DIVERGENCE OF NON-RANDOM FLUCTUATION IN FIRST PASSAGE PERCOLATION

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Abstract. We study non-random fluctuation in the first passage percolation on \( \mathbb{Z}^d \) and show that it diverges for any dimension. We also prove the divergence of the non-random shape fluctuation, which was conjectured in [Yu Zhang. The divergence of fluctuations for shape in first passage percolation. Probab. Theory. Related. Fields. 136(2) 298–320, 2006].

1. Introduction

First Passage Percolation is a dynamical model of infection, which was introduced by Hammersley and Welsh [14]. The model has received much interests both in mathematics and physics because it has rich structures from the viewpoint of the random metric and it is related to the KPZ-theory [18]. See [2] on the background and related topics.

We consider the first passage percolation (FPP) on the lattice \( \mathbb{Z}^d \) with \( d \geq 2 \). The model is defined as follows. The vertices are the elements of \( \mathbb{Z}^d \). Let us denote by \( E^d \) the set of edges:

\[
E^d = \{ \{v, w\} : v, w \in \mathbb{Z}^d, |v - w|_1 = 1 \},
\]

where we set \( |v - w|_1 = \sum_{i=1}^d |v_i - w_i| \) for \( v = (v_1, \cdots, v_d) \), \( w = (w_1, \cdots, w_d) \). Note that we consider non-oriented edges in this paper, i.e., \( \{v, w\} = \{w, v\} \) and we sometimes regard \( \{v, w\} \) as a subset of \( \mathbb{Z}^d \) with a slight abuse of notation. We assign a non-negative random variable \( \tau_e \) on each edge \( e \in E^d \), called the passage time of the edge \( e \). The collection \( \tau = \{\tau_e\}_{e \in E^d} \) is assumed to be independent and identically distributed with common distribution \( F \).

A path \( \gamma \) is a finite sequence of vertices \( (x_1, \cdots, x_l) \subset \mathbb{Z}^d \) such that for any \( i \in \{1, \cdots, l-1\} \), \( \{x_i, x_{i+1}\} \in E^d \). Given an edge \( e \in E^d \), we write \( e \in \gamma \) if there exists \( i \in \{1, \cdots, l-1\} \) such that \( e = \{x_i, x_{i+1}\} \).

Given a path \( \gamma \), we define the passage time of \( \gamma \) as

\[
T(\gamma) = \sum_{e \in \gamma} \tau_e.
\]

For \( x \in \mathbb{R}^d \), we set \([x] = ([x_1], \cdots, [x_d])\) where \([a]\) is the greatest integer less than or equal to \( a \). Given two vertices \( v, w \in \mathbb{R}^d \), we define the first passage time between vertices \( v \) and \( w \) as

\[
T(v, w) = \inf_{\gamma : [v] \to [w]} T(\gamma),
\]

where the infimum is taken over all finite paths \( \gamma \) starting at \([v]\) and ending at \([w]\). A path \( \gamma \) from \( v \) to \( w \) is said to be optimal if it attains the first passage time, i.e., \( T(\gamma) = T(v, w) \). We define \( G(t) = \{x \in \mathbb{R}^d : \exists T(0, x) \leq t\} \).

By Kingman’s subadditive ergodic theorem, if \( \mathbb{E}\tau_e < \infty \), for any \( x \in \mathbb{R}^d \), there exists a non-random constant \( g(x) \geq 0 \) such that

\[
\mathbb{E}^x T(0, x) \geq t \geq 0.
\]

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(1.1) \[ g(x) = \lim_{t \to \infty} t^{-1} T(0,tx) = \lim_{t \to \infty} t^{-1} \mathbb{E}[T(0,tx)] \quad a.s. \]

This \( g(x) \) is called the \textit{time constant}. Note that, by the subadditivity, if \( x \in \mathbb{Z}^d \), then \( g(x) \leq \mathbb{E}T(0,x) \) and moreover for any \( x \in \mathbb{R}^d \), \( g(x) \leq \mathbb{E}T(0,x) + 2d \mathbb{E} \tau_c \). It is easy to check the homogeneity and convexity: \( g(\lambda x) = \lambda g(x) \) and \( g(rx + (1-r)y) \leq rg(x) + (1-r)g(y) \) for \( \lambda \in \mathbb{R} \), \( r \in [0,1] \) and \( x, y \in \mathbb{R}^d \). It is well-known that if \( F(0) < p_c(d) \), then \( g(x) > 0 \) for any \( x \neq 0 \) \[19\]. Therefore, if \( F(0) < p_c(d) \), then \( g : \mathbb{R}^d \to \mathbb{R}_{\geq 0} \) is a norm.

