Quantum Brownian Motion of a Macroscopic Object in a General Environment

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Abstract

For the purpose of understanding the quantum behavior such as quantum decoherence, fluctuations, dissipation, entanglement and teleportation of a mesoscopic or macroscopic object interacting with a general environment, we derive here a set of exact master equations for the reduced density matrix of \( N \) interacting harmonic oscillators in a heat bath with arbitrary spectral density and temperature. Two classes of problems of interest to us which these equations can be usefully applied to are that of the quantum dynamics of nanoelectromechanical oscillators and the entanglement evolution of multi-partite macroscopic states such as quantum superposition of mirrors in a high Q cavity. To address a key conceptual issue for macroscopic quantum phenomena we examine the conditions for an assumption often implicitly made in these studies to be valid, namely, that the quantum behavior of a macroscopic object in an environment can be accurately represented by only treating the dynamics of its center-of-mass variable. We also mention how these results can be used to calculate the uncertainty principle governing a macroscopic object at finite temperature.

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I. INTRODUCTION

In an earlier paper [1] we showed the derivation of an exact master equation for two coupled quantum harmonic oscillators interacting bilinearly with a common environment made up of \( n \) harmonic oscillators at an arbitrary temperature for a general spectral density function. (This is referred to as a ‘general environment’ in [2].) This equation can be applied to the analysis of model problems in macroscopic quantum phenomena (MQP) involving two harmonic oscillators, either mechanical such as the superposition of two mirrors [3, 4, 5], or one of them of electromagnetic or superconducting flux origin, as in nanoelectromechanical (NEM) resonators [6] or SQUID-resonator [7, 29].

In this paper we take a small step towards treating the quantum properties of a mesoscopic or macroscopic object in a general environment by providing the technical base for such studies. We consider a system modeled by \( N \) harmonic oscillators (NHO) interacting with a heat bath consisting of \( n \) harmonic oscillators (HOB). The aim is to delineate the conditions upon which the mechanical and statistical mechanical properties of this quantum object can be described in terms of its center-of-mass (COM) variable by a master equation for the reduced density matrix with the bath variables integrated out, and for such conditions, derive an exact master equation for a bath with arbitrary spectral density and temperature. The motivation for this work has both conceptual and practical underpinnings. At the conceptual level we want to examine the validity of an implicit assumption made in many MQP investigations, namely, that the quantum mechanical behavior of a macroscopic object like the NEM or a C60 molecule [8, 9], placed in interaction with an environment, behavior such as quantum decoherence [10], fluctuations and dissipation, entanglement and teleportation, can be captured by its COM behavior. For convenience we refer to this as the ‘COM axiom’. This assertion is intuitively reasonable, as one might expect it to be true from normal-mode decompositions familiar in classical mechanics, but when particles (NHO) interact with each other (such as in a quantum bound state problem) in addition to interacting with their common environment, all expressed in terms of the reduced density matrix, it is not such a clear-cut result. At least we have not seen a proof of it [30].

Another important issue is the demarkation between microscopic, mesoscopic and macroscopic. Assuming that the object is made up of \( N \) physically relevant quantum particles (e.g. atoms, forgetting about the tighter-bound substructures), starting with \( N = 1 \) which we refer to as microscopic, the question is: At what number of \( N \) will one begin to describe the object as mesoscopic with a qualitatively distinct behavior from microscopic, and likewise for macroscopic? In clas-
sical statistical mechanics, this issue underlies the important attempt to derive from molecular Hamiltonian dynamics (with deterministic chaos) the thermodynamic and kinetic properties of a gas (with $N$ molecules), such as transport functions, and their dynamics, which possesses salient dissipative and time-asymmetric features. To provide a quantitative analysis of such issues one needs to work with a stochastic equation for the $N$ particles so as to be able to see the cross-over behavior between any two of these three regimes as one varies $N$, and its thermodynamic behavior as one varies the temperature $T$. In this paper we derive such an equation for NHO in HOB, but will leave the analysis of this theoretical issue to later investigations [12].

On the applied side, the master equation for NHO in a HOB is useful for a range of problems which are of experimental interests. Detection of small displacements of a NEM resonator by a superconducting single electron transistor (SSET) or a biased quantum point contact (BQPC) [6] is a useful scheme for probing the appearance of quantum properties of oscillators, such as decoherence and entanglement [14], noise and fluctuations [15], the standard quantum limit [16] and the uncertainty principle for quantum open systems at finite temperatures [17, 18]. It has been shown that in certain well-accessible regimes both the SSET and BQPC [19, 20] detection devices behave like a thermal bath (albeit in some regimes it shows strong back-action effects which have been suggested as a creative way to cool the resonator [21, 22]). Master and Fokker-Planck equations have been derived for these systems in [19, 23].

Another interesting setup is a linear array of NEM resonators. Studying the entanglement transport in chains of mechanical oscillators, Eisert et. al. [24] showed that the motions of distant oscillators can be entangled without the need for control of individual oscillators and without assuming any direct interaction between them. Our master equation for $N$ harmonic oscillators in a general environment can be usefully applied to this problem to check on the distinct robustness of entanglement in the canonical coordinates found by these authors.

Technically, our derivation of the master equation for the NHO system is similar to the simple method we used in [1] for the case of two coupled harmonic oscillators in a HOB. The task rests on finding a suitable canonical transformation which preserves the structure of the Poisson brackets. In Sec. 2 we define our model and outline the procedure. In Sec. 3 we write down the evolution of the density matrix and derive the master equation which has the same form as the Hu-Paz-Zhang (HPZ) equation [2, 25, 26, 27]. In Sec. 4, we derive the corresponding Fokker-Planck equation. In Sec. 5 we discuss the general features of these equations. The overall characteristics of the results derived in this paper may posteriori be deduced from intuitive reasoning. However, the fine points present in the full solution are not so easily obtained. They are needed to address the validity of
the COM axiom we posed above, i.e., under what conditions can one presume that the center of mass coordinate of a macroscopic object is the one most sensitive to the environmental influence? We discuss this issue in the last section. In the Appendices we derive the canonical transformation for a general NHO system, and show the explicit construction procedures for the cases of $N = 2 - 5$.

