Scalar products of elementary distributions

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Abstract

The field of real numbers being extended as a larger commutative field, we investigate the possibility of defining a scalar product for the distributions of finite discrete support. Then we focus on the most simple possible extension (which is an ordered field), we provide explicit formulas for this scalar product, and we exhibit a structure of positive definite inner-product space. In a one dimensional application to the Schroedinger equation, the distributions supported by the origin are embedded into a bra-ket vector space, where the "singular" potential describing point interaction is defined in a natural way. A contact with the hyperreal numbers that arise in nonstandard analysis is possible but not essential, our extensions of $\mathbb{R}$ and $\mathbb{C}$ being obtained by a quite elementary method.

1 Introduction, Notation

Many difficulties of quantum mechanics stem from the need to consider unnormalizable state vectors, which implies going out of a strict Hilbert space formulation. To some extent, using Dirac’s formalism permits to ignore this complication and all physicists are familiar with wave "functions" that can actually be distributions, but this approach runs into the problem of their norm.

A more rigorous formulation resorts to the concept of "rigged Hilbert space" [1] which considers generalized eigenvectors; however these more general state vectors still have a divergent norm in the usual sense.

The difficulty has two sources:

a) The usual scalar product is defined through the integral of a product of two distributions, and this product may be ill-defined.

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b) Even if the multiplication of these distributions is well-defined, the integral may be divergent.

In view of case a) valuable work has been performed by mathematicians in order to define the product of singular distributions. But the obligation to circumvent the well-known Schwartz’s impossibility theorem results in several complications that are discouraging for the physicist. Some authors propose a commutative product which cannot be always associative [2]. Colombeau and followers [3] succeeded in constructing a differential algebra which is associative, but at the price of including new objects that are not always distributions.

Nevertheless, we can make the following observation: Leaving apart some specific problems of field theory, what is most basically needed for quantum mechanics is a scalar product of distributions, rather than a multiplication which would produce another distribution.

According to this remark, in this paper we radically avoid to consider the multiplication of distributions, and shall directly resort to their scalar product.

But it seems that no consistent picture can be obtained within the framework of real (or complex) numbers. Indeed intuition suggests that some "infinite quantity" should be introduced. This situation naturally leads to an enlargement of the field of real (or complex) numbers, by embedding this field into some suitable commutative and associative algebra (if possible a field) of which some element somehow represents this "infinity". Naturally we expect that this new algebra of scalars is of infinite dimension over the usual scalars.

At a more elementary level, we made an attempt some years ago [4] defining a scalar product for distributions of which the support is a finite number of isolated points. This naive approach already allowed for considering a space of states endowed with a sesquilinear form (which actually was a Hermitian form). But quantum mechanics requires that at least some states have a positive definite norm, whereas the algebra of scalars considered in this early work was completely lacking of any ordering relation (except in its trivial restriction to the reals).

Therefore, in the present work we aim at building a new scalar product which takes on values in the complexified of a totally ordered field. For the sake of physical applications it is obviously desirable that this new scalar product, as much as possible, mimicks several nice properties of the inner product in Hilbert spaces.

Our main tool is the introduction of new quantities that can be either infinitely large or infinitesimal. Such an extension was advocated as soon as in 1975 by M.O.Farrukh [5] who proposed to formulate quantum mechanics in a non-standard Hilbert space. According to his approach, the divergent lengths of unnormalizable states are re-defined as infinitely large non-standard numbers. In his framework however, the status of distributions suffers from some complications (for instance, all representations of distributions as pointwise-defined non-standard functions coincide except on an infinitesimal neighborhood of zero). In the same spirit, see a recent work by Almeida and Teixeira [6].

But all the works carried out along this line assume the knowlege of nonstandard (n.s.) analysis [7] which in turn requires mastering mathematical logics, the theory of ultra filters, and the Transfer Principle! In general it can be observed that the theory of nonstandard Hilbert spaces somehow avoids the concept of distribution, to
a large extent replaced by that of pointwise defined, nonstandard-valued, function of a n.s. variable.

In contradistinction, the generalized numbers involved in the present work are constructed by elementary methods, by-passing all the sophistications of mathematical logics. Moreover we restrict ourselves to functions and distributions that depend on a real (and standard) variable, though they can be linearly combined with help of coefficients that are generalized numbers. In this framework we naturally keep treating the distributions on their own right, and consider differentiation in the sense of distributions.

In Section 2 we recall some elementary results about bra-ket vector spaces in general and, without specifying the field of scalars, we consider the possibility of constructing a Hermitian form \(<\cdot,\cdot>\) on the space of the distributions concentrated at the origin.

In Section 3 we focus on the field \(\mathbb{F}^C\) (which is in some sense the most simple extension of \(\mathbb{C}\)) and construct a positive definite Hermitian form \((\cdot,\cdot)\) on the space of polynomials. We give explicit formulas for use in calculations and we generalize \((\cdot,\cdot)\) to more general regular functions that are the Fourier transformed of the distributions with finite discrete support. This situation feeds us back with a Hermitian form \(<\cdot,\cdot>\) defined on the space of these distributions.

Finally we display in Section 4 an application to the Schrödinger equation in one dimension with point interaction, and we construct the solutions.

### 1.1 Terminology

The words "scalar product" have been loosely employed in the literature, including in previous works of the author. We need here a more precise terminology [9].

Let \(\mathbb{L} \subset \mathbb{K}\) be commutative fields with an involution \(*\), such that \(\mathbb{K}\) is a field extension of \(\mathbb{L}\). Let \(\mathcal{A}, \mathcal{B}\) be vector spaces on \(\mathbb{L}\). The binary map

\[
<\cdot,\cdot>: \quad \mathcal{A} \times \mathcal{B} \mapsto \mathbb{K}
\]

that is \(<a,b> \in \mathbb{K}\) when \(a \in \mathcal{A}\) and \(b \in \mathcal{B}\), is a sesquilinear map when it is antilinear in \(a\) and linear in \(b\).

When \(\mathcal{A} = \mathcal{B}\) it may happen that in addition we have \(<a,b>^* = <b,a>\) for all couple \(a,b\). In this case \(<\cdot,\cdot>\) is a Hermitian map: \(\mathcal{A} \times \mathcal{A} \mapsto \mathbb{K}\).

When further we have \(\mathbb{K} = \mathbb{L}\), this map is actually a Hermitian form on \(\mathcal{A}\) and we say that \(\mathcal{A}\) is a bra-ket vector space. In some sense a bra-ket vector space can be seen as an inner-product space which may be degenerated. However, in agreement with a widespread but not universal convention [9] [10] we reserve the name inner-product space for non-degenerate cases, as follows:

We say that \(\mathcal{A}\) endowed with the Hermitian form \(<\cdot,\cdot>\) is an inner-product space in the weak sense when

\[
<u,v> = 0 \quad \forall v \quad \text{implies} \quad u = 0
\]  \(1\)
We say that $\mathcal{A}$ endowed with the Hermitian form is an inner-product space in the strong sense when there is no neutral vector, in other words
\[ <u, u> = 0 \quad \text{implies} \quad u = 0 \quad (2) \]

It is easy to check that condition (2) implies (1). Indeed, if there exists $u$ such that $<u, v> = 0 \forall v$, in particular $<u, u>$ vanishes, thus $u$ itself must be zero, according to (2). The converse is not true.

It is more interesting to distinguish between the weak and the strong definitions when $\mathbf{K}$ is the complexified of an ordered extension $\mathbf{B}$ of $\mathbf{R}$, say $\mathbf{K} = \mathbf{B}^C$ with an obvious notation (we assume that the ordering is compatible with addition and multiplication).

In this case, the weak definition refers to a space with indefinite metric, because it allows for positive, negative or vanishing values of $<u, u>$.

In contrast the strong definition refers to a space with definite metric; in this case, the space $\mathcal{A}$ endowed with the form $<.,.>$ is a Hermitian vector space.

The following result is straightforward and will be useful in the sequel,

**Proposition 1** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be bra-ket vector spaces (on $\mathbf{L} \subset \mathbf{K}$), with $\mathbf{K}$-valued Hermitian forms $<.,.>_1$ and $<.,.>_2$ respectively, and such that $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$. Defining further two sesquilinear maps
\[ <.,.>: \mathcal{H}_1 \times \mathcal{H}_2 \hookrightarrow \mathbf{K} \]
\[ <.,.>: \mathcal{H}_2 \times \mathcal{H}_1 \hookrightarrow \mathbf{K} \]

amounts to construct a sesquilinear map: $\mathcal{H}_1 \oplus \mathcal{H}_2 \hookrightarrow \mathbf{K}$. When, in addition, we have that $<f_1, f_2>^* = <f_2, f_1>$ for all $f_1 \in \mathcal{H}_1, f_2 \in \mathcal{H}_2$, then this sesquilinear map actually is a Hermitian map, and if $\mathbf{L} = \mathbf{K}$, we get on $\mathcal{H}_1 \oplus \mathcal{H}_2$ a Hermitian form which encompasses $<.,.>_1$ and $<.,.>_2$.

The direct sum is not orthogonal unless $<.,.>$ and $<.,.>$ both vanish.

