TECHNICAL REPORT

Fundamental lemmas for the determination of optimal control strategies for a class of single machine family scheduling problems

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Abstract
Four lemmas, which constitute the theoretical foundation necessary to determine optimal control strategies for a class of single machine family scheduling problems, are presented in this technical report. The scheduling problem is characterized by the presence of sequence-dependent batch setup and controllable processing times; moreover, the generalized due-date model is adopted in the problem. The lemmas are employed within a constructive procedure (proposed by the Author and based on the application of dynamic programming) that allows determining the decisions which optimally solve the scheduling problem as functions of the system state. Two complete examples of single machine family scheduling problem are included in the technical report with the aim of illustrating the application of the fundamental lemmas in the proposed approach.
1 Introduction

In [1], a class of single machine family scheduling problems (mainly characterized by multiclass jobs, generalized due-dates, and controllable processing times) has been formalized as an optimal control problem. Its solution consists of optimal control strategies which are functions of the system state, and therefore they are able to provide the optimal decisions for any actual machine behavior (the single machine is assumed to be unreliable and then perturbations, such as breakdowns, generic unavailabilities, and slowdowns, may affect the nominal behavior of the system). However, the scheduling problem in [1] has been solved under the assumption that, for each class of jobs, any unitary tardiness cost is greater than the unitary cost related to the deviation from the nominal service time. In order to remove such a strong hypothesis and to extend the scheduling model by adding setup times and, especially, setup costs, new fundamental lemmas have been defined. They are employed within the constructive procedure proposed in [2] that solves, from a control-theoretic perspective, a single machine scheduling problem with sequence-dependent batch setup and controllable processing times.

This technical report is organized as follows. Some preliminary definitions are reported in section 2. The four new lemmas are presented in sections 3, together with their complete proofs. Nine numerical examples aiming at illustrating how lemmas 1 and 2 work are in section 4. Finally, sections 5 and 6 present two complete example which explain the application of the procedure proposed in [2] to two single machine family scheduling problems (the latter with setup).

2 Definitions

Definition 1. Consider a function $f(x)$ which is continuous, nondecreasing, and piece-wise linear function of the independent variable $x$. Let $f(x)$ be characterized by $M \geq 1$ changes of slopes and let $\gamma_i$, $i = 1, \ldots, M$, be the values of the horizontal axis at which the slope changes ($\gamma_{i+1} > \gamma_i$, $\forall i = 1, \ldots, M - 1$). In this connection, let $\mu_0$ be the slope in interval $(-\infty, \gamma_1)$, $\mu_i$ be the slope in interval $[\gamma_i, \gamma_{i+1})$, $i = 1, \ldots, M - 1$, and $\mu_M$ be the slope in interval $[\gamma_M, +\infty)$ ($\mu_{i+1} \neq \mu_i$, $\forall i = 0, \ldots, M - 1$). Moreover, it is assumed $f(x) = 0$ for any $x \leq \gamma_1$; then, $\mu_0 = 0$. An example of function $f(x)$ following this definition is in figure 1.

![Example of function $f(x)$.](image)

Definition 2. With reference to $f(x)$, as defined by definition 1, let $f(x + t)$ be a continuous, nondecreasing, and piece-wise linear function of the independent variable $x$, parameterized by the real value $t$. An example of function $f(x + t)$ following this definition is in figure 2.
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\[ f(x + t) = \mu_0 - \mu_1 x - \mu_2 x^2 - \mu_3 x^3 - \mu_4 x^4 - \mu_5 x^5 - \mu_6 x^6 - \mu_7 x^7 - \mu_8 x^8 - \mu_9 x^9 - \mu_{10} x^{10} \]

\[ g(x) = \begin{cases} 
-\nu (x - x_2) & x \in [x_1, x_2) \\
0 & x \notin [x_1, x_2) 
\end{cases} \quad (1) \]

Definition 3. Consider a function \( g(x) \) which is noncontinuous, nonincreasing, and piece-wise linear function of the independent variable \( x \). Let \( g(x) \) be defined as

An example of function \( g(x) \) following this definition is in figure 3.

\[ g(x) = \begin{cases} 
-\nu & x \in [x_1, x_2) \\
0 & x \notin [x_1, x_2) 
\end{cases} \]

Figure 3: Example of function \( g(x) \).

3 Lemmas

In connection with functions \( f(x) \) and \( g(x) \) as defined by definitions 1 and 3 let:

- \( A \) be the set of indices \( i, i \in \{1, \ldots, M\} \), such that \( \mu_{i-1} < \nu \) and \( \mu_i \geq \nu \); in this connection, let \( |A| \) be the cardinality of set \( A \) and, if \( |A| > 0 \), let \( a_j, j = 1, \ldots, |A| \), be the generic element of set \( A \); thus, \( \gamma_{a_j}, j = 1, \ldots, |A| \), are the value of the horizontal axis at which the slope of \( f(x) \) changes from a value less than \( \nu \) to a value greater than or equal to \( \nu \);

- \( B \) be the set of indices \( i, i \in \{1, \ldots, M\} \), such that \( \mu_{i-1} \geq \nu \) and \( \mu_i < \nu \); in this connection, let \( |B| \) be the cardinality of set \( B \) and, if \( |B| > 0 \) let \( b_j, j = 1, \ldots, |B| \), be the generic element of set \( B \); thus, \( \gamma_{b_j}, j = 1, \ldots, |B| \), are the value of the horizontal axis at which the slope of \( f(x) \) changes from a value greater than or equal to \( \nu \) to a value less than \( \nu \).

Since it has been assumed \( \mu_0 = 0 \), then \( a_j < b_j \forall j = 1, \ldots, |B| \) and \( b_j < a_{j+1} \forall j = 1, \ldots, |A| - 1 \). Moreover, \( |A| - |B| \leq 1 \) being \( |A| = |B| \) if \( \mu_M < \nu \) and \( |A| = |B| + 1 \) if \( \mu_M \geq \nu \).
**Lemma 1.** Let \( f(x + t) \) be a continuous nondecreasing piece-wise linear function of \( x \), parameterized by \( t \), as defined by definition 2 and let \( g(x) \) be a noncontinuous function of \( x \), as defined by definition 3.

In case \(|B| \geq 1\), let \( \Omega \) be the set of time instants \( \{\omega_1, \ldots, \omega_j, \ldots, \omega_{|B|}\} \) in which any value \( \omega_j, j = 1, \ldots, |B| \), is obtained by executing algorithm 1. Each value \( \omega_j, j = 1, \ldots, |B| \), is either finite or nonfinite. Let \( T \) be the set of time instants \( \{t_1^*, t_2^*, \ldots, t_Q^*\} \) which is obtained from \( \Omega \) by removing all nonfinite values from it, that is

\[
T = \Omega \setminus \{\omega_j : \omega_j = +\infty, j = 1, \ldots, |B|\}
\]

Let \( Q \) be the cardinality of set \( T \); it is obviously \( 1 \leq Q \leq |B| \). In case \(|B| = 0\), it is \( T = \emptyset \) and \( Q = 0 \).

Then, the function of \( t \)

\[
x^*(t) = \arg \min_{x_1 \leq x \leq x_2} \{ f(x + t) + g(x) \}
\]

is a nonincreasing, possibly noncontinuous, piece-wise linear function of \( t \) defined as

if \( Q = 0 \) : \( x^*(t) = x_e(t) \)

if \( Q = 1 \) : \( x^*(t) = \begin{cases} x_e(t) & t < t_1^* \\ x_e(t) & t \geq t_1^* \end{cases} \)

if \( Q > 1 \) : \( x^*(t) = \begin{cases} x_e(t) & t < t_1^* \\ x_q(t) & t^*_q \leq t < t_{q+1}^*, \quad q = 1, \ldots, Q - 1 \\ x_e(t) & t \geq t_Q^* \end{cases} \)

where \( x_e(t) \), \( x_q(t) \), and \( x_e(t) \) are the following functions of \( t \):

- \( x_e(t) \) is a continuous nonincreasing piece-wise linear functions of \( t \) defined as:

  if \( t_1^* > \gamma_{a_1} - x_1 \) \( \rightarrow \) \( x_e(t) = \begin{cases} x_2 & t < \gamma_{a_1} - x_2 \\ x_1 & t \geq \gamma_{a_1} - x_1 \end{cases} \)

- \( x_q(t) \) is a continuous nonincreasing piece-wise linear functions of \( t \) defined as:

  if \( t_q^* < \gamma_{a(q)+1} - x_2 \) \( \rightarrow \) \( x_q(t) = \begin{cases} x_2 & t < \gamma_{a(q)+1} - x_2 \\ x_1 & t \geq \gamma_{a(q)+1} - x_1 \end{cases} \)

  and \( t_{q+1}^* > \gamma_{a(q)+1} - x_1 \)

- \( x_e(t) \) is a continuous nonincreasing piece-wise linear functions of \( t \) defined as:

  if \( t_q^* > \gamma_{a(q)+1} - x_2 \) \( \rightarrow \) \( x_e(t) = \begin{cases} x_2 & t < \gamma_{a(q)+1} - x_2 \\ x_1 & t \geq \gamma_{a(q)+1} - x_1 \end{cases} \)

  and \( t_{q+1}^* \leq \gamma_{a(q)+1} - x_1 \)

having assumed (for notational convenience) \( t_Q^* = -\infty \) when \( Q = 0 \).
In (5) and (7), \( l(q), q = 1, \ldots, Q, \) is a mapping function which provides the index \( j \in \{1, \ldots, |B|\} \) of the value \( t^*_j \) in the set \( \Omega \), that is, \( l(q) = j \Leftrightarrow \omega_j = t^*_j \). In this connection, it is always \( l(Q) = |B| \) and, in case \(|A| > |B|\), it turns out \( l(Q) + 1 = |A| \). Moreover, it is assumed, for notational convenience, \( l(0) = 0 \).

**Algorithm 1.** Determination of the time instant \( \omega_j, j = 1, \ldots, |B| \), at which, in case \( \omega_j < +\infty \), the function \( x^c(t) \) jumps in an upward direction.

**SECTION A – INITIALIZATION**

1. \( \gamma_0 = -\infty \)
2. \( h_0 = 0 : \gamma_{h_0} \leq \gamma_{h_0} - (x_2-x_1) < \gamma_0 + 1 \)
3. \( a_0 = 1 \)
4. \( k \leq M : \gamma_k < \gamma_k + (x_2-x_1) \leq \gamma_k + 1 \)
5. if \( j = |B| \) and \( |A| = |B| \) then
6. \( a_{j+1} = M + 1 \)
7. end if
8. for \( p = h \) to \( k \)
9. \( \mu_p = \mu_p - \nu \)
10. end for
11. \( \tau = \gamma_j - (x_2-x_1) \)
12. \( \theta = \gamma_j \)
13. \( d = \max \{0, \tilde{\mu}_p(\gamma_{h+1} - \tau)\} \)
14. if \( h < b_j - 1 \) then
15. \( p = h + 1 \) to \( b_j - 1 \)
16. \( d = \max \{0, d + \tilde{\mu}_p(\gamma_{h+1} - \gamma_p)\} \)
17. end for
18. end if
19. \( \lambda = h \)
20. \( \xi = i \)

**SECTION B – FIRST LOOP**

while \( h < b_j \) and \( i < a_{j+1} \)

\[ \psi = \min \{ \gamma_{h+1} - \tau, \gamma_{i+1} - \theta \} \] 
if \( \gamma_{h+1} - \tau \leq \gamma_{i+1} - \theta \) then
\( \lambda = h + 1 \) end if
if \( \gamma_{h+1} - \tau \geq \gamma_{i+1} - \theta \) then
\( \xi = i + 1 \) end if
\( \delta = \max \{0, \tilde{\mu}_p[\gamma_{h+1} - (\tau + \psi)]\} \) if \( \lambda < b_j - 1 \) then
\( p = \lambda + 1 \) to \( b_j - 1 \)
\( \delta = \max \{0, \delta + \tilde{\mu}_p(\gamma_{h+1} - \gamma_p)\} \) end if
end if
if \( \xi = b_j \) then
\( \delta = \delta + \tilde{\mu}_p[(\theta + \psi) - \gamma_j] \)
else if \( \xi = a_{j+1} \) then
\( \delta = \delta + \sum_{\gamma_p = b_j} \tilde{\mu}_p(\gamma_{h+1} - \gamma_p) \)
else
\( \delta = \delta + \sum_{\gamma_p = b_j} \tilde{\mu}_p(\gamma_{h+1} - \gamma_p + \tilde{\mu}_p[(\theta + \psi) - \gamma_j] \) end if
if \( \delta \leq 0 \) then
\( a_0 = 0 \)
end if
end if
end while

**SECTION C – SECOND LOOP**

while \( h < b_j \)
\( \psi = \gamma_{h+1} - \tau \)
\( \lambda = h + 1 \)
if \( \lambda < b_j \) then
\( \delta = \max \{0, \tilde{\mu}_p[\gamma_{h+1} - (\tau + \psi)]\} \)
else if \( \lambda < b_j - 1 \) then
\( \delta = \max \{0, \delta + \tilde{\mu}_p(\gamma_{h+1} - \gamma_p) \) end if
end if
\( a_0 = 0 \)
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Proof. The function $x^o(t)$ can be obtained by analyzing the shape of the function $f(x + t) + g(x)$ in the interval $[x_1, x_2]$, with $t$ moving from $-\infty$ to $+\infty$. The proof consists of seven parts:

1. in the first part, it is proven that, when $|B| = 0$, $x^o(t)$ has the structure provided by (4a), with $x_e(t)$ provided by (7a) (if $l(Q) < |A|$) or (7c) (if $l(Q) = |A|$);

2. in the second part, it is proven that, when $\gamma_{b_1} - \gamma_{a_1} > (x_2 - x_1), \forall j = 1, \ldots, |B|$, $|B| > 0$, and $\gamma_{a_{i+1}} - \gamma_{b_j} > (x_2 - x_1), \forall j = 1, \ldots, |B|, |A| - 1, |A| > 1$, $x^o(t)$ has the structure provided by (4b) or (4c), with $x_e(t)$ provided by (5a), $x_\epsilon(t), q = 1, \ldots, Q - 1, Q > 1$, provided by (6a), and $x_e(t)$ provided by (7a) (if $l(Q) < |A|$) or (7c) (if $l(Q) = |A|$);

3. in the third part, it is shown that the number of jump discontinuities in $x^o(t)$ may be less than $|B|$, that is, they are $Q \leq |B|$, and the conditions for which a jump discontinuity does not exist in connection with a specific abscissa $\gamma_j - t, j \in \{1, \ldots, |B|\}$ are provided;

4. in the fourth part, it is proven that, even if the assumptions considered in the second part do not hold for some $j \in \{1, \ldots, |B|\}$ or $j \in \{1, \ldots, |A| - 1\}$, it is sufficient that $t_{q+1} > \gamma_{a_1} - x_1, or t_{q} < \gamma_{a_{i+1}} - x_2 and t_{q+1} > \gamma_{a_{i+1}} - x_1, q \in \{1, \ldots, Q - 1\}, Q > 1, or t_{Q} < \gamma_{a_{i+1}} - x_2, to guarantee that $x_e(t)$ has the structure provided by (5a), $x_\epsilon(t)$ has the structure provided by (5a), and $x_e(t)$ has the structure provided by (7a) (if $l(Q) < |A|$), respectively;

5. in the fifth part, it is proven that, if $t_{q} \geq \gamma_{a_{i+1}} - x_2, q \in \{1, \ldots, Q - 1\}$ or $t_{Q} \geq \gamma_{a_{i+1}} - x_2, then there is at $t = t_{q}$ a discontinuity in $x^o(t)$ at which it jumps upwardly from $x_1$ to $t + \gamma_{a_{i+1}} \leq \gamma_{b_1} - t \leq x_2, q \in \{1, \ldots, Q\}, that, is, $x_\epsilon(t)$ in (4c) has the structure of (6a) or (6d) or $x_e(t)$ in (4b) and (4c) has the structure of (6b);

6. in the sixth part, it is proven that, if $t_{q+1} \leq \gamma_{a_1} - x_1 or t_{Q+1} \leq \gamma_{a_{i+1}} - x_1, q \in \{1, \ldots, Q - 1\}$, then there is at $t = t_{q+1}$ a discontinuity in $x^o(t)$ at which it jumps upwardly from $t + \gamma_{a_{i+1}} \geq x_1 to x_2, q \in \{1, \ldots, Q\}, that, is, $x_\epsilon(t)$ in (4b) and (4c) has the structure of (5b) or $x_e(t)$ in (4c) has the structure of (6c) or (6d);

7. in the seventh and last part, algorithm [1], which allows determining time instants $\omega_j, j = 1, \ldots, |B|$, is described.

First part

Consider the case $|B| = 0$, which implies $\Omega = \emptyset$ and then $T = \emptyset and Q = 0$. Moreover, $l(0) = 0$. If $|A| = 0$ as well, all the slopes of $f(x + t) + g(x)$ are less than $\nu$; then, $f(x + t) + g(x)$ is a strictly decreasing function of $x, \forall t$. In this case, the minimum of the function $f(x + t) + g(x)$, with respect to $[x_1, x_2]$, is always obtained at $x_1$. Thus, in this case, $x^o(t)$ has the structure provided by (4a), with $x_e(t)$ provided by (7c), being $l(Q) = |A|$.

If $|A| > 0$, it is definitely $|A| = 1 = l(Q) + 1$. In this case, the slopes of $f(x + t)$ are less than $\nu$ in the interval $(-\infty, \gamma_{a_{i+1}})$ and greater than or equal to $\nu$ in $[\gamma_{a_{i+1}}, +\infty)$. This case is very similar to that considered in

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When $t$ is such that $x_2 < \gamma_{a(t)} - t$, that is, $t < \gamma_{a(t)} - x_2$), the minimum with respect to $x$, $x_1 \leq x \leq x_2$, of $f(x + t) + g(x)$ is obtained at $x_2$. When $t$ is such that $x_1 < \gamma_{a(t)} - t \leq x_2$ (that is, $\gamma_{a(t)} - x_2 \leq t < \gamma_{a(t)} - x_1$), the function $f(x + t) + g(x)$ is strictly decreasing in $[x_1, \gamma_{a(t)} - t]$ and nondecreasing in $[\gamma_{a(t)} - t, x_2]$; then, it has a minimum, with respect to $[x_1, x_2]$, in $\gamma_{a(t)} - t$; when $t$ increases in the interval $[\gamma_{a(t)} - x_2, \gamma_{a(t)} - x_1)$, the minimum decreases (with unitary speed) from $x_2$ to $x_1$. Finally, when $t$ is such that $x_1 \geq \gamma_{a(t)} - t$ (that is, $t \geq \gamma_{a(t)} - x_1$), the minimum is obtained at $x_1$. Thus, in this case, $x_{\ast}(t)$ has the structure provided by (44), with $x_1(t)$ provided by (7a), being $l(Q) < |A|$. Note that, since $t^*_{Q} = -\infty$ when $Q = 0$, it is $t^* Q < \gamma_{a(t)} - x_2$ for sure.

Second part

Consider the case $|A| > 1$ and $|B| > 0$, and assume $\gamma_{b_j} - \gamma_{a_j} > (x_2 - x_1), \forall j = 1, \ldots, |B|$, and $\gamma_{a_j + 1} - \gamma_{b_j} > (x_2 - x_1), \forall j = 1, \ldots, |A| - 1$. Under such hypotheses, the function $x_{\ast}(t)$ is defined as follows.

1. When $t$ is such that the slopes of $f(x + t)$ in the interval $[x_1, x_2]$ are less than $\nu$, that is, $\forall t < \gamma_{a_1} - x_2$, $\forall t \in [\gamma_{a_1} - x_2, \gamma_{a_1} - x_2), j = 1, \ldots, |A| - 1$, and $\forall t \geq \gamma_{b_j} - x_1$ if $|A| = |B|$, the minimum of the function $f(x + t) + g(x)$, with respect to $[x_1, x_2]$, is obtained at $x_2$, since $f(x + t) + g(x)$ is strictly decreasing in $[x_1, x_2]$.

2. When $t$ is such that $\gamma_{a_j} - t \in [x_1, x_2], j = 1, \ldots, |A|$, that is, $\forall t \in [\gamma_{a_2} - x_2, \gamma_{a_2} - x_1), j = 1, \ldots, |A| - 1$, the function $f(x + t) + g(x)$ is strictly decreasing in $[x_1, \gamma_{a_j} - t]$ and nondecreasing in $[\gamma_{a_2} - t, x_2]$; then, it has a minimum, with respect to $[x_1, x_2]$, in $\gamma_{a_j} - t$; when $t$ increases in the interval $[\gamma_{a_2} - x_2, \gamma_{a_2} - x_1)$, the minimum decreases (with unitary speed) from $x_2$ to $x_1$.

3. When $t$ is such that the slope of $f(x + t)$ in the interval $[x_1, x_2]$ is greater than or equal to $\nu$, that is, $\forall t \in [\gamma_{a_1} - x_2, \gamma_{a_1} - x_2), j = 1, \ldots, |B|$, and $\forall t \geq \gamma_{a_1} - x_1$ if $|A| > |B|$, the minimum of the function $f(x + t) + g(x)$, with respect to $[x_1, x_2]$, is obtained at $x_1$, since $f(x + t) + g(x)$ is nondecreasing in $[x_1, x_2]$.

4. When $t$ is such that $\gamma_{b_j} - t \in [x_1, x_2], j = 1, \ldots, |B|$, that is, $\forall t \in [\gamma_{b_2} - x_2, \gamma_{b_2} - x_1), j = 1, \ldots, |B|$, the function $f(x + t) + g(x)$ is nondecreasing in $[x_1, \gamma_{b_j} - t]$ and strictly decreasing in $[\gamma_{b_2} - t, x_2]$; then, it has a maximum, with respect to $[x_1, x_2]$, in $x = \gamma_{b_j} - t$, and the minimum is obtained either at $x_1$ or $x_2$, depending on the values $f(x_1 + t) + g(x_1)$ and $f(x_2 + t) + g(x_2)$, $t \in [\gamma_{b_2} - x_2, \gamma_{b_2} - x_1)$ (the minimum is obtained at $x_{\ast} = x_1$ if $f(x_1 + t) + g(x_1) < f(x_2 + t) + g(x_2)$ and at $x_{\ast} = x_2$ otherwise). In this connection, note that:

- when $t = \gamma_{b_j} - x_2$, it is certainly $f(x_1 + t) + g(x_1) \leq f(x_2 + t) + g(x_2)$;
- when $t$ increases in the interval $[\gamma_{b_j} - x_2, \gamma_{b_j} - x_1)$, the value of $f(x_1 + t) + g(x_1)$ increases or remains constant and the value of $f(x_2 + t) + g(x_2)$ decreases;
- when $t = \gamma_{b_j} - x_1$, it is certainly $f(x_1 + t) + g(x_1) > f(x_2 + t) + g(x_2)$.

Thus, according to the previous “rules”, since $\mu_0 = 0$ the function $x_{\ast}(t)$ is $x_2$ at the beginning (rule 1), decreases with slope $-1$ in the interval $[\gamma_{a_1} - x_2, \gamma_{a_1} - x_1)$ (rule 2), is equal to $x_1$ from $\gamma_{a_1} - x_1$ to $\omega_1$, at which it jumps to $x_2$ (rule 3), $x_{\ast}'(t)$ remains equal to $x_2$ from $\omega_1$ up to $\gamma_{a_2} - x_2$ (rules 4 and 1), then it decreases with slope $-1$ in the interval $[\gamma_{a_2} - x_2, \gamma_{a_2} - x_1)$ (rule 2), is equal to $x_1$ from $\gamma_{a_2} - x_1$ to $\omega_2$, at which jumps to $x_2$ (rules 3 and 4), and so on. In its last part, the function $x_{\ast}(t)$ is $x_1$ if $|A| > |B|$ or $x_2$ if $|A| = |B|$, in accordance with rules 3 and 4.
is obtained). Let \( d = f(x_2 + t) + g(x_2) - f(x_1 + t) + g(x_1) \). Without considering the upward movement of the function (which is not important for the determination of \( \omega_j \)), when \( t \) increases the function moves leftward and \( d \) is reduced. As an example, in figure 5 the same function \( f(x + t) + g(x) \) is illustrated when \( t = \gamma_{b_j} - x_1 \). It is evident that \( \omega_j \) is the time instant at which \( d \) is null.

\[
\Delta f(x + t) + g(x) \text{ when } t = \gamma_{b_j} - x_1
\]

\[
\Delta f(x + t) + g(x) \text{ when } t = \gamma_{b_j} - x_2
\]

On the basis of such considerations, with the considered assumptions, to compute \( \omega_j \) it is possible to use the following algorithm (which is not formally described, but the reader can refer to the description of algorithm 1 which generalizes the following one).

**SECTION A – INITIALIZATION**

1. \( h \in \{a_j, \ldots, b_j - 1\} : \gamma_h \leq \gamma_{b_j} - (x_2 - x_1) < \gamma_{h+1} \)
2. \( i = b_j \)
3. if \( j = |B| \text{ and } |A| = |B| \) then
4. \( a_{j+1} = M + 1 \)
5. \( \gamma_{M+1} = +\infty \)
6. end if
7. \( k \in \{b_j, \ldots, a_{j+1} - 1\} : \gamma_k < \gamma_{b_j} + (x_2 - x_1) \leq \gamma_{k+1} \)
8. for \( p = h \) to \( k \) do
9. \( \tilde{\mu}_p = \mu_p - \nu \)
10. end for
11. \( \tau = \gamma_{b_j} - (x_2 - x_1) \)
12. \( \theta = \gamma_{b_j} \)
13. \( d = \tilde{\mu}_h(\gamma_{h+1} - \tau) \)
14. if \( h < b_j - 1 \) then
15. \( d = d + \sum_{p=h+1}^{b_j-1} \tilde{\mu}_p(\gamma_{p+1} - \gamma_p) \)
This algorithm provides, for any \( j = 1, \ldots, |B| \), the time instant \( \omega_j \) at which a jump discontinuity in \( x^2(t) \) occurs. Since \( \omega_j < +\infty \) for all \( j = 1, \ldots, |B| \), then \( T = \Omega \), being \( \Omega = \{ \omega_1, \omega_2, \omega_j, \omega |B| \} \). Moreover, \( Q = |B| \), \( t^* = \omega_q \), and \( l(q) = q \), \( \forall q = 1, \ldots, Q \). Then, it is possible to write \( t^* \in [h_{(i)} - t, h_{(i)} - t] \), \( h_{(i)} - t > x_1 \), \( \forall q = 1, \ldots, Q \), and \( \gamma_{(i)} > x_1 \), \( \forall q = 1, \ldots, Q \) if \( l(q) < |A| \) or \( \forall q = 1, \ldots, Q - 1 \) if \( l(q) = |A| \). Then, \( x^2(t) \) has the structure provided by (43) or (44), with \( x_i(t) \) provided by (5a), \( x_q(t) \), \( q = 1, \ldots, Q - 1 \), \( Q > 1 \), provided by (5a), and \( x_q(t) \) provided by (7a) if \( l(q) < |A| \) or (7c) if \( l(q) = |A| \).

Third part

It has been shown in the second part of the proof that, under the assumptions \( \gamma_{b_j} < \gamma_{a_j} > (x_2 - x_1) \), \( \forall j = 1, \ldots, |B| \), \( |B| > 0 \), and \( \gamma_{a_{j+1}} - \gamma_{b_j} > (x_2 - x_1) \), \( \forall j = 1, \ldots, |A| - 1 \), \( |A| > 1 \), there exists, for each value \( b_j \), \( j = 1, \ldots, |B| \), a finite value \( \omega_j \in \gamma_{b_j} - x_2, \gamma_{b_j} - x_1 \) at which \( x^2(t) \) jumps to \( x_2 \). In other words, there are \( |B| \) points of discontinuity in the function \( x^2(t) \). In the presence of a narrower intervals, this is not necessarily true.

As a matter of fact, in connection with two consecutive time intervals \( \gamma_{a_{j+1}} - \gamma_{a_j} \) and \( \gamma_{a_{j+1}} - \gamma_{a_j} \), which are such that \( \gamma_{b_{j+1}} - \gamma_{b_j} < (x_2 - x_1) \), when, for any \( t \in [\gamma_{a_{j+1}} - x_2, \min(\gamma_{b_{j+1}} - x_1, \gamma_{b_{b_j}} - x_1)] \), at least one of the two conditions \( f(x_2 + t) + g(x_2) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t) \) and \( f(x_2 + t) + g(x_2) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}}) \) is satisfied, then the local minimum at \( \gamma_{b_{j+1}} - t \) is never the absolute minimum in the interval \( [x_1, x_2] \). Then, in this case, the presence of the abcissa \( \gamma_{b_j} \), at which the slope of \( f(x + t) \) changes from a value greater than or equal to \( \nu \) to a value less than \( \nu \), does not cause the function \( x^2(t) \) to jump in an upward direction.

To show this, consider the example of function \( f(x + t) + g(x) \) illustrated in figure (5a), when \( t = \gamma_{b_j} - x_2 \), in which it is \( 2 \gamma_{b_{j+1}} - \gamma_{b_j} < (x_2 - x_1) \). When \( t = \gamma_{b_j} - x_2 \), the minimum with respect to \( [x_1, x_2] \) is obtained at \( x_1 \). If \( f(x_1 + t) + g(x_1) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t) \) when \( t = \gamma_{a_{j+1}} - x_2 \) (see figure (6b)), that is, if \( f(\gamma_{a_{j+1}} - x_2 + x_1) + g(x_1) < f(\gamma_{a_{j+1}}) \), then the minimum certainly remains at \( x_1 \) when \( t \) increases in the interval \( [\gamma_{b_{j+1}} - x_2, \gamma_{b_{j+1}} - x_2] \), since \( f(x + t) + g(x) \) is strictly decreasing in \( [\gamma_{b_{j+1}} - t, \gamma_{b_{j+1}} - t] \).

When \( t \) increases in the interval \( [\gamma_{a_{j+1}} - x_2, \gamma_{a_{j+1}} - x_2) \), if \( f(x_1 + t) + g(x_1) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t) \) for all \( t \) in such an interval, then the minimum is once more at \( x_1 \), since \( f(x_1 + t) + g(x) \) is nondecreasing in \( [\gamma_{a_{j+1}} - t, \gamma_{b_{j+1}} - t] \). When \( t \) increases in the interval \( [\gamma_{b_{j+1}} - x_2, \gamma_{b_{j+1}} - x_2) \), if at least one of the two conditions \( f(x_2 + t) + g(x_2) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t) \) and \( f(x_2 + t) + g(x_2) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t) \) is satisfied, then the minimum is at \( x_1 \) if \( f(x_1 + t) + g(x_1) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t) \) or \( x_2 \) or \( \gamma_{a_{j+1}} - t \) (if \( f(x_1 + t) + g(x_1) \geq f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t) \)). More specifically, the minimum jumps from \( x_1 \) to \( x_2 \) (or to \( \gamma_{a_{j+1}} - t \) when \( t \) is such that \( f(x_1 + t) + g(x_1) = f(x_2 + t) + g(x_2) \) (see figure (6c))), however, such a jump in an upward direction has to be associated with abcissa \( \gamma_{b_j} \), and not with \( \gamma_{b_j} \).

Note that, assumption \( \gamma_{b_{j+1}} - \gamma_{b_j} < (x_2 - x_1) \) is a necessary condition, because in case \( \gamma_{b_{j+1}} - \gamma_{b_j} > (x_2 - x_1) \) there is definitely a time instant \( t \) at which the local minimum at \( \gamma_{b_{j+1}} - t \) is the absolute minimum in the interval \( [x_1, x_2] \).
Algorithm 1 determines in the section C ("second loop") if the local minimum at $\gamma_{a_{j+1}} - t$ is the absolute minimum in the interval $[x_1, x_2]$. Such a part of the algorithm (rows 99–111 and 114–119) moves the function $f(x + t) + g(x)$ in a leftward direction (by increasing the time variable $\tau$) until that the value of the function at the local minimum $\gamma_{a_{j+1}} - t$ is lower than or equal to the value of the function at $x_1$ or, equivalently, until that $f(x_1 + t) + g(x_1) \geq f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t)$. When this happens, it results $\delta \leq 0$. At that point, the algorithm determines (at rows 119–123) if there is a value of the function $f(x + t) + g(x)$ in $(\gamma_{a_{j+1}} - t, x_2] \subset [x_1, x_2]$ which is lower than the value of the function at the local minimum $\gamma_{a_{j+1}} - t$ or, equivalently, if it exists $t$ such that $f(x_1 + t) + g(x_2) < f(\gamma_{a_{j+1}}) + g(\gamma_{a_{j+1}} - t)$. In the algorithm, such a lower value exists when $\phi < 0$; in this case, $\omega_j$ is set to the nonfinite value $+\infty$ and the algorithm ends since there is no more the possibility that the local minimum becomes an absolute minimum (since $f(x + t) + g(x)$ is strictly decreasing in $[\gamma_{b_{j+1}} - t, \gamma_{a_{j+1}} - t]$).

In conclusion, it has been shown in this part of the proof that some of the abscissae $\gamma_{b_j} - t$, $j \in \{1, \ldots, |B|\}$, may not cause a jump discontinuity in $x^\phi(t)$. Then, $x^\phi(t)$ has a number of discontinuities (at which it jumps in an upward direction) equal to $Q \leq |B|$. In accordance with the notation adopted in algorithm 1, values $\omega_j < +\infty$, $j \in \{1, \ldots, |B|\}$, are those actually corresponding to jump discontinuities. These $Q$ values are denoted as $\tau^*_q$, $q = 1, \ldots, Q$. The link between values $\omega_j$ and $\tau^*_q$ is represented by the mapping function $l(q) = j$, $j \in \{1, \ldots, |B|\}$, $q = 1, \ldots, Q$ (that is, $l(q) = j$ $\Leftrightarrow$ $\omega_j = \tau^*_q$).

From now on, only the $Q$ abscissae $\gamma_{b_1} - t$, $q = 1, \ldots, Q$, and the $Q + 1$ abscissae $\gamma_a$, and $\gamma_{a_{l+1}} - t$, $q = 1, \ldots, Q$, will be taken into account, to prove that, when $Q \geq 1$, $x^\phi(t)$ has the structure provided by (41b) or (44), with $x_0(t)$ provided by one of the (45), $x_0(t)$, $q = 1, \ldots, Q - 1$, $Q > 1$, provided by one of the (46), and $x_0(t)$ provided by one of the (47).

Fourth part
Consider the case $Q \geq 1$, and assume that $\gamma_{a_l} - \gamma_{b_1} < (x_2 - x_1)$ and $\gamma_{a_{l+1}} - \gamma_{a_0} < (x_2 - x_1)$ for some $q \in \{1, \ldots, Q\}$. An example of such a case is illustrated in figure 17. Note that, in accordance with the considerations made in the third part of the proof, regarding the time instants which actually produce a jump discontinuity in $x^\phi(t)$, such assumptions imply $l(q) + 1 = l(q + 1)$ and $l(q) + 2 = l(q + 1) + 1 = l(q + 2)$.
The condition $t^*_q < \gamma_{a(i+1)} - x_2$ means that a discontinuity occurs at $t = t^*_q$, at which $x^2(t)$ jumps to $x_2$, from either $x_1$ or $-t^*_q + \gamma_{a(i)}$ (depending if $t^*_q > \gamma_{a(i)} - x_1$ or $t^*_q \leq \gamma_{a(i)} - x_1$, respectively). In figure 7(a), $f(x + t) + g(x)$ when $t = t^*_q$ is illustrated. In accordance with the rules discussed in the second part of the proof, when $t$ increases from $t^*_q$ to $\gamma_{a(i+1)} - x_2$, the minimum remains at $x_2$. Moreover, when $t$ increases from $\gamma_{a(i+1)} - x_2$ on, the minimum decreases with unitary speed from $x_2$ towards $x_1$.

