ON $p$-HARMONIC MAPS AND CONVEX FUNCTIONS

GIONA VERONELLI

Abstract. We prove that, in general, given a $p$-harmonic map $F : M \to N$ and a convex function $H : N \to \mathbb{R}$, the composition $H \circ F$ is not $p$-subharmonic. By assuming some rotational symmetry on manifolds and functions, we reduce the problem to an ordinary differential inequality. The key of the proof is an asymptotic estimate for the $p$-harmonic map under suitable assumptions on the manifolds.

1. Introduction and statement of the main result

A twice differentiable map $F : M \to N$ between Riemannian manifolds is said to be $p$-harmonic, $p > 1$, if it is a solution of the system

$$
\tau_p(F) := \text{div}(|dF|^{p-2} dF) = 0.
$$

The vector field $\tau_p(F)$ along $F$ is named the $p$-tension field of $F$ and, whenever $N = \mathbb{R}$, it is denoted by $\Delta_p$ and called the $p$-laplacian of $F$. In the special situation $p = 2$, the 2-tension field is traditionally denoted by $\tau(F)$ and the 2-laplacian reduces to the ordinary Laplace-Beltrami operator $\Delta$ of the underlying manifold. Moreover, a 2-harmonic map is simply called a harmonic map.

It is well known that, given a harmonic map between Riemannian manifolds $F : M \to N$ and a convex function $H : N \to \mathbb{R}$, the composition $H \circ F : M \to \mathbb{R}$ is a subharmonic function, namely $\Delta(H \circ F) \geq 0$. As a matter of fact this property can be used to characterize the harmonicity of $F$; [3]. This is extremely useful since, for example, Liouville type theorems for harmonic maps into targets supporting a convex function can be obtained directly from results in linear potential theory of real valued functions. Such Liouville conclusions, in turn, have topological consequences e.g. on the homotopy class of maps with finite energy from a geodesically complete domain; see [8] and references therein. It is also known, [9], that $p$-harmonic maps are the natural candidates for the extension of the above mentioned topological results to maps with finite higher energies; see also [10] for further topological aspects of $p$-harmonic maps. In this respect, one is led to inquire whether the composition of a $p$-harmonic map with a convex function is $p$-subharmonic and, therefore, if the non-linear potential theory of real-valued functions suffices to get the desired conclusions. It is

Date: April 29, 2009.

2000 Mathematics Subject Classification. 58E20, 53C43.

Key words and phrases. $p$-harmonic map, convex function.
folklore that, in general, this is not the case so that one is forced to follow different paths; see e.g. [1], [6], [7]. However, to the best of our knowledge, counterexamples are not yet available in the literature. The present paper aims to fill this lack.

From now on, we let $M_g$ and $N_j$ be $(n + 1)$-dimensional Riemannian manifolds with rotationally symmetric metrics defined as

$$
M_g = ([0, +\infty) \times S^n, ds^2 + g^2(s) d\theta^2)
$$

$$
N_j = ([0, +\infty) \times S^n, dt^2 + j^2(t) d\theta^2),
$$

where $g, j \in C^2([0, +\infty))$ satisfy

(1) \quad g(0) = j(0) = 0, \quad g'(0) = j'(0) = 1, \quad g(s), j(t) > 0 \text{ for } s, t > 0,

and $(S^n, d\theta^2)$ is the Euclidean $n$-sphere with its standard metric. We say that the $C^2$ map $F : M_g \to N_j$ is rotationally symmetric if

$$
F(s, \theta) = (f(s), \theta) \quad \forall s > 0, \theta \in S^n,
$$

for some function $f \in C^2([0, +\infty))$. Similarly, by a $C^2$ rotationally symmetric real valued function on $N_j$ we mean a function $H : N_j \to \mathbb{R}$ of the form

$$
H(t, \theta) = h(t) \quad \forall t > 0, \theta \in S^n,
$$

for some $h \in C^2([0, +\infty))$.

We shall prove the following

**Theorem 1.** Consider two rotationally symmetric $(n+1)$-dimensional manifolds $M_g$, $N_j$. Suppose that $(n + 1) > p > \max \{2, n\}$ and assume that the warping functions $g, j \in C^2([0, +\infty))$ have the form

$$
g(s) = (s + \delta^{-\frac{1}{p-1}})^\delta - \delta^{-\frac{\delta}{p-1}}, \quad j(t) = (t + \sigma^{-\frac{1}{p-1}})^\sigma - \sigma^{-\frac{\sigma}{p-1}},
$$

where $\delta > (p - n)^{-1} > 1$ and $0 < \sigma < 1$. Then, there exist a $C^2$ rotationally symmetric $p$-harmonic map $F : M_g \to N_j$ and a sequence $\{s_k\}_{k=1}^\infty \to +\infty$, such that

$$
\Delta_p(H \circ F)(s_k, \theta) < 0
$$

for every rotationally symmetric convex function $H : N_j \to \mathbb{R}$, provided the corresponding $h \in C^2([0, +\infty))$ satisfies $h'(t) > 0$ for $t > 0$.

