The Casimir effect in light-front quantization

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Abstract

We show that the standard result for the Casimir force between conducting plates at rest in an inertial frame can be computed in light-front quantization. This is not the same as light-front analyses where the plates are at “rest” in an infinite momentum frame. In that case, Lenz and Steinbacher have shown that the result does not agree with the standard result for plates at rest. The two important ingredients in the present analysis are a careful treatment of the boundary conditions, inspired by the work of Almeida et al. on oblique light-front coordinates, and computation of the ordinary energy density, rather than the light-front energy density.

\[\text{[Based on a talk contributed to the Lightcone 2014 workshop, Raleigh, North Carolina, May 26-30, 2014.]}\]
I. INTRODUCTION

The Casimir effect is a force between conducting plates that arises from the exclusion of vacuum modes by boundary conditions at the plates \[1\]. The vacuum energy density between the plates differs from the free density. This defines a separation-dependent effective potential for the plates, and variation with respect to the separation yields the force. The necessary energy density is obtained from a sum over the allowed modes. The analysis is simplified by working with a massless scalar field and periodic boundary conditions, in place of the complexity of quantum electrodynamics.

Contrary to what has been thought, the standard result for the Casimir force between conducting plates at rest in an inertial frame can be computed in light-front quantization \[2\]. This is not the same as light-front analyses where the plates are at “rest” in an infinite momentum frame \[3\], where the result does not agree. Placement of the plates in the correct frame is critical.

The light-front analysis has two important ingredients. One is a careful treatment of the boundary conditions, inspired by the work of Almeida et al. \[4\] on oblique light-front coordinates. The other is the computation of the ordinary energy density, rather than the light-front energy density. The key point here is to focus on the physics; calculations of the same physical effect in different coordinate systems must yield the same result.

We define light-front coordinates \[5, 6\] as \(x^+ = t + z\) for time and \(\vec{x} = (x^-, \vec{x}_\perp)\) for space, with \(x^- \equiv t - z\) and \(\vec{x}_\perp = (x, y)\). The light-front energy is \(p^- = E - p_z\), and the light-front momentum is \(p = (p^+, \vec{p}_\perp)\), with \(p^+ \equiv E + p_z\) and \(\vec{p}_\perp = (p_x, p_y)\). The mass-shell condition \(p^2 = m^2\) becomes \(p^- = (m^2 + p^2_\perp)/p^+\).

The natural choice for the analysis of plates at rest in an inertial frame would be equal-time coordinates, with periodicity of the field along one spatial direction. The “natural” situation for light-front coordinates is to have spatial periodicity in \(x^-\), but this corresponds to plates moving with the speed of light. Lenz and Steinbacher considered this arrangement \[3\]. Almeida et al. \[4\] used oblique light-front coordinates with \(\bar{x}^0 = t + z\) and \(\bar{x}^3 = z\), and boundary conditions periodic in \(z\), to obtain the correct result. Both attempts implied that light-front quantization is deficient in some way; instead, light-front coordinates are just harder to use.

In addition to using boundary conditions appropriate to plates at rest in an inertial frame, one must calculate the true vacuum energy. This is not the light-front energy \(p^-\). The equal-time energy \(E\) is what determines the effective potential and, therefore, the force. Other examples of where this choice matters can be found in the variational analysis of \(\phi^4\) theory \[11\] and the calculation of thermodynamic partition functions. In particular, a partition function should be computed for a system in contact with a heat bath at rest, not at the speed of light \[8, 9\], by use of \(e^{-\beta E}\), not \(e^{-\beta p^-}\).

The standard result for the Casimir force comes from the computation of the expectation value for the energy density, which can be written as a sum over zero-point energies

\[
\langle \mathcal{H} \rangle = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int \frac{d^2p_\perp}{(2\pi)^2} E_n, \tag{1.1}
\]

with \(E_n = \sqrt{p^2_\perp + \left(\frac{2\pi n}{L}\right)^2}\). The sum can be regulated by a heat-bath factor \(e^{-\Lambda E_n}\), to obtain

\[
\langle \mathcal{H} \rangle = \frac{3}{2\pi^2 L^4} - \frac{\pi^2}{90L^4}, \tag{1.2}
\]
The second term is the regulator-independent effective potential and determines the Casimir force. We now consider how this can be done in light-front coordinates.

