A proof of Culter’s theorem on the existence of periodic orbits in polygonal outer billiards

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Abstract

We discuss a recent result by C. Culter: every polygonal outer billiard has a periodic trajectory.

This note is an exposition of a theorem proved by Chris Culter, then an undergraduate student; he obtained this result as a participant of the 2004 Penn State REU program.¹ A complete account of Culter’s work involving a more general class of maps and a more detailed analysis of their periodic orbits will appear in his paper, currently in progress.

An outer billiard table is a compact convex domain $P$. Pick a point $x$ outside $P$. There are two support lines from $x$ to $P$; choose one of them, say, the right one from the view-point of $x$, and reflect $x$ in the support point. One obtains a new point, $y$, and the transformation $T : x \mapsto y$ is the outer (a.k.a. dual) billiard map, see figure 1. The map $T$ is not defined if the support line has a segment in common with the outer billiard table. In this note, $P$ is a convex $n$-gon; the set of points for which $T$ or any of its iterations is not defined is contained in a countable union of lines and has zero measure. For ease of exposition, we assume that $P$ has no parallel sides.

Outer billiards were introduced in [7] and popularized in [5, 6]; we refer to [1, 14, 15] for surveys. Here we are concerned with the existence of periodic

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trajectories of the outer billiard map. For the conventional, inner, billiards it is an outstanding open problem whether every polygon has a periodic billiard path. The best result so far is a theorem of R. Schwartz: every obtuse triangle with the obtuse angle not greater than 100° has a periodic trajectory, see \cite{9, 10, 11}. Note also that both inner and outer polygonal billiards on the sphere $S^2$ may have no periodic trajectories at all, see \cite{3}.

It will be convenient to consider the second iteration $T^2$ of the outer billiard map. Connecting the consecutive points of a periodic trajectory of $T^2$, one obtains a closed polygonal line. The number of turns made by this line about the billiard table is called the rotation number. The main result is as follows.

**Theorem 1** The map $T^2$ has a periodic trajectory that lies outside of any compact neighborhood of $P$ and has rotation number 1.

**Proof of Theorem.** For every outer billiard, not necessarily polygonal, the asymptotic dynamics of the map $T^2$ at infinity has the following description; see the sited surveys or \cite{16, 17, 18}. A bird’s eye view of a outer billiard is almost a point and the map $T$ is almost the reflection in this point. More precisely, after rescaling, the distance between a point $x$ and $T^2(x)$ is very small, and the evolution of a point under $T^2$ appears a continuous clockwise motion along a centrally symmetric curve $R$.

In our case, $R$ is a convex $2n$-gon, and each vector $(x, T^2(x))$ belongs to a finite set \{$\pm v_1, \ldots, \pm v_n$\}. These vectors are as follows. For every direction, other than the directions of the sides, there exists a pair of parallel support
lines to \( P \); the vector \( v_i \) is twice the vector connecting the respective support vertices of \( P \), see figure 2. For example, if \( P \) is a triangle then \( R \) is an affine-regular hexagon.

![Diagram](image)

**Figure 2**: The second iteration of the outer billiard map

Consider the polygon \( R \) (defined up to dilation). To every side of \( R \) there corresponds “time”, the ratio of the length of this side to the magnitude of the respective vector \( v_i \). One obtains a collection of “times” \((t_1, \ldots, t_k)\), defined up to a common factor. The polygon \( P \) is called quasi-rational if all these numbers are rational multiples of each other. For example, lattice polygons are quasi-rational and so are affine-regular ones. It is known that the orbits of the outer billiard about a quasi-rational polygon are bounded, see [2, 4, 8] or the cited surveys. Recently R. Schwartz proved that polygonal outer billiards may have orbits escaping to infinity [12, 13].

The actual map \( T^2 \), sufficiently far away from \( P \), is a piece-wise parallel translation through the vectors \( \pm v_1, \ldots, \pm v_n \). The discontinuities are \( 2n \) rays: the clockwise extensions of the sides of \( P \) and the reflections of these rays in the opposite vertices of \( P \) (a vertex opposite to a side is the one farthest from it). The lines containing these \( 2n \) rays form \( n \) strips \( S_1, \ldots, S_n \) whose intersection contains \( P \), see figure 3.

