The effect of noise intensity on stochastic parabolic equations

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Abstract

In the present paper, the effect of noise intensity on stochastic parabolic equations is discussed. We focus on the effect of noise on the energy solutions of the stochastic parabolic equations. By utilising Itô’s formula and the energy estimate method, we obtain the excitation indices of the energy solutions $u$ at any finite time $t$. Furthermore, we improve certain existing results in the literature by presenting a comparably simple method to show those existing results.

Keywords: Stochastic parabolic equations; Itô’s formula; the energy estimate method.

AMS subject classifications (2010): 35K20, 60H15, 60H40.

1 Introduction

In recent years, many authors attempt to explore the role of the noise in various dynamical equations in both analytical and numerical aspects. For example, noise can make the solution smooth [10], can prevent singularities in linear transport equations [9], can prevent collapse of Vlasov-Poisson point charges [6], and also can induce singularities (finite time blow up of solutions) [3, 4, 19]. In the present paper, we focus on the effect of noise on parabolic equations.

The concept of “Intermittency” is the property that the solution $u(t, x)$ develops extreme oscillations at certain values of $x$, typically when $t$ is going to be large. Intermittency was announced first (1949) by Batchelor and Townsend in a WHO conference in Vienna [1], and slightly later by Emmons [8] in the context of boundary layer turbulence. Meanwhile, intermittency has been observed in an enormous number of scientific disciplines. For example, intermittency is observed as “spikes” and “shocks” in neural activity and in finance, respectively. Tuckwell [24] contains a gentle introduction to SPDEs in neuroscience.

Recently, Khoshnevisan-Kim in [14] [15] considered the following stochastic heat equation

$$\frac{\partial}{\partial t} u(t, x) = \mathcal{L}u(t, x) + \lambda \sigma(u(t, x))\xi(t, x)$$

(1.1)

where $t \in (0, \infty)$ stands for the time variable, $x \in G$ the space variable with $G$ being a given nice state space, such as $\mathbb{R}$, $\mathbb{Z}$ (a discrete set) or a finite interval like $[0, 1]$, and the initial data value $u_0 : G \to \mathbb{R}$ is deterministic (i.e., non random) and is well behaved. The operator $\mathcal{L}$ acts on the

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spatial variable $x \in G$ only, and is taken to be the generator of a nice Markov stochastic process on $G$, and $\xi$ denotes space-time white noise on $(0, \infty) \times G$. Whereby, $\lambda > 0$ is a constant and the coefficient $\sigma : \mathbb{R} \to \mathbb{R}$ is supposed to be a Lipschitz continuous function.

Let $u$ be a mild solution of (1.1) with given initial data $u_0$. Set

$$u(t) := u(t, \cdot) : D \to \mathbb{R}, \quad t \in [0, \infty)$$

and then define

$$E_t(\lambda) := \sqrt{\mathbb{E} \left( ||u(t)||^2_{L^2(G)} \right)},$$

which stands for the energy of the solution at time $t$. In papers [14, 15, 11, 18], the authors showed that the energy $E_t(\lambda)$ behaviors like $\exp(\text{const} \cdot \lambda^q)$, for certain fixed positive constant $q$, as $\lambda \uparrow \infty$. In order to do so, the following two quantities have been introduced

$$\varphi(t) := \lim \inf_{\lambda \to \infty} \frac{\log \log E_t(\lambda)}{\log \lambda}, \quad \bar{\varphi}(t) := \lim \sup_{\lambda \to \infty} \frac{\log \log E_t(\lambda)}{\log \lambda}.$$

Clearly, $\varphi$ and $\bar{\varphi}$ represent the lower and upper excitation indices of $u$ at time $t$, respectively. In many interesting cases, $\varphi(t)$ and $\bar{\varphi}(t)$ are exactly equal, and they do not depend on the time variable $t \in [0, \infty)$. In such situations, we tacitly write $\varphi$ for that common value, just for simplicity.

In paper [14, Khoshnevisan-Kim proved that

(i) If $G$ is discrete, then $\bar{\varphi}(t) \leq 2$ for all $t \geq 0$. Furthermore, it hold that $\bar{\varphi} = 2$ if

$$l_\sigma := \inf_{z \in \mathbb{R} \setminus \{0\}} \frac{|\sigma(z)|}{z} > 0.$$

(ii) Suppose that $G$ is connected and (1.3) holds, then $\varphi(t) \geq 4$ for all $t \geq 0$, provided that in addition either $G$ is non compact or $G$ is compact, metrizable, and has more than one element.

