Reduction of fourth order ordinary differential equations to second and third order Lie linearizable forms

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Abstract.
Meleshko presented a new method for reducing third order autonomous ordinary differential equations (ODEs) to Lie linearizable second order ODEs. We extended his work by reducing fourth order autonomous ODEs to second and third order linearizable ODEs and then applying the Ibragimov and Meleshko linearization test for the obtained ODEs. The application of the algorithm to several ODEs is also presented.

1 Introduction

First order ODEs can always be linearized [1] by point transformations [2]. Lie [3] showed that all linearizable second order ODEs must be cubically semi-linear, i.e.

\[ y'' + a_1(x,y)y'^3 - a_2(x,y)y'^2 + a_3(x,y)y' - a_4(x,y) = 0 \],

(1)

the coefficients \(a_1, a_2, a_3, a_4\) satisfy an over-determined integrable system of four constraints involving two auxiliary functions, which Tresse wrote in more usable form [4]

\[
3(a_1a_3) - 3a_4a_{1y} - 6a_1a_{4y} - 2a_2a_{2x} + a_2a_{3y} - 3a_{1xx} + 2a_{2xy} - a_{3yy} = 0 ,
3(a_4a_2) - 3a_1a_{4x} - 6a_4a_{1x} - 2a_3a_{3y} + a_3a_{2x} + 3a_{4yy} - 2a_{3xy} + a_{2xx} = 0 .
\]

(2)

We call such equations Lie linearizable.

Chern [5, 6] and Grebot [7, 8] extended the linearization programme to the third order using contact and point transformations respectively to obtain linearizability criteria for equations reducible to the forms \(u'''(t) = 0\) and \(u'''(t) + u(t) = 0\). It was shown [9] that there are three classes of third order ODEs that are linearizable by point transformations, viz. those that reduce to the above two forms or \(u'''(t) + \alpha(t)u(t) = 0\). Neut and Petitot [10] dealt with the general third order...
ODEs. Ibragimov and Meleshko (IM) [11] used the original Lie procedure [3] of point transformation to determine the linearizability criteria for third order ODEs. They showed that any third order ODE \( y''' = f(x, y, y', y'') \) obtained from a linear equation \( u''' + \alpha(t)u = 0 \) by means of point transformations \( t = \varphi(x, y), u = \psi(x, y) \), must belong to one of the following two types of equations.

**Type I:** If \( \varphi_y = 0 \) the equations that are linearizable are of the form

\[
y''' + (a_1y' + a_0)y'' + b_3y'^3 + b_2y'^2 + b_1y' + b_0 = 0 .
\]

(3)

**Type II:** If \( \varphi_y \neq 0 \), set \( r(x, y) = \frac{\varphi_x}{\varphi_y} \), equations are of the form

\[
y''' + \frac{1}{y' + r}[-3(y'')^2 + (c_2y'^2 + c_1y' + c_0)y''
\]

\[
+d_3y'^3 + d_4y'^4 + d_3y'^3 + d_2y'^2 + d_1y' + d_0] = 0 ,
\]

(4)

where all coefficients \( a_i, b_i, c_i, d_i \), being the functions of \( x \) and \( y \), satisfy certain constraint requirements. Afterwards Ibragimov, Meleshko and Suksern [12, 13] used the point and contact transformations to determine the criteria for the linearizability of fourth order scalar ODEs. Meleshko [14] provided a simple algorithm to reduce third order ODEs of the form \( y''' = f(y, y', y'') \) to second order ODEs. If the reduced equations satisfy the Lie linearizability criteria, they can then be solved by linearization. Meleshko showed that a third order ODE is reducible to the second order linearizable ODE if it is of the form

\[
y''' + A(y, y')y'^3 + B(y, y')y'^2 + c(y, y')y'' + D(y, y') ,
\]

(5)

where the coefficients \( A, B, C, D \) satisfy certain constraints.

