On weighted Bergman spaces of a domain with Levi-flat boundary II

Masanori Adachi

Abstract

Holomorphic functions on the maximal Grauert tube of a hyperbolic compact Riemann surface are studied. It is shown that their weighted Bergman spaces are infinite dimensional for arbitrary weight order greater than $-1$ in spite of the fact that they do not admit any non-constant bounded holomorphic functions. The key ingredient of the proof is a computation of weighted norms of analytic continuations of eigenfunctions of the Laplacian in terms of the hypergeometric function. This result complements our previous work (Adachi in Trans Am Math Soc 374(10):7499–7524, 2021) where it was shown that the space of geodesic segments on the Riemann surface has exactly the same property.

Keywords

Weighted Bergman space · Grauert tube · Analytic continuation · Eigenfunction · Hypergeometric function

Mathematics Subject Classification

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1 Introduction

We shall continue our study [3,4] on function theory over 1-convex domains with Levi-flat boundary. The starting point of this paper is the following question raised by Akira Fujiki [15], which asks the validity of a “reflection principle” about Levi-flat real hypersurfaces.

**Question 1.1** Let $X$ be a compact complex manifold and $\Omega \subset X$ a 1-convex domain bounded by $C^\infty$-smooth Levi-flat real hypersurface $\partial \Omega$. Is the complement $\Omega' := X \setminus \overline{\Omega}$ also 1-convex?

A Levi-flat real hypersurface $M$ in a complex manifold $X$ is defined to be a real hypersurface that locally divides $X$ into Stein domains, that is, any point of $M$ has an open neighborhood $U$ in $X$ such that $U \setminus M$ is Stein. Hence, a closed real hypersurface $M \subset X$ must be Levi-flat if $X \setminus M$ is 1-convex. The above question by Fujiki asks its partial converse. So far, no counterexample has been known and all the known examples of 1-convex domains with Levi-flat boundary have 1-convex complements.

When $\dim X = 2$, there are some supporting evidences for Question 1.1. We have several kinds of examples where Question 1.1 is affirmative (cf. [24, Chapter 5], [5, §4]). If we replace 1-convexity with a stronger convexity called Takeuchi 1-convexity (see [14] for its definition), the question is known to be affirmative: If $\Omega$ is Takeuchi 1-convex, the holomorphic normal bundle of $\partial \Omega$ is leafwise positive (see [2]), and then Brunella’s construction [11] (see also [1,8,23]) shows that $\Omega'$ is also Takeuchi 1-convex. In the remaining case, where $\Omega$ is 1-convex but not Takeuchi 1-convex, Canales’ dichotomy theorem [12, Theorem 3.4] tells us that the Levi foliation of $\partial \Omega$ admits an invariant transverse measure. We do not know, however, how to make use of the invariant transverse measure to transfer convexity of $\Omega$ into $\Omega'$. When $\dim X \geq 3$, we have less supporting evidence: we do not even know examples of 1-convex domains with Levi-flat boundary except for those of Nemirovski type [21] (see also [24, Chapter 5]). In fact, non-existence results have been studied well in higher dimension. Ohsawa’s theorem [22] says that compact Kähler manifolds of dimension $\geq 3$ cannot contain 1-convex domain with real-analytic Levi-flat boundary, and Brinkman’s theorem [10] says that a Takeuchi 1-convex domain with Levi-flat boundary does not exist in complex manifolds of dimension $\geq 3$. 

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Masanori Adachi
adachi.masanori@shizuoka.ac.jp

Department of Mathematics, Faculty of Science, Shizuoka University, 836 Ohya, Suruga-ku, Shizuoka 422-8529, Japan
In this paper, motivated by Question 1.1, we shall study a quantitative problem by restricting ourselves to the case where \( \dim X = 2 \) and \( \Omega \) is Takeuchi 1-convex.

**Question 1.2** Let \( X \) be a compact complex surface and \( \Omega \subset X \) a Takeuchi 1-convex domain bounded by \( C^\infty \)-smooth Levi-flat real hypersurface \( \partial \Omega \). The complement of \( \Omega \) is denoted by \( \Omega' := X \setminus \overline{\Omega} \). Suppose that the weighted Bergman space \( A\alpha^2(\Omega) \) is infinite dimensional for some weight order \( \alpha > -1 \). Is \( A\alpha^2(\Omega') \) also infinite dimensional?

Here the weighted Bergman space \( A\alpha^2(\Omega) \) should be understood as the topological vector space defined by
\[
A\alpha^2(\Omega) := \left\{ f : \Omega \to \mathbb{C}, \text{holomorphic} \mid \int_\Omega |f|^2(-\rho)^\alpha dV < \infty \right\}
\]
where \( \rho \) is a smooth defining function of \( \Omega \), \( dV \) is a volume form of \( X \), and \( \alpha > -1 \). Note that its inner product depends on the choice of \( \rho \) and \( dV \), but its topology does not in our setting. Roughly speaking, Question 1.2 asks Question 1.1 for holomorphic functions with growth condition along \( \partial \Omega \). In the setting of Question 1.2, it is known that the Diederich–Fornæss index is a useful device (see [2,6]). The Diederich–Fornæss index is a useful device to solve the \( \overline{\partial} \)-equation on weakly pseudoconvex domain with \( \partial \)-equation on weakly pseudoconvex domain with boundary control (cf. [7]), hence, we may hope that Question 1.2 could be affirmative.

