Neutrino condensation from a New Higgs Interaction

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We study the consequences of having a new interaction between neutrinos and a Higgs scalar. We find that there are two possible attractive channels in the resulting effective 4-fermi theory which lead to a neutrino condensation in cosmic neutrinos and the creation of a neutrino superfluid at low temperatures and finite density. We find that at the minimum of the effective potential $V$ the condensates are mostly made up of pairs of left-left + right-right composites, with a slight admixture of left-right + right-left composites. The effective theory is sensitive to the cutoff mass $m_H$ (the mass of the Higgs particle) in the sense that if one wants to use the almost pole-like contribution coming from the fermi surface to calculate the effective potential, one needs to renormalize the coupling constant so that the dependence of the variables of the theory on the cutoff is only logarithmic.

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I. INTRODUCTION

It is virtually certain that the cosmic neutrino background (CNB) exists, even though it has yet to be detected, lurking at a slightly lower temperature than its more renowned cousin, the cosmic microwave background. If and when the CNB is amenable to observation, it may well exhibit some interesting dynamical properties.

One reason so to suppose is the tantalizing numerical coincidence expressed by $\Lambda M_p^2 = m^4_\nu$. Here $\Lambda$ is the measured cosmological constant, $M_p$ is the Planck length, and, as a plausible assumption, we take the neutrino mass $m_\nu$ to be of the same order of magnitude as the measured neutrino mass differences. Another way of stating this result is that the scale of dark energy density is given by the neutrino mass.

This may be a pure coincidence, of no further significance than, say, the Koide relation \[1\], or the fact that the proton to electron mass ratio is $6\pi^5$. Nevertheless, late in the last millennium, it prompted Caldi and one of the present authors to conjecture \[2\] that the cosmological vacuum was home to a neutrino condensate, as a way of seeing why the neutrino mass and the dark energy might be connected. The rough argument was based on assuming an effective 4-neutrino interaction at low energies. If a $<\nu\nu>$ condensate formed, then schematically a neutrino mass would be generated by terms of the form $\nu\nu <\nu\nu>$, and the pure condensate term $<\nu\nu><\nu\nu>$ would contribute to the cosmological constant. Here we are being very generic: $<\nu\nu>$ is meant to stand for any type of neutrino-neutrino or neutrino-antineutrino pairing, as dictated by whatever attractive interactions exist that could induce a condensate to form. Of course, even if this scenario is realized, one still has to investigate whether it can lead to the simple numerical relationship mentioned above.

As a first step, the authors of \[2\] examined the effective 4-neutrino interaction due to the exchange of the $Z$ boson. They looked for pairing of the superfluid neutrino-neutrino type, in the presence of a chemical potential. If there is an attractive channel, a solution to the relevant gap equation is guaranteed, because, in the absence of a gap, the interaction becomes infinite as one approaches the Fermi surface. However, the authors found that no attractive channel exists, perhaps not surprising in view of the observation that vector exchange produces repulsion between like charges.

A few years later, the subject was advanced by the work of Kapusta \[3\], who considered the exchange of the Higgs boson, which indeed produces an attractive channel. As we shall see below, there are in fact two such channels; Kapusta examined the one that couples left-handed to right-handed neutrinos. Once again, if one adds a chemical potential one finds a solution to the gap equation, and hence evidence for a condensate. However, the coupling $g$ of the Higgs boson to neutrinos (assuming that that is how the neutrinos get their mass) is exceedingly small, since the vacuum expectation value of the Higgs is already determined by the Standard Model. From the gap equation, one infers that the size of the condensate depends exponentially on the coupling: $<\nu\nu> \propto e^{-1/g^2}$ so the magnitude of a condensate generated in this way is unfortunately totally negligible.

Lately, there has been increased interest in various forms of non-standard neutrino interactions \[4\], invoked to address a range of issues. One possibility is the existence of a new, “neutrinophilic” Higgs-like boson \[5\]. It is hypothesized to couple preferentially to neutrinos, in the same manner as the ordinary Higgs would, but with a much lower value for the vacuum expectation value, thus permitting the coupling constant to be similar in magnitude to the coupling of the ordinary Higgs to the charged leptons. As a result, it could generate a condensate similar to the one found by Kapusta, but with a considerably larger value.

In the following analysis we explore this possibility in the simplified situation where there is only one flavor of neutrino. As in past work, we look for a pairing of superfluid type. We note that the relevant Fierz rearrangement allows for condensates with two different sets of quantum numbers, the Lorentz-invariant matrix structure, $i\gamma^1\gamma^3$, which does not flip the handedness of the neutrino, and the structure $\gamma^2\gamma^5$, which does. It is the latter that was considered in \[3\] for the case of the ordinary Higgs. In our analysis, we shall find, for a range of the parameters, that the former condensate in fact dominates the dynamics.

In section II, we shall derive the gap equations in mean-field approximation, using the well-known Hubbard-Stratonovich procedure. In section III, we shall analyze these equations to determine the extrema of the effective potential. We shall discuss our results in Section IV.
II. DYNAMICS OF NEUTRINOS WHEN THERE IS INTERACTION WITH A NEW HIGGS PARTICLE

As described in the introduction, we assume for simplicity, one species of neutrino interacting with a Higgs scalar described by the Lagrangian:

\[ \mathcal{L}_s = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi) - V[\phi], \]

\[ V[\phi] = -\mu^2 \phi^2 / 2 + h \phi^4 / 4, \]  

(2.1)

with \( \mu^2 > 0 \). The Higgs potential gives the scalar particle a vacuum expectation value

\[ \phi_0 = v = \sqrt{\mu^2 / h}, \]  

(2.2)

and a tree-level mass

\[ m_h^2 = \frac{d^2 V}{d\phi^2} |_{\phi = v} = 2h v^2. \]  

(2.3)

We assume that the neutrino is a Dirac particle having both right and left handed components. In the Dirac representation

\[ \psi_L = \frac{1}{\sqrt{2}} (1 - \gamma_5) \psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \nu_L \\ -\nu_L \end{pmatrix}, \]  

(2.4)

\[ \psi_R = \frac{1}{\sqrt{2}} (1 + \gamma_5) \psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \nu_R \\ \nu_R \end{pmatrix}, \]  

(2.5)

where \( \nu_R \) and \( \nu_L \) are two component spinors so that

\[ \psi = \frac{1}{\sqrt{2}} \left( \nu_R + \nu_L \right). \]  

(2.6)

The neutrino Lagrangian including interaction with the Higgs particle is given by

\[ \mathcal{L}_f = \bar{\psi} i\gamma^\mu \partial_\mu \psi - g \bar{\psi} \psi. \]  

(2.8)

The leading effect of the Higgs particle is to give a Dirac mass to the neutrinos: \( m = gv \), where \( v = \langle \phi \rangle \), as well as induce an effective four-fermi interaction between the neutrinos. This leads to a low energy effective neutrino Lagrangian:

\[ \mathcal{L}_f = \frac{i}{2} (\bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi) - m \bar{\psi} \psi \right) + \frac{g^2}{m_h^2} (\bar{\psi} \psi)^2. \]  

(2.9)

We can write this as

\[ \mathcal{L}_f = \frac{i}{2} \left( \psi^\dagger A \psi - \psi A^T \psi^\dagger \right) + \mathcal{M}_{\alpha\beta\gamma\delta} \psi^\dagger_\alpha \psi_\beta \psi^\dagger_\gamma \psi_\delta, \]  

(2.10)

where choosing the metric diagonal \( \{1, -1, -1, -1\} \)

\[ A = i \partial_0 + i \gamma^0 \gamma^k \partial_k - \gamma^0 m. \]  

(2.11)

We are interested in rearranging the last term in Eq. (2.10) in order to understand whether this low energy effective interaction can lead to the usual “Cooper pairs”, \( \psi^\dagger \psi^\dagger \) and \( \psi \psi \) found in superfluidity. To do this one makes a Fierz re-ordering of the 4-Fermi interaction, which means we want to write

\[ \mathcal{M}_{\alpha\beta\gamma\delta} = \sum_\lambda \eta_\lambda Q^__(\lambda) Q^{*__(\lambda)} \]  

(2.12)
so that
\[ M_{\alpha\beta\gamma\delta} \psi^\dagger_\alpha \psi_\beta \psi^\dagger_\gamma \psi_\delta = - \sum_\lambda \eta_\lambda \psi^\dagger_\alpha Q^{(\lambda)}_{\alpha\gamma} \psi_\gamma Q^{*(\lambda)}_{\beta\delta} \psi_\delta. \] (2.13)

The general Fierz transformation is based on the fact that there are sixteen independent $4 \times 4$ matrices $O^i$ which can be written in terms of five types of terms: scalar 1 vector $\gamma^\mu$, tensor $\sigma^{\mu\nu}$, psudoscalar $\gamma^5$ and axial vector $\gamma^5 \gamma^\mu$. So we can write
\[ \bar{\psi}_{1a} M_{ab} \bar{\psi}_{2b} \bar{\psi}_{3c} N_{cd} \psi_{4d} = \bar{\psi}_{1a} \bar{\psi}_{3c} \psi_{4d} M_{ab} N_{cd}. \] (2.14)