II. LIGHT-FRONT ANALYSIS

Our goal is to impose periodic boundary conditions on a neutral massless scalar field and compute the vacuum energy density. The light-front mode expansion for the scalar field is

\[ \phi = \int \frac{dp}{\sqrt{16\pi^3 p^+}} \left\{ a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x} \right\}, \]

with the modes quantized such that

\[ [a(p), a^\dagger(p')] = \delta(p - p'). \]

For plates perpendicular to the \( z \) axis, placed at \( z = 0 \) and \( z = L \), the periodicity imposed is \( \phi(z + L) = \phi(z) \). In light-front coordinates, this periodicity is

\[ \phi(x^+ + L, x^- - L, \vec{x}_\perp) = \phi(x^+, x^-, \vec{x}_\perp), \]

which implies \(-p^+L/2 + p^-L/2 = 2\pi n \) or \( p^2/p^+ - p^+ = \frac{4\pi}{L}n \) with \( n \) any integer between \(-\infty \) and \( \infty \). The positive solution of this constraint is

\[ p^+_n \equiv \frac{2\pi}{L}n + \sqrt{\left(\frac{2\pi}{L}n\right)^2 + p^2_\perp}. \]

Then \( n = -\infty \) corresponds to \( p^+ = 0 \), and \( n = \infty \) to \( p^+ = \infty \).

A discrete mode expansion can be constructed, with use of discrete annihilation operators

\[ a_n(\vec{p}_\perp) = \sqrt{\frac{dp^+}{dn}} a(p^+_n, \vec{p}_\perp), \]

where \( \frac{dp^+}{dn} = \frac{2\pi}{L} \frac{p^+_n}{En} \). For these operators, the commutation relation becomes

\[ [a_n(\vec{p}_\perp), a^\dagger_{n'}((\vec{p}_\perp)')] = \delta_{nn'}\delta(\vec{p}_\perp - \vec{p}_\perp'). \]

The integration over \( p^+ \) becomes a sum over \( n \):

\[ \int dp^+ = \int \frac{dp^+}{dn}dn \to \sum_n \frac{dp^+}{dn}. \]

These yield

\[ \phi(x^+ = 0) = \frac{1}{\sqrt{2L}} \sum_n \int d^2p_\perp \left\{ a_n(\vec{p}_\perp)e^{-ip^+_n x^-/2 + ip_\perp \cdot \vec{x}_\perp} + a^\dagger_n(\vec{p}_\perp)e^{ip^+_n x^-/2 - ip_\perp \cdot \vec{x}_\perp} \right\}. \]

The leading \( \frac{1}{\sqrt{2L}} \) factor is consistent with normalization on the interval \([-2L, 0] \) in \( x^- \).
For the free scalar, the light-front energy and longitudinal momentum densities are $\mathcal{H}^- = \frac{1}{2} |\partial_\perp \phi|^2$ and $\mathcal{H}^+ = 2 |\partial_\perp \phi|^2$. Their vacuum expectation values are

$$
\langle 0 | \mathcal{H}^- | 0 \rangle = \frac{1}{4L} \sum_{n,n'} \int \frac{d^2p_\perp d^2p_\perp'}{(2\pi)^2} \vec{p}_\perp \cdot \vec{p}_\perp' \langle 0 | a_n(\vec{p}_\perp) a_{n'}(\vec{p}_\perp') | 0 \rangle
$$

$$
= \frac{1}{4L} \sum_n \int \frac{d^2p_\perp}{(2\pi)^2} p_\perp^2,
$$

$$
\langle 0 | \mathcal{H}^+ | 0 \rangle = \frac{2}{2L} \sum_{n,n'} \int \frac{d^2p_\perp d^2p_\perp'}{(2\pi)^2} \frac{p_\perp^2 p_{n'}^2}{4} \langle 0 | a_n(\vec{p}_\perp) a_{n'}(\vec{p}_\perp') | 0 \rangle
$$

$$
= \frac{1}{4L} \sum_n \int \frac{d^2p_\perp}{(2\pi)^2} (p_\perp^2)^2.
$$

The energy density, relative to light-cone coordinates, is

$$
\mathcal{E}_{\text{LF}} \equiv \frac{1}{2} (\langle 0 | \mathcal{H}^- | 0 \rangle + \langle 0 | \mathcal{H}^+ | 0 \rangle)
$$

$$
= \frac{1}{8L} \sum_n \int \frac{d^2p_\perp}{(2\pi)^2} (2E_n^2 + 2\frac{2\pi}{L} n E_n).
$$

The second term is zero, because it is proportional to $\sum_{n=\pm\infty} n = 0$, leaving

$$
\mathcal{E}_{\text{LF}} = \frac{1}{4L} \sum_n \int \frac{d^2p_\perp}{(2\pi)^2} E_n.
$$

However, we still need to relate this to the energy density $\mathcal{E}$ relative to equal-time coordinates.

