On Classical and Bayesian Asymptotics in Stochastic Differential Equations with Random Effects having Mixture Normal Distributions

Trisha Maitra and Sourabh Bhattacharya

Abstract

Delattre et al. (2013) considered a system of stochastic differential equations (SDE’s) in a random effects set-up. Under the independent and identical (iid) situation, and assuming normal distribution of the random effects, they established weak consistency of the maximum likelihood estimators (MLE’s) of the population parameters of the random effects.

In this article, respecting the increasing importance and versatility of normal mixtures and their ability to approximate any standard distribution, we consider the random effects having finite mixture of normal distributions and prove asymptotic results associated with the MLE’s in both independent and identical (iid) and independent but not identical (non-iid) situations. Besides, we consider iid and non-iid set-ups under the Bayesian paradigm and establish posterior consistency and asymptotic normality of the posterior distribution of the population parameters.

It is important to note that Delattre et al. (2016) also assumed the SDE set-up with normal mixture distribution of the random effect parameters but considered only the iid case and proved only weak consistency of the MLE under an extra, strong assumption as opposed to strong consistency that we are able to prove without the extra assumption. Furthermore, they did not deal with asymptotic normality of MLE or the Bayesian asymptotics counterpart which we investigate in details.

Keywords: Asymptotic normality; Finite mixture of normals; Maximum likelihood estimator; Posterior consistency; Random effects; Stochastic differential equations.

1 Introduction

Data pertaining to inter-individual variability and intra-individual variability with respect to continuous time can be modeled through systems of stochastic differential equations (SDE’s) consisting of random effects. In this regard, Delattre et al. (2013), Maitra and Bhattacharya (2015), Maitra and Bhattacharya (2016) investigate asymptotic inference in the context of systems of SDE’s of the following form:

\[ dX_i(t) = \phi_i b(X_i(t))dt + \sigma(X_i(t))dW_i(t), \quad \text{with} \quad X_i(0) = x^i, \quad i = 1, \ldots, n, \tag{1.1} \]

where, for \( i = 1, \ldots, n \), the stochastic process \( X_i(t) \) is assumed to be continuously observed on the time interval \([0, T_i]\) with \( T_i > 0 \) known, and corresponding to the \( i \)-th process initial values \( \{x^i; \ i = 1, \ldots, n\} \) are also assumed to be known. Here \( \{\phi_i; \ i = 1, \ldots, n\} \) are random effect parameters independent of the Brownian motions \( \{W_i(\cdot); \ i = 1, \ldots, n\} \). The above authors assume that \( \phi_i \) are independently and identically distributed (iid) with common distribution \( g(\varphi, \theta)d\nu(\varphi) \) where \( g(\varphi, \theta) \) is a density with respect to a dominating measure \( \nu \) on \( \mathbb{R}^d \) for all \( \theta \) (\( \mathbb{R} \) is the real line and \( d \) is the dimension). Here the unknown parameter to be estimated is \( \theta \in \Omega \subset \mathbb{R}^p \) \((p \geq 2d)\). In particular, the above authors assume that \( g(\varphi, \theta) \) is the Gaussian density with unknown means and covariance matrix, which are to be learned from the data and the model for classical inference (see Delattre et al. (2013), Maitra and Bhattacharya (2016)), and from a combination of the data, model and the prior for Bayesian inference (see Maitra and Bhattacharya (2015)). Statistically, the \( i \)-th process \( X_i(\cdot) \) corresponds to the \( i \)-th individual and the corresponding random effect is \( \phi_i \). The following conditions (see Delattre et al. (2013), Maitra and Bhattacharya (2015) and Maitra and Bhattacharya (2016)) that we assume ensure existence of solutions of (1.1):

\[(H1) \quad (i) \quad b(\cdot) \text{ and } \sigma(\cdot) \text{ are } C^1 \text{ (differentiable with continuous first derivative) on } \mathbb{R} \text{ satisfying } b^2(x) \leq K(1 + x^2) \text{ and } \sigma^2(x) \leq K(1 + x^2) \text{ for all } x \in \mathbb{R}, \text{ for some } K > 0.\]