II. THE MODEL

For the investigation of environmental influences on the quantum properties of a macroscopic object we consider the quantum Browninan motion (QBM) of $N$ identical harmonic oscillators with mutual interactions coupled to a collection of $n$ oscillators making up the environment [28]. The generalized QBM Hamiltonian is:

$$H_{\text{sys}} = \sum_{i=1}^{N} \left( \frac{1}{2} M \ddot{x}_i^2 + \frac{1}{2} M \Omega^2 x_i^2 \right) + \sum_{i,j=1}^{N} V_{ij}(x_i - x_j)$$

$$H_{\text{bath}} = \sum_{i=1}^{n} \left( \frac{1}{2} m \dot{q}_i^2 + \frac{1}{2} m \omega_i^2 q_i^2 \right)$$

$$H_{\text{int}} = \sum_{i=1}^{N} \sum_{j=1}^{n} C_{ij}(x_i q_j)$$

For simplicity, let us assume that $C_{ij} = C_j, \forall i = 1,...,N$ i.e. all the system harmonic oscillators couple to the bath with equal strength. Hence the interacting Hamiltonian can be rewritten more compactly as

$$H_{\text{int}} = \sum_{i=1}^{N} \sum_{j=1}^{n} C_{ij}(x_i q_j) = \left( \sum_{i=1}^{N} x_i \right) \left( \sum_{j=1}^{n} C_j q_j \right)$$

With this generalized $N$ harmonic oscillator (NHO) QBM model, we can show the following:

(1) For any given finite $N$ sets of canonical coordinates $(x_i, P_i), i = 1,...,N$, we will give a procedure (or algorithm) to construct another set of coordinates $(\tilde{X}_i, \tilde{P}_i), i = 1,...,N$ where $\tilde{X}_1 = \frac{1}{N}(x_1 + ... + x_N), \tilde{P}_1 = P_1 + ... + P_N$ are the center-of-mass coordinate and total momentum of the system respectively. This transformation matrix $\tilde{T}$ which transforms $\vec{x}$ to $\vec{X}$, i.e. $\tilde{X}_i = \tilde{T}_{ij} x_j$, satisfies $|\det \tilde{T}| = 1$. The choice of $\{\tilde{X}_i\}$ is not unique. However, our method gives a systematic way of construction and is thus quite useful. We also provide the transformation for the effective masses. The explicit construction is shown in Appendix A.
The quadratic part of the system Hamiltonian can be shown to transform as
\[
\sum_{i=1}^{N} \left( \frac{P_i^2}{2M} + \frac{1}{2} M \Omega^2 x_i^2 \right) = \sum_{i=1}^{N} \left( \frac{\tilde{P}_i^2}{2\tilde{M}_i} + \frac{1}{2} \tilde{M}_i \Omega^2 \tilde{X}_i^2 \right)
\]
where \( \tilde{M}_1 = \sum_{i=1}^{N} M = NM \) is the total mass of the system.

Under this transformation, the Poisson brackets and hence the commutation relations are preserved,
\[
[x_i, P_j] = [\tilde{X}_i, \tilde{P}_j] = i\hbar \delta_{ij}, \quad [x_i, x_j] = [P_i, P_j] = [\tilde{X}_i, \tilde{X}_j] = [\tilde{P}_i, \tilde{P}_j] = 0.
\]
Hence our transformation is a canonical one. Most importantly, we will show that the interaction potential between the \( N \) harmonic oscillators in the new coordinate system is independent of \( \tilde{X}_1 \), the center-of-mass coordinate.

**Lemma:** If the potentials \( V_{ij} \) among the system oscillators \( x_i \) and \( x_j \) are functions of \( x_i - x_j \) only, then \( \frac{\partial V_{ij}}{\partial \tilde{X}_1} = 0 \). Hence the total potential \( \sum_{i,j; i \neq j}^{N} V_{ij}(x_i - x_j) \) is independent of \( \tilde{X}_1 \). The proof of the lemma will be given in Appendix B.

Combining all the above properties, we can rewrite the original Hamiltonian as follows:
\[
H_{\text{sys}} = \sum_{i=1}^{N} \left( \frac{1}{2} M x_i^2 \right) + \frac{1}{2} M \Omega^2 x_i^2 + \sum_{i,j; i \neq j}^{N} V_{ij}(x_i - x_j)
\]
\[
= \sum_{i=1}^{N} \left( \frac{\tilde{P}_i^2}{2\tilde{M}_i} + \frac{1}{2} \tilde{M}_i \Omega^2 \tilde{X}_i^2 \right) + \tilde{V}(\tilde{X}_2, ..., \tilde{X}_N)
\]
\[
= \frac{\tilde{P}_1^2}{2\tilde{M}_1} + \frac{1}{2} \tilde{M}_1 \Omega^2 \tilde{X}_1^2 + \sum_{i=2}^{N} \left( \frac{\tilde{P}_i^2}{2\tilde{M}_i} + \frac{1}{2} \tilde{M}_i \Omega^2 \tilde{X}_i^2 \right) + \tilde{V}(\tilde{X}_2, ..., \tilde{X}_N)
\]
\[
= \tilde{H}_1 + \tilde{H}_2,
\]
where
\[
\tilde{H}_1 = \frac{\tilde{P}_1^2}{2\tilde{M}_1} + \frac{1}{2} \tilde{M}_1 \Omega^2 \tilde{X}_1^2,
\]
\[
\tilde{H}_2 = \sum_{i=2}^{N} \left( \frac{\tilde{P}_i^2}{2\tilde{M}_i} + \frac{1}{2} \tilde{M}_i \Omega^2 \tilde{X}_i^2 \right) + \tilde{V}(\tilde{X}_2, ..., \tilde{X}_N).
\]

\[
H_{\text{int}} = \sum_{i=1}^{N} \sum_{j=1}^{n} C_{ij}(x_i q_j) = \left( \sum_{i=1}^{N} x_i \right) \left( \sum_{j=1}^{n} C_{j} q_j \right) = \tilde{X}_1 (N \sum_{j=1}^{n} C_{j} q_j)
\]
Note that the heat bath has the spectral density $J(\omega)$:

$$J(\omega) = \pi \sum_{j=1}^{n} \frac{\tilde{C}_j^2}{2m_j \omega_j} \delta(\omega - \omega_j).$$

(14)

which differs from the original heat bath density by a numerical factor $N^2$.

The total Hamiltonian can then be written as

$$H_{\text{tot}} = H_{\text{sys}} + H_{\text{bath}} + H_{\text{int}}$$

$$= \tilde{H}_1 + \tilde{H}_2 + H_{\text{bath}} + H_{\text{int}}$$

$$= H_{\text{cm}} + H' + H_{\text{bath}}.$$  

$H_{\text{cm}} = \tilde{H}_1 + H_{\text{int}}, H' = \tilde{H}_2$. Note that $[H', H_{\text{cm}}] = [H', H_{\text{bath}}] = 0$.

### III. DENSITY MATRIX AND MASTER EQUATION

In our derivation, we shall make the following two assumptions: (1) The system and the environment are initially uncorrelated. (2) The heat bath is initially in a thermal equilibrium state at temperature $T = (k_B \beta)^{-1}$.

