DONE: Distributed Newton-type Method for Federated Edge Learning

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Abstract—There is growing interest in applying distributed machine learning to edge computing, forming federated edge learning. Compared with conventional distributed machine learning in a datacenter, federated edge learning faces non-independent and identically distributed (non-i.i.d.) and heterogeneous data, and the communications between edge workers, possibly through distant locations with unstable wireless networks, are more costly than their local computational overhead. In this work, we propose a distributed Newton-type algorithm (DONE) with fast convergence rate for communication-efficient federated edge learning. First, with strongly convex and smooth loss functions, we show that DONE can produce the Newton direction approximately in a distributed manner by using the classical Richardson iteration on each edge worker. Second, we prove that DONE has linear-quadratic convergence and analyze its computation and communication complexities. Finally, the experimental results with non-i.i.d. and heterogeneous data show that DONE attains the same performance as the Newton’s method. Notably, DONE requires considerably fewer communication iterations compared to the distributed gradient descent algorithm and outperforms DANE, a state-of-the-art, in the case of non-quadratic loss functions.

Index Terms—Distributed optimization, Federated learning, Edge computing, Newton’s method.

I. INTRODUCTION

Based on the increasing adoption of mobile and Internet-of-Things (IoT) devices, Cisco estimates that nearly 847 ZB of data will be generated at the network edge by 2021, compared to the global datacenter traffic that will only reach 19.5 ZB by 2021 [1]. Thus, there is a paradigm shift in terms of “big data” generation and storage from large-scale datacenters to edge networks, which fuels the booming trend of intelligence at the edge [2]–[4].

Traditional centralized machine learning procedures are gradually becoming inadequate as the computational and storage capacities of individual machines fall short of the quantity of data involved in learning. The shift to parallel and distributed approaches follows logically from this: they offer more storage, more scalable computing power, and can be set up and implemented in various geographical locations. Such architectures – most notably MapReduce [5], Apache Spark [6], [7], etc. – have been developed quickly to adapt to large-scale distributed machine learning in datacenters.

The existence of (i) powerful edge computing with data abundance and (ii) successful datacenter-type distributed machine learning architectures raises a natural question: Can we apply large-scale distributed machine learning to edge computing networks? To address this, we identify two key obstacles to overcome. First, while data sources of datacenter-type distributed machine learning are centrally collected, shuffled, and evenly distributed to worker nodes to form i.i.d. data sources [8], the data at edge networks are collected separately. Edge data sources are, therefore, generally non-i.i.d., heterogeneous in terms of sizes, and cannot be sent over a network for shuffling (i.e., non-shareable data). These properties of edge data sources share similarities with those of cross-device federated learning originally proposed for smart devices [9], thus distributed learning at edge networks is called federated edge learning.

Second, identifying the key “bottleneck” in the learning performance will differentiate federated edge learning from other distributed learning schemes. Typical distributed machine learning schemes involve local computation of data stored in each worker node and global communication between workers and an aggregator. We see that datacenter-type distributed machine learning and cross-device federated learning are two extremes: while the former involves learning with optimized computing nodes, data shuffling, and communication networks, the latter requires a massive number of participating devices with constrained computing, storage, and communication capacities. On the other hand, the storage and computing capacities of federated edge learning can be comparable to the former, whereas its communication environments are similar to those of the latter due to notable physical distances between edge workers, multi-hop transmission, and different types of communication medium (e.g., wireless/optical/wireline/mmWave.) Compared with local computation cost at each edge worker, the cost of edge communication is considerably higher in terms of speed, delay, and energy consumption [2], [4], [10], [11]; thus, it is...
often considered as the bottleneck specific to federated edge learning.

Our goal is to develop a communication-efficient distributed algorithm, which only uses a minimal number of communication rounds to reach a certain precision for convergence and handles non-i.i.d. and heterogeneous data across edge workers. In the literature, distributed variations of first-order methods, including stochastic gradient descent (SGD) accelerated SGD, variance reduced SGD, stochastic coordinate descent methods and dual coordinate ascent algorithms, are found most frequently. These methods use only local gradient information, which can be smoothly implemented in a distributed manner and requires little local computation. Second-order algorithms like the Newton’s method, which have also been well studied in literature, on the other hand, use both gradient and curvature information. Their advantage over the first-order counterparts is that by finding a “better” descent direction, they can lead to a substantially faster convergence rate, therefore requiring much fewer communication rounds to find a solution.

In the context of federated edge learning, although the Newton’s method comes at a cost of more local computation per iteration, this is gradually remedied by the improvement in computing power of edge workers and becoming less significant over time. Even though the Newton’s method is not friendly to distributed implementations due to its descent direction structure involving complicated inverse matrix-vector product, we design an algorithm that can overcome this issue and is summarized as follows.

- We propose a distributed Newton-type algorithm (DONE) for federated edge learning. When the loss function of the learning task is strongly convex and smooth, the gist of DONE is exploiting the classical Richardson iteration to enable the Newton’s method to be distributively implementable on the edge workers.
- Theoretically, we show that DONE has a local linear-quadratic convergence rate. We also analyze the computation and communication complexities of DONE. Specifically, with generalized linear models and carefully chosen parameters, the computation complexity of DONE is comparable to that of the first-order method GD.
- Experimentally, using linear regression and multinomial logistic regression tasks on non-i.i.d. and heterogeneous data, we show that DONE has almost the same performance as the true Newton’s method and therefore significantly reduces the number of communication iterations compared to first-order GD. DONE also outperforms DANE, a state-of-the-art distributed approximate Newton’s algorithm, in a multinomial logistic regression task.

The rest of this paper is organized as follows. In Section II we will review the related works. In Section III we will present the system model and DONE. The convergence and complexity analysis of DONE will be in Section IV. Section V will show experimental settings and results, and Section VI concludes our work.

II. RELATED WORK

First-order distributed optimization methods. First-order methods find a descent direction based on gradient information. They are, in general, parallelizable, easy to implement, and are the most common in practice. Stochastic gradient descent (SGD) [16–18], accelerated SGD [19], variance reduction SGD [20, 21], stochastic coordinate descent [22–25], and dual coordinate ascent [25–27] are examples of this type of algorithms. Recently, [28] proposes the use of varying workers per round to tackle the high cost of specialized hardware (such as GPUs) in distributed SGD training. As mentioned earlier, nevertheless, the trade-off for the less computation at the edge, i.e., finding only the gradient and possibly some additional information, is the requirement of many iterations until convergence. Communication overheads become a bigger problem in multi-hop edge networks where the bandwidth varies or is limited among workers.