We can think the term $M_{ab} N_{cd}$ as the $ac$ component of a $4 \times 4$ matrix:
\[ M_{ab} N_{cd} = \begin{bmatrix} K_{db} \end{bmatrix}^{ac} = C_{ij} \begin{bmatrix} O_{ij} \end{bmatrix}^{ac}, \] (2.15)
where $O_{ij} = (O_i)^{-1}$. Then writing $C_{ij} = C_{ij} O_{ij}^b$ and using
\[ Tr[O_i O_k] = 4 \delta_i^k, \] (2.16)
we obtain:
\[ (\bar{\psi} \psi)^2 = - \frac{1}{4} \sum_{\alpha=1}^{6} \eta_\alpha (\psi^\dagger O^{\alpha} \psi^\dagger)(\psi O^{\alpha*} \psi), \] (2.17)
and we have used the fact that only the antisymmetric matrices of the sixteen $O^i$ can contribute since the $\psi_a$ anticommute. Here $\eta_\alpha = \pm 1$. One finds for the six non-vanishing $O^\alpha$
\[ \eta_\alpha = -1 \text{ for } \gamma^1, \gamma^3, \gamma^0 \gamma^5, \sigma^{02}, \eta_\alpha = +1 \text{ for } \gamma^2 \gamma^5, \sigma^{13}. \] (2.18)

Here we have ($\mu \neq \nu$)
\[ \sigma^{\mu\nu} = i \gamma^\mu \gamma^\nu. \] (2.19)

Since we are interested in neutrino condensation we will focus on the attractive channels. For those cases $O^{\alpha*} = -O^\alpha$, and (ignoring the repulsive channels),
\[ (\bar{\psi} \psi)^2 \rightarrow \frac{1}{4} [(\psi^\dagger \sigma^{13} \psi^\dagger)(\psi \sigma^{13} \psi) + (\psi^\dagger \gamma^2 \gamma^5 \psi^\dagger)(\psi \gamma^2 \gamma^5 \psi)]. \] (2.20)

So that
\[ M_{\alpha\beta\gamma\delta} \rightarrow \sum_\lambda Q^{(\lambda)}_{\alpha\gamma} Q^{*(\lambda)}_{\beta\delta}, \] (2.21)
with
\[ Q^{(1)} = i \kappa \gamma^1 \gamma^3, Q^{(2)} = \kappa \gamma^2 \gamma^5, \] (2.22)
and $\kappa = \frac{g^2}{4m^2}$. Note that $Q$ is already proportional to $\kappa$. To make contact with the work of Kapusta, we can write the $\gamma^\mu$ in terms of the Pauli matrices. We find that
\[ \sigma^{13} = i \gamma^1 \gamma^3 = -\begin{pmatrix} \sigma_2, & 0 \\ 0, & \sigma_2 \end{pmatrix}, \quad (\sigma_2)_{ij} = -i \epsilon_{ij}, \] (2.23)
so that
\[ [(\psi^\dagger \sigma^{13} \psi^\dagger)(\psi \sigma^{13} \psi) \rightarrow (\nu_R^\dagger \sigma_2 \psi_R^\dagger + \nu_L^\dagger \sigma_2 \psi_L^\dagger)(\nu_R \sigma_2 \nu_R + \nu_L \sigma_2 \nu_L) \] (2.24)

(2.25)
On the other hand

\[ \gamma^2 \gamma^5 = \begin{pmatrix} \sigma_2, & 0 \\ 0, & -\sigma_2 \end{pmatrix}, \quad (2.26) \]

so that in terms of right-handed and left-handed neutrinos, the attractive channels are:

\[ \left( \psi^\dagger \gamma^2 \gamma^5 \psi \right) \left( \psi^\gamma^2 \gamma^5 \psi \right) \rightarrow \]

\[ (\nu_R^\dagger \sigma_2 \nu_L^\dagger + \nu_L^\dagger \sigma_2 \nu_R^\dagger)(\nu_R \sigma_2 \nu_L + \nu_L \sigma_2 \nu_R) \]  
\[ = 4(\nu_R^\dagger \sigma_2 \nu_L^\dagger)(\nu_R \sigma_2 \nu_L) \quad (2.29) \]

Kapusta only studies the second possibility where the condensate is a \( RL \) composite. We will find that in our more general framework, this solution is only a relative minimum at an endpoint of the generalized \( \theta \) space of solutions.

We now implement the Hubbard-Stratonovich procedure by adding auxiliary fields to the action in such a way as to cancel the terms that are quartic in the neutrino fields. This does not change the action, since if we perform the integration over these fields in the path integral we recover the original action. Thus we add to \( L \) the terms

\[ -\frac{1}{\kappa^2} \sum_\lambda \left( B^{(\lambda)\dagger} - \kappa Q^{(\lambda)}_{\alpha \gamma} \psi^{\dagger}_\alpha \psi^{\gamma} \right) \left( B^{(\lambda)} + \kappa Q^{(\lambda)}_{\beta \delta} \psi^\dagger_\beta \psi^\delta \right) \]  
\[ \text{which then cancels the quartic interaction in Eq. (2.9), to yield:} \]

\[ L_f = \frac{1}{2} (\psi^\dagger A \psi - \psi A^T \psi^\dagger) - \frac{1}{\kappa^2} \sum_\lambda B^{(\lambda)\dagger} B^{(\lambda)} + \psi^\dagger B \psi^\dagger + \psi B^\dagger \psi, \quad (2.31) \]

where

\[ B = \frac{1}{\kappa} \sum_\lambda B^{(\lambda)} Q^{(\lambda)}; \quad B^\dagger = -\frac{1}{\kappa} \sum_\lambda B^{(\lambda)\dagger} Q^{(\lambda)} \]

\[ B = (B^{(1)} i \gamma^1 \gamma^3 + B^{(2)} i \gamma^2 \gamma^5) \quad B^\dagger = (B^{(1)} i \gamma^1 \gamma^3 + B^{(2)} i \gamma^2 \gamma^5), \quad (2.32) \]

since \( Q^{(\lambda)} = -Q^{(\lambda)} \).

Thinking of \((\psi, \psi^\dagger)\) as a column vector \( \Psi \), we can represent \( L_f \) as

\[ L_f = -\frac{1}{\kappa^2} \sum_\lambda B^{(\lambda)\dagger} B^{(\lambda)} + \Phi^\dagger S^{-1} \Phi. \quad (2.33) \]

One is now able to perform the fermion path integral by making the translation

\[ \psi = \chi + \alpha \psi^\dagger = \chi + \psi^\dagger \alpha^T \]  
\[ \text{with} \]

\[ \alpha = \frac{1}{2} (B^\dagger)^{-1} A^T; \quad \alpha^T = -\frac{1}{2} A (B^\dagger)^{-1} \quad (2.35) \]

to obtain

\[ \int dB^\dagger dB \exp [\Gamma_{eff}(B^\dagger, B)], \quad (2.36) \]

where

\[ \Gamma_{eff} = -\int d^4 x \left( \frac{1}{\kappa^2} \sum_\lambda B^{(\lambda)} B^{(\lambda)} + \frac{i}{2} \text{Tr} \log \left[ 1 + 4 A^{-1} B (A^T)^{-1} B^\dagger \right] \right). \quad (2.37) \]
If we introduce a chemical potential $\mu$ we have

$$A = i\partial_0 + i\gamma^0 \gamma^k \partial_k - \gamma^0 m - \mu. \quad (2.38)$$

In the Dirac representation

$$i\gamma^0 \gamma^k = \sigma^0 k = i\alpha^k, \quad (2.39)$$

where

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}. \quad (2.40)$$

The transpose is given by

$$A^T = -i\partial_0 - i(\alpha^i)^T \partial_i - \gamma^0 m - \mu. \quad (2.41)$$

Here

$$\tilde{\alpha} = (\alpha^1, \alpha^2, \alpha^3) \quad \tilde{\alpha}^T = (\alpha^1, -\alpha^2, \alpha^3). \quad (2.42)$$

In Fourier space we have

$$A(x, y) = \int \frac{d^4p}{(2\pi)^4} \left[ (p_0 - \mu) - \tilde{\alpha} \cdot \vec{p} - m\gamma^0 \right] e^{-ip(x-y)} \quad (2.43)$$

or

$$A(x, y) = \int \frac{d^4p}{(2\pi)^4} \left[ \gamma^0 (\gamma^\mu p_\mu - m) - \mu \right] e^{-ip(x-y)}. \quad (2.44)$$

The naive inverse is

$$A^{-1}(x, y) = \int \frac{d^4p}{(2\pi)^4} \left[ (\gamma^\mu p_\mu + m)\gamma^0 - \mu \right] e^{-ip(x-y)}. \quad (2.45)$$

Here $p^2 = p_0^2 - \vec{p} \cdot \vec{p}$. However to give the correct interpretation of the chemical potential $\mu$ one needs to introduce an $i\epsilon$ prescription, so that

$$A^{-1}(x, y) = \int \frac{d^4p}{(2\pi)^4} \left[ (\gamma^\mu p_\mu + m)\gamma^0 - \mu \right] e^{-ip(x-y)} \quad (2.46)$$

or

$$A^{-1}(x, y) = \int \frac{d^4p}{(2\pi)^4} \left[ (p_0 - \mu) + \tilde{\alpha} \cdot \vec{p} + m\gamma^0 \right] e^{-ip(x-y)} \quad (2.47)$$

or

$$A^{-1}(x, y) = \int \frac{d^4p}{(2\pi)^4} \left[ (\tilde{p}_0 - \mu) - \tilde{\alpha} \cdot \vec{p} - m\gamma^0 \right]^{-1} e^{-ip(x-y)} \quad (2.48)$$

where

$$\tilde{p}_0 = p_0 + i\text{sgn}p_0. \quad (2.49)$$

To get the transpose of this we need to take the transpose of the numerator and also since $x \leftrightarrow y$ we need to also change $p^\mu \rightarrow -p^\mu$ to keep the same definition of the Fourier transform. So we get:

$$(A^{-1})^T(x, y) = -\int \frac{d^4p}{(2\pi)^4} \left[ (\tilde{p}_0 + \mu) + \tilde{\alpha}^T \cdot \vec{p} - m\gamma^0 \right] e^{-ip(x-y)} \quad (2.50)$$
Now consider the quantity
\[ X = 4A^{-1}B(A^T)^{-1}B^\dagger. \] (2.51)
We are interested in the case where the \( B^{(i)} \) and the \( B^{(i)\dagger} \) are constants. This is appropriate for evaluating the effective potential \( V_{\text{eff}} \), given by
\[ \Gamma_{\text{eff}} = -V_{\text{eff}} \int d^4x, \] (2.52)
whose minima determine the allowed vacuum states of the theory.

