ON THE D-AFFINITY OF FLAG VARIETIES IN POSITIVE CHARACTERISTIC

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Abstract. Let \( G \) be a simple simply connected algebraic group of type \( B_2 \) over an algebraically closed field \( k \) of odd characteristic. We prove that the flag variety \( G/B \) is D-affine. This extends an earlier result of Andersen and Kaneda \[2\].

1. Introduction

Let \( X \) be a smooth algebraic variety over an algebraically closed field \( k \), and let \( \mathcal{D}_X \) be the sheaf of differential operators on \( X \). Then \( X \) is said to be D-affine if the following two conditions hold: (i) for any \( \mathcal{D}_X \)-module \( M \) that is quasi-coherent over \( \mathcal{O}_X \) the natural morphism \( \mathcal{D}_X \otimes_{\mathcal{O}_X} \Gamma(\mathcal{D}_X) \rightarrow \Gamma(M) \) is onto, and (ii) \( H^i(X, \mathcal{D}_X) = 0 \) for \( i > 0 \). Let \( G \) be a semisimple algebraic group over \( k \) and \( P \) a parabolic subgroup of \( G \). If \( k \) is of characteristic zero then the well-known Beilinson–Bernstein localization theorem \[3\] states that \( G/P \) is D-affine. Much less is known when \( k \) is of characteristic \( p > 0 \). Haastert \[4\] showed that projective spaces and the flag variety of the group \( SL_3 \) are D-affine, and Langer \[7\] proved the D-affinity for odd-dimensional quadrics if the characteristic of \( k \) is greater than the dimension of variety (while even-dimensional quadrics turn out to be not D-affine). Even earlier, Kashiwara and Lauritzen \[6\] produced a counterexample to the D-affinity: their result implies that the flag variety of the group \( SL_5 \) is not D-affine in any characteristic. Nevertheless, the question about which flag varieties are D-affine in positive characteristic remains open; nothing was known except the above cases. In the present paper we show that the flag variety of the group \( Sp_4 \) is D-affine in odd characteristic. By \[4\], it is sufficient to prove that \( H^i(Sp_4/B, \mathcal{D}_{Sp_4/B}) = 0 \) for \( i > 0 \). This is achieved by showing that all the terms of the \( p \)-filtration on the sheaf \( \mathcal{D}_{Sp_4/B} \) have vanishing higher cohomology groups, thus extending an earlier result of Andersen and Kaneda \[2\], where they showed the cohomology vanishing of the first term of the \( p \)-filtration (in any characteristic). Contrary to their representation theoretic approach, we use simple geometric arguments to reduce the problem to computing cohomology groups of line bundles on the flag variety \( Sp_4/B \). Cohomology of line bundles on flag varieties in the rank two case are well understood thanks to Andersen’s et al. work (see \[1\] for a recent survey and \[5\] for a comprehensive treatment); working out the cohomology groups in question completes the proof. However, for the sake of consistency with our approach and convenience of the reader, we explicitly show all the necessary vanishings without the use of general theory.

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2. Preliminaries

Let \( k \) be an algebraically closed field of odd characteristic \( p > 0 \), and \( V \) be a symplectic vector space of dimension 4 over \( k \). Let \( G \) be the symplectic group \( \text{Sp}_4 \) over \( k \); the root system of \( G \) is of type \( B_2 \). Let \( B \) be a Borel subgroup of \( G \). Consider the flag variety \( G/B \). The group \( G \) has two parabolic subgroups \( P_\alpha \) and \( P_\beta \) that correspond to the simple roots \( \alpha \) and \( \beta \), the root \( \beta \) being the long root. The homogeneous spaces \( G/P_\alpha \) and \( G/P_\beta \) are isomorphic to the 3-dimensional quadric \( Q_3 \) and \( \mathbb{P}^3 \), respectively. Denote \( q \) and \( \pi \) the two projections of \( G/B \) onto \( Q_3 \) and \( \mathbb{P}^3 \). The line bundles on \( G/B \) that correspond to the fundamental weights \( \omega_\alpha \) and \( \omega_\beta \) are isomorphic to \( \pi^*O_{\mathbb{P}^3}(1) \) and \( q^*O_{Q_3}(1) \), respectively. The canonical line bundle \( \omega_{G/B} \) corresponds to the weight \( -2\rho = -2(\omega_\alpha + \omega_\beta) \) and is isomorphic to \( \pi^*O_{\mathbb{P}^3}(-2) \otimes q^*O_{Q_3}(-2) \). The projection \( \pi \) is the projective bundle over \( \mathbb{P}^3 \) associated to a rank two vector bundle \( N \) over \( \mathbb{P}^3 = \mathbb{P}(V) \), and the projection \( q \) is the projective bundle associated to the spinor bundle \( U_2 \) on \( Q_3 \). The bundle \( N \) is symplectic, that is there is a non-degenerate skew-symmetric pairing \( \wedge^2 N \to O_{\mathbb{P}^3} \) that is induced by the given symplectic structure on \( V \). There is a short exact sequence on \( \mathbb{P}^3 \):

\[
0 \to O_{\mathbb{P}^3}(-1) \to \Omega^1_{\mathbb{P}^3}(1) \to N \to 0,
\]

while the spinor bundle \( U_2 \), which is also isomorphic to the restriction of the rank two universal bundle on \( \text{Gr}_{2,4} = Q_4 \) to \( Q_3 \), fits into a short exact sequence on \( Q_3 \):

\[
0 \to U_2 \to V \otimes O_{Q_3} \to U_2^* \to 0.
\]

