Research Article

Generalizations on Some Hermite-Hadamard Type Inequalities for Differentiable Convex Functions with Applications to Weighted Means

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Some new Hermite-Hadamard type inequalities for differentiable convex functions were presented by Xi and Qi. In this paper, we present new generalizations on the Xi-Qi inequalities.

1. Introduction

The Hermite-Hadamard inequality [1–3] states that if \( f \) is a convex function on \([a, b] \), then

\[
f(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}. \tag{1}
\]

Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be differentiable on \( I \) and \([a, b] \subseteq I \) with \( a < b \). Below we recall some Hermite-Hadamard type inequalities.

In 1998, Dragomir and Agarwal [4] showed that (i) if \( |f'(x)| \) is convex on \([a, b] \), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2(p+1)^{1/p}} \left( \frac{\left| f'(a) \right|^{p/(p-1)} + \left| f'(b) \right|^{p/(p-1)}}{2} \right)^{(p-1)/p}.
\]

(3)

and (ii) if \( |f'(x)|^{p/(p-1)} \) is convex on \([a, b] \) with \( p > 1 \), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left( \frac{\left| f'(a) \right|^q + \left| f'(b) \right|^q}{2} \right)^{1/q}.
\]

(2)

In 2000, Pearce and Pečarić [5] showed that if \( |f'(x)|^q \) is convex on \([a, b] \) with \( q \geq 1 \), then
\[
|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx| \leq \frac{b-a}{4} \left(\left|f'(a)\right|^p + \left|f'(b)\right|^p\right)^{\frac{1}{p}}.
\]

(4)

In 2004, Kirmaci [6] showed that if \(|f'(x)|^{p/(p-1)}\) is convex on \([a, b]\) with \(p > 1\), then

\[
|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx| \leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{1/p} \left(\left|f'(a)\right|^p + \left|f'(b)\right|^p\right)^{\frac{1}{p}}.
\]

(5)

In 2010, Sarikaya et al. [7] showed that if \(f' \in L[a, b]\) and \(|f'(x)|^q\) is convex on \([a, b]\) with \(q \geq 1\), then

\[
\left|\frac{1}{6} \left[ f(a) + f(b) + 4f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{5(b-a)}{72} \times \left[ \left( \frac{61\left|f'(a)\right|^q + 29\left|f'(b)\right|^q}{90} \right)^{\frac{1}{q}} + \left( \frac{29\left|f'(a)\right|^q + 61\left|f'(b)\right|^q}{90} \right)^{\frac{1}{q}} \right].
\]

(6)

In 2012, Xi and Qi [8] showed that if \(\lambda, \mu \in [0, 1]\) and if \(f' \in L[a, b]\) and \(|f'(x)|^q\) is convex on \([a, b]\) with \(q \geq 1\), then

\[
\left|\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{b-a}{8} \left(1 - 2\lambda + 2\lambda^2\right)^{\frac{1}{1-q}} \times \left[ \left(4 - 9\lambda + 12\lambda^2 - 2\lambda^3\right)\left|f'(a)\right|^q + (2 - 3\lambda + 2\lambda^3)\left|f'(b)\right|^q \right] \times (6^{-1})^{\frac{1}{q}}
\]

\[
+ \frac{b-a}{8} \left(1 - 2\mu + 2\mu^2\right)^{\frac{1}{1-q}} \times \left[ \left(2 - 3\mu + 2\mu^3\right)\left|f'(a)\right|^q + (4 - 9\mu + 12\mu^2 - 2\mu^3)\left|f'(b)\right|^q \right] \times (6^{-1})^{\frac{1}{q}},
\]

(7)

Moreover, for other results involving the Hermite-Hadamard type inequalities, we also refer to [9–23].

In this paper, we generalize the Xi-Qi inequalities.
2. Preliminaries

Lemma 1. Let $\lambda, \mu \in \mathbb{R}$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on $I$ and $[a,b] \subseteq I$ with $a < b$. Assume that $f' \in L[a,b]$ and $0 < e < b - a$. Then
\[
eq \lambda f(a) + (b - a - e) \mu f(b) + \left[ e (1 - \lambda) + (b - a - e) (1 - \mu) \right] f(a + e) - \int_a^{a+e} f(x) \, dx\]
\[
+ \left[ e^2 (1 - \lambda - t) f'(ta + (1 - t) (a + e))
+ (b - a - e)^2 (\mu - t) \right]
\times f'(t(a + e) + (1 - t) b) \, dt.
\]
(8)

Proof. Integrating by part and changing variable, we have
\[
\int_0^1 e^2 (1 - \lambda - t) f'(ta + (1 - t) (a + e)) \, dt
= \int_0^1 e (\lambda + t - 1) f(ta + (1 - t) (a + e)) \, dt
= \left[ e(\lambda + t - 1) f(ta + (1 - t) (a + e)) \right]_{t=0}^{t=1}
- e \int_0^1 f(ta + (1 - t) (a + e)) \, dt
= e\lambda f(a) + e(1 - \lambda) f(a + e) - \int_a^{a+e} f(x) \, dx,
\]
\[
\int_0^1 (b - a - e)^2 (\mu - t) f'(t(a + e) + (1 - t) b) \, dt
= \int_0^1 (b - a - e)(t - \mu) f(t(a + e) + (1 - t) b) \, dt
= \left[ (b - a - e)(t - \mu) f(t(a + e) + (1 - t) b) \right]_{t=0}^{t=1}
- (b - a - e) \int_0^1 f(t(a + e) + (1 - t) b) \, dt
= (b - a - e)(1 - \mu) f(a + e)
+ (b - a - e) \mu f(b) - \int_a^{a+e} f(x) \, dx.
\]
(9)

Thus,
\[
\int_0^1 e^2 (1 - \lambda - t) f'(ta + (1 - t) (a + e))
+ (b - a - e)^2 (\mu - t) \right]
\times f'(t(a + e) + (1 - t) b) \, dt
= e\lambda f(a) + (b - a - e) \mu f(b)
+ \left[ e (1 - \lambda) + (b - a - e) (1 - \mu) \right] f(a + e) - \int_a^{a+e} f(x) \, dx.
\]
(10)

Lemma 2 (see [8]). Let $s > 0$ and $0 \leq \xi \leq 1$. Then
\[
\int_0^1 \xi t^s \, dt = \frac{\xi^{s+1} + (1 - \xi)^{s+1}}{s + 1},
\]
(11)
\[
\int_0^1 t \xi t^s \, dt = \frac{\xi^{s+2} + (s + 1 + \xi)(1 - \xi)^{s+1}}{(s + 1) (s + 2)}.
\]

3. Main Results

Theorem 3. Let $\lambda, \mu \in [0, 1]$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on $I$ and $[a,b] \subseteq I$ with $a < b$. Assume that $f' \in L[a,b]$ and $0 < e < b - a$. If $|f'(x)|^q$ is convex on $[a,b]$ with $q \geq 1$, then
\[
\left| e\lambda f(a) + (b - a - e) \mu f(b)
+ \left[ e (1 - \lambda) + (b - a - e) (1 - \mu) \right] f(a + e) - \int_a^{a+e} f(x) \, dx \right|
\leq e^2 \left( \frac{1 - \lambda^2 + \mu^2}{2} \right)^{1/(1-q)}
\times \left[ \frac{1}{6} \left[ 3 - 6\lambda + 6\lambda^2 - \frac{e}{b - a} \left( 2 - 3\lambda + 2\lambda^3 \right) \right] |f'(a)|^q
+ \frac{1}{6} \left[ \frac{e}{b - a} \left( 2 - 3\lambda + 2\lambda^3 \right) \right] |f'(b)|^q \right]^{1/q}
+ (b - a - e)^2 \left( \frac{\mu^2 + (1 - \mu)^2}{2} \right)^{1/(1-q)}
\times \left[ \frac{1}{6} \left[ 1 - \frac{e}{b - a} \right] \left[ 2 - 3\mu + 2\mu^3 \right] \right] |f'(a)|^q
+ \frac{1}{6} \left[ 1 - 3\mu + 6\mu^2 - 2\mu^3 + \frac{e}{b - a} \left( 2 - 3\mu + 2\mu^3 \right) \right]
\times |f'(b)|^q \right]^{1/q}.
\]
(12)

Proof. Suppose that $|f'(x)|^q$ is convex on $[a,b]$ with $q \geq 1$. By Lemma 1, we have
\[
\left| e\lambda f(a) + (b - a - e) \mu f(b)
+ \left[ e (1 - \lambda) + (b - a - e) (1 - \mu) \right] f(a + e) - \int_a^{a+e} f(x) \, dx \right|
\leq e^2 \left( \frac{1 - \lambda^2 + \mu^2}{2} \right)^{1/(1-q)}
\times \left| f'(ta + (1 - t) (a + e)) \right| dt
\]
(13)
\[
+ (b - a - e)^2 \int_0^1 |\mu - t| \left| f'(t(a + e) + (1 - t) b) \right| dt.
\]
Case \((q = 1)\). By the convexity of \(|f'(x)|\) and Lemma 2, we have
\[
\int_0^1 |1 - \lambda - t| |f'(ta + (1 - t)(a + \epsilon))| \, dt
= \int_0^1 |1 - \lambda - t|
\times |f'(a)| \, dt
\leq \int_0^1 |1 - \lambda - t|
\times \left( \left| 1 - \frac{\epsilon}{b-a} \right| \int a + \frac{\epsilon}{b-a} b \right) \, dt
\leq \int_0^1 |1 - \lambda - t|
\times \left( \left| 1 - \frac{\epsilon}{b-a} \right| f'(a) + \frac{\epsilon}{b-a} f'(b) \right) \, dt
= \left[ \left( 1 - \frac{\epsilon}{b-a} \right) |f'(a)| - \frac{\epsilon}{b-a} |f'(b)| \right]
\times \int_0^1 |1 - \lambda - t| \, dt
\leq \int_0^1 |1 - \lambda - t| \, dt
\times \left( \left| 1 - \frac{\epsilon}{b-a} \right| f'(a) + \frac{\epsilon}{b-a} f'(b) \right)
\leq \frac{\epsilon^2}{6} \left[ 3 - 6\lambda + 6\lambda^2 - \frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(a)|
+ \frac{\epsilon^2}{6} \left[ \frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(b)|
+ \frac{(b-a-\epsilon)^2}{6}
\times \left[ \left( 1 - \frac{\epsilon}{b-a} \right) (2 - 3\lambda + 2\lambda^3) \right] |f'(a)|
+ \frac{(b-a-\epsilon)^2}{6}
\times \left[ 1 - 3\mu + 6\mu^2 - 2\mu^3 + \frac{\epsilon}{b-a} (2 - 3\mu + 2\mu^3) \right] |f'(b)|.
\]
Thus,
\[
\left| e\lambda f(a) + (b-a-\epsilon) \mu f(b) \right|
+ \left( \epsilon (1-\lambda) + (b-a-\epsilon)(1-\mu) \right)
\times f(a+\epsilon) - \int_a^b f(x) \, dx \right| \leq \frac{e^2}{6} \left[ 3 - 6\lambda + 6\lambda^2 - \frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(a)|
+ \frac{\epsilon^2}{6} \left[ \frac{\epsilon}{b-a} (2 - 3\lambda + 2\lambda^3) \right] |f'(b)|
+ \frac{(b-a-\epsilon)^2}{6}
\times \left[ \left( 1 - \frac{\epsilon}{b-a} \right) (2 - 3\lambda + 2\lambda^3) \right] |f'(a)|
+ \frac{(b-a-\epsilon)^2}{6}
\times \left[ 1 - 3\mu + 6\mu^2 - 2\mu^3 + \frac{\epsilon}{b-a} (2 - 3\mu + 2\mu^3) \right] |f'(b)|.
\]

Case \((q > 1)\). By Hölder’s inequality, we have
\[
\left| e\lambda f(a) + (b-a-\epsilon) \mu f(b) \right|
+ \left( \epsilon (1-\lambda) + (b-a-\epsilon)(1-\mu) \right)
\times f(a+\epsilon) - \int_a^b f(x) \, dx \right| \leq e^2 \left( \int_0^1 |1 - \lambda - t| \, dt \right)^{1/(1/q)}
\]
\[
\begin{align*}
&\times \left( \int_0^1 |1-\lambda-t| |f'(ta+(1-t)(a+e))|^{q} dt \right)^{1/q} \\
&+ (b-a-e)^2 \left( \int_0^1 |\mu-t|^{1-(1/q)} dt \right)^{1/q} \\
&\times \left( \int_0^1 |\mu-t| |f'(t(a+e)+(1-t)b)|^{q} dt \right)^{1/q} \\
&\leq \int_0^1 |\mu-t|^{1-(1/q)} dt \\
&\times \left[ \left( t - \frac{et}{b-a} \right) |f'(a)|^q + \left( 1 - \left( t - \frac{et}{b-a} \right) \right) |f'(b)|^q \right] dt \\
&= \left[ \left( 1 - \frac{e}{b-a} \right) |f'(a)|^q - \left( 1 - \frac{e}{b-a} \right) |f'(b)|^q \right] \int_0^1 |1-\lambda-t| dt \\
&+ \left[ \frac{e}{b-a} |f'(a)|^q - \frac{e}{b-a} |f'(b)|^q \right] \int_0^1 t |1-\lambda-t| dt \\
&= \left[ \left( 1 - \frac{e}{b-a} \right) |f'(a)|^q + \frac{e}{b-a} |f'(b)|^q \right] \frac{(1-\lambda)^2 + \lambda^2}{2} \\
&\geq \left[ \left( 1 - \frac{e}{b-a} \right) |f'(a)|^q + \frac{e}{b-a} |f'(b)|^q \right] \frac{(1-\lambda)^3 + (3-\lambda) \lambda^2}{6} \\
&\geq \frac{1}{6} \left[ 3 - 6\lambda + 6\lambda^2 - \frac{e}{b-a} (2-3\lambda + 2\lambda^3) \right] |f'(a)|^q \\
&+ \frac{1}{6} \left[ \frac{e}{b-a} (2-3\lambda + 2\lambda^3) \right] |f'(b)|^q.
\end{align*}
\]

By the convexity of $|f'(x)|^q$ and Lemma 2, we have

\[
\begin{align*}
&\int_0^1 |1-\lambda-t| |f'(ta+(1-t)(a+e))|^{q} dt \\
&= \int_0^1 |1-\lambda-t| \left[ \left( \frac{e}{b-a} \right) a + \frac{e(1-t)}{b-a} b \right]^{q} dt \\
&\leq \int_0^1 |1-\lambda-t| \\
&\times \left[ \left( \frac{e}{b-a} \right) |f'(a)|^q + \frac{e(1-t)}{b-a} |f'(b)|^q \right] dt \\
&= \left[ \left( \frac{e}{b-a} \right) |f'(a)|^q + \frac{e}{b-a} |f'(b)|^q \right] \int_0^1 |1-\lambda-t| dt \\
&+ \left[ \frac{e}{b-a} |f'(a)|^q - \frac{e}{b-a} |f'(b)|^q \right] \int_0^1 t |1-\lambda-t| dt \\
&= \left[ \left( \frac{e}{b-a} \right) |f'(a)|^q + \frac{e}{b-a} |f'(b)|^q \right] \frac{(1-\lambda)^2 + \lambda^2}{2} \\
&\geq \left[ \left( \frac{e}{b-a} \right) |f'(a)|^q + \frac{e}{b-a} |f'(b)|^q \right] \frac{(1-\lambda)^3 + (3-\lambda) \lambda^2}{6} \\
&\geq \frac{1}{6} \left[ 3 - 6\lambda + 6\lambda^2 - \frac{e}{b-a} (2-3\lambda + 2\lambda^3) \right] |f'(a)|^q \\
&+ \frac{1}{6} \left[ \frac{e}{b-a} (2-3\lambda + 2\lambda^3) \right] |f'(b)|^q,
\end{align*}
\]

Thus,

\[
\begin{align*}
&\int_0^1 |\mu-t| |f'(t(a+e)+(1-t)b)|^{q} dt \\
&= \int_0^1 |\mu-t| \left[ \left( t - \frac{et}{b-a} \right) a + \left( 1 - \left( t - \frac{et}{b-a} \right) \right) b \right]^{q} dt \\
&\leq \int_0^1 |\mu-t| \\
&\times \left[ \left( t - \frac{et}{b-a} \right) |f'(a)|^q + \left( 1 - \left( t - \frac{et}{b-a} \right) \right) |f'(b)|^q \right] dt \\
&= \left[ \left( \frac{e}{b-a} \right) |f'(a)|^q + \frac{e}{b-a} |f'(b)|^q \right] \int_0^1 |\mu-t| dt \\
&+ \left[ \frac{e}{b-a} |f'(a)|^q - \frac{e}{b-a} |f'(b)|^q \right] \int_0^1 t |\mu-t| dt \\
&= \left[ \left( \frac{e}{b-a} \right) |f'(a)|^q + \frac{e}{b-a} |f'(b)|^q \right] \frac{\mu^3 + (2+\mu)(1-\mu)^2}{6} \\
&\geq \frac{1}{6} \left[ \left( \frac{e}{b-a} \right) (2-3\mu + 2\mu^2) \right] |f'(a)|^q \\
&+ \frac{1}{6} \left[ 3 - 3\mu + 6\mu^2 - 2\mu^3 + \frac{e}{b-a} (2-3\mu + 2\mu^3) \right] |f'(b)|^q.
\end{align*}
\]
\[\frac{1}{6} \left[ (1 - \frac{e}{b-a}) (2 - 3\mu + 2\mu^3) \right] |f'(a)|^q + \frac{1}{6} \left[ 1 - 3\mu + 6\mu^2 - 2\mu^3 + \frac{e}{b-a} (2 - 3\mu + 2\mu^3) \right] \times |f'(b)|^q \] ^{1/q}.

(18)

This proof is completed.

It is easy to notice that if we put \( \varepsilon = (b-a)/2 \) in Theorem 3 then we get the following.

**Corollary 5.** Let \( \lambda, \mu \in [0,1] \) and let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be differentiable on \( I' \) and \( [a,b] \subseteq I \) with \( a < b \). Assume that \( f' \in L[a,b] \). If \( f'(x) \)^q is convex on \( [a,b] \) with \( q \geq 1 \), then

\[ \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f \left( \frac{a + b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{b-a}{8} \left( 1 - 2\lambda + 2\lambda^3 \right)^{1-1/q} \times \left[ \left( (4 - 9\lambda + 12\lambda^2 - 2\lambda^3) \right] |f'(a)|^q \right. \\
+ \left( 2 - 3\lambda + 2\lambda^3 \right) |f'(b)|^q \times (6)^{-1} \right]^{1/q} \\
+ \frac{b-a}{8} \left( 1 - 2\mu + 2\mu^2 \right)^{1-1/q} \times \left[ \left( (2 - 3\mu + 2\mu^3) \right] |f'(a)|^q \right. \\
+ (4 - 9\mu + 12\mu^2 - 2\mu^3) \left. |f'(b)|^q \right) \times (6)^{-1} \right]^{1/q}. \]

(19)

One can easily check that if we put \( \varepsilon = (b-a)/3 \) in Theorem 3, then we get the following.

**Corollary 6.** Let \( \lambda, \mu \in [0,1] \) and let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be differentiable on \( I' \) and \( [a,b] \subseteq I \) with \( a < b \). Assume that \( f' \in L[a,b] \). If \( f'(x) \)^q is convex on \( [a,b] \) with \( q \geq 1 \), then

\[ \frac{2\lambda f(a) + \mu f(b)}{3} + \frac{2 - \lambda}{3} f \left( \frac{a + 2b}{3} \right) \\
- \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{4(b-a)}{9} \left( \left( 1 - \frac{1}{q} \right) |f'(a)|^q \right. \\
+ \left. \left( \frac{1}{9} \left( 5 - 12\mu + 18\mu^2 - 4\mu^3 \right) \right] |f'(b)|^q \right) \times (6)^{-1} \times (6)^{-1} \right]^{1/q}. \]

(20)

It is easy to notice that if we put \( \lambda = \mu = 0 \) in Theorem 3 then we get the following.

**Corollary 7.** Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be differentiable on \( I' \) and \( [a,b] \subseteq I \) with \( a < b \). Assume that \( f' \in L[a,b] \) and \( 0 < \varepsilon < b-a \). If \( f'(x) \)^q is convex on \( [a,b] \) with \( q \geq 1 \), then

\[ 2^{1-1/q} (b-a) \left| f(a + \varepsilon) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \varepsilon^2 \left( \left( \frac{1}{2} \left( \frac{1}{3} \right) \right] \right] |f'(a)|^q \right. \\
+ \left. \left( \frac{1}{3} \left( \frac{1}{3} \right) \right] \right] \right. |f'(b)|^q \right) \times (6)^{-1} \times (6)^{-1} \right]^{1/q}. \]

(22)
It is easy to notice that if we put $\lambda = \mu = 1/2$ in Theorem 3 then we get the following.

**Corollary 8.** Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on $I'$ and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$ and $0 < \epsilon < b - a$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$
4^{1-1/q} \left| ef(a) + (b-a-\epsilon)f(b) + (b-a)f(a+\epsilon) \right| \leq \epsilon^2 \left( \frac{1}{4} \frac{e}{8(b-a)} \right) f'(a)^q + \epsilon \frac{e}{8(b-a)} f'(b)^q \right)^{1/q} + (b-a-\epsilon)^2 
$$

$$
\times \left( \left( \frac{1}{8} \frac{e}{8(b-a)} \right) f'(a)^q + \left( \frac{1}{8} + \frac{e}{8(b-a)} \right) f'(b)^q \right)^{1/q}.
$$

(23)

It is easy to notice that if we put $\lambda = \mu = 1$ in Theorem 3 then we get the following.

**Corollary 9.** Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on $I'$ and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$ and $0 < \epsilon < b - a$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$
2^{1-1/q} \left| ef(a) + (b-a-\epsilon)f(b) - \int_a^b f(x) \, dx \right| \leq \epsilon^2 \left( \frac{1}{3} \frac{e}{6(b-a)} \right) f'(a)^q + \epsilon \frac{e}{6(b-a)} f'(b)^q \right)^{1/q} + (b-a-\epsilon)^2 
$$

$$
\times \left( \left( \frac{1}{6} \frac{e}{6(b-a)} \right) f'(a)^q + \left( \frac{1}{3} + \frac{e}{6(b-a)} \right) f'(b)^q \right)^{1/q}.
$$

(24)

**Theorem 10.** Let $\lambda, \mu \in [0, 1]$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on $I'$ and $\{a, b\} \subseteq I$ with $a < b$. Assume that $f' \in L[a, b]$ and $0 < \epsilon < b - a$. If $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$
\left| e\lambda f(a) + (b-a-\epsilon)\mu f(b) \right| + \left[ e(1-\lambda) + (b-a-\epsilon)(1-\mu) \right] 
$$

$$
\times f(a+\epsilon) - \int_a^b f(x) \, dx \leq \epsilon^2 \left\{ \frac{1}{6} \left[ 3-6\lambda + 6\lambda^2 - \frac{e}{b-a} (2-3\lambda + 2\lambda^3) \right] f'(a) \right| 
$$

$$
+ \frac{1}{6} \left[ \frac{e}{b-a} (2-3\lambda + 2\lambda^3) \right] f'(b) \right| \right.
$$

$$
+ (b-a-\epsilon)^2 
$$

$$
\times \left\{ \left[ \frac{1}{6} \left( 1-\frac{e}{b-a} \right) (2-3\mu + 2\mu^3) \right] f'(a) \right| 
$$

$$
+ \frac{1}{6} \left[ 1-3\mu + 6\mu^2 - 2\mu^3 + \frac{e}{b-a} (2-3\mu + 2\mu^3) \right] f'(b) \right| \right.
$$

$$
= \epsilon^2 \left\{ \left[ \frac{(1-\lambda)^2 + \lambda^2}{2} - \frac{e}{b-a} (2+\lambda) (1-\lambda)^2 + \lambda^3 \right] \right.
$$

$$
\times f'(a) \right| + \left( \frac{e}{b-a} (2+\lambda) (1-\lambda)^2 + \lambda^3 \right] \right.
$$

$$
\times f'(b) \right| \right.
$$

(25)

**Proof.** Suppose that $|f'(x)|^q$ is convex on $[a, b]$ with $q \geq 1$. If $q = 1$, then, by Theorem 3, we have

$$
\left| e\lambda f(a) + (b-a-\epsilon)\mu f(b) \right| + \left[ e(1-\lambda) + (b-a-\epsilon)(1-\mu) \right] 
$$

$$
\times f(a+\epsilon) - \int_a^b f(x) \, dx \leq \epsilon^2 \left\{ \frac{1}{6} \left[ 3-6\lambda + 6\lambda^2 - \frac{e}{b-a} (2-3\lambda + 2\lambda^3) \right] f'(a) \right| 
$$

$$
+ \frac{1}{6} \left[ \frac{e}{b-a} (2-3\lambda + 2\lambda^3) \right] f'(b) \right| \right.
$$

$$
+ (b-a-\epsilon)^2 
$$

$$
\times \left\{ \left[ \frac{1}{6} \left( 1-\frac{e}{b-a} \right) (2-3\mu + 2\mu^3) \right] f'(a) \right| 
$$

$$
+ \frac{1}{6} \left[ 1-3\mu + 6\mu^2 - 2\mu^3 + \frac{e}{b-a} (2-3\mu + 2\mu^3) \right] f'(b) \right| \right.
$$

$$
= \epsilon^2 \left\{ \left[ \frac{(1-\lambda)^2 + \lambda^2}{2} - \frac{e}{b-a} (2+\lambda) (1-\lambda)^2 + \lambda^3 \right] \right.
$$

$$
\times f'(a) \right| + \left( \frac{e}{b-a} (2+\lambda) (1-\lambda)^2 + \lambda^3 \right] \right.
$$

$$
\times f'(b) \right| \right.
$$

(25)
\[
\begin{align*}
  &+ (b - a - e)^2 \\
  &\times \left\{ \left[ \left( 1 - \frac{e}{b - a} \right) \frac{\mu^3 + (2 + \mu) (1 - \mu)^2}{6} \right] |f'(a)| \\
  &+ \left[ \frac{\mu^2 + (1 - \mu)^2}{2} - \left( 1 - \frac{e}{b - a} \right) \right] \frac{\mu^3 + (2 + \mu) (1 - \mu)^2}{6} \right\} |f'(b)|.
\end{align*}
\]

Next, we suppose that \( q > 1 \). By Lemma 1 and Hölder’s inequality, we have
\[
\begin{align*}
  &\left| e \lambda f(a) + (b - a - e) \mu f(b) \\
  &+ \left[ \epsilon (1 - \lambda) + (b - a - e) (1 - \mu) \right] \\
  &\times f(a + e) - \int_a^b f(x) \, dx \right| \\
  &\leq \epsilon^2 \int_0^t \left| 1 - \lambda - t \right| \left| f'(t (a + 1 - t) (a + e)) \right| \, dt \\
  &+ (b - a - e)^2 \int_0^1 \left| \mu - t \right| \left| f'(t (a + 1 - t) (a + e)) + (1 - t) b \right| \, dt \\
  \leq \epsilon^2 \left( \int_0^t \left( \int_0^1 \left| 1 - \lambda - t \right| \left| f'(t (a + 1 - t) (a + e)) \right| \, dt \right) \right)^{(1 - 1/q)} \\
  &\times \left( \int_0^1 \left| \mu - t \right|^q \right) \\
  \left( \int_0^1 \left| f'(t (a + 1 - t) (a + e)) \right|^q \, dt \right)^{1/q} \\
  &+ (b - a - e)^2 \left( \int_0^1 \left| \mu - t \right|^q \right) \\
  \left( \int_0^1 \left| f'(t (a + 1 - t) (a + e)) \right|^q \, dt \right)^{1/q} \\
  = \epsilon^2 \left( \int_0^1 \left| 1 - \lambda - t \right| \left| f'(t (a + 1 - t) (a + e)) \right| \, dt \right)^{(1 - 1/q)} \\
  &\times \left( \int_0^1 \left| \mu - t \right|^q \right) \\
  \left( \int_0^1 \left| f'(t (a + 1 - t) (a + e)) \right|^q \, dt \right)^{1/q} \\
  + (b - a - e)^2 \\
  \times \left( \int_0^1 \left| \mu - t \right|^q \right) \\
  \left( \int_0^1 \left| f'(t (a + 1 - t) (a + e)) \right|^q \, dt \right)^{1/q}.
\end{align*}
\]

By the convexity of \( |f'(x)|^q \) and Lemma 2, we have
\[
\begin{align*}
  &\int_0^1 \left| 1 - \lambda - t \right| \left| f'(t (a + 1 - t) (a + e)) \right| \, dt \\
  = \int_0^1 \left| 1 - \lambda - t \right| \left| f'\left( \left( 1 - \frac{e (1 - t)}{b - a} \right) a + \frac{e (1 - t)}{b - a} b \right) \right|^q \, dt \\
  \leq \int_0^1 \left| 1 - \lambda - t \right|^q \\
  \times \left[ \left( 1 - \frac{e (1 - t)}{b - a} \right) \left| f'(a) \right|^q + \frac{e (1 - t)}{b - a} \left| f'(b) \right|^q \right] \, dt
\end{align*}
\]

\[
\begin{align*}
  = &\left[ \left( 1 - \frac{e}{b - a} \right) \left| f'(a) \right|^q + \frac{e}{b - a} \left| f'(b) \right|^q \right] \\
  \times \int_0^1 |t - \lambda - t|^q \, dt \\
  + \left[ \frac{e}{b - a} \left| f'(a) \right|^q - \frac{e}{b - a} \left| f'(b) \right|^q \right] \\
  \times \int_0^1 t |t - \lambda - t|^q \, dt \\
  = &\left[ \left( 1 - \frac{e}{b - a} \right) \left| f'(a) \right|^q + \frac{e}{b - a} \left| f'(b) \right|^q \right] \\
  \times \frac{(1 - \lambda)^{q+1} + \lambda^{q+1}}{q + 1} \\
  + \left[ \frac{e}{b - a} \left| f'(a) \right|^q - \frac{e}{b - a} \left| f'(b) \right|^q \right] \\
  \times \frac{(1 - \lambda)^{q+2} + (q + 2 - \lambda) \lambda^{q+1}}{q + 1 (q + 2)} \\
  \times \left[ \left( \frac{e}{b - a} \right) \left( 1 + \lambda + (1 - \lambda)^{q+1} \right) \left( q + 1 \right) \left( q + 2 \right) \right] \\
  \times \frac{(1 + \lambda + q) \left( 1 - \lambda \right)^{q+2} + \lambda^{q+2}}{q + 1 (q + 2)} \left| f'(a) \right|^q \\
  + \left[ \left( \frac{e}{b - a} \right) \left( 1 + \lambda + q \right) \left( 1 - \lambda \right)^{q+1} \right] \left| f'(b) \right|^q.
\end{align*}
\]
It is easy to notice that if we put $\varepsilon = (b-a)/2$ in Theorem 10 then we get the following.

**Corollary 11** (see [8]). Let $\lambda, \mu \in [0,1]$ and let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on $I$ and $[a,b] \subseteq I$ with $a < b$. Assume that $f' \in L[a,b]$. If $|f'(x)|^q$ is convex on $[a,b]$ with $q \geq 1$, then

$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f \left( \frac{a + b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right|$$

$$\leq \frac{b-a}{4} \left( \frac{1}{2q+1} \right)^{1/q} \times \left[ \left( (q+3-\lambda)(1-\lambda)q^{q+1} + (q+4-\lambda) \lambda q^{q+1} \right) \left| f'(a) \right|^q \right]$$

$$+ \left( (q+1+\lambda)(1-\lambda)q^{q+1} + \lambda q^{q+2} \right) \left| f'(b) \right|^q \right]^{1/q}.$$

(29)

One can easily check that if we put $\varepsilon = (b-a)/3$ in Theorem 10 then we get the following.

**Corollary 12.** Let $\lambda, \mu \in [0,1]$ and let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on $I$ and $[a,b] \subseteq I$ with $a < b$. Assume that $f' \in L[a,b]$. If $|f'(x)|^q$ is convex on $[a,b]$ with $q \geq 1$, then

$$\left| \frac{\lambda f(a) + 2\mu f(b)}{3} + \frac{2 - \lambda - 2\mu}{3} f \left( \frac{2a + b}{3} \right) \right.$$

$$- \frac{1}{b-a} \int_a^b f(x) \, dx \right|$$

$$\leq \frac{b-a}{9} \times \left\{ \left( \frac{5+2q-\lambda}{3}(1-\lambda)q^{q+1} + (6+3q-\lambda) \lambda q^{q+1} \right) \left| f'(a) \right|^q \right.$$}

$$\left. + \left( \frac{1+\lambda+q}{3}(1-\lambda)q^{q+1} + \lambda q^{q+2} \right) \left| f'(b) \right|^q \right\}^{1/q}.$$

(30)
$$\frac{4(b-a)}{9} \times \left\{ \frac{2\mu a^{q+2} + (q + 1 + \mu)(1 - \mu)^{q+1}}{3(q+1)(q+2)} |f'(a)|^q 
+ \frac{(6 + 3q - 2\mu) \mu a^{q+1} + (4 + q - 2\mu)(1 - \mu)^{q+1}}{3(q+1)(q+2)} 
\times |f'(b)|^q \right\}^{1/q}.$$  

Corollary 13. Let $\lambda, \mu \in [0, 1]$ and let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on $I'$ and $[a, b] \subseteq I$ with $a < b$. Assume that $f'' \in L[a, b]$. If $|f''(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\frac{(b-a) f(a) + \mu f(b)}{3} + \left( 1 - \frac{2\lambda}{3} - \frac{\mu}{3} \right) f\left( \frac{a + 2b}{3} \right) \leq \frac{4(b-a)}{9} \times \left\{ \frac{(4 - q - 2\lambda)(1 - \lambda)^{q+1} + (6 + 3q - 2\lambda) \lambda^{q+1}}{3(q+1)(q+2)} 
\times |f'(a)|^q + \frac{2(1 + \lambda + q)(1 - \lambda)^{q+1} + 2\lambda a^{q+2}}{3(q+1)(q+2)} 
\times |f'(b)|^q \right\}^{1/q} + \frac{b-a}{9} \times \left\{ \frac{\mu a^{q+2} + (q + 1 + \mu)(1 - \mu)^{q+1}}{3(q+1)(q+2)} |f'(a)|^q 
+ \frac{(6 + 3q - \mu) \mu a^{q+1} + (5 + 2q - \mu)(1 - \mu)^{q+1}}{3(q+1)(q+2)} 
\times |f'(b)|^q \right\}^{1/q}.$$  

Corollary 14. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on $I'$ and $[a, b] \subseteq I$ with $a < b$. Assume that $f'' \in L[a, b]$ and $0 < \epsilon < b - a$. If $|f''(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\left| (b-a) f(a + \epsilon) - \int_a^b f(x) \, dx \right| \leq \epsilon^2 \left\{ \frac{1}{q+1} - \frac{\epsilon}{(b-a)(q+2)} \right\} |f'(a)|^q$$

$$+ \frac{\epsilon}{(b-a)(q+2)} |f'(b)|^q \right\}^{1/q} + \frac{b-a-\epsilon}{9} \times \left\{ \left( 1 - \frac{\epsilon}{b-a} \right) \left( \frac{1}{q+1} \right) |f''(a)| \right\}^{1/q} + \frac{\epsilon}{2(b-a)} \left( \frac{1}{q+1} \right) |f''(b)| \right\}^{1/q}.$$  

Corollary 15. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable on $I'$ and $[a, b] \subseteq I$ with $a < b$. Assume that $f'' \in L[a, b]$ and $0 < \epsilon < b - a$. If $|f''(x)|^q$ is convex on $[a, b]$ with $q \geq 1$, then

$$\left| e f(a) + (b-a-\epsilon) f(b) + (b-a) f(a + \epsilon) - \int_a^b f(x) \, dx \right| \leq \frac{\epsilon}{2(q+1)} \left\{ \left( 1 - \frac{\epsilon}{2(b-a)} \right) \right\} |f''(a)| \right\}^{1/q} + \frac{\epsilon}{2(b-a)} \left( \frac{1}{q+1} \right) |f''(b)| \right\}^{1/q} + \frac{b-a-\epsilon}{2} \left( \frac{1}{q+1} \right) \left\{ \left( 1 - \frac{\epsilon}{2(b-a)} \right) \right\} |f''(a)| \right\}^{1/q} + \frac{\epsilon}{2(b-a)} \left( \frac{1}{q+1} \right) |f''(b)| \right\}^{1/q}.$$  

It is easy to notice that if we put $\lambda = \mu = 0$ in Theorem 10 then we get the following.

It is easy to notice that if we put $\lambda = \mu = 1$ in Theorem 10 then we get the following.
Corollary 16. Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be differentiable on \( I^* \) and \( \{a, b\} \subseteq I \) with \( a < b \). Assume that \( f' \in L[a, b] \) and \( 0 < \epsilon < b - a \). If \( f'(x)^q \) is convex on \( [a, b] \) with \( q \geq 1 \), then

\[
\left| ef'(a) + \frac{b-a}{b-a} f(b) - \int_a^b f(x) \, dx \right| \leq \epsilon^2 \left( \frac{1}{q+1} \right)^{1/q} \left\{ \left(1 - \frac{\epsilon}{a-b}(q+2)\right) \left| f'(a)^q \right| + \frac{\epsilon}{a-b}(q+2) \left| f'(b)^q \right| \right\}^{1/q} + (b-a-e)^2 \left( \frac{1}{q+1} \right)^{1/q} \times \left\{ \left(1 - \frac{\epsilon}{b-a} \right) \left( \frac{1}{q+2} \right) \left| f'(a)^q \right| + \left(1 - \left(1 - \frac{\epsilon}{b-a} \right) \left( \frac{1}{q+2} \right) \right) \left| f'(b)^q \right| \right\}^{1/q}.
\]

(35)

4. Applications

In this section, we suppose that \( \{s, q\} \subseteq [1, \infty) \) and \( \{a, b, \omega_a, \omega_b\} \subseteq (0, \infty) \) with \( a < b \). Let \( 0 < \epsilon < b - a \).

The weighted arithmetic mean of data \( \{a, b\} \) with weight \( \{\omega_a, \omega_b\} \) is defined by

\[
A(a, b; \omega_a, \omega_b) = \frac{\omega_a a + \omega_b b}{\omega_a + \omega_b}.
\]

(36)

The weighted geometric mean of data \( \{a, b\} \) with weight \( \{\omega_a, \omega_b\} \) is defined by

\[
G(a, b; \omega_a, \omega_b) = a^{\omega_a/(\omega_a+\omega_b)} b^{\omega_b/(\omega_a+\omega_b)}.
\]

(37)

The generalized logarithmic mean of data \( \{a, b\} \) is defined by

\[
L(a, b) = \left( \frac{b^{s+1} - a^{s+1}}{(s+1)(b-a)} \right)^{1/s}.
\]

(38)

The identric mean of data \( \{a, b\} \) is defined by

\[
I(a, b) = \frac{1}{e} \left( \frac{b}{a} \right)^{(b-a)/(b-a)}.
\]

(39)

Applying Corollary 7 with \( f(x) = x^s \) on \((0, \infty)\), we get the following:

\[
2^{1-1/q} \frac{b-a}{s} \left| A^s(a, b; b-a, e, e) - L^s(a, b) \right| \leq \epsilon^2 \left( \frac{a^{(s-1)q}}{2} + \frac{e \left( b^{(s-1)q} - a^{(s-1)q} \right)}{3(b-a)} \right)^{1/q} + (b-a-e)^2 \left( \frac{2a^{(s-1)q} + b^{(s-1)q}}{6} + \frac{e \left( b^{(s-1)q} - a^{(s-1)q} \right)}{3(b-a)} \right)^{1/q}.
\]

(40)

Applying Corollary 9 with \( f(x) = x^s \) on \((0, \infty)\), we get the following:

\[
2^{1-1/q} \frac{b-a}{s} \left| A^s(a, b; b-a, e, e) - L^s(a, b) \right| \leq \epsilon^2 \left( \frac{a^{(s-1)q}}{2} + \frac{e \left( b^{(s-1)q} - a^{(s-1)q} \right)}{3(b-a)} \right)^{1/q} + (b-a-e)^2 \left( \frac{2a^{(s-1)q} + b^{(s-1)q}}{6} + \frac{e \left( b^{(s-1)q} - a^{(s-1)q} \right)}{3(b-a)} \right)^{1/q}.
\]

(41)

Applying Corollary 14 with \( f(x) = x^s \) on \((0, \infty)\), we get the following:

\[
2^{1-1/q} \frac{b-a}{s} \left| A^s(a, b; b-a, e, e) - L^s(a, b) \right| \leq \epsilon^2 \left( \frac{a^{(s-1)q}}{2} + \frac{e \left( b^{(s-1)q} - a^{(s-1)q} \right)}{3(b-a)} \right)^{1/q} + (b-a-e)^2 \left( \frac{2a^{(s-1)q} + b^{(s-1)q}}{6} + \frac{e \left( b^{(s-1)q} - a^{(s-1)q} \right)}{3(b-a)} \right)^{1/q}.
\]

(42)

Applying Corollary 7 with \( f(x) = \ln x \) on \((0, \infty)\), we get the following:

\[
2^{1-1/q} \frac{b-a}{s} \left| \ln \left( \frac{G(a, b; b-a, e)}{I(a, b)} \right) \right| \leq \epsilon^2 \left( \frac{a^{q}}{2} + \frac{e(b^q - a^q)}{3(b-a)} \right)^{1/q} + (b-a-e)^2 \left( \frac{2a^q + b^q}{6} + \frac{e(b^q - a^q)}{3(b-a)} \right)^{1/q}.
\]

(43)

Applying Corollary 9 with \( f(x) = \ln x \) on \((0, \infty)\), we get the following:

\[
2^{1-1/q} \frac{b-a}{s} \left| \ln \left( \frac{G(a, b; b-a, e)}{I(a, b)} \right) \right| \leq \epsilon^2 \left( \frac{a^{q}}{2} + \frac{e(b^q - a^q)}{3(b-a)} \right)^{1/q} + (b-a-e)^2 \left( \frac{2a^q + b^q}{6} + \frac{e(b^q - a^q)}{3(b-a)} \right)^{1/q}.
\]

(44)
Applying Corollary 16 with $f(x) = x'$ on $(0, \infty)$, we get the following:

$$
\frac{(q + 1)^{1/q}}{s} \left| A \left( d', b', \epsilon, b - a - \epsilon \right) - L' \left( a, b \right) \right|
\leq \epsilon^2 \left( \frac{a^{(s-1)q} + b^{(s-1)q}}{q + 2} \right) + \frac{\epsilon (b^{(s-1)q} - a^{(s-1)q})}{(b - a) (q + 2)} \right) ^ {1/q}
+ \frac{\epsilon (b^{(s-1)q} - a^{(s-1)q})}{(b - a) (q + 2)} \right) ~.
\nonumber
(45)

Applying Corollary 14 with $f(x) = \ln x$ on $(0, \infty)$, we get the following:

$$
(b - a) \left| \ln \left( \frac{A(a, b; b - a - \epsilon, \epsilon)}{I(a, b)} \right) \right|
\leq \epsilon^2 \left( \frac{a^{-q} + \epsilon (b^{-q} - a^{-q})}{q + 1} \frac{\epsilon (b^{-q} - a^{-q})}{(b - a) (q + 2)} \right) ^ {1/q}
+ \frac{\epsilon (b^{-q} - a^{-q})}{(b - a) (q + 2)} ~.
\nonumber
(46)

Applying Corollary 16 with $f(x) = \ln x$ on $(0, \infty)$, we get the following:

$$
\frac{(q + 1)^{1/q}}{s} \left| \ln \left( \frac{G(a, b; \epsilon, b - a - \epsilon)}{I(a, b)} \right) \right|
\leq \epsilon^2 \left( \frac{a^{-q} + \epsilon (b^{-q} - a^{-q})}{q + 1} \frac{\epsilon (b^{-q} - a^{-q})}{(b - a) (q + 2)} \right) ^ {1/q}
+ \frac{\epsilon (b^{-q} - a^{-q})}{(b - a) (q + 2)} ~.
\nonumber
(47)

Conflicts of Interest

The author declares that there is no conflict of interests regarding the publication of this paper.

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