COSMOLOGICAL EXPANSION IN THE
RANDALL-SUNDRUM WARPED COMPACTIFICATION

J.M. CLINE
McGill University, 3600 University St.
Montréal, Québec H3A 2T8, Canada
E-mail: jcline@physics.mcgill.ca

The cosmology of a brane-universe embedded in a higher dimensional bulk space-
time presents some peculiarities not seen in ordinary (3 + 1) dimensional gravity. I
summarize the current understanding, with emphasis on the suggestion by Randall
and Sundrum that the bulk is 5-D anti-deSitter space, leading to a solution of the
weak scale hierarchy problem.

1 Large Versus Small Extra Dimensions

In the last few years there has been a revival of interest in the idea of extra
dimensions, first proposed by Kaluza and Klein. The new realization of Arkani-
Hamed, Dvali and Dimopoulos (ADD) was that the extra dimensions could be
macroscopically large if one assumed that our (3 + 1) dimensional universe
is a slice (a 3-brane) of the higher dimensional bulk. The particles of the
standard model should be restricted to the brane so that no light Kaluza-
Klein (KK) excitations exist, which otherwise would have already been seen.
Gravity, however, can propagate in the extra bulk dimensions (otherwise they
would have no observable consequences whatsoever). The effect of the extra
dimensions can only be seen on distance scales less than the order of their size.

With $N$ extra compact dimensions of size $R$, Newton’s gravitational force law
for two masses $m_1$ and $m_2$, separated by a distance $r$, is modified to

$$ F = \frac{\Gamma(3+N)}{4\pi(3+N)/2} \left( \frac{m_1 m_2}{M^{2+N} r^N} \right), \quad r \ll R, \quad (1) $$

where $M$ is the new quantum gravity scale appearing in the Einstein-Hilbert
action for gravity in $4 + N$ dimensions,

$$ S = -\frac{1}{2} M^{2+N} \int d^4 x d^N y \sqrt{|g|} R, \quad (2) $$

and $y_I$ parametrize the extra dimensions. At larger separations, the gravita-
tional flux is no longer diluted by spreading out in the bulk, so the force reverts
to its usual form,

$$ F = \frac{1}{8\pi} \left( \frac{m_1 m_2}{M_p^2 r^2} \right), \quad r \gg R, \quad (3) $$
involving the 4-D Planck mass $M_p$. Deviations of (3) from its $1/r^2$ form have only been tested at separations greater than a millimeter or so, showing that $R$ could be as large as 1 mm. This is obviously far bigger than the limit which exists if the standard model particles are allowed to propagate in the bulk, $R \lesssim 10^{-3}$ fm.

The relationship between the 4-D and the $(4 + N)$-D gravity scales can easily be deduced by requiring that the action (2) reduce to the usual one after integrating over the extra dimensions. Let us suppose that the line element in the full spacetime has the simple form

$$ds^2 = a^2(y_i) \left( dt^2 - \sum_{i=1}^{3} dx_i dx_i \right) - b^2(y_i) \sum_{I=1}^{N} dy_I dy_I. \quad (4)$$

Then, if the brane is located at $y_i = 0$, the relationship is

$$M^2_p = M^{N+2} \int \left( \frac{a(y_i)}{a(0)} \right)^2 b^N(y_i) d^N y. \quad (5)$$

In ADD, the geometry is assumed to be factorizable, meaning that $a$ does not depend upon $y_i$; hence the integral just gives the volume of the extra dimensions: $M^2_p = M^{N+2} V_N$. Since $V_N$ can be quite large (mm$^3$), this has the interesting consequence that $M$ could be at the TeV scale, yet be consistent with the much higher scale of $M_p$. This opens the mind-boggling possibility that all new particle physics, including quantum gravity, could become accessible at the LHC. Moreover we have a partial explanation of the weak scale hierarchy problem, the question of why $M_p$ is 16 orders of magnitude larger than the $W$ boson mass. It is not really a solution because one is left with the annoying question of why $R$ is so much larger than the natural scale, $1/M_p$. The exact size depends on the number of extra dimensions. If $N = 1$ it is not possible to obtain $M \sim 1$ TeV because $R$ is too large; demanding that $R = 1$ mm gives $M \sim 10^8$ GeV. But for $N = 2$, the TeV scale emerges just as the experimental bound on $R$ is saturated, and for higher dimensions it can be attained with smaller sizes, $R \sim 100$ fm in the case of $N = 6$.

