Loop Groups and Twin Buildings

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Dedicated to John Stallings on the occasion of his 65th birthday.

Abstract

In these notes we describe some buildings related to complex Kac-Moody groups. First we describe the spherical building of $\text{SL}_n(\mathbb{C})$ (i.e. the projective geometry $\mathbb{P} \mathbb{G} (\mathbb{C}^n)$) and its Veronese representation. Next we recall the construction of the affine building associated to a discrete valuation on the rational function field $\mathbb{C}(z)$. Then we describe the same building in terms of complex Laurent polynomials, and introduce the Veronese representation, which is an equivariant embedding of the building into an affine Kac-Moody algebra. Next, we introduce topological twin buildings. These buildings can be used for a proof — which is a variant of the proof by Quillen and Mitchell [27] — of Bott periodicity which uses only topological geometry. At the end we indicate very briefly that the whole process works also for affine real almost split Kac-Moody groups.

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1 Introduction

We briefly recall the definition of Coxeter complexes and buildings; for more details, we refer to the books by Brown [8], Ronan [35], Scharlau [41], and Tits [48]. For our purposes, a simplicial complex is poset $(\leq )$ whose elements are called simplices, with the following two properties: any two simplices $X \leq Y$ have a unique infimum $X \cup Y$, and for any $X \leq Y$, the poset $x = fY$ 2

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is order-isomorphic to the power set \((2^F; \cdot)\) of some finite set \(F\); the cardinality of this set \(F\) is called the rank of the simplex \(X\). Note that the rank of a simplex differs by one from the dimension of its geometric realization; a \(k\)-simplex has rank \(k + 1\). The rank of a simplicial complex is the maximum of the ranks of its simplices.

1.1 Coxeter complexes

Let \(I\) be a finite set with \(r\) elements, and let \((m_{ij})\) be a symmetric matrix indexed by \(I \times I\), with entries in \(\mathbb{N}\) for all \(i \neq j\), and \(m_{ii} = 1\) for all \(i\). Such a Coxeter matrix is determined by its Coxeter graph; the vertices of this graph are the elements of \(I\), and two vertices \(i; j\) are joined by \(m_{ij}\) edges, or by one edge labeled \(m_{ij}\). The corresponding \(\text{Coxeter system} (\mathcal{W}; S)\) is the group \(\mathcal{W}\) with generating set \(S = \{s_i j : 2 \mid 1 \leq i \leq j \leq r\}\) and relators \((s_i s_j)^{m_{ij}}\). Associated to such a Coxeter system is a simplicial complex, the \(\text{Coxeter complex} = (\mathcal{W}; S)\), which is defined as follows. For \(J \subseteq I\) let \(\mathcal{W}_J\) denote the subgroup generated by the elements \(s_j\), for \(j \in J\). The simplices of \((\mathcal{W}; S)\) are the cosets \(w\mathcal{W}_J\) for \(J \subseteq I\); the partial ordering is the reversed inclusion, i.e.

\[
\text{if and only if } g\mathcal{W}_J \subseteq h\mathcal{W}_J.
\]

The group \(\mathcal{W}\) acts regularly on the maximal simplices of this simplicial complex (by left translations). The type of a coset \(w\mathcal{W}_J\) is

\[
\text{type}(w\mathcal{W}_J) = I \setminus J;
\]

a subset \(J \subseteq I\) is called spherical if \(\mathcal{W}_J\) is finite; if \(I\) itself is spherical, then \(\mathcal{W}\) is called spherical.

The geometric realization of a spherical Coxeter complex of rank \(k\) is a triangulated \(k - 1\)-sphere. We define a \(\mathcal{W}\)-invariant double-coset valued distance function on \(\mathcal{W}\) as follows:

\[
\ell_{\mathcal{W}_J}(w \mathcal{W}_J; v \mathcal{W}_K) = \mathcal{W}_J w \mathcal{W}_K v \mathcal{W}_K.
\]

Coxeter groups have nice geometric properties; in particular, the word problem can be solved. Let \(A\) be an abelian group, and let \(\alpha : S \longrightarrow A\) be a function. We require that \(\alpha(s_i) = \alpha(s_j)\) holds whenever \(m_{ij}\) is finite and odd. Then there is a well-defined extension \(\alpha : \mathcal{W} \longrightarrow A\), which is defined as \(\alpha(s_i s_k \ldots s_j) = \alpha(s_i) + \ldots + \alpha(s_j)\) for a reduced (minimal) expression \(w = s_i s_k \ldots s_j\), \(S\) the \(\alpha\)-length. In the special case \(A = \mathbb{Z}\), with \(\alpha(s_i) = 1\) for all \(i\), we obtain the usual length function

\[
\ell : \mathcal{W} \longrightarrow \mathbb{Z};
\]

In general, the set of generators \(S\) is not uniquely determined by the abstract group \(\mathcal{W}\), so it is important to consider the pair \((\mathcal{W}; S)\); the question to which extent \(S\) is determined by the group \(\mathcal{W}\) is treated in M"uhlherr [30] and Brady, McCammond, M"uhlherr & Neumann [6], cp. also Charney & Davis [10] for related results.
1.2 Buildings Let $\mathcal{C}$ be a simplicial complex, and let $\mathfrak{W} = (W; S)$ be a Coxeter complex. A simplicial injection $\iota : \mathcal{C} \to \mathcal{D}$ is called a chart, and its image $A = \iota(\mathcal{C})$ is called an apartment. The complex $\mathcal{D}$ is called a building (of type $\mathfrak{m}_{ij}$ and rank $r$) if there exists a collection $A$ of apartments with the following properties.

**Bld$_1$** For any two simplices $X; Y \in \mathcal{C}$, there exists an apartment $A \in A$ with $X; Y \in A$.

**Bld$_2$** Given two charts $\iota_i : \mathcal{C} \to \mathcal{D}$, for $i = 1; 2$, there exists an element $w \in W$ such that $\iota_1(\mathcal{C}) = \iota_2(\mathcal{C})$ holds for all $X \in \mathcal{C}$.

A simplex of maximal rank $r$ is also called a chamber; the set of all chambers is $\text{Cham}(\mathcal{C})$. For any subset $X \subseteq \mathcal{C}$, we let $\text{Cham}(X)$ denote the set of all chambers contained in $X$. The building is thick if every simplex of rank $r \leq 1$ is contained in at least three chambers. **All buildings in this paper will be thick.** A spherical building is a building with finite apartments.

It follows from the axioms that there is a well-defined double-coset valued distance function $d : S \to \mathcal{W} \backslash \mathcal{W} = \mathcal{W} \backslash \mathcal{W} = \mathcal{W} \backslash \mathcal{W}$ whose restriction to any apartment is given by the function defined above in 1.1. The restriction of $\text{Cham}(\mathcal{C})$ to $\text{Cham}(\mathcal{C})$ is Tits’ more familiar $W$-valued distance function; buildings can also be characterized by properties of the distance function, see Ronan’s book [35].

Let $N^r_1$ denote the standard $r$-simplex $Q^r$. The two axioms yield a simplicial surjection, the type function (the ‘accordion map’) $\text{type} : N^r_1 \to \mathcal{D}$ whose restriction to any apartment agrees with the type function defined above. The type function is characterized (up to automorphisms of $N^r_1$) by the fact that its restriction to every simplex of $\mathcal{C}$ is injective. A simplex $X \in \mathcal{C}$ is called spherical if $I \cap \text{type}(\mathcal{C})$ is spherical.

For non-spherical buildings the apartment system $A$ is in general not unique (there is always a unique maximal apartment system), but the isomorphism type of the apartments and the Coxeter system $(W; S)$ is uniquely determined by the simplicial complex $\mathcal{C}$.

1.3 Automorphisms The automorphism group $\text{Aut}(\mathcal{C})$ consists of all simplicial automorphisms of $\mathcal{C}$; it has a normal subgroup $\text{Spe}(\mathcal{C})$ consisting of all type-preserving automorphisms. An action of a group $G$ on $\mathcal{C}$ is a homomorphism $G \to \text{Aut}(\mathcal{C})$. We say that $G$ acts transitively on $\mathcal{C}$ if it acts transitively on the set $\text{Cham}(\mathcal{C})$ of chambers.

**Strongly transitive actions and BN-pairs** A group $G$ is said to act strongly transitively on $\mathcal{C}$ (with respect to $A$) if $G$ acts as a group of special automorphisms on $\mathcal{C}$ such that $\text{STA}_G$ $G$ acts transitively on the set $A$ of apartments.
If $A$ is an apartment, then the set-wise stabilizer $N$ of $A$ acts transitively on the chambers in $A$.

Let $C$ be a chamber in an apartment $A$, let $B = G_C$ denote the stabilizer of $C$, and $N$ the set-wise stabilizer of $A$. Then $(B ; N)$ is a so-called $B$-pair for the group $G$; the Weyl group $W = N = (N \setminus B)$ acts regularly on $A$ and is isomorphic to the Coxeter group of $G$. The stabilizers of the simplices contained in $C$ are called standard parabolic subgroups of $G$.

### 1.4 Panels and residues
Let $X$ be a simplex of type $J$. The residue of $X$ is the poset $Res(X) = X = \{Y \subseteq X \mid Y \neq X\}$. The residue is order-isomorphic to the link of $X$, and thus can be identified with a subcomplex of $X$ (although strictly speaking, $Res(X)$ is not a subcomplex). It is a basic but important fact that residues are again buildings; the corresponding Coxeter complex is obtained from the restricted Coxeter matrix $(m_{ij})_{J_1 \cup J}$. The type of the residue $Res(X)$ is $I \cap J$, and its rank is $\text{card}(I \cap J)$. A residue $S_i = Res(X_i)$ of rank 1 and type $I \cap J$ is called an $i$-panel.

The following observation is very simple, but useful.

### 1.5 Lemma
Suppose that $G$ acts as a group of special automorphisms on a building of rank $r$; let $C$ be a chamber. Then $G$ acts transitively on $C$ if and only if the following holds for all $r$ simplices $X_1; \ldots; X_r$ of rank $r$:

the stabilizer $G_{X_1}$ acts transitively on the panel $Res(X_1)$.

Tits pointed out that transitive groups acting on buildings of higher rank can be represented as amalgams.

### 1.6 Theorem
(Tits [49] 2.3) Let $G$ be a group acting transitively and type-preserving on an irreducible building of rank $r \geq 3$ (i.e., we assume that the Coxeter diagram is connected and has at least 3 vertices). Let $X$ be a chamber, let $X_1; \ldots; X_r$, denote its subsimplices of rank $r \geq 1$, and $X_{r+1}, r+2 \ldots$ its $r$ subsimplices of rank $r \geq 2$. Then the $2^{r+1} + 1$ different $G$-stabilizers of these simplices with their natural inclusions form a diagram (a simple 2-complex of groups, cp. Bridson & Haefliger [7] II.12 and III.C) — the corresponding poset is the set of all subsets of $I$ with at most two elements) whose limit is $G$. 
For example, a building of rank 4 with \( I = f1; 2; 3; 4g \) yields a complex of groups as follows, if we put \( G_{x,y} = G_y \) for short.

In general, the geometric realization of this 2-complex of groups is the cone over the first barycentric subdivision of the 1-skeleton of an \( r \)-simplex.

For group amalgamations related to twin buildings (cp. Section 4) see Abramenko & M"uhl-herr [2].

2 **The spherical case: projective space and \( \text{SL}_n(\mathbb{C}) \)**

In this section we describe the spherical building obtained from \( n \)-dimensional complex projective space, some related groups, and the Veronese representation. Given a ring \( R \), we let \( R(k) \) denote the matrix algebra of all \( k \times k \)-matrices with entries in \( R \).

**The spherical building \( (\mathbb{C}^n) \)**

The Grassmannian of \( k \)-spaces in \( \mathbb{C}^n \) is the complex projective variety

\[
G_{r_k}(\mathbb{C}^n) = fU \subseteq \mathbb{C}^n | \dim(U) = kg;
\]

A partial flag in \( \mathbb{C}^n \) is a nested sequence of subspaces \( 0 < U_{j_1} < U_{j_2} < \cdots < U_j \subseteq \mathbb{C}^n \), with \( \dim(U_j) = j \). Such a partial flag can be viewed as a map

\[
f1; \ldots; n \rightarrow U \rightarrow G_{r_1}(\mathbb{C}^n) \rightarrow \cdots \rightarrow G_{r_n}(\mathbb{C}^n);
\]
with \( \dim(U_j) = j \) and \( U_j = U_k \) for \( j < k \). There is a natural order \( \prec \) on the collection \( (\mathbb{C}^n) \) of all partial flags: if \( U \) and \( U^0 \) are partial flags with \( J \prec J_0 \prec \ldots \prec J_g \), then \( U \prec U^0 \) if and only if \( U^{0,j} = U \). The resulting poset is a simplicial complex of rank \( n+1 \), the spherical building

\[
(\mathbb{C}^n) = \big( (\mathbb{C}^n); \prec \big);
\]

Of course, \( (\mathbb{C}^n) \) is precisely the same as the \( n \)-1-dimensional projective geometry \( \mathbb{P} \mathbb{G}(\mathbb{C}^n) \) in a different guise.

Given an ordered basis \( (v_1; \ldots; v_n) \) of \( \mathbb{C}^n \), we may consider the maximal flag \( U = U^c(v_1; \ldots; v_n) \), where \( U_1 = \text{span}_{\mathbb{C}} f v_1 \ldots v_n g \), and the apartment \( \mathbb{A}^c f v_1 \ldots v_n g \) consisting of all flags obtained as partial flags from the \( n! \) distinct maximal flags (chambers) \( U^c(v_i; \ldots; v_n) \), where \( 2 \leq \text{Sym}(n) \). It is not difficult to check that \( (\mathbb{C}^n) \) together with this collection of apartments is a spherical building. As a simplicial complex, \( \mathbb{A}^c f v_1 \ldots v_n g \) is a triangulation of the sphere \( S^n \). In fact, let \( N_n \) denote the standard \( n \)-1-simplex, and \( \partial(N_n) \) its boundary; then \( \mathbb{A}^c f v_1 \ldots v_n g \) is simplicially isomorphic to the first barycentric subdivision \( = \text{Sd}(\partial(N_n)) \). The corresponding Coxeter group is the symmetric group \( \hat{W} = \text{Sym}(n) \), given by the presentation

\[
\hat{W} = s_1; \ldots; s_n \quad \text{where} \quad m_{ij} = \begin{cases} 2 & \text{if } i = j \\ 3 & \text{if } i < j \text{ and } j < i \\ 1 & \text{else.} \end{cases}
\]

The involution \( s_i \) is the transposition \( (i \leftrightarrow i+1) \); the Coxeter diagram is

\[
\mathbb{A}_{n-1}:
\]

\( n \) nodes. The type of a flag

\[
U : J \to \mathbb{G} \mathbb{L}(\mathbb{C}^n) \]

is \( \text{type}(U) = J \), the set of the dimensions of the subspaces occurring in \( U \). In \( (\mathbb{C}^n) \), a panel \( S \) is determined by a flag of the form

\[
U_1 < U_2 < \ldots < U_j < U_{j+1} < \ldots < U_n ;
\]

and there is a natural bijection \( S \to \mathbb{G} \mathbb{L}(U_{j+1}=U_j) = \mathbb{C} \mathbb{P}^1 = \mathbb{C} [ f1 g ] \) onto a projective line.

**Parabolics in** \( \mathbb{S} \mathbb{L}_n(\mathbb{C}) \)

There are natural actions of the groups \( \mathbb{G} \mathbb{L}_n(\mathbb{C}) \) and \( \mathbb{S} \mathbb{L}_n(\mathbb{C}) \) on the building \( (\mathbb{C}^n) \) (by type preserving automorphisms), and it is not difficult to see that these actions are strongly transitive. The
SL_n(C)-stabilizer of a partial flag is thus a parabolic subgroup. The maximal parabolics are the stabilizers of the flags of rank 1, i.e. of subspaces 0 < V < C^n. Such a maximal parabolic is conjugate to one of the standard maximal parabolics

\[ P^i = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in GL(n) \text{ with } A B C = 1; \det(A) \det(C) = 1; \]

and

\[ G_n(C^n) = SL_n(C) = P^i; \]

The minimal parabolics or Borel subgroups are the stabilizers of maximal flags, i.e. chambers in \((C^n)\). The Borel subgroups are conjugate to the standard Borel subgroup

\[ B = P^i \setminus \mathfrak{p}^i = SL_n(C) = \begin{pmatrix} B & \ldots & C \\ \vdots & \ddots & \vdots \\ 0 & \ldots & a_n \end{pmatrix} \in GL(n); \]

consisting of all unimodular upper triangular matrices. The corresponding homogeneous space is the complex flag variety

\[ FL(C^n) = CHam(C^n) = \{ U_1, \ldots, U_n \}; \text{ such that } 0 < U_1 < \ldots < U_n < C^n \}

\[ G \in (C^n) = SL_n(C) = B; \]

The anisotropic real structure of \( SL_n(C) \)

We define a semi-linear involution \( \tau \) on \( \mathfrak{sl}(n) \) by putting \( X = X^\top \) (conjugate transpose). Recall that the Lie algebra \( \mathfrak{sl}(n) \) consists of all traceless \( n \times n \)-matrices. Let

\[ \mathfrak{su}(n) = \{ X \in \mathfrak{sl}(n) \}; X + X^\top = 0 \} \quad \text{and} \quad \mathfrak{p}_n = \{ X \in \mathfrak{sl}(n) \}; X^\top = X \} \]

The decomposition \( \mathfrak{sl}(n) = \mathfrak{su}(n) \oplus \mathfrak{p}_n \) is called the Cartan decomposition of the Lie algebra \( \mathfrak{sl}(n) \). On the group level, let

\[ SU(n) = \{ g \in SL_n(C); g X g^{-1} = X \} \]

denote the group of fixed elements of the involution \( \{ g \gamma ; g \} \in Aut_n(SL_n(C)) \). Both the involution \( X \gamma = X \) in the Lie algebra and the involution \( g \gamma = g \) in the group are called Cartan involutions. (In terms of algebraic groups, we have defined an almost simple \( R \)-group scheme \( G \) such that \( G(R) = SU(n) \) and \( G(C) = SL_n(C) \). The Galois group of \( C = R \) acts on the group \( SL_n(C) \) of \( C \)-points of \( G \), and \( SU(n) \) is the group of fixed elements. The group scheme \( G \) is \( R \)-anisotropic: no
parabolic of $G$ is defined over $\mathbb{R}$. For semi-simple $\mathbb{R}$-algebraic groups, 'anisotropic' is the same as 'compact'.)

More geometrically, the group $SU(n)$ can be described as follows. Let $\langle x; y \rangle = x_1 y_1 + \cdots + x_n y_n$ denote the standard hermitian form on $\mathbb{C}^n$. This form induces a map $G_{\mathbb{R}}(\mathbb{C}^n) \to G_{\mathbb{R}}(\mathbb{C}^n)$, $\forall \gamma \neq 1$, for all $k$, which extends in a natural way to an involution $\gamma$ on $\mathbb{C}^n$. Then $SU(n)$ is the centralizer of this involution, $SU(n) = C_{\mathbb{C}^n} = SL_n(\mathbb{C})/\gamma$:

Classically, the involution $\gamma$ is called the standard elliptic polarity on the complex projective geometry $PG(\mathbb{C}^n)$; the associated Riemannian symmetric space $X = SL_n(\mathbb{C})/SU(n)$ is — as a subset of $Aut()$ — precisely the space of all elliptic polarities (i.e. the space of all positive definite hermitian forms on $\mathbb{C}^n$).

Using Gram-Schmidt orthonormalization, one shows that $SU(n)$ acts transitively on the flag variety $Fl(\mathbb{C}^n)$. Bernhard Mühlherr pointed out that there is a different proof which uses Lemma 1.5, and which carries over to the Kac-Moody case described later.

2.1 Lemma The group $SU(n)$ acts transitively on $Fl(\mathbb{C}^n)$.

Proof. Let $U = U^C(e_1; \cdots; e_n) = (U_1; \cdots; U_n)$ denote the maximal flag arising from the standard basis of $\mathbb{C}^n$. The $SU(n)$-stabilizer of a panel $S = Res(U_1; \cdots; U_{i-1}; U_{i+1}; \cdots; U_n)$ containing $U$ is isomorphic to $SU(2) \times T^{n-2}$; it induces the transitive group $SO(3) = SU(2) \times T^{n-2}$ on the panel (we let $T^k = U(1)$ denote the compact torus of rank $k$; in terms of algebraic groups, $T^k$ is the group $T^k(\mathbb{R})$ of $\mathbb{R}$-points of an anisotropic $\mathbb{R}$-torus $T$ of rank $k$). The assertion follows with Lemma 1.5.

For $n = 4$, we have by Theorem 1.6 the following complex of groups which represents $SU(4)$ as an amalgam (we indicate the nonzero entries of a matrix by $\neq$, and all matrices are assumed to be unimodular and unitary).
Here

\[ \begin{align*}
\mathbb{B} & = \mathbb{B} & \mathbb{B} & = \mathbb{B} \\
\mathbb{1} & = \mathbb{1} & \mathbb{1} & = \mathbb{1} \\
\mathbb{C} & = \mathbb{C} & \mathbb{A} & = \mathbb{A}
\end{align*} \]

\[ \mathbb{B} = \mathbb{B} \mathbb{A} \mathbb{B} \]

\[ \mathbb{1} = \mathbb{1} \mathbb{1} \mathbb{1} \mathbb{1} \]

\[ \mathbb{C} = \mathbb{C} \mathbb{A} \mathbb{C} \]

\[ \mathbb{A} = \mathbb{A} \mathbb{A} \mathbb{A} \mathbb{A} \]

In group theoretic terms, we thus have

\[ \text{SL}_n (\mathbb{C}) = \text{SU}(n) \mathbb{B} \quad \text{SU}(n) \setminus \mathbb{B} = \mathbb{T} \]

\[ \text{SL}_n (\mathbb{C}) = \text{SU}(n) \mathbb{P} \quad \text{SU}(n) \setminus \mathbb{P} = S(\mathbb{U}(n) \mathbb{U}(n)) \]

In particular,

\[ \text{Fil}(\mathbb{C}) = \text{SU}(n) = \mathbb{T} \mathbb{1} \quad \text{Gr}_k(\mathbb{C}) = \text{SU}(n) = S(\mathbb{U}(k) \mathbb{U}(n, k)) \]

**Knarr’s construction**

Let \( j (\mathbb{C}^n) \) denote the *geometric realization* of the simplicial complex \( (\mathbb{C}^n) \). There are various ways to topologize this set. One possibility is the *weak topology* determined by the simplices: by definition, a subset \( \mathbb{J} (\mathbb{C}^n) \) is closed in the weak topology if and only if its intersection with every simplex is closed. We denote the resulting topological space by \( j (\mathbb{C}^n) \) \( j \text{weak} \); there are other nice topologies, all of which yield the same weak homotopy type for the space \( j (\mathbb{C}^n) \) \( j \text{cp} \) Bridson & Haefliger [7] I.7. The Solomon-Tits Theorem asserts in our situation that there is a homotopy equivalence

\[ j (\mathbb{C}^n) \text{weak} \simeq S^n \mathbb{X} \]

where \( \mathbb{X} \) is a discrete set of cardinality \( 2^{\mathbb{A}} \) and \( \mathbb{X} + \) its one-point compactification. (In general, the Solomon-Tits Theorem says that for a spherical building of rank \( \mathbb{r} \),

\[ j j \text{weak} \simeq S^{\mathbb{r}} \mathbb{X} + \]

where \( \mathbb{X} \) is a discrete space whose cardinality is \( \text{card}(\mathbb{X}) = \text{card}(\mathbb{A} \mathbb{C} \mathbb{A} \mathbb{G}) \) for some fixed chamber \( \mathbb{C} \). The action of the automorphism group \( \text{SL}_n (\mathbb{C}) \) on the top-dimensional homology group \( H_{\mathbb{r} - 1} (j j \text{weak}) \) of this complex is called the *Steinberg representation*. See Ronan [35] App. 4 for more details and further references.)
However, this construction neglects the natural topology of \( \mathbb{C}^n \). Consider the following construction. Fix an \((n-2)\)-simplex \( N \) and label its vertices as \( 1; 2; \ldots; n \). There is a natural surjection

\[ \mathbb{F} \mathcal{I}(\mathbb{C}^n) \to \mathbb{N}^{n-2} \to \mathbb{C}^n \]

which maps \( \mathbb{F} U \mathcal{I} \to \mathbb{N}^{n-2} \to \mathbb{C}^n \) to the geometric realization of the simplex of \( \mathbb{C}^n \) spanned by the vertices \( U_1; U_2; \ldots; U_n \) of the given flag \( U \), in such a way that the \( i \)-th vertex of \( \mathbb{F} U \mathcal{I} \to \mathbb{N}^{n-2} \to \mathbb{C}^n \) is identified with \( U_i \). There is a natural compact topology on \( \mathbb{F} \mathcal{I}(\mathbb{C}^n) \to \mathbb{N}^{n-2} \to \mathbb{C}^n \), and we endow \( \mathbb{C}^n \) with the quotient topology. We denote the resulting space by \( \mathbb{C}^n_{Knarr} \) (because Knarr — inspired by Mitchell [27] — introduced it first in [21] for compact buildings of rank 2). It can be shown that there is a homeomorphism

\[ \mathbb{C}^n_{Knarr} = S^{n-2} \],

see Knarr loc.cit. We will prove this in the next section, using the Veronese representation of \( \mathbb{C}^n \); a more general result is stated in Section 7.

The Knarr construction works for general topological buildings. If the topology on the chamber set of a spherical building of rank \( r \) satisfies certain natural conditions (e.g. the inclusions between its Schubert varieties should be cofibrations), then \( \mathbb{C}^n_{Knarr} = S^{r-1} \wedge O^+ \), where \( O \) is the set of all chambers opposite to a fixed chamber, and \( O^+ \) its one-point compactification. For the special case of a discrete topology on the chamber set, this is precisely the Solomon-Tits Theorem. There is a well-developed theory of compact spherical buildings (the case of rank 2 is worked out in Kramer [23], and the results proved there extend immediately to the case of higher rank); the result for the homotopy type of \( \mathbb{C}^n_{Knarr} \) can be proved in much greater generality, see [23] Sec. 3.3. If the building is spherical, irreducible, compact, connected, and of rank at least 3 (see Section 7 for definitions), then by the results of Burns & Spatzier [9] and Grundhöfer, Knarr & Kramer [14], [15], [16], the space \( \mathbb{C}^n_{Knarr} \) can be identified with the visual boundary of \( X \) (1.), cp. Bridson & Haefliger [7] II.8., of a Riemannian symmetric space \( X \); for \( \mathbb{C}^n \), the symmetric space in question is \( X = \text{SL}_n(\mathbb{C})/\text{SU}(n) \); the same conclusion holds for buildings of rank 2, provided that the automorphism group acts transitively on the flags, see Grundhöfer, Knarr & Kramer [14], [15], [16].

**The Veronese representation of \( \mathbb{C}^n \)**

We endow \( \mathbb{C}^n \) with the positive definite hermitian form

\[ \langle X; Y \rangle = \text{tr}(X^* Y) \].

Consider the subspace \( H \) consisting of all hermitian matrices in \( \mathbb{C}^n \). Every matrix \( X \) has a unique decomposition

\[ X = X^{\text{tbs}} + \frac{\text{tr}(X)}{n} 1 \]
into a traceless hermitian matrix $X^{\text{cls}}$ and a real multiple of the identity matrix. The adjoint action of $\text{SU}(n)$ on $H(\mathfrak{n})$ is the action by conjugation,

$$X \cdot g X g :$$

We have an orthogonal $\text{SU}(n)$-invariant splitting $H(\mathfrak{n}) = \mathfrak{p}_n \oplus 1 \mathbb{R}$ (recall that $\mathfrak{p}_n = \mathfrak{f} \times 2 H(\mathfrak{n})$; $\mathfrak{g} \equiv \mathfrak{t} \mathfrak{r}(\mathfrak{x}) = 0$). Suppose that $X \in \mathfrak{h}(\mathfrak{n})$ is a projector, i.e. that

$$X^2 = X :$$

If $X \notin 0; 1$, then the minimal polynomial of $X$ is $X(\mathfrak{t}) = \mathfrak{t}(\mathfrak{t} - 1)$, and $\mathfrak{tr}(X) = k$, for some $k \geq 2; \cdots; m$. The kernel $\mathfrak{V}$ of $X$ is then an $n - k$-dimensional subspace of $\mathbb{C}^n$. In this way, we obtain an $\text{SU}(\mathfrak{n})$-equivariant 1-1 correspondence between elements of $G \mathfrak{r}_2(\mathfrak{n})$ and self-adjoint projectors with trace $k$ which is given by the map $X \cdot X^{\text{cls}} = X \cdot \frac{\mathfrak{tr}(X)}{1}$. The map $X \cdot \mathfrak{tr}(X)$ is $\text{SU}(\mathfrak{n})$-equivariant; in this way, we obtain an embedding $\text{Gr}(\mathfrak{k} \subset \mathbb{C}^n) \hookrightarrow \mathfrak{p}_n$ as follows. For $V \subset \mathbb{C}^n$ let $X_V$ denote the unique self-adjoint projector with $\ker(X_V) = V$, and put $\mathfrak{v} = \langle X \rangle^{\text{cls}}$. The elliptic polarity $\mathfrak{v}$ is built-in: the other eigenspace of $X_V$ is the image $\mathfrak{v}^\perp$ of $V$ under the elliptic polarity. Even better, the incidence can be seen in $\mathfrak{p}_n$: two self-adjoint operators $\mathfrak{v}$, $\mathfrak{w}$ are incident if and only if the euclidean distance $\mathfrak{d}(\mathfrak{v}, \mathfrak{w})$ attains the minimum possible value $\mathfrak{d}_{\text{min}}(\mathfrak{w}) = \mathfrak{d}(\mathfrak{G} \mathfrak{r}_1(\mathfrak{n}) ; \mathfrak{G} \mathfrak{r}_2(\mathfrak{n}))$. If $U$ is a flag in $\mathbb{C}^n$, and if $\mathfrak{v} \in \mathfrak{F}^2 n$ then we map $\langle U ; \mathfrak{p} \rangle \simeq \mathfrak{F}^2 \mathbb{C}^n \to \mathbb{C}^n$ to the hermitian operator

$$\mathfrak{H} = \mathfrak{p}_1(\mathfrak{U}_1) \mathfrak{p}_n :$$

In this way we obtain an $\text{SU}(\mathfrak{n})$-equivariant injection

$$\mathfrak{j}: \mathfrak{C} \to \mathfrak{p}_n :$$

We call this the Veronese representation of the building $\mathfrak{C}$. The image of the flag space $\mathfrak{F} \mathfrak{L}(\mathfrak{n})$ in $\mathfrak{p}_n$ is an isoparametric submanifold (we identify a chamber with the barycenter of its geometric realization); the images of the partial flag varieties are parallel focal submanifolds in this isoparametric foliation, cp. Palais & Terng [32], Thorbergsson [46], Knarr & Kramer [22]. (The corresponding construction for the real projective geometry $\mathbb{P} \mathfrak{G}(\mathbb{R}^3)$ leads to the classical Veronese embedding of $\mathbb{R}^2 \to S^4$, whence the name.)

The following variation of the map $\mathfrak{j}$ is also useful. For a nonzero matrix $X \in \mathcal{M}(\mathfrak{n})$, put $\mathfrak{X} \equiv \mathfrak{j} X$, where $\mathfrak{j} X$ denotes the euclidean norm, and consider the map $\mathfrak{b} : \langle U ; \mathfrak{p} \rangle \simeq \mathfrak{U} \mathfrak{P}(\mathfrak{n}) \to \mathfrak{P}_n$. Then it is not difficult to see that

$$\mathfrak{b}(\mathfrak{j} X) = S^{n^2 - 2} \mathfrak{p}_n = \mathbb{R}^{n^2 - 1} ;$$
the map is injective, since we can recover a flag \((U_{k_1}; \ldots; U_{k_r})\) from its image \(X \in \mathcal{P}_n\) as follows: the hermitian matrix \(X\) has eigenvalues \(1 < 2 < \cdots < r\), and \(U_{k_i} = \ker(X - 1)\). The surjectivity follows from the fact that every hermitian matrix can be diagonalized under the \(SU(n)\)-action, because the image of the Veronese representation contains certainly all diagonal traceless matrices of norm 1; these are precisely the images of the simplices in the apartment \(A^C \langle e_1; \ldots; e_n \rangle \). Since \(i\) is continuous on \(j(C^n) = S^{n-2}\):

### 3 The affine building of \(C(z)\)

In this section we describe the affine building associated to the discrete valuation on the rational function field \(C(z)\). This building is discussed in considerably more detail in the books by Brown [8] and Ronan [35].

**Lattices in \(C(z)^n\)**

We let \(L = C(z)\) denote the field of fractions of the polynomial ring \(C[z]\). Thus, \(L\) is the field of rational functions on the complex projective line \(C \mathbb{P}^1 = C[z]/(f1, g)\). For \(c \in 2 C \mathbb{P}^1\) we let \(O_c \subset L\) denote the subring of all rational functions which don’t have a pole in \(c\) and \(m_c = \ker(0_c)\) the maximal ideal of \(O_c\). Evaluation at \(c\) yields a map \(ev_c : O_c \rightarrow C, f \mapsto f(c)\) with kernel \(m_c\), and we obtain exact sequences

\[
0 \rightarrow m_c \rightarrow O_c \xrightarrow{ev_c} C \rightarrow 0
\]

and

\[
1 \rightarrow SL_n(m_c) \rightarrow SL_n(O_c) \xrightarrow{ev_c} SL_n(C) \rightarrow 1
\]

(the group \(SL_n(m_c)\) is defined to be the kernel of the evaluation map \(ev_c\)). We may view the elements of \(SL_n(L)\) as *rational maps* from \(C \mathbb{P}^1 \) into \(SL_n(C)\). Note also that every element \(q \neq 0\) of \(L\) can be expressed in the form

\[
q = \frac{(z - c)^{k}f}{g};
\]

with \(k \in Z\), \(f, g \in C[z]\), and \(f(c) \neq 0 \neq g(c)\). We put \(c(q) = k\), and \(c(0) = 1\). The map

\[
c : L \rightarrow Z[\ f1, g]
\]
is a *discrete valuation* on \( L \) (with some modifications for \( c = 1 \)). Note that \( O_c = f q \cdot O \cdot c(q) = 0 \). There is nothing special about the choice of \( 2 \cdot CP^1 \), and we put \( c = 0 \) for the remainder of this section.

The group \( SL_n(L) \) acts on the projective geometry \( \mathbb{P}G(L^n) \) in very much the same way as \( SL_n(C) \) on \( \mathbb{P}G(C^n) \), and we could consider the spherical building \( (L^n; O_0) \). But now we introduce a different geometry for this group, the *affine building* \( (L^n; O_0) \). Given an \( L \)-basis \( v_1, \ldots, v_n \) of \( L^n \), we have the free \( O_0 \)-module

\[
M = \text{span}_{O_0} f v_1, \ldots, v_n g = v_1 O_0 + \# O_0
\]

of rank \( n \) generated by these basis vectors. We call \( M \) an \( O_0 \)-*lattice*, and we let \( \text{Lat}_n(L; O_0) \) denote the collection of all such lattices. (The following simple observation is useful. If \( M \) \( L^n \) is a free \( O_0 \)-module of rank \( k \), with \( O_0 \)-basis \( f v_1, \ldots, v_k g \), then \( f v_1, \ldots, v_k g \) is linearly independent over \( L \), because \( L \) is the field of fractions of \( O_0 \). Thus, the \( O_0 \)-lattices are precisely the free \( O_0 \)-modules of rank \( n \) in \( L^n \).) Evidently, the group \( GL_n(L) \) acts transitively on \( \text{Lat}_n(L; O_0) \); the \( GL_n(L) \)-stabilizer of the \( O_0 \)-module \( M_0 \) spanned by the canonical basis \( e_1, \ldots, e_n \) of \( L^n \) is the group \( GL_n(O_0) \), and therefore

\[
\text{Lat}_n(L; O_0) = G L_n(L) / G L_n(O_0).
\]

We call two lattices \( M \) \( 2 \) \( \text{Lat}_n(L; O_0) \) *projectively equivalent* if \( M = q M \) \( 2 \) \( L \). In view of the factorization of \( q \) given above, this is clearly equivalent with the condition that \( M = z^k M \) \( 2 \) holds for some \( k \) \( 2 \). The projective equivalence class of \( M \) \( 2 \) \( \text{Lat}_n(L; O_0) \) is denoted by

\[
[M] = f z^k M \mid k \in \mathbb{Z}
\]

Thus we have obtained an action of the projective groups \( \mathbb{P}G L_n(L) \) and \( \mathbb{P}SL_n(L) \) on the set

\[
f [M] \mid M \in 2 \text{Lat}_n(L; O_0) g
\]

of projective equivalence classes of \( O_0 \)-lattices.

**The action of \( SL_n(L) \) and the type function**

The group \( SL_n(L) \) is *not* transitive on set of projective equivalence classes of \( O_0 \)-lattices. Let

\[
M_0 = \text{span}_{O_0} f e_1, \ldots, e_n g
\]

denote the \( O_0 \)-module spanned by the canonical basis \( e_1, \ldots, e_n \) of \( L^n \). Suppose that \( g(M_0) = M \) \( 0 \), for some \( g \) \( 2 \) \( GL_n(L) \). Since \( O_0 (\det \theta)) = 0 \) for all \( h \) \( 2 \) \( GL_n(O_0) \) (because \( O_0 (q) = 0 \) if \( q \) \( 2 \) \( O_0 \)
is a unit), the number \( \det(g) \) depends only on the module \( M^0 \). The determinant of the map
\[ z^k : v \rightarrow z^k \cdot v \] is \( \det(z^k) = z^{kn} \). Thus we have a well-defined map
\[ \text{type}(M^0) = (\det(g)) + nZ \] which is \( SL_n(L) \)-invariant. Note also that the stabilizers agree,
\[ SL_n(L)_M = SL_n(L)_{M^0} \]
since \( \det(z^k) \neq 1 \) for \( k \neq 0 \). We put
\[ V_i = fM \in M_2 \text{Lat}_n(L;O_0); \text{type}(M) = ig: \]
Let
\[ M_i = \text{span}_{O_0} \{ v_1; \ldots; v_i; e_{i+1}; \ldots; e_n \} \]
denote the \( O_0 \)-module spanned by the vectors \( v_1; \ldots; v_i; e_{i+1}; \ldots; e_n \). Then \( [M_1] \in V_i \).

3.1 Lemma The action of \( SL_n(L) \) on \( V_i \) is transitive.

Proof. If \( [M] = [g[M_1]] \in V_i \) for some \( g \in GL_n(L) \), then \( n \neq \text{odd} \) \( n \), whence \( \det(z^kg) = 0 \) for a suitable \( k \neq 0 \). Put \( g^0 = z^kg \), then \( [M] = [g^0[M_1]] \) and \( \det(g^0) = 0 \), whence \( \det(g^0) \neq 0 \). Finally, put \( h = \text{diag}(1; \ldots; 1) \). Then \( h \) fixes \( M_i \), and thus \( g^0h \) maps \( [M_1] \) to \( [M] \).

These \( n \) different \( O_0 \)-modules \( M_0; \ldots; M_{n-1} \) thus form a cross-section for the action of \( SL_n(L) \) on the set of projective equivalence classes of \( O_0 \)-lattices, and we put
\[ P_0^i = SL_n(L)_{M^0} = SL_n(O_0) \]
\[ P_i^1 = SL_n(L)_M = \begin{pmatrix} A & z \\ zC & D \end{pmatrix}, A 2 O_0(1); \begin{pmatrix} A & B \\ C & D \end{pmatrix} 2 SL_n(O_0) \]
for \( i = 1; \ldots; n \). The sets \( V_0; \ldots; V_{n-1} \) play the same rôles as the Grassmannians \( G_{r_j}(\mathbb{C}^n) \) in Section 1.1, and the stabilizers \( P_0^r; \ldots; P_{n-1}^r \) play the same rôles as the standard maximal parabolics \( P^1; \ldots; P^{n-1} \) in \( SL_n(\mathbb{C}) \). However, there is one fundamental difference: the stabilizers \( P_i^1, i = 0; \ldots; n-1 \), are conjugate to \( P_0^1 \) in \( GL_n(L) \),
\[ P_i^1 = g_i SL_n(O_0) g_i^{-1} \]
where \( g_i = \text{diag}(1; \ldots; 1; 1; \ldots; 1) 2 GL_n(L) \).
Incidence, periodic flags, and apartments

We define an incidence relation $I$ on the set $fM_j | M \in \text{Lat}_n (L; O_0) g$ as follows:

$$M \sqcup M' \overset{\text{def}}{=} zM \quad z^k M \quad M \quad \text{for some } k \geq 2 \in \mathbb{Z}.$$  

This relation is symmetric, since $zM = z^k M \quad M$ implies that $z^M = z^k M \quad M$. Clearly, $SL_n (L)$ acts by incidence preserving automorphisms on this incidence geometry. Similarly as before, we can use the incidence relation to construct a poset $(L_n; O_0)$, the affine building of $(L_n; O_0)$; the elements of $(L_n; O_0)$ are the sets of pairwise incident elements of $V_0$.

Note also that $M_i \sqcup M_j$ holds for all $i \neq j$. The chambers are of the following form. Let $B = (v_1; \ldots; v_n)$ be an ordered basis of $L^n$, and put $M^B = \text{span}_{O_0} f v_1; \ldots; v_i; v_{i+1}; \ldots; v_n g$. Then $M (B) = f M^B_0; \ldots; M^B_{n-1} g$ is a maximal flag, which can be viewed as an infinite sequence of free $O_0$-modules

$$>^1 M^B_0 > M^B_0 > M^B_1 > \ldots >^n M^B_1 > zM^B_0 > z^2 M^B_1 >$$

of rank $n$. The quotient of two consecutive modules in this chain is a one-dimensional complex vector space. Note that $(L_n; O_0)$ has rank $n$.

The collection of all chambers is called the periodic flag variety $F(L_n; O_0)$. The stabilizer of the chamber $f M_0; \ldots; M_{n-1} g$ is the Borel group

$$B_{\text{rat}} = P_{\text{rat}} \setminus P_{\text{rat}}$$

Given a basis $v_1; \ldots; v_n$ of $L^n$, we define the standard apartment $A_{\text{rat}} = f v_1; \ldots; v_n g$ as the collection of all partial flags obtained from the maximal flags $M (B)$, where $B$ runs through the collection of all bases of the form $B = (v^{k_1} v_{1}; \ldots; v^{k_n} v_{n})$, for $\text{Sym} (n)$ and $k_1; \ldots; k_n \geq 2 \in \mathbb{Z}$. The set-wise $SL_n (L)$-stabilizer $N_{\text{rat}}$ of $A_{\text{rat}}$ is the collection of all unimodular permutation matrices with entries in $L^1$, and the element-wise stabilizer $T_{\text{rat}}$ of $A_{\text{rat}}$ is the collection of all diagonal unimodular matrices with entries in $O_0$. The quotient is the affine Weyl group $\tilde{W} = N_{\text{rat}} = T_{\text{rat}} = A^L_{n-1}$ of $SL_n (L)$. As a simplicial complex, $A^L_{n-1}$ is a triangulation of $R^n$.

3.2 Theorem The simplicial complex $(L_n, O_0)$ is a building of rank $n$ and type $A^L_{n-1}$; as an apartment system, we may choose the set $A^L_0 = f A_0^L f v_1; \ldots; v_n g$ as a basis for $L^ng$. The Coxeter diagram is $A^L_{n-1}$;
(n nodes).

Proof. Probably the easiest way to see that this is a building is to verify that \(B_{\text{rat}} \backslash N_{\text{rat}}\) is a \(B \backslash N\) pair for the group \(\text{SL}_n(L)\); cp. Brown [8] Ch. V.8 and Ronan [35] Ch. 9.2.

It is not difficult to see that the residue of \([M_i]\) is isomorphic to \(\mathbb{C}^n\), for \(i = 0; \ldots; n - 1\). Thus the panels in \((L^n; O_0)\) are again isomorphic to the complex projective line \(\mathbb{C}P^1\).

One final remark. We have constructed the affine building \((L^n; O_0)\) related to the discrete valuation \(\nu_0\). There is nothing special about the point \(0 \in \mathbb{C}P^1\); if we choose a different point \(c \in \mathbb{C}P^1\), then we obtain a different affine building \((L^n; O_c)\). These buildings are pairwise isomorphic; in fact they are permuted by the group \(\text{PSL}_2(\mathbb{C})\). Each of these building has a distinct collection of parabolics, so \(\text{SL}_n(L)\) contains an uncountable set of \(B \backslash N\) pairs.

4 The twinning over \(\mathbb{C}[z; 1/z]\)

Now we describe the buildings \((L^n; O_0)\) and \((L^n; O_1)\) in a slightly different way, replacing the field \(\mathbb{C}(z)\) by the ring \(\mathbb{C}[z; 1/z]\). This section owes much to the paper [4] by Abramenko & Van Maldeghem. So let

\[A = \mathbb{C}[z; 1/z] = \big\langle \text{f}_{0,x} \cdot jx \big| 2 < C g\right\rangle\]

denote the ring of all rational functions which are holomorphic on \(\mathbb{C}P^1\). This is a subring of \(L\), the ring of Laurent polynomials. Note that \(A \setminus O_0 = \mathbb{C}[z]\) and \(A \setminus O_1 = \mathbb{C}[1/z]\). Similarly as before, we let \(\text{Lat}_n(A; \mathbb{C}[z])\) denote the collection of all free \(\mathbb{C}[z]\)-modules in \(A^n\) which are spanned by an \(A\)-basis, and \(\text{Lat}_n(A; \mathbb{C}[1/z])\) the collection of all free \(\mathbb{C}[1/z]\)-modules spanned by \(A\)-bases. Thus

\[\text{Lat}_n(A; \mathbb{C}[z]) = \big\langle \text{f}_{0,j}E_{0,j} \big| 2 < \text{GL}_n(A)g\right\rangle\quad\text{and}\quad\text{Lat}_n(A; \mathbb{C}[1/z]) = \big\langle \text{f}_{0,j}E_{0,j} \big| 2 < \text{GL}_n(A)g\right\rangle\]

where \(E_{0,j}^+ = \text{span}_{\mathbb{C}[z]}(e_1; \ldots; e_i; e_{i+1}; \ldots; e_n; g)\) and \(E_{0,j}^- = \text{span}_{\mathbb{C}[1/z]}(e_1; \ldots; e_i; e_{i+1}; \ldots; e_n; g)\). For \(E \in \text{Lat}_n(A; \mathbb{C}[z])\) [\(\text{Lat}_n(A; \mathbb{C}[1/z])\)] we put as before

\[E_j = \text{f}_{x} E_{j} \big| 2 < \text{Z} g\right\rangle\]

and

\[E_{j}^+ = \text{span}_{\mathbb{C}[z]}(fz_1; \ldots; z_i; e_{i+1}; \ldots; e_n; g)\quad\text{and}\quad E_{j}^- = \text{span}_{\mathbb{C}[1/z]}(fz_1; \ldots; z_i; e_{i+1}; \ldots; e_n; g)\]

and

\[V_j = \big\langle \text{f}_{x} E_{j} \big| 2 < \text{SL}_n(A)g\right\rangle\]
The incidence is also defined as before. If \( [E] \supseteq V_i \) and \( [E^0] \supseteq V_j \), then
\[
[E \mid [E^0] \overset{\text{def}}{=} z^1E z^kE^0E \quad \text{for some } k \geq 2 \text{ in } Z.
\]

Thus we obtain two simplicial complexes \( ^+ (\mathbb{A}^n) \) and \( (\mathbb{A}^n) \) which are isomorphic under the map induced by the ring automorphism \( z \mapsto 1 \). The set of all maximal simplices is denoted by \( F^1(\mathbb{A}^n) \).

Now there is a canonical map \( \text{Lat}_n(\mathbb{A}; C[\{2\}]) \longrightarrow \text{Lat}_n(\mathbb{L}; O_0) \) which maps \( E \) to \( \text{span}_{O_0}(E) \), and a similar map \( \text{Lat}_n(\mathbb{A}; C[1 = 2]) \longrightarrow \text{Lat}_n(\mathbb{L}; O_1) \); these maps induce canonical \( \text{SL}_n(\mathbb{A}) \)-equivariant simplicial maps \( ^+ (\mathbb{A}^n) \longrightarrow (\mathbb{L}^n; O_0) \) and \( (\mathbb{A}^n) \longrightarrow (\mathbb{L}^n; O_1) \).

4.1 Proposition The two maps \( ^+ (\mathbb{A}^n) \longrightarrow (\mathbb{L}^n; O_0) \) and \( (\mathbb{A}^n) \longrightarrow (\mathbb{L}^n; O_1) \) are isomorphisms and thus \( ^+ (\mathbb{A}^n) \), \( (\mathbb{A}^n) \) are buildings; the group \( \text{SL}_n(\mathbb{A}) \) acts transitively on both buildings.

Proof. For the proof we note that the \( \text{SL}_n(\mathbb{A}) \)-stabilizer of the \( n \) 2-simplex \( B_n = E_0 \cup \ldots \cup E_n \) induces the transitive group \( \text{PSL}_2(\mathbb{C}) \) on the corresponding panel. Thus \( \text{SL}_n(\mathbb{A}) \) acts transitively on \( (\mathbb{L}^n; O_0) \) by Lemma 1.5. Since \( \text{SL}_n(\mathbb{A}) \) has the same stabilizers both in \( ^+ (\mathbb{A}^n) \) and in \( (\mathbb{L}^n; O_0) \) (see below), we obtain the claimed isomorphism \( ^+ (\mathbb{A}^n) = (\mathbb{L}^n; O_0) \). The involution \( z \mapsto 1 \) on the ring \( \mathbb{A} \) and the projective line \( \mathbb{P}^1 = C \{1 \} = \mathbb{R} \) normalizes \( \text{SL}_n(\mathbb{A}) \), and we obtain \( (\mathbb{A}^n) = (\mathbb{L}^n; O_1) \).

If \( B = (v_1; \ldots; v_n) \) is an \( \mathbb{A} \)-basis for \( \mathbb{A}^n \), then we have similarly as before the chamber \( E^+ (B) = E^0 \cup \ldots \cup E_n \) \( g \)-corresponding to the modules \( E^+_1 \overset{\text{mod}}{=} \text{span}_C(\{2\})fzv_1; \ldots; zv_1 v_i; \ldots; v_i g \), and the apartment \( ^+ (\mathbb{A}) f \) \( v_1; \ldots; v_n g \) obtained from the \( \mathbb{A} \)-bases \( (z^k e_{(i)}; \ldots; z^k e_{(i)}) \); similarly, we obtain an apartment \( ^+ (\mathbb{A}) f \) \( v_1; \ldots; v_n g \) for the building \( \mathbb{A}^n \). It can be checked that \( \text{SL}_n(\mathbb{A}) \) acts strongly transitively on \( ^+ (\mathbb{A}^n) \) and \( (\mathbb{A}^n) \) with respect to these apartment systems

\[
A = fA \overset{\mathbb{A}}{=} f v_1; \ldots; v_n g \overset{\mathbb{A}}{=} v_1; \ldots; v_n g
\]

Note that the apartment system \( ^+ (\mathbb{A}^n) \) is strictly smaller than \( \mathbb{A}^n \).

4.2 Proposition The group \( \text{SL}_n(\mathbb{A}) \) acts strongly transitively both on \( ^+ (\mathbb{A}^n) \) and on \( (\mathbb{A}^n) \).

Proof. This follows from the fact that \( \text{SL}_n(\mathbb{A}) \) acts transitively on pairs of opposite chambers in \( ^+ (\mathbb{A}^n) \) \( (\mathbb{A}^n) \), cp. Abramenko & Van Maldeghem [4]; the opposition relation is defined in the next section.

We put
\[
P^+_\alpha = P^+_\alpha \setminus \text{SL}_n(\mathbb{A}) \quad \text{and} \quad B^+_\alpha = B^+_\alpha \setminus \text{SL}_n(\mathbb{A}).
\]
Thus
\[ P_{a_{ij}}^{\alpha \gamma} = SL_n (C[z]) \quad \text{and} \quad P_{a_{kj}}^{\gamma \alpha} = g_i (P_{a_{ij}}^{\alpha \gamma}) g_j \]
where \( g_i = \text{diag}(z; \ldots; z; 1; \ldots; 1) \) as before. These are the parabolics corresponding to \( + (\mathbb{A}^n) \); there are similar parabolics for \( - (\mathbb{A}^n) \).

**Twin buildings**

The notion of a twin building was developed by Ronan and Tits in order to supply geometries for Kac-Moody groups; we refer to Abramenko [1], Abramenko & Ronan [3] Abramenko & Van Maldeghem [4], M"uhlherr [28] [29], M"uhlherr & Ronan [31], Ronan [36], Ronan & Tits [37] and Tits [51], [50] for more details about twin buildings.

Let \((W; S)\) be a Coxeter system, and let \((+; -)\) be a pair of buildings with this given Coxeter system. The \(\tilde{W}\) -valued distance in \(\widetilde{W} = \text{Cham}(+; -)\) is denoted \(d_{\tilde{W}}(C; D)\). A *twinning* of \(+; -\) is a \(W\) -valued codistance function

\[ :\text{Cham}(+; -) \rightarrow \tilde{W} ; \]

subject to the following axioms. *The intuitive idea is that objects with a small codistance are far apart.*

**Tw** The relation \( (C; D ) = (D ; C ) \) holds for all chambers \(C \neq D\) (the codistance is 'symmetric').

**Tw** Let \(w \neq \tilde{w}\) and \(s \neq \tilde{s}\), and suppose that \( \langle w, s \rangle = \langle \tilde{w}, \tilde{s} \rangle \) (i.e. that \(w\) has a reduced expression with \(s\) as the last letter). If \(C \neq D\) are chambers with \( (C; D ) = w\) and \( (D ; E ) = s\), then \( (C; E ) = w s\) (all chambers \(E\) in the \(s\)-panel through \(D\) are 'further away' from \(C\)).

**Tw** If \(C \neq D\) are chambers with codistance \( (C; D ) = w\), and if \(s \neq \tilde{s}\), then there exists a chamber \(E \neq D\), with \( (D; E ) = s\), and with \( (C; E ) = w s\) (the codistance leads to galleries).

Note that there is a symmetry in the axioms if we exchange the signs '+' and '-'. Sometimes we will state a result for a specific choice of the signs; the corresponding result for the opposite choice of signs follows in the same way.

It is clear how to extend the codistance to a double coset valued codistance

\[ :S \widetilde{W} ; n \tilde{W} = \tilde{W}_K \rightarrow J; K : \]

The *opposition relation* \(\text{op} \quad + \) is defined as follows: two chambers are opposite if their codistance is 1; this relation extends in a natural way to the simplices. The codistance can be used to
sync the type functions in both buildings in such a way that opposite vertices have the same type, and we will assume this to be done.

A (special) automorphism of a twin building is a pair \((g^+; g^-)\) of automorphisms \(g \in \text{Spec}(\mathbb{C}^+)\) which preserves the codistance, \((C^+; g(C^-)) = (g^+(C^+); g(C^-))\).

### 4.3 Example

Let \(\mathfrak{B} \mathfrak{n}\) be a spherical building, let \(w_0\) denote the unique longest element in the Coxeter group \(W\) of \(\mathfrak{B} \mathfrak{n}\), put \(g^+ = \mathbb{C}^+\), and \((C; D) = (w_0(C); g^+(D))\). The resulting geometry is a twin building.

All twin buildings with spherical halves arise in this way, see Tits [50]. Twin buildings are natural generalizations of spherical buildings; they share many of the particular geometric properties of spherical buildings. The twinning is in general not determined by the pair \((g^+; g^-)\); a pair of buildings (e.g. a pair of trees) can admit many non-equivalent twinnings.

The group \(\text{SL}_n(\mathbb{A})\) induces a twinning on the pair \((\mathbb{A}^+, \mathbb{A}^-)\); the codistance can be defined as follows. The group \(\text{SL}_n(\mathbb{A})\) has a Birkhoff decomposition (a Bruhat twin decomposition) as

\[
\text{SL}_n(\mathbb{A}) = B_{\mathbb{A}^+} N B_{\mathbb{A}^-} = -fB_{\mathbb{A}^+}w B_{\mathbb{A}^-}jw 2 N = (B_{\mathbb{A}^+} \setminus B_{\mathbb{A}^-})g
\]

where \(N\) is the set-wise stabilizer of the twin apartment \((\mathbb{A}^+, \mathbb{A}^-)\) = \(B_{\mathbb{A}^+}\) and \(N = (B_{\mathbb{A}^+} \setminus B_{\mathbb{A}^-}) = W\). We may use the Birkhoff decomposition to define the codistance as

\[
(gB_{\mathbb{A}^+}; hB_{\mathbb{A}^-}) = w \quad \text{if and only if} \quad B_{\mathbb{A}^+}g^{-1}hB_{\mathbb{A}^-} = B_{\mathbb{A}^+}w B_{\mathbb{A}^-}.
\]

The following Theorem is 'folklore'. A nice proof is given in Abramenko & Van Maldeghem [4].

### 4.4 Theorem

The triple \((\mathbb{A}^+, \mathbb{A}^-)\) is a twin building, and the group \(\text{SL}_n(\mathbb{A})\) acts as a strongly transitive group of automorphisms. Two chambers \((C^+; C^-)\) and \((\mathbb{A}^+, \mathbb{A}^-)\) are opposite if and only if they arise from lattices which are 'back to back', i.e. if there exists an ordered \(\mathbb{A}\)-basis \((\mathbb{A}_1; \cdots; \mathbb{A}_n)\) such that \(C^+ = E^+(\mathbb{A}_1; \cdots; \mathbb{A}_n)\) and \(C^- = E^-((\mathbb{A}_1; \cdots; \mathbb{A}_n))\).

### 5 Loop groups

There is one big difference between \(\text{SL}_n(\mathbb{L})\) and \(\text{SL}_n(\mathbb{A})\) which is due to the fact that \(\mathbb{A}\) is a ring, while \(\mathbb{L}\) is a field: the group \(\text{SL}_n(\mathbb{L})\) is almost simple, whereas the group \(\text{SL}_n(\mathbb{A})\) is a semi-direct product.
Based loops

Fix \( c \in \mathbb{C} \) and consider the evaluation map \( ev_c : SL_n(\mathbb{A}) \rightarrow SL_n(\mathbb{C}) \) given by

\[
ev_c : f(z) \mapsto f(c).
\]

We denote the kernel of this map by \( \Delta_g(SL_n(\mathbb{C});c) \) and we put \( \Delta_g(SL_n(\mathbb{C}));1 = \Delta_g(SL_n(\mathbb{C}));1 \) for short. The injection \( SL_n(\mathbb{C}) \rightarrow SL_n(\mathbb{A}) \) leads to a split exact sequence

\[
1 \rightarrow \Delta_g(SL_n(\mathbb{C});c) \rightarrow SL_n(\mathbb{A}) \xrightarrow{ev_c} SL_n(\mathbb{C}) \rightarrow 1;
\]

hence \( SL_n(\mathbb{A}) \) is a semi-direct product. (There is an obvious \( SL_n(\mathbb{A}) \)-action on \( SL_n(\mathbb{A}) \) given by \( f(z) \mapsto f(az) \); under this action, the collection of normal subgroups \( \Delta_g(SL_n(\mathbb{C});c) \) for \( c \in \mathbb{C} \) is permuted transitively. Thus one is lead to the semi-direct product \( SL_n(\mathbb{A}) \times \mathbb{C} \).

5.1 Lemma For every \( c \in \mathbb{C} \), the group \( \Delta_g(SL_n(\mathbb{C});c) \) acts transitively on each of the sets \( \mathbb{C}^n \).

Proof. We clearly have \( SL_n(\mathbb{C}) \rightarrow SL_n(\mathbb{C}[z]) = P^0_{\Delta_g}, \) and thus

\[
P^0_{\Delta_g} \Delta_g(SL_n(\mathbb{C});c) \rightarrow SL_n(\mathbb{C}) \Delta_g(SL_n(\mathbb{C});c) = SL_n(\mathbb{A});
\]

Now \( P^0_{\Delta_g} = \Delta_g P^0_{\Delta_g} \Delta^{-1} \Delta_g(SL_n(\mathbb{C});c) \) is \( \Delta_g \)-invariant, whence

\[
P^0_{\Delta_g} \Delta_g(SL_n(\mathbb{C});c) = \Delta_g P^0_{\Delta_g} \Delta^{-1} \Delta_g(SL_n(\mathbb{C});c) = SL_n(\mathbb{A});
\]

There is a natural map \( SL_n(\mathbb{A}) \rightarrow C^1(S^1;GL_n(\mathbb{C})) = L_{d1} SL_n(\mathbb{C}) \) into the set \( L_{d1} SL_n(\mathbb{C}) \) of smooth maps from the unit circle into \( SL_n(\mathbb{C}) \); this map is obtained by viewing the elements of \( SL_n(\mathbb{A}) \) as maps from \( S^1 \) into \( SL_n(\mathbb{C}) \). If \( f = \int_0^1 f(z)dz \) is \( 2 \) \( SL_n(\mathbb{A}) \), then

\[
f_k = \frac{1}{2} \int_0^1 \frac{1}{z^{k+1}} f(z)dz.
\]

Therefore, the map into \( L_{d1} SL_n(\mathbb{C}) \) is an injection. From now on, we denote the group \( SL_n(\mathbb{A}) \) also by \( L_{\Delta_g} SL_n(\mathbb{C}) \). The subgroup \( \Delta_g(SL_n(\mathbb{C});c) \) is thus the subgroup \( L_{\Delta_g} SL_n(\mathbb{C}) \setminus \Delta_1 SL_n(\mathbb{C});c \) of \( c \)-based algebraic loops (by \( c \)-based we mean that the base-point of \( S^1 \) is \( c \) — the base point of the group is always the identity element).
The Cartan involution

Consider the semi-linear involution $\#$ on the complex vector space $A(n)$ which is given by

$$f = X f_k z^k \quad f^\# = X f_k z^k.$$  

The subgroup of all elements $g \in SL_n(A)$ with $g^\# = g$ is denoted by $L_{a\gamma}SU(n)$. This terminology is motivated by the fact that

$$L_{a\gamma}SU(n) = L_{d1}SU(n) \setminus L_{a\gamma}SL_n(C).$$

To see this, note that for $z \in S^1$ we have $z^\# = z$, hence $f(z)f(z) = 1$ holds for all $z \in S^1$ if and only if $f \in L_{a\gamma}SU(n)$. For $c \in S^1$ we have a semi-direct decomposition

$$L_{a\gamma}SU(n) = SU(n) \rtimes_{a\gamma}(SU(n); c)$$

as before, and an $S^1$-action which is given by $f(z) 7^! f(az)$.

5.2 Lemma We have $L_{a\gamma}SU(n) \setminus SL_n(C[z]) = SU(n)$.

Proof. Let $f = f_0 + f_1 z + \ldots z^k 2 SL_n(C[z])$. Then $f^\# = f_k z^k + z f^1 + f_0$. If $f^\# = f^1 2 SL_n(C[z])$, then $f_i = 0$ for $i \neq 1$.

In particular,

$$B_{a\gamma}^+ \setminus L_{a\gamma}SU(n) = T_{a\gamma}^n \quad \text{and} \quad P_{a\gamma}^{i+} \setminus L_{a\gamma}SU(n) = S(U(i) U(n i))$$

(as in Section 2, $T^k$ denotes a compact torus of rank $k$). There is a more geometric description of the group $L_{a\gamma}SU(n)$. The map $A^n \rightarrow A^n$ induces isomorphisms $+ (A^n) \rightarrow (A^n)$ and $(A^n) \rightarrow + (A^n)$,

and $L_{a\gamma}SU(n) = C \cap_{SL_n(A)}(\#)$. There is a corresponding Cartan decomposition of the loop algebra $sl_n(A)$ into eigenspaces of the involution $\#$,

$$sl_n(A) = L_{a\gamma}su(n) \times;$$

where X denotes the traceless hermitian matrices in $A(n)$.
5.3 Theorem  The group $L_{a_k} \text{SU}(n)$ acts transitively on the periodic flags.

$L_{a_k} \text{SL}_n(\mathbb{C}) = B_{a_k}^+ L_{a_k} \text{SU}(n)$;  \quad \text{Flag}(^{+} (\mathfrak{A}^n)) = L_{a_k} \text{SU}(n) = T^n$.

Proof. The proof is exactly the same as in 2.1. The $L_{a_k} \text{SU}(n)$-stabilizer of a panel is isomorphic to $\text{SU}(2)$ and induces the transitive group $\text{SO}(3)$ on the panel.

In particular, $L_{a_k} \text{SU}(n)$ acts transitively on $V^+_0$. Since $L_{a_k} \text{SU}(n) M_0 = \text{SU}(n)$, the group $a_k \text{SU}(n)$ acts regularly on $V^+_0$.

$V^+_0 = L_{a_k} \text{SU}(n) = \text{SU}(n) = a_k \text{SU}(n)$;

and

$\text{Flag}(^{+} (\mathfrak{A}^n)) = a_k \text{SU}(n) = \text{SU}(n) = T^n$.

Note that the proof above (which is due to Bernhard M"uhlherr) is much simpler than the classical proof given e.g. in Pressley & Segal [34] Theorem 8.3.2.

6 The affine Veronese representation of $^{+} (\mathfrak{A}^n)$

In this section we construct an equivariant embedding $^{+} (\mathfrak{A}^n) \longrightarrow \mathfrak{sl}_n(\mathfrak{A})$ which is very similar to the finite-dimensional Veronese representation $(\mathbb{C}^n) \longrightarrow \mathfrak{p}_n \longrightarrow \mathfrak{sl}_n(\mathbb{C})$. Recall that we presented the flags in $(\mathbb{C}^n)$ as certain hermitian operators. Similarly, we want to associate an operator to the $\mathbb{C}[z]$-module $E_0$. To this end we consider the first order linear differential operator

$A \longrightarrow A; \quad f \longmapsto z \partial_z f = \frac{\partial f}{\partial z}$.

If $f = \sum_k f_k z^k$, then $z \partial_z f = \sum_k k f_k z^k$. Thus

$\ker(z \partial_z) = \begin{cases} \mathbb{C} & \text{for } n = 2 \mathbb{Z} \\ 0 & \text{for } 2 \mathbb{Z} \end{cases}$

In particular,

$\mathbb{C}[z] = \ker(z \partial_z + k)$ and $\mathbb{C}[1+z] = \ker(z \partial_z + k)$.
The operator \( z \otimes_z \) extends in an obvious way to \( A_n \) and to the matrix algebra \( A_n(\mathfrak{a}_n) \). For \( f \in A_n \) we put

\[
Df = z \otimes_z f.
\]

Let \( k = \text{diag}(1; \ldots; 1; 0; \ldots; 0) \in \mathfrak{h}(\mathfrak{a}_n) \) and let \( \text{tls}_k = k \cdot \mathfrak{h}(\mathfrak{a}_n) \) denote its traceless image in \( \mathfrak{p}_n \).

Thus

\[
E^+_k = \text{span}_{\mathbb{C}[z]} f e_1; \ldots; e_{k-1}; e_k; e_{k+1}; \ldots; e_n; g
\]

\[
= \ker(\text{D}_k \oplus 1; k \cdot \mathfrak{h}(\mathfrak{a}_n))
\]

\[
= \ker(\text{D}_k \oplus \text{tls}_k(k + \frac{1}{n})1);
\]

the elements of the flag varieties \( V^+_k \) correspond bijectively to the \( \mathfrak{sl}_2(\mathfrak{a}_n) \)-conjugates of \( D \cdot \text{tls}_k \).

Let \( g \in \mathfrak{sl}_2(\mathfrak{a}_n) \). Then

\[
0 = z \otimes_z (g^* g) = (z \otimes_z g^*) g^* + g \otimes_z (g^*)^*;
\]

Therefore \( g \otimes \mathfrak{sl}_2(\mathfrak{a}_n) f = g(z \otimes_z (g^*)^*) f + g^* D f = D f (z \otimes_z g^*) g^* f \), whence \( g \otimes \mathfrak{sl}_2(\mathfrak{a}_n) g^* = D \cdot (z \otimes_z g^*) g^* \), and

\[
g \otimes \text{tls}_k g^* = D \cdot (z \otimes_z g^*) g^* g \cdot \text{tls}_k g^*.
\]

We put

\[
X = fX \in \mathfrak{sl}_2(\mathfrak{a}_n) jX^* = X g;
\]

This is an infinite-dimensional real vector space, and \( (z \otimes_z g^*) g^* \cdot X \) for all \( g \in \mathfrak{sl}_2(\mathfrak{a}_n) \). We construct an \( \mathfrak{sl}_2(\mathfrak{a}_n) \)-equivariant embedding of \( \mathfrak{a}_n^* \) into the infinite-dimensional real vector space \( X \cdot \mathfrak{sl}_2(\mathfrak{a}_n) \) as follows.

\[
V^+_k \longrightarrow X \quad \text{RD}
\]

\[
[gE_k^+] \cdot \mathfrak{sl}_2(\mathfrak{a}_n) g^* = D \cdot (z \otimes_z g^*) g^* g \cdot \text{tls}_k g^*;
\]

We extend this mapping to the geometric realization \( j^* \mathfrak{a}_n^* \) of \( \mathfrak{a}_n^* \) in the canonical way, and we call the resulting \( \mathfrak{sl}_2(\mathfrak{a}_n) \)-equivariant map

\[
\overline{j^* \mathfrak{a}_n^*} j \longrightarrow X \quad \text{RD} \quad X
\]

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the affine Veronese representation of $^+ (\mathbb{A}^n)$. Explicitly, the Veronese representation of the vertex $[gE_k]$ is

$$(gE_k) = g \wedge_k g^\# + (z \otimes g) g^\#: $$

Note that the group $L_{\mathfrak{a}_\mathfrak{h}} SU (n)$ acts through gauge transformations

$$X \mapsto gX g^\# + (z \otimes g) g^\#$$
on $X$. Similarly as in the spherical case, it is not difficult to prove that injects the geometric realization $j^+ (\mathbb{A}^n) j$ into $X$. The partial flags in the building $^+ (\mathbb{A}^n)$ correspond thus to certain operators $D$ on $2 X \times R D$ with finite-dimensional nontrivial eigenspaces. Note also that $\#$ swaps $^+ (\mathbb{A}^n)$ and $(\mathbb{A}^n)$, so is at the same time a Veronese representation for $(\mathbb{A}^n)$ with exactly the same image.

### 6.1 Caveat

The affine Veronese representation is not surjective. Let $a(z) = z + 1 = z$ and $r \in R$. The differential equation

$$(z \otimes ra)f = f$$

has the solution $f(z) = e^{r(1 - z)}z \ \text{const.}$. This function is holomorphic on $C$ if and only if $r$ is an integer, but it is not meromorphic on $C$. Let $f \in A$, so $f \in A$. Thus, the traceless diagonal matrix $X = \text{diag}(a; \ldots; a; (1 - n)a) 2 X$ does not represent a flag of the building because $D X$ has no nontrivial eigenspaces.

The space $X$ is a subspace of the loop algebra $\mathfrak{s}_h (\mathbb{A})$, which in turn is contained in the semi-direct product $\mathfrak{s}_h (\mathbb{A}) \subset C D$. So $X \times R D$ plays a very similar rôle as $\mathfrak{p}_n \subset \mathfrak{s}_h (\mathbb{C})$. Let $Q$ denote the collection of all barycenters of images of chambers in $X$. The closure $M$ of $Q$ in the Hilbert space completion of the real pre-Hilbert space $X$ is an infinite-dimensional isoparametric submanifold. See Pinkall & Thorbergsson [33] and Heintze, Palais, Terng & Thorbergsson [19] for more information. The set $Q$ coincides with the subset $Q (\varphi) \ M$ introduced in [18] by Heintze & Liu. In Groß, Heintze, Kramer & Mühlherr [12] we prove that all known isoparametric submanifolds of rank at least 3 in Hilbert spaces arise in a uniform way from Veronese representations of twin buildings.

### 7 Topological geometry and Bott periodicity

In this section we propose a definition of topological twin buildings. Since spherical buildings are twin buildings, this is at the same time a definition of topological spherical buildings. Definitions of spherical topological buildings have been proposed by Burns & Spatzier [9], Jäger [20], Kühne...
[25], and myself; Mitchell [27] proposes an ad hoc definition of topological BN-pairs. For spherical buildings of rank 2 there is a well-established theory, see Salzmann [39], Salzmann et al. [40], Schroth [42], Grundhöfer & Knarr [13], Grundhöfer & Van Maldeghem [17], Grundhöfer, Knarr & Kramer [14], and Kramer [23], [24]. The starting point is always a topology on the set of vertices of the building (i.e. on the 0-simplices). Using the type function, the simplices of higher rank can be interpreted as ordered tuples of vertices, and thus one obtains a topology on \( \mathcal{B} \); the question then is which maps should be continuous. Burns & Spatzier [9] require only that \( \text{Cham}(\ ) \) should be closed, i.e. that every net of chambers, viewed as a net of \( r \)-tuples of vertices, converges to some chamber. Moreover, they claim that for the classical geometries (projective spaces) this agrees with the traditional notion of a topological geometry, see loc.cit p. 1. This is definitely not true, and it is not difficult to construct perverse topologies on nice geometries which satisfy their condition nevertheless. However, in the compact spherical case, their definition is the correct one (and that’s the only instance where they need it in their work [9]), see Proposition 7.5 below.

Compactness or local compactness in the non-spherical case leads to locally finite buildings (finite panels), and this in turn is related to locally compact CAT(0)-spaces; however, these matter are not within the scope of the present article.

Similarly as in Jäger [20], our notion of a topological building asks for the continuity of certain projections. The fact that this definition makes sense for twin buildings was pointed out by Bernhard Mühlherr during a meeting in Oberwolfach back in 1992; then, we planned to write a joint paper with Martina Jäger on projections and topologies in buildings which, however, never came to existence. This section is a first approximation of what we had in mind.

The fact that topological buildings can be used to prove Bott periodicity is particularly appealing, since topological \( K \) -theory is an important ingredient in topological geometry; many classification results in Salzmann et al. [40] and in Kramer [23] depend in an essential way on Bott periodicity.

### Projections in (twin) buildings

Tits defined projections in (spherical) buildings in [48]; a modern account based on metric properties of buildings is given in Dress & Scharlau [11]. Let \( C ; D \) be chambers in a building \( \mathcal{B} \), let \( (C ; D) = w, \) and let \( w = s_{i_1} \ldots s_{i_r} \) be a reduced (minimal) expression for \( w \) in terms of the generating set \( S \). Then there exists a unique minimal gallery \( G = (C_0 ; C_1 ; \ldots ; C_r = D) \) of type \( (s_{i_1} ; \ldots ; s_{i_r}) \), consisting of chambers \( C_0 ; \ldots ; C_r \), such that \( (C_{k-1} ; C_k) = s_{i_k} \) holds for \( k = 1 , \ldots , r \). Now let \( X \) be a simplex and let \( C \) be a chamber. Then there exists a unique chamber \( E \) in \( \mathcal{B} \) which we denote

\[
E = \text{proj}_X C;
\]

the projection of \( C \) onto \( X \), with the following property: for every chamber \( D \) in \( \mathcal{B} \), and for every minimal gallery \( G \) starting at \( C \) and ending at \( D \), the first chamber in \( G \) which is contained in
Res(\(X\)) is \(E\), the gate of Res(\(X\)) with respect to \(C\).

Note that \(\text{proj}_X C = C\) if \(C \in \text{Cham}(\text{Res}(X))\). If \(Y\) is an arbitrary simplex, then there exists a unique simplex \(Z\) which is contained in some chamber in Res(\(X\)), such that

\[
\text{Cham}(\text{Res}(Z)) = \text{proj}_X \text{Cham}(\text{Res}(Y));
\]

and we put

\[
Z = \text{proj}_X Y:
\]

Now suppose that \((\varepsilon; \theta; \gamma)\) is a twin building, that \(X\) is a spherical simplex (recall from Section 1 that this means that Res(\(X\)) is spherical), and that \(C \in \text{Cham}(\text{Res}(X))\) is a chamber. Then there exists a unique chamber \(E \in \text{Res}(X)\) which maximizes the numerical codistance function \(D_{\varepsilon}^{\theta}(C; D)\) on \(\text{Cham}(\text{Res}(X))\), see Ronan [36] (4.1). Intuitively, a 'small' codistance corresponds to a 'big' distance, so \(E\) is the chamber 'closest' to \(C\); note that \(D_{\varepsilon}^{\theta}(C; D)\) is bounded above because \(X\) is spherical.

The chamber \(E\) is again denoted

\[
E = \text{proj}_X C;
\]

and the projection \(\text{proj}_X Y\) of an arbitrary simplex \(Y\) onto \(X\) is defined exactly as before (but only for spherical \(X\)).
7.1 Schubert cells
Let $C_0$ be a chamber in a building, let $J$, $I$, and let $wW = W$. The set

$$C_{wW}(C_0) = \{ X \mid X = wWJ \}$$

is called a Schubert cell in . Schubert cells in general buildings have no special structure (e.g. if is a tree), but Schubert cells in halves of twin buildings (and in particular Schubert cells in spherical buildings) have a nice product structure, i.e. they admit coordinates (labels).

7.2 Coordinatizing a half twin
Let $C_0$ and $D_0$ be a pair of opposite chambers in a twin building. These two chambers determine a unique apartment $A$ (half of the twin apartment spanned by $(C_0; D_0)$, see e.g. Ronan [36] 2.8). Let $wW$ and let $w = s_i^1 s_i^r$ be a reduced expression. Let $E = C_{w}(C_0)$, and let $C_0; \ldots; C_r = E$ be a (necessarily minimal) gallery of type $(s_i^1; \ldots; s_i^r)$. Let $D_0; \ldots; D_r$ be the unique minimal gallery of the same type in the apartment $A$. Then $C_k \cap D_k$ holds for $k = 0; \ldots; r$. We define $r$ coordinates $(X_1; \ldots; X_r)$ by $X_k = \text{proj}_{D_k \cup D_{k-1}} C_k$.

Note that $X_k \in \text{res}(D_k \cup D_{k-1}) \cap \text{proj}_{D_k}$. Now the point is that step by step, the whole gallery can be recovered from these coordinates:

$$C_1 = \text{proj}_{D_0 \cup C_1} X_1$$
$$C_2 = \text{proj}_{D_1 \cup C_2} X_2$$
$$C_3 = \text{proj}_{D_2 \cup C_3} X_2 \text{ etc.}$$

since $C_{k-1}$ and $s_i^r$ determine $C_{k-1} \cup C_k$, the information needed is only $C_0$, the coordinates $(X_1; \ldots; X_r)$, and the reduced expression $(s_i^1; \ldots; s_i^r)$. Note also that different reduced expressions for a
Schubert cell lead to different coordinates; our coordinatization process depends on a choice of a reduced expression for every element \( w \in W \). In the case of spherical buildings of rank two, the different expressions of the longest element in the Coxeter group and the resulting 'changes of coordinates' lead to Van Maldeghem’s coordinatizing rings [52].

The process above yields coordinates for the Schubert cells \( C_w (C_0) \). For a coset \( wW \) we may assume that \( w \) is the unique shortest coset representative. The canonical 'forgetful' map \( C_w (C_0) -\longrightarrow C_{wW} (C_0) \) which sends a chamber \( C \) to the unique subsimplex \( C_{\langle \alpha \rangle} \) of type \( \alpha \) contained in \( C \) is a bijection; in fact, \( C = \text{proj}_{C_{\langle \alpha \rangle}} C_0 \). The Schubert cell \( C_{wW} (C_0) \) can also directly be coordinatized, by the same method as described above. In any case we see that each Schubert cell is in a natural correspondence with a finite product of punctured panels; for \( + (\mathbb{A}^n) \) we see that the Schubert cell \( C_{wW} (C_0) \) bijects onto \( \mathbb{C}^m \), where \( m \) is the length of the shortest coset representative of \( wW \).

### Topological twin buildings

Let \(( + ; \quad ; \quad )\) be a twin building. Suppose that there is a Hausdorff topology on the set of vertices (the 0-simplices) of both buildings. The simplices of type \( J \) can be regarded as \( J \)-tuples of vertices; in this way, the topology on the vertices determines a topology on both buildings. For \( J; K \) and \( w \in W \) we put

\[
D_{wW}^{JK} = \{ (X; Y) \mid \text{type}(X) = I \cap J; \text{type}(Y) = I \cap K; \quad (X; Y) = W \cdot wW \cdot K \}.
\]

#### 7.3 Definition

A twin building is called a topological twin building if the following condition is satisfied:

**TTB** If \( J \) is spherical (i.e. if \( W \) is finite) and \( K \) is arbitrary, then \( (X; Y) \) \( \mapsto \) \( \text{proj}_K Y \) is continuous on the set \( D_{wW}^{JK} \).

The condition \( (X; Y) = W \cdot W \cdot K \) means that \( X \) and \( Y \) are almost opposite, i.e. that there exist chambers \( C \quad X \) and \( D \quad Y \) with \( C \ \text{op} \ D \).

#### 7.4 Remark

For spherical buildings of rank 2 and for projective spaces, this is the common notion of a topological building, cp. Kramer [23] and Kühne & Löwen [26], Kühne [25].

In the compact spherical case, there is a nice criterion due to Grundhöfer and Van Maldeghem:

**7.5 Proposition** If \( \quad \) is spherical, and if the topology on the vertex set is compact, then \( \quad \) is a topological building if and only if the chamber set is compact.

**Proof.** This follows from the closed graph theorem for maps into compact spaces, see Grundhöfer & Van Maldeghem [17].

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We just mention the following result.

**7.6 Proposition** Let \((\wedge^n;\wedge^i;\wedge^j)\) be a topological twin building, and let \(X\) be spherical. Then \(R\) is in a natural way a topological building. If \(Z\) is another simplex of the same type as \(X\) (in either half of the twin building), then \(R\) is continuously isomorphic to \(R\).

**Proof.** Pick \(Y\) opposite \(X\). Then \(\text{proj}\) induces an isomorphism \(R = \text{proj}(Y)\). Let \(U;V\) be simplices with maximal \(\text{dist}_X\) distance. We need to show that \(\text{proj}\) is continuous. But \(U = \text{proj}_U U\) depends continuously on \(U\), and \(\text{proj}_U U = \text{proj}_V U\). For the second claim one uses the following fact which is not difficult to prove (see the proof by Tits [48] 3.30 in the spherical case): given simplices \(X;X\) (in the same half of the building), there exists a simplex \(X\) in the other half which is opposite both to \(X\) and to \(X\).

**7.7 Example** The building \((C^n;\wedge^n;\wedge^i;\wedge^j)\), with the natural topology on its vertex set \(G_\frac{r}{2}\) of \(C_n\) is another simplex of the same type as \(X\) (in the other half of the building), there exists a simplex \(X\) in the other half which is opposite both to \(X\) and to \(X\);

**7.8 Theorem** The twin building \((\wedge^n;\wedge^i;\wedge^j)\) is a topological twin building.

**Sketch of proof.** Let \((C;D)\) be a pair of opposite chambers, let \(X\) be a spherical building of type \(I\) and \(Y\) of type \(J\). Since \(SL_n(\mathbb{A})\) acts strongly transitively on the twin building, it acts transitively on the set \(D_{W_{J,K}}\), and \(D_{W_{J,K}} = SL_n(\mathbb{A})\). Now \(SL_n(\mathbb{A})\) fixes \(Z = \text{proj}_W Y\), and we have a continuous map \(SL_n(\mathbb{A})\) induces a topology on the set \(V_0\) of vertices of \(\wedge^n(\mathbb{A})\) and similarly on the vertices of \(\wedge^n(\mathbb{A})\).

**7.9 Proposition** The Schubert varieties \(C_{w,J}(C_0)\) in \(\wedge^n(\mathbb{A})\) are CW-complexes, with Poincaré series

\[P_{vW,J,wW,J} = \text{proj}(vW,J)\]

**Sketch of proof.** Let \(w\) be a shortest coset representative for \(vW,J\), and let \(G_{s_1;:\ldots;sl}(C_0)\) denote the collection of all (possibly stammering) galleries of type \(s_1;:\ldots;sl\) (for some fixed reduced expression \(s_1;:\ldots;sl\) for \(w\)), starting with the chamber \(C_0\). It is not difficult that to show that \(G_{s_1;:\ldots;sl}(C_0)\) is an...
iterated $\mathbb{CP}^1$-bundle (sometimes called a Bott-Samelson cycle); the total space is a smooth manifold of dimension $2m$ ($w W J = 2^r(w)$). These gallery spaces are also known as Bott-Samelson desingularizations of Schubert varieties. Consider the endpoint map

$$G_{s_1, \ldots, s_r} (C_0) \longrightarrow C_{w W J} (C_0):$$

The non-stammering galleries in $G_{s_1, \ldots, s_r} (C_0)$ are mapped bijectively onto the Schubert cell $C_{w W J} (C_0)$. There is a canonical injection

$$G_{s_1, \ldots, s_r} (C_0) \longrightarrow G_{s_1, \ldots, s_r} (C_0)$$

(by stammering at the end). The stammering galleries in $G_{s_1, \ldots, s_r} (C_0)$ are either of this type, or galleries which don’t stammer at the end, but somewhere before the end. Using this and the fact that

$$G_{s_1, \ldots, s_r} (C_0) \longrightarrow G_{s_1, \ldots, s_r} (C_0)$$

is a $\mathbb{CP}^1$-bundle, one can use induction on $^r(w)$ to define a cellular map

$$e^2 \quad e^{2^r(w)} \quad 2 \longrightarrow G_{s_1, \ldots, s_r} (C_0)$$

which maps the boundary of the cell $e^2 \quad e^{2^r(w)} \quad 2$ onto the stammering galleries, see Mitchell [27] Thm. 2.22 and Kramer [23] Sec. 4.1. Now is an attaching map for a $2^r(w)$-cell.

It follows that the based loop group $\text{loops} SU(n)$ has a CW decomposition with Poincaré series

$$P \quad \text{where } W \quad \text{is the affine Weyl group of type } \mathbb{A}_n \quad \text{generated by } s_1; \ldots; s_n, \text{ and } A_n \quad \text{the subgroup generated by } s_1; \ldots; s_n.$$ 30

**Knarr’s construction for $^+ (\mathbb{A}_n)$**

We apply Knarr’s construction to the the halves of the topological twin building ($^+ (\mathbb{A}_n); \quad (\mathbb{A}_n); \quad$). The main ideas can be found in Mitchell’s paper [27], although our approach (which is the same as Knarr’s approach [21]) is more based on geometric properties (i.e. the coordinatization of twin buildings), whereas Mitchell makes strong use of the $B N$-pair. Let $j$ denote the geometric realization of $^+ (\mathbb{A}_n)$. By Theorem 7.8, there is a canonical topology on the flag space $\text{Flag}(^+ (\mathbb{A}_n))$ and we endow $j$ with the quotient topology induced by the map $\text{Flag}(^+ (\mathbb{A}_n)) \rightarrow$ $\mathbb{A}_n \quad \rightarrow \quad j$ The resulting space is denoted $j$ Knarr.
More generally, assume that \(( +; i; j)\) is a topological twin building, and that the panels are topological spheres. For example, in \(( +; 0; 2)\) the panels are homeomorphic to \(\mathbb{C}P^1 = S^2\). Moreover, panels of the same type are homeomorphic by Proposition 7.6. Let \(m \langle s_i \rangle\) denote the topological dimension of the \(i\)-panels in \(C\). It is proved in Kramer [23] Prop. 2.0.2 that \(m \langle s_i \rangle = m \langle s_j \rangle\) holds whenever \(m_{ij}\) is odd. Thus we obtain a \(Z\)-length \(m : W \rightarrow Z\). For a Schubert cell \(C_w \langle C_0 \rangle\), we have

\[
C_w \langle C_0 \rangle = R^m (w) ;
\]

**7.10 Proposition** (cp. Knarr [21] and Mitchell [27] 2.16) Let \(( +; i; j)\) be a topological twin building. Assume that the panels are topological spheres. For any \(w \in W\) let \(X_w\) denote the image of \(C_w \langle C_0 \rangle\) \(\mathbb{N}^{r-1} j i n j + \delta_{w_{n a r r}}\). Then for each \(w \in W\), the set \(fX_v j v \ w g\) is contractible.

*Sketch of proof.* The proof is by induction on the length \('(w)'\). Note that \(X_1 = fC_0 g \mathbb{N}^{r-1} j i s\) contractible. Moreover,

\[
C_w \langle C_0 \rangle = fC_u \langle C_0 \rangle j u \ w g = S^m (w) ;
\]

because this quotient is the one-point compactification of the Schubert cell \(C_w \langle C_0 \rangle\).

The inductive step is accomplished as follows. First of all, \(fX_v j v \ w g\) is contractible for all \(u \ w\). Then it is not hard to see that \(fX_v j v \ w g = fX_v j v \ w g\) is also contractible. This implies that \(fX_v j v \ w g\) is contractible. Finally, \(fX_v j v \ w g\) is homotopy equivalent to a wedge of such contractible spaces and hence itself contractible.

**7.11 Corollary** If \(( +; i; j)\) is spherical, then \(j + \delta_{w_{n a r r}}\) is homeomorphic to a sphere of dimension \(m \langle w_{0} \rangle + r 1\), where \(w_{0}\) is the unique longest element in the Coxeter group \(W\), and \(r\) is the rank of the building. In the non-spherical case, \(j + \delta_{w_{n a r r}}\) is contractible.

*Sketch of proof.* In the spherical case, let \(w_{0} \in W\) denote the longest element; this is at the same time the unique maximal element in the Bruhat order. Then \(X_{w_{0}} = fX_{v} j v \ w_{0} g = R^m (w_{0}) \ R^{r 1}\), \(s o X_{w_{0}} = fX_{v} j v \ w_{0} g = S^m (w_{0}) \times S^{r-1}\), and this space is homotopy equivalent to \(j + \delta_{w_{n a r r}}\) since \(fX_{v} j v \ w_{0} g\) is contractible. Finally, it is not difficult to see that \(X_{w_{0}}\) is a manifold, cp. Kramer [23] Prop. 4.2.1, hence \(j + \delta_{w_{n a r r}}\) is a compact manifold (of dimension at least 5) homotopy equivalent to a sphere, and thus to homeomorphic to \(S^m (w_{0}) \times S^{r-1}\) by the proof of the generalized Poincaré conjecture due to Smale [43], Stallings [44] and Zeemann [53]. In the non-spherical case, \(j + \delta_{w_{n a r r}}\) is a limit of contractible spaces and hence itself contractible.

The Theorem above can be proved in much greater generality; it suffices to assume that the panels are compact, connected, and of finite covering dimension. Under these assumptions, the Corollary holds up to homotopy equivalence, see Kramer [23] Sec. 3.3. On the other hand, if the vertices of the building are endowed with the discrete topology, then the Theorem leads to a quick proof of the Solomon-Tits Theorem.


**Bott periodicity**

The crucial step in the proof of Bott periodicity is the following observation. We have

\[ \text{Res}(E_0) = (\mathbb{C}^n); \]

and thus a natural map \( G_r(\mathbb{C}^n) \rightarrow V_k \). But the first terms in the Poincaré series for these two spaces agree; the first few shortest coset representatives for \( W = W_J \) and \( W_K = W_{J \setminus K} \) are the same, where \( J = \{2; \ldots; n\} \) and \( K = \{2; \ldots; k \ 1; k + 1; \ldots; n\} \). For example, we have the following cell structure for \( n = 4 \). Put \( I = \{1; 2; 3; 4\} \) and let \( J = \{2; 3; 4\} \) and \( K = \{1; 2; 4\} \); thus we have the following Coxeter groups.

For \( W_{f_1; 2; \beta; 4g} = W_{f_4; 2g} \) we have the following Bruhat order for the shortest coset representatives, cp. Scharlau [41] 2.5.
Thus we see that $G_{r_2}(\mathbb{C}^4)$ has a cell decomposition as $e^0 \cdot e^2 \cdot e^4 \cdot e^6 \cdot e^8$. The Bruhat order for the shortest coset representatives of $W_{f_1, f_2, f_3, f_4} = W_{f_2, f_3, f_4}$ starts as

![Diagram](image)

(and continues infinitely to the right). Accordingly, the cell structure of $V_2 = \text{alg SU}(4)$ is $e^0 \cdot e^2 \cdot e^4 \cdot e^6 \cdot e^8$; in particular, the 5-skeleton of $V$ the same as for $G_{r_2}(\mathbb{C}^4)$. The general result is as follows.

**7.12 Proposition** Suppose that $n = 2k$ is even. Then the inclusion $G_{r_k}(\mathbb{C}^{2k}) \xrightarrow{\sim} V_k$ is a 2k-equivalence.

The proof for Bott periodicity is now as follows. If $k$ is large, then $G_{r_k}(\mathbb{C}^{2k})$ is a good approximation for the classifying space $BU$; so in the limit, we obtain a homotopy equivalence $BU \to \text{alg SU}$. There is one problem, though; we have only considered the space $\text{alg SU}(n)$ consisting of all based loops which can be expressed as Laurent polynomials, whereas topologists consider the space $\text{cts SU}(n)$ of all continuous (or smooth) based loops. So the proof is not yet finished. At this point it is convenient to introduce Quillen’s *space of special paths*

$$S = \mathbb{R}^7 \times \text{alg SU}(n) \times \text{SU}(n;1) \times \mathbb{R}^0;$$

where $\mathbb{R}^7 \times \text{alg SU}(n)$ acts on $S$. But $\mathbb{R}^7 \times \text{alg SU}(n)$ can also be identified with the image of the chamber $f[E_0; \ldots; E_n; 1]$ under the Veronese representation. This identification extends in a natural way to an $\text{alg SU}(n)$-equivariant homeomorphism

$$S \xrightarrow{\sim} j + (\mathbb{A}^n)_{\text{narr}} \times X$$

33
as follows: a path $S$ is mapped to its 'logarithmic derivative' $\frac{1}{2} i t \frac{d}{dt} (t) (t)^{1}$ which is a path in the tangent space $T_{1} SU(n)$. (Conversely, a smooth path in the tangent space can be read as a differential equation whose solution starting at 1 is a path in the Lie group.) If we put $z = e^{2 i t}$, then

$$\frac{d}{dt} g(z) e^{2 i t} g^\# (1) = 2 i (z \partial_z g(z)) e^{2 i t} g^\# (1) + g(z) X e^{2 i t} g^\# (1)$$

$$\frac{d}{dt} g(z) e^{2 i t} g^\# (1) = g(z) e^{2 i t} g^\# (1)^{1} = 2 i (z \partial_z g(z)) g^\# (z) + g(z) X g^\# (z):$$

7.13 Lemma The map

7 ! $\frac{1}{2} i t \frac{d}{dt} (t) (t)^{1}$

is an $L_{alg} SU(n)$-equivariant homeomorphism $S \xrightarrow{\text{j}^+(\mathbb{A}^{n})_{\mathbb{K}narr}} X$, where the action on $\mathbb{A}^{n}$ is the standard one, and the action on Quillen’s space of special paths is by $g(e^{2 i t}) (tg^{-1}(1))$. In particular, $S$ is contractible. The endpoint map $(SU(n); 1) \xrightarrow{\text{(B;\mathcal{P})}} SU(n)$ yields the universal bundle

$\text{ctsSU}(n) \xrightarrow{\text{aB}} SU(n)$

(where $\text{cts}$ refers to the group of continuous loops) and as a subbundle we have

$\text{aB} SU(n) \xrightarrow{\text{cts}} S \xrightarrow{\text{SU}(n)}$:

Since the total spaces of both bundles are contractible, the inclusion $\text{aB} SU(n) \xrightarrow{\text{cts}} SU(n)$ is a weak (and therefore also a strong) homotopy equivalence. We have proved Quillen’s following result.

7.14 Theorem (Quillen) The orbit space $\mathbb{A}^{n}$ $\xrightarrow{\text{aB} SU(n)}$ is homeomorphic to $SU(n)$. 2

Recall that $G_{\mathbb{R}} (\mathbb{C}^{n})$ is a good approximation of the classifying space $BU$ in small dimensions.

7.15 Corollary (Unitary Bott Periodicity) The inclusion

$G_{\mathbb{R}} (\mathbb{C}^{2k}) \xrightarrow{\text{aB}} SU(k) \xrightarrow{\text{cts}} SU(k)$

is a $2k$-equivalence. In the limit, the natural map

$BU \xrightarrow{\text{cts}} SU$ is a homotopy equivalence.

This implies in particular the famous Bott isomorphisms $2k (U) = 0$ and $2k+1 (U) = 2$, for all $k \geq 0$. Note that we have also proved that the $\text{aB} SU(n)$-orbit space map

$\mathbb{A}^{n}$ $\xrightarrow{\text{aB} SU(n)}$ SU(n)

is a universal classifying bundle for $\text{aB} SU(n)$. 34
8 Real forms and compact symmetric spaces

So far, we have discussed the group $\text{SL}_n(A)$ which is the proper generalization of the complex group $\text{SL}_n(C)$. In this last section we consider briefly how real groups fit into the picture; more details can be found in Mitchell [27] and in Groß, Heintze, Kramer & M"uhlherr [12]. Consider the involution given by

$$X \mapsto X^{-1} \quad X \in \text{SL}_n(A).$$

The group of $-\text{fixed elements in } \text{SL}_n(A)$ is $\text{SL}_n(R[z;1=z])$, and there is a corresponding twin building over $R[z;1=z]$ which is defined exactly in the same way as the one over $C[z;1=z]$ considered so far. Note however that we cannot interpret the elements of $\text{SL}_n(R[z;1=z])$ as loops in $\text{SL}_n(R)$. Instead, we view the elements of $\text{SL}_n(R[z;1=z])$ as paths

$$[0;1] \rightarrow L_{ahl} \text{SL}_n(C) \quad t \mapsto g(e^{it}z).$$

These paths have the special property that they start and end in $\text{SL}_n(R)$. If we intersect $\text{SL}_n(R[z;1=z])$ with $L_{ahl} \text{SU}(n)$, then we obtain the group

$$\text{SL}_n(R[z;1=z]) \setminus L_{ahl} \text{SU}(n)$$

consisting of paths in $\text{SU}(n)$ which start and end in $\text{SO}(n)$. Similarly as before, this group is homotopy equivalent with the based loop space

$$\text{cts} \text{(SU}(n) = \text{SO}(n)).$$

These loop spaces of compact Riemannian symmetric spaces play an important rôle in topology. They can be used to prove the other versions of Bott periodicity (real and quaternionic), see Mitchell [27].

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