Stability and convergence in discrete convex monotone dynamical systems

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Abstract. We study the stable behaviour of discrete dynamical systems where the map is convex and monotone with respect to the standard positive cone. The notion of tangential stability for fixed points and periodic points is introduced, which is weaker than Lyapunov stability. Among others we show that the set of tangentially stable fixed points is isomorphic to a convex inf-semilattice, and a criterion is given for the existence of a unique tangentially stable fixed point. We also show that periods of tangentially stable periodic points are orders of permutations on $n$ letters, where $n$ is the dimension of the underlying space, and a sufficient condition for global convergence to periodic orbits is presented.

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1. Introduction

Many natural dynamical systems preserve a type of ordering on the state space. Such dynamical systems are called monotone and often display rather simple behaviour. In the last couple of decades monotone dynamical systems have been studied intensively, see [15] for an up-to-date survey. Ground-breaking work on monotone dynamical systems was done by Hirsch [13, 14], who showed, among others, that in a continuous-time strongly monotone dynamical system, almost all precompact orbits converge to the set of equilibrium points. In a discrete-time strongly monotone dynamical system, one has generic convergence to periodic orbits under appropriate conditions on the map; see [9, 12, 24]. Various additional conditions have been studied to obtain convergence of all orbits instead of almost all orbits. A type of concavity condition, also called subhomogeneity, has received a great deal of

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attention; see [2, 16, 19, 30, 31]. The concavity condition makes the dynamical system nonexpansive, which allows one to prove strikingly detailed results concerning their behaviour.

In this paper, we study discrete-time dynamical systems

\[ x^{k+1} = f(x^k) \quad \text{for } k = 0, 1, 2, \ldots, \tag{1} \]

where \( f: \mathcal{D} \to \mathcal{D} \) is a convex monotone map on \( \mathcal{D} \subseteq \mathbb{R}^n \) preserving the partial ordering induced by the standard positive cone. Such dynamical systems are in general not nonexpansive. We introduce the notion of tangential stability for fixed points and periodic points, which is weaker than Lyapunov stability. It turns out that this notion is the right one to prove a variety of results concerning the stable behaviour of monotone convex dynamical systems, which are of comparable detail as the ones for monotone nonexpansive dynamical systems.

In particular, we show that the tangentially stable fixed point set is isomorphic to a convex inf-semilattice in \( \mathbb{R}^n \). We also give a criterion for the existence of a unique tangentially stable fixed point. In addition, tangentially stable periodic orbits are analysed and a condition is presented under which there is global convergence to Lyapunov stable periodic orbits. Among others it is shown that the periods of tangentially stable periodic points divide the cyclicity of the critical graph, which implies that the periods are orders of permutations on \( n \) letters. However, the periods of unstable periodic orbits can be arbitrary large.

The results are a continuation of [1] in which the first two authors studied discrete-time dynamical systems (1), where \( f: \mathbb{R}^n \to \mathbb{R}^n \) is not only convex and monotone, but also additively subhomogeneous. The extra subhomogeneity condition makes the dynamical system nonexpansive under the sup-norm [8]. The nonexpansiveness property severely constrains the complexity of its behaviour [18, 22] and makes all fixed points and periodic orbits Lyapunov stable. It also ensures that the subdifferential of \( f \) at a fixed point consists of row-stochastic matrices. Without the additively homogeneity condition, the subdifferential merely consists of stable nonnegative matrices, which makes the analysis more subtle.

Motivating examples of discrete convex monotone dynamical systems arise in Markov decision processes and game theory as value iteration schemes; see [1] and the references therein. The results in this paper extend results for Markov decision processes with substochastic transition matrices to arbitrary nonnegative matrices. In particular, they apply to Markov decision processes with negative discount rates; see [28]. Discrete convex monotone dynamical systems are also used in static analysis of programs by abstract interpretation [7], i.e., automatic verification of variables in computer programs. They also appear in the theory of discrete event systems [4], statistical mechanics [23], and in the analysis of imprecise Markov chains [10]. At the end of Section 2 we give several explicit examples.
The paper contains nine sections. In Section 2 several basic definitions and properties of convex monotone maps are collected. Subsequently, in Section 3, various degrees of stability of fixed points of convex monotone maps are discussed and the notion of tangential stability is introduced. In Section 4 several preliminary results concerning stable nonnegative matrices are given. Among others, rectangular sets of stable nonnegative matrices are studied. In Section 5, we analyse tangentially stable fixed points and introduce the critical graph of a monotone convex map. In Section 6 we collect several preliminary results concerning convex monotone positively homogeneous maps that are needed in the analysis of the geometry of the tangentially stable fixed point set. Section 7 contains the main result on the geometry of the tangentially stable fixed point set. Section 8 concerns tangentially stable periodic points and their periods. In the final section, a criterion is given under which each orbit of a discrete-time convex monotone dynamical systems converges to a Lyapunov stable periodic orbit.

2. Basic properties of convex monotone maps

Let \( \mathbb{R}_+^n = \{ x \in \mathbb{R}^n : x_i \geq 0 \text{ for } 1 \leq i \leq n \} \) denote the standard positive cone. The cone \( \mathbb{R}_+^n \) induces a partial ordering on \( \mathbb{R}^n \) by \( x \leq y \) if \( y - x \in \mathbb{R}_+^n \). We write \( x \ll y \) if \( y - x \) is in the interior of \( \mathbb{R}_+^n \). In particular, we say that \( x \) is positive if \( 0 \ll x \). For \( x, y \in \mathbb{R}^n \) we also use the notation \( x \geq y \) and \( x \gg y \) with the obvious interpretation. A set \( X \subseteq \mathbb{R}^n \) is called bounded from above if there exists \( u \in \mathbb{R}^n \) such that \( x \leq u \) for all \( x \in X \). Similarly, we say that \( X \subseteq \mathbb{R}^n \) is bounded from below if there exists \( l \in \mathbb{R}^n \) such that \( l \leq x \) for all \( x \in X \). The partially ordered vector space \( (\mathbb{R}^n, \leq) \) is a vector lattice, where the binary relations \( \land \) and \( \lor \) are defined as follows. For \( x, y \in (\mathbb{R}^n, \leq) \), \( x \land y \) is the greatest lower bound of \( x \) and \( y \), so \( (x \land y)_i = \min\{x_i, y_i\} \) for \( 1 \leq i \leq n \), and \( x \lor y \) is the least upper bound of \( x \) and \( y \), so \( (x \lor y)_i = \max\{x_i, y_i\} \) for \( 1 \leq i \leq n \).

A map \( f : D \to \mathbb{R}^m \), where \( D \subseteq \mathbb{R}^n \), is called monotone if for each \( x, y \in D \) with \( x \leq y \) we have that \( f(x) \leq f(y) \). It is called strongly monotone if \( x \leq y \) and \( x \neq y \) implies that \( f(x) \ll f(y) \). A map \( f : D \to \mathbb{R}^m \), where \( D \subseteq \mathbb{R}^n \) is convex, is called convex if

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \text{for all } 0 \leq \lambda \leq 1 \text{ and } x, y \in D.
\]

In other words, \( f : D \to \mathbb{R}^m \) is convex if each coordinate function is convex in the usual sense. The reader may note that our notion of monotonicity is different from the one commonly used in convex analysis [25].

The orbit of \( x \in D \) under a map \( f : D \to D \) is given by \( \mathcal{O}(x; f) = \{ f^k(x) : k = 0, 1, 2, \ldots \} \). We say that \( x \in D \) is a periodic point of \( f : D \to D \) if \( f^p(x) = x \) for some integer \( p \geq 1 \), and the minimal \( p \geq 1 \) with this property is called the period of \( x \) under \( f \).

Let \( M_{m,n} \) denote the set of all \( m \times n \) real matrices, and let \( P_{m,n} \) be the set of all nonnegative matrices in \( M_{m,n} \). A matrix \( P \in P_{m,n} \) is called positive if \( p_{ij} > 0 \) for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). Given a matrix \( M \in M_{m,n} \), we
denote its rows by $M_1, \ldots, M_m \in \mathcal{M}_{1,n}$, and we identify $M$ with the $m$-tuple $(M_1, \ldots, M_m)$. So, $\mathcal{M}_{m,n}$ is identified with the $m$-fold direct product

$$\mathcal{M}_{m,n} = \mathcal{M}_{1,n} \times \cdots \times \mathcal{M}_{1,n}.$$  

We say that $\mathcal{R} \subseteq \mathcal{M}_{m,n}$ is rectangular if $\mathcal{R}$ can be written as $\mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_m$, where $\mathcal{R}_1, \ldots, \mathcal{R}_m$ are nonempty subsets of $\mathcal{M}_{1,n}$. Furthermore, it is convenient to introduce the following matrix notation. Given $M \in \mathcal{M}_{m,n}$, $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, n\}$, we write $M_{IJ}$ to denote the $|I| \times |J|$ submatrix of $M$ with row indices in $I$ and column indices in $J$. Likewise, given $x \in \mathbb{R}^n$ and $K \subseteq \{1, \ldots, n\}$, we write $x_K \in \mathbb{R}^K$ to denote the vector in $\mathbb{R}^K$ obtained by restricting $x$ to its coordinates in $K$.

For a convex map $f : \mathcal{D} \to \mathbb{R}^m$, where $\mathcal{D} \subseteq \mathbb{R}^n$ is open and convex, the subdifferential of $f$ at $v \in \mathcal{D}$ is defined by

$$\partial f(v) = \{M \in \mathcal{M}_{m,n} : f(x) - f(v) \geq M(x - v) \text{ for all } x \in \mathcal{D}\}. \quad (2)$$

In the following proposition several basic facts concerning the subdifferential are collected; cf. [25, Theorem 23.4].

**Proposition 2.1.** If $f$ is a convex map from an open convex subset $\mathcal{D} \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$, then for each $v \in \mathcal{D}$, the set $\partial f(v)$ is nonempty, compact, convex and rectangular.

We note that the rectangularity of $\partial f(v)$ follows directly from the fact that $f(x) - f(v) \geq M(x - v)$ is equivalent to $f_i(x) - f_i(v) \geq M_i(x - v)$ for all $1 \leq i \leq m$. If, in addition, the map is monotone, then $\partial f(v)$ consists of nonnegative matrices as the following proposition shows.

**Proposition 2.2.** If $f : \mathcal{D} \to \mathbb{R}^m$, where $\mathcal{D} \subseteq \mathbb{R}^n$ is open and convex, is a convex monotone map, then $\partial f(v) \subseteq \mathcal{P}_{m,n}$ for all $v \in \mathcal{D}$. Moreover, if $f$ is strongly monotone, then each $P \in \partial f(v)$ is positive.

**Proof.** If $v \in \mathcal{D}$, then there exists $\mathcal{U}$ open neighbourhood of 0 such that $v - u \in \mathcal{D}$ for all $u \in \mathcal{U}$. Now if $u \geq 0$, with $u \in \mathcal{U}$, and $P \in \partial f(v)$, then $0 \geq f(v - u) - f(v) \geq -Pu$, as $f$ is monotone. Thus, $Px \geq 0$ for all $x \in \mathbb{R}^n_+$ and hence $P \in \mathcal{P}_{m,n}$. We note that if $f$ is strongly monotone, $u \geq 0$ and $u \neq 0$, then $0 \geq f(v - u) - f(v) \geq -Pu$. This implies that $Px \gg 0$ for all $x \in \mathbb{R}^n_+ \setminus \{0\}$ and therefore $P$ is positive. □

For a convex map $f : \mathcal{D} \to \mathbb{R}^m$, where $\mathcal{D} \subseteq \mathbb{R}^n$ is open and convex, and $v \in \mathcal{D}$, the one-sided directional derivative of $f$ at $v$ is given by

$$f'_v(y) = \lim_{\varepsilon \downarrow 0} \frac{f(v + \varepsilon y) - f(v)}{\varepsilon}. \quad (3)$$

The map $f'_v : \mathbb{R}^n \to \mathbb{R}^m$ is well defined, convex, finite valued and positively homogeneous, meaning that $f'_v(\lambda x) = \lambda f'_v(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$; see [25, Theorem 23.1]. Moreover, $f'_v$ is monotone (because it is defined as a pointwise limit of monotone maps). We shall occasionally need the following representation of $f'_v$:

$$f'_v(y) = \sup_{P \in \partial f(v)} Py \quad \text{for } y \in \mathbb{R}^n \quad (4)$$
(see [25, Theorem 23.4]). If \( f : \mathcal{D} \to \mathcal{D} \), where \( \mathcal{D} \subseteq \mathbb{R}^n \) is open and convex, then for each \( v \in \mathcal{D} \) we also have that

\[
(f'_v)^k = f'_{f^{k-1}(v)} \circ \cdots \circ f'_v \circ f'_v
\]

(see [1, Lemma 4.3]).

Throughout the remainder of the exposition, we shall make the following assumption on the domain of convex monotone maps \( f : \mathcal{D} \to \mathcal{D} \).

**Hypothesis 2.3.** The set \( \mathcal{D} \subseteq \mathbb{R}^n \) is open and convex.

Although in some results more general domains can be treated, we restrict ourselves to this case, as it simplifies the presentation.

