A new uniform lower bound on Weil–Petersson distance

Yunhui Wu

Received: 18 May 2021 / Accepted: 5 May 2022 / Published online: 1 June 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
In this paper we study the injectivity radius based at a fixed point along Weil–Petersson geodesics. We show that the square root of the injectivity radius based at a fixed point is 0.3884-Lipschitz on Teichmüller space endowed with the Weil–Petersson metric. As an application we reprove that the square root of the systole function is uniformly Lipschitz on Teichmüller space endowed with the Weil–Petersson metric, where the Lipschitz constant can be chosen to be 0.5492. Applications to asymptotic geometry of moduli space of Riemann surfaces for large genus will also be discussed.

Mathematics Subject Classification 30F60 · 53C21 · 32G15

1 Introduction
Let $S_g$ be a closed surface of genus $g$ ($g \geq 2$), and $\mathcal{T}_g$ be the Teichmüller space of $S_g$. Let $\text{Teich}(S_g)$ be the space $\mathcal{T}_g$ endowed with the Weil–Petersson metric. The mapping class group $\text{Mod}(S_g)$ acts on $\text{Teich}(S_g)$ by isometries. The moduli space $\mathcal{M}_g$ of $S_g$ endowed with the Weil–Petersson metric, is realized as the quotient $\text{Teich}(S_g)/\text{Mod}(S_g)$. Let $\mathcal{M}_{-1}$ be the space of complete Riemannian metrics on $S_g$ of constant Gauss curvature $-1$. It is known that $\mathcal{T}_g = \mathcal{M}_{-1}/\text{Diff}_0(S_g)$ where $\text{Diff}_0(S_g)$ is the group of diffeomorphisms of $S_g$ isotopic to the identity. Let $p \in S_g$ be fixed and $\tilde{X} \in \mathcal{M}_{-1}$ be a hyperbolic metric on $S_g$. The injectivity radius $\text{inj}_{\tilde{X}}(p)$ of $\tilde{X}$ at $p$ is half of the length of a shortest nontrivial geodesic loop based at $p$. The geodesic loop based at $p$ realizing $\text{inj}_{\tilde{X}}(p)$ may not be unique. It is known that $\text{inj}_{\tilde{X}}(p)$ is bounded from above by a positive constant only depending on $g$. Let

$$\pi : \mathcal{M}_{-1} \to \mathcal{T}_g$$

be the natural projection. In order to transfer the quantity $\text{inj}_{\tilde{X}}(p)$ onto Weil–Petersson geodesics in $\mathcal{T}_g$, we make the following definition. First we recall in [13, 14] that for a smooth path $c(t) \subseteq \mathcal{M}_{-1}$, we say $c(t)$ is a horizontal curve if for each $t$, there exists a holomorphic quadratic differential $\Psi(t)$ of $c(t)$ such that the variation of hyperbolic metrics

Communicated by Peter Topping.

Yunhui Wu
yunhui_wu@mail.tsinghua.edu.cn

1 Tsinghua University, Haidian District, Beijing 100084, China
satisfies $\frac{dc(t)}{dt} = \Re\Psi(t)$. A smooth path in $T_g$ can always be lifted onto a horizontal curve in $\mathcal{M}_{-1}$. Throughout this paper we always assume that parameters are proportional to arc-length parameters for both geodesics in hyperbolic surfaces and smooth Weil–Petersson paths in Teichmüller space of Riemann surfaces. It is not well-defined for $\text{inj}(c)(p)$ on $T_g$ because point $p$ is clearly not invariant by diffeomorphisms of $S_g$. To fix this problem, we first pick the Weil–Petersson geodesic joining $X$ and $Y$, and lift this Weil–Petersson geodesic onto a horizontal curve $c : [0,1] \to \mathcal{M}_{-1}$. Then we consider the injectivity radius function $\text{inj}(c)(p)$ along $c([0,1])$ which is well-defined. Now we define

**Definition** Fix a point $p \in S_g \ (g \geq 2)$. For any $X, Y \in T_g$, we define

$$\left| \sqrt{\text{inj}_X(p)} - \sqrt{\text{inj}_Y(p)} \right| := \sup_c \left| \sqrt{\text{inj}_{c(0)}(p)} - \sqrt{\text{inj}_{c(1)}(p)} \right|$$

where $c : [0,1] \to \mathcal{M}_{-1}$ runs over all smooth horizontal curves with $\pi(c(0)) = X$, $\pi(c(1)) = Y$ and $\pi(c([0,1])) \subset T_g$ is the Weil–Petersson geodesic joining $X$ and $Y$.

The definition above actually does not depend on the choice of $p$. One may see the following remark for an equivalent definition.

**Remark** It is known that any two horizontal lifts in $\mathcal{M}_{-1}$ of a smooth curve in $T_g$ differ by an element in $\text{Diff}_0(S_g)$ (see e.g. [18, Chapter 2]), and the group $\text{Diff}_0(S_g)$ acts transitively on $S_g$. So the definition above is equivalent to

$$\left| \sqrt{\text{inj}_X(p)} - \sqrt{\text{inj}_Y(p)} \right| := \sup_{q \in S_g} \left| \sqrt{\text{inj}_{c'(0)}(q)} - \sqrt{\text{inj}_{c'(1)}(q)} \right|$$

where $c' : [0,1] \to \mathcal{M}_{-1}$ is a horizontal lift of the Weil–Petersson geodesic joining $X$ and $Y$.

In this paper, we show that

**Theorem 1.1** Fix a point $p \in S_g \ (g \geq 2)$. Then for any $X, Y \in T_g$,

$$\left| \sqrt{\text{inj}_X(p)} - \sqrt{\text{inj}_Y(p)} \right| \leq 0.3884 \ \text{dist}_{wp}(X, Y)$$

where $\text{dist}_{wp}$ is the Weil–Petersson distance.

**Remark** Rupflin and Topping in [13, Section 2] showed that

$$\left| \sqrt{\text{inj}_X(p)} - \sqrt{\text{inj}_Y(p)} \right| \leq c(g) \ \text{dist}_{wp}(X, Y)$$

where $c(g) > 0$ is a constant depending on $g$. Our approach is similar to that of Rupflin and Topping, but using a detailed analysis of injectivity radius along shortest geodesic loops and a recent uniform bound for harmonic Beltrami differentials on thin parts [6], we are able to obtain the above uniform bound independent of $g$. The Lipschitz constant 0.3884 above is not optimal. More refined arguments in this proof can improve this uniform constant. In general, it is difficult to measure the Weil–Petersson distance on $T_g$. One may see [1–3, 7, 10, 13, 15, 16, 20, 22] for related bounds on Weil–Petersson distances.

Recall that the systole $\ell_{sys}(X)$ of $X \in T_g$ is the length of a shortest nontrivial closed geodesic in $X$. Which is also the same as $2 \min_{p \in X} \text{inj}_X(p)$ where $\tilde{X} \in \mathcal{M}_{-1}$ is a hyperbolic metric on $S_g$ with $\pi(\tilde{X}) = X \in T_g$. As a direct application of Theorem 1.1, we prove
Corollary 1.2  For any $X, Y \in \mathcal{T}_g$ ($g \geq 2$),
\[
\sqrt{\ell_{\text{sys}}(X)} - \sqrt{\ell_{\text{sys}}(Y)} \leq 0.5492 \text{dist}_{wp}(X, Y).
\]

Proof  Without loss of generality, one may assume that
\[
\ell_{\text{sys}}(X) \geq \ell_{\text{sys}}(Y).
\]
Let $c: [0, 1] \to \mathcal{M}_{-1}$ be a horizontal curve with $\pi(c(0)) = X$, $\pi(c(1)) = Y$ and $\pi(c([0, 1])) \subset \mathcal{T}_g$ is the Weil–Petersson geodesic joining $X$ and $Y$. Let $\alpha \subset c(1)$ be a shortest closed geodesic. So for any $p \in \alpha$, we have
\[
2 \text{inj}_{c(1)}(p) = \ell_{\text{sys}}(Y) \quad \text{and} \quad 2 \text{inj}_{c(0)}(p) \geq \ell_{\text{sys}}(X).
\]
Then by Theorem 1.1 we get
\[
\sqrt{\ell_{\text{sys}}(X)} - \sqrt{\ell_{\text{sys}}(Y)} \leq 2 \text{inj}_{c(0)}(p) - 2 \text{inj}_{c(1)}(p) \leq 0.5492 \text{dist}_{wp}(X, Y)
\]
as desired. \qed

Remark  (1) It was shown in [22] that
\[
\sqrt{\ell_{\text{sys}}(X)} - \sqrt{\ell_{\text{sys}}(Y)} \leq K \text{dist}_{wp}(X, Y)
\]
where $K > 0$ is a uniform (implicit) constant independent of $g$.

(2) Very recently, Bridgeman–Bromberg in [1] show that the uniform constant $K$ above can be chosen to be $\frac{1}{5}$ by a completely different method.

Both the proofs in [22] and [1] rely on certain uniform bound for the Weil–Petersson norm $||\nabla(\ell_{\alpha}(X))||_{wp}$ of the Weil–Petersson gradient $\nabla(\ell_{\alpha}(X))$ of the geodesic length function $\ell_{\alpha}(\cdot)$ on Teichmüller space, where $\alpha \subset X$ is a systolic curve (one may also see [23] for a different proof). In this paper, our proof is totally different without any estimation on $||\nabla(\ell_{\alpha}(X))||_{wp}$. Moreover, we are able to obtain the explicit Lipschitz constant above to be 0.5492, which can be improved by more careful arguments for the proof of Theorem 1.1.

The Weil–Petersson completion of the moduli space $\mathcal{M}_g$ is compact which is homeomorphic to the Deligne–Mumford compactification of the moduli space of Riemann surfaces. In particular, the moduli space $\mathcal{M}_g$ has finite Weil–Petersson diameter and inradius. Cavendish–Parlier [7] showed that for large genus the ratio $\frac{\text{diam}(\mathcal{M}_g)}{\sqrt{g}}$ is bounded below by a uniform positive constant and above by a uniform constant multiple of $\ln(g)$. It is an open problem that whether the Weil–Petersson diameter diam$(\mathcal{M}_g)$ of $\mathcal{M}_g$ is uniformly comparable to $\sqrt{\ln(g)}$. Recall that the Weil–Petersson inradius $\text{InRad}(\mathcal{M}_g)$ of $\mathcal{M}_g$ is defined as
\[
\text{InRad}(\mathcal{M}_g) := \max_{X \in \mathcal{M}_g} \text{dist}_{wp}(X, \partial \mathcal{M}_g)
\]
where $\partial \mathcal{M}_g$ is the boundary of $\mathcal{M}_g$ consisting of nodal surfaces. It was shown in [22] that as $g \to \infty$, the Weil–Petersson inradius $\text{InRad}(\mathcal{M}_g)$ is uniformly comparable to $\sqrt{\ln(g)}$. More precisely, there exists a uniform constant $K' > 0$ independent of $g$ such that
\[
K' \leq \frac{\text{InRad}(\mathcal{M}_g)}{\sqrt{\ln(g)}} \leq \sqrt{4\pi}
\]
where the uniform (implicit) constant $K'$ depends on the work of Buser–Sarnak [4]. The following question is natural and interesting.
Question 1 Does \( \lim_{g \to \infty} \frac{\ln \text{Rad}(\mathcal{M}_g)}{\sqrt{\ln g}} \) exist? If exists, what is its value?

Set
\[
\text{sys}(g) = \max_{X \in \mathcal{M}_g} \ell_{\text{sys}}(X).
\]
It is known that
\[
\ell_{\text{sys}}(X) \leq 2 \ln(4g - 2)
\]
for all \( X \in \mathcal{M}_g \). Buser and Sarnak in [4] showed that
\[
\text{sys}(g) \geq U \ln(g)
\]
for some uniform constant \( U > 0 \). Moreover, they also showed that there exists a sequence \( \{g_k\}_{k \geq 1} \) of positive integers tending to infinity such that for each \( g_k \), there exists a closed hyperbolic surface \( \mathcal{X}_{g_k} \) of genus \( g_k \) with
\[
\ell_{\text{sys}}(\mathcal{X}_{g_k}) \geq \frac{4 \ln(g_k)}{3} - U'
\]
where \( U' > 0 \) is a uniform constant independent of \( g \). Thus, the quantity \( \text{sys}(g) \) is uniformly comparable to \( \ln(g) \) as \( g \to \infty \). Moreover
\[
\frac{4}{3} \leq \limsup_{g \to \infty} \frac{\text{sys}(g)}{\ln(g)} \leq 2.
\]
By applying the proof of Theorem 1.1 it is not hard to see that
\[
2.0472 \leq \liminf_{g \to \infty} \frac{\ln \text{Rad}(\mathcal{M}_g)}{\sqrt{\ln g}} \leq 2.5066
\]
and
\[
2.3696 \leq \limsup_{g \to \infty} \frac{\ln \text{Rad}(\mathcal{M}_g)}{\sqrt{\ln g}} \leq 3.5449.
\]
In this paper we show that

Theorem 1.3 The following limit holds:
\[
\lim_{g \to \infty} \frac{\ln \text{Rad}(\mathcal{M}_g)}{\sqrt{\ln g}} = \sqrt{2\pi} \sim 2.5066.
\]

Remark (1) Theorem 1.3 was firstly obtained in [1] by Bridgeman and Bromberg. We are grateful to M. Bridgeman for kindly sharing their latest version of [1].

(2) The proof of [1] relies on bounds of \( ||\nabla(\ell_\alpha(X))||_{w_\alpha} \) in terms of functions on collars. Our proof is a slightly refined argument of the proof of [22, Theorem 1.1] where we bound \( ||\nabla(\ell_\alpha(X))||_{w_\alpha} \) in terms of functions on hyperbolic disks. In both cases, \( \alpha \) is a systolic curve of \( X \).

Theorem 1.3 reduces Question 1 to study the following one which has no metric involved on \( \mathcal{M}_g \).

Question 2 Does \( \lim_{g \to \infty} \frac{\text{sys}(g)}{\ln(g)} \) exist? If exists, what is its value?
Recall that for any hyperbolic surface \( X \in \mathcal{M}_g \), the Bers’ constant \( B_g(X) \) at \( X \) is the smallest positive number such that there exist \( (3g-3) \) disjoint simple closed geodesics \( \{ \gamma_i \}_{i=1}^{3g-3} \) on \( X \) with

\[
\max_{1 \leq i \leq (3g-3)} \ell_{\gamma_i}(X) \leq B_g(X).
\]

It is known \([5, \text{Chapter 5}]\) that

\[
\sqrt{6g - 2} \leq \sup_{X \in \mathcal{M}_g} B_g(X) \leq 26(g - 1).
\]

Buser \([5]\) conjectures that \( \sup_{X \in \mathcal{M}_g} B_g(X) \) is uniformly comparable to \( \sqrt{g} \).

Fix any \( L > 0 \), we set the subset \( \mathcal{MC}(\leq L) \subseteq \mathcal{M}_g \) as

\[
\mathcal{MC}(\leq L) := \left\{ X \in \mathcal{M}_g ; B_g(X) \leq L \right\}.
\]

So \( \mathcal{MC} \left( \leq \sup_{X \in \mathcal{M}_g} B_g(X) \right) = \mathcal{M}_g \).

Let \( L_g = \epsilon \ln(g) \) for large \( g \) and small enough \( \epsilon > 0 \).

By applying \([11]\) and Theorem 1.1, we show that for large enough \( g \), the \( 1.0511\sqrt{\ln(g)} \)-neighbourhood of \( \mathcal{MC}(\leq L_g) \) can be arbitrarily small in \( \mathcal{M}_g \) in the following sense.

**Theorem 1.4** For any small enough \( \epsilon > 0 \) and if \( L_g = \epsilon \ln(g) \), then

\[
\lim_{g \to \infty} \frac{\text{Vol}_{WP} \left( \left\{ X \in \mathcal{M}_g ; \text{dist}_{wp}(X, \mathcal{MC}(\leq L_g)) > 1.0511\sqrt{\ln(g)} \right\} \right)}{\text{Vol}_{WP}(\mathcal{M}_g)} = 1
\]

where \( \text{Vol}_{WP}(\cdot) \) is the Weil–Petersson volume.

**Plan of the paper.** Section 2 provides some necessary background and the basic properties on two-dimensional hyperbolic geometry and Teichmüller theory. In Sect. 3 we prove two bounds for the injectivity radius along a shortest geodesic loop based at a fixed point. A technical inequality is provided in Sect. 4 which is crucial in the proof of Theorem 1.1. In Sect. 5 we finish the proof of Theorem 1.1. Theorem 1.3 is shown in Sect. 6. And we prove Theorem 1.4 in Sect. 7.

### 2 Preliminaries

In this section we will set up the notations and provide some necessary background on two-dimensional hyperbolic geometry and Teichmüller theory of Riemann surfaces.

#### 2.1 Injectivity radius at a point

Let \( X \) be a closed hyperbolic surface. Since the curvature of \( X \) is \(-1\), the conjugate radius at any point of \( X \) is infinity. Thus for any point \( p \in X \), the injectivity radius \( \text{inj}_X(p) \) of \( X \) at \( p \) is half of the length of a shortest nontrivial geodesic loop based at \( p \). Let

\[
\sigma : [0, 2 \text{inj}_X(p)] \to X
\]

be such a shortest geodesic loop with \( \sigma(0) = \sigma(2 \text{inj}_X(p)) = p \) of arc-length parameter. Then

(1) the restriction \( \sigma : [0, \text{inj}_X(p)] \to X \) is a minimizing geodesic;

(2) the restriction \( \sigma : [\text{inj}_X(p), 2 \text{inj}_X(p)] \to X \) is also a minimizing geodesic.
For any $r > 0$, we let

$$B(p; r) := \{ q \in X; \text{dist}(q, p) < r \}$$

be the open geodesic ball centered at $p$ of radius $r$. The open ball $B(p; \text{inj}_X(p))$ is an embedded hyperbolic open disk of radius $\text{inj}_X(p)$. By the Gauss–Bonnet formula we know that $\text{Area}(X) = 4\pi (g - 1)$. Thus,

$$\text{Area}(B(p; \text{inj}_X(p))) = 2\pi \left( \cosh(\text{inj}_X(p)) - 1 \right) \leq 4\pi (g - 1)$$

which implies that for any $p \in X$,

$$\text{inj}_X(p) \leq \ln(4g - 2). \quad (2.1)$$

We remark here that for all $g \geq 2$, Buser and Sarnak in [4] constructed a closed surface $X_g$ of genus $g$ such that

$$\inf_{p \in X_g} \text{inj}_{X_g}(p) \geq U \ln(g)$$

for some uniform constant $U > 0$ independent of $g$.

### 2.2 Teichmüller space and Weil–Petersson metric.

We denote by $S_g$ an oriented closed surface of genus $g$ ($g \geq 2$). The Uniformization Theorem implies that the surface $S_g$ admits hyperbolic metrics of constant curvature $-1$. We let $T_g$ be the Teichmüller space of surfaces of genus $g$, which we consider as the equivalence classes under the action of the group $\text{Diff}_0(S_g)$ of diffeomorphisms isotopic to the identity of the space of hyperbolic surfaces $X = (S_g, \sigma(z)dz^2)$. The tangent space $T_X T_g$ at a point $X = (S_g, \sigma(z)dz^2)$ is identified with the space of harmonic Beltrami differentials on $X$, i.e.,

forms on $X$ expressible as $\mu = \overline{\psi}/\sigma$ where $\psi \in Q(X)$ is a holomorphic quadratic differential on $X$. The pointwise norm $|\mu| : X \rightarrow \mathbb{R}^{\geq 0}$ gives a continuous nonnegative function on $X$. Let $z = x + iy$ and $d\text{Area} = \sigma(z)dx dy$ be the volume form. The Weil–Petersson metric is the Hermitian metric on $T_g$ arising from the Petersson scalar product

$$\langle \varphi, \psi \rangle = \int_X \frac{\varphi \cdot \overline{\psi}}{\sigma^2} d\text{Area}$$

via duality. We will concern ourselves primarily with its Riemannian part $g_{WP}$. Throughout this paper we denote by $\text{Teich}(S_g)$ the Teichmüller space endowed with the Weil–Petersson metric. By definition it is easy to see that the mapping class group $\text{Mod}(S_g)$ acts on $\text{Teich}(S_g)$ as isometries. Thus, the Weil–Petersson metric descends to a metric, also called the Weil–Petersson metric, on the moduli space of Riemann surfaces $\mathcal{M}_g$ which is defined as $T_g / \text{Mod}(S_g)$. Throughout this paper we also denote by $\mathcal{M}_g$ the moduli space endowed with the Weil–Petersson metric. One may refer to [9, 21] for more details on Weil–Petersson geometry.

### 2.3 Uniform bounds on harmonic Beltrami differentials

In this subsection we recall two uniform bounds on the pointwise norm of any harmonic Beltrami differential in terms of the injectivity radius at a point. We first refer to a function $C(r)$ introduced by Teo in [17] which is given by

$$C(r) = \left( \frac{4\pi}{3} \left( 1 - \left( \frac{4e^r}{(1 + e^r)^2} \right)^3 \right) \right)^{-\frac{1}{2}}. \quad (2.2)$$
It follows that $C(r)$ is decreasing with respect to $r$ and as $r$ tends to zero we have

$$C(r) = \frac{1}{\sqrt{\pi r}} + O(1).$$

Furthermore $C(r)$ tends to $\frac{1}{\sqrt{\pi}}$ as $r$ tends to infinity. The following property follows by a Taylor expansion of $\mu$ on a hyperbolic disk of radius $r > 0$.

**Proposition 2.1** (Teo, [17, Prop 3.1] or [24, Prop 2.10]) Let $X$ be a closed hyperbolic surface and $\mu$ be a harmonic Beltrami differential on $X$. Then for any $p \in X$,

$$|\mu(p)|^2 \leq \left( C(\text{inj}_X(p)) \right)^2 \int_{B(p; r)} |\mu(z)|^2 \, d\text{Area}(z), \quad \forall \ 0 < r \leq \text{inj}_X(p)$$

where the constant $C(\cdot)$ is given by (2.2).

Proposition 2.1 is useful when the injectivity radius at a point is uniformly bounded from below, especially as the injectivity radius goes to infinity. For the case that the injectivity radius at a point is small, we will use the following recent result, which follows by a detailed analysis on the Fourier expansion of $\mu$ on a collar of a short closed geodesic. More precisely,

**Proposition 2.2** (Bridgeman–Wu, [6, Prop 1.1]) Let $X$ be a closed hyperbolic surface and $\mu$ be a harmonic Beltrami differential on $X$. Then for any $p \in X$ with $\text{inj}_X(p) \leq \text{arcsinh}(1)$,

$$|\mu(p)|^2 \leq \frac{\int_X |\mu(z)|^2 \, d\text{Area}(z)}{\text{inj}_X(p)}.$$

### 3 Two bounds on injectivity radius

Let $X$ be a closed hyperbolic surface of genus $g \geq 2$ and $\gamma \subset X$ be a non-trivial simple loop. There always exists a unique closed geodesic, still denoted by $\gamma$, representing this loop. The Collar Lemma says that it has a tubular neighborhood which is a topological cylinder with a standard hyperbolic metric. And the width of this cylinder, only depending on the length of $\gamma$, goes to infinity as the length of $\gamma$ goes to 0. First we recall the following version of the Collar Lemma which will be applied.

**Theorem 3.1** [5, Theorem 4.1.1] Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be disjoint simple closed geodesics on a closed hyperbolic Riemann surface $X$ of genus $g$, and $\ell(\gamma_i)$ be the length of $\gamma_i$. Then $m \leq 3g - 3$ and we can define the collar of $\gamma_i$ by

$$C(\gamma_i) = \{ x \in X_g; \ \text{dist}(x, \gamma_i) \leq w(\gamma_i) \}$$

where

$$w(\gamma_i) = \text{arcsinh} \left( \frac{1}{\sinh \frac{1}{2} \ell(\gamma_i)} \right)$$

is the half width of the collar.

Then the collars are pairwise disjoint for $i = 1, \ldots, m$. Each $C(\gamma_i)$ is isomorphic to a cylinder $(\rho, t) \in [-w(\gamma_i), w(\gamma_i)] \times S^1$, where $S^1 = \mathbb{R}/\mathbb{Z}$, with the metric

$$ds^2 = d\rho^2 + \ell(\gamma_i)^2 \cosh^2 \rho \, dt^2. \quad (3.1)$$

And for a point $(\rho, t)$, the point $(0, t)$ is its projection on the geodesic $\gamma_i$, the absolute value $|\rho|$ is the distance to $\gamma_i$, $t$ is the coordinate on $\gamma_i \cong S^1$. 
As the length $\ell(\gamma)$ of the central closed geodesic goes to 0, the width
\[ e^{w(\gamma)} \sim \frac{4}{\ell(\gamma)} \tag{3.2} \]
which tends to infinity. In this paper, we mainly deal with the case that $\ell(\gamma)$ is small and so $w(\gamma)$ is large.

Now we recall another version of the Collar Lemma which provides useful information on the injectivity radius at a point.

**Theorem 3.2** [5, Theorem 4.1.6] Let $\beta_1, \ldots, \beta_k$ be the set of all simple closed geodesics of length $\leq 2 \arcsinh(1)$ on a closed hyperbolic Riemann surface $X$ of genus $g$. Then $k \leq 3g - 3$, and the followings hold.

1. The geodesics $\beta_1, \ldots, \beta_k$ are pairwise disjoint
2. $\text{inj}_X(p) > \arcsinh(1)$ for any $p \in X \setminus (\bigcup_{i=1}^{k} C(\beta_i))$.
3. If $p \in C(\beta_i)$, and $d(p) = \text{dist}(p, \partial C(\beta_i))$, then
   \[
   \sinh(\text{inj}_X(p)) = \cosh\left(\frac{\ell(\beta_i)}{2}\right) \cosh(d(p)) - \sinh(d(p)), \tag{3.3}
   \]
   \[
   \sinh(\text{inj}_X(p)) = \sinh\left(\frac{\ell(\beta_i)}{2}\right) \cosh(\text{dist}(p, \beta_i)) \tag{3.4}
   \]

By comparing the total area of all these standard collars $C(\beta_i)$’s and the total area $4\pi(g - 1)$ of $X$, the set $X \setminus (\bigcup_{i=1}^{k} C(\beta_i))$ is always non-empty. And for any point $q \in \partial C(\beta_i)$, by continuity or (3.3) we know that $\text{inj}_X(q) \geq \arcsinh(1)$. Now we study the injectivity radius along shortest geodesic loops (may not smooth at base points). First we consider the case that the base point is contained in a collar with a central closed geodesic of length $\leq 2 \arcsinh(1)$.

**Proposition 3.3** Let $X$ be a closed hyperbolic surface. For any $p \in X$ with $\text{inj}_X(p) \leq \arcsinh(1)$, we let $\sigma : [0, 2 \text{inj}_X(p)] \to X$ be a shortest nontrivial geodesic loop based at $p$. Then for any $s \in [0, 2 \text{inj}_X(p)]$, we have
\[
(\sqrt{2} - 1) \text{inj}_X(p) \leq \text{inj}_X(\sigma(s)) \leq \text{inj}_X(p).
\]

**Proof** Since $\text{inj}_X(p) \leq \arcsinh(1)$, by Theorem 3.1 and 3.2 one may assume that $\beta$ is the unique simple closed geodesic of length $\leq 2 \arcsinh(1)$ such that $p \in C(\beta)$. So the shortest geodesic loop
\[
\sigma([0, 2 \text{inj}_X(p)]) \subset C(\beta)
\]
otherwise it follows by Theorem 3.2 that $\sigma([0, 2 \text{inj}_X(p)])$ contains a point $q \notin C(\beta)$ with $\text{inj}_X(q) > \arcsinh(1)$ implying that $\ell(\sigma) > 2 \arcsinh(1)$ which is a contradiction.

Now we first show the right hand side inequality: up to a conjugation one may lift $\beta$ onto the imaginary axis $\mathbb{i} \cdot \mathbb{R}^+$ in the upper half plane $\mathbb{H}$, and the deck transformation corresponding to $\beta$ is $A(z) = e^{\ell(\beta)}z$. By Theorem 3.1 one may let $\bar{\beta}$ be a lift of $p$ with
\[
\text{dist}_\mathbb{H}(\bar{p}, \mathbb{i} \cdot \mathbb{R}^+) = \text{dist}(p, \beta) \leq \arcsinh\left(\frac{1}{\sinh \frac{1}{2}\ell(\beta)}\right).
\]
Then the lift $\bar{\sigma} : [0, 2 \text{inj}_X(p)] \to \mathbb{H}$ of $\sigma$ based at $\bar{p}$ is the geodesic joining $\bar{p}$ and $A(\bar{p})$ in $\mathbb{H}$. By the convexity of distance functions on $\mathbb{H}$ we know that for any $s \in [0, 2 \text{inj}_X(p)]$,
\[
\text{dist}_\mathbb{H}(\bar{\sigma}(s), \mathbb{i} \cdot \mathbb{R}^+) \leq \text{dist}_\mathbb{H}(\bar{p}, \mathbb{i} \cdot \mathbb{R}^+) = \text{dist}_\mathbb{H}(A(\bar{p}), \mathbb{i} \cdot \mathbb{R}^+)
\]
which implies that
\[
\text{dist}(\sigma(s), \beta) \leq \text{dist}(p, \beta). \tag{3.5}
\]

Then it follows by (3.4) that
\[
\text{inj}_X(\sigma(s)) \leq \text{inj}_X(p)
\]
for all \( s \in [0, 2\text{inj}_X(p)] \).

Next we show the other side inequality. Let \( s \in [0, 2\text{inj}_X(p)] \). Then
\[
\text{dist}(p, \sigma(s)) \leq \text{inj}_X(p) \leq \text{arcsinh}(1). \tag{3.6}
\]

We finish the proof by considering the following two cases.

Case (1). \( \text{dist}(p, \beta) > \text{dist}(p, \sigma(s)) \). Let \( \pi(\sigma(s)) \in \beta \) with
\[
\text{dist}(\sigma(s), \pi(\sigma(s))) = \text{dist}(\sigma(s), \beta).
\]
By the triangle inequality we have
\[
\text{dist}(\sigma(s), \beta) \geq \text{dist}(\pi(\sigma(s)), p) - \text{dist}(p, \sigma(s)) \geq \text{dist}(p, \beta) - \text{dist}(p, \sigma(s)) > 0.
\]
Then it follows by (3.4) and (3.6) that
\[
\sinh(\text{inj}_X(\sigma(s))) = \sinh\left(\frac{\ell(\beta)}{2} \cosh(\text{dist}(\sigma(s), \beta))\right)
\geq \sinh\left(\frac{\ell(\beta)}{2} \cosh(\text{dist}(p, \beta) - \text{dist}(p, \sigma(s)))\right)
\geq \left(\sinh\left(\frac{\ell(\beta)}{2} \cosh(\text{dist}(p, \beta))\right)\right)^e - \text{dist}(p, \sigma(s))
\geq \sinh(\text{inj}_X(p)) \cdot e^{-\text{arcsinh}(1)}
> \sinh\left(e^{-\text{arcsinh}(1)} \cdot \text{inj}_X(p)\right).
\]
Thus, we have
\[
\text{inj}_X(\sigma(s)) \geq e^{-\text{arcsinh}(1)} \text{inj}_X(p) = \left(\sqrt{2} - 1\right) \text{inj}_X(p). \tag{3.7}
\]
Which completes the proof for this case.

Case (2). \( \text{dist}(p, \beta) \leq \text{dist}(p, \sigma(s)) \). Then by (3.6) we have
\[
\text{dist}(p, \beta) \leq \text{dist}(p, \sigma(s)) \leq \text{arcsinh}(1). \tag{3.8}
\]
By (3.1) of Theorem 3.1, we have that the closed curve (not a geodesic loop) based at \( p \) with equidistance \( \text{dist}(p, \beta) \) to \( \beta \) has length \( \ell(\beta) \cosh(\text{dist}(p, \beta)) \). Which together with (3.8) implies that
\[
\text{inj}_X(p) < \frac{\ell(\beta)}{2} \cosh(\text{arcsinh}(1)) = \frac{\sqrt{2}}{2} \ell(\beta).
\]
Thus, we have
\[
\text{inj}_X(\sigma(s)) \geq \frac{\ell(\beta)}{2} > \frac{\sqrt{2}}{2} \text{inj}_X(p). \tag{3.9}
\]
Then the conclusion follows by (3.7) and (3.9) because \( \frac{\sqrt{2}}{2} > (\sqrt{2} - 1) \).
Now we consider the case that the base point has injectivity radius larger than arcsinh(1). Let $\beta$ be a simple closed geodesic in $X$ of length
\[ \ell(\beta) \leq 2 \text{arcsinh}(1). \]
The boundary $\partial \mathcal{C}(\beta)$ of the collar $\mathcal{C}(\beta)$ are two disjoint closed curves homotopic to $\beta$. By (3.1) we know that for each component $\beta'$ of $\partial \mathcal{C}(\beta)$, the length $\ell(\beta')$ of $\beta'$ is
\[ \ell(\beta') = \ell(\beta) \cosh(\omega(\beta)) = \ell(\beta) \cdot \cosh \left( \text{arcsinh} \left( \frac{1}{\sinh \frac{1}{2} \ell(\beta)} \right) \right). \]
A simple computation shows that
\[ \ell(\beta') = \ell(\beta) \frac{1}{\sinh \frac{1}{2} \ell(\beta)} \sqrt{1 + \left( \sinh \frac{1}{2} \ell(\beta) \right)^2} \leq 2 \sqrt{2}. \]

Now we are ready to state the result for the other case.

**Proposition 3.4** Let $X$ be a closed hyperbolic surface. For any $p \in X$ with $\text{inj}_X(p) > \text{arcsinh}(1)$, we let $\sigma : [0, 2 \text{inj}_X(p)] \rightarrow X$ be a shortest nontrivial geodesic loop based at $p$. Then
\[ \min_{s \in [0, 2 \text{inj}_X(p)]} \text{inj}_X(\sigma(s)) \geq \text{arcsinh}(1) > \ln \left( e^{-\sqrt{2}} + \sqrt{e^{-2\sqrt{2}} + 1} \right) \sim 0.2407. \]

**Proof** If the geodesic loop $\sigma$ does not intersect with any standard collar with central closed geodesic of length less than or equal to $2 \text{arcsinh}(1)$, it follows by Theorem 3.2 that
\[ \min_{s \in [0, 2 \text{inj}_X(p)]} \text{inj}_X(\sigma(s)) \geq \text{arcsinh}(1) > \ln \left( e^{-\sqrt{2}} + \sqrt{e^{-2\sqrt{2}} + 1} \right) \sim 0.2407. \]

Now we assume that
\[ \sigma \cap \mathcal{C}(\beta) \neq \emptyset \]
for some simple closed geodesic $\beta$ of length less than or equal to $2 \text{arcsinh}(1)$.

**Claim:** $\forall q \in (\sigma \cap \mathcal{C}(\beta))$, $\text{dist}(q, \partial \mathcal{C}(\beta)) \leq \sqrt{2}$.
If the claim above is true, then it follows by (3.3) in Theorem 3.2 that
\[
\sinh(\text{inj}_X(q)) = \cosh \left( \frac{\ell(\beta)}{2} \right) \cosh(\text{dist}(q, \partial \mathcal{C}(\beta))) - \sinh(\text{dist}(q, \partial \mathcal{C}(\beta))) \\
\geq \cosh(\text{dist}(q, \partial \mathcal{C}(\beta))) - \sinh(\text{dist}(q, \partial \mathcal{C}(\beta))) \\
= e^{-\text{dist}(q, \partial \mathcal{C}(\beta))} \\
\geq e^{-\sqrt{2}}
\]
which implies that
\[ \text{inj}_X(q) \geq \ln \left( e^{-\sqrt{2}} + \sqrt{e^{-2\sqrt{2}} + 1} \right) \sim 0.2407. \]
Since $\beta$ is an arbitrary closed geodesic of length less than or equal to $2 \text{arcsinh}(1)$ such that $\sigma \cap \mathcal{C}(\beta) \neq \emptyset$, by Theorem 3.2 we know that
\[ \min_{s \in [0, 2 \text{inj}_X(p)]} \text{inj}_X(\sigma(s)) \geq \min \{ \text{arcsinh}(1), 0.2407 \} = 0.2407. \]
Now we prove the claim.

Proof of Claim. Suppose for contradiction that

$$\operatorname{dist}(q, \partial C(\beta)) > \sqrt{2}$$

for some $q \in (\sigma \cap C(\beta))$. We let $t_1 > 0$ be the first time when $\sigma$ meets $C(\beta)$, and $t_2 > 0$ be the last time when $\sigma$ meets $C(\beta)$. That is,

$$t_1 := \min \{ s \in [0, 2 \operatorname{inj}_X(p)]; \sigma(s) \in C(\beta) \}$$

and

$$t_2 := \max \{ s \in [0, 2 \operatorname{inj}_X(p)]; \sigma(t) \in C(\beta) \}.$$ 

Since $\operatorname{inj}_X(\sigma(0)) = \operatorname{inj}_X(\sigma(2 \operatorname{inj}_X(p))) = \operatorname{inj}_X(p) > \arcsinh(1)$, we have that $t_1 > 0$ and $t_2 < 2 \operatorname{inj}_X(p)$. Clearly we have $\sigma(t_1) \cup \sigma(t_2) \in \partial C(\beta)$. If $\sigma(t_1)$ and $\sigma(t_2)$ are on the same component of the boundary $\partial C(\beta)$, then we have

$$t_2 - t_1 \geq \operatorname{dist}(\sigma(t_1), q) + \operatorname{dist}(q, \sigma(t_2)) > 2 \sqrt{2}.$$ 

If $\sigma(t_1)$ and $\sigma(t_2)$ are on the different components of the boundary $\partial C(\beta)$, by symmetry of the standard collar $C(\beta)$ we also have

$$t_2 - t_1 \geq 2 \operatorname{dist}(q, \partial C(\beta)) > 2 \sqrt{2}.$$ 

So we always have

$$t_2 - t_1 > 2 \sqrt{2}. \quad (3.11)$$

Now we finish the argument by considering the following two cases.

Case (1). $t_1 = \ell(\sigma([0, t_1])) \leq \ell(\sigma([t_2, 2 \operatorname{inj}_X(p)])) = 2 \operatorname{inj}_X(p) - t_2$. Let $\beta'$ be the component of the boundary $\partial C(\beta)$ with $\sigma(t_1) \in \beta'$, and we parametrize $\beta'$ such that $\beta'(0) = \beta'(\ell(\beta')) = \sigma(t_1)$. Consider the closed curve $\sigma'$ based at $p$ as following.

$$\sigma'(s) := \begin{cases} 
\sigma(s), & s \in [0, t_1]; \\
\beta'(s - t_1), & s \in [t_1, t_1 + \ell(\beta')]; \\
\sigma(2t_1 + \ell(\beta') - s), & s \in [t_1 + \ell(\beta'), 2t_1 + \ell(\beta')].
\end{cases}$$

The closed curve $\sigma'$ is freely homotopic to $\beta$. So $\sigma'$ is nontrivial. By (3.10) and (3.11), the length of $\sigma'$ satisfies

$$\ell(\sigma') = 2t_1 + \ell(\beta') \leq t_1 + (2 \operatorname{inj}_X(p) - t_2) + 2 \sqrt{2} < 2 \operatorname{inj}_X(p)$$

which is a contradiction since $\sigma$ is a shortest nontrivial geodesic loop based at $p$.

Case (2). $t_1 = \ell(\sigma([0, t_1])) \geq \ell(\sigma([t_2, 2 \operatorname{inj}_X(p)])) = 2 \operatorname{inj}_X(p) - t_2$. Let $\beta''$ be the component of the boundary $\partial C(\beta)$ with $\sigma(t_2) \in \beta''$. Similarly, we consider the closed curve $\sigma''$ based at $p$ which is defined as

$$\sigma'' = \sigma([t_2, 2 \operatorname{inj}_X(p)]) \cup \beta'' \cup \sigma([t_2, 2 \operatorname{inj}_X(p)]).$$

Then the closed curve $\sigma''$ is freely homotopic to $\beta$. So $\sigma''$ is nontrivial. By (3.10) and (3.11), the length of $\sigma''$ satisfies

$$\ell(\sigma'') = 2(2 \operatorname{inj}_X(p) - t_2) + \ell(\beta'')$$
\[ \leq t_1 + (2 \text{inj}_X(p) - t_2) + 2\sqrt{2} < 2 \text{inj}_X(p) \]

which is a contradiction since \( \sigma'' \) is a nontrivial closed loop based at \( p \).

The proof is complete. \( \square \)

4 One useful inequality

In this section we prove the following property which is crucial in the proof of Theorem 1.1.

**Proposition 4.1** Let \( X \) be a hyperbolic surface. For any \( p \in X \) we let \( \sigma : [0, 2 \text{inj}_X(p)] \to X \) be a shortest nontrivial geodesic loop based at \( p \). Assume that

\[ \inf_{s \in [0, 2 \text{inj}_X(p)]} \text{inj}_X(\sigma(s)) \geq 2\varepsilon_0 \]

for some uniform constant \( \varepsilon_0 > 0 \). Then for any function \( f \geq 0 \) on \( X \), we have

\[ \int_0^{2\text{inj}_X(p)} \left( \int_{B(\sigma(s);\varepsilon_0)} f \, d\text{Area} \right) \, ds \leq 12\varepsilon_0 \int_{\text{N}_{\varepsilon_0}(\sigma)} f \, d\text{Area} . \]

Where \( B(\sigma(s);\varepsilon_0) = \{ q \in X; \text{dist}(q, \sigma(s)) < \varepsilon_0 \} \) and \( \text{N}_{\varepsilon_0}(\sigma) \) is the \( \varepsilon_0 \)-neighbourhood of \( \sigma \), i.e.,

\[ \text{N}_{\varepsilon_0}(\sigma) = \{ z \in X; \text{dist}(x, \sigma([0, 2 \text{inj}_X(p)])) < \varepsilon_0 \} \]

We split the proof into several parts.

First since \( \sigma : [0, 2 \text{inj}_X(p)] \to X \) is a shortest nontrivial geodesic loop based at \( p \), it is known that both the two restrictions \( \sigma : [0, \text{inj}_X(p)] \to X \) and \( \sigma : [\text{inj}_X(p), 2 \text{inj}_X(p)] \to X \) are minimizing geodesics. For any \( s \in (0, 2 \text{inj}_X(p)) \), we let \( \vec{n}(s) \) be an unit normal vector of \( \sigma \) at \( \sigma(s) \). Consider the foliation \( \{ \exp_{\sigma(s)}(t \cdot \vec{n}(s)) \}_{t \in (-\varepsilon_0, \varepsilon_0), \ t \in (0, 2 \text{inj}_X(p))} \) along \( \sigma \) where \( \exp_{\sigma(s)}(\cdot) \) is the standard exponential map at \( \sigma(s) \). Set

\[ m = \left[ \frac{\text{inj}_X(p)}{\varepsilon_0} \right] \]

to be the largest integer of the number \( \frac{\text{inj}_X(p)}{\varepsilon_0} \).

Now we assume that

\[ m \geq 2. \]

Set

1. for \( 1 \leq i \leq m - 1, \)

\[ R_i = \bigcup_{s \in [(i-1)\varepsilon_0, i\varepsilon_0)} \bigcup_{t \in (-\varepsilon_0, \varepsilon_0)} \exp_{\sigma(s)}(t \cdot \vec{n}(s)) \]

and

\[ R_m = \bigcup_{s \in [(m-1)\varepsilon_0, \text{inj}_X(p)]} \bigcup_{t \in (-\varepsilon_0, \varepsilon_0)} \exp_{\sigma(s)}(t \cdot \vec{n}(s)) \]

(see Fig. 1).
2. for $1 \leq i \leq m - 1$,

$$R_i' = \bigcup_{s \in (2\text{inj}_X(p) - i\varepsilon_0, 2\text{inj}_X(p) - (i-1)\varepsilon_0]} \bigcup_{t \in (-\varepsilon_0, \varepsilon_0)} \exp_{\sigma}(t \cdot \vec{n}(s))$$

and

$$R_m' = \bigcup_{s \in [\text{inj}_X(p), 2\text{inj}_X(p) - (m-1)\varepsilon_0]} \bigcup_{t \in (-\varepsilon_0, \varepsilon_0)} \exp_{\sigma}(t \cdot \vec{n}(s)).$$

Lemma 4.2 With the notations above,

(1) $\bigcup_{i=1}^m \left( R_i \cup R_i' \right) \subseteq \mathcal{N}_{\varepsilon_0}(\sigma)$;

(2) for all $1 \leq i \neq j \leq m$, we have

$$R_i \cap R_j = \emptyset \quad \text{and} \quad R_i' \cap R_j' = \emptyset.$$

Proof Part (1) is clear.

For Part (2), we first prove

$$R_i \cap R_j = \emptyset \text{ for } 1 \leq i \neq j \leq m.$$

Suppose for contradiction that there would exist a point $q \in R_i \cap R_j$ for some $i \neq j \in [1, m]$. Recall that $\sigma : [0, \text{inj}_X(p)] \to X$ is a minimizing geodesic. By construction one may assume that $q_i \in R_i$ and $q_j \in R_j$ such that

(i) the geodesic triangle $\Delta(p_i, p_j, q)$ with vertices $p_i, p_j$ and $q$ has at least two interior $\frac{\pi}{2}$-angles;

(ii) $\max\{\text{dist}(p_i, q), \text{dist}(p_j, q)\} < \varepsilon_0$.

It follows by the triangle inequality and Part (ii) above that

$$\text{dist}(p_i, p_j) < 2\varepsilon_0$$

which implies that the geodesic triangle $\Delta(p_i, p_j, q) \subset B(p_i; 2\varepsilon_0)$. Recall that $\text{inj}_X(p_i) \geq 2\varepsilon_0$. So $\Delta(p_i, p_j, q) \subset B(p_i; 2\varepsilon_0)$ bounds a disk. By the Gauss–Bonnet formula [5] we know that the total interior angle of $\Delta(p_i, p_j, q)$ is less than $\pi$, which contradicts Part (i).

The proof for $R_i' \cap R_j' = \emptyset$ is the same as above. \qed

Remark 4.3 It is possible that for all $1 \leq i \leq m$, $R_i \cap R_i' \neq \emptyset$. 

Fig. 1 $R_i$’s and $R_m$
Lemma 4.4 With the notations above,

\[(1) \int_{0}^{\varepsilon_0} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Area} \right) ds \leq \varepsilon_0 \int_{\mathcal{N}_{\varepsilon_0}(\sigma)} f \, d\text{Area}. \]

\[(2) \int_{(m-1)\varepsilon_0}^{\varepsilon_0} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Area} \right) ds \leq 2\varepsilon_0 \int_{\mathcal{N}_{\varepsilon_0}(\sigma)} f \, d\text{Area}. \]

**Proof** It follows by the triangle inequality that

\[ \bigcup_{0 \leq s \leq \varepsilon_0} B(\sigma(s); \varepsilon_0) \subset \mathcal{N}_{\varepsilon_0}(\sigma) \text{ and } \bigcup_{(m-1)\varepsilon_0 \leq s \leq \varepsilon_0} B(\sigma(s); \varepsilon_0) \subset \mathcal{N}_{\varepsilon_0}(\sigma) \]

which implies (1), and

\[ \int_{(m-1)\varepsilon_0}^{\varepsilon_0} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Area} \right) ds \leq \left( \text{inj}_X(p) - (m - 1)\varepsilon_0 \right) \int_{\mathcal{N}_{\varepsilon_0}(\sigma)} f \, d\text{Area} \]

\[ \leq 2\varepsilon_0 \int_{\mathcal{N}_{\varepsilon_0}(\sigma)} f \, d\text{Area}. \]

The proof is complete. \(\square\)

Now we are ready to prove Proposition 4.1.

**Proof of Proposition 4.1** If

\[ m = \left\lceil \frac{\text{inj}_X(p)}{\varepsilon_0} \right\rceil \leq 5, \]

then \(2 \text{inj}_X(p) \leq 12\varepsilon_0\). Since \(\bigcup_{0 \leq s \leq 2\text{inj}_X(p)} B(\sigma(s); \varepsilon_0) \subset \mathcal{N}_{\varepsilon_0}(\sigma)\), we have

\[ \int_{0}^{2\text{inj}_X(p)} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Area} \right) ds \leq 12\varepsilon_0 \int_{\mathcal{N}_{\varepsilon_0}(\sigma)} f \, d\text{Area} \]

which completes the proof.

Now we always assume that

\[ m = \left\lceil \frac{\text{inj}_X(p)}{\varepsilon_0} \right\rceil \geq 6. \]

From Lemma 4.4 we have

\[ \int_{0}^{\text{inj}_X(p)} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Area} \right) ds = \int_{0}^{\varepsilon_0} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Area} \right) ds + \]

\[ \sum_{i=1}^{m-2} \int_{i\varepsilon_0}^{(i+1)\varepsilon_0} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Area} \right) ds + \int_{(m-1)\varepsilon_0}^{\varepsilon_0} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Area} \right) ds \]

\[ \leq 3\varepsilon_0 \int_{\mathcal{N}_{\varepsilon_0}(\sigma)} f \, d\text{Area} + \sum_{i=1}^{m-2} \int_{i\varepsilon_0}^{(i+1)\varepsilon_0} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Area} \right) ds. \tag{4.1} \]

For each \(1 \leq i \leq m - 2\), by the triangle inequality we know that

\[ \bigcup_{s=i\varepsilon_0}^{(i+1)\varepsilon_0} B(\sigma(s); \varepsilon_0) \subset (R_i \cup R_{i+1} \cup R_{i+2}). \]
Thus, by Lemma 4.2 we have
\[
\sum_{i=1}^{m-2} \int_{i \in \mathbb{E}_0} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Area} \right) ds \leq \sum_{i=1}^{m-2} \varepsilon_0 \left( \int_{R_i \cup R_{i+1} \cup R_{i+2}} f \, d\text{Area} \right)
\]
\[
= \varepsilon_0 \sum_{i=1}^{m-2} \left( \int_{R_i} f \, d\text{Area} + \int_{R_{i+1}} f \, d\text{Area} + \int_{R_{i+2}} f \, d\text{Area} \right)
\]
\[
\leq 3\varepsilon_0 \int_{\bigcup_{i=1}^{m} R_i} f \, d\text{Area}
\]
\[
\leq 3\varepsilon_0 \int_{\mathcal{N}(\sigma)} f \, d\text{Area}
\]
which together with (4.1) implies that
\[
\int_{0}^{\text{inj}_X(p)} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Area} \right) ds \leq 6\varepsilon_0 \int_{\mathcal{N}(\sigma)} f \, d\text{Area}.
\] (4.2)

Restricted on the geodesic \(\sigma([\text{inj}_X(p), 2\text{inj}_X(p)])\) and replace each \(R_i\) by \(R_i'\), one may also apply the same argument above to get
\[
\int_{\text{inj}_X(p)}^{2\text{inj}_X(p)} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Area} \right) ds \leq 6\varepsilon_0 \int_{\mathcal{N}(\sigma)} f \, d\text{Area}.
\] (4.3)

Then the conclusion follows by (4.2) and (4.3).

\[\square\]

**Remark 4.5** The two main ingredients in the proof above are:

1. the two restrictions \(\sigma : [0, \text{inj}_X(p)] \to X\) and \(\sigma : [\text{inj}_X(p), 2\text{inj}_X(p)] \to X\) are minimizing geodesics;
2. the total interior angle of any geodesic triangle, which bounds a disk, is less than or equal to \(\pi\).

Actually in the argument above, if one replaces the foliation
\[
\{\exp_{\sigma(s)}(t \cdot \vec{n}(s))\}_{t \in (-\varepsilon_0, \varepsilon_0), t \in (0, 2\text{inj}_X(p))}
\] along \(\sigma\) by
\[
\{\exp_{\sigma(s)(t \cdot \vec{w}(s))}\}_{t \in (-\varepsilon_0, \varepsilon_0), t \in (0, 2\text{inj}_X(p))}, \quad \text{\(\vec{w}(s)\) is unit and normal to } \sigma'(s),
\]
then it follows by the same argument above that one may generalize Proposition 4.1 to higher dimensions. More precisely, we have

**Proposition 4.6** Let \(M\) be a complete Riemannian manifold of nonpositive curvature. For any \(p \in M\) we let \(\sigma : [0, 2\text{inj}_M(p)] \to M\) be a shortest nontrivial geodesic loop based at \(p\). Assume that
\[
\inf_{s \in [0, 2\text{inj}_M(p)]} \text{inj}_X(\sigma(s)) \geq 2\varepsilon_0
\]
for some uniform constant \(\varepsilon_0 > 0\). Then for any function \(f \geq 0\) on \(M\), we have
\[
\int_{0}^{2\text{inj}_M(p)} \left( \int_{B(\sigma(s); \varepsilon_0)} f \, d\text{Vol} \right) ds \leq 12\varepsilon_0 \int_{\mathcal{N}_{\varepsilon_0}(\sigma)} f \, d\text{Vol}.
\]
Where \( \mathcal{N}_{\varepsilon_0}(\sigma) \) is the \( \varepsilon_0 \)-neighbourhood of \( \sigma \), i.e.,
\[
\mathcal{N}_{\varepsilon_0}(\sigma) = \{ z \in X; \text{dist}(x, \sigma([0, 2 \text{inj}_M(p)])) < \varepsilon_0 \}.
\]

5 Proof of Theorem 1.1

In this section we prove Theorem 1.1.

Let \( c : [0, T_0] \rightarrow \mathcal{M}_{-1} \) be a smooth horizontal path of Weil–Petersson arc-length parameter where \( T_0 > 0 \) is a constant. For any \( t \in (0, T_0) \), here one may view the tangent vector \( c'(t) \) as a harmonic Beltrami differential on the hyperbolic surface \( c(t) \in \mathcal{M}_{-1} \). Since \( t \) is an arc-length parameter,
\[
\int_{c(t)} |c'(t)(z)|^2 \text{dArea}(z) = 1
\]
for all \( t \in (0, T_0) \). It is known from [13] that this function \( \text{inj}_{c(t)}(p) \) is differentiable almost everywhere in \( (0, T_0) \) and the Fundamental Theorem of Calculus holds for \( \text{inj}_{c(t)}(p) \) along \( c \). Now we recall the following two lemmas from [13], whose proofs are outlined here for completeness.

**Lemma 5.1** (Rupflin–Topping, [13, Lemma 2.1]) Let \( c : [0, T_0] \rightarrow \mathcal{M}_{-1} \) be a smooth horizontal path of Weil–Petersson arc-length parameter where \( T_0 > 0 \) is a fixed constant. Then for any \( p \in S_g \), the function \( \text{inj}_{c(t)}(p) : [0, T_0] \rightarrow \mathbb{R}^{>0} \) is locally Lipschitz.

**Proof** For completeness we outline the proof here. One may see the proof of [13, Lemma 2.1] for more details. Let \( I \subset (0, T_0) \) be any sub-interval. Since \( c(t) \) is smooth, for any \( t_1, t_2 \in I \) there exists a constant \( C > 0 \) such that for all \( t_1, t_2, t \in I \),
\[
||c(t_1) - c(t_2)||_{c(t)} \leq C \cdot |t_1 - t_2|
\]
where \( ||c(t_1) - c(t_2)||_{c(t)} \) is the norm of the difference of two hyperbolic metrics \( c(t_1) \) and \( c(t_2) \) at \( c(t) \). Let \( \sigma_i (i = 1, 2) : [0, 2 \text{inj}_{c(t)}(p)] \rightarrow c(t) \) be a shortest geodesic loop based at \( p \) respectively. Without loss of generality one may assume that \( \text{inj}_{c(t_1)}(p) \geq \text{inj}_{c(t_2)}(p) \). Since,
\[
2 \text{inj}_{c(t)}(p) \leq \int_0^{2 \text{inj}_{c(t)}(p)} \sqrt{\langle \sigma'_2(s), \sigma'_2(s) \rangle_{c(t)}} ds
\]
\[
\leq \frac{1}{2} \int_0^{2 \text{inj}_{c(t)}(p)} \left( \langle \sigma'_2(s), \sigma'_2(s) \rangle_{c(t)} + 1 \right) ds.
\]
Thus, we have
\[
2(\text{inj}_{c(t_1)}(p) - \text{inj}_{c(t_2)}(p)) \leq \frac{1}{2} \int_0^{2 \text{inj}_{c(t_2)}(p)} \left( \langle \sigma'_2(s), \sigma'_2(s) \rangle_{c(t_1)} - 1 \right) ds
\]
\[
\leq \frac{1}{2} \int_0^{2 \text{inj}_{c(t_2)}(p)} ||c(t_2) - c(t_1)||_{c(t)} ds
\]
\[
\leq C \text{inj}_{c(t_2)}(p)|t_1 - t_2|
\]
\[
\leq C \ln(4g - 2)|t_1 - t_2|
\]
which completes the proof. \( \square \)
Lemma 5.2 (Rupflin–Topping, [13, Lemma 2.5]) Let $c : [0, T_0] \to \mathcal{M}_{-1}$ be a smooth horizontal path of Weil–Petersson arc-length parameter where $T_0 > 0$ is a fixed constant. For any $p \in S_g$, and suppose that the function $\text{inj}_{c(t)}(p) : [0, T_0] \to \mathbb{R}^{>0}$ is differentiable at $t = t_0 \in (0, T_0)$. Then for any shortest geodesic loop $\sigma : [0, 2 \text{inj}_{c(t_0)}(p)] \to c(t_0)$ based at $p$,

$$\frac{d}{dt} \text{inj}_{c(t)}(p)\bigg|_{t=t_0} = \frac{1}{4} \int_0^{2 \text{inj}_{c(t_0)}(p)} c'(t_0)(\sigma'(s), \sigma'(s)) ds.$$ 

**Proof** For completeness we also provide the proof here. Set

$$f(t) := \frac{1}{2} \int_0^{2 \text{inj}_{c(t_0)}(p)} \sqrt{(\sigma'(s), \sigma'(s))}c(t_0) ds - \text{inj}_{c(t)}(p).$$

So we have $f(t) \geq 0$ and $f(t_0) = 0$. Then the conclusion follows by that $f'(t_0) = 0$. □

Now we prove the key estimation in the proof of Theorem 1.1.

Proposition 5.3 Let $c : [0, T_0] \to \mathcal{M}_{-1}$ be a smooth horizontal path of Weil–Petersson arc-length parameter where $T_0 > 0$ is a fixed constant. For any $p \in S_g$, and suppose that the function $\text{inj}_{c(t)}(p) : [0, T_0] \to \mathbb{R}^{>0}$ is differentiable at $t = t_0 \in (0, T_0)$. Then we have

$$\left|\frac{d}{dt}\sqrt{\text{inj}_{c(t)}(p)}\right|_{t=t_0} \leq 0.3884.$$ 

Remark It was shown in [13, Lemma 2.2] by Rupflin and Topping that $K_g$ where $K_g > 0$ is a constant depending on $g$. Our essential improvement for Proposition 5.3 is that this constant $K_g$ can be chosen to be uniform.

Proof of Proposition 5.3 We follow the same idea as the proof of [13, Lemma 2.2], and prove it by two cases. Let $\sigma : [0, 2 \text{inj}_{c(t_0)}(p)] \to c(t_0)$ be a shortest geodesic loop based at $p$. Thus, it follows by Lemma 5.2 that

$$\left|\frac{d}{dt}\text{inj}_{c(t)}(p)\right|_{t=t_0} \leq \frac{1}{4} \int_0^{2 \text{inj}_{c(t_0)}(p)} |c'(t_0)(\sigma'(s), \sigma'(s))| ds \leq \frac{1}{4} \int_0^{2 \text{inj}_{c(t_0)}(p)} |c'(t_0)(\sigma(s))| ds. \tag{5.1}$$

Either (1). $\text{inj}_{c(t_0)}(p) \leq \arcsinh(1)$ or (2). $\text{inj}_{c(t_0)}(p) > \arcsinh(1)$. We finish the proof by considering these two cases.

Case (1). $\text{inj}_{c(t_0)}(p) \leq \arcsinh(1)$. By Proposition 3.3 we know that for any $s \in [0, 2 \text{inj}_{c(t_0)}(p)]$,

$$\left(\sqrt{2} - 1\right)\text{inj}_{c(t_0)}(p) \leq \text{inj}_{c(t_0)}(\sigma(s)) \leq \text{inj}_{c(t_0)}(p) \leq \arcsinh(1). \tag{5.2}$$
Then one may apply Proposition 2.2 to get that for any \( s \in [0, 2 \text{inj}_{c(t_0)}(p)] \),

\[
|c'(t_0)(\sigma(s))| \leq \frac{1}{\text{inj}_{c(t_0)}(\sigma(s))} \int_{c(t_0)} |c'(t_0)(z)|^2 \, d\text{Area}(z)
\]

where in the last inequality we apply the fact that \( t \) is an arc-length parameter. Thus, by (5.1), (5.2) and (5.3) we have

\[
\left| \frac{d}{dt} \text{inj}_{c(t)}(p) \right|_{t=0} = \frac{1}{4} \int_0^{2 \text{inj}_{c(t_0)}(p)} \frac{1}{\sqrt{\text{inj}_{c(t_0)}(\sigma(s))}} \, ds
\]

which implies that

\[
\left| \frac{d}{dt} \sqrt{\text{inj}_{c(t)}(p)} \right|_{t=0} \leq \frac{1}{4 \sqrt{2} - 1} \sim 0.3884. \tag{5.4}
\]

**Case (2).** \( \text{inj}_{c(t_0)}(p) > \text{arcsinh}(1) \). First by Proposition 3.4 we know that

\[
\min_{s \in [0, 2 \text{inj}_{X}(\sigma(s))]} \text{inj}_{X}(\sigma(s)) \geq 0.2407. \tag{5.5}
\]

Let

\[
0 < r_0 \leq \frac{0.2407}{2} \sim 0.1203.
\]

One may apply Proposition 2.1 to get that for any \( s \in [0, 2 \text{inj}_{c(t_0)}(p)] \),

\[
|c'(t_0)(\sigma(s))| \leq C(r_0) \int_{B_{c(t_0)}(\sigma(s); r_0)} |c'(t_0)(z)| \, d\text{Area}(z) \tag{5.6}
\]

where \( B_{c(t_0)}(\sigma(s); r_0) = \{q \in c(t_0) : \text{dist}(q, \sigma(s)) < r_0\} \). Then by (5.1) and (5.6) we have

\[
\left| \frac{d}{dt} \text{inj}_{c(t)}(p) \right|_{t=0} \leq \frac{C(r_0)}{4} \int_0^{2 \text{inj}_{c(t_0)}(p)} \sqrt{\int_{B_{c(t_0)}(\sigma(s); r_0)} |c'(t_0)(z)|^2 \, d\text{Area}(z)} \, ds
\]

\[
\leq \frac{C(r_0)}{4} \sqrt{2 \text{inj}_{c(t_0)}(p)} \times \left( \int_0^{2 \text{inj}_{c(t_0)}(p)} \left( \int_{B_{c(t_0)}(\sigma(s); r_0)} |c'(t_0)(z)|^2 \, d\text{Area}(z) \right) \right) \, ds
\]

where in the last inequality we apply the Cauchy–Schwarz inequality. In light of (5.5), we now apply Proposition 4.1. Let \( X = c(t_0) \), \( \varepsilon_0 = r_0 \) and \( f(z) = |c'(t_0)(z)|^2 \geq 0 \) in Proposition 4.1, then it follows by Proposition 4.1 that

\[
\left| \frac{d}{dt} \text{inj}_{c(t)}(p) \right|_{t=0} \leq \frac{\sqrt{6r_0}}{2} C(r_0) \sqrt{\text{inj}_{c(t_0)}(p)} \sqrt{\int_{N_{r_0}(\sigma)} |c'(t_0)(z)|^2 \, d\text{Area}(z)}
\]

\[
\leq \frac{\sqrt{6r_0}}{2} C(r_0) \sqrt{\text{inj}_{c(t_0)}(p)}
\]
where in the last inequality we apply that \( c(t) \) is of Weil–Petersson arc-length parameter. Thus, we have
\[
\left| \frac{d}{dt} \sqrt{\text{inj}_{c(t)}(p)} \right|_{t=t_0} \leq \frac{\sqrt{6}}{4} C(r_0) \sqrt{r_0}.
\] (5.7)

Recall that \( C(r) = \frac{1}{\sqrt{\pi r}} + O(1) \) as \( r \to 0 \). Let \( r_0 \to 0 \) in (5.7) we get
\[
\left| \frac{d}{dt} \sqrt{\text{inj}_{c(t)}(p)} \right|_{t=t_0} \leq \frac{\sqrt{6}}{4\sqrt{\pi}} \sim 0.3454.
\] (5.8)

Then the conclusion follows by (5.4) and (5.8).

Now we are ready to prove Theorem 1.1.

**Theorem 5.4** (Theorem 1.1) Fix a point \( p \in S_g \) \((g \geq 2)\). Then for any \( X, Y \in T_g \),
\[
\left| \sqrt{\text{inj}_X(p)} - \sqrt{\text{inj}_Y(p)} \right| \leq 0.3884 \text{dist}_{wp}(X, Y)
\]
where \( \text{dist}_{wp} \) is the Weil–Petersson distance.

**Proof of Theorem 1.1** For any \( X, Y \in \text{Teich}(S_g) \), by Wolpert [19] there exists a unique Weil–Petersson geodesic \( c : [0, \text{dist}_{wp}(X, Y)] \to \text{Teich}(S_g) \) of arc-length parameter such that \( c(0) = X \) and \( c(\text{dist}_{wp}(X, Y)) = Y \). We lift \( c \) onto a horizontal curve \( c : [0, \text{dist}_{wp}(X, Y)] \to \mathcal{M}_{-1} \) such that the projection \( \pi(c(t)) = c(t) \) for all \( t \in [0, \text{dist}_{wp}(X, Y)] \). By Lemma 5.1 we know that the injectivity radius \( \text{inj}_{c(t)}(p) : [0, \text{dist}_{wp}(X, Y)] \to \mathbb{R}^+ \) is locally Lipschitz. Then we apply the Fundamental Theorem of Calculus and Proposition 5.3 to get
\[
\left| \sqrt{\text{inj}_{c(0)}(p)} - \sqrt{\text{inj}_{c(1)}(p)} \right| = \left| \int_0^{\text{dist}_{wp}(X, Y)} \frac{d}{dt} \left( \sqrt{\text{inj}_{c(t)}(p)} \right) dt \right| \\
\leq \left| \int_0^{\text{dist}_{wp}(X, Y)} \frac{d}{dt} \left( \sqrt{\text{inj}_{c(t)}(p)} \right) dt \right| \\
\leq 0.3884 \text{dist}_{wp}(X, Y)
\]
which implies the conclusion by letting \( c \) runs over all such horizontal lifts of the Weil–Petersson geodesic \( c \) joining \( X \) and \( Y \).

If we only restrict the proof of Theorem 1.1 on the Case (2) in the proof of Proposition 5.3, one may get

**Corollary 5.5** Fix a point \( p \in S_g \) \((g \geq 2)\) and let \( X, Y \in T_g \). Assume that for some horizontal curve \( c \subset \mathcal{M}_{-1} \) such that the projection \( \pi(c) \) is the Weil–Petersson geodesic joining \( X \) and \( Y \), and
\[
\min_{t \in [0, \text{dist}_{wp}(X, Y)]} \text{inj}_{c(t)}(p) > \text{arcsinh}(1),
\]
then we have
\[
\left| \sqrt{\text{inj}_X(p)} - \sqrt{\text{inj}_Y(p)} \right| \leq 0.3454 \text{dist}_{wp}(X, Y).
\]
6 The geometry of $\mathcal{M}_g$ for large genus

In this section we study the asymptotic geometry of $\mathcal{M}_g$ for large genus.

Before proving Theorem 1.3, we recall several things in [22]. Let $\alpha \subset X$ be a simple closed geodesic. Up to a conjugation one may lift $\alpha$ to the imaginary axis $i\mathbb{R}^+$ in the upper half plane $\mathbb{H}$. A special case of Riera’s formula [12, Theorem 2] says that

$$\left\langle \nabla \ell_{\alpha}, \nabla \ell_{\alpha}\right\rangle_{wp}(X) = \frac{2}{\pi} (\ell_{\alpha} + \sum_{B \in \{\langle A \rangle \backslash \Gamma / \langle A \rangle - id\}} (u \ln \frac{u + 1}{u - 1} - 2)) \quad (6.1)$$

where $u = \cosh(\text{dist}_{\mathbb{H}}(i\mathbb{R}^+, B \circ i\mathbb{R}^+))$ and the double-coset of the identity element is omitted from the sum.

From now on, we always assume that $\alpha$ is a systolic curve of $X$ with large length, more precisely,

$$\ell_{\alpha}(X) = \ell_{\text{sys}}(X) \geq 8. \quad (6.2)$$

As in [22, Page 1327], we know that for any $B \in \{\langle A \rangle \backslash \Gamma / \langle A \rangle - id\}$ there exists a unique point $p_B \in B \circ (i\mathbb{R}^+)$ such that

$$\text{dist}_{\mathbb{H}}(p_B, i\mathbb{R}^+) = \text{dist}_{\mathbb{H}}(B \circ (i\mathbb{R}^+), i\mathbb{R}^+).$$

By [22, Lemma 4.6] and [22, Lemma 4.8], one may choose a representative $B' \in \langle A \rangle \backslash \Gamma - id$ for $B$ such that

1. $\text{dist}_{\mathbb{H}}(p_{B'}, i\mathbb{R}^+) \geq \frac{\ell_{\text{sys}}(X)}{4} \geq 2$;
2. $1 \leq r_{B'} \leq e^{\ell_{\text{sys}}(X)}$

where $p_{B'} = (r_{B'}, \theta_{B'})$ in polar coordinate be the nearest projection point on $B' \circ (i\mathbb{R}^+)$ from $i\mathbb{R}^+$. For $z = (r, \theta) \in \mathbb{H}$ given in polar coordinate where $\theta \in (0, \pi)$, the hyperbolic distance between $z$ and the imaginary axis $i\mathbb{R}^+$ is

$$\text{dist}_{\mathbb{H}}(z, i\mathbb{R}^+) = \ln |\csc \theta + |\cot \theta||. \quad (6.3)$$

Which implies

$$e^{-2 \text{dist}_{\mathbb{H}}(z, i\mathbb{R}^+)} \leq \sin^2 \theta = \frac{\text{Im}^2(z)}{|z|^2} \leq 4 e^{-2 \text{dist}_{\mathbb{H}}(z, i\mathbb{R}^+)}. \quad (6.4)$$

Now we consider the geodesic balls $\{B(p_B; 1)\}_{B \in \langle A \rangle \backslash \Gamma - id}$ of radius 1 in $\mathbb{H}$.

**Lemma 6.1** For any $B_1 \neq B_2 \in \langle A \rangle \backslash \Gamma - id$,

$$B(p_{B_1}; 1) \cap B(p_{B_2}; 1) = \emptyset.$$

**Proof** It follows by the triangle inequality and [22, Lemma 4.6]. □

**Lemma 6.2**

$$\bigcup_{B \in \langle A \rangle \backslash \Gamma - id} B(p_B; 1) \subset \left\{(r, \theta) \in \mathbb{H}; e^{-1} \leq r \leq e^{\ell_{\text{sys}}(X)+1} \text{ and } \sin(\theta) \leq 2 e^{-\frac{\ell_{\text{sys}}(X)}{8}} \right\}.$$

**Proof** For any $z = (r, \theta) \in (p_B; 1)$ where $B \in \{\langle A \rangle \backslash \Gamma - id\}$ is arbitrary, since $1 \leq r_B \leq e^{\ell_{\text{sys}}(X)}$, by the triangle inequality we clearly have

$$e^{-1} \leq r \leq e^{\ell_{\text{sys}}(X)+1}.$$
Now we control the angle $\theta$. Since $\text{dist}_H(p_B, i\mathbb{R}^+) \geq \frac{\ell_{\text{sys}}(X)}{4}$, by the triangle inequality we have that for any $z = (r, \theta) \in B(p_B; 1)$,

$$\text{dist}_H(z, i\mathbb{R}^+) \geq \text{dist}_H(p_B, i\mathbb{R}^+) - \text{dist}_H(p_B, z) \geq \frac{\ell_{\text{sys}}(X)}{4} - 1 \geq \frac{\ell_{\text{sys}}(X)}{8}.$$ 

Then by (6.3) we know that

$$\sin(\theta) \leq 2e^{-\frac{\ell_{\text{sys}}(X)}{8}}$$

which completes the proof. \hfill \square

Now we follow the same argument of the proof of [22, Proposition 4.4] to prove the following property with an effective leading constant.

**Proposition 6.3** Let $X \in \mathcal{M}_g$ with $\ell_{\text{sys}}(X) \geq 8$. Then for any curve $\alpha \subset X$ with $\ell_{\alpha}(X) = \ell_{\text{sys}}(X)$ there exists a uniform constant $C > 0$ independent of $g$ such that

$$||\nabla \ell_{\alpha}(X)||_{wp}^2 \leq 2\pi \ell_{\text{sys}}(X) \left( 1 + Ce^{-\frac{\ell_{\text{sys}}(X)}{8}} \right).$$

That is

$$||\nabla \sqrt{\ell_{\alpha}(X)}||_{wp} \leq \frac{1}{\sqrt{2\pi}} \sqrt{1 + Ce^{-\frac{\ell_{\text{sys}}(X)}{8}}}.$$

**Proof** We will apply (6.1) of Riera to finish the proof. First we know that

$$\lim_{u \to \infty} \frac{u \ln \frac{u+1}{u-1} - 2}{u^2} = \frac{2}{3}.$$ 

Similar as [22, Equation (4.5)], since $\ell_{\text{sys}}(X) \geq 8$, the quantity $u$ in Equation (6.1) satisfies $u \geq \cosh(2) > 1$. Thus, it follows by (6.1) that there exists a uniform constant $C_2 > 0$ independent of $g$ such that

$$\langle \nabla \ell_{\alpha}, \nabla \ell_{\alpha} \rangle_{wp}(X) \leq \frac{2}{\pi} \left( \ell_{\alpha} + C_2 \sum_{B \in \{A\}/\Gamma(A) - \text{id}} e^{-2 \text{dist}_H(i\mathbb{R}^+, B \circ (i\mathbb{R}^+))} \right).$$

As introduced above one may choose $p_B = (r_B, \theta_B) \in B \circ (i\mathbb{R}^+)$ such that

1. $\text{dist}_H(p_B, i\mathbb{R}^+) \geq \frac{\ell_{\text{sys}}(X)}{4} \geq 2$;
2. $1 \leq r_B \leq e^{\ell_{\text{sys}}(X)}$;
3. $\text{dist}_H(p_B, i\mathbb{R}^+ + i\mathbb{R}^+) = \text{dist}_H(B \circ (i\mathbb{R}^+), i\mathbb{R}^+)$. 

Then, we have

$$||\nabla \ell_{\alpha}(X)||_{wp}^2 \leq \frac{2}{\pi} \left( \ell_{\alpha} + C_2 \sum_{B \in \{A\}/\Gamma(A) - \text{id}} e^{-2 \text{dist}_H(i\mathbb{R}^+, p_B)} \right). \quad (6.5)$$
It is known from [20, Lemma 2.4] or [22, Lemma 2.1] that the function $e^{-2 \text{dist}_{H}(i\mathbb{R}^+, z)}$ has the mean value property. More precisely, it follows by [20, Lemma 2.4] or [22, Lemma 2.1] that there exists a uniform constant $C_3 > 0$ such that

$$e^{-2 \text{dist}_{H}(i\mathbb{R}^+, p_B)} \leq C_3 \int_{B_H(p_B; 1)} e^{-2 \text{dist}_{H}(z, i\mathbb{R}^+)} \, d\text{Area}(z).$$

By Lemma 6.1 we know that the geodesic balls $\{B_H(p_B; 1)\}_{B \in \{\langle A \rangle \setminus \langle A \rangle \to i\mathbb{R}^+\}}$ are pairwise disjoint. Thus, we have

$$\sum_{B \in \{\langle A \rangle \setminus \langle A \rangle \to i\mathbb{R}^+\}} e^{-2 \text{dist}_{H}(i\mathbb{R}^+, p_B)} \leq C_3 \sum_{B \in \{\langle A \rangle \setminus \langle A \rangle \to i\mathbb{R}^+\}} \int_{B_H(p_B; 1)} e^{-2 \text{dist}_{H}(z, i\mathbb{R}^+)} \, d\text{Area}(z) \leq C_3 \int_{\bigcup_{B \in \{\langle A \rangle \setminus \langle A \rangle \to i\mathbb{R}^+\}} B_H(p_B; 1)} e^{-2 \text{dist}_{H}(i\mathbb{R}^+, p_B)} \, d\text{Area}(z).$$

(6.6)

It follows by (6.4) and Lemma 6.2 that

$$\int_{\bigcup_{B \in \{\langle A \rangle \setminus \langle A \rangle \to i\mathbb{R}^+\}} B_H(p_B; 1)} e^{-2 \text{dist}_{H}(i\mathbb{R}^+, p_B)} \, d\text{Area}(z) \leq \int_{\sin(\theta) \leq 2e^{-\ell_{\text{sys}}(X)/2}} e^{\ell_{\text{sys}}(X)} \sin^2 \theta \, d\text{Area}(z) \leq 2 \int_{0}^{\arcsin(2e^{-\ell_{\text{sys}}(X)/2})} \int_{e^{-1}}^{e^{\ell_{\text{sys}}(X)/2}} \frac{\sin^2 \theta \, dr \, d\theta}{r^2 \sin^2 \theta} e^{-\ell_{\text{sys}}(X)} \left(\ell_{\text{sys}}(X) + 2\right) \leq C_4 \ell_{\text{sys}}(X) e^{-\ell_{\text{sys}}(X)/8}$$

(6.7)

where $C_4 > 0$ is a uniform constant.

Thus, it follows by (6.5)–(6.7) that

$$||\nabla \ell_\alpha(X)||^2_{up} \leq \frac{2}{\pi} \ell_{\text{sys}}(X) \left(1 + C_2 C_3 C_4 e^{-\ell_{\text{sys}}(X)/8}\right).$$

(6.8)

Then the conclusion follows by choosing

$$C = C_2 C_3 C_4 > 0$$

which is a uniform constant independent of $g$. \qed

Recall that

$$\text{sys}(g) = \max_{X \in \mathcal{M}_g} \ell_{\text{sys}}(X).$$

As $g \to \infty$, by Buser–Sarnak [4] we know that $\text{sys}(g)$ is uniformly comparable to $\ln(g)$. In particular

$$\text{sys}(g) \to \infty \text{ as } g \to \infty.$$

For any multicurve $\gamma \subset S_g$, we denote by $\mathcal{M}_g^\gamma$ the stratum of $\mathcal{M}_g$ whose pinching curves are $\gamma$. Wolpert in [20] applied Riera’s formula [12, Theorem 2] to give an upper bound for
the Weil–Petersson distance from any $X \in \mathcal{M}_g$ to $\mathcal{M}_g^\gamma$ in terms of the length $\ell_\gamma(X)$ of $\gamma$ at $X$. More precisely,

**Theorem 6.4** [Wolpert, [20, Section 4]] For any $X \in \mathcal{M}_g$,

$$\text{dist}_{wp}(X, \mathcal{M}_g^\gamma) \leq \sqrt{2\pi \ell_\gamma(X)}.$$

Now we are ready to prove Theorem 1.3.

**Theorem 6.5** (=Theorem 1.3) The following limit holds:

$$\lim_{g \to \infty} \frac{\text{InRad}(\mathcal{M}_g)}{\sqrt{\text{sys}(g)}} = \sqrt{2\pi} \sim 2.5066.$$

**Proof of Theorem 1.3** For the upper bound, we follow the same argument as the proof of the upper bound of [22, Theorem 1.1]. For any hyperbolic surface $X \in \mathcal{M}_g$, let $\alpha \subset X$ with $\ell_{\text{sys}}(X) = \ell_\alpha(X)$. Let $\mathcal{M}_g^\alpha$ be a stratum of $\mathcal{M}_g$ whose pinching curve is $\alpha$. Then it follows by Theorem 6.4 that

$$\text{dist}_{wp}(X, \mathcal{M}_g^\alpha) \leq \sqrt{2\pi \ell_\alpha(X)} \leq \sqrt{2\pi \text{sys}(g)}$$

which implies that

$$\text{dist}_{wp}(X, \partial \mathcal{M}_g) \leq \sqrt{2\pi \text{sys}(g)}.$$

Since $X \in \mathcal{M}_g$ is arbitrary, we have

$$\limsup_{g \to \infty} \frac{\text{InRad}(\mathcal{M}_g)}{\sqrt{\text{sys}(g)}} \leq \sqrt{2\pi} \sim 2.5066. \quad (6.9)$$

For the lower bound, we follow a similar idea as in [1]. Let $X \in \mathcal{M}_g$ with $\ell_{\text{sys}}(X) = \text{sys}(g)$.

Let $c : [0, \text{InRad}(\mathcal{M}_g)] \to \mathcal{M}_g$ be a Weil–Petersson geodesic of arc-length parameter realizing $\text{InRad}(\mathcal{M}_g)$, i.e.,

1. $c(0) = X$;
2. $c(t) \in \mathcal{M}_g$ for all $t \in [0, \text{InRad}(\mathcal{M}_g)]$;
3. $c(\text{InRad}(\mathcal{M}_g)) \in \partial \mathcal{M}_g$.

For any fixed number

$$T > 8,$$

by continuity one may assume that $t_g \in (0, \text{InRad}(\mathcal{M}_g))$ such that for large enough $g$,

$$\min_{t \in [0, t_g]} \ell_{\text{sys}}(c(t)) = \ell_{\text{sys}}(c(t_g)) = T > 8. \quad (6.10)$$

By [22, Lemma 3.4] we know that $\ell_{\text{sys}}(c(\cdot))$ is piecewise smooth. So one may apply the Fundamental Theorem of Calculus and the Cauchy–Schwarz inequality to get

$$\left| \sqrt{\text{sys}(g)} - \sqrt{T} \right| = \left| \sqrt{\ell_{\text{sys}}(X)} - \sqrt{\ell_{\text{sys}}(c(t_g))} \right| = \left| \int_0^{t_g} \langle \nabla \sqrt{\ell_{\text{sys}}(c(t))}, c'(t) \rangle_{wp} dt \right|.$$
\[ \leq \int_0^{t_g} \left\| \nabla \ell_{\text{sys}}(c(t)) \right\|_{\text{wp}} \, dt \leq \frac{t_g}{\sqrt{2\pi}} \sqrt{\left( 1 + Ce^{-\frac{T}{8}} \right)} \]  

(6.11)

where we apply Proposition 6.3 and (6.10) in the last inequality above, and the uniform constant \( C > 0 \) is from Proposition 6.3. Clearly we have

\[ t_g \leq \text{InRad}(\mathcal{M}_g). \]

Recall that \( \text{sys}(g) \to \infty \) as \( g \to \infty \). Thus, it follows by (6.11) that

\[
\liminf_{g \to \infty} \frac{\text{InRad}(\mathcal{M}_g)}{\sqrt{\text{sys}(g)}} \geq \liminf_{g \to \infty} \frac{t_g}{\sqrt{\text{sys}(g)}} \geq \frac{\sqrt{2\pi}}{\sqrt{\left( 1 + Ce^{-\frac{T}{8}} \right)}}.
\]

Since \( T \geq 8 \) is arbitrary, let \( T \to \infty \) we get

\[
\liminf_{g \to \infty} \frac{\text{InRad}(\mathcal{M}_g)}{\sqrt{\text{sys}(g)}} \geq \sqrt{2\pi} \sim 2.5066. \quad (6.12)
\]

Then the conclusion follows by (6.9) and (6.12). \( \square \)

Remark 6.6 The argument for Theorem 1.3 also works for the moduli space \( \mathcal{M}_{g,n} \) of Riemann surfaces with punctures where \( n > 0 \). We only consider the closed case in this paper for simplicity.

Remark 6.7 The argument of Theorem 1.3 above highly relies on large genus. Bromberg and Bridgeman in [1] proved the following surprising result: for all \( g, n \) with \( 3g + n - 3 < 0 \),

\[ \text{InRad}(\mathcal{M}_{g,n}) \geq 0.94\sqrt{2\pi \text{sys}(g, n)} \]

where \( \text{sys}(g, n) = \max_{X \in \mathcal{M}_{g,n}} \ell_{\text{sys}}(X) \).

Similarly, we define the Weil–Petersson inradius \( \text{InRad}(T_g) \) of the Teichmüller space \( T_g \) as

\[ \text{InRad}(T_g) := \max_{X \in T_g} \text{dist}_{\text{wp}}(X, \partial T_g) \]

where \( \partial T_g \) is the boundary of \( T_g \) consisting of nodal surfaces. The proof of Theorem 1.3 also gives that

Corollary 6.8 The following limit holds:

\[ \lim_{g \to \infty} \frac{\text{InRad}(T_g)}{\sqrt{\text{sys}(g)}} = \sqrt{2\pi}. \]
7 Proof of Theorem 1.4

For any $L > 0$ which may depend on the genus $g$, recall that for any $X \in \mathcal{M}C(\leq L)$ there exists a pants decomposition $\mathcal{P}$ of $X$ such that the length satisfies

$$\max_{\alpha \in \mathcal{P}} \ell(\alpha) \leq L.$$ 

The following lemma tells that the largest radius of embedded hyperbolic disk in $X$ is bounded above by a function of $L$. More precisely,

**Lemma 7.1** For any $X \in \mathcal{M}C(\leq L)$,

$$\max_{p \in X} \text{inj}_X(p) < \frac{L}{2} + \ln(6).$$  

**Proof** For any $p \in X \in \mathcal{M}C(\leq L)$, one may assume that $p \in P$ where $P$ is a pant whose three boundary closed geodesics all have length $\leq L$. Take two copies of $P$ and we double them into a closed hyperbolic surface $X_2$ of genus 2 (here one may take any twist along these three closed geodesics). In $X_2$, by (2.1) we have

$$\text{inj}_{X_2}(p) \leq \ln(6).$$

Let $B_{X_2}(p; \text{inj}_{X_2}(p)) := \{ q \in X_2; \text{dist}(q, p) < \text{inj}_{X_2}(p) \}$ be the hyperbolic open disk in $X_2$ centered at $p$ of radius $\text{inj}_{X_2}(p)$. We finish the proof by considering the following two cases.

- **Case (1).** $B_{X_2}(p; \text{inj}_{X_2}(p)) \cap \partial P = \emptyset$. For this case we clearly have
  
  $$\text{inj}_X(p) = \text{inj}_{X_2}(p) \leq \ln(6).$$

- **Case (2).** $B_{X_2}(p; \text{inj}_{X_2}(p)) \cap \partial P \neq \emptyset$. For this case, one may assume
  
  $$p_0 \in B_{X_2}(p; \text{inj}_{X_2}(p)) \cap \alpha$$

for some component $\alpha$ of $\partial P$. We parametrize $\alpha$ such that $\alpha(0) = \alpha(\ell(\alpha)) = p_0$. Let $\sigma : [0, \text{dist}(p, p_0)] \to P \subset X$ be a shortest geodesic of $X$ joining $p$ and $p_0$. Consider the closed curve $\sigma'$ based at $p$ as following.

$$\sigma'(s) := \begin{cases} \sigma(s), & 0 \leq s \leq \text{dist}(p, p_0) \\ \alpha(s - \text{dist}(p, p_0)), & \text{dist}(p, p_0) \leq s \leq \text{dist}(p, p_0) + \ell(\alpha) \\ \alpha(2 \text{dist}(p, p_0) + \ell(\alpha) - s), & \text{dist}(p, p_0) + \ell(\alpha) \leq s \leq 2 \text{dist}(p, p_0) + \ell(\alpha) \end{cases}$$

This closed curve $\sigma' \subset P \subset X$ is freely homotopic to $\alpha$. So it is nontrivial. Thus, we have

$$\text{inj}_X(p) < \frac{\ell(\sigma')}{2} = \text{dist}(p, p_0) + \frac{\ell(\alpha)}{2} \leq \ln(6) + \frac{L}{2}.$$  

(7.2)

Then the conclusion follows by these two cases.  

Now we recall the following result of Mirzakhani which roughly says that as $g \to \infty$, almost all hyperbolic surfaces of genus $g$ contain an embedded hyperbolic disk of radius $\ln(6)$. More precisely,

**Theorem 7.2** [Mirzakhani, [11, Theorem 4.5]]

$$\lim_{g \to \infty} \frac{\text{Vol}_{WP} \left( \left\{ X \in \mathcal{M}g; \max_{p \in X} \text{inj}_X(p) \geq \frac{\ln(g)}{6} \right\} \right)}{\text{Vol}_{WP}(\mathcal{M}g)} = 1.$$
Now we are ready to prove Theorem 1.4.

**Theorem 7.3** (= Theorem 1.4) For any small enough $\epsilon > 0$ and if $L_g = \epsilon \ln(g)$, then

$$\lim_{g \to \infty} \frac{\text{Vol}_{WP} \left( \{ X \in \mathcal{M}_g; \ \text{dist}_{wp} \left( \{ X, \mathcal{MC}(\leq L_g) \right) > 1.0511\sqrt{\ln(g)} \} \right)}{\text{Vol}_{WP}(\mathcal{M}_g)} = 1$$

where $\text{Vol}_{WP}(\cdot)$ is the Weil–Petersson volume.

**Proof of Theorem 1.4** Set

$$\mathcal{MR} := \left\{ X \in \mathcal{M}_g; \ \max_{p \in X} \text{inj}_X(p) \geq \frac{\ln(g)}{6} \right\}.$$  

For any $X \in \mathcal{MR}$ and $Y \in \mathcal{MC}(\leq L_g)$, one may let $p \in S_g$ such that

$$\text{inj}_X(p) \geq \frac{\ln(g)}{6}.$$

Then it follows by Theorem 1.1 and Lemma 7.1 that

$$\text{dist}_{wp}(X, Y) \geq \frac{\sqrt{\text{inj}_X(p) - \sqrt{\text{inj}_Y(p)}}}{0.3884} \geq \frac{\sqrt{\ln(g) - \sqrt{3L_g} + 6\ln(6)}}{0.3884\sqrt{6}}.$$  

Since $\frac{1}{0.3884\sqrt{6}} \sim 1.051102 > 1.0511$ and $L_g = \epsilon \ln(g)$, we have that for large enough $g$ and small enough $\epsilon > 0$,

$$\text{dist}_{wp}(X, Y) > 1.0511\sqrt{\ln(g)}. \quad (7.3)$$

Which implies that for large enough $g$,

$$\mathcal{MR} \subset \left\{ X \in \mathcal{M}_g; \ \text{dist}_{wp}(X, \mathcal{MC}(\leq L_g)) > 1.0511\sqrt{\ln(g)} \right\}.$$  

Then the conclusion follows by Theorem 7.2.

**Remark 7.4** For $L > 0$, we say $X \in \mathcal{M}_g$ has total pants length at least $L$ if for any pants decomposition $\mathcal{P}$ of $X$, $\sum_{\alpha \in \mathcal{P}} \ell(\alpha) \geq L$. Guth, Parlier and Young in [8] showed that for any $\epsilon > 0$,

$$\lim_{g \to \infty} \frac{\text{Vol}_{WP} \left( \{ X \in \mathcal{M}_g; \ X \text{ has total pants length at least } g^{\frac{2}{3} - \epsilon} \} \right)}{\text{Vol}_{WP}(\mathcal{M}_g)} = 1.$$  

Clearly we have that for any $X \in \mathcal{MC}(\leq L_g)$, the hyperbolic surface $X$ has total pants length at most $(3g - 3)L_g$. We do not know too much information on the least total pants length of $Y \in \mathcal{M}_g$ with $\text{dist}_{wp}(Y, \mathcal{MC}(\leq L_g)) > 1.0511\sqrt{\ln(g)}$.

**Acknowledgements** The author would like to thank Martin Bridgeman, Ran Ji and Scott Wolpert for helpful conversations on this paper, and thank to Melanie Rupflin, Peter Topping and Shing-Tung Yau for their interests. He especially would like to thank Peter Topping for useful discussions on the Definition on Page 2. He is also very grateful to anonymous referees for their helpful comments. This work is supported by the NSFC Grant No. 12171263 and a grant from Tsinghua University.

\[\text{Springer}\]
References

1. Bridgeman, M., Bromberg, K.: Strata separation for the Weil-Petersson completion and gradient estimates for length functions. J. Topol. Anal. (2022). https://doi.org/10.1142/S1793525321500667
2. Bridgeman, M., Brock, J., Bromberg, K.: Schwarzian derivatives, projective structures, and the Weil-Petersson gradient flow for renormalized volume. Duke Math. J. 168(5), 867–896 (2019)
3. Brock, J.F.: The Weil-metric and volumes of 3-dimensional hyperbolic convex cores. J. Am. Math. Soc. 16(3), 495–535 (2003)
4. Buser, P., Sarnak, P.: On the period matrix of a Riemann surface of large genus. Invent. Math. 117(1), 27–56. With an appendix by J. H. Conway and N. J. A, Sloane (1994)
5. Buser, P.: Geometry and spectra of compact Riemann surfaces. Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA (2010) Reprint of the 1992 edition
6. Bridgeman, M., Yunhui, W.: Uniform bounds on harmonic Beltrami differentials and Weil-Petersson curvatures. J. Reine Angew. Math. 770, 159–181 (2021)
7. Cavendish, W., Parlier, H.: Growth of the Weil-Petersson diameter of moduli space. Duke Math. J. 161(1), 139–171 (2012)
8. Guth, L., Parlier, H., Young, R.: Pants decompositions of random surfaces. Geom. Funct. Anal. 21(5), 1069–1090 (2011)
9. Imayoshi, Y., Taniguchi, M.: An Introduction to Teichmüller Spaces. Springer-Verlag, Tokyo (1992). Translated and revised from the Japanese by the authors
10. Kojima, S., McShane, G.: Normalized entropy versus volume for pseudo-Anosovs. Geom. Topol. 22(4), 2403–2426 (2018)
11. Mirzakhani, M.: Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus. J. Differ. Geom. 94(2), 267–300 (2013)
12. Riera, G.: A formula for the Weil-Petersson product of quadratic differentials. J. Anal. Math. 95, 105–120 (2005)
13. Rupflin, M., Topping, P.M.: Horizontal curves of hyperbolic metrics. Calc. Var. Partial Di ffer. Equ. 57(4), Paper No. 106, 17 (2018)
14. Schlenker, J.-M.: Teichmüller harmonic map flow into nonpositively curved targets. J. Differ. Geom. 108(1), 135–184 (2018)
15. Schlenker, J.-M.: The renormalized volume and the volume of the convex core of quasifuchsian manifolds. Math. Res. Lett. 20(4), 773–786 (2013)
16. Teo, L.-P.: The Weil-Petersson geometry of the moduli space of Riemann surfaces. Proc. Am. Math. Soc. 137(2), 541–552 (2009)
17. Tromba, A.J.: Teichmüller theory in Riemannian geometry. Lectures in Mathematics. ETH Zürich, Birkhäuser Verlag, Basel (1992). Lecture notes prepared by Jochen Denzler
18. Wolpert, S.A.: Geodesic length functions and the Nielsen problem. J. Differ. Geom. 25(2), 275–296 (1987)
19. Wolpert, S.A.: Behavior of geodesic-length functions on Teichmüller space. J. Differ. Geom. 79(2), 277–334 (2008)
20. Wolpert, S.A.: Families of Riemann Surfaces and Weil–Petersson Geometry. CBMS Regional Conference Series in Mathematics, vol. 113, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI (2010)
21. Wu, Y.: Growth of the Weil–Petersson inradius of moduli space. Annales de l’Institut Fourier 69(3), 1309–1346 (en) (2019)
22. Wu, Y.: Systole functions and Weil–Petersson geometry. Preprint
23. Wolf, M., Wu, Y.: Uniform bounds for Weil–Petersson curvatures. Proc. Lond. Math. Soc. (3) 117(5), 1041–1076 (2018)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.