(iii) For every $\theta \geq 4$ there exist models of the triple $(G, \mathcal{L}, u_0)$ for which $\varphi = \theta$. One such model is that $\mathcal{L} := -(-\Delta)^{\frac{\alpha}{2}}$ (the generator of a symmetric $\alpha$-stable Lévy process) for $1 < \alpha \leq 2$.

In [15], Khoshnevisan-Kim considered the following problem for the stochastic evolution equation

$$\begin{cases}
\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial x^2} u(t, x) + \lambda \sigma(u(t, x)) \dot{w}(t, x), & 0 < x < L, t > 0, \\
u(t, 0) = u(t, L) = 0, & t > 0, \\
u(0, x) = u_0(x),
\end{cases}$$

where $\dot{w}$ is a space-time white noise, $L > 0$ is fixed, $u_0(x) \geq 0$ is a non-random, bounded continuous function and $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with $\sigma(0) = 0$. Let

$$l_\sigma := \inf_{z \in \mathbb{R} \setminus \{0\}} \frac{|\sigma(z)|}{z} > 0, \quad L_\sigma := \sup_{z \in \mathbb{R} \setminus \{0\}} \frac{|\sigma(z)|}{z} > 0.$$

They derived the following

$$\frac{l_\sigma^2 t}{2} \leq \lim \inf_{\lambda \to \infty} \frac{1}{\lambda^2} \log E_t(\lambda), \quad \lim \sup_{\lambda \to \infty} \frac{1}{\lambda^4} \log E_t(\lambda) \leq 8L_\sigma^4 t.$$

More recently, Foondun-Joseph [11] complemented the results of [15], that is, they obtained $\varphi = 4$. It is easy to see that a mild solution $u$ of (1.5) which is adapted to the natural filtration of the white noise $\dot{w}$ and satisfies the following mild formulation of the evolution equation

$$u(t, x) = (\mathcal{G}_D u)(t, x) + \lambda \int_0^t \int_0^L p_D(t - s, x, y) \sigma(u(s, y)) \dot{w}(dsdy),$$

(1.7)
the authors only consider the expression into random parabolic equations. Moreover, it is not hard to find that, in the earlier results, given above. We will obtain the similar result to [11] by changing the stochastic parabolic equations where \( D \) and \( p > 0 \) for

\[ D = \{0, L\} \]. They used the estimate of kernel \( p_D(t, x, y) \) and a new Gronwall’s inequality to prove that \( \varepsilon = 4 \). Using similar method, Liu-Tian-Foondun [18] considered the fractional Laplacian on a bounded domain.

Now, given a complete probability space endowed with a filtration \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}) \), let us consider the following linear stochastic differential equation (SDE) with \( \lambda > 0 \) being given before

\[
dX_t = \lambda X_t dB_t, \quad t > 0, \quad X_0 = x \in D.
\]

For simplicity, we assume that \( B(t) \) is a standard one-dimensional Brownian motion on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}) \). It is easy to see that the unique solution of the above SDE is explicitly given by

\[
X_t = x e^{-\frac{x^2}{2} t} e^{\lambda B(t)}, \quad t \in [0, +\infty).
\]

Direct calculations then show that

\[
\mathbb{E}[X_t] = x e^{-\frac{x^2}{2} t} e^{\lambda B(t)} = x; \quad \mathbb{E}[X^2_t] = x^2 e^{-\lambda^2 t} e^{2 \lambda B(t)} = x^2 e^{2 \lambda B(t)};
\]

\[
\mathbb{E}[X^p_t] = x^p e^{-\frac{x^2}{2} t} e^{\lambda B(t)} = x^p e^{\lambda^p B(t) - \frac{1}{2} \lambda^2 t}
\]

for \( p > 1 \). This then implies that for \( p > 1 \)

\[
\lim_{\lambda \to \infty} \frac{\log \log \left( \mathbb{E}[X^p_t] \right)}{\log \lambda} = 2,
\]

which yields that the excitation index of \( X_t \) is 2. This is clearly different from the results obtained in [11] [15] [25], where the authors proved the excitation index of \( u(t, x) \) of (1.5) is 4 for \( x \in [\varepsilon, L - \varepsilon] \) (\( \varepsilon \) is a sufficiently small constant). A natural and very interesting question then appeared to be that is there some kind of solutions of (1.5) with the associated indices being 2? This motivates us to initiate the present paper.