\( X[p] \) is the integrand of the vacuum polarization graph at zero momentum transfer. In momentum space we have
\[
X[p] = -4 \left[ \frac{1}{[(\hat{p}_0 - \mu) - \vec{\alpha} \cdot \vec{p} - m\gamma_0]} (B^{(1)} i\gamma^1 \gamma^3 + B^{(2)} i\gamma^2 \gamma^5) \times \frac{1}{[(\hat{p}_0 + \mu) - \vec{\alpha}^T \cdot \vec{p} + m\gamma_0]} (B^{(1)\dagger} i\gamma^1 \gamma^3 + B^{(2)\dagger} i\gamma^2 \gamma^5) \right]. \] (2.53)

Using the (anti-)commutation relations for the \( \gamma \) matrices, and performing the matrix algebra, we find that
\[
X[p] = -4 \left[ \frac{1}{[(\hat{p}_0 - \mu) - \vec{\alpha} \cdot \vec{p} - m\gamma_0]} \times \frac{1}{[(\hat{p}_0 + \mu) + \vec{\alpha} \cdot \vec{p} + m\gamma_0]} \right] M_1 + \frac{1}{[(\hat{p}_0 + \mu) - \vec{\alpha} \cdot \vec{p} + m\gamma_0]} M_2 , \] (2.54)

where
\[
M_1 = B^{(1)} B^{(1)\dagger} 1 - \gamma^0 B^{(1)} B^{(2)\dagger} , \] (2.55)
and
\[
M_2 = B^{(2)} B^{(2)\dagger} 1 - \gamma^0 B^{(2)} B^{(1)\dagger} . \] (2.56)

As we shall see below \( X[p] \) plays a crucial role in determining the effective potential.

### III. EFFECTIVE POTENTIAL

To obtain the effective potential we take the \( B^{(i)} \) to be constant, and from
\[ \Gamma_{\text{eff}} = -V_{\text{eff}} \int d^4x, \] (3.1)
we have
\[
V_{\text{eff}} = \left( \frac{1}{\kappa^2} \sum_\lambda B^{(\lambda)} B(\lambda) + \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \log [1 + X[p]] \right) . \] (3.2)

This naive effective potential has both quadratic divergences and logarithmic divergences. However, since this is only an effective theory, valid up to the mass of the Higgs particle, we will instead think of this theory having an effective cutoff \( \Lambda \approx m_h \). It will, however, be useful to define a “renormalized” coupling constant \( \kappa_R^2 \) so that the theory only logarithmically depends on the cutoff. It is convenient to parametrize the condensates as follows:
\[
B^{(1)} = R \cos \theta e^{i\phi_1} ; \quad B^{(2)} = R \sin \theta e^{i\phi_2} , \] (3.3)
where we take \( 0 \leq \theta \leq \pi/2 \). Then we obtain, letting \( \phi = \phi_1 - \phi_2 \)
\[
M_1 = B^{(1)} B^{(1)\dagger} 1 - \gamma^0 B^{(1)} B^{(2)\dagger} \]
\[ = R^2 \left( \cos^2 \theta 1 - \frac{\gamma^0}{2} \sin 2\theta e^{i\phi} \right) \]
\[ = M_{1a} + \gamma^0 M_{1b} . \] (3.4)
\[ M_2 = B^{(2)} B^{(2)\dagger} 1 - \gamma_0 B^{(2)} B^{(1)\dagger} \]
\[ = R^2 \left( \sin^2 \theta 1 - \frac{\gamma_0}{2} \sin 2\theta e^{-i\phi} \right) \]
\[ \equiv M_{2a} + \gamma_0 M_{1b} \quad (3.5) \]

When \( \theta = 0, M_1 = R^2 1, M_2 = 0 \), whereas when \( \theta = \pi/2, M_1 = 0, M_2 = R^2 1 \). We notice that when \( \sin 2\theta = 0, M_1, M_2 \) are independent of \( \phi \). This occurs at the special cases \( \theta = 0, M_1 = R^2 \) and \( \theta = \pi/2, M_2 = R^2 \).

We can write
\[ X[p] = A^{-1} Z[p] \quad (3.6) \]
with
\[ A = \tilde{p}_0 - \mu - \vec{\alpha} \cdot \vec{p} - m\gamma_0, \quad (3.7) \]
whence
\[ V_{\text{eff}} = \left( \frac{1}{\kappa^2} R^2 + \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \log \left[ 1 + A^{-1} Z[p] \right] \right. \quad (3.8) \]

Taking the derivative with respect to \( \theta, \phi \) we obtain
\[ \frac{\partial V_{\text{eff}}}{\partial \theta} = \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ (A + Z)^{-1} \frac{\partial Z}{\partial \theta} \right], \]
\[ \frac{\partial V_{\text{eff}}}{\partial \phi} = \frac{i}{2} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ (A + Z)^{-1} \frac{\partial Z}{\partial \phi} \right]. \quad (3.9) \]

Taking the derivative with respect to \( R^2 \) we obtain since \( \partial Z/\partial R^2 = Z/R^2 \),
\[ \frac{\partial V_{\text{eff}}}{\partial R^2} = \frac{1}{\kappa^2} - \frac{i}{4\pi i R^2} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ (A + Z)^{-1} \frac{\partial Z}{\partial R^2} \right] \]
\[ = \frac{1}{\kappa^2} - \frac{i}{4\pi i R^2} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ (A + Z)^{-1} Z \right]. \quad (3.10) \]

Our strategy is now to write
\[ (A + Z)^{-1} = \frac{N_0}{D_0}, \quad (3.11) \]
and
\[ Z = \frac{N}{D}, \quad (3.12) \]
where
\[ N_0 = \nu_0 - \nu_1 \vec{\alpha} \cdot \vec{p} - \nu_2 \gamma^0 - \nu_3 \gamma^0 \vec{\alpha} \cdot \vec{p}, \]
\[ D_0 = \nu_0^2 - \nu_1^2 \vec{p}^2 - \nu_2^2 + \nu_3^2 \vec{p}^2, \]
\[ N = \rho_0 + \rho_1 \vec{\alpha} \cdot \vec{p} + \rho_2 \gamma^0 + \rho_3 \gamma^0 \vec{\alpha} \cdot \vec{p}, \]
\[ D = (\tilde{p}_0 + \mu)^2 - \vec{p}^2 - m^2. \quad (3.13) \]

Here the \( \nu_i \) and \( \rho_i \) are determined by the definition of \( (A + Z)^{-1} \) and \( Z \). Using these expressions we can compute the integrands in the gap equations above. In particular, we find that
\[ DD_0 = (\tilde{p}_0 - \lambda_+)(\tilde{p}_0 - \lambda_-)(\tilde{p}_0 + \lambda_+)(\tilde{p}_0 + \lambda_-), \quad (3.15) \]
where
\[ \lambda_{\pm} = (\beta \pm 2\sqrt{\gamma})^{1/2}, \quad (3.16) \]
and
\[ \begin{align*}
\beta &= \mu^2 + \omega_p^2 + 4R^2; \quad \omega_p^2 = p^2 + m^2; \\
\gamma &= p^2(\mu^2 + 4R^2 \sin^2 \theta) + (\mu m - 2R^2 \sin 2\theta \cos \phi)^2 \\
&= (\omega_p^2 - m^2)(\mu^2 + 2R^2(1 - \cos 2\theta)) + (\mu m - 2R^2 \sin 2\theta \cos \phi)^2.
\end{align*} \]

This will enable us in what follows to perform the \( p_0 \) integral by contour integration.