In the present paper we extend Meleshko’s procedure to the fourth order ODEs in the cases that the equations do not depend explicitly on the independent or the dependent variable (or both) to reduce it to third (respectively second) order equations. Once the order is reduced we can apply the IM (or Lie) linearization test. If the reduced third (or second) order ODE satisfies the IM (or Lie) linearization test, then after finding a linearizing transformation, the general solution of the original equation is obtained by quadrature. So this method is effective in the sense that it reduces many ODEs, that cannot be linearized, to lower order linearizable forms. This is one of the motivations for studying this method. Another hope for the study of the linearization problem is that by using it we may be able to provide a complete classification of ODEs according to the number of arbitrary initial conditions that can be satisfied [17].
2 Equations reducible to linearizable forms

Meleshko had only treated the special case of independence of $x$ for third order ODE. We include independence of $y$ for completeness before proceeding to the fourth order.

Third order ODEs independent of $y$

Taking $y'$ as the independent variable $u(x)$, we convert the ODE

$$y''' = f(x, y', y'') ,$$

(6)

to the second order ODE

$$u'' = f(x, u, u') ,$$

(7)

which is linearizable by Lie’s criteria if it is cubically semi-linear with the coefficients satisfying conditions (2).

Hence (7) is reducible to second order linearizable form if and only if

$$f(x, y', y'') = -c(x, y')y''^3 + g(x, y')y''^2 - h(x, y')y'' + d(x, y') ,$$

(8)

with the coefficients satisfying

$$3(ch)_x - 3dcy' - 6cd_y' - 2ggx + gh_y' - 3c_{xx} + 2g_{xy} - h_{yy'} = 0 ,$$

$$3(dg)_y - 3cd_x - 6dc_x - 2h_{xy} + hgx + 3d_{yy'} - 2h_{xy'} + g_{xx} = 0 .$$

(9)

Fourth order ODEs independent of $y$

Since the variable $y$ is missing, by taking $y'$ as the new dependent variable $u(x)$, the ODE

$$y^{(4)} = f(x, y', y'', y''') ,$$

(10)

is reduced to third order ODE

$$u''' = f(x, u, u', u'') .$$

(11)

Equation (11) is linearizable for the type I of Ibragimov and Meleshko’s criteria if and only if

$$f(x, y', y'', y''') = -(a_1y'' + a_0)y''^3 - b_3y''^3 - b_2y''^2 - b_1y'' - b_0 ,$$

(12)

with the coefficients $a_i = a_i(x, y') , (i = 0, 1)$ and $b_j = b_j(x, y') , (j = 0, 1, 2, 3)$,
satisfying the conditions
\begin{align*}
a_{0y'} - a_{1x} &= 0 , \\
(3b_1 - a_0^2 - 3a_0x)y' &= 0 , \\
3a_{1x} + a_0a_1 - 3b_2 &= 0 , \\
3a_{1y'} + a_1^2 - 9b_3 &= 0 , \\
(9b_1 - 6a_0x - 2a_0^2)a_{1x} + 9(b_{1x} - a_1b_0)y' + 3b_{1y'}a_0 - 27b_{0y'y'} &= 0 .
\end{align*}

Also the necessary and sufficient conditions for (11) to be linearizable for the type II of Ibragimov and Meleshko’s criteria are
\begin{equation}
f(x, y', y'', y''') = \frac{-1}{y'' + r}[-3(y''')^2 + (c_2y''^2 + c_1y'' + c_0)y'''
+ d_5y''^5 + d_4y''^4 + d_3y''^3 + d_2y''^2 + d_1y'' + d_0] ,
\end{equation}
and the coefficients \(c_i = c_i(x, y')\), \((i = 0, 1, 2)\), \(d_j = d_j(x, y')\), \((j = 0, 1, 2, 3, 4, 5)\) and \(r = r(x, y')\) have to satisfy constraint equations which can be produced simply by replacing \(y\) by \(y'\) for the type II constraint equations in [11].