*Trisha Maitra is a PhD student and Sourabh Bhattacharya is an Associate Professor in Interdisciplinary Statistical Research Unit, Indian Statistical Institute, 203, B. T. Road, Kolkata 700108. Corresponding e-mail: sourabh@isical.ac.in.
Almost surely for each \( i \geq 1, \)
\[
\int_0^{T_i} \frac{b^2(X_i(s))}{\sigma^2(X_i(s))} ds < \infty.
\]

\textbf{Delattre et al.} (2013) show that the likelihood, depending upon \( \theta \), admits a relatively simple form involving the following sufficient statistics:
\[
U_i = \int_0^{T_i} \frac{b(X_i(s))}{\sigma^2(X_i(s))} dX_i(s), \quad V_i = \int_0^{T_i} \frac{b^2(X_i(s))}{\sigma^2(X_i(s))} ds, \quad i = 1, \ldots, n. \quad (1.2)
\]

The exact likelihood is given by
\[
L(\theta) = \prod_{i=1}^n \lambda_i(X_i, \theta), \quad (1.3)
\]
where
\[
\lambda_i(X_i, \theta) = \int_{\mathbb{R}} g(\varphi, \theta) \exp \left( \varphi U_i - \frac{\varphi^2}{2} V_i \right) d\nu(\varphi). \quad (1.4)
\]

For the Gaussian distribution of \( \phi_i \) with mean \( \mu \) and variance \( \omega^2 \), that is, with \( g(\varphi, \theta) d\nu(\varphi) \equiv N(\mu, \omega^2) \), it is easy to obtain the following form of \( \lambda_i(X_i, \theta) \) (see Delattre et al. (2013)):
\[
\lambda_i(X_i, \theta) = \frac{1}{(1 + \omega^2 V_i)^{1/2}} \exp \left[ - \frac{V_i}{2(1 + \omega^2 V_i)} \left( \mu - \frac{U_i}{V_i} \right)^2 \right] \exp \left( \frac{U_i^2}{2V_i} \right), \quad (1.5)
\]
where \( \theta = (\mu, \omega^2) \in \Omega \subset \mathbb{R} \times \mathbb{R}^+ \). As in Delattre et al. (2013), Maitra and Bhattacharya (2016) and Maitra and Bhattacharya (2015) here also we assume that

\textbf{(H2)} \( \Omega \) is compact.

Delattre et al. (2013) consider \( x^i = x \) and \( T_i = T \) for \( i = 1, \ldots, n \), so that the set-up boils down to the \( iid \) situation, and investigate asymptotic properties of the \textit{MLE} of \( \theta \), providing proofs of consistency and asymptotic normality. As an alternative, Maitra and Bhattacharya (2016) verify the regularity conditions of existing results in general set-ups provided in Schervish (1995) and Hoadley (1971) to prove asymptotic properties of the \textit{MLE} in this \textit{SDE} set-up in both \( iid \) and non-\( iid \) cases. Interestingly, this alternative way of verification of existing general results allowed Maitra and Bhattacharya (2016) to come up with stronger results under weaker assumptions, compared to Delattre et al. (2013).

Maitra and Bhattacharya (2015), for the first time in the literature, established Bayesian asymptotic results \textit{SDE}-based random effects model, for both \( iid \) and non-\( iid \) set-ups. Specifically, considering prior distributions \( \pi(\theta) \) of \( \theta \), they established asymptotic properties of the corresponding posterior
\[
\pi_n(\theta|X_1, \ldots, X_n) = \frac{\pi(\theta) \prod_{i=1}^n \lambda_i(X_i|\theta)}{\int_{\psi \in \Omega} \pi(\psi) \prod_{i=1}^n \lambda_i(X_i|\psi)d\psi} \quad (1.6)
\]
as the sample size \( n \) tends to infinity, through the verification of regularity conditions existing in Choi and Schervish (2007) and Schervish (1995).

In this article, we extend the asymptotic works of Maitra and Bhattacharya (2016) and Maitra and Bhattacharya (2015) assuming that the random effects are modeled by finite mixtures of normal distributions. The importance and generality of such mixture models are briefly discussed in Section 1.1

### 1.1 Need for mixture distribution for the random effects parameters

The need for validation of our asymptotic results established in Maitra and Bhattacharya (2016) and Maitra and Bhattacharya (2015) for a bigger class of distributions corresponding to the random effect
parameter $\phi$ leads us to consider finite mixture of normal distributions as the distribution of $\phi$, since any continuous density can be approximated arbitrarily accurately by an appropriate normal mixture.

