REGULARITY AND CLASSIFICATION OF SOLUTIONS TO STATIC HARTREE EQUATIONS INVOLVING FRACTIONAL LAPLACIANS

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ABSTRACT. In this paper, we are concerned with the fractional order equations (1) with Hartree type \( H^\frac{d}{2} \)-critical nonlinearity and its equivalent integral equations (3). We first prove a regularity result which indicates that weak solutions are smooth (Theorem 1.2). Then, by applying the method of moving planes in integral forms, we prove that positive solutions \( u \) to (1) and (3) are radially symmetric about some point \( x_0 \in \mathbb{R}^d \) and derive the explicit forms for \( u \) (Theorem 1.3 and Corollary 1). As a consequence, we also derive the best constants and extremal functions in the corresponding Hardy-Littlewood-Sobolev inequalities (Corollary 2).

1. Introduction. In this paper, we consider the following \( H^\frac{d}{2} \)-critical fractional order equation with Hartree type nonlinearity

\[
\begin{aligned}
\left\{(\Delta)^{\frac{d}{2}} u = \left( \frac{1}{|x|^{2\alpha}} \ast |u|^2 \right) u, \quad x \in \mathbb{R}^d, \\
u \in H^\frac{d}{2} (\mathbb{R}^d), \quad u(x) > 0, \quad x \in \mathbb{R}^d,
\end{aligned}
\]

(1)

where \( 0 < \alpha < \frac{d}{2} \) and \( d \geq 1 \). The weak solutions \( u \) of (1) are defined in the distributional sense, that is, \( u \in H^\frac{d}{2} (\mathbb{R}^d) \) and satisfies

\[
\int_{\mathbb{R}^d} (-\Delta)^{\frac{d}{2}} u(x)(-\Delta)^{\frac{d}{2}} \phi(x)dx = \int_{\mathbb{R}^d} \left( \frac{1}{|x|^{2\alpha}} \ast |u|^2 \right) u(x)\phi(x)dx
\]

(2)

for any \( \phi \in C_0^\infty (\mathbb{R}^d) \), where the fractional Laplacians are defined by Fourier transform as usual and \( H^\frac{d}{2} (\mathbb{R}^d) \) is the inhomogeneous Sobolev space with the norm \( \| u \|_{H^\frac{d}{2} (\mathbb{R}^d)} := \| u \|_{L^2(\mathbb{R}^d)} + \| (\Delta)^{\frac{d}{2}} u \|_{L^2(\mathbb{R}^d)} \) while the homogeneous Sobolev norm

\[
\| u \|_{H^\frac{d}{2} (\mathbb{R}^d)} := \| (\Delta)^{\frac{d}{2}} u \|_{L^2(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} |\xi|^{2\alpha} |\mathcal{F}[u](\xi)|^2 d\xi \right)^{\frac{1}{2}}.
\]

2010 Mathematics Subject Classification. Primary: 35R11, 35J91; Secondary: 35B06, 35B65.

Key words and phrases. Fractional Laplacians, positive solutions, radial symmetry, uniqueness, regularity, Hartree type nonlinearity, methods of moving planes in integral forms.

The first author was supported by the NNSF of China (No. 11501021), the second author was supported by the NNSF of China (No. 11301166).

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One should observe that both the fractional Laplacians $(-\Delta)^{\alpha/2}$ and the Hartree type nonlinearity are nonlocal in our equation (1), which is quite different from most of the known results in previous literature. The equation (1) is equivalent to the following integral equation

$$u(x) = \int_{\mathbb{R}^d} \frac{R_{\alpha,d}}{|x - y|^{d - \alpha}} \left( \int_{\mathbb{R}^d} \frac{|u(z)|^2}{|y - z|^{2\alpha}} dz \right) u(y) dy,$$

(3)

where $R_{\alpha,d} := \frac{\Gamma\left(\frac{d - \alpha}{2}\right)}{\pi^{\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right)}$. The equations (1) and (3) are $\dot{\mathcal{H}}^{\frac{\alpha}{2}}$-critical in the sense that both the equations (1), (3) and the $\dot{\mathcal{H}}^{\frac{\alpha}{2}}$ norm are invariant under the scaling $u_{\rho}(x) = \rho^\frac{d - \alpha}{2} u(\rho x)$.

The fractional Laplacian is a nonlocal integral operator. It can be used to model diverse physical phenomena, such as anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics, and relativistic quantum mechanics of stars (see [13, 3] and the references therein). It also has various applications in probability and finance (see [1, 2] and the references therein). In particular, the fractional Laplacian can also be understood as the infinitesimal generator of a stable Lévy process (see [1]).

The solution $u$ to problem (1) is also a ground state or a stationary solution to the following $\dot{\mathcal{H}}^{\frac{\alpha}{2}}$-critical focusing fractional order dynamic Schrödinger-Hartree equation

$$i\partial_t u + (-\Delta)^{\alpha/2} u = \left(\frac{1}{|x|^{2\alpha}} + |u|^2\right) u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

(4)

where $0 < \alpha < \frac{d}{2}$. The Hartree equation has many interesting applications in the quantum theory of large systems of non-relativistic bosonic atoms and molecules (see, e.g. [18]). The Schrödinger equations and Hartree equations with Laplacians or poly-Laplacians have been quite intensively studied, please refer to [24, 35, 36] and the references therein, in which the ground state solution can be regarded as a crucial criterion or threshold for global well-posedness and scattering in the focusing case. Therefore, the classification of solutions to (1) plays an important and fundamental role in the study of the focusing fractional order and higher order Hartree equation (4). PDEs of the type (1) also arise in the Hartree-Fock theory of the nonlinear Schrödinger equations (see [26]).

The qualitative properties of solutions to fractional order or higher order elliptic equations have been extensively studied, for instance, see Chen, Fang and Yang [7], Chen, Li and Li [10], Chen, Li and Ou [11], Chang and Yang [6], Fang and Chen [17], Lin [28], Wei and Xu [41] and the references therein. In [11], Chen, Li and Ou classified all the positive solutions to the PDE

$$(-\Delta)^{\alpha/2} u = u^{\frac{d+\alpha}{d-\alpha}}$$

(5)

and its equivalent integral equation by developing the method of moving planes in integral forms. There are also lots of literatures on the qualitative properties of solutions to Hartree and Choquard equations of second order and higher order, please see e.g. Cao and Dai [5], Lei [21], Liu [32], Moroz and Schaftingen [37], Ma and Zhao [34], Xu and Lei [42] and the references therein. Liu proved in [32] the classification results for positive solutions to (1) with $\alpha = 2$, Xu and Lei [42] derived the classification results for (1) with $\alpha = \frac{d}{3}$, some regularity lifting results for positive solutions were also obtained in [21, 32]. The authors in [21, 32, 42] have used the idea of considering the equivalent systems of integral equations instead,
which was initially used by Ma and Zhao [34]. In [5], Cao and Dai classified all the positive $C^4$ solutions to the $H^2$-critical bi-harmonic equation (1) with $\alpha = 4$, they also derived Liouville theorem in the subcritical cases.

In this paper, we will study the regularity and classification of weak solutions to the equation (1). We will first establish the following equivalence relationship between the PDE (1) and the integral equation (3).

