Krichever-Novikov Vertex Algebras on Compact Riemann Surfaces

Lu Ding∗ Shikun Wang†
Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, People’s Republic of China

Abstract

We give a notation of Krichever-Novikov vertex algebras on compact Riemann surfaces which is a bit weaker, but quite similar to vertex algebras. As example, we construct Krichever-Novikov vertex algebras of generalized Heisenberg algebras on arbitrary compact Riemann surfaces, which are reduced to be Heisenberg vertex algebra when restricted on Riemann spheres.

1 Introduction

Vertex operator algebras are a class of algebras, whose structures arose naturally from vertex operator constructions of representations of affine Lie algebras and in the work of Frenkel-Lepowsky-Meurman and Borcherds on the “moonshine module” for the Monster finite simple group [1] [2] [3] [9] [10]. They are (rigorous) mathematical counterparts of chiral algebras in 2-dimensional conformal field theory in physics. A vertex algebra consists in principle of a state-field correspondence

\[ V \rightarrow \text{End}(V) \left[ [z, z^{-1}] \right] \]

that maps any element \( a \) to a field \( Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^n \) with \( a_n \in \text{End}(V) \) which satisfies vacuum axiom, translation axiom and locality axiom. Typical examples of vertex algebras are the Heisenberg vertex algebra, affine Kac-Moody vertex algebra and lattice vertex algebra.

The goal of the present paper is the construction, as we hope, of regular analogues of vertex algebras, connected with Riemann surfaces of genus \( g \geq 0 \). It’s not surprising that the space \( \text{End}(V) \left[ [z, z^{-1}] \right] \) should be replaced by a certain space which encodes the geometry of the Riemann surface. We consider its important subspace—the associative algebra \( \mathbb{C}[z, z^{-1}] \) of Laurent polynomials firstly. According to the view of Krichever and Novikov ([16], [17], [18], [20]), if we identify it with the algebra of meromorphic functions on Riemann sphere which are holomorphic outside zero and infinite, then it is easily generalized on higher genus Riemann surfaces to the associative algebra of meromorphic functions which are holomorphic outside two distinguished points. [16] cited that the new associative algebra has a basis \( \{ A_n(P) | n \in Z' \} \) by Riemann-Roch theorem, where \( Z' \) takes the integer set or half-integer set depending on whether the genus \( g \) is even or odd. If we define \( (A(P))^n \) by \( A_{n+\frac{1}{2}}(P) \), then it is the space \( \mathbb{C}[A(P), A(P)^{-1}] \) which is the extension of the algebra of Laurent polynomials. Hence, the space \( \text{End}(V) \left[ [z, z^{-1}] \right] \) is naturally extended on any higher genus Riemann surface to \( \text{End}(V) \left[ [A(P), A(P)^{-1}] \right] \) in which elements are the form \( \sum_{n \in \mathbb{Z}} a_n (A(P))^n \)

∗Email: dinglu@amss.ac.cn
†Email: wsk@amss.ac.cn
or $\sum_{n \in \mathbb{Z}} a_n A_n (P)$. Obviously, the axioms in vertex algebras need to be reformulated in a suitable way as well. Here, we consider the locality axiom only. Note that the locality axiom
\[(z - w)^N [Y (a, z), Y (b, w)] = 0, N \gg 0\]
is equivalent to
\[(z^n - w^n)^N [Y (a, z), Y (b, w)] = 0, \forall n \in \mathbb{Z}, N \gg 0.
\]
As explained above, on a Riemann surface, the status of $z^n$ can be replaced by some meromorphic function $A_m (P)$. Then the locality axiom of KN vertex algebra is reformulated as
\[[A_m (P) - A_m (Q)]^N [Y (a, P), Y (b, Q)] = 0, \forall m \in \mathbb{Z}', N \gg 0.
\]
Analogously, the derivative, the function $\frac{1}{z - w}$, and distance function are reformulated as Lie derivative, Szegö kernel and the function defined by the level line respectively.

Using the data above, we can give the extention of the vertex algebra which is called the Krichever-Novikov vertex algebra with respect to a compact Riemann surface and the two distinguished points in general positions. For briefly, we also call it the KN vertex algebra if no confusion.

As examples of KN vertex algebras, we construct KN vertex algebras of generalized Heisenberg algebras on compact Riemann surfaces. When restricted on Riemann spheres, they are the Heisenberg vertex algebras.

The paper is organized as follows. Section 2 gives a brief reviews of KN basis and results needed later in order to make this paper self-contained. Section 3 sets up the notations and gives the definition of vertex algebras on compact Riemann surfaces. In section 4, we construct the Heisenberg vertex algebras on Riemann surfaces. Also, we introduce the Szegö kernel and the Level line used to prove the data we constructed satisfy the axioms of a vertex algebra on a Riemann surface. In the last section, we will see that the generalized Heisenberg vertex algebras on Riemann surfaces with genus zero is the usual Heisenberg vertex algebra.

In the forthcoming papers, Kac-Moody KN vertex algebra and Virasoro KN vertex algebra on a Riemann surface will be given. And the structures of KN vertex operator algebras on Riemann surfaces will be discussed later.

## 2 Notation

In this section, we recall some results on Riemann surfaces, and then reduce some results which are needed later.

### 2.1 Krichever-Novikov bases

Let $\mathcal{M}$ be a compact Riemann surface of genus $g$, $S_+, S_-$ two distinguished points in general position.

The KN bases [16] [17] [18] are certain bases for the spaces $\mathcal{F}^\lambda$ of meromorphic tensors of weight $\lambda$ on the Riemann surface $\mathcal{M}$ which are holomorphic outside $S_+$ and $S_-$. For integer $\lambda \neq 0, 1$ and $g > 1$, the Riemann-Roch theorem guarantees the existence and uniqueness of meromorphic tensors of conformal weight $\lambda$ which are holomorphic outside $S_+$ and $S_-$, and have the following behavior in a neighborhood of $S_+$ and $S_-:
\[ f_{\lambda,n} (z_\pm) = \varphi_{\lambda,n}^\pm z_\pm^{s_\lambda} (1 + O (z_\pm)) (dz_\pm)^\lambda, \]
\[ (1) \]
where $s_\lambda = \frac{g}{2} - \lambda (g - 1), \varphi_{\lambda,n}^+ = 1, \varphi_{\lambda,n}^- \neq 0$. Here $(dz_\pm)^\lambda$ means $\left( \frac{\partial}{\partial z_\pm} \right)^\lambda$ for $\lambda < 0, z_+$ and $z_-$ are local coordinates at small neighborhoods of $S_+$ and $S_-$ respectively which satisfy $z_+ (S_+) = 0$.
and \( z_- (S_-) = 0 \). The index \( n \) in Eq. (1) takes either integer or half-integer values depending on whether \( g \) is even or odd.

