The group law for the Jacobi variety of plane curves

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We extend the group law of curves of degree three by chords and tangents to the Jacobi variety of plane curves of degree \( n \geq 4 \) by replacing points by point groups and lines by algebraic curves. The curves are nonsingular or have simple singularities. In the cases \( n = 4, 5, 6 \) we have a close analogy to the case \( n = 3 \). We describe an algorithm using Groebner bases.

Keywords: Jacobi variety, divisors, algorithms

1 Introduction

The Jacobi variety of an algebraic curve of genus \( g \) plays a central role in algebraic geometry. Explicit descriptions of the group law play a less important role in the history of the subject. With the development of cryptography algorithms arose for the group law. In 1987 D.G. Cantor described the group law of a hyperelliptic curve in the context of cryptography in analogy to the Gauß composition of quadratic forms, cf. [4, 14].

Later group laws of other classes of plane curves were described. The papers [9, 10, 12] are based on the analogy of the Jacobian group with class groups in number theory. Their methods are restricted to curves with a special type of infinite points. Other papers are concerned with special types of curves (e.g. Picard curves [7, 8], \( C_{ab} \)-curves [1]). The papers [17, 13] describe general ideas for an algorithm for arbitrary plane curves. They are based on the theory of plane curves. These algorithms are not practical from a computational point of view.

The content of this paper is an algorithm for smooth plane curves or of plane curves with simple double points. We give an elementary presentation using projective curves, Riemann surfaces and commutative algebra. We consider arbitrary complex nonsingular plane \( n \)-curves \( C \) \( (n \geq 4) \) with an arbitrary zero point. We describe a geometric group law intersecting \( C \) with certain algebraic \( m \)-curves. For \( n = 4, 5, 6 \) we intersect \( C \) with \( (n - 1) \)-curves and we have a very close analogy to the group law of an elliptic curve. On the basis of the geometric group law we give an algebraic description. We represent divisors by certain homogeneous ideals and describe the group law by ideal operations. All operations are rational and can be carried out by Groebner basis operations. Infinite points do not play a special role. The construction works also for curves with simple double points. We give an example for a curve of genus 2. In the last sections we discuss hyperelliptic curves, Picard curves and the case of fields of characteristic \( p \).

Because of the use of Groebner bases it is difficult to give a realistic complexity analysis. We used the computer algebra program Mathematica 4.0 forming the computations.
2 Nonsingular curves and intersections

Consider a nonsingular projective plane curve $C$ of degree $n \geq 4$ defined by the irreducible polynomial

$$F(x, y, z) := \sum_{i+j \leq n} a_{ij} x^i y^j z^{n-i-j}.$$ 

One can consider $C$ as a Riemann surface of genus $g = \frac{(n-1)(n-2)}{2}$ (i.e. $g = 3, 6, 10, \ldots$). In every point one of the coordinates $x, y, z$ defines a complex chart. We have a $g$-dimensional space of holomorphic differentials. Every holomorphic differential admits an explicit representation in the form

$$G(x, y, z) \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ x & y & z \\ dx \ dy \ dz \end{vmatrix} \gamma_1 F_x + \gamma_2 F_y + \gamma_3 F_z,$$

where $G(x, y, z)$ is a homogeneous polynomial of degree $n - 3$ and $\gamma_1, \gamma_2, \gamma_3$ are three complex numbers chosen depending on $x, y, z$ such that the denominator is not zero (cf. [5]). Furthermore let $H(P_1 + \cdots + P_t)$ be the subspace of holomorphic differentials whose zero divisors contain the points $P_1, \ldots, P_t$.

Now we consider $m$-curves $C_m$. It is allowed that $C_m$ has multiple irreducible components. First we consider the case $1 \leq m \leq n - 1$. $C_m$ has $\frac{(m+1)(m+2)}{2}$ parameters. Given $b_m := \frac{m(m+3)}{2}$ points on $C$ we have a (not necessary unique) $C_m$-curve through these points. (A multiple intersection point will correspond to a multiple contact of $C_m$ with $C$). Because of the irreducibility of $C$ an $m$-curve $C_m$ has $mn$ intersections with $C$. Therefore we have

$$c_m := mn - \frac{m(m+3)}{2} = \frac{m(2n - m - 3)}{2} = g - \left( \frac{n - m - 1}{2} \right)$$

further intersections.

For $n = 4, 5, 6$ we will be interested in the cases $m = n - 1, n - 2$ and $n - 3$. Here we have $c_{n-1} = g, c_{n-2} = g, c_{n-3} = g - 1$ and $b_{n-1} = g + 2n - 2, b_{n-2} = g + n - 2, b_{n-3} = g - 1$.

For $n > 6$ we will need also curves $C_m$ of degree $m \geq n$. In this case we have the difficulty that it is possible, that $C$ is an irreducible component of $C_m$. For $m \geq n$ we fix a monomial $x^k y^l z^n - k - l$ of $F$ with $a_{kl} \neq 0$. Then we consider the $m$-curves

$$\sum_{(i, j) \text{ with } x^k y^l z^n - k - l} \beta_{i,j,m-i-j} x^i y^j z^{m-i-j}.$$ 

They form a linear system of dimension $\left( \frac{m+2}{2} \right) - \left( \frac{m-n+2}{2} \right) - 1 = mn - g$ of curves without common components with $C$. Given $b_m := mn - g$ points on $C$ we find a $C_m$-curve of this special form through these points. The curves $C$ and $C_m$ have no common components and therefore they have $mn$ intersections, i.e. we have $c_m = g$ further intersections.

