Quantum Inozemtsev model, quasi-exact solvability and $\mathcal{N}$-fold supersymmetry

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Abstract

Inozemtsev models are classically integrable multi-particle dynamical systems related to Calogero-Moser models. Because of the additional $q^6$ (rational models) or $\sin^2 2q$ (trigonometric models) potentials, their quantum versions are not exactly solvable in contrast with Calogero-Moser models. We show that quantum Inozemtsev models can be deformed to be a widest class of partly solvable (or quasi-exactly solvable) multi-particle dynamical systems. They posses $\mathcal{N}$-fold supersymmetry which is equivalent to quasi-exact solvability. A new method for identifying and solving quasi-exactly solvable systems, the method of pre-superpotential, is presented.
1 Introduction

In this paper we address the problem of the relationship among classical integrability, quantum integrability and quantum partial integrability (or quasi-exactly solvability) within the multi-particle dynamical systems by taking a wide class of explicit examples, Inozemtsev models. We demonstrate that the Inozemtsev models (with degenerate potentials, that is non-elliptic potentials) can be made quantum partly solvable (or QES, quasi-exactly solvable). For this purpose we present a simple new formulation of QES systems of single as well as multiple degrees of freedom. We also show that the notions of higher derivative (or non-linear or $\mathcal{N}$-fold) supersymmetry and quasi-exact solvability are equivalent. In other words, Inozemtsev models provide plentiful examples of multi-particle $\mathcal{N}$-fold supersymmetry.

Inozemtsev models [1]-[5] are generalisation of Calogero-Moser models [6, 7, 8],[9, 10],[12, 13] associated with the root system of $BC$ type and $A$ type. They belong to the category of ‘twisted’ Calogero-Moser models [12, 14]. Like all the Calogero-Moser models they are classically integrable for all the four types of potentials, elliptic, hyperbolic, trigonometric and rational in the sense that their equations of motion can be expressed in Lax pair forms.

We start by asking a general and naturally vague question: to which extent does classical integrability imply quantum integrability in multi-particle dynamical systems? As is well-known, the converse, that quantum integrability always implies classical integrability, is true in multi-particle dynamical systems, since the quantum system is the $\hbar$ deformation of the classical one. We do not know a way to answer this question abstractly by starting from the pure notion of classical integrability, although some attempts have been made to construct quantum conserved quantities as a deformation of classical ones in the framework of perturbed conformal field theory (see, for example, [15]).

However, the experience of Calogero-Moser models, the widest class of integrable multi-particle systems ever known, tells that the classical integrability is very close to quantum integrability. The quantum integrability is proved universally, that is for all the root systems including the non-crystallographic ones, for Calogero-Moser models with degenerate potentials [16, 17], namely those with trigonometric, hyperbolic and rational potentials.

Thus we are naturally led to the question of quantum integrability of Inozemtsev models with degenerate potentials. Rational Inozemtsev models have an additional potential of sixth degree polynomial in $q$ (coordinates), see [3,10], on top of the Calogero-Moser potentials.
Trigonometric Inozemtsev models have an additional $\sin^2 2q$ potential, see (4.11), on top of the Calogero-Moser potentials. These additional potentials destroy the mechanism for providing quantum conserved quantities, the so-called ‘sum-to-zero’ condition of the second member of the Lax pair [18, 16].

The very interactions ($q^6$, $\sin^2 2q$, etc) that constitute obstructions for quantum integrability of Inozemtsev models are known to play essential roles in the quasi-exactly solvable one particle quantum mechanics. This leads to a conjecture that at least a certain class of Inozemtsev models can be made quasi-exactly solvable. We demonstrate that supersymmetrisable (see (2.3) and (2.4)) Inozemtsev models can be deformed to quasi-exactly solvable systems which are characterised by an integer deformation parameter $\mathcal{M}$. It is also shown that the concepts of quasi-exact solvability and higher derivative [20] or non-linear [21] or $\mathcal{N}$-fold [22] supersymmetry (with a typical relationship $\mathcal{N} = \mathcal{M} + 1$) are equivalent.

This paper is organised as follows. In section two we present the basic tool for investigating quasi-exactly solvable systems which we call the method of pre-superpotential. We briefly summarise the classical Inozemtsev models in comparison with Calogero-Moser models. In section three we demonstrate the quasi-exact solvability of a single particle rational $BC$ type Inozemtsev model based on a new method of employing a pre-superpotential $W$. This provides the most general single particle QES system with $q^6$ potential known to date. The equivalence of quasi-exact solvability and $\mathcal{N}$-fold supersymmetry is generally established. Other related notions, the “Bethe Ansatz” type equations [23], ODE spectral equivalence [24, 25] and Bender-Dunne polynomials [26] are simply explained from the new point of view. Section four deals with the quasi-exact solvability of a single particle trigonometric $BC$ type Inozemtsev model. In section five we discuss the rational $A$ type Inozemtsev model with $q^4$ potential, which provides an example of a spontaneously broken $\mathcal{N}$-fold supersymmetry. Through section six to section eight, various Inozemtsev models (rational $BC$ type, trigonometric $BC$ type and trigonometric $A$ type) are shown to be quasi-exactly solvable and the generators of $\mathcal{N}$-fold supersymmetries are identified. The QES of quantum Inozemtsev models is the consequence of exact solvability of quantum Calogero-Moser models and QES of the added single particle type interactions. The final section is devoted for comments and discussion. Appendix A gives the classical Lax pairs for the $BC$ type and $A$ type Inozemtsev models in the same notation as used in the main text. Appendix B is for the details of the lower triangularity of trigonometric Calogero-Moser interactions which are necessary for
establishing quasi-exact solvability of trigonometric Inozemtsev models.

2 Basic tool and Classical Inozemtsev models

2.1 Basic tool

One basic tool for showing the existence of some exact eigenfunctions (quasi-exact solvability) is the following simple fact. Let \( W = W(q) \) be a real smooth function of the coordinate(s), then trivial differentiation formulas \( (p = -i\partial/\partial q) \)

\[
p^2 e^W = - \left[ \left( \frac{\partial W}{\partial q} \right)^2 + \frac{\partial^2 W}{\partial q^2} \right] e^W, \quad \sum_{j=1}^{r} p_j^2 e^W = - \sum_{j=1}^{r} \left[ \left( \frac{\partial W}{\partial q_j} \right)^2 + \frac{\partial^2 W}{\partial q_j^2} \right] e^W,
\]

imply that \( e^W \) is an eigenfunction of the Hamiltonian \( H \) with eigenvalue 0:

\[
H e^W = 0, \quad H = \frac{1}{2} p^2 + \frac{1}{2} \left[ \left( \frac{\partial W}{\partial q} \right)^2 + \frac{\partial^2 W}{\partial q^2} \right], \quad H = \frac{1}{2} \sum_{j=1}^{r} p_j^2 + \frac{1}{2} \sum_{j=1}^{r} \left[ \left( \frac{\partial W}{\partial q_j} \right)^2 + \frac{\partial^2 W}{\partial q_j^2} \right], \quad (2.1)
\]

so long as it is square integrable

\[
\int e^{2W} dq < \infty. \quad (2.2)
\]

This is the simplest example of quasi-exact solvability. Looked differently, one might say this is a property of ‘factorised’ Hamiltonians

\[
H = \frac{1}{2} \left( p - i \frac{\partial W}{\partial q} \right) \left( p + i \frac{\partial W}{\partial q} \right), \quad H = \frac{1}{2} \sum_{j=1}^{r} \left( p_j - i \frac{\partial W}{\partial q_j} \right) \left( p_j + i \frac{\partial W}{\partial q_j} \right), \quad (2.3)
\]

together with a differential operator(s) that annihilate the state

\[
\left( p + i \frac{\partial W}{\partial q} \right) e^W = 0, \quad \left( p_j + i \frac{\partial W}{\partial q_j} \right) e^W = 0, \quad j = 1, \ldots, r. \quad (2.4)
\]

This fact can also be considered as the very base of supersymmetric quantum mechanics.

