Some effects of the noise intensity upon non-linear stochastic heat equations on $[0, 1]$

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Abstract

Various effects of the noise intensity upon the solution $u(t, x)$ of the stochastic heat equation with Dirichlet boundary conditions on $[0, 1]$ are investigated. We show that for small noise intensity, the $p$-th moment of $\sup_{x \in [0, 1]} |u(t, x)|$ is exponentially stable, however, for large one, it grows at least exponentially. We also prove that the noise excitation of the $p$-th energy of $u(t, x)$ is 4, as the noise intensity goes to infinity. We formulate a common method to investigate the lower bounds of the above two different behaviors for large noise intensity, which are hard parts in [8], [10] and [16].

Keywords: stochastic heat equation, Dirichlet boundary condition, space-time white noise, excitation index, exponential stability, growth rate

2010 Mathematics Subject Classification: Primary 60H15, 60H25; Secondary 35R60, 60K37.

1 Introduction and main results

We are interested in various behaviors of the following stochastic heat equation relative to $\lambda$:

$$
\begin{aligned}
\partial_t u(t, x) &= \frac{1}{2} \Delta u(t, x) + \lambda \sigma(u(t, x)) \dot{w}(t, x), \quad t > 0, \quad x \in (0, 1), \\
u(0, x) &= u_0(x), \quad x \in (0, 1),
\end{aligned}
$$

(1.1)

where $\lambda > 0$ is a positive number, $\sigma$ is a non-random measurable function defined on $\mathbb{R}$ and $\dot{w}(t, x)$ is a Gaussian space-time noise on $[0, \infty) \times [0, 1]$. Such equation is closely connected to the parabolic Anderson model (as $\sigma(u) = u$, see [3]), the stochastic Burger’s equation[11] [13] and the Kardar-Parisi-Zhang (KPZ) equation [11] [14]. Hence some crucial properties, such as the weak intermittency of the solution, are actively studied, see [4], [9], [17] and references therein.

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In this paper, we are mainly interested in \( (1.1) \) with homogeneous Dirichlet boundary condition, i.e., \( u(t, 0) = u(t, 1) = 0 \). Some of our results will also hold for \( (1.1) \) with homogeneous Neumann boundary condition \( \partial_t u(t, 0) = \partial_t u(t, 1) = 0 \) and we will state them in form of remarks.

According to [10] and [16], the parameter \( \lambda > 0 \) in \( (1.1) \) will be called the level of noise or noise intensity, which is regarded as the inverse temperature. The solution \( u(t, x) \) can be though of the partition function of a continuous space-time random polymer, see [2] for more explanations.

In this paper, two kinds of the behaviors of the solution relative to noise intensity \( \lambda \) will be studied. To explain our aims and motivations in detail, let us first introduce some notation and the definition of the solution \( (1.1) \). Let

\[
\{F_t\}_{t \geq 0} \quad \text{be adapted and continuous random field}
\]

\[
\text{filtration generated by the } \{w(t, x); t \geq 0, x \in [0, 1]\}, \text{ see } [22]. \text{ In this paper, we will always assume the following assumption A is satisfied:}
\]

\begin{enumerate}
\item [(A.1)] The initial value \( u_0 \) is non-random and continuous on \([0, 1]\). Furthermore, we assume that the Lebesgue measure of the set \( \text{supp}(u_0) \cap [\gamma, 1 - \gamma] \) is strictly positive, where \( \text{supp}(u_0) \) denotes the support of \( u_0 \) and \( \gamma \in (0, 1/4) \) is fixed hereafter.
\item [(A.2)] \( \sigma(0) = 0 \) and \( \sigma \) is Lipschitz continuous, that is, there exists \( K_\sigma > 0 \) such that for all \( u, v \in \mathbb{R} \),
\[
|\sigma(u) - \sigma(v)| \leq K_\sigma |u - v|.
\]
\end{enumerate}

Let us recall the definition of the solution to \( (1.1) \). Based on the definition introduced in [22], a random field \( \{u_\lambda(t, x); t \geq 0, x \in [0, 1]\} \) is said to be a mild solution of \( (1.1) \) with the homogeneous Dirichlet boundary condition if it is \( F_t \)-adapted and continuous in \((t, x)\), and further it satisfies the following integral equation with probability one

\[
u(t, x) = \int_0^1 g_D(t, x, y)u_0(y)dy + \int_0^t \int_0^1 g_D(t - s, x, y)\lambda \sigma(u(s, y))w(dsdy)
\]

\[
:= D_1(t, x) + D_{2, \lambda}(t, x),
\]

where \( g_D(t, x, y) \) denotes the fundamental solution (or heat kernel) of the linear part of the stochastic heat equation \( (1.1) \) with Dirichlet boundary condition \( u(t, 0) = u(t, 1) = 0 \). Similarly, an \( F_t \)-adapted and continuous random field \( \{u_\lambda(t, x); t \geq 0, x \in [0, 1]\} \) is said to be a mild solution of \( (1.1) \) with homogeneous Neumann boundary condition if \( (1.2) \) is satisfied almost surely replaced \( g_D(t, x, y) \) by the Neumann kernel \( g_N(t, x, y) \). For the introduction to stochastic partial differential equations, we also refer the reader to [5] for more information.

Since our topics are closely depending on the noise intensity \( \lambda \), we will denote by \( u_\lambda(t, x) \) the solution of \( (1.1) \) with homogeneous Dirichlet boundary condition. Let \( p \geq 2 \) in this paper and then any real valued measurable function \( u \) defined on \([0, 1]\), let \( \|u\|_{L^p} \) denote its \( L^p \)-norm on \([0, 1]\). Recalling that for \( p = \infty \), \( \|u\|_{L^\infty} = \text{ess sup}_{x \in [0, 1]} |u(x)| \).

One of our main aims is to study the exponential stability of the solution for fixed \( \lambda \), which is widely studied, because of its importance in applications. One of the important and hard problem for stability is to calculate the Lyapunov exponents. For stochastic parabolic partial differential equations driven by a finite dimensional
Gaussian noise, we refer the reader to [18] and [23]. However, it seems very hard for (1.1), see [15]. Recently, in [10], the authors proved that if the level of the noise is small, then the $p$-th absolute moment of $u_\lambda(t, x)$ is exponentially stable, however, for large enough $\lambda$, the $p$-th absolute moment of $u_\lambda(t, x)$ becomes unstable and grows at least exponentially. We generalized the main results in [10] in two ways. One is to show the exponential stability of the $p$-th moment of $\|u_\lambda(t)\|_{L^\infty}$, instead of $|u_\lambda(t, x)|$, and the other is that an innovative method to show the lower bound of the growth rate of the solution for large, but fixed $\lambda$, see Theorem 1.3 below.

Our method can also be applied to study the excitability of the noise as $\lambda \to \infty$ for each $t > 0$, which is our second main topic. The non-linear noise excitability of stochastic heat equations is initially introduced in [16] and restudied in [8], see also [17] for other research. To study this kind of problem, in [16] the authors implemented a projection method to prove that as $\lambda \to \infty$, the $L^2$-energy of the solution grows at least as $\exp(\kappa_1 \lambda^2)$, and at most as $\exp(\kappa_2 \lambda^4)$. But the authors predicted that the lower bound may be $\exp(\kappa_1 \lambda^4)$, instead of $\exp(\kappa_1 \lambda^2)$, that is, the noise excitation may be equal to 4, same as that for the large number of intermittent complex systems [17]. To fill this gap, a renewal approach is introduced in [8], which is essentially depends on the short time estimate of the heat kernel. They first proved that the noise excitation is 4 for small time, and then extended it to each fixed $t > 0$. However, their method can not be applied to study the larger time behavior as $t \to \infty$ for fixed $\lambda$, which is the first goal of this paper. Our main motivation is of this observation and a method is introduced to study both kinds of behaviors above. It seems that our technique can be applied to (fractional) stochastic heat equations on general bounded domain of $\mathbb{R}^d$ driven by white or colored noises, which is currently being considered.

From now, we will state our main results. The first is going to show that the $p$-th moment of $\|u_\lambda(t)\|_{L^\infty}$ grows exponentially fast at $t \to \infty$.

For each $\beta \in \mathbb{R}$ and $p \geq 2$, let us denote by $B_{p, \beta}$ the class of all the $\mathcal{F}_t$-adapted and continuous random field $\{u(t, x), t \geq 0, x \in [0, 1]\}$ satisfying

$$\sup_{t \geq 0} \mathbb{E}[e^{\beta t} \|u_\lambda(t)\|^p_{L^\infty}] < \infty.$$  

For each $u \in B_{p, \beta}$, we set

$$\|u_\lambda\|_{p, \beta} = \left(\sup_{t \geq 0} \mathbb{E}[e^{\beta t} \|u_\lambda(t)\|^p_{L^\infty}]\right)^{\frac{1}{p}}. \quad (1.3)$$

Then it is easy to know that $(B_{p, \beta}, \|u_\lambda\|_{p, \beta})$ is a Banach space.