To conclude this section, we briefly discuss several examples of convex monotone maps. In the theory of Markov decision processes one considers monotone convex maps \( f : \mathbb{R}^n \to \mathbb{R}^n \) of the form

\[
f_i(x) = \sup_{j \in A_i} r^j_i + p^j \cdot x \quad \text{for } i = 1, \ldots, n.
\]

(5)

Here \( r^j_i \in \mathbb{R} \) and \( p^j \) is a substochastic vector for each \( j \in A_i \). The results in this paper apply to the case where \( p^j \) is merely a nonnegative vector.

Other examples arise in the study of systems of polynomial equations, \( x = P(x) \), where \( P = (P_1, \ldots, P_n) \) and each \( P_i \) is a polynomial with variables \( x_1, \ldots, x_n \) and nonnegative coefficients. Looking for a positive solution of \( x = P(x) \) is equivalent to finding a fixed point of the map \( f(x) = \text{Log} \circ P \circ \text{Exp} \), where \( \text{Exp}(x_1, \ldots, x_n) = (e^{x_1}, \ldots, e^{x_n}) \) and \( \text{Log} \) denotes its inverse. We can write \( P_i(x) = \sum_{j \in A_i} a_{ij} x^j \), where \( A_i \subseteq \mathbb{N}^n \) is a finite set and each \( a_{ij} \geq 0 \), with the convention that \( x^j = x_1^{j_1} \cdots x_n^{j_n} \) for \( j = (j_1, \ldots, j_n) \in \mathbb{N}^n \). Then

\[
f_i(x) = \log \left( \sum_{j \in A_i} a_{ij} \exp(j \cdot x) \right).
\]

(6)

Such “log-exp” functions are not only monotone, but also convex; see [26, Example 2.16]. More generally, we could allow \( A_i \) to be a subset \( \mathbb{R}^n_+ \), instead of \( \mathbb{N}^n \). This yields a class of functions \( P_i \) which are usually called posynomials [6]. Posynomials play a role in static analysis of programs by abstract interpretation [7]. Note that the example in (5) can be obtained as a limit of posynomials by setting \( a_{ij} = e^{\beta r^j_i} \) and

\[
f_i^\beta(x) := \beta^{-1} \log \left( \sum_{j \in A_i} \exp(\beta (r^j_i + p^j \cdot x)) \right).
\]

If \( \beta \) tends to \( +\infty \), then \( f_i^\beta \) converges to the map (5); see [32].

### 3. Stability of fixed points

Recall that a fixed point \( v \in \mathcal{D} \) of \( f : \mathcal{D} \to \mathcal{D} \) is **Lyapunov stable** if for each neighbourhood \( \mathcal{U} \) of \( v \), there exists a neighbourhood \( \mathcal{V} \subseteq \mathcal{D} \) of \( v \) such that
$v \in V$ and $f^k(V) \subseteq U$ for all $k \geq 1$. An $n \times n$ matrix $P$ is called stable if all the orbits of $P$ are bounded. Stable matrices have the following well-known characterizations.

**Proposition 3.1.** For a matrix $P$ the following assertions are equivalent.

(i) $P$ is stable.
(ii) There exists a norm on $\mathbb{R}^n$ such that the induced matrix norm of $P$ is at most one.
(iii) All the eigenvalues of $P$ have modulus at most one and all the eigenvalues of modulus one are semisimple.
(iv) The origin is a Lyapunov stable fixed point of $P$.

In the analysis of convex monotone dynamical systems, it is useful to distinguish various notions of stability that are weaker than the classical Lyapunov stability.

**Proposition 3.2.** If $f: D \to D$ is a convex monotone map with a fixed point $v \in D$, then the following assertions

(i) $v$ is a Lyapunov stable fixed point,
(ii) there exists a neighbourhood $V \subseteq D$ of $v$ such that every orbit of $x \in V$ is bounded,
(iii) there exists a neighbourhood $V \subseteq D$ of $v$ such that every orbit of $x \in V$ is bounded from above,
(iv) every orbit of $f'_v: \mathbb{R}^n \to \mathbb{R}^n$ is bounded,
(v) every orbit of $f'_v: \mathbb{R}^n \to \mathbb{R}^n$ is bounded from above,
(vi) each $P \in \partial f(v)$ is stable,
(vii) every orbit of $f'_v: \mathbb{R}^n \to \mathbb{R}^n$ is bounded from below,
(viii) every orbit of $f: D \to D$ is bounded from below,

satisfy the following implications:

$$(i) \iff (ii) \iff (iii) \Rightarrow (iv) \iff (v) \Rightarrow (vi) \Rightarrow (vii) \Rightarrow (viii).$$

**Proof.** The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(iv) \Rightarrow (v)$ are trivial. We start by showing that (iii) implies (v). Let $x \geq 0$ be such that $v + x \in V$. Remark that $f(v + x) \geq f(v) + f'_v(x) = v + f'_v(x)$. Since $f$ is monotone, we deduce that

$$f^k(v + x) \geq v + (f'_v)^k(x) \quad \text{for all } k \geq 1,$$

and hence $O(x; f'_v)$ is bounded from above. As $f'_v$ is monotone and positively homogeneous, there exist for each $y \in \mathbb{R}^n$, a vector $x \geq 0$ and a scalar $\lambda > 0$ such that $y \leq \lambda x$ and $v + x \in V$. Thus, we get that $f'_v$ has all its orbits bounded from above.

Next we prove that (v) implies (vi). If $P \in \partial f(v)$, then we know by (4) that $f'_v(x) \geq Px$. As $f'_v$ is monotone, we get that $(f'_v)^k(x) \geq P^kx$ for all $k \geq 1$ and $x \in \mathbb{R}^n$ and therefore $P$ has all its orbits bounded from above. This implies that $P$ has all its orbits bounded, since $P$ is linear.

Suppose that $\partial f(v)$ contains a stable matrix $P$. We deduce from (4) that $(f'_v)^k(x) \geq P^kx$ for all $k \geq 1$ and $x \in \mathbb{R}^n$. As $P$ is stable, this implies that $O(x; f'_v)$ is bounded from below, which shows that (vi) implies (vii).
To see that (vii) implies (viii), let \( x \in \mathcal{D} \) and note that, by (7),
\[
f^k(x) \geq v + (f'_0)^k(x - v)
\]
for all \( k \geq 1 \).

As \( \mathcal{O}(x - v; f'_0) \) is bounded from below, we get that \( \mathcal{O}(x; f) \) is also bounded from below.

Note that (v) implies (vi) and (vi) implies (vii), so that (iv) and (v) are equivalent. Similarly, (iii) implies (viii), so that (ii) and (iii) are equivalent. It remains to show that (ii) implies (i).

If (ii) holds, there exists \( u \gg 0 \) such that \( v + \lambda u \in \mathcal{D} \) for all \( |\lambda| \leq 1 \) and \( \mathcal{O}(v + u; f) \) is bounded. Let \( k \geq 1 \) and note that for \( 0 < \lambda < 1 \),
\[
\lambda(f^k(v + u) - v) = \lambda(f^k(v + u) - v) + (1 - \lambda)(f^k(v) - v)
\]
\[
\geq f^k(\lambda(v + u) + (1 - \lambda)v) - v
\]
\[
= f^k(v + \lambda u) - v.
\]

Take \( P \in \partial f(v) \) and remark that \( f(x) \geq P(x - v) + v \). This implies that
\[
f^k(x) \geq P^k(x - v) + v \quad \text{for all} \quad x \in \mathcal{D}.
\]
Hence
\[
f^k(v - \lambda u) - v \geq -\lambda P^k u \quad \text{for} \quad 0 < \lambda < 1.
\]
(8)

As (ii) implies (vi), we know that \( P \) is stable. Define a norm \( \|\cdot\|_u \) on \( \mathbb{R}^n \) by
\[
\|x\|_u = \inf \{\alpha > 0: -\alpha u \leq x \leq \alpha u\}
\]
and let
\[
\gamma = \sup_{k \geq 1} \left\{ \|P^k u\|_u, \|f^k(v + u) - v\|_u \right\}.
\]

Note that \( \gamma < \infty \), as \( P \) is stable and \( \mathcal{O}(v + u; f) \) is bounded. Let \( \lambda := \|x - v\|_u \), so that \( v - \lambda u \leq x \leq v + \lambda u \). If \( \lambda < 1 \), we get that \( -\lambda \gamma u \leq f^k(x) - v \leq \lambda \gamma u \) for each \( k \geq 1 \). Thus, \( \|f^k(x) - v\|_u \leq \gamma \lambda = \gamma \|x - v\|_u \) for all \( x \in \mathcal{D} \) such that \( \|x - v\|_u < 1 \) and for all \( k \geq 1 \). Hence \( v \) is a Lyapunov stable fixed point.

**Example 1.** Let \( f: \mathbb{R}^n \to \mathbb{R}^n \) be given by \( f(x) = \max\{0, x + x^2\} \) for \( x \in \mathbb{R}^n \). Then \( f'_0(x) = \max\{0, x\} \) and hence every orbit of \( f'_0 \) is bounded. However, the orbit of each \( x > 0 \) is unbounded under \( f \). This show that (iv) does not imply (iii).

**Example 2.** This example shows that (vi) does not imply (v). Consider \( h(p) = -p \log p - (1 - p) \log(1 - p) \) for \( p \in [0, 1] \), and define \( g: \mathbb{R}^2 \to \mathbb{R}^2 \) by
\[
g(x) = \sup_{p \in [0, 1]} px_1 + h(p)x_2 \quad \text{for} \quad x = (x_1, x_2) \in \mathbb{R}^2,
\]
which is the Legendre–Fenchel transform of \(-h\). Note that \( g(x) = x_2 \log(1 + e^{x_1/x_2}) \), for \( x_2 > 0 \). Indeed, put \( x_2 = 1 \) and consider
\[
\frac{d}{dp}(px_1 + h(p)) = 0.
\]
Solving for \( p \) gives \( p = e^{x_1}/(1 + e^{x_1}) \), so that \( g(x_1, 1) = \log(1 + e^{x_1}) \). As \( g \) is positively homogeneous, \( g(x) = x_2 \log(1 + e^{x_1/x_2}) \) for \( x_2 > 0 \).
Now define \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) by
\[
f_1(x) = \begin{cases} 
  \max\{0, x_1\} & \text{if } x_2 \leq 0, \\
  x_2 \log(1 + e^{x_1/x_2}) & \text{if } x_2 > 0
\end{cases}
\]
and \( f_2(x) = x_2 \) for all \( x = (x_1, x_2) \in \mathbb{R}^2 \). If \( x_2 > 0 \), we get that \( f_1^k(x) = x_2 \log(k + e^{x_1/x_2}) \to \infty \), as \( k \to \infty \). Thus, not all orbits of \( f_0' = f \) are bounded from above. But \( \partial f(0) \) consists of matrices of the form
\[
\begin{pmatrix}
p & s \\
0 & 1
\end{pmatrix}, \quad \text{where } 0 \leq s \leq h(p) \text{ and } 0 \leq p \leq 1.
\]
As \( h(1) = 0 \), all these matrices are stable.

**Example 3.** To see that (vii) does not imply (vi), we consider the map given by
\[
R(A) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
and \( f_2(x) = x_2 \) for all \( x = (x_1, x_2) \in \mathbb{R}^2 \). If \( x_2 > 0 \), we get that \( f_1^k(x) = x_2 \log(k + e^{x_1/x_2}) \to \infty \), as \( k \to \infty \). Thus, not all orbits of \( f_0' = f \) are bounded from above. But \( \partial f(0) \) consists of matrices of the form
\[
\begin{pmatrix}
p & s \\
0 & 1
\end{pmatrix}, \quad \text{where } 0 \leq s \leq h(p) \text{ and } 0 \leq p \leq 1.
\]
As \( h(1) = 0 \), all these matrices are stable.

**Example 4.** To prove that (viii) does not imply (vii), consider \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) given by
\[
f \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} e^{x_1} + e^{x_2} - 2 \\ x_2 \end{array} \right) \; \text{ for } x = (x_1, x_2) \in \mathbb{R}^2.
\]
Then
\[
f_0' \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)
\]
so that \( O(-u; f_0') \) is unbounded from below for \( u \gg 0 \). But clearly every orbit of \( f \) is bounded from below.

We use the following notion of stability which is weaker than ordinary Lyapunov stability according to Proposition 3.2.

**Definition 3.3.** Let \( f : D \to D \) be a convex monotone map and let \( v \in D \) be a fixed point of \( f \). We say that \( v \) is a *tangentially stable*, or \( t \)-stable, fixed point if \( f'_v \) has all its orbits bounded from above. Similarly, we call a periodic point \( \xi \in D \) with period \( p \) *tangentially stable* if it is a \( t \)-stable fixed point of \( f^p \).

We note that if \( v \) is a \( t \)-stable periodic point of \( f \) with period \( p \), then \( f^m(v) \) is also \( t \)-stable for each \( 0 < m < p \). Indeed, as
\[
(f^p)'_v = (f^{p-m})'_v \circ (f^m)'_v \quad \text{and} \quad (f^p)'_{f^m(v)} = (f^m)'_v \circ (f^{p-m})'_{f^m(v)},
\]
we get that
\[
(f^p)'_{f^m(v)} = (f^m)'_v \circ (f^{p-m})'_{f^m(v)} = (f^{p-m})'_{f^m(v)} \circ (f^p)'_v.
\]
Thus, every point in the orbit of a \( t \)-stable periodic point is also \( t \)-stable. We denote by \( \mathcal{E}(f) \) the set of all \( t \)-stable fixed points of \( f \) and we let
\[
\mathcal{E}_t(f) = \{ v \in \mathcal{E}(f) : v \text{ is } t \text{-stable}\}.
\]
The subdifferential of a t-stable fixed point consists of nonnegative stable matrices by Proposition 3.2. In the next section we collect some results concerning stable matrices that will be useful in the analysis.

