Quantum-correlation breaking channels, broadcasting scenarios, and finite Markov chains

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One of the classical results concerning quantum channels is the characterization of entanglement-breaking channels [1, 2]. The latter are defined as channels which turn any bipartite state into a separable (non-entangled) one. The main result of Ref. [1] states that a channel Λ is entanglement breaking if and only if its Choi-Jamiołkowski state (i.e. its witness) Λ(ρ+) is a separable state (P+, denotes the projector on the maximally entangled state, see Eq. (2)). However, it is known that quantum correlations are more general than entanglement (see e.g. Ref. [3] and references therein).

To our knowledge, the characterization from Ref. [1] has not yet been refined to a case when a channel breaks more general quantum correlations, i.e. transforms any state into a state that does not possess some type of quantum correlations (see however Ref. [4] where partial results were obtained). Here we show that such a refinement is indeed possible for channels mapping (when applied to one subsystem) into a separable (non-entangled) one. The answer is fully affirmative in the case of breaking quantum correlations down to the, so called, QC (Quantum-Classical) type, while it is no longer true in the CC (Classical-Classical) case. The corresponding channels turn out to be measurement maps. Our study also reveals an unexpected link between quantum state and local correlation broadcasting and finite Markov chains. We present a possibility of broadcasting via non von Neumann measurements, which relies on the Perron-Frobenius Theorem. Surprisingly, this is not the typical generalized C-NOT gate scenario, appearing naturally in this context.

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There is a well-known result concerning a characterization of entanglement-breaking channels [1, 2]. The latter are defined as channels which turn any bipartite state (when applied to one subsystem) into a separable (non-entangled) one. The main result of Ref. [1] states that a channel Λ is entanglement breaking if and only if its Choi-Jamiołkowski state (i.e. its witness) 1 ⊗ Λ(ρ+) is a separable state (P+, denotes the projector on the maximally entangled state, see Eq. (2)). However, it is known that quantum correlations are more general than entanglement (see e.g. Ref. [3] and references therein).

To our knowledge, the characterization from Ref. [1] has not yet been refined to a case when a channel breaks more general quantum correlations, i.e. transforms any state into a state that does not possess some type of quantum correlations (see however Ref. [4] where partial results were obtained). Here we show that such a refinement is indeed possible for channels mapping (when applied to one subsystem) into a separable (non-entangled) one, so called, QC state. Such channels turn out to be quantum-to-classical measurement maps [5]. Moreover, we show that a similar statement does not hold in the case of broadcasting of correlations.

Recall that a QC (or more precisely QAQB) state is a bipartite state of a form

\[
\sigma_{QC} = \sum_i p_i \sigma_i^A \otimes |e_i\rangle_B \langle e_i|,
\]

where \(\sigma_i\)'s are states at Alice's side, \(\{e_i\}\) is an orthonormal basis on Bob's side (possibly different from the computational basis \(\{|i\}\)), and \(p_i\)'s are probabilities. In the analogous way one defines a CQ (more precisely CAQB) state, where the classical part (projectors on the orthonormal basis) is located at Alice's side.

Throughout the work we will always assume that Λ is a trace-preserving, completely positive map, i.e. a channel, and

\[
P_+ := |\psi_+\rangle \langle \psi_+| = \frac{1}{d} \sum_{i,j} |ii\rangle \langle jj|
\]

is the projector on the maximally entangled state \(\psi_+\) and \(\{|ij\}\) is a fixed computational product basis. We prove the following

**Theorem 1** For any channel Λ its Choi-Jamiołkowski state 1 ⊗ Λ(ρ+) is a QC state if and only if 1 ⊗ Λ(\(\varphi_{AB}\)) is a QC state for any bipartite state \(\varphi_{AB}\).

**Proof.** We propose to call the above type of channels QC-type channels. In order to setup the notation and

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methods (cf. Ref. [1]), we present a detailed proof. In one direction the implication is obvious. To prove it in the other one, assume that the state $\mathbb{I} \otimes \Lambda(P_+) = \text{QC}$.

\[ \mathbb{I} \otimes \Lambda(P_+) = \sum_i p_i \sigma_i \otimes |e_i\rangle\langle e_i|, \]  

(3)

From the inversion formula for the Choi-Jamiołkowski isomorphism [10]

\[ \Lambda(A) = d \text{Tr}_A[W_A(A^T \otimes \mathbb{I})], \]  

(4)

where $W_A = \mathbb{I} \otimes \Lambda(P_+)$ and the transpose is defined in the computational basis $\{|i\rangle\}$, it follows that

\[ \Lambda(\varrho) = d \sum_i p_i \text{Tr}(\varrho \sigma_i^T)|e_i\rangle\langle e_i|, \]  