The condition $t^*_{q+1} > \gamma_{a(i+1)} - x_1$ means that the minimum, which is decreasing, reaches $x_1$ when $t = \gamma_{a(i+1)} - x_1$, and remains at $x_1$ in the interval $[\gamma_{a(i+1)} - x_1, t^*_{q+1}]$. At $t = t^*_{q+1}$, a discontinuity occurs, at which $x^2(t)$ jumps from $x_1$, to either $x_2$ or $-t^*_{q+1} + \gamma_{a(i+1)}$ (depending if $t^*_{q+1} < \gamma_{a(i+1)} - x_2$ or $t^*_{q+1} \geq \gamma_{a(i+1)} - x_2$, respectively).

Then, in case $t^*_q < \gamma_{a(i+1)} - x_2$ and $t^*_{q+1} > \gamma_{a(i+1)} - x_1$, the function $x^2(t)$ between time instants $t^*_q$ and $t^*_{q+1}$ has the structure provided by (5a).

When $q = 0$, the function $f(x + t) + g(x)$ is strictly decreasing in $(\infty, \gamma_{a_1} - t)$, then, the minimum is obtained at $x_2$ for all $t < \gamma_{a_1} - x_2$. When $t$ increases from $\gamma_{a_1} - x_2$ on, the minimum decreases with unitary speed from $x_2$ towards $x_1$. The condition $t^*_q > \gamma_{a_1} - x_1$ means that the minimum, which is decreasing, reaches $x_1$ when $t = \gamma_{a_1} - x_1$, and remains at $x_1$ in the interval $[\gamma_{a_1} - x_1, t^*_q]$. At $t = t^*_q$ a discontinuity occurs, at which $x^2(t)$ jumps from $x_1$, to either $x_2$ or $-t^*_q + \gamma_{a_1}$ (depending if $t^*_q < \gamma_{a_2} - x_2$ or $t^*_q \geq \gamma_{a_2} - x_2$, respectively). Then, in case $t^*_q > \gamma_{a_1} - x_1$, the function $x^2(t)$ before time instant $t^*_q$ has the structure provided by (5a).

When $q = Q$, if $t^*_Q < \gamma_{a(Q+1)} - x_2$, then a discontinuity occurs at $t = t^*_Q$, at which $x^2(t)$ jumps to $x_2$, from either $x_1$ or $-t^*_Q + \gamma_{a(Q)}$ (depending if $t^*_Q > \gamma_{a(Q)} - x_1$ or $t^*_Q \leq \gamma_{a(Q)} - x_1$, respectively). In accordance with the previous considerations, the minimum remains at $x_2$ in the interval $[t^*_Q, \gamma_{a(Q+1)} - x_2]$, decreases with unitary speed in the interval $[\gamma_{a(Q+1)} - x_2, \gamma_{a(Q+1)} - x_1]$, and remains at $x_1$ from $t = \gamma_{a(Q+1)} - x_1$ on, since $f(x + t) + g(x)$ is nondecreasing in $[\gamma_{a(Q+1)} - x_1, \infty)$. Then, in case $t^*_Q < \gamma_{a(Q+1)} - x_2$, the function $x^2(t)$ after time instant $t^*_Q$ has the structure provided by (7a).

Fifth part

Consider the case $Q \geq 1$, and assume that $\gamma_{a(i)} - \gamma_{b(i)} < (x_2 - x_1)$ and $\gamma_{a(i+1)} - \gamma_{a(i)} > (x_2 - x_1)$ for some $q \in \{1, \ldots, Q - 1\}$. If $t^*_q \geq \gamma_{a(i+1)} - x_2$, then $f(x_1 + t) + g(x_1) \leq f(x_2 + t) + g(x_2)$ when $t = \gamma_{a(i+1)} - x_2$, that is, $f(\gamma_{a(i+1)} - x_2 + \gamma_{a(i)}) + g(\gamma_{a(i+1)}) \leq f(\gamma_{a(i+1)} - x_2 + \gamma_{a(i)}) + g(\gamma_{a(i)} - x_1)$, as in the case illustrated in figure 7(a). When $t$ increases from $\gamma_{a(i+1)} - x_2$ on, the local minimum at $\gamma_{a(i+1)} - t$ decreases with unitary speed from $x_2$ towards $x_1$. Thus, $t^*_q$, which corresponds to the finite value $\omega_{a(q)}$, is the time instant at which $f(x_1 + t) + g(x_1) = f(\gamma_{a(i+1)} - x_2 + \gamma_{a(i)}) + g(\gamma_{a(i+1)} - t)$, as it is illustrated in figure 7(b). At $t^*_q$, the minimum within $[x_1, x_2]$ jumps from $x_1$ to $-t^*_q + \gamma_{a(i+1)}$. Then, $x_2(t)$ in (6b) has the structure of (6b) or (6d) (depending on the value $t^*_q+1$, as discussed in the following part of the proof).

Consider now the same case in which $\gamma_{a(i+1)} - \gamma_{b(i)} < (x_2 - x_1)$, for some $q \in \{1, \ldots, Q - 1\}$, but without any assumption about the interval $[\gamma_{a(i)}, \gamma_{a(i+1)}]$. In this case, when $t = \gamma_{a(i+1)} - x_2$, one or more local minima are present in the interval $[x_1, x_2]$, as in the cases illustrated in figure 9(a). In accordance with the considerations made in the third part of the proof, regarding the time instants which actually produce a jump discontinuity in $x^2(t)$, if $t^*_q \geq \gamma_{a(i+1)} - x_2$, then, when $t = \gamma_{a(i+1)} - x_2$, the global minimum in $[x_1, x_2]$ is at absissa $\gamma_{a(i+1)} - t$ (see again figure 9(a)). As before, when $t$ increases from $\gamma_{a(i+1)} - x_2$ on, the local minimum at $\gamma_{a(i+1)} - t$
Consider the case

The same considerations can be made when

minimum jumps from \( \gamma \) on the value \( f \)

makes in the third part of the proof, regarding the time instants which actually produce a jump discontinuity in \( f \) or \( g \) (depending on the value \( t^*_q+1 \)).

Figure 8: Example of function \( f(x + t) + g(x) \), (a) when \( t = \gamma_{a(q+1)} - x_2 \), and (b) when \( t = t^*_q \).

decreases with unitary speed from \( x_2 \) towards \( x_1 \), and \( t^*_q \) is the time instant at which \( f(x_1 + t) + g(x_1) = f(\gamma_{a(q+1)}) + g(\gamma_{a(q+1)} - t) \), as it is illustrated in figure 8(b). At \( t^*_q \), the minimum within \([x_1, x_2]\) jumps from \( x_1 \) to \(-t^*_q + \gamma_{a(q+1)} \). Then, \( x_q(t) \) in (4c) has the structure of (6b) or (6d) (depending on the value \( t^*_q+1 \)).

Figure 9: Example of function \( f(x + t) + g(x) \), (a) when \( t = \gamma_{a(q+1)} - x_2 \), and (b) when \( t = t^*_q \).

The same considerations can be made when \( q = Q \), in the case \( l(Q) < |A| \). If \( t^*_Q \geq \gamma_{a(q+1)} - x_2 \), at \( t^*_Q \) the minimum within \([x_1, x_2]\) jumps from \( x_1 \) to \(-t^*_Q + \gamma_{a(q+1)} \), and then \( x_q(t) \) in (4d) or (4e) has the structure of (7b).

Sixth part

Consider the case \( Q \geq 1 \), and assume that \( \gamma_{b(q+1)} - \gamma_{a(q+1)} < (x_2 - x_1) \) and \( \gamma_{a(q+1)} - \gamma_{a(q+1)} > (x_2 - x_1) \) for some \( q \in \{1, \ldots, Q - 1\} \). When \( t = \gamma_{b(q+1)} - x_2 \), the minimum within \([x_1, x_2]\) (which is decreasing with unitary speed since \( t \) was equal to \( \gamma_{a(q+1)} - x_2 \)) is at \( \gamma_{a(q+1)} - t \), as in the case illustrated in figure 10(a). If \( t^*_q+1 \leq \gamma_{a(q+1)} - x_1 \), then the minimum jumps to \( x_2 \) before than (or exactly when) it reaches \( x_1 \), that is, the minimum jumps from \( \gamma_{a(q+1)} - t \geq x_1 \) to \( x_2 \). \( t^*_q+1 \) is the time instant at which \( f(\gamma_{a(q+1)}) + g(\gamma_{a(q+1)} - t) = f(x_2 + t) + g(x_2) \), as it is illustrated in figure 8(b). Then, \( x_q(t) \) in (4f) has the structure of (6c) or (6d) (depending on the value \( t^*_q \), as discussed in the previous part of the proof).

Consider now the same case in which \( \gamma_{b(q+1)} - \gamma_{a(q+1)} < (x_2 - x_1) \), for some \( q \in \{1, \ldots, Q - 1\} \), but without any assumption about the interval \([\gamma_{a(q+1)} - \gamma_{a(q+1)} + 1] \). In this case, when \( t = \gamma_{b(q+1)} - x_2 \), one or more local minima are present in the interval \([x_1, x_2]\), as in the cases illustrated in figure 11(a). In accordance with the considerations made in the third part of the proof, regarding the time instants which actually produce a jump discontinuity in \( x^\circ(t) \), the global minimum in \([x_1, x_2]\) is at abscissa \( \gamma_{a(q+1)} - t \), as before. Then, also in this case, if \( t^*_q+1 \leq \gamma_{a(q+1)} - x_1 \), then the minimum jumps, at \( t^*_q+1 \), from \( \gamma_{a(q+1)} - t \geq x_1 \) to \( x_2 \). In conclusion, \( x_q(t) \) in (4f) has the structure of (6c) or (6d) (depending on the value \( t^*_q \)).

The same considerations can be made in connection with time instant \( t^*_1 \), when \( \gamma_{b(1)} - \gamma_{a_1} < (x_2 - x_1) \). In this case, if \( t^*_1 \leq \gamma_{a_1} - x_1 \), then the minimum jumps from \( \gamma_{a_1} - t \geq x_1 \) to \( x_2 \), and then \( x_q(t) \) in (4f) or (4e) has the structure of (6c) or (6d) (depending on the value \( t^*_q \)).
The algorithm which computes the value $\omega_j$, in correspondence with abscissa $\gamma_b_j$, considers the function $f(x) + g(x)$ and the “window” $[\gamma_b_j - (x_2 - x_1), \gamma_b_j]$, which is moved rightward to find the instant at which the minimum of the function within the window “jumps” from the left bound to the right bound, as discussed in the previous parts of the proof. Note that, considering the function $f(x) + g(x)$ and the window $[\gamma_b_j - (x_2 - x_1), \gamma_b_j]$, which are included in the interval $[x_1, x_2]$, that could “enter” the window when it moves rightward, are determined (rows $1 \div 5$); the slopes of $f(x) + g(x)$ are computed for any of those segments (rows $6 \div 11$); the initial values of the left and right bounds $\tau$ and $\theta$ are set (rows $12 \div 13$), and the initial value of $d$ is calculated (rows $14 \div 19$). Note that, the min operator in the determination of $d$ is necessary to compute $d$ when the minimum within $[\gamma_b_j - (x_2 - x_1), \gamma_b_j]$ (that is, at the beginning) is not obtained.

### Seventh part

The algorithm which computes the value $\omega_j$, in correspondence with abscissa $\gamma_b_j$, considers the function $f(x) + g(x)$ and the “window” $[\gamma_b_j - (x_2 - x_1), \gamma_b_j]$, which is moved rightward to find the instant at which the minimum of the function within the window “jumps” from the left bound to the right bound, as discussed in the previous parts of the proof. Note that, considering the function $f(x) + g(x)$ and the window $[\gamma_b_j - (x_2 - x_1), \gamma_b_j]$ is equivalent to consider the function $f(x + t) + g(x)$ with $t = \gamma_b_j - x_2$ and the window $[x_1, x_2]$. Basically, to determine the time instant at which the minimum within the window jumps in an upward direction, if it exists (as discussed in the third part of the proof), the algorithm moves the window rightward until the difference $d$ between the value of $f(x) + g(x)$ at the right bound $\theta$ of the window (or at the local minimum which is the nearest to the right bound) and its value at left bound $\tau$ of the same window (or at the current global minimum within $[\tau, \theta]$) becomes null. Since $f(x) + g(x)$ is a piece-wise linear function, the window is repeatedly moved of intervals whose lengths $\psi$ correspond to the lengths on the abscissa axis of the segments of the function. At each step, the new difference $\delta$ is computed and, if $\delta$ turns out to be null or negative, then the minimum has jumped to the right bound; this also means that the time instant $\omega_j$ is within the last rightward movement, that is, $\omega_j \in [\tau - x_1, \tau - x_1 + \psi]$.

In the “Section A – Initialization” part of the algorithm, the segments of the piece-wise linear function $f(x) + g(x)$ which are included in the interval $[\gamma_b_j - (x_2 - x_1), \gamma_b_j]$, and those of the interval $[\gamma_b_j, \gamma_b_j + (x_2 - x_1)]$ that could “enter” the window when it moves rightward, are determined (rows $1 \div 5$); the slopes of $f(x) + g(x)$ are computed for any of those segments (rows $6 \div 11$); the initial values of the left and right bounds $\tau$ and $\theta$ are set (rows $12 \div 13$), and the initial value of $d$ is calculated (rows $14 \div 19$). Note that, the min operator in the determination of $d$ is necessary to compute $d$ when the minimum within $[\gamma_b_j - (x_2 - x_1), \gamma_b_j]$ (that is, at the beginning) is not obtained.

Figure 10: Example of function $f(x + t) + g(x)$, (a) when $t = \gamma_b_{(q+1)} - x_2$ and (b) when $t = t^*_q + 1$.

Figure 11: Example of function $f(x + t) + g(x)$, (a) when $t = \gamma_b_{(q+1)} - x_2$ and (b) when $t = t^*_q + 1$. 
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at the left bound but is obtained at an abscissa greater than \(\gamma_{b_j} - (x_2 - x_1)\) (as for example, in the cases illustrated in figures [10](a) and [11](a)).

In the “Section B – First loop” of the algorithm, the while loop allows moving, segment-by-segment, the window \([\tau, \theta]\) leftward. At each step of the while loop, the length \(\psi\) of the next rightward movement is determined (row 23) and the new difference \(\delta\) is computed (rows 30–42). If \(\delta \leq 0\), then \(\omega_j\) can be determined (through one of the equations at rows 59, 61, 64 and 68); otherwise all values and indexes are updated (rows 72–76) and another step of the loop is executed. It is worth noting that several equations to compute \(\omega_j\) must be provided because of the possible presence of local minima within the moving window \([\tau, \theta]\); in this connection, values \(\chi\) (rows 48–51) are the relative value at the local minima (relative with respect to the value at the left bound of the window), and \(m\) (rows 52–56) is the relative value at the global minimum; note also that all local minima, if present, are before \(\gamma_{b_j} \in [\tau, \theta]\).

In the first loop, the window is moved until its right bound reaches abscissa \(\gamma_{a_{j+1}}\). This means that, if \(\omega_j\) is determined within the first loop, then the new minimum is definitely obtained at \(x_2\), since \(f(x) + g(x)\) is strictly decreasing in \([\gamma_{b_j}, \gamma_{a_{j+1}}]\). In case the minimum did not jump during the first loop (or, equivalently, if \(\omega_j\) has not been determined during the first loop), then the algorithm executes another loop in which, again, the window is moved rightward; the difference with respect to the first loop is that now the local minimum of \(f(x) + g(x)\) at \(\gamma_{a_{j+1}}\) is within the window.

In the “Section C – Second loop” of the algorithm, as before, the while loop allows moving, segment-by-segment, the window \([\tau, \theta]\) leftward and, at each step of the while loop, the length \(\psi\) of the next rightward movement is determined (row 80) and the new difference \(\delta\) is computed (rows 82–92). If \(\delta \leq 0\), then a nonfinite or a finite value of \(\omega_j\) is determined (respectively at rows 106 or 112); otherwise all values and indexes are updated (rows 115–117) and another step of the loop is executed. In this second loop, the window is moved until its left bound reaches abscissa \(\gamma_{b_j}\), but the algorithm certainly exits before then.

It is important to observe that when it results \(\delta \leq 0\), it is necessary to analyze the shape of \(f(x) + g(x)\) in the last part of the window, that is, from \(\gamma_{a_{j+1}}\), to \(\theta\); as a matter of fact, it is possible that, when the value of \(f(x) + g(x)\) at the abscissa \(\gamma_{a_{j+1}}\) becomes lower than or equal to all the values in \([\tau, \gamma_{a_{j+1}}]\), it is not the global minimum in \([\tau, \theta]\) because a lower value is obtained in \((\gamma_{a_{j+1}}, \theta]\) (such a lower value exists when \(\phi\), determined at rows 98–104, is negative); this is the case in which the presence of a local maximum at \(\gamma_{b_j}\) do not produce a jump discontinuity in \(x^\circ(t)\), as discussed in the third part of the proof. In this case, \(\omega_j\) is conventionally set to \(+\infty\).

This concludes the proof. \(\Box\)

Lemma \(\Xi\) is still valid when \(f(x) = e \neq 0\) for any \(x \leq \gamma_1\). Moreover, Lemma \(\Xi\) can be easily extended to consider the more general case in which the slope of function \(f(x)\) is not null at the beginning, that is, \(\mu_0 \neq 0\).

**Lemma 2.** With reference to the functions \(f(x + t)\) and \(g(x)\), as considered in Lemma \(\Xi\) and to the function \(x^\circ(t) = \arg \min_x \{f(x + t) + g(x)\}, x_1 \leq x \leq x_2\), provided by Lemma \(\Xi\) itself, the function

\[
h(t) = f(x^\circ(t) + t) + g(x^\circ(t))
\]

is a continuous, nondecreasing, and piece-wise linear function of the independent variable \(t\), that can be obtained by \(f(x + t)\) and \(x^\circ(t)\) as follows:

- \(h(t)\) is equal to \(f(x_2 + t)\) for all \(t\) in which \(x^\circ(t) = x_2\);
- \(h(t)\) is a linear segment with slope \(\nu\) for all \(t\) in which \(x^\circ(t)\) decreases with slope \(-1\); the vertical alignments of such segments are such that \(h(t)\) is a continuous function;
- \(h(t)\) is equal to \(f(x_1 + t) + \nu(x_2 - x_1)\) for all \(t\) in which \(x^\circ(t) = x_1\).

Then:

\[
\text{if } Q = 0 : \quad h(t) = \begin{cases} f(x_2 + t) & \forall t : x^\circ(t) = x_2 \\ \nu t + [f(\gamma_{a_1}) - \nu(\gamma_{a_1} - x_2)] & \forall t : x^\circ(t) \neq \{x_1, x_2\} \\ f(x_1 + t) + \nu(x_2 - x_1) & \forall t : x^\circ(t) = x_1 \end{cases}
\]

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if $Q = 1$ :  

$$h(t) = \begin{cases} 
  f(x_2 + t) & \forall t : x^2(t) = x_2 \\
  \nu t + f(\gamma_{a_1}) - \nu(\gamma_{a_1} - x_2) & \forall t < t^*_1 : x^2(t) \neq \{x_1, x_2\} \\
  \nu t + f(\gamma_{a_{(q+1)}}) - \nu(\gamma_{a_{(q+1)}} - x_2) & \forall t \geq t^*_1 : x^2(t) \neq \{x_1, x_2\} \\
  f(x_1 + t) + \nu(x_2 - x_1) & t : x^2(t) = x_1 
\end{cases}$$  

(9b)

if $Q > 1$ :  

$$h(t) = \begin{cases} 
  f(x_2 + t) & \forall t : x^2(t) = x_2 \\
  \nu t + f(\gamma_{a_1}) - \nu(\gamma_{a_1} - x_2) & \forall t < t^*_1 : x^2(t) \neq \{x_1, x_2\} \\
  \nu t + f(\gamma_{a_{(q+1)}}) - \nu(\gamma_{a_{(q+1)}} - x_2) & \forall t \in [t^*_{q}, t^*_{q+1}) : x^2(t) \neq \{x_1, x_2\} \\
  \nu t + f(\gamma_{a_{(q+1)}}) - \nu(\gamma_{a_{(q+1)}} - x_2) & \forall t \geq t^*_q : x^2(t) \neq \{x_1, x_2\} \\
  f(x_1 + t) + \nu(x_2 - x_1) & t : x^2(t) = x_1 
\end{cases}$$  

(9c)

Proof. When $x = x_2$, it is $g(x) = 0$ (see figure 3); then, when $x^2(t) = x_2$, function $h(t) = f(x_2 + t)$. Instead, when $x = x_1$, it is $g(x) = \nu(x_2 - x)$ (see again figure 3); then, when $x^2(t) = x_1$, function $h(t) = f(x_1 + t) + \nu(x_2 - x_1)$.

When $\gamma_{a_1} - x_2 < t < \gamma_{a_1} - x_1$, with regards to (55), $x^2(t)$ passes linearly (with unitary speed) from the value $x_2$ at $t = \gamma_{a_1} - x_2$ to the value $x_1$ at $t = \gamma_{a_1} - x_1$; then, in the same interval, function $h(t)$ passes, with the same dynamics (that is, linearly), from the value $f(x_2 + \gamma_{a_1} - x_2) = f(\gamma_{a_1})$ (at $t = \gamma_{a_1} - x_2$) to the value $f(x_1 + \gamma_{a_1} - x_2) + \nu(x_2 - x_1)$ (at $t = \gamma_{a_1} - x_1$), the segment which joins such values belongs to the line $\nu t + f(\gamma_{a_1}) - \nu(\gamma_{a_1} - x_2)$.

In analogous way, when $\gamma_{a_{(q+1)}} - x_2 < t < \gamma_{a_{(q+1)}} - x_1$, with regards to (55) or (72), $x^2(t)$ passes linearly (with unitary speed) from the value $x_2$ at $t = \gamma_{a_{(q+1)}} - x_2$ to the value $x_1$ at $t = \gamma_{a_{(q+1)}} - x_1$, function $h(t)$ passes linearly from the value $f(\gamma_{a_{(q+1)}})$ (at $t = \gamma_{a_{(q+1)}} - x_2$) to the value $f(\gamma_{a_{(q+1)}}) + \nu(x_2 - x_1)$ (at $t = \gamma_{a_{(q+1)}} - x_1$), the segment which joins such values belongs to the line $\nu t + f(\gamma_{a_{(q+1)}}) - \nu(\gamma_{a_{(q+1)}} - x_2)$.

Note that, when $l(Q) = |A|$ there isn’t any $t^*_Q$ such that $x^2(t) \neq \{x_1, x_2\}$ (since $x_1(t) = x_2$ for any $t \geq t^*_Q$, in accordance with (72)). Then, in this case, the term $\nu t + f(\gamma_{a_1}) - \nu(\gamma_{a_1} - x_2)$ in (9b) and the term $\nu t + f(\gamma_{a_{(q+1)}}) - \nu(\gamma_{a_{(q+1)}} - x_2)$ in (9c) have not to be considered as a part of the function $h(t)$.

Lemma 3. Let $f_1(x + t)$ and $f_2(x + t)$ be two continuous nondecreasing piece-wise linear functions of $x$, parameterized by $t$, as defined by definition 2. The sum function

$$s(x + t) = f_1(x + t) + f_2(x + t)$$  

(10)

is still a continuous nondecreasing piece-wise linear functions of $x$, parameterized by $t$, which is in accordance with definition 2.

Proof. It is evident that the sum of two continuous piece-wise linear functions of the same argument is a continuous piece-wise linear function of that argument as well; moreover, since all slopes in $f_1(x + t)$ and $f_2(x + t)$ are nonnegative, the slope in a generic segment of $s(x + t)$ is nonnegative as well, because it is the sum of two specific (nonnegative) slopes of $f_1(x + t)$ and $f_2(x + t)$; finally, since the initial slope of both $f_1(x + t)$ and $f_2(x + t)$ is
null, also \( s(x + t) \) has initial slope null. Then, \( s(x + t) \) is a continuous nondecreasing piece-wise linear functions of \( x \), parameterized by \( t \) (which is in accordance with definition\(^2\)). This concludes the proof.

**Lemma 4.** Let \( f_1(x) \) and \( f_2(x) \) be two continuous nondecreasing piece-wise linear functions of \( x \), as defined by definition\(^1\). The min function

\[
m(x) = \min \{ f_1(x), f_2(x) \}
\]

is still a continuous nondecreasing piece-wise linear functions of \( x \), which is in accordance with definition\(^1\).

**Proof.** It is evident that the minimum of two continuous piece-wise linear functions of the same argument is a continuous piece-wise linear function of that argument as well; moreover, since all slopes in \( f_1(x) \) and \( f_2(x) \) are nonnegative, the slope in a generic segment of \( m(x) \) is nonnegative as well, because it corresponds to the slope of one segment of \( f_1(x) \) or one segment of \( f_2(x) \); finally, since the initial slope of both \( f_1(x) \) and \( f_2(x) \) is null, also \( m(x) \) has initial slope null. Then, \( m(x) \) is a continuous nondecreasing piece-wise linear functions of \( x \) (which is in accordance with definition\(^1\)). This concludes the proof.
4 Examples

4.1 Example 1

Consider the following functions $f(x + t)$ and $g(x)$ (depicted in the same graphic).

Algorithm 1 provides $\omega_1 = 13.3$ and $\omega_2 = 25.6$. Then, by applying lemma 1 (taking into account $f(x + t)$, instead of $f(x)$, and $g(x)$) the following function $x^\circ(t)$ is obtained.

$$x^\circ(t) = \begin{cases} x_s(t) & t < 13.3 \\ x_1(t) & 13.3 \leq t < 25.6 \\ x_e(t) & t \geq 25.6 \end{cases}$$

with

$$x_s(t) = \begin{cases} 8 & t < 10 \\ -t + 18 & 10 \leq t < 13.3 \end{cases}$$

$$x_1(t) = \begin{cases} 8 & 13.3 \leq t < 18 \\ -t + 26 & 18 \leq t < 22 \\ 4 & 22 \leq t < 25.6 \end{cases}$$

$$x_e(t) = \begin{cases} -t + 32 & 25.6 \leq t < 28 \\ 4 & t \geq 28 \end{cases}$$

Note that, $T = \{13.3, 25.6\}$, that is, $t_1^* = 13.3$ and $t_2^* = 25.6$, and $Q = 2$. Since $T = \Omega$, the mapping function is basically $l(1) = 1$ and $l(2) = 2$. Moreover, $t_1^* \leq \gamma_a_1 - x_1 = 14$ (then, $x_s(t)$ has the structure of (5b)), $t_1^* < \gamma_a_2 - x_2 = 18$ and $t_2^* > \gamma_a_3 - x_1 = 22$ (then, $x_1(t)$ has the structure of (6a)), and $t_2^* \geq \gamma_a_3 - x_2 = 24$ (then, $x_e(t)$ has the structure of (7b)). The graphical representation of $x^\circ(t)$ is the following.

By applying lemma 2 the following function $h(t) = f(x^\circ(t) + t) + g(x^\circ(t))$ is obtained.
In accordance with lemma 2 and, in particular, with (9c), function $h(t)$ is

$$ h(t) = \begin{cases} 
  f(8 + t) & \forall t : x^\circ(t) = 8 \\
  t - 9 & \forall t < 13.3 : x^\circ(t) \neq \{4, 8\} \quad (\Rightarrow 10 < t < 13.3) \\
  t - 12 & \forall t \in [13.3, 25.6] : x^\circ(t) \neq \{4, 8\} \quad (\Rightarrow 18 < t < 22) \\
  t - 10 & \forall t \geq 25.6 : x^\circ(t) \neq \{4, 8\} \quad (\Rightarrow 25.6 < t < 28) \\
  f(4 + t) + 4 & \forall t : x^\circ(t) = 4 
\end{cases} $$
4.2 Example 2

Consider the following functions \( f(x + t) \) and \( g(x) \) (depicted in the same graphic).

Algorithm \( \Pi \) provides \( \omega_1 = 11 \), \( \omega_2 = 14 \), and \( \omega_3 = 23 \). Then, by applying lemma \( \Pi \) (taking into account \( f(x + t) \), instead of \( f(x) \), and \( g(x) \)) the following function \( x^\circ(t) \) is obtained.

\[
x^\circ(t) = \begin{cases} 
  x_s(t) & t < 11 \\
  x_1(t) & 11 \leq t < 14 \\
  x_2(t) & 14 \leq t < 23 \\
  x_e(t) & t \geq 23
\end{cases}
\]

with

\[
x_s(t) = \begin{cases} 
  8 & t < 8 \\
  -t + 16 & 8 \leq t < 11 \\
  x_1(t) = -t + 19
\end{cases}
\]

\[
x_2(t) = \begin{cases} 
  8 & 14 \leq t < 18 \\
  -t + 26 & 18 \leq t < 22 \\
  4 & 22 \leq t < 23 \\
  x_e(t) = \begin{cases} 
  8 & 23 \leq t < 27 \\
  -t + 35 & 27 \leq t < 31 \\
  4 & t \geq 31
\end{cases}
\end{cases}
\]

Note that, \( T = \{11, 14, 23\} \), that is, \( t_1 = 11 \), \( t_2 = 14 \), and \( t_3 = 23 \), and \( Q = 3 \). Since \( T = \Omega \), the mapping function is basically \( l(1) = 1 \), \( l(2) = 2 \), and \( l(3) = 3 \). Moreover, \( t_1^* \leq \gamma_{o_a} - x_1 = 12 \) (then, \( x_s(t) \) has the structure of \( (5\mathbf{a}) \)), \( t_2^* \geq \gamma_{o_a} - x_2 = 11 \) and \( t_3^* \leq \gamma_{o_a} - x_1 = 15 \) (then, \( x_1(t) \) has the structure of \( (5\mathbf{b}) \)), \( t_2^* < \gamma_{o_a} - x_2 = 18 \) and \( t_3^* > \gamma_{o_a} - x_1 = 22 \) (then, \( x_2(t) \) has the structure of \( (6\mathbf{a}) \)), and \( t_3^* < \gamma_{o_a} - x_2 = 27 \) (then, \( x_e(t) \) has the structure of \( (7\mathbf{a}) \)). The graphical representation of \( x^\circ(t) \) is the following.

By applying lemma \( \mathbf{2} \) the following function \( h(t) = f(x^\circ(t) + t) + g(x^\circ(t)) \) is obtained.
In accordance with lemma 2 and, in particular, with (9c), function $h(t)$ is

$$h(t) = \begin{cases} 
  f(8 + t) & \forall t : x^0(t) = 8 \\
  t - 8 & \forall t < 11 : x^0(t) \neq \{4, 8\} \quad (\Rightarrow 8 < t < 11) \\
  t - 8 & \forall t \in [11, 14) : x^0(t) \neq \{4, 8\} \quad (\Rightarrow 11 < t < 14) \\
  t - 10 & \forall t \in [14, 23) : x^0(t) \neq \{4, 8\} \quad (\Rightarrow 18 < t < 22) \\
  t - 12 & \forall t \geq 23 : x^0(t) \neq \{4, 8\} \quad (\Rightarrow 27 < t < 31) \\
  f(4 + t) + 4 & \forall t : x^0(t) = 4 
\end{cases}$$
4.3 Example 3

Consider the following functions $f(x + t)$ and $g(x)$ (depicted in the same graphic).

Algorithm 1 provides $\omega_1 = 9.5$ and $\omega_2 = 12.5$. Then, by applying lemma 1 (taking into account $f(x + t)$, instead of $f(x)$, and $g(x)$) the following function $x^0(t)$ is obtained.

$$x^0(t) = \begin{cases} x_s(t) & t < 9.5 \\ x_1(t) & 9.5 \leq t < 12.5 \\ x_e(t) & t \geq 12.5 \end{cases}$$

with

$$x_s(t) = \begin{cases} 8 & t < 8 \\ -t + 16 & 8 \leq t < 9.5 \end{cases}$$

$$x_1(t) = \begin{cases} 8 & 9.5 \leq t < 11 \\ -t + 19 & 11 \leq t < 12.5 \end{cases}$$

$$x_e(t) = \begin{cases} 8 & 12.5 \leq t < 27 \\ -t + 35 & 27 \leq t < 31 \\ 4 & t \geq 31 \end{cases}$$

Note that, $T = \{9.5, 12.5\}$, that is, $t_1^* = 9.5$ and $t_2^* = 12.5$, and $Q = 2$. Since $T = \Omega$, the mapping function is basically $l(1) = 1$ and $l(2) = 2$. Moreover, $t_1^* \leq \gamma_{a_1} - x_1 = 12$ (then, $x_s(t)$ has the structure of (5b)), $t_2^* < \gamma_{a_2} - x_2 = 11$ and $t_2^* \leq \gamma_{a_2} - x_1 = 15$ (then, $x_1(t)$ has the structure of (6c)), and $t_2^* < \gamma_{a_3} - x_2 = 27$ (then, $x_e(t)$ has the structure of (7a)). The graphical representation of $x^0(t)$ is the following.

By applying lemma 2 the following function $h(t) = f(x^0(t) + t) + g(x^0(t))$ is obtained.
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In accordance with lemma 2 and, in particular, with (9c), function $h(t)$ is

$$h(t) = \begin{cases} f(8 + t) & \forall t : x^o(t) = 8 \\ 1.5t - 12 & \forall t < 9.5 : x^o(t) \neq \{4, 8\} \quad (\Rightarrow 8 < t < 9.5) \\ 1.5t - 13.5 & \forall t \in [9.5, 12.5) : x^o(t) \neq \{4, 8\} \quad (\Rightarrow 11 < t < 12.5) \\ 1.5t - 25.5 & \forall t \geq 12.5 : x^o(t) \neq \{4, 8\} \quad (\Rightarrow 27 < t < 31) \\ f(4 + t) + 6 & \forall t : x^o(t) = 4 \end{cases}$$

Figure 20: Example 3 – Function $h(t) = f(x^o(t) + t) + g(x^o(t))$. 
4.4 Example 4

Consider the following functions $f(x + t)$ and $g(x)$ (depicted in the same graphic).

Algorithm 1 provides $\omega_1 = 10.3$ and $\omega_2 = 22.53$. Then, by applying lemma 1 (taking into account $f(x + t)$, instead of $f(x)$, and $g(x)$) the following function $x^o(t)$ is obtained.

$$x^o(t) = \begin{cases} 
  x_s(t) & t < 10.3 \\
  x_1(t) & 10.3 \leq t < 22.53 \\
  x_e(t) & t \geq 22.53 
\end{cases}$$

with

$$x_s(t) = \begin{cases} 
  8 & t \leq 8 \\
  -t + 16 & 8 \leq t < 10.3 
\end{cases}, \quad x_1(t) = \begin{cases} 
  8 & 10.3 \leq t < 13 \\
  -t + 21 & 13 \leq t < 18 \\
  3 & 18 \leq t < 22.53 
\end{cases}, \quad x_e(t) = 8$$

Note that, $T = \{10.3, 22.53\}$, that is, $t_1^* = 10.3$ and $t_2^* = 22.53$, and $Q = 2$. Since $T = \Omega$, the mapping function is basically $l(1) = 1$ and $l(2) = 2$. Moreover, $t_1^* \leq \gamma a_1 - x_1 = 13$ (then, $x_s(t)$ has the structure of $5b$), $t_1^* < \gamma a_2 - x_2 = 13$ and $t_2^* > \gamma a_2 - x_1 = 18$ (then, $x_1(t)$ has the structure of $6a$); since $l(Q) = |A| = 2$, the function $x_e(t)$ has the structure of $7a$. The graphical representation of $x^o(t)$ is the following.

By applying lemma 2 the following function $h(t) = f(x^o(t) + t) + g(x^o(t))$ is obtained.
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Figure 23: Example 4 – Function \( h(t) = f(x^\circ(t) + t) + g(x^\circ(t)) \).