The equation which extremizes the potential with respect to \( \phi \) can be written as
\[ \frac{\partial V_{\text{eff}}}{\partial \phi} = -8 \int \frac{dp_0 \ d^3p}{2\pi i (2\pi)^3} \left[ \frac{c_{0\phi}}{4\mathcal{D}D_0} \right] = 0, \]
where \( \mathcal{D}D_0 \) is given by (3.15) and
\[ \hat{c}_{0\phi} = 8R^2 \sin 2\theta \sin \phi f[p, p_0, \mu, R^2] \]
\[ f[p, p_0, \mu, R^2] = -m \mu \]
\[ 2 \sin(2\theta) \cos(\phi) \left( m^2(p^2 - R^2) + p^4 - p^2(\mu + p_0)^2 + R^2(\mu + p_0)^2 \right) \]
\[ m^2 + p^2 - (\mu + p_0)^2. \]

Hence we can extremize the potential with respect to \( \phi \) by choosing \( \sin \phi = 0, \cos \phi = \pm 1 \). We will restrict ourselves to the case \( \cos \phi = \pm 1 \) in what follows. With that assumption, we can rewrite \( \gamma \) as
\[ \gamma = p^2(\mu^2 + 4R^2 \sin^2 \theta) + (\mu m \cos \phi - 2R^2 \sin 2\theta)^2. \]

We notice that the potential depends on the product of sign \( \mu \) and \( \cos \phi \) which can be \( \pm 1 \). Writing
\[ \mu \cos \phi = |\mu|\eta = \rho \]
with \( \eta = \pm 1 \), we have
\[ \gamma = p^2(\mu^2 + 4R^2 \sin^2 \theta) + (\rho \rho - 2R^2 \sin 2\theta)^2. \]

We will consider separately the two case \( \eta = \pm 1 \). We will find when we study \( V_{\pm}[R^2, \theta] \) that apart from the endpoints \( \theta = 0, \pi/2 \), where \( V_{+} = V_{-} \), the potential with \( \eta = -1 \) always has higher energy than the potential with \( \eta = +1 \). So treating both cases by using the parameter \( \rho = |\mu|\eta \) we have
\[ \frac{\partial V_{\text{eff}}}{\partial R^2} = \frac{1}{\kappa^2} - 8 \int \frac{dp_0 \ d^3p}{2\pi i (2\pi)^3} \left[ \frac{c_0}{4R^2\mathcal{D}D_0} \right], \]
with \( \mathcal{D}D_0 \) given by (3.15) and now
\[ \hat{c}_0 = -\hat{N}(\bar{p}_0, p) \equiv N(p^2) - \bar{p}_0^2 \]
\[ = \mu^2 + m^2 + 2\rho m \sin 2\theta + p^2 \cos 2\theta + 4R^2 \cos^2 2\theta. \]

We can write this expression as
\[ \frac{\partial V_{\text{eff}}}{\partial R^2} = \frac{1}{\kappa^2} + 8 \int \frac{dp_0 \ d^3p}{2\pi i (2\pi)^3} \left[ \frac{\hat{N}(\bar{p}_0, p)}{\mathcal{D}D_0} \right]. \]

Doing the \( p_0 \) integral by contour integration and closing the contour in the upper half plane we obtain
\[ \frac{\partial V_{\text{eff}}}{\partial R^2} = \frac{1}{\kappa^2} - 8 \int \frac{d^3p}{(2\pi)^3} \left[ \frac{\hat{N}(\lambda_+, p)}{2\lambda_+}(\lambda_+^2 - \lambda_-^2) - \frac{\hat{N}(\lambda_-, p)}{2\lambda_-}(\lambda_+^2 - \lambda_-^2) \right]. \]

Writing \( \hat{N}(\lambda_\pm, p) = \lambda_\pm^2 - N(p^2) \) we can express this as:
\[
\frac{\partial V_{\text{eff}}}{\partial R^2} = \frac{1}{\kappa^2} - \frac{2}{\pi^2} \int_0^{p_{\text{max}}} p^2 dp \left[ \frac{N(p^2)}{\lambda_+ \lambda_-} \frac{1}{\lambda_+ + \lambda_-} + \frac{1}{\lambda_+ + \lambda_-} \right] \\
\equiv \frac{1}{\kappa^2} - \frac{1}{\pi^2} \int_0^{p_{\text{max}}} dp I(p, R^2, \theta, q_i).
\]

(3.27)

We can define a renormalized coupling constant \(1/\kappa^2_R\) as the value of \(\frac{\partial V_{\text{eff}}}{\partial R^2}\) when \(\mu = R^2 = 0\) and \(\theta = 0\).

\[
\frac{1}{\kappa^2_R} = \frac{1}{\kappa^2} - \frac{1}{\pi^2} \int_0^{p_{\text{max}}} dp I(p, R^2 = 0, \theta = 0, \mu = 0, m) \\
= \frac{1}{\kappa^2} - \Sigma(m, H).
\]

(3.28)

Here \(H = p_{\text{max}} = m_H\). Explicitly we have

\[
\Sigma(H, m) = \frac{m^2 \left( -\frac{2H}{\sqrt{H^2 + m^2}} + 2 \log \left( \sqrt{H^2 + m^2} + H \right) - \log \left( m^2 \right) \right)}{\pi^2},
\]

(3.29)

which displays the logarithmic dependence of the coupling constant on the cutoff \(m_H\). One has that

\[
\Sigma(H = 20, m = 1) = .54526.
\]

(3.30)

The relations between \(\kappa\) and \(\kappa_R\) are given by

\[
\kappa^2_R = \frac{\kappa^2}{1 - \kappa^2 \Sigma(H, m)}; \quad \kappa^2 = \frac{\kappa^2_R}{1 + \kappa^2_R \Sigma(H, m)}.
\]

(3.31)

Choosing \(H = 20, m = 1\), we get the curve in Fig. 1 relating \(\kappa^2\) and \(\kappa^2_R\).

Choosing \(V_{\text{eff}}(R^2 = 0) = 0\) and integrating with respect to \(R^2\) we obtain

\[
V = \frac{R^2}{\kappa^2} - \frac{1}{\pi^2} \int_0^{R^2} dR^2 \int_0^{p_{\text{max}}} dp I(p, R^2, \theta, q_i).
\]

(3.32)

In terms of the renormalized coupling constant:

\[
V = \frac{R^2}{\kappa^2_R} - \frac{1}{\pi^2} \int_0^{R^2} dR^2 \int_0^{p_{\text{max}}} dp \left[ I(p, R^2, \theta, q_i) - I(p, R^2 = 0, \theta = 0, \mu = 0, m) \right].
\]

(3.33)
The gap equation is obtained from the place in $R^2$ where the potential is a minimum so that

$$
\frac{1}{\kappa^2} = \frac{1}{\pi^2} \int_0^{p_{\text{max}}} dp \, \mathcal{I}(p, R^2 = \Delta^2/4, \theta, m, \mu).
$$

(3.34)

Renomalizing the coupling constant we obtain the renormalized gap equation,

$$
\frac{1}{\kappa^2_R} = \frac{1}{\pi^2} \int_0^{p_{\text{max}}} dp \, \mathcal{I}_{\text{sub}}(p, R^2 = \Delta^2/4, \theta, m, \mu),
$$

(3.35)

where we use the subtracted integrand. For fixed values of $\theta, m, \mu$ this equation gives the relation between the mass of the gap $\Delta$ and the inverse the renormalized coupling constant $\kappa^2_R$.

We want the value of $\theta$ that gives the deepest potential. When $\sin \phi = 0$, one finds

$$
\frac{\partial V_{\text{eff}}}{\partial \theta} = -8 \int \frac{dp_0 \, d^3p}{(2\pi)^3} \left[ \frac{c_{0\theta}}{4DD_0} \right]
$$

(3.36)

where

$$
c_{0\theta} = 4R^2 \left( 4\rho_m \cos(2\theta) - 2p^2 \sin(2\theta) - 8R^2 \sin(2\theta) \cos(2\theta) \right).
$$

(3.37)

$$
\frac{\partial V_{\text{eff}}}{\partial \theta} = 2 \int \frac{d^3p}{(2\pi)^3} \left[ \frac{c_{0\theta}}{2\lambda_+ (\lambda_+^2 - \lambda_-^2)} - \frac{c_{0\theta}}{2\lambda_- (\lambda_+^2 - \lambda_-^2)} \right]
$$

$$
= -\frac{1}{2\pi^2} \int_0^{p_{\text{max}}} p^2 dp \frac{c_{0\theta}}{\lambda_+ \lambda_- (\lambda_+ + \lambda_-)}.
$$

(3.38)

At the stationary points

$$
0 = -\frac{1}{2\pi^2} \int_0^{p_{\text{max}}} p^2 dp \frac{4R^2 \left( 4\rho_m \cos(2\theta) - 2p^2 \sin(2\theta) - 8R^2 \sin(2\theta) \cos(2\theta) \right)}{\lambda_+ \lambda_- (\lambda_+ + \lambda_-)}.
$$

(3.39)

We observe from eqs. (3.27) and (3.38) that the denominator of each of the integrands contains a factor of $\lambda_-$. From the expressions for $\beta$ and $\gamma$ we see that, when $R^2 = 0$, $\lambda_-$ vanishes at the point $\omega_p = |\mu|$ (this is the Fermi surface), and therefore the integral has a logarithmic singularity as $R^2 \to 0$. It is this fact that guarantees a solution to the $R^2$ gap equation no matter how small $\kappa^2$ may be, provided $|\mu| > m$.

One might also conjecture that the integral in eq. (3.39) is dominated by the region around $\omega_p = |\mu|$, in which case one would conclude that

$$
\tan 2\theta^* \approx \frac{2\eta |\mu| m}{|\mu|^2 - m^2}.
$$

(3.40)

However, the term in the numerator proportional to $p^2$ produces a quadratic divergence in the integral as the cutoff tends to infinity; hence we might expect that the integrand is dominated not by the region around $\omega_p = |\mu|$ but rather by $p^4$ near the cutoff $m_H^2$. We now present numerical evidence to show that, if we use the unrenormalized coupling, the integrand has a small peak near $\omega_p = |\mu|$ but is larger near the cutoff; however, if we use the renormalized coupling, the integrand is dominated by the peak at the Fermi surface, as one would anticipate on physical grounds.