**Fourth order ODEs independent of \(x\)**

The transformation \(y' = u(y)\) will transform autonomous ODE of the fourth order
\begin{equation}y^{(4)} = f(y, y', y'', y''') ,\end{equation}
into the equation
\begin{equation}u^3u'' + 4u^2u'u'' + uu'^3 - f(y, u, uu', u^2u'' + uu'^2) = 0 ,\end{equation}
which is a third order ODE in \((y, u)\). It is linearizable by Ibragimov Meleshko’s criteria if it is of the form (3) i.e,
\begin{equation}
f(y, u, uu', u''u^2 + uu'^2) = -u^3[(a_1u' + a_0)u'' + b_3u'^3 + b_2u'^2 + b_1u' + b_0]
+ 4u^2u'u'' + uu'^3 ,
\end{equation}
where \(a_i = a_i(y, u)\), \((i = 0, 1)\) and \(b_j = b_j(y, u)\), \((j = 0, 1, 2, 3)\). With this (16) takes the form
\begin{equation}
u'' + (a_1u' + a_0)u'' + b_3u'^3 + b_2u'^2 + b_1u' + b_0 = 0 .
\end{equation}
Transforming (18) into a fourth order ODE with \(x\) as independent variable and \(y\) as dependent variable:
\begin{equation}y^{(4)} + (A_1y'' + A_0)y''' + B_3y'^3 + B_2y'^2 + B_1y'' + B_0 = 0 ,\end{equation}
where
\[ A_i = A_i(y, y') \; (i = 0, 1) \; ; \; B_j = B_j(y, y') \; (j = 0, 1, 2, 3) \] (20)
subject to the identification of coefficients
\[
\begin{align*}
    a_1 &= A_1 + \frac{4}{y'} , \quad a_0 = \frac{A_0}{y'} , \quad b_3 = B_3 + \frac{A_1}{y'} + \frac{1}{y'^2} , \\
    b_2 &= \frac{B_2}{y'} + \frac{A_0}{y'^2} , \quad b_1 = \frac{B_1}{y'^2} , \quad b_0 = \frac{B_0}{y'^3} ,
\end{align*}
\] (21)
with the constraints
\[
\begin{align*}
    y'^2 A_{1y} - y' A_{0y'} + A_0 &= 0 , \\
    y'^2 (-3 A_{0y''}) + y' (3 B_{1y'} + 3 A_{0y} - 2 A_0 A_{0y'}) + (-6 B_1 + 2 A_0^2) &= 0 , \\
    y'^2 (3 A_{1y}) + y' (A_0 A_1 - 3 B_2) + A_0 &= 0 , \\
    y'^2 (3 A_{1y'} - 9 B_3 + A_1^2) - y' A_1 - 5 &= 0 , \\
    y'^4 (-6 A_{0y} A_{1y}) + y'^3 (9 B_1 A_{1y} - 2 A_0^2 A_{1y} + 9 B_{1y'} y') + y'^2 (-18 B_{1y} - 9 A_1 B_{0y'}) \\
    -9 B_{0y} A_{1y'} + 3 A_0 B_{1y'} - 27 B_{0y'} y') + y' (27 A_1 B_0 - 6 A_0 B_1 + 126 B_{0y'}) \\
    -180 B_0 &= 0 .
\end{align*}
\] (22)

Also in order to make (16) linearizable of type II of Ibragimov and Meleshko’s criteria we have to take
\[
f(y, u, uu', u^2 u'' + uu'^2) &= -\frac{u^3}{u' + r} [-3(u'')^2 + (c_2 u'^2 + c_1 u' + c_0) u''] \\
+&d_5 u'^5 + d_4 u'^4 + d_3 u'^3 + d_2 u'^2 + d_1 u' + d_0] + 4 u'^2 u'' u' + uu'^3 ,
\] (23)
where \( c_i = c_i(y, u) \; (i = 0, 1, 2) \; , \; d_j = d_j(y, u) \; (j = 0, 1, 2, 3, 4, 5) \; \) and \( r = r(y, u) \).

