On the Valuation of Discrete Asian Options in High Volatility Environments

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ABSTRACT
In this paper, we are concerned with the Monte Carlo valuation of discretely sampled arithmetic and geometric average options in the Black-Scholes model and the stochastic volatility model of Heston in high volatility environments. To this end, we examine the limits and convergence rates of asset prices in these models when volatility parameters tend to infinity. We observe, on the one hand, that asset prices, as well as their arithmetic means converge to zero almost surely, while the respective expectations are constantly equal to the initial asset price. On the other hand, the expectation of geometric means of asset prices converges to zero. Moreover, we elaborate on the direct consequences for option prices based on such means and illustrate the implications of these findings for the design of efficient Monte-Carlo valuation algorithms. As a suitable control variate, we need among others the price of such discretely sampled geometric Asian options in the Heston model, for which we derive a closed-form solution.

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1. Introduction

Asian options are options on certain types of averages of the underlying’s prices. They are used as types of risk-reducing options, since the process of averages of asset prices is less volatile than the asset price process itself. From this point of view, Asian options seem to be attractive financial instruments. A seminal work on this subject is the paper of Kemna and Vorst (1990), where besides ordinary time-discrete and time-continuous arithmetic average options, the related geometric average options were introduced.

It is well-known that for geometric Asian options closed-form solutions can easily be derived in the Black-Scholes model, see e.g., Zhang (1998) and Boyle and Potapchik (2008). Yet, already in the Black-Scholes model and even more so in sophisticated asset pricing models as e.g., Heston’s stochastic volatility model, it is very difficult to obtain similar results for arithmetic Asian options, and one usually relies on either approximated prices, respectively on a valuation based on a suitable Monte-Carlo algorithm (compare...
again Zhang (1998); Boyle and Potapchik (2008), or on other (numerical) methods. For instance, Geman and Yor (1993) give the Laplace transform of the time-integral of a geometric Brownian motion, which can be used to compute the price of an arithmetic Asian option. Moreover, a suitable PDE method for the pricing of such options can be found in Vecer (2001). In this spirit, in Kemna and Vorst (1990), the closed-form prices of geometric Asian calls (puts) were used as lower (upper) bounds for the prices of arithmetic Asian calls (puts).

In this paper, we are concerned with the Monte Carlo valuation of discretely sampled arithmetic and geometric average options in the Black-Scholes model and the stochastic volatility model of Heston in high volatility environments, which is a very relevant question from a practical and a theoretical point of view.

On a formal level, we share common features with the literature strand on large deviations respectively asymptotic analysis in stochastic volatility models, see for instance (Guliashvili and Stein (2010); Jacquier and Mijatović (2014); Jacquier, Keller-Ressel, and Mijatović (2013)). Guliashvili and Stein (2010) thereby derive asymptotic expansions for the density of the stock price and for the implied volatility in the Stein-Stein model and the Heston model. Jacquier and Mijatović (2014) and Jacquier, Keller-Ressel, and Mijatović (2013) derive large deviation principles for affine stochastic volatility models (with and without jumps) for large times, also deducing asymptotics for implied volatilities as well as option prices. The difference is that in addition to large deviations theory, we can derive an almost sure convergence of the stock price instead of a point-wise convergence of the characteristic function.

However, to the best of our knowledge, the pricing of Asian options with respect to high volatility parameters has barely investigated. One notable exception is Carr, Ewald, and Xiao (2008), in which the authors show that in the Black–Scholes model, the price of an arithmetic average Asian call with fixed strike is a strictly increasing function of the volatility parameter \( \sigma > 0 \). Yet, besides this qualitative result, no explicit form of the high volatility limit \( \sigma \to \infty \) is available for the prices of Asian options. Moreover, this analysis has only been carried out in the framework of the Black-Scholes model. Of course, this limiting behaviour is directly connected to the limit of the underlying share price(s) of the investigated options. Therefore, in our paper, we aim at closing this gap

- by determining the limiting behaviour of the underlying share price in the Black-Scholes model for the volatility parameter \( \sigma \to \infty \),
- by determining the limiting behaviour of the underlying share price in the stochastic volatility model of Heston for the long term variance \( \theta \to \infty \) and (the starting value of) the instantaneous variance \( \nu_0 \to \infty \),
- and by applying these results to the concrete valuation of Asian options, both in the Black-Scholes and in the Heston model.

In order to obtain the limiting behaviour of the stock price in Heston’s stochastic volatility model, we first have to deal with the high volatility limit of the underlying variance process, i.e., the well-known Cox-Ingersoll-Ross (CIR) process, which we provide as well in Section 2, including the exact rates of convergence.

In Section 3, we are then concerned with the concrete valuation of Asian options when volatility levels of the underlying asset prices become high (i.e., tend to infinity), both for
the Black-Scholes and the Heston model. In particular, we deduce explicit limits for the prices of discrete arithmetic Asian options (Section 3.1), discrete geometric Asian options (Section 3.2) and discrete arithmetically averaged options (Section 3.3). These limiting behaviours have implications for the choice of the right control variate when valuing these options with the help of Monte Carlo simulation. As corresponding control variate, we need among others the price of a discrete geometric Asian option in the Black-Scholes and in the Heston model. Whereas this price is well-known in the Black-Scholes model, as far as we can tell, it is not in the Heston model. Therefore we derive a closed-form solution in Heston’s model in Section 4.

To exemplify our results, in Section 5, we give examples of Monte-Carlo valuations of Asian options using different control variates and increasing volatilities under both, the Black-Scholes and Heston model. It turns out that simulation under high volatilities is extremely sensitive to the right choice of the control variate.

1.1. Framework

We consider a general discounted asset price process \( S^{(\Sigma)} = (S_t^{(\Sigma)})_{t \in [0, T]} \) on a finite time horizon \([0, T]\) depending on the \( \mathbb{R}^m \)-valued vector \( \Sigma \) of positive volatility parameters with \( S_0^{(\Sigma)} \equiv s_0 \) for all \( \Sigma \). Since we are mainly interested in the dependence of the asset price on \( \Sigma \), we add this parameter as a superscript. Moreover, we let

\[
|\Sigma| := \left( \sum_{i=1}^{m} \sigma_i^2 \right)^{1/2},
\]

where \( \sigma_1, \ldots, \sigma_m \) denote volatility parameters of the underlying stochastic processes, depending on the concrete model we are investigating.

We always assume an arbitrage-free market model and work under the market’s chosen risk-neutral martingale measure \( Q \), for which the discounted price process is a martingale w.r.t. \( Q \) and the filtration \((\mathcal{F}_t)_{0 \leq t \leq T}\), i.e.,

\[
S_{t_1}^{(\Sigma)} = \mathbb{E}[S_{t_2}^{(\Sigma)} | \mathcal{F}_{t_1}]
\]

for all \( 0 \leq t_1 \leq t_2 \leq T \). In particular,

\[
\mathbb{E}[S_T^{(\Sigma)}] = s_0.
\]

1.2. Observations in the Black-Scholes Model

In the Black-Scholes model (cf. Black and Scholes 1973), the asset price \( S^{(\sigma)} \) is given by

\[
S_t^{(\sigma)} = s_0 \exp(-\sigma^2 t/2 + \sigma W_t),
\]

where \((W_t)\) is a standard Brownian motion, and where we assumed zero interest rates. Formally, we therefore let \( \Sigma \equiv \sigma \).

To motivate what is coming, we record the following obvious result:
Proposition 1.1 (High Volatility Share Price Convergence Rate in the Black-Scholes Model): We have

\[
\lim_{\sigma \to \infty} \frac{\log(S_t^{(\sigma)})}{\sigma^2} = -\frac{t}{2} \quad \text{a.s. for all } t > 0.
\]

In particular, the share price \( S_t^{(\sigma)} \) converges to zero almost surely as \( \sigma \to \infty \).

**Proof:** From (3) we immediately obtain

\[
\frac{\log(S_t^{(\sigma)})}{\sigma^2} = \frac{\log(s_0)}{\sigma^2} - \frac{t}{2} + \frac{W_t}{\sigma}.
\]

The assertion now follows from

\[
\frac{W_t}{\sigma} \to 0 \quad \text{a.s. if } \sigma \to \infty \text{ for all } t > 0.
\]

This means that in the Black-Scholes model for \( t > 0 \) the asset price \( S_t^{(\sigma)} \) almost surely converges to zero if the Black volatility \( \sigma \) goes to infinity.

2. The Heston Model

The question we want to pursue for the first part of this paper is if an analogue result to Proposition 1.1 holds within the stochastic volatility model of Heston (cf. Heston 1993), which is given by the set of stochastic differential equations

\[
dS_t^{(\Sigma)} = \sqrt{v_t^{(\Sigma)} S_t^{(\Sigma)}} dW_t^{(1)}, \tag{4}
dv_t^{(\Sigma)} = \kappa (\theta - v_t^{(\Sigma)}) dt + \eta \sqrt{v_t^{(\Sigma)}} dW_t^{(2)}, \tag{5}
\]

with correlated Brownian motions \( d\langle W^{(1)}, W^{(2)} \rangle_t = \rho \, dt \). In order to emphasize the dependence of the process \( \nu \) on \( \nu_0 \) and \( \theta \), we use the notation \( (v_t^{(\Sigma)}) \) with \( \Sigma \equiv (\nu_0, \theta) \) for all \( t > 0 \). The constants \( \nu_0, \kappa, \theta, \eta \) are assumed to be positive and to satisfy the Feller condition \( 2\kappa \theta \geq \eta^2 \). In what follows, we are interested in the behaviour of the processes \( (v_t^{(\Sigma)}) \) and \( (S_t^{(\Sigma)}) \) when \( |\Sigma| = \sqrt{\nu_0^2 + \theta^2} \to \infty \).

