Potts model with \( q = 3 \) and \( 4 \) states on directed Small-World network

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Monte Carlo simulations are performed to study the two-dimensional Potts models with \( q = 3 \) and \( 4 \) states on directed Small-World network. The disordered system is simulated applying the Heat bath Monte Carlo update algorithm. A first-order and second-order phase transition is found for \( q = 3 \) depending on the rewiring probability \( p \), but for \( q = 4 \) the system presents only a first-order phase transition for any value \( p \). This critical behavior is different from the Potts model on a square lattice, where the second-order phase transition is present for \( q \leq 4 \) and a first-order phase transition is present for \( q > 4 \).

Keywords: Monte Carlo simulation, spins, networks, Ising, Potts.

It was conjectured by Harris \([1]\) that the sign of the critical exponent of the specific heat \( \alpha \) determines whether spin systems are affected or not by randomness. For positive values of \( \alpha \) the system with randomness or impurities has a critical behavior different from the pure system case. For negative values of \( \alpha \), on the other hand, the critical behavior of the system should be the same for both pure and impure cases. In particular, for two-dimensional regular lattices, the ferromagnetic Potts model with \( q \) states displays first order phase transitions for \( q > 4 \) \([2,3]\), while the pure ferromagnetic three-state Potts model has \( \alpha = 1/3 \), hence, according to the aforementioned criterion we expect to find a different behavior for a random interaction system. However, Picco \([4]\) and Lima et al. \([5–8]\) studied this model with different type of disorder and did not find any relevant difference from the pure case.

The \( q \)-state Potts model has been studied in scale-free networks by Igloi and Turban \([9]\) and depending on the value of \( q \) and of the degree-exponent \( \gamma \) first- and second-order phase transitions were found. This model was also studied by Lima \([10]\) on directed Barabási-Albert(BA) networks, where only first-order phase transition has been obtained for any \( q \)-values with connectivity \( z = 2 \) and \( z = 7 \) of the directed BA network. Here, we studied the Potts model with \( q = 3 \) and \( 4 \) states. We also calculate the critical exponents ratio \( \beta/\nu \) and \( \gamma/\nu \) for second-order phase transitions that appears due to the SW disorder.

We consider the ferromagnetic Potts model with \( q = 3 \) and \( q = 4 \), on directed small-world networks where every site of a directed small-world network of size \( N = L \times L \) have spin variables \( \sigma \) taking values 1, 2, 3 and 1, 2, 3, 4 for \( q = 3 \) and \( 4 \), respectively. With \( L \) being the side of a square lattice. In this network, created by Sánchez et al. \([11]\) (see Fig. 1), we start from a two-dimensional square lattice consisting of sites linked to their four nearest neighbors by both outgoing and incomplete links. Then, with probability \( p \), we reconnect nearest-neighbors outgoing links to a different site chosen randomly. After repeating this process for every link, we are left with a network with a density \( p \) of SW directed links. Therefore, with this procedure every site will have exactly four outgoing links and different (random) number of incoming links. The time evolution of this system is given by a single spin-flip like dynamics with a probability \( p_i : \)

\[
p_i = \frac{1}{1 + \exp(2E_i/k_BT)}.
\]

The Hamiltonian of a \( q \)-states ferromagnetic Potts model with \( q \) states is given by:

\[ H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j \]

where \( J > 0 \) is the coupling constant between nearest neighbors, \( \sigma_i \) are the spin variables, and \( \langle ij \rangle \) denotes the sum over nearest-neighbor pairs.
model can be written as

\[ H = -J \sum_{<i,j>} \delta_{\sigma_i \sigma_j}, \]

where \( \delta \) is the Kronecker delta function, and the sum runs over all neighbors of \( \sigma_i \).

The simulations have been performed applying the HeatBath update algorithm on different lattice sizes: \( N = 64, 256, 1024, 4096, \) and 16384. For each system size quenched averages over the connectivity disorder are approximated by averaging over \( R = 40 \) independent realizations. For each simulation we have started with a uniform configuration of spins (the results are independent of the initial configuration). We ran \( 4 \times 10^4 \) Monte Carlo steps (MCS) per spin with \( 2 \times 10^4 \) configurations discarded for thermalization.

In studying the critical behavior of the model using the HeatBath algorithm we define the variable \( e = E/N \), where \( E \) is the energy of system, and the magnetisation of system \( M = (q, \max|n_i| - N)/(q - 1) \), where \( n_i \leq N \) denote the number of spins with ‘orientation’ \( i = 1, ..., q \). From the fluctuations of \( e \) measurements we can compute: the average of \( e \), the specific heat \( C \) and the fourth-order cumulant of \( e \),

\[ u(T) = [< E >]_{av}/N, \]

\[ C(T) = K^2N[< e^2 > - < e^2 >^2]_{av}, \]

\[ B(T) = \left[ 1 - \frac{< e^4 >}{3 < e^2 >^2} \right]_{av}, \]

the temperature can be defined as \( T = J/k_B K \), where \( k_B \) is the Boltzmann constant. Similarly, we can derive from the magnetization measurements the average magnetization \( (m = M/N) \), the susceptibility, and the magnetic cumulants,

\[ m(T) = [< |m| >]_{av}, \]

\[ \chi(T) = KN[< m^2 > - < |m| >^2]_{av}, \]

\[ U_4(T) = \left[ 1 - \frac{< m^4 >}{3 < m^2 >^2} \right]_{av}. \]

where in all the above equations \( < ... > \) stands for a thermodynamic average and \( [... ]_{av} \) for an average over the 40 realizations.

