Five-dimensional supergravity and the hyperbolic Kac–Moody algebra $G_2^H$

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Abstract
Motivated by the recent analysis of the $E_{10}$ sigma model for the study of M-theory, we study a one-dimensional sigma model associated with the hyperbolic Kac–Moody algebra $G_2^H$ and its link to $D = 5, \mathcal{N} = 2$ pure supergravity, which closely resembles in many ways $D = 11$ supergravity. The bosonic equations of motion and the Bianchi identity for $D = 5$ pure supergravity match the equations of the level $\ell \leq 3$ truncation of the $G_2^H$ sigma model up to higher level terms, just as they do for the $D = 11$ case. We also compute low level root and outer multiplicities in the $A_3$ decomposition, and indeed find singlets at $\ell = 4k, k = 2, 3, \ldots$ corresponding to the scaling of $E_{11+k}$ terms, although the missing singlet at $\ell = 4$ remains a puzzle.

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1. Introduction

The recent study of gravity and supergravity solutions near a spacelike singularity has revealed that their oscillatory behaviour is related in a way not well understood to the consistency of superstring and M-theories. A typical observation is that in the BKL limit [1] pure gravity ceases to be chaotic in dimensions $D \geq 11$ [2], but the oscillatory behaviour is restored if gravity is coupled to a 3-form [3]. The criterion of whether the given theory is chaotic or not can be summarized in a word: hyperbolicity [4].

In the $D = 11$ supergravity case, the behaviour of the logarithmic scale factors of a metric is described as a billiard motion in the Weyl chamber of the hyperbolic Kac–Moody algebra $E_{10}$ [5]. A systematic analysis of this ‘cosmological billiard’ was carried out in [6], in which it was shown that the billiard dynamics is asymptotically equivalent to a one-dimensional sigma model associated with a corresponding hyperbolic Kac–Moody group (see [7–13] for the pioneering works on the billiard approach). Moreover, agreement was found between the equations of motion of $D = 11$ supergravity and those of the $E_{10}$ sigma model up to
height $\leq 29$ in the framework of the $A_9$ 'level' decomposition of $E_{10}$ [14–16] (see also [17] for $A_d$ decomposition of the very extended Kac–Moody algebras), which has led to the conjecture that even information on higher order corrections of M-theory is encoded in the infinite towers of roots of $E_{10}$. (See [18] for the relevance of $E_{10}$ in M-theory, and [19] for a different M-theory interpretation of the imaginary roots. We should also mention that a conceptually different proposal on the role of $E_{11}$ in M-theory had been made in [20], and the formulation based on the very extended algebras was elaborated in [21].) Similar analyses were also done in massive IIA [22] and IIB supergravities [23].

More recently, further evidence supporting this conjecture was given in the $E_{10}/A_9$ decomposition analysis [24], in which a series of $A_9$ singlets, whose existence was suggested by the scaling behaviour of higher derivative corrections of the form $E R^N$, were indeed found at levels $\ell = 10k, k = 1, 2, \ldots$. This predicted that the higher order corrections are allowed only for $N = 4, 7, \ldots$, which was recently confirmed to be consistent with the string duality [25].

As was emphasized in [24], the correspondence between the wall forms associated with the higher curvature corrections and singlet representations of the relevant subalgebra appears to be a special property of M-theory and $E_{10}$, which is not shared by, for instance, pure Einstein gravity and $AE_d$ [18]. Therefore it will be interesting to explore if there is any other such supergravity model which also possesses singlets in the decomposition at the locations expected from the scaling behaviour of the higher curvature corrections. In this paper, we use $D = 5$ pure supergravity [26] as our example to investigate.

It is known that $D = 11$ supergravity [27] and $D = 5$ pure supergravity are very similar [28] in many ways. In particular, the dimensional reduction to three dimensions can be done in a very similar manner in both theories to obtain $E_{8(8)}/SO(16)$ [29, 30] and $G_{2(2)}/SO(4)$ [28] nonlinear sigma models, respectively. Moreover, it was shown [31] that $D = 5$ pure supergravity is also one of the special classes of theories that exhibits chaotic behaviour in the BKL limit, in which the billiard is the Weyl chamber of $G_2^H$, the canonical hyperbolic extension of $G_2$. The similarity of $E_8$ and $G_2$ was noted in [32]. Therefore we consider, in place of $E_{10}$, a one-dimensional sigma model associated with $G_2^H$, and study its relation to $D = 5$ pure supergravity. We will see again that there is a strong parallelism. We will show that the bosonic equations of motion and the Bianchi identity for $D = 5$ pure supergravity match the equations of the level $\ell \leq 3$ truncation of the $G_2^H$ sigma model up to a few higher level terms, just as they do [16] for the $D = 11$ case. We also compute low level root and outer multiplicities in the $A_3$ decomposition up to height $\leq 40$ and $\leq 60$, respectively, and find singlets at $\ell = 4k, k = 2, 3, \ldots$, which precisely corresponds to the scaling of $E R^{k+1}$ terms in $D = 5$ pure supergravity. However, it turns out that there are no singlets at $\ell = 4$, which is puzzling because there is a corresponding on-shell 1-loop divergence; this will be discussed in section 5.

2. The hyperbolic Kac–Moody algebra $G_2^H$

The Kac–Moody algebra $G_2^H$ is defined to be generated by multiple commutators of the Chevalley generators $\{e_i, f_i, h_i\}$ ($i = 0, 1, 2, 3$) with the relations (following the conventions of [33])

$$
[h_i, e_j] = A_{ij}e_j, \quad [h_i, f_j] = -A_{ij}f_j, \quad [e_i, f_j] = \delta_{ij}h_i, \quad (\text{ad } e_i)^{-A_{ij}}(e_j) = 0, \quad \text{and} \quad (\text{ad } f_j)^{-A_{ij}}(f_j) = 0 \quad \text{if} \ i \neq j.
$$

(2.1)
where $A_{ij}$ ($i, j = 0, 1, 2, 3$) is the Cartan matrix

$$A_{ij} = \frac{2(\alpha_i | \alpha_j)}{\alpha_i | \alpha_i} = \begin{pmatrix} 2 & -3 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{2.2}$$

(figure 1). $\alpha_i$ are the simple roots. $G^H_L$ has a regular subalgebra $A_3$ whose simple roots consist of $\{\alpha_1, \alpha_2, \alpha_3\}$. We decompose the whole set of $G^H_L$ roots into irreducible orbits of this $A_3$ action.

