5d $E_n$ Seiberg-Witten curve via toric-like diagram

Sung-Soo Kim  
sungsoo.kim@kias.re.kr

and

Futoshi Yagi  
fyagi@kias.re.kr

School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea.

Abstract

We consider 5d $Sp(1)$ gauge theory with $E_{N_f+1}$ global symmetries based on toric(-like) diagram constructed from $(p,q)$-web with 7-branes. We propose a systematic procedure to compute the Seiberg-Witten curve for generic toric-like diagram. For $N_f = 6,7$ flavors, we explicitly compute the Seiberg-Witten curves for 5d $Sp(1)$ gauge theory, and show that these Seiberg-Witten curves agree with already known $E_{7,8}$ results. We also discuss a generalization of the Seiberg-Witten curve to rank-$N$ cases.
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1 Introduction

Various aspects of supersymmetric gauge theories have been studied via branes, and rich physics associated with branes has been revealed. Wide class of four-dimensional \( \mathcal{N} = 2 \) supersymmetric gauge theories are given by D4-NS5 brane setup, whose asymptotic distance of the NS5 branes gives 1-loop correction to the gauge coupling constant. The condition that NS5 branes do not intersect gives the asymptotic free/conformal condition of the gauge theory. Furthermore, the M-theory uplift of the corresponding brane setup gives Seiberg-Witten (SW) curve [1].

Five-dimensional uplift of this derivation of SW curve from branes was studied in [2] for the case with simple Lie groups like \( SU, SO, Sp \), and extended to more general setup by using \((p, q)\) 5-brane web in [3]. The method with \((p, q)\) 5-brane web gives a systematic and simple procedure to derive the SW curve, using the dual graph of the 5-brane web whose vertices correspond to the non-vanishing coefficients of the polynomial describing the curve. In [3], it was discussed that five-dimensional \( Sp(1) \) (\( \simeq SU(2) \)) gauge theories could be studied up to four flavors by this \((p, q)\) 5-brane web\(^1\). As five-dimensional theories are given as natural uplift of four-dimensional gauge theories, studying the case with more than four flavors via the \((p, q)\) 5-branes seems to have difficulty. As discussed, for example, in [5], more flavors lead to the intersection of NS5-branes analogously to the four-dimensional case, which makes it hard to interpret.

On the other hand, five-dimensional \( Sp(1) \) gauge theory with \( N_f \) flavors is expected to have nontrivial UV fixed point up to \(^2 N_f \leq 7 \) [6], where the global symmetry enhancement to \( E_{N_f+1} \) group is realized. Such class of isolated conformal field theories (CFT) has been studied in various different methods. Recently, the global symmetry enhancement was explicitly checked in [7] by computing the superconformal index (up to \( N_f = 5 \)), and confirmed later up to \( N_f = 6, 7 \) in [8]. It was also shown that the Nekrasov partition function is invariant under the

\(^1\)In this paper, we do not consider O-planes. If one introduces it, 5-brane web can describes up to six flavors [4].

\(^2\)UV fixed point of the five-dimensional \( Sp(1) \) theory with eight flavors is believed to be six-dimensional. Although this case is also interesting, we do not study in this paper.
enhanced $E_{N_f+1}$ symmetry when one expands it in terms of the properly redefined Coulomb moduli parameter [9], where fiber-base duality [3, 10, 11, 12] plays an important role.

The SW curve for corresponding four-dimensional theory with $E_{6,7,8}$ global symmetries was studied in [13, 14]. These curves were obtained from dimensional analysis and symmetry argument without referring to specific field theory description at UV. Uplift of the SW curve to five dimensions [15] and to six dimensions [16, 17] was performed via the effective action of the E-string theory, which is obtained by compactification to the corresponding local del Pezzo surface or half K3 manifold. The toric diagram of this Calabi-Yau geometry can be reinterpreted as the dual graph of the $(p, q)$ 5-brane web [18]. It followed that the SW curve for five-dimensional $Sp(1)$ gauge theory with $N_f \leq 4$ flavors was reproduced as mirror curve of the corresponding Calabi-Yau geometry [19]. While it is toric for $N_f \leq 5$, the corresponding local del Pezzo surface for $N_f = 6, 7$ is non-toric, and thus there have been difficulties finding corresponding $(p, q)$ 5-brane web diagrams.

Such obstacles seem avoidable in the brane setup introduced in [20] which thus opens up a possibility to analyze the $Sp(1)$ gauge theory with more than four flavors in the $(p, q)$ 5-brane setup. It is $[p, q]$ 7-branes at infinity that resolve the non-toric nature of dual diagrams, each of which binds an arbitrary number of $(p, q)$ 5-branes. Some of the bound 5-branes “jump over” other 5-branes in such a way not to break the s-rule. The brane setup in [20] was originally introduced as a five-dimensional uplift of isolated CFTs discussed in [21]. It includes five-dimensional CFT of $E_6$, $E_7$ and $E_8$ global symmetries, which are expected to be identified as the UV fixed point of the $Sp(1)$ gauge theories with five, six and seven flavors, respectively.

It was also known that that the five-dimensional $Sp(1)$ gauge theory with $N_f$ flavors can be realized by adding $N_f$ D7-branes inside the 5-brane loop of the pure $Sp(1)$ brane setup, and that the global symmetry enhancement to $E_{N_f+1}$ can be shown by using the monodromy properties of 7-branes [22, 23]. More intuitive and relevant explanation connecting D7-branes inside the 5-brane loop to 7-branes at infinity leading to $E_6$ symmetry was presented in [24] for $N_f = 5$ where the brane setup with $E_{N_f+1}$ symmetry studied in [20] can be derived by properly pulling out all the 7-branes outside by the Hanany-Witten effect.

Using this brane setup, the SW curve for $Sp(1)$ with five flavors was computed in [20, 24] and is in agreement with the aforementioned result [15, 16, 17]. Moreover, by generalizing the relation between toric $(p, q)$ 5-brane web and toric diagram discussed in [18], the superconformal index/Nekrasov partition function in [7, 8] were reproduced from the computations of topological string partition function [24, 25, 26]. Throughout this paper, such dual graph of the original $(p, q)$ 5-brane web diagram, introduced in [20], we call “toric-like diagram” as the counterpart of the toric diagram.

\footnote{For $N_f = 3, 4, 5$, the geometry becomes toric only when we choose the specific complex structure. For $N_f = 0, 1, 2$, the geometry is toric for arbitrary complex structure.}
Along the direction that these recent developments allow some of the theories obtained by compactifying non-toric Calabi-Yau geometry to be analyzed with the technology developed for toric Calabi-Yau geometry or \((p, q)\) 5-brane web, in this paper, we would like to further develop the \((p, q)\) 5-brane configuration by computing the SW curve for the five-dimensional theory with \(N_f = 6, 7\) flavors based on the toric-like diagram.

The rest of this paper is organized as follows: In section 2, we review the derivation of SW curve of five-dimensional \(Sp(1)\) gauge theory with \(N_f = 5\) flavors [24], which is identified as five-dimensional \(T_3\) theory. Throughout this review, we clarify the generic procedure to compute SW curve from toric-like diagram. In section 3 and 4, we compute the SW curve for the theory with \(N_f = 6\) and \(N_f = 7\), respectively, which turns out to agree with the known result [15, 16, 17]. In section 5, we study mass decoupling limit to reproduce the SW curve for lower flavors from the curve for higher flavors, especially from \(N_f = 7\) to \(N_f = 6\). In section 6, we consider the SW curve obtained from the toric-like web diagram corresponding to higher rank \(E_n\) theory given in [20] and shows that the rank-2 curve actually factorizes into the two copies of the SW curve for the rank-1 \(E_n\) theory. We then conclude and discuss the observed relation\(^4\) analogous to the special case of \(\mathcal{N} = 2\) dualities [21]. In Appendices, we give various complimentary computation and results of the SW curves for \(Sp(1)\) theory with \(N_f\) flavors.

2 5d Seiberg-Witten curve from toric-like diagram

In this section, after reviewing the SW curve of the 5d \(T_3\) theory, which is identified as 5d \(Sp(1)\) theory with five flavors, we propose a procedure to derive the SW curve from generic toric-like diagram.

2.1 5d \(T_3\) theory

5d version of 4d \(T_N \equiv T[A_{N-1}]\) theories was studied based on the web or dual toric diagrams [20] where it describes M-theory compactified on a non-compact CY threefold. The dual toric diagram is obtained by associating a vertex to each face of 5-brane junctions. For a single juction, \(T_1\), it corresponds to \(\mathbb{C}^3\) and for multijuctions, \(T_N\), it corresponds to \(\mathbb{C}^3/(\mathbb{Z}_N \times \mathbb{Z}_N)\). Upon compactification on \(S^1\), it gives rise to 4d \(T_N\) constructed in [21].

We now briefly review 5d \(T_3\) theory in relation with its SW curve. A detail analysis for \(T_3\) theory has been done in [24]. Rather than summarizing the result of [24], we here point out salient features of the analysis and then use this theory to give an intuitive idea on how SW curves for \(E_7\) and \(E_8\) can be derived.

\(^4\) Although this relation should be closely related to [27], we give slightly different interpretation. Our observation is closer to the one in [24].
As shown in [20], the 5d uplift of rank-1 4d $T_N$ theories is well fit into the multi-junction of the 5d $(p, q)$ web. 4d Minahan-Nemenschansky’s isolated superconformal theories with the exceptional $E_n$ symmetry can be uplifted and studied in this framework. For instance, in 5d, the $N_f = 5$ superconformal theory with $E_6$ global symmetry at the UV fixed point corresponds to $T_3$ theory; the $N_f = 6, 7$ superconformal theory with $E_7$ global symmetry corresponds to a Higgsed $T_{4,6}$ (to keep one Coulomb modulus) theory, respectively.

$E_n$ symmetries realized in the framework is not manifest, only subgroup of $E_n$ is manifest. For example, consider $T_3$ theory. It has the following web diagram or corresponding dual toric diagram:

![Toric diagram for $E_6$](image1)

Figure 1: A toric diagram for $E_6$

As the diagram indicates, it has manifest $SU(3)^3$ global symmetry which is a maximal compact subgroup of $E_6$

$$E_6 \supset SU(3) \times SU(3) \times SU(3). \quad (2.1)$$

We explain from the Hanany-Witten transition (as well as monodromy of 7-brane branch cut) this diagram is related to the diagram of five flavors ($N_f = 5$). In the $(p, q)$ web configuration, the matters are represented by semi-infinite $(1, 0)$ 5-branes. See Figure 2. One introduces $[p, q]$ 7-branes such that $(p, q)$ 5-branes can end without breaking supersymmetry. Let us imagine a web diagram with five $(1, 0)$ 5-branes ending on 7-branes (denoted by $\otimes$).

![Brane configuration with five flavors leading to $T_3$-diagram](image2)

Figure 2: A brane configuration with five flavors leading to $T_3$-diagram.

For convenience, we put three on the left and two on the right. This is the leftmost web diagram configuration in Figure 2. In order to avoid colliding of 7-branes, we bring down the
[0, 1] 7-brane filled in red. Recall that this [0, 1] 7-brane has a branch cut denoted by the dashed line. When the [0, 1] 7-brane passes through a \((p, q)\) 5-brane, the charge of the 5-brane changes as it experiences monodromy due to the [0, 1] 7-brane. For instance, \((1, 0)\) 5-brane charge is altered to \((1, 1)\), which is depicted in the middle of Figure 2. As it is brought to further down, it becomes the web diagram of \(T_3\), which is the rightmost Figure 2.

It is interesting to see what happens to the web diagram if one pushes up the [0, 1] 7-brane rather than bringing it down. See Figure 3.

![Figure 3: A brane configuration with five flavors leading to a toric-like diagram.](image)

As explained in [20], the [0, 1] 7-brane jumps over the \((1, 1)\) 5-brane when crossing, and another \((0, 1)\) 5-brane is created to attach to the [0, 1] 7-brane (the Hanany-Witten effect). The resultant dual toric-like diagram involves a new kind of dot (white dot) to indicate this jumping phenomenon associated with binding multiple 5-branes attached to a single 7-brane. In this way, the number of the Coulomb moduli remains unaltered. This dual toric-like diagram is called a dot diagram in [20].

If the white dot above were a black dot, then it would increase the number of both the dimension of the Coulomb moduli and triangulation, hence it would be a toric diagram for \(SU(3)\) gauge theory. Notice that in this specially tuned web diagram or toric-like diagram in Figure 3, it appears as having six flavors as there are six \((1, 0)\) 5-branes and thus manifest global symmetry is no longer \(SU(3)^3\), it is rather \(S[U(3) \times U(3)] \times SU(2)\). Turning a black dot into a white dot can be interpreted as a procedure of Higgsing. But we call it a special tuning, as it can be understood as a procedure keeping dimension of the Coulomb moduli to be one. It thus leads to a dual picture of seeing the \(E_6\) theory as a special tuning of \(SU(3)\) gauge theory with six flavors.

It is clear that the toric diagram for the \(T_3\) theory can be given in a different way through the Hanany-Witten transition as explained. This means that as one writes the SW curve based on a toric diagram, two SW curves obtained from two different toric diagrams should be related by the Hanany-Witten effect. We emphasize that the way that the Hanany-Witten effect is realized in the SW curve is a coordinate transformation. It is also worth noting that
as a toric diagram shows manifest global symmetry, one can find different manifest symmetry by the Hanany-Witten transition. As we will show later, we find that the Hanany-Witten transition in the (tuned) $T_N$ theories gives rise to compact subgroup of $E_n$ symmetry.

We now consider the construction of the SW curve. For this, we compactify the theory on a circle, and T-dualize it to become IIA theory. We then uplift it to M-theory. The curve then describes M5 brane configuration embedded in $\mathbb{R}^2 \times T^2$. Given a toric diagram, say Figure 1, the SW curve takes the form as

$$\sum_{ij} c_{ij} t^i w^j = 0, \quad (2.2)$$

with the SW one-form

$$\lambda_{SW} = \log t \ d(\log w). \quad (2.3)$$

Here, $c_{ij}$ in (2.2) is the non-vanishing coefficient that corresponds to the $(i, j)$ dot in the toric diagram.

![Figure 4: The configuration for the SW curve of $T_3$ toric diagram](image)

In Figure 4, the configuration for the $T_3$ toric diagram is given. The unknown coefficients $c_{ij}$ are determined from the boundary conditions. Here, we impose the following boundary condition

$$t = N_1, N_2, N_3 \quad \text{as} \quad w \to 0$$

$$w = M_1, M_2, M_3 \quad \text{as} \quad t \to 0$$

$$w = -L_1 w, -L_2 w, -L_3 w \quad \text{as} \quad |t| \sim |w| \to \infty, \quad (2.4)$$

where the first, second and third lines correspond to three NS5-branes, three D5-branes and three (1,1) 5-branes, respectively, in the original type IIB picture. See Figure 4. These
conditions impose the constraint to the coefficients are follows:

$$\sum_{i=0}^{3} c_{i0} t^i = c_{30}(t - N_1)(t - N_2)(t - N_3),$$

$$\sum_{j=0}^{3} c_{0j} w^j = c_{03}(w - M_1)(w - M_2)(w - M_3),$$

$$\sum_{j=0}^{3} c_{i,3-i} t^i w^{3-i} = c_{03}(w + L_1 t)(w + L_2 t)(w + L_3 t). \quad (2.5)$$

In order for these conditions to be consistent, we find that the following compatibility conditions are necessary

$$M_1 M_2 M_3 = N_1 N_2 N_3 \cdot L_1 L_2 L_3. \quad (2.6)$$

This means that one out of nine equations in (2.5) is used not to determine the coefficient but to determine this compatibility condition. Since the SW curve does not change when we multiply an identical constant to all the coefficient, this degree of freedom can be also used to determine one coefficient as we like. Including this, we can determine 9 coefficients out of 10 in (2.2). The undetermined coefficient $c_{11}$ corresponds to the internal dot in the toric diagram in Figure 4 and is not affected by the boundary condition. This coefficient $c_{11}$ is interpreted as the Coulomb moduli parameter and we denote it as $U$.

Using the degrees of freedom of rescaling $t$ and $w$, we can further impose the conditions on $N_i$ and $M_i$. Together with the compatibility condition, it is convenient to impose

$$M_1 M_2 M_3 = N_1 N_2 N_3 = L_1 L_2 L_3. \quad (2.7)$$

It is then straightforward to find the curve corresponding to the toric diagram, Figure 4:

$$w^3 - \sum_i M_i w^2 + \sum_i L_i w^2 t + \sum_i M_i^{-1} w + U w t \nonumber$$

$$+ \sum_i L_i^{-1} w t^2 - 1 + \sum_i N_i^{-1} t - \sum_i N_i t^2 + t^3 = 0. \quad (2.8)$$

The procedure to move from the diagram in Figure 1 to the one in Figure 3 by moving the [0,1] 7-brane at $t = N_3$ upward by using Hanany-Witten effect can be realized by the coordinate transformation

$$w = W(t - N_3). \quad (2.9)$$

With this transformation, the SW curve (2.8) can be rewritten as

$$(t - N_3)^2 W^3 + (t - N_3) \left( \sum_i L_i t - \sum_i M_i \right) W^2 \nonumber$$

$$+ \left( \sum_i L_i^{-1} t^2 + U t + \sum_i M_i^{-1} \right) W + (t - N_1)(t - N_2) = 0, \quad (2.10)$$
where we have divided entire equation by the factor \((t - N_3)\). Although the corresponding diagram in Figure 3 includes a white dot, we observe that the non-vanishing coefficients still corresponds to the dots of the toric-like diagram also in this case, which is the same as the rule for usual toric diagrams. However, compared to the SW curve corresponding to the usual toric diagram, we find that the coefficients are tuned to be specific values. That is, some extra conditions are imposed to this SW curve due to the white dot. Writing the left hand side of (2.10) as \(\sum_{i,j} c_{ij} t^i W^j\), it is straightforward to see that this SW curve satisfies the following relation, which we interpret as the constraint coming from the boundary conditions:

\[
\begin{align*}
\sum_{i=0}^{2} c_{3i} t^i &\propto (t - N_3)^2, \\
\sum_{i=0}^{2} c_{2i} t^i &\propto (t - N_3), \\
\sum_{i=0}^{2} c_{1i} t^i &\propto (t - N_1)(t - N_2), \\
\sum_{j=0}^{3} c_{2j} W^j &\propto (W - L_1)(W - L_2)(W - L_3), \\
\sum_{j=0}^{3} c_{0j} W^j &\propto \left( W - \frac{M_1}{N_3} \right) \left( W - \frac{M_2}{N_3} \right) \left( W - \frac{M_3}{N_3} \right).
\end{align*}
\] (2.11)

As depicted in Figure 5, what is related to the white dots in the toric-like diagram is the first line of (2.11). The first relation, which gives the leading behavior at \(W \to \infty\), is consistent with the two NS5-branes are coincident, bound by one [0,1] 7-brane. Furthermore, as for the second, which gives the subleading contribution in the region \(W \to \infty\), we would like to
interpret that this is the consequence that one out of two coincident NS5-branes jump over another 5-brane. The remaining three relations are more straightforward to interpret. They correspond to the boundary conditions for the three external D5-branes at \( t \to 0 \), three more external D5-branes at \( t \to \infty \), and two NS5-branes at \( W \to 0 \).

It is worth noting that these boundary conditions (2.11) together with our convention (2.7) are enough to reproduce the SW curve (2.10). This example gives us intuition what kind of boundary condition we should impose, if we have white dots in the toric-like diagram, which we discuss in the next subsection.

### 2.2 General procedure

Based on the computation of the SW curve for \( E_6 \) theory, we propose a systematic procedure of deriving the SW curve from any given toric-like diagram.

First, if all the vertices are black dots, it is actually the usual toric diagram and it is known that the SW curve is given by

\[
\sum_{(i,j) \in \text{vertices}} c_{ij} t^i w^j = 0, \tag{2.12}
\]

where \((i, j)\) are summed over all the vertices in the diagram. The coefficients \(c_{ij}\) corresponding to the dots on the boundary edge of the toric diagram are determined by the boundary condition for the external \((p, q)\) 5-branes. These boundary conditions are given in the form

\[
w^p t^{-q} \sim \tilde{m}_n^{(p,q)} \quad \text{at} \quad |w|^q \sim |t|^p \to \infty, \tag{2.13}
\]

where this \(\tilde{m}_n^{(p,q)}\) corresponds to the "mass parameter"\(^5\). For example, the first line in (2.4) is obtained by identifying \(p = 0, q = -1\) and \(\tilde{m}_n^{(-1,0)} = N_n\). These boundary conditions give the constraints

\[
\sum_{(i,j) \atop ip + jq = N_{i,j}} c_{ij} t^i q^j \propto \prod_n \left( w^p - \tilde{m}_n^{(p,q)} t^q \right), \tag{2.14}
\]

where \(N_{i,j}\) is the maximum of the value \(ip + jq\) among all the combination of \((i, j)\) corresponding to the vertex in the toric diagram. In other words, the \((i, j)\) is summed over the vertex on the boundary edge which is perpendicular to the considered \((p, q)\) 5-brane. We impose this boundary condition corresponding to all the external \((p, q)\) 5-branes and solve for \(c_{i,j}\). Note that in order to have solution, there is one constraint among all the mass parameters.