4. Stable nonnegative matrices

With an $n \times n$ nonnegative matrix $P = (p_{ij})$ we associate a directed graph $G(P)$ on $n$ nodes, in the usual way, by letting an arrow go from node $i$ to $j$ if $p_{ij} > 0$. We say that a node $i$ has access to a node $j$ if there is a (directed) path in $G(P)$ from $i$ to $j$. Using the notion of access, one defines an equivalence relation $\sim$ on $\{1, \ldots , n\}$ by $i \sim j$ if $i$ has access to $j$ and vice versa. The equivalence classes are called classes of $P$. A nonnegative matrix $P$ is called irreducible if it has only one class. Otherwise, it is said to be reducible.

The spectral radius of $P$ is given by

$$\rho(P) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } P\}.$$

If $P$ is a stable nonnegative matrix, we call a class $C$ of $P$ critical if $\rho(P_{CC}) = 1$. Recall that in Perron–Frobenius theory a class $C$ of $P$ is called basic if $\rho(P_{CC}) = \rho(P)$. Thus, all critical classes of a stable nonnegative matrix are basic by Proposition 3.1. The following proposition is a direct consequence of [27, Theorem 3] (see also [29, Corollary 3.4]) and Proposition 3.1.

**Proposition 4.1.** If $P$ is a nonnegative stable matrix and $C$ and $C'$ are two distinct critical classes of $P$, then no $i \in C$ has access to any $j \in C'$.

Moreover, we have the following general fact concerning nonnegative matrices (cf. [11, Chapter XIII, Section 3.4]).

**Proposition 4.2.** If $P$ is a nonnegative matrix, then $\rho(P') \leq \rho(P)$ for all principal submatrices $P'$ of $P$. If $P$ is reducible, then the equality holds for at least one principal submatrix $P'$ of $P$ with $P' \neq P$.

Using these proposition we now prove the following useful normal form for stable nonnegative matrices.

**Proposition 4.3.** If $P$ is a nonnegative stable $n \times n$ matrix, then there exist a permutation matrix $\Pi$ and a unique partition of $\{1, \ldots , n\}$ into disjoint sets $U$, $C$, $D$ and $I$ such that

(i)

$$\Pi^T P \Pi = \begin{pmatrix} P_{UU} & P_{UC} & P_{UD} & P_{UI} \\ 0 & P_{CC} & P_{CD} & 0 \\ 0 & 0 & P_{DD} & 0 \\ 0 & 0 & P_{ID} & P_{II} \end{pmatrix},$$

where $C$ is the disjoint union of the critical classes $C_1, \ldots , C_r$ of $P$,

(ii) $P_{CC}$ is block diagonal, with blocks $P_{C_i C}$ for $1 \leq i \leq r$,

(iii) every $i \in U$ has access to some $j \in C$, and for every $i \in D$ there exists $j \in C$ that has access to $i$,

(iv) $I = \{1, \ldots , n\} \setminus (U \cup C \cup D)$.

Moreover, in that case, we have $\rho(P_{UU}) < 1$, $\rho(P_{DD}) < 1$ and $\rho(P_{II}) < 1$. 
**Figure 1.** Partition associated with a stable matrix.

**Proof.** Let $C$ be the union of the critical classes $C_1, \ldots, C_r$ of $P$. Then $P_{CC}$ is block diagonal by Proposition 4.1. Let $U$ be the set of nodes of $G(P)$ not in $C$ that have access to some $i \in C$. In addition, let $D$ be the set of nodes $i$ in $G(P)$ that are not in $C$, but from which there exists $j \in C$ such that $j$ has access to $i$. We remark that $U \cap D = \emptyset$. Indeed, if $i \in U \cap D$, then there exist $j_1, j_2 \in C$ such that $i$ has access to $j_1$ and $j_2$ has access to $i$. By Proposition 4.1, $j_1$ and $j_2$ are both in a single class, say $C_m$. But this implies that $i \in C_m$, which is a contradiction. In fact, the same argument shows that $P_{DU} = 0$, $P_{CU} = 0$ and $P_{DC} = 0$.

Now let $I = \{1, \ldots, n\} \setminus (U \cup C \cup D)$. By definition, no node in $I$ has access to a node in $C$, nor can it be accessed by a node in $C$. Hence $P_{IC} = 0$ and $P_{CI} = 0$. Furthermore, the definition of $U$ and $D$ implies that $P_{IU} = 0$ and $P_{DI} = 0$. Thus, the sets $U$, $C$, $D$ and $I$ communicate as in Figure 1 and hence there exists a permutation matrix $\Pi$ such that $\Pi^T P \Pi$ satisfies (9).

To prove the final assertion, we remark that if $\rho(P_{UU}) = 1$, then there exists a critical class $U^* \subseteq U$ such that $\rho(P_{U^* U^*}) = \rho(P_{UU}) = 1$ and $P_{U^* U^*}$ is irreducible by Proposition 4.2, which is a contradiction. In exactly the same way it can be shown that $\rho(P_{DD}) < 1$ and $\rho(P_{II}) < 1$. □

The sets of nodes $U$, $C$, $D$ and $I$ in $G(P)$ are, respectively, called upstream nodes, critical nodes, downstream nodes and independent nodes. By using the normal form and the Perron–Frobenius theorem we now prove the following assertion.

**Proposition 4.4.** If $P$ is nonnegative stable $n \times n$ matrix and $Pz \leq z$, then

(i) $P_{CC}z_C = z_C$,
(ii) $z_D = 0$ and $(Pz)_{C \cup D} = z_{C \cup D}$,
(iii) $z_I \geq 0$,
(iv) if, in addition, $z_S \geq 0$ for some $S \subseteq \{1, \ldots, n\}$ that contains at least one element of each critical class of $P$, then $z \geq 0$.

**Proof.** As $Pz \leq z$, it follows from Proposition 4.3 that $P_{DD}z_D \leq z_D$. Since $\rho(P_{DD}) < 1$, we get that $(P_{DD})^k z_D \to 0$ as $k \to \infty$, and hence $z_D \geq 0$. This implies that

$$z_C \geq P_{CC}z_C + P_{CD}z_D \geq P_{CC}z_C.$$
Let $C_1, \ldots, C_r$ be the critical classes of $P$. As $P_{C_i C_i}$ is nonnegative and irreducible, it follows from the Perron–Frobenius theorem [11] that there exists for each $1 \leq i \leq r$ a positive $m^i \in \mathbb{R}^{C_i}$ such that $m^i P_{C_i C_i} = m^i$. Put $m = (m_1, \ldots, m_r) \in \mathbb{R}^C$ and remark that, as $P_{CC}$ is block diagonal, $m P_{CC} = m$. Multiplication by $m$ from the left in (10) gives
\[ m z_C \geq m P_{CC} z_C + m P_{CD} z_D \geq m P_{CC} z_C = m z_C. \]
As $m$ is positive, we deduce that
\[ P_{CD} z_D = 0 \quad \text{and} \quad P_{CC} z_C = z_C, \tag{11} \]
which proves (i).

Recall that $z_D \geq 0$. To show (ii), we assume by way of contradiction that $z_j > 0$ for some $j \in D$. By definition, there exists a path $(i_1, \ldots, i_q)$ in $\mathcal{G}(P)$ with $i = i_1 \in C_i, i_2, \ldots, i_q \in D$, and $j = i_q$. Since $P z \leq z$ we get that
\[ (P_{CD} z_D)_i = P_{iD} z_D \geq P_{i_1 i_2} z_{i_2} \geq P_{i_1 i_2} P_{i_2 i_3} z_{i_3} \geq \cdots \geq \left( \prod_{k=1}^{q-1} P_{i_k i_{k-1}} \right) z_{i_q} > 0, \]
which contradicts (11) and hence $z_D = 0$. By Proposition 4.3(i), we also find that $(P z)_{C \cup D} = z_{C \cup D}$.

To prove that $z_j \geq 0$, we remark that $z_I \geq P_{ID} z_D + P_{II} z_I = P_{II} z_I$ by (ii). As $\rho(P_{II}) < 1$, we get that $z_I \geq P_{II}^k z_I \to 0$ as $k \to \infty$, so that $z_I \geq 0$.

Finally, assume that $S \subseteq \{1, \ldots, n\}$ contains at least one element in each critical class of $P$ and $z_S \gg 0$. Remark that $z_U \geq P_{UU} z_U + P_{UC} z_C + P_{IZ} z_I \geq P_{UU} z_U + P_{UC} z_C$, since $z_I \geq 0$. As $\rho(P_{UU}) < 1$, we get that $(I - P_{UU})^{-1} = \sum_{k \geq 0} P_{UU}^k$ is nonnegative and
\[ z_U \geq (I - P_{UU})^{-1} P_{UC} z_C. \tag{12} \]
As each $P_{C_i C_i}$ is irreducible and $\rho(P_{C_i C_i}) = 1$, it follows from the Perron–Frobenius theorem that $z_C$ is a multiple of the unique positive eigenvector of $P_{C_i C_i}$. By assumption $z_C$ has at least one positive coordinate, and hence $z_C \geq 0$. It now follows from (12) that $z_U \geq 0$ and therefore $z \geq 0$. \hfill \Box

For $z \in \mathbb{R}^n$ and $S \subseteq \{1, \ldots, n\}$ we simply say that $z = 0$ on $S$, or $z$ is zero on $S$, if $z_S = 0$. Similar terminology will be used for $z_S \geq 0$ and $z_S \gg 0$.

Given a collection $\mathcal{P}$ of nonnegative $n \times n$ matrices, we define
\[ \mathcal{G}(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} \mathcal{G}(P), \tag{13} \]
and we observe that the following lemma holds.

**Lemma 4.5.** If $\mathcal{P}$ is a convex set of nonnegative $n \times n$ matrices, then there exists $M \in \mathcal{P}$ such that $\mathcal{G}(M) = \mathcal{G}(\mathcal{P})$.

**Proof.** Since the number of edges in $\mathcal{G}(\mathcal{P})$ is finite, there exists a finite set $\mathcal{F} \subseteq \mathcal{P}$ such that $\mathcal{G}(\mathcal{F}) = \mathcal{G}(\mathcal{P})$. Define $M = |\mathcal{F}|^{-1} \sum_{Q \in \mathcal{F}} Q$ and note that $M \in \mathcal{P}$, as $\mathcal{P}$ is convex. Moreover, $\mathcal{G}(M) = \mathcal{G}(\mathcal{F}) = \mathcal{G}(\mathcal{P})$. \hfill \Box
For a stable nonnegative matrix $P$, we let $N^c(P)$ denote the set of critical nodes of $G(P)$ and we let $G^c(P)$ denote the restriction of $G(P)$ to $N^c(P)$. For a collection of stable nonnegative $n \times n$ matrices, $\mathcal{P}$, we define

$$N^c(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} N^c(P) \quad \text{and} \quad G^c(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} G^c(P).$$

(14)

Using these concepts we can now present the main theorem of this section.

**Theorem 4.6.** If $\mathcal{P}$ is a convex rectangular set of stable nonnegative $n \times n$ matrices, then there exists $M \in \mathcal{P}$ such that $G^c(M) = G^c(\mathcal{P})$.

**Proof.** The assertion is trivial if $G^c(\mathcal{P})$ is empty. Let $\mathcal{F}$ be a finite set of matrices in $\mathcal{P}$ such that $G^c(\mathcal{F}) = G^c(\mathcal{P})$. For each $k \in N^c(\mathcal{P})$ we let

$$Q_k = \{P_k : P \in \mathcal{F} \text{ and } k \in N^c(P)\}.$$

For $k \notin N^c(\mathcal{P})$ we pick an arbitrary $P \in \mathcal{P}$ and put $Q_k = \{P_k\}$. Subsequently we define an $n \times n$ nonnegative matrix $M$ by

$$M_k = |Q_k|^{-1} \sum_{q \in Q_k} q \quad \text{for } 1 \leq k \leq n.$$

As $\mathcal{P}$ is convex and rectangular, $M \in \mathcal{P}$ and hence $G^c(M) \subseteq G^c(\mathcal{P})$.

We claim that $G^c(\mathcal{P}) \subseteq G(M)$ by construction. Indeed, if $(i, j)$ is an arc in $G^c(\mathcal{P})$, then there exists $P \in \mathcal{F}$ such that $(i, j)$ is an arc in $G^c(P)$. This implies that $i \in N^c(P) \subseteq N^c(\mathcal{P})$ and $P_i \in Q_i$. As $P_{ij} > 0$, we get that

$$M_{ij} = |Q_i|^{-1} \sum_{q \in Q_i} q_j \geq \frac{P_{ij}}{|Q_i|} > 0.$$

Thus, $G^c(M) \subseteq G^c(\mathcal{P}) \subseteq G(M)$, so that

$$G^c(\mathcal{P})|_{N^c(M)} = G^c(M).$$

(Here $G^c(\mathcal{P})|_{N^c(M)}$ denotes the restriction of the graph $G^c(\mathcal{P})$ to the nodes in $N^c(M)$.) As $N^c(M) \subseteq N^c(\mathcal{P})$, it remains to prove that $N^c(\mathcal{P}) \subseteq N^c(M)$ to establish the equality $G^c(M) = G^c(\mathcal{P})$. To show the inclusion we use the following claim.