Any root of $G^H_L$ can be written as

$$\alpha = \ell \alpha_0 + \sum_{j=1}^{3} m^j \alpha_j \tag{2.3}$$

with all non-negative or all non-positive integers $\ell$ and $m^j$. The coefficient $\ell$ is called the level [14] of $\alpha$. By definition, the $A_3$ action does not change $\ell$; the whole set of roots at each level $\ell$ are decomposed into a direct sum of $A_3$ representations, which are specified by their Dynkin labels $p_k(\Lambda) \equiv (\alpha_k | \Lambda)$ ($k = 1, 2, 3$) of some highest weight $\Lambda$.

We can proceed exactly in the same way as the $E_{10}/A_9$ or $AE_3/A_2$ decomposition. For example, the Chevalley generator $f_0$ is a root vector with root $-\alpha_0$, which is the highest weight vector of $A_3$ with Dynkin label $(p_1, p_2, p_3) = (1, 0, 0)$; all the level $\ell = -1$ roots have one-to-one correspondence with the $A_3$ weights, and hence are components of an $A_3$ vector. In the next section, we will see that it is identified as the (spatial part of the) $U(1)$ gauge field in $D = 5$ simple supergravity.

One can also derive constraints in order for a negative root $-\alpha$ (2.3) to be a highest weight of some representation of the $A_3$ subalgebra, similarly to [14]. The result is, for $\Lambda = -\alpha$,

$$p_1 = \ell - 2m^1 + m^2 \geq 0,$$

$$p_2 = m^1 - 2m^2 + m^3 \geq 0,$$

$$p_3 = m^2 - 2m^3 \geq 0, \tag{2.4}$$

and

$$|\alpha|^2 = -\frac{1}{12} \ell^2 + \frac{1}{4} (3p_1^2 + 4p_2^2 + 3p_3^2 + 4p_1p_2 + 4p_2p_3 + 2p_1p_3) \leq 2. \tag{2.5}$$

Using these constraints one can easily show that there is a unique dominant weight $(p_1, p_2, p_3) = (0, 1, 0)$ at $\ell = -2$, which corresponds to an $A_3$ rank-2 antisymmetric tensor. We will see that it corresponds to the electromagnetic dual of the gauge field.

At $\ell = -3$, there are two solutions $(p_1, p_2, p_3) = (1, 1, 0)$ and $(0, 0, 1)$ to the constraints (2.4) and (2.5). As we will see, however, the latter is not a highest weight, and there is again a unique representation at $\ell = -3$. It carries three mixed symmetric and antisymmetric indices and is identified as a dual graviton in section 3.

These three lowest representations in our $G^H_L/A_3$ decomposition show a striking resemblance to those in the $E_{10}/A_9$ decomposition [14] in their index structures, being a reflection of the similarity between $D = 11$ and $D = 5$ supergravities.
we would have to display more roots for the same given maximum height (positive roots here.\footnote{We could equally compute multiplicities of negative roots. However, although the number of $A_2$-dominant $G_2^H$ roots at level $\ell$ is of course the same as that at level $-\ell$, the height of the former is generally larger than the latter, so we would have to display more roots for the same given maximum height (=40).}) is no difference between $G_2^H$ outer multiplicities of the $A_5$ levels. We could equally compute multiplicities of $G_2^H$ roots at level $\ell = 4$ We could equally compute multiplicities of $G_2^H$ roots at level $\ell = 1$ there is a unique representation at each level. These dominant weights are (0, 1, 0) and (1, 0, 0) with root multiplicities 1 and 2, respectively. The former representation contains (1, 0, 0) as one of the weights, with weight multiplicity being precisely 2. Therefore the outer multiplicity (that is, how many times the representation occurs) of (1, 0, 0) is zero.\footnote{The level decomposition for the ‘very extended’ version of $G_2$ has been worked out in \cite{2}. At low levels there is no difference between $G_2^H$ and their result. In particular, one can find the supergravity fields as well as the same outer multiplicities of the $A_3$ singlets at $\ell = 4$ and $8$. We thank Axel Kleinschmidt for pointing this out.}

Since the constraints can only provide necessary conditions for the highest weights, in order to further compute the decomposition, we employ Peterson’s recursive formula for root multiplicities \cite{33}

$$\langle \beta | \beta - 2\rho \rangle c_\beta = \sum_{\beta', \beta'' \in \mathbb{Q}^*} (\beta' | \beta'') c_{\beta'} c_{\beta''},$$

(2.6)

where $c_\beta = \sum_{n \geq 1} n^{-1} \text{mult}(\beta/n)$ and $\mathbb{Q}^* = \sum_{i=0}^{\infty} \mathbb{Z}_i \alpha_i$. We have used Mathematica to obtain root multiplicities of $G_2^H$ with low heights. Results of a sample computation are shown in table 2, in which multiplicities of all the positive roots $\alpha = \ell \alpha_0 + m^1 \alpha_1 + m^2 \alpha_2 + m^3 \alpha_3$ that are dominant as an $A_1$ weight (that is, all the Dynkin labels $p_i$ are non-negative) are listed up to height $\leq 40$.\footnote{When compared with our $G_2^H$ roots, the ordering of Dynkin labels should be reversed since we are decomposing positive roots here.}