Another propose of our paper is to introduce a comparably simpler method to prove the result of [11] in a simple case. That is, we consider the following stochastic parabolic equations

\[
\begin{aligned}
&\left\{ \begin{array}{ll}
du(t, x) = \Delta u(t, x) dt + \lambda u(t, x) dB_t, & x \in D, \ t > 0, \\
&u|_{\partial D} = 0, \\
&u(0, x) = u_0(x),
\end{array} \right.
\end{aligned}
\]

where \( D \subset \mathbb{R}^n \ (n \geq 1) \), \( B_t \) is a standard one-dimensional Brownian motion on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}) \) as given above. We will obtain the similar result to [11] by changing the stochastic parabolic equations into random parabolic equations. Moreover, it is not hard to find that, in the earlier results, the authors only consider the expression \( \sqrt{\mathbb{E} \left( \|u(t)\|_{L^2(G)}^2 \right)} \), and we can consider the expression

\[
\left[ \mathbb{E} \left( \|u(t)\|_{L^p(G)}^p \right) \right]^{1/p}, \quad p > 0,
\]

which is clearly an interesting generalisation.

In this paper, we will focus on the noise excitability of energy solution for some parabolic equations. We obtain a new result regarding the noise excitability, that is, \( \varepsilon = 2 \) under the same condition as in [11] when the noise is only the time white noise (not the space-time white noise). The contribution of our paper is that we consider energy solutions (comparing to [11] where mild solutions are considered).

The rest of the paper is organised as follows. In Section 2, some preliminaries and main results are given. Section 3 is devoted to the proofs of the main results. In Section 4, we consider a special noise case of (1.7) and we discuss noise excitability of stochastic equations involving nonlocal operators.
2 Preliminaries and two main results

Inspired by [14, 15, 11, 20, 23], in this paper, we consider the simple case

\[
\begin{aligned}
\frac{du}{dt}(t, x) &= \Delta u(t, x) dt + \lambda \sigma(u(t, x)) dB_t, \quad x \in D, t > 0, \\
|u|_{\partial D} &= 0, \quad t > 0, \\
u(0, x) &= u_0(x), \quad x \in D
\end{aligned}
\]

(2.1)

where \( D \subset \mathbb{R}^n (n \geq 1) \) is a bounded domain, and \( B_t \) denotes one dimensional Brownian motion.

The existence of solutions of (2.1) was obtained by [2].

The main results of this paper are formulated as the following

**Theorem 2.1** Assume that (1.6) holds. The noise excitation index of the energy solution to (2.1) with initial data \( u_0(x) \geq 0 (\neq 0) \) is 2.

If the one-dimensional Brownian motion is replaced by \( Q \)-Wiener process, where \( Q \) is a trace class operator on \( L^2(D) \), the result of Theorem 2.1 still holds. In fact, for the following equation (i.e., the case driven by time white noise), we have the following result.

\[
\begin{aligned}
\frac{\partial}{\partial t} u(t, x) &= \Delta u(t, x) + \lambda \sigma(u(t, x)) \dot{w}(t, x) \\
|u|_{\partial D} &= 0, \quad t > 0, \\
u(0, x) &= u_0(x), \quad x \in D
\end{aligned}
\]

(2.2)

where \( D \subset \mathbb{R}^n (n \geq 1) \) is a bounded domain and \( w(t, x) \) is a \( Q \)-Wiener process. The existence of solutions of (2.2) was also obtained by [2].

**Theorem 2.2** Assume that (1.6) holds. Let \( w(t, x) \) be a \( Q \)-Wiener process with covariance

\[ E[w(t, x)w(s, y)] = (t \wedge s)q(x, y), \quad s, t \in (0, +\infty), x, y \in D \]

where \( q : D \times D \to \mathbb{R} \) is the kernel of the trace class operator \( Q : L^2(D) \to L^2(D) \).

