Integrating existing cone-shaped and projection-based cardinal direction relations and a $TCSP^\star$-like decidable generalisation

Amar Isli
Fachbereich Informatik, Universität Hamburg,
Vogt-Kölln-Strasse 30, D-22527 Hamburg, Germany
isli@informatik.uni-hamburg.de

Abstract. Integrating different knowledge representation languages is clearly an important topic. This allows, for instance, for a unified representation of knowledge coming from different sources, each source using one of the integrated languages for its knowledge representation. This is of special importance for QSR languages, for such a language makes only a finite number of distinctions: integrating QSR languages may be looked at as an answer to the well-known poverty conjecture. With these considerations in mind, we consider the integration of Frank’s cone-shaped and projection-based calculi of cardinal direction relations, well-known in QSR. The more general, integrating language we consider is based on convex constraints of the qualitative form $r(x, y)$, with $r$ being a cone-shaped or projection-based cardinal direction atomic relation, or of the quantitative form $(\alpha, \beta)(x, y)$, with $\alpha, \beta \in [0, 2\pi]$ and $(\beta - \alpha) \in [0, \pi]$: the meaning of the quantitative constraint, in particular, is that point $x$ belongs to the (convex) cone-shaped area rooted at $y$, and bounded by angles $\alpha$ and $\beta$. The general form of a constraint is a disjunction of the form 

$$r_1 \lor \cdots \lor r_n, (\alpha_1, \beta_1) \lor \cdots \lor (\alpha_n, \beta_n)(x, y),$$

with $r_i(x, y)$, $i = 1\ldots n_1$, and $(\alpha_i, \beta_i)(x, y)$, $i = 1\ldots n_2$, being convex constraints as described above; the meaning of such a general constraint is that, for some $i = 1\ldots n_1$, $r_i(x, y)$ holds, or, for some $i = 1\ldots n_2$, $(\alpha_i, \beta_i)(x, y)$ holds.

$^\star$ TCSPs stands for Temporal Constraint Satisfaction Problems, a well-known constraint-based temporal framework [8].

$^\star\star$ This work was supported partly by the EU project “Cognitive Vision systems” (CogVis), under grant CogVis IST 2000-29375.

1 Qualitative Spatial Reasoning.
holds. A conjunction of such general constraints is a TCSP-like CSP, which we will refer to as an SCSP (Spatial Constraint Satisfaction Problem). An effective solution search algorithm for an SCSP will be described, which uses (1) constraint propagation, based on a composition operation to be defined, as the filtering method during the search, and (2) the Simplex algorithm, guaranteeing completeness, at the leaves of the search tree. The approach is particularly suited for large-scale high-level vision, such as, e.g., satellite-like surveillance of a geographic area.

1 Introduction

Knowledge representation (KR) systems allowing for the representation of both qualitative knowledge and quantitative knowledge are more than needed by modern applications, which, depending on the level of detail of the knowledge to be represented, may feel happy with a high-level, qualitative language, or need to use a low-level, quantitative language. Qualitative languages suffer from what Forbus et al. refer to as the poverty conjecture (which corresponds more or less to Habel’s argument that such languages suffer from not having “the ability to refine discrete structures if necessary”), but have the advantage of behaving computationally better. On the other hand, quantitative languages do not suffer from the poverty conjecture, but have a slow computational behaviour. Thus, such a KR system will feel happier when the knowledge at hand can be represented in a purely qualitative way, for it can then get rid of heavy numeric calculations, and restrict its computations to symbols manipulation, consisting, in the case of constraint-based languages in the style of the Region-Connection Calculus RCC-8 [21], mainly in computing a closure under a composition table.

An important question raised by the above discussion is clearly how to augment the chances of a qualitative/quantitative KR system to remain at the qualitative level. Consider, for instance, QSR constraint-based, RCC-8-like languages. Given the poverty conjecture, which corresponds to the fact that such a language can make only a finite number of distinctions, reflected by the number of its atomic relations, one way of answering the question could be to integrate more than one QSR language within the same KR system. The knowledge at hand is then handled in a quantitative way only
in the extreme case when it can be represented by none of the QSR languages which the system integrates.

One way for a KR system, such as described above, to reason about its knowledge is to start with reasoning about the qualitative part of the knowledge, which decomposes, say, into n components, one for each of the QSR languages the system integrates. For RCC-8-like languages, this can be done using a constraint propagation algorithm such as the one in [1]. If in either of the n components, an inconsistency has been detected, then the whole knowledge has been detected to be inconsistent without the need of going into low-level details. If no inconsistency has been detected at the high, qualitative level, then the whole knowledge needs translation into the unifying quantitative language, and be processed in a purely quantitative way. But even when the high-level, qualitative computations fail to detect any inconsistency, they still potentially help the task of the low-level, purely quantitative computations. The situation can be compared to standard search algorithms in CSPs, where a local-consistency pre-processing is applied to the whole knowledge to potentially reduce the search space, and eventually detect the knowledge inconsistency, before the actual search for a solution starts.

With the above considerations in mind, we consider the integration of Frank’s cone-shaped and projection-based calculi of cardinal direction relations [10], well-known in QSR. A complete decision procedure for the projection-based calculus is known from Ligozat’s work [17]. For the other calculus, based on a uniform 8-sector partition of the plane, making it more flexible and cognitively more plausible, no such procedure is known. For each of the two calculi, the region of the plane associated with each of the atomic relations is convex, and given by the intersection of two half-planes. As a consequence, each such relation can be equivalently written as a conjunction of linear inequalities on variables consisting of the coordinates of the relation’s arguments. We consider a more general, qualitative/quantitative language, which, at the basic level, expresses convex constraints of the form $r(x, y)$, where $r$ is a cone-shaped or projection-based atomic relation of cardinal directions, or of the form $(\alpha, \beta)(x, y)$, with $\alpha, \beta \in [0, 2\pi)$ and $(\beta - \alpha) \in [0, \pi]$: the meaning of $(\alpha, \beta)(x, y)$, in particular, is that point $x$ belongs to the (con
vex) cone-shaped area rooted at \( y \), and bounded by angles \( \alpha \) and \( \beta \). We refer to such constraints as basic constraints: qualitative basic constraint in the first case, and quantitative basic constraint in the second. A conjunction of basic constraints can be solved by first applying constraint propagation, based on a composition operation to be defined, which is basically the spatial counterpart of composition of two TCSP constraints [8]. If the propagation detects no inconsistency then the knowledge is translated into a system of linear inequalities, and solved with the well-known Simplex algorithm. The preprocessing of the qualitative component of the knowledge can be done with a constraint propagation algorithm such as the one in [1], and needs the composition tables of the cardinal direction calculi, which can be found in [10].

