Universal characteristics of one-dimensional non-Hermitian superconductors

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Abstract
We establish a non-Bloch band theory for one-dimensional (1D) non-Hermitian topological superconductors. The universal physical properties of non-Hermitian topological superconductors are revealed based on the theory. According to the particle-hole symmetry, there exist reciprocal particle and hole loops of generalized Brillouin zone. The critical point of quantum phase transition, where the energy gap closes, appears when the particle and hole loops intersect at Bloch points. If the non-Hermitian system has non-Hermitian skin effects, the non-Hermitian skin effect should be the $\mathbb{Z}_2$ skin effect: the corresponding eigenstates of particle and hole localize at opposite ends of an open chain, respectively. The non-Bloch band theory is applied to two examples, non-Hermitian $p$- and $s$-wave topological superconductors. In terms of Majorana Pfaffian, a $\mathbb{Z}_2$ non-Bloch topological invariant is defined to establish the non-Hermitian bulk-boundary correspondence for the non-Hermitian topological superconductors.

Keywords: non-Hermitian skin effect, non-Bloch bulk-boundary correspondence, generalized Brillouin zone, topological superconductors

(Some figures may appear in colour only in the online journal)

1. Introduction
Non-Hermitian systems [1–10], described by non-Hermitian Hamiltonians, exhibit a new physical world beyond Hermitian systems from classic [11–18] to quantum [19–35] in recent years. The non-Hermitian topological insulators exhibit non-Hermitian skin effect [7, 36–50], the eigenstates localize at the boundaries under open-boundary conditions (OBC) rather than extend over the bulk, which has been confirmed by experiments [13, 14, 21, 51, 52]. The non-Hermitian skin effect has no counterpart in the Hermitian systems, and cannot be explained by the conventional bulk-boundary correspondence (BBC) in the framework of Bloch theory. To reveal the novel BBC, a non-Bloch band theory for non-Hermitian topological insulator is introduced, in which the concept of Brillouin zone (BZ) in the Bloch theory is extended to generalized Brillouin zone (GBZ) [7, 36, 49, 53–59]. The real-valued wave numbers on BZ are generalized into complex-valued wave numbers on GBZ. Correspondingly, non-Bloch topological invariants can be defined in the GBZ [7, 36, 53], which describe fingerprints of the non-Hermitian topological insulators like the topological invariants for the Hermitian topological insulators.

Following the studies on non-Hermitian topological insulator, non-Hermitian topological superconductor has attracted much attention very recently [60–65]. Topological superconductor is a cousin of topological insulator in the topological family. In a Hermitian topological superconductor, there exist Majorana zero modes (MZMs) at topological defects...
such as boundaries, vortices or domain walls [66–71]. The robust MZMs hold the promising applications in fault-tolerant quantum computing [72–74]. A few preliminary researches indicate that the non-Hermitian superconductors indeed have exotic phenomena, such as robust MZMs [75–92] and non-Hermitian skin effect [93]. However, the universal physical characteristics of the non-Hermitian superconductor systems have not been uncovered far away, because a proper non-Bloch band theory for the non-Hermitian systems has not been established yet.

In this work, we establish a non-Bloch band theory for 1D non-Hermitian superconductors and reveal the universal properties, which does not depend on specific systems but only depends on the particle-hole symmetry (PHS). Because of the PHS, the GBZ of the systems are determined as paired reciprocal loops. Remarkably, the non-Hermitian skin effect of the superconductors is a $Z_2$ skin effect. The energy gap of a non-Hermitian superconductor under OBC closes only when the paired loops of GBZ intersect at Bloch points (BPs) with real-valued non-Bloch wave numbers. These universal properties are clarified in two simple non-Hermitian two-band $p$- and four-band $s$-wave superconductor systems. The GBZ of the two systems are one and two reciprocal loops, respectively. A $Z_2$ skin effect is observed in the two non-Hermitian systems, where the corresponding eigenstates of particle and hole localize at opposite ends of an open chain. In addition, a $Z_2$ non-Bloch topological invariant is defined in terms of the Pfaffian of the Majorana representation matrix, and the non-Hermitian BBC of the non-Hermitian superconductors are established.