As an example of the difference of the potentials when $\eta = \pm 1$ we plot in Fig. 2 the two potentials as a function of $\theta$ for $\kappa^2 = 1$, $m = 1$, $\mu = 1.5$, $m_H = 20$. We see that the red curve for $\eta = -1$ is above the blue curve for $\eta = +1$ and that the blue curve displays the minimum near $\theta = 0$.

The position of the minimum $\theta^*$ is sensitive to $m^2$ and proportional to it. For example, if we choose $m = 2$ the position of the minimum increases by a factor of 4 from the origin. The potential also gets shallower. This is seen in Fig. 3. Having determined $\theta^*$ from the unrenormalized equations, we turn our attention to the unrenormalized and renormalized gap equation integrands. The unrenormalized integrand $\mathcal{I}(p, R^2 = \Delta^2/4, \theta^*, m, \mu)$ has a peak at $p^2 + m^2 = \mu^2$. For $m = 1, \mu = 1.1, m_H = 20$ we have that $\theta^* = 0.2$ so that $\mathcal{I}$ has the behavior shown in Fig. 4. This curve shows that if we are going to just keep the pole approximation we should use the renormalized coupling constant.
FIG. 2. $V(\theta)$ as a function of $\theta$ for $\mu = 1.1$, $m = 1$, $m_H = 20$, and $\eta = 1$ (blue), $\eta = -1$ (red).

FIG. 3. $V(\theta)$ as a function of $\theta$ for $m = 2$, $\mu = 2.5$, $M = 1/1000$, $m_H = 20$.

A. results for $\eta = +1$

When $\eta = +1$, we find that the minimum in $\theta$ of the potential occurs near $\theta = 0$. We also find that for fixed $m, \mu$ the place where $dV/d\theta = 0$ does not change very much with $M$ when $M < 1/100$. Also the value of $\theta_{\text{min}}$ slightly decreases as we decrease $M$ from 1/100 to 1/1000. See Fig. 5.

Similarly if we keep $M$ fixed ($M = 1/1000$) and increase $\mu$ the minimum in $\theta$ occurs at larger and larger $\theta$. This is shown in Fig. 6. Note that the value of $\theta$ that extremizes the potential (i.e. $\theta^*$) does not depend on the coupling parameter $1/\kappa^2$. When $\eta = +1$ we find that the dependence of $\kappa^2$ on $\theta$ near the minimum at $\theta = 0.02$ is shown in Fig. 7.

Once we have determined the value of $\theta_{\text{min}}$, we can evaluate the effective potential $V_+(\kappa^2, M)$ for fixed $\mu, m_H$ for different $\theta$ around the minimum value $\theta_{\text{min}}$. The value of $\theta_{\text{min}}$ is sensitive to $\mu$, but the potential is always deepest at $\theta_{\text{min}}$. Choosing $m = 1$, $m_H = 20$ we show how $V$ changes as we go from $\theta = 0$ to $\theta = \theta_{\text{min}}$ to $\theta = 2\theta_{\text{min}}$ for two values of $\mu$ namely 1.1 and 1.5. We have slightly changed the value of $\kappa^2 \approx 0.024$ in these two plots so as to keep the minimum in both cases to be near $M = 1/1000$. The results for $\mu = 1.1$ are shown in Fig. 8. The results for $\mu = 1.5$ are shown in Fig. 9.

B. Results when $\eta = -1$

When $\eta = -1$, the stationary point of the potential in $\theta$ occurs a little below $\theta = \pi/2$. However it is now a maximum. Again the position of this maximum is not sensitive to $M$ when $M \leq 1/100$ and is close to $\theta = \pi/2$. As we decrease $M$ the position of the maximum gets a little smaller. In Fig. 10 we show the results for $\mu = 1.1$, $m = 1$, $m_H = 20$, and $M = 1/100$, 1/1000. The lowest value of $M$ leads to the curve furthest to the left. Note that in the region near $\partial V/\partial \theta = 0$ the slope is negative, so there is a relative minimum at $\theta = \pi/2$. When $\eta = -1$, $dV/d\theta$ has positive slope (as well as being > 0) for $0 < \theta < \pi/4$ and negative slope for $\pi/4 < \theta^*$. $V$ has a relative minimum at the endpoint $\theta = 0$, see Fig. 11 and a maximum at $\theta = \theta^* \approx \pi/2 - \delta$. This curve is higher than the curve for $\eta = +1$ which was shown previously. The details near the maxima are shown on the right hand side of Fig. 11.
IV. APPROXIMATE ANALYTIC CALCULATIONS FOR THE GENERAL CASE WHEN $R^2 << m^2$

If we consider the gap equation, and expand in $R^2$ and looking at the largest contribution to the integral, it comes from $\lambda_-$. In fact when $R^2 = 0$, $(\lambda_-)^2 = (\omega - \mu)^2$ so as a first approximation we can change variables to $\xi = (\omega - \mu)$ and assume that everything in the numerator and denominator apart from $\lambda_-$ has approximately their value at $\xi = 0$

We want to approximately solve the gap equation:

$$\frac{1}{\kappa^2} = \frac{2}{\pi^2} \int_0^{p_{max}} p^2 dp \left[ \frac{N(p^2)}{\lambda_+ \lambda_-} + \frac{1}{\lambda_+ + \lambda_-} \right]$$

(4.1)

when $R << 1, \mu > 0$. One finds when $R << 1$ and if the integral is dominated by contribution from the Fermi
\( \kappa^2 \) as a function of \( \theta \) for \( m = 1, M = 1/1000, m_H = 20 \), and \( \mu = 1.1 \).

\[
\kappa^2 \approx \sqrt{\mu^2 + m^2 + 2\mu m \eta \sin 2\theta + (\mu^2 - m^2) \cos 2\theta + 4R^2 \cos^2 2\theta}.
\]

FIG. 7.

\( V \) as a function of \( M \) for \( m = 1, \mu = 1.1, m_H = 20 \) for \( \theta = 0, 0.02, 0.04 \), shown in red, blue and green. \( \eta = +1 \).

Here we have used the result from \( \frac{\partial V}{\partial \eta} \) that if we only keep the pole like contributions

\[
\tan 2\theta = \frac{2\mu m \eta}{(\mu^2 - m^2)}.
\]

FIG. 8.

\( V \) as a function of \( M \) for \( m = 1, \mu = 1.5, m_H = 20 \), for \( \theta = 0, 0.02, 0.04 \), shown in red,blue and green. \( \eta = +1 \).

FIG. 9.
Eq. 4.2 shows that $N$ is proportional to $R^2$ if $\eta = -1$ in the “pole” approximation. Thus when $\eta = -1$, one cannot make this “drastic” approximation and satisfy the gap equation. See appendix A for a further discussion.

Also in the denominator in this approximation, $\lambda_+ \to \sqrt{2\mu^2}$, $\lambda_+ + \lambda_- \to \sqrt{2\mu^2}$. If we now calculate $\lambda_-$ as a series in $R^2$ for small $R^2$, we have for $\mu > 0$ and $\eta = \pm 1$,

$$\sqrt{\gamma} = \mu_0 - \frac{2m^2 R^2 (2\mu^2 + m^2 - \omega^2)}{\mu_0 (\mu^2 + m^2)}; \eta = 1$$

$$= \mu_0 + \frac{2\mu R^2 (m^2 + \omega^2)}{\omega (\mu^2 + m^2)}; \eta = -1. \quad (4.4)$$

Letting $\omega = \mu + \xi$ we have to leading order

$$\lambda_- = \sqrt{\xi^2 + \frac{4 (m^2 + \mu^2)}{\mu^2} R^2}; \eta = 1$$

$$= \sqrt{\frac{4\xi R^2 (m^2 - \mu^2)}{\mu (\mu^2 + m^2)} + \xi^2}; \eta = -1. \quad (4.5)$$

Again we find that the $R^2$ expansion is very different for $\eta = -1$.

One now wants to change variables from $p$ to $\xi = \omega_+ - \mu$. When $p = 0$, $\xi = \xi_{\text{min}} = m - \mu$. Since $p_{\text{max}} = m_H$, $\xi_{\text{max}} = \sqrt{m_H^2 + m^2} - \mu$.

$$\frac{2\kappa^2}{\pi^2} \int_0^{m_H} p^2 \, dp = \frac{2\kappa^2}{\pi^2} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi (\xi + \mu) \sqrt{\xi^2 + (\xi + \mu)^2 - m^2}. \quad (4.6)$$
First let us look at $\eta = +1$. The gap equation becomes, when we evaluate all the terms except $1/\lambda$ at $\xi = 0$

$$
1 = \frac{\mu^2 - m^2 (\mu^2 + m^2)}{\mu} \kappa^2 \frac{\mu}{\pi^2} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi \frac{1}{\sqrt{\xi^2 + \Delta^2}}
$$

$$
= \frac{\mu^2 - m^2 (\mu^2 + m^2)}{\mu} \kappa^2 \frac{\mu}{\pi^2} \log \left( \frac{4 \epsilon_{\text{max}} |\epsilon_{\text{min}}|}{\Delta^2} \right)
$$

(4.7)

where $\epsilon_{\text{max}} = \sqrt{m^2 + m^2 - \mu} \approx m_H$ and $\epsilon_{\text{min}} = m - \mu$.

$$
\Delta^2 = 4 \left( \frac{m^2 + \mu^2}{\mu^2} \right) R^2
$$

(4.8)

We see we need $\mu^2 > m^2$ for a solution. For the case $\eta = -1$, for the reasons discussed in the appendix, the “drastic” approximation is not valid and we must solve the gap equation numerically. The position for the minimum in $M = R^2$ is at

$$
M^* = m_H |m - \mu| e^{-\frac{\mu^2}{\sqrt{m^2 - m_H^2}}}
$$

(4.9)

V. SPECIAL CASE OF HAVING A SINGLE CONDENSATE

Here we study the two extreme case of $\theta = 0$ and $\theta = \pi/2$. This is a useful check since many of the integrals for these two cases can be done analytically and compared with the numerical results. For the same values of the parameters we will show that the effective potential for $\theta = 0$ is always lower than the effective potential for $\theta = \pi/2$.

