Towards a functorial description of quantum relative entropy

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Abstract. A Bayesian functorial characterization of the classical relative entropy (KL divergence) of finite probabilities was recently obtained by Baez and Fritz. This was then generalized to standard Borel spaces by Gagné and Panangaden. Here, we provide preliminary calculations suggesting that the finite-dimensional quantum (Umegaki) relative entropy might be characterized in a similar way. Namely, we explicitly prove that it defines an affine functor in the special case where the relative entropy is finite. A recent non-commutative disintegration theorem provides a key ingredient in this proof.

Keywords: Bayesian inversion · disintegration · optimal hypothesis.

1 Introduction and outline

In 2014, Baez and Fritz provided a categorical Bayesian characterization of the relative entropy of finite probability measures using a category of hypotheses [1]. This was then generalized to standard Borel spaces by Gagné and Panangaden in 2018 [5]. An immediate question remains as to whether or not the quantum (Umegaki) relative entropy [12] has a similar characterization.\footnote{The ordinary Shannon and von Neumann entropies were characterized in [2] and [7], respectively, in a similar categorical setting.} The purpose of the present work is to begin filling this gap by using the recently proved non-commutative disintegration theorem [9].

The original motivation of Baez and Fritz came from Petz’ characterization of the quantum relative entropy [11], which used a quantum analogue of hypotheses known as conditional expectations. Although Petz’ characterization had some minor flaws, which were noticed in [1], we believe Petz’ overall idea is correct when formulated on an appropriate category of non-commutative probability spaces and non-commutative hypotheses. In this article, we show how the Umegaki relative entropy defines an affine functor that vanishes on the sub-category of non-commutative optimal hypotheses for faithful states. The chain rule for quantum conditional entropy is a consequence of functoriality. The non-faithful case will be addressed in future work, where we hope to provide a characterization of the quantum relative entropy as an affine functor.
2 The categories of hypotheses and optimal hypotheses

In this section, we introduce non-commutative analogues of the categories from [1]. All \( C^* \)-algebras here are finite-dimensional and unital. All \( * \)-homomorphisms are unital unless stated otherwise. \( \mathcal{M}_n \) denotes the algebra of \( n \times n \) matrices. If \( V : \mathbb{C}^n \to \mathbb{C}^n \) is a linear map, \( \text{Ad}_V : \mathcal{M}_n \to \mathcal{M}_n \) denotes the linear map sending \( A \) to \( VAV^\dagger \), where \( V^\dagger \) is the adjoint (conjugate transpose) of \( V \). Linear maps between algebras are written with squiggly arrows \( \rightsquigarrow \), while \( \ast \)-homomorphisms are written as straight arrows \( \to \). The acronym CPU stands for “completely positive unital.” If \( A \) and \( B \) are matrix algebras, then \( \text{tr}_A : A \otimes B \rightsquigarrow B \) denotes the \textit{partial trace over} \( A \) and is the unique linear map determined by \( \text{tr}_A(A \otimes B) = \text{tr}(A)B \) for \( A \in A \) and \( B \in B \). If \( \omega \) is a state on \( A \), its quantum entropy is denoted by \( S(\omega) \) (cf. [7, Definition 2.20]).

\[ \text{Definition 1.} \text{ Let } \text{NCFStat} \text{ be the category of non-commutative probability spaces, whose objects are pairs } (A, \omega), \text{ with } A \text{ a } C^*-\text{algebra and } \omega \text{ a state on } A. \text{ A morphism } (B, \xi) \to (A, \omega) \text{ is a pair } (F, Q) \text{ with } F : B \to A \text{ a } \ast\text{-homomorphism and } Q : A \rightsquigarrow B \text{ a CPU map (called a hypothesis), such that } \omega \circ F = \xi \text{ and } Q \circ F = \text{id}_B. \]

The composition rule in \( \text{NCFStat} \) is given by

\[ (C, \zeta) \xrightarrow{\langle G, R \rangle} (B, \xi) \xrightarrow{(F, Q)} (A, \omega) \implies (C, \zeta) \xrightarrow{(F \circ G, R \circ Q)} (A, \omega). \]

Let \( \text{NCFP} \) be the subcategory of \( \text{NCFStat} \) with the same objects but whose morphisms are pairs \( (F, Q) \) as above and \( Q \) is an optimal hypothesis, i.e. \( \xi \circ Q = \omega \).

The subcategories of \( \text{NCFStat}^{\text{op}} \) and \( \text{NCFP}^{\text{op}} \) consisting of commutative \( C^* \)-algebras are equivalent to the categories \( \text{FinStat} \) and \( \text{FP} \) from [1] by stochastic Gelfand duality (cf. [6, Sections 2.5 and 2.6], [4], and [9, Corollary 3.23]).

\[ \text{Notation 1.} \text{ On occasion, the notation } A, B, \text{ and } C \text{ will be used to mean } \]

\[ A := \bigoplus_{x \in X} \mathcal{M}_{m_x}, \quad B := \bigoplus_{y \in Y} \mathcal{M}_{n_y}, \quad \text{and} \quad C := \bigoplus_{z \in Z} \mathcal{M}_{a_z}, \]

where \( X, Y, Z \) are finite sets, often taken to be ordered sets \( X = \{1, \ldots, s\}, \)
\( Y = \{1, \ldots, t\}, \) \( Z = \{1, \ldots, u\} \) for convenience (cf. [9, Section 5] and/or [7, Example 2.2]). Note that every element of \( A \) (and analogously for \( B \) and \( C \) is of the form \( A = \bigoplus_{x \in X} A_x \), with \( A_x \in \mathcal{M}_{m_x} \). Furthermore, \( \omega, \xi, \text{ and } \zeta \) will refer to states on \( A, B, \) and \( C \), respectively, with decompositions of the form

\[ \omega = \sum_{x \in X} p_x \text{tr}(\rho_x \cdot), \quad \xi = \sum_{y \in Y} q_y \text{tr}(\sigma_y \cdot), \quad \text{and} \quad \zeta = \sum_{z \in Z} r_z \text{tr}(\tau_z \cdot). \]

If \( Q : A \rightsquigarrow B \) is a linear map, its \( xy \) component \( Q_{yx} \) is the linear map obtained from the composite \( \mathcal{M}_{m_x} \hookrightarrow A \xrightarrow{Q} B \to \mathcal{M}_{n_y} \), where the first and last maps are the (non-unital) inclusion and projection, respectively.
Definition 2. Let $A$ and $B$ be as in Notation 1. A morphism $(B, \xi) \xrightarrow{(F, Q)} (A, \omega)$ in NCFinStat is said to be in standard form iff there exist non-negative integers $c_{yx}^F$ such that (cf. [7, Lemma 2.11] and [3, Theorem 5.6])

$$F(B) = \bigoplus_{x \in X} \bigoplus_{y \in Y} \left(1_{c_{x}^F} \otimes B_y \right) \equiv \bigoplus_{x \in X} \text{diag} \left(1_{c_{x}^F} \otimes B_1, \ldots, 1_{c_{x}^F} \otimes B_t \right) \quad \forall B \in B,$$

which is a direct sum of block diagonal matrices. The number $c_{yx}^F$ is called the multiplicity of $M_{n_y}$ in $M_{m_x}$ associated to $F$. In this case, each $A_x \in M_{m_x}$ will occasionally be decomposed as $A_x = \sum_{y,y' \in Y} E_{y'y}^{(t)} \otimes A_{x,yy'}$, where $\{E_{y'y}^{(t)}\}$ denote the matrix units of $M_t$ and $A_{x,yy'}$ is a $(c_{yx}^F, c_{yx}^F)$ matrix. If $F$ is in standard form and if $\omega$ and $\xi$ are states on $A$ and $B$ (as in Notation 1) such that $\xi = \omega \circ F$, then (cf. [7, Lemma 2.11] and [9, Proposition 5.67])

$$q_y \sigma_y = \sum_{x \in X} p_{xt} \text{tr}_{M_{c_{xy}^F}} (\rho_{xy}). \quad (2.1)$$

The standard form of a morphism will be useful later for proving functoriality of relative entropy, and it will allow us to formulate expressions more explicitly in terms of matrices.

Lemma 1. Given a morphism $(B, \xi) \xrightarrow{(F, Q)} (A, \omega)$ in NCFinStat, with $A$ and $B$ be as in Notation 1, there exists a unitary $U \in A$ such that $(B, \xi) \xrightarrow{(\text{Ad}_{U^1} \circ F, Q \circ \text{Ad}_{U})} (A, \omega \circ \text{Ad}_{U})$ is a morphism in NCFinStat that is in standard form. Furthermore, if $(F, Q)$ is in NCFP, then $(\text{Ad}_{U^1} \circ F, Q \circ \text{Ad}_{U})$ is also in NCFP.

Proof. First, $(\text{Ad}_{U^1} \circ F, Q \circ \text{Ad}_{U})$ is in NCFinStat for any unitary $U$ because

$$(\omega \circ \text{Ad}_{U}) \circ (\text{Ad}_{U^1} \circ F) = \xi \quad \text{and} \quad (Q \circ \text{Ad}_{U}) \circ (\text{Ad}_{U^1} \circ F) = \text{id}_B,$$

so that the two required conditions hold. Second, the fact that a unitary $U$ exists such that $F$ is in the form in Definition 2 is a standard fact regarding (unital) $*$-homomorphisms between direct sums of matrix algebras [3, Theorem 5.6]. Finally, if $(F, Q)$ is in NCFP, which means $\xi \circ Q = \omega$, then $(\text{Ad}_{U^1} \circ F, Q \circ \text{Ad}_{U})$ is also in NCFP because $\xi \circ (Q \circ \text{Ad}_{U}) = (\xi \circ Q) \circ \text{Ad}_{U} = \omega \circ \text{Ad}_{U}$.

Although the composite of two morphisms in standard form is not necessarily in standard form, a permutation can always be applied to obtain one. Furthermore, a pair of composable morphisms in NCFinStat can also be simultaneously rectified. This is the content of the following lemmas.

Lemma 2. Given a composable pair $(C, \zeta) \xrightarrow{(G, R)} (B, \xi) \xrightarrow{(F, Q)} (A, \omega)$ in NCFinStat, there exist unitaries $U \in A$ and $V \in B$ such that

$$(C, \zeta) \xrightarrow{(\text{Ad}_{U^1} \circ G, R \circ \text{Ad}_{U})} (B, \xi \circ \text{Ad}_{V}) \xrightarrow{(\text{Ad}_{U^1} \circ F \circ \text{Ad}_{V}, \text{Ad}_{U^1} \circ Q \circ \text{Ad}_{V})} (A, \omega \circ \text{Ad}_{U})$$

is a pair of composable morphisms in NCFinStat that are both in standard form.
Proof. By Lemma 1, there exists a unitary $V \in \mathcal{B}$ such that

$$(\mathcal{C}, \zeta) \xrightarrow{(\text{Ad}_{V^1} \circ G, R \circ \text{Ad}_{V^1})} (\mathcal{B}, \xi \circ \text{Ad}_{V^1}) \xrightarrow{(\text{F} \circ \text{Ad}_{V^1}, \text{Ad}_{V^1} \circ Q)} (\mathcal{A}, \omega)$$

is a composable pair of morphisms in $\text{NCFinStat}$ with the left morphism in standard form. The right morphism is indeed in $\text{NCFinStat}$ because

$$\omega \circ (F \circ \text{Ad}_{V^1}) = (\omega \circ F) \circ \text{Ad}_{V} = \xi \circ \text{Ad}_{V} \quad \text{and}$$

$$(\text{Ad}_{V^1} \circ Q) \circ (F \circ \text{Ad}_{V^1}) = \text{Ad}_{V^1} \circ (Q \circ F) \circ \text{Ad}_{V} = \text{Ad}_{V^1} \circ \text{Ad}_{V} = \text{id}_\mathcal{G}.$$  

Then, applying Lemma 1 again, but to the new morphism on the right, gives a unitary $U$ satisfying the conditions claimed.

**Lemma 3.** Given a composable pair $(\mathcal{C}, \zeta) \xrightarrow{(G, R)} (\mathcal{B}, \xi) \xrightarrow{(F, Q)} (\mathcal{A}, \omega)$ in $\text{NCFinStat}$, each in standard form, there exist permutation matrices $P_x \in \mathcal{M}_{m_x}$ such that

$$(\mathcal{C}, \zeta) \xrightarrow{(\text{Ad}_{P^1} \circ \text{F}, \text{R} \circ \text{Ad}_{P^1})} (\mathcal{A}, \omega \circ \text{Ad}_{P})$$

is also in standard form, where $P := \bigoplus_{x \in X} P_x$ and the multiplicities $c_{G,F}^{G,F}$ of $\text{Ad}_{P^1} \circ G \circ F$ are given by $c_{G,F}^{G,F} = \sum_{y \in Y} c_{x^{G,F}}^{G,F} y_x.$

**Proof.** The composite $F \circ G$ is given by

$$F(G(C)) = F \left( \bigoplus_{y \in Y} \bigoplus_{z \in Z} \left( \mathcal{I}_{c_{x^{G,F}}^{G,F}} \otimes C_z \right) \right)_{B_y} = \bigoplus_{x \in X} \bigoplus_{y \in Y} \left( \mathcal{I}_{c_{x^{G,F}}^{G,F}} \otimes \left( \mathcal{I}_{c_{x^{G,F}}^{G,F}} \otimes C_z \right) \right)_{A_x}.$$  

The matrix $A_x$ takes the more explicit form (with zeros in unfilled entries)

$$A_x = \text{diag} \left( \begin{array}{c} \mathcal{I}_{c_{x^{G,F}}^{G,F}} \otimes C_{1} \\ \vdots \\ \mathcal{I}_{c_{x^{G,F}}^{G,F}} \otimes C_{n} \end{array} \right).$$

From this, one sees that the number of times $C_z$ appears on the diagonal is $\sum_{y \in Y} c_{x^{G,F}}^{G,F} y_x.$ However, the positions of $C_z$ are not all next to each other. Hence, a permutation matrix $P_x$ is needed to put them into standard form.

**Notation 2** Given a composable pair $(\mathcal{C}, \zeta) \xrightarrow{(G, R)} (\mathcal{B}, \xi) \xrightarrow{(F, Q)} (\mathcal{A}, \omega)$ in standard form as in Lemma 3, the states $\zeta \circ R$ and $\xi \circ Q$ will be decomposed as

$$\zeta \circ R = \sum_{y \in Y} q_y^R \text{tr} \left( \sigma_y^R \cdot \right) \quad \text{and} \quad \xi \circ Q = \sum_{x \in X} p_x^Q \text{tr} \left( \rho_x^Q \cdot \right).$$

**Lemma 4.** Given a morphism $(F, Q)$ in standard form as in Notation 2 such that all states are faithful, there exist strictly positive matrices $\alpha_{yx} \in \mathcal{M}_{c_{x^{G,F}}^{G,F}}$ for all $x \in X$ and $y \in Y$ such that

$$\text{tr} \left( \sum_{x \in X} \alpha_{yx} x \right) = 1 \quad \forall \ y \in Y, \quad p_x^Q \rho_x^Q = \bigoplus_{y \in Y} (\alpha_{yx} \otimes q_y \sigma_y) \quad \forall \ x \in X,$$  

and

$$Q_{yx}(A_x) = \text{tr}_{\mathcal{M}_{c_{x^{G,F}}^{G,F}}} \left( (\alpha_{yx} \otimes 1_n_y) A_x_{x:y} \right) \quad \forall \ y \in Y, \ A_x \in \mathcal{M}_{m_x}, \ x \in X.$$
Proof. Because $Q$ and $R$ are disintegrations of $(F, \xi \circ Q, \xi)$ and $(G, \zeta \circ R, \zeta)$, respectively, the claim follows from the non-commutative disintegration theorem [9, Theorem 5.67] and the fact that $F$ is an injective $*$-homomorphism. The $\alpha_{yx}$ matrices are strictly positive by the faithful assumption.

If $(C, \zeta) \xrightarrow{(G,R)} (B, \xi) \xrightarrow{(F,Q)} (A, \omega)$ is composite, then

$$\zeta \circ R \circ Q = \sum_{x \in X} \text{tr} \left( \bigoplus_{y \in Y} \alpha_{yx} \otimes q^R_y \sigma^T_y \right) \cdot$$

(2.2)

3 The relative entropy as a functor

Definition 3. Set $\text{RE} : \text{NCFFinStat} \to \mathbb{B}(-\infty, \infty]$ to be the assignment that sends a morphism $(B, \xi) \xrightarrow{(F,Q)} (A, \omega)$ to $S(\omega \| \xi \circ Q)$ (the assignment is trivial on objects). Here, $\mathbb{B}M$ is the one object category associated to any monoid $M$, $S(\cdot \| \cdot)$ is the relative entropy of two states on the same $C^*$-algebra, which is defined on an ordered pair of states $(\omega, \omega')$, with $\omega \leq \omega'$ (meaning $\omega(a^*a) = 0$ implies $\omega(a^*a) = 0$), on $A = \bigoplus_{x \in X} M_{m_x}$ by

$$S(\omega \| \omega') := \text{tr} \left( \bigoplus_{x \in X} p_x \rho_x \left( \log(p_x \rho_x) - \log(p_x' \rho_x') \right) \right),$$

where $0 \log 0 := 0$ by convention. If $\omega \not\preceq \omega'$, then $S(\omega \| \omega') := \infty$.

Lemma 5. Using the notation from Definition 3, the following facts hold.

(a) $\text{RE}$ factors through $\mathbb{B}[0, \infty]$.
(b) $\text{RE}$ vanishes on the subcategory $\text{NCFP}$.
(c) $\text{RE}$ is invariant with respect to changing a morphism to standard form, i.e., in terms of the notation introduced in Lemma 1,

$$\text{RE} \left( (B, \xi) \xrightarrow{(F,Q)} (A, \omega) \right) = \text{RE} \left( (B, \xi) \xrightarrow{(\text{Ad}_U \circ F, Q \circ \text{Ad}_U)} (A, \omega \circ \text{Ad}_U) \right).$$

Proof. Left as an exercise.

Proposition 1. For a composite pair $(C, \zeta) \xrightarrow{(G,R)} (B, \xi) \xrightarrow{(F,Q)} (A, \omega)$ in $\text{NCFFinStat}$ (with all states and CPU maps faithful),

$$S(\omega \| \zeta \circ R \circ Q) = S(\omega \| \xi \circ Q) + S(\xi \| \zeta \circ R) = \text{RE} \left( (F \circ G, R \circ Q) \right) = \text{RE} \left( (G, R) \right) + \text{RE} \left( (F, Q) \right).$$

The morphisms of $\mathbb{B}M$ from that single object to itself equals the set $M$ and the composition is the monoid multiplication. Here, the monoid is $(-\infty, \infty]$ under addition (with the convention that $a + \infty = \infty$ for all $a \in (-\infty, \infty]$.

Faithfulness guarantees the finiteness of all expressions. More generally, our proof works if the appropriate absolute continuity conditions hold. Also, note that the “conditional expectation property” in [11] is a special case of functoriality applied to a composite pair of morphisms of the form $(C, \text{id}_C) \xrightarrow{(F,R)} (B, \xi) \xrightarrow{(F,Q)} (A, \omega)$, where $! : C \to B$ is the unique unital linear map. Indeed, Petz’ $A, B, E, \omega|_{A}, \varphi|_{A}$, and $\varphi$, are our $B, A, Q, \xi, R$, and $R \circ Q$, respectively ($\omega$ is the same).
Proof. By Lemma 5, it suffices to assume \((F, Q)\) and \((G, R)\) are in standard form. To prove the claim, we expand each term. First,\(^4\)

\[
S(\omega \parallel \xi \circ Q) \overset{\text{Lem} \; 4}{=} -S(\omega) - \sum_{x \in X} \sum_{y \in Y} \text{tr} \left( p_x \rho_x \log \left( \bigoplus_{y \in Y} \alpha_{yx} \otimes q_y \sigma_y \right) \right)
\]

\[
= -S(\omega) - \sum_{x \in X} \sum_{y \in Y} \text{tr} \left( p_x \rho_{x:y} \left( \log(\alpha_{yx}) \otimes \mathbb{1}_{n_y} \right) \right) \tag{3.1}
\]

\[
- \sum_{x \in X} \sum_{y \in Y} \text{tr} \left( p_x \text{tr}_{M_c F \xi} (\rho_{x:y}) \log(q_y \sigma_y) \right).
\]

The last equality follows from the properties of the trace, partial trace, and logarithms of tensor products. By similar arguments,

\[
S(\xi \parallel \zeta \circ R) \overset{(2.1)}{=} \sum_{x \in X} \sum_{y \in Y} \text{tr} \left( p_x \text{tr}_{M_c F \xi} (\rho_{x:y}) \log(q_y \sigma_y) \right)
\]

\[
- \sum_{x \in X} \sum_{y \in Y} \text{tr} \left( p_x \text{tr}_{M_c F \xi} (\rho_{x:y}) \log(q_y \sigma_y) \right) \tag{3.2}
\]

and

\[
S(\omega \parallel \zeta \circ R \circ Q) \overset{(2.2)}{=} -S(\omega) - \sum_{x \in X} \sum_{y \in Y} \text{tr} \left( p_x \rho_{x:y} \left( \log(\alpha_{yx}) \otimes \mathbb{1}_{n_y} \right) \right)
\]

\[
- \sum_{x \in X} \sum_{y \in Y} \text{tr} \left( p_x \text{tr}_{M_c F \xi} (\rho_{x:y}) \log(q_y \sigma_y) \right) \tag{3.3}
\]

Hence, \((3.1) + (3.2) = (3.3)\), which proves the claim.

Example 1. The usual chain rule for the quantum conditional entropy is a special case of Proposition 1. To see this, set \(A := M_{d_A}, B := M_{d_B}, C := M_{d_C}\) with \(d_A, d_B, d_C \in \mathbb{N}\). Given a density matrix \(\rho_{ABC}\) on \(A \otimes B \otimes C\), we implement subscripts to denote the associated density matrix after tracing out a subsystem.

The chain rule for the conditional entropy states

\[
H(AB \mid C) = H(A \mid BC) + H(B \mid C), \tag{3.4}
\]

where \((\text{for example})\)

\[
H(B \mid C) := \text{tr}(\rho_{BC} \log \rho_{BC}) - \text{tr}(\rho_C \log \rho_C)
\]

is the quantum conditional entropy of \(\rho_{BC}\) given \(\rho_C\). One can show that

\[
\text{RE}((F \circ G, R \circ Q)) = H(AB \mid C) + \log(d_A) + \log(d_B),
\]

\[
\text{RE}((G, R)) = H(B \mid C) + \log(d_B), \quad \text{and} \quad \text{RE}((F, Q)) = H(A \mid BC) + \log(d_A)
\]

\(^4\) Equation (3.1) is a generalization of Equation (3.2) in \([1]\), which plays a crucial role in proving many claims. We will also use it to prove affinity of \(\text{RE}\).
Proposition 1 does not fully prove functoriality of RE. One still needs to check functoriality in case one of the terms is infinite (e.g., if $S(\omega \parallel \zeta \circ R \circ Q) = \infty$, then at least one of $S(\xi \parallel \zeta \circ R)$ or $S(\omega \parallel \xi \circ Q)$ must be infinite, and conversely). This will be addressed in future work. In the remainder, we prove affinity of RE.

**Definition 4.** Given $\lambda \in [0, 1]$, set $\overline{\lambda} := 1 - \lambda$. The $\lambda$-weighted convex sum $\lambda(A, \omega) \oplus \overline{\lambda}(\overline{A}, \overline{\omega})$ of objects $(A, \omega)$ and $(\overline{A}, \overline{\omega})$ in NCFinStat is given by the pair $(\lambda(A) \oplus \overline{\lambda}(\overline{A}), \lambda(\omega) \oplus \overline{\lambda}(\overline{\omega}))$ whenever $A \in A, \overline{A} \in \overline{A}$. The convex sum $\lambda(F, Q) \oplus \overline{\lambda}(\overline{F}, \overline{Q})$ of $(B, \xi)$ and $(\overline{B}, \overline{\xi})$ is the morphism $(F \oplus \overline{F}, Q \oplus \overline{Q})$. A functor NCFinStat $\xrightarrow{\mathcal{L}} \mathbb{B}[0, \infty]$ is **affine** if $\mathcal{L}(\lambda(F, Q) \oplus \overline{\lambda}(\overline{F}, \overline{Q})) = \lambda \mathcal{L}(F, Q) + \overline{\lambda} \mathcal{L}(\overline{F}, \overline{Q})$ for all pairs of morphisms in NCFinStat and $\lambda \in [0, 1]$.

**Proposition 2.** Let $(B, \xi) \xrightarrow{(F, Q)} (A, \omega)$ and $(\overline{B}, \overline{\xi}) \xrightarrow{\overline{(F, Q)}} (\overline{A}, \overline{\omega})$ be two morphisms for which $\text{RE}(F, Q)$ and $\text{RE}(\overline{F}, \overline{Q})$ are finite. Then $\text{RE}(\lambda(F, Q) \oplus \overline{\lambda}(\overline{F}, \overline{Q})) = \lambda \text{RE}(F, Q) + \overline{\lambda} \text{RE}(\overline{F}, \overline{Q})$.

**Proof.** When $\lambda \in \{0, 1\}$, the claim follows from the convention $0 \log 0 = 0$. For $\lambda \in (0, 1)$, temporarily set $\mu := \text{RE}(\lambda(F, Q) \oplus \overline{\lambda}(\overline{F}, \overline{Q}))$. Then

$$
\mu \overset{\text{(3.1)}}{=} \sum_{x, \overline{\sigma} \in \mathcal{X}} \text{tr} \left( \lambda p_x \rho_x \log(\lambda p_x \rho_x) + \sum_{\overline{\tau} \in \mathcal{X}} \text{tr} \left( \overline{\lambda} \overline{p}_{\overline{\tau}} \overline{\rho}_{\overline{\tau}} \log(\overline{\lambda} \overline{p}_{\overline{\tau}} \overline{\rho}_{\overline{\tau}}) \right) \right) \\
- \sum_{x, y} \left[ \text{tr} \left( \lambda p_x \rho_{x;yy} \log(\alpha_{yx} \otimes I_{A_Y}) \right) + \text{tr} \left( \lambda p_x \text{tr}_{M_{\overline{F}, \overline{\tau}}} (\rho_{x;yy} \log(\lambda q_{yy} \sigma_y)) \right) \right] \\
- \sum_{x, \overline{\tau}, \overline{\sigma}} \left[ \text{tr} \left( \overline{\lambda} \overline{p}_{\overline{\tau}} \overline{\rho}_{\overline{\tau}y} \log(\overline{\alpha}_{\overline{\tau};yy} \otimes I_{\overline{A}_Y}) \right) + \text{tr} \left( \overline{\lambda} \overline{p}_{\overline{\tau}} \text{tr}_{M_{\overline{\tau}}}(\overline{\rho}_{\overline{\tau}y} \log(\overline{\lambda} \overline{p}_{\overline{\tau}} \overline{\rho}_{\overline{\tau}y})) \right) \right],
$$

where we have used bars to denote analogous expressions for the algebras, morphisms, and states with bars over them. From this, the property $\log(ab) = \log(a) + \log(b)$ of logarithms is used to complete the proof.

In summary, we have taken the first steps towards illustrating that the quantum relative entropy may have a functorial description along similar lines to those of the classical one in [1]. Using the recent non-commutative disintegration theorem [9], we have proved parts of affinity and functoriality of the relative entropy. The importance of functoriality comes from the connection between the
quantum relative entropy and the reversibility of morphisms [10, Theorem 4]. For example, optimal hypotheses are Bayesian inverses [8, Theorem 8.3], which admit stronger compositional properties [8, Propositions 7.18 and 7.21] than alternative recovery maps in quantum information theory [13, Section 4]. In future work, we hope to prove functoriality (without any faithfulness assumptions), continuity, and a complete characterization.

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5 One must assume faithfulness for some of the calculations in [13, Section 4]. The compositional properties in [8, Proposition 7.21], however, need no such assumptions.