IMPLEMENTATION OF PELLET’S THEOREM

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Abstract

Pellet’s theorem determines when the zeros of a polynomial can be separated into two regions, based on the presence or absence of positive roots of an auxiliary polynomial, but does not provide a method to verify its conditions or to compute the roots of the auxiliary polynomial when they exist. We derive an explicit condition for these roots to exist and, when they do, propose efficient ways to compute them. A similar auxiliary polynomial appears for the generalized Pellet theorem for matrix polynomials and it can be treated in the same way.

Key words: Pellet, zero, root, polynomial, matrix polynomial

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1 Introduction

Pellet’s theorem, a classical result from 1881, derives conditions under which the zeros of a polynomial can be divided into two groups, according to their magnitudes. It is a direct consequence of Rouché’s theorem and is stated as follows.

Theorem 1.1 ([6], [4, Th.(28,1), p.128]) Given the polynomial
\[ p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \]
with complex coefficients, \( n \geq 3 \), \( 1 \leq k \leq n - 1 \), and \( a_0 a_k \neq 0 \). Let the polynomial
\[ \phi(x) = x^n + |a_{n-1}| x^{n-1} + \cdots + |a_k + 1| x^{k+1} - |a_k| x^k + |a_{k-1}| x^{k-1} + \cdots + |a_0| \]
have two distinct positive roots \( r \) and \( R \), \( r < R \). Then \( p \) has exactly \( k \) zeros in or on the circle \( |z| = r \) and no zeros in the annular ring \( r < |z| < R \).

We note that, by Descartes’ rule, \( \phi \) has either two or no positive roots. Although the function \( \phi \) depends on \( k \), this parameter remains fixed throughout the paper, and it will therefore be omitted from the notation to prevent unnecessary clutter.

Recently, ([2], [5]) a generalized Pellet theorem was derived for matrix polynomials, which have received a lot of attention recently because of their application in several engineering fields ([7]). Matrix polynomials are polynomials whose coefficients are matrices instead of scalars. They occur in polynomial eigenvalue problems, which consist of finding a nonzero eigenvector \( v \), corresponding to an eigenvalue \( z \) satisfying \( P(z)v = 0 \), where
\[ P(z) = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_0, \]
with $A_j \in \mathcal{M}_{m \times m}$ for $j = 0, \ldots, n$, and with $\det(P(z))$ not identically zero. If $A_n$ is singular then $P$ has infinite eigenvalues and if $A_0$ is singular then zero is an eigenvalue. There are $nm$ eigenvalues, including possibly infinite ones. The finite eigenvalues are the solutions of $\det(P(z)) = 0$. The following theorem generalizes Pellet’s theorem to matrix polynomials.

**Theorem 1.2 (Generalized Pellet theorem.)** ([2], [5]) Let

$$P(z) = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_1 z + A_0$$

be a matrix polynomial with $n \geq 2$, $A_j \in \mathcal{M}_{m \times m}$ for $j = 0, \ldots, n$, and $A_0 \neq 0$. Let $A_k$ be invertible for some $k$ with $1 \leq k \leq n - 1$, and let the polynomial

$$g(x) = ||A_n||x^n + ||A_{n-1}||x^{n-1} + \cdots + ||A_{k+1}||x^{k+1} - ||A_k^{-1}||^{-1}x^k + ||A_{k-1}||x^{k-1} + \cdots + ||A_1||x + ||A_0||,$$

where the norm can be any vector-induced norm, have two distinct positive roots $r$ and $R$, $r < R$. Then $P$ has exactly $km$ zeros in or on the disk $|z| = r$ and no zeros in the annular ring $r < |z| < R$.

Its generalization widens the usefulness of Pellet’s theorem considerably, as the computation of nonlinear eigenvalues is much more costly than the computation of polynomial zeros, making easily computed bounds more valuable.