This gives the ground state wavefunction of the quantum Calogero-Moser models. In other words, all the (quantum integrable) Calogero-Moser models can be described by pre-superpotentials \( W \) \[10, 18\]. To sum up, if a Hamiltonian can be expressed in terms of \( W \) as (2.1) or (2.3) up to a constant, the existence of one eigenfunction is guaranteed save the square normalisability. Throughout this paper we call function \( W \) a pre-superpotential.
2.2 Classical models

Here we present classical Inozemtsev models together with Calogero-Moser models for comparison. The dynamical variables are canonical coordinates \( \{q_j\} \) and their canonical conjugate momenta \( \{p_j\} \). We denote them by \( r \)-dimensional vectors \( q \) and \( p \) with standard inner product:

\[
q = (q_1, \ldots, q_r) \in \mathbb{R}^r, \quad p = (p_1, \ldots, p_r) \in \mathbb{R}^r, \quad p^2 = p \cdot p = p_1^2 + \cdots + p_r^2.
\]

As is well-known for a root system \( \Delta \) (rank \( r \)) and four types of potentials, elliptic, trigonometric, hyperbolic and rational, Calogero-Moser and Inozemtsev models are classically completely integrable. In this paper we discuss only those models based on classical root systems, that is \( A \) type and \( BC \) (and \( D \)) type and degenerate potentials, that is trigonometric, hyperbolic and rational potentials. Since algebraic structures are almost the same for the trigonometric and hyperbolic potential models, we discuss trigonometric case as a representative. Among various types of Inozemtsev models \([1]-[5]\) we focus our attention to the supersymmetrisable models, namely to those models whose Hamiltonians can be collectively expressed in terms of a pre-superpotential \( W = W(q) \) \([18]\) as

\[
H = \frac{1}{2} p^2 + \frac{1}{2} \sum_{j=1}^{r} \left( \frac{\partial W}{\partial q_j} \right)^2, \quad (2.5)
\]

or ‘factorisable’ at the classical level:

\[
H = \frac{1}{2} \sum_{j=1}^{r} \left( p_j - i \frac{\partial W}{\partial q_j} \right) \left( p_j + i \frac{\partial W}{\partial q_j} \right). \quad (2.6)
\]

Each specific model in this class is given by a choice of \( W \), which are listed as below.

2.3 \( BC \) type Calogero-Moser models

The rational model pre-superpotential \( W \)

\[
W_{C-M} = g_M \sum_{j<k}^{r} \{ \log |q_j - q_k| + \log |q_j + q_k| \} + g_S \sum_{j=1}^{r} \log |q_j|, \quad (2.7)
\]

contains two real coupling constants \( g_M \) for the middle roots (length\(^2\)=2) and \( g_S \) for the short roots (length\(^2\)=1). The trigonometric model pre-superpotential \( W \)

\[
W_{C-M} = g_M \sum_{j<k}^{r} \{ \log |\sin(q_j - q_k)| + \log |\sin(q_j + q_k)| \} + g_S \sum_{j=1}^{r} \log |\sin q_j| + g_T \sum_{j=1}^{r} \log |\sin 2q_j|, \quad (2.8)
\]

\]
has one more coupling constant than the rational case, $g_L$ for the long roots (length$^2$=4).
For the rational potential, the long roots and short roots are essentially the same. When $g_L = 0$ and $g_S = 0$ the models belong to the $D_r$ root system. If $g_L = 0$ and $g_S \neq 0$ ($g_L \neq 0$ and $g_S = 0$) the models belong to the $B_r$ ($C_r$) root system. Throughout this paper we put the scale factor in the trigonometric functions to unity for simplicity.

2.4 A type Calogero-Moser models

For $A$ type models, it is customary to express the roots by embedding in a space with one higher dimensions. We will discuss $A_{r-1}$ models with $r$ degree of freedom. Since all the roots have the same length, $A$ type models have only one real coupling constant $g$. The rational model pre-superpotential $W$ is given by

$$W_{C-M} = g \sum_{j<k} \log |q_j - q_k|, \quad (2.9)$$

whereas the trigonometric pre-superpotential $W$ reads

$$W_{C-M} = g \sum_{j<k} \log |\sin(q_j - q_k)|. \quad (2.10)$$

2.5 BC type Inozemtsev models

The rational supersymmetrisable $BC$ type Inozemtsev model \[4, 5\] has two more real coupling constants, $a$ and $b$, than the corresponding Calogero-Moser model,

$$W = -\sum_{j=1}^r \left(\frac{a}{4}q_j^4 + \frac{b}{2}q_j^2\right) + W_{C-M}(2.7), \quad (2.11)$$

which leads to degree six polynomial potentials. The trigonometric supersymmetrisable $BC$ type Inozemtsev model \[4, 5\] has also two more real coupling constants, $a$ and $b$, than the corresponding Calogero-Moser model,

$$W = \sum_{j=1}^r \left(-\frac{a}{2}\cos 2q_j + \frac{b}{2}\log |\cot q_j|\right) + W_{C-M}(2.8). \quad (2.12)$$

2.6 A type Inozemtsev models

The rational supersymmetrisable $A$ type Inozemtsev model \[4, 5\] has two more real coupling constants, $a$ and $b$, than the corresponding Calogero-Moser model,

$$W = \sum_{j=1}^r \left(\frac{a}{3}q_j^3 + \frac{b}{2}q_j^2\right) + W_{C-M}(2.9), \quad (2.13)$$
which leads to degree four polynomial potentials. The trigonometric A type Inozemtsev model \[1\]–\[3\] has only one more real coupling constant, \(a\), than the corresponding Calogero-Moser model, 

\[
W = \sum_{j=1}^{r} \left( -\frac{a}{2} \cos 2q_j \right) + W_{C-M}(2.10) .
\]  

(2.14)

It should be emphasised that these additional interactions of the supersymmetrisable Inozemtsev models are all ‘single particle’ type. In the following three sections \[3\]–\[5\] we will investigate the characteristic single particle dynamics of supersymmetrisable Inozemtsev models.

3 Rational \(BC\) type Inozemtsev model with one degree of freedom

Let us start with a pre-superpotential \(W = W(q)\) which is decomposed into two parts:

\[
W = W_0 + W_1 ,
\]

(3.1)

\[
W_0 = -\left( \frac{a}{4} q^4 + \frac{b}{2} q^2 \right) + g_S \log |q| , \quad a > 0 , \quad g_S > 0 ,
\]

(3.2)

\[
W_1 = \sum_{k=1}^{\mathcal{M}} \log |q^2 - \xi_k| ,
\]

(3.3)

in which \(\mathcal{M}\) is an arbitrary non-negative integer and \(\{\xi_k\}\)’s are distinct but as yet undetermined parameters. The first part \(W_0\) corresponds to the ‘single particle interactions’ of the \(BC\) type rational Inozemtsev model, which is an even function of \(q\). The added part gives rise to an arbitrary polynomial in \(q^2\) of degree \(\mathcal{M}\) in \(e^W = e^{W_0} \prod_{k=1}^{\mathcal{M}} (q^2 - \xi_k)\). Since

\[
\left( \frac{\partial W}{\partial q} \right)^2 + \frac{\partial^2 W}{\partial q^2} = \left( \frac{\partial W_0}{\partial q} \right)^2 + \frac{\partial^2 W_0}{\partial q^2} + 2 \frac{\partial W_0}{\partial q} \frac{\partial W_1}{\partial q} + \left( \frac{\partial W_1}{\partial q} \right)^2 + \frac{\partial^2 W_1}{\partial q^2} ,
\]

we will evaluate the terms containing \(W_1\):

\[
2 \frac{\partial W_0}{\partial q} \frac{\partial W_1}{\partial q} + \left( \frac{\partial W_1}{\partial q} \right)^2 + \frac{\partial^2 W_1}{\partial q^2} 
\]

\[
= 2 \left\{ -(aq^3 + bq) + \frac{g_S}{q} \right\} \sum_{k=1}^{\mathcal{M}} \frac{2q}{q^2 - \xi_k} + \sum_{k=1}^{\mathcal{M}} \frac{2}{q^2 - \xi_k} + 8q^2 \sum_{k<l} \frac{1}{q^2 - \xi_k q^2 - \xi_l} .
\]

(3.4)

This is a meromorphic function in \(q^2\) with at most simple poles. We demand that the residues of the simple poles, \(q^2 = \xi_k, k = 1, \ldots, \mathcal{M}\) should all vanish \[23\], which results in a set of
rational ("Bethe ansatz" type) equations for \( \{\xi_k\} \)’s:

\[
2 \left\{ -(a\xi_k^2 + b\xi_k) + g_S \right\} + 1 + 4\xi_k \sum_{l \neq k} \frac{1}{\xi_k - \xi_l} = 0, \quad k = 1, \ldots, M. \tag{3.5}
\]

Then expression (3.4) becomes a linear polynomial in \( q^2 \), which is easy to evaluate

\[
(3.4) = -4aMq^2 - 4bM - 4a \sum_{k=1}^{M} \xi_k.
\]

Thus we arrive at

\[
\left( \frac{\partial W}{\partial q} \right)^2 + \frac{\partial^2 W}{\partial q^2} = \left( \frac{\partial W_0}{\partial q} \right)^2 + \frac{\partial^2 W_0}{\partial q^2} - 4aMq^2 - 2E_1, \tag{3.6}
\]

\[
E_1 = 2bM + 2a \sum_{k=1}^{M} \xi_k. \tag{3.7}
\]

It should be emphasised that except for the constant term \( E_1 \) the expression (3.6) is independent of the parameters \( \{\xi_k\} \)’s introduced in \( W_1 \).