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\(^5\)The "mass parameter" here is not physical mass of the gauge theory. It is typically the exponential of some linear combination of masses and inverse gauge coupling with some shift.
Generic toric-like diagram are obtained by converting some of the black dots into white dots in the toric diagram. This procedure corresponds to the tuning of the coefficients $c_{i,j}$. Therefore, the SW curve is still the same form as (2.12) but more conditions are added to the coefficients. Suppose that $n$-th $(p, q)$ 7-brane binds $k_n$ external $(p, q)$ 5-brane. In this case, the boundary condition is generalized to the following sets of constraint

$$
\sum_{(i,j)} c_{i,j} t^i q^j \propto \prod_n (w^p - \tilde{m}_{n}^{(p,q)} t^q)^{k_n},
$$

$$
\sum_{(i,j)} c_{i,j} t^i q^j \propto \prod_n (w^p - \tilde{m}_{n}^{(p,q)} t^q)^{\max(k_n-1,0)},
$$

$$
\sum_{(i,j)} c_{i,j} t^i q^j \propto \prod_n (w^p - \tilde{m}_{n}^{(p,q)} t^q)^{\max(k_n-\ell,0)},
$$

$$\cdots$$

These sets of constraints are understood as natural generalization of the first two conditions in (2.11).

2.3 Comment on general procedure

In the previous subsection, we proposed the procedure to derive SW curve from generic toric-like diagram. In this subsection, we discuss that this procedure can be understood as a natural 5d uplift of the SW curve for the following D4-NS5 brane setup with flavor D6-branes [1].

First, we review the related result in [1]. Suppose that we have $n+1$ NS5-branes labeled by $\alpha = 0, 1, \cdots, n$. Between $\alpha$-th NS5-brane and $(\alpha - 1)$-th NS5-brane, $k_{\alpha}$ color D4 branes are suspended $(\alpha = 1, \cdots, n)$. Moreover, $(i_{\alpha} - i_{\alpha-1})$ D6-branes exist between $\alpha$-th NS5-brane and $(\alpha - 1)$-th NS5-brane at the place $v = e_a$ with $a = i_{\alpha-1} + 1, \cdots, i_{\alpha}$, where we put $i_0 = 0$. In this setup, no D4-branes are attached to any of the D6-branes. The D6-branes are uplifted to the Taub-NUT space in the M-theory, which is defined by

$$
\frac{y}{z} = \prod_{a=1}^{i_n} (v - e_a) = \prod_{s=1}^{n} J_s(v),
$$

embedded in a space $\mathbb{C}^3$ with three complex coordinates $^6 y$, $z$, and $v$, where we put

$$J_s(v) = \prod_{a=i_{s-1} + 1}^{i_s} (v - e_a).$$

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^6Our $z$ corresponds to $z^{-1}$ in [1].
It is known that the SW curve is given in the following form:

\[ y^{n+1} + g_1(v)y^n + g_2(v)J_1(v)y^{n-1} + g_3(v)J_1(v)^2J_2(v)y^{n-2} \]
\[ + \cdots + g_\alpha(v) \prod_{s=1}^{\alpha-1} J_s(v)^{\alpha-s} \cdot y^{n+1-\alpha} + \cdots + f \prod_{s=1}^n J_s(v)^{n+1-s} = 0, \quad (2.18) \]

where \( g_\alpha(v) \) are polynomials of degree \( k_\alpha \). Or, if we change the coordinate from \( y \) to \( z \) by using (2.16), it gives

\[ \prod_{s=1}^n J_s^s \cdot z^{n+1} + g_1(v) \prod_{s=2}^n J_s^{s-1} \cdot z^n + g_2(v) \prod_{s=3}^n J_s^{s-2} \cdot z^{n-1} \]
\[ + \cdots + g_\alpha(v) \prod_{s=\alpha+1}^n J_s^{s-\alpha} \cdot z^{n+1-\alpha} + \cdots + g_{n-1}(v)J_n(v)z + f = 0. \quad (2.19) \]

In order to connect this results with our proposal, we reinterpret this in slightly different way. Instead of placing the D6-branes between the NS5-branes without any D4 branes attached, we can move these D6-branes horizontally to infinity using the Hanany-Witten effect. Suppose that we move all the D6-branes to the direction of the outside of the 0-th NS5-brane. Through this process, the \( (\alpha - i_{\alpha-1}) \) D6-branes originally placed between \( \alpha \)-th NS5-brane and \( (\alpha - 1) \)-th NS5-brane pass through NS5-branes \( \alpha \) times, and thus each D6 brane binds \( \alpha \) D4-branes. The bound D4-branes jump over NS5-branes properly in such a way to avoid breaking s-rule. In this setup, since D6-branes are placed at infinity, the space is not Taub NUT space anymore but just flat \( \mathbb{C}^2 \).

We reinterpret (2.19) as the M5-brane configuration exactly in this situation, where the space is now flat \( \mathbb{C}^2 \) spanned by the two coordinate \( v \) and \( z \). All the D6-branes now exist at the region \( z \to \infty \), and the first term in (2.19) is consistent with the situation that \( (i_s - i_{s-1}) \) D6-branes bind \( s \) D4-branes each. The second term has also the factor \( J_s \) but with one power less. As we decrease the power of \( z \), the power of \( J_s \) also reduces one by one. We claim that this is the counterpart of our proposal (2.15). What is generalized in our proposal is that we consider not only for D7-branes but also for arbitrary \([p, q]\) 7-branes. This generalization appears only when we uplift to five dimensions.

We note that if we reinterpret like that, we find that (2.16), which originally defined the multi Taub NUT space, can be seen as the coordinate transformation to move the external D4-branes from one side to the other side by the Hanany-Witten effect, which is the analogue of (2.9). Then, other expression of the curve (2.18) is also consistent with this interpretation, where the \( (i_\alpha - i_{\alpha-1}) \) D6-branes originally placed between \( \alpha \)-th NS5-brane and \( (\alpha - 1) \)-th NS5-brane bind \( n+1 - \alpha \) D4-branes each. Analogous consistency check is also possible for our examples dealt in this paper. That is, even after such coordinate transformation, our proposal (2.15) is still satisfied.
3  \( E_7 \) Seiberg-Witten curve

We now compute the SW curve for \( N_f = 6 \) based on toric-like diagram. Let us first start by adding a flavor to the brane configuration for \( N_f = 5 \), for instance, the first diagram of Figure 2. As shown for the \( N_f = 5 \) case leading to \( T_3 \) diagram, it is then straightforward to obtain a tuned \( T_4 \) diagram, via successive use of the Hanany-Witten transition.

![Diagram](image)

Figure 6: \( N_f = 6 \) brane configuration (left) and tuned \( T_4 \) diagram after Hanany-Witten transitions (right).

The corresponding toric-like diagram is given by

![Diagram](image)

Figure 7: A toric-like diagram with \( E_7 \) symmetry. It can be viewed as a tuned \( T_4 \) diagram. It has manifest \( SU(4) \times SU(4) \times SU(2) \) symmetry.

This is a \( T_4 \) diagram that is specially tuned to have only one Coulomb modulus. It has manifest global symmetry \( SU(4) \times SU(4) \times SU(2) \). It is a maximal compact subgroup of \( E_7 \). If the SW curve is written based on this toric-like diagram, then the curve is expressed in terms of the characters of \( SU(4) \times SU(4) \times SU(2) \). In addition, the curve is supposed to be invariant under the discrete Weyl group of \( E_7 \).

As shown in Figure 1, the curve can be seen in various ways through the Hanany-Witten transition. For instance, one can push a 7-brane upward instead of bringing it downward. This gives a rectangular shape toric-like diagram. See Figure 8.
Figure 8: Another toric-like diagram for $E_7$. It has manifest $S[U(4) \times U(4)]$ symmetry. This can be viewed as a tuned toric diagram for $SU(4)$ gauge theory with eight flavors.

We note that as it is obtained via Hanany-Witten transition, very the same diagram can also be obtained from the tuned $T_4$ diagram above, by taking upward one of the white dot in the bottom. This toric-like diagram for $E_7$ has manifest $S[U(4) \times U(4)]$ symmetry. It is then natural to see $N_f = 6$, $E_7$ theory can be viewed as a a special case of toric diagram for $SU(4)$ gauge theory with $N_f = 8$ flavors.

Figure 9: Another web diagram for $E_7$. We use $\otimes$ to denote the 7-brane that combine 5-branes. It has manifest $S[U(4) \times U(4)]$ symmetry. This can be viewed as a tuned toric diagram for $SU(4)$ gauge theory with eight flavors.
3.1 $N_f = 6$ Seiberg-Witten curve from M-theory

We now compute the SW curve for $N_f = 6$ based on toric-like diagram, which is of the form of the polynomial

\[ \sum_{i=0}^{2} \sum_{j=0}^{4} c_{ij} t^i w^j = 0. \]  

(3.1)

The boundary conditions are given as follows. The polynomial at asymptotic region behaves

\[ t \to 0 : \quad \sum_{j=0}^{4} c_{0j} w^j = c_{04} \prod_{j=1}^{4} (w - \tilde{m}_j), \]

\[ t \to \infty : \quad \sum_{j=0}^{4} c_{2j} t^2 w^j = c_{24} t^2 \prod_{j=5}^{8} (w - \tilde{m}_j), \]  

(3.2)

and the white dots yield degenerate polynomials

\[ w \to 0 : \quad \sum_{i=0}^{2} c_{i0} t^i = c_{20} (t - t_2)^2; \quad \sum_{i=0}^{2} c_{i1} t^i \propto (t - t_2), \]

\[ w \to \infty : \quad \sum_{i=0}^{2} c_{i4} t^i w^4 = w^4 c_{24} (t - t_1)^2; \quad \sum_{i=0}^{2} c_{i3} t^i \propto (t - t_1), \]  

(3.3)
It follows that the dots of the toric-like diagram, especially, \( c_{00}, c_{04}, c_{24}, c_{20} \), are interrelated so that

\[
c_{00} = c_{20} t_2^2 = c_{04} \prod_{i=5}^{8} \tilde{m}_i, \quad c_{04} = c_{24} t_1^2, \quad c_{20} = c_{24} \prod_{i=1}^{4} \tilde{m}_i, \tag{3.4}
\]

which lead to the compatibility condition

\[
\frac{t_2^2}{t_1^2} = \frac{\prod_{i=5}^{8} \tilde{m}_i}{\prod_{j=1}^{4} \tilde{m}_j}. \tag{3.5}
\]

There are 15 dots in the above toric-like diagram. This means that we have 15 non-zero coefficients in the SW curve curve (3.1). From the boundary conditions, eight parameters \( \tilde{m}_i \) associated black dots give 8 conditions as it yields two degree four polynomials (3.2); degenerated polynomials (3.3) associated white dots gives 6 conditions (three conditions on each boundary). Due to the compatibility condition, not all \( t_i \) and \( m_i \) are independent. In total, the boundary conditions taking into account the compatibility condition yield 13 conditions. It means out of 15 coefficients, one can determine 13 coefficients. Among two remaining undetermined coefficients, one can be identified as an overall constant and the other plays a role of the Coulomb modulus \( U \) of the theory which is the black dot in the middle of the toric diagram, \( c_{12} \).

As shifting along the \( t- \) and \( w- \)axes is irrelevant, one can say that there are three rescaling degrees of freedom one can freely choose (overall constant, shifts in \( t \) and \( w \) coordinates). We take the following: For an overall constant,

\[
c_{24} = 1. \tag{3.6}
\]

For the rescaling of \( t \),

\[
t_1 = 1, \tag{3.7}
\]

For the rescaling of \( w \), it is convenient to set the center of mass position of \( m_i \)'s to be unity

\[
\prod_{i=1}^{8} \tilde{m}_i = 1. \tag{3.8}
\]

We note that, together with (3.5), these rescalings yield \( t_2 = \prod_{i=5}^{8} \tilde{m}_i \).

With a little calculation, one finds that

\[
\prod_{i=1}^{4} (w - \tilde{m}_i) t^2 + k(w) t + \prod_{i=5}^{8} (w - \tilde{m}_i) = 0, \tag{3.9}
\]
where
\[ k(w) = -2w^4 + \chi_{\mu_1}^{SU(8)} w^3 + U w^2 + \chi_{\mu_7}^{SU(8)} w - 2. \] (3.10)

Here we used the choice (3.8) enabling us to express in terms as \( \chi_{\mu_i} \) the characters of the fundamental weights \( \mu_i \) of \( SU(8) \),

\[ \chi_{\mu_i}^{SU(8)} \equiv \sum_{k_1<k_2<\ldots<k_i} \tilde{m}_{k_1} \tilde{m}_{k_2} \cdots \tilde{m}_{k_i}. \] (3.11)

By performing the coordinate transformation
\[ t \rightarrow \frac{T}{\prod_{i=1}^4 (w - \tilde{m}_i)}, \] (3.12)
we can write the SW curve in an \( SU(8) \) manifest way as
\[ T^2 + k(w) T + \prod_{i=1}^8 (w - \tilde{m}_i) = 0, \] (3.13)
or
\[ T^2 + (-2w^4 + \chi_{\mu_1} w^3 + U w^2 + \chi_{\mu_7} w - 2)T + w^8 - \chi_{\mu_1} w^7 + \chi_{\mu_2} w^6 - \chi_{\mu_3} w^5 + \chi_{\mu_4} w^4 - \chi_{\mu_5} w^3 + \chi_{\mu_6} w^2 - \chi_{\mu_7} w + 1 = 0, \] (3.14)
where we have dropped the superscript of the characters, \( \chi_{\mu_i} \equiv \chi_{\mu_i}^{SU(8)}. \)

As a coordinate transformation is realized as the Hanany-Witten transition in toric(-like) diagram, (3.12) corresponds to moving all the \([1,0]\) 7-branes on the right hand sides to the left.

\(^7\)Dynkin diagram for \( SU(8) \) is given by

\[ \begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 & \nu_7
\end{array} \]

and \( \mu_i \) are the fundamental weight corresponding to the Dynkin label.
3.2 $E_7$ invariance

In this subsection, we discuss $E_7$ symmetry of the SW curve with six flavors. The curve (3.14) that we obtained for the $N_f = 6$ case is manifestly $SU(8)$ invariant, which is a maximal compact subgroup of $E_7$, and is expected to be $E_7$ invariant. One way to check the $E_7$ invariance is to check whether the curve (3.14) is invariant under the Weyl invariance of $E_7$. This can be done, but in practice it is not so straightforward because it involves mixing of coordinate transformations. Another way, which is more direct, is to compare (3.14) to the curve which is written in a $E_7$ manifest way [15, 16, 17] by examining the modular function called $j$-invariant of the elliptic curve.

To this end, we rewrite the $SU(8)$ invariant curve (3.14) in a form that is easier to extract the $j$-invariant

$$y^2 = (-4U + \chi_{\mu_1}^2 - 4\chi_{\mu_2}) w^4 + (2U\chi_{\mu_1} + 4\chi_{\mu_3} - 4\chi_{\mu_7}) w^3 + (U^2 + 2\chi_{\mu_1}\chi_{\mu_7} - 4\chi_{\mu_4} + 8) w^2 + (2U\chi_{\mu_7} - 4\chi_{\mu_1} + 4\chi_{\mu_3}) w - 4U - 4\chi_{\mu_6} + \chi_{\mu_7}^2$$

(3.15)

where

$$y = T - w^4 + \frac{1}{2}\chi_{\mu_1} w^3 + \frac{1}{2}U w^2 + \frac{1}{2}\chi_{\mu_7} w - 1.$$ 

(3.16)
The $E_7$ manifest curve [16, 17], that we want to compare, is of the following form

$$y^2 = 4x^3 + (-u^2 + 4\chi_{\mu_1}^{E_7} - 100)x^2 + \left((2\chi_{\mu_2}^{E_7} - 12\chi_{\mu_7}^{E_7})u + 4\chi_{\mu_3}^{E_7} - 4\chi_{\mu_6}^{E_7} - 64\chi_{\mu_1}^{E_7} + 824\right)x$$

$$+ 4u^4 + 4\chi_{\mu_7}^{E_7}u^3 + (4\chi_{\mu_6}^{E_7} - 8\chi_{\mu_1}^{E_7} + 92)u^2 + (4\chi_{\mu_3}^{E_7} - 4\chi_{\mu_1}^{E_7}\chi_{\mu_7}^{E_7} - 20\chi_{\mu_2}^{E_7} + 116\chi_{\mu_7}^{E_7})u$$

$$+ 4\chi_{\mu_4}^{E_7} - \chi_{\mu_2}^{E_7}\chi_{\mu_2}^{E_7} + 4\chi_{\mu_3}^{E_7}\chi_{\mu_1}^{E_7} - 40\chi_{\mu_3}^{E_7} + 36\chi_{\mu_6}^{E_7} + 248\chi_{\mu_1}^{E_7} - 2232,$$

(3.17)

where $\chi_{\mu_i}^{E_7}$ is the character of the fundamental weight $\mu_i$ of $E_7$ which is associated to the node of the $E_7$ Dynkin diagram as

$$\begin{align*}
\circ_{\mu_2} & \quad \circ_{\mu_3} & \quad \circ_{\mu_4} & \quad \circ_{\mu_5} & \quad \circ_{\mu_6} & \quad \circ_{\mu_7} \\
\mu_1 & & & & & \\
\end{align*}$$

Expressed in the standard Weierstrass form, this $E_7$ manifest curve is of degree three polynomial in $x$. On the other hand, the our $SU(8)$ manifest curve is quartic in $w$. In order to compare both, we use the $j$-invariant of elliptic curve

$$j = \frac{g_3^3}{g_2^2 - 27g_3^2}. \quad (3.18)$$

For generic cubic and quartic polynomials, the forms of $g_2$ and $g_3$ are given in Appendix C

We first compare the massless case by taking all the mass parameters $\tilde{m}_i$ to unity ($\tilde{m}_i = e^{-\beta m_i} \rightarrow 1$). This means that the character of the fundamental weights becomes the dimension of the corresponding fundamental weights: For $E_7$,

$$\chi_{\mu_1}^{E_7} \rightarrow 133, \quad \chi_{\mu_2}^{E_7} \rightarrow 912, \quad \chi_{\mu_3}^{E_7} \rightarrow 8645,$$

$$\chi_{\mu_4}^{E_7} \rightarrow 365750, \quad \chi_{\mu_5}^{E_7} \rightarrow 27664, \quad \chi_{\mu_6}^{E_7} \rightarrow 1539, \quad \chi_{\mu_7}^{E_7} \rightarrow 56.$$

(3.19)

and the curve (3.17) is written as

$$y^2 = 4x^3 + (432 - u^2)x^2 + (1152u + 20736)x + 4u^4 + 224u^3 + 5184u^2 + 69120u + 442368.$$

(3.20)

The corresponding $j$-invariants is given by

$$\frac{(u - 36)^3}{1728(u - 52)}.$$

(3.21)

For $SU(8)$, the characters again become the dimensions of the representations

$$\chi_{\mu_1} = \chi_{\mu_7} \rightarrow 8, \quad \chi_{\mu_2} = \chi_{\mu_6} \rightarrow 28, \quad \chi_{\mu_3} = \chi_{\mu_5} \rightarrow 56, \quad \chi_{\mu_4} \rightarrow 70,$$

(3.22)
and the curve (3.15) is written as
\[ y^2 = (U + 12) \left[ -4w^4 + 16w^3 + (U - 12)w^2 + 16w - 4 \right]. \] (3.23)

The corresponding \( j \)-invariants for this is given by
\[ \frac{(U - 36)^3}{1728(U - 52)}, \] (3.24)
which coincide with that of \( E_7 \) manifest curve. From this, we can identify the Coulomb moduli parameter \( U \) with \( u \) used in [17].

With this identification of the Coulomb modulus and agreement of the \( j \)-invariant for massless case, one can check a generic massive case. Although it is tedious, it is straightforward to see that the \( j \)-invariant for the \( E_7 \) manifest curve (3.17) exactly coincides with the \( j \)-invariant for the \( SU(8) \) manifest curve (3.15) by implementing the decomposition of the \( E_7 \) fundamental weights into the \( SU(8) \) fundamental weights listed in Appendix D.1. (We list the form of \( g_2 \) and \( g_3 \) for the \( SU(8) \) manifest curve (3.15) in Appendix D.) Therefore, our expression (3.15) of the SW curve for \( N_f = 6 \) flavors describes the \( E_7 \) curve, although it is not manifestly \( E_7 \) invariant.