**Proposition 2** Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be inner-product spaces such that $\mathcal{E}_1 \cap \mathcal{E}_2 = \{0\}$. There is a unique orthogonal direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$ and this direct sum is an inner-product space.

In the situation considered here, it is essential that $\mathcal{E}_1$ and $\mathcal{E}_2$ are mutually orthogonal. Otherwise, there is no unicity and the last statement might be wrong.

**Extension of the scalars**

In the course of this article we generally start with some vector space $\mathcal{X}$, on the field $\mathbf{C}$ of complex numbers.

Then we extend the field of the scalars, by imbedding the real numbers into a commutative algebra, say $\mathbf{B}$ which, as an algebra, is a vector space on the reals but can be complexified in the usual way. By complexification of $\mathbf{B}$ we obtain $\mathbf{B}^C$ equipped with the involution $*$ (we say that $\mathbf{B}^C$ is an involutive extension of $\mathbf{C}$). So
\[ \mathbf{R} \hookrightarrow \mathbf{B}, \quad \mathbf{C} \hookrightarrow \mathbf{B}^C \]
The elements of $\mathcal{B}$ (resp. $\mathcal{B}^C$) will be called extra-real (resp. extra-complex). This extension entails the possibility of making linear combinations of the elements of $X$ by extra-complex numbers, which uniquely defines [8] a module over $\mathcal{B}^C$, denoted as $X^\sharp$. When $\mathcal{B}$ actually is a field, its complexified also is a field, say $\mathcal{B}^C = \mathbb{K}$. In this case $X^\sharp$ is a vector space on $\mathbb{K}$ and any Hermitian map

$$X \times X \rightarrow \mathcal{B}^C$$

gets extended as a Hermitian form

$$X^\sharp \times X^\sharp \rightarrow \mathcal{B}^C$$

We shall carry out this procedure in various spaces of functions and distributions. Note that $(X_1 \oplus X_2)^\sharp = X_1^\sharp \oplus X_2^\sharp$.

1.2 Notation

$\mathcal{S}$ is the space of test functions (Schwartz space) and $\mathcal{S}'$ is the space of tempered distributions.

d/dx and $D$ denote differentiation respectively in the sense of distributions and in the sense of functions.

The Heaviside step function is noted as $\eta(x)$. Thus $\eta(-x) = 1 - \eta(x)$.

$\mathcal{P}$ is the space of polynomials in the single variable $x$, with complex coefficients.

$\mathcal{Q}$ is the space spanned by finite complex combinations of the various monomial waves $x^m e^{i ax}$, where $m$ is a non-negative integer and $a \in \mathbb{R}$. Fixing $a$ defines $\mathcal{Q}_a$.

Let $\Delta$ be the space of the distributions that have a compact discrete support. Let $\Delta_a$ be the vector space (over $\mathbb{C}$) of the distributions that have their support concentrated at the point $x = a$.

$\mathcal{P}^\sharp$ is the space of polynomials in the single variable $x$, with extra-complex coefficients, similar conventions define $\mathcal{Q}^\sharp$ and $\Delta_a^\sharp$.

For all $f \in \mathcal{S}'$ we consider the translations:

$$f(x) \mapsto f(x + a)$$

and we define also the phase translations:

$$f(x) \mapsto e^{ibx} f(x)$$

where $a$ and $b$ are arbitrary real numbers.

Let us also introduce space reflection: $f(x) \mapsto f(-x)$

**Fourier automorphism**

After the Fourier transformation $f(x) \mapsto F(u) = (2\pi)^{-\frac{1}{2}} \int e^{-iux} f(x) \, dx$, if we change the name of the variable in $F$, we obtain $\overline{f} = F(x)$, say for any distribution,

$$\overline{f} = \frac{1}{\sqrt{2\pi}} \int e^{-ixy} f(y) \, dy$$

(3)

Thus $\overline{f} = f(-x)$. Parity is preserved by the Fourier automorphism: when $f(x) = \pm f(-x)$ then we have that $\overline{f}(x) = \pm \overline{f}(-x)$.
The well-known properties of the Fourier transformation entail for all integer \( n \geq 0 \),

\[ \hat{x}^n = i^n \sqrt{2\pi} \delta^{(n)}(x) \]  

More generally

\[ \hat{x}^n e^{ian} = i^n \sqrt{2\pi} \delta^{(n)}(x - a) \]  

which displays the one-to-one map of \( Q \) onto \( \Delta \), such that \( Q_a \) is mapped onto \( \Delta_a \).

### 1.3 Linear operators in bra-ket vector spaces

Consider a bra-ket vector space \( \mathcal{H} \) on a field \( \mathbf{K} \). Unless otherwise specified, all the operators we consider acting in \( \mathcal{H} \) are linear and everywhere defined.

If \( \Omega \) is an operator in \( \mathcal{H} \), we say that \( \Omega^\dagger \) is a symmetric of \( \Omega \) provided

\[ \langle \Omega u, v \rangle = \langle u, \Omega^\dagger v \rangle \]  

for all \( u, v \in \mathcal{H} \). In general, unicity is not guaranteed. Self-symmetric operators are simply called symmetric.

To any ordered couple \( A, B \) where \( A \) and \( B \) are in \( \mathcal{H} \), we associate the linear operator \( A \times B \) defined as follows:

\[ A \times B \ w = \langle B, w \rangle A \quad \forall w \in \mathcal{H} \]

An operator of this type will be called a ket-bra operator in obvious reminiscence of the heuristic Dirac’s notation \( |A><B| \).

It is a simple algebraic exercise to check that \( B \times A \) is symmetric to \( A \times B \).

In particular:

*any operator of the form \( u \times u \) (projective operator) is self-symmetric.*

**Idempotent operators**

Let \( \mathcal{H} \) be a bra-ket vector space and \( R \) a linear operator such that \( R^2 = 1 \). For any \( f \in \mathcal{H} \) we have the unique decomposition \( f = f^+ + f^- \) where the even and odd parts of \( f \) with respect to \( R \), are defined as follows

\[ Rf^\pm = \pm f^\pm \]

The question arises as to know if \( R \) is symmetric with respect to the Hermitian form; it is easy to see that

**Proposition 3** \( R \) is symmetric in \( \mathcal{H} \) endowed with \( <\ ,\ > \) iff the even vectors are orthogonal to the odd ones.

Indeed, consider \( f, g \) in \( \mathcal{H} \). Assume for a moment that \( g \) is odd and \( f \) is even. Then we can write

\[ f = \frac{1}{2} (1 + R)f, \quad g = \frac{1}{2} (1 - R)g \]

thus

\[ <f, g> = <\frac{1}{2} (1 + R)f, \frac{1}{2} (1 - R)g> = \frac{1}{4} <f, (1 + R)(1 - R)g> = 0 \]
Conversely let us now assume that every even vector is orthogonal to every odd one. Consider any two vectors, say \( f, g \), in \( \mathcal{H} \), and split them into even/odd parts. We can write \( Rf = f^+ - f^- \), \( Rg = g^+ - g^- \) thus

\[
<Rf, g> = <f^+, g^-> - <f^-, g^+>
\]

\[
<f, Rg> = <f^+, g^-> - <f^-, g^+>
\]

But these quantities are equal, thus \( R \) is symmetric. []

The above proposition can be applied to space reflection. Physically it is satisfactory that space reflection be symmetric, and from a mathematical point of view it is very natural to generalize the odd/even orthogonality, from square integrable functions to distributions.

**In the sequel we shall consider linear spaces made of functions or distributions, and \( R \) will be the space reflection: \( Rf(x) = f(-x) \).**

### 2 General Results

In this Section we recollect several elementary results that can be easily derived without specifying which commutative field of scalars is taken as extension of \( \mathbb{C} \). We remember that any nontrivial candidate for this extension is necessarily of infinite dimension on \( \mathbb{C} \).

The words "scalar product" have been loosely employed in the literature, including in previous works of the author. We need here a more precise terminology.

In order to control whether standard results are recovered as special cases, it is convenient to set

**Definition**

Let \( \mathcal{H} \) be a space of functions or distributions; it is a vector space on \( \mathbb{C} \) ( resp. on \( \mathbb{K} \)). Assume that \( \mathcal{H} \) is endowed with a Hermitian map (resp. Hermitian form )

\[
\mathcal{H} \times \mathcal{H} \mapsto \mathbb{K},
\]

denoted as \( <.,.> \). **We say that this map (resp. form ) is admissible if**

\[
<u, v> = \int_{-\infty}^{\infty} u^* v dx
\]

**whenever the integrand in the right-hand side is well defined and the integral converges to a finite complex number.**

Admissibility requires that, in the sense of \( <.,.> \) the odd distributions are orthogonal to the even ones.

#### 2.1 Distributions supported by the origin

The most elementary distributions which are not trivial (i.e. not defined by a locally integrable function) have a discrete compact support. In particular consider \( \Delta_0 \). It is clear that \( \Delta_0 \) is stable by space reflection and by differentiation.
Note that $R$ and the operator $d/dx$, understood in the sense of distributions, are
everywhere defined in $\Delta_0$.

Note that, in the Hilbert framework, the skew symmetry of $d/dx$ (understood in
the sense of functions) was ensured by the fast decrease (at infinity) of the square
integrable functions.

