Bianchi type \(I\) string cosmological solutions via Hojman symmetry

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Abstract

In this paper, we find exact string cosmological solutions for Bianchi type \(I\) cosmology, by using Hojman symmetry approach. The string cosmology under consideration includes a dilaton field \(\psi\) with the potential \(W(\psi)\), and a totally antisymmetric field strength \(H_{\mu\nu\rho}\) which is specifically defined in terms of the scale factor \(a(t)\). We show that for this string cosmology, Hojman conserved quantities exist using which new exact solutions for the scale factor and the scalar field are obtained for some specific potentials \(W(\psi)\).

1 Introduction

String theory has major objectives, the most recent one of which is the development of a well-defined cosmological framework to upgrade the conventional cosmology near the Planck scale. This is the scale at which most of the major cosmological problems arise and hence one needs a very sophisticated insight to resolve them. The cosmology at this era is offered by string theory as “String Cosmology” which is supposed to provide a sufficient understanding of the very early universe and a subsequent graceful exit towards the conventional “Hot Big Bang cosmology”. Based on the recent observations, we have several compelling reasons indicating that the dynamics of early universe might have been profoundly affected by the presence of spatial anisotropies near the Planck scale [1]. In other words, it seems that the assumption of a Friedmann-Robertson-Walker (FRW) isotropic evolution of the universe may hinder some important aspects of the very early universe. Motivated by this argument in the study of string cosmology, the people have relaxed the requirement of spatial isotropy over the 4D spatially homogeneous FRW universe [2]-[9] and considerable attention has been focused on the spatially homogeneous but not necessarily isotropic 4D string backgrounds. We consider this class as the string cosmology counterpart of the vacuum Bianchi-type models [10]-[15]. Vacuum Bianchi-type models involves 4D spatially homogeneous, but not isotropic, spacetimes which satisfy at least the lowest-order string beta-function equations [16]. These models generalize all possible FRW cosmological models and provide the best models available for understanding the impacts of anisotropy on the dynamics of early universe.

Symmetries and the corresponding conserved quantities, in general, help to simplify the dynamics and give rise to exact solutions for physical systems under consideration. Noether symmetry is a well known example which is widely used in different aspects of classical and quantum field theory, as well as in general relativity and black hole physics. Noether symmetry approach has been received great amount of attention in the context of cosmology [18]-[21]. Imposing Noether symmetry on the point-like Lagrangians, associated to the equations of motion of a cosmological model, allows one to find the first integrals of the equations of motion. Recently, an alternative approach to the Noether symmetry approach has been received attention, so called Hojman symmetry approach, which can

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provide us with a method to find new exact solutions\cite{22}. Unlike the Noether symmetry approach, in the Hojman symmetry approach Lagrangian and Hamiltonian functions are not required to find the exact solutions, rather the symmetry vectors and the corresponding conserved charges are obtained by using the equations of motion, without referring to Lagrangian or Hamiltonian. It turns out that these two approaches may give rise to different conserved quantities as well as different exact solutions. Recently, parallel to the Noether symmetry, the Hojman symmetry has also been extensively used to study some models of extended theories of gravity and cosmology\cite{23}-\cite{27}. In this paper, we study and apply the Hojman symmetry for some string cosmological models, to obtain new conserved charges and exact solutions. The organization of the paper is following. In Sec. 2, we briefly review the main points of the String Cosmology. In Sec. 3, we have a summary look at the Hojman symmetry approach and in Sec. 4, we use Hojman symmetry method in Bianchi type I string cosmological model.

2 Preliminaries in String Cosmology

Cosmology is a framework to describe the evolution of the Universe, starting from its very beginning at high energy regime of Planck scale, namely quantum gravity era. On the other hand, string theory is considered as the only known consistent theory of quantum gravity, at Planck energy scale. Therefore, it is appealing to study the very early Universe in the context of string theory, and conversely, the very early Universe seems to be a very natural place to establish some predictions for string models (for more details see\cite{28}). In this line of study, string-dilatonic cosmology has come out from the low energy limit of superstring theory, and some exact solutions has been obtained in homogeneous isotropic backgrounds\cite{29}-\cite{32}. Moreover, it has been shown that the gravity may couple, in addition to scalar field, to an antisymmetric second rank Kalb-Ramon tensor, so called $B$-field. In this model, at low energy limit with a very weak coupling, one may describe a string dominated universe by the tree level effective action including only massless modes, namely tensor modes (graviton) and scalar modes (dilaton), the exact solutions of which for homogeneous and anisotropic string cosmological models are studied in 4D\cite{16} and 5D\cite{17}. However, at high energy limit with strong coupling, the tensor modes and scalar modes experience considerable growths\cite{33}. Therefore, true string cosmological description of the universe, at high energy regime of Planck scale, necessitates the use of perturbative corrections in string cosmology, including the stringy type $\alpha'$-expansion and the quantum loop expansion in string coupling\cite{34},\cite{35},\cite{36},\cite{37}.