In order to deduce the limiting behaviour of the share price process (4) in the Heston model, we have to consider the high volatility behaviour of the Cox-Ingersoll-Ross process (5), which goes originally back to Cox et al. (1985).

**Theorem 2.1 (High Volatility Convergence Rates of the Heston Variance Process):** We have

\[
\lim_{\nu_0 \to \infty} \frac{v_t^{(\Sigma)}}{\nu_0} = \exp(-\kappa t), \quad \lim_{\theta \to \infty} \frac{v_t^{(\Sigma)}}{\theta} = 1 - \exp(-\kappa t) \quad \text{a.s.}
\]

for all \( t > 0 \). In particular, the variance \( v_t^{(\Sigma)} \) converges to infinity almost surely as either or both the initial variance \( \nu_0 \) or the long term variance \( \theta \) go to infinity.
**Proof:** The proof will be accomplished in several steps.

**Definition of processes.**

Let

\[ x_t^{(v_0)} := v_t^{(\Sigma)} / v_0 \quad \text{and} \quad y_t^{(\theta)} := v_t^{(\Sigma)} / \theta. \]

For some fixed \( p > 2 \) denote

\[ u(t) := \mathbb{E}[|\exp(-\kappa t) - x_t^{(v_0)}|^p] \quad \text{and} \quad v(t) := \mathbb{E}[|1 - \exp(-\kappa t) - y_t^{(\theta)}|^p]. \]

**Calculation of moments.**

Note that Proposition A.1 of the Appendix implies

\[
\mathbb{E}[|x_s^{(v_0)}|^{p/2}] = \frac{1}{v_0^{p/2}} \mathbb{E}[|v_s^{(\Sigma)}|^{p/2}] \leq C \left( \frac{c_1 \theta + c_2 v_0}{v_0} \right)^{p/2},
\]

\[
\mathbb{E}[|y_s^{(\theta)}|^{p/2}] = \frac{1}{\theta^{p/2}} \mathbb{E}[|v_s^{(\Sigma)}|^{p/2}] \leq C \left( \frac{c_1 \theta + c_2 v_0}{\theta} \right)^{p/2}
\]

for some constants \( C, c_1, \) and \( c_2 \) independent of \( v_0 \) and \( \theta \). In particular, these moments are bounded, when \( v_0 \) or \( \theta \) go to infinity, respectively.

**Convergence in \( L_p \).**

Integrating the definition of the variance process (5) we obtain

\[
x_t^{(v_0)} - \exp(-\kappa t) = \frac{\kappa \theta t}{v_0} + \kappa \int_0^t (\exp(-\kappa s) - x_s^{(v_0)}) \, ds + \frac{\eta}{\sqrt{v_0}} \int_0^t \sqrt{x_s^{(v_0)}} \, dW_s^{(2)},
\]

\[
y_t^{(\theta)} - 1 + \exp(-\kappa t) = y_0^{(\theta)} + \kappa \int_0^t (1 - \exp(-\kappa s) - y_s^{(\theta)}) \, ds + \frac{\eta}{\sqrt{\theta}} \int_0^t \sqrt{y_s^{(\theta)}} \, dW_s^{(2)}.
\]

Taking the \( L_p \)-norm and using the triangle inequality for the \( L_p \)-norm, we obtain

\[
\mathbb{E}[|x_t^{(v_0)} - \exp(-\kappa t)|^p]^{1/p} \leq \frac{\kappa \theta t}{v_0} + \kappa \mathbb{E} \left[ \left| \int_0^t (\exp(-\kappa s) - x_s^{(v_0)}) \, ds \right|^p \right]^{1/p}
\]

\[
+ \frac{\eta}{\sqrt{v_0}} \mathbb{E} \left[ \left| \int_0^t \sqrt{x_s^{(v_0)}} \, dW_s^{(2)} \right|^p \right]^{1/p}, \tag{6}
\]

\[
\mathbb{E}[|y_t^{(\theta)} - 1 + \exp(-\kappa t)|^p]^{1/p} \leq y_0^{(\theta)} + \kappa \mathbb{E} \left[ \left| \int_0^t (1 - \exp(-\kappa s) - y_s^{(\theta)}) \, ds \right|^p \right]^{1/p}
\]

\[
+ \frac{\eta}{\sqrt{\theta}} \mathbb{E} \left[ \left| \int_0^t \sqrt{y_s^{(\theta)}} \, dW_s^{(2)} \right|^p \right]^{1/p}. \tag{7}
\]
Now writing \( c_p := (p(p-1)/2)^{1/2} \), the last terms in both inequalities can be estimated using It’s rule as in Exercise 3.25 in Karatzas and Shreve (1991):

\[
E \left[ \left| \int_0^t \sqrt{x_s^{(v_0)}} \, dW_s^{(2)} \right|^p \right]^{1/p} \leq c_p E \left[ \left( \int_0^t |x_s^{(v_0)}| \, ds \right)^{p/2} \right]^{1/p}
\]

\[
\leq c_p \left( \int_0^t E[|x_s^{(v_0)}|^{p/2}] \, ds \right)^{1/2},
\]

\[
E \left[ \left| \int_0^t \sqrt{y_s^{(\theta)}} \, dW_s^{(2)} \right|^p \right]^{1/p} \leq c_p E \left[ \left( \int_0^t |y_s^{(\theta)}| \, ds \right)^{p/2} \right]^{1/p}
\]

\[
\leq c_p \left( \int_0^t E[|y_s^{(\theta)}|^{p/2}] \, ds \right)^{1/2}.
\]

Note that the last inequality in both lines follows from Jessen’s inequality (see Proposition A.2 in the Appendix).

For the second terms on the right hand side of (6) and (7) using Jessen’s inequality again, we obtain respectively

\[
E \left[ \left| \int_0^t (\exp(-\kappa s) - x_s^{(v_0)}) \, ds \right|^p \right]^{1/p} \leq \int_0^t E[|\exp(-\kappa s) - x_s^{(v_0)}|]^{1/p} \, ds,
\]

\[
E \left[ \left| \int_0^t (1 - \exp(-\kappa s) - y_s^{(\theta)}) \, ds \right|^p \right]^{1/p} \leq \int_0^t E[|1 - \exp(-\kappa s) - y_s^{(\theta)}|]^{1/p} \, ds.
\]

Putting all estimates together, we have shown that

\[
u(t)^{1/p} \leq \frac{a}{v_0^{1/2}} + b \int_0^t u(s)^{1/p} \, ds \quad \text{and} \quad \nu(t)^{1/p} \leq \frac{a}{\theta^{1/2}} + b \int_0^t v(s)^{1/p} \, ds
\]

for \( v_0 \) and \( \theta \) large enough, respectively, and for constants \( a \) and \( b \) that do not depend on \( v_0 \) and \( \theta \). Using the fact that \( (x+y)^p \leq 2^{p-1}(x^p + y^p) \) and Hölder’s inequality, we are in the situation to apply Gronwall’s inequality (Proposition A.3) for (possibly larger) constants \( a \) and \( b \) to obtain

\[
u(t) \leq \frac{a}{v_0^{p/2}} \exp(bt) \quad \text{and} \quad \nu(t) \leq \frac{a}{\theta^{p/2}} \exp(bt).
\]

This proves \( L_p \)-convergence.

*Almost sure convergence along 1, 2, \ldots*

Using the Borel-Cantelli Lemma A.4, we obtain almost sure convergence for values \( v_0 = 1, 2, \ldots \) and \( \theta = 1, 2, \ldots \). To this end, let

\[
A_{v_0}(\epsilon) := \{ \omega : |x_t^{(v_0)} - \exp(-\kappa t)| > \epsilon \}
\]

where \( t > 0 \). From what we just proved and Markov’s inequality, we get

\[
P[A_{v_0}(\epsilon)] \leq \frac{u(t)}{\epsilon} \leq \frac{a}{\epsilon v_0^{p/2}} \exp(bt).
\]
Since for \( p > 2 \), the sum \( \sum_{n=0}^{\infty} n^p/2 \) converges, by the Borel-Cantelli Lemma A.4, we must have

\[
x_t^{(n)} \to \exp(-\kappa t) \quad \text{a.s.}
\]

An analogue argument shows the almost sure convergence along \( \theta = 1, 2, \ldots \) for \( y_t^{(\theta)} \).

