Optimal Allocation of Water and Sanitation Facilities To Prevent Communicable Diarrheal Diseases in Senegal Under Partial Interference

Chan Park\textsuperscript{a}, Guanhua Chen\textsuperscript{b}, Menggang Yu\textsuperscript{b}, and Hyunseung Kang\textsuperscript{a}

\textsuperscript{a}Department of Statistics, University of Wisconsin–Madison
\textsuperscript{b}Department of Biostatistics and Medical Informatics, University of Wisconsin–Madison

Abstract

For several decades, Senegal has faced inadequate water, sanitation, and hygiene (WASH) facilities in households, contributing to persistent, high levels of communicable diarrheal diseases. Unfortunately, the ideal WASH policy where every household in Senegal installs WASH facilities is impossible due to logistical and budgetary concerns. This work proposes to estimate an optimal allocation rule of WASH facilities in Senegal by combining recent advances in personalized medicine and partial interference in causal inference. Our allocation rule helps public health officials in Senegal decide what fraction of total households in a region should get WASH facilities based on block-level and household-level characteristics. We characterize the excess risk of the allocation rule and show that our rule outperforms other allocation policies in Senegal.

Keywords: Causal inference, Continuous treatment, Demographic and Health Survey, Excess risk, Optimal treatment rule

1 Introduction

1.1 Motivation: Household Sanitation and Preventing Communicable Diarrheal Disease in Senegal

In 2013, the United Nations’ International Children’s Emergency Fund and the World Health Organization launched the Integrated Global Action Plan for Pneumonia and Diarrhea (GAPPD) to reduce under-5 mortality rates in developing countries by 2030 (\textit{World Health Organization, 2013}). To achieve this goal, GAPPD laid out specific targets for developing countries to achieve by 2030, notably reducing incidence of severe diarrhea by 75\% compared to existing country-specific levels in 2010. Prior works have shown that improving household access to water, sanitation, and hygiene (WASH) resources are critical to reduce rates of diarrhea-related diseases (Esrey et al., 1985;
Clasen et al., 2007), especially among children (Daniels et al., 1990; McMichael, 2019). But, existing allocation policies of WASH resources are often informed by political or bureaucratic factors and not evidence-based (Chapter 5 of Evans and Mara, 2011; Section 3 of UNICEF, 2016; UN-Water and WHO, 2017; Appiah-Brempong et al., 2018), contributing to inefficiency and sub-optimal outcomes (Easterly, 2002; Shapiro, 2017). The main theme of this paper is to use a data-driven approach to allocate WASH resources in Senegal, one of the countries in the GAPPD coalition with high incidence of diarrhea.

For several decades, Senegal has faced inadequate WASH resources. As recently as 2020, Senegal lags behind other sub-Saharan nations in terms of having well-maintained sanitation and handwashing facilities (UN-Water, 2021). The lack of WASH resources in Senegal has contributed to high, persistent levels of risk factors associated with children’s diarrheal diseases (Okeke, 2009; Thiam et al., 2017) and some (Black et al., 2019; GBD 2019 Under-5 Mortality Collaborators, 2021) project that Senegal may not meet the aforementioned GAPPD targets for diarrhea incidence.

For Senegal, the ideal WASH policy would be to exhaustively install WASH facilities in every household. But, this is an impossible task for logistical and budgetary concerns, with Senegal’s government facing a budget gap of $756 million for WASH facilities in 2019 (USAID/WASH-FIN, 2020). Consequently, Senegalese policymakers and international development organizations need a more targeted allocation of WASH facilities where given a pre-defined target, say the GAPPD target, decide what fraction of households in a census block should get WASH facilities. A bit more formally, given (a) a target goal for diarrhea incidence, say the desired, minimum diarrhea-free incidence is \( T \in [0, 1] \), and (b) characteristics about a census block and households within the census block, determine (i) whether the block needs any WASH facilities and if so, (ii) what fraction \( \alpha \in [0, 1] \) of households in the census block needs WASH facilities to achieve the desired goal. For example, a block that is economically well-off may only need zero (i.e. \( \alpha = 0 \)) WASH facilities to meet the threshold \( T \) whereas another block that is economically worse-off may need 70% (i.e. \( \alpha = 0.7 \)) of its households to have WASH facilities to meet the same threshold. Critically, installing a sufficient fraction of WASH facilities in a census block may have a multiplying, positive spillover effect for all households in the same census block, akin to reaching herd immunity in infectious disease settings. Our proposed allocation rule takes this last part into account by combining recent methods in personalized medicine and partial interference in causal inference.

To train our allocation rule, we use the data provided by the Senegal’s Demographic and Health Survey (DHS) (ANSD and ICF, 2020). Briefly, the DHS is a survey of demographics, socioeconomic, and health-related information from developing countries in order to inform policies on international development and aid. For Senegal, the DHS contains information about (i) the number of WASH facilities, defined as having a private water source with a flushable toilet in a house, (ii) children’s diarrhea status, and (iii) other demographic and socioeconomic characteristics, all measured at the household level. We use the data from 2014 to 2017, which consist of 13,556 total households from 1,027 census blocks, to train our allocation rule and test its performance on the 2018 data; see
1.2 Prior Works & Our Contribution

Our work broadly fits into the literature on optimal treatment regimes (Murphy, 2003; Robins, 2004; Zhao et al., 2009, 2012; Zhang et al., 2012a,b; Chakraborty et al., 2010; Qian and Murphy, 2011; Laber et al., 2014; Moodie et al., 2014; Song et al., 2015), notably works by Laber and Zhao (2015), Chen et al. (2016), and Schulz and Moodie (2021) under non-binary treatment. However, there are some important differences between the existing work on optimal treatment regimes and our work. First, as noted above and based on prior works on WASH allocation policies (Overbo et al., 2016; Benjamin-Chung et al., 2018), having a sufficient number of households with WASH facilities can have a social multiplier effect for a region by reducing the chance that pathogens responsible for diarrhea contaminate communal water sources. This phenomenon is known as partial interference in causal inference (Cox, 1958; Rubin, 1986; Sobel, 2006; Hudgens and Halloran, 2008) where interference occurs within clusters, but not across clusters. Under partial interference, the optimal regime cannot be determined by only studying one study unit’s characteristics because other study units’ treatment values may affect the outcome of the said unit. Recent works by Su et al. (2019) and Viviano (2021) considered Q- and A-learning to optimally assign treatment under interference. But, Su et al. (2019) required the response model to be linear and Viviano (2021) restricted the rule to be parametric in the application. Also, both works used a binary treatment variable. In contrast, our work differs from these two works as well as existing works in optimal treatment regimes as follows. First, we use nonparametric, doubly robust estimators and work with a continuous treatment variable (i.e. $\alpha \in [0,1]$). Second, as we empirically show in Section 5, there are minimal, if any, negative externalities of installing WASH facilities with respect to children’s diarrhea status and we incorporate this useful, side information to place a monotonic, shape constraint on the response function. Third, often in public policy, there is a minimum, pre-defined target that the policymaker wants to achieve (e.g. GAPPD’s minimum targets for diarrhea incidence) and, to our knowledge, the current literature on optimal treatment regime do not use this information (Li, 2018).

The paper presents two allocation rules under partial interference with a continuous treatment and target threshold. The first allocation rule is a modest extension of the indirect method (Moodie et al., 2012; Chakraborty and Moodie, 2013) to partial interference where only the outcome model under partial interference is used to indirectly estimate the allocation rule; we call this the indirect rule in the paper. The second allocation rule (and our recommended rule for practice) directly estimates the allocation rule by recasting our problem as an instance of estimating an optimal (continuous) dose in Chen et al. (2016) with weights derived from partial interference and a threshold term on the response function; we call this the direct rule. In particular, we propose a novel loss function, which is a non-trivial modification of the loss function used in continuous dosing rules under O-learning, to accommodate the potential overall outcome (Hudgens and Halloran,
2008; Tchetgen Tchetgen and VanderWeele, 2012), a causal quantity in partial interference that measures the combined direct and spillover effect of treatment; see Section 3.2.

One important limitation of our work is that the estimated rule only informs what fraction of households in a census block should get WASH facilities. The rule does not tell which households in the census block should get WASH facilities. For example, if our rule suggests that a block needs 60% of its households to get WASH facilities in order to achieve at least 70% diarrhea-free incidence, the rule does not suggest which households in the block should get WASH facilities. This as well as other limitations of our methodology or data are discussed throughout the text.

2 Setup

2.1 Review: Notation, Partial Interference, and Overall Potential Outcome

Consider the study unit to be households and each household belongs to one of the census blocks in Senegal. Let \( N \) be the number of census blocks and let each census block be indexed by \( i = 1, \ldots, N \). For each block \( i \), let \( n_i \) be the number of households from block \( i \) and let each household be indexed by \( j = 1, \ldots, n_i \). We assume \( n_i \) is bounded above by a constant \( M \). Practically, this is approximately plausible in our data as the average block size is 13.2, which is much smaller than the number of blocks (i.e. 1,027). Also, the Senegal DHS imposes this assumption in their stratified sampling strategy where a fixed number of houses are sampled from each block (ANSD and ICF, 2020). Theoretically, we use this assumption to allow standard asymptotics under partial interference to work; see Section 3.4.

For each household \( j \) in block \( i \), we observe \( O_{ij} = (Y_{ij}, A_{ij}, X_{ij})^\top \) where \( Y_{ij} \) is a binary indicator on whether all children in household \( ij \) are diarrhea-free (i.e. \( Y_{ij} = 1 \)) or not (i.e. \( Y_{ij} = 0 \)), \( A_{ij} \) is a binary indicator on whether the household has a WASH facility (i.e. \( A_{ij} = 1 \)) or not (i.e. \( A_{ij} = 0 \)), and \( X_{ij} \) consists of various household and block characteristics, specifically: block size, indicator on whether census block \( i \) is located in an urban area, number of members in household \( j \), number of children in household \( j \), indicator on whether both parents do not have jobs, indicators on whether parents have ever attended schools, mother’s age, and average age of children in household \( j \). We assume that the dimension of \( X_{ij} \), which is nine in our dataset, is bounded as \( N \) increases.

For each census block \( i \), let \( O_i = (Y_i, A_i, X_i)^\top \) be the collection of all household data from block \( i \) where \( Y_i = (Y_{i1}, \ldots, Y_{in_i})^\top \), \( A_i = (A_{i1}, \ldots, A_{in_i})^\top \), and \( X_i = (X_{i1}, \ldots, X_{in_i})^\top \). Let \( \overline{O}_i = (\overline{Y}_i, \overline{A}_i, \overline{X}_i)^\top \) be the averaged block-level variables where \( \overline{Y}_i = n_i^{-1} \sum_{j=1}^{n_i} Y_{ij} \), \( \overline{A}_i = n_i^{-1} \sum_{j=1}^{n_i} A_{ij} \), and \( \overline{X}_i = n_i^{-1} \sum_{j=1}^{n_i} X_{ij} \). Also, let \( A_{i(-j)} \) be the vector of treatment for all households in block \( i \) that excludes household \( j \) and let \( S_i = \sum_{j=1}^{n_i} A_{ij} \). Without loss of generality, we assume larger outcomes are preferred.

We use the potential outcomes notation of Neyman (1923) and Rubin (1974) to define causal effects. Let \( \mathcal{A}(t) \) be a collection of \( t \)-dimensional binary vectors (e.g., \( \mathcal{A}(2) = \{ (0, 0), (0, 1), (1, 0), (1, 1) \} \)). For a treatment vector \( a_i \in \mathcal{A}(n_i) \), let \( Y_{ij}(a_i) = Y_{ij}(a_{i(-j)}) \) be the potential outcome of household
In block \(i\) and \(\overline{Y}_i^{(a_i)} = n_i^{-1} \sum_{j=1}^{n_i} Y_{ij}^{(a_i)}\) be the average of household potential outcomes in block \(i\).

Critically, departing from the usual setup in optimal treatment regimes (Chakraborty and Moodie, 2013; Kosorok and Moodie, 2015), the potential outcome \(Y_{ij}^{(a_{ij}, a_{i(-j)})}\) allows for partial interference from other households in block \(i\). Following the works on partial interference (Hudgens and Halloran, 2008; Tchetgen Tchetgen and VanderWeele, 2012), we define the expected overall potential outcome under policy \(\alpha \in (0,1)\) as

\[
\tau(\alpha) = E\left\{ \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{a_i \in A(n_i)} Y_{ij}^{(a_i)} \pi(a_i; \alpha) \right\}, \quad \pi(a_i; \alpha) = \prod_{j=1}^{n_i} \alpha^{a_{ij}}(1-\alpha)^{1-a_{ij}}.
\]

In words, \(\tau(\alpha)\) measures the effect of intervening on a census block with a policy that assigns treatment to households in the block with probability \(\alpha\). VanderWeele and Tchetgen Tchetgen (2011) and Halloran (2019) show that the overall potential outcome in (1) measures the entire effect of a policy applied at the block-level by accounting for both the direct effect and the spillover effect of treatment. In particular, VanderWeele and Tchetgen Tchetgen (2011) showed that the contrast between two overall potential outcomes can be decomposed into the direct and spillover effects and Halloran (2019) showed various examples of using the overall potential outcome to measure the overall effectiveness of a public health policy under interference. We use the overall potential outcome as the target outcome to maximize in our optimal allocation rule.

Finally, we use the following notations for norms, limits, and common functions. Let \(\|v\|_q\) be the \(q\)-norm of a vector for \(q \geq 1\) and let \(\|f\|_{P,q} = \left\{ \int \|f(o_i)\|_q^2 dP(o_i) \right\}^{1/q}\) be the \(L_q(P)\)-norm of a function \(f\) for \(q \geq 1\). For a set \(\mathcal{D} \subset \{1, \ldots, N\}\), let \(\mathcal{D}^c\) be its complement. Let \([\cdot]\) be the floor function. Finally, let \(O(\cdot), O_P(\cdot), o(\cdot),\) and \(o_P(\cdot)\) be the usual big-\(O\) and small-\(O\) notations.