1.1. \textbf{non-random fluctuation.} Hammersley and Welsh \[14\] have proved that \( \frac{1}{N} T(0, N \mathbf{e}_1) \) converges to \( g(\mathbf{e}_1) \) in probability when \( d = 2 \). This statement was strengthened by Kingman \[16\] as stated in \(1.1\). Since then, the rate of this convergence becomes one of the basic problems in this model. The difference \( T(0, x) - g(x) \) can be naturally divided into the \textit{random fluctuation} part and \textit{non-random fluctuation} part as follows:

\[
T(0, x) - g(x) = \underbrace{T(0, x) - \mathbb{E}T(0, x)}_{\text{random}} + \underbrace{\mathbb{E}T(0, x) - g(x)}_{\text{non-random}}.
\]

Let us briefly review the earlier works. It is widely believed that there exist universal constants \( \chi(d), \chi'(d) \geq 0 \) such that for any \( x \in \mathbb{R}^d \), as \( t \to \infty \),

\[
T(0,tx) - \mathbb{E}T(0,tx) \sim \sqrt{\text{Var}(T(0,tx))} \sim t^{\chi(d)} \quad \text{and} \quad \mathbb{E}T(0,tx) - g(tx) \sim t^{\chi'(d)},
\]

in a suitable sense. This “universal” means that these values are independent of distributions. To state the previous works precisely, we introduce four relevant quantities:

\[
\chi(d) = \inf \liminf_{t \to \infty} \frac{\log \text{Var}(T(0,tx))}{2 \log t}, \quad \chi'(d) = \inf \liminf_{t \to \infty} \frac{\log \text{Var}(T(0,tx))}{2 \log t},
\]

\[
\bar{\chi}(d) = \sup \limsup_{t \to \infty} \frac{\log \text{Var}(T(0,tx))}{2 \log t}, \quad \bar{\chi}'(d) = \sup \limsup_{t \to \infty} \frac{\log \text{Var}(T(0,tx))}{2 \log t}.
\]

Due to the works of Kesten \[20\], it is (the best currently) known that \( 0 \leq \chi(d) \leq \bar{\chi}(d) \leq 1/2 \) under the condition that the second moment of \( \tau \) is finite. On the other hand, Newman and Piza showed that \( \bar{\chi}(2) \geq 1/8 \) for useful distributions under an exponential moment condition \[21\], where useful distributions are defined in \(1.3\) below.

Let us move on to the previous researches on the non-random fluctuation. Alexander found the relationship between \( \bar{\chi}(d) \) and \( \chi'(d) \) and he proved \( \bar{\chi}'(d) \leq 1/2 \) with an exponential moment condition \[1\], which was later relaxed to a low moment condition \[11\]. For the lower bounds, it is proved that \( \chi'(d) \geq -1 \) \[20\] and \( \bar{\chi}'(d) \geq -1/2 \) \[3\] with an exponential moment condition.

Remarkably, it was shown in \[3\] that \( \chi(d) \) and \( \chi'(d) \) in \(1.2\) are actually the same under the assumption of the existence of \( \chi(d) \) in a suitable sense. In fact, it is expected that they have the exactly same growth \[12\] \[15\]. As a consequence, the above four quantities should be all the same, which are called the \textit{fluctuation exponent} collectively. From the KPZ-theory, it is conjectured that \( \chi(2) = \chi'(2) = 1/3 \). However for other dimensions, the values are unknown. Some physicists predicted that for sufficiently large dimension, \( \chi(d) = 0 \) \[9\] \[13\] \[22\]. If it is correct, the further problem can be conceivable whether the random fluctuation and non-random fluctuation diverge or not. In this paper, we prove that the latter diverges for any dimension \( d \geq 2 \), which is the first result around related models. Accordingly, we believe that the former does so.

We restrict our attention to the following class of distributions. A distribution \( F \) is said to be \textit{useful} if

\[
\mathbb{P}(\tau_c = F^-) < \begin{cases} p_c(d) & \text{if } F^- = 0 \\ \bar{p}_c(d) & \text{otherwise}, \end{cases}
\]

\[1.3\]
where \( p_c(d) \) and \( \bar{p}_c(d) \) stand for the critical probabilities for \( d \)-dimensional percolation and oriented percolation model, respectively and \( F^- \) is the infimum of the support of \( F \). Note that if \( F \) is continuous, i.e., \( \mathbb{P}(\tau_e = a) = 0 \) for any \( a \in \mathbb{R} \), then \( F \) is useful.

