Local field-interaction approach to the Dirac monopole

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Abstract

We introduce the local field interaction approach to Dirac magnetic monopoles. Our analysis reveals two physically different types of a monopole. The first type is free of singularity, and the field angular momentum plays an essential role in the interaction. The second type is described as an endpoint of an invisible semi-infinite flux tube (a Dirac string). Notably, a different phase factor \((-1)^n\) exists between the two types where \(n\) is the quantum number of the field angular momentum. Our study provides a realistic description of the two types of monopoles. Various aspects of these monopoles are discussed, including the Maxwell dual of the Dirac string, exchange symmetry, and an analogy to the Coriolis interaction.

1. Introduction

Dirac [1] originally showed that the existence of a magnetic monopole is consistent with quantum theory if the magnetic charge \(g\) satisfies the quantization condition

\[
eg g/(2\hbar) = n,
\]

where \(e\) is the electric charge of another particle and \(n\) is an integer. The quantum mechanical description of a charge under the vector potential \(A\) generated by the monopole inevitably includes a singularity (or discontinuity) in \(A\) because \(\nabla \cdot (\nabla \times A) = 0\). Equation (1) is derived using the single-valuedness of the wave function. Recently, by adopting a local field interaction (LFI) approach, we developed a quantum theory of electromagnetic interaction that does not involve \(A\) [2, 3]. Classical electrodynamics and the topological Aharonov–Bohm effect are successfully reproduced in the LFI theory. In addition, the remarkable consequences of the LFI theory concerning the locality [4, 5] and the gauge symmetry [6] were also revealed. In the LFI approach, the role of the potential is replaced by the field momentum produced by the charge and the external magnetic field \(B\). This approach has generated a revival of the debates on the locality of the Aharonov–Bohm effect (see e.g., [7, 8]).

In this work, we apply the LFI theory to the problem of a charge interacting with a magnetic monopole and show that two types of monopole, namely Type I and Type II, can be derived (see figure 1). It should be noted that the difference is not merely a mathematical artifact but a physical reality. The LFI approach involves replacing the vector potential by the field momentum. In the case of a charge–monopole pair, however, the field momentum vanishes and thus fails to describe the interaction between the two particles. We show that the interaction is mediated by the field angular momentum produced by two particles (electric charge and magnetic monopole). The field angular momentum plays the same role as the spin and is essential for constructing the singularity-free description of the monopole ("Type I"). Alternatively, it is also possible to describe the monopole with a Dirac string in the LFI approach ("Type II"). In monopoles of the latter type, the interaction between the two particles is produced by the field momentum confined inside the string. The Type-II monopole is equivalent to the original description of the Dirac monopole, which has a singular string. The two different types of monopole reveal duality in the classical and quantum equations of motion. Notably, the two types are not completely equivalent: an additional phase factor \((-1)^n\) appears in the quantum state of Type I. The implication thereof is discussed in detail.

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Note that we do not consider singularity-free monopoles in the context of non-Abelian gauge group and spontaneous symmetry breaking \cite{9, 10}. Our discussion is restricted to the original Dirac monopole.

The paper is organized as follows. In section 2, we derive the Lagrangian for a charge-monopole pair using the LFI approach. We point out that two different types of monopole are possible, depending on whether a Dirac string is present. Section 3 presents the evaluation of the quantum mechanical phase shifts for each type. The field angular momentum and the singularity play major roles in the different phase factors of the two types. In section 4, the corresponding Hamiltonians are derived for each of the two types and their duality is analyzed. Section 5 discusses a few intriguing aspects derived from our approach. A notable property of the Type-II monopole is found from the Maxwell duality: the electric charge can also be described as an endpoint of singularity. The exchange symmetry of the charge-monopole composite particles is derived for the two types. In addition, the formal equivalence between the Type-I monopole and the Coriolis interaction is demonstrated. Section 6 concludes the paper.