In this case, the noise \( \dot{w}(t, x) \) is white in time and colored in the space variable. Assume that \( 0 < \sup_{x \in D} q(x, x) \leq q_1 < \infty \), then, the upper excitation index of the solution to (2.2) with initial data \( u_0(x) \geq 0 \) is 2. Furthermore, if \( \sigma \geq 0 \) (or \( \leq 0 \)) and there is a positive real number \( q_0 > 0 \) such that \( q_0 < \inf_{x, y \in D} q(x, y) \), then the excitation index of the solution to (2.2) with initial data \( u_0(x) \geq 0, \neq 0 \) is 2.

**Remark 2.1** Now, we give the reason why we can not consider the case that \( g(u) \) satisfies local Lipschitz condition. More precisely, consider the following general case

\[
\begin{aligned}
du(t) &= [Au(t) + f(u(t))] dt + \lambda \sigma(u(t)) dw(t), \quad t > 0, \\
u(0) &= u_0
\end{aligned}
\]

(2.3)

where \( A \) is a divergence operator, \( f \) and \( \sigma \) satisfy the local Lipschitz condition. For example, let \( f(u) \geq au^{1+\alpha} \) and \( \sigma(u) = u^m \). Then the solutions of (2.3) will blow up in finite time (see [4, 19]). Moreover, the largest existence time \( T \to 0 \) as \( \lambda \to \infty \). So we cannot consider problem (2.3).

3 The proofs of our main results

In this section, we will prove Theorem 2.1 and Theorem 2.2 by using energy method. Let us first prove Theorem 2.1.
Proof of Theorem 2.1. By using the idea of [20, 23], one can prove that there exists a unique energy solution. It follows from the results of [19] that the energy solution will keep positive if the initial data \( u_0 \geq 0 \) almost surely. We divide our proof into two steps.

Step 1: \( \bar{\sigma}(t) = 2 \).

By Itô formula, we have
\[
\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + 2 \int_0^t \langle \Delta u(s, x), u(s, x) \rangle ds + 2\lambda \int_0^t \int_D u(s, x)\sigma(u(s, x)) dx dB_s + \lambda^2 \int_0^t \int_D \sigma^2(u(s, x)) dx ds.
\]

Integrating by parts shows that
\[
\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 - 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2\lambda \int_0^t \int_D u(s, x)\sigma(u(s, x)) dx dB_s + \lambda^2 \int_0^t \int_D \sigma^2(u(s, x)) dx ds,
\]
which implies
\[
E\|u(t)\|_{L^2}^2 \leq E\|u_0\|_{L^2}^2 + \lambda^2 E\int_0^t \int_D \sigma^2(u(s, x)) dx ds
\]
\[
\leq E\|u_0\|_{L^2}^2 + L_\sigma \lambda^2 E\int_0^t E\|u(s)\|_{L^2}^2 dx ds.
\]
It follows from Gronwall’s inequality that
\[
E\|u(t)\|_{L^2}^2 \leq E\|u_0\|_{L^2}^2 e^{L_\sigma \lambda^2 t},
\]
which implies that \( \bar{\sigma}(t) \leq 2 \).

Step 2: \( \underline{\sigma}(t) = 2 \).

In order to get the lower bounded, let us consider the following eigenvalue problem for the elliptic equation
\[
\begin{cases}
-\Delta \phi = \lambda \phi, & \text{in } D, \\
\phi = 0, & \text{on } \partial D.
\end{cases}
\]

Since all the eigenvalues are strictly positive, increasing and the eigenfunction \( \phi \) corresponding to the smallest eigenvalue \( \lambda_1 \) does not change sign in domain \( D \), as shown in [13], then one can normalise it in such a way that
\[
\phi(x) > 0 \text{ in } D, \quad \int_D \phi(x) dx = 1.
\]

Noting that under the assumptions of Theorem 2.1, the solutions of (1.5) will remain positive, thus we can consider \( (u(t), \phi) \) due to the fact that \( (u(t), \phi) > 0 \). Denote \( \hat{u}(t) := (u(t), \phi) \). By
applying Itô's formula to $\hat{u}^2(t)$ and making use of (3.3), we get
\begin{align*}
\hat{u}^2(t) &= (u_0, \phi)^2 - 2\lambda_1 \int_0^t \hat{u}^2(s)ds + 2\lambda \int_0^t \int_D \hat{u}(s)\sigma(u(s,x))\phi(x)dxdB_s \\
&\quad + \lambda^2 \int_0^t \int_D \sigma^2(u(s,x))\phi^2(x)dxds \\
&\geq (u_0, \phi)^2 - 2\lambda_1 \int_0^t \hat{u}^2(s)ds + 2\lambda \int_0^t \int_D \hat{u}(s)\sigma(u(s,x))\phi(x)dxdB_s \\
&\quad + \lambda^2 I_{\sigma}^2 \int_0^t \int_D u^2(s,x)\phi^2(x)dxds \\
&\geq (u_0, \phi)^2 - 2\lambda_1 \int_0^t \hat{u}^2(s)ds + 2\lambda \int_0^t \int_D \hat{u}(s)\sigma(u(s,x))\phi(x)dxdB_s \\
&\quad + \lambda^2 I_{\sigma}^2 \int_0^t \hat{u}^2(s)ds.
\end{align*}
(3.4)