**Theorem 1.** Suppose that \( F \) is useful and \( \mathbb{E}[\tau^2_\epsilon(\log \tau_\epsilon)_+] < \infty \). Let \( x_d \in \partial \mathbb{B}_d \) such that there exist \( x_1 \in \mathbb{R}^d \) and \( r > 0 \) such that \( B(x_1, r) \subset \mathbb{B}_d \) and \( x_d \in \partial B(x_1, r) \), where \( B(x, r) = \{ y \in \mathbb{R}^d \mid d(x, y) \leq r \} \). Then there exists a sequence \( x_n \in \mathbb{Z}^d \) such that \( x_n/|x_n| \rightarrow x_d/|x_d| \) and

\[
\lim_{n \rightarrow \infty} \mathbb{E}|T(0, x_n) - g(x_n)| = \infty.
\]

In particular, by Jensen inequality,

\[
\lim_{n \rightarrow \infty} \mathbb{E}|T(0, x_n) - g(x_n)| = \infty.
\]

**Remark 1.** Let \( R = \sup \{ r > 0 \mid B(0, r) \subset \mathbb{B}_d \} \). We take an arbitrary point \( x_d \in \partial B_d \cap \partial B(0, R) \) (see Figure 1). Then \( x_d \) satisfies the assumption in Theorem 1.

We will prove Theorem 1 as a corollary of Theorem 2. Let \( \mathbb{B}_d = \{ x \in \mathbb{R}^d \mid g(x) \leq 1 \} \). We consider the fluctuation of \( G(t) \) from \( t \mathbb{B}_d \).

**Definition 1.** For \( l > 0 \) and a subset \( \Gamma \) of \( \mathbb{R}^d \) containing the origin \( \mathbb{R}^d \), let

\[
\Gamma^{-}_l = \{ v \in \Gamma \mid d(v, \Gamma^c) \geq l \} \ \text{and} \ \Gamma^{+}_l = \{ v \in \mathbb{R}^d \mid d(v, \Gamma) \leq l \},
\]

where \( d \) is the Euclidean distance. Given three sets \( A, B, C \subset \mathbb{R}^d \), we define the fluctuation of \( A \) from \( B \) inside \( C \) as

\[
F_C(A, B) = \inf \{ \delta > 0 \mid B_{-\delta} \cap C \subset A \cap C \subset B_{+\delta} \cap C \}.
\]

**Remark 2.** If \( A, B, C \) are convex subset, \( F_C(A, B) \) is coincide with the Hausdorff distance \( d_H(A \cap C, B \cap C) \). Although they do not coincide in general, the same proofs still work with a suitable modification and the results below hold even when we replace \( F_C(A, B) \) by \( d_H(A \cap C, B \cap C) \).

To consider the directional fluctuation, we define the following cone.

**Definition 2.** Given \( \theta \in \mathbb{R}^d \) and \( r > 0 \), let

\[
L(\theta, r) = \{ a \cdot v \mid a \in [0, \infty), \ v \in B(\theta, r) \},
\]

where \( B(x, r) \) is the closed ball whose center is \( x \) and radius is \( r \).

Note that if \( r > 2 \), \( L(\theta, r) \) is the entire \( \mathbb{R}^d \). Let us consider the divergence of the non-random shape fluctuation \( F(G(t), t \mathbb{B}_d) \), which was predicted in Remark 2 of [22].

**Theorem 2.** Suppose that \( F \) is useful and \( \mathbb{E}[\tau^2_\epsilon(\log \tau_\epsilon)_+] < \infty \). Let \( x_d \in \partial \mathbb{B}_d \) such that there exist \( x_1 \in \mathbb{R}^d \) and \( r > 0 \) such that \( B(x_1, r) \subset \mathbb{B}_d \) and \( x_d \in \partial B(x_1, r) \). Then for any \( r > 0 \), there exists \( c > 0 \) such that for any sufficiently large \( t \),

\[
F_{L(\{ x_d \}, \{ r \})}(G(t), t \mathbb{B}_d) \geq c(\log \log t)^{1/d}.
\]

1.2. **Notation and terminology.** This subsection collects useful notations and terminologies for the proof.

- It is useful to extend the definition of Euclidean distance \( d(\cdot, \cdot) \) as

\[
d(A, B) = \inf \{ d(x, y) \mid x \in A, \ y \in B \} \quad \text{for} \ A, B \subset \mathbb{R}^d.
\]

When \( A = \{ x \} \), we write \( d(x, B) \).