Here, we will assume the 4-dimensional spacetime curvatures below the string-Planck scale and describe the evolution of the universe by the following effective action\cite{38}

$$S_{\text{eff}} = \int d^4x \sqrt{-g} e^\psi \left( R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + \partial_\mu \psi \partial^\mu \psi - W(\psi) \right),$$

where $g_{\mu\nu}$ is the background metric, non-minimally coupled to a dilaton field $\psi$ with the potential $W(\psi)$, and $H_{\mu\nu\rho}$ is a totally antisymmetric field strength which is defined in terms of the antisymmetric Kalb-Ramon field $B_{\mu\nu}$ as

$$H_{\mu\nu\rho} = \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu} + \partial_\mu B_{\nu\rho}.$$  

The field equations can be derived as follows (see\cite{2} and\cite{3})

$$R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \partial_\mu \psi \partial^\mu \psi - 2 \nabla_\mu \nabla^\mu \psi - W(\psi) - \frac{dW}{d\psi} = 0,$$  

$$R_{\mu\nu} - \frac{1}{4} H_{\mu\nu}^2 - \nabla_\mu \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} \frac{dW}{d\psi} = 0,$$

$$\nabla_\mu \left( e^\psi H_{\mu\nu\rho} \right) = 0.$$  

We take the flat Friedmann-Robertson-Walker (FRW) background metric

$$dS^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2).$$
where \( a(t) \) is the scale factor. In the case of homogeneous cosmological backgrounds, \( \psi \) should also be a monotonic function of time \( t \). Also, according to \[16\] and with no loss of generality, we may consider the 3-form
\[
H = a^2(t) \, dt \wedge dx \wedge dy,
\]
for the \( H \) field which is satisfied by the equation (2).

### 3 Brief review on Hojman conservation method

Hojman symmetry was proposed in 1992 \[22\], a brief review of which is as follows. Consider a set of second-order differential equations
\[
\ddot{q}^i = F^i(q^j, \dot{q}^j, t), \quad i, j = 1, 2, \ldots n
\]
where \( q^i \) denotes the coordinates, \( F^i \) denotes the forces, and a dot denotes derivative with respect to time \( t \). If this equation has a symmetry vector \( X^i = X^i(q^j, \dot{q}^j, t) \), then it has to satisfy the following equation (\[23\], \[24\])
\[
\frac{d^2 X^i}{dt^2} - \frac{\partial F^i}{\partial q^j} \dot{X}^j - \frac{\partial F^i}{\partial \dot{q}^j} \frac{dX^j}{dt} = 0,
\]
where
\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + F^i \frac{\partial}{\partial \dot{q}^i}.
\]
The symmetry vector \( X^i \) has the property that under the infinitesimal transformation
\[
\dot{q}^i = q^i + \epsilon X^i(q^j, \dot{q}^j, t),
\]
the solutions \( q^i \) of Eq. (8) are mapped into the solutions \( \dot{q}^i \) of the same equations (up to \( \epsilon^2 \) terms) \[25\], \[26\]. With this property of the symmetry vector \( X^i \), the Hojman conserved quantities are defined through the following theorem \[27\]:

**Theorem:**
1. If the force \( F^i \) satisfies the following equation
\[
\frac{\partial F^i}{\partial \dot{q}^i} = 0,
\]
then
\[
Q = \frac{\partial X^i}{\partial q^i} + \frac{\partial}{\partial q^i} \left( \frac{dX^i}{dt} \right),
\]
is a conserved quantity for Eq. (8), i.e. \( \frac{dQ}{dt} = 0 \),

2. and if \( F^i \) satisfies
\[
\frac{\partial F^i}{\partial \dot{q}^i} = - \frac{d}{dt} \ln \gamma,
\]
then
\[
Q = \frac{1}{\gamma} \frac{\partial(\gamma X^i)}{\partial q^i} + \frac{\partial}{\partial \dot{q}^i} \left( \frac{dX^i}{dt} \right),
\]
is a conserved quantity, where \( \gamma \) is merely a function of \( q^i \).
4 Bianchi type I String Cosmology via Hojman symmetry