At $\ell = 0$, we can see the highest weight of the adjoint representation of $A_3$ as expected. For $\ell = 1, 2$ there is a unique representation at each level. These dominant weights are minus the lowest weights in the corresponding representations at $\ell = -1, -2$ in table 1. At $\ell = 3$, we have $(0, 1, 1)$ and $(1, 0, 0)$ with root multiplicities 1 and 2, respectively. The former representation contains (1, 0, 0) as one of the weights, with weight multiplicity being precisely 2. Therefore the outer multiplicity (that is, how many times the representation occurs) of (1, 0, 0) is zero.\footnote{The level decomposition for the ‘very extended’ version of $G_2$ has been worked out in \cite{2}. At low levels there is no difference between $G_2^H$ and their result. In particular, one can find the supergravity fields as well as the same outer multiplicities of the $A_3$ singlets at $\ell = 4$ and $8$. We thank Axel Kleinschmidt for pointing this out.}

From this table, we observe the following two further similarities to the $A_9$ decomposition of $E_{10}$. First, $G_2^H$ has three towers of roots with $A_3$ Dynkin labels $(n, 0, 1), (n, 1, 0)$ and $(n, 1, 1)$ at levels $\ell = 3n + 1, 3n + 2$ and $3n + 3$, respectively, with root multiplicity 1. (Their outer multiplicities are also 1.) They are the $G_2^H$ analogues of the three series of $E_{10}$ roots with $A_9$ labels $(0, 0, 1, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 1, 0, 0, 0)$ and $(1, 0, 0, 0, 0, 0, 0, 1, 0, 0)$.\footnote{When compared with our $G_2^H$ roots, the ordering of Dynkin labels should be reversed since we are decomposing positive roots here.}

Thus we may say that there is also ‘enough room’ \cite{14} in $G_2^H$ roots for the spatial gradients of $D = 5$ supergravity fields identified at $\ell \leq 3$, just in the same manner as $E_{10}$ contains the spatial gradients of $D = 11$ supergravity.

The second interesting observation is that there are a series of $A_3$ weights $(0, 0, 0)$ at levels $\ell = 4k$, $k = 1, 2, \ldots$ with root coefficients $(\ell, m^1, m^2, m^3) = (4k, 3k, 2k, k)$. (In fact, the first one ($k = 1$) turns out to have outer multiplicity zero.) It was found in \cite{24} that $E_{10}$ has a series of singlets in the $A_9$ decomposition, and such roots are proportional to the ‘wall form’ of higher order curvature corrections of $D = 11$ supergravity. In section 4 we will discuss the relevance of these $A_3$ singlets to higher order corrections to $D = 5$ supergravity.
### Table 2. A sample computation of root multiplicities of $G_2^H$ at low height.

| $(\ell, m^1, m^2, m^3)$ | $3\alpha^2$ | Mult | $(p_1, p_2, p_3)$ | $(\ell, m^1, m^2, m^3)$ | $3\alpha^2$ | Mult | $(p_1, p_2, p_3)$ |
|--------------------------|-----------|------|------------------|--------------------------|-----------|------|------------------|
| $(0, 1, 1, 1)$           | 6         | 1    | $(1, 0, 1)$      | $(10, 10, 7, 4)$         | 2         | 1    | $(3, 0, 1)$      |
| $(1, 1, 1, 1)$           | 2         | 1    | $(0, 0, 1)$      | $(10, 9, 8, 4)$          | 2         | 1    | $(0, 3, 0)$      |
| $(2, 2, 2, 1)$           | 2         | 1    | $(0, 1, 0)$      | $(10, 9, 7, 5)$          | 2         | 1    | $(1, 0, 3)$      |
| $(3, 3, 2, 1)$           | 0         | 2    | $(1, 0, 0)$      | $(11, 10, 7, 4)$         | -16       | 21   | $(2, 0, 1)$      |
| $(4, 3, 2, 1)$           | -4        | 3    | $(0, 0, 0)$      | $(11, 9, 7, 5)$          | -10       | 9    | $(0, 0, 3)$      |
| $(3, 3, 3, 2)$           | 6         | 1    | $(0, 1, 1)$      | $(12, 10, 7, 4)$         | -30       | 135  | $(1, 0, 1)$      |
| $(4, 4, 3, 2)$           | 2         | 1    | $(1, 0, 1)$      | $(11, 10, 8, 4)$         | -10       | 9    | $(0, 1, 2)$      |
| $(5, 4, 3, 2)$           | -4        | 3    | $(0, 0, 1)$      | $(13, 10, 7, 4)$         | -40       | 378  | $(0, 0, 1)$      |
| $(5, 5, 4, 2)$           | 2         | 1    | $(1, 1, 0)$      | $(12, 10, 8, 4)$         | -24       | 66   | $(0, 2, 0)$      |
| $(6, 5, 4, 2)$           | -6        | 6    | $(0, 1, 0)$      | $(11, 11, 8, 4)$         | 2         | 1    | $(3, 1, 0)$      |
| $(6, 6, 4, 2)$           | 0         | 2    | $(2, 0, 0)$      | $(11, 10, 8, 5)$         | -4        | 3    | $(1, 1, 2)$      |
| $(6, 5, 4, 3)$           | 0         | 2    | $(0, 0, 2)$      | $(12, 11, 8, 4)$         | -18       | 32   | $(2, 1, 0)$      |
| $(7, 6, 4, 2)$           | -10       | 9    | $(1, 0, 0)$      | $(12, 10, 8, 5)$         | -18       | 32   | $(0, 1, 2)$      |
| $(8, 6, 4, 2)$           | -16       | 21   | $(0, 0, 0)$      | $(13, 11, 8, 5)$         | -34       | 199  | $(1, 1, 0)$      |
| $(6, 6, 5, 3)$           | 6         | 1    | $(1, 1, 1)$      | $(12, 12, 8, 4)$         | 0         | 2    | $(0, 0, 0)$      |
| $(7, 6, 5, 3)$           | -4        | 3    | $(0, 1, 1)$      | $(12, 11, 8, 5)$         | -12       | 14   | $(0, 2, 0)$      |
| $(7, 7, 5, 3)$           | 2         | 1    | $(2, 0, 1)$      | $(12, 10, 8, 6)$         | 0         | 2    | $(0, 0, 4)$      |
| $(8, 7, 5, 3)$           | -10       | 9    | $(1, 0, 1)$      | $(14, 11, 8, 4)$         | -46       | 702  | $(0, 1, 0)$      |
| $(9, 7, 5, 3)$           | -18       | 32   | $(0, 0, 1)$      | $(13, 12, 8, 4)$         | -22       | 48   | $(3, 0, 0)$      |
| $(8, 7, 6, 3)$           | -4        | 3    | $(0, 2, 0)$      | $(13, 11, 8, 5)$         | -28       | 99   | $(1, 0, 2)$      |
| $(8, 8, 6, 3)$           | 2         | 1    | $(2, 1, 0)$      | $(12, 11, 9, 5)$         | -6        | 6    | $(1, 2, 1)$      |
| $(8, 7, 6, 4)$           | 2         | 1    | $(0, 1, 2)$      | $(14, 12, 8, 4)$         | -40       | 378  | $(0, 2, 0)$      |
| $(9, 8, 6, 3)$           | -12       | 14   | $(1, 1, 0)$      | $(14, 11, 8, 5)$         | -40       | 378  | $(0, 0, 2)$      |
| $(10, 8, 6, 3)$          | -22       | 48   | $(0, 1, 0)$      | $(13, 11, 9, 5)$         | -22       | 48   | $(0, 2, 1)$      |
| $(9, 9, 6, 3)$           | 0         | 2    | $(3, 0, 0)$      | $(12, 12, 9, 5)$         | 6         | 1    | $(3, 1, 1)$      |
| $(9, 8, 6, 4)$           | -6        | 6    | $(1, 0, 2)$      | $(12, 11, 9, 6)$         | 6         | 1    | $(1, 1, 3)$      |
| $(10, 9, 6, 3)$          | -16       | 21   | $(2, 0, 0)$      | $(15, 12, 8, 4)$         | -54       | 1559 | $(1, 0, 0)$      |
| $(10, 8, 6, 4)$          | -16       | 21   | $(0, 0, 2)$      | $(13, 12, 9, 5)$         | -16       | 21   | $(2, 1, 1)$      |
| $(9, 8, 7, 4)$           | 0         | 2    | $(0, 2, 1)$      | $(13, 11, 9, 6)$         | 0         | 10   | $(0, 1, 3)$      |
| $(11, 9, 6, 3)$          | -28       | 99   | $(1, 0, 0)$      | $(16, 12, 8, 4)$         | -64       | 3786 | $(0, 0, 0)$      |
| $(9, 9, 7, 4)$           | 6         | 1    | $(2, 1, 1)$      | $(14, 12, 9, 5)$         | -34       | 199  | $(1, 1, 1)$      |
| $(12, 9, 6, 3)$          | -36       | 258  | $(0, 0, 0)$      | $(13, 13, 9, 5)$         | 2         | 1    | $(4, 0, 1)$      |
| $(10, 9, 7, 4)$          | -10       | 9    | $(1, 1, 1)$      | $(13, 12, 10, 5)$        | -4        | 3    | $(1, 3, 0)$      |
| $(11, 9, 7, 4)$          | -22       | 48   | $(0, 1, 1)$      | $(13, 12, 9, 6)$         | -4        | 3    | $(2, 0, 3)$      |