**Theorem 1.1.** Assume $d \geq 1$ and $0 < \alpha < \frac{d}{2}$. If $u \in H^{\frac{2d}{d+2\alpha}}(\mathbb{R}^d)$ is a weak solution to PDE (1), then it satisfies the integral equation (3), and vice versa.

By Theorem 1.1, we only need to investigate the integral equation (3) and the corresponding results for PDE (1) can be derived as a consequence immediately.

Applying the regularity lifting lemma by contracting operators (see [9, 33]), we prove the following regularity lifting theorem for (3) which indicates that weak solutions are smooth.

**Theorem 1.2.** Assume $d \geq 1$ and $0 < \alpha < \frac{d}{2}$. Suppose $u \in L^{\frac{2d}{d-\alpha}}(\mathbb{R}^d)$ is a solution of (3), then $u \in H^{\alpha,p}(\mathbb{R}^d)$ for any $\frac{2d}{d-\alpha} \leq p < \infty$, in particular, $u \in C(\mathbb{R}^d)$.

Moreover, if $1 \leq \alpha < \frac{d}{4}$, then $u \in C^{[\alpha]}(\mathbb{R}^d)$, where $[\alpha]$ denotes the least integer which is larger or equal to $\alpha$.

**Remark 1.** In general, if $0 < \alpha < \frac{d}{2}$ and let $u \in L^{p_0}(\mathbb{R}^d)$ for some $p_0 \in \left(\frac{d}{d-\alpha}, \frac{d}{\alpha}\right)$ be a solution of (3), then it is clear from the proof of Theorem 1.2 (see Section 3) that $u \in H^{\alpha,p}(\mathbb{R}^d)$ for any $p_0 \leq p < \infty$ and all the conclusions in Theorem 1.2 also hold true.

**Remark 2.** When $\alpha \geq 1$, the regularity result $u \in C^{[\alpha]}(\mathbb{R}^d)$ in Theorem 1.2 is not optimal. Lei [22] has established the regularity results about the Choquard equation, he has proved some integrable solution of Choquard equation belongs to $C^\infty(\mathbb{R}^d)$ as long as $\alpha \geq 1$. By using the arguments in Lei [22] or developing the method in the proof of Theorem 1.2 further, we believe that our regularity result in Theorem 1.2 can also be lifted up to $u \in C^\infty(\mathbb{R}^d)$.

Next, we will apply the method of moving planes (see [39, 19, 20, 4, 8]) in integral forms due to Chen, Li and Ou [11, 12] to classify all the positive solutions of (3). The methods of moving planes was initially invented by Alexanderoff in the early 1950s. Later, it was further developed by Serrin [39], Gidas, Ni and Nirenberg [19, 20], Caffarelli, Gidas and Spruck [4], Chen and Li [8], Li and Zhu [25], Li [23], Lin [28], Chen, Li and Ou [11], Chen, Li and Li [10] and many others. For more literatures on the classification of solutions and Liouville type theorems for various PDE and IE problems via the methods of moving planes or spheres, please refer to [5, 7, 9, 6, 14, 15, 16, 17, 31, 34, 41] and the references therein.

Our classification result for (3) is the following theorem.

**Theorem 1.3.** Assume $d \geq 1$ and $0 < \alpha < \frac{d}{2}$. Suppose $u \in L^{\frac{2d}{d-\alpha}}(\mathbb{R}^d)$ is a positive solution of (3). Then, $u$ is radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^d$, in particular, the positive solution $u$ must assume the following form

$$u(x) = \mu \frac{d-\alpha}{x} Q(\mu(x-x_0))$$

for some $\mu > 0$,

where $Q(x) = \sqrt{\frac{1}{\Gamma_{d-\alpha} I(\alpha) I\left(\frac{d-\alpha}{d}\right) \left(\frac{1}{\Gamma(\xi)}\right)^{\frac{d-\alpha}{2}}}}$ with $I(s) := \frac{\pi^{\frac{d-2}{2}} I\left(\frac{d-2}{2}\right)}{\Gamma(d-s)}$ for $0 < s < \frac{d}{2}$. 

Combining Theorem 1.2, Remark 1 and Theorem 1.3 for integral equation (3) with Theorem 1.1, we conclude the following corollary for the PDE (1) immediately.

**Corollary 1.** Assume \( d \geq 1 \) and \( 0 < \alpha < \frac{d}{2} \). If \( u \in H^\frac{\alpha}{2} (\mathbb{R}^d) \) is a weak solution of (1), then \( u \in H^{\alpha,p}(\mathbb{R}^d) \) for any \( 2 \leq p < \infty \), in particular, \( u \in C(\mathbb{R}^d) \); moreover, if \( 1 \leq \alpha < \frac{d}{2} \), then \( u \in C^{[\alpha]}(\mathbb{R}^d) \). If \( u \in H^\frac{\alpha}{2} (\mathbb{R}^d) \) is a positive weak solution of (1), then \( u \) is radially symmetric and monotone decreasing about some point \( x_0 \in \mathbb{R}^d \), in particular, the positive solution \( u \) must assume the following form

\[
u(x) = \mu \frac{d-\alpha}{d-\alpha} Q(\mu(x - x_0)) \quad \text{for some } \mu > 0.
\]

The classification of the solutions to (1) would provide the best constants and extremal functions for the corresponding Hardy-Littlewood-Sobolev inequality (see [27]). We define the norm \( \|u\|_{L^p} := \|(V_{\alpha} \ast |u|^2)|u|^2\|^\frac{1}{2} \|_{L^p(\mathbb{R}^d)} \) for \( 0 < \alpha < \frac{d}{2} \) with potential \( V_{\alpha} = \frac{1}{|x|^{\alpha}} \). For any \( u \in H^\frac{\alpha}{2} (\mathbb{R}^d) \), we have the following Hardy-Littlewood-Sobolev inequality (see [27, 40])

\[
\|u\|_{L^p} \leq S_{\alpha,d}^{-1}\|(-\Delta)^\frac{\alpha}{2} u\|_{L^2(\mathbb{R}^d)},
\]

where the best constant \( S_{\alpha,d} \) is given by

\[
S_{\alpha,d} = \inf_{u \in H^\frac{\alpha}{2} (\mathbb{R}^d), u \neq 0} \frac{\|(-\Delta)^\frac{\alpha}{2} u\|_{L^2(\mathbb{R}^d)}}{\|u\|_{L^p}}.
\]

The equation (1) is the corresponding Euler-Lagrange equation for the minimization problem described in (7). By using the concentration-compactness arguments in [29, 30] and the uniqueness of spherically symmetric positive solutions of the Euler-Lagrange equation (1) derived in Corollary 1, we obtain that

\[
\mathcal{M} = \left\{ e^{i\theta} \mu \frac{d-\alpha}{d-\alpha} Q(\mu(x - y)) : \forall \theta \in (-\pi, \pi), \mu > 0, y \in \mathbb{R}^d \right\}
\]

is the set of minimizers for \( S_{\alpha,d} \). Since minimization problem (7) can be attained by the extremal function \( Q \), one can deduce from the definition of \( S_{\alpha,d} \) and equation (1) that

\[
\|(-\Delta)^\frac{\alpha}{2} Q\|_{L^2(\mathbb{R}^d)} = S_{\alpha,d}\|Q\|_{L^p}, \quad \|(-\Delta)^\frac{\alpha}{2} Q\|_{L^2(\mathbb{R}^d)}^2 = \|Q\|_{L^p}^4.
\]

Therefore, the best constant \( S_{\alpha,d} \) for Hardy-Littlewood-Sobolev inequality (6) can be calculated explicitly as

\[
S_{\alpha,d} = \|(-\Delta)^\frac{\alpha}{2} Q\|_{L^2(\mathbb{R}^d)}^\frac{1}{4} = \|Q\|_{L^p}.
\]

By further calculations, we can deduce from (9) the following corollary.