However, both the classical and generalized Pellet theorems do not explain how their assumptions can be verified, nor do they provide a method to compute the roots $r$ and $R$ when the assumptions are satisfied. In addition, as will be explained below, we require certain properties of any iterates generated in such computations. Our sole purpose is to address these shortcomings, thereby improving the applicability of these theorems. We stress that we focus on the implementation of Pellet’s theorem, if and when it is used. When and where it is worthwhile to use the theorem is a consideration that lies well outside the scope of this work.

Because the real polynomials $\phi$ and $g$ are of the exact same form, the techniques we will develop apply to both Theorem 1.1 and Theorem 1.2. To keep matters simple, we will henceforth refer only to Theorem 1.1 with the understanding that all results carry over in a straightforward way to Theorem 1.2.

Figure 1 shows a few typical scenarios for the function $\phi$ in Theorem 1.1. On the left $\phi$ increases at first, then decreases, intersects the x-axis and then increases while crossing the x-axis again; in the middle, $\phi$ behaves as on the left, but does not decrease enough to have positive roots; on the right $\phi$ increases monotonically. The presence or lack of positive roots needs to be detected first, and, if there are such roots, then they need to be computed with a method that generates iterates that are themselves proper bounds, which means that $r$ and $R$ need to be approached from inside the interval $[r, R]$. In this way, the numerical process can be stopped at any time with an upper bound on $r$ and a lower bound on $R$, allowing for an inexact solution while still providing correct bounds on the two groups of $k$ and $n - k$ zeros of $P$. We consider this a key property to be satisfied and, although there exist many methods to compute the real roots of a polynomial, none of them accomplishes this.
We derive an easily computable criterion for the absence or presence of positive roots of \( \phi \) in Theorem 1.1, which has the important additional advantage of providing an adequate starting point in \([r, R]\) for the computation of the positive roots of \( \phi \). This allows us to reformulate Pellet's theorem in a more useful way and to propose a framework for generating efficient methods to compute the roots, when they exist, from inside \([r, R]\) for a particular given value of \( k \). The complexity of detecting whether \( \phi \) has positive roots or not is, as we will see later, \( O(n) \). This means that, if the theorem needs to be applied for every value of \( k \), with \( 1 \leq k \leq n - 1 \), then this complexity becomes \( O(n^2) \), which makes it more efficient to first use a result by [1]. In [1], it was shown that if \( \phi \) has two real roots for a particular value of \( k \), then \( k \) must be the abscissa of a vertex of the Newton polygon associated with \( p \). Moreover, computing these vertices only costs \( O(n \log n) \) operations ([3]). In such a case, we would first compute the abscissae \( k_1, ..., k_m \) of the vertices of the associated Newton polygon and then compute the roots \( r \) and \( R \) only for the values of \( k \in \{k_1, ..., k_m\} \). Once the abscissae have been computed, the complexity of detecting the positive roots is then \( O(mn) \). Typically, \( m \ll n \) since only polynomials with very special coefficients have more than just a few values of \( k \) for which Pellet’s theorem can be applied.

From now on, we compute the roots of \( \phi \) for a given value of \( k \) which remains fixed throughout this work. Detection of the roots and the determination of a starting point is the subject of Section 2, while a strategy to compute the roots is developed in Section 3. We believe that the ideas behind our techniques are general enough to be useful in other situations as well.

![Figure 1: Typical scenarios for \( \phi \).](image)

