Strong Stationary Duality for Möbius monotone Markov chains: examples

Pawel Lorek∗†  
Ryszard Szekli∗ ‡

University of Wrocław

January 22, 2014

Abstract

We construct strong stationary dual chains for Ising model on a circle, non-symmetric random walk on square lattice and a random walk on hypercube. The strong stationary dual chains are all sharp and have the same state space as original chains. We use Möbius monotonicity of these chains with respect to natural orderings of the corresponding state spaces. This method provides an easy way to find eigenvalues in the Ising model and for a random walk on hypercube.

Keywords: Markov chains; stochastic monotonicity; eigenvalues; Möbius monotonicity; strong stationary duality; strong stationary times; separation distance; mixing time; Ising model; hypercube

AMS MSC 2010: 60J10; 06A06; 60G40

1 Introduction

Consider an ergodic Markov chain \( X = (X_n)_{n \geq 0} \) on a discrete (finite or countable) state space \( E \) with transition matrix \( P \) and initial distribution \( \nu \). One way of studying the speed of convergence of \( X \) to its stationary distribution \( \pi \) is to find (and bound its tail) so-called Strong Stationary Time (SST), i.e. such a stopping time \( T \) (\( T \) implicitly depends on \( \nu \)) that it is independent from \( X_T \) and \( X_T \) has distribution \( \pi \). SST’s were introduced by Aldous and Diaconis [2, 3], who also gave examples of SST and their applications. Many examples can also be found in Diaconis [4]. First examples of SST’s were created by ad hoc methods. A general approach was invented by Diaconis and Fill [5] who introduced dual processes. They showed that for \( X \) there always exists absorbing, so-called Strong Stationary Dual (SSD) absorbing chain \( X^* \), such that its time to absorption \( T^* \) is equal, in distribution, to a SST \( T \) for \( X \). Their proof is an existence type argument which does not show how to construct a dual chain in general. They showed one tractable case [5, Theorem 4.6], where the state space is linearly ordered. Under the condition of stochastic monotonicity (related to the linear order) of the corresponding time-reversed chain (and some assumptions on the initial distribution) they gave a recipe of how to construct a dual chain on the same state space. A special, and important, case is a stochastically monotone birth-and-death chain for which the dual chain is an absorbing birth-and-death chain.

Strong stationary dual chains have a variety of applications. Diaconis and Fill [6] gave an extension of this theory to countable state spaces. Fill [11] gave a stochastic proof of a well-known theorem (usually attributed to Keilson), which states that the first passage time from 0 to \( M \) of a stochastically monotone birth-and-death process on \( \{0, \ldots, M\} \) is equal, in distribution, to a sum of geometric random variables related to the spectral values of \( X \). Similar results for

∗Work of both authors supported by NCN Research Grant DEC-2011/01/B/ST1/01305.
†Email: Pawel.Lorek@math.uni.wroc.pl
‡Email: Ryszard.Szekli@math.uni.wroc.pl
continuous time birth-and-death processes were obtained by Diaconis and Miclo [7]. Diaconis and Saloff-Coste [8] studied cut-off phenomena for birth-and-death chains using SSD theory.

All the mentioned examples above (although very interesting) somehow rely on Theorem 4.6 of [5] which involves linearly ordered states space. That is why most of the known examples are related to birth-and-death chains. The main underlying assumption is (classical) stochastic monotonicity of the time-reversed chain. Although this monotonicity is defined also for partially ordered state spaces, it is not sufficient for an analogous construction of a SSD chain as in Diaconis and Fill [5]. Lorek and Szekli [15] gave a recipe of how to construct dual chains on partially ordered state spaces with a special feature that the duals have the same state space as original chains. The assumption of the classical stochastic monotonicity was replaced by the assumption of Möbius monotonicity. This extension (to partially ordered state spaces) opens a new way of finding SSD chains defined for not linearly ordered state spaces. The purpose of this paper is to get a new SSD insight to some classical examples of finite state Markov chains. In section 2 we recall needed definitions and facts about Möbius monotone chains. In section 3 we present strong stationary duals for an Ising model on a circle, non-symmetric random walk on a square lattice, and for a random walk on hypercube. We find eigenvalues in the Ising model and for a random walk on hypercube immediately from the form of the duals - since they are connected by a square lattice, and for a random walk on a hypercube. We find eigenvalues in the Ising model and for a random walk on hypercube immediately from the form of the duals - since they are related to birth-and-death chains. The main underlying assumption is (classical) stochastic monotonicity of the time-reversed chain. Although this monotonicity is defined also for partially ordered state spaces, it is not sufficient for an analogous construction of a SSD chain as in Diaconis and Fill [5] which involves linearly ordered states space. That is why most of the known examples are related to birth-and-death chains. The main underlying assumption is (classical) stochastic monotonicity of the time-reversed chain. Although this monotonicity is defined also for partially ordered state spaces, it is not sufficient for an analogous construction of a SSD chain as in Diaconis and Fill [5].

In section 2 we recall needed results on SSD and Möbius monotone chains. For a more complete material on duality see Diaconis and Fill [5], and for results on Möbius monotone chains, see Lorek and Szekli [15].

2 Möbius monotonicity and duality

In this section we recall needed results on SSD and Möbius monotone chains. For a more complete material on duality see Diaconis and Fill [5], and for results on Möbius monotone chains, see Lorek and Szekli [15].

2.1 Strong Stationary Duality

For an ergodic Markov chain \( X = (X_n)_{n \geq 0} \) with the transition matrix \( P \) and initial distribution \( \nu \), we are interested in bounding a distance between \( \nu P^k \) (a distribution of a chain at step \( k \)) and its stationary distribution \( \pi \). Often used distance is the total variation distance \( d_{TV}(\nu P^k, \pi) = \max_{A \subseteq \mathbb{E}} |\nu P^k(A) - \pi(A)| \). Another useful distance is the separation distance \( s(\nu P^k, \pi) = \max_{e \in \mathbb{E}} (1 - \nu P^k(e)/\pi(e)) \). For random times \( T \) which are SST, Aldous and Diaconis [5] show that \( d_{TV}(\nu P^k, \pi) \leq s(\nu P^k, \pi) \leq P(T > n) \).

Let \( X^* \) be a Markov chain with transition matrix \( P^* \), initial distribution \( \nu^* \) and a state space \( E^* \), with an absorbing state \( e^*_a \). Let \( \Lambda = \Lambda(e^*), e^* \in E^*, e \in E \) be a stochastic kernel (called a link), such that \( \Lambda(e^*_a, \cdot) = \pi \), for \( e^*_a \in E^* \). \( X^* \) is a Strong Stationary Dual (SSD) chain for \( X \) if

\[
\nu = \nu^* \Lambda \quad \text{and} \quad \Lambda P = P^* \Lambda. \tag{2.1}
\]

Diaconis and Fill [5] proved that the absorption time \( T^* \) of \( X^* \) is a SST for \( X \). Thus, the problem of finding SST for \( X^* \) translates into the problem of studying the absorption time of \( X^* \).

**Definition 2.1.** Strong Stationary Dual chain \( X^* \) is called sharp if \( s(\nu P^n, \pi) = P(T^* > n) \).

**Remark 2.1.** The relation (2.1) implies that for finite \( E \) and \( E^* \), \( P \) and \( P^* \) have the same set of eigenvalues.

It turns out, that in some examples we can easily identify the eigenvalues of \( P^* \), and thus, by the above remark, we will also obtain the eigenvalues of \( P \) which are usually not easy to obtain directly.
2.2 Duality for Möbius monotone chains

In this section we recall how to construct a SSD chain for finite partially ordered state spaces. We shall consider a finite state space $E = \{e_1, \ldots, e_M\}$ with a partial ordering $\preceq$. From the very beginning we shall choose an enumeration of $E$ such that $e_i \preceq e_j$ implies $i < j$ (which is always possible). We call such an enumeration consistent with $\preceq$. With this enumeration the partial ordering can by represented by an upper-triangular, 0-1 valued matrix $C$. The inversion $C^{-1}$ represents (in the incidence algebra) the so called Möbius function, usually denoted by $\mu$, see Rota [17]. The Möbius function allows for the following calculus: it is possible to recover $f$ from the relation $F(e) = \sum_{e \preceq e_i} f(e)$, namely $f(e_i) = \sum_{e \preceq e_i} \mu(e_i, e) F(e)$.

**Definition 2.2.** Let $P$ be a transition matrix with enumeration of states consistent with $C$. We say that $P$ (or alternatively, $X$) is $\downarrow$-Möbius monotone ($\uparrow$-Möbius monotone) if $C^{-1}PC \succeq 0$ ($C^TPC \geq 0$) (each entry is nonnegative).

We say that $f : E \to \mathbb{R}$ is $\downarrow$-Möbius monotone ($\uparrow$-Möbius monotone) if $f(C^T)^{-1} \geq 0$ ($fC^{-1} \geq 0$).

In terms of the transition probabilities, we have

- $\downarrow$-Möbius monotonicity: $\forall(e_i, e_j \in E) \sum_{e \preceq e_i} \mu(e_i, e) P(e, \{e_j\}^\uparrow) \geq 0$,
- $\uparrow$-Möbius monotonicity: $\forall(e_i, e_j \in E) \sum_{e \preceq e_j} P(e, \{e_i\}^\downarrow) \mu(e, e_j) \geq 0$,

where $\{e_j\}^\downarrow = \{e : e \preceq e_j\}$, $\{e_j\}^\uparrow = \{e : e \succeq e_j\}$, and $P(e, A) = \sum_{e' \in A} P(e, e')$.

We recall the SSD result of Lorek and Szekli [15] ($\hat{X}$ denotes the time-reversed process).

**Theorem 2.1** (Lorek and Szekli [15]). Let $X$ be an ergodic Markov chain on a finite state space $E = \{e_1, \ldots, e_M\}$, which is partially ordered with $\preceq$, and has a unique maximal state $e_M$. For the stationary distribution $\pi$ and an initial distribution $\nu$ we assume that

(i) $g(e) = \frac{\nu(e)}{\pi(e)}$ is $\downarrow$-Möbius monotone,

(ii) $\hat{X}$ is $\downarrow$-Möbius monotone.

Then there exists a Strong Stationary Dual chain $X^*$ on $E^* = E$ with link being a truncated stationary distribution $\Lambda(e_i, e_j) = I(e_i \preceq e_j) \frac{\pi(e_j)}{P(e_j)}$, where $H(e_j) = \sum_{e \preceq e_j} \pi(e)$. The initial distribution and transitions of $X^*$ are given, respectively, by

$$
\nu^*(e_i) = H(e_i) \sum_{e \preceq e_i} \mu(e_i, e)g(e),
$$

$$
\mathbf{P}^*(e_i, e_j) = \frac{H(e_j)}{H(e_i)} \sum_{e \preceq e_j} \mu(e_j, e)\mathbf{P}(e, \{e_j\}^\downarrow).
$$

**Remark 2.2.** Following Remark 2.39 of Diaconis and Fill [5] and the terminology used there, the Strong Stationary Dual $X^*$ in Theorem [2.1] is sharp, and the corresponding strong stationary time is the time to stationarity, i.e., $s(\nu \mathbf{P}^n, \pi) = P(T > n)$. The reason for this is that $\Lambda(e^*, e_M) = 0$ for all, $e^* \neq e_M \in E^*$.

**Remark 2.3.** Theorem [2.1] is stated for $\downarrow$-Möbius monotonicity, but it can be similarly stated for $\uparrow$-Möbius monotonicity (see Corollary 3.1 in [15]). The other formulation is potentially useful, because a chain can be, e.g., $\downarrow$-Möbius monotone but not $\uparrow$-Möbius monotone.

**Remark 2.4.** The assumption on the initial distribution is not very restrictive, for example if $e_1$ is a unique minimal state and $\nu = \delta_{e_1}(\cdot)$, then the assumption is fulfilled, and also $\nu^* = \delta_{e_1}(\cdot)$. For simplicity of presentation, in all subsequent examples the initial distribution will be the single atom at the minimal element (this assumption may be relaxed).
In order to find and use the above constructed SSD chains one has to find an appropriate ordering (w.r.t which the chain is Möbius monotone). It is worth mentioning, that for linearly ordered state space Möbius monotonicity is equivalent to the usual stochastic monotonicity, in general partially ordered spaces this is not the case. It turns out that for partially ordered spaces some natural orderings work. A non-symmetric random walk on the unit cube is an example presented in [15]. In the next section we shall give new examples.

3 Möbius monotone Markov chains: examples

3.1 Ising model on a circle

Let $G = (V, E)$ be a finite graph. Elements of state space $\mathbb{E} = \{-1, 1\}^V$ are called configurations, and for $e \in \mathbb{E}$ the value $e(v)$ is called the spin at vertex $v$. For a given configuration $e$ its energy is defined as

$$\mathcal{H}(e) = -\sum_{\{x,y\} \in E} e(x) \cdot e(y),$$

where the sum is over all edges of the graph. For $\beta \geq 0$, the Ising model on the graph $G$ with parameter $\beta$ is the probability measure on $\mathbb{E}$ given by

$$\pi(e) = \frac{e^{-\beta \mathcal{H}(e)}}{Z_\beta}, \quad (3.1)$$

where $Z_\beta = \sum_{e \in \mathbb{E}} e^{-\beta \mathcal{H}(e)}$ is a normalizing constant. The parameter $\beta$ has a physical interpretation as the inverse of the temperature of the configuration. Note, that for $\beta = 0$ (equivalent to infinite temperature), every spin configuration is equally likely, i.e., it is the same as setting spin at each vertex to -1 or +1 with probability 1/2 independently. In general, $\beta$ represents the influence of energy $\mathcal{H}$ on $\pi$.

This model has focused a lot of attention in the context of speed of convergence to equilibrium of particle systems. Propp and Wilson [16] introduced Coupling From The Past algorithm and used it to show how to draw exact sample from (3.1) in the case of square lattice. Recently Ding and Peres [9] showed that for Ising models on each graph it takes at least $(1/4 + o(1))n \log n$ steps for the Glauber dynamics to mix, where $n$ is the corresponding number of vertices. In Ding and Peres [10] a simple proof for the bound $n \log n/2$ was presented.

We shall consider the Ising model on a circle. We set $V = \{0, \ldots, N-1\}$ and $E = \{\{i, (i+1) \mod N\} : i = 0, \ldots, N-1\}$. The distribution (3.1) in this case can be written as

$$\pi(e) = \frac{1}{Z_\beta} \exp \left( \beta \sum_{i=0}^{N-1} e(i)e(i+1) \right). \quad (3.2)$$

(we always mean vertex number modulo $N$). We build a Gibbs sampler for this model with stationary distribution (3.2). This chain has the state space $\mathbb{E} = \{-1, 1\}^V$ and its dynamics can be described as follows

- Given a configuration $e$ at step $n$, i.e., $X_n = e$, choose a vertex $i \in \{0, \ldots, N-1\}$ with probability $1/N$.
- Take $U_{n+1}$, a random variable with the uniform distribution $U(0,1)$, independent of $U_i$, $i \leq n$. Update the spin at vertex $k$ in the following way

$$X_{n+1}(i) = \begin{cases} +1 & \text{if } U_{n+1} < \frac{e^{2\beta(k_+(i,e) - k_-(i,e))}}{e^{2\beta(k_+(i,e) - k_-(i,e))} + 1}, \\ -1 & \text{otherwise}, \end{cases}$$

where $k_+(i,e)$ is the number of neighbours of vertex $i$, in configuration $e$, with spin values +1, and $k_-(i,e)$ is the number of neighbours of vertex $i$, in configuration $e$, with spin values -1.
Note that, for the circle, we have \(k_+(i,e), k_-(i,e) \in \{0, 1, 2\}\), and \(k_+(i,e) + k_-(i,e) = 2\). The chain \(X\) constructed in this way is reversible. Moreover, \(X\) can be viewed as a random walk on \(N\)-dimensional cube, where the probability of changing coordinate \(i\) depends on the values of the neighbouring coordinates.

It turns out that if we consider the coordinate-wise ordering, i.e., \(e \preceq e'\) if \(e(i) \leq e'(i)\) for every vertex \(i \in V\), then \(X\) is \(\sim\)-Möbius monotone. Let \(M := 2^{|V|} = 2^N\). Denote by \(e_i\) the state with all spins equal to \(-1\) (minimal state), and by \(e_M\) the state with all spins equal to \(+1\) (maximal state). We identify \(E = \{-1, 1\}^V\) with the enumerated set \(\{e_1, \ldots, e_M\}\), where the enumeration is consistent with \(\preceq\). Applying Theorem 2.1 we obtain

**Theorem 3.1.** Consider the Gibbs sampler \(X\) for the Ising model on the circle with vertices \(V = \{0, \ldots, N-1\}\), and edges \(E = \{(i,i+1 \mod N) : i = 0, \ldots, N-1\}\). Assume that \(X\) starts with the configuration \(e_1\). Then, there exists sharp SSD chain \(X^* = (X^*_n)_{n \geq 0}\) on the state space \(E^* = E\), with the state \(e_M\) being the absorbing one, starting with probability 1 at \(e_1\), and having transition probabilities for \(e, e' \in \{e_1, \ldots, e_M\}\)

\[
P^*(e, e') = \begin{cases} 
0 & \text{if } e > e' \\
\frac{1}{N} S(e) & \text{if } e = e' \\
\frac{H(e')}{H(e)} \frac{1}{N} \left(1 - \frac{e^{2\beta(k_+(j,e)-k_-(j,e))}}{e^{2\beta(k_+(j,e)-k_-(j,e))} + 1}\right) & \text{if } e' = e + s_j, \ e(j) = -1
\end{cases}
\]

where \(s_j = (0, \ldots, 0, 2, 0, \ldots, 0)\) (2 on the \(j\)-th coordinate), \(S(e) = \sum_{i=0}^{N-1} 1\{e(i) = 1\}\), and \(H(e) = \sum_{e' \leq e} \pi(e')\).

Note that our SSD chain \(X^*\) jumps, with probability 1, only to greater or equal states in the ordering \(\preceq\), thus its eigenvalues are the entries on the diagonal of the matrix \(P^*\) written using the enumeration of the states consistent with this ordering. Therefore the eigenvalues are \(\{0, 1/N, 2/N, \ldots, 1\}\), and from Remark 2.1 these values are also the eigenvalues of the original transition matrix \(P\) for the Gibbs sampler \(X\). The multiplicity of eigenvalue \(k/N\) is \(\binom{N}{k}\).

Consider one dimensional projection \(Z^*_n := S(X^*_n)\). Note that \(Z^* = (Z^*_n)_{n \geq 0}\) is also a Markov chain, since, for all \(j = 1 \ldots 2^N\), we have

\[
P^*(e, [e_j]) = P^*(e', [e_j]) \quad \text{for all } e \sim_S e',
\]

where \(e \sim_S e'\) if \(S(e) = S(e')\), \(e, e' \in E\), and \([e_j] := \{e : S(e) = S(e_j)\}\), \(j = 0, \ldots, N - 1\) denote the equivalence classes of the relation \(\sim_S\).

With \(X^*_0 = e_1\), and thus with \(Z^*_0 = 0\), the time to absorption \(T^*\) for \(X^*\) is the same as the time to absorption at \(N\) for \(Z^*\). Note that \(Z^*\) is a pure-birth chain on the state space \(\{0, \ldots, N\}\), with birth rates \(\lambda_k = 1 - k/N\), \(k = 0, \ldots, N - 1\), and absorption at \(N\). The time to absorption \(T^*\) is thus equal, in distribution, to \(\sum_{i=0}^{N-1} Y_i\), where \(Y_i, i = 0, \ldots, N - 1\) are independent random variables, and \(Y_i\) has geometric distribution on \(\{1, \ldots\}\) with the success parameter \(p_i = 1 - \frac{i}{N}\).

A coupon-collector argument shows that, for \(n = N \log N + cN\), we have (equality because of Remark 2.2)

\[
s(\delta_e, P^n, \pi) = P(T^* > n) \leq e^{-c}, \quad c > 0.
\]

### 3.2 Random walk on weighted directed graph

Consider a random walk on a directed weighted graph \(G = (V, E)\) with vertices \(V = \{v_1, v_2, \ldots, v_n\}\), edges \(E = \{(i,j) : \text{edge from } v_i \text{ to } v_j\}\) and with a weighting function \(w : E \to [0, \infty)\). Denote by \(w_{i,j}\) the nonnegative weight of the directed edge from node \(v_i\) to \(v_j\). If there is no edge between these nodes, i.e., \((i,j) \notin E\), then \(w_{i,j} = 0\). We allow \(w_{i,i}\) to be nonzero.

Let \(\mathcal{N}(i) = \{j : (i,j) \in E\}\) be a set of neighbours of node \(v_i\). Random walk may be viewed as a process of sequential vertex visiting. We assume that weights are normalized, i.e., for all
we have $w_{i,i} + \sum_{r \in \mathcal{N}(i)} w_{i,r} = 1$. The probability of a single step from node $i$ to $j$ is then given by $P(i, j) = w_{i,j}$.

In this section we consider the following example: Let $V = \{0, 1, \ldots, N\}^2$ with edges

$$(x_1, y_1), (x_2, y_2) \in E \iff |x_1 - x_2| + |y_1 - y_2| = 1 \quad (3.4)$$

for $x_1, x_2, y_1, y_2 \in \{0, \ldots, N\}$. Thus, for each node there are at most four edges in four directions: up, down, left, right plus a possible self-loop. The weighting function depends only on the direction in the following way: for $((x_1, y_1), (x_2, y_2)) \in E$ and nonnegative parameters $\lambda_1, \lambda_2, \mu_1, \mu_2$ such that $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 \leq 1$

$$w_{((x_1, y_1), (x_2, y_2))} = \begin{cases} 
\lambda_1 & \text{if } x_2 = x_1 + 1, y_2 = y_1, \\
\mu_1 & \text{if } x_2 = x_1 - 1, y_2 = y_1, \\
\lambda_2 & \text{if } x_2 = x_1, y_2 = y_1 + 1, \\
\mu_2 & \text{if } x_2 = x_1, y_2 = y_1 - 1, \\
1 - \sum_{(x,y) \in \mathcal{N}((x_1, y_1))} w_{((x_1, y_1), (x, y))} & \text{if } x_2 = x_1, y_2 = y_1.
\end{cases} \quad (3.5)$$

We associate weights directly with one step probabilities:

$$P((x_1, y_1), (x_2, y_2)) = w_{((x_1, y_1), (x_2, y_2))}.$$ 

Roughly speaking, we consider a random walk on square lattice $\{0, \ldots, N\}^2$, at each step we can move (if feasible): right with probability $\lambda_1$, left with probability $\mu_1$, up with probability $\lambda_2$ and down with probability $\mu_2$. With remnant probability we stay at a given vertex. For convenience, we let $\rho_1 := \lambda_1 / \mu_1$, and $\rho_2 := \lambda_2 / \mu_2$. Denote the transition matrix of a corresponding Markov chain $X$ by $P$. The chain is time-reversible (i.e. $P = \tilde{P}$) and has (time-reversibility equations can be easily checked) the stationary distribution on $V$

$$\pi((x, y)) = C^{-1} \rho_1^x \rho_2^y$$

for $(x, y) \in V = \{0, \ldots, N\}^2$, where the normalizing constant $C$ for $\rho_1 \neq 1$ and $\rho_2 \neq 1$ is given by

$$C = \frac{1 - \rho_1^{N+1}}{1 - \rho_1} \cdot \frac{1 - \rho_2^{N+1}}{1 - \rho_2},$$

and $C$ for other cases can be obtained by obvious modifications.

We shall use the coordinate-wise partial ordering $(x_1, y_1) \preceq (x_2, y_2) \iff x_1 \leq x_2$ and $y_1 \leq y_2$. Then we have unique minimal element $e_1 = (0, 0)$ and the maximal one $e_M = (N, N)$, where $M = (N + 1)^2$. It turns out that $X$ is M"{o}bius monotone for any set of parameters $\lambda_1, \mu_1, \lambda_2, \mu_2 > 0$, such that $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 \leq 1$, and applying Theorem 2.1 we have:

**Theorem 3.2.** Let $X$ be a random walk on directed weighted graph with $G = (V, E)$, with $V = \{0, \ldots, N\}^2$, and $E$ given in $(3.4)$, weights given in $(3.5)$ and with positive parameters $\lambda_1 \neq \mu_1$, $\lambda_2 \neq \mu_2$, such that $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 \leq 1$. Assume, that $X$ starts at $e_1 = (0, 0)$. Then there exists sharp SSD chain $X^*$ which is an absorbing Markov chain (with $e_M = (N, N)$ being the single absorbing state) on the state space $E^* = E = \{0, \ldots, N\}^2$, starting at $e_1 = (0, 0)$, with
the following transition probabilities (for \(x, x', y, y' \in \{0, \ldots, N\}\))

\[
\mathbf{P}^*((x, y), (x', y')) = \begin{cases} 
\frac{1 - \rho^2 + 2}{1 - \rho^2}, \mu_1 & \text{if } x' = x + 1, \ y' = y \\
\frac{1 - \rho^2 + 2}{1 - \rho^2}, \mu_2 & \text{if } y' = y + 1, \ x' = x \\
\frac{1 - \rho^2}{1 - \rho^2}, \lambda_2 & \text{if } x' = x, \ y' = y - 1, \ y' \neq N \\
\frac{1 - \rho^2}{1 - \rho^2}, \lambda_1 & \text{if } y' = y, \ x' = x - 1, \ x' \neq N \\
1 - (\lambda_1 + \lambda_2 + \mu_1 + \mu_2) & \text{if } x' = x, \ y' = y, \ (x, y) \in \{0, \ldots, N - 1\}^2 \\
1 - (\lambda_2 + \mu_2) & \text{if } x' = x, \ y' = y, \ y = N, \ x \in \{0, \ldots, N - 1\} \\
1 - (\lambda_1 + \mu_1) & \text{if } x' = x, \ y' = y, \ x = N, \ y \in \{0, \ldots, N - 1\} \\
1 & \text{if } x' = x = y = y = N 
\end{cases}
\]

(3.6)

Thus, the SSD chain \(X^*\) is again a chain on \(E\), with feasible moves in the same directions as \(X\) except for movements on the upper borders of this square lattice. Once the chain hits the border \((\cdot, N)\) (or \((N, \cdot)\)), then it can only move left or right (up or down) until it hits the absorbing state \((N, N)\). Note that probability of changing \(i\)-th coordinate, \(i = 1, 2\), is independent of the value of \((3 - i)\)-th coordinate. The chain \(X^*\), for a suitable selection of the parameters, can have a drift towards the absorbing state. Note that the case \(\rho_1 = 1\), and/or \(\rho_2 = 1\) can be obtained by obvious modifications in computing \(H(x, y)\) (see the proof in section 4.2).

One can study the time to absorption \(T^*\) in the following way: it is the time of hitting a border \((\cdot, N)\) or \((N, \cdot)\) plus the time for the one dimensional birth-and-death chain with birth probability \(\lambda_1\) and death probability \(\mu_1\) (or \(\lambda_2\) and \(\mu_2\) respectively) to reach the state \(N\) (worst cases scenarios can be used).

3.3 Random change of single coordinate on a cube

Let us consider a discrete time Markov chain \(X\) with state space \(E = \{0, \ldots, k\}^n\), which evolves in the following way: it stays with probability \(1/2\) or (with probability \(1/2\)) for one coordinate chosen uniformly, it changes uniformly its value to any other different value. In terms of the transition probabilities, for \(e = (e(1), \ldots, e(n)) \in E\), \(e(i) \in \{0, \ldots, k\}\), we set

\[
P(e, e') = \begin{cases} 
\frac{1}{2} & \text{if } e = e' \\
\frac{1}{2nk} & \text{if for some } i \ e(i) \neq e'(i) \text{ and } e(j) = e'(j), \ j \neq i \\
0 & \text{otherwise}
\end{cases}
\]

(3.7)

Since \(P\) is symmetric, the corresponding stationary distribution is uniform, i.e.,

\[
\pi(e) = \frac{1}{(k + 1)^n}, \ e \in E.
\]

The motivation for this example comes from DNA sequence alignment. Given \(n\) sequences of length \(k + 1\) the task is to find points of references in each one such that, starting reading sequence \(i\) from it’s reference point \(r(i)\) we have the biggest agreement in all sequences. Since the state space is huge (of size \((k + 1)^n\)), often Monte Carlo methods are used. One constructs a chain such that its stationary distribution assigns higher mass to states with high agreements.

The chain given in (3.7) is a simplified version of such a chain.

The chain \(X\) can be seen as an extension of the standard lazy random walk on the unit cube (obtained for \(k = 1\)). Using the coordinate-wise ordering \(\leq\) on \(E\), it turns out that \(X\) (which is reversible) is M"obius monotone. For this ordering, the state \(e_1 = (0, \ldots, 0)\) is the minimal state.
and \( e_M = (k, \ldots, k) \) is the maximal state (with \( M = (k+1)^n \)), where we use an enumeration of \( E \) consistent with \( \preceq \). Applying Theorem 2.1 we obtain.

**Theorem 3.3.** Consider the chain \( X \) described above, on state space \( E = \{0, \ldots, k\}^n \), with transition probabilities given in (3.7). Assume that \( X \) starts at \( e_1 \). Then, there exists sharp SSD chain \( X^* \) on the state space \( E^* = E \), with the state \( e_M \) being the absorbing one, starting with probability 1 at \( e_1 \), and having transition probabilities, for all \( A \subseteq \{1, \ldots, n\}, \ j \notin A \)

\[
P^*(e^{(k)}_{A}, e^{(k)}_{A,j}) = \frac{(k+1)}{2nk},
\]

\[
P^*(e^{(k)}_{A}, e^{(k)}_{A}) = \frac{n(k-1) + |A|(k+1)}{2nk},
\]

where \( e^{(k)}_{A} = (e(1), \ldots, e(n)) \) with \( e(i) = k \) if \( i \in A \) and \( e(i) = 0 \) if \( i \notin A \), and all other transitions have probability 0.

Note that SSD chain \( X^* \) jumps, with probability 1, only to greater or equal states in the ordering \( \preceq \), thus its eigenvalues are the entries on the diagonal of the matrix \( P^* \) written using an enumeration of the states consistent with this ordering. The states which can be traversed by \( X^* \) are of the form \( e^{(k)}_{A} \), which means that \( X^* \) can be identified with a random walk on the unit cube \( \{0, k\}^n \). Again, by Remark 2.1 the eigenvalues of \( P \) are the same as diagonal entries of \( P^* \), i.e.,

\[
\frac{n(k-1) + i(k+1)}{2nk}, \quad i = 0, 1, \ldots, n.
\]

Similarly as in the Ising model example, we can consider the time to absorption of one dimensional projection \( Z^*_i := S(X^*_i) \), where \( S(e) = \sum_{i=1}^n 1(e(i) = k) \). If \( Z^*_0 = 0 \), then the time to absorption \( T^* \) of \( Z^*_i \) is the same as for \( X^*_i \), and is distributed as the sum of independent variables \( \sum_{i=0}^{n-1} Y_i \), where \( Y_i \) has geometric distribution with the success parameter \( p_i = \frac{(n-i)(k+1)}{2nk} \). For the expected absorption time we have

\[
ET^* = \frac{1}{p_i} = \sum_{i=0}^{n-1} \frac{1}{p_i} - \sum_{i=0}^{n} \frac{2nk}{k+1} = \sum_{i=1}^{n} \frac{1}{i} \leq \frac{2k}{k+1}(n+1)\log n
\]

For the variance of \( T^* \) we have

\[
VarT^* = \sum_{i=0}^{n-1} \frac{1-p_i}{p_i^2} = \frac{2nk}{(k+1)^2} \sum_{i=0}^{n-1} \frac{nk-n+ki+i}{(n-i)^2}
\]

\[
= \frac{2nk}{(k+1)^2} \left[ nk \sum_{i=0}^{n-1} \frac{1}{(n-i)^2} + k \sum_{i=0}^{n-1} \frac{i}{(n-i)^2} - \sum_{i=0}^{n} \frac{1}{i} \right] \leq \left( \frac{2nk}{k+1} \right) \frac{n^2}{6},
\]

where in \(^(*) \) we used the following inequalities

\[
\sum_{i=0}^{n-1} \frac{1}{(n-i)^2} \leq \frac{n^2}{6}, \quad \sum_{i=0}^{n-1} \frac{i}{(n-i)^2} \leq n \frac{n^2}{6}.
\]

By Remark 2.2 and from Chebyshev’s inequality, we have that after \( m = \frac{2k}{k+1}(n+1)\log n + c\frac{2k}{k+1} \sqrt{n}, c \geq 0 \) steps we have

\[
s(\nu P^m, \pi) = P(T > m) \leq P(T - ET \leq c\sqrt{Var}) \leq P(|T - ET| \leq c\sqrt{Var}) \leq \frac{1}{c^2}.
\]
4 Proofs

4.1 Proof of Theorem 3.1

The Möbius function for the coordinate-wise ordering is given by

$$\mu(e, e') = \begin{cases} (-1)^{S(e') - S(e)} & \text{if } e \preceq e', \\ 0 & \text{otherwise}, \end{cases}$$

where $S(e) = \sum_{i=0}^{N-1} 1\{e(i) = +1\}$. For convenience, let us define

$$f(i, e) := \frac{e^{2\beta(k_+(i, e) - k_-(i, e))}}{e^{2\beta(k_+(i, e) - k_-(i, e))} + 1},$$

so that the probability of choosing vertex $i$ and updating it to +1 in configuration $e$ is $\frac{1}{N} f(i, e)$. Let us define also

$$e_i^r := (e(0), \ldots, e(i-1), r, e(i+1), \ldots, e(N-1)), \ r \in \{-1, +1\}$$

and we will shortly write $e_i^−$ for $e_i^{−1}$ and $e_i^+$ for $e_i^{+1}$. Moreover, by $\mathcal{N}(i)$ we denote the neighbors of vertex $i$, i.e., $\mathcal{N}(i) = \{i-1, i+1\}$. We will directly apply Theorem 2.1. We can replace $\tilde{P}$ by $P$, since this chain is reversible.

First, let us calculate $P^*(e_i^+, e_i^-)$.

$$\frac{H(e_i^+)}{H(e_i^-)} P^*(e_i^+, e_i^-) = \sum_{e \preceq e_i^-} \mu(e_i^−, e) P(e, \{e_i^+\}^i)$$

$$= P(e_i^−, \{e_i^+\}^i) + \sum_{j \in \mathcal{N}(i), j \neq i} \sum_{j \in \mathcal{N}(i), j \neq i} P(e_i^+, +s_j, \{e_i^+\}^i) - \sum_{j \in \mathcal{N}(i), j \neq i} P(e_i^+, s_j, \{e_i^+\}^i)$$

Since $e_i^− + s_i = e_i^+$, we have

$$\frac{H(e_i^+)}{H(e_i^-)} P^*(e_i^+, e_i^-) = P(e_i^−, \{e_i^+\}^i) - P(e_i^− + s_i, \{e_i^+\}^i)$$

$$- \sum_{j \in \mathcal{N}(i), j \neq i} \frac{1}{N} [1 - f(j, e_i^+ + s_j)] - \sum_{j \in \mathcal{N}(i), j \neq i} \frac{1}{N} [1 - f(j, e_i^− + s_j)]$$

$$+ \sum_{j \in \mathcal{N}(i), j \neq i} \frac{1}{N} [1 - f(j, e_i^+ + s_j)] + \sum_{j \in \mathcal{N}(i), j \neq i} \frac{1}{N} [1 - f(j, e_i^− + s_j)]. \quad (4.1)$$

Note that for $j \notin \mathcal{N}(i)$ we have $k_+(j, e_i^− + s_j) = k_+(j, e_i^+ + s_j)$ and $k_−(j, e_i^− + s_j) = k_−(j, e_i^+ + s_j)$, thus $f(j, e_i^− + s_j) = f(j, e_i^+ + s_j)$ in this case. The corresponding terms with summation over $j \notin \mathcal{N}(i)$ cancel out. For the first two terms we have

$$P(e_i^−, \{e_i^+\}^i) = 1 - \sum_{j \in \mathcal{N}(i), j \neq i} \frac{1}{N} f(j, e_i^− + s_j) - \sum_{j \in \mathcal{N}(i), j \neq i} \frac{1}{N} f(j, e_i^− + s_j)$$

and

$$P(e_i^+, \{e_i^+\}^i) = 1 - \sum_{j \in \mathcal{N}(i), j \neq i} \frac{1}{N} f(j, e_i^+ + s_j) - \sum_{j \in \mathcal{N}(i), j \neq i} \frac{1}{N} f(j, e_i^+ + s_j).$$
Again, for \( j \notin \mathcal{N}(i) \), we have \( k_+(j, e_i^- + s_j) = k_+(j, e_i^+) + s_j \), and \( k_-(j, e_i^- + s_j) = k_-(j, e_i^- + s_j) \), thus \( f(j, e_i^- + s_j) = f(j, e_i^+ + s_j) \). The corresponding terms again cancel out in \( \mathbf{P}(e_i^+, \{e_i^+\}^*) - \mathbf{P}(e_i^-, \{e_i^-\}^*) \). Plugging in the remaining sums to (4.11) we obtain
\[
\frac{H(e_i^+)}{H(e_i^-)} \mathbf{P}^*(e_i^+, e_i^-) = - \sum_{j \in \mathcal{M}(i)} \frac{1}{N} f(j, e_i^- + s_j) - \sum_{j \in \mathcal{M}(i)} \left[ 1 - f(j, e_i^- + s_j) \right]
+ \sum_{j \in \mathcal{M}(i)} \frac{1}{N} f(j, e_i^+ + s_j) + \sum_{j \in \mathcal{M}(i)} \left[ 1 - f(j, e_i^+ + s_j) \right].
\]
Now all the terms on the right hand side in the above equality cancel out, therefore \( \mathbf{P}^*(e_i^+, e_i^-) = 0 \).

Next, we get immediately
\[
\mathbf{P}^*(e_i^-, e_i^+) = \frac{H(e_i^+)}{H(e_i^-)} \sum_{e \in e_i^-} \mu(e, i) P(e, \{e_i^+\}^*) = \frac{H(e_i^+)}{H(e_i^-)} \frac{1}{N} f(i, e_i^-).
\]
We have yet to calculate the probability of staying at \( e \).
\[
\mathbf{P}^*(e, e) = \sum_{e \in e} \mu(e, e') \mathbf{P}(e', \{e\}^*) = \mathbf{P}(e, \{e\}^*) - \sum_{j \in \mathcal{M}(i)} \mathbf{P}(e + s_j, e)
= 1 - \sum_{j \in \mathcal{M}(i)} \mathbf{P}(e, e + s_j) - \sum_{j \in \mathcal{M}(i)} \mathbf{P}(e + s_j, e)
= 1 - \sum_{j \in \mathcal{M}(i)} \frac{1}{N} f(j, e) - \sum_{j \in \mathcal{M}(i)} \frac{1}{N} \left[ 1 - f(j, e + s_j) \right]
= 1 - \sum_{j \in \mathcal{M}(i)} \frac{1}{N} f(j, e) + \sum_{j \in \mathcal{M}(i)} \frac{1}{N} f(j, e + s_j) - \sum_{j \in \mathcal{M}(i)} \frac{1}{N}
= 1 - \frac{N - S(e)}{N} = \frac{S(e)}{N}.
\]
Summing up, we obtain \( \mathbf{P}^* \) given in (3.3).

### 4.2 Proof of Theorem 3.2

We start with a detailed expression for the transition probabilities of \( \mathbf{X} \)

\[
\mathbf{P}((x, y), (x', y')) = \begin{cases} 
\lambda_1 & \text{if } x' = x + 1 \leq N, y' = y \\
\lambda_2 & \text{if } x' = x, y' = y + 1 \leq N \\
\mu_1 & \text{if } x' = x - 1 \geq 0, y' = y \\
\mu_2 & \text{if } y' = y - 1 \geq 0, x' = x \\
1 - (\lambda_1 + \lambda_2 + \mu_1 + \mu_2) & \text{if } x' = x > 0, y' = y > 0 \\
1 - (\lambda_1 + \lambda_2 + \mu_1) & \text{if } x' = x > 0, y' = y = 0 \\
1 - (\lambda_1 + \lambda_2 + \mu_2) & \text{if } x' = x = 0, y' = y > 0 \\
1 - (\mu_1 + \mu_2) & \text{if } x' = x = y = y' = N \\
1 - (\mu_1 + \mu_2 + \lambda_1) & \text{if } x' = x > 0, y' = y = N \\
1 - (\mu_1 + \mu_2 + \lambda_2) & \text{if } x' = x = N, y' = y > 0 \\
1 - (\lambda_1 + \lambda_2) & \text{if } x' = x = y = y' = 0 
\end{cases}
\]
In a standard way we can check that $X$ is reversible and the stationary distribution is given by

$$\pi((x, y)) = C^{-1} \rho^x \rho^y$$

where $C$ is the normalizing constant, and $\rho_i = \lambda_i / \mu_i$, $i = 1, 2$. For the coordinate-wise ordering

$$(x, y) \leq (x', y') \iff x \leq x'$ and $y \leq y',$$

with the minimal state $e_1 = (0, 0)$, and the maximal state $e_M = (N, N)$, $(M = (N+1)^2)$ directly from Proposition 5 in [17], we find the corresponding Möbius function:

- $\mu((x, y), (x, y)) = 1$
- $\mu((x, y), (x + 1, y)) = -1$ if $x + 1 \leq N$
- $\mu((x, y), (x, y + 1)) = -1$ if $y + 1 \leq N$
- $\mu((x, y), (x + 1, y + 1)) = 1$ if $x + 1 \leq N$ and $y + 1 \leq N$
- $\mu((x, y), (x + 1, y + 1)) = 0$ otherwise.

For

$$H(x, y) = C^{-1} \sum_{x' \leq x} \rho_i^{x'} \sum_{y' \leq y} \rho_j^{y'} = C^{-1} (1 - \rho_1)^{-1} (1 - \rho_2)^{-1} (1 - \rho_1^x) (1 - \rho_2^y),$$

we shall compute

$$P^*((x, y), (x_2, y_2)) = \frac{H(x_2, y_2)}{H(x, y)} \sum_{(x', y') \geq (x_2, y_2)} \mu((x_2, y_2), (x', y')) \tilde{P}((x', y'), \{(x, y)\}^1). \tag{4.2}$$

Set

$$S := \sum_{(x', y') \geq (x_2, y_2)} \mu((x_2, y_2), (x', y')) \tilde{P}((x', y'), \{(x, y)\}^1).$$

Note that in order to prove that $\tilde{X}$ is $^1$-Möbius monotone it is enough to show that $S \geq 0$. Since $X$ is reversible, we take $P$ instead of $\tilde{P}$ in the above formula. We shall consider all possible transitions, case by case.

- (inside lattice, up $x$ direction)
  $$x_2 = x + 1, \ y_2 = y$$
  $$S = \sum_{(x', y') \geq (x + 1, y)} \mu((x + 1, y), (x', y')) P((x', y'), \{(x, y)\}^1)$$
  where $\mu$ will be non-zero only in the following cases
  $$\mu((x + 1, y), (x + 1, y)) = 1, \ \mu((x + 1, y), (x + 1, y + 1)) = -1,$$
  $$\mu((x + 1, y), (x + 2, y)) = -1, \ \mu((x + 1, y), (x + 2, y + 1)) = 1.$$  
  Combining these cases with the values of $P((x', y'), \{(x, y)\}^1)$ we get
  $$S = \mu((x + 1, y), (x + 1, y)) P((x + 1, y), \{(x, y)\}^1) - 1 \cdot 0 - 1 \cdot 0 + 1 \cdot 0 = \mu_1,$$

- (inside lattice, up $y$ direction)
  $$x_2 = x, \ y_2 = y + 1$$
  in a similar way as above, we get
  $$S = \mu((x, y + 1), (x, y + 1)) P((x, y + 1), \{(x, y)\}^1) - 1 \cdot 0 - 1 \cdot 0 + 1 \cdot 0 = \mu_2,$$
• (inside lattice, down $x$ direction)

\[ x_2 = x - 1 \geq 0, \quad y_2 = y \]

using the formula for $S$ we have

\[
S = \mu((x - 1, y), (x - 1, y))P((x - 1, y), \{(x, y)\}^1) + \mu((x - 1, y), (x, y))P((x, y), \{(x, y)\}^1) \\
+ \mu((x - 1, y), (x, y + 1))P((x, y + 1), \{(x, y)\}^1)\mu((x - 1, y), (x, y + 1))P((x, y + 1), \{(x, y)\}^1)
\]

\[ = 1 \cdot (1 - \lambda_2) - 1 \cdot (1 - \lambda_2 - \lambda_1) - 1 \cdot \mu_2 + 1 \cdot \mu_2 = \lambda_1, \]

• (inside lattice, down $y$ direction)

\[ x_2 = x, \quad y_2 = y - 1 \geq 0 \]

\[
S = \mu((x, y - 1), (x, y - 1))P((x, y - 1), \{(x, y)\}^1) + \mu((x, y - 1), (x, y))P((x, y), \{(x, y)\}^1) \\
+ \mu((x, y - 1), (x + 1, y - 1))P((x + 1, y - 1), \{(x, y)\}^1)\mu((x, y - 1), (x + 1, y))P((x + 1, y), \{(x, y)\}^1)
\]

\[ = 1 \cdot (1 - \lambda_1) - 1 \cdot (1 - \lambda_2 - \lambda_1) - 1 \cdot \mu_1 + 1 \cdot \mu_1 = \lambda_2, \]

• (inside lattice, down on both axes)

\[ x_2 = x - 1 \geq 0, \quad y_2 = y - 1 \geq 0 \]

\[
S = \mu((x - 1, y - 1), (x - 1, y - 1))P((x - 1, y - 1), \{(x, y)\}^1) + \mu((x - 1, y - 1), (x - 1, y))P((x - 1, y), \{(x, y)\}^1) \\
+ \mu((x - 1, y - 1), (x, y - 1))P((x, y - 1), \{(x, y)\}^1)\mu((x - 1, y - 1), (x, y))P((x, y), \{(x, y)\}^1)
\]

\[ = 1 \cdot 1 - 1 \cdot (1 - \lambda_2) - (1 - \mu_1) + \mu_1 + \mu_2 = 0. \]

In a similar way it is possible to check that inside the lattice the only one remaining movement with positive probability is the feedback movement

• (feedback inside lattice)

\[ x_2 = x > 0, \quad y_2 = y > 0 \]

\[
P^*((x, y), (x, y)) = 1 - \lambda_1 - \lambda_2 - \mu_1 - \mu_2 = P((x, y), (x, y)), \]

• (upper border, up $x$ direction)

\[ x_2 = x + 1 \leq N, \quad y_2 = y = N \]

\[
S = \mu((x + 1, N), (x + 1, N))P((x + 1, N), \{(x, N)\}^1) = \mu_1, \]

• (upper border, down $y$ direction)

\[ x_2 = x < N, \quad y_2 = N, \quad y_2 = N - 1 \]

\[
S = \mu((x, N - 1), (x, N - 1))P((x, N), \{(x, y)\}^1) + \mu((x, N - 1), (x, N))P((x, y), \{(x, y)\}^1) \\
+ \mu((x, N - 1), (x + 1, N - 1))P((x + 1, N - 1), \{(x, y)\}^1)\mu((x, N - 1), (x + 1, N))P((x + 1, N), \{(x, y)\}^1)
\]

\[ = 1 \cdot (1 - \lambda_1) - 1 \cdot (1 - \lambda_2) - 1 \cdot \mu_1 + 1 \cdot \mu_1 = 0, \]

• (upper border, down $x$ direction)

\[ x_2 = x - 1 \geq 0, \quad y_2 = y = N \]

\[
S = \mu((x - 1, N), (x - 1, N))P((x - 1, N), \{(x, y)\}^1) + \mu((x - 1, N), (x, N))P((x, N), \{(x, y)\}^1)
\]

\[ = +1 \cdot 1 - 1 \cdot (1 - \lambda_1) = \lambda_1, \]
• (lower border, up x direction)
  \[ x_2 = x + 1 \leq N, \ y_2 = y = 0 \]
  \[ S = \mu((x + 1, 0), (x + 1, 0)P((x + 1, 0), \{(x, 0)\}^+) = \mu_1, \]

• (lower border, down x direction)
  \[ x_2 = x - 1 \geq 0, \ y_2 = 0 \]
  \[ S = \mu((x, 0), (x-1, 0)) = \mu((x-1, 0), (x-1, 0))P((x-1, 0), \{(x, 0)\}^+) + \mu((x-1, 0), (x, 0))P((x, 0), \{(x, 0)\}^+) \]
  \[ + \mu((x - 1, 0), (x - 1, 1)P((x - 1, 1), \{(x, 0)\}^+) + \mu((x - 1, 0), (x, 1))P((x, 1), \{(x, 0)\}^+) \]
  \[ = 1 \cdot (1 - \lambda_2) - 1 \cdot (1 - \lambda_1 - \lambda_2) - 1 \cdot \mu_2 + \mu_2 = \lambda_1, \]

• (lower border, up y direction)
  \[ x_2 = x \geq 0, \ y_2 = 1, \ y = 0 \]
  \[ S = \mu((x, 1), (x, 1))P((x, 1), \{(x, 0)\}^+) = \mu_2. \]

In a similar way we get

• (right border, up y)
  \[ x_2 = x = N, \ y_2 = y + 1 \leq N, \ S\mu_2, \]

• (right border, down y)
  \[ x_2 = x = N, \ y_2 = y - 1 \leq N, \ S = \lambda_2, \]

• (right border, down x)
  \[ x_2 = N - 1, \ x = N, \ y_2 = y < N, \ S = 0, \]

• (left border, up y)
  \[ x_2 = x = 0, \ y_2 = y + 1 \leq N, \ S = \mu_2, \]

• (left border, up x)
  \[ x_2 = x + 1 \leq N, \ y_2 = y, \ S = \mu_1, \]

• (left border, down y)
  \[ x_2 = x = 0, \ y_2 = y - 1 \geq N, \ S = \lambda_2, \]

• (absorbing state)
  \[ x_2 = x = N, \ y_2 = y = N, \ S = 1. \]

• (feedback movements)
  for all \((x, y) \in \{0, \ldots, N_1\}^2\), \( S = 1 - (\lambda_1 + \lambda_2 + \mu_1 + \mu_2), \)
  for \(x = N, \) and \(y \in \{0, \ldots, N - 1\}, \ S = 1 - (\lambda_2 + \mu_2), \)
  for \(y = N, \) and \(x \in \{0, \ldots, N - 1\}, \ S = 1 - (\lambda_1 + \mu_1). \)

Now using (4.2), and using values of \( H(x, y) \) we obtain \( P^* \) given in (3.6).
4.3 Proof of Theorem 3.3

Consider the coordinate-wise ordering

\[ \mathbf{e} = (\mathbf{e}(1), \ldots, \mathbf{e}(n)) \preceq (\mathbf{e}'(1), \ldots, \mathbf{e}'(n)) = \mathbf{e}' \iff \mathbf{e}(i) \leq \mathbf{e}'(i), i = 1, \ldots, n. \]

Again, for this ordering with minimal element \( \mathbf{e}_1 = (0, \ldots, 0) \) and maximal element \( \mathbf{e}_M = (k, \ldots, k) \) (with \( M = (k + 1)^n \)), directly from Proposition 5 in Rota [17], we find the corresponding Möbius function

\[
\mu((\mathbf{e}(1), \ldots, \mathbf{e}(n)), (\mathbf{e}(1) + d_1, \ldots, \mathbf{e}(n) + d_n)) = \begin{cases} (-1)^{\sum_{i=1}^{n} d_i} & d_i \in \{0, 1\}, \ i = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}
\]

For \( H(\mathbf{e}) = \sum_{\mathbf{e}' \preceq \mathbf{e}} \pi(\mathbf{e'}) = |\{\mathbf{e}' : \mathbf{e}' \preceq \mathbf{e}\}| \cdot 1/(k + 1)^n \), we shall compute directly transitions of the dual chain (2.2) from Theorem 2.1. Note, that in order to prove that \( \bar{\mathbf{X}} \) is \( 1 \)-Möbius monotone, it is enough to show that all summands in (2.2) are non-negative. We take \( \mathbf{P} \) instead of \( \bar{\mathbf{P}} \) since this chain is reversible.

For convenience, we shall consider states of the following form

\[ \mathbf{e}_A^{(k)} = (\mathbf{e}_A^{(k)}(1), \ldots, \mathbf{e}_A^{(k)}(k)), \ A \subseteq \{1, \ldots, n\}, \]

with \( \mathbf{e}_A^{(k)}(i) = k \) if \( i \in A \) and 0 otherwise. Note, that there are \( (k + 1)^{|A|} \) states smaller or equal (w.r.t. \( \preceq \)) to \( \mathbf{e}_A^{(k)} \), and we have

\[
H(\mathbf{e}_A^{(k)}) = \frac{(k + 1)^{|A| + |j|}}{(k + 1)^{|A|}} = k + 1 \text{ for } j \notin A. \tag{4.3}
\]

Let us calculate transitions of the dual chain from state \( \mathbf{e}_A^{(k)} \). We shall use \( s_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) with 1 at position \( i \). For the probability of staying at this state we get

\[
\mathbf{P}^*(\mathbf{e}_A^{(k)}, \mathbf{e}_A^{(k)}) = 1 \cdot \sum_{\mathbf{e} \preceq \mathbf{e}_A^{(k)}} \mu(\mathbf{e}_A^{(k)}, \mathbf{e}) \mathbf{P}(\mathbf{e}, \mathbf{e}_A^{(k)})^\downarrow
\]

\[
= \mu(\mathbf{e}_A^{(k)}, \mathbf{e}_A^{(k)}) \mathbf{P}(\mathbf{e}_A^{(k)}, \mathbf{e}_A^{(k)})^\downarrow + \sum_{i \in A} \mu(\mathbf{e}_A^{(k)}, \mathbf{e}_A^{(k)} + s_i) \mathbf{P}(\mathbf{e}_A^{(k)} + s_i, \mathbf{e}_A^{(k)})^\downarrow
\]

\[
= 1 \cdot \left( \frac{1}{2} + \frac{k}{2nk} \right) - \frac{1}{2nk} = \frac{1}{2} + \frac{k|A|}{2nk} - \frac{n - |A|}{2nk} = \frac{n(k - 1) + |A|(k + 1)}{2nk},
\]

since \( \mathbf{P}(\mathbf{e}_A^{(k)} + s_i, \mathbf{e}_A^{(k)})^\downarrow = \mathbf{P}(\mathbf{e}_A^{(k)} + s_i, \mathbf{e}_A^{(k)}) \).

Now, for the probability of transition from \( \mathbf{e}_A^{(k)} \) to \( \mathbf{e}_{A \cup \{j\}}^{(k)} \), \( j \notin A \) we obtain

\[
\mathbf{P}^*(\mathbf{e}_A^{(k)}, \mathbf{e}_{A \cup \{j\}}^{(k)}) = \frac{H(\mathbf{e}_{A \cup \{j\}}^{(k)})}{H(\mathbf{e}_A^{(k)})} \sum_{\mathbf{e} \preceq \mathbf{e}_{A \cup \{j\}}^{(k)}} \mu(\mathbf{e}_{A \cup \{j\}}^{(k)}, \mathbf{e}) \mathbf{P}(\mathbf{e}, \mathbf{e}_{A \cup \{j\}}^{(k)})^\downarrow.
\]

The only state \( \mathbf{e} \) for which \( \mathbf{P}(\mathbf{e}, (\mathbf{e}_{A \cup \{j\}}^{(k)})^\downarrow > 0 \) is \( \mathbf{e} = \mathbf{e}_{A \cup \{j\}}^{(k)} \), thus (using (4.3)) we have

\[
\mathbf{P}^*(\mathbf{e}_A^{(k)}, \mathbf{e}_{A \cup \{j\}}^{(k)}) = (k + 1) \mu(\mathbf{e}_{A \cup \{j\}}^{(k)}, \mathbf{e}_{A \cup \{j\}}^{(k)}) \mathbf{P}(\mathbf{e}_{A \cup \{j\}}^{(k)}, \mathbf{e}_{A \cup \{j\}}^{(k)})^\downarrow = \frac{k + 1}{2nk}.
\]

This completes our argument since all other transitions have probability 0, which is clear from the following summation

\[
\mathbf{P}^*(\mathbf{e}_A^{(k)}, \mathbf{e}_{A \cup \{j\}}^{(k)}) + \sum_{j \in A^c} \mathbf{P}^*(\mathbf{e}_A^{(k)}, \mathbf{e}_{A \cup \{j\}}^{(k)}) = \frac{n(k - 1) + |A|(k + 1)}{2nk} + (n - |A|) \cdot \frac{k + 1}{2nk}
\]
\[ \frac{n(k - 1) + n(k + 1) + |A|(k + 1) - |A|(k + 1)}{2nk} = 1. \]

Note that the dual chain starts at the minimal state which is also of the form \( e_A^{(k)} \), namely with \( A = \emptyset \).

### References

[1] Aldous, D.J. Finite-Time Implications of Relaxation Times for Stochastically Monotone Processes. *Probab. Th. Rel. Fields*, 77, 137-145, (1988)

[2] Aldous, D.J., Diaconis, P. Shuffling cards and stopping times. *American Mathematical Monthly*, 93, 333-348, (1986)

[3] Aldous, D.J., Diaconis, P. Strong uniform times and finite random walks. *Advances in Applied Mathematics*, 8, 69–97, (1987)

[4] Diaconis, P. *Group Representations in Probability and Statistics*. IMS, Hayward, CA. (1988)

[5] Diaconis, P., Fill, J.A. Strong stationary times via a new form of duality. *The Annals of Probability*, 18, 1483-1522, (1990)

[6] Diaconis, P., Fill, J.A. Examples for the theory of strong stationary duality with countable state spaces. *Probability in the Engineering and Informational Sciences*, 4, 157–180, (1990)

[7] Diaconis, P. and Miclo, L. On times to quasi-stationarity for birth and death processes. *Journal of Theoretical Probability* 22, 558-586, (2009)

[8] Diaconis, P. and Saloff-Coste, L. Separation cut-offs for birth and death chains. *The Annals of Applied Probability* 16(4), 2098–2122, (2006)

[9] Ding, J. and Peres, Y. Mixing time for the Ising model: A uniform lower bound for all graphs. *Ann. Inst. H. Poincaré Probab. Statist.*, 47(4), 1020–1028, (2011)

[10] Ding, J. and Peres, Y. Mixing time for the Ising model: A uniform lower bound for all graphs. *arXiv:0909.5162v2 [math.PR]* (2013)

[11] Fill, J. A. The passage time distribution for a birth-and-death chain:Strong stationary duality gives a first stochastic proof. *Journal of Theoretical Probability* 22, 543-557, (2009a)

[12] Fill, J. A. On hitting times and fastest strong stationary times for skip-free and more general chains. *Journal of Theoretical Probability* 22(3), 587-600, (2009)

[13] Hayes, T. P. and Sinclair, A. A general bound for mixing of single-site dynamics on graphs. *The Annals of Applied Probability* 17(3), 931–952, (2007)

[14] Lorek, P. , Szekli, R. Computable bounds on the spectral gap for unreliable Jackson networks. Submitted to *Journal of Applied Probability*. *arXiv:1101.0332 [math.PR]* (2013)

[15] Lorek, P., Szekli, R. Strong stationary duality for Möbius monotone Markov chains. *Queueing Systems*, 71, 79–95, (2012)

[16] Propp, J. G. and Wilson, D. B. Exact sampling with coupled Markov chains and applications to statistical mechanics *Random Structures & Algorithms*, 9(1), 223–252, (1996)

[17] Rota, G. C. On the Foundations of Combinatorial Theory I. Theory of Möbius Functions. *Z. Wahrscheinlichkeitstheorie* 2, 340–368, (1964)