The goal of this paper is to provide a supporting evidence for Question 1.2, a concrete example where the conclusion of Question 1.2 is valid. Let us explain the construction of our example briefly. Take a compact Riemann surface \( \Sigma \) of genus \( \geq 2 \) and uniformize it by the unit disk \( \mathbb{D} \) in \( \mathbb{C} \), \( \Sigma = \mathbb{D}/\Gamma \), where \( \Gamma \) is a Fuchsian group. We regard \( \Gamma \) acting on the Riemann sphere \( \mathbb{CP}^1 \) and define a ruled surface \( X := \mathbb{D} \times \mathbb{CP}^1/\Gamma \) where \( \Gamma \) acts on \( \mathbb{D} \times \mathbb{CP}^1 \) diagonally:
\[
\gamma \cdot (z, w) = (\gamma z, \gamma w)
\]
for \( \gamma \in \Gamma \), \((z, w) \in \mathbb{D} \times \mathbb{CP}^1 \). The first projection induces the ruling map \( X \to \Sigma \). This \( X \) contains a real-analytic Levi-flat real hypersurface \( M := \mathbb{D} \times \partial \mathbb{D}/\Gamma \) and \( X \setminus M \) splits into two domains \( \Omega := \mathbb{D} \times \mathbb{D}/\Gamma \) and \( \Omega' := \mathbb{D} \times (\mathbb{CP}^1 \setminus \mathbb{D})/\Gamma \).

It has been known since [13] that \( \Omega \) is 1-connected and \( \Omega' \) is Stein, in particular, both are Takeuchi 1-convex. The domain \( \Omega' \) may be referred to as the space of geodesic segments on \( \Sigma \) since the equivalence class \([\{z, w\}] \in \mathbb{D} \times \mathbb{D}/\Gamma \) determines a geodesic segment joining \([z], [w] \in \Sigma \). The domain \( \Omega' \) is biholomorphic to the Grauert tube of maximal radius \( \pi/2 \) of the hyperbolic surface \( \Sigma \) (see [3]). More details will be explained in Sect. 2.2 with a different coordinate.

In our previous work [4], we have shown that the weighted Bergman space \( A\alpha^2(\Omega) \) of \( \Omega \) is infinite dimensional for any weight order \( \alpha > -1 \) and dense in the space of holomorphic functions \( \mathcal{O}(\Omega) \) in the compact-open topology. The main idea in [4] was extending jets of holomorphic functions along \( D \), the maximal compact complex analytic subset in \( \Omega \), to holomorphic functions on \( \Omega \) with minimal \( L^2 \) norm. By identifying those jets with holomorphic differentials over \( \Sigma \), we showed that the optimal extension is given by an integral operator \( I \),
\[
I(\psi)(z, w) = \frac{1}{B(\alpha, \alpha)} \int_{\mathbb{D}} \frac{(w - \tau)(\tau - z)}{w - \tau} \psi(\tau)(d\tau)^\alpha
\]
where \( \tau \) is the coordinate of the universal cover \( \mathbb{D} \) of \( \Sigma \), \( \psi = \psi(\tau)(d\tau)^N \) is a holomorphic \( N \)-differential on \( \Sigma \), and \( B(\rho, q) \) is the beta function.

To confirm that Question 1.2 is affirmative in our example, instead of studying optimal \( L^2 \) jet extension, we analytically continue each eigenfunction \( f \) of \( \partial \)-Laplacian of \( \Sigma \) to a holomorphic function \( I\alpha(f) \) on \( \Omega' \).

**Main Theorem** Let \( \Sigma \) be a compact Riemann surface of genus \( \geq 2 \) equipped with the hyperbolic metric, and \( \Omega' \) the Grauert tube of maximal radius \( \pi/2 \) of the hyperbolic surface \( \Sigma \). Take an orthonormal basis \( \{f_k\}_k^\infty \) of \( L^2(\Sigma) \) consisting of eigenfunctions of \( \partial \)-Laplacian \( \square_0 \) of \( \Sigma \). Then we have the followings:

1. Each eigenfunction \( f_k \) extends to a holomorphic function \( I\alpha(f_k) \) on \( \Omega' \).
2. \( I\alpha(f_k) \in A\alpha^2(\Omega') \) for any \( \alpha > -1 \).
3. \( \{I\alpha(f_k)\} \) form a topological basis of \( A\alpha^2(\Omega') \).
4. For any \( \alpha > -1 \), \( A\alpha^2(\Omega') \) is dense in \( \mathcal{O}(\Omega') \) in the compact-open topology.

We remark that the first point is a classical fact. For instance, its proofs (in generalized settings) can be found in Krötz and Schlichtkrull [17] from the viewpoint of complex crowns and in Zelditch [27] (see also [28]) from the viewpoint of Grauert tubes of real-analytic Riemannian manifolds. Our contribution is the other points, which follow from a computation of weighted \( L^2 \) norms of analytic continuations of eigenfunctions over the maximal Grauert tube. For a particular choice of weight function and volume form (see Sect. 2.2 for the choice), we can compute the exact value of the weighted \( L^2 \) norms in terms of the hypergeometric function:
\[
\|I\alpha(f_k)\|_{L^2}^2 = c_{k, \alpha} \pi, \quad c_{k, \alpha} := 2F_1\left(1 + \sqrt{1 - 8\lambda_k}, 1 - \sqrt{1 - 8\lambda_k}; 2 + \alpha; 1 \right)
\]
where \( \lambda_k \) denotes the eigenvalue of \( f_k \), \( \lambda_k = \lambda_k f_k \). We achieve this computation by the same way as in our previous work [4], namely, we determine the Taylor coefficients of the extension inductively with exact value of their \( L^2 \) norms.
This elementary approach gives another proof for the first point.
We also remark that analogous statements for third and fourth points have been known in the theory of Grauert tubes. Let $Y$ be a closed real-analytic Riemannian manifold and $Y_r$ the Grauert tube of $Y$ of radius $r > 0$. We consider $Y_r$ with sufficiently small $r$, where the boundary $\partial Y_r$ is strictly pseudoconvex. Based on a result that was stated by Boutet de Monvel [9], whose full proofs can be found in Zelditch [27], Lebeau [18] and Stenzel [25], it is known that eigenfunctions of the Laplacian on $Y$ are analytically continued to $Y_r$ and form a topological basis for the $L^2$ Hardy space $A^2_\omega(Y_r)$ or unweighted Bergman space $A_0^2(Y_r)$ (See also [26,28]). Our main theorem may be seen as a refinement of these general results, studying Grauert tubes of a closed hyperbolic surface and analyzing the analytic continuation of the Poisson kernel for our theorem by the approach taken in these previous works, a local frame $\psi, \phi$. Similarly, for $L^2$ Hardy space $(/\Omega^1)$, we write locally