To verify this transition order for this model, we apply finite-size scaling (FSS) \[12\]. Initially we search for the minima of the fourth-order parameter of Eq. (5).

This quantity gives a qualitative as well as a quantitative description of the order of the transition \[13\]. It is known \[14\] that this parameter takes a minima value \( B_{\min} \) at effective transition temperature \( T_c(N) \). One can show \[13\] that for a second-order transition \( \lim_{N \to \infty} (2/3 - B_{\min}) = 0 \), even at \( T_c \), while at a first-order transition the same limit using the same quantity is small and \( (2/3 - B_{\min}) \neq 0 \).

A more quantitative analysis can be carried out through the FSS of the \( C \) fluctuation \( C_{\max} \), the susceptibility maxima \( \chi_{\max} \) and the minima of the Binder parameter \( B_{\min} \).

If the hypothesis of a first-order phase transition is correct, we should then expect, for large systems sizes, an asymptotic FSS behavior of the form \[12,13\],

\[ C_{\max} = a_C + b_C N + ..., \]

\[ \chi_{\max} = a_\chi + b_\chi N + ..., \]

\[ B_{\min} = a_B + b_B / N + ..., \]

if the hypothesis of a second-order phase transition is correct, we should then expect, for large systems sizes, an asymptotic FSS behavior of the form

\[ C = C_{\text{reg}} + L^{\alpha/\nu} f_C(x)[1 + ...], \]

\[ m = L^{-\beta/\nu} f_m(x)[1 + ...], \]

\[ \chi = L^{\gamma/\nu} f_\chi(x)[1 + ...], \]

\[ \frac{dU_4}{dT} = L^{1/\nu} f_{U_4}(x)[1 + ...], \]

where \( C_{\text{reg}} \) is a regular background term, \( \nu, \alpha, \beta, \) and \( \gamma \) are the usual critical exponents, and \( f_i(x) \) are FSS functions with

\[ x = (T - T_c)L^{1/\nu}, \]

being the scaling variable, and the brackets \( [1 + ...] \) indicate corrections-to-scaling terms. Therefore, from the size dependence of \( M \) and \( \chi \) we obtain the exponents \( \beta/\nu \) and \( \gamma/\nu \), respectively. The maxima value of susceptibility also scales as \( L^{\gamma/\nu} \).

For each value of \( q \), we apply the finite size scaling technique \[12\], and the same procedure is done for systems with different number of sites \( N = 64, 256, 1024, 4096, \) and 16384. The critical temperature for infinite size system is estimated by using the fourth-order magnetization (Binder) cumulant.

In Fig. 2, we show the dependence of the energy \( u \) and magnetization \( m \) on the temperature \( T \), obtained from simulations on directed SW network with lattice size \( L = 8, 16, 32, 64 \), and 128 and the rewiring probability \( p = 0.0, 0.1, \) and 0.9. The shape of \( m(T) \) and energy \( u \) curve, for
the particular parameters used ($N=16384$ and $q=3$), suggests the existence of a second-order phase transition in the system for $p=0.0$ and $p=0.1$, and a first-order phase transition in the system for $p=0.9$. The phase transition occurs at the value of the critical parameter $T_c$. The energetic Binder cumulant as a function of the reduced temperature $T$ is shown in Fig. 3 for $p=0.1$ and 0.9 and different lattice sizes ($L=8$ to 128). From the figure one can see a typical second-order phase transition (for a large system $B_c(T) \rightarrow 2/3$) and a first-order phase transition is observed for $p=0.1$ and 0.9, respectively.

In Fig. 4, the difference $2/3 - B_{\text{min}}$ is shown as a function of the parameter $1/\sqrt{N}$ for $p=0.1$ and $p=0.9$. For $p=0.1$, a second-order transition takes place since the $\lim_{N \rightarrow \infty} (2/3 - B_{i,\text{min}}) = 0$, even at $T_c$. However, for $p=0.9$ a first-order transition is observed, because one has $(2/3 - B_{i,\text{min}}) \neq 0$.

We display the scalings for natural logarithm for the dependence of the magnetization $m$ on inflection point at $K=T_c(L)$ and $p=0.1$ for $q=3$ in the Figure 5. The slopes of curves correspond to the exponent ratio $\beta/\nu$ according to Eq. 13. The obtained exponents are $\beta/\nu = 0.24(5)$. The exponents ratio $\gamma/\nu$ are obtained from the slopes of the straight lines with $\gamma/\nu = 1.5(1)$ for SW, as presented in Fig. 6 and obtained from Eq. 14. The results present a reliable indication in favor of the Harris criterium, error bars are only statistical, and much larger systems might give different exponents, also, the exponents ratio $\beta/\nu$ and $\gamma/\nu$ obey the hyper-scaling law $\gamma/\nu + 2\beta/\nu = d$.