Newton-type distributed optimization methods. Different from first-order methods, Newton-type methods use not only gradient but also curvature information to find a descent direction. Existing distributed Newton-type algorithms include DANE [13], AIDE [21], DiSCO [29], and GIANT [30]. Both DANE (including its inexact version AIDE) and GIANT are approximate Newton-type methods. To approximate the Newton direction, DANE requires each worker to solve a well-designed local optimization problem, whereas GIANT uses the harmonic mean (instead of the true arithmetic mean) of the Hessian matrix. On the other hand, DiSCO is an inexact damped Newton method using a preconditioned conjugate gradient algorithm, which requires multiple communication exchanges between the workers to find the Newton’s direction. Our proposed DONE can be considered as an inexact distributed Newton-type method because DONE can find an exact Newton direction when local computation rounds go to infinity, and thus a finite computation rounds for practical purpose will produce an inexact version of Newton direction.

Federated edge learning. Learning-at-the-edge has proliferated for two main reasons. First, the requirement of real-time processing of data in many applications today does not allow that operation to be done in a distant, centralized server, but rather near the source of data. Second, smart devices and/or edge workers are equipped with large storage capacity and high-performance computing power to perform expensive tasks such as model training or statistical inference, which were infeasible in the past. This has motivated research into intelligence at the edge [2–4]. Recently, cross-device federated learning has achieved success in some applications, such as in the next-character prediction in keyboards on mobile devices [31]. An example of cross-device federated learning algorithm that has been studied extensively lately is the FedAvg [9], which includes a large network of highly diverse devices participating in learning a global predictive model. On the other hand, one of the early adopters of federated edge learning is [32], which proposes the use GD for a FedAvg-type
algorithm. In such distributed computing methods, the cost of communication is generally significant and poses serious problems in operation, especially in networks where limited bandwidth and high latency are prominent. There is always a trade-off between computation and communication: in order for the training process uses less communications, the amount of computational processing in edge workers/devices must increase. The investigation of more communication-efficient methods in federated learning has attracted much attention from the research community in the last few years [8].

III. DONE: DISTRIBUTED NEWTON-TYPE METHOD FOR FEDERATED EDGE LEARNING

A. Optimization Problem

In the context of federated edge learning, there are $n$ edge workers, located at different sites and communicating with an edge aggregator to learn a model $w^*$ which is a solution to the following problem

$$
\min_{w \in \mathbb{R}^d} f(w) = \frac{1}{n} \sum_{i=1}^n f_i(w),
$$

where $f_i(w)$ is the loss function of worker $i$. Each worker $i$ has a local dataset containing a collection of $D_i$ samples $\{a_{ij}, y_{ij}\}_{j=1}^{D_i}$, where $a_{ij} \in \mathbb{R}^d$ is an input of and $y_{ij} \in \mathbb{R}$ can be a target response (or label). Then the (regularized) loss function of each worker $i$ is the average of losses on its data point as follows

$$
f_i(w) = \frac{1}{D_i} \sum_{j=1}^{D_i} l(w, (a_{ij}, y_{ij})) + \frac{\lambda}{2} \|w\|^2,
$$

where the regularization term $\frac{\lambda}{2} \|w\|^2$ is added to improve stability and generalization. Some examples of the loss function are: linear regression with $l(w, (a_{ij}, y_{ij})) = \frac{1}{2} \| (a_{ij}, w) - y_{ij} \|^2$, $y_{ij} \in \mathbb{R}$, and logistic regression with $l(w, (a_{ij}, y_{ij})) = \log(1 + \exp(-y_{ij}(a_{ij}, w)))$, $y_{ij} \in \{-1, 1\}$, where $\langle x, y \rangle$ denotes the inner product of vectors $x$ and $y$.

Assumption 1. The function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is twice continuously differentiable, $L$-smooth, and $\lambda$-strongly convex, $\lambda > 0$, i.e.,

$$
\lambda I \preceq (\nabla^2 f(w) \succeq LI, \quad \forall w \in \mathbb{R}^d,
$$

where $\nabla^2 f(w)$ and $I \in \mathbb{R}^{d \times d}$ denote the Hessian of $f$ at $w$ and identity matrix, respectively.

We use Assumption 1 throughout this paper. Let $\|x\|$ and $\|A\|$ denote the Euclidean norm of vector $x \in \mathbb{R}^d$ and the spectral norm of the matrix $A \in \mathbb{R}^{d \times d}$, respectively. We note that strong convexity and smoothness in Assumption 1 can be found in a wide range of loss function examples such as linear regression and logistic regression as above. We also denote $\kappa = \frac{\lambda}{L}$ the condition number of $f$, where large $\kappa$ means $f$ is ill-conditioned, which may requires high computational complexity to optimize.

B. Challenges to Distributed Newton’s Method

We first review the Newton’s method [13] to solve (1) as follows

$$
w_{t+1} = w_t - \left(\nabla^2 f(w_t)\right)^{-1} \nabla f(w_t)
= w_t - \left(\frac{1}{n} \sum_{i=1}^n \nabla^2 f_i(w_t)\right)^{-1} \left(\frac{1}{n} \sum_{i=1}^n \nabla f_i(w_t)\right).
$$

One might be tempted to allow each worker to send $\nabla f_i(w_t)$ and $\nabla^2 f_i(w_t)$ to the aggregator for updating (3). However, the crux to designing a distributed Newton’s method is Hessian-free computation and inverse-Hessian-free computation. Indeed, sending Hessians of size $O(d^2)$ over the network or computing the inverse Hessian with complexity $O(n^3 d^2 + d^3)$ is considered impractical considering high-dimensional feature vector or “big data” size.

There are some approaches allowing each worker to approximately calculate $(\nabla^2 f_i(w_t))^{-1} \nabla f(w_t)$ [15], [30] and send this vector to the edge aggregator for the following update

$$
w_{t+1} = w_t - \frac{1}{n} \sum_{i=1}^n (\nabla^2 f_i(w_t))^{-1} \nabla f_i(w_t).
$$

However, $\frac{1}{n} \sum_{i=1}^n (\nabla^2 f_i(w_t))^{-1} \nabla f_i(w_t)$ can be far from the true Newton direction $(\nabla^2 f(w_t))^{-1} \nabla f(w_t)$.