Let us now formulate the first main result of this paper, which is about the existence and uniqueness of the solution in the Banach space $B_{p, \beta}, \beta < 0$ and then gives a upper bound of the growth rate for any $\lambda > 0$.

**Theorem 1.1.** Let $p > 2$. Then there exists $\beta_0 < 0$ such that for any $\beta < \beta_0$, the equation (1.1) has a unique mild solution $u_\lambda(t, \cdot) \in B_{p, \beta}$. 

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In particular, the growth of the solution in time $t$ is at most in an exponential rate in the $p$-moment sense. Precisely speaking, for any $\beta < \beta_0$

$$\limsup_{t \to \infty} \frac{1}{t} \log E[\|u_\lambda(t)\|_{L^\infty}^p] \leq -\beta.$$

**Remark 1.1.**

(i) The condition $\sigma(0) = 0$ is not required and our result holds for any Lipschitz continuous function $\sigma$.

(ii) From the proof of this theorem, it is known that the same result holds for (1.1) with homogeneous Neumann boundary condition.

(iii) In stead of the $L^\infty$-norm of $u_\lambda(t, x)$, the similar behavior of the $p$-th absolute moment of $u_\lambda(t, x)$ is initially investigated in [9] for stochastic heat equations on $\mathbb{R}$, and then is restudied in [10] on a bounded domain.

In the next theorem, we are going to show that if $\lambda$ is small enough, then the $p$-th moment of $\|u_\lambda(t)\|_{L^\infty}$ is exponentially stable.

**Theorem 1.2.** There exists $\lambda_L$ such that for $\lambda \in (0, \lambda_L)$,

$$-\infty < \limsup_{t \to \infty} \frac{1}{t} \log E[|u_\lambda(t, x)|^p] \leq \limsup_{t \to \infty} \frac{1}{t} \log E[\|u_\lambda(t)\|_{L^\infty}^p] < 0. \quad (1.4)$$

**Remark 1.2.**

(i) In fact, we can prove that there exists a $\beta \in (0, (2 - \alpha)\pi^2)$ for some $\alpha \in (2/p, 1)$ such that for all $\lambda \in (0, \lambda_L)$ and all $t \geq 0$, $E[\|u(t)^p\|_{L^\infty}] \leq e^{-\beta t}$, see the proof of this theorem in Section 2.

(ii) It is easy to know that for $\lambda < \lambda_L$, the solution $u_\lambda$ is not weakly intermittent. According to [9], we recall that the solution $u_\lambda$ is of weak intermittence if for any $p \geq 2$,

$$\limsup_{t \to \infty} \frac{1}{t} \log E[|u_\lambda(t, x)|^p] \in (0, \infty).$$

(iii) The lower bound can be easily proved by Jensen’s inequality and Theorem 1.1 [10]. As a by-product of the proof of Theorem 1.3 below, we have another proof of it, see Section 3.

(iv) We point out that in this paper we will not take $\lambda \to 0$, which is the problem of the large deviations principle for small noises(Freidlin-Wentzell large deviation principle), see [26] and references therein.

It is now natural to ask what will happen for the solution $u_\lambda(t, x)$ with large noise intensity $\lambda$ as $t \to \infty$. It is recently studied in [10]. On the other hand, as we mentioned in the above, we are also interested in the excitation of non-linear noise, see [16], [17] and [8]. The proof for the lower bound is hard, as we stated above and different methods are introduced respectively in [8] and [16]. It seems that there is no relation between the method for the lower bound in [10] for growth rate as $t \to \infty$ and that in [8] for the noise excitability. However, we believe that the large time behavior for large, but fixed $\lambda$ and the excitation of non-linear noise are essentially same, and thus there must be a common approach to study both phenomena. As expected, we can find such common approach, see Theorem 1.3 below, which is our
main contribution. Our approach essentially depends on the lower bound of the global estimate for $g_D(t, x, y)$, see Lemma 3.1.

Before we state our key theorem, let us further impose the next assumption on the coefficient $\sigma$:

(A3) For any $u \in \mathbb{R}$,  
$$K_L|u| \leq \sigma|u|,$$

where $K_L > 0$ is a constant. It is clear that $K_U \geq K_L$ is required.

Then we have the next theorem, which plays a key role in this paper.

**Theorem 1.3.** If further (A3) is fulfilled, then there exist two constants $\kappa_1 > 0$ and $\kappa_2 > 0$ such that for all $t > 0$

$$\inf_{x \in [\gamma, 1-\gamma]} \mathbb{E}[|u_\lambda(t, x)|^2] \geq \kappa_1 \exp(-2\pi^2 t + \kappa_2 \lambda^4 K_L^4 t). \quad (1.5)$$

As the first application of Theorem 1.3, we will research on the lower bound of the growth rate of the $p$-th absolute moment of $u_\lambda(t, x)$ with a large noise intensity $\lambda$ as $t \to \infty$.

**Theorem 1.4.** Under the assumptions in Theorem 1.3, there exists $\lambda_U > \lambda_L$ such that for all $\lambda \in (\lambda_U, \infty)$ and $x \in [\gamma, 1-\gamma]$

$$0 < \lim \inf_{t \to \infty} \frac{1}{t} \log \mathbb{E}[|u_\lambda(t, x)|^p] \leq \lim \sup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[\|u_\lambda(t)\|_{L^p}] < \infty. \quad (1.6)$$

**Remark 1.3.** The phenomena in Theorem 1.2 and Theorem 1.4 are peculiar to (1.1) with Dirichlet boundary condition, which are not satisfied for Neumann boundary condition, see [10]. These display some kind of competition between the noise and dissipativity of the Dirichlet Laplacian.

For $p \geq 2$, let us introduce the $p$-th energy $\mathcal{E}_p(t, \lambda)$ relative to the solution $u_\lambda(t, x)$ of (1.1) at time $t > 0$ as below:

$$\mathcal{E}_p(t, \lambda) = (\mathbb{E}[\|u_\lambda(t)\|_{L^p}^p])^{\frac{1}{p}}.$$

We remark that when $p = 2$, $\mathcal{E}_p(t, \lambda)$ is called $L^2$-energy in [8, 10, 16], and [17] and we generalize it to the definition of $p$-th energy.

**Corollary 1.5.** Suppose the assumptions in Theorem 1.4 are fulfilled. Let $\lambda_L$ and $\lambda_U$ be the same constants as that appeared in Theorem 1.2 and Theorem 1.4 respectively. Then for $\lambda < \lambda_L$,

$$-\infty < \lim \sup_{t \to \infty} \frac{1}{t} \log \mathcal{E}_p(t, \lambda) < 0,$$

and for $\lambda > \lambda_U$

$$0 < \lim \inf_{t \to \infty} \frac{1}{t} \log \mathcal{E}_p(t, \lambda) < \infty. \quad (1.7)$$
Let us now turn to study the non-linear noise excitability of the stochastic heat equation (1.1) by letting the noise intensity $\lambda$ go to infinity for each $t > 0$ as another application of Theorem 1.3.

**Theorem 1.6.** Under the assumptions in Theorem 1.3, for all $t > 0$, there is constant $c_p > 0$ such that

\[
\begin{aligned}
c_pK^d_L t &\leq \liminf_{\lambda \to \infty} \lambda^{-4}K_L \log \left( \inf_{x \in [\gamma, 1-\gamma]} \mathbb{E}[|u_\lambda(t, x)|^p] \right) \\
&\leq \limsup_{\lambda \to \infty} \lambda^{-4} \log \left( \sup_{x \in [0,1]} \mathbb{E}[|u_\lambda(t, x)|^p] \right) \\
&\leq c_p^{-1}K^d_U t
\end{aligned}
\]

Then the following theorem exhibits the quantitative behavior of the noise excitability for $p$-th energy.

**Corollary 1.7.** Under the assumptions in Theorem 1.6, for all $t > 0$, there is constant $c_p$ such that

\[
\begin{aligned}
c_pK^d_L t &\leq \liminf_{\lambda \to \infty} \lambda^{-4}K_L \log \mathcal{E}_p(t, \lambda) \\
&\leq \limsup_{\lambda \to \infty} \lambda^{-4} \log \mathcal{E}_p(t, \lambda) \\
&\leq c_p^{-1}K^d_U t.
\end{aligned}
\]

**Remark 1.4.** (i) The results in Theorem 1.6 and Corollary 1.7 still hold for (1.1) with Neumann boundary condition, see [8] and [16].