- Let \( F^- \) and \( F^+ \) be the infimum and supremum of the support of \( F \), respectively:

\[
F^- = \inf \{ \delta > 0 \mid \mathbb{P}(\tau_e < \delta) > 0 \}, \quad F^+ = \sup \{ \delta > 0 \mid \mathbb{P}(\tau_e > \delta) > 0 \}.
\]

- We simply write \( \log^{(2)} x = \log \log x \).
2. **Proof of the Divergence of the Non-random Fluctuation**

The heuristic behind the proof for $\sup_{x \in \mathbb{Z}^d} |\mathbb{E}T(0, x) - g(x)| = \infty$ is the following. Given $x \in \mathbb{Z}^d$, observe that $2(\mathbb{E}T(0, x) - g(x)) \geq \mathbb{E}T(0, x) + \mathbb{E}T(x, 2x) - \mathbb{E}T(0, 2x)$, by using the facts $\mathbb{E}T(0, 2x) - 2g(x) \geq 0$ and $\mathbb{E}T(0, x) = \mathbb{E}T(x, 2x)$. Therefore, noting that $T(0, x) + T(x, 2x) \geq T(0, 2x)$, it suffices to find a vertex $x \in \mathbb{Z}^d$ such that $\Delta(x) = T(0, x) + T(x, 2x) - T(0, 2x)$ is sufficiently large with some probability. However, this strategy does not work directly because $\Delta(x)$ is still complicated object. Instead, we first suppose that $\sup_{x \in \mathbb{Z}^d} |\mathbb{E}T(0, x) - g(x)| < \infty$ and we will find a vertex where $\Delta(x) > 0$ with probability greater than 1, which leads to a contradiction.

2.1. **Proof of Theorem** Let $B \subset \mathbb{R}^d$ be a convex subset and $x_d \in \partial B$. Suppose that there exists $x_1 \in \mathbb{R}^d$ and $r > 0$ such that $B(x_1, r) \subset B$ and $x_d \in \partial B(x_1, r)$. Let $L$ be an unique tangent plane of $\partial B(x_1, r)$ at $x_d$. Then there exists $K > 0$ such that for any $t > 0$ and $y \in tL$ with $|y - tx_d| \leq \sqrt{t}$,

$$d(y, \partial(tE_d)) \leq K.$$

Proof. By the rotation and translation, it suffices to prove it in the case where $d = 2$, $x_1 = re_2$ and $x_d = 0$ (See Figure 1). Then $L = \{(x, 0) \mid x \in \mathbb{R}\}$. Note that $\partial tB(x_1, r)$ can be expressed by a function $y = tr - t\sqrt{r^2 - (x/t)^2}$ and if $|x| \leq \sqrt{t}$, $tr - t\sqrt{r^2 - (x/t)^2} \leq K$ with some constant $K > 0$ independent of $t$. Since $\partial tE_d$ is between $tL$ and $\partial(tB(x_1, r))$, $d(y, \partial(tE_d)) \leq K$ follows.

Note that $L$ is also a tangent plane of $\partial B_d$ at $x_d$. Let $K > 0$ to be chosen later. Suppose that

$$\lim_{t \to \infty} \frac{F_{t(x_d, r)}(G(t), tE_d)}{(\log(t^{d/2}))^{1/d}} = 0,$$

and we shall derive a contradiction. Then for any $\epsilon > 0$, we can take a positive sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \uparrow \infty$ as $n \to \infty$ and

$$F_{t(x_d, r)}(G(t_n), t_nE_d) \leq \epsilon(\log(t^{d/2}))^{1/d}.$$

for any $n \in \mathbb{N}$.

One can find a finite subset $S_n$ of $t_nL$ such that the following hold:

$$\#S_n = [(\log t_n)^{1/8}],
\text{if } a \neq b \in S_n, |a - b| \geq t_n^{1/2}(\log t_n)^{-1/8},
\text{for any } a \in S_n, |a - t_nx_d| \leq t_n^{1/2}.$$

Given $a, b, y \in \mathbb{R}^d$, we define $T(a, y, b) = T(a, y) + T(y, b)$, which is the first passage time from $a$ to $b$ passing through $y$.

![Figure 1](image-url)

Left: Figure of $x_d$ and $L$.
Right: The schematic picture of Step 2 in the proof of Lemma
Lemma 2. Under the assumption of (2.2), if we take $K > 0$ sufficiently large independent of $h$, for any sufficiently large $n \in \mathbb{N}$ and $y \in S_n$, 
\[ \mathbb{E}T(0, 2t_n x_d) \leq \mathbb{E}T(0, y, 2t_n x_d) \leq K \epsilon (\log (t_n)^{1/d} + \mathbb{E}T(0, 2t_n x_d)). \]