In our context we turn to distributions and we observe that (with respect to the
behavior at infinity) the elements of $\Delta$ offer an obvious analogy with the $C^\infty$
functions of fast decrease. Thus it is natural to demand that $d/dx$ (now understood in
the sense of distributions) be skew-symmetric with respect to the scalar product.
This property would render the momentum operator $-id/dx$ symmetric, which is
desirable for the sake of applications to quantum mechanics; we shall see later some
implications of this requirement. But let us first consider other operators every-
where defined in $\Delta_0$. Multiplication by $x$ maps $\Delta_0$ into itself, according to the
well-known formula

$$x\delta^{(m)} = -m\delta^{(m-1)}$$  \hspace{1cm} (6)

Of course we have the commutator $[d/dx, x] = 1$ as usual.

Let $B$ be some commutative algebra, extension of $R$, hence $B^C$ involutive extension
of $C$. So $R \subset B$, $C \subset B^C$. Suppose that we have a Hermitian form $\Delta_0^* \times \Delta_0^* \mapsto
\mathbb{B}^C$ denoted as $<.,.>$, such that $\frac{d}{dx}$ is skew symmetric, say

$$< \frac{d\phi}{dx}, \psi > = -< \phi, \frac{d\psi}{dx} >$$

It is convenient to define $\zeta_k \in B^C$ as follows, for all integer $k = 0, 1, 2, 3 \cdots$

$$\zeta_k = < \delta^{(k)}, \delta >$$  \hspace{1cm} (7)

Skew symmetry of $\frac{d}{dx}$ immediately entails that $\zeta_k^* = (-)^k \zeta_k$. It is easy to verify:

**Proposition 4** $\zeta_{2m+1} = 0$, $\forall m \geq 0$ iff

$$< \delta^{(2m+1)}, \delta^{(2n)} > = 0$$  \hspace{1cm} (8)

for all integers $m, n \geq 0$. In other words, $\zeta_{2m+1}$ vanishes $\forall m$, iff the odd distribu-
tions concentrated at the origin are orthogonal to the even ones.

Indeed we note that, owing to the skew symmetry of $d/dx$

$$< \delta^{(2m+1)}, \delta^{(2n)} > = < (d/dx)^{2n} \delta^{(2m+1)}, \delta > = < \delta^{(2m+2n+1)}, \delta >$$

$$< \delta^{(2m+1)}, \delta^{(2n)} > = \zeta_{2(m+n)+1}$$  \hspace{1cm} (9)

Now assume that any odd distribution in $\Delta_0^*$ is orthogonal to the all the even ones,
and vice-versa. Then the left-hand side of (9) vanishes, which means that also
$\zeta_{2(m+n)+1}$ vanishes for all $m, n$. But any $p \geq 0$ is of the form $p = m + n$ for some
couple $m, n$. Thus every $\zeta_{2p+1}$ vanishes.
Conversely, if we now assume that $\zeta_k$ vanishes for all odd indices; it follows that $\delta^{(2m+1)}$ is orthogonal to $\delta^{(2n)}$. But in $\Delta^\sharp_0$ any odd (resp. even) element is a finite combination of distributions like $\delta^{(2m+1)}$ (resp. $\delta^{(2n)}$). Hence the odd/even orthogonality. \]

Naturally the structure of our Hermitian form depends on the detailed shape of the various $\zeta_k$ as elements of a specified algebra of scalars. However the following result is general

**Theorem 1** Let $\mathcal{B}^C$ be some commutative algebra, involutive extension of $\mathcal{C}$. Suppose that the space $\Delta^\sharp_0$ is endowed with a $\mathcal{B}^C$-valued Hermitian form, such that $R$ is symmetric and $\frac{d}{dx}$ is skew-symmetric. Then we have that

$$<\delta^{(m)},\delta^{(n)}> = (-)^m \zeta_{m+n} = (-)^n \zeta_{m+n}$$

(10)

Proof According to Propo.3 the even vectors are orthogonal to the odd ones. On the one hand Propo.4 tells that $\zeta_{2p+1}$ vanishes. On the other hand we know that $\zeta^*_2 = \zeta_2$. Finally every $\zeta_p$ is extra-real.

Skew symmetry of $d/dx$ entails that for all integer $p \geq 0$

$$<\delta^{(m)},\delta^{(n)}> = - <\delta^{(m+1)},\delta^{(n-1)}> = <\delta^{(m+2)},\delta^{(n-2)}> = \ldots$$

$$<\delta^{(m)},\delta^{(n)}> = \cdots (-)^p <\delta^{(m+p)},\delta^{(n-p)}>$$

$$<\delta^{(m)},\delta^{(n)}> = (-)^p <\delta^{(m+p)},\delta^{(n-p)}>$$

where $p = 0, \ldots, n$. In particular, when $p = n$ we obtain

$$<\delta^{(m)},\delta^{(n)}> = (-)^n \zeta_{m+n}$$

By Hermiticity of the inner product we also have $<\delta^{(m)},\delta^{(n)}> = <\delta^{(n)},\delta^{(m)}>^*$, or (by exchange of $m$ with $n$)

$$<\delta^{(n)},\delta^{(m)}> = <\delta^{(m)},\delta^{(n)}>^* = (-)^n \zeta^*_{m+n}$$

which is extra-real. \]

Remark

Formulae (8)(10) are not sufficient for the determination of a Hermitian form on $\Delta_0$. Indeed all the scalar products $<\delta^{(m)},\delta> = \zeta_m$ remain to be defined. To this end, various inequivalent choices are possible. For instance, eqs (10) were postulated in [4], but with the assumption that all the $\zeta_{2k}$ were independent in $\mathcal{B}^C$. In the present article this particular assumption is abandoned in favor of a more appropriate choice; see subsection 3.4 below.

In contrast to the above result, we have this general impossibility result:

**Theorem 2** If space reflection is symmetric whereas $d/dx$ is skew symmetric, then the multiplicative operator $x$ cannot be symmetric in $\Delta^\sharp_0$. 

9
Proof
Evaluating $\langle x\delta, \delta' \rangle$ and $\langle \delta, x\delta' \rangle$ we first find that symmetry of $x$ would make $\zeta_0$ to vanish.

Then consider any positive integer $k$. From (6) we have on the one hand

$\langle x^{(k)}, \delta^{(k+1)} \rangle = -k\delta^{(k-1)}, \delta^{(k+1)} \rangle = -k < \delta^{(k-1)}, \delta^{(k+1)} > = -k(-)^{k-1}\zeta_{2k}$

But on the other hand

$\langle \delta^{(k)}, x\delta^{(k+1)} \rangle = -\delta^{(k)}, -(k+1)\delta \rangle = -(k+1) < \delta^{(k)}, \delta^{(k)} >$

and from (10)

$\langle \delta^{(k)}, x\delta^{(k+1)} \rangle = -(k+1)(-)^k < \delta^{(2k)}, \delta \rangle = -(k+1)(-)^k\zeta_{2k}$

Symmetry of operator $x$ would imply

$-k(-)^{k-1}\zeta_{2k} = -(k+1)(-)^k\zeta_{2k}

(2k+1)\zeta_{2k} = 0$

Hence $\zeta_{2k} = 0$, which would make the scalar product to vanish identically.

This result tells that, if we demand that odd vectors are orthogonal to the even ones, we cannot endow the space of the distributions supported by the origin with a scalar product where $x$ and the momentum operator $p = -id/dx$ are simultaneously symmetric. Our choice is definitely in favor of the latter, because the most basic ingredient of energy is its kinetic part, which involves momentum (and not position).

Remark I. In [4] we pointed out a similar situation, for a particular case where $\mathbb{C}$ was extended into a ring of polynomials in arbitrarily many indeterminates. The Note 11 of that article contains a guess about more generality.

Remark II. In a recent article, Almeida and Teixeira [6] have put forward “position operators” which by-pass this impossibility [6]. But their operators (defined by spectral decomposition) cannot be strictly identified with our multiplicative operator $x$ (in fact, in their framework, there are several “position” operators).

Remark III. In spite of the above theorem, we note that multiplication by a smooth function of polynomial growth, say $V(x)$ defines a symmetric operator, provided that at the origin all the derivatives of $V$ vanish.

3 Ordered Extension of the Scalars

So far we have not specified $\mathbf{B}$. The most simple extension of the scalars consists in replacing $\mathbb{R}$ by a ring of polynomials with real coefficients (resp. $\mathbb{C}$, complex coefficients).

In a previous work we had considered a ring of polynomials in infinitely many indeterminates [4]. It would be more reasonable to employ the ring $\mathbb{P}$ of polynomials in one single real variable $X$, with real coefficients (this ring is an algebra over the field of real numbers). It is clear that real (resp. complex) numbers can be identified...
with constant polynomials with a real (resp. complex) value; that is \( \mathbb{R} \subset \mathbb{P} \) and \( \mathbb{C} \subset \mathbb{P}^C \).

Still, in a ring of polynomials, the division is generally not possible. In order to deal with a field, that is an algebra with division, we must at least consider rational functions, of the form \( P/Q \) where \( P \) and \( Q \) are polynomials. Let \( \mathbb{F} \) be the field of real rational functions of the indeterminate \( X \). For any element of \( \mathbb{F} \) we can speak of the sign at infinity \[11\] which allows to define a total (non Archimedean) ordering of \( \mathbb{F} \).