2. Field-interaction Lagrangian of a charge-monopole pair

In the standard potential-based approach, the dynamics of a charge ($e$) under an external magnetic field ($\mathbf{B}$) is described by the Lagrangian

$$\mathcal{L}_A = \mathcal{L}_0 + \frac{e}{c} \mathbf{\dot{r}} \cdot \mathbf{A},$$

where $\mathcal{L}_0 = -mc\sqrt{c^2 - \mathbf{\dot{r}} \cdot \mathbf{r}}$ is the kinetic part of the charged particle with mass $m$. The vector potential $\mathbf{A}$ in the interaction term is replaced by the field momentum ($\mathbf{\Pi}$) in the LFI theory, and the Lagrangian of the system is given by \cite{2, 3}

$$\mathcal{L}' = \mathcal{L}_0 + \mathbf{\dot{r}} \cdot \mathbf{\Pi}$$

where the field momentum,

$$\mathbf{\Pi} = \frac{1}{4\pi c} \int \mathbf{E}_e \times \mathbf{B} \, d^3\mathbf{x},$$

is generated by the overlap between the electric field ($\mathbf{E}_e$) of charge $e$ and the external $\mathbf{B}$. In the absence of the Dirac-string-type singularity ("Type I" in figure 1(a)), $\mathbf{\Pi} = 0$ for the charge-monopole pair (see e.g., section 6.12 of [11]), in which case the Lagrangian (3) fails to describe the charge-monopole interaction.

The appropriate LFI Lagrangian for the charge-monopole pair is achieved by including the rotational degree of freedom. For the angular velocity $\mathbf{\dot{\psi}}$ of the charge, we introduce the Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \frac{\mathbf{\dot{\psi}} \cdot \mathbf{S}}{2},$$

Figure 1. Electric charge $e$ and a magnetic monopole $g$ (a) without singularity ("Type I"), and (b) with a Dirac string attached to the latter ("Type II").
where \( \mathbf{S} \) is the electromagnetic field angular momentum produced by \( e \) and \( g \), see e.g., [11]:

\[
\mathbf{S} = -\frac{e\mathbf{\dot{r}}}{c}.
\]

(4b)

Here, \( \mathbf{S} \) plays the same role as the particle spin, as is discussed later. A microscopic derivation of this Lagrangian is produced by the monopole. The classical equation of motion corresponds to the Lagrange equation for the angle variable \( \psi \),

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0,
\]

(5)

leads to the equation of motion

\[
\frac{d}{dt} (\mathbf{L} + \mathbf{S}) = 0,
\]

(6a)

where \( \mathbf{L} = \frac{\partial \mathcal{L}_0}{\partial \dot{\psi}} \) is the kinetic angular momentum of the charge. This indicates the conservation of the total angular momentum. It can be rewritten in a more familiar form involving the Lorentz force, as

\[
\frac{d}{dt} \mathbf{p} = \frac{e}{c} \mathbf{\dot{r}} \times \mathbf{B},
\]

(6b)

where \( \mathbf{p} \equiv \frac{\partial \mathcal{L}_0}{\partial \dot{\mathbf{r}}} \) is the kinetic momentum of the charge, and \( \mathbf{B} = g\mathbf{\hat{r}}/r^2 \) is the magnetic field generated by the monopole.

The above derivation of the equation of motion (6) demonstrates the validity of the Lagrangian (4) in describing the charge interacting with the magnetic monopole. Now, we present a microscopic derivation of the Lagrangian (4) based on the LFI approach [2, 3]. For simplicity, we do not consider the motion of the monopole at this stage. (The Lagrangian including the motion of the monopole is discussed in section VI-A). In the LFI approach, a charged particle subject to an external electromagnetic field is described by the Lagrangian:

\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{em}},
\]

(7a)

where

\[
\mathcal{L}_{\text{em}} = \frac{1}{8\pi} \int F_{\mu\nu} F^{\mu\nu} d^3x,
\]

(7b)

represents the interaction between the field of charge and the external field (denoted by the field tensors \( F_{\mu\nu}^{(\text{em})} \) and \( F^{\mu\nu} \), respectively). In our case, \( F^{\mu\nu} \) is produced by the monopole.