Proof. Because $g$ is a norm, the triangular inequality leads to $g(2t_n x_d) \leq g(y) + g(2t_n x_d - y)$ for any $y \in S_n$. By the reflection symmetry, we have $B(2x_d - x_1, r) \subset \{ x \in \mathbb{R}^d \mid g(2x_d - x) \leq 1 \}$. By Lemma 1 there exist $y_1 \in t_n B_d$ and $y_2 \in \{ x \in \mathbb{R}^d \mid g(2x_d - x) \leq t_n \}$ such that $|y - y_1|, |y - y_2| \leq K$. Since $g(x) \leq 2d \mathbb{E}[\tau_e | x]$, for any $x \in \mathbb{R}^d$, we obtain for sufficiently large $n$, 
\[ g(y) + g(2t_n x_d - y) - 4dK \mathbb{E}[\tau_e] \leq g(y_1) + g(y_2) \leq 2t_n = g(2t_n x_d). \]

By Lemma 1 for any $y \in S_n$, there exist $y_1, y_2 \in \mathbb{R}^d \cap L(x_d, r)$ such that $|y_1 - y|, |y_2 - y| \leq K$ and $g(y_1) = g(2x_d - y_2) = t_n$. Under the assumption (2.2), there exist $y_1', y_2' \in G(t_n)$ such that $|y_1' - y_1|, |y_2' - y_2| \leq \epsilon (\log (t_n)^{1/d})$. Note that $|y_1' - y_2'| \leq 2K + \epsilon (\log (t_n)^{1/d})$ and, in particular, 
\[ \mathbb{E}[T(y_1', y_2')] \leq 4d \mathbb{E}[\tau_e] (K + \epsilon (\log (t_n)^{1/d}). \]

This yields 
\[ \mathbb{E}[T(0, y, 2t_n x_d)] \leq g(y) + g(2t_n x_d - y) + 4d \mathbb{E}[\tau_e] (K + \epsilon (\log (t_n)^{1/d}) \leq g(2t_n x_d) + \frac{1}{2} K \epsilon (\log (t_n)^{1/d}). \]

Since $g(2t_n x_d) \leq 2d \mathbb{E}[\tau_e] + \mathbb{E}(0, 2t_n x_d)$, it follows that 
\[ \mathbb{E}(0, 2t_n x_d) \leq \mathbb{E}(0, y, 2t_n x_d) \]
\[ \leq \frac{1}{2} K \epsilon (\log (t_n)^{1/d}) + g(2t_n x_d) \]
\[ \leq K \epsilon (\log (t_n)^{1/d}) + \mathbb{E}(0, 2t_n x_d). \]

□

Lemma 3. Under the assumption of (2.2), for any sufficiently large $n \in \mathbb{N}$ and $y \in S_n$, 
\[ \mathbb{P}\{ T(0, y, 2t_n x_d) < T(0, 2t_n x_d) + 2K \epsilon (\log (t_n)^{1/d}) \} \geq \frac{1}{2}. \]

Proof. By Lemma 2, and the fact $T(a, y, b) \geq T(a, b)$, we have 
\[ \mathbb{E}[T(0, 2t_n x_d)] + K \epsilon (\log (t_n)^{1/d}) \]
\[ \geq \mathbb{E}[T(0, y, 2t_n x_d)] \]
\[ \geq \mathbb{E}[T(0, y, 2t_n x_d)] + 2K \epsilon (\log (t_n)^{1/d}; T(0, y, 2t_n x_d) \geq T(0, 2t_n x_d)] + 2K \epsilon (\log (t_n)^{1/d}] \]
\[ + \mathbb{E}[T(0, 2t_n x_d); T(0, y, 2t_n x_d) < T(0, 2t_n x_d) + 2K \epsilon (\log (t_n)^{1/d})] \]
\[ = \mathbb{E}[T(0, 2t_n x_d)] + 2K \epsilon (\log (t_n)^{1/d}) \mathbb{P}(T(0, y, 2t_n x_d) \geq T(0, 2t_n x_d) + 2K \epsilon (\log (t_n)^{1/d}). \]

Rearranging it, we obtain 
\[ \mathbb{P}(T(0, y, 2t_n x_d) < T(0, 2t_n x_d) + 2K \epsilon (\log (t_n)^{1/d}) > 1/2. \]

□

The following is a crucial property of a useful distribution.

Lemma 4. If $F$ is useful, there exists $\delta > 0$ and $D > 0$ such for any $v, w \in \mathbb{Z}^d$, 
\[ \mathbb{P}(T(v, w) < (F^- + \delta) | v - w|_1) \leq e^{-D |v - w|_1}. \]

For a proof of this lemma, see Lemma 5.5 in [1].