#### A. The density matrix

The density matrix for the total system develops in time under the unitary evolutionary operator:

$$\rho(t) = \exp \left[ -\frac{i}{\hbar} H_{\text{tot}} t \right] \rho(0) \exp \left[ \frac{i}{\hbar} H_{\text{tot}} t \right]$$

$$= \exp \left[ -\frac{i}{\hbar} \frac{(H_{\text{cm}} + H' + H_{\text{bath}}) t}{\hbar} \right] \rho(0) \exp \left[ \frac{i}{\hbar} \frac{(H_{\text{cm}} + H' + H_{\text{bath}}) t}{\hbar} \right]$$

$$= \exp \left[ -\frac{i}{\hbar} \frac{H' t}{\hbar} \right] \exp \left[ -\frac{i}{\hbar} \frac{(H_{\text{cm}} + H_{\text{bath}}) t}{\hbar} \right] \rho(0) \exp \left[ \frac{i}{\hbar} \frac{(H_{\text{cm}} + H_{\text{bath}}) t}{\hbar} \right] \exp \left[ \frac{i}{\hbar} \frac{H' t}{\hbar} \right]$$

(15)

In the third equality, we have to use the condition that $[H_{\text{cm}}, H'] = [H_{\text{bath}}, H'] = 0$. (Note that for general operators $A, B$, we have the Baker-Campbell-Hausdorff formula: $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[[B,[B,A]]]+\ldots)}$)

If we define

$$\tilde{\rho}(t) = \exp \left[ -\frac{i}{\hbar} \frac{(H_{\text{cm}} + H_{\text{bath}}) t}{\hbar} \right] \rho(0) \exp \left[ \frac{i}{\hbar} \frac{(H_{\text{cm}} + H_{\text{bath}}) t}{\hbar} \right]$$

(16)
then

$$\rho(t) = \exp \left[ -i \frac{H' t}{\hbar} \right] \tilde{\rho}(t) \exp \left[ i \frac{H' t}{\hbar} \right]$$  \hspace{1cm} (17)$$

$$\rho_r = Tr_{\text{bath}} \rho(t) = \exp \left[ -i \frac{H' t}{\hbar} \right] Tr_{\text{bath}} \tilde{\rho}(t) \exp \left[ i \frac{H' t}{\hbar} \right]$$ \hspace{1cm} (18)$$

In a similar manner as in [1], we can get the exact master equation for the arbitrary $N$ oscillators.

**B. The Master equation**

Tracing over the heat bath leads us to a HPZ type master equation [2] for $\tilde{X}_1, \tilde{P}_1$:

$$\dot{\rho}_r = \frac{1}{i\hbar} [\tilde{H}_1, \rho_r] + \frac{a(t)}{2i\hbar} [\tilde{X}_1^2, \rho_r] + \frac{b(t)}{2i\hbar} [\tilde{X}_1, \{\tilde{P}_1, \rho_r\}] + \frac{c(t)}{\hbar^2} [\tilde{X}_1, \tilde{P}_1, \rho_r] - \frac{d(t)}{\hbar^2} [\tilde{X}_1, [\tilde{X}_1, \rho_r]]$$ \hspace{1cm} (19)$$

where $\tilde{H}_1$ is the Hamiltonian for $\tilde{X}_1, \tilde{P}_1$ only. This has the same form as the HPZ equation for the QBM of a single harmonic oscillator $\tilde{X}_1, \tilde{P}_1$ interacting with a general heat bath. Note that coefficients satisfy the same equations as listed in Refs [2] or [26], but with different coupling constants and masses.

From the evolution equation (17), the required master equation for $\rho_r(t)$ is obtained,

$$\dot{\rho}_r = \frac{1}{i\hbar} [H_{\text{sys}}, \rho_r] + \frac{a(t)}{2i\hbar} [\tilde{X}_1^2, \rho_r] + \frac{b(t)}{2i\hbar} [\tilde{X}_1, \{\tilde{P}_1, \rho_r\}] + \frac{c(t)}{\hbar^2} [\tilde{X}_1, \tilde{P}_1, \rho_r] - \frac{d(t)}{\hbar^2} [\tilde{X}_1, [\tilde{X}_1, \rho_r]]$$ \hspace{1cm} (20)$$

The only difference between Eq. (20) and Eq. (19) is that the unitary evolution is modified by the remaining $N-1$ fictitious harmonic oscillators $(\tilde{X}_j, \tilde{P}_j), j = 2, 3, ..., N$.

In terms of the original variables $x_1, ..., x_N, P_1, ..., P_N$, we get

$$\dot{\rho}_r = \frac{1}{i\hbar} [H_{\text{sys}}, \rho_r] + \frac{a(t)}{2N^2i\hbar} [(x_1 + ... + x_N)^2, \rho_r] + \frac{b(t)}{2N^2i\hbar} [x_1 + ... + x_N, \{P_1 + ... + P_N, \rho_r\}] + \frac{c(t)}{N\hbar^2} [x_1 + ... + x_N, [P_1 + ... + P_N, \rho_r]] - \frac{d(t)}{N^2\hbar^2} [x_1 + ... + x_N, [x_1 + ... + x_N, \rho_r]]$$ \hspace{1cm} (21)$$

This exact master equation for the $N$ coupled harmonic oscillators in a general environment is the main result of this paper.

**IV. FOKKER-PLANCK EQUATION**

In this section, we present an alternative, but useful form of the master equation derived in the last section. We also provide the explicit expressions for the coefficients appearing in (20).
A. Fokker-Planck equation

In terms of the Wigner function, the above master equation takes the form:

\[
\frac{\partial \tilde{W}}{\partial t} = -\sum_{i=1}^{N} \left( \frac{P_i \partial \tilde{W}}{M \partial x_i} - M\Omega^2 x_i \frac{\partial \tilde{W}}{\partial P_i} \right) + M\Omega(t)(x_1 + \ldots + x_N) \left( \frac{\partial}{\partial P_1} + \ldots + \frac{\partial}{\partial P_N} \right) \tilde{W} + 2\Gamma(t) \left( \frac{\partial}{\partial P_1} + \ldots + \frac{\partial}{\partial P_N} \right) \left[ (P_1 + \ldots + P_N) \tilde{W} \right] + \Sigma(t) \left( \frac{\partial}{\partial P_1} + \ldots + \frac{\partial}{\partial P_N} \right)^2 \tilde{W} + \Delta(t) \left( \frac{\partial}{\partial P_1} + \ldots + \frac{\partial}{\partial P_N} \right) \left( \frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_N} \right) \tilde{W}
\]  
(22)

Note that the Wigner function is related to the reduced density matrix in the following way:

\[
\tilde{W}(x_1, \ldots, x_N, P_1, \ldots, P_N, t) = \frac{1}{(2\pi)^N} \int du_1 \ldots du_N e^{i(u_1 P_1 + \ldots + u_N P_N)/\hbar} \rho_r \left( x_1 - \frac{u_1}{2}, \ldots, x_N - \frac{u_N}{2}; x_1 + \frac{u_1}{2}, \ldots, x_N + \frac{u_N}{2}, t \right)
\]

where we identify in Eq. (22):

\[
a(t) = M\Delta\Omega(t), \\
b(t) = 2\Gamma(t), \\
c(t) = \Delta(t), \\
d(t) = \Sigma(t)
\]

