Transport Theory of Massless Fields

Stanisław Mrówczyński

Soltan Institute for Nuclear Studies,
ul. Hoża 69, PL - 00-681 Warsaw, Poland
and Institute of Physics, Pedagogical University,
ul. Leśna 16, PL - 25-509 Kielce, Poland

Using the Schwinger-Keldysh technique we discuss how to derive the transport equations for the system of massless quantum fields. We analyse the scalar field models with quartic and cubic interaction terms. In the $\phi^4$ model the massive quasiparticles appear due to the self-interaction of massless bare fields. Therefore, the derivation of the transport equations strongly resembles that one of the massive fields, but the subset of diagrams which provide the quasiparticle mass has to be resummed. The kinetic equation for the finite width quasiparticles is found, where, except the mean-field and collision terms, there are terms which are absent in the standard Boltzmann equation. The structure of these terms is discussed. In the massless $\phi^3$ model the massive quasiparticles do not emerge and presumably there is no transport theory corresponding to this model. It is not surprising since the $\phi^3$ model is anyhow ill defined.

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1E-mail: MROW@FUW.EDU.PL
1 Introduction

Transport theory is a very convenient tool to study many-body nonequilibrium systems, nonrelativistic and relativistic as well. The kinetic equations which play a central role in the transport approach can be usually derived by means of simple heuristic arguments similar to those which were used by Ludwig Boltzmann over hundred years ago when he introduced his famous equation. However, such arguments fail when one studies a system of very complicated dynamics as the quark-gluon plasma governed by QCD. In spite of some efforts, the general structure of collision terms in the transport equations of the quark-gluon plasma remains unknown. In such a situation one has to refer to a formal scheme which allows to derive the transport equation directly from the underlying quantum field theory. The formal scheme is also needed to specify the limits of the kinetic approach. Indeed, the derivation shows the assumptions and approximations which lead to the transport theory, and hence the domain of its applicability can be established.

The so-called Schwinger-Keldysh formulation of the quantum field theory provides a very promising basis to derive the transport equation. Kadanoff and Baym developed the technique for nonrelativistic quantum systems, which has been further generalized to relativistic ones. We mention here only the papers which went beyond the mean field or Vlasov approximation and provide a more or less systematic analysis of the collision terms.

The treatment of the massless fields, which are crucial for the gauge theories as QED or QCD, is particularly difficult when the transport equations are derived. Except the well known infrared divergences which plague the perturbative expansion, there is a specific problem of nonequilibrium massless fields. The inhomogeneities in the system cause the off-mass-shell propagation of particles and then the perturbative analysis of the collision terms appears hardly tractable. More specifically, it appears very difficult, if possible at all, to express the field self-energy as the transition matrix element squared and consequently we loose the probabilistic character of the kinetic theory. The problem is absent for the massive fields when the system is assumed homogeneous at the inverse mass or Compton scale. This is a natural assumption within the transport theory which anyway deals with the quantities averaged over a certain scale which can be identified with the Compton one.

The problem of the massless nonequilibrium fields has not been fully recognized in the existing literature. One has usually assumed, explicitly or implicitly, the on-mass-shell propagation or equivalently the ideal homogeneity of the system. We find such an approach unsatisfactory and suggest a systematic solution of the problem. The basic idea is rather obvious.

The fields which are massless in vacuum gain an effective mass in a medium due to the interaction. Therefore, the minimal scale at which the transport theory works is not an inverse bare mass, which is infinite for massless fields, but the inverse effective one. The starting point of the perturbative computation should be no longer free fields but the interacting ones. In physical terms, we postulate existence of the massive quasiparticles and look for their transport equation.
At technical level, we begin with the lagrangian of the massless fields and make a formal trick which is well known in the quantum field theory at finite temperature, see, e.g., [2, 3, 4, 5, 6, 7, 8, 9]. Namely, the auxiliary mass term is added to the free lagrangian and then is subtracted due to a redefinition of the interaction one. As a result the subset of diagrams which contributes to the mass, which is determined in a self consistent way, is effectively resummed in the perturbative expansion. A somewhat similar technique was applied in the kinetic theory in [10].

In this paper we show how the suggested method works for the self-interacting scalar fields. We discuss in detail the $\phi^3$ and $\phi^4$ models which appear to be qualitatively different. We successfully derive the transport equations for the $\phi^4$ model and show why the method does not work for the $\phi^3$ case. Our discussion closely follows the scheme of derivation which was earlier developed for the massive fields: self-interacting scalar fields [7] and the spinor fields interacting with the scalar and vector ones [8].

The main steps of the derivation are the following. We define the contour Green function with the time arguments on the contour in a complex time plane. This function is a key element of the Schwinger-Keldysh approach. After discussing its properties and relevance for nonequilibrium systems, we write down the exact equations of motion i.e. the Dyson-Schwinger equations. Assuming the macroscopic quasi-homogeneity of the system, we perform the gradient expansion and the Wigner transformation. Then, the pair of Dyson-Schwinger equations are converted into the transport and mass-shell equations both satisfied by the Wigner function. The later equation allows one to identify the initially introduced fictitious mass with the effective one generated by the interaction. We further perform the perturbative analysis showing how the Vlasov terms and the collisional ones emerge. Finally we define the distribution functions of standard probabilistic interpretation and find the transport equations satisfied by these functions.

Throughout the article we use natural units where $\hbar = c = 1$. The signature of the metric tensor is $(+, -, -, -)$. As far as possible we keep the convention of Bjorken and Drell [20].

2 Preliminaries

We consider the system of massless scalar fields with the lagrangian density of the form

$$L(x) = \frac{1}{2} \partial^\mu \phi(x) \partial^\mu \phi(x) - \frac{g}{n!} \phi^n(x),$$

(1)

where $n$ equals 3 or 4. The renormalization counterterms are omitted in the lagrangian. We introduce an auxiliary position dependent mass $m_*(x)$ which can be treated as an external field. Specifically, we redefine the lagrangian as

$$L_m(x) = \frac{1}{2} \partial^\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m_*^2(x) \phi^2(x) + L_I(x)$$
with the interaction term

\[ \mathcal{L}_I(x) = \frac{1}{2} m^2_*(x) \phi^2(x) - \frac{g}{n!} \phi^n(x) . \]

The fields which satisfy the equation of motion

\[ (\partial^2 + m^2_*(x)) \phi(x) = 0 , \]

represent free quasiparticles with mass \( m_* \). We observe that it is not \textit{a priori} clear whether massive quasiparticles emerge due to the field self-interaction. It is even less clear whether the limit of free quasiparticles exist. As will be shown it is indeed the case for the \( \phi^4 \) model, but not for the \( \phi^3 \) one.

We write down the energy-momentum tensor defined as

\[ T^{\mu\nu}(x) = \partial^\mu \phi(x) \partial^\nu \phi(x) - g^{\mu\nu} \mathcal{L}(x) . \]

Subtracting the total derivative

\[ \frac{1}{4} \partial^\mu \partial^\nu \phi^2(x) - g^{\mu\nu} \frac{1}{4} \partial^\sigma \partial_\sigma \phi^2(x) , \]

we get the energy-momentum tensor which for the free fields is of the form convenient for our purposes i.e.

\[ T_0^{\mu\nu}(x) = - \frac{1}{4} \phi(x) \leftrightarrow ^\mu \leftrightarrow ^\nu \phi(x) . \]

The fields are assumed here to satisfy the equation of motion (2).

### 3 Green functions

The central role in our considerations plays the contour Green function defined as

\[ i\Delta(x,y) \overset{\text{def}}{=} \langle \tilde{T} \phi(x) \phi(y) \rangle \]

where \( \langle ... \rangle \) denotes the ensemble average at time \( t_0 \) (usually identified with \(-\infty\)); \( \tilde{T} \) is the time ordering operation along the directed contour shown in Fig. 1. The parameter \( t_{\text{max}} \) is shifted to \(+\infty\) in the calculations. The time arguments are complex with an infinitesimal positive or negative imaginary part, which locates them on the upper or on the lower branch of the contour. The ordering operation is defined as

\[ \tilde{T} \phi(x) \phi(y) \overset{\text{def}}{=} \Theta(x_0, y_0) \phi(x) \phi(y) + \Theta(y_0, x_0) \phi(y) \phi(x) , \]

where \( \Theta(x_0, y_0) \) equals 1 if \( x_0 \) succeeds \( y_0 \) on the contour, and equals 0 when \( x_0 \) precedes \( y_0 \).

If the field is expected to develop a finite expectation value, as it happens when the symmetry is spontaneously broken, the contribution \( \langle \phi(x) \rangle \langle \phi(y) \rangle \) is subtracted
in the right-hand-side of the equation defining the Green function, see e.g. [7, 8]. Then, one concentrates on the field fluctuations around the expectation values. Since \( \langle \phi(x) \rangle \) is expected to vanish in the models defined by the lagrangians (1) we neglect this contribution in the Green function definition.