### 3. $G_2^H$ sigma model and $D = 5$ supergravity

#### 3.1. $G_2^H$ generators for $\ell \leq 3$

In [16] the comparison was made between the equations of motion of the $E_{10}/K(E_{10})$ sigma model and those of $D = 11$ supergravity, in which the decomposition of $E_{10}$ under $A_9$ representations was used to show their matching up to ‘level’ $\ell \leq 3$. In this section, we will do a similar analysis for a $G_2^H/K(G_2^H)$ sigma model and $D = 5$ supergravity to find, again, a very similar result.

We write, as in the case of $A_9$ in $E_{10}$, the generators of the $A_3$ subalgebra as

$$[K^a, b, K^c, d] = \delta^c_b K^a_{d} - \delta^d_b K^c_{a}$$

with, in our case, $a, b, \ldots = 1, \ldots, 4$. We also take two conjugate $A_3 = sl(4)$ vectors $E^a$, $F_a$ transforming

$$[K^a, b, E^c] = \delta^c_b E^a, \quad [K^a, b, F_c] = -\delta^c_b F_a$$
where $K = K^1 + \cdots + K^4$. They are identified as the elements of $G^H_2$ which belong to the root spaces with $\ell = \pm 1$. One can then take the Chevalley generators as

\begin{align}
\epsilon_0 &= F_1, & h_0 &= -3K^1 + K, \\
\epsilon_i &= K^i_{i+1}, & f_i &= K^{i+1}_i, & h_i &= K^i_i - K^{i+1}_{i+1}
\end{align}

(i = 1, 2, 3). Note that no summation is taken over the repeated indices in the definition of $h_i$. We further define the $\ell = \pm 2$ and $\pm 3$ generators as

\begin{align}
E_{ab} &= [E^a, E^b], & F_{ab} &= -[F^a, F^b], \\
E_{[a|b]} &= [E^a, E^b], & F_{[a|b]} &= -[F^a, F^b].
\end{align}

then

\begin{align}
[F^a, E^{b_1 b_2}] &= 8\delta^b_a E^{b_1 b_2}, \\
[F^a, E^{[b_1 b_2]}] &= -16 \left( 3\delta^b_{[a_1} K^{b_2]} - \delta^{b_1 b_2}_{a_1 a_2} K \right), \\
[F^a, E^{b_1 [b_2}] &= 2 \left( \delta^b_a E^{b_1 b_2} - \delta^{b_1 b_2}_a E^{b_1 b_2} \right), \\
[F^a, E^{b_1 b_2}] &= 16 \left( \delta^b_{[a_1} \delta^{b_2}_{a_2]} E^{b_1 b_2} - \delta^{[b_1 b_2]} a_1 a_2 E^{b_1 b_2} \right), \\
[F^a, E^{[b_1 b_2]}] &= 48 \left[ 3 \left( \delta^c_{a_1} \delta^{b_1}_{a_2} K^{b_2}_{a_3} - \delta^c_{a_1} \delta^{b_2}_{a_2} K^{b_1}_{a_3} - \delta^{b_1 b_2}_{a_1 a_2} K^{c}_{a_3} + \delta^{b_1 b_2}_{a_1 a_2} K^{c}_{a_3} \right) + \left( \delta^c_{a_1} \delta^{b_1}_{a_2} \delta^{b_2}_{a_3} - \delta^c_{a_1} \delta^{b_1}_{a_2} \delta^{b_2}_{a_3} \right) K \right].
\end{align}

