Photon-efficient grey-level image projection by the generalized phase contrast method

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Abstract. The generalized phase contrast (GPC) method is analysed in the context of projecting two-dimensional grey-level images. A rectangular phase contrast filter is introduced, instead of the usually applied circular filter, to correspond better with the rectangular aperture characteristic of commercially available spatial light modulators. A simple expression is derived for determining the appropriate phase-only modulation to convert an initial plane wave illumination into a desired output intensity distribution. Numerical simulations demonstrate excellent image reconstruction by GPC using the derived phase-to-intensity mapping. Optical throughput is predicted to be as high as 87% for optimum image quality, and may reach 98% with some trade-off in image fidelity.
1. Introduction

Generalized phase contrast (GPC) is a powerful technique for transforming the spatial phase information of an optical beam into a spatial intensity distribution without absorbing light. It may be seen as a generalization of the Zernike phase contrast (ZPC) method [1]. While ZPC is restricted to a small-scale phase approximation, GPC extends the domain to the full phase range of an optical cycle.

The basic principle of GPC involves separating a light beam into its Fourier components using a lens. The on-axis, low spatial frequency components are shifted in phase using a small wave retarder or phase contrast filter (PCF). A second lens then recombines the high and low spatial frequency components. The introduced phase shift causes the different components to interfere and produces an intensity distribution according to the phase information carried by the higher spatial frequencies [2]. This concept has been applied by others to high-resolution quantitative phase measurements in microscopy [3].

The availability of programmable phase-only spatial light modulators (SLMs) has also allowed GPC to be demonstrated as a highly useful method for the efficient projection of arbitrary and reconfigurable images. In particular, GPC has been applied with great success in the projection of binary light patterns through arbitrary-NA microscope objectives for real-time three-dimensional (3D) manipulation of microscopic particles [4], including live biological cells [5] and potential microtool structures [6]. The key advantages of GPC for real-time 3D optical micromanipulation are efficient utilization of the incident light through phase-only modulation, and rapid update of the generated light patterns due to the low computational overhead [7]. These same features make the GPC method a suitable scheme for projecting 2D grey-level images.

Phase-only modulation can create intensity patterns without blocking or absorbing light. It is therefore attractive as an inherently more efficient alternative to amplitude modulation based light-valve systems for image and video projection. Computer generated holography (CGH) is perhaps the most popular approach to image projection by phase-only modulation. However, the CGH method is notorious for being computationally expensive [8] and demanding in terms of the required high space-bandwidth product. Some researchers have even gone to the extent of designing specialized hardware to deal with the cumbersome iterative phase-retrieval.
algorithms [9]. Others have taken the route of exchanging spatial bandwidth in the SLM, for temporal bandwidth during hologram computation [10]. With the GPC method however, it is possible to address the appropriate phase patterns directly, without need for such compromises [11]. GPC perhaps represents the most feasible solution to the problem of real-time video projection by phase-only modulation.

Other than image projection, GPC has also been used successfully to demonstrate phase-only parallel decryption of scrambled phase patterns [12, 13]. Encoded information was obscured in a fixed binary phase mask or on a reconfigurable SLM device. The overlay of a second phase pattern, the decrypting key, allowed the retrieval of the information as a binary intensity distribution. The extension of such a system to use multiple phase-levels and grey-level imaging would increase both the security of the system and density of information storage.

Earlier mathematical analyses of the GPC method were either limited to 1D [11], or considered only for circular apertures at the object plane [2]. Further, interest in optical micro-manipulation applications also led to an emphasis on binary- and ternary-level images [14]. Here, a new analysis of the GPC method is presented that considers a rectangular illumination aperture together with a rectangular PCF. Such investigation is particularly relevant in the context of efficient utilization of commercially available SLM devices. Imposition of a circular truncated illumination results in a significant portion of the typically rectangular active SLM area not being illuminated and therefore, not utilized.