### 3.3 4d limit of 5d \( E_7 \) Seiberg-Witten curve

From the 5d curve (3.9)
\[ \prod_{i=1}^{4} (w - \tilde{m}_i) t^2 + (w^4 + \chi^{SU(8)}_{\mu_1} w^3 + U w^2 + \chi^{SU(8)}_{\mu_7} w - 2) t + \prod_{i=5}^{8} (w - \tilde{m}_i) = 0, \] (3.25)
whose corresponding toric-like diagram is of rectangular shape given in Figure 8, we discuss 4d limit of 5d theory which is to take zero radius limit of the compactified circle. We associate the radius of the circle \( \beta \) and then the 5d coordinate \( w \) and mass parameters \( \tilde{m}_i \) are related to the 4d coordinate \( v \) and masses \( m_i \) as
\[ w = e^{-\beta v}, \quad \text{and} \quad \tilde{m}_i = e^{-\beta m_i}. \] (3.26)

To take zero size limit of the radius \( \beta \), we expand the Coulomb moduli parameter \( U \) in five dimensions as
\[ U = \sum_{k=0}^{\infty} u_k \beta^k. \] (3.27)

Expansion of the curve (3.25) leads to consistent conditions determining the expansion coefficient of the 5d Coulomb moduli parameter \( U \), and non-trivial relation occurs at order \( \beta^4 \)
which gives rise to 4d SW curve

\[ t^2 \prod_{i=1}^{4} (v - m_i) + t \left( -2v^4 - D_2 v^2 + D_3 v + u \right) + \prod_{i=5}^{8} (v - m_i) = 0, \]  

(3.28)

where \( D_n \equiv \sum_{i_1 < \ldots < i_n} m_{i_1} \cdots m_{i_n} \) are the symmetric product and the 4d Coulomb moduli parameter \( u \) appears at order \( \beta^4 \) in the expansion of \( U \)

\[ U = -12 + 2D_2 \beta^2 + \left( u + \frac{1}{6}(2D_2 - D_2) \right) \beta^4 + O(\beta^5). \]  

(3.29)

![Figure 12: Sphere with three punctures which corresponds to \( E_7 \) CFT.](image)

In the following, we check that the 4d SW curve (3.28) is exactly the SW curve for the 4d \( E_7 \) CFT found in [21, 20], which is given by the quadruple cover of the sphere with three punctures with specific type. See Figure 12. For later convenience, we reparametrize the mass parameters as

\[ m = \frac{1}{4} \sum_{i=1}^{4} m_i = -\frac{1}{4} \sum_{i=5}^{8} m_i, \]

\[ \hat{m}_i = m_i - m \quad (i = 1, 2, 3, 4), \quad \hat{m}_i = m_i + m \quad (i = 5, 6, 7, 8) \]  

(3.30)

By changing the coordinate as

\[ v = xt + m \frac{t + 1}{t - 1}, \]  

(3.31)

we can write the curve in the way

\[ x^4 + \sum_{n=2}^{4} \phi_n(t)x^{4-n} = 0, \]  

(3.32)
with SW one-form $\lambda = xdt$. Here, $\phi_n(t)$ has poles at $t = 0, 1, \infty$ where the three punctures exist. The residues at each pole are given by

\[
\begin{align*}
\{\hat{m}_5, \hat{m}_6, \hat{m}_7, \hat{m}_8\} & \text{ at } t = 0, \\
\{-2m, -2m, 2m, 2m\} & \text{ at } t = 1, \\
\{\hat{m}_1, \hat{m}_2, \hat{m}_3, \hat{m}_4\} & \text{ at } t = \infty,
\end{align*}
\]

which we identify as mass parameters. This is consistent with the type of each puncture. The type of each puncture can be further checked by looking at the order of the pole of $\phi_n$ at each puncture when we turn off the mass parameters associated with the corresponding puncture. The expected order of $\phi_n$ are given by “$n$—(height of the $n$-th boxes)”, where we label the boxes in the Young diagram in such a way that the height of the box does not decrease. See [21] for detail. Denoting the order of the pole of $\phi_n$ as $p_n$, we can explicitly check

\[
\begin{align*}
(p_2, p_3, p_4) &= (1, 2, 3) \text{ at } t = 0 \text{ when } \hat{m}_5 = \hat{m}_6 = \hat{m}_7 = \hat{m}_8 = 0, \\
(p_2, p_3, p_4) &= (1, 1, 2) \text{ at } t = 1 \text{ when } m = 0, \\
(p_2, p_3, p_4) &= (1, 2, 3) \text{ at } t = \infty \text{ when } \hat{m}_1 = \hat{m}_2 = \hat{m}_3 = \hat{m}_4 = 0.
\end{align*}
\]

This is again consistent with the type of punctures. Thus, we have checked that our 4d curve (3.28) agree with that of the 4d $E_7$ CFT.

## 4 $E_8$ Seiberg-Witten curve

In this section, we consider the SW curve for $Sp(1)$ gauge theory with $N_f = 7$ flavors. We begin by adding one more flavor brane to the $N_f = 6$ brane configuration. Via successive applications of the Hanany-Witten transition, it is straightforward to see that it leads to a tuned $T_6$ diagram as in Figure 13.
Figure 13: $N_f = 7$ brane configuration (left) and tuned $T_6$ diagram after Hanany-Witten transitions (right)

Figure 14: A toric-like diagram with $E_8$ symmetry. It can be viewed as a tuned $T_6$ diagram.

The corresponding toric-like diagram is given in Figure 14. As this toric-like diagram is a tuned $T_6$ diagram, it has manifest symmetry of $SU(6) \times SU(3) \times SU(2)$ which is a maximal compact subgroup of $E_8$. If we compute the corresponding SW curve, then the curve will be expressed such that manifest symmetry is $SU(6) \times SU(3) \times SU(2)$. As explained earlier, by performing the Hanany-Witten transition, the manifest symmetry structure is changed to another subgroup of $E_8$. For instance, if we perform the Hanany-Witten transition on one of 7-branes combining three 5-branes on the bottom of the toric-like diagram, Figure 14, one gets the corresponding toric-like diagram of a rectangular shape, Figure 15. It has manifest symmetry of $S[U(6) \times U(3)]$. In particular, we find that it is convenient to work with a rectangular shape of toric-like diagram when we compute the SW curve.

---

8We note that although this toric-like diagram is a tune $T_6$, the number of white dots inside is different from the tuned $T_6$ in [20]. Depending on how one triangulates while keeping one Coulomb modulus, an interior black dot near the boundary can be turned to a white dot.
4.1 \( N_f = 7 \) Seiberg-Witten curve from M-theory

![Diagram](image)

Figure 15: A coefficient in toric like diagram for \( E_8 \)

We now compute the SW curve based on toric-like diagram of a rectangular shape for the \( N_f = 7 \) case. We emphasize again the structure of the SW curve with the white dots. The curve does not depend on the detail of triangulation, but it does depend on white dots on the boundary edges. The white dots on the boundary yield degenerate polynomials of the curve and more importantly the white dots next to the edge also give rise to the degenerate polynomials of one less degree than those corresponding to white dots on the boundary, as explained in section 2.2.

We first write the curve corresponding to Figure 15 as

\[
\sum_{i=0}^{3} \sum_{j=0}^{6} c_{ij} t^i w^j = 0,
\]

where the nonvanishing coefficients \( c_{ij} \) will be determined from the boundary conditions. Given the asymptotic values as \( \tilde{m} \) and \( t \), the boundary conditions for Figure 15 are given as follows:

\[
t \to \infty : \quad c_{36} w^6 + c_{35} w^5 + c_{34} w^4 + c_{33} w^3 + c_{32} w^2 + c_{31} w + c_{30} = c_{36} \prod_{j=1}^{6} (w - \tilde{m}_j),
\]

\[
t \to 0 : \quad c_{06} w^6 + c_{05} w^5 + c_{04} w^4 + c_{03} w^3 + c_{02} w^2 + c_{01} w + c_{00} = c_{06} \prod_{i=7}^{9} (w - \tilde{m}_i)^2;
\]

\[
c_{16} w^6 + c_{15} w^5 + c_{14} w^4 + c_{13} w^3 + c_{12} w^2 + c_{11} w + c_{10} \propto \prod_{i=7}^{9} (w - \tilde{m}_i),
\]
and

\[
\begin{align*}
    w \to \infty : & \quad c_{36} t^3 + c_{26} t^2 + c_{16} t + c_{06} = c_{36} (t - t_1)^3; \\
    c_{35} t^3 + c_{25} t^2 + c_{15} t + c_{05} & \propto (t - t_1)^2; \\
    c_{34} t^3 + c_{24} t^2 + c_{14} t + c_{04} & \propto (t - t_1), \\
    w \to 0 : & \quad c_{30} t^3 + c_{20} t^2 + c_{10} t + c_{00} = c_{30} (t - t_2)^3; \\
    c_{31} t^3 + c_{21} t^2 + c_{11} t + c_{01} & \propto (t - t_2)^2; \\
    c_{32} t^3 + c_{22} t^2 + c_{12} t + c_{02} & \propto (t - t_2). \\
\end{align*}
\]

\[\text{(4.4)}\]

The dots or coefficients of the toric-like diagram are interrelated, especially \(c_{00}, c_{06}, c_{36}, c_{30}\), are constrained so that

\[
c_{00} = -c_{30} t_2^3 = c_{06} \prod_{i=1}^{9} \tilde{m}_i, \quad c_{06} = -c_{36} t_1^3, \quad c_{30} = c_{36} \prod_{i=1}^{6} \tilde{m}_i,
\]

\[\text{(4.6)}\]

leading to the compatibility condition

\[
\frac{t_2^3}{t_1^3} = \frac{\prod_{i=1}^{9} \tilde{m}_i}{\prod_{j=1}^{6} \tilde{m}_j}.
\]

\[\text{(4.7)}\]

We have 28 dots or 28 coefficients to be determined. Let us count conditions from the boundaries: The boundary condition for \(t \to \infty\) gives 6 conditions (4.2); the boundary condition for \(t \to 0\) yields a degree two and degree one degenerate polynomials (4.3) which gives 9 conditions; the boundary condition for \(w \to \infty\) yields a degree three, two, one degenerate polynomials (4.4) which gives 6 conditions; likewise the boundary condition for \(w \to 0\) gives 6 conditions (4.5). Taking into account the compatibility condition (4.7), in total, these give 26 conditions. Two undermined coefficients are an overall constant and one Coulomb modulus \(U\), and thus one can completely determine the curve (4.1).

For convenience, as for three rescaling degrees of freedom, we choose

\[
t_2 = 1, \quad \prod_{i=1}^{9} \tilde{m}_i = 1, \quad c_{36} = 1,
\]

\[\text{(4.8)}\]

which are for the rescaling of \(t\), the rescaling of \(w\), and the overall rescaling, respectively. With a little calculation, one finds the SW curve corresponding to the toric-like diagram, Figure 15, is determined as

\[
\begin{align*}
(t - S_6)^3 w^6 - S_1 (t - S_6)^2 (t - 2 T_1 S_6 S_1^{-1}) w^5 \\
+ S_2 (t - S_6) \left( t^2 - S_2^{-1} [S_1 S_6 T_1 + S_5 + S_6 T_2 + S_6^2 T_2] t + S_2^{-1} S_6^2 (T_1^2 + 2 T_2) \right) w^4 \\
+ \left( - S_3 t^3 + U t^2 - [S_1 S_6^2 T_2 + S_5 S_6 T_1 + 3 S_6 + 3 S_6^2 + S_6^2 T_1 T_2 + S_6^3 T_1 T_2] t + 2 S_6^3 T_1 T_2 + 2 S_6^2 \right) w^3 \\
+ S_4 (t - 1) \left( t^2 - S_4^{-1} [S_1 S_6 + S_5 S_6 T_2 + S_6 T_1 + S_6^2 T_1] t + S_4^{-1} S_6^2 (S_6 T_2^2 + 2 T_1) \right) w^2 \\
- S_5 (t - 1)^2 (t - 2 S_5^{-1} S_6^2 T_2) w + S_6 (t - 1)^3 = 0,
\end{align*}
\]

\[\text{(4.9)}\]
where
\[ S_1 = \sum_{i=1}^{6} \tilde{m}_i, \quad S_2 = \sum_{i,j=1, i<j}^{6} \tilde{m}_i \tilde{m}_j, \quad S_3 = \sum_{i_1, i_2, i_3}^{6} \tilde{m}_{i_1} \tilde{m}_{i_2} \tilde{m}_{i_3}, \ldots, \quad S_6 = \prod_{i=1}^{6} \tilde{m}_i = t_1, \]
\[ T_1 = \sum_{i=7}^{9} \tilde{m}_i, \quad T_2 = \sum_{i,j=7, i<j}^{9} \tilde{m}_i \tilde{m}_j, \quad T_3 = \tilde{m}_7 \tilde{m}_8 \tilde{m}_9, \quad S_6 T_3 = 1. \tag{4.10} \]

We note that this curve is invariant under \( S[U(6) \times U(3)] \).

In terms of the character of the fundamental weights of \( SU(9) \)
\[ \chi_n = \sum_{i=0}^{n} S_{n-i} T_i \quad (S_0 = 1 = T_0, \quad S_{n>6} = 0 = T_{n>3}, \quad \chi_9 = S_6 T_3 = 1), \tag{4.11} \]
the curve is expressed as
\[
\begin{align*}
&\left[ \prod_{i=1}^{6} (w - \tilde{m}_i) \right] t^3 - S_6 \left[ 3w^6 - 2\chi_1 w^5 + (\chi_2 + \chi_8) w^4 + U w^3 + (\chi_1 + \chi_7) w^2 - 2\chi_8 w + 3 \right] t^2 \\
&+ S_6^2 \left[ (3w^3 - \chi_1 w^2 + \chi_8 w - 3) \prod_{j=7}^{9} (w - \tilde{m}_j) \right] t - S_6^3 \prod_{i=7}^{9} (w - \tilde{m}_i)^2 = 0. \tag{4.12}
\end{align*}
\]

We now take the coordinate transformation
\[ t \to -S_6 \prod_{i=7}^{9} \left( \frac{w - \tilde{m}_i}{T} \right), \tag{4.13} \]
to obtain an \( SU(9) \) manifest curve
\[
\begin{align*}
&T^3 + \left( 3w^3 - \chi_1 w^2 + \chi_8 w - 3 \right) T^2 \\
&+ \left( 3w^6 - 2\chi_1 w^5 + (\chi_2 + \chi_8) w^4 + U w^3 + (\chi_1 + \chi_7) w^2 - 2\chi_8 w + 3 \right) T \\
&+ w^9 - \chi_1 w^8 + \chi_2 w^7 - \chi_3 w^6 + \chi_4 w^5 - \chi_5 w^4 + \chi_6 w^3 - \chi_7 w^2 + \chi_8 w - 1 = 0. \tag{4.14}
\end{align*}
\]

As this transformation corresponds to the Hanany-Witten transition putting all the dots in the one side, the corresponding toric-like diagram is given in Figure 16.

\footnote{Dynkin diagram for \( SU(9) \) is given by 
\[
\begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8
\end{array}
\]
and \( \mu_i \) are the fundamental weight associated with the Dynkin label.}
Figure 16: A toric-like diagram for $E_8$ with manifest $SU(9)$ symmetry

4.2 $E_8$ invariance

We now compare the $SU(9)$ manifest SW curve for the $N_f = 7$ case to the known $E_8$ manifest curve [16, 17] to check $E_8$ invariance of the curve \((4.14)\). To this end, with

$$\tilde{T} = \frac{1}{w} \left( T - w^3 + \frac{1}{3} \chi_1 w^2 - \frac{1}{3} \chi_8 w + 1 \right), \quad (4.15)$$

we rewrite \((4.14)\) as

$$\tilde{T}^3 + \left[ \left( -\frac{\chi_1^2}{3} + \chi_2 - \chi_8 \right) w^2 + \left( U + \frac{2\chi_1 \chi_8}{3} + 6 \right) w - \frac{\chi_8^2}{3} - \chi_1 + \chi_7 \right] \tilde{T}$$

$$+ \left( -U - \frac{2}{27} \chi_1^3 + \frac{1}{3} \chi_2 \chi_1 - \frac{1}{3} \chi_8 \chi_1 - \chi_3 - 3 \right) w^3$$

$$+ \left( \frac{U}{3} \chi_1 + \frac{2}{9} \chi_8 \chi_1^2 + \chi_1 + \frac{\chi_8^2}{3} + \chi_4 - \chi_7 - \frac{\chi_2 \chi_8}{3} \right) w^2$$

$$+ \left( -\frac{1}{3} U \chi_8 - \frac{1}{3} \chi_1^2 - \frac{2}{9} \chi_8^2 \chi_1 + \frac{1}{3} \chi_7 \chi_1 + \chi_2 - \chi_5 - \chi_8 \right) w$$

$$+ U + \frac{2\chi_8^3}{27} + \frac{\chi_1 \chi_8}{3} - \frac{\chi_7 \chi_8}{3} + \chi_6 + 3 = 0. \quad (4.16)$$

Observe that this curve is of mutually degree 3 polynomials in $\tilde{T}$ and $w$. One can convert this into the standard Weierstrass form which makes it easier to compare to the known $E_8$ manifest curve (For an explicit coordinate transformation to the Weierstrass form, see, for example, [28]).
The $E_8$ manifest curve [16, 17] is given by

$$y^2 = 4x^3 + \left[ -u^2 + 4\chi_{E_8}^\mu \mu_1 - 100\chi_{E_8}^\mu + 9300 \right] x^2 + \left[ (2\chi_{E_8}^\mu - 12\chi_{E_8}^\mu_7 - 70\chi_{E_8}^\mu_1 + 1840\chi_{E_8}^\mu_8 - 115010)u + 4\chi_{E_8}^\mu - 4\chi_{E_8}^\mu_4 - 64\chi_{E_8}^\mu_1 \chi_{E_8}^\mu_8 + 824(\chi_{E_8}^\mu_8)^2 - 112\chi_{E_8}^\mu_2 + 680\chi_{E_8}^\mu_3 + 8024\chi_{E_8}^\mu_8 - 205744\chi_{E_8}^\mu_8 + 9606776 \right] x + 4u^5 + (4\chi_{E_8}^\mu - 992)u^4 + (4\chi_{E_8}^\mu - 12\chi_{E_8}^\mu_1 - 68\chi_{E_8}^\mu_8 + 93620)u^3 + (4\chi_{E_8}^\mu - 8\chi_{E_8}^\mu_1 \chi_{E_8}^\mu_8 + 92(\chi_{E_8}^\mu_8)^2 - 28\chi_{E_8}^\mu_2 + 540\chi_{E_8}^\mu_4 + 2320\chi_{E_8}^\mu_8 + 30608\chi_{E_8}^\mu_8 - 3823912)u^2 + (4\chi_{E_8}^\mu - 4\chi_{E_8}^\mu_1 \chi_{E_8}^\mu_5 - 20\chi_{E_8}^\mu_2 \chi_{E_8}^\mu_8 + 116\chi_{E_8}^\mu_7 \chi_{E_8}^\mu_8 + 8(\chi_{E_8}^\mu_1)^2 - 52\chi_{E_8}^\mu_3 - 416\chi_{E_8}^\mu_5 + 1436\chi_{E_8}^\mu_1 \chi_{E_8}^\mu_8 - 17776(\chi_{E_8}^\mu_8)^2 + 4180\chi_{E_8}^\mu_2 + 16580\chi_{E_8}^\mu_1 - 182832\chi_{E_8}^\mu_1 + 1103956\chi_{E_8}^\mu_8 + 18130536 \right] u + 4\chi_{E_8}^\mu_4 - (\chi_{E_8}^\mu_2)^2 + 4(\chi_{E_8}^\mu_1)^2 \chi_{E_8}^\mu_8 - 40\chi_{E_8}^\mu_1 \chi_{E_8}^\mu_8 + 36\chi_{E_8}^\mu_8 \chi_{E_8}^\mu_8 + 248\chi_{E_8}^\mu_1 (\chi_{E_8}^\mu_8)^2 - 2232(\chi_{E_8}^\mu_8)^3 + 2\chi_{E_8}^\mu_3 \chi_{E_8}^\mu_8 - 232\chi_{E_8}^\mu_5 + 224\chi_{E_8}^\mu_1 \chi_{E_8}^\mu_8 + 1124\chi_{E_8}^\mu_2 \chi_{E_8}^\mu_8 - 6580\chi_{E_8}^\mu_7 \chi_{E_8}^\mu_8 - 457(\chi_{E_8}^\mu_1)^2 + 4980\chi_{E_8}^\mu_3 + 8708\chi_{E_8}^\mu_8 - 88136\chi_{E_8}^\mu_1 \chi_{E_8}^\mu_8 + 1129964(\chi_{E_8}^\mu_8)^2 - 146282\chi_{E_8}^\mu_2 + 66612\chi_{E_8}^\mu_3 + 6123126\chi_{E_8}^\mu_1 - 104097420\chi_{E_8}^\mu_8 + 2630318907, \quad (4.17)
$$

where the characters of $E_8$, $\chi_{E_8}^\mu$, are associated with the fundamental weights $\mu_i$ assigned to the $E_8$ Dynkin diagram as follows:

$$E_8 \begin{array}{cccccccc}
\circ_{\mu_2} \\
\circ_{\mu_1} & \circ_{\mu_3} & \circ_{\mu_4} & \circ_{\mu_5} & \circ_{\mu_6} & \circ_{\mu_7} & \circ_{\mu_8}
\end{array}
$$

We first check the $j$-invariant for massless cases, where the character becomes the dimension of the representation. For the $E_8$ manifest curve, it reads

$$y^2 = 4x^3 - (u^2 + 14968)x^2 + 8(41644u + 1022767)x + 4u^5 - 76392u^3 + 9420308u^2 - 777374372u + 12225915472. \quad (4.18)
$$

The $j$-invariant is then

$$\frac{u^2}{1728(u - 432)}. \quad (4.19)
$$

For the $SU(9)$ manifest curve, the corresponding curve is expressed as

$$t^3 + (U + 60)wt - (U + 60)w^3 + 3(U + 60)w^2 - 3(U + 60)w + (U + 60) = 0, \quad (4.20)
$$

and the corresponding $j$-invariant is

$$\frac{(U + 60)^2}{1728(U - 372)}. \quad (4.21)\]
Note that two \( j \)-invariants, (4.19) and (4.21), look different, but a shift in the Coulomb moduli parameter can make them coincide with each other. In fact, if one checks the \( j \)-invariant for the case with generic masses, in order to compare two curves, one needs to shift the Coulomb moduli parameters by a constant \(^{10}\)

\[ U \to u - 60. \]  

(4.25)

With this shift of the Coulomb moduli parameter, one can write (4.16) in the standard Weierstrass form

\[ y^2 = 4x^3 - g_2^{SU(9)}x - g_3^{SU(9)}, \]  

(4.26)

where

\[
\begin{align*}
g_2^{SU(9)} &= \frac{1}{12} u^4 - \frac{2}{3} \left( 2349 + \chi_1 \chi_2 - 27 \chi_3 + \chi_1 \chi_5 - 27 \chi_6 + \chi_2 \chi_7 - 25 \chi_1 \chi_8 \\
&\quad + \chi_4 \chi_8 + \chi_7 \chi_8 \right) u^2 + \mathcal{O}(u), \\
g_3^{SU(9)} &= \frac{1}{216} u^6 - 4u^5 - \frac{1}{18} \left( 15579 + \chi_1 \chi_2 + 45 \chi_3 + \chi_1 \chi_5 + 45 \chi_6 + \chi_2 \chi_7 \\
&\quad + 47 \chi_1 \chi_8 + \chi_4 \chi_8 + \chi_7 \chi_8 \right) u^4 + \mathcal{O}(u^3).
\end{align*}
\]  

(4.27)

The complete expressions of \( g_2 \) and \( g_3 \) are complicated and long, so we list them in Appendix E, from which one can compute the \( j \)-invariant. As the global symmetry for \( N_f = 7 \) flavors is expected to be \( E_8 \), this \( j \)-invariant should coincide with that from the \( E_8 \) manifest curve \([17, 28]\)

\[ y^2 = 4x^3 - g_2^{E_8}x - g_3^{E_8}, \]  

(4.28)

where

\[
\begin{align*}
g_2^{E_8} &= \frac{1}{12} u^4 - \left( \frac{2}{3} \chi_1^{E_8} - \frac{50}{3} \chi_8^{E_8} + 1550 \right) u^2 + \mathcal{O}(u), \\
g_3^{E_8} &= \frac{1}{216} u^6 - 4u^5 - \left( \frac{1}{18} \chi_1^{E_8} + \frac{47}{18} \chi_8^{E_8} - \frac{5177}{6} \right) u^4 + \mathcal{O}(u^3).
\end{align*}
\]  

(4.29)

\(^{10}\)We can also compare (4.21) to the \( j \)-invariant of \( SO(16) \) manifest SW curve for \( E_8 \) \([15]\), which reads in the massless case,

\[ y^2 = x^3 + u_N^2 x^2 - 2u_N^5. \]  

(4.22)

The corresponding \( j \)-invariant is

\[ \frac{u_N^2}{54u_N - 729}. \]  

(4.23)

We then find that the two curves are related by a constant shift in the Coulomb modulus

\[ U \to 4(8u_N - 15). \]  

(4.24)
For the explicit form of $g_2^{E_8}$ and $g_3^{E_8}$, see Appendix B.