Choose an origin \( O \) inside \( P \) and consider the lines \( L_1, \ldots, L_n \) through \( O \) parallel to the sides of \( P \). Fix the above described polygon \( R \) so that \( O \) is its center. Denote by \( qR \) the dilation of \( R \) with coefficient \( q \). These polygons can be constructed by choosing a starting point on \( L_1 \), drawing the line in the direction \( v_1 \) until its intersection with \( L_2 \), then drawing the line in the direction \( v_2 \) until its intersection with \( L_3 \), etc.

Let \( p_1, \ldots, p_n \) be positive integers. Denote by \( Q(p_1, \ldots, p_n) \) the centrally...
Figure 3: The lines $L_i$, the strips $S_i$ and the polygon $R$

symmetric $2n$-gon whose sides are given by the vectors 

$$p_1v_1, p_2v_2, \ldots, p_nv_n, -p_1v_1, \ldots, -p_nv_n$$

and whose center is $O$. We wish to show that, for an appropriate choice of $p_1, \ldots, p_n$, the polygon $Q(p_1, \ldots, p_n)$ is an orbit of the map $T^2$. For this, the vertices of $Q(p_1, \ldots, p_n)$ should lie inside the strips $S_i$ (the opposite vertices in the same strip).

Clearly, there is $\varepsilon > 0$ (depending only on $P$ and the choice of the origin) such that if the vertices of an $2n$-gon $Q$ are $\varepsilon$-close to the respective vertices of a polygon $qR$ then the vertices of $Q$ lie inside the strips $S_i$. We claim that there exist arbitrarily large real $q$ and integers $p_1, \ldots, p_n$ such that the respective vertices of $qR$ and $Q(p_1, \ldots, p_n)$ are within $\varepsilon$ from each other.

For the claim to hold, it will suffice to have

$$|qt_i - p_i| < \delta, \quad i = 1, \ldots, n$$

(1)

where $\delta > 0$ is a small enough constant. Indeed, the first vertex of the
polygon $qR$ is
\[-\frac{1}{2} \sum_{1}^{n} q t_i v_i,
\]
whereas that of the polygon $Q(p_1, \ldots, p_n)$ is
\[-\frac{1}{2} \sum_{1}^{n} p_i v_i,
\]
and similarly for the other vertices.

Finally, consider the torus $T^n = \mathbb{R}^n / \mathbb{Z}^n$, and let $F_t$ be the constant flow with the vector $(t_1, \ldots, t_n)$. Then (1) means that $F_q(O)$ is $\delta$-close to $O$ where $O = (0, \ldots, 0)$. Indeed, the flow $F_t$ is either periodic, and then $F_q(O) = O$ for $q$ forming an arithmetic progression, or quasi-periodic and thus returning arbitrarily close to the initial point infinitely often. □

**Remarks.** 1. A composition of a number of central symmetries is either a central symmetry or a parallel translation. It follows that a $k$-periodic point of the outer billiard map about a polygon has a polygonal neighborhood consisting of periodic points with period $k$ or $2k$ (the latter holds if $k$ is odd).

2. The density of the numbers $q$ satisfying (1) is positive. One can deduce that the lower density of the set of periodic trajectories described in Theorem [1] is also positive.

3. A periodic trajectory of the polygonal outer billiard map is called stable if, under an arbitrary small perturbation of the outer billiard polygon $P$, the trajectory is also perturbed but not destroyed. A criterion for stability is known, see [14]. Enumerate the vertices of $P$ counterclockwise as $A_1, \ldots, A_n$. An even-periodic orbit of the dual billiard map is encoded by the sequence vertices in which the consecutive reflections occur. One obtains a cyclic word $W$ in the letters $A_1, \ldots, A_n$. The orbit is stable if and only if each appearance of every letter in an odd position in $W$ is balanced by its appearance in an even position. By this criterion, the periodic trajectories of Theorem 1 are stable.

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