*Almost sure convergence for the whole parameter set.*

Finally, we can apply the Comparison Theorem (Karatzas and Shreve 1991, Chapter 5, Proposition 2.18)) to conclude that

\[
x_t^{(n+1)} \leq x_t^{(n)} \quad \text{and} \quad y_t^{(n+1)} \leq y_t^{(n)} \quad \text{a.s.}
\]

whenever \( n \leq n_0 < n+1 \) or \( n \leq \theta < n+1 \), which proves the convergence for \( v_0 \to \infty \) and \( \theta \to \infty \). Here we used the function \( h(x) = \sqrt{x} \) in (Karatzas and Shreve 1991, (2.24)) and the facts that \( x_0^{(n_0)} = 1, \theta/(n+1) < \theta/v_0 \leq \theta/n, \) and \( y_0^{(n+1)} = v_0/(n+1) < y_0^{(\theta)} = v_0/\theta \leq y_0^{(n)} = v_0/n \) and \( \kappa(1-y) \) does not depend on the parameter \( \theta \).

We are now ready to prove the analogous result to Proposition 1.1 for the Heston model.

**Theorem 2.2 (High Volatility Share Price Convergence Rates in the Heston Model):** We have

\[
\lim_{v_0 \to \infty} \frac{\log(S_t^{(\Sigma)})}{v_0} = \frac{\exp(-\kappa t) - 1}{2\kappa}, \quad \lim_{\theta \to \infty} \frac{\log(S_t^{(\Sigma)})}{\theta} = 1 - \frac{\exp(-\kappa t) - \kappa t}{2\kappa} \quad \text{a.s.}
\]

for all \( t > 0 \). In particular, the share price \( S_t^{(\Sigma)} \) converges to zero almost surely as either or both the initial variance \( v_0 \) or the long term variance \( \theta \) go to infinity.

**Proof:** Taking the corresponding limits, and using dominated convergence\(^3\), Theorem 2.1 implies

\[
\lim_{v_0 \to \infty} \frac{1}{v_0} \int_0^t \sqrt{v_s^{(\Sigma)}} \, dW_s^{(1)} = 0, \quad \lim_{\theta \to \infty} \frac{1}{\theta} \int_0^t \sqrt{v_s^{(\Sigma)}} \, dW_s^{(1)} = 0
\]

almost surely for all \( t > 0 \).

From the asymptotic behaviour of the variance as in Theorem 2.1 and again by dominated convergence, we moreover obtain

\[
\lim_{v_0 \to \infty} \frac{1}{v_0} \int_0^t v_s^{(\Sigma)} \, ds = \frac{1 - \exp(-\kappa t)}{\kappa} > 0,
\]

\[
\lim_{\theta \to \infty} \frac{1}{\theta} \int_0^t v_s^{(\Sigma)} \, ds = \frac{\kappa t - (1 - \exp(-\kappa t))}{\kappa} > 0
\]

almost surely for all \( t > 0 \).
From the integrated defining equation (4) of the asset price, we see that

\[
\frac{\log(S_t^{(\Sigma)})}{\nu_0} = \frac{\log(s_0)}{\nu_0} - \frac{1}{2\nu_0} \int_0^t \nu(s) ds + \frac{1}{\nu_0} \int_0^t \sqrt{\nu(s)} dW_s^{(1)}
\]

and

\[
\frac{\log(S_t^{(\Sigma)})}{\theta} = \frac{\log(s_0)}{\theta} - \frac{1}{2\theta} \int_0^t \nu(s) ds + \frac{1}{\theta} \int_0^t \sqrt{\nu(s)} dW_s^{(1)}.
\]

Taking limits and plugging in Equations (8) and (9), the results follow. Note that again, by dominated convergence, we are allowed to interchange limit and integral.

Notably, for the limit \( \nu_0 \to \infty \), we obtain in the special case of \( \kappa = 0 \) that

\[
\lim_{\nu_0 \to \infty} \frac{\log(S_t^{(\Sigma)})}{\nu_0} = -\frac{t}{2},
\]

almost surely for all \( t > 0 \), and thus recovering the result of Proposition 1.1.

**Remark 2.1:** As noted in the introduction, our analysis shares certain similarities with the literature on asymptotic analysis and large deviations. Notably, Gulisashvili and Stein (2010) give explicit formulae for the leading term for the density in the asymptotic expansion of the time average of the squared volatility process and the density of the stock price, whereas our focus lies on the exact almost sure limiting values and convergence rates of the Heston variance and stock price processes (cf. Theorems 2.1 and 2.2) when volatility is high. Moreover, due to the well-known scaling properties of Brownian motion, there is a close relation between large-time and large-volatility limits. With this in mind, the application of the ideas from Jacquier and Mijatović (2014) and Jacquier, Keller-Ressel, and Mijatović (2013) to the large volatility case seems to be a promising future research direction.

### 3. Application to the Pricing of Asian Options

In the following we will consider Asian options with strike price(s) \( K > 0 \), discretely monitored at time points \( t_0 = 0 < t_1 < \cdots < t_n = T \), where \( n \geq 2 \). As it is known from general option pricing theory, the (non-discounted) price at time \( t = 0 \) of any option payoff profile \( \Phi(S^{(\Sigma)}) \) with (remaining) lifetime \([0, T]\) and depending on \( S^{(\Sigma)} \) is equal to its (risk-neutral) expectation \( \mathbb{E}[\Phi(S^{(\Sigma)})] \).

Rephrasing the definition of weak convergence to the setting of option payoffs, we have the following:

**Lemma 3.1:** Let \( \Phi : \mathbb{R}^n \to \mathbb{R} \) be a bounded and continuous payoff profile. Assume that \( \Phi(S^{(\Sigma)}) \) converges weakly to a real-valued random variable \( \Phi \) as \( |\Sigma| \to \infty \). Then

\[
\lim_{|\Sigma| \to \infty} \mathbb{E}[\Phi(S^{(\Sigma)})] = \mathbb{E}[\Phi].
\]
In particular, for every non-negative continuous function $\Lambda : \mathbb{R}^n \to \mathbb{R}$ such that $\Lambda(S^{(\Sigma)})$ converges weakly to $\Lambda$, we have

$$\lim_{|\Sigma| \to \infty} \mathbb{E} \left[ (K - \Lambda(S^{(\Sigma)}))^+ \right] = \mathbb{E} \left[ (K - \Lambda)^+ \right]. \quad (10)$$

Moreover, if $\lambda_0 := \lim_{|\Sigma| \to \infty} \mathbb{E}[\Lambda(S^{(\Sigma)})]$ exists we have

$$\lim_{|\Sigma| \to \infty} \mathbb{E} \left[ (\Lambda(S^{(\Sigma)}) - K)^+ \right] = \lambda_0 - K + \mathbb{E} \left[ (K - \Lambda)^+ \right].$$

**Proof:** The first part follows by dominated convergence as the payoff profile is bounded. For the second part, note that

$$0 \leq (K - \Lambda(S^{(\Sigma)}))^+ \leq K$$

and

$$|(K - \lambda_1)^+ - (K - \lambda_2)^+| \leq |\lambda_1 - \lambda_2|,$$

hence the map $\lambda \to (K - \lambda)^+$ is bounded and continuous and (10) follows from the first part.

The last equation is a consequence of the put-call-parity

$$(\lambda - K) = (\lambda - K)^+ - (K - \lambda)^+. \quad \square$$

Throughout the remainder of this section, we assume that the asset price process $S^{(\Sigma)}$ satisfies the condition

$$\lim_{|\Sigma| \to \infty} S_t^{(\Sigma)} = 0 \quad (11)$$

in probability for all $t > 0$.

The validity of this condition is a straight-forward consequence of Proposition 1.1 in the Black-Scholes-Merton model and of Theorem 2.2 in Heston’s stochastic volatility model.

**Remark 3.1:** Note that in direct contrast to (11), by the martingale property of the asset price we have that

$$\lim_{|\Sigma| \to \infty} \mathbb{E}[S_t^{(\Sigma)}] = \mathbb{E}[S_t^{(\Sigma)}] = s_0.$$

Intuitively this is caused by the fact that the density of the log-normal distribution has more mass close to zero for higher values of volatility. These facts, together with (11), have direct consequences on the pricing of options and the simulation of asset prices, which we investigate in the following sections.
3.1. Pricing of Arithmetic Asian Options

In this paragraph, we consider discretely monitored arithmetic average options with prices of the underlying given at time points \( t_1, \ldots, t_n \) and payoffs

- \( (\frac{1}{n} \sum_{j=1}^{n} S_{t_j} - K)^+ \) for the call,
- \( (K - \frac{1}{n} \sum_{j=1}^{n} S_{t_j}^{(\Sigma)})^+ \) for the put.