We next review the key ingredient that helps DONE overcome these challenges.

C. Richardson Iteration Review

The purpose of the following Richardson iteration is to find $x^*$ satisfying the linear system $A x^* = b$, assuming $A \in \mathbb{R}^{d \times d}$ is a symmetric and positive definite matrix.

$$
x_k = (I - \alpha A)x_{k-1} + \alpha b, \quad k = 1, 2, \ldots
$$

Let $\lambda_{\text{max}}(A)$ and $\lambda_{\text{min}}(A)$ denote the largest and smallest eigenvalues of $A$, respectively. The Richardson iteration converges, i.e., $\lim_{k \rightarrow \infty} x_k = x^* = A^{-1} b$, if and only if

$$
0 < \alpha < \frac{2}{\lambda_{\text{max}}(A)},
$$

which ensures $\|I - \alpha A\| < 1$. Furthermore, if we choose

$$
\alpha = \frac{2}{\lambda_{\text{max}}(A) + \lambda_{\text{min}}(A)},
$$

the convergence rate of Richardson’s iteration is as follows

$$
\|x_k - x^*\| \leq \left(\frac{\lambda_{\text{max}}(A) - \lambda_{\text{min}}(A)}{\lambda_{\text{max}}(A) + \lambda_{\text{min}}(A)}\right)^k \|x_0 - x^*\|
= \left(\frac{\kappa_A - 1}{\kappa_A + 1}\right)^k \|x_0 - x^*\|,
$$

where $\kappa_A \equiv \frac{\lambda_{\text{max}}(A)}{\lambda_{\text{min}}(A)}$ is the condition number of $A$. The details of Richardson iteration analysis can be found in [33]. From
another viewpoint, the Richardson iteration is equivalent to using GD method to solve the following quadratic problem
\[ \min_{y \in \mathbb{R}^d} \frac{1}{2} \langle y, Ay \rangle - \langle y, b \rangle. \]

**D. DONE Algorithm**

The proposed DONE is presented in Alg. [1]. We see that \(2T\) is the number of communication iterations between the aggregator and edge workers. We note that the exact global gradient \(\nabla f(w_{t})\) is required at every worker (line 5), which is the key to the convergence of DONE to be shown later. \(R\) is the number of local computations at each edge worker to compute \(d_{i,t}^R\) (line 8) based on its local data.

The gist of **DONE**: Even though each worker uses the Richardson iteration to obtain its local Newton direction \(d_{i,t}^R\), their average \(d_{t}^R\) at the aggregator converges to the global Newton direction. Thus, the update (6) is the exact Newton's method when \(R \to \infty\). To prove this, we first present the most important result for the design of DONE.

**Theorem 1.** Considering a symmetric and positive definite matrix \(A = \frac{1}{n} \sum_{i=1}^{n} A_i\), \(A_i \in \mathbb{R}^{d \times d}\), and \(x_k\) and \(x_{i,k}\) follow the Richardson iteration as follows
\[
\begin{align*}
    x_k &= (I - \alpha A)x_{k-1} + \alpha b \\
    x_{i,k} &= (I - \alpha A_i)x_{i,k-1} + \alpha b, \quad k = 1, 2, \ldots
\end{align*}
\]

with \(0 < \alpha < \frac{2}{\lambda_{\max}(A)}\), then we have:
\[
\begin{align*}
    \lim_{k \to \infty} x_k &= A^{-1}b \\
    \lim_{k \to \infty} x_{i,k} &= A_i^{-1}b
\end{align*}
\]

**Theorem 2.** When \(\|1 - \alpha A\| < 1\), \(\|1 - \alpha A_i\| < 1\), and \(\alpha \leq \frac{1}{(k+1)}\), we have \(\|x_k - \frac{1}{n} \sum_{i=1}^{n} x_{i,k}\| \leq O((\|b\| + \|x_0\|)\|A^2 - \frac{1}{n} \sum_{i=1}^{n} A_i^2\|)\) when \(x_{i,0} = x_0, \forall i = 1, \ldots, n\).

Observing that \(d_{i,t}^R, \nabla^2 f_{i}(w_{i}),\) and \(-\nabla f(w_{i})\) of DONE play the same roles as \(x_{i,k}, A_i,\) and \(b\) in the above theorem, respectively, we have:

**Corollary 1.** With Assumption [2] and if \(0 < \alpha < \frac{2}{L}\), DONE is a distributed Newton's method with
\[
\lim_{R \to \infty} d_{t}^R = -\left(\nabla^2 f(w_{t})\right)^{-1}\nabla f(w_{t}).
\]

**Remarks:**
- The only parameter of DONE that needs to be fine-tuned is \(\alpha\). The condition of \(\alpha\) in Corollary [1] is sufficient to the convergence of \(d_{i,t}^R\) to the global Newton direction. However, it does not mean that this \(\alpha\) also satisfies \(\|I - \alpha \nabla^2 f_{i}(w_{i})\| < 1, \forall i\), and thus the Richardson iteration of \(d_{i,t}^R\) is not guaranteed to converge for all worker \(i\). The optimal value of \(\alpha\) is according to [4]. However, computing the eigenvalues of \(\nabla^2 f(w_{i})\) is impractical since it requires accessing to all workers' data. In practice, a sufficiently small \(\alpha\) always works, but very small \(\alpha\) can lead to slow convergence (c.f. Section [V]).