Similarly let \( \mathbb{F}^C \) be the field of complex rational functions of the variable \( X \). We have \( \mathbb{R} \subset \mathbb{P} \subset \mathbb{F} \), \( \mathbb{C} \subset \mathbb{P}^C \subset \mathbb{F}^C \). For all \( a \in \mathbb{F}^C \) we have \( a^*a \in \mathbb{F} \) and \( a^*a > 0 \) unless \( a = 0 \). See Appendix.

### 3.1 A Positive Scalar Product

From now on we assume that the field \( \mathcal{B} \) is simply \( \mathbb{F} \), hence \( \mathcal{K} = \mathbb{F}^C \).

Our main goal remains the construction of a scalar product for the elementary distributions characterized by a discrete finite support. But for convenience of the exposition, it is more easy to start with polynomials \( f(x) \). They are trivial as distributions but, through Fourier automorphism, they happen to be one-to-one connected with the distributions supported at the origin \(^1\).

Let us for a moment consider arbitrary complex functions of a real variable \( x \). When \( f(x) \) and \( g(x) \) are \( \mathbb{C} \)-valued square integrable functions, their usual scalar product \( \int f^*g \, dx \) is the limit, for \( X \to \infty \), of the integral

\[ I(X, f, g) = \int_{-X}^{X} f^*g \, dx \] \( (11) \)

When \( f(x) \) and \( g(x) \) are not square integrable, the usual scalar product may be divergent, or without a limit. This may occur in particular when \( f \) and \( g \) are polynomials in the variable \( x \). In this case, the integral above deserves to be considered on its own right. So we define a Hermitian map

\[ \mathcal{P} \times \mathcal{P} \mapsto \mathbb{P}^C \subset \mathbb{F}^C \]

which depends on \( X \) through the formula

\[ (f, g) = I(X, f, g) \] \( (12) \)

Invoking the primitive \( G(x) = \int_{0}^{x} f^*g \, dx \) we observe that \( I(X, f, g) \) is an odd polynomial in the variable \( X \) (this would not be the case for a more general choice of the functions \( f \) and \( g \)).

\(^1\)It must be clearly understood that polynomials in the variable \( X \) and polynomials in the variable \( x \) play very different roles: whereas \( \mathbb{P} \) is just a device for extending the scalars, \( \mathcal{P} \) is seriously taken as a functional space.
In order to check admissibility let us investigate the cases where the usual scalar product exists. We obtain

**Proposition 5** Let \( f(x), g(x) \) be complex-valued polynomials. Then \( \int_{-\infty}^{\infty} f^* g \, dx \) is finite iff \( I(X, f, g) \) vanishes for all \( X \), which occurs iff \( f^* g \) is an odd polynomial.

Indeed \( \lim_{X \to \infty} I(X, f, g) \) is finite iff \( I(X, f, g) \) is a constant. Being an odd function, this constant is necessarily zero. Conversely, if \( I(X, f, g) \) vanishes for all \( X \) then \( G \) is even, thus \( f^* g \) is odd, which makes its integral (from \(-\infty \) to \( \infty \)) to vanish. []

To summarize:

\( (f, g) \) always coincides with \( \int f^* g \, dx \) when the latter converges. (in this case \( I(X, f, g) \) does not depend on \( X \)).

Now extending the scalars from \( \mathbb{C} \) to \( \mathbb{F}^C \) we replace \( \mathcal{P} \) by \( \mathcal{P}^\sharp \) but stick to the definition (12).

For instance any element of \( \mathcal{P}^\sharp \) can be written as

\[
  f = f_0(X) + f_1(X) \, x + \cdots + f_D(X) \, x^D
\]

where \( D \) is integer and \( f_0, f_1, \cdots f_D \) are complex-valued rational fractions of the only variable \( X \).

Here equation(12) defines a Hermitian form, say

\[
  \mathcal{P}^\sharp \times \mathcal{P}^\sharp \mapsto \mathbb{F}^C
\]

Now \( f \) and \( g \) depend not only on \( x \) but also on \( X \). They are polynomials in \( x \) but rational fractions in \( X \). According to this scheme equation (11) defines a rational fraction of \( X \) which does not depend on \( x \). We can say that the scalar product of two polynomials always has a meaning as an extra-complex number, i.e. an element of \( \mathbb{F}^C \). Note that \( (f, f) \) is always in \( \mathbb{F} \), we say that it is extra-real. Moreover, for \( f = g \) we have the following property

**Proposition 6** The scalar product of a polynomial by itself is non-negative in \( \mathbb{F} \)

Proof The integral \( I(X) \) in equation(12) is a rational function of the variable \( X \), with no more poles than those of \( f \) and \( g \), which form a finite set of points. This function \( I(X) \) is defined for all \( X \) larger than the maximum of these poles, and when \( f = g \), its value is obviously a non-negative real number. Therefore \( I(X) \) is positive as a rational function in the sense of the ordering in \( \mathbb{F} \), see Appendix. []

**Theorem 3** Let \( f \in \mathcal{P}^\sharp \). Then \( (f, f) \) vanishes iff \( f = 0 \).

Proof In the present context, the vanishing of \( I(X, f, f) \) means that this rational fraction is zero for all \( X \), and the vanishing of \( f \) means that \( f \) is identically zero, for all values of \( X \) and \( x \).
If we had only \( I(X_0, f, f) = 0 \) for some fixed value \( X_0 \) of \( X \) we could say that 
\[
|f(X_0, x)| \text{ vanishes on the interval } x \in [-X_0, X_0],
\]
in other words \( f(X, x) \), function of two independent variables, vanishes on the segment of the \( X, x \) plane defined by the conditions \( X = X_0, \quad x \in [-X_0, X_0] \).

But our assumption is stronger; it implies that \( I(X, f, f) \) vanishes for every positive \( X \). Therefore the function \( f(X, x) \) vanishes on infinitely many segments of the above type, browsing all the domain defined by the conditions \(-X \leq x \leq X\) (this domain is limited by the straight lines \( X = x \) and \( X = -x \)). Since \( f(X, x) \) is rational in \( X \) and polynomial in \( x \), its vanishing on this domain implies that \( f \) is identically zero.

**Corollary**

\( \mathcal{P}^\sharp \) endowed with the Hermitian form (12) is a positive definite inner product space so that \( \mathcal{P}^\sharp \) is a Hermitian vector space on \( \mathbb{F}^C \).

### 3.2 Explicit Formulas

The monomials \( x^m \), with \( m \) a nonnegative integer, form a countable basis of \( \mathcal{P} \) (and also of \( \mathcal{P}^\sharp \)). Thus insofar as \( \mathcal{P} \) only is concerned, it is sufficient to compute the brackets \((x^m, x^n)\) for all integers \( m, n \geq 0 \) with help of (12). One finds, with \( m, n \geq 0 \)
\[
(x^m, x^n) = [1 + (-1)^{m+n}] \frac{X^{m+n+1}}{m+n+1} \tag{13}
\]
Remark: this quantity vanishes whenever \( m + n \) is odd.

**Remarks**

The multiplicative operator \( x \) is everywhere defined in \( \mathcal{P}^\sharp \) and symmetric with respect to the scalar product (12).

The derivative operator and the multiplication by \( x \) are everywhere defined on \( \mathcal{P}^\sharp \). Moreover \( \mathcal{P}^\sharp \) is obviously stable by the translations \( f(x) \mapsto f(x+h) \) with real \( h \).

But our scalar product of polynomials is not invariant under translations. Similarly, we observe that the derivative operator is not skew-symmetric with respect to the scalar product in \( \mathcal{P}^\sharp \).

However the multiplicative operator \( x \) is symmetric on \( \mathcal{P}^\sharp \).

### 3.3 Polynomial Waves

**Definitions**

The function \( x^m e^{i ax} \), where \( k \) is a nonnegative integer and \( a \in \mathbb{R} \), will be referred to as

*the monomial wave of degree \( m \) and wave-number \( a \).*

Linear complex combinations of monomial waves, with a finite number of terms, but possibly including different degrees and different wave-numbers, will be called *polynomial waves*; they form a linear space \( \mathcal{Q} \). Such functions are generally not bounded at infinity thus, in the position representation of quantum mechanics they are generally not eligible as wave functions of physical systems (with the remarkable
exception of plane waves). However polynomial waves are of some interest because, through the Fourier transformation, they are in one-to-one correspondence with the elements of $\Delta$ (resp. $\Delta^\sharp$).

Admitting extra-complex combinations, in other words taking the coefficients in $F^C$, provides a vector space on the field $F^C$, referred to as $Q^\sharp$.

We have these two interesting special cases:

Ordinary polynomials correspond to the wave number zero,

Plane waves correspond to the degree zero, by finite linear combination they span a vector space denoted as $W$.

The monomial waves with wave number $a$ span a vector space denoted as $Q^\sharp_a$. The elements of $Q^\sharp_a$ are obtained from those of $P^\sharp$ by the phase-translation: $f \mapsto e^{iax}f$.

In particular we have $Q^\sharp_0 = P^\sharp$.