There is no closed-form solution for Asian options on the arithmetic mean, thus they are priced with the help of Monte-Carlo simulation. Usually, these simulations use the price of Asian options on the geometric mean as control variate, compare Section 3.2. However, we know a few things about their behaviour in the high volatility limit. We deduce the following result:

**Corollary 3.2:** Assume that \( S^{(\Sigma)} \) satisfies (11). Then the price of a discrete arithmetic average put converges to the strike as the volatility goes to infinity, i.e.,

\[
\lim_{|\Sigma| \to \infty} \mathbb{E} \left[ \left( K - \frac{1}{n} \sum_{j=1}^{n} S_{t_j}^{(\Sigma)} \right)^+ \right] = K.
\]

The price of a generalized arithmetic average call converges to the spot as the volatility goes to infinity, i.e.,

\[
\lim_{|\Sigma| \to \infty} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{j=1}^{n} S_{t_j}^{(\Sigma)} - K \right)^+ \right] = s_0.
\]

**Proof:** We use \( \Lambda(S^{(\Sigma)}) = \frac{1}{n} \sum_{j=1}^{n} S_{t_j}^{(\Sigma)} \) in Lemma 3.1. By the martingale property (2), we immediately obtain for all \( \Sigma \) that

\[
\mathbb{E}[\Lambda(S^{(\Sigma)})] = \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} S_{t_j}^{(\Sigma)} \right] = s_0.
\]

Hence \( \lambda_0 = \lim_{|\Sigma| \to \infty} \mathbb{E}[\Lambda(S^{(\Sigma)})] = s_0 \). Moreover, it follows from (11) that \( \lim_{|\Sigma| \to \infty} \Lambda(S^{(\Sigma)}) = \lim_{|\Sigma| \to \infty} \frac{1}{n} \sum_{j=1}^{n} S_{t_j}^{(\Sigma)} = 0 \) and therefore \( \Lambda = 0 \). The results are thus an immediate consequence of Lemma 3.1.

**Remark 3.2:** Note that this is one of the crucial findings of the paper. In particular, the price of the call might be counter-intuitive, as for large values of \( K \), its price should tend to zero. This is also in line with the reasoning of Remark 3.1.

3.2. Pricing of Geometric Asian Options

In this paragraph, we consider discretely monitored geometric average options with prices of the underlying given at time points \( t_1, \ldots, t_n \) and payoffs

- \( (\frac{1}{n} \sum_{j=1}^{n} S_{t_j}^{(\Sigma)} - K)^+ \) for the call,
- \( (K - \frac{1}{n} \sum_{j=1}^{n} S_{t_j}^{(\Sigma)})^+ \) for the put.
• \(((\prod_{j=1}^{n} S_{t_{j}}^{(\Sigma)})^{1/n} - K)^{+}\) for the call,
• \((K - (\prod_{j=1}^{n} S_{t_{j}}^{(\Sigma)})^{1/n})^{+}\) for the put.

For shortening notation, we write

\[ G_{n}^{(\Sigma)} := \left( \prod_{j=1}^{n} S_{t_{j}}^{(\Sigma)} \right)^{1/n} . \]

First, we note the following obvious result:

**Lemma 3.3:** If \( S^{(\Sigma)} \) satisfies (11), then

\[ \lim_{|\Sigma| \to \infty} G_{n}^{(\Sigma)} = 0 \]

in probability.

However, in contrast to the expectation of the arithmetic mean, the expectation of the geometric mean converges to zero in this setting. This convergence also holds in expectation, which is needed for the actual option pricing problem.

**Lemma 3.4:** Assume that \( S^{(\Sigma)} \) satisfies (11). Then we obtain

\[ \lim_{|\Sigma| \to \infty} \mathbb{E}[G_{n}^{(\Sigma)}] = 0. \]

We prove Lemma 3.4 for the Heston Model only. The Black-Scholes-case is then obtained as the special case with \( \nu_{0} = \sigma^{2} \) and \( \eta = \kappa = 0 \). For ease of notation, we skip the dependencies on \( \Sigma \) during the proof, and we formally set \( t_{0} = 0 \).

**Proof:** We start with the case of \( n = 2 \). For \( t_{2} > t_{1} \), we obtain in the Heston model:

\[
S_{t_{2}} = S_{0} \exp \left( -\frac{1}{2} \int_{0}^{t_{1}} \nu_{s} \, ds + \int_{0}^{t_{1}} \sqrt{\nu_{s}} \, dW_{s} \right) \cdot \exp \left( -\frac{1}{2} \int_{t_{1}}^{t_{2}} \nu_{s} \, ds + \int_{t_{1}}^{t_{2}} \sqrt{\nu_{s}} \, dW_{s} \right).
\]

Therefore

\[
(S_{t_{1}}, S_{t_{2}})^{1/2} = S_{0} \exp \left( -\frac{1}{2} \int_{0}^{t_{1}} \nu_{s} \, ds + \int_{0}^{t_{1}} \sqrt{\nu_{s}} \, dW_{s} \right) \exp \left( -\frac{1}{4} \int_{t_{1}}^{t_{2}} \nu_{s} \, ds + \frac{1}{2} \int_{t_{1}}^{t_{2}} \sqrt{\nu_{s}} \, dW_{s} \right).
\]

Assuming Feller’s condition to hold we can always find suitable parameters \( \kappa, \theta, \eta \) such that Novikov’s condition is satisfied in Heston’s model (compare Theorem 3.5 in Wong...
Therefore we obtain with $\alpha > 0$ in a first step

$$\mathbb{E}[(S_{t_1} S_{t_2})^{1/2}] = S_0 \cdot \mathbb{E} \left[ \exp \left( -\frac{1}{4} \int_{t_1}^{t_2} \nu_s \, ds + \frac{1}{2} \int_{t_1}^{t_2} \sqrt{\nu_s} \, dW_s \right) \right]$$

$$= S_0 \cdot \mathbb{E} \left[ \exp \left( -\frac{\alpha}{4} \int_{t_1}^{t_2} \nu_s \, ds \right) \right] \cdot \exp \left( -\frac{(1-\alpha)}{4} \int_{t_1}^{t_2} \nu_s \, ds + \frac{1}{2} \int_{t_1}^{t_2} \sqrt{\nu_s} \, dW_s \right).$$

Moreover, by Hölder’s inequality for $p, q > 0$ such that $1/p + 1/q = 1$, we get

$$\mathbb{E}[(S_{t_1} S_{t_2})^{1/2}] \leq S_0 \cdot \left( \mathbb{E} \left[ \exp \left( -\frac{\alpha}{4} p \int_{t_1}^{t_2} \nu_s \, ds \right) \right] \right)^{1/p} \cdot \left( \mathbb{E} \left[ \exp \left( -\frac{(1-\alpha)}{4} q \int_{t_1}^{t_2} \nu_s \, ds + \frac{1}{2} \int_{t_1}^{t_2} q \sqrt{\nu_s} \, dW_s \right) \right] \right)^{1/q}.$$

Then, with the admissible choice $p = 3$, $q = 3/2$ and $\alpha = 1/4$ one has

$$\mathbb{E}[(S_{t_1} S_{t_2})^{1/2}] \leq S_0 \cdot \left( \mathbb{E} \left[ \exp \left( -\frac{3}{16} \int_{t_1}^{t_2} \nu_s \, ds \right) \right] \right)^{1/3} \cdot \left( \mathbb{E} \left[ \exp \left( -\frac{9}{32} \int_{t_1}^{t_2} \nu_s \, ds + \frac{3}{4} \int_{t_1}^{t_2} \sqrt{\nu_s} \, dW_s \right) \right] \right)^{2/3}$$

$$= S_0 \cdot \left( \mathbb{E} \left[ \exp \left( -\frac{3}{16} \int_{t_1}^{t_2} \nu_s \, ds \right) \right] \right)^{1/3}.$$

where we have again used Theorem 3.5 in Wong and Heyde (2006), respectively Theorem 3.4 in Desmettre, Leobacher, and Rogers (2021) in the last line.

Thus we obtain the limiting behaviour by a direct application of Theorem 2.1 and a dominated convergence argument that

$$\lim_{|\Sigma| \to \infty} \mathbb{E}[(S_{t_1} S_{t_2})^{1/2}] = \lim_{|\Sigma| \to \infty} S_0 \cdot \left( \mathbb{E} \left[ \exp \left( -\frac{3}{16} \int_{t_1}^{t_2} \nu_s \, ds \right) \right] \right)^{1/3} = 0.$$ 

For the general case note that

$$G_n = (S_{t_1})^{1/n} \cdots (S_{t_n})^{1/n}.$$
Therefore, by the generalized Hölder inequality with $p = n/2 > 0$ and $q_j = n > 0$, $j = 3, \ldots, n$ such that $1/p + \sum_{j=3}^{n} 1/q_j = 2/n + \sum_{j=3}^{n} 1/n = 1$, we obtain

$$
\mathbb{E}[G_n] = \mathbb{E} \left[ (S_{t_1} S_{t_2})^{1/n} \cdot \prod_{j=3}^{n} S_{t_j}^{1/n} \right] \\
\leq \mathbb{E} \left[ (S_{t_1} S_{t_2})^{(1/n)(n/2)} \cdot \prod_{j=3}^{n} \mathbb{E}[S_{t_j}^{(1/n)\cdot n}]^{1/n} \right] \\
= \mathbb{E} \left[ (S_{t_1} S_{t_2})^{1/2} \cdot \prod_{j=3}^{n} \mathbb{E}[S_{t_j}]^{1/n} \right] = \mathbb{E} \left[ (S_{t_1} S_{t_2})^{1/2} \cdot (S_0)^{(n-2)/n} \right],
$$

where we have used once more Theorem 3.5 in Wong and Heyde (2006), respectively Theorem 3.4 in Desmettre, Leobacher, and Rogers (2021). The remaining part

$$
\mathbb{E}[((S_{t_1} S_{t_2})^{1/2})^{2/n}] = \mathbb{E}[(S_{t_1} S_{t_2})^{1/n}]
$$

can be dealt with exactly in the same way as in the case $n = 2$, and thus we obtain in an analogue manner, again using Theorem 2.1 and a dominated convergence argument that

$$
\lim_{|\Sigma| \to \infty} \mathbb{E}[G_n] = \lim_{|\Sigma| \to \infty} (S_0)^{(n-2)/n} \cdot \mathbb{E} \left[ \exp \left( -\frac{3}{16} \int_{t_1}^{t_2} \nu_s \, ds \right) \right]^{2/(3n)} = 0.
$$