(ii) Analogously to [17], let us introduce the noise excitation index of the solution $u_\lambda(t, x)$ relative to $p$-th energy $\mathcal{E}_p(t, \lambda)$. If for each $t > 0$,

\[
\lim_{\lambda \to \infty} \frac{\log \log \mathcal{E}_p(t, \lambda)}{\log \lambda}
\]

exists, then its limit denoted by $e_p(t)$ is called the noise excitation index of $p$-th energy. If furthermore, $e_p(t)$ does not depend on $t$, then the common value denoted by $e_p$ is called the index of nonlinear noise excitation of the $p$-th energy $\mathcal{E}_p(t, \lambda)$. It is clear from the above theorem that $e_p = 4$ which is independent of $p, p \in [2, \infty)$. The definition of noise excitation index is initially introduced by D. Khoshnevisan, K. Kim in [17] and we refer the reader to this paper for its significance.

The paper is organized as follows: In Section 2, we give proofs of Theorem 1.1 and Theorem 1.2 based on some lemmas. In Section 3, we first state a lower bound of the global time estimate for the heat kernel and then prove our important result, Theorem 1.3. Proofs of Theorem 1.4 and Theorem 1.6 and their corollaries are formulated in Section 4 and Section 5 respectively. In the end, for the reader’s convenience, we write down a version of Garsia-Rodemich-Rumsey theorem in Appendix, which is cited in Section 2.
2 Proof of Theorem 1.1 and Theorem 1.2

Since we consider the mild solution \( u_\lambda(t, x) \) of (1.1) with homogeneous Dirichlet boundary condition, most of our calculations depend on various estimates of \( g_D(t, x, y) \). We will recall necessary properties of \( g_D(t, x, y) \) when they are required.

Firstly, by the spectral theory, it is well-known that

\[
g_D(t, x, y) = 2 \sum_{n=1}^{\infty} e^{-(n\pi)^2 t} \sin(n\pi x) \sin(n\pi y), \quad x, y \in [0, 1].
\]

(2.1)

It is also easy to know that \( t > 0 \) and \( x, y \in [0, 1] \),

\[
0 \leq g_D(t, x, y) \leq g(t, x, y)
\]

where \( g(t, x, y) \) denotes the transition probability density of some one-dimensional standard Brownian motion.

Different from the study of the long time behaviour of \( E[|u_\lambda(t, x)|^p] \) in [10], to study that of \( E[\|u(t)\|_{L^\infty}] \), the estimate of the derivative of \( g_D(t, x, y) \) is vital, that is

\[
|\partial_x g_D(t, x, y)| \leq K_1 t^{-1} e^{-K_2 (x-y)^2 / t}.
\]

(2.3)

where \( K_1 \) and \( K_2 \) are two generic positive constants.

Before we state the proof of Theorem 1.1 we will give some lemmas. Let us first formulate the famous Kolmogorov’s regularity theorem with its brief proof for the reader’s convenience and our purpose.

Lemma 2.1. (Kolmogorov’s regularity theorem) Let \( \{u(x)\}_{x \in [0, 1]} \) be a real valued stochastic process. If there exist \( p \geq 1 \) and positive constants \( K, \delta \) such that

\[
E[|u(x) - u(y)|^p] \leq K|x - y|^{1+\delta}.
\]

(2.4)

then we have that for each \( \epsilon \in (0, \min\{\delta, 1\}) \), there exists a positive constant \( \kappa \) depending only on \( p, \delta, \epsilon \) such that

\[
|u(x) - u(y)| \leq \kappa B^{1/p} |x - y|^{\frac{\delta - \epsilon}{p}},
\]

(2.5)

where \( B = B(\epsilon, \delta) \) is the positive random variable defined by

\[
B = \int_0^1 \int_0^1 \frac{|u(x) - u(y)|^p}{|x - y|^{2+\delta-\epsilon}} dxdy.
\]

(2.6)

In particular, the stochastic process \( \{u(x)\}_{x \in [0, 1]} \) has a \( \frac{\delta - \epsilon}{p} \)-Hölder continuous modification.

Proof. This lemma is a modification of Corollary 1.2 [22]. Similarly, we state its proof briefly using the celebrated analytic inequality introduced by Garsia, Rodemich and Rumsey, see the original research paper [12], Theorem 1.1 [22] or Theorem 6.1 in Appendix. Let us consider \( \Phi(x) = |x|^p, \quad x \in \mathbb{R} \) and \( \phi(x) = |x|^{\frac{2+\delta-\epsilon}{p}}, \quad x \in \mathbb{R} \).
It is clear that the two functions $\Phi(x)$ and $\phi(x)$ satisfy the conditions in Theorem 6.1. We can now rewrite the random variable $B$ defined by (2.6) as

$$B = \int_0^1 \int_0^1 \Phi\left(\frac{u(x) - u(y)}{\phi(|x-y|)}\right) dx dy.$$

Taking the expectation of $B$, and combining with the condition (2.4) on $u(x)$, we can easily arrive at

$$\mathbb{E}[B] = K \int_0^1 \int_0^1 |x-y|^{-1-\epsilon} dx dy = \frac{2K}{\epsilon(1+\epsilon)} < \infty,$$

(2.7)

where $\epsilon \in (0, \min\{\delta, 1\})$ has been used. Thus, we can apply the Garsia-Rodemich-Rumsey Theorem, see Theorem 6.1. Using Theorem 6.1 and the integration by parts, we obtain that

$$|u(x) - u(y)| \leq 8 \int_0^{|x-y|} B_\lambda^{\frac{1}{p}} u^{-\frac{2}{p}} d\left(u^\frac{2+\lambda - \epsilon}{p}\right)$$

$$= 8B_\lambda^{\frac{1}{p}} |x-y|^{\frac{1}{p}} + \frac{2}{p} \int_0^{|x-y|} u^{-\frac{1}{p}} du$$

$$= 8B_\lambda^{\frac{1}{p}} (1 + (\delta - \epsilon)^{-1}) |x-y|^{\frac{1}{p}}.$$

Taking $\kappa = 8(1 + (\delta - \epsilon)^{-1})$, the estimate of (2.5) is proved. Finally, the existence of $\frac{1}{p}$-Hölder continuous modification for the stochastic process $\{u(x)\}_{x \in [0,1]}$ is obvious from (2.6). Consequently, our proof is completed.

**Remark 2.1.** As we said in the above proof, we mainly imitated the approach for Corollary 1.2[22]. So it may be considered that our proof is insignificant. In fact, the proofs for Theorem 1.1 and Theorem 1.2 essentially depends on the estimates of $K$ and $B$ relative to the solution of $u_\lambda(t,x)$. However, we can not know the concrete form of the right hand side of (2.5), if we do not read the proof of Corollary 1.2[22] carefully. On the other hand, although the similar result to (2.5) in Corollary 1.2[22] is better that (2.5), our result is concise and easy to be applied.

**Lemma 2.2.** Assume $\alpha \in (0,1)$. Then there exists a constant $C > 0$ depending on $\alpha$ such that for any $\beta < 0$

$$\sup_{t \geq 0, x \in [0,1]} \int_0^t e^{\beta s} s^{-\alpha} \int_0^1 \left|g_D(s, x, y)\right|^{2-\alpha} dy ds \leq C |\beta|^\frac{\alpha - 1}{2}.$$

**Proof.** This proof is very easy by noting that the $g_D(t,x,y) \leq g(t,x,y)$. In fact,
by (2.2) and the conditions $\alpha \in (0,1)$ and $\beta < 0$, we have

$$\int_0^t e^{\beta s} s^{-\alpha} \int_0^1 |g_D(s,x,y)|^{2-\alpha} dy ds$$

$$\leq \int_0^t e^{\beta s} s^{-\alpha} \int_0^1 |g(s,x,y)|^{2-\alpha} dy ds$$

$$=(2\pi)^{\frac{2-\alpha}{2}(2-\alpha)^{-\frac{1}{2}}} \int_0^1 |g(s,x,y)|^{2-\alpha} dy ds$$

$$\leq (2\pi)^{\frac{2-\alpha}{2}(2-\alpha)^{-\frac{1}{2}}} |\beta| \int_0^\infty e^{-s^2} ds$$

$$=(2\pi)^{\frac{2-\alpha}{2}} (2-\alpha)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\right) |\beta|^{2-\alpha},$$

where $\Gamma$ denotes the Gamma function. Thus, we can conclude the proof. \hfill $\Box$

For any $u \in B_{p,\beta}$, let us define the mapping $S u(t,x)$ by the stochastic convolution

$$S u(t,x) = \int_0^t \int_0^1 g_D(t-s,x,y) u(s,y) w(ds,dy).$$

In the following two lemmas, we will show that $S$ maps $B_{p,\beta}$ to itself and it is contractive for some $\beta$, respectively.