As shown in (4.27) and (4.29), the leading terms of $g_2$ and $g_3$ for the $SU(9)$ manifest curve and the $E_8$ manifest curve coincide. This implies that one can directly compare terms of $g_2$ and $g_3$ for two cases, which yields agreement of the $j$-invariants. This comparison can be done, as in the $N_f = 6$ case, by decomposing the fundamental weights of $E_8$ into the fundamental weights of $SU(9)$.

We remark that although the decomposition of the $E_8$ weights to the $SU(9)$ ones, in principle, can be performed with help of computer programs, e.g. a Mathematica package like LieART [29] or a computer algebra package LiE [30, 31], it takes a lot of time and resources as the dimension of some weights of $E_8$ is big. With these algebra applications, we have explicitly decomposed relatively lower dimensional weights such as $\chi_1^{E_8}, \chi_2^{E_8}, \chi_3^{E_8}, \chi_6^{E_8}, \chi_7^{E_8}$ and $\chi_8^{E_8}$, and found agreement on $g_2$ as well as the part in $g_3$ that do not involve $\chi_4^{E_8}, \chi_5^{E_8}$. For example, $\chi_5^{E_8}$ and $\chi_4^{E_8}$ appear in the coefficient of the $u$ and $u^0$ terms of $g_3^{E_8}$, except for the these terms of $g_3$ we found agreement. This gives a very strong support on $E_8$ invariance of the $SU(9)$ manifest curve.

For other two higher dimensional weights of $E_8$, we have not completed, as $\chi_4^{E_8}, \chi_5^{E_8}$ are of dimension 6, 899, 079, 264, and 146, 325, 270, respectively. We instead assumed that $g_3$ for both cases do agree and then obtain the decompositions of $\chi_4^{E_8}, \chi_5^{E_8}$ into the weights of $SU(9)$. See E.1 for explicit form the decomposition. We also find the expression of the decomposition of $\chi_4^{E_8}, \chi_5^{E_8}$ does have correct dimension. Based on agreement on $g_2$ expressions as well as massless case, correction dimensions, and uniqueness of the decomposition, we claim that $g_3$ for the $SU(9)$ manifest and $E_8$ curves agree. Hence, we conclude that one can obtain the $SU(9)$ manifest curve (4.26) from the $E_8$ manifest curve (4.28) by decomposing the fundamental weights of $E_8$ into those of $SU(9)$.

4.3 4d limit of 5d $E_8$ Seiberg-Witten curve

We start with our SW curve which is of a manifest $S[U(3) \times U(6)]$ invariant form (4.12). By redefining $-\frac{t}{S_8}$ by $t$, we get

$$\left[ \prod_{i=1}^{6} (w - \tilde{m}_i) \right] t^3 + \left[ 3w^6 - 2\chi_1 w^5 + (\chi_2 + \chi_8) w^4 + U w^3 + (\chi_1 + \chi_7) w^2 - 2\chi_8 w + 3 \right] t^2$$

$$+ \left[ (3w^3 - \chi_1 w^2 + \chi_8 w - 3) \prod_{j=7}^{9} (w - \tilde{m}_j) \right] t + \prod_{i=7}^{9} (w - \tilde{m}_i)^2 = 0,$$

where $w = e^{-\beta w}$ and $\tilde{m}_i = e^{-\beta m_i}$.

Upon reduction to 4d by expanding $w$ and $\tilde{m}_i$ in $\beta$ as well as the 5d Coulomb moduli parameter $U$ as a polynomial of $\beta$ as (3.27), we obtain 4d SW curve at the order of $\beta^6$, 

30
expressed in terms of the symmetric product $D_n \equiv \sum_{i_1 < \cdots < i_n} m_{i_1} \cdots m_{i_n}$, as

$$t^3 \prod_{i=1}^{6} (v - m_i) + t^2 \left[ 3v^6 + 2D_2v^4 - 2D_3v^3 + D_4v^2 - D_5v + u^{4d} \right]$$

$$+ t \left( 3v^3 + D_2v - D_3 \right) \prod_{j=7}^{9} (v - m_j) + \prod_{j=7}^{9} (v - m_j)^2 = 0,$$  \hspace{1cm} (4.31)

where

$$U = 12D_2\beta^2 - D_2^2\beta^4 + \left( u^{4d} - \frac{1}{60} \left( 24D_6 - 4D_4D_2 + 3D_3^2 - 2D_2^3 \right) \right) \beta^6$$  \hspace{1cm} (4.32)

![Figure 17: Sphere with three punctures which corresponds to $E_8$ CFT.](image)

In the following, we check that the 4d SW curve (4.31) is exactly the SW curve for the 4d $E_8$ CFT found in [21, 20], which is given by the sextuple cover of the sphere with three punctures of specific type. See Figure 17. For convenience, we reparametrize the mass parameters as

$$m = \frac{1}{6} \sum_{i=1}^{6} m_i = -\frac{1}{6} \sum_{i=5}^{8} m_i,$$

$$\hat{m}_i = m_i - m \quad (i = 1, 2, 3, 4, 5, 6), \quad \hat{m}_i = m_i + 2m \quad (i = 7, 8, 9).$$  \hspace{1cm} (4.33)

By changing the coordinate as

$$v = xt - m \frac{t - 2}{t + 1},$$  \hspace{1cm} (4.34)

we can write the curve in the way

$$x^6 + \sum_{n=2}^{6} \phi_n(t)x^{6-n} = 0,$$  \hspace{1cm} (4.35)
with SW one-form $\lambda = xdt$. Here, $\phi_n(t)$ has poles at $t = 0, -1, \infty$ where the three punctures exist. The residues at each pole are given by

$$
\{\hat{m}_7, \hat{m}_7, \hat{m}_8, \hat{m}_8, \hat{m}_9, \hat{m}_9\} \quad \text{at} \quad t = 0,
$$

$$
\{-3m, -3m, -3m, 3m, 3m, 3m\} \quad \text{at} \quad t = -1,
$$

$$
\{\hat{m}_1, \hat{m}_2, \hat{m}_3, \hat{m}_4, \hat{m}_5, \hat{m}_6\} \quad \text{at} \quad t = \infty,
$$

which we identify as mass parameters. This is consistent with the type of each puncture. The type of each puncture can be further checked by looking at the order of the pole of $\phi_n$ at each puncture when we turn off the mass parameters associated with the corresponding puncture. Denoting the order of the pole of $\phi_n$ as $p_n$, we can explicitly check

$$
(p_2, p_3, p_4, p_5, p_6) = (1, 2, 2, 3, 4) \quad \text{at} \quad t = 0 \quad \text{when} \quad \hat{m}_7 = \hat{m}_8 = \hat{m}_9 = 0,
$$

$$
(p_2, p_3, p_4, p_5, p_6) = (1, 1, 2, 2, 3) \quad \text{at} \quad t = -1 \quad \text{when} \quad m = 0,
$$

$$
(p_2, p_3, p_4, p_5, p_6) = (1, 2, 3, 4, 5) \quad \text{at} \quad t = \infty \quad \text{when} \quad \hat{m}_1 = \hat{m}_2 = \cdots = \hat{m}_6 = 0. \quad (4.37)
$$

This is again consistent with the type of punctures. Thus, we have checked that our 4d curve (4.31) agree with that of the 4d $E_8$ CFT.

## 5 Mass decoupling limit

In this section, we discuss “mass decoupling” limit of 5d theory with $E_8$ ($N_f = 7$). Here masses are positions of semi-infinite $D7$ branes in $(p, q)$ web. As in 4d, one can take large mass limit such that the flavor associated with large mass decouples which yields that rank of global symmetry group is reduced to lower one. As toric-like diagrams of $E_7$ theory can be naturally embedded into toric-like diagrams of $E_8$ theory, one expects that mass decoupling limit of $E_8$ toric-diagram leads to a $E_7$ toric-like diagram.

For mass decoupling limit from $E_8$ to $E_7$ SW curve, consider for the $SU(9)$ manifest curve for $N_f = 7$ flavors, Figure 16. We take the following scaling limit

$$
\tilde{m}_1 \to L^{-1} \tilde{m}_1, \quad \tilde{m}_9 \to L, \quad \tilde{m}_i \to \tilde{m}_i \quad (i = 2, \cdots, 8),
$$

and

$$
U \to LU, \quad w \to w, \quad t \to t. \quad (5.2)
$$

This scaling leads that the fundamental weight of $SU(9)$ scales like

$$
\chi_i \sim L^{U(7)} \chi_{i-1} \quad (i = 2, \cdots, 8), \quad (5.3)
$$
where $\chi_0^{U(7)} \equiv 1$ and $\chi_i^{U(7)}$ are the $U(7)$ fundamental characters. For instance, $\chi_1^{U(7)} = \sum_{i=2}^{8} \tilde{m}_i$ and $\chi_7^{U(7)} = \prod_{i=2}^{8} \tilde{m}_i$. It follows from the $SU(9)$ traceless condition that in this scaling one obtains $SU(8)$ traceless condition $\prod_{i=1}^{8} \tilde{m}_i = 1$, and thus $\chi_7^{U(7)} = \tilde{m}_1^{-1}$.

By taking large $L$ limit, we find that the $SU(9)$ manifest SW curve (4.14) becomes

$$
\left( -w + \chi_7^{U(7)} \right) T^2 + \left( -2w^4 + (\chi_1^{U(7)} + \chi_7^{U(7)})w^3 + Uw^2 + (1 + \chi_6^{U(7)})w - 2\chi_7^{U(7)} \right) T
$$

$$
- w^7 + \chi_1^{U(7)} w^6 - \chi_2^{U(7)} w^5 + \chi_3^{U(7)} w^4 - \chi_4^{U(7)} w^3 + \chi_5^{U(7)} w^2 - \chi_6^{U(7)} w + \chi_7^{U(7)} = 0.
$$

(5.4)

This has $S[U(7) \times U(1)]$ which can be also read off from the corresponding toric-like diagram below.

Figure 18: $E_7$ toric-like diagram of a manifest $S[U(7) \times U(1)]$ symmetry

We now take the Hanany-Witten transition to move the 7-brane on the right to the left. It leads to the same $SU(8)$ manifest toric-like diagram as Figure 11. As Hanany-Witten effect is realized as a coordinate transformation, we have

$$
T \rightarrow (-w + \chi_7^{U(7)})^{-1} t,
$$

(5.5)

which gives

$$
t^2 + \left( -2w^4 + (\chi_1^{U(7)} + \chi_7^{U(7)})w^3 + Uw^2 + (1 + \chi_6^{U(7)})w - 2\chi_7^{U(7)} \right) t
$$

$$
+ \left( -w + \chi_7^{U(7)} \right) \left( -w^7 + \chi_1^{U(7)} w^6 - \chi_2^{U(7)} w^5 + \chi_3^{U(7)} w^4 - \chi_4^{U(7)} w^3 + \chi_5^{U(7)} w^2 - \chi_6^{U(7)} w + \chi_7^{U(7)} \right) = 0.
$$

(5.6)

In order to compare to (3.14), we redefine the coordinate

$$
t \rightarrow \tilde{m}_1^{-1} T, \quad w \rightarrow \tilde{m}_1^{-\frac{1}{4}} w,
$$

(5.7)
which leads to the the $SU(8)$ manifest curve for $N_f = 6$ (3.14) with the decomposition between the fundamental characters of $S[U(7) \times U(1)]$ and $SU(8)$

$$
\chi_n^{U(7)} \tilde{m}_1^2 + \chi_{n-1}^{U(7)} \tilde{m}_1^2 = \chi_n^{SU(8)},
$$

(5.8)

where $\chi_0^{U(7)} = 1$, $\chi_7^{U(7)} = \tilde{m}^{-1}$, and $n = 1, \ldots , 7$.

We note that there exits another scaling of mass parameters which is equivalent up to $E_8$ Weyl transformation

$$
\tilde{m}_i \rightarrow L^{\frac{2}{3}} \tilde{m}_i \ (i = 1, 2, 3), \quad \tilde{m}_j \rightarrow L^{-\frac{1}{3}} \tilde{m}_j \ (j = 4, \cdots , 9),
$$

(5.9)
as well as

$$
U \rightarrow LU, \quad w \rightarrow L^{-\frac{1}{3}} w, \quad T \rightarrow T.
$$

(5.10)

As the shows two distinct scaling behaviors, it leads to the following decomposition of $SU(9)$ characters into $SU(3) \times SU(6)$ characters:

$$
\chi_n \rightarrow L^{\frac{2n}{3}} \chi_n^{SU(3)} \ (n = 1, 2, 3), \quad \chi_n \rightarrow L^{-\frac{n}{3}} \chi_{9-n}^{SU(6)} \ (n = 4, \cdots , 8),
$$

(5.11)
satisfying $\chi_3^{SU(3)} = 1 = \chi_6^{SU(6)}$. Taking large $L$ limit, the $E_8$ SW curve (4.14) is expressed as

$$
T^3 + \left( - \chi_1^{SU(3)} w^2 + \tilde{\chi}_1^{SU(6)} w - 3 \right) T^2
+ \left( \chi_2^{SU(3)} w^4 + U w^3 + (\chi_1^{SU(3)} + \chi_2^{SU(6)}) w^2 - 2 \tilde{\chi}_1^{SU(6)} w + 3 \right) T
- w^6 + \tilde{\chi}_5^{SU(6)} w^5 - \tilde{\chi}_4^{SU(6)} w^4 + \tilde{\chi}_3^{SU(6)} w^3 - \tilde{\chi}_2^{SU(6)} w^2 + \tilde{\chi}_1^{SU(6)} w - 1 = 0,
$$

(5.12)

which is of manifest $SU(6) \times SU(3)$ symmetry, a maximal compact subgroup of $E_7$. The corresponding toric-like diagram is given by Figure 19.

![Figure 19: Toric-like diagram of $E_7$ theory with a manifest $SU(6) \times SU(3)$ symmetry](image)
One can show that this diagram is equivalent to toric-like diagram for $E_7$ global symmetry by applying the Hanany-Witten transition several times.

The mass decoupling limit from $E_7$ to $E_6$ and to lower $E_n$ can be done in a similar fashion, as lower $E_n$ toric (or toric-like) diagram is embedded into higher $E_n$ diagram. For example, $E_6$ rectangular shape toric-like diagram in Figure 3 is embedded to $E_7$ toric-like diagram, Figure 8. For $N_f \leq 4$, mass decoupling limit is straightforward and given in Appendix A.

6 Rank-$N$ $E_n$ curve

Toric-like diagrams for higher rank $E_n$ theories are proposed in [20] based on symmetry, dimension of the Higgs branch as well as the Coulomb branch. As in the rank-1 case, they are embedded in $T_N$. More precisely, the rank-$N$ $E_6$, $E_7$, and $E_8$ theories are embedded in $T_{3N}$, $T_{4N}$, and $T_{6N}$, respectively, such that $N$ 5-branes are bound together with the same 7-branes as in the rank-1 case, on each side of the multi-junction. This means that the number of 7-branes does not change regardless of rank of the gauge group, $Sp(N)$, and hence global symmetry is still $E_n$.

In this section, we consider the SW curve for the higher rank $E_n$ theories based on toric-like diagram. Computing the curve for the corresponding toric-like diagram is straightforward, following the properties of the white dots in the previous sections, so here we do not give details of computation, rather we sketch how the computation can be done. As explained before, to find the SW curve, it is convenient to implement the Hanany-Witten transition on the (tuned) $T_N$ diagram so that the resultant diagram is of a rectangular shape.

![Figure 20](image)

Figure 20: (Left) Rank-1 $E_7$ toric-like diagram, (Right) Rank-2 $E_7$ toric-like diagram

Consider toric-like diagram for rank-1 and rank-2 $E_7$ curves above. The toric-like diagram
for rank-1 (on the left of Figure 20) is the same one as Figure 8. The black dot in the middle of the rank-1 diagram corresponds to the Coulomb modulus. The toric-like diagram for rank-2 (on the right of Figure 20) is obtained as follows: Given two dots which are next to each other along the outer edges of the rank-1 diagram, one inserts a white dot such that, from the point of view of \((p, q)\)-web, the number of semi-infinite 5-branes is doubled while they are combined with the same 7-branes. The dots inside of the rank-2 diagram are introduced such that it does not break the s-rule, and dimension of the Coulomb moduli gets doubled to account for rank-2. This procedure of making multi-junction is also applicable to rank-\(N\) diagrams with any \(N_f(\leq 7)\) flavors.

To compute the SW curve

\[
\sum_{i=0}^{4} \sum_{j=0}^{8} c_{ij} t^i w^j = 0, \tag{6.1}
\]

we write the boundary conditions

\[
t \to 0 : \quad \sum_{j=0}^{8} c_{0j} w^j = c_{08} \prod_{j=1}^{4} (w - \tilde{m}_j)^2; \quad \sum_{j=0}^{8} c_{1j} t w^j \propto t \prod_{j=1}^{4} (w - \tilde{m}_j), \tag{6.2}
\]

\[
t \to \infty : \quad \sum_{j=0}^{8} c_{4j} t^4 w^j = c_{48} t^4 \prod_{j=5}^{8} (w - \tilde{m}_j)^2; \quad \sum_{j=0}^{8} c_{3j} t^3 w^j \propto t^3 \prod_{j=5}^{8} (w - \tilde{m}_j), \tag{6.3}
\]

and

\[
w \to 0 : \quad \sum_{i=0}^{4} c_{i0} t^i = c_{40} (t - t_2)^4; \quad \sum_{i=0}^{4} c_{i1} t^i w \propto c_{41} w(t - t_2)^3;
\]

\[
\sum_{i=0}^{4} c_{i2} t^i w^2 \propto c_{42} w^2(t - t_2)^2; \quad \sum_{i=0}^{4} c_{i3} t^i w^3 \propto c_{43} w^3(t - t_2), \tag{6.4}
\]

\[
w \to \infty : \quad \sum_{i=0}^{4} c_{i8} t^i w^8 = c_{48} w^8(t - t_1)^4; \quad \sum_{i=0}^{4} c_{i7} t^i w^7 \propto w^7(t - t_1)^3;
\]

\[
\sum_{i=0}^{4} c_{i6} t^i w^6 \propto w^6(t - t_1)^2; \quad \sum_{i=0}^{4} c_{i5} t^i w^5 \propto w^5(t - t_1). \tag{6.5}
\]

As explained in (3.5), not all parameters are independent but they are constrained by the same compatibility condition as for the rank-1 case

\[
\frac{t_2^2}{t_1^2} = \frac{\prod_{i=5}^{8} \tilde{m}_i}{\prod_{j=1}^{4} \tilde{m}_j}.
\]

We note that unlike rank-1 case, it turns out that there is another set of coefficients which is related each other again by this compatibility condition.
Let us count the number of the coefficients and the conditions from the boundaries. There are 45 dots (or non-vanishing coefficients) in the rank-2 $E_7$ toric-like diagram. For the boundary conditions $t \to 0$ and $t \to \infty$, one finds $(8 + 4) \times 2 = 24$ conditions; for the boundary conditions $w \to 0$ and $w \to \infty$, one finds $(4 + 3 + 2 + 1) \times 2 = 20$ conditions. Recall that the compatibility condition indicates that not all boundary conditions are independent. For rank-2, there are two sets of conditions, as mentioned above. The compatibility condition for rank-2 tells us that the number of independent conditions is 42 instead of 44. Hence, one is left with 3 undetermined coefficients; One of them is an overall constant (or rescaling) and the other two are two Coulomb moduli for the rank-2 theory.