### 2 Detection of the roots

Our strategy to determine if \( \phi \) has positive roots is to first transform it to a strictly convex function \( \chi \) with the same roots or absence of roots as \( \phi \). Denoting \( \chi \)'s unique minimizer by \( x^* \), we conclude that \( \phi \) has two positive roots if and only if \( \phi(x^*) < 0 \). We note that this minimizer does not, in general, correspond to the minimizer of \( \phi \) (if it exists). This leads to the following theorem.
Theorem 2.1 Given the polynomial
\[ \phi(x) = x^n + \eta_{n-1}x^{n-1} + \cdots + \eta_k x^{k+1} + \eta_k x^k + \eta_{k-1} x^{k-1} + \cdots + \eta_0, \]
with \( \eta_j \geq 0 \) (\( j = 1, \ldots, n \)), \( \eta_0 \eta_k \neq 0 \), \( n \geq 3 \), and \( 1 \leq k \leq n-1 \). Then \( \phi \) has two positive roots if and only if \( \phi(x^*) < 0 \), where \( x^* \) is the unique positive root of the polynomial
\[ \chi(x) = (n-k)x^n + (n-k-1)\eta_{n-1}x^{n-1} + \cdots + \eta_k x^{k+1} - \eta_k x^k - \eta_{k-1} x^{k-1} - \cdots - k\eta_0. \]

Proof. We define \( \psi(x) \) for \( x \in (0, +\infty) \) as
\[ \psi(x) = x^{-k}\phi(x) = x^{n-k} + \eta_{n-1}x^{n-k-1} + \cdots + \eta_k x^{k+1} - \eta_k x^k - \eta_{k-1} x^{k-1} + \cdots + \eta_0 x^{-k}, \]
and observe that it is a strictly convex function with the same positive roots or lack thereof as \( \phi \). This function will therefore have a unique positive minimizer, which we denote by \( x^* \), and it will have two positive roots if and only if \( \psi(x^*) < 0 \), which is equivalent to \( \phi(x^*) < 0 \). Because \( x^* \) is the unique minimizer of a strictly convex differentiable function, it can be obtained as the unique positive solution of \( \psi'(x) = 0 \). Since
\[ \psi'(x) = (n-k)x^{n-k-1} + (n-k-1)\eta_{n-1}x^{n-k-2} + \cdots + \eta_k - \eta_{k-1} x^{-2} - \cdots - k\eta_0 x^{-k-1}, \]
and since \( \psi'(x) = 0 \iff x^{k+1}\psi'(x) = 0 \), the statement of the theorem follows. \( \square \)

Defining
\[ \sigma(x) = x^n + \eta_{n-1}x^{n-1} + \cdots + \eta_k x^{k+1} + \eta_{k-1} x^{k-1} + \cdots + \eta_0, \]
we remark that Theorem 2.1 can also be obtained from the observation that the existence of \( u > 0 \) for which \( \phi(u) < 0 \) is equivalent to the existence of \( u > 0 \) for which \( |a_k| > \sigma(u)/u^k \). The function in the right-hand side of this inequality is strictly convex and has a unique minimizer \( u^* \). Then \( \sigma(u^*)/(u^*)^k \) represents the lowest strict threshold value for \( |a_k| \) to guarantee the existence of some \( u > 0 \) for which \( \phi(u) < 0 \).

Since \( x^* \) is independent of the value of \( \eta_k \), it is possible to establish a criterion for \( \eta_k \) that guarantees positive roots for \( \phi \), and that depends only on the other coefficients. Incorporating this in Pellet’s theorem results in the following more explicit version of that same theorem.

Theorem 2.2 Given the polynomial \( p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \) with complex coefficients, \( a_0a_k \neq 0 \), and \( n \geq 3 \). Let \( 1 \leq k \leq n-1 \),
\[ \phi(x) = x^n + |a_{n-1}|x^{n-1} + \cdots + |a_{k+1}|x^{k+1} - |a_k|x^k - |a_{k-1}|x^{k-1} + \cdots + |a_0|, \]
\[ \sigma(x) = x^n + |a_{n-1}|x^{n-1} + \cdots + |a_{k+1}|x^{k+1} + |a_{k-1}|x^{k-1} + \cdots + |a_0|, \]
and let \( x^* \) be the unique positive root of the polynomial
\[ \chi(x) = (n-k)x^n + (n-k-1)|a_{n-1}|x^{n-1} + \cdots + |a_{k+1}|x^{k+1} - |a_k|x^k - |a_{k-1}|x^{k-1} - 2|a_{k-2}|x^{k-2} - \cdots - k|a_0|. \]
If
\[ |a_k| > \frac{\sigma(x^*)}{(x^*)^k}, \]
(1)
then the polynomial \( \phi \) has two distinct positive roots \( r \) and \( R \) with \( r < R \), and the polynomial \( p \) has exactly \( k \) zeros in or on the circle \( |z| = r \) and no zeros in the annular ring \( r < |z| < R \).