3 Jacobi varieties and Reduced Divisors

The Jacobi variety of $C$ is the Abelian group

$$Jac(C) = \text{Div}^0(C)/\text{Div}^P(C).$$
Here $\text{Div}^0(C)$ denotes the group of divisors of degree 0 and $\text{Div}^P(C)$ is the subgroup of principal divisors (i.e. the zeros and poles of analytic functions), cf. [11]. If $G(x, y, z)$ defines an $m$-curve $C_m$, then $G(x, y, 1)$ defines a meromorphic function with zeros at the finite intersections of $C_m$ and with poles of order $m - h$ at the infinite points $P$ of $C$ if $P$ is a $h$-fold intersection with $C_m$. Let $D_\infty$ be the divisor of the infinite points of $C$. Then we have $P_1 + \cdots + P_{mn} \sim mD_\infty$, if $P_1, \cdots, P_{mn}$ are the (possibly infinite) intersections of $C_m$ and $C$.

We fix an arbitrary point $P_0$ on $C$. We call a divisor $D$ of the form $D = P_1 + \cdots + P_t - tP_0$ with $P_1, \cdots, P_t \neq P_0$ a semireduced divisor. We call this semireduced divisor $D$ a reduced divisor if there is no divisor $D' = P'_1 + \cdots + P'_s - sP_0$ with $P'_1, \cdots, P'_s \neq P_0$, $D' \sim D$ and $s < t$.

**Proposition 1** We find in every divisor class of $\text{Jac}(C)$ a unique reduced divisor with $t \leq g$.

**Proof:** Existence: Let $D \in \text{Div}^0(C)$. By the Riemann Roch theorem we have $\dim L(D + gP_0) \geq g - g + 1 = 1$. It follows

$$D + gP_0 + (f) = P_1 + \cdots + P_g$$

with $(f) \in L(D + gP_0)$ and certain $P_i \in C$. Therefore we have $D \sim P_1 + \cdots + P_g - gP_0$, i.e. the set of semireduced divisors with $t \leq g$ is not empty and we can find reduced divisors. 

Uniqueness: Let $D = P_1 + \cdots + P_t - tP_0 \sim D' = Q_1 + \cdots + Q_t - tP_0$ be two different reduced divisors with $P_i, Q_i \neq P_0$. There is a function $f$ with $(f) = D - D'$. Then the divisor $D' + (f - f(P_0))$ has the form $R_1 + \cdots + R_s - sR_0$ with $s < t$. This is a contradiction.

\[ \text{Remark:} \] One can choose $P_0$ arbitrarily, but the structure of a reduced divisor might vary with this choice. For the reason of simplicity it can be useful to choose for $P_0$ an exceptional point (flex, rational point, infinite point etc.). 

**Remark:** We mention that not all semireduced divisors with $t \leq g$ are reduced. In the generic case we have $t = g$ for a reduced divisor.

We have the following abstract characterisation of reduced divisors.

**Proposition 2** The following three assertions are equivalent.

(i) The divisor $D = P_1 + \cdots + P_t - tP_0$ with $t \leq g$ and $P_1, \cdots, P_t \neq P_0$ is reduced.

(ii) $\dim H(P_1 + \cdots + P_t) = g - t$.

(iii) The dimension of the linear system of $(n - 3)$-curves, which vanish on $P_1, \cdots, P_t$ does not exceed $g - t$.

**Proof.** The proposition is a consequence of the Riemann Roch theorem. 

For $n = 4, 5, 6$ we have an explicit description of the set of reduced divisors.

**Proposition 3** (i) Let $n = 4$. The divisor $D = P_1 + \cdots + P_t - tP_0$ with $t \leq g = 3$ and $P_1, \cdots, P_t \neq P_0$ is reduced if and only if $t = 3$ and $P_1, P_2, P_3$ lie not on a projective line or $t < 3$.

(ii) Let $n = 5$. The divisor $D = P_1 + \cdots + P_t - tP_0$ with $t \leq g = 6$ and $P_1, \cdots, P_t \neq P_0$ is reduced if and only if no 6 points lie on a conic section and no 4 points lie on a line.

(iii) Let $n = 6$. The divisor $D = P_1 + \cdots + P_t - tP_0$ with $t \leq g = 10$ and $P_1, \cdots, P_t \neq P_0$ is reduced if and only if no 10 points lie on a cubic and no 9 points form the intersection of two cubics and no 8 points lie on a conic section and no 5 points lie on a line.
Proof. $n \leq 2d + 2$ points of the projective plane fail to impose independent conditions on curves of degree $d$ if and only if either $d + 2$ of the points are collinear or $n = 2d + 2$ and the $n$ points lie on a conic, cf. [b]. The proposition follows from this fact, from the Cayley-Bacharach theorem, cf. also [6] and from Proposition 2.

Remark: An explicit characterisation of reduced divisors for arbitrary $n$ is associated to the classification problem of special divisors on Riemann surfaces or to the Cayley-Bacharach conjectures for algebraic curves. However there is an algorithm to determine the reduced divisor (cf. below).

4 The construction of the reduced divisor

Let

$$D = D^+ - D^- = P_1 + \cdots + P_s - Q_1 - \cdots - Q_s$$

with $P_i, Q_i \in C$ be an arbitrary divisor of degree zero.

At first we consider an $m$-curve with a polynomial $G(x, y, z) \neq 0 \mod F$ with a minimal $m \geq n - 2$ such that

$$s + g \leq b_m := \begin{cases} 
\frac{m(m+3)}{2} & \text{for } m < n \\
mn - g & \text{for } m \geq n 
\end{cases}$$

through the $s$ points of $D^+$ and $(b_m - s)P_0$. Because of our discussion at the end of section 2 the polynomial $G(x, y, z) \neq 0 \mod F$ exists and there are $g$ remaining intersections $R_1, \cdots, R_g$. We have

$$D^+ + (b_m - s)P_0 + R_1 + \cdots + R_g \sim mD_\infty.$$

Then we consider another $m$-curve $G'(x, y, z) \neq 0 \mod F$ through the $s + g$ points of $D^- + R_1 + \cdots + R_g$ and $(b_m - s - g)P_0$. Because this curve is not necessary unique we require a maximal additional contact $\alpha$ at $P_0$. Let $S_1, \cdots, S_{g-\alpha}$ be the remaining intersections not equal to $P_0$. We have

$$D^- + R_1 + \cdots + R_g + (b_m - s - g + \alpha)P_0 + S_1 + \cdots + S_{g-\alpha} \sim mD_\infty.$$

It follows

$$D^+ - D^- \sim S_1 + \cdots + S_{g-\alpha} - (g - \alpha)P_0.$$

Proposition 4

$$\overline{D} := S_1 + \cdots + S_{g-\alpha} - (g - \alpha)P_0$$

is the reduced divisor for $D$.