We also use four other Green functions with real time arguments:

\[
\begin{align*}
    i\Delta^>(x, y) & \equal \langle \phi(x)\phi(y) \rangle, \\
    i\Delta^<(x, y) & \equal \langle \phi(y)\phi(x) \rangle, \\
    i\Delta^c(x, y) & \equal \langle T^c\phi(x)\phi(y) \rangle, \\
    i\Delta^a(x, y) & \equal \langle T^a\phi(x)\phi(y) \rangle,
\end{align*}
\]

where \( T^c(T^a) \) prescribes (anti-)chronological time ordering:

\[
\begin{align*}
    T^c\phi(x)\phi(y) & \equal \Theta(x_0 - y_0)\phi(x)\phi(y) + \Theta(y_0 - x_0)\phi(y)\phi_c(x), \\
    T^a\phi(x)\phi(y) & \equal \Theta(y_0 - x_0)\phi(x)\phi(y) + \Theta(x_0 - y_0)\phi(y)\phi(x).
\end{align*}
\]

These functions are related to the contour Green functions in the following manner:

\[
\begin{align*}
    \Delta^c(x, y) & \equiv \Delta(x, y) \text{ for } x_0, y_0 \text{ from the upper branch}, \\
    \Delta^a(x, y) & \equiv \Delta(x, y) \text{ for } x_0, y_0 \text{ from the lower branch}, \\
    \Delta^>(x, y) & \equiv \Delta(x, y) \text{ for } x_0 \text{ from the upper branch and } y_0 \text{ from the lower one}, \\
    \Delta^<(x, y) & \equiv \Delta(x, y) \text{ for } x_0 \text{ from the lower branch and } y_0 \text{ from the upper one}.
\end{align*}
\]

One easily finds the identities which directly follow from the definitions

\[
\begin{align*}
    \Delta^c(x, y) & = \Theta(x_0 - y_0)\Delta^>(x, y) + \Theta(y_0 - x_0)\Delta^<(x, y), \\
    \Delta^a(x, y) & = \Theta(y_0 - x_0)\Delta^>(x, y) + \Theta(x_0 - y_0)\Delta^<(x, y).
\end{align*}
\]

One also observes that

\[
\begin{align*}
    \left( i\Delta^>(x, y) \right) \dagger & = i\Delta^>(x, y), \\
    \left( i\Delta^a(x, y) \right) \dagger & = i\Delta^c(x, y),
\end{align*}
\]

where \( \dagger \) denotes hermitian conjugation, i.e. complex conjugation with an exchange of the Green function arguments. Because the fields are real, the functions \( i\Delta^>(x, y) \) satisfy the relation

\[
\Delta^>(x, y) = \Delta^<(y, x).
\]
It appears convenient to introduce the retarded (+) and advanced (−) Green functions

\[ \Delta^\pm(x, y) \overset{\text{def}}{=} \pm (\Delta^>(x, y) - \Delta^<(x, y)) \Theta(\pm x_0 \mp y_0) . \]  

One immediately finds the identity

\[ \Delta^+(x, y) - \Delta^-(x, y) = \Delta^>(x, y) - \Delta^<(x, y) . \]  

Let us now briefly discuss the physical interpretation of the Green functions. The function \( \Delta^c(x, y) \) describes the propagation of disturbance in which a single particle is added to the many-particle system in space-time point \( y \) and then is removed from it in a space-time point \( x \). An antiparticle disturbance is propagated backward in time. The meaning of \( \Delta^a(x, y) \) is analogous but particles are propagated backward in time and antiparticles forward. In the zero density limit \( \Delta^c(x, y) \) coincides with the Feynman propagator.

The physical meaning of functions \( \Delta^>(x, y) \) and \( \Delta^<(x, y) \) is more transparent when one considers the Wigner transform defined as

\[ \Delta^\approx(X, p) \overset{\text{def}}{=} \int d^4x e^{ipx} \Delta^\approx(x + \frac{1}{2}u, x - \frac{1}{2}u) . \]  

Then, the free-field energy-momentum tensor \( \langle T_{\mu \nu}(X) \rangle \) averaged over ensemble can be expressed as

\[ \langle T_{\mu \nu}(X) \rangle = \int \frac{d^4p}{(2\pi)^4} p^\mu p^\nu i\Delta^<(X, p) . \]  

One recognizes the standard form of the energy-momentum tensor in the kinetic theory with the function \( i\Delta^<(X, p) \) giving the density of particles with four-momentum \( p \) in a space-time point \( X \). Therefore, \( i\Delta^<(X, p) \) can be treated as a quantum analog of the classical distribution function. Indeed, the function \( i\Delta^<(X, p) \) is hermitian. However it is not positively definite and the probabilistic interpretation is only approximately valid. One should also observe that, in contrast to the classical distribution functions, \( i\Delta^<(X, p) \) can be nonzero for the off-mass-shell four-momenta.

### 4 Green Function Equations of Motion

The Dyson-Schwinger equations satisfied by the contour Green function are

\[ [\partial_x^2 + m_x^2(x)] \Delta(x, y) = -\delta^{(4)}(x, y) + \int_C d^4x' \Pi(x, x') \Delta(x', y) , \]  

\[ [\partial_y^2 + m_y^2(y)] \Delta(x, y) = -\delta^{(4)}(x, y) + \int_C d^4x' \Delta(x, x') \Pi(x', y) , \]

where \( \Pi(x, y) \) is the self-energy; the integration over \( x'_0 \) is performed on the contour and the function \( \delta^{(4)}(x, y) \) is defined on the contour as

\[ \delta^{(4)}(x, y) = \begin{cases} 
\delta^{(4)}(x - y) & \text{for } x_0, y_0 \text{ from the upper branch}, \\
0 & \text{for } x_0, y_0 \text{ from the different branches}, \\
-\delta^{(4)}(x - y) & \text{for } x_0, y_0 \text{ from the lower branch}.
\end{cases} \]
Let us split the self-energy into three parts as

\[ \Pi(x, y) = \Pi_\delta(x, y) + \Pi^>(x, y)\Theta(x_0, y_0) + \Pi^<(x, y)\Theta(y_0, x_0). \]

As we shall see later, \( \Pi_\delta \) provides a dominant contribution to the mean-field while \( \Pi^> \) determines the collision terms in the transport equations. Therefore, we call \( \Pi_\delta \) the mean-field self-energy and \( \Pi^> \) the collisional self-energy.

With the help of the retarded and advanced Green functions and the retarded and advanced self-energies defined in an analogous way, the equations (10) and (11) can be rewritten as

\[
\left[ \partial_x^2 + m^2 \right] \Delta^\pm(x, y) = \int d^4x' \left[ \Pi^\pm(x, x') \Delta^\pm(x', y) + \Pi^\mp(x, x') \Delta^\mp(x', y) \right],
\]

(12)

\[
\left[ \partial_y^2 + m^2 \right] \Delta^\pm(x, y) = \int d^4x' \left[ \Delta^\pm(x, x') \Pi^\pm(x', y) + \Delta^\mp(x, x') \Pi^\mp(x', y) \right],
\]

(13)

where all time integrations run from \(-\infty\) to \(+\infty\).

Let us also write down the equations satisfied by the functions \( \Delta^\pm \)

\[
\left[ \partial_x^2 + m^2 \right] \Delta^\pm(x, y) = \int d^4x' \left[ \Pi^\pm(x, x') \Delta^\pm(x', y) + \Pi^\mp(x, x') \Delta^\mp(x', y) \right],
\]

(14)

\[
\left[ \partial_y^2 + m^2 \right] \Delta^\pm(x, y) = \int d^4x' \left[ \Delta^\pm(x, x') \Pi^\pm(x', y) + \Delta^\mp(x, x') \Pi^\mp(x', y) \right].
\]

(15)

5 Towards Transport Equations

The transport equations are derived under the assumption that the Green functions and the self-energies depend weakly on the sum of their arguments, and that they are significantly different from zero only when the difference of their arguments is close to zero. To express these properties it is convenient to define a new set of variables as

\[ \Delta(X, u) \equiv \Delta(X + \frac{1}{2}u, X - \frac{1}{2}u). \]

For homogeneous systems, the dependence on \( X = (x + y)/2 \) drops out entirely due to the translational invariance and \( \Delta(x, y) \) depends only on \( u = x - y \). For weakly inhomogeneous, or quasihomogeneous systems, the Green functions and self-energies are assumed to vary slowly with \( X \). We additionally assume that the Green functions
and self-energies are strongly \textit{peaked} near \( u = 0 \). The effective mass \( m_*(x) \) is simply assumed to be weakly dependent on \( x \).

We will now convert the equations \([12, 13]\) into transport equations by implementing the above approximation and performing the Wigner transformation \([8]\) for all Green functions and self-energies. This is done using the following set of translation rules which can be easily derived:

\[
\begin{align*}
\int d^4x' f(x, x')g(x', y) & \longrightarrow f(X, p)g(X, p) \\
& \quad + \frac{i}{2} \left[ \frac{\partial f(X, p)}{\partial p_\mu} \frac{\partial g(X, p)}{\partial X^\mu} - \frac{\partial f(X, p)}{\partial X^\mu} \frac{\partial g(X, p)}{\partial p_\mu} \right], \\
h(x)g(x, y) & \longrightarrow h(X)g(X, p) - \frac{i}{2} \frac{\partial h(X)}{\partial X^\mu} \frac{\partial g(X, p)}{\partial p_\mu}, \\
h(y)g(x, y) & \longrightarrow h(X)g(X, p) + \frac{i}{2} \frac{\partial h(X)}{\partial X^\mu} \frac{\partial g(X, p)}{\partial p_\mu}, \\
\partial^\mu_x f(x, y) & \longrightarrow (-ip^\mu + \frac{1}{2} \partial^\mu) f(X, p), \\
\partial^\mu_y f(x, y) & \longrightarrow (ip^\mu + \frac{1}{2} \partial^\mu) f(X, p).
\end{align*}
\]

Here \( X \equiv (x + y)/2, \partial^\mu \equiv \frac{\partial}{\partial X^\mu} \) and the functions \( f(x, y) \) and \( g(x, y) \) satisfy the assumptions discussed above. The function \( h(x) \) is assumed to be weakly dependent on \( x \).