To summarise, given combined qualitative/quantitative conjunctive knowledge, expressed as a conjunction of basic constraints, the reasoning methodology we propose works in three steps:

1. First apply (qualitative) constraint propagation [1] to each of the qualitative components of the knowledge.
2. If no inconsistency has been detected by the previous step, then translate the qualitative knowledge into quantitative knowledge, so that the whole knowledge gets expressed in the unifying quantitative language; then apply (quantitative) constraint propagation to the whole, based on a composition operation to be defined later.
3. If no inconsistency has been detected by the previous step, then translate the knowledge into a conjunction of linear inequalities, and apply the complete Simplex procedure to decide whether the knowledge is consistent.

The general form of a constraint is \((s_1 \lor \cdots \lor s_n)(x, y)\), which we also represent as \(\{s_1, \ldots, s_n\}(x, y)\), where \(s_i(x, y)\), for all \(i \in \{1, \ldots, n\}\), is a basic constraint, either qualitative or quantitative. The meaning of such a general constraint is that, either of the \(n\) basic constraints is satisfied, i.e., \(s_1(x, y) \lor \cdots \lor s_n(x, y)\). A general constraint is qualitative if it is the disjunction of qualitative basic constraints of one type, cone-shaped or projection-based; it is quantitative otherwise. The language can be looked at as the spatial counterpart of Dechter
et al.’s TCSPs \cite{8}: the domain of a TCSP variable is $\mathbb{R}$, symbolising continuous time, whereas the domain of an SCSP variable is the cross product $\mathbb{R} \times \mathbb{R}$, symbolising the continuous 2-dimensional space.

The reasoning module of our KR system involves thus two known techniques: constraint propagation, based on composition of two basic constraints, and the Simplex algorithm. If both basic constraints are qualitative, and both cone-shaped or both projection-based, then their composition is given by existing composition tables \cite{10}. Otherwise, the basic constraints are considered as quantitative, and their composition is computed in a way to be defined later (similar to composition of two convex TCSP constraints \cite{8}).

Some emphasis on our approach to knowledge representation is needed. Researchers working on purely quantitative languages use arguments such as the poverty conjecture in \cite{9} to criticise qualitative reasoning in general, and QSR in particular. On the other hand, QIR\textsuperscript{2} researchers argue that quantitative reasoning goes often too much into unnecessary details, which is reflected by ideas such as “make only as many distinctions as necessary” \cite{3}, borrowed to naïve physics \cite{14}. Our approach is a conciliating one, and is meant to satisfy both tendencies. It consists of combining QIR languages known to be sufficient for a large number of applications, with a subsuming quantitative language. The number of QIR languages may be, as in the present work, more than just one, to allow potential applications high chances to remain at the high-level, qualitative languages for their knowledge representation. QIR researchers are satisfied since they have the possibility of using only the qualitative part of the language. On the other hand, if an application needs more expressiveness than is allowed by any of the QIR sublanguages, then the unifying quantitative language is there to satisfy it.

Current research shows clearly the importance of developing decidable constraint-based spatial languages: specialising an $\mathcal{ALC}(D)$-like Description Logic (DL) \cite{2}, so that the roles are temporal immediate-successor (accessibility) relations, and the concrete domain is generated by a decidable constraint-based spatial language, such as

---

\footnote{We use QIR and QnR as shorthands for Qualitative Reasoning and for Quantitative Reasoning, respectively.}
an RCC-8-like qualitative spatial RA [21], or a combined qualitative/quantitative language such as the one to be described in this paper, leads to a computationally well-behaving family of languages for spatial change in general, and for motion of spatial scenes in particular:

1. Deciding satisfiability of an $\mathcal{ALC}(\mathcal{D})$ concept with respect to (w.r.t.) a cyclic TBox is, in general, undecidable (see, for instance, [18]).
2. In the case of the spatio-temporalisation, however, if we use what is called weakly cyclic TBoxes in [15], then satisfiability of a concept w.r.t. such a TBox is decidable. The axioms of a weakly cyclic TBox capture the properties of modal temporal operators. The reader is referred to [15] for details.

Spatio-temporal theories such as the ones defined in [15] can be seen as single-ontology spatio-temporal theories, in the sense that the concrete domain represents only one type of spatial knowledge (e.g., RCC-8 relations if the concrete domain is generated by RCC-8). The calculus to be defined can, of course, generate such a single-ontology spatio-temporal theory; but with the disadvantage that the concrete domain would be heterogeneous, in the sense that it would group together two qualitative languages and a unifying quantitative language, which some applications might not find clean. We could, instead, use a 3-ontology spatio-temporal theory: two ontologies of the theory generated by the cone-shaped and the projection-based calculi of cardinal direction relations, the third ontology by the unifying quantitative language.

2 Constraint satisfaction problems

A constraint satisfaction problem (CSP) of order $n$ consists of:

1. a finite set of $n$ variables, $x_1, \ldots, x_n$;
2. a set $U$ (called the universe of the problem); and
3. a set of constraints on values from $U$ which may be assigned to the variables.
An $m$-ary constraint is of the form $R(x_{i_1}, \ldots, x_{i_m})$, and asserts that the values $a_{i_1}, \ldots, a_{i_m}$ assigned to the variables $x_{i_1}, \ldots, x_{i_m}$, respectively, are so that the $m$-tuple $(a_{i_1}, \ldots, a_{i_m})$ belongs the $m$-ary relation $R$ (an $m$-ary relation over the universe $U$ is any subset of $U^m$).

An $m$-ary CSP is one of which the constraints are $m$-ary constraints. We will be concerned exclusively with binary CSPs.