**Corollary 2.** The best constant in the Hardy-Littlewood-Sobolev inequality (6) is

\[
S_{\alpha,d} = R_{\alpha,d}^\frac{\alpha+2d}{4\alpha} I(\alpha)^{-\frac{1}{2}} \left[ I\left(\frac{d-\alpha}{2}\right)\right]^{-\frac{1}{4}} \frac{R_{\alpha,d}^\frac{\alpha}{2}}{S_{\alpha,d}^\frac{d-\alpha}{d}},
\]

where \( R_{\alpha,d} := \left( \frac{1}{2\pi} \right)^\frac{\alpha}{2} \frac{\Gamma(d)}{\Gamma(d-\alpha)} \right)^{\frac{d-\alpha}{d}} \left( \frac{\Gamma\left(\frac{d+\alpha}{2}\right)\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{d+\alpha-\alpha}{2}\right)} \right)^{\frac{d-\alpha}{d}} \) is the best constant in the Sobolev inequality

\[
\|f\|_{L^d(\mathbb{R}^d)} \leq \tilde{S}_{\alpha,d} \|(-\Delta)^\frac{\alpha}{2} f\|_{L^2(\mathbb{R}^d)}
\]

for any \( f \in H^\frac{\alpha}{2} (\mathbb{R}^d) \).
The rest of our paper is organized as follows. In Section 2, we establish the equivalence between the PDE (1) and the integral equation (3), and thus prove Theorem 1.1. In Section 3, we will prove the regularity lifting theorem for (3), that is, Theorem 1.2. Section 4 is devoted to carrying out the proof of Theorem 1.3 and deriving the complete classification results for positive solutions to the integral equation (3).

In what follows, the notation $C$ denotes a general positive constant that may depend on $d$, $\alpha$ and the solution $u$ itself. The constant $C$ will usually change from line to line.

2. Proof of Theorem 1.1. We will show the equivalence between the PDE (1) and the integral equation (3) in this section.

Proof. (i) Suppose $0 < \alpha < \frac{d}{2}$ and $u \in H^\frac{\alpha}{2}(\mathbb{R}^d)$ is a weak solution to (1). For any $\phi \in C^\infty_0(\mathbb{R}^d)$, let

$$
\psi(x) := (-\Delta)^{-\frac{\alpha}{2}} \phi(x) = \int_{\mathbb{R}^d} \frac{R_{\alpha,d}}{|x-y|^{d-\alpha}} \phi(y)dy,
$$

then $\psi \in H^\alpha(\mathbb{R}^d) \subset H^\frac{\alpha}{2}(\mathbb{R}^d)$ and satisfy $(2\pi|\xi|)^\alpha \mathcal{F}[\psi](\xi) = \mathcal{F}[\phi](\xi)$ (see [40]), where

$$
\mathcal{F}[f](\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x \cdot \xi}dx
$$

is the Fourier transform of $f$. By the definition (2) of weak solution to (1), we have

$$
\int_{\mathbb{R}^d} (-\Delta)^\frac{\alpha}{2} u(-\Delta)^\frac{\alpha}{2} \psi dx = \int_{\mathbb{R}^d} (2\pi|\xi|)^\alpha \mathcal{F}[u]\mathcal{F}[\psi]d\xi = \int_{\mathbb{R}^d} \left( \frac{1}{|x|^{2\alpha}} * |u|^2 \right) u(x)\psi(x)dx.
$$

Integrating by parts of the left hand side and exchanging the order of integration of the right hand side yield that

$$
\int_{\mathbb{R}^d} u(x)\phi(x)dx = \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \frac{R_{\alpha,d}}{|x-y|^{d-\alpha}} \left( \int_{\mathbb{R}^d} \frac{|u(z)|^2}{|y-z|^{2\alpha}}dz \right)(y)u(y)dy \right] \phi(x)dx.
$$

Since $\phi \in C^\infty_0(\mathbb{R}^d)$ is arbitrary, we can deduce from the above formula that the weak solution $u$ also satisfies the integral equation (3).

(ii) Assume $0 < \alpha < \frac{d}{2}$ and $u \in H^\frac{\alpha}{2}(\mathbb{R}^d)$ is a solution to the integral equation (3). Applying Fourier transform to both sides of (3) (see [40]), we get

$$
\mathcal{F}[u](\xi) = (2\pi|\xi|)^{-\alpha} \mathcal{F}\left[ \left( \frac{1}{|x|^{2\alpha}} * |u|^2 \right) u \right](\xi).
$$

Thus, for any function $\phi \in C^\infty_0(\mathbb{R}^d)$, we have

$$
\int_{\mathbb{R}^d} (-\Delta)^\frac{\alpha}{2} u(-\Delta)^\frac{\alpha}{2} \phi dx = \int_{\mathbb{R}^d} \mathcal{F}\left[ \left( \frac{1}{|x|^{2\alpha}} * |u|^2 \right) u \right]\mathcal{F}[\phi]d\xi
$$

$$
= \int_{\mathbb{R}^d} \left( \frac{1}{|x|^{2\alpha}} * |u|^2 \right) u(x)\phi(x)dx,
$$

which implies that $u \in H^\frac{\alpha}{2}(\mathbb{R}^d)$ is also a weak solution to the PDE (1).

This finishes the proof of Theorem 1.1.
3. Proof of Theorem 1.2. Applying the following regularity lifting lemma by contracting operators (see [9, 39]), we will prove Theorem 1.2 in this section.

Lemma 3.1. (Regularity Lifting) Suppose $V$ is a Hausdorff topological vector space and there are two extended norms defined on $V$, namely, $\| \cdot \|_X, \| \cdot \|_Y : V \to [0, \infty]$. Let $X := \{ v \in V : \|v\|_X < \infty \}$ and $Y := \{ v \in V : \|v\|_Y < \infty \}$. Assume $X$ and $Y$ are complete under the norm $\| \cdot \|_X, \| \cdot \|_Y$ respectively and the convergence in $X$ or $Y$ implies the convergence in $V$. Let $T$ be a contraction map from $X$ and $Y$ into themselves. Suppose $f \in X$ and there exists $g \in Z := X \cap Y$ such that $f = Tf + g$ in $X$. Then $f$ also belongs to $Z$.

Lemma 3.2. (Hardy-Littlewood-Sobolev inequality, [27, 40]) Let $d \geq 1$, $0 < s < d$ and $1 < p < q < \infty$ be such that $\frac{d}{q} = \frac{d}{p} - s$. Then we have

$$\left\| \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-s}} \, dy \right\|_{L^q(\mathbb{R}^d)} \leq C_{d,s,p,q} \|f\|_{L^p(\mathbb{R}^d)}$$

for all $f \in L^p(\mathbb{R}^d)$.