It should be noted that, in the paper [3] cited above, the author also considers a special category of harmonic maps, the harmonic morphisms, which pull back germs of harmonic functions on the target to harmonic functions in the domain. It is proved that harmonic morphisms are characterized by a weakly horizontal conformality condition. Recently, [5], such a characterization has been extended to the $p$-harmonic setting, $p > 2$. It turns out that the $p$-tension field of the composition of a $p$-harmonic morphism with a generic function enjoys a very special decomposition. Accordingly one has that $p$-harmonic morphisms pull back $p$-subharmonic functions (hence
convex functions) to \( p \)-subharmonic functions. Such a special decomposition, however, fails to be true in general for a \( p \)-harmonic map, and the rotationally symmetric realm provides concrete examples.

2. Preliminary results

The proof of Theorem 1 relies on a number of preliminary facts on rotationally symmetric \( p \)-harmonic maps, ranging from explicit formulas up to existence results and companion asymptotic estimates. In all that follows, notations are those introduced in Theorem 1.

Some fundamental formulas. The \( p \)-tension field of the map \( F \), on the subset of \( M_g \) where \( |dF| \neq 0 \), writes as
\[
\tau_p(F) = \text{div}( |dF|^{p-2} dF ) = |dF|^{p-2} \left\{ \tau(F) + i_{d\text{lg}|dF|^{p-2}dF} \right\},
\]
where \( i \) denotes the interior product on 1-forms. Using the rotational symmetry condition we have
\[
|dF|^2 (s) = \left\{ \left( f'(s) \right)^2 + n \frac{j^2(f(s))}{g^2(s)} \right\}.
\]
Furthermore, the tension field of \( F \) takes the expression
\[
\tau(F) = \left\{ f''(s) + \frac{n}{g'(s)} \left[ g(s)g'(s)f'(s) - j(f(s))j'(f(s)) \right] \right\} \frac{\partial}{\partial t} \big|_{f(s)},
\]
Combining this latter with (2), therefore gives
\[
\tau_p(F) = |dF|^{p-2} (s) \left\{ \left[ f''(s) + \frac{n}{g'^2(s)} \left( g(s)g'(s)f'(s) - j(f(s))j'(f(s)) \right) \right]
\]
\[+ (p-2) |dF|^{-2} (s) f'(s) f''(s) \left[ f'(s) f''(s) \right] \right\} \frac{\partial}{\partial t} \big|_{f(s)} = 0,
\]
provided \( F \) is \( p \)-harmonic. Now, we want to compute the \( p \)-laplacian of the composition \( H \circ F \). Using (2) with \( F \) replaced by \( H \circ F \), and setting
\[
K(s) = |d(H \circ F)|^{p-2} = |h'(f(s)) f'(s)|^{p-2}
\]
we conclude
\[
\Delta_p(H \circ F) = K(s) \left\{ h'(f(s)) f''(s) + h''(f(s)) (f'(s))^2 \right\}
\]
\[+ n g^{-1}(s) g'(s) f'(s) h'(f(s))
\]
\[+ (p-2) \left[ h'(f(s)) f''(s) + h''(f(s)) (f'(s))^2 \right] \bigg),
\]
on the subset
Finally, we recall that

\[
\text{Hess}(H)(t, \theta) = h''(t) dt^2 + j'(t) j(t) h'(t) d\theta^2.
\]

Since the function \(j(t)\) defined in Theorem 1 is positive and strictly increasing, the above expression gives us that the convexity of \(H\) is equivalent to the set of conditions

\[
\begin{cases}
   h''(t) \geq 0 \\
   h'(t) \geq 0,
\end{cases}
\forall t > 0.
\]

Existence results and asymptotic estimates. The existence of rotationally symmetric \(p\)-harmonic maps has been investigated by several authors. Here, we recall the following theorem which encloses in a single statement Lemma 2.5, Theorem 2.11, Proposition 3.1 and Theorem 3.2 in [1] (see also Corollary 3.22 in [1]).