Applying these translation rules to eqs. \([12, 13]\), we obtain

\[
\begin{align*}
\left[ \frac{1}{4} \partial^2 - ip^\mu \partial_\mu - p^2 + m_*(X) - \Pi_\delta(X) - \frac{i}{2} \partial_\mu \left( m_*(X) - \Pi_\delta(X) \right) \partial^\mu \right] & \Delta^\tilde{\zeta}(X, p) \\
& = \Pi^\tilde{\zeta}(X, p) \Delta^{-}(X, p) + \Pi^+(X, p) \Delta^{\tilde{\zeta}}(X, p) \\
& + \frac{i}{2} \left\{ \Pi^\tilde{\zeta}(X, p), \Delta^{-}(X, p) \right\} + \frac{i}{2} \left\{ \Pi^+(X, p), \Delta^{\tilde{\zeta}}(X, p) \right\}, \quad (16)
\end{align*}
\]

\[
\begin{align*}
\left[ \frac{1}{4} \partial^2 + ip^\mu \partial_\mu - p^2 + m_*(X) - \Pi_\delta(X) + \frac{i}{2} \partial_\mu \left( m_*(X) - \Pi_\delta(X) \right) \partial^\mu \right] & \Delta^{\tilde{\zeta}}(X, p) \\
& = \Delta^\tilde{\zeta}(X, p) \Pi^{-}(X, p) + \Delta^+(X, p) \Pi^\tilde{\zeta}(X, p) \\
& + \frac{i}{2} \left\{ \Delta^\tilde{\zeta}(X, p), \Pi^{-}(X, p) \right\} + \frac{i}{2} \left\{ \Delta^+(X, p), \Pi^\tilde{\zeta}(X, p) \right\}, \quad (17)
\end{align*}
\]

where we have introduced the Poisson-like bracket defined as

\[
\begin{align*}
\left\{ C(X, p), D(X, p) \right\} & \equiv \frac{\partial C(X, p)}{\partial p_\mu} \frac{\partial D(X, p)}{\partial X^\mu} - \frac{\partial C(X, p)}{\partial X^\mu} \frac{\partial D(X, p)}{\partial p_\mu}.
\end{align*}
\]

The kinetic theory deals only with averaged system characteristics. Thus, one usually assumes that the system is homogeneous on a scale of the Compton wave length.
of the quasiparticles. In other words, the characteristic length of inhomogeneities is assumed to be much larger than the inverse mass of quasiparticles Therefore, we impose the condition
\[ \left| \Delta^\pm(X, p) \right| \gg \left| \frac{1}{m^*_s} \partial^2 \Delta^\pm(X, p) \right| , \]
which leads to the quasiparticle approximation. As discussed in the next section and in the Appendix, the requirement (18) renders the off-shell contributions to the Green functions \( \Delta^\pm \) negligible. Thus, we deal with the quasiparticles having on-mass-shell momenta. Unfortunately, the assumption (18) cannot be applied to massless particles and for this reason we have introduced the effective mass \( m^*_s \).

Let us now take the difference and the sum of eqs. (16) and (17), where the \( \partial^2 \) terms have been neglected due to the quasiparticle approximation (18). Then, one gets
\[
\left[ p^\mu \partial_\mu + \frac{1}{2} \partial_\mu (m^2_s)(X) - \Pi_\delta(X) \right] \partial^\mu (X, p) \Delta^\pm(X, p) \\
= \frac{i}{2} \left( \Pi^>(X, p) \Delta^<(X, p) - \Pi^<(X, p) \Delta^>(X, p) \right) \\
- \frac{1}{4} \left\{ \Pi^>(X, p), \Delta^+(X, p) + \Delta^-(X, p) \right\} \\
- \frac{1}{4} \left\{ \Pi^+(X, p) + \Pi^+(X, p), \Delta^\pm(X, p) \right\} ,
\]
\[ (19) \]
\[
\left[ - p^2 + m^2_s(X) - \Pi_\delta(X) \right] \Delta^\pm(X, p) \\
= \frac{1}{2} \left( \Pi^>(X, p) \Delta^+(X, p) + \Delta^-(X, p) \right) + \left( \Pi^+(X, p) + \Pi^-(X, p) \right) \Delta^\pm(X, p) \\
+ \frac{i}{4} \left\{ \Pi^>(X, p), \Delta^<(X, p) \right\} - \frac{i}{4} \left\{ \Pi^<(X, p), \Delta^>(X, p) \right\} ,
\]
\[ (20) \]
where we have used the identity (7) applied to the Green functions and self-energies.

One recognizes eq. (19) as a transport equation while eq. (20) as a so-called mass-shell equation. We will write down these equation in a more compact way. From the definition (5) one finds that
\[
\Delta^\pm(X, p) = \pm \frac{1}{2} \left( \Delta^>(X, p) - \Delta^<(X, p) \right) \\
+ \frac{1}{2 \pi i} P \int d\omega' \frac{\Delta^>(X, \omega', p) - \Delta^<(X, \omega', p)}{\omega - \omega'} .
\]
\[ (21) \]
The first term in the r.h.s is antihermitian while the second one is hermitian. Thus, we introduce
\[
\text{Im} \Delta^\pm(X, p) \equiv \pm \frac{1}{2 \pi} \left( \Delta^>(X, p) - \Delta^<(X, p) \right) ,
\]
\[ (22) \]
\[ \text{Re}\Delta^\pm(X, p) \equiv \frac{1}{2\pi i} \mathcal{P} \int d\omega' \frac{\Delta^\geq(X, \omega', p) - \Delta^\leq(X, \omega', p)}{\omega - \omega'} . \]  

(23)

With the help of (23) and analogous formulas for \( \Pi^\pm \) the equations (19, 20) can be rewritten as

\[
\begin{align*}
\{ p^2 - m^2_s(X) + \Pi_\delta(X) + \text{Re}\Pi^+(X, p), \Delta^\geq(X, p) \} \\
= i \left( \Pi^\geq(X, p) \Delta^\leq(X, p) - \Pi^\leq(X, p) \Delta^\geq(X, p) \right) \\
- \left\{ \Pi^\geq(X, p), \text{Re}\Delta^+(X, p) \right\} ,
\end{align*}
\]

(24)

\[
\begin{align*}
\left[ p^2 - m^2_s(X) + \Pi_\delta(X) + \text{Re}\Pi^+(X, p) \right] \Delta^\geq(X, p) &= -\Pi^\geq(X, p) \text{Re}\Delta^+(X, p) \\
- \frac{i}{4} \left\{ \Pi^\geq(X, p), \Delta^\leq(X, p) \right\} + \frac{i}{4} \left\{ \Pi^\leq(X, p), \Delta^\geq(X, p) \right\} .
\end{align*}
\]

(25)

In the case of fields with the finite bare mass the gradient terms in the right-hand-sides of eqs. (24, 25) are small and are usually neglected. When the bare fields are massless, as those studied here, there is no reason to neglect the gradient terms. The equation analogous to (24) was earlier derived in (2, 3).

It appears useful to write down the transport and mass-shell equations satisfied by the retarded and advanced Green functions. Starting with eqs. (14, 15) one finds

\[
\begin{align*}
\{ p^2 - m^2_s(X) + \Pi_\delta(X) + \Pi^\pm(X, p), \Delta^\pm(X, p) \} &= 0 , \\
\left[ p^2 - m^2_s(X) + \Pi_\delta(X) + \Pi^\pm(X, p) \right] \Delta^\pm(X, p) &= 1 .
\end{align*}
\]

(26)

(27)

We observe that the gradient terms drop out entirely in eq. (27). Nevertheless the equation holds within the first order of gradient expansion. Due to the absence of the gradients, eq. (27) can be immediately solved as

\[ \Delta^\pm(X, p) = \frac{1}{p^2 - m^2_s(X) + \Pi_\delta(X) + \Pi^\pm(X, p)} . \]

(28)

One notices that \( \Delta^\pm \) of the form (28) solves not only eq. (27) but eq. (26) as well. Indeed, any function \( f \) of \( K \) satisfies the equation \( \{ K, f(K) \} = 0 \).

The real and imaginary parts of \( \Delta^\pm(X, p) \), which are needed in our further considerations, are

\[
\begin{align*}
\text{Re}\Delta^\pm(X, p) &= \frac{p^2 - m^2_s(X) + \Pi_\delta(X) + \text{Re}\Pi^+(X, p)}{(p^2 - m^2_s(X) + \Pi_\delta(X) + \text{Re}\Pi^+(X, p))^2 + (\text{Im}\Pi^+(X, p))^2} , \\
\text{Im}\Delta^\pm(X, p) &= \frac{\pm\text{Im}\Pi^+(X, p)}{(p^2 - m^2_s(X) + \Pi_\delta(X) + \text{Re}\Pi^+(X, p))^2 + (\text{Im}\Pi^+(X, p))^2} .
\end{align*}
\]

(29)

(30)
6 Free quasiparticles

Before further analysis the equations obtained in the previous section we consider here a very important limit which corresponds to the free quasiparticles. Specifically, we assume that \( \Pi_\delta = \Pi^\pm = 0 \). Then, the equations (24, 25) read

\[
\left[ p^\mu \partial_\mu + \frac{1}{2} \partial_\mu m^2_\ast(X) \partial^\mu_p \right] \Delta^\pm_0(X, p) = 0 ,
\]

\[
[p^2 - m^2_\ast] \Delta_0^\pm(X, p) = 0 .
\]

Although the quasiparticles are assumed to be free, the transport equation is of the Vlasov not of the free form. This is a simple consequence of the \( X \)–dependence of the effective mass.