Using Lemma 3.1 and Lemma 3.2, we will first prove the following Lemma.

Lemma 3.3. Assume $d \geq 1$ and $0 < \alpha < \frac{d}{3}$. Suppose $u \in L^{\frac{2d}{\alpha}}(\mathbb{R}^d)$ is a solution of (3), then $u \in L^p(\mathbb{R}^d)$ for any $\frac{2d}{\alpha} \leq p \leq \infty$.

Proof. Let $P(x) := \frac{1}{|x|^d} \ast |u|^2$. By Hardy-Littlewood-Sobolev inequality (see Lemma 3.2), $u \in L^{\frac{2d}{\alpha}}(\mathbb{R}^d)$ implies that $P \in L^{\frac{d}{\alpha}}(\mathbb{R}^d)$. For a large $L > 0$ to be determined later, we define

$$P_L(x) := P(x)1_{\{x \in \mathbb{R}^d, P(x) \geq L\}} \quad \text{and} \quad P_B(x) := P(x) - P_L(x),$$

where $1_A$ denotes the characteristic function of set $A$. Define the linear operator $T_L$ by $(T_L v)(x) := \int_{\mathbb{R}^d} \frac{R_{\alpha,d}}{|x-y|^d} P_L(y)v(y) \, dy$, then the solution $u \in L^{\frac{2d}{\alpha}}(\mathbb{R}^d)$ to (3) satisfies

$$u(x) = T_L u(x) + T_B(x), \quad (11)$$

where $T_B(x) := \int_{\mathbb{R}^d} \frac{R_{\alpha,d}}{|x-y|^d} P_B(y)u(y) \, dy$.

For any $\frac{d}{\alpha} < p < \infty$ and any function $v \in L^p(\mathbb{R}^d)$, we deduce from the Hardy-Littlewood-Sobolev and Hölder inequalities that

$$\|T_L v\|_{L^p(\mathbb{R}^d)} \leq C \|P_L v\|_{L^{\frac{2d}{\alpha}}(\mathbb{R}^d)} \leq C \|P_L\|_{L^{\frac{d}{\alpha}}(\mathbb{R}^d)} \|v\|_{L^p(\mathbb{R}^d)}. \quad (12)$$

One can choose a large number $L$, such that $C \|P_L\|_{L^{\frac{d}{\alpha}}(\mathbb{R}^d)} \leq \frac{1}{2}$, and hence arrive at

$$\|T_L v\|_{L^p(\mathbb{R}^d)} \leq \frac{1}{2} \|v\|_{L^p(\mathbb{R}^d)}. \quad (13)$$

That is, the linear operator $T_L$ is a contracting operator from $L^p$ to $L^p$ for any $\frac{d}{\alpha} < p < \infty$.

For $0 < \alpha < \frac{d}{3}$, since $P_B(x)$ is bounded, Hardy-Littlewood-Sobolev inequality yields

$$\|T_B\|_{L^{\frac{2d}{\alpha}}(\mathbb{R}^d)} \leq C \|P_B u\|_{L^{\frac{2d}{\alpha}}(\mathbb{R}^d)} \leq C \|u\|_{L^{\frac{2d}{\alpha}}(\mathbb{R}^d)}, \quad (14)$$

moreover, one can deduce from (11) and (13) that

$$\|T_B\|_{L^{\frac{2d}{\alpha}}(\mathbb{R}^d)} \leq \|u\|_{L^{\frac{2d}{\alpha}}(\mathbb{R}^d)} + \|T_L u\|_{L^{\frac{2d}{\alpha}}(\mathbb{R}^d)} \leq \frac{3}{2} \|u\|_{L^{\frac{2d}{\alpha}}(\mathbb{R}^d)}, \quad (15)$$

Finally, we discuss the case of $\alpha = \frac{d}{3}$.
4. Sobolev embedding. For any \( \frac{2d}{d-\alpha} \leq p \leq \frac{2d}{d-3\alpha} \). As a consequence, we can infer from (13) and Lemma 3.1 that

\[
u \in L^p(\mathbb{R}^d), \quad \forall \frac{2d}{d-\alpha} \leq p \leq \frac{2d}{d-3\alpha} \quad \text{if} \quad 0 < \alpha < \frac{d}{3}.
\]

Then, a completely similar argument as above yields that

\[
\begin{aligned}
&u \in L^p(\mathbb{R}^d), \quad \forall \frac{2d}{d-\alpha} \leq p < \infty \quad \text{if} \quad 3\alpha < d \leq 5\alpha, \\
&u \in L^p(\mathbb{R}^d), \quad \forall \frac{2d}{d-\alpha} \leq p \leq \frac{2d}{d-2\alpha} \quad \text{if} \quad 5\alpha < d.
\end{aligned}
\]

Continuing in this way, we add 2\( \alpha \) to the dimension \( d \) at each step and finally get

\[
u \in L^p(\mathbb{R}^d), \quad \forall \frac{2d}{d-\alpha} \leq p < \infty, \quad \forall \, d > 3\alpha,
\]

which also implies that \( P(x) := |x|^{-2\alpha} * |u|^2 \in L^p(\mathbb{R}^d) \) for any \( \frac{d}{3} \leq r < \infty \).

Now, in order to prove Lemma 3.3, we only need to show that \( u \in L^\infty(\mathbb{R}^d) \) for \( 0 < \alpha < \frac{d}{3} \). To this end, we decompose \( u \) into two parts:

\[
u(x) = \int_{B_1(x)} \frac{R_{\alpha,d}P(y)}{|x-y|^{d-\alpha}} u(y) dy + \int_{\mathbb{R}^d \setminus B_1(x)} \frac{R_{\alpha,d}P(y)}{|x-y|^{d-\alpha}} u(y) dy
\]

for arbitrary \( x \in \mathbb{R}^d \). Since \( 0 < \alpha < \frac{d}{3} \), the boundedness of \( u_1 \) and \( u_2 \) follow directly from Hölder inequality and (18):

\[
|u_1(x)| \leq R_{\alpha,d} \left( \int_{B_1(x)} \frac{1}{|x-y|^{2d/(d-\alpha)}} dy \right)^{2d/(d-\alpha)} \| P \|_{L^{2d/(d-\alpha)}(\mathbb{R}^d)} \| u \|_{L^{2d/(d-\alpha)}(\mathbb{R}^d)} \leq C,
\]

\[
|u_2(x)| \leq R_{\alpha,d} \left( \int_{\mathbb{R}^d \setminus B_1(x)} \frac{1}{|x-y|^{2d/(d-\alpha)}} dy \right)^{2d/(d-\alpha)} \| P \|_{L^{2d/(d-\alpha)}(\mathbb{R}^d)} \| u \|_{L^{2d/(d-\alpha)}(\mathbb{R}^d)} \leq C
\]

for any \( x \in \mathbb{R}^d \). Therefore, \( u \in L^\infty(\mathbb{R}^d) \). This finishes the proof of Lemma 3.3. □