$$\langle\eta, \theta\rangle_{/\Omega^1} = \int_{/\Omega^1} \eta \cdot \theta dV, \quad \langle\phi, \psi\rangle_{/\Omega^1} = \int_{/\Omega^1} \phi \cdot \psi dV$$

respectively. Their inner products are given by

$$\langle\phi, \psi\rangle_{/\Omega^1} = \int_{/\Omega^1} \phi \cdot \psi dV, \quad \langle\eta, \theta\rangle_{/\Omega^1} = \int_{/\Omega^1} \eta \cdot \theta dV$$

resulting in the corresponding norms denoted by $\|\cdot\|_{/\Omega^1}$. When $L$ is the trivial line bundle, we simply write $L^2(\Sigma) := L^{(0,0)}_2(\Sigma, L)$ and omit the subscript $L$ for its inner product and norm.

We will use the Chern connection $\partial L + \overline{\partial}$ of $L$ and the $\partial$-Laplacians $\square_L = \partial^* L \partial L$ and $\square_L = \partial^* \partial L$ where $\partial^* L$ denotes the $L^2$ adjoint of $\partial L: L^{(0,0)}_2(\Sigma, L) \rightarrow L^{(1,0)}_2(\Sigma, L)$, which is defined in the sense of distributions. Note that the spectrums of our $\partial$-Laplacians consist of non-negative eigenvalues since we are working on closed manifold $\Sigma$. For $L$-valued smooth $(1,0)$-form $\eta$, $\partial^* L$ has local expression

$$\partial^* L \eta = -\frac{1}{g} \frac{\partial}{\partial \eta} \overline{\partial} e U$$

on $U$ since the Chern connection is given by

$$\partial L = \left( \frac{\partial}{\partial \eta} + \frac{\log h_U}{g^U} \phi_U \right) e U \otimes dz$$

for a smooth section $\phi$ of $L$, and

$$\langle\langle\partial L \phi, \eta\rangle\rangle_L = \int_{/\Omega^1} \int_{/\Omega^1} h_U \left( \frac{\partial}{\partial \eta} + \frac{\log h_U}{g^U} \phi_U \right) \eta \cdot \theta dV \otimes d\bar{V}$$

The $\partial$-Laplacians therefore have local expressions

$$\square^{(0)}_L \phi = \partial^* L \partial L \phi = -\frac{1}{g} \frac{\partial}{\partial \eta} \overline{\partial} e U \phi_U,$$

$$\square^{(1)}_L \eta = \left( \frac{\partial}{\partial \eta} + \frac{\log h_U}{g^U} \phi_U \right) e U \otimes dz.$$

We will use later a special case of the Nakano identity

$$\square^{(0)}_L \phi = \square^{(0)}_L \phi + 1 \frac{\partial^2 (\log h_U)}{\partial \eta \partial \bar{\eta}} \phi_U e U,$$

where $\square^{(0)}_L = \partial^* L \overline{\partial}$ denotes the $\partial$-Laplacian for sections of $L$.

\[ \square^{(0)}_L \phi = \square^{(0)}_L \phi + \frac{1}{g^U} \frac{\partial^2 (\log h_U)}{\partial \eta \partial \bar{\eta}} \phi_U e U, \]
From now on, we will assume that $\Sigma$ is of genus $\geq 2$ and fix its universal covering $\mathbb{D} \to \Sigma$, where $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$. We refer the standard coordinate of the universal covering $\mathbb{D}$ as the uniformizing coordinate of $\Sigma$. We equip $\Sigma$ with the hyperbolic metric $g$, whose fundamental form is expressed as

$$\omega_g = ig(z)dz \wedge d\overline{z}, \quad g(z) := \frac{2}{(1 - |z|^2)^2},$$

in the uniformizing coordinate $z$. Later, we will consider the power of canonical line bundle $K^{\otimes n}_\Sigma$ and equip it with the Hermitian metric $h_n$ induced from the hyperbolic metric. When we trivialize $K^{\otimes n}_\Sigma$ by $(dz)^{\otimes n}$ in the uniformizing coordinate, the Hermitian metric $h_n$ is locally expressed by $g^{-n}$. The Chern connection of $(K^{\otimes n}_\Sigma, h_n)$ and the $\partial$-Laplacian for sections of $K^{\otimes n}_\Sigma$ will be denoted by $\partial_n$ and $\square_n = \partial_n^* \partial_n$ respectively.