Due to eq. (32), \( \Delta^\pm_0(X, p) \) is proportional to \( \delta(p^2 - m^2) \), and consequently free quasiparticles are always on mass-shell. If the quasiparticle approximation is not applied, the mass-shell equation gets the form

\[
\left[ \frac{1}{4} \partial^2 - p^2 + m^2_\ast(X) \right] \Delta_0^\pm(X, p) = 0 ,
\]

and the off-shell contribution to the Green function \( \Delta_0^\pm \) is nonzero. A detailed discussion of the quasiparticle approximation is given in the Appendix.

We also discuss the (anti-)chronological Green functions \( \Delta_0^{\pm(a)} \) in the limit of free quasiparticles. One easily finds their equations of motion as

\[
\left[ p^\mu \partial_\mu + \frac{1}{2} \partial_\mu m^2_\ast(X) \partial^\mu_p \right] \Delta_0^{\pm(a)}(X, p) = 0 ,
\]

\[
[p^2 - m^2_\ast] \Delta_0^{\pm(a)}(X, p) = 1 .
\]

For the antichronological function \( \Delta^a \), the right hand side of the mass-shell equation equals \(-1\) instead of \(+1\). The solution of these equation can be written as

\[
\Delta^a_0(X, p) = \frac{1}{p^2 - m^2_\ast + i0^+} + \Theta(-p_0)\Delta^\geq_0(X, p) + \Theta(p_0)\Delta^\leq_0(X, p) ,
\]

where \( \Delta^\geq_0(X, p) \) is assumed to satisfy eqs. (31, 32). It is worth mentioning that any function which depends on \( (X, p) \) through \( (p^2 - m^2) \) solves the Vlasov equation (31). \( \Delta^a_0(X, p) \) obeys the initial condition of the standard Feynman propagator. It also satisfies the relation (3).

The antichronological Green function is

\[
\Delta^a_0(X, p) = \frac{-1}{p^2 - m^2_\ast - i0^+} + \Theta(p_0)\Delta^\geq_0(X, p) + \Theta(-p_0)\Delta^\leq_0(X, p) .
\]

Knowing \( \Delta^a_0 \) and \( \Delta^\pm_0 \) one immediately gets the retarded and advanced functions

\[
\Delta^\pm_0(X, p) = \frac{1}{p^2 - m^2_\ast \pm ip_00^+} ,
\]

(33)
which obey the respective initial conditions. Confronting the expressions of $\Delta^\pm$ for free and interacting quasiparticles, one finds that

\[
\text{Im}\Pi^+(X,p) = -\text{Im}\Pi^-(X,p) = \begin{cases} 
0^+ & \text{for } p_0 > 0 , \\
0^- & \text{for } p_0 < 0 ,
\end{cases}
\]

in the limit of free quasiparticles.

It appears useful to express the Green functions $\Delta^<\geqslant$ through the distribution function $f_0$ as

\[
\Theta(p_0) i \Delta^<\geqslant(X, p) = \Theta(p_0) 2\pi \delta(p^2 - m^2) f_0(X, p) = \frac{\pi}{E_p} \delta(E_p - p_0) f_0(X, p) ,
\]

where $E_p \equiv \sqrt{p^2 + m^2}$. This equation should be treated as a definition of $f_0$.

Due to the relation (5) we have

\[
\Delta^<\geqslant(X, p) = \Delta^>\geqslant(X, -p) ,
\]

and consequently

\[
\Theta(p_0) i \Delta^>\geqslant(X, -p) = \Theta(p_0) 2\pi \delta(p^2 - m^2) f_0(X, p) = \frac{\pi}{E_p} \delta(E_p - p_0) f_0(X, p) .
\]

In that way we express the positive energy part of $\Delta^<\geqslant(X, p)$ and the negative energy part of $\Delta^>\geqslant(X, p)$ through $f_0(X, p)$. We extend these expressions to the whole energy domain using the identity (7). With the help of the explicit form of the retarded and advanced functions (33) we get the formula

\[
i \Delta^<\geqslant(X, p) - i \Delta^<\geqslant(X, p) = 2\pi \delta(p^2 - m^2) (\Theta(p_0) - \Theta(-p_0)) ,
\]

which is discussed in detail in the next section.

Combining eqs. (35, 36) with (37) one finds the desired expression of the Green functions $\Delta^>\geqslant$ in terms of the distribution function $f_0$. Namely,

\[
i \Delta^<\geqslant(X, p) = \frac{\pi}{E_p} \delta(E_p - p_0) f_0(X, p) + \frac{\pi}{E_p} \delta(E_p + p_0) (f_0(X, -p) + 1) ,
\]

\[
i \Delta^>\geqslant(X, p) = \frac{\pi}{E_p} \delta(E_p - p_0) (f_0(X, p) + 1) + \frac{\pi}{E_p} \delta(E_p + p_0) f_0(X, -p) .
\]

When the system is in thermodynamical equilibrium the distribution functions reads

\[
f_0^{\text{eq}}(p) = \frac{1}{e^{\beta \mu p^\mu} - 1} ,
\]

where $\beta \mu \equiv u^\mu / T$ with $u^\mu$ being the hydrodynamical velocity and $T$ the temperature. In the local equilibrium the two parameters are $X$-dependent.
7 Spectral function

In this section we introduce one more function which appears useful in the analysis of the interacting systems. The spectral function \( A \) is defined as

\[
A(x, y) \equiv i \Delta^>(x, y) - i \Delta^<(x, y).
\]

Thus,

\[
A(x, y) = \langle [\phi(x), \phi(y)] \rangle,
\]

where \([\phi(x), \phi(y)]\) denotes the field commutator.

Due to the equal time commutation relations

\[
[\phi(t, x), \phi(t, y)] = 0, \quad [\dot{\phi}(t, x), \phi(t, y)] = -i \delta^{(3)}(x - y),
\]

with the dot denoting the time derivative, the Wigner transformed spectral function satisfies the two identities

\[
\int \frac{dp_0}{2\pi} A(X, p) = 0, \quad \int \frac{dp_0}{2\pi} p_0 A(X, p) = 1.
\]  

(41)

One also sees that (cf. eq. (22))

\[
A(X, p) = \mp 2 \text{Im} \Delta^+(X, p).
\]  

(42)

Finally, we observe that the identity (43), which holds for the real fields, provides the relation

\[
A(X, p) = -A(X, -p).
\]

From the transport and mass-shell equations (24, 25) one immediately finds the equations for \( A(X, p) \) as

\[
\left\{ p^2 - m^2(X) + \Pi(X) + \text{Re} \Pi^+(X, p), A(X, p) \right\} = 2 \left\{ \text{Im} \Pi^+(X, p), \text{Re} \Delta^+(X, p) \right\}, \tag{43}
\]

\[
\left[ p^2 - m^2(X) + \Pi(X) + \text{Re} \Pi^+(X, p) \right] A(X, p) = 2 \text{Im} \Pi^+(X, p) \text{Re} \Delta^+(X, p). \tag{44}
\]

Substituting \( \text{Re} \Delta^+(X, p) \) from eq. (29) into the algebraic equation (44) we find its solution as

\[
A(X, p) = \frac{2 \text{Im} \Pi^+(X, p)}{(p^2 - m^2(X) + \Pi(X) + \text{Re} \Pi^+(X, p))^2 + (\text{Im} \Pi^+(X, p))^2}. \tag{45}
\]

Then, one easily shows that the function of the form (45) solves eq. (43) as well. In fact, the spectral function (45) can be found directly from eq. (29) due to the relation (42).
The spectral function of the free quasiparticles can be, obviously, obtained from (45) but the limit should be taken with care. We firstly write the spectral function (45) as

\[ A(X, p) = \frac{i}{p^2 - m^2_\pm(X) + \Pi_\delta(X) + \text{Re}\Pi^+(X, p) + i\text{Im}\Pi^+(X, p)} \]

Then we take the limit \( \Pi \to 0 \) keeping in mind the condition (34). Using the well-known identity

\[ \frac{1}{x \pm i0^+} = P \frac{1}{x} \mp i\pi \delta(x) , \]

we get the spectral function of noninteracting quasiparticles as

\[ A_0(X, p) = \frac{2\pi\delta(p^2 - m^2_\pm(X))(\Theta(p_0) - \Theta(-p_0))}{(E^2_\pm(X, p) - p^2)} , \] (46)

which, of course, coincides with (37).

Let us also consider a specific approximate form of the spectral function. If the condition

\[ p^2 + m^2_\pm(X) - \Pi_\delta(X) - \text{Re}\Pi^+(X, p) \gg |\text{Im}\Pi^+(X, p)| \]

is satisfied, \( A \) as a function of \( p_0 \) is close to zero everywhere except two narrow regions around \( p_0 = \pm E_\pm \) \( (E_\pm > 0) \) which solve the equations

\[ E^2_\pm(X, p) = p^2 + m^2_\pm(X) - \Pi_\delta(X) - \text{Re}\Pi^+(X, p_0 = \pm E_\pm, p) \] .

Then, the function (45) can be approximated as

\[ A(X, p) = \frac{1}{E_+(X, p)} \frac{\Gamma_+(X, p)}{(E_+(X, p) - p_0)^2 + \Gamma^2_+(X, p)} - \frac{1}{E_-(X, p)} \frac{\Gamma_-(X, p)}{(E_-(X, p) + p_0)^2 + \Gamma^2_-(X, p)} , \] (47)

with

\[ \Gamma_\pm(X, p) \equiv \pm \frac{\text{Im}\Pi^+(X, p_0 = \pm E_\pm, p)}{2E_\pm(X, p)} . \] (48)

One easily checks that the spectral function of the form (47) satisfies the sum rules (11).