In accordance with lemma\ref{lemma} and, in particular, with \eqref{9c}, function \( h(t) \) is

\[
\begin{align*}
   h(t) = \begin{cases} 
   f(8 + t) & \forall t : x^\circ(t) = 8 \\
   t - 7 & \forall t < 10.3 : x^\circ(t) \neq \{3, 8\} \quad (\Rightarrow 8 < t < 10.3) \\
   t - 9 & \forall t \in [10.3, 22.53) : x^\circ(t) \neq \{3, 8\} \quad (\Rightarrow 13 < t < 18) \\
   f(3 + t) + 5 & \forall t : x^\circ(t) = 3 
   \end{cases}
\end{align*}
\]
4.5 Example 5

Consider the following functions \( f(x + t) \) and \( g(x) \) (depicted in the same graphic).

Algorithm 1 provides \( \omega_1 = 11 \) and \( \omega_2 = 19.6 \). Then, by applying lemma 1 (taking into account \( f(x + t) \), instead of \( f(x) \), and \( g(x) \)) the following function \( x^o(t) \) is obtained.

\[
x^o(t) = \begin{cases} 
  x_s(t) & t < 11 \\
  x_1(t) & 11 \leq t < 19.6 \\
  x_e(t) & t \geq 19.6 
\end{cases}
\]

with

\[
x_s(t) = \begin{cases} 
  8 & t < 9 \\
  -t + 17 & 9 \leq t < 11 
\end{cases} \quad x_1(t) = \begin{cases} 
  8 & 11 \leq t < 15 \\
  -t + 23 & 15 \leq t < 19.6 \\
  4 & 19 \leq t < 19.6 
\end{cases}
\]

\[
x_e(t) = \begin{cases} 
  -t + 25 & 19.6 \leq t < 21 \\
  4 & t \geq 21 
\end{cases}
\]

Note that, \( T = \{11, 19.6\} \), that is, \( t^*_1 = 11 \) and \( t^*_2 = 19.6 \), and \( Q = 2 \). Since \( T = \Omega \), the mapping function is basically \( l(1) = 1 \) and \( l(2) = 2 \). Moreover, \( t^*_1 \leq \gamma_a - x_1 = 13 \) (then, \( x_s(t) \) has the structure of (5b)), \( t^*_1 < \gamma_a - x_2 = 15 \) and \( t^*_2 > \gamma_a - x_1 = 19 \) (then, \( x_1(t) \) has the structure of (6a)), and \( t^*_2 \geq \gamma_a - x_2 = 17 \) (then, \( x_e(t) \) has the structure of (7b)). The graphical representation of \( x^o(t) \) is the following.

By applying lemma 2 the following function \( h(t) = f\left(x^o(t) + t\right) + g\left(x^o(t)\right) \) is obtained.
In accordance with lemma \ref{lemma2} and, in particular, with (9c), function \( h(t) \) is

\[
h(t) = \begin{cases} 
  f(8 + t) & \forall t : x^o(t) = 8 \\
  0.5 t - 0.5 & \forall t < 11 : x^o(t) \neq \{4, 8\} \quad (\Rightarrow 9 < t < 11) \\
  0.5 t - 1.5 & \forall t \in [11, 19.6) : x^o(t) \neq \{4, 8\} \quad (\Rightarrow 15 < t < 19) \\
  0.5 t - 0.5 & \forall t \geq 19.6 : x^o(t) \neq \{4, 8\} \quad (\Rightarrow 19.6 < t < 21) \\
  f(4 + t) + 2 & \forall t : x^o(t) = 4 
\end{cases}
\]
4.6 Example 6

Consider the following functions $f(x)$ and $g(x)$ (depicted in the same graphic).

Algorithm 1 provides $\omega_1 = +\infty$ and $\omega_2 = 19.1\overline{16}$. Then, by applying lemma 1 (taking into account $f(x + t)$, instead of $f(x)$, and $g(x)$) the following function $x^\circ(t)$ is obtained.

$$x^\circ(t) = \begin{cases} x_s(t) & t < 19.1\overline{16} \\ x_e(t) & t \geq 19.1\overline{16} \end{cases}$$

with

$$x_s(t) = \begin{cases} 8 & t < 12 \\ -t + 20 & 12 \leq t < 16 \\ 4 & 16 \leq t < 19.1\overline{16} \end{cases} \quad x_e(t) = \begin{cases} 8 & 19.1\overline{16} \leq t < 28 \\ -t + 36 & 28 \leq t < 32 \\ 4 & t \geq 32 \end{cases}$$

Note that, $T = \{19.1\overline{16}\}$, that is, $t^*_1 = 19.1\overline{16}$, and $Q = 1$. The mapping function provides $l(1) = 2$. Moreover, $t^*_1 > \gamma_{a_1} - x_1 = 16$ (then, $x_s(t)$ has the structure of (5a)) and $t^*_1 < \gamma_{a_3} - x_2 = 28$ (then, $x_e(t)$ has the structure of (7a)). The graphical representation of $x^\circ(t)$ is the following.

By applying lemma 2 the following function $h(t) = f(x^\circ(t) + t) + g(x^\circ(t))$ is obtained.
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Figure 29: Example 6 – Function $h(t) = f(x^\circ(t) + t) + g(x^\circ(t))$.

In accordance with lemma 2 and, in particular, with (9b), function $h(t)$ is

$$h(t) = \begin{cases} 
  f(8 + t) & \forall t : x^\circ(t) = 8 \\
  t - 12 & \forall t < 19.16 : x^\circ(t) \neq \{4, 8\} \quad (12 < t < 16) \\
  t - 14.5 & \forall t \geq 19.16 : x^\circ(t) \neq \{4, 8\} \quad (28 < t < 32) \\
  f(4 + t) + 4 & \forall t : x^\circ(t) = 4
\end{cases}$$
4.7 Example 7

Consider the following functions \( f(x) \) and \( g(x) \) (depicted in the same graphic).

Algorithm 1 provides \( \omega_1 = 19.5 \) and \( \omega_2 = 21.3 \). Then, by applying lemma 1 (taking into account \( f(x + t) \), instead of \( f(x) \), and \( g(x) \)) the following function \( x^\circ(t) \) is obtained.

\[
x^\circ(t) = \begin{cases} 
  x_a(t) & t < 19.5 \\
  x_1(t) & 19.5 \leq t < 21.3 \\
  x_e(t) & t \geq 21.3 
\end{cases}
\]

with

\[
x_a(t) = \begin{cases} 
  8 & t < 12 \\
  -t + 20 & 12 \leq t < 16 \\
  4 & 16 \leq t < 19.5 
\end{cases} \\
x_1(t) = \begin{cases} 
  -t + 25 & 19.5 \leq t < 21 \\
  4 & 21 \leq t < 21.3 
\end{cases} \\
x_e(t) = \begin{cases} 
  8 & 21.3 \leq t < 28 \\
  -t + 36 & 28 \leq t < 32 \\
  4 & t \geq 32 
\end{cases}
\]

Note that, \( T = \{19.5, 21.3\} \), that is, \( t_1^* = 19.5 \) and \( t_2^* = 21.3 \), and \( Q = 2 \). Since \( T = \Omega \), the mapping function is basically \( l(1) = 1 \) and \( l(2) = 2 \). Moreover, \( t_1^* \geq \gamma_a - x_1 = 16 \) (then, \( x_a(t) \) has the structure of (5b)), \( t_1^* \geq \gamma_{a_1} - x_2 = 17 \) and \( t_2^* \geq \gamma_{a_2} - x_1 = 21 \) (then, \( x_1(t) \) has the structure of (6b)), and \( t_2^* \leq \gamma_{a_4} - x_2 = 28 \) (then, \( x_e(t) \) has the structure of (7a)). The graphical representation of \( x^\circ(t) \) is the following.

By applying lemma 2 the following function \( h(t) = f(x^\circ(t) + t) + g(x^\circ(t)) \) is obtained.
In accordance with lemma 2 and, in particular, with (9c), function $h(t)$ is

$$h(t) = \begin{cases} 
  f(8 + t) & \forall t : x^\circ(t) = 8 \\
  t - 12 & \forall t < 19.5 : x^\circ(t) \neq \{4, 8\} \ (\Rightarrow 12 < t < 16) \\
  t - 5 & \forall t \in [19.5, 21.3] : x^\circ(t) \neq \{4, 8\} \ (\Rightarrow 19.5 < t < 21) \\
  t - 5.5 & \forall t \geq 21.3 : x^\circ(t) \neq \{4, 8\} \ (\Rightarrow 28 < t < 32) \\
  f(4 + t) + 4 & \forall t : x^\circ(t) = 4 
\end{cases}$$
4.8 Example 8

Consider the following functions $f(x)$ and $g(x)$ (depicted in the same graphic).

Algorithm\[1\] provides $\omega_1 = +\infty, \omega_2 = +\infty,$ and $\omega_3 = 18.83$. Then, by applying lemma\[1\] (taking into account $f(x + t)$, instead of $f(x)$, and $g(x)$) the following function $x^o(t)$ is obtained.

$$x^o(t) = \begin{cases} x_s(t) & t < 18.83 \\ x_e(t) & t \geq 18.83 \end{cases}$$

with

$$x_s(t) = \begin{cases} 12 & t < 8 \\ -t + 20 & 8 \leq t < 16 \\ 4 & 16 \leq t < 18.83 \end{cases}$$

$$x_e(t) = \begin{cases} 12 & 18.83 \leq t < 26 \\ -t + 38 & 26 \leq t < 34 \\ 4 & t \geq 34 \end{cases}$$

Note that, $T = \{18.83\}$, that is, $t^*_1 = 18.83$, and $Q = 1$. The mapping function provides $l(1) = 3$. Moreover, $t^*_1 > \gamma a_1 - x_1 = 16$ (then, $x_s(t)$ has the structure of (5a)) and $t^*_1 < \gamma a_4 - x_2 = 26$ (then, $x_e(t)$ has the structure of (7a)). The graphical representation of $x^o(t)$ is the following.

By applying lemma\[2\] the following function $h(t) = f(x^o(t) + t) + g(x^o(t))$ is obtained.
In accordance with lemma 2 and, in particular, with (9b), function $h(t)$ is

$$h(t) = \begin{cases} 
  f(12 + t) & \forall t : x^o(t) = 12 \\
  t - 8 & \forall t < 18.83 : x^o(t) \neq \{4, 12\} \quad (\Rightarrow 8 < t < 16) \\
  t - 9.5 & \forall t \geq 18.83 : x^o(t) \neq \{4, 12\} \quad (\Rightarrow 26 < t < 34) \\
  f(4 + t) + 8 & \forall t : x^o(t) = 4 
\end{cases}$$
4.9 Example 9

Consider the following functions $f(x)$ and $g(x)$ (depicted in the same graphic).

![Graph of functions $f(x)$ and $g(x)$](image)

Algorithm 1 provides $\omega_1 = 18$, $\omega_2 = +\infty$, and $\omega_3 = 18.4$ (the application of algorithm 1 is reported in the following, for each value of $j$). Then, by applying lemma 1 (taking into account $f(x + t)$, instead of $f(x)$, and $g(x)$) the following function $x^e(t)$ is obtained.

$$x^e(t) = \begin{cases} 
x_s(t) & t < 18 \\
x_1(t) & 18 \leq t < 18.4 \\
x_e(t) & t \geq 18.4
\end{cases}$$

with

$$x_s(t) = \begin{cases} 
12 & t < 4 \\
-t + 16 & 4 \leq t < 12 \\
4 & 12 \leq t < 18
\end{cases} \quad x_1(t) = -t + 25 \quad x_e(t) = \begin{cases} 
12 & 18.4 \leq t < 26 \\
-t + 38 & 26 \leq t < 34 \\
4 & t \geq 34
\end{cases}$$

Note that, $T = \{18, 18.4\}$, that is, $t^1_1 = 18$ and $t^2_1 = 18.4$, and $Q = 2$. The mapping function provides $l(1) = 1$ and $l(2) = 3$. Moreover, $t^1_1 \geq \gamma_\alpha_1 - x_1 = 12$ (then, $x_s(t)$ has the structure of (5a)), $t^1_1 \geq \gamma_\alpha_2 - x_2 = 13$ and $t^2_2 \leq \gamma_\alpha_3 - x_1 = 21$ (then, $x_1(t)$ has the structure of (6a)), and $t^1_2 < \gamma_\alpha_4 - x_2 = 26$ (then, $x_e(t)$ has the structure of (7a)). The graphical representation of $x^e(t)$ is the following.

![Graph of function $x^e(t)$](image)

By applying lemma 2 the following function $h(t) = f(x^e(t) + t) + g(x^e(t))$ is obtained.
In accordance with lemma 2 and, in particular, with (9c), function $h(t)$ is

$$h(t) = \begin{cases} 
  f(12 + t) & \forall t : x^\circ(t) = 12 \\
  t - 4 & \forall t < 18 : x^\circ(t) \neq \{4, 12\} \ (\Rightarrow 4 < t < 12) \\
  t - 1 & \forall t \in [18, 18.4) : x^\circ(t) \neq \{4, 12\} \ (\Rightarrow 18 < t < 18.4) \\
  t - 5.75 & \forall t \geq 18.4 : x^\circ(t) \neq \{4, 12\} \ (\Rightarrow 26 < t < 34) \\
  f(4 + t) + 8 & \forall t : x^\circ(t) = 4
\end{cases}$$
5 Application to the single machine scheduling

Consider a single machine scheduling problem in which 1 job of class $P_1$ and 2 jobs of class $P_2$ must be executed. The due dates, the marginal tardiness costs of jobs, the processing time bounds and the marginal deviation costs of jobs are the:

| Class | $\alpha$ | $\beta$ | $p_{l}^{\text{low}}$ | $p_{n}^{\text{nom}}$ |
|-------|----------|--------|---------------------|---------------------|
| $P_1$ | 0.5      | 1      | 1                   | 4                   |
| $P_2$ | 0.25     | 1      | 1                   | 2                   |

No setup is required between the execution of jobs of different classes. The evolution of the system state can be represented by the following diagram.

![State diagram](image)

Figure 39: State diagram in the case of two classes of jobs, where $N_1 = 1$ and $N_2 = 2$.

The application of dynamic programming, in conjunction with the new lemmas, provides the following optimal control strategies.

Stage 2 – State $[1 \ 1 \ t_2]^T$

In state $[1 \ 1 \ t_2]^T$ the unique job of class $P_1$ has been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable $\tau$ only (which corresponds to the processing time $p_{t_2}$), is

$$\alpha_{2,2} \max\{t_2 + \tau - dd_{2,2}, 0\} + \beta_{2}(p_{t_2}^{\text{nom}} - \tau) + J_{1,2}(t_3)$$

that can be written as $f(p_{t_2} + t_2) + g(p_{t_2})$ being

$$f(p_{t_2} + t_2) = 0.75 \cdot \max\{p_{t_2} + t_2 - 20, 0\}$$

$$g(p_{t_2}) = \begin{cases} 2 - p_{t_2}, & p_{t_2} \in [1, 2] \\ 0, & p_{t_2} \notin [1, 2] \end{cases}$$

the two functions illustrated in figure 40.
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It is possible to apply lemma 1 (note that \( f(p_{t_2,2} + t_2) \) follows definition 2 and \( g(p_{t_2,2}) \) follows definition 3), which provides the optimal processing time

\[
p_{t_2,2}^o(t_2) = \arg \min_{1 \leq p_{t_2,2} \leq 2} \{ f(p_{t_2,2} + t_2) + g(p_{t_2,2}) \} = x_e(t_2)
\]

illustrated in figure 41, being \( x_e(t_2) \) the function \( x_e(t_2) = 2 \).

\( p_{t_2,2}^o(t_2) \) and \( x_e(t_2) \) are in accordance with \( 43 \) and \( 76 \), respectively. Note that, in this case, \( A = 0, B = 0, |A| = |B| = 0 \), and then there is no need of executing algorithm 1. Taking into account the mandatory decision about the class of the next job to be executed; the optimal control strategies for this state are

\[
\begin{align*}
\delta^o_1(1, 1, t_2) &= 0 \quad \forall t_2 \\
\delta^o_2(1, 1, t_2) &= 1 \quad \forall t_2 \\
\tau^o(1, 1, t_2) &= p_{t_2,2}^o(t_2) = 2 \quad \forall t_2 \\
pt_{t_2,2}^o(t_2) &\equiv \tau^o(1, 1, t_2)
\end{align*}
\]

Figure 41: Optimal control strategy \( \tau^o(1, 1, t_2) \) (service time) in state \([1 1 \ t_2]^T\).

The optimal cost-to-go

\[
J_{1,1}^o(t_2) = f(p_{t_2,2}^o(t_2) + t_2) + g(p_{t_2,2}^o(t_2))
\]

illustrated in figure 42, is provided by lemma 2.

\[
J_{1,1}^o(t_2) = f(p_{t_2,2}^o(t_2) + t_2) + g(p_{t_2,2}^o(t_2))
\]

Figure 42: Optimal cost-to-go \( J_{1,1}^o(t_2) \) in state \([1 1 \ t_2]^T\).

Stage 2 – State \([0 2 \ t_2]^T\)

In state \([0 2 \ t_2]^T\) all jobs of class \( P_2 \) have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable \( \tau \) only (which corresponds to the processing time \( p_{t_1,1} \)), is

\[
\alpha_{1,1} \max\{t_2 + \tau - dd_{1,1}, 0\} + \beta_1 (p_{t_1}^{nom} - \tau) + J_{1,2}^o(t_3)
\]
that can be written as \( f(pt_{1,1} + t_2) + g(pt_{1,1}) \) being

\[
f(pt_{1,1} + t_2) = 0.5 \cdot \max\{pt_{1,1} - 10, 0\}
\]

\[
g(pt_{1,1}) = \begin{cases} 
4 - pt_{1,1} & \text{if } pt_{1,1} \in [1, 4) \\
0 & \text{if } pt_{1,1} \notin [1, 4)
\end{cases}
\]

the two functions illustrated in figure 44.

**Figure 43:** Functions \( f(pt_{1,1} + t_2) \) and \( g(pt_{1,1}) \) in state \([0 \ 2 \ t_2]^T\).

It is possible to apply lemma 1 (note that \( f(pt_{1,1}, t_2) \) follows definition 2 and \( g(pt_{1,1}) \) follows definition 3), which provides the optimal processing time

\[
pt_{1,1}^o(t_2) = \arg\min_{p_{1,1} \in [1, 4]} \{ f(pt_{1,1} + t_2) + g(pt_{1,1}) \} = x_e(t_2)
\]

illustrated in figure 44 being \( x_e(t_2) \) the function

\[
x_e(t_2) = 4
\]

\( pt_{1,1}^o(t_2) \) and \( x_e(t_2) \) are in accordance with \( 43 \) and \( 76 \), respectively. Note that, in this case, \( A = 0, B = 0, |A| = |B| = 0 \), and then there is no need of executing algorithm 1. Taking into account the mandatory decision about the class of the next job to be executed; the optimal control strategies for this state are

\[
\delta_1^o(0, 2, t_2) = 1 \quad \forall t_2 \\
\delta_2^o(0, 2, t_2) = 0 \quad \forall t_2 \\
\tau^o(0, 2, t_2) = pt_{1,1}^o(t_2) = 4 \quad \forall t_2
\]

**Figure 44:** Optimal control strategy \( \tau^o(0, 2, t_2) \) (service time) in state \([0 \ 2 \ t_2]^T\).

The optimal cost-to-go

\[
J_{0,2}^o(t_2) = f(pt_{1,1}^o(t_2) + t_2) + g(pt_{1,1}^o(t_2))
\]

illustrated in figure 45 is provided by lemma 2.

**Figure 45:** Optimal cost-to-go \( J_{0,2}^o(t_2) \) in state \([0 \ 2 \ t_2]^T\).
remarking that, in the current state, the decision about the class of the next job to be executed is mandatory, since the unique job of class $\gamma_3$ in state $\text{Stage 1 – State } [0 \ t_1]^T$ be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable $\tau$ only (which corresponds to the processing time $pt_{2,1}$), is

$$\alpha_{2,1} \max \{t_1 + \tau - dd_{2,1}, 0\} + (pt_{2,1}^{\text{nom}} - \tau) + J_{1,1}^p(t_2)$$

that can be written as $f(pt_{2,1} + t_1) + g(pt_{2,1})$ being

$$f(pt_{2,1} + t_1) = 0.25 \cdot \max \{pt_{2,1} + t_1 - 12, 0\} + J_{1,1}^p(pt_{2,1} + t_1)$$

$$g(pt_{2,1}) = \begin{cases} 2 - pt_{2,1} & pt_{2,1} \in [1, 2] \\ 0 & pt_{2,1} \notin [1, 2] \end{cases}$$

the two functions illustrated in figure 46.

![Figure 46: Functions $f(pt_{2,1} + t_1)$ and $g_{2,1}(pt_{2,1})$ in state $[0 \ t_1]^T$.](image)

Using figure 46, we find $pt_{2,1}^\circ(t_1) \equiv \tau^*(1,0,t_1)$ illustrated in figure 47, being $x_e(t_1)$ the function

$$x_e(t_1) = \begin{cases} 2 & t_1 < 16 \\ -t_1 + 18 & 16 \leq t_1 < 17 \\ 1 & t_1 \geq 17 \end{cases}$$

and

$$pt_{2,1}^\circ(t_1)$$

$pt_{2,1}^\circ(t_1)$ and $x_e(t_1)$ are in accordance with (4a) and (7a), respectively. Note that, in this case, $A = \{2\}$, $|A| = 1$, $\gamma_{a_1} = 18$; moreover, since $B = \emptyset$ and $|B| = 0$, there is no need of executing algorithm 1. It is worth again remarking that, in the current state, the decision about the class of the next job to be executed is mandatory, since the unique job of class $P_1$ has been completed. Then,

$$\delta_1^1(1,0,t_1) = 0 \quad \forall \ t_1 \quad \delta_2^1(1,0,t_1) = 1 \quad \forall \ t_1$$

$$\tau^*(1,0,t_1) = pt_{2,1}^\circ(t_1) = \begin{cases} 2 & t_1 < 16 \\ -t_1 + 18 & 16 \leq t_1 < 17 \\ 1 & t_1 \geq 17 \end{cases}$$
The optimal cost-to-go
\[ J_{1,0}^o(t_1) = f(pt_{2,1}(t_1) + t_1) + g(pt_{2,1}(t_1)) \]
illustrated in figure 48 is provided by lemma 2.

Figure 48: Optimal cost-to-go \( J_{1,0}^o(t_1) \) in state [1 0 t_1]^T.

**Stage 1 – State [0 1 t_1]^T**

In state [0 1 t_1]^T, the cost function to be minimized, with respect to the (continuous) decision variable \( \tau \) and to the (binary) decision variables \( \delta_1 \) and \( \delta_2 \) is

\[ \delta_1 \alpha_{1,1} \max\{t_1 + \tau - dd_{1,1}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + J_{1,1}^o(t_2) + \delta_2 \alpha_{2,2} \max\{t_1 + \tau - dd_{2,2}, 0\} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + J_{0,2}^o(t_2) \]

**Case i)** in which it is assumed \( \delta_1 = 1 \) (and \( \delta_2 = 0 \)).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \( \tau \) which corresponds to the processing time \( pt_{1,1} \), the following function

\[ \alpha_{1,1} \max\{t_1 + \tau - dd_{1,1}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + J_{1,1}^o(t_2) \]

that can be written as \( f(pt_{1,1} + t_1) \) being

\[ f(pt_{1,1} + t_1) = 0.5 \cdot \max\{pt_{1,1} + t_1 - 10, 0\} + J_{1,1}^o(pt_{1,1} + t_1) \]

\[ g(pt_{1,1}) = \begin{cases} 4 - pt_{1,1} & pt_{1,1} \in [1, 4) \\ 0 & pt_{1,1} \notin [1, 4) \end{cases} \]

the two functions illustrated in figure 49.

It is possible to apply lemma 1 (note that \( f(pt_{1,1} + t_1) \) follows definition 2 and \( g(pt_{1,1}) \) follows definition 3), which provides the function

\[ pt_{1,1}^o(t_1) = \min_{\substack{t_1 \leq pt_{1,1} \leq 4}} \{ f(pt_{1,1} + t_1) + g(pt_{1,1}) \} = x_e(t_1) \]

illustrated in figure 50 being \( x_e(t_1) \) the function

\[ x_e(t_1) = \begin{cases} 4 & t_1 < 14 \\ -t_1 + 18 & 14 \leq t_1 < 17 \\ 1 & t_1 \geq 17 \end{cases} \]
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\[ f(pt_1, 1) + t_1 \]
\[ g(pt_1, 1) \]

Figure 49: Functions \( f(pt_1, 1) \) and \( g(pt_1, 1) \) in state \([0 1 t_1]^T\).

\[ pt_{i, 1}(t_1) \]

Figure 50: Function \( pt_{i, 1}(t_1) \).

\[ pt_{i, 1}(t_1) \text{ and } x_{\alpha}(t_1) \text{ are in accordance with (4a) and (7a), respectively. Note that, in this case, } A = \{2\}, |A| = 1, \gamma_{\alpha_1} = 18; \text{ moreover, since } B = \emptyset \text{ and } |B| = 0, \text{ there is no need of executing algorithm 1.} \]

The conditioned cost-to-go

\[ J_{0, 1}(t_1 | \delta_1 = 1) = f \left( pt_{i, 1}(t_1) + t_1 \right) + g \left( pt_{i, 1}(t_1) \right) \]

illustrated in figure 51 is provided by lemma 2.

\[ J_{0, 1}(t_1 | \delta_1 = 1) \]

Figure 51: Conditioned cost-to-go \( J_{0, 1}(t_1 | \delta_1 = 1) \).

Case ii) in which it is assumed \( \delta_2 = 1 \) (and \( \delta_1 = 0 \)).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \( \tau \) which corresponds to the processing time \( pt_{2, 2} \), the following function

\[ \alpha_{2, 2} \max \{t_1 + \tau - dd_{2, 2}, 0\} + \beta_2 (pt_{2, 2}^{\text{nom}} - \tau) + J_{0, 2}^0(t_2) \]

that can be written as \( f(pt_{2, 2} + t_1) + g(pt_{2, 2}) \) being

\[ f(pt_{2, 2} + t_1) = 0.75 \cdot \max \{pt_{2, 2} + t_1 - 20, 0\} + J_{0, 2}^0(pt_{2, 2} + t_1) \]

\[ g(pt_{2, 2}) = \begin{cases} 
2 - pt_{2, 2} & pt_{2, 2} \in [1, 2] \\
0 & pt_{2, 2} \notin [1, 2]
\end{cases} \]
the two functions illustrated in figure 52.

![Figure 52: Functions $f(p_{t_2} + t_1)$ and $g(p_{t_2})$ in state $[0 1 t_1]T$.](image)

It is possible to apply lemma 1 (note that $f(p_{t_2} + t_1)$ follows definition 2 and $g(p_{t_2})$ follows definition 3), which provides the function

$$p_{t_2}^0(t_1) = \arg \min_{1 \leq p_{t_2} \leq 2} \{ f(p_{t_2} + t_1) + g(p_{t_2}) \} = x_e(t_1)$$

illustrated in figure 53, being $x_e(t_1)$ the function

$$x_e(t_1) = \begin{cases} 2 & t_1 < 18 \\ -t_1 + 20 & 18 \leq t_1 < 19 \\ 1 & t_1 \geq 19 \end{cases}$$

$p_{t_2}^0(t_1)$ and $x_e(t_1)$ are in accordance with (4a) and (7a), respectively. Note that, in this case, $A = \{ 2 \}, |A| = 1, \gamma_{a_1} = 20$; moreover, since $B = \emptyset$ and $|B| = 0$, there is no need of executing algorithm 1.

![Figure 53: Function $p_{t_2}^0(t_1)$.](image)

The conditioned cost-to-go

$$J_{0,1}^\delta(t_1 | \delta_2 = 1) = f(p_{t_2}^0(t_1) + t_1) + g(p_{t_2}^0(t_1))$$

illustrated in figure 54 is provided by lemma 2.

![Figure 54: Conditioned cost-to-go $J_{0,1}^\delta(t_1 | \delta_2 = 1)$.](image)

In order to find the optimal cost-to-go $J_{0,1}^\delta(t_1)$, it is necessary to carry out the following minimization

$$J_{0,1}^\delta(t_1) = \min \{ J_{0,1}^\delta(t_1 | \delta_1 = 1), J_{0,1}^\delta(t_1 | \delta_2 = 1) \}$$
which provides, in accordance with lemma 4, the function illustrated in figure 55.

![Figure 55: Optimal cost-to-go $J_{0,1}(t_1)$ in state $[0 1 t_1]^T$.](image)

Since $J_{0,1}(t_1 | \delta_1 = 1)$ is the minimum in $(-\infty, 16)$ and $J_{0,1}(t_1 | \delta_2 = 1)$ is the minimum in $[16, +\infty)$, the optimal control strategies for this state are

$$
\delta_1^*(0, 1, t_1) = \begin{cases} 
1 & t_1 < 16 \\
0 & t_1 \geq 16
\end{cases}
$$

$$
\delta_2^*(0, 1, t_1) = \begin{cases} 
0 & t_1 < 16 \\
1 & t_1 \geq 16
\end{cases}
$$

$$
\tau^*(0, 1, t_1) = \delta_1^*(0, 1, t_1) pt_{1,1}^n(t_1) + \delta_2^*(0, 1, t_1) pt_{2,2}^n(t_1) = \begin{cases} 
4 & t_1 < 14 \\
-1 + 18 & 14 \leq t_1 < 16 \\
2 & 16 \leq t_1 < 18 \\
-1 + 20 & 18 \leq t_1 < 19 \\
1 & t_1 \geq 19
\end{cases}
$$

illustrated in figures 56, 57, and 58 respectively.

![Figure 56: Optimal control strategy $\delta_1^*(0, 1, t_1)$ in state $[0 1 t_1]^T$.](image)

![Figure 57: Optimal control strategy $\delta_2^*(0, 1, t_1)$ in state $[0 1 t_1]^T$.](image)

![Figure 58: Optimal control strategy $\tau^*(0, 1, t_1)$ (service time) in state $[0 1 t_1]^T$.](image)

**Stage 0 – State $[0 0 t_0]^T$ (initial state)**

In state $[0 0 t_0]^T$, the cost function to be minimized, with respect to the (continuos) decision variable $\tau$ and to the (binary) decision variables $\delta_1$ and $\delta_2$ is

$$
\delta_1 \left[ \alpha_{1,1} \max\{t_0 + \tau - dd_{1,1}, 0\} + \beta_1 (pt_{1,1}^{nom} - \tau) + J_{1,0}^n(t_1) \right] +
\delta_2 \left[ \alpha_{2,1} \max\{t_0 + \tau - dd_{2,1}, 0\} + \beta_2 (pt_{2,2}^{nom} - \tau) + J_{0,1}^n(t_1) \right]
$$
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Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable $\tau$ which corresponds to the processing time $pt_{1,1}$, the following function

$$
\alpha_{1,1} \max \{t_0 + \tau - dd_{1,1}, 0\} + \beta_1 (pt_{1,1}^{\text{nom}} - \tau) + J_{0,1}^0(t_1)
$$

that can be written as $f(pt_{1,1} + t_0) + g(pt_{1,1})$ being

$$
f(pt_{1,1} + t_0) = 0.5 \cdot \max \{pt_{1,1} + t_0 - 10, 0\} + J_{0,1}^0(pt_{1,1} + t_0)
$$

$$
g(pt_{1,1}) = \begin{cases} 
4 - pt_{1,1} & pt_{1,1} \in [1, 4) \\
0 & pt_{1,1} \notin [1, 4) 
\end{cases}
$$

the two functions illustrated in figure 59.

It is possible to apply lemma 1 (note that $f(pt_{1,1} + t_0)$ follows definition 2 and $g(pt_{1,1})$ follows definition 3), which provides the function

$$
pt_{1,1}^\circ(t_0) = \arg \min_{1 \leq pt_{1,1} \leq 4} \{f(pt_{1,1} + t_0) + g(pt_{1,1})\} = x_e(t_0)
$$

illustrated in figure 60 being $x_e(t_0)$ the function

$$
x_e(t_0) = \begin{cases} 
4 & t_1 < 12 \\
-t_1 + 16 & 12 \leq t_1 < 15 \\
1 & t_1 \geq 15 
\end{cases}
$$

$pt_{1,1}^\circ(t_0)$ and $x_e(t_0)$ are in accordance with (4a) and (7a), respectively. Note that, in this case, $A = \{2\}$, $|A| = 1$, $\gamma_{a_1} = 16$; moreover, since $B = \emptyset$ and $|B| = 0$, there is no need of executing algorithm 1.

The conditioned cost-to-go

$$
J_{0,0}^0(t_0 | \delta_1 = 1) = f(pt_{1,1}^\circ(t_0) + t_0) + g(pt_{1,1}^\circ(t_0))
$$

illustrated in figure 61 is provided by lemma 2.
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Figure 61: Conditioned cost-to-go $J^\circ_{0,0}(t_0 \mid \delta_1 = 1)$.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{2,1}$, the following function

$$
\alpha_{2,1} \max\{t_0 + \tau - dd_{2,1}, 0\} + \beta_2 (pt_{2,1}^{\text{nom}} - \tau) + J^\circ_{0,1}(t_1)
$$

that can be written as $f(pt_{2,1} + t_0) + g(pt_{2,1})$ being

$$
f(pt_{2,1} + t_0) = 0.25 \cdot \max\{pt_{2,1} + t_0 - 12, 0\} + J^\circ_{0,1}(pt_{2,1} + t_0)
$$

$$
g(pt_{2,1}) = \begin{cases} 
2 - pt_{2,1} & pt_{2,1} \in [1, 2] \\
0 & pt_{2,1} \notin [1, 2] 
\end{cases}
$$

the two functions illustrated in figure 62.

Figure 62: Functions $f(pt_{2,1} + t_0)$ and $g(pt_{2,1})$ in state $[0 0 t_0]^T$.

