Pure State Tomography with Fourier Transformation

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Extracting information from quantum devices has long been a crucial problem in the field of quantum mechanics. By performing elaborate measurements, quantum state tomography, an important and fundamental tool in quantum science and technology, can be used to determine unknown quantum states completely. In this study, methods to determine multi-qubit pure quantum states uniquely and directly are explored. Two adaptive protocols are proposed, with their respective quantum circuits. Herein, two or three observables are sufficient, while the number of measurement outcomes are either the same or fewer than those in existing methods. Additionally, experiments on the IBM 5-qubit quantum computer, as well as numerical investigations, demonstrate the feasibility of the proposed protocols.

1. Introduction

The superposition and entanglement properties of quantum states account for the various elegant quantum algorithms[1–5] and efficient communication protocols in quantum information processing.[6–10] A quantum bit resembles an arbitrary state of a spinning coin. A qubit (pure state) is defined as \( a|0\rangle + b|1\rangle \), where \( a \) and \( b \) are complex numbers, and \( |a|^2 + |b|^2 = 1 \). When a quantum computer with an initial state \( |0\rangle^\otimes N \) is operated, we obtain an \( N \)-qubit pure quantum state, if unaffected by noise. All of its possible states can be defined with a superposed mathematical form, given by \( \sum_{k=1}^{2^N} a_k |k\rangle \), where \( \{a_k\} \) are complex numbers. The readout of the coefficients \( \{a_k\} \) is important in several applications. This function is a benchmark in verifying the performance of quantum machines, producing desirable quantum states.[11] For example, fidelity is calculated to confirm whether the state currently being transported is close to the initial state, for certain quantum communication protocols.[12,13] Moreover, the entropy and the entanglement measure of the states can be directly computed, provided the states are known.[14–17]

Quantum state tomography aims at obtaining an unknown quantum state by measuring a group of identical quantum states.[18–21] First, a few applicable observables are designed and implemented (or physical quantity measurement settings are designed) to measure the quantum states. Then, the frequency of outcomes in each measurement is recorded to compute the probability distributions of measurements, or the expectation values of the observables. With these data, different reconstruction methods can be used to estimate the unknown quantum states. Certain important indicators relevant to the time and cost of the process exist, such as the number of observables, complexity of the reconstruction algorithms, and number of measurement outcomes.

Pursuing a minimal number of observables has a long history. With the inception of quantum mechanics to define quantum states, the problem of unique determination has been generated. Pauli examined whether a wave function can be determined using position and momentum distributions.[22] However, this is impossible as identical probability distributions may correspond to different wave functions.[23] Furthermore, Asher Peres considered a finite version, as described in his book.[24] A wave function is analogous to a pure \( d \)-dimensional quantum state (qudit). The position and momentum observables are analogous to the two orthonormal bases \( B_0 \) and \( B_1 \), which can be transformed by performing Fourier transformation. Peres conjectured that the two orthogonal bases mentioned above would suffice, except in the case of an ambiguous set. Flammia et al., proved that an ambiguous set cannot be a measure zero set.[25] According to some of the previous studies,[26–32] four orthonormal bases are sufficient to uniquely determine an unknown pure qudit. Although the complex mathematical expressions of the projected basis states[33,34] hinder the physical implementation, a constant number of observables to uniquely determine the pure quantum states is still attractive, especially in an exponential dimension \( d = 2^N \), in a qubit system. Decreasing the number of observables is beneficial in physical implementation. The changes in different physical observables are caused by the changes in the measurement setups, which could result in unwanted alterations to the data, prolonged measurements, and impaired noise assumptions.[35] Therefore, it is preferable to carry out the physical implementation with as few observables as possible.

The complexity of reconstruction algorithm affects runtime to process estimation of outputs of certain quantum device, for example, the 10-qubit experiment in a superconducting platform.[14] Instead of performing a complex computation using the entire data to determine the perfectly matched one, a novel concept of direct reconstruction is proposed herein. Sequential weak and strong measurements are enhanced to directly determine the value of each density matrix for the mixed quantum states.[35–39]
The number of measurement outcomes is important as it affects the time consumed to obtain all experimental data. Although using as few measurement outcomes as possible is preferable, a lower limit exists, because the number of outcomes strongly influences the ability of the protocol to determine a pure qudit. It has been proved that there should be at least 4d − 5 (5d − 7) outcomes to uniquely determine a pure qudit among the pure (mixed) quantum states.\textsuperscript{[10,40]} With the adaptive strategy, a measure zero set from all the pure states can be neglected, and thus, only 3d − 2 outcomes are required.\textsuperscript{[25,41,42]} When a qudit is measured using an observable, d outcomes may appear if there are no ancilla qubits, with each outcome corresponding to a projected eigenstate. The different eigenstates of an observable should be orthogonal. This constraint makes the observable design more complicated. Goyeneche et al., designed five adaptive orthogonal bases corresponding to 3d eigenstates, to uniquely determine an unknown pure qudit state.\textsuperscript{[43]} Zambrano et al., constructed three orthogonal bases, where 2d eigenstates were used to determine 2d−1 candidates and the rest of the projected eigenstates were used to find out an estimation through a likelihood function.\textsuperscript{[44]} Accordingly, the following question needs to be addressed: Is it possible to use a few precise measurements to uniquely and directly determine an unknown pure quantum state with fewer number of observables?

In this study, we added an auxiliary qubit and further employed the adaptive strategy to address this challenge. Herein, two protocols are proposed to uniquely and directly determine finite-dimensional pure quantum states with two or three observables. The measurements projected onto the orthogonal basis \( B_0 \) are used to determine the amplitudes. One or two observables connected via a partial Fourier transform are constructed to determine the phases, which can additionally overcome the frequent changes in the measurement setups. Besides, the number of measurement outcomes for protocol 2 is increased to 80% of the outcomes for the protocols proposed by Goyeneche et al. Moreover, we present the circuit implementation of N-qubit pure states. Two Fourier transformations between the auxiliary qubit and the Nth qubit are required, in addition to a global shift operation. Furthermore, details of the numerical experiments on mixed quantum states and the experimental demonstration on IBM Quantum Experience are provided, thereby proving the feasibility of both the proposed protocols.

2. Main Result

An N-qubit pure state \( |\psi\rangle \) in \( d = 2^N \)-dimensional Hilbert space \( \mathcal{H}_d \) is given by,

\[
|\psi\rangle = \sum_{k=0}^{d-1} a_k e^{i\theta_k} |k\rangle
\]

where \( a_k \) and \( \theta_k \) are the amplitude and phase of \( |\psi\rangle \), respectively, with \( a_k \geq 0 \) and \( \theta_k \in [-\pi, \pi] \).

The canonical basis \( B_0 = \{|0\rangle, \ldots, |d-1\rangle\} \) is sufficient to determine the amplitudes, \( \{a_k\} \). Consider that the measurements are carried out repeatedly \( M \) times. \( n_k \) is recorded as the frequency of the state, collapsed into \( |k\rangle \). Based on the Born rule, when \( M \) is sufficiently large, probability \( P_k = |\langle k|\psi\rangle|^2 = n_k/M \approx a_k^2 \).

The Fourier transformation circuit of \( B_0 \) is direct and has many applications such as Shor’s algorithm\textsuperscript{[3]} and HHL algorithm.\textsuperscript{[45]} But it cannot uniquely and directly determine the phases \( \theta_k \).

Therefore, we introduce an auxiliary qubit. A new orthogonal basis \( C_1 \), or the two orthogonal bases \( D_1 \) and \( D_2 \) can uniquely and directly determine the phases. The coding rule is presented below. The basis states of the compound system \( \mathcal{H}_d \otimes \mathcal{H}_d \) are \(|0\rangle|k\rangle, |1\rangle|k\rangle : k = 0 \ldots, d - 1 \). The states \(|j\rangle|k\rangle \) are encoded as \(|dj + k\rangle \) of \( \mathcal{H}_{2d} \), where \( j \in \{0, 1\}, k \in \{0, \ldots, d - 1\} \).

Protocol 1. A new projective measurement onto the orthonormal basis \( C_1 \) is sufficient to determine the phases uniquely and directly.

\[
C_1 = \left\{ |k_1\rangle|j\rangle, |k_2\rangle|j\rangle, |k_3\rangle|j\rangle, |k_4\rangle|j\rangle : 0 \leq k \leq \left\lfloor \frac{d - 2}{2} \right\rfloor \right\}
\]

Figure 1 illustrates the transformation from \(|0\rangle, |1\rangle \otimes B_0 \) into \( C_1 \). Generally, \(|k_1\rangle = |2k\rangle \), \(|k_2\rangle = |2k + 1\rangle \), \(|k_3\rangle = |2k + 1 + d\rangle \), and \(|k_4\rangle = |2k + 2 + d\rangle \). The symbol \( \otimes \) represents the modulo \( d \) operation. The subscript \( j \) is used to denote the FT. The states can be represented as follows

\[
\begin{align*}
|k_1\rangle_j &= (|k_1\rangle + |k_2\rangle + |k_3\rangle + |k_4\rangle)/2 \\
|k_2\rangle_j &= (|k_1\rangle + i|k_2\rangle - |k_3\rangle - i|k_4\rangle)/2 \\
|k_3\rangle_j &= (|k_1\rangle - |k_2\rangle + |k_3\rangle - |k_4\rangle)/2 \\
|k_4\rangle_j &= (|k_1\rangle - i|k_2\rangle - |k_3\rangle + i|k_4\rangle)/2
\end{align*}
\]

For the reconstruction process, we consider \(|\Phi_0\rangle = |0\rangle|\psi\rangle \) and \(|\Phi_1\rangle = |1\rangle|\psi\rangle \). Repeatedly measure the states \( |\Phi_0\rangle \) and \(|\Phi_1\rangle \) with the projective measurement on \( C_1 \) and record the frequency.
The first phase bases Protocol 2. Two new projective measurements onto the orthogonal basis of measurement outcomes. The probability is denoted as $P_{(k)}(P_{(k)})$, when the state $|\Phi_k\rangle$ ($|\Phi_k\rangle$) collapses into the state $|k\rangle$.

The following equations are established

$$\begin{align*}
\cos(\theta_{k+1} - \theta_k) &= \frac{4P_{(k+1)} - P_{(k)} - P_{(k+2)}}{2\sqrt{P_{(k)}P_{(k+2)}}} \\
\sin(\theta_{k+1} - \theta_k) &= \frac{4P_{(k+1)} - P_{(k)} - P_{(k+2)}}{2\sqrt{P_{(k)}P_{(k+2)}}} \\
\cos(\theta_{k+2} - \theta_k) &= \frac{4P_{(k+2)} - P_{(k+1)} - P_{(k+2)}}{2\sqrt{P_{(k+1)}P_{(k+2)}}} \\
\sin(\theta_{k+2} - \theta_k) &= \frac{4P_{(k+2)} - P_{(k+1)} - P_{(k+2)}}{2\sqrt{P_{(k+1)}P_{(k+2)}}}
\end{align*}$$

(4)

The first phase $\theta_k$ is set as zero for the freedom in choosing the global phase. All the phases can be gradually determined.

Protocol 2. Two new projective measurements onto the orthonormal bases $D_1$ and $D_2$ are required to determine the phases uniquely and directly

$$
D_1 = \{(2k+1)\rangle, |2k+1\rangle, |2k+1+1\rangle, |2k+1+d\rangle\}
$$

$$
D_2 = \{(2k+1)\rangle, |2k+2\rangle, |2k+1+d\rangle, |2k+2+d\rangle\}
$$

(5)

where $0 \leq k \leq \lfloor N/2 \rfloor$, and the addition of labels is modulo 2d.

Figure 2 illustrates the transformation from $\{0\}, \{1\} \odot B_0$ into $D_1$ and $D_2$. The subscripts $\tau_1$ and $\tau_2$ denote the FT on the three elements. For example, the basis states in $D_1$ are

$$|2k\rangle = (2k + w(2k + 1) + w^2(2k + d))/\sqrt{3}$$

where $w = \exp(2\pi/3)$.

The product state $|0\rangle \otimes |\phi\rangle$ is measured with the projective measurements on bases $D_1$ and $D_2$. The probability is recorded as $P_{(k)}(P_{(k)})$, when the state $|\Phi_k\rangle$ collapses into the $k$th measurement outcome of each basis. With a similar analysis, $\cos(\theta_{k+1} - \theta_1)$ and $\cos(\theta_{k+1} - \theta_1 - 2\pi/3)$ can be estimated and thereby, $k = 0, \ldots, d - 1$. The value of $\theta_{k+1} - \theta_k$ can be determined by $P_{(k+1)}$ and $P_{(k+2)}$, further, $\theta_{k+2} - \theta_{k+1}$ can be determined by $P_{(k+2)}$ and $P_{(k+1)}$.

The above-mentioned protocols need one or two observables to determine the phases uniquely and directly. As far as we know, these protocols use the minimum number of observables among the existing ones. Protocol 1 produces at most 5d measurement outcomes, similar to the protocol proposed by Goyeneche et al. [43].

Protocol 2 needs two settings, however, the number of measurement outcomes is only 80% of the former ones, theoretically. As shown in Figure 2, the unknown pure state $|0\rangle \otimes |\phi\rangle$, in $H_2 \otimes H_4$ will never collapse into the states $|5\rangle$, $|6\rangle$, and $|7\rangle$, in $H_4$. In general, $|\Phi_k\rangle^2 = 0$, for $k = d + 1, \ldots, 2d - 1$. These projected states $|k\rangle$, $k = d + 1, \ldots, 2d - 1$ are simply designed to make the basis complete. When experiments are carried out, errors may occur in the quantum state preparations and measurements. In this case, if the auxiliary qubit is strictly $|0\rangle$ and the projected states are not completely superposed with states $|k\rangle$, $k = 0, \ldots, d - 1$, the corresponding probabilities will vanish. If not, the measurement outcomes for projected states $|k\rangle$, $k = d + 1, \ldots, 2d - 1$ can be observed.

3. Circuit Implementation for N-Qubit Pure State

The quantum circuits to implement the bases for both the protocols were designed. The projective measurement on a set of orthonormal bases can be translated into a unitary operation with canonical projective measurements onto $B_0$. The operation transforming the bases $\{0\}, \ldots, \{d - 1\}$ into $\{|\psi_0\rangle, \ldots, |\psi_{d-1}\rangle\}$ can be denoted as $U$. It is established that

$$tr[\rho|\psi_i\rangle \langle \psi_i|] = tr[\rho(U|k\rangle \langle k|U^\dagger)] = tr(U^\dagger \rho U|k\rangle \langle k|)$$

(7)

The left side of the expression represents the directly measured probability of the unknown state with basis $\{|\psi_0\rangle, \ldots, |\psi_{d-1}\rangle\}$. It is equal to the right side of the expression, obtained by performing the operation $U^\dagger$, followed by the canonical measurement. For N-qubits, the projection onto the canonical basis $B_0$ is implemented by the Pauli Z measurement at each qubit.

Figures 3 and 4 shows the quantum circuits for protocols 1 and 2, respectively. The last line is labeled as auxiliary system and all the measurement settings are on the canonical Pauli Z.

For protocol 1, the basis $C_1$ is obtained by using partial Fourier transformation and the conditional shift operation, as illustrated in Figure 1. $U^{(0)}_N$ denotes the quantum version of the increment gate for N-qubit. Further, $U^{(0)}_N$ denotes the quantum version of the increment gate for N-qubit.
Thus, the conditional shift operation $T_1$ is $|0⟩⟨0| \otimes I + |1⟩⟨1| \otimes U^N_{N+1}$. Let $U^G_N = (U^N_{N+1})^\dagger$. The circuit implementation is shown in Figure 5. The partial Fourier transformation $U_2$ is a two-qubit operation on the auxiliary qubit and the $N$th qubit. The implementation of its conjugate is shown in Figure 6.

For protocol 2, the basis $D_i$ is obtained by operating a two-qubit gate $U_i$ in Equation (8), from the canonical basis.

$$U_i = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & w & w^2 & 0 \\ 1 & w^2 & w & 0 \\ 0 & 0 & 0 & \sqrt{3} \end{pmatrix}$$

(8)

The decompositions are expressed as

$$V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\frac{j}{3}} & e^{\frac{j}{3}} \\ e^{\frac{j}{3}} & e^{-\frac{j}{3}} \end{pmatrix}, V_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} & \sqrt{1} \\ \sqrt{1} & -\sqrt{2} \end{pmatrix}$$

(9)

The shift operation $T_{N+1}$ at $N + 1$ qubits system is $U^G_{N+1}$. The basis $D_i$ is transformed into $D_j$, through this operation.

The quantum circuits for both protocols share a universal form. There are two parts for each protocol. One is related to the partial Fourier transformation between the auxiliary qubit and the $N$th qubit. The other is related to the quantum version of reduction gate, $U^G_1$. As shown in Figure 3, $I^\otimes N \otimes |0⟩⟨0| + U^G_N \otimes |1⟩⟨1| = (I^\otimes N \otimes X) \times U^G_{N+1}$. The gate $U^G_{N+1}$ is fundamental in classical computation and is easily realized with one high-dimension photon system.\(^{[46]}\) It is used in the protocol proposed by Goyeneche et al.\(^{[41]}\) For a general physical system, it is decomposed into the circuits shown in Figure 5. Using Lemma 6.1 and 7.2 described in ref. [47], each multi-qubit conditional gate can be decomposed into $O(N^3)$ one-qubit gates and controlled NOT gates. The tomography circuits for $N + 1$-qubit can be obtained from $N$-qubit by performing an additional operation. The qubits in the circuits for $N$-qubit pure state tomography are relabeled as $q_1, \ldots, q_N$. The new $(N + 1)$th qubit and the auxiliary qubit are relabeled as $q_{N+1}$ and $a$, respectively. The additional operation is denoted as $X^{q_{N+1}}$, acting on qubit $q_0$. If the qubits $q_1, \ldots, q_N, a$ are all 1, the qubit $q_0$ will flip. If not, the qubit $q_0$ will remain.

### 4. Simulation

Currently, as quantum devices are still in the NISQ era, noise is inevitably introduced during information processing. To test the feasibility of the proposed protocols, their performances were investigated with a white noise model, which causes decoherence effects. Therefore, the states considered are no longer pure quantum states.

$$\rho = \{1 - \lambda\} |\phi⟩⟨\phi| + \frac{\lambda}{2^n} I$$

(10)

where $\lambda$ is the measure of the noise in the device, $I$ is the maximixation state induced by the noisy quantum device, and $|\phi⟩$ is the state to be reconstructed.

In the numerical computation performed, $\rho$ is randomly prepared and reconstructed with both protocols 1 and 2. Specifically, there are a total of 100 testing quantum states, drawn from a uniform distribution on Hilbert space. This is implemented by repeatedly applying an operator to $(1 - \lambda) |0⟩\ldots |0⟩, |0⟩\ldots |0⟩ + \frac{\lambda}{2^n} I$, where the operator is distributed uniformly according to the Haar measure. The mixed initial state can be realized by driving $|0⟩\ldots |0⟩$, through a non-unitary channel. Then, the fidelities $F$
Figure 7. Simulation (colored online) for protocols 1a,c and 2b,d. a) show the fidelity calculated using $\rho$ and its simulation, respectively. c,d) The fidelity calculated using $|\phi\rangle\langle\phi|$ and its simulation, respectively is shown. The fidelity varies with the $\lambda$ value, the decoherence strength due to the actual quantum device, and the size of the system $N$. The averaged fidelity of 100 random states is shown as a bold line. Percentiles are shown as shaded bands around the central median lines.

are calculated by estimating the phases and amplitudes of $\rho$, according to protocols 1 and 2, with the size of the system scaling from 1 to 7, denoted as $N$.

$$F = \text{tr}(\sqrt{\rho\sqrt{\rho}})^2$$

(11)

The results are shown in Figure 7. In this simulation, different noise levels with $\lambda = 0$ (green), 0.01 (red), and 0.02 (blue) were studied. As indicators of the probability distribution of fidelity, both the averaged and the percentile fidelities were calculated. Averaged fidelity, which is the mean of the fidelities of 100 random states, is indicated by bold lines, whereas the dashed, dotted, and solid lines refer to the different noise levels. Percentiles, shown as shaded bands around the central median lines, are also marked with different color depths. A darker shaded area denotes a value closer to the median points. Thus, percentiles of $i \times 10\% (i = 1, \ldots, 9)$ are plotted. Remarkably, for Figure 7a,b, $\rho$ is the same as that given in Equation (10); however, for Figure 7c,d, a decoherence-free simulation is conducted, where $\rho = |\phi\rangle\langle\phi|$.

The fidelities tend to decay with increasing $\lambda$ and $N$ in both the protocols similarly. Moreover, the averaged fidelity and percentiles jointly indicate the shape of the probability distribution of fidelity, as the mean points are always below the median ones. In other words, in most cases, a high-quantity estimation is possible, but the outliers occasionally deviate significantly. We term this distribution as a “good” probability. Additional noises will cause a wider probability distribution of the fidelity. Therefore, if the introduced noise is controllable and limited, both protocols are feasible for producing comparatively good results.

Rigorously speaking, one cannot conduct perfect measurements, as assumed in the aforementioned simulations. Therefore, an additional numerical experiment was performed to evaluate the inaccuracy in estimating the phases and amplitudes with a limited number of measurements. $M$ is the number of projective measurements and takes the value $10^k (k = 1, \ldots, 6)$ in this simulation. Therefore, $\lambda$ was set as 0.003, and the fidelity for 100 initial random states was then calculated, as generated in the first simulation. The results are shown in Figure 8, with $N = 2$ (green), 3 (red), and 4 (blue). The averaged fidelity and percentiles are presented in the same manner as in the first simulation.

The results present a similar consequence as that shown in Figure 7, where the indicators averaged fidelity and percentiles assist a good probability distribution of fidelity, measured with the proposed protocols. Additionally, when $M$ grows, the results approach $1 - \lambda/2^N \approx 1$, which is the assumption for a perfect measurement. Therefore, the inaccuracy in probability will produce little effect on the results, if $M$ takes a sufficiently large value.

5. IBMQ Simulation

The proposed protocols were tested on IBM quantum computer ibmq-manila (ibmq) and ibmq-qasm-simulator (simulator) with
for the initial states, each qubit on ibmq is initialized at $|0\rangle$. The recorded frequencies of the outcomes and measurements at each qubit. The ideal fidelity is given by $1 - \lambda / 2^N \approx 1$. The averaged fidelity of 100 random states is indicated as a bold line. Percentiles are shown as shaded bands around the central median lines.

In protocol 1, the fidelity of 100 random states simulated by the simulator (ibmq). In protocol 2, the fidelity $99.89\%$ (80.13\%) is obtained from the simulator data (ibmq).

The fidelity is close to 1 if the data collected by ibmq-qasm simulator is used. This is consistent with the previous numerical experiment. The gap between the fidelities of simulator and ibmq can be attributed to the noise when operated on the quantum computer, such as decoherence and depolarization. For protocol 2, the fidelity decreases to 80.13\% when ibmq is used. The main discrepancy appears in the frequencies of the second circuits, as listed in Table 2. This is due to the low accuracy of three-qubit Toffoli gate. For the chain structure of the IBM-manila superconducting chip, a comparable deep quantum circuit is required. This gate can be overlapped if the global quantum increment gate, which is potentially accessible with spatially encoded qubits in the optical system, can be implemented directly.[46]

6. Discussion
With the development of quantum technologies, efforts are being devoted toward large-scale, fault-tolerant quantum computers. It is, therefore, important to design circuits to uniquely and directly read out the produced quantum states. Besides, many applications require the determination of the properties of large-scale complex quantum systems with limited measurement settings, outcomes, and computation resources.

In this study, we designed two adaptive protocols to determine $N$-qubit pure states. The number $N$ denotes the size of the quantum computer, which is arbitrary in the proposed protocols. Pure states are the output states of a quantum computer in the absence of noise. If the noise is sufficiently small, the output is close to a pure state. On the one hand, traditional Pauli measurements $Z^{\otimes 2}$ are used to estimate amplitudes. On the other hand, all the phase differences are uniquely and directly calculated with four probabilities obtained via the additional measurement settings for protocols 1 and 2. Theoretically, the number of measurement settings, as well as the number of measurement outcomes, decreased under protocol 2. Furthermore, quantum circuits for both these protocols were designed, where the quantum version of the reduction gate and the related two-qubit Fourier transformation were required. Numerical and actual experiments on IBMQ indicated the feasibility of both these protocols.
The unitary matrices $V_1$ and $V_2$ are $U(\pi/2, -\pi/2, 2\pi/3)$ and $U(2 \arccos \sqrt{2/3}, 0, \pi)$, respectively. The other parts of the circuits are used to determine the phases.

### Table 1. Frequency of protocol 1.

| Outcome | Simulator 1 | Ibmq 1 | Simulator 2 | Ibmq 2 |
|---------|-------------|--------|-------------|--------|
| 0       | 2114        | 2246   | 2000        | 1953   |
| 1       | 1028        | 945    | 998         | 1165   |
| 2       | 2056        | 2022   | 2035        | 1763   |
| 3       | 1014        | 980    | 1041        | 1042   |

### Table 2. Frequency of protocol 2.

| Outcome | Simulator 1 | Ibmq 1 | Simulator 2 | Ibmq 2 |
|---------|-------------|--------|-------------|--------|
| 0       | 2768        | 2616   | 2751        | 1867   |
| 1       | 696         | 683    | 685         | 654    |
| 2       | 2654        | 2488   | 681         | 891    |
| 3       | 697         | 685    | 700         | 987    |

These adaptive protocols can be used for the determination of all unknown $N$-qubit pure states. Notably, we tackled the case where all the amplitudes of $|\phi\rangle$ were nonzero. With randomly chosen $|\phi\rangle$, there are two ways for the other states to be in a zero measure set. The bases to determine the phases can be designed on the projected subspace, or permutation operations can be performed in advance to ensure that the front amplitudes of the new state are nonzero. In future research, the design of several fixed and extendable circuits for all quantum states, either pure or mixed, needs to be explored. Our protocols are efficient as they provide designs requiring less measurement resources such as settings and outcomes; this is expected to benefit the readout of pure states for future quantum computers.

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### Conflict of Interest

The authors declare no conflict of interest.

### Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

### Keywords

Fourier transformation, $N$-qubit pure state, projective measurements, quantum circuits, quantum state tomography

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