### 2.2 Assumptions for Causal Identification and Estimation

To identify the overall potential outcome in (1), we make the following assumptions.

(A1) **Consistency:** \(Y_{ij} = \sum_{a_i \in A(n_i)} \mathbb{1}(A_i = a_i) Y_{ij}^{(a_i)}\).

(A2) **Conditional Ignorability:** \(Y_{ij}^{(a_i)} \perp A_i \mid X_i\) for all \(a_i \in A(n_i)\).

(A3) **Overlap:** There exists a constant \(c > 0\) satisfying \(c < P(A_i = a_i \mid X_i) < 1 - c\) for any \(a_i \in A(n_i)\) and \(X_i\).

Assumptions (A1)-(A3) are natural extensions of consistency/Stable Unit Value Treatment Assumption (SUTVA) (Rubin, 1976, 1978), conditional ignorability, and overlap to partial interference; see Imbens and Rubin (2015) and Hernán and Robins (2020) for textbook discussions. Under Assumptions (A1)-(A3), we can identify the overall potential outcome in (1) from the observed...
data as

$$\tau(\alpha) = E\left\{ \frac{1}{n_i} \sum_{j=1}^{n_i} E(Y_{ij} \mid A_i = a_i, X_i) \pi(a_i ; \alpha) \right\} = E\left\{ \sum_{a_i \in A(n_i)} E(Y_i \mid A_i = a_i, X_i) \pi(a_i ; \alpha) \right\}.

(2)$$

While Assumptions (A1)-(A3) are sufficient to identify the overall potential outcome, actually estimating it, specifically the response function $E(Y_i \mid A_i, X_i)$ in (2), may be challenging due to the curse of dimensionality and some assumptions are required to obtain consistent estimators. As such, we make two additional assumptions about the response function. The first assumption is a type of “exposure mapping” in Aronow and Samii (2017) from Hudgens and Halloran (2008), van der Laan (2014), Sofrygin and van der Laan (2016), and Ogburn et al. (2017) where conditional on all the covariates in block $i$, including household-level covariates, the average response in block $i$ is a function of $\bar{A}_i$, a lower-dimensional summary of the treatment variable $A_i$.

(A4) Conditional Stratified Interference: Given $X_i$, we have $E(Y_i \mid A_i, X_i) = \mu(\bar{A}_i, X_i)$ for any $A_i$.

Assumption (A4) is closely related to the stratified interference assumption of Hudgens and Halloran (2008) without any covariates where after fixing the proportion of treatment $\bar{A}_i$, the average outcome in a cluster is invariant to who actually received treatment. Critically, Assumption (A4) does not imply (a) a parametric model between $\bar{A}_i$ and $Y_i$ or (b) that block-level summaries of the observed data, say $\bar{Y}_i, \bar{A}_i$, and $\bar{X}_i$, are always sufficient to estimate $\mu$ or other functions needed for our analysis. For example, if a block has 20 households, the assumption allows the average response of block $i$ to be of the form $E(Y_i \mid A_i, X_i) = f(\bar{A}_i) + g(X_{i1}, \ldots, X_{i20})$ where $f$ and $g$ can be parametric or nonparametric functions; note that if the function $g$ is known a priori (and that’s a big if), we can use block-level data of the form $\bar{Y}_i, \bar{A}_i$, and $g(X_i)$ to estimate $\mu$, but not necessarily to estimate other functional for our analysis, say the propensity score. Assumption (A4) also allows for interactions between $\bar{A}_i$ and the pre-treatment covariates. On the other hand, if the average response of block $i$ depends on the treatment status of a few, focal households, say $E(Y_i \mid A_i, X_i) = \beta_0 + \beta_1 A_{i1} + f(\bar{A}_i) + g(X_{i1}, \ldots, X_{i20})$, Assumption (A4) is violated because the average response of block $i$ depends on the treatment status of household 1. Theoretically, variants of Assumption (A4) have been used in works on partial interference to achieve consistency, non-degenerate asymptotic limits (e.g. asymptotic normality or asymptotic mixtures of normal distributions), and/or consistent estimation of variance (Hong and Raudenbush, 2006; Liu and Hudgens, 2014; van der Laan, 2014; Basse and Feller, 2018; Bargagli-Stoffi et al., 2020; Egami, 2021; Forastiere et al., 2021); including the recent work on optimal treatment regimes under interference (Viviano, 2021). For additional discussions of the assumption in the context of our data and how to visually inspect this assumption, see Section 5.2.

The second assumption formalizes a useful side information about the relationship between diarrheal outcomes and WASH facilities.
(A5) **Monotonic Response:** \( \mu(\bar{A}_i, X_i) \) is non-decreasing in \( \bar{A}_i \) for any \( X_i \).

Assumption (A5) implies that the proportion of diarrhea-free children in a census block is a non-decreasing function of the proportion of households with WASH facilities. In general, Assumption (A5) would be plausible if the treatment is harmless, but would be violated if some study units have severe side effects from the treatment. For the Senegal study, this assumption is supported by prior works on WASH policies and by empirical observations; see Section 5.2. Overall, both Assumptions (A4) and (A5) are used to consistently estimate the optimal rule and unlike Assumptions (A1)-(A3), they are not required for identifying the overall potential outcome.

### 2.3 Problem Statement

We now formalize the problem of estimating the optimal allocation rule. First, under Assumptions (A1)-(A4), the expected overall outcome can be rewritten as

\[
\tau(\alpha) = E \left\{ \sum_{s=0}^{n_i} \left( \begin{array}{c} n_i \\ s \end{array} \right) \mu \left( \frac{s}{n_i}, X_i \right) \alpha^s (1 - \alpha)^{n_i - s} \right\}.
\]

Let \( \Theta \) be a collection of functions with the domain equal to the support of \( X_i \) and the range equal to \([0, 1]\), i.e. \( \Theta = \{ \theta \mid \theta(x_i) \in [0, 1] \} \). In words, a function \( \theta \in \Theta \) is one possible allocation rule where it outputs the proportion of WASH facilities that block \( i \) should have based on block \( i \)'s characteristics. Then, we define the value of the allocation rule \( \theta \) as the expected overall outcome when the policy \( \alpha \) equals \( \theta(x_i) \), i.e.

\[
\tau(\theta(X_i)) = E \left[ \sum_{s=0}^{n_i} \left( \begin{array}{c} n_i \\ s \end{array} \right) \mu \left( \frac{s}{n_i}, X_i \right) \theta(X_i)^s \{1 - \theta(X_i)\}^{n_i - s} \right]. \tag{3}
\]

Let \( T \in [0, 1] \) be the target outcome informed by public policy goals, say the proportion of diarrhea-free children should be at least 70%. Given the target \( T \) and Assumptions (A1)-(A5), the optimal allocation rule that achieves \( T \) is the rule \( \theta^*(x_i) \) where

\[
\theta^*(x_i) = \inf_{\alpha \in [0, 1]} \left\{ \alpha \left| \sum_{s=0}^{n_i} \left( \begin{array}{c} n_i \\ s \end{array} \right) \mu \left( \frac{s}{n_i}, x_i \right) \alpha^s (1 - \alpha)^{n_i - s} \geq T \right\}. \tag{4}
\]

In words, \( \theta^*(x_i) \) is the minimum proportion of WASH facilities needed in a census block with characteristics \( x_i \) in order to achieve the target, diarrhea-free incidence level of \( T \); we refer to \( \theta^* \) as the **optimal minimal allocation rule** (OMAR). If the set in (4) is empty, we let \( \theta^*(x_i) = 1 \) since under (A5), this maximizes the expected average outcome. The rest of the paper discusses the details of estimating the function \( \theta^* \).
3 Estimation of the Optimal Minimum Allocation Rule

3.1 An Indirect Approach Via Outcome Modeling

We first present a solution based on an existing solution by only modeling the outcome (Moodie et al., 2012; Chakraborty and Moodie, 2013). Formally, let $\hat{\mu}$ be the estimated $\mu$; see below for details on how to estimate this function. Then, an estimate of $\theta$ for a given $x_i$, denoted as $\hat{\theta}_{\text{IND}}(x_i)$, is obtained by replacing $\mu$ with $\hat{\mu}$ in (4) and doing a grid search over $\alpha$, i.e.

$$\hat{\theta}_{\text{IND}}(x_i) = \inf_{\alpha \in [0,1]} \left\{ \alpha \left| \sum_{s=0}^{n_i} \binom{n_i}{s} \hat{\mu} \left( \frac{s}{n_i}, x_i \right) \alpha^s (1 - \alpha)^{n_i - s} \geq T \right\}. \tag{5}$$

We make a few remarks about the indirect allocation rule $\hat{\theta}_{\text{IND}}$, all of which are well-known either in the literature on optimal treatment regimes or partial interference. First, $\hat{\theta}_{\text{IND}}$ does not yield a closed, functional form of the allocation rule. Second, similar to prior observations of indirect methods under no interference, $\hat{\theta}_{\text{IND}}$ may yield sub-optimal performance compared to our preferred approach below that directly estimate $\theta^*$ (Zhao et al., 2012) and we reconfirm this observation under partial interference in Sections 4 and 5. Third, unlike indirect methods under no interference, naively using some popular nonparametric (or parametric) methods to estimate $\mu$ where $Y_i$ is the dependent variable and $(A_i, X_i)$ are the independent variables will likely be infeasible because $X_i$ have different dimensions across different blocks. For instance, our dataset has 9 covariates per household and if block $i = 1$ has 5 households while block $i = 2$ has 10 households, the dimension of $X_1$ would be $9 \cdot 5 = 45$ while the dimension of $X_2$ would be $9 \cdot 10 = 90$. To resolve this and especially if the dataset is large, one can use a common trick in recent, supervised machine learning methods based on recurrent neural nets where we pad variable length sequential data by adding zeros so that all sequential data have identical length (Goodfellow et al., 2016, Chapter 10). Alternatively and especially when the dataset is small or moderately large, works in partial interference (van der Laan, 2014; Sofrygin and van der Laan, 2016; Ogburn et al., 2017) often modify the original $X_i$ to have a common dimension $d$ in order to use popular nonparametric regression models, say kernel regression, that require the independent variables to have identical dimensions across observations. Some common ways of modifying $X_i$ in the literature include (a) averaging household-level covariates, (b) taking a random subsample of a fixed number of households from each block, (c) using covariates from the most “extreme” households, say $X_i = (X_{ij}^{(\text{max})}, X_{ij}^{(\text{min})})$ where $X_{ij}^{(\text{max})}$ and $X_{ij}^{(\text{min})}$ are the collection of maximum and minimum values, respectively, or (d) generating a $d$-dimensional, low-rank summary of $X_i$, to name a few. Given that our data has 13,556 households, we also use the latter approach for estimating the indirect and direct rules, specifically when we estimate the direct rule using off-the-shelf implementation of support vector machines (SVM) with a Gaussian kernel. Finally, we remark that by unifying the dimensions of $X_i$, there is an implicit modeling assumption about $\mu$, but this assumption is method-dependent (e.g. SVM versus recurrent neural nets or other methods that allow for variable length independent variables) and
unlike Assumptions (A1)-(A5), is not fundamental for identification and/or consistent estimation.

3.2 A Direct Approach Via Risk Minimization

In this section, we propose our direct allocation rule based on directly estimating the OMAR in (4). We do this by recasting OMAR as a solution to a risk minimization problem with a specialized loss function $L(\theta, O_i)$ tailored for partial interference. Unlike the above approach, by reframing the original problem as a risk minimization problem, we can use methods in empirical risk minimization to obtain a direct estimate of $\theta^*$. Formally, for a given $t \in \mathbb{R}$ and data $O_i$, consider the following loss function

$$L(t, O_i) = \begin{cases} \nu(0, O_i) + \delta - \delta e^t & \text{if } -\infty < t < 0 \\ \nu(t, O_i) + \delta - \delta e^{t+1} & \text{if } 0 \leq t \leq 1 \\ \nu(1, O_i) + \delta - \delta e^{t+1} & \text{if } 1 < t < \infty \end{cases}$$

$$\nu(t, O_i) = \sum_{s=0}^{n_i} \left( \frac{n_i}{s} \right) \psi_{DR}(s, O_i) \sum_{\ell=0}^{n_i-s} \left( \frac{n_i-s}{\ell} \right) \frac{(-1)^{\ell+1}}{\ell + s + 1} - \mathcal{T}t + C_0$$

where $\delta > 0$ is any positive constant and $C_0$ is any large constant guaranteeing $\nu(t, O_i) \geq 0$. The function $e(s \mid X_i) = P(S_i = s \mid X_i)$ ($s = 0, 1, \ldots, n_i$) is the propensity score where the number of treated individuals is used as the treatment variable. The function $\psi_{DR}$ is motivated from the influence function of the doubly robust estimator of the overall causal effect under partial interference (Liu et al., 2019). In particular, we can use other $\psi$s, say the influence function for the inverse probability-weighted (IPW) estimator (Tchetgen Tchetgen and VanderWeele, 2012; Liu et al., 2016) or the outcome regression estimator; see Section A.1 of the supplementary material for details.

The reason behind the peculiar form of the loss function is due to the following result.

**Lemma 3.1** (Equivalence Lemma). The optimal minimum proportion rule $\theta^*$ is equivalent to the minimizer of the $R(\theta) = \mathbb{E}\{L(\theta(X_i), O_i)\}$, i.e. $R(\theta^*) \leq R(\theta)$ for all $\theta \in \Theta$.

Methodologically, Lemma 3.1 provides a direct approach to estimating $\theta^*$ by solving for the minimizer of the risk function $R(\theta)$. In particular, this connection to risk minimization allows investigators to use a variety of methods under empirical risk minimization with our loss function $L$. As a concrete example which we use in our empirical analysis, consider an SVM estimator of $\theta^*$ over a reproducing kernel Hilbert space (RKHS). Let $\mathcal{K}: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be the reproducing kernel function over the space $X_i$ that is symmetric, continuous, and positive definite. As mentioned in Section 3.1, SVMs require a unified dimension in its arguments to work “off-the-shelf,” notably to have a proper kernel, and as such, we assume $X_i$ (and the corresponding kernel) has dimension $d$. Also, we will use the Gaussian kernel $\mathcal{K}(x_i, x_j) = \exp\left(-\|x_i - x_j\|_2^2/\gamma_N^2\right)$ where $\gamma_N$ is the
bandwidth parameter, but other kernels are possible. Let $H_K$ be the RKHS associated with $K$ so that if $\theta(x_i) \in H_K$, $\theta(x_i)$ can be written as $\theta(x_i) = w^T \phi(x_i) + b$ with a weight vector $w$, a scalar shift term $b$, and a reproducing kernel function $\phi$ satisfying $\phi(x_i)^T \phi(x_j) = K(x_i, x_j)$. Also, using the representer theorem (Kimeldorf and Wahba, 1970; Schölkopf et al., 2001), the estimated rule can be written as $\tilde{\theta}(x_i) = \sum_{j=1}^N \tilde{\eta}_j K(x_i, x_j) + \tilde{b}$ where $\tilde{\eta}_j$ and $\tilde{b}$ are the solutions to the following problem:

$$
\min_{\theta \in H_K} \left\{ \frac{1}{N} \sum_{i=1}^N L(\theta(X_i), O_i) + \frac{\lambda_N}{2} \|\theta\|_{H_K}^2 \right\} = \min_{\eta, b} \left\{ \frac{1}{N} \sum_{i=1}^N L(k_i^T \eta + b, O_i) + \frac{\lambda_N}{2} \eta^T K \eta \right\}.
$$

Here, $\|\theta\|_{H_K}$ is a seminorm of $\theta$ in $H_K$, $\lambda_N$ is a regularization parameter, $\eta = [\eta_1, \ldots, \eta_N]^T \in \mathbb{R}^N$, $k_i = [K(x_i, x_1), \ldots, K(x_i, x_N)]^T \in \mathbb{R}^N$ ($i = 1, \ldots, N$), and $K = [k_1, \ldots, k_N] \in \mathbb{R}^{N \times N}$. Note that the optimization problem in (7) is nonconvex, but this particular nonconvex function can be efficiently solved; see Section A.3 of the supplementary material for details. Finally, we winsorize the estimated rule $\tilde{\theta}$ from (7) to satisfy the bound restrictions for policy $\alpha$, i.e. $\tilde{\theta} = \mathcal{W}(\tilde{\theta})$ where $\mathcal{W}(\tilde{\theta}) = \tilde{\theta}^+ \{ \tilde{\theta} \in [0, 1] \} + 1\{ \tilde{\theta} > 1 \}$. Investigators can then use winsorized rule $\tilde{\theta}$ to optimally allocate WASH facilities.

We conclude by briefly discussing how to choose the tuning parameters $\gamma$ and $\lambda$ in the SVM; for additional details and examples, see Steinwart and Christmann (2008) and Hastie et al. (2009). Recall that $\gamma$ controls the bandwidth of the SVM kernel whereas $\lambda$ is the regularization term for the SVM coefficients $\eta$ in (7). While there are theoretically optimal values of $\gamma$ and $\lambda$ (see Theorem 3.1 below), in practice, cross-validation is often used to choose these values where we would start with a grid of values for $\gamma, \lambda$ and find the value that minimizes the cross-validated risk. Our empirical analysis also uses cross-validation to choose $\gamma, \lambda$ and Section A.4 of the supplementary materials lays out the details.

### 3.3 Estimating the Nuisance Functions

This section discusses how to estimate the nuisance functions inside the loss function in (6), specifically the outcome regression and the propensity score. For the outcome regression $\mu(\tilde{a}_i, x_i)$, as mentioned in Section 3.1, we suggest a nonparametric approach, say a supervised machine learning method or an ensemble of machine learning methods via the super learner algorithm (van der Laan et al., 2007; Polley and van der Laan, 2010); see Section A.2 of the supplementary materials for some examples. The estimated outcome regression is denoted as $\hat{\mu}$. For the propensity score $e(s | x_i)$, we suggest two approaches, a nonparametric approach and a parametric approach. The nonparametric approach is similar to fitting the outcome regression model where we use $A_{ij}$ and $X_{ij}$ as the dependent variable and the independent variables, respectively, inside of the super learner to estimate the propensity score $P(A_{ij} = a_{ij} | X_{ij} = x_{ij})$. Then, we use the estimated propensity
score to obtain an estimated $e(s \mid x_i)$ as follows.

$$\hat{e}(s \mid x_i) = \sum_{a_i} \prod_{j=1}^{n_i} \hat{P}(A_{ij} = a_{ij} \mid X_{ij} = x_{ij}).$$

Implicitly, the estimate $\hat{e}$ assumes that the propensity score $P(A_i = a_i \mid X_i = x_i)$ can be decomposed into a product of individual propensity scores $P(A_{ij} = a_{ij} \mid X_{ij} = x_{ij})$. If this assumption is implausible or the estimated $\hat{e}$ is numerically unstable due to the product term, an alternative approach is to use a parametric model. Specifically, suppose we bin $\bar{A}_i$ into $(M + 1)$ equi-spaced bins and consider the following ordinal regression model for $\hat{e}$.

$$P\left(\bar{A}_i \leq \frac{t}{M+1} \mid x_i\right) = \expit(\beta_0 + x_i^T \beta), \; \beta_{00} \leq \beta_{01} \leq \ldots \leq \beta_{0M}, \; t = 0, \ldots, M.$$

We then estimate $(\beta_{00}, \ldots, \beta_{0M}, \hat{\beta})$ via the likelihood principle and the estimate of $e$ is simply the plug-in estimates of $\beta$, i.e., $\hat{e}(s \mid x_i) = \expit(\hat{\beta}_0 + x_i^T \hat{\beta})$, where

$$\hat{\beta} = \hat{\theta} = \left[\hat{\theta}(\ell) \mid \ell \in \{1, 2\}\right].$$

Specifically, suppose we split the data into two folds $D_1$ and $D_2$ and let $\hat{\mu}(\ell), \hat{e}(\ell), \hat{L}(\ell), \ell \in \{1, 2\}$ be the estimated outcome regression, propensity score, and loss function, respectively, by using $D_\ell$. We estimate the OMAR by solving the SVM in (7) where $\hat{L}(\ell)$ is evaluated in $D_\ell$, i.e.,

$$\hat{\theta}(\ell)(x_{(\ell),i}) = \hat{\eta}^\top \eta + \hat{b}(\ell)$$

where

$$\hat{(\eta_\ell, \hat{b}(\ell))} = \arg\min_{\eta, b} \left\{ \frac{1}{N/2} \sum_{i \in D_\ell} \hat{L}(\ell)(\hat{k}_{(\ell),i} \eta + b, O_{(\ell),i}) + \frac{\lambda N}{2} \eta^\top K(\ell) \eta \right\}.$$

Here $O_{(\ell),i}$ is the $i$-th observation in $D_\ell$, $k_{(\ell),i} = [K(x_{(\ell),i}, x_{(\ell),1}), \ldots, K(x_{(\ell),i}, x_{(\ell),N/2})]^\top (i \in D_\ell)$, and $K(\ell) = [k_{(\ell),1}, \ldots, k_{(\ell),N/2}]$. Again, we take the winsorized rule $\hat{\theta}(\ell)$ to make sure the optimal rule is bounded between 0 and 1, i.e., $\hat{\theta}(\ell) = W(\hat{\theta}(\ell))$. Investigators may use either $\hat{\theta}_{(1)}, \hat{\theta}_{(2)}$, or the winsorized average of the two rules, i.e., $W(\{\hat{\theta}_{(1)} + \hat{\theta}_{(2)}\}/2)$, as the final estimated OMAR. Also, it’s common to repeat cross-fitting multiple times and aggregate the final estimates to remove finite-sample effects from sample splitting; see Section A.5 in the supplementary materials for details.

### 3.4 Theoretical Properties

To study the performance of the estimate rule $\hat{\theta}(\ell)$ from the previous section, we introduce the following assumptions about the estimated nuisance functions $\hat{\theta}(\ell)$ and $\hat{e}(\ell)$ ($\ell = 1, 2$).

1. **Overlap of $\hat{e}(\ell)$**: There exists a constant $c' > 0$ such that $c' < \hat{e}(\ell)(s \mid X_i) < 1 - c'$ for any $s = 0, 1, \ldots, n_i$ and $X_i$. 

\[ 11 \]
(E2) Bounded \( \hat{\mu}_{(-\ell)} \): \( \hat{\mu}_{(-\ell)}(s/n_i, X_i) \) is bounded for any \( s = 0, 1, \ldots, n_i \) and \( X_i \).

(E3) Consistency of \( \hat{\ell}_{(-\ell)} \) and \( \hat{\mu}_{(-\ell)} \): As \( N \to \infty \), we have \( \lambda_{e,N} := \| \hat{\lambda}_{(-\ell)}(s | X_i) - e(s | X_i) \|_{P,2} = o(1) \) and \( \lambda_{\mu,N} := \| \hat{\lambda}_{(-\ell)}(s/n_i, X_i) - \mu(s/n_i, X_i) \|_{P,2} = o(1) \) for any \( s \) with probability \( 1 - \Delta N \) and \( \Delta N = o(1) \).

Assumption (E1) implies that the estimated propensity score \( \hat{\ell}(s | X_i) \) satisfies overlap and Assumption (E2) implies that the estimated outcome regression is bounded. For the Senegal DHS, Assumption (E2) is satisfied because the outcome is binary. Assumption (E3) is satisfied so long as Assumption (E1) implies that the estimated propensity score \( \hat{\ell}(s | X_i) \) satisfies overlap and Assumption (E2) is satisfied because the outcome is binary. Assumption (E3) is satisfied so long as the estimated nuisance functions converge to the true nuisance functions. We remark that similar, but more stringent, conditions are introduces in previous works (e.g. Chernozhukov et al. (2018)) to obtain a \( \sqrt{N} \)-consistent estimator whereas we only need convergence at some rate because our goal is characterize the estimator’s excess risk, a common metric of performance in the optimal treatment regime literature (Zhao et al., 2012; Chen et al., 2016; Díaz et al., 2018). In particular, Assumption (E3) holds even if both nuisance functions are estimated at rates slower than \( N^{-1/4} \).

Theorem 3.1 states the excess risk of \( \hat{\theta}_{(-\ell)} \) under Assumptions (A1)-(A5) and (E1)-(E3).

**Theorem 3.1.** Suppose that Assumptions (A1)-(A5) and (E1)-(E3) hold and \( \theta^* \) belongs to a Besov space on \( \mathbb{R}^d \) with smoothness parameter \( \beta > 0 \); see Section B.3 of the supplementary materials for the definition of a Besov space. If \( \gamma_N \propto N^{-1/(2\beta+d)} \) and \( \lambda_N \propto N^{-(\beta+d)/(2\beta+d)} \), we have \( R(\hat{\theta}_{(-\ell)}) - R(\theta^*) = O_P(N^{-\beta/(2\beta+d)}) + O_P(r_{e,N}r_{\mu,N}) \).

The upper bound on the excess risk is determined by two quantities, the convergence rate of the SVM (i.e. \( N^{-\beta/(2\beta+d)} \)) and the convergence rate of the estimated nuisance functions (i.e. \( r_{e,N}r_{\mu,N} \)). The convergence rate of the SVM depends on the smoothness of the true OMAR \( \theta^* \) and if \( \theta^* \) is very smooth (i.e. \( \beta \to \infty \) and \( d \) is fixed), the rate of the excess risk bound approximates the rate \( N^{-1/2} \) from Li (2018) if the nuisance functions are estimated at \( N^{-1/4} \). Also, in the same scenario, our excess risk is better than the \( N^{-1/4} \) rate from Chen et al. (2016) because unlike their work, our work finds an optimal rule that reaches the desired target \( T \) and we have an additional monotonicity constraint (A5).

### 4 Simulation

We conduct a small simulation study to assess the finite-sample performance of the different rules. We generate \( N = 500 \) clusters and, for block \( i \), we randomly generate the size of cluster \( n_i \in \{2, \ldots, 10\} \) with equal probability. For unit \( j \) in block \( i \), we generate six unit-level covariates \( (W_{ij1}, \ldots, W_{ij6}) \) and three block-level covariates \( (C_{i1}, C_{i2}, C_{i3}) \) independently from \( W_{ijk} \sim N(0,1) \) (\( k = 1, \ldots, 4 \)), \( W_{ijk} \sim \text{Ber}(0.5) \) (\( k = 5, 6 \)), and \( C_{ik} \sim N(0,1) \) (\( k = 1, 2, 3 \)), respectively. For each
block \( i \), we generate the number of treated units as follows:

\[
S_i \mid X_i \sim \text{Bin}(n_i, \expit\{-2 + 1.51(\bar{W}_{i1} \geq 0.5) + 2.5\bar{W}_{i2}^2 + 2\bar{W}_{i5} - 2C_{i1}\}) \, , \quad \bar{W}_{ik} = \sum_j W_{ijk}/n_i .
\]

The outcome model is given below:

\[
Y_i \mid (A_i, X_i) \sim N\left(0.4 + (0.2 + 0.05C_{i1}^2)\bar{A}_i + 0.0251(\bar{W}_{i1}\bar{W}_{i2} \geq 1) + 0.025\bar{W}_{i1}\bar{W}_{i5}, \frac{0.25^2}{n_i}\right).
\]

The propensity score and the outcome regression model satisfy Assumptions (A1)-(A5). For the direct rule, we unify the dimension of \( X_i \) with \( \bar{X}_i \) to use the SVM, run 5-fold cross-validation for the SVM parameters, and do median-adjustment from 25 cross-fitted estimators. For the indirect rule, we use two methods to estimate \( \mu \), a linear regression with main and second-order interaction terms of \((\bar{A}_i, \bar{X}_i)\) regularized by the Lasso penalty (Tibshirani, 1996) and random forests (Breiman, 2001); these methods are referred to as Lasso and RF, respectively. For the target outcome \( T \), we vary between 0.45 and 0.55. We evaluate the performance of each method on a test set with 500 blocks. We repeat this entire process five times and obtain 2,500 OMAR estimates for each method and \( T \).

Figure 4.1 shows the absolute deviations between the estimated and the true OMARs. Our direct rule shows the best performance where its absolute deviation between the predicted and the true OMARs are uniformly the smallest across all threshold values \( T \). Also, the absolute deviations under our direct rule has the smallest standard errors.

![Figure 4.1: Boxplots of the Absolute Deviation Between the Predicted and True OMARs.](image)

(x and y-axes show the threshold \( T \) and the absolute deviance between the predicted and true OMARs, respectively.

Next, we assess the methods using classification performance measures. Specifically, given an OMAR \( \theta \), we define the true positive (TP), true negative (TN), false positive (FP), and false
negative (FN) quantities.

\[ TP = \sum_{i \in D_{\text{test}}} 1\{Y_i > T, \bar{A}_i > \theta(X_i)\} , \quad TN = \sum_{i \in D_{\text{test}}} 1\{Y_i \leq T, \bar{A}_i \leq \theta(X_i)\} , \]
\[ FP = \sum_{i \in D_{\text{test}}} 1\{Y_i \leq T, \bar{A}_i > \theta(X_i)\} , \quad FN = \sum_{i \in D_{\text{test}}} 1\{Y_i > T, \bar{A}_i \leq \theta(X_i)\} . \]

Given these definitions of true and false positives/negatives, we use the following classification performance measures: accuracy, F1 score, inverse F1 score and the Matthews correlation coefficient (MCC) (Matthews, 1975) which are defined below.

\[ \text{Accuracy} = \frac{TP + TN}{|D_{\text{test}}|} , \quad \text{F1} = \frac{2TP}{2TP + FP + FN} , \quad \text{Inverse-F1} = \frac{2TN}{2TN + FP + FN} , \]
\[ \text{MCC} = \frac{TP \times TN - FP \times FN}{\sqrt{(TP + FP) \times (TP + FN) \times (TN + FP) \times (TN + FN)}} . \]

For each classification performance measure, a larger value means better performance. Figure 4.2 shows the result for each method. We also plot the true OMAR for comparison. The true OMAR generally performs the best across most all four measures of performance and all thresholds \( T \), justifying that the four measures are useful for evaluating the performance of the rules. Among the estimated rules, our direct rule generally performs well across most \( T \) and the three classification performance measures other than F1 score. Combining with 4.1, we would generally recommend our direct rule in terms of bias and classification performance measures.

![Figure 4.2](image)

Figure 4.2: Evaluation of OMARs Using Different Measures of Classification Performance. \( x \) and \( y \)-axes show the threshold \( T \) and the value of the measure that is specified at the top of each plot, respectively. A larger value in the \( y \) axis implies better performance.

5 Optimal Allocation of WASH Facilities in Senegal

5.1 Background

We use the proposed methods to estimate the optimal allocation of WASH facilities in Senegal. Specifically, we use the 2014-2018 Senegal DHS, which used a two-stage stratified survey design where households are nested under census blocks. As a reminder, we treat the census blocks as
clusters and households within each census block as individual study units. We restrict the sample to households with complete data on $O_{ij} = (Y_{ij}, A_{ij}, X_{ij})$; see our setup section in Section 2.1 for the exact definition of the observed variables. For training the allocation rule, we use the 2014-2017 DHS, which contains $N = 1,027$ census blocks with 13,556 households. For testing the allocation rule, we use the 2018 DHS, which contains 213 census blocks with 2,859 households. In particular, for the 2018 test set, we vary the diarrhea-free target threshold $T$ from 0.64 to 0.73, which is just slightly above and below the diarrhea-free incidence level of 0.689 between 2014 and 2017. Then, for each target threshold $T$, the estimated OMAR reports the minimum proportion of WASH facilities needed to have at least $T$ proportion of households in a census block to be diarrhea free.

5.2 Discussion on Assumptions (A1)-(A5)

We take a moment to discuss the plausibility of Assumptions (A1)-(A5) in the Senegal DHS. Assumption (A1) is plausible as long as households in different census blocks do not interact with each other. In the data, 99.15% of the census blocks are geographically far apart from each other; the average/median distance among 55,578 (99.15%) pairs of census blocks in the 2018 Senegal DHS is 245.04km and 230.17km, respectively, while only 192 (0.85%) pairs of census blocks have distance smaller than 10km. Given this, we find that the partial interference assumption is plausible where interference likely occurs between households in the same census block and not across different census blocks. For Assumptions (A2) and (A3), we checked covariate balance and overlap by using the binning approach in Hirano and Imbens (2004), Kluve et al. (2012) and Flores et al. (2012) for a continuous treatment variable. Overall, all 9 observed covariates are balanced across different bins of treatment values and overlap is reasonable; see Section A.6 of the supplementary materials for details.

Assumption (A4) is plausible so long as the average number of diarrhea-free children in a census block can be reasonably approximated by census block-level summaries. However, the assumption may fail if a few households’ presence (or absence) of WASH facilities is driving the incidence of diarrhea in the entire block, say if these WASH-less households are located near communal water sources. Unfortunately, the data does not contain information about the location of households. Instead, in Section A.6 of the supplementary material, we visually diagnose the assumption using a residual plot of the predicted values of the mean block-level response versus the observed $Y_i$. While the diagnostic is not perfect, we find the predicted means do not show trends across the x-axis and Assumption (A4) seems plausible.

Finally, for Assumption (A5), many prior works (Esrey et al., 1985; Daniels et al., 1990; Clasen et al., 2007; Ejemot-Nwadiaro et al., 2015; McMichael, 2019) suggest that installing WASH facilities will not have a negative impact on incidence of diarrhea; however, it may have a negative effect on other, non-health outcomes. Also, when we empirically assess Assumption (A5), we find that the monotonicity assumption is rarely violated in the Senegal DHS and if violated, the deviation from monotonicity is small; see Section A.6 of the supplementary material for details.
5.3 Results

Finding 1: Our direct rule is more accurate.

For all of our empirical results below, we use the same estimators and classification performance measures used in the simulation study in Section 4 except we cross-fit 100 times. Figure 5.1 shows the four classification performance measures. In terms of accuracy, F1 score, and MCC, our direct rule performs better than the indirect rules across all thresholds $T$ except very small values of $T$. For inverse F1, the direct rule performs well for small thresholds and shows comparable performance for large thresholds.

![Figure 5.1: Evaluation of the Estimated OMARs in 2018 Senegal DHS. $x$ and $y$-axes show the threshold $T$ and the magnitude of each classification performance measure, respectively. Higher $y$ values indicate better performance.](image)

To help understand the results in Figure 5.1, Figure 5.2 shows the allocation rules at four values of the thresholds. Specifically, the heatmap shows the weighted averages of the estimated OMARs aggregated to 45 Senegalese administrative regions where the weight is based on the number of households in a census block (i.e. $n_i$); see Section A.7 of the supplementary material for more details behind the heatmaps. For each $T$, our direct rule is generally more diffused than the other two indirect rules, implying that our direct rule generally prefers a more balanced, nation-wide approach to allocating WASH facilities across Senegal compared to the other rules. Also, since our direct rule is more accurate than the other rules, the finding broadly suggests that the optimal strategy to allocate WASH facilities in Senegal is a diffuse approach compared to a targeted, all-or-nothing approach, where neighbors of an area with a high estimated OMAR should also receive a relatively large amount of WASH facilities.

Finding 2: Our direct rule is more cost-efficient.

Figure 5.3 shows the weighted averages of estimated OMARs needed to achieve the target outcome level $T$, weighted by the size of the census blocks. When $T$ is less than 0.695, the two indirect rules suggest that threshold $T$ can be achieved with fewer WASH facilities. On the contrary, when $T$ is greater than 0.695, our direct rule suggests that threshold $T$ can be achieved with fewer WASH facilities.
Figure 5.2: Estimated OMARs Across 45 Administrative Regions in 2018 Senegal DHS. Each column corresponds to one of the four threshold values $T$. Each row corresponds to one of the allocation rules. Each heatmap shows the average of the estimated OMAR. Darker (lighter) blue color indicates that the estimated average OMAR in the administrative region is larger (smaller).

Figure 5.3: Averages of Estimated OMARs in 2018 Senegal DHS. $x$ and $y$-axes show the threshold $T$ and the average of estimated OMARs across census blocks.

Combining our finding on the accuracy of our direct rule, our direct allocation rule is better than the other, indirect methods in terms of correctly targeting the right regions of Senegal in order to achieve the highest accuracy. To better explain this, consider again Figure 5.2. While our direct rule requires more resources than the other rules for thresholds $T$ less than 0.695, the direct rule is more accurate than competing methods, suggesting that the indirect methods are targeting the wrong region of Senegal or devoting less resources than what’s necessary to achieve the threshold. This can be seen in the first two columns of Figure 5.2 where the Lasso and random
forest are heavily targeting three administrative regions of Senegal whereas the direct rule is diffusely targeting multiple regions, which ultimately resulted in higher accuracy. Also, for thresholds greater than 0.695, our direct rule is using less resources and has higher accuracy than competing methods, suggesting that the other indirect methods are, again, targeting the wrong regions of Senegal or overusing resources than what’s necessary to achieve the target threshold. In particular, in the last two columns of Figure 5.2, the random forest is over-allocating WASH facilities across most of Senegal, even in areas that may not need additional WASH facilities to achieve threshold $\mathcal{T}$. In contrast, our direct rule is generally allocating less resources throughout while simultaneously achieving higher accuracy.

**Finding 3:** Compared to current policy recommendations, our direct rule uses different characteristics to allocate WASH.

In international development, one of the prevailing advice on allocation of WASH facilities is to target rural areas (ESARO, 2019; World Bank Group, 2019). Specifically, these works suggest that nations use proportion of rural areas in a region to allocate WASH facilities, where a large proportion of rural areas would indicate that a large number of WASH facilities is needed. However, as depicted in Figure 5.4, our direct rule suggests this should not be the main strategy to allocate WASH facilities. Instead, the allocation rule should focus on average household size in a census block. For example, the regions in Figure 5.4 marked with an asterisk (*) are mostly rural areas but are associated low OMAR estimates. In contrast, the same regions have higher average household size and is associated with higher OMAR estimates. In general, our direct rule is more closely correlated with the average household size, where a census block with many large households need more WASH facilities. This correlation between the OMAR and average household size also agrees with prior works in infectious diseases where incidence of diarrheal disease is strongly correlated with household size (Yilgwan et al., 2012; Beyene and Melku, 2018; Omona et al., 2020). In short, if the target outcome is to reduce incidence of diarrhea among children, our direct rule suggests that household size may be a better indicator compared to the proportion of rural area.

6 Discussion

In this paper, we consider optimal allocation of WASH facilities in Senegal when partial interference is present. Specifically, given a target objective for the response, say that the proportion of diarrhea-free children must be at least 70%, the estimated rule reports the proportion of WASH facilities necessary to achieve this objective. We consider two potential solutions, one based on a modest extension of an indirect method and the other based on a direct method using a novel loss function. In particular, for the direct method, we frame our problem as a solution to a risk minimization problem and use a SVM, which is based on empirical risk minimization, to find the optimal allocation rule. We then use the methods to train the optimal allocation of WASH fa-
Figure 5.4: Estimated OMARs (left), Household Size (middle), and Proportion of Rural Areas (right) in the 2018 Senegal DHS. Deeper (lighter) blue color indicates a larger (smaller) value. The two regions with an asterisk (*) are mostly rural, but are associated with low OMARs and small household sizes.

cilities by using the 2014-2018 Senegal DHS. We show that our direct rule is more accurate and cost-efficient than the other methods. We also show the difference between our direct rule and what’s currently recommended for WASH allocation in international development.

We end by providing some guidelines on how to use our direct rule and interpret its results in practice. First, one may simply maximize the overall potential outcome without having to set a target $T$, especially if Assumption (A5) no longer holds. We conjecture that our approach can be modified to find the optimal treatment rule that maximized equation (3), but the convergence rate is likely slower than the convergence rate discussed in Section 3.2. Second, as mentioned in the very beginning, our OMAR lets policymakers know the proportion of households in a census block that need WASH facilities. The rule does not provide how to allocate the given amount of WASH facilities within a census block. Such an allocation rule would require granular-level data on household-to-household interactions along with a plausible model for such interactions. Given that the Senegal DHS does not contain such information, we leave this as future work. Third, similar to any method in causal inference with observational data, the estimated OMAR for Senegal makes untestable assumptions that cannot be validated with the observed data. This includes not only the usual identification Assumptions (A1)-(A3), but also modeling Assumptions (A4)-(A5) to allow for consistent estimation of the overall potential outcome. While our aforementioned diagnostics showed no reason for concern with respect to these assumptions, the diagnostics are not perfect. In particular, should any of these assumptions be violated, it’s likely that our estimated OMAR may provide misleading allocations of WASH facilities. Nevertheless, so long as these assumptions are plausible, first-order approximations of the underlying truth, we believe our direct rule provides a useful, data-driven approach to allocate public resources related to infectious diseases for a nation.
Supplementary Material

This document contains supplementary materials for “Optimal Allocation of Water and Sanitation Facilities To Prevent Communicable Diarrheal Diseases in Senegal Under Partial Interference.” Section A presents additional results related to the main paper. Section B proves lemmas and theorems stated in the paper.

A Details of the Main Paper

A.1 Inverse Probability-Weighted and Outcome Regression-based Loss Functions

We introduce the IPW and outcome regression-based loss functions. Specifically, we can replace \( \psi_{DR} \) in (6b) in the main paper with the following functions.

\[
\psi_{IPW}(s, O_i) = \frac{\overline{Y}_i \mathbb{1}(S_i = s)}{e(s | X_i)}, \quad \psi_{OR}(s, O_i) = \mu \left( \frac{s}{n_i}, X_i \right).
\]

A.2 Machine Learning Methods Used for the Outcome Regression Estimation

As candidate machine learning methods, we include the following methods and R packages in our super learner library: linear regression (\texttt{glm}), Lasso/elastic net (\texttt{glmnet} (Friedman et al., 2010)), spline (\texttt{earth} (Friedman, 1991), \texttt{polspline} (Kooperberg, 2020)), generalized additive model (\texttt{gam} (Hastie and Tibshirani, 1986)), boosting (\texttt{xgboost} (Chen and Guestrin, 2016), \texttt{gbm} (Greenwell et al., 2019)), random forest (\texttt{ranger} (Wright and Ziegler, 2017)), and neural net (\texttt{RSNNS} (Bergmeir and Benítez, 2012)).

A.3 Computation: Training the Support Vector Machine in (7) in the Main Paper

This section presents the computational details on training the support vector machine in (7) in the main paper. The algorithm to train SVMs under this type of nonconvex function is already discussed in prior works (An and Tao, 1997; Chen et al., 2016) and we present a summary of it for completeness. Also, to keep the notation clear, the discussion below assumes that the nuisance functions are known, but the identical computation algorithm is used to train the SVM when the nuisance functions are estimated.

To start off, we can decompose the loss function in equation (6) in the main paper into the difference of two convex function \( L_+(t, O_i) \) and \( L_-(t, O_i) \), i.e. \( L(t, O_i) = L_+(t, O_i) - L_-(t, O_i) \) for
any $t$ and $O_i$ where

$$L_+(t, O_i) = \begin{cases} 
\nu_+(0, O_i) - 2\delta t & \text{if } -\infty < t < 0 \\
\nu_+(t, O_i) & \text{if } 0 \leq t \leq 1 \\
\nu_+(1, O_i) + (\bar{\delta} + 2\delta)(t - 1) & \text{if } 1 < t < \infty 
\end{cases}$$

$$L_-(t, O_i) = \begin{cases} 
\nu_-(0, O_i) - 2\delta t - \delta + \delta e^t & \text{if } -\infty < t < 0 \\
\nu_-(t, O_i) & \text{if } 0 \leq t \leq 1 \\
\nu_-(1, O_i) + (\bar{\delta} + 2\delta)(t - 1) - \delta + \delta e^{1-t} & \text{if } 1 < t < \infty 
\end{cases}$$

$$\nu_\pm(t, O_i) = \sum_{s=0}^{n_i} \sum_{\ell=0}^{n_i-s} \binom{n_i}{s} \binom{n_i-s}{\ell} \left\{ \frac{\psi_{DR}(s, O_i)(-1)^\ell}{\ell + s + 1} \right\}_{\pm} t^{t+s+1} + (T)_{\pm} t$$

Here, $(a)_+ = \max(a, 0)$, $(a)_- = -\min(a, 0)$, and $\bar{\delta}$ is chosen as the maximum of the left derivatives of $\nu_+(t, O_i)$ and $\nu_-(t, O_i)$ at $t = 1$, i.e., $\bar{\delta} = \max\left\{ \lim_{t \downarrow 0} \nabla \nu_+(1 - \epsilon, O_i), \lim_{t \downarrow 0} \nabla \nu_-(1 - \epsilon, O_i) \right\}$ and $\nabla \nu_\pm(t, O_i)$ is the derivative of $\nu_\pm(t, O_i)$ with respect to $t$. Critically, the two loss functions $L_+$ and $L_-$ are convex and non-decreasing in $t$.

Given the decomposition of the loss function into the difference of two convex functions, we use the DC algorithm (An and Tao, 1997), which is an iterative algorithm, to solve the original non-convex optimization problem; see Algorithm 1 for details.

**Algorithm 1 DC Algorithm**

**Require:** Initialize values $\eta^{(0)} \in \mathbb{R}^N$, $b^{(0)} \in \mathbb{R}$. Set iteration number to zero, $j \leftarrow 0$.
1: Precompute the gradient $\nabla L_-(t, O_i)$ where

$$\nabla L_-(t, O_i) = \begin{cases} 
\frac{\partial}{\partial t} L_-(t, O_i) & t \neq 0, 1 \\
\frac{1}{2} \lim_{\epsilon \uparrow 0} \left\{ \frac{\partial}{\partial \epsilon} L_-(t + \epsilon, O_i) + \frac{\partial}{\partial \epsilon} L_-(t - \epsilon, O_i) \right\} & t = 0, 1 
\end{cases}$$

2: repeat
3: Let $\eta^{(j+1)}$ and $b^{(j+1)}$ be the solution to the following convex optimization problem.

$$\begin{bmatrix} \eta^{(j+1)} \\ b^{(j+1)} \end{bmatrix} \in \arg \min_{\eta, b} \left\{ \frac{1}{N} \sum_{i=1}^{N} \left[ L_+(k_i^T \eta + b, O_i) - \nabla L_-(k_i^T \eta^{(j)}) + b^{(j)}, O_i \right] (b + k_i^T \eta) + \frac{\lambda N}{2} \eta^T K \eta \right\}$$

4: $j \leftarrow j + 1$
5: until convergence
6: return $(\hat{\eta}, \hat{b}) \leftarrow (\eta^{(j)}, b^{(j)})$.

To initiate the DC algorithm, we choose the initial value as follows. First, for each $i$, let the solution be $r_i$, i.e., $r_i = \arg \min_{t \in [0, 1]} L(t, O_i)$ which can be obtained from a grid-search. In words, $r_i$ is an approximate of $\hat{\theta}(x_i)$ that are found by a grid-search. But, since $r_i$ is bounded in the unit interval, it may not be a suitable approximate of $\hat{\theta}(x_i)$, the SVM solution before the
winsorization. As a consequence, directly using $r_i$ to construct initial points may lead to an estimate rule shrinking to a certain value, i.e. a rule does not reflect the heterogeneity induced by $\bar{x}_i$. To stretch $r_i$ outside of the unit interval, we fit a linear regression where non-0 and non-1 $r_i$s are regressed on $\psi_{DR}(\sum_{j=1}^{m_i} A_{ij}, O_i)$. For clusters having $r_i = 1$, we use the larger value between the predicted regression value and 1; similarly, for clusters having $r_i = 0$, we use the smaller between the predicted regression value and 0. Non-0 and non-1 $r_i$s are used without regression adjustment. We denote the regression-adjusted $r_i$ as $\hat{r}_i$. Second, we take $b(0) = \sum_{i=1}^{N} \hat{r}_i / N$ and $\eta(0)$ as a vector satisfying $\hat{r}_i = k_i^T \eta(0) + b(0)$ for all $i$; i.e. $\hat{r} = K \eta(0) + b(0) \mathbf{1}$ where $\hat{r} = [\hat{r}_1, \ldots, \hat{r}_N]^\top \in \mathbb{R}^N$ and $\mathbf{1} = [1, \ldots, 1]^\top \in \mathbb{R}$. Even though the kernel matrix $K$ is invertible due to the positive definiteness of the kernel function $\mathcal{K}$, the inverse of $K$ cannot be obtained due to the numerical singularity. Under such case, we add a tiny value to diagonal of $K$ until its inverse can be obtained. In line 1, $\nabla L_-$ is a subgradient of $L_-$ that accounts for the non-differentiability of $L_-$ at $t = 0$ and $t = 1$.

The convex optimization in line 3 can be solved by using many standard algorithms and softwares. The iteration stops when $\| (\eta^{(j+1)}, b^{(j+1)}) - (\eta^{(j)}, b^{(j)}) \|_2$ drops below some threshold value.

We remark that because the objective function in (7) in the main paper is bounded below, the algorithm will always converge in finite steps (An and Tao, 1997; Chen et al., 2016).

A.4 Details of Cross-validation

We present the details on how to choose the SVM parameters $\gamma$ and $\lambda$. We consider a set of candidate values for $(\gamma_\ell, \lambda_\ell)$ where $\ell = 1, \ldots, K$. Without loss of generality, let the estimation data fold be $\mathcal{D}_1 = \mathcal{D}_2^c$ and, as a consequence, observations in $\mathcal{D}_2$ is used to evaluate the estimated loss function $\hat{L}_{(-1)}(t, O_\ell)$ for $i \in \mathcal{D}_2$. We further split $\mathcal{D}_2$ into training and tuning sets, denoted by $\mathcal{D}_{2,\text{train}}$ and $\mathcal{D}_{2,\text{tuning}}$, respectively, based on the number of cross-validation folds. For each candidate parameter $(\gamma_\ell, \lambda_\ell)$, we estimate the direct OMAR rule $\hat{\theta}_{\text{train}}(X_i; \ell)$ by only using the training set $\mathcal{D}_{2,\text{train}}$ and obtain the empirical risk using the tuning set $\mathcal{D}_{2,\text{tuning}}$. The optimal parameters $(\gamma^*, \lambda^*)$ are the minimizer of the average of the empirical risks across the tuning sets, i.e.

$$(\gamma^*, \lambda^*) = \arg \min_{\ell = 1, \ldots, K} \frac{1}{|\mathcal{D}_{2,\text{tuning}}|} \sum_{i \in \mathcal{D}_{2,\text{tuning}}} \hat{L}_{(-1)}(\hat{\theta}_{\text{train}}(X_i; \ell), O_\ell).$$

A.5 Details of Cross-fitting Procedures

We discuss the details on how to negate the impact of a particular realization of the cross-fitting procedure. Once we split the data into two folds $\mathcal{D}_1$ and $\mathcal{D}_2$, we obtain two direct rules $\hat{\theta}_{(-\ell)}$ for $k = 1, 2$ where $\mathcal{D}_1^c$ is used as the estimation data fold and $\mathcal{D}_\ell$ is used as the evaluation data fold. Investigators may use either $\hat{\theta}_{(-1)}$ or $\hat{\theta}_{(-2)}$ as the final estimate of the OMAR, denoted by $\hat{\theta}(F)$. However, we recommend to use $\hat{\theta}(F)(x) = \mathcal{W}((\hat{\theta}_{(-1)} + \hat{\theta}_{(-2)})/2)(x)$, the winsorized rule of the average of two unwinsorized rules, for the new $\bar{x}$ as the estimate of the OMAR to fully use the
A.6 Assessment of Assumptions in the Main Paper

We present the details on how to assess covariate balance and overlap. First, Algorithm 2 shows the details on the covariate balance assessment. In the Senegal DHS, \( q_1 = 0.22 \) and \( q_2 = 0.8 \), and we use the median of the propensity score estimates from 100 cross-fitting procedures. As a consequence, we obtain the unadjusted/adjusted \( t \)-statistics in Figure A.1. Overall, the covariate balance was satisfied except for the case under cluster size and \( A_2 \), but the violation is not severe based on the magnitude of the \( t \)-statistic 2.045.

Algorithm 2 Assessment of Covariate Balance

1. Let \( \mathcal{A}_i : \mathcal{A}_1 = \{ i : \mathcal{A}_i \in R_1 := [0, q_1) \}, \mathcal{A}_2 = \{ i : \mathcal{A}_i \in R_2 := (q_1, q_2) \}, \) and \( \mathcal{A}_3 = \{ i : \mathcal{A}_i \in R_3 := (q_2, 1] \} \) where \( q_1 \) and \( q_2 \) are tertiles of \( \mathcal{A}_i \) in the training data.
2. Obtain unadjusted \( t \)-statistics that compare the distribution of \( X_i \) between \( \mathcal{A}_k \) and \( \mathcal{A}_k^c \).
3. For each \( i \), we obtain the median of the estimated propensity scores \( \hat{e}_k^{(\text{median})}(X_i) = \hat{P}(\mathcal{A}_i \in R_k | X_i) \) (\( i = 1, \ldots, N, k = 1, 2, 3 \)) from 100 cross-fitting procedures.
4. Let \( \mathcal{E}_{j,k} \) (\( j = 1, \ldots, 10, k = 1, 2, 3 \)) be the collection of indices \( i \) so that \( \hat{e}_k^{(\text{median})}(X_i) \) belongs to the \( j \)th bin of \( \{ \hat{e}_k^{(\text{median})}(X_i) | i \in \{ 1, \ldots, N \} \} \) where 10 bins are defined by deciles of \( \{ \hat{e}_k^{(\text{median})}(X_i) | i \in \mathcal{A}_k \} \).
5. Obtain \( t \)-statistics that compare the distribution of \( X_i \) between \( \mathcal{E}_{j,k} \cap \mathcal{A}_k \) and \( \mathcal{E}_{j,k} \cap \mathcal{A}_k^c \).
6. Aggregate the \( t \)-statistics obtained in Step 5 with weights from the size of \( \mathcal{E}_{1,k}, \ldots, \mathcal{E}_{10,k} \).
7. Obtain adjusted \( t \)-statistics by taking the median of \( t \)-statistics in Step 6 across \( T \) cross-fitting procedures.

Second, we assess the overlap assumption based on Algorithm 3. Again, in the Senegal DHS, \( q_1 = 0.22 \) and \( q_2 = 0.8 \), and we use the median of the propensity score estimates from 100 cross-fitting procedures. Figure A.2 shows histograms that visually assess the overlap assumption. Based on the histograms, the overlap assumption seems to be satisfied or to be not severely violated.
Algorithm 3 Assessment of Overlap

1: Let $A_1 = \{ i \mid \overline{A_i} \in R_1 := [0, q_1] \}$, $A_2 = \{ i \mid \overline{A_i} \in R_2 := (q_1, q_2] \}$, and $A_3 = \{ i \mid \overline{A_i} \in R_3 := (q_2, 1] \}$ where $q_1$ and $q_2$ are tertiles of $\overline{A_i}$ in the training data.

2: For each $i$, we obtain the median of the estimated propensity scores $\hat{e}_k^{(\text{median})}(X_i) = \hat{P}(\overline{A_i} \in R_k \mid X_i)$ ($i = 1, \ldots, N, k = 1, 2, 3$) from 100 cross-fitting procedures.

3: Compare histograms of $\hat{e}_k^{(\text{median})}(X_i)$ for $A_k$ and $A_k^*$ for each $k = 1, 2, 3$.

Next we visually check Assumption (A4) via residual plots. Specifically, let $\hat{\epsilon}_i^{(\text{median})} = Y_i - \hat{\mu}^{(\text{median})}(A_i, X_i)$ be the residuals where $\hat{\mu}^{(\text{median})}$ is the median of the outcome regression from 100 cross-fitting procedures. We compare the residuals across the outcome regression estimate and the regressors $(A_i, X_i)$ and check whether the residuals deviate from zero in Figure A.3. In general, the residuals are close to zero across the regressors, implying that the outcome regression under Assumption (A4) is not severely violated.

Lastly, Assumption (A5) can be empirically assessed by two means. First, let $\hat{\mu}_c$ be the outcome regression estimate obtained from $c$th ($c = 1, \ldots, C$) cross-fitting estimate. We obtain the mean $\bar{\mu}(s/n_i, X_i) = \sum_{c=1}^C \hat{\mu}_c(s/n_i, X_i)/C$ for each $s$ and $X_i$ where $\hat{\mu}_c$ does not use the data from census block $i$. Then, for each census block $i$, we vary $s$ from 0 to $n_i$ (in integer increments) and count
the number of times monotonicity is violated. Second, we measure the worst-case slope of the estimated \( \hat{\mu} \) as follows. Let \( V_{si} = \tilde{\mu}(s/n_i, X_i) - \tilde{\mu}((s - 1)/n_i, X_i) \) (\( s = 1, \ldots, n_i \)) be the signed variation of \( \mu \) between \( [s-1, s] \) and let \( TV_i = \sum_{s=1}^{n_i} |V_{si}| \) be the total variation of \( \mu \) between \([0, n_i]\). We compute the relative magnitude of the slopes that are decreasing compared to the total variation of \( \hat{\mu} \), i.e. \( \frac{\sum_{i=1}^{N} \sum_{s=1}^{n_i} \mathbb{1}(V_{si} < 0) |V_{si}|}{\sum_{i=1}^{N} TV_i} \). Overall, under the first assessment, we found that the monotonicity is violated 1.35% of the time and under the second assessment, the relative magnitude of decreasing slopes is \( 1.89 \times 10^{-3} \). In short, the empirical validations show that the monotonicity assumption is rarely violated in the Senegal DHS and if violated, the deviation from monotonicity is small.

### A.7 Details of Figures 5.2-5.4 in the Main Paper

We additionally describe how we draw Figures 5.2-5.4 in the main paper. The reported estimated OMARs in Figures 5.2 and 5.4 are weighted average of the estimated OMARs in each administrative region where weights are the number of households in a census block, i.e. census block size \( n_i \). That is, the values represent \( \bar{\theta}_g \)'s, which are defined as

\[
\bar{\theta}_g = \frac{\sum_{i \in D_{2018}} \mathbb{1}\{i \in \text{administrative area } g\} \cdot n_i \cdot \hat{\theta}(x_i)}{\sum_{i \in D_{2018}} \mathbb{1}\{i \in \text{administrative area } g\} \cdot n_i}
\]

where \( D_{2018} \) is the collection of census blocks in the 2018 Senegal DHS. In words, \( \bar{\theta}_g \) is the proportion of households in administrative area \( g \) that require WASH facilities. Similarly, the average
household sizes in Figure 5.4 represent
\[
\bar{x}_{g, \text{Household Size}} = \frac{\sum_{i \in D_{2018}} 1 \{ i \in \text{administrative area } g \} \cdot n_i \cdot \bar{x}_{i, \text{Household Size}}}{\sum_{i \in D_{2018}} 1 \{ i \in \text{administrative area } g \} \cdot n_i}.
\]
Again, \( \bar{x}_{g, \text{Household Size}} \) is the average household wise in administrative area \( g \). The proportions of rural area in Figure 5.4 represent
\[
\bar{x}_{g, \text{Rural}} = \frac{\sum_{i \in D_{2018}} 1 \{ i \in \text{administrative area } g \} \cdot c_{i, \text{Rural}}}{\sum_{i \in D_{2018}} 1 \{ i \in \text{administrative area } g \}}.
\]
Here \( \bar{x}_{g, \text{Rural}} \) is the proportion of the rural census blocks in administrative area \( g \). Note that \( \bar{y}_g \), \( \bar{x}_{g, \text{Household Size}} \), and \( \bar{x}_{g, \text{Rural}} \) do not address the geographical distance between census regions in different administrative areas. But, we believe that these statistics are geographically meaningful summaries to highlight the heterogeneity across administrative areas; see Figures 1 and 2 of Houngbonon et al. (2021) for similar summary statistics aggregated at Senegalese administrative areas.

Lastly, Figure 5.3 shows the weighted average of the estimated OMARs across all 45 administrative areas where weights are the number of households in a census block, i.e. census block size \( n_i \). That is, \( y \)-axis represents \( \bar{y} \), which is defined as
\[
\bar{y} = \frac{\sum_{i \in D_{2018}} n_i \cdot \hat{\theta}(x_i)}{\sum_{i \in D_{2018}} n_i}.
\]
In words, \( \bar{y} \) is the proportion of households in Senegal that require WASH facilities.

**B Proof of Lemmas and Theorems**

**B.1 Useful Lemmas**

**Lemma B.1.** Suppose that \( \theta^* \) belongs to a Besov space on \( \mathbb{R}^d \) with smoothness parameter \( \beta > 0 \), i.e. \( \mathcal{B}_1^\beta(\mathbb{R}^d) = \{ \theta \in L_\infty(\mathbb{R}^d) \mid \sup_{r > 0} t^{-\beta} \{ \omega_r, L_1(\mathbb{R}^d) \{ \theta, t \} \} < \infty, r > \beta \} \) where \( \omega_r \) is the modulus of continuity of order \( r \). Then, for any \( \epsilon > 0 \) and \( d/(d+\tau) < p < 1 \) with \( \tau > 0 \), we have the following excess risk bound of \( \hat{\theta} \) with probability not less than \( 1 - 3e^{-\tau} \).

\[
R(\hat{\theta}) - R(\theta^*) \leq c_1 \lambda_N \gamma_N^{-d} + c_2 \gamma_N^{-\beta} + c_3 \left( \gamma_N^{-d(1-p)/(1+\epsilon)} \lambda_N^{p} \right)^{-\frac{1}{1-\epsilon}} + c_4 N^{-1/2} \tau^{1/2} + c_5 N^{-1} \tau
\]

**Lemma B.2.** Let \( \hat{R}_{(-\ell)}(\theta) = \mathbb{E}\{ \hat{L}_{(-\ell)}(\theta(X_i, O_i) \mid D_{-\ell}^i) \} \) be the estimated risk function where the expectation is taken with respect to \( O_i \) while \( \hat{L}_{(-\ell)} \) is considered as a fixed function which is clarified by denoting \( D_{-\ell}^i \) in the conditioning statement. Under conditions in Assumption (A1)-(A5) and (E1)-(E3) in the main paper, we have \( |R(\theta) - \hat{R}_{(-\ell)}(\theta)| = O_P(r_N) \) where \( r_N = r_{e,N} \) if the inverse
probability-weighted loss function is used, \( r_N = r_{\mu,N} \) if the outcome regression loss function is used, and \( r_N = r_{e, N \mu_N} \) if the doubly robust loss function is used.

### B.2 Proof of Lemma 3.1

Let \( \Theta^{[0,1]} = \{ f \mid f(x_i) \in [0, 1] \} \) be the collection of rules ranging over the unit interval and \( W(\theta) \in \Theta^{[0,1]} \) be the winsorized function of \( \theta \in \Theta \) over the unit interval, i.e.

\[
W(\theta)(x_i) = 0 \cdot 1\{\theta(x_i) < 0\} + \theta(x_i) \cdot 1\{0 \leq \theta(x_i) \leq 1\} + 1 \cdot 1\{1 < \theta(x_i)\}.
\]

From the definition of \( L \), we find \( L(0, O_i) \leq L(t, O_i) \) for any \( t \in (-\infty, 0) \) and \( L(1, O_i) \leq L(t, O_i) \) for any \( t \in (1, \infty) \). As a consequence, for any rule \( \theta \in \Theta \) satisfies \( L(W(\theta)(X_i), O_i) \leq L(\theta(X_i), O_i) \) and \( R(W(\theta)) \leq R(\theta) \). This implies that \( \theta' \), the minimizer of \( R \), must belong to \( \Theta^{[0,1]} \).

For any function \( \theta \in \Theta^{[0,1]} \), we find \( L(\theta(X_i), O_i) = \nu(\theta(X_i), O_i) \) and \( R(\theta) = R^{[0,1]}(\theta) \triangleq E\{\nu(\theta(X_i), O_i)\} \) due to the constructions of \( L \) and \( R \). Combining the above results, we observe the following relationship.

\[
\arg \min_{\theta \in \Theta} R(\theta) = \arg \min_{\theta \in [0,1]} R(\theta) = \arg \min_{\theta \in \Theta^{[0,1]}} R^{[0,1]}(\theta)
\]

Thus, it suffices to show that \( \theta^* \) defined in (4) in the main paper minimizes \( R^{[0,1]} \), which is represented as follows.

\[
R^{[0,1]}(\theta) = E\{\nu(\theta(X_i), O_i)\}
\]

\[
= E \left[ \int_0^1 E \left[ \sum_{s=0}^{n_i} \binom{n_i}{s} \nu(s, O_i) \mid X_i \right] \alpha^s (1-\alpha)^{n_i-s} - T \right] \mathbb{1}\{\alpha \leq \theta(X_i)\} d\alpha
\]

\[
= E \left[ \int_0^1 \left\{ \sum_{s=0}^{n_i} \binom{n_i}{s} \mu \left( \frac{s}{n_i}, X_i \right) \alpha^s (1-\alpha)^{n_i-s} - T \right\} \mathbb{1}\{\alpha \leq \theta(X_i)\} \right] d\alpha.
\]

The first and second identities are trivial from the definition of \( R^{[0,1]} \) and \( \nu \). The third identity is from \( E \{\psi(s, O_i) \mid X_i\} = \mu(s/n_i, X_i) \) for \( s = 0, 1, \ldots, n_i \) and \( k \in \{\text{IPW, OR, DR}\} \). From the last representation, it is straightforward that \( R^{[0,1]}(\theta) \) is minimized at \( \theta' \) that satisfies the following conditions.

\[
\sum_{s=0}^{n_i} \binom{n_i}{s} \mu \left( \frac{s}{n_i}, X_i \right) \alpha^s (1-\alpha)^{n_i-s} \leq T \quad \text{for all} \quad \alpha \in [0, \theta'(X_i)],
\]

\[
\sum_{s=0}^{n_i} \binom{n_i}{s} \mu \left( \frac{s}{n_i}, X_i \right) \alpha^s (1-\alpha)^{n_i-s} \geq T \quad \text{for all} \quad \alpha \in [0, \theta'(X_i)].
\]

As a consequence, \( \theta' \) agrees with \( \theta^* \) defined in (4) in the main paper. This concludes the proof.
B.3 Proof of Lemma B.1

The proof of B.1 is similar to that of Theorem 2 of Chen et al. (2016) which use Theorem 7.23 of Steinwart and Christmann (2008) and Theorem 2.2 and 2.3 of Eberts and Steinwart (2013). For completeness, we present a full exposition to our setting below. We first introduce Theorem 7.23 of Steinwart and Christmann (2008).

**Theorem 7.23. (Oracle Inequality for SVMs Using Benign Kernels; Steinwart and Christmann (2008))** Let $L$ be a loss function having non-negative value. Also, let $\mathcal{H}_K$ be a separable RKHS of a measurable kernel $K$ over $\mathcal{X}$ = supp($X_i$) $\subset \mathbb{R}^d$. Let $P$ be a distribution on $O_i$. Furthermore, suppose the following conditions are satisfied.

1. (C1) For all $(t, o_i)$, there exists a constant $B > 0$ satisfying $L(t, o_i) \leq B$.
2. (C2) $L(t, o_i)$ is locally Lipschitz continuous with respect to $t$.
3. (C3) For all $(t, o_i)$, we have $L(W_{\alpha_0}(t), o_i) \leq L(t, o_i)$ where $W_{\alpha_0}(t) = t \cdot 1 \{ |t| \leq c_0 \} + \text{sign}(t)c_0 \cdot 1 \{ c_0 < |t| \}$.
4. (C4) $E\{ (L(W_{\alpha_0}(\theta)(X_i), O_i) - L(\theta^*(X_i), O_i))^2 \} \leq V \cdot E\{ (L(W_{\alpha_0}(\theta)(X_i), O_i) - L(\theta^*(X_i), O_i))^v \}$ is satisfied for constant $v \in [0, 1]$, $V \geq B^{2-v}$, and for all $\theta \in \mathcal{H}_K$.
5. (C5) For fixed $N \geq 1$, there exists constants $\mu \geq 0$ such that the dyadic entropy number $E_{D_X \sim P_X^N}[e_i(\text{identity map} : \mathcal{H}_K \to L_2(D_X))] \leq a \cdot \mu^{-\frac{1}{2p}} (i \geq 1)$ where $e_i(A)$ is the entropy number of $A$.

We fix $\theta_0 \in \mathcal{H}_K$ and a constant $B_0 \geq B$ such that $L(\theta_0(x_i), o_i) \leq B_0$ for any $o_i$. Then, for all fixed $\tau > 0$ and $\lambda_N$, the SVM using $\mathcal{H}_K$ and $L$ satisfies

$$\lambda_N \|\theta\|^2_{\mathcal{H}_K} + R(W_{\alpha_0}(\theta)) - R(\theta^*) \leq 9 \{ \lambda_N \|\theta_0\|^2_{\mathcal{H}_K} + R(\theta_0) - R(\theta^*) \} + K_0 \left( \frac{a^{2p}}{\lambda_N N} \right)^{\frac{1}{2p - p + vp}} + 3 \left( \frac{72V\tau}{N} \right)^{\frac{1}{2-p}} + \frac{15B_0\tau}{N} .$$

(8)

with probability $P^N$ not less than $1 - 3e^{-\tau}$, where $K_0 \geq 1$ is a constant only depending on $p$, $c_0$, $B$, $v$, and $V$.

We verify Assumptions (C1)-(C5) as follows:

**Verification of Assumption (C1):** From Assumptions (A1)-(A5) and $\overline{\mu}_i \in [0, 1]$, we find that $\psi_{IPW}$, $\psi_{OR}$, and $\psi_{DR}$ are bounded and, as a consequence, $\nu$ in (6a) in the main paper is bounded. As a result, $L$ in (6) in the main paper is bounded as well.
Verification of Assumption (C2): We find the derivative of \( L \) in (6) in the main paper is
\[
\nabla L(t, o_i) = \begin{cases} 
\delta e^t & t \in (-\infty, 0) \\
\sum_{s=0}^{n_t} \psi_t(s, o_i)t^s(1-t)^{n_t-s} - \mathcal{T} & t \in (0, 1) \\
\delta e^{1-t} & t \in (0, \infty) 
\end{cases}
\]
and we find that \( \nabla L(t, O_i) \) is bounded for all \( t \) except \( t = 0, 1 \). Moreover, \( L(t, o_i) \) is continuous at \( t = 0 \) and \( t = 1 \). Thus, \( L(t, o_i) \) is locally Lipschitz continuous with Lipschitz constant \( B' = \sup_{(t, o_i)} \nabla L(t, o_i) \).

Verification of Assumption (C3): We take \( c_0 = 1 \). It is trivial that \( L(W_{c_0}(\theta), o_i) \leq L(\theta, o_i) \) from the form of \( L \) in (6) in the main paper.

Verification of Assumption (C4): Note that \( L(t, o_i) \leq B \). Thus, we find
\[
\mathbb{E}\left[ \{L(W_{c_0}(\theta)(X_i), O_i) - L(\theta^*(X_i), O_i)\}^2 \right] \leq 2\mathbb{E}\left[ \{L(W_{c_0}(\theta)(X_i), O_i)\}^2 + \{L(\theta^*(X_i), O_i)\}^2 \right] \leq 4B^2.
\]
We take \( v = 0 \) and \( V = 4B^2 \) and the condition is satisfied.

Verification of Assumption (C5): Since we use the Gaussian kernel, we can directly use Theorem 7.34 of Steinwart and Christmann (2008) which is given below.

**Theorem 7.34.** (Entropy Numbers for Gaussian Kernels; Steinwart and Christmann (2008)) Let \( \nu \) be a distribution on \( \mathbb{R}^d \) having tail exponent \( \tau \in (0, \infty] \). Then, for all \( \epsilon > 0 \) and \( d/(d+\tau) < p < 1 \), there exists a constant \( c_{\epsilon,p} \geq 1 \) such that
\[
e_i(\text{identity map} : \mathcal{H}_K \rightarrow L_2(\nu)) \leq c_{\epsilon,p}\gamma_i\left(\frac{(1-p)(1+\epsilon)d}{2p}\right)^{1-\frac{1}{2p}}
\]
for all \( i \geq 1 \) and \( \gamma \in (0, 1] \).

Therefore, Assumption (C5) holds with \( a = c_{\epsilon,p}\gamma \frac{(1-p)(1+\epsilon)d}{2p} \).

As a consequence, the result in equation (8) holds with \( c_0 = 1, v = 1, V = 4B^2, B_0 = B, a = c_{\epsilon,p}\gamma \frac{(1-p)(1+\epsilon)d}{2p} \). Moreover, we find \( L(W(\theta)(x_i), o_i) \leq L(W_{c_0=1}(\theta)(x_i), o_i) \) since \( L(0, o_i) \leq L(t, o_i) \) for \( t \in [-1, 0] \) and this leads \( R(W(\theta)) \leq R(W_{c_0=1}(\theta)) \). Thus, we find the following result holds
with probability $P^N$ not less than $1 - 3e^{-\tau}$.

\[
R(\tilde{\theta}) - R(\theta^*) \leq R(W_{\theta_0=1}(\tilde{\theta})) - R(\theta^*) \\
\leq \lambda_N \|\tilde{\theta}\|^2_{\mathcal{H}_K} + R(W_{\theta_0=1}(\tilde{\theta})) - R(\theta^*) \\
\leq 9\left\{ \lambda_N \|\theta_0\|^2_{\mathcal{H}_K} + R(\theta_0) - R(\theta^*) \right\} + K_0 \left\{ \frac{9^{-1-p}(1+c)d}{\lambda_N^p N} \right\}^{\frac{1}{2p}} + 36\sqrt{2B\sqrt{\frac{\tau}{N}}} + 15B\frac{\tau}{N}.
\]

The above result holds for any $\theta_0 \in \mathcal{H}_K$, so we can further bound the approximation error $\lambda_N \|\theta_0\|^2_{\mathcal{H}_K} + R(\theta_0) - R(\theta^*)$ by choosing $\theta_0$ in a specific way which is presented in Eberts and Steinwart (2013). We first define a function $Q_{r,\gamma} : \mathbb{R}^d \rightarrow \mathbb{R}$ as

\[
Q_{r,\gamma}(z) = \sum_{j=1}^{\lfloor r \rfloor} (-1)^{j-1} \frac{1}{j^d} \frac{2}{\gamma^2} K_{\gamma^2/2}(z) \ , \ K_{\gamma}(z) = \exp \left\{ -\gamma^2 \|z\|_2^2 \right\}
\]

for $r \in \{1, 2, \ldots\}$ and $\gamma > 0$. Since the range of $\theta^*$ is bounded between $[0, 1]$, we find $\theta^* \in L_2(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$. Thus, we can define $\theta_0$ by convolving $Q_{r,\gamma}$ with $\theta^*$ as follows (Eberts and Steinwart, 2013).

\[
\theta_0(x_i) = (Q_{r,\gamma} * \theta^*)(x_i) = \int_{\mathbb{R}^d} Q_{r,\gamma}(x_i - z) \theta^*(z) dz.
\]

Next, we introduce theorem 2.2 and 2.3 of Eberts and Steinwart (2013).

**Theorem 2.2.** Let us fix some $q \in [1, \infty)$. Furthermore, assume that $P_X$ is a distribution on $\mathbb{R}^d$ that has a Lebesgue density $f_X \in L_p(\mathbb{R}^d)$ for some $p \in [1, \infty]$. Let $\theta : \mathbb{R}^d \rightarrow \mathbb{R}$ be such that $\theta \in L_q(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$. Then, for $r \in \{1, 2, \ldots\}$, $\gamma > 0$, and $s \geq 1$ with $1 = s^{-1} + p^{-1}$, we have

\[
\|Q_{r,\gamma} * \theta - \theta\|_{L_q(P_X)}^q \leq C_{r,q} \cdot \|f_X\|_{L_p(\mathbb{R}^d)} \cdot \omega_q^{\gamma/2}(\theta, \gamma/2)
\]

where $C_{r,q}$ is a constant only depending on $r$ and $q$.

**Theorem 2.3.** Let $\theta \in L_2(\mathbb{R}^d)$, $\mathcal{H}_K$ be the RKHS of the Gaussian kernel $K$ with parameter $\gamma > 0$, and $Q_{r,\gamma}$ be defined by (10) for a fixed $r \in \{1, 2, \ldots\}$. Then we have $Q_{r,\gamma} * \theta \in \mathcal{H}_K$ with

\[
\|Q_{r,\gamma} * \theta\|_{\mathcal{H}_K} \leq (\gamma \sqrt{\pi})^{-\frac{d}{2}} (2^r - 1) \|\theta\|_{L_2(\mathbb{R}^d)}.
\]

Moreover, if $\theta \in L_\infty(\mathbb{R}^d)$, we have $|Q_{r,\gamma} * \theta| \leq (2^r - 1) \|\theta\|_{L_\infty(\mathbb{R}^d)}$. 

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As a result, we obtain
\[
\begin{align*}
\lambda_N \| \theta_0 \|_{H^K}^2 + R(\theta_0) - R(\theta^*) \\
= \lambda_N \| Q_{r, \gamma_N} \ast \theta^* \|_{H^K}^2 + R(Q_{r, \gamma_N} \ast \theta^*) - R(\theta^*) \\
\leq \lambda_N (\gamma_N \sqrt{\pi})^{-d} (2^r - 1)^2 \| \theta^* \|_{L_2(\mathbb{R}^d)}^2 + R(Q_{r, \gamma_N} \ast \theta^*) - R(\theta^*) \\
\leq \lambda_N (\gamma_N \sqrt{\pi})^{-d} (2^r - 1)^2 \| \theta^* \|_{L_2(\mathbb{R}^d)}^2 + B' \cdot \| Q_{r, \gamma_N} \ast \theta^* - \theta^* \|_{L_1(P_X)} \\
\leq \lambda_N (\gamma_N \sqrt{\pi})^{-d} (2^r - 1)^2 \| \theta^* \|_{L_2(\mathbb{R}^d)}^2 + B' \cdot C_{r, 1} \cdot \| f_X \|_{L_1(\mathbb{R}^d)} \cdot \omega_{r, L_1(\mathbb{R}^d)}(\theta, \gamma_N/2) \\
\leq \lambda_N (\gamma_N \sqrt{\pi})^{-d} (2^r - 1)^2 \| \theta^* \|_{L_2(\mathbb{R}^d)}^2 + B' \cdot C_{r, 1} \cdot \| f_X \|_{L_1(\mathbb{R}^d)} \gamma_N^\beta
\end{align*}
\]

The first equality is from the construction of \( \theta_0 \). The first inequality is from Theorem 2.3 of Eberts and Steinwart (2013). The second inequality is from Lipschitz continuity of \( L \). The third inequality is from Theorem 2.2 of Eberts and Steinwart (2013) with \( q = s = 1 \) and \( p = \infty \). The last inequality holds for some constant \( c' \) since \( \theta^* \in \mathcal{B}_1(\mathbb{R}^d) \) implies \( \omega_{r, L_1(\mathbb{R}^d)}(\theta^*, \gamma_N/2) \leq c' \gamma_N^\beta \) from the definition of a Besov space. Combining the results in (9) and (11), we have the following result

\[
R(\hat{\theta}) - R(\theta^*) \leq 9 \left\{ \lambda_N (\gamma_N \sqrt{\pi})^{-d} (2^r - 1)^2 \| \theta^* \|_{L_2(\mathbb{R}^d)}^2 + B' \cdot C_{r, 1} \cdot \| f_X \|_{L_1(\mathbb{R}^d)} \gamma_N^\beta \right\} \\
+ K_0 \left\{ \frac{\gamma_N^{(1-p)(1+\epsilon)d}}{\lambda_N^p N} \right\} \frac{1}{1-p} + 36 \sqrt{2} B \sqrt{\frac{\tau}{N}} + 15 B \frac{\tau}{N} \\
\leq c_1 \lambda_N \gamma_N^{-d} + c_2 \gamma_N^\beta + c_3 \left\{ \frac{\gamma_N^{(1-p)(1+\epsilon)d}}{\lambda_N^p N} \right\} \frac{1}{1-p} + c_4 N^{-1/2} \tau^{1/2} + c_5 N^{-1} \tau
\]

**B.4 Proof of Lemma B.2**

We find the following result for \( t \in [0, 1] \).

\[
\left| \hat{L}_{(s)}(t, O_i) - L(t, O_i) \right| = \left| \hat{\nu}_{(s)}(t, O_i) - \nu(t, O_i) \right| \\
= \left| \sum_{s=0}^{n_i} \left( \frac{n_i}{s} \right) \left( \hat{\psi}_s(k, O_i) - \psi_s(k, O_i) \right) \right| \\
\leq \sum_{s=0}^{n_i} \left( \frac{n_i}{s} \right) \left| \hat{\psi}_s(k, O_i) - \psi_s(k, O_i) \right| \\
\leq C' \max_{s \in \{0, \ldots, M\}} \left| \hat{\psi}_s(k, O_i) - \psi_s(k, O_i) \right|
\]

for some generic constant \( C' \). The last inequality is from \( t \in [0, 1] \) and bounded \( n_i \). Also, we find the following result for \( t \in (-\infty, 0) \).

\[
\left| \hat{L}_{(s)}(t, O_i) - L(t, O_i) \right| = \left| \hat{\nu}_{(s)}(0, O_i) - \nu(0, O_i) \right| = 0.1
\]
Finally, we find the following result for \( t \in (1, \infty) \) for all \( s = 0, 1, \ldots, n_i \).

\[
\left| \hat{L}_{(-\ell)}(t, O_i) - L(t, O_i) \right| = \left| \hat{\psi}_{(-\ell)}(1, O_i) - \nu(1, O_i) \right| \leq C' \max_{s \in \{0, \ldots, M\}} \left| \hat{\psi}_k(s, O_i) - \psi_k(s, O_i) \right|
\]

where constant \( C' \) is given in (12). Therefore, for any \( \theta \), we find

\[
\left| R(\theta) - \hat{R}_{(-\ell)}(\theta) \right| \leq E \left[ \left| L(\theta(X_i), O_i) - \hat{L}_{(-\ell)}(\theta(X_i), O_i) \right| \right]_{D_\ell} \\
\leq E \left[ \left| L(\theta(X_i), O_i) - \hat{L}_{(-\ell)}(\theta(X_i), O_i) \right|^2 \right]_{D_\ell}^{1/2} \\
\leq C' E \left[ \max_{s \in \{0, \ldots, M\}} \left| \hat{\psi}_k(s, O_i) - \psi_k(s, O_i) \right|^2 \right]_{D_\ell}^{1/2} \\
\leq C' \max_{s \in \{0, \ldots, M\}} \left\| \hat{\psi}_k(s, O_i) - \psi_k(s, O_i) \right\|_{P, 2}
\]

The first inequality is from the definition of \( R \). The second inequality is from the Jensen’s inequality. The third inequality is from the above results. The last inequality is from the definition of \( \| \cdot \|_{P, 2} \). Therefore, it suffices to bound \( \| \hat{\psi}_k(s, O_i) - \psi_k(s, O_i) \|_{P, 2} \) for all three types of \( \psi_k \) where \( k \in \{ \text{IPW, OR, DR} \} \).

First, the difference between \( \hat{\psi}_{\text{IPW}}(s, O_i) \) and \( \psi_{\text{IPW}}(s, O_i) \) is

\[
\left| \hat{\psi}_{\text{IPW}}(s, O_i) - \psi_{\text{IPW}}(s, O_i) \right| = \left| \frac{Y_i 1(S_i = s)}{\hat{e}_{(-\ell)}(s \mid X_i)} - \frac{Y_i 1(S_i = s)}{e(s \mid X_i)} \right| \\
\leq \left| \frac{Y_i 1(S_i = s)}{\hat{e}_{(-\ell)}(s \mid X_i)} \right| \frac{\left| e(s \mid X_i) - \hat{e}_{(-\ell)}(s \mid X_i) \right|}{e(s \mid X_i)} \\
\leq \left| e(s \mid X_i) - \hat{e}_{(-\ell)}(s \mid X_i) \right|_{cc'}
\]

The upper bound is from the bounded outcome and Assumptions (A3) and (E1). Thus, we find \( \| \hat{\psi}_{\text{IPW}}(s, O_i) - \psi_{\text{IPW}}(s, O_i) \|_{P, 2} \leq \| e(s \mid X_i) - \hat{e}_{(-\ell)}(s \mid X_i) \|_{P, 2}/(cc') = O(r_{e, N}) \) with probability greater than \( 1 - \Delta_N \).

Second, we study the difference between \( \hat{\psi}_{\text{OR}} \) and \( \psi_{\text{OR}} \) which is \( \| \hat{\psi}_{\text{OR}}(s, O_i) - \psi_{\text{OR}}(s, O_i) \| = \| \hat{\mu}_{(-\ell)}(s/n_i, X_i) - \mu(s/n_i, X_i) \|_{P, 2} = O(r_{\mu, N}) \) with probability greater than \( 1 - \Delta_N \).

Lastly, we prove the result when \( \psi_{\text{DR}} \) is chosen. From (12), we have

\[
-C' \left[ \hat{\psi}_{\text{DR}}(s, O_i) - \psi_{\text{DR}}(s, O_i) \right] \leq \hat{L}_{(-\ell)}(t, O_i) - L(t, O_i) \leq C' \left[ \hat{\psi}_{\text{DR}}(s, O_i) - \psi_{\text{DR}}(s, O_i) \right]
\]
where the sign of $C'$ is chosen to satisfy the inequality above. The expectation of $\hat{\psi}_{\text{DR}} - \psi_{\text{DR}}$ is

$$E\left\{ \hat{\psi}_{\text{DR}}(s, O_i) - \psi_{\text{DR}}(s, O_i) \mid \mathcal{D}^c_{\ell} \right\}$$

$$= E\left\{ \frac{Y_i - \hat{\mu}_{(-\ell)}(s/n_i, X_i)}{\hat{e}_{(-\ell)}(s \mid X_i)} - \frac{Y_i - \mu(s/n_i, X_i)}{e(s \mid X_i)} \right\} 1(S_i = s) + \hat{\mu}_{(-\ell)}(s/n_i, X_i) - \mu(s/n_i, X_i) \mid \mathcal{D}^c_{\ell}$$

$$= E\left[ \frac{\{\mu(s/n_i, X_i) - \hat{\mu}_{(-\ell)}(s/n_i, X_i)\} e(s \mid X_i)}{\hat{e}_{(-\ell)}(s \mid X_i)} + \hat{\mu}_{(-\ell)}(s/n_i, X_i) - \mu(s/n_i, X_i) \mid \mathcal{D}^c_{\ell} \right]$$

$$= E\left[ \{\mu(s/n_i, X_i) - \hat{\mu}_{(-\ell)}(s/n_i, X_i)\} e(s \mid X_i) - \hat{e}_{(-\ell)}(s \mid X_i) \mid \mathcal{D}^c_{\ell} \right].$$

The equalities are straightforward from the definition of $\psi_{\text{DR}}$ and the law of total expectation. Since $c' \leq \hat{e}_{(-\ell)}$, we find

$$E\left[ \frac{\{\mu(s/n_i, X_i) - \hat{\mu}_{(-\ell)}(s/n_i, X_i)\} e(s \mid X_i)}{\hat{e}_{(-\ell)}(s \mid X_i)} \right] \leq \frac{1}{c'} E\left[ \frac{\{\mu(s/n_i, X_i) - \hat{\mu}_{(-\ell)}(s/n_i, X_i)\} e(s \mid X_i)}{\hat{e}_{(-\ell)}(s \mid X_i)} \mid \mathcal{D}^c_{\ell} \right]

\leq \frac{1}{c'} E\left[ \{\mu(s/n_i, X_i) - \hat{\mu}_{(-\ell)}(s/n_i, X_i)\} \ell(s/n_i, X_i) \mid \mathcal{D}^c_{\ell} \right]

\leq \frac{1}{c'} \ell_{\text{max}} \sup_{s \in \{0, \ldots, M\}} \left\{ \ell_{\text{max}} (s/n_i, X_i) \right\}_{P,2} \left\{ e(s \mid X_i) - \hat{e}_{(-\ell)}(s \mid X_i) \right\}_{P,2}.

The first inequality is straightforward. The second inequality is from the Hölder’s inequality. From the last line, we find $E\left\{ \hat{\psi}_{\text{DR}}(s, O_i) - \psi_{\text{DR}}(s, O_i) \mid \mathcal{D}^c_{\ell} \right\} = O(r_{e,N} r_{\mu,N}).$ As a result, we have the following results with probability greater than $1 - \Delta_N$.

$$\left| R(\theta) - \hat{R}_{(-\ell)}(\theta) \right| = \left| E\left\{ L(\theta(X_i), O_i) - \hat{L}_{(-\ell)}(\theta(X_i), O_i) \mid \mathcal{D}^c_{\ell} \right\} \right|

\leq C' \max_{s \in \{0, \ldots, M\}} E\left\{ \hat{\psi}_{\text{DR}}(s, O_i) - \psi_{\text{DR}}(s, O_i) \mid \mathcal{D}^c_{\ell} \right\}

\leq \frac{C}{c'} \max_{s \in \{0, \ldots, M\}} \left\{ \ell_{\text{max}} (s/n_i, X_i) \right\}_{P,2} \left\{ e(s \mid X_i) - \hat{e}_{(-\ell)}(s \mid X_i) \right\}_{P,2}

= O(r_{e,N} r_{\mu,N}).$$

Combining the established results, we have the following results with probability greater than $1 - \Delta_N$.

$$\left| R(\theta) - \hat{R}_{(-\ell)}(\theta) \right| = \begin{cases} O(r_{e,N}) & \text{if } \psi_{\text{TPW}} \text{ is chosen} \\
O(r_{\mu,N}) & \text{if } \psi_{\text{OR}} \text{ is chosen} \\
O(r_{e,N} r_{\mu,N}) & \text{if } \psi_{\text{DR}} \text{ is chosen} \end{cases}.$$

This implies $\left| R(\theta) - \hat{R}_{(-\ell)}(\theta) \right| = O_P(r_N)$ where $r_N = r_{e,N}$ if the inverse probability-weighted loss function is used, $r_N = r_{\mu,N}$ if the outcome regression loss function is used, and $r_N = r_{e,N} r_{\mu,N}$ if the doubly robust loss function is used.
B.5 Proof of Theorem 3.1 in the Main Paper

Before we present the proof, we start with defining the risk function and the OMAR associated with the estimated loss function. Let \( \hat{R}_{(-\ell)}(\theta) = E\{ \hat{L}_{(-\ell)}(\theta(X_i), O_i) \mid D_{\ell}^e \} \) be the estimated risk function where the expectation is taken with respect to \( O_i \) while \( \hat{L}_{(-\ell)} \) is considered as a fixed function which is clarified by denoting \( D_{\ell}^e \) in the conditioning statement. Accordingly, let \( \theta^*_(-\ell) \) be the approximated OMAR which is the minimizer of \( \hat{R}_{(-\ell)}(\theta) \), i.e. \( \hat{R}_{(-\ell)}(\theta^*_(-\ell)) \leq \hat{R}_{(-\ell)}(\theta) \) for all \( \theta \in \Theta \). Using \( \theta^*_(-\ell) \) as the intermediate quantities, we can establish the excess risk of \( \hat{\theta}_{(-\ell)} \).

We decompose the excess risk as follows.

\[
\left| R(\hat{\theta}_{(-\ell)}) - R(\theta^*) \right| = \left| R(\hat{\theta}_{(-\ell)}) - \hat{R}_{(-\ell)}(\hat{\theta}_{(-\ell)}) \right| + \left| \hat{R}_{(-\ell)}(\hat{\theta}_{(-\ell)}) - \hat{R}_{(-\ell)}(\theta^*_(-\ell)) \right| + \left| \hat{R}_{(-\ell)}(\theta^*_(-\ell)) - R(\theta^*) \right| .
\]

In the rest of the proof, we bound terms (A), (B), and (C).

From Lemma 3.1 in the main paper, we find the upper bound of (A).

\[
\left| R(\hat{\theta}_{(-\ell)}) - \hat{R}_{(-\ell)}(\hat{\theta}_{(-\ell)}) \right| = O_P(r_N) .
\]

Next, we bound of (B). Since \( \hat{L}_{(-\ell)} \) satisfies Assumptions (C1)-(C5) and \( \theta^*_(-\ell) \) belongs to a Besov space \( \mathcal{B}_1^{\beta}(\mathbb{R}^d) \), Theorem B.1 can be applied. Hence, we find

\[
\left| \hat{R}_{(-\ell)}(\hat{\theta}_{(-\ell)}) - \hat{R}_{(-\ell)}(\theta^*_(-\ell)) \right| = O_P\left(N^{-\frac{\beta}{4\beta+4d}}\right) .
\]

Lastly, we bound (C). Since \( \theta^* \) is the minimizer of \( R(\theta) \), we find

\[
\hat{R}_{(-\ell)}(\theta^*_(-\ell)) = R(\theta^*_(-\ell)) + \hat{R}_{(-\ell)}(\theta^*_(-\ell)) - R(\theta^*_(-\ell)) \geq R(\theta^*) + \hat{R}_{(-\ell)}(\theta^*_(-\ell)) - R(\theta^*_(-\ell)) ,
\]
and this implies \( \hat{R}_{(-\ell)}(\theta^*_(-\ell)) - R(\theta^*_(-\ell)) \leq \hat{R}_{(-\ell)}(\theta^*_(-\ell)) - R(\theta^*) \). Similarly, since \( \theta^*_(-\ell) \) is the minimizer of \( \hat{R}_{(-\ell)}(\theta) \), we find

\[
R(\theta^*) = \hat{R}_{(-\ell)}(\theta^*) + R(\theta^*) - \hat{R}_{(-\ell)}(\theta^*) \geq \hat{R}_{(-\ell)}(\theta^*_(-\ell)) + R(\theta^*) - \hat{R}_{(-\ell)}(\theta^*_(-\ell)) ,
\]
and this implies \( R(\theta^*) - \hat{R}_{(-\ell)}(\theta^*) \leq R(\theta^*) - \hat{R}_{(-\ell)}(\theta^*_(-\ell)) \), i.e. \( \hat{R}_{(-\ell)}(\theta^*_(-\ell)) - R(\theta^*) \leq \hat{R}_{(-\ell)}(\theta^*) - R(\theta^*) \). Combining two results, we have

\[
\left| \hat{R}_{(-\ell)}(\theta^*_(-\ell)) - R(\theta^*) \right| \leq \max \left\{ \left| \hat{R}_{(-\ell)}(\theta^*_(-\ell)) - R(\theta^*_(-\ell)) \right| , \left| \hat{R}_{(-\ell)}(\theta^*_(-\ell)) - R(\theta^*_(-\ell)) \right| \right\} .
\]

From Lemma 3.1 in the main paper, the right hand side of the above term is \( O_P(r_N) \). As a result, we find an upper bound of (C), which is \( \left| \hat{R}_{(-\ell)}(\theta^*_(-\ell)) - R(\theta^*) \right| = O_P(r_N) \). This concludes the desired result.
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