### Algorithm 1 Distributed Newton-type Method (DONE)

1. **input:** \(T, R, \alpha, w_0\)
2. **for** \(t = 0\) to \(T - 1\) **do**
3. Aggregator sends \(w_t\) to all edge workers
4. **for** all edge workers \(i = 1, \ldots, n\) in parallel **do**
5. Send \(\nabla f_i(w_t)\) to the edge aggregator, then receives the aggregated gradient \(\nabla f(w_t) = \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(w_t)\) sent back by the edge aggregator.
6. Set \(d_{i,0}^R = 0\)
7. **for** \(r = 1\) to \(R\) **do**
8. \(d_{i,t}^R = (I - \alpha \nabla^2 f_{i}(w_t))d_{i,t}^{R-1} - \alpha \nabla f(w_t)\)
9. **end for**
10. Send \(d_{i,t}^R\) to the aggregator
11. **end for**
12. Aggregator receives \(d_{i,t}^R\) from all workers and updates:
\[
   w_{t+1} = w_t + d_{t}^R, \quad \text{where} \quad d_{t}^R = \frac{1}{n} \sum_{i=1}^{n} d_{i,t}^R. \quad (6)
\]
13. **end for**

- In this work, the theory of DONE is applicable to full data passing for each worker and full participation of all edge workers. However, using mini-batches is common in machine learning to reduce the computation bottleneck at the worker. Another critical issue is the straggler effect, in which the run-time in each iteration is limited by the slowest worker (the straggler) because heterogeneous edge workers compute and communicate at different speeds. Thus, choosing a subset of participating workers in aggregation phase is a practical approach to reduce the straggler effect. In Section [V], we will experimentally show that DONE works well with mini-batches and sampling of a subset of workers [7] when approximating the Hessian \(\nabla^2 f_{i}(w_{t})\) and mitigating straggler effect, respectively.

- It is obvious that the larger \(R\) and \(T\), the higher computation and communication complexities of DONE, yet the better accuracy and small optimal gap, for the Newton direction and convergence, respectively. In the next section, we will quantifiy how much \(R\) and \(T\) affect the convergence of DONE.

### IV. Convergence and Complexity Analysis

In this section, we will provide the convergence and complexity analysis of DONE, and compare it with distributed GD and DANE [13].

#### A. Convergence analysis

We see that each edge worker has \(R\) computation rounds, and the complexity at each round is calculating \(d_{i,t}^R\) on line 8.

It is straightforward to see that the bottleneck of this step is performing the matrix-vector product with \(O(D_i \cdot d^2)\) by uniformly at random choosing a subset of \(B\) data points from \(D_i\) samples and a subset of \(S \leq n\) workers, respectively.
computation complexity. However, this complexity can be reduced to \( O(D_i \cdot d) \) with a special class of generalized linear model (GLM) \(^{34}\) having the linear term \( \langle a_j, w_t \rangle \) in its loss function, e.g., regularized linear regression or logistic regression. Indeed, from \(^2\), the Hessian of \( f_t \) in these learning tasks is

\[
\nabla^2 f_i(w_t) = \frac{1}{D_i} \sum_{j=1}^{D_i} \beta a_j a_j^T + \lambda I,
\]

where \( \beta \) is a scalar that depends on \( \langle a_j, w_t \rangle \). Therefore, the matrix-vector product \( \nabla^2 f_i(w_t) d_{i,t}^{-1} \) becomes

\[
\frac{1}{D_i} \sum_{j=1}^{D_i} \beta a_j a_j^T d_{i,t}^{-1},
\]

which contains only vector-vector products with computation complexity \( O(D_i \cdot d) \). Therefore, with GLM, the computation complexity of each worker \( i \) using DONE is \( O(D_i \cdot d \cdot R \cdot T) \), which only scales linearly with respect to \( (w.r.t) \) the data size and feature dimension.

For the ease of presentation, we denote \( \nabla f_i(w_t) \) for the \( t \)-th iteration of the algorithm until a good neighborhood is found, common to use the damped Newton phase for some initial convergence is guaranteed for any initialization point to enable global convergence of Newton-type method, i.e., \( f_t \) approximated by a quadratic function, e.g.,\( M \)

\[
f(0) = f(w) + \nabla f(w)^	op (x - w) + \frac{1}{2} (x - w)^	op H (x - w),
\]

where \( \beta \) is a scalar that depends on \( \langle a_j, w_t \rangle \). Therefore, with GLM, the computation complexity of each worker \( i \) using DONE is \( O(D_i \cdot d \cdot R \cdot T) \), which only scales linearly with respect to \( (w.r.t) \) the data size and feature dimension.

Before analyzing the effects of \( R \) and \( T \) on the convergence of DONE, we need the following standard assumption for the analysis of a Newton method \(^{13}\).

**Assumption 2.** The Hessian of \( f \) is \( M \)-Lipschitz continuous, i.e., \( \|\nabla^2 f(w) - \nabla^2 f(w')\| \leq M \|w - w'\|, \forall w, w' \in \mathbb{R}^d \).

Here, the value of \( M \) measures how well \( f \) can be approximated by a quadratic function, e.g., \( M = 0 \) for linear regression. In this work, we focus on the local convergence of DONE in a similar way to the Newton method. In practice, to enable global convergence of Newton-type methods, i.e., convergence is guaranteed for any initialization point \( w_0 \), it is common to use the damped Newton phase for some initial iterations of the algorithm until a good neighborhood is found \(^{13}\). In the damped Newton phase, the edge aggregator will update \( w_{t+1} = w_t + \eta_t d_t^R \) with a step size \( \eta_t \) (found by using the backtracking line search, for example) to ensure \( d_t^R \) is a descent direction. When the damped Newton phase finishes and the algorithm enters the pure Newton phase, we can set \( \eta_t = 1 \) until convergence. For our experimental testing in Section \(^X\), \( \eta_t = 1 \) always works in all iterations.

For practical DONE, we need to choose a finite value for \( R \), which means that \( d_t^R \) is an approximation of the Newton direction. We next define a parameter \( \delta \) that measures how well \( d_t^R \) approximates the true Newton direction. In the sequel, for the ease of presentation, we denote \( g_t = \nabla f(w_t), H_t = \nabla^2 f(w_t), \) and \( \hat{d}_t = -H_t^{-1} g_t \).

**Definition 1.** \( d_t^R \) is called a \( \delta \)-approximate \( \hat{d}_t \) if

\[
\|\hat{d}_t - d_t^R\| \leq \delta \|\hat{d}_t\|.
\]

In other words, \( \delta \) captures the inexact level of the approximate solution \( d_t^R \) to the equation \( H_t x_t = g_t \) with the true solution \( x^* = \hat{d}_t \).

The following result shows the relationship between \( R \) and \( \delta \), and how the \( \delta \)-approximation affects the convergence rate of DONE.

**Lemma 1.** Let Assumptions \(^1\) and \(^2\) hold. For an arbitrary small \( \delta > 0 \) and \( \alpha = \frac{2}{\lambda} \), if we set

\[
R = O\left(\kappa \log \frac{1}{\delta}\right)
\]

then DONE has

\[
\|w_{t+1} - w^*\| \leq \frac{M}{2\lambda} \|w_t - w^*\|^2 + \delta \|w_t - w^*\|.
\]

The proof appears in Appendix \(^A\). While the conventional Newton’s method is well known to have local quadratic convergence \(^{13}\), Lemma \(^1\) shows that DONE has a linear-quadratic convergence rate. The quadratic term in this lemma is the same as in the conventional Newton’s method, whereas the linear term is due to the \( \delta \)-approximation of Newton direction. It is obvious that \( R \to \infty \) when \( \delta \to 0 \), and thus the linear term disappears, recovering the quadratic convergence of the exact Newton method. Lemma \(^1\) also shows that \( R \) increases when \( \kappa \) is large, i.e., more computation rounds are needed because the Hessian \( H_t \) is ill-conditioned.

**B. Complexity analysis**

We next address the communication complexity \( O(T) \) for the local convergence of DONE to \( w^* \).

**Definition 2.** We define \( w_T \) an \( \epsilon \)-optimal solution to \(^1\) if

\[
\|w_T - w^*\| \leq \epsilon.
\]

From Lemma \(^1\), we see that there are regimes in which DONE has linear or quadratic convergence rate, and these regimes depend on the \( \delta \) setting and the initialization \( w_0 \) in a neighborhood of \( w^* \). In the following, we will identify two extreme regimes.

In the first regime where the \( \delta \)-approximation error dominates the bound in \(^1\), i.e., \( \delta \|w_0 - w^*\| \geq \max\left\{ \frac{\epsilon}{\delta}, \frac{M}{2\lambda} \|w_0 - w^*\|^2 \right\} \), then DONE has linear convergence.

**Theorem 3.** For any \( \gamma \in (0, 1) \), if \( \|w_0 - w^*\| \leq \frac{\gamma A}{\lambda} \) and we set \( \delta = \gamma / 2 \), then DONE has linear convergence

\[
\|w_{t+1} - w^*\| \leq \gamma \|w_t - w^*\|
\]

and its communication complexity is

\[
T = O\left(\log \frac{1}{\epsilon}\right).
\]

**Proof:** We will use induction to prove \(^3\). From Lemma \(^1\) we first prove the base case

\[
\|w_1 - w^*\| \leq \frac{M}{2\lambda} \|w_0 - w^*\|^2 + \delta \|w_0 - w^*\|
\]

\[
\leq \frac{\gamma}{2} \|w_0 - w^*\| + \frac{\gamma}{2} \|w_0 - w^*\|,
\]

where

\[
\|w_1 - w^*\| \leq \frac{M}{2\lambda} \|w_0 - w^*\|^2 + \delta \|w_0 - w^*\|
\]

\[
\leq \frac{\gamma}{2} \|w_0 - w^*\| + \frac{\gamma}{2} \|w_0 - w^*\|,
\]
where the second inequality is due to the initialization condition $\|w_0 - w^*\| \leq \frac{\lambda}{M}$ and $\delta = \frac{\lambda}{2}$. Next assuming that (8) is true at the iteration $t > 1$, from Lemma [1] we have
\[
\|w_{t+1} - w^*\| \\
\leq \frac{M}{2\lambda}\|w_t - w^*\|^2 + \delta\|w_t - w^*\| \\
\leq \frac{M}{2\lambda}\|w_t - w^*\|^2 + \delta\|w_t - w^*\| + \frac{\gamma}{2}\|w_t - w^*\| \\
\leq \frac{M}{2\lambda}\|w_t - w^*\|^2 + \frac{\gamma}{2}\|w_t - w^*\| \\
\leq \frac{M}{2\lambda}\|w_t - w^*\|^2 + \frac{\gamma}{2}\|w_t - w^*\|,
\]
where the third inequality is by applying (8) recursively and the last inequality is due to $\|w_0 - w^*\| \leq \frac{\lambda}{M}$.

Applying (8) recursively to have
\[
\|w_{t+1} - w^*\| \leq \gamma t\|w_0 - w^*\| \leq \epsilon,
\]
then we can obtain
\[
T \geq \frac{\log\|w_0 - w^*\|/\epsilon}{\log 1/\gamma}.
\]

In the second regime where the $\delta$-approximation error is negligible, i.e., $\delta\|w_0 - w^*\| \leq \frac{\lambda}{2} \leq \frac{1}{2\lambda}\|w_0 - w^*\|^2$, then DONE has quadratic convergence.

**Theorem 4.** If
\[
\|w_0 - w^*\| \leq \sqrt{\frac{\lambda}{2M}} \text{ and } \delta\sqrt{\frac{\lambda}{2M}} \leq \epsilon/2,
\]
then DONE has quadratic convergence
\[
\|w_{t+1} - w^*\| \leq \frac{M}{\lambda}\|w_t - w^*\|^2
\]
and its communication complexity is
\[
T = O\left(\log\log\frac{1}{\epsilon}\right).
\]

**Proof:** From Lemma [1] we have
\[
\|w_{t+1} - w^*\| \leq \frac{M}{2\lambda}\|w_t - w^*\|^2 + \delta\|w_0 - w^*\| \\
\leq \frac{M}{2\lambda}\|w_t - w^*\|^2 + \epsilon/2 \\
\leq \frac{M}{\lambda}\|w_t - w^*\|^2,
\]
where the first inequality is by [10] and the second one is by the regime condition. Applying the above recursively to have
\[
\|w_{t+1} - w^*\| \leq \left(\sqrt{\frac{M}{\lambda}\|w_0 - w^*\|}\right)^{2t} \leq \epsilon,
\]
then we obtain (11).

We see that controlling $\delta$ gives a trade-off between computation and communication complexities, where very small $\delta$ increases the computation iterations but may significantly reduce the communication cost with quadratic convergence, and vice versa. With the target to reduce the communication cost, we have the following corollary by combining all previous results:

**Corollary 2.** With generalized linear models (GLM), by choosing $\delta\sqrt{\frac{\lambda}{2M}} = \frac{\epsilon}{2}$ for DONE, the computation complexity $O(D_i \cdot d \cdot R \cdot T)$ of worker $i$ is
\[
O\left(D_i \cdot d \cdot \kappa \cdot \log\left(\frac{\kappa\sqrt{\lambda/M}}{\epsilon}\right) \cdot \log\log\frac{1}{\epsilon}\right).
\]

**C. Comparison with other distributed methods**

If a first-order method such as GD is used for federated edge learning, the edge aggregator first receives $\nabla f_i(w_t), \forall i$, updates
\[
w_{t+1} = w_t - \eta \left(\frac{1}{n} \sum_{i=1}^{n} \nabla f_i(w_t)\right),
\]
and then sends $w_{t+1}$ to all workers for next iteration. By choosing $\eta = \frac{\epsilon}{\sqrt{M}}$, it has been shown that the communication complexity of GD for $\epsilon$-optimal solution [14, Theorem 2.1.5] is $\kappa \log \frac{1}{\epsilon}$, and thus the computation complexity of each worker $i$ is
\[
O\left(D_i \cdot d \cdot \kappa \cdot \log\frac{1}{\epsilon}\right).
\]

Since $\log\log\frac{1}{\epsilon}$ can be considered as a constant in practice [13], we see that DONE and GD have almost the same computation complexity (for learning GLM) when $\kappa \sqrt{\lambda/M}$ is not extremely large. We note that this distributed GD [12] is a common approach in data-center-type distributed machine learning [35, 36], which is different from FedAvg-type GD in [9, 32]. While FedAvg-type GD (or SGD) allows each device/worker to update its local model multiple times using local gradient information (thus its alternative name is Local GD/SGD [37]), the update in (12) requires each client/worker to calculate its gradient once and send to the aggregator. Compared to first-order methods, without multiple local model updates, DONE is more in line with distributed GD in (12) than FedAvg-type GD algorithms.

Regarding other related Newton-type methods, DiSCO [29] requires several communication rounds between the aggregator and workers even for one Newton direction update, whereas GIANT [30] needs data shuffling. Thus, both DiSCO and

| Method   | Complexity | Metric                     |
|----------|------------|----------------------------|
| DONE     | $O\left(\log\frac{1}{\epsilon}\right)$ | $\|w_t - w^*\| \leq \epsilon$ (Local convergence) |
| DANE     | $O\left(\frac{\kappa\sqrt{\lambda/M}}{\epsilon}\cdot\log\log\frac{1}{\epsilon}\right)$ | $f(w_t) - f(w^*) \leq \epsilon$ (Global convergence) |
| GD       | $O\left(\kappa\log\frac{1}{\epsilon}\right)$ | $\|w_t - w^*\| \leq \epsilon$ (Global convergence) |
GIANT are only applicable to datacenter-type distributed machine learning. On the other hand, DANE [15], an approximate Newton-type method comparable to DONE, has been shown to be a good candidate for federated learning with several variants [38, 39]. In DANE, two communication rounds are used in one global iteration. In the first round, the workers compute the local gradients $\nabla f_i(w_t)$, which are then aggregated into the global gradient $\nabla f(w_t)$. In the second round, each worker solves a special local optimization using $f_i(w)$ and $\nabla f(w_t)$. It has been shown in [15] that to achieve an $\epsilon$-accurate solution, DANE requires $O\left(\frac{\kappa^2 \log (dn) \log \frac{1}{\epsilon}}{\kappa} \right)$ iterations. Therefore, the computational complexity of each worker is

$$O\left(\kappa^2 d \log (dn) \log \frac{1}{\epsilon} \right).$$

A comparison of communication complexities of DONE, distributed GD, and DANE for a ridge regression task (quadratic loss) is summarized in Table I.

V. EXPERIMENTAL RESULTS AND DISCUSSION

In this section, we evaluate the performance of DONE when the data sources across edge workers are non-i.i.d. and have heterogeneous size. We first verify our theoretical findings, then compare DONE with the Newton’s method [5], DANE [15], and GD [12]. The experimental outcomes show that the results not only correspond well with the theory but also achieve the performance improvement over DANE and GD in terms of training loss convergence rate and test accuracy in various settings.

The definition of $\epsilon$-accuracy in [15] is $f(w_t) - f(w^*) \leq \epsilon$.

Fig. 1: Effects of various values of $\alpha$ and $R$ on synthetic ($\kappa = 10^2$).

Fig. 2: Effects of various values of $\alpha$ and $R$ on MNIST.

Fig. 3: Effects of $R$ on synthetic with different values of $\kappa$ ($\alpha = 0.1$).

A. Experimental Settings

We consider both classification and regression tasks using real (MNIST) and synthetic datasets, respectively. For both datasets, we distribute the complete data to $n = 32$ edge workers and randomly split the data to two parts: 75% for training and 25% for testing.

MNIST [40]: A handwritten digit dataset including 70,000 samples and 10 labels. In order to simulate a heterogeneous
and non-i.i.d. data setting, each worker has only 3 labels and the data sizes vary in the range [219, 3536], following the power law in [11].

**Synthetic:** To generate non-i.i.d. data, each edge worker $i$ has a collection of $D_i$ samples $(a_j, y_j)_{j=1}^{D_i}$ following the linear regression model $y_j = (w^*, a_j) + c$ with $w^*, a_j \in \mathbb{R}^d, c \in \mathbb{R}$, $a_j \sim \mathcal{N}(0, \sigma_i^2 \Sigma)$, where $\sigma_j \sim \mathcal{U}(1, 10)$, $c \sim \mathcal{N}(0, 1)$, and $\Sigma \in \mathbb{R}^{d \times d}$ is a diagonal covariance matrix with $\Sigma_{i,i} = i^{-\tau}$, $i = 1, \ldots, d$. We control the condition number $\kappa$ by setting $\tau = \log(d)$; therefore, $\kappa = d^\tau$ is the ratio between the maximum and minimum covariance values of $\Sigma$. In addition, to model a heterogeneous setting, each edge has a different data size in the range [540, 5630].

In our experiment, the regression task on the synthetic dataset uses a linear regression model with mean squared error loss, whereas the classification task on MNIST uses a multinomial logistic regression model with cross-entropy loss. The loss function $f_i$ is defined below:

- Mean squared error loss for linear regression (synthetic dataset):
  $$f_i(w) = \frac{1}{D_i} \sum_{j=1}^{D_i} ((a_j, w) - y_j)^2 + \frac{\lambda}{2} \| w \|_2^2.$$

- Cross-entropy loss for multinomial logistic regression for $C$ classes (MNIST):
  $$f_i(w) = \frac{1}{D_i} \sum_{j=1}^{D_i} \sum_{c=1}^{C} 1(y_{j,c} = c) \log \left( \frac{\exp((a_j, w_c))}{\sum_{k=1}^{C} \exp((a_j, w_k))} \right) + \frac{\lambda}{2} \sum_{c=1}^{C} \| w_c \|_2^2.$$

We implement all algorithms using PyTorch version 1.5.0 [42].

**B. Effect of parameters: $\alpha$, $R$, and $\kappa$**

We first verify our theoretical findings by showing the impact of wide ranges of values of $\alpha$, $R$, and $\kappa$ on the convergence of DONE in Figs. [1] [2] and [3]. To monitor the effect of $\alpha$, we fix the value of $R$ and vice versa. The results in Figs. [1] and [2] for synthetic and MNIST datasets, respectively, demonstrate that there exists a sufficiently small $\alpha$ allowing DONE to converge. Besides, larger $\alpha$ speeds up the convergence of DONE until a threshold, which leads to divergence. Note that in linear regression, we can obtain $\lambda_{i,\text{min}}$ and $\lambda_{i,\text{max}}$, the smallest and largest eigenvalues of the Hessian of $f_i$ and $\alpha_i = \lambda_{i,\text{min}} + \lambda_{i,\text{max}}$ for each worker $i$. Then we choose $\alpha \leq \min_i \{\alpha_i\}$. On the other hand, larger $R$ also benefits DONE with faster convergence but requires more local computations. In Fig. [3] we observe the behavior of DONE with a wide range values of $\kappa$ including small value, i.e., 10, medium values, i.e., $(10^2, 10^3)$, and significant large value, i.e., $10^4$. It can be seen that given the same error tolerance, larger $\kappa$ requires larger $R$ to obtain the descent direction, which verifies the theory. Hence, both $R$ and $\alpha$ should be tuned carefully depending on $\kappa$ to balance the local computations and communications.
C. Performance comparison with different distributed algorithms

In this sub-section, we compare DONE with the Newton’s method, GD, and DANE. For a fair comparison, we fine-tune the hyper-parameters w.r.t. the highest accuracy and stability of each algorithm. For DANE, we fix \( \eta = 1 \) and chose the best regularization parameter \( \mu \) in \( \{0, \lambda, 3\lambda\} \). We also use Richardson iteration for the true Newton’s method \( ^3 \) since it is impractical to do the inverse Hessian (especially for multinomial logistic regression). In comparison to Newton, two separate cases are considered: when Newton has the same values of \( \alpha \) and \( R \) with DONE, and when Newton has fine-tuned \( \alpha \) and \( R \). Even though our analysis for DONE requires computing full-batch Hessian-gradient products, we also consider an additional case where we sample a mini-batch in each round to approximate the true Hessian-gradient product.

The experimental results on both datasets in Figs. 4 and 5 show that when \( R \) is sufficiently large and \( \alpha \) is small enough, DONE has similar convergence rate to that of the Newton’s method. The difference only occurs when the value of \( \alpha \) is increased, especially in the case of MNIST dataset. In this case, Newton has a higher threshold of \( \alpha \) than that of DONE (in case of synthetic, both Newton and DONE have the same threshold of \( \alpha \)). Therefore, larger \( \alpha \) allows Newton to converge faster than DONE in Fig. 5. Interestingly, with a sufficiently large mini-batch size \( B_i = B = 256, \forall i \), using the full-batch size \( D_i \), and using mini-batch size \( B_i \), we can almost achieve the same performance. The effect of mini-batch size also depends on the learning tasks. Specifically, a mini-batch size \( B_i = 128 \) produces good performance as full-batch size for linear regression in Fig. 4 but considerably hurts the performance of DONE for multinomial logistic regression Fig. 5. Regarding the linear regression task, DONE, Newton, and DANE perform similarly and converge significantly faster than the first-order GD method. The difference can be clearly noticed in the case of multinomial logistic regression, where DONE outperforms both DANE and GD. Compared to DANE, the improvement in test accuracy and training loss are approximately 0.2% and 7.3%, respectively. The corresponding figures are 0.9% and 20% when compared to GD.

D. Effect of sampling a subset of edge workers

In practice, it is critical to reduce the straggler effect; thus, we also consider a scenario when a subset of workers \( S \) is selected randomly for aggregation. We keep using the same experimental setting as above, which involves 32 edge workers in total, and randomly select the value of \( S \) in the set \( \{8, 16, 24, 32\} \). In Figs. 6 and 7, DONE converges in all choices of \( S \) on both learning tasks. Especially, larger \( S \) allows DONE to converge faster with linear regression. On the other hand, with logistic regression, the performance deterioration when reducing the subset size is negligible, showing that DONE can perform well with worker-subset sampling.

VI. Conclusion

In this work, we develop a distributed Newton-type algorithm (DONE) suitable for federated edge learning. We show that DONE effectively approximates the true Newton direction using the Richardson iteration when the loss functions are strongly convex and smooth. Additionally, we specify that DONE has linear-quadratic convergence and provide its computation and communication complexity analysis. Finally, we experimentally verify our theoretical findings and compare our approach with the distributed GD method and DANE, an approximate distributed Newton-type algorithm.

APPENDIX

A. Proof of Lemma 7

Using triangle inequality, we have

\[
\|w_{t+1} - w^*\| = \|w_t - w^* + d_t^R\| \\
\leq \|w_t - w^* + \hat{d}_t\| + \|d_t^R - \hat{d}_t\|. 
\]

Using the classical Newton’s method analysis \( ^{13}, ^{14} \), we bound \( T_1 \) using the \( M \)-Lipschitz continuous Hessian in Assumption 2

\[
T_1 = \|H_t^{-1}(H_t(w_t - w^*) - g_t)\| \\
\leq \|H_t^{-1}\|\|H_t(w_t - w^*) - g_t\| \\
\leq \frac{1}{\lambda}\|w_t - w^*\|\left|\int_{t=0}^{1} [\nabla^2 f(w_t) - \nabla^2 f(w_t + t(w^* - w_t))] dt\right| \\
\leq \frac{1}{\lambda}\|w_t - w^*\|\int_{t=0}^{1} \|\nabla^2 f(w_t) - \nabla^2 f(w_t + t(w^* - w_t))\| dt \\
\leq \frac{1}{\lambda}\|w_t - w^*\|\int_{t=0}^{1} M\|w_t - w^*\| t dt \\
\leq \frac{M}{2\lambda}\|w_t - w^*\|^2. 
\]

We next bound \( T_2 \), which is due to the \( \delta \)-approximation error.

\[
T_2 = \|H_t^{-1}g_t - d_t^R\| \\
\leq \frac{(\kappa - 1)}{\kappa + 1} R\|H_t^{-1}g_t\| \\
\leq \delta\|H_t^{-1}g_t\| \\
\leq \delta\|H_t^{-1}\|\|g_t\| \\
\leq \delta\frac{\lambda}{\lambda}\|\nabla f(w_t) - \nabla f(w^*)\| \\
\leq \delta\frac{L}{\lambda}\|w_t - w^*\|,
\]

where the first inequality is due to the convergence rate of Richardson iteration \( ^3 \), the second is by choosing

\[
R = \frac{\log \frac{1}{3}}{\log \frac{\kappa + 1}{\kappa - 1}},
\]

and the last two inequalities are by Assumption 1.

We can further refine the value \( R \). When \( \kappa \) is small or medium, we have \( 1/\log \frac{\kappa + 1}{\kappa - 1} \) is approximate to a constant.

Check DANE \[15\] for the meaning of these parameters.
But when \( \kappa \) is large, e.g., \( \kappa \gg 10 \), we have \( 1/\log \frac{\kappa+1}{\kappa} \approx \frac{\kappa}{\kappa+1} \). Combining two cases, we have

\[
R = O(\kappa \log \frac{1}{\delta}).
\]

**B. Proof of Theorem 7**

By Richardson iteration convergence, it is straightforward to have the first result \( \lim_{k \to \infty} x_k = A^{-1} b \) and \( \lim_{k \to \infty} x_{i,k} = A_i^{-1} b \).

**C. Proof of Theorem 8**

Using Richardson, we have:

\[
x_k = \left( I - \alpha A \right) x_{k-1} + \alpha b
= (I - \alpha A)^K x_0 + \sum_{k=0}^{K} (I - \alpha A)^k \alpha b,
\]

\[
x_{i,k} = (I - \alpha A_i) x_{i,k-1} + \alpha b
= (I - \alpha A_i)^K x_{i,0} + \sum_{k=0}^{K} (I - \alpha A_i)^k \alpha b
\]

\[
\frac{1}{n} \sum_{i=1}^{n} x_{i,k} = \frac{1}{n} \sum_{i=1}^{n} \left( (I - \alpha A_i)^K x_{i,0} + \sum_{k=0}^{K} (I - \alpha A_i)^k \alpha b \right)
= \frac{1}{n} \sum_{i=1}^{n} \left( (I - \alpha A_i)^K x_{i,0} \right) + \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{k=0}^{K} (I - \alpha A_i)^k \alpha b \right).
\]

The expanded \( (I - \alpha A)^k \) as form of taylor series:

\[
(I - \alpha A)^k = \binom{k}{0} I + \binom{k}{1} (-\alpha A)^1 + \frac{\binom{k}{2}}{2} (-\alpha A)^2 + ...
= I - \alpha k A + \alpha^2 \frac{k(k-1)}{2} A^2 + ...
\]

Then we have:

\[
\sum_{k=0}^{K} (I - \alpha A)^k = (K + 1) I - \alpha \frac{(K + 1)K}{2} A + \alpha^2 \frac{(K + 1)K(K - 1)}{3!} A^2 + ...
\]

For any matrix \( A \) and a constant \( \alpha \) satisfy the condition \( \|I - \alpha A\| < 1 \), \( (15) \) and \( (16) \) converge absolutely for any complex number \( k \). Now we express \( (15) \) and \( (16) \) using big O notation respectively:

\[
(I - \alpha A)^k = I - \alpha k A + O(\alpha^2 A^2 K^2)
\]

\[
\sum_{k=0}^{K} (I - \alpha A)^k = (K + 1) I - \alpha \frac{(K + 1)K}{2} A + O(\alpha^2 A^2 (K + 1)^3)
\]

So \( (13) \) and \( (14) \) can be written as follows.

\[
x_k = (I - \alpha k A + O(\alpha^2 A^2 K^2)) x_0
+ \left( (K + 1) I - \frac{(K + 1)K}{2} \alpha A + O(\alpha^2 A^2 (K + 1)^3) \right) \alpha b,
\]

\[
\frac{1}{n} \sum_{i=1}^{n} x_{i,k} = \frac{1}{n} \sum_{i=1}^{n} \left( (I - \alpha k A_i) x_{i,0} + \frac{1}{n} \sum_{i=1}^{n} O(\alpha^2 A_i^2 K^2) x_{i,0} \right)
+ \frac{1}{n} \sum_{i=1}^{n} \left( (K + 1) I - \frac{(K + 1)K}{2} \alpha A_i \right) \alpha b
+ \frac{1}{n} \sum_{i=1}^{n} O(\alpha^2 A_i^2 (K + 1)^3) \alpha b
\]

if \( A = \frac{1}{n} \sum_{i=1}^{n} A_i \) and \( x_0 = x_{i,0} \) we have:

\[
||x_i - \frac{1}{n} \sum_{i=1}^{n} x_{i,k}|| \leq O(\alpha^2 (K + 1)^2 ||A^2 - \frac{1}{n} \sum_{i=1}^{n} A_i^2 ||||x_0||)
+ O(\alpha^3 (K + 1)^3 ||A^2 - \frac{1}{n} \sum_{i=1}^{n} A_i^2 ||||b||)
\]

By choosing \( \alpha = \frac{1}{K+1} \):

\[
||x_i - \frac{1}{n} \sum_{i=1}^{n} x_{i,k}|| \leq O\left(||A^2 - \frac{1}{n} \sum_{i=1}^{n} A_i^2 ||(||b|| + ||x_0||)\right)
\]