(5)

and hence

\[ \mathbb{I} \otimes \Lambda(\varrho AB) = d \sum_k p_k \text{Tr}_B(\varrho AB \mathbb{I} \otimes \sigma_k^T) \otimes |e_k\rangle\langle e_k|, \]  

(6)

for an arbitrary bipartite state $\varrho AB$. We define unnormalized residual states

\[ \tilde{\sigma}^A_k := d p_k \text{Tr}_B(\varrho AB \mathbb{I} \otimes \sigma_k^T), \]  

(7)

and their traces

\[ \tilde{\varrho}_k := \text{Tr} \tilde{\sigma}^A_k = d p_k \text{Tr}_B(\varrho AB \mathbb{I} \otimes \sigma_k^T). \]  

(8)

We show that $\sum_k \tilde{\varrho}_k = 1$. From the assumption that $\Lambda$ is trace-preserving, it follows that

\[ \text{Tr}_B[\mathbb{I} \otimes \Lambda(P_+)] = \frac{1}{d} \sum_{i,j} |i\rangle\langle j| \text{Tr}_A(|i\rangle\langle j|) \]  

\[ = \frac{1}{d} \sum_{i,j} |i\rangle\langle j| \text{Tr}_A(|i\rangle\langle i|) = \frac{1}{d} \mathbb{I}, \]  

(9)

On the other hand, the QC assumption [3] implies that

\[ \text{Tr}_B[\mathbb{I} \otimes \Lambda(P_+)] = \sum_k p_k \sigma_k, \]  

(10)

and consequently

\[ \sum_k p_k \sigma_k = \mathbb{I} \otimes \mathbb{I}. \]  

(11)

Thus, the collection $\{dp_i\sigma_i\}$, or equivalently its transposition

\[ E_i := dp_i \sigma_i^T, \]  

(12)

forms a POVM, which together with Eq. (8) implies that

\[ \sum_k \tilde{\varrho}_k = d \text{Tr}_A(\varrho AB \mathbb{I} \otimes \sum_k p_k \sigma_k^T) = \text{Tr} \varrho AB = 1. \]  

(13)

Hence, Eq. (8) may be rewritten as

\[ \mathbb{I} \otimes \Lambda(\varrho AB) = \sum_k \tilde{\varrho}_k \tilde{\sigma}^A_k \otimes |e_k\rangle\langle e_k|, \]  

(14)

with $\tilde{\varrho}_k := \tilde{\sigma}^A_k / \text{Tr} \tilde{\sigma}^A_k = \varrho AB / \tilde{\varrho}_k$, which is a QC state.

We remark that Thm. [1] will not in general be true if one changed the QC state to a CQ one, keeping the form of the Choi-Jamiołkowski isomorphism. Indeed, if $\mathbb{I} \otimes \Lambda(P_+) = \sum_i p_i |e_i\rangle\langle e_i| \otimes \sigma_i$, then from Eq. (1) it follows that $\Lambda(\varrho) = d \sum_i p_i |\varrho e_i\rangle\langle e_i| \otimes \sigma_i$ and $\mathbb{I} \otimes \Lambda(\varrho AB) = d \sum_i p_i \text{Tr}_B(\varrho AB \mathbb{I} \otimes |e_i\rangle\langle e_i| \otimes \sigma_i)$, which is in general a separable state but not a QC nor CQ one. As an example, consider $\Lambda^{CQ}$ as a von Neumann measurement in the standard basis on a qubit. Obviously, $\mathbb{I} \otimes \Lambda(P_+)$ is a CQ state, since it is CC. Now consider a two-qubit state $\varrho AB$ which is an unbiased mixture of the projectors corresponding to two vectors $|\psi_+1\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$ and $|\psi_+2\rangle = 1/\sqrt{2}(|+\rangle + |0\rangle)$, where $|\psi_+1\rangle := 1/2(|0\rangle + |1\rangle)$ and $|\psi_+2\rangle := 1/2|+\rangle$. But $\tilde{\varrho}_1 \neq \tilde{\varrho}_2$, breaking the necessary condition for $\mathbb{I} \otimes \Lambda(\varrho AB)$ to be a CQ state.

As expected from the general results of Ref. [1] on entanglement breaking channels, Eqs. (6), (14) and (15) imply that the action of QC-type channel $\sigma^{CC}$ consist of a POVM-measurement followed by a state preparation, but the preparation is always done in the same orthonormal basis $\{|e_i\rangle\}$

\[ \Lambda(\varrho) = \sum_i \text{Tr}(\varrho E_i)|e_i\rangle\langle e_i|. \]  

(15)

The later plays a role of a classical register, so that every QC-type channel is in fact a quantum-to-classical measurement map $\mathbb{B}$: $\Lambda(\varrho)$ gives the state of a measuring apparatus after the measurement of $\{E_i\}$ on a system in the state $\varrho$. In the light of this observation, Thm. [1] states that a channel is a measurement map if and only if (iff) its Choi-Jamiołkowski state is a QC state.