It is possible to apply lemma 1 (note that $f(pt_{2,1} + t_0)$ follows definition 2 and $g(pt_{2,1})$ follows definition 3), which provides the function

$$
pt_{2,1}^\circ(t_0) = \arg \min_{1 \leq pt_{2,1} \leq 2} \{ f(pt_{2,1} + t_0) + g(pt_{2,1}) \} = \begin{cases} 
x_s(t_0) & t_0 < 14.5 \\
x_e(t_0) & t_0 \geq 14.5 
\end{cases}
$$

illustrated in figure 63 in which 14.5 is the value $\omega_1$ determined by applying algorithm 1 and being $x_s(t_0)$ and $x_e(t_0)$ the functions

$$
x_s(t_0) = \begin{cases} 
2 - t_0 + 14 & t_0 < 12 \\
12 & 12 \leq t_0 < 13 \\
13 & 13 \leq t_0 < 14.5 
\end{cases}
$$
\[ x_e(t_0) = \begin{cases} 2 & 14.5 \leq t_0 < 16 \\ -t_0 + 18 & 16 \leq t_0 < 17 \\ 1 & t_0 \geq 17 \end{cases} \]

\[ p t^2_{2,1}(t_0), x_s(t_0), \text{and } x_e(t_0) \text{ are in accordance with (4b), (5a), and (7a), respectively.} \]

\[ pt^2_{2,1}(t_0) \]

\[ \omega_1 \text{ is determined by applying algorithm 1 as follows.} \]

A = \{3, 5\} \quad |A| = 2 \quad \gamma_{a_1} = 14 \quad \gamma_{a_2} = 18 \quad x_1 = 1 \quad x_2 = 2

B = \{4\} \quad |B| = 1 \quad \gamma_{b_1} = 16

With \( j = 1 \), the “Section A – Initialization” part of the algorithm provides:

\begin{align*}
\text{[row 1]:} & \quad \gamma_0 = -\infty \\
\text{[row 2]:} & \quad h \geq 0 : \gamma_h \leq 16 - (2 - 1) < \gamma_{h+1} \quad \Rightarrow \quad h = 3 \\
\text{[row 3]:} & \quad i = 4 \\
\text{[row 4]:} & \quad \gamma_1 = +\infty \\
\text{[row 5]:} & \quad k \leq 6 : \gamma_k < 16 + (2 - 1) \leq \gamma_{k+1} \quad \Rightarrow \quad k = 4 \\
\text{[row 6]:} & \quad \text{condition: } j = |B| \text{ and } |A| = |B| (1 = 1 \text{ and } 2 = 1) \text{ is false} \\
\text{[row 10]:} & \quad \begin{aligned} \bar{\mu}_3 &= 1.25 - 1 = 0.25 \\
\bar{\mu}_4 &= 0.75 - 1 = -0.25 \end{aligned} \\
\text{[row 12]:} & \quad \tau = 16 - (2 - 1) = 15 \\
\text{[row 13]:} & \quad \theta = 16 \\
\text{[row 14]:} & \quad d = \max \{0, 0.25 \cdot (16 - 15)\} = 0.25 \\
\text{[row 15]:} & \quad \text{condition: } h < b_j - 1 (3 < 4 - 1) \text{ is false} \\
\text{[row 20]:} & \quad \lambda = 3 \\
\text{[row 21]:} & \quad \xi = 4
\end{align*}

Since condition: \( h < b_1 \) and \( i < a_2 \) (4 < 5 and 3 < 4) [row 22] is true, the “Section B – First Loop” part of the algorithm is executed:

\begin{align*}
\text{[row 23]:} & \quad \psi = \min \{16 - 15, 18 - 16\} = 1 \\
\text{[row 24]:} & \quad \text{condition: } \gamma_{h+1} - \tau \leq \gamma_{i+1} - \theta (16 - 15 \leq 18 - 16) \text{ is true} \\
\text{[row 25]:} & \quad \lambda = 3 + 1 = 4 \\
\text{[row 27]:} & \quad \text{condition: } \gamma_{h+1} - \tau \geq \gamma_{i+1} - \theta (16 - 15 \geq 18 - 16) \text{ is false} \\
\text{[row 30]:} & \quad \delta = \max \{0, -0.25 \cdot [18 - (15 + 1)]\} = 0 \\
\text{[row 31]:} & \quad \text{condition: } \lambda < b_j - 1 (3 < 4 - 1) \text{ is false} \\
\text{[row 36]:} & \quad \text{condition: } \xi = b_j (4 = 4) \text{ is true} \\
\text{[row 37]:} & \quad \delta = 0 - 0.25 \cdot [(16 + 1) - 16] = -0.25 \\
\text{[row 43]:} & \quad \text{condition: } \delta \leq 0 (-0.25 \leq 0) \text{ is true}
\end{align*}
[row 44]: \( a_0 = 0 \)

[row 45]: \( r \geq 1 : a_{r-1} \leq 3 < a_r \implies r = 2 \)

[row 46]: condition: \( r \leq j (2 \leq 1) \) is false

[row 68]: \( \omega_1 = 15 - 1 + \frac{0.25}{0.25 + 0.25} = 14.5 \)

[row 69]: exit algorithm

The conditioned cost-to-go

\[
J_{\delta_0,0}(t_0 | \delta_2 = 1) = f(pt^\circ_{2,1}(t_0) + t_0) + g(pt^\circ_{2,1}(t_0))
\]

illustrated in figure 64, is provided by lemma 2.

\[
\text{Figure 64: Conditioned cost-to-go } J_{\delta_0,0}(t_0 | \delta_2 = 1).
\]

In order to find the optimal cost-to-go \( J_{\delta_0,0}(t_0) \), it is necessary to carry out the following minimization

\[
J_{\delta_0,0}(t_0) = \min \left\{ J_{\delta_0,0}(t_0 | \delta_1 = 1), J_{\delta_0,0}(t_0 | \delta_2 = 1) \right\}
\]

which provides, in accordance with lemma 4, the function illustrated in figure 65.

\[
\text{Figure 65: Optimal cost-to-go } J_{\delta_0,0}(t_0) \text{ in state } [0 \ 0 \ t_0]^T.
\]

Since \( J_{\delta_0,0}(t_0 | \delta_1 = 1) \) is the minimum in \( (-\infty, 15.5] \) and \( J_{\delta_0,0}(t_0 | \delta_2 = 1) \) is the minimum in \([15.5, +\infty)\), the optimal control strategies for this state are

\[
\delta^1_1(0, 0, t_0) = \begin{cases} 1 & t_0 < 15.5 \\ 0 & t_0 \geq 15.5 \end{cases} \quad \delta^2_2(0, 0, t_0) = \begin{cases} 0 & t_0 < 15.5 \\ 1 & t_0 \geq 15.5 \end{cases}
\]

\[
\tau^\circ(0, 0, t_0) = \delta^1_1(0, 0, t_0) pt^\circ_{1,1}(t_0) + \delta^2_2(0, 0, t_0) pt^\circ_{2,1}(t_0) = \begin{cases} 4 & t_0 < 12 \\ -t_0 + 16 & 12 \leq t_0 < 15 \\ 2 & 15 \leq t_0 < 15.5 \\ 2 & 15.5 \leq t_0 < 16 \\ -t_0 + 18 & 16 \leq t_0 < 17 \\ 1 & t_0 \geq 17 \end{cases}
\]
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illustrated in figures 66, 67, and 68 respectively.

Figure 66: Optimal control strategy \( \delta_1(0, 0, t_0) \) in state \([0 0 t_0]^T\).

Figure 67: Optimal control strategy \( \delta_2(0, 0, t_0) \) in state \([0 0 t_0]^T\).

Figure 68: Optimal control strategy \( \tau^o(0, 0, t_0) \) (processing time) in state \([0 0 t_0]^T\).

Since the two conditional costs-to-go have the same value in the interval \([10, 13]\), the following functions represent alternative optimal control strategies for the considered state

\[
\delta_1^o(0, 0, t_0) = \begin{cases} 
1 & t_0 \leq 10 \\
0 & 10 \leq t_0 < 13 \\
1 & 13 \leq t_0 < 15.3 \\
0 & t_0 > 15.3
\end{cases}
\]

\[
\delta_2^o(0, 0, t_0) = \begin{cases} 
0 & t_0 \leq 10 \\
1 & 10 \leq t_0 < 13 \\
0 & 13 \leq t_0 < 15.3 \\
1 & t_0 > 15.3
\end{cases}
\]

\[
\tau^o(0, 0, t_1) = \delta_1^o(0, 0, t_0) pt_1^o(0, 0, t_0) + \delta_2^o(0, 0, t_0) pt_2^o(0, 0, t_0) = \begin{cases} 
4 & t_0 < 10 \\
2 & 10 \leq t_0 < 12 \\
-t_0 + 14 & 12 \leq t_0 < 13 \\
-t_0 + 16 & 13 \leq t_0 < 15.3 \\
1 & 15.3 \leq t_0 < 15.5 \\
2 & 15.5 \leq t_0 < 16 \\
-t_0 + 18 & 16 \leq t_0 < 17 \\
1 & t_0 \geq 17
\end{cases}
\]

Such functions are illustrated in figures 69, 70, and 71 respectively.

Figure 69: Alternative optimal control strategy \( \delta_1^o(0, 0, t_0) \) in state \([0 0 t_0]^T\).

Figure 70: Alternative optimal control strategy \( \delta_2^o(0, 0, t_0) \) in state \([0 0 t_0]^T\).

Figure 71: Alternative optimal control strategy \( \tau^o(0, 0, t_0) \) (service time) in state \([0 0 t_0]^T\).

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6 Application to the single machine scheduling – Example with setup

Consider a single machine scheduling problem in which 4 jobs of class $P_1$ and 3 jobs of class $P_2$ must be executed. The due dates, the marginal tardiness costs of jobs, the processing time bounds and the marginal deviation costs of jobs are:

| Job Class | Due Date $d$ | Marginal Tardiness Cost $\alpha$ | Processing Time Bound $p$ |
|-----------|-------------|---------------------------------|--------------------------|
| $P_1$     | $1, 1$      | $0.75, 0.5, 1.5, 0.5$           | $19, 24, 29, 41$         |
| $P_2$     | $1, 2$      | $2, 1, 1$                        | $21, 24, 38$             |

A setup is required between the execution of jobs of different classes. Setup times and costs are:

| Setup Class | Setup Time $s$ | Setup Cost $c$ |
|-------------|----------------|--------------|
| $P_1$       | $0, 0.5$       | $0, 0$       |
| $P_2$       | $1, 0.5$       | $1, 0$       |

The evolution of the system state can be represented by the following diagram.

Figure 72: State diagram in the case of two classes of jobs, where $N_1 = 4$ and $N_2 = 3$, with setup.

The 32 states (from $S_0$ to $S_{31}$) in the 7 stages are:
Remark: In the following, the time variables $t_j$, $j = 0, \ldots, 7$, will be considered $\in \mathbb{R}$, that is, also negative values are taken into account. Negative values of $t_j$ can be considered when the strategies are determined in advance with respect to the initial time instant 0 at which the processing of the jobs starts. In this case, it is possible to exploit the optimal control strategies determined for the negative values of $t_j$ to start the execution of the jobs as soon as they become available, even before 0.

**Stage 7 – State $[4\ 3\ 2\ t_7]^T$ (S31)**

No decision has to be taken in state $[4\ 3\ 2\ t_7]^T$. The optimal cost-to-go is obviously null, that is

$$J_{4,3,2}^o(t_7) = 0$$

**Stage 7 – State $[4\ 3\ 1\ t_7]^T$ (S30)**

No decision has to be taken in state $[4\ 3\ 1\ t_7]^T$. The optimal cost-to-go is obviously null, that is

$$J_{4,3,1}^o(t_7) = 0$$

**Stage 6 – State $[3\ 3\ 2\ t_6]^T$ (S29)**

In state $[3\ 3\ 2\ t_6]^T$ all jobs of class $P_2$ have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable $\tau$ only (which corresponds to the processing time $pt_{1,4}$), is

$$\alpha_{1,4} \max\{t_6 + st_{1,1} + \tau - dd_{1,4}, 0\} + \beta_1 (pt_{1,4}^{nom} - \tau) + sc_{2,1} + J_{4,3,1}^o(t_7)$$

that can be written as $f(pt_{1,4} + t_6) + g(pt_{1,4})$ being

$$f(pt_{1,4} + t_6) = 0.5 \cdot \max\{pt_{1,4} + t_6 - 40.5, 0\} + 1$$

$$g(pt_{1,4}) = \begin{cases} 8 - pt_{1,4} & pt_{1,4} \in [4, 8] \\ 0 & pt_{1,4} \notin [4, 8] \end{cases}$$

The function $pt_{1,4}^c(t_6) = \arg\min_{pt_{1,4}}\{f(pt_{1,4} + t_6) + g(pt_{1,4})\}$, with $4 \leq pt_{1,4} \leq 8$, is determined by applying Lemma 4. It is

$$pt_{1,4}^c(t_6) = x_c(t_6) \quad \text{with} \quad x_c(t_6) = 8 \quad \forall t_6$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta_{7}^1 (3, 3, 2, t_6) = 1 \quad \forall t_6 \quad \delta_{7}^2 (3, 3, 2, t_6) = 0 \quad \forall t_6$$
The optimal control strategy \( \tau^\circ(3, 3, 2, t_6) \) is illustrated in Figure 73.

![Figure 73: Optimal control strategy \( \tau^\circ(3, 3, 2, t_6) \) in state \([3, 3, 2, t_6]^T\).](image1)

The optimal cost-to-go \( J^\circ_3(3, 3, 2, t_6) \) is provided by lemma 2. It is specified by the initial value 1, by the abscissa \( \gamma_1 = 32.5 \) at which the slope changes, and by the slope \( \mu_1 = 0.5 \) in the interval \([32.5, +\infty)\).

![Figure 74: Optimal cost-to-go \( J^\circ_3(3, 3, 2, t_6) \) in state \([3, 3, 2, t_6]^T\).](image2)

**Stage 6 – State \([3, 3, 1, t_6]^T\) (S28)**

In state \([3, 3, 1, t_6]^T\) all jobs of class \( P_2 \) have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continus) decision variable \( \tau \) only (which corresponds to the processing time \( pt_{1,4} \)), is

\[
\alpha_{1,4} \max\{t_6 + st_{1,1} + \tau - dd_{1,4}, 0\} + \beta_1 (pt_{1,4}^{\text{nom}} - \tau) + sc_{1,1} + J^\circ_{4,3,1}(t_7)
\]

that can be written as \( f(pt_{1,4} + t_6) + g(pt_{1,4}) \) being

\[
f(pt_{1,4} + t_6) = 0.5 \cdot \max\{pt_{1,4} + t_6 - 41, 0\}
\]

\[
g(pt_{1,4}) = \begin{cases} 8 - pt_{1,4} & pt_{1,4} \in [4,8] \\ 0 & pt_{1,4} \notin [4,8] \end{cases}
\]

The function \( pt_{1,4}^\circ(t_6) = \arg\min_{pt_{1,4}} \{ f(pt_{1,4} + t_6) + g(pt_{1,4}) \} \), with \( 4 \leq pt_{1,4} \leq 8 \), is determined by applying lemma 1. It is

\[
pt_{1,4}^\circ(t_6) = x_e(t_6) \quad \text{with} \quad x_e(t_6) = 8 \quad \forall t_6
\]

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

\[
\delta^\circ_1(3, 3, 1, t_6) = 1 \quad \forall t_6 \quad \delta^\circ_2(3, 3, 1, t_6) = 0 \quad \forall t_6
\]

\[
\tau^\circ(3, 3, 1, t_6) = 8 \quad \forall t_6
\]

The optimal control strategy \( \tau^\circ(3, 3, 1, t_6) \) is illustrated in Figure 75.

![Figure 75: Optimal control strategy \( \tau^\circ(3, 3, 1, t_6) \) in state \([3, 3, 1, t_6]^T\).](image3)
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Stage $6$ – State $[4 \ 2 \ 2 \ t_6]^T$ ($S27$)

In state $[4 \ 2 \ 2 \ t_6]^T$ all jobs of class $P_1$ have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable $\tau$ only (which corresponds to the processing time $pt_{2,3}$), is

$$\alpha_{2,3} \max\{t_6 + st_{2,2} + \tau - dd_{2,3} , 0\} + \beta_{2} (pt_{2,3}^{\text{nom}} - \tau) + sc_{2,2} + J_{3,3,1}^5(t_7)$$

that can be written as $f(pt_{2,3} + t_6) + g(pt_{2,3})$ being

$$f(pt_{2,3} + t_6) = \max\{pt_{2,3} + t_6 - 38, 0\}$$

$$g(pt_{2,3}) = \begin{cases} 1.5 \cdot (6 - pt_{2,3}) & pt_{2,3} \in [4, 6) \\ 0 & pt_{2,3} \notin [4, 6) \end{cases}$$

The function $pt_{2,3}^{\text{opt}}(t_6) = \arg \min_{pt_{2,3}} \{f(pt_{2,3} + t_6) + g(pt_{2,3})\}$, with $4 \leq pt_{2,3} \leq 6$, is determined by applying lemma $4$. It is

$$pt_{2,3}^{\text{opt}}(t_6) = x_e(t_6) \quad \text{with} \quad x_e(t_6) = 6 \quad \forall t_6$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta_{2}^7(4, 2, 2, t_6) = 0 \quad \forall t_6 \quad \delta_{2}^7(4, 2, 2, t_6) = 1 \quad \forall t_6$$

$$\tau_{2}^7(4, 2, 2, t_6) = 6 \quad \forall t_6$$

The optimal control strategy $\tau_{2}^7(4, 2, 2, t_6)$ is illustrated in figure $77$.

The optimal cost-to-go $J_{2,1}^5(t_6) = f(pt_{2,1}^{\text{opt}}(t_6) + t_6) + g(pt_{2,3}^{\text{opt}}(t_6))$, illustrated in figure $78$, is provided by lemma $2$. It is specified by the initial value $0$, by the abscissa $\gamma_1 = 32$ at which the slope changes, and by the slope $\mu_1 = 1$ in the interval $[32, +\infty)$. 

![Figure 75: Optimal control strategy $\tau^5(3, 3, 1, t_6)$ in state $[3 \ 3 \ 1 \ t_6]^T$.](image)

![Figure 76: Optimal cost-to-go $J_{3,3,1}^5(t_6)$ in state $[3 \ 3 \ 1 \ t_6]^T$.](image)
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Figure 77: Optimal control strategy $\tau \circ (4, 2, 2, t_6)$ in state $[4 \; 2 \; 2 \; t_6]^T$.

Figure 78: Optimal cost-to-go $J \circ (4, 2, 2, t_6)$ in state $[4 \; 2 \; 2 \; t_6]^T$.

Stage 6 – State $[4 \; 2 \; 1 \; t_6]^T$ (S26)

In state $[4 \; 2 \; 1 \; t_6]^T$ all jobs of class $P_1$ have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable $\tau$ only (which corresponds to the processing time $pt_{2,3}$), is

$$
\alpha_{2,3} \max \{t_6 + st_{1,2} + \tau - dd_{2,3} , 0]\} + \beta_2 (pt_{2,3}^{1,2} - \tau) + sc_{1,2} + J_{4,3,2}^\circ(t_7)
$$

that can be written as $f(pt_{2,3} + t_6) + g(pt_{2,3})$ being

$$
f(pt_{2,3} + t_6) = \max \{pt_{2,3} + t_6 - 37 , 0\} + 0.5
$$

$$
g(pt_{2,3}) = \begin{cases} 
1.5 \cdot (6 - pt_{2,3}) & \text{if } pt_{2,3} \in [4, 6) \\
0 & \text{otherwise}
\end{cases}
$$

The function $pt_{2,3}^\circ(t_6) = \arg \min_{pt_{2,3}} \{ f(pt_{2,3} + t_6) + g(pt_{2,3}) \}$, with $4 \leq pt_{2,3} \leq 6$, is determined by applying lemma[1]. It is

$$
pt_{2,3}^\circ(t_6) = x_e(t_6) \quad \text{with} \quad x_e(t_6) = 6 \quad \forall t_6
$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$
\delta_7^\circ(4, 2, 1, t_6) = 0 \quad \forall t_6 \quad \delta_2^\circ(4, 2, 1, t_6) = 1 \quad \forall t_6
$$

$$
\tau^\circ(4, 2, 1, t_6) = 6 \quad \forall t_6
$$

The optimal control strategy $\tau^\circ(4, 2, 1, t_6)$ is illustrated in figure[79].

Figure 79: Optimal control strategy $\tau^\circ(4, 2, 1, t_6)$ in state $[4 \; 2 \; 1 \; t_6]^T$. 
The optimal cost-to-go \( J^2_{3,2,1}(t_0) = f(pt^2_{3,3}(t_0) + t_0) + g(pt^2_{3,3}(t_0)) \), illustrated in figure [30], is provided by lemma [2]. It is specified by the initial value 0.5, by the abscissa \( \gamma_1 = 31 \) at which the slope changes, and by the slope \( \mu_1 = 1 \) in the interval \([31, +\infty)\).

**Stage 5 – State \([2 \ 3 \ 2 \ t_5]^T (S25)\)**

In state \([2 \ 3 \ 2 \ t_5]^T\) all jobs of class \( P_2 \) have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable \( \tau \) only (which corresponds to the processing time \( pt_{1,3} \)), is

\[
\alpha_{1,3} \max\{t_5 + st_{2,1} + \tau - dd_{1,3}, 0\} + \beta_1 (pt_{1,3}^{\text{nom}} - \tau) + se_{2,1} + J^2_{3,3,1}(t_6)
\]

that can be written as \( f(pt_{1,3} + t_5) + g(pt_{1,3}) \) being

\[
f(pt_{1,3} + t_5) = 1.5 \cdot \max\{pt_{1,3} + t_5 - 28.5, 0\} + 1 + J^2_{3,3,1}(pt_{1,3} + t_5 + 0.5)
\]

\[
g(pt_{1,3}) = \begin{cases} 8 - pt_{1,3} & pt_{1,3} \in [4,8) \\ 0 & pt_{1,3} \notin [4,8) \end{cases}
\]

The function \( pt_{1,3}^{opt}(t_5) = \arg \min_{pt_{1,3}} \{ f(pt_{1,3} + t_5) + g(pt_{1,3}) \} \), with \( 4 \leq pt_{1,3} \leq 8 \), is determined by applying lemma [1]. It is

\[
pt_{1,3}^{opt}(t_5) = x_c(t_5) \quad \text{with} \quad x_c(t_5) = \begin{cases} 8 & t_5 < 20.5 \\ -t_5 + 28.5 & 20.5 \leq t_5 < 24.5 \\ 4 & t_5 \geq 24.5 \end{cases}
\]

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

\[
\delta_1(2,3,2,t_5) = 1 \quad \forall t_5 \quad \delta_2(2,3,2,t_5) = 0 \quad \forall t_5
\]

\[
\tau^c(2,3,2,t_5) = \begin{cases} 8 & t_5 < 20.5 \\ -t_5 + 28.5 & 20.5 \leq t_5 < 24.5 \\ 4 & t_5 \geq 24.5 \end{cases}
\]

The optimal control strategy \( \tau^c(2,3,2,t_5) \) is illustrated in figure [31].

![Figure 80: Optimal cost-to-go \( J^2_{3,2,1}(t_0) \) in state \([4 \ 2 \ 1 \ t_0]^T\).](image)

![Figure 81: Optimal control strategy \( \tau^c(2,3,2,t_5) \) in state \([2 \ 3 \ 2 \ t_5]^T\).](image)
The optimal cost-to-go $J^{\pi}_{2,3,2}(t_5) = f(pt^{\pi}_{1,3}(t_5) + t_5) + g(pt^{\pi}_{2,3}(t_5))$, illustrated in figure 82, is provided by lemma 2. It is specified by the initial value 1, by the set $\{20.5, 24.5, 28.5\}$ of abscissae $\gamma_i$, $i = 1, \ldots, 3$, at which the slope changes, and by the set $\{1, 1.5, 2\}$ of slopes $\mu_i$, $i = 1, \ldots, 3$, in the various intervals.

**Stage 5 – State $[2 \ 3 \ 1 \ t_5]^T$** $(S24)$

In state $[2 \ 3 \ 1 \ t_5]^T$ all jobs of class $P_2$ have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable $\tau$ only (which corresponds to the processing time $pt_{1,3}$), is

$$\alpha_{1,3} \max\{t_5 + st_{1,1} + \tau - dd_{1,3}, 0\} + \beta_{1} (pt^{\text{nom}}_{1} - \tau) + sc_{1,1} + J^{\pi}_{3,3,1}(t_6)$$

that can be written as $f(pt_{1,3} + t_5) + g(pt_{1,3})$ being

$$f(pt_{1,3} + t_5) = 1.5 \cdot \max\{pt_{1,3} + t_5 - 29, 0\} + J^{\pi}_{3,3,1}(pt_{1,3} + t_5)$$

$$g(pt_{1,3}) = \begin{cases} 8 - pt_{1,3} & \text{if } pt_{1,3} \in [4,8) \\
0 & \text{if } pt_{1,3} \notin [4,8) \end{cases}$$

The function $pt^\pi_{1,3}(t_5) = \arg \min_{pt_{1,3}} \{f(pt_{1,3} + t_5) + g(pt_{1,3})\}$, with $4 \leq pt_{1,3} \leq 8$, is determined by applying lemma 1. It is

$$pt^\pi_{1,3}(t_5) = x_3(t_5) \quad \text{with} \quad x_3(t_5) = \begin{cases} 8 & t_5 < 21 \\
-t_5 + 29 & 21 \leq t_5 < 25 \\
4 & t_5 \geq 25 \end{cases}$$

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

$$\delta^\pi_1(2,3,1,t_5) = 1 \quad \forall \ t_5 \quad \delta^\pi_2(2,3,1,t_5) = 0 \quad \forall \ t_5$$

$$\tau^\pi(2,3,1,t_5) = \begin{cases} 8 & t_5 < 21 \\
-t_5 + 29 & 21 \leq t_5 < 25 \\
4 & t_5 \geq 25 \end{cases}$$

The optimal control strategy $\tau^\pi(2,3,1,t_5)$ is illustrated in figure 83.
The conditioned cost-to-go provided by lemma 2. It is specified by the initial value 1.5, by the set at which the slope changes, and by the set of slopes \( \mu_i, i = 1, \ldots, 3 \), in the various intervals.

**Stage 5 – State \([3 \ 2 \ 2 \ t_5]^T\) (S23)**

In state \([3 \ 2 \ 2 \ t_5]^T\), the cost function to be minimized, with respect to the (continuous) decision variable \( \tau \) and to the (binary) decision variables \( \delta_1 \) and \( \delta_2 \) is

\[
\delta_1 \left[ \alpha_{1,4} \max \{ t_5 + st_{2,1} + \tau - dd_{1,4}, 0 \} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + se_{2,1} + J_{4,2,1}^2(t_0) \right] + \nonumber \\
+ \delta_2 \left[ \alpha_{2,3} \max \{ t_5 + st_{2,2} + \tau - dd_{2,3}, 0 \} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + se_{2,2} + J_{3,3,2}^2(t_0) \right]
\]

**Case i)** in which it is assumed \( \delta_1 = 1 \) and \( \delta_2 = 0 \).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \( \tau \) which corresponds to the processing time \( pt_{1,4} \), the following function

\[
\alpha_{1,4} \max \{ t_5 + st_{2,1} + \tau - dd_{1,4}, 0 \} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + se_{2,1} + J_{4,2,1}^2(t_0)
\]

that can be written as \( f(pt_{1,4}, t_5) + g(pt_{1,4}) \) being

\[
f(pt_{1,4}, t_5) = 0.5 \cdot \max \{ pt_{1,4} + t_5 - 40.5, 0 \} + 1 + J_{4,2,1}^2(pt_{1,4} + t_5 + 0.5)
\]

\[
g(pt_{1,4}) = \begin{cases} 8 - pt_{1,4} & \text{if } pt_{1,4} \in [4, 8] \\ 0 & \text{if } pt_{1,4} \notin [4, 8] \end{cases}
\]

The function \( pt_{1,4}^o(t_5) = \arg \min_{pt_{1,4}} \{ f(pt_{1,4}, t_5) + g(pt_{1,4}) \} \), with \( 4 \leq pt_{1,4} \leq 8 \), is determined by applying lemma \[2\] It is (see figure \[85\])

\[
pt_{1,4}^o(t_5) = x_e(t_5) \quad \text{with} \quad x_e(t_5) = \begin{cases} 8 & t_5 < 22.5 \\ -t_5 + 30.5 & 22.5 \leq t_5 < 26.5 \\ 4 & t_5 \geq 26.5 \end{cases}
\]

Figure 85: Optimal processing time \( pt_{1,4}^o(t_5) \), under the assumption \( \delta_1 = 1 \) in state \([3 \ 2 \ 2 \ t_5]^T\).
Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $\pt_{2,3}$, the following function

$$\alpha_{2,3} \max\{\tau + s_{2,2} + \tau - dd_{2,3}, 0\} + \beta_2 (p_{2,3}^\text{nom} - \tau) + sc_{2,2} + J_{3,3,2}^0(\tau)$$

that can be written as $f(\pt_{2,3} + t_5) + g(\pt_{2,3})$ being

$$f(\pt_{2,3} + t_5) = \max\{\pt_{2,3} + t_5 - 38, 0\} + J_{3,3,2}^0(\pt_{2,3} + t_5)$$

$$g(\pt_{2,3}) = \begin{cases} 1.5 \cdot (6 - \pt_{2,3}) & \pt_{2,3} \in [4, 6) \\ 0 & \pt_{2,3} \notin [4, 6) \end{cases}$$

The function $\pt_{2,3}^0(t_5) = \arg \min_{\pt_{2,3}} \{f(\pt_{2,3} + t_5) + g(\pt_{2,3})\}$, with $4 \leq \pt_{2,3} \leq 6$, is determined by applying lemma 1. It is (see figure 86).

$$\pt_{2,3}^0(t_5) = x_e(t_5) \quad \text{with} \quad x_e(t_5) = \begin{cases} 6 & t_5 < 32 \\ -t_5 + 38 & 32 \leq t_5 < 34 \\ 4 & t_5 \geq 34 \end{cases}$$

Figure 86: Optimal processing time $\pt_{2,3}^0(t_5)$, under the assumption $\delta_2 = 1$ in state $[3 2 2 t_5]^T$.

The conditioned cost-to-go $J_{3,2,2}^0(t_5 \mid \delta_2 = 1) = f(\pt_{2,3}^0(t_5) + t_5) + g(\pt_{2,3}^0(t_5))$, illustrated in figure 87, is provided by lemma 2. It is specified by the initial value 1, by the set $\{26.5, 32\}$ of abscissae $\gamma_i$, $i = 1, \ldots, 2$, at which the slope changes, and by the set $\{0.5, 1.5\}$ of slopes $\mu_i$, $i = 1, \ldots, 2$, in the various intervals.

In order to find the optimal cost-to-go $J_{3,2,2}^0(t_5)$, it is necessary to carry out the following minimization

$$J_{3,2,2}^0(t_5) = \min \left\{ J_{3,2,2}^0(t_5 \mid \delta_1 = 1), J_{3,2,2}^0(t_5 \mid \delta_2 = 1) \right\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 88.

The function $J_{3,2,2}^0(t_5)$ is specified by the initial value 1, by the set $\{26.5, 32\}$ of abscissae $\gamma_i$, $i = 1, \ldots, 2$, at which the slope changes, and by the set $\{0.5, 1.5\}$ of slopes $\mu_i$, $i = 1, \ldots, 2$, in the various intervals.

Since $J_{3,2,2}^0(t_5 \mid \delta_2 = 1)$ is always the minimum (see again figure 87), the optimal control strategies for this state are

$$\delta_1^0(3, 2, 2, t_5) = 0 \quad \forall t_5 \quad \delta_2^0(3, 2, 2, t_5) = 1 \quad \forall t_5$$
Stage 5 – State \([3\ 2\ 1\ t_5]^T\) (S22)

In state \([3\ 2\ 1\ t_5]^T\), the cost function to be minimized, with respect to the (continuos) decision variable \(\tau\) and to the (binary) decision variables \(\delta_1\) and \(\delta_2\) is

\[
\delta_1 \left[ \alpha_{1,4} \max\{t_5 + s t_{1,1} + \tau - d d_{1,4}, 0\} + \beta_1 (p t_{1,4}^{\text{nom}} - \tau) + s c_{1,1} + J_{2,2,1}^2(t_6) \right] + \\
+ \delta_2 \left[ \alpha_{2,3} \max\{t_5 + s t_{1,2} + \tau - d d_{2,3}, 0\} + \beta_2 (p t_{2,3}^{\text{nom}} - \tau) + s c_{1,2} + J_{3,3,2}^3(t_6) \right]
\]

**Case i)** in which it is assumed \(\delta_1 = 1\) (and \(\delta_2 = 0\)).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable \(\tau\) which corresponds to the processing time \(p t_{1,4}\), the following function

\[
\alpha_{1,4} \max\{t_5 + s t_{1,1} + \tau - d d_{1,4}, 0\} + \beta_1 (p t_{1,4}^{\text{nom}} - \tau) + s c_{1,1} + J_{2,2,1}^2(t_6)
\]

that can be written as \(f(p t_{1,4} + t_5) + g(p t_{1,4})\) being

\[
f(p t_{1,4} + t_5) = 0.5 \cdot \max\{p t_{1,4} + t_5 - 41, 0\} + J_{2,2,1}^2(p t_{1,4} + t_5)
\]

\[
g(p t_{1,4}) = \begin{cases} 
8 - p t_{1,4} & p t_{1,4} \in [4, 8) \\
0 & p t_{1,4} \notin [4, 8)
\end{cases}
\]

The function \(p t_{1,4}^2(t_5) = \arg \min_{p t_{1,4}} \{f(p t_{1,4} + t_5) + g(p t_{1,4})\}\), with \(4 \leq p t_{1,4} \leq 8\), is determined by applying lemma\[H\]. It is (see figure\[90\])

\[
p t_{1,4}^2(t_5) = x_e(t_5) \quad \text{with} \quad x_e(t_5) = \begin{cases} 
8 & t_5 < 23 \\
-t_5 + 31 & 23 \leq t_5 < 27 \\
4 & t_5 \geq 27
\end{cases}
\]
The conditioned cost-to-go $J_{3,2,1}^\circ(t_5 \mid \delta_1 = 1) = f(pt_{2,3}^\circ(t_5) + t_5) + g(pt_{2,3}^\circ(t_5))$, illustrated in figure 92 is provided by lemma 2. It is specified by the initial value 0.5, by the set \{23, 37\} of abscissae $\gamma_i$, $i = 1,\ldots,2$, at which the slope changes, and by the set \{1, 1.5\} of slopes $\mu_i$, $i = 1,\ldots,2$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{2,3}$, the following function

$$
\alpha_{2,3} \max\{t_5 + s_{1,2} + \tau - dd_{2,3}, 0\} + \beta_2 (pt_{2,3}^{\text{nom}} - \tau) + sc_{1,2} + J_{3,3,2}^\circ(t_6)
$$

that can be written as $f(pt_{2,3} + t_5) + g(pt_{2,3})$ being

$$
f(pt_{2,3} + t_5) = \max\{pt_{2,3} + t_5 - 37, 0\} + 0.5 + J_{3,3,2}^\circ(pt_{2,3} + t_5 + 1)
$$

$$
g(pt_{2,3}) = \begin{cases} 1.5 \cdot (6 - pt_{2,3}) & pt_{2,3} \in [4, 6] \\ 0 & pt_{2,3} \notin [4, 6] \end{cases}
$$

The function $pt_{2,3}^\circ(t_5) = \arg\min_{pt_{2,3}} \{f(pt_{2,3} + t_5) + g(pt_{2,3})\}$, with $4 \leq pt_{2,3} \leq 6$, is determined by applying lemma 1. It is (see figure 91)

$$
pt_{2,3}^\circ(t_5) = x_6(t_5) \quad \text{with} \quad x_6(t_5) = \begin{cases} 6 & t_5 < 31 \\ -t_5 + 37 & 31 \leq t_5 < 33 \\ 4 & t_5 \geq 33 \end{cases}
$$

The conditioned cost-to-go $J_{3,2,1}^\circ(t_5 \mid \delta_2 = 1) = f(pt_{2,3}^\circ(t_5) + t_5) + g(pt_{2,3}^\circ(t_5))$, illustrated in figure 92 is provided by lemma 2. It is specified by the initial value 1.5, by the set \{25.5, 31\} of abscissae $\gamma_i$, $i = 1,\ldots,2$, at which the slope changes, and by the set \{0.5, 1.5\} of slopes $\mu_i$, $i = 1,\ldots,2$, in the various intervals.

Figure 90: Optimal processing time $pt_{3,1}^\circ(t_5)$, under the assumption $\delta_1 = 1$ in state $[3\ 2\ 1\ t_5]^T$.

Figure 91: Optimal processing time $pt_{2,3}^\circ(t_5)$, under the assumption $\delta_2 = 1$ in state $[3\ 2\ 1\ t_5]^T$.

Figure 92: Conditioned costs-to-go $J_{3,2,1}^\circ(t_5 \mid \delta_1 = 1)$ and $J_{3,2,1}^\circ(t_5 \mid \delta_2 = 1)$ in state $[3\ 2\ 1\ t_5]^T$. 
In order to find the optimal cost-to-go $J_{3,2,1}^* (t_5)$, it is necessary to carry out the following minimization

$$J_{3,2,1}^* (t_5) = \min \{ J_{3,2,1}^* (t_5 \mid \delta_1 = 1), J_{3,2,1}^* (t_5 \mid \delta_2 = 1) \}$$

which provides, in accordance with lemma\[4] the continuous, nondecreasing, piecewise linear function illustrated in figure 93.

![Figure 93: Optimal cost-to-go $J_{3,2,1}^* (t_5)$ in state [3 2 1 t5].](image)

The function $J_{3,2,1}^* (t_5)$ is specified by the initial value $0.5$, by the set \{ 23, 24, 25.5, 31 \} of abscissae $\gamma_i$, $i = 1, \ldots, 4$, at which the slope changes, and by the set \{ 1, 0, 0.5, 1.5 \} of slopes $\mu_i$, $i = 1, \ldots, 4$, in the various intervals.

Since $J_{3,2,1}^* (t_5 \mid \delta_1 = 1)$ is the minimum in $(-\infty, 24)$, and $J_{3,2,1}^* (t_5 \mid \delta_2 = 1)$ is the minimum in $[24, +\infty)$, the optimal control strategies for this state are

$$\delta_1^* (3, 2, 1, t_5) = \begin{cases} 1 & t_5 < 24 \\ 0 & t_5 \geq 24 \end{cases} \quad \delta_2^* (3, 2, 1, t_5) = \begin{cases} 0 & t_5 < 24 \\ 1 & t_5 \geq 24 \end{cases}$$

$$\tau^* (3, 2, 1, t_5) = \begin{cases} 8 & t_5 < 23 \\ -t_5 + 31 & 23 \leq t_5 < 24 \\ 6 & 24 \leq t_5 < 31 \\ -t_5 + 37 & 31 \leq t_5 < 33 \\ 4 & t_5 \geq 33 \end{cases}$$

The optimal control strategy $\tau^* (3, 2, 1, t_5)$ is illustrated in figure 94.

![Figure 94: Optimal control strategy $\tau^* (3, 2, 1, t_5)$ in state [3 2 1 t5].](image)

**Stage 5 – State [4 1 2 t5]T (S21)**

In state [4 1 2 t5]T all jobs of class $P_1$ have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable $\tau$ (which corresponds to the processing time $pt_{2,2}$), is

$$\alpha_{2,2} \max \{ t_5 + st_{2,2} + \tau - dd_{2,2} , 0 \} + \beta_{2} (pt_{2,2}^{nm} - \tau) + sc_{2,2} + J_{4,2,2}^* (t_6)$$

that can be written as $f (pt_{2,2} + t_5) + g (pt_{2,2})$ being

$$f (pt_{2,2} + t_5) = \max \{ pt_{2,2} + t_5 - 24 , 0 \} + J_{4,2,2}^* (pt_{2,2} + t_5)$$
The optimal cost-to-go slope changes, and by the set lemma 2. It is specified by the initial value 0, by the set
Stage the optimal control strategy

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

The function \( p \tau^2(t_5) = x_\delta(t_5) \) with \( x_\delta(t_5) = \begin{cases} 6 & t_5 < 26 \\ -t_5 + 32 & 26 \leq t_5 < 28 \\ 4 & t_5 \geq 28 \end{cases} \)

The optimal control strategy \( \tau^2(4, 1, 2, t_5) \) is illustrated in figure 95.

The optimal cost-to-go \( J^\tau_4(1, 2, t_5) = f(p \tau^2(t_5) + t_5) + g(p \tau^2(t_5)) \), illustrated in figure 96, is provided by lemma 2. It is specified by the initial value 0, by the set \{ 18, 26, 28 \} of abscissae \( \gamma_i, i = 1, \ldots, 3 \), at which the slope changes, and by the set \{ 1, 1.5, 2 \} of slopes \( \mu_i, i = 1, \ldots, 3 \), in the various intervals.

Figure 95: Optimal control strategy \( \tau^2(4, 1, 2, t_5) \) in state \([4 1 2 t_5]^T\).

Figure 96: Optimal cost-to-go \( J^\tau_4(1, 2, t_5) \) in state \([4 1 2 t_5]^T\).

Stage 5 – State \([4 1 1 t_5]^T \) (S20)

In state \([4 1 1 t_5]^T \) all jobs of class \( P_1 \) have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable \( \tau \) only (which corresponds to the processing time \( p \tau^2(t_5) \)), is

\[
\alpha_{2,2} \max\{t_5 + s1.2 + \tau - dd_{2,2}, 0\} + \beta_2 (p \tau_{2,2}^{nom} - \tau) + sc_1.2 + J^\tau_2(4, 2, t_5)
\]

that can be written as \( f(p \tau^2(t_5) + t_5) + g(p \tau^2(t_5)) \) being

\[
f(p \tau^2(t_5) + t_5) = \max\{p \tau^2(t_5) + t_5 - 23, 0\} + 0.5 + J^\tau_4(4, 2, t_5 + 1)
\]
The optimal cost-to-go slope changes, and by the set lemma 2. It is specified by the initial value 0.5, by the set Stage 4 – State [1 3 2 4]T (S19)

In state [1 3 2 4]T all jobs of class P3 have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable τ only (which corresponds to the processing time pt,2), is

\[ \alpha_{1,2} \max\{t_4 + s t_{2,1} + \tau - dd_{1,2} + 0\} + \beta_{1} (pt_{1,2}^{\text{nom}} - \tau) + sc_{2,1} + J_{2,3,1}^p(t_5) \]

that can be written as

\[ f(pt_{1,2} + t_4) = 0.5 \cdot \max\{pt_{1,2} + t_4 - 23.5, 0\} + 1 + J_{2,3,1}^p(pt_{1,2} + t_4 + 0.5) \]
The optimal cost-to-go which the slope changes, and by the set lemma 2. It is specified by the initial value 1, by the set variable \( \delta \).

The function \( pt_{1,2}^\circ(t_4) = \arg \min_{pt_{1,2}} \{ f(pt_{1,2} + t_4) + g(pt_{1,2}) \} \), with \( 4 \leq pt_{1,2} \leq 8 \), is determined by applying lemma 1. It is

\[
g(pt_{1,2}) = \begin{cases} 8 - pt_{1,2} & pt_{1,2} \in [4, 8) \\ 0 & pt_{1,2} \notin [4, 8) \end{cases}
\]

The function \( pt_{1,2}^\circ(t_4) = x_e(t_4) \) with \( x_e(t_4) = \begin{cases} 8 & t_4 < 12.5 \\ -t_4 + 20.5 & 12.5 \leq t_4 < 16.5 \\ 4 & t_4 \geq 16.5 \end{cases} \)

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

\[
\delta_1^\circ(1, 3, 2, t_4) = 1 \quad \forall t_4 \quad \delta_2^\circ(1, 3, 2, t_4) = 0 \quad \forall t_4
\]

\[
\tau^\circ(1, 3, 2, t_4) = \begin{cases} 8 & t_4 < 12.5 \\ -t_4 + 20.5 & 12.5 \leq t_4 < 16.5 \\ 4 & t_4 \geq 16.5 \end{cases}
\]

The optimal control strategy \( \tau^\circ(1, 3, 2, t_4) \) is illustrated in figure 99.

![Figure 99: Optimal control strategy \( \tau^\circ(1, 3, 2, t_4) \) in state \([1 \ 3 \ 2 \ t_4]^T\).](image)

The optimal cost-to-go \( J_{1,3,2}^\circ(t_4) = f(pt_{1,2}^\circ(t_4) + t_4) + g(pt_{1,2}^\circ(t_4)) \), illustrated in figure 100, is provided by lemma 2. It is specified by the initial value 1, by the set \{ 12.5, 19.5, 20.5, 24.5 \} of abscissae \( \gamma_i \), \( i = 1, \ldots, 4 \), at which the slope changes, and by the set \{ 1, 1.5, 2, 2.5 \} of slopes \( \mu_i \), \( i = 1, \ldots, 4 \), in the various intervals.

![Figure 100: Optimal cost-to-go \( J_{1,3,2}^\circ(t_4) \) in state \([1 \ 3 \ 2 \ t_4]^T\).](image)

**Stage 4 – State \([1 \ 3 \ 1 \ t_4]^T\) (S18)**

In state \([1 \ 3 \ 1 \ t_4]^T\) all jobs of class \( P_2 \) have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable \( \tau \) only (which corresponds to the processing time \( pt_{1,2} \)), is

\[
\alpha_{1,2} \max \{ t_4 + s_1, 0 \} + \beta_1 (pt_{1,2}^{\text{nom}} - \tau) + \gamma_1 + J_{2,3,1}^\circ(t_5)
\]

that can be written as \( f(pt_{1,2} + t_4) + g(pt_{1,2}) \) being

\[
f(pt_{1,2} + t_4) = 0.5 \cdot \max \{ pt_{1,2} + t_4 - 24, 0 \} + J_{2,3,1}^\circ(pt_{1,2} + t_4)
\]
The optimal cost-to-go lemma 2. It is specified by the initial value 0, by the set the slope changes, and by the set 

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

\[
\delta_1(1, 3, 1, t_4) = 1 \quad \forall t_4 \quad \delta_2(1, 3, 1, t_4) = 0 \quad \forall t_4
\]

\[
\tau^o(1, 3, 1, t_4) = \left\{ \begin{array}{ll} 
8 & t_4 < 13 \\
-t_4 + 21 & 13 \leq t_4 < 17 \\
4 & t_4 \geq 17 
\end{array} \right.
\]

The optimal control strategy \( \tau^o(1, 3, 1, t_4) \) is illustrated in figure 101.

![Figure 101: Optimal control strategy \( \tau^o(1, 3, 1, t_4) \) in state \([1 3 1 t_4]^T\).](image)

The optimal cost-to-go \( J_{g,3,1}^o(t_4) = f(pt_{1,2}^o(t_4) + t_4) + g(pt_{1,2}^o(t_4)) \), illustrated in figure 102, is provided by lemma 2. It is specified by the initial value 0, by the set \{ 13, 20, 21, 25 \} of abscissae \( \gamma_i \), \( i = 1, \ldots, 4 \), at which the slope changes, and by the set \{ 1, 1.5, 2, 2.5 \} of slopes \( \mu_i \), \( i = 1, \ldots, 4 \), in the various intervals.

![Figure 102: Optimal cost-to-go \( J_{g,3,1}^o(t_4) \) in state \([1 3 1 t_4]^T\).](image)

**Stage 4 – State \([2 2 2 t_4]^T \) (S17)**

In state \([2 2 2 t_4]^T \), the cost function to be minimized, with respect to the (continuous) decision variable \( \tau \) and to the (binary) decision variables \( \delta_1 \) and \( \delta_2 \) is

\[
\delta_1 \left[ \alpha_{1,3} \max \{ t_4 + st_{2,1} + \tau - dd_{1,3}, 0 \} + \beta_{1} (pt_{1,2}^{\text{nom}} - \tau) + sc_{2,1} + J_{g,3,2,1}(t_5) \right] + \\
\delta_2 \left[ \alpha_{2,3} \max \{ t_4 + st_{2,2} + \tau - dd_{2,3}, 0 \} + \beta_{2} (pt_{2,2}^{\text{nom}} - \tau) + sc_{2,2} + J_{g,3,3,2}(t_5) \right]
\]

**Case i)** in which it is assumed \( \delta_1 = 1 \) (and \( \delta_2 = 0 \)).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \( \tau \) which corresponds to the processing time \( pt_{1,3} \), the following function

\[
\alpha_{1,3} \max \{ t_4 + st_{2,1} + \tau - dd_{1,3}, 0 \} + \beta_{1} (pt_{1,2}^{\text{nom}} - \tau) + sc_{2,1} + J_{g,3,2,1}(t_5)
\]
that can be written as \( f(pt_{1,3} + t_4) + g(pt_{1,3}) \) being

\[
f(pt_{1,3} + t_4) = 1.5 \cdot \max\{pt_{1,3} + t_4 - 28.5, 0\} + 1 + J_{3,2,1}^2(pt_{1,3} + t_4 + 0.5)
\]

\[
g(pt_{1,3}) = \begin{cases} 
8 - pt_{1,3} & \text{if } pt_{1,3} \in [4, 8] \\
0 & \text{if } pt_{1,3} \notin [4, 8]
\end{cases}
\]

The function \( pt_{1,3}^o(t_4) = \arg \min_{pt_{1,3}} \{ f(pt_{1,3} + t_4) + g(pt_{1,3}) \} \), with \( 4 \leq pt_{1,3} \leq 8 \), is determined by applying lemma \([8]\). It is (see figure \[104\]).

\[
pt_{1,3}^o(t_4) = \begin{cases} 
x_s(t_4) & t_4 < 15.5 \\
x_e(t_4) & t_4 \geq 15.5
\end{cases}
\text{ with } x_s(t_4) = \begin{cases} 
8 & t_4 < 14.5 \\
-t_4 + 22.5 & 14.5 \leq t_4 < 15.5
\end{cases}
\text{ and } x_e(t_4) = \begin{cases} 
8 & 15.5 \leq t_4 < 20.5 \\
-t_4 + 28.5 & 20.5 \leq t_4 < 24.5 \\
4 & t_4 \geq 24.5
\end{cases}
\]

Figure 103: Optimal processing time \( pt_{1,3}^o(t_4) \), under the assumption \( \delta_1 = 1 \) in state \( [2 \ 2 \ 2 \ t_4]^T \).

The conditioned cost-to-go \( J_{3,2,2}^2(t_4 \mid \delta_1 = 1) = f(pt_{1,3}^o(t_4) + t_4) + g(pt_{1,3}^o(t_4)) \), illustrated in figure \[105\] is provided by lemma \([2]\). It is specified by the initial value 1.5, by the set \( \{ 14.5, 15.5, 17, 20.5, 24.5, 26.5 \} \) of abscissae \( \gamma_i, i = 1, \ldots, 6 \), at which the slope changes, and by the set \( \{ 1, 0, 0.5, 1, 2, 3 \} \) of slopes \( \mu_i, i = 1, \ldots, 6 \), in the various intervals.

**Case ii)** in which it is assumed \( \delta_2 = 1 \) (and \( \delta_1 = 0 \)).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \( \tau \) which corresponds to the processing time \( pt_{2,3} \), the following function

\[
\alpha_{2,3} \max\{t_4 + st_{2,2} + \tau - dd_{2,3}, 0\} + \beta_2 (pt_{2,3}^{\text{nom}} - \tau) + sc_{2,2} + J_{3,2,2}^o(t_5)
\]

that can be written as \( f(pt_{2,3} + t_4) + g(pt_{2,3}) \) being

\[
f(pt_{2,3} + t_4) = \max\{pt_{2,3} + t_4 - 38, 0\} + J_{3,2,2}^o(pt_{2,3} + t_4)
\]

\[
g(pt_{2,3}) = \begin{cases} 
1.5 \cdot (6 - pt_{2,3}) & \text{if } pt_{2,3} \in [4, 6] \\
0 & \text{if } pt_{2,3} \notin [4, 6]
\end{cases}
\]

The function \( pt_{2,3}^o(t_4) = \arg \min_{pt_{2,3}} \{ f(pt_{2,3} + t_4) + g(pt_{2,3}) \} \), with \( 4 \leq pt_{2,3} \leq 6 \), is determined by applying lemma \([8]\). It is (see figure \[104\]).

\[
pt_{2,3}^o(t_4) = x_e(t_4) \text{ with } x_e(t_4) = \begin{cases} 
6 & t_4 < 18.5 \\
-t_4 + 24.5 & 18.5 \leq t_4 < 20.5 \\
4 & t_4 \geq 20.5
\end{cases}
\]

The conditioned cost-to-go \( J_{2,2,2}^2(t_4 \mid \delta_2 = 1) = f(pt_{2,3}^o(t_4) + t_4) + g(pt_{2,3}^o(t_4)) \), illustrated in figure \[105\] is provided by lemma \([2]\). It is specified by the initial value 1, by the set \( \{ 14.5, 18.5, 24.5, 34 \} \) of abscissae \( \gamma_i, i = 1, \ldots, 4 \), at which the slope changes, and by the set \( \{ 1, 1.5, 2, 3 \} \) of slopes \( \mu_i, i = 1, \ldots, 4 \), in the various intervals.

In order to find the optimal cost-to-go \( J_{2,2,2}^2(t_4) \), it is necessary to carry out the following minimization

\[
J_{2,2,2}^2(t_4) = \min \{ J_{2,2,2}^2(t_4 \mid \delta_1 = 1), J_{2,2,2}^2(t_4 \mid \delta_2 = 1) \}
\]
Fundamental lemmas for the determination of optimal control strategies for a class of single machine family scheduling problems

Figure 104: Optimal processing time $p_{2,3}^*(t_4)$, under the assumption $\delta_2 = 1$ in state $[2 \, 2 \, 2 \, t_4]^T$.

Figure 105: Conditioned costs-to-go $J_{2,2,2}^*(t_4 \mid \delta_1 = 1)$ and $J_{2,2,2}^*(t_4 \mid \delta_2 = 1)$ in state $[2 \, 2 \, 2 \, t_4]^T$.

Figure 106: Optimal cost-to-go $J_{2,2,2}^*(t_4)$ in state $[2 \, 2 \, 2 \, t_4]^T$.

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 106.

The function $J_{2,2,2}^*(t_4)$ is specified by the initial value 1, by the set $\{14.5, 16, 17, 20.5, 24.5, 26.5, 32.25, 34\}$ of abscissae $\gamma_i$, $i = 1, \ldots, 8$, at which the slope changes, and by the set $\{1, 0, 0.5, 1, 2, 3, 2, 3\}$ of slopes $\mu_i$, $i = 1, \ldots, 8$, in the various intervals.

Since $J_{2,2,2}^*(t_4 \mid \delta_1 = 1)$ is the minimum in $[16, 32.25)$, and $J_{2,2,2}^*(t_4 \mid \delta_2 = 1)$ is the minimum in $(-\infty, 16)$ and in $[32.25, +\infty)$, the optimal control strategies for this state are

$$\delta_1^*(2, 2, 2, t_4) = \begin{cases} 0 & t_4 < 16 \\ 1 & 16 \leq t_4 < 32.25 \\ 0 & t_4 \geq 32.25 \end{cases}$$

$$\delta_2^*(2, 2, 2, t_4) = \begin{cases} 1 & t_4 < 16 \\ 0 & 16 \leq t_4 < 32.25 \\ 1 & t_4 \geq 32.25 \end{cases}$$

$$\tau^*(2, 2, 2, t_4) = \begin{cases} 6 & t_4 < 16 \\ 8 & 16 \leq t_4 < 20.5 \\ -t_4 + 28.5 & 20.5 \leq t_4 < 24.5 \\ 4 & t_4 \geq 24.5 \end{cases}$$

The optimal control strategy $\tau^*(2, 2, 2, t_4)$ is illustrated in figure 107.
Stage 4 – State $[2 \ 2 \ 1 \ t_4]^T$ ($S_16$)

In state $[2 \ 2 \ 1 \ t_4]^T$, the cost function to be minimized, with respect to the (continuos) decision variable $\tau$ and to the (binary) decision variables $\delta_1$ and $\delta_2$ is

$$\delta_1 \left[ \alpha_{1,3} \max \{ t_4 + st_{1,1} + \tau - dd_{1,3}, 0 \} + \beta_1 (pt_1^{\text{nom}} - \tau) + sc_{1,1} + J_{3,2,1}^\circ(t_5) \right] +$$

$$+ \delta_2 \left[ \alpha_{2,3} \max \{ t_4 + st_{1,2} + \tau - dd_{2,3}, 0 \} + \beta_2 (pt_2^{\text{nom}} - \tau) + sc_{1,2} + J_{2,3,2}^\circ(t_5) \right]$$

**Case i)** in which it is assumed $\delta_1 = 1$ and $\delta_2 = 0$.

In this case, it is necessary to minimize, with respect to the (continuos) decision variable $\tau$ which corresponds to the processing time $pt_{1,3}$, the following function

$$\alpha_{1,3} \max \{ t_4 + st_{1,1} + \tau - dd_{1,3}, 0 \} + \beta_1 (pt_1^{\text{nom}} - \tau) + sc_{1,1} + J_{3,2,1}^\circ(t_5)$$

that can be written as $f(pt_{1,3} + t_4) + g(pt_{1,3})$ being

$$f(pt_{1,3} + t_4) = 1.5 \cdot \max \{ pt_{1,3} + t_4 - 29, 0 \} + J_{3,2,1}^\circ(pt_{1,3} + t_4)$$

$$g(pt_{1,3}) = \begin{cases} 8 - pt_{1,3} & \text{if } pt_{1,3} \in [4, 8) \\ 0 & \text{if } pt_{1,3} \notin [4, 8) \end{cases}$$

The function $pt_{1,3}^\circ(t_4) = \arg \min_{pt_{1,3}} \{ f(pt_{1,3} + t_4) + g(pt_{1,3}) \}$, with $4 \leq pt_{1,3} \leq 8$, is determined by applying lemma [1]. It is (see figure 108)

$$pt_{1,3}^\circ(t_4) = \begin{cases} x_n(t_4) & t_4 < 16 \\ x_c(t_4) & t_4 \geq 16 \end{cases}$$

with $x_n(t_4) = \begin{cases} 8 & t_4 < 15 \\ -t_4 + 23 & 15 \leq t_4 < 16 \end{cases}$, and $x_c(t_4) = \begin{cases} 8 & 16 \leq t_4 < 21 \\ -t_4 + 29 & 21 \leq t_4 < 25 \\ 4 & t_4 \geq 25 \end{cases}$

**Case ii)** in which it is assumed $\delta_2 = 1$ and $\delta_1 = 0$.

In this case, it is necessary to minimize, with respect to the (continuos) decision variable $\tau$ which corresponds to the processing time $pt_{2,3}$, the following function

$$\alpha_{2,3} \max \{ t_4 + st_{1,2} + \tau - dd_{2,3}, 0 \} + \beta_2 (pt_2^{\text{nom}} - \tau) + sc_{1,2} + J_{2,3,2}^\circ(t_5)$$

Figure 107: Optimal control strategy $\tau^\circ(2, 2, 2, t_4)$ in state $[2 \ 2 \ 2 \ t_4]^T$.

Figure 108: Optimal processing time $pt_{1,3}^\circ(t_4)$, under the assumption $\delta_1 = 1$ in state $[2 \ 2 \ 1 \ t_4]^T$.

The conditioned cost-to-go $J_{2,3,2}^\circ(t_4 \mid \delta_1 = 1) = f(pt_{1,3}^\circ(t_4) + t_4) + g(pt_{1,3}^\circ(t_4))$, illustrated in figure 110, is provided by lemma [2]. It is specified by the initial value 1.5, by the set $\{ 15, 16, 17.5, 21, 25, 27 \}$ of abscissae $\gamma_i$, $i = 1, \ldots, 6$, at which the slope changes, and by the set $\{ 1, 0, 0.5, 1, 2, 3 \}$ of slopes $\mu_i$, $i = 1, \ldots, 6$, in the various intervals.
that can be written as $f(pt_{2,3} + t_4) + g(pt_{2,3})$ being

$$f(pt_{2,3} + t_4) = \max\{pt_{2,3} + t_4 - 37, 0\} + 0.5 + J^0_{2,3,2}(pt_{2,3} + t_4 + 1)$$

$$g(pt_{2,3}) = \begin{cases} 1.5 \cdot (6 - pt_{2,3}) & \text{if } pt_{2,3} \in [4, 6) \\ 0 & \text{if } pt_{2,3} \notin [4, 6) \end{cases}$$

The function $pt^2_{2,3}(t_4) = \arg\min_{pt_{2,3}}\{f(pt_{2,3} + t_4) + g(pt_{2,3})\}$, with $4 \leq pt_{2,3} \leq 6$, is determined by applying lemma 2. It is (see figure 109)

$$pt^2_{2,3}(t_4) = x_c(t_4) \quad \text{with} \quad x_c(t_4) = \begin{cases} 6 & t_4 < 17.5 \\ -t_4 + 23.5 & 17.5 \leq t_4 < 19.5 \\ 4 & t_4 \geq 19.5 \end{cases}$$

The conditioned cost-to-go $J^2_{2,2,1}(t_4 | \delta_2 = 1) = f(pt^2_{2,3}(t_4) + t_4) + g(pt^2_{2,3}(t_4))$, illustrated in figure 110, is provided by lemma 3. It is specified by the initial value 1.5, by the set $\{13.5, 17.5, 23.5, 33\}$ of abscissae $\gamma_i$, $i = 1, \ldots, 4$, at which the slope changes, and by the set $\{1, 1.5, 2, 3\}$ of slopes $\mu_i$, $i = 1, \ldots, 4$, in the various intervals.

In order to find the optimal cost-to-go $J^2_{2,2,1}(t_4)$, it is necessary to carry out the following minimization

$$J^2_{2,2,1}(t_4) = \min \{J^2_{2,2,1}(t_4 | \delta_1 = 1), J^2_{2,2,1}(t_4 | \delta_2 = 1)\}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 111.

The function $J^2_{2,2,1}(t_4)$ is specified by the initial value 0.5, by the set $\{15, 16, 17.5, 21, 25, 27\}$ of abscissae $\gamma_i$, $i = 1, \ldots, 6$, at which the slope changes, and by the set $\{1, 0, 0.5, 1, 2, 3\}$ of slopes $\mu_i$, $i = 1, \ldots, 6$, in the various intervals.

Since $J^2_{2,2,1}(t_4 | \delta_1 = 1)$ is always the minimum (see again figure 110), the optimal control strategies for this state are

$$\delta^*_1(2, 2, 1, t_4) = \begin{cases} 8 & t_4 < 15 \\ -t_4 + 23 & 15 \leq t_4 < 16 \\ 8 & 16 \leq t_4 < 21 \\ -t_4 + 29 & 21 \leq t_4 < 25 \\ 4 & t_4 \geq 25 \end{cases}$$

$$\tau^0(2, 2, 1, t_4) = \begin{cases} 8 & t_4 < 15 \\ -t_4 + 23 & 15 \leq t_4 < 16 \\ 8 & 16 \leq t_4 < 21 \\ -t_4 + 29 & 21 \leq t_4 < 25 \\ 4 & t_4 \geq 25 \end{cases}$$
The optimal control strategy \(\tau^o(2, 2, 1, t_4)\) is illustrated in figure 112.

**Stage 4 – State \([3 \ 1 \ 2 \ t_4]^T\) (S15)**

In state \([3 \ 1 \ 2 \ t_4]^T\), the cost function to be minimized, with respect to the (continuos) decision variable \(\tau\) and to the (binary) decision variables \(\delta_1\) and \(\delta_2\) is

\[
\delta_1 \left[ \alpha_{1,4} \max \{ t_4 + st_{2,1} + \tau - dd_{1,4}, 0 \} + \beta_1 (pt_1^{\text{nom}} - \tau) + sc_{2,1} + J_{4,1,1}^o(t_5) \right] + \\
\delta_2 \left[ \alpha_{2,2} \max \{ t_4 + st_{2,2} + \tau - dd_{2,2}, 0 \} + \beta_2 (pt_2^{\text{nom}} - \tau) + sc_{2,2} + J_{3,2,2}^o(t_5) \right]
\]

\(\delta_1 = 1\) and \(\delta_2 = 0\).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable \(\tau\) which corresponds to the processing time \(pt_{1,4}\), the following function

\[
f(pt_{1,4} + t_4) = 0.5 \cdot \max \{ pt_{1,4} + t_4 - 40.5, 0 \} + 1 + J_{4,1,1}^o(pt_{1,4} + t_4 + 0.5)
\]

\[
g(pt_{1,4}) = \begin{cases} 
8 - pt_{1,4} & \text{if } pt_{1,4} \in [4, 8] \\
0 & \text{if } pt_{1,4} \notin [4, 8]
\end{cases}
\]

The function \(pt_{1,4}^e(t_4) = \arg \min_{pt_{1,4}} \{ f(pt_{1,4} + t_4) + g(pt_{1,4}) \}\), with \(4 \leq pt_{1,4} \leq 8\), is determined by applying lemma [1]. It is (see figure 113).

\[
pt_{1,4}^e(t_4) = x_e(t_4) \quad \text{with} \quad x_e(t_4) = \begin{cases} 
8 & t_4 < 8.5 \\
-t_4 + 16.5 & 8.5 \leq t_4 < 12.5 \\
4 & t_4 \geq 12.5
\end{cases}
\]

The conditioned cost-to-go \(J_{3,1,2}^o(t_4 \mid \delta_1 = 1) = f(pt_{1,4}^e(t_4) + t_4) + g(pt_{1,4}^e(t_4))\), illustrated in figure 115, is provided by lemma [2]. It is specified by the initial value 1.5, by the set \{ 8.5, 20.5, 22.5, 36.5 \} of abscissae \(\gamma_i\), \(i = 1, \ldots, 4\), at which the slope changes, and by the set \{ 1, 1.5, 2.5 \} of slopes \(\mu_i\), \(i = 1, \ldots, 4\), in the various intervals.
at which the slope changes, and by the set provided by lemma 2. It is specified by the initial value 1, by the set

**Case ii)**

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{2,2}$, the following function

$$
\alpha_{2,2} \max\{t_4 + st_{2,2} + \tau - dd_{2,2}, 0\} + \beta_{2} (pt_{2,2}^{\text{nom}} - \tau) + sc_{2,2} + J_{3,2,2}^{\circ}(t_5)
$$

that can be written as $f(pt_{2,2} + t_4) + g(pt_{2,2})$ being

$$
f(pt_{2,2} + t_4) = \max\{pt_{2,2} + t_4 - 24, 0\} + J_{3,2,2}^{\circ}(pt_{2,2} + t_4)
$$

$$
g(pt_{2,2}) = \begin{cases} 
1.5 \cdot (6 - pt_{2,2}) & \text{if } pt_{2,2} \in [4, 6) \\
0 & \text{if } pt_{2,2} \notin [4, 6)
\end{cases}
$$

The function $pt_{2,2}^{\circ}(t_4) = \min_{pt_{2,2}} \{f(pt_{2,2} + t_4) + g(pt_{2,2})\}$, with $4 \leq pt_{2,2} \leq 6$, is determined by applying lemma \[1\]. It is (see figure 114)

$$
pt_{2,2}^{\circ}(t_4) = x_e(t_4) \quad \text{with} \quad x_e(t_4) = \begin{cases} 
6 & t_4 < 20.5 \\
-t_4 + 26.5 & 20.5 \leq t_4 < 22.5 \\
4 & t_4 \geq 22.5
\end{cases}
$$

**Figure 114:** Optimal processing time $pt_{2,2}^{\circ}(t_4)$, under the assumption $\delta_2 = 1$ in state $[3 1 2 t_4]^T$.

The conditioned cost-to-go $J_{3,1,2}^{\circ}(t_4 \mid \delta_2 = 1) = f(pt_{2,2}^{\circ}(t_4) + t_4) + g(pt_{2,2}^{\circ}(t_4))$, illustrated in figure 115, is provided by lemma \[2\]. It is specified by the initial value 1, by the set \{ 18, 20.5, 28 \} of abscissae $\gamma_i, i = 1, \ldots, 3$, at which the slope changes, and by the set \{ 1, 1.5, 2.5 \} of slopes $\mu_i, i = 1, \ldots, 3$, in the various intervals.

**Figure 115:** Conditioned costs-to-go $J_{3,1,2}^{\circ}(t_4 \mid \delta_1 = 1)$ and $J_{3,1,2}^{\circ}(t_4 \mid \delta_2 = 1)$ in state $[3 1 2 t_4]^T$.

In order to find the optimal cost-to-go $J_{3,1,2}^{\circ}(t_4)$, it is necessary to carry out the following minimization

$$
J_{3,1,2}^{\circ}(t_4) = \min \{ J_{3,1,2}^{\circ}(t_4 \mid \delta_1 = 1), J_{3,1,2}^{\circ}(t_4 \mid \delta_2 = 1) \}
$$

which provides, in accordance with lemma \[4\] the continuous, nondecreasing, piecewise linear function illustrated in figure 116.
The optimal control strategy

Stage

are

Since

the (binary) decision variables

In state

The function

at which the slope changes, and by the set

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Case i)

In this case, it is necessary to minimize, with respect to the (continuos) decision variable the processing time

The function

is always the minimum (see again figure [115], the optimal control strategies for this state are



The optimal control strategy \( \tau^o(3, 1, 2, t_4) \) is illustrated in figure [117]

![Figure 117: Optimal control strategy \( \tau^o(3, 1, 2, t_4) \) in state \([3 \ 1 \ 2 \ t_4]^T\).](image)

**Stage 4 – State \([3 \ 1 \ 1 \ t_4]^T\) (S14)**

In state \([3 \ 1 \ 1 \ t_4]^T\), the cost function to be minimized, with respect to the (continuos) decision variable \( \tau \) and to the (binary) decision variables \( \delta_1 \) and \( \delta_2 \) is

\[
\delta_1 \left[ \alpha_{1,4} \max\{t_4 + st_{1,1} + \tau - dd_{1,4}, 0\} + \beta_1 (pt_{1}^{nom} - \tau) + sc_{1,1} + J_{2,1,1}^o(t_5) \right] + \\
+ \delta_2 \left[ \alpha_{2,2} \max\{t_4 + st_{1,2} + \tau - dd_{2,2}, 0\} + \beta_2 (pt_{2}^{nom} - \tau) + sc_{1,2} + J_{3,2,2}^o(t_5) \right]
\]

**Case i)** in which it is assumed \( \delta_1 = 1 \) (and \( \delta_2 = 0 \)).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable \( \tau \) which corresponds to the processing time \( pt_{1,4} \), the following function

\[
\alpha_{1,4} \max\{t_4 + st_{1,1} + \tau - dd_{1,4}, 0\} + \beta_1 (pt_{1}^{nom} - \tau) + sc_{1,1} + J_{4,1,1}^o(t_5)
\]

that can be written as \( f(pt_{1,4} + t_4) + g(pt_{1,4}) \) being

\[
\begin{align*}
f(pt_{1,4} + t_4) &= 0.5 \cdot \max\{pt_{1,4} + t_4 - 41, 0\} + J_{4,1,1}^o(pt_{1,4} + t_4) \\
g(pt_{1,4}) &= \begin{cases} \\
8 - pt_{1,4} & pt_{1,4} \in [4, 8) \\
0 & pt_{1,4} \notin [4, 8) \end{cases}
\end{align*}
\]
The function $pt^{\infty}_{i,4}(t_4) = \arg \min_{pt_{1,4}} \{ f(pt_{1,4} + t_4) + g(pt_{1,4}) \}$, with $4 \leq pt_{1,4} \leq 8$, is determined by applying lemma\[1\] It is (see figure 118)

$$pt^{\infty}_{1,4}(t_4) = x_c(t_4) \quad \text{with} \quad x_c(t_4) = \begin{cases} 8 & t_4 < 9 \\ -t_4 + 17 & 9 \leq t_4 < 13 \\ 4 & t_4 \geq 13 \end{cases}$$

![Figure 118: Optimal processing time $pt^{\infty}_{i,4}(t_4)$, under the assumption $\delta_i = 1$ in state $[3 \ 1 \ 1 \ t_4]^T$.](image)

The conditioned cost-to-go $J^{\infty}_{0,1,1}(t_4 \mid \delta_1 = 1) = f(pt^{\infty}_{i,4}(t_4) + t_4) + g(pt^{\infty}_{i,4}(t_4))$, illustrated in figure 120 is provided by lemma[2] It is specified by the initial value 0.5, by the set \{ 9, 21, 23, 37 \} of abscissae $\gamma_1$, $i = 1, \ldots, 4$, at which the slope changes, and by the set \{ 1, 1.5, 2, 2.5 \} of slopes $\mu_i$, $i = 1, \ldots, 4$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{2,2}$, the following function

$$\alpha_{22} \max\{ t_4 + st_{1,2} + \tau - dd_{2,2}, 0 \} + \beta_2 (pt^{\infty}_{2,2} - \tau) + sc_{1,2} + J^{\infty}_{3,2,2}(t_5)$$

that can be written as $f(pt_{2,2} + t_4) + g(pt_{2,2})$ being

$$f(pt_{2,2} + t_4) = \max\{ pt_{2,2} + t_4 - 23, 0 \} + 0.5 + J^{\infty}_{5,2,2}(pt_{2,2} + t_4 + 1)$$

$$g(pt_{2,2}) = \begin{cases} 1.5 \cdot (6 - pt_{2,2}) & pt_{2,2} \in [4,6) \\ 0 & pt_{2,2} \notin [4,6) \end{cases}$$

The function $pt^{\infty}_{2,2}(t_4) = \arg \min_{pt_{2,2}} \{ f(pt_{2,2} + t_4) + g(pt_{2,2}) \}$, with $4 \leq pt_{2,2} \leq 6$, is determined by applying lemma[3] It is (see figure 119)

$$pt^{\infty}_{2,2}(t_4) = x_c(t_4) \quad \text{with} \quad x_c(t_4) = \begin{cases} 6 & t_4 < 19.5 \\ -t_4 + 25.5 & 19.5 \leq t_4 < 21.5 \\ 4 & t_4 \geq 21.5 \end{cases}$$

![Figure 119: Optimal processing time $pt^{\infty}_{2,2}(t_4)$, under the assumption $\delta_2 = 1$ in state $[3 \ 1 \ 1 \ t_4]^T$.](image)

The conditioned cost-to-go $J^{\infty}_{3,1,1}(t_4 \mid \delta_2 = 1) = f(pt^{\infty}_{2,2}(t_4) + t_4) + g(pt^{\infty}_{2,2}(t_4))$, illustrated in figure 120 is provided by lemma[2] It is specified by the initial value 1, by the set \{ 17, 19.5, 27 \} of abscissae $\gamma_1$, $i = 1, \ldots, 3$, at which the slope changes, and by the set \{ 1, 1.5, 2.5 \} of slopes $\mu_i$, $i = 1, \ldots, 3$, in the various intervals.