The polynomial \( \chi \) appearing in the two previous theorems, is strictly convex to the right of its positive root, so that, to compute it, Newton’s method or a suitably accelerated version of it can be used with guaranteed monotonic convergence from the right of the root. As a starting point, any easily computable upper bound on the roots of \( \phi \) can be used. Such a bound can be found with the following theorem.

**Theorem 2.3** Let

\[
\omega(x) = \eta_n x^n + \eta_{n-1} x^{n-1} + \cdots + \eta_m x^m - \eta_\ell x^\ell - \eta_{\ell-1} x^{\ell-1} - \cdots - \eta_0 ,
\]

with \( \eta_j \geq 0 \) \( (j = 1, \ldots, n) \), \( m > \ell \), \( \eta_0 \eta_\ell \eta_m \eta_n \neq 0 \), and \( n \geq 3 \). Then the unique positive root \( x^* \) of \( \omega \) satisfies the following.

(1) If \( \omega(1) < 0 \), then

\[
1 < x^* \leq \left( \frac{\eta_\ell + \eta_{\ell-1} + \cdots + \eta_0}{\eta_n + \eta_{n-1} + \cdots + \eta_m} \right)^{\frac{1}{m-\ell}} .
\]

(2) If \( \omega(1) > 0 \), then

\[
0 < x^* \leq \left( \frac{\eta_\ell + \eta_{\ell-1} + \cdots + \eta_0}{\eta_n + \eta_{n-1} + \cdots + \eta_m} \right)^{\frac{1}{n}} < 1 .
\]

**Proof.** By Descartes’ rule of signs, the polynomial \( \omega \) has a single positive root \( x^* \) and is therefore negative for \( 0 \leq x < x^* \). If \( \omega(1) = 0 \), then \( x^* = 1 \). If \( \omega(1) < 0 \), then \( x^* > 1 \), and for \( x > 1 \) we have that

\[
\omega(x) \geq (\eta_n + \eta_{n-1} + \cdots + \eta_m) x^m - (\eta_\ell + \eta_{\ell-1} + \cdots + \eta_0) x^\ell ,
\]

with the equality holding for \( x = 1 \). The unique positive root of the right-hand side then provides an upper bound on the positive root of \( \omega \).

If \( \omega(1) > 0 \), then \( x^* < 1 \), and for \( x < 1 \) we have that

\[
\omega(x) \geq (\eta_n + \eta_{n-1} + \cdots + \eta_m) x^n - (\eta_\ell + \eta_{\ell-1} + \cdots + \eta_0) x^\ell ,
\]

with equality for \( x = 1 \). The unique positive root of the right-hand side once again provides an upper bound on the positive root of \( \omega \). That it is less than 1 follows from \( \omega(1) > 0 \). This concludes the proof. \( \square \)

Applying this theorem to the aforementioned polynomial \( \chi \) directly yields the following corollary.
Corollary 2.1 Let
\[ \chi(x) = (n-k)x^n + (n-k-1)\eta_{n-1}x^{n-1} + \cdots + \eta_{k+1}x^{k+1} - \eta_{k-1}x^{k-1} - 2\eta_{k-2}x^{k-2} - \cdots - k\eta_0, \]
with \( \eta_j \geq 0 \) (\( j = 1, \ldots, n \)), \( \eta_0 \neq 0 \), \( n \geq 3 \), and \( 1 \leq k \leq n-1 \). Then the unique positive root \( x^* \) of \( \chi \) satisfies the following.