### 2.2 Weighted $L^2$-norms and Bergman spaces of the Grauert tube

To analyze holomorphic functions on the maximal Grauert tube of hyperbolic surface $\Sigma$, we use the following coordinates. The standard coordinate of the bidisk $\mathbb{D} \times \mathbb{D}$ in $\mathbb{C}^2$ is denoted by $(z, w)$. We also use another non-holomorphic coordinate $(z, t)$ given by

$$\mathbb{D} \times \mathbb{D} \ni (z, w) \mapsto \left( z = \frac{w - \overline{z}}{1 - zw}, w = \frac{t + \overline{z}}{1 +zt} \right) \in \mathbb{D} \times \mathbb{D},$$

whose inverse transformation is

$$\mathbb{D} \times \mathbb{D} \ni (z, t) \mapsto (z, w) \in \mathbb{D} \times \mathbb{D}.$$

Note that this coordinate change identifies the totally real subset $\Delta' := \{(z, \overline{z}) | z \in \mathbb{D}\}$ in $(z, w)$-coordinate with a horizontal disk $\mathbb{D} \times \{0\}$ in $(z, t)$-coordinate.

Recall that our Riemann surface $\Sigma$ is of genus $\geq 2$ and we fixed a universal covering $\mathbb{D} \to \Sigma$. Let $\Gamma$ be the Deck transformation group of the universal covering. Note that $\Gamma$ is a Fuchsian group and consists of rational fractional transformations. We are going to study the quotient space $\Omega' := \mathbb{D} \times \mathbb{D} / \Gamma$, where $\Gamma$ acts on $\mathbb{D} \times \mathbb{D}$ by

$$\gamma \cdot (z, w) = (\gamma z, \overline{\gamma w})$$

for each $\gamma \in \Gamma$, $(z, w) \in \mathbb{D} \times \mathbb{D}$. Here $\overline{\gamma w} := \overline{\gamma} \overline{w}$. Let $X := \mathbb{D} \times \mathbb{C}P^1 / \Gamma$ where $\Gamma$ acts on $\mathbb{D} \times \mathbb{C}P^1$ in the same way. The first projection of $\mathbb{D} \times \mathbb{C}P^1$ induces a holomorphic submersion $\pi : X \to \Sigma$, which is a holomorphic $\mathbb{C}P^1$ bundle, and $\pi | \Omega'$ is a holomorphic $\mathbb{D}$-bundle.

The space of geodesic segments on $\Sigma$, which was studied in our previous work [4] is biholomorphic to $\Omega := X \setminus \overline{\Omega'}$. The open complex surface $\Omega'$ is identified with the Grauert tube of hyperbolic surface $(\Sigma, g)$ of maximal radius $\pi/2$ in the sense of Guillemin–Stenzel [16] and Lempert–Szöke [20]. More details¹ can be found in [3]. Note that $\Omega'$ and $\Omega$ are only “half” biholomorphic: the map $(z, w) \mapsto (z, 1/w)$ on $\mathbb{D} \times \mathbb{C}P^1$ induces a diffeomorphism between them but is not holomorphic. In spite of this situation, we will observe some similarity between the spaces of holomorphic functions on $\Omega'$ and $\Omega$.

We shall use a Hermitian metric $G$ on $\Omega'$ whose fundamental form is expressed as

$$\omega_G := \frac{2i dz \wedge d\overline{z}}{(1 - |z|^2)^2} + \frac{(1 - |z|^2)^2}{|1 - Zw|^4} \frac{i}{2} dw \wedge d\overline{w},$$

in $(z, w)$-coordinate. Note that $\omega_G$ is defined so that $\omega_G|_{\Delta} = idt \wedge d\overline{t}/2$. The well-definedness of $\omega_G$ on $\Omega'$ follows from a direct computation. Using the volume form induced from $G$,

$$dV = \frac{1}{2!}(\omega_G)^2 = \frac{4}{|1 - Zw|^4} \frac{i}{2} dz \wedge d\overline{z} \wedge \frac{i}{2} dw \wedge d\overline{w},$$

and the weight function of the form $\delta^\alpha, \alpha > -1$, where

$$\delta = 1 - |t|^2 = 1 - \left| \frac{w - \overline{z}}{1 - zw} \right|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - zw|^2},$$

we define the weighted $L^2$ inner product of measurable functions on $\Omega'$ by

$$\langle (F_1, F_2) \rangle_\alpha := \frac{1}{\Gamma(\alpha + 1)} \int_{\Omega'} F_1 F_2 \delta^\alpha dV$$

for measurable functions $F_1, F_2$ on $\Omega'$ and $\alpha > -1$. The weighted Lebesgue space is the Hilbert space

$$L^2_\alpha(\Omega') := \{ F : \text{measurable on} \ \Omega' \ | \ \| F \|_\alpha := \langle (F, F) \rangle_\alpha < \infty \}$$

¹ We would like to correct an error in [3]. The uniqueness statement in [3, Fact 2] was not correctly stated. The complex structure of the Grauert tube $X$ is uniquely determined up to diffeomorphisms, namely, only the biholomorphism type of $X$ is unique.
and the weighted Bergman space is its closed subspace defined by
\[ A^2_\alpha(\Omega') := L^2_\alpha(\Omega') \cap \mathcal{O}(\Omega'). \]

3 Analytic continuation from the totally real subset

We identify the totally real subset \( \Delta' = \{(z, \bar{z}) \mid z \in \mathbb{D}\} \) with the unit disk \( \mathbb{D} \) by the first projection, and \( D' := \Delta'/\Gamma \subset \Omega' \) with \( \Sigma \) as real-analytic surfaces. Then \( D' \) is a determining set for holomorphic functions on \( \Omega' \), namely, if two holomorphic functions on \( \Omega' \) agree on \( D' \), they must coincide on \( \Omega' \). On the other hand, it is a classical fact that any eigenfunction \( f \) of the hyperbolic Laplacian on \( \Sigma \), regarded as a function on \( D' \), is analytically continued to a holomorphic function \( I'(f) \) on \( \Omega' \). In this section, we give another proof for this fact by using the Taylor expansion of \( I'(f) \) along \( D' \). Our method of the proof also gives an explicit value of the weighted \( L^2 \) norm of \( I'(f) \), and we will see that the weighted Bergman spaces are infinite dimensional for any weight order \( \alpha > -1 \) and dense subsets of the space of holomorphic functions \( \mathcal{O}(\Omega') \) in the compact-open topology.