**Theorem 2.** Suppose that \(p > 2\) and assume that there exist constants \(a > 0\) and \(\delta > 1\) with \(n\delta > p - 1\) such that \(g, j \in C^2(0, \infty)\),

\[
j(t) > 0, \quad 0 \leq j'(t) \leq a \quad \forall t > 0,
\]

and

\[
g(s) \sim s^\delta, \quad g'(s) > 0 \quad \text{for large } s,
\]

where \(g\) and \(j\) satisfy the conditions in (1). Then, for any \(\alpha > 0\), there is a bounded solution \(f \in C^2[0, +\infty)\) to equation (4) such that \(f(0) = 0\), \(f'(0) = \alpha\) and \(f(s), f'(s) > 0\) for all \(s > 0\).

**Remark 3.** Note that Theorem 2 and the assumption \(h'(t) > 0\) imply that \((s, \theta) \in M_+\) and \(K(s) \neq 0\) for every \(s > 0\).

We now want to obtain an asymptotic estimate for \(f'(s)\). The following lemma, which is modeled on Corollary 3.13 in [1], will play a crucial role.

**Lemma 4.** Suppose that \((n + 1) > p > \max\{2, n\}\) and assume that there exist constants \(a > 0\) and \(\delta > 1\) with \(\delta > (p-n)^{-1}\), such that \(g, j \in C^1(0, \infty)\),

\[
j(t) > 0, \quad 0 < j'(t) \leq a \quad \forall t > 0,
\]

and

\[
g(s) \sim C_1 s^\delta, \quad g'(s) > 0 \quad \text{for large } s, \quad C_1 > 0,
\]

where \(g\) and \(j\) satisfy the conditions in (1). Then all positive solutions to equation (7) satisfy

\[
f'(s) \sim D s^{-\delta(n-(p-2))}, \quad \text{as } s \to +\infty,
\]

for some positive constant \(D\).
Proof. Let us begin by recalling the following estimate which will be useful later (see (3.7) in [1])

\[
g^n(s)|dF|^{p-2}(s)f'(s) \leq \hat{C} \left( \int_{s_0}^s r^{(n-2)\delta}|dF|^{p-2}(r) f(r) dr + 1 \right),
\]

for \( s \geq s_0 \), where \( s_0 \) is some positive constant. From equations (2) and (3), we get

\[
f'(s)(|dF|^{p-2}(s))' = |dF|^{p-2}(s) \left[ \frac{n j'(f(s))j(f(s))}{g^2(s)} - f''(s) - \frac{ng'(s)f'(s)}{g(s)} \right],
\]

from which we obtain that

\[
(g^n|dF|^{p-2}f')'(s) = n|dF|^{p-2}(s)g^{n-2}(s)j(f(s))j'(f(s)) \geq 0, \quad \forall s > 0.
\]

Hence \((g^n|dF|^{p-2}f')'\) is non-decreasing and the following limit holds

\[
(g^n|dF|^{p-2}f')'(s) \rightarrow P \in (0, +\infty), \quad \text{for } s \rightarrow +\infty.
\]

We claim that the limit \( P \) is finite. By contradiction suppose \( P = +\infty \), then there exists a sequence \( \{S_N\}_{N=1}^{\infty} \) such that

\[
S_N \rightarrow +\infty \quad \text{and} \quad (g^n|dF|^{p-2}f')(S_N) = N,
\]

which implies, for all \( s \leq S_N \),

\[
g^n(s)(f'(s))^{p-1} \leq g^n(s)|dF|^{p-2}(s)f'(s) \leq N
\]

and

\[
f'(s) \leq N^{\frac{1}{p-1}} g^{-\frac{n}{p-1}}(s) \leq CN^{\frac{1}{p-1}} s^{-\frac{n\delta}{p-1}}.
\]

Moreover, since \( p < n + 1 \) and \( \delta > (p - n)^{-1} \) imply \( n\delta > (p - 1) \), we can apply Theorem 2 to deduce that \( f'(s) > 0 \) and \( f(s) \) is bounded. Thus

\[
f(s) \rightarrow \hat{c} > 0 \quad \text{as } s \rightarrow +\infty,
\]