Let us give a bit more explanation for the compatibility condition. Along the four boundary edges, there are 24 dots, and we have 24 conditions from the boundary conditions above. These boundary conditions interrelate the 24 coefficients. It turns that these 24 conditions do not have solutions unless the compatibility condition is satisfied. Given this compatibility condition, 23 out of 24 conditions are independent. The undetermined coefficient is associated with choice of overall rescaling.

Now consider the dots $c_{11}, c_{21}, c_{31}$ along the next-to-boundary edge on the bottom. As there are three dots here and we have 3 conditions from the property of the white dots, the coefficients $c_{11}, c_{21}, c_{31}$ are all determined. The same logic applied to the dots $c_{17}, c_{27}, c_{37}$ along the next-to-boundary edge on the top. We then consider the vertical five dots $c_{12}, \cdots, c_{16}$, we know there are only 4 conditions from the boundary conditions (6.2). Similarly, we have the vertical five dots $c_{32}, \cdots, c_{36}$ and 4 conditions (6.3). We also have three dots $c_{32}, c_{22}, c_{12}$ and 2 conditions from (6.4). Likewise, three dots $c_{36}, c_{26}, c_{16}$ and 2 conditions from (6.5). In total 12 dots with

Figure 21: Rank-2 $E_7$ toric-like diagram

Let us give a bit more explanation for the compatibility condition. Along the four boundary edges, there are 24 dots, and we have 24 conditions from the boundary conditions above. These boundary conditions interrelate the 24 coefficients. It turns that these 24 conditions do not have solutions unless the compatibility condition is satisfied. Given this compatibility condition, 23 out of 24 conditions are independent. The undetermined coefficient is associated with choice of overall rescaling.

Now consider the dots $c_{11}, c_{21}, c_{31}$ along the next-to-boundary edge on the bottom. As there are three dots here and we have 3 conditions from the property of the white dots, the coefficients $c_{11}, c_{21}, c_{31}$ are all determined. The same logic applied to the dots $c_{17}, c_{27}, c_{37}$ along the next-to-boundary edge on the top. We then consider the vertical five dots $c_{12}, \cdots, c_{16}$, we know there are only 4 conditions from the boundary conditions (6.2). Similarly, we have the vertical five dots $c_{32}, \cdots, c_{36}$ and 4 conditions (6.3). We also have three dots $c_{32}, c_{22}, c_{12}$ and 2 conditions from (6.4). Likewise, three dots $c_{36}, c_{26}, c_{16}$ and 2 conditions from (6.5). In total 12 dots with
12 conditions. However, here the coefficients along this rectangular are in interrelated, just like how the compatibility condition (3.5) was obtained. It thus gives another undetermined coefficient, which is related to the Coulomb moduli for the rank-2 case. Recall that the dot in the middle of the diagram is not determined, and this dot is another undetermined coefficient $c_{24}$. Together, these two unknown coefficients account for two Coulomb moduli of the rank-2 theory. With suitable identification of these Coulomb moduli parameters, for instance, with $U_1 + U_2$ for the first undermined coefficient and $U_1 U_2$ for $c_{24}$, one finds that the rank-2 SW curve is factorized as the product of the rank-1 SW curves of different Coulomb moduli parameters

$$\left( \prod_{i=1}^{4} (w - \tilde{m}_i) t^2 + (-2 w^4 + \chi_{\mu_1}^{SU(8)} w^3 + U_1 w^2 + \chi_{\mu_7}^{SU(8)} w - 2) t + \prod_{i=5}^{8} (w - \tilde{m}_i) \right)$$

$$\times \left( \prod_{i=1}^{4} (w - \tilde{m}_i) t^2 + (-2 w^4 + \chi_{\mu_1}^{SU(8)} w^3 + U_2 w^2 + \chi_{\mu_7}^{SU(8)} w - 2) t + \prod_{i=5}^{8} (w - \tilde{m}_i) \right) = 0. \quad (6.6)$$

We note that even though rank-2 in this case means $Sp(2)$ and the $Sp(2)$ gauge theories contain an additional antisymmetric hypermultiplet compare with $Sp(1)$ theory, the rank-2 SW curve (6.6) does not describe a generic $Sp(2)$ SW curve with antisymmetric hypermultiplet, rather it describes the $Sp(2)$ SW curve with the vanishing mass of antisymmetric hypermultiplet.

Generalization to rank-$N$ is straightforward. Toric-like diagram for rank-$N$ is a generalization of the rank-2 diagram. The compatibility condition is the exactly the same as that for rank-1. As in rank-2 case, $N$ boundary conditions are redundant for rank-$N$ case.

For instance, if the number of dots of the corresponding toric-like diagram for rank-$N$ is $n$, then there are $n - 1$ conditions from the boundary conditions but taking into account the compatibility conditions, the number of independent conditions is $n - 1 - N$. Hence, among $n$ coefficients of the SW curve, $n - 1 - N$ coefficients are fully specified by the parameters of the theory, and the undermined $N + 1$ coefficients are one overall rescaling and $N$ Coulomb moduli parameters.

The product of SW curve for rank-1 satisfies the boundary conditions for the toric-like diagram of rank-$N$ theory proposed by [20]. We thus claim that the SW curve for rank-$N$ theory is also factorized as the product form of the rank-1 SW curves

$$SW_N(U_1, U_2, \ldots, U_N) = \prod_{i=1}^{N} SW_i(U_i), \quad (6.7)$$

where we denoted $SW_n$ as the SW curve for rank-$n$ with $E_{\nu_f+1}$ symmetry, and $U_i$ as the corresponding Coulomb moduli parameters. This describes the $Sp(N)$ SW curve of $E_{\nu_f+1}$ symmetry with the vanishing masses of antisymmetric hypermultiplet.
7 Summary and discussion

In this paper, we have proposed a systematic procedure to compute the Seiberg-Witten (SW) curve from generic toric-like diagram. Using this method, we have computed the SW curve for five-dimensional $\mathcal{N} = 1$ $Sp(1)$ gauge theory with $N_f = 6, 7$ flavors, which are expected to have UV fixed with $E_7, E_8$ global symmetry, respectively. Although $E_n$ global symmetry does not look manifest in our expression at first sight, various subgroup of it can be seen by simple coordinate transformations realizing the Hanany-Witten transition. Our SW curves are computed in totally different method from the previously known $E_n$ manifest results, which are computed by using E-string effective action. By comparing the $j$ invariants, or by performing coordinate transformation to express as the Weierstrass standard form, we find that our result agrees with the known results. Mass decoupling limits are discussed to connect the SW curve for the theory with less flavor. 4d limit reproduces 4d SW curve, as expected.

We have also computed the SW curve for five-dimensional $Sp(2)$ gauge theory with six fundamental flavors and one massless antisymmetric tensor, which is identified as rank-2 $E_7$ theory, and have shown that it reduces to just the two copies of SW curve for $Sp(1)$ gauge theory with six fundamental flavors. Our results strongly implies that the SW curve for the five-dimensional $Sp(N)$ gauge theory with $N_f$ fundamental flavors and one massless antisymmetric tensor, which is rank-$N$ $E_{N_f+1}$ CFT, is also factorized into $N$ copies of the SW curve of the rank-1 $E_{N_f+1}$ CFT.

As our method is applicable to any toric-like diagram, there will be more applications to various theories. One of the interesting directions is the five-dimensional uplift of class S theory. The toric diagram for the five-dimensional uplift of the $T_N$ theory has been given in [20], and the corresponding SW curve has been studied in [20, 24]. By replacing some of the full punctures of $T_4$ and $T_6$ with certain type of degenerate punctures, $E_7$ and $E_8$ theories are obtained which is studied in the paper. It is straightforward to write down toric-like diagram corresponding to the sphere with three punctures of arbitrary type, which is the counterpart of the “pants” in the context of pants decomposition, and thus computing the corresponding SW curve can be done. We have already shown in this paper that the SW curve for some of such example can be factorized. It would be interesting to consider classification of the 5d uplifts of the pants. Moreover, it is natural to expect that 5d uplift of class S theories can be obtained by connecting such pants analogously to the pants decomposition in the 4d class S theories. The corresponding SW curve can be computed based on the method developed in this paper.

When we consider the 5d uplift of class S theory, an important issue is to uplift the “$\mathcal{N} = 2$ dualities” [21], which is generalization of electric magnetic duality of the $SU(2)$ SW theory or Argyres-Seiberg duality for $SU(3)$ theory with six flavors. Related issue has been addressed in various papers including [24, 27, 32]. Especially, it is pointed out in [24], that the SW curve
of $E_6$ CFT is obtained by tuning some of the parameters of the SW curve for $SU(3)$ gauge theory with six flavors with proper identification of the mass parameters, which we expect to be related to Argyres-Seiberg duality. Using the result of this paper, we can find several more examples analogous to this.

![Diagram of $E_7$ CFT construction](image)

Figure 22: Construction of $E_7$ CFT. Start from $SU(4)$ with eight flavors, take the strong coupling limit, and Higgs one of the punctures.

As an example, 4d $E_7$ CFT is constructed from $SU(4)$ gauge theory with eight flavors with $N = 2$ dualities and Higgsing procedure to reduce the order of a puncture. See Figure 22. Especially, it means that the SW curve for 4d $E_7$ CFT is obtained from the SW curve for 4d $SU(4)$ gauge theory with eight flavors. This analogue can be seen in five dimensions. From the toric-like diagram for $E_7$ theory in Figure 10, one would find that it corresponds to the toric diagram for $SU(4)$ with 8 flavors if all the dots are black. Changing black dots to white dots corresponds to tuning the parameters in a specific way that is discussed in this paper. Therefore, we observe that the SW curve for 5d $E_7$ is obtained from the SW curve for 5d $SU(4)$ with eight flavors by tuning parameters. Especially, this tuning turns out to include strong coupling limit. It is, therefore, natural to expect that our observation is related to the construction of 4d $E_7$ CFT, where the $\mathcal{N} = 2$ dualities play essential role.

![Diagram of $E_8$ CFT construction](image)

Figure 23: Construction of $E_8$ CFT. Start from $SU(6)^2$ theory with 6+6 flavors, take the weak coupling limit in the S-dual frame, and Higgs two of the punctures.

Another example is the construction of the $E_8$ CFT from $SU(6) \times SU(6)$ gauge theory with 6+6 flavors. There exists such construction in 4d level. See Figure 23. From the toric-like diagram for 5d $E_8$ theory in Figure 15, we observe that the SW curve for $E_8$ CFT is obtained from that for the $SU(6) \times SU(6)$ gauge theory with 6+6 flavors in five dimensions. This observation also implies that topological string amplitude for $E_8$ theory computed in [26] can be obtained as a limit of that for $N_f = 6$ $SU(6) \times N_f = 6$ $SU(6)$ theory.
Although we saw nice observation which is expected to be related to the $\mathcal{N} = 2$ duality, it is not clear how this observation is related to what is discussed in [27]. We would like clarify this point in the future.

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**A Seiberg-Witten curves for $N_f \leq 4$ flavors**

In this appendix, derivation of the Seiberg-Witten curve from toric-diagram.

**A.1 $N_f = 0$ with $E_1$ or $\tilde{E}_1$ flavor symmetry**

**A.1.1 $E_1$ symmetry**

![Figure 24: Toric web diagram for $N_f = 0$ of $E_1$ flavor symmetry](image)

The SW curve for the pure case ($N_f = 0$) is of the form

$$c_{01} w + c_{10} t + c_{11} tw + c_{12} tw^2 + c_{21} t^2 w = 0.$$  \hspace{1cm} (A.1)

Though we have thus five unknowns from the beginning, including three rescalings, one is left with only two parameters which are one Coulomb modulus and the dynamical scale $\Lambda_0$. We
show how one can identify them below. We start with asymptotic boundary conditions:

\[
\begin{align*}
|w| & \sim |t| \text{ large : } c_{12}tw^2 + c_{21}t^2w = c_{12}tw(w + \ell_1 t) \\
|w^{-1}| & \sim |t| \text{ large : } c_{10}t + c_{21}t^2w = c_{21}t(\ell_2 + tw) \\
|w| & \sim |t^{-1}| \text{ large : } c_{12}tw^2 + c_{01}w = c_{12}w(tw + \ell_3) \\
|w^{-1}| & \sim |t^{-1}| \text{ large : } c_{01}w + c_{10}t = c_{01}(w + \ell_4 t). 
\end{align*}
\]

(A.2)

The compatibility condition is then

\[ \ell_1 \ell_2 = \ell_3 \ell_4. \]  

(A.3)

As for three rescalings degrees of freedom, we take

\[ c_{01} = c_{21} = 1, \quad c_{10} = c_{12} = \ell_2 = \ell_1^{-1} = \ell_4, \quad c_{11} \propto U, \]

(A.4)

and the SW curve for \( N_f = 0 \) is expressed in terms of two parameters, the Coulomb modulus \( U \), and \( \ell_1 \):

\[ w + \ell_1^{-1}t \left( w^2 + Uw + 1 \right) + t^2w = 0. \]

(A.5)

The \( \ell_1 \) is in fact related to the dynamical scale \( \Lambda_0 \). We explain how to relate it to the dynamical scale. By referring to the topological vertex results, the dynamical scale is determined as the geometric mean of differences \( \Delta t \) evaluated in asymptotic values of \( t \) for given \( w \). This is consistent with the dependence of the dynamical scale over energy scales

\[
\sqrt{\left( \frac{\ell_3 w_1^{-1}}{\ell_1^{-1} w_1} \right) \left( \frac{\ell_2 w_2^{-1}}{\ell_4^{-1} w_2} \right)} = \left( \frac{\ell_1 \ell_3}{\ell_2 \ell_4} \right)^{\frac{1}{2}} w_2 w_1^{-1} \bigg|_{w_1 = w_2 = w_0} = (2\pi R\Lambda_0)^{2N_c - N_f}. \]

(A.6)

We then find

\[ \ell_1 = \left( 2\pi R\Lambda_0 \right)^2, \]

(A.7)

and thus the SW curve for \( N_f = 0 \) is written as

\[ w + \frac{t}{\left( 2\pi R\Lambda_0 \right)^2} \left( w^2 + Uw + 1 \right) + t^2w = 0. \]

(A.8)

This has \( E_1 \) flavor symmetry.
There exists an inequivalent toric diagram which has different asymptotic boundary conditions compared with the above $E_1$ case. It is known as $\tilde{E}_1$ theory. The asymptotic boundary conditions are given by

\[
\begin{align*}
|w^{-1}| \sim |t^2| & \text{ large : } c_{10} t + c_{22} t^2 w^2 = c_{22} t (t w^2 + \ell'_2) \quad \Rightarrow \quad c_{10} = \ell'_2 c_{22}, \\
|w| & \text{ large : } c_{12} t w^2 + c_{22} t^2 w^2 = c_{22} t w^2 (-t_1 + t) \quad \Rightarrow \quad c_{12} = -t_1 c_{22}, \\
|w| \sim |t^{-1}| & \text{ large : } c_{12} t w^2 + c_{01} w = c_{12} w (t w + \ell_3) \quad \Rightarrow \quad c_{01} = \ell_3 c_{12}, \\
|w^{-1}| \sim |t^{-1}| & \text{ large : } c_{01} w + c_{10} t = c_{01} (w + \ell_4 t) \quad \Rightarrow \quad c_{10} = \ell_4 c_{01}, \quad (A.9)
\end{align*}
\]

subject to the compatibility condition

\[
\ell'_2 = -t_1 \ell_3 \ell_4. \quad (A.10)
\]

We choose the three rescaling degrees of freedom to be

\[
c_{12} = c_{10}, \quad c_{22} = 1 = c_{01}, \quad (A.11)
\]

The dynamical scale $\tilde{\Lambda}_0$ is

\[
\ell_4 = (2\pi R \tilde{\Lambda}_0)^{-2}. \quad (A.12)
\]

One then obtains the SW curve for $\tilde{E}_1$ as

\[
w + \frac{t}{(2\pi R \tilde{\Lambda}_0)^2} (w^2 + \tilde{U} w + 1) + t^2 w^2 = 0. \quad (A.13)
\]

We note that the 4d limit of $E_1$ and $\tilde{E}_1$ gives the same SW curve at $O(\beta^0)$,

\[
1 + t^2 + \frac{t (u + v^2)}{\Lambda_0} = 0. \quad (A.14)
\]
A.1.3 $E_0$ Seiberg-Witten curve

It is interesting to take a limit that leads to the $E_0$ SW curve from the $\tilde{E}_1$ curve. For this, we decouple the coefficient $c_{12}$ or the $tw^2$-term. Instead of (A.13), we start a generic form of the $\tilde{E}_1$ curve

$$c_{01}(w + \ell_4 t + U tw + \frac{1}{\ell_3} tw^2 + \frac{\ell_4}{\ell_2} t^2 w^2) = 0,$$  \hspace{1cm} (A.15)

We then take $\ell_3 \to \infty, t_1 \to 0$ while $-t_1 \ell_3$ fixed, which gives

$$w + \ell_4 t + U tw + \frac{\ell_4}{\ell_2} t^2 w^2 = 0.$$  \hspace{1cm} (A.16)

Using the remaining rescaling degrees of freedom associated with $t$ and $w$, we can fix $\ell_4 = 1, \ell_2' = 1$, yielding the $E_0$ SW curve

$$w + t + U tw + t^2 w^2 = 0.$$  \hspace{1cm} (A.17)

A.2 $N_f = 1$

![Toric web diagram](image)

Figure 26: Toric web diagram for $N_f = 1$ of $E_2$ flavor symmetry

For $N_f = 1$, we start with

$$c_{01} w + c_{10} t + c_{11} tw + c_{12} tw^2 + c_{21} t^2 w + c_{22} t^2 w^2 = 0.$$  \hspace{1cm} (A.18)

The boundary conditions are

- $|w^{-1}| \sim |t|$ large : $c_{10} t + c_{21} t^2 w = c_{21} t(\ell_2 + tw) \quad \Rightarrow \quad c_{10} = \ell_2 c_{21}$,
- $|w| \sim |t^{-1}|$ large : $c_{12} tw^2 + c_{01} w = c_{12} w(tw + \ell_3) \quad \Rightarrow \quad c_{01} = \ell_3 c_{12}$,
- $|w^{-1}| \sim |t^{-1}|$ large : $c_{01} w + c_{10} t = c_{01}(w + \ell_4 t) \quad \Rightarrow \quad c_{10} = \ell_4 c_{01}$.  \hspace{1cm} (A.19)
In addition to this, there is an extra contribution from a flavor:

\[ c_{21} t^2 w + c_{22} t^2 w^2 = c_{22} t^2 w (\tilde{m}_1 + w) \]
\[ c_{12} t w^2 + c_{22} t^2 w^2 = c_{22} t w^2 (-t_1 + t) \]
\[ \Rightarrow \quad c_{21} = -\tilde{m}_1 c_{22}, \]
\[ c_{12} = -t_1 c_{22}. \]

(A.20)

The compatibility condition is

\[ \frac{\tilde{m}_1}{t_1} \ell_2 = \ell_3 \ell_4. \]

(A.21)

As in the \( N_f = 0 \) case, for three rescaling degrees of freedom, we take

\[ c_{01} = c_{21} = 1 \quad \text{and} \quad c_{10} = c_{12}, \]

(A.22)

or equivalently,

\[ \frac{t_1}{\ell_2} = \tilde{m}_1, \quad t_1 \ell_3 = \tilde{m}_1 \quad \text{(or} \quad \ell_2 = \ell_4), \]

(A.23)

With dynamical scale is given by

\[ t_1^2 \tilde{m}_1^{-\frac{3}{2}} = (2\pi R_B \Lambda_1)^{-3}, \]

(A.24)

the SW curve take the following form

\[ w + t_1 \tilde{m}_1^{-1} t \left( w^2 + U w + 1 \right) + t^2 w - \tilde{m}_1^{-1} t^2 w^2 = 0. \]

(A.25)

**A.2.1 \( E_1 \) limit**

Figure 27: One obtains \( E_1 \) curve from the \( E_2 \) curve by taking \( \tilde{m}_1 \to \infty \) and \( t_1 \to \infty \), while \( \tilde{m}_1 t_1^{-1} = \ell_1 \) fixed.

It is clear from Figure 27 that by taking the mass decoupling limit

\[ \tilde{m}_1 \to \infty \quad \text{(while} \quad t_1^{-1} \tilde{m}_1 = \ell_1 \text{ fixed)}, \]

(A.26)

one reproduces the \( N_f = 0, E_1 \) curve, (A.5). It follows that the condition (A.23) becomes the condition for \( N_f = 0 \), (A.4). The dynamical scale for \( N_f = 1 \) and \( N_f = 0 \) are related by

\[ \tilde{m}_1^\frac{1}{3} (2\pi R \Lambda_1)^3 = (2\pi R \Lambda_0)^4, \]

(A.27)

which relates (A.24) to (A.7).
A.2.2 \( \tilde{E}_1 \) limit

Figure 28: One obtains \( \tilde{E}_1 \) curve from the \( E_2 \) curve by taking \( \tilde{m}_1 \to 0 \) and \( \ell_2 \to \infty \), while \( -\tilde{m}_1 \ell_2 = \ell'_2 \) fixed.