As a direct consequence, we obtain:

**Corollary 3.5:** Assume that $S^{(\Sigma)}$ satisfies (11). Then the price of a discrete geometric average put converges to the strike as the volatility goes to infinity, i.e.,

$$
\lim_{|\Sigma| \to \infty} \mathbb{E}[(K - G_n^{(\Sigma)})^+] = K.
$$

The price of a discrete geometric average call converges to zero as the volatility goes to infinity, i.e.,

$$
\lim_{|\Sigma| \to \infty} \mathbb{E}[(G_n^{(\Sigma)} - K)^+] = 0.
$$

**Proof:** We use $\Lambda(S^{(\Sigma)}) = G_n^{(\Sigma)}$ in Lemma 3.1. By Lemma 3.4, $\lambda_0 = 0$. Moreover, it follows from Lemma 3.3 that $\lim_{|\Sigma| \to \infty} (\Lambda)(S^{(\Sigma)}) = 0$ and therefore, $\Lambda = 0$. The results are thus an immediate consequence of Lemma 3.1.

For completeness, and as it will be used in Section 5 as control variate, we recall that the price of the discretely sampled geometric Asian call option in the Black-Scholes model
(compare e.g., Zhang 1998) is given by

\[ \pi_{dage}^{BS}(0) = G \Phi(d_1) - K \Phi(d_2), \]

where

\[ d_{1,2} = \frac{\log(G/K) \pm \sigma^2 T^{(1)}}{\sigma \sqrt{T^{(1)}}}, \quad G = S_0 e^{-\frac{\sigma^2}{2}(T^{(0)} - T^{(1)})}, \]

\[ T^{(0)} = \frac{1}{n} \sum_{j=1}^{n} t_j, \quad T^{(1)} = \frac{1}{n^2} \sum_{j,k=1}^{n} \min(t_j, t_k), \]

and \( \Phi \) is the cumulative normal density function. Analogously, for the put we have in the Black-Scholes model, using the same notations, again assuming zero dividends, zero interest rates and that we only consider points in time \( t_j = 1, \ldots, n \) that lie in the future:

\[ \pi_{dagep}^{BS}(0) = -G \Phi(-d_1) + K \Phi(-d_2). \]

### 3.3. Pricing of Arithmetically Averaged Options

In this paragraph, we investigate arithmetically averaged options, where the average of the payoffs

\[ (S_{t_j}^{(\Sigma)} - K)^+ \quad \text{and} \quad (K - S_{t_j}^{(\Sigma)})^+ \]

for \( t_0 = 0 < t_1 < \cdots < t_n \leq T \) of plain vanilla options is considered, i.e.,

- \( \frac{1}{n} \sum_{j=1}^{n} (S_{t_j}^{(\Sigma)} - K)^+ \) is the payoff of an arithmetically averaged call,
- \( \frac{1}{n} \sum_{j=1}^{n} (K - S_{t_j}^{(\Sigma)})^+ \) is the payoff of an arithmetically averaged put.

The put-call-parity \( (\lambda - K) = (\lambda - K)^+ - (K - \lambda)^+ \) immediately carries over to these types of options:

\[ \frac{1}{n} \sum_{j=1}^{n} S_{t_j}^{(\Sigma)} - K = \frac{1}{n} \sum_{j=1}^{n} (S_{t_j}^{(\Sigma)} - K)^+ - \frac{1}{n} \sum_{j=1}^{n} (K - S_{t_j}^{(\Sigma)})^+. \]

For what follows, we first note the obvious fact that the functional \( \Phi : \mathbb{R}^n \to \mathbb{R} \) defined by

\[ \Phi(f) := \frac{1}{n} \sum_{j=1}^{n} (K - f_{t_j})^+ \]

is bounded and continuous. We obtain the following result.
Corollary 3.6: Assume that $S^{(\Sigma)}$ satisfies (11). Then the price of a generalized arithmetically averaged put converges to the strike as the volatility goes to infinity, i.e.,

$$
\lim_{|\Sigma| \to \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} (K - S^{(\Sigma)}_{tj})^+ \right] = K.
$$

The price of a generalized arithmetically averaged call converges to the spot as the volatility goes to infinity, i.e.,

$$
\lim_{|\Sigma| \to \infty} \mathbb{E} \left[ \frac{1}{n} \sum_{j=1}^{n} (S^{(\Sigma)}_{tj} - K)^+ \right] = s_0.
$$

Proof: The first result is again an application of Lemma 3.1 to the functional

$$
\Phi(S^{(\Sigma)}) = \frac{1}{n} \sum_{j=1}^{n} (K - f_{tj})^+,
$$

which is as noted above bounded and continuous. The second result follows from the put-call-parity (12). □

3.4. Summary of Limit Values

We summarize our findings in Table 1.

We wish to stress again that these results hold true for both, the Black-Scholes model and the Heston model, represented by the cases $\Sigma \equiv \sigma$ and $\Sigma \equiv (\nu_0, \theta)$.

Remark 3.3: In chapter 4 of Delbaen and Schachermayer (2006), the limit $\sigma \to \infty$ of the price of a plain European call is investigated and found to be equal to $s_0$. By the put-call parity this implies a fortiori that the plain European put is worth exactly $K$ in the limit $\sigma \to \infty$. We note that this behaviour is consistent with that of the arithmetic Asian call and put and even of the geometric Asian put. In contrast to that, the geometric Asian call behaves rather differently.

4. The Price of a Discrete Geometric Asian Option in Heston’s Model

For the Monte-Carlo valuation in Section 5, we need as a corresponding control variate the price of a discrete geometric Asian option under Heston’s model. To this end, we follow an approach similar to Kim and Wee (2014) to derive a closed-form solution. We work in the Heston model as given in (4) and (5). We will, however, omit the dependency on $\Sigma$.

The (non-discounted) price at time $t_0 = 0$ of such an option is given as

$$
\pi_{\text{dgac}}^{\text{Heston}}(0) := \mathbb{E}[(G_n - K)^+]
$$

We first derive a formula for the geometric mean $G_n := (\prod_{j=1}^{n} S_{tj})^{1/n}$ and the (final) stock price $S_T$ in the Heston model.
Proposition 4.1: We have

\[
\log G_n = \log S_0 + \frac{\rho}{\eta} \left( \frac{1}{n} \sum_{j=1}^{n} v_{t_j} - v_0 \right) - \frac{\kappa \theta \rho}{\eta} \left( \frac{1}{n} \sum_{j=1}^{n} t_j \right) \\
+ \left( \frac{\kappa \rho}{\eta} - \frac{1}{2} \right) \left( \sum_{j=1}^{n} \left( 1 - \frac{j - 1}{n} \right) \int_{t_{j-1}}^{t_j} \nu_\tau \, d\tau \right) \\
+ (1 - \rho^2)^{1/2} \left( \sum_{j=1}^{n} \left( 1 - \frac{j - 1}{n} \right) \int_{t_{j-1}}^{t_j} \sqrt{\nu_\tau} \, dW_\tau \right),
\]

and in particular

\[
\log S_T = \log S_0 + \frac{\rho}{\eta} (\nu_T - v_0) - \frac{\kappa \theta \rho}{\eta} T \\
+ \left( \frac{\kappa \rho}{\eta} - \frac{1}{2} \right) \int_0^T \nu_\tau \, d\tau + (1 - \rho^2)^{1/2} \int_0^T \sqrt{\nu_\tau} \, dW_\tau.
\]

Here, as before, \( W \) is a Brownian motion independent of \( W^{(2)} \) such that \( W^{(1)}_t = \sqrt{1 - \rho^2} W_t + \rho W^{(2)}_t \) for all \( t \).