This means that, for each set of solutions \( \{\xi_k\} \)’s (with real \( \sum \xi_k \) and up to the ordering), we have an eigenstate

\[
e^W = e^{W_0} \prod_{k=1}^{M} (q^2 - \xi_k) = q^{gs} e^{-(\frac{a}{2}q^4 + \frac{b}{2}q^2)} \prod_{k=1}^{M} (q^2 - \xi_k) \tag{3.8}
\]

with eigenvalue \( E_1, (3.7) \), of the Hamiltonian

\[
H = \frac{1}{2} p^2 + \frac{1}{4} \left[ \left( \frac{\partial W_0}{\partial q} \right)^2 + \frac{\partial^2 W_0}{\partial q^2} \right] - 2aMq^2, \tag{3.9}
\]

\[
= \frac{1}{2} p^2 + \frac{1}{4} q^2 (a^2 q^2 + b^2) + \frac{gs(g_s - 1)}{2q^2} - a(\frac{3}{2} + 2M + g_S)q^2 - \frac{b}{2}(1 + 2g_S). \tag{3.10}
\]

It has \( q^6, q^4, q^2 \) and \( 1/q^2 \) potentials and a part of the coefficients of the quadratic potential is quantised. Because of the singular centrifugal term \( 1/q^2 \), we restrict the function space to the half line, \( (0, +\infty) \). The restriction on the coupling constants, \( a > 0 \) is for securing the square integrability of \( e^W \) at \( q = +\infty \) and \( g_S > 0 \) for finiteness at \( q = 0 \).

The above result implies that by adding a single term \(-2aMq^2\) to the Hamiltonian the single particle \( BC \) type rational Inozemtsev model can be made quasi-exactly solvable, that is a finite number of eigenstates together with their eigenvalues can be obtained exactly by algebraic means. The very term \(-\frac{a}{4}q^4\) in \( W_0 \) that obstructs quantum integrability is
instrumental for the introduction of the additional term $-2a\mathcal{M}q^2$. The eigenfunction \((3.8)\) of the above quasi-exactly solvable system belongs to a “polynomial space”

$$\mathcal{V}_\mathcal{M} = \text{Span} \left[ 1, q^2, \ldots, q^{2k}, \ldots, q^{2\mathcal{M}} \right] e^{W_0}. \quad (3.11)$$

In other words, the Hamiltonian \((3.10)\) leaves this polynomial space invariant:

$$H\mathcal{V}_\mathcal{M} \subseteq \mathcal{V}_\mathcal{M}. \quad (3.12)$$

Therefore these “polynomial space” can be called the exactly solvable sector of the system.

It is elementary to see that the “polynomial space” \(\mathcal{V}_\mathcal{M}\) is annihilated by an \(\mathcal{N} = \mathcal{M} + 1\) st order differential operator \(P_\mathcal{N}\):

$$P_\mathcal{N} = \prod_{k=0}^{N-1} \left( D + \frac{i}{q} q^k \right) \left( D + \frac{i(N-1)}{q} \right) \cdots \left( D + \frac{i}{q} \right) D, \quad (3.13)$$

$$P_\mathcal{N}\mathcal{V}_\mathcal{M} = 0, \quad D = p + i \frac{\partial W_0}{\partial q}. \quad (3.14)$$

Since \(P_\mathcal{N}\) is an \(\mathcal{N} = \mathcal{M} + 1\) st order differential operator, it is obvious that \(\mathcal{V}_\mathcal{M}\) gives the entire solution space of a differential equation

$$P_\mathcal{N} y = 0.$$ 

This differential operator together with its hermitian conjugate defines a higher derivative \cite{20} or non-linear \cite{21} or an \(\mathcal{N}\)-fold \cite{22} supersymmetry generated by

$$Q = P_\mathcal{N} \psi^\dagger, \quad Q^\dagger = P_\mathcal{N}^\dagger \psi, \quad (3.15)$$

in which \(\psi\) and \(\psi^\dagger\) are fermion annihilation and creation operators. The “polynomial space” \(\mathcal{V}_\mathcal{M}\) is characterised as the zero-modes of \(Q\) and \(Q^\dagger\)

$$Q\mathcal{V}_\mathcal{M} = Q^\dagger\mathcal{V}_\mathcal{M} = 0, \quad (3.16)$$

which is the generalisation of the property of the ground state of the ordinary \((\mathcal{N} = 1)\) supersymmetric quantum mechanics. The second equality \(Q^\dagger\mathcal{V}_\mathcal{M} = 0\) is trivial since \(\mathcal{V}_\mathcal{M}\) has zero fermion number.

The structure of the exactly solvable sector can be better understood by making a similarity transformation of \(H\) by \(e^{W_0}\) (see \cite{19} for example):

$$\tilde{H} = e^{-W_0}H e^{W_0} = \frac{1}{2}p^2 - \frac{\partial W_0}{\partial q} \frac{\partial}{\partial q} - 2a\mathcal{M}q^2,$$

$$= \frac{1}{2}p^2 + (aq^3 + bq - \frac{gs}{q}) \frac{\partial}{\partial q} - 2a\mathcal{M}q^2. \quad (3.17)$$
Then the above “polynomial space” (3.11) is mapped to a genuine polynomial space

\[ \tilde{V}_M = \text{Span}\left[1, q^2, \ldots, q^{2k}, \ldots, q^{2M}\right], \quad (3.18) \]

whose invariance under \( \tilde{H} \) (3.17)

\[ \tilde{H}\tilde{V}_M \subseteq \tilde{V}_M \]

is rather elementary to verify. If one substitutes an expansion

\[ \tilde{\Psi} = \sum_{k=0}^{M} \alpha_k q^{2k}, \quad \alpha_0 = 1 \]

into the eigenvalue equation

\[ \tilde{H}\tilde{\Psi} = E\tilde{\Psi} \]

one obtains a three term recursion relation for \( \{\alpha_k\} \)’s

\[ (k + 1)(2k + 1 + g_S)\alpha_{k+1} = (2kb - E)\alpha_k + 2a(k - M - 1)\alpha_{k-1}. \quad (3.19) \]

This determines \( \alpha_k \) as a polynomial in \( E \) of degree \( k \) which is a Bender-Dunne polynomial [26] in the naivest sense. The condition \( \alpha_{M+1} = 0 \) gives the characteristic equation of \( \tilde{H} \):

\[ \alpha_{M+1} = 0 \Leftrightarrow (2Mb - E)\alpha_M - 2a\alpha_{M-1} = 0 \Leftrightarrow \det(\tilde{H} - E) = 0. \quad (3.20) \]

In the rest of this section let us discuss the relationship between quasi-exact solvability and \( \mathcal{N} \)-fold supersymmetry in the general context. This applies to the other cases discussed in the following sections as well. The \textit{exactly solvable sector} of a quasi-exactly solvable theory is characterised by its “polynomial space”

\[ V_M = \text{Span}\left[1, h, \ldots, h^k, \ldots, h^M\right] e^{W_{\text{gen}}}, \quad (3.21) \]

which is invariant under Hamiltonian

\[ H_{\text{gen}}V_M \subseteq V_M. \quad (3.22) \]

In these formulas the subscript “gen” in \( H \) and \( W \) stands for ‘generic’ and the function \( h = h(q) \) need not be a polynomial in \( q \). (For example, in the trigonometric \( BC \) type Inozemtsev model (see section 4) \( h(q) = \sin^2 q \).) It is straightforward to verify that the “polynomial space” \( V_M \) is annihilated by an \( \mathcal{N} = M + 1 \) st order differential operator \( P_{\mathcal{N}} \):

\[ P_{\mathcal{N}} = \prod_{k=0}^{\mathcal{N}-1} (D_{\text{gen}} + ikE(q)), \quad D_{\text{gen}} = p + i \frac{\partial W_{\text{gen}}}{\partial q}, \quad E(q) \equiv \frac{h''(q)}{h'(q)}, \quad (3.23) \]

\[ P_{\mathcal{N}}V_M = 0. \quad (3.24) \]
As above, the “polynomial space” $V_M$ gives the entire solution space of the differential equation $P_N y = 0$. One could summarise the situation as the exactly solvable sector of a quasi-exactly solvable dynamics is characterised as the states annihilated by the generators $Q$ and $Q^\dagger$ of an $\mathcal{N}$-fold supersymmetry.

