969
[9] Gary Chartrand, Linda Eroh, Mark A. Johnson, Ortrud R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, 105(1-3)(2000), 99-113.