An approach to the quantization of black-hole quasi-normal modes

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(Dated: December 20, 2013)

In this work we present a systematic procedure to quantize the black-hole quasi-normal modes. Using the field theoretic Lagrangian, we map the quasi-normal modes of black holes to the Bateman-Feshbach-Tikochinsky oscillator and Caldirola-Kanai Lagrangian. We then discuss the conditions under which the black-hole quasi-normal modes form a complete set of modes. Using the system-bath model, similar to the Bateman-Feshbach-Tikochinsky oscillator, we show that the black-hole quasi-normal modes can be quantized and obtain the general form of the generating functional.

PACS numbers: 04.30.-w,04.62.+v,04.60.-m,04.70.-s,04.70.Dy

I. INTRODUCTION

Resonant modes play a central role in the phenomena of energy flow among coupled systems as they provide characteristic information about the physical system. In the case of gravitating systems, these resonant modes are commonly referred as quasi-normal modes. These are damped perturbations about a fixed background (black-holes or Neutron stars) that propagate to spatial infinity.

To understand the importance of the black-hole quasi-normal modes and why they have attracted attention over the last four decades, let us consider a non-spherical collapse of a gravitational object that results in a slightly perturbed black hole. The black hole will then reach its quiscent state by radiating away the perturbations in the form of gravitational waves. The radiation form the quasi-normal modes spectrum of the black hole. One can therefore define the quasi-normal modes of a black hole as single-frequency oscillations of the black hole spacetime that satisfy ingoing boundary conditions at the black hole event horizon and outgoing boundary conditions at spatial infinity. The real part (that corresponds to the frequency of the oscillation) and complex part (that corresponds to the damping rate) of the frequencies are independent of the initial perturbations and depend only on the properties of the black holes.

In the above way of describing the dynamics and the modes emanating to infinity, quasi-normal modes are assumed to be purely classical and are not quantized. However, there have been indications that these carry some information about quantum gravity. Firstly, QNM has been shown to be an useful tool in understanding AdS/CFT correspondence. In other words, it has been shown that there is a one-to-one mapping of the damping time scales (evaluated via simple QNM techniques) of black holes in Anti-de Sitter spacetimes and the thermalization time scales of the corresponding conformal field theory (which are, in general, difficult to compute) [5, 6]. Secondly, using Bohr’s correspondence principle “the transition frequencies at high quantum numbers equate the classical oscillation frequencies” it was realised that one could possibly identify the transition frequency between different black-hole mass with the black-hole quasi-normal mode frequencies [5, 7]. This indeed has provided the correct black-hole entropy spectrum like other more rigorous semiclassical quantization approaches [10].

It is important to note that although the quasi-normal frequencies are derived using quantum mechanical techniques, however, by definition these are purely classical (see, for instance [2, 3]). The question which we address in this work is the following: If these modes are indeed quantum mechanical (which do not preserve the total probability at later times) can they yield directly the quasi-normal mode frequencies and whether the asymptotic frequencies match with the classical calculations? Also, can the quantization procedure provide the link between the quasi-normal frequencies and the black-hole entropy spectrum.

To authors knowledge, the quantization of black-hole quasi-normal modes was attempted by Kim [11]. Kim heuristically maps the black-hole quasi-normal modes to a dissipative system, in particular, Feshbach-Tikochinsky oscillator and showed that the two — black-hole quasi-normal modes and Feshbach-Tikochinsky oscillator — systems have the same group structure and, hence, maps the quantum states of Feshbach-Tikochinsky oscillator to that of quasi-normal modes.

However, there are several drawbacks in Kim’s approach: Firstly, Feshbach-Tikochinsky oscillator has two modes. One that decays in time and other that gets amplified in time. It is natural to map the decay modes to quasi-normal frequencies, however, as pointed by Kim, it is not clear what is the role of the amplifying modes? Secondly, unlike the normal modes, the quasi-normal modes do not form a complete set of basis vectors in the sense that the system cannot be described as a sum over its QNMs, unless the black hole potential satis-
We discuss our results and present some concluding remarks. In Sec. (VI) we derive the quasi-normal modes. In Sec. (III), we consider a rectangular barrier and obtain the quasi-normal frequencies for this model potential. In order to illustrate the conditions introduced in Sec. (III), we discuss the second point above by showing that the recent approach by Galley [17] can be used to map different equivalent Lagrangians of the damped harmonic oscillator. More importantly, we use the field theoretic analogy to obtain the mapping and address the first point. We give a physical interpretation for the doubling of the modes in any dissipative system. In Sec. (III), we discuss the second point above by looking at the conditions necessary for the completeness of these modes. Here we present those conditions.