By using this transformation, the master equation can be easily obtained:

\[
\frac{i\hbar}{\partial t} \rho_r = -\frac{\hbar^2}{2M} \left( \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_N^2} - \frac{\partial^2}{\partial y_1^2} + \ldots - \frac{\partial^2}{\partial y_N^2} \right) \rho_r + \frac{1}{2} M\Omega^2(x_1^2 + \ldots + x_N^2 - y_1^2 - \ldots - y_N^2) \rho_r \\
+ \frac{1}{2} M\delta\Omega^2(t)(x_1 + \ldots + x_N - y_1 - \ldots - y_N) \frac{1}{2} (x_1 + \ldots + x_N + y_1 + \ldots + y_N) \rho_r \\
- iM\Gamma(t)(x_1 + \ldots + x_N - y_1 - \ldots - y_N) \frac{1}{2} \left( \frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_N} - \frac{\partial}{\partial y_1} + \ldots - \frac{\partial}{\partial y_N} \right) \rho_r \\
- iM\Sigma(t)(x_1 + \ldots + x_N - y_1 - \ldots - y_N) \frac{1}{2} \left( \frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_N} + \frac{\partial}{\partial y_1} + \ldots + \frac{\partial}{\partial y_N} \right) \rho_r
\]

(28)

B. Coefficients

The coefficients \(a(t), b(t), c(t), d(t)\) appearing in (21) or (22) can be constructed in terms of the elementary functions \(u_i(s), i = 1, 2\). The \(u_i(s)\) are defined as the functions that satisfy the following homogeneous integro-differential equation:

\[
\ddot{\Sigma}(s) + \Omega^2 \Sigma(s) + \frac{N^2}{M} \int_0^s d\lambda \eta(s - \lambda) \Sigma(\lambda) = 0
\]

(29)
with the boundary conditions:

\[ u_1(s = 0) = 1, \quad u_1(s = t) = 0, \quad (30) \]

and

\[ u_2(s = 0) = 0, \quad u_2(s = t) = 1. \quad (31) \]

where

\[ \eta(s) = - \int_0^\infty d\omega I(\omega) \sin(\omega s) \quad (32) \]

is the dissipation kernel and \( I(\omega) = \frac{1}{N^2} J(\omega) \) is the spectral density of the environment. Note that the numerical pre-factor before the integration in this equation is different from that defined in HPZ \[2\]. This is the main difference induced by the presence of \( N \) harmonic oscillators.

Let \( G_1(s, \tau) \) be the Green function which satisfies the following equation:

\[ \frac{d^2}{ds^2} G_1(s, \tau) + \Omega^2 G_1(s, \tau) + \frac{N^2}{M} \int_0^s d\tau \eta(s - \tau) G_1(s, \tau) = \delta(s - \tau), \quad (33) \]

where \( G_1(s, \tau) \) as a function of \( s \) satisfies the following initial conditions:

\[ G_1(s = 0, \tau) = 0, \quad \frac{d}{ds} G_1(s, \tau)|_{s=0} = 0. \quad (34) \]

The Green function \( G_2(s, \tau) \) is defined analogously. The coefficients can then be written as

\[ a(t) = N \int_0^t ds \eta(t - s) \left( u_2(s) - \frac{u_1(s) \dot{u}_2(t)}{\dot{u}_1(t)} \right), \quad (35) \]

\[ b(t) = \frac{N}{M} \int_0^t ds \eta(t - s) \frac{u_1(s)}{\dot{u}_1(t)}. \quad (36) \]

\[ c(t) = \frac{\hbar}{N M} \int_0^t d\lambda G_1(t, \lambda) \nu(t - \lambda) \]

\[ - \frac{N^2 \hbar}{M^2} \int_0^t ds \int_s^t d\tau \int_0^\tau d\lambda \eta(t - s) G_1(t, \lambda) G_2(t, \tau) \nu(\tau - \lambda), \quad (37) \]

and

\[ d(t) = \frac{\hbar}{N} \int_0^t d\lambda G_1^t(t, \lambda) \nu(t - \lambda) \]

\[ - \frac{N^2 \hbar}{M} \int_0^t ds \int_s^t d\tau \int_0^\tau d\lambda \eta(t - s) G_1^t(t, \lambda) G_2(t, \tau) \nu(\tau - \lambda). \quad (38) \]

where \( \nu(s) \) is defined as

\[ \nu(s) = \int_0^{+\infty} d\omega I(\omega) \coth\left( \frac{\omega}{2\hbar\beta} \right) \cos(\omega s). \quad (39) \]

which is the noise kernel of the environment. Here a prime denotes taking the derivative with respect to the first variable of \( G_1(s, \tau) \).
V. DISCUSSIONS

We end with a few technical remarks followed by two conceptual points, one referring to the COM axiom for the quantum dynamics of macroscopic objects, and the other, to the generalized standard quantum limit.

A. Technical Remarks

First, note that although we can show $[\tilde{X}_1, \tilde{H}_2] = [\tilde{P}_1, \tilde{H}_2] = 0$ and hence $[\tilde{H}_1, \tilde{H}_2] = 0$, the $(N - 1)$ set of variables $\tilde{X}_j, j = 2, ..., N$ will in general not commute with $\tilde{H}_2$ because of the potential $\tilde{V}(\tilde{X}_2, ..., \tilde{X}_N)$.

Here we assume that all the $N$ particles are of the same mass $M$ and have the same eigenfrequency $\Omega$. They both couple to the environment with equal strength. So these $N$ harmonic oscillators are "identical" particles. If the $N$ system oscillators have different masses, this becomes more involved.

In the proof that the potential is independent of $\tilde{X}_1$, we made no assumption about the functional form of the potential. All that was needed was that the potentials are functions of $x_i - x_j$ only. This is a reasonable assumption valid for many interesting physical situations. Hence the range of applicability of our result is by no means overly restrictive.

B. The COM Axiom for quantum dynamics of macroscopic objects

We now address the question raised in the beginning, i.e., on the validity of representing the quantum behavior of a macroscopic object by its center-of-mass dynamics, which we referred to as the ‘COM axiom’ for quantum dynamics of many body systems. To do this we consider a more general type of coupling between the system and the environment, e.g., coupling of the form $f(x_i)q_j$ instead of $x_iq_j$, and examine if the COM variable dynamics separates from the reduced variable dynamics.

For this purpose, let us note that if the function $f(x)$ has the property $\sum_{i=1}^{N} f(x_i) = \tilde{f}(\tilde{X}_1) + g(\tilde{X}_2, ..., \tilde{X}_N)$, for example $f(x) = x$ or $f(x) = x^2$, one can split the coupling between the system and environment into couplings containing the COM coordinate and the relative coordinates. Tracing out the environmental degrees of freedom $q_i$, one can easily get the influence action which characterizes the effect of the environment on the system.

However, the coarse-graining made by tracing out the environmental variables $q_i$ does not
necessarily lead to the separation of the COM and the relative coordinates in the effective action. When they are mixed up and can no longer be written as the sum of these two contributions, the form of the master equation will be radically altered as it would contain both the relative coordinate and the center-of-mass coordinate dynamics. One can work out how much of a change this would bring about in the COM dynamics, but at least we could say that when the coupling is not in these forms the COM axiom for macroscopic quantum dynamics no longer holds.

Therefore we can conclude that for the N harmonic oscillators QBM model, the coupling between the system and the environment need be bi-linear, in the form \( x_i q_j \), for this axiom to hold. In that case, one can say that the quantum evolution of a macroscopic object in a general environment is completely described by the dynamics of the center-of-mass variable obeying a master equation of the HPZ type.