Definition 3. Let $c > 0$ be a fixed constant. A $y \in S_n$ is said to be black if for any $a, b \in B(y, c (\log (t_n)^{1/d})$ satisfying $|a - b| \geq \frac{c}{2} (\log (t_n)^{1/d}$, $T(a, b) \geq (F^- + \delta) |a - b|_1$. A $y \in S_n$ is said to be good if $T(0, y, 2t_n x_d) < T(0, 2t_n x_d) + 2K \epsilon (\log (t_n)^{1/d}$ and $y$ is black.
Note that by Lemma 4 we have
\[
\lim_{n \to \infty} \inf_{S_n} \min_{y \in S_n} \mathbb{P}(y \text{ is black}) = 1,
\]
where $S_n$ runs over all subset of $t_n L$ satisfying (2.3). Combining it with Lemma 3, we have that for sufficiently large $n \in \mathbb{N}$,
\[
\mathbb{P}(y \text{ is good}) \geq 3/8.
\]

**Lemma 5.** Under the assumption of (2.2), independent of the choice of $S_n$, we have the following: for any sufficiently large $n \in \mathbb{N}$,
\[
\mathbb{P}(\{y \in S_n | y \text{ is good}\} \geq 2S_n/4) \geq 1/8.
\]

**Proof.** By (2.7), we obtain
\[
\frac{3}{8}S_n \leq \mathbb{E}[\{y \in S_n | y \text{ is good}\}]
\]
\[
\leq \frac{1}{2}S_n \mathbb{P}(\{y \in S_n | y \text{ is good}\} \geq 2S_n/4) + \frac{S_n}{8}.
\]
Rearranging it, the proof is completed. \hfill \Box

We define three events $\mathcal{A}_1, \mathcal{A}_2,$ and $\mathcal{A}_3$ as
\[
\mathcal{A}_1 = \{\{y \in S_n | y \text{ is good}\} \geq 2S_n/4\},
\]
\[
\mathcal{A}_2 = \{\forall a, b \in B(0, t_n^2) \text{ satisfying } |a - b| \geq t_n^{1/4}, T(a, b) \geq (F^- + \delta)|a - b|_1\},
\]
\[
\mathcal{A}_3 = \{\forall y \in S_n, \max_{z = 0, 2t_n x_d} |T(z, y) - \mathbb{E}[T(z, y)]| \leq t_n^{1/2}(\log t_n)^{-1/4}\}.
\]

Lemma 5 and Lemma 4 lead to
\[
\lim_{n \to \infty} \inf_{S_n} \mathbb{P}(\mathcal{A}_1) = 1/8,
\]
\[
\lim_{n \to \infty} \inf_{S_n} \mathbb{P}(\mathcal{A}_2) = 1,
\]
Moreover, we have the following.

**Lemma 6.**
\[
\lim_{n \to \infty} \inf_{S_n} \mathbb{P}(\mathcal{A}_3) = 1.
\]

**Proof.** We use the sublinear variance \[5, 6, 10\]: Under the assumption $\mathbb{E}[\tau_0^2(\log \tau_0)_+] < \infty$, there exists $C > 0$ depending only on $F$ and $d$ such that for any $x \in \mathbb{R}^d$,
\[
\mathbb{V}(T(0, x)) \leq C \frac{|x|}{\log |x|}.
\]
Then by the Chebyshev’s inequality and the union bound, we have
\[
\mathbb{P}(\exists y \in S_n \text{ such that } \max_{z = 0, 2t_n x_d} |T(z, y) - \mathbb{E}[T(z, y)]| \leq t_n^{1/2}(\log t_n)^{-1/4})
\]
\[
\leq 2\frac{1}{2}S_n \max_{y \in S_n} \mathbb{P}(|T(0, y) - \mathbb{E}[T(0, y)]| \leq t_n^{1/2}(\log t_n)^{-1/4})
\]
\[
\leq 2C'(\log t_n)^{1/8}(\log t_n)^{-1/2} \to 0,
\]
where $C'$ is a constant depending only on $d$ and $F$. \hfill \Box

We set $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$. Note that for sufficiently large $n \in \mathbb{N}$, independent of the choice of $S_n$, we have
\[
\mathbb{P}(\mathcal{A}) \geq 1/16.
\]
Given $y \in S_n$, let us define $\mathcal{A}_y = \{\forall z \in S_n \text{ with } z \neq y, T(0, y, 2t_n x_d) < T(0, z, 2t_n x_d)\}$. 