For \( \lambda = 0 \), the behavior is modified with respect to Eq. (1). Let \( A_n, |n| \geq \frac{g}{2} + 1 \), be the unique function which has the Laurent expansion in a neighborhood of \( S_\pm \):

\[
A_n (z_\pm) = \alpha_n^{\pm} z_\pm^{n - \frac{g}{2}} (1 + O (z_\pm)), \tag{2}
\]

where \( \alpha_n^+ = 1, \alpha_n^- \) is some nonzero complex number. As before, \( n \) is integer or half-integer depending on the parity of \( g \). For \( n = -\frac{g}{2}, \ldots, \frac{g}{2} - 1 \) we take the function with the following behavior!!!

\[
A_n (z_+) = \alpha_n^+ z_+^{n - \frac{g}{2}} (1 + O (z_+)) \quad \text{and} \quad A_n (z_-) = \alpha_n^- z_-^{n - \frac{g}{2} - 1} (1 + O (z_-)), \tag{3}
\]

where \( \alpha_n^+, \alpha_n^- \) are required as before. For \( n = \frac{g}{2} \), choose \( A_{z_+} = 1 \).

For \( \lambda = 1 \), we take the basis of one-forms as follows: in the range \( |n| \geq \frac{g}{2} + 1 \), \( \omega^n = f_{1-n} \) with \( f_{1-n} \) given by (1); for \( n = -\frac{g}{2}, \ldots, \frac{g}{2} - 1 \), those is specified by the local series

\[
\omega^n (z_+) = \beta_n^+ z_+^{n - \frac{g}{2} - 1} (1 + O (z_+)) dz_+ \quad \text{and} \quad \omega^n (z_-) = \beta_n^- z_-^{n + \frac{g}{2}} (1 + O (z_-)) dz_- \tag{4}
\]

Here, choose \( \beta_n^+ = 1 \) to fix \( \omega^n \) and take \( \omega^\frac{g}{2} \) as the Abelian differential of the third kind with simple poles in \( S_\pm \) and residues \( \pm 1 \), normalized in such a way that its periods over all cycles be purely imaginary.

In the case \( g = 1 \), the existence of a nonzero holomorphic one-form \( \xi \) with \( \xi (z_+) = (1 + o (z_+)) dz_+ \) enables us to construct a series of \( \lambda \)-forms which are holomorphic outside \( S_\pm \) by zero-forms:

\[
f_{\lambda,n} = A_n \xi^\lambda,
\]

where the \( A_n \)'s are defined by Equations (2) and (3).

For \( g = 0 \), when the compact Riemann surface is the ordinary completion of the complex plane, and \( S_\pm \) are the points \( z = 0 \) and \( z = \infty \), then the functions \( A_n \) coincide with \( z^n \), the generators of the Laurent basis, and \( f_{\lambda,n} (z) = z^n - \lambda (dz)^\lambda \) for \( \lambda \in \mathbb{Z} \).

Let \( f_{0,n} = A_n, f_{1,n} = \omega^{-n} \) and \( Z' = \mathbb{Z} + \frac{g}{2} \). On any compact Riemann surface, by Riemann-Roch theorem, \( \{ f_{\lambda,n} | n \in Z' \} \) makes up a basis of \( \mathcal{F}^\lambda \) which is called the Krichever-Novikov basis (KN basis) with weight \( \lambda (\mathbb{Z} - \mathbb{Z}) \).

Since all small cycles are homologous around \( S_+ \) (respectively, \( S_- \)) and \( \omega \cdot \eta \) is holomorphic away from \( S_+ \) and \( S_- \) for any \( \omega \in \mathcal{F}^\lambda, \eta \in \mathcal{F}^{1-\lambda} \), then the integral around the two points doesn’t depend on the choice of cycles. Hence, we can define residue operator by integral as follows.

**Definition 1**

\[
\text{Res}_{S_\pm} : \mathcal{F}^\lambda \times \mathcal{F}^{1-\lambda} \rightarrow \mathbb{C}, \quad \omega, \eta \mapsto \frac{1}{2\pi i} \oint_{c_{s_\pm}} \omega \cdot \eta,
\]

where \( c_{s_+}, (c_{s_-}) \) is a cycle homologous to a small cocycle around the points \( S_+ (S_-) \).

For simplicity, we denote \( \text{Res}_{S_+} \) by \( \text{Res} \) if there is no confusion. Define \( f^\lambda_\delta \) by \( f_{\lambda,-n} \) with \( n \in Z' \), \( \delta^m_\psi \) by 1 if \( n = m \) and 0 otherwise as usual. By direct calculus, we have the following important proposition.

**Proposition 2** \( \text{Res} [f_{\lambda,n} (P) f^{m}_{1-\lambda} (P)] = \delta^m_n \), where \( n, m \in Z' \).
2.2 Lie derivative

Definition 3 Define Lie derivative of a tensor field $g(P)$ with respect to a tangent vector field $\zeta (P)$ locally as

$$\nabla_{\zeta (P)} g(P) |_{U(P)} = \nabla_{\zeta (z)} \left( g(z) \frac{dz}{dz} \right) = \left( \zeta (z) \frac{dg(z)}{dz} + \lambda g(z) \frac{d\zeta (z)}{dz} \right) (dz)^{\lambda},$$

where $z$ is a local coordinate of the neighborhood $U(P)$ with $P \in \mathcal{M}$.

Definition 4 Denote $f_{-1, \omega_{-1}} (P)$ by $e_{-1, \omega_{-1}} (P)$ or $e(P)$, and $\nabla e_{\omega_{-1}} (P)$ by $\nabla$. Let $\langle \cdot, \cdot \rangle$ be the coupling of 1-form and tangent vector field.

Lemma 5 If $g(P) \in F^{\lambda}$, then

1. $\nabla \langle g(P), e(P) \rangle = \langle \nabla g(P), e(P) \rangle$;
2. $\nabla g(P) = d \langle g(P), e(P) \rangle$.

Proposition 6 $\nabla \omega^n (P) = \sum_{m \in \mathbb{Z}} \xi_m \omega^m$, where $\xi_m = - \text{Res} \left( \langle \omega^n (P), e_{\omega_{-1}} (P) \rangle dA_m (P) \right)$.

Proof. By the second part of Lemma 5 and Proposition 2 we have

$$\xi_m = \text{Res} \left( \nabla \omega^n (P) A_m (P) \right) = \text{Res} \left( d \langle \omega^n (P), e(P) \rangle A_m (P) \right) = \text{Res} \left( d \langle [\omega^n (P), e(P)] A_m (P) \rangle - \langle \omega^n (P), e(P) \rangle dA_m (P) \right).$$

Note that the first term is zero, we complete the proof. ■

Lemma 7 For any $n \in \mathbb{Z}'$, $\sum_{u_1, u_2, \ldots, u_k} \xi_{u_1} \xi_{u_2} \cdots \xi_{u_k}$ is

1. $(-n + \frac{g}{2} - 1) \cdots (-n + \frac{g}{2} - k)$, if $u_k = n + k$;
2. 0, if $u_k > n + k$;
3. 0, if $n = \frac{g}{2} - k, \ldots, \frac{g}{2} - 1$.