C. Generalized standard quantum limit

Another important issue of great interest to experimentalists is the generalized uncertainty relation of an N-body system at finite temperature. In our simple model the coupling between the system of NHO and the environment is only through the center-of-mass coordinate. If there is no mutual interaction between these N harmonic oscillators, only the center-of-mass coordinate is coupled to the environment and the remaining \((N - 1)\) degrees of freedom are orthogonal. Hence the uncertainty function for the whole system will simply be \( U_{HZ} \times (U_1)^{N-1} \), where \( U_{HZ} \) is the uncertainty function for the 1HO QBM case given in [17] and \( U_1 \) represents the quantum Heisenberg uncertainty relation for one pair of relative coordinate canonical variables \((\tilde{X}_i, \tilde{P}_i)\). Here we want to point out that in this situation, the effect of the environment, say the temperature of the heat bath, enters only through the center-of-mass coordinate in the form of \( U_{HZ} \), the remaining \((N - 1)\) pair of canonical conjugate variables are subject to quantum Heisenberg uncertainty relation \( U_1 \).

The time evolution and temperature dependence of the uncertainty function for an Ohmic bath was studied in great detail in [17, 18].

If inter-particle interactions \( V_{ij} \neq 0 \) exist among the N particles, then the uncertainty function \( U_{HZ} \times (U_1)^N \) will be modified. Due to the interactions between the N harmonic oscillators, the uncertainty relation governing the \((N - 1)\) pairs of relative coordinate variables \( (\tilde{X}_i, \tilde{P}_i) \) might be squeezed and rotated. However, if the number of harmonic oscillators \( N \) is large, and the interacting potential among the N-particles are short-ranged, or that the forces amongst them are very strong, so that the characteristic frequencies of these ‘hard modes’ are much higher than that
of the natural frequency of the COM modes, under these conditions, the quanta corresponding to the motion of the relative coordinates are not easily excited and the leading order contribution to the uncertainty function will be dominated by the center-of-mass degree of freedom. The details can be worked out from a perturbation analysis on the present results. (See, e.g., [25] for treating one form of interaction.)

D. Conclusion

In this paper, we outlined the procedure to find a canonical transformation to transform from the individual coordinates \((x_i, P_i)\) to the collective coordinates \((\tilde{X}_i, \tilde{P}_i)\), \(i = 1, ..., N\) where \(\tilde{X}_1, \tilde{P}_1\) are the center-of-mass coordinate and momentum respectively. We then proved that the potential \(V_{ij}(x_i - x_j)\) is independent of the center-of-mass coordinate \(\tilde{X}_1\). Then following the simple derivation of the master equation for 2HO in our previous work we showed that the system with variables \((\tilde{X}_1, \tilde{P}_1)\) obeys a master equation of the same form as the HPZ equation. We gave the details of derivation of an exact non-Markovian master equation for the reduced density matrix of this system constructed with the heat bath variables integrated out. We stress that this result for the \(N\) mutually interacting harmonic oscillators in a general environment is more than just a normal mode decomposition problem as in classical mechanics because there are interactions between the individual system oscillators and collective interaction with the quantum environment.

This result is expected to be useful for the study of entanglement dynamics of multipartite particles and quantum to classical transition issues. Finally, we established a relation between the center-of-mass coordinate of \(N\) body harmonic oscillators and the well-known one oscillator QB model. This provides a key step in establishing a microscopic theory for macroscopic quantum phenomena, a topic we intend to pursue further in the future.

VI. APPENDIX

A. Construction of the canonical transformation for general \(N\).

Given a system of \(N\) identical harmonic oscillators with equal mass \(M\) and intrinsic frequency \(\Omega\),

\[ H_{N0} = \sum_{i}^{N} \frac{P_i^2}{2M} + \sum_{i}^{N} \frac{1}{2} M \Omega^2 x_i^2, \]

Note that \(P_i = M\dot{x}_i = \frac{\partial H_{N0}}{\partial \dot{x}_i}\) and \([x_i, P_j] = i\hbar \delta_{ij}\).
Goal: Find out \( \{ \tilde{X}_i \}, \{ \tilde{P}_i \}, \text{and} \{ \tilde{M}_i \} \) such that

\[
H_{N0} = \sum_i^N \frac{P_i^2}{2M} + \sum_i^N \frac{1}{2} M \Omega^2 x_i^2 = \sum_i^N \frac{\tilde{P}_i^2}{2\tilde{M}_i} + \sum_i^N \frac{1}{2} \tilde{M}_i \Omega^2 \tilde{X}_i^2,
\]

where \([\tilde{X}_i, \tilde{P}_j] = i\hbar \delta_{ij} = [x_i, P_j] \) and \( \tilde{X}_1 = \frac{1}{\tilde{N}} (x_1 + x_2 + \ldots + x_N), \tilde{P}_1 = (P_1 + P_2 + \ldots + P_N) \) and \( \tilde{M}_1 = N \tilde{M} \) are the center-of-mass coordinate, total momentum, and total mass of the system respectively. Note that it’s easy to see that \([\tilde{X}_1, \tilde{P}_1] = i\hbar \).

B. The explicit construction procedure:

1. \( N = 2 \):

\[
\begin{align*}
\tilde{X}_1 &= \frac{1}{2} (x_1 + x_2), \quad \tilde{X}_2 = x_1 - x_2, \\
\tilde{P}_1 &= P_1 + P_2, \quad \tilde{P}_2 = \frac{1}{2} (P_1 - P_2), \quad [\tilde{X}_1, \tilde{P}_2] = i\hbar \delta_{ij}.
\end{align*}
\]

\( (\tilde{M}_1, \tilde{M}_2) = (2M, \frac{M}{2}) \)

Note that \( \tilde{P}_i = \tilde{M}_i \tilde{X}_i \) and \( P_i = M \dot{x}_i \).

\[
\begin{align*}
\frac{P_1^2}{2M} + \frac{P_2^2}{2M} + \frac{1}{2} M \Omega^2 x_1^2 + \frac{1}{2} M \Omega^2 x_2^2 &= \frac{\tilde{P}_1^2}{2(2M)} + \frac{\tilde{P}_2^2}{2(2M)} + \frac{1}{2} (2M) \Omega^2 \tilde{X}_1^2 + \frac{1}{2} (\frac{M}{2}) \Omega^2 \tilde{X}_2^2.
\end{align*}
\]

(40)

(41)

2. \( N = 3 \):

: We can go from canonical variables \((x_1, x_2, x_3)\) to \((\tilde{X}_1 = X_{cm}, \tilde{X}_2, \tilde{X}_3)\) with the following transformation: Viewing \( \{ x_i \} \) and \( \{ \tilde{X}_i \} \) as orthogonal base of the 3 dimensional vector space, our goal is to find a \( 3 \times 3 \) linear transformation matrix \( T_{ij} \), such that \( \tilde{X}_i = T_{ij} x_j \) with \( \det(T_{ij}) = 1 \). A convenient choice is that since we have already construct the transformation of \( N = 2 \) case, we can just choose \( \tilde{X}_1 = \frac{1}{3} (x_1 + x_2 + x_3) \) and \( \tilde{X}_2 = x_1 - x_2 \). Since \( \tilde{X}_3 \) must perpendicular with \( \tilde{X}_1, \tilde{X}_2 \), we have \( \tilde{X}_3 = a_3 (x_1 + x_2 - 2x_3) \). Since \( \det(T) = 1 \), we then have \( a_3 = \frac{1}{2} \).