For a moving charged particle with velocity \( \mathbf{\dot{r}} \), the interaction part of the Lagrangian (7b) can be expressed as

\[
\mathcal{L}_{\text{em}} = \mathbf{r} \cdot \mathbf{\Pi} [2, 3].
\]

However, as mentioned above, \( \mathbf{\Pi} = 0 \) in our system. That is, the translational motion does not produce a coupling with the monopole. Instead, a nonvanishing field interaction is generated in the rotational degree of freedom. The rotation of charge, with its angular velocity \( \dot{\psi} \), produces a magnetic field \( \mathbf{B}_e(x) \) at a position \( x \) (see figure 2) in the form

\[
\mathbf{B}_e(x) = \frac{1}{c} \dot{\psi} (\mathbf{\hat{r}} \times x) \times \mathbf{E}_e(x),
\]

where \( \mathbf{E}_e(x) \) is the electric field of the charge. This expression of \( \mathbf{B}_e \) is obtained by evaluating the magnetic field in a uniformly rotating frame with angular velocity \( \dot{\psi} \). The nonvanishing term in the interaction Lagrangian (7b) originates from the \( \mathbf{B}_e \cdot \mathbf{B} \) coupling, and we obtain

\[
\mathcal{L}_{\text{em}} = \dot{\psi} \mathbf{\hat{r}} \cdot \mathbf{S},
\]

(8a)

where

\[
\mathbf{S} = \frac{1}{4\pi c} \int \mathbf{x} \times (\mathbf{E}_e \times \mathbf{B}) d^3\mathbf{x}
\]

(8b)

corresponds to the field angular momentum. \( \mathbf{S} \) is reduced to equation (4b) for the magnetic field of the monopole. The Lagrangian (4) derived above does not include any singularity and is classified as ‘Type I’.

It is also possible to describe the monopole attached to a Dirac string (‘Type II’) in the LFI approach with the Lagrangian (3). The string is an invisible tube of the magnetic flux of \( 4\pi g \) terminated at the origin (figure 1(b)). For the string located at \( \theta = \pi \), an evaluation of equation (3b) shows that the field momentum

\[
\mathbf{\Pi}(r) = \frac{eg}{c} \left[ 1 - \frac{\cos \theta}{r \sin \theta} \right] \frac{\mathbf{\hat{r}}}{c}.
\]

(9)

is generated inside the string. The classical equation of motion (6b) is derived from the Lagrangian (3) with \( \mathbf{\Pi} \) of equation (9), demonstrating the duality of the two different types represented by the Lagrangians (3) and (4),
respectively. The Type-II monopole described by the Lagrangian (3) is equivalent to the original problem of the Dirac monopole with singular vector potential.

The location of the Dirac string gives rise to a problem of arbitrariness. The value of the field momentum depends on the position of the string. However, two field momenta $\Pi'$ and $\Pi$ with different locations of the string are related by the gauge transformation $\Pi' = \Pi + \nabla \chi$ with a single-valued scalar function $\chi$. This property of the gauge transformation is also present in the vector potential of the ordinary formulation (see e.g., section 6.12 of [11]). In any case, the physical observables are independent of the position of the string.

3. Quantum theory of monopoles with two types of the field-interaction Lagrangian

Here, we apply the Lagrangians derived in section II (equations (3) and (4)) to quantum theory. We consider an arbitrary closed loop in the path of the charge (figure 3). In Type I (figure 3(a)), the phase shift ($\varphi$) generated by the monopole is evaluated from the interaction term of the Lagrangian (4):

$$\varphi = \frac{1}{\hbar} \oint \hat{\psi} \cdot S dt = \frac{1}{\hbar} \oint S \cdot d\hat{\psi}. \quad (10)$$
For an arbitrary loop with a solid angle $\Omega$, we find
\[ \varphi = \frac{S}{\hbar} (\Omega - 2\pi). \] (11)

The phase shift can also be evaluated from the opposite side of the solid angle, $\Omega = -(4\pi - \Omega)$:
\[ \tilde{\varphi} = S(\Omega - 2\pi) / \hbar. \] The equivalence of the two phases, with modulo $2\pi (\varphi = \varphi + 2n\pi$ with integer $n$), imposes the quantization of the field angular momentum
\[ S = eg / e = n\hbar / 2. \] (12)

This is exactly the Dirac quantization for the electric and magnetic charges. With this quantization, the phase shift is given by
\[ \varphi = \frac{eg}{\hbar c} \Omega - n\pi. \] (13)