**Claim.** If $C$ is the set of nodes of a strongly connected component of $G^c(\mathcal{P})$, then $\rho(M_{CC}) = 1$.

If we assume the claim for the moment and take $i \in N^c(\mathcal{P})$, then there exists a strongly connected component $C$ in $G^c(\mathcal{P})$ such that $i \in C$. Clearly there exists a class $C^*$ of $M$ such that $C \subseteq C^*$ and hence $1 = \rho(M_{CC}) \leq \rho(M_{C^*C^*})$ by Proposition 4.2 and by the claim. This implies that $C^*$ is a critical class of $M$ and therefore $i \in N^c(M)$.

To prove the claim we consider a nonlinear map $g : \mathbb{R}^C_+ \to \mathbb{R}^C_+$ given by

$$g_k(y) = \sup_{q \in Q_k} q_c y \quad \text{for } k \in C \text{ and } y \in \mathbb{R}^C_+.$$

We begin by constructing an eigenvector $u \gg 0$ for $g$. As $g$ is monotone, positively homogeneous and continuous, we can use the Brouwer fixed point
theorem to find $u \in \mathbb{R}^C_+$, with $u \neq 0$, and $\lambda \geq 0$ such that $g(u) = \lambda u$ (see [3, pp. 152–154] or [17, p. 201]). Since $F$ is finite, $Q_k$ is finite, and hence the sup is attained for $u$ and $k \in C$, say by $q^k \in Q_k$. Now let $Q$ be the $n \times n$ nonnegative matrix with $Q_k = q^k$ for all $k \in C$ and $Q_k$ is some element in $Q_k$ for all $k \notin C$. As $P$ is rectangular, $Q \in \mathcal{P}$.

Moreover, $Q_{CC}u = g(u) = \lambda u$ and $\rho(Q_{CC}) \leq \rho(Q) \leq 1$, as $Q$ is stable. Thus, we find that $\lambda \leq 1$.

Now note that if $(i_1, i_2)$ is an arrow in $G^c(\mathcal{P})$, then there exists $q \in Q_{i_1}$ with $q_{i_2} > 0$. This implies that if $x \in \mathbb{R}^C_+$ and $x_{i_2} > 0$, then
\begin{equation}
   g_{i_1}(x) \geq q_{i_2} x_{i_2} > 0.
\end{equation}

Since $C$ is a strongly connected component of $G^c(\mathcal{P})$, there exists a path from any $i$ to any $j$ in $C$. Recall that $u \in \mathbb{R}^C_+$ and $u \neq 0$. Hence there exists $i \in C$ such that $u_i > 0$.

Now let $(i_0, i_1, \ldots, i_r)$ be a path in $G^c(\mathcal{P}) \setminus C$ from $i = i_0$ to $j = i_r$. By (15), $g_{i_{r-1}}(u) \geq q_i u_{i_r} > 0$, and $g_{i_{r-2}}(u) \geq q_{i_{r-1}} g_{i_r}(u) \geq q_{i_{r-1}} q_{i_r} u_{i_r} > 0$. By repeating the argument we get that
\begin{equation}
   u_i = g_i^r(u) = g_i^r(u) \geq \left( \prod_{k=1}^r q_{i_k} \right) u_{i_r} > 0.
\end{equation}

Thus, $u_i > 0$ for all $i \in C$.

Let $k \in C$ and $q \in Q_k$. Then there exists $P \in F$ such that $q = P_k$ and $k \in N^c(P)$. Moreover, there exists a critical class $C'$ of $P$ with $k \in C' \subseteq C$ and $\rho(P_{C'C'}) = 1$. We note that
\begin{equation}
   P_{C'C'}u_{C'} \leq P_{C'C}u \leq g(u)_{C'} = \lambda u_{C'} \leq u_{C'},
\end{equation}
as $P_l \in Q_l$ for all $l \in C'$. Since $u$ is positive on $C'$, it follows from Proposition 4.4(i) that $P_{C'C'}u_{C'} = u_{C'}$, so that $\lambda = 1$ and $P_{C'C}u = u_{C'}$ by (16).

Therefore, if $k \in C$ and $q \in Q_k$, then $Q_{CC}u = P_{kCC}u = u_k$, so that $M_{kCC}u = u_k$ for all $k \in C$. From this we conclude that $M_{CC}u = u$ and hence $\rho(M_{CC}) = 1$, which proves the claim.

$\Box$

5. Tangentially stable fixed points

By using the results from the previous section we can now start analysing the t-stable fixed points of monotone convex maps. To begin, we have the following lemma.

**Lemma 5.1.** If $f : \mathcal{D} \to \mathcal{D}$ is a convex monotone map and $v$ and $w$ are t-stable fixed points of $f$, then $G^c(\partial f(v)) = G^c(\partial f(w))$.

**Proof.** From Propositions 2.1, 2.2 and 3.2, it follows that $\partial f(v)$ and $\partial f(w)$ are convex rectangular sets of stable nonnegative $n \times n$ matrices. By Theorem 4.6 there exists $M \in \partial f(v)$ such that $G^c(M) = G^c(\partial f(v))$. Moreover,
\begin{equation}
   w - v = f(w) - f(v) \geq M(w - v),
\end{equation}
so that $w - v = M(w - v)$ on $C \cup D$, by Proposition 4.4. (Here $C$ and $D$ are the critical nodes and the downstream nodes of $M$.) This implies that
\begin{equation}
   f_i(v) - f_i(w) = (v - w)_i = M_i(v - w) \quad \text{for all } i \in C \cup D.
\end{equation}
From this equality we deduce that
\[ f_i(x) - f_i(w) = f_i(x) - f_i(v) + f_i(v) - f_i(w) \]
\[ \geq M_i(x - v) + M_i(v - w) \]
\[ = M_i(x - w) \]
for all \( x \in D \) and \( i \in C \cup D \). Thus, \( M_i \in \partial f_i(w) \) for all \( i \in C \cup D \). Now let \( P \in \partial f(w) \) and define \( Q \in P_{n,n} \) by
\[ Q_i = \begin{cases} 
M_i & \text{if } i \in C \cup D, \\
P_i & \text{otherwise.} 
\end{cases} \]
As \( \partial f(w) \) is rectangular, \( Q \in \partial f(w) \). Clearly \( G^c(f(w)) \supseteq G^c(Q) \supseteq G^c(M) = G^c(\partial f(v)) \). By interchanging the roles of \( v \) and \( w \), we obtain the desired equality. \( \Box \)

By Lemma 5.1 we can define for a convex monotone map \( f : D \to D \) with a t-stable fixed point \( v \in D \), the set of critical nodes of \( f \) and the critical graph of \( f \), respectively, by

\[ N^c(f) = N^c(\partial f(v)) \quad \text{and} \quad G^c(f) = G^c(\partial f(v)). \]

**Lemma 5.2.** Let \( f : D \to D \) be a convex monotone map and let \( v \in D \) be a t-stable fixed point of \( f \). Let \( S \subseteq \{1, \ldots, n\} \) be a set that contains at least one node in each connected component of \( G^c(\partial f(v)) \). If \( w \in D \) is a fixed point of \( f \) and \( v \leq w \) on \( S \), then \( v \leq w \).

**Proof.** By Theorem 4.6 there exists \( M \in \partial f(v) \) such that \( M = G^c(\partial f(v)) \). Then \( w - v = f(w) - f(v) \geq M(w - v) \) and \( w - v \geq 0 \) on \( S \), so that \( w - v \geq 0 \) by Proposition 4.4(iv).

\( \Box \)

From the previous lemma we immediately deduce the following theorem for t-stable fixed points.

**Theorem 5.3.** Let \( f : D \to D \) be a convex monotone map and let \( v, w \in D \) be t-stable fixed points of \( f \). If \( v = w \) on a set \( S \subseteq \{1, \ldots, n\} \) that has at least one node in each strongly connected component of \( G^c(f) \), then \( v = w \). In particular, the t-stable fixed point is unique if \( N^c(f) \) is empty.

To analyse the geometry of the t-stable fixed point set \( E_t(f) \) and t-stable periodic points, we need some preliminary results concerning convex monotone positively homogeneous maps.

### 6. Positively homogeneous maps

If \( h : \mathbb{R}^n \to \mathbb{R}^n \) is a monotone convex positively homogeneous map, then 0 is a fixed point and we can associate with \( h \) a graph \( G(h) \) by

\[ G(h) = G(\partial h(0)). \]

If, in addition, 0 is t-stable, then we define
\[ A(h) = \{ i : \text{there exists a path in } G(h) \text{ from } i \text{ to some } j \in N^c(h) \} \]
and we put $B(h) = \{1, \ldots, n\} \setminus A(h)$. Convex monotone positively homogeneous maps, which have 0 as a t-stable fixed point, have the following properties.

**Lemma 6.1.** Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a convex monotone positively homogeneous map, with 0 as a t-stable fixed point. Write $C = N^c(h)$, $A = A(h)$ and $B = B(h)$, and identify each $x \in \mathbb{R}^n$ with $(x_A, x_B) \in \mathbb{R}^A \times \mathbb{R}^B$. Then $C \subseteq A$ and the map $h$ can be rewritten in the form

$$h(x_A, x_B) = (h_A(x_A, x_B), h_B(x_B)),$$

where $h_A : \mathbb{R}^A \times \mathbb{R}^B \to \mathbb{R}^A$ and $h_B : \mathbb{R}^B \to \mathbb{R}^B$ are convex monotone positively homogeneous maps. Moreover, if $h_A : \mathbb{R}^A \to \mathbb{R}^A$ is given by

$$h_A(y) = h_A(y, 0) \quad \text{for all } y \in \mathbb{R}^A,$$

then for each $y \in \mathbb{R}^A_+$ with $y_C \gg 0$ and each $i \in A$, there exists $k \geq 1$ such that $(h_A^k)_i(y) > 0$.

**Proof.** Since $h$ is convex and positively homogeneous, $h'_0 = h$ and $h(x) = \sup_{P \in \partial h(0)} P x$ for all $x \in \mathbb{R}^n$. Since $B = \{1, \ldots, n\} \setminus A$, we can write $h$ in the form

$$h(x_A, x_B) = (h_A(x_A, x_B), h_B(x_A, x_B)).$$

We note that $P_{BA} = 0$ for all $P \in \partial h(0)$. Indeed, otherwise there exists $j \in B$ that has access to some node $i \in A$ in $G(P)$. But this implies that there exists a path from $j$ to a node in $C$ in $G(h)$, as $G(P) \subseteq G(h)$, which contradicts $j \in B$. Since $\partial h(0)$ is rectangular,

$$h_B(x_A, x_B) = \sup_{P \in \partial h(0)} P_{BA} x_A + P_{BB} x_B = \sup_{P \in \partial h(0)} P_{BB} x_B.$$

Thus, $h_B(x_A, x_B)$ is of the form $h_B(x_B)$, and therefore $h$ can be rewritten as

$$h(x_A, x_B) = (h_A(x_A, x_B), h_B(x_B))$$

for all $(x_A, x_B) \in \mathbb{R}^A \times \mathbb{R}^B$.

To prove the last assertion, we let $y \in \mathbb{R}^A_+$ be such that $y_C \gg 0$ and $i \in A$. By Lemma 4.5 there exists $P \in \partial h(0)$ such that $G(P) = G(h)$. We have that $h_A(z) = h_A(z, 0) \geq P_{AA} z$ for all $z \in \mathbb{R}^A$. Hence $(h^A)_k(y) \geq (P_{AA})^k y$ for all $k \geq 1$. Since there exists a path from $i \in A$ to some node $j \in C$, say with length $m \geq 1$, we get that $(h^A)_k(y)_i \geq (P_{AA})^m_{ij} y_j > 0$. \hfill \Box

By using the previous lemma we prove the following proposition.

**Proposition 6.2.** If $h : \mathbb{R}^n \to \mathbb{R}^n$ is a convex monotone positively homogeneous map, with 0 as a t-stable fixed point, then $h$ has a fixed point $v$ such that $v \gg 0$ on $A(h)$ and $v = 0$ on $B(h)$.

**Proof.** By Theorem 4.6 there exists $M \in \partial h(0)$ such that $G^c(M) = G^c(h)$. Let $C_1, \ldots, C_r$ denote the critical classes of $M$, so $C = C_1 \cup \cdots \cup C_r$. By the Perron–Frobenius theorem [11] there exists for each $1 \leq i \leq r$ a positive eigenvector $u^i \in \mathbb{R}^{C_i}$ such that $M_{C_i, C_i} u^i = u^i$. Define $u \in \mathbb{R}^n$ by $u_{C_i} = u^i$ for
1 ≤ i ≤ r and u_j = 0 if j /∈ C. Clearly u ≥ 0, u ≫ 0 on C, and Mu ≥ u. This implies that

\[ h(u) ≥ Mu ≥ u ≥ 0. \] (17)

As 0 is t-stable and h = h'_0, we know that \( O(u; h) \) is bounded. Moreover, it follows from (17) that \( (h^k(u))_k \) is increasing and hence \( v = \lim_{k \to \infty} h^k(u) \) exists. Obviously \( v \) is a fixed point of \( h \) and \( v ≫ u \), so that \( v \gg 0 \) on C. Remark that \( u_B = 0 \), because \( B \cap C = \emptyset \). Therefore, \( v_B = 0 \) by Lemma 6.1 and hence \( v_A \) is a fixed point of \( h^A \). By the second part of Lemma 6.1 we obtain that \( v_A \gg 0 \), as \( v_C \gg 0 \).