By Lemma 3.3, we have \( P(x) := |x|^{-2\alpha} * |u|^2 \in L^p(\mathbb{R}^d) \) for any \( \frac{d}{\alpha} \leq p < \infty \), and hence the integral equation (3) implies that \( (-\Delta)^{\frac{\alpha}{2}} u \in L^p(\mathbb{R}^d) \) for any \( \frac{2d}{d+\alpha} \leq p < \infty \). Thus \( u \in H^{\alpha,p}(\mathbb{R}^d) \) for any \( \frac{2d}{d+\alpha} \leq p < \infty \), in particular, \( u \in C(\mathbb{R}^d) \) by Sobolev embedding. Moreover, if \( 1 \leq \alpha < \frac{d}{2} \), then \( \nabla u \in L^p(\mathbb{R}^d) \) for any \( \frac{2d}{d+\alpha} \leq p < \infty \). By (3), Hardy-Littlewood-Sobolev and Hölder inequalities, we have

\[
\nabla (-\Delta)^{\frac{\alpha}{2}} u = 2 \left( \frac{1}{|x|^{2\alpha}} * (u \nabla u) \right) u + \left( \frac{1}{|x|^{2\alpha}} * |u|^2 \right) \nabla u \in L^p(\mathbb{R}^d), \quad \forall \frac{2d}{d+\alpha} \leq p < \infty.
\]

Therefore, \( u \in H^{\alpha+1,p}(\mathbb{R}^d) \) for any \( \frac{2d}{d+\alpha} \leq p < \infty \), in particular, \( u \in C^{(\alpha)}(\mathbb{R}^d) \) by Sobolev embedding.

This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3. In this section, we will carry out our proof of Theorem 1.3. For clarity of presentation, we divide this Section into two Sub-sections.
4.1. **Radial symmetry of positive solutions.** We apply the method of moving planes in integral forms (see [11, 12]) to integral equation (3) and carry out the process of moving plane in the $x_1$ direction. For this purpose, we need some definitions.

Let $\lambda$ be an arbitrary real number and let the moving plane be

$$T_\lambda := \{ x \in \mathbb{R}^d : x_1 = \lambda \}.$$  

We denote

$$\Sigma_\lambda := \{ x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : x_1 < \lambda \},$$

and let

$$x^\lambda := (2\lambda - x_1, x_2, \cdots, x_d)$$

be the reflection of $x$ about the plane $T_\lambda$, and define

$$u_\lambda(x) := u(x^\lambda), \quad P(x) := \frac{1}{|x|^{2\alpha}} * |u|^2, \quad P_\lambda(x) := P(x^\lambda).$$

Let $\omega_\lambda(x) := u_\lambda(x) - u(x)$. By properly exploiting some global properties of the integral equation, we will first show that, for $\lambda$ sufficiently negative,

$$\omega_\lambda(x) \geq 0, \quad \forall \ x \in \Sigma_\lambda. \quad (20)$$

Then, we start moving the plane $T_\lambda$ from near $x_1 = -\infty$ to the right as long as (20) holds, until its limiting position and finally derive symmetry. Therefore, the moving plane process can be divided into two steps.

**Step 1. Start moving the plane from near $x_1 = -\infty$.** Define

$$\Sigma^-_\lambda := \{ x \in \Sigma_\lambda | \omega_\lambda(x) < 0 \}. \quad (21)$$

We will show that for $\lambda$ sufficiently negative, $\Sigma^-_\lambda$ must be empty.

One can observe that, for any $x \in \Sigma_\lambda$,

$$u(x) - u_\lambda(x)$$

$$= \int_{\Sigma_\lambda} \left( \frac{R_{\alpha,d}}{|x-y|^{d-\alpha}} - \frac{R_{\alpha,d}}{|x^\lambda - y|^{d-\alpha}} \right) (P(y)u(y) - P_\lambda(y)u_\lambda(y)) dy$$

$$= \int_{\Sigma_\lambda} \left( \frac{R_{\alpha,d}}{|x-y|^{d-\alpha}} - \frac{R_{\alpha,d}}{|x^\lambda - y|^{d-\alpha}} \right) P(y)(u(y) - u_\lambda(y)) dy$$

$$+ \int_{\Sigma_\lambda} \left( \frac{R_{\alpha,d}}{|x-y|^{d-\alpha}} - \frac{R_{\alpha,d}}{|x^\lambda - y|^{d-\alpha}} \right) (P(y) - P_\lambda(y)) u_\lambda(y) dy$$

$$= \int_{\Sigma_\lambda} \left( \frac{R_{\alpha,d}}{|x-y|^{d-\alpha}} - \frac{R_{\alpha,d}}{|x^\lambda - y|^{d-\alpha}} \right) P(y)(u(y) - u_\lambda(y)) dy$$

$$+ \int_{\Sigma_\lambda} \left( \frac{R_{\alpha,d}}{|x-y|^{d-\alpha}} - \frac{R_{\alpha,d}}{|x^\lambda - y|^{d-\alpha}} \right) \times$$

$$\left\{ \int_{\Sigma_\lambda} \left( \frac{1}{|y-z|^{2\alpha}} - \frac{1}{|y^\lambda - z|^{2\alpha}} \right) \left( |u(z)|^2 - |u_\lambda(z)|^2 \right) dz \right\} u_\lambda(y) dy,$$
Therefore, we get from the above formula that, for any $x \in \Sigma^-$,

$$0 < u(x) - u_\lambda(x) \leq \int_{\Sigma^-} \frac{R_{\alpha,d}}{|x-y|^{d-\alpha}} P(y) (u(y) - u_\lambda(y)) dy$$

(23)

$$+ \int_{\Sigma^+} \left( \frac{R_{\alpha,d}}{|x-y|^{d-\alpha}} - \frac{R_{\alpha,d}}{|x-y'-\alpha|^{d-\alpha}} \right) \left[ \int_{\Sigma^-} \frac{1}{|y-z|^{\alpha}} \left( |u(z)|^2 - |u_\lambda(z)|^2 \right) dz \right] u_\lambda(y) dy$$

$$\leq \int_{\Sigma^-} \frac{R_{\alpha,d}}{|x-y|^{d-\alpha}} P(y) (u(y) - u_\lambda(y)) dy$$

$$+ \int_{\Sigma^+} \frac{2R_{\alpha,d}}{|x-y|^{d-\alpha}} \left[ \int_{\Sigma^-} \frac{\omega_\lambda(z) u(z)}{|y-z|^{2\alpha}} dz \right] u_\lambda(y) dy.$$  

We apply Hardy-Littlewood-Sobolev inequality twice and Hölder inequality to the above (23) and obtain, for any $\max \{ \frac{d}{d-\alpha}, \frac{2d}{d-\alpha} \} < q < \infty$,

$$\|\omega_\lambda\|_{L^q(\Sigma^-)} \leq C \|P\omega_\lambda\|_{L^\frac{d}{d+\alpha}(\Sigma^-)} + C \left( \int_{\Sigma^-} \frac{\omega_\lambda(z) u(z)}{|y-z|^{2\alpha}} dz \right) u_\lambda(y) \|_{L^\frac{d}{d+\alpha}(\Sigma^-)}$$