Let \( D_{G/B} \) be the sheaf of differential operators on \( G/B \). By Theorem 4.4.1 of [4] flag varieties are quasi \( D \)-affine, that is every \( D \)-module on a flag variety is \( D \)-generated by its global sections. This implies that the \( D \)-affinity of \( G/B \) will follow if the sheaf \( D_{G/B} \) has vanishing higher cohomology groups. The main result of the paper is the following theorem:

**Theorem 2.1.**

\[ H^i(G/B, D_{G/B}) = 0 \]

for \( i > 0 \).

**Proof.** Let \( F^n : G/B \to G/B \) be the \( n \)-th absolute Frobenius morphism. By Theorem 1.2.4 of [4] there is an isomorphism of sheaves \( D_{G/B} = \bigcup_{n \geq 1} \text{End}(F^n_*O_{G/B}) \). Fix \( i \geq 0 \). Clearly, \( H^i(G/B, \text{End}(F^n_*O_{G/B})) = 0 \) for all \( n \geq 1 \) implies \( H^i(G/B, D_{G/B}) = 0 \). The statement will follow from Theorem 2.2 below, whose proof occupies the next two sections.

**Theorem 2.2.**

\[ H^i(G/B, \text{End}(F^n_*O_{G/B})) = 0 \]

for \( i > 0 \) and \( n \geq 1 \).

The method used in [4] and [2] was to identify the sheaf \( \text{End}(F^n_*O_{G/B}) \) with an equivariant vector bundle on \( G/B \) associated to the induced module \( \text{Ind}^{G_n \times B}_{B}(2(p^n - 1)\rho) \), where \( G_n \) is the \( n \)-th Frobenius kernel, and to study an appropriate filtration on such a module. Our main tool is a short exact sequence from [11] that relates the Frobenius pushforwards of the structure sheaves on the total space of a \( \mathbb{P}^1 \)-bundle and on the base variety.
3. Proof of Theorem 2.2

Recall the short exact sequence from [11] mentioned above. Assume given a smooth variety $S$ and a locally free sheaf $\mathcal{E}$ of rank 2 on $S$. Let $X = \mathbb{P}_S(\mathcal{E})$ be the projective bundle over $S$ and $\pi : X \to S$ the projection. Denote $O_{\pi}(-1)$ the relative invertible sheaf. One has $\pi_{\ast}O_{\pi}(1) = \mathcal{E}^\ast$.

**Lemma 3.1.** For any $n \geq 1$ there is a short exact sequence of vector bundles on $X$:

\[
\begin{align*}
0 & \to \pi^\ast F^n_{\pi}O_S \to F^n_{\pi}O_X \to \pi^\ast(F^n_{\pi}(D^{p^n - 2}\mathcal{E} \otimes \det \mathcal{E}) \otimes \det \mathcal{E}^\ast)) \otimes O_{\pi}(-1) \to 0.
\end{align*}
\]

Here $D^k\mathcal{E} = (S^k\mathcal{E})^\ast$ is the $k$-th divided power of $\mathcal{E}$.

For convenience of the reader we recall the proof.

**Proof.** Let $D^k(X)$ be the bounded derived category of coherent sheaves on $X$, and denote $[1]$ the shift functor. By [8], for any object $A \in D^k(X)$ there is a distinguished triangle:

\[
\begin{align*}
\cdots & \to \pi^\ast R^\bullet \pi_{\ast}A \to A \to \pi^\ast(\tilde{A}) \otimes O_{\pi}(-1) \to \pi^\ast R^\bullet \pi_{\ast}A[1] \to \cdots.
\end{align*}
\]

The object $\tilde{A}$ can be found by tensoring the triangle (11) with $O_{\pi}(-1)$ and applying the functor $R^\ast \pi_{\ast}$ to the obtained triangle. Given that $R^\ast \pi_{\ast}O_{\pi}(-1) = 0$, we get an isomorphism:

\[
\begin{align*}
R^\ast \pi_{\ast}A \otimes O_{\pi}(-1)) & \simeq \tilde{A} \otimes R^\ast \pi_{\ast}O_{\pi}(-2).
\end{align*}
\]

One has $R^\ast \pi_{\ast}O_{\pi}(-2) = \det \mathcal{E}[-1]$. Tensoring both sides of the isomorphism (5) with $\det \mathcal{E}^\ast$, we get:

\[
\begin{align*}
\tilde{A} = R^\ast \pi_{\ast}(A \otimes O_{\pi}(-1)) \otimes \det \mathcal{E}^\ast[1].
\end{align*}
\]