On the other hand, let us suppose that one has a pair of Hamiltonians $H_{\text{gen}}$ and $H_{\text{gen}}^+$ which are intertwined by $P_N$:

$$P_N H_{\text{gen}} - H_{\text{gen}}^+ P_N = 0. \tag{3.25}$$

Let $V_M$ be the space of solutions of the differential equation $P_N y = 0$, which is finite dimensional. Then from (3.25) one obtains $P_N H_{\text{gen}} V_M = 0$ and thus deduces that the finite dimensional space $V_M$ is invariant under $H_{\text{gen}}$: $H_{\text{gen}} V_M \subseteq V_M$, (3.22). One could summarise this as the quasi-exact solvability of $H_{\text{gen}}$ is a consequence of the $\mathcal{N}$-fold supersymmetry and the intertwining relation (3.25). The spectral equivalence of $H_{\text{gen}}$ and $H_{\text{gen}}^+$ holds outside of $V_M$ as in the ordinary ($\mathcal{N} = 1$) supersymmetric quantum mechanics.

4 Trigonometric $BC$ type Inozemtsev model with one degree of freedom

Here we consider a one dimensional quantum mechanical system with the following super potential $W$ in a finite interval $[0, \pi/2]$:

$$W = W_0 + W_1, \tag{4.1}$$

$$W_0 = -\frac{a}{2} \cos 2q + \frac{b}{2} \log |\cot q| + g_S \log |\sin q|, \tag{4.2}$$

$$g_S > 0, \quad g_S > \frac{b}{2} > 0, \tag{4.3}$$

$$W_1 = \sum_{k=1}^{\mathcal{M}} \log |\sin^2 q - \xi_k|, \tag{4.4}$$

in which $W_0$ part is obtained by retaining the single particle part of the trigonometric $BC$ type Inozemtsev model (2.12). It is an even function of $q$ and it reduces to a well-known “double sine-Gordon” quantum mechanics [27] if only the first term $\frac{a}{2} \cos 2q$ is kept. Here we have not included the long root term $g_L \log |\sin 2q|$ in (2.5) since it can be expressed as a linear combination of $\log |\cot q|$ and $\log |\sin q|$ terms. As in the previous section, we evaluate the terms containing $W_1$ in $(\partial W/\partial q)^2 + \partial^2 W/\partial q^2$:

$$2 \left( a \sin 2q - \frac{b}{\sin 2q} + g_S \cot q \right) \sum_{k=1}^{\mathcal{M}} \frac{2 \sin q \cos q}{\sin^2 q - \xi_k}$$
\[ +8 \sin^2 q \cos^2 q \sum_{k<l} \frac{1}{\sin^2 q - \xi_k \sin^2 q - \xi_l} + 2 \cos 2q \sum_{k=1}^{M} \frac{1}{\sin^2 q - \xi_k}, \quad (4.5) \]

which is a meromorphic function in \( x = \sin^2 q \) with at most simple poles:

\[
(4.5) = 2 \left( 4ax(1-x) - b + 2gs(1-x) \right) \sum_{k=1}^{M} \frac{1}{x - \xi_k} + 8x(1-x) \sum_{k<l} \frac{1}{x - \xi_k x - \xi_l} + 2(1-2x) \sum_{k=1}^{M} \frac{1}{x - \xi_k}. \quad (4.6) \]

As in the previous case we demand that the residues at the simple poles \( x = \xi_k, k = 1, \ldots, M \) should all vanish. This requires that \( \{\xi_k\} \)’s should obey a set of rational equations

\[
(4a\xi_k + 2gs)(1 - \xi_k) - b + 1 - 2\xi_k + 4\xi_k(1 - \xi_k) \sum_{l \neq k} \frac{1}{\xi_k - \xi_l} = 0, \quad k = 1, \ldots, M. \quad (4.7)\]

Then expression \((4.6)\) becomes a linear function in \( x = \sin^2 q \) which is easy to evaluate

\[
(4.7) = -8aM \sin^2 q - 8a \sum_{k=1}^{M} \xi_k - 4M(g_s + M). \]

Thus we arrive at

\[
\left( \frac{\partial W}{\partial q} \right)^2 + \frac{\partial^2 W}{\partial q^2} = \left( \frac{\partial W_0}{\partial q} \right)^2 + \frac{\partial^2 W_0}{\partial q^2} - 8aM \sin^2 q - 2E_1, \quad (4.8) \]

\[
E_1 = 4a \sum_{k=1}^{M} \xi_k + 2M(g_s + M). \quad (4.9) \]

Again, except for the constant term the expression \((4.8)\) is independent of the parameters \( \{\xi_k\} \)’s. Thus for each real solution of \((4.7)\), we have an eigenfunction

\[
e^W = e^{W_0} \prod_{k=1}^{M} (\sin^2 q - \xi_k) = (\sin q)^{gs}(\cot q)^{\frac{k}{2}} e^{-a \cos 2q} \prod_{k=1}^{M} (\sin^2 q - \xi_k), \quad (4.10)\]

with eigenvalue \( E_1, (4.9) \), of the Hamiltonian

\[
H = \frac{1}{2} p^2 + \frac{1}{2} \left( \frac{\partial W_0}{\partial q} \right)^2 + \frac{\partial^2 W_0}{\partial q^2} - 4aM \sin^2 q. \quad (4.11) \]

The eigenfunction \((4.10)\) is square integrable in \([0, \pi/2]\) due to the restriction on the parameters \((4.3)\). In other words, the above Hamiltonian \((4.11)\) is quasi-exactly solvable. The eigenfunction \((4.10)\) belongs to a “polynomial space”

\[
\mathcal{V}_{M} = \text{Span} \left[ 1, \sin^2 q, \ldots, (\sin q)^{2k}, \ldots, (\sin q)^{2M} \right] e^{W_0}, \quad (4.12) \]
which is invariant under the action of the Hamiltonian \( H \mathcal{V}_M \subseteq \mathcal{V}_M \). It is easy to see that \( \mathcal{V}_M \) is annihilated by an \( N = M + 1 \) st order differential operator \( P_N \):

\[
P_N = \prod_{k=0}^{N-1} (D + i2k \cot 2q), \quad D = p + i \frac{\partial W_0}{\partial q}, \tag{4.13}
\]

\[
P_N \mathcal{V}_M = 0. \tag{4.14}
\]

Thus the general statements in the previous section concerning the quasi-exact solvability and \( N \)-fold supersymmetry also hold in this case.