Notably, most SLM devices are further limited by inter-pixel dead space that primarily introduces higher-diffraction-order replicas at the Fourier plane, as well as additional zero-order signal [15]. Such issues are not directly considered here, but can be overcome in practice by utilizing non-pixelated devices such as an optically-addressed SLM.

The succeeding section begins with a description of the generic optical system for GPC image projection. It then continues with a mathematical analysis of GPC, wherein a simple expression for the phase-to-intensity mapping of GPC with rectangular apertures is derived. This result is later tested via numerical simulations of 2D grey-level image projections. The images produced using different PCF sizes are compared in terms of image fidelity and light efficiency.

2. GPC with a rectangular aperture

2.1. The generic optical system

Figure 1 illustrates the generic optical system being considered. Two lenses are arranged in a standard 4-f lens configuration. A collimated light beam illuminates a phase-only SLM at the front focal plane of the first lens. Aside from introducing a spatial phase modulation, \( \phi (x, y) \), the SLM also imposes a rectangular truncation window. Assuming a unit-amplitude monochromatic plane wave illumination, the spatial field distribution entering the 4-f system takes the form

\[
a (x, y) = \text{rect} \left( \frac{x}{\Delta x}, \frac{y}{\Delta y} \right) \exp[i\phi (x, y)]
\]

where \( \Delta x \) and \( \Delta y \) are the dimensions of the SLM window.

The spatial filter located at the Fourier plane is similar to the PCF considered in previous analyses [2, 11]. Here however, a rectangular, rather than circular, phase retarder is introduced.
Figure 1. 4-f imaging system for grey-level image projection by the GPC method. In the analysis, rectangular formats are considered for both the SLM and the PCF. An input phase, \( \phi(x, y) \), is mapped to an output intensity distribution \( |o(x', y')|^2 \).

to match the rectangular window of the SLM. A phase-only filter is ideal for maximum light throughput. This is described in the spatial frequency coordinates, \( f_x \) and \( f_y \), by

\[
H(f_x, f_y) = 1 + \left[ \exp(i\theta) - 1 \right] \text{rect}(f_x/\Delta f_x, f_y/\Delta f_y)
\]

where \( \theta \) is the phase retardation introduced by the filter with dimensions \( \Delta f_x \) and \( \Delta f_y \) (in the spatial frequency domain).

The goal of the succeeding analysis is to determine the appropriate input phase distribution, \( \phi(x, y) \), that will yield a target output intensity, \( |o(x', y')|^2 \) at the image plane behind the second lens.

2.2. Phase-to-intensity mapping

Within the framework of scalar diffraction theory \[16\], the actions of consecutive lenses in a 4-f set-up are described by two consecutive Fourier transforms, \( F\{ \} \). As a mathematical convenience, the second Fourier transform is performed as an inverse Fourier transform, \( F^{-1}\{ \} \), with inverted spatial coordinates \( (x', y') \). Taking into account the filter function in (2), the field distribution at the image plane is expressed as

\[
o(x', y') = a(x', y') + \left[ \exp(i\theta) - 1 \right] c(x', y')
\]

where

\[
c(x', y') = F^{-1}\{ \text{rect}(f_x/\Delta f_x, f_y/\Delta f_y) F\{ a(x, y) \} \}
= |c(x', y')| \exp \left[ i\phi_c(x', y') \right]
\]

\( c(x', y') \) represents the Fourier components of \( a(x, y) \) that experience a phase shift through the PCF. In the context of GPC, \( c(x', y') \) is conveniently interpreted as a reference wave that interferes with \( a(x', y') \) at the image plane. Within the bounds of \( \text{rect}(x'/\Delta x, y'/\Delta y) \), the resulting intensity distribution is

\[
|o(x', y')|^2 = 1 + 4|c(x', y')| \sin(\theta/2) \left\{ |c(x', y')| \sin(\theta/2) \right. \\
+ \sin \left[ \phi(x', y') - \phi_c(x', y') - \theta/2 \right] \}
\]