So we can define a Hermitian form on each $Q^\sharp_a$ by imposing

\[ (x^m e^{iax}, x^n e^{iax}) = (x^m, x^n) \]  

say according to (13)

\[ (x^m e^{iax}, x^n e^{iax}) = \left[ 1 + (-1)^{m+n} \right] \frac{x^{m+n+1}}{m+n+1} \]  

Observing that $Q^\sharp_a \cap Q^\sharp_b = \{0\}$ for $a \neq b$, it is natural to look after a Hermitian form defined on the direct sum $Q^\sharp_a \oplus Q^\sharp_b$.

By formula (15) the scalar product is first extended, for every real number $a$, to the space $Q^\sharp_a$ spanned by the functions $x^m e^{iax}$.

This requirement alone would not completely define the form on the whole $Q^\sharp$, but we naturally also impose admissibility i.e. we also demand that $(x^m e^{iax}, x^n e^{ibx})$ reduces to $\int x^{m+n} e^{i(b-a)x} dx$ whenever this integral exists as a convergent integral in the standard framework, which occurs for $a \neq b$, since in the sense of distributions, setting $m + n = r$, $a - b = c$ we have

\[ \int x^r e^{-icx} dx = 2\pi(-i)^{-r} \delta^{(r)}(c) \]

which is zero for nonvanishing $c$. Therefore formula (15) is completed by

\[ (x^m e^{iax}, x^n e^{ibx}) = 0, \quad \forall a \neq b \]  

which states that $Q^\sharp_a$ and $Q^\sharp_b$ are orthogonal for $a \neq b$.

In view of Proposition 1 it is now clear that $(f, g)$ is defined for all $f, g \in Q^\sharp$.

**Proposition 7** For all $f \in Q^\sharp$ the scalar product $(f, f)$ is positive, unless it is zero which corresponds to a vanishing $f$.

The proof is straightforward. Indeed it was proved for $f \in P^\sharp$ and is trivially extended to any $Q^\sharp_a$ by phase translation. Then if $f = f_1 + \cdots + f_r$ with $f_k \in Q^\sharp_a$, orthogonality of the various $Q^\sharp_a$ with distinct indices entails $(f, f) = \sum (f_k, f_k)$.

This results ensures this

**Corollary**

$Q^\sharp$ endowed with $(.,.)$ is a (positive) inner product space.

It can be read off (15) (16) that
Proposition 8 The multiplicative operator \( x \) is everywhere defined on \( Q^\sharp \) and is symmetric with respect to the inner product.

Note also that the derivative operator \( d/dx \) is everywhere defined on \( Q^\sharp \). Moreover \( Q^\sharp \) is obviously stable by the translations \( f(x) \mapsto f(x + h) \) with real \( h \).

But our inner product of polynomial waves is not invariant under translations, although it remains invariant in the particular case of plane waves, as can be read off (15) (16) by making \( m = n = 0 \).

Similarly, we observe that in general the derivative operator \( d/dx \) is not skew-symmetric with respect to the inner product in \( Q^\sharp \) (for instance with \( m = 2, n = 0 \) we find \( \left( \frac{d}{dx} x^2, 1 \right) = 2X^3/3 \) whereas \( (x^2, \frac{d}{dx} 1) \) vanishes), although it remains skew-symmetric in the particular case of plane waves, say

\[
\text{Proposition 9} \quad \text{Differentiation maps } W \text{ into itself and is skew-symmetric in } W. \\
\]

Indeed we find \( \left( \frac{d}{dx} e^{iax}, e^{ibx} \right) = -ia(e^{iax}, e^{ibx}) \) and \( \left( e^{iax}, \frac{d}{dx} e^{ibx} \right) = ib(e^{iax}, e^{ibx}) \).

According to (16) both are zero if \( a \neq b \), and if \( a = b \) then we get \( \left( \frac{d}{dx} e^{iax}, e^{iax} \right) = -ia(e^{iax}, e^{iax}) \) and \( \left( e^{iax}, \frac{d}{dx} e^{iax} \right) = ia(e^{iax}, e^{iax}) \), which proves the skew-symmetry of \( d/dx \) with respect to the inner product of plane waves.[^]

Remark: The space of polynomial waves is stable by differentiation, by multiplication by \( x \), by translations and by the phase-translations.

3.4 The space of distributions with compact discrete support

For every \( a \in \mathbb{R} \) let \( \Delta_a \) be the space of the distributions

\[
\phi = \phi_0 \delta(x-a) + \phi_1 \delta'(x-a) + \cdots + \phi_r \delta^{(r)}(x-a)
\]

(17)

with constant complex coefficients, where the nonnegative integer \( r \) depends on \( \phi \). The multiplication of two such distributions is not defined; we wish however to define a scalar product of them.

With this goal in mind we observe that, defining

\[
\Delta = \bigoplus_{a \in \mathbb{R}} \Delta_a
\]

(18)

hence, by extension of the scalars

\[
\Delta^\sharp = \bigoplus_{a \in \mathbb{R}} \Delta_a^\sharp
\]

(19)

we have \( Q \cap \Delta = \{0\} \), \( Q^\sharp \cap \Delta^\sharp = \{0\} \) and \( \Delta \) (resp. \( \Delta^\sharp \)) is in one-to-one correspondence with \( Q \) (resp. \( Q^\sharp \)). Namely the Fourier automorphism sends \( x^m e^{iax} \in Q \) to \( i^m \sqrt{2\pi} \delta^m(x-a) \), formula (5). So \( f \in Q \mapsto f \in \Delta \) and \( Q_a \mapsto \Delta_a \).
As well as the monomials waves span $\mathcal{Q}$ (resp. $\mathcal{Q}^\sharp$), the concentrated distributions $\delta^{(m)}(x-a)$ with arbitrary $a$ in the real line, span $\Delta$ (resp. $\Delta^\sharp$). Thus defining a Hermitian map on $\Delta$ (resp. a Hermitian form on $\Delta^\sharp$) will be straightforward; we proceed as follows.

The brackets in $\Delta$ are automatically deduced from the brackets in $\mathcal{Q}$, since any distribution $\phi$ in $\Delta$ is the Fourier image of a polynomial wave in $\mathcal{Q}$, say $\phi = \mathcal{F}$ and we define

$$< \mathcal{F}, \mathcal{G} > = (f,g)$$

(20)

So doing, we endow each $\Delta^\sharp_a$ with a structure isomorphic to that of $\mathcal{Q}^\sharp_a$, and $\Delta^\sharp$ with a structure isomorphic to that of $\mathcal{Q}^\sharp$.

**Proposition 10** In $\Delta^\sharp$ endowed with $< \ldots, \ldots >$, the scalar product of a distribution $\phi$ by itself is always positive unless it vanishes, which happens only if $\phi$ is zero.

The proof is straightforward by Fourier duality since, according to Proposition 9, it is true for $\mathcal{Q}^\sharp$.

From Proposition 8 and well-known properties of the Fourier transform, we can state

**Proposition 11** The differentiation operator $\frac{d}{dx}$ acting in $\Delta^\sharp$ is everywhere defined and skew-symmetric with respect to the scalar product $< \ldots, \ldots >$.

Accordingly, the momentum operator $-i \frac{d}{dx}$ acting in $\Delta^\sharp$ is symmetric.

We have seen previously that $\frac{d}{dx}$ acting in $\mathcal{Q}^\sharp$ is not skew-symmetric; similarly, the multiplicative operator $x$ acting in $\Delta^\sharp$ is not symmetric.

We now turn to explicit formulas. Using the linearity of the Fourier transform, we easily derive from (4)

$$< \delta^{(m)}, \delta^{(n)} > = \frac{i^{m-n}}{2\pi} (x^m, x^n)$$

(21)

Now take (13) into account. The bracket in the r.h.s. of (13) vanishes for odd $m+n$ and equals 2 for for even $m+n$. Let us evaluate $< \delta^{(m)}, \delta^{(n)} >$ in the latter case, setting $m+n = 2k$. We obtain

$$< \delta^{(m)}, \delta^{(n)} > = (-)^{k-n} \frac{X^{2k+1}}{(2k+1)\pi}$$

(22)

In particular we can write

$$< \delta^{(2p)}, \delta > = (-)^p \frac{X^{2p+1}}{(2p+1) \pi}$$

(23)

Let us introduce this notation

$$\delta_{<0>}^{(m)} = < \delta^{(m)}, \delta >$$

(24)
Hence the formulas

\[ \delta^{(2p)}_{<0>} = (-)^p \frac{X^{2p+1}}{(2p + 1)\pi} \]  

(25)
in particular

\[ X = \pi \delta_{<0>} \]  

(26)
In agreement with the convention made in equation (7), we shall equally make use of the notation

\[ \delta^{(2p)}_{<0>} = \zeta_{2p} \]  

(27)
and we can set \( \zeta_{2p+1} = 0 \). With this convention it is not difficult to compare the present formalism with an early (and inequivalent) approach that we proposed many years ago [4]. An important difference between the choices of B in Ref. [4] and here respectively, is that the choice proposed in [4] was an algebra of polynomials depending on infinitely many variables. That algebra was somehow "too large" for physical purposes.

In contrast, in the present work we employ the rational functions of a single variable \( X \) (thus including the polynomials in \( X \)), which is a field and has a total ordering.