3.1 Relations among Taylor coefficients

Let \( F = F(z, w) \in \mathcal{O}(\mathbb{D} \times \mathbb{D}) \). We work in the coordinate \((z, t), \) and let \( \tilde{F}(z, t) := F(z, w(z, t)) \). Since \( F \) is holomorphic, the smooth function \( \tilde{F} \) enjoy
\[
0 = \frac{\partial}{\partial z} F(z, w) = \frac{\partial}{\partial z} \tilde{F}(z, t(z, w)) = \frac{\partial}{\partial z} \tilde{F}(z, w - \frac{\bar{z}}{1 - z \bar{w}}) = \frac{\partial \tilde{F}}{\partial z} + \frac{\partial \tilde{F}}{\partial t} \frac{-1}{1 - z \bar{w}} = \frac{\partial \tilde{F}}{\partial z} - \frac{1 + zt}{1 - |z|^2} \frac{\partial \tilde{F}}{\partial t}.
\]

Let us denote the Taylor coefficients of \( F \) along \( \Delta' \) computed in the coordinate \((z, t)\) by
\[
f_n(z) := \frac{1}{n!} \frac{\partial^n}{\partial t^n} \tilde{F}(z, 0).
\]

Then they enjoy
\[
0 = \frac{1}{n!} \frac{\partial^n}{\partial t^n} \left( \frac{\partial \tilde{F}}{\partial z} - \frac{1 + zt}{1 - |z|^2} \frac{\partial \tilde{F}}{\partial t} \right) \bigg|_{t=0} = \frac{\partial f_n}{\partial z} - \frac{z}{1 - |z|^2} \frac{\partial f_n}{\partial t} \bigg|_{t=0} - \frac{n + 1}{1 - |z|^2} f_{n+1}
\]
\[
= \frac{n}{1 - |z|^2} f_n - \frac{n + 1}{1 - |z|^2} f_{n+1}
\]

for \( n \geq 0 \).

Now we assume that our \( F \in \mathcal{O}(\mathbb{D} \times \mathbb{D}) \) is invariant under the action of \( \Gamma \). Then, for each \( \gamma = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \in \Gamma < \text{PSU}(1, 1) \cong \text{Aut}(\mathbb{D}), |\alpha|^2 - |\beta|^2 = 1 \), we have
\[
\tilde{F}(z, t) = F(z, w(z, t)) = F(\gamma z, \gamma w(z, t)) = \tilde{F}(\gamma z, t(\gamma z, \gamma w(z, t))) = \tilde{F}(\gamma z, \frac{\beta z + \alpha}{\beta \bar{z} + \alpha} t)
\]

since
\[
\gamma w(z, t) = \frac{\alpha t + \beta}{\beta t + \alpha} + \frac{\bar{\beta} z + \alpha}{\beta \bar{z} + \alpha} t = (\alpha z + \beta t) + (\alpha \bar{z} + \beta)
\]

and
\[
t(\gamma z, \gamma w(z, t)) = \frac{\gamma w(z, t) - \gamma z}{1 - \gamma z \cdot \gamma w(z, t)} = \frac{\beta z + \alpha}{\beta \bar{z} + \alpha} \gamma w(z, t) - (\alpha z + \beta)
\]

\[
= \frac{\beta z + \alpha}{\beta \bar{z} + \alpha} t.
\]

Hence, the Taylor coefficients satisfy
\[
f_n(z) = \frac{1}{n!} \frac{\partial^n}{\partial t^n} \tilde{F}(z, t) \bigg|_{t=0} = \frac{1}{n!} \frac{\partial^n}{\partial t^n} \tilde{F}(\gamma z, \frac{\beta z + \alpha}{\beta \bar{z} + \alpha} t) \bigg|_{t=0} = \left( \frac{\beta z + \alpha}{\beta \bar{z} + \alpha} \right)^n f_n(\gamma z).
\]

Since \( \gamma^* dz = (\beta z + \alpha)^{-2} dz \), we can easily see that
\[
\psi_n := \frac{\sqrt{2 \pi}}{1 - |z|^2} \tilde{F}(z, t) \bigg|_{t=0} = \left( \frac{\beta z + \alpha}{\beta \bar{z} + \alpha} \right)^n f_n(z) g(z)^{n/2} \tilde{F}(z, t) \bigg|_{t=0}
\]

is a \( \Gamma \)-invariant differential form on \( \mathbb{D} \), hence, defines a \((n, 0)\)-differential on \( \Sigma \). We call \( \psi_n \) the \( n \)-th associated differential of \( F \).