It is well-known that the equation of motion containing the Regge-Wheeler potential can be mapped to Huen equation whose solutions are yet unknown [18, 19]. In Sec. (IV), we model the Regge-Wheeler potential to Poischl-Teller potential whose solutions are known [20]. In order to illustrate the conditions introduced in Sec. (III), we consider a rectangular barrier and obtain the quasi-normal modes frequencies for this model potential. Sec. (V) deals with the actual quantization procedure and we derive the quasi-normal frequencies. In Sec. (VI) we discuss our results and present some concluding remarks.

Through out this work we set \( \hbar = 1 \).

**II. EQUIVALENCE OF DIFFERENT APPROACHES: CLASSICAL**

Recently, Galley has come up with a new formulation for the nonconservative systems [17]. One of the crucial feature of Galley’s approach is the formal doubling of the variables. While this formal doubling of variables has been the feature for understanding dissipative oscillators, however, there is no physical meaning associated to these extra variables. In this section, we show that using the field theoretic concepts it is possible to physical identify the doubling of the modes in any dissipative system. In specific, we use Galley’s approach to obtain a one-to-one mapping between the Bateman-Feshbach-Tikochinsky system and Caldirola-Kanai (details can be found in Appendix A).

A free scalar field in a flat spacetime is governed by the Klein-Gordon equation

\[
[\Box + m^2] \phi(x, t) = 0
\]  

(2.1)

with \( m \) being the mass of the field. Fourier transforming Eq. (2.1) in the scalar variables, we obtain

\[
\ddot{y}_k(t) + (k^2 + m^2)y_k(t) = 0
\]  

(2.2)

where

\[
y_k(t) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{-ik\cdot x}\phi(x, t).
\]

Eq. (2.2) is the equation of motion of a simple harmonic oscillator with the frequency being \( \omega_k \equiv \sqrt{k^2 + m^2} \). The scalar field can therefore be considered as a collection of an infinite number of simple harmonic oscillators with densely spaced frequencies. We now ask the question “What happens to the field if we include damping?” i.e., what would be nature of the field decomposed in terms of oscillators instead of satisfying Eq. (2.2)

\[
\ddot{y}_k(t) + \omega_k^2y_k + \gamma \dot{y}_k = 0
\]

where \( \gamma \) is a constant. To answer this question we start with a Lagrangian density of the form

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \Phi^D \partial^\mu \Phi^D + \partial^\mu \Phi^D \partial_\mu \Phi^D \right) + \alpha(x, t) \Phi^D \gamma^\mu \partial_\mu \Phi^D
\]

(2.3)

where \( \Phi^D = \Phi^1 \gamma^0 \) and \( \Phi = (\phi_1, \phi_2, \ldots, \phi_n)^T \) is a n-component wavefunction, with scalar fields as components. The \( \gamma^a \), \( a = 0, 1, 2, \ldots, n \) are the Dirac matrices in n-dimensions.

For simplicity, let us look at \((1+1)-dimensions\). The Dirac matrices in two dimensions are

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Therefore in \((1+1)-dimensions\), the Euler-Lagrange equation of the Lagrangian density in Eq. (2.3) is

\[
\partial_\mu \partial^\mu \Phi + \alpha(x, t) \gamma^\mu \partial_\mu \Phi = 0
\]

(2.4)

where \( \Phi = (\phi_1, \phi_2)^T \) or equivalently

\[
\partial_x^2 \phi_1 - \partial_t \phi_1 - \alpha(x, t)[\delta_1 \phi_2 + \partial_t \phi_1] = 0
\]

(2.5a)

\[
\partial_x^2 \phi_2 - \partial_t \phi_2 - \alpha(x, t)[\delta_1 \phi_1 - \partial_t \phi_2] = 0
\]

(2.5b)

Rewriting \( \phi_1 \) and \( \phi_2 \) as \( \phi_+ = (\phi_1 + \phi_2)/\sqrt{2} \) and \( \phi_- = (\phi_1 - \phi_2)/\sqrt{2} \), we obtain

\[
\partial_\mu \partial^\mu \phi_+ - \alpha(x, t) \partial_0 \phi_+ = \alpha(x, t) \partial_1 \phi_-
\]

(2.6a)

\[
\partial_\mu \partial^\mu \phi_- + \alpha(x, t) \partial_0 \phi_- = \alpha(x, t) \partial_1 \phi_+
\]

(2.6b)