For a monotone positively homogeneous map \( h: \mathbb{R}^n \to \mathbb{R}^n \), the spectral radius is defined by

\[ \tau(h) = \sup \{ \lambda ≥ 0: h(x) = \lambda x \text{ for some } x \in \mathbb{R}^n_+ \setminus \{0\} \}. \] (18)

**Proposition 6.3.** Let \( h: \mathbb{R}^n \to \mathbb{R}^n \) be a convex monotone positively homogeneous map with 0 as a t-stable fixed point. If \( h_B: \mathbb{R}^B \to \mathbb{R}^B \) is as in Lemma 6.1, then \( \tau(h_B) < 1 \).

**Proof.** Assume by way of contradiction that \( \tau(h_B) = r ≥ 1 \). Then there exists \( v ≥ 0 \) with \( v \neq 0 \) such that \( h_B(v) = rv ≥ v \). But \( h_B(v) = \sup_{P \in \partial h(0)} P_{BB}v \) and \( \partial h(0) \) is a rectangular compact set of stable nonnegative matrices. Hence \( h_B(v) = Q_{BB}v = rv \) for some \( Q \in \partial h(0) \). This implies that there exists a class \( K \) of \( Q \) such that \( K ⊆ B \) and \( \rho(Q_{KK}) = r ≥ 1 \). As \( Q \) is stable, \( r = 1 \), and hence \( K ⊆ N^c(Q) ⊆ N^c(h) ⊆ A \), which is a contradiction. \( □ \)

It was shown by Nussbaum [21, Theorem 3.1] that \( \tau(h) = \tau'(h) \), where \( \tau'(h) \) is the so-called Collatz–Wielandt spectral radius of a monotone positively homogeneous map \( h: \mathbb{R}^n \to \mathbb{R}^n \), which is given by

\[ \tau'(h) = \inf \{ \mu ≥ 0: h(x) ≤ \mu x \text{ for some } x ≫ 0 \}. \] (19)

Thus, Proposition 6.3 has the following consequence.

**Corollary 6.4.** If \( h: \mathbb{R}^n \to \mathbb{R}^n \) is a convex monotone positively homogeneous map, with 0 as a t-stable fixed point, and \( h_B \) is as in Lemma 6.1, then there exist \( 0 < \lambda < 1 \) and \( w ≫ 0 \) such that \( h_B(w) ≤ \lambda w \).

We conclude this section by showing that a monotone convex positively homogeneous map with 0 as a t-stable fixed point is nonexpansive with respect to a polyhedral norm. Recall that a norm on \( \mathbb{R}^n \) is called polyhedral if its unit ball is a polyhedron.

**Theorem 6.5.** Let \( h: \mathbb{R}^n \to \mathbb{R}^n \) be a monotone convex positively homogeneous map with 0 as a t-stable fixed point, then there exist \( v ≫ 0 \) and \( \alpha > 0 \) such that \( h \) is nonexpansive with respect to the polyhedral norm

\[ \|x\|_v = \max_{i \in A(h)} |x_i/v_i| + \alpha \max_{i \in B(h)} |x_i/v_i| \quad \text{for } x \in \mathbb{R}^n, \] (20)

where \( A(h) \) and \( B(h) \) are as in Lemma 6.1.
Proof. We use the same notation as in Lemma 6.1. By Proposition 6.2, $h$ has an eigenvector $u \in \mathbb{R}^n$ such that $u \gg 0$ on $A = A(h)$ and $u = 0$ on $B = B(h)$. Moreover, by Corollary 6.4 there also exist $0 < \lambda < 1$ and $w \in \mathbb{R}^B$ such that $w \gg 0$ and $h_B(w) \leq \lambda w$. Let $v \in \mathbb{R}^A \times \mathbb{R}^B$ be defined by $v = u$ on $A$ and $v = w$ on $B$. Further, let $W$ be the diagonal matrix with $v$ as its diagonal and define $g: \mathbb{R}^n \to \mathbb{R}^n$ by $g(x) = (W^{-1} \circ h \circ W)(x)$ for all $x \in \mathbb{R}^n$. It follows from Lemma 6.1 that we can write $g$ in the form

$$g(x) = (g_A(x_A, x_B), g_B(x_B)),$$

where $g_A: \mathbb{R}^A \times \mathbb{R}^B \to \mathbb{R}^B$ and $g_B: \mathbb{R}^B \to \mathbb{R}^B$ are convex monotone positive homogeneous maps. Moreover, $g_A(\mu 1, 0) = \mu 1$ and $g_B(\mu 1) \leq \mu 1$ for all $\mu \geq 0$, where $1$ denotes the vector with all coordinates unity.

We write $\| \cdot \|_\infty$ to denote the sup-norm, so $\|x\|_\infty = \max_i |x_i|$. Remark that

$$g(x)_B - g(y)_B \leq g(x - y)_B = g_B((x - y)_B) \leq \lambda \| (x - y)_B \|_\infty 1,$$

as $g$ is convex, monotone and positively homogeneous. By interchanging the roles of $x$ and $y$, we deduce that

$$\|g(x)_B - g(y)_B\|_\infty \leq \lambda \|x_B - y_B\|_\infty \quad \text{for all } x, y \in \mathbb{R}^n. \quad (21)$$

Subsequently, we remark that there exists a constant $C > 0$ such that $g_A(0, x_B) \leq C \|x_B\|_\infty 1$, as $g_A$ is continuously and positively homogeneous. (Recall that $g(x) = \sup_{P \in \partial g(0)} Px$ for each $x \in \mathbb{R}^n$.) Thus, for each $x \in \mathbb{R}^n$,

$$g(x)_A = g_A(x_A, x_B) \leq g_A(x_A, 0) + g_B(0, x_B) \leq \|x_A\|_\infty 1 + C \|x_B\|_\infty 1. \quad (22)$$

As $g$ is convex,

$$g(x) \leq g(x - y) + g(y), \quad (23)$$

so that $g(x) - g(y) \leq g(x - y)$ for all $x, y \in \mathbb{R}^n$. As $g(0) = 0$, we have that $-g(y) \leq g(-y)$, and hence it follows from (22) that

$$\|g(x)_A\|_\infty \leq \|x_A\|_\infty + C \|x_B\|_\infty \quad \text{for all } x \in \mathbb{R}^n. \quad (24)$$

It follows from (21), (24) and (25) that

Now let $\alpha > C/(1 - \lambda)$ and define $\| \cdot \|'$ on $\mathbb{R}^n$ by

$$\|x\|' = \|x_A\|_\infty + \alpha \|x_B\|_\infty \quad \text{for all } x \in \mathbb{R}^A \times \mathbb{R}^B.$$ 

It now follows from (21), (24) and (25) that

$$\|g(x) - g(y)\|' = \|g(x)_A - g(y)_A\|_\infty + \alpha \|g(x)_B - g(y)_B\|_\infty$$

$$\leq \|(x - y)_A\|_\infty + C \|(x - y)_B\|_\infty + \alpha \lambda \|(x - y)_B\|_\infty$$

$$\leq \|(x - y)_A\|_\infty + \alpha \|(x - y)_B\|_\infty$$

$$= \|x - y\|'.$$

Finally, we recall that $g \circ W^{-1} = W^{-1} \circ g$, so that $h$ is nonexpansive with respect to $\|W^{-1}(\cdot)\|' = \| \cdot \|_v$ and we are done. \qed

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7. The tangentially stable fixed point set

Throughout this section we assume, in addition to Hypothesis 2.3, that the domain $\mathcal{D} \subseteq \mathbb{R}^n$ satisfies the following property.

**Hypothesis 7.1.** The domain $\mathcal{D} \subseteq \mathbb{R}^n$ is a downward set, i.e., if $x \in \mathcal{D}$ and $y \leq x$, then $y \in \mathcal{D}$.

Given a convex monotone map $f: \mathcal{D} \to \mathcal{D}$, we define $\mathcal{E}^+(f) = \{z \in \mathcal{D}: f(z) \leq z\}$. As $f$ is convex, $\mathcal{E}^+(f)$ is convex. There exists a natural projection from $\mathcal{E}^+(f)$ onto $\mathcal{E}(f)$ if $f$ has a t-stable fixed point (cf. [1, Lemma 3.3]).

**Lemma 7.2.** If $f: \mathcal{D} \to \mathcal{D}$ is a convex monotone map with a t-stable fixed point, then

$$f^\omega(z) = \lim_{k \to \infty} f^k(z)$$

exists and $f^\omega(z) = z$ on $N_c(f)$ for each $z \in \mathcal{E}^+(f)$. In addition, the map $f^\omega: \mathcal{E}^+(f) \to \mathcal{E}(f)$ is a surjective convex monotone projection, i.e., $(f^\omega)^2 = f^\omega$.

**Proof.** Since $f: \mathcal{D} \to \mathcal{D}$ has a t-stable fixed point, all orbits of $f$ are bounded from below by Proposition 3.2. Therefore, $f^\omega(z) = \lim_{k \to \infty} f^k(z)$ exists for all $z \in \mathcal{E}^+(f)$, as $(f^k(z))_k$ is a nonincreasing sequence and $\mathcal{D}$ is downward. By continuity of $f$, $f^\omega(z)$ is a fixed point of $f$.

Let $v$ be a t-stable fixed point of $f$ and $z \in \mathcal{E}^+(f)$. By Theorem 4.6, there exists $Q \in \partial f(v)$ such that $\mathcal{G}^c(Q) = \mathcal{G}^c(\partial f(v)) = \mathcal{G}^c(f)$. We also have that

$$z - v \geq f(z) - v = f(z) - f(v) \geq Q(z - v).$$

From Proposition 4.4 it follows that $z - v = Q(z - v)$ on $N_c(f) = N_c(Q)$ and hence $z = f(z)$ on $N_c(f)$. Replacing $z$ by $f^k(z)$ in the previous argument gives $f^{k+1}(z) = f^k(z) = \cdots = z$ on $N_c(f)$ for all $k \geq 1$. Thus, $f^\omega(z) = \lim_{k \to \infty} f^k(z) = z$ on $N_c(f)$. Clearly, $f^\omega(x) = x$ if $x \in \mathcal{E}(f)$, so that $f^\omega: \mathcal{E}^+(f) \to \mathcal{E}(f)$ is onto and $(f^\omega)^2 = f^\omega$. Moreover, as $f^\omega$ is the pointwise limit of $(f^k)_k$, $f^\omega$ is a convex monotone map. \hfill \Box

The fixed point set $\mathcal{E}(f)$ can be naturally equipped with a binary operation $\wedge_f$ that turns $(\mathcal{E}(f), \wedge_f)$ into an inf-semilattice, if $f: \mathcal{D} \to \mathcal{D}$ has a t-stable fixed point. The relation $\wedge_f$ on $\mathcal{E}(f)$ is defined by

$$x \wedge_f y = \lim_{k \to \infty} f^k(x \wedge y).$$

We note that if $x, y \in \mathcal{E}(f)$, then $f(x \wedge y) \leq f(x) = x$ and $f(x \wedge y) \leq f(y) = y$, so that $f(x \wedge y) \leq x \wedge y$. As $f: \mathcal{D} \to \mathcal{D}$ has all its orbits bounded from below and $\mathcal{D}$ is downward, the limit (28) exists. To prove that $(\mathcal{E}(f), \wedge_f)$, is an inf-semilattice one has to show that $\wedge_f$ is associative, symmetric and reflexive, which is a simple exercise. It also follows from Lemma 7.2 that if we put $C = N_c(f)$ and define $r_C: \mathcal{E}(f) \to \mathbb{R}^C$ by $r_C(x) = x_C$ for all $x \in \mathcal{E}(f)$, then $r_C(\mathcal{E}(f))$ is an inf-semilattice in $\mathbb{R}^C$, where $\wedge$ is the infimum operation induced by the partial ordering $\leq$ on $\mathbb{R}^C$. Indeed, if $x, y \in \mathcal{E}(f)$ and $v \in \mathcal{D}$ is a t-stable fixed point of $f: \mathcal{D} \to \mathcal{D}$, then there exists $M \in \partial f(v)$ such
that $\mathcal{G}^c(M) = \mathcal{G}^c(f)$. As $f(x \wedge y) \leq f(x) = x$ and $f(x \wedge y) \leq f(y) = y$, $f(x \wedge y) \leq x \wedge y$, so that $f^k(x \wedge y) \leq f^{k-1}(x \wedge y)$ for all $k \geq 1$. This implies that

$$f^{k-1}(x \wedge y) - v \geq f^k(x \wedge y) - f(v) \geq M(f^{k-1}(x \wedge y) - v).$$

By Proposition 4.4, we get that

$$f^{k-1}(x \wedge y) - v = f^k(x \wedge y) - f(v) = f^k(x \wedge y) - v$$
on $C$ for all $k \geq 1$. Hence

$$r_C(x \wedge y) = (x \wedge y)_C = \lim_{k \to \infty} f^k(x \wedge y)_C = (x \wedge y)_C = r_C(x) \wedge r_C(y),$$

so $(r_C(\mathcal{E}(f)), \wedge)$ is an inf-semilattice in $\mathbb{R}^C$. The difference between $(\mathcal{E}(f), \wedge_f)$ and $(r_C(\mathcal{E}(f)), \wedge)$ is illustrated by the following simple example. Consider

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So, $P$ is a projection. Clearly, $\mathcal{E}(P) = \text{span}\{(1/2, 1, 0), (1/2, 0, 1)\}$, but $\mathcal{E}(P)$ is not an inf-semilattice with respect to $\wedge$, as $(1/2, 0, 0) = (1/2, 1, 0) \wedge (1/2, 0, 1)$ is for instance not in $\mathcal{E}(P)$. In this case $N^c(P) = \{2, 3\}$ and $\text{span}\{(1, 0), (0, 1)\}$ is an inf-semilattice with respect to $\wedge$.