\( f(s) < \hat{c} \) for all \( s \) and \( f(s) > \hat{c}/2 \) for \( s \) large enough. Now,

\[
|dF|^2(s) \leq CN^{\frac{2}{p-1}} s^{-\frac{2n\delta}{p-1} + \frac{n^2 f^2(s)}{g^2(s)}}
\]

\[
\leq C \max \left\{ N^{\frac{2}{p-1}} s^{-\frac{2n\delta}{p-1}}; s^{-2\delta} \right\} \leq CN^{\frac{2}{p-1}} s^{-2\delta},
\]

since \( n > (p - 1) \). Hence, from (5), (9) and (10), we get

\[
N = (g^n|dF|^{p-2}f')(S_N) \leq \hat{C} \left( \int_{S_0}^{S_N} r^{(n-2)\delta}|dF|^{p-2}(r) f(r) dr + 1 \right)
\]

\[
\leq C \left( N^{\frac{p-2}{p-1}} \int_{S_0}^{S_N} r^{-\delta(p-n)} dr + 1 \right) = o(N), \quad \text{as } s \rightarrow +\infty,
\]

since \( p > n \) and \( \delta(p-n) > 1 \). Contradiction. Then

\[
f'(s) \sim P|dF|^{2-p}(s)g^{-n}(s),
\]
for some positive constant $P < \infty$.

Now, we need an asymptotic estimate for $|dF|$. Note that

$$0 \leq \frac{(f'(s))^2 g^2(s)}{nj^2(f(s))} \leq \frac{CS^{\frac{2\delta}{p-1}}s^{2\delta}}{nj^2(s^{\frac{\delta}{2}})} = CS^{2\delta(1-\frac{n}{p-1})} \to 0, \quad \text{as } s \to +\infty,$$

since $p - 1 < n$. Therefore

$$\lim_{s \to +\infty} \frac{|dF|^2(s)}{nj^2(f(s))g^2(s)} = \lim_{s \to +\infty} \frac{(f'(s))^2 g^2(s)}{nj^2(f(s))} + 1 = 1,$$

proving that

$$|dF|^2(s) \sim nj^2(f(s))g^2(s) \sim \frac{n\hat{c}^2}{C_1^2} s^{-2\delta}, \quad \text{as } s \to +\infty.$$

Using this information into (11) we conclude

$$f'(s) \sim Ds^{-\delta(n-(p-2))}, \quad \text{with } D := PC_1^{-n} \left( \frac{C_1^2}{nj^2(\hat{c})} \right)^{\frac{p-2}{2}} > 0,$$

where $n > p - 1 > p - 2$. \hfill \Box

### 3. Proof of Theorem 1

Observe that the warping functions $g$ and $j$ defined as in Theorem 1 satisfy the assumptions of Theorem 2 and Lemma 4. Then, there exists a rotationally symmetric $p$-harmonic map $F(s, \theta) = (f(s), \theta) : M_g \to N_j$ where $f(s)$ is a positive, bounded, increasing function which satisfies (11) and the asymptotic estimates (7) and (9).

Now, multiplying (11) by $h'(f(s))$, we get

$$h'(f(s))f''(s) + ng^{-1}(s)h'(f(s))g'(s)f'(s)$$

$$= ng^{-2}(s)h'(f(s))j(f(s))j'(f(s)) - (p-2) |dF|^2 [h'(f(s))f''(s)(f'(s))^2$$

$$+ng^{-2}(s)j(f(s))j'(f(s))h'(f(s))(f'(s))^2 - ng^{-3}(s)j^2(f(s))g'(s)h'(f(s))f'(s)].$$

and inserting the latter into (13) we obtain

$$\Delta_p(H \circ F) = K(s)\hat{K}(s) \{A_1(s) + A_2(s) + A_3(s)\},$$

\hfill (13)
where we have set
\[ K(s) = \left| h'(f(s)) f'(s) \right|^p > 0, \quad \forall s > 0; \]
\[ \tilde{K}(s) := \frac{n j(f(s)) h'(f(s))}{|dF|^2(s) g^2(s)} > 0, \quad \forall s > 0; \]
\[ A_1(s) := j'(f(s)) \left[ (3 - p)(f'(s))^2 + \frac{n j^2(f(s))}{g^2(s)} \right]; \]
\[ A_2(s) := (p - 2) j'(f(s)) \left[ \frac{g'(s) f'(s)}{g(s)} + f''(s) \right]; \]
\[ A_3(s) := (p - 1)(f'(s))^2 h''(f(s)) \frac{|dF|^2(s) g^2(s)}{n j(f(s)) h'(f(s))}. \]

**Remark 5.** In the harmonic case \( p = 2 \), (13) reduces to
\[ \Delta(H \circ F) = (f'(s))^2 h''(f(s)) + \frac{n}{g^2(s)} j(f(s)) j'(f(s)) h'(f(s)) \]
which is always nonegative when \( H \) is convex, as we observed in the Introduction.