Let now $A$ be the vector bundle $F^n_{\pi}O_X$. The triangle (11) becomes in this case:

\[
\begin{align*}
\cdots & \to \pi^\ast R^\bullet \pi_{\ast}F^n_{\pi}O_X \to F^n_{\pi}O_X \to \pi^\ast(\tilde{A}) \otimes O_{\pi}(-1) \to \pi^\ast R^\bullet \pi_{\ast}F^n_{\pi}O_X[1] \to \cdots.
\end{align*}
\]

where $\tilde{A} = R^\ast \pi_{\ast}(F^n_{\pi}O_X \otimes O_{\pi}(-1)) \otimes \det \mathcal{E}^\ast[1]$. Recall that for a coherent sheaf $\mathcal{F}$ on $X$ one has an isomorphism $R^i \pi_{\ast}F^n_{\pi}\mathcal{F} = F^n_{\pi}R^i \pi_{\ast}\mathcal{F}$, the Frobenius morphism being finite and commuting with arbitrary morphisms. Therefore,

\[
R^\ast \pi_{\ast}F^n_{\pi}O_X = F^n_{\pi}R^\ast \pi_{\ast}O_X = F^n_{\pi}O_S.
\]

On the other hand, by the projection formula one has $R^\ast \pi_{\ast}(F^n_{\pi}O_X \otimes O_{\pi}(-1)) = R^\ast \pi_{\ast}(F^n_{\pi}O_{\pi}(-p^n)) = F^n_{\pi}R^\ast \pi_{\ast}O_{\pi}(-p^n)$. The relative Serre duality for $\pi$ gives:

\[
R^\ast \pi_{\ast}O_{\pi}(-p^n) = D^{p^n - 2}\mathcal{E} \otimes \det \mathcal{E}[-1].
\]

Let $\tilde{\mathcal{E}}$ be the vector bundle $D^{p^n - 2}\mathcal{E} \otimes \det \mathcal{E}$. Putting these isomorphisms together we see that the triangle (11) can be rewritten as follows:

\[
\begin{align*}
\cdots & \to \pi^\ast F^n_{\pi}O_S \to F^n_{\pi}O_X \to \pi^\ast(F^n_{\pi}\tilde{\mathcal{E}} \otimes \det \mathcal{E}^\ast)) \otimes O_{\pi}(-1) \to \cdots.
\end{align*}
\]

Therefore, the above distinguished triangle is in fact a short exact sequence of vector bundles on $X$:

\[
\begin{align*}
0 & \to \pi^\ast F^n_{\pi}O_S \to F^n_{\pi}O_X \to \pi^\ast(F^n_{\pi}\tilde{\mathcal{E}} \otimes \det \mathcal{E}^\ast)) \otimes O_{\pi}(-1) \to 0.
\end{align*}
\]

$\blacksquare$
We will use the projection \( q : G/B \to Q_3 \) to compute the bundle \( F^n_x O_{G/B} \). Applying Lemma 3.1 in this case, we get a short exact sequence:

\[
0 \to q^* F^n_x O_{Q_3} \to F^n_x O_{G/B} \to q^*(F^n_x (D^n-2U_2(-1)) \otimes O_{Q_3}(1)) \otimes O_q(-1) \to 0.
\]

Here \( O_q(-1) = \pi^* O_{P^3}(-1) \) is the relative line bundle with respect to the projection \( q \). Apply the functor \( \text{Hom}(-, F^n_x O_{G/B}) \) to this sequence. Consider first the groups \( \text{Ext}^i(q^* F^n_x O_{Q_3}, F^n_x O_{G/B}) \).

By adjunction we get an isomorphism:

\[
\text{Ext}^i(q^* F^n_x O_{Q_3}, F^n_x O_{G/B}) = \text{Ext}^i(F^n_x O_{Q_3}, F^n_x O_{Q_3}).
\]

Indeed, \( R^i q_* F^n_x O_{G/B} = F^n_x R^i q_* O_{G/B} = F^n_x O_{Q_3} \).

**Lemma 3.2.** \( \text{Ext}^i(F^n_x O_{Q_3}, F^n_x O_{Q_3}) = 0 \) for \( i > 0 \) and \( n \geq 1 \).

**Proof.** For \( n = 1 \) this follows from [9]. For quadrics of arbitrary dimension an explicit decomposition of the Frobenius pushforward of a line bundle was found in [7]; in particular, this implies Lemma 3.2. However, it is worth giving an independent proof that is based on the argument from [9]; the proof of Theorem 2.2 is just an extension of it. Recall (Lemma 2.3, *loc.cit.*) that there is an isomorphism of cohomology groups:

\[
\text{Ext}^i(F^n_x O_{Q_3}, F^n_x O_{Q_3}) = H^i(Q_3 \times Q_3, (F^n \times F^n)^*(i_* O_{\Delta}) \otimes (O_{Q_3} \otimes \omega_{Q_3}^{1-p^n}))
\]