From Eqs. (2.6) we see that the fields \( \phi_+ \) and \( \phi_- \) are not free but coupled to each other. We can now define a Fourier-like transform

\[
F_0(\varphi) = \int_{-\infty}^{\infty} e^{i\omega x} \varphi dx.
\]
Let \( F_\Omega (\psi) = (y_1(t), y_2(t))^T \). Taking the \( F_\Omega \) transform of Eqs. (2.6) we obtain
\[
\begin{align*}
\ddot{y}_1 + \omega^2 y_1 + \alpha \dot{y}_2 - \alpha y_1 &= 0 \\
\ddot{y}_2 + \omega^2 y_2 + \alpha \dot{y}_1 - \alpha y_2 &= 0
\end{align*}
\]
which can be further simplified to
\[
\begin{align*}
\ddot{y}_+ + (\omega^2 - \alpha \omega) y_+ + \alpha \dot{y}_+ &= 0 \\
\ddot{y}_- + (\omega^2 - \alpha \omega) y_- - \alpha \dot{y}_- &= 0
\end{align*}
\]
where \( y_+ = (y_1 + y_2)/2 \) and \( y_- = (y_1 - y_2)/2 \). Eqs. (2.8) are similar to the equations of motion of the Bateman-Feshbach-Tikochinsky system. Therefore the fields \( \phi_+ \) and \( \phi_- \) can considered as infinite collection of coupled oscillators - one of which is dissipative and the other is amplified.

In Appendix (A) we have show that Galley’s approach can also lead to Caldirola-Kanai system. From the Lagrangian \( \mathcal{L}_5 \), one notices that the \( x \) particle has positive kinetic energy and \( y \) particle with negative kinetic energy. Using the field theory analogy, it is possible to identify \( x \) to the particle and \( y \) to the anti-particle. In other words, the decaying mode can be interpreted to the particle while the growing mode to the antiparticle. The full understanding of this mapping is still under investigation.

### III. QUANTIZATION OF QUASI-NORMAL MODES

Before we go into the quantization, it may be useful to look at the crucial differences between the normal and quasi-normal modes and the issues involved in quantization of the quasi-normal modes: Let us consider a system whose motion is defined by an equation of the Sturm-Liouville type, for example a finite vibrating string, whose motion is given by the wave equation. Since we can write the Sturm-Liouville type equation as a self-adjoint operator and can use the concept of normal modes to describe the dynamics of the system. Each normal mode is an oscillation in which all the components of the system move with the same frequency and the same phase [21]. The normal modes form a complete set and we can write down the most general motion of the system as a superposition of the normal modes [21].

Now suppose we introduce dissipation to this system, for example we couple a semi-infinite string to the finite string via a spring [22]. Such a system can dissipate energy away to infinity via radiation. The dynamics of the system are described by quasi-normal modes. Quasi-normal modes, on the other hand, form complete sets only under some special conditions [12]. Therefore a system of quasi-normal modes can be quantized only if those conditions are satisfied. It should be noted that both canonical [23, 24] and path integral [25] quantizations of QNMs of leaky optical cavities have been attempted.

In the rest of this section, we discuss the conditions required for the completeness of the quasi-normal modes and definition of the inner product and normalization of the quasi-normal modes.

#### A. Completeness of QNMs

We use the Klein-Gordon (KG) equation to describe wave propagation in curved space
\[
\hat{D}\phi(x, t) \equiv \left[ \partial_t^2 - \partial_x^2 + V(x) \right] \phi(x, t) = 0. \quad (3.1)
\]
Assuming the time dependency of the form \( e^{-i\omega t} \), the KG equation reduces to a Schrödinger-like equation
\[
\hat{D}\phi(x) \equiv \left[ \partial_x^2 + \omega^2 - V(x) \right] \phi(x) = 0. \quad (3.2)
\]
The QNMs are the eigenfunctions \( f(x, t) \) of the operator \( \hat{D} \) or equivalently the eigenfunctions \( f(x) \) of the operator \( \hat{D} \), that satisfy the condition
\[
f(x) \sim e^{i\omega |x|}, \quad |x| \rightarrow \infty. \quad (3.3)
\]
If the potential \( V(x) \) is finite everywhere and vanishes sufficiently rapidly as \( |x| \rightarrow \infty \) and there are spatial discontinuities in \( V(x) \) marking a “cavity” \( T \), then for \( x \in T \), \( \{ f_j \} \) is a complete set and we can express the time evolution of the wavefunction as a sum over the QNMs,
\[
\phi(x, t) = \sum_j a_j f_j(x) e^{-i \omega_j t} \quad \text{or} \quad \phi(x, t = 0) = \sum_j a_j f_j \quad [12].
\]