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In order to achieve good contrast in the output image, the phase-to-intensity mapping should be capable of producing a zero-value output for some input phase value \( \phi_o(x', y') \). That is, following (3),

\[
0 = \exp \left[ i\phi_o(x', y') \right] + \left[ \exp (i\theta) - 1 \right] c(x', y')
\]  

Equation (6) is satisfied if and only if \( |c(x', y')| \) approaches a constant

\[
|c(x', y')| \to \frac{1}{2|\sin(\theta/2)|}.
\]

Similarly, a constraint on \( \phi_c(x', y') \) can be derived by substituting (7) in (5), and again applying the zero-output condition:

\[
\phi_c(x', y') \to \phi_o(x', y') - \theta/2 \pm \pi/2,
\]

with ‘\( \pm \)’ arising due to the term \( \sin(\theta/2)/|\sin(\theta/2)| \). Thus, \(+\pi/2\) is taken when \( \sin(\theta/2) \) is positive, and \(-\pi/2\) otherwise. Substituting (7) and (8) into (5) leads to the much simplified expression

\[
|o(x', y')|^2 = 4\sin^2 \left\{ \left[ \phi(x', y') - \phi_o(x', y') \right]/2 \right\}
\]  

and the corresponding phase-to-intensity mapping,

\[
\phi(x', y') = 2\sin^{-1} \left[ \frac{|o(x', y')|}{2} \right] + \phi_o(x', y').
\]

Notably, \( |o(x', y')|^2 \) is not entirely arbitrary, but is constrained by energy conservation. Assuming the ideal case where total power through the output aperture is equal to the power through the input aperture, and given the unit-amplitude field described by (1), the target pattern can be expressed in terms of a normalized intensity distribution

\[
|o(x', y')|^2 = \Delta x \Delta y |o_{\text{norm}}(x', y')|^2,
\]

where

\[
\int_S |o_{\text{norm}}(x', y')|^2 \, dx' \, dy' = 1,
\]

and \( S \) is the surface defined by \( \text{rect}(x/\Delta x, y/\Delta y) \). Thus,

\[
\phi(x', y') = 2\sin^{-1} \left[ \frac{(\Delta x \Delta y)^{1/2} |o_{\text{norm}}(x', y')|}{2} \right] + \phi_o(x', y'),
\]

where \( \phi_o(x', y') \) is a phase constant that defines the zero-output phase value. This mapping serves as a basis for designing an input phase distribution \( \phi(x', y') \) to produce a desired output intensity distribution \( |o(x', y')|^2 \) with an arbitrary PCF-shift \( \theta \). Although (13) is not explicitly \( \theta \)-dependent, the zero-output phase \( \phi_o(x', y') \) depends on \( \theta \) and \( \phi_c(x', y') \), as shown in (8).

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Notably, (7) suggests that the ideal reference wave \( c(x', y') \) is a plane wave, or at least has a uniform amplitude dictated by \( \theta \). This would provide uniform contrast over the entire aperture of the output image. However, the actual features of \( c(x', y') \) are determined by the PCF aperture, and the input phase distribution itself. In particular, aperture size influences the flatness of the reference wave. A smaller aperture leads to a more planar wave-front, but at the expense of less energy being carried by \( c(x', y') \) [2]. The input distribution \( a(x, y) \) may also add distortions to the reference wave if low spatial frequency components, other than the dc component, are projected through the PCF in the Fourier plane.

The next section describes how the reference wave can be approximated by a plane wave. Succeeding that, the particular case of a phase shift \( \theta = \pi \) is considered. Some useful parameters such as the zero-output phase value and the optimal size of the PCF aperture are calculated.