A natural question arises if one can refine Thm. [1] even more to the so-called CC states, i.e. states of a form

\[ \sigma^{CC} = \sum_{i,j} p_{ij} |e_i\rangle\langle e_i| \otimes |f_j\rangle\langle f_j|, \]  

(16)

where $\{e_i\}$ and $\{f_j\}$ are orthonormal bases on Alice’s and Bob’s side correspondingly, and $p_{ij}$ is a classical joint probability distribution. It turns out that as stated, Thm. [1] does not specify down to such a case, as even if $\mathbb{I} \otimes \Lambda(P_+)$ is a CC state, $\mathbb{I} \otimes \Lambda(\varrho AB)$ is generically a QC state. To see this, assume that

\[ \mathbb{I} \otimes \Lambda(P_+) = \sum_{i,j} p_{ij} |e_i\rangle\langle e_i| \otimes |f_j\rangle\langle f_j|. \]  

(17)

From the inversion formula [1] one then obtains that

\[ \Lambda(\varrho) = \sum_j \text{Tr}(\varrho E_j)|f_j\rangle\langle f_j|, \]  

(18)

\[ \mathbb{I} \otimes \Lambda(\varrho AB) = \sum_j \text{Tr}_B(\varrho AB \mathbb{I} \otimes E_j) \otimes |f_j\rangle\langle f_j|. \]  

(19)
What is quite important is that the POVM condition (17) consequences for broadcasting of states and correlations.

Similarly to the QC case, trace-preserving property of \( A \) implies that \( \{ E_i \} \) form a POVM, \( \sum_j E_j = \mathds{1} \) (cf. Eqs. (19)). However, in this case the POVM elements necessarily pairwise commute

\[
[E_i, E_j] = 0,
\]

since by Eq. (20) they correspond to a measurement in one fixed basis, but they need not form a von Neumann measurement, as in general \( E_j \)'s may overlap

\[
E_j E_{j'} = \sum_i p_{ij} p_{ij'} |e_i^* \rangle \langle e_i^*| \neq \delta_{jj'} E_j.
\]

What is quite important is that the POVM condition \( \sum_j E_j = \mathds{1} \), puts some constraints on \( p_{ij} \):

\[
\sum_{i,j} p_{ij} |e_i^* \rangle \langle e_i^*| = \frac{1}{d} \Rightarrow p_i := \sum_j p_{ij} = \frac{1}{d},
\]

which in turn implies that the numbers

\[
p_{jj} := dp_{ij}
\]

are in fact conditional probabilities: \( \sum_j p_{jj} = 1 \) for any \( i \). Thus, the matrix \( P^A := [p^A_{jj}] \) is a stochastic matrix [\( \mathcal{P} \)] and

\[
E_j = \sum_i p_{jj} |e_i^* \rangle \langle e_i^*|.
\]

From a probabilistic point of view, a stochastic matrix defines a finite Markov chain [\( \mathcal{M} \)]: it provides transition probabilities between the sites. Hence, with every CC-type channel satisfying (17) there is an associated finite Markov chain and vice versa—with every \( d \)-site Markov chain and orthonormal bases \( \{ e_i \} \) one can associate a CC-type channel through the formulas (18) and (25). In what follows we will also associate a finite Markov chain with a general QC-type channel and investigate the consequences for broadcasting of states and correlations.

The state (23) is obviously a QC state. It will be a CC state iff there exists a common basis \( \{ \tilde{e}_i \} \) such that

\[
\frac{1}{p_j} \text{Tr}_B (g_{AB} \mathds{1} \otimes E_j) = \sum_i p_{ij} |\tilde{e}_i \rangle \langle \tilde{e}_i|,
\]

for every \( j \), where \( p_j := \text{Tr}(g_{AB} \mathds{1} \otimes E_j) \) and \( p_{ij} := (1/p_j)(\tilde{e}_i |\text{Tr}_B (g_{AB} \mathds{1} \otimes E_j)| \tilde{e}_i \rangle \). Condition (26) means that all the Alice residual states, to which Bob steers via his measurement

\[
g^A_j := \frac{1}{p_j} \text{Tr}_B (g_{AB} \mathds{1} \otimes E_j),
\]

are simultaneously diagonalizable, or equivalently

\[
g^A_j, g^A_{j'} = 0
\]

for all \( j, j' \) (cf. Eq. (19)).