The experimental constraints on large extra dimensions come from the effects of the KK excitations of the graviton, which can be very light, $m_n = n/R$ for integer $n$. The only thing which saves these particles from being easily discovered is their weak interactions; like the ordinary graviton, their couplings are suppressed by $1/M^2_p$ (as opposed to $1/M^2$). Consequently the bounds from accelerator physics are rather weak: $M \gtrsim$ several TeV. Astrophysics gives better constraints, at least for $N = 1$ and 2. One such bound comes from requiring that supernova 1987A not cool too quickly by graviton emission.
giving $M \gtrsim 100$ TeV for $N = 1$ and $M \gtrsim 5$ TeV for $N = 2$. In the early universe, KK gravitons can be produced by thermal processes, and decay slowly into photons that would distort the cosmic gamma ray background unless $M$ obeys bounds similar to the supernova ones. Actually, the last-mentioned bound is quite generous toward the ADD scenario because it assumes that, by some miracle, the universe is already free from primordially produced KK gravitons at temperatures near 1 MeV—a necessary condition since otherwise the gamma rays produced by their decays would destroy deuterium and consequently the successful predictions of big bang nucleosynthesis. It is quite difficult to justify this assumption. Benakli and Davidson showed that if $M = 1$ TeV, the reheat temperature after inflation would have to be no greater than 0.1 GeV, for $N = 6$; for smaller $N$ the bound is even more stringent. This is well below what is needed for electroweak baryogenesis, which is generally considered to be the lowest temperature mechanism available. Therefore baryogenesis presents a major challenge to the ADD idea.

Randall and Sundrum (RS) have suggested another way of solving the hierarchy problem with an extra dimension, which avoids the difficulties encountered by ADD. They considered just a single extra dimension, compactified on an orbifold $S_1/Z_2$, a circle modded by $Z_2$. The coordinate is in the range $y \in [-1, 1]$, with the endpoints identified and with $y \leftrightarrow -y$ being the orbifold symmetry. One places a 3-brane at each of the orbifold fixed points, $y = 0$ and $y = 1$. They have equal and opposite tensions, $\pm \sigma$ (tension is the 4-D energy density, which has the same form as a 4-D cosmological constant). In addition there is a 5-D cosmological constant in the bulk, $\Lambda$. The stress-energy tensor is therefore

$$T_{\mu\nu} = (g_{\mu\nu} - n_\mu n_\nu) \sigma (\delta(y) - \delta(y - 1)) / b + \Lambda g_{\mu\nu}, \quad (6)$$

where $n_\mu$ is the normal to the branes (hence the brane tensions make no contribution to $T_{yy}$). A static solution to the 5-D Einstein equations exists if

$$\Lambda = -\frac{\sigma^2}{6M_p^2}, \quad (7)$$

and it has the form of eq. (3) with

$$a(y) = e^{-ky}; \quad k = |\Lambda/\sigma| \quad (8)$$

and $b$, the size of the extra dimension, being undetermined. Using eq. (3), one finds that $M_p$ is related to the 5-D gravity scale by

$$M_p^2 = \frac{M_3^3}{k^2} (1 - e^{-2kb}). \quad (9)$$
The dramatic consequence of this solution is that if one considers the
Lagrangian for a particle confined to the brane at \( y = 1 \), it takes its canonical
form only after a Weyl rescaling of the field by the “warp factor” \( a(1) = e^{-kb} \). If
the Lagrangian originally had \( M_p \) as the mass scale for the particle, it becomes
rescaled by \( \exp(-kb) \). One can take all the parameters \( M, \Lambda, \sigma \) and \( k \) to be of
order \( M_p \) to the appropriate power; then if \( b \sim 36/k \sim 36/M_p \), one obtains TeV
scale masses on the \( y = 1 \) brane (henceforth called the TeV brane). Clearly
\( bk \sim 36 \) is a much more moderate hierarchy than \( M_p/M \sim 10^{16} \), so this
constitutes an attractive possible explanation of the weak scale. Furthermore
the extra dimension is still small, so the KK gravitons can be sufficiently heavy
to present no difficulties in the early universe.