1. If \( \chi(1) < 0 \), then
   \[ 1 < x^* \leq \left( \frac{\eta_{k-1} + 2\eta_{k-2} + \cdots + k\eta_0}{(n-k) + (n-k-1)\eta_{n-1} + \cdots + \eta_{k+1}} \right)^{1/2}. \tag{2} \]

2. If \( \chi(1) > 0 \), then
   \[ 0 < x^* \leq \left( \frac{\eta_{k-1} + 2\eta_{k-2} + \cdots + k\eta_0}{(n-k) + (n-k-1)\eta_{n-1} + \cdots + \eta_{k+1}} \right)^{1/n} < 1. \tag{3} \]

The bounds can be adjusted like in Theorem 2.3 if \( \eta_{k+1}, \eta_{k-1} \), or more coefficients vanish.

There are several possible strategies for finding a proper starting point to begin the computation of the roots of \( \phi \) if they exist: one can periodically compute \( \text{sgn}(\phi) \) at an iterate, and use that iterate as a starting point if \( \text{sgn}(\phi) = -1 \), or one can simply first compute \( x^* \) and only then start the computation of the roots of \( \phi \), using \( x^* \) as a starting point (assuming, as we did, that \( \phi(x^*) < 0 \)). The complexity of this detection phase is \( O(n) \).

We remark here that there exist other methods to detect if a polynomial has real roots, such as, e.g., Sturm sequences. These methods have similar complexity, but they generally do not produce an appropriate starting point in \([r, R]\), which is essential.

3 Computation of the roots

In this section we assume that \( \phi \) has two positive roots \( r \) and \( R \) and that we are given a point \( \bar{x} \) such that \( r \leq \bar{x} \leq R \), which can be obtained in the way explained at the end of the previous section. We then propose a method to compute the roots iteratively, with the iterates converging monotonically to the roots from inside the interval \([r, R]\). The main idea behind this method is to approximate \( \phi \) at a given point by a similar but simpler function that dominates \( \phi \) on \([r, R]\). Its roots will be approximations to the roots of \( \phi \), and the method then continues iteratively from those approximations. The following theorem derives the approximation to \( \phi \).

Theorem 3.1 Let the polynomial
\[ \phi(x) = x^n + \eta_{n-1}x^{n-1} + \cdots + \eta_{k+1}x^{k+1} - \eta_{k-1}x^{k-1} - \cdots + \eta_0, \]
with \( \eta_j \geq 0 \) (\( j = 1, \ldots, n \)), \( \eta_0 \eta_k \neq 0 \), \( n \geq 3 \), and \( 1 \leq k \leq n-1 \), have two positive roots \( r \) and \( R \) with \( r < R \) and let
\[ \phi_1(x) = x^n + \eta_{n-1}x^{n-1} + \cdots + \eta_{k+1}x^{k+1}, \tag{4} \]
\[ \phi_2(x) = -\eta_kx^k + \eta_{k-1}x^{k-1} + \cdots + \eta_1x + \eta_0. \tag{5} \]
Then for \( r \leq \bar{x} \leq R \), the trinomial 
\[ f(x) = \alpha x^n - \beta x^k + \gamma , \]
where 
\[ \alpha = \frac{1}{n} \bar{x}^{1-n} \phi'_1(\bar{x}) > 0, \]
\[ \beta = -\frac{1}{k} \bar{x}^{1-k} \phi'_2(\bar{x}) > 0, \]
\[ \gamma = \phi(\bar{x}) - \bar{x} \left( \frac{1}{n} \phi'_1(\bar{x}) + \frac{1}{k} \phi'_2(\bar{x}) \right) > 0, \]
has two positive zeros \( r_1 \) and \( r_2 \) with \( r \leq r_1 < r_2 \leq R \), and \( f(x) \geq \phi(x) \) for \( x \geq 0 \).