Notably, we find a different phase shift ($\varphi'$) from the Type-II Lagrangian (3),
\[ \varphi' = \frac{1}{\hbar} \oint \Pi \cdot dt = \frac{eg}{\hbar c} \Omega, \] (14)

which corresponds to the geometric phase, or the Aharonov–Bohm phase generated by the magnetic flux of the monopole. The phase shift $\varphi'$ can also be obtained from the usual potential-based Lagrangian (2). The additional phase shift $-n\pi$ in Type I (equation (14)) reflects the fermionic (bosonic) nature of the field angular momentum for odd (even) values of $n$ (equation (12)). For odd values of $n$, this gives rise to an additional phase factor of $-1$. This phase factor originates from the field angular momentum and is unrelated to the intrinsic spin of each particle. Note that this $(-1)^n$ in the phase factor can also be derived from the ‘spin approach’ adopted in [12], although it was not explicitly analyzed there.

The difference between the two phases, $\varphi$ (equation (14)) and $\varphi'$ (equation (14)), is also closely related to the absence or presence of a Dirac string. The Type-II monopole is not an isolated particle but emerges as an endpoint of the string. This implies that another monopole with opposite magnetic charge $-g$ exists somewhere at a large distance from the system. This gives rise to additional field angular momentum of $n\hbar / 2$ and the phase shift of $n\pi$, which cancels $-n\pi$ in equation (14). This cancellation cannot be avoided, even when the string stretches to infinity, and explains the difference between $\varphi$ (equation (14)) and $\varphi'$ (equation (14)).

Our analysis based on the LFI approach clearly suggests that there could be two different types of magnetic monopole: (i) ‘Type I’ without singularity, which can be described in terms of the interaction between the ‘spin’ of the charged particle and the field angular momentum (equation (4)); (ii) ‘Type II’ with a singular Dirac string where the motion of the charge couples to the field momentum localized in the string (equations (3) and (9)). Remarkably, this classification is not merely a mathematical construction for describing the same system. For odd multiples of the field angular momentum, the two types can be distinguished by the presence (absence) of the additional phase shift $n\pi$ in $\varphi$ ($\varphi'$) of Type I (Type II).

4. Hamiltonian and the duality of the two types of monopoles

4.1. Hamiltonian

Derivation of the Hamiltonian from the Lagrangian of the system via a Legendre transformation is straightforward. For Type I (equation (4)), four independent variables are necessary, $q_i = (r, \vec{v})$, composed of the distance from the origin ($r$) and the three-dimensional angle vector $\vec{v}$. The Hamiltonian is obtained from the relation $H = \sum_i \dot{q}_i p_i - L$, where $p_i = \partial L / \partial \dot{q}_i$ is the conjugate momentum. We find
\[ H = \sqrt{c^2 \left( p_r^2 + \frac{(J - S)\hat{e}^2}{r^2} \right) + m^2 c^4}, \] (15)

where $p_r = \partial L / \partial \dot{r}$ and $J = \partial L / \partial \dot{\vec{v}}$ denote the radial component of the canonical momentum and the angular momentum vector, respectively. When applied to quantum theory, $J = (\hbar / i) \partial / \partial \vec{v}$, and $S$ becomes a spinor satisfying the quantization condition of equation (12). In the nonrelativistic limit, the Hamiltonian is reduced to
\[ H = mc^2 + \frac{1}{2m} \left( \frac{p_r^2}{r^2} + \frac{(J - S)^2}{r^2} \right). \] (16)

In the present case, $S$ is parallel to $r$ and satisfies $(J - S)^2 = J^2 - S^2$. Under this condition, the Hamiltonian (16) is equivalent to that adopted in the spin approach to the monopole in [12]. The ‘spin Hamiltonian’ in [12] was introduced for consistency in the classical equation of motion. In contrast to this, we presented a microscopic derivation of the Hamiltonian ((15) and (16)) above. The Type-II Hamiltonian, denoted by $H'$, can also be obtained from the Lagrangian (3) as
\[ H' = \sqrt{c^2(p - \Pi)^2 + m^2c^4}, \]  
(17)

where \( p = \partial L/\partial \dot{r} \) is the canonical momentum. This Hamiltonian contains a singularity (Dirac string) in \( \Pi \) (see e.g., equation (9)).