2. Non-Bloch band theory and $Z_2$ skin effect

For a Hermitian superconductor, the periodic-boundary condition (PBC) energies of the Bloch Hamiltonian $H(k)$ with real-valued wave numbers $k$ are asymptotically identical to the energies of the system under OBC. The Majorana edge modes of a topological phase can be characterized by a topological invariant of the periodic system, which is named bulk-boundary correspondence. Therefore, the Bloch Hamiltonian captures the properties of the system under OBC. In contrast to Hermitian case, the Bloch Hamiltonian of a non-Hermitian superconductor no longer always gives the properties of the system under OBC, such as the spectra and non-Hermitian skin effect. To understand the properties of the open system, it is necessary to go beyond the Bloch band theory by generalizing the real-valued wave numbers $k$ to a complex plane. Then, a non-Bloch Hamiltonian $H(\beta)$ can be obtained by generalization of the Bloch Hamiltonian with $\beta := e^{ik}$. Here, $\beta$ is extended from a unit circle in Hermitian superconductors to the entire complex plane in non-Hermitian superconductors. We can determined $\beta$ with the complex eigenvalues by the characteristic equation:

$$f(\beta, E) = \det[H(\beta) - E] = 0,$$

which is an even-order ($2M$) algebraic equation in terms of $\beta$, in generally. We number the solutions $\beta_i (i = 1, 2, \ldots, 2M)$ of the eigenvalue equation so as to satisfy $|\beta_1| \leq |\beta_2| \leq \ldots \leq |\beta_{2M}|$. Then, the continuum bands condition is given by [7, 53, 58]:

$$|\beta_M| = |\beta_{M+1}|,$$

and the trajectories of $\beta_M$ and $\beta_{M+1}$ give the GBZ $C_\beta$. In the conception of the GBZ, the non-Bloch Hamiltonian $H(\beta)$ gives the spectra and the eigenstates of the open system. The non-Hermiticity unifies time-reversal symmetry with $TH^\ast(\beta)T^{-1} = H(\beta^{-1})$ and PHS with $CH^T(\beta)C^{-1} = -H(\beta^{-1})$. In addition, the non-Hermitian Bogoliubov-de Gennes (BdG) Hamiltonian for superconductors and superfluids satisfy the PHS [56]:

$$CH^T(\beta)C^{-1} = -H(\beta^{-1}),$$

with $C^2 = \pm 1$. Thus, we only consider the non-Hermitian superconductors with PHS. When $\beta$ is a solution of the equation (1) for a eigenvalue $E$, we have

$$\det[H(\beta) - E] = \det[-CH^T(\beta^{-1})C^{-1} - E] = \det[H(\beta^{-1}) + E] = 0,$$

which implies that $\beta^{-1}$ is a solution for a eigenvalue $-E$. Then, the $\beta$ of the GBZ must satisfy

$$\beta(E) = \beta^{-1}(-E),$$

which indicates that the GBZs of the corresponding particle and hole are paired and reciprocal to each other as shown in figure 1(a). When the particle loop is larger than the BZ, the hole loop will be smaller than the BZ. The corresponding solutions $\beta_{c,i}$ and $\beta_{h,i}$ are the solutions of characteristic equation $f(\beta, E = 0) = 0$ without changing the order in terms of $\beta$. Thus, the corresponding solutions $\beta_{c,i}$ satisfy $\beta_{c,i} = \beta_{c,2M-i+1}$. On account of the GBZ condition (equation (2)), the gap closes the GBZs of the corresponding particle and hole loops of GBZ intersect at BPs with:

$$|\beta_c| = 1.$$
mapped to a hole eigenvector \(|\beta(-E), -E\rangle\) under the particle-hole transformation with
\[
|\beta(E), E\rangle = C|\beta(-E), -E\rangle.
\]
In non-Bloch band theory, the eigenstates localize at the right end for \(|\beta| > 1\) and at the left end for \(|\beta| < 1\) \([7, 45, 58]\). Thus, the corresponding eigenstates of particle and hole localize at the opposite ends of an open chain when \(|\beta(E)| \neq 1\) as shown in figure 1(b). Therefore, the non-Hermitian skin effect of a superconductor are \(Z_2\) skin effect which is protected by the PHS.

Due to the above non-Hermitian features, a non-Hermitian superconductor under OBC is no longer captured by a Bloch Hamiltonian, but by a non-Bloch Hamiltonian. Then, a topological invariant also should be defined on the GBZ to characterize the topological properties of the system. To be concrete, we perform the non-Bloch band theory into two typical systems: non-Hermitian two bands \(p\)- and four bands \(s\)-wave superconductors. With the help of GBZ, a non-Bloch topological invariant is defined in terms of the Majorana Pfaffian to establish the non-Hermitian BBC for the non-Hermitian superconductors.