**Proof.** Our goal is to approximate \( \phi \) at a certain point \( r \leq \bar{x} \leq R \), for which \( \phi(\bar{x}) < 0 \), by a function \( f \) that agrees at this point with \( \phi \) in (at least) function and first derivative values. In addition, \( f \) needs to dominate \( \phi \) for \( x \geq 0 \). Clearly, a straightforward linear approximation is not possible as \( \phi \) is composed of terms that are both convex and concave.

We solve this problem by constructing separate approximations for \( \phi_1 \) and \( \phi_2 \), whose sum is \( \phi \), and which were defined in the statement of the theorem.

The transformation of variables \( w = x^n \) transforms \( \phi_1(x) \) into 
\[ \phi_1(w^{1/n}) = w + \eta_{n-1} w^{\frac{n-1}{n}} + \cdots + \eta_{k+1} w^{\frac{k+1}{n}}. \]
This is a concave function of \( w \), so that it is dominated by its linear approximation \( \alpha_1 w + \alpha_2 \) at a point \( \tilde{w} = \bar{x}^n \). A straightforward calculation shows that 
\[ \alpha_1 = \left( \frac{1}{n} w^{\frac{1-n}{n}} \right) \phi'_1(\tilde{w}^{1/n}) = \frac{1}{n} \bar{x}^{1-n} \phi'_1(\bar{x}) , \]
\[ \alpha_2 = \phi_1(\tilde{w}^{1/n}) - \alpha_1 \tilde{w} = \phi_1(\bar{x}) - \frac{1}{n} \bar{x} \phi'_1(\bar{x}) , \]
where the derivatives are with respect to \( x \). Clearly, \( \alpha_1 > 0 \) and 
\[ \alpha_2 = \phi_1(\bar{x}) - \frac{1}{n} \bar{x} \phi'_1(\bar{x}) = \frac{1}{n} \left( \eta_{n-1} \bar{x}^{n-1} + 2 \eta_{n-2} \bar{x}^{n-2} + \cdots + (n-k-1) \eta_{k+1} \bar{x}^{k+1} \right) > 0 . \]

We have obtained an approximation to \( \phi_1(x) \) of the form \( \alpha_1 x^n + \alpha_2 \), with \( \alpha_1 x^n + \alpha_2 \geq \phi_1(x) \) for \( x \geq 0 \).

On the other hand, the transformation \( y = x^k \) transforms \( \phi_2(x) \) into 
\[ \phi_2(y^{1/k}) = -\eta_k y + \eta_{k-1} y^{\frac{k-1}{k}} + \cdots + \eta_1 y^{1/k} + \eta_0 , \]
a concave function of \( y \). It is therefore dominated by its linear approximation \( \beta_1 y + \beta_2 \) at a point \( \tilde{y} = \bar{x}^k \), with 
\[ \beta_1 = \left( \frac{1}{k} y^{\frac{1-k}{k}} \right) \phi'_2(\tilde{y}^{1/k}) = \frac{1}{k} \bar{x}^{1-k} \phi'_2(\bar{x}) , \]
\[ \beta_2 = \phi_2(\tilde{y}^{1/k}) - \beta_1 \tilde{y} = \phi_2(\bar{x}) - \frac{1}{k} \bar{x} \phi'_2(\bar{x}) . \]
Since
\[ \phi_2(\bar{x}) - \frac{1}{k} \bar{x} \phi_2'(\bar{x}) = \frac{1}{k} \left( \eta_{k-1} \bar{x}^{k-1} + 2\eta_{k-2} \bar{x}^{k-2} + \cdots + (k-1)\eta_1 \bar{x} + \eta_0 \right) , \]
we have that \( \beta_2 = \phi_2(\bar{x}) - \frac{1}{k} \bar{x} \phi_2'(\bar{x}) > 0 \), from which \( \phi_2(\bar{x}) > \phi_2'(\bar{x}) \). Because \( \phi(\bar{x}) = \phi_1(\bar{x}) + \phi_2(\bar{x}) \leq 0 \), and therefore \( \phi_2(\bar{x}) \leq -\phi_1(\bar{x}) < 0 \), this means that \( \phi_2'(\bar{x}) < 0 \), so that \( \beta_1 < 0 \), and we have obtained an approximation to \( \phi_2(x) \) of the form \( \beta_1 x^{k+1} + \beta_2 \), with \( \beta_1 x^{k+1} + \beta_2 \geq \phi_2(x) \) for \( x \geq 0 \). Consequently, our first-order approximation \( f \) to \( \phi \) at \( \bar{x} \) is given by the trinomial
\[ f(x) = \alpha_1 x^n + \beta_1 x^{k+1} + \alpha_2 + \beta_2 , \]
which corresponds to \( \Theta \) in the statement of the theorem. It satisfies \( f(x) \geq \phi(x) \), is of the same form as \( \phi \), and, since \( f(\bar{x}) \leq 0 \), it has two roots \( r_1 \) and \( r_2 \) with \( r_1 < r_2 \leq R \).