### 2.3. Approximating the reference wave

As in previous analyses of the GPC method [2, 11], the following approximation for the reference wave is useful:

\[
c(x', y') = \mathcal{F}^{-1}\{\text{rect}(f_x/\Delta f_x, f_y/\Delta f_y)\mathcal{F}\{a(x', y')\}\} \approx \bar{a} g(x', y'),
\]

where \( g(x', y') \) is a plane wave truncated by the object aperture at the input plane, and again by the filter aperture at the Fourier plane,

\[
g(x', y') = \mathcal{F}^{-1}\{\text{rect}(f_x/\Delta f_x, f_y/\Delta f_y)\mathcal{F}\{\text{rect}(x/\Delta x, y/\Delta y)\}\}
\]

\( \bar{a} \) is the complex spatial average of the input field \( a(x, y) \),

\[
\bar{a} = |\bar{a}| \exp (i\phi_{\bar{a}}) = \int_S \frac{\exp(i\phi(x, y))}{\Delta x \Delta y} \, dx \, dy.
\]

This approximation is most appropriate when there is a distinct separation between high and low spatial frequency components of \( a(x, y) \) at the Fourier plane. It essentially assumes that only the dc component of \( a(x, y) \) contributes to the phase-shifted reference wave \( c(x', y') \). All other image information is then carried in higher spatial frequencies that do not pass through the PCF centre.

The appendix shows the evaluation and expansion of \( g(x', y') \) into an infinite series. The largest contribution to \( g(x', y') \) is from the spatially non-varying term. It is thus approximated as

\[
g(x', y') \approx \frac{4}{\pi^2} \text{Si}(\pi \eta_x) \text{Si}(\pi \eta_y),
\]

where \( \text{Si}(\cdot) \) is the sine integral function [17], and \( \eta_x \) is the ratio of the width of the PCF to the width of the central lobe of the field distribution, i.e. sinc distribution, along the corresponding axis at the Fourier plane,

\[
\eta_x = \Delta x \Delta f_x / 2.
\]
Approximation (17) also suggests that

$$\phi_c(x', y') \approx \bar{\alpha}$$ \hspace{1cm} (19)

A convenient situation would be for this phase to vanish, i.e. $\bar{\alpha}$ becomes real-valued. In fact, this is realizable as (9) implies that the phase-to-intensity mapping (13) is arbitrary to within the complex conjugate of $[\phi(x', y') - \phi_o(x', y')]$. Thus, there is enough flexibility to tune the input phase distribution such that

$$\int_S \sin[\phi(x', y')] \, dx' \, dy' = 0$$ \hspace{1cm} (20)

and

$$\bar{\alpha} = 0.\hspace{1cm} (21)$$

2.4. Projection example: $\theta = \pi$

As a practical example, a PCF that provides $\theta = \pi$ is particularly convenient for image projection as it leads to

$$\phi_o(x', y') = 0$$ \hspace{1cm} (22)

following (8), (19), and (21). A phase distribution is thus easily drawn according to

$$\phi(x', y') = 2\sin^{-1}\left[\frac{(\Delta x \Delta y)^{1/2} |\alpha_{\text{norm}}(x', y')|}{2}\right]$$ \hspace{1cm} (23)

and then condition (20) is satisfied by taking the phase conjugate of alternate pixels. Assuming the initial phase pattern is slowly varying, as in a picture with smooth features, such a checkered pattern of phase-flipping should closely approach the desired zero-phase average. The checkered pattern also provides the additional benefit of introducing a high-frequency phase modulation to the input signal. Such modulation enforces the validity of the approximation introduced in (14).

While assuming a value for $\theta$ implies a particular thickness for the PCF, the appropriate transverse dimensions remain to be determined. Ideally, the magnitude of (14) should satisfy (7), i.e. $|c(x', y')| \approx \bar{\alpha}g(x', y') = 1/2$ for $\theta = \pi$. $g(x', y')$ depends only on the aperture of the PCF, while $\bar{\alpha}$ is determined by the information content of the image. Strictly then, an optimal aperture will need to be matched to a particular image. However, (16) shows that $\bar{\alpha}$ defined as a normalized quantity, reaching only values between 0 and 1. The mean of this range, $\bar{\alpha} = 1/2$, thus seems like a reasonable approximation. This leads to a requirement that $g(x', y') = 1$. Plotting the sine integral functions in (17) then suggests that the optimal dimensions for a rectangular PCF satisfy $\eta_x = \eta_y \approx 0.62$.

