ASYMPTOTIC FREEDOM OF RANK 4 TENSOR GROUP FIELD THEORY

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Recently, a rank four tensor group field theory has been proved renormalizable. We provide here the key points on the renormalizability of this model and its UV asymptotic freedom.

Keywords: Random tensors; Renormalization; β-function; Asymptotic freedom.

1. Introduction

Random tensor models have been introduced in the early 90’s as a natural generalization of matrix models. Random matrices defined a very successful framework for addressing the quantization of gravity in 2D via random triangulations. Until recently, the essential tool used for achieving all analytical results in matrix models, namely the t’Hooft 1/N expansion, was crucially missing for higher rank extension of these models. Indeed, Gurau for a particular class of models called colored succeeded in defining such a 1/N expansion. Important results follow rapidly this breakthrough. To mention a few of these, the statistical analysis of the colored models shows a phase transition with computable critical exponent, and universal behavior of these tensors find a sense, and, finally, one defines the first renormalizable tensor models of rank higher than 2. We refer such a category of quantum field theory to as Tensor Group Field Theories (TGFT’s).

The present work concerns the first of these renormalizable models addressing the generation of 4D simplicial pseudo-manifold via a quantum...
field theoretical formalism. We provide here an overview of the renormalizability\textsuperscript{18} and UV asymptotic freedom\textsuperscript{20} of this model. Despite the fact that a contact with a quantization of gravity is not yet understood at this stage, such a framework is definitely useful for addressing future investigations on emergent geometries.\textsuperscript{1,4} 

2. Renormalizability of the model

Consider a complex rank four tensor over the group $U(1)$, $\varphi : U(1)^4 \rightarrow \mathbb{C}$ which admits the decomposition in Fourier modes as

$$\varphi(h_1, h_2, h_3, h_4) = \sum_{p_j \in \mathbb{Z}} \varphi[p_1, p_2, p_3, p_4] e^{ip_1 \theta_1} e^{ip_2 \theta_2} e^{ip_3 \theta_3} e^{ip_4 \theta_4}$$

where $h_i \in U(1)$, $\theta_i \in [0, 2\pi)$ and $p_j$ are momentum indices. We denote $\varphi[p_1, p_2, p_3, p_4] = \varphi_{1,2,3,4}$ and assume no symmetry under permutation of the arguments for this tensor.

The kinetic term of the model, written in momentum space, reads

$$S^{\text{kin, 0}} = \sum_{p_j} \varphi_{1,2,3,4} \left[ \sum_{s=1}^{4} (p_s)^2 + m^2 \right] \varphi_{1,2,3,4}$$

where the sum is performed over all momentum values $p_j$. Clearly, such a kinetic term is inferred from a Laplacian dynamics which can be motivated in different ways.\textsuperscript{1,24} To this kinetic term, we associate a Gaussian measure $d\mu_C[\varphi]$ with covariance $C = (\sum_s p_s^2 + m^2)^{-1}$.

Trace invariant objects\textsuperscript{16} define the interaction part of the model:

$$S_{6;1} = \sum_{p_j} \varphi_{1,2,3,4} \bar{\varphi}_{1',2,3,4} \varphi_{1',2',3',4'} \bar{\varphi}_{1''2'',3'',4''} + \text{perm.}$$

$$S_{6;2} = \sum_{p_j} \varphi_{1,2,3,4} \bar{\varphi}_{1',2',3',4'} \varphi_{1',2',3',4'} \bar{\varphi}_{1''2'',3'',4''} + \text{perm.}$$

$$S_{4;1} = \sum_{p_j} \varphi_{1,2,3,4} \bar{\varphi}_{1',2,3,4} \varphi_{1',2',3',4'} \bar{\varphi}_{1',2',3',4'} + \text{perm.}$$

$$S_{4;2} = \left[ \sum_{p_j} \bar{\varphi}_{1,2,3,4} \varphi_{1,2,3,4} \right] \left[ \sum_{p_j'} \bar{\varphi}_{1',2',3',4'} \varphi_{1',2',3',4'} \right]$$

where the sum (+perm.) is over all 24 permutations of the four color indices. The flow of $S_{6;2}$ generates a $\phi^4$-type interaction that we will refer to as anomalous term (5) and so should be part of our action.

Feynman graphs have a tensor structure. Lines are four-stranded and represent propagators (Figure 1) whereas vertices are nonlocal objects (as
Fig. 1. The propagator.

depicted in Figure 2 and Figure 3).