An integration over a finite volume between the plates yields

$$
\mathcal{E} = \frac{1}{LL_\perp} \int_{-L_\perp}^0 dx^- \int_0^{L_\perp} d^2x_\perp \mathcal{E}_{\text{LF}}.
$$

A change of variable from $x^-$ to $z = (x^+ - x^-)/2$ at fixed $x^+$ brings

$$
\mathcal{E} = \frac{1}{LL_\perp^2} \int_0^L 2dx^- \int_0^{L_\perp} d^2x_\perp \mathcal{E}_{\text{LF}} = 2\mathcal{E}_{\text{LF}}.
$$

Thus, the energy density is

$$
\mathcal{E} = \frac{1}{2L} \sum_n \int \frac{d^2p_\perp}{(2\pi)^2} E_n,
$$

which matches exactly the standard result. It can be regulated with the same heat-bath factor $e^{-\Lambda E_n}$, as appropriate for a system in contact with a heat bath at rest in an inertial frame.

For the transverse case, where the plate are separated in a direction transverse to the $z$ axis, the direct implementation of light-front coordinates by Lenz and Steinbacher does yield the correct result. This is not surprising, because plates separated in the transverse direction can be at rest in an inertial frame. However, there could be concern that the
additional steps introduced here will somehow destroy this agreement, and we need to check that a consistent result is still obtained.

Let the periodicity be in the $x$ direction, with the plates at $x = 0$ and $x = L_\perp$, to require $\phi(x^+, x^-, x + L_\perp, y) = \phi(x^+, x^-, x, y)$. The momentum component $p_x$ is then restricted to the discrete values $p_n \equiv 2\pi n/L_\perp$. We define discrete annihilation operators

$$a_n(p^+, p_y) = \sqrt{\frac{2\pi}{L}} a(p^+, p_n, p_y),$$

(2.16)

with the commutation relation

$$[a_n(p^+, p_y), a_{n'}(p'^+, p'_y)] = \delta_{nn'}\delta(p^+ - p'^+)\delta(p_y - p'_y).$$

(2.17)

The scalar field is again a discrete sum

$$\phi(x^+ = 0) = \frac{1}{\sqrt{L_\perp}} \sum_n \int \frac{dp^+ dp_y dp'^+ dp'_y}{\sqrt{8\pi^2 p^+ p'^+}} \left\{ a_n(p^+, p_y) e^{-ip^+ x^+/2 + ip_n x + ip_y y} + a_n^\dagger(p^+, p_y) e^{ip^+ x^+/2 - ip_n x - ip_y y} \right\}.$$ (2.18)

The leading factor is consistent with the normalization on the interval $[0, L_\perp]$ in $x$. The energy density is again constructed from the sum of the minus and plus components

$$\langle 0| \mathcal{H}^- |0\rangle = \frac{1}{2L_\perp} \sum_{nn'} \int \frac{dp^+ dp_y dp'^+ dp'_y}{8\pi^2 \sqrt{p^+ p'^+}} (p_n p_{n'} + p_y p'_y) \langle 0|a_n(p^+, p_y) a_{n'}^\dagger(p'^+, p'_y)|0\rangle$$

$$= \frac{1}{2L_\perp} \sum_n \int \frac{dp^+ dp_y}{8\pi^2} p_n^2 + p_y^2 p^+,$$

(2.19)

and

$$\langle 0| \mathcal{H}^+ |0\rangle = \frac{2}{L_\perp} \sum_{nn'} \int \frac{dp^+ dp_y dp'^+ dp'_y}{8\pi^2 \sqrt{p^+ p'^+}} \langle 0|a_n(p^+, p_y) a_{n'}^\dagger(p'^+, p'_y)|0\rangle$$

$$= \frac{1}{2L_\perp} \sum_n \int \frac{dp^+ dp_y}{8\pi^2} p^+.$$

(2.20)