3. Numerical simulations of GPC 2D image projection

Eight-bit grey-level images, each $256 \times 256$ pixels in size, were used to test the accuracy of the derived phase-to-intensity mapping with $\theta = \pi$. Appropriate phase patterns were calculated based on (23) and checkered phase-flipping was applied to satisfy (20). The phase distributions

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were quantized to 256 phase-levels between \([0, 2\pi]\) to emulate the finite bit-depth of a practical SLM device.

The actions of the optical system, as illustrated in figure 1, on the modulated light were simulated to yield the output intensity distribution. This was easily accomplished with the aid of an efficient fast Fourier transform (FFT) algorithm [18]. Finally, the mean-square error (MSE = \(\frac{1}{\text{total number of pixels}} \sum (\text{original pixel value} - \text{rescaled output pixel value})^2\)) of each output image was measured with respect to the corresponding original image. The brightness of output images were rescaled before MSE-comparison such that the integrated brightness matched that of the corresponding original images.

Figures 2(a) and (b) show some representative output images. The corresponding original images are shown in figures 2(c) and figures 2(c) and (d), respectively. Presented images employed \(9 \times 9\)-pixel PCFs. Given the 256-pixel image aperture and \(2048 \times 2048\) pixels FFT array employed, this PCF size corresponded to \(\eta = 0.5625\). Visual comparison suggested excellent image reconstruction, consistent with the low MSE scores. Light projection was also very efficient, with 87% of the input light contributing to the final image in both examples. These two images were particularly well-suited to the GPC image projection method. The many sharply contrasting details in the images translated to dominantly high spatial frequency content, thus making the approximation in (14) highly applicable.

Other GPC image projections under the similar conditions are shown in figures 3(a) and (b), with corresponding original images shown in figures 3(c) and (d). These images featured larger flat areas and a lack of sharply contrasting edges. Consequently, reconstruction quality was expected to be poorer as significant image information was not well separated from the dc spatial frequency component. MSE scores were indeed nominally higher, but subjective image quality still remained very good. Errors manifested mainly as degraded contrast, particularly close to edges of the images. Diffraction efficiency was also slightly reduced to 85% (figure 3(a), ‘flower’) and 84% (figure 3(b), ‘mallet’).

Figures 4(a)–(d) demonstrate the effect of PCF size on image quality. Larger PCF sizes were used in figures 4(a) \((\eta = 2.5625, 41 \times 41\)-pixels\), and (b) \((\eta = 0.9375, 15 \times 15\)-pixels\), in comparison to the best-quality image in figure 4(c) \((\eta = 0.5625, 9 \times 9\)-pixels\). Figure 4(d) utilized the smallest PCF \((\eta = 0.1875, 3 \times 3\)-pixels\). Visual comparison revealed some image quality degradation, primarily contrast errors, introduced by using PCF sizes much larger or much smaller than the optimal size predicted by (21), i.e. \(\eta \approx 0.62\).

Variations were further emphasized by the corresponding absolute error \(|\text{original pixel value} - \text{rescaled output pixel value}|\) maps in figures 5(a)–(d). Perhaps the most interesting details presented in these figures were the dark outlines in the background of the image, indicating robust reconstruction of edge features. Edge information is carried by high spatial frequency components, and fits well under the approximations of the previous derivations. On the other hand, figures 5(a) and (b) revealed the limitations of the planar reference wave assumption for larger PCF apertures.