In this section, we will apply the Hojman symmetry approach on the string cosmological system introduced in section 2. Our starting point is the Lagrangian density

$$L = e^\psi \left( R - \frac{1}{12} H_{\mu \nu \rho} H^{\mu \nu \rho} + \partial_{\mu} \psi \partial^{\mu} \psi - W(\psi) \right).$$  \hspace{1cm} (13)

By considering the transformations \([40]\)

$$\varphi(\psi) = 2\sqrt{2} e^\frac{\psi}{2}, \quad F(\varphi) = e^\psi = \frac{1}{8} \varphi^2, \quad V(\varphi) = e^\psi W(\psi),$$  \hspace{1cm} (14)

we obtain the equivalent Lagrangian density

$$L = F(\varphi) R - \frac{1}{12} F(\varphi) H_{\mu \nu \rho} H^{\mu \nu \rho} + \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - V(\varphi).$$  \hspace{1cm} (15)

Using the FRW metric (6) and the 3-form (7), the point-like Lagrangian density (15) reads as

$$L = -6F(\varphi)a\dot{a}^2 - 6F'(\varphi)a^2\dot{\varphi} - a^3\dot{\varphi}^2 - a^3 Z(\varphi),$$  \hspace{1cm} (16)

where \(Z(\varphi) = V(\varphi) - \frac{1}{2} F(\varphi)\). The Euler-Lagrange equations corresponding to (16) are obtained

$$\ddot{a} + 2 \frac{\dot{a}^2}{a} + 4 \frac{F'}{F} \frac{\dot{a}}{a} \dot{\varphi} + 2 \frac{F'}{F} \ddot{\varphi} + \left( 2 \frac{F''}{F} - \frac{1}{F} \right) \dot{\varphi}^2 - \frac{1}{F} Z(\varphi) = 0,$$

$$\ddot{\varphi} + 3 \frac{F'}{a} \ddot{a} + 3 \frac{F'}{a} \dot{a}^2 + 3 \frac{\dot{a}}{a} \dot{\varphi} - \frac{1}{2} Z'(\varphi) = 0,$$  \hspace{1cm} (17)

subject to the zero energy constraint

$$-6Faa\dot{a}^2 - 6F'a\dot{a}^2 \dot{\varphi} - a^3\dot{\varphi}^2 + a^3 Z(\varphi) = 0,$$  \hspace{1cm} (19)

where \(^'\) denotes the differentiation \(h'(y) = dh/dy\).

Now, it is possible to use the following conformal transformations \([22]\) and \([39]\)

$$\tilde{a} = \sqrt{2F} a, \quad \tilde{\varphi} = \sqrt{2F} \varphi, \quad \frac{d\tilde{\varphi}}{d\tilde{a}} = \sqrt{\frac{3F'^2 - F}{2F^2}} \frac{d\varphi}{dt},$$  \hspace{1cm} (20)

under which the Lagrangian (16) takes on the form, corresponding to a minimally coupled scalar field, as follows

$$\tilde{L} = 3\tilde{a}\ddot{\tilde{a}} - \frac{\dot{\tilde{a}}^2}{a^2} \ddot{\tilde{\varphi}} - \tilde{a}^3 \tilde{Z}(\tilde{\varphi}),$$  \hspace{1cm} (21)

where \(\tilde{Z}(\tilde{\varphi}) = \frac{Z(\varphi)}{F}\). The Euler-Lagrange equations corresponding to (21) are also obtained as

$$2 \frac{\tilde{a}}{a} \frac{\dot{\tilde{a}}}{a} + \frac{\dot{\tilde{a}}^2}{a^2} + \frac{1}{4} \dot{\tilde{\varphi}}^2 - \tilde{Z}(\tilde{\varphi}) = 0,$$  \hspace{1cm} (22)

$$\ddot{\tilde{\varphi}} - \frac{1}{2} \dot{\tilde{\varphi}}^2 - 6 \frac{\dot{\tilde{a}}^2}{a^2} - 6 \frac{\dot{\tilde{a}}}{a} + 3 \frac{\dot{\tilde{a}}}{a} \dot{\tilde{\varphi}} + 2 \left( \tilde{Z}' + 2\tilde{Z} \right) = 0,$$  \hspace{1cm} (23)