We can take a distinct mass decoupling limit such that \( \tilde{m}_1 \to 0 \), keeping \( -\tilde{m}_1 \ell_2 \) fixed. In this case, the relation (A.21) is expressed as

\[
-t_1 \ell_3 \ell_4 = -\tilde{m}_1 \ell_2 \quad (\equiv \ell'_2). \tag{A.28}
\]

In this limit, the \( t^2 w \)-term drops out and the SW curve (A.25) becomes

\[
w - t_1 \ell_2^{-1} \ell_4 t \left( w^2 + \tilde{U} w - \ell_4^{-1} \ell_2 \right) - \ell_2^{-1} \ell_4 t^2 w^2 = 0. \tag{A.29}
\]

With the choice \( -t_1^{-1} \ell'_2 = 1 \) and \( \ell_2^{-1} \ell_4 = 1 \), or equivalently \( c_{12} = c_{10} \) and \( c_{01} = c_{22} \), the SW curve is written in term of two parameters, \( \ell_4, \tilde{U} \),

\[
w + \ell_4 t \left( 1 + \tilde{U} w + w^2 \right) + t^2 w^2 = 0, \tag{A.30}
\]

which is nothing but the SW curve for \( \tilde{E}_1 \), (A.13).

A.3 \( N_f = 2 \)

Figure 29: Toric and web diagrams for \( N_f = 2 \) of \( E_3 \) flavor symmetry
For $N_f = 2$, we start with
\[
c_0 w + c_{10} t + c_{11} tw + c_{12} tw^2 + c_{21} t^2 w + c_{22} t^2 w^2 + c_{20} t^2 = 0. \tag{A.31}
\]

The boundary conditions are given by
\[
\begin{align*}
|w| \text{ large:} & \quad c_{12} tw^2 + c_{22} t^2 w^2 = c_{22} t^2 w^2(-t_1 + t) \\
|w| \text{ small:} & \quad c_{10} t + c_{20} t^2 = c_{20} t(-t_2 + t) \\
|w| \sim |t^{-1}| \text{ large:} & \quad c_{12} tw^2 + c_{01} w = c_{12} w(tw + \ell_3) \\
|w^{-1}| \sim |t^{-1}| \text{ large:} & \quad c_{01} w + c_{10} t = c_{01} (w + \ell_4 t), \tag{A.32}
\end{align*}
\]
and the following extra boundary condition: When $|t|$ is very large,
\[
c_{20} t^2 + c_{21} t^2 w + c_{22} t^2 w^2 = c_{22} t^2 (w - \tilde{m}_1)(w - \tilde{m}_2). \tag{A.33}
\]

The compatibility condition is
\[
\tilde{m}_1 t_1^{-1} \tilde{m}_2 t_2 = \ell_3 \ell_4. \tag{A.34}
\]

For three rescaling degrees of freedom, we choose
\[
c_{01} = c_{21} = 1, \quad c_{10} = c_{12}, \tag{A.35}
\]
or equivalently
\[
\frac{t_1}{t_2} = \tilde{m}_1 \tilde{m}_2, \quad t_1 \ell_3 = \tilde{m}_1 + \tilde{m}_2. \tag{A.36}
\]
The dynamical scale is given by
\[
\left(\frac{t_1 t_2 \ell_4}{\ell_3}\right)^\frac{1}{2} = (2\pi R_B \Lambda_2)^{-2}, \tag{A.37}
\]
or equivalently,
\[
t_1^2 \tilde{m}_1^{-\frac{1}{2}} \tilde{m}_2^{-\frac{1}{2}} (\tilde{m}_1 + \tilde{m}_2)^{-1} = (2\pi R_B \Lambda_2)^{-2}. \tag{A.38}
\]
The SW curve for $N_f = 2$ is then
\[
w + t \frac{t_1}{\tilde{m}_1 + \tilde{m}_2} (w^2 + Uw + 1) - t^2 \frac{1}{\tilde{m}_1 + \tilde{m}_2} (w - \tilde{m}_1) (w - \tilde{m}_2) = 0. \tag{A.39}
\]
In the mass decoupling limit $m_2 \to \infty$, correspondingly $\tilde{m}_2 \to 0$, while keeping $\tilde{m}_2 t_2 = \ell_2$ fixed, it is straightforward to see that the SW curve for $N_f = 2$, (A.39), becomes that for $N_f = 1$, (A.25). The condition that we used, (A.36) becomes the condition for $N_f = 1$, (A.23), and the dynamical scales for $N_f = 2$ and $N_f = 1$ are related in this decoupling limit as
\[
\tilde{m}_2^{-\frac{1}{2}} (2\pi R \Lambda_2)^{3} = (2\pi R \Lambda_1)^{3} \tag{A.40}
\]
which relates (A.38) to (A.24).
For $N_f = 3$, we start with

$$c_0 w + c_{10} t + c_{11} tw + c_{12} tw^2 + c_{21} t^2 w + c_{22} t^2 w^2 + c_{20} t^2 + c_{02} w^2 = 0. \quad (A.41)$$

The boundary conditions are given by

- $|w|$ small: $c_{10} t + c_{20} t^2 = c_{20} t(-t_2 + t)$,
- $|w^{-1}| \sim |t^{-1}|$ large: $c_{01} w + c_{10} t = c_{01} (w + \ell_4 t)$,
- $|t|$ large: $c_{20} t^2 + c_{21} t^2 w + c_{22} t^2 w^2 = c_{22} t^2 (w - \tilde{m}_1)(w - \tilde{m}_2). \quad (A.42)$

and when $|w|$ large, the boundary condition is given by

$$c_{02} w^2 + c_{12} t w^2 + c_{22} t^2 w^2 = c_{22} w^2 (t - t_1)(t - t_3), \quad (A.43)$$

and the boundary condition for $|t|$ small ($|w|$ not small) is given by

$$c_{01} w + c_{02} w^2 = c_{02} w (w - \tilde{m}_3). \quad (A.44)$$

The compatibility is given by

$$\tilde{m}_1 t_1^{-1} \tilde{m}_2 t_2 = \tilde{m}_3 t_3 \ell_4. \quad (A.45)$$

For three rescaling degrees of freedom, we choose

$$c_{01} = c_{21} = 1, \quad c_{10} = c_{12}, \quad (A.46)$$

or equivalently

$$\frac{t_1 + t_3}{t_2} = \tilde{m}_1 \tilde{m}_2, \quad t_1 t_3 = \frac{\tilde{m}_1 + \tilde{m}_2}{\tilde{m}_3}. \quad (A.47)$$
The dynamical scale is given by

\[
\left(\frac{t_1 t_2 \ell_4}{t_3}\right)^{\frac{1}{2}} = (2\pi R_B \Lambda_3)^{-1},
\]  

(A.48)

or

\[
t_1 = \left(\frac{\tilde{m}_1 + \tilde{m}_2}{\tilde{m}_3}\right)^{\frac{1}{2}} \left(\frac{(\tilde{m}_1 \tilde{m}_2 \tilde{m}_3)^{\frac{1}{2}}}{2\pi R \Lambda_3} - 1\right)^{\frac{1}{2}},
\]  

(A.49)

The SW curve for \(N_f = 3\) is then

\[
w + t \left(\frac{t_1}{\tilde{m}_1 + \tilde{m}_2} + \frac{1}{t_1 \tilde{m}_3}\right) \left(w^2 + U w + 1\right) - t^2 \frac{1}{\tilde{m}_1 + \tilde{m}_2} \left(w - \tilde{m}_1\right) \left(w - \tilde{m}_2\right) - \frac{1}{\tilde{m}_3} w^2 = 0.
\]  

(A.50)

In the mass decoupling limit \(\tilde{m}_3 \to \infty\) and \(t_3 \to 0\), while \(\tilde{m}_3 t_3 = \ell_3\) fixed, the SW curve for \(N_f = 3\), (A.57), becomes that for \(N_f = 2\), (A.50), and the condition that we used, (A.47) becomes the condition for \(N_f = 3\), (A.36), and the dynamical scales for \(N_f = 3\) and \(N_f = 2\) are related in this decoupling limit as

\[
\tilde{m}_3^{\frac{1}{2}} (2\pi R \Lambda_3) = (2\pi R \Lambda_2)^2
\]  

(A.51)

which relates (A.49) to (A.38).

A.5 \(N_f = 4\)

For \(N_f = 4\), we start with

\[
c_{01} w + c_{10} t + c_{11} t w + c_{12} t w^2 + c_{21} t^2 w + c_{22} t^2 w^2 + c_{20} t^2 + c_{02} w^2 + c_{00} = 0.
\]  

(A.52)
The boundary conditions are

\[ |w| \text{ large : } c_{02}w^2 + c_{12}tw^2 + c_{22}t^2w^2 = c_{22}w^2(t - t_1)(t - t_3), \]
\[ |w| \text{ small : } c_{20}t^2 + c_{10}t + c_{00} = c_{20}(t - t_2)(t - t_4), \]
\[ |t| \text{ large : } c_{20}t^2 + c_{21}t^2w + c_{22}t^2w^2 = c_{22}t^2(w - \tilde{m}_1)(w - \tilde{m}_2), \]
\[ |t| \text{ small : } c_{02}w^3 + c_{01}w + c_{00} = c_{02}(w - \tilde{m}_3)(w - \tilde{m}_4). \] (A.53)

The compatibility condition is given by

\[ \tilde{m}_1 t_1^{-1} \tilde{m}_2 t_2 = \tilde{m}_3 t_3 \tilde{m}_4 t_4^{-1}. \] (A.54)

For three rescaling degrees of freedom, we choose

\[ c_{01} = c_{21} = 1, \quad c_{10} = c_{12}, \] (A.55)

or equivalently,

\[ \frac{t_1 + t_3}{t_2 + t_4} = \tilde{m}_1 \tilde{m}_2, \quad t_1 t_3 = \frac{\tilde{m}_1 + \tilde{m}_2}{\tilde{m}_3 + \tilde{m}_4}. \] (A.56)

The SW curve for \( N_f = 4 \) is then

\[ -\frac{1}{\tilde{m}_3 + \tilde{m}_4} (w - \tilde{m}_3) (w - \tilde{m}_4) + t \left( \frac{t_1}{\tilde{m}_1 + \tilde{m}_2} + \frac{1}{t_1 (\tilde{m}_3 + \tilde{m}_4)} \right) (w^2 + Uw + 1) \]
\[ -t^2 \frac{1}{\tilde{m}_1 + \tilde{m}_2} (w - \tilde{m}_1) (w - \tilde{m}_2) = 0. \] (A.57)

In this case, there is no dynamical scale but one can define the gauge coupling as geometric average of \( t_i \)

\[ \left( \frac{t_1 t_2}{t_3 t_4} \right)^{\frac{1}{2}} = q^{-1}. \] (A.58)

It follows from (A.54), (A.56), and (A.58) that \( t_1 \) are expressed in terms of masses and the gauge coupling as

\[ t_1 = \left( \frac{\tilde{m}_1 + \tilde{m}_2}{\tilde{m}_3 + \tilde{m}_4} \right)^{\frac{1}{2}} \left( \frac{q^{-1} S^\frac{1}{2} - 1}{1 - q S^\frac{1}{2}} \right)^{\frac{1}{2}}, \quad S \equiv \tilde{m}_1 \tilde{m}_2 \tilde{m}_3 \tilde{m}_4. \] (A.59)

The SW curve is also written in a symmetric way as

\[ t^2 (w - \tilde{m}_1)(w - \tilde{m}_2) - t (t_1 + t_3) (w^2 + Uw + 1) + t_1 t_3 (w - \tilde{m}_3)(w - \tilde{m}_4) = 0, \] (A.60)

with

\[ t_1 + t_3 = \left( \frac{\tilde{m}_1 + \tilde{m}_2}{\tilde{m}_3 + \tilde{m}_4} \right)^{\frac{1}{2}} q^{-\frac{1}{2}} S^\frac{1}{2} \left( \frac{S^\frac{1}{2} - q S}{S^\frac{1}{2} - q} \right)^{\frac{1}{2}} \left( 1 + q \frac{S^\frac{1}{2} - q}{1 - q S^\frac{1}{2}} \right). \] (A.61)
In the mass decoupling limit where $\tilde{m}_4 \to 0$ and $t_4 \to 0$, while $\tilde{m}_4 t_4^{-1} = \ell_4$ fixed, the SW curve for $N_f = 4$, (A.57), becomes naturally the SW curve for $N_f = 3$, (A.50), and the condition that we used, (A.56), becomes the condition for $N_f = 3$, (A.47). We note that in this mass decoupling limit, dynamical scales for $N_f = 4$ and $N_f = 3$ are related as

$$q^{-1} \tilde{m}_4^{1/2} = (2\pi R \Lambda_3)^{-1}, \quad (A.62)$$

which is consistent with that (A.59) for $N_f = 4$ turns into that of $N_f = 3$, (A.49). We note that all the parameters including dynamical scales except for $q$ are not physical. This is due to our choice of three rescaling degrees of freedom. The choice, $c_{01} = c_{21} = 1$ and $c_{10} = c_{12}$, we have chosen, makes mass decoupling limit easier. We comment that the SW curve with unphysical parameters can be related to the SW curve with physical ones by coordinate transformation below.

**SW curve with physical masses**

Different choices for three rescaling degrees of freedom lead to a differently looking curve which is related by coordinate transformation. We consider a choice that the position of the exponential of masses $\tilde{m}_i$ is measured from the center of the Coulomb branch. With $c_{22} = 1 = t_4$ in Figure 31, the SW curve is given by

$$t^2 (w - \tilde{m}_1')(w - \tilde{m}_2') - t \left( \tilde{m}_1' \tilde{m}_2'(1 + qS'^{-1/2})w^2 + U'w + (1 + qS'^{1/2})\tilde{m}_1' \tilde{m}_2' \right) + qS'^{-1/2} \tilde{m}_1'^2 \tilde{m}_2'^2 (w - \tilde{m}_3')(w - \tilde{m}_4') = 0, \quad (A.63)$$

where $S' \equiv \tilde{m}_1' \tilde{m}_2' \tilde{m}_3' \tilde{m}_4'$. In the 4d limit, by expanding $w \equiv e^{-\beta v}$ and $\tilde{m}_i' \equiv e^{-\beta m_i}$ with respect to $\beta$ while keeping $t$ and $q$ as they are, one finds [33, 12]

$$t^2 (v - m_1)(v - m_2) + t \left( - (1 + q)v^2 + q \sum_{i=1}^{4} m_i + u \right) + q(v - m_3)(v - m_4) = 0, \quad (A.64)$$

where the masses $m_i$ are physical masses. The coordinate transformation that connects (A.63) to (A.60) is as follows:

$$w \to Tw, \quad \tilde{m}_i' \to T\tilde{m}_i, \quad T = \left(\frac{q - S^1_2}{qS - S^0_2}\right)^{1/2},$$

$$t \to \sqrt{qS^{-1/2}T^2 \tilde{m}_1^2 \tilde{m}_2^2 (\tilde{m}_3 + \tilde{m}_4)/(\tilde{m}_1 + \tilde{m}_2)} \ t, \quad (A.65)$$

where $S$ is the product of masses given in (A.59).
A.6 Higher rank curve

The toric-like diagram for higher rank-$N$ with $N_f$ flavors can be obtained in the same way as explained section 6. The corresponding SW curve is again factorized as the product of that of rank-1. As an example, we show the toric-like diagram for rank-2 $E_1$ theory ($N_f = 0$) and the corresponding $(p, q)$ web in Figure 32.

![Toric-like diagram for rank-2 E1 theory](image)

Figure 32: Toric-like diagram for rank-2 $E_1$ theory and corresponding web diagram.

B The Seiberg-Witten curves from $E_8$ to $E_0$

We here list the result of decomposition of the characters of $E_n$ into $E_{n-1} \times U(1)$ and then factoring out the $U(1)$ part, introduced in \[17\]:

(i) Rescale the all variables

$$(u, x, y) \rightarrow (Lu, L^2x, L^3y),$$

$$(x_{\mu_1}, x_{\mu_2}, x_{\mu_3}, x_{\mu_4}, x_{\mu_5}, x_{\mu_6}, x_{\mu_7}, x_{\mu_8}) \rightarrow (L^2x_{\mu_1}, L^3x_{\mu_2}, L^4x_{\mu_3}, L^5x_{\mu_4}, L^6x_{\mu_5}, L^7x_{\mu_6}, L^8x_{\mu_7}, L^9x_{\mu_8}),$$

where the power of $L$ in the character is the marks of $E_8$.

(ii) Set $x_{\mu_8}$ to 1 when reducing from $E_8$ to $E_7$, and take $L \rightarrow \infty$ limit, and

(iii) When reducing from $E_7$ to $E_6$, likewise the corresponding character to be unity while keeping the scaling.

The $E_n$ manifest curve is of the standard Weierstrass form \[17, 28\]

$$y^2 = 4x^3 - g_2^E x - g_3^E,$$

where $g_2$ and $g_3$ are given according to $E_n$ as follows.
For $E_8$:

$$ g^E_2 = \frac{1}{12} u^4 - \left( \frac{2}{3} \chi_1 - \frac{50}{3} \chi_8 + 1550 \right) u^2 - \left( -70 \chi_1 + 2 \chi_2 - 12 \chi_7 + 1840 \chi_8 - 115010 \right) u \\
+ \frac{4}{3} \chi_1 \chi_1 - \frac{8}{3} \chi_2 \chi_8 - 1824 \chi_1 + 112 \chi_2 - 4 \chi_3 + 4 \chi_6 \\
- 680 \chi_7 + 28 \chi_8 \chi_8 + 50744 \chi_8 - 2399276, $$

$$ g^E_3 = \frac{1}{216} u^6 - \left( \frac{1}{18} \chi_1 + \frac{47}{18} \chi_8 - \frac{5177}{6} \right) u^4 \\
- \left( -\frac{107}{6} \chi_1 - 1 \chi_1 \chi_2 + 3 \chi_7 - \frac{1580}{3} \chi_8 + \frac{504215}{6} \right) u^3 \\
- \left( -\frac{2}{9} \chi_2 \chi_8 - \frac{20}{3} \chi_1 \chi_8 + \frac{5866}{3} \chi_1 - \frac{112}{3} \chi_2 + \frac{1}{3} \chi_3 \\
+ \frac{11}{3} \chi_8 - \frac{1450}{3} \chi_7 + 196 \chi_8 \chi_8 + 39296 \chi_8 - \frac{12673792}{3} \right) u^2 \\
- \left( \frac{81}{9} \chi_1 \chi_1 - \frac{20}{3} \chi_2 \chi_2 + 718 \chi_1 \chi_8 - \frac{270736}{3} \chi_1 \chi_8 - \frac{10}{3} \chi_2 \chi_8 + 2630 \chi_2 - 52 \chi_3 + 4 \chi_5 \\
- 416 \chi_6 + 16 \chi_8 \chi_8 + 25880 \chi_8 - \frac{7328}{3} \chi_8 \chi_8 - \frac{3841382}{3} \chi_8 + 107263286 \right) u \\
- \frac{8}{27} \chi_1 \chi_1 \chi_1 + \frac{28}{3} \chi_1 \chi_3 \chi_3 + 1065 \chi_1 \chi_1 \chi_3 - \frac{118}{3} \chi_1 \chi_2 \chi_2 + \frac{4}{3} \chi_1 \chi_3 \chi_3 - \frac{4}{3} \chi_1 \chi_8 \chi_8 \\
+ \frac{8}{3} \chi_1 \chi_7 + \frac{40}{9} \chi_1 \chi_3 \chi_8 + \frac{19264}{3} \chi_1 \chi_8 - \frac{4521802}{3} \chi_1 \chi_2 - \frac{572}{3} \chi_2 \chi_8 \chi_8 \\
+ \frac{59482}{8} \chi_2 + \frac{20}{3} \chi_3 \chi_8 - 1880 \chi_3 - 4 \chi_4 + 232 \chi_5 - \frac{8}{3} \chi_6 \chi_8 - \frac{2740}{3} \chi_7 \chi_8 \\
+ \frac{460388}{3} \chi_7 - \frac{136}{27} \chi_8 \chi_8 \chi_8 - \frac{205492}{3} \chi_8 \chi_8 - \frac{5856940}{3} \chi_8 + 1091057493. $$

(B.3)

For $E_7$:

$$ g^E_2 = \frac{1}{12} u^4 - \left( \frac{2}{3} \chi_7 - \frac{50}{3} \right) u^2 - \left( 2 \chi_7 - 12 \chi_7 \right) u + \frac{4}{3} \chi_7 \chi_1 - \frac{8}{3} \chi_7 - 4 \chi_3 + 4 \chi_6 + \frac{28}{3}, $$

$$ g^E_3 = \frac{1}{216} u^6 - \left( \frac{1}{18} \chi_1 + \frac{47}{18} \chi_7 \right) u^4 - \left( \frac{1}{6} \chi_2 + 3 \chi_7 \right) u^3 + \left( \frac{2}{9} \chi_7 \chi_1 + \frac{20}{9} \chi_1 - \frac{1}{3} \chi_3 - \frac{3}{3} \chi_6 - \frac{196}{9} \right) u^2 \\
+ \left( \frac{2}{3} \chi_7 \chi_2 + \frac{10}{3} \chi_2 - 4 \chi_5 - 16 \chi_7 \right) u - \frac{8}{27} \chi_7 \chi_1 \chi_1 - \frac{28}{9} \chi_7 \chi_1 \chi_7 - \frac{4}{3} \chi_1 \chi_3 \\
- \frac{4}{3} \chi_1 \chi_6 + \frac{40}{9} \chi_1 \chi_2 + 2 \chi_1 \chi_3 - 4 \chi_4 - \frac{8}{3} \chi_6 - \frac{136}{27}. $$