Lemma 7. If we take \( h > 0 \) sufficiently small depending on \( c \), for any \( y \in S_n \), the following holds:

\[
\mathbb{P}(A_y) \geq \mathbb{P}(\forall e \in B(y, c(\log \log t_n)^{1/4}), \; \tau_e \leq F^- + \delta/2) \mathbb{P}(A \cap \{ y \text{ is good } \}).
\]

Proof. We use the resampling argument in [4]. Let \( \tau^* = \{ \tau_e^* \}_{e \in E^c} \) be independent copy of \( \{ \tau_e \}_{e \in E^c} \). We enlarge the probability space so that we can measure the event both for \( \tau \) and \( \tau^* \) and we still denote the joint probability measure by \( \mathbb{P} \). We define \( \tilde{\tau} = \{ \tilde{\tau}_e \}_{e \in E^c} \) as

\[
\tilde{\tau}_e = \begin{cases} 
\tau_e^* & \text{if } e \in B(y, c(\log^2 t_n)^{1/4}) \\
\tau_e & \text{otherwise.}
\end{cases}
\]

Note that the distributions of \( \tau \) and \( \tilde{\tau} \) are the same under \( \mathbb{P} \) since \( \tau \) and \( \tau^* \) are independent. Thus \( \mathbb{P}(A_y) = \mathbb{P}(A_y) \), where \( A_y \) is the same condition as \( A_y \) for \( \tilde{\tau} \). We write \( \tilde{T}(a, b) \) for the first passage time from \( a \) to \( b \) with respect to \( \tilde{\tau} \). We define \( \tilde{T}(a, y, b) \) similarly. Since the right hand side of (2.16) equals to

\[
\mathbb{P}(\forall e \in B(y, c(\log \log t_n)^{1/4}), \; \tau_e \leq F^- + \delta/2, \; A \cap \{ y \text{ is good } \}),
\]

it suffices to show that the event in (2.17) implies \( A_y \). To do this, we suppose that \( \tau \) and \( \tilde{\tau} \) are in this event.

Step 1 \( (\tilde{T}(0, y, 2t_n x_d) + 4Kc(\log^2 t_n)^{1/4} < T(0, y, 2t_n x_d)) \)

We take an arbitrary optimal path \( \gamma = (\gamma_i)_{i=1}^l \subset \mathbb{Z}^d \) for \( T(0, y, 2t_n x_d) \). Let \( s = \min \{ i \in \{1, \ldots, l\} | \gamma_i \in B(y, c(\log^2 t_n)^{1/4}) \} \) and \( f = \max \{ i \in \{1, \ldots, l\} | \gamma_i \in B(y, c(\log^2 t_n)^{1/4}) \} \).

By the assumption, we have

\[
\tilde{T}(0, y, 2t_n x_d) \leq T(0, \gamma_s) + T(\gamma_f, 2t_n x_d) + |\gamma_f - \gamma_s|_1 (F^- + \delta/2).
\]

On the other hand, \( y \) is black, and we have

\[
T(0, y, 2t_n x_d) \geq T(0, \gamma_s) + T(\gamma_f, 2t_n x_d) + (|\gamma_f - \gamma_s|_1 \vee |y| - \gamma_s|_1)(F^- + \delta)
\]

Since \( |\gamma_s - |y|| \geq \frac{c}{2d}(\log t_n)^{1/4} \), we have

\[
\tilde{T}(0, y, 2t_n x_d) + 4Kc(\log^2 t_n)^{1/4} < T(0, y, 2t_n x_d).
\]

Step 2 \( (\tilde{T}(0, y, 2t_n x_d) < \tilde{T}(0, z, 2t_n x_d) \text{ for } \forall z \in S_n \text{ with } z \neq y) \)

Let \( z \in S_n \) with \( z \neq y \). We first suppose that \( \tilde{T}(0, z, 2t_n x_d) \leq T(0, z, 2t_n x_d) \). Then, since we resample the configurations only on \( B(y, c(\log^2 t_n)^{1/4}) \), any optimal path \( \gamma = (\gamma_i)_{i=1}^l \) for \( \tilde{T}(0, z, 2t_n x_d) \) must touch with \( B(y, c(\log^2 t_n)^{1/4}) \), i.e., there exists \( i \in \{1, \ldots, l\} \) such that \( \gamma_i \in B(y, c(\log^2 t_n)^{1/4}) \). By the definition, \( [z] \) is included in \( \gamma \) and let \( j \in \{1, \ldots, l\} \) be \( \gamma_j = [z] \). Without loss of generality, we can suppose that \( i < j \). Then, by the condition of \( A_2 \), it is easy to check that

\[
\tilde{T}(0, z) \geq \tilde{T}(0, y) + \frac{1}{2} (F^- + \delta) t_n^{1/2}(\log t_n)^{-1/8}.
\]