subject to the zero energy constraint

$$-3 \frac{\dot{\tilde{a}}^2}{a^2} + \frac{1}{4} \dot{\tilde{\varphi}}^2 + \tilde{Z}(\tilde{\varphi}) = 0.$$  \hspace{1cm} (24)
Combining (22) and (23) with (24) leads to

\[
\begin{align*}
2 \bar{a} \ddot{\bar{a}} - 2 \dot{\bar{a}}^2 + \frac{1}{2} \dot{\phi}^2 &= 0, \\
2 \bar{Z}' + 4 \bar{Z} &= \frac{\ddot{\phi} - \frac{1}{2} \dot{\phi}^2 - 6 \dot{\bar{a}}^2 - 6 \frac{\ddot{\bar{a}}}{\bar{a}} + 3 \frac{\ddot{\phi}}{\bar{a}}}{\ddot{\phi}} + \frac{3}{4} \ddot{\phi}^2
\end{align*}
\]

respectively. The property that \( \bar{a}(t) \) and \( \bar{\phi}(\bar{t}) \) are invertible functions of time \( t \), helps us to simplify the dynamics by reducing two above dynamical equations to one dynamical equation. In doing so, we define \( x = \ln \bar{a} \), \( \bar{\phi}(\bar{t}) = \bar{\phi}(x(\bar{t})) \), and use them in Eqs. (25) and (29) to obtain the following dynamical equations

\[
\ddot{x} = -\frac{1}{4} \dot{\bar{\phi}}^2(x) \dot{x}^2,
\]

and

\[
\frac{\bar{Z}'}{\bar{Z}} = \frac{1}{2} \left[ -\dot{\phi}'(x) + \frac{4}{\dot{\phi}^2(x) - 12} \ddot{\phi}'(x) \right].
\]

Now, these equations are in their most suitable forms to be studied in the context of Hojman symmetry approach. Assuming the one dimensional vector \( X(x, \dot{x}) \), independent of explicit time \( t \), the equation of symmetry vector \( X \) is obtained as

\[
\left( \frac{\partial^2 X}{\partial x^2} + f'(x)X + f(x) \frac{\partial X}{\partial x} \right) + \dot{x}^2 f^2(x) \frac{\partial^2 X}{\partial x \partial \dot{x}} - \dot{x} \left( 2f(x) \frac{\partial^2 X}{\partial x \partial \dot{x}} + f'(x) \frac{\partial X}{\partial \dot{x}} \right) = 0,
\]

where

\[
f(x) = \frac{1}{4} \dot{\phi}^2(x).
\]

From equation (27), we can recognize that \( F(x, \dot{x}) = -f(x)\dot{x}^2 \), thus

\[
\gamma(x) = \gamma_0 e^{2 \int f(x) dx}.
\]

In general, the differential equation for vector \( X \) is difficult to solve, so the authors in [23] tried some ansatz for solving the equation (29) which we consider them in the following.

### 4.1 \( X \sim X(\dot{x}) \)

By considering \( X \sim X(x) \), the corresponding differential equation for symmetry vector \( X \) according to (29) takes the form

\[
\frac{d^2 X}{dx^2} + f'(x)X + f(x) \frac{dX}{dx} = 0,
\]

or

\[
\frac{d}{dx} \left( f(x)X + \frac{dX}{dx} \right) = 0,
\]

which is nothing but \( \frac{dQ}{dx} = 0 \), according to Eq. (12), so we will not consider this choice. For the choice \( X = X(\dot{x}) \), the equation of symmetry vector \( X \) reads as Euler equation

\[
f'(x)X + \dot{x}^2 f^2(x) \frac{d^2 X}{dx^2} - \dot{x} f'(x) \frac{dX}{dx} = 0,
\]

from which \( X \) and \( f(x) \) are obtained respectively as follows

\[
X = A_1 \dot{x} + A_2 \dot{x}^n,
\]

\[
f(x) = -\frac{1}{nx + f_0},
\]

respectively.
where $A_1, A_2, f_0$ and $n$ are constant parameters, and the Hojman conserved quantity reads as

$$Q = 2f(x)\dot{x}^n - f(x)n(n+1)\dot{x}^n.$$  \hspace{1cm} (37)

We can easily verify that the symmetry vectors $X \sim \dot{x}$ and $X \sim \dot{x}^{-2}$ give rise to vanishing conserved charge, namely $Q = 0$. Therefore, we may discard $n = 1, -2$ cases.