Proof. By the part (1) of Lemma 5 and Proposition 3 we have

$$\nabla^k \langle \omega^n (P), e(P) \rangle = \xi_{u_1} \xi_{u_2} \cdots \xi_{u_k} \langle \omega^{u_k} (P), e(P) \rangle,$$

Here, and henceforth, repeated indices are summed in the integer or half-integer set depending on the parity of the genus of the Riemann surface. From Section 2.1 we know the equation above has the following local behaviour near the point $S_i$

$$\left( 1 + o(z) \right) \left( \frac{\partial}{\partial z} \right)^k \left[ z^{-n + \frac{g}{2} - 1} \right] = \xi_{u_1} \xi_{u_2} \cdots \xi_{u_k} z^{-u_k} \left[ 1 + o(z) \right].$$

That is,

$$(-n + \frac{g}{2} - 1) \cdots (-n + \frac{g}{2} - k) z^{-n + \frac{g}{2} - k} \left[ 1 + o(z) \right] = \xi_{u_1} \xi_{u_2} \cdots \xi_{u_k} z^{-u_k} \left[ 1 + o(z) \right].$$

Compare the coefficients of $z^m$ of the two sides of the above equation, we can get the result. ■
2.3 Szegő kernel, Level line and formal delta function

Here, we introduce Szegő kernel, level line briefly \cite{4}. By the data, we construct and research on formal delta functions on compact Riemann surfaces.

Let \( \mathcal{M} \) be a compact Riemann surface with genus \( g \), \( S_+ \) and \( S_- \) are distinct points in general position.

**Definition 8** Define the Szegő kernel by \( S(P, Q) = \frac{E(P, P_+)^{\theta(P, Q, u)} E(Q, P_+)^{\theta(Q, P, u)}}{E(P, Q)^2 E(Q, P)^2} \), where \( E(P, Q) \) is the Schottky-Klein prime form, \( u = gP_+ - P_+ - \Delta \).

\( S(P, Q) \) is a holomorphic \((0, 1)\) form outside \( P = Q \) on \( \mathcal{M} \times \mathcal{M} \). In a neighborhood of \( P = Q \),

\[
S(P, Q) |_{U(P=Q)} = \frac{1}{z(P) - z(Q)} \left[ 1 + o(z(P) - z(Q))^2 \right] dz(Q).
\]

Suppose \( \rho \) is the unique differential of the third kind with pure imaginary periods, that is, \( \rho \) has poles of order 1 at \( S_{\pm} \), \( \text{Res}_{S_{\pm}} \rho = \pm 1 \), and has purely imaginary period. Let \( P_0 \) be an arbitrary reference point on \( \mathcal{M} \) different from \( S_{\pm} \). The contours \( c_\tau \) are defined as the level lines of the function, i.e.

\[
c_\tau = \{ P \in \mathcal{M} | \text{Re} \int_{P_0}^{P} \rho = \tau \}.
\]

For \( \tau \to \pm \infty \), the contours become small circles around \( S_{\pm} \).

For \( g = 0 \), the Szegő kernel is \( S(z, w) = \frac{1}{z - w} dw \), the level lines defined by \( \text{Re} \int_{P_0}^{P} \frac{1}{z} dz \) are circles around the origin in the complex plane, where \( P_0 \in \mathbb{C} \setminus \{ 0 \} \). In the regions of \( \tau (z) > \tau (w) \) and \( \tau (w) > \tau (z) \), the Szegő kernel has expansions \( \Sigma_{n \in \mathbb{N}} z^n w^{n-1} dw \) and \( -\Sigma_{n \in \mathbb{N}} z^n w^{n-1} dw \) respectively. Their minus is \( \Sigma_{n \in \mathbb{Z}} z^n w^{n-1} dw \) which is similar to the formal delta function \( \delta(z, w) = \Sigma_{n \in \mathbb{Z}} z^n w^{n-1} \).

For higher genus Riemann surfaces, we hope to get the formal delta functions using the similar way. Denote the expansion of \( S(P, Q) \) in \( \tau (Q) > \tau (P) \) and \( \tau (P) > \tau (Q) \) by \( i_{Q, P} S(P, Q) \) and \( i_{P, Q} S(P, Q) \) respectively. In \cite{16, 18} and \cite{23}, we know

\[
i_{P, Q} S_1(P, Q) = \sum_{n < \frac{3}{4}} A_n(P) \omega^n(Q), \quad i_{Q, P} S_1(P, Q) = - \sum_{n \geq \frac{3}{4}} A_n(P) \omega^n(Q) . \tag{5}
\]

Denote their minus by \( \triangle(P, Q) \). Clearly, \( \triangle(P, Q) = \sum_{n \in \mathbb{Z}} A_n(P) \omega^n(Q) \). We call it the formal delta function on the Riemann surface not only because the way of its construction is similar to the formal delta function \( \delta(z, w) \)'s, but also it has the complete analogous property of delta function.

**Proposition 9** If \( f \in \mathcal{F}^0, g \in \mathcal{F}^1 \), then

\[
\text{Res}_{Q = S_+} (\triangle(P, Q) f(Q)) = f(P), \quad \text{Res}_{Q = S_+} (\triangle(P, Q) g(P)) = g(Q) .
\]

**Proof.** It’s easy to prove by Proposition\cite{2}.

Recall in vertex algebras, the following properties of the formal delta function is very useful in proving the locality axiom

\[
\partial_w \delta(z, w) = -\partial_w \delta(z, w), \quad (z^n - w^n)^m \partial^{m+1}_z \delta(z, w) = 0 . \tag{6}
\]

From the below Propositions, we know, as we hope, the formal delta function on the Riemann surface also has the analogous properties in which \( \partial_z \) and \( z^n \) are replaced by \( \nabla_P \) and \( A_n(P) \) respectively.

**Proposition 10** \( \nabla_P \triangle(P, Q) = -\nabla_Q \triangle(P, Q) \).
Hence be got by the complete way. Note that

\[ \text{Proof.} \]

By Eq. 7, we assume it’s true when \( M \times M \). Using the same method, we have

\[ \forall \ m \]

\[ \text{Proposition 11} \]

Since \( \text{Res} \left( A_m (P) \omega^k (P) \right) = \delta^k_m \), we get \( \alpha_{mn} = \text{Res} \left[ A_u (P) A_n (P) \omega^m (P) \right] \).

Using the same method, we have

\[ A_u (Q) \omega^m (Q) = \alpha_{um} \omega^m (Q). \]

Hence

\[ [A_u (P) - A_u (Q)] \triangledown (P, Q) = 0 \]

\[ \text{Proposition 11} \]

Differentiating the formula in the above proposition with respect to \( P \), and multiplying the result by \( (A_u (P) - A_u (Q)) \), we obtain

\[ [A_u (P) - A_u (Q)]^2 d_P \triangledown (P, Q) = 0, \tag{7} \]

where \( d_P \triangledown (P, Q) \) means \( d (A_n (P)) \omega^m (Q). \)

\[ \text{Theorem 12} \]

For any \( u \in Z' \), \( m, n \in Z_+ \), \( [A_u (P) - A_u (Q)]^{m+n} \left( \nabla^m_P \nabla^N_Q d_P \triangledown (P, Q) \right) = 0 \) on \( \mathcal{M} \times \mathcal{M}. \)

\[ \text{Proof.} \]

By Eq. \( 7 \) we assume it’s true when \( m + n = \tilde{N} \) with \( \tilde{N} \in Z_+. \) We will prove the theorem by induction. When \( m + n = \tilde{N} + 1 \), there exists a pair of non-negative integers \( (M, N) \) such that \( (m, n) = (M + 1, N) \) or \( (M, N + 1) \). We only prove the theorem for the first case, the other case can be got by the complete way. Note that \( [A_u (P) - A_u (Q)]^{M+N+3} \left( \nabla^M_P \nabla^N_Q d_P \triangledown (P, Q) \right) \) is equal to

\[ \nabla_P \left( [A_u (P) - A_u (Q)]^{M+N+3} \nabla^M_P \nabla^N_Q d_P \triangledown (P, Q) \right) - \left( \nabla_P [A_u (P) - A_u (Q)]^{M+N+3} \right) \nabla^M_P \nabla^N_Q d_P \triangledown (P, Q). \]

By our inductive assumption, the first term is zero. The second term is

\[ -(M + N + 3) \nabla_P A_u (P) [A_u (P) - A_u (Q)]^{M+N+2} \nabla^M_P \nabla^N_Q d_P \triangledown (P, Q). \]

Using our inductive assumption again, we got the result.
3 \textbf{KN vertex algebras on compact Riemann surfaces}

3.1 KN Fields and derivatives

Recall in a vertex algebra, a formal series $a(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ is a field if for any $v \in V$, $a_n v = 0$, $n \ll 0$. In other words, the formal series is a formal Laurent series.