### 4.2. Duality

The duality of the two types of monopoles is already apparent in the Lagrangian approach in the previous sections. The two Lagrangians representing each type (equations (4) and (3)) provide the same classical dynamics and the quantum phase shift for an arbitrary closed path except the additional \( -n\pi \) (equation (14)) in Type-I monopole. In the following, the duality of the two types is derived in a general way from the Hamiltonians \( H \) (equation (15)) and \( H' \) (equation (17)).

Let \( u_{i\rightarrow f} \equiv \langle \mathbf{r}_i, t_i | \mathbf{r}_f, t_f \rangle \) be the transition amplitude from an initial \( (\mathbf{r}_i, t_i) \) to the final \( (\mathbf{r}_f, t_f) \) spacetime locations for a Type-I system. In the Feynman path-integral representation, it reads

\[ u_{i\rightarrow f} = \int_{\mathcal{C}} e^{i\int \mathcal{L} dt} dt, \]
(18)

where \( \mathcal{L} \) is the Type-I Lagrangian (4). The same transition amplitude, namely, \( u'_{i\rightarrow f} \), can be defined for a Type-II system where \( \mathcal{L} \) in equation (18) is replaced by \( \mathcal{L}' \) of the Lagrangian (3).

Let \( u_i(u'_i) \) be the transition amplitude in the Type-I (Type-II) system for a closed path, that is, \( u_i = u_{i\rightarrow f} \) \( (u'_i = u'_{i\rightarrow f}) \) for \( \mathbf{r}_f = \mathbf{r}_i \) with one-loop rotation. We find

\[ u'_i = (-1)^n u_i, \]
(19)

from the relation between \( \varphi \) (equation (14)) and \( \varphi' \) (equation (14)). The transition amplitude of equation (18) is also expressed in the Hamiltonian representation as

\[ u_{i\rightarrow f} = \langle \mathbf{r}_f | e^{-iH(t_f - t_i)/\hbar} | \mathbf{r}_i \rangle \]

\[ = \sum_n \psi_i(\mathbf{r}_i) \psi^*_f(\mathbf{r}_f) e^{-iE_n(t_f - t_i)/\hbar}, \]
(20)

where \( E_n \) and \( \psi_n \) denote the eigenvalue and eigenfunction of \( H \), respectively. We find that the two types are related by a unitary transformation (U)

\[ \psi \rightarrow \psi' = U\psi \quad \text{along with} \quad H \rightarrow H' = UHU^\dagger, \]
(21)

and the transition amplitude in Type-II representation can be expressed as

\[ u'_{i\rightarrow f} = \sum_l U^\dagger(\mathbf{r}_i) U(\mathbf{r}_f) \psi_l(\mathbf{r}_i) \psi^*_l(\mathbf{r}_f) e^{-iE_l(t_f - t_i)/\hbar}, \]
(22)

The general condition of the unitary transformation \( U \) can be imposed from equation (19). By writing

\[ U(\mathbf{r}) = e^{-i\int \mathbf{a} \cdot d\mathbf{r}}, \]
(23a)

we obtain

\[ \oint \mathbf{a} \cdot d\mathbf{r} = n\pi. \]
(23b)

from equation (19). For example, the unitary operation

\[ U = e^{-i\theta \hat{\mathbf{n}}/\hbar} \]
(23c)

transforms \( H \) (Type I) into \( H' \) (Type II) (see also [12]).