If we define a new variable as $y = -(nx + f_0) > 0$, then using (28) and (30), we obtain

$$\bar{\varphi} = \varphi_c \mp \frac{4}{n} \sqrt{y},$$  \hspace{1cm} (38)

$$\bar{Z}(\bar{\varphi}) = \lambda \bar{\varphi}^3 - \frac{16}{3n^2} \lambda \bar{\varphi}^2 - 1,$$  \hspace{1cm} (39)

where $\lambda = -\frac{3n^2}{16} \bar{Z}_0 \left(\frac{n^2}{16}\right)^{\frac{2-n}{n}}$. On the other hand, using the equations (12) and (31), for the conserved quantity we have

$$Q_0 = \dot{x}^n \frac{nx + f_0}{nx + f_0}.$$  \hspace{1cm} (40)

As was mentioned in [23], for $\dot{x}^n > 0$ and $\dot{x}^n < 0$ one obtains $Q_0 < 0$ and $Q_0 > 0$, respectively. On the other hand, since $\dot{x}$ can be negative or positive, we can assume $n$ to be an integer. Using $y = -(nx + f_0)$, we have

$$\dot{y} = (-n)^2 y|Q_0|.$$  \hspace{1cm} (41)

The solution of this equation is obtained as

$$y(t) = \left[\left(1 - \frac{1}{n}\right) (y_0 - n|Q_0|^\frac{1}{n} t)\right]^{\frac{n}{n-1}}.$$  \hspace{1cm} (42)

Finally, by using the transformations (14), (20) and using the equation (42), we obtain

$$W(\psi) = 4\lambda e^\psi \left[\psi^4 - \frac{16}{3n^2} \lambda \psi^2 - 2\right] + \frac{1}{2},$$  \hspace{1cm} (43)

$$\psi(\tau) = \mp \frac{4}{n} \left(1 - \frac{1}{n}\right)^{\frac{1}{2(n-1)}} \bar{\varphi}^\frac{4}{n} - 1,$$  \hspace{1cm} (44)

$$a(\tau) = \pm \frac{\sqrt{2}}{2} e^{-\frac{2a}{n}} e^{\pm \frac{2}{n} \left(\frac{n}{1-n}\right) t} e^{-\frac{1}{n} \left(\frac{n}{1-n}\right) \tau} e^{-\frac{1}{n} \left(\frac{n}{1-n}\right) \tau}.$$  \hspace{1cm} (45)

where $\tau = y_0 - n|Q_0|^\frac{1}{n} t$.

### 4.2 $X(x, \dot{x}) \sim \dot{x}g(x)$

According to [23], we consider another ansatz $X(x, \dot{x}) \sim \dot{x}g(x)$, where $g(x)$ is an arbitrary function. By this assumption, in order for $X$ to be the symmetry vector, we obtain

$$f(x) = \frac{1}{4} \dot{\phi}^2(x) = \frac{g''(x)}{g'(x)} > 0.$$  \hspace{1cm} (46)

By solving this equation we find $\phi$ in terms of $x$ as

$$\phi(x) = \phi_c \pm 2 \int \sqrt{\frac{g''(x)}{g'(x)}} dx.$$  \hspace{1cm} (47)

Also, the Hojman conserved quantity is obtained as

$$\dot{x}g'(x) = Q_0.$$  \hspace{1cm} (48)
On the other hand, using equations \([16]\) and \([28]\) gives rise to
\[
\ddot{Z}(x) = \dot{Z}_0 g''(x) - 3g'(x) g'''(x),
\]
which can be rewritten as
\[
\ddot{Z}(x) = \dot{Z}_0 (f - 3 e^{-2f_{2,0}} f'(x') dx').
\]

Now, by considering some ansatzs for \(g(x)\), we find some exact solutions in the following.

i. \(g(x) = \lambda e^{\frac{1}{2} \alpha^2 x}\) (\(\alpha, \lambda\) are constants)

By this choice, Eq.\([47]\) becomes
\[
\tilde{\phi}(x) = \tilde{\phi}_c \pm \sqrt{2} \alpha (x - x_0),
\]
where \(\tilde{\phi}_c\) and \(x_0\) are constants. Also, using the equations \([48], [50]\) and \([51]\) yields
\[
x(t) = x_0 + \frac{2}{\alpha^2} \ln \left( 1 + e^{-\frac{1}{2} \alpha^2 x_0} \frac{Q_0}{\lambda} (t - t_0) \right),
\]
and
\[
\ddot{Z}(\tilde{\phi}) = \dot{Z}_0 \frac{2(\alpha^2 - 6)}{\lambda^2 \alpha^4} e^{-\frac{1}{2} \alpha^2 (\phi - \phi_c)}.
\]