Taking mean norm then yields that
\begin{align*}
\mathbb{E}\hat{u}^2(t) &\geq \mathbb{E}(u_0, \phi)^2 - 2\lambda_1 \int_0^t \mathbb{E}\hat{u}^2(s)ds + \lambda^2 I_{\sigma}^2 \int_0^t \mathbb{E}\hat{u}^2(s)ds.
\end{align*}
(3.5)

By the comparison principle, we know that
\begin{align*}
\mathbb{E}\hat{u}^2(t) \geq \mathbb{E}(u_0, \phi)^2 e^{(\lambda^2 I_{\sigma}^2 - 2\lambda_1)t}.
\end{align*}

Due to
\begin{align*}
\hat{u}^2(t) = (u, \phi)^2 \leq \|\phi\|_{L^\infty} \|u\|_{L^2}^2,
\end{align*}
we have $\bar{u}(t) \geq 2$. So we have $e = 2$. \hfill \Box

**Outline of the proof of Theorem 2.2.** Similar to the proof of Theorem 2.1, equation (2.1) has a unique positive energy solution.

From (3.1), we have
\begin{align*}
\|u(t)\|_{L^2}^2 &= \|u_0\|_{L^2}^2 + 2 \int_0^t \int_D (\Delta u(s, x), u(s, x))dxds + 2\lambda \int_0^t \int_D u(s, x)\sigma(u(s, x))w(xds) \\
&\quad + \lambda^2 \int_0^t \int_D q(x, x)\sigma^2(u(s, x))dxds \\
&\leq \|u_0\|_{L^2}^2 + 2\lambda \int_0^t \int_D u(s, x)\sigma(u(s, x))w(xds) + q_1 \lambda^2 L_\sigma \int_0^t \int_D u^2(s, x)dxds.
\end{align*}

Then taking expectation on both sides and using Grönwall’s inequality, we have $\bar{e}(t) \leq 2$.

Similar to the proof of Theorem 2.1 we have further
\begin{align*}
\hat{u}^2(t) &= (u_0, \phi)^2 - 2\lambda_1 \int_0^t \hat{u}^2(s)ds + 2\lambda \int_0^t \int_D \hat{u}(s)\sigma(u(s,x))\phi(x)w(dx, ds) \\
&\quad + \lambda^2 \int_0^t \int_D \sigma(u(s,x))\phi(x)q(x, y)\sigma(u(s,y))\phi(y)dxdyds \\
&\geq (u_0, \phi)^2 - 2\lambda_1 \int_0^t \hat{u}^2(s)ds + 2\lambda \int_0^t \int_D \hat{u}(s)\sigma(u(s,x))\phi(x)w(dx, ds) \\
&\quad + \lambda^2 q_0 \int_0^t \int_D \sigma(u(s,x))\phi(x)\sigma(u(s,y))\phi(y)dxdyds \\
&\geq (u_0, \phi)^2 - 2\lambda_1 \int_0^t \hat{u}^2(s)ds + 2\lambda \int_0^t \int_D \hat{u}(s)\sigma(u(s,x))\phi(x)w(dx, ds) \\
&\quad + \lambda^2 q_0 I_{\sigma}^2 \int_0^t \hat{u}^2(s)ds.
\end{align*}
(3.6)
which implies that \( g(t) \geq 2 \). So we have \( g = 2 \). □

**Remark 3.1**  
1. We have considered the problem with higher space dimensions as we study the equations perturbed by a noise white in time and colored in the space variable. While in papers [14, 15, 11, 18], the authors only considered one space dimension due to their equations are driven by space time white noise.