#### B. Inner product and normalization

Let \( \phi(x, t) \) and \( \psi(x, t) \) be two wavefunctions in the space spanned by \( \{ f_j(x, t) \} \) and let \( \{ f_j \} \) be complete. Therefore \( \phi(x, t = 0) = \sum_j a_j f_j(x) \) and \( \psi(x, t = 0) = \sum_j b_j f_j(x) \). Defining the two-component wavefunction \( \phi = (\phi, \partial_t \phi)^T \) we give the inner product between the two wavefunctions as [26, 28]
\[
\langle \phi | \psi \rangle = i \left\{ \int_{-a}^{a} dx \left[ \phi \partial_t \psi + (\partial_t \phi) \psi \right] + \left[ \phi(-a) \psi(-a) + \phi(a) \psi(a) \right] \right\}, \quad (3.4)
\]
where \( |a| \) is a finite value. The inner product gives the normalization relation between the QNMs
\[
\langle f_j | f_j \rangle = 2 \omega_j \int_{-a}^{a} f_j^2(x) dx + i \left[ f_j^2(-a) + f_j^2(a) \right] \quad (3.5)
\]
where \( f_j = (f_j, -i \omega_j f_j)^T \) and the expansion coefficients are given by
\[
a_j = \frac{\langle \phi_m | f_j \rangle}{\langle f_j | f_j \rangle}, \quad b_j = \frac{\langle \psi_m | f_j \rangle}{\langle f_j | f_j \rangle} \quad (3.6)
\]
IV. MODEL POTENTIALS

Taking into account the restrictions set on the potential in Sec. (III), we will use potentials that have support in the interval, \( I \equiv [-a, a] \). In this section we will describe two such potentials and the associated QNMs.

The rectangular barrier is given below as an illustration of the method as it gives exact quasi-normal modes frequencies while the modified Pöschl-Teller potential has all the features of Regge-Wheeler potential \[20\].

A. Rectangular barrier

The rectangular barrier potential is given by

\[
V(x) = \begin{cases} 
V_0 & x \in [-a, a] \\
0 & \text{otherwise}. 
\end{cases}
\] (4.1)

For this barrier potential the boundary conditions that should be satisfied by the QNMs are reduced to

\[
\partial_x f(x) = \pm i\omega f(x), \quad x = \pm a
\]

The eigenfunctions of \( D \) are given by Eq. (4.1)

\[
f(x) = \begin{cases} 
P e^{i\omega x}, & x > a \\
Q e^{ikx} + R e^{-ikx}, & a \geq x \geq -a \\
S e^{-i\omega x}, & -a > x,
\end{cases}
\] (4.2)

where \( k = \sqrt{\omega^2 - V_0} \) and \( P, Q, R \) and \( S \) are constants. Maintaining the continuity of \( f(x) \) and \( \partial_x f(x) \) we obtain

\[
\left( \frac{k - \omega}{k + \omega} \right)^2 e^{i\omega a} = 1
\]

which is simplified to

\[
k = i\omega_0 \cos(ka) \quad \text{or} \quad k = -i\omega_0 \sin(ka), \quad (4.3)
\]

where \( \omega_0 = V_0^{1/2} \). Depending on the sign of \( V_0 \), Eq. (4.3) gives different sets of QNM frequencies \[24\].

1. Negative \( V_0 \)

For \( V_0 < 0 \), \( \omega_0 \) is purely imaginary, \( \omega_0 = i|\omega_0| \). Therefore from Eq. (4.3)

\[
k = -|\omega_0| \cos(ka) \quad \text{or} \quad k = |\omega_0| \sin(ka). \quad (4.4)
\]

Eq. (4.4) sometimes have solutions for real \( k \) if \( |k| \leq |\omega_0| \). Since \( \omega^2 = k^2 + \omega_0^2 = k^2 - |\omega_0|^2 \leq 0 \), therefore the values of \( \omega \) are purely imaginary corresponding to bound states of the potential and not quasi-normal modes.

However, if \( \text{Im}(k) > 0 \), then \( \cos(ka) \approx \exp(-ik)/2 \) and \( \sin(ka) \approx -\exp(-ik)/2 \). In such a situation the approximate values of the quasi-normal frequencies are given by the \( j \)-th solutions of

\[
2k \approx -|\omega_0| e^{-ika}. \quad (4.5)
\]

2. Positive \( V_0 \)

For \( V_0 > 0 \), \( \omega_0 \) is real. If \( \omega = i|\omega| \), corresponding to a purely damped mode, then from Eq. (4.3)

\[
|k| = \omega_0 \cosh(|k|a) \quad \text{or} \quad k = 0. \quad (4.6)
\]