Finally, by using the conformal transformations, we find
\[
a(t) = \pm \frac{2c}{\sqrt{2}} e^{x_0} \left( 1 + e^{-\frac{1}{2} \alpha^2 x_0} \frac{Q_0}{\lambda} (t - t_0) \right) \frac{2 \sqrt{2}}{\alpha^2},
\]
\[
\psi(t) = \ln \left[ \frac{1}{fc^2} \left( 1 + e^{-\frac{1}{2} \alpha^2 x_0} \frac{Q_0}{\lambda} (t - t_0) \right) \frac{2 \sqrt{2}}{\alpha} \right],
\]
and
\[
W(\psi) = \frac{1}{4} W_0 \left( \frac{1}{2} \alpha^2 - 3 \right) e^{\frac{\sqrt{2}}{2} \alpha} e^{(1 + \frac{\sqrt{2}}{2} \alpha) \psi} + \frac{1}{2},
\]

ii. \(g(x) = \frac{(f_0 + x)^{1+\alpha}}{1+\alpha}, \quad \alpha > 0\)

Here \(f_0\) and \(\alpha\) are constants parameters. Again, from Eqs. \([17], [43], [50]\) and \([51]\) we find respectively \(\tilde{\phi}(x), x(t)\) and \(\ddot{Z}(\tilde{\phi})\) as follows
\[
\tilde{\phi}(x) = \tilde{\phi}_c \pm 4 \sqrt{\alpha} \left( \sqrt{f_0 + x} \right) \mp 4 \sqrt{\alpha} \left( \sqrt{f_0 + x_0} \right),
\]
\[
x(t) = -f_0 + \left[ (\alpha + 1)Q_0 (t - t_0) + (f_0 + x_0)^{1+\alpha} \right]^{\frac{1}{1+\alpha}},
\]
and
\[
\ddot{Z}(\tilde{\phi}) = \lambda \Phi^{\alpha - 4} - \frac{16}{3} \lambda \alpha^3 \Phi^{-4\alpha - 2},
\]
where \(\Phi = \tilde{\phi}(x) - \tilde{\phi}_c \pm 4 \sqrt{\alpha} \left( \sqrt{f_0 + x_0} \right)\). Finally, using the conformal transformations yields the scale factor, the dilaton field and the generic potential, respectively as follows
\[
a(\tau) = \sqrt{2} e^{-f_0} (\pm \frac{2}{c})^{\frac{1}{2}} e^{\mp 2 \sqrt{\alpha} ((f_0 + x_0)^{1+\alpha + \tau})^{\frac{1}{2 (1+\alpha)}}} e^{((f_0 + x_0)^{1+\alpha + \tau})^\frac{1}{1+\alpha}},
\]
\( \psi(\tau) = \varphi_0 \pm 4\sqrt{\alpha} \left( (f_0 + x_0)^{1+\alpha} + \tau \right)^{1/(1+\alpha)}, \quad (61) \)

and

\( W(\psi) = 4 \left[ \lambda \psi^{-2\alpha-1} - \frac{3\lambda}{16\alpha^2} \psi^{-2\alpha+1} \right] + \frac{1}{2}. \quad (62) \)

where \( \tau = (1 + \alpha) Q_0 (t - t_0) \).

iii. \( g(x) = -\sqrt{2} \alpha (x + e^{2\alpha^2 x}) \)

Here \( \alpha \) is a constant parameter. Analogous to the previous cases,

\( \tilde{\phi}(x) = \tilde{\phi}_c \pm \frac{2\sqrt{2}}{\alpha} \text{arcsinh}(\sqrt{2} \alpha e^{\alpha^2 x}), \quad (63) \)

\( x(\tau) = \frac{1}{2\alpha^2} \left( \tau - \text{LambertW}(2\alpha^2 e^\tau) \right), \quad (64) \)