For a higher genus Riemann surface, as we explained in the introduction, the space $\text{End}(V) [[z, z^{-1}]]$ is replaced by $\text{End}(V) \left[[A(P), A(P)^{-1}]\right]$, and the formal series $a(z)$ is replaced by $a(P) = \sum_{n \in \mathbb{Z}} a_n A_n(P)$. The new formal series is called a KN field if for any $v \in V$, there exists a number $N$ such that $a_n v = 0$ for $n \leq N$.

Recall the derivative on the field $a(z)$ is defined by $\partial_z$ which acts on functions $z^n$ directly. Since $\nabla_{\partial_z}$ is the extension of $\partial_z$ on any Riemann surface, then the derivative of a KN field $a(P)$ is defined according to Lie derivative, that is, $\nabla a(P) = \sum_{n \in \mathbb{Z}} a_n \nabla A_n(P)$.

3.2 KN vertex algebras

Using the above data, we can give the notation of a KN vertex algebra on a compact Riemann surface.

\textbf{Definition 13} Suppose $\mathcal{M}$ is a compact Riemann surface with genus $g$, $S_+$ and $S_-$ are two distinguished points in general positions. A \textbf{Krichever-Novikov vertex algebra} $(\mathcal{M}, V, S_{\pm}, T, \{0\})$ (KN vertex algebra) is a collection of data

1. Space of states: $V$ is a $\mathbb{Z}$--graded vector space (graded by weights)
   \begin{equation*}
   V = \bigoplus_{n \in \mathbb{Z}} V_n, \text{ for } v \in V_n, n = \text{wt} \{v\}
   \end{equation*}
   where $\dim V_n < \infty$ for $n \in \mathbb{Z}$ and $V_n = 0$ for $n$ sufficiently negative;
2. Vacuum vector: $\{0\} \in V_0$;
3. Translation operator: a linear operator $T : V \rightarrow V$;
4. KN vertex operators: a linear operation
   \begin{equation*}
   Y(\cdot, P) : V \rightarrow \text{End}(V) [[A^+(P), A^-(P)]]
   \end{equation*}
   taking each $a \in V$ to a KN field acting on $V$,
   \begin{equation*}
   Y(a, P) = \sum_{n \in \mathbb{Z}'} a_n A_n(P), \ Z' = \mathbb{Z} + \frac{g}{2}.
   \end{equation*}

And these data are subject to the following axioms:

1. Vacuum axiom: $Y(\{0\}, P) = \text{Id}_V$. Furthermore, for any $a \in V$, $Y(a, P) \{0\}$ has a well-defined value at $P = S_+$. When $a$ is a homogeneous element with weight $k$, then
   \begin{equation*}
   Y(a, P) \{0\} |_{P = S_+} \equiv a \mod \bigoplus_{n < k} V_n.
   \end{equation*}

2. Translation axiom: $T \{0\} = 0$. For any nonzero space $V_n$, there always exists nonzero element $a$ in $V_n$ such that
   \begin{equation*}
   [T, Y(a, P)] = \nabla Y(a, P).
   \end{equation*}

3. Locality axiom: For any $a, b \in V$, there exists a positive integer $N$ such that
   \begin{equation*}
   F_u(P, Q)^N [Y(a, P), Y(b, Q)] = 0,
   \end{equation*}
   on $\mathcal{M} \times \mathcal{M}$, where $F_u(P, Q) = A_u(P) - A_u(Q)$ for any $u \in Z'$. 

7
4 KN vertex algebras of generalized Heisenberg algebras

In this section, we will construct a KN vertex algebras of the generalized Heisenberg algebra on an arbitrary compact Riemann surface as an example. Firstly, we recall the definition of the generalized Heisenberg algebra ([16]-[18]).

Definition 14

By a generalized Heisenberg algebra $\hat{A}$ connected with a compact Riemann surface $M$ with genus $g$ and a pair of points $S_+$ and $S_-$ in a general position is meant an algebra generated by generators $A_n$ and a central element $1$ with relations

$$[A_n, A_m] = \gamma_{nm} 1, \quad [A_n, 1] = 0,$$

where the numbers $\gamma_{nm}$ are defined as

$$\gamma_{nm} = \text{Res}_{P_=(S_+)} (A_m(P) \, dA_n(P)),$$

where $A_n(P), n \in Z'$ are the KN basis connected with $S_+$ and $S_-$. For $g = 0$, $S_\pm$ are the points 0 and $\infty$, the Lie brackets become

$$[A_n, A_m] = -n\delta_{n+m, 0}, \quad [A_n, 1] = 0.$$

Hence on a Riemann sphere, the generalized Heisenberg algebra becomes the ordinary Heisenberg algebra.

Now, we construct a representation of the generalized Heisenberg algebra. Note that $\gamma_{nm} = 0$ when $n, m \geq g_2$, then we get a commutative subalgebra $\hat{A}_+ \subset \hat{A}$ generated by $A_n$ and $1$ with $n \geq g_2$ which has a trivial 1-dimensional representation $C$. Hence the algebra has an induced representation of the generalized Heisenberg algebra

$$V = \text{Ind}_{\hat{A}_+}^\hat{A} C = U(\hat{A}) \otimes_{U(\hat{A}_+)} C.$$

Clearly, any element in $\hat{A}$ can be seen as an endomorphism of $V$. Let $|0\rangle = 1 \otimes 1$. By the Poincaré-Birkhoff-Witt theorem, $V$ has a PBW basis

$$W = \left\{ A_{-j_1 + \frac{g}{2}} \cdots A_{-j_n + \frac{g}{2}|0\rangle} \mid j_1 \geq j_2 \geq \cdots \geq j_n \geq 1, j_i \in N \right\},$$

and $V$ is a $Z$-graded vector space by defining the weight of the element $A_{-j_1 + \frac{g}{2}} \cdots A_{-j_n + \frac{g}{2}|0\rangle}$ in the basis as $j_1 + \cdots + j_n$. Denote the weight of a homogeneous element $v$ by $\text{wt}(v)$.

Remark 15 $V$ is graded only as a vector space, not a representation. In fact, as a representation, $V$ is quasi-graded.