Indeed, as is well-known, the unknown data generating process can be flexibly modeled by a mixture of parametric distributions. In fact, when a single parametric family can not provide a satisfactory model due to local variations in the observed data, mixture models can handle quite complex distributions by the appropriate choice of its components. So mixture distributions can be thought as the basis approximation to the unknown distributions. That finite mixture of normals with enough components can approximate any multivariate density, is established by Norets and Pelenis (2012). As infinite degree of smoothness and wide range of flexibility of mixture of normal densities allow us to model any unknown smooth density, it has been used for various inference problems including cluster analysis, density estimation, and robust estimation (see, for example, Banfield and Raftery (1993), Lindsay (1995), Roeder and Wasserman (1997)). Mixture models have also been adopted in hazards models (Louvada-Neto et al. (2002)), structural equation models (Zhu and Lee (2001)), analysis of proportions (Brooks (2001)), disease mapping (Green and Richardson (2002)), neural networks (Bishop (1995)), and in many more areas. Wirjanto and Xu (2009) provide a survey on recent developments and applications of normal mixture models in empirical finance.

Apart from the applicabilities of mixture models to various well-known statistical problems, it is worth noting that mixtures have important roles to play even in random effects models. Indeed, since mixture models are appropriate for data that are expected to arise from different sub-populations, these provide interesting extension of ordinary random effects models to random effects models associated with multiple sub-populations. From the authors’ perspective, this provides a particularly sound motivation for considering mixture distributions for the random effects.

It is important to note that system of $SDE$’s where the random effect parameters are samples from Gaussian mixtures, has also been considered by Delattre et al. (2016). However, asymptotically, their contribution is confined to only proving weak consistency of the $MLE$ in the $iid$ set-up. In contrast, we are able to prove strong and weak consistency of the $MLE$ in the $iid$ and non-$iid$ cases, respectively, asymptotic normality of the $MLE$ in both $iid$ and non-$iid$ set-ups, and consistency and asymptotic normality of the Bayesian posterior distribution.

For finite samples Delattre et al. (2016) recommend the standard $EM$ algorithm (Dempster et al. (1977)) for computation of $MLE$ and the standard Bayes Information Criterion ($BIC$) (see, for example, Kass and Raftery (1995)) for selecting the number of mixture components (see Leroux (1992) and Keribin (2000) for $BIC$ related to mixture models). In the Bayesian set-up it is natural to consider the number of mixture components to be unknown and place a prior on this unknown quantity. Recently, Das and Bhattacharya (2017) have proposed a novel and efficient transformation based variable-dimensional Markov chain Monte Carlo method to simulate from variable-dimensional distributions; they refer to this method as Trandimensional Transformation based Markov chain Monte Carlo (TTMCMC) and demonstrate the success of this method and its supremacy over the erstwhile reversible jump Markov chain Monte Carlo (RJMCMC) method (Green (1995), Richardson and Green (1997)) specifically in the case of mixtures with unknown number of components. Hence, we recommend TTMCMC for the Bayesian version of our $SDE$ mixture problem.

There is another issue to be remarked about, namely, the well-known label-switching problem associated with mixtures, which is also persistent in our $SDE$ set-up. Indeed, as is evident from (2.1) and (2.2), the likelihood remains the same even if the labels of $\{(a_k, \beta_k) : k = 1, \ldots, M\}$ are permuted, showing non-identifiability of the likelihood with respect to label-switching of the parameter components. Under somewhat restrictive assumption (see (H4) of Delattre et al. (2016)), Delattre et al. (2016) show that if two such $SDE$ mixtures with two sets of parameters are the same, then the two sets of parameters are also the same. However, that simply does not even touch the label-switching problem, and the weak consistency of $MLE$ proved by Delattre et al. (2016) in the $iid$ set-up holds up to label-switching. In contrast, none of our results require assumption (H4) of Delattre et al. (2016), which are still unique up to label-switching. Specifically, in the $iid$ case we have been able to prove strong consistency of the $MLE$ under weaker assumption compared to weak consistency proved by
Delattre et al. (2016) under stronger assumption. Finally, it is important to remark in this context that Das and Bhattacharya (2017) argue (see Section S-6 of their supplement) that avoidance of label-switching is not desirable from the clustering perspective.