An algorithm to compute \( r \) and \( R \) can now be based on Theorem 3.1 in a standard way: starting from any initial point on the interval \((r, R)\), \( f \)'s largest root can be used as the next iterate in the computation of \( R \). Convergence is monotonic from the left. An analogous algorithm is obtained for \( r \) by considering the smallest root of \( f \). In this case, convergence is monotonic from the right. The function \( \phi' \) has either one root, which must lie in \((r, R)\), or two roots, one in \((0, r)\) and one in \((r, R)\), so that \( \phi'(r) \phi'(R) \neq 0 \). It is then a technical exercise to show that the order of convergence of these algorithms is quadratic.

However, to make the aforementioned algorithms implementable, we need a method to compute the roots of \( f \) itself from inside the interval determined by its roots. The basis for such a method is provided by the following theorem, which derives a first order approximation to \( f \) that dominates it, and whose roots can be computed explicitly.

**Theorem 3.2** Let the trinomial \( f(x) = \alpha x^n - \beta x^k + \gamma \) with \( \alpha, \beta, \gamma > 0 \), \( n \geq 3 \), and \( 1 \leq k \leq n-1 \), have two positive roots \( r_1 \) and \( r_2 \) with \( r_1 < r_2 \). Then for \( r_1 \leq \bar{x} \leq r_2 \), the function
\[ h(x) = \frac{\alpha \delta}{\epsilon - x^k} - \beta x^k + \gamma , \tag{7} \]
where
\[ \delta = \frac{k}{n} \bar{x}^{k+n} > 0 , \]
\[ \epsilon = \frac{n+k}{n} \bar{x}^k > \bar{x}^k , \]
has two positive zeros \( s_1 \) and \( s_2 \) with \( r_1 \leq s_1 < s_2 \leq r_2 \), and \( h(x) \geq f(x) \) for \( 0 \leq x < (1 + n/k)\bar{x} \).

**Proof.** With the transformation of variables, \( y = x^k \), \( f \) can be written as \( f(y^{1/k}) = \alpha y^n - \beta y + \gamma \). If the function \( y^{n/k} \) is approximated at \( \bar{y} = \bar{x}^k \) to first order by \( \delta/(\epsilon - y) \), then this is equivalent to approximating \( y^{-n/k} \) to first order by \((-1/\delta)y + \epsilon/\delta \). Since \( y^{-n/k} \)
is convex and its approximation is linear, this implies that \((-1/\delta)y + \epsilon/\delta \leq y^{−n/k}\), and therefore that \(\delta/(\epsilon - y) \geq y^{n/k}\), or

\[
\frac{\delta}{\epsilon - x^k} \geq x^n.
\]