Proof. From the above relations it follows $D \sim \overline{D}$. Furthermore we consider the divisor

$$D_1 := D^- + R_1 + \cdots + R_g + (b_m - s - g + \alpha)P_0 - (m - n + 3)D_\infty.$$

Lemma 5 All differentials of $H(D_1)$ have the form

$$G_m(x, y, z) \frac{\begin{vmatrix} 
\gamma_1 & \gamma_2 & \gamma_3 \\
x & y & z \\
dx & dy & dz 
\end{vmatrix}}{z^{m-n+3}} = (\gamma_1F_x + \gamma_2F_y + \gamma_3F_z)$$

with $\deg G_m = m$. 


Proof. Case 1 (the positive part of $D_1$ does not contain infinite points): Meromorphic differentials are unique determined by its principal part up to a holomorphic differential. The space of principal parts is $(m-n+3)n$-dimensional. Therefore the space of differentials with poles at most at $(m-n+3)D_\infty$ has the dimension $(m-n+3)n + g = mn - g + 1$. Otherwise the space of degree $m$ polynomials $G \mod F$ has the dimension $mn - g + 1$ for $m \geq n - 3$ (cf. above). Therefore every differential with poles at most at $(m-n+3)D_\infty$ has the above form.

Case 2 ($\alpha$ points of the positive part $D_1^+$ and the negative part $D_1^-$ of $D_1$ cancel): Analogously the space of differentials with negative part at most at $D_1^-$ has the dimension $mn - g + 1 - \alpha$. On the other side the space of degree $m$ polynomials $G \mod F$ with zeros at the $\alpha$ common points of $D_1^+$ and $D_1^-$ has also the dimension $mn - g + 1 - \alpha$.

For $G_m = G'$ we obtain a differential in $H(D_1)$. Because $C'$ has maximal contact at $P_0$ with $C$ we have no further differential in $H(D_1)$. Therefore we have

$$\dim H(D_1) = 1.$$ 

Because $D_1 + S_1 + \cdots + S_{g-\alpha}$ is the divisor of poles and zeros of the differential with $G_m = G'$ it is a canonical divisor. By the Riemann Roch theorem it follows

$$\dim H(S_1 + \cdots + S_{g-\alpha}) = \dim L(D_1) =$$

$$\dim H(D_1) = (s + g + b_m - s - g + \alpha - (m-n+3)n) - g + 1 + 1 =$$

$$mn - (m-n+3)n - 2g + \alpha + 2 = \alpha.$$ 

Therefore $\overline{D}$ is reduced according to Proposition 2.

5 The group law

We represent the elements of the Jacobian by reduced divisors. For a description of the group law it is sufficient to reduce $-D$ and $D_1 + D_2$ if $D, D_1, D_2$ are reduced divisors.

1. The inverse divisor. Let $D = P_1 + \cdots + P_g - gP_0$ where $P_1, \cdots, P_g = P_0$ is allowed. In this case we apply the above construction to $D^+ = gP_0$ and $D^- = P_1 + \cdots + P_g$.

2. The addition of reduced divisors. Let $D_1 = P_1 + \cdots + gP_0$ and $D_2 = P_{g+1} + \cdots + P_{2g} - gP_0$. In this case we apply the above construction to $D^+ = D_1 + \cdots + P_{2g}$ and $D^- = 2gP_0$.

For $n = 4, 5, 6$ we can carry out the construction with $(n-1)$-curves. Therefore we have a close analogy to the group law of cubic curves. In these cases we have $b_{n-1} = g + 2n - 2 \geq 2g$. Instead of the $(n-2)$-curves of the cubic case we consider $(n-1)$-curves. We construct an $(n-1)$-curve through $P_1, \cdots, P_{2g}, (2n-2-g)P_0$. We have the remaining points $R_1, \cdots, R_g$. Then we construct an $(n-1)$-curve through $R_1, \cdots, R_g$ with highest contact at $P_0$ and obtain the remaining points $S_1, \cdots, S_t$. Then $S_1 + \cdots + S_t - tP_0$ is the reduced divisor for $D_1 + D_2$. Analogously, for $n \geq 7$ we carry out the construction with $m$-curves $G(x, y, z) \not\equiv 0 \mod F$ with $m \geq n$ and $b_m \geq 2g$ (i.e. $mn \geq 3g$).

6 Algebraic description - The ideal-divisor-correspondence

1. The affine case. We remark that there is a one-to-one correspondence between ideals of the quotient ring $\mathbb{C}[x, y]/I_C$ of polynomial functions on the affine curve $C$ and the
ideals of $\mathbb{C}[x,y]$ with $I \supset I_C$ where $I_C = (F(x,y,1))$. $\mathbb{C}[x,y]/I_C$ is one-dimensional and a Dedekind ring, cf. [2].

Now let $I$ be an ideal with $I \supset I_C$. Then $I$ is zerodimensional. Because $\mathbb{C}[x,y]/I_C$ is a Dedekind ring we have the unique primary decomposition

$$I = \cap(I_{P_i}^{m_i}, I_C)$$

where $P_i = (x_i,y_i)$ and $I_{P_i} = ((x-x_i), (y-y_i), I_C)$, cf. [2]. We associate to $I$ the effective divisor

$$D_I := \sum m_i P_i .$$

Conversely, let $D := \sum m_i P_i$ be an effective divisor of finite points. Then we associate to $D$ the ideal

$$I_D := \cap(I_{P_i}^{m_i}, I_C) .$$

Therefore we have a one-to-one correspondence of effective divisors and ideals with $I \supset I_C$.