Define the space of states of KN vertex algebra by $V$, the translation operator by the actions

$$T|0\rangle = 0 \quad \text{and} \quad [T, A_n] = \sum_{\xi_n \in Z'} \xi_n A_n,$$

where $\xi_n = -\text{Res}_{P=(S_+)} \left( \langle \omega^n(P), e_{\frac{g}{2}+1}(P) \rangle \, dA_n(P) \right)$, and KN vertex operators by:

$$Y(|0\rangle, P) = Id_V,$$

$$Y(A_{\frac{g}{2}+1}|0\rangle, P) = \sum_{n \in Z'} A_n(\omega^n(P), e(P)),$$

$$Y(A_{\frac{g}{2}+m}|0\rangle, P) = \frac{1}{m!} \nabla^m Y \left( A_{\frac{g}{2}+1}|0\rangle, P \right).$$
For simplicity, denote \( Y \left( A_{\frac{2}{m} - 1}\right), P \) by \( A(P) \) sometimes. For elements as \( A_{-j_1 + \frac{2}{m} - 1} \cdots A_{-j_n + \frac{2}{m} - 1}\) with \( n > 1 \), before defining their KN vertex operators, we need to give the notation of normal ordered product.

Now we recall the case in a vertex algebras, the normal ordered product of two fields is defined as

\[
: a (w) b (w) := Res_{z=0} \left[ a (z) b (w) i_{z,w} \frac{1}{z-w} - b (w) a (z) i_{w,z} \frac{1}{z-w} \right],
\]

that is,

\[
: a (w) b (w) := \sum_{n<0} a_n w^{-n-1} b (z) + b (w) \sum_{n\geq 0} a_n w^{-n-1}.
\]

Since \( i_{P, Q} \) and \( i_{Q, P} \) are the extensions of \( i_{z, w} \) and \( i_{w, z} \) respectively, the simple and natural way is to use some analogous form of Eq.11 to define the normal ordered product. Now, we give the notation of normal ordered product to be

\[
: a (Q) b_\mu (Q) := Res_{P=S_+} (i_{P, Q} S_1 (P, Q) a (P) b_\mu (Q) - i_{Q, P} S_1 (P, Q) b_\mu (Q) a (P)).
\]

By direct calculus, we have

\[
: a (Q) b_\mu (Q) := \sum_{n<\frac{2}{m}} a_n \omega^n (Q) b_\mu (Q) + b_\mu (Q) \sum_{n\geq \frac{2}{m}} a_n \omega^n (Q)
\]

which is similar to the equation[12] Through the normal ordered product is defined only on the pair of 1-form and \( \lambda \)-form series, it is enough for us.

For the fields \( a' (P) = a (P) (e (P)) \) and \( b_\mu' (Q) = b_\mu (Q) (e^\mu (Q)) \), we define their normal ordered product to be

\[
: a' (Q) b_\mu' (Q) := : a (Q) b_\mu (Q) (e^{\mu + 1} (Q))
\]

where \( e^\mu (Q) = e (Q)^{\otimes \mu}, a (P) (e (P)) \) means the coupling of 1-form series \( a (P) \) and the tangent field \( e (P) \), that is, \( \sum_{n \in \mathbb{Z}} a_n \omega^n (P) e (P) \), \( b_\mu (Q) (e^\mu (Q)) \) and \( : a (Q) b_\mu (Q) (e^{\mu + 1} (Q)) \) have the same meaning. Now we can define the KN vertex operators as follows

\[
Y \left( A_{-j_1 + \frac{2}{m} - 1} \cdots A_{-j_n + \frac{2}{m} - 1}\right), P \) \) := \[ Y \left( A_{-j_1 + \frac{2}{m} - 1}\right), P \) \[ \cdots \[ Y \left( A_{-j_n + \frac{2}{m} - 1}\right), P \) \]

It’s not difficult to prove the KN vertex operators are KN fields. We make our focus on proving the axioms of KN vertex algebra.

### 4.1 Vacuum axiom, transation axiom

The statement \( Y (|0\rangle, P) = ID \) follows from our definition. The remainder of the vacuum axiom follows by induction. We start with the case \( A_{\frac{2}{m} - m - 1}|0\rangle \) with \( m \geq 0 \). Note

\[
Y \left( A_{\frac{2}{m} - m - 1}\right), P \) \[ |0\rangle = \frac{1}{m!} \nabla^m Y \left( A_{\frac{2}{m} - 1}|0\rangle, P \) \[ |0\rangle
\]

\[
= \frac{1}{m!} \sum_{n \leq \frac{2}{m} - 1} A_n \langle \nabla^m \omega^n (P), e (P) |0\rangle
\]

\[
= \frac{1}{m!} \sum_{n \leq \frac{2}{m} - 1} A_n \xi_{u_1} \cdots \xi_{u_{m-1}} \langle \omega^m (P), e (P) |0\rangle.
\]

9
If $A_n \epsilon_n^{u_1} \cdots \epsilon_n^{u_m-1} \neq 0$, then $n \leq \frac q 2 - 1 - m$ by Lemma 7(3), correspondingly, $u_m \leq \frac q 2 - 1$ by Lemma 7(2). Hence the KN vertex operator action on $|0\rangle$ is well defined at $P = S_+$. More precisely,

$$Y \left( A_{\frac q 2 - m - 1}|0\rangle, P \right) |0\rangle \bigg|_{P=S_+} = \frac 1 {m!} \sum_{n \leq \frac q 2 - 1 - m} \sum_{u_m \leq \frac q 2 - 1} A_n \epsilon_n^{u_1} \cdots \epsilon_n^{u_m-1} z^{-u_m+\frac q 2 - 1} (1 + o(z)) |0\rangle_{z=0}$$

By Lemma 7(2), the sum above is not zero only when $n = \frac q 2 - 1 - m$. Using Lemma 7(1), we got

$$Y \left( A_{\frac q 2 - m - 1}|0\rangle, P \right) |0\rangle \bigg|_{P=S_+} = A_{\frac q 2 - m - 1}|0\rangle.$$

Define the level of the element $A_{\frac q 2 - n_m - 1} \cdots A_{\frac q 2 - n_1 - 1}|0\rangle$ by $m$. Then for any element of level one, we have proved its KN vertex operator satisfies the vacuum axiom. For the higher levels, we will prove it by induction as follows.

**Theorem 16** For any element $a \in V$, $Y (a, P) |0\rangle$ is well defined at $P = S_+$. Moreover, if $a$ is a homogeneous element in $V$ with weight $m$, then $Y (a, P) |0\rangle \big|_{P=S_+} \equiv a \mod \oplus_{k<m} V_k|0\rangle$.

**Proof.** Assume that it is true for any element of level $m$ in the basis $W$. That is, for any element $A_{\frac q 2 - n_m - 1} \cdots A_{\frac q 2 - n_1 - 1}|0\rangle$ in $V$ with $n_m \geq n_{m-1} \geq \cdots \geq n_1 \geq 0$, we have

$$Y \left( A_{\frac q 2 - n_m - 1} \cdots A_{\frac q 2 - n_1 - 1}|0\rangle, P \right) |0\rangle \bigg|_{P=S_+} = A_{\frac q 2 - n_m - 1} \cdots A_{\frac q 2 - n_1 - 1}|0\rangle + v|0\rangle,$$

where $v \in \oplus_{k<n_1+\cdots+n_m+m} V_k$.