A. $B^{(2)} = 0$, $M_1 = B^{(1)} B^{(1)}$

When $B^{(2)} = 0$ we have that

$$
V_{\text{eff}} = \left( \frac{1}{\kappa^2} B^{(1)\dagger} B^{(1)} + i \right) \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \log [1 + X[p]]
$$

(5.1)

where

$$
X[p] = -4 M_1 \left( [\not{p} - \mu] - \vec{\alpha} \cdot \vec{p} - m \gamma^0 \right) \left( [\not{p} + \mu] + \vec{\alpha} \cdot \vec{p} + m \gamma^0 \right)
$$

$$
= -4 M_1 \not{J}_1[p].
$$

(5.2)

Doing the $p_0$ integral by contour integration we find that Eq. (3.20) reduces to

$$
\frac{\partial V}{\partial M_1} = \frac{1}{\kappa^2} - 2 \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{1}{\sqrt{4 M_1 + (\mu + \omega_p)^2}} + \frac{1}{\sqrt{4 M_1 + (\mu - \omega_p)^2}} \right].
$$

(5.3)

The gap equation for $\Delta$ for $\theta = 0$ reduces to

$$
\frac{1}{\kappa^2} = 2 \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{1}{\sqrt{\Delta^2 + (\mu + \omega_p)^2}} + \frac{1}{\sqrt{\Delta^2 + (\mu - \omega_p)^2}} \right].
$$

(5.4)

We can define a renormalized coupling constant from

$$
\frac{1}{\kappa_R} = \frac{d^2 V}{d B_1 d B_1^{\dagger}} \bigg|_{\mu = M_1 = 0}
$$

$$
= \frac{1}{\kappa^2} - 4 \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{1}{\omega_p} \right].
$$

(5.5)
The unrenormalized gap equation can be written:

\[ 1 = \frac{\kappa^2}{\pi^2} \int_0^{m_H} p^2 \, dp \left[ \frac{1}{\sqrt{\Delta^2 + (\mu + \omega_p)^2}} + \frac{1}{\sqrt{\Delta^2 + (\mu - \omega_p)^2}} \right], \]

so that the renormalized gap equation reads:

\[ 1 = \frac{\kappa^2_R}{\pi^2} \int_0^{m_H} p^2 \, dp \left[ \frac{1}{\sqrt{\Delta^2 + (\mu + \omega_p)^2}} + \frac{1}{\sqrt{\Delta^2 + (\mu - \omega_p)^2}} - \frac{2}{\omega_p} \right]. \]  

This integral can be used to define a function of four variables:

\[ f(\mu, m, m_H, \Delta^2) = \int_0^{m_H} p^2 \, dp \left[ \frac{1}{\sqrt{\Delta^2 + (\mu + \omega_p)^2}} + \frac{1}{\sqrt{\Delta^2 + (\mu - \omega_p)^2}} - \frac{2}{\omega_p} \right], \]

so that the value of the coupling that allows a solution of the gap equation for given value of \( \mu, m, M_H, M = \Delta^2/4 \) is given by

\[ \kappa^2_R = \frac{\pi^2}{f(\mu, m, m_H, M)}. \]

The integral gets a large contribution near the Fermi surfaces \( \omega_p = \pm \mu \). Let us consider the case where \( \mu > 0 \) so the large contribution to the integral will be from the second term. So now change variables to \( \xi = \omega_p - \mu \). When \( p = 0, \xi = \xi_{\text{min}} = m - \mu \). Since \( p_{\text{max}} = m_H \) \( \xi_{\text{max}} = \sqrt{m_H^2 + m^2} - \mu \). The gap equation now becomes

\[ 1 = \frac{\kappa^2}{\pi^2} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi \left( \xi + \mu \right) \sqrt{(\xi + \mu)^2 - m^2} \left[ \frac{1}{\sqrt{(\xi + \mu)^2 + \xi^2}} + \frac{1}{\sqrt{(\xi + \mu)^2 + (2\mu + \xi)^2}} \right], \]

and the renormalized gap equation is now:

\[ 1 = \frac{\kappa^2}{\pi^2} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi (\xi + \mu) \sqrt{(\xi + \mu)^2 - m^2} \left[ \frac{1}{\sqrt{(\xi + \mu)^2 + \xi^2}} + \frac{1}{\sqrt{(\xi + \mu)^2 + (2\mu + \xi)^2}} - \frac{2}{\xi + \mu} \right] = \frac{\kappa^2_R}{\pi^2} f(\mu, m, m_H, M). \]

Now at large \( \xi \)

\[ h_1(\xi, \mu, m, M) \to -\frac{-2\mu^2 + m^2 + 4M}{\xi}. \]  

There is a logarithmic tail to \( f \) coming from this \( 1/\xi \) behavior as well as a large contribution coming from the sharp peak near the Fermi surface where \( \omega_p \approx \mu \) or \( \xi \approx 0 \). This is seen in our plot of \( h_1[\xi] \) for \( m = 1, \mu = 1.5, M = 10^{-3} \) shown in Fig. 12. The logarithmic tail is proportional to \( 2\mu^2 - m^2 \) so this contribution gets larger as we increase \( \mu^2 \) relative to \( m^2 \). We can see how \( \kappa^2_R \) depends on \( \mu \) for fixed \( m_H, M, m \) by plotting

\[ \kappa^2_R = \frac{\pi^2}{f(\mu, m, m_H, M)}. \]  

If we fix \( m = 1, M = 1/1000, H = m_H = 20 \), then \( \kappa^2_R \) depends on \( \mu \) as shown in Fig. 13. If we now keep \( \mu \) fixed at 1.5 and vary \( M \), keeping \( m = 1, m_H = 20 \) we get the result shown in Fig. 14.

The effective potential is given by

\[ V = \frac{M}{\kappa^2_R} - \int_0^M dM' \frac{f(\mu, m, m_H, M')}{\pi^2}. \]  

(5.6)
For example if we choose to keep the minimum of $V$ at $M = 1/1000$ by changing $\kappa$ and keeping $m = 1, m_H = 20$ and varying $\mu = 1.1, 1.3, 1.5$, we find that the potential in terms of $\kappa^2 R$ deepens as a function of $\mu$. This is seen in Fig. 15 where the increasing values of $\mu$ correspond to the black, green and red curves. The relevant $\kappa^2 R$ are 1.1, 1.708, 1.503, with $\kappa R$ decreasing as we increase $\mu$ keeping the position of the minimum fixed at $10^{-3}$.

The results shown in Fig. 15 were identical whether we did both integrals numerically, or one analytically and the second numerically using Mathematica.
1. Analytic approximations

We can get an approximate analytic answer for the gap equation when \( \Delta \ll m \) by approximating the \( p^2 \) term in the integrand at the Fermi surface \( \omega_p = \sqrt{p^2 + m^2} = \pm \mu \). Let us consider the case where \( \mu > 0 \). Changing variables to \( \xi = \omega_p - \mu \) when \( p = 0; \xi = \xi_{\text{min}} = m - \mu \). Since \( p_{\text{max}} = m_H \xi_{\text{max}} = \sqrt{m_H^2 + m^2} - \mu \). The renormalized gap equation in these variables becomes

\[
1 = \frac{\kappa^2 R}{\pi^2} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi \frac{1}{\xi + \mu} \left[ \frac{1}{\sqrt{\Delta^2 + \xi^2}} + \frac{1}{\sqrt{\Delta^2 + (2\mu + \xi)^2}} - \frac{2}{\xi + \mu} \right] - \frac{1}{\xi + \mu}.
\]

When \( \Delta \ll m \) the integrand is highly peaked around \( \xi = 0 \). If we replace the prefactor by its value at \( \xi = 0 \) we get the approximation

\[
1 = \frac{\kappa^2 R}{\pi^2} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi \frac{1}{\xi + \mu} g_1(\xi, \mu, m, M) = \frac{\kappa^2 R}{\pi^2} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi \frac{1}{\xi + \mu} g_2(\xi, \mu, m, M, H),
\]

where we have used the shorthand \( H = m_H \). We can evaluate \( g_2 \) analytically to obtain

\[
g_2(\mu, m, M, H) = \mu \sqrt{\mu^2 - m^2} \times \left( \log \left( \sqrt{4M + y^2 + y} + \log \left( 2\mu + \sqrt{4M + (2\mu + y)^2 + y} \right) - 2\log(\mu + y) \right) \bigg|_{y = \xi_{\text{max}}} \right).
\]

The integrands \( h_1 \) and \( h_2 \) are quite similar except for the \( 1/\xi \) tail in \( h_1 \). This can be seen in Fig. 16. This leads to the result that \( g_1 > g_2 \). In terms of the value of \( \kappa^2 \), keeping only the pole contribution and using the equation

\[
\kappa^2_{\text{pole}} = \frac{\pi^2}{g_2(\mu, m, M, H)},
\]

we get the result shown in Fig. 17. The result of not including the logarithmic tail is that the approximate value of \( \kappa^2_R \) overestimates \( \kappa^2 \). The qualitative behavior is the same as the numerical solution.