Fig. 2. Vertices of the type $\phi_6^{(1)}$ from $S_{6;1}$ (left) and of the type $\phi_6^{(2)}$ from $S_{6;1}$ (right).

Fig. 3. Vertices of the type $\phi_4^{(1)}$ from $S_{4;1}$ (left) and the anomalous term $\phi_4^{(2)} = (\phi^2)^2$ from $S_{4;2}$ (right).

Introducing an ultraviolet cutoff $\Lambda$ and counterterms, one proves\textsuperscript{18} that the model described by (2) and (3)-(5) is renormalizable at all orders of perturbation theory. This statement is proved by multiscale analysis\textsuperscript{23} and the study of the graph topology. We review now the main ingredients of this proof.

- A slice decomposition: Use the Schwinger representation of the propagator $C = \int_0^\infty e^{-\alpha \left( \sum_i p_i^2 + m^2 \right)} d\alpha$, to slice this quantity as $C = \sum_{i=0}^\infty C_i$ where $C_i = \int_{M^{-2i+2}}^{M^{-2i}} e^{-\alpha \left( \sum_i p_i^2 + m^2 \right)} d\alpha \leq K M^{-2i} e^{M^{-2i} \left( \sum_i p_i^2 + m^2 \right)}$, for $M > 0$.

- The multi-scale expansion of a graph amplitude using the above bound yields the crude power-counting at a momentum attribution $\mu$,
\[ A_{\mathcal{G};\mu} \leq K^n \prod_{i,k} M^{\omega_d(G)} \] where \( K, n \) are some constants, \( G^i_k \) are called the quasi-local subgraphs\(^{23} \) of \( \mathcal{G} \), \( \omega_d(G) = -2L(G) + F_{\text{int}}(G) \) is the degree of divergence of the graph \( G \), \( L(G) \) the number of internal lines and \( F_{\text{int}}(G) \) the number of closed loops or faces of the subgraph \( G \). Quasi-local subgraphs define the generalized notion of locality here.

- A refined power-counting theorem: A graph \( \mathcal{G} \) in this theory admits a five-color extension \( \mathcal{G}_{\text{color}} \) which is itself a rank four tensor graph. A jacket \( J \) of \( \mathcal{G}_{\text{color}} \) is a ribbon subgraph of \( \mathcal{G}_{\text{color}} \) defined by a color cycle \((0abcd)\) up to a cyclic permutation. The jacket \( \tilde{J} \) is the closed jacket obtained from \( J \) after closing all external legs present in \( J \). The boundary \( \partial \mathcal{G} \) of the graph \( \mathcal{G} \) is the closed graph defined by vertices corresponding to external legs and by lines corresponding to external strands of \( \mathcal{G} \).\(^8 \) It is, presently, a vacuum rank 3 colored graph. A boundary jacket \( J_{\partial} \) is a jacket of \( \partial \mathcal{G} \).

The anomalous vertex \((\phi^2)^2\) is disconnected from the point of view of its strands. We define connected component graphs by reducing such vertices to twice \( \phi^2 \)-vertices.

The divergence degree any connected graph \( \mathcal{G} \) is given by

\[
\omega_d(G) = -\frac{1}{3} \left[ \sum_{J} g_J - \sum_{J_{\partial}} g_{J_{\partial}} \right] - \left[ C_{\partial \mathcal{G}} - 1 \right] - V_4 - 2[V_2 + V_2''] - \frac{1}{2} [N_{\text{ext}} - 6] \tag{6}
\]

where \( g_J \) and \( g_{J_{\partial}} \) are the genus of \( J \) and \( J_{\partial} \), respectively, \( C_{\partial \mathcal{G}} \) is the number of connected components of the boundary graph \( \partial \mathcal{G} \); the first sum is performed on all closed jackets \( J \) of \( \mathcal{G}_{\text{color}} \) and the second sum is performed on all boundary jackets \( J_{\partial} \) of \( \partial \mathcal{G} \); \( V_4 \) is the number of \( \phi^4 \) vertices, \( V_2 \) the number of vertices of the type \( \phi^2 \) (mass counterterms), \( V_2'' = 2V_4' \) is twice the number of vertices of type \( \phi^4 \) \((\phi^2)^2\), \( N_{\text{ext}} \) its number of external legs.