(24)

$$\leq C \|P\|_{L^\frac{d}{d+\alpha}(\Sigma^-)} \|\omega_\lambda\|_{L^q(\Sigma^-)} + C \|\omega_\lambda u\|_{L^r(\Sigma^-)} \|u\|_{L^{\frac{2d}{2d+(d-\alpha)q}}(\mathbb{R}^d)}$$

$$\leq C \left[ \|P\|_{L^\frac{d}{d+\alpha}(\Sigma^-)} + \|\omega_\lambda\|_{L^q(\Sigma^-)} \|u\|_{L^{\frac{2d}{2d+(d-\alpha)q}}(\mathbb{R}^d)} \right] \|\omega_\lambda u\|_{L^r(\Sigma^-)},$$

where $r = \frac{2dq}{2d-(d-\alpha)q}$ and $s = \frac{2dq}{2d+(d-\alpha)q}$. Since $u \in L^{\frac{2d}{2d+(d-\alpha)q}}(\mathbb{R}^d)$ and $P \in L^{\frac{d}{d+\alpha}}(\mathbb{R}^d)$, we can choose $R > 0$ sufficiently large, such that for $\lambda \leq -R$, we have

$$C \left[ \|P\|_{L^{\frac{d}{d+\alpha}}(\Sigma^-)} + \|\omega_\lambda\|_{L^q(\Sigma^-)} \|u\|_{L^{\frac{2d}{2d+(d-\alpha)q}}(\mathbb{R}^d)} \right] \leq \frac{1}{2}. $$

(25)

Therefore, (24) and (25) imply that $\|\omega_\lambda\|_{L^q(\Sigma^-)} = 0$, and hence $\mu(\Sigma^-) = 0$ for $\lambda \leq -R$. Furthermore, we can deduce from (22) that, for any $x \in \Sigma^-$,

$$u(x) - u_\lambda(x) \leq 0,$$

thus $\Sigma^-$ is empty and (20) holds for $\lambda \leq -R$. This completes Step 1.

**Step 2. Move the plane to the limiting position to derive symmetry.** Now we move the plane $T_\lambda$ to the right as long as (20) holds. Let

$$\lambda_0 := \sup \{ \lambda \in \mathbb{R} \mid \omega_\rho \geq 0 \text{ in } \Sigma_\rho, \forall \rho \leq \lambda \}.$$  

(26)

By applying a entirely similar argument as in Step 1, we can also start moving the plane from near $x_1 = +\infty$ to the left, thus we must have $\lambda_0 < \infty$. Now, we will show that

$$\omega_{\lambda_0}(x) \equiv 0, \quad \forall x \in \Sigma_{\lambda_0}.$$  

(27)

We prove (27) by contradiction arguments. Suppose on contrary that $\omega_{\lambda_0} \geq 0$ but $\omega_{\lambda_0}$ is not identically zero in $\Sigma_{\lambda_0}$. In fact, by (22), we have $\omega_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$. We will obtain a contradiction with (26) via showing that the plane $T_{\lambda}$ can be moved
a little bit further to the right, more precisely, there exists an \( \varepsilon > 0 \) small enough such that \( \omega_\lambda \geq 0 \) in \( \Sigma_\lambda \) for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon] \).

It can be clearly seen from (24) and (25) in Step 1 that, our primary task is to prove that, one can choose \( \varepsilon > 0 \) sufficiently small such that, for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon] \),

\[
\|P\|_{\mathcal{L}^{\frac{d}{d-n}}(\Sigma_\lambda^{-})} + \|u\|_{\mathcal{L}^{\frac{d}{d-n}}(\mathbb{R}^d)} \leq \frac{1}{2C},
\]

(28)

where the constant \( C \) is the same as in (24) and (25).

Since \( u \in \mathcal{L}^{\frac{d}{d-n}}(\mathbb{R}^d) \) and \( P \in \mathcal{L}^{\frac{d}{d-n}}(\mathbb{R}^d) \), we can choose \( R > 0 \) large enough, such that

\[
\left( \int_{|x| \geq R} |P(x)|^\frac{d}{d-n} \, dx \right)^\frac{1}{\frac{d}{d-n}} + \|u\|_{\mathcal{L}^{\frac{d}{d-n}}(\mathbb{R}^d)} \left( \int_{|x| \geq R} |u(x)|^\frac{2d}{d-n} \, dx \right)^\frac{d-n}{2d} < \frac{1}{4C}.
\]

(29)

Now fix this \( R \), in order to derive (28), we only need to show that

\[
\lim_{\lambda \to \lambda_0^+} \mu(\Sigma_\lambda^{-} \cap B_R(0)) = 0.
\]

(30)

To this end, we define \( E_x := \{ x \in \Sigma_{\lambda_0} \cap B_R(0) \mid \omega_{\lambda_0}(x) > \delta \} \) and \( F_\delta := (\Sigma_{\lambda_0} \cap B_R(0)) \setminus E_\delta \) for any \( \delta > 0 \), and let \( D_\lambda := (\Sigma_\lambda \setminus \Sigma_{\lambda_0}) \cap B_R(0) \) for any \( \lambda > \lambda_0 \). Then, one can easily verify the following properties:

\[
\lim_{\delta \to 0^+} \mu(F_\delta) = 0, \quad \lim_{\lambda \to \lambda_0^+} \mu(D_\lambda) = 0;
\]

\[
\Sigma_\lambda^{-} \cap B_R(0) = \Sigma_\lambda^{-} \cap (E_\delta \cup F_\delta \cup D_\lambda) \subset (\Sigma_\lambda^{-} \cap E_\delta) \cup F_\delta \cup D_\lambda.
\]

(31)

For an arbitrarily fixed \( \eta > 0 \), by (31), one can choose a \( \delta > 0 \) small enough such that \( \mu(F_\delta) \leq \eta \). For this fixed \( \delta \), we are to prove

\[
\lim_{\lambda \to \lambda_0^+} \mu(\Sigma_\lambda^{-} \cap E_\delta) = 0.
\]

(32)

Indeed, one can observe that \( u(x^{\lambda_0}) - u(x^{\lambda}) = \omega_{\lambda_0}(x) - \omega_{\lambda}(x) > \delta \) for all \( x \in \Sigma_\lambda^{-} \cap E_\delta \). It follows that \( (\Sigma_\lambda^{-} \cap E_\delta) \subset G_\delta := \{ x \in B_R(0) \mid |u(x^{\lambda_0}) - u(x^\lambda)| > \delta \} \). By Chebyshev inequality, we get

\[
\mu(G_\delta) \leq \frac{1}{\delta} \int_{G_\delta} |u(x^{\lambda_0}) - u(x^\lambda)|^\frac{2d}{d-n} \, dx
\]

\[
\leq \frac{1}{\delta} \int_{\mathbb{R}^d} |u(x) - u(x + 2(\lambda - \lambda_0)e_1)|^\frac{2d}{d-n} \, dx,
\]

where \( e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^d \), and hence \( \lim_{\lambda \to \lambda_0^+} \mu(G_\delta) = 0 \), from which (33) follows.

Therefore, by (31), (32) and (33), we have

\[
\lim_{\lambda \to \lambda_0^+} \mu(\Sigma_\lambda^{-} \cap B_R(0)) \leq \mu(F_\delta) \leq \eta.
\]

(34)

Since \( \eta > 0 \) is arbitrarily chosen, (30) follows immediately from (34). Combining (30) with (29), we arrive at (28).