There is a resolution of the sheaf \( i_* O_{\Delta} \) (Lemma 3.1, [9]):

\[
0 \to U_2 \otimes U_2(-2) \to \Psi_2 \otimes O_{Q_3}(-2) \to \Psi_1 \otimes O_{Q_3}(-1) \to O_{Q_3} \otimes O_{Q_3} \to i_* O_{\Delta} \to 0,
\]

This is a particular case of Kapranov’s resolution of the diagonal for quadrics. Put \( \Psi_0 = O_{Q_3} \) and \( \Psi_3 = U_2 \). When \( k \) is of characteristic zero, the bundles \( \Psi_i \) for \( i = 1, 2 \) can explicitly be described as follows: the bundle \( \Psi_1 \) is isomorphic to the restriction of \( \Omega^1(1) \) on \( P^4 \) to \( Q_3 \), the quadric \( Q_3 \) being naturally embedded into \( P^4 = P(W) \), and the bundle \( \Psi_2 \) fits into the short exact sequence

\[
0 \to \Omega^2_{P^4}(2) \otimes O_{Q_3} \to \Psi_2 \to O_{Q_3} \to 0.
\]

Let us check that the same descriptions of \( \Psi_1 \) and \( \Psi_2 \) are valid when the characteristic of \( k \) is an odd prime. This amounts to computation of cohomology groups. Indeed, for any coherent sheaf \( E \) on \( Q_3 \) there is a standard spectral sequence converging to \( E \), and whose \( E_1 \)-term is equal to \( H^j(Q_3, E \otimes O_{Q_3}(j)) \otimes \Psi_{-j} \) for \( j = -2, \ldots, 0 \), and \( H^j(Q_3, E \otimes U_2(-2)) \otimes \Psi_{-3} \) for \( j = -3 \). Taking \( E \) to be \( \Omega^1_{P^4}(1) \otimes O_{Q_3} \) or \( \Omega^2_{P^4}(2) \otimes O_{Q_3} \) and computing the terms of spectral sequence we see that the cohomology groups in question are the same as in characteristic zero, thus arriving at the above resolutions for \( \Psi_1 \) and \( \Psi_2 \).

Arguing as in the proof of Theorem 3.2 of [9], we conclude that Lemma 3.2 follows from the following statement:

**Proposition 3.1.** \( H^i(Q_3, F^n U_2) = 0 \) for \( i \neq 2 \) and \( n \geq 1 \).

**Proof.** Denote \( O_{\pi}(-1) \) the relative line bundle with respect to the projection \( \pi : G/B = P(N) \to G/P_\alpha = P^3 \). Consider the short exact sequence

\[
0 \to \pi^* O_{P^3}(-1) \to q^* U_2 \to O_{\pi}(-1) \to 0.
\]
Applying the functor $F^{n*}$ to it we get:

\[(18) \quad 0 \to \pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n) \to q^* F^{n*} \mathcal{U}_2 \to \mathcal{O}_\pi(-p^n) \to 0.\]

First, one has $H^i(G/B, \pi^* \mathcal{O}_{\mathbb{P}^3}(-p^n)) = H^i(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-p^n)) = 0$ for $i \neq 3$. Let us show that $H^i(G/B, \mathcal{O}_\pi(-p^n)) = 0$ for $i < 2$. Indeed, $H^0(G/B, \mathcal{O}_\pi(-p^n)) = 0$, the line bundle $\mathcal{O}_\pi(-k)$ being not effective for any $k$. Let us consider $H^1$. One has $R^i \pi_* \mathcal{O}_\pi(-p^n) = \mathcal{D}^{p^n-2} \mathbb{N}[-1]$. Thus, $H^4(G/B, \mathcal{O}_\pi(-p^n)) = H^0(\mathbb{P}^3, \mathcal{D}^{p^n-2} \mathbb{N})$. For any $k > 0$ there is a short exact sequence on $G/B$:

\[(19) \quad 0 \to \mathcal{O}_\pi(-k) \to \pi^* \mathcal{D}^k \mathbb{N} \to \pi^* \mathcal{D}^{k-1} \mathbb{N} \otimes \mathcal{O}_\pi(1) \to 0.\]

It is obtained from the relative Euler sequence

\[(20) \quad 0 \to \mathcal{O}_\pi(-1) \to \pi^* \mathbb{N} \to \mathcal{O}_\pi(1) \to 0\]