Averaged together, these fix $\mathcal{E}_{LF}$ to be

$$\mathcal{E}_{LF} = \frac{1}{2L_\perp} \sum_n \int \frac{dp^- dp_y dp^+ p^- + p^+}{8\pi^2} \delta\left(p^- - \frac{p_n^2 + p_y^2}{p^+}\right).$$

(2.21)

The delta function is equivalent to the mass-shell condition

$$\delta(p^- - (p_n^2 + p_y^2)/p^+) = p^+ \delta(p^2) = p^+ \delta(E^2 - E_n^2),$$

(2.22)

with $E_n = \sqrt{\left(\frac{2\pi}{L_\perp} n\right)^2 + p_x^2 + p_y^2}$. The form of the delta function motivates a conversion to the equal-time variables $E = (p^+ + p^-)/2$ and $p_z = (p^+ - p^-)/2$, which yields

$$\mathcal{E}_{LF} = \frac{1}{2L_\perp} \sum_n \int \frac{2dE dp_z dp_y}{8\pi^2} E (E + p_z) \frac{1}{2E_n} \delta(E - E_n).$$

(2.23)
The $E$ integral can be done trivially. The contribution from the $p_z$ term is zero, because that part of the $p_z$ integral is odd. This yields the same result as a calculation of the light-front energy density alone; the difference is just the contributions proportional to $p_z$, which integrate to zero.

This determines the energy density relative to light-front coordinates as

$$E_{LF} = \frac{1}{4L_\perp} \sum_n \int \frac{dp_z dp_y}{(2\pi)^2} E_n.$$  

(2.24)

The transformation to the energy density relative to equal-time coordinates is again just multiplication by two. Therefore, we find in the transverse case

$$E = \frac{1}{2L_\perp} \sum_n \int \frac{dp_z dp_y}{(2\pi)^2} E_n.$$  

(2.25)

which matches the usual equal-time result and is of the same form as in the longitudinal case.

III. SUMMARY

We have computed the Casimir effect for parallel plates in light-front coordinates and obtained the standard result, by remaining true to the physics of plates at rest. This is not a simple constraint in light-front coordinates, but is physically correct. Also important for the calculation was the focus on the equal-time energy as the true vacuum energy that determines the effective potential and therefore the Casimir force. This new derivation of the Casimir effect demonstrates that the physics of the effect is independent of the coordinate choice, as it must be, and that light-front quantization is not deficient in its treatment of such vacuum effects. The derivation provides additional confidence in the usefulness and applicability of the light-front approach.

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[1] Casimir HBG (1948) On the attraction between two perfectly conducting plates. Proc. K. Ned. Akad. Wet. 51: 793-795
[2] Chabysheva SS, Hiller JR (2013) Light-front analysis of the Casimir effect. Phys. Rev. D 88: 085006
[3] Lenz F, Steinbacher D (2003) The Casimir effect on the light cone. Phys. Rev. D 67: 045010
[4] Almeida T, Alves VS, Alves DT, Perez S, Rodrigues PLM (2013) Light front Casimir effect. Phys. Rev. D 87: 065028
[5] Dirac PAM (1949) Forms of relativistic dynamics. Rev. Mod. Phys. 21: 392-399
[6] For reviews of light-cone quantization, see Burkardt M (2002) Light front quantization. Adv. Nucl. Phys. 23: 1-74; Brodsky SJ, Pauli H-C, Pinsky SS (1998) Quantum chromodynamics and other field theories on the light cone. Phys. Rep. 301: 299-486
[7] Harindranath A, Vary JP (1988) Variational calculation of the spectrum of two-dimensional $\phi^4$ theory in light-front field theory. Phys. Rev. D 37: 3010–3013

[8] Elser S, Kalloniatis AC (1996) QED in (1+1)-dimensions at finite temperature: A Study with light cone quantization. Phys. Lett. B 375: 285-291

[9] Hiller JR, Proestos Y, Pinsky S, Salwen N (2004) $N = (1, 1)$ super Yang-Mills theory in 1+1 dimensions at finite temperature. Phys. Rev. D 70: 065012