From the last inequality of (24), we have, for any \( \max\{ \frac{d}{d-n}, \frac{2d}{d-n} \} < q < \infty \),

\[
\|\omega_\lambda\|_{L_q(\Sigma_\lambda^{-})} \leq C \left[ \|P\|_{\mathcal{L}^{\frac{d}{d-n}}(\Sigma_\lambda^{-})} + \|u\|_{\mathcal{L}^{\frac{d}{d-n}}(\mathbb{R}^d)} \|u\|_{\mathcal{L}^{\frac{d}{d-n}}(\Sigma_\lambda^{-})} \right] \|\omega_\lambda\|_{L_q(\Sigma_\lambda^{-})}.
\]

By (28) and the above inequality, we deduce that, there exists an \( \varepsilon > 0 \) sufficiently small such that, for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon] \), \( \|\omega_\lambda\|_{L_q(\Sigma_\lambda^{-})} = 0 \), thus \( \mu(\Sigma_\lambda^{-}) = 0 \). Furthermore, by (22), we have \( \Sigma_\lambda^{-} \) is actually empty, hence \( \omega_\lambda \geq 0 \) in \( \Sigma_\lambda \) for all
\( \lambda \in [\lambda_0, \lambda_0 + \varepsilon] \). This contradicts with the definition (26) of \( \lambda_0 \). Therefore, (27) must hold. In fact, one can also easily verify that \( P(x) = P_{\lambda_0}(x) \) for all \( x \in \Sigma_{\lambda_0} \).

Since the equation (3) is invariant under rotation, the \( x_1 \) direction can be chosen arbitrarily, we conclude that the positive solution \( u \in L^\frac{2d}{d-2} (\mathbb{R}^d) \) must be radially symmetric and monotone decreasing about some point \( x_0 \in \mathbb{R}^d \).

### 4.2. Uniqueness of positive solutions.

Now we will show the uniqueness of positive solution modulo scalings and translations. We first prove the following lemma.

**Lemma 4.1.** For every \( \mu > 0 \) and \( y \in \mathbb{R}^d \), \( \mu^{\frac{d-2}{2}} Q(\mu(x-y)) = \) a solution to (3), where \( Q(x) = \sqrt{R_{\alpha,d}(I(2\pi|x|))} \frac{1}{1+|x|^2} (\frac{1}{1+|x|^2})^{\frac{d-\alpha}{2}} \) with \( I(s) := \pi^\frac{d}{2} \Gamma(\frac{d+\alpha}{2}) \Gamma(d-s) \) for \( 0 < s < \frac{d}{2} \).

**Proof.** By the invariance of the equation (3), we may assume \( \mu = 1 \) and \( y = 0 \). By using the Fourier transforms of the kernels of Riesz and Bessel potentials, we will show that \( Q(x) \) is a solution to (3). From [40], we know that the Fourier transforms of the kernel for Riesz potentials \( \frac{1}{|x|^\alpha} \) is \( \beta(2\pi|x|)^{-\alpha} \) in the sense that

\[
\int_{\mathbb{R}^d} |\xi|^{-d+\alpha} \hat{\phi}(\xi) d\xi = \int_{\mathbb{R}^d} \beta(2\pi|x|)^{-\alpha} \mathcal{F}[\phi](x) dx,
\]

where \( \beta(\alpha) = \pi^{\frac{d}{2}} \Gamma(\frac{d+\alpha}{2}) \Gamma(d-\alpha) \) and \( 0 < \alpha < d \); and the kernel of Bessel potential is given by

\[
G_\alpha(x) = \frac{1}{(4\pi)^\frac{d}{2} \Gamma(\frac{d}{2})} \int_0^\infty e^{-\frac{\pi x^2 s^2}{4}} e^{-\frac{s}{\pi} \delta \frac{\alpha d}{2}} d\delta \quad \text{with} \quad \mathcal{F}[G_\alpha](x) = (1+4\pi^2|x|^2)^{-\frac{d}{2}},
\]

where \( \alpha > 0 \). By (35), (36) and integral variable substitution, we get for any \( 0 < s < \frac{d}{2} \),

\[
\int_{\mathbb{R}^d} \frac{1}{|x-y|^{2\alpha}} (\frac{1}{1+|y|^2}) \frac{1}{1+|x-y|^2} dx = \int_{\mathbb{R}^d} \frac{1}{|y|^{2\alpha}} (\frac{1}{1+|x|^2}) \frac{1}{1+|x|^2} dy
\]

\[
= \beta(d-2s) \int_{\mathbb{R}^d} \left( 2\pi |y| \right)^{-(d-2s)} \mathcal{F} \left( \frac{y e^{-2\pi i x y}}{1+|x|^2} \right) dy
\]

\[
= \beta(d-2s) \int_{\mathbb{R}^d} \frac{1}{|y|^{d-2s}} \Gamma(d-s) \int_0^\infty e^{-\frac{\pi x^2 s^2}{4}} \frac{e^{-\frac{s}{\pi} \delta \frac{\alpha d}{2}}}{\delta} e^{-2\pi i x y} dy
\]

\[
= \beta(d-2s) \Gamma(s) \frac{\Gamma(d-s)}{2^{d-2s} \Gamma(d-s)} \int_0^\infty e^{-\frac{s}{\pi} \delta \frac{\alpha d}{2}} \frac{\Gamma(s)}{\delta} e^{-2\pi i x y} dy
\]

\[
= \beta(d-2s) \Gamma(s) \frac{1}{\Gamma(d-s)} \left( \frac{1}{1+|x|^2} \right)^s = \frac{\pi^{\frac{d}{2}} \Gamma(d-\alpha)}{\Gamma(d-s)} \left( \frac{1}{1+|x|^2} \right)^s =: I(s) \left( \frac{1}{1+|x|^2} \right)^s.
\]

By (37) and direct calculations, we get

\[
\int_{\mathbb{R}^d} \frac{R_{\alpha,d}}{|x-y|^{d-\alpha}} \left[ \int_{\mathbb{R}^d} \frac{1}{|y-z|^{2\alpha}} \left| Q(z) \right|^2 dz \right] Q(y) dy = \int_{\mathbb{R}^d} \frac{1}{R_{\alpha,d} I(\alpha) I(d-\alpha)} \left[ \int_{\mathbb{R}^d} \frac{1}{|y|^2} \left( \frac{1}{1+|y|^2} \right)^{\frac{d-\alpha}{2}} dy \right] dy
\]

\[
= \sqrt{R_{\alpha,d} I(\alpha) I(d-\alpha)} \left( \frac{1}{1+|x|^2} \right)^\frac{d-\alpha}{2} = Q(x).
\]
Therefore, $Q$ solves the equation (3). This completes the proof of Lemma 4.1. □

In subsection 4.1, we have proved that any positive solution $u \in L^{\frac{2d}{d-\alpha}}(\mathbb{R}^d)$ to (3) must be radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^d$. Now we show that $u$ has the desired asymptotic behavior at $\infty$, that is, it satisfies

$$\lim_{|x| \to \infty} |x|^{d-\alpha}u(x) = u_\infty < \infty$$

(39)

for some positive number $u_\infty$.