The last equation can be conveniently written as

\begin{align}
[F^a, E^{[b_1 b_2]}] &= 48 \left( 3X_{c[a_1} K^{b_2]} a_3 + X_{c[a_1} a_2 K + 3X_{a_1[a_2]} b K c - X_{a_1[a_2]} K \right),
\end{align}

for $X^a_{[a_1 b]} = 0$.

### 3.2 $G^H_2$ sigma model equations of motion

Let $\theta$ be the Chevalley involution and define the transpose operation $T$ as $T = -\theta$. Let $\mathcal{V}(t)$ be a formal exponentiation of a $t$-dependent element of $G^H_2$. Let

\begin{align}
\mathcal{V}^{-1} \delta \mathcal{V} = \mathcal{L} + \mathcal{P}_t
\end{align}

with $\mathcal{L} = -\mathcal{D}_t$, $\mathcal{P}_t = \mathcal{P}_t$. Using the invariant bilinear form $\langle \cdot, \cdot \rangle$ of the Kac–Moody algebra, we define the standard coset $G^H_2 / K (G^H_2)$ sigma model Lagrangian as $[14, 16, 21]$

\begin{align}
\mathcal{L} = \frac{1}{2} \mathcal{V}^{-1} \langle \mathcal{P}_t, \mathcal{P}_t \rangle
\end{align}

with the lapse parameter $n^{-1}$, where $G^H_2$ is the corresponding Kac–Moody group, and $K (G^H_2)$ is the (formal) maximal compact subgroup whose Lie algebra is spanned by the ‘antisymmetric’ elements with respect to the above-defined transposition $T$. The equation of motion derived from this Lagrangian is

\begin{align}
n \delta_t (n^{-1} \mathcal{L}) + [\mathcal{D}_t, \mathcal{P}_t] = 0.
\end{align}

where

\begin{align}
\mathcal{D}_t &= Q_t^{(0)} \ast L + \frac{1}{2} P_t^{(1)} \ast (E^{(1)} - F^{(1)}) + \cdots, \\
\mathcal{P}_t &= P_t^{(0)} \ast S + \frac{1}{2} P_t^{(1)} \ast (E^{(1)} + F^{(1)}) + \cdots,
\end{align}

(a = 1, \ldots, 4). The relation between $F^a$ and $E^b$ is

\begin{align}
[F^a, E^b] = -3K^b + \delta^b_a K.
\end{align}
with \( L_{ab} = \frac{1}{2}(K^a_b - K^b_a) \), \( S_{ab} = \frac{1}{2}(K^a_b + K^b_a) \), and omitting the subscript \( t \)

\[
\begin{align*}
\mathcal{Q}^{(0)} & \ast L = \mathcal{Q}_a^{(0)} L_{ab}, \\
p^{(0)} & \ast S = P^{(0)}_{ab} S_{ab}, \\
p^{(1)} & \ast E^{(1)} = P_a^{(1)} E^a, \\
p^{(2)} & \ast E^{(2)} = \frac{1}{2} P^{(2)}_{ab} E^{ab}, \\
p^{(3)} & \ast E^{(3)} = \frac{1}{3!} P^{(3)}_{a_{[bc]}^e} E^{a_{|bc|}.}
\end{align*}
\]

(3.12)

\( \mathcal{D}^{(0)} \) is defined to act on an \( A_3 \) vector \( V_a \) as

\[
\begin{align*}
\mathcal{D}^{(0)} V_a & \equiv \partial V_a + (Q^{(0)}_{ab} - P^{(0)}_{ab}) V_b, \\
\end{align*}
\]

(3.13)

and extends to tensors in the same way as an ordinary covariant derivative. There is no distinction between the upper and lower indices. Then we have the following:

\(
\ell = 0 \) equation

\[
\begin{align*}
\mathcal{D}^{(0)} \left( \mathcal{D}^{(0)} \right)^{n-1} P^{(0)}_{ab} & = -\frac{3}{2} \left( p^{(1)}_{a_{[c]}^e} - \frac{1}{2} \delta_{ab} p^{(1)}_{c} \right) - 6 \left( p^{(2)}_{a_{[c]}^e} - \frac{1}{2} \delta_{ab} p^{(2)}_{c} \right) P^{(2)}_{cb} \\
& - 2 P^{(3)}_{a_{[c]}^e} + P^{(3)}_{a_{[bc]}^e} + \delta_{ab} P^{(3)}_{c} P^{(3)}_{d}.
\end{align*}
\]

(3.14)

\( \ell = 1 \) equation

\[
\begin{align*}
\mathcal{D}^{(0)} \left( \mathcal{D}^{(0)} \right)^{n-1} P^{(1)}_{ab} & = -4 P^{(2)}_{a_{[c]}^e} + 4 P^{(3)}_{a_{[bc]}^e} P^{(2)}_{ab}.
\end{align*}
\]

(3.15)

\( \ell = 2 \) equation

\[
\begin{align*}
\mathcal{D}^{(0)} \left( \mathcal{D}^{(0)} \right)^{n-1} P^{(2)}_{ab} & = P^{(3)}_{a_{[bc]}^e} P^{(1)}_{e}.
\end{align*}
\]