Furthermore we have the following Lemma.

**Lemma 6** Let $D, D'$ be effective divisors of finite points and let $f \in I_D$. Then we have $(f) \geq D$ and

$$(I_D, I_{D'}, I_C) = I_{D+D'}$$

and

$$(f), I_C : I_D = I_{(f) - D} .$$

Here $(f)$ denotes both, the ideal generated by $f(x,y)$ and the divisor of the analytic continuation of the meromorphic function $f(x,y)$.

**Proof.** The proof follows from the above primary decomposition and the relations $I_{P}^P \cap I_{Q}^Q = I_{P}^P I_{Q}^Q \bmod I_C$ for $P \neq Q$ and $I_{P}^P \cap I_{P}^Q = I_{P}^{\max (p,q)} \bmod I_C$. ●

2. The projective case. We suppose that $[0 : 1 : 0] \notin C$. Now let $D = D_\infty + D_\infty$ be an effective divisor with the finite part $D_\infty$ and the infinite part $D_\infty$. We define

$$I_D^h := H_z(I_{D_\infty}) \cap H_x(I_{D_\infty})$$

where $I_{D_\infty} \subset \mathbb{C}[x,y]$, $I_{D_\infty} \subset \mathbb{C}[y,z]$ and $H_z, H_x$ are the homogenisations with respect to $z, x$, respectively.

Conversely, let $I$ be a homogeneous ideal of $\mathbb{C}[x,y,z]$ containing the curve ideal, i.e. $I \supset (F(x,y,z))$. Then we define

$$D_I := D_{A_x(I)} + D_{(A_x(I), z)}$$

where $A_x, A_z$ are the affinisations with respect to $z, x$.

We have $D_{I_D} = D$. We remark that for ideals $I_1, I_2$ with $I_1, I_2 \supset (F(x,y,z))$ the relations $I_1(D_1) = I$ and $I_1 I_2 = I_{D_1 + D_2}$ are in general not valid.

Now we define a product $\odot$ for homogeneous ideals $I_1, I_2$ which corresponds to the addition of divisors. We form the ideal product of the corresponding affine ideals. In order to include infinite points we consider products with respect to two affinisations with $z, x = 1$. The intersection of the corresponding homogenisations will contain all curves with intersection divisor $\geq D_{I_1} + D_{I_2}$. I.e. we define

$$I_1 \odot I_2 := H_z((A_x(I_1) \cdot A_z(I_2), I_C)) \cap H_x((A_x(I_1) \cdot A_z(I_2), I_C^z))$$

where $I_C^z := A_x(H_z(I))$ for ideals $I \subset \mathbb{C}[x,y]$. A generalized ideal quotient $\oslash$ is defined by

$$(G) \oslash I := H_z((A_x((G)), I_C) : A_x(I)) \cap H_x((A_x((G)), I_C^z) : A_x(I)) .$$
PROPOSITION 7 Let $D, D'$ be effective divisors and let $G \in I_D$. Then we have $(G) \geq D$ and

\[ I^h_{D+D'} = I^h_D \odot I^h_{D'} \quad \text{and} \quad I^h_{(G)-D} = I^h_{(G)} \odot I^h_D. \]

Proof. The proof follows from the fact, that the left and the right side are equal to the homogeneous ideal of all curves with intersection divisor $\geq D_{I_1} + D_{I_2}$ and $\geq (G) - D_1$, respectively.

7 Reduction and the group law

Let $D = D^+ - D^-$ with $\deg D^+ = \deg D^- = s$ be a divisor of degree zero and let $I^+ = I^h_{D^+}$, $I^- = I^h_{D^-}$. Furthermore we consider the homogeneous ideals

\[ I_r := I^h_{rP_0} \]

of forms with an $r$-fold point $P_0$. We choose $m$ such that $b_m \geq s + g$. Then we choose an arbitrary element $G$ in $I^+ \odot I_{b_m-s} = I^h_{D^++(b_m-s)P_0}$ of degree $m$ and we form $J = (G) \odot (I^+ \odot I_{b_m-s})$. Then we determine the number $\alpha$ such that $(J \odot I^-) \cap I_{b_m-s+g+\alpha}$ contains exactly one Groebner basis element $G'$ of degree $m$ with respect to a degree order. We form $I_{red} = (G') \odot ((J \odot I^-) \cap I_{b_m-s+g+\alpha}) = I_{red} = I^h_{S_1, \ldots, S_g - \alpha}$ where $S_1, \ldots, S_g - \alpha = (g-\alpha)P_0$ is the reduced divisor of $D$.

Remark: Given $I^+, I^-$, $t$ one can carry out the determination of $I^+ \odot I_{b_m-s}$, $J = (G) \odot (I^+ \odot I_{b_m-s})$, $(J \odot I^-) \cap I_r$ ($r \geq b_m-s+g$) and $(G') \odot ((J \odot I^-) \cap I_{b_m-s+g+\alpha})$ by Groebner basis calculations, cf. [3].

Now we can describe the group law. In order to add the divisors $D_1 = P_1 + \cdots + P_{g-gP_0}$ and $D_2 = P_{g+1} + \cdots + P_{2g} - gP_0$ we apply the above construction to $D^+ = P_1 + \cdots + P_{2g}$ and $D^- = 2gP_0$. In order to determine the ideal for $-D$ of $D = P_1 + \cdots + P_g - gP_0$ (where $P_1, \ldots, P_g = P_0$ is allowed) we apply the above construction to $D^+ = gP_0$ and $D^- = P_1 + \cdots + P_g$.

8 An Example

We consider the 4-curve $C$ with $x^4 + y^4 = 2z^4$ with $g = 3$. Let $P_0 = (1,1,1)$ and $P_1 = (1,-1,1)$. We reduce the divisor $D = 6P_1 - 6P_0$.