Reasoning as in the proof of (12) above, we obtain
\[ A_1(s) \sim n j'(\hat{c}) j^2(\hat{c}) s^{-2\delta}, \]
and
\[ A_3(s) \sim \frac{(p - 1) D^2 h''(\hat{c})}{h'(\hat{c})} j'(\hat{c}) s^{-2\delta(n-(p-2))}, \]
as \( s \to +\infty \). Moreover, according to l'Hôpital rule we have
\[ 1 = \lim_{s \to +\infty} \frac{f'(s)}{s^{-(n-(p-2))}} \leq \limsup_{s \to +\infty} \frac{f''(s)}{-\delta(n - (p - 2)) D s^{-\delta(n-(p-2))-1}}. \]

Thus, for every \( \epsilon > 0 \) there exists a sequence \( \{s_k\}_{k=1}^{\infty} \) such that \( s_k \to +\infty \) and
\[ f''(s_k) \leq -\delta(n - (p - 2)) D s_k^{-\delta(n-(p-2))-1} (1 - \epsilon). \]
Since
\[ \frac{g'(s) f'(s)}{g(s)} \sim \delta D s^{-\delta(n-(p-2))-1}, \quad \text{as } s \to +\infty, \]
we have
\[ A_2(s_k) \leq (p - 2) j'(\hat{c}) \left\{ (1 + \epsilon) \delta D s_k^{-\delta(n-(p-2))-1} \right. \]
\[ - (1 - \epsilon) \delta(n - (p - 2)) D s_k^{-\delta(n-(p-2))-1} \right\}. \]
for \( k \) large enough. Now recall that, by the assumptions on \( n \) and \( p \), it holds
\[ D \delta(1 - (n - (p - 2))) < 0. \]
Therefore, we can choose
\[ 0 < \epsilon < (n + 1 - p)/(n + 3 - p) \]
in order to ensure that, for every \( k \) large enough,
\[
A_2(s_k) < 0.
\]
Finally note that, as \( s_k \to +\infty \), \( A_1(s_k) \) and \( A_3(s_k) \) decay faster than \( A_2(s_k) \) because, by the assumptions on \( \delta, n \) and \( p \),
\[
-2\delta(n - (p - 2)) < -1 - \delta(n - (p - 2)), \quad -2\delta < -1 - \delta(n - (p - 2)).
\]
According to [13], this shows that, for \( k \) large enough, \( \Delta_p(H \circ F)(s_k) < 0 \), as requested.

**Acknowledgement.** The author is deeply grateful to his advisor Stefano Pigola for his guidance and constant encouragement during the preparation of the manuscript.

**References**

[1] L.-F. Cheung, C.-K. Law, M.-C. Leung, J.B. McLeod, *Entire solutions of quasilinear differential equations corresponding to \( p \)-harmonic maps*. Nonlinear Anal. 31 (1998), no. 5-6, 701–715.

[2] L.-F. Cheung, P.-F. Leung, *A remark on convex functions and \( p \)-harmonic maps*. Geom. Dedicata 56 (1995), 269–270.

[3] T. Ishihara, *A mapping of Riemannian manifolds which preserves harmonic functions*. J. Math. Kyoto Univ. 19 (1979), 215–229.

[4] M.-C. Leung, *Positive solutions of second order quasilinear equations corresponding to \( p \)-harmonic maps*. Nonlinear Anal. 31 (1998), no. 5-6, 717–733.

[5] E. Loubeau, *\( p \)-harmonic morphisms*. Differential Geom. Appl. 12 (2000), 219–229.

[6] S. Kawai, *\( p \)-harmonic maps and convex functions*. Geom. Dedicata 74 (1999), 261–265.

[7] S. Pigola, M. Rigoli, A.G. Setti, *Constancy of \( p \)-harmonic maps of finite \( q \)-energy into non-positively curved manifolds*. Math. Z. 258 (2008), no. 2, 347–362.

[8] R. Schoen, S.T. Yau, *Lectures on Harmonic Maps*. Lectures Notes in Geometry and Topology, Volume II, International Press.

[9] S.W. Wei, *The minima of the \( p \)-energy functional*. Elliptic and parabolic methods in geometry (Minneapolis 1994), pp. 171–203. A K Peters, Wellesley (1996).

[10] S.W. Wei, *Representing homotopy groups and spaces of maps by \( p \)-harmonic maps*. Indiana Univ. Math. J. 47 (1998), 625–670.

DIPARTIMENTO DI MATematica, UNIVERSITÀ DI Milano, VIA Saldini 50, I-20133 Milano, ITALY

E-mail address: giona.veronelli@unimi.it