Figure 6(a) plots the MSE behaviour as a function of \(\eta\) for the four images presented. In all cases, the lowest MSE score was achieved by the PCF size closest to \(\eta = 0.62\). This was \(9 \times 9\)-pixels with \(\eta = 0.5625\). Only odd-sized filters were used to preserve the symmetry of the system, leading to coarse steps between \(\eta\)-values. Finer steps are possible by either reducing the image size, or increasing the FFT array, e.g. to \(4096 \times 4096\)-pixels. However, the configuration used sufficiently demonstrated high-quality reconstruction of the images.
Figure 2. Simulated projections of images (a) ‘bird’ and (b) ‘lettuce’ by the GPC method using 9 × 9-pixel PCF sizes, i.e. $\eta = 0.5625$. MSEs were calculated with respect to the corresponding original images, (c) and (d). Images are 256 × 256-pixels, with 8-bit grey-level depth.

An interesting system trade-off between image quality and power throughput is suggested by the previous graph and the plot of image gain against PCF size in figure 6(b). Here, gain is defined as the ratio of integrated brightness of an image projected by GPC phase modulation, to the expected brightness of the same image projected by pure amplitude modulation, assuming the same incident light power. This value represented the light efficiency of the GPC method compared to a traditional light-valve type projection system.

As seen in the plot, gain increased as PCF size was increased over the best image quality condition. Adjusting the PCF size was analogous to tuning the beam ratio in an interferometer to increase fringe visibility. Gain reached values as high as 2.44, corresponding to a diffraction
Figure 3. Simulated projections of images (a) ‘flower’ and (b) ‘mallet’ by the GPC method using $9 \times 9$-pixel PCF sizes, i.e. $\eta = 0.5625$. MSEs were calculated with respect to the corresponding original images, (c) and (d). Images are $256 \times 256$-pixels, with 8-bit grey-level depth.

efficiency of 98%. Although the MSE score rapidly worsened for the larger PCF sizes, the image was still clearly recognizable, as in figures 4(a) and (b). This may prove useful in some applications where light efficiency and power throughput might be considered primary over image fidelity, for example in fully parallel laser marking.

Notably, very small PCF sizes also indicated relatively large gain values, but only because these represented minimal perturbation of the input field. Figures 4(d) and 5(d) show the poor image reconstruction with a PCF width of 3-pixels ($\eta = 0.1875$).
4. Conclusion

Previous analyses of the GPC method dealt primarily with circular apertures and circular PCFs. Here, we presented a mathematical analysis of a GPC optical system utilizing a rectangular aperture and PCF. This improved understanding paves the way for more efficient utilization of the rectangular format common to commercially available SLMs.

In particular, we derived a simple expression for directly calculating the appropriate phase modulation to produce an arbitrary 2D grey-level image by the GPC method. Excellent image
reconstruction in the numerical simulations showed GPC to be a promising technique for efficient grey-level image projection. Predicted light diffraction efficiencies reached 87% for optimal image quality. Light efficiencies as high as 98% are possible by adjustment of the PCF size, but with some trade-offs against image quality.

Figure 5. Absolute error maps of image projections by the GPC method emphasize the contrast errors introduced by using various PCF sizes. Image projections (figures 4(a)–(d)) are compared to the original image (figure 3(c)), absolute error $= |\text{original pixel value} - \text{rescaled output pixel value}|$. 
Figure 6. (a) MSE of simulated image projections (green triangles, ‘flower’; orange squares, ‘mallet’; red crosses, ‘lettuce’; blue diamonds, ‘bird’) by the GPC method using varying PCF sizes. (b) Light efficiency of the same projected images measured as a ratio of the integrated brightness at the output to that expected from purely amplitude modulation. Curves serve only as guides to the eye.

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Appendix. Derivation of series expansion of \( g(x', y') \)

In previous analyses [2, 11], \( g(x', y') \) was introduced as the synthetic reference wave in a common path interferometer. It essentially describes a plane wave truncated by the object aperture at the input plane, and again by the filter aperture at the Fourier plane. Given rectangular apertures at both planes, it is expressed as

\[
g(x', y') = F^{-1} \left\{ \text{rect}(f_x/\Delta f_x, f_y/\Delta f_y) F\{\text{rect}(x/\Delta x, y/\Delta y)\} \right\}.
\]  

(A.1)