If there exists \( i' > j \) such that \( \gamma_{i'} \in B(y, c(\log^2 t_n)^{1/4}) \), by \( \tilde{T}(\gamma_i, y, \gamma_{i'}) < 2dc(\log^2 t_n)^{1/4} \) and the condition \( A_2 \), \( \tilde{T}(0, y, 2t_n x_d) < \tilde{T}(0, z, 2t_n x_d) \) as desired. Thus, without loss of generality, we suppose that for any \( i' > j \), we may assume \( \gamma_{i'} \notin B(y, c(\log^2 t_n)^{1/4}) \). Since we change the configurations only on \( B(y, c(\log^2 t_n)^{1/4}) \), we have \( \tilde{T}(z, 2t_n x_d) = T(z, 2t_n x_d) \). By the condition \( A_3 \), it yields

\[
\tilde{T}(y, 2t_n x_d) - \tilde{T}(z, 2t_n x_d)
\]

\[
\leq \tilde{T}(y, 2t_n x_d) - ET(y, 2t_n x_d) + |ET(z, 2t_n x_d) - ET(y, 2t_n x_d)| + ET(z, 2t_n x_d) - T(z, 2t_n x_d)
\]

\[
\leq 3t_n^{1/2}(\log t_n)^{-1/4}.
\]

Together with (2.18), this gives

\[
\tilde{T}(0, y, 2t_n x_d) < \tilde{T}(0, z, 2t_n x_d).
\]
We now turn to the case \( \tilde{T}(0, z, 2t_n x_d) \geq T(0, z, 2t_n x_d) \). Then, since
\[
T(0, y, 2t_n x_d) - 2K\epsilon (\log^{(2)} t_n)^{1/d} < T(0, 2t_n x_d)
\]
by the goodness of \( y \), Step 1 implies \( \tilde{T}(0, y, 2t_n x_d) < \tilde{T}(0, z, 2t_n x_d) \). Thus the proof is completed.

Since \( \{A_y\}_{y \in S_n} \) are disjoint from each other, if we take \( c \) sufficiently small, by Lemma\(^7\) we get
\[
1 \geq \sum_{y \in S_n} P(A_y) \geq \sum_{y \in S_n} \mathbb{P}(\forall c \subset B(y, c(\log^{(2)} t_n)1/d), \; \tau_c \leq F^- + \delta/2)\mathbb{P}(\{ y \text{ is good } \}; \; A) \geq \mathbb{E}[\sharp\{ y \in S_n \mid y \text{ is good } \} \cap A] \min_{y \in S_n} \mathbb{P}(\forall c \subset B(y, c(\log^{(2)} t_n)1/d), \; \tau_c \leq F^- + \delta/2).
\]
Since \( \sharp\{ y \in S_n \mid y \text{ is good } \} \geq \frac{2S_n}{4} \) on the event \( A \), by (2.15), this is further bounded from below by
\[
\frac{2S_n}{4} \mathbb{P}(A)(\log t_n)^{-c} \geq (\log t_n)^{1/32}.
\]
If we take \( n \) sufficiently large, we have a contradiction and the proof of Theorem\(^2\) is completed.

3. PROOF OF THE DIVERGENCE OF THE NON-RANDOM FLUCTUATION

Proof of Theorem\(^7\). We first prove that for any \( \epsilon > 0 \), \( \sup_{x \in \mathbb{Z}^{d} \cap L(x_d, \epsilon)} |\mathbb{E}T(0, x) - g(x)| < \infty \) leads to a contradiction. Note that \( \sup_{x \in \mathbb{R}^{d} \cap L(x_d, \epsilon)} |\mathbb{E}T(0, x) - g(x)| < \infty \), since \( g(x) \leq 2d\mathbb{E}\tau_c|x| \). We set \( k > 0 \) as
\[
k = \sup_{x \in \mathbb{R}^{d} \cap L(x_d, \epsilon)} |\mathbb{E}T(0, x) - g(x)| < \infty.
\]
By the definition of \( G(t) \), we obtain for any \( t > k \),
\[
(t - k)B_d \cap L(x_d, \epsilon) \subset G(t) \cap L(x_d, \epsilon) \subset (t + k)B_d \cap L(x_d, \epsilon).
\]
Therefore, writing \( \text{diam}(B_d) = \sup\{d(x, y) \mid x, y \in B_d\} \), we have
\[
F_{L(x_d, \epsilon)}(G(t), tB_d) \leq k\text{diam}(B_d),
\]
which contradicts Theorem\(^2\).

Therefore, for any \( m \in \mathbb{N} \), we can find a sequence \( \{x_n^{[m]}\}_{n \in \mathbb{N}} \subset L(x_d, 1/m) \cap \mathbb{Z}^{d} \) such that \( |\mathbb{E}T(0, x_n^{[m]}) - g(x_n^{[m]})| \geq n \). Let us define \( x_n = x_n^{[m]} \). Then
\[
\lim_{n \to \infty} |\mathbb{E}T(0, x_n) - g(x_n)| = \infty \text{ and } \lim_{n \to \infty} x_n/|x_n| = x_d/|x_d|.
\]

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