The \( k = 0 \) mode is not a true quasi-normal frequency, the other equation gives the physical quasi-normal frequencies

\[
|k|a = \omega_0 a \cosh(|k|a). \quad (4.7)
\]

For small \( \omega_0 a \) there are two quasi-normal frequencies, one of which is given by the perturbative expression

\[
k = i\omega_0 \left\{ 1 + \frac{1}{2}(\omega_0 a)^2 + \frac{13}{24}(\omega_0 a)^4 + \mathcal{O}((\omega_0 a)^6) \right\}. \quad (4.8)
\]

or

\[
\omega = i\omega_0(\omega_0 a) \left\{ 1 + \frac{2}{3}(\omega_0 a)^2 + \frac{4}{5}(\omega_0 a)^4 + \mathcal{O}((\omega_0 a)^6) \right\}.
\]

No such representations exist for the other quasi-normal mode. However if \( \omega_0 a \approx 0.663 \), then the two quasi-normal modes merge and for \( \omega_0 a \gtrsim 0.663 \) there are no quasi-normal modes.

In this case, if \( \text{Im}(k) > 0 \) then the quasi-normal frequencies are the \( j \)-th solutions of

\[
2k \approx -i\omega_0 e^{-ika}. \quad (4.9)
\]

B. Modified Pöschl-Teller potential

The Pöschl-Teller potential is a continuous function over the real line \((-\infty, \infty)\). The QNMs of this potential forms a complete set, in the sense that the solutions at late times can be represented by an infinite sum of the QNMs \[20\]. However the quantization scheme that we will be presenting in the next section requires that spatial discontinuities be present in the potential. We therefore artificially introduce such discontinuities in the potential by making it go to zero for \( x > a \) for a large \( a \) where the
where the factors \( \nu \) and \( c \) can be adjusted to approximate actual black hole potentials \([21]\), subject to the condition \((\nu^2 + c^2) > 0\). Here we have considered the quasi-exactly solvable form of the Poschl-Teller potential \([31, 32]\), a special form of the more general Scarf II potential. The Schrödinger - like equation for the Poschl-Teller potential is

\[
\frac{d^2 f(x)}{dx^2} + \left( \omega^2 - \frac{1}{4\nu} (\nu^2 + c^2) \right) \times \text{sech}^2(\sqrt{\nu}x) f(x) = 0, \quad x \in [-a, a] \tag{4.10}
\]

and

\[
\frac{d^2 f(x)}{dx^2} + \omega^2 f(x) = 0, \quad \text{otherwise.}
\]

We can approximate the QNM boundary condition as

\[
f(x) = (\cosh(\sqrt{\nu}x))^{(\nu+ic)/2\nu}, \quad x = \pm a.
\]

With these boundary conditions, we rewrite the wavefunction as \( f = (\cosh(\sqrt{\nu}x))g(x) \) \([32]\). Introducing the new variable \( y = \sinh(\sqrt{\nu}x) \) Eq. (4.11) becomes

\[
\frac{d^2 g(y)}{dy^2} + \frac{iyc - 2IC}{(y^2 + 1)\nu} \frac{dg(y)}{dy} - \frac{c^2 - 2IC - \nu(4\omega^2 + \nu)}{4(y^2 + 1)\nu^2} g(y) = 0. \tag{4.12}
\]

From Eq. (4.12) using the asymptotic iteration method we get the QNM frequencies as \([32, 33]\)

\[
\omega_n = \pm \left( \frac{c}{2\sqrt{\nu}} - i \frac{(2n + 1)\sqrt{\nu}}{2} \right) \tag{4.13}
\]

and the QNMs as

\[
f_n(x) \approx (\cosh(\sqrt{\nu}x))^{(\nu+ic)/2\nu} P_n^{(\alpha, \beta)}(i \sinh(\sqrt{\nu}x))
\]

where \( P_n^{(\alpha, \beta)}(x) \) are the Jacobi Polynomials.