Proof: Using It’s formula, it follows from the defining Equation (4) of \( S_t \) that

\[
\log S_t = \log S_0 + \int_0^t \sqrt{\nu_\tau} \, dW^{(1)}_\tau - \frac{1}{2} \int_0^t \nu_\tau \, d\tau. \tag{13}
\]

On the other hand, by writing out the defining Equation (5) of \( \nu_t \), we obtain

\[
\nu_t = v_0 + \kappa \int_0^t (\theta - \nu_\tau) \, d\tau + \eta \int_0^t \sqrt{\nu_\tau} \, dW^{(2)}_\tau. \tag{14}
\]

Multiplying (14) by \( \rho/\eta \) and subtracting this from (13) yields the second equation of the claim by exploiting the relation on \( W^{(1)}, W^{(2)}, \) and \( W \). For the first claimed equation we observe that

\[
\log G_n = \frac{1}{n} \sum_{j=1}^{n} \log S_{t_j} \quad \text{and} \quad \sum_{j=1}^{n} \int_0^{t_j} f(\tau) \, d\tau = \sum_{j=1}^{n} (n - j + 1) \int_{t_{j-1}}^{t_j} f(\tau) \, d\tau
\]

and use the derived formula for \( \log S_t \).
For a complex number \( s \) let

\[
\psi(s) := \mathbb{E}[\exp(s \log G_n)].
\]

Writing

\[
A_1 := \log S_0 - \frac{\rho v_0}{\eta} - \frac{\kappa \theta \rho}{\eta} \left( \frac{1}{n} \sum_{j=1}^{n} t_j \right),
\]

\[
A_2 := \frac{\rho}{\eta} \left( \frac{1}{n} \sum_{j=1}^{n} v_{t_j} \right) + \left( \frac{\kappa \rho}{\eta} - \frac{1}{2} \right) \sum_{j=1}^{n} \left( 1 - \frac{j - 1}{n} \right) \int_{t_{j-1}}^{t_j} v_{\tau} \, d\tau,
\]

\[
A_3 := (1 - \rho^2)^{1/2} \sum_{j=1}^{n} \left( 1 - \frac{j - 1}{n} \right) \int_{t_{j-1}}^{t_j} \sqrt{v_{\tau}} \, dW_{\tau},
\]

and using a \( \sigma \)-field \( \mathcal{G} \) generated by \( \mathcal{F} \) and \( W_t^{(2)} \) with \( 0 < t \leq T \), we have

\[
\psi(s) = \exp(s A_1) \cdot \mathbb{E}[\exp(s A_2) \cdot \mathbb{E}[\exp(s A_3) \mid \mathcal{G}]].
\]

(15)

The last expectation can be handled by the following folklore result.

**Proposition 4.2:** If \( f \in L_2[0, t] \) and \( W \) is a Brownian motion, then we have

\[
\mathbb{E}\left[ \exp \left( \int_0^t f(\tau) \, dW_{\tau} \right) \right] = \exp \left( \frac{1}{2} \int_0^t f^2(\tau) \, d\tau \right).
\]

**Proof:** The proof follows from the fact that \( \int_0^t f(\tau) \, dW_{\tau} \) is normally distributed with mean 0 and variance \( \int_0^t f^2(\tau) \, d\tau \) and the fact that \( \mathbb{E}[\exp(X)] = \exp(\sigma^2/2) \) for any normally distributed random variable \( X \) with mean 0 and variance \( \sigma^2 \).

Note that by taking the conditional expectation on \( \mathcal{G} \), the Brownian motion \( W_t^{(2)} \) and hence \( v_t \) are deterministic in the integral and hence the same reasoning as in the proof of Proposition 4.2 shows that

\[
\mathbb{E}[\exp(s A_3) \mid \mathcal{G}] = \exp \left( \frac{1 - \rho^2}{2} \sum_{j=1}^{n} \frac{s^2}{n} \left( 1 - \frac{j - 1}{n} \right)^2 \int_{t_{j-1}}^{t_j} v_{\tau} \, d\tau \right).
\]

We can hence rewrite the expression in the outer expectation in (15) as

\[
\exp \left( \frac{\rho}{\eta} \sum_{j=1}^{n} v_{t_j} + \sum_{j=1}^{n} \left( \frac{\kappa \rho}{\eta} - \frac{1}{2} \right) s \left( 1 - \frac{j - 1}{n} \right) + \frac{1 - \rho^2}{2} s^2 \left( 1 - \frac{j - 1}{n} \right)^2 \int_{t_{j-1}}^{t_j} v_{\tau} \, d\tau \right).
\]
We have to calculate the expectation of an expression of the form
\[
\exp \left( \sum_{j=1}^{n} x_j \int_{t_{j-1}}^{t_j} \nu \, d\tau + \sum_{j=1}^{n} y_n \nu_{t_j} \right)
\]
where
\[
x_j = \left( \frac{\kappa \rho}{\eta} - \frac{1}{2} \right) s \left( 1 - \frac{j - 1}{n} \right) + \frac{1 - \rho^2}{2} s^2 \left( 1 - \frac{j - 1}{n} \right)^2, \quad y_n = \frac{\rho}{\eta} s.
\]
We will now use the factorized conditional expectation to successively calculate this expectation. To this end, we use Proposition 5.1 in Kraft (2005).

**Proposition 4.3 (Laplace transform):** Let \( t \leq u \) and
\[
f(t, u, x, y) := \exp \left( x \int_{t}^{u} \nu \, d\tau + y \nu_u \right).
\]
We have
\[
\mathbb{E}[f(t, u, x, y) \mid \nu_t] = \exp(A(t, u, x, y) + B(t, u, x, y) \nu_t),
\]
where
\[
A(t, u, x, y) = \frac{\kappa \theta}{\eta^2} ((\kappa - a(x))(u - t) - 2 \ln \left( \frac{1 - C(t, u, x, y)}{1 - b(x, y)} \right)),
\]
\[
B(t, u, x, y) = \frac{1}{\eta^2} \frac{(\kappa + a(x))C(t, u, x, y) - \kappa + a(x)}{C(t, u, x, y) - 1}.
\]
Furthermore, we have
\[
a(x) := \sqrt{\kappa^2 - 2x\eta^2},
\]
\[
b(x, y) := \frac{y\eta^2 - \kappa + a(x)}{y\eta^2 - \kappa - a(x)},
\]
\[
C(t, u, x, y) := b(x, y) \exp(-a(x)(u - t)).
\]

**Proof:** See (Kraft 2005, Proposition 5.1).\[\square\]

We can now rewrite \( \psi \) as
\[
\psi(s) = \exp(sA_1) \cdot \mathbb{E} \left[ \prod_{j=1}^{n} f(t_{j-1}, t_j, x_j, y_n) \right].
\]
Recursively we define coefficients \( z_1, \ldots, z_n \) by \( z_n := y_n \) and \( z_{j-1} = y_n + B(t_{j-1}, t_j, x_j, z_j) \). Then applying the proposition above recursively, we obtain
\[
\psi(s) = \exp(sA_1) \cdot \prod_{j=1}^{n} \exp(A(t_{j-1}, t_j, x_j, z_j)) \exp(B(0, t_1, x_1, z_1) \nu_0).
\]
We are now ready to derive the closed formula of a discrete fixed strike geometric Asian call with strike \(K\), given by

\[
\pi_{Heston\ dgac}^{\text{Heston}}(0) = \mathbb{E}[(G_n - K)^+] = \mathbb{E}[G_n \chi_{\{G_n > K\}}] - K \mathbb{Q}[G_n > K].
\]

We introduce the probability measure \(\mathbb{Q}^*\) by

\[
d\mathbb{Q}^* = \frac{G_n}{\mathbb{E}[G_n]} d\mathbb{Q}.
\]

By definition of \(\mathbb{Q}^*\) it follows that

\[
\mathbb{E}[G_n \chi_{\{G_n > K\}}] = \mathbb{Q}^*[G_n > K] \cdot \mathbb{E}[G_n].
\]

Now by the definition of \(\psi\) we have

\[
\mathbb{E}[G_n] = \psi(1).
\]

Moreover, \(\psi(is)\) is the characteristic function of \(\ln(G_n)\) and the inversion formula for the characteristic function yields

\[
\mathbb{Q}[G_n > K] = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left( \psi(is) \frac{\exp(-is \ln(K))}{is} \right) \, ds.
\]

Finally, the characteristic function under \(\mathbb{Q}^*\) of \(G_n\) is given as

\[
\mathbb{E}^*[\exp(is \ln(G_n))] = \frac{\mathbb{E}[\exp(is \ln(G_n)G_n)]}{\mathbb{E}[G_n]} = \frac{\psi(1 + is)}{\psi(1)}.
\]

Thus, it follows again from the inversion formula for the characteristic function that

\[
\mathbb{Q}^*[G_n > K] = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left( \frac{\psi(1 + is)}{\psi(1)} \frac{\exp(-is \ln(K))}{is} \right) \, ds.
\]

To summarize, we obtain that

\[
\pi_{Heston\ dgac}^{\text{Heston}}(0) = \frac{\psi(1) - K}{2} + \frac{1}{\pi} \int_0^\infty \Re \left( (\psi(1 + is) - K\psi(is)) \frac{\exp(-is \ln(K))}{is} \right) \, ds.
\]