\( \tilde{Z}(\tilde{\Phi}) = \tilde{Z}_0 \left[ (2\alpha^2 - 3) \text{sech}^4 \left( \frac{1}{2} \alpha \tilde{\Phi} \right) - 2\alpha^2 \text{sech}^6 \left( \frac{1}{2} \alpha \tilde{\Phi} \right) \right], \quad (65) \)

where \( \phi_c \) is a constant parameter, \( \tilde{\Phi} = \pm \frac{\sqrt{2}}{2} (\tilde{\phi} - \tilde{\phi}_c) \) and \( \tau = 2\alpha^2 Q_0 (t - t_0) \). Also, using the conformal transformations here again we find

\( a(\tau) = \pm e^{-\frac{1}{2} \phi_c} e^{\mp \frac{\sqrt{2}}{\alpha} \text{arcsinh}(\sqrt{2} \alpha e^{\alpha^2 x})} e^{\frac{1}{2\alpha^2} \left( \tau - \text{LambertW}(2\alpha^2 e^\tau) \right)}, \quad (66) \)

\( \psi(\tau) = \psi_0 \pm \frac{2\sqrt{2}}{\alpha} \text{arcsinh} \left( \sqrt{2} \alpha e^{\frac{1}{2} \alpha \tilde{\Phi}} \text{LambertW}(2\alpha^2 e^\tau) \right), \quad (67) \)

and for generic potential

\( W(\psi) = 4\tilde{Z}_0 \left[ (2\alpha^2 - 3) \text{sech}^4 (\psi_c \pm \frac{\sqrt{2}}{4} \alpha \psi) - 2\alpha^2 \text{sech}^6 (\psi_c \pm \frac{\sqrt{2}}{4} \alpha \psi) \right] e^{2\psi} + \frac{1}{2} e^\psi \quad (68) \)

where \( \psi_0 = \ln \frac{1}{2} + \tilde{\phi}_c \) and \( \psi_c = \pm \frac{\sqrt{2}}{2} \alpha \ln 2 \mp \frac{\sqrt{2}}{2} \alpha \tilde{\phi}_c \) is a constant parameter.

5 Conclusion

In this paper, we have studied a string cosmological model with the background metric of Bianchi type I, accompanied by a totally antisymmetric field strength, non-minimally coupled to a dilaton field having a potential term. Using the Hojman symmetry approach, we have fixed the dilaton field potentials and obtained the corresponding exact solutions for the scale factor and the dilaton field.

References

[1] M. Bucher, International Journal of Modern Physics D. 24, (2015) 1530004.

[2] E. Fradkin and A. Tseytlin, Nucl. Phys. B 261 (1985) 1; C. Callan, D. Friedan, E. Martinec and M. Perry, Nucl. Phys. B 262 (1985) 593; R.R. Metsaev, A.A. Tseytlin, Nucl. Phys. B 293 (1987) 385.

[3] D. J. Gross and J. D. Sloan, Nucl. Phys., 13291, 41 (1987).

[4] R. Myers, Phys. Lett. B 199 (1987) 371.
[5] I. Antoniadis, C. Bachas, J. Ellis and D. V. Nanopoulos, Phys. Lett. B 211 (1988) 393; Nucl. Phys. B 328 (1989) 117.

[6] M. Muller, Nucl. Phys. B 337 (1990) 37.

[7] R. Brandenberger and C. Vafa, Nucl. Phys. B 316 (1988) 319; A. A. Tseytlin and C. Vafa, Nucl. Phys. B 372 (1992) 443.

[8] E. J. Copeland, A. Lahiri and D. Wands, Phys. Rev. D 50 (1994) 4868; String cosmology with a time-dependent antisymmetric-tensor potential, preprint Sussex-Ast/94/10-1, hep/th/9410136.

[9] M. Gasperini and R. Ricci, Homogeneous conformal string backgrounds, preprint DFrT-02/95, hep/9501055.

[10] J. M. A. H. MacCallum, Anisotropic and inhomogeneous relativistic cosmologies, in General relativity - An Einstein centenary survey, ed. S.W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, 1979): S. Barrow and D.H. Sonoda, Phys. Rep. 139 (1986) I.

[11] A. Taub, Ann. Math. 53 (1951) 472.

[12] G. E. R. Ellis and M.A.H. MacCallum, Commun. Math Phys. 12 (1969) 108; 19 (1970) 31.

[13] C. W. Misner, Phys. Rev. Lett. 22 (1969) 1071.

[14] M. Ryan and L. Shepley, Homogeneous relativistic cosmologies (Princeton Univ. Press, Princeton, 1975); D. Kramer et al., Exact solutions of Einstein’s field equations (Cambridge Univ. Press, Cambridge, 1980).