**Lemma 2.3.** Assume $\beta < 0$ and $p > 2$. Then for each $\alpha \in (2/p,1)$, there exists a constant $C = C(p,\alpha) > 0$ independent of $\beta$ such for all $u \in B_{p,\beta}$,

$$\|Su\|_{p,\beta} \leq C \|u\|_{p,\beta} \left( |\beta|^{-\frac{p(1-\alpha)}{4}} + |\beta|^{-p/4} \right) \tag{2.8}$$

**Proof.** The key point of this proof is the application of Lemma 2.1. By Burkholder’s inequality, for any $x,y \in [0,1]$ and $t \geq 0$

$$\mathbb{E}[|Su(t,x)(t,x) - S(t,y)|^p]$$

$$\leq \kappa(p) \mathbb{E} \left[ \left( \int_0^t \int_0^1 (g(t-s,x,z) - g(t-s,y,z))^2 u^2(s,z) dz ds \right)^{p/2} \right]$$

where $\kappa(p)$ denotes the optimal constant in Burkholder’s $L^p(\Omega)$-inequality for continuous square-integrable martingales, see [7]. Using the continuous version of Minkowski’s inequality, see, for example, Theorem 6.2.7 [21], and the above estimate, we have that

$$\mathbb{E}[|Su(t,x) - Su(t,y)|^{2/p}]$$

$$\leq \kappa(p) \left( \int_0^t \int_0^1 (g(t-s,x,z) - g(t-s,y,z))^2 \mathbb{E}[|u(s,z)|^{2/p}] dz ds \right)^{p/2}.\]
Applying the mean value theorem and (2.3), we obtain that for each $\alpha \in (0, 1)$
\[ \mathbb{E}[|Su(t, x) - Su(t, y)|^p] \leq \kappa^p(p) \left( \int_0^1 \int_0^1 |\partial_x g_D(t - s, x + \theta(x - y), z)|d\theta \right)^\alpha 
\times |g(t - s, x, z) - g(t - s, y, z)|^{2 - \alpha} \mathbb{E}[|u(s, z)|^{2/p}dzds]^{p/2} |x - y|^\alpha p/2 
:= K_1(t)|x - y|^\alpha p/2. \tag{2.9} \]

From now, we will assume that $\alpha p > 2$. We point out that because of $p > 2$, it is possible for us to choose $\alpha \in (\frac{2}{p}, 1)$ such that $\alpha p > 2$. Applying Lemma 2.1 to $Su(t, x)$, we deduce from (2.9) that for all $t \geq 0$ and all $x, y \in [0, 1]$
\[ |Su(t, x) - Su(t, y)| \leq \kappa B(t)^{\frac{1}{p}}|x - y|^{\frac{q}{2} - \frac{1+\epsilon}{p}}, \tag{2.10} \]
where $\kappa$ is same as that in (2.3) in Lemma 2.1.

\[ B(t) = \int_0^1 \int_0^1 \frac{|Su(t, x) - Su(t, y)|^p}{|x - y|^{1+\alpha p/2 - \epsilon}}dxdy \]

and $\epsilon \in (0, \min\{\alpha p/2-1, 1\})$. It is valuable to point out that the constant $\kappa$ appeared in (2.10) does not depend on time $t$, which is very important for our goal. Let $y = y_0 \in [0, 1]$ be fixed. Then from (2.10) and noting that $\frac{q}{2} - \frac{1+\epsilon}{p} > 0$, we can deduce that for any $t \geq 0$ and $x \in [0, 1]$
\[ |Su(t, x)| \leq \kappa B(t)^{1/p}|x|^{\frac{q}{2} - \frac{1+\epsilon}{p}} + |Su(t, y_0)|, \]
which implies that for any $t > 0$
\[ \sup_{x \in [0, 1]} |Su(t, x)| \leq \kappa B(t)^{1/p} + |Su(t, y_0)|. \]

Let us now take the $p$-th moments of the both sides of the above inequality and use the inequality $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p), \ a, b \in \mathbb{R}$, we obtain that
\[ \mathbb{E} \left[ \sup_{x \in [0, 1]} |Su(t, x)|^p \right] \leq 2^{p-1}(\kappa^p \mathbb{E}[B(t)] + \mathbb{E}[|Su(t, y_0)|^p]). \tag{2.11} \]

By (2.9) and analogously to (2.7), we know that there is a positive constant $C_1$ depending on $\epsilon$ such that for all $t \geq 0$,
\[ \mathbb{E}[B(t)] \leq C_1 K_1(t). \tag{2.12} \]

Recalling that $K_1(t)$ is defined in (2.3). Let us now pay attention to the estimate of $K_1(t)$. From (2.3), it follows that
\[ \left| \int_0^1 \partial_x g_D(t - s, x + \theta(x - y), z)d\theta \right| \leq K_1 t^{-1}. \]
Thus, by the definition of $K_1(t)$, see (2.9) above and Lemma 2.2, it follows that for all $t \geq 0$ and any $x, y \in [0, 1]$

$$K_1(t) \leq \kappa^p(p) C_2 \left( \int_0^t \int_0^1 (t-s)^\alpha |g(t-s, x, z) - g(t-s, y, z)|^{2-\alpha} \right. \left. \times \mathbb{E}[\|u(s)\|^p_{L^\infty}]^{2/p}dzds \right)^{p/2}$$

$$\leq \kappa^p(p) C_2 \left( \int_0^t \int_0^1 (t-s)^\alpha |g(t-s, x, z) - g(t-s, y, z)|^{2-\alpha} \times (e^{\beta(t-s)} e^{\beta(t-s)} \mathbb{E}[\|u(s)\|^p_{L^\infty}])^{2/p}dzds \right)^{p/2}$$

$$\leq \kappa^p(p) C_2 e^{-\beta t} \|u\|^p_{p, \beta} \left( \int_0^t \int_0^1 e^{\beta s} |s^{\alpha} - g(s, y, z)|^{2-\alpha} dzds \right)^{p/2}$$

$$\leq \kappa^p(p) C_3 e^{-\beta t} \|u\|^p_{p, \beta} \frac{\beta^{(\alpha - 1)}}{4},$$

where $u \in B_{p, \beta}$ has been used for the third line, and $C_2$, $C_3$ are two generic constants depending on $p$ and $\alpha$.

We can more easily give the estimate of the term $\mathbb{E}[S u(t, y_0)|^p]$ in (2.11). In fact, similarly to (2.13), by using Minkowski’s inequality, Burkholder’s inequality and the semigroup property $\int_0^1 g_D^2(s, y_0, y)dy = g_D(2s, y_0, y_0)$, we deduce that

$$\mathbb{E}[S u(t, y_0)|^p] \leq \kappa^p(p) \left( \int_0^t \int_0^1 g_D^2(t-s, y_0, y) \mathbb{E}[\|u(s, y)\|^p_{L^\infty}]^{2/p}dyds \right)^{p/2}$$

$$\leq \kappa^p(p) e^{-\beta t} \left( \int_0^t \int_0^1 g_D^2(s, y_0, y) e^{2\beta(s-t)} \mathbb{E}[e^{2\beta s} \|u(s)\|^p_{L^\infty}]^{2/p}dyds \right)^{p/2}$$

$$\leq \kappa^p(p) e^{-\beta t} \|u\|^p_{p, \beta} \left( \int_0^t g_D(2s, y_0, y_0) e^{2\beta s} ds \right)^{p/2}$$

$$\leq \kappa^p(p) e^{-\beta t} \|u\|^p_{p, \beta} \left( \int_0^\infty (4\pi s)^{-1/2} e^{\beta s} ds \right)^{p/2}$$

$$\leq \kappa^p(p) e^{-\beta t} \|u\|^p_{p, \beta} \beta^{-p/4} \left( (4\pi)^{-1/2} \frac{1}{2} \right)^{p/2}$$

$$= 2^{-p/2} \kappa^p(p) e^{-\beta t} \|u\|^p_{p, \beta} \beta^{-p/4}. $$

Consequently, plugging (2.13) and (2.14) into (2.11), we deduce that there exists a constant $C_4 = C_4(p, \alpha) > 0$ such that

$$\mathbb{E} \left[ \sup_{x \in [0, 1]} |S u(t, x)|^p \right] \leq C_4 e^{-\beta t} \|u\|^p_{p, \beta} \left( \beta^{-p(1-\alpha)/4} + |\beta|^{-p/4} \right).$$

Multiplying both sides of the above inequality by $e^{\beta t}$ and then taking the supremum for $t \geq 0$, we go to

$$\|S u\|^p_{p, \beta} \leq C_4 \|u\|^p_{p, \beta} \left( \beta^{-p(1-\alpha)/4} + |\beta|^{-p/4} \right),$$

which completes the proof of (2.8).
Lemma 2.4. Assume $\beta < 0$ and $p > 2$. Then for each $\alpha \in (2/p, 1)$, there exists a constant $C = C(p, \alpha) > 0$ independent of $\beta$ such for any $u, v \in B_{p, \beta}$,

$$\|Su - Sv\|_{p, \beta}^p \leq C\|u - v\|_{p, \beta}^p (|\beta|^{-\frac{11}{4}} + |\beta|^{-p/4})$$