Next, we study the case where $q=4$. In Fig. 7, as in the Fig. 2, we show the dependence of the magnetization $m$ and energy $u$ on the temperature $T$, obtained from simulations on directed with lattice size $L=8, 16, 32, 64$, and 128 with ($L \times L = N$) sites and the rewiring probability $p=0.0$, $p=0.1$, and $p=0.9$. The shape of $m(T)$ and energy $u$ curve, for a given value of $N=16384$ sites and $q=4$, suggests the presents of the second-order phase
FIG. 5. Display of the magnetisation at the inflection point versus the size system $L$ for $p = 0.1, q = 3$.

FIG. 6. Logarithmic plots of the susceptibility at $T_c$ versus the size system $L$ for $p = 0.1, q = 3$.

FIG. 7. The same plot of Fig. 2, but now for $q = 4$.

FIG. 8. The same plot of Fig. 5, but now for $q = 4$.

transition in the system for $p = 0.0$, but also suggests the presence of the first-order phase transition in the system for $p = 0.1$ and $0.9$.

In Fig. 8, as in the Fig. 5, we plot the difference $2/3 - B_{min}$ as a function of the parameter $1/\sqrt{N}$ for different probabilities $p = 0.1$ and $p = 0.9$. Unlike the $q = 3$ case, for both values $p = 0.1$ and $0.9$ a first-order transition is observed, because $(2/3 - B_{i, min}) \neq 0$.

In conclusion, we have presented simulations for Potts model with $q = 3$, and 4 states on directed SW network. The disordered system is simulated applying the Heat-Bath Monte Carlo update algorithm. The Potts model with $q = 3$ does display a second-order phase for rewiring probability $p = 0.1$, with exponent ratio $\beta/\nu = 0.24(5)$ and $\gamma/\nu = 1.5(1)$ that are different of the Potts model on a regular lattice, where, the specific-heat exponent $\alpha = 2/3$ is a good candidate for a change of the critical exponents, that agree with the Harris criterium [1] and obey the hyper-scaling law $\gamma/\nu + 2\beta/\nu = d$ and for case of $p = 0.9$ we have a first-order phase transition. In the case $q = 4$ both values here studied rewiring probability $p = 0.1$ and $0.9$ present a first-order phase transition as showed in the Fig. 7 and 8, that again agree with Harris criteria. In summary, the behavior of Potts model for $q = 3$ and $4$, here studied, is due to the directed links of the SW networks, where can have short and long range interaction.

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[1] A.B. Harris, J. Phys. C 7, 1671 (1974).
[2] F.Y. Wu, Rev. Mod. Phys. 54, 235 (1982).
[3] C. Tsallis, Phys. Rep. 268, 305 (1996).
[4] M. Picco, Phys. Rev. B 54, 1493 (1996).
[5] F.W.S. Lima, U.M.S. Costa, M.P. Almeida, and J.S. Andrade Jr., Eur. Phys. J. B 17, 111 (2000).
[6] F. W. S. Lima, U. L. Fulco, and R. N. Costa Filho, Phys. Rev. E 71, 036105 (2005).
[7] F. W. S. Lima, R. N. Costa Filho, and U. M. S. Costa, J. Mag. Mag. Mat. 270, 182 (2004).
[8] F. W. S. Lima, U. M. S. Costa, and R. N. Costa Filho, Physica A 387, 1545 (2008).
[9] F. Igloi and L. Turban, Phys. Rev. E66, 036140 (2002), cond-mat/0206522.
[10] F. W. S. Lima, Commun. Comput. Phys. 2, 358-366 (2007).
[11] Alejandro D. Sanchez, Juan M. Lopez, and Miguel A. Rodriguez, Phys. Rev. Lett. 88, 048701-1 (2002).
[12] See Finite Size Scaling and Numerical Simulation of Statistical Systems, edited by V. Privman (World Scientific, Singapore, 1990).
[13] M.S.S. Challa, D. P. Landau, K. Binder, Phys. Rev. B, 34, 1841 (1986).
[14] W. Janke, Phys. Rev. B 47, 14757 (1993).
[15] K. Binder, D. J. Herrmann, Monte-Carlo Simulation in Statistical Phys., (Springer-Verlag, Berlin, 1988), p. 61-62.
[16] W. Janke, Mohammad Katoot and R. Villanova, Phys. Rev. B, 49, 9644 (1994).
[17] W. Janke, R. Villanova, Phys. Lett. A 209, 179 (1995).
[18] F.W.S. Lima, J.E. Moreira, J.S. Andrade Jr., and U.M.S. Costa, Eur. Phys. J. B 13, 107 (2000).