Let us now analyse the t-stable fixed point set in more detail. We shall prove the following theorem.

**Theorem 7.3.** If $f : D \to D$ is a convex monotone map with a t-stable fixed point, then $(\mathcal{E}_t(f), \wedge_f)$ is an inf-semilattice and $(r_C(\mathcal{E}_t(f)), \wedge)$ is a convex inf-semilattice in $\mathbb{R}^C$, where $C = N^c(f)$.

But first we give two preliminary lemmas.

**Lemma 7.4.** Let $f : D \to D$ be a convex monotone map with a t-stable fixed point. Let $z \in \mathcal{E}^+(f)$ and $w = f^w(z)$ be given by (26). Write $S = \{i : w_i < z_i\}$ and $E = \{i : w_i = z_i\}$. Then the map $h = f_w^*$ can be written in the form

$$h(x_S, x_E) = (h_S(x_S, x_E), h_E(x_E)).$$

(29)

where $h_S : \mathbb{R}^S \times \mathbb{R}^E \to \mathbb{R}^S$ and $h_E : \mathbb{R}^E \to \mathbb{R}^E$ are convex positively homogeneous maps. Moreover, the map $h_S : \mathbb{R}^S \to \mathbb{R}^S$, given by $h_S(y) = h_S(y, 0)$ for $y \in \mathbb{R}^S$, satisfies $\tau(h_S) < 1$.

**Proof.** Since $h(x) = \sup_{P \in \partial f^w} Px$ for all $x \in \mathbb{R}^n$ and $f(z) \leq z$, we get that

$$z - w \geq f(z) - w \geq h(z - w).$$

(30)

As $z - w \geq 0$ and $z = w$ on $E$, we find that $h(z - w) = 0$ on $E$. Now suppose that $y \in \mathbb{R}^n$ is such that $y_E = 0$. Then there exists $\lambda > 0$ such that $y \leq \lambda(z - w)$, as $z - w \gg 0$ on $S$. This implies that $h(y) \leq \lambda h(z - w) \leq \lambda(z - w)$,
and hence \( h(y) \leq 0 \) on \( E \) if \( y = 0 \) on \( E \). Now let \( y_S, y'_S \in \mathbb{R}^S \) and \( y_E \in \mathbb{R}^E \), and remark that

\[
h(y_S, y_E)_E \leq h(y_S - y'_S, 0)_E + h(y'_S, y_E)_E \leq h(y'_S, y_E)_E,
\]
as \( h \) is positively homogeneous and convex. Thus, \( h(y_S, y_E)_E \) is independent of \( y_S \) and hence \( h \) can be written in the form (29).

Let \( h^S: \mathbb{R}^S \to \mathbb{R}^S \) be given by \( h^S(y) = h_S(y, 0) \) for all \( y \in \mathbb{R}^S \) and denote \( v = (z - w)_S \). Then \( v \gg 0 \) and by (30),

\[
v \geq h(z - w)_S \geq h^S(v).
\]

This implies that \( \tau(h^S) \leq 1 \), as \( \tau(h^S) = \tau'(h^S) \) by [21, Theorem 3.1]. (Here \( \tau(\cdot) \) and \( \tau'(\cdot) \) are as in (18) and (19), respectively.)

Assume by way of contradiction that \( \tau(h^S) = 1 \). Then there exists \( u \in \mathbb{R}_+^S \) such that \( u \neq 0 \) and \( h^S(u) = u \). Then \( \eta = (u, 0) \in \mathbb{R}^S \times \mathbb{R}^E \) satisfies \( h(\eta) = \eta \) by (29). Since \( \partial f(w) \) is a compact rectangular set of nonnegative matrices and \( h(\eta) = \sup_{P \in \partial f(w)} P \eta \), there exists \( P \in \partial f(w) \) such that \( \eta = h(\eta) = P \eta \). Let \( S' = \{ i : \eta_i \neq 0 \} \) and remark that \( S' \subseteq S \) and \( S' \) is a union of classes of \( P \). To proceed, we need the notion of a final class. A class of a nonnegative matrix is called final if it has no access to any other class. It is known (see [5, Theorem 3.10]) that a nonnegative matrix \( M \) has a positive eigenvector if and only if each final class of \( M \) is basic. Clearly \( P_{S'F} \) has a final class, say \( F \). As \( P_{S'F} \eta_{S'} = \eta_{S'} \) and \( \eta_{S'} \gg 0 \), we find that \( F \) is a basic class of \( P_{S'F} \), and hence \( \rho(P_{FF}) = \rho(P_{S'F}) = 1 \). By (30) we have that

\[
(z - w)_F \geq (f(z - w)_F \geq h(z - w)_F \geq (P(z - w))_F \geq P_{FF}(z - w)_F. \tag{31}
\]

By the Perron–Frobenius theorem, there exists \( m \gg 0 \) in \( \mathbb{R}^F \) such that \( mP_{FF} = m \). This implies that \( m(z - w)_F \geq mP_{FF}(z - w)_F = m(z - w)_F \), and hence \( (z - w)_F = P_{FF}(z - w)_F \). Thus, \( f(z)_F = z_F \) by (31). Similarly, we deduce that \( f^k(z)_F = z_F \) for all \( k \geq 1 \). Indeed,

\[
z - w \geq f^k(z) - w \geq (f^k)'_w(z - w) = (f'_w)^k(z - w) = h^k(z - w)
\]

and

\[
h^k(z - w)_F \geq (P^k)_F(z - w)_F \geq (P_{FF})_F(z - w)_F.
\]

Recall that \( w_F = \lim_{k \to \infty} f^k(z)_F \) and therefore \( w_F = z_F \). But this implies that \( F \subseteq E \), which contradicts the fact that \( F \subseteq S' \subseteq S \). Thus, we conclude that \( \tau(h^S) < 1 \).

\[\square\]

**Lemma 7.5.** Let \( h, h_E, h^S, S \) and \( E \) be as in Lemma 7.4. If \( h_E: \mathbb{R}^E \to \mathbb{R}^E \) has all its orbits bounded from above, then \( h \) has all its orbits bounded from above.

**Proof.** Since \( h: \mathbb{R}^n \to \mathbb{R}^n \) is monotone, it suffices to prove that \( O(x; h) \) is bounded from above for all \( x \in \mathbb{R}^n_+ \). As \( h \) can be written in the form (29), we know that \( \{ h^k(x)_E : k \geq 0 \} = \{ h^k_E(x_E) : k \geq 0 \} \) is bounded from above. It therefore remains to be shown that \( \{ h^k(x)_S : k \geq 0 \} \) is bounded from above.


Since \( \tau'(h^S) = \tau(h^S) < 1 \), there exist \( u \in \mathbb{R}^+_+ \) and \( 0 < \alpha < 1 \) such that \( u \gg 0 \) and \( h^S(u) \leq \alpha u \). For \( y \in \mathbb{R}^S \), we define a norm by

\[
\|y\|_u = \max_{i \in S} |y_i/u_i|.
\]

For each \( y \in \mathbb{R}^S_+ \) we have that \( y \leq \|y\|_u u \), so that

\[
\|h^S(y)\|_u \leq \|h^S(\|y\|_u u)\|_u = \|h^S(u)\|_u \|y\|_u \leq \alpha \|y\|_u,
\]
as \( h^S \) is positively homogeneous and monotone. Since \( \{h^k(x)_E : k \geq 0\} \) is bounded from above, there exists \( v \gg 0 \) in \( \mathbb{R}^E \) such that \( h^k(x)_E \leq v \) for all \( k \geq 0 \). This implies that

\[
h(0, h^k(x)_E)_S \leq h(0, v)_S \leq \gamma u
\]
for some \( \gamma > 0 \). Now using the fact that \( h \) is positively homogeneous and convex, we get that

\[
0 \leq h^{k+1}(x)_S \leq h(h^k(x)_S, 0)_S + h(0, h^k(x)_E)_S \leq h^S(h^k(x)_S) + \gamma u,
\]
so that

\[
\|h^{k+1}(x)_S\|_u \leq \|h^S(h^k(x)_S)\|_u + \gamma \leq \alpha \|h^k(x)_S\|_u + \gamma.
\]

By induction we obtain

\[
\|h^{k+1}(x)_S\|_u \leq \alpha^k \|x_S\|_u + \frac{\gamma}{1 - \alpha},
\]
which shows that \( \{h^k(x)_S : k \geq 0\} \) is bounded from above. \( \square \)

Let us now prove Theorem 7.3.

**Proof of Theorem 7.3.** To prove that \( (\mathcal{E}(f), \wedge_f) \) is an inf-semilattice, it suffices to show that if \( x, y \in \mathcal{E}(f) \), then \( x \wedge_f y \in \mathcal{E}(f) \), as \( (\mathcal{E}(f), \wedge_f) \) is an inf-semilattice. So, suppose that \( x, y \in \mathcal{E}_i(f) \). Put \( z = x \wedge y \) and let \( w = x \wedge_f y \). We need to show that \( h = f'_w : \mathbb{R}^n \to \mathbb{R}^n \) has all its orbits bounded from above by Proposition 3.2. By Lemma 7.4, we can write \( h \) in the form of (29), since \( f(z) \leq z \). We also get that \( \tau(h^S) < 1 \). By Lemma 7.5, it is sufficient to prove that \( h_E : \mathbb{R}^E \to \mathbb{R}^E \) has all its orbit bounded from above. For each \( i \in E \) we have that

\[
f_i(w) = w_i = x_i \wedge y_i = f_i(x) \wedge f_i(y),
\]
as \( x, y, w \in \mathcal{E}(f) \). Note that \( w \leq x \wedge y \) implies that

\[
w + \varepsilon u \leq (x \wedge y) + \varepsilon u = (x + \varepsilon u) \wedge (y + \varepsilon u)
\]
for all \( u \in \mathbb{R}^n \) and \( \varepsilon > 0 \). Now let \( u \in \mathbb{R}^n \) and \( \varepsilon > 0 \) sufficiently small so that \( x + \varepsilon u, y + \varepsilon u \in D \). Then

\[
f_i(w + \varepsilon u) \leq f_i(x + \varepsilon u) \wedge f_i(y + \varepsilon u),
\]
since \( f_i \) is monotone, and hence

\[
f_i(w + \varepsilon u) - f_i(w) \leq f_i(x + \varepsilon u) \wedge f_i(y + \varepsilon u) - f_i(x) \wedge f_i(y)
\]
\[
\leq \max \{f_i(x + \varepsilon u) - f_i(x), f_i(y + \varepsilon u) - f_i(y)\}
\]
for all $i \in E$. This implies that
\[ f'_w(u) = \lim_{\varepsilon \downarrow 0} \frac{f_i(w + \varepsilon u) - f_i(w)}{\varepsilon} \leq \max\{f'_x(u)_i, f'_y(u)_i\} \tag{32} \]
for all $i \in E$.

Applying the same argument for the map $f^k$ and using the fact that $(f^k)'_v = (f'_v)^k$ for all $v \in \mathcal{E}(f)$, we obtain that
\[ (f'_w)^k(u)_i \leq \max\{(f'_x)^k(u)_i, (f'_y)^k(u)_i\} \]
for all $u \in \mathbb{R}^n$, $i \in E$ and $k \geq 1$. But $x$ and $y$ are $t$-stable and therefore
\[ (h_E)^k(s) = (h^k(0, s))_E = (f'_w)^k(0, s)_E \]
is bounded from above as $k \to \infty$ for all $s \in \mathbb{R}^E$. Thus, we conclude that $w$ is a $t$-stable fixed point of $f$.

Recall that $(r_C(\mathcal{E}(f)), \wedge)$ is an inf-semilattice in $\mathbb{R}^C$, where $C = N^c(f)$, and $r_C(x \wedge y) = r_C(x) \wedge r_C(y)$ for all $x, y \in \mathcal{E}(f)$. This implies that $(r_C(\mathcal{E}_t(f)), \wedge)$ is an inf-semilattice in $\mathbb{R}^C$. To show that $r_C(\mathcal{E}_t(f))$ is convex we apply the same technique as before.