Proof. The proof is essentially same as that of Lemma 2.3. So we will only write down the different parts and leave the details to the reader. Form the definition of $Su$, it follows that

$$Su(t, x) - Sv(t, x) = \int_0^t \int_0^1 g_D(t - s, x, z)(u(s, z) - v(s, z))w(ds, dy).$$

Then, similarly to (2.13),

$$\mathbb{E}[|Su(t, x) - Sv(t, x) - (Su(t, y) - Sv(t, y))|^p] \leq \kappa^p(p)|x - y|^{\alpha p/2} \left(\int_0^t \int_0^1 \int_0^1 |\partial_x g_D(t - s, x + \theta(y - x), z)]^\alpha \right. \times |g(t - s, x, z) - g(t - s, y, z)|^{2-\alpha} \mathbb{E}[|u(s, z) - v(s, z)|^{2p}dzds]^{p/2}$$

$$:= K_2(t)|x - y|^{\alpha p/2}.$$ 

Noting that $\alpha \in (2/p, 1)$ and using Lemma 2.1, there exists a constant $C_1 = C_1(p, \alpha) > 0$ such that

$$\mathbb{E}\left[\sup_{x \in [0, 1]} |Su(t, x) - Sv(t, x)|^p\right] \leq C_1(K_2(t) + \mathbb{E}[|Su(t, y_0) - Sv(t, y_0)|^p]), y_0 \in [0, 1],$$

(2.15)

On the analogy of (2.13) and (2.14), we can deduce that for constants $C_2$ and $C_3$ depending only on $p$ and $\alpha$, $t \geq 0$ and any $x, y \in [0, 1]$,

$$K_2(t) \leq C_2 e^{-\beta t} \|u - v\|_{p, \beta}^p |\beta|^\frac{p\alpha - 1}{4},$$

(2.16)

and

$$\mathbb{E}[|Su(t, y_0) - Sv(t, y_0)|^p] \leq C_3 e^{-\beta t} \|u - v\|_{p, \beta}^p |\beta|^{-p/4}.$$ (2.17)

Thus, we can easily obtain our result, by plugging (2.16) and (2.17) into (2.15). $$\square$$

Let us now begin to formulate the proof of Theorem 1.1 by using Lemma 2.3 and Lemma 2.4.

Proof. (Proof of Theorem 1.1) Without loss of the generality, let us suppose that $\lambda = 1$ in this part. Since $u_0$ is continuous on $[0, 1]$ and non-random, we see that for any $\beta < 0$

$$\|D_1(t, x)\|_{p, \beta}^p = \sup_{t \geq 0} e^{\beta t} \left|\int_0^1 g_D(t, x, y)u_0(y)dy\right|^p \leq \|u_0\|_{L^\infty} \sup_{t \geq 0} e^{\beta t} \left|\int_0^1 g(t, x, y)dy\right|^p \leq \|u_0\|_{L^\infty}^p,$$
which implies for any \( \beta < 0 \), \( D_1(t, x) \in B_{p,\beta} \); recalling that \( D_1(t, x) \) is defined in (1.2).

Consider the following operator \( T \) on \( B_{p,\beta} \) by

\[
Tu(t, x) = D_1(t, x) + S(\sigma(u(t, x))).
\]

By (A2), it is known that \( |\sigma(u)| \leq KU|u| \). Hence, by Lemma 2.3 for any \( u(t, x) \in B_{p,\beta} \),

\[
\|Tu\|^p_{p,\beta} \leq \|u_0\|^p_{L^\infty} + C K_U^p \|u\|^p_{p,\beta}(\beta |\beta|^{-\frac{p(1-\alpha)}{4}} + |\beta|^{-p/4}) < \infty,
\]

which \( \alpha \in (2/p, 1) \) and \( C \) is the constant in Lemma 2.3. Thus, we have that \( T \) maps \( B_{p,\beta} < 0 \) into itself for any \( \beta \).

Analogously, from Lemma 2.4 it follows that for each \( \alpha \in (2/p, 1) \) and for any \( u, v \in B_{p,\beta} \),

\[
\|Tu - Tv\|^p_{p,\beta} = K_U^p \|Su - Sv\|^p_{p,\beta} \leq C K_U^p \|u - v\|^p_{p,\beta}(\beta |\beta|^{-\frac{p(1-\alpha)}{4}} + |\beta|^{-p/4}),
\]

where \( C \) is the constant in Lemma 2.4. Recall that \( C \) depends on \( p, \alpha \), which is independent of \( \beta \). Since (2.18) is satisfied for any \( \beta < 0 \), and noting that \( \alpha \in (2/p, 1) \), we choose a \( \beta_0 < 0 \) such that for any \( \beta < \beta_0 \),

\[
CK_U^p(\beta |\beta|^{-\frac{p(1-\alpha)}{4}} + |\beta|^{-p/4}) < 1,
\]

which implies that for any \( \beta < \beta_0 < 0 \), the operator \( T \) on the Banach space \( B_{p,\beta} \) is contractive. Consequently, the existence and uniqueness of the solution \( u \) in \( B_{p,\beta} \) is proved.

The second part is a direct conclusion of the above proof, by noting that for any \( \beta < \beta_0 \), \( \|Tu\|_{p,\beta} < \infty \).

From now, we are going to give the proof of Theorem 1.2. To do it, we will state another property of \( g_D(t, x, y) \). From the concrete form \( g_D(t, x, y) \), see (2.1), it is easy to see that there exists a constant \( K_3 > 0 \) such that for any \( t \geq 1 \),

\[
g_D(t, x, y) \leq K_3 e^{-\pi^2 t}.
\]

The next lemma is required.

**Lemma 2.5.** Assume \( \alpha \in (0, 1) \) and \( \beta \in (0, (2-\alpha)\pi^2) \). Then there exists a constant \( C > 0 \) depending on \( \alpha \), such that for any \( t \geq 0 \) and \( x \in [0, 1] \)

\[
\int_0^t e^{\beta s} s^{-\alpha} \int_0^1 |g_D(s, x, y)|^{2-\alpha} dy ds \leq C \left( \beta^{\alpha-1} + \frac{1}{(2-\alpha)\pi^2 - \beta} \right).
\]
Proof. This proof can be easily completed by using the properties (2.2) and (2.19) of $g_D(t, x, y)$. In fact, noting that (2.2), we have

$$
\int_0^1 e^{\beta s} s^{-\alpha} \int_0^1 |g_D(s, x, y)|^{2-\alpha} dy ds \leq \int_0^1 e^{\beta s} s^{-\alpha} \int_0^1 |g(s, x, y)|^{2-\alpha} dy ds \\
= (2\pi)^{\frac{\alpha-1}{2}} (2-\alpha)^{-\frac{1}{2}} \int_0^1 e^{\beta s} s^{-\frac{1+\alpha}{2}} ds \\
\leq (2\pi)^{\frac{\alpha-1}{2}} (2-\alpha)^{-\frac{1}{2}} \beta^{\frac{\alpha-1}{2}} \int_0^{2\pi^2} e^{\beta s} s^{-\frac{1+\alpha}{2}} ds.
$$

Since $\alpha \in (0, 1)$, the integrand $e^{\beta s} s^{-\frac{1+\alpha}{2}}$ above is integrable on $[0, 2\pi^2]$, from (2.20), there exists a constant $C_1 > 0$ depending only on $\alpha$ such that

$$
\int_0^1 e^{\beta s} s^{-\alpha} \int_0^1 |g_D(s, x, y)|^{2-\alpha} dy ds \leq C_1 \beta^{\frac{\alpha-1}{2}}. \tag{2.21}
$$

On the other hand, by (2.19) and $\beta \in [0, (2-\alpha)\pi^2)$, we see that

$$
\int_1^{\infty} e^{\beta s} s^{-\alpha} \int_0^1 |g_D(s, x, y)|^{2-\alpha} dy ds \\
\leq K_3^{2-\alpha} \int_1^{\infty} e^{\beta s} s^{-\alpha} \int_0^1 e^{(2-\alpha)\pi^2 s} dy ds \\
= K_3^{2-\alpha} \int_1^{\infty} e^{(\beta-(2-\alpha)\pi^2) s} s^{-\alpha} ds \\
\leq K_3^{2-\alpha} \int_1^{\infty} e^{(\beta-(2-\alpha)\pi^2) s} ds \\
\leq C_2 ((2-\alpha)\pi^2 - \beta)^{-1},
$$

where $C_2 > 0$ is a constant depending on $\alpha$. As a consequence of (2.21) and (2.22), we can complete our proof. \qed

In the following, we will formulate the proof of Theorem 1.2.

