Effects of geometry and topology on some quantum mechanical systems

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Abstract

We study the behaviour of a nonrelativistic quantum particle interacting with different potentials in the spacetimes of topological defects. We find the energy spectra and show how they differ from their free-space values.

PACS numbers: 03.65.-w, 03.65.Ge, 04.90.Te

I. Introduction

The study of quantum systems under the influence of a gravitational field has been an exciting research field. It has been known that the energy eigenstates of an atom which interacts with a gravitational field are affected by the local spacetime curvature. As a result of this interaction, an observer at rest with respect to the atom would see a change in its spectrum, which would depend on the features of the spacetime. The problem of finding these shifts of the energy levels under the influence of a gravitational field is of considerable theoretical interest as well as possible observational. These shifts in the energy spectrum due to the gravitational field are different from the ones produced by the electromagnetic field present, for example, near white dwarfs and neutron starts. In fact, it
was already shown that in the Schwarzschild geometry, the level spacing of the gravitational effect is different from that of the well-known first order Stark and Zeeman effects, and therefore, in principle, it would be possible to separate the electromagnetic and gravitational perturbations of the spectrum [3]. Other investigations concerning this interesting subject is to use loosely bound Rydberg atoms in curved spacetime to study the gravitational shift in the energy spectrum [4].

The first experiment which showed the gravitational effect on a wave function was performed by Colella, Overhauser and Werner [5] by measure of the quantum mechanical phase difference of two neutron beams induced by a gravitational field. Another gravitational effect that appears in quantum interference due to a gravitational field is the neutrino oscillations [6].

The general theory of relativity, as a metric theory, predict that gravitation is manifested as curvature of spacetime. This curvature is characterized by the Riemann tensor $R^\alpha_{\beta\gamma\delta}$.

It is of interest to know how the curvature of spacetime at the position of the atom affects its spectrum. On the other hand, we know that there are connections between topological properties of the space and local physical laws in such a way that the local intrinsic geometry of the space is not sufficient to describe completely the physics of a given system. Therefore, it is also important to investigate the role played by a nontrivial topology, for example, on a quantum system. As examples of these investigations we can mention the calculation of the topological scattering amplitude in the context of quantum mechanics on a cone [7] and the interaction of a quantum system with conical singularities [8,9]. Therefore, an atom placed in a gravitational field is no longer exclusively influenced by its interaction with the local curvature; the topology plays a role.

Then, the problem of finding how the energy spectrum of an atom placed in a gravitational field is perturbed by this background has to take into account the geometrical structure and topological features of the spacetime under consideration. In other words, the dynamic of quantum systems is determined by the curvature and also by the topology of the background spacetime.

According to standard quantum mechanics, the motion of a charged particle can be
influenced by electromagnetic fields in regions from which the particle is rigorously excluded [10]. In this region the electromagnetic field vanishes and just inside a thin flux tube it does not vanish. This phenomenon has come to be called Aharonov-Bohm effect after a seminal paper by Aharonov and Bohm [10]. It was shown that in the quantum scattering in accordance with the Aharonov-Bohm effects problem this background leads to a non-trivial scattering, which was already confirmed experimentally [11].

The analogue of the electromagnetic Aharonov-Bohm effect set up in the gravitational framework is the background spacetime of a cosmic string [12,13] in which the geometry is flat everywhere apart from a symmetry axis. In this scenario we also have a gravitational Aharonov-Bohm effect for bound states [8,14].

Cosmic strings [12,13] and monopoles [15] are exotic topological defects [16] which may have been formed at phase transitions in the very early history of the Universe. Up to the moment no direct observational evidence of their existence has been found, but the richness of the new ideas they brought along to general relativity seems to justify the interest in the study of these structures.

The gravitational field of a cosmic string is quite remarkable; a particle placed at rest around a straight, infinite, static string will not be attracted to it; there is no local gravity. The spacetime around a cosmic string is locally flat but not globally. The external gravitational field due to a cosmic string may be approximately described by a commonly called conical geometry. Due to this conical geometry a cosmic string can induce several effects like, for example, gravitational lensing [17], pair production [18], electrostatic self-force [19] on an electric charge at rest, bremsstrahlung radiation [20] and the so-called gravitational Aharonov-Bohm effect [21].

The spacetime of a point global monopole has also some unusual properties. It possesses a deficit solid angle $\Delta = 32\pi^2 G\eta^2$, $\eta$ being the energy scale of symmetry breaking. Test particles in this spacetime experiences a topological scattering by an angle $\pi \frac{1}{2}$ irrespective of their velocity and their impact parameter. The effects produced by the point global monopole are due to the deficit solid angle which determines the curvature and the topological features of this spacetime.
In this paper we shall study the energy shift associated with a non-relativistic quantum particle interacting with an harmonic oscillator and a Coulomb potential with these systems placed in the background spacetime of a cosmic string, and determine how the nontrivial topology of this background spacetime perturbs the energy spectra. The influence of the conical geometry on the energy eigenvalues manifests as a kind of gravitational Aharonov-Bohm effect for bound states [8,14], whose analogue in the electromagnetic case shows that [22] that the bound state energy dependents on the external magnetic flux in a region from which the electron is excluded.