Note that \( \tilde{P}_3 = \tilde{M}_3 \tilde{X}_3 \) hence proportional to \( \tilde{X}_3 \), thus we have \( \tilde{P}_3 = b_3 (P_1 + P_2 - 2P_3) \). From \([\tilde{X}_3, \tilde{P}_3] = i\hbar \), we have \( b_3 = \frac{1}{3} \).

\[
\begin{align*}
\tilde{X}_1 &= \frac{1}{3} (x_1 + x_2 + x_3) = X_{cm}, \quad \tilde{X}_2 = x_1 - x_2, \quad \tilde{X}_3 = \frac{1}{2} (x_1 + x_2 - 2x_3),
\end{align*}
\]

13
\[ \tilde{P}_1 = (P_1 + P_2 + P_3), \quad \tilde{P}_2 = \frac{1}{2}(P_1 - P_2), \quad \tilde{P}_3 = \frac{1}{3}(P_1 + P_2 - 2P_3) \]

and we have

\[ [\tilde{X}_i, \tilde{P}_j] = i\hbar \delta_{ij}, \quad [\tilde{X}_i, \tilde{X}_j] = [\tilde{P}_i, \tilde{P}_j] = 0. \]

\[ (\tilde{M}_1, \tilde{M}_2, \tilde{M}_3) = (2M, \frac{M}{2}, \frac{M}{3} = \frac{b}{a}M) \]

\[ \frac{\tilde{P}_1^2}{2M} + \frac{\tilde{P}_2^2}{2M} + \frac{\tilde{P}_3^2}{2M} + \frac{1}{2}M\Omega^2 x_1^2 + \frac{1}{2}M\Omega^2 x_2^2 + \frac{1}{2}M\Omega^2 x_3^2 \]

\[ = \frac{\tilde{P}_1^2}{2(3M)} + \frac{\tilde{P}_2^2}{2(\frac{M}{2})} + \frac{\tilde{P}_3^2}{2(\frac{M}{3})} + \frac{1}{2}(3M)\Omega^2 \tilde{X}_1^2 + \frac{1}{2}M\Omega^2 \tilde{X}_2^2 + \frac{1}{2}(\frac{2M}{3})\Omega^2 \tilde{X}_3^2 \]

3. \( N = 4 \):

Note that in this case \( N = 4 = 2 \times 2 \). We can make use of our result of \( N = 2 \).

Since we require that \( \tilde{X}_1 = \frac{1}{4}(x_1 + x_2 + x_3 + x_4) \) in this case. We may first define \( y_1 = \frac{1}{2}(x_1 + x_2), \ y_2 = \frac{1}{2}(x_3 + x_4), \) and \( y_3 = (x_1 - x_2), y_4 = (x_3 - x_4) \). That means that we treat the whole 4 oscillators as two pairs of oscillators using the center-of-mass and relative coordinates of each pair. The corresponding conjugate momentum are \( P_{y1} = (P_1 + P_2), P_{y2} = (P_3 + P_4), P_{y3} = \frac{1}{2}(P_1 - P_2), P_{y4} = \frac{1}{2}(P_3 - P_4) \). With this transformation, we have \([y_i, P_{yj}] = i\hbar \delta_{ij} \). The effective masses for \( y_i \) would be \((2M, 2M, \frac{M}{2}, \frac{M}{2}) \). It’s easy to see that \( y_3 \) is perpendicular to all the other \( y_i \), and so is \( y_4 \). Hence \( y_3, y_4 \) are perpendicular to any linear combination of \( y_1 \) and \( y_2 \). That means by changing from \( x_i \) to \( y_i \) we can decompose the original 4 dimensional vector space generated by \( x_i \) into direct sums of a two dimensional vector space generated by \( y_1 \) and \( y_2 \) and two one dimensional vector spaces generated by \( y_3 \) and \( y_4 \) respectively.

We can then make use of \( N = 2 \) result for \( y_1, y_2 \). Hence \( \tilde{X}_1 = \frac{1}{2}(y_1 + y_2) = \frac{1}{4}(x_1 + x_2 + x_3 + x_4), \tilde{X}_2 = y_1 - y_2 = \frac{1}{2}(x_1 + x_2 - x_3 - x_4) \). The corresponding \( \tilde{P}_1 = P_{y1} + P_{y2} = P_1 + P_2 + P_3 + P_4, \tilde{P}_2 = \frac{1}{2}(P_{y1} - P_{y2}) = \frac{1}{2}(P_1 + P_2 - P_3 - P_4) \). The effective masses are \((2(2M), \frac{1}{2}(2M)) = (4M, M) \).

Combing with the other two one dimensional vector spaces generated by \( y_3 \) and \( y_4 \), we then
have the following transformation

\[ \tilde{X}_1 = \frac{1}{2}(y_1 + y_2) = \frac{1}{4}(x_1 + x_2 + x_3 + x_4), \]
\[ \tilde{X}_2 = y_1 - y_2 = \frac{1}{2}(x_1 + x_2 - x_3 - x_4) \]
\[ \tilde{X}_3 = y_3 = x_1 - x_2, \]
\[ \tilde{X}_4 = y_4 = x_3 - x_4 \]  

(42)

with corresponding conjugate momentum

\[ \tilde{P}_1 = P_{y_1} + P_{y_2} = P_1 + P_2 + P_3 + P_4, \]
\[ \tilde{P}_2 = \frac{1}{2}(P_{y_1} - P_{y_2}) = \frac{1}{2}(P_1 + P_2 - P_3 - P_4), \]
\[ \tilde{P}_3 = P_{y_3} = \frac{1}{2}(P_1 - P_2), \]
\[ \tilde{P}_4 = P_{y_4} = \frac{1}{2}(P_3 - P_4) \]  

(43)

and we have

\[ [\tilde{X}_i, \tilde{P}_j] = i\hbar\delta_{ij}, \quad [\tilde{X}_i, \tilde{X}_j] = [\tilde{P}_i, \tilde{P}_j] = 0. \]

\[ (\tilde{M}_1, \tilde{M}_2, \tilde{M}_3, \tilde{M}_4) = (4M, M, \frac{M}{2}, \frac{M}{2}) \]

This procedure can be easily applied to any other cases where the number of oscillators are even. For example if \( N = 2k \). We can first define \( y_1 = \frac{1}{2}(x_1 + x_2), y_2 = \frac{1}{2}(x_3 + x_4), ..., y_k = \frac{1}{2}(x_{2k-1} + x_{2k}) \) and \( y_{k+1} = (x_1 - x_2), y_{k+2} = (x_3 - x_4), ..., y_{2k} = (x_{2k-1} - x_{2k}) \). The conjugate momentum are \( P_{y_1} = (P_1 + P_2), P_{y_2} = (P_3 + P_4), ..., P_{y_{k+1}} = \frac{1}{2}(P_1 - P_2), P_{y_{k+2}} = \frac{1}{2}(P_3 - P_4), ..., P_{y_{2k}} = \frac{1}{2}(P_{2k-1} - P_{2k}) \) and the corresponding effective masses are \( (2M, 2M, ..., 2M, \frac{M}{2}, \frac{M}{2}, ..., \frac{M}{2}) \).