For the element $A_{\frac q 2 - n_1 - 1} A_{\frac q 2 - n_m - 1} \cdots A_{\frac q 2 - n_1 - 1}|0\rangle$ with $n_m \geq n_{m-1} \geq \cdots \geq n_1 \geq 0$, by the definition of KN normal ordered product and Proposition 6 we have

$$Y \left( A_{\frac q 2 - n_1 - 1} A_{\frac q 2 - n_m - 1} \cdots A_{\frac q 2 - n_1 - 1}|0\rangle, P \right) = \frac 1 {n!} : \nabla^n A (P) Y \left( A_{\frac q 2 - n_m - 1} \cdots A_{\frac q 2 - n_1 - 1}|0\rangle, P \right) :$$

$$= \frac 1 {n!} : A_k \epsilon_k^{u_1} \cdots \epsilon_k^{u_n-1} \langle \omega^{u_n} (P), e (P) \rangle Y \left( A_{\frac q 2 - n_m - 1} \cdots A_{\frac q 2 - n_1 - 1}|0\rangle, P \right) :$$

$$= Y \left( A_{\frac q 2 - n_1 - 1} \cdots A_{\frac q 2 - n_1 - 1}|0\rangle, P \right) \sum_{u_n \geq \frac q 2} A_k \epsilon_k^{u_1} \cdots \epsilon_k^{u_n-1} \frac {\langle \omega^{u_n} (P), e (P) \rangle Y \left( A_{\frac q 2 - n_m - 1} \cdots A_{\frac q 2 - n_1 - 1}|0\rangle, P \right)} {n!}$$

$$+ \sum_{u_n < \frac q 2} A_k \epsilon_k^{u_1} \cdots \epsilon_k^{u_n-1} \frac {\langle \omega^{u_n} (P), e (P) \rangle Y \left( A_{\frac q 2 - n_m - 1} \cdots A_{\frac q 2 - n_1 - 1}|0\rangle, P \right)} {n!}.$$  

For $u_n \geq \frac q 2$, if $\epsilon_k^{u_1} \cdots \epsilon_k^{u_n-1} \neq 0$, then by Lemma 7(2), $k \geq u_n - n \geq \frac q 2 - n$. Using the lemma(3), we know $k$ should be bigger than $\frac q 2$. Then the first sum kills $|0\rangle$, and by our inductive assumption, the second sum gives a series with positive powers of $z$ with the constant term

$$A_k \epsilon_k^{u_1} \cdots \epsilon_k^{u_n-1} \frac {\langle \omega^{u_n} (P), e (P) \rangle Y \left( A_{\frac q 2 - n_m - 1} \cdots A_{\frac q 2 - n_1 - 1}|0\rangle, P \right)} {n!}.$$  

If $\epsilon_k^{u_1} \cdots \epsilon_k^{u_n-1} \neq 0$, then $k \geq \frac q 2 - n - 1$ by Lemma 7(2), correspondingly, $k = \frac q 2 - n - 1$
by the third part of the lemma. Hence

\[
Y \left( A_{\frac{n}{2} - n_{1}} A_{\frac{n}{2} - n_{m} - 1} \cdots A_{\frac{n}{2} - m_{1} - 1} |0\rangle, P \right) |0\rangle_{P=S_{+}} = \left( \frac{\xi_{n_1}^{n_{1} - 1} \xi_{n_2}^{n_{2} - 1} \cdots \xi_{n_{m}}^{n_{m} - 1}}{n!} A_{\frac{n}{2} - n_{1} - 1} + \sum_{k \geq 1} a^{k} A_k \right) \left( A_{\frac{n}{2} - n_{m} - 1} \cdots A_{\frac{n}{2} - m_{1} - 1} |0\rangle + v|0\rangle \right)
\]

where \( a^{k} = \frac{\xi_{n_1}^{k} \xi_{n_2}^{k} \cdots \xi_{n_{m}}^{k}}{n!} \). Here, the last equation is got by Lemma (7). Because weights of homogeneous elements in \( A_{\frac{n}{2} - n} v|0\rangle \) and \( A_{k} \left( A_{\frac{n}{2} - n_{m} - 1} \cdots A_{\frac{n}{2} - m_{1} - 1} |0\rangle + v|0\rangle \right) \) are smaller than \( n_1 + \cdots + n_{m} + n + m + 1 \), then

\[
Y \left( A_{\frac{n}{2} - n_{1}} A_{\frac{n}{2} - n_{m} - 1} \cdots A_{\frac{n}{2} - m_{1} - 1} |0\rangle, P \right) |0\rangle_{P=S_{+}} = A_{\frac{n}{2} - n_{1}} A_{\frac{n}{2} - n_{m} - 1} \cdots A_{\frac{n}{2} - m_{1} - 1} |0\rangle \mod (\oplus_{k < n_{1} + \cdots + n_{m} + n + m + 1} V_{k}) |0\rangle.
\]

Hence for any homogeneous element \( a \) with level \( m + 1 \), the vacuum axiom is satisfied.

Now, we prove the translation axiom.

For \( |0\rangle \in V_{0} \), from \( T|0\rangle = 0 \) and \( Y([0], P) = I d_{V} \), we have \( [T, Y([0], P)] = \nabla Y([0], P) \).

For \( A_{\frac{n}{2} - 1} |0\rangle \in V_{1} \), we have

\[
\left[ T, A_{\frac{n}{2} - 1} |0\rangle \right] = \sum_{n \in Z'} [T, A_{n}] \langle \omega^{n} (P), e(P) \rangle
\]

where the first and second equation are got by the fact \( [T, A_{n}] = \sum_{u \in Z'} \xi_{n}^{u} A_{u} \) and the equation \( \nabla \omega^{n} (P) = \xi_{n}^{n_{1}} \omega^{n} (P) \) respectively. For \( A_{-n + \frac{n}{2} - 1} |0\rangle \in V_{n+1} \), from

\[
\left[ T, Y \left( A_{-n + \frac{n}{2} - 1} |0\rangle, P \right) \right] = \frac{1}{n!} \left[ T, \nabla^{n} A(P) \right] = \frac{1}{n!} \nabla^{n} \left[ T, A(P) \right]
\]

and the equation (15), we get

\[
\left[ T, Y \left( A_{-n + \frac{n}{2} - 1} |0\rangle, P \right) \right] = \frac{1}{n!} \nabla^{n+1} A(P) = \nabla Y \left( A_{-n + \frac{n}{2} - 1} |0\rangle, P \right).
\]

Hence, for any \( V_{n} \) with \( n \in Z_{+} \), there exists an element \( a \) in \( V_{n} \) such that

\[
[T, Y(a, P)] = \nabla Y(a, P).
\]

### 4.2 Locality axiom

Recall the state space has a basis

\[
W = \{ A_{\frac{n}{2} - n_{1} - 1} \cdots A_{\frac{n}{2} - n_{k} - 1} |0\rangle | k \geq 0, n_{1} \geq \cdots \geq n_{k} \geq 0 \}.
\]
and the element $A_{2^n-1} \cdots A_{2^{n_k}-1}(0)$ in $W$ has level $k$.

Note that for any element $a$ in $W$ with level $k$, $Y(a, P)$ has the form $\sum a_{\eta_1 \cdots \eta_k} \omega^{n_1} (P) \cdots \omega^{n_k} (P) (e^k (P))$.