The analysis in sections III and IV clearly indicates that both types of monopoles are consistent with quantum theory under the same Dirac quantization condition. We cannot predict the type of the real monopoles. In principle, the type of the real monopole, once it is discovered, can be distinguished (for odd \( n \) only) by measuring the phase factor. On the other hand, for effective monopoles, their types can be classified in our scheme. It is found that a spin (pseudospin) system behaves similar to a charged particle under the magnetic field of a monopole, as manifested in Berry’s phase [13] (see [14, 15] for a review). This case belongs to Type I, where its spin (pseudospin) is represented by the spinor in the Hamiltonian (equation (15)). The Type-II system includes effective monopoles produced by analogues of the Dirac string: an endpoint of a long solenoid or others of a similar nature. Examples can be found in various systems such as spin ice [16–18] and synthetic magnetic field [19], etc.
5. Discussion

5.1. Various configurations of the singularity
As described above, a notable difference exists between the two types of monopole description: the absence (presence) of a singularity in the form of a Dirac string in Type I (Type II). In either case, the singularity is unobservable. Nevertheless, we can gain insight into this problem by considering the dynamics of both particles on an equal footing and exchanging the role of electric (e) and magnetic (g) charges, especially in Type II.

The symmetry of the electrodynamics under Maxwell duality transformation (see e.g., [20]) leads to an intriguing consequence on the Type-II monopole as described below. In the Maxwell dual (Figure 4(a)) of the original Type-II system (Figure 4(b)), the ‘Dirac string’ is attached to the electric charge. In other words, electric charge emerges as an endpoint of an invisible string of the electric flux. The field momentum \( \Pi \) in the dual configuration is generated by the overlap between the electric string and the magnetic field generated by the monopole. In both cases of figures 4(a), (b), the Lagrangian of the system is expressed in the form

\[
\mathcal{L}' = \mathcal{L}_0 + (\dot{r} - \dot{r}_g) \cdot \Pi,
\]

where \( \mathcal{L}_0 \) includes the kinetic part of both the particles, and \( \dot{r}_g \) is the velocity of the monopole g. The field momentum depends on the location of the strings. For the particular case in which the string is located at \( \theta = \pi \), the momentum is expressed by equation (9) in both the original (Figure 4(a)) and dual configurations (Figure 4(b)). Moreover, it is also possible that both particles are the endpoints of the corresponding strings (Figure 4(c)). In this case, the Lagrangian is given by

\[
\mathcal{L}' = \mathcal{L}_0 + \dot{r} \cdot \Pi_r - \dot{r}_g \cdot \Pi_e,
\]

involving two different field momenta, \( \Pi_r \) and \( \Pi_e \), which are localized inside the magnetic and electric Dirac strings, respectively. Irrespective of the configuration, the physics of all cases (figures 4(a), (b), (c)) is equivalent, leading to the same classical equation of motion and the quantum phase shift (equation (14)).

The problem is much simpler in Type I because the Maxwell dual is identical to the original system itself. The dynamics of the two particles can be described on an equal footing by generalizing the Lagrangian (4) as

\[
\mathcal{L} = \mathcal{L}_0 + (\dot{\psi} - \dot{\psi}_e) \cdot S,
\]

where \( \dot{\psi}_e \) represents the angular velocity of the monopole. Apparently, the system is identical upon the exchange of e and g in the absence of a Dirac string.

5.2. Exchange symmetry of charge-monopole composites
Let us consider two identical charge-monopole composite particles with each one located at \( r_1 \) and \( r_2 \). The wave function of the system may be written as \( \Psi = \Psi(r, \eta_1, \eta_2) \), where \( r = r_1 - r_2 \) and \( \eta_i \) (\( i = 1, 2 \)) represents the internal state of each particle. For the exchange of the two particles (represented by the operator \( P \)), we show that

\[
P \Psi(r, \eta_1, \eta_2) \equiv \Psi(-r, \eta_2, \eta_1) = (-1)^{2s+n} \Psi(r, \eta_1, \eta_2)
\]

where \( s \) and \( n \) are the intrinsic spin and the Dirac quantum number (equation (1)) of the composite, respectively. We find that this result is independent of the types of the monopoles. Interestingly, we arrive at the result (equation (27)) in different ways for Type-I and Type-II monopoles as is shown below.
In general, the two Lagrangians \( \mathcal{L} \) and \( \mathcal{L}_0 \) with the relation
\[
\mathcal{L} = \mathcal{L}_0 + \frac{d\Lambda}{dt}
\]  
(28a)

exhibit a particular symmetry. With the aid of the Feynman path-integral formulation, the transition amplitudes \( u \) and \( u_0 \) (associated with the Lagrangians \( \mathcal{L} \) and \( \mathcal{L}_0 \), respectively) have the relation
\[
u = u_0 e^{i\Delta \alpha},
\]  
(28b)