We regard the \((n, 0)\)-differential \( \psi_n \) as a smooth section of \( K^{\otimes n}_{\mathbb{D}} \) and rewrite Equation (2) using the Chern connection with respect to the hyperbolic metric. We have
\[
\frac{\partial}{\partial z} \psi_n = \left( \frac{\partial}{\partial z} + \frac{\partial \log g^{-n}}{\partial z} \right) \left( \frac{f_n(z) g(z)^{n/2}}{\sqrt{2 \pi}} \right) dz^{\otimes (n+1)}
\]
\[
= \left( \frac{\partial}{\partial z} - \frac{n z}{1 - |z|^2} \frac{1}{f_n(z)} \right) g^{n/2} dz^{\otimes (n+1)}
\]
\[
= \frac{n + 1}{1 - |z|^2} f_n(z) g^{n/2} dz^{\otimes (n+1)} = \frac{n + 1}{\sqrt{2}} \psi_{n+1}
\]

for \( n \geq 0 \).
3.2 $L^{2}$ norms of Taylor coefficients

Let $f$ be an eigenfunction of the hyperbolic Laplacian $\Box_{0} = \Box_{0}^{(0)}$ on $\Sigma$ with eigenvalue $\lambda \geq 0$. Note that $f$ is real analytic from the ellipticity of the hyperbolic Laplacian. By letting $\psi_{0} := f$ and define

$$\psi_{n+1} := \sqrt{\frac{\lambda}{n+1}} \partial_{n} \psi_{n}$$

inductively for $n \geq 0$. We see that all the $\psi_{n}$’s are also eigensections of the $\Box_{n}$.

**Lemma 3.1** The smooth section $\psi_{n}$ of $K_{\Sigma}^{\otimes n}$ is eigensection of the $\Box_{n}$ with eigenvalue $\lambda + n(n+1)/2$.

**Proof** It is enough to show that

$$\Box_{n+1} \partial_{n} \psi - \partial_{n} \Box_{n} \psi = (n+1) \partial_{n} \psi$$

for any smooth section $\psi$ of $K_{\Sigma}^{\otimes n}$ since the claim follows from this by induction on $n$. From the Nakano identity (1) and the fact that the hyperbolic metric is Einstein–Kähler, $\omega_{g} = i \partial \bar{\partial} \log g$, we have

$$\Box_{n+1} \partial_{n} \psi = \Box_{n+1} \partial_{n} \psi = \Box_{n+1} ( -g \partial_{n} \omega_{g} \psi )$$

Computing the first term in the uniformizing coordinate, we have

$$\Box_{n+1} \partial_{n} \psi = \Box_{n+1} \partial_{n} \psi = \Box_{n+1} \left( -g \partial_{n} \omega_{g} \psi \right)$$

where we temporarily wrote $\Box_{n} = \psi_{n} d \partial^{\otimes n}$. This completes the proof. \□

From this Lemma, we can compute the $L^{2}$ norm of $\psi_{n}$ explicitly.

**Proposition 3.2** For $n \geq 1$, we have

$$\| \psi_{n} \|^{2}_{K_{\Sigma}^{\otimes n}} = \frac{\Gamma(n+1)}{(n!)^{2}} \| \psi_{0} \|^{2}.$$ 

**Proof** From Lemma 3.1, we have

$$\| \psi_{k+1} \|^{2}_{K_{\Sigma}^{\otimes (k+1)}} = \frac{2}{(k+1)^{2}} \| \partial_{k} \psi_{k} \|^{2}_{K_{\Sigma}^{\otimes (k+1)}}$$

where we temporarily wrote $\Box_{n} = \psi_{n} d \partial^{\otimes n}$. This completes the proof. \□

3.3 Convergence of the formal extension

Let $F$ be a measurable function on $\Omega'$, and we assume that $F$ is holomorphic along all the fibers of $\pi | \Omega': \Omega' \to \Sigma$. Then, in the same manner as in Sect. 3.1, we can define associated differentials $\{ \psi_{n} \}$ as measurable $(n, 0)$-differentials on $\Sigma$ by expanding $F$ on each fiber of $\pi | \Omega'$ at the intersection with $D'$. The weighted $L^{2}$ norm of $F$ can be expressed by the $L^{2}$ norms of $\psi_{n}$’s:

$$\| F \|_{\alpha}^{2} = \pi \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \| \psi_{n} \|^{2}_{K_{\Sigma}^{\otimes n}} \frac{\Gamma(n+1)}{\Gamma(n+2+\alpha)}$$

for any $\alpha > -1$, $F \in A_{\alpha}^{2}(\Omega')$, which gives the desired analytic continuation of $F$. First we confirm that $F \in L_{\alpha}^{2}(\Omega')$.

**Proposition 3.3** The sequence of partial sums $\{ F_{n} := \sum_{m=0}^{n} f_{m}(z)t^{m} \}_{n}$ is Cauchy in $L_{\alpha}^{2}(\Omega')$ for any $\alpha > -1$.

**Proof** First note that $F_{n}$ is invariant under the action of $\Gamma$ on $\mathbb{D} \times \mathbb{D}$ in view of Sect. 3.1. We identify $F_{n}$ with a smooth function on $\Omega'$. From Proposition 3.2 and Eq. (7), we can compute, for any $n \geq 0$,

$$\| F_{n} \|_{\alpha}^{2} = \pi \sum_{m=0}^{n} \| \psi_{m} \|^{2}_{K_{\Sigma}^{\otimes m}} \frac{\Gamma(m+1)}{\Gamma(m+2+\alpha)}$$

We show that, for any $\alpha > -1$, $F \in A_{\alpha}^{2}(\Omega')$, which gives the desired analytic continuation of $F$. First we confirm that $F \in L_{\alpha}^{2}(\Omega')$.
where \( \lambda \geq 0 \) is the eigenvalue of \( f \), \( \square_0^0 f = \lambda f \). Hence, it is enough to show the convergence of the series

\[
\sum_{m=0}^{\infty} \frac{1}{m!} \Gamma(m + 2 + \alpha) \left( \frac{1+\sqrt{1-8\lambda}}{2} \right)_m \left( \frac{1-\sqrt{1-8\lambda}}{2} \right)_m \frac{1}{m!},
\]

where \( (N)_m := N(N + 1) \ldots (N + m - 1) \) and \( (N)_0 := 1 \). This is nothing but a special value of the hypergeometric function

\[
\mathbf{2F}_1 \left( \frac{1+\sqrt{1-8\lambda}}{2}, \frac{1-\sqrt{1-8\lambda}}{2}; \frac{1}{2} \right),
\]

which is finite since \( \alpha > -1 \). \( \square \)

Next we show that this \( F \) is actually holomorphic. We remark that \( F_0 \) is only smooth function in general; Proposition 3.3 does not imply that \( F \) is holomorphic.