Let us investigate the set \( CC(A) \) of states \( g_{AB} \) which solve the above condition, i.e. lead to a CC state via (19) for a given CC-type channel \( A \). We are able to state what follows

- Obviously \( P_1 \in CC(A) \), by the very assumption (17), but it also contains mixtures of pure states with the following Schmidt decompositions:

\[
\psi_{AB}(\tilde{e}, \tilde{e}) := \sum_i c_i |\tilde{e}_i \rangle_A \otimes |e_i^* \rangle_B,
\]

where \( \tilde{e} \in \mathbb{R}_+^d \), \( \sum_i c_i^2 = 1 \), \( \{ \tilde{e}_i \} \) is some arbitrary basis, and \( \{ e_i \} \) is the fix basis from Eq. (25). Indeed, the states (27) for \( |\psi(\tilde{e}, \tilde{e})\rangle \langle \psi(\tilde{e}, \tilde{e})| \) read:

\[
p_j g^A_j = \sum_i p^A_{jj} c_i^2 |\tilde{e}_i \rangle \langle \tilde{e}_i|,
\]

from which there appears a stratified structure of convex sets generated by (27); mixing is allowed only within the states with the same, fixed \( \{ \tilde{e}_i \} \), thus generating convex subsets \( K(\tilde{e}) \). Partial unitaries \( U_A \otimes \mathds{1} \) transform between different \( K(\tilde{e}) \)'s. Furthermore, inside each \( K(\tilde{e}) \) there is a hierarchy of convex sets with increasing Schmidt number (11). This hierarchy is preserved by \( U_A \otimes \mathds{1} \). A schematic representation of this set is given in Fig. 1. Note that both \( \psi_+ \) and its local orbit \( U_A \otimes U_B \psi_+ \) are of the form (27), as \( U_A \otimes U_B \psi_+ = (U_A U_B \otimes \mathds{1}) \psi_+ \) and

\[
K(\tilde{e}) := \text{conv}\{ |\psi(\tilde{e}, \tilde{e})\rangle \langle \psi(\tilde{e}, \tilde{e})|; \{ \tilde{e}_i \} - \text{fixed}\}
\]
If we think of Alice and Bob as of Environment and Apparatus respectively, then
we broadcast states (29) (for an arbitrary \{\lambda_i\} such that after the measure-
ment, described by Thm. 1 by every conditional probability, follows from the fact that
\text{Tr}(\phi_j E_i) = \lambda_i(\phi)
for every i. This is equivalent to the following eigenvalue problem
\sum_j p_{ij} \lambda_j(\phi) = \lambda_i(\phi)
for a d \times d stochastic matrix
\begin{align*}
P(\phi) := \{p_{ij}(\phi)\}, \quad p_{ij}(\phi) := \langle \phi_j | E_i \phi_j \rangle.
\end{align*}
That this is a stochastic matrix, or equivalently a matrix of conditional probabilities, follows from the fact that
E_i’s form a POVM by Eqs. (11) and (12):
\sum_j p_{ij}(\phi) = \langle \phi_j | (\sum_i E_i) \phi_j \rangle = \langle \phi_j | \phi_j \rangle = 1
for every \( j \). By the celebrated Perron-Frobenius Theorem, the above eigenvalue problem has at least one non-negative, normalized solution \( \lambda_\phi(\phi) \), from which we construct through Eq. (32) the desired state \( \rho_\phi(\phi) \). Moreover, this solution is unique if the matrix \( P(\phi) = [p_{ij}(\phi)] \) is primitive, i.e. is irreducible and possesses exactly one eigenvector of the maximum modulus (equal to one in our case), which in turn is equivalent to that all the entries of the \((d^2 - 2d + 2)\)-th power of \( P(\phi) \) are non-zero. We now construct from \( \Lambda^{\text{QC}} \) a new channel (cf. Eq. (15))

\[
\Lambda^{(N)}(\rho) := \sum_i \text{Tr}(gE_i)\langle e_i | \otimes \cdots \otimes | e_i | \rangle_{e_i},
\]

(37)

which by condition N-copy spectrum-broadcasts the state \( \rho_\phi(\phi) \) (or equivalently N-copy broadcast its eigenvalues).

Since the basis \{\( \phi_i \)\} above was arbitrary, we obtain from the Perron-Frobenius Theorem that there exists a spectrum-broadcastable state in any basis (the states in different bases can be equal though, e.g. when the bases differ only by a permutation). For the basis \{\( e_i \)\}, associated with \( \Lambda^{\text{QC}} \) by the QC-condition, the corresponding state \( \rho_\phi(\phi) \) will be a fixed point of \( \Lambda^{\text{QC}} \): \( \Lambda^{\text{QC}}(\rho_\phi(\phi)) = \rho_\phi(\phi) \) by Eqs. (15) and (33). Thus \( \Lambda^{(N)}(\rho_\phi(\phi)) = \sum_j \lambda_j(\langle e_j | \otimes \cdots \otimes | e_j | \rangle_{e_j}) \) is a full N-copy broadcast state of \( \rho_\phi(\phi) \).