Considering the form (23) and converting (16) into fourth order with \( x \) as independent and \( y \) as dependent variable, we have
\[
\begin{align*}
    y^{(4)} + \frac{1}{y'' + r_0} [-3(y''')^2 + (C_2 y'''' + C_1 y''' + C_0) y'''] \\
+&D_5 y'^5 + D_4 y'^4 + D_3 y'^3 + D_2 y'^2 + D_1 y' + D_0] &= 0 ,
\end{align*}
\] (24)
where
\[
\begin{align*}
    C_i &= C_i(y, y') \; (i = 0, 1, 2) \; , \; D_j &= D_j(y, y') \; (j = 0, 1, 2, 3, 4, 5) \; , \; r_0 &= r_0(y, y') ,
\end{align*}
\]
subject to the identification of coefficients
\[
\begin{align*}
    c_2 &= C_2 - \frac{2}{y'} , \quad c_1 = C_1 + \frac{4 r_0}{y'} , \quad c_0 = \frac{C_0}{y'^2} , \quad d_5 = \frac{D_5}{y'^5} , \\
    d_4 &= D_4 + \frac{C_2}{y'} - \frac{2}{y'^2} , \quad d_3 = \frac{D_3}{y'} + \frac{C_1}{y'} + \frac{4 r_0}{y'^2} - \frac{3 r_0}{y'^3} , \\
    d_2 &= \frac{D_2}{y'^2} + \frac{C_0}{y'^3} , \quad d_1 = \frac{D_1}{y'^3} , \quad d_0 = \frac{D_0}{y'^4} , \quad r = \frac{r_0}{y'} ,
\end{align*}
\] (25)
with the constraints (43)–(51) (presented in the appendix).

**Fourth order ODEs independent of** $x$ **and** $y$

By considering $y'$ as independent and $y''$ as dependent variable, we convert the equation

$$y^{(4)} = f(y', y'', y''')$$

into a second order ODE:

$$u^2u'' + uu'^2 = f(y', u, uu').$$

(27)

For (27) to be Lie-linearizable we must have

$$f(y', u, uu') = -u^2[A(y', u)u'^3 + B(y', u)u'^2 + C(y', u)u' + D(y', u)] + uu'^2.$$  

(28)

Hence (26) takes the form

$$y^{(4)} + a(y', y'')y''' + b(y', y'')y'' + c(y', y'')y' + d(y', y'') = 0,$$

(29)

where $a, b, c$ and $d$ must satisfy the constraints:

$$
\begin{align*}
&\quad (3a_{y'y'})y'^4 + (2bb'y' - 3ca_{y'} - 3ac_{y'} - 2b'y'y'')y'^3 + (2b'y' - bc'y')y'^2 \\
&+3a_{y'y''} + 6ad_{y''} - c_{y'y''})y'^2 + (bc - 9ad - 3c_{y'})y'' - c = 0, \\
&\quad (by'y')y'^4 + (by'c + 3dy'y'b - 3dy'y'a - 6a_{y'd} - 2c_{y'y'})y'^3 + (c_{y'} + 3dy') \\
&- 6bd + 3b_{y'y''}d - 2cc_{y'} + 3d_{y'y''})y'^2 + (2c^2 - 6d - 12d_{y'})y'' + 15d = 0.
\end{align*}

(30)

Thus we have the following theorems.

**Theorem 1.** Equation (19) is reduced to the third order linearizable form if and only if it obeys (22).

**Theorem 2.** Equation (24) is reduced to the third order linearizable form if and only if it obeys (43)–(51) (presented in the appendix II).

**Theorem 3.** Equation (29) is reduced to the second order linearizable form if and only if it obeys (30).

**Remark:** If we have a fourth order ODE of the form

$$y^{(4)} = -f(x, y)y'^5 + 10\frac{y''y^m}{y'} - 15\frac{y'^3}{y'^2},$$

(31)

with $f(x, y)$ linear in $x$, then we can convert it to a linear ODE $x^{(4)} = f(x, y)$ by simply taking $x$ as dependent and $y$ as independent variables.
3 Illustrative Examples