For any two binary relations $R$ and $S$, $R \cap S$ is the intersection of $R$ and $S$, $R \cup S$ is the union of $R$ and $S$, $R \circ S$ is the composition of $R$ and $S$, and $R^\sim$ is the converse of $R$; these are defined as follows:

- $R \cap S = \{(a, b) : (a, b) \in R \text{ and } (a, b) \in S\}$,
- $R \cup S = \{(a, b) : (a, b) \in R \text{ or } (a, b) \in S\}$,
- $R \circ S = \{(a, b) : \text{for some } c, (a, c) \in R \text{ and } (c, b) \in S\}$,
- $R^\sim = \{(a, b) : (b, a) \in R\}$.

Three special binary relations over a universe $U$ are the empty relation $\emptyset$ which contains no pairs at all, the identity relation $I_U = \{(a, a) : a \in U\}$, and the universal relation $\top_U = U \times U$.

Composition and converse for binary relations were introduced by De Morgan [5,6].

### 2.1 Constraint matrices

A binary constraint matrix of order $n$ over $U$ is an $n \times n$-matrix, say $B$, of binary relations over $U$ verifying the following:

- $(\forall i \leq n)(B_{ii} \subseteq I_U)$ (the diagonal property),
- $(\forall i, j \leq n)(B_{ij} = (B_{ji})^\sim)$ (the converse property).

A binary CSP $P$ of order $n$ over a universe $U$ can be associated with the following binary constraint matrix, denoted $B^P$:

1. Initialise all entries to the universal relation: $(\forall i, j \leq n)((B^P)_{ij} \leftarrow \top_U)$
2. Initialise the diagonal elements to the identity relation: $(\forall i \leq n)((B^P)_{ii} \leftarrow I_U)$
3. For all pairs $(x_i, x_j)$ of variables on which a constraint $(x_i, x_j) \in R$ is specified: $(B^P)_{ij} \leftarrow (B^P)_{ij} \cap R, (B^P)_{ji} \leftarrow ((B^P)_{ij})^\sim$.

We make the assumption that, unless explicitly specified otherwise, a CSP is given as a constraint matrix.
2.2 Strong $k$-consistency, refinement

Let $P$ be a CSP of order $n$, $V$ its set of variables and $U$ its universe. An instantiation of $P$ is any $n$-tuple $(a_1, a_2, \ldots, a_n)$ of $U^n$, representing an assignment of a value to each variable. A consistent instantiation is an instantiation $(a_1, a_2, \ldots, a_n)$ which is a solution: $(\forall i, j \leq n)((a_i, a_j) \in (B^P)_{ij})$. $P$ is consistent if it has at least one solution; it is inconsistent otherwise. The consistency problem of $P$ is the problem of verifying whether $P$ is consistent.

Let $V' = \{x_{i_1}, \ldots, x_{i_j}\}$ be a subset of $V$. The sub-CSP of $P$ generated by $V'$, denoted $P_{V'}$, is the CSP with $V'$ as the set of variables, and whose constraint matrix is obtained by projecting the constraint matrix of $P$ onto $V'$: $(\forall k, l \leq j)((B^{P_{V'}})_{kl} = (B^P)_{i_ki_l})$. $P$ is $k$-consistent \cite{11,12} (see also \cite{4}) if for any subset $V'$ of $V$ containing $k - 1$ variables, and for any variable $X \in V$, every solution to $P_{V'}$ can be extended to a solution to $P_{V' \cup \{X\}}$. $P$ is strongly $k$-consistent if it is $j$-consistent, for all $j \leq k$.

1-consistency, 2-consistency and 3-consistency correspond to node-consistency, arc-consistency and path-consistency, respectively \cite{19,20}. Strong $n$-consistency of $P$ corresponds to what is called global consistency in \cite{7}. Global consistency facilitates the important task of searching for a solution, which can be done, when the property is met, without backtracking \cite{12}.

A refinement of $P$ is a CSP $P'$ with the same set of variables, and such that: $(\forall i, j)((B^{P'})_{ij} \subseteq (B^P)_{ij})$.

3 Frank’s calculi of cardinal direction relations

Frank’s models of cardinal directions in 2D \cite{10} are illustrated in Figure 1. They use a partition of the plane into regions determined by lines passing through a reference object, say $S$. Depending on the region a point $P$ belongs to, we have $No(P, S)$, $NE(P, S)$, $Ea(P, S)$, $SE(P, S)$, $So(P, S)$, $SW(P, S)$, $We(P, S)$, $NW(P, S)$, or $Eq(P, S)$, corresponding, respectively, to the position of $P$ relative to $S$ being north, north-east, east, south-east, south, south-west, west, north-west, or equal. Each of the two models can thus be seen as a binary Relation Algebra (RA), with nine atoms. Both use a global horizontal/vertical,
left-right/bottom-up reference frame, which we suppose to be a Cartesian coordinate system \((O, x', y')\). The coordinate system so chosen clearly verifies the fact that, on the one hand, the \(x\)-axis \(x'\) is parallel to, and has the same orientation as the West-East directed line of Frank’s projection-based model, and, on the other hand, the \(y\)-axis \(y'\) is parallel to, and has the same orientation as the South-North directed line of the same model — \(O\) is the intersection of the \(x\)- and \(y\)-axes.

To differentiate between the two models, we use the underscore \(cs\) for the cone-shaped model, and the underscore \(pb\) for the projection-based model. Thus, from now on, (1) we refer to the cone-shaped model as \(\mathcal{CDA}_{cs}\), and to the projection-based model as \(\mathcal{CDA}_{pb}\); and (2) we denote the atoms of \(\mathcal{CDA}_{cs}\) as \(No_{cs}, NE_{cs}, Ea_{cs}, SE_{cs}, So_{cs}, SW_{cs}, We_{cs}, NW_{cs}\), and \(Eq_{cs}\), and the atoms of \(\mathcal{CDA}_{pb}\) as \(No_{pb}, NE_{pb}, Ea_{pb}, SE_{pb}, So_{pb}, SW_{pb}, We_{pb}, NW_{pb}\), and \(Eq_{pb}\). 

A \(\mathcal{CDA}_{cs}\) (resp. \(\mathcal{CDA}_{pb}\)) relation is any subset of the set of all \(\mathcal{CDA}_{cs}\) (resp. \(\mathcal{CDA}_{pb}\) atoms. A \(\mathcal{CDA}_{cs}\) (resp. \(\mathcal{CDA}_{pb}\)) relation is said to be atomic if it contains one single atom (a singleton set); it is said to be the \(\mathcal{CDA}_{cs}\) (resp. \(\mathcal{CDA}_{pb}\)) universal relation if it contains all the \(\mathcal{CDA}_{cs}\) (resp. \(\mathcal{CDA}_{pb}\)) atoms. When no confusion raises, we may omit the brackets in the representation of an atomic relation.