All the above obviously applies to CC-type channels, as a subclass of QC-type ones. However, as already mentioned, with any CC-type channel \( \Lambda \) there is a naturally associated stochastic matrix \( P^{(j)} \) by Eqs. (17)-(21), without a need of an additional basis \{\( e_i^* \)\} of Eq. (25) plays its role. The corresponding solution \( \lambda^\Lambda_\phi \equiv \lambda_\phi(e^\ast) \) of (34), \( \sum_i p^{(j)}_{ij} \lambda^\Lambda_i = \lambda^\Lambda_j \), and the state \( \rho_\phi(e^\ast) \) are now intrinsic characteristics of the channel. Note that for an arbitrary basis \{\( \phi_i \)\}, Eq. (33) reads

\[
\sum_{i,k} p^{\Lambda}_{ijk} |U_{ik}|^2 \lambda_k(\phi) = \lambda_j(\phi),
\]

(38)

where \( \phi_j = U e_j^\ast \) and \(|U|^2 := \langle e_j^\ast | U e_k^\ast \rangle^2 \) is a doubly-stochastic matrix. By the Birkhoff Theorem every such a matrix is a convex combination of at most \( d^2 - 2d + 2 \) distinct permutation matrices \( P_\sigma \), \( \sigma \in \mathfrak{S}_d \) and hence

\[
p_{ij}(\phi) = \sum_{\sigma \in \mathfrak{S}_d} \lambda_\sigma p^{\Lambda}_{ijk}(P_\sigma)k_j = \sum_{\sigma \in \mathfrak{S}_d} \lambda_\sigma p^{\Lambda}_{i\sigma^{-1}(j)},
\]

(39)

while for a general QC-type channel there will also be a “coherent” part:

\[
p_{ij}(U\phi) = \sum_{\sigma \in \mathfrak{S}_d} \lambda_\sigma p_{i\sigma^{-1}(j)}(\phi) + \sum_{k \neq l} U_{k,j} U_{l,j}|k|_{E_i}|l|.
\]

(40)

The existence of fully broadcastable state(s) \( \rho_\phi(e^\ast) \) for any QC-type channel is in some way surprising, as the measurements described by such channels are in general not von Neumann measurements, but POVMs (cf. Eq. (15)). The existence of a whole family of spectrum-broadcastable states is perhaps even more surprising. Note, however, that spectrum-broadcastability is a far weaker condition than full state broadcasting. By the same reason, although the broadcasting channel \( \Lambda^{(N)} \) is the same for every basis—it depends only on \( \Lambda \), we do not contradict the no-go theorem for state broadcasting from Ref. 6.

From a probabilistic point of view, the existence of (spectrum-)broadcastable states follows from the fact that one can associate a finite Markov process with the problem through Eq. (34), and by the Perron-Frobenius Theorem each such a process possesses a stationary distribution. The (spectrum-)broadcastable states are constructed precisely from this distribution.

Let us continue the above analysis and study the implications of the Ergodic Theorem for finite Markov chains. For a stochastic matrix \( P \) (cf. Eq. (33)) and \( \tilde{T} := (1, \ldots, 1) \). Note that the limiting matrix elements are the same for each column index \( i \). Asymptotically the probability for the process to be at site \( j \) does not depend on the initial site \( i \). As a consequence, the limiting distribution of the process \( P^\infty := \sum_j P^\infty_j p_j \) will not depend on the initial distribution \( p_j \):

\[
\sum_j P^\infty_j p_j = \lambda_i.
\]

(42)

Consider now the \( r \)-th power of a QC-type channel \( \Lambda \):

\[
\Lambda^r(\rho) = \sum_{i,j} P^r(\rho)^{ij} \text{Tr}(gE_i)\langle e_i | \rangle_{e_i},
\]

(43)

where \( P^r(\rho) \) is defined through Eq. (33). By the Ergodic Theorem, the limit \( \lim_{r \to \infty} \Lambda^r := \Lambda^\infty \) exists iff the matrix \( P(\rho) \) is primitive. By Eqs. (13) and (15), \( \Lambda^\infty \) is then a constant channel, analogously to (42)

\[
\Lambda^\infty(\rho) = \rho_\phi(e^\ast)
\]

(44)

for any state \( \rho \). Indeed, \( \Lambda^\infty(\rho) = \sum_i \lambda_i(\langle e_i | \text{Tr}(gE_i)\langle e_i | \rangle_{e_i}) = \langle \text{Tr}(g) \rangle \sum_i \lambda_i(\langle e_i | \rangle_{e_i}) = \rho_\phi(e^\ast) \) (cf. Eq. (32)). As a consequence, \( \Lambda^\infty \) breaks all correlations \( I \otimes \Lambda^\infty(\rho_{AB}) = g_B \otimes \rho_\phi \otimes g_B = \text{Tr}_{B\bar{B}} \rho_{AB} \).