(3.16)

and also the trivial \( \ell = 3 \) equation \( \mathcal{D}^{(0)} \left( \mathcal{D}^{(0)} \right)^{n-1} P^{(3)}_{a_{[bc]}^e} = 0. \)

3.3. Comparison with \( D=5 \) supergravity

The bosonic Lagrangian for \( D = 5, \mathcal{N} = 2 \) pure gravity is

\[
\mathcal{L}_{\text{SUGRA}} = E \left( R - \frac{3}{2} F_{MN} F^{MN} \right) - \frac{1}{2} \epsilon^{MPQR} F_{MN} F_{PQ} A_R.
\]

(3.17)

Here \( M, N, \ldots \) are the \( D = 5 \) curved indices. Below we also use the flat indices \( A, B, \ldots = 0, \ldots, 4 \). We have taken an unconventional normalization for the vector kinetic term so that the equations of motion are simplified. The relevant equations are as follows:

\( \text{Einstein’s equation} \)

\[
R_{AB} = \frac{3}{2} F_A^C F_{BC} - \frac{1}{2} \eta_{AB} F^2.
\]

(3.18)

\( \text{Maxwell’s equation} \)

\[
D_A F^{AB} = \frac{1}{2} \epsilon^{BCDEF} F_{CD} F_{EF}.
\]

(3.19)

\( \text{Bianchi identity} \)

\[
D_A F_{BC} = 0.
\]

(3.20)

We take the pseudo-Gaussian gauge

\[
E_M^A = \begin{pmatrix} N & 0 \\ 0 & e^a_m \end{pmatrix}
\]

(3.21)
(det $e^a_m = e$, $n = Ne^{-1}$; $m$ and $a = 1, \ldots, 4$ are the four-dimensional curved and flat indices, respectively) and make the following identification:

$$P_{ab}^{(0)} = N \omega_{ab}^0, \quad Q_{ab}^{(0)} = N \omega_{ab}^0, \quad P_{a}^{(1)} = N F_{a},$$

(3.22)

The coefficients of anholonomy $\Omega_{AB}^C \equiv e, n \equiv \Omega_{n}^\equiv N e_{m}^n$ and $a = 1, \ldots, 4$ are the four-dimensional curved and flat indices, respectively) and make the following identification:

$$P_{0}^{ab} = N \omega_{ab}^0, Q_{0}^{ab} = N \omega_{ab}^0, P_{a}^{(1)} = N F_{a},$$

(3.22)

The coefficients of anholonomy $\Omega_{ABC} \equiv 2E_{A}^{\mu}E_{B}^{\nu}\partial_M E_{N}^l C$ are decomposed as

$$\Omega_{abc} \equiv \tilde{\Omega}_{abc} + \frac{1}{3} \Omega_{abc} \delta_{bc}, \quad \Omega_{a} \equiv \Omega_{abc} \delta_{bc}$$

(3.23)

and set $\Omega_a = 0$ as was done in [16]. Strictly speaking, four among the 24 components of $P_{a}^{(3)}$ cannot be identified in the last equation of (3.22) since the corresponding roots are zero norm, and therefore the complete matching fails above height $\geq 6$. Up to this incompleteness, it can be shown that

- Einstein’s equation coincides with the $\ell = 0$ equation up to terms $-\frac{1}{2} (\partial_c \tilde{\Omega}_{abc} + \partial_c \tilde{\Omega}_{bca} + \tilde{\Omega}_{a} \delta_{bc})$ (extra in $DF$);
- Maxwell’s equation coincides with the $\ell = 1$ equation up to a term $-N^{-1} \partial_b (N \omega_{b})$ (extra in $DF$);
- the Bianchi identity coincides with the $\ell = 2$ equation up to a term $2N^{-1} \partial_a (N \omega_{a})$ (extra in $DF$).

All the terms that do not match are similar in their structure to those in the $D = 11$ case, and can be regarded as higher level contributions by the same scaling argument as in [16].

4. Singlet representations and higher order corrections

4.1. Wall forms for higher order corrections

In the BKL limit, the cosmological billiard of $D = 5$ supergravity coincides with a scaling limit of the $\mbox{G}^{H}_{2}$ sigma model. This follows from the general theorem for the cosmological billiard [6] proven using the Iwasawa decomposition:

$$e^a_m = \exp(-\beta^a) \cdot N^a_m$$

(4.1)

(m = 1, \ldots, 4). In the limit, the off-diagonal degrees of freedom $N^a_m$ tend to freeze asymptotically, leaving a one-dimensional sigma model with metric

$$\sum_{a,b=1}^{4} G_{ab} \beta^a \beta^b = \sum_{a=1}^{4} (\beta^a)^2 - \left( \sum_{a=1}^{4} \beta^a \right)^2,$$

(4.2)

which is coupled to sharp ‘wall’ potentials forming a billiard. The inverse metric $G^{ab}$ is transformed by the ‘wall form matrix’ [31]

$$U_{ia} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

(4.3)

(i = 0, 1, 2, 3; a = 1, 2, 3, 4) to the matrix of bilinear pairing of $G^{H}_{2}$ roots

$$\sum_{a,b=1}^{4} U_{ia} U_{jb} G^{ab} = (\alpha_i | \alpha_j),$$

(4.4)

7 In this section, summation is not taken over the repeated indices but written explicitly.
where the lhs is given by (2.2) with a normalization \((\alpha_1|\alpha_1) = 2\). In the BKL limit, the leading behaviour of the supergravity metric is governed by a billiard motion in a chamber enclosed by sharp exponential potentials of the wall forms given by \(U_{ia}\) \[31\].