8 Perturbative expansion

As discussed in e.g. [5, 6, 11] the contour Green functions admit a perturbative expansion very similar to that known from vacuum field theory with essentially the
same Feynman rules. However, the time integrations do not run from \(-\infty\) to \(+\infty\), but along the contour shown in Fig. 1. The right turning point of the contour \((t_{max})\) must be above the largest time argument of the evaluated Green function. In practice, \(t_0\) is shifted to \(-\infty\) and \(t_{max}\) to \(+\infty\). The second difference is the appearance of tadpoles, \(i.e.\) loops formed by single lines, which give zero contribution in the vacuum case. A tadpole corresponds to a Green function with two equal space-time arguments. Since the Green function \(\Delta(x,y)\) is not well defined for \(x=y\) we ascribe the function \(-i\Delta^<(x,x)\) to each tadpole. The rest of Feynman rules can be taken from the textbook of Bjorken and Drell \([20]\).

In this section we consider the lowest-order contributions to the self-energies. It should be stressed that the Green functions, which are represented by the lines of the Feynman diagrams, correspond to those of free quasiparticles not of noninteracting fields.

\(\phi^4\) model

The lowest-order contribution to the self-energy which is associated with the graphs from Fig. 2 is

\[
\Pi(x,y) = \delta^{(4)}(x,y) \left( m_s^2(x) - \frac{ig}{2} \Delta^<_0(x,x) \right) ,
\]

giving

\[
\Pi_\delta(x) = m_s^2(x) - \frac{ig}{2} \Delta^<_0(x,x) , \tag{49}
\]

and

\[
\Pi^>(x,y) = \Pi^<(x,y) = 0 .
\]

Substituting \(\Delta^<_0\) given by eq. (38), where \(\Delta^<_0\) is expressed through the distribution function, into (49) one finds \(\Pi_\delta\) as

\[
\Pi_\delta(x) = m_s^2(x) - \frac{g}{2} \int \frac{d^3p}{(2\pi)^3 2E_p} \left[2f_0(x,p) + 1\right] .
\]

As seen the integral is quadratically divergent in the vacuum limit when \(f_0 \to 0\). This type of divergence, which appears due to the zero-mode fluctuations, is well known in the field theory. We remove it away by subtracting the vacuum value from \(\Pi_\delta\). Thus one gets

\[
\Pi_\delta(x) = m_s^2(x) - \frac{g}{2} \int \frac{d^3p}{(2\pi)^3 2E_p} f_0(x,p) . \tag{50}
\]

We compute \(\Pi_\delta\) for the equilibrium system when the distribution function is given by eq. (33) with \(u^\mu = (1,0,0,0)\). For \(T \gg m_s\) we get after elementary integration the well-known result, see e.g. \([17]\),

\[
\Pi_\delta(x) = m_s^2(x) - \frac{gT^2(x)}{24} , \tag{51}
\]

where the temperature is \(x\)-dependent in the case of local equilibrium.
The lowest-order contribution to the self-energy corresponding to the graphs from Fig. 3a and 3b is

\[ \Pi_a(x, y) = \delta^{(4)}(x, y) \left[ m_\ast^2 - \frac{i}{2} g^2 \int d^4 x' \Delta_0(x', x) \Delta_0^<(x', x') \right]. \]

Locating the argument \( x \) on the upper branch of the contour, one finds

\[ \Pi_{\delta}(x) = m_\ast^2 - \frac{i}{2} g^2 \int d^4 x' \left[ \Delta_0^<(x', x) - \Delta_0^<(x', x') \right] \Delta_0^<(x', x'), \]

where the time integration runs from \(-\infty\) to \(+\infty\). Observing that \( \Delta^c - \Delta^< = \Delta^+ \) we get

\[ \Pi_{\delta}(x) = m_\ast^2 - \frac{i}{2} g^2 \int d^4 x' \Delta_0^+(x', x) \Delta_0^<(x', x'). \]

Using the explicit form of \( \Delta_0^+ \) and expressing \( \Delta_0^< \) through the distribution function, one finds

\[ \Pi_{\delta}(x) = m_\ast^2 - \frac{i}{2} g^2 \int d^4 x' d^4 p \frac{d^3 k}{(2\pi)^4} \int \frac{d^3 q}{(2\pi)^4} E_k e^{-ip(x'-x)} f_0(\mathbf{k}, \mathbf{q}). \]

In the case of global equilibrium, when \( f_0(x, k) \) is independent of \( x \), the integral from (52) can be computed as

\[ \Pi_{\delta} = m_\ast^2 + \frac{g^2}{2m_\ast^2} \int \frac{d^3 k}{(2\pi)^3 E_k} f_0(\mathbf{k}) . \]

In the limit \( T \gg m_\ast \) one finally finds

\[ \Pi_{\delta} = m_\ast^2 + \frac{g^2 T^2}{24 m_\ast^2} . \]

The graph from Fig. 3c corresponds to

\[ \Pi_b(x, y) = -\frac{i}{2} g^2 \Delta_0(x, y) \Delta_0(y, x), \]

and it gives

\[ \Pi^<(x, y) = -\frac{i}{2} g^2 \Delta_0^<(x, y) \Delta_0^<(y, x) . \]

Substituting \( \Delta^< \) expressed through the distribution function to eq. (53) one finds

\[ \Pi^<(X, p) = \frac{i}{2} g^2 \int \frac{d^3 k}{(2\pi)^3 E_k} \int \frac{d^3 q}{(2\pi)^3 E_q} (2\pi)^4 \delta^{(3)}(p + q - k) \left[ \delta(p_0 - E_k - E_q) f_0(X, \mathbf{k}) f_0(X, -\mathbf{q}) + \delta(p_0 - E_k + E_q) f_0(X, \mathbf{k}) (f_0(X, \mathbf{q}) + 1) + \delta(p_0 + E_k - E_q) (f_0(X, -\mathbf{k} + 1) f_0(X, -\mathbf{q}) + \delta(p_0 + E_k + E_q) (f_0(X, -\mathbf{k} + 1) (f_0(X, \mathbf{q}) + 1) \right] , \]
\( \Pi^<(X, p) = \frac{i g^2}{2} \int \frac{d^3k}{(2\pi)^3 2E_k} \frac{d^3q}{(2\pi)^3 2E_q} (2\pi)^4 \delta^{(3)}(p + q - k) \)
\[
\left[ \delta(p_0 - E_k - E_q) (f_0(X, k) + 1) (f_0(X, -q) + 1) \\
+ \delta(p_0 - E_k + E_q) (f_0(X, k) + 1) f_0(X, q) \\
+ \delta(p_0 + E_k - E_q) f_0(X, -k) (f_0(X, -q) + 1) \\
+ \delta(p_0 + E_k + E_q) f_0(X, -k) f_0(X, q) \right]. \\
\text{(57)}
\]

Let us observe here that due to the trivial kinematical reasons \( \Pi^<(X, p) = \Pi^<(X, p) = 0 \) for on-mass-shell four-momenta i.e. when \( p^2 = m^2 \).

We further compute \( \Pi^<(X, p) \) for \( p = 0 \). In the case of equilibrium when the distribution function is of the form (40) and, as previously, \( u^\mu = (1, 0, 0, 0) \), one finds \( \Pi^<(X, p) \) for \( p = (\pm \omega, 0) \) with \( \omega > 0 \) as
\[
\Pi^>(X, \omega, 0) = \Pi^<(X, -\omega, 0) \\
= -\frac{i g^2}{16\pi} \Theta(\omega - 2m_*) \sqrt{1 - \frac{4m_*^2}{\omega^2}} \left( \frac{1}{\exp \left( \frac{\beta \omega}{2} \right)} - 1 \right)^2, \\
\text{(58)}
\]
\[
\Pi^<(X, \omega, 0) = \Pi^>(X, -\omega, 0) \\
= -\frac{i g^2}{16\pi} \Theta(\omega - 2m_*) \sqrt{1 - \frac{4m_*^2}{\omega^2}} \left( \frac{\exp \left( \frac{\beta \omega}{2} \right)}{\exp \left( \frac{\beta \omega}{2} \right) - 1} \right)^2. \\
\text{(59)}
\]

We also compute
\[
\text{Im}\Pi^+(X, \pm \omega, 0) = \frac{1}{2\pi} \left( \Pi^>(X, \pm \omega, 0) - \Pi^<(X, \pm \omega, 0) \right) \\
= \frac{\pm g^2}{16\pi} \Theta(\omega - 2m_*) \sqrt{1 - \frac{4m_*^2}{\omega^2}} \left( \frac{1}{\exp \left( \frac{3\beta \omega}{2} \right) - 1} \right).
\]

One sees that the condition (54) is indeed satisfied by the perturbative self-energy when \( g \to 0 \) and \( m_* \) is kept fixed.

### 9 Quasiparticle mass

The structure of eqs. (24, 25) motivates the definition of the quasiparticle mass as a solution of the equation
\[
\Pi_\delta(X) + \text{Re}\Pi^+(X, p_0 = m_*, p = 0) = 0. \\
\text{(60)}
\]

The definition is not Lorentz invariant but statistical systems usually break such an invariance. In the case of global equilibrium there is, for example, a preferential reference frame related to the thermostat.

Let us now look for the explicit expression of \( m_* \) in the lowest nontrivial order in \( g \) within the two models under consideration.
\( \phi^4 \) model

The only nonvanishing contribution to the self-energy in the lowest order of perturbative expansion comes from \( \Pi_\delta \), which is given by eq. (50). Therefore, the effective mass is a solution of the equation

\[
m^2_*(x) = \frac{g}{2} \int \frac{d^3 p}{(2\pi)^3 E_p} f_0(x, p).
\]

One should keep in mind that \( E_p \) and \( f_0 \) depend on \( m_* \).