In order to investigate the structure of the *exactly solvable sector* we make as before the similarity transformation of \( H \) by \( e^{W_0} \):

\[
\widetilde{H} = e^{-W_0} H e^{W_0} = \frac{1}{2} p^2 - \frac{\partial W_0}{\partial q} \frac{\partial}{\partial q} - 4aM \sin^2 q, \]

\[
= \frac{1}{2} p^2 - \left( a \sin 2q - \frac{b}{\sin 2q} + g_S \cot q \right) \frac{\partial}{\partial q} - 4aM \sin^2 q. \tag{4.15}
\]

The eigenfunction of \( \widetilde{H} \) in the polynomial space \( \tilde{\mathcal{V}}_M = \text{Span} \left[ 1, \sin^2 q, \ldots, (\sin q)^{2M} \right] \), can be obtained by substituting an expansion \( \tilde{\Psi} = \sum_{k=0}^{M} \alpha_k (\sin q)^{2k} \); \( \alpha_0 = 1 \) into the eigenvalue equation \( \widetilde{H} \tilde{\Psi} = E \tilde{\Psi} \). This again leads to a three term recursion relation for “Bender-Dunne” polynomials \( \{ \alpha_k(E) \} \)’s:

\[
(k + 1)(2k + 1 - b + 2g_S)\alpha_{k+1} = (2k(1 - 2a + 2g_S) - E) \alpha_k + 4a(k - M - 1) \alpha_{k-1}. \tag{4.16}
\]

Again \( \alpha_k(E) \) is a polynomial in \( E \) of degree \( k \) and the condition \( \alpha_{M+1} = 0 \) gives the characteristic equation for \( \widetilde{H} \):

\[
\alpha_{M+1} = 0 \iff (2M(1 - 2a + 2g_S) - E) \alpha_M - 4a \alpha_{M-1} = 0 \iff \det(\widetilde{H} - E) = 0.
\]

5 Rational A type Inozemtsev model with one degree of freedom

This is an interesting example which fails to achieve quasi-exact solvability due to the lack of square integrability of the eigenfunction. It is interesting to know how far the algebraic procedures go in parallel with the previous cases. We start with the following pre-superpotential \( W \) which is obtained by retaining the single particle part of \( W \) in (2.13):

\[
W = W_0 + W_1, \quad W_0 = \frac{a}{3} q^3 + \frac{b}{2} q^2, \quad W_1 = \sum_{k=1}^{M} \log |q - \xi_k|. \tag{5.1}
\]
This is cubic in $q$ and leads to a quartic potential of $q$, see (5.3). The terms containing $W_1$ in $(\partial W/\partial q)^2 + \partial^2 W/\partial q^2$ are:

$$2(aq^2 + bq) \sum_{k=1}^{M} \frac{1}{q - \xi_k} + 2 \sum_{k<l} \frac{1}{q - \xi_k q - \xi_l}. \tag{5.2}$$

From the requirement of vanishing residue at $q = \xi_k$, we obtain rational equations

$$a\xi_k^2 + b\xi_k + \sum_{l \neq k} \frac{1}{\xi_k - \xi_l} = 0, \quad k, l = 1, \ldots, M, \tag{5.3}$$

and the expression (5.2) reads

$$\text{(5.2)} = 2aMq + 2a \sum_{k=1}^{M} \xi_k + 2bM.$$

Thus for each real solution $\{\xi_k\}$ of (5.3) we obtain an “eigenfunction”

$$e^W = e^{W_0} \prod_{k=1}^{M} (q - \xi_k) = e^{(\frac{3}{2}q^3 + \frac{3}{2}q^2)} \prod_{k=1}^{M} (q - \xi_k) \tag{5.4}$$

of a Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2} \left[ \left( \frac{\partial W_0}{\partial q} \right)^2 + \frac{\partial^2 W_0}{\partial q^2} \right] + aMq,$$

$$= \frac{1}{2}p^2 + \frac{1}{2}q^2(aq + b)^2 + a(M + \frac{1}{2})q + \frac{b}{2} \tag{5.5}$$

with energy

$$E = -a \sum_{k=1}^{M} \xi_k - bM. \tag{5.6}$$

Let us recall the simple facts about the limiting case of $a = 0$ and $b = -\omega$, $\omega > 0$. Then the Hamiltonian (5.5) becomes that of the simple harmonic oscillator with angular frequency $\omega$ and the equations (5.3) determine $\{\xi_k\}$’s as the zeros of Hermite polynomials [28], with scaling by $\sqrt{\omega}$. This results in the well-known eigenfunction with Hermite polynomials (5.4) and the spectrum $E = \omega M$ (5.6). (The zero-point energy $\omega/2$ is contained in the Hamiltonian (5.3).) Thus at least for $b < 0$ and $|a/b| \ll 1$, that is the quartic and the accompanying cubic terms in the potential can be considered as ‘perturbations’, it is expected that the equations (5.3) have real solutions and the above solution generating method would work. However, the “eigenfunction” (5.4) is not square integrable in the region $(-\infty, +\infty)$ for whichever choice of the sign of $a \neq 0$. One might be tempted to restrict the region to a half line, say
(0, +∞), by introducing a singular potential at the origin, for example, by adding a term $g \log |q|$ to $W_0$, as in the example in section 3. However, this cannot remedy the situation because of the wrong parity of the $aq^3$ term.

The $\mathcal{N} = \mathcal{M} + 1$ st order differential operator $P_\mathcal{N}$ annihilating the “polynomial space”

$$\mathcal{V}_\mathcal{M} = \text{Span} \left[ 1, q, \ldots, q^k, \ldots, q^\mathcal{M} \right] e^{W_0},$$

(5.7)

has a very simple form:

$$P_\mathcal{N} = D^\mathcal{N}, \quad D = p + i \frac{\partial W_0}{\partial q}. \quad (5.8)$$

In this case the $\mathcal{N}$-fold supersymmetry generated by $Q = P_\mathcal{N} \psi$ and $Q^\dagger = P_\mathcal{N}^\dagger \psi$ is spontaneously broken for $a \neq 0$.

The difference in the algebraic structure from the quasi-exactly solvable case discussed in section 3 becomes clearer by making the similarity transformation of $H$ by $e^{W_0}$:

$$\tilde{H} = e^{-W_0} H e^{W_0} = \frac{1}{2} p^2 - \frac{\partial W_0}{\partial q} \frac{\partial}{\partial q} + a\mathcal{M} q,$$

$$= \frac{1}{2} p^2 - (aq^2 + bq) \frac{\partial}{\partial q} + a\mathcal{M} q. \quad (5.9)$$

This maps $q^k$ to $q^{k+1}$, $q^k$ and $q^{k-2}$:

$$\tilde{H} q^k = a(\mathcal{M} - k) q^{k+1} - bkq^k - \frac{k(k-1)}{2} q^{k-2}.$$

Since the last term is $q^{k-2}$ instead of $q^{k-1}$, the three term recursion relations for the coefficients $\{\alpha_k\}$s in a series solution $\tilde{\Psi} = \sum_{k=0}^{\mathcal{M}} \alpha_k q^k$, $\alpha_0 = 1$ for the eigenvalue equation $\tilde{H} \tilde{\Psi} = E \tilde{\Psi}$ do not hold any more. The $\{\alpha_k(E)\}$s are no longer degree $k$ polynomial in $E$.

We will not discuss “trigonometric A type Inozemtsev model with one degree of freedom”, since the single particle part of (2.14) is simply $W = -\frac{1}{2} \cos 2q$, that is the “double sine-Gordon” quantum mechanics. This is a well-known example of quasi-exactly solvable dynamics [27] and is a special case of the model treated in section 4.

6 Rational BC type Inozemtsev model

Following the results of the single particle case in section 3, we consider the following Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{r} p_j^2 + \frac{1}{2} \sum_{j=1}^{r} \left[ \left( \frac{\partial W_0}{\partial q_j} \right)^2 + \frac{\partial^2 W_0}{\partial q_j^2} \right] - 2a_{\mathcal{M}} \sum_{j=1}^{r} q_j^2, \quad (6.1)$$

15
in which $\mathcal{M}$ is an arbitrary non-negative integer and $W_0$ is given by (2.11):

$$W_0 = -\sum_{j=1}^{r} \left( \frac{a_j^4}{4} + \frac{b_j^2}{2} \right) + W_{C-M},$$  \hspace{1cm} (6.2)

$$W_{C-M} = g_M \sum_{j<k} \left\{ \log |q_j - q_k| + \log |q_j + q_k| \right\} + g_S \sum_{j=1}^{r} \log |q_j|,$$

$$a > 0, \quad g_S > 0, \quad g_M > 0. \quad (6.3)$$

The Hamiltonian as well as $W_0$ are Coxeter (Weyl) invariant of the $BC_r$ root system. The only difference with the classical supersymmetrisable Inozemtsev model is the added quadratic terms proportional to $\mathcal{M}$. We will consider the model in the principal Weyl chamber

$$q_1 > q_2 > \cdots > q_r > 0. \quad (6.4)$$

A special case of this Hamiltonian with one free parameter other than $\mathcal{M}$ (plus an invisible overall scale factor) was discussed in [29].