(B.4)

For $E_6$:

$$ g^E_2 = \frac{1}{12} u^4 - \frac{2}{3} \chi_1 u^2 - \left( 2 \chi_2 - 12 \right) u + \frac{4}{3} \chi_1 \chi_1 - 4 \chi_3 + 4 \chi_6, $$

$$ g^E_3 = \frac{1}{216} u^6 - \left( \frac{1}{18} \chi_1 + \frac{47}{18} \chi_7 \right) u^4 - \left( \frac{1}{6} \chi_2 + 3 \right) u^3 + \left( \frac{2}{9} \chi_7 \chi_1 - \frac{1}{3} \chi_3 - \frac{11}{3} \chi_6 \right) u^2 + \left( \frac{2}{3} \chi_1 \chi_2 - 4 \chi_5 \right) u \\
- \frac{8}{27} \chi_1 \chi_1 - \chi_1 \chi_3 - \frac{4}{3} \chi_1 \chi_6 + \chi_2 \chi_2 - 4 \chi_4. $$

(B.5)
For $E_5 = SO(10)$:

$$
g_2^{E_5} = \frac{1}{12} u^4 - \frac{2}{3} \chi_1 E_3 u^2 - 2 \chi_2 E_2 u + \frac{4}{3} \chi_1 E_3 \chi_1 - 4 \chi_3 E_3 + 4,$$

$$
g_3^{E_5} = \frac{1}{216} u^6 - \frac{1}{18} \chi_1 E_5 u^4 - \frac{1}{3} \chi_2 E_2 u^3 + \left( \frac{2}{9} \chi_1 E_5 \chi_1 - \frac{1}{3} \chi_3 E_5 \chi_1 \right) u^2 + \left( \frac{2}{3} \chi_1^2 \chi_2 - 4 \chi_3 E_5 \right) u
- \frac{8}{27} \chi_1 E_3 \chi_1 + \frac{4}{3} \chi_1 E_3 \chi_1 + \chi_2 E_2 - 4 \chi_4 E_4. \quad (B.6)
$$

For $E_4 = SU(5)$:

$$
g_2^{E_4} = \frac{1}{12} u^4 - \frac{2}{3} \chi_1 E_4 u^2 - 2 \chi_2 E_2 u + \frac{4}{3} \chi_1 E_4 \chi_1 - 4 \chi_3 E_4,$$

$$
g_3^{E_4} = \frac{1}{216} u^6 - \frac{1}{18} \chi_1 E_5 u^4 - \frac{1}{3} \chi_2 E_2 u^3 + \left( \frac{2}{9} \chi_1 E_5 \chi_1 - \frac{1}{3} \chi_3 E_5 \chi_1 \right) u^2 + \left( \frac{2}{3} \chi_1^2 \chi_2 - 4 \chi_3 E_4 \right) u
- \frac{8}{27} \chi_1 E_3 \chi_1 \chi_1 + \frac{4}{3} \chi_1 E_3 \chi_1 + \chi_2 E_2 - 4 \chi_4 E_4. \quad (B.7)
$$

For $E_3 = SU(3) \times SU(2)$:

$$
g_2^{E_3} = \frac{1}{12} u^4 - \frac{2}{3} \chi_1 E_3 u^2 - 2 \chi_2 E_2 u + \frac{4}{3} \chi_1 E_3 \chi_1 - 4 \chi_3 E_3,$$

$$
g_3^{E_3} = \frac{1}{216} u^6 - \frac{1}{18} \chi_1 E_3 u^4 - \frac{1}{3} \chi_2 E_2 u^3 + \left( \frac{2}{9} \chi_1 E_3 \chi_1 - \frac{1}{3} \chi_3 E_3 \chi_1 \right) u^2 + \frac{2}{3} \chi_1 E_3 \chi_1 + \chi_2 E_2 - 4 \chi_4 E_4. \quad (B.8)
$$

For $E_2 = SU(2) \times U(1)$:

$$
g_2^{E_2} = \frac{1}{12} u^4 - \frac{2}{3} \chi_1 E_2 u^2 - 2 \chi_2 E_2 u + \frac{4}{3} \chi_1 E_2 \chi_1 - 4, \quad (B.9)
$$

$$
g_3^{E_2} = \frac{1}{216} u^6 - \frac{1}{18} \chi_1 E_3 u^4 - \frac{1}{6} \chi_2 E_2 u^3 + \left( \frac{2}{9} \chi_1 E_2 \chi_1 - \frac{1}{3} \right) u^2 + \frac{2}{3} \chi_1 E_3 \chi_1 + \chi_2 E_2 E_2 u - \frac{8}{27} \chi_1 E_3 \chi_1 \chi_1 + \frac{4}{3} \chi_1 E_2 + \chi_2 E_2 E_2. \quad (B.10)
$$

For $E_1 = SU(2)$: (by removing $\mu_2$ from $E_2$)

$$
g_2^{E_1} = \frac{1}{12} u^4 - \frac{2}{3} \chi_1 E_1 u^2 - 2 u + \frac{4}{3} \chi_1 E_1 \chi_1, \quad (B.10)
$$

$$
g_3^{E_1} = \frac{1}{216} u^6 - \frac{1}{18} \chi_1 E_1 u^4 - \frac{1}{6} \chi_2 E_1 u^3 + \left( \frac{2}{9} \chi_1 E_1 \chi_1 - \frac{1}{3} \right) u^2 + \frac{2}{3} \chi_1 E_1 \chi_1 + \chi_2 E_1 E_1 u - \frac{8}{27} \chi_1 E_1 \chi_1 \chi_1 + \frac{4}{3} \chi_1 E_1 + 1. \quad (B.11)
$$

For $\tilde{E}_1 = U(1)$: (by removing $\mu_1$ from $E_2$)

$$
g_2^{\tilde{E}_1} = \frac{1}{12} u^4 - \frac{2}{3} u^2 - 2 \tilde{\chi}_2 \tilde{E}_1 u + \frac{4}{3},
$$

$$
g_3^{\tilde{E}_1} = \frac{1}{216} u^6 - \frac{1}{18} u^4 - \frac{1}{6} \tilde{\chi}_2 \tilde{E}_1 u^3 - \frac{1}{9} u^2 + \frac{2}{3} \tilde{\chi}_2 \tilde{E}_1 u + \frac{28}{27} + \tilde{\chi}_2 \tilde{E}_1 \chi_2. \quad (B.11)
$$

For $E_0$: (from $\tilde{E}_1$)

$$
g_2^{E_0} = \frac{1}{12} u^4 - 2 u, \quad g_3^{E_0} = \frac{1}{216} u^6 - \frac{1}{6} u^3 + 1. \quad (B.12)\)
In summary, one stars from $E_8$ curve and obtains lower $E_n$ curves:

\[ E_8 \to E_7 \to E_6 \to E_5 \to E_4 \to E_3 \to E_2 \to \tilde{E}_1 \to E_0 \to E_1. \]  
(B.13)

All the holomorphic SW one form is of the standard form:

\[ \omega_{SW} = \frac{dx}{y}. \]  
(B.14)

\section*{C The $j$-invariant}

The Weierstrass form for elliptic curve is given

\[ y^2 = 4z^3 - g_2z - g_3, \]  
(C.1)

then the $j$-invariant which is $SL(2, \mathbb{Z})$ invariant is define by

\[ j(\tau) = \frac{g_2^3}{g_3^3 - 27g_2^2}, \]  
(C.2)

where the denominator is proportional to the discriminant of the Weierstrass form. For (C.1), $\Delta = 16(g_2^3 - 27g_3^2)$, and thus the denominator of the $j$-invariant is $\frac{\Delta}{16}$.

For an elliptic curve given by

\[ y^2 = Ax^3 + Bx^2 + Cx + D, \]  
(C.3)

let us find how the $j$-invariant is given. It is straightforward to rewrite (C.3) into the standard Weierstrass form

\[ \tilde{y}^2 = 4\tilde{x}^3 - \frac{4}{3A^2}(B^2 - 3AC)\tilde{x} - \frac{4}{27A^3}(9ABC - 2B^3 - 27A^2D), \]  
(C.4)

where $\tilde{y} = \frac{y}{\sqrt{A/2}}$, $\tilde{x} = x + \frac{B}{3A}$, yielding

\[ g_2 = \frac{4}{3A^2}(B^2 - 3AC), \quad g_3 = \frac{4}{27A^3}(9ABC - 2B^3 - 27A^2D). \]  
(C.5)

As the discriminant $\Delta$ for a cubic equation $Ax^3 + Bx^2 + Cx + D = 0$ is given by

\[ \Delta = B^2C^2 - 4AC^3 - 4B^3D - 27A^2D^2 + 18ABC\cdot D, \]  
(C.6)

one finds that

\[ g_2^3 - 27g_3^2 = \frac{16}{A^4}\Delta. \]  
(C.7)
An elliptic curve may also be expressed as a quartic polynomial

\[ y^2 = ax^4 + bx^3 + cx^2 + dx + e. \]  \hspace{1cm} (C.8)

The forms of \( g_2 \) and \( g_3 \) for the curve are given by

\[ g_2 = \frac{4}{a^3}(c^2 - 3bd + 12ae), \quad g_3^3 - 27g_2^2 = \frac{16}{a^6}\Delta, \]  \hspace{1cm} (C.9)

where the discriminant for the quartic equation is given by

\[
\Delta = 256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 + 144ab^2ce^2 - 6ab^2d^2e - 80abcde \\
+ 18abcd^3 + 16ac^4e - 4ac^3d^2 - 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2.
\]

**D The Weierstrass from for \( SU(8) \) manifest curve of \( E_7 \) theory**

Following [28], we rewrite (3.15) into the standard Weierstrass form

\[ y^2 = 4x^3 - g_2x - g_3, \]  \hspace{1cm} (D.1)

where

\[
g_2 = -\frac{4}{3} \left( -64 - 48\chi_1\chi_3 + 64\chi_4 - 16\chi_4^2 + 48\chi_3\chi_5 + 48\chi_1\chi_6 - 192\chi_2\chi_6 + 16\chi_1\chi_7 \\
+ 16\chi_1\chi_3\chi_7 - 48\chi_5\chi_7 - 16\chi_1^2\chi_7^2 + 48\chi_2\chi_7^2 + 24\chi_1^2u - 192\chi_2u + 24\chi_1\chi_5u \\
- 192\chi_6u + 24\chi_3\chi_7u + 24\chi_7^2u - 208u^2 + 8\chi_4u^2 + 8\chi_1\chi_7u^2 - u^4 \right). \]  \hspace{1cm} (D.2)
and
\[ g_3 = \frac{8}{27} \left( -512 - 216 \chi_1^4 + 864 \chi_1^2 \chi_2 - 576 \chi_1 \chi_3 + 768 \chi_4 + 288 \chi_1 \chi_3 \chi_4 - 384 \chi_4^2 + 64 \chi_4^3 \right) \]  
(D.3)

\[ + 432 \chi_1^2 \chi_5 - 1728 \chi_1 \chi_2 \chi_5 + 576 \chi_3 \chi_5 - 288 \chi_3 \chi_4 \chi_5 - 216 \chi_2^2 \chi_5^2 + 864 \chi_2 \chi_5^2 \]
\[ - 1152 \chi_1^2 \chi_6 + 4608 \chi_2 \chi_6 + 864 \chi_3^2 \chi_6 + 576 \chi_1^2 \chi_4 \chi_6 - 2304 \chi_2 \chi_4 \chi_6 + 192 \chi_1 \chi_7 \]
\[ - 144 \chi_1^3 \chi_7 + 96 \chi_1 \chi_4 \chi_7 - 96 \chi_1^2 \chi_4 \chi_7 - 576 \chi_5 \chi_7 + 144 \chi_1 \chi_3 \chi_5 \chi_7 + 288 \chi_4 \chi_5 \chi_7 \]
\[ - 288 \chi_1^3 \chi_6 \chi_7 + 1152 \chi_1 \chi_2 \chi_6 \chi_7 - 1728 \chi_3 \chi_6 \chi_7 + 336 \chi_1^2 \chi_7^2 - 1152 \chi_2 \chi_7^2 - 216 \chi_3 \chi_7^2 \]
\[ - 96 \chi_1^2 \chi_4 \chi_7^2 + 576 \chi_2 \chi_4 \chi_7^2 - 144 \chi_1 \chi_5 \chi_7^2 + 864 \chi_6 \chi_7^2 + 64 \chi_1^3 \chi_7^3 - 288 \chi_1 \chi_2 \chi_7^3 \]
\[ + 432 \chi_3 \chi_7^3 - 216 \chi_4^2 - 576 \chi_1^2 \chi_7 u + 4608 \chi_2 \chi_7 u + 864 \chi_3^2 \chi_7 u + 720 \chi_1 \chi_4 \chi_7 u - 2304 \chi_2 \chi_4 \chi_7 u \]
\[ - 1440 \chi_1 \chi_5 u - 144 \chi_1 \chi_4 \chi_5 u + 864 \chi_2^2 \chi_5 u + 4608 \chi_6 \chi_5 u + 864 \chi_1 \chi_3 \chi_6 \chi_5 u - 2304 \chi_4 \chi_6 \chi_5 u \]
\[ - 144 \chi_1^3 \chi_7 u + 288 \chi_1 \chi_2 \chi_7 \chi_7 u - 1440 \chi_3 \chi_4 \chi_7 u - 144 \chi_1 \chi_5 \chi_7 u \]
\[ + 864 \chi_2 \chi_5 \chi_7 u + 288 \chi_1 \chi_6 \chi_7 u - 576 \chi_1^2 \chi_7^2 u - 2304 \chi_2 \chi_4 \chi_7^2 u - 144 \chi_1 \chi_2 \chi_7^3 u \]
\[ + 4416 u^2 + 792 \chi_1 \chi_3 u^2 - 2112 \chi_4 u^2 - 48 \chi_4^2 u^2 + 72 \chi_3 \chi_5 u^2 + 2 \chi_3 \chi_4 \chi_5 u^2 + 576 \chi_2 \chi_6 u^2 \]
\[ - 456 \chi_1 \chi_5 u^2 - 24 \chi_1 \chi_4 \chi_7 u^2 + 792 \chi_5 \chi_7 u^2 - 48 \chi_1 \chi_7^2 u^2 + 72 \chi_2 \chi_5 \chi_7^2 u^2 + 36 \chi_1^2 \chi_7^3 u^2 \]
\[ + 576 \chi_2 u^3 + 36 \chi_1 \chi_5 u^3 + 576 \chi_6 u^3 + 36 \chi_3 \chi_7 u^3 + 36 \chi_1 \chi_5 u^3 + 552 u^4 + 12 \chi_4 u^4 + 12 \chi_1 \chi_7 u^4 - u^6 \).

D.1  Decomposition of the $E_7$ characters into $SU(8)$

We list decomposition of the characters $\chi_i^{E_7} \equiv \chi_i^{E_7}$ of $E_7$ fundamental weights into the characters $\chi_i$ of $SU(8)$ fundamental weights

\[ \begin{array}{cccccc}
E_7 & \text{Dynkin diagram} & -1 & +1 & 2 & 3 \\
\chi_i^{E_7} & -1 & +1 & \chi_1 \chi_7 + \chi_4 & \chi_2^{E_7} & \chi_1^2 + \chi_2^2 + \chi_3 \chi_7 + \chi_1 \chi_5 - 2 \chi_2 - 2 \chi_6 & \chi_3^{E_7} & -1 - 2 \chi_4 + 3 \chi_3 \chi_5 + 2 \chi_3 \chi_6 - 3 \chi_2 \chi_6 - \chi_1 \chi_7 + \chi_1 \chi_4 \chi_7 + 2 \chi_2 \chi_7^2 & \\
\chi_4^{E_7} & -2 + 2 \chi_2^2 - \chi_1 \chi_3 + 2 \chi_4 - \chi_2^2 + \chi_3 \chi_5 - 3 \chi_1 \chi_2 \chi_5 + 2 \chi_3 \chi_5 + \chi_2 \chi_5^2 - \chi_2 \chi_6 & + 3 \chi_2 \chi_6 + 2 \chi_3 \chi_6 + 2 \chi_1 \chi_4 \chi_6 - 4 \chi_2 \chi_4 \chi_6 - \chi_1 \chi_3 \chi_6 + \chi_2 \chi_6^2 + 2 \chi_1 \chi_7 - \chi_2 \chi_3 \chi_7 - \chi_5 \chi_7 + \chi_1 \chi_3 \chi_5 \chi_7 - 3 \chi_3 \chi_6 \chi_7 - 2 \chi_2 \chi_7^2 + 2 \chi_2 \chi_4 \chi_7^2 + 3 \chi_3 \chi_7^3 & \\
\chi_5^{E_7} & \chi_3^3 + \chi_2 \chi_4 - 3 \chi_2 \chi_4 - \chi_1 \chi_5 + \chi_3 \chi_6 - 3 \chi_1 \chi_3 \chi_6 - 3 \chi_4 \chi_6 - \chi_3 \chi_7 + \chi_2 \chi_5 \chi_7 + 4 \chi_2 \chi_7 & \chi_6^{E_7} & -1 + \chi_1 \chi_3 - 2 \chi_4 + \chi_2 \chi_6 + \chi_5 \chi_7 & \chi_7^{E_7} & \chi_2 + \chi_6.
\end{array} \] 

(D.4)
E  The Weierstrass from for $SU(9)$ manifest curve of $E_8$ theory

Following [28], we rewrite (4.16) with $U \rightarrow u - 60$ into the standard Weierstrass form

$$y^2 = 4x^3 - g_2x - g_3,$$

where

$$g_2 = \frac{1}{972} \left( 16(\chi_1^2 - 3\chi_2 + 3\chi_8)^2(3\chi_1 - 3\chi_7 + \chi_8^2)^2 + 16(3\chi_1 - 3\chi_7 + \chi_8^2) \left[ 2\chi_1^2\chi_8 + 3(3\chi_4 - 3\chi_7 - \chi_2\chi_8 + \chi_8^2) + 3\chi_1(-57 + u) \right]^2 - 8(\chi_1^2 - 3\chi_2 + 3\chi_8)(3\chi_1 - 3\chi_7 + \chi_8^2) \left( 2\chi_1\chi_8 + 3(-54 + u) \right)^2 + (-162 + 2\chi_1\chi_8 + 3u)^4 - 16(\chi_1^2 - 3\chi_2 + 3\chi_8) \left[ 2\chi_1^2\chi_8 + 3(3\chi_4 - 3\chi_7 - \chi_2\chi_8 + \chi_8^2) + 3\chi_1(-57 + u) \right] \times (-1539 + 27\chi_6 + 9\chi_1\chi_8 - 9\chi_7\chi_8 + 2\chi_8^2 + 27u) - 8 \left[ 2\chi_1^2\chi_8 + 3(3\chi_4 - 3\chi_7 - \chi_2\chi_8 + \chi_8^2) + 3\chi_1(-57 + u) \right] (2\chi_1\chi_8 + 3(-54 + u)) \times (3\chi_1^2 - 9\chi_2 + 9\chi_5 - 3\chi_1\chi_7 - 171\chi_8 + 2\chi_1\chi_8^2 + 3\chi_8u) + 16(\chi_1^2 - 3\chi_2 + 3\chi_8)(3\chi_1^2 - 9\chi_2 + 9\chi_5 - 3\chi_1\chi_7 - 171\chi_8 + 2\chi_1\chi_8^2 + 3\chi_8u)^2 + 8 \left[ 2\chi_1\chi_8 + 3(-54 + u) \right] \left[ -1539 + 27\chi_6 + 9\chi_1\chi_8 - 9\chi_7\chi_8 + 2\chi_8^2 + 27u \right] \times (2\chi_1^3 - 9\chi_1(\chi_2 - \chi_8) + 27(-57 + \chi_3 + u) \right] - 16(3\chi_1 - 3\chi_7 + \chi_8^2)(3\chi_1^2 - 9\chi_2 + 9\chi_5 - 3\chi_1\chi_7 - 171\chi_8 + 2\chi_1\chi_8^2 + 3\chi_8u) \left[ 2\chi_1^3 - 9\chi_1(\chi_2 - \chi_8) + 27(-57 + \chi_3 + u) \right],$$