Consequently, \(h(x) \geq f(x)\) for \(x \geq 0\) and, because \(h(\bar{x}) = f(\bar{x}) \leq 0\), it has two positive roots \(s_1\) and \(s_2\) that satisfy \(r_1 \leq s_1 < s_2 \leq r_2\). The constants \(\delta\) and \(\epsilon\) are computed from the first order approximation conditions

\[
\frac{\delta}{\epsilon - \bar{y}} = \bar{y}^{n/k}
\]

and

\[
\frac{\delta}{(\epsilon - \bar{y})^2} = \frac{n}{k} \bar{y}^{n/k-1},
\]

as

\[
\delta = \frac{k}{n} \bar{y}^{1+n/k} = \frac{k}{n} x^{k+n},
\]

\[
\epsilon = \frac{n+k}{n} \bar{y} = \frac{n+k}{n} x^k.
\]

We note that \(h(x)\) becomes unbounded as \(x \to (1+n/k)\bar{x}\). This concludes the proof. □

Figure 2 shows the functions \(\phi \leq f \leq h\) for the polynomial

\[
q(z) = z^8 + z^7 + 3z^6 + \frac{1}{2}z^4 + 15z^3 - 2z^2 + (i + 1)z - 4,
\]

with \(k = 3\) and \(\bar{x} = 1.02\).

The approximation \(h\) of \(f\) in Theorem 3.2 leads to an iterative method for the computation of the roots of \(f\), by computing the roots of \(h\) and then using the smallest and largest of those roots as the next iterates for the computation of \(r_1\) and \(r_2\), respectively. As was the case for the roots of \(\phi\), we obtain quadratic and monotonic convergence to the roots of \(f\), although now the roots of the approximation can be computed explicitly since they are the roots of a quadratic. This follows by setting \(h(x) = 0\), which is equivalent to

\[
\beta x^{2k} - (\epsilon\beta + \gamma)x^k + (\alpha\delta + \epsilon\gamma) = 0,
\]

a quadratic equation in \(x^k\). That both roots of this quadratic are real follows directly from the properties of \(h\). Once available, the appropriate root of \(f\) serves as the next iterate in the computation of the roots of \(\phi\).

The main computational effort in our approach to compute the positive roots of \(\phi\) in Theorem 1.1 for a fixed parameter \(k\) is concentrated in the computation of the coefficients of the trinomial \(f\), which requires \(O(n)\) arithmetic operations. The computation of the roots of \(f\) itself is far less costly. Furthermore, the iterative process can be stopped at any moment, since each iterate provides a correct bound on the corresponding root of \(\phi\). This is precisely what we set out to obtain.

Example.

The motivation for this work was the absence of a method converging from inside the interval \([r, R]\), which means that there is no equivalent method to compare our methods.
to. Instead we will illustrate the method outlined above at the hand of an example, namely, the polynomial \( q \) defined in (3). For this polynomial, we computed the roots of the corresponding real polynomial \( \phi \) for \( k = 3 \) to a relative accuracy of \( 10^{-12} \). The steps involved in this process with the corresponding number of iterations are listed below. All computations are carried out to the same aforementioned relative accuracy of \( 10^{-12} \).

**Step 1:** Compute a starting point with Corollary 2.1 for the computation of \( x^* \), the positive root of \( \chi \).

**Step 2:** Compute \( x^* \) (4 Newton steps).

**Step 3:** Starting from \( x^* \), compute the roots of \( \phi \) with the help of the trinomials \( f \), (6 iterations with the method based on Theorem 3.1 for each root).

The number of iterations, necessary to compute the largest root of \( f \) with the method from Theorem 3.2, was 9, 7, 5, 3, 2, and 1, corresponding to the six times such a root needed to be computed. For the smallest root, we obtained 6, 5, 4, 2, 1, and 1 iterations. We observed that this decrease in the number of iterations required for the roots of \( f \) is typical, regardless of the degree of the polynomial.

The number of iterations in the computation of the roots of \( \phi \) does not vary significantly with increasing degree, although the number of Newton steps necessary for the computation of \( x^* \) tends to increase with increasing degree and may require an accelerated Newton method. On the other hand, it is usually not necessary to accurately compute \( x^* \).

![Figure 2: The functions \( \phi \leq f \leq h \) for \( q(z), k = 3, \text{ and } \bar{x} = 1.02. \)](image)

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