In order to show the quasi-exact solvability of the Hamiltonian (6.1) we have to demonstrate that a certain “exactly solvable sector” is invariant under $H$. As before let us first define a “polynomial space”

$$V_\mathcal{M} = \text{Span}_{0 \leq n_j \leq \mathcal{M}, \ 1 \leq j \leq r} \left[ (q_1^2)^{n_1} \cdots (q_j^2)^{n_j} \cdots (q_r^2)^{n_r} \right] e^{W_0}. \quad (6.5)$$

The “exactly solvable sector” is the permutation $(q_j \leftrightarrow q_k)$ invariant subspace of $V_\mathcal{M}$:

$$V_\mathcal{M}^{G_{BC}} = \{ v \in V_\mathcal{M} | g v = v, \ \forall g \in G_{BC} \}, \quad (6.6)$$

in which $G_{BC}$ is the Coxeter (Weyl) group of the $BC$ root system. This fact can be seen easily, as in the single particle case, by similarity transformation

$$\tilde{H} = e^{-W_0} H e^{W_0} = \frac{1}{2} \sum_{j=1}^{r} p_j^2 - \sum_{j=1}^{r} \frac{\partial W_0}{\partial q_j} \frac{\partial}{\partial q_j} - 2a\mathcal{M} \sum_{j=1}^{r} q_j^2,$$

$$= \frac{1}{2} \sum_{j=1}^{r} p_j^2 - \sum_{j=1}^{r} \frac{\partial W_{C-M}}{\partial q_j} \frac{\partial}{\partial q_j} + \sum_{j=1}^{r} (a q_j^3 + b q_j) \frac{\partial}{\partial q_j} - 2a\mathcal{M} \sum_{j=1}^{r} q_j^2, \quad (6.7)$$

$$\tilde{V}_\mathcal{M} = \text{Span}_{0 \leq n_j \leq \mathcal{M}, \ 1 \leq j \leq r} \left[ (q_1^2)^{n_1} \cdots (q_j^2)^{n_j} \cdots (q_r^2)^{n_r} \right]. \quad (6.8)$$

The proof of the invariance of $\tilde{V}_\mathcal{M}^{G_{BC}}$ under the added terms $\sum_{j=1}^{r} (a q_j^3 + b q_j) \frac{\partial}{\partial q_j} - 2a\mathcal{M} \sum_{j=1}^{r} q_j^2$ is essentially the same as in the single particle case. As for the Calogero-Moser part, $W_{C-M}$, it always decreases the power $\sum_{j=1}^{r} n_j$ by one unit. The Coxeter (Weyl) invariance is necessary
and sufficient so that the result remains a polynomial, without developing unwanted poles. Thus the invariance of $\tilde{\mathcal{V}}^G_{BC}$ under $\tilde{H}$ and the quasi-exact integrability of $H$ is proved. It is obvious that the elements of $\mathcal{V}^G_{BC}$ are square integrable since

$$e^{W_0} = e^{-\sum_{j=1}^r (\frac{4a}{q_j} + \frac{b}{q_j^2})} \prod_{j=1}^r (q_j)^{g_S} \prod_{j<k} (q_j^2 - q_k^2)^{g_M}$$

(6.9)

and the restriction on the parameters (6.3) and the integration region (6.4) secure finiteness at infinity and at the boundaries of the Weyl chambers.

Thus we have shown that the rational $BC$ type Inozemtsev model can be made quasi-exactly solvable by adding properly quantised quadratic terms $-2a_M \sum_{j=1}^r q_j^2$. The above “polynomial space” $\mathcal{V}_M$ is annihilated by the following $r$ different $\mathcal{N} = M + 1$ st order commuting differential operators $P^{(j)}_N$:

$$P^{(j)}_N = \prod_{k=0}^{N-1} (D_j + \frac{i}{q_j}), \quad D_j = p_j + i \frac{\partial W_0}{\partial q_j}.$$  

(6.10)

$$P^{(j)}_M \mathcal{V}_M = 0, \quad [P^{(j)}_N, P^{(k)}_N] = 0, \quad j, k = 1, \ldots, r.$$  

(6.11)

The $\mathcal{N}$-fold supersymmetry is generated by

$$Q = \sum_{j=1}^r P^{(j)}_N \psi_j^\dagger, \quad Q^\dagger = \sum_{j=1}^r (P^{(j)}_N)^\dagger \psi_j,$$

(6.12)

in which $\psi_j$ and $\psi_j^\dagger$ are the annihilation and creation operators of the $j$-th fermion.

### 7 Trigonometric $BC$ type Inozemtsev model

Following the results of the single particle case in section 4, we consider the following Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^r p_j^2 + \frac{1}{2} \sum_{j=1}^r \left[ \left( \frac{\partial W_0}{\partial q_j} \right)^2 + \frac{\partial^2 W_0}{\partial q_j^2} \right] - 4a_M \sum_{j=1}^r \sin^2 q_j,$$

(7.1)

in which $M$ is an arbitrary non-negative integer and $W_0$ is given by (2.12):

$$W_0 = \sum_{j=1}^r \left( -\frac{a}{2} \cos 2q_j + \frac{b}{2} \log |\cot q_j| \right) + W_{C-M},$$

(7.2)

$$W_{C-M} = g_M \sum_{j<k}^r \{ \log |\sin(q_j - q_k)| + \log |\sin(q_j + q_k)| \}$$

$$+ g_S \sum_{j=1}^r \log |\sin q_j|.$$  

(7.3)
Let us recall here the argument in section 4 for dropping \( g_L \log |\sin 2q_j| \) term in favour of \( b \log |\cot q_j| \) term. The Hamiltonian as well as \( W_0 \) are Coxeter (Weyl) invariant of the \( BC_r \) root system. The only difference with the classical supersymmetrisable Inozemtsev model is the added \( \sin^2 q_j \) terms proportional to \( M \). We will consider the quantum mechanical model in the principal Weyl alcove

\[
\frac{\pi}{2} > q_1 > q_2 > \cdots > q_r > 0,
\]

(7.5)
due to the periodicity and Coxeter (Weyl) invariance of the model. As in the single particle case, we will demonstrate that a certain “exactly solvable sector” is invariant under \( H \). As before let us first define a “polynomial space”

\[
\mathcal{V}_M = \text{Span}_{0 \leq n_j \leq M, \ 1 \leq j \leq r} \left[ (\sin^2 q_1)^{n_1} \cdots (\sin^2 q_j)^{n_j} \cdots (\sin^2 q_r)^{n_r} \right] e^{W_0},
\]

(7.6)
or equivalently:

\[
\mathcal{V}_M = \text{Span}_{-M \leq n_j' \leq M, \ 1 \leq j \leq r} \left[ \cos 2 \left( \sum_{j=1}^{r} n_j' q_j \right) \right] e^{W_0}.
\]

(7.7)
The “exactly solvable sector” is the permutation \((q_j \leftrightarrow q_k)\) invariant subspace of \( \mathcal{V}_M \):

\[
\mathcal{V}^{G_{BC}}_M = \{ v \in \mathcal{V}_M | gv = v, \ \forall g \in G_{BC} \},
\]

(7.8)
in which \( G_{BC} \) is the Coxeter (Weyl) group of the \( BC \) root system. All these functions are square integrable, since the integration region is finite and the possible singularities in \( e^{W_0} \) at the boundary \( (7.3) \)

\[
e^{W_0} = e^{-\frac{a}{2} \sum_{j=1}^{r} \cos 2q_j} \prod_{j=1}^{r} (\sin q_j)^{g_s} (\cot q_j)^{b} \prod_{k<l}^{r} \sin(q_k - q_l) \sin(q_k + q_l) \sum_{j=1}^{r} \sin^2 q_j,
\]

(7.9)
are taken care of by the restriction on the parameters \((7.4)\). The quasi-exact solvability can be shown by the similarity transformation

\[
\tilde{H} = e^{-W_0} H e^{W_0} = \frac{1}{2} \sum_{j=1}^{r} p_j^2 - \sum_{j=1}^{r} \frac{\partial W_0}{\partial q_j} \frac{\partial}{\partial q_j} - 4aM \sum_{j=1}^{r} \sin^2 q_j,
\]

(7.10)

\[
\tilde{V}_M = \text{Span}_{0 \leq n_j \leq M, \ 1 \leq j \leq r} \left[ (\sin^2 q_1)^{n_1} \cdots (\sin^2 q_j)^{n_j} \cdots (\sin^2 q_r)^{n_r} \right],
\]

(7.11)

\[
= \text{Span}_{-M \leq n_j' \leq M, \ 1 \leq j \leq r} \left[ \cos 2 \left( \sum_{j=1}^{r} n_j' q_j \right) \right].
\]

(7.12)
The corresponding “exactly solvable sector” is the permutation \((q_i \leftrightarrow q_k)\) invariant subspace of \(\tilde{V}_M\):

\[
\tilde{V}_M^{G_{BC}} = \{ v \in \tilde{V}_M | g v = v, \quad \forall g \in G_{BC}\}.
\]

(7.13)

As for the Calogero-Moser part, the above Hamiltonian is lower triangular \([16]\) in the basis \((B.6)\) of \(\tilde{V}_M^{G_{BC}}\). The lower triangularity is stronger than the invariance of the polynomial space under the Hamiltonian. An outline of the proof is given in Appendix B. As for the added part

\[- \sum_{j=1}^{r} \left(a \sin 2q_j - \frac{b}{\sin 2q_j}\right) \frac{\partial}{\partial q_j} - 4aM \sum_{j=1}^{r} \sin^2 q_j,
\]

the proof of the invariance of the polynomial space (7.11) is essentially the same as in the single particle case. Thus the quasi-exact solvability of the trigonometric \(BC\) type Inozemtsev model is established.