Let us summarize: we have defined a scalar product on each of the following spaces \( \mathcal{P}^\sharp, \Delta_0^\sharp, \Delta^\sharp, \mathcal{Q}^\sharp \). In addition it is obvious that \( \mathcal{P}^\sharp \subset \mathcal{Q}^\sharp \) and \( \Delta_0^\sharp \subset \Delta^\sharp \), and the scalar product in \( \Delta_0^\sharp \) is consistent with the scalar product in \( \Delta^\sharp \). Similarly the scalar product in \( \mathcal{P}^\sharp \) gets generalized as the scalar product in \( \mathcal{Q}^\sharp \).

Moreover \( \mathcal{Q}^\sharp \cap \Delta^\sharp = \{0\} \) and the Fourier transformation establishes a one-to-one correspondance between \( \mathcal{P}^\sharp \) and \( \Delta^\sharp \).

In most physical applications, we shall have to unify the inner-product space \( \Delta^\sharp \) with some suitable space of "ordinary functions", in order to build at least a bracket vector space. As an illustration of the method we consider below the potential for point interaction located at the origin. In this simple case, the only singularity is at \( x = 0 \), and it is sufficient to unify \( \Delta_0^\sharp \) with a suitable space of functions.

### 4 The one-center point interaction in one dimension

As an illustration of the method we consider, in one dimension, the potential for point interaction located at the origin. In this simple case, the only singularity is at \( x = 0 \). This potential has a long story [12]. Its contribution is often represented by the heuristic expression \( \alpha \delta(x) \) which is singular in the usual setting. The most popular trick consists in omitting this term (hence keeping the free-particle Schroedinger operator) whereas \textit{ad hoc} boundary conditions are imposed to the wave function. In contrast, Albeverio \textit{et al} [13] explicitly write down the potential as the operator which multiplies the wave function by the characteristic function of an infinitesimal domain around the origin. Their approach resorts to the most sophisticated (and powerful) tools of n.s. analysis.

Here also we shall write down an expression for the potential, in a much more simple framework however.
Physical intuition suggests that "nothing special can happen" outside the origin, therefore it is natural to take the view that concentrated distributions like $\delta^{(m)}(x-a)$ where $a \neq 0$ play no role in this problem. Insofar as concentrated distributions are concerned, it is therefore sufficient that our space of states include $\Delta^\#_0$. In the same spirit we shall also include ordinary functions that are smooth everywhere except perhaps for a possible discontinuity at $x = 0$.

In order to be more precise we first introduce left/right square integrable functions as follows. In the standard framework, we say that a function $f$ is respectively right or left square integrable when $\int_0^\infty f^* f \, dx$ or $\int_{-\infty}^0 f^* f \, dx$ converges. Any square integrable function is both right and left square integrable, and vice versa. But there are functions that are only right (resp. only left) square integrable.

We say that a smooth function $f$ belongs to $I^+$ (resp. $I^-$) when $f$ and all its derivatives are right (resp. left ) square integrable. Now if $f_+$ (resp. $f_-$) denotes the restriction of $f$ at the interval $[-\infty, 0]$ (resp. $[0, \infty]$) it is clear that $f_+$ belongs to the Hilbert space $L^2([0, \infty], dx)$ whereas $f_-$ belongs to the Hilbert space $L^2([-\infty, 0], dx)$. These Hilbert spaces are respectively endowed with the complex-valued inner products $(\cdot, \cdot)_+$ and $(\cdot, \cdot)_-$. These Hilbert spaces are respectively endowed with the complex-valued inner products $(\cdot, \cdot)_+$ and $(\cdot, \cdot)_-$. Finally we introduce $J_0$ as the space of the functions that can be written as

$$h(x) = f \eta + g(1 - \eta)$$

(28)

with $f \in I^+$ and $g \in I^-$. So $h$ is a function defined almost everywhere, actually everywhere but at the origin $^2$, and it defines a distribution.

Although $h$ is generally not defined at the origin, we extend it to this point, with an average value

$$h_0 = \frac{1}{2} ( \lim_{x \to 0^+} h + \lim_{x \to 0^-} h)$$

in other words

$$h_0 = \frac{1}{2} (f(0) + g(0))$$

(29)

so that $h_0$ coincides with $h(0)$ in the particular case where $f = g$.

From now on, for typographic convenience, we systematically write, respectively $f_0, g_0$, etc. for $f(0), g(0)$ etc. whenever $f(x) \in I^+$ or $g(x) \in I^-$. This convention $^3$ is consistent with formula (29).

Notice that for a given $h(x)$ in $J_0$ neither $f$ nor $g$ is unique; only their restrictions $f_+$ and $g_-$ are unique. Clearly $J_0$ is endowed with an inner product; if $h_1$ and $h_2$ belong to $J_0$, with an obvious notation similar to that of equation (28) we have

$$(h_1, h_2) = \int_{-\infty}^\infty h_1^* h_2 \, dx = (f_1, f_2)_+ + (g_1, g_2)_-$$

We observe that, although $J_0$ does not coincide with $L^2$, it includes $\mathcal{S}$ which is dense in $L^2$.

Definition

$^2$In this article "almost everywhere" means everywhere except at $x = 0$.

$^3$this notation should not be confused with the notation using indices for the coefficients in the development of a distribution with compact discrete support.
For any function \( h \in J_0 \) as in (28), \( \theta h \) is the \textit{jump of } \( h \) \textit{across the origin}, say \( \theta h = f_0 - g_0 \).

\( J_0 \) is stable by \( D \), since \( Dh = \eta f' + (1 - \eta) g' \).

In contrast, although \( h \) obviously defines a distribution, we observe however that its distributional derivative cannot belong to \( J_0 \). As well known

\[ h' = Dh + (\theta h) \delta \quad (30) \]

**Proposition 12** Let \( F = \eta u(x) + (1 - \eta) s(x), \ G = \eta v(x) + (1 - \eta) t(x), \) be two elements of \( J_0 \). Then we have that \( (DF, G) + (F, DG) = -u_0^* v_0 + s_0^* t_0 \)

Indeed we have

\[ DF = \eta u' + (1 - \eta) s', \quad DG = \eta v' + (1 - \eta) t' \]
\[ \theta F = u_0 - s_0, \quad \theta G = v_0 - t_0 \]
\[ F_0 = \frac{1}{2}(u_0 + s_0) \quad G_0 = \frac{1}{2}(v_0 + t_0) \]

Let us compute

\[ (DF, G) = (u', v)_+ + (s', t)_-, \quad (F, DG) = (u, v')_+ + (s, t')_- \]

On the one hand

\[ (u', v)_+ + (u, v')_+ = \int_0^\infty u^* v dx + \int_{-\infty}^0 u^* v' dx \]

where \( u, v \) are \textit{right} square integrable; integrating by parts yields

\[ (u', v)_+ + (u, v')_+ = -u_0^* v_0 \]

On the other hand, in \( (s', t)_- + (s, t')_- \) the functions \( s, t \) are \textit{left} square integrable; a similar procedure yields

\[ (s', t)_- + (s, t')_- = s_0^* t_0 \]

Adding these two formulas achieves the proof. \[ \]

We shall work within the direct sums \( \Delta_0 \oplus J_0 \) and its extension \( \Delta^\#_0 \oplus J^\#_0 \) which are stable by \( d/dx \).

The most general element of \( \Delta_0 \oplus J_0 \) takes on the form

\[ \Phi = \phi + F, \quad \phi \in \Delta_0, \ F \in J_0 \quad (31) \]

In the sense of distributions, all the derivatives \( F^{(n)} = \frac{d^n F}{dx^n} \) are in \( \Delta_0 \oplus J_0 \), thus all the derivatives

\[ \Phi^{(n)} = \phi^{(n)} + F^{(n)} \]

also belong to \( \Delta_0 \oplus J_0 \).
Definition

Let $\Phi \in \Delta_0 \oplus J_0$. The pseudo-value of $\Phi$ at the origin is the quantity

$$\Phi_{<0>} = F_0 + <\delta, \phi>$$

(32)

where $F_0$ is the average value of $F$ at the origin, like in formula (29).

This definition is a generalization of the convention made in (24).

By extension of the scalars, Proposition 12 and the definition of average value carry over to $J_0^\sharp$; the definition of pseudo-value carries over to $(\Delta_0 \oplus J_0)^\sharp$.

We want to define a Hermitian form on $(\Delta_0 \oplus J_0)^\sharp$; to this end it will be sufficient to define first a Hermitian map $(\Delta_0 \oplus J_0) \times (\Delta_0 \oplus J_0) \mapsto \mathbb{C}$.

So let

$$F = \eta u(x) + (1 - \eta) s(x), \quad G = \eta v(x) + (1 - \eta) t(x)$$

(33)

be two arbitrary elements of $J_0$. For the sake of admissibility we impose that $<F, G>$ is simply $\int_{-\infty}^{+\infty} F^* G \, dx$.

According to Proposition 1, since the various $\delta^{(m)}$ are a basis of $\Delta_0$, we must give a meaning to all the brackets $<\delta^{(m)}, G>$ and $<F, \delta^{(m)}>$ where $F$ or $G \in J_0$.