If we denote $\tilde{Y}(a, P) = \sum a_{\eta_1 \cdots \eta_k} \omega^{n_1} (P) \cdots \omega^{n_k} (P)$, then $Y(a, P) = \tilde{Y}(a, P) (e^k (P))$. Since for arbitrary two elements $a, b$ in $W$ with level $(a) = k, (b) = k'$,

$$[Y(a, P), Y(b, P)] = \left[ \tilde{Y}(a, P), \tilde{Y}(b, Q) \right] \left( e^k (P) \otimes e^{k'} (Q) \right),$$

then the locality axiom holds on the generalized Heisenberg KN vertex algebra if the following equalities are true

$$F_u^N (P, Q) \left[ \tilde{Y}(a, P), \tilde{Y}(b, Q) \right] = 0, \forall u \in Z', N \gg 0. \tag{16}$$

In the following, we will prove that for any $a, b \in W$, $\tilde{Y}(a, P), \tilde{Y}(b, Q)$ satisfy the condition (16), which is also called locality.

Firstly, we check that (16) holds for $a = b = A_{2^{n_1}-1}$. Since

$$\left[ \tilde{Y} \left( A_{2^{n_1}-1}(0), P \right), \tilde{Y} \left( A_{2^{n_1}-1}(0), Q \right) \right] = [A_n, A_m] \omega^n (P) \omega^m (Q) \gamma_{nm} \omega^n (P) \omega^m (Q)$$

and

$$d_\Delta (P, Q) = [dA_m (P)] \omega^n (Q) \gamma_{nm} \omega^n (P) \omega^m (Q) = - \gamma_{nm} \omega^n (P) \omega^m (Q),$$

where $\gamma_{nm} = \text{Res}(A_m (P) dA_n (P))$, then

$$\left[ \tilde{Y} \left( A_{2^{n_1}-1}(0), P \right), \tilde{Y} \left( A_{2^{n_1}-1}(0), Q \right) \right] = - d_\Delta (P, Q).$$

By Formula (7), we have

$$F_u^2 (P, Q) \left[ \tilde{Y} \left( A_{2^{n_1}-1}(0), P \right), \tilde{Y} \left( A_{2^{n_1}-1}(0), Q \right) \right] = 0, \forall u \in Z'.$$

According to Theorem (12) and

$$\left[ \tilde{Y} \left( A_{2^{n_1}-m}(0), P \right), \tilde{Y} \left( A_{2^{n_1}-m}(0), Q \right) \right] = \frac{1}{n_1!} \left[ \nabla^n \tilde{Y} \left( A_{2^{n_1}-1}(0), P \right), \nabla^m \tilde{Y} \left( A_{2^{n_1}-1}(0), Q \right) \right] = \frac{1}{n_1!} \nabla^n \tilde{Y} \left( A_{2^{n_1}-1}(0), P \right), \nabla^m \tilde{Y} \left( A_{2^{n_1}-1}(0), Q \right),$$

we get $\tilde{Y} \left( A_{2^{n_1}-1}(0), P \right)$ and $\tilde{Y} \left( A_{2^{n_1}-m}(0), Q \right)$ are local.

Finally, locality of any two formal series $\tilde{Y}(a, P)$ and $\tilde{Y}(b, Q)$ with $a, b \in W$ follows by an induction from locality of $\tilde{Y} \left( A_{2^{n_1}-1}(0), P \right)$ and $\tilde{Y} \left( A_{2^{n_1}-m}(0), Q \right)$ using the following theorem. Hence the locality of any two fields can be got because $W$ is a basis of the state space.

**Theorem 17 (Generalized Doung’s Lemma)** Let $g^m (P) = \omega^{m_1} (P) \cdots \omega^{m_k} (P)$ with $m_i \in Z'$. If $a (P) = \sum_{n \in Z} a_n \omega^n (P)$, $b_\mu (P) = \sum_{m \in Z} b_m g^m (P)$, $c_\gamma (P) = \sum_{n \in Z} c_m g^m (P)$ are mutual local, then $a (P) b_\mu (P) :$ and $c_\gamma (P) :$ are mutual local in $\mathcal{M} \times \mathcal{M}$.

**Proof.** By assumption, we may find $N$ so that for all $m \geq N$,

$$F_u (P, Q)^m a (P) b_\mu (Q) = F_u (P, Q)^m b_\mu (Q) a (P), \tag{17}$$

$$F_u (P, Q)^m b_\mu (P) c_\gamma (Q) = F_u (P, Q)^m c_\gamma (Q) b_\mu (P), \tag{18}$$

$$F_u (P, Q)^m a (P) c_\gamma (Q) = F_u (P, Q)^m c_\gamma (Q) a (P). \tag{19}$$

We wish to find an integer $M$ such that

$$F_u (Q, R)^M : a (Q) b_\mu (Q) : c_\gamma (R) = F_u (Q, R)^M : a (Q) b_\mu (Q) : c_\gamma (R).$$
By Formula (13), this will follow from the statement

\[ F_u (Q, R)^M [-i_{P,Q} S_1 (P, Q) a (P) b_\mu (Q) + i_{Q,P} S_1 (P, Q) b_\mu (Q) a (P)] c_\gamma (R) \]

(20)

\[ = F_u (Q, R)^M c_\gamma (R) [-i_{P,Q} S_1 (P, Q) a (P) b_\mu (Q) + i_{Q,P} S_1 (P, Q) b_\mu (Q) a (P)]. \]

Let us take

\[ \Delta = 0, \]

(19)

\[ Z = 0, \]

(18)

\[ = 0, \]

(21)

where the last equality is got by the fact \( \Delta (P, Q) F_u (Q, P) = 0 \). For the terms with \( 0 \leq k \leq N \), by

(17) – (19), we get

\[ F_u (Q, R)^N F_u (P, R)^k [-i_{P,Q} S_1 (P, Q) a (P) b_\mu (Q) + i_{Q,P} S_1 (P, Q) b_\mu (Q) a (P)] c_\gamma (R) \]

(22)

\[ = F_u (Q, R)^N F_u (P, R)^k c_\gamma (R) [-i_{P,Q} S_1 (P, Q) a (P) b_\mu (Q) + i_{Q,P} S_1 (P, Q) b_\mu (Q) a (P)]. \]

The same phenomena occurs on the right side of (20) : the terms with \( N \leq k \leq 2N \) will vanish, and the other terms give us the same expression as what we now have on the left hand side. Thus we have establish (20), and hence the theorem. □

4.3 Generalized Heisenberg KN vertex algebras on Riemann spheres

Recall that on a Riemann sphere, choose \( S_\pm = 0 \) and \( S_- = \infty \), then \( A_n (z) = z^n, \omega^n (z) = z^{-n-1} dz, e (z) = \frac{\partial}{\partial z} \) with \( z \in \mathbb{C} \cup \{ \infty \} \). Since

\[ \gamma_{nm} = -Res_{P=S_+} (A_n (z) dA_m (z)) = -Res_{z=0} (m z^{n+m-1} dz) = -m \delta_n^m = n \delta_n^m, \]

then the Generalized Heisenberg algebra on \( \mathbb{C}P^1 \) is generated by \( A_n \) and a central element 1 with relations

\[ [A_n, A_m] = n \delta_n^m 1, \ [A_n, 1] = 0, \]

which is indeed the structure of the Heisenberg algebra.