On the other hand, in the analysis of the \(E_{10}\) sigma model, the higher order curvature corrections \(ER^1, ER^2, \ldots\) are identified to correspond to some negative roots of \(E_{10}\) which are singlets under the \(A_3\) decomposition. The identification was made by estimating the scaling behaviour of the terms of the form \(ER^N\). In our \(D = 5\) supergravity case, the contribution of these terms is similarly estimated to be

\[
ER^N \propto \exp(2(N - 1)\sigma), \quad \sigma = \sum_{a=1}^{4} \beta^a.
\]  

By a change of basis using \(U_{ia}\), the linear form \(\sigma\) can be written in terms of simple roots of \(G_2^H\) as

\[
\sigma = 4\alpha_0 + 3\alpha_1 + 2\alpha_2 + 1\alpha_3.
\]  

Therefore \((N - 1)\sigma\) is always on the root lattice for any \(N \in \mathbb{Z}\). Since our linear form \(\sigma\) is invariant under permutation of spatial indices, as is the case for the \(D = 11\) supergravity billiard, we expect that \((N - 1)\sigma\) correspond to singlets in the \(A_3\) decomposition for the correspondence between \(D = 5\) supergravity and the \(G_2^H\) sigma model to hold. Amazingly, the series of singlets that we found in the last section have precisely such root coefficients \((\ell, m^1, m^2, m^3) = (4k, 3k, 2k, k)\)!

4.2. Outer multiplicities

For a given root of \(G_2^H\), its outer multiplicity is computed as its root multiplicity minus the sum of weight multiplicities of representations which contains the root as a non-highest weight. For example, at \(\ell = 4\) there is a root \((\ell, m^1, m^2, m^3) = (4, 3, 2, 1)\), which is an \(A_3\) singlet \((p_1, p_2, p_3) = (0, 0, 0)\) with root multiplicity 3. But at the same level there is another root \((\ell, m^1, m^2, m^3) = (4, 4, 3, 2)\) with Dynkin label \((p_1, p_2, p_3) = (1, 0, 1)\), and, in fact, the representation with highest weight \((1, 0, 1)\) has a weight \((0, 0, 0)\) with weight multiplicity 3, implying that the outer multiplicity of \((4, 3, 2, 1)\) is zero.

On the other hand, at \(\ell = 8\) the root \((\ell, m^1, m^2, m^3) = (8, 6, 4, 2)\) has root multiplicity 21. Other (positive, \(A_3\)-dominant) roots at \(\ell = 8\) are

| \((\ell, m^1, m^2, m^3)\) | \((p_1, p_2, p_3)\) | Root multiplicity | Weight contained |
|-----------------|-----------------|-----------------|-----------------|
| (8, 6, 4, 2)    | (0, 0, 0)       | 21              | \((0, 0, 0)\)   |
| (8, 7, 5, 3)    | (1, 0, 1)       | 9               | \((1, 0, 1) + 2(0, 0, 0)\) |
| (8, 7, 6, 3)    | (0, 2, 0)       | 3               | \((0, 2, 0) + 2(1, 0, 1) + 3(0, 0, 0)\) |
| (8, 8, 6, 3)    | (2, 1, 0)       | 1               | \((0, 2, 0) + 2(1, 0, 1) + 3(0, 0, 0)\) |
| (8, 7, 6, 4)    | (0, 1, 2)       | 1               | \((0, 2, 0) + 2(1, 0, 1) + 3(0, 0, 0)\) |

From the two rows at the bottom one can see that the outer multiplicity of \((0, 2, 0)\) is \(3 - 1 - 1 = 1\). Then the outer multiplicity of \((1, 0, 1)\) is \(9 - 2 - 2 - 1 = 4\). Finally, the outer multiplicity of the singlet is computed to be \(21 - 3 - 2 - 4 \times 3 = 1\). Thus we see that there is a singlet at \(\ell = 8\).
In principle, one can similarly proceed to higher levels, but the computation becomes more tedious. Alternatively, one can use the equations of (11.11) in [33] to count the outer multiplicities directly. Namely, we expand the ‘denominator’ \( F \) defined as\(^8\) 
\[
R = \prod_{\alpha > 0} (1 - e^{-\alpha})^{\text{mult}(\alpha)},
\]
\[
F = -\log(e^{\rho} R),
\]
not in the monomials of exponential of roots, but in the \( A_3 \) irreducible characters directly. To do this, we first write \( F \) as 
\[
F = -\rho + \sum_{\alpha > 0} \text{mult}(\alpha) \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\alpha}
\]
\[
= -\rho + \sum_{n=1}^{\infty} \left( X | e^{-\alpha_1} e^{-\alpha_2} \right),
\]
(4.7)
where 
\[
X = \sum_{\alpha > 0} \text{mult}(\alpha) e^{-\alpha}.
\]

We separate the level \( \ell = 0 \) piece \( X_0 \) from the rest \( X - X_0 \); the former is given by 
\[
X_0 = e^{-\alpha_1} + e^{-\alpha_2} + e^{-\alpha_1-\alpha_2} + e^{-\alpha_1} + e^{-\alpha_2} + e^{-\alpha_1-\alpha_2},
\]
while the latter is invariant under the action of the \( A_3 \) Weyl subgroup and is expanded as 
\[
X - X_0 = M_{(1,0,0,0)} \Theta_{(1,0,0,0)} + M_{(0,1,0,0)} \Theta_{(0,1,0,0)} + \cdots,
\]
(4.10)
where 
\[
\Theta_{(i,j,k,n)} = \frac{\mathcal{D}_{(i+k+n,j+k+n,k+n,n)}}{\mathcal{D}_{(0,0,0,0)}}, \quad \mathcal{D}_{(n_1,n_2,n_3,n_4)} = \det \left( x_{i,j,k,n}^{n_1+n_2+n_3+n_4} \right)
\]
(4.12)
is the \( GL(4) \) character associated with the partition \( (i + j + k + n + 3, j + k + n + 2, k + n + 1, n) \). If we use the relation 
\[
(x_1, x_2, x_3, x_4) = (e^{-\alpha_0}, e^{-\alpha_1-\alpha_2}, e^{-\alpha_0-\alpha_1-\alpha_2}, e^{-\alpha_0-\alpha_1-\alpha_2-\alpha_3})
\]
in (4.12), then we find that \( \Theta_{(p_1,p_2,p_3)} \) is precisely the \( A_3 \) character of Dynkin label \( (p_1, p_2, p_3) \) at level \( -\ell = -(p_1 + 2p_2 + 3p_3 + 4p_4) \). Thus we can write \( F \) as a linear function of unknown coefficients \( M_{(p_1,p_2,p_3,p_4)} \). Plugging this expression into the relation [33] 
\[
\sum_{i,j=0}^{3} B_{ij} \left( \frac{\partial F}{\partial \alpha_i} \frac{\partial F}{\partial \alpha_j} - \frac{\partial^2 F}{\partial \alpha_i \partial \alpha_j} \right) = (\rho|\rho)
\]
(4.14)
with 
\[
B_{ij} = (\alpha_i|\alpha_j) = \begin{pmatrix}
\frac{3}{2} & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}
\]
(4.15)
and solving it by induction on the height, we can compute outer multiplicities of roots directly. In this way, we have computed outer multiplicities of all the \( A_3 \) representations appearing up to height \( \leq 60 \). Since it would not be very illuminating to list a number of pages of data on the decomposition, we will only display the list of outer multiplicities of the singlets in table 3.