In order to find the optimal cost-to-go $J^{\infty}_{3,1,1}(t_4)$, it is necessary to carry out the following minimization

$$J^{\infty}_{3,1,1}(t_4) = \min \left\{ J^{\infty}_{3,1,1}(t_4 \mid \delta_1 = 1), J^{\infty}_{3,1,1}(t_4 \mid \delta_2 = 1) \right\}$$

which provides, in accordance with lemma[4] the continuous, nondecreasing, piecewise linear function illustrated in figure 121.
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Figure 120: Conditioned costs-to-go $J_{3,1,1}^0(t_4 | \delta_1 = 1)$ and $J_{3,1,1}^0(t_4 | \delta_2 = 1)$ in state $[3 1 1 t_4]^T$.

Figure 121: Optimal cost-to-go $J_{3,1,1}^0(t_4)$ in state $[3 1 1 t_4]^T$.

The function $J_{3,1,1}^0(t_4)$ is specified by the initial value 0.5, by the set \{ 9, 10, 17, 19.5, 27 \} of abscissae $\gamma_i$, $i = 1, \ldots, 5$, at which the slope changes, and by the set \{ 1, 0, 1, 1.5, 2.5 \} of slopes $\mu_i$, $i = 1, \ldots, 5$, in the various intervals.

Since $J_{3,1,1}^0(t_4 | \delta_1 = 1)$ is the minimum in $(-\infty, 10)$, and $J_{3,1,1}^0(t_4 | \delta_2 = 1)$ is the minimum in $[10, +\infty)$, the optimal control strategies for this state are

$$
\delta_1^*(3,1,1,t_4) = \begin{cases} 
1 & t_4 < 10 \\
0 & t_4 \geq 10
\end{cases} \quad \delta_2^*(3,1,1,t_4) = \begin{cases} 
0 & t_4 < 10 \\
1 & t_4 \geq 10
\end{cases}
$$

$$
\tau^*(3,1,1,t_4) = \begin{cases} 
8 & t_4 < 9 \\
-t_4 + 17 & 9 \leq t_4 < 24.5 \\
6 & 10 \leq t_4 < 19.5 \\
-t_4 + 25.5 & 19.5 \leq t_4 < 21.5 \\
4 & t_5 \geq 21.5
\end{cases}
$$

The optimal control strategy $\tau^*(3,1,1,t_4)$ is illustrated in figure 122.

Figure 122: Optimal control strategy $\tau^*(3,1,1,t_4)$ in state $[3 1 1 t_4]^T$.

Stage 4 – State $[4 0 1 t_4]^T$ (S13)

In state $[4 0 1 t_4]^T$ all jobs of class $P_1$ have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable $\tau$ only (which corresponds to the processing time $p_{2,1}$), is

$$
\alpha_{2,1} \max\{t_4 + st_{1,2} + \tau - dd_{2,1}, 0\} + \beta_2 (p_{2}^{nom} - \tau) + sc_{1,2} + J_{4,1,2}^0(t_5)
$$
that can be written as \( f(pt_{2,1} + t_4) + g(pt_{2,1}) \) being

\[
f(pt_{2,1} + t_4) = 2 \cdot \max\{pt_{2,1} + t_4 - 20, 0\} + 0.5 + J_{t_{1,2}}^0(pt_{2,1} + t_4 + 1)
\]

\[
g(pt_{2,1}) = \begin{cases} 
1.5 \cdot (6 - pt_{2,1}) & \text{if } pt_{2,1} \in [4, 6] \\
0 & \text{if } pt_{2,1} \notin [4, 6]
\end{cases}
\]

The function \( pt_{2,1}^o(t_4) = \arg \min_{pt_{2,1}} \{f(pt_{2,1} + t_4) + g(pt_{2,1})\} \), with \( 4 \leq pt_{2,1} \leq 6 \), is determined by applying lemma I. It is

\[
pt_{2,1}^o(t_4) = x_o(t_4) \quad \text{with} \quad x_o(t_4) = \begin{cases} 
6 & t_4 < 14 \\
-t_4 + 20 & 14 \leq t_4 < 16 \\
4 & t_4 \geq 16
\end{cases}
\]

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategy for this state are

\[
\delta^o(4, 0, 1, t_4) = \begin{cases} 
0 & \forall t_4 \\
1 & \forall t_4
\end{cases}
\]

\[
\tau^o(4, 0, 1, t_4) = \begin{cases} 
6 & t_4 < 14 \\
-t_4 + 20 & 14 \leq t_4 < 16 \\
4 & t_4 \geq 16
\end{cases}
\]

The optimal control strategy \( \tau^o(4, 0, 1, t_4) \) is illustrated in figure 123.

Figure 123: Optimal control strategy \( \tau^o(4, 0, 1, t_4) \) in state \([4 0 1 t_4]^T\).

The optimal cost-to-go \( J_{t_{1,2}}^o(t_4) = f(pt_{2,1}^o(t_4) + t_4) + g(pt_{2,1}^o(t_4)) \), illustrated in figure 124, is provided by lemma 2. It is specified by the initial value 0.5, by the set \( \{11, 14, 16, 21, 23\} \) of abscissae \( \gamma_i, i = 1, \ldots, 5 \), at which the slope changes, and by the set \( \{1, 1.5, 3, 3.5, 4\} \) of slopes \( \mu_i, i = 1, \ldots, 5 \), in the various intervals.

Figure 124: Optimal cost-to-go \( J_{t_{1,2}}^o(t_4) \) in state \([4 0 1 t_4]^T\).

Stage 3 – State \([0 3 2 t_3]^T\) (S12)

In state \([0 3 2 t_3]^T\) all jobs of class \( P_3 \) have been completed; then the decision about the class of the next job to be executed is mandatory. The cost function to be minimized in this state, with respect to the (continuous) decision variable \( \tau \) only (which corresponds to the processing time \( pt_{1,1} \)), is

\[
\alpha + \max\{t_3 + st_{2,1} + \tau - dd_{1,1}, 0\} + \beta_1 (pt_{1,2}^{nom} - \tau) + sc_{2,1} + J_{t_{1,3,1}}(t_4)
\]
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that can be written as \( f(pt_{1,1} + t_3) + g(pt_{1,1}) \) being

\[
f(pt_{1,1} + t_3) = 0.75 \cdot \max\{pt_{1,1} + t_3 - 18.5, 0\} + 1 + J^{o}_{1,3,1}(pt_{1,1} + t_4 + 0.5)
\]

\[
g(pt_{1,1}) = \begin{cases} 8 - pt_{1,1} & \text{if } pt_{1,1} \in [4,8) \\ 0 & \text{if } pt_{1,1} \notin [4,8) \end{cases}
\]

The function \( pt^{o}_{1,1}(t_3) = \arg \min_{pt_{1,1}} \{ f(pt_{1,1} + t_3) + g(pt_{1,1}) \} \), with \( 4 \leq pt_{1,1} \leq 8 \), is determined by applying lemma 1. It is

\[
pt^{o}_{1,1}(t_3) = x_{e}(t_3) \quad \text{with} \quad x_{e}(t_3) = \begin{cases} 8 & t_3 < 4.5 \\ -t_3 + 12.5 & 4.5 \leq t_3 < 8.5 \\ 4 & t_3 \geq 8.5 \end{cases}
\]

Taking into account the mandatory decision about the class of the next job to be executed, the optimal control strategies for this state are

\[
\delta^{1}(0,3,2,t_3) = 1 \quad \forall t_3 \quad \delta^{2}(0,3,2,t_3) = 0 \quad \forall t_3
\]

\[
\tau^{o}(0,3,2,t_3) = \begin{cases} 8 & t_3 < 4.5 \\ -t_3 + 12.5 & 4.5 \leq t_3 < 8.5 \\ 4 & t_3 \geq 8.5 \end{cases}
\]

The optimal control strategy \( \tau^{o}(0,3,2,t_3) \) is illustrated in figure 125.

![Figure 125: Optimal control strategy \( \tau^{o}(0,3,2,t_3) \) in state \([0 \ 3 \ 2 \ t_3]^T\).](image)

The optimal cost-to-go \( J^{o}_{0,3,2}(t_3) = f(pt^{o}_{1,1}(t_3) + t_4) + g(pt_{1,1}^{o}(t_3)) \), illustrated in figure 126, is provided by lemma 2. It is specified by the initial value 1, by the set \{ 4.5, 14.5, 15.5, 16.5, 20.5 \} of abscissae \( \gamma_i, i = 1, \ldots, 5 \), at which the slope changes, and by the set \{ 1, 1.75, 2.25, 2.75, 3.25 \} of slopes \( \mu_i, i = 1, \ldots, 5 \), in the various intervals.

![Figure 126: Optimal cost-to-go \( J^{o}_{0,3,2}(t_3) \) in state \([0 \ 3 \ 2 \ t_3]^T\).](image)

Stage 3 – State \([1 \ 2 \ 2 \ t_3]^T\) (S11)

In state \([1 \ 2 \ 2 \ t_3]^T\), the cost function to be minimized, with respect to the (continuous) decision variable \( \tau \) and to the (binary) decision variables \( \delta_i \) is

\[
\delta_{1} \left[ a_{1,2} \max\{ t_3 + st_{2,1} + \tau - dd_{1,2}, 0 \} + \beta_{1} (pt_{1}^{\text{nom}} - \tau) + sc_{2,1} + J^{o}_{2,2,1}(t_4) \right] + \\
\delta_{2} \left[ a_{2,3} \max\{ t_3 + st_{2,2} + \tau - dd_{2,3}, 0 \} + \beta_{2} (pt_{2}^{\text{nom}} - \tau) + sc_{2,2} + J^{o}_{1,3,2}(t_4) \right]
\]
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Case i) in which it is assumed \( \delta_1 = 1 \) (and \( \delta_2 = 0 \)).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \( \tau \) which corresponds to the processing time \( pt_{1,2} \), the following function

\[
\alpha_{1,2} \max\{t_3 + st_{2,1} + \tau -dd_{1,2} , 0\} + \beta_{1} (pt_{1}^\text{nom} - \tau) + sc_{2,1} + J^o_{2,2,1}(t_4)
\]

that can be written as \( f(pt_{1,2} + t_3) + g(pt_{1,2}) \) being

\[
f(pt_{1,2} + t_3) = 0.5 \cdot \max\{pt_{1,2} + t_3 - 23.5 , 0\} + 1 + J^o_{2,2,1}(pt_{1,2} + t_3 + 0.5)
\]

\[
g(pt_{1,2}) = \begin{cases} 8 - pt_{1,2} & pt_{1,2} \in [4,8) \\ 0 & pt_{1,2} \notin [4,8) \end{cases}
\]

The function \( pt_{1,2}^o(t_3) = \arg \min_{pt_{1,2}} \{ f(pt_{1,2} + t_3) + g(pt_{1,2}) \} \), with \( 4 \leq pt_{1,2} \leq 8 \), is determined by applying lemma \([1]\). It is (see figure \([27]\))

\[
pt_{1,2}^o(t_3) = \begin{cases} x_s(t_3) & t_3 < 7.5 \\ x_e(t_3) & t_3 \geq 7.5 \end{cases}
\]

with \( x_s(t_3) = \begin{cases} 8 & t_3 < 6.5 \\ -t_3 + 14.5 & 6.5 \leq t_3 < 7.5 \end{cases} \)

and \( x_e(t_3) = \begin{cases} 8 & 7.5 \leq t_3 < 12.5 \\ -t_3 + 20.5 & 12.5 \leq t_3 < 16.5 \\ 4 & t_3 \geq 16.5 \end{cases} \)

![Figure 127: Optimal processing time](image)

The conditioned cost-to-go \( J^o_{1,2,2}(t_3 | \delta_1 = 1) = f(pt_{1,2}^o(t_3) + t_3) + g(pt_{1,2}^o(t_3)) \), illustrated in figure \([29]\) is provided by lemma \([2]\). It is specified by the initial value 1, by the set \{ 6.5, 7.5, 9, 12.5, 19.5, 20.5, 22.5 \} of abscissae \( \gamma_i, i = 1, \ldots, 7 \), at which the slope changes, and by the set \{ 1, 0, 0.5, 1, 1.5, 2.5, 3.5 \} of slopes \( \mu_i, i = 1, \ldots, 7 \), in the various intervals.

Case ii) in which it is assumed \( \delta_2 = 1 \) (and \( \delta_1 = 0 \)).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \( \tau \) which corresponds to the processing time \( pt_{2,3} \), the following function

\[
\alpha_{2,3} \max\{t_3 + st_{2,2} + \tau -dd_{2,3} , 0\} + \beta_{2} (pt_{2}^\text{nom} - \tau) + sc_{2,2} + J^o_{1,3,2}(t_4)
\]

that can be written as \( f(pt_{2,3} + t_3) + g(pt_{2,3}) \) being

\[
f(pt_{2,3} + t_3) = \max\{pt_{2,3} + t_3 - 38 , 0\} + J^o_{1,3,2}(pt_{2,3} + t_3)
\]

\[
g(pt_{2,3}) = \begin{cases} 1.5 \cdot (6 - pt_{2,3}) & pt_{2,3} \in [4,6) \\ 0 & pt_{2,3} \notin [4,6) \end{cases}
\]

The function \( pt_{2,3}^o(t_3) = \arg \min_{pt_{2,3}} \{ f(pt_{2,3} + t_3) + g(pt_{2,3}) \} \), with \( 4 \leq pt_{2,3} \leq 6 \), is determined by applying lemma \([1]\). It is (see figure \([28]\))

\[
pt_{2,3}^o(t_3) = x_e(t_3) \quad \text{with} \quad x_e(t_3) = \begin{cases} 6 & t_3 < 13.5 \\ -t_3 + 19.5 & 13.5 \leq t_3 < 15.5 \\ 4 & t_3 \geq 15.5 \end{cases}
\]

The conditioned cost-to-go \( J^o_{1,2,2}(t_3 | \delta_2 = 1) = f(pt_{2,3}^o(t_3) + t_3) + g(pt_{2,3}^o(t_3)) \), illustrated in figure \([29]\) is provided by lemma \([2]\). It is specified by the initial value 1, by the set \{ 6.5, 13.5, 16.5, 20.5, 34 \} of abscissae \( \gamma_i \).
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Figure 128: Optimal processing time \( p^{\text{t}_2}_{1,2,3}(t_3) \), under the assumption \( \delta_2 = 1 \) in state \([1 \ 2 \ 2 \ t_3]^T\).

Figure 129: Conditioned costs-to-go \( J^{\text{t}_2}_{1,2,2}(t_3 \mid \delta_1 = 1) \) and \( J^{\text{t}_2}_{1,2,2}(t_3 \mid \delta_2 = 1) \) in state \([1 \ 2 \ 2 \ t_3]^T\).

\( \iota = 1, \ldots, 5 \), at which the slope changes, and by the set \( \{ 1, 1.5, 2, 2.5, 3.5 \} \) of slopes \( \mu_i \), \( i = 1, \ldots, 5 \), in the various intervals.

In order to find the optimal cost-to-go \( J^{\text{t}_2}_{1,2,2}(t_3) \), it is necessary to carry out the following minimization

\[
J^{\text{t}_2}_{1,2,2}(t_3) = \min \{ J^{\text{t}_2}_{1,2,2}(t_3 \mid \delta_1 = 1), J^{\text{t}_2}_{1,2,2}(t_3 \mid \delta_2 = 1) \}
\]

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 130.

Figure 130: Optimal cost-to-go \( J^{\text{t}_2}_{1,2,2}(t_3) \) in state \([1 \ 2 \ 2 \ t_3]^T\).

The function \( J^{\text{t}_2}_{1,2,2}(t_3) \) is specified by the initial value 1, by the set \( \{ 6.5, 8, 9, 12.5, 19.5, 20.5, 22.5, 30.25, 34 \} \) of abscissae \( \gamma_i \), \( i = 1, \ldots, 9 \), at which the slope changes, and by the set \( \{ 1, 0, 0.5, 1, 1.5, 2.5, 3.5, 2.5, 3.5 \} \) of slopes \( \mu_i \), \( i = 1, \ldots, 9 \), in the various intervals.

Since \( J^{\text{t}_2}_{1,2,2}(t_3 \mid \delta_1 = 1) \) is the minimum in \([8, 30.25]\), and \( J^{\text{t}_2}_{1,2,2}(t_3 \mid \delta_2 = 1) \) is the minimum in \(( -\infty, 8) \) and in \([30.25, +\infty)\), the optimal control strategies for this state are

\[
\delta^*_1(1, 2, 2, t_3) = \begin{cases} 0 & t_3 < 8 \\ 1 & 8 \leq t_3 < 30.25 \\ 0 & t_3 \geq 30.25 \end{cases} \quad \delta^*_2(1, 2, 2, t_3) = \begin{cases} 1 & t_3 < 8 \\ 0 & 8 \leq t_3 < 30.25 \\ 1 & t_3 \geq 30.25 \end{cases}
\]

\[
\tau^*(1, 2, 2, t_3) = \begin{cases} 6 & t_3 < 8 \\ 8 & 8 \leq t_3 < 12.5 \\ -t_3 + 20.5 & 12.5 \leq t_3 < 16.5 \\ 4 & t_3 \geq 16.5 \end{cases}
\]
The optimal control strategy $\tau^o(1, 2, 2, t_3)$ is illustrated in figure 131.

![Figure 131: Optimal control strategy $\tau^o(1, 2, 2, t_3)$ in state $[1 2 2 t_3]^T$.](image)

**Stage 3 – State $[1 2 1 t_3]^T$ (S10)**

In state $[1 2 1 t_3]^T$, the cost function to be minimized, with respect to the (continuous) decision variable $\tau$ and to the (binary) decision variables $\delta_1$ and $\delta_2$ is

$$
\begin{align*}
\delta_1 \left[ \alpha_{1,2} \max\{t_3 + st_{1,1} + \tau - dd_{1,2}, 0\} + \beta_1 (pt_{1,2}^{\text{nom}} - \tau) + s c_{1,1} + J_{2,2,1}^g(t_4) \right] + \\
+ \delta_2 \left[ \alpha_{2,3} \max\{t_3 + st_{1,2} + \tau - dd_{2,3}, 0\} + \beta_2 (pt_{2,3}^{\text{nom}} - \tau) + s c_{1,2} + J_{3,3,2}^g(t_4) \right]
\end{align*}
$$

Case i) in which it is assumed $\delta_1 = 1$ and $\delta_2 = 0$.

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{1,2}$, the following function

$$
\alpha_{1,2} \max\{t_3 + st_{1,1} + \tau - dd_{1,2}, 0\} + \beta_1 (pt_{1,2}^{\text{nom}} - \tau) + s c_{1,1} + J_{2,2,1}^g(t_4)
$$

that can be written as $f(pt_{1,2} + t_3) + g(pt_{1,2})$ being

$$
f(pt_{1,2} + t_3) = 0.5 \cdot \max\{pt_{1,2} + t_3 - 24, 0\} + J_{2,2,1}^g(pt_{1,2} + t_3)
$$

$$
g(pt_{1,2}) = \begin{cases} 8 - pt_{1,2} & pt_{1,2} \in [4, 8] \\ 0 & pt_{1,2} \notin [4, 8] \end{cases}
$$

The function $pt_{1,2}^o(t_3)$ is the minimum of $f(pt_{1,2} + t_3) + g(pt_{1,2})$, with $4 \leq pt_{1,2} \leq 8$, determined by applying lemma [8]. It is (see figure 132)

$$
pt_{1,2}^o(t_3) = \begin{cases} x_1(t_3) & t_3 < 8 \\ x_2(t_3) & t_3 \geq 8 \end{cases}
$$

with $x_1(t_3) = \begin{cases} 8 & 8 \leq t_3 < 13 \\ -t_3 + 21 & 13 \leq t_3 < 17 \\ 4 & t_3 \geq 17 \end{cases}$

and $x_2(t_3) = \begin{cases} 8 & \text{if } t_3 < 7 \\ -t_3 + 15 & 7 \leq t_3 < 8 \end{cases}$

![Figure 132: Optimal processing time $pt_{1,2}^o(t_3)$, under the assumption $\delta_1 = 1$ in state $[1 2 1 t_3]^T$.](image)

The conditioned cost-to-go $J_{2,2,1}^g(t_3 | \delta_1 = 1) = f(pt_{1,2}^o(t_3) + t_3) + g(pt_{1,2}^o(t_3))$, illustrated in figure 134, is provided by lemma [2]. It is specified by the initial value 0.5, by the set { 7, 8, 9.5, 13, 20, 21, 23 } of abscissae $\gamma_i$, $i = 1, \ldots, 7$, at which the slope changes, and by the set { 1, 0, 0.5, 1, 1.5, 2.5, 3.5 } of slopes $\mu_i$, $i = 1, \ldots, 7$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ and $\delta_1 = 0$.

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{2,3}$, the following function

$$
\alpha_{2,3} \max\{t_3 + st_{1,2} + \tau - dd_{2,3}, 0\} + \beta_2 (pt_{2,3}^{\text{nom}} - \tau) + s c_{1,2} + J_{3,3,2}^g(t_4)
$$
that can be written as \( f(pt_{2,3} + t_3) + g(pt_{2,3}) \) being
\[
f(pt_{2,3} + t_3) = \max\{pt_{2,3} + t_3 - 37,0\} + 0.5 + J_{1,3,2}^0(pt_{2,3} + t_3 + 1)
\]
\[
g(pt_{2,3}) = \begin{cases} 
1.5 \cdot (6 - pt_{2,3}) & pt_{2,3} \in [4,6) \\
0 & pt_{2,3} \notin [4,6)
\end{cases}
\]
The function \( pt_{2,3}^o(t_3) = \arg \min_{pt_{2,3}} \{ f(pt_{2,3} + t_3) + g(pt_{2,3}) \} \), with \( 4 \leq pt_{2,3} \leq 6 \), is determined by applying lemma[1]. It is (see figure 133)

\[
pt_{2,3}^o(t_3) = x_c(t_3) \quad \text{with} \quad x_c(t_3) = \begin{cases} 
6 & t_3 < 12.5 \\
-t_3 + 18.5 & 12.5 \leq t_3 < 14.5 \\
4 & t_3 \geq 14.5
\end{cases}
\]

Figure 133: Optimal processing time \( pt_{2,3}^o(t_3) \), under the assumption \( \delta_2 = 1 \) in state \([1 2 1 t_3]^T\).

The conditioned cost-to-go \( J_{1,2,1}^0(t_3 \mid \delta_2 = 1) = f(pt_{2,3}^o(t_3) + t_3) + g(pt_{2,3}^o(t_3)) \), illustrated in figure 134, is provided by lemma[2]. It is specified by the initial value 1.5, by the set \{ 5.5, 12.5, 15.5, 19.5, 33 \} of abscissae \( \gamma_i, i = 1, \ldots, 5 \), at which the slope changes, and by the set \{ 1, 1.5, 2, 2.5, 3.5 \} of slopes \( \mu_i, i = 1, \ldots, 5 \), in the various intervals.

In order to find the optimal cost-to-go \( J_{1,2,1}^0(t_3) \), it is necessary to carry out the following minimization
\[
J_{1,2,1}^0(t_3) = \min \{ J_{1,2,1}^0(t_3 \mid \delta_1 = 1), J_{1,2,1}^0(t_3 \mid \delta_2 = 1) \}
\]
which provides, in accordance with lemma[4] the continuous, nondecreasing, piecewise linear function illustrated in figure 135.

The function \( J_{1,2,1}^0(t_3) \) is specified by the initial value 0.5, by the set \{ 7, 8, 9.5, 13, 20, 21, 23 \} of abscissae \( \gamma_i, i = 1, \ldots, 7 \), at which the slope changes, and by the set \{ 1, 0, 0.5, 1, 1.5, 2.5, 3.5 \} of slopes \( \mu_i, i = 1, \ldots, 7 \), in the various intervals.

Since \( J_{1,2,1}^0(t_3 \mid \delta_1 = 1) \) is always the minimum (see again figure 134), the optimal control strategies for this state are

\[
\delta_1^o(1, 2, 1, t_3) = 1 \quad \forall t_3 \quad \delta_2^o(1, 2, 1, t_3) = 0 \quad \forall t_3
\]

\[
\tau^o(1, 2, 1, t_3) = \begin{cases} 
8 & t_3 < 7 \\
-t_3 + 15 & 7 \leq t_3 < 8 \\
8 & 8 \leq t_3 < 13 \\
-t_3 + 21 & 13 \leq t_3 < 17 \\
4 & t_3 \geq 17
\end{cases}
\]
The optimal control strategy \( \tau^*(1, 2, 1, t_3) \) is illustrated in figure [136]

**Stage 3 – State \([2 \ 1 \ 2 \ t_3]^T\) \((59)\)**

In state \([2 \ 1 \ 2 \ t_3]^T\), the cost function to be minimized, with respect to the (continuous) decision variable \( \tau \) and to the (binary) decision variables \( \delta_1 \) and \( \delta_2 \) is

\[
\begin{align*}
\delta_1 & [\alpha_{1,3} \max\{t_3 + st_{2,1} + \tau - dd_{1,3}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{2,1} + J_{3,1,1}^0(t_4)] + \\
& + \delta_2 [\alpha_{2,2} \max\{t_3 + st_{2,2} + \tau - dd_{2,2}, 0\} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + sc_{2,2} + J_{2,2,2}^0(t_4)]
\end{align*}
\]

*Case i)* in which it is assumed \( \delta_1 = 1 \) (and \( \delta_2 = 0 \)).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \( \tau \) which corresponds to the processing time \( pt_{1,3} \), the following function

\[
\alpha_{1,3} \max\{t_3 + st_{2,1} + \tau - dd_{1,3}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{2,1} + J_{3,1,1}^0(t_4)
\]

that can be written as \( f(pt_{1,3} + t_3) + g(pt_{1,3}) \) being

\[
f(pt_{1,3} + t_3) = 1.5 \cdot \max\{pt_{1,3} + t_3 - 28.5, 0\} + 1 + J_{3,1,1}^0(pt_{1,3} + t_3 + 0.5)
\]

\[
g(pt_{1,3}) = \begin{cases} 
8 - pt_{1,3} & pt_{1,3} \in [4,8) \\
0 & pt_{1,3} \notin [4,8) 
\end{cases}
\]

The function \( pt_{1,3}^o(t_3) = \arg \min_{pt_{1,3}} \{ f(pt_{1,3} + t_3) + g(pt_{1,3}) \} \), with \( 4 \leq pt_{1,3} \leq 8 \), is determined by applying lemma[1] It is (see figure[137])

\[
pt_{1,3}^o(t_3) = \begin{cases} 
8 & t_3 < 1.5 \\
x_a(t_3) & 1.5 \leq t_3 < 8.5 \\
x_e(t_3) & 8.5 \leq t_3 < 12.5 \\
4 & t_3 \geq 12.5 
\end{cases}
\]

with \( x_a(t_3) = \begin{cases} 
8 & t_3 < 0.5 \\
-t_3 + 8.5 & 0.5 \leq t_3 < 1.5 
\end{cases} \)

and \( x_e(t_3) = \begin{cases} 
8 & t_3 < 1.5 \\
-t_3 + 16.5 & 1.5 \leq t_3 < 8.5 \\
8.5 & t_3 \geq 8.5 
\end{cases} \)

The conditioned cost-to-go \( J_{2,1,2}^o(t_3 | \delta_1 = 1) = f(pt_{3,3}^o(t_3) + t_3) + g(pt_{3,3}^o(t_3)) \), illustrated in figure [139] is provided by lemma[2] It is specified by the initial value 1.5, by the set \{ 0.5, 1.5, 8.5, 15, 22.5, 24.5 \} of abscissae
Provided by lemma 2. It is specified by the initial value 1, by the set $\gamma_i, i = 1, \ldots, 6$, at which the slope changes, and by the set \{1, 0, 1, 1.5, 2.5, 4\} of slopes $\mu_i, i = 1, \ldots, 6$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{2,2}$, the following function

$$\alpha_{2,2} \max \{t_3 + st_{2,2} + \tau - d_{2,2} \cdot 0\} + \beta_2 (pt_{2,2}^{\text{nom}} - \tau) + sc_{2,2} + J^2_{2,2,2}(t_4)$$

that can be written as $f(pt_{2,2} + t_3) + g(pt_{2,2})$ being

$$f(pt_{2,2} + t_3) = \max \{pt_{2,2} + t_3 - 24, 0\} + J^2_{2,2,2}(pt_{2,2} + t_3)$$

$$g(pt_{2,2}) = \begin{cases} 1.5 \cdot (6 - pt_{2,2}) & pt_{2,2} \in [4, 6] \\ 0 & pt_{2,2} \notin [4, 6] \end{cases}$$

The function $pt_{2,2}^2(t_3) = \arg \min_{pt_{2,2}} \{f(pt_{2,2} + t_3) + g(pt_{2,2})\}$, with $4 \leq pt_{2,2} \leq 6$, is determined by applying lemma[1]. It is (see figure 138)

$$pt_{2,2}^2(t_3) = x_e(t_3) \quad \text{with} \quad x_e(t_3) = \begin{cases} 6 & t_3 < 18 \\ -t_3 + 24 & 18 \leq t_3 < 20 \\ 4 & t_3 \geq 20 \end{cases}$$

The conditioned cost-to-go $J^2_{2,1,2}(t_3 | \delta_2 = 1) = f(pt_{2,2}^2(t_3) + t_3) + g(pt_{2,2}^2(t_3))$, provided by lemma[2]. It is specified by the initial value 1, by the set \{8.5, 10, 11, 14.5, 18, 20, 20.5, 22.5, 28.25, 30\} of abscissae $\gamma_i, i = 1, \ldots, 10$, at which the slope changes, and by the set \{1, 0, 0.5, 1, 1.5, 2, 3, 4, 3, 4\} of slopes $\mu_i, i = 1, \ldots, 10$, in the various intervals.

In order to find the optimal cost-to-go $J^2_{2,1,2}(t_3)$, it is necessary to carry out the following minimization

$$J^2_{2,1,2}(t_3) = \min \{J^2_{2,1,2}(t_3 | \delta_1 = 1), J^2_{2,1,2}(t_3 | \delta_2 = 1)\}$$
which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 140.

![Optimal cost-to-go $J_{2,1,2}(t_3)$ in state $[2 1 2 t_3]^T$.](image)

The function $J_{2,1,2}(t_3)$ is specified by the initial value 1, by the set \{ 8.5, 10, 11, 14.5, 18, 20, 20.5, 22.5, 24.16, 24.5, 28.75, 30 \} of abscissae $\gamma_i$, $i = 1, \ldots, 12$, at which the slope changes, and by the set \{ 1, 0, 0.5, 1, 1.5, 2, 3, 4, 2.5, 4, 3, 4 \} of slopes $\mu_i$, $i = 1, \ldots, 10$, in the various intervals.

Since $J_{2,1,2}(t_3 | \delta_1 = 1)$ is the minimum in [24.16, 28.75], and $J_{2,1,2}(t_3 | \delta_2 = 1)$ is the minimum in $(-\infty, 24.16)$ and in [28.75, $+\infty$], the optimal control strategies for this state are

$$
\delta_1^\ast(2, 1, 2, t_3) = \begin{cases} 0 & t_3 < 24.16 \\ 1 & 24.16 \leq t_3 < 28.75 \\ 0 & t_3 \geq 28.75 \end{cases}
$$

$$
\delta_2^\ast(2, 1, 2, t_3) = \begin{cases} 1 & t_3 < 24.16 \\ 0 & 24.16 \leq t_3 < 28.75 \\ 1 & t_3 \geq 28.75 \end{cases}
$$

$$
\tau^\ast(2, 1, 2, t_3) = \begin{cases} 6 & t_3 < 18 \\ -t_3 + 24 & 18 \leq t_3 < 20 \\ 4 & t_3 \geq 20 \end{cases}
$$

The optimal control strategy $\tau^\ast(2, 1, 2, t_3)$ is illustrated in figure 141.

![Optimal control strategy $\tau^\ast(2, 1, 2, t_3)$ in state $[2 1 2 t_3]^T$.](image)

Stage 3 – State $[2 1 1 t_3]^T$ (S8)

In state $[2 1 1 t_3]^T$, the cost function to be minimized, with respect to the (continuous) decision variable $\tau$ and to the (binary) decision variables $\delta_1$ and $\delta_2$ is

$$
\delta_1 [\alpha_{1,3} \max \{t_3 + st_{1,1} + \tau - dd_{1,3}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{1,1} + J_{3,1,1}(t_4)] + \\
+ \delta_2 [\alpha_{2,2} \max \{t_3 + st_{1,2} + \tau - dd_{2,2}, 0\} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + sc_{1,2} + J_{2,2,2}(t_4)]
$$

Case i) in which it is assumed $\delta_1 = 1$ and $\delta_2 = 0$.

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{1,3}$, the following function

$$
\alpha_{1,3} \max \{t_3 + st_{1,1} + \tau - dd_{1,3}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{1,1} + J_{3,1,1}(t_4)
$$

that can be written as $f(pt_{1,3} + t_3) + g(pt_{1,3})$ being

$$
f(pt_{1,3} + t_3) = 1.5 \cdot \max \{pt_{1,3} + t_3 - 29, 0\} + J_{3,1,1}(pt_{1,3} + t_3)
$$
The function $p^{\circ}_{t_{1,3}}(t_3) = \arg\min_{p_{t_{1,3}}} \{ f(p_{t_{1,3}} + t_3) + g(p_{t_{1,3}}) \}$, with $4 \leq p_{t_{1,3}} \leq 8$, is determined by applying lemma [1]. It is (see figure 142)

$$

\begin{align*}
  g(pt_{1,3}) = \begin{cases} 
  8 - pt_{1,3} & pt_{1,3} \in [4, 8) \\
  0 & pt_{1,3} \notin [4, 8)
  \end{cases}
\end{align*}

$$

The function $p^{\circ}_{t_{1,3}}(t_3) = \arg\min_{p_{t_{1,3}}} \{ f(pt_{1,3} + t_3) + g(pt_{1,3}) \}$, with $4 \leq pt_{1,3} \leq 8$, is determined by applying lemma [1]. It is (see figure 142)

$$

\begin{align*}
  p^{\circ}_{t_{1,3}}(t_3) &= \begin{cases} 
  x_e(t_3) & t_3 < 2 \\
  x_e(t_3) & t_3 \geq 2
  \end{cases} \\
  \text{with } x_e(t_3) &= \begin{cases} 
  8 & t_3 < 1 \\
  -t_3 + 9 & 1 \leq t_3 < 2
  \end{cases}
\end{align*}

\begin{align*}
  \text{and } x_e(t_3) &= \begin{cases} 
  8 & 1.5 \leq t_3 < 9 \\
  -t_3 + 17 & 9 \leq t_3 < 13 \\
  4 & t_3 \geq 13
  \end{cases}
\end{align*}

Figure 142: Optimal processing time $p^{\circ}_{t_{1,3}}(t_3)$, under the assumption $\delta_1 = 1$ in state $[2 \ 1 \ 1 \ t_3]^T$.