Consider $\Phi, \Psi$ as in (31), say $\Psi = \psi + G$. Since $<\delta^{(m)}, \delta^{(n)}>$ was already defined, inducing the expression for $<\phi, \psi>$, the only thing which remains to be done consists in defining $<\phi, G>$ and $<F, \psi>$.

By analogy with the case of smooth functions, we postulate

$$<\delta^{(m)}, G> = (-)^m (G^{(m)})_{<0>} \quad m \geq 0$$

(34)

In order to get a Hermitian map we are obliged to complete by the conjugate formula, say

$$<F, \delta^{(n)}> = (-)^n (F^{*^{(n)}})_{<0>} \quad m \geq 0$$

(35)

Then, since the distributions $\delta^{(m)}$ are a basis of $\Delta_0$, the above formulas automatically define the sesquilinear maps

$$\Delta_0 \times J_0 \mapsto \mathbb{C}, \quad J_0 \times \Delta_0 \mapsto \mathbb{C}$$

hence (with help of Proposition 1) the Hermitian map $(\Delta_0 \oplus J_0) \times (\Delta_0 \oplus J_0) \mapsto \mathbb{C}$. Finally extending the algebra of the scalars provides the Hermitian form on $(\Delta_0 \oplus J_0)^\sharp$. For instance (34) and (35) keep being valid when $F$ and $G$ belong to $J_0^\sharp$, etc.

The form is admissible: we recover the inner product in $\Delta_0^\sharp$ as a particular case, and also the ordinary scalar product $(\delta^{(m)}, G)$ in the simple case where $G$ is an everywhere defined and smooth square integrable function.

Let us now exhibit a nice property of $d/dx$ in the bra-ket space $(\Delta_0 \oplus J_0)^\sharp$. It is convenient to observe first the following
Proposition 13 For $\phi \in \Delta^\sharp_0$ and $G \in J^\sharp_0$ we have

$$<\phi, G'> + <\phi', G> = 0 \quad (36)$$

Proof: On the one hand writing formula (34) for $m + 1$ yields

$$<\delta^{(m+1)}_\phi, G> = (-)^m G^m_{<0>}$$

(37)

On the other hand, writing (34) for $G'$ yields

$$<\delta^m(G'), G> = (-)^{m+1} G^m_{<0>}$$

(38)

Say $\phi = \sum \phi_m \delta^{(m)}$ hence $\phi' = \sum \phi_m \delta^{(m+1)}$ we get

$$<\phi, G'> = \sum (-)^m \phi_m G^m_{<0>},$$

$$<\phi', G> = \sum (-)^{m+1} \phi_m G^m_{<0>}$$

but $(-)^m$ and $(-)^{m+1}$ are opposite.

We are in a position to derive this result

Theorem 4 Defining the Hermitian form with help of the formulas (34) (35) en-
sures that $d/dx$ is skew-symmetric in $(\Delta_0 \oplus J_0)^2$

Proof

Consider in $(\Delta_0 \oplus J_0)^2$ two elements $\Phi = \phi + F$, $\Psi = \psi + G$ where $\phi$ and $\psi$ are in $\Delta^\sharp_0$ and $F, G$ as in (33), with $u, v \in \mathcal{I}^+$ and $s, t \in \mathcal{I}^-$. Compute $<\Phi', \Psi> + <\Phi, \Psi'>$.

We get

$$<\Phi', \Psi> = <\phi', \psi> + <\phi', G> + <F', \psi> + <F', G>$$

$$<\Phi, \Psi'> = <\phi, \psi'> + <\phi, G'> + <F, \psi'> + <F, G'>$$

Taking into account the skew-symmetry of $d/dx$ in $\Delta^\sharp_0$ yields a first cancellation, we are left with

$$<\Phi', \Psi> + <\Phi, \Psi'> =$$

$$<\phi', G> + <F', \Psi> + <F', G> + <\phi, G'> + <\phi, \psi'> + <F, \psi'> + <F, G'>$$

(39)

Now let us calculate the contribution of $<F', G> + <F, G'>$. According to the rule (30) we find

$$<F', G> + <F, G'> = <DF + \theta F \delta, G> + <F, DG + \theta G \delta>$$

$$<F', G> + <F, G'> = <DF, G> + <F, DG> + (\theta F)^* G_0 + \theta G F_0^*$$

(40)

But $\theta F = u_0 - s_0$, $\theta G = v_0 - t_0$, hence with help of Proposition 12

$$<F', G> + <F, G'> = -u_0^* v_0 + s_0^* t_0 + (u_0^* - s_0^*) G_{<0>} + (v_0 - t_0) F_{<0>}$$

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Remember that 
\[ F_0 = \frac{1}{2}(u_0 + s_0), \quad G_0 = \frac{1}{2}(v_0 + t_0) \]
thus 
\[ <F', G> + <F, G'> = -u_0^*v_0 + s_0^*t_0 + \frac{1}{2}(u_0^* - s_0^*)(v_0 + t_0) + \frac{1}{2}(v_0 - t_0)(u_0^* + s_0^*) \]
which identically vanishes, so 
\[ <F', G> + <F, G'> = 0 \] (41)
We remain with 
\[ <\Phi', \Psi> + <\Phi, \Psi'> = <\phi', G> + <\phi, G'> + <F', \psi> + <F, \psi'> \] (42)
which vanishes in view of Proposition 13.

4.1 Solving the Schroedinger equation.

We are now in a position to consider the Schroedinger equation with a simple example of point interaction.
In view of the considerations made in subsection 1.3 let us first write down the "singular" potential in terms of the ket-bra operator \( \delta \times \delta \).
We limit ourselves to the case of an attractive potential so we write the wave equation as 
\[ y'' + \lambda y = \alpha(\delta \times \delta)y \] (43)
where \( \alpha \) is a negative real number, say \( \alpha = -2\beta \). Having in mind physical applications we look for a real standard eigenvalue \( \lambda \). Here \( y \) is supposed to be an element of \( J_0 \oplus \Delta_0^\sharp \) and the derivatives are meant in the sense of distributions.
Since we seek solutions in \( (J_0 \oplus \Delta_0)^\sharp \) we write 
\[ y = f \eta + (1 - \eta) g + \phi \] (44)
where \( f \in \mathcal{I}^+, \quad g \in \mathcal{I}^-, \quad \phi \in \Delta_0^\sharp \).
Since \( \eta' = \delta, \quad (1 - \eta)' = -\delta \), we obtain 
\[ y'' = f''\eta + g''(1 - \eta) + 2(f' - g')\delta + (f - g)\delta + \phi' \]
But \( h\delta' = h_0\delta' - h_0\delta \) for all function \( h \). Hence 
\[ y'' = f''\eta + g''(1 - \eta) + (f_0' - g_0')\delta + (f_0 - g_0)\delta' + \phi'' \] (45)
\[ y'' + \lambda y = (f'' + \lambda f)\eta + (g'' + \lambda g)(1 - \eta) + (f_0' - g_0')\delta + (f_0 - g_0)\delta' + \phi'' + \lambda \phi \] (46)
Let us now evaluate the contribution of the potential, 
\[ \alpha (\delta \times \delta)y = \alpha <\delta, y > \delta \]
\[ <\delta, y >= \frac{1}{2}f_0 + \frac{1}{2}g_0 + <\delta, \phi > \]
\[ \alpha (\delta \times \delta) y = \frac{\alpha}{2} (f_0 + g_0) \delta + \alpha < \delta, \phi > \delta \] (47)

Note that (45)(46) and (47) hold true for all \( y \) of the form (44) irrespective of the Schrödinger equation. But now insert (46) and (47) into (43); we identify and get separately

\[ (f'' + \lambda f) \eta + (g'' + \lambda g)(1 - \eta) = 0 \] (48)

\[ \frac{\alpha}{2} (f_0 + g_0) \delta + \alpha < \delta, \phi > \delta = (f'_0 - g'_0) \delta + (f_0 - g_0) \delta' + \phi'' + \lambda \phi \] (49)

**Remark**

Since \( \phi \) is a finite sum, we can write

\[ \phi = \sum_{n=0}^{r} \phi_n \delta^{(n)} \] (50)

and it can be read off (49) that the development of the distribution \( \phi'' + \lambda \phi \) cannot involve derivatives of \( \delta \) higher than \( \delta' \).

**Proposition 14** In (44) necessarily \( \phi \) vanishes

The proof is *ab absurdo*. Assume that \( \phi \) is actually of order \( r \), in other words \( \phi_r \) does not vanish. Differentiating (50) we obtain \( \phi'' = \sum_{n=0}^{r} \phi_n \delta^{(n+2)} \), hence the development

\[ \phi'' + \lambda \phi = \lambda \phi_0 \delta + \lambda \phi_1 \delta' + \lambda \phi_2 \delta'' + \cdots + \lambda \phi_r \delta^{(r)} \]

\[ + \phi_0 \delta' + \cdots + \phi_{r-2} \delta^{(r-1)} + \phi_{r-1} \delta^{(r+1)} + \phi_r \delta^{(r+2)} \]

In view of the Remark above, it is clear that \( \phi_r \) must vanish, contrary to our assumption.