2. The Laplace operator \( \Delta \) can be substituted by the divergent operator \( A \).

### 4 A special case and the noise excitability for nonlocal operators

In this section, we consider the following problem

\[
\begin{aligned}
    du(t, x) &= \Delta u(t, x)dt + \lambda u(t, x)dB_t, \quad x \in D, t > 0, \\
    u|_{\partial D} &= 0, \quad t > 0, \\
    u(0, x) &= u_0(x), \quad x \in D,
\end{aligned}
\]

where \( D \subset \mathbb{R}^n \) \((n \geq 1)\), \( B_t \) is a standard one-dimensional Brownian motion on a stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\).

We first give a equivalent equation to (4.1).

**Lemma 4.1** Let \( u \) be a weak solution of (4.1). Then the function \( v \) defined via

\[
v(t, x) := e^{-\lambda B_t}u(t, x), \quad t > 0, \quad x \in D
\]

solves the following deterministic equations for random field \( v(t, x) \)

\[
\begin{aligned}
    \frac{\partial}{\partial t}v(t, x) &= \Delta v(t, x) - \frac{\lambda^2}{2}v(t, x), \quad x \in D, t > 0, \\
    v|_{\partial D} &= 0, \quad t > 0, \\
    v(0, x) &= u_0(x), \quad x \in D.
\end{aligned}
\]

The proof of this lemma is standard, see e.g. the proof of Proposition 1.1 of [7]. We therefore omit it here.

**Theorem 4.1** Let \( u \) be a weak solution of (4.1) with deterministic initial data \( u_0 \) satisfying

\[
c_1 \leq u_0(x) \leq c_2, \quad \forall x \in D,
\]

where \( c_i, i = 1, 2, \) are positive constants. Then we have, for \( p > 0 \),

\[
2 \leq \liminf_{\lambda \to \infty} \frac{\log \log \mathcal{E}_t(\lambda)}{\log \lambda} \leq \limsup_{\lambda \to \infty} \frac{\log \log \mathcal{E}_t(\lambda)}{\log \lambda} \leq 2,
\]

where \( \mathcal{E}_t(\lambda) := \mathbb{E} \left( \|u(t)\|_{L^p(D)}^p \right)^{1/p} \).

**Proof.** It follows from Lemma 4.1 that the solutions of (4.1) can be expressed as

\[
u(t, x) = e^{\lambda B_t}v(t, x).
\]

It follows from the classical parabolic theory that the solutions \( v \) of (4.2) can be written as

\[
v(t, x) = e^{\frac{\lambda^2 t}{2}}(e^{t\Delta}u_0)(x) = e^{\frac{\lambda^2 t}{2}} \int_D p_D(t, x-y)u_0(y)dy, \quad a.s.,
\]
where \( p_D(t, x) \) is the kernel function of the Dirichlet Laplacian \( \Delta \) on \( D \). By using (4.3), we have
\[
\hat{c}_1 e^{\frac{x^2}{2}t} \leq v(t, x) \leq \hat{c}_2 e^{\frac{x^2}{2}t}, \quad \text{a.s.,}
\]
which implies that
\[
\mathbb{E} \left[ \|u(t)\|_p^p(D) \right] = \mathbb{E} \left[ \|v(t)\|e^{\lambda B_t} \|_p(D) \right] \\
\geq \hat{c}_1 e^{\frac{x^2}{2}pt} e^{\lambda B_t} \\
= \hat{c}_1 e^{\frac{x^2}{2}pt} e^{\lambda B_t}
\]
and
\[
\mathbb{E} \left[ \|u(t)\|_p^p(D) \right] = \mathbb{E} \left[ \|v(t)\|e^{\lambda B_t} \|_p(D) \right] \\
\leq \hat{c}_2 e^{\frac{x^2}{2}pt} e^{\lambda B_t} \\
= \hat{c}_2 e^{\frac{x^2}{2}pt} e^{\lambda B_t}.
\]
Combining the above two inequalities, we get the desired result. The proof is thus complete. □

Next, we will consider the following initial value problem for nonlocal equations
\[
\begin{aligned}
\frac{\partial}{\partial t} u(t, x) &= -(-\Delta)^{\frac{\alpha}{2}} u(t, x) + \lambda \sigma(u(t, x)) \dot{\omega}(t, x), \quad x \in \mathbb{R}, \ t > 0 \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R},
\end{aligned}
\tag{4.5}
\]
where \( \alpha \in (1, 2], (-\Delta)^{\frac{\alpha}{2}} \) is the \( L^2 \)-generator of a symmetric \( \alpha \)-stable process \( X_t \) such that \( \mathbb{E} \exp(i \xi \cdot X_t) = \exp(-t|\xi|^{\alpha}) \); \( \{\dot{\omega}(t, x)\}_{t \geq 0, x \in \mathbb{R}} \) denotes the space-time white noise. In paper \([13]\), the authors considered the equation (1.3) on bounded domain. Here we would like to generalise the result to the situation of whole spatial space.