### V. PATH INTEGRAL QUANTIZATION

Classical treatments of non-conservative or open system often use the cavity-bath model to describe such systems. The system of interest forms the cavity and the surroundings forms the bath with infinite degrees of freedom. Energy is exchanged between the cavity and the bath via some interaction. The complete system is described by a Lagrangian of the form \( L(q, \dot{q}, Q, \dot{Q}) = L_c(q, \dot{q}) + L_b(Q, \dot{Q}) + L_{int}(q, \dot{q}, Q, \dot{Q}) \), where \( L_c(q, \dot{q}) \) describes the cavity, \( L_b(Q, \dot{Q}) \) the bath and \( L_{int}(q, \dot{q}, Q, \dot{Q}) \) gives the interaction between the two \([34]\). The structure of the Lagrangian shows that the quantization would involve both the cavity \((q, \dot{q})\) and the bath \((Q, \dot{Q})\) degrees of freedom. However we are not interested in the time evolution of the bath. It is therefore desirable to eliminate \( Q \) and \( \dot{Q} \) variables and compute the expectation value of any observable in terms of \( q \) and \( \dot{q} \) only. Feynman and Vernon showed that the path integral formalism is very effective in this regard \([35]\). The general idea is to write down the generating functional or the density matrix of the whole system and then integrate out the bath degrees of freedom. This leaves us a reduced density matrix which incorporates the effect of the bath on the cavity and is expressed in terms of the \( q \) and \( \dot{q} \) variables only \([35, 36]\).

As mentioned in Sec. III, QNMs are the exponentially damped eigensolutions of the operator \( \tilde{D} \) and form a complete set for the potentials with discontinuities, like those mentioned in Sec. IV. Moreover the spatial discontinuities in potentials allow us to treat the space around the source of the QNMs as an open system or cavity coupled to a bath, the interval \( I \) being the cavity and everything outside \( I \) forms the bath; the gravitational waves carry off energy from the cavity to the bath. The QNMs can therefore be used for exact eigenfunction expansions in the cavity which is analogous to the normal mode expansions in Hermitian systems. If we eliminate the bath modes then we can second quantize the open system in terms of the damped QNMs only. In this section we will use the Feynman-Vernon formalism as employed in \([25]\) to quantize QNMs of the model potentials.

#### A. Integrating out the bath modes

For the time being we consider that the universe is restricted to \([-\lambda, \lambda]\). We will later take \( \lambda \to \infty \). Though the cavity, \([-a, a]\) is an open system, the entire universe is closed and is defined by the Lagrangian

\[
L = \int_X dx L = \frac{1}{2} \int_X dx \left[ (\partial_t \phi)^2 - (\partial_x \phi)^2 - V(x)\phi^2 \right].
\]

Introducing Euclidean time, \( \tau = it \), we rewrite the Lagrangian,

\[
L_E = -\frac{1}{2} \int_X dx \left[ (\partial_t \phi)^2 + (\partial_x \phi)^2 + V(x)\phi^2 \right]. \tag{5.1}
\]

The Euclidean action for the universe is

\[
S_E = -\frac{1}{2} \int_0^\beta d\tau \int_X dx \left[ (\partial_t \phi)^2 + (\partial_x \phi)^2 + V(x)\phi^2 \right]. \tag{5.2}
\]

where \( \beta \) is the inverse temperature. We also consider a source \( \chi \) that interacts only with the cavity fields via
Therefore the configuration space generating functional of the cavity-bath system is
\[ W[\chi] = \frac{1}{Z} \int \mathcal{D}\phi \ e^{-S[\phi\partial_{\tau}\phi]+(\phi,\chi)}. \] (5.4)

The real field \( \phi \) satisfies the boundary conditions \( \phi(-\lambda-a,0) = \phi(a+\lambda,0) = 0 \) and \( \phi(x,0) = \phi(x,\beta) \). We now separate the generating functional into cavity and bath factors,
\[ Z^{-1} \int \mathcal{D}\phi = Z_c^{-1} \int \mathcal{D}\phi_c \times Z_b^{-1} \int \mathcal{D}\phi_b, \]
the latter running over the fields in the region \([\lambda-a, \lambda+a] \cup [\lambda, \lambda+a]\), with given boundary values \( \phi(a,\tau) \) and \( \phi(-a,\tau) \). \( Z^{-1} \), \( Z_c^{-1} \) and \( Z_b^{-1} \) are normalizing factors. The bath factor turns out to be

\[ W_b = \frac{1}{Z_b} \int \mathcal{D}\phi_b \exp \left[ \frac{1}{2} \int_0^\beta d\tau \int_{-a}^{-a} dx \left( (\partial_x\phi)^2 + (\partial_x\phi)^2 \right) + \frac{1}{2} \int_0^\beta d\tau \int_{-a}^{-a} dx \left( (\partial_x\phi)^2 + (\partial_x\phi)^2 \right) \right]. \] (5.5)