where the phase shift
\[
\Delta \alpha = \frac{1}{\hbar} \int \mathcal{A} \cdot d\Lambda
\]  
(28c)
is independent of the path taken in the configuration space. The indices \( i \) and \( f \) represent the initial and final points in the configuration space of the system, respectively. Equation (28) is valid in general and is not limited to the particular problem of the monopoles considered here. A typical example is the gauge symmetry where the gauge field \( A_{\mu} \) is given by \( A_{\mu} = \partial_{\mu} \Lambda \). Equation (28) is also useful for deriving the exchange phase factor of the charge-monopole composites.

The Lagrangian of two identical charge-monopole composites can be written as
\[
\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{\text{int}},
\]  
(29)

For Type I, \( \mathcal{L}_i \) \( (i=1,2) \) represents each composite described by the Lagrangian of equation (4). The inter-cluster interaction is given by
\[
\mathcal{L}_{\text{int}} = (\hat{\psi}_1 - \hat{\psi}_2) \cdot [\mathbf{S}(\mathbf{r}) - \mathbf{S}(-\mathbf{r})],
\]  
(30)

where \( \hat{\psi}_i \) \( (i=1,2) \) is the angular velocity of each particle, and \( \mathbf{S}(\mathbf{r}) = -(e\gamma/c) \mathbf{r} \) is the field angular momentum produced by the inter-cluster interaction. For an exchange of two particles, \( (\hat{\psi}_1 - \hat{\psi}_2) \cdot \mathbf{r} = 0 \), and thus \( \mathcal{L}_{\text{int}} = 0 \): The system can be regarded as two independent composite particles. Each particle contains the intrinsic spin \( s \) and field angular momentum of \( n/2 \) in units of \( \hbar \). The field angular momentum is also represented by a spinor (see equations (15) and (16)) in the Type-I charge-monopole cluster. Therefore, the wave function has the symmetry of equation (27).

Derivation of the exchange symmetry for the Type-II system is equivalent to that presented in [21] analyzed with the vector potential \( \mathbf{A} \). The Lagrangian of two identical composite particles is also given in the form of equation (29). In our formulation, each term \( \mathcal{L}_i \) \( (i=1,2) \) for the composite particle involving the Type-II monopole is described by the field-interaction Lagrangian (3). We find that the inter-cluster interaction is
\[
\mathcal{L}_{\text{int}} = \mathbf{r} \cdot [\mathbf{\Pi}(\mathbf{r}) - \mathbf{\Pi}(-\mathbf{r})],
\]  
(31)

where \( \mathbf{\Pi}(\mathbf{r}) \) \( (\mathbf{\Pi}(-\mathbf{r})) \) is the field momentum produced by the charge at \( \mathbf{r} \) \( (\mathbf{r}) \) interacting with the Type-II monopole at \( \mathbf{r} \). (Note that the two field momenta can be set identical without affecting the result). It is straightforward to show that \( \mathbf{\Pi}(\mathbf{r}) - \mathbf{\Pi}(-\mathbf{r}) \) is curl-free: \( \mathbf{\Pi}(\mathbf{r}) - \mathbf{\Pi}(-\mathbf{r}) = \nabla \Lambda \), and thus \( \mathcal{L}_{\text{int}} = d\Lambda/dt \). Therefore, the wave function of the system is given in the form
\[
\Psi(\mathbf{r}, \eta_1, \eta_2) = e^{i\eta_0} \Psi_0(\mathbf{r}, \eta_1, \eta_2).
\]  
(32)

Unlike the Type-I system, the wave function \( \Psi_0 \) associated with \( \mathcal{L}_1 + \mathcal{L}_2 \) does not include the field angular momentum in its internal state, and satisfies
\[
P \Psi_0(\mathbf{r}, \eta_1, \eta_2) = (-1)^2 \Psi_0(\mathbf{r}, \eta_1, \eta_2).
\]  
(33)

Applying equation (28), we obtain the interaction-induced phase shift,
\[
\Delta \alpha = \frac{1}{\hbar} \int_C [\mathbf{\Pi}(\mathbf{r}) - \mathbf{\Pi}(-\mathbf{r})] \cdot d\mathbf{r}
\]  
(34)

where \( C \) is a path for the exchange, and therefore the wave function of the system has the symmetry of equation (27). In summary, the exchange symmetry of the composite particle satisfies the relation (27) for both Type-I and Type-II monopoles. An interesting feature is that the phase factor \( (-1)^{2l+n} \) is established in different ways for the two types.