The conditioned cost-to-go $J_{2,1,1}^2(t_2 \mid \delta_1 = 1) = f(p^{\circ}_{t_{2,2}}(t_2) + t_3) + g(p^{\circ}_{t_{2,2}}(t_3))$, illustrated in figure 144, is provided by lemma [2]. It is specified by the initial value 0.5, by the set $\{ 1, 2, 9, 15.5, 23, 25 \}$ of abscissae $\gamma_i$, $i = 1, \ldots, 6$, at which the slope changes, and by the set $\{ 1, 0, 1, 1.5, 2.5, 4 \}$ of slopes $\mu_i$, $i = 1, \ldots, 6$, in the various intervals.

**Case ii)** in which it is assumed $\delta_2 = 1$ and $\delta_1 = 0$.

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{2,2}$, the following function

$$

\alpha_{2,2} \max\{t_3 + st_{1,2} + \tau - dd_{2,2}, 0\} + \beta_2 (pt_{2,2}^{\text{nom}} - \tau) + sc_{1,2} + J_{2,2,2}^2(t_4)
$$

that can be written as $f(pt_{2,2} + t_3) + g(pt_{2,2})$ being

$$

f(pt_{2,2} + t_3) = \max\{pt_{2,2} + t_3 - 23, 0\} + 0.5 + J_{2,2,2}^2(pt_{2,2} + t_3 + 1)
$$

and

$$

\begin{align*}
  g(pt_{2,2}) &= \begin{cases} 
  1.5 \cdot (6 - pt_{2,2}) & pt_{2,2} \in [4, 6) \\
  0 & pt_{2,2} \notin [4, 6)
  \end{cases}
\end{align*}

$$

The function $p^{\circ}_{t_{2,2}}(t_3) = \arg\min_{pt_{2,2}} \{ f(pt_{2,2} + t_3) + g(pt_{2,2}) \}$, with $4 \leq pt_{2,2} \leq 6$, is determined by applying lemma [1]. It is (see figure 143)

$$

\begin{align*}
  p^{\circ}_{t_{2,2}}(t_3) &= x_e(t_3) \\
  \text{with } x_e(t_3) &= \begin{cases} 
  6 & t_3 < 17 \\
  -t_3 + 23 & 17 \leq t_3 < 19 \\
  4 & t_3 \geq 19
  \end{cases}
\end{align*}

$$

Figure 143: Optimal processing time $p^{\circ}_{t_{2,2}}(t_3)$, under the assumption $\delta_2 = 1$ in state $[2 \ 1 \ 1 \ t_3]^T$.

The conditioned cost-to-go $J_{2,1,1}^2(t_2 \mid \delta_2 = 1) = f(p^{\circ}_{t_{2,2}}(t_2) + t_3) + g(p^{\circ}_{t_{2,2}}(t_3))$, illustrated in figure 144, is provided by lemma [2]. It is specified by the initial value 1.5, by the set $\{ 7.5, 9, 10, 13.5, 17, 19, 19.5, 21.5, 27.25, 29 \}$ of abscissae $\gamma_i$, $i = 1, \ldots, 10$, at which the slope changes, and by the set $\{ 1, 0, 0.5, 1, 1.5, 2, 3, 4, 3, 4 \}$ of slopes $\mu_i$, $i = 1, \ldots, 10$, in the various intervals.
Figure 144: Conditioned costs-to-go $J_{2,1,1}^2(t_3 \mid \delta_1 = 1)$ and $J_{2,1,1}^2(t_3 \mid \delta_2 = 1)$ in state $[2 \ 1 \ 1 \ t_3]^T$.

In order to find the optimal cost-to-go $J_{2,1,1}^2(t_3)$, it is necessary to carry out the following minimization

$$J_{2,1,1}^2(t_3) = \min \{ J_{2,1,1}^2(t_3 \mid \delta_1 = 1), J_{2,1,1}^2(t_3 \mid \delta_2 = 1) \}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 145.

Figure 145: Optimal cost-to-go $J_{2,1,1}^2(t_3)$ in state $[2 \ 1 \ 1 \ t_3]^T$.

The function $J_{2,1,1}^2(t_3)$ is specified by the initial value 0.5, by the set $\{ 1, 2, 9, 11, 13.5, 17, 19.5, 20, 23, 25 \}$ of abscissae $\gamma_i$, $i = 1, \ldots, 11$, at which the slope changes, and by the set $\{ 1, 0, 1, 0.5, 1, 1.5, 2, 3, 1.5, 2.5, 4 \}$ of slopes $\mu_i$, $i = 1, \ldots, 11$, in the various intervals.

Since $J_{2,1,1}^2(t_3 \mid \delta_1 = 1)$ is the minimum in $(-\infty, 2)$, in $[9, 11]$, and in $[20, 23, +\infty)$, and $J_{2,1,1}^2(t_3 \mid \delta_2 = 1)$ is the minimum in $[2, 9]$ and in $[11, 20, 23]$, the optimal control strategies for this state are

$$\delta^2_1(2,1,1,t_3) = \begin{cases} 1 & t_3 < 2 \\ 0 & 2 \leq t_3 < 9 \\ 1 & 9 \leq t_3 < 11 \\ 0 & 11 \leq t_3 < 20 \bar{a} \\ 1 & t_3 \geq 20 \bar{a} \end{cases} \quad \delta^2_2(2,1,1,t_3) = \begin{cases} 0 & t_3 < 2 \\ 1 & 2 \leq t_3 < 9 \\ 0 & 9 \leq t_3 < 11 \\ 1 & 11 \leq t_3 < 20 \bar{a} \\ 0 & t_3 \geq 20 \bar{a} \end{cases}$$

$$\tau^2(2,1,1,t_3) = \begin{cases} 8 & t_3 < 1 \\ -t_3 + 9 & 1 \leq t_3 < 2 \\ 6 & 2 \leq t_3 < 9 \\ -t_3 + 17 & 9 \leq t_3 < 11 \\ 6 & 11 \leq t_3 < 17 \\ -t_3 + 23 & 17 \leq t_3 < 19 \\ 4 & t_3 \geq 19 \end{cases}$$

The optimal control strategy $\tau^2(2,1,1,t_3)$ is illustrated in figure 146.
Case ii) In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$, the following function

$$\delta_1 [\alpha_{1,4} \max\{t_3 + st_{1,1} + \tau - \tau - dd_{1,4}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{1,1} + J_{4,0,1}^{\text{nom}}(t_4)] +$$

$$+ \delta_2 [\alpha_{2,1} \max\{t_3 + st_{1,2} + \tau - \tau - dd_{2,1}, 0\} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + sc_{1,2} + J_{3,1,2}^{\text{nom}}(t_4)]$$

**Stage 3 – State $[3 \ 0 \ 1 \ t_3]^T$ (S7)**

In state $[3 \ 0 \ 1 \ t_3]^T$, the cost function to be minimized, with respect to the (continuous) decision variable $\tau$ and to the (binary) decision variables $\delta_1$ and $\delta_2$ is

$$\delta_1 [\alpha_{1,4} \max\{t_3 + st_{1,1} + \tau - dd_{1,4}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{1,1} + J_{4,0,1}^{\text{nom}}(t_4)] +$$

$$+ \delta_2 [\alpha_{2,1} \max\{t_3 + st_{1,2} + \tau - dd_{2,1}, 0\} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + sc_{1,2} + J_{3,1,2}^{\text{nom}}(t_4)]$$

**Case i)** in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{1,4}$, the following function

$$\alpha_{1,4} \max\{t_3 + st_{1,1} + \tau - dd_{1,4}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{1,1} + J_{4,0,1}^{\text{nom}}(t_4)$$

that can be written as $f(pt_{1,4} + t_3) + g(pt_{1,4})$ being

$$f(pt_{1,4} + t_3) = 0.5 \cdot \max\{pt_{1,4} + t_3 - 41, 0\} + J_4^{nom}(pt_{1,4} + t_3)$$

$$g(pt_{1,4}) = \begin{cases} 8 - pt_{1,4} & \text{if } pt_{1,4} \in [4, 8] \\ 0 & \text{if } pt_{1,4} \notin [4, 8] \end{cases}$$

The function $pt_{1,4}^{\text{opt}}(t_3) = \arg \min_{pt_{1,4}} \{ f(pt_{1,4} + t_3) + g(pt_{1,4}) \}$, with $4 \leq pt_{1,4} \leq 8$, is determined by applying lemma [1]. It is (see figure 147)

$$pt_{1,4}^{\text{opt}}(t_3) = x(t_3) \quad \text{with} \quad x(t_3) = \begin{cases} 8 & t_3 < 3 \\ -t_3 + 23 & 3 \leq t_3 < 7 \\ 4 & t_3 \geq 7 \end{cases}$$

Figure 147: Optimal processing time $pt_{1,4}^{\text{opt}}(t_3)$, under the assumption $\delta_1 = 1$ in state $[3 \ 0 \ 1 \ t_3]^T$.

The conditioned cost-to-go $J_{3,0,1}^{\text{nom}}(t_3 \mid \delta_1 = 1) = f(pt_{1,4}^{\text{opt}}(t_3) + t_3) + g(pt_{1,4}^{\text{opt}}(t_3))$, illustrated in figure 149, is provided by lemma [2]. It is specified by the initial value 0.5, by the set $\{ 3, 10, 12, 17, 19, 37 \}$ of abscissae $\gamma_i$, $i = 1, \ldots, 6$, at which the slope changes, and by the set $\{ 1, 1.5, 3, 3.5, 4, 4.5 \}$ of slopes $\mu_i$, $i = 1, \ldots, 6$, in the various intervals.

**Case ii)** in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{2,1}$, the following function

$$\alpha_{2,1} \max\{t_3 + st_{1,2} + \tau - dd_{2,1}, 0\} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + sc_{1,2} + J_{3,1,2}^{\text{nom}}(t_4)$$

that can be written as $f(pt_{2,1} + t_3) + g(pt_{2,1})$ being

$$f(pt_{2,1} + t_3) = 2 \cdot \max\{pt_{2,1} + t_3 - 20, 0\} + 0.5 + J_{3,1,2}^{\text{nom}}(pt_{2,1} + t_3 + 1)$$

Figure 146: Optimal processing time $\tau^\circ(2, 1, 1, t_3)$ in state $[2 \ 1 \ 1 \ t_3]^T$. 

![Figure 146: Optimal processing time $\tau^\circ(2, 1, 1, t_3)$ in state $[2 \ 1 \ 1 \ t_3]^T$.](image-url)
Fundamental lemmas for the determination of optimal control strategies for a class of single machine family scheduling problems

\[ g(p_{t,2,1}) = \begin{cases} 
1.5 \cdot (6 - p_{t,2,1}) & p_{t,2,1} \in [4, 6) \\
0 & p_{t,2,1} \notin [4, 6) 
\end{cases} \]

The function \( p_{t,2,1}(t_3) = \arg \min_{p_{t,2,1}} \{ f(p_{t,2,1} + t_3) + g(p_{t,2,1}) \} \), with \( 4 \leq p_{t,2,1} \leq 6 \), is determined by applying lemma 1. It is (see figure 148)

\[ p_{t,2,1}(t_3) = x_e(t_3) \quad \text{with} \quad x_e(t_3) = \begin{cases} 
6 & t_3 < 13.5 \\
-t_3 + 19.5 & 13.5 \leq t_3 < 15.5 \\
4 & t_3 \geq 15.5 
\end{cases} \]

Figure 148: Optimal processing time \( p_{t,2,1}(t_3) \), under the assumption \( \delta_2 = 1 \) in state \([3 \ 0 \ 1 \ t_3]^T\).

The conditioned cost-to-go \( J_{3,0,1}^o(t_3 \mid \delta_1 = 1) = f(p_{t,2,1}(t_3) + t_3) + g(p_{t,2,1}(t_3)) \), illustrated in figure 149, is provided by lemma 2. It is specified by the initial value 1.5, by the set \{ 11, 13.5, 16, 23 \} of abscissae \( \gamma_i \), \( i = 1, \ldots, 4 \), at which the slope changes, and by the set \{ 1, 1.5, 3.5, 4.5 \} of slopes \( \mu_i \), \( i = 1, \ldots, 4 \), in the various intervals.

Figure 149: Conditioned costs-to-go \( J_{3,0,1}^o(t_3 \mid \delta_1 = 1) \) and \( J_{3,0,1}^o(t_3 \mid \delta_2 = 1) \) in state \([3 \ 0 \ 1 \ t_3]^T\).

In order to find the optimal cost-to-go \( J_{3,0,1}^o(t_3) \), it is necessary to carry out the following minimization

\[ J_{3,0,1}^o(t_3) = \min \{ J_{3,0,1}^o(t_3 \mid \delta_1 = 1) , J_{3,0,1}^o(t_3 \mid \delta_2 = 1) \} \]

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 150.

Figure 150: Optimal cost-to-go \( J_{3,0,1}^o(t_3) \) in state \([3 \ 0 \ 1 \ t_3]^T\).

The function \( J_{3,0,1}^o(t_3) \) is specified by the initial value 0.5, by the set \{ 3, 4, 11, 13.5, 16, 23 \} of abscissae \( \gamma_i \), \( i = 1, \ldots, 6 \), at which the slope changes, and by the set \{ 1, 0, 1, 1.5, 3.5, 4.5 \} of slopes \( \mu_i \), \( i = 1, \ldots, 6 \), in the various intervals.
Since \( J^\diamond_{0,0,1}(t_3 \mid \delta_1 = 1) \) is the minimum in \((-\infty, 4)\), and \( J^\diamond_{0,0,1}(t_3 \mid \delta_2 = 1) \) is the minimum in \([4, +\infty)\), the optimal control strategies for this state are

\[
\delta^1_3(3, 0, 1, t_3) = \begin{cases} 
1 & t_3 < 4 \\
0 & t_3 \geq 4 
\end{cases}
\]

\[
\delta^2_3(3, 0, 1, t_3) = \begin{cases} 
0 & t_3 < 4 \\
1 & t_3 \geq 4 
\end{cases}
\]

The optimal control strategy \( \tau^o(3, 0, 1, t_3) \) is illustrated in figure 146.

Figure 151: Optimal control strategy \( \tau^o(3, 0, 1, t_3) \) in state \([3 0 1 t_3]^T\).

**Stage 2 – State \([0 2 2 t_2]^T\) (S6)**

In state \([0 2 2 t_2]^T\), the cost function to be minimized, with respect to the (continuous) decision variable \( \tau \) and to the (binary) decision variables \( \delta_1 \) and \( \delta_2 \) is

\[
\delta_1 \left[ \alpha_{1,1} \max\{t_2 + st_{2,1} + \tau - dd_{1,1}, 0\} + \beta_1 (pt_{1,1}^{\text{nom}} - \tau) + sc_{2,1} + J^\diamond_{1,2,1}(t_3) \right]
\]

\[
+ \delta_2 \left[ \alpha_{2,3} \max\{t_2 + st_{2,2} + \tau - dd_{2,3}, 0\} + \beta_2 (pt_{2,1}^{\text{nom}} - \tau) + sc_{2,2} + J^\diamond_{3,3,2}(t_3) \right]
\]

Case i) in which it is assumed \( \delta_1 = 1 \) (and \( \delta_2 = 0 \)).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \( \tau \) which corresponds to the processing time \( pt_{1,1} \), the following function

\[
\alpha_{1,1} \max\{t_2 + st_{2,1} + \tau - dd_{1,1}, 0\} + \beta_1 (pt_{1,1}^{\text{nom}} - \tau) + sc_{2,1} + J^\diamond_{1,2,1}(t_3)
\]

that can be written as \( f(pt_{1,1} + t_2) + g(pt_{1,1}) \) being

\[
f(pt_{1,1} + t_2) = 0.75 \cdot \max\{pt_{1,1} + t_2 - 18.5, 0\} + 1 + J^\diamond_{1,2,1}(pt_{1,1} + t_2 + 0.5)
\]

\[
g(pt_{1,1}) = \begin{cases} 
8 - pt_{1,1} & pt_{1,1} \in [4, 8) \\
0 & pt_{1,1} \notin [4, 8)
\end{cases}
\]

The function \( pt^o_{1,1}(t_2) = \arg \min_{pt_{1,1}} \{ f(pt_{1,1} + t_2) + g(pt_{1,1}) \} \), with \( 4 \leq pt_{1,1} \leq 8 \), is determined by applying lemma 1. It is (see figure 142)

\[
pt^o_{1,1}(t_2) = \begin{cases} 
x_2(t_2) & t_2 < -0.5 \\
x_3(t_2) & t_2 \geq -0.5
\end{cases}
\]

with

\[
x_2(t_2) = \begin{cases} 
8 & t_2 < -1.5 \\
-t_2 + 6.5 & -1.5 \leq t_2 < -0.5
\end{cases}
\]

and

\[
x_3(t_2) = \begin{cases} 
8 & t_2 < -1.5 \\
-t_2 + 12.5 & 4.5 \leq t_2 < 8.5 \\
4 & t_2 \geq 8.5
\end{cases}
\]

The conditioned cost-to-go \( J^\diamond_{0,2,2}(t_2 \mid \delta_1 = 1) = f(pt^o_{1,1}(t_2) + t_2) + g(pt^o_{1,1}(t_2)) \), illustrated in figure 154, is provided by lemma 2. It is specified by the initial value 1.5, by the set \{ -1.5, -0.5, 1, 4.5, 14.5, 15.5, 16.5, 18.5 \} of abscissae \( \gamma_i, i = 1, \ldots, 8 \), at which the slope changes, and by the set \{ 1, 0, 0.5, 1, 1.75, 2.25, 3.25, 4.25 \} of slopes \( \mu_i, i = 1, \ldots, 8 \), in the various intervals.
Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable $\tau$, which corresponds to the processing time $pt_{2,3}$, the following function

$$
\alpha_{2,3} \max \{t_2 + s_{2,2} + \tau - dd_{2,3}, 0\} + \beta_{2} (pt_{2,3}^{nom} - \tau) + s_{c2,2} + J_{0,3,2}^{o}(t_3)
$$

that can be written as $f(pt_{2,3} + t_2) + g(pt_{2,3})$ being

$$
f(pt_{2,3} + t_2) = \max \{pt_{2,3} + t_2 - 38, 0\} + J_{0,3,3}^{o}(pt_{2,3} + t_2)
$$

$$
g(pt_{2,3}) = \begin{cases} 
1.5 \cdot (6 - pt_{2,3}) & \text{if } pt_{2,3} \in [4, 6) \\
0 & \text{otherwise} 
\end{cases}
$$

The function $pt_{2,3}^{o}(t_2) = \arg \min_{pt_{2,3}} \{f(pt_{2,3} + t_2) + g(pt_{2,3})\}$, with $4 \leq pt_{2,3} \leq 6$, is determined by applying lemma [1]. It is (see figure 153)

$$
pt_{2,3}^{o}(t_2) = x_{c}(t_2) \quad \text{with} \quad x_{c}(t_2) = \begin{cases} 
6 & t_2 < 8.5 \\
-t_2 + 14.5 & 8.5 \leq t_2 < 10.5 \\
4 & t_2 \geq 10.5 
\end{cases}
$$

The conditioned cost-to-go $J_{0,2,2}^{o}(t_2 \mid \delta_2 = 1) = f(pt_{2,3}^{o}(t_2) + t_2) + g(pt_{2,3}^{o}(t_2))$, illustrated in figure 154, is provided by lemma [2]. It is specified by the initial value 1, by the set $\{-1.5, 8.5, 10.5, 11.5, 12.5, 16.5, 34\}$ of abscissae $\gamma_i$, $i = 1, \ldots, 7$, at which the slope changes, and by the set $\{1, 1.5, 1.75, 2.25, 2.75, 3.25, 4.25\}$ of slopes $\mu_i$, $i = 1, \ldots, 7$, in the various intervals.

In order to find the optimal cost-to-go $J_{0,2,2}^{o}(t_2)$, it is necessary to carry out the following minimization

$$
J_{0,2,2}^{o}(t_2) = \min \{J_{0,2,2}^{o}(t_2 \mid \delta_1 = 1), J_{0,2,2}^{o}(t_2 \mid \delta_2 = 1)\}
$$

which provides, in accordance with lemma [4] the continuous, nondecreasing, piecewise linear function illustrated in figure 155.
Fundamental lemmas for the determination of optimal control strategies for a class of single machine family scheduling problems

The function \( J_{0,2,2}^\circ(t_2) \) is specified by the initial value 1, by the set \( \{ -1.5, 0, 1, 4.5, 14.5, 15.5, 16.5, 18.5, 29.25, 34 \} \) of abscissae \( \gamma_i, i = 1, \ldots, 10 \), at which the slope changes, and by the set \( \{ 1, 0, 0.5, 1, 1.75, 2.25, 3.25, 4.25, 3.25, 4.25 \} \) of slopes \( \mu_i, i = 1, \ldots, 10 \), in the various intervals.

Since \( J_{0,2,2}^\circ(t_2 | \delta_1 = 1) \) is the minimum in \([0, 29.25] \), and \( J_{0,2,2}^\circ(t_2 | \delta_2 = 1) \) is the minimum in \((-\infty, 0) \) and in \([29.25, +\infty) \), the optimal control strategies for this state are

\[
\delta_1^\circ(0, 2, 2, t_2) = \begin{cases} 0 & t_2 < 0 \\ 1 & 0 \leq t_2 < 29.25 \\ 0 & t_2 \geq 29.25 \end{cases} \quad \delta_2^\circ(0, 2, 2, t_2) = \begin{cases} 1 & t_2 < 0 \\ 0 & 0 \leq t_2 < 29.25 \\ 1 & t_2 \geq 29.25 \end{cases}
\]

\[
\tau^\circ(0, 2, 2, t_2) = \begin{cases} 6 & t_2 < 0 \\ 8 & 0 \leq t_2 < 4.5 \\ -t_2 + 12.5 & 4.5 \leq t_2 < 8.5 \\ 4 & t_2 \geq 8.5 \end{cases}
\]

The optimal control strategy \( \tau^\circ(0, 2, 2, t_2) \) is illustrated in figure 156.

Stage 2 – State \([1 \ 1 \ 2 \ t_2]^T \) (S5)

In state \([1 \ 1 \ 2 \ t_2]^T \), the cost function to be minimized, with respect to the (continuos) decision variable \( \tau \) and to the (binary) decision variables \( \delta_1 \) and \( \delta_2 \) is

\[
\delta_1 \left[ \alpha_{1,2} \max\{t_2 + st_{2,1} + \tau - dd_{1,2}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{2,1} + J_{2,1,1}^\circ(t_3) \right] + \\
\quad \delta_2 \left[ \alpha_{2,2} \max\{t_2 + st_{2,2} + \tau - dd_{2,2}, 0\} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + sc_{2,2} + J_{2,2,2}^\circ(t_3) \right]
\]

Case i) in which it is assumed \( \delta_1 = 1 \) (and \( \delta_2 = 0 \)).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable \( \tau \) which corresponds to the processing time \( pt_{1,2} \), the following function

\[
\alpha_{1,2} \max\{t_2 + st_{2,1} + \tau - dd_{1,2}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{2,1} + J_{2,1,1}^\circ(t_3)
\]

that can be written as \( f(pt_{1,2} + t_2) + g(pt_{1,2}) \) being

\[
f(pt_{1,2} + t_2) = 0.5 \cdot \max\{pt_{1,2} + t_2 - 23.5, 0\} + 1 + J_{2,1,1}^\circ(pt_{1,2} + t_2 + 0.5)
\]

Figure 155: Optimal cost-to-go \( J_{0,2,2}^\circ(t_2) \) in state \([0 \ 2 \ 2 \ t_2]^T \).

Figure 156: Optimal control strategy \( \tau^\circ(0, 2, 2, t_2) \) in state \([0 \ 2 \ 2 \ t_2]^T \).
The conditioned cost-to-go \( J_{\tilde{t}_{1,2}}(t_2 | \delta_1 = 1) = f(p_{t_{1,2}}(t_2) + t_2) + g(p_{t_{1,2}}) \), with \( 4 \leq p_{t_{1,2}} \leq 8 \), is determined by applying lemma \( \text{[157]} \). It is (see figure \( \text{[157]} \)).

\[
g(p_{t_{1,2}}) = \begin{cases} 8 - p_{t_{1,2}} & p_{t_{1,2}} \in [4, 8] \\ 0 & p_{t_{1,2}} \notin [4, 8] \end{cases}
\]

The function \( p_{t_{1,2}}(t_2) = \arg \min_{p_{t_{1,2}}} \{ f(p_{t_{1,2}} + t_2) + g(p_{t_{1,2}}) \} \), with \( 4 \leq p_{t_{1,2}} \leq 8 \), is determined by applying lemma \( \text{[157]} \). It is (see figure \( \text{[157]} \)).

\[
p_{t_{1,2}}^2(t_2) = \begin{cases} x_2(t_2) & t_2 < -6.5 \\ x_1(t_2) & -6.5 \leq t_2 < 2.5 \\ x_e(t_2) & t_2 \geq 2.5 \end{cases}
\]

with \( x_2(t_2) = \begin{cases} 8 & t_2 < -7.5 \\ -t_2 + 0.5 & -7.5 \leq t_2 < -6.5 \end{cases} \)

\[
x_1(t_2) = \begin{cases} 8 & t_2 < -6.5 \\ -t_2 + 8.5 & -6.5 \leq t_2 < 0.5 \\ 2.5 & 0.5 \leq t_2 < 2.5 \end{cases}
\]

and \( x_e(t_2) = \begin{cases} 8 & 2.5 \leq t_2 < 5 \\ -t_2 + 13 & 5 \leq t_2 < 9 \\ 4 & t_2 \geq 9 \end{cases} \)

Figure 157: Optimal processing time \( p_{t_{1,2}}^2(t_2) \), under the assumption \( \delta_1 = 1 \) in state \( [1 \ 1 \ 2 \ t_2]^T \).

The conditioned cost-to-go \( J_{\tilde{t}_{1,2}}(t_2 | \delta_1 = 1) = f(p_{t_{1,2}}(t_2) + t_2) + g(p_{t_{1,2}}) \), illustrated in figure \( \text{[159]} \), is provided by lemma \( \text{[2]} \). It is specified by the initial value 1.5, by the set \( \{-7.5, -6.5, 0.5, 2.5, 5, 12.5, 14.5, 15, 16, 16.5, 18.5, 19.5, 20.5\} \) of abscissae \( \gamma_i, i = 1, \ldots, 12 \), at which the slope changes, and by the set \( \{1, 0.1, 0.5, 1, 1.5, 2, 3, 1.5, 2.5, 3, 4.5\} \) of slopes \( \mu_i, i = 1, \ldots, 12 \), in the various intervals.

Case ii) in which it is assumed \( \delta_2 = 2 \) (and \( \delta_1 = 0 \)).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \( \tau \) which corresponds to the processing time \( p_{t_{2,2}} \), the following function

\[
o_{2,2} \max\{t_2 + st_{2,2} + \tau - dd_{2,2}, 0\} + \beta_2 (p_{t_{2,2}} - \tau) + sc_{2,2} + J_{\tilde{t}_{2,2}}(t_3)
\]

that can be written as \( f(p_{t_{2,2}} + t_2) + g(p_{t_{2,2}}) \) being

\[
f(p_{t_{2,2}} + t_2) = \max\{p_{t_{2,2}} + t_2 - 24, 0\} + J_{\tilde{t}_{1,2}}(p_{t_{2,2}} + t_2)
\]

\[
g(p_{t_{2,2}}) = \begin{cases} 1.5 \cdot (6 - p_{t_{2,2}}) & p_{t_{2,2}} \in [4, 6] \\ 0 & p_{t_{2,2}} \notin [4, 6] \end{cases}
\]

The function \( p_{t_{2,2}}^2(t_2) = \arg \min_{p_{t_{2,2}}} \{ f(p_{t_{2,2}} + t_2) + g(p_{t_{2,2}}) \} \), with \( 4 \leq p_{t_{2,2}} \leq 6 \), is determined by applying lemma \( \text{[158]} \). It is (see figure \( \text{[158]} \)).

\[
p_{t_{2,2}}^2(t_2) = x_e(t_2) \quad \text{with} \quad x_e(t_2) = \begin{cases} 6 & t_2 < 13.5 \\ -t_2 + 19.5 & 13.5 \leq t_2 < 15.5 \\ 4 & t_2 \geq 15.5 \end{cases}
\]

Figure 158: Optimal processing time \( p_{t_{2,2}}^2(t_2) \), under the assumption \( \delta_2 = 1 \) in state \( [1 \ 1 \ 2 \ t_2]^T \).

The conditioned cost-to-go \( J_{\tilde{t}_{1,2}}(t_2 | \delta_2 = 1) = f(p_{t_{2,2}}^2(t_2) + t_2) + g(p_{t_{2,2}}^2(t_2)) \), illustrated in figure \( \text{[159]} \), is provided by lemma \( \text{[2]} \). It is specified by the initial value 1, by the set \( \{0.5, 2, 3, 6.5, 13.5, 16.5, 18.5, 20, 26.25, \ldots\} \),
Fundamental lemmas for the determination of optimal control strategies
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Figure 159: Conditioned costs-to-go $J_{1,1,2}(t_2 | \delta_1 = 1)$ and $J_{1,1,2}(t_2 | \delta_2 = 1)$ in state $[1, 1, 2, t_2]^T$.

30 \} of abscissae $\gamma_i, i = 1, \ldots, 10$, at which the slope changes, and by the set \{ 1, 0, 0.5, 1, 1.5, 2.5, 3.5, 4.5, 3.5, 4.5 \} of slopes $\mu_i, i = 1, \ldots, 10$, in the various intervals.

In order to find the optimal cost-to-go $J_{1,1,2}(t_2)$, it is necessary to carry out the following minimization

$$J_{1,1,2}(t_2) = \min \{ J_{1,1,2}(t_2 | \delta_1 = 1), J_{1,1,2}(t_2 | \delta_2 = 1) \}$$

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure [160].

Figure 160: Optimal cost-to-go $J_{1,1,2}(t_2)$ in state $[1, 1, 2, t_2]^T$.

The function $J_{1,1,2}(t_2)$ is specified by the initial value 1, by the set \{ 0.5, 2, 3, 6.5, 13.5, 16.5, 18.5, 20, 26.25, 30 \} of abscissae $\gamma_i, i = 1, \ldots, 10$, at which the slope changes, and by the set \{ 1, 0.5, 1, 1.5, 2.5, 3.5, 4.5, 3.5, 4.5 \} of slopes $\mu_i, i = 1, \ldots, 10$, in the various intervals.

Since $J_{1,1,2}(t_2 | \delta_2 = 1)$ is always the minimum (see again figure [159]), the optimal control strategies for this state are

$$\delta_1^o(1, 1, 2, t_2) = 0 \quad \forall t_2 \quad \delta_2^o(1, 1, 2, t_2) = 1 \quad \forall t_2$$

$$\tau^o(1, 1, 2, t_2) = \begin{cases} 6 & t_2 < 13.5 \\ -t_2 + 19.5 & 13.5 \leq t_2 < 15.5 \\ 4 & t_2 \geq 15.5 \end{cases}$$

The optimal control strategy $\tau^o(1, 1, 2, t_2)$ is illustrated in figure [161].

Figure 161: Optimal control strategy $\tau^o(1, 1, 2, t_2)$ in state $[1, 1, 2, t_2]^T$. 

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Stage 2 – State $[1 \ 1 \ 1 \ t_2]T$ (S4)

In state $[1 \ 1 \ 1 \ t_2]T$, the cost function to be minimized, with respect to the (continuos) decision variable $\tau$ and to the (binary) decision variables $\delta_1$ and $\delta_2$ is

$$
\delta_1 [\alpha_{1,2} \max\{t_2 + st_{1,1} + \tau - dd_{1,2} , 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{1,1} + J_{2,1,1}^{\tau}(t_3)] + \\
+ \delta_2 [\alpha_{2,2} \max\{t_2 + st_{1,2} + \tau - dd_{2,2} , 0\} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + sc_{1,2} + J_{1,2,2}^{\tau}(t_3)]
$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable $\tau$ which corresponds to the processing time $pt_{1,2}$, the following function

$$
\alpha_{1,2} \max\{t_2 + st_{1,1} + \tau - dd_{1,2} , 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{1,1} + J_{2,1,1}^{\tau}(t_3)
$$

that can be written as $f(pt_{1,2} + t_2) + g(pt_{1,2})$ being

$$
f(pt_{1,2} + t_2) = 0.5 \cdot \max\{pt_{1,2} + t_2 - 24 , 0\} + J_{2,1,1}^{\tau}(pt_{1,2} + t_2)
$$

$$
g(pt_{1,2}) = \begin{cases} 
8 - pt_{1,2} & \text{if } pt_{1,2} \in [4, 8] \\
0 & \text{if } pt_{1,2} \notin [4, 8]
\end{cases}
$$

The function $pt_{1,2}^{*}(t_2) = \arg \min_{pt_{1,2}} \{f(pt_{1,2} + t_2) + g(pt_{1,2})\}$, with $4 \leq pt_{1,2} \leq 8$, is determined by applying lemma[1]. It is (see figure[162]).

$$
pt_{1,2}^{*}(t_2) = \begin{cases} 
x_a(t_2) & t_2 < -6 \\
x_1(t_2) & -6 \leq t_2 < 3 \\
x_c(t_2) & t_2 \geq 3
\end{cases}
$$

with

$$
x_a(t_2) = \begin{cases} 
8 & t_2 < -7 \\
-t_2 + 1 & -7 \leq t_2 < -6
\end{cases}
$$

$$
x_1(t_2) = \begin{cases} 
8 & -t_2 + 9 \\
-t_2 + 13.5 & 1 \leq t_2 < 3
\end{cases}
$$

and

$$
x_c(t_2) = \begin{cases} 
8 & 3 \leq t_2 < 5.5 \\
-t_2 + 19.5 & 5.5 \leq t_2 < 9.5 \\
4 & t_2 \geq 9.5
\end{cases}
$$

Figure 162: Optimal processing time $pt_{1,2}^{*}(t_2)$, under the assumption $\delta_1 = 1$ in state $[1 \ 1 \ 1 \ t_2]T$.

The conditioned cost-to-go $J_{1,1,1}^{\tau}(t_2 \mid \delta_1 = 1) = f(pt_{1,2}^{*}(t_2) + t_2) + g(pt_{1,2}^{*}(t_2))$, illustrated in figure[164], is provided by lemma[2]. It is specified by the initial value 0.5, by the set $\{-7, -6, 1, 3, 5.5, 13, 15, 15.5, 16.5, 19, 20, 21\}$ of abscissas $\gamma_i$, $i = 1, \ldots, 12$, at which the slope changes, and by the set $\{1, 0, 1, 0, 1, 2, 3, 5, 3, 4, 5\}$ of slopes $\mu_i$, $i = 1, \ldots, 12$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable $\tau$ which corresponds to the processing time $pt_{2,2}$, the following function

$$
\alpha_{2,2} \max\{t_2 + st_{1,2} + \tau - dd_{2,2} , 0\} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + sc_{1,2} + J_{1,2,2}^{\tau}(t_3)
$$

that can be written as $f(pt_{2,2} + t_2) + g(pt_{2,2})$ being

$$
f(pt_{2,2} + t_2) = \max\{pt_{2,2} + t_2 - 23 , 0\} + 0.5 + J_{1,2,2}^{\tau}(pt_{2,2} + t_2 + 1)
$$

$$
g(pt_{2,2}) = \begin{cases} 
1.5 \cdot (6 - pt_{2,2}) & pt_{2,2} \in [4, 6] \\
0 & pt_{2,2} \notin [4, 6]
\end{cases}
$$
The function \( pt_{2,2}(t_2) = \arg \min_{pt_{2,2}} \{ f(pt_{2,2} + t_2) + g(pt_{2,2}) \} \), with \( 4 \leq pt_{2,2} \leq 6 \), is determined by applying lemma 1. It is (see figure 163)
\[
pt_{2,2}(t_2) = x_e(t_2) = \begin{cases} 
6 & t_2 < 12.5 \\
-t_2 + 18.5 & 12.5 \leq t_2 < 14.5 \\
4 & t_2 \geq 14.5
\end{cases}
\]

Figure 163: Optimal processing time \( pt_{2,2}(t_2) \), under the assumption \( \delta_2 = 1 \) in state \([1 1 1 t_2]^T\).