Proof. (Proof of Theorem 1.2) By Jensen’s inequality and Theorem 1.1 [10], we easily have

$$
-\infty < \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[|u_\lambda(t, x)|^p].
$$

Thus, the main task is to give the proof for the upper bound. Let us assume that $\alpha \in (2/p, 1)$ and $\beta \in (0, (2-\alpha)\pi^2)$ in this part. Recalling the definition of $\|u_\lambda\|_{p, \beta}$, see (1.3), it is sufficient to show for some $\beta \in (0, (2-\alpha)\pi^2)$, there exist $\lambda_L > 0$, such that for any $\lambda \in (0, \lambda_L)$, the following holds:

$$
\|u_\lambda\|_{p, \beta} < \infty. \tag{2.23}
$$
Let us first consider the term $D_{2,\lambda}(t, x)$ appeared in (1.2), whose estimate is essentially different from that formulated by M. Foondun and E. Nualart [10] as we will see below. Since $D_{2,\lambda}(t, x) = S(\lambda \sigma(u_\lambda(t, x)))$ and then by (2.11) and (2.12), we have that for any $y_0 \in [0, 1]$

$$
\mathbb{E}\left[ \sup_{x \in [0, 1]} |D_{2,\lambda}(t, x)|^p \right] \leq 2^{p-1} \lambda^p K_U^p (\kappa^p \mathbb{E}[B(t)] + \mathbb{E}[|Su(t, y_0)|^p])
$$

(2.24)

$$
\leq 2^{p-1} \lambda^p K_U^p (\kappa^p C_1(t) + \mathbb{E}[|Su(t, y_0)|^p]),
$$

where $K_1(t)$ is defined in (2.9). Let us now give the estimate of $K_1(t)$ based on Lemma 2.5. In fact, similarly to (2.13), by Lemma 2.5 we can easily see that for all $t \geq 0$ and any $x, y \in [0, 1]$

$$
K_1(t) \leq \kappa^p(p) C_2 \left( \int_0^t \int_0^1 (t-s) \alpha |g(t-s, x, z) - g(t-s, y, z)|^{2-\alpha} \mathbb{E}[\|u(s)\|_{L^\infty}^{2/p}] ds dz \right)^{p/2}
$$

$$
\leq \kappa^p(p) C_3 (\beta^p + ((2-\alpha)\pi^2 - \beta)^{-1}) e^{-\beta t} \|u\|_{p, \beta}^p,
$$

(2.25)

where $C_2$ and $C_3$ are generic positive constants and $\alpha \in (0, 1)$ has been used. On the other hand, it is easier to see that

$$
\mathbb{E}[|Su(t, y_0)|^p] \leq \kappa^p(p) \left( \int_0^t \int_0^1 g^2(t-s, y_0, y) \lambda^2 \mathbb{E}[\|u_\lambda(s, y)\|_{L^\infty}^{2/p}] ds dy \right)^{p/2}
$$

(2.26)

$$
\leq \kappa^p(p) \left( \int_0^t \int_0^1 g^2(t-s, y_0, y) e^{-\beta(t-s)} e^{-\beta(t-s)} \lambda^2 \mathbb{E}[\|u(s)\|_{L^\infty}^{2/p}] ds dy \right)^{p/2}
$$

$$
\leq \kappa^p(p) e^{-\beta t} \|u\|_{p, \beta}^p \left( \int_0^t g^2(2s, y_0, y) e^{\beta s} ds \right)^{p/2},
$$

Similarly to Lemma 2.5 we can easily show that if $\beta \in (0, 2\pi^2)$, then

$$
\sup_{t \geq 0, y_0 \in [0, 1]} \int_0^t g(2s, y, y) e^{\beta s} ds < C_3 \left( \frac{1}{2\pi^2 - \beta} + \frac{1}{\sqrt{\beta}} \right).
$$

Thus, combining this with (2.26), we see

$$
\mathbb{E}[|Su(t, y_0)|^p] \leq C_4 \kappa^p(p) \left( \frac{1}{2\pi^2 - \beta} + \frac{1}{\sqrt{\beta}} \right) e^{-\beta t} \|u\|_{p, \beta}^p.
$$

(2.27)

Consequently, plugging (2.25) and (2.27) into (2.24), we have that there exists a constant $C_5$ depending on $\beta \in (0, (2-\alpha)\pi^2)$ such that for all $t \geq 0$

$$
\mathbb{E}\left[ \sup_{x \in [0, 1]} |D_{2,\lambda}(t, x)|^p \right] \leq C_5 \lambda^p K_U^p e^{-\beta t} \|u\|_{p, \beta}^p,
$$

and then we can get that for each $\beta \in (0, (2-\alpha)\pi^2)$, $D_{2,\lambda}(t, \cdot) \in B_{p, \beta}$. 

\[\text{Page 15}\]
Let us now turn to consider $D_1(t, x)$, which is a easy part. Using (2.2) and (2.19), it is easy to see that for any $\beta \in (0, \pi^2)$,

$$\sup_{t \geq 0, x, y \in [0, 1]} \int_0^1 e^{\beta t} g_D(t, x, y) dy < \infty,$$

Since $u_0 \in C([0, 1])$, the above estimate tells us that $\|D_1(t, \cdot)\|_{p, \beta} \leq \|u_0\|_{L^\infty}$, that is, $D_1(t, \cdot) \in B_{p, \beta}$ for $\beta \in (0, p \pi^2)$.

Consequently, we proved that the solution $u_\lambda(t, x) \in B_{p, \beta}$ for each $\beta \in (0, (2 - \alpha)\pi^2)$, which is equivalent to (2.23). Therefore, the proof is completed.

3 Proof of Theorem 1.3

The proof of this theorem is essentially depends on the global behavior of the lower bound for the heat kernel $g_D(t, x, y)$. Such estimate is very important and has been studied actively. For large time and short time, we, for example, refer the reader to [6] and [24] respectively. For our aim, the global behavior for $g_D(t, x, y)$ is needed, which is studied by many authors for different domains, please see [19], [20] and [25]. Based on their research, we will state in the next corollary using our notation and omit its proof.

Lemma 3.1. There exist two strictly positive constants $\kappa_1$ and $\kappa_2$ such that for any $x, y \in [\gamma, 1 - \gamma], \gamma \in (0, 1/4)$

$$g_D(t, x) \geq \kappa_1 \exp(-\pi^2 t) \exp \left(-\kappa_2 \frac{|x - y|^2}{t}\right)(t^{-\frac{1}{2}}1_{(0, \gamma^2]}(t) + 1_{(\gamma^2, \infty)}(t)), \ t > 0.$$

Remark 3.1. In the papers of [19], [20] and [25], sharp bounds of both sides of Dirichlet heat kernel for the Laplacian on bounded domains with different conditions are discussed. To prove our result, the above lower bound of $g_D(t, x, y)$ is enough. So we do not state the corresponding upper bound, see [19], [20] and [25] and the references therein.

Let us now formulate the proof of Theorem 1.3 based on the above corollary.

Proof. (Proof of Theorem 1.3) Under our assumptions, by the positivity of $g_D(t, x, y)$ and Lemma 3.1, it is easy to see that for all $x \in [\gamma, 1 - \gamma],

$$D_1(t, x) = \int_0^1 g_D(t, x, y) u_0(y) dy \geq \inf_{x, y \in [\gamma, 1 - \gamma]} g_D(t, x, y) \int_\gamma^{1 - \gamma} u_0(y) dy \geq C_1 \inf_{x, y \in [\gamma, 1 - \gamma]} \exp(-\pi^2 t) \exp \left(-\kappa_2 \frac{|x - y|^2}{t}\right)(t^{-\frac{1}{2}}1_{(0, \gamma^2]}(t) + 1_{(\gamma^2, \infty)}(t)) \geq C_1 \exp(-\pi^2 t) \exp(-\kappa_2 t^{-1})(t^{-\frac{1}{2}}1_{(0, \gamma^2]}(t) + 1_{(\gamma^2, \infty)}(t)), \ t > 0.$$
where $C_1 = \kappa_1 \int_0^{1-\gamma} u_0(y)dy > 0$ by the assumption (A1).