by taking first its $k$-th symmetric power and then passing to the dual (since the bundle $\mathbb{N}$ is symplectic, it is self-dual, that is $\mathbb{N} = \mathbb{N}^*$). We saw above that the line bundle $\mathcal{O}_\pi(-k)$ did not have global sections. Using the sequence (19) for $k = p^n - 2, p^n - 3, \ldots, 1$ and descending induction, we see that $H^0(G/B, \pi^* \mathcal{D}^k \mathbb{N}) = H^0(\mathbb{P}^3, \mathcal{D}^k \mathbb{N}) = 0$. This implies $H^i(\mathbb{Q}_3, F^{n*} \mathcal{U}_2) = 0$ for $i < 2$. By Serre duality $H^3(\mathbb{Q}_3, F^{n*} \mathcal{U}_2) = H^0(\mathbb{Q}_3, F^{n*} \mathcal{U}_2^* \otimes \omega_{\mathbb{Q}_3})$. Recall that $\omega_{\mathbb{Q}_3} = \mathcal{O}_{\mathbb{Q}_3}(-3)$. Dualizing the sequence (18) and tensoring it with $\omega_{\mathbb{Q}_3}$, we see that the bundle $F^{n*} \mathcal{U}_2^* \otimes \omega_{\mathbb{Q}_3}$ is an extension of two line bundles, both of which are non-effective. Thus, $H^3(\mathbb{Q}_3, F^{n*} \mathcal{U}_2) = 0$. Finally, one gets a short exact sequence:

\[(21) \quad 0 \to H^2(\mathbb{Q}_3, F^{n*} \mathcal{U}_2) \to H^2(G/B, \mathcal{O}_\pi(-p^n)) \to H^3(G/B, \pi^* \mathcal{O}(-p^n)) \to 0,\]

and the statement follows.

**Remark 3.1.** The (non)-vanishing of the first cohomology group of a line bundle on arbitrary flag variety was completely determined by H.H.Andersen (cf. \[1\], 2.3). The line bundle $L_\chi = \mathcal{O}_\pi(-p^n)$ corresponds to the weight $\chi = p^n \omega_\alpha - p^n \omega_\beta$. Using Andersen’s criterion one immediately checks the vanishing of $H^1(G/B, L_\chi)$.

Next step is the following vanishing:

**Lemma 3.3.**

\[(22) \quad Ext^i(q^* (F^{n*}_* (\mathcal{D}^{p^n-2} \mathcal{U}_2(-1)) \otimes \mathcal{O}_{\mathbb{Q}_3}(1)) \otimes \mathcal{O}_(-1), F^{n*}_* \mathcal{O}_{G/B}) = 0\]

for $i > 0$ and $n \geq 1$.

Clearly, Lemma [3,2] and Lemma [3,3] will imply Theorem [2,2].

**Proof.** By the projection formula, one has:

\[(23) \quad Ext^i(q^* (F^{n*}_* (\mathcal{D}^{p^n-2} \mathcal{U}_2(-1)) \otimes \mathcal{O}_{\mathbb{Q}_3}(1)) \otimes \mathcal{O}_(-1), F^{n*}_* \mathcal{O}_{G/B}) =
\quad = Ext^i(F^{n*}_* (\mathcal{D}^{p^n-2} \mathcal{U}_2(-1)) \otimes \mathcal{O}_{\mathbb{Q}_3}(1), F^{n*}_* S^{p^n} \mathcal{U}_2).\]

Indeed,

\[(24) \quad R^* q_s (F^{n*}_* \mathcal{O}_{G/B} \otimes \mathcal{O}_(-1)) = R^* q_s (F^{n*}_* \mathcal{O}_{G/B} \otimes \pi^* \mathcal{O}_{\mathbb{P}^3}(1)) = R^* q_s F^{n*}_s \pi^* \mathcal{O}_{\mathbb{P}^3}(p^n) =
\quad = F^{n*}_s R^* q_s \pi^* \mathcal{O}_{\mathbb{P}^3}(p^n) = F^{n*}_s S^{p^n} \mathcal{U}_2.\]
Recall that a right adjoint functor to $F_*^n$ on a smooth variety $X$ over $k$ is given by the formula:

\[(25) \quad F_*^n(?) = F_*^{n*} (?) \otimes \omega_{X}^{1-p^n}.\]

Therefore,

\[(26) \quad Ext^i(F_*^n(D^{p^n-2}U_2(-1)) \otimes O_{\mathcal{Q}_3}(1), F_*^nS^{p^n}U_2^n) =\]

\[\quad = Ext^i(D^{p^n-2}U_2(-1), F_*^{n*}F_*^nS^{p^n}U_2^n \otimes O_{\mathcal{Q}_3}(-p^n) \otimes \omega_{\mathcal{Q}_3}^{1-p^n}).\]

We have $O_{\mathcal{Q}_3}(-p^n) \otimes \omega_{\mathcal{Q}_3}^{1-p^n} = O_{\mathcal{Q}_3}(2p^n - 3)$. Finally,

\[(27) \quad Ext^i(D^{p^n-2}U_2(-1), F_*^{n*}F_*^nS^{p^n}U_2^n \otimes O_{\mathcal{Q}_3}(2p^n - 3)) = H^i(\mathcal{Q}_3, F_*^{n*}F_*^nS^{p^n}U_2^n \otimes S^{p^n-2}U_2^n(2p^n - 2)),\]

and there is an isomorphism of cohomology groups (Corollary 2.1, [11]):

\[(28) \quad H^i(\mathcal{Q}_3, F_*^{n*}F_*^nS^{p^n}U_2^n \otimes S^{p^n-2}U_2^n(2p^n - 2)) =\]

\[\quad = H^i(\mathcal{Q}_3 \times \mathcal{Q}_3, (F^n \times F^n)^*(i_\ast \mathcal{O}_\Delta) \otimes (S^{p^n}U_2 \otimes S^{p^n-2}U_2(2p^n - 2)).\]