Here \(\psi\) is given for \(s \in \mathbb{C}\) by

\[
\psi(s) = \exp(sA_1) \cdot \prod_{j=1}^n \exp(A(t_{j-1}, t_j, x_j, z_j)) \exp(B(0, t_1, x_1, z_1) \nu_0)
\]

with

\[
A_1 = \log S_0 - \frac{\rho \nu_0}{\eta} - \frac{\kappa \theta}{\eta} \left( \frac{1}{n} \sum_{j=1}^n t_j \right),
\]

\[
B(t_1, \ldots, t_n, x_1, \ldots, x_n, z_1, \ldots, z_n) = \frac{1}{n} \sum_{j=1}^n x_j + \sum_{j=1}^{n-1} \sqrt{2} \eta \left( \frac{t_{j+1} - t_j}{\nu_0} \right) \sqrt{D(t_{j+1}, t_j) z_j}.
\]
and

\[ a(x) = \sqrt{\kappa^2 - 2x\eta^2}, \]
\[ b(x, y) = -\eta^2 + \kappa - a(x), \]
\[ A(t, u, x, y) = \frac{2\kappa \theta}{\eta^2} \left( 2(\kappa - a(x))(u - t) - \ln \frac{C(t, u, x, y)}{b(x, y) - 1} \right), \]
\[ B(t, u, x, y) = -\frac{(\kappa + a(x))C(t, u, x, y) + 2\kappa}{\eta^2 C(t, u, x, y)} \]
\[ C(t, u, x, y) = 1 - b(x, y) \exp(-a(x)(u - t)). \]

Finally for \( j = 1, \ldots, n \) and \( s \in \mathbb{C} \) we let

\[ y_n = \frac{\rho}{\eta} \frac{s}{n}, \quad x_j = \left( \frac{\kappa \rho}{\eta} - \frac{1}{2} \right) s \left( 1 - \frac{j - 1}{n} \right) + \frac{1 - \rho^2}{2} s^2 \left( 1 - \frac{j - 1}{n} \right)^2, \]
\[ z_n = y_n, \quad z_{j-1} = y_n + B(t_{j-1}, t_j, x_j, z_j). \]

The put price follows by put-call-parity and the fact that \( \mathbb{E}[G_n] = \psi(1) \) as

\[
\pi_{\text{Heston}}(0) = \frac{K - \psi(1)}{2} + \frac{1}{\pi} \int_0^{\infty} \Re \left( \psi(1 + is) - K \psi(is) \frac{\exp(-is \ln(K))}{is} \right) ds.
\]

5. Consequences for Monte-Carlo Valuation

In this section, we illustrate with numerical experiments how the limiting behaviour of Asian option prices as summarized in Table 1 influences the choice of a correct control variate when valuing discrete Asian options with the help of Monte Carlo simulation, both for the Black-Scholes and the Heston model.

5.1. The Right Choice of the Control Variate

In general, for a given random variable \( X \), the expectation \( \mathbb{E}[X] \) is calculated as the average

\[ \sum_{i=1}^{N} \frac{X(\omega_i)}{N}, \]

where the random numbers \( \omega_i \) are sampled independently from a suitable distribution (compare e.g., Korn, Korn, and Kroisandt 2010, Section 3.2). To improve accuracy and speed up calculations, a control variate \( Y \) can be used, for which \( \mathbb{E}[Y] \) is explicitly known.

| Option price for \( |\Sigma| \to \infty \) | Arithmetic Asian | Geometric Asian | Averaged |
|--------------------------------------|-----------------|-----------------|-----------------|
| Call | Put | Call | Put | Call | Put |
| \( s_0 \) | \( K \) | \( 0 \) | \( K \) | \( s_0 \) | \( K \) |
and whose distribution is close to that of \( X \). Then \( \mathbb{E}[X] \) can be approximated (see e.g., Korn, Korn, and Kroisandt 2010, Section 3.3) by

\[
\sum_{i=1}^{N} \frac{X(\omega_i) - Y(\omega_i)}{N} + \mathbb{E}[Y].
\]

When calculating prices of arithmetic Asian options, it is quite common to use a corresponding geometric Asian option as control variate \( Y \), see Kemna and Vorst (1990, Section 4). However, the limiting behaviours proved in the previous sections, indicate that this might lead to inaccurate results, both for the Black-Scholes and the Heston model. This holds true, in particular, for call options, since for these, the payoff function is unbounded for increasing volatility. For \( |\Sigma| \) large enough, the geometric mean however is close to zero, both pointwise and in expectation, as it follows from Lemmata 3.3 and 3.4. Therefore the payoff function of the geometric Asian call remains bounded. From this, we deduce that for large volatilities the Monte-Carlo calculation of put prices is more accurate and one should use the put-call-parity for the calculation of call prices.

For a call, it is more feasible to use other control variates. In Kamizono et al. (2004), more control variates are investigated, in particular, the averaged options defined in Section 3.3 prove to be a reliable tool in our case as well. In Zhan and Cheng (2004), a suitably weighted sum of the geometric Asian option, the averaged option, and the stock price on the first sample date is used.

To emphasize our point, we carried out simulations of Asian option prices in both the Black-Scholes and the Heston model. For these examples, we used an Asian call on the arithmetic/geometric mean of spots with a strike set 1.0 over a time span of 5 years. We used an underlying with a spot of 1.0 (i.e., at the money options), no dividends and varying volatilities as given in the first column of the respective table, and for simplicity, a market with a flat zero interest rate curve. We used 1,000,000 simulations and give the simulated standard deviation as a tolerance in the tables.

The results of the simulations are summarized in Table 2. Throughout, we consider four simulations with different control variates:

- **None** plain Monte-Carlo simulation without control variate
- **Kemna-Vorst** original Kemna-Vorst control variate, namely the corresponding option on the geometric mean
- **Modified Kemna-Vorst** a corrected version of the previous by the difference of arithmetic and geometric mean
- **Averaged** a simplified variant of the control variate suggested in Kamizono et al. (2004), namely the corresponding arithmetically averaged option.

### 5.2. Simulation Results in the Black-Scholes Model

Note that the true option prices remain unknown in all cases. We can see, however, in Table 2, that for a volatility parameter \( \sigma = 500\% \) the option prices converge. As can be seen from Table 1, the theoretical limiting call and put price for an at the money option are both 1. This exact result is only obtained for the averaged control variate with nearly zero standard deviation for all volatility levels.
Table 2. Simulated option prices under the Black-Scholes model using different control variates and volatilities for strike 1.

| σ   | Control variate          | Call | None | Kemna-Vorst | Modified Kemna-Vorst | Averaged |
|-----|--------------------------|------|------|-------------|----------------------|---------|
| 25% | 0.2570 ± 0.0006          |      | 0.2567 ± 0.0001 | 0.2566 ± 0.0000 | 0.2566 ± 0.0000 |
| 100%| 0.8279 ± 0.0374          |      | 0.8214 ± 0.0332 | 0.7990 ± 0.0002 | 0.7991 ± 0.0001 |
| 200%| 0.3134 ± 0.0514          |      | 0.3250 ± 0.0501 | 0.9853 ± 0.0001 | 0.9853 ± 0.0001 |
| 500%| 0.0000 ± 0.0000          |      | 0.0000 ± 0.0000 | 1.0000 ± 0.0000 | 1.0000 ± 0.0000 |
| Put | None                     |      | Kemna-Vorst      | Modified Kemna-Vorst | Averaged |
| 25% | 0.2566 ± 0.0003          |      | 0.2566 ± 0.0000 | 0.2567 ± 0.0001 | 0.2566 ± 0.0000 |
| 100%| 0.7989 ± 0.0003          |      | 0.7990 ± 0.0002 | 0.8214 ± 0.0332 | 0.7991 ± 0.0001 |
| 200%| 0.9853 ± 0.0001          |      | 0.9853 ± 0.0001 | 0.3250 ± 0.0501 | 0.9853 ± 0.0001 |
| 500%| 1.0000 ± 0.0000          |      | 1.0000 ± 0.0000 | 1.0000 ± 0.0000 | 1.0000 ± 0.0000 |

For the modified Kemna-Vorst control variate, we see a significant improvement for call options, while the original Kemna-Vorst control variate is good for puts. Actually using Kemna-Vorst for puts and modified Kemna-Vorst for calls yields nearly the same results as the averaged control variate.

To explain the limiting behaviour in the other cases, remember that for the high volatility cases the distribution of the stock price becomes extremely skewed (almost everywhere convergence towards 0 versus an expectation which is constantly equal to 1). So most of the outcomes of the simulation will be close to zero and only the outliers will make the expectation work. But, while still the Monte-Carlo simulation will converge to the correct value, the number of simulations for this will become excessively high and the empirical standard deviation of the sample itself will no longer be meaningful for the degree of convergence.

To make the Monte-Carlo simulation work again, we have to offset the skewed distribution with an equally skewed distributed variable as control variate.

5.3. Simulation Results in the Heston Model

As expected, the same observations can be made in the case of the Heston model. Here we investigated the convergence for \( \nu_0 \to \infty \) and \( \theta \to \infty \) separately, as given in Tables 3 and 4 for the in the money options.

It can, in particular, be observed, that convergence is much slower for \( \nu_0 \to \infty \) than for \( \theta \to \infty \). It must also be noted, that both \( \nu_0 \) and \( \theta \) are variances as compared to \( \sigma \), which is
Table 4. Simulated option prices under the Heston model using different control variates and growing long-term variances for strike 1.

| θ       | Control variate       | None            | Kemna-Vorst     | Modified Kemna-Vorst | Averaged          |
|---------|-----------------------|-----------------|-----------------|----------------------|-------------------|
| 50%     | 0.6213 ± 0.0056       | 0.6256 ± 0.0037 | 0.6324 ± 0.0001 | 0.6323 ± 0.0001      |
| 100%    | 0.7252 ± 0.0189       | 0.7358 ± 0.0156 | 0.7862 ± 0.0002 | 0.7861 ± 0.0001      |
| 10000%  | 0.0000 ± 0.0000       | 0.0000 ± 0.0000 | 1.0000 ± 0.0000 | 1.0000 ± 0.0000      |
| 1000000%| 0.0000 ± 0.0000       | 0.0000 ± 0.0000 | 1.0000 ± 0.0000 | 1.0000 ± 0.0000      |

the volatility (root of variance). Therefore the order of magnitude of the parameters must be different.