3. Non-Hermitian Kitaev model

The nonreciprocal Kitaev model describes a non-Hermitian spinless \(p\)-wave superconductor, which can be written in real-space as:
\[
H = \sum_{i} \left[-t_L c_i^\dagger c_{i+1} + t_R c_i^\dagger c_{i+1} + \Delta c_i^\dagger c_i + \Delta c_{i+1}^\dagger c_{i+1}^\dagger - \mu c_i^\dagger c_i\right],
\]
where \(c_i^\dagger (c_i)\) is the creation (annihilation) operator on site \(i\). \(t_L (t_R)\), \(\Delta\) and \(\mu\) are real parameters and denote the left (right) hopping amplitude, pairing amplitude and chemical potential respectively. When \(t_L = t_R\) (Hermitian case), the OBC energies are asymptotically identical to the PBC energies. The MZMs locate at the ends of an open chain when \(\mu \in (-2, 2)\) \([67]\).

When \(t_L \neq t_R\), the OBC energies will be different from PBC energies, especially in the imaginary parts. Figures 2(a) and (b) show the difference for the real and imaginary part, where the black(gray) curves represent the cases of OBC(PBC). The values of other parameters are \(t_L = 1.3, t_R = 0.7\) and \(\Delta = 0.7\). The imaginary part of the energies does not change with the chemical potential under PBC, but sharply collapse under OBC. Meanwhile, it is found that there are MZMs appearing in the open chain. Figures 2(c)-(e) show three typical energy spectra with \(\mu = 0.7, 2.0\) and 3.0, respectively. The OBC energy spectra dramatically collapses compared with the PBC energy spectra. Figures 2(f)-(h) show the corresponding eigenstates of an open Kitaev chain for figures 2(c)-(e).

To exactly capture the non-Hermitian features, the Hamiltonian in equation (8) can be mapped to a non-Bloch Hamiltonian \(H(\beta)\) expressed as:
\[
H(\beta) = h_0(\beta) + h_1(\beta)\sigma_y + h_2(\beta)\sigma_z,
\]
where \(h_0(\beta) = \delta(\beta - \beta^{-1})\), \(h_1(\beta) = i\Delta(\beta - \beta^{-1})\) and \(h_2(\beta) = \mu + i(\beta + \beta^{-1})\) with \(2t = t_L + t_R\) and \(2\Delta = t_L - t_R\). The Hamiltonian \(H(\beta)\) preserves the PHS: \(\sigma_x H(\beta^{-1})\sigma_y = -H(\beta)\), which ensures that the eigenvalues appear in pairs: \(E_{\pm}(\beta) = h_0(\beta) \pm \sqrt{h_1^2(\beta) + h_2^2(\beta)}\).

The characteristic equation of the Hamiltonian in equation (9) has four solutions. According to equation (2), the GBZ of the non-Hermitian Kitaev model is given by the trajectories of \(\beta_2\) and \(\beta_3\) with \(|\beta_2| = |\beta_3|\). Figures 3(a)-(c) show the GBZ of the non-Hermitian Kitaev chain for figures 2(c)-(e), respectively. The loops of particles \(C^p_\beta\) and holes \(C^p_\beta\) are reciprocal to each other with \(\beta_p = \beta_{p}^{-1}\). To reveal the gap closing, figures 3(d)-(f) show the absolute energy spectra in an open chain on the GBZ of figures 3(a)-(c). The inner loop of GBZ \(C^p_\beta\) gives the energy spectra of holes with \(\text{Re}(E(\beta_p)) < 0\), the out loop of GBZ \(C^p_\beta\) gives the energy spectra of particles with \(\text{Re}(E(\beta_p)) > 0\).