For the equilibrium system with \( T \gg m_* \) we immediately get the well known result:

\[
m^2_*(x) = \frac{g^2 T^2(x)}{24}.
\]

One sees that the condition \( T \gg m_* \) is automatically satisfied in the perturbative limit where \( g^{-1} \gg 1 \).

\( \phi^3 \) model

Now we have the nonzero contributions to the self-energy not only from \( \Pi_\delta \) (52) but from \( \Pi^\pm \) (56, 57) as well. The real part of \( \Pi^+ \) which enters eq. (60) is given by the equation analogous to (23). Thus,

\[
\text{Re} \Pi^+(X, m_*, 0) = \frac{1}{2\pi i} \text{P} \int d\omega \frac{\Pi^>(X, \omega, 0) - \Pi^<(X, \omega, 0)}{m_* - \omega}.
\]

Using eqs. (58, 59) one finds

\[
\text{Re} \Pi^+(X, m_*, 0) = -\frac{g^2}{16\pi^2} m_* \int_{2m_*}^\infty \frac{d\omega}{\omega^2 - m_*^2} \sqrt{1 - \frac{4m_*^2}{\omega^2}} \frac{1}{\exp \left( \frac{\beta \omega}{2} \right) - 1}.
\]

= \frac{-g^2}{32\pi^2} I(\beta m_*),

(63)

where

\[
I(a) \equiv \int_1^\infty \frac{dx}{x^2 - 1/4} \frac{\sqrt{1 - 1/x^2}}{e^{ax} - 1}.
\]

Thus, in the case of global equilibrium with \( T \gg m_* \) the equation (60) reads

\[
m^2_* + \frac{g^2}{24} \frac{T^2}{m_*^2} - \frac{g^2}{32\pi^2} I(\beta m_*) = 0.
\]

(64)

One observes that

\[
I(a) < \frac{1}{a} \int_1^\infty \frac{dx}{x^2} = \frac{1}{a}.
\]
Thus, \( I(\beta m_*) < T/m_* \) and consequently the absolute value of the third term in l.h.s of eq. (64) is much smaller than that of the second one in the limit of \( T \gg m_* \). Therefore, there is no real positive \( m_* \) which satisfies eq. (64). It means that the massive quasiparticles do not emerge in the massless \( \phi^3 \) model. It is not surprising since the potential \( V(\phi) = \frac{g}{3!} \phi^3 \) from the lagrangian (1) has no minimum, even a local one. So, we do not consider the \( \phi^3 \) model any more and concentrate on the \( \phi^4 \) model.

Having the mass of quasiparticles we can determine their dispersion relation. Since all self-energies except \( \Pi^\delta(X) \) vanish in the lowest order of the perturbative expansion, the mass-shell equation (25) coincides with that one of the free quasiparticles (32). Thus, we have explicitly shown that in the first order of \( g \), the \( \phi^4 \) model provides the system of free quasiparticles described by the transport equation of the Vlasov form i.e.

\[
[p^\mu \partial_\mu + \frac{1}{2} \partial_\mu m_*^2(X) \partial_\nu]f_0(X, p) = 0 ,
\]

with the effective mass given by eq. (61).

10 Higher order self-energy

We discuss here the \( g^2 \) contributions to the self-energy in the \( \phi^4 \) model, which are represented by graphs shown in Fig. 4. The bubble is again related to the effective mass. The contributions corresponding to the diagrams 4a and 4b can be easily computed. However, they are pure real and the only effect of these contributions is a higher order modification of the effective mass. Thus, we do not explicitly calculate these diagrams but instead we analyse the contribution 4c which provides a qualitatively new effect.

The graph from Fig. 4c gives the contour self-energy as

\[
\Pi^\zeta(x, y) = \frac{g^2}{6} \Delta_0(x, y) \Delta_0(y, x) \Delta_0(x, y) ,
\]

and consequently

\[
\Pi^\zeta(x, y) = \frac{g^2}{6} \Delta_0^\zeta(x, y) \Delta_0^\zeta(y, x) \Delta_0^\zeta(x, y) .
\]

Substituting \( \Delta_0^\zeta \) expressed through the distribution function \( f_0 \) as in eqs. (38) and (39) one finds the self-energy \( \Pi^\zeta \) in the form

\[
\Pi^\zeta(X, p) = \frac{i g^2}{6} \int \frac{d^3k}{(2\pi)^3 E_k} \frac{d^3q}{(2\pi)^3 E_q} \frac{d^3r}{(2\pi)^3 E_r} \frac{(2\pi)^4 \delta(3)}{(p + q - k - r)} \times [\delta(p_0 - E_k - E_q - E_r) f_0^k f_0^q f_0^r] ,
\]

with the effective mass given by eq. (61).
\[
\Pi^\geq(X,p) = \frac{ig^2}{6} \int \frac{d^3k}{(2\pi)^32E_k} \frac{d^3q}{(2\pi)^32E_q} \frac{d^3r}{(2\pi)^32E_r} (2\pi)^4\delta^{(3)}(p + q - k - r) \\
\times \left[ \delta(p_0 + E_k + E_q + E_r) f_0^{-k} f_0^{-q} f_0^{-r} + (f_0^{-k} f_0^{-q} f_0^{-r} + 1) \\
+ \delta(p_0 + E_k + E_q + E_r) f_0^{k} f_0^{q} f_0^{r} + (f_0^{k} f_0^{q} f_0^{r} + 1) \right],
\]

with

\[
f_0^{k} \equiv f_0(X,k), \quad f_0^{-k} \equiv f_0(X,-k).
\]

It is important to notice that in contrast to the similar expressions of the \(\phi^3\) model i.e. eqs. (66) and (57), the self-energies (57) and (66) are nonzero not only for the off-shell but for on-shell momentum \(p\) as well. However, the number of terms from eq. (67) or (66) which contribute to \(\Pi^\geq\) is reduced when \(p^2 = m^2\). Indeed, eqs. (66, 67) simplify in this case as

\[
\Theta(p_0) \Pi^\leq(X,p) = \frac{ig^2}{6} \int \frac{d^3k}{(2\pi)^32E_k} \frac{d^3q}{(2\pi)^32E_q} \frac{d^3r}{(2\pi)^32E_r} (2\pi)^4\delta^{(3)}(p + q - k - r) \\
\times \left[ \delta(p_0 - E_k - E_q + E_r) f_0^{k} f_0^{-q} f_0^{-r} + (f_0^{k} f_0^{-q} f_0^{-r} + 1) \\
+ \delta(p_0 - E_k + E_q - E_r) f_0^{k} f_0^{q} f_0^{r} + (f_0^{k} f_0^{q} f_0^{r} + 1) \right],
\]

\[
\Theta(p_0) \Pi^\geq(X,p) = \frac{ig^2}{6} \int \frac{d^3k}{(2\pi)^32E_k} \frac{d^3q}{(2\pi)^32E_q} \frac{d^3r}{(2\pi)^32E_r} (2\pi)^4\delta^{(3)}(p + q - k - r)
\]
\[ \times \left[ \delta(p_0 - E_k - E_q + E_r) (f_0^k + 1) (f_0^{-q} + 1) f_0^{-r} \\
+ \delta(p_0 - E_k + E_q - E_r) (f_0^k + 1) f_0^{q} (f_0^{r} + 1) \\
+ \delta(p_0 + E_k - E_q - E_r) f_0^{-k} (f_0^{-q} + 1) (f_0^{r} + 1) \right] \]
\[ = \frac{ig^2}{2} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3r}{(2\pi)^3} \frac{\delta^{(4)}(p + q - k - r)}{2E_k 2E_q 2E_r} \]
\[ \times \left( f_0^q (f_0^k + 1) (f_0^{r} + 1) - (f_0^q + 1) f_0^{k} f_0^{r} \right) , \quad (69) \]

where \( p_0 = \sqrt{m^2 + p^2} \).

The self-energies \( \Pi \) provide, through the equations analogous to (22) and (23), the \( \text{Re} \Pi^+ \) and \( \text{Im} \Pi^+ \) which enter the transport (24) and mass-shell (25) equations. The imaginary part of \( \Pi^+ \) is of particular interest. Due to the finite value of \( \text{Im} \Pi^+ \), the spectral function (45) is no longer delta-like but it is of the Breit-Wigner shape. Thus, the quasiparticles are of finite life time. For the on-mass-shell momenta with \( p_0 > 0 \) the imaginary part of \( \Pi^+ \) equals

\[ \text{Im} \Pi^+(X, p) = \frac{1}{2i} \left( \Pi^>(X, p) - \Pi^<(X, p) \right) \]
\[ = \frac{g^2}{4} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3r}{(2\pi)^3} \frac{\delta^{(4)}(p + q - k - r)}{2E_k 2E_q 2E_r} \]
\[ \times \left( f_0^q (f_0^k + 1) (f_0^{r} + 1) - (f_0^q + 1) f_0^{k} f_0^{r} \right) . \quad (70) \]

This function was computed for the equilibrium distribution in [17, 19].

### 11 Interacting quasiparticles

In this chapter we discuss the dispersion relation of the interacting quasiparticles and then define the respective distribution function.

Having the self-energies calculated in \( g^2 \) order we can determine the quasiparticle dispersion relation in this order. As previously, the quasiparticle mass is found as a solution of the equation (50). Therefore, the singular self-energy \( \Pi_\delta(X) \) as well as \( \text{Re} \Pi^+(X, m_*, 0) \) are included in \( m^2_\delta(X) \). To avoid the double counting, the expression

\[ p^2 - m^2_\delta(X) + \Pi_\delta(X) + \text{Re} \Pi^+(X, p) \]

is everywhere replaced by

\[ p^2 - m^2_\delta(X) + \text{Re} \tilde{\Pi}^+(X, p) , \]

with

\[ \text{Re} \tilde{\Pi}^+(X, p) \equiv + \text{Re} \Pi^+(X, p) - \Theta(p_0) \text{Re} \Pi^+(X, m_*, 0) - \Theta(-p_0) \text{Re} \Pi^+(X, -m_*, 0) . \]
The dispersion relation is given by the equation
\[ p^2 - m^2(X) + \text{Re} \Pi^+(X, p) = 0, \]
but according to eq. (45), which determines the spectral function, the quasiparticles are of the finite width and the relation (71) gives only the most probable quasiparticle four-momentum. We call the four-momenta, which satisfy the relation (71), as ‘on-shell’, however one should keep in mind that the meaning of this term differs for the finite and zero width quasiparticles.