3.1 CSPs of cardinal direction relations on 2D points

We define a $\mathcal{CDA}_{cs}$-CSP (resp. $\mathcal{CDA}_{pb}$-CSP) as a CSP of which the constraints are $\mathcal{CDA}_{cs}$ (resp. $\mathcal{CDA}_{pb}$) relations on pairs of the variables. The universe of such a CSP is the set $\mathbb{R}^2$ of 2D points.

A $\mathcal{CDA}_{cs}$-matrix (resp. $\mathcal{CDA}_{pb}$-matrix) of order $n$ is a binary constraint matrix of order $n$ of which the entries are $\mathcal{CDA}_{cs}$ (resp. $\mathcal{CDA}_{pb}$) relations. The constraint matrix associated with a $\mathcal{CDA}_{cs}$-CSP (resp. $\mathcal{CDA}_{pb}$-CSP) is a $\mathcal{CDA}_{cs}$-matrix (resp. $\mathcal{CDA}_{pb}$-matrix).

A scenario of such a CSP is a refinement $P'$ such that all entries of the constraint matrix of $P'$ are atomic relations. A CSP of cardinal direction relations that does not include the empty constraint, which indicates a trivial inconsistency, is strongly 2-consistent. A $\mathcal{CDA}$-CSP is a CSP which is either a $\mathcal{CDA}_{cs}$-CSP or a $\mathcal{CDA}_{pb}$-CSP.

An atomic $\mathcal{CDA}$-CSP is a $\mathcal{CDA}$-CSP which is its own unique scenario (i.e., of which all entries of the constraint matrix are atomic relations).

3.2 Solving a $\mathcal{CDA}$-CSP

A simple adaptation of Allen’s constraint propagation algorithm \[1\] can be used to achieve path consistency (hence strong 3-consistency) for a CSP of cardinal direction relations, thanks to composition tables of the calculi which can be found in \[10\]. Applied to such a CSP, say $P$, such an adaptation would repeat the following steps until either stability is reached or the empty relation is detected (indicating inconsistency):

1. Consider a triple $(X_i, X_j, X_k)$ of variables verifying $(B^P)_{ij} \not\subseteq (B^P)_{ik} \circ (B^P)_{kj}$
2. $(B^P)_{ij} \leftarrow (B^P)_{ij} \cap (B^P)_{ik} \circ (B^P)_{kj}$
3. If $((B^P)_{ij} = \emptyset)$ then exit (the CSP is inconsistent).

Path consistency is complete for atomic $\mathcal{CDA}_{pb}$-CSPs \[17\]. Given this, Ladkin and Reinefeld’s solution search algorithm \[16\] can be used to search for a solution, if any, or otherwise report inconsistency, of a general $\mathcal{CDA}_{pb}$-CSP. However, no such result is known for atomic $\mathcal{CDA}_{cs}$-CSPs. But even so, we still can apply the search
algorithm in [16] to search for a path-consistent scenario of a \( CDA_{cs} \)-CSP, if such a refinement exists, or report inconsistency otherwise. The main result of the present work implies that we can solve the consistency problem of an atomic \( CDA_{cs} \)-CSP, by first translating it into a conjunction of linear inequalities on variables consisting of the coordinates of the point-variables of the \( CDA_{cs} \)-CSP. This means that for a general \( CDA_{cs} \)-CSP, we can use the search algorithm in [16] augmented with the Simplex algorithm to decide its consistency problem. The basic idea is to apply the algorithm in [16] as it is, and, whenever it succeeds to find a path-consistent scenario (the algorithm is then at the level of a leaf of the search tree), check, using the Simplex algorithm, whether that scenario is consistent, by translating it into a conjunction of linear inequalities. If the conjunction of linear inequalities is consistent then the corresponding scenario is consistent, and is thus a consistent scenario of the input \( CDA_{cs} \)-CSP. If the conjunction is inconsistent, then the search for a possible consistent scenario has to continue. This is illustrated in Figure 2.

Fig. 2. A consistent scenario search function for \( CDA_{cs} \)-CSPs.
4 Temporal Constraint Satisfaction Problems — TCSPs

TCSPs have been proposed in [8] as an extension of (discrete) CSPs [19,20] to continuous variables.

Definition 1 (TCSP [8]). A TCSP consists of (1) a finite number of variables ranging over the universe of time points; and (2) Dechter, Meiri and Pearl’s constraints (henceforth DMP constraints) on the variables.

A DMP constraint is either unary or binary. A unary constraint has the form \( R(Y) \), and a binary constraint the form \( R(X,Y) \), where \( R \) is a subset of the set \( \mathbb{R} \) of real numbers, seen as a unary relation in the former case, and as a binary relation in the latter case, and \( X \) and \( Y \) are variables ranging over the universe of time points: the unary constraint \( R(Y) \) is interpreted as \( Y \in R \), and the binary constraint \( R(X,Y) \) as \( (Y - X) \in R \). A unary constraint \( R(Y) \) may be seen as a special binary constraint if we consider an origin of the World (time 0), represented, say, by a variable \( X_0 \): \( R(Y) \) is then equivalent to \( R(X_0, Y) \). Unless explicitly stated otherwise, we assume, in the rest of the paper, that the constraints of a TCSP are all binary.

Definition 2 (STP [8]). An STP (Simple Temporal Problem) is a TCSP of which all the constraints are convex, i.e., of the form \( R(X,Y) \), \( R \) being a convex subset of \( \mathbb{R} \).

The universal relation for TCSPs in general, and for STPs in particular, is the relation consisting of the whole set \( \mathbb{R} \) of real numbers: the knowledge \( (Y - X) \in \mathbb{R} \), expressed by the DMP constraint \( \mathbb{R}(X,Y) \), is equivalent to “no knowledge”. The identity relation is the (convex) set reducing to the singleton \( \{0\} \): the constraint \( \{0\}(X,Y) \) “forces” variables \( X \) and \( Y \) to be equal.