Apply $F_*^{n*} \times F_*^{n*}$ to the resolution $[15]$. Denote $C^\ast$ the complex, whose terms are $C^j = F_*^{n*}\Psi_{-j} \otimes F_*^{n*}O_{\mathcal{Q}_3}(j)$ for $j = -2, -1, 0$ and $C^{-3} = F_*^{n*}U_2 \otimes F_*^{n*}U_2(-2)$. Tensor $C^\ast$ with the bundle $S^{p^n}U_2 \otimes S^{p^n-2}U_2(2p^n - 2)$. Then the complex $C^\ast \otimes (S^{p^n}U_2 \otimes S^{p^n-2}U_2(2p^n - 2))$ computes the cohomology group in the right hand side of (28).

**Lemma 3.4.** $H^i(\mathcal{Q}_3 \times \mathcal{Q}_3, C^j \otimes (S^{p^n}U_2 \otimes S^{p^n-2}U_2(2p^n - 2)) = 0$ for $i > -j$ and $n \geq 1$.

Clearly, this implies $H^i(\mathcal{Q}_3 \times \mathcal{Q}_3, (F^n \times F^n)^*(i_\ast \mathcal{O}_\Delta) \otimes (S^{p^n}U_2 \otimes S^{p^n-2}U_2(2p^n - 2)) = 0$ for $i > 0$ and $n \geq 1$, and hence Lemma 3.3. The proof of Lemma 3.4 is broken up into a series of propositions below.

**Proposition 3.2.** $H^i(\mathcal{Q}_3 \times \mathcal{Q}_3, C^j \otimes (S^{p^n}U_2 \otimes S^{p^n-2}U_2(2p^n - 2)) = 0$ for $i > -j$, where $j = -1, 0$ and $n \geq 1$.

**Proof.** Indeed, $S^{k}U_2 = R^k\pi_*\pi^*O_{\mathcal{P}_3}(k)$ for $k \geq 0$, and $S^{p^n-2}U_2(2p^n - 2) = R^q\pi_*\pi^*O_{\mathcal{P}_3}(p^n - 2) \otimes q^*O_{\mathcal{Q}_3}(2p^n - 2)$. Both line bundles $\pi^*O_{\mathcal{P}_3}(p^n)$ and $\pi^*O_{\mathcal{P}_3}(p^n - 2) \otimes q^*O_{\mathcal{Q}_3}(2p^n - 2)$ are effective, so using the projection formula, the Kempf vanishing and the Künneth formula, we see immediately that

\[(29) \quad H^i(\mathcal{Q}_3 \times \mathcal{Q}_3, S^{p^n}U_2 \otimes S^{p^n-2}U_2(2p^n - 2)) = 0\]

for $i > 0$. Further, the bundle $\Psi_1 = \mathcal{O}_{\mathcal{Q}_3}(1) \otimes O_{\mathcal{Q}_3}$ has a resolution:

\[(30) \quad 0 \to \mathcal{O}_{\mathcal{Q}_3}(1) \otimes O_{\mathcal{Q}_3} \to \mathcal{W}^* \otimes O_{\mathcal{Q}_3} \to O_{\mathcal{Q}_3}(1) \to 0,\]

Tensoring this sequence with $S^{p^n-2}U_2^*$ and using once again the Kempf vanishing and the Künneth formula we get:

\[(31) \quad H^i(\mathcal{Q}_3 \times \mathcal{Q}_3, (F^n\Psi_1 \otimes S^{p^n}U_2^*) \otimes S^{p^n-2}U_2^*(2p^n - 2)) = 0.\]

for $i > 1$. □

**Proposition 3.3.** $H^i(\mathcal{Q}_3 \times \mathcal{Q}_3, (F^n\Psi_2 \otimes S^{p^n}U_2^*) \otimes S^{p^n-2}U_2^*(-2)) = 0$ for $i > 2$ and $n \geq 1$. □
Proof. Propositions 3.3 and 3.4 below ensure that for \( n \geq 1 \) one has \( H^i(Q_3, S^{p^n-2}U(-2)) = 0 \) for \( i \neq 1 \) and \( H^i(Q_3, F^{n*} \Psi_2 \otimes S^{p^n}U_2) = 0 \) for \( i > 1 \). The Künneth formula finishes the proof. \( \square \)

**Proposition 3.4.** \( H^i(Q_3 \times Q_3, (F^{n*}U_2 \otimes S^{p^n}U_2^*) (F^{n*}U_2 \otimes S^{p^n-2}U_2^*) = 0 \) for \( i > 3 \) and \( n \geq 1 \).