One notices that \( \omega_d(G) \) does not depend on vertices of the type \( \phi^6 \).

Finally, the following table lists all primitively divergent graphs:

| \( N_{\text{ext}} \) | \( V_2 + V_2'' \) | \( V_4 \) | \( \sum_{J_{\partial}} g_{J_{\partial}} \) | \( C_{\partial \mathcal{G}} - 1 \) | \( \sum_{J} g_J \) | \( \omega_d(G) \) |
|-----------------|----------------|----------------|------------------|------------------|----------------|----------------|
| 6               | 0              | 0              | 0                | 0                | 0               | 0              |
| 4               | 0              | 0              | 0                | 0                | 0               | 1              |
| 4               | 0              | 1              | 0                | 0                | 0               | 0              |
| 4               | 0              | 0              | 1                | 0                | 0               | 0              |
| 2               | 0              | 0              | 0                | 0                | 0               | 2              |
| 2               | 0              | 1              | 0                | 0                | 0               | 1              |
| 2               | 0              | 2              | 0                | 0                | 0               | 0              |
| 2               | 0              | 0              | 0                | 0                | 0               | 0              |
| 2               | 1              | 0              | 0                | 0                | 0               | 0              |

Table 1: List of primitively divergent graphs

Call graphs satisfying \( \sum_J g_J = 0 \) “melonic” graphs or simply “mel-
ons". Thus, some graphs in Table 1 are melons with melonic boundary.

3. \( \beta \)-functions

We will discuss the \( \beta \)-functions of the model given by the interaction

\[
S = \frac{1}{3} \lambda_{6;1} S_{6;1} + \lambda_{6;2} S_{6;2} + \frac{1}{2} \lambda_{4;1} S_{4;1} + \frac{1}{2} \lambda_{4;2} S_{4;2}
\]

(7)

where some symmetry factors are introduced. It turns out that the behavior of the renormalized coupling constants \( \lambda_{6;1/2}^{\text{ren}} \) in the UV are those significant for this model.

The \( \beta \)-functions of the \( \phi^6 \) are evaluated with two ingredients: (1) the truncated and amputated one particle irreducible (1PI) 6-point functions \( \Gamma_{6;1/2}(\cdot) \) the external data of which are designed in the form of the initial (bare) interaction; (2) the wave function renormalization given by

\[
Z = 1 - \partial_{b_1^2} \Sigma(b_1, b_2, b_3, b_4) = 0,
\]

where \( \Sigma(b_1, b_2, b_3, b_4) = \langle \varphi_{1,2,3,4} \rangle^I_{\text{1PI}} \) is the so-called self-energy or sum of all amputated 1PI two-point functions. We identify all relevant graphs related to the six- and two-point functions using Table 1.

The following ratios encode the \( \beta \)-functions of the coupling constants:

\[
\lambda_{6;1/2}^{\text{ren}} = - \frac{\Gamma_{6;1/2}(0,0,0,0,0,0,0,0,0,0,0,0)}{Z^2}
\]

(8)

At two loops (for the first) and four loops (for the second), one finds that the renormalized coupling constants satisfy the equations

\[
\lambda_{6;2}^{\text{ren}} = \lambda_{6;2} + 2\lambda_{6;2} \lambda_{6;1} S^1 + 3\lambda_{6;2}^2 [S^1 + S^{12}] + O(\lambda^3)
\]

(9)

\[
\lambda_{6;1}^{\text{ren}} = \lambda_{6;1} + 8\lambda_{6;1}^3 S + O(\lambda_{6;1}^4)
\]

(10)

where \( S^{1,12} \) and \( S \) are formal log-divergent sums. The \( \beta \)-functions of this model at this order of perturbation are given by

\[
\beta_{6;2} = 3 \quad \beta_{6;1; (12)} = 2 \quad \beta_{6;1} = 8
\]

(11)

Assuming positive coupling constants, (9)-(11) show that the overall model is asymptotically free in UV.

In fact, the complete study of all renormalized coupling equations shows that there exists a UV fixed manifold for the model given by

\[
\lambda_{6;1/2} = 0 \quad \forall \lambda_{4;1} \quad \lambda_{4;2} = 0
\]

(12)

Perturbing the system around this fixed manifold by adding small quantities to the bare couplings, the renormalized coupling constants increase in the infrared (IR). This fact generally hints at a phase transition, a scenario very promising for finding new models in the continuum.
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