**Proposition 3.4** The function \( F \) constructed above is holomorphic in \( \Omega' \).

**Proof** From a standard reduction argument as in [4, Proposition 5.3], it is enough to show that \( ||\bar{\partial}F_n||_\alpha \to 0 \) as \( n \to \infty \) for some \( \alpha > -1 \). Since Equations (2) and (4) are equivalent for each \( n \), we have

\[
\bar{\partial}F_n = \sum_{m=0}^{n} \left( \frac{\partial f_m}{\partial \bar{\sigma}} - \frac{mz}{1-|z|^2} f_m - \frac{m+1}{1-|z|^2} f_{m+1} \right) i^m \, d\bar{\sigma}
\]

Hence, a computation similar to [4, Proposition 5.3] yields

\[
||\bar{\partial}F_n||_\alpha^2 = (n+1)^2 \int_{\Omega'} \left| \frac{f_{n+1}(z)}{1-|z|^2} \right|^2 |\bar{\sigma}|^2 |d\bar{\sigma}|^2 \, dV
\]

\[
= (n+1)^2 \int_R \left| f_{n+1}(z) \right|^2 |\bar{\sigma}|^2 |d\bar{\sigma}|^2 \, dV
\]

\[
= \frac{\pi(n+1)^2}{4} \left( \frac{\Gamma(n+1) \Gamma(\alpha+1)}{\Gamma(n+2+\alpha)} \right) \left( \psi_n^{(k)} \right)^2 \left( \psi_n^{(l)} \right)^2 \left( \kappa_{\lambda(k)} \right)^2
\]

where \( R \subset \mathbb{D} \) is a fundamental domain of the universal covering \( \mathbb{D} \to \Sigma \). Now we take \( \alpha = 2 \). Then it follows from Proposition 3.2 that

\[
||\bar{\partial}F_n||_2^2 = \frac{\pi(n+1)^2}{4} \prod_{k=1}^{n+1} \frac{(2\lambda + k(k-1))}{((n+1)!)^2} ||\psi_0||^2 \frac{n! 2!}{(n+3)!}
\]

\[
= \frac{\pi}{2} ||\psi_0||^2 \prod_{k=1}^{n+1} \frac{(2\lambda + k(k-1))}{n!(n+3)!}
\]

\[
= \frac{\pi}{2} ||\psi_0||^2 \frac{(2\lambda + n(n+1))}{(n+2)(n+3)} \prod_{k=1}^{n+1} \frac{(2\lambda + k(k-1))}{n!(n+1)!}
\]

\[
\to \frac{\pi}{2} ||\psi_0||^2 \cdot 1 \cdot 0 = 0
\]

as \( n \to \infty \) since the proof of Proposition 3.3 also shows that

\[
\lim_{n \to \infty} \prod_{k=1}^{n+1} \frac{(2\lambda + k(k-1))}{n!(n+2+\alpha)} = 0
\]

for any \( \alpha > -1 \). \( \square \)

### 3.4 Proof of main theorem

Now we complete the proof of our main theorem.

**Proof of main theorem** Let \( \{ f_k \}_{k=0}^{\infty} \) be an orthonormal basis of \( L^2(\Sigma) \) consisting of eigenfunctions of \( \partial \)-Laplacian \( \square_0^0 \) of \( \Sigma \) with \( \square_0^0 f_k = \lambda_k f_k \). From Propositions 3.3 and 3.4, we see that each eigenfunction \( f_k \) extends to a holomorphic function \( f_k' \) on \( \Omega' \) and \( f_k' \) has finite weighted \( L^2 \) norm. We have proved (i) and (ii).

To prove (iii), we shall show that \( \langle I'(f_k), \Pi' \rangle \) forms an orthonormal basis of \( A_{\lambda}^2(\Omega') \). Let us check the orthogonality \( \langle I'(f_k), \Pi' \rangle = 0 \) when \( k \neq l \). In the same way as Eq. (7), we can express this inner product as

\[
\langle I'(f_k), \Pi' \rangle = \pi \Gamma(\alpha+1) \sum_{n=0}^{\infty} \langle \psi_n^{(k)}, \psi_n^{(l)} \rangle \kappa_{\lambda(k)} \kappa_{\lambda(l)}
\]

where \( \{ \psi_n^{(k)} \} \) denote the associated differentials of \( f_k. \)

From Equations (4) and (6), we have

\[
\langle \psi_{n+1}^{(k)}, \psi_{n+1}^{(l)} \rangle \kappa_{\lambda(k)} \kappa_{\lambda(l)} = \frac{2}{(n+1)^2} \langle \partial_{\lambda} \psi_{n}^{(k)}, \partial_{\lambda} \psi_{n}^{(l)} \rangle \kappa_{\lambda(n+1)}
\]

\[
= \frac{2}{(n+1)^2} \langle \psi_{n}^{(k)}, \psi_{n}^{(l)} \rangle \kappa_{\lambda(n+1)}
\]

\[
= \frac{2}{(n+1)^2} \left( \lambda_k + \frac{n(n+1)}{2} \right) \langle \psi_{n}^{(k)}, \psi_{n}^{(l)} \rangle \kappa_{\lambda(n+1)}
\]

(8)
Since \( \langle \psi_0^{(k)}, \psi_0^{(l)} \rangle = \langle f_k, f_l \rangle = 0 \), we see that 
\( \langle \psi_n^{(k)}, \psi_n^{(l)} \rangle \rangle = 0 \) for any \( n \). The orthogonality of \( I'(f_k) \) and \( I'(f_l) \) follows.