In the case of a point global monopole we are concerned with the similar proposal. We will investigate the interactions of a non-relativistic quantum particle with the Kratzer [23] and the Morse potentials [24] placed in the background spacetime of a global monopole. In this case we also determine the shifts in the energy levels.

These modifications in the energy spectra as compared to the simplest situation of empty flat Minkowski spacetime, could be used, in principle, as a probe of the presence of these defects in the cosmos. The possibility of using this effects for such a purpose is interesting because at present our observational knowledge of the existence of such defects is quite limited and this effect offers one more possibility to detect these topological defects.

To this end let us consider that a non-relativistic particle living a curved spacetime is described by the Schrödinger equation which should take the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2\mu} \nabla_{LB}^2 \psi + V(r), \quad (1)$$

where $\nabla_{LB}^2$ is the Laplace-Beltrami operator the covariant version of the Laplacian given by $\nabla_{LB}^2 = g^{-\frac{1}{2}} \partial_i \left(g^{ij} g^{\frac{1}{2}} \partial_j \right)$, with $i, j = 1, 2, 3; g = \det (g_{ij})$ stands for the determinant of the metric $g_{ij}; \mu$ is the mass of the particle and $V(r)$ is an external potential. Throughout this paper we will use units in which $c = 1$.

II. Harmonic oscillator in the spacetime of a cosmic string

The line element corresponding to the cosmic string spacetime is given in cylindrical
coordinates by \[3\]
\[ds^2 = -dt^2 + d\rho^2 + \alpha^2 \rho^2 d\theta^2 + dz^2,\] 
(2)
where \(\rho \geq 0\) and \(0 \leq \varphi \leq 2\pi\), the parameter \(\alpha = 1 - 4G\bar{\mu}\) runs in the interval \([0, 1]\), \(\bar{\mu}\) being the linear mass density of the cosmic string. The string is situated on the \(z\)-axis. In the special case \(\alpha = 1\) we obtain the Minkowski space in cylindrical coordinates. This metric has a cone-like singularity at \(\rho = 0\). In other words, the curvature tensor of the metric (2), considered as a distribution, is of the form
\[R^{12}_{12} = 2\pi \frac{\alpha - 1}{\alpha} \delta^2(\rho),\] 
(3)
where \(\delta^2(\rho)\) is the two-dimensional Dirac \(\delta\)-function.

Now, let us consider the Schrödinger equation in the metric (2) which is given by
\[-\frac{\hbar^2}{2\mu} \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\alpha^2 \rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right] \psi(t, \rho, \theta, z) + V(\rho, z)\psi(t, \rho, \theta, z) = i\hbar \frac{\partial}{\partial t} \psi(t, \rho, \theta, z),\] 
(4)
where \(V(\rho, z)\) is the potential energy corresponding to a three-dimensional harmonic oscillator which is assumed to be
\[V(\rho, z) = \frac{1}{2} \mu w^2 \left( \rho^2 + z^2 \right).\] 
(5)
In what follows, we shall solve the above Schrödinger equation with the interaction potential given by Eq. (5). To do this let us use the method of separation of variables by searching for solutions of the form
\[\psi(t, \rho, \theta, z) = \frac{1}{\sqrt{2\pi}} e^{-iEt+im\theta} R(\rho) Z(z),\] 
(6)
with \(m\) an integer. Using Eq. (6), Eq. (4) yields to the following ordinary differential equations for \(R(\rho)\) and \(Z(z)\)
\[-\frac{\hbar^2}{2\mu} \left[ \frac{1}{R(\rho)} \frac{\partial^2 R(\rho)}{\partial \rho^2} + \frac{1}{R(\rho)\rho} \frac{\partial R(\rho)}{\partial \rho} - \frac{m^2}{\alpha^2 \rho^2} \right] + \frac{1}{2} \mu w^2 \rho^2 = \epsilon,\] 
(7)
and
\[-\frac{\hbar^2}{2\mu Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} + \frac{1}{2} \mu w^2 z^2 = \varepsilon_z,\] 
(8)
where $\epsilon$ is a separation constant and such that
\[ \epsilon + \varepsilon_z = E. \] (9)

Equation (8) is the Schrödinger equation for a particle in the presence of one-dimensional harmonic oscillator potential, and then we have the well-known results
\[ \varepsilon_z = \left( n_z + \frac{1}{2} \right) \hbar \omega; \quad n_z = 0, 1, 2, ... \] (10)
with
\[ Z(z) = 2^{-\frac{n_z}{\alpha}} (n_z !)^{\frac{1}{2}} \left( \frac{\mu \omega}{\hbar \pi} \right)^{\frac{1}{4}} e^{-\frac{\mu \omega z^2}{2\hbar}} H_{n_z} \left( \sqrt{\frac{\mu \omega}{\hbar}} z \right), \] (11)
where $H_{n_z}$ denotes a Hermite Polynomial of degree $n_z$.

Solving Eq. (7) we get
\[ R(\rho) = \exp \left( -\frac{\tau}{2} \rho^2 \right) \rho^{\frac{|m|}{\alpha}} F(\rho), \] (12)
where $\tau = \frac{m \omega}{\hbar}$ and
\[ F(\rho) = \,_1 F_1 \left( \frac{1}{2} + \frac{|m|}{2\alpha} - \frac{\mu \epsilon}{2\hbar^2 \tau}, \frac{A}{2}; \tau \rho^2 \right) \] (13)
is the degenerate hypergeometric function, with $A = 1 + 2 \frac{|m|}{\alpha}$.

In order to have normalizable wavefunction, the series in Eq. (13) must be a polynomial of degree $n_\rho$, and therefore
\[ \frac{1}{2} + \frac{|m|}{2\alpha} - \frac{\mu \epsilon}{2\hbar^2 \tau} = -n_\rho; \quad n_\rho = 0, 1, 2, ... \] (14)
With this condition, we obtain the following result
\[ \epsilon = \hbar \omega \left( 1 + \frac{|m|}{\alpha} + 2n_\rho \right). \] (15)
If we substitute Eqs. (15) and (10) into (9) we get, finally, the energy eigenvalues
\[ E_N = \hbar \omega \left( N + \frac{|m|}{\alpha} + \frac{3}{2} \right), \] (16)
where $N = 2n_\rho + n_z$. 
From this expression we can conclude that the presence of the cosmic string breaks the degeneracy of the energy levels.

The complete eigenfunctions are then given by
\[
\psi(t, \rho, \theta, z) = C_{Nm} e^{-iE_N t} e^{-\frac{i\rho^2}{2\alpha}} F_1\left(1 + \frac{|m|}{2\alpha} - \frac{\mu e}{2h^2 \alpha}, 2; \frac{\rho^2}{\tau^2}\right) \\
e^{i m \theta} (n_z!)^{-\frac{1}{2}} \left(\frac{\mu w}{\hbar \pi}\right)^{-\frac{1}{2}} e^{-\frac{\mu w}{\hbar z^2}} H_{n_z}\left(\sqrt{\frac{\mu w}{\hbar}} z\right),
\]
where \(C_{Nm}\) is a normalization constant.

The results corresponding to the energy eigenvalues and eigenfunctions given by Eqs. (16) and (17), respectively, recover the ones for the flat spacetime in the limit \(\alpha \to 1\) as it should be.

As an estimation of the effect of the cosmic string on the energy shift, let us consider \(\alpha \approx 0.999999\), which corresponds to GUT cosmic strings. In this case the shift in the energy spectrum between the first two levels increases of about \(10^{-5}\%\) as compared with the flat Minkowski spacetime case.

### III. Coulomb potential in the spacetime of a cosmic string

Now, let us determine the energy eigenvalues of a particle in the presence of a Coulomb potential in the spacetime of a cosmic string. To do this let us consider the exterior metric of an infinitely long straight and static string in spherical coordinates. It reads as
\[
ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + \alpha^2 r^2 \sin^2 \theta d\varphi^2,
\]
with \(0 < r < \infty, 0 < \theta < \pi, 0 \leq \varphi \leq 2\pi\). In the special case \(\alpha = 1\) we obtain the Minkowski space in spherical coordinates.

In this case, the time-independent Schrödinger equation is
\[
-\frac{\hbar^2}{2\mu r^2} \left[2r \frac{\partial}{\partial r} + \frac{r^2 \partial^2}{\partial r^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \theta^2} + \frac{1}{\alpha^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}\right] \psi(r, \theta, \varphi) \\
+ V(r) \psi(r, \theta, \varphi) = E \psi(r, \theta, \varphi).
\]
This partial differential equation can be solved by finding separated solutions of the form
\[
\psi(r, \theta, \varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} R(r) \Theta(\theta),
\] (20)

for which \( m = 0, \pm 1, \pm 2, \ldots \). When we substitute Eq. (20) into Eq. (19) we thus obtain

\[
\begin{split}
\left[-\frac{\hbar^2}{2\mu} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + r^2 V(r) + \lambda\right] R(r) = E R(r) \\
\frac{\hbar^2}{2\mu} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{m^2}{\alpha^2 \sin^2 \theta} \right] \Theta(\theta) + \lambda \Theta(\theta) = 0,
\end{split}
\] (21)

indicating a neat separation of the variables \( r \) and \( \theta \), with \( \lambda \) being the separation constant.

If we change the function \( R(r) \) by \( u(r) \), introducing

\[ u(r) = rR(r), \] (23)

we find that \( u(r) \) obeys the radial equation

\[
-\frac{\hbar^2}{2\mu} \frac{\partial^2 u(r)}{\partial r^2} + V(r)_{\text{eff}} u(r) = Eu(r),
\] (24)

where \( V_{\text{eff}} \) is the effective potential experienced by the particle and given by

\[ V(r)_{\text{eff}} = V(r) + \frac{\lambda}{r^2}. \] (25)

Equation (24) is identical in form with the one-dimensional Schrödinger equation for a potential \( V(r) \) with the addition of the term \( \frac{\lambda}{r} \) to the potential energy.

Now, we turn to Eq. (22). By a change of variables \( \xi = \cos \theta; \xi \in [-1, 1]; \theta \in [0, \pi] \), and introducing \( F(\xi) = \Theta(\theta) \), Eq. (22) is transformed into

\[
\left(1 - \xi^2\right) \frac{d^2 F(\xi)}{d\xi^2} - 2\xi \frac{dF(\xi)}{d\xi} - \left(\frac{2\mu\lambda}{\hbar^2} + \frac{m^2}{\alpha^2 (1 - \xi^2)}\right) F(\xi) = 0.
\] (26)

Equation (26) is a kind of associated Legendre equation with a dependence on a parameter \( \alpha \), which lies between 0 and 1. Let us call it then generalized associated Legendre equation. It becomes an equation with eigenvalue \( \frac{2\mu\lambda}{\hbar^2} \), if we demand that the solution be finite at the singular points \( \xi = \pm 1 \).

A convenient method to obtain the solutions of Eq. (26) is by the investigation of its behaviour at the singular points \( \xi = \pm 1 \). To do this let us translate the origin to \( \xi = 1 \) and introduce a new variable \( z = 1 - \xi \).
Then, Eq. (26) turns into

\[ z(2-z)\frac{d^2 F(z)}{dz^2} + 2(1-z)\frac{dF(z)}{dz} - \left(\frac{2\mu\lambda}{\hbar^2} + \frac{m^2}{\alpha^2 z(2-z)}\right) F(z) = 0. \] (27)

The solution of Eq. (27) can be expanded in a power series

\[ F(z) = z^s \sum_{k=0}^{\infty} a_k z^k. \] (28)

Substituting Eq. (28) into (27) we get the indicial equation

\[ a_0 \left(s(s-1)4z^s + 4sz^s - \frac{m^2}{\alpha^2 z^s}\right) = 0, \] (29)

which implies that

\[ s = \frac{m}{2\alpha}, \]

and therefore

\[ F(\xi) = (1-\xi)^\frac{m}{2\alpha} f(\xi), \] (30)

where \( f(\xi) \) is an analytic function which does not vanish at \( \xi = 1 \).

In order to investigate the behaviour of \( F(\xi) \) in the neighborhood of \( \xi = -1 \) let us introduce the following substitution \( z = 1 + \xi \). Then we get that \( F(\xi) = (1+\xi)^\frac{m}{2\alpha} g(\xi) \), and therefore, the accepted solution may have the form

\[ F(\xi) = \left(1-\xi^2\right)^\frac{m}{2\alpha} G(\xi), \] (31)

where \( G(\xi) \) is an analytic function in all space, except when \( z \to \infty \), and is different from zero for \( \xi = \pm 1 \).

From Eqs. (26) and (31), we get

\[ \left(1-\xi^2\right)\frac{d^2 G(\xi)}{d\xi^2} - 2(m_{(a)} + 1)\xi \frac{dG(\xi)}{d\xi} - \left(m_{(a)}^2 + m_{(a)} + \bar{\lambda}\right) G(\xi) = 0, \] (32)

where

\[ m_{(a)} \equiv \frac{m}{\alpha}; \quad \bar{\lambda} = \frac{2\mu\lambda}{\hbar^2}. \]
The substitution of Eq. (31) into Eq. (32) yields the recursion relation

\[ a_{n+2} = \frac{n(n-1) + 2(m_0 + 1)n + \lambda + m_0(m_0 + 1)}{(n+1)(n+2)} a_n, \]  

(33)

where \( n \) is an integer \( \geq 0 \). In order to acceptable eigenfunctions, the series must terminate at some finite value of \( n \). According to Eq. (33), this will happen if \( \lambda \) has the value

\[ \lambda = l_0(l_0 + 1), \]  

(34)

where

\[ l_0 = m_0 + n. \]

Therefore, Eq. (26) turns into

\[ (1 - \xi^2) \frac{d^2 F(\xi)}{d \xi^2} - 2\xi \frac{d F(\xi)}{d \xi} - l_0(l_0 + 1) F(\xi) - \frac{m^2}{\alpha^2(1 - \xi^2)} F(\xi) = 0, \]  

(35)

whose solutions are the generalized associated Legendre functions

\[ F_{l_0}^{m_0}(\xi) = \frac{1}{2^{l_0} l_0!} \left( 1 - \xi^2 \right)^{\frac{m_0}{2}} \frac{d^{m_0 + l_0}}{d \xi^{m_0 + l_0}} \left[ (\xi^2 - 1)^{l_0} \right]. \]  

(36)

Now, we turn to Eq. (24), substitute the expression for \( \lambda \) given by Eq. (34) and consider \( V(r) = -\frac{k}{r} \). We thus obtain

\[ \frac{d^2 u(r)}{d u^2} + 2k \frac{\mu u(r)}{r \hbar^2} - \beta^2 u(r) - \frac{1}{r^2} \left[ l_0(l_0 + 1) \right] u(r) = 0, \]  

(37)

where

\[ \beta^2 = -\frac{2\mu E_n r}{\hbar^2}; \quad E_n < 0. \]  

(38)

Equation (37) is a confluent hypergeometric equation whose solution is given by

\[ u(r) = {}_1 F_1 \left( l_0 + 1 - \frac{k^2 \mu}{\beta \hbar^2} 2 + 2l_0 + 2\beta r \right). \]  

(39)

This function is divergent, unless

\[ 1 + l_0 - \frac{\mu k^2}{\beta \hbar^2} = -n_r; \quad n_r = 0, 1, 2, ... \]  

(40)
Then, from the previous condition we find the energy eigenvalues

\[ E_{n_r} = -\frac{\mu k^2}{2\hbar^2} \left[ l(\alpha) + n'_{(r)} \right]^{-2}; \quad n'_{(r)} = 1, 2, 3, \ldots, \]  

where \( n'_{(r)} = 1 + n_r \). Note that the energy levels become more and more spaced as \( \alpha \) tends to 1, which means that the shift in the energy levels due to the presence of the cosmic string increases with the increasing of the angular deficit.

From the expression for energy given by Eq. (41) we can notice that the levels without a \( z \)-component of the angular momentum are not shifted relative to the Minkowski case. Except these levels, all the other ones are degenerated.

In this present case the energy of the particle in the presence of the cosmic string increases as compared with the flat spacetime value. This increasing is of about \( 4 \times 10^{-3}\% \) for the first two energy levels.

### VI. Kratzer potential in the spacetime of a global monopole

The solution corresponding to a global monopole in a \( O(3) \) broken symmetry model has been investigated by Barriola and Vilenkin [15].

Far away from the global monopole core we can neglect the mass term and as a consequence the main effects are produced by the deficit solid angle. The respective metric in Einstein’s theory of gravity can be written as [15]

\[ ds^2 = -dt^2 + dr^2 + b^2 r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \]  

where \( b^2 = 1 - 8\pi G \eta^2 \), \( \eta \) being the energy scale of symmetry breaking.

Now, let us consider a particle interacting with a Kratzer potential and placed in the background spacetime given by metric (42).

The Kratzer potential has the form [23]

\[ V(r) = -2D \left( \frac{A}{r} - \frac{1}{2} \frac{A^2}{r^2} \right), \]  

where \( A \) and \( D \) are positive constants.
In order to determine the energy spectrum let us write the Schrödinger equation in the background spacetime of a global monopole given by line element (42). In this case the Schrödinger equation can thus be reduced to

$$-\frac{\hbar^2}{2\mu b^2 r^2} \left[ 2r b^2 \frac{\partial}{\partial r} + b^2 r^2 \frac{\partial^2}{\partial r^2} - \mathbf{L}^2 - 2D \left( \frac{A}{r} - \frac{1}{2} \frac{A^2}{r^2} \right) \right] \psi(r) = E \psi(r),$$  \hspace{1cm} (44)

where $\mathbf{L}$ is the usual orbital angular momentum operator. We begin by using the standard procedure for solving Eq. (44) which consists in the separation of the eigenfunctions as

$$\psi_{m,l}(r) = R_l(r) Y^m_l(\theta, \varphi).$$  \hspace{1cm} (45)

Substitution of Eq. (45) into Eq. (44) leads to

$$-\frac{\hbar^2}{2\mu} \frac{d^2 g_l(r)}{dr^2} - 2D \left( \frac{A}{r} - \frac{1}{2} \frac{A^2}{r^2} \right) g_l(r) + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{b^2 r^2} g_l(r) = E g_l(r),$$  \hspace{1cm} (46)

where $g_l(r) = r R_l(r)$.

Analyzing Eq. (46) when $r \to 0$ and $r \to \infty$ we find that its solution can be written as

$$g_l(r) = r^{\lambda_l} e^{-\bar{\beta} r} F_l(r),$$  \hspace{1cm} (47)

where

$$\lambda_l = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4 \left( \frac{2\mu DA^2}{\hbar^2} + \frac{l(l+1)}{b^2} \right)}$$  \hspace{1cm} (48)

and

$$\bar{\beta} = -\frac{2\mu E}{\hbar^2}, \ E < 0.$$  \hspace{1cm} (49)

Substituting Eq. (47) into Eq. (46) and making use of Eqs. (38) and (48) we obtain the equation for $F(z)$

$$z \frac{d^2 F(z)}{dz^2} + (2\lambda_l - z) \frac{dF(z)}{dz} - \left( \lambda_l - \frac{2mAD}{\bar{\beta} \hbar^2} \right) F(z) = 0,$$  \hspace{1cm} (50)

where $z = 2\bar{\beta} r$.

The solution of this differential equation is the confluent hypergeometric function

$$\, _1F_1 \left( \lambda_l - \frac{z^2}{2A}, 2\lambda_l; 2\bar{\beta} r \right), \text{ where } \gamma^2 = \frac{2\mu DA^2}{\hbar^2}.$$
Therefore, the solution for the radial function \( g_l(r) \) is given by
\[
g_l(r) = r^{\lambda_l} e^{-\bar{\beta}r_1} F_1 \left( \lambda_l - \frac{\gamma^2}{\bar{\beta}A}, 2\lambda_l ; 2\bar{\beta}r \right). \tag{51}
\]

In order to make \( g_l(r) \) vanishes for \( r \to \infty \), the confluent hypergeometric function may increase not faster than some power of \( r \), that is, the function must be a polynomial. To fulfill this condition we must have
\[
\lambda_l - \frac{\gamma^2}{\bar{\beta}A} = -\bar{n}_r, \quad \bar{n}_r = 0, 1, 2, \ldots, \tag{52}
\]
which implies that the eigenvalues are
\[
E_{l, \bar{n}_r} = -\frac{\hbar^2}{2\mu A^2} \gamma^4 \left( \bar{n}_r + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{l(l+1)}{b^2}} + \gamma^2 \right)^{-2}. \tag{53}
\]

Let us consider the parameter \( \gamma \gg 1 \), which is a situation valid for the majority of cases. Then we may expand Eq. (53) into powers of \( \frac{1}{\gamma} \), and obtain the approximate result
\[
E_{l, \bar{n}_r} \approx D \left( -1 + 2 \left( \frac{\bar{n}_r + \frac{1}{2}}{\gamma} \right) + \left( \frac{1}{4} + \frac{l(l+1)}{b^2} \right) \gamma^2 \right.
-3 \left( \frac{\bar{n}_r + \frac{1}{2}}{\gamma^2} \right)^2 - 3 \left( \frac{\bar{n}_r + \frac{1}{2}}{\gamma^2} \right) \left( \frac{1}{4} + \frac{l(l+1)}{b^2} \right) \gamma^3 \right). \tag{54}
\]

Introducing the momentum of inertia defined by
\[
\Theta = \mu A^2, \tag{55}
\]
and using the classical frequency for small harmonic vibrations which is given by
\[
w = \sqrt{\frac{2D}{A^2 \mu}}, \tag{56}
\]
we thus find that Eq. (54) transforms into
\[
E_{l, \bar{n}_r} \approx \frac{w^2 \Theta}{2} + \left( \bar{n}_r + \frac{1}{2} \right) \hbar w + \frac{\hbar^2}{2\Theta} \left( \frac{1}{4} + \frac{l(l+1)}{b^2} \right)
-3 \left( \bar{n}_r + \frac{1}{2} \right)^2 \frac{\hbar^2}{2\Theta} - 3 \left( \bar{n}_r + \frac{1}{2} \right) \left( \frac{1}{4} + \frac{l(l+1)}{b^2} \right) \frac{\hbar^3}{w\Theta^2}. \tag{57}
\]

From Eq. (52) we can get the energy of dissociation of a molecule by taking \( \bar{n}_r = 0 \) and \( l = 0 \), in which case there is no dependence on the presence of the monopole. This result mean that the energy of dissociation is no affected by the presence of the global monopole.
From the above results given by Eqs. (53) and (57), we can see that when \( b = 1 \) we recover the well-known result corresponding to a particle submitted to the Kratzer potential \[23\] in Minkowski spacetime as expected. It is worthy noticing from expression for the energy given by Eq. (53) that even in the case in which the \( z \)-component of the angular momentum vanishes the energy level is shifted relative to the Minkowski case.

As an estimation of the effect of the global monopole on the energy spectrum, let us consider a stable global monopole configuration for which \( \eta = 0.19m_p \), where \( m_p \) is Planck mass. In this situation the shift in the energy spectrum between the first two levels in this background spacetime decreases of about 350% as compared with the Minkowski spacetime. For symmetry breaking at grand unification scale, the typical value of \( 8\pi G\eta^2 \) is around \( 10^{-6} \) and in this case decreasing in the energy shift is of about 1%.

V. Morse potential in the spacetime of a global monopole

Now, let us consider the Morse potential \[24\] which is given by

\[
V(r) = D \left( e^{2\beta x} - 2e^{-\beta x} \right); \quad x = \frac{r - r_0}{r_0}; \quad \beta > 0.
\] (58)

In order to determine of the energy levels in a simply way, let us consider a particular situation that corresponds to low-energy vibration. Hence, in this case, we can expand this potential around \( r = r_0 \) (or \( x = 0 \)) and obtain the approximate result

\[
V(r) = D \left( -1 + \beta^2 x^2 + \ldots \right)
\]

\[
\cong -D + \frac{\beta^2}{r_0^2} (r - r_0)^2.
\] (59)

Introducing a frequency \( w \) defined by

\[
w^2 = \frac{2D\beta^2}{Mr_0^2},
\] (60)

we can write Eq. (59) as
\[ V(r') \approx -D + \frac{1}{2} M w^2 r'^2, \tag{61} \]

where \( r' = r - r_0 \).

This potential corresponds to the one of an isotropic harmonic oscillator minus a constant. Substituting this potential into the radial part the Schrödinger, we get

\[ \frac{d^2 g(r')}{dr'^2} - \frac{2M^2}{\hbar^2} V(r') g(r') - l(l + 1) g(r') + \frac{2M}{\hbar^2} E_M g(r') = 0, \tag{62} \]

where

\[ g(r') = r'R(r'). \]

The solution of Eq. (62) is given by

\[ g(r') = \exp \left( -\frac{1}{2} M \frac{w}{\hbar} r'^2 \right) r'^{\frac{l}{2} + \frac{1}{2}} \sqrt{1 + \frac{4}{b^2} l(l+1)} F(r'). \tag{63} \]

Substituting Eq. (63) into Eq. (62) we obtain

\[ r' \frac{d^2 F(r')}{dr'^2} + \left[ c - \frac{2Mw}{\hbar} r'^2 \right] \frac{dF(r')}{dr'} + \left[ P - \frac{2Mw}{\hbar} \right] r' F(r') = 0, \tag{64} \]

with

\[ c = 1 + \sqrt{1 + \frac{4}{b^2} l(l+1)} \text{ and } P = \frac{2Mw}{\hbar^2} (E_M + D) - \frac{Mw}{\hbar} \sqrt{1 + \frac{4}{b^2} l(l+1)} \]

Performing the change of variables

\[ M w r'^2 = \rho, \tag{65} \]

and defining \( \frac{P}{\hbar} = x \), we can write Eq. (64) as

\[ x \frac{d^2 F(x)}{dx^2} + \left[ \frac{c}{2} - x \right] \frac{dF(x)}{dx} + \left[ \frac{P}{4Mw} - \frac{1}{2} \right] F(x) = 0, \tag{66} \]

whose solution is

\[ F(x) = _1 F_1 \left( \frac{1}{2} - \frac{\hbar P}{4Mw}, \frac{c}{2}, x \right), \tag{67} \]

or, in terms of variable \( r' \), we get
\[ F(r') = \frac{1}{2} F_{1} \left( \frac{1}{2} - \frac{1}{2h\omega} (E_M + D) + \frac{1}{4} \sqrt{1 + \frac{4}{b^2}(l+1)}; \frac{1}{2} + \frac{1}{2} \sqrt{\frac{4}{b^2}(l+1)} \frac{Mw'\gamma^2}{\hbar} \right). \] (68)

Note that this solution is divergent, unless the condition

\[ \frac{1}{2} - \frac{1}{2h\omega} (E_M + D) + \frac{1}{4} \sqrt{1 + \frac{4}{b^2}(l+1)} = -n_M; \quad n_M = 0, 1, 2, 3, \ldots, \] (69)

is fulfilled.

From Eq. (69) we find the energy levels

\[ E_M = \hbar \omega \left[ N_{lM} + \frac{3}{2} \right] - D, \] (70)

where

\[ N_{lM} = \frac{1}{2} \left( \sqrt{1 + \frac{4}{b^2}(l+1)} - 1 \right) + 2n_M. \] (71)

Now, let us consider an additional term in the Morse potential which corresponds to the rotation energy. Since, we can write \( V(r) \) as

\[ V(r) = D \left( e^{-\beta x} - e^{-\beta x} \right) + V'(r), \] (72)

where

\[ V'(r) = \frac{\hbar^2}{2M} \frac{(l + 1)}{b^2r^2} = \frac{l(l + 1)}{b^2\gamma^2} \frac{D}{(x + 1)^2}; \quad \gamma^2 = \frac{2MDr_0^2}{\hbar^2}. \] (73)

Hence we can expand \( V'(r) \) and get the result

\[ V'(r) \approx \frac{l(l + 1)}{b^2\gamma^2} D \left( c_0 + c_1 e^{-\beta x} + c_2 e^{-2\beta x} \right), \] (74)

for \(|x| \ll 1\), where the coefficients \( c_0, c_1 \) and \( c_2 \) are given by

\[ c_0 = 1 - \frac{3}{\beta} + \frac{3}{\beta^2}; \quad c_1 = \frac{4}{\beta} - \frac{6}{\beta^2}; \quad c_2 = -\frac{1}{\beta} + \frac{3}{\beta^2}. \] (75)

Substituting Eq. (74) into Eq. (72) and then into Eq. (62), we get

\[ \frac{d^2g(x)}{dx^2} + \left[ -\beta_1^2 + 2e^{-\beta x}\gamma_1 - e^{-2\beta x}\gamma_2^2 \right] g(x) = 0 \] (76)

where
\[
\beta^2_1 = l \left(1 + \frac{1}{b^2}\right) c_0 - 2 \frac{M E_M r_0^2}{\hbar} ; \quad E_M < 0. \tag{77}
\]

\[
\gamma^2_1 = \gamma^2 - \frac{c_1 l (l + 1)}{b^2}. \tag{78}
\]

\[
\gamma^2_2 = \gamma^2 + c_2 l \frac{(l + 1)}{b^2}. \tag{79}
\]

Now, let us introduce the following change of variable

\[y = \xi e^{-\beta x},\tag{80}\]

and a function \(R(y)\) such that

\[g(y) = \frac{R(y)}{\sqrt{y}}.\tag{81}\]

The equation for \(R(y)\) is

\[y^2 \frac{d^2 R(y)}{dy^2} + \frac{\gamma^2_1}{\gamma^2_2} y R(y) + \left[ \frac{1}{4} - \frac{1}{4} y^2 - \frac{\beta^2_1}{\beta^2} \right] R(y) = 0, \tag{82}\]

whose solution is given by

\[R(y) = \exp\left(-\frac{1}{2} y\right) y^{\frac{1}{2} + \frac{\beta_1}{\beta}} F(y),\tag{83}\]

with \(F(y)\) obeying the Kummer equation

\[y \frac{d^2 F(y)}{dy^2} + \left(1 + 2 \left| \frac{\beta_1}{\beta} \right| - y \right) \frac{dF(y)}{dy} - \left( \frac{1}{2} + \left| \frac{\beta_1}{\beta} \right| - \frac{\gamma^2_1}{\gamma^2_2} \right) F(y) = 0, \tag{84}\]

whose solution is the Kummer function

\[F(y) = M \left( \frac{1}{2} + \left| \frac{\beta_1}{\beta} \right| - \frac{\gamma^2_1}{\gamma^2_2} , 1 + 2 \left| \frac{\beta_1}{\beta} \right| ; y \right). \tag{85}\]

This solution diverges, unless we have

\[\frac{1}{2} + \left| \frac{\beta_1}{\beta} \right| - \frac{\gamma^2_1}{\gamma^2_2} = -n; \quad n = 0, 1, 2, ... \tag{86}\]

This relation leads thus to the energy levels
\[ E_M = \frac{\hbar^2}{2Mr_0^2} \left[ -\gamma^2 + 2\gamma\beta \left( n + \frac{1}{2} \right) - \beta^2 \left( n + \frac{1}{2} \right)^2 - l^2 (l + 1)^2 \frac{(c_1 + c_2)^2}{4b^2\gamma^2} + l \frac{(l+1)}{b^2} (c_0 + c_1 + c_2) - l \frac{(l+1)}{b^2} (c_1 + c_2) \left( n + \frac{1}{2} \right) \right]. \] (87)

which was obtained by use of the approximate result

\[ \frac{\gamma_1}{\gamma_2} = \left[ \gamma^2 - \frac{c_1}{2} l \frac{(l+1)}{b^2} \right] \left[ \gamma^2 - \frac{c_2}{2} l \frac{(l+1)}{b^2} \right]^{-\frac{1}{2}} \]
\[ \approx \gamma \left( 1 - l \frac{(l+1)}{2b^2\gamma^2} (c_1 + c_2) \right), \]

Finally, substituting the values of the coefficients \( c_0, c_1 \) and \( c_2 \) given by Eq. (75) into Eq. (87) we find

\[ E_M = \frac{\hbar^2}{2Mr_0^2} \left[ -\gamma^2 + 2\gamma\beta \left( n + \frac{1}{2} \right) - \beta^2 \left( n + \frac{1}{2} \right)^2 + l \frac{(l+1)}{b^2} \right. \]
\[ - \frac{3}{\beta\gamma} \left( n + \frac{1}{2} \right) l \frac{(l+1)}{b^2} - \frac{9 (\beta - 1)^2}{4\gamma^2} l^2 \frac{(l+1)^2}{b^2} \left. \right]. \] (88)

which reduces to the result of flat spacetime when \( b = 1 \) as expected.

In the case of a particle interacting with a Morse potential in the background spacetime of a global monopole we can consider two different situations. In the first, only the vibration energy is taken into account and in this case the decreasing in the energy goes from \( 3 \times 10^{-5}\% \) to 32\% in the cases of GUT and stable monopoles, respectively, as compared to the flat spacetime background. In the second situation, we consider also the corrections due to the rotational energy and in this case we observe that there is a decrease in the energy that goes from \( 8 \times 10^{-4}\% \) to 85\% compared to Minkowski spacetime case, for GUT and stable monopoles, respectively.

I. FINAL REMARKS

In the spacetime of a cosmic string we studied the behaviour of a particle interacting with an harmonic oscillator and a Coulomb potentials. The quantum dynamics of such a single particle depends on the nontrivial topological features of the cosmic string spacetime. The presence of the defect shift the energy levels in both cases relative to Minkowski

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spacetime case. These gravitational effects can be understood as a kind of Aharonov-Bohm effect for bound states and as a consequence extends the Aharonov-Bohm effect beyond electromagnetic theory to the gravitational field.

In the case of the molecular potentials of Kratzer and Morse in the presence of a global monopole the shifts in the energy levels are due to the combined effects of the curvature and the nontrivial topology determined by the deficit solid angle corresponding to this spacetime.

The numerical estimations concerning the energy shifts for different potentials in the backgrounds of a cosmic string and a global monopole are all measurable, in principle. An important application of the results concerning the modifications in the energy spectra could be found in the astrophysical context in which these modifications enters the interpretation of the spectroscopical data and these could be used as a probe of the presence of a cosmic string or a global monopole in the universe.

Although it seems difficult to provide verification of the effects considered with current experimental detectability, we believe that investigation of systems in which both quantum effects and gravitational effects associated with curvature and topology come into play is important.

It is interesting to draw the attention to the fact that the presence of cosmic string in the cases considered increases the energy which is in contrast to the case of the presence of a monopole in which situation the energy decreases as compared with the flat Minkowski spacetime background.

It is worthy commenting that the study of a quantum system in a nontrivial gravitational background like the ones corresponding to a cosmic string and to a global monopole may shed some light on the problems of combining quantum mechanics and general relativity, in situation in which nontrivial topological aspects of the background spacetime under consideration a represent.
ACKNOWLEDGMENTS

We acknowledge Conselho Nacional de Desenvolvimento Científico e Tecnológico for partial financial support.
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