Evaporation of a packet of quantized vorticity

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A recent experiment has confirmed the existence of quantized turbulence in superfluid $^3$He-B and suggested that turbulence is inhomogenous and spreads away from the region around the vibrating wire where it is created. To interpret the experiment we study numerically the diffusion of a packet of quantized vortex lines which is initially confined inside a small region of space. We find that reconnections fragment the packet into a gas of small vortex loops which fly away. We determine the time scale of the process and find that it is in order of magnitude agreement with the experiment.

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The decay of superfluid turbulence at very low temperatures raises the fundamental question of the existence near absolute zero of an energy cascade from large to small scales. In the case of ordinary turbulence, the nonlinear terms of the Navier - Stokes equation distribute the energy over various scales without changing the total energy. The process leads to Richardson’s cascade of bigger eddies breaking up into smaller eddies, until the wavenumber is large enough that kinetic energy is dissipated by viscous forces. In the case of superfluid turbulence, recent work indicates that generation of sound plays the role of ‘sink’ of kinetic energy. Furthermore it appears that the generation of small scales occurs via creation of helical waves of higher and higher wavenumbers on the quantized vortex filaments (Kelvin waves), and via creation of small vortex loops. The nonlinear mechanism behind both processes is vortex reconnection, either indirectly (reconnections create cusps which relax into large amplitude Kelvin waves) or directly (reconnections create small vortex loops).

The aim of this Letter is to show that the formation of small vortex loops is particularly important if the turbulence is not homogeneous, a case which is less investigated than homogeneous turbulence but is relevant to experiments performed at the lowest temperatures. In the case of $^4$He many experiments have been performed above 1.3K but much less is known about lower temperatures. In the most relevant study turbulence was produced by oscillating a grid at temperatures as low as 20mK (0.01$T_c$, where $T_c$ is the critical temperature). Although there is no direct evidence, it is reasonable to expect that the vortex tangle was localized in the region of the grid. The detection was based on trapping ions on the quantized vortices, and measurements of the collected charge indicated that the total amount of turbulence in the cell decayed in time. In the case of $^3$He-B, direct studies of vortices are typically done in a rotating cryostat, and an indirect observation of turbulence has been confirmed only recently. In this experiment turbulence was created by vibrating a small wire shaped like a half-circle at temperatures around 0.11$T_c$. Again, we expect that turbulence was localized in the region of the vibrating wire. Additional wires were used to detect the Andreev reflection of quasi-particles from the turbulent superfluid velocity field. The experimental set-up could not tell unambiguously whether the growth time of the screening reflects the intrinsic temporal decay of the vorticity or the spatial evolution of the vortex tangle away from the region in which it was created. Nevertheless, the authors argued from the temperature dependence of the effect that it is more likely that a spatial evolution of the vortex tangle took place; typically, the spatial and temporal scales were 1 mm and 1 sec respectively.

In both experiments, the lack of flow visualization makes the interpretation of the data difficult, so numerical simulations can give insight into the problem. Thus motivated, the simple question which we address is: what is the fate of a tangle of quantized vortex loops initially confined in a small region? Can any physical processes cause the packet of quantized vorticity to diffuse out and spread in space?

In a classical Navier - Stokes fluid, diffusion of vorticity is caused by viscous forces. In the absence of viscosity (a perfect Euler fluid) vortex reconnections are not possible and the initial topology is frozen into the fluid; vortex loops distort but remain linked to each other, so conservation of helicity prevents the packet from spreading. In a superfluid, vortex reconnections are possible, so the question which we ask is whether the packet of quantized vorticity remains localized in the initial region or not, and if not, how fast it diffuses away. Additional motivation arises from the recent result that a vortex tangle’s geometrical and topological complexity (the amount of twists and links of the turbulent structure) seems to be related to its ability to diffuse in the time scale under consideration.

We represent a quantized vortex filament as a space curve $s = s(\xi,t)$ where $\xi$ is arclength and $t$ is time. In the absence of friction the velocity $d\mathbf{s}/dt$ of the filament at the point $\mathbf{s}$ is given by

$$\frac{d\mathbf{s}}{dt} = \frac{\Gamma}{4\pi} \int \frac{(r-s) \times \mathbf{dr}}{|r-s|^3},$$

(1)

where $\Gamma$ is the quantum of circulation and the Biot - Savart integral extends over all vortex filaments present in the flow (we do not apply the local induction approximation). By changing the discretization along the filaments we
tested that the results do not depend on the reconnection procedure (details of the algorithm are in reference \[14\]). Since our model is incompressible, we have no transformation of kinetic energy (length of vortex line) into sound, an effect which can be studied using the Gross - Pitaevski model \[1\]. A small loss of vortex length occurs due to the numerical reconnection procedure \[3\] but does not affect our results.

Using the above model, we have performed numerical experiments to determine the spatial - temporal evolution of localized packets of vorticity. Although we used \(^4\)He’s parameter values, the results can be reinterpreted for \(^3\)He by rescaling the units of time and length according to the different values of \(\Gamma\). The vortex filament method, originally developed for \(^4\)He, is equally valid for \(^3\)He-B despite the much larger vortex core size than \(^4\)He, because the length scales of the calculation (eg distance between discretization points on the same filament and distance between filaments) are still much bigger than the vortex core. Typically our initial condition consists of a given number \(N_0\) of circular vortex rings, whose centres and orientations are randomly generated, initially confined in a sphere of radius \(S_0\). Other kinds of initial conditions have been discussed in the related literature \[13\], notably that of random vortex network, but there is no reason to believe that they apply to our case and we know too little of the details of how a vibrating wire or grid generates quantized vorticity to be more realistic. Fortunately it is known \[13\] that any simple configuration which is almost isotropic quickly evolves in an turbulent tangle independent of the initial state, and by numerically experimenting we found that our results do not depend on how we start the calculation. For example we tried replacing many small circular rings with few longer Fourier knots (trefoil-like curves which wrap around themselves few times before closing \[10\]). This changed drastically the initial topology, but the same results were found as for rings. Unlike previous numerical simulations of superfluid turbulence (performed either with periodic boundary conditions or in channels with rigid walls), our calculations are carried out in an infinite volume. At each time, quantities which are useful to describe the vortex packet are: the total length \(\Lambda\), the number of loops \(N\), the radius of the confining sphere \(S\), the vortex line density \(L = 3A/4\pi S^3\), the average inter-vortex distance \(\delta = L^{−1/2}\), and the average vortex loop length in the packet \(D\). These quantities depend on \(t\) and we use the subscript zero to denote initial values.

The time evolution of the small vortex packet shown in Figure 1 is typical. The initial vortex rings (here \(N_0 = 20\)) interact, become distorted and reconnect. The evolution of the vortex packet is determined by the balance between self-reconnections and reconnections between different loops \[3\]. During the initial coalescence phase the reconnections between different loops dominate and the number of separate loops decreases (self-reconnections and reconnections between different loops \[3\]). During the initial coalescence phase the reconnections interact, become distorted and reconnect. The evolution of the vortex packet is determined by the balance between self-reconnections and reconnections between different loops \[3\].

The interesting question is what determines the characteristic timescale for the vortex packet to evaporate. Since one important parameter of the problem is certainly the quantum of circulation, \(\Gamma\), to obtain a time scale \(\ell^2/\Gamma\) we must identify the relevant length scale \(\ell\). There are only two length scales in the problem: the size of the packet, \(S\), and the average distance between the vortices, \(\delta\), which both change with time. For the sake of simplicity, hereafter we refer to the initial values \(S_0\) and \(\delta_0\). The reason is that the definitions of \(S\) and \(\delta\) at later times are somewhat arbitrary, as they are sensitive to the presence of small (fast) loops in a particular numerical calculation. The use of the initial values simplifies the analysis and lets us concentrate on the simple issue of whether we can predict the evolution of the packet given initial length \(\Lambda_0\) and size \(S_0\).

Figure 2 shows how a typical distribution of loop lengths changes with time. At \(t = 0\) the distribution is on the 11th bin (all \(N = 30\) loops have the same length \(d = 0.067\)cm by construction). As time proceeds the distribution moves to the 3rd bin (centred at \(d \approx 0.013\)cm). Note the direct cascade from the initial peak at the right to the final peak at the left without creation of intermediate length scales. The position and height of the initial peak depends.
on the initial configuration (if we have few longer Fourier knots at the place of many small circular loops, the initial peak is smaller and more to the right). What is universal is the creation of the final left peak, which happens in all our simulations.

Now we analyze how the average loop length, \( D = \langle d \rangle \), depends on \( t \). For the sake of clarity, we normalize \( D \) using the maximum value \( D_{\text{max}} \) achieved in each particular run. If we plot \( D/D_{\text{max}} \) versus \( t \) we note an initial increase (coalescence) followed by a decrease (evaporation) which eventually remains constant as separate loops fly away in all directions (escape). The time scale for the packet to evaporate ranges in the interval \( 0.05\text{sec} < t < 2\text{sec} \), depending on the particular run. In Figure 3 we plot \( D/D_{\text{max}} \) versus the scaled time \( t/\tau \) where

\[
\tau = \frac{\delta_0^2}{\Gamma}.
\]

It is apparent that curves corresponding to the evolution of different vortex packets now overlap, and evaporation takes place within the shorter interval \( 1.5 < t/\tau < 2 \). If we plotted the same graph by scaling \( t \) with \( S_0 \) rather than \( \delta_0 \) the curves would be very separate. Figure 3 therefore suggests that the characteristic time scale of evaporation is of the order of \( \tau_0^2/\Gamma \). Physically, \( \tau \) represents the time scale of reconnections. In fact, from the quantization of vorticity \( \oint \mathbf{v} \cdot d\mathbf{l} = \Gamma \), we estimate that the typical speed inside the packet is of the order of \( v_\lambda = \Gamma/\delta \) hence the typical reconnection time is of the order of \( \delta/v_\lambda = \delta^2/\Gamma = \tau \). Note that, since the distribution of values of \( \delta \) is large, some filaments reconnect earlier, which is evident in Figure 1 at the beginning of the run. The insert in Figure 3 shows the evolution in space and time of different vortex packets. Because of the above mentioned difficulty with the definition of \( S \), we use the more robust quantity \( S' \), defined as the radius of the sphere which contains half the total vortex length. After the evaporation, the packet becomes a gas of loops which fly to infinity, so we expect \( S' \approx vt \) where \( v \) is the speed of the typical loop. From (2) we have, for \( t > \tau \),

\[
\frac{S'}{\delta_0} \approx \left( \frac{L}{4\pi} \right) \left( \frac{t}{\tau} \right),
\]

where \( L \) is a term with a weak logarithmic dependence on \( \delta_0 \). The insert of Figure 3 confirms that \( S'/\delta_0 \) and \( t/\tau \) are proportional. The evolution of all packets are similar and collapse onto the same curve, as shown by the solid line which represents Eq. (3).

We can now interpret the \(^3\)He-B turbulence experiments \[^1\]. In \(^3\)He we have \( \Gamma = h/2m_3 = 6.6 \times 10^{-4}\text{cm}^2/\text{sec} \) and \( a \approx 10^{-9}\text{cm} \). The tangle is created by a vibrating NbTi filament bent into an approximately semicircular shape with diameter 0.3cm, so we assume that the initial vortex packet has dimension \( S_0 \approx 0.1\text{cm} \). We note that the time scale \( S_0^2/\Gamma \approx 15\text{sec} \) is far too large to have relevance to what is observed. The number of vortices required to produce the observed barrier to the quasi - particles is estimated by the authors to correspond to a flow of order \( v_\lambda \geq 0.1\text{cm/sec} \), hence, from \( v_\lambda = \Gamma/2\pi \delta_0 \), we estimate \( \delta_0 \geq 10^{-3}\text{cm/sec} \), and we conclude that the vortex line density must be of the order of \( L_0 \leq 10^6\text{cm}^{-2} \). The characteristic time scale for the vortex packet to evaporate in a gas of small rings is therefore of the order of \( \tau = \delta_0^2/\Gamma \approx 1.5 \times 10^{-5}\text{sec} \). The small loops fly away with speed \( v_\lambda \approx 0.4\text{cm/sec} \) estimated from equation (2) since we know that \( R \geq 10^{-6}\text{cm} \). This result, that the velocity of expansion of the quantized vorticity is of the order of 1mm/sec, is consistent with the observation that the vortex tangle spreads over the distance of 1mm in the time of approximately 1sec.

In conclusion, we have shown that a packet of quantized vorticity, initially localized in a small region of space, evaporates \[^7\] and diffuses away as a gas of small vortex loops on the time scale of order \( \tau = 1/\Gamma \). Application of this scenario to the recent turbulent \(^3\)He-B experiment yields order of magnitude agreement with the observed spatio - temporal evolution. This cascade to small loops is similar to an idea originally proposed by Feynman \[^3\]. To pursue this study in the context of \(^4\)He it would be interesting to use the Gross - Pitaevskii model to determine whether small vortex loops radiate phonons.

Finally, our results should be of interest in fluid dynamics. Firstly, we have found a peculiar form of diffusion in what is actually an inviscid fluid. Secondly, we have found a mechanism to transfer energy to small scales. Thirdly, we have shown that, as far as helicity is concerned, the superfluid represents a different, third benchmark to study, besides the traditional Navier - Stokes and Euler fluids.

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Figure 1:
Evolution of a small vortex packet of quantized vorticity (data corresponding to the crosses in Figure 3).

Figure 2:
Number \( N \) of vortex loop of given length at different times corresponding to the upward triangles in Figure 3. Note the direct cascade from the right peak to the left one (the small peak in the middle is due to some larger loops left at the centre of the evaporated packet).

Figure 3:
Normalized average loop length \( D/D_{\text{max}} \) vs \( t/\tau \). The values of \( N_0, S_0 (cm) \), \( L_0 (cm^{-2}) \) and \( D_{\text{max}} (cm) \) for each run are: Stars: 30, 0.090, 1252 and 0.38; Crosses: 20, 0.090, 835 and 0.22; White circles: 30, 0.045, 5007 and 0.139; Black squares: 25, 0.018, 26080 and 0.042; Downward triangles: 25, 0.027, 11590 and 0.081; White squares: 60, 0.090, 2504 and = 0.339. Black circles: 4, 0.018, 31460 and 0.194 (in this case the initial condition consists of few long Fourier knots, so, unlike the other runs, reconnections immediately increase the number of separate loops and \( D \) is maximum at \( t = 0 \)).

The insert (\( S'/\delta_0 \) vs \( t/\tau \)) shows how the evolution of different packets scale together. The solid line shows Eq. (4).