Eq. (71) can be easily solved if \( \text{Re} \Pi^+ \) provides only a small correction to the free quasiparticle dispersion relation. Then, one finds the on-shell momentum as
\[ p^\pm = (\pm E^\pm, \mathbf{p}) \]
with
\[ E^\pm = \sqrt{m^2(X) + \mathbf{p}^2 + \text{Re} \Pi^+(X, \pm \sqrt{m^2(X) + \mathbf{p}^2}, \mathbf{p})}. \]

The distribution function \( f(X, p) \) of the interacting quasiparticles is defined in a way analogous to eq. (35) i.e.
\[ \Theta(p_0) i \Delta^<(X, p) = \Theta(p_0) A(X, p) f(X, p) \]
where \( A(X, p) \) is the spectral function (45). In contrast to the case of free quasiparticles, \( f(X, p) \) depends not on the three-vector \( \mathbf{p} \) but on the four-vector \( p \). Due to the identities
\[ \Delta^<(X, p) = \Delta^>(X, -p), \quad A(X, p) = i \Delta^>(X, p) - i \Delta^<(X, p), \]
we have
\[ i \Delta^>(X, p) = \Theta(p_0) A(X, p) (f(X, p) + 1) - \Theta(-p_0) A(X, p) f(X, -p), \]
\[ i \Delta^<(X, p) = \Theta(p_0) A(X, p) f(X, p) - \Theta(-p_0) A(X, p) (f(X, -p) + 1). \]

There is a very important property of \( \Delta^\pm \) expressed in the from (74, 75). Namely, if the Green functions \( \Delta^\pm \) satisfy the transport equation (24) and the spectral function solves the equation (44), the mass-shell equation of \( \Delta^\pm \), i.e. eq. (25), is satisfied automatically in the 0-th order of the gradient expansion. Let us derive this result.

The transport and mass-shell equations (24, 25) with the gradient terms neglected read
\[ 0 = \Pi^>(X, p) \Delta^<(X, p) - \Pi^<(X, p) \Delta^>(X, p), \]
\[ \left[ p^2 - m^2(X) + \text{Re} \Pi^+(X, p) \right] \Delta^<(X, p) = -\Pi^<(X, p) \text{Re} \Delta^+(X, p). \]
Substituting \( \Delta^\pm \) in the form (74, 75) into the first equation and taking only the terms corresponding to \( p_0 > 0 \), the equation is
\[ 0 = A(x, p) \left[ \Pi^>(X, p) f(X, p) - \Pi^<(X, p) (f(X, p) + 1) \right]. \]
Now we substitute $\Delta^<$ given by (73) into the mass-shell equation and get
\[
\left[ p^2 - m^2_\ast(X) + \text{Re} \Pi^+(X, p) \right] A(X, p) f(X, p) = -\Pi^< (X, p) \text{Re} \Delta^+(X, p) ,
\]
where $p_0$ is assumed to be positive. Using the spectral function equation (44), the equation (77) is manipulated to the form
\[
\text{Re} \Delta^+(x, p) \left[ \Pi^< (X, p) f(X, p) - \Pi^< (X, p) \left( f(X, p) + 1 \right) \right] = 0 .
\]
One sees that if $f$ solves eq. (76), it automatically satisfies eq. (78). Similar considerations can be easily repeated for $p_0 < 0$ and then for $\Delta^<$ with $p_0 > 0$ and $p_0 < 0$.

Let us observe that the quasiparticles studied in this paper are narrow. Indeed, the effective mass (61) is of order $g^{1/2}$ while the width of the Breit-Wigner distribution given by eq. (48) is proportional to $g$ in power at least $3/2$. Thus, the width of quasiparticles is much smaller than their mass ($g$ is obviously assumed to be small). Due to this property we will often refer to the case of zero-width quasiparticles or on-mass-shell momenta.

### 12 Transport equation

The distribution function $f$ satisfies the transport equation which can be obtained from eq. (24) for $\Delta^>$ or $\Delta^<$. After using eq. (43) one finds
\[
A(X, p) \left\{ p^2 - m^2_\ast(X) + \text{Re} \Pi^+(X, p), f(X, p) \right\}
\]
\[
= iA(X, p) \left( \Pi^>(X, p) f(X, p) - \Pi^< (X, p) \left( f(X, p) + 1 \right) \right)
\]
\[
+ if(X, p) \left\{ \Pi^>(X, p), \text{Re} \Delta^+(X, p) \right\}
\]
\[
- i(f(X, p) + 1) \left\{ \Pi^< (X, p), \text{Re} \Delta^+(X, p) \right\} ,
\]
where $p_0 > 0$. We have also used here the following property of the Poisson-like brackets:
\[
\{ A, B C \} = \{ A, B \} C + \{ A, C \} B .
\]
Since eq. (79) is one of the main results of this paper we discuss it in detail.

The left-hand-side of eq. (79) is a straightforward generalization of the drift term of the standard relativistic transport equation. Computing the Poisson-like bracket and imposing the mass-shell constraint (71) one finds the familiar structure
\[
\frac{1}{2} \Theta(p_0) \left\{ p^2 - m^2_\ast(X) + \text{Re} \Pi^+(X, p), f(X, p) \right\}
\]
\[
= E_p^+ \left( \frac{\partial}{\partial t} + v \nabla \right) f(X, p) + \nabla V(X) \nabla_p f(X, p)
\]
where
\[
V(X) \equiv m^2_\ast(X) - \text{Re} \Pi^+(X, p) ,
\]
and the velocity \( \mathbf{v} \) equals \( \partial E_p^+ / \partial \mathbf{p} \) with the energy \( E_p^+ \) given by eq. (72).

Let us now analyse the right-hand-side of eq. (79). Since the quasiparticles of interest are narrow, we take into account only those terms contributing to the self-energies \( \Pi^\pm \) which are nonzero for the on-mass-shell momenta. The other terms are negligibly small. Then, \( \Pi^\pm \) from the transport equation (79) are given by the formulas analogous to eqs. (68,69) with \( f \) instead of \( f_0 \). Consequently,

\[
\Theta(p_0) \Pi^- (X, p) = \frac{i g^2}{2} \int \frac{d^4 k A_k^+}{(2\pi)^4} \frac{d^4 q A_q^+}{(2\pi)^4} \frac{d^4 r A_r^+}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p + q - k - r)
\]

\[
\times \left( f^q + 1 \right) f^k f^r ,
\]

(80)

\[
\Theta(p_0) \Pi^+ (X, p) = \frac{i g^2}{2} \int \frac{d^4 k A_k^+}{(2\pi)^4} \frac{d^4 q A_q^+}{(2\pi)^4} \frac{d^4 r A_r^+}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p + q - k - r)
\]

\[
\times \left( f^q (f^k + 1) (f^r + 1) \right) ,
\]

(81)

where

\[
A_k^+ \equiv \Theta(k_0) A(X, k) .
\]

One sees that in the limit of zero-width quasiparticles

\[
\frac{d^4 k A_k^+}{(2\pi)^4} \rightarrow \frac{d^3 k}{(2\pi)^3 2E_k} .
\]

The first term in r.h.s of the transport equation (79) is very similar to the standard collision term of the relativistic transport equation [21]. Indeed,

\[
i \left( \Pi^+ (X, p) f(X, p) - \Pi^- (X, p) (f(X, p) + 1) \right)
\]

(82)

\[
= \frac{g^2}{2} \int \frac{d^4 k A_k^+}{(2\pi)^4} \frac{d^4 q A_q^+}{(2\pi)^4} \frac{d^4 r A_r^+}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p + q - k - r)
\]

\[
\times \left( f^p + 1 \right) (f^q + 1) f^k f^r - f^p f^q (f^k + 1) (f^r + 1) .
\]

The last two terms from r.h.s of eq. (79), analogous to those found long time ago in [2, 3], are absent in the usual transport equation. We are going to show that in the local equilibrium, when the collision term (82) vanish, we reproduce the standard collisionless equation if the four-momentum is on mass-shell.

As well known [21], the standard collision term, which emerges from (82) when the quasiparticle width tends to zero (cf. eq. (46)), vanishes for the local equilibrium distribution function of the form (40). Following [21], one easily shows that the collision term (82) also vanishes for the distribution function (40) with the particle momentum \( p \) no longer constrained by the mass-shell condition.