Notes

1. Another exception is Mendonca et al. (2018), but in the context of Barrier options, which is outside the scope of our paper.
2. Equivalently we can assume that the riskless interest rate \( r \) satisfies \( r = 0 \) and equally consider \( S(t) \) defined to be the discounted asset price process \( \tilde{S} = e^{-rt}S(t) \).
3. Note therefore that \( \mathbb{E}[^{\sup_{0 \leq s \leq T} v_s^2}] < \infty \) for any \( p \geq 0 \) and \( T > 0 \); see e.g., (Hambly and Kolliopoulus 2019, Lemma 3.2).
4. Moreover, Theorem 3.4 in Desmettre, Leobacher, and Rogers (2021) treats the case of a violated Feller condition as well.
5. We assume zero dividends, zero interest rates and that we only consider points in time \( t_j = 1, \ldots, n \) that lie in the future seen from now on.
6. We recall that this limiting behaviour of option prices is a direct consequence of the high volatility limiting behaviour for the corresponding asset prices which were deduced in Sections 1.2 and 2.

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References

Black F., and Scholes M.. 1973. “The Pricing of Options and Corporate Liabilities.” Journal of Political Economy 81: 637–654.
Boyle P., and Potapchik A.. 2008. “Prices and Sensitivities of Asian Options: A Survey.” Insurance: Mathematics and Economics 42: 189–211.
Carr P., Ewald C.-O., and Xiao Y.. 2008. “On the Qualitative Effect of Volatility and Duration on Prices of Asian Options.” Finance Research Letters 5: 162–171.
Cox J. C., Ingersoll J., Jonathan E., and Ross S. A.. 1985. “A Theory of the Term Structure of Interest Rates.” Econometrica 53 (2): 385–407.
Delbaen F., and Schachermayer W.. 2006. The Mathematics of Arbitrage. Berlin: Springer Finance.
Desmettre S., Leobacher G., and Rogers L. C. G.. 2021. “Change of Drift in One-dimensional Diffusions.” Finance and Stochastics 25 (2): 359–381.
Geman H., and Yor M..1993. “Bessel Processes, Asian Options, and Perpetuities.” Mathematical Finance 3 (4): 349–375.
Gronwall T. H. 1919. “Note on the Derivative with Respect to a Parameter of the Solutions of a System of Differential Equations.” Annals of Mathematics 20: 292–296.
Guliashvili A., and Stein E. M..2010. “Asymptotic Behavior of the Stock Price Distribution Density and Implied Volatility in Stochastic Volatility Models.” Applied Mathematics and Optimization 61: 287–315.
Hambly B., and Kolliopoulos N.. 2019. “Stochastic Evolution Equations for Large Portfolios of Stochastic Volatility Models.” https://arxiv.org/abs/1701.05640.
Heston S. L. 1993. “A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options.” The Review of Financial Studies 6 (2): 327–343.
Jacquier A., Keller-Ressel M., and Mijatović A.. 2013. “Large Deviations and Stochastic Volatility Models with Jumps: Asymptotic Implied Volatility for Affine Models.” Stochastics 85 (2): 321–345.
Jacquier A., and Mijatović A.. 2014. “Large Deviations for the Extended Heston Model: The Large-time Case.” Asia-Pacific Financial Markets 21: 263–280.
Korn R., Korn E., and Kroisandt G.. 2010. Monte Carlo Methods and Models in Finance and Insurance. Boca Raton: Chapman & Hall/CRC.
Kraft H. 2005. “Optimal Portfolios and Heston’s Stochastic Volatility Model: An Explicit Solution for Power Utility.” Quantitative Finance 5 (3): 303–313.
Mendonca K., Kontosakos V. E., Pantelous A. A., and Zuev K. M.. 2018. “Efficient Pricing of Barrier Options on High Volatility Assets using Subset Simulation.” https://arxiv.org/abs/1803.03364.
Shiryaev A. 1996. Probability. 2nd ed. New York, NY: Springer.
Vecer J. 2001. “A New PDE Approach for Pricing Arithmetic Average Asian Options.” Journal of Computational Finance 4 (4): 105–113.
Wong B., and Heyde C.. 2006. “On Changes of Measure in Stochastic Volatility Models.” Journal of Applied Mathematics and Stochastic Analysis 2006: 1–13.
Zhan H., and Cheng Q.. 2004. “A New Multiple Control Variate Estimator for Asian Options.” Acta Scientiarum Naturalium Universitatis Pekinensis 40 (1): 5–11.
Zhang P. 1998. Exotic Options. 2nd ed. Singapore: World Scientific Publishing Co Pte Ltd.
Appendix. Auxiliary results

Proposition A.1 (Moments of the CIR process, Cox et al. 1985): Let $v_t$ be given by

$$dv_t = \kappa(\theta - v_t)\,dt + \eta \sqrt{v_t}\,dW_t.$$ 

Then for $n \in \mathbb{N}$ and $t > 0$ the moments of $v_t$ are given by

$$m_n(v_t) := \mathbb{E}[|v_t|^n] = \frac{(n - 1)!((d + n\lambda)}{2c^n} + \sum_{k=1}^{n-1} \frac{(n - 1)!}{2c^k(n - k)!} (d + k\lambda)m_{n-k}(v_t).$$

In particular

$$m_n(v_t) = O(d + \lambda)^n.$$ 

Here $d := 4\kappa\theta/\eta^2$ degrees of freedom and noncentrality parameter $\lambda := 2cv_0 \exp(-\kappa t)$, where $c := 2\kappa/(\eta^2(1 - \exp(-\kappa t)))$.

Proof: It is well known, that a suitable multiple of $v_t$ is noncentral chi-square distributed. In particular, $2cv_t$ is noncentral chi-square distributed with $d := 4\kappa\theta/\eta^2$ degrees of freedom and noncentrality parameter $\lambda := 2cv_0 \exp(-\kappa t)$, where $c := 2\kappa/(\eta^2(1 - \exp(-\kappa t)))$. Therefore the moments of $2cv_t$ are given by

$$m_n(2cv_t) = 2^n c^n \mathbb{E}[|v_t|^n] = 2^{n-1}(n - 1)!((d + n\lambda) + \sum_{k=1}^{n-1} \frac{(n - 1)!2^{k-1}}{(d + k\lambda)m_{n-k}(2cv_t)}. $$

Division by $2^n c^n$ yields the first part of the result.

Dividing again by $(d + \lambda)^n$ gives

$$m_n(v_t) = \frac{(d + n\lambda)}{(d + \lambda)^n} \frac{1}{2c^n} \frac{1}{2c^k(n - k)!} (d + k\lambda)m_{n-k}(v_t) \frac{1}{(d + \lambda)^{n-k} (d + \lambda)^{k-1}}.$$ 

The second part now follows by induction. 

Proposition A.2 (Jessen’s inequality, Jessen 1933): Let $0 < u \leq v < \infty$ and assume that $f$ is a $\mu \times \nu$-measurable scalar-valued function on $M \times N$. Then applying the triangle inequality for integrals to the $L_{v/u}$-valued function $F : s \rightarrow |f(s, \cdot)|^u$, it follows that

$$\left(\int_M \left(\int_N |f(s, t)|^u \,d\mu(s)\right)^{\nu/u} \,dv(t)\right)^{1/v} \leq \left(\int_M \left(\int_N |f(s, t)|^\nu \,dv(t)\right)^{u/v} \,d\mu(s)\right)^{1/u}$$

whenever the right-hand integral is finite.

Proposition A.3 (Gronwall inequality, Gronwall 1919): Assume that the function $u : [0, T] \rightarrow \mathbb{R}$ is continuous and satisfies

$$u_t \leq a + b \int_0^t u_s \,ds \quad \text{for all } t \in [0, T].$$

Then

$$u_t \leq a \exp(bt) \quad \text{for all } t \in [0, T]. \quad \text{(A1)}$$

For assessing almost sure convergence, the Lemma of Borel Cantelli is a useful tool. We provide the following implication (cf. Shiryaev 1996) which is suitable for our purposes:

Theorem A.4 (Borel-Cantelli Characterization of Almost Sure Convergence): Let $\{X_n\}$ be a sequence of random variables and $X$ be a limit random variable. Suppose that for $\epsilon > 0$, $A_n(\epsilon)$ is the
event defined as

\[ A_n(\epsilon) = \{ \omega : |X_n(\omega) - X(\omega)| > \epsilon \}. \]

Then the following holds:

If \( \sum_{n=1}^{\infty} \mathbb{P}[A_n(\epsilon)] = \sum_{n=1}^{\infty} \mathbb{P}[|X_n - X| > \epsilon] < \infty \) then \( X_n \to X \) a.s. \hspace{1cm} (A2)