The functions involved in (A.1) are separable in \( x \) and \( y \). Thus for compactness, equivalent 1D expressions are momentarily considered, where

\[
g(x', y') = g_x(x') g_y(y')
\]  

(A.2)

and

\[
g_x(x') = F^{-1} \{ \text{rect}(f_x/\Delta f_x) F\{\text{rect}(x/\Delta x)\} \}.
\]  

(A.3)

The Fourier transform of \( \text{rect}(x/\Delta x) \) is well known, and the inverse Fourier transform is written explicitly in its integral form,

\[
g_x(x') = \Delta x \int_{-\Delta f_x/2}^{\Delta f_x/2} \text{sinc}(\Delta xf_x) \exp(-i2\pi x'_f_x) df_x.
\]  

(A.4)
Further, the exponential function is expanded as a Maclaurin series, and the sinc function written out explicitly as well,

\[ g_x(x') = \Delta x \sum_{n=0}^{\infty} \frac{(-i2\pi x')^n}{n!} \int_{-\Delta f_s/2}^{\Delta f_s/2} f_x^n \frac{\sin(\pi \Delta x' f_x)}{\pi \Delta x' f} \, df_x. \]  

(A.5)

Separating the \( n = 0 \) term of the summation, then resetting the dummy index, \( n \), such that the lower limit is again 0 yields

\[ g_x(x') = \Delta x \int_{-\Delta f_s/2}^{\Delta f_s/2} \frac{\sin(\pi \Delta x f_x)}{\pi \Delta x f_x} \, df_x + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-i2\pi x)^{n+1}}{(n+1)!} \int_{-\Delta f_s/2}^{\Delta f_s/2} f_x^n \sin(\pi \Delta x f_x) \, df_x. \]  

(A.6)

The integral in the second term of (A.6) is solved by successive integration-by-parts and leads to an \( n + 1 \)-long series of alternating sines and cosines. These, in turn, are expressed compactly in terms of complex exponentials as

\[ g_x(x') = \Delta x \int_{-\Delta f_s/2}^{\Delta f_s/2} \frac{\sin(\pi \Delta x f_x)}{\pi \Delta x f_x} \, df_x + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-i2\pi x)^{n+1}}{(n+1)!} \left\{ \sum_{m=0}^{n} \frac{n! f_x^{n-m}}{(n-m)! (\pi \Delta x)^{m+1}} (-1)^m \right\} \int_{-\Delta f_s/2}^{\Delta f_s/2} f_x^n \sin(\pi \Delta x f_x) \, df_x. \]  

(A.7)

The largest contribution to \( g_x(x') \) comes from the leading term in (A.7)

\[ K_x = \Delta x \int_{-\Delta f_s/2}^{\Delta f_s/2} \frac{\sin(\pi \Delta x f_x)}{\pi \Delta x f_x} \, df_x. \]  

(A.8)

The integrand is an even function, thus it only needs to be evaluated over half the interval. A slight change in the variables to \( \psi_x = \pi \Delta x f_x \) places the integral into the standard form of the sine integral function, \( \text{Si}(\cdot) \) [17],

\[ K_x = \frac{2}{\pi} \int_{0}^{\pi \Delta x \Delta f_s/2} \frac{\sin(\psi_x)}{\psi_x} \, d\psi_x = \frac{2}{\pi} \text{Si}(\pi \Delta x \Delta f_s/2). \]  

(A.9)

Further substituting \( \eta_x \) for the ratio of the PCF width to the width of the central sinc lobe at the Fourier plane [2], i.e.

\[ \eta_x = \Delta x \Delta f_s/2, \]  

(A.10)

yields

\[ K_x = \frac{2}{\pi} \text{Si}(\pi \eta_x). \]  

(A.11)

The synthetic reference wave can be approximated by this lowest order term, in 2D it is

\[ g(x', y') \approx \frac{4}{\pi^2} \text{Si}(\pi \eta_x) \text{Si}(\pi \eta_y). \]  

(A.12)
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