Figure 3(b) shows the GBZ of critical case between the topologically nontrivial and trivial phases. The two reciprocal loops of GBZ intersect at a BP with \(\beta_\xi = -1\), where the energy gap closes with \(E(\beta_\xi) = 0\) as shown in figure 3(e). This can be explained as following. The non-Bloch Hamiltonian of the non-Hermitian Kitaev model has an additional symmetry \(H^*(\beta) = H(\beta^*)\), which requires an additional condition on the critical values \(\beta_\xi\) with \(\beta_{\xi}^* = \beta_\xi\). Thus, the gap closing condition in equation (6) becomes \(\beta_\xi = \pm 1\). With help of the condition, we can determine the critical chemical potential by \(E_{\pm}(\beta_\xi) = 0\). For positive chemical potential case, the energy gap closes at \(\mu_c = 2\) with \(\beta_\xi = -1\). While, the reciprocal loops of GBZ can intersect at \(\beta_\xi = 1\) and the energy gap closes at \(\mu_c = -2\) for negative chemical potential.
The $Z_2$ skin effect of the non-Hermitian Kitaev model can be well understood by the conception of GBZ. The eigenstates of particle $|\beta_p, E\rangle$ and hole $|\beta_h, -E\rangle$ satisfy $|\beta_p, E\rangle = \sigma_x |\beta_h, -E\rangle$ with $\beta_p = \beta_h^{-1}$. Therefore, a right localized particle eigenstate with $|\beta| > 1$ will be mapped to a corresponding left localized hole eigenstate with $|\beta| < 1$. Then, the non-Hermitian Kitaev model exhibits a $Z_2$ skin effect, which is protected by the PHS.

To characterize the emergence of the MZMs, we rewrite the non-Bloch Hamiltonian into the Majorana representation. With the help of GBZ, a $Z_2$ non-Bloch topological invariant is given in terms of the Pfaffian of the Majorana representation matrix:

\[ \nu = \text{sgn} [\text{Pf}(B(0)) \text{Pf}(B(\pi))] = \pm 1, \tag{10} \]

here, $\pm 1$ corresponds to topologically trivial and nontrivial. The non-Bloch Majorana representation matrix is $iB(\beta)U^\dagger = h_0(\beta) - h_\pi(\beta)\sigma_x - h_\beta(\beta)\sigma_y$ with unitary transformation matrix $U$:

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}. \tag{11} \]

$\beta_{p/h}(\beta_{p/h,\pi})$ are the values of $\beta_{p/h}$ with $\arg \beta_{p/h} = 0 (\arg \beta_{p/h} = \pi)$ on the particle/hole loop of GBZ. $B(0) = (B(\beta_0) + B(\beta_0, \pi))/2$ and $B(\pi) = (B(\beta_\pi) + B(\beta_\pi, \pi))/2$ are the averages of all the loops of the GBZ. Then, we can determine that the superconductor is topologically nontrivial with $\nu = -1$ for $\mu \in [-2, 2]$ and trivial with $\nu = 1$ for other region. The robust MZMs exist at the ends of an open chain when the phase is topologically nontrivial. The emergence of the robust MZMs can be characterized by the $Z_2$ non-Bloch topological invariant. A non-Hermitian BBC is well-established for the non-Hermitian Kitaev model based on the GBZ. Therefore, the non-Hermitian $p$-wave superconductor can be characterized by the non-Bloch band theory and exhibits the $Z_2$ skin effect. The topological phase transition occurs when the gap closes at BPs.

4. Non-Hermitian s-wave superconductor

The universal characteristics of the non-Hermitian superconductors do not depend on the type of pairing, the number of the bands, and so on, but only on particle-hole symmetry. Here, we investigate the $s$-wave topological superconductor [94, 95] with nonreciprocal hopping. The non-Hermitian spinfull $s$-wave superconductor can be described by the Hamiltonian in real space:

\[ \text{Hamiltonian in real space} \]
The superconductor exhibits a $Z_2$ skin effect. There are five intersections in the loops of GBZ. The corresponding particle loop (red curve) and hole loop (blue curve) only intersect at a BP with $\beta_{B} = -1$. As shown in figure 4(d), the gap closes only when the corresponding particle and hole loops intersect at a BP with $\beta_{B} = -1$ for $\mu > 0$ and $\beta_{B} = 1$ for $\mu < 0$. Then, the gap closes at $h_{\alpha} = \sqrt{ (\mu + 2)^2 + \Delta^2 }$. As increasing of the Zeeman field, the topological phase transition from topologically trivial to nontrivial NH superconductor occurs at $h_{\alpha} = \min(h_{\alpha})$.