An interesting situation arises when Eq. (33) has more than one solution, i.e. when a QC-type channel \( \Lambda^{QC} \) (spectrum-)broadcasts more than one state. Probabilistically, this means that the Markov process, corresponding to \( \Lambda^{QC} \), possesses more than one stationary distribution. This happens when the process splits into two or more disconnected processes. Algebraically this means that the trans-
According to the Perron-Frobenius Theorem, each of any correspondingly (each of them is normalized). Clearly, for any \( \sigma \), where \( \mathbf{0} = \mathbf{0} \), \( \mathbf{0} = \mathbf{0} \), \( \mathbf{0} \), \( \mathbf{0} \), \( \mathbf{0} \) is an eigenvalue-

\[ \lambda_{k+1}, \ldots, \lambda_{d}\] 

This is an example of the case where any state from the convex combination \( p_{w}(1) + (1 - p)q_{w}(2) \) can be (spectrum-)broadcasted. Clearly, this example generalizes to more than a binary combination of states if the matrix \( P(\phi) \) decomposes into more than two components: if the number of terms (degeneracy) in Eq. (15) is \( D \), there exists a \( D \)-dimensional simplex of states (spectrum-)broadcastable by \( \mathcal{L}^{(C)} \) (cf. Eq. (37)). The most degenerate case is of course when \( D = d \), i.e. when the transition matrix \( P(\phi) = 1 \), so that the Markov process is trivial—there are no transitions between states, which happens when the POVM is in fact a von Neumann measurement in \( \{ \phi_{\alpha} \} \). \( E_{i} = |\phi_{\alpha}\rangle \langle \phi_{\alpha}| \).

One can continue the above analysis and consider local broadcasting of correlations. From the general No-Local-Broadcasting Theorem from Ref. 2, we know that the only locally broadcastable states are the CC ones. Let us thus consider a family of CC states, built from the stationary solutions \( g_{(m)}^{(n)}(\phi) \) corresponding to a degenerate transition matrix \( P(\phi): \)

\[ \theta_{AB}(\pi; \phi) := \sum_{m,n=1}^{D} \pi_{mn} g_{(m)}^{(n)}(\phi) \otimes g_{(n)}^{(m)}(\phi) \]

Thus

\[ \Delta_{A} \otimes \Delta_{B}(\varrho_{AB}) = (\Delta_{A} \otimes 1)(1 \otimes \Delta_{B}) \varrho_{AB} \]

\[ = \sum_{i,j} \text{Tr}_{A}[E_{i}^{A} \varrho_{AB} E_{j}^{B}] |\phi_{i} \rangle \langle \phi_{i}| \otimes |f_{j} \rangle \langle f_{j}| \]

From Thm. 3 it then follows that for any basis \( \{ \phi_{\alpha}^{AB} \} \), \( \alpha = 1, \ldots, d \), of \( \mathcal{H}_{A} \otimes \mathcal{H}_{B} \) (the spaces \( \mathcal{H}_{A}, \mathcal{H}_{B} \) need not be the same now) there exists a state \( \varrho_{AB}(\phi_{\alpha}^{AB}) \), build from a stationary distribution of the stochastic matrix \( \mathcal{L}^{(C)} \).

The analysis of state broadcasting and the product state as well: \( P(\phi_{\alpha}^{AB}) = P(\phi_{\alpha}) \otimes P(\phi_{\alpha}) \) and \( P(\phi_{\alpha}^{AB}) \) is primitive iff both \( P(\phi_{\alpha}) \) and \( P(\phi_{\alpha}) \) are, i.e. \( \Delta_{A} \otimes \Delta_{B} \)-spectrum-broadcast only one state each. In such a case, the product state \( \varrho_{AB}(\phi_{\alpha}^{AB}) = \varrho_{A}(\phi_{\alpha}^{A}) \otimes \varrho_{B}(\phi_{\alpha}^{B}) \) is the only state that can be spectrum-broadcasted and there is no local broadcasting of classical correlations—the spectrum of \( \varrho_{AB}(\phi_{\alpha}^{AB}) \) is a product, \( \lambda_{ij}(\phi_{\alpha}^{AB}) = \lambda_{ij}(\phi_{\alpha}^{A}) \lambda_{ij}(\phi_{\alpha}^{B}) \). If, however, at least one channel spectrum-broadcasts more than one state, then there exists a family of locally spectrum-broadcastable correlated CC states, built analogously as in Eq. (15). A concrete example of such a situation is presented in the Appendix, Eqs. (43,44). When it comes to local full state broadcasting, by Thm. 3 it is guaranteed for \( \varrho_{AB}(\varphi_{e}, f) \), which is a CC state in the bases \( \{ e_{i} \} \), \( \{ f_{j} \} \) (cf. Eqs. (17,18)), in accordance with the general results of Ref. 2. Again, if both matrices \( P(\varphi_{e}) \) and \( P(\varphi_{f}) \) are primitive, \( \varrho_{AB}(\varphi_{e}, f) \) is a product state with no correlations. However, if at least one of \( P(\varphi_{e}) \) or \( P(\varphi_{f}) \) is
not primitive, by the above construction there will be a family of locally broadcastable correlated CC states $\varrho_{AB}(\pi, e, f)$.