[15] E. W. Kolb and M. S. Turner, The Early Universe: Reprints (Addison-Wesley, N.Y., 1988); The Early Universe (Addison-Wesley, N.Y., 1990).

[16] N. A. Batakis and A. A. Kehagias, Anisotropic space-times in homogeneous string cosmology, Nucl. Phys. B 449 (1995) 248; N. A. Batakis, A new class of homogeneous string backgrounds, Phys. Lett. B353, (1995) 450.

[17] B. Mojaveri and A. Rezaei-Aghdam, International Journal of Modern Physics A Vol. 27, No. 07, 1250032 (2012).

[18] S. Capozziello, M. De Laurentis, R. Myrzakulov, Int. J. Geom. Meth. Mod. Phys. 12, 05, (2015) 1550065.

[19] S. Capozziello, M. De Laurentis, R. Myrzakulov, Int. J. Geom. Meth. Mod. Phys. 12, N09, (2015) 1550095.

[20] K. Myrzakulov, P. Tsyba, R. Myrzakulov, Noether symmetry in F(T) gravity with f-essence, arXiv:[1601.07357].

[21] A. Aslam, M. Jamil, R. Myrzakulov. Phys. Scr., 88, (2013) 025003

[22] S. A. Hojman, J. Phys. A: Math. Gen 25, (1992) L291.

[23] S. Capozziello and M. Roshan, Phys. Lett. B, 726, (2013) 471.

[24] M. Paolella and S. Capozziello, Phys. Lett. A, 379, (2015) 1304.

[25] A. Paliathanasis, P. G. L. Leach. Comment on the Hojman conservation quantities in Cosmology.

[26] Hao Wei, Ya-Nan Zhou, Hong-Yu Li, Xiao-Bo Zou. Astrophys. Space Sci., 360, (2015) 6; I. A. Bizyaev, A. V. Borisov, I. S. Mamaev. SIGMA, 12, (2016) 012.
[27] Hao Wei, Hong-Yu Li, Xiao-Bo Zou. Nucl. Phys. B, 903, (2016) 132.

[28] H. Nastase. Cosmology and String Theory (Springer, 2019).

[29] S. Capozziello, R. DE Ritis and C. RUBANO: Phys. Lett. A, 177, 8 (1993).

[30] S. Capozziello and R. DE Ritis: Int. J. Mod. Phys. D, 2, 373 (1993).

[31] S. Capozziello, R. DE Ritis and P. Scudellaro: Int. J. Mod. Phys. D, 2, 463 (1993).

[32] S. Capozziello and R. DE Ritis: Int. J. Mod. Phys. D, 2, 367 (1993).

[33] G. Veneziano, Phys. Lett. B 265 (1991) 287, M. Gasperini, G. Veneziano, Mod. Phys. Lett. A 8 (1993) 3701; M. Gasperini, G. Veneziano, Phys. Rev. D 50 (1994) 2519.

[34] M. Gasperini, M. Maggiore, and G. Veneziano, Nucl. Phys. B 494, 315 (1997).

[35] N. A. Batakis, Phys. Lett. B 353, 450 (1995); N. A. Batakis and A. A. Kehagias, Nucl. Phys. B 449, 248 (1995); N. A. Batakis, Phys. Lett. B 353, 39 (1995); J. D. Barrow and M. P. Da, browski, Phys. Rev. D 55, 630 (1997).

[36] J. D. Barrow and K. E. Kunze, Phys. Rev. D 56, 741 (1997).

[37] F. Naderi, A. Rezaei-Aghdam and F. Darabi, Phys. Rev. D98 (2018)026009; F. Naderi and A. Rezaei-Aghdam, Nucl. Phys. B923(2017) 416.

[38] K. Kikkawa and M. Yamasaki, Phys. Lett. B149, 357 (1984); G. Veneziano, Phys. Lett. B265 (1991) 287; E. Witten, Phys. Lett. B149, 4; K.A. Meissner and G. Veneziano, Phys. Lett. B267 (1991) 33; M. Gasperini, Astroparticle Phys. 1 (1993) 317; M. Gasperini and G. Veneziano, Phys. Lett. B272 (1991) 277.

[39] G. Allemandi, M. Capone, S. Capozziello, M. Francaviglia, Gen. Relativ. Gravit. 38 (2005) 33.

[40] S. Capozziello, R. DE Ritis, C. Rubano and P. Scudellaro: RIVISTA DEL NUOVO CIMENTO, Vol. 19 (1996) N. 4.