5 A spatial counterpart of TCSPs: Spatial Constraint Satisfaction Problems (SCSPs)

We now provide a spatial counterpart of TCSPs, which we refer to as SCSPs — Spatial Constraint Satisfaction Problems. The domain
of an SCSP variable is the cross product \( \mathbb{R} \times \mathbb{R} \), which we look at as the set of points of the 2-dimensional space. As for a TCSP, an SCSP will have unary constraints and binary constraints, and unary constraints can be interpreted as special binary constraints by choosing an origin of the 2-dimensional space —space \((0, 0)\). This will be explained shortly.

**Definition 3 (SCSP).** An SCSP consists of (1) a finite number of variables ranging over the universe of points of the 2-dimensional space (henceforth 2D-points); and (2) SCSP constraints on the variables.

An SCSP constraint is either unary or binary, and either basic or disjunctive. A basic constraint is (1) of the form \( e(x, y) \), \( e \) being equality, (2) of the qualitative form \( \langle \overline{r} \rangle^3(x, y) \) or \( \langle \overline{r} \rangle^3(x) \), depending on whether it is binary or unary, \( r \) being a cone-shaped or projection-based atomic relation of cardinal directions other than equality, \( \overline{i}, \overline{j} \in \{0, 1\} \), or (3) of the quantitative form \( \langle \overline{i} \alpha, \beta \rangle^3(x, y) \) (binary) or \( \langle \overline{i} \alpha, \beta \rangle^3(x) \) (unary), with \( \alpha, \beta \in [0, 2\pi) \), \( (\beta - \alpha) \in [0, \pi] \), \( \overline{i}, \overline{j} \in \{0, 1\} \). \( ^0 \) and \( ^1 \) stand, respectively, for the left open bracket ( and the left close bracket [. Similarly, \( ^0 \) and \( ^1 \) stand, respectively, for the right open bracket ) and the right close bracket ]. A graphical illustration of a quantitative basic constraint is provided in Figure 3.

### 5.1 Translating a qualitative basic constraint into a quantitative basic constraint

A qualitative basic relation \( \langle \overline{r} \rangle^3 \) includes (resp. excludes) its lower bound, which is a half-line, if \( \overline{i} = 1 \) (resp. \( \overline{i} = 0 \)); it includes (resp. excludes) its upper bound, which is also a half-line, if \( \overline{j} = 1 \) (resp. \( \overline{j} = 0 \)). This means that the version of Frank’s relations of cardinal directions we are using is such that, the region associated with an atom (see Figure 1) may include both, one or none of its delimiting half-lines.

We remind the reader that we have chosen our Cartesian system of coordinates, \((O, x’x, y’y)\), in such a way that, on the one hand, the \(x\)-axis \(x’x\) is parallel to, and has the same orientation as the West-East
directed line of Frank’s projection-based model, and, on the other hand, the y-axis $y'y$ is parallel to, and has the same orientation as the South-North directed line of the same model. The x-axis $x'x$ is the origin of angles, and the anticlockwise orientation is the positive orientation for angles. Given that we use the set $[0, 2\pi)$ as the universe of angles, if two angles $\alpha$ and $\beta$ are so that $\alpha > \beta$, the interval $\langle \alpha, \beta \rangle$ will represent the union $\langle \alpha, 2\pi \rangle \cup [0, \beta \rangle$. Furthermore, given any $\alpha, \beta \in [0, 2\pi)$, the difference $\beta - \alpha$ will measure the (angular) distance of $\beta$ relative to $\alpha$: the length, in radians, of the anticlockwise “walk” from $\alpha$ to $\beta$ (this is, in other words, the size of anticlockwise sector determined by $[\alpha, \beta]$).

The atom $No_{cs}$, for instance, is bounded by the lines whose angular distances from the x-axis are $\frac{\pi}{8}, \frac{7\pi}{8}$, for the lower bound, and $\frac{\pi}{2}, \frac{3\pi}{2}$, for the upper bound (see Figure IIV left). The qualitative basic constraint $\langle No_{cs}\rangle(x, y)$ is thus equivalent to the quantitative basic constraint $\langle No_{cs}\rangle^{\pi}(x, y)$. The atom $NE_{pb}$ is associated with the region bounded by angles $0$ and $\frac{\pi}{2}$: the constraint $\langle NE_{pb}\rangle(x, y)$ can thus equivalently be represented as the quantitative basic constraint $\langle NE_{pb}\rangle^{\pi}(x, y)$. In a similar way, $\langle No_{pb}\rangle^{\pi}(x, y)$ is equivalent to $\langle No_{pb}\rangle(x, y)$. The other qualitative basic constraints, either cone-shaped or projection-based, are translated in a similar way. The situation is summarised in the table below.

| $\mathcal{CD}$ basic constraint | Translation | $\mathcal{CD}$ basic constraint | Translation |
|---------------------------------|-------------|---------------------------------|-------------|
| $\langle No_{cs}\rangle(x, y)$  | $\langle \frac{\pi}{8}, \frac{7\pi}{8}\rangle(x, y)$ | $\langle No_{pb}\rangle(x, y)$ | $\langle 0, \frac{\pi}{2}\rangle(x, y)$ |
| $\langle NE_{cs}\rangle(x, y)$  | $\langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle(x, y)$ | $\langle NE_{pb}\rangle(x, y)$ | $\langle 0, \frac{\pi}{2}\rangle(x, y)$ |
| $\langle EA_{cs}\rangle(x, y)$  | $\langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle(x, y)$ | $\langle EA_{pb}\rangle(x, y)$ | $\langle 0, 0\rangle(x, y)$ |
| $\langle SE_{cs}\rangle(x, y)$  | $\langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle(x, y)$ | $\langle SE_{pb}\rangle(x, y)$ | $\langle \frac{\pi}{2}, 0\rangle(x, y)$ |
| $\langle So_{cs}\rangle(x, y)$  | $\langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle(x, y)$ | $\langle So_{pb}\rangle(x, y)$ | $\langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle(x, y)$ |
| $\langle SW_{cs}\rangle(x, y)$  | $\langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle(x, y)$ | $\langle SW_{pb}\rangle(x, y)$ | $\langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle(x, y)$ |
| $\langle W_{cs}\rangle(x, y)$   | $\langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle(x, y)$ | $\langle W_{pb}\rangle(x, y)$ | $\langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle(x, y)$ |
| $\langle NW_{cs}\rangle(x, y)$  | $\langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle(x, y)$ | $\langle NW_{pb}\rangle(x, y)$ | $\langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle(x, y)$ |