When \( \sigma \) satisfies global Lipschitz continuous condition, it is routine to show that (4.5) has a unique global mild solution, see e.g. the monographs \([2, 21, 17]\), as well as Dalang \([5]\) and Foondun-Khosnevisan \([12]\). It is easy to see that the mild solution of (4.3) fulfills the following mild formulation
\[
u(t, x) = \int_{\mathbb{R}} p(t, x - y)u_0(y)dy + \lambda \int_0^t \int_{\mathbb{R}} p(t - s, x - y)\sigma(u(s, y))w(dsdy),
\tag{4.6}
\]
where \( p(t, x) \) is the transition density function of the symmetric \( \alpha \)-stable process \( X_t \).

Before we state our main results, we recall some properties of the kernel function (transition density function) \( p(t, x) \).

**Proposition 4.1** (22) The transition density \( p(t, \cdot) \) of a strictly \( \alpha \)-stable process satisfies
(i) \( p(st, x) = t^{-1/\alpha} p(s, t^{-1/\alpha}x) \);
(ii) For \( t \) large enough such that \( p(t, 0) \leq 1 \) and \( a > 2 \), we have
\[
p(t, (x - y)/a) \geq p(t, x)p(t, y), \quad \text{for all} \ x \in \mathbb{R};
\]
(iii) \( p(t, x) \approx t^{-1/\alpha} \leq \frac{a}{|x|^{1+\alpha}}. \)

By using Proposition 4.1 it is easy to verify that
\[
\int_{\mathbb{R}} p(t, x)p(s, x)dx = p(t + s, 0). \tag{4.7}
\]
In particular, \( \|p(t, \cdot)\|_p^2_{L^2(\mathbb{R})} = p(2t, 0) \).

Let
\[
\delta_1(\lambda) = \sqrt{\mathbb{E} \left[ \|u_t\|_p^2_{L^2(\mathbb{R})} \right]},
\]
where
Theorem 4.2 Assume that (1.6) holds and
\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} p(t, x - y)u_0(y)dy \right)^2 dx \leq \mu,
\]
where \(\mu\) is a positive constant, then the noise excitation index of solution to (1.5) with initial data \(u_0(x) \geq 0(\neq 0)\) is \(2\alpha/(\alpha - 1)\).

Remark 4.1 We remark the condition (4.8) does indeed make sense. Let us give an example. Taking \(u_0(x) = \delta_{x_0}(x)\) for a fixed \(x_0 \in \mathbb{R}\), we have
\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} p(t, x - y)u_0(y)dy \right)^2 dx = \int_{\mathbb{R}} p^2(t, x - x_0)dx = p(2t, 0) =: \mu < \infty.
\]

Another example is that \(u_0\) can be taken a function with compact support, such as the indicator function of a closed interval \([-l, l]\) for arbitrarily fixed \(l > 0\), that is, \(u_0(x) = 1_{[-l,l]}(x)\). Namely, \(u_0(x) = 1\) for \(x \in [-l, l]\) and \(u_0(x) = 0\) for \(x \not\in [-l, l]\). For a fixed \(t > 0\), it follows from Proposition 4.1 that there is an \(x_0 > 0\) such that
\[
p(t, x - y) \leq \frac{C}{|x - l|^{1+\alpha}} \quad \text{for } x > x_0, y \in [-l, l];
p(t, x - y) \leq \frac{C}{|x + l|^{1+\alpha}} \quad \text{for } x < -x_0, y \in [-l, l],
\]
where \(C > 0\) is a constant. Direct calculations then show that
\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} p(t, x - y)u_0(y)dy \right)^2 dx = \int_{\mathbb{R}} \left( \int_{-l}^{l} p(t, x - y)dy \right)^2 dx \\
\leq \frac{C^2x_0}{t^{2/\alpha}} + 2\int_{0}^{\infty} \frac{C^2}{|x - l|^{2+2\alpha}}dx + 2\int_{-\infty}^{-x_0} \frac{C^2}{|x + l|^{2+2\alpha}}dx \\
=: \mu(x_0, \alpha) < \infty.
\]

