The Optimal Global Estimates and Boundary Behavior for Large Solutions to the $k$-Hessian Equation

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Abstract In this paper, we consider the $k$-Hessian equation

$$S_k(D^2 u) = b(x)f(u) \quad \text{in } \Omega, \quad u = +\infty \quad \text{on } \partial \Omega,$$

where $\Omega$ is a smooth, bounded, strictly convex domain in $\mathbb{R}^N$ with $N \geq 2$, $b \in C^\infty(\Omega)$ is positive in $\Omega$ and may be singular or vanish on the boundary, $f \in C^\infty(0, +\infty) \cap C[0, +\infty)$ (or $f \in C^\infty(\mathbb{R})$) is positive and increasing on $[0, +\infty)$ (or $\mathbb{R}$) and satisfies the Keller–Osserman type condition. We first prove an upper and lower solution method of classical $k$-convex large solutions to the above equation, and then we study the optimal global estimates and boundary behavior of large solutions. In particular, we investigate the asymptotic behavior of such solutions when the parameters on $b$ tend to the corresponding critical values and infinity.

Keywords $k$-Hessian equation, boundary blow-up problem, the upper and lower solution method, the optimal global estimates, asymptotic behavior

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1 Introduction

In this paper, we establish the optimal global estimates and boundary behavior of classical $k$-convex solutions to the following $k$-Hessian problem

$$S_k(D^2 u) = S_k(\lambda_1, \ldots, \lambda_N) = b(x)f(u) \quad \text{in } \Omega, \quad u = +\infty \quad \text{on } \partial \Omega, \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a smooth, bounded, strictly convex domain, $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of the Hessian $D^2 u(x) = (\frac{\partial^2 u(x)}{\partial x_i \partial x_j})_{N \times N}$ of $u \in C^2(\Omega)$, the
last condition \( u = +\infty \) on \( \partial \Omega \) means that \( u(x) \to +\infty \) as \( d(x) := \text{dist}(x, \partial \Omega) \to 0 \) and the solution is called large solution, blow-up solution or explosive solution. The \( b \) and \( f \) satisfy

\[(b_1) \ b \in C^\infty(\Omega) \text{ is positive in } \Omega; \]

\[(S_1) \ f \in C^\infty(0, +\infty) \cap C[0, +\infty) \text{ with } f(0) = 0 \text{ and } f \text{ is strictly increasing on } [0, +\infty) \text{ (or } (S_{01}) \ f \in C^\infty(\mathbb{R}), f(t) > 0, \forall t \in \mathbb{R}, \text{ and } f \text{ is strictly increasing on } \mathbb{R}). \]

We see from \([5]\) and \([35]\) that

\[S_k(\lambda_1, \ldots, \lambda_N) = \sum_{1 \leq i_1 < \cdots < i_k \leq N} \lambda_{i_1} \cdots \lambda_{i_k}.\]

For \( k \in \{1, \ldots, N\} \), let \( \Gamma_k \) be the component of \( \{ \lambda \in \mathbb{R}^N : S_k(\lambda) > 0 \} \subset \mathbb{R}^N \) containing the positive cone \( \Gamma^+ := \{ \lambda \in \mathbb{R}^N : \lambda_i > 0, i = 1, \ldots, N \} \). It follows from \([4]\) that \( \Gamma^+ = \Gamma_N \subset \cdots \subset \Gamma_1 \). From Definition 1.1 of \([34]\), we see that for \( k \in \{1, \ldots, N\}, u \in C^2(\Omega) \) is (strictly) \( k \)-convex if \( S_i(D^2 u) = S_i(\lambda_1, \ldots, \lambda_N) (> \geq 0 \text{ in } \Omega \text{ for } i = 1, \ldots, k \). Let \( \Omega \subset \mathbb{R}^N \) be an open set with \( C^2 \)-boundary, then from Definition 1.2 of \([34]\), we see that \( \Omega \) is (strictly) convex if \( S_i(\kappa_1, \ldots, \kappa_{N-1}) (> \geq 0 \text{ on } \partial \Omega \text{ for } i = 1, \ldots, N - 1, \text{ where } \kappa_i(\bar{x}) (i = 1, \ldots, N - 1) \text{ are the principal curvatures of } \partial \Omega \text{ at } \bar{x}. \)

When \( k = 1 \), problem (1.1) is the following semilinear elliptic problem

\[\Delta u = b(x)f(u) \text{ in } \Omega, \quad u = +\infty \text{ on } \partial \Omega. \quad (1.2)\]

Problem (1.2) arises from many branches of mathematics and applied mathematics and has been discussed extensively by many authors in different contexts, please refer to \([1, 12, 21–23, 26, 32]\), and the references therein.

When \( k = N \), problem (1.1) is the Monge–Ampère problem

\[\det(D^2 u) = b(x)f(u) \text{ in } \Omega, \quad u = +\infty \text{ on } \partial \Omega. \quad (1.3)\]

Problem (1.3) arose in Riemannian geometry and was considered by Cheng and Yau \([6, 7]\) for \( f(u) = \exp(Ku) \) in bounded convex domains and for \( f(u) = \exp(2u) \) in unbounded domains. When \( b \in C^\infty(\Omega) \) is positive in \( \Omega \) and \( f(u) = u^\gamma \ (\gamma > N) \) or \( f(u) = \exp(u) \), Lazer and McKenna \([24]\) showed the existence, uniqueness and global estimates of strictly convex \( C^\infty \)-solutions to problem (1.3). Matero \([28]\) treated the more general case for bounded strictly convex domains, generalizing a result of Keller \([21]\) and Osserman \([32]\) for the Laplacian operator. Then problem (1.3) was further studied by Mohammed \([31]\), Cîrstea and Trombetti \([9]\), Yang and Chang \([37]\) and Zhang and Du \([38]\). In particular, Huang \([15]\) extended and developed the results of \([9]\) to the case of problem (1.1). Let \( \Omega \) be a domain in \( \mathbb{R}^N \) and \( h \) be a positive function in \( \Omega \times \mathbb{R} \times \mathbb{R}^N \).
Guan and Jian [14] studied the following boundary blow-up problem for the Monge–Ampère equation

$$\det(D^2u) = h(x, u, \nabla u) \text{ in } \Omega, \quad u = +\infty \text{ on } \partial \Omega.$$  (1.4)

In [14], various existence and nonexistence were established and the optimal growth condition of \( h(x, z, p) \) was given for the existence of strictly convex solutions to problem (1.4). Then, the results were generalized by Jian [18] to the case of \( k \)-Hessian equation. Using radial function methods and techniques of ordinary differential inequality, Jian [18] constructed various barrier functions (super-solution and sub-solution), then proved existence and nonexistence theorems using those barriers and theory of viscous solutions.

Very recently, Zhang and Feng [40] studied the existence and global estimates of \( k \)-convex solutions to problem (1.1) by using the following structure condition

$$\lim_{t \to +\infty} J(t) = E^+_{f, \infty}$$
$$\lim_{t \to 0^+} J(t) = E^0_f,$$  (1.5)

where \( E^0_f \in (0, +\infty), E^+_{f, \infty} \in (0, +\infty] \) and

$$J(t) := \left( (F(t))^{1/(k+1)} \right)' \int_t^{+\infty} (F(s))^{-1/(k+1)} ds.$$  (1.6)

When \( f(t) = \exp(t) \), we see by a direct calculation that \( f(t) \) does not satisfy (1.5). In fact, (1.5) is just right when \( f \) satisfies (S₁), but it is inappropriate when \( f \) satisfies (S₀₁). Most recently, Zhang [45] by introducing some new local structure conditions established the optimal global estimates and boundary behavior of strictly convex solutions to problem (1.3) when \( f \) satisfies (S₁) (or (S₀₁)). Then, when \( b \in C^\infty(\bar{\Omega}) \) is positive in \( \Omega \), Mi and Chen [29] studied the boundary behavior of large solutions to problem (1.1). For other related works, please refer to [2, 10, 13, 16, 17, 19, 20, 27, 39, 41–44], and the references therein.

Inspired by the above results, in this paper we establish the optimal global estimates and asymptotic behavior of \( k \)-convex solutions to problem (1.1). We first prove the upper and lower solution method of classical \( k \)-convex solution to problem (1.1), and then we investigate the optimal global estimates and asymptotic behavior of \( k \)-convex solutions to problem (1.1). Some results in [45] can be improved in this paper. Especially, we establish the optimal global estimate of \( k \)-convex solutions to problem (1.1) when \( f \) is regularly varying at positive infinity with the critical index \( k \).

To obtain Theorems 2.3–2.5 and Theorem 2.7, we assume that \( f \) satisfies the following conditions, not necessarily simultaneously:

\((f_1)\quad \int_t^{+\infty} (f(s))^{-1/k} ds < +\infty, \ t > 0 \ (or \ t \in \mathbb{R}) \ and \)

$$I(t) := ((f(t))^{1/k})' \int_t^{+\infty} (f(s))^{-1/k} ds, \quad t > 0 \ (or \ t \in \mathbb{R});$$
(f_2) there exists $C_f^{+\infty} \in (0, +\infty)$ such that $\lim_{t \to +\infty} I(t) = C_f^{+\infty}$;

(f_3) $f$ satisfies (S_1), $\int_0^1 (f(s))^{-1/k} ds = +\infty$, and there exists $C_f^0 \in (0, +\infty)$ such that $\lim_{t \to 0^+} I(t) = C_f^0$;

(f_4) $f$ satisfies (S_01) and there exists $C_f^{-\infty} \in (0, +\infty)$ such that $\lim_{t \to -\infty} I(t) = C_f^{-\infty}$.

By (S_01), we see that $\int_{-\infty}^a (f(s))^{-1/k} ds = +\infty$.

**Remark 1.1.** Some basic examples of $f$ in (f_1)-(f_2) and (f_3) (or (f_4)) are

(i) if $f(t) = t^\gamma$, $t \geq 0$ with $\gamma > k$, then $C_f^{+\infty} = C_f^0 = \frac{\gamma}{\gamma - k}$;

(ii) if $f(t) = \exp(t)$, $t \in \mathbb{R}$, then $C_f^{+\infty} = C_f^{-\infty} = 1$;

(iii) if $f(t) \sim (-t)^\gamma$ as $t \to -\infty$, where $\gamma < 0$, then $C_f^{-\infty} = \frac{-\gamma}{\gamma - k}$.

To obtain Theorem 2.6, we assume that $f$ satisfies (S_1) or $f$ satisfies (S_01) with

$$
\int_{-\infty}^0 f(s) ds < +\infty. \tag{1.7}
$$

Moreover, we further assume that $f$ satisfies the following conditions, not necessarily simultaneously:

(f_5) $\int_t^{+\infty} (F(s))^{-1/(k+1)} ds < +\infty$, where

$$
F(t) = \begin{cases} 
\int_0^t f(s) ds, t > 0, & \text{if (S_1) holds,} \\
\int_{-\infty}^t f(s) ds, t \in \mathbb{R}, & \text{if (S_01) holds with (1.7);}
\end{cases} \tag{1.8}
$$

(f_6) $\lim_{t \to +\infty} J(t) = +\infty$, where $J$ is given by (1.6) and $F$ is given by (1.8);

(f_7) $f$ satisfies (S_1),

$$
\int_0^1 (F(s))^{-1/(k+1)} ds = +\infty, \quad F(t) = \int_0^t f(s) ds,
$$

and there exists $E_f^0 \in (0, +\infty)$ such that (1.5) holds;

(f_8) $f$ satisfies (S_01) with (1.7), and there exists $E_f^{-\infty} \in (0, +\infty)$ such that

$$
\lim_{t \to -\infty} J(t) = E_f^{-\infty}, \quad \text{where $F(t) = \int_{-\infty}^t f(s) ds$.}
$$

By (S_01), we see that $\int_{-\infty}^a (F(s))^{-1/(k+1)} ds = +\infty$ for $a \in \mathbb{R}$.

**Remark 1.2.** Some basic examples of $f$ in (f_5), (f_7) (or (f_8)) are

(i) if $f(t) \sim t^\gamma$ as $t \to 0^+$, where $\gamma > k$, then $E_f^0 = \frac{\gamma + 1}{\gamma - k}$;

(ii) if $f(t) \sim (-t)^\gamma$ as $t \to -\infty$, where $\gamma < -1$, then $E_f^{-\infty} = \frac{\gamma + 1}{\gamma - k}$;

(iii) if $f(t) \sim \exp(t)$ as $t \to -\infty$, then $E_f^{-\infty} = 1$. 
2 Main Results

2.1 Optimal Global Estimates

(I) Let $\psi$ be uniquely determined by
\[
\int_{\psi(t)}^{+\infty} (f(s))^{-1/k} ds = t, \quad t > 0. \tag{2.1}
\]

We note that
(i) $(S_1)$ (or $(S_{01})$), $(f_1)-(f_2)$ imply $\psi(t) \to +\infty$ if and only if $t \to 0^+$;
(ii) $(S_1)$, $(f_1)$ and $(f_3)$ imply $\psi(t) \to 0^+$ if and only if $t \to +\infty$;
(iii) $(S_{01})$, $(f_1)$ and $(f_4)$ imply $\psi(t) \to -\infty$ if and only if $t \to +\infty$.

(II) Since $\Omega$ is smooth, bounded, strictly convex domain in $\mathbb{R}^N$, we see from [24] that there exists a function $\phi \in C^\infty(\bar{\Omega})$ with the following properties:
\[
\phi(x) < 0, \quad x \in \Omega, \quad \phi|_{\partial\Omega} = 0, \quad \nabla\phi(x) \neq 0, \quad x \in \partial\Omega
\]
and $\phi$ is positive definite in $\bar{\Omega}$. Let
\[
v = -\phi \tag{2.2}
\]
and assume that
\[
\max_{x \in \bar{\Omega}} v(x) < 1.
\]
It is clear that $D^2v$ is negative definite in $\bar{\Omega}$. In fact, for any positive constant $\rho \in (0, 1]$, we can always take $\phi$ such that
\[
\max_{x \in \bar{\Omega}} v(x) < \rho.
\]

(III) Let $\mathcal{L}$ denote the set of Karamata functions defined on $(0, 1]$ by
\[
\tilde{L}(t) = c \exp \left( \int_0^1 \frac{y(s)}{s} ds \right), \quad c > 0, \quad y \in C(0, 1] \text{ and } y(t) \to 0 \text{ as } t \to 0^+.
\]
We see from Proposition 4.3 that $\tilde{L}$ is normalized slowly varying at zero.

Our results can be summarized as follows.

Theorem 2.1. Let $f(t) = t^\gamma$, $t \geq 0$ with $\gamma > k$, $b$ satisfy $(b_1)$ and $(b_2)$ there exist positive constants $b_1, b_2$ ($b_1 \leq b_2$) and $\lambda > -1 - k$ such that for any $x \in \Omega$,
\[
b_1(v(x))^\lambda \leq b(x) \leq b_2(v(x))^\lambda,
\]
then problem (1.1) has a unique classical $k$-convex solution $u_\lambda$ satisfying
\[ m_0(v(x))^{-\alpha} \leq u_\lambda(x) \leq M_0(v(x))^{-\alpha}, \quad x \in \Omega, \]
where
\[ \alpha = \frac{k + 1 + \lambda}{\gamma - k}, \quad m_0 = \left( \frac{b_2}{\alpha^k c_0} \right)^{\frac{1}{k-\gamma}}, \quad M_0 = \left( \frac{b_1}{\alpha^k c_0} \right)^{\frac{1}{k-\gamma}} \] (2.3)
with
\[ c_0 = c_0(\lambda) = \min_{x \in \Omega} \omega_0(x), \quad C_0 = C_0(\lambda) = \max_{x \in \Omega} \omega_0(x), \]
where
\[ \omega_0(x) = v(x)(-1)^k S_k(D^2 v(x)) + (\alpha + 1)\mathcal{A}(x), \] (2.4)
where
\[ \mathcal{A}(x) := \sum_{i=1}^{C_N^k} (-1)^k \det(v_{x_i,x_j})(-\nabla v_i(x))^T B(v_i(x))\nabla v_i(x), \quad x \in \bar{\Omega}, \] (2.5)
where $B(v_i)$ denotes the inverse of the $i$-th principal submatrix $(v_{x_i,x_j})$, $\det(v_{x_i,x_j})$ denotes the determinant of $(v_{x_i,x_j})$ and
\[ \nabla v_i = (v_{x_{i1}}, \ldots, v_{x_{ik}})^T, \quad i = 1, \ldots, C_N^k \] and $C_N^k := \frac{N!}{(N-k)!k!}$.
Moreover, we have
\[ \lim_{\lambda \to (-k-1)^+} u_\lambda(x) = 0, \quad \lim_{\lambda \to +\infty} u_\lambda(x) = +\infty \] (2.6)
and
\[ \left( \frac{b_2}{c_1} \right)^{\frac{1}{k-\gamma}} \leq \liminf_{\lambda \to (-k-1)^+} \frac{u_\lambda(x)}{\alpha^k C_0^{\gamma-k}} \leq \limsup_{\lambda \to (-k-1)^+} \frac{u_\lambda(x)}{\alpha^k C_0^{\gamma-k}} \leq \left( \frac{b_1}{C_1} \right)^{\frac{1}{k-\gamma}} \] (2.7)
both hold uniformly for $x \in \Omega_1$ which is an arbitrary compact subset of $\Omega$, where
\[ c_1 = \min_{x \in \Omega} \omega_1(x), \quad C_1 = \max_{x \in \Omega} \omega_1(x) \] (2.8)
with
\[ \omega_1(x) = v(x)(-1)^k S_k(D^2 v(x)) + \mathcal{A}(x), \] (2.9)
where $\mathcal{A}$ is given by (2.5). In particular, for fixed $x \in \Omega$, we have
\[ \frac{1}{b_2^{\frac{1}{k-\gamma}}} \leq \liminf_{\lambda \to +\infty} \frac{u_\lambda(x)}{\alpha^k C_0^{\gamma-k}(v(x))^{-\alpha}} \]
and
\[ \limsup_{\lambda \to +\infty} \frac{u_\lambda(x)}{\alpha^k C_0^{\gamma-k}(v(x))^{-\alpha}} \leq \frac{1}{b_1^{\frac{1}{k-\gamma}}}. \]
Theorem 2.2. Let \( f(t) = \exp(t), t \in \mathbb{R} \), \( b \) satisfy \((b_1)-(b_2)\), then problem (1.1) has a unique classical \( k \)-convex solution \( u_{\lambda} \) satisfying
\[
m_1 - (\lambda + k + 1) \ln v(x) \leq u_{\lambda}(x) \leq M_1 - (\lambda + k + 1) \ln v(x), \quad x \in \Omega,
\]
where
\[
m_1 = \ln c_1 + k \ln(\lambda + k + 1) - \ln b_2, \quad M_1 = \ln C_1 + k \ln(\lambda + k + 1) - \ln b_1 \quad (2.10)
\]
and \( c_1, C_1 \) are given by (2.8). Moreover, we have
\[
\lim_{\lambda \to (-k-1)^+} u_{\lambda}(x) = -\infty, \quad \lim_{\lambda \to +\infty} u_{\lambda}(x) = +\infty \quad (2.11)
\]
and
\[
\lim_{\lambda \to (-k-1)^+} \frac{u_{\lambda}(x)}{\ln(k + 1 + \lambda)} = k, \quad \lim_{\lambda \to +\infty} \frac{u_{\lambda}(x)}{(k + 1 + \lambda) \ln v(x)} = -1 \quad (2.12)
\]
both hold uniformly for \( x \in \Omega_1 \) which is an arbitrary compact subset of \( \Omega \).

Theorem 2.3. Let \( f \) satisfy \((S_1) \) (or \((S_{01})\)), \((f_1)-(f_2)\) and \((f_3) \) (or \((f_4)\)), \( b \) satisfy \((b_1)-(b_2)\) with
\[
(h_0 - 1)\eta + 1 > 0, \quad \eta = \frac{k+1+\lambda}{k} \quad \text{and} \quad (2.13)
\]
then problem (1.1) has a \( k \)-convex solution \( u_{\lambda} \) satisfying
\[
\psi(\tau_2 \eta^{-1} v^\eta(x)) \leq u_{\lambda}(x) \leq \psi(\tau_1 \eta^{-1} v^\eta(x)), \quad x \in \Omega, \quad (2.14)
\]
where \( \eta = \frac{k+1+\lambda}{k} \) and
\[
h_0 := \begin{cases} 
\inf_{t > 0}((f(t))^{1/k})' \int_{t}^{+\infty} (f(s))^{-1/k} ds > 0, & \text{if } (f_3) \text{ holds}, \\
\inf_{t \in \mathbb{R}}((f(t))^{1/k})' \int_{t}^{+\infty} (f(s))^{-1/k} ds > 0, & \text{if } (f_4) \text{ holds},
\end{cases} \quad (2.15)
\]
\( \tau_1 \) and \( \tau_2 \) are given by
\[
\tau_1^k \max_{x \in \Omega} \omega_2(\tau_1, x) = b_1, \quad \tau_2^k \min_{x \in \Omega} \omega_2(\tau_2, x) = b_2 \quad (2.16)
\]
with
\[
\omega_2(\tau_l, x) = v(x)(-1)^k S_k(D^2 v(x)) + (\Psi(\tau_l \eta^{-1}(v(x))^\eta))\eta + 1 - \eta \omega(x) \quad (2.17)
\]
and
\[
\Psi(t) = -\frac{\psi''(t)}{\psi'(t)}, \quad t > 0, \quad l = 1, 2, \quad (2.18)
\]
where \( \mathcal{A} \) is given by (2.5). Moreover, we have
\[
\begin{align*}
&\left\{ \begin{array}{ll}
\lim_{\lambda \to (-k-1)^+} \max_{x \in \Omega_1} u_\lambda(x) = 0, & \text{if } (f_3) \text{ holds}, \\
\lim_{\lambda \to (-k-1)^+} \max_{x \in \Omega_1} u_\lambda(x) = -\infty, & \text{if } (f_4) \text{ holds,} \\
\lim_{\lambda \to +\infty} \min_{x \in \Omega_1} u_\lambda(x) = +\infty, & \text{if } (f_3) \text{ (or } (f_4) \text{) holds with } h_0 \geq 1.
\end{array} \right.
\end{align*}
\]

In particular, if \((f_3)\) holds, then we further have
\[
\left( \frac{b_2}{c_1} \right)^{1-C_0^f/k} \leq \liminf_{\lambda \to (-k-1)^+} \frac{u_\lambda(x)}{\psi(\eta^{-1})} \leq \limsup_{\lambda \to (-k-1)^+} \frac{u_\lambda(x)}{\psi(\eta^{-1})} \leq \left( \frac{b_1}{C_1} \right)^{1-C_0^f/k} \tag{2.19}
\]
uniformly for \( x \in \Omega_1 \) which is an arbitrary compact subset of \( \Omega \); if \((f_4)\) holds, then we further have
\[
\left( \frac{b_1}{C_1} \right)^{1-C_f^{-\infty}/k} \leq \liminf_{\lambda \to (-k-1)^+} \frac{u_\lambda(x)}{\psi(\eta^{-1})} \leq \limsup_{\lambda \to (-k-1)^+} \frac{u_\lambda(x)}{\psi(\eta^{-1})} \leq \left( \frac{b_2}{c_1} \right)^{1-C_f^{-\infty}/k} \tag{2.20}
\]
uniformly for \( x \in \Omega_1 \).

It follows from (2.19) and (2.20) that if \( C_f^0 = 1 \) (or \( C_f^{-\infty} = 1 \)), then for any \( x \in \Omega \), it holds
\[
\lim_{\lambda \to (-k-1)^+} \frac{u_\lambda(x)}{\psi(\eta^{-1})} = 1.
\]

**Remark 2.1.** If \((f_4)\) holds, then by Lemma 5.4 (i), we have \( h_0 \leq 1 \). So, in Theorem 2.3, \((f_4)\) with \( C_f^{-\infty} = 1 \) and (2.13) for any \( \eta > 0 \) imply that \( h_0 = 1 \).

**Theorem 2.4.** Let \( f \) satisfy \((S_1)\) (or \((S_{01})\)), \((f_1)-(f_2)\) and \((f_3)\) (or \((f_4)\)), \( b \) satisfy \((b_1)\) and the following condition
\[
(b_3) \text{ there exist } \mu > 1 \text{ and positive constants } b_1, b_2 \text{ (} b_1 \leq b_2 \text{) such that for any } x \in \Omega,
\]
\[
b_1(v(x))^{-k-1}(-\ln v(x))^{-k\mu} \leq b(x) \leq b_2(v(x))^{-k-1}(-\ln v(x))^{-k\mu}.
\]

Then problem (1.1) has a classical \( k \)-convex solution \( u_\mu \) satisfying for any \( x \in \Omega \),
\[
\psi(\tau_4(\mu-1)^{-1}(-\ln v(x))^{1-\mu}) \leq u_\mu(x) \leq \psi(\tau_3(\mu-1)^{-1}(-\ln v(x))^{1-\mu}), \tag{2.21}
\]
where \( \tau_3 \) and \( \tau_4 \) are given by
\[
\tau_3^k \max_{x \in \Omega} \omega_3(\tau_3, x) = b_1, \quad \tau_4^k \min_{x \in \Omega} \omega_3(\tau_4, x) = b_2 \tag{2.22}
\]
with
\[
\omega_3(\tau_l, x) = v(x)(-1)^k S_k(D^2v(x)) + (\Psi(\tau_l, \mu - 1)^{-1}(-\ln v(x))^{1-\mu}) \\
\times (\mu - 1)(-\ln v(x))^{-1} + 1 - \mu(-\ln v(x))^{-1}) \mathcal{A}(x), \quad l = 3, 4,
\] (2.23)

where \( \Psi \) is given by (2.18) and \( \mathcal{A} \) is given by (2.5). Moreover, we have
\[
\begin{cases}
\lim_{\mu \to 1^+} \max_{x \in \Omega_1} u_\mu(x) = 0, & \text{if } (f_3) \text{ holds}, \\
\lim_{\mu \to 1^+} \max_{x \in \Omega_1} u_\mu(x) = -\infty, & \text{if } (f_4) \text{ holds}, \\
\lim_{\mu \to +\infty} \min_{x \in \Omega_1} u_\mu(x) = +\infty, & \text{if } (f_3) \text{ (or } (f_4) \text{) holds}.
\end{cases}
\]

In particular, if \((f_3)\) holds, then we further have
\[
\left(\frac{b_2}{c_2}\right)^{\frac{1-C^0}{k}} \leq \liminf_{\mu \to 1^+} \frac{u_\mu(x)}{\psi((\mu - 1)^{-1})} \leq \limsup_{\mu \to 1^+} \frac{u_\mu(x)}{\psi((\mu - 1)^{-1})} \leq \left(\frac{b_1}{C_2}\right)^{\frac{1-C^0}{k}}
\] (2.24)

uniformly for \(x \in \Omega_1\) which is an arbitrary compact subset of \(\Omega\); if \((f_4)\) holds, then we further have
\[
\left(\frac{b_1}{C_2}\right)^{\frac{1-C^{-\infty}}{k}} \leq \liminf_{\mu \to 1^+} \frac{u_\mu(x)}{\psi((\mu - 1)^{-1})} \\
\leq \limsup_{\mu \to 1^+} \frac{u_\mu(x)}{\psi((\mu - 1)^{-1})} \leq \left(\frac{b_2}{c_2}\right)^{\frac{1-C^{-\infty}}{k}}
\] (2.25)

uniformly for \(x \in \Omega_1\), where
\[
c_2 = \min_{x \in \Omega} \hat{\omega}_2(x), \quad C_2 = \max_{x \in \Omega} \hat{\omega}_2(x)
\]
with
\[
\hat{\omega}_2(x) = v(x)(-1)^k S_k(D^2v(x)) + (1 - (\ln v(x))^{-1}) \mathcal{A}(x),
\]
where \( \mathcal{A} \) is given by (2.5).

It follows from (2.24) and (2.25) that if \(C_f^0 = 1\) (or \(C_f^{-\infty} = 1\)), then for any \(x \in \Omega\), it holds
\[
\lim_{\mu \to 1^+} \frac{u_\mu(x)}{\psi((\mu - 1)^{-1})} = 1.
\]

**Theorem 2.5.** Let \( f \) satisfy \((S_1)\) (or \((S_{01})\)), \((f_1)-(f_2)\) and \((f_3)\) (or \((f_4)\)), \( b \) satisfy \((b_1)\) and the following condition
Remark 2.2. In Theorem 2.5, if \( b_1, b_2 \) \((b_1 \leq b_2)\), \( \lambda \geq -k - 1 \) and some function \( \tilde{L} \in \mathcal{L} \) such that
\[
b_1(v(x))^\lambda \tilde{L}^k(v(x)) \leq b(x) \leq b_2(v(x))^\lambda \tilde{L}^k(v(x)), \quad x \in \Omega.
\]
If we further assume that
\[
kh_0 + (1 + \lambda)(h_0 - 1) > 0 \quad (2.26)
\]
in \(b_4\), where \( h_0 \) is given by \( (2.15) \), then problem \((1.1)\) has a classical \( k \)-convex solution \( u \) satisfying
\[
\psi\left(\tau_6 \int_0^{v(x)} s^{\frac{1+k}{k}} \tilde{L}(s) ds\right) \leq u(x) \leq \psi\left(\tau_5 \int_0^{v(x)} s^{\frac{1+k}{k}} \tilde{L}(s) ds\right), \quad x \in \Omega, \quad (2.27)
\]
where \( \tau_5 \) and \( \tau_6 \) are given by
\[
\tau_5 = \max_{x \in \Omega} \omega_4(\tau_5, x), \quad \tau_6 = \min_{x \in \Omega} \omega_4(\tau_6, x) = b_1, \quad \tau_6 = \min_{x \in \Omega} \omega_4(\tau_6, x) = b_2 \quad (2.28)
\]
with
\[
\omega_4(\tau_1, x) = v(x)(-1)^k S_k(D^2v(x)) + \left[\Psi\left(\tau_1 \int_0^{v(x)} s^{\frac{1+k}{k}} \tilde{L}(s) ds\right)\right] - \frac{\lambda + 1}{k} - \frac{v(x)L'(v(x))}{L(v(x))} \omega(x), \quad l = 5, 6, \quad (2.29)
\]
where \( \Psi \) is given by \((2.18)\) and \( \omega \) is given by \((2.5)\).

Remark 2.3. In Theorem 2.5, if \( \lambda = -k - 1 \) and \( \tilde{L}(v(x)) = (-\ln v(x))^{-k\mu} \) with \( \mu > 1 \), then the global estimate \((2.27)\) is the same as \((2.21)\). If \( \lambda > -k - 1 \) and \( \tilde{L} \equiv 1 \), then the estimate \((2.27)\) is the same as \((2.14)\).

Remark 2.4. In Theorem 2.5, if \( f(u) = u^\gamma \) with \( \gamma > k \), then the \( k \)-convex solution \( u \) satisfies for any \( x \in \Omega \),
\[
\left[\frac{(\gamma - k)\tau_6}{k} \int_0^{v(x)} s^{\frac{1+k}{k}} \tilde{L}(s) ds\right]^\frac{k}{\gamma - 1} \leq u(x) \leq \left[\frac{(\gamma - k)\tau_5}{k} \int_0^{v(x)} s^{\frac{1+k}{k}} \tilde{L}(s) ds\right]^\frac{k}{\gamma - 1}
\]
and if \( f(u) = \exp(u) \), then the \( k \)-convex solution \( u \) satisfies for any \( x \in \Omega \),
\[
-k \left[\ln \left(\frac{\tau_6}{k} \int_0^{v(x)} s^{\frac{1+k}{k}} \tilde{L}(s) ds\right)\right] \leq u(x) \leq -k \left[\ln \left(\frac{\tau_5}{k} \int_0^{v(x)} s^{\frac{1+k}{k}} \tilde{L}(s) ds\right)\right].
\]
In fact, by Lemma 5.2 (ii)–(iii), we see that the conditions \((S_1)\) (or \((S_{01})\)), \((f_1)\)–\((f_2)\) imply that \(f \in NRV_\gamma\) with \(\gamma > k\) or \(f\) is rapidly varying to positive infinity at positive infinity. Next, we will show the optimal global estimate of \(k\)-convex solutions to problem \((1.1)\) when \(f \in RV_k\).

Let \(f\) satisfy \((S_1)\) (or \((S_{01})\) with \((1.7)\)) and \((f_5)\), and \(\varphi\) be uniquely determined by

\[
\int_{\varphi(t)}^{+\infty} ((k + 1)F(s))^{-1/(k+1)} ds = t,
\]

where

\[
F(t) = \int_t^\infty f(s) ds \text{ with } \zeta = \begin{cases} 
0, & \text{if } (S_1) \text{ holds}, \\
-\infty, & \text{if } (S_{01}) \text{ holds with } (1.7).
\end{cases}
\]

We note that

(i) \((S_1)\) (or \((S_{01})\) with \((1.7)\)), \((f_5)\) imply \(\varphi(t) \to +\infty\) if and only if \(t \to 0^+\);

(ii) \((S_1)\), \((f_5)\) and \((f_7)\) imply \(\varphi(t) \to 0^+\) if and only if \(t \to +\infty\);

(iii) \((S_{01})\) with \((1.7)\), \((f_5)\) and \((f_8)\) imply \(\varphi(t) \to -\infty\) if and only if \(t \to +\infty\).

Our result can be summarized as follows.

**Theorem 2.6.** Let \(f\) satisfy \((S_1)\) (or \((S_{01})\) with \((1.7)\)), \((f_5)-(f_6)\) and \((f_7)\) (or \((f_8)\)), \(b\) satisfy \((b_1)\) and \((b_4)\) with \(-k - 1 < \lambda < 0\), then problem \((1.1)\) has a classical \(k\)-convex solution \(u\) satisfying

\[
\varphi \left[ \tau_8 \left( \int_0^{v(x)} s^{1+\lambda \over k} \tilde{L}(s) ds \right)^{k \over k+1} \right] \leq u(x) \leq \varphi \left[ \tau_7 \left( \int_0^{v(x)} s^{1+\lambda \over k} \tilde{L}(s) ds \right)^{k \over k+1} \right], \quad x \in \Omega,
\]

where \(\tau_7\) and \(\tau_8\) are given by

\[
\tau_7^{k+1} \max_{x \in \Omega} \omega_5(\tau_7, x) = b_1, \quad \tau_8^{k+1} \min_{x \in \Omega} \omega_5(\tau_8, x) = b_2,
\]

with

\[
\omega_5(\tau_1, x) = \left( \frac{k}{k+1} \right)^k \left\{ \Phi \left( \tau_1 \left( \int_0^{v(x)} s^{1+\lambda \over k} \tilde{L}(s) ds \right)^{k \over k+1} \right) v(x) (-1)^k S_k (D^2 v(x)) \right. \\
+ \left. \left[ \frac{(v(x))^{k+1+\lambda \over k} \tilde{L}(v(x))}{\int_0^{v(x)} s^{1+\lambda \over k} \tilde{L}(s) ds} + \Phi \left( \tau_1 \left( \int_0^{v(x)} s^{1+\lambda \over k} \tilde{L}(s) ds \right)^{k \over k+1} \right) \right] \right\} \mathcal{A}(x)
\]

and

\[
\Phi \left( \tau_1 \left( \int_0^{v(x)} s^{1+\lambda \over k} \tilde{L}(s) ds \right)^{k \over k+1} \right)
\]
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\[
= - \frac{\varphi'(\tau_l(\int_0^v(x) s^{\frac{1+\lambda}{k}} L(s) ds)^{\frac{k}{k+1}})}{\varphi''(\tau_l(\int_0^v(x) s^{\frac{1+\lambda}{k}} L(s) ds)^{\frac{k}{k+1}})} \tau_l(\int_0^v(x) s^{\frac{1+\lambda}{k}} L(s) ds)^{\frac{k}{k+1}}, \quad l = 7, 8.
\]

2.2 The Exact Boundary Behavior

Let \( \Lambda \) denote the set of all positive monotonic functions \( \theta \in C^1(0, \delta_0) \cap L^1(0, \delta_0) \) which satisfy

\[
\lim_{t \to 0^+} \frac{d}{dt} \left( \frac{\Theta(t)}{\theta(t)} \right) = D_\theta \in [0, +\infty), \quad \Theta(t) = \int_0^t \theta(s) ds.
\]

The set \( \Lambda \) was first introduced by Cîrstea and Rădulescu [8] for non-decreasing functions and by Mohammed [30] for non-increasing functions to study the boundary behavior and uniqueness of solutions for boundary blow-up elliptic problems. Our result can be summarized as follows.

**Theorem 2.7.** Let \( f \) satisfy (\( S_1 \)) (or (\( S_{01} \))), (\( f_1 \))−(\( f_2 \)), \( b \) satisfy (\( b_1 \)) and the following condition

\( (b_5) \) there exist positive constants \( b_1, b_2 \) (\( b_1 \leq b_2 \)) and some function \( \theta \in \Lambda \) such that

\[
b_1 := \liminf_{d(x) \to 0} \frac{b(x)}{\theta^{k+1}(d(x))} \leq \limsup_{d(x) \to 0} \frac{b(x)}{\theta^{k+1}(d(x))} =: b_2.
\]

If we further assume that

\( C^{-C_\infty}_f > 1 \)

or

\( C^{-C_\infty}_f = 1 \) and \( D_\theta > 0 \),

then any classical \( k \)-convex solution \( u \) to problem (1.1) satisfies

\[
\tau_9^{1-C^+_\infty}_f \leq \liminf_{d(x) \to 0} \frac{u(x)}{\psi((\Theta(d(x)))^{\frac{k+1}{k}})} \leq \limsup_{d(x) \to 0} \frac{u(x)}{\psi((\Theta(d(x)))^{\frac{k+1}{k}})} \leq \tau_9^{1-C^+_\infty}_f, \quad (2.33)
\]

where \( \psi \) is uniquely determined by (2.1),

\[
\tau_9 = \frac{k}{k+1} \left( \frac{b_1 k}{((k+1)(C^+_\infty - 1) + kD_\theta)M_{k-1}^+} \right)^{1/k}
\]

and

\[
\tau_{10} = \frac{k}{k+1} \left( \frac{b_2 k}{((k+1)(C^+_\infty - 1) + kD_\theta)M_{k-1}^-} \right)^{1/k}
\]
with
\[ M^+_{k-1} = \max_{\bar{x} \in \partial \Omega} S_{k-1}(\kappa_1(\bar{x}), \ldots, \kappa_{N-1}(\bar{x})) \]
and
\[ M^-_{k-1} = \min_{\bar{x} \in \partial \Omega} S_{k-1}(\kappa_1(\bar{x}), \ldots, \kappa_{N-1}(\bar{x})). \]

In particular, if \( C^+_f \infty = 1 \), then
\[ \lim_{d(x) \to 0} \frac{u(x)}{\psi((\Theta(d(x)))^{k+1})} = 1. \]

3 Some Preliminary Results

In this section, we collect some well-known results for the convenience of later utilization and reference.

**Lemma 3.1** [18, Lemma 2.1]. Suppose that \( \Omega \subset \mathbb{R}^N \) is a bounded domain, and \( u, v \in C^2(\Omega) \) are \( k \)-convex. If
(i) \( \phi_1(x, z, q) \geq \phi_2(x, z, q), \forall (x, z, q) \in \Omega \times \mathbb{R} \times \mathbb{R}^N; \)
(ii) \( S_k(D^2u) \geq \phi_1(x, u, Du) \) and \( S_k(D^2v) \leq \phi_2(x, v, Dv) \) in \( \Omega; \)
(iii) \( u \leq v \) on \( \partial \Omega; \)
(iv) \( \partial_z \phi_1(x, z, p) > 0 \) or \( \partial_z \phi_2(x, z, p) > 0, \) then \( u \leq v \) in \( \Omega. \)

The following interior estimate for derivatives of smooth solutions is a simple variant of Lemma 2.2 in [24] and can be proved by the idea of Theorem 3.1 and Remark 3.1 of [36].

**Lemma 3.2.** Let \( \Omega \) be a bounded \((k-1)\)-convex domain in \( \mathbb{R}^N \) with \( N \geq 2 \) and \( \partial \Omega \in C^\infty. \) Let \( \beta \in [-\infty, \infty) \) and \( h \in C^\infty(\Omega \times (\beta, \infty)) \) with \( h(x, u) > 0 \) for \( (x, u) \in \bar{\Omega} \times (\beta, \infty) \). Let \( u \in C^\infty(\bar{\Omega}) \) be a solution of the following problem
\[ S_k(D^2u) = h(x, u), \quad x \in \Omega, \quad u|_{\partial \Omega} = c = \text{constant} \]
with \( \beta < u(x) < c \) in \( \Omega. \) Let \( D \subset \Omega \) be a subdomain of \( \Omega \) and assume that \( \beta < \beta_1 \leq u(x) \leq \beta_2 \) for \( x \in \bar{D} \) and let \( \tau > 1 \) be an integer. Then there exists a constant \( C \) which depends only on \( \beta_1, \beta_2, \tau, \) bounds for the derivatives of \( h(x, u) \) for \( (x, u) \in \bar{D} \times [\beta_1, \beta_2] \) and \( \text{dist}(D, \partial \Omega) \) such that
\[ \|u\|_{C^\tau(D)} \leq C. \]

**Lemma 3.3** [13, Theorem 1.1] [34, Theorem 1.1]. Let \( \Omega \) be an open domain in \( \mathbb{R}^N \) with \( C^\infty \)-boundary and let \( h(x, t) \) be a \( C^\infty \)-function such that \( h > 0 \) and \( h_t \geq 0 \) in \( \Omega \times \mathbb{R}. \) Then the following problem
\[ S_k(D^2u) = h(x, u), \quad x \in \Omega, \quad u|_{\partial \Omega} = \tilde{\phi} \in C(\partial \Omega) \]
has a unique $k$-convex solution provided there exists a $k$-convex strict subsolution $v$ in $\bar{\Omega}$, i.e., a $k$-convex function $v$ such that $v|_{\partial \Omega} = \tilde{\phi}$ and $S_k(D^2v) \geq h(x,v) + \delta$ in $\Omega$, for some $\delta > 0$.

**Lemma 3.4** [15, Proposition 2.1] [40, Lemma 2.3]. Let $u \in C^2(\Omega)$ be such that all of the principal submatrices of $(u_{x_ix_j}(x))$ are invertible for $x \in \Omega$, and let $h$ be a $C^2$-function defined on an interval containing the range of $u$. Then

$$S_k(D^2h(u)) = S_k(D^2u)(h'(u))^k + (h'(u))^{k-1}h''(u) \sum_{i=1}^{C_N^k} \det(u_{x_ix_j}) \nabla u_i^T B(u_i) \nabla u_i,$$

where $A^T$ denotes the transpose of the matrix $A$, $B(u_i)$ denotes the inverse of the $i$-th principal submatrix $(u_{x_ix_j})$, $\det(u_{x_ix_j})$ denotes the determinant of $(u_{x_ix_j})$ and

$$\nabla u_i = (u_{x_i1}, \ldots, u_{x_ik})^T, \quad i = 1, \ldots, C_N^k$$

and $C_N^k = \frac{N!}{(N-k)!k!}$.

For any $\delta > 0$, we define

$$\Omega_\delta := \{ x \in \Omega : 0 < d(x) < \delta \}. \quad (3.1)$$

Since $\Omega$ is smooth, for positive integer $m \geq 2$, we can always take $\tilde{\delta} > 0$ such that (please refer to Lemmas 14.16 and 14.17 in [12])

$$d \in C^m(\Omega_\tilde{\delta}), \quad |\nabla d(x)| = 1, \quad \forall x \in \Omega_\tilde{\delta}. \quad (3.2)$$

**Lemma 3.5** [15, Corollary 2.3]. Let $\Omega$ be a bounded domain with $C^m$-boundary ($m \geq 2$) and $h$ be a $C^2$-function on $(0, \tilde{\delta})$. Assume that $x \in \Omega_\tilde{\delta}$ and $\bar{x} \in \partial \Omega$ is the nearest point to $x$, i.e., $d(x) = |x - \bar{x}|$, then

$$S_k(D^2h(d(x))) = (-h'(d(x)))^k S_k(\epsilon_1, \ldots, \epsilon_{N-1}) + (-h'(d(x)))^{k-1}h''(d(x))S_{k-1}(\epsilon_1, \ldots, \epsilon_{N-1}),$$

where

$$\epsilon_i = \frac{\kappa_i(\bar{x})}{1 - \kappa_i(\bar{x})d(x)}, \quad i = 1, \ldots, N - 1$$

and $\kappa_i(\bar{x})$, $i = 1, \ldots, N - 1$ are the principal curvatures of $\partial \Omega$ at $\bar{x}$.

4 Some Basic Facts from Karamata Regular Variation Theory

Some basic facts from Karamata regular variation theory are given in this section, please refer to, for instance, [3, 11, 25, 33] and Zhang’s paper [45].
Definition 4.1. A positive continuous function $f$ defined on $[a, +\infty)$, for some $a > 0$, is called \textbf{regularly varying at positive infinity} with index $\rho$, written $f \in RV_\rho$, if for each $\xi > 0$ and some $\rho \in \mathbb{R}$,

$$\lim_{t \to +\infty} \frac{f(\xi t)}{f(t)} = \xi^\rho. \tag{4.1}$$

In particular, when $\rho = 0$, $f$ is called \textbf{slowly varying at infinity}. Clearly, if $f \in RV_\rho$, then $\hat{L}(t) := f(t)/t^\rho$ is slowly varying at positive infinity.

Similarly, we define the regularly varying functions at zero and at negative infinity as follows.

Definition 4.2. A positive continuous function $g$ defined on $(0, a)$ for some $a > 0$, is \textbf{regularly varying at zero} with index $\rho$ (write $g \in RVZ_\rho$) if $t \mapsto g(1/t)$ belongs to $RV_{-\rho}$.

We see from Definition 4.2 that $g \in RVZ_\rho$ ($\rho \in \mathbb{R}$), if for each $\xi > 0$,

$$\lim_{t \to 0^+} \frac{g(\xi t)}{g(t)} = \xi^\rho. \tag{4.2}$$

Definition 4.3. A positive continuous function $h$ defined on $(-\infty, a)$ for some $a < 0$, is \textbf{regularly varying at negative infinity} with index $\rho$ if $t \mapsto h(-t)$ is regularly varying at positive infinity with index $\rho$.

Proposition 4.1 (Uniform Convergence Theorem). If $f \in RV_\rho$, then (4.1) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$. If $g \in RVZ_\rho$, then (4.2) holds uniformly for $\xi \in [c_1, c_2]$ with $0 < c_1 < c_2$.

Proposition 4.2 (Representation Theorem). A function $L$ is slowly varying at positive infinity if and only if it may be written in the form

$$L(t) = \nu(t)\exp \left( \int_{a_1}^{t} \frac{y(s)}{s} ds \right), \quad t \geq a_1$$

for some $a_1 > 0$, where the functions $\nu$ and $y$ are continuous and for $t \to +\infty$, $y(t) \to 0$ and $\nu(t) \to c$, with $c > 0$. We call that

$$\hat{L}(t) = c\exp \left( \int_{a_1}^{t} \frac{y(s)}{s} ds \right), \quad t \geq a_1$$

is \textbf{normalized} slowly varying at positive infinity and $f(t) := t^\rho \hat{L}(t)$, $t \geq a_1$ is \textbf{normalized} regularly varying at positive infinity with index $\rho$ (and write $f \in NRV_\rho$).
Proposition 4.3 (Representation Theorem). A function $L$ is slowly varying at zero if and only if it may be written in the form

$$L(t) = \nu(t)\exp\left(\int_{t}^{a_1} \frac{y(s)}{s} ds\right), \quad t \leq a_1,$$

for some $a_1 > 0$, where the functions $\nu$ and $y$ are continuous and for $t \to 0^+$, $y(t) \to 0$ and $\nu(t) \to c$, with $c > 0$. We call that

$$\hat{L}(t) = c\exp\left(\int_{t}^{a_1} \frac{y(s)}{s} ds\right), \quad t \leq a_1$$

is normalized slowly varying at zero and

$$g(t) := t^\rho \hat{L}(t), \quad t \leq a_1$$

is normalized regularly varying at zero with index $\rho$ (and write $g \in NRVZ_\rho$).

Proposition 4.4 (Representation Theorem). A function $L$ is slowly varying at negative infinity if and only if it may be written in the form

$$L(t) = \nu(t)\exp\left(\int_{t}^{a_1} \frac{y(s)}{s} ds\right), \quad t \leq a_1,$$

for some $a_1 < 0$, where the functions $\nu$ and $y$ are continuous and for $t \to -\infty$, $y(t) \to 0$ and $\nu(t) \to c$, with $c > 0$. We call that

$$\hat{L}(t) = c\exp\left(\int_{t}^{a_1} \frac{y(s)}{s} ds\right), \quad t \leq a_1$$

is normalized slowly varying at negative infinity and $h(t) := (-t)^\rho \hat{L}(t), \; t \leq a_1$ is normalized regularly varying at negative infinity with index $\rho$.

Proposition 4.5. A function $f \in C^1[a_1, \infty)$ for some $a_1 > 0$ belongs to $NRV_\rho$ if and only if

$$\lim_{t \to +\infty} \frac{tf'(t)}{f(t)} = \rho.$$

A function $g \in C^1(0, a_1]$ for some $a_1 > 0$ belongs to $NRVZ_\rho$ if and only if

$$\lim_{t \to 0^+} \frac{tg'(t)}{g(t)} = \rho.$$

A function $h \in C^1(-\infty, a_1]$ for some $a_1 < 0$ is normalized regularly varying at negative infinity with index $\rho$ if and only if

$$\lim_{t \to -\infty} \frac{th'(t)}{h(t)} = \rho.$$
Proposition 4.6. Let functions $L, L_1$ be slowly varying at zero and at negative infinity, respectively, then
(i) for every $\rho > 0$ and $t \to 0^+$, $t^\rho L(t) \to 0$ and $t^{-\rho} L(t) \to +\infty$;
(ii) for $\rho > 0$ and $t \to -\infty$, $(-t)^{-\rho} L_1(t) \to 0$ and $(-t)^{\rho} L_1(t) \to +\infty$.

Proposition 4.7 (Asymptotic Behavior). Let $L$ and $L_1$ be slowly varying at positive infinity and at zero respectively, and $a_1$ be a positive constant, then
(i) $\int_a^\infty s^\rho L(s)ds \sim (-\rho - 1)^{-1} t^{1+\rho} L(t), \ t \to +\infty$, for $\rho < -1$;
(ii) $\int_0^a s^\rho L_1(s)ds \sim (-\rho - 1)^{-1} t^{1+\rho} L_1(t), \ t \to 0^+$, for $\rho < -1$;
(iii) $\int_a^\infty s^\rho L(s)ds \sim (\rho + 1)^{-1} t^{1+\rho} L(t), \ t \to +\infty$, for $\rho > -1$;
(iv) $\int_0^t s^\rho L_1(s)ds \sim (\rho + 1)^{-1} t^{1+\rho} L_1(t), \ t \to 0^+$, for $\rho > -1$.

5 Auxiliary Results

In this section, we collect some useful results, which are necessary in the proofs of our results.

Lemma 5.1 [23, Lemma 2.1]. Let $g \in C^1(0, +\infty)$ be positive on $(0, +\infty)$, $g'(t) \leq 0, \forall t \in (0, +\infty)$ and $\lim_{t \to 0^+} g(t) = +\infty$, then

$$\lim_{t \to 0^+} \frac{\int_t^1 g(s)ds}{g(t)} = 0.$$  

Lemma 5.2. Let $f$ satisfy $(S_1)$ (or $(S_{01})$) and $(f_1)$, then
(i) if $(f_2)$ holds, then $C_f^{+\infty} \geq 1$ and \(\lim_{t \to +\infty} \frac{(f(t))^{1/k}}{t} \int_t^{+\infty} (f(s))^{-1/k} ds = C_f^{+\infty} - 1\);
(ii) $(f_2)$ holds with $C_f^{+\infty} > 1$ if and only if $f \in NRV_\gamma$ with $\gamma = \frac{kC_f^{+\infty}}{C_f^{+\infty} - 1}$;
(iii) if $(f_2)$ holds with $C_f^{+\infty} = 1$, then for any $\gamma > 0$, $\lim_{t \to +\infty} \frac{f(t)}{t^\gamma} = +\infty$.

Proof. (i) We see

$$I(t) = ((f(t))^{1/k})' \int_t^{+\infty} (f(s))^{-1/k} ds, \ t > 0.$$  

Integrating $I$ from $a > 0$ to $t > a$ and integration by parts, we obtain that

$$\int_a^t I(s)ds = \int_a^t \left( ((f(s))^{1/k})' \int_s^{+\infty} (f(\tau))^{-1/k} d\tau \right) ds$$  

$$= (f(t))^{1/k} \int_t^{+\infty} (f(s))^{-1/k} ds - (f(a))^{1/k} \int_a^{+\infty} (f(s))^{-1/k} ds + t - a.$$  

It follows by the L’Hospital’s rule that

$$\lim_{t \to +\infty} \frac{(f(t))^{1/k}}{t} \int_t^{+\infty} (f(s))^{-1/k} ds = \lim_{t \to +\infty} I(t) - 1 = C_f^{+\infty} - 1 \geq 0.$$
So, we obtain (i) holds.

(ii) Necessity. A straightforward calculation shows that

\[
\lim_{t \to +\infty} \frac{f'(t)}{f(t)} = \lim_{t \to +\infty} \frac{k((f(t))^{1/k})' \int_t^{+\infty} (f(s))^{-1/k} ds}{(f(t))^{1/k} \int_t^{+\infty} (f(s))^{-1/k} ds} = \frac{kC_f^+}{C_f^+ - 1}.
\] (5.1)

It follows by Proposition 4.5 that \( f \in NRV_{\gamma} \) with \( \gamma = \frac{kC_f^+}{C_f^+ - 1} \).

Sufficiency. By Proposition 4.2, we see that there exist \( a_0 > 0 \) and \( L_{+\infty} \in NRV_0 \cap C^1[a_0, +\infty) \) such that \( f(t) = t^3 L_{+\infty}(t), t \in [a_0, +\infty) \). By Proposition 4.7 (i) and a straightforward calculation, we obtain

\[
\lim_{t \to +\infty} (t^{\gamma/k}(L_{+\infty}(t))^{1/k})' \int_t^{+\infty} s^{-\gamma/k}(L_{+\infty}(s))^{-1/k} ds = \frac{\gamma}{\gamma - k} = C_f^+.
\]

(iii) It follows by the similar calculation as (5.1) that \( \lim_{t \to +\infty} \frac{f'(t)}{f(t)} = +\infty \).

Therefore, for an arbitrary \( \gamma > 0 \), there exists \( t_0 > 0 \) such that \( \frac{f(t)}{t^{\gamma}} > (\gamma + 1)t^{-1}, t \in [t_0, +\infty) \). Integrating it from \( t_0 \) to \( t > t_0 \), we obtain \( \frac{f(t)}{t^{\gamma}} > \frac{f(t_0)}{t_0^{\gamma}} t, t > t_0 \). Letting \( t \to +\infty \), we obtain (iii) holds.

\[ \square \]

**Lemma 5.3.** Let \( f \) satisfy (S1) and (f1), then

(i) if (f3) holds, then \( C_f^0 \geq 1 \) and \( \lim_{t \to 0^+} \frac{(f(t))^{1/k}}{t} \int_t^{+\infty} (f(s))^{-1/k} ds = C_f^0 - 1 \);

(ii) (f3) holds with \( C_f^0 > 1 \) if and only if \( f \in NRVZ_{\gamma} \) with \( \gamma = \frac{kC_f^0}{C_f^0 - 1} \);

(iii) if (f3) holds with \( C_f^0 = 1 \), then for any \( \gamma > 0 \), \( \lim_{t \to 0^+} \frac{f(t)}{t^{\gamma}} = 0 \).

**Proof.** (i) By Lemma 5.1, we see that \( \lim_{t \to 0^+} (f(t))^{1/k} \int_t^{+\infty} (f(s))^{-1/k} ds = 0 \). Integrating \( I \) from 0 to \( t > 0 \) and integration by parts, we obtain that

\[
\int_0^t I(s) ds = \int_0^t \left( ((f(s))^{1/k})' \int_s^{+\infty} (f(\tau))^{-1/k} d\tau \right) ds
\]

\[
= (f(t))^{1/k} \int_t^{+\infty} (f(s))^{-1/k} ds + t.
\]

It follows by the L’Hospital’s rule that

\[
0 \leq \lim_{t \to 0^+} (f(t))^{1/k} \int_t^{+\infty} (f(s))^{-1/k} ds = \lim_{t \to 0^+} I(t) - 1 = C_f^0 - 1.
\]

So, we obtain (i) holds.
(ii) Necessity. A straightforward calculation shows that

\[
\lim_{t \to 0^+} \frac{tf'(t)}{f(t)} = \lim_{t \to 0^+} \frac{k ((f(t))^{1/k})' \int_{t}^{+\infty} (f(s))^{-1/k} ds}{(f(t))^{1/k} \int_{t}^{+\infty} (f(s))^{-1/k} ds} = \frac{kC_f^0}{C_f^0 - 1}.
\]

(5.2)

It follows by Proposition 4.5 that \( f \in NRVZ_\gamma \) with \( \gamma = \frac{kC_f^0}{C_f^0 - 1} \).

Sufficiency. By Proposition 4.3, we see that there exist \( a_0 > 0 \) and \( L_0 \in NRVZ_0 \cap C^1(0,a_0] \) such that \( f(t) = t^\gamma L_0(t), t \in (0,a_0] \).

It follows by Proposition 4.5 and Proposition 4.6(i) that

\[
\lim_{t \to 0^+} \frac{(f(t))^{1/k} f'(t)}{kf(t)} \int_{a_0}^{+\infty} (f(s))^{-1/k} ds = 0.
\]

This combined with Proposition 4.5 and Proposition 4.7(ii) implies that

\[
\lim_{t \to 0^+} \frac{(f(t))^{1/k} \int_{t}^{+\infty} (f(s))^{-1/k} ds}{(f(t))^{1/k} \int_{t}^{+\infty} (f(s))^{-1/k} ds}
\]

\[
= \lim_{t \to 0^+} \frac{(f(t))^{1/k} f'(t)}{kf(t)} \left( \int_{a_0}^{\infty} (f(s))^{-1/k} ds + \int_{t}^{a_0} (f(s))^{-1/k} ds \right)
\]

\[
= \lim_{t \to 0^+} \frac{1}{k} \frac{(f(t))^{1/k} f'(t)t}{f(t)} \int_{t}^{a_0} (f(s))^{-1/k} ds = \frac{\gamma}{\gamma - k} = C_f^0.
\]

(iii) It follows by the similar calculation as (5.2) that \( \lim_{t \to 0^+} \frac{tf'(t)}{f(t)} = +\infty \).

Therefore, for any \( \gamma > 0 \), there exists \( t_0 > 0 \) such that \( \frac{f(t)}{f(t_0)} > (\gamma + 1)t^{-1}, t \in (0,t_0] \). Integrating it from \( t > 0 \) to \( t_0 \), we obtain \( f(t)t^{-\gamma} < \frac{f(t_0)t}{t_0 + 1}, t \in (0,t_0] \).

Letting \( t \to 0 \), we obtain (iii) holds. \( \square \)

**Lemma 5.4.** Let \( f \) satisfy \((S_0)\) and \((f_1)\), then

(i) if \((f_4)\) holds, then \( C_f^{-\infty} \leq 1 \) and \( \lim_{t \to -\infty} \frac{(f(t))^{1/k}}{t} \int_{t}^{+\infty} (f(s))^{-1/k} ds = C_f^{-\infty} - 1; \)

(ii) \((f_4)\) holds with \( C_f^{-\infty} < 1 \) if and only if \( f \) is normalized regularly varying at negative infinity with index \( \gamma \), where \( \gamma = \frac{kC_f^{-\infty}}{C_f^{-\infty} - 1}; \)

(iii) if \((f_4)\) holds with \( C_f^{-\infty} = 1 \), then for any \( \gamma > 0 \), \( \lim_{t \to -\infty} \frac{f(t)}{(-t)^{-\gamma}} = 0. \)

**Proof.** (i) Take \( a \in \mathbb{R} \). Integrating \( I \) from \( t \) to \( a > t \) and integration by parts, we obtain

\[
\int_{t}^{a} I(s) ds = (f(a))^{1/k} \int_{a}^{+\infty} (f(s))^{-1/k} ds - (f(t))^{1/k} \int_{t}^{+\infty} (f(s))^{-1/k} ds + a - t.
\]
It follows by the L'Hospital's rule that
\[
0 \geq \lim_{t \to -\infty} \frac{(f(t))^{1/k}}{t} \int_t^{+\infty} (f(s))^{-1/k} ds = -1 + \lim_{t \to -\infty} I(t) = -1 + C_f^{-\infty}.
\]
So, we obtain (i) holds.

(ii) Necessity. A straightforward calculation shows that
\[
\lim_{t \to -\infty} \frac{tf'(t)}{f(t)} = \frac{k((f(t))^{1/k})' \int_t^{+\infty} (f(s))^{-1/k} ds}{(f(t))^{1/k} \int_t^{+\infty} (f(s))^{-1/k} ds} = \frac{kC_f^{-\infty}}{C_f^{-\infty} - 1}.
\] (5.3)

It follows by Proposition 4.5 that \( f \) is normalized regularly varying at negative infinity with index \( \gamma \), where \( \gamma = \frac{kC_f^{-\infty}}{C_f^{-\infty} - 1} \).

Sufficiency. By Proposition 4.4, we see that there exist \( a_0 < 0 \) and a slowly varying function at negative infinity \( L_\infty \in C^1(-\infty, a_0] \) such that \( f(t) = (-t)^\gamma L_\infty(t), t \in (-\infty, a_0] \). It follows by Proposition 4.5 and Proposition 4.6 (ii) that
\[
\lim_{t \to -\infty} \frac{(f(t))^{1/k} f'(t)}{kf(t)} \int_{a_0}^{+\infty} (f(s))^{-1/k} ds = 0.
\]
This combined with Proposition 4.5 and Proposition 4.7 (iii) implies that
\[
\lim_{t \to -\infty} \frac{(f(t))^{1/k} \int_t^{+\infty} (f(s))^{-1/k} ds}{(f(t))^{1/k} \int_t^{+\infty} (f(s))^{-1/k} ds} = \frac{\gamma}{\gamma - k} = C_f^{-\infty}.
\]

(iii) It follows by the similar calculation as (5.3) that \( \lim_{t \to -\infty} \frac{tf'(t)}{f(t)} = -\infty \).

Therefore, for any \( \gamma > 0 \), there exists \( t_0 < 0 \) such that \( \frac{f'(t)}{f(t)} > -(\gamma + 1) t^{-1} \), \( t \in (-\infty, t_0] \). Integrating it from \( t \) to \( t_0 \), we obtain \( (-t)^\gamma f(t) < f(t_0)(-t_0)^\gamma + 1(-t)^{-1}, t \in (-\infty, t_0] \). Letting \( t \to -\infty \), we obtain (iii) holds.

**Lemma 5.5.** Let \( f \) satisfy (S\(_1\)) (or (S\(_{01}\))) and (f\(_1\)), \( \psi \) be uniquely determined by (2.1), then
(i) \( \psi'(t) = -(f(\psi(t)))^{1/k} \) and \( \psi''(t) = \frac{1}{k}(f(\psi(t))) \frac{2-k}{k} f'(\psi(t)), t > 0; \)
(ii) if (f\(_2\)) holds, then \( \lim_{t \to 0^+} \frac{t \psi'(t)}{\psi(t)} = 1 - C_f^{+\infty} \) and \( \lim_{t \to 0^+} \frac{t \psi''(t)}{\psi(t)} = -C_f^{+\infty}; \)
(iii) if (S\(_1\)) and (f\(_3\)) hold, then \( \lim_{t \to +\infty} \frac{t \psi'(t)}{\psi(t)} = 1 - C_f^0, \lim_{t \to +\infty} \frac{t \psi''(t)}{\psi(t)} = -C_f^0; \)
(iv) if (S\(_{01}\)) and (f\(_4\)) hold, then \( \lim_{t \to +\infty} \frac{t \psi'(t)}{\psi(t)} = 1 - C_f^{-\infty}, \lim_{t \to +\infty} \frac{t \psi''(t)}{\psi(t)} = -C_f^{-\infty}. \)
Proof. (i) By a direct calculation, we obtain (i) holds.

(ii) By (i) and Lemma 5.2 (i), we have

$$\lim_{t \to 0^+} \frac{t \psi''(t)}{\psi(t)} = - \lim_{t \to 0^+} \frac{1}{t} \int_{\psi(t)}^{+\infty} (f(s))^{-1/k} ds$$

Moreover, by (i) and \((f_2)\), we have

$$\lim_{t \to 0^+} \frac{t \psi'''(t)}{\psi'(t)} = - \lim_{t \to 0^+} \frac{1}{k} \frac{1}{(f\psi(t))^{1/k}} f'(\psi(t)) \int_{\psi(t)}^{+\infty} (f(s))^{-1/k} ds$$

(iii) By (i) and Lemma 5.3 (i), we have

$$\lim_{t \to +\infty} \frac{t \psi'(t)}{\psi(t)} = - \lim_{t \to +\infty} \frac{1}{k} \frac{1}{(f\psi(t))^{1/k}} f'(\psi(t)) \int_{\psi(t)}^{+\infty} (f(s))^{-1/k} ds$$

Moreover, by (i) and \((f_3)\), we have

$$\lim_{t \to +\infty} \frac{t \psi''(t)}{\psi'(t)} = - \lim_{t \to +\infty} \frac{1}{k} \frac{1}{(f\psi(t))^{1/k}} f'(\psi(t)) \int_{\psi(t)}^{+\infty} (f(s))^{-1/k} ds$$

(iv) By (i) and Lemma 5.4 (i), we have

$$\lim_{t \to +\infty} \frac{t \psi'(t)}{\psi(t)} = - \lim_{t \to +\infty} \frac{1}{k} \frac{1}{(f\psi(t))^{1/k}} f'(\psi(t)) \int_{\psi(t)}^{+\infty} (f(s))^{-1/k} ds$$

Moreover, by (i) and \((f_4)\), we have

$$\lim_{t \to +\infty} \frac{t \psi''(t)}{\psi'(t)} = - \lim_{t \to +\infty} \frac{1}{k} \frac{1}{(f\psi(t))^{1/k}} f'(\psi(t)) \int_{\psi(t)}^{+\infty} (f(s))^{-1/k} ds$$
Lemma 5.6. Let $f$ satisfy $(S_1)$ (or $(S_{01})$ with (1.7)) and $(f_5)$, $F$ be defined by (2.31), then

(i) $\lim_{t \to +\infty} \frac{(F(t))^{1/k}}{t} = +\infty$;

(ii) if $f \in RV_k$, then

$$\lim_{t \to +\infty} \frac{(F(t))^{1/(k+1)}}{t} \int_t^{+\infty} (F(s))^{-1/(k+1)} ds = +\infty$$

and

$$\lim_{t \to +\infty} \left( (F(t))^{1/(k+1)} \right)^{\prime} \int_t^{+\infty} (F(s))^{-1/(k+1)} ds = +\infty.$$

Proof. (i) If (i) is false, then there exist constant $c_*$ and an increasing sequence of real numbers $\{s_i\}_{i=1}^{+\infty}$ satisfying $\lim_{i \to +\infty} s_i = +\infty$ and $2s_{i-1} \leq s_i$, $i = 1, 2, \cdots$ such that $(F(s_i))^{-1/k} \geq 1/(s_i c_*)$. A direct calculation shows that

$$+\infty > \int_{s_0}^{+\infty} (F(s))^{-1/k} ds \geq \sum_{i=1}^{+\infty} \int_{s_{i-1}}^{s_i} (F(s))^{-1/k} ds \geq \sum_{i=1}^{+\infty} \int_{s_{i-1}}^{s_i} \frac{1}{s_i c_*} ds = +\infty.$$

This is a contradiction. So, (i) holds.

(ii) Since $\lim_{t \to +\infty} \int_0^t \frac{f(s) ds}{tf(t)} = 0$, by the Lebesgue’s dominated convergence theorem, we obtain

$$\lim_{t \to +\infty} \frac{F(t)}{tf(t)} = \lim_{t \to +\infty} \left( \int_0^t \frac{f(s) ds}{tf(t)} + \frac{\int_0^t f(s) ds}{tf(t)} \right)$$

$$= \lim_{t \to +\infty} \int_0^t \frac{f(s) ds}{tf(t)} = \lim_{t \to +\infty} \int_0^1 \frac{f(t \tau)}{f(t)} d\tau = \int_0^1 \tau^{k-1} d\tau = \frac{1}{k+1}. \quad (5.4)$$

It follows by using the L’Hospital’s rule that

$$\lim_{t \to +\infty} \frac{(F(t))^{1/(k+1)}}{t} \int_t^{+\infty} (F(s))^{-1/(k+1)} ds$$

$$= \lim_{t \to +\infty} \frac{\int_t^{+\infty} (F(s))^{-1/(k+1)} ds}{t(F(t))^{-1/(k+1)}} = \lim_{t \to +\infty} \left( \frac{1}{k+1} \frac{tf(t)}{F(t)} - 1 \right)^{-1} = +\infty. \quad (5.5)$$

Combining (5.4) and (5.5), we have

$$\lim_{t \to +\infty} \left( (F(t))^{1/(k+1)} \right)^{\prime} \int_t^{+\infty} (F(s))^{-1/(k+1)} ds = +\infty.$$

\(\square\)

Lemma 5.7. Let $f$ satisfy $(S_1)$ and $(f_5)$, $F$ be defined by (2.31). Then

(i) if $(f_7)$ holds, then $E^0_f \geq 1$ and $\lim_{t \to 0^+} \frac{(F(t))^{1/(k+1)}}{t} \int_t^{+\infty} (F(s))^{-1/(k+1)} ds =$

$$= \lim_{t \to +\infty} \frac{(F(t))^{1/(k+1)}}{t} \int_t^{+\infty} (F(s))^{-1/(k+1)} ds = +\infty.$$
$E_f^0 - 1$;

(ii) $(f_7)$ holds with $E_f^0 > 1$ if and only if $f \in RVZ_\gamma$ with $\gamma = \frac{kE_f^0 + 1}{E_f^0 - 1}$;

(iii) if $(f_7)$ holds with $E_f^0 = 1$, then for any $\gamma > 0$, $\lim_{t \to 0^+} \frac{F(t)}{t^\gamma} = 0$.

**Proof.** (i) We see

$$J(t) = \left((F(t))^{1/(k+1)}\right)' \int_t^{+\infty} (F(s))^{-1/(k+1)}ds, \quad t > 0.$$  

By Lemma 5.1, we see that $\lim_{t \to 0^+} (F(t))^{1/(k+1)} \int_t^{+\infty} (F(s))^{-1/(k+1)}ds = 0$. Integrating $J$ from 0 to $t > 0$ and integration by parts, we obtain

$$\int_0^t J(s)ds = (F(t))^{1/(k+1)} \int_t^{+\infty} (F(s))^{-1/(k+1)}ds + t, \quad t > 0.$$  

It follows by the L’Hospital’s rule that

$$0 \leq \lim_{t \to 0^+} \frac{(F(t))^{1/(k+1)}}{t} \int_t^{+\infty} (F(s))^{-1/(k+1)}ds = \lim_{t \to 0^+} J(t) - 1 = E_f^0 - 1.$$  

So, we obtain (i) holds.

(ii) Necessity. A straightforward calculation shows that

$$\lim_{t \to 0^+} \frac{F(t)}{tf(t)} = \lim_{t \to 0^+} \frac{1}{k+1} \frac{(F(t))^{1/(k+1)} \int_t^{+\infty} (F(s))^{-1/(k+1)}ds}{tJ(t)} = \frac{E_f^0 - 1}{(k+1)E_f^0}.$$  

(5.6)

It follows by Proposition 4.5 that $F \in NRVZ_{(k+1)E_f^0/(E_f^0 - 1)}$. This implies that there exist $a_1 > 0$ and $\hat{L}_0 \in NRVZ_0 \cap C^1(0,a_1]$ such that $F(t) = t^{(k+1)E_f^0/(E_f^0 - 1)}\hat{L}_0(t)$, $t \in (0,a_1]$. By a simple calculation, we obtain

$$f(t) = t^\gamma \left(\frac{(k+1)E_f^0}{E_f^0 - 1} + \frac{t\hat{L}'_0(t)}{\hat{L}_0(t)}\right)\hat{L}_0(t) \quad \text{with} \quad \gamma = \frac{kE_f^0 + 1}{E_f^0 - 1}.$$  

It follows by Proposition 4.3 that $f \in RVZ_\gamma$.

**Sufficiency.** Since $f \in RVZ_\gamma$, by the Lebesgue’s dominated convergence theorem, we obtain

$$\lim_{t \to 0^+} \frac{F(t)}{tf(t)} = \lim_{t \to 0^+} \frac{\int_0^t f(s)ds}{tf(t)} = \lim_{t \to 0^+} \int_0^1 \frac{f(t\tau)}{f(t)}d\tau = \int_0^1 \tau^\gamma d\tau = \frac{1}{\gamma + 1}.$$  

So, $F \in NRVZ_{\gamma + 1}$. By Proposition 4.3, we see that there exist $a_1 > 0$ and $\hat{L}_0 \in C^1(0,a_1] \cap NRVZ_0$ such that $F(t) = t^{\gamma + 1}\hat{L}_0(t)$, $t \in (0,a_1]$. It follows by Proposition 4.5 and Proposition 4.6 (i) that

$$\lim_{t \to 0^+} \frac{(F(t))^{1/(k+1)}f(t)}{(k+1)F(t)} \int_{a_1}^{+\infty} (F(s))^{-1/(k+1)}ds = 0.$$  

This combined with Proposition 4.5 and Proposition 4.7 (ii) implies that
\[
\lim_{t \to 0^+} ((F(t))^{1/(k+1)})' \int_t^{+\infty} (F(s))^{-1/(k+1)} ds \\
= \lim_{t \to 0^+} \frac{(F(t))^{1/(k+1)}}{(k+1)F(t)} \left( \int_t^{+\infty} (F(s))^{-1/(k+1)} ds + \int_{t}^{a_1} (F(s))^{-1/(k+1)} ds \right) \\
= \lim_{t \to 0^+} \frac{(F(t))^{1/(k+1)}}{(k+1)t} \int_t^{a_1} (F(s))^{-1/(k+1)} ds = \frac{\gamma + 1}{\gamma - k} = E_f^0.
\]

(iii) By the similar argument as Lemma 5.3 (iii), we obtain (iii) holds.

\[\square\]

**Lemma 5.8.** Let \( f \) satisfy (S\(_{01}\)) with (1.7) and (f\(_5\)), \( F \) be defined by (2.31), then
(i) if (f\(_8\)) holds, then \( E_f^{-\infty} \leq 1, \lim_{t \to -\infty} \frac{(F(t))^{1/(k+1)}}{t} \int_t^{+\infty} (F(s))^{-1/(k+1)} ds = E_f^{-\infty} - 1; \)
(ii) (f\(_8\)) holds with \( E_f^{-\infty} < 1 \) if and only if \( F \) is normalized regularly varying at negative infinity with index \( \gamma \), where \( \gamma = \frac{E_f^{-\infty}(k+1)}{E_f^{-\infty} - 1} \); 
(iii) if (f\(_8\)) holds with \( E_f^{-\infty} = 1 \), then for any \( \gamma > 0 \), \( \lim_{t \to -\infty} \frac{F(t)}{-t^{\gamma - k}} = 0. \)

**Proof.** Take \( a \in \mathbb{R} \). Integrating \( J \) from \( t \) to \( a > t \) and integration by parts, we obtain
\[
\int_t^a J(s) ds = \int_t^a \left( ((F(s))^{1/(k+1)})' \int_s^{+\infty} (F(\tau))^{-1/(k+1)} d\tau \right) ds \\
= (F(a))^{1/(k+1)} \int_a^{+\infty} (F(s))^{-1/(k+1)} ds \\
- (F(t))^{1/(k+1)} \int_t^{+\infty} (F(s))^{-1/(k+1)} ds + a - t.
\]
It follows by the L’Hospital’s rule that
\[
0 \geq \lim_{t \to -\infty} \frac{(F(t))^{1/(k+1)}}{t} \int_t^{+\infty} (F(s))^{-1/(k+1)} ds = -1 + \lim_{t \to -\infty} J(t) = E_f^{-\infty} - 1.
\]

(ii) Necessity. By the similar calculation as (5.6), we obtain the necessity holds.

Sufficiency. Since \( F \) is normalized regularly varying at negative infinity with index \( \gamma = \frac{E_f^{-\infty}(k+1)}{E_f^{-\infty} - 1} \), by Proposition 4.4, we see that there exist \( a_1 < 0 \) and a normalized slowly varying function at negative infinity \( \hat{L}_{-\infty} \in C^1(-\infty, a_1] \) such that \( F(t) = (-t)^{\gamma} \hat{L}_{-\infty}(t), t \in (-\infty, a_1]. \) The rest of the proof is similar to the proof of Lemma 5.7 (ii), so we omit it here.

(iii) By the similar argument as (iii) of Lemma 5.7, we obtain (iii) holds. \[\square\]
Lemma 5.9. Let $f$ satisfy $(S_1)$ (or $(S_{01})$ with (1.7)) and $(f_5)$, $\varphi$ be uniquely determined by (2.30), then

(i) $\varphi'(t) = -((k+1)F(\varphi(t)))^{1/(k+1)}$, $\varphi''(t) = ((k+1)F(\varphi(t)))^{1-k/(k+1)}f(\varphi(t))$ and
\[ (-\varphi'(t))^{k-1}\varphi''(t) = f(\varphi(t)), \quad t > 0; \]

(ii) if $(f_6)$ holds, then $\lim_{t \to 0^+} \frac{\varphi'(t)}{t \varphi''(t)} = 0$;

(iii) if $(f_7)$ holds, then $\lim_{t \to +\infty} \frac{\varphi'(t)}{t \varphi''(t)} = -(E_f^0)^{-1}$;

(iv) if $(f_8)$ holds, then $\lim_{t \to +\infty} \frac{\varphi'(t)}{t \varphi''(t)} = -(E_f^{-\infty})^{-1}$.

Proof. By a direct calculation, we obtain (i) holds.

(ii) By (i) and $(f_6)$, we have
\[ \lim_{t \to 0^+} \frac{\varphi'(t)}{t \varphi''(t)} = - \lim_{t \to +\infty} \left( \frac{((F(t))^{1/(k+1)})'}{t} \int_t^{+\infty} (F(s))^{-1/(k+1)} ds \right)^{-1} = 0. \]

(iii) By (i) and $(f_7)$, we have
\[ \lim_{t \to +\infty} \frac{\varphi'(t)}{t \varphi''(t)} = - \lim_{t \to 0^+} \left( \frac{((F(t))^{1/(k+1)})'}{t} \int_t^{+\infty} (F(s))^{-1/(k+1)} ds \right)^{-1} = -(E_f^0)^{-1}. \]

(iv) By (i) and $(f_8)$, we have
\[ \lim_{t \to +\infty} \frac{\varphi'(t)}{t \varphi''(t)} = - \lim_{t \to +\infty} \left( \frac{((F(t))^{1/(k+1)})'}{t} \int_t^{+\infty} (F(s))^{-1/(k+1)} ds \right)^{-1} = -(E_f^{-\infty})^{-1}. \]

\[ \square \]

Lemma 5.10 [41, Lemma 2.1]. Let $\theta \in \Lambda$, we have

(i) $\lim_{t \to 0^+} \frac{\Theta'(t)}{\Theta(t)} = 0$ and $\lim_{t \to 0^+} \frac{\Theta(t)\Theta'(t)}{\Theta(t)^2} = 1 - D_{\theta}$;

(ii) if $D_{\theta} > 0$, then $\theta \in NRVZ(1-D_{\theta})/D_{\theta}$; if $D_{\theta} = 1$, then $\theta$ is slowly varying at zero; if $D_{\theta} = 0$, then for any $\gamma > 0$, $\lim_{t \to 0^+} \frac{\theta(t)}{t^\gamma} = 0$.

6 Optimal Global Behavior of Large Solutions

In this section, we give the proofs of Theorems 2.1–2.6. Basic to our subsequent discussions are the following two lemmas.

We first introduce a sub-supersolution method of $k$-convex solution to problem (1.1).

Definition 6.1. A function $u \in C^2(\Omega)$ is called subsolution to problem (1.1) if $u$ is $k$-convex in $\Omega$ and satisfies
\[ S_k(D^2u(x)) \geq b(x)f(u(x)), \quad x \in \Omega, \ u|_{\partial \Omega} = +\infty. \]
Definition 6.2. A function \( u \in C^2(\Omega) \) is called supersolution to problem (1.1) if \( \overline{u} \) is \( k \)-convex in \( \Omega \) and satisfies

\[
S_k(D^2 \overline{u}(x)) \leq b(x)f(\overline{u}(x)), \quad x \in \Omega, \quad \overline{u}|_{\partial \Omega} = +\infty.
\]

Lemma 6.1. Let \( b \) satisfy (b_1), \( f \) satisfy (S_1) (or (S_01)) and the functions \( h_1, h_2 \in C^2(0, \max_{x \in \Omega} v(x)] \), where \( v \) is given by (2.2). Assume that \( \overline{u} = h_1(v) \) and \( \underline{u} = h_2(v) \) are positive (or may be sign-changing) classical supersolution and subsolution respectively to problem (1.1) and satisfy \( \underline{u} \leq \overline{u} \) in \( \Omega \), then problem (1.1) has at least one \( k \)-convex solution \( u \in C^\infty(\Omega) \) in the order interval \([\underline{u}, \overline{u}]\).

Proof. Since \( \underline{u} = h_2(v) \) is a subsolution to problem (1.1), we have

\[
S_k(D^2 \underline{u}(x)) \geq b(x)f(\underline{u}(x)), \quad x \in \Omega.
\]

Take \( \varepsilon > 0 \) and let \( w_\varepsilon(x) := u(x) - \varepsilon \), then it is clear that \( w_\varepsilon(x) \) is also a subsolution to problem (1.1). In fact, if (S_1) holds, we can take \( \varepsilon > 0 \) small enough such that \( w_\varepsilon \) is positive in \( \Omega \). So, we have

\[
S_k(D^2 w_\varepsilon(x)) > b(x)f(w_\varepsilon(x)), \quad x \in \Omega. \tag{6.1}
\]

Let \( \{\sigma_n\}_{n=1}^{+\infty} \) be a strictly increasing sequence of positive numbers such that \( \sigma_n \to +\infty \) as \( n \to +\infty \), and let

\[
\Omega_n^\varepsilon := \{x \in \Omega : w_\varepsilon(x) < \sigma_n\} \quad \text{and} \quad \Omega_n := \{x \in \Omega : u(x) < \sigma_n\}.
\]

Since any level surface of \( u \) is a level surface of \( v \), for each \( n \geq 1 \), \( \partial \Omega_n^\varepsilon \) and \( \partial \Omega_n \) are strictly convex \( C^\infty \)-submanifold of \( \mathbb{R}^N \) of dimension \( N - 1 \).

When (S_1) holds, we take \( \varepsilon_0 > 0 \) such that \( w_{\varepsilon_0} \) is positive in \( \Omega \) and define

\[
a_* := \min_{x \in \Omega} w_{\varepsilon_0}(x).
\]

For this case, we extend \( f \) from \( f \in C^\infty[a_*, +\infty) \) to \( \tilde{f} \in C^\infty(\mathbb{R}) \), where \( \tilde{f} \) is increasing on \( \mathbb{R} \) and \( \tilde{f} = f \) on \( [a_*, +\infty) \).

Next, we still denote \( \tilde{f} \) by \( f \) for convenience.

Taking \( \varepsilon < \varepsilon_0 \), it is clear that \( w_\varepsilon > w_{\varepsilon_0} \) in \( \Omega \). We see from (6.1) that

\[
S_k(D^2 w_\varepsilon(x)) > b(x)f(w_\varepsilon(x)), \quad x \in \Omega_n^\varepsilon.
\]

By Lemma 3.3, there exists a \( k \)-convex function \( u_n^\varepsilon \) for \( n \geq 1 \) such that

\[
S_k(D^2 u_n^\varepsilon(x)) = b(x)f(u_n^\varepsilon(x)), \quad x \in \Omega_n^\varepsilon, \quad u_n^\varepsilon|_{\partial \Omega_n^\varepsilon} = \sigma_n. \tag{6.2}
\]

We conclude from Lemma 3.1 that

\[
w_\varepsilon \leq u_n^\varepsilon \quad \text{in} \quad \Omega_n^\varepsilon, \quad w_\varepsilon \leq u_{n+1}^\varepsilon \quad \text{in} \quad \Omega_{n+1}^\varepsilon \tag{6.3}
\]
and
\[ u_n^\varepsilon \leq \bar{u} \quad \text{in } \Omega_n^\varepsilon. \]

Moreover, by the definitions of \( \Omega_n^\varepsilon \) and \( \Omega_n \), we see that for any \( \varepsilon > 0 \) the following hold
\[
\bar{\Omega}_n^\varepsilon \subset \Omega_{n+1}^\varepsilon, \quad \bar{\Omega}_n \subset \Omega_{n+1}, \quad \Omega_n \subset \Omega_n^\varepsilon \quad \text{and} \quad \bigcup_{n=1}^{\infty} \Omega_n^\varepsilon = \bigcup_{n=1}^{\infty} \Omega_n = \Omega. \quad (6.4)
\]

Combining (6.3) and (6.4), we obtain \( u_{n+1}^\varepsilon \geq w_\varepsilon = u_n^\varepsilon \) on \( \partial \Omega_n^\varepsilon \). It follows from Lemma 3.1 that
\[
u_{n+1}^\varepsilon \leq u_n^\varepsilon \leq u_{n+1}^\varepsilon \leq \bar{u} \quad \text{in } \Omega_n. \quad (6.5)
\]

Let \( \{\varepsilon_n\}_{n=1}^{+\infty} \) with \( \varepsilon_1 < \varepsilon_0 \) be a strictly decreasing sequence of positive numbers such that \( \varepsilon_n \to 0 \) as \( n \to +\infty \). By (6.5), we have for fixed \( \varepsilon_i \),
\[
w_{\varepsilon_i} \leq u_n^\varepsilon_i \leq u_{n+1}^\varepsilon_i \leq \bar{u} \quad \text{in } \Omega_n. \quad (6.6)
\]

This implies that for every \( x \in \Omega \),
\[
u_i(x) := \lim_{n \to +\infty} u_n^\varepsilon_i(x) \text{ exists}
\]
and
\[
\min_{x \in \Omega_n} w_{\varepsilon_1}(x) \leq \nu_i(x) \leq \max_{x \in \Omega_n} \bar{u}(x), \quad x \in \bar{\Omega}_n.
\]

Fix an integer \( m \). For any \( n > m \), we have
\[
\bar{\Omega}_m \subset \Omega_n
\]
and
\[
\text{dist}(\bar{\Omega}_m, \partial \Omega_{m+1}) \leq \text{dist}(\bar{\Omega}_m, \partial \Omega_n) \leq \text{dist}(\Omega_m, \partial \Omega). \]

Since \( u_n^\varepsilon_i \) is a \( k \)-convex solution to (6.2), by Lemma 3.2, we obtain that for fixed integer \( j \geq 3 \), there exists a positive constant \( C_{j,m} \) (corresponding to \( j \) and \( m \)) independent of \( n \) such that for any \( n \geq m \), it holds
\[
\|u_n^\varepsilon_i\|_{C^j(\bar{\Omega}_m)} \leq C_{j,m}.
\]

By Arzelà–Ascoli’s theorem, we can take a subsequence of \( \{u_n^\varepsilon_i\}_{n=1}^{+\infty} \), still denoted by itself, such that \( u_n^\varepsilon_i \to \nu_i \) in \( C^{j-1}(\bar{\Omega}_m) \). Hence, for any \( x \in \bar{\Omega}_m \), the following holds
\[
S_k(D^2 u_i(x)) = \lim_{n \to \infty} S_k(D^2 u_n^\varepsilon_i(x)) = b(x) \lim_{n \to \infty} f(u_n^\varepsilon_i(x)) = b(x) f(u_i(x)).
\]
Since $j, m$ are arbitrary, $u_{n}^{\varepsilon_{i}}$ is a $k$-convex solution of (6.2) and
\[
\lim_{d(x) \to 0} w_{\varepsilon_{1}}(x) = +\infty,
\]
we obtain that $u^{i} \in C^{\infty}(\Omega)$ is a $k$-convex solution to problem (1.1).

On the other hand, we note that
\[
u_{i, n}(x) \leq u_{i, n}^{\varepsilon_{i} + 1}(x) \leq \overline{u}(x), \quad x \in \Omega_{n}.
\]
This combined with (6.6) shows that
\[
u_{\varepsilon_{i}}(x) \leq u^{i}(x) \leq u^{i + 1}(x) \leq \overline{u}(x), \quad x \in \Omega.
\]
So, we have
\[
u(x) := \lim_{i \to +\infty} u^{i}(x) \text{ exists and } \nu_{\varepsilon_{i}}(x) \leq u(x) \leq \overline{u}(x), \quad x \in \Omega.
\]
Passing to $i \to +\infty$, we obtain $\nu(x) \leq u(x) \leq \overline{u}(x), \quad x \in \Omega$. By the same argument as the above, we see that $u \in C^{\infty}(\Omega)$ is $k$-convex and satisfies (1.1).

**Lemma 6.2.** Let $\mathcal{I}$ be an arbitrary interval and $h(x, t)$ be a continuous function on $\overline{\Omega} \times \mathcal{I}$, then
\[
t \mapsto \max_{x \in \overline{\Omega}} h(x, t) \quad \text{and} \quad t \mapsto \min_{x \in \overline{\Omega}} h(x, t)
\]
are continuous on $\mathcal{I}$.

**Proof.** \(\forall t_{0} \in \mathcal{I}\), we show that $t \mapsto \max_{x \in \overline{\Omega}} h(x, t)$ is continuous at $t_0$. Otherwise, there exist constant $\varepsilon_{0} > 0$ and a sequence of numbers $\{t_{n}\}_{n=1}^{+\infty}$ satisfying $t_{n} \to t_{0}$ as $n \to +\infty$ such that
\[
\max_{x \in \Omega} h(x, t_{n}) - \max_{x \in \Omega} h(x, t_{0}) > \varepsilon_{0}. \quad (6.7)
\]
Since $h(x, t_{0})$ is continuous in $\overline{\Omega}$, we can take $x_{0} \in \overline{\Omega}$ such that
\[
\max_{x \in \overline{\Omega}} h(x, t_{0}) = h(x_{0}, t_{0}). \quad (6.8)
\]
In the same way, we can take $x_{n} \in \overline{\Omega}$ such that
\[
\max_{x \in \overline{\Omega}} h(x, t_{n}) = h(x_{n}, t_{n}). \quad (6.9)
\]
It follows by (6.7)–(6.9) that
\[
|h(x_{n}, t_{n}) - h(x_{0}, t_{0})| > \varepsilon_{0}. \quad (6.10)
\]
Since $\bar{\Omega}$ is a bounded closed domain in $\mathbb{R}^N$, $\{x_n\}_{n=1}^{+\infty}$ has a convergent subsequence. For convenience, we still denote the subsequence by $\{x_n\}_{n=1}^{+\infty}$. So, there exists $x_* \in \bar{\Omega}$ such that $\lim_{n \to +\infty} (x_n, t_n) \to (x_*, t_0)$. This together with (6.10) implies that
$$
|h(x_*, t_0) - h(x_0, t_0)| \geq \varepsilon_0. \tag{6.11}
$$

Recalling (6.9), we see that $h(x_n, t_n) \geq h(x_0, t_n)$. Letting $n \to +\infty$, we obtain $h(x_*, t_0) \geq h(x_0, t_0)$. This together with (6.11) implies that $\max_{x \in \bar{\Omega}} h(x, t_0) < h(x_*, t_0)$. This is a contradiction with (6.8).

By similar arguments as the above, we can obtain $t \mapsto \min_{x \in \bar{\Omega}} h(x, t)$ is continuous on $\mathcal{X}$.

### 6.1 Proof of Theorem 2.1

**Proof.** The definition of $v$ implies that $(v_{x_i x_j})$ is negative definite in $\bar{\Omega}$. So, each principal submatrix of size $k$ of $(v_{x_i x_j})$ is also negative definite. It follows that there exist positive constants $e_1$ and $e_2$ ($e_1 < e_2$) such that
$$
eq e_1 \|\nabla v_i(x)\|^2 \leq (-\nabla v_i(x))^T B(v_i(x)) \nabla v_i(x) \leq e_2 \|\nabla v_i(x)\|^2 \tag{6.12}
$$
and
$$
\Delta v_i = \sum_{j=1}^{k} v_{x_i x_j}(x) < 0, \quad x \in \bar{\Omega}.
$$

From Hopf’s maximum principle, there exist positive constants $e$ and $\delta_1$ such that
$$
\|\nabla v_i\|^2 > e \quad \text{in} \quad \Omega_{\delta_1}, \tag{6.13}
$$
where $\Omega_{\delta_1}$ is defined as shown in (3.1). Combining (6.12)–(6.13), we obtain that $\mathcal{A}$ is nonnegative in $\bar{\Omega}$ and positive in $\Omega_{\delta_1}$. So, we have
$$
\omega_0 > 0 \quad \text{in} \quad \bar{\Omega},
$$
where $\omega_0$ is given by (2.4).

Let $u_\lambda(x) = m_0(v(x))^{-\alpha}, \ x \in \Omega$, where $m_0$ and $\alpha$ are given by (2.3). By $(b_1)$–$(b_2)$ and Lemma 3.4, we obtain
$$
S_k(D^2 u_\lambda(x)) = (m_0 \alpha^k v(x))^{-\alpha} \omega_0(x)
\geq c_0 (m_0 \alpha^k v(x))^{-\alpha - 1} \omega_0(x)
= c_0 \alpha^k m_0^{k-\gamma} (v(x))^\lambda (m_0(v(x))^{-\alpha})^\gamma
\geq b(x) u_\lambda(x), \quad x \in \Omega, \tag{6.14}
$$
i.e., $u_\lambda$ is a subsolution to problem (1.1) in $\bar{\Omega}$. Moreover, by a similar calculation as (6.14), we see that $S_i(D^2 u_\lambda) > 0$ in $\Omega$ for $i = 1, \ldots, N$. This implies that $u_\lambda$ is strictly convex in $\Omega$. 
In a similar way, we can show that $\overline{u}_\lambda = M_0v^{-\alpha}$ is a strictly convex supersolution to problem (1.1) in $\Omega$, where $M$ is given by (2.3). Since $u_\lambda \leq \overline{u}_\lambda$ in $\Omega$, by Theorem 4.2 of [34] and Lemma 6.1, we obtain that problem (1.1) has a unique classical $k$-convex solution $u_\lambda \in C^\infty(\Omega)$ satisfying

$$u_\lambda(x) \leq u_\lambda(x) \leq \overline{u}_\lambda(x), \quad x \in \Omega.$$  

(6.15)

We see by Lemma 6.2 that $c_0$ and $C_0$ are continuous on $[-k-1, +\infty)$, and $c_0(-k-1) = c_1$ and $C_0(-k-1) = C_1$, where $c_1$ and $C_1$ are given by (2.8). Let $\Omega_1$ be an arbitrary compact subset of $\Omega$. Passing to $\lambda \to -k - 1$ and $\lambda \to +\infty$, we obtain that (2.6)–(2.7) hold.

6.2 Proof of Theorem 2.2

Proof. By the similar argument as in the proof of Theorem 2.1, we see that

$$\omega_1 > 0 \quad \text{in } \bar{\Omega},$$

where $\omega_1$ is given by (2.9).

Let $u_\lambda(x) = m_1 - (k + 1 + \lambda)\ln v(x), x \in \Omega$, where $m_1$ is given by (2.10). By (b1)–(b2) and Lemma 3.4, we obtain

$$S_k(D^2u_\lambda(x)) = (k + 1 + \lambda)^k(v(x))^\lambda(v(x))^{-(k+1+\lambda)}\omega_1(x) \geq c_1(k + 1 + \lambda)^k \exp(-m_1)(v(x))^\lambda \exp(u_\lambda(x)) \geq b(x) \exp(u_\lambda(x)), \quad x \in \Omega,$$

(6.16)
i.e., $u_\lambda$ is a subsolution to problem (1.1) in $\Omega$. Moreover, by a similar calculation as (6.16), we see that $u_\lambda$ is strictly convex in $\Omega$.

In a similar way, we can show that $\overline{u}_\lambda = M_1 - (k + 1 + \lambda)\ln v$ is a strictly convex supersolution in $\Omega$, where $M_1$ is given by (2.10). Obviously, $u_\lambda \leq \overline{u}_\lambda$ in $\Omega$. So, by Lemma 6.1, we obtain that problem (1.1) has a classical $k$-convex solution $u_\lambda \in C^\infty(\Omega)$ satisfying (6.15). By a direct calculation, we see that (2.11)–(2.12) hold.

6.3 Proof of Theorem 2.3

Proof. If (f1)–(f3) hold, then by Lemma 5.5 (ii)–(iii), we see that

$$0 < h_0 := \inf_{t>0} \Psi(t) \leq \min\{C_f^0, C_f^{+\infty}\} \leq \max\{C_f^0, C_f^{+\infty}\} \leq \sup_{t>0} \Psi(t) =: H_0 < +\infty.$$  

(6.17)

If (f1)–(f2) and (f4) hold, then by Lemma 5.5 (ii) and (iv), we obtain

$$0 < h_0 := \inf_{t>0} \Psi(t) \leq C_f^{-\infty} \leq 1 \leq C_f^{+\infty} \leq \sup_{t>0} \Psi(t) =: H_0 < +\infty.$$  

(6.18)
Moreover, by a direct calculation, we have

\[ \Psi(t) = \frac{1}{k} f^{\frac{1-k}{k}}(\psi(t)) f'(\psi(t)) t. \]  

(6.19)

Combining (6.17)–(6.19), (2.13) and a similar argument as in the proof of Theorem 2.1, we obtain that there exist positive constants \( \beta_1 \) and \( \beta_2 \) (\( \beta_1 < \beta_2 \)) such that for any \( \tau > 0 \), it holds

\[ \beta_1 < \omega_2(\tau, x) < \beta_2, \quad \forall \ x \in \bar{\Omega}, \]  

(6.20)

where \( \omega_2 \) is given by (2.17).

On the other hand, we see from Lemma 6.2 that \( \tau \mapsto \tau^k \max_{x \in \Omega} \omega_2(\tau, x) \) and \( \tau \mapsto \tau^k \min_{x \in \Omega} \omega_2(\tau, x) \) are continuous functions on \((0, +\infty)\). This together with (6.20) implies that

\[ \tau^k \beta_1 \leq \tau^k \min_{x \in \Omega} \omega_2(\tau, x) \leq \tau^k \max_{x \in \Omega} \omega_2(\tau, x) \leq \tau^k \beta_2. \]

The existence theorem of zero point of continuous function implies that there exist positive constants \( \tau_1 \) and \( \tau_2 \) (\( \tau_1 \leq \tau_2 \)) such that (2.16) holds.

Let \( u_\lambda(x) = \psi(\tau_2 \eta^{-1} (v(x))^{\eta}) \), \( x \in \Omega \), where \( \eta \) is given by (2.15). By \((b_1)-(b_2)\) and Lemma 3.4, we obtain

\[ S_k(D^2 u_\lambda(x)) = \tau_2^k (-\psi'(\tau_2 \eta^{-1} (v(x))^{\eta}))^k (v(x))^{(\eta-1)k-1} \omega_2(\tau_2, x) \]

\[ \geq \tau_2^k \min_{x \in \Omega} \omega_2(\tau_2, x) \cdot ((v(x))^{\lambda} f(\psi(\tau_2 \eta^{-1} (v(x))^{\eta}))) \]

\[ = b_2(v(x))^{\lambda} f(\psi(\tau_2 \eta^{-1} (v(x))^{\eta})) \]

\[ \geq b(x) f(\psi(\tau_2 \eta^{-1} (v(x))^{\eta})), \quad x \in \Omega, \]  

(6.21)

i.e., \( u_\lambda \) is a subsolution to problem (1.1) in \( \Omega \). Moreover, by a similar calculation as (6.21), we see that \( u_\lambda \) is strictly convex in \( \Omega \).

In a similar way, we can show that \( \bar{u}_\lambda = \psi(\tau_1 \eta^{-1} v^{\eta}) \) is a strictly convex supersolution to problem (1.1) in \( \Omega \). Obviously, \( u_\lambda \leq \bar{u}_\lambda \) in \( \Omega \). So, by Lemma 6.1, we obtain that problem (1.1) has a classical \( k \)-convex solution \( u_\lambda \in C^\infty(\Omega) \) satisfying (6.15).

Take \( \eta_* > 0 \) and let

\[ \overline{\omega}(x, \eta) = v(x)(-1)^k S_k(D^2 v(x)) + ((H_0 - 1)\eta + 1) \mathcal{A}(x), \quad (x, \eta) \in \Omega \times [0, \eta_*] \]

and

\[ \omega(x, \eta) = v(x)(-1)^k S_k(D^2 v(x)) + ((h_0 - 1)\eta + 1) \mathcal{A}(x), \quad (x, \eta) \in \bar{\Omega} \times [0, \eta_*]. \]
So, we have
\[
\left( \frac{b_1}{\max_{(x,\eta)\in\Omega \times [0,\eta_*]} \omega(x,\eta)} \right)^{1/k} \leq \tau_1 \leq \tau_2 \leq \left( \frac{b_2}{\min_{(x,\eta)\in\Omega \times [0,\eta_*]} \omega(x,\eta)} \right)^{1/k}.
\]

If (f_3) holds, then by Lemma 5.5 (iii) and Proposition 4.1, we see that
\[
\lim_{\eta \to 0^+} \frac{\psi(\tau_j \eta^{-1}(v(x))^\eta)}{\psi(\eta^{-1})} = \tau_j^{1-C_f^0}, \quad j = 1, 2 \tag{6.22}
\]
uniformly for \( x \in \Omega_1 \). If (f_4) holds, then by Lemma 5.5 (iv) and Proposition 4.1, we see that
\[
\lim_{\eta \to 0^+} \frac{\psi(\tau_j \eta^{-1}(v(x))^\eta)}{\psi(\eta^{-1})} = \tau_j^{1-C_f^{-\infty}}, \quad j = 1, 2 \tag{6.23}
\]
uniformly for \( x \in \Omega_1 \). Thus, (6.22) (or (6.23)) implies that (2.19) (or (2.20)) holds. On the other hand, it follows by
\[
\psi((b_2/c_1)^{1/k} \eta^{-1}) \leq \min_{x \in \Omega_1} \psi(\tau_2 \eta^{-1}(v(x))^\eta) \text{ and } \lim_{\eta \to +\infty} \psi((b_2/c_1)^{1/k} \eta^{-1}) = +\infty,
\]
that \( \lim_{\eta \to +\infty} \min_{x \in \Omega_1} u_\lambda(x) = +\infty. \)

6.4 Proof of Theorem 2.4

Proof. Without loss of generality, we assume
\[
\max_{x \in \Omega} v(x) < \exp(-\mu).
\]
This implies that \( 1 - \mu(-\ln v(x))^{-1} > 0, x \in \bar{\Omega} \). By (6.17)–(6.19) and a similar argument as in the proof of Theorem 2.1, we obtain that there exist positive constants \( \beta_3 \) and \( \beta_4 \) (\( \beta_3 < \beta_4 \)) such that for any \( \tau > 0 \), it holds
\[
\beta_3 < \omega_3(\tau, x) < \beta_4, \quad x \in \bar{\Omega},
\]
where \( \omega_3 \) is given by (2.23).

By the similar argument as in the proof of Theorem 2.3, we see that there exist positive constants \( \tau_3 \) and \( \tau_4 \) (\( \tau_3 \leq \tau_4 \)) such that (2.22) holds.

Let \( u_{\mu} = \psi(\tau_4(\mu - 1)^{-1}(-\ln v(x))^{1-\mu}), \quad x \in \Omega \). By (b_1), (b_3) and Lemma 3.4, we obtain
\[
S_k(D^2 u_{\mu}(x)) = \tau_4^k(-\psi'(\tau_4(\mu - 1)^{-1}(-\ln v(x))^{1-\mu}))^k \times (v(x))^{-(k+1)(-\ln v(x))^{-\mu}k} \omega_3(\tau_4, x)
\]
This implies that when \( u_\mu \) is a subsolution to problem (1.1) in \( \Omega \). Moreover, by a similar calculation as (6.24), we see that \( u_\mu \) is strictly convex in \( \Omega \).

In a similar way, we can show that \( \bar{u}_\mu \) is a strictly convex supersolution to problem (1.1) in \( \Omega \). Obviously, \( u_\mu \leq \bar{u}_\mu \) in \( \Omega \). So, by Lemma 6.1, we obtain that problem (1.1) has a classical \( k \)-convex solution \( u_\mu \in C^\infty(\Omega) \) satisfying

\[
u_\mu(x) \leq u_\mu(x) \leq \bar{u}_\mu(x), \quad x \in \Omega.
\]

Take \( \mu_\ast > 1 \) and we may as well assume that \( \max_{x \in \bar{\Omega}} v(x) < \exp(-\mu_\ast) \). Define

\[
\varpi_\ast(x, \mu) = v(x)(-1)^k S_k(D^2 v(x)) + (H_0(\mu - 1)(-\ln v(x))^{-1} + 1 - \mu(-\ln v(x))^{-1} \mathcal{A}(x), \quad (x, \mu) \in \bar{\Omega} \times [1, \mu_\ast]
\]

and

\[
\varpi_\ast(x, \mu) = v(x)(-1)^k S_k(D^2 v(x)) + (h_0(\mu - 1)(-\ln v(x))^{-1} + 1 - \mu(-\ln v(x))^{-1} \mathcal{A}(x), \quad (x, \mu) \in \bar{\Omega} \times [1, \mu_\ast],
\]

where \( H_0 \) and \( h_0 \) are given by (6.17) and (6.18). So, we have

\[
\left(\frac{b_1}{\max_{(x, \mu) \in \bar{\Omega} \times [0, \mu_\ast]} \varpi_\ast(x, \mu)}\right)^{1/k} \leq \tau_3 \leq \left(\frac{b_2}{\min_{(x, \mu) \in \bar{\Omega} \times [0, \mu_\ast]} \varpi_\ast(x, \mu)}\right)^{1/k}.
\]

The rest of the proof is similar to that of Theorem 2.3 and the proof is omitted here. \( \Box \)

### 6.5 Proof of Theorem 2.5

**Proof.** By Proposition 4.7 (iv), we have

\[
\lim_{d(x) \to 0} \frac{(v(x))^{\frac{k+1+\lambda}{k}} \tilde{L}(v(x))}{\int_0^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds} = \frac{k + 1 + \lambda}{k}.
\]

This implies that when \( d(x) \to 0 \),

\[
\frac{(v(x))^{\frac{k+1+\lambda}{k}} \tilde{L}(v(x))h_0}{\int_0^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds} - \frac{\lambda + 1}{k} - \frac{v(x)\tilde{L}'(v(x))}{\tilde{L}(v(x))} \to \frac{k h_0 + (1 + \lambda)(h_0 - 1)}{k}.
\]

Since (2.26) holds, without loss of generality, we always assume that

\[
\min_{x \in \Omega} \left(\frac{(v(x))^{\frac{k+1+\lambda}{k}} \tilde{L}(v(x))}{\int_0^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds} h_0 - \frac{\lambda + 1}{k} - \frac{v(x)\tilde{L}'(v(x))}{\tilde{L}(v(x))}\right) > 0.
\]
Combining above with (6.17)–(6.19) and a similar argument as in the proof of Theorem 2.1, we obtain that there exist positive constants $\beta_5$ and $\beta_6$ ($\beta_5 < \beta_6$) such that for any $\tau > 0$, it holds

$$\beta_5 < \omega_4(\tau, x) < \beta_6, \quad x \in \Omega,$$

where $\omega_4$ is given by (2.29).

By the similar argument as in the proof of Theorem 2.3, we see that there exist positive constants $\tau_5$ and $\tau_6$ ($\tau_5 \leq \tau_6$) such that (2.28) holds.

Let $u(x) = \psi(\tau_6 \int_0^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds), \quad x \in \Omega$. By (b$_1$), (b$_4$) and Lemma 3.4, we obtain

$$S_k(D^2 u(x)) = \tau_6^k \left( - \psi' \left( \tau_6 \int_0^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds \right) \right)^k (v(x))^\lambda \tilde{L}^k(v(x)) \omega_4(\tau_6, x)$$

$$\geq \tau_6^k \min_{x \in \Omega} \omega_4(\tau_6, x) \cdot (\langle v(x) \rangle^\lambda \tilde{L}^k(v(x)) \omega_4(\tau_6, x))$$

$$= b_2(v(x))^\lambda \tilde{L}^k(v(x)) f(u(x)) \geq b(x) f(u(x)), \quad x \in \Omega,$$

i.e., $u$ is a subsolution to problem (1.1) in $\Omega$. Moreover, by a straightforward calculation, we have for all $l = 1, \ldots, N$,

$$S_l(D^2 u(x)) = \tau_6^l \left( - \psi' \left( \tau_6 \int_0^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds \right) \right)^l (v(x))^\lambda \tilde{L}^l(v(x)) (v(x))^{\frac{(l-k)(1+\lambda)}{k}}$$

$$\times \left[ v(x) \cdot (-1)^l S_l(D^2 v(x)) + \left( \frac{1}{v(x)} \int_0^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds \right) \right] \gg_2(x) > 0, \quad x \in \Omega.$$

This implies that $u$ is strictly convex in $\Omega$.

In a similar way, we can show that $\bar{u} = \psi(\tau_5 \int_0^v s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds)$ is a strictly convex supersolution in $\Omega$. Obviously, $u \leq \bar{u}$ in $\Omega$. By Lemma 6.1, we see that problem (1.1) has a classical $k$-convex solution $u \in C^\infty(\Omega)$ satisfying

$$u(x) \leq u(x) \leq \bar{u}(x), \quad x \in \Omega. \quad (6.26)$$

\[\square\]

### 6.6 Proof of Theorem 2.6

**Proof.** By (6.25) and $\lambda < 0$, without loss of generality, we assume that $v(x)$ satisfies

$$\left( \frac{(v(x))^{\frac{k+1+\lambda}{k}} \tilde{L}(v(x))}{(k+1) \int_0^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds} \right) - \frac{\lambda + 1}{k} - \frac{v(x) \tilde{L}'(v(x))}{\tilde{L}(v(x))} > 0, \quad \forall x \in \Omega. \quad (6.27)$$
On the other hand, by Lemma 5.9 (ii)–(iv), we see that

\[
0 = \inf_{t > 0} \Phi(t) \leq \sup_{t > 0} \Phi(t) < +\infty. \tag{6.28}
\]

It follows from (6.27)–(6.28) that

\[
\sup_{(\tau, x) \in (0, 1] \times \Omega} \omega_5(\tau, x) < +\infty, \tag{6.29}
\]

where \(\omega_5\) is given by (2.32). Fix \(\tau \in (0, 1]\). By the similar argument as in the proof of Theorem 2.1, we have

\[
\min_{x \in \Omega} \omega_5(\tau, x) > 0.
\]

This together with (6.29) implies that there exists \(\tau \in (0, 1]\) such that

\[
\tau^{k+1} \min_{x \in \Omega} \omega_5(\tau, x) < b. \tag{6.30}
\]

Take \(\delta_1 > 0\) such that \(\mathcal{A}\) is positive in \(\Omega_{\delta_1}\). We see from Lemma 5.9 (ii)–(iv) that

\[
0 < \inf_{(\tau, x) \in [1, +\infty) \times (\Omega \setminus \Omega_{\delta_1})} \Phi \left( \tau \left( \int_0^{v(x)} s^{\lambda+1} \tilde{L}(s) ds \right)^{\frac{k}{\lambda+1}} \right) \leq \sup_{(\tau, x) \in [1, +\infty) \times (\Omega \setminus \Omega_{\delta_1})} \Phi \left( \tau \left( \int_0^{v(x)} s^{\lambda+1} \tilde{L}(s) ds \right)^{\frac{k}{\lambda+1}} \right) < +\infty.
\]

This, combined with (6.27), shows that

\[
0 < \inf_{(\tau, x) \in [1, +\infty) \times \Omega} \omega_5(\tau, x) < \sup_{(\tau, x) \in [1, +\infty) \times \Omega} \omega_5(\tau, x) < +\infty. \tag{6.31}
\]

Fix \(\tau \in [1, +\infty)\). By the similar argument as in the proof of Theorem 2.1, we have \(\min_{x \in \Omega} \omega_5(\tau, x) > 0\). This together with (6.31) implies that there exists \(\tau \in [1, +\infty)\) such that

\[
\tau^{k+1} \min_{x \in \Omega} \omega_5(\tau, x) > b. \tag{6.32}
\]

We conclude from (6.30), (6.32) and Lemma 6.2 that there exists \(\tau_8 \in (\underline{\tau}, \tau)\) such that

\[
\tau_8^{k+1} \min_{x \in \Omega} \omega_5(\tau_8, x) = b.
\]

By the similar argument as the above, we can show that there exists positive constant \(\tau_7\) with \(\tau_7 \leq \tau_8\) such that

\[
\tau_7^{k+1} \max_{x \in \Omega} \omega_5(\tau_7, x) = b_1.
\]
Let \( \underline{u}(x) = \varphi(\tau_{8}(\int_{0}^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds)^{\frac{k}{k+1}}) \), \( x \in \Omega \). By (b_1), (b_4) and Lemma 3.4, we obtain
\[
S_{k}(D^{2}\underline{u}(x)) = \tau_{8}^{k+1}\left(-\varphi'\left(\tau_{8}\left(\int_{0}^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds\right)^{\frac{k}{k+1}}\right)\right)^{k-1} \\
\times \varphi''\left(\tau_{8}\left(\int_{0}^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds\right)^{\frac{k}{k+1}}\right)\omega_{5}(\tau_{8}, x) \\
\geq \tau_{8}^{k+1} \min_{x \in \Omega} \omega_{5}(\tau_{8}, x) \cdot ((v(x))^{\lambda} \tilde{L}^{k}(v(x)) f(\underline{u}(x))) \\
= b_{2}(v(x))^{\lambda} \tilde{L}^{k}(v(x)) f(\underline{u}(x)) \geq b(x) f(\underline{u}(x)), \quad x \in \Omega,
\]
i.e., \( \underline{u} \) is a subsolution to problem (1.1) in \( \Omega \). Moreover, by a straightforward calculation, we have for all \( l = 1, \ldots, N, \)
\[
S_{l}(D^{2}\underline{u}(x)) = \tau_{8}^{l+1}\left(\frac{k}{k+1}\right)^{k} \left(-\varphi'\left(\tau_{8}\left(\int_{0}^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds\right)^{\frac{k}{k+1}}\right)\right)^{l-1} \\
\times \varphi''\left(\tau_{8}\left(\int_{0}^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds\right)^{\frac{k}{k+1}}\right)\omega_{l}(\tau_{8}, x) \\
\times \left(\int_{0}^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds\right)^{\frac{k-l-1}{k+1}} \left\{\Phi\left(\int_{0}^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds\right)^{\frac{k}{k+1}}\right\} \\
\times v(x)(-1)^{l} S_{l}(D^{2}v(x)) + \left[\frac{(v(x))^{\frac{k+1+\lambda}{k}} \tilde{L}(v(x))}{\int_{0}^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds}\right. \\
+ \left.\frac{(v(x))^{\frac{k+1+\lambda}{k}} \tilde{L}(v(x))}{(k+1) \int_{0}^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds}\right] \omega_{l}(x) \bigg\} > 0, \quad x \in \Omega.
\]
This implies that \( \underline{u} \) is strictly convex in \( \Omega \).

In a similar way, we can show that \( \bar{u} = \varphi(\tau_{7}(\int_{0}^{v(x)} s^{\frac{1+\lambda}{k}} \tilde{L}(s) ds)^{\frac{k}{k+1}}) \) is a strictly convex supersolution in \( \Omega \). Obviously, \( u \leq \bar{u} \) in \( \Omega \). By Lemma 6.1, we see that problem (1.1) has a classical \( k \)-convex solution \( u \in C^{\infty}(\Omega) \) satisfying (6.26).

\section{The Exact Boundary Behavior of Large Solutions}

In this section, we prove Theorem 2.7.

\textbf{Proof.} Let \( \varepsilon \in (0, \frac{b_{1}}{2(1+C_{0})}) \), \( C_{0} > b_{2} \) and
\[
\xi_{-\varepsilon} = \tau_{9}(1 - (1 + C_{0})\varepsilon/b_{1})^{1/k}, \quad \xi_{+\varepsilon} = \tau_{10}(1 + (1 + C_{0})\varepsilon/b_{2})^{1/k}.
\]
It follows that
\[ \tau_9 \left( \frac{1}{2} \right)^{1/k} < \xi_{-\varepsilon} < \xi_{+\varepsilon} < \tau_{10} \left( \frac{3}{2} \right)^{1/k}. \]

As (3.1), we define
\[ \Omega_{\delta_*} = \{ x \in \Omega : 0 < d(x) < \delta_* \}, \]
where \( \delta_* \in (0, \min\{\delta_0, \tilde{\delta}\}) \) and \( \tilde{\delta} \) is given in (3.2).

Next, we consider the following two cases.

**Case 1.** \( \theta \) is non-increasing on \((0, \delta_0)\). From Lemma 5.5 (ii) and Lemma 5.10 (i), we see that corresponding to \( \varepsilon \), there exists sufficiently small constant \( \delta_\varepsilon \in (0, \delta_* / 2) \) such that \( x \in \Omega_{2\delta_\varepsilon} \) and \( r \in (0, \delta_\varepsilon) \), the following hold

\[ \frac{M_{k-1}^-}{1 + \varepsilon} < S_{k-1}(\epsilon_1, \ldots, \epsilon_{N-1}) < \frac{M_{k-1}^+}{1 - \varepsilon}, \]
\[ b_1 - C_0 \varepsilon < \frac{b(x)}{\theta^{k+1}(d(x))} < b_2 + C_0 \varepsilon, \]

\[ \mathfrak{T}_1(x, \Theta_r^\pm(d(x))) := \frac{k + 1}{k} \Psi \left( \xi_{\mp\varepsilon}(\Theta_r^\pm(d(x)))^{\frac{k+1}{k}} \right) - \frac{1}{k} \frac{\theta'(d(x))}{\theta^2(d(x))} > 0, \]

\[ \mathfrak{T}_2(x, \Theta_r^\pm(d(x))) := \left| \left( \frac{(k + 1)\xi_{\mp\varepsilon}}{k} \right)^k \frac{\Theta(d(x))}{\theta(d(x))} S_k(\epsilon_1, \ldots, \epsilon_{N-1})(1 + \varepsilon) \right. \]
\[ \left. + \left( \frac{(k + 1)\xi_{\mp\varepsilon}}{k} \right)^k \frac{k + 1}{k} \left( \Psi \left( \xi_{\mp\varepsilon}(\Theta_r^\pm(d(x)))^{\frac{k+1}{k}} \right) \right) \right. \]
\[ \left. - C_f^\pm \right) M_{k-1}^\pm \left( \frac{(k + 1)\xi_{\mp\varepsilon}}{k} \right)^k \left( \frac{\theta'(d(x))}{\theta^2(d(x))} \right) \]
\[ \left. - (1 - D_{\theta}) \right) M_{k-1}^\pm < \varepsilon, \]

where
\[ \Psi \left( \xi_{\mp\varepsilon}(\Theta_r^\pm(d(x)))^{\frac{k+1}{k}} \right) = \frac{\psi''(\xi_{\mp\varepsilon}(\Theta_r^\pm(d(x)))^{\frac{k+1}{k}})}{\psi'(\xi_{\mp\varepsilon}(\Theta_r^\pm(d(x)))^{\frac{k+1}{k}})} \]
and
\[ \Theta_r^\pm(d(x)) := \Theta(d(x)) \mp \Theta(r) > 0. \]

Take \( \sigma \in (0, \delta_\varepsilon) \) and define
\[ D^- := \Omega_{2\delta_\varepsilon} \setminus \bar{\Omega}_\sigma, \quad D^+ := \Omega_{2\delta_\varepsilon - \sigma}. \]
Let
\[ \overline{u}_\varepsilon(x) = \psi \left( \xi_{-\varepsilon}(\Theta^{-}_\sigma(d(x)))^{\frac{k+1}{k}} \right), \quad x \in D^-_\sigma, \]
\[ \underline{u}_\varepsilon(x) = \psi \left( \xi_{+\varepsilon}(\Theta^{+}_\sigma(d(x)))^{\frac{k+1}{k}} \right), \quad x \in D^+_\sigma. \]

By (7.1)–(7.3) and a straightforward calculation, we have
\[
S_k(D^2\overline{u}_\varepsilon(x)) - b(x)f(\overline{u}_\varepsilon(x)) \\
= \left( - \psi' \left( \xi_{-\varepsilon}(\Theta^{-}_\sigma(d(x)))^{\frac{k+1}{k}} \right) \right)^k \theta^{k+1}(d(x)) \left\{ \left( \frac{(k+1)\xi_{-\varepsilon}}{k} \right)^k \Theta^{-}_\sigma(d(x)) \theta(d(x)) \right\} \\
\times S_k(\epsilon_1, \ldots, \epsilon_{N-1}) + \left( \frac{(k+1)\xi_{-\varepsilon}}{k} \right)^k \left[ \frac{k+1}{k} \Psi(\xi_{-\varepsilon}(\Theta^{-}_\sigma(d(x)))^{\frac{k+1}{k}}) - \frac{1}{k} \right] \\
- \frac{\theta'(d(x))\Theta^{-}_\sigma(d(x))}{\theta^2(d(x))} \left\{ S_{k-1}(\epsilon_1, \ldots, \epsilon_{N-1}) - \frac{b(x)}{\theta^{k+1}(d(x))} \right\} \\
\leq f(\overline{u}_\varepsilon(x))\theta^{k+1}(d(x))(1-\varepsilon)^{-1} \left\{ \left( \frac{(k+1)\xi_{-\varepsilon}}{k} \right)^k \Theta(d(x)) \theta(d(x)) S_k(\epsilon_1, \ldots, \epsilon_{N-1})(1-\varepsilon) \right\} \\
+ \left( \frac{(k+1)\xi_{-\varepsilon}}{k} \right)^k \frac{k+1}{k} \left[ \Psi(\xi_{-\varepsilon}(\Theta^{-}_\sigma(d(x)))^{\frac{k+1}{k}}) - C^+_{f\infty} \right] M^+_k - \left( \frac{(k+1)\xi_{-\varepsilon}}{k} \right)^k \right\} \left( \frac{(k+1)\xi_{-\varepsilon}}{k} \right)^k \xi_{-\varepsilon} \\
\times \left( \frac{\theta'(d(x))\Theta(d(x))}{\theta^2(d(x))} - (1-D_\theta) \right) M^+_k + \left( \frac{k+1}{k} \right)^k \xi_{-\varepsilon} \\
\times \left( \frac{(k+1)C^+_{f\infty} + kD_\theta - (k+1)}{k} \right) M^+_k - (b_1 - C_0\varepsilon) \right\} \\
\leq f(\overline{u}_\varepsilon(x))\theta^{k+1}(d(x))(1-\varepsilon)^{-1} \left\{ \left( \frac{(k+1)\xi_{-\varepsilon}}{k} \right)^k \xi_{-\varepsilon} \left[ (k+1)C^+_{f\infty} + kD_\theta - (k+1) \right] \right\} \\
\times M^+_k - b_1 + (1+C_0)\varepsilon \leq 0,
\]
i.e., \( \overline{u}_\varepsilon \) is a supersolution to Eq. (1.1) in \( D^-_\sigma \). Moreover, by (7.2), we see that for all \( i = 1, \ldots, N, \)
\[
S_i(D^2\overline{u}_\varepsilon(x)) = \left( - \psi' \left( \xi_{-\varepsilon}(\Theta^{-}_\sigma(d(x)))^{\frac{k+1}{k}} \right) \right)^i(\Theta^{-}_\sigma(d(x)))^{\frac{i-k}{k}} \theta^{i+1}(d(x)) \\
\times \left\{ \left( \frac{(k+1)\xi_{-\varepsilon}}{k} \right)^i \Theta^{-}_\sigma(d(x)) \theta(d(x)) S_i(\epsilon_1, \ldots, \epsilon_{N-1}) \right\} \\
+ \left( \frac{(k+1)\xi_{-\varepsilon}}{k} \right)^i \left[ \frac{k+1}{k} \Psi(\xi_{-\varepsilon}(\Theta^{-}_\sigma(d(x)))^{\frac{k+1}{k}}) \right] \\
- \frac{1}{k} \left[ \frac{\theta'(d(x))\Theta^{-}_\sigma(d(x))}{\theta^2(d(x))} \right] S_{i-1}(\epsilon_1, \ldots, \epsilon_{N-1}) > 0, \quad x \in D^-_\sigma.
\]
This implies that \( \overline{u}_\varepsilon \) is strictly convex in \( D^-_\sigma \).
In a similar way, we can show that \( u_\varepsilon \) is a strictly convex subsolution to Eq. (1.1) in \( D_+^\sigma \).

**Case 2.** \( \theta \) is non-decreasing on \((0, \delta_0)\). Since \( \lim_{t \to 0^+} \Theta(t) = 0 \), for convenience, we define

\[
\Theta_0^+(d(x)) := \Theta(d(x))
\]

in (7.4). From Lemma 5.5 (ii) and Lemma 5.10 (i), we see that corresponding to \( \varepsilon \), there exists sufficiently small constant \( \delta_0 \) in (7.4). From Lemma 5.5 (ii) and Lemma 5.10 (i), we see that corresponding to \( \varepsilon \), there exists sufficiently small constant \( \delta_0 \) in (7.4). From Lemma 5.5 (ii) and Lemma 5.10 (i), we see that corresponding to \( \varepsilon \), there exists sufficiently small constant \( \delta_0 \) in (7.4). From Lemma 5.5 (ii) and Lemma 5.10 (i), we see that corresponding to \( \varepsilon \), there exists sufficiently small constant \( \delta_0 \) in (7.4).

Take \( \sigma \in (0, \delta_0) \) and define

\[
\bar{u}_\varepsilon(x) = \psi\left(\xi_{-\varepsilon}(\Theta(d_- (x)))^{\frac{k+1}{k}}\right), \quad x \in D_-,
\]

\[
\tilde{u}_\varepsilon(x) = \psi\left(\xi_{+\varepsilon}(\Theta(d_+ (x)))^{\frac{k+1}{k}}\right), \quad x \in D_+,
\]

where \( D_+^\sigma \) are defined as shown in (7.5) and

\[
d_-(x) := d(x) - \sigma, \quad d_+(x) := d(x) + \sigma.
\]

A straightforward calculation shows that

\[
S_k(D^2\bar{u}_\varepsilon(x)) - b(x)f(\bar{u}_\varepsilon(x))
\]

\[
= \left(- \psi'\left(\xi_{-\varepsilon}(\Theta(d_- (x)))^{\frac{k+1}{k}}\right)\right)^k \Theta^{k+1}(d_- (x)) \left\{ \left(\frac{(k+1)\xi_{-\varepsilon}}{k}\right)^k \Theta(d_- (x)) + (1 - D_\theta)M_{k-1}^+ \right\}
\]

\[
\times \left\{ S_k(\varepsilon_1, \ldots, \varepsilon_N) + \left(\frac{(k+1)\xi_{-\varepsilon}}{k}\right)^k \frac{k+1}{k} \Psi\left(\xi_{-\varepsilon}(\Theta(d_- (x)))^{\frac{k+1}{k}}\right) \right\}
\]

\[
\leq f(\bar{u}_\varepsilon(x))\psi^{k+1}(d_-(x))(1 - \varepsilon)^{-1} \left( \left(\frac{(k+1)\xi_{-\varepsilon}}{k}\right)^k \Theta(d_- (x)) \right) S_k(\varepsilon_1, \ldots, \varepsilon_N) - C_f^{+\infty}M_{k+1}^+ \]

\[
\times \left( 1 - \varepsilon \right) + \left(\frac{(k+1)\xi_{-\varepsilon}}{k}\right)^k \frac{k+1}{k} \left( (1 - D_\theta)M_{k-1}^+ \right)
\]

\[
- \left(\frac{(k+1)\xi_{-\varepsilon}}{k}\right)^k \left( \frac{\theta'(d_- (x))\Theta(d_- (x))}{\theta^2(d_- (x))} \right)
\]

\[
+ \left(\frac{(k+1)\xi_{-\varepsilon}}{k}\right)^k \left( (k+1)C_f^{+\infty} + kD_\theta - (k+1) \right)
\]

\[
\leq f(\bar{u}_\varepsilon(x))\theta^{k+1}(d_-(x))(1 - \varepsilon)^{-1} \left( \left(\frac{(k+1)\xi_{-\varepsilon}}{k}\right)^k \frac{(k+1)C_f^{+\infty} + kD_\theta - (k+1)}{k} \right)
\]

\[
\times M_{k-1}^+ - C_1 + (1 + C_0)\varepsilon \right\} \leq 0,
\]

i.e., \( \bar{u}_\varepsilon \) is a supersolution to Eq. (1.1) in \( D_+^\sigma \). By the similar argument as the above, we can show \( \bar{u}_\varepsilon \) is strictly convex in \( D_+^\sigma \).
In a similar way, we can show that $u_\varepsilon$ is a strictly convex subsolution to Eq. (1.1) in $D_+^\sigma$.

For Case 1 and Case 2, let $u$ be an arbitrary $k$-convex solution to problem (1.1). We assert that there exists a large constant $M > 0$ such that

$$u(x) \leq \overline{u}_\varepsilon(x) + M, \ x \in D_+^\sigma \quad \text{and} \quad \underline{u}_\varepsilon(x) \leq u(x) + M, \ x \in D_-^\sigma.$$  \hspace{1cm} (7.6)

In fact, we can take a constant $M > 0$ independent of $\sigma$ such that

$$u(x) \leq \overline{u}_\varepsilon(x) + M, \ x \in \{x \in \Omega : d(x) = 2\delta_\varepsilon\},$$

$$\underline{u}_\varepsilon(x) \leq u(x) + M, \ x \in \{x \in \Omega : d(x) = 2\delta_\varepsilon - \sigma\}.$$  \hspace{1cm} (7.7)

Moreover, we also see that

$$u(x) < \overline{u}_\varepsilon(x) = +\infty, \ x \in \{x \in \Omega : d(x) = \sigma\};$$

$$\underline{u}_\varepsilon(x) < u(x) = +\infty, \ x \in \partial \Omega.$$  

This implies that we can take a sufficiently small positive constant $\rho (\rho < \delta_\varepsilon)$ such that

$$\sup_{x \in D_-^\sigma} u(x) \leq \overline{u}_\varepsilon(x), \ x \in D_-^\sigma \setminus \tilde{D}_-^\sigma \quad \text{and} \quad \sup_{x \in D_+^\sigma} \underline{u}_\varepsilon(x) \leq u(x), \ D_+^\sigma \setminus \tilde{D}_+^\sigma, \hspace{1cm} (7.8)$$

where

$$\tilde{D}_-^\sigma = \Omega_{2\delta_\varepsilon} \setminus \bar{\Omega}_{(1+\rho)\sigma}$$ \quad and \quad $$\tilde{D}_+^\sigma = \Omega_{2\delta_\varepsilon - \sigma} \setminus \bar{\Omega}_\rho.$$  

By $(S_1)$ (or $(S_{01})$), we note that $\overline{u}_\varepsilon + M$ and $u + M$ are both supersolutions in $\tilde{D}_-^\sigma$ and $\tilde{D}_+^\sigma$, respectively. It follows from (7.7)–(7.8) and Lemma 3.1 that

$$u(x) \leq \overline{u}_\varepsilon(x) + M, \ x \in \tilde{D}_-^\sigma \quad \text{and} \quad \underline{u}_\varepsilon(x) \leq u(x) + M, \ x \in \tilde{D}_+^\sigma.$$  

This together with (7.8) implies that (7.6) holds. Hence, letting $\sigma \to 0$, we have for $x \in \Omega_{2\delta_\varepsilon},$

$$\frac{u(x)}{\psi(\xi_{-\varepsilon}(\Theta(d(x)))^{\frac{k+1}{k}})} \leq 1 + \frac{M}{\psi(\xi_{-\varepsilon}(\Theta(d(x)))^{\frac{k+1}{k}})},$$

and

$$\frac{u(x)}{\psi(\xi_{+\varepsilon}(\Theta(d(x)))^{\frac{k+1}{k}})} \geq 1 - \frac{M}{\psi(\xi_{+\varepsilon}(\Theta(d(x)))^{\frac{k+1}{k}})}.$$  

Consequently, we have

$$\limsup_{d(x) \to 0} \frac{u(x)}{\psi(\xi_{-\varepsilon}(\Theta(d(x)))^{\frac{k+1}{k}})} \leq 1 \quad \text{and} \quad \liminf_{d(x) \to 0} \frac{u(x)}{\psi(\xi_{+\varepsilon}(\Theta(d(x)))^{\frac{k+1}{k}})} \geq 1.$$
It follows from Lemma 5.5 (ii) that

$$\limsup_{d(x) \to 0} \frac{u(x)}{\psi((\Theta(d(x)))^{\frac{k+1}{k}})} = \limsup_{d(x) \to 0} \frac{u(x)}{\psi(\xi_{-\varepsilon}(\Theta(d(x)))^{\frac{k+1}{k}})} \times \lim_{d(x) \to 0} \frac{\psi(\xi_{-\varepsilon}(\Theta(d(x)))^{\frac{k+1}{k}})}{\psi((\Theta(d(x)))^{\frac{k+1}{k}})} \leq \xi_{-\varepsilon}^{1-C_j^{-\infty}}$$

$$\liminf_{d(x) \to 0} \frac{u(x)}{\psi((\Theta(d(x)))^{\frac{k+1}{k}})} = \liminf_{d(x) \to 0} \frac{u(x)}{\psi(\xi_{+\varepsilon}(\Theta(d(x)))^{\frac{k+1}{k}})} \times \lim_{d(x) \to 0} \frac{\psi(\xi_{+\varepsilon}(\Theta(d(x)))^{\frac{k+1}{k}})}{\psi((\Theta(d(x)))^{\frac{k+1}{k}})} \geq \xi_{+\varepsilon}^{1-C_j^{+\infty}}.$$

Passing to $\varepsilon \to 0$, we obtain (2.33) holds.

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