We choose $m = 3$. It is sufficient to consider affine ideals. ($C$ has no infinite intersections with all curves occurring during the calculation.) We obtain by Groebner basis calculations with respect to lexicographic order

\[ I^+ = \{1+6y+15y^2+20y^3+15y^4+6y^5+y^6, 102-x+524y+1092y^2+1141y^3+598y^4+126y^5\}, \]

\[ I_3 = \{-1 + 3y - 3y^2 + y^3, -1 + x + 5y - 3y^2\}, \]

\[ (I^+I_3, I_C) = \{-1 - 3y + 8y^3 + 6y^4 - 6y^5 - 8y^6 + 3y^8 + y^9, 1289 - 128x + 2492y - 2016y^2 - 7476y^3 - 806y^4 + 7476y^5 + 3080y^6 - 2492y^7 - 1419y^8\}. \]

With respect to a degree order we find an element of degree 3 in $(I^+I_3, I_C)$

\[ f = 2612 - 3078x + 378x^2 + 281x^3 + 478y - 1912xy + 1195x^2y - 1286y^2 + 1093xy^2 + 239y^3. \]
Furthermore

\( J := ((f), I_C) : (I^+ I_3) = \{698405268857+63573554837y-10585774871y^2+10861966941y^3, \)

\[ 26870776349 + 254869376165x + 103445986821y + 10861966941y^2 \}, \]

\[ -6626x^2y - 735502y^2 + 70382xy^2 + 325163y^3 \],

\[ I_6 = \{1-6y+15y^2-20y^3-15y^4+6y^5+y^6, -102+x+524y-1092y^2+1141y^3-598y^4+126y^5 \}, \]

\( (J_6, I_C) = \{698405268857-354956264305y+6651081164962y^2-425994925918y^3-3049132563596y^4+7186609158448y^5-5447186836050y^6+328254039678y^7 - 662303791517y^8 + 10861966941y^9, \)

\[ -9377644423051557041062363432972616453181055355827939921760183+218588416297803358805434835328036429801750462271550638009646820x +3966094251028137403139367162629560152398671810731266834522828y9-55359919759638769561177902366625266643132129388264389587928932370y^2 +109584419337196973244704249837673133664254808013142794005682y^3+4883833459174203386208854616203642163940121543827434792929080y^4 +54821260785215328240108587994328013884947131311449528021917262y^5 +272507862737893198252063713723235299066716213131652109697839214y^6 +8584455924362109011538689603037701794199996130679164705227470y^7 +17705688476915973915960794135661897722344673148355783307303y^8 \}. \]

There is an element of degree 3 in \((J_6, I_C)\)

\[ g = 683086 - 414993x - 636078x^2 + 356233x^3 - 259643y + 67767xy. \]

Finally we obtain

\[ I_{red} = ((g), I_C) : (J_6) = \{45944281343+377260313207y+408415639297y^2+134215744153y^3, \]

\[ -53515118937 + 13173978910x - 225487128300y - 134215744153y^2 \}. \]

Because the minimal element of \(I_{red}\)

\[ -53515118937 + 13173978910x - 225487128300y - 134215744153y^2 \]

with respect to a degree order is of degree 2 > n – 3 the ideal \(I_{red}\) is reduced. The ideal \(I_{red}\) corresponds to the reduced divisor

\[ ( -0.82409 - 0.62806i, -1.31975 + 0.06425i, 1) \]

\[ (+-0.82409 +0.62806i, -1.31975 -0.06425i, 1) +(-1.18524, -0.40347, 1) -3(1,1,1) \sim D. \]

9 Curves with simple double points

The above construction applies analogously to curves with simple singularities. Here we consider the case of \(n\)-curves \(C\) with \(d\) finite simple double points \(D_1, \ldots, D_d \neq P_0\) with \(F_x = F_y = 0, F_{xx}F_{yy} - F_{xy}^2 \neq 0\). Furthermore we suppose \([0, 1, 0] \notin C\). To every point \(D_i\) correspond two points \(D_i^+, D_i^-\) on the Riemann surface of \(C\) of genus \(g = \frac{(n-1)(n-2)}{2} - d\).

Let

\[ \Delta := D_1^+ + D_1^- + \cdots + D_d^+ + D_d^- \]

be the double point divisor of \(C\). Now consider the divisor

\[ D = D^+ - D^- = P_1 + \cdots + P_s - Q_1 - \cdots - Q_s \]

of degree zero. We consider an \(m\)-curve with a polynomial \(G(x, y, z) \neq 0 \mod F\) with

\[ s + g + d \leq b_m := \begin{cases} \frac{m(m+3)}{2} & \text{for } m < n \\ \frac{mn - (n-1)(n-2)}{2} & \text{for } m \geq n \end{cases} \]

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through the $s$ points of $D^+, \mathcal{D}_1, \cdots, \mathcal{D}_d$ and $(b_m - s - d)p_0$. We have
\[
mn - s - 2d - (b_m - s - d) = mn - b_m - d = \frac{(n-1)(n-2)}{2} - d = g
\]
remaining intersections $R_1, \cdots, R_g$, i.e.
\[
D^+ + \Delta + (b_m - s - d)p_0 + R_1 + \cdots + R_g \sim mD_\infty.
\]
Then we consider an $m$-curve $G'(x, y, z) \neq 0 \mod F$ through the $s$ points of $D^-$ and through $R_1, \cdots, R_g, \mathcal{D}_1, \cdots, \mathcal{D}_d, (b_m - s - d - g)p_0$. We require a maximal additional contact $\alpha$ at $p_0$. Let $S_1, \cdots, S_{g-\alpha}$ be the remaining intersections not equal to $p_0$. We have
\[
D^- + \Delta + R_1 + \cdots + R_g + (b_m - s - d - g + \alpha)p_0 + S_1 + \cdots + S_{g-\alpha} \sim mD_\infty.
\]
It follows
\[
D^+ - D^- \sim \mathcal{D} := S_1 + \cdots + S_{g-\alpha} - (g - \alpha)p_0.
\]
Analogously to Proposition 4 one shows that $\mathcal{D}$ is the reduced divisor of $D$.

For an algebraic description of the reduction let $\mathcal{I}_\Delta$ be the ideal of all adjoint curves through the $\mathcal{D}_1, \cdots, \mathcal{D}_d$. For a finite simple double point $\mathcal{D}_i$ we have two taylor series expansions
\[
y - y_i = a_1^\pm (x - x_i) + a_2^\pm (x - x_i)^2 + \cdots
\]
for the two branches of $\mathcal{D}_i^\pm$ (or $x - x_i = b_1^\pm (y - y_i) + b_2^\pm (y - y_i)^2 + \cdots$, if $b_1^\pm = 0$).
Then
\[
\mathcal{I}_{mD_i^\pm} := H_z(((x - x_i, y - y_i)^m), y - y_i - a_1^\pm (x - x_i) - \cdots - a_{m-1}^\pm (x - x_i)^{m-1})
\]
are the ideals of polynomials with an $m$-fold common point with the branch of $\mathcal{D}_i^\pm$. Now let
\[
D = m_1P_1 + \cdots + m_pP_p + \sum_i (m_i^+D_i^+ + m_i^-D_i^-)
\]
be a divisor with different ordinary points $P_1, \cdots, P_p$. We consider the ideal of curves with intersection divisor $\geq \Delta + D$
\[
\mathcal{I}_{\Delta+D}^h := (\mathcal{I}_{m_1P_1}^h \cap \cdots \cap \mathcal{I}_{m_pP_p}^h) \cap \bigcap_i (\mathcal{I}_{m_i^+D_i^+}^h \cap \mathcal{I}_{m_i^-D_i^-}^h).
\]
Now we define an ideal product $\odot_\Delta$ for ideals $\mathcal{I}_1 = \mathcal{I}_{\Delta+D_1}^h, \mathcal{I}_2 = \mathcal{I}_{\Delta+D_2}^h$ which corresponds to the addition of the divisors $\mathcal{D}_1$ and $\mathcal{D}_2$.
\[
\mathcal{I}_1 \odot_\Delta \mathcal{I}_2 := H_z((A_x(\mathcal{I}_1) \cdot A_x(\mathcal{I}_2), I_C : I_\Delta) \bigcap H_x((A_x(\mathcal{I}_1) \cdot A_x(\mathcal{I}_2), I_C : A_x(I_\Delta))
\]
where $I_\Delta^C := A_x(F)$. $\mathcal{I}_1 \odot_\Delta \mathcal{I}_2$ contains all curves with intersection divisor $\geq \Delta + D_1 + D_2$.

A generalized ideal quotient is given by
\[
(G) \odot_\Delta \mathcal{I} := H_z((A_x((G)) \cdot I_\Delta, I_C : A_x(I)) \bigcap H_x((A_x((G)) \cdot A_x(I_\Delta), I_C : A_x(I))
\]

**Proposition 8** Let $D, D'$ be effective divisors and let $G \in \mathcal{I}_{\Delta+D}$. Then we have $(G) \geq \Delta + D$ and
\[
\mathcal{I}_{\Delta+D+D'}^h = \mathcal{I}_{\Delta+D}^h \odot_\Delta \mathcal{I}_{\Delta+D'}^h \quad \text{and} \quad \mathcal{I}_{(G)-D}^h = \mathcal{I}_{(G)}^h \odot_\Delta \mathcal{I}_{\Delta+D}^h.$
Proof. The proof follows from the fact, that the left and the right side are equal to the homogeneous ideal of all curves with intersection divisor $\Delta + D_{1} + D_{2}$ and $\geq (G) - D_{1}$, respectively.

Now let $D = D^+ - D^-$ with $D^+, D^- \geq 0$, $\deg D^+ = \deg D^- = s$ be a divisor of degree zero. Furthermore let $I^+ := I^h_{\Delta + D^+}$, $I^- := I^h_{\Delta + D^-}$, $I_r := I^h_{P_0}$. We choose $m$ such that $b_m \geq s + d + g$. Then we choose an arbitrary element $G$ in $I^+ \cap I_{b_m-s-d}(P_0)$ of degree $m$ and we form $J = (G) \cap_{\Delta} (I^+ \cap I_{b_m-s-d}) = I^h_{\Delta + r_1 + \ldots + r_2}$. We determine the number $r \geq b_m - s - d - g$ such that $(J \cap_{\Delta} I^-) \cap I_r$ contains exactly one Groebner basis element $G'$ of degree $m$ with respect to a degree order. We form $I_{red} = (G') \cap_{\Delta} ((J \cap_{\Delta} I^-) \cap I_r)$. We obtain $I_{red} = I^h_{\Delta + P_1 + \ldots + P_t}$ with $t = g - r + (b_m - s - d - g)$ where $P_1 + \ldots + P_t - tP_0$ is the reduced divisor for $D$.

10 An Example

We consider the hyperelliptic 4-curve $C$ with $x^4 - y^4 = 30xyz^2$ with $g = 2$. $C$ has one simple double point $D^+_0 = (0, 0, 1)$ with the Taylor series expansions

$$y = \frac{x^3}{30} - \frac{x^{11}}{24300000} + \cdots, \quad x = \frac{-y^3}{30} + \frac{y^{11}}{24300000} - \cdots.$$  

We choose $P_0 := (1, 1, 0)$ and we consider the points $P_1 = (4, 2, 1)$, $P_2 = (1, -1, 0)$. We apply the reduction to ideals $I^+, I^-$, which correspond to the divisor $2D^+_0 - P_1 - P_2$.

We have

$I^+ = I^h_{\Delta + 2D^+_0} = (x^3, y)$,

and

$I^- = I^h_{\Delta + P_1 + P_2} = (xz - 2yz, -xy - y^2 + 6yz, -x^2 + y^2 + 6yz, y^2z - 2yz^2)$.

We form

$I^h_{\Delta + 2D^+_0 + 2P_0} = I^+ \cap (x - y, z^2) = (-xy + y^2, z^2, x^3 - x^2y, x^3z^2)$.

We choose $G = -xy + y^2$. We obtain the quotient

$J = (G) \cap_{\Delta} I^h_{\Delta + 2D^+_0 + 2P_0} = (x^2, xy, y^2) = I^h_{\Delta + R_1 + R_2}$.

We remark that $R_1 + R_2 = D^+_0 + D^-_0$. Now we form

$I^h_{\Delta + R_1 + R_2 + P_1 + P_2} = I^- \cap J = (-x^2 + xy + 2y^2, x^2z - 2xyz, x^3 + x^2y - 6x^2z, x^3z - 4x^2z^2)$.

We choose $G' = x^2 - xy - 2y^2$. Then we obtain the quotient

$(G') \cap_{\Delta} I^h_{\Delta + R_1 + R_2 + P_1 + P_2} = (xz - 2yz, -xy - y^2 - 6yz, x^2 - y^2 + 6yz, y^2z + 2yz^2) = I^h_{\Delta + S_1 + S_2} = I_{red}$.

We obtained $I_{red}$ from $I^+, I^-$ by rational operations. We remark that $S_1 + S_2 = (-4, -2, 1) + P_2$. 




11 Hyperelliptic curves

In this section we present an algorithm for hyperelliptic curves $C$ of genus $g$ in the standard form

$$y^2 = a(x - x_1)(x - x_2) \cdots (x - x_{2g+1}) =: h(x)$$

with different $x_i$, $a \neq 0$. The projectivisation has a single singular (nonsimple) infinite point $P_\infty = (0, 1, 0)$. We choose $P_0 := P_\infty$. Hyperelliptic curves have an involution $x \rightarrow -y$. Let $\mathcal{I}$ and $\overline{\mathcal{D}}$ be the image of the ideal $\mathcal{I}$ and the divisor $\mathcal{D}$ with respect to this involution. Then we have $(x_1, y_1) + \cdots + (x_n, y_n) = (x_1, -y_1) + \cdots + (x_n, -y_n)$.

Because $C$ has only one infinite point it is sufficient to consider affine ideals. However, the selection of the interpolating curves requires a modification.

Let $D = D^+ - D^- \sim D^+ + \overline{D^-} - 2\deg(D^-)P_0$ be a divisor of finite points. We have $J := I_{D^+ + \overline{D^-}} = I_{D^+ + \overline{D^-}}$. We replace the degree order by the weighted degreelexicographic order $\deg_{2g+1,2}$ with $\deg_{2g+1,2}(x^ay^b) := (2g + 1)a + 2b$ and $y > x$. Now we determine the minimal element $f = p(x) + q(x)y$ of $J$ with respect to this order. Then $C$ and the curve $f(x, y) = 0$ have $\deg_{2g+1,2}(f)$ finite intersections with $p^2(x) - h(x)q^2(x) = 0$ and $y = p(x)/q(x)$. Let $(x_i, y_i)$, $i = 1, \ldots, t := \deg_{2g+1,2}(f) - \deg(D^+) - \deg(D^-)$ be the remaining finite intersections. It follows

$$(f) = D^+ + \overline{D^-} + (x_1, y_1) + \cdots + (x_t, y_t) - \deg_{2g+1,2}(f)P_0,$$

i.e.

$$D - (\deg(D^+) - \deg(D^-))P_0 \sim (x_1, -y_1) + \cdots + (x_t, -y_t) - tP_0 =: D_1 - tP_0.$$

**Lemma 9** $D_1 - tP_0$ is the reduced divisor for $D - \deg(D)P_0$.

**Proof:** The proof follows from the fact that $t := \deg_{2g+1,2}(f) - \deg(D^+) - \deg(D^-)$ is minimal if $\deg_{2g+1,2}(f)$ is minimal. $\blacksquare$

The divisor $D_1$ corresponds to the ideal

$$I_{\text{red}} = I_{D_1} = (f, I_C): (I_{D^+} + I_{\overline{D^-}}).$$

**Remark:** Contrarily to Cantor’s algorithm we can describe our algorithm by this single formula. Our algorithm uses a reduction function of the general form $f = p(x) + yq(x)$. In contrast Cantor’s algorithm uses reduction functions of the special form $y - p(x)$ several times (cf. the example below).

12 Picard curves

In this section we present an algorithm for Picard curves $C$

$$y^3 = a(x - x_1)(x - x_2)(x - x_3)(x - x_4) =: h(x)$$

with four different $x_i$, $a \neq 0$. The projectivisation has a single 4-fold infinite point $P_\infty = (0, 1, 0)$. We choose $P_0 := P_\infty$. $C$ has only one infinite point and it is sufficient to consider affine ideals. Similarly to the case of hyperelliptic curves we replace the degree
order by the weighted degree-lexicographic order with \( deg_{4,3} := (x^a y^b) := 4a + 3b \) and \( y > x \).