Thus we can, without loss of generality, suppose that a basic constraint is of the form $e(x, y)$, or of the quantitative form $\langle e(x, y) \rangle$. A disjunctive constraint is of the form $[S_1 \lor \cdots \lor S_n](x, y)$ (binary) or $[S_1 \lor \cdots \lor S_n](x)$ (unary), with $S_k(x, y)$ and $S_k(x)$, $k = 1 \ldots n$, being basic constraints as described above: in the binary case, the

\footnote{The reader should be convinced that the constraint $\langle \frac{\pi}{2}, \frac{3\pi}{2}\rangle(x, y)$ is consistent iff $i = j = 1$. A similar remark applies to the other qualitative basic constraints built from a 1-dimensional projection-based atom (EA_{pb}, W_{pb} and So_{pb}).}
meaning of such a disjunctive constraint is that, for some \( k = 1 \ldots n \), \( S_k(x, y) \) holds; similarly, in the unary case, the meaning is that, for some \( k = 1 \ldots n \), \( S_k(x) \) holds. A unary constraint \( R(x) \) may be seen as a special binary constraint if we consider an origin of the World (space \((0,0)\)), represented, say, by a variable \( x_0 \): \( R(x) \) is then equivalent to \( R(x, x_0) \). Unless explicitly stated otherwise, we assume, in the rest of the paper, that the constraints of an SCSP are all binary.

An SCSP constraint, \( R(x, y) \), is convex if, given an instantiation \( y = a \) of \( y \), the set of points \( x \) satisfying \( R(x, a) \) is a convex subset of the plane. A universal SCSP constraint is an SCSP constraint of the form \([0,2\pi)](x, y)\): the knowledge consisting of such a constraint is equivalent to “no knowledge”, i.e., any instantiation \((a,b)\) of the pair \((x,y)\) satisfies it. A universal constraint is also a convex constraint. A convex SCSP is an SCSP of which all the constraints are convex. Given its similarity to an STP (Simple Temporal Problem) \[\mathbb{S}\], we refer to a convex SCSP as an SSP (Simple Spatial Problem). An SCSP is basic if, for all pairs \((x,y)\) of variables, the SCSP includes a basic constraint of the form \( R(x, y) \) or \( R(y, x) \). We refer to a basic SCSP as a BSP (Basic Spatial Problem).

The standard path consistency procedure for binary CSPs is guided by three algebraic operations, the converse of a constraint, the composition of two constraints, and the intersection of two constraints. These are defined below for SCSPs.

### 5.2 The converse of an SCSP constraint

The converse of an SCSP relation \( R \) is the SCSP relation \( R^\sim \) such that, for all \( x, y \), \( R(x, y) \) iff \( R^\sim(y, x) \). We refer to the constraint \( R^\sim(y, x) \) as the converse of the constraint \( R(x, y) \). The converse of \( e(x, y) \) is clearly \( e(y, x) \). The converse of an SCSP quantitative basic relation \( \langle \alpha, \beta \rangle^\mathbb{S}(x,y) \) is the SCSP quantitative basic relation \( \langle \alpha+\pi, \beta+\pi \rangle^\mathbb{S}(y,x) \), which can be explained by the simple fact that, given any instantiation \((x,y) = (a,b)\) of the pair \((x,y)\) satisfying the constraint \( \langle \alpha, \beta \rangle^\mathbb{S}(x,y) \), the angle of the \( x \)-axis with the directed line \((ba)\) is obtained by adding \( \pi \) to the angle of the \( x \)-axis with the directed line \((ab)\).
Fig. 3. Graphical interpretation of the basic constraint $\langle s, t \rangle^2(X, Y)$: Given $Y$, the set of points $X$ satisfying the constraint $\langle s, t \rangle^2(X, Y)$ is the cone-shaped area centred at $Y$, whose lower bound (open if $t = 0$, close otherwise) and upper bound (open if $t = 0$, close otherwise) are, respectively, the half-lines whose angular distances from the $x$-axis, with respect to anticlockwise orientation, are $s$ and $t$. 
5.3 The composition of two SCSP constraints

The composition of two SCSP relations $R$ and $S$, $R \circ S$, is the most specific relation $T$ such that, for all $x$, $y$, $z$, if $R(x, y)$ and $S(y, z)$ then $T(x, z)$. We refer to the constraint $T(x, z)$ as the composition of the constraints $R(x, y)$ and $S(y, z)$.

We describe how to compute the composition of two basic constraints, from which derives the composition of two general SCSP constraints.$^4$ Clearly, $e \circ R = R \circ e = R$, for all SCSP relation $R$. For the general case, let $R = \langle \iota \alpha_1, \beta_1 \rangle \bar{\iota}$ and $S = \langle \iota_2 \alpha_2, \beta_2 \rangle \bar{\iota}$. The result here is that, if $\beta_1 < \alpha_2 < \beta_1 + \pi$ and $\beta_2 > \beta_1 + \pi$, then $R \circ S$ is the universal relation $[0, 2\pi)$, which means that, in such a case, given the knowledge $R(x, y)$ and $S(y, z)$, no knowledge can be inferred on the extreme variables $x$ and $z$. Otherwise, $R \circ S$ is obtainable in very much the same way as the composition of two (convex) intervals of the real line (cyclicity of the universe $[0, 2\pi)$ of angles is a bit tedious but manageable). Basically, the result is $\langle \iota \alpha, \beta \rangle \bar{\iota}$, where $\alpha$ is the minimum, in a certain sense, of $\alpha_1$ and $\alpha_2$, $\beta$ is the maximum of $\beta_1$ and $\beta_2$, $\iota$ is the logical AND of $\iota_1$ and $\iota_2$, and $\bar{\iota}$ is the logical AND of $\bar{\iota}_1$ and $\bar{\iota}_2$, 0 and 1 being interpreted as FALSE and TRUE, respectively —left to the reader.