Before we conclude, let us digress on a nature of some multipartite QC states. We assume that e.g. Bob holds two (possibly different) subsystems and that the joint state is $Q_{A}C_{BB'}$, that is

$$\varrho_{ABB'} = \sum_{\alpha} p_{\alpha} |e_{\alpha}\rangle_{BB'} \langle e_{\alpha}|,$$  \hspace{1cm} (52)

where $\{e_{\alpha}\}$ is a basis in $\mathcal{H}_{B} \otimes \mathcal{H}_{B'}$, labeled by $\alpha$. It is not necessarily a product basis—for the definition of a $Q_{A}C_{BB'}$ state it is enough that it is orthonormal. What is interesting is that simultaneously forcing both reductions $\varrho_{AB} := \text{Tr}_{B'}\varrho_{ABB'}$ and $\varrho_{AB'} := \text{Tr}_{B}\varrho_{ABB'}$ to be $Q_{A}C_{B}$ and $Q_{A}C_{B'}$, respectively:

$$\varrho_{AB} = \sum_{i} \lambda_{i} |e_{i}\rangle_{B} \langle e_{i}|,$$  \hspace{1cm} (53)

$$\varrho_{AB'} = \sum_{i'} \pi_{i'} |f_{i'}\rangle_{B'} \langle f_{i'}|$$  \hspace{1cm} (54)

does not force $\varrho_{ABB'}$ to be $Q_{A}C_{BB'}$ and $Q_{A}C_{BB}$ simultaneously (we may label such a class by $Q_{A}C_{B}C_{B'}$), i.e. $\{e_{\alpha}\}$ in (52) still need not be a product basis. As a simple example consider $\mathcal{H}_{B} = \mathcal{H}_{B'} = \mathbb{C}^{2}$ and $\{e_{\alpha}\}_{\alpha=1,...,4}$ the Bell basis. Then obviously both reductions $\varrho_{AB} \varrho_{AB'}$ are product, $1/2(\sum_{\alpha} p_{\alpha} |e_{\alpha}\rangle_{B} \langle e_{\alpha}| \otimes \mathbb{I})$, and hence trivially $Q_{A}C_{B}$ and $Q_{A}C_{B'}$, but the whole state $\varrho_{ABB'}$ is not $Q_{A}C_{B}C_{B'}$.

In some sense a converse of the above observation is also true: there exist $Q_{A}C_{BB'}$ states with a product basis on $BB'$, which are nevertheless not $Q_{A}C_{B}C_{B'}$, or, equivalently, both reductions $\text{Tr}_{B'}\varrho_{ABB'}$ and $\text{Tr}_{B}\varrho_{ABB'}$ are not $Q_{A}C_{B}$ and $Q_{A}C_{B'}$, respectively. As an example of such a state consider $\mathcal{H}_{B} = \mathcal{H}_{B'} = \mathbb{C}^{3}$, and choose as $\{e_{\alpha}\}_{\alpha=1,...,9}$ in (52) the “nonlocality without entanglement” $3 \otimes 3$ basis from Ref. [13]. Then both $\text{Tr}_{B'}\varrho_{ABB'}$ and $\text{Tr}_{B}\varrho_{ABB'}$ will contain an overcomplete set on $B$ and $B'$ side respectively.

In conclusion, we have provided a refinement of the characterization of entanglement breaking channels from Ref. [1] to more general quantum correlations and connected it to measurement maps, quantum state/correlations broadcasting, and finite Markov chains. We have considered two classes of channels—the ones that (i) break quantum correlations by turning them into the QC form and (ii) that fully break quantum correlations by turning them into CC ones. We have shown that a channel belongs to the first class if it turns a maximally entangled state into a QC state or equivalently it is represented by a measure-and-prepare scheme, where the outcomes of a POVM measurement are followed by a preparation of states from some specific orthonormal basis. In other words, it is a quantum-to-classical measurement map (i.e. it gives the state of the Apparatus after tracing the System).