We then make use of the transformations for the \( N = k \) case to \( \{y_i\}, i = 1, ..., k \). By doing this we can have \( \{\tilde{X}_i\}, \{\tilde{P}_i\}, \{\tilde{M}_i\}, i = 1, ..., k \). Together with the \( \tilde{X}_j = y_j, \tilde{P}_j = P_j, j = k + 1, ..., 2k \), we then have the complete transformation for \( N = 2k \) case.

4. \( N = 5 \):

Because \( 5 = 4 + 1 \), we can make use of the result for \( N = 4 \) case. But note that in our construction we require that \( \tilde{X}_1 = \frac{1}{6}(x_1 + x_2 + x_3 + x_4 + x_5) \) in this case, hence the corresponding
\( \tilde{P}_1 = (P_1 + P_2 + P_3 + P_4 + P_5) \) and effective mass is 5\( M \). The corresponding \( \tilde{X}_2, \tilde{X}_3, \tilde{X}_4 \) and \( \tilde{P}_2, \tilde{P}_3, \tilde{P}_4 \) and effective masses are the same as those in the case of the \( N = 4 \) case. The remaining \( \tilde{X}_5 \) has to be perpendicular to \( \tilde{X}_i, i = 1, 2, 3, 4 \) and it’s thus obvious that it must be a linear combination of the \( \tilde{X}_i' = \frac{1}{4}(x_1 + x_2 + x_3 + x_4) \) in \( N = 4 \) case and \( x_5 \) such that it is perpendicular to \( \tilde{X}_1 = \frac{1}{5}(x_1 + x_2 + x_3 + x_4 + x_5) \). It’s easy to see that \( \tilde{X}_5 = a_5(x_1 + x_2 + x_3 + x_4 - 4x_5) \). Hence the corresponding conjugate momentum is \( \tilde{P}_5 = b_5(P_1 + P_2 + P_3 + P_4 - 4P_5) \). With the requirement that \( \tilde{X}_i = T_{ij}x_j \) and \( \det(T_{ij}) = 1 \), we have \( a_5 = \frac{1}{4} \). From the requirement \([ \tilde{X}_5, \tilde{P}_5 ] = ih \), we then have \( b_5 = \frac{1}{5} \). The corresponding effective mass is then given by \( \tilde{M}_5 = \frac{b_5}{a_5}M = \frac{4}{5}M \).

Hence the transformation for \( N = 5 \) is given by

\[
\begin{align*}
\tilde{X}_1 &= \frac{1}{5}(x_1 + x_2 + x_3 + x_4 + x_5), \\
\tilde{X}_2 &= \frac{1}{2}(x_1 + x_2 - x_3 - x_4) \\
\tilde{X}_3 &= x_1 - x_2, \\
\tilde{X}_4 &= x_3 - x_4, \\
\tilde{X}_5 &= \frac{1}{4}(x_1 + x_2 + x_3 + x_4 - 4x_5),
\end{align*}
\]

(44)

with corresponding conjugate momentum

\[
\begin{align*}
\tilde{P}_1 &= P_1 + P_2 + P_3 + P_4 + P_5, \\
\tilde{P}_2 &= \frac{1}{2}(P_1 + P_2 - P_3 - P_4), \\
\tilde{P}_3 &= \frac{1}{2}(P_1 - P_2), \\
\tilde{P}_4 &= \frac{1}{2}(P_3 - P_4), \\
\tilde{P}_5 &= \frac{1}{5}(P_1 + P_2 + P_3 + P_4 - 4P_5),
\end{align*}
\]

(45)

and we have

\[
[\tilde{X}_i, \tilde{P}_j] = ih\delta_{ij}, \quad [\tilde{X}_i, \tilde{X}_j] = [\tilde{P}_i, \tilde{P}_j] = 0.
\]

\[
(\tilde{M}_1, \tilde{M}_2, \tilde{M}_3, \tilde{M}_4, \tilde{M}_5) = (5M, M, \frac{M}{2}, \frac{M}{2}, \frac{4M}{5}).
\]

This procedure can be easily applied to any other cases where the number of oscillators are odd. For example if \( N = 2k+1 \). We can make use of the results for \( N = 2k \). Since we already have \( \{x_i\} \) to \( \{\tilde{X}_i\}, i = 1, ..., 2k \) and these \( \tilde{X}_i \) are already orthogonal to each other. With the addition of \( x_{2k+1} \) we
only need to change \( \tilde{X}_1 = \frac{1}{2k+1} (x_1 + x_2 + \ldots + x_{2k} + x_{2k+1}) \) and it’s obvious that the \( \tilde{X}_2k+1 \) must have the structure \( \tilde{X}_{2k+1} = a_{2k+1}(x_1 + x_2 + \ldots + x_{2k} - 2k x_{2k+1}) \). With the requirement that \( \tilde{X}_i = T_{ij} x_j \) and \( \det(T_{ij}) = 1 \), we have \( a_{2k+1} = \frac{1}{2k+1} \). Similarly \( \tilde{P}_{2k+1} = b_{2k+1}(P_1 + P_2 + \ldots + P_{2k} - 2k P_{2k+1}) \). The requirement \( [\tilde{X}_{2k+1}, \tilde{P}_{2k+1}] = i\hbar \) gives \( b_{2k+1} = \frac{1}{2k+1} \) hence the corresponding effective mass is \( \tilde{M}_{2k+1} = \frac{b_{2k+1}}{a_{2k+1}} M = \frac{2kM}{2k+1} = \frac{N-1}{N} M \).

We have thus provided a procedure to explicitly construct the canonical transformations for the canonical coordinates for any finite number of harmonic oscillators. The transformation for the effective masses are also given. With this canonical transformation, the structure of the Poisson brackets are preserved.

C. Proof of the Lemma

Lemma: If the potentials \( V_{ij} \) among the system oscillators \( x_i \) and \( x_j \) are functions of \( x_i - x_j \) only, then \( \frac{\partial V_{ij}}{\partial X_1} = 0 \). Hence the total potential \( \sum_{i,j;i\neq j}^N V_{ij} (x_i - x_j) \) is independent of \( \tilde{X}_1 \).