The above “polynomial space” \(V_M\) (7.6) is annihilated by the following \(r\) different \(N = M + 1\) st order commuting differential operators \(P^{(j)}_N\):

\[
P^{(j)}_N = \prod_{k=0}^{N-1} (D_j + 2ik \cot 2q_j), \quad D_j = p_j + i \frac{\partial W_0}{\partial q_j},
\]

(7.14)

\[
P^{(j)}_N V_M = 0, \quad [P^{(j)}_N, P^{(k)}_N] = 0, \quad j, k = 1, \ldots, r.
\]

(7.15)

The \(N\)-fold supersymmetry is generated by

\[
Q = \sum_{j=1}^{r} P^{(j)}_N \psi_j^\dagger, \quad Q^\dagger = \sum_{j=1}^{r} (P^{(j)}_N)^\dagger \psi_j,
\]

in which \(\psi_j\) and \(\psi_j^\dagger\) are the annihilation and creation operators of the \(j\)-th fermion.

8 Trigonometric \(A\) type Inozemtsev model

This has a much simpler Hamiltonian than the previous one:

\[
H = \frac{1}{2} \sum_{j=1}^{r} p_j^2 + \frac{1}{2} \sum_{j=1}^{r} \left[ \left( \frac{\partial W_0}{\partial q_j} \right)^2 + \frac{\partial^2 W_0}{\partial q_j^2} \right] - 4aM \sum_{j=1}^{r} \sin^2 q_j,
\]

(8.1)

in which \(M\) is an arbitrary non-negative integer and \(W_0\) is given by (2.14):

\[
W_0 = -\frac{a}{2} \sum_{j=1}^{r} \cos 2q_j + W_{C-M}, \quad W_{C-M} = g \sum_{j<k} \log |\sin(q_j - q_k)|.
\]

(8.2)
There are only two real coupling constants $a$ and $g$ and we require
\[ g > 0 \] (8.3)
for square integrability of the eigenfunctions of the form (8.3). The Hamiltonian and $W_0$ are invariant under any transpositions $(q_j, p_j) \leftrightarrow (q_k, p_k)$, which form the Coxeter group of $A_{r-1}$. Thus we consider the quantum mechanics in the fundamental Weyl alcove:
\[ \pi > q_1 > q_2 > \cdots > q_r > 0. \] (8.4)
Reflecting the simple form of the Hamiltonian, the “exactly solvable sector” which is left invariant under $H$ has a simpler structure than those in the previous cases. Let us first define a space of truncated Fourier series with two units:
\[ V_M = \text{Span}_{-M \leq n_j \leq M, \ 1 \leq j \leq r} \left[ e^{2i \sum_{j=1}^r n_j q_j} \right] e^{W_0}. \] (8.5)
The “exactly solvable sector” is the permutation $(q_j \leftrightarrow q_k)$ invariant subspace of $V_M$:
\[ \bar{V}_M = \{ v \in V_M | gv = v, \ \forall g \in G_A \}, \] (8.6)
in which $G_A$ is the Coxeter (Weyl) group of the $A$ type root system. The quasi-exact solvability of the Hamiltonian (8.1) is again easily verified by the similarity transformation:
\[ \bar{H} = e^{-W_0} H e^{W_0} = \frac{1}{2} \sum_{j=1}^r p_j^2 - \sum_{j=1}^r \frac{\partial W_0}{\partial q_j} \frac{\partial}{\partial q_j} - 4aM \sum_{j=1}^r \sin^2 q_j, \]
\[ = \frac{1}{2} \sum_{j=1}^r p_j^2 - \sum_{j=1}^r \frac{\partial W_{C-M}}{\partial q_j} \frac{\partial}{\partial q_j} - a \sum_{j=1}^r \sin 2q_j \frac{\partial}{\partial q_j} - 4aM \sum_{j=1}^r \sin^2 q_j, \] (8.7)
\[ \bar{V}_M = \text{Span}_{-M \leq n_j \leq M, \ 1 \leq j \leq r} \left[ e^{2i \sum_{j=1}^r n_j q_j} \right]. \] (8.8)
The lower triangularity of the Calogero-Moser part of the Hamiltonian in the basis (8.8) was proven originally by Sutherland [7]. For the additional part
\[ -a \sum_{j=1}^r \sin 2q_j \frac{\partial}{\partial q_j} - 4aM \sum_{j=1}^r \sin^2 q_j, \]
the proof that it leaves the space of the truncated Fourier series (8.8) invariant is rather elementary.

The above “polynomial space” $V_M$ (8.5) is annihilated by the following $r$ different $\mathcal{N} = 2M + 1$ st order commuting differential operators $P_{\mathcal{N}}^{(j)}$:
\[ P_{\mathcal{N}}^{(j)} = \prod_{k=-M}^M (D_j + 2ik), \quad D_j = p_j + i \frac{\partial W_0}{\partial q_j}, \] (8.9)
\[ P_{\mathcal{N}}^{(j)} V_M = 0, \quad [P_{\mathcal{N}}^{(j)}, P_{\mathcal{N}}^{(k)}] = 0, \quad j, k = 1, \ldots, r. \] (8.10)
The $\mathcal{N}$-fold supersymmetry is generated by

$$Q = \sum_{j=1}^{r} P^{(j)}_{\mathcal{N}} \psi_j, \quad Q^\dagger = \sum_{j=1}^{r} (P^{(j)}_{\mathcal{N}})^\dagger \psi_j,$$

in which $\psi_j$ and $\psi_j^\dagger$ are the annihilation and creation operators of the $j$-th fermion.

We will not discuss the quasi-exact solvability of multiparticle rational $A$ type Inozemtsev model, since its wavefunctions are not square integrable as seen in section 5. One could say that the $\mathcal{N}$-fold supersymmetry is spontaneously broken in this case.

Let us summarise that the quasi-exact solvability of quantum Inozemtsev models discussed in sections 6, 7 and 8 is a consequence of the exact solvability of the quantum Calogero-Moser models and the quasi-exact solvability of the added single particle like interactions.

9 Comments and discussion

It should be stressed that the present method for showing quasi-exact solvability of single particle systems developed in section 3 to 5 does not depend on any existing methods or criteria for QES, see for example, [30]. As shown in section 5 it also gives the known exact solutions when the QES system tends to the harmonic oscillator.

An interesting question along the line of arguments in this paper is the ‘hierarchy problem’ of quasi-exactly solvable systems, as in the completely integrable systems. For example, the Inozemtsev models have higher conserved quantities $\text{Tr}(L^k)$ obtained from the Lax pairs in Appendix A. They define new classical and quantum Hamiltonian systems. Can the quantum version of the higher members of the hierarchy be deformed to be quasi-exactly solvable?

In so far, elliptic Calogero-Moser models defied various attempts to construct quantum theory based on a Hilbert space, although existence of mutually commuting operators are known for the $A$ type models [31]. In analogy with the present arguments, it is quite natural to expect the quantum elliptic Calogero-Moser models to be quasi-exactly solvable [32, 33] rather than exactly integrable.