Surprisingly, a similar question in the case of the second class of channels becomes even more interesting: the analogy to entanglement-breaking channels now fails and one cannot characterize the channels from the second class only by their actions on the maximally entangled state. However, a characterization from a different perspective seems possible. First of all, it turns out that the POVMs, constituting the channels, are mutually commuting and arise from a stochastic matrix, thus making a connection to finite Markov chains. Second, the set of bipartite states that are mapped into the CC form is more complicated.

Our analysis of the ability to broadcast quantum states and correlations by QC-type channels reveals an interesting application of the Perron-Frobenius Theorem. The existence of a family of spectrum-broadcastable states and at least one fully broadcastable state, even if the POVM measurement is not of the von Neumann type, follows from the fact that each finite Markov process possesses a stationary distribution. This broadcasting scheme, albeit in general substantially weaker than the standard broadcasting of e.g. Refs. [3] [8], surprisingly goes beyond the simple C-NOT scenario. The connection between broadcasting and finite Markov chains is, to our knowledge, quite unexpected and will be a subject of a further research.

In fact, perfect broadcasting operations applied so far corresponded to a scenario where to a given input CC state $\varrho_{AB} = \sum_{i,j} p_{ij} |i\rangle_{A} \otimes |j\rangle_{B}$ one locally applies the generalized C-NOT gates $U_{i,j} : = |i\rangle_{A} \otimes |j\rangle_{B}$. Application of the Perron-Frobenius Theorem presented in this work goes beyond this simple scenario.

We believe that the current work opens new perspectives for an analysis of the measurement problem and state/correlations broadcasting. Especially interesting seems possibility to study quantum decoherence in terms of broadcasting.

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**Appendix**

Consider the following example. Let

$$P^{(1)} := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$  \hspace{1cm} (A.1)
for some fixed basis \( \{ \phi_i \} \) and \( P^{(2)} \) be an arbitrary irreducible bistochastic matrix on \( \mathbb{R}^3 \), say:

\[
P^{(2)} := \begin{bmatrix}
\frac{1}{3} & \frac{2}{3} & 0 \\
\frac{3}{4} & \frac{1}{4} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{bmatrix}
\]  

(A.2)

for the same basis. Since we know that any matrix \( A \in M_{d \times d}(\mathbb{R}) \) with non-negative elements is irreducible iff \( (1 + A)^{d-1} \) has all elements non-negative, we may easily check that both matrices are irreducible. The unique Perron vector of \( P^{(1)} \) is just \( \lambda^{(1)} = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]^T \). The unique eigenvector of the irreducible bistochastic matrix is of course \( \lambda^{(2)} = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}]^T \). Consider now the stochastic matrix \( P := P^{(1)} \otimes P^{(2)} \) on \( \mathbb{R}^6 \) Then any state of the form \( \varrho_{AB}(\pi) = \sum_{m,n=1}^{2} \pi_{mn} \varrho_{A}^{(m)} \otimes \varrho_{B}^{(n)} \) with \( \varrho_{A}^{(1)} := \text{diag}[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}] \) and \( \varrho_{B}^{(2)} := \text{diag}[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}] \) can be spectrum/full broadcasted by the product of the channels \( \Lambda^{(N)} \), defined in Eq. (37).

Even simpler example with two different channels can be constructed to illustrate spectrum/full broadcasting of correlations. Namely, consider two bistochastic matrices of the form:

\[
P^A := \begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} \\
1 & 0 & 0
\end{bmatrix} 
\]  

(A.3)

and

\[
P^B := \begin{bmatrix}
\frac{2}{3} & 0 & \frac{1}{3} \\
\frac{1}{3} & 0 & \frac{2}{3} \\
0 & 1 & 0
\end{bmatrix} 
\]  

(A.4)

for some basis \( \{ \phi_i \} \). They are clearly reducible. Finding their Perron vectors and defining \( \varrho_{AB}(\pi) := \sum_{m,n=1}^{2} \pi_{mn} \varrho_{A}^{(m)} \otimes \varrho_{B}^{(n)} \) as \( \varrho_{A}^{(1)} := \text{diag}[0, \frac{1}{2}, \frac{1}{2}] \), \( \varrho_{A}^{(2)} := [0, 0, 1] \), \( \varrho_{B}^{(1)} := \text{diag}[\frac{1}{3}, 0, \frac{2}{3}] \), \( \varrho_{B}^{(2)} := [0, 0, 1] \), we see that \( \varrho_{AB}(\pi) \) is locally broadcastable by the map \( \Lambda_{A}^{(N)} \otimes \Lambda_{B}^{(N)} \) where \( \Lambda_{A} \), \( \Lambda_{B} \) are defined again through (37).

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