Let $x, y \in \mathcal{E}_t(f)$ and $0 < \lambda < 1$. Put $z = \lambda x + (1-\lambda)y$. Since $f$ is convex,
\[ f(z) \leq \lambda f(x) + (1-\lambda)f(y) = \lambda x + (1-\lambda)y = z. \]

Hence $w = f^\omega(z) = \lim_{k \to \infty} f^k(z)$ exists and is a fixed point of $f$ with $w_C = z_C = \lambda x_C + (1-\lambda)y_C$. We need to show that $w$ is $t$-stable. Let $h = f'_w$ and recall that it suffices to show that all the orbits of $h_E$ are bounded from above by Lemma 7.5. Let $i \in E$ and note that as $x, y, w \in \mathcal{E}(f)$,
\[ f^k_i(w) = w_i = \lambda x_i + (1-\lambda)y_i = \lambda f^k_i(x) + (1-\lambda)f^k_i(y). \]

Clearly $w \leq \lambda x + (1-\lambda)y$, so that
\[ w + \varepsilon u \leq \lambda(x + \varepsilon u) + (1-\lambda)(y + \varepsilon u) \]
for all $u \in \mathbb{R}^n$ and $\varepsilon > 0$. Now fix $u \in \mathbb{R}^n$. Then for all $\varepsilon > 0$ sufficiently small, we have that
\[ f^k_i(w + \varepsilon u) \leq f^k_i(\lambda(x + \varepsilon u) + (1-\lambda)(y + \varepsilon u)) \leq \lambda f^k_i(x + \varepsilon u) + (1-\lambda)f^k_i(y + \varepsilon u) \]
for all $k \geq 1$. Thus,
\[ f^k_i(w + \varepsilon u) - f^k_i(w) \leq \lambda(f^k_i(x + \varepsilon u) - f^k_i(x)) + (1-\lambda)(f^k_i(y + \varepsilon u) - f^k_i(y)), \]
and hence
\[ (f^k)'_w(u)_i \leq \lambda(f^k)'_x(u)_i + (1-\lambda)(f^k)'_y(u)_i \]
for all $k \geq 1$ and $i \in E$. As $x, y \in \mathcal{E}_t(f)$, the right-hand side is bounded from above as $k \to \infty$. Therefore, $((f'_w)^k(0, s)_E)_k = ((h_E)^k(s))_k$ is bounded from above for all $s \in \mathbb{R}^E$, which shows that $w$ is $t$-stable. \qed
We note that if $\mathcal{E}_t(f)$ is compact, then it is a connected set. To show that, it suffices to prove that $r_{C}^{-1}$ is continuous, as $r_{C}(\mathcal{E}_t(f))$ is convex. Note that $r_{C}$ is one-to-one on $\mathcal{E}_t(f)$ by Theorem 5.3. So, let $y_k \to y$ in $r_{C}(\mathcal{E}_t(f))$, $r_{C}(x_k) = y_k$ for all $k$, and $r_{C}(x) = y$. If $(x_{k_i})_i$ is a subsequence of $(x_k)_k$ and $x_{k_i} \to z$, then $z \in \mathcal{E}_t(f)$ by compactness. Moreover, $r_{C}(z) = y$, which implies that $z_{C} = x_{C}$. Thus, by Theorem 5.3, $z = x$, and hence $r_{C}^{-1}$ is continuous.

Another consequence of Theorem 7.3 is the following. Recall that $O(\xi) = \{\xi, f(\xi), \ldots, f^{p-1}(\xi)\} \subseteq D$ is a Lyapunov stable periodic orbit of $f$ if for all neighbourhoods $U_i$ of $f^i(\xi)$, $i = 0, \ldots, p-1$, there exist neighbourhoods $V_i$ of $f^i(\xi)$, $i = 0, \ldots, p-1$, such that $f^{kp+i}(y) \in U_i$ for all $y \in V_i$ and for all $k \geq 0$.

**Corollary 7.6.** If $f : D \to D$ is a convex monotone map with a t-stable periodic point $\xi \in D$, then $f$ has a t-stable fixed point. Moreover, if $\xi$ has a Lyapunov stable orbit, then $f$ has a Lyapunov stable fixed point.

**Proof.** Let $\xi \in D$ be a t-stable periodic point of $f$ with period $p$. Put $g = f^p$ and note that $f^k(\xi)$ is a t-stable fixed point of $g$ for all $0 \leq k < p$. Let $z = \xi \wedge f(\xi) \wedge \cdots \wedge f^{p-1}(\xi)$. Clearly $g(z) \leq z$ and hence $u := g^\omega(z)$ exists and is a t-stable fixed point of $g$ by Theorem 7.3. As $f^k(z) \leq z$ for all $0 \leq k < p$, we have that

$$f^k(u) = f^k(g^\omega(z)) = g^\omega(f^k(z)) \leq u$$

for all $0 \leq k < p$. In particular, $f(u) \leq u$ and

$$u = g(u) = f^p(u) \leq f(u) \leq u,$$

so that $f(u) = u$. Moreover, as $(f'_u)^p = g'_u$ and $f'_u$ is continuous, we conclude that $u$ is a t-stable fixed point of $f$.

Now assume that $\xi$ has a Lyapunov stable orbit. For each $i = 0, \ldots, p-1$, there exists a neighbourhood $V_i$ of $f^i(\xi)$ such that the orbit of each $y \in V_i$ is bounded from above. Let $W_i = \{x \in D : x \leq y$ for some $y \in V_i\}$ and put $W = \bigcup_{i=0}^{p-1} W_i$. Note that, as $f^i(\xi) \in V_i$, $u \in W$ and $W$ is a neighbourhood of $u$. As $f$ is monotone and the orbit of each $y \in V_i$, $i = 0, \ldots, p-1$, is bounded from above under $f$, the orbit of each $w \in W$ is bounded from above, and hence $u$ is a Lyapunov stable fixed point.

We also have the following result.

**Lemma 7.7.** Suppose $f : D \to D$ is a convex monotone map and $v$ and $w$ are fixed points with $w \leq v$. If $w \ll v$ or $v$ is Lyapunov stable, then $w$ is Lyapunov stable.

**Proof.** Suppose that $w \ll v$ and let $V = \{x \in D : x \ll z\}$ be an open neighbourhood of $w$, where $w \ll z \ll v$. For each $x \in V$ we have that $x \ll v$, so that $f^k(x) \leq f^k(v) = v$ for all $k \geq 0$. Thus, the orbit of $x$ is bounded from above and hence $w$ is Lyapunov stable by Proposition 3.2.

Now assume that $w \leq v$ and $v$ is Lyapunov stable. Then there exists a neighbourhood $U$ of $v$ such that the orbit of each $y \in U$ is bounded from
above. Let $W = \{x \in \mathcal{D}: x \leq y \text{ for some } y \in U\}$. Note that $W$ is a neighbourhood of $w$, since $v \in U$ and $w \leq v$. As $f$ is monotone, the orbit of each $x \in W$ is bounded from above under $f$, and hence $w$ is Lyapunov stable. 

We conclude this section by showing that every $t$-stable fixed point of a convex monotone piecewise affine map is Lyapunov stable. Recall that a map $f: \mathbb{R}^n \to \mathbb{R}^n$ is piecewise affine if $\mathbb{R}^n$ can be partitioned into finitely many polyhedra in such a way that the restriction of $f$ to each polyhedron is an affine map.

**Corollary 7.8.** If $f: \mathbb{R}^n \to \mathbb{R}^n$ is a convex monotone piecewise affine map, then every $t$-stable fixed point of $f$ is Lyapunov stable.

**Proof.** Since $f$ is piecewise affine, we can find by [1, Lemma 6.4] a neighborhood $\mathcal{W}$ of 0 such that $f(v + x) = f(v) + f'(v)x$ for all $x \in \mathcal{W}$. By Theorem 6.5 there exists a norm under which $f'(v)$ is nonexpansive. Since $f'(v)(0) = 0$, we can take any open ball, $\mathcal{B}$, around 0 for this norm, and get $f'(v)(\mathcal{B}) \subseteq \mathcal{B}$. By taking $\mathcal{B}$ of sufficiently small radius, we can guarantee that $\mathcal{B} \subseteq \mathcal{W}$. Since $f(v) = v$, we find for all $x \in \mathcal{B}$ that $f(v + x) = v + f'(v)x \in v + \mathcal{B}$, which shows that $f(v + \mathcal{B}) \subseteq v + \mathcal{B}$. Since this inclusion holds for all balls $\mathcal{B}$ of sufficiently small radius, $v$ is a Lyapunov stable fixed point of $f$. 

\[\square\]

8. Tangentially stable periodic points

For a directed graph $\mathcal{G}$ and integer $k \geq 1$, we let $\mathcal{G}^k$ be the directed graph that has the same nodes as $\mathcal{G}$ with an arrow from node $i$ to node $j$ if and only if there exists a directed path of length $k$ in $\mathcal{G}$ from $i$ to $j$. There exists the following relation between $\mathcal{G}^c(f^k)$ and $(\mathcal{G}^c(f))^k$.

**Theorem 8.1.** If $f: \mathcal{D} \to \mathcal{D}$ is a convex monotone map with a $t$-stable fixed point, then $\mathcal{G}^c(f^k) = (\mathcal{G}^c(f))^k$ for all $k \geq 1$.

To prove this theorem we reduce it to a special case, which was analysed in [1]. Recall that $g: \mathbb{R}^n \to \mathbb{R}^n$ is called additively homogeneous if $g(x + \lambda \mathbf{1}) = g(x) + \lambda \mathbf{1}$ for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. (Here $\mathbf{1}$ is the vector in $\mathbb{R}^n$ with all coordinates unity.) The map $g$ is said to be additively subhomogeneous if $g(x + \lambda \mathbf{1}) \leq g(x) + \lambda \mathbf{1}$ for all $x \in \mathbb{R}^n$ and $\lambda \geq 0$. The following theorem for convex monotone additively homogeneous maps is proved in [1, Theorem 4.1].

**Theorem 8.2 (see [1]).** If $g: \mathbb{R}^n \to \mathbb{R}^n$ is a convex monotone additively homogeneous map with a fixed point, then $\mathcal{G}^c(g^k) = (\mathcal{G}^c(g))^k$ for all $k \geq 1$.

We note that every fixed point of a monotone additively subhomogeneous map $g: \mathbb{R}^n \to \mathbb{R}^n$ is stable, because $g$ is nonexpansive with respect to the sup-norm [8]. Using a standard “cemetery state” argument, see [1, Section 1.4], we derive the following consequence of Theorem 8.2.

**Corollary 8.3.** If $g: \mathbb{R}^n \to \mathbb{R}^n$ is a convex monotone additively subhomogeneous map with a fixed point, then $\mathcal{G}^c(g^k) = (\mathcal{G}^c(g))^k$ for all $k \geq 1$. 

\[\square\]
Proof. Define \( h: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) by
\[
    h(x, x_{n+1}) = \left( x_{n+1}1 + g(x - x_{n+1}1) \right)_{x_{n+1}}
\]
for all \((x, x_{n+1}) \in \mathbb{R}^{n+1}\). It is easy to verify that \( h \) is a convex monotone additively homogeneous map. Let \( v \in \mathbb{R}^n \) be a fixed point of \( g \) and remark that \( w = (v, 0) \) is a fixed point of \( h \). Due to the triangular structure of \( h \), we see that \( \partial g(v) = \partial h(w)_{JJ} \), where \( J = \{1, \ldots, n\} \) and \( \mathcal{G}^c(h) \) is the union of \( \mathcal{G}^c(g) \) with the loop \( \{(n+1), (n+1)\} \).

Clearly we observe that \( h \) is \( k \)-additively homogeneous for all \( k \). Due to this form we have that
\[
    h^k(x, x_{n+1}) = \left( x_{n+1}1 + g^k(x - x_{n+1}1) \right)_{x_{n+1}}
\]
for all \((x, x_{n+1}) \in \mathbb{R}^{n+1}\). It follows from Theorem 8.2 that \( \mathcal{G}^c(h^k) = (\mathcal{G}^c(h))^k \) for all \( k \geq 1 \). By considering the subgraph on the nodes \( \{1, \ldots, n\} \), we find that \( \mathcal{G}^c(g^k) = (\mathcal{G}^c(g))^k \) for all \( k \geq 1 \). ◻

We shall use the previous corollary to prove the following proposition, which is the key ingredient in the proof of Theorem 8.1.

**Proposition 8.4.** If \( f: \mathcal{D} \to \mathcal{D} \) is a convex monotone map with a \( t \)-stable fixed point \( v \in \mathcal{D} \), then \( \mathcal{G}^c((f')^k) = (\mathcal{G}^c(f'))^k \) for all \( k \geq 1 \).

Proof. Recall that \( f': \mathbb{R}^n \to \mathbb{R}^n \) is a convex monotone positivity homogeneous map. Write \( h = f' \) and let \( A, B, C, h_A, h_B \) and \( h^A \) be as in Lemma 6.1. Similar notation will be used for \( h^k \); so, \( (h^k)_A, (h^k)_B \) and \( (h^k)^A \). By definition of \( A \), we know that \( \mathcal{G}^c(h^A) = \mathcal{G}^c(h) \). As \( v \) is a \( t \)-stable fixed point of \( f 

0 \) is \( t \)-stable for \( h \). It follows from Lemma 6.1 that we can write \( h \) in the form
\[
    h(x_A, x_B) = (h_A(x_A, x_B), h_B(x_B)),
\]
where \( h_A: \mathbb{R}^A \times \mathbb{R}^B \to \mathbb{R}^A \) and \( h_B: \mathbb{R}^B \to \mathbb{R}^B \) are convex monotone positively homogeneous maps. Due to this form we have that \( (h^k)_B = (h_B)^k \) and \( (h^k)^A = (h^A)^k \), where \( h_A: \mathbb{R}^A \to \mathbb{R}^A \) is given by \( h_A(x_A) = h_A(x_A, 0) \). This implies that \( \mathcal{G}^c((h^A)^k) = \mathcal{G}^c(h^k) \).