Proof of the Lemma: The reason is very simple: regarding the original \{\( x_i \), \( i = 1, ..., N \) as an orthonormal coordinate basis which spans a \( N \)-dimensional vector space. My construction for \{\( \tilde{X}_i \), \( i = 1, ..., N \) is another set of complete orthogonal basis which contains the center-of-mass coordinate \( X_{cm} = \frac{1}{N} (x_1 + \ldots + x_N) \) as \( \tilde{X}_1 \). In the \{\( x_i \)\} basis, \( \tilde{X}_1 = \frac{1}{N} (1,1,\ldots,1) \) and \( x_i - x_j = (0,0,\ldots,1,0,\ldots,-1,0,\ldots,0) \) with the i-th entry equals to 1 and the j-th entry equals to -1. Hence it is obvious that \( x_i - x_j \) is orthogonal to \( X_{cm} = \tilde{X}_1 \). Hence \( x_i - x_j \) belongs to the \((N - 1)\)-dimensional vector space spanned by \{\( \tilde{X}_j \), \( j = 2, ..., N \) of the new coordinates. Because \{\( \tilde{X}_i \), \( i = 1, ..., N \) is an orthogonal basis of the \( N \)-dimensional vector space \( V^N \), \{\( \tilde{X}_j \), \( j = 2, ..., N \) is an orthogonal basis for the \((N-1)\)-dimensional subspace \( W^{N-1} \). Let \( L \) be the 1-dimensional vector space spanned by \( \tilde{X}_1 \), then \( V^N = L \oplus W^{N-1}, L \cap W^{N-1} = 0 \). Hence \( (x_i - x_j) \in W^{N-1} \) and can be expressed in terms of \{\( \tilde{X}_j \), \( j = 2, ..., N \) uniquely. Since we assume that the potentials are functions of only \( x_i - x_j \), hence the potentials are functions of \{\( \tilde{X}_j \), \( j = 2, ..., N \) only and are thus independent of \( \tilde{X}_1 \). Q.E.D.

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[1] C. H. Chou, Ting Yu and B. L. Hu, “Exact Master Equation and Quantum Decoherence for Two Harmonic Oscillators in a General Environment” quant-ph/0703088

[2] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D 45, 2843 (1992)

[3] W. Marshall, C. Simon, R. Penrose and D. Bouwmeester, Phys. Rev. Lett. 91, 130401 (2003).

[4] A. Bassi, E. Ippoliti and S. L. Adler, Phys. Rev. Lett. 94 030401 (2005).

[5] J. Z. Bernad, L. Diosi, and T. Geszti, Phys. Rev. Lett. 97, 250404 (2006)

[6] M. P. Blencke, Phys. Rep. 395, 159 (2004) and references therein.

[7] E. Buks and M. P. Blencowe, “Decoherence and Recoherence in a Vibrating RF SQUID” quant-ph/0607106

[8] M. Arndt, O. Nairz, J. Vos-Andreae, C. Keller, G. van der Zouw and A. Zeilinger, Nature 401, 680 (1999).

[9] B. Brezger, L. Hackermiller, S. Uttenthaler, J. Petschinka, M. Arndt, and A. Zeilinger, Phys. Rev. Lett 88, 100404 (2002).

[10] S. Bose, K. Jacobs and P. L. Knight, Phys. Rev. A 59, 3204 (1999).

[11] S. Bose, Phys. Rev. Lett. 96 060402 (2006).

[12] T. Yu, C.H. Chou and B.L. Hu, work in progress.

[13] K. Shiokawa and B. L. Hu, Phys. Rev. A 70 (2004) 062106

[14] A. D. Armour, M. P. Blencowe and K. C. Schwab, Phys. Rev. lett. 88, 148301 (2002).

[15] D. Mozyrsky, I. Martin, and M. B. Hastings, Phys. Rev. Lett. 92, 018303 (2004).

[16] M.D. LaHaye, O. Buu, B. Camarota, K.C. Schwab, Science 304, 74 (2004)

[17] B. L. Hu and Y. Zhang, Mod. Phys. Lett. A8, 3575 (1993); Int. J. Mod. Phys. 10, 4537 (1995)

[18] A. Anderson and J. J. Halliwell, Phys. Rev. D48, 2753 (1993). A. Anastopoulos and J. J. Halliwell, Phys. Rev. D 5, 6870 (1995).

[19] M. P. Blencowe, J. Imbers and A. D. Armour, New J. Phys. 7, 236 (2005).

[20] M. B. Hastings, I. Martin and D. Mozyrsky, Phys. Rev. B 68, 035101(2003).

[21] A. Naik, O. Buu, M. D. LaHaye, A. D. Armour, A. A. Clerk, M. P. Blencowe and K. C. Schwab, Nature 443, 193 (2005).

[22] A. A. Clerk, S. Bennett, New J. Phys. 7, 238 (2005).

[23] D. Rodrigues and Andrew Armour, “Quantum master equation descriptions of a nanomechanical res-
onator coupled to a single-electron transistor” cond-mat/0510236

[24] J. Eisert, M. B. Plenio, S. Bose, and J. Hartley, Phys. Rev. Lett. 93, 190402 (2004).
[25] B. L. Hu, J. P. Paz, and Y. Zhang, Phys. Rev. D 47, 1576 (1993).
[26] J. J. Halliwell and T. Yu, Phys. Rev. D 53, 2012 (1996).
[27] W. T. Strunz and T. Yu, Phys. Rev. A 69, 052115 (2004).

[28] For a single harmonic oscillator coupled to a bath of harmonic oscillators, in addition to those cited before [2, 25, 26] of special relevance to our problem, here is a list of sample references: R. Feynman and F. L. Vernon, Ann. Phys. (N.Y.) 24, 118 (1963); A. O. Caldeira and A. J. Leggett, Physica A 121, 587 (1983); V. Hakim and V. Ambegaokar, Phys. Rev. A 32, 423 (1985); F. Haake and R. Reibold, Phys. Rev. A 32, 2462 (1985); W. G. Unruh and W. H. Zurek, Phys. Rev. D 40, 1071 (1985); H. Grabert, P. Schramm, and G. L. Ingold, Phys. Rep. 168, 115 (1988); J.J. Halliwell and A. Zoupas, Phys. Rev. D 52, 7294 (1995).

[29] A level reduction scheme need be applied to one of the two oscillators, see, e.g., [13], thus turning it into a two-level system. The corresponding transformed equation can be applied to problems modeled by a qubit-oscillator interaction (see, e.g., [11]).

[30] In realistic situations, just saying that the object contains N particles is not enough; there are layers of structure involved. Are the particles molecules, atoms, nucleons or quarks? The coupling strength between constituents at each level of structure (e.g., inter-atomic) compared to that structure’s coupling with the environment (e.g., atom- field in cavity QED, where the field is taken to be the environment) will determine the relative weight of each level of structure’s partaking of the macroscopic object’s overall quantum behavior. It is for this reason that in atom-optical (ev scale) physics we don’t usually mention quarks of the GeV energy scale. If we are interested in atomic scale processes we may refer to atoms as physically ‘relevant’ particles and all sub-level particles as ‘irrelevant’ (borrowing terminology from projection operator formalism in nonequilibrium statistical mechanics), as far as their contribution to the macroscopic quantum phenomena at the atomic physics energy scale is concerned. It is with this assumption that one can view a C60 molecule passing through a double slit showing the familiar diffraction pattern as a single quantum particle, its size notwithstanding.