**Proof.** Similarly to the above lemma, Propositions 3.3 and 4.1 below imply that for \( n \geq 1 \) one has \( H^i(Q_3, F^{n*}U_2 \otimes S^{p^n}U_2^*) = 0 \) for \( i > 1 \) and \( H^i(Q_3, F^{n*}U_2 \otimes S^{p^n-2}U_2^*) = 0 \) for \( i > 2 \). We are done by Künneth. \( \square \)

4. End of the proof

**Proposition 4.1.** \( H^i(Q_3, S^{p^n-2}U_2^*(-2)) = 0 \) for \( i \neq 1 \) and \( n \geq 1 \).

**Proof.** One has \( H^i(Q_3, S^{p^n-2}U_2^*(-2)) = H^i(G/B, \pi_* O_{p^3}(p^n-2) \otimes q^* O_{Q_3}(-2)) \). Recall that \( \omega_{G/B} = \pi^* O_{p^3}(-2) \otimes q^* O_{Q_3}(-2) \). By Serre duality one has

\[
(32) \quad H^i(G/B, \pi_* O_{p^3}(p^n-2) \otimes q^* O_{Q_3}(-2)) = H^{4-i}(G/B, \pi_* O_{p^3}(-p^n))^*,
\]

and the last group is isomorphic to \( H^{4-i}(\mathbb{P}^3, O_{p^3}(-p^n))^* \) that can be non-zero only if \( i = 1 \).

**Proposition 4.2.** \( H^i(Q_3, F^{n*} \Psi_2 \otimes S^{p^n}U_2^*) = 0 \) for \( i > 1 \) and \( n \geq 1 \).

**Proof.** Consider the sequence:

\[
(33) \quad 0 \to \Omega^2_{\mathbb{P}^4}(2) \otimes O_{Q_3} \to \wedge^2 V^* \otimes O_{Q_3} \to \Omega^1_{\mathbb{P}^4}(2) \otimes O_{Q_3} \to 0.
\]

Tensor it with \( S^{p^n}U_2^* \). Since \( H^i(Q_3, S^{p^n}U_2^*) = H^i(G/B, \pi_* O_{p^3}(p^n)) = 0 \) for \( i > 0 \), we see that the statement will follow if we show \( H^i(Q_3, F^{n*} \Omega^1_{p^4}(2) \otimes O_{Q_3} \otimes S^{p^n}U_2^*) = 0 \) for \( i > 0 \). Recall that \( Q_3 \subset \mathbb{P}^4 = \mathbb{P}(W) \). Consider the adjunction sequence tensored with \( O_{Q_3}(-1) \):

\[
(34) \quad 0 \to \mathcal{T}_{Q_3}(-1) \to \mathcal{T}_{\mathbb{P}^4} \otimes O_{Q_3}(-1) \to O_{Q_3}(1) \to 0.
\]

Recall that if the characteristic \( p \) is odd then the bundle \( \mathcal{T}_{Q_3}(-1) \) is self-dual, that is \( \mathcal{T}_{Q_3}(1) = \Omega^1_{Q_3}(1) \) on \( Q_3 \) (see, for instance, [10], Lemma 4.1). Dualizing the above sequence and tensoring it then with \( O_{Q_3}(1) \), we get:

\[
(35) \quad 0 \to O_{Q_3} \to \Omega^1_{p^4}(2) \otimes O_{Q_3} \to \mathcal{T}_{Q_3} \to 0.
\]

Consequently, the statement will follow from \( H^i(Q_3, F^{n*} \mathcal{T}_{Q_3} \otimes S^{p^n}U_2^*) = 0 \) for \( i > 0 \). Since \( p \) is odd, one has \( \mathcal{T}_{Q_3} = S^2U_2^* \). Consider the universal exact sequence:

\[
(36) \quad 0 \to U_2 \to V \otimes O_{Q_3} \to U_2^* \to 0.
\]

Recall that \( \det U_2 = O_{Q_3}(-1) \). Taking the symmetric square of this sequence and then applying the functor \( F^{n*} \), we get:

\[
(37) \quad 0 \to O_{Q_3}(-p^n) \to F^{n*}U_2 \otimes F^{n*}V \to F^{n*}S^2V \otimes O_{Q_3} \to F^{n*}S^2U_2^* \to 0.
\]

Tensor it with \( S^{p^n}U_2^* \). Proposition 3.3 below states that \( H^i(Q_3, S^{p^n}U_2^* \otimes F^{n*}U_2) = 0 \) for \( i > 1 \) and \( n \geq 1 \). It is sufficient therefore to show that \( H^3(Q_3, S^{p^n}U_2^* \otimes O_{Q_3}(-p^n)) = 0 \), or, equivalently, that \( H^3(G/B, O_{\pi}(-p^n)) = 0 \). Indeed, there is an isomorphism of line bundles:

\[
(38) \quad q^* O_{Q_3}(p^n) = O_{\pi}(p^n) \otimes \pi^* O_{p^3}(p^n).
\]
Hence, by the projection formula:

\[ H^3(Q_3, S^{2n}U_2^* \otimes O_{Q_3}(-p^n)) = H^3(Q_3, q_* \pi^* O_{P^3}(p^n) \otimes q^* O_{Q_3}(-p^n)) = H^3(G/B, O_{\pi}(-p^n)). \]

Considering the sequence (18) and applying Proposition \ref{prop:cohomology_sequence} we get the statement.