5.4 The intersection of two SCSP constraints

The intersection of two SCSP relations $R$ and $S$, $R \cap S$, is the SCSP relation $T$ such that, for all $x, y$, the conjunction $R(x, y) \land S(x, y)$ is equivalent to $T(x, y)$. We refer to the constraint $T(x, y)$ as the intersection of the constraints $R(x, y)$ and $S(x, y)$. Clearly, we have $R \cap S = S \cap R$ (commutativity).

We describe how to compute the intersection of two basic constraints, from which derives the intersection of two general SCSP constraints.

$e \cap e = e; e \cap \langle \iota \alpha, \beta \rangle \bar{\iota} = e$ if $\iota = j = 1$; and $e \cap \langle \iota \alpha, \beta \rangle \bar{\iota} = \emptyset$ if $\iota = 0$ or $j = 0$. $\langle \iota \alpha_1, \beta_1 \rangle \bar{\iota} \cap \langle \iota_2 \alpha_2, \beta_2 \rangle \bar{\iota} = [\alpha_1, \alpha_1] \cup [\beta_1, \beta_1]$ if $\alpha_1 = \beta_2 = \beta_1 - \pi = \alpha_2 - \pi$ and $\iota_1 = j_1 = \iota_2 = j_2 = 1$; otherwise, $\langle \iota \alpha_1, \beta_1 \rangle \bar{\iota} \cap \langle \iota_2 \alpha_2, \beta_2 \rangle \bar{\iota}$ is obtainable, again, in very much the same

$^4$ The reader should keep in mind that a quantitative basic constraint $\langle \iota \alpha, \beta \rangle \bar{\iota}(x, y)$ is so that, the difference $\beta - \alpha$ belongs to $[0, \pi]$.
way as the intersection of two (convex) intervals of the real line — left to the reader.

5.5 Translating an SCSP constraint into a conjunction of linear inequalities

We now provide a translation of a quantitative basic constraint into (a conjunction of) linear inequalities. We will then be able to translate any SSP (thus, any BSP) into a conjunction of linear inequalities, and solve it with the well-known Simplex algorithm. Constraint propagation, based on the algebraic operations we have defined, and the Simplex can be combined in a solution search algorithm for general SCSPs: constraint propagation will be used at the internal nodes of the search space, as a filtering procedure, and the Simplex at the level of the leaves, as a completeness-guaranteeing procedure (the SCSP at the level of a leaf is a path-consistent SSP, but since we know nothing about completeness of path-consistency for SSPs, we need to translate into linear inequalities and solve with the Simplex).

Given a point $X$ of the plane, we denote by $(x_X, y_X)$ its coordinates. The translation of $e(X, Y)$ is obvious: $x_X - x_Y \leq 0 \land y_Y - y_X \leq 0 \land y_Y - y_X \leq 0 \land y_Y - y_X \leq 0$. For the translation of the quantitative basic constraint $\langle \alpha, \beta \rangle \mathcal{S}(X, Y)$, we consider: the left half-plane (open if $\iota = 0$, close otherwise) delimited by the directed line through $Y$, whose angular distance from the $x$-axis is $\alpha$; the right half-plane (open if $\jmath = 0$, close otherwise) delimited by the directed line through $Y$, whose angular distance from the $x$-axis is $\beta$; and the close right half-plane delimited by the directed line through $Y$, whose angular distance from the $x$-axis is $\alpha + \frac{\pi}{2}$ (see Figure 3 for details). We denote the three half-planes by $\text{lhp}(Y, \alpha, \iota)$, $\text{rhp}(Y, \beta, \jmath)$ and $\text{crhp}(Y, \alpha + \frac{\pi}{2})$, respectively. It is now easy to see that the constraint $\langle \alpha, \beta \rangle \mathcal{S}(X, Y)$ is equivalent to $X \in \text{lhp}(Y, \alpha, \iota) \cap \text{rhp}(Y, \beta, \jmath)$, if $\alpha \neq \beta$; and to $X \in \text{lhp}(Y, \alpha, \iota) \cap \text{rhp}(Y, \beta, \jmath) \cap \text{crhp}(Y, \alpha + \frac{\pi}{2})$, if $\alpha = \beta$.

Thus, all we need is to show how to represent with a linear inequality each of the following assertions on two points $X$ and $Y$ of the plane:

A1 $X$ lies within the open left half-plane delimited by the directed line through $Y$, whose angular distance from the $x$-axis is $\alpha$. 

18
A2 X lies within the close left half-plane delimited by the directed line through Y, whose angular distance from the x-axis is α.
A3 X lies within the open right half-plane delimited by the directed line through Y, whose angular distance from the x-axis is α.
A4 X lies within the close right half-plane delimited by the directed line through Y, whose angular distance from the x-axis is α.

We refer to assertions A1, A2, A3 and A4 as \(X \in \text{ohlhp}(Y, \alpha), X \in \text{clhp}(Y, \alpha), X \in \text{orhp}(Y, \alpha)\) and \(X \in \text{crhp}(Y, \alpha)\), respectively. We refer to the line through Y, whose angular distance from the x-axis is α, as D. We consider eight cases: \(\alpha = 0, 0 < \alpha < \frac{\pi}{2}, \alpha = \frac{\pi}{2}, \frac{\pi}{2} < \alpha < \pi, \alpha = \pi, \pi < \alpha < \frac{3\pi}{2}, \alpha = \frac{3\pi}{2}, \frac{3\pi}{2} < \alpha < 2\pi\). Since, for all \(\alpha\) such that \(\pi \leq \alpha < 2\pi\) (equivalent to \(0 \leq \alpha - \pi < \pi\)), we have

1. \(X \in \text{ohlhp}(Y, \alpha)\) iff \(X \in \text{orhp}(Y, \alpha - \pi)\),
2. \(X \in \text{clhp}(Y, \alpha)\) iff \(X \in \text{crhp}(Y, \alpha - \pi)\),
3. \(X \in \text{orhp}(Y, \alpha)\) iff \(X \in \text{ohlhp}(Y, \alpha - \pi)\), and
4. \(X \in \text{crhp}(Y, \alpha)\) iff \(X \in \text{clhp}(Y, \alpha - \pi)\).

we can restrict the study to the first four cases, \(\alpha = 0, 0 < \alpha < \frac{\pi}{2}, \alpha = \frac{\pi}{2}, \frac{\pi}{2} < \alpha < \pi\). The result is given by the table below, where, given an angle \(\alpha\), \(\text{tg}\alpha\) denotes the tangent of \(\alpha\).