This solution to the hierarchy problem requires that we are living on the
negative tension brane (taken to be at \( y = 1 \)). The positive tension brane at
\( y = 0 \) has no such suppression of its masses, so it is referred to as the Planck
brane, and must constitute a kind of hidden sector.

2 Effect of Extra Dimensions on Cosmological Expansion

For a while it appeared that cosmology could provide an interesting constraint
on large extra dimensions. Binétruy, Deffayet and Langlois (BDL) considered
the cosmological expansion of 3-brane universes in a 5-D bulk and found solu-
tions in which the Hubble expansion rate in the brane was related to the
energy density \( \rho \) on the brane by

\[
H = \frac{\dot{a}}{a} = \frac{\rho}{6M^3},
\]

in contrast to the usual Friedmann equation, \( H \propto \sqrt{\rho} \). Although other authors
had found inflationary solutions with this property, BDL were the first to point
out that it would be a problem for later cosmology. Especially, such a modifi-
cation to the expansion rate would probably drastically alter the predictions
of big bang nucleosynthesis.

It is not difficult to see from the 5-D Einstein equations, \( G_{\mu\nu} = M^{-3}T_{\mu\nu} \),
why one gets the unusual dependence of \( H \) on \( \rho \). Consider the 00 component,

\[
\frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) = \frac{a^2}{b^2} \left( \frac{a''}{a} + \frac{a'}{a} \left( \frac{a'}{a} - \frac{b'}{b} \right) \right) + \frac{1}{3M^3} T_{00}.
\]

To obtain the delta functions in \( T_{00} \), eq. (3), \( a'(y) \) must be discontinuous at
both branes, and the discontinuity is proportional to the total energy density on
the branes. Moreover the orbifold symmetry (as well as common sense) requires
that \(a(y)\) be symmetric about either brane, so that \(a'(1 - \epsilon) = -a'(1 + \epsilon)\). This implies that \(a'(y)\) itself is linearly proportional to the brane tension \(\sigma\). Now consider the \(yy\) component, which has no delta functions:

\[
\left( \frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} = 2 \frac{a'^2}{\dot{a}^2} \left( \frac{a'}{a} \right)^2 - \frac{1}{3M^3} T_{yy}
\]  

(12)

Recalling that \(H = \frac{\dot{a}}{a}\), clearly \(H^2\) will get contributions proportional to \((a'/a)^2 \sim \sigma^2\) as well as \(\Lambda\). In fact, if we allow for a cosmological constant in the bulk and extra energy densities \(\rho_P\) and \(\rho_T\), in addition to the respective tensions \(\sigma\) and \(-\sigma\) on the Planck and TeV branes, the complete expression becomes:

\[
H^2 = \frac{(\sigma + \rho_P)^2}{36M^6} + \frac{\Lambda}{6M^3} = \frac{(-\sigma + e^{-2kb}\rho_T)^2}{36M^6} + \frac{\Lambda}{6M^3}
\]

(13)

It was noticed\(^9\)\(^1\)\(^0\) that by tuning \(\sigma\) to cancel the contribution from \(\Lambda\) in the limit \(\rho_i = 0\), one obtains an expression for \(H\) which at leading order in \(\rho\) has the desired \(\sqrt{\rho}\) form, plus small fractional corrections of order \(\rho/M^4\). Not surprisingly, in retrospect, this tuning is precisely the same condition (7) required by RS to obtain their solution. (Any deviation from this condition results in an effective 4-D cosmological constant and therefore a nonstatic solution.) But at the time we first noticed this coincidence, it was striking to us, since we were unaware of RS and had thus come upon the condition (7) starting from a completely different motivation from that of RS.

However, all is not well with the cosmological solution leading to eq. (13). For one thing, the energy densities on the two branes are constrained, \(\rho_T = -e^{-2kb}\rho_P\). Moreover, this implies that \(\rho_T < 0\), i.e., that the energy density of matter on the TeV brane is negative, a physically unacceptable situation. Thus, although cosmology appeared to be normal on the Planck brane (for densities \(\rho_P \ll M^4\)), not so on the TeV brane, where the hierarchy problem is solved. This seemed to present a problem for the RS proposal\(^9\)\(^1\)\(^3\).