Since \( e (z) = \frac{\partial}{\partial z} \), then \( \nabla_e (z) = \partial_z \) when acts on meromorphic functions. Corresponding, the Lie derivative on a field \( a (z) = \sum_{n \in \mathbb{Z}} a_n A_n (z) \) is \( \nabla_e (z) a (z) = \sum_{n \in \mathbb{Z}} a_n \partial_z (A_n (z)). \)

Because \( Z' = \mathbb{Z} + \frac{\omega}{2} = \mathbb{Z} \) and \( \langle \omega^n (z), e (z) \rangle = z^{-n-1} \), then the vertex operators are defined as

\[ Y (A_{-1} |0 \rangle, z) = \sum_{n \in \mathbb{Z}} A_n \langle \omega^n (z), e (z) \rangle = \sum_{n \in \mathbb{Z}} A_n z^{-n-1} \]

\[ Y (A_{-m-1} |0 \rangle, z) = \frac{1}{m!} \sum_{n \in \mathbb{Z}} A_n \nabla^m \langle \omega^n (z), e (z) \rangle \]

\[ = \frac{1}{m!} \sum_{n \in \mathbb{Z}} A_n \partial_z^m z^{-n-1} = \frac{1}{m!} \partial_z^m Y (A_{-1} |0 \rangle, z), \]

13
and
\[ Y(A_{-n_1} \cdots A_{-n_m-1} |0, z) =: Y(A_{-n_1} |0, z) \cdots Y(A_{-n_m-1} |0, z) Y(A_{-n_m-1} |0, z) \cdots. \]

Next, we proceed to consider the form of KN normal ordered product. Since
\[ Y(A_{-m-1} |0, z) = \sum_{n \in \mathbb{Z}} \frac{1}{n!} A_n \omega^n(z), e(z) b(z) = \sum_{n \in \mathbb{Z}} (-n)_A \omega^{n+m}(z), e(z) b(z), \]
then by the definition of KN normal ordered product, for any KN field \( b(z) \),
\[ Y(A_{-m-1} |0, z) b(z) : = \sum_{n+m \leq -1} (-n)_A \omega^{n+m}(z), e(z) b(z) + \sum_{n+m \geq 0} (-n)_A \omega^{n+m}(z), e(z) A_n \]
\[ = \sum_{n+m \leq -1} (-m)_A z^{-n-m-1} b(z) + \sum_{n+m \geq 0} (-m)_A z^{-n-m-1} A_n \]
\[ = Y(A_{-m-1} |0, z) b(z) + b(z) Y(A_{-m-1} |0, z), \]

For any KN field \( b(z) \), where \( Y(a, z)_+ := \sum_{a < 0} a_n z^{-n-1}, Y(a, z)_- := \sum_{a \geq 0} a_n z^{-n-1} \). Clearly the KN normal ordered product is same as normal ordered product in the vertex algebra. Hence KN vertex operators of the KN vertex algebra on a Riemann sphere are same as in the Heisenberg vertex algebra.

Recall the transition operator \( T \) is defined by the actions \( T |0 = 0 \) and \( [T, A_m] = \sum_{n \in \mathbb{Z}} c_n^m A_n \). Since, on a Riemann sphere, \( c_n^m = -Res(z^{-n-1}dz^m) = -m \delta_{m-1} \), then \( T \) is same as in the vertex algebra.

Hence, the generalized Heisenberg KN vertex algebra on a Riemann sphere is same as the Heisenberg vertex algebra.

**Acknowledgement** 18 The first author would like to thank Professor Chongying Dong for his various kinds of help when she researched in University of California, Santa Cruz.

**References**

[1] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Natl. Acad. Sci. USA 83 (1986), 3068-3071.

[2] R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, Invent. Math. 109 (1992),405-444.

[3] A. Beilinson and V. Drinfeld, Chiral algebra, Colloq. Publ. 51, AMS (2004).

[4] L. Bonora, A. Lugo, M. Matone, J. Russo, A global operator formalism on higher genus Riemann surfaces: b-c systems, Commun. Math. Phys. 123, 329-352 (1989).

[5] J. H. Conway and S. P. Norton, Monstrous moonshine, Bull. London Math. Soc. 11 (1979), 308-339.

[6] Lu Ding, Zhouru Zheng, P-Twisted Affine Lie Algebras and Their Associated Vertex Algebras, Commun. Theor. Phys. 50 7-12.

[7] H. Earkas, I. Kra, Riemann surfaces, Berlin, Heidelberg, New York: Springer 1980.

[8] J. D. Fay, Theta functions on Riemann surfaces, Springer Lecture Notes in Math. 352 (1972).
[9] I. Frenkel, J. Lepowsky and A. Meurman, A natural representation of the Fischer-Griess Monster with the modular function J as character, Proc. Natl. Acad. Sci. USA 81 (1984), 3256-3260.

[10] I. Frenkel, J. Lepowsky and A. Meurman, Vertex operator algebras and the Monster, Pure and Appl. Math. Vol. 134, Academic Press, Boston (1988).

[11] Edward Frenkel and David Ben-Zvi, Vertex algebras and algebraic curves, Math. Surv. Mon. Vol. 88, AMS (2004).

[12] I. Frenkel, J. Lepowsky and A. Meurman, Vertex operator calculus, in: Mathematical Aspects of String Theory, Proc. 1986 conference, San Diego, ed. by S.-T. Yau, World Scientific Singapore (1987), 150-188.

[13] R. C. Gunning, Lectures on Riemann surfaces, Princeton University Press, 1966.

[14] R. C. Gunning, Lectures on vector bundles over Riemann surfaces, Princeton university press, 1967.

[15] V. Kac, Vertex algebras for beginners, Second Edition, AMS (1998).

[16] I. M. Krichever, S. P. Novikov, Algebras of virasoro type, Riemann surfaces and structures of the theory of solitons, Funktional Anal. i. Prilozhen. 21 (1987), no. 2, 46.

[17] I. M. Krichever, S. P. Novikov, Virasoro type algebras, Riemann surfaces and strings in Minkowski space, Funktional Anal. i. Prilozhen. 21 (1987), no. 4, 47.

[18] I. M. Krichever, S. P. Novikov, Algebras of Virasoro type, energy-momentum tensors and decompositions of operators on Riemann surfaces, Funktional Anal. i. Prilozhen. 23 (1989), no. 1, 19-23.

[19] Y.-Z. Huang, Two-dimensional conformal geometry and vertex operator algebras, Progress in Mathematics 148, Birkhauser, Boston, 1997.

[20] K. Linde, Towards vertex algebras of Krichever-Novikov type, Part I. Preprint.

[21] K. Linde, Global vertex algebras on Riemann surfaces.

[22] O. Sheinman, Modules with highest weight for affine Lie algebras on Riemann surfaces, Funct. Anal. Appl. 29 (1995), no.1, 44-45.

[23] O. Sheinman, A fermionic model of representations of affine Krichever-Novikov algebras, Funct. Anal. Appl. 35 (2001), no.3, 209-219.

[24] O. Sheinman, Affine Krichever-Novikov algebras, Their representations and Applications, Preprint math. RT/0304020.

[25] Han-ying Guo, Ji-Sheng, Jian-min Shen, Shi-kun Wang and Qi-huang Yu, The algebras of meromorphic vector fields and their realization on the spaces of meromorphic λ-differentials on Riemann surfaces. I, 1990, J. Phys. A: Math. Gen. 23 379-384.

[26] Y. Zhu, Global vertex operators on Riemann surfaces, Comm. Math Phys. 165 (1994) 485-531.