5.3. Analogy to the Coriolis interaction
Finally, we point out the formal equivalence between the Type-I monopole and the Coriolis interaction. This equivalence provides useful insight into the physics of magnetic monopoles and their analogues. Take an object of mass \( m \) at point \( P \) in a uniformly rotating frame with angular velocity \( \omega_0 \) (figure 5). In the nonrelativistic limit,
the Lagrangian of the object can be written as (see e.g., [22])
\[
\mathcal{L} = \frac{1}{2} m \mathbf{\dot{r}} \cdot \mathbf{\dot{r}} + \frac{1}{2} m |\mathbf{\omega}_0 \times \mathbf{r}|^2 + \mathcal{L}_c, \tag{35a}
\]
where the ‘Coriolis’ term \( \mathcal{L}_c \) is
\[
\mathcal{L}_c = m \mathbf{\dot{r}} \cdot \mathbf{\omega}_0 \times \mathbf{r}. \tag{35b}
\]
The position \( \mathbf{r} \) of the particle is specified by the locally flat coordinates with the basis vectors \( \mathbf{\hat{e}}_1 \) and \( \mathbf{\hat{e}}_2 \) (see figure 5): \( \mathbf{r} = u \mathbf{\hat{e}}_1 + v \mathbf{\hat{e}}_2 \). The angle vector \( \mathbf{\psi} \) with \( |\mathbf{\psi}| = \arctan(v/u) \) is perpendicular to \( \mathbf{r} \), and the Coriolis interaction of equation (35b) can be rewritten as
\[
\mathcal{L}_c = \mathbf{\dot{\psi}} \cdot \mathbf{L}_0, \tag{35c}
\]
where \( \mathbf{L}_0 = m r^2 \mathbf{\omega}_0 \) corresponds to the angular momentum due to the rotation of the Lab frame with angular velocity \( \mathbf{\omega}_0 \). This form of the Coriolis Lagrangian is equivalent to the interaction Lagrangian of a Type-I monopole (equation (4)). The field angular momentum \( \mathbf{S} \) in the charge-monopole pair plays the same role as the mechanical angular momentum \( \mathbf{L}_0 \) in the Coriolis interaction (equation (35c)).

In the Coriolis interaction, a geometric phase shift already appears at the classical level: the precession angle \( \varphi = 2\pi \Omega \) (\( \Omega \) is the solid angle) of a Foucault pendulum for one rotation of the frame. An interesting coincidence is found between this precession angle and \( j \) in equation (14) for \( S = \hbar (n = 2) \).

6. Conclusion

Adopting the local field interaction approach, we showed that the Dirac monopole can be classified into two different types. Notably, the difference is not merely a mathematical artifact but a physical reality, which arises from the presence or absence of a Dirac string. The duality of the two types was analyzed, and it revealed that the two types yield identical results except a quantum phase factor \((-1)^n\) for one loop rotation depending on the quantum number \( n \) of the field angular momentum. Notably, this factor can be detected for odd \( n \) in real and effective monopoles. We also pointed out the formal equivalence of the Type-I monopole with the Coriolis interaction. For Type II, the Maxwell duality gives rise to various possibilities of the Dirac string configuration: Both the electric and magnetic charges may emerge as endpoints of the corresponding strings. The exchange symmetry was analyzed for identical particles of charge-monopole composites. In both types, the wave function has a symmetry factor of \((-1)^{2s+n}\), including the intrinsic spin \( s \) and the field angular momentum. Notably, our approach provides a physically realistic description of the interaction between a charge and magnetic a monopole. This is possible owing to the locality of the field-interaction theory [3, 6].

Data availability statement

No new data were created or analysed in this study.
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