**Proposition 4.3.** \( H^i(Q_3, S^{2n}U_2^* \otimes F^{n*}U_2^*) = 0 \) for \( i > 1 \) and \( n \geq 1 \).

**Proof.** Apply \( F^{n*} \) to the sequence (35):

\[ 0 \to F^{n*}U_2 \to F^{n*}V \otimes O_{Q_3} \to F^{n*}U_2^* \to 0. \]

Tensoring the sequence (39) with \( S^{2n}U_2^* \), we obtain:

\[ 0 \to F^{n*}U_2 \otimes S^{2n}U_2^* \to F^{n*}V \otimes S^{2n}U_2^* \to F^{n*}U_2^* \otimes S^{2n}U_2^* \to 0. \]

We saw above that the higher cohomology groups of \( S^{2n}U_2^* \) vanish. Hence, from the long exact cohomology sequence it is sufficient to show that \( H^i(Q_3, S^{2n}U_2^* \otimes F^{n*}U_2^*) = 0 \) for \( i > 0 \). Take the dual to the sequence (18):

\[ 0 \to O_{\pi}(p^n) \to q^* F^{n*}U_2^* \to \pi^* O_{\mathbb{P}^3}(p^n) \to 0. \]

Tensor this with \( \pi^* O_{\mathbb{P}^3}(p^n) \):

\[ 0 \to O_{\pi}(p^n) \otimes \pi^* O_{\mathbb{P}^3}(p^n) \to \pi^* O_{\mathbb{P}^3}(p^n) \otimes q^* F^{n*}U_2^* \to \pi^* O_{\mathbb{P}^3}(2p^n) \to 0. \]

Using the isomorphism (38) we get:

\[ 0 \to q^* O_{\mathbb{P}^3}(p^n) \to \pi^* O_{\mathbb{P}^3}(p^n) \otimes q^* F^{n*}U_2^* \to \pi^* O_{\mathbb{P}^3}(2p^n) \to 0. \]

Applying to this sequence the functor \( q_* \), and using the projection formula and the isomorphism \( R^* q_* \pi^* O_{\mathbb{P}^3}(k) = S^k U_2^* \) for \( k \geq 0 \), we obtain:

\[ 0 \to O_{Q_3}(p^n) \to S^{2n}U_2^* \otimes F^{n*}U_2^* \to S^{2n}U_2^* \to 0. \]

The leftmost and rightmost terms of the above sequence have vanishing higher cohomology, hence the statement of the lemma.

**Proposition 4.4.** \( H^3(Q_3, S^{2n-2}U_2^*(-2) \otimes F^{n*}U_2^*) = 0 \) for \( n \geq 1 \).

**Proof.** One has \( H^i(Q_3, S^{2n-2}U_2^*(-2) \otimes F^{n*}U_2^*) = H^i(G/B, \pi^* O_{\mathbb{P}^3}(p^n - 2) \otimes q^* O_{Q_3}(-2) \otimes q^* F^{n*}U_2^*). \) By Serre duality we get:

\[ H^3(G/B, \pi^* O_{\mathbb{P}^3}(p^n - 2) \otimes q^* O_{Q_3}(-2) \otimes q^* F^{n*}U_2^*) = H^1(G/B, \pi^* O_{\mathbb{P}^3}(-p^n) \otimes q^* F^{n*}U_2^*). \]

Tensor the sequence (10) with \( \pi^* O_{\mathbb{P}^3}(-p^n) \). One has:

\[ \cdots \to H^1(G/B, F^{n*}V \otimes \pi^* O_{\mathbb{P}^3}(-p^n)) \to H^1(G/B, \pi^* O_{\mathbb{P}^3}(-p^n) \otimes q^* F^{n*}U_2^*) \to H^2(G/B, \pi^* O_{\mathbb{P}^3}(-p^n) \otimes q^* F^{n*}U_2^*) \to \cdots. \]

Clearly, \( H^1(G/B, F^{n*}V \otimes \pi^* O_{\mathbb{P}^3}(-p^n)) = H^1(P^3, O_{\mathbb{P}^3}(-p^n)) \otimes F^{n*}V = 0. \) Let us show that \( H^2(G/B, \pi^* O_{\mathbb{P}^3}(-p^n) \otimes q^* F^{n*}U_2^*) = 0. \) Indeed, tensoring the sequence (18) with \( \pi^* O_{\mathbb{P}^3}(-p^n) \) we see that \( H^i(G/B, \pi^* O_{\mathbb{P}^3}(-p^n) \otimes q^* F^{n*}U_2^*) = 0 \) if \( i \neq 3 \), hence the statement.


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