The rest of our paper is structured as follows. In Section 2 we describe the likelihood corresponding to normal mixture distribution of the random effects, and in Sections 3 and 4 we investigate asymptotic properties of MLE in the iid and non-iid contexts, respectively. In Sections 5 and 6 we investigate Bayesian asymptotics in the iid and non-iid set-ups, respectively. We summarize our work and provide concluding remarks in Section 7. Moreover, in Section S-3 of the supplement we briefly discuss extension of our asymptotic results to the multidimensional random effects set-up.

Notationally, “a.s. →”, “P →” and “L →” denote convergence “almost surely”, “in probability” and “in distribution”, respectively.

2 Likelihood with respect to normal mixture distribution of the random effects

We consider the following normal mixture form of the distribution of \( \phi \):

\[
g(\varphi, \theta) d\nu(\varphi) \equiv \sum_{k=1}^{M} a_k N(\mu_k, \omega_k^2),
\]

such that \( a_k \geq 0 \) for \( k = 1, \ldots, M \) and \( \sum_{k=1}^{M} a_k = 1 \). We assume that \( \theta = (\gamma, \beta) \) where \( \gamma = (a_1, a_2, \ldots, a_M) \) and \( \beta = (\beta_1, \ldots, \beta_M) \), such that \( \beta_i = (\mu_i, \omega_i^2); i = 1, \ldots, M \). Here, for all \( k = 1, \ldots, M \), \( (\mu_k, \omega_k^2) \in \Omega_\beta \subset \mathbb{R} \times \mathbb{R}^+ \). We assume that \( \Omega_\beta \) is compact, so that, denoting the simplex by the compact space \( \Omega_\gamma \) on which \( \gamma \) lie, it follows that \( \Omega = \Omega_\gamma \times \Omega_\beta^M \) is compact, in accordance with (H2). Our likelihood corresponding to the \( i \)-th individual is

\[
\lambda_i(X_i, \theta) = \sum_{k=1}^{M} a_k f(X_i|\beta_k),
\]

where

\[
f(X_i|\beta_k) = \frac{1}{(1 + \omega_k^2 V_i)^{1/2}} \exp \left[ - \frac{V_i}{2(1 + \omega_k^2 V_i)} \left( \mu_k - \frac{U_i}{V_i} \right)^2 \right] \exp \left( \frac{U_i^2}{2 V_i} \right).
\]

Assuming independence of the individuals conditional on the parameters, it then follows that the complete likelihood is the product of (2.1) over the \( n \) individuals, having the form (1.3).

Note that, due to non-identifiability of the mixture form of the subject-wise likelihood (2.1), our asymptotic results will be unique up to label switching (see, for example, Redner (1981)).

3 Consistency and asymptotic normality of MLE in the iid set-up

3.1 Strong consistency of MLE

Consistency of the MLE under the iid set-up can be verified through the verification of the regularity conditions of the following theorem (Theorems 7.49 and 7.54 of Schervish (1995)); for our purpose we present the version for compact \( \Omega \).

**Theorem 1 (Schervish (1995))** Let \( \{X_n\}_{n=1}^{\infty} \) be conditionally iid given \( \theta \) with density \( \lambda_1(x|\theta) \) with respect to a measure \( \nu \) on a space \( (\mathcal{X}, \mathcal{B}^1) \). Fix \( \theta_0 \in \Omega \), and define, for each \( S \subseteq \Omega \) and \( x \in \mathcal{X} \),

\[
Z(S, x) = \inf_{\psi \in S} \log \frac{\lambda_1(x|\theta_0)}{\lambda_1(x|\psi)}.
\]
Assume that for each \( \theta \neq \theta_0 \), there is an open set \( N_\theta \) such that \( \theta \in N_\theta \) and that \( E_{\theta_0} Z(N_\theta, X_i) > -\infty \). Also assume that \( \lambda_1(x|\cdot) \) is continuous at \( \theta \) for every \( \theta \), a.s. \( [P_{\theta_0}] \). Then, if \( \theta_n \) is the MLE of \( \theta \) corresponding to \( n \) observations, it holds that \( \lim_{n \to \infty} \theta_n = \theta_0 \), a.s. \( [P_{\theta_0}] \).