Noting that $\inf_{t \in (0, \kappa^2]} t^{-\frac{1}{2}} \exp(-\kappa_2 t^{-1}) > 0$ and $\exp(-\kappa_2 t^{-1}) \geq \exp(-\kappa_2 \gamma^{-2})$ for $t \geq \gamma^2$, the above estimate (3.1), implies that there is a constant $C_2 > 0$ such that for all $t \geq 0$ and $x \in [\gamma, 1-\gamma]$,

$$\int_0^t g_D(t, x, y)u_0(y)dy \geq C_2 \exp(-\pi^2 t) \tag{3.2}$$

By Ito’s isometry and the assumption $|\sigma(u)| \geq K_L |u|, u \in \mathbb{R}$, we have that

$$\mathbb{E}[|D_{2,\lambda}(t, x)|^2] = \int_0^t \int_0^1 g_D^2(t - s, x, y)\mathbb{E}[|\lambda \sigma(u_\lambda(s, y))|^2]dyds \tag{3.3}$$

\[ \geq \lambda^2 K_L^2 \int_0^t \int_0^1 g_D^2(t - s, x, y)\mathbb{E}[|u_\lambda(s, y)|^2]dyds. \]

From now, let us deal with the term

$$\int_0^t \int_0^1 g_D^2(t, x, y)\mathbb{E}[|u_\lambda(s, y)|^2]dyds$$

appeared in the last inequality. For brevity, let us define $h(t) = \inf_{x \in [\gamma, 1-\gamma]} \mathbb{E}[|u_\lambda(t, x)|^2]$. Using Lemma 3.3 if $x \in [\gamma, 1-\gamma]$, then we have

$$\int_0^t \int_0^1 g_D^2(t, x, y)\mathbb{E}[|u_\lambda(s, y)|^2]dyds \tag{3.4}$$

$$\geq \int_0^t \int_0^{1-\gamma} g_D^2(t - s, x, y)h(s)dyds$$

$$\geq \int_0^t \int_0^{1-\gamma} \kappa_1^2 \exp(-2\pi^2(t - s)) \exp \left( -2\kappa_2 \frac{|x - y|^2}{t - s} \right)$$

$$\times \left( (t - s)^{-\frac{1}{2}} 1_{(0,\gamma]}(t - s) + 1_{(\gamma^2,\infty)}(t - s) \right) h(s)dyds$$

$$= \int_0^t \int_0^{1-\gamma} \kappa_1^2 \exp(-2\pi^2(t - s)) \exp \left( -2\kappa_2 \frac{|x - y|^2}{t - s} \right)$$

$$\times \left( (t - s)^{-\frac{1}{2}} 1_{(0,\gamma]}(t - s) + 1_{(\gamma^2,\infty)}(t - s) \right) h(s)dyds$$

Set $A(x; s, t) := [\gamma, 1-\gamma] \cap \{ y : |y - x| \geq \sqrt{t - s} \}$, $s \leq t$ and then noting that $\gamma \in (0, 1/4)$, we can show that for any $x \in [\gamma, 1-\gamma], |A(x; s, t)| \geq \sqrt{t - s}$. Consequently, noting that for $y \in A(x; s, t), |y - x| \leq \sqrt{t - s}$ and the non-increasing of $e^{-x}$, for $s \in (t - \gamma^2, t)$, we have

$$\int_0^{1-\gamma} \exp(-2\pi^2(t - s)) \exp \left( -2\kappa_2 \frac{|x - y|^2}{t - s} \right) (t - s)^{-1}dy \tag{3.5}$$

$$\geq \int_{A(x; t, s)} \exp(-2\pi^2(t - s)) \exp \left( -2\kappa_2 \frac{|x - y|^2}{t - s} \right) (t - s)^{-1}dy$$

$$\geq \exp(-2\kappa_2) \exp(-2\pi^2(t - s))(t - s)^{-1}A(x; t, s)$$

$$\geq \exp(-2\kappa_2) \exp(-2\pi^2(t - s))(t - s)^{-\frac{1}{2}}.$$
On the other hand, if \( s \in (0, t - \gamma^2] \), then for any \( x, y \in [\gamma, 1 - \gamma] \),
\[
\exp \left( -2\kappa_2 \frac{|x - y|^2}{t - s} \right) \geq \exp \left( -2\kappa_2 \frac{(1 - 2\gamma)^2}{t - s} \right) \geq \exp(-2\kappa_2\gamma^{-2}) > 0.
\]
Thus, it is easy to see that if \( s \in (0, t - \gamma^2] \), then for any \( x \in [\gamma, 1 - \gamma] \)
\[
\int_{\gamma}^{1-\gamma} \kappa_1^2 \exp(-2\pi^2(t - s)) \exp \left( -2\kappa_2 \frac{|x - y|^2}{t - s} \right) dy \\
\geq \exp(-2\kappa_2\gamma^{-2}) \exp(-2\pi^2(t - s)).
\]
Inserting (3.5) and (3.6) into (3.4), we see that there exists a constant \( H \) such that for any \( t > C_0 \geq \lambda \) and then, noting that \( 1 \geq \gamma(t - s)^{-\frac{1}{2}} \) for \( t - s \geq \gamma^2 \), we deduce that
\[
\int_{0}^{t} \int_{0}^{1} g_D^2(t, x, y) \mathbb{E}[[u_{\lambda}(s, y)]^2] dy ds \\
\geq C_3 \int_{0}^{t} \exp(-2\pi^2(t - s)) \left( (t - s)^{-\frac{1}{2}} \mathbb{1}_{(0, \gamma^2]}(t - s) + 1_{(\gamma^2, \lambda]}(t - s) \right) h(s) ds,
\]
and then, noting that \( 1 \geq \gamma(t - s)^{-\frac{1}{2}} \) for \( t - s \geq \gamma^2 \), we deduce that
\[
\int_{0}^{t} \int_{0}^{1} g_D^2(t, x, y) \mathbb{E}[[u_{\lambda}(s, y)]^2] dy ds \geq C_4 \int_{0}^{t} \exp(-2\pi^2(t - s))(t - s)^{-\frac{1}{2}} h(s) ds
\]
holds for some constant \( C_4 > 0 \).
Consequently, combining (3.3) with the above estimate, we have
\[
\mathbb{E}[[D_{2\lambda}(t, x)]^2] \geq C_4 \lambda^2 K_L^2 \int_{0}^{t} \exp(-2\pi^2(t - s))(t - s)^{-\frac{1}{2}} h(s) ds. \quad (3.7)
\]
Noting that under our assumptions, Ito’s isometry gives that
\[
\mathbb{E}[[u_{\lambda}(t, x)]^2] = \left( \int_{0}^{1} g_D(t, x, y) u_0(y) dy \right)^2 + \mathbb{E}[[D_{2\lambda}(t, x)]^2].
\]
Using (3.2) and (3.7), we have that for any \( x \in [\gamma, 1 - \gamma] \),
\[
\mathbb{E}[[u_{\lambda}(t, x)]^2] \geq C_2^2 \exp(-2\pi^2 t) + C_4 \lambda^2 K_L^2 \int_{0}^{t} \exp(-2\pi^2(t - s))(t - s)^{-\frac{1}{2}} h(s) ds,
\]
which implies that for any \( t > 0 \)
\[
h(t) \geq C_2^2 \exp(-2\pi^2 t) + C_4 \lambda^2 K_L^2 \int_{0}^{t} \exp(-2\pi^2(t - s))(t - s)^{-\frac{1}{2}} h(s) ds. \quad (3.8)
\]
Let us now define \( H(t) = \exp(2\pi^2 t) h(t) \) and then from (3.8), we get the following relation for \( H(t) \): for any \( t > 0 \)
\[
H(t) \geq C_2^2 + C_4 \lambda^2 K_L^2 \int_{0}^{t} (t - s)^{-\frac{1}{2}} H(s) ds.
\]
Finally, owing to Gronwall’s inequality, we can easily obtain that
\[
H(t) \geq C_2^2 \exp(C_4^2 \lambda^4 K_L^4 t), \quad t > 0,
\]
which completes our proof.
4 Proof of Theorem \textbf{1.4} and Corollary \textbf{1.5}

The first aim of this part is to formulate the proof of Theorem \textbf{1.4} based on Theorem \textbf{1.3} and the second one is to prove Corollary \textbf{1.5} as the application of Theorem \textbf{1.4}.

Proof. (Proof of Theorem \textbf{1.4}) Owing to Theorem \textbf{1.1} and \(|u_\lambda(t, x)| \leq \|u_\lambda\|_{L^\infty}\), it is sufficient for us to verify the lower bound, i.e., for any \(\lambda \in (\lambda_U, \infty)\) and \(x \in [\gamma, 1-\gamma]\)

\[
0 < \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{E}[|u_\lambda(t, x)|^p].
\] (4.1)

Jensen’s inequality tells us that for any \(p > 2\),

\[
\mathbb{E}[|u_\lambda(t, x)|^p]^{1/p} \leq \mathbb{E}[|u_\lambda(t, x)|^{p/2}]
\]

and thus it is enough for us to show that there exists a large enough \(\lambda_U\), (4.1) is satisfied when \(p = 2\). However, this is a quick result of Theorem \textbf{1.3}. In fact, since (1.5) holds for any \(t > 0\), we easily know that

\[
\log \left( \inf_{x \in [\gamma, 1-\gamma]} \mathbb{E}[|u_\lambda(t, x)|^2] \right) \geq \log \kappa_1 + (\kappa_2 \lambda^4 K_L^4 - 2\pi^2)t, \ t \geq 0.
\]

Dividing both sides of the above inequality by \(t\) and taking the infimum limit as \(t \to \infty\), we see that

\[
\liminf_{t \to \infty} \frac{1}{t} \log \left( \inf_{x \in [\gamma, 1-\gamma]} \mathbb{E}[|u_\lambda(t, x)|^2] \right) \geq \kappa_2 \lambda^4 K_L^4 - 2\pi^2.
\]

Let us take \(\lambda_U = \left( \frac{2\pi^2}{\kappa_2 K_L^4} \right)^{1/4}\). Then, from the above inequality, we have that for all \(\lambda > \lambda_U\), (4.1) holds for \(p = 2\).

\textbf{Proof. (Proof of Corollary \textbf{1.5})} The first part comes immediately from Theorem \textbf{1.2} by noting that for any \(p \geq 2\), \(\mathbb{E}[\|u_\lambda(t)\|_{L^p}^p] \leq \mathbb{E}[\|u_\lambda(t)\|_{L^\infty}^p]\).

Let us now consider the proof of the second part. By Fubini’s theorem and Jensen’s inequality, for any \(p > 2\),

\[
\mathbb{E} \left[ \int_0^1 |u_\lambda(t, x)|^p dx \right] \geq \int_0^1 \mathbb{E}[u_\lambda^2(t, x)]^{p/2} dx
\]

\[
\geq \int_{\gamma}^{1-\gamma} \inf_{x \in [\gamma, 1-\gamma]} \mathbb{E}[u_\lambda^2(t, x)]^{p/2} dx
\]

\[
= (1 - 2\gamma) \inf_{x \in [\gamma, 1-\gamma]} \mathbb{E}[u_\lambda^2(t, x)]^{p/2}.
\]

Combining the above estimate with Theorem \textbf{1.3} we deduce that for all \(t > 0\)

\[
\log \mathcal{E}_p(t, \lambda) \geq \frac{1}{p} \log \left( (1 - 2\gamma) \inf_{x \in [\gamma, 1-\gamma]} \mathbb{E}[u_\lambda^2(t, x)]^{p/2} \right) \geq \frac{1}{p} \log \left( (1 - 2\gamma) \kappa_1^{p/2} \right) + \frac{1}{2} (\kappa_2 \lambda^4 K_L^4 - 2\pi^2)t.
\] (4.2)

Finally, dividing both sides by \(t\) and taking the infimum limit as \(t \to \infty\), we see that the lower bound holds for any \(\lambda > \lambda_U\).
5 Proof of Theorem 1.6 and Corollary 1.7

Proof. (Proof of Theorem 1.6) It is first task to proof the upper bound. It is easier than the proofs of Theorem 1.1 and Theorem 1.2. By the assumption (A1) on \( u_0 \) and the estimate (2.2) of \( g_D(t, x, y) \), it is easy to know that there exists a constant \( C_1 \) such that for all \( t > 0 \),

\[
\sup_{x \in [0, 1]} D_1(t, x) \leq C_1.
\]

By a similar approach used in the proof of Theorem 1.1, we can prove that there is a constant \( C_2 > 0 \) such that for each \( t > 0 \) and any \( x \in [0, 1] \),

\[
\mathbb{E}[|D_{2, \lambda}(t, x)|^{p/2}] \leq C_2 \lambda^2 K^2_\lambda \int_0^t g_D^p(t - s, x, y) \mathbb{E}[|u_\lambda(s, y)|^{p/2}] \, dy \, ds
\]

\[
\leq C_2 \lambda^2 K^2_\lambda \int_0^t g_D^p(t - s, x, y) \sup_{y \in [0, 1]} \mathbb{E}[|u_\lambda(s, y)|^{p/2}] \, dy \, ds
\]

\[
\leq C_2 \lambda^2 K^2_\lambda \int_0^t g_D(2(t - s), x, x)) \sup_{y \in [0, 1]} \mathbb{E}[|u_\lambda(s, y)|^{p/2}] \, ds
\]

\[
\leq C_2 \lambda^2 K^2_\lambda \int_0^t \frac{1}{\sqrt{4\pi(t - s)}} \sup_{y \in [0, 1]} \mathbb{E}[|u_\lambda(s, y)|^{p/2}] \, ds.
\]

Hence, by Minkowski’s inequality and the above estimates, there exist constant \( C_3 \) and \( C_4 \) such that for any \( t > 0 \)

\[
\sup_{x \in [0, 1]} \mathbb{E}[|u_\lambda(t, x)|^{p/2}] \leq C_3 + C_4 \lambda^2 K^2_\lambda \int_0^t \frac{1}{\sqrt{(t - s)}} \sup_{y \in [0, 1]} \mathbb{E}[|u_\lambda(s, y)|^{p/2}] \, ds,
\]

and then for any \( t > 0 \)

\[
\sup_{x \in [0, 1]} \mathbb{E}[|u_\lambda(t, x)|^{p/2}] \leq C_3 + 2C_3C_4 \sqrt{t} + C_4^2 \lambda^4 K^4_\lambda \int_0^t \mathbb{E}[|u_\lambda(s, y)|^{p/2}] \, ds,
\]

As a consequence, owing to Gronwall inequality, we have for all \( t > 0 \)

\[
\sup_{x \in [0, 1]} \mathbb{E}[|u_\lambda(t, x)|^{p/2}] \leq (C_3 + 2C_3C_4 \sqrt{t}) \exp(C_4^2 \lambda^4 K^4_\lambda t),
\]

which immediately implies the upper bound.

Let us now begin to state the proof of the lower bound, i.e., for some \( c_p > 0 \),

\[
c_p K^2_L t \leq \liminf_{\lambda \to \infty} \lambda^{-4} K_L \log \left( \inf_{x \in [\gamma, 1-\gamma]} \mathbb{E}[|u_\lambda(t, x)|^p] \right).
\]

It is also a direct conclusion of Theorem 1.3. From Jensen’s inequality, it is sufficient for us to show that the above lower bound holds for \( p = 2 \). By Theorem 1.3 for any \( t > 0 \)

\[
\log \left( \inf_{x \in [\gamma, 1-\gamma]} \mathbb{E}[u_\lambda^2(t, x)] \right) \geq \log \kappa_1 + (\kappa_2 \lambda^4 K^2_L - 2\pi^2)t.
\]

Let us first divide both sides of the above inequality by \( \lambda^4 \) and then take the infimum limit as \( \lambda \to \infty \), the lower bound can be proved immediately. \( \square \)
In the end, let us state the proof of Corollary 1.7 as the application of Theorem 1.6.

Proof. (Proof of Corollary 1.7) The lower bound can be easily deduced from (1.2), appeared in the proof of Corollary 1.5. In fact, dividing both sides of (1.2) by $\lambda^4$ and then taking the infimum limit as $\lambda \to \infty$, we can easily obtain the lower bound for each $t > 0$.

On the other hand, noting that

\[ \mathbb{E}[\|u(t)\|_{L^p}^p] \leq \sup_{x \in [0,1]} \mathbb{E}[|u_\lambda(t,x)|^p], \]

the upper bound is obtained immediately by Theorem 1.6. As a result, the proof of this corollary is completed.

\[ \square \]

6 Appendix

According to [12] and [22], let us rewrite the celebrated Garsia-Rodemich-Rumsey theorem for our purpose. Let $\Phi : \mathbb{R} \to [0, \infty)$ be a Young function (a convex and even function with $\Phi(0) = 0$ and $\lim_{x \to \infty} \Phi(x) = \infty$) and let $\phi : [0,1] \to [0, \infty)$ be continuous and increasing with $\phi(0) = 0$.

Theorem 6.1. Let $\Phi$ and $\phi$ be defined as above. If $f$ is a measurable function on $[0,1]$ such that

\[ \int_0^1 \int_0^1 \Phi\left(\frac{f(x)-f(y)}{\phi(|x-y|)}\right) dx dy = B < \infty, \]

then

\[ |f(x) - f(y)| \leq 8 \int_0^{|x-y|} \Phi^{-1}\left(\frac{B}{u}\right) du \ a.e. \ x, y \in [0,1]. \]

Acknowledgements

The author was supported in part by Grant-in-Aid for Young Scientist (B) 25800060 and Grant-in-Aid for Scientific Research (C) 24540198 from Japan Society for the Promotion of Science(JSPS).

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