Let \( D = D^+ - D^- \) be a divisor of finite points. We determine the minimal element \( f = p(x) + q(x)y + r(x)y^2 \) of \( I_{D^+} \) with respect to the above order. We have

\[
(f) = D^+ + R_1 + \cdots + R_t - \deg_{4,3}(f)P_0
\]

with \( t := \deg_{4,3}(f) - \deg(D^+) \) remaining finite points \( D' = R_1 + \cdots + R_t \) whose \( x \)-coordinates are zeros of the polynomial

\[
\begin{vmatrix}
  p & q & r \\
  hr & p & q \\
  hq & hr & p
\end{vmatrix} = p^3 + q^3h + r^3h^2 - 3pqrh
\]

of degree \( t \). We have

\[
I_{D'} = (f, I_C) : I_{D^+}.
\]

We determine the minimal element \( g \) of \( I_{D'^-} = I_{D^+}I_D^- \) with respect to the above order. We have

\[
(g) = D' + D^- + S_1 + \cdots + S_q - \deg_{4,3}(g)P_0
\]

with \( q := \deg_{4,3}(g) - \deg(D') - \deg(D^-) \) remaining finite points \( D'' = S_1 + \cdots + S_q \). It follows

\[
D'' - qP_0 \sim -D' - D^- + (\deg_{4,3}(f) - q)P_0 \sim D - (\deg(D^+) - \deg(D^-))P_0.
\]

**Lemma 10** \( D'' - qP_0 \) is the reduced divisor for \( D - \deg(D)P_0 \).

**Proof:** The proof follows from the fact that \( q \) is minimal if \( \deg_{4,3}(g) \) is minimal. \( \bullet \)

We have

\[
I_{\text{red}} = I_{D''} = (g, I_C) : I_{D^+D^-}.
\]

An algorithm for the Jacobian group of this curve is discussed in [17, 18]. Contrarily to our algorithm these algorithms use the concrete structure of the curve and require the distinction of many different cases. Our algorithm for Picard curves has a straightforward generalization to superelliptic curves

\[
y^m = a(x - x_1) \cdots (x - x_n)
\]

with different \( x_i, a \neq 0 \) and \( \text{lcm}(m, n) = 1 \).

13 The case of characteristic \( p \)

In this last section we make some remarks about the case of characteristic \( p \). Using the theory of [15] one can show that our algorithm has an analogue if we replace \( \mathbb{C} \) by a field \( k \) of characteristic \( p \). \( k \) has the algebraic closure \( \overline{k} \). The divisor group \( Div(C) \) is the free Abelian group consisting of formal finite sums \( \sum_{P \in C(\overline{k})} m_P P \) with \( m_P \in \mathbb{Z} \).

A divisor is defined over \( k \) if it is fixed by the natural Galois action of \( \text{Gal}(\overline{k}; k) \). The divisors defined over \( k \) form the subgroup \( Div_k(C) \). Analogously one defines the group \( Div_0^k(C) \). Principal divisors are defined as zeros and poles of rational functions \( f(x, y, z) \) where \( G, H \) \((H \notin (F(x, y, z)) \) are homogeneous polynomials of equal degree with
coefficients in \( k \). They form a subgroup \( \text{Div}_k^0(C) \subset \text{Div}_k^0(C) \). We define \( \text{Jac}_k(C) := \text{Div}_k^0(C)/\text{Div}_k^P(C) \). Furthermore we can define the analog notion of an reduced divisor.

Let \( D \) be an element of \( \text{Div}_k(C) \) without infinite points. Then the polynomials \( p(x, y) \in k[x, y] \) with \( (p) \supseteq D \) form an ideal \( I_D \subset k[x, y] \). One can show that there is a one-to-one correspondence between ideals \( I \) of \( k[x, y] \) with \( I \supset I_C \) and divisors \( D \) of \( \text{Div}_k(C) \) of finite points.

Now let \( C \) be a hyperelliptic curve or a Picard curve. The affine part of the above hyperelliptic curves is smooth for \( p \neq 2 \) and the above Picard curves are smooth for \( p \neq 3 \). This case is interesting in view of applications in cryptography. Let \( I^+, I^- \) be two ideals of \( k[x, y] \) with divisors \( D^+, D^- \). We apply the corresponding algorithm of the two previous sections to \( I^+, I^- \). Because all operations are rational we obtain an ideal \( I_{\text{red}} \) of \( k[x, y] \). We have an analogue of Lemma 9.10. Therefore \( I_{\text{red}} \) corresponds to a reduced divisor.

**Example:** Let \( C \) be the hyperelliptic curve \( y^2 = (x - 3)(x - 2)(x - 1)x(x + 1)(x + 2)(x + 3) =: h(x) \) for \( k = F_{17} \) and let \( D_1 = (4, 5) + (5, 8) + (6, 4), D_2 = (7, 5) + (10, 3) + (11, 1) \). We have \( I_{D_1} = (x^3 + 2x^2 + 6x + 16, 5x^2 + 9x + 8 - y) =: (a_1, b_1 - y) \) and \( I_{D_2} = (x^3 + 6x^2 + 2x + 12, 11x^2 + 5x + 9 - y) =: (a_2, b_2 - y) \).

1. **Cantor’s algorithm:** We have \( s_1a_1 + s_2a_2 = 1 \) with certain \( s_1, s_2 \). We obtain the composed ideal \( I_{D_1}I_{D_2} \) with the basis \( (a, b - y) := (a_1a_2, (s_1a_1b_2 + s_2a_2b_1 - y)\text{mod}a_1a_2) = (x^6 + 8x^5 + 3x^4 + 13x^3 + 2x^2 + 5, x^5 + 7x^4 + 2x^3 + 6x^2 + 5x + 5 - y) \). The reduction process gives the ideals of equivalent divisors \( (a', b' - y) := (\frac{b^2 - h}{a^2}, (b - y)\text{mod}a') = (x^4 + 6x^3 + 2x^2 + 5x + 5, 6x^3 + x^2 + 5x - y) \) and \( (a'', b'' - y) := (\frac{b^2 - h}{a^2}, (b' - y)\text{mod}a'') = (x^3 + 9x^2 + 2x^2 + 13 - y) \).

The last ideal corresponds to the reduced divisor.

2. **Our algorithm:** We obtain for \( I_{D_1}I_{D_2} \) the Groebner basis \( (x^6 + 8x^5 + 3x^4 + 13x^3 + 2x^2 + 5, 11x^4 + 9x^3 + 2x^2 + 9 + y(x + 1)) =: (\cdot, f) \) with respect to the weighted lexicographic order and \( (f, y^2 - h(x)) : (I_{D_1}I_{D_2}) = (x^3 + 9x^2 + 2x^2 + 13 - y) \).

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