One observes that in the local equilibrium the collisional self-energies can be written as

\[
\Pi^+ (X, p) = 2i \text{Im} \Pi^+(X, p) (f^\text{eq}(X, p) + 1) ,
\]

(83)
and 
\[ \Pi^<(X,p) = 2i\text{Im}\Pi^+(X,p) f^{eq}(X,p). \]

The transport equation (79) then simplifies to
\[
A(X,p) \left\{ p^2 - m^2_\ast(X) + \text{Re}\Pi^+(X,p), f^{eq}(X,p) \right\} = 2\text{Im}\Pi^+(X,p) \left\{ f^{eq}(X,p), \text{Re}\Delta^+(X,p) \right\}. 
\]

Using eqs. (29) and (45) one manipulates this equation to the form
\[
\text{Im}\Pi^+(X,p) \left\{ p^2 - m^2_\ast(X) + \text{Re}\Pi^+(X,p), f^{eq}(X,p) \right\} = \left\{ p^2 - m^2_\ast(X) + \text{Re}\Pi^+(X,p) \right\} \left\{ \text{Im}\Pi^+(X,p), f^{eq}(X,p) \right\}. 
\]

As seen, the term on the r.h.s drops down for the on-mass-shell momenta and then we reproduce the usual Vlasov equation i.e.
\[
\left\{ p^2 - m^2_\ast(X) + \text{Re}\Pi^+(X,p), f^{eq}(X,p) \right\} = 0. 
\]

The role of the two unusual terms from r.h.s of the transport equation (79) beyond the local equilibrium is rather unclear and needs further studies.

13 Summary and concluding remarks

We have discussed in this paper the nonequilibrium features of the massless fields. The derivation of the kinetic equation in such a case faces serious difficulties because there is no natural length scale over which the system inhomogeneities can be integrated over. As known the transport theory deals with the quantities averaged over an elementary phase-space cell of the minimal size given by the particle Compton wavelength.

The fields with the zero bare mass usually gain an effective mass due to the self-interaction. Therefore, we have introduced the auxiliary mass term in the lagrangian and then, the transport theory has been derived in a way very similar to the earlier studied \cite{4, 5} case of massive fields. However, due to the position dependence of the effective mass, the limit of the noninteracting quasiparticles corresponds to the Vlasov rather than the free particle case. The smallness of the effective mass has also forced us to take into account some extra gradient terms which are usually neglected in the transport equation.

We have considered in detail the $\phi^3$ and $\phi^4$ models which appear to be very different from each other. In the $\phi^4$ model the effective mass is generated in the lowest nontrivial order of the perturbative expansion. In contrast, the massive quasiparticles do not emerge in the $\phi^3$ model and most probably there is no transport limit of this model which, as well known, is anyhow ill defined.
Within the $\phi^4$ model we have derived the transport equation for the finite width quasiparticles. The distribution function has been defined in such a way that the mass-shell constraint is automatically satisfied (in the gradient zeroth order). We have found except the mean-field and collision terms the specific ones which are absent in the standard transport equation. However, in the case of local equilibrium we have been able to reproduce the usual collisionless equation if the four-momentum is on the mass shell.

The massless fields play a crucial role in the gauge theories such as Quantum Chromodynamics. We believe that the methods developed in this study will be useful in the discussion of the transport theory of quarks and gluons. Till know only the mean-field limit of such a theory is well understood [22, 23]. In spite of some efforts [24, 25], the systematic derivation of the collision terms is still missing. The problem of the off-mass-shell propagation, which on phenomenological level has been proven to play a very important role [26], is also unsettled.

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Appendix

We discuss here the quasiparticle approximation for the system of noninteracting fields. To simplify the discussion the bare mass $m$ is assumed to be nonzero or equivalently $m_*$ is treated as a constant. The transport equation and the mass-shell constraint read

$$p \cdot \partial \Delta_0^>(X, p) = 0 ,$$  \hspace{1cm} (85)

$$\left[ \frac{1}{4} \partial^2 - p^2 + m^2 \right] \Delta_0^>(X, p) = 0 .$$  \hspace{1cm} (86)

These equations, which directly follow from the field equation of motion (3) with $m = m_*$, are exact in the case of the massive free fields - the gradient expansion is not needed to derive them.

The mass-shell constraint (86) shows that the function $\Delta_0^>(X, p)$ is indeed nonzero for the off-shell momenta i.e. $\Delta_0^>(X, p) \neq 0$ for $p^2 \neq m^2$. This result looks surprising if one keeps in mind that the field, which solves the equation of motion (2), is in a sense on mass-shell. The field is the sum of the plane waves

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega}} \left( e^{-ikx} a(k) + e^{ikx} a^*(k) \right) ,$$  \hspace{1cm} (87)
where \( k \equiv (\omega, \mathbf{k}) \) with \( \omega \equiv \sqrt{k^2 + m^2} \). Thus, \( k^2 = m^2 \). Substituting the field (87) into the \( \Delta_{\psi,0}(X, p) \) definition, one finds that the off-shell contribution to \( \Delta_{\psi,0}(X, p) \) comes from the interference of the positive and negative energy parts present in eq. (87).

Let us consider when such a contribution can be neglected.

One easily shows that the transport equation (85) is solved by the function which depends on the four-position \( X = (t, \mathbf{x}) \) only through \( \mathbf{x} - vt \), i.e.

\[
\Delta_{\psi,0}(X, p) = F(\mathbf{x} - vt, p) .
\]

The quasiparticle condition (18) applied to the function \( F \) reads

\[
|F(\mathbf{x} - vt, p)| \gg \frac{1}{m^2} \left| (v_i v_j - \delta_{ij}) \frac{\partial^2 F(\mathbf{x} - vt, p)}{\partial(\mathbf{x} - vt)_i \partial(\mathbf{x} - vt)_j} \right| .
\]

If this condition is satisfied for every \( \mathbf{x} \) at a given moment of time, say \( t_0 \), it is satisfied at any time. In other words, if the initial condition at \( t_0 \) is sufficiently homogeneous that the quasiparticle approximation can be applied, then this approximation is applicable at any time - the system remains homogeneous.

The question arises whether \( \Delta_{\psi,0} \), which simultaneously solves the transport (85) and mass-shell (86) equations, can satisfy the quasiparticle condition. We introduce the Fourier transformed function \( \tilde{\Delta}_{\psi,0}(Q, p) \) defined as

\[
\tilde{\Delta}_{\psi,0}(Q, p) \overset{\text{def}}{=} \int d^4 X \, e^{iQ \cdot X} \, \Delta_{\psi,0}(X, p) .
\]

The equations corresponding to (85) and (86), respectively, read

\[
p \cdot Q \, \tilde{\Delta}_{\psi,0}(Q, p) = 0 ,
\]

\[
[-\frac{1}{4} Q^2 - p^2 + m^2] \, \tilde{\Delta}_{\psi,0}(Q, p) = 0 .
\]

They are both solved by

\[
i \tilde{\Delta}_{\psi,0}(Q, p) = \delta(p \cdot Q) \, \delta(-\frac{1}{4} Q^2 - p^2 + m^2) \, A(Q, p) ,
\]

with \( A(Q, p) \) controlled by the initial condition. Since \( i \Delta_{\psi,0}(X, p) \) is real, \( A(Q, p) \) has the property

\[
A(Q, p) = A^*(-Q, p) .
\]

The solution of the equations (85) and (86) satisfies the quasiparticle condition (18) when

\[
|A(Q, p)| \gg \left| \frac{Q^2}{m^2} A(Q, p) \right| ,
\]

or equivalently \( A(Q, p) \neq 0 \) only for \( Q^2 \ll m^2 \).
It is instructive to consider the explicit solution of eqs. (85) and (86) in 1 + 1 dimensions. Using (89) we get

\[ i \Delta_0^{\tilde Q}(X, p) = \int \frac{d^2Q}{(2\pi)^2} e^{-iQ \cdot X} \delta(p \cdot Q) \delta(-\frac{1}{4}q^2 - p^2 + m^2) A(Q, p), \]

\[ = \left[ \Theta(-p^2) + \Theta(p^2 - m^2) \right] \frac{1}{(2\pi)^2|p^2|} \sqrt{\frac{p^2}{p^2 - m^2}} \times \left( e^{-i\tilde Q \cdot X} A(\tilde Q, p) + e^{i\tilde Q \cdot X} A(-\tilde Q, p) \right), \]

(92)

where \( \tilde Q \) denotes the two-vector

\[ \tilde Q \equiv 2|p_0|\sqrt{\frac{p^2 - m^2}{p^2}} (\frac{p_1}{p_0}, 1). \]

Keeping in mind the property (90) the solution (92) can be rewritten as

\[ i \Delta_0^{\tilde Q}(X, p) = \left[ \Theta(-p^2) + \Theta(p^2 - m^2) \right] \left( f(p) \sin(\tilde Q \cdot X) + g(p) \cos(\tilde Q \cdot X) \right), \]

(93)

where \( f(p) \) and \( g(p) \) are the real functions of \( p \) determined by the initial condition.

The quasiparticle condition (18) is satisfied by (93) if

\[ |\tilde Q^2| = 4|p^2 - m^2| \ll m^2. \]

One also sees that \( \Delta_0^{\tilde Q}(X, p) \sim \delta(p^2 - m^2) \) only for \( \tilde Q = 0 \). In other words, the function \( \Delta_0^{\tilde Q}(X, p) \) is strictly zero for the off-mass-shell momenta when the system is exactly homogeneous. If we are interested in the weakly nonhomogeneous systems, the functions are nonzero for \( p^2 > m^2 \). Equivalently, if \( p^2 \approx m^2 \) then \( p^2 > m^2 \) but not \( p^2 < m^2 \).

The properties of the function \( \Delta_0^{\tilde Q} \) in 1 + 1 dimension can be trivially generalized to 3 + 1 case showing the limitations of the quasiparticle approximation.
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Figure Captions

**Fig. 1.** The contour along the time axis for an evaluation of the operator expectation values.

**Fig. 2.** The lowest-order diagrams of the self-energy in the $\phi^4$ model. The bubble in a) denotes the additional interaction due to the effective mass.

**Fig. 3.** The lowest-order diagrams of the self-energy in the $\phi^3$ model. The bubble in a) denotes the additional interaction due to the effective mass.

**Fig. 4.** The second-order diagrams of the self-energy in the $\phi^4$ model.