Let \( \xi = x + a \) for the first integral in Eq. (5.5) and \( \eta = x - a \) for the second. Fourier-expansion of the bath modes in terms of the Matsubara frequencies \( 2\pi m/\beta \) gives us

\[ \phi(\eta,\tau) = \frac{1}{\beta} \sum_m \left[ \phi_m(a) \frac{\lambda - \eta}{\lambda} + \sum_{u=1}^\infty \phi_m(\eta) \sin \left( \frac{\pi u \eta}{\lambda} \right) \right] e^{-i\nu_m \tau}, \] (5.6a)
\[ \phi(\xi,\tau) = \frac{1}{\beta} \sum_m \left[ \phi_m(-a) \frac{\lambda + \xi}{\lambda} + \sum_{u=1}^\infty \phi_m(\xi) \sin \left( \frac{\pi u \xi}{\lambda} \right) \right] e^{-i\nu_m \tau}. \] (5.6b)

\( \nu_m = 2\pi m/\beta \) are the bosonic Matsubara frequencies. Substituting Eqs. (5.6) into the respective integrals in Eq. (5.5), we obtain

\[ W_b = \exp \left[ \frac{1}{2\beta} \sum_m \{ |\phi_m(a)|^2 - |\phi_m(-a)|^2 \} \right] \times \left( \frac{\lambda \nu_m^2}{3} - \frac{1}{\lambda} \sum_{u=1}^\infty \frac{2(\lambda \nu_m^2/\pi u^2)}{\nu_m^2 + \pi^2 u^2/\lambda} \right). \] (5.7)

Therefore effective generating functional (or reduced density matrix) of the cavity fields is

\[ W_c^R[\chi] = \frac{1}{Z_c} \int \mathcal{D}\phi_c \exp \left[ (\phi_m,\chi_m) - \frac{1}{2\beta} \sum_m \int_{-a}^{a} dx \left[ |(\partial_x\phi_m)|^2 + |(\partial_x\phi_m)|^2 \right] + V(x) |\phi_m|^2 + |\nu_m| (|\phi_m(0)|^2 - |\phi_m(a)|^2)^2 \right]. \] (5.8)

### B. QNM expansion and path integral quantization

We expand the cavity and the source fields in terms of the QNMs, \( \phi_m(x,\tau) = \sum_j a_{jm} f_j \) and \( \chi_m(x,\tau) = \sum_j b_{jm} f_j \). We temporarily assume that \( V(x) = 0 \).
0, \ x \in [-a,a]. Therefore using the orthogonality condi-

tion given by Eq. (3.7) in Eq. (5.8) we obtain

\[ W^R_c[\chi]_0 = \exp \left[ \frac{1}{8\beta} \sum_{j,m} b_j b_{-m} \left( \frac{\theta(m)\omega_k}{i\omega_k + \nu_m} + \frac{\theta(-m)\omega_k}{i\omega_k - \nu_m} \right) \right]. \tag{5.9} \]

We calculated this path integral by temporarily assum-
ing that \( V(x) = 0, \ x \in [-a,a]. \) Now for the actual case of \( V(x) \neq 0, \ x \in [-a,a] \) we can calculate the cav-

ity generating functional from Eq. (5.9) by perturbation techniques \cite{23}. Let \( a_{jm} \to \beta \partial/2\omega_j \partial b_{j,-m}. \) Then the generating functional for the cavity is

\[ W^R_c[\chi] = \frac{1}{Z_c} \exp \left[ -\frac{\beta}{2} \sum_{m_1,m_2,j_1,j_2} \int_{-a}^{a} dx \ V(x) f_{j_1}(x) f_{j_2}(x) \frac{1}{2\omega_{j_1}\omega_{j_2}} \frac{\partial}{\partial b_{j_1,m_1}} \frac{\partial}{\partial b_{j_2,m_2}} \right] W^R_c[\chi]_0 \tag{5.10} \]

where we can substitute \( V(x) = V_0 \) for the rectangular barrier and \( V(x) = (1/4\nu)(\nu^2 + \nu^2) \) sech\(^2(\sqrt{\nu}x) \) for the modified Pöschl-Teller potential. Therefore Eq. (5.10) gives the generating functional of the cavity in terms of the QNMs of the system only. As we have calculated the QNMs for our model potentials in Sec. VI we can just replace them in Eq. (5.10) to obtain the final algebraic forms of the generating functional.

VI. CONCLUSION

In this paper we have presented two descriptions of the QNMs - one classical and the other quantum mechanical. In the classical picture we have drawn analogy between the QNMs and the damped simple harmonic oscillator. We have shown that there’s a connection between the system of QNMs and the Bateman oscillator and therefore the study of one system may give valuable information about the other. The damped harmonic oscillator is considered to be coupled to an amplified one and the system therefore follows the Bateman equations of motion.