By Proposition 6.2, there exists \( v_A \gg 0 \) in \( \mathbb{R}^A \) such that \( h^A(v_A) = v_A \). Let \( W \) be the \(|A| \times |A| \) diagonal matrix with \( W_{ii} = (v_A)_i \) for all \( i \in A \). Now define \( g: \mathbb{R}^A \to \mathbb{R}^A \) by
\[
    g(x) = (W^{-1} \circ h^A \circ W)(x) \quad \text{for all } x \in \mathbb{R}^A.
\]
Clearly \( \mathcal{G}^c((h^A)^k) = \mathcal{G}^c(g^k) \) for all \( k \geq 1 \). Moreover, \( g(0) = 0 \) and \( g(1) = 1 \), so that
\[
    g(x + \lambda 1) \leq g(x) + \lambda 1
\]
for all \( x \in \mathbb{R}^A \) and \( \lambda \geq 0 \), as \( g \) is convex and positively homogeneous. By Corollary 8.3, we get that \( \mathcal{G}^c(g^k) = (\mathcal{G}^c(g))^k \) for all \( k \geq 1 \). From this it follows that
\[
    \mathcal{G}^c(h^k) = \mathcal{G}^c((h^A)^k) = \mathcal{G}^c(g^k) = (\mathcal{G}^c(g))^k = (\mathcal{G}^c(h^A))^k = (\mathcal{G}^c(h))^k,
\]
which completes the proof. ◻
By using Proposition 8.4 it is now easy to prove Theorem 8.1.

**Proof of Theorem 8.1.** Let \( f: \mathcal{D} \to \mathcal{D} \) be a convex monotone map with a t-stable fixed point \( v \in \mathcal{D} \). Then 0 is a t-stable fixed point of \( f^t \). For each \( k \geq 1 \), we have that \( \partial f^k(v) = \partial(f^k)'_v(0) \), so that

\[
G^c(f^k) = G^c(\partial f^k(v)) = G^c(\partial(f^k)'_v(0)) = G^c((f^k)'_v).
\]

Thus, it follows from Proposition 8.4 that

\[
G^c(f^k) = G^c((f^k)'_v) = G^c((f^k)'_v)^k = (G^c(f)^k)^k
\]

for each \( k \geq 1 \), and we are done. \( \square \)

To analyse the periods of t-stable periodic points, we need to recall the notion of cyclicity of a graph. The cyclicity of a strongly connected directed graph \( G \), denoted \( c(G) \), is the greatest common divisor of the lengths of its circuits. The cyclicity of a general directed graph \( G \) is given by

\[
c(G) = \text{lcm} \{ c(G_i) : G_i \text{ is a strongly connected component of } G \}.
\]

The cyclicity of a nonnegative stable matrix \( P \) is defined by \( c(P) = c(G^c(P)) \).

Note that \( c(P) \) is the order of a permutation on \( n \) letters, where \( n \) is the size of the matrix. The following consequence of the Perron–Frobenius theorem concerning the cyclicity of stable nonnegative matrices can be found in [20, Theorem 9.1].

**Theorem 8.5 (see [20]).** If \( P \) is a stable nonnegative matrix, then the period of each periodic point of \( P \) divides \( c(P) \).

For a convex monotone map \( f: \mathcal{D} \to \mathcal{D} \) with a t-stable fixed point, we define the cyclicity of \( f \) by \( c(f) = c(G^c(f)) \).

**Theorem 8.6.** If \( \mathcal{D} \subseteq \mathbb{R}^n \) is downward and \( f: \mathcal{D} \to \mathcal{D} \) is a convex monotone map with a t-stable fixed point, then the period of each t-stable periodic point of \( f \) divides \( c(f) \). In particular, the period of each t-stable periodic point is the order of a permutation on \( n \) letters.

**Proof.** Let \( v \in \mathcal{D} \) be a t-stable fixed point of \( f \) and let \( \xi \in \mathcal{D} \) be a t-stable periodic point of \( f \) with period \( p \). Put \( g = f^c(f) \) and note that \( c(g) = 1 \). Indeed, by Theorem 8.1 we get that \( G^c(g) = G^c(f^c(f)) = (G^c(f))^c(f) = \bigcup_{i=1}^{s} G^c_i(f) \), where \( G_1, \ldots, G_s \) are the disjoint strongly connected components of \( G^c(f) \). Let \( c_i = c(G_i) \) for \( 1 \leq i \leq s \) and note that \( c_i \) divides \( c(f) \). It is well known that if \( G_i \) is a strongly connected graph with cyclicity \( c_i \), then \( c(G_i^{k_c}) = 1 \) for all \( k \geq 1 \) (see [5, Section 2]). In particular, we get that \( c(G_i^{c(f)}) = 1 \) for all \( 1 \leq i \leq s \). Thus,

\[
c(g) = c\left( \bigcup_{i=1}^{s} G^c_i(f) \right) = \text{lcm} \left\{ c\left( G_i^c(f) \right) : 1 \leq i \leq s \right\} = 1.
\]
By Theorem 4.6 there exists $M \in \partial g(v)$ such that $G^c(M) = G^c(g)$, so $c(M) = c(g) = 1$. Now let $C$, $D$, $U$ and $I$ be as in Proposition 4.3. As $g(v) = v$ and $g(x) - g(v) \geq M(x - v)$ for all $x \in D$, we get that
\[
g^k(x) \geq M^k(x - v) + v \quad \text{for all } x \in D \text{ and } k \geq 1. \tag{33}
\]
In particular,
\[
\xi = g^p(\xi) \geq M^p(\xi - v) + v, \tag{34}
\]
so that $\xi - v \geq M^p(\xi - v)$. It follows that $\xi - v = M^p(\xi - v)$ on $C \cup D$ and $\xi - v = 0$ on $D$ from Proposition 4.4. This implies that $g^p(\xi) \geq \xi$ on $C \cup D$.

Put $F = C \cup D$ and $G = U \cup I$. By definition, we have that $M_{FG} = 0$, and so $M_{FG}^k = 0$ for all $k \geq 1$. From this we deduce that
\[
(\xi - v)_F = (M^p(\xi - v))_F = (M_{FG})^p(\xi - v)_F = M_{FF}^p(\xi - v)_F.
\]
The matrix $M$ is stable and has cyclicity one. Therefore, the matrix $M_{FG}$ is also stable and has cyclicity one. This implies that any periodic point of $M_{FG}$ must have period one by Theorem 8.5. Thus, we find that $(\xi - v)_F = M_{FF}^p(\xi - v)_F$, so that $(\xi - v)_F = M_{FF}^k(\xi - v)_F$ for all $k \geq 1$. Since $M_{FG} = 0$, we deduce that
\[
(\xi - v) = M^k(\xi - v) \quad \text{on } F.
\]
From (33) it follows that $g^k(\xi) \geq \xi$ on $F$. Let $z = \xi \land g(\xi) \land \cdots \land g^{p-1}(\xi)$. Clearly $z \leq \xi$ and $z = \xi$ on $F$. As $g(z) \leq z$, it follows from Lemma 7.2 that $g^\omega(z) = \lim_{k \to \infty} g^k(z)$ exists and $g^\omega(z) = z$ on $C$. Thus, $g^\omega(z) = \xi$ on $C$. Note that $g^\omega(z)$ and $\xi$ are fixed points of $g^p$, and $\xi$ is a $t$-stable fixed point of $g^p = f^{pc(f)}$. Therefore, it follows from Lemma 5.2 that $g^\omega(z) \geq \xi$. As $g^\omega(z) \leq z \leq \xi$, we conclude that $g^\omega(z) = z = \xi$. Hence $f^{pc(f)}(\xi) = g(\xi) = \xi$
from which we conclude that the period of $\xi$ divides $c(f)$. \hfill \Box

Remark that if $f$ is strongly monotone convex map with a t-stable fixed point $v \in D$, then every $P \in \partial f(v)$ is positive by Proposition 2.2. Hence $c(f) = 1$ in that case, and therefore $f$ has no t-stable periodic points except its t-stable fixed points. We also like to point out that in Theorem 8.6 the t-stability assumptions are essential. Indeed, consider
\[
A = \begin{pmatrix}
\cos \alpha & \sin \alpha & 0 \\
-\sin \alpha & \cos \alpha & 0 \\
0 & 0 & b
\end{pmatrix}
\quad \text{and} \quad P = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
-\alpha & -\alpha & 1
\end{pmatrix},
\]
then
\[
B = PA P^{-1} \sim \begin{pmatrix}
1 & \alpha b & b - 1 \\
\alpha(b - 2) & 1 & b - 1 \\
\alpha(b - 1) & \alpha(b - 1) & b
\end{pmatrix}
\]
when $\alpha > 0$ is sufficiently small. Thus, for $b > 2$, the matrix $B$ is nonnegative for all sufficiently small $\alpha > 0$. Now if $\alpha = 2\pi/p$, then $A$ has a periodic point of period $p$ and hence $B$ has one too. This shows that a monotone convex map may have (unstable) periodic orbits with arbitrary large periods. We also remark that if $\alpha > 0$ is an irrational multiple of $\pi$, then $B$ has an unstable bounded orbit that does not converge to a periodic orbit.
9. Global convergence and nonexpansiveness

In this final section, we give a condition under which every orbit of a convex monotone map \( f: D \rightarrow D \), where \( D = \mathbb{R}^n \), converges to a Lyapunov stable periodic orbit. To present it we need the notion of the recession map \( \hat{f} \) of \( f \), which can be defined by

\[
\hat{f}(x) = \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} f(\lambda x) \quad \text{for all } x \in \mathbb{R}^n.
\]

Since \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is convex, \( \hat{f}(x) \in (\mathbb{R} \cup \{\infty\})^n \) exists for each \( x \in \mathbb{R}^n \) and is equal to

\[
\hat{f}(x) = \sup_{y \in \mathbb{R}^n} f(y + x) - f(y)
\]  
(35)

(see [25, Theorem 8.2]). As \( \hat{f} \) is the pointwise limit of a convex monotone map, \( \hat{f} \) is also convex and monotone. The next theorem shows that if \( \hat{f} \) has all its orbits bounded from above, then \( f \) is nonexpansive with respect to a polyhedral norm.

**Theorem 9.1.** If \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a convex monotone map and the recession map \( h = \hat{f} \) has all its orbits bounded from above, then \( f \) is nonexpansive with respect to the norm \( \| \cdot \|_v \) given in Theorem 6.5.

**Proof.** We remark that \( \hat{f} \) is a convex, monotone and positively homogeneous map, which has 0 as a t-stable fixed point, because \( (\hat{f})_0 = \hat{f} \) and \( \hat{f} \) has all its orbits bounded from above. Moreover,

\[
-\hat{f}(-x) \leq f(y + x) - f(y) \leq \hat{f}(x)
\]  
(36)

for all \( x, y \in \mathbb{R}^n \) by (35).

Let \( \| \cdot \|_v \) be the polyhedral norm from Theorem 6.5 and remark that \( \hat{f} \) is nonexpansive with respect to \( \| \cdot \|_v \). Clearly \( \|u\|_v \leq \|w\|_v \) if \( u, w \in \mathbb{R}^n \) are such that \( 0 \leq |u| \leq |w| \), where \( |z| = (|z_1|, \ldots, |z_n|) \). Therefore, it follows from (36) that

\[
\|f(y + x) - f(y)\|_v \leq \max\{\|\hat{f}(-x)\|_v, \|\hat{f}(x)\|_v\} \leq \|x\|_v
\]

for all \( x, y \in \mathbb{R}^n \). Thus, \( f \) is also nonexpansive with respect to \( \| \cdot \|_v \). □

This theorem has the following consequence.

**Corollary 9.2.** If \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a convex monotone map and \( f \) is nonexpansive with respect to some norm on \( \mathbb{R}^n \), then \( f \) is nonexpansive with respect to a polyhedral norm.

**Proof.** We note that \( \hat{f}(x) = \lim_{\lambda \rightarrow \infty} f(\lambda x)/\lambda \) for all \( x \in \mathbb{R}^n \). As \( f \) is nonexpansive with respect to some norm, \( \hat{f} \) will be nonexpansive with respect to the same norm. This implies that \( \hat{f} \) has all its orbits bounded from above, since \( \hat{f}(0) = 0 \). From Theorem 9.1 we conclude that \( f \) is nonexpansive with respect to a polyhedral norm. □
It was proved in [22] that if a map $f: \mathbb{R}^n \to \mathbb{R}^n$ is nonexpansive with respect to a polyhedral norm, then every bounded orbit of $f$ converges to a periodic orbit. Moreover, if the unit ball of the polyhedral norm has $N$ facets, then the period of each periodic point of a nonexpansive map does not exceed $\max_k 2^k \left(\frac{\lfloor N/2 \rfloor}{k}\right)$; see [18]. By using these results, the following global convergence theorem can be proved.

**Theorem 9.3.** If $f: \mathbb{R}^n \to \mathbb{R}^n$ is a convex monotone map, with a fixed point, and the recession map $\hat{f}$ has all its orbits bounded from above, then every orbit of $f$ converges to a Lyapunov stable periodic orbit of $f$ whose period divides $c(f)$.

**Proof.** As $\hat{f}$ has all its orbits bounded from above, we know by Theorem 9.1 that $f$ is nonexpansive with respect to a polyhedral norm, so that all periodic orbits of $f$ are Lyapunov stable. Since $f$ has a fixed point, it follows from [22] that every orbit of $f$ converges to a periodic orbit. Proposition 3.2 implies that every periodic point of $f$ is $t$-stable, and hence the period of each periodic point divides $c(f)$ by Theorem 8.6, which completes the proof. □

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