Lemma 4.2 ([18, Proposition 2.6]) Let \(T \leq \infty\) and \(\beta > 0\). Suppose that \(f(t)\) is a nonnegative, locally integrable function satisfying
\[
f(t) \geq c_1 + k \int_{0}^{t} (t - s)^{\beta - 1}f(s)ds, \quad \text{for all } 0 \leq t \leq T,
\]
where \(c_1\) is some positive constant. Then for any \(t \in (0, T]\), we have the following
\[
\liminf_{k \to \infty} \frac{\log \log f(t)}{\log k} \geq \frac{1}{\beta}.
\]

When the inequality ([19]) is reversed with the second inequality in ([13]), we have
\[
\limsup_{k \to \infty} \frac{\log \log f(t)}{\log k} \leq \frac{1}{\beta}.
\]

Proof of Theorem 4.1 By using mild formulation and Itô isometry, we have the following
\[
E|u(t, x)|^2 = \int_{\mathbb{R}} p(t, x - y)u_0(y)dy + \lambda^2 \int_{0}^{t} \int_{\mathbb{R}} p^2(t - s, x - y)E|\sigma(u(s, y))|^2dyds. \quad (4.10)
\]
Integrating over \(\mathbb{R}\) and by Fubini lemma, we get
\[
E\|u(t)\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} p(t, x - y)u_0(y)dy \right)^2 dx \\
+ \lambda^2 \int_{0}^{t} \int_{\mathbb{R}} \|\sigma(u(s, y))\|^2 \left( \int_{\mathbb{R}} p^2(t - s, x - y)dx \right) dyds. \quad (4.11)
\]
By using (1.6), (4.7) and (4.8), we have
\[
E\|u(t)\|_{L^2(\mathbb{R})}^2 \leq \mu + \lambda^2 \int_0^t p(2(t-s),0) \int_{\mathbb{R}} E|\sigma(u(s,y))|^2 dy ds
\leq \mu + \lambda^2 L_2^2 \int_0^t \frac{C}{(t-s)^{1/\alpha}} E\|u(s)\|_{L^2(\mathbb{R})}^2 ds.
\] (4.12)

By Lemma 4.1, (4.12) then implies that
\[
\limsup_{\lambda \to \infty} \frac{\log \log E\|u(t)\|_{L^2(\mathbb{R})}^2}{\log \lambda} \leq \frac{2\alpha}{\alpha - 1}.
\] (4.13)

That is, \( \overline{g}(t) \leq 2\alpha/(\alpha - 1) \).

Next, we prove \( \underline{g}(t) \geq 2\alpha/(\alpha - 1) \). First, it follows from (4.10) that
\[
E|u(t,x)|^2 \geq \lambda^2 \int_0^t \int_{\mathbb{R}} p^2(t-s,x-y)E|\sigma(u(s,y))|^2 dy ds.
\]
Integrating over \( \mathbb{R} \) with utilising Fubini lemma, we get
\[
E\|u(t)\|_{L^2(\mathbb{R})}^2 \geq \lambda^2 \int_0^t \int_{\mathbb{R}} E|\sigma(u(s,y))|^2 \left( \int_{\mathbb{R}} p^2(t-s,x-y) dx \right) dy ds
\geq \lambda^2 t^2 \sigma^2 \int_0^t \frac{C}{(t-s)^{1/\alpha}} E\|u(s)\|_{L^2(\mathbb{R})}^2 ds.
\] (4.14)

Again, by Lemma 4.1, (4.14) then implies that
\[
\liminf_{\lambda \to \infty} \frac{\log \log E\|u(t)\|_{L^2(\mathbb{R})}^2}{\log \lambda} \geq \frac{2\alpha}{\alpha - 1}.
\] (4.15)

That is, \( \underline{g}(t) \geq 2\alpha/(\alpha - 1) \).

Combining (4.13) and (4.15), we thus complete the proof of Theorem 4.1. \( \square \)

**Remark 4.2** When \( \alpha = 2 \), (4.15) was obtained in Khoshnevisan-Kim [14].

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