To get the usual type gap equation for \( \Delta^2 \) we can make some further approximations.

\[
\log \left( \sqrt{\Delta^2 + \xi_{\text{max}}^2 + \xi_{\text{max}}} \right) \to \log 2\xi_{\text{max}}.
\]
In the second log one has that $\xi_{\text{min}} < 0$ so when we expanding the log for small $\Delta$ we obtain
\[
\left( \sqrt{\xi_{\text{min}}^2 + \xi_{\text{min}}} \right) + \frac{\Delta^2}{2 \sqrt{(\xi_{\text{min}})^2}} + O(\Delta^3) \approx \frac{\Delta^2}{2 \sqrt{(\xi_{\text{min}})^2}}. \tag{5.20}
\]
So we approximately get:
\[
\log \left( \sqrt{\Delta^2 + (\xi_{\text{min}})^2 + \xi_{\text{min}}} \right) \to \frac{\Delta^2}{2 \sqrt{\xi_{\text{min}}^2}}. \tag{5.21}
\]
\[
1 = \frac{\kappa^2}{\pi^2} \mu \sqrt{\mu^2 - m^2} \log\left[ \frac{4 \xi_{\text{max}} \sqrt{\xi_{\text{min}}^2}}{\Delta^2} \right]. \tag{5.22}
\]
This leads to the approximate equation
\[
\Delta^2 = 4m_h (\mu - m) e^{-\frac{\Delta^2}{\pi^2 \sqrt{\mu^2 - m^2}}}, \tag{5.23}
\]
since $\xi_{\text{max}} \approx m_h$ and $\xi_{\text{min}} = m - \mu < 0$. If we start from the equation
\[
\frac{\partial V}{\partial M} = \frac{1}{\kappa^2_R} - \frac{1}{\pi^2} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi \left( \xi + \mu \right) \sqrt{(\xi + \mu)^2 - m^2} \left[ \frac{1}{\sqrt{4M + \xi^2}} + \frac{1}{\sqrt{4M + (2\mu + \xi)^2}} - \frac{2}{\xi + \mu} \right], \tag{5.24}
\]
which can be used to determine the behavior of the integrands as a function of $\xi$ for $\mu = 1.5, m = 1, M = 10^{-3}, h_3$ is blue.
FIG. 18. Potential as a function of $M$ for $\mu = 1.2$ (black), 1.3 (red), 1.5 (green) keeping the minimum at $M = .001$ and choosing the approximate $\kappa^2$ which are 8.96, 6.6, 4.257.

and approximate the answer for small $M$ by letting $\xi = 0$ in the pre-factor so that

$$\frac{\partial V}{\partial M} \approx \frac{1}{\kappa^2} - \mu \sqrt{\mu^2 - m^2} \frac{1}{\pi^2} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d\xi \left( \frac{1}{\sqrt{4M + \xi^2}} + \frac{1}{\sqrt{4M + (2\mu + \xi)^2}} - \frac{2}{\xi + \mu} \right),$$

(5.25)

then we can get an explicit approximate answer for $V$ by integrating $\frac{\partial V}{\partial M}$ over $M$.

$$V_{\text{pole}} = \frac{M}{\kappa^2} - \int_0^M dM \ g_2(\mu, m, M, m_0)$$

(5.26)

The potential we get from ignoring the tail contributions is shown in Fig. 18. This has the same general behavior as the numerically evaluated potential of getting deeper as a function of $\mu$ but is much shallower than the actual potential shown in Fig. 15.

B. $B^{(1)} = 0, \ M_2 = B^{(2)}B^{(2)*}$

When $M_1 = 0$, then instead we have

$$X[p] = -4M_2 \mathcal{J}^2[p],$$

(5.27)

where

$$T^2 = (\mathcal{J}^2[p])^{-1} = [(\tilde{p}_0 + \mu) - \tilde{\alpha} \cdot \tilde{p} + m\gamma^0] [(\tilde{p}_0 - \mu) - \tilde{\alpha} \cdot \tilde{p} - m\gamma^0] = \tilde{p}_0^2 + \tilde{p} \cdot \tilde{p} - m^2 - \mu^2 - 2(\tilde{p}_0 + m\gamma^0)(\tilde{\alpha} \cdot \tilde{p}) - 2\mu m\gamma^0.$$

(5.28)

For this case, we have that

$$\frac{\partial V}{\partial M_2} = \frac{1}{\kappa^2} + 2i \int \frac{d^4p}{(2\pi)^4} \text{Tr} \frac{1}{4M_2 - T^2},$$

(5.29)

Doing the $p_0$ integral by contour integration we find that Eq. 3.26 reduces to

$$\frac{\partial V}{\partial M_2} = \frac{1}{\kappa^2} - \frac{1}{\pi^2} \int_0^{p_{\text{max}}=m_0} p^2 dp \left[ \frac{(p^2 + \sqrt{\gamma})}{\sqrt{\gamma} \sqrt{\beta + 2\sqrt{\gamma}}} - \frac{(p^2 - \sqrt{\gamma})}{\sqrt{\gamma} \sqrt{\beta - 2\sqrt{\gamma}}} \right]$$

$$= \frac{1}{\kappa^2} - \frac{1}{\pi^2} \int_0^{m_0} p^2 dp \ I(p, M_2, m, \mu) \equiv \frac{1}{\kappa^2} - \frac{1}{\pi^2} \ f_2(m, \mu, M, m_0).$$

(5.30)

Now again let $4M_2 = \Delta^2$ be the place where the potential is stationary, i.e. $\frac{\partial V}{\partial M_2} = 0$. 
This leads to the gap equation:

\[ 1 = \frac{\kappa^2}{\pi^2} f_2(m, \mu, M, m_H) \]  

(5.31)

where \( 4M^2 \to \Delta^2 \). We can interpret this equation as determining \( \kappa^2[m, \mu, M, m_H] \)

\[ \kappa^2[m, \mu, M, m_H] = \frac{\pi^2}{f_2(m, \mu, M, m_H)}. \]  

(5.32)

If we fix \( m = 1, M = 1/1000, m_H = 20 \), then \( \kappa^2 \) depends on \( \mu \) as shown in fig 19. Keeping instead \( \mu \) fixed at 1.5 and varying \( M \), keeping \( m = 1, m_H = 20 \) we get the result shown in Fig. 20.

We can rewrite the gap equation in terms of the renormalized coupling constant defined as:

\[ \left. \frac{\partial V}{\partial M^2} \right|_{\mu=0, M=0} = \frac{1}{\kappa R^2}. \]  

(5.33)

so that the renormalized gap equation is given by

\[ \frac{1}{\kappa R^2} = \frac{1}{\pi^2} \int_0^{m_H} dp \left[ I(p, M_2, m, \mu) - I(p, M_2 = 0, m, \mu = 0) \right] \]

\[ = \frac{1}{\pi^2} \int_0^{m_H} dp \tilde{I}(p, M_2, m, \mu). \]  

(5.34)

The function \( \tilde{I} \) is peaked at the Fermi surface as seen in Fig. 21.
FIG. 21. $\tilde{I}$ as a function of $p$ for $\mu = 1.1, m = 0, M = 1/1000$.

FIG. 22. Potential as a function of $M^2$ for $\kappa^2 = 1.00265, \mu = 1.5, m = 1, m_H = 20$.

The potential is then given by

$$V_2 = \frac{M_2}{\kappa^2} - \frac{1}{\pi^2} \int_0^{M_2} dM' f_2(\mu, m, M', m_H),$$

or in renormalized form:

$$V_2 = \frac{M_2}{\kappa_R^2} - \frac{1}{\pi^2} \int_0^{M_2} dM' \{f_2(\mu, m, M', m_H) - f_2(m/2, m, 0, m_H)\}.$$

This potential gives the same result as the general potential at $\theta = \pi/2, \eta = \pm 1$. The result for $V_2$ evaluated at $\kappa^2 = 1.00265, \mu = 1.5$ is shown in Fig. 22.

For $V_2$ the potential is quite sensitive to $\mu$. If we choose $\kappa$ to keep the minimum at .001 for $\mu = 1/5$, we obtain the results shown in Fig 23 where the (blue, red, green) curves are for $\mu = (1.4, 1.5, 1.6)$. Here $\kappa^2 = .9597$. We see that for $V_2$ the shape of $V_2$ is much more sensitive to the parameters than $V_1$ was.