There were several attempts to solve this problem. In one it was observed that by decompactifying the orbifold\(^1\)\(^2\) placing the TeV brane at the position required by the hierarchy problem \((y \approx 36/kb)\), and giving it a tension between 0 and \(-\sigma/2\), one could obtain the normal Hubble rate on the TeV brane with a positive energy density\(^4\). However this solution involved simultaneous expansion of the bulk, which is unacceptable for late time cosmology because a growing \(b(t)\) leads to a Planck mass which is increasing in time, according to eq. (1). Hence gravity would be getting weaker on the TeV brane, contrary to stringent constraints on the time variation of Newton’s constant. In another

\(^a\)the factor \(e^{-2kb}\) was first pointed out by ref. 5
attempt it was pointed out that the normal Friedmann equation would ensue if the \( y_\mu \) component of \( T_{\mu\nu} \) was allowed to have a rather complicated dependence on the bulk coordinate \( y \), rather than being a constant (\( \Lambda \)). The origin of such a dependence seemed obscure (but see below).

It was recently shown that both of the cosmological problems of brane universe models—the artificial relation between the energy densities on the two branes, and the generically “wrong” form for the Friedmann equation—can be solved by introducing a mechanism for insuring that the size of the extra dimension remains fixed while the branes expand. Recall that this degree of freedom was completely undetermined in the RS model, meaning that it corresponds to a modulus, i.e., a massless field, in this context called the radion. This is problematic in itself, because it implies a fifth force, as in scalar-tensor theories of gravity, which in the present case has couplings suppressed by the TeV rather than the Planck scale. In the absence of a mechanism for stabilizing this modulus (see ref. for such a mechanism), it is not surprising that a fine-tuning between the brane energies as in eq. (13) should be needed to insure that the bulk does not expand along with the branes.

Somewhat less obvious is the fact that the normal Friedmann equation also results if \( b \) (the size of the compact dimension) is stabilized. Heuristically, this occurs because the bulk cosmological constant \( \Lambda \) is now replaced by a potential for \( b \), \( V(b) \). Since the \( y_\mu \) component of Einstein’s equation comes from the variation of the action with respect to \( b \), a new term appears in \( T_{y\mu} \),

\[
T_{y\mu} = b^2 (V(b) + bV'(b)).
\] (14)

Therefore the \( G_{y\mu} \) equation, which we used in the argument above to obtain \( H \propto \rho \), is no longer available for fixing the magnitude of \( H \); rather it determines \( b \) because \( b \) no longer sits at the bottom of the potential during a period of cosmological expansion, but is slightly shifted. Moreover ref. showed that the \( y \)-dependent stress-energy needed in to get the correct Friedmann equation automatically arises from the stabilization of the radion.

In it is shown that, if the radion is stabilized, then as long as the excess energy densities \( \rho_i \) (above and beyond those needed in eq. 12 to get a static solution) are small compared to the cutoff scales (\( M_p \) on the Planck brane and 1 TeV on the TeV brane), the two branes expand at approximately the same rate, given by

\[
H^2 = \frac{8\pi G}{3} \left( \rho_r + e^{-4kb} \rho_T + \int_0^1 dy b e^{-4kby} \rho_{\text{bulk}}(y) \right).
\] (15)

This expression can be derived from the effective 4-D Lagrangian obtained by integrating over the extra dimension in the background of the RS metric.
Notice the factor of $e^{-4kb}$ multiplying $\rho_T$. This is precisely the same redshifting of mass scales that applies to all masses on the TeV brane. Thus $\rho_T$ represents the bare value of the energy density, presumably of order $M_p^4$, while $e^{-4kb}\rho_T$ is the physically observed value. No such suppression occurs for matter living on the Planck brane. Therefore, if the expansion of the universe is to be dominated by matter on our (TeV) brane, it is necessary to demand that the Planck brane (and the bulk) be devoid of matter. Fortunately this does not seem to pose a major challenge: inflation will effectively empty out the Planck brane, as long as it harbors no nearly massless particles and the reheat temperature is significantly below the cutoff. Although it is tempting to suggest that $\rho_p$ could be the dark matter of the universe, it is difficult to see how it could be made sufficiently small, if it is not zero.