Suppose the asymptotic property (39) does not hold. Assume $x^1$ and $x^2$ are any two different points in $\mathbb{R}^d$ and let $x^0$ be the midpoint of the line segment $\overline{x^1x^2}$. Consider the Kelvin type transform centered at $x^0$:

$$U(x) = \frac{1}{|x-x^0|^{d-\alpha}}u\left(\frac{x-x^0}{|x-x^0|^2} + x^0\right).$$

Then $U(x)$ must have a singularity at $x^0$ and $U \in L^{\frac{2d}{d-\alpha}}(\mathbb{R}^d)$ is also a positive solution to integral equation (3). By the same arguments as in subsection 4.1, we can deduce that $U$ must be radially symmetric and monotone decreasing about its singular point $x^0$ and hence $u(x^1) = u(x^2)$. Since $x^1$ and $x^2$ are arbitrarily chosen in $\mathbb{R}^d$, $u$ must be constant, thus $u \equiv 0$, which contradicts with $u > 0$.

By the asymptotic behavior (39) and the assumption $u \in L^{\frac{2d}{d-\alpha}}(\mathbb{R}^d)$, we infer that $u \in L^p(\mathbb{R}^d)$ for any $\frac{d}{d-\alpha} < p \leq \frac{2d}{d-\alpha}$. By completely similar arguments as in the proof of Theorem 1.2 (see Remark 1), it is easy to deduce that for $0 < \alpha < \frac{d}{2}$,

$$u \in L^p(\mathbb{R}^d) \ \forall \ \frac{d}{d-\alpha} < p \leq \infty \ \text{and} \ u \in H^{\alpha,p}(\mathbb{R}^d) \ \forall \ \frac{d}{d-\alpha} < p < \infty,$$

in particular, $u \in C(\mathbb{R}^d)$.

Let $u \in L^{\frac{2d}{d-\alpha}}(\mathbb{R}^d)$ be a positive solution to (3), then $u$ must be continuous, radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^d$ and satisfy

$$\lim_{|x| \to \infty} |x|^{d-\alpha}u(x) = u_\infty < \infty.$$ 

We define $\tilde{u}(x) := \mu^{-\frac{d-\alpha}{d}}\tilde{u}(\mu^{-1}x + x_0)$ with $\mu = \left[\frac{u(x_0)}{u_\infty}\right]^\frac{1}{\frac{d-\alpha}{d}}$, then $\tilde{u} \in C(\mathbb{R}^d)$ is also a positive solution to (3) with radial symmetry about the origin and $\lim_{|x| \to \infty} |x|^{d-\alpha}\tilde{u}(x) = \tilde{u}_\infty = \tilde{u}(0)$. We will show that

$$\tilde{u}(x) \equiv Q(x), \quad x \in \mathbb{R}^d.$$ (40)

We will use the arguments from Chen, Li and Ou [11] (which is an analytic method that is different from Lieb’s approach in [27]) to prove (40). This method is also relevant to Ou [38].

First, by slightly modifying the proof of Lemma 3.1 in [11], we can prove the following lemma, which reveals an essential property shared by $\tilde{u}$ and the standard solution $Q$.

**Lemma 4.2.** Assume $v$ is a positive solution to (3) centered at 0, which satisfies the assumptions of Theorem 1.3. Let $a \in \mathbb{R}^d$ and $s^{d-\alpha}v(a) = v_\infty$, then

$$v(sx + a) = \frac{1}{|x|^{d-\alpha}}v\left(\frac{sx}{|x|^2} + a\right).$$ (41)

**Proof.** We first assume $a = 0$. Let $x^0 \neq 0$ be an arbitrarily fixed point in $\mathbb{R}^d$ and $e := \frac{x^0}{|x^0|}$. Define

$$w(x) = \frac{1}{|x|^{d-\alpha}}v\left(s\left(\frac{x}{|x|^2} - e\right)\right),$$ (42)
then one easily verifies that \( w(0) = s^{\alpha-d}v_\infty = v(0) = w(e) \) and \( s^{\frac{d-\alpha}{2}} w \in L^{\frac{2d}{d+\alpha}}(\mathbb{R}^d) \) is also a positive solution to (3). Therefore, \( w \) must be radially symmetric with respect to some point \( \bar{x} \) that lies on the hyperplane \((e)^+ + \frac{1}{2}e\) through \( \frac{1}{2}e \) which is perpendicular to \( e \). Furthermore, since \( v \) is radially symmetric about \( 0 \), for any \( \frac{1}{2} < r < \infty \), consider two different points \( y^1, y^2 \in B_r(0) \cap B_r(e) \), we can deduce from (42) that

\[
  w(y^1) = \frac{1}{|y^1|^{d-\alpha}} v\left( s\left( \frac{y^1}{|y^1|^2} - e \right) \right) = \frac{1}{|y^2|^{d-\alpha}} v\left( s\left( \frac{y^2}{|y^2|^2} - e \right) \right) = w(y^2). \tag{43}
\]

Therefore, \( w(x) = w(|x - \frac{1}{2}e|) \) on the hyperplane \((e)^+ + \frac{1}{2}e\), and hence \( \bar{x} = \frac{1}{2}e \) and \( w \) is actually radially symmetric about \( \frac{1}{2}e \). There exists some \( \theta \in (\frac{1}{2}, \frac{3}{2}) \) such that \( |x^0| = \frac{1}{2}\theta \), then (42) implies that

\[
  w\left( \frac{1}{2} - \theta \right) e = \frac{1}{|\frac{1}{2} - \theta|^{d-\alpha}} v\left( s\left( \frac{\frac{1}{2} + \theta}{\frac{1}{2} - \theta} e \right) \right) = w\left( \frac{1}{2} + \theta \right) e = \frac{1}{|\frac{1}{2} + \theta|^{d-\alpha}} v\left( s\left( \frac{\frac{1}{2} - \theta}{\frac{1}{2} + \theta} e \right) \right),
\]

from which we deduce that

\[
  v(sx^0) = \frac{1}{|sx^0|^{d-\alpha}} v\left( \frac{sx^0}{|sx^0|^2} \right). \tag{44}
\]

If \( a \neq 0 \), then (41) follows by considering \( v(\cdot + a) \) instead of \( v \) itself. The proof of Lemma 4.2 is finished.

Using Lemma 4.2 and similar arguments as in the proof of Lemma 3.2 in [11], we can also infer that, if \( v \) is a positive solution to (3) satisfying the assumptions of Theorem 1.3 with symmetric center \( p \), then

\[
  v(p)v_\infty = \frac{1}{R_{\alpha,d}I(\alpha)I(\frac{d-\alpha}{2})}. \tag{45}
\]

Then, using Lemma 4.2 and (45), similar argument as in [11] (see p. 338 in [11]) yields that \( \tilde{u}(x) \leq Q(x) \) in \( \mathbb{R}^d \). Since (45) implies \( \tilde{u}(0) = Q(0) \) and \( \tilde{u}, Q \) are continuous solutions to integral equation (3), we must have

\[
  \tilde{u}(x) \equiv Q(x), \quad x \in \mathbb{R}^d. \tag{46}
\]

Therefore, we have proved (40) and the positive solution \( u(x) \equiv \mu^{\frac{2-\alpha}{\alpha}}Q(\mu(x - x_0)) \) in \( \mathbb{R}^d \).

This concludes the proof of Theorem 1.3.

Acknowledgments. The authors are grateful to the referees for their careful reading and valuable comments and suggestions that improved the presentation of the paper.

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Received May 2017; revised November 2017.

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