As discussed earlier, $V_2$ is the the value of $V[\phi]$ when $\theta = \pi/2$ and $\cos \phi = \pm 1$. The potential when $\eta = \mu \cos \phi/|\mu|$ is $+1$ the potential is always lower than the potential when $\eta = -1$. For $\eta = -1$, $V_2$ is an unstable minimum to changes in $\phi$ and then goes to the minimum at at $\eta = -1$ which is unstable to lowering $\theta$. For example we can compare the endpoints for two cases. One where there is a minimum in $V_2$, and one where where is a minimum in $V_1$. In the first case there is no stable condensate, and in the second the condensate stabilizes at $\theta_{\text{min}}$ which is near $V_1$.

1. Analytic Approximation

The integral over $p$ for the gap equation gets the largest contribution near the Fermi Surfaces $\omega_p = \pm \mu$. Let us consider the case where $\mu > 0$ so the major contribution to the integral will be from the second term. So now change
FIG. 23. Potential as a function of $M$ for $\kappa^2 = .9597, \mu = (1.4, 1.5, 1.6), m = 1, m_H = 20$.

FIG. 24. Comparison of $V_1$ and $V_2$ in two cases, when there are relative minima in $V_2$ (upper figures) and $V_1$ (lower figures) respectively keeping the values of parameters $\kappa, \mu, m$ the same for $V_1$ and $V_2$. We see that $V_1$ is always lower than $V_2$

variables to $\xi = \omega_p - \mu$. When $p = 0; \xi = \xi_{min} = m - \mu$. Since $p_{max} = m_H, \xi_{max} = \sqrt{m_H^2 + m^2} - \mu$.

$$\frac{\kappa^2}{\pi^2} \int_0^{m_H} p^2 dp = \frac{\kappa^2}{\pi^2} \int_{\xi_{min}}^{\xi_{max}} d\xi (\xi + \mu) \sqrt{(\xi + \mu)^2 - m^2}. \quad (5.37)$$

Now

$$\gamma = \mu^2 (\xi + \mu)^2 \left[1 + \Delta^2 \left(\frac{(\xi + \mu)^2 - m^2}{\mu^2 (\xi + \mu)^2}\right)\right], \quad (5.38)$$

so that at small $\Delta$ we have

$$\sqrt{\gamma} = \mu (\xi + \mu) \left[1 + \Delta^2 \left(\frac{(\xi + \mu)^2 - m^2}{2\mu^2 (\xi + \mu)^2}\right)\right]. \quad (5.39)$$

Since

$$\beta = (\xi + \mu)^2 + \mu^2 + \Delta^2, \quad (5.40)$$
then at small $\Delta$ we have

$$\beta - 2\sqrt{\gamma} = \xi^2 + \Delta^2 f(\mu, m, \xi),$$

(5.41)

where

$$f(\mu, m, \xi) = 1 - \left(\frac{(\xi + \mu)^2 - m^2}{\mu(\xi + \mu)}\right).$$

(5.42)

Now we are interested in the peak where $\xi = 0$. Near $\xi = 0$ we have

$$\sqrt{\gamma} \to \mu^2 + \Delta^2 \frac{(\mu^2 - m^2)}{\mu^2},$$

$$\sqrt{\gamma} - p^2 = m^2 + \Delta^2 \frac{(\mu^2 - m^2)}{\mu^2}.$$  

(5.43)

For small $\Delta^2$,

$$\frac{\sqrt{\gamma} - p^2}{\sqrt{\gamma}} = \frac{m^2}{\mu^2}.$$  

(5.44)

Thus

$$f(\mu, m, \xi \approx 0) = \frac{m^2}{\mu^2},$$

(5.45)

so that

$$\frac{1}{\sqrt{\beta - 2\sqrt{\gamma}}} \approx \frac{1}{\sqrt{\xi^2 + \Delta^2 m^2/\mu^2}} \equiv \frac{1}{\sqrt{\xi^2 + \Delta_1^2}}$$

(5.46)

Also near $\xi = 0$

$$(\xi + \mu)\sqrt{(\xi + \mu)^2 - m^2} \to \mu\sqrt{\mu^2 - m^2}$$

(5.47)

The approximate equation for the gap equation becomes

$$1 = \frac{\kappa^2 m^2}{\pi^2} \frac{\sqrt{\mu^2 - m^2}}{\mu} \int_{\xi_{\min}}^{\xi_{\max}} d\xi \frac{1}{\sqrt{\xi^2 + \Delta_1^2}}$$

(5.48)

or

$$1 = \frac{\kappa^2 m^2}{\pi^2} \frac{\sqrt{\mu^2 - m^2}}{\mu} \log\left(\frac{4\xi_{\max}\sqrt{\xi_{\min}^2}}{\Delta_1^2}\right).$$

(5.49)

This is similar to our previous result for the $M_1$ condensate with an extra factor of $\frac{m^2}{\mu^2}$ and with $\Delta \to \Delta_1 = m\Delta/\mu$. This leads to the approximate equation

$$\Delta_1^2 = 4m_h(\mu - m)e^{-\frac{\mu^2}{\kappa^2m^2\sqrt{\mu^2 - m^2}}}. $$

(5.50)

The minimum of the potential at $M_2 = \Delta^2/4$ occurs at

$$M_2 = \Delta^2/4 = \frac{\mu^2}{m^2}m_h(\mu - m)e^{-\frac{\mu^2}{\kappa^2m^2\sqrt{\mu^2 - m^2}}}.$$ 

(5.51)

In the same approximation we obtain for $dV/dM_2$

$$\frac{\partial V}{\partial M_2} \approx \frac{1}{\kappa^2} \frac{-\mu\sqrt{\mu^2 - m^2}}{\pi^2} \int_{\xi_{\min}}^{\xi_{\max}} d\xi \left[ \frac{m^2}{\mu^2} \frac{1}{\sqrt{4M_2 m^2/\mu^2 + \xi^2}} + \frac{(2\mu^2 - m^2)}{\mu^2} \frac{1}{\sqrt{(2\mu + \xi)^2 + 4M_2(2\mu^2 - m^2)/\mu^2}} \right]$$

(5.52)
Performing the integrals we obtain:

$$\frac{\partial V}{\partial M_2} = \frac{1}{\kappa^2} - \frac{\sqrt{\mu^2 - m^2}}{\pi^2 \mu} \times$$

$$\left[ m^2 \log \left( \sqrt{m_h^2 + \frac{4m^2 M_2}{\mu^2}} + m_h \right) - m^2 \log \left( \sqrt{\frac{4m^2 M_2}{\mu^2}} + (m - \mu)^2 + (m - \mu) \right) \right.$$

$$+ (m^2 - 2\mu^2) \left( \log \left( \mu + \sqrt{M_2 \left( 8 - \frac{4m^2}{\mu^2} \right)} + (\mu + m)^2 + m \right) \right. $$

$$\left. - \log \left( (m_h + 2\mu)^2 + M_2 \left( 8 - \frac{4m^2}{\mu^2} \right) + m_h + 2\mu \right) \right) \right]$$

(5.53)

For the parameters $m = 1, \mu = 2, m_h = 100$, if we want the minimum to occur at $M_2 = 0.1$, we obtain from the gap equation that $\kappa = 0.587605$. For this choice of parameters we get (subtracting $V(0)$) that $V[M_2]$ has the behavior shown in Fig 25.

VI. CONCLUSIONS

We considered a simple model for the neutrino mass based on having a separate Higgs particle of mass $m_H$ coupling to one species of Dirac neutrinos. By introducing composite fields connected with the two attractive channels we obtain the effective potential for the composite fields by making a Hubbard-Stratonovich transformation and integrating out the underlying fermion fields. We then keep the leading order term in the loop expansion of the resulting path integral expressed in terms of the composite fields [6]. At finite density we find that there are two possible condensates having different quantum numbers, coming from this interaction when viewed in the “s” channel, having a mixing angle $\theta$. What we find is that the theory favors a very small mixing angle, if we want the mass of the condensate not to be too large for a given value of the coupling constant. The second relative minimum solution, which has the condensate at the endpoint solution made only of the second condensate, is very sensitive to the values of the coupling as well as $\mu$. This solution always has higher energy than the true minimum. The predominant condensate is of the form (in two component notation) $(\nu_R \sigma_2 \nu_R) + (\nu_L \sigma_2 \nu_L)$. In obtaining the effective potential we realized that an often used approximation for the gap equation, keeping the leading contribution to the integrals coming from the fermi surface, is unreliable. It underestimates the integrals dramatically, even though the theory has an effective cutoff which is the mass of the Higgs particle $m_H$.

Related work can be found in references [7] and [8].
Appendix A: failure of the drastic approximation when $\eta = -1$

In the case $\eta = -1$, because the numerator of the integrand vanishes when $R^2 = 0$, the drastic approximation has the form

$$\frac{1}{\tilde{g}^2} = \Delta^2 \int_{\xi_{\min}}^{\xi_{\max}} d\xi \frac{1}{\sqrt{\Delta^2 + \xi^2}}$$

$$= \Delta^2 \left( \log \left( \sqrt{\Delta^2 + \xi_{\max}^2} + \xi_{\max} \right) - \log \left( \sqrt{\Delta^2 + \xi_{\min}^2} + \xi_{\min} \right) \right)$$

(A1)

where $\tilde{g}$ is a suitably defined coupling constant (proportional to $\kappa$ and $\Delta$ is a suitably defined gap (proportional to $R$).

However, this vitiates the drastic approximation, because the rhs no longer blows up as $\Delta \to 0$; in fact, it vanishes in that limit. Thus for $\eta = -1$, one has to use the full gap equations, (3.43) and (3.47) and evaluate them numerically to see if the gap equation can be satisfied.

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