3 Outlook

The Randall-Sundrum proposal for solving the hierarchy problem with a small extra dimension looks compatible with most cosmological requirements. Unlike the ADD scenario of large extra dimensions, it does not suffer from the problem of light KK gravitinos wreaking havoc with nucleosynthesis and the cosmic gamma ray background. Furthermore it might have a plausible string theoretic origin, perhaps being related to the 5D-anti-deSitter space/conformal field theory correspondence and the holographic principle. There remain a few puzzles. One is the apparent possibility of a “dark radiation” term in the Friedmann equation

$$H^2 = \frac{8\pi G}{3} \left( \text{usual terms} + \frac{C}{a^4} \right)$$

which arises as an initial condition, due to the fact that the solutions to 5-D gravity have an additional constant of integration relative to 4-D. Does this oddity also disappear when the extra dimension is stabilized? Another problem is inflation, which typically requires the presence of an intermediate scale, $M_i \sim 10^{13}$ GeV, to get the right magnitude of density perturbations, $\delta \rho / \rho \sim M_i / M_p$. No such intermediate scale exists if the TeV scale is the true cutoff on our brane. A third interesting question is how to generalize the RS scenario to higher dimensions, which is just beginning to be explored.

References

1. N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B429 (1998) 263; Phys. Rev. D59 (1999) 086004;
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. B436 (1998) 257.
2. G. F. Giudice, R. Rattazzi and J. D. Wells, Nucl. Phys. B544, 3 (1999);
   T. Han, J. D. Lykken and R. Zhang, Phys. Rev. D59, 105006 (1999);
3. S. Cullen and M. Perelstein, Phys. Rev. Lett. 83, 268 (1999);
   V. Barger, T. Han, C. Kao and R. J. Zhang, Phys. Lett. B461, 34 (1999).
4. L. J. Hall and D. Smith, Phys. Rev. D60, 085008 (1999).
5. K. Benakli and S. Davidson, Phys. Rev. D60, 025004 (1999).
6. L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999).
7. P. Binétruy, C. Deffayet and D. Langlois, hep-th/9905012.
8. A. Lukas, B. A. Ovrut, D. Waldram, Phys. Rev. D60 (1999) 086001;
   N. Kaloper, Phys. Rev. D60, 123506 (1999);
   T. Nihei, Phys. Lett. B465, 81 (1999).
   H. B. Kim and H. D. Kim, hep-th/9909053.
9. C. Csaki, M. Graesser, C. Kolda and J. Terning, Phys. Lett. B462, 34 (1999).
10. J. M. Cline, C. Grojean and G. Servant, Phys. Rev. Lett. 83, 4245 (1999).
11. C. Csaki, M. Graesser, L. Randall and J. Terning, hep-ph/9911406.
12. L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999);
    J. Lykken and L. Randall, hep-th/9908076.
13. J. Cline, C. Grojean and G. Servant, hep-ph/9909496, to appear in Phys.
    Lett. B (2000).
14. P. Kanti, I. I. Kogan, K. A. Olive and M. Pospelov, Phys. Lett. B468, 31 (1999).
15. W. D. Goldberger and M. B. Wise, hep-ph/9911457.
16. W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. 83, 4922 (1999).
17. U. Ellwanger, hep-th/9909103.
18. J. Cline, talk given at “Current Issues in String Cosmology” workshop,
    IHES, Bures-sur-Yvette, France, 21 June, 1999.
19. P. Kanti, I. I. Kogan, K. A. Olive and M. Pospelov, hep-ph/9912266.
20. C. Grojean, J. Cline and G. Servant, hep-th/9910081.
21. H. Verlinde, hep-th/9906182.
22. P. Kraus, JHEP 9912, 011 (1999) [hep-th/9910149];
    P. Binétruy, C. Deffayet, U. Ellwanger and D. Langlois, hep-th/9910219;
    E. E. Flanagan, S. H. Tye and I. Wasserman, hep-ph/9910498;
    S. Mukohyama, hep-th/9911167.
23. A. G. Cohen and D. B. Kaplan, hep-th/9910132.
   C. Csaki, J. Erlich, T. J. Hollowood and Y. Shirman, hep-th/0001033.
   Z. Chacko and A. E. Nelson, hep-th/9912186.