We used the path integral formalism prescribed by Feynman and Vernon to develop our quantum theory. This is because the system-bath model that we used is in spirit similar to Bateman’s dual oscillator system and the physics can therefore be intuitively understood. The final form of the generating functional presented in Sec. VI can be used to derive interesting physical quantities, such as the free energy, defined as, \( F_c^R[\chi] \equiv \ln W^R_c[\chi] \) and entropy. We hope to present the applications of the generating functional in a future article.

Acknowledgments

The authors thank Sashideep Gutti for discussions. SP thanks ISER-TVM for hospitality for carrying out the initial part of the work and is supported by Junior Research Fellowship of CSIR, India. KR is supported by the INSPIRE fellowship, DST, India. The work is sup-
ported by Max Planck-India Partner Group on Gravity and Cosmology. SS is partially supported by Ramanujan Fellowship of DST, India.

Appendix A: Caldirola-Kanai System from Galley’s approach

Consider a simple harmonic oscillator with variable co-
ordinate, \( q(t) \) and natural frequency, \( \omega, \) coupled to \( N \) number of simple harmonic oscillators, \( Q(\omega_n). \) The equations of motion are

\[ M_n \ddot{Q}(\omega_n) + M_n \omega_n^2 Q(\omega_n) = \lambda(\omega_n)q \tag{A1a} \]

\[ m\ddot{q} + m\omega^2 q = \sum_{n=1}^{N} \lambda(\omega_n)Q(\omega_n). \tag{A1b} \]

We double the degrees of freedom, \( q \to (q_1,q_2) \) and \( Q(\omega_n) \to (Q_1(\omega_n),Q_2(\omega_n)) \) and define new coordinates

\[ q_\pm = q_1 \pm \frac{q_2}{\sqrt{2}}, \quad Q_\pm(\omega_n) = \frac{Q_1(\omega_n) \pm Q_2(\omega_n)}{\sqrt{2}}. \]

Using the retarded and advanced Green’s functions we write the equations of motion in terms of these new co-
ordinates

\[ Q_+(\omega_n, t) = \frac{Q^h(\omega_n, t)}{M_n} \int_{t_1}^{t_2} dt' G_{r,n}(t - t') q_+(t') \]  

(A2a)

\[ Q_-(\omega_n, t) = \frac{\lambda(\omega_n)}{M_n} \int_{t_1}^{t_2} dt' G_{a,n}(t - t') q_-(t') \]  

(A2b)

where \( G_{r,n}(G_{a,n}) \) is the retarded (advanced) Green’s function of the \( n \)th oscillator and \( Q^h(\omega_n) \) is a solution of the homogeneous equation. Using these solutions we can write the effective Lagrangian for the \( q \)-system as

\[ L = m\ddot{q}_+ - m\omega^2 q_+ + K(q_+, \dot{q}_+) \]  

(A3)

where \( K(q_{\pm}, \dot{q}_{\pm}, t) = q_{\pm} \sum_n \lambda(\omega_n) Q^h(\omega_n, t) + \int_{t_1}^{t_2} q_{\pm}(t') G(t - t')q_{\pm}(t') dt' \) and \( G(t - t') = \sum_n \chi_n^2/(M_n \omega_n) \sin \omega_n(t - t') \). Let \( Q^h(\omega_n, t) = 0 \), \( M_n = M \) and \( \lambda_n = \lambda(\omega_n) = \omega_n \). Then in the limit \( N \to \infty \) and assuming that the frequencies are dense in \( R_+ \) we get

\[ G(t - t') = -\frac{\lambda^2}{M} \sum_n d \frac{d}{dt'} \cos \omega_n(t - t') = -\alpha \frac{d}{dt'} \delta(t - t') \]

where \( \alpha \) is a constant. Therefore the effective Lagrangian becomes

\[ L = m\ddot{q}_+ - m\omega^2 q_+ - \alpha \dot{q}_+ q_-. \]  

(A4)

We define \( x = (q_+ - e^{\alpha t} q_-)/\sqrt{2} \) and \( y = (q_+ + e^{-\alpha t} q_-)/\sqrt{2} \). In terms of these new coordinates the effective Lagrangian becomes

\[ L = \frac{e^{\alpha t}}{2}(\dot{x}^2 - \omega^2 x^2) - \frac{e^{-\alpha t}}{2}(\dot{y}^2 - \omega^2 y^2) + m \frac{d}{dt} \left( \frac{\lambda}{2} y^2 e^{-\alpha t} \right). \]  

(A5)

The Lagrangian in Eq. (A5) describes a dual-Caldirola-Kanai system. Therefore we see that the harmonic oscillator coupled to a bath of infinite number of oscillators (Eq. (A1)), the Bateman dual system (Eq. (A3)) and the Caldirola-Kanai system are all intimately connected.
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