### 3.1.1 Verification of strong consistency of MLE in our SDE set-up

Let the true set of parameter be \( \theta_0 = (\gamma_0, \beta_0) \) where \( \gamma_0 = (a_{0,1}, a_{0,2}, \ldots, a_{0,M}) \) and \( \beta_0 = (\beta_{0,1}, \beta_{0,2}, \ldots, \beta_{0,M}) \), such that \( \beta_{0,i} = (\mu_{0,i}, \omega_{0,i}^2) \) for \( i = 1, \ldots, M \).

To verify the conditions of Theorem 1 in our case, we note that for any \( x \), \( \lambda_1(x|\theta) = \sum_{k=1}^M a_k f(x|\beta_k) \) where \( f(x|\beta_k) \) given by (2.2) is clearly continuous in \( \beta_k \), implying continuity of \( \lambda_1(x|\theta) \) in \( \theta \). Also, it follows from the proof of Proposition 7 of Delattre et al. (2013) that

\[
\log f(x|\beta_{0,k}) = \frac{1}{2} \log \left( \frac{1 + \omega_{0,k}^2 V}{1 + \omega_{0,k}^2 V} \right) + \frac{1}{2} \left( \frac{\omega_{0,k}^2 - \omega_1^2}{1 + \omega_{0,k}^2 V} \right) \left( \frac{U}{1 + \omega_{0,k}^2 V} \right)^2 \left( \frac{1 + \omega_{0,k}^2}{\omega_1^2} \right)
\]

Hence,

\[
f(x|\beta_{0,k}) \leq \exp(C_1(U, V, \beta_1, \beta_{0,k})) f(x|\beta_1).
\]

Now,

\[
\left| \log \frac{\lambda_1(x|\theta)}{\lambda_1(x|\theta_0)} \right| = \log \frac{\sum_{k=1}^M a_{0,k} f(x|\beta_{0,k})}{\sum_{k=1}^M a_k f(x|\beta_k)} \leq \log \left| \frac{\sum_{k=1}^M a_{0,k} f(x|\beta_{0,k})}{\sum_{k=1}^M a_k f(x|\beta_1)} \right|
\]

Hence,

\[
\left| \log \frac{\lambda_1(x|\theta)}{\lambda_1(x|\theta_0)} \right| \leq \left| \log \sum_{k=1}^M a_{0,k} \exp(C_1(U, V, \beta_1, \beta_{0,k})) \right| a_1 + |\log a_1| \leq \sum_{k=1}^M |C_1(U, V, \beta_1, \beta_{0,k})| + |\log a_1| \leq \sum_{k=1}^M C_1(U, V, \beta_1, \beta_{0,k}) + |\log a_1| \quad \text{(as } C_1 > 0 \text{)}.
\]
The last inequality is due to the following result of Atienza et al. (2005) that \( \sum_{k=1}^{M} a_k = 1 \), with \( a_k \geq 0 \) implies
\[
\left| \log \sum_{k=1}^{M} a_k f_k \right| \leq \sum_{k=1}^{M} \left| \log f_k \right|
\]
for \( f_k > 0; \ k = 1, \ldots, M \).

Taking \( N_\theta \) to be any open subset of the relevant compact parameter space containing \( \theta \), and noting that
\[
E_{\theta_0} \left[ \inf_{\theta \in N_\theta} \frac{\lambda_1(X|\theta_0)}{\lambda_1(X|\theta)} \right] \geq E_{\theta_0} \left[ \inf_{\theta \in N_\theta} \frac{\lambda_1(X|\theta_0)}{\lambda_1(X|\theta)} \right] = -E_{\theta_0} \left[ \sup_{\theta \in N_\theta} \frac{\lambda_1(X|\theta_0)}{\lambda_1(X|\theta)} \right]
\]
it is sufficient to establish that \( E_{\theta_0} \left[ \sup_{\theta \in N_\theta} \left| \log \left( \frac{\lambda_1(X|\theta_0)}{\lambda_1(X|\theta)} \right) \right| \right] < +\infty \) in order to conclude that \( E_{\theta_0} Z(N_\theta, X_i) > -\infty \).