On the one hand (48) splits into these two statements

\[ (f''_+ + \lambda f_+) = 0, \quad (g''_- + \lambda g_-) = 0 \] (51)

On the other hand, in view of the above Proposition we can rewrite (49) simply as

\[ \frac{\alpha}{2} (f_0 + g_0) \delta = (f'_0 - g'_0) \delta + (f_0 - g_0) \delta' \] (52)

which provides us with two boundary conditions

\[ \frac{\alpha}{2} (f_0 + g_0) = (f'_0 - g'_0) \] (53)

\[ f_0 = g_0 \] (54)

Up to arbitrary and irrelevant (smooth) modifications of \( f_- \) and \( g_+ \), the most general solution to (51) is, if \( \lambda \geq 0 \)

\[ f = A^+ \exp(i \sqrt{\lambda} x) + A^- \exp(-i \sqrt{\lambda} x) \]

\[ g = B^+ \exp(i \sqrt{\lambda} x) + B^- \exp(-i \sqrt{\lambda} x) \]
and if $\lambda < 0$, setting $\lambda = -\rho$,

$$f = a^+ \exp(\sqrt{\rho}x) + a^- \exp(-\sqrt{\rho}x)$$

$$g = b^+ \exp(\sqrt{\rho}x) + b^- \exp(-\sqrt{\rho}x)$$

where $A^\pm, B^\pm, a^\pm, b^\pm$ are in $\mathbf{F}$. But $f$ (resp. $g$) is supposed to be right (resp. left) square integrable, thus **within $\mathcal{J}_0^\# \oplus \Delta_0^\#$ no eigenstate corresponds to $\lambda \geq 0$**.

Let us turn to the possibility of a negative $\lambda$. Since $f$ (resp. $g$) is supposed to be right (resp. left) square integrable, it is clear that $a^+$ and $b^-$ must vanish. Therefore

$$f = a^- \exp(-\sqrt{\rho}x), \quad g = b^+ \exp(\sqrt{\rho}x) \quad (55)$$

Hence

$$f_0 = a^-, \quad g_0 = b^+, \quad f_0' = -a^- \sqrt{\rho}, \quad g_0' = b^+ \sqrt{\rho}$$

Now condition (54) yields $a^- = b^+$ and (discarding the trivial solution $y = 0$) condition (53) yields $\alpha = -2\sqrt{\beta}$. Thus the only possibility is

$$y = \text{const.} \ (e^{-\beta x} \eta(x) + e^{\beta x} \eta(-x)) \quad (56)$$

The constant factor is arbitrary, but standard solutions correspond to choosing this factor in $\mathbf{C}$.

Conversely, it is easy to check that (56) actually satisfies (43) provided $\lambda = -\beta^2 = -\frac{\alpha^2}{4}$. This calculation uses the formula

$$< \delta, \ e^{-\beta x} \eta(x) > = < \delta, \ e^{\beta x} \eta(-x) > = \frac{1}{2}$$

obtained from (29). [] We summarize:

**Theorem 5** In $(\mathcal{J}_0 \oplus \Delta_0)^\#$, the linear operator $-\frac{d^2}{dx^2} + \alpha(\delta \times \delta)$, with real negative $\alpha = -2\beta$, has the eigenvalue $-\frac{1}{4}\alpha^2$ and the eigenvector given by (56).

## 5 Conclusion and outlook

In this work a positive definite Hermitian form was constructed for the space of distributions with compact discrete support. The values of this form belong to a field extension of $\mathbf{C}$ which, in spite of its simplicity, allows for considering some infinitesimals and some infinitely large numbers. This extension is in an obvious sense minimal, and further extensions are certainly desirable. An amusing open question is how far can we go on doing n.s. calculus with elementary methods that ignore n.s. analysis.

Let us emphasize that here the distributions are taken for what they usually are in the environment of standard quantum mechanics, and *not replaced* by functions $\cdot \mathbf{R} \mapsto \cdot \mathbf{R}$ with infinitesimal support [2]. In this matter we remain rather conservative: the distributions we consider can be seen as $\mathbf{F}^\mathbf{C}$-valued linear functionals,
but their argument $x$ runs inside the standard real line $\mathbb{R}$. It would be interesting to make a contact with some results of reference [2] but this difference in the status of the distributions renders such a task difficult.

The structure of bra-ket vector space seems to be the most general framework available for implementing the machinery of Dirac’s formalism in a flexible but rigorous way; in most physical situations a richer structure would be desirable, at least that of inner-product space, thus more work is needed in this direction.

For elementary applications to the Schroedinger equation, the differential operator $d/dx$ must be understood in the sense of distribution theory and some subspace of $\Delta^2$ is to be unified with a suitable space of functions, both spaces getting imbedded into a larger vector space equipped with a Hermitian form. Further investigation is needed in order to determine the cases where this form remains positive definite. Naturally this line of research has obvious limitations: we avoided any kind of topological considerations, preferring to focus on the direct computational mechanism provided by the bra-ket framework.

However we expect that in several cases this approach will legitimate a lot of heuristic calculations that would seem meaningless otherwise.

As a very simple example we considered the point interaction in one dimension. The potential was treated in a natural and intuitive manner, being symmetric and everywhere defined as a linear operator. Playing with cancellations of infinite quantities, we re-derived the usual eigenvalue and eigenfunction corresponding to the bound state. Naturally we remain aware of the possibility to attain the same result through the sophisticated methods of n.s. analysis displayed in [13], but one of our goals was precisely to provide an elementary formulation available to every theoretical physicist.

\section{APPENDIX}

\textbf{Minimal extension of the real numbers.}

We take $\mathcal{B} = \mathcal{F}$ where $\mathcal{F}$ is the field of real rational functions in one real variable $X$ (rational functions being characterized by their form $f = P/Q$ where $P$ and $Q$ are polynomials).

For large enough $X$ any nonvanishing rational function $f$ takes on a definite sign, so that speaking of the \textit{sign of $f$ toward infinity} makes sense. We say that $f$ is positive or negative (toward infinity) according to this sign. Moreover this sign is compatible with the multiplication in $\mathcal{F}$, that is $\text{sign}(f)\text{sign}(g) = \text{sign}(fg)$, which permits to define a \textit{total ordering} among all the rational functions, just by saying that $f > g$ iff $f - g$ is positive toward infinity.

Most usual properties of the real numbers carry over to the field of rational fractions. For instance: $f > 0$ and $g > 0$ imply $f + g > 0$.

When $f > 0$, having $g < h$ implies $fg < fh$. When both $f, g > 0$, then also $fg$ is positive; moreover having that $f > g$ implies that $f^2 > g^2$.

Then it is possible to define an absolute value: $||f|| = f$ if $f \geq 0$ and $||f|| = -f$ otherwise. It is clear that $||f||$ is a positive extra-real unless $f = 0$, and that every
square of a non-vanishing rational fraction is positive.

Finally we can check that for all couple \( f, g \) we have \( ||a + b|| \leq ||a|| + ||b|| \).

Any rational fraction \( f(X) \) has a degree\(^4\) noted as \( \deg f \), which is non-negative for polynomials. We have these inequalities

\[
\deg(a + b) \leq \max(\deg a, \deg b) \\
\deg(ab) = \deg a + \deg b
\]

obvious for polynomials, then easily extended to rational functions.

As well known, every \( f \in \mathbb{F} \) admits a unique decomposition \( f = P + \epsilon \) where \( P(X) \) is a polynomial and \( \epsilon(X) \) is a rational function of negative degree.

According to the ordering we have \( \epsilon(X) < c < X^p \) whenever \( p \) is a positive integer and \( c \) is a positive constant, so we can interpret the positive powers of \( X \) as "infinitely large" numbers, and \( \epsilon \) as "infinitesimal".

Let \( a \) be the constant term in \( P(X) \). Then \( a + \epsilon \) can be called the finite part of \( f \), and \( a \) its standard part. This possibility of extracting a "standard part" seems to be a particular feature of \( \mathbb{F} \). It would probably not survive in further extensions from \( \mathbb{F} \) to a larger ordered field.

Although the present scheme has been introduced by elementary methods [11], it mimicks the behavior of the "hyperreal numbers" considered in the framework of n.s. analysis. The reader who is familiar with n.s. analysis can understand this as follows:

Let \( \kappa \in {}^*\mathbb{R} \) denote any infinitely large and positive hyperreal. Then the structure of \( \mathbb{F} \) is isomorphic to the class formed by all the hyperreal numbers that can be written as a rational function of \( \kappa \) (this class is an algebra on \( \mathbb{R} \)).

Due to the arbitrariness in the choice of \( \kappa \) we conclude that \( \mathbb{F} \) can be imbedded into \( {}^*\mathbb{R} \) in infinitely many different ways. But there is no preferred correspondance. Note that, in contrast to n.s. analysis, our approach has no concept of "infinite integer".

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An important difference between the choices of \( \mathcal{B} \) in Ref. [4] and here respectively, is that the choice proposed in [4] was an algebra of polynomials

\(^4\text{When } P,Q \text{ are polynomials, } \deg P/Q = \deg P - \deg Q\)
depending on infinitely many variables. In contrast, in the present work we consider polynomials depending on a single variable. The algebra considered in [4] was somehow "too large" for physical purposes. In the one considered now, the concept of positivity is recovered.

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