Now let us prove the density. Let \( F \in A_b^2(\Omega') \) that is orthogonal to all the \( I'(f_k) \)'s, and \( \{ \psi_n \} \) the associated differentials of \( F \). We expand \( F|D' = \psi_0 \) by the orthonormal basis \( \{ f_k \}_{k=0}^\infty \) of \( L^2(\Sigma) \), \( \psi_0 = \sum_{k=0}^\infty \alpha_k f_k \) in \( L^2(\Sigma) \). Then, by computing in the same way as Eq. (8), Proposition 3.2 implies

\[
\langle \psi_n^{(k)}, \psi_n^{(l)} \rangle \rangle = \frac{2\lambda_k + n(n-1)}{n^2} \langle \psi_n^{(k)}, \psi_n^{(l)} \rangle \rangle, \\
= \frac{\prod_{l=1}^n (2\lambda_k + l(l-1))}{(n!)^2} \langle \psi_0, f_k \rangle, \\
= \| \psi_n^{(k)} \|^2_{K^\infty} \alpha_k.
\]

Hence, it follows that

\[
0 = \langle F, I'(f_k) \rangle \rangle = \pi \Gamma(\alpha + 1) \sum_{n=0}^\infty \frac{\langle \psi_n, \psi_n \rangle \rangle}{\Gamma(n+2+\alpha)} = \pi c_{k,0} \alpha_k,
\]

that is, \( \alpha_k = 0 \). Therefore, \( F|D' = \psi_0 = 0 \) and \( F \) must vanish in \( \Omega' \). This completes the proof for (iii).

To prove (iv), let us look at \( \Omega'_\varepsilon := \{ p \in \Omega' \mid d'(p) > \varepsilon \} \), the Grauert tube of \( \Sigma \) of radius \( \arccos \sqrt{\varepsilon} \) (see [3] for the computation of radius), where \( \varepsilon > 0 \). Note that, in the same way as Equation (7), the unweighted \( L^2 \) norm with respect to \( dV \) is

\[
\| F \|^2_{L^2(\Omega'_\varepsilon)} = \int_{\Omega'_\varepsilon} |F|^2 dV = \pi \sum_{n=0}^\infty \| \psi_n \|^2_{K^\infty} \frac{1}{n+1} - \varepsilon)^{n+1},
\]

where \( \{ \psi_n \} \) denote the associated differentials of a holomorphic function \( F \) on \( \Omega'_\varepsilon \). By repeating the argument above over \( \Omega'_\varepsilon \), we see that \( \{ I'(f_k)|\Omega'_\varepsilon \} \) form an orthogonal basis of the unweighted Bergman space \( A_b^2(\Omega'_\varepsilon) \). It follows from Cauchy’s estimate that any holomorphic function on \( \Omega' \) can be uniformly approximated in \( \Omega'_\varepsilon \) by linear combinations of \( I'(f_k) \)'s.

\( \square \)

4 Application: Liouville property of the Grauert tube

As an application, we shall give yet another proof for the triviality of \( L^2 \) Hardy space of \( \Omega' \), which was announced in [3, Remark 3.5].

Corollary 4.1 The \( L^2 \) Hardy space \( A_{b}^2(\Omega') \) consists only of constant functions.

Recall that the \( L^2 \) Hardy space of \( \Omega' \) is defined by

\[
A_{b}^2(\Omega') := \{ f \in \mathcal{O}(\Omega') \mid \| f \|_{L^2} < \infty \}.
\]

where

\[
\| f \|_{L^2} = \lim_{\varepsilon \to 0} \| f \|^2_{L^2(\Omega'_\varepsilon)} = \pi \sum_{k=0}^\infty \| \psi_n \|^2.
\]

This Liouville property is, in fact, a direct corollary of Hopf’s ergodicity theorem, or follows from the boundary behavior of the Grauert tube function of \( \Omega' \). We refer the interested reader to [3] for these details.

Proof of Corollary 4.1 Let \( F \in \mathcal{O}(\Omega') \). Using (iv) of Main Theorem, we can expand \( F = \sum_{k=0}^\infty \alpha_k I'(f_k) \) on \( \Omega'_\varepsilon \) for any \( \varepsilon > 0 \). Then we can compute the Hardy norm of \( F \) as follows:

\[
\| F \|^2_{L^2(\Omega'_\varepsilon)} = \int_{\Omega'_\varepsilon} |F|^2 dV = \pi \sum_{k=0}^\infty |c_k|^2 \lim_{\varepsilon \to 0} \int_{\Omega'_\varepsilon} |I'(f_k)|^2 dV = \pi \sum_{k=0}^\infty \frac{|c_k|^2}{\varepsilon} \int_{\Omega'_\varepsilon} \frac{1+\sqrt{1-8\varepsilon}+\sqrt{1-8\varepsilon}}{2+\varepsilon} ; 1.
\]

If \( \| F \|_{L^2} < \infty \), the coefficients \( \alpha_k \) must vanish except for \( k \) with \( \lambda_k = 0 \). Since \( \square_0 \) -harmonic function is constant on \( \Sigma \), \( F \) must be a constant function. \( \square \)

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