Now, \( E_{\beta,0,k} \left( \frac{U}{1+\omega^2_{0,k}V} \right)^2 \) and \( E_{\beta,0,k} \left( \frac{U}{1+\omega^2_{0,k}V} \right) \) are finite due to Lemma 1 of Delattre et al. (2013), which imply that \( E_{\theta_0} \left[ \sup_{\theta \in N_\theta} \left| \log \left( \frac{\lambda_1(X|\theta_0)}{\lambda_1(X|\theta)} \right) \right| \right] < +\infty \), so that \( E_{\theta_0} Z(N_\theta, X_i) > -\infty \). Hence, \( \hat{\theta}_n \xrightarrow{a.s.} \theta_0 \) \( [P_{\hat{\theta}_0}] \). We summarize the result in the form of the following theorem:

**Theorem 2** Assume the iid setup and conditions (H1) and (H2). Then the MLE is strongly consistent in the sense that \( \hat{\theta}_n \xrightarrow{a.s.} \theta_0 \) \( [P_{\hat{\theta}_0}] \).

### 3.2 Asymptotic normality of MLE

To verify asymptotic normality of MLE we invoke the following theorem provided in Schervish (1995) (Theorem 7.63):

**Theorem 3 (Schervish (1995))** Let \( \Omega \) be a subset of \( \mathbb{R}^n \), and let \( \{X_n\}_{n=1}^{\infty} \) be conditionally iid given \( \theta \) each with density \( \lambda_1(\cdot|\theta) \). Let \( \hat{\theta}_n \) be an MLE. Assume that \( \hat{\theta}_n \xrightarrow{P} \theta \) under \( P_0 \) for all \( \theta \). Assume that \( \lambda_1(x|\theta) \) has continuous second partial derivatives with respect to \( \theta \) and that differentiation can be passed under the integral sign. Assume that there exists \( H_r(x, \theta) \) such that, for each \( \theta_0 \in \text{int}(\Omega) \) and each \( k, j, \)
\[
\sup_{\|\theta - \theta_0\| \leq r} \left| \frac{\partial^2}{\partial \theta_k \partial \theta_j} \log \lambda_{X_1|\theta}(x|\theta_0) - \frac{\partial^2}{\partial \theta_k \partial \theta_j} \log \lambda_{X_1|\theta}(x|\theta) \right| \leq H_r(x, \theta_0), \tag{3.5}
\]
with
\[
\lim_{r \to 0} E_{\theta_0} H_r(X, \theta_0) = 0. \tag{3.6}
\]
Assume that the Fisher information matrix \( I(\theta) \) is finite and non-singular. Then, under \( P_{\theta_0} \),
\[
\sqrt{n} \left( \hat{\theta}_n - \theta_0 \right) \xrightarrow{d} \mathcal{N} \left( 0, I^{-1}(\theta_0) \right). \tag{3.7}
\]

### 3.2.1 Verification of the above regularity conditions for asymptotic normality in our SDE set-up

In Section 3.1.1 we proved almost sure consistency of the MLE \( \hat{\theta}_n \) in the SDE set-up. Hence, \( \hat{\theta}_n \xrightarrow{P} \theta \) under \( P_0 \) for all \( \theta \).

We assume,

(H3) For some real constant \( K > 0 \),
\[
\frac{b^2(x)}{\sigma^2(x)} < K(1 + x^\tau), \quad \text{for some} \quad \tau \geq 1
\]
Note that (H3) implies moments of all orders of \( V_i \) (given by (1.2)) for all \( i = 1, \ldots, n \) are finite. Hence, along with Lemma 1 of Delattre et al. (2013), (H3) implies existence of all order moments of \( \frac{V_i}{\hat{V}_i} \) for all \( i = 1, \ldots, n \). Also note that, since for \( k \geq 1, E[\phi_i]^{2k} < \infty \), for all \( i = 1, \ldots, n \), because of normal mixture distribution, Proposition 1 of Delattre et al. (2013) implies that for all \( T > 0 \), \( \sup_{t \in [0, T]} E[\chi_i(t)]^{2k} < \infty \) for all \( k \geq 1 \) and \( i = 1, \ldots, n \).

That differentiation can be passed under the integral sign in our case, is proved in Section S-1 of the supplement. For the third order derivatives of \( \log \lambda \) note that

\[
\frac{\partial^3 \log \lambda}{\partial \theta_r \partial \theta_s \partial \theta_t} = \frac{1}{\lambda} \frