Extrapolated New Hermitian and Skew-Hermitian Splitting Method for Non-Hermitian Positive Definite Linear System

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Abstract. Extrapolated new Hermitian and skew-Hermitian splitting method is presented for solving non-Hermitian and normal positive definite linear systems. We theoretically prove that this method is convergent in three different ranges of the parameters \( \alpha \) and \( \omega \). Specially, the convergence results of the NHSS method are also obtained, which generalize the conclusions in [2]. The numerical example further verifies the results.

1. Introduction

Many problems in scientific computing result in a system of linear equations

\[
Ax = b
\]

where \( A \in \mathbb{C}^{n \times n} \) is a non-Hermitian positive definite matrix, \( x, b \in \mathbb{C}^n \) and \( x \) is an unknown vector and \( b \) is a given vector. The systems usually are solved by the iterative methods. Let \( A = M - N \) and \( M \) be nonsingular, then the basic iterative method is

\[
x^{(k+1)} = T x^{(k)} + c, k = 0, 1, 2, \ldots
\]

where \( c = M^{-1} b, T = M^{-1} N \) is the iterative matrix, whose spectral radius decides whether the iterative method converges and even the convergence speed. So, efficient matrix splittings of the coefficient matrix \( A \) are required.

Extrapolated methods can improve the performance of the basic iterative methods. The extrapolated method corresponding to (2) is defined as following:
\[ x^{(k+1)} = T_\omega x^{(k)} + \omega M^{-1}b, k = 0,1,2,\ldots, \]  

(3)

where \( T_\omega = (1 - \omega)I + \omega T \) is the iterative matrix, \( \omega \in \mathbb{R} \) is the extrapolated parameter.

Obviously, if \( \omega = 0, T_0 = I \) and \( x^{(k+1)} = x^{(k)}, k = 0,1,2,\ldots \). So, in the paper, we let \( \omega \neq 0 \). In (3), if \( \omega = 1 \), the extrapolated method would return to (2).

The extrapolated method (3) is completely consistent with the method (2), which be used to improve and speed up the method (2), even when it is divergent.

To solve the large sparse non-Hermitian positive definite linear systems, Bai, Golub and Ng [1] proposed the HSS iterative method as following

\[
\begin{align*}
(\alpha I + H)x^{(k+\frac{1}{2})} &= (\alpha I - S)x^{(k)} + b, \\
(\alpha I + S)x^{(k+\frac{1}{2})} &= (\alpha I - H)x^{(k+\frac{1}{2})} + b,
\end{align*}
\]

(4)

Where \( H = \frac{1}{2}(A + A^*) \) and \( S = \frac{1}{2}(A - A^*) \) are the Hermitian and skew-Hermitian parts of \( A \), respectively, and \( \alpha \) is a given positive constant. In [1], the convergence property of the HSS method is given.

Recently, for the non-Hermitian and normal positive definite linear systems with strong Hermitian parts, the NHSS (new Hermitian and skew-Hermitian) method has been established in [2] by Pour and Goughery, which has the following iterative scheme

\[
\begin{align*}
Hx^{(k+\frac{1}{2})} &= -Sx^{(k)} + b, \\
(\alpha I + H)x^{(k+\frac{1}{2})} &= (\alpha I - S)x^{(k+\frac{1}{2})} + b,
\end{align*}
\]

(5)

where \( \alpha \) is a given positive constant.

It is pointed out in [2] if the coefficient matrix \( A \) is normal positive definite, and the eigenvalues of \( H \) and \( S \) meet some conditions, the NHSS iteration converges to the unique solution of the linear system (1) for any \( \alpha > 0 \).

Further application and generalization of the HSS iterative methods can be founded in [3]-[16], such as the NSS, the PSS, the GHSS, the MHSS, the PHSS, the AHSS, the LHSS and so on. One can see [17] for a comprehensive survey on the HSS method.

In this paper, the extrapolated NHSS method, abbreviated as ENHSS, will be proposed, and the convergence of the method will be concentrated on. Convergence results will be gotten, which demonstrate the convergence range of the method is closed related with the values of \( \alpha \) and \( \omega \). That is, when the parameters \( \alpha \) and \( \omega \) are in certain range, the extrapolated NHSS method is convergent. Specially, the convergence of the NHSS is only dependent on the choice of \( \alpha \).
2. Convergence Analysis of ENHSS Method

From (3) and (5), we can obtain the iterative matrix of the extrapolated NHSS method

\[ T_{\omega,\alpha} = (1 - \omega) I + \omega (\alpha I + H)^{-1} (\alpha I - S) H^{-1} (-S) \]

\[ = (1 - \omega) I + \omega T(\alpha), \]

where \( T(\alpha) = (\alpha I + H)^{-1} (\alpha I - S) H^{-1} (-S) \) is the iterative matrix of the NHSS method (5), \( \alpha \in \mathbb{R}, \omega \in \mathbb{R} \).

Let \( x \in \mathbb{C}^n \) and \( \|x\|_2 = 1 \), for the convenience of later discussions, denote

\[ b = x^* H x, c = x^* SH^{-1} S x, d = x^* H^{-1} S x, \]

and

\[ \eta_M = \max_{\|x\|_2 = 1} |x^* SH^{-1} S x|, \quad \eta_m = \min_{\|x\|_2 = 1} |x^* SH^{-1} S x|, \]

\[ \xi_M = \max_{\|x\|_2 = 1} |x^* (H - SH^{-1} S) x|, \quad \xi_m = \min_{\|x\|_2 = 1} |x^* (H - SH^{-1} S) x|, \]

\[ \delta_M = \max_{\|x\|_2 = 1} |x^* H^{-1} S x|, \quad \delta_m = \min_{\|x\|_2 = 1} |x^* H^{-1} S x|, \]

\[ \zeta_M = \max_{\|x\|_2 = 1} |x^* H x|, \quad \zeta_m = \min_{\|x\|_2 = 1} |x^* H x|. \]

Obviously, when \( A \) is non-Hermitian and normal positive definite matrix, \( H \) is a Hermitian positive definite matrix, and \( SH^{-1} S \) is a Hermitian negative definite matrix, and \( H^{-1} S \) is a skew-Hermitian matrix. So \( b > 0, c > 0, d \) is a pure imaginary number.

For the extrapolated method (3), the following theorem is given in [18].

**Theorem 1** The extrapolated method (3) is convergent if and only if (1) or (2) holds,

1. \( x_j < 1, j = 1, 2, \ldots, \eta \), and \( 0 < \omega < \min \frac{2(1 - x_j)}{(1 - x_j)^2 + y_j^2}, \)

2. \( x_j > 1, j = 1, 2, \ldots, \eta \), and \( 0 > \omega > \max \frac{2(1 - x_j)}{(1 - x_j)^2 + y_j^2}, \)

where \( x_j + iy_j, j = 1, 2, \ldots, \eta \) are all of the eigenvalues of the matrix \( T \).

From Theorem 1, we can easily conclude the following corollary.

**Corollary 1** Let \( x_j + iy_j, j = 1, 2, \ldots, n \), be all of the eigenvalues of the matrix \( T \). If \( x_j < 1 \) for every \( j = 1, 2, \ldots, n \), \( \rho(T) < 1 \) holds.

**Proof.** Firstly, when \( x_j < 1 \) for every \( j = 1, 2, \ldots, n \), we can prove that \( \rho(T) < 1 \) if and
only if for every \( j \), \( \frac{2(1-x_j)}{(1-x_j)^2 + y_j^2} > 1 \). If for every \( j \), \( \frac{2(1-x_j)}{(1-x_j)^2 + y_j^2} > 1 \), we can easily concluded \( x_j^2 + y_j^2 < 1 \) for \( j = 1, 2, \cdots, n \), that is \( \rho(T) < 1 \). Conversely, if \( \rho(T) < 1 \), that is \( x_j^2 + y_j^2 < 1 \) for every \( j = 1, 2, \cdots, n \). So \( (1-x_j-1)^2 + y_j^2 < 1 \), that is \( (1-x_j)^2 + y_j^2 < 2(1-x_j) \), thus \( \frac{2(1-x_j)}{(1-x_j)^2 + y_j^2} > 1 \).

In Theorem 1 (1), let \( \omega = 1 \), we can obtain that if \( x_j < 1 \) for all \( j = 1, 2, \cdots, n \), \( \rho(T) < 1 \) will hold.

For the ENHSS method, we have the following convergence theorem.

**Theorem 2** Let \( A \) be a non-Hermitian and normal positive definite matrix, \( H = \frac{1}{2}(A + A^*) \) and \( S = \frac{1}{2}(A - A^*) \) be its Hermitian and skew-Hermitian parts. If \( \alpha \) and \( \omega \) are in one of the following ranges, the extrapolated NHSS method (6) is convergent.

\[
\begin{align*}
(1) \quad & \alpha > -\xi\omega \quad \text{and} \quad 0 < \omega < \frac{2(\alpha + \xi\omega)(\alpha + \xi\omega)}{(\alpha + \xi\omega)^2 + (\alpha \omega)^2}; \\
(2) \quad & \alpha < -\xi\omega \quad \text{and} \quad 0 < \omega < \frac{2(\alpha + \xi\omega)(\alpha + \xi\omega)}{(\alpha + \xi\omega)^2 + (\alpha \omega)^2}; \\
(3) \quad & -\xi\omega < \alpha < -\xi\omega \quad \text{and} \quad 0 > \omega > \frac{2(\alpha + \xi\omega)(\alpha + \xi\omega)}{(\alpha + \xi\omega)^2 + (\alpha \omega)^2}, \quad \text{if} \quad -\xi\omega < -\xi\omega.
\end{align*}
\]

**Proof.** From (6), we know

\[
T(\alpha) = (\alpha I + H)^{-1}(\alpha I - S)H^{-1}(-S)
\]

Let \( \lambda \in C \) be an arbitrary eigenvalue of \( T(\alpha) \). \( x \in C^n \) be the corresponding eigenvector satisfying \( \|x\|_2 = 1 \). Then

\[
-(\alpha I + H)^{-1}(\alpha I - S)H^{-1}Sx = \lambda x,
\]

That is

\[
-(\alpha I - S)H^{-1}Sx = \lambda(\alpha I + H)x.
\]

So,

\[
x'(SH^{-1}S - \alpha H^{-1}S)x = \lambda x'(\alpha I + H)x.
\]

Thus
\[ \lambda = \frac{x^* S H^{-1} S x - ax^* H^{-1} S x}{\alpha + x^* H x} \]

Therefore,

\[ \text{Re}(\lambda) = \frac{x^* S H^{-1} S x}{\alpha + x^* H x} = c, \quad \text{Im}(\lambda) = \frac{i a x^* H^{-1} S x}{\alpha + x^* H x} = \frac{i a d}{\alpha + b}. \]

And

\[ \frac{2(1 - \text{Re}(\lambda))}{(1 - \text{Re}(\lambda))^2 + (\text{Im}(\lambda))^2} = \frac{2(\alpha + b - c)(\alpha + b)}{(\alpha + b - c)^2 + \alpha^2 (id)^2}. \]

Obviously, \( \text{Re}(\lambda) < 1 \), that is \( \frac{c}{\alpha + b} < 1 \).

(1) When \( \alpha + b > 0 \), then \( c < \alpha + b \), thus \( \alpha > -b \) and \( \alpha > c - b \). Because of \( c < 0 \), we have \( \alpha > -b \) for all \( x \) satisfied \( \|x\|_2 = 1 \), so \( \alpha > -\zeta_m \). And

\[
\min_{\lambda \in \Lambda(T(\alpha))} \frac{2(1 - \text{Re}(\lambda))}{(1 - \text{Re}(\lambda))^2 + (\text{Im}(\lambda))^2} \\
= \min_{\|x\|_2} \frac{2(\alpha + b - c)(\alpha + b)}{(\alpha + b - c)^2 + \alpha^2 (id)^2} \\
\geq \frac{2(\alpha + \xi_m)(\alpha + \zeta_m)}{(\alpha + \xi_m)^2 + (\alpha \delta_m^2)},
\]

Therefore, from Theorem 1(1), we know in the situation (1), the extrapolated NHSS method is convergent.

(2) When \( \alpha + b < 0 \), then \( c > \alpha + b \), thus \( \alpha < -b \) and \( \alpha < c - b \). Because of \( c < 0 \), we have \( \alpha < c - b \) for all \( x \) satisfied \( \|x\|_2 = 1 \), therefore \( \alpha < -\zeta_m \). And from \( \alpha + b < 0, \alpha + b - c < 0 \), the following formula will hold.

\[
\min_{\lambda \in \Lambda(T(\alpha))} \frac{2(1 - \text{Re}(\lambda))}{(1 - \text{Re}(\lambda))^2 + (\text{Im}(\lambda))^2} \\
= \min_{\|x\|_2} \frac{2(\alpha + b - c)(\alpha + b)}{(\alpha + b - c)^2 + \alpha^2 (id)^2} \\
\geq \frac{2(\alpha + \xi_m)(\alpha + \zeta_m)}{(\alpha + \xi_m)^2 + (\alpha \delta_m^2)},
\]

So, from Theorem 1(1), we have the result (2).

(3) When \( \text{Re}(\lambda) > 1 \), that is \( \frac{c}{\alpha + b} > 1 \), because of \( c < 0 \), we know \( \alpha + b < 0 \) and \( c < \alpha + b \). It is obviously that \( \alpha < -b \) and \( \alpha > c - b \). Thus \( c - b < \alpha < -b \) for all \( x \) satisfied \( \|x\|_2 = 1 \), so \( -\zeta_m < \alpha < -\zeta_m \). And
Therefore, from Theorem 1(2), we know when \( \alpha \) and \( \omega \) are in the range (3), the extrapolated NHSS method is convergent.

Although \( \zeta < \xi_M \) is not necessarily true. So \( -\zeta < -\xi_M \) is not always true. That is, for some specific linear equations, there may be no the third convergence range.

**Theorem 3** Let \( A \) be a non-Hermitian and normal positive definite matrix, \( H = \frac{1}{2}(A + A') \) and \( S = \frac{1}{2}(A - A') \) be its Hermitian and skew-Hermitian parts. Then the iterative method (5) is convergent when \( \alpha > -\zeta_m \) or \( \alpha < -\xi_M \).

**Proof.** From the proof of Theorem 2, we know when \( \alpha > -\zeta_m \) or \( \alpha < -\xi_M \), for every \( j = 1, 2, \cdots, n \),

\[ x_j = \text{Re}(\lambda_j) = \frac{c}{\alpha + b} < 1. \]

So from Corollary 1, we have that the NHSS iterative method is convergent.

Since for the range (3) in Theorem 2, \( \omega \neq 1 \), the NHSS method does not converge under this condition.

Because \( \zeta_m < \xi_M \), the ranges \( \alpha > -\zeta_m \) and \( \alpha < -\xi_M \) are disjoint. Theorem 3 shows that the NHSS iterative method is convergent for two disjoint ranges of \( \alpha \) as above.

**Corollary 2** Let \( A \) be a non-Hermitian and normal positive definite matrix, \( H = \frac{1}{2}(A + A') \) and \( S = \frac{1}{2}(A - A') \) be its Hermitian and skew-Hermitian parts. Then the iterative method (5) is convergent when \( \alpha > -k_{\min} \) or \( \alpha < -u_{\max} \), where

\[
\min_{k \in \Lambda(H)} \{k_1, k_2, \cdots, k_n\}, u_{\max} = \min_{u_i \in \Lambda(H-SH^{-1}S)} \{u_1, u_2, \cdots, u_n\}, \Lambda(\cdot) \text{ is the set of the spectrum of the corresponding matrix}.
\]

**Proof.** (a) Because \( H \) is Hermitian and positive definite matrix, it can be diagonalized. Let all of the eigenvalues of the matrix \( H \) be \( k_1, k_2, \cdots, k_n \), and \( k_{\min} \) be the minimum of them, respectively. Then for any \( x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{C}^n \) and \( \|x\|_2 = 1 \), we have

\[
x^*Hx = k_1x_1^2 + k_2x_2^2 + \cdots + k_nx_n^2 \geq k_{\min}x_1^2 + k_{\min}x_2^2 + \cdots + k_{\min}x_n^2 = k_{\min}.
\]

So \( \zeta_m = \min_{\alpha} \|x^*Hx\|_1 = k_{\min} \). Therefore, from Theorem 3, we know when \( \alpha > -k_{\min} \), the iterative method (5) is convergent.
(b) Because of the positive definiteness of the matrix $H$ and the negative definiteness of the matrix $SH^{-1}S$, we have $H - SH^{-1}S$ is a Hermitian and positive definite matrix. Let all of the eigenvalues of the matrix $H - SH^{-1}S$ be $u_1, u_2, \ldots, u_n$, and $u_{\text{max}}$ be the maximum of them, respectively. Then for any $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$ and $\|x\|_2 = 1$, we have

$$x^*(H - SH^{-1}S)x = u_1x_1^2 + u_2x_2^2 + \cdots + u_nx_n^2 \leq u_{\text{max}}x_1^2 + u_{\text{max}}x_2^2 + \cdots + u_{\text{max}}x_n^2 = u_{\text{max}}.$$ 

So $\xi_m = \max_{H^{-1}S} |x^*(H - SH^{-1}S)x| = u_{\text{max}}$. Therefore, from Theorem 3, we know when $\alpha < -u_{\text{max}}$, the iterative method (5) is convergent.

From Corollary 2, the NHSS iterative method is convergent for $\alpha > 0$. It generalizes Theorem 2.1 in [2], in which, the convergence of the NHSS method is dependent on not only $\alpha$, but also the eigenvalues of the matrices $H$ and $S$ when $\alpha > 0$.

3. Numerical Experiments

To illustrate the effectiveness of the conclusions, we consider the sparse nonsymmetric of linear equation (1) with coefficient matrix

$$A = \text{tridiag}(\eta_k^{(1)}I, T, \mu_k^{(1)}I) \in \mathbb{R}^{n \times n}$$

which is a $N \times N$ block tridiagonal matrix. Here $T = \text{tridiag}(\eta_k^{(2)}, \mu_0^{(0)}, \mu_k^{(2)}) \in \mathbb{R}^{N \times N}$ is a tridiagonal matrix. The right side is $b = Ab_0$, where $b_0 = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$. The entries of the matrix $A \in \mathbb{R}^{n \times n}$ are defined by the formulas

$$\mu^{(0)} = 4(\alpha + \theta\sigma h^2), \quad \mu^{(1)} = -(\alpha + \frac{1}{2}h\beta), \quad \mu^{(2)} = -(\alpha - \frac{1}{2}h\delta),$$

$$\eta_i^{(1)} = -(\alpha + \frac{1}{2}h\beta), \quad \eta_i^{(2)} = -(\alpha + \frac{1}{2}h\delta), \quad 1 \leq i \leq N,$$

where $n = N \times N$, $h = 1/(N + 1), \beta = ih/10, \delta_j = (jh)^2$. In fact this system of linear difference arises from central difference discretization of a second order partial differential equation (for more details see [19]). We consider the example, in which $\alpha = 1, \theta = 10^3, \sigma = 64$.

The spectral radius of the iteration matrix is a good criterion for convergence. Tables 1 and 2 show the spectral radius $\rho(T_{o,o})$ of the extrapolated NHSS method for the ranges (1) and (2) in Theorems 2, respectively.
Table 1. Spectral Radius $\rho(T_{\omega, \alpha})$ for the Range (1)

| $n$ | $-\zeta_m$ | $\alpha_1$ | $\omega_{\text{sup1}}$ | $\omega_1$ | $\rho_1$ |
|-----|------------|------------|----------------|----------|--------|
| 32  | -235.0957  | -230       | 0.7803          | 0.7653   | 0.2347 |
| 40  | -152.3017  | -150       | 0.4478          | 0.4432   | 0.5568 |
| 48  | -106.6302  | -100       | 0.9074          | 0.8563   | 0.1437 |
| 56  | -78.7994   | -78        | 0.1819          | 0.1802   | 0.8198 |
| 64  | -60.5963   | -60        | 0.1388          | 0.1258   | 0.8742 |

Table 2. Spectral Radius $\rho(T_{\omega, \alpha})$ for the Range (2)

| $n$ | $-\xi_M$ | $\alpha_2$ | $\omega_{\text{sup2}}$ | $\omega_2$ | $\rho_2$ |
|-----|----------|------------|----------------|----------|--------|
| 32  | -243.0604| -245       | 0.0767          | 0.0578   | 0.9422 |
| 40  | -160.2789| -165       | 0.2765          | 0.2468   | 0.7532 |
| 48  | -114.6143| -115       | 0.0042          | 0.0036   | 0.9964 |
| 56  | -86.7876  | -88        | 0.0347          | 0.0345   | 0.9655 |
| 64  | -68.5872  | -70        | 0.0451          | 0.0490   | 0.9571 |

Figure 1 depicts the curve of the spectral radius of the NHSS iterative method with the different $\alpha \in (-\zeta_m, +\infty)$ in Theorem 3 when $n = 32$. Figure 2 depicts that of $\alpha \in (-\infty, -\xi_M)$. Figure 3 depicts the curves of the spectral radius of the NHSS method for $\alpha$ in two disjoint ranges in Theorem 3 when $n = 32$.

Figure 1. Curve of the Spectral Radius of NHSS Method with $\alpha \in (-\zeta_m, +\infty)$
Figures 1 shows the spectral radius of the NHSS method is decreasing as \( \alpha \) increasing in the interval \((-\zeta_m, +\infty)\). And at the left endpoint of the interval \((-\zeta_m, +\infty)\), the spectral radius drops very quickly. Figures 2 shows the spectral radius of the NHSS method is increasing as \( \alpha \) increasing in the interval \((-\infty, -\xi_M)\). And at the right endpoint of the interval \((-\infty, -\xi_M)\), the spectral radius rises very quickly. From Figure 3, we can see the NHSS method is convergent for almost all \( \alpha \in \mathbb{R} \) when \( A \) is a non-Hermitian and normal positive definite matrix.

**Figure 2** Curve of the Spectral Radius of NHSS Method with \( \alpha \in (-\infty, -\xi_M) \).

**Figure 3.** Curve of the Spectral Radius of NHSS Method with \( \alpha \) in Disjoint Intervals

### 4. Conclusion

In the paper, the extrapolated NHSS method has been presented and the convergence performance of it has been analyzed. The convergence result shows that the iterative method is convergent when the parameters are in the certain ranges. As specially, the convergence theorems for the NHSS method have been given, which generalizes the related conclusions in [2].

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