and

$$g_3 = \frac{1}{1574464} \left( -64(\chi_1^2 - 3\chi_2 + 3\chi_8)^3(3\chi_1 - 3\chi_7 + \chi_8^2)^3 + 192(\chi_1^2 - 3\chi_2 + 3\chi_8) \times (3\chi_1^2 - 3\chi_7 + \chi_8^2) \left[ 2\chi_1^2\chi_8 + 3(3\chi_4 - 3\chi_7 - \chi_2\chi_8 + \chi_8^2) + 3\chi_1(-57 + u) \right]^2 + 48(\chi_1^2 - 3\chi_2 + 3\chi_8)^2(3\chi_1 - 3\chi_7 + \chi_8^2)^2(2\chi_1\chi_8 + 3(-54 + u))^2 + 24(3\chi_1^2 - 3\chi_7 + \chi_8^2) \left[ 2\chi_1^2\chi_8 + 3(3\chi_4 - 3\chi_7 - \chi_2\chi_8 + \chi_8^2) + 3\chi_1(-57 + u) \right]^2 \times (2\chi_1\chi_8 + 3(-54 + u))^2 - 12(\chi_1^2 - 3\chi_2 + 3\chi_8)(3\chi_1 - 3\chi_7 + \chi_8^2)(-162 + 2\chi_1\chi_8 + 3u)^4 + (-162 + 2\chi_1\chi_8 + 3u)^6 - 96(\chi_1^2 - 3\chi_2 + 3\chi_8)^2 \times (3\chi_1^2 - 3\chi_7 + \chi_8^2) \left[ 2\chi_1^2\chi_8 + 3(3\chi_4 - 3\chi_7 - \chi_2\chi_8 + \chi_8^2) + 3\chi_1(-57 + u) \right] \times (-1539 + 27\chi_6 + 9\chi_1\chi_8 - 9\chi_7\chi_8 + 2\chi_8^2 + 27u) - 32 \left[ 2\chi_1^2\chi_8 + 3(3\chi_4 - 3\chi_7 - \chi_2\chi_8 + \chi_8^2) + 3\chi_1(-57 + u) \right]^3 \times (-1539 + 27\chi_6 + 9\chi_1\chi_8 - 9\chi_7\chi_8 + 2\chi_8^2 + 27u) + 72(\chi_1^2 - 3\chi_2 + 3\chi_8) \times (2\chi_1\chi_8 + 3(-54 + u))^2 \times (-1539 + 27\chi_6 + 9\chi_1\chi_8 - 9\chi_7\chi_8 + 2\chi_8^2 + 27u) + 32(\chi_1^2 - 3\chi_2 + 3\chi_8)^3 \times (-1539 + 27\chi_6 + 9\chi_1\chi_8 - 9\chi_7\chi_8 + 2\chi_8^2 + 27u)^2 - 240(\chi_1^2 - 3\chi_2 + 3\chi_8)(3\chi_1 - 3\chi_7 + \chi_8^2) \times
\[ \times \left[ 2\chi_1^2\chi_8 + 3(3\chi_4 - 3\chi_7 - \chi_2\chi_8 + \chi_8^2) + 3\chi_1(-57 + u) \right] \\
\times (2\chi_1\chi_8 + 3(-54 + u))(3\chi_1^2 - 9\chi_2 + 9\chi_5 - 3\chi_1\chi_7 - 171\chi_8 + 2\chi_1\chi_8^2 + 3\chi_8u) \\
-12\left[ 2\chi_1^2\chi_8 + 3(3\chi_4 - 3\chi_7 - \chi_2\chi_8 + \chi_8^2) + 3\chi_1(-57 + u) \right](2\chi_1\chi_8 + 3(-54 + u))^3 \\
\times (3\chi_1^2 - 9\chi_2 + 9\chi_5 - 3\chi_1\chi_7 - 171\chi_8 + 2\chi_1\chi_8^2 + 3\chi_8u) - 96(\chi_1^2 - 3\chi_2 + 3\chi_3)^2 \\
\times (2\chi_1\chi_8 + 3(-54 + u))(-1539 + 27\chi_6 + 9\chi_1\chi_8 - 9\chi_7\chi_8 + 2\chi_8^3 + 27u) \\
\times (3\chi_1^2 - 9\chi_2 + 9\chi_5 - 3\chi_1\chi_7 - 171\chi_8 + 2\chi_1\chi_8^2 + 3\chi_8u)^2 + 24(2\chi_1^2\chi_8 + 3(3\chi_4 - 3\chi_7 - \chi_2\chi_8 + \chi_8^2) \\
+ 3\chi_1(-57 + u))^2(3\chi_1^2 - 9\chi_2 + 9\chi_5 - 3\chi_1\chi_7 - 171\chi_8 + 2\chi_1\chi_8^2 + 3\chi_8u)^2 + 24(\chi_1^2 - 3\chi_2 + 3\chi_3) \\
\times (2\chi_1\chi_8 + 3(-54 + u))^2(3\chi_1^2 - 9\chi_2 + 9\chi_5 - 3\chi_1\chi_7 - 171\chi_8 + 2\chi_1\chi_8^2 + 3\chi_8u)^2 \\
- 96(\chi_1^2 - 3\chi_2 + 3\chi_3)^2\left[ 2\chi_1^2\chi_8 + 3(3\chi_4 - 3\chi_7 - \chi_2\chi_8 + \chi_8^2) + 3\chi_1(-57 + u) \right](2\chi_1\chi_8 + 3(-54 + u)) \\
\times (2\chi_1^2 - 9\chi_1(\chi_2 - \chi_8) + 27(-57 + \chi_3 + u)) + 48(\chi_1^2 - 3\chi_2 + 3\chi_3)(3\chi_1 - 3\chi_7 + \chi_8^2) \\
\times (2\chi_1\chi_8 + 3(-54 + u))(-1539 + 27\chi_6 + 9\chi_1\chi_8 - 9\chi_7\chi_8 + 2\chi_8^3 + 27u) \\
\times \left[ 2\chi_1^3 - 9\chi_1(\chi_2 - \chi_8) + 27(-57 + \chi_3 + u) \right] - 20(2\chi_1\chi_8 + 3(-54 + u))^3 \\
\times (-1539 + 27\chi_6 + 9\chi_1\chi_8 - 9\chi_7\chi_8 + 2\chi_8^3 + 27u) \left[ 2\chi_1^3 - 9\chi_1(\chi_2 - \chi_8) + 27(-57 + \chi_3 + u) \right] \\
- 96(\chi_1^2 - 3\chi_2 + 3\chi_3)(3\chi_1 - 3\chi_7 + \chi_8^2)^2(3\chi_1^2 - 9\chi_2 + 9\chi_5 - 3\chi_1\chi_7 - 171\chi_8 + 2\chi_1\chi_8^2 + 3\chi_8u) \\
\times (2\chi_1^2 - 9\chi_1(\chi_2 - \chi_8) + 27(-57 + \chi_3 + u)) + 72(3\chi_1 - 3\chi_7 + \chi_8^2)(2\chi_1\chi_8 + 3(-54 + u))^2 \\
\times (3\chi_1^2 - 9\chi_2 + 9\chi_5 - 3\chi_1\chi_7 - 171\chi_8 + 2\chi_1\chi_8^2 + 3\chi_8u)(2\chi_1^3 - 9\chi_1(\chi_2 - \chi_8) + 27(-57 + \chi_3 + u)) \\
+ 48\left[ 2\chi_1^2\chi_8 + 3(3\chi_4 - 3\chi_7 - \chi_2\chi_8 + \chi_8^2) + 3\chi_1(-57 + u) \right](-1539 + 27\chi_6 + 9\chi_1\chi_8 - 9\chi_7\chi_8 + 2\chi_8^3 + 27u) \\
\times (3\chi_1^2 - 9\chi_2 + 9\chi_5 - 3\chi_1\chi_7 - 171\chi_8 + 2\chi_1\chi_8^2 + 3\chi_8u) \left[ 2\chi_1^3 - 9\chi_1(\chi_2 - \chi_8) + 27(-57 + \chi_3 + u) \right] \\
- 32(3\chi_1^2 - 9\chi_2 + 9\chi_5 - 3\chi_1\chi_7 - 171\chi_8 + 2\chi_1\chi_8^2 + 3\chi_8u)^2 \left[ 2\chi_1^3 - 9\chi_1(\chi_2 - \chi_8) + 27(-57 + \chi_3 + u) \right] \\
+ 32(3\chi_1 - 3\chi_7 + \chi_8^2)^3(2\chi_1^3 - 9\chi_1(\chi_2 - \chi_8) + 27(-57 + \chi_3 + u))^2 \\
- 8(-1539 + 27\chi_6 + 9\chi_1\chi_8 - 9\chi_7\chi_8 + 2\chi_8^3 + 27u)^2 \left[ 2\chi_1^3 - 9\chi_1(\chi_2 - \chi_8) + 27(-57 + \chi_3 + u) \right]^2 \right) \quad \text{(E.3)} \]

### E.1 Decomposition of the $E_8$ characters into $SU(9)$

We list decomposition of the characters $\chi_{i}^{E_8} \equiv \chi_{\mu_{i}}^{E_8}$ of $E_8$ fundamental weights into the characters $\chi_{i}$ of $SU(9)$ fundamental weights. As mentioned in section 4.2, $\chi_4$ and $\chi_5$ are determined such that (E.3) agrees with (B.3).
\[
\begin{align*}
\chi_1^{E_s} &= -1 + \chi_1 x_2 - 2 x_3 + \chi_1 x_5 - 2 x_6 + \chi_2 x_7 + \chi_4 x_8 + \chi_7 x_8 \\
\chi_2^{E_s} &= \chi_1^3 + \chi_1^2 x_4 - 4(\chi_1 x_2 - \chi_3) + \chi_1 x_2 x_6 - 2(\chi_2 x_7 - \chi_1 x_8) + \chi_1 x_2 \\
&\quad - \chi_1 x_3 x_8 + \chi_1 x_4 x_7 - 4(\chi_1 x_5 - \chi_6) + 3 \chi_1 x_5 - \chi_1 x_6 x_8 + \chi_1 (\chi_7)^2 \\
&\quad + \chi_1 x_8 - 4(\chi_1 x_8 - 1) + \chi_2^2 x_8 - 2 \chi_2 x_4 + \chi_2 x_5 x_8 + 2 \chi_2 x_7 \\
&\quad - 4(\chi_4 x_8 - \chi_3) - 3 \chi_3 x_6 + \chi_3 x_7 x_8 - 5 \chi_3 + 4 x_4 x_5 + 4 x_4 x_8 + \chi_5 (x_8)^2 \\
&\quad - 4(\chi_7 x_8 - \chi_6) - 5 \chi_6 + \chi_7 x_8 + \chi_8^3 - 3 + \chi_2 x_4 - \chi_1 x_5 - 5 \chi_7 x_8 + 4 x_8 \\
\chi_3^{E_s} &= \chi_2^3 + \chi_3 x_4 - 5 \chi_1 x_2 x_3 + 5 \chi_1^2 - \chi_2 x_4 + \chi_1 x_5 + \chi_1^2 x_5 - \chi_2^2 x_5 \\
&\quad - \chi_1 x_4 x_5 - \chi_4 x_5 + \chi_2 x_5^2 - 2 \chi_1 x_2 x_6 + 5 \chi_3 x_6 + \chi_1^2 x_4 x_6 - 2 \chi_2 x_4 x_6 \\
&\quad - 3 \chi_1 x_5 x_6 + 5 \chi_6^2 + 2 \chi_2 x_7 + \chi_1 x_2 x_7 - \chi_2^2 x_3 x_7 - \chi_2 x_3 x_7 + \chi_1 x_4 x_7 \\
&\quad + \chi_4 x_7 - \chi_5 x_7 + \chi_1 x_2 x_5 x_7 - 2 \chi_3 x_5 x_7 - \chi_2 x_6 x_7 - \chi_1 x_7^2 + \chi_1 x_3 x_7^2 \\
&\quad - \chi_4 x_7^2 + \chi_3^2 - \chi_2^2 x_8 + \chi_1 x_3 x_8 + 4 x_8 + \chi_1 x_2 x_4 x_8 - 3 \chi_3 x_4 x_8 \\
&\quad - (\chi_1)^2 x_5 x_8 + \chi_2 x_3 x_8 + \chi_1 x_4 x_5 x_8 + \chi_1 x_6 x_8 + (\chi_2)^2 x_6 x_8 \\
&\quad - 2 \chi_1 x_3 x_6 x_8 - 4 x_4 x_6 x_8 - 2 \chi_1 x_7 x_8 + \chi_2 x_4 x_7 x_8 + \chi_1 x_5 x_7 x_8 - 5 \chi_6 x_7 x_8 \\
&\quad + \chi_2 (x_7)^2 x_8 - \chi_1 x_4 (x_8)^2 + \chi_3 x_5 (x_8)^2 - \chi_2 x_6 (x_8)^2 + \chi_4 x_7 (x_8)^2 + \chi_6 (x_8)^3 \\
\chi_5^{E_s} &= 3 + 2 \chi_2 x_4 + \chi_4 + \chi_4^2 x_5 + \chi_5^2 - 2 \chi_1 x_3 + \chi_2 x_5^2 + \chi_3^2 - 3 \chi_6 + \chi_3^2 x_6 - 3 \chi_2 x_4 x_6 \\
&\quad + \chi_2^2 x_5 x_6 - 4 \chi_2 x_5 x_6 - 4 \chi_2 x_7 - 2 \chi_2 x_4 x_7 + \chi_2 x_7 + 2 \chi_5 x_7 + \chi_2 x_4 x_5 x_7 \\
&\quad + 2 \chi_2 x_4 x_7 + \chi^2 x_5^2 + \chi_4 x_4^2 - 2 \chi_2 x_5 x_7^2 + \chi_4^2 (-1 + (-1 + \chi_3) x_6 + \chi_2^2 - \chi_5 x_7) - \chi_2^2 x_8 \\
&\quad - \chi_2^2 x_8 - 2 \chi_2 x_5 x_8 - \chi_2^2 x_8 - 2 \chi_2 x_6 x_8 + 6 \chi_4 x_6 x_8 + 2 \chi_7 x_8 + 2 \chi_2 x_4 x_7 x_8 \\
&\quad + \chi_4 x_5 x_7 x_8 - \chi_2 x_7^2 x_8 + 2 \chi_2 x_8^2 + \chi_4 x_4 x_8 - \chi_2^2 x_8 + x_2 x_6 x_8 - 4 \chi_4 x_7 x_8 \\
&\quad + 2 \chi_2 x_5 x_7 x_8 - \chi_8^3 - \chi_2 x_4 x_8^3 + \chi_4 x_4^3 + \chi_4 x_4 - 8 \chi_6 - 4 \chi_7 x_8 + x_2^3 + \chi_2^2 (-\chi^2 + \chi_4 x_6 + 2 \chi_7 \\
&\quad + \chi_3 x_7 + \chi_3 x_6 x_7 + \chi_2 x_8 - 4 \chi_4 x_6 x_8 - x_3 x_8 - 6 \chi_6 x_8 + 2 \chi_2 (-4 \chi_5 + \chi_4 x_7 + \chi_8) \\
&\quad + \chi_3 (\chi_2^2 - (-2 + 2 \chi_3 + \chi_6) x_8)) + \chi_3 (-3 + 8 \chi_2^2 + 2 \chi_2 x_7 - 3 \chi_5 x_7 + \chi_4 + \chi_5 x_8 + 4 \chi_7 x_8 + \chi_5 \chi_7^2 \\
&\quad - \chi_5^3 + \chi_4 (4 \chi_5 + \chi_7^2 + 3 \chi_8 - 2 \chi_6 x_8) + \chi_6 (-3 - 2 \chi_2 x_7 - 6 \chi_7 x_8 + \chi_2^3 + \chi_5^3) \\
&\quad + \chi_1 (6 \chi_3 x_5 + \chi_2^2 (-1 + \chi_6) + 3 \chi_5 x_6 - 2 \chi_3 x_5 x_6 - \chi_2^2 x_7 - \chi_2^2 - 2 \chi_3 x_7 - 3 \chi_8 + \chi_3 x_8 + \chi_6 x_8 \\
&\quad + \chi_3 x_6 x_8 - 2 \chi_5 x_7 x_8 + \chi_3 x_5 x_7 x_8 + \chi_2^2 x_8 + 2 \chi_7 x_8 + \chi_2^2 (-\chi_7 + \chi_5 x_8) \\
&\quad + \chi_4 (2 \chi_7 + \chi_2^2 x_8 - \chi_8 (x_5 + (-2 + 3 + 2 \chi_6) x_8)) + \chi_2 (2 - 4 \chi_5 + 2 \chi_5 x_7 - \chi_7 x_8 \\
&\quad + \chi_3 x_7 x_8 - 5 \chi_5 x_8 + 4 \chi_5 (x_5 - (2 + \chi_6) x_8) + \chi_6 (4 - 6 \chi_3 + \chi_7 x_8)) \\
\chi_6^{E_s} &= -1 + 3 \chi_3 - \chi_2 x_4 + \chi_4 x_4 + \chi_1 x_5 + \chi_2^2 x_5 - 2 \chi_1 x_3 x_5 - 2 \chi_4 x_5 + 3 \chi_6 + \chi_3^2 x_6 - 4 \chi_1 x_2 x_6 \\
&\quad + 6 \chi_3 x_6 + 2 \chi_2 x_4 x_6 - \chi_1 x_5 x_6 + \chi_2^2 x_5 x_7 - 2 \chi_2 x_3 x_7 - 5 \chi_5 x_7 + \chi_3 x_5 x_7 + \chi_2 x_6 x_7 - 2 \chi_2 x_6 x_7 \\
&\quad + 2 \chi_1 x_7^2 + \chi_2 x_8 - 2 \chi_1 x_3 x_8 + 4 x_8 + \chi_1 x_2 x_4 x_8 - \chi_3 x_4 x_8 - \chi_1^2 x_5 x_8 + \chi_3^2 x_8 - 2 \chi_1 x_6 x_8 \\
&\quad + \chi_1 x_3 x_6 x_8 - 2 \chi_4 x_6 x_8 + \chi_1 x_2 x_7 x_8 - 4 \chi_3 x_7 x_8 + \chi_1 x_5 x_7 x_8 - \chi_1^2 x_8 + \chi_2 x_3 x_8 \\
&\quad - \chi_1 x_4 x_8 + \chi_2 x_6 x_8^2 + \chi_3 x_3^3 \\
\end{align*}
\]
\[ \chi_{E_8}^{E_7} = \chi_1 \chi_7 + \chi_1 \chi_8 - 2 \chi_1 \chi_5 + \chi_1 \chi_6 \chi_8 + 2 \chi_1 \chi_8 - 4 (\chi_1 \chi_8 - 1) + 2 \chi_2 \chi_4 \]

\[ - 2 \chi_2 \chi_7 + \chi_2 \chi_8^2 + \chi_3 \chi_6 - \chi_3 - \chi_6 - 2 - \chi_2 \chi_4 + \chi_1 \chi_5 + \chi_5 \chi_7 - \chi_4 \chi_8 \]  \hspace{9cm} (E.9)

\[ \chi_{E_8}^{E_6} = - 1 + \chi_3 + \chi_6 + \chi_1 \chi_8 \]  \hspace{9cm} (E.10)

\[ \chi_{A_4}^{E_8} = - 3 + 6 \chi_3 - \chi_3^2 + \chi_1 \chi_4 + \chi_3^2 + 5 \chi_4 \chi_5 - 3 \chi_4 \chi_6 \chi_6 - 7 \chi_3 \chi_6^2 + 9 \chi_3 \chi_6^2 - \chi_5^2 + \chi_1 \chi_2 + \chi_1 \chi_6 \]

\[ + \chi_5 \chi_7 - 2 \chi_3 \chi_5 \chi_7 - 2 \chi_3 \chi_6 \chi_7 - 2 \chi_3 \chi_6 \chi_7 - 7 \chi_3 \chi_5 \chi_6 \chi_7 + 3 \chi_3 \chi_4 \chi_6^2 \]

\[ + 2 \chi_4 \chi_6 \chi_7 - \chi_5^2 - \chi_3 \chi_5^2 + \chi_3 \chi_5^2 - 4 \chi_4 \chi_8 - \chi_3 \chi_4 \chi_8 + 2 \chi_3 \chi_4 \chi_8 - \chi_3 \chi_4 \chi_8 \]

\[ - 3 \chi_3 \chi_7 + \chi_3 \chi_7 + 3 \chi_4 \chi_6 \chi_8 + 5 \chi_3 \chi_4 \chi_6 \chi_8 \]

\[ + 4 \chi_4 \chi_7 + 2 \chi_3 \chi_4 \chi_6 \chi_8 + 4 \chi_4 \chi_7 + 2 \chi_3 \chi_4 \chi_6 \chi_8 - 3 \chi_3 \chi_4 \chi_7 \]

\[ - 4 \chi_4 \chi_7 + 3 \chi_4 \chi_7 + 2 \chi_3 \chi_4 \chi_7 + 2 \chi_3 \chi_4 \chi_7 - 4 \chi_4 \chi_7 + 2 \chi_3 \chi_4 \chi_7 \]

\[ \chi_7 (1 - \chi_6 + 2 \chi_6 + 2 \chi_7 + 2 \chi_7) \]

\[ + 3 \chi_3 \chi_6 \chi_7 + \chi_3 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_8 + 4 \chi_5 \chi_6 \chi_8 + 2 \chi_5 \chi_6 \chi_8 + 2 \chi_5 \chi_6 \chi_8 - 2 \chi_5 \chi_6 \chi_8 \]

\[ - \chi_5 \chi_6 \chi_8 - \chi_5 \chi_6 \chi_8 - \chi_5 \chi_6 \chi_8 + \chi_5 \chi_6 \chi_8 + \chi_5 \chi_6 \chi_8 - 4 \chi_5 \chi_6 \chi_8 - 4 \chi_5 \chi_6 \chi_8 \]

\[ + \chi_4 (2 + \chi_6 - 6 \chi_7 + \chi_8 (x_5 + x_8))) + \chi_4 (2 + \chi_6 - 6 \chi_7 + \chi_8 (x_5 + x_8)) + \chi_4 (2 + \chi_6 - 6 \chi_7 + \chi_8 (x_5 + x_8)) \]

\[ + \chi_2 (x_5 + 2 \chi_5 + 2 \chi_6 + 2 \chi_7 + 2 \chi_7) \]

\[ + 3 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 \]

\[ + 3 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 \]

\[ + 3 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 \]

\[ + 3 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 \]

\[ + 3 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 \]

\[ + 3 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 \]

\[ + 3 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 \]

\[ + 3 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 \]

\[ + 3 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 \]

\[ + 3 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 - 2 \chi_5 \chi_6 \chi_7 + 2 \chi_5 \chi_6 \chi_7 \]  \hspace{9cm} (E.11)
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