| \(X \in \text{ohlhp}(Y, \alpha)\) | \(\alpha = 0\) | \(0 < \alpha < \frac{\pi}{2}\) | \(\alpha = \frac{\pi}{2}\) | \(\frac{\pi}{2} < \alpha < \pi\) |
|---------------------------------|----------------|------------------|-----------------|----------------|
| \(y_x > y_y\)                  |                 | \(y_x - y_y > \text{tg}\alpha.(x_x - x_y)\) | \(y_x < y_y\) | \(y_x - y_y > \text{tg}\alpha.(x_x - x_y)\) |
| \(X \in \text{clhp}(Y, \alpha)\) |                 | \(y_x - y_y \geq \text{tg}\alpha.(x_x - x_y)\) | \(y_x \leq y_y\) | \(y_x - y_y \geq \text{tg}\alpha.(x_x - x_y)\) |
| \(X \in \text{orhp}(Y, \alpha)\) | \(y_x < y_y\) | \(y_x - y_y < \text{tg}\alpha.(x_x - x_y)\) | \(y_x > y_y\) | \(y_x - y_y < \text{tg}\alpha.(x_x - x_y)\) |
| \(X \in \text{crhp}(Y, \alpha)\) | \(y_x \leq y_y\) | \(y_x - y_y \leq \text{tg}\alpha.(x_x - x_y)\) | \(y_x \geq y_y\) | \(y_x - y_y \leq \text{tg}\alpha.(x_x - x_y)\) |

6 Summary

We have provided a qualitative/quantitative constraint-based, TCSP-like language for reasoning about relative position of points of the 2-dimensional space. The language, SCSPs (Spatial Constraint Satisfaction Problems), subsumes two existing qualitative calculi of relations of cardinal directions [10], and is particularly suited for applications of large-scale high-level vision, such as, e.g., satellite-like surveillance of a geographic area. We have provided all the required tools for the implementation of the presented work; in particular, the
algebraic operations of converse, intersection and composition, which are needed by path consistency. An adaptation of a solution search algorithm, such as, e.g., the one in [16] (see also [8]), which would use path consistency as the filtering procedure during the search, can be used to search for a path consistent BSP refinement of an input SCSP. But, because we know nothing about completeness of path consistency for BSPs, even when a path consistent BSP refinement exists, this does not say anything about consistency of the original SCSP. To make the search complete for SCSPs, we have proposed to augment it with the Simplex algorithm, by translating, whenever a leaf of the search space is successfully reached, the corresponding path consistent BSP into a conjunction of linear inequalities, which can be solved with the well-known Simplex algorithm.
References

1. J F Allen. Maintaining knowledge about temporal intervals. *Communications of the Association for Computing Machinery*, 26(11):832–843, 1983.
2. F Baader and P Hanschke. A scheme for integrating concrete domains into concept languages. In *Proceedings of the 12th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 452–457, Sydney, 1991. Morgan Kaufmann.
3. A G Cohn. Qualitative spatial representation and reasoning techniques. In *Proceedings KI: German Annual Conference on Artificial Intelligence*, volume 1303 of *Lecture Notes in Artificial Intelligence*, pages 1–30, Freiburg, Germany, 1997. Springer-Verlag.
4. M C Cooper. An Optimal k-Consistency Algorithm. *Artificial Intelligence*, 41(1):89–95, 1989.
5. A De Morgan. On the syllogism, no. iv, and on the logic of relations. *Trans. Cambridge Philos. Soc.* 10, pages 331–358, 1864.
6. A De Morgan. *On the Syllogism and other Logical Writings*. Yale University Press, New Haven, 1966.
7. R Dechter. From local to global consistency. *Artificial Intelligence*, 55:87–107, 1992.
8. R Dechter, I Meiri, and J Pearl. Temporal constraint networks. *Artificial Intelligence*, 49:61–95, 1991.
9. K D Forbus, P Nielsen, and B Faltings. Qualitative spatial reasoning: The clock project. *Artificial Intelligence*, 51:417–471, 1991.
10. A U Frank. Qualitative spatial reasoning about distances and directions in geographic space. *Journal of Visual Languages and Computing*, 3:343–371, 1992.
11. E C Freuder. Synthesizing constraint expressions. *Communications of the Association for Computing Machinery*, 21:958–966, 1978.
12. E C Freuder. A sufficient condition for backtrack-free search. *Journal of the Association for Computing Machinery*, 29:24–32, 1982.
13. C Habel. Representing space and time: Discrete, dense or continuous? is that the question? In C Eschenbach and W Heydrich, editors, *Parts and Wholes — Integrity and Granularity*, pages 97–107. 1995.
14. P J Hayes. The second naive physics manifesto. In J R Hobbs and R C Moore, editors, *Formal Theories of the Commonsense World*, pages 1–36. Ablex, 1985.
15. A Isli. Bridging the gap between modal temporal logic and constraint-based QSR as a spatio-temporalisation of ALC(D) with weakly cyclic TBoxes. Technical Report FBI-HH-M-311/02, Fachbereich Informatik, Universität Hamburg, 2002. Downloadable from http://kogs-www.informatik.uni-hamburg.de/~isli/home-Publications-TR.html and from http://arXiv.org/abs/cs.AI/0307040.
16. P Ladkin and A Reinefeld. Effective Solution of qualitative Constraint Problems. *Artificial Intelligence*, 57:105–124, 1992.
17. G Ligozat. Reasoning about cardinal Directions. *Journal of Visual Languages and Computing*, 9(1):23–44, 1998.
18. C Lutz. Combining interval-based temporal reasoning with general TBoxes. *Artificial Intelligence*, ...(,)...,-,..., 2003. In Press.
19. A K Mackworth. Consistency in Networks of Relations. *Artificial Intelligence*, 8:99–118, 1977.
20. U Montanari. Networks of Constraints: fundamental Properties and Applications to Picture Processing. *Information Sciences*, 7:95–132, 1974.

21. D Randell, Z Cui, and A Cohn. A spatial Logic based on Regions and Connection. In *Proceedings KR-92*, pages 165–176, San Mateo, 1992. Morgan Kaufmann.