The conditioned cost-to-go \( J_{1,1,1}^\circ(t_2 \mid \delta_1 = 1) = f(pt_{2,2}(t_2) + t_2) + g(pt_{2,2}(t_2)) \), illustrated in figure 164, is provided by lemma 2. It is specified by the initial value 1.5, by the set \( \{-0.5, 1, 2, 5.5, 12.5, 15.5, 17.5, 19, 25.25, 29\} \) of abscissae \( \gamma_i, i = 1, \ldots, 10 \), at which the slope changes, and by the set \( \{1, 0, 0.5, 1.5, 2.5, 3.5, 4.5, 3.5, 4.5\} \) of slopes \( \mu_i, i = 1, \ldots, 10 \), in the various intervals.

Figure 164: Conditioned costs-to-go \( J_{1,1,1}^\circ(t_2 \mid \delta_1 = 1) \) and \( J_{1,1,1}^\circ(t_2 \mid \delta_2 = 1) \) in state \([1 1 1 t_2]^T\).

In order to find the optimal cost-to-go \( J_{1,1,1}^\circ(t_2) \), it is necessary to carry out the following minimization
\[
J_{1,1,1}^\circ(t_2) = \min \{ J_{1,1,1}^\circ(t_2 \mid \delta_1 = 1), J_{1,1,1}^\circ(t_2 \mid \delta_2 = 1) \}
\]
which provides, in accordance with lemma 3, the continuous, nondecreasing, piecewise linear function illustrated in figure 165.

Figure 165: Optimal cost-to-go \( J_{1,1,1}^\circ(t_2) \) in state \([1 1 1 t_2]^T\).

The function \( J_{1,1,1}^\circ(t_2) \) is specified by the initial value 0.5, by the set \( \{-7, -6, 1, 3, 5.5, 13, 15, 15.5, 17.25, 19, 20, 21\} \) of abscissae \( \gamma_i, i = 1, \ldots, 10 \), at which the slope changes, and by the set \( \{1, 0, 0.5, 1, 1.5, 2, 2.5, 1.5, 2.5, 3, 4.5\} \) of slopes \( \mu_i, i = 1, \ldots, 10 \), in the various intervals.
Since \( J_{1,1}^\tau(t_2 | \delta_1 = 1) \) is the minimum in \((-\infty, -6)\), in \([1, 15.5)\), and in \([17.25, +\infty)\), and \( J_{1,1}^\tau(t_2 | \delta_2 = 1) \) is the minimum in \([-6, -1)\) and in \([15.5, 17.25)\), the optimal control strategies for this state are

\[
\delta_1^\tau(1,1,1,t_2) = \begin{cases} 
1 & t_2 < -6 \\
0 & -6 \leq t_2 < 1 \\
1 & 1 \leq t_2 < 15.5 \\
0 & 15.5 \leq t_2 < 17.25 \\
1 & t_2 \geq 17.25 
\end{cases} \quad \delta_2^\tau(1,1,1,t_2) = \begin{cases} 
0 & t_2 < -6 \\
1 & -6 \leq t_2 < 1 \\
0 & 1 \leq t_2 < 15.5 \\
1 & 15.5 \leq t_2 < 17.25 \\
0 & t_2 \geq 17.25 
\end{cases}
\]

\( \tau^\circ(1,1,1,t_2) \) is illustrated in figure 166.

**Figure 166: Optimal control strategy \( \tau^\circ(1,1,1,t_2) \) in state \([111 t_2]^T \).**

**Stage 2 - State \([2 0 1 t_2]^T \) (S3)**

In state \([2 0 1 t_2]^T \), the cost function to be minimized, with respect to the (continuous) decision variable \( \tau \) and to the (binary) decision variables \( \delta_1 \) and \( \delta_2 \) is

\[
\delta_1 \left[ \alpha_{1,3} \max \{ t_2 + st_{1,1} + \tau - dd_{1,3} , 0 \} + \beta_1 ( pt_{1,3}^{\text{nom}} - \tau) + sc_{1,1} + J_{2,0,1}^\tau(t_3) \} + \\
\delta_2 \left[ \alpha_{2,1} \max \{ t_2 + st_{1,2} + \tau - dd_{2,1} , 0 \} + \beta_2 ( pt_{1,2}^{\text{nom}} - \tau) + sc_{1,2} + J_{2,1,2}^\tau(t_3) \} \right]
\]

**Case i) in which it is assumed \( \delta_1 = 1 \) (and \( \delta_2 = 0 \)).**

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \( \tau \) which corresponds to the processing time \( pt_{1,3} \), the following function

\[
\alpha_{1,3} \max \{ t_2 + st_{1,1} + \tau - dd_{1,3} , 0 \} + \beta_1 ( pt_{1,3}^{\text{nom}} - \tau) + sc_{1,1} + J_{2,0,1}^\tau(t_3)
\]

that can be written as \( f(pt_{1,3} + t_2) + g(pt_{1,3}) \) being

\[
f(pt_{1,3} + t_2) = 1.5 \cdot \max \{ pt_{1,3} + t_2 - 29 , 0 \} + J_{2,0,1}^\tau(pt_{1,3} + t_2)
\]

\[
g(pt_{1,3}) = \begin{cases} 
8 - pt_{1,3} & \text{pt}_{1,3} \in [4,8] \\
0 & \text{pt}_{1,3} \notin [4,8] 
\end{cases}
\]

The function \( pt_{1,3}^{\circ} \) is determined by applying lemma \( \text{[1]} \) It is (see figure 167)

\[
pt_{1,3}^{\circ}(t_2) = \begin{cases} 
x_a(t_2) = t_2 < -4 \\
x_e(t_2) = t_2 \geq -4 
\end{cases}
\]

with \( x_a(t_2) = \begin{cases} 
8 & t_2 < -5 \\
-4 & -5 \leq t_2 < -4 \\
3 & t_2 < 7 \\
4 & t_2 \geq 7 
\end{cases} \)

and \( x_e(t_2) = \begin{cases} 
8 & -4 \leq t_2 < 3 \\
-4 & -4 \leq t_2 < -5 \\
3 & t_2 < 7 \\
4 & t_2 \geq 7 
\end{cases} \)
The conditioned cost-to-go $J^2_{2,0,1}(t_2 \mid \delta_1 = 1) = f(pt^2_{2,1}(t_2) + t_2) + g(pt^2_{2,1}(t_2) + t_2)$, illustrated in figure [169] is provided by lemma 2. It is specified by the initial value 0.5, by the set $\{-5, -4, 3, 9.5, 12, 19, 25\}$ of abscissae $\gamma_i$, $i = 1, \ldots, 7$, at which the slope changes, and by the set $\{1, 0, 1, 1.5, 3.5, 4.5, 6\}$ of slopes $\mu_i$, $i = 1, \ldots, 7$, in the various intervals.

Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{2,1}$, the following function

$$
\alpha_{2,1} \max\{t_2 + st_{1,2} + \tau - dd_{2,1}, 0\} + \beta_2 (pt^{2\text{nom}}_{2} - \tau) + sc_{1,2} + J^2_{2,1,2}(t_3)
$$

that can be written as $f(pt_{2,1} + t_2) + g(pt_{2,1})$ being

$$
f(pt_{2,1} + t_2) = 2 \cdot \max\{pt_{2,1} + t_2 - 20, 0\} + 0.5 + J^2_{2,1,2}(pt_{2,1} + t_2 + 1)
$$

$$
g(pt_{2,1}) = \begin{cases} 1.5 \cdot (6 - pt_{2,1}) & pt_{2,1} \in [4, 6] \\ 0 & pt_{2,1} \notin [4, 6] \end{cases}
$$

The function $pt^2_{2,1}(t_2) = \arg\min_{pt_{2,1}} \{f(pt_{2,1} + t_2) + g(pt_{2,1})\}$, with $4 \leq pt_{2,1} \leq 6$, is determined by applying lemma 1. It is (see figure 168)

$$
pt^2_{2,1}(t_2) = x_e(t_2)
$$

with $x_e(t_2) = \begin{cases} 6 & t_2 < 11 \\ -t_2 + 17 & 11 \leq t_2 < 13 \\ 4 & t_2 \geq 13 \end{cases}$

The conditioned cost-to-go $J^2_{2,0,1}(t_2 \mid \delta_2 = 1) = f(pt^2_{2,1}(t_2) + t_2) + g(pt^2_{2,1}(t_2))$, illustrated in figure [169] is provided by lemma 2. It is specified by the initial value 1.5, by the set $\{1.5, 3, 4, 7.5, 11, 15, 15.5, 16, 17.5, 19, 19.5, 23.75, 25\}$ of abscissae $\gamma_i$, $i = 1, \ldots, 13$, at which the slope changes, and by the set $\{1, 0, 0.5, 1, 1.5, 2, 3, 5, 6, 4.5, 6, 5, 6\}$ of slopes $\mu_i$, $i = 1, \ldots, 13$, in the various intervals.

In order to find the optimal cost-to-go $J^2_{2,0,1}(t_2)$, it is necessary to carry out the following minimization

$$
J^2_{2,0,1}(t_2) = \min \{J^2_{2,0,1}(t_2 \mid \delta_1 = 1), J^2_{2,0,1}(t_2 \mid \delta_2 = 1)\}
$$

which provides, in accordance with lemma 3 the continuous, nondecreasing, piecewise linear function illustrated in figure [170].

The function $J^2_{2,0,1}(t_2)$ is specified by the initial value 0.5, by the set $\{-5, -4, 3, 5, 7.5, 11, 15, 15.5, 16, 17.5, 19, 19.5, 21, 25\}$ of abscissae $\gamma_i$, $i = 1, \ldots, 14$, at which the slope changes, and by the set $\{1, 0, 1, 0.5, 1, 1.5, 2, 3, 5, 6, 4.5, 6, 4.5, 6\}$ of slopes $\mu_i$, $i = 1, \ldots, 14$, in the various intervals.
Fundamental lemmas for the determination of optimal control strategies for a class of single machine family scheduling problems

Since $J_{2,0,1}(t_2 | \delta_1 = 1)$ is the minimum in $(-\infty, -4)$, in $[3, 5)$, and in $[21.3, +\infty)$, and $J_{2,0,1}(t_2 | \delta_2 = 1)$ is the minimum in $[-4, 3]$ and in $[5, 21.3)$, the optimal control strategies for this state are

$$
\delta_1^{*}(2, 0, 1, t_2) = \begin{cases} 
1 & t_2 < -4 \\
0 & -4 \leq t_2 < 3 \\
1 & 3 \leq t_2 < 5 \\
0 & 5 \leq t_2 < 21.3 \\
1 & t_2 \geq 21.3 
\end{cases}
$$

$$
\delta_2^{*}(2, 0, 1, t_2) = \begin{cases} 
0 & t_2 < -4 \\
1 & -4 \leq t_2 < 3 \\
0 & 3 \leq t_2 < 5 \\
1 & 5 \leq t_2 < 21.3 \\
0 & t_2 \geq 21.3 
\end{cases}
$$

$$
\tau^{*}(2, 0, 1, t_2) = \begin{cases} 
8 & t_2 < -5 \\
-t_2 + 3 & -5 \leq t_2 < -4 \\
6 & -4 \leq t_2 < 3 \\
-t_2 + 11 & 3 \leq t_2 < 5 \\
6 & 5 \leq t_2 < 11 \\
-t_2 + 17 & 11 \leq t_2 < 13 \\
4 & t_2 \geq 13 
\end{cases}
$$

The optimal control strategy $\tau^{*}(2, 0, 1, t_2)$ is illustrated in figure 171.
Stage 1 – State \([0\ 1\ 2\ t_1]^T\) (S2)

In state \([0\ 1\ 2\ t_1]^T\), the cost function to be minimized, with respect to the (continuous) decision variable \(\tau\) and to the (binary) decision variables \(\delta_1\) and \(\delta_2\) is

\[
\delta_1 [\alpha_{1,1} \max\{t_1 + st_{2,1} + \tau - dd_{1,1}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{2,1} + J_{1,1,1}^e(t_2)] + \\
\delta_2 [\alpha_{2,2} \max\{t_1 + st_{2,2} + \tau - dd_{2,2}, 0\} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + sc_{2,2} + J_{0,2,2}^e(t_2)]
\]

**Case i) in which it is assumed \(\delta_1 = 1\) (and \(\delta_2 = 0\)).**

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \(\tau\) which corresponds to the processing time \(pt_{1,1}\), the following function

\[
\alpha_{1,1} \max\{t_1 + st_{2,1} + \tau - dd_{1,1}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{2,1} + J_{1,1,1}^e(t_2)
\]

that can be written as \(f(pt_{1,1} + t_1) + g(pt_{1,1})\) being

\[
f(pt_{1,1} + t_1) = 0.75 \cdot \max\{pt_{1,1} + t_1 - 18.5, 0\} + 1 + J_{1,1,1}^e(pt_{1,1} + t_1 + 0.5)
\]

\[
g(pt_{1,1}) = \begin{cases} 
8 - pt_{1,1} & \text{if } pt_{1,1} \in [4, 8] \\
0 & \text{if } pt_{1,1} \notin [4, 8]
\end{cases}
\]

The function \(pt_{1,1}^e(t_1) = \arg \min_{pt_{1,1}} \{ f(pt_{1,1} + t_1) + g(pt_{1,1}) \}\), with \(4 \leq pt_{1,1} \leq 8\), is determined by applying lemma[1]. It is (see figure[172]).

\[
pt_{1,1}^e(t_1) = \begin{cases} 
x_a(t_1) & t_1 < -14.5 \\
x_1(t_1) & -14.5 \leq t_1 < -5.5 \\
x_c(t_1) & t_1 \geq -5.5
\end{cases}
\]

with \(x_a(t_1) = \begin{cases} 
8 & t_1 < -15.5 \\
-7.5 - t_1 & -15.5 \leq t_1 < -14.5
\end{cases}\)

\[
x_1(t_1) = \begin{cases} 
8 & -t_1 - 0.5 \\
-7.5 & -t_1 < -5.5
\end{cases}\]

and \(x_c(t_1) = \begin{cases} 
8 & -t_1 + 5 \\
-3 & -t_1 < 1 \\
4 & t_1 \geq 1
\end{cases}\)

Figure 172: Optimal processing time \(pt_{1,1}^e(t_1)\), under the assumption \(\delta_1 = 1\) in state \([0\ 1\ 2\ t_1]^T\).

\[
J_{0,1,2}^e(t_1 \mid \delta_1 = 1) = f(pt_{1,1}^e(t_1) + t_1) + g(pt_{1,1}^e(t_1)),
\]

illustrated in figure[174] is provided by lemma[2]. It is specified by the initial value 1.5, by the set \([-15.5, -14.5, -7.5, -5.5, -3, 8.5, 10.5, 11, 12.75, 14.5, 15.5, 16.5\} of abscissae \(\gamma_i, i = 1, \ldots, 12\), at which the slope changes, and by the set \([1, 0.1, 0.5, 1, 1.5, 2, 2.5, 1.5, 3.25, 3.75, 5.25]\} of slopes \(\mu_i, i = 1, \ldots, 12\), in the various intervals.

Case ii) in which it is assumed \(\delta_2 = 1\) (and \(\delta_1 = 0\)).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable \(\tau\) which corresponds to the processing time \(pt_{2,2}\), the following function

\[
\alpha_{2,2} \max\{t_1 + st_{2,2} + \tau - dd_{2,2}, 0\} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + sc_{2,2} + J_{0,2,2}^e(t_2)
\]

that can be written as \(f(pt_{2,2} + t_1) + g(pt_{2,2})\) being

\[
f(pt_{2,2} + t_1) = \max\{pt_{2,2} + t_1 - 24, 0\} + J_{0,2,2}^e(pt_{2,2} + t_1)
\]

\[
g(pt_{2,2}) = \begin{cases} 
1.5 \cdot (6 - pt_{2,2}) & pt_{2,2} \in [4, 6] \\
0 & pt_{2,2} \notin [4, 6]
\end{cases}
\]
The function $pt^\circ_{2,2}(t_1) = \arg \min_{pt_{2,2}} \{ f(pt_{2,2} + t_1) + g(pt_{2,2}) \}$, with $4 \leq pt_{2,2} \leq 6$, is determined by applying lemma [1]. It is (see figure 173)

$$pt^\circ_{2,2}(t_1) = x_e(t_1) = \begin{cases} 6 & t_1 < 8.5 \\ -t_1 + 14.5 & 8.5 \leq t_1 < 10.5 \\ 4 & t_1 \geq 10.5 \end{cases}$$

Figure 173: Optimal processing time $pt^\circ_{2,2}(t_1)$, under the assumption $\delta_2 = 1$ in state $[0\ 1\ 2 \ t_1]^T$.

The conditioned cost-to-go $J^\circ_{0,1,2}(t_1 | \delta_2 = 1) = f(pt^\circ_{2,2}(t_1) + t_1) + g(pt^\circ_{2,2}(t_1))$, illustrated in figure 174, is provided by lemma [2]. It is specified by the initial value 1, by the set $\{-7.5, -6, -5, -1.5, 8.5, 10.5, 11.5, 12.5, 14.5, 20, 25.25, 30\}$ of abscissae $\gamma_i, i = 1, \ldots, 12$, at which the slope changes, and by the set $\{1, 0, 0.5, 1, 1.5, 1.75, 2.25, 3.25, 4.25, 5.25, 4.25, 5.25, 4.25, 5.25\}$ of slopes $\mu_i, i = 1, \ldots, 12$, in the various intervals.

Figure 174: Conditioned costs-to-go $J^\circ_{0,1,2}(t_1 | \delta_1 = 1)$ and $J^\circ_{0,1,2}(t_1 | \delta_2 = 1)$ in state $[0\ 1\ 2 \ t_1]^T$.

In order to find the optimal cost-to-go $J^\circ_{0,1,2}(t_1)$, it is necessary to carry out the following minimization

$$J^\circ_{0,1,2}(t_1) = \min \{ J^\circ_{0,1,2}(t_1 | \delta_1 = 1), J^\circ_{0,1,2}(t_1 | \delta_2 = 1) \}$$

which provides, in accordance with lemma [3], the continuous, nondecreasing, piecewise linear function illustrated in figure 175.

Figure 175: Optimal cost-to-go $J^\circ_{0,1,2}(t_1)$ in state $[0\ 1\ 2 \ t_1]^T$.

The function $J^\circ_{0,1,2}(t_1)$ is specified by the initial value 1, by the set $\{-7.5, -6, -5, -1.5, 8.5, 10.5, 11.5, 12.5, 14.5, 15, 15.5, 16.5, 17.5, 20, 25.25, 30\}$ of abscissae $\gamma_i, i = 1, \ldots, 16$, at which the slope changes, and by the set $\{1, 0, 0.5, 1, 1.5, 1.75, 2.25, 3.25, 4.25, 5.25, 3.75, 5.25, 4.25, 5.25, 4.25, 5.25, 5.25\}$ of slopes $\mu_i, i = 1, \ldots, 16$, in the various intervals.
Since $J_{0,1,2}^0(t_1 \mid \delta_1 = 1)$ is the minimum in $[15, 17.5)$, and $J_{0,1,2}^1(t_1 \mid \delta_2 = 1)$ is the minimum in $(-\infty, 15)$ and in $[17.5, +\infty)$, the optimal control strategies for this state are

$$
\delta^1_1(0, 1, 2, t_1) = \begin{cases} 
0 & t_1 < 15 \\
1 & 15 \leq t_1 < 17.5 \\
0 & t_1 \geq 17.5
\end{cases}
\quad
\delta^2_2(0, 1, 2, t_1) = \begin{cases} 
1 & t_1 < 15 \\
0 & 15 \leq t_1 < 17.5 \\
1 & t_1 \geq 17.5
\end{cases}
$$

The optimal control strategy $\tau^0(0, 1, 2, t_1)$ is illustrated in figure 176.

**Figure 176: Optimal control strategy $\tau^0(0, 1, 2, t_1)$ in state $[0 \ 1 \ 2 \ t_1]^T$.**

**Stage 1 – State $[1 \ 0 \ 1 \ t_1]^T$ (S1)**

In state $[1 \ 0 \ 1 \ t_1]^T$, the cost function to be minimized, with respect to the (continuos) decision variable $\tau$ and to the (binary) decision variables $\delta_1$ and $\delta_2$ is

$$
\delta_1 \left[ \alpha_{1,2} \max\{t_1 + s_{1,1} + \tau - dd_{1,2}, 0\} + \beta_1 (pt_1^{nom} - \tau) + sc_{1,1} + J_{2,0,1}^2(t_2) \right] + \\
+ \delta_2 \left[ \alpha_{2,1} \max\{t_1 + s_{1,2} + \tau - dd_{2,1}, 0\} + \beta_2 (pt_2^{nom} - \tau) + sc_{1,2} + J_{1,1,2}^2(t_2) \right]
$$

Case i) in which it is assumed $\delta_1 = 1$ (and $\delta_2 = 0$).

In this case, it is necessary to minimize, with respect to the (continuos) decision variable $\tau$ which corresponds to the processing time $pt_{1,2}$, the following function

$$
\alpha_{1,2} \max\{t_1 + s_{1,1} + \tau - dd_{1,2}, 0\} + \beta_1 (pt_1^{nom} - \tau) + sc_{1,1} + J_{2,0,1}^2(t_2)
$$

that can be written as $f(pt_{1,2} + t_1) + g(pt_{1,2})$ being

$$
f(pt_{1,2} + t_1) = 0.5 \cdot \max\{pt_{1,2} + t_1 - 24, 0\} + J_{2,0,1}^2(pt_{1,2} + t_1)
$$

$$
g(pt_{1,2}) = \begin{cases} 
8 - pt_{1,2} & pt_{1,2} \in [4, 8) \\
0 & pt_{1,2} \not\in [4, 8)
\end{cases}
$$

The function $pt_{1,2}^2(t_1) = \arg \min_{pt_{1,2}} \{f(pt_{1,2} + t_1) + g(pt_{1,2})\}$, with $4 \leq pt_{1,2} \leq 8$, is determined by applying lemma[1]. It is (see figure 177).

$$
pt_{1,2}^2(t_1) = \begin{cases} 
x_6(t_1) & t_1 < -12 \\
x_7(t_1) & t_1 \geq -12 \leq t_1 < -3 \\
x_6(t_1) & t_1 > -3
\end{cases}
$$

with

$$
x_6(t_1) = \begin{cases} 
8 & t_1 < -13 \\
-t_1 - 5 & -13 \leq t_1 < -12
\end{cases}
$$

$$
x_7(t_1) = \begin{cases} 
8 & t_1 < -3 \\
-t_1 + 7.5 & -3 \leq t_1 < 3.5 \\
4 & t_1 \geq 3.5
\end{cases}
$$

The conditioned cost-to-go $J_{0,1}^0(t_1 \mid \delta_1 = 1) = f(pt_{1,2}^2(t_1) + t_1) + g(pt_{1,2}^2(t_1))$, illustrated in figure 178, is provided by lemma[2]. It is specified by the initial value 0.5, by the set $\{ -12, -5, -3, -0.5, 7, 11, 11.5, 12, 13.5, 15, 16, 15.5, 17.5, 20, 21 \}$ of abscissae $\gamma_i, i = 1, \ldots, 15$, at which the slope changes, and by the set $\{ 1, 0.5, 1, 0.5, 1, 1.5, 2, 3, 5, 6, 4.5, 6, 4.5, 5, 6.5 \}$ of slopes $\mu_i, i = 1, \ldots, 15$, in the various intervals.
Case ii) in which it is assumed $\delta_2 = 1$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{2,1}$, the following function

$$\alpha_{2,1} \max\{t_1 + st_{1,2} + \tau - dd_{2,1}, 0\} + \beta_2 \left(\frac{p_{2,1}^{\text{nom}}}{2} - \tau\right) + sc_{1,2} + J_{1,1,2}^0(t_2)$$

that can be written as $f(pt_{2,1} + t_1) + g(pt_{2,1})$ being

$$f(pt_{2,1} + t_1) = 2 \cdot \max\{pt_{2,1} + t_1 - 20, 0\} + 0.5 + J_{1,1,2}^0(pt_{2,1} + t_1 + 1)$$

$$g(pt_{2,1}) = \begin{cases} 
1.5 \cdot (6 - pt_{2,1}) & pt_{2,1} \in [4, 6] \\
0 & pt_{2,1} \notin [4, 6]
\end{cases}$$

The function $pt_{2,1}^0(t_1) = \arg\min_{pt_{2,1}}\{f(pt_{2,1} + t_1) + g(pt_{2,1})\}$, with $4 \leq pt_{2,1} \leq 6$, is determined by applying lemma\[1\]. It is (see figure 178)

$$pt_{2,1}^0(t_1) = x_e(t_1) \quad \text{with} \quad x_e(t_1) = \begin{cases} 
6 & t_1 < 6.5 \\
-t_1 + 12.5 & 6.5 \leq t_1 < 8.5 \\
4 & t_1 \geq 8.5
\end{cases}$$

The conditioned cost-to-go $J_{1,0,1}^0(t_1 \mid \delta_2 = 1) = f(pt_{2,1}^0(t_1) + t_1) + g(pt_{2,1}^0(t_1))$, illustrated in figure 179 is provided by lemma\[2\]. It is specified by the initial value 1.5, by the set $\{-6.5, -5, -4, -0.5, 0.5, 1, 1.5, 2.5, 3.5, 4.5, 5.5, 6.5\}$ of abscissae $\gamma_i, i = 1, \ldots, 11$, at which the slope changes, and by the set $\{1, 0, 0.5, 1, 1.5, 2.5, 3.5, 4.5, 5.5, 6.5\}$ of slopes $\mu_i, i = 1, \ldots, 11$, in the various intervals.

In order to find the optimal cost-to-go $J_{1,0,1}^0(t_1)$, it is necessary to carry out the following minimization

$$J_{1,0,1}^0(t_1) = \min\{J_{1,0,1}^0(t_1 \mid \delta_1 = 1), J_{1,0,1}^0(t_1 \mid \delta_2 = 1)\}$$

which provides, in accordance with lemma\[3\] the continuous, nondecreasing, piecewise linear function illustrated in figure 180.
The function $J^\tau_{0,1}(t_1)$ is specified by the initial value 0.5, by the set $\{-13,-12,-5,-3,-0.5,7,11,11.5,13.5,15,16,21.25,25\}$ of abscissae $\gamma_i$, $i = 1, \ldots, 13$, at which the slope changes, and by the set $\{1,0,1,0.5,1,1.5,2,2.5,3.5,4.5,6.5,5.5,6.5\}$ of slopes $\mu_i$, $i = 1, \ldots, 13$, in the various intervals.

Since $J^\tau_{0,1}(t_1 | \delta_1 = 1)$ is the minimum in $(-\infty,12)$ and in $[-5,11.5]$, and $J^\tau_{0,1}(t_1 | \delta_2 = 1)$ is the minimum in $[-12,-5)$ and in $[11.5,\infty)$, the optimal control strategies for this state are

$$
\delta_1^\tau(1,0,1,t_1) = \begin{cases} 
1 & t_1 < -12 \\
0 & -12 \leq t_1 < -5 \\
1 & -5 \leq t_1 < 11.5 \\
0 & t_1 \geq 11.5 
\end{cases} 
\quad \delta_2^\tau(1,0,1,t_1) = \begin{cases} 
0 & t_1 < -12 \\
1 & -12 \leq t_1 < -5 \\
0 & -5 \leq t_1 < 11.5 \\
1 & t_1 \geq 11.5 
\end{cases} 
$$

The optimal control strategy $\tau^\delta(1,0,1,t_1)$ is illustrated in figure 181.

**Stage 0 – State $[0\ 0\ 0\ t_0]^T$ ($S0$)**

In the initial state $[0\ 0\ 0\ t_0]^T$, the cost function to be minimized, with respect to the (continuous) decision variable $\tau$ and to the (binary) decision variables $\delta_1$ and $\delta_2$ is

$$
\delta_1[\alpha_{1,1} \max\{t_0 + st_{0,1} + \tau - dd_{1,1}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{0,1} + J^\tau_{1,0,1}(t_1)] + \\
+ \delta_2[\alpha_{2,1} \max\{t_0 + st_{0,2} + \tau - dd_{2,1}, 0\} + \beta_2 (pt_{2}^{\text{nom}} - \tau) + sc_{0,2} + J^\tau_{0,1,2}(t_1)]
$$

**Case i) in which it is assumed $\delta_1 = 1$ and $\delta_2 = 0$.**

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{1,1}$, the following function

$$
\alpha_{1,1} \max\{t_0 + st_{0,1} + \tau - dd_{1,1}, 0\} + \beta_1 (pt_{1}^{\text{nom}} - \tau) + sc_{0,1} + J^\tau_{1,0,1}(t_1)
$$
that can be written as $f(pt_{1,1} + t_0) + g(pt_{1,1})$ being
\[
f(pt_{1,1} + t_0) = 0.75 \cdot \max\{pt_{1,1} + t_0 - 19, 0\} + J_{0,1}(pt_{1,1} + t_0)
\]
\[
g(pt_{1,1}) = \begin{cases} 8 - pt_{1,1} & \text{if } pt_{1,1} \in [4, 8] \\ 0 & \text{if } pt_{1,1} \notin [4, 8] \end{cases}
\]
The function $pt_{1,1}^2(t_0) = \arg \min_{pt_{1,1}} \{f(pt_{1,1} + t_0) + g(pt_{1,1})\}$, with $4 \leq pt_{1,1} \leq 8$, is determined by applying lemma 2. It is specified by the initial value $0.5$, by the set of abscissae $\gamma_i$, $i = 1, \ldots, 14$, at which the slope changes, and by the set of slopes $\mu_i$, $i = 1, \ldots, 14$, in the various intervals.

**Figure 182** Optimal processing time $pt_{1,1}^2(t_0)$, under the assumption $\delta_1 = 1$ in the initial state $[0 0 0 t_0]^T$.

The conditioned cost-to-go $J_{0,0,0}(t_0 | \delta_1 = 1) = f(pt_{1,1}^2(t_0) + t_0) + g(pt_{1,1}^2(t_0))$, illustrated in figure 2. It is specified by the initial value $0.5$, by the set of abscissae $\gamma_i$, $i = 1, \ldots, 14$, at which the slope changes, and by the set of slopes $\mu_i$, $i = 1, \ldots, 14$, in the various intervals.

**Case ii)** In which it is assumed $\delta_2 = 2$ (and $\delta_1 = 0$).

In this case, it is necessary to minimize, with respect to the (continuous) decision variable $\tau$ which corresponds to the processing time $pt_{2,1}$, the following function
\[
\alpha_{2,1} \max\{t_0 + st_{0,2} + \tau - dd_{2,1}, 0\} + \beta_{2} (pt_{2,1}^{\text{nom}} - \tau) + sc_{0,2} + J_{0,1,2}(t_1)
\]
that can be written as $f(pt_{2,1} + t_0) + g(pt_{2,1})$ being
\[
f(pt_{2,1} + t_0) = 2 \cdot \max\{pt_{2,1} + t_0 - 21, 0\} + J_{0,1,2}(pt_{2,1} + t_0)
\]
\[
g(pt_{2,1}) = \begin{cases} 1.5 \cdot (6 - pt_{2,1}) & \text{if } pt_{2,1} \in [4, 6] \\ 0 & \text{if } pt_{2,1} \notin [4, 6] \end{cases}
\]
The function $pt_{2,1}^2(t_0) = \arg \min_{pt_{2,1}} \{f(pt_{2,1} + t_0) + g(pt_{2,1})\}$, with $4 \leq pt_{2,1} \leq 6$, is determined by applying lemma 2. It is (see figure 183).

\[
pt_{2,1}^2(t_0) = x_e(t_0) \quad \text{with} \quad x_e(t_0) = \begin{cases} 6 & t_0 < 2.5 \\ -t_0 + 8.5 & 2.5 \leq t_0 < 4.5 \\ 4 & t_0 \geq 4.5 \end{cases}
\]

The conditioned cost-to-go $J_{0,0,0}(t_0 | \delta_2 = 1) = f(pt_{2,1}^2(t_0) + t_0) + g(pt_{2,1}^2(t_0))$, illustrated in figure 184, is provided by lemma 2. It is specified by the initial value $1$, by the set of abscissae $\gamma_i$, $i = 1, \ldots, 17$, at which the slope changes, and by the set of slopes $\mu_i$, $i = 1, \ldots, 17$, in the various intervals.

In order to find the optimal cost-to-go $J_{0,0,0}(t_0)$, it is necessary to carry out the following minimization
\[
J_{0,0,0}(t_0) = \min \{J_{0,0,0}(t_0 | \delta_1 = 1), J_{0,0,0}(t_0 | \delta_2 = 1)\}
\]
Figure 183: Optimal processing time $p^{\circ}_{2,1}(t_0)$, under the assumption $\delta_2 = 1$ in the initial state $[0 0 0 t_0]^T$.

Figure 184: Conditioned costs-to-go $J^{0,0,0}(t_0 | \delta_1 = 1)$ and $J^{0,0,0}(t_0 | \delta_2 = 1)$ in the initial state $[0 0 0 t_0]^T$.

Figure 185: Optimal cost-to-go $J^{0,0,0}(t_0)$ in the initial state $[0 0 0 t_0]^T$.

which provides, in accordance with lemma 4, the continuous, nondecreasing, piecewise linear function illustrated in figure 185.

The function $J^{0,0,0}(t_0)$ is specified by the initial value 0.5, by the set \{-21, -20.5, -13.5, -12, -11, -7.5, 2.5, 6.5, 7.5, 8.5, 10.5, 11, 11.5, 12.5, 13.5, 15, 17, 21.25, 26\} of abscissae $\gamma_i, i = 1, \ldots, 19,$ at which the slope changes, and by the set \{1, 0, 1, 0, 0.5, 1, 1.5, 1.75, 2.25, 3.25, 4.25, 3.25, 3.75, 5.25, 4.25, 5.25, 7.25, 6.25, 7.25\} of slopes $\mu_i, i = 1, \ldots, 19,$ in the various intervals.

Since $J^{0,0,0}(t_0 | \delta_1 = 1)$ is the minimum in $(-\infty, -20.5)$, and $J^{0,0,0}(t_0 | \delta_2 = 1)$ is the minimum in $[-20.5, +\infty)$ (see again figure 184), the optimal control strategies for the initial state are

$$\delta^{0}(0, 0, 0, t_0) = \begin{cases} 1 & t_0 < -20.5 \\ 0 & t_0 \geq -20.5 \end{cases}$$

$$\delta^{2}(0, 0, 0, t_0) = \begin{cases} 1 & t_0 < -20.5 \\ 0 & t_0 \geq -20.5 \end{cases}$$

$$\tau^{\circ}(0, 0, 0, t_0) = \begin{cases} 8 & t_0 < -21 \\ -t_0 - 13 & -21 \leq t_0 < -20.5 \\ 6 & -20.5 \leq t_0 < 2.5 \\ -t_0 + 8.5 & 2.5 \leq t_0 < 4.5 \\ 4 & t_0 \geq 4.5 \end{cases}$$

The optimal control strategy $\tau^{\circ}(0, 0, 0, t_0)$ is illustrated in figure 186.
Figure 186: Optimal control strategy $\tau^*(0, 0, 0, t_0)$ in the initial state $[0 \ 0 \ 0 \ t_0]^T$.

References

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[2] Davide Giglio. Optimal control strategies for single machine family scheduling with sequence-dependent batch setup and controllable processing times. *Journal of Scheduling*. Under review.