\(^8\) In exercise 11.11 of [33], \( R \) should read \( e^{\rho} R \), while it is correct in first and second editions.
invariant containing it will also have this formula one can confirm that, unlike \( D \) does not vanish for \( D = 5 \) pure supergravity. Thus the correspondence between a singlet and a quantum correction fails at \( \ell = 4 \), although there are three roots there. Despite this, however, the other higher level singlets which otherwise exist seem remarkable, and we may say that the hyperbolic Kac–Moody algebra \( G_2^H \) has much chance of playing a crucial role in \( D = 5 \) pure supergravity as \( E_{10} \) has been conjectured to govern the dynamics of M-theory.

Of course, \( D = 5 \) supergravity is a non-renormalizable theory with little power of prediction for higher curvature corrections. If the conjecture is also true in \( D = 5 \), what is the quantum counterpart of it? Although it is not so far known how to realize \( D = 5 \) pure supergravity as a string compactification or any strong coupling limit thereof [38, 39], turning on flux might cure this problem. More ambitiously, in view of the strong resemblance to the

### Table 3. Outer multiplicities of the \( A_3 \) singlets.

| Height | \((\ell, m^1, m^2, m^3)\) | \((p_1, p_2, p_3)\) | Outer multiplicity |
|--------|-----------------|-----------------|-------------------|
| 10     | (4, 3, 2, 1)    | (0, 0, 0)       | 0                 |
| 20     | (8, 6, 2, 2)    | (0, 0, 0)       | 1                 |
| 30     | (12, 9, 2, 3)   | (0, 0, 0)       | 7                 |
| 40     | (16, 12, 2, 4)  | (0, 0, 0)       | 59                |
| 50     | (20, 15, 2, 5)  | (0, 0, 0)       | 549               |
| 60     | (24, 18, 2, 6)  | (0, 0, 0)       | 5924              |

5. Conclusions

We have studied a one-dimensional sigma model associated with the hyperbolic Kac–Moody algebra \( G_2^H \), and its possible link to \( D = 5 \), \( N = 2 \) pure supergravity. We have confirmed the matching between the bosonic equations of motion and the Bianchi identity for \( D = 5 \) pure supergravity and the equations of the \( G_2^H \) sigma model with levels truncated to \( \ell \leq 3 \), to the same extent as the matching checked for the \( D = 11 \) supergravity and the \( E_{10} \) sigma model, in the sense that the terms that do not match have similar structures in both models and can be regarded as coming from roots with higher levels.

We have also studied the \( A_3 \) decomposition of \( G_2^H \) at low levels. We have found three (presumably infinite) towers of roots which can be identified as the spatial gradients of the three lowest level fields. This is, again, the same observation as was already seen in the \( A_9 \) decomposition of \( E_{10} \). We have also found a (again, presumably infinite) series of singlets at levels \( \ell = 4k \) for \( k = 2, 3, \ldots \), which is consistent with the scaling of the higher curvature corrections of the form \( E R^{k+1} \). This is a reasonable result because, in contrast to the M-theory case, we expect corrections of the form \( E R^{k+1} \) for every positive integer \( k \) in \( D = 5 \) supergravity. Thus \( D = 5 \) pure supergravity is the first example of theories with less supercharges than \( D = 11 \) supergravity/M-theory that shows evidence of the link between the higher order corrections and infinitely many roots of a certain Kac–Moody algebra.

However, what is puzzling is the absence of any singlets at \( \ell = 4 \), since this would predict the absence of \( E R^2 \) corrections in \( D = 5 \) pure supergravity. The anomaly cancellation argument [34] analogous to the \( D = 11 \) case requires that in \( D = 5 \) pure supergravity there must be a gravitational Chern–Simons coupling proportional to \( A \wedge \text{Tr} R^2 \) [28] (such a term is also known to arise in Calabi–Yau compactifications of M-theory [35, 36]) and the super-invariant containing it will also have \( E R^2 \) terms. More explicitly, a general formula for 1-loop divergences in supergravity in diverse dimensions has been known for a long time [37]. Using this formula one can confirm that, unlike \( D = 11 \) supergravity, the coefficient \( a_1 \) of \( R^2_{MNPQ} \) does not vanish for \( D = 5 \) pure supergravity. Thus the correspondence between a singlet and a quantum correction fails at \( \ell = 4 \), although there are three roots there. Despite this, however, the other higher level singlets which otherwise exist seem remarkable, and we may say that the hyperbolic Kac–Moody algebra \( G_2^H \) has much chance of playing a crucial role in \( D = 5 \) pure supergravity as \( E_{10} \) has been conjectured to govern the dynamics of M-theory.
supergravity of distinguished character, it would be interesting to see if it can be realized as a matrix-like theory [40, 41] with quarter supercharges.

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