5. Conclusion and discussion

We established a non-Bloch band theory for 1D non-Hermitian superconductors. Based on the theory, a series of universal physical characteristics have been uncovered, such as two reciprocal loops, $Z_2$ skin effect. These results will be crucial for understanding superconductivity under ubiquitous environment couplings. The two non-Hermitian superconductor models can be realized in cold atoms [96, 97] and topolectric circuits [98, 99]. Our results can be extended to non-Hermitian superconductors with other symmetry such as the Floquet non-Hermitian superconductors. In the presence of periodic driving, the Floquet Hamiltonian also preserves a PHS. The skin modes of the Floquet non-Hermitian superconductors are $Z_2$ skin modes. Furthermore, the periodicity in quasienergy will enrich the non-Hermitian features in non-Hermitian superconductors.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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Figure 4. (a) The energy spectra in an open $s$-wave non-Hermitian superconductor chain under OBC (red and blue) and PBC (gray). (b) The eigenstates of the open chain. (c) The two paired reciprocal particle and hole loops of GBZ (solid curves) and BZ (dashed curves). The corresponding particle and hole loops of GBZ intersect at a BP when the gap closes. (d) The OBC energy spectra on the GBZ. The values of the parameters are $t_{\alpha} = 1.4$, $t_{B} = 0.6$, $\alpha_{R} = 0.8$, $\Delta = 1.2$, $h = 2$ and $\mu = 3.6$. 

\[
H = H_{I} + H_{\alpha} + H_{\text{sc}} + H_{\text{z}} \\
H_{I} = \sum_{j, \sigma \in \uparrow, \downarrow} \left[ -t_{\alpha} c_{j}^{\dagger} c_{j+1, \sigma} - t_{\text{B}} c_{j+1, \sigma} c_{j, \sigma} - \mu c_{j, \sigma}^{\dagger} c_{j, \sigma} \right], \\
H_{\alpha} = \alpha_{R} \sum_{j} \left[ c_{j, \uparrow}^{\dagger} c_{j+1, \downarrow}^{\dagger} + c_{j, \downarrow}^{\dagger} c_{j+1, \uparrow}^{\dagger} - c_{j, \uparrow}^{\dagger} c_{j+1, \downarrow} - c_{j, \downarrow}^{\dagger} c_{j+1, \uparrow} \right], \\
H_{\text{sc}} = \sum_{j} \Delta \left[ c_{j, \uparrow}^{\dagger} c_{j, \downarrow}^{\dagger} - c_{j, \downarrow} c_{j, \uparrow} \right], \\
H_{\text{z}} = h \sum_{j} \left[ c_{j, \uparrow}^{\dagger} c_{j, \downarrow} + c_{j, \downarrow}^{\dagger} c_{j, \uparrow} \right],
\]

where $c_{j}^{\dagger}$ ($c_{j}$) is the creation (annihilation) operator on site $j$, $\alpha_{R}$ and $h$ are real parameters and denote the strength of spin–orbit coupling and Zeeman field, respectively.

Figure 4(a) shows the OBC energies and PBC energies with $t_{\alpha} = 1.4$, $t_{B} = 0.6$, $\alpha_{R} = 0.8$, $\Delta = 1.2$, $h = 2$ and $\mu = 3.6$. The four bands OBC energies dramatically collapses comparing with the PBC energies. The eigenstates of particles and holes localize at the opposite ends of an open chain as shown in figure 4(b).

To understand the non-Hermitian features, the Hamiltonian in equation (12) is rewritten into a non-Bloch Hamiltonian $H(\beta) = \frac{1}{2} \sum_{j} \left[ -\delta_{\beta} (\beta - \beta^{-1}) c_{j}^{\dagger} c_{j} + i (\beta + i \beta^{-1} - \mu) c_{j}^{\dagger} c_{j} + i \alpha_{R} (\beta - \beta^{-1}) \tau \sigma_{j} + h \tau c_{j} - \Delta \tau c_{j} \right]$, where $\beta$ and $\tau$ are the Pauli matrices in spin and particle-hole space, respectively. The non-Bloch Hamiltonian preserves the PHS with $\tau H(\beta) \tau = -H(\beta^{-1})$. The GBZ $C_{\beta}$ is given by the trajectories of $\beta_{\alpha}$ and $\beta_{h}$ with $|\beta_{\alpha}| = |\beta_{h}|$ as shown in figure 4(c). The GBZ are two paired reciprocal loops with two out particle loops $(|\beta_{\alpha}| > 1)$ and corresponding two inner hole loops $(|\beta_{h}| < 1)$. Therefore,