Acknowledgements

R. S. is partially supported by the Grant-in-aid from the Ministry of Education, Culture, Sports, Science and Technology, priority area (#707) “Supersymmetry and unified theory of
Appendix A: Lax pairs for classical Inozemtsev models

Here we present the Lax pairs for classical Inozemtsev models in the same notation as is used in the main text as evidence for their classical integrability. As mentioned in Introduction only certain subset of classical Inozemtsev models can be made quasi-exactly solvable. We focus on the supersymmetrisable Inozemtsev models for simplicity of presentation. For the full content of Lax pairs of classical inozemtsev models we refer to the original paper by Inozemtsev and Meshchryakov [5] and for the universal Lax pairs for Calogero-Moser models in general, see [11, 18].

The Lax pair consists of a pair of $2r \times 2r$ ($r$ rank) matrices $L$ and $M$ such that the canonical equations of motion can be expressed in a matrix form

$$\dot{L} = [L, M]$$

and a sufficient number of classical conserved quantities can be obtained as as the trace of powers of $L$, $\text{Tr}(L^k)$. The Hamiltonian is $H \propto \text{Tr}(L^2)$.

A.1 BC type models

The following Lax pair applies for the models presented in subsection 2.5 for the rational as well as trigonometric models for proper choice of functions, $x, \nu$, etc. as listed below. The pair of matrices decomposes into diagonal and off-diagonal matrices:

$$L = P + X, \quad M = D + Y.$$  \quad (A.1)

The diagonal matrices $P$ and $D$ are of the form

$$P = \sum_{j=1}^{r} p_j (E_{j,j} - E_{j+r,j+r}), \quad D = \sum_{j=1}^{r} D_j (E_{j,j} + E_{j+r,j+r}),$$  \quad (A.2)

in which $E_{j,k}$ is the usual matrix unit $(E_{j,k})_{lm} = \delta_{lj}\delta_{mk}$. The diagonal-free matrices $X$ and $Y$ have the form

$$X = ig_M \sum_{j \neq k} x(q_j - q_k) E_{j,k} + ig_M \sum_{j \neq k} x(q_j + q_k) E_{j,k+r}$$
\[ + ig_M \sum_{j \neq k} x(-q_j - q_k) E_{j+r,k} + ig_M \sum_{j \neq k} x(-q_j + q_k) E_{j+r,k+r} \]
\[ + 2i \sum_j \nu(q_j) E_{j+j+r} - 2i \sum_j \nu(q_j) E_{j+r,j}, \quad (A.3) \]

\[ Y = ig_M \sum_{j \neq k} y(q_j - q_k) E_{j,k} + ig_M \sum_{j \neq k} y(q_j + q_k) E_{j,k+r} \]
\[ + ig_M \sum_{j \neq k} y(-q_j - q_k) E_{j+r,k} + ig_M \sum_{j \neq k} y(-q_j + q_k) E_{j+r,k+r} \]
\[ + i \sum_j \nu'(q_j) E_{j,j+r} + i \sum_j \nu'(q_j) E_{j+r,j}. \quad (A.4) \]

The diagonal elements of \( D \) are given by
\[ D_j = -ig_M \sum_{k \neq j}^r (z(q_j - q_k) + z(q_j + q_k)) - i \sum_{j=1}^r \tau(q_j). \quad (A.5) \]

Some functions are related to each other:
\[ y(u) = dx(u)/du, \quad z(u) = x(u)^2 + \text{const.}, \quad \tau(u) = 2x(2u)\nu(u) + \text{const.} \quad (A.6) \]

The rational and trigonometric models correspond to the following choice of functions:

1. Rational model,
\[ x(u) = \frac{1}{u}, \quad z(u) = \frac{1}{u^2}, \quad \nu(u) = -(au^3 + bu) + \frac{g_S}{u}, \quad (A.7) \]
where \( a, b \) and \( g_S \) are real coupling constants.

2. Trigonometric model,
\[ x(u) = \cot u, \quad z(u) = \frac{1}{\sin^2 u}, \quad \nu(u) = a \sin 2u - \frac{b}{\sin 2u} + g_S \cot u, \quad (A.8) \]
where \( a, b \) and \( g_S \) are real coupling constants.

The functions \( x \) and \( \nu \) correspond to those appearing in \( \partial W/\partial q \).

### A.2 A type models

The Lax pair is again a pair of \( 2r \times 2r \) \((r - 1 \text{ is the rank})\) matrices, with the decomposition
\[ L = P + X, \quad M = D + Y, \]
in which \( P \) and \( D \) are diagonal matrices of the form
\[ P = \sum_{j=1}^r p_j (E_{j,j} - E_{j+r,j+r}), \quad D = \sum_{j=1}^r D_j (E_{j,j} + E_{j+r,j+r}) \]
and $X$ and $Y$ are diagonal-free matrices of the form

$$X = ig \sum_{j \neq k} x(q_j - q_k) E_{j,k} + ig \sum_{j \neq k} x(-q_j + q_k) E_{j+r,k+r}$$

$$+ 2 \sum_j \kappa(q_j) E_{j,j+r} + 2 \sum_j \kappa(-q_j) E_{j+r,j},$$

(A.9)

$$Y = ig \sum_{j \neq k} y(q_j - q_k) E_{j,k} + ig \sum_{j \neq k} y(-q_j + q_k) E_{j+r,k+r}$$

$$+ \sum_j \kappa'(q_j) E_{j,j+r} + \sum_j \kappa'(-q_j) E_{j+r,j}.$$

(A.10)

The diagonal elements of $D$ are given by

$$D_j = -ig \sum_{k \neq j} z(q_j - q_k).$$

(A.11)

The rational and trigonometric models correspond to the following choice of functions:

1. Rational model,

$$x(u) = \frac{1}{u}, \quad z(u) = \frac{1}{u^2}, \quad \kappa(u) = au^2 + bu,$$

where $a$ and $b$ are arbitrary real coupling constants.

2. Trigonometric model,

$$x(u) = \frac{1}{\sin u}, \quad z(u) = \frac{1}{\sin^2 u}, \quad \kappa(u) = a \sin 2u,$$

where $a$ is the arbitrary real coupling constant.

**Appendix B: Lower triangularity**

Here we show the details of the argument that Calogero-Moser part of the similarity transformed Hamiltonian (7.10) is lower triangular in the basis (7.13). The triangularity of the trigonometric Calogero-Moser model is proved universally in [16] by using the Coxeter (Weyl) invariant basis:

$$\phi_\lambda(q) \equiv \sum_{\mu \in W(\lambda)} e^{2\mu \cdot q},$$

(B.1)

in which $\lambda$ is a dominant weight

$$\lambda = \sum_{j=1}^r m_j \lambda_j, \quad m_j \in \mathbb{Z}_+, \quad \lambda_j : \text{fundamental weight}$$

(B.2)
and \( W(\lambda) \) is the orbit of \( \lambda \) by the action of the Weyl group:

\[
W(\lambda) = \{ \mu \in \Lambda(\Delta) \mid \mu = g(\lambda), \ \forall g \in G_\Delta \}, \ \Lambda(\Delta) : \text{weight lattice of } \Delta.
\] (B.3)

The above \( \phi_\lambda(q) \) is Coxeter invariant. The set of functions \( \{\phi_\lambda\} \) has an order >:

\[
|\lambda|^2 > |\lambda'|^2 \Rightarrow \phi_\lambda > \phi_{\lambda'}.
\] (B.4)

For the \( BC_r \) root system, the set of weights \( W(\lambda) \) is symmetric:

\[
\mu \in W(\lambda) \Leftrightarrow -\mu \in W(\lambda).
\] (B.5)

Thus the Coxeter invariant basis (B.1) for \( BC \) type root system can be rewritten as

\[
\phi'_\lambda(q) \equiv \sum_{\mu \in W(\lambda)} \cos(2\mu \cdot q).
\] (B.6)

All the fundamental weights, except for the spinor weights of \( B_r \) are integral. That is, so long as the above dominant weight \( \lambda \) contains the fundamental spinor weights in even multiples, all the \( \mu \cdot q \) in (B.1) have the form

\[
\mu \cdot q = \sum_{j=1}^{r} k_j q_j, \quad k_j \in \mathbb{Z}_+.
\]

Therefore the basis (B.6) has the form (7.13) used in section 7. Moreover, after the application of \( \tilde{H} \) on the basis functions (7.13), those appearing as lower terms have also the same property of having even number of fundamental spinor weights and can be expressed in the same form (7.13). This completes the proof of the lower triangularity of the Hamiltonian (7.10) in the space (7.13).

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