Example 1. The nonlinear fourth order ODE
\[ y^{(4)} - y'' y''' - 3y^2 y''' + 2y^3 y'' + 3y^5 = 0 , \] (32)
cannot be linearized by point or contact transformation. It has the form (19) with the coefficients \( A_1 = -1/y' \), \( A_0 = -3y' \), \( B_3 = B_2 = 0 \), \( B_1 = 2y^2 \), \( B_0 = 3y^5 \). One can verify that these coefficients satisfy the conditions (22). The transformation \( y' = u(y) \) will reduce this ODE to the third order linearizable ODE
\[ u''' + \frac{3}{2} u'' + 3u'' - \frac{3}{2} u^2 + 2u' + 3u = 0 . \] (33)

By using transformation equations in [11], we arrive at the transformation \( t = e^y \), \( s = u^2 \) which maps (33) to the linear third order ODE \( s''' - \frac{2}{3} s = 0 \), whose solution is given by \( s = c_1 t^{-1} + s^2 \{c_2 \cos(\sqrt{2} \ln t) + c_3 \sin(\sqrt{2} \ln t)\} \), where \( c_i \) are arbitrary constants. By using the above transformation we get the solution of (33) given by
\[ u = \pm \sqrt{c_1 e^{-y} + c_2 y \{c_2 \cos(\sqrt{2} y) + c_3 \sin(\sqrt{2} y)\}} \]. Hence the general solution of (32) is obtained by taking the quadrature
\[ \int \frac{dy}{\sqrt{c_1 e^{-y} + c_2 y \{c_2 \cos(\sqrt{2} y) + c_3 \sin(\sqrt{2} y)\}}} = \pm x + c_4 , \] (34)
where \( c_i \) are arbitrary constants.

Example 2. The nonlinear ODE
\[ y^2 y'^2 y^{(4)} - 10y^2 y'' y''' - 3y y^3 y''' + 15y^2 y''^2 + 9y y^2 y'^2 + 3y^4 y'' = 0 , \] (35)
is of the form (19) with the coefficients \( A_1 = -\frac{10}{y'} \), \( A_0 = -\frac{3y'}{y} \), \( B_3 = \frac{15}{y^2} \), \( B_2 = \frac{9}{y} \), \( B_1 = \frac{3y^2}{y^3} \), \( B_0 = 0 \) satisfying the conditions (22). So it is reduced to the third order linearizable ODE
\[ y^2 u^2 u''' - 3y u^2 u'' - 6y^2 u u'' + 3u^2 u' + 6y u u'^2 + 6y^2 u'^3 = 0 , \] (36)
with \( y \) as independent and \( u \) as dependent variable. The transformation \( t = y^2 \), \( s = \frac{1}{u} \), reduces (36) to the linear third order ODE \( s''' = 0 \), whose solution is \( s = c_1 t^2 + c_2 t + c_3 \). Now one only needs to solve the equation \( y' = 1/(c_1 y^4 + c_2 y^2 + c_3) \), where \( c_i \) are arbitrary constants. Hence, the general solution of (35) is given by
\[ x = c_1 y^5 + c_2 y^3 + c_3 y + c_4 \).

Example 3. The ODE
\[ y' y^{(4)} - 3y y''' + 6y^2 y'^2 y''' - 4y y'' y''' - y'y^5 = 0 , \] (37)
has 2 symmetries. It is of the form (24) with the coefficients \( r_0 = 0, C_2 = 6y'^2 - \frac{A}{y}, \) \( C_1 = C_0 = 0, D_5 = -1, D_4 = D_3 = D_2 = D_1 = D_0 = 0, \) obey the conditions (43)–(51). So it is reducible to linearizable third order ODE
\[
    u''' + \frac{1}{u'}[-3u''^2 - yu'^2] = 0 .
\]
The transformation \( t = u, s = y, \) will convert the nonlinear ODE (38) to the linear ODE \( s''' + s = 0 \) with solution
\[
    s = c_1 e^{-t} + c_2 e^{\frac{t}{2}} \cos t + c_3 e^{\frac{t}{2}} \sin t .
\]
Finally to find the solution of (37), we only need to solve
\[
    y = c_1 e^{-y'} + c_2 e^{\frac{y'}{2}} \cos y' + c_3 e^{\frac{y'}{2}} \sin y' .
\]

**Example 4.** The nonlinear ODE
\[
y'' y^{(4)} + y''' - y'' y''' ,
\]
is of the form (29) and the coefficients \( a = \frac{1}{y''}, b = -\frac{1}{y''}, c = -y''', d = 0, \) that satisfy conditions (30). So it is reduced to the linearizable second order ODE \( u'' + u'^3 - u' = 0. \) By using the transformation \( t = u, s = e^y, \) we can reduce it to linear ODE \( s'' - s = 0, \) whose solution is given by \( s = c_1 e^t + c_2 e^{-t}, \) where \( c_i \) are arbitrary constants. So that solution of (41) is obtained by solving the second order ODE
\[
e^{y'} = c_1 e^{-y'} + c_2 e^{y''} ,
\]
where \( c_i \) are arbitrary constants.

## 4 Concluding Remarks

Nonlinear ODEs are difficult to solve but, if they can be converted to linear ones by invertible transformations, they can be solved. Hence linearization plays a significant role in the theory of ODEs. In this paper we have presented criteria for fourth order autonomous ODEs to be reducible to linearizable third and second order ODEs. There are certain fourth order ODEs, not depending explicitly on the independent variable, which cannot be linearized by point or contact transformations but can be reducible to linearizable third order ODEs by Meleshko’s method. The solution of the original equation is then obtained by a quadrature. Various fourth order ODEs with fewer symmetries can be reduced to linearizable form by this procedure. The class of ODEs linearizable by this method is not included in the Ibragimov and Meleshko classes or conditionally linearizable classes [15, 16] of
the ODEs (though there can be an overlap but it is not contained in that either). The reason is that it is not linearizable but reducible to linearizable form. In Lie’s programme there is no definite statement available for the cases when the ODEs are not linearizable. By the recent developments this gap may be filled. By using the concept of Meleshko linearization a new class of scalar ODEs may be defined on the basis of initial conditions to be satisfied by ODEs.

Appendix

\[(r_0 C_1 - 6r_0 y)y'^2 + (6r_0 r_0 y' + 4r_0^2 - r_0^2 C_2 - C_0)y' - 4r_0^2 = 0, \quad (43)\]

\[(C_2 y - C_1 y')y'^3 + (r_0 C_2 y + C_2 r_0 y - 4r_0 y' - 6r_0 y')y'^2 + (10 r_0 y' + 4r_0 - C_2 r_0) y' - 8r_0 = 0, \quad (44)\]

\[(-6r_0^2 C_1 y - 54(r_0 y')^2 + 18r_0 y_{0y} + 18r_0 r_0 y C_1 - 2r_0^2 C_1^2) y'^8 + (3r_0^3 C_1 y' + 48r_0^2 y_0 y - 3r_0^2 C_2 y - 36r_0^2 y_{0y} - 6r_0^2 y_0 y C_2 - 18r_0^2 y_{0y} C_1 + 2r_0^2 C_1 C_2 - 16r_0^2 C_1) y'^7 + (-60r_0^3 y_{0y} + 9r_0^4 C_2 y' - 42r_0^2 y_{0y} - 36r_0^2 y_{0y} C_2 + 14r_0^3 C_1 - 32r_0^4 + 8r_0^2 C_2 + 4r_0^4 C_2^2 + 18r_0^4 D_4) y'^6 + (44r_0^5 + 72r_0^2 y_{0y} - 18r_0^2 y_{0y} - 7r_0^3 C_2) y'^5 + (-20r_0^4) y'^4 - 72r_0^5 D_5 = 0, \quad (45)\]

\[(-12r_0 C_1 y + 18 r_0 y_{0y} + 18r_0 y C_1 - 4r_0 C_1^2) y'^8 + (9r_0^2 C_1 y' - 48r_0 y_0 y - 27r_0^2 C_2 y - 36r_0 y_{0y} - 18r_0 y + 72r_0 y C_2 - 24r_0 y_{0y} C_2 - 18r_0 y_{0y} C_1 - 18r_0 r_0 y_{0y} - 32r_0^2 C_1 - 2r_0^2 C_2 y - 36r_0^2 y_{0y} + 33r_0^3 C_2 y' + 6r_0 y_{0y} + 18r_0^2 C_1 - 21r_0^2 y_{0y} C_2 + 18r_0^2 y_{0y} C_2 - 64r_0^3 + 4r_0^2 C_1 - 8r_0^3 C_2 + 20r_0^3 C_2^2 + 2r_0^3 D_4) y'^6 + (52r_0^3 + 6r_0^2 y_{0y} + 13r_0^2 C_2) y'^5 + (-22r_0^3) y'^4 - 27r_0^4 D_5 = 0, \quad (46)\]

\[(-3C_1 y - C_1^2) y'^8 + (3r_0 C_1 y' - 12r_0 y - 21r_0 C_2 y - 8r_0 C_1 + 15r_0 y C_2 - 5r_0 C_1 C_2) y'^7 + (-9d_2 + 12r_0 r_0 y' + 21r_0^2 C_2 y - 30r_0 y - 15r_0 r_0 y' C_2 + 10r_0 C_1 - 20r_0^2 C_2 + 14r_0^2 C_2^2 + 54r_0^2 D_4 - 16r_0^2) y'^6 + (-9C_0 + 28r_0^2 + 30r_0 r_0 y' + 13r_0^2 C_2) y'^5 + (-40r_0^2) y'^4 - 180r_0^3 D_5 = 0, \quad (47)\]

\[(-3C_2 y - C_1 C_2) y'^7 + (-3D_3 + 4C_1 + 3r_0 C_2 y - 4r_0 C_2 - 2r_0 C_2 + 12r_0 D_4) y'^6 + (-4r_0 + 4r_0 C_2) y'^5 + (-r_0) y'^4 - 30r_0^2 D_5 = 0, \quad (48)\]
\[-188y + 18C_{1y}y' + 3C_{2y}C_{1y} - 72C_{2yy} - 39C_{2y}C_{2y} \] 
\[+ 72r_{0y}y' + 12C_{2}r_{0y} - 6C_{1y} + 36r_{0}C_{2y}y' - 3r_{0}C_{2}C_{2y} + 72r_{0y}C_{2y} \] 
\[+ 33C_{2y}r_{0y} + 108D_{4}r_{0y} + 54r_{0}D_{4}y + 36r_{0}C_{2}^{2} + 18r_{0}C_{2y}y' \] 
\[+ (-168r_{0y} - 12r_{0}C_{2} - 138r_{0}C_{2y} - 24C_{2}r_{0y} - 33r_{0}C_{2}^{2} - 36r_{0}D_{4})y^{5} \] 
\[+ (168r_{0} - 228r_{0}C_{2} + 60r_{0y})y^{5} + (-120r_{0})y^{4} + (270D_{5}r_{0y} \] 
\[+ 270r_{0}D_{5y})y^{2} + (54r_{0}^{2}D_{5y} - 810r_{0}r_{0y}D_{5})y' + 2160r_{0}^{2}D_{5} = 0, \] (49)

and

\[-H_{y}y^{2} + (3Hr_{0y} + r_{0}H_{y})y' - 3Hr_{0} = 0, \] (50)

where

\[
H = (D_{4y} + \frac{1}{3}C_{2y}y' + \frac{2}{3}C_{2}C_{2y} + \frac{2}{3}C_{2}D_{4} + \frac{4}{27}C_{2}^{3}) \]
\[+ \frac{1}{y}(-\frac{4}{3}C_{2y} + \frac{2}{3}C_{2}^{2} - \frac{4}{3}D_{4} - \frac{8}{9}C_{2}) \]
\[+ \frac{1}{y^{2}}(-\frac{5}{9}C_{2} + \frac{1}{y^{3}}(\frac{40}{27}) + \frac{1}{y^{6}}(-2D_{5y} - \frac{2}{3}C_{2}D_{5}) \]
\[+ \frac{1}{y^{6}}(-3r_{0}D_{5y} - 5D_{5}r_{0y} - 2r_{0}C_{2}D_{5} - \frac{8}{3}r_{0}D_{5}) \]
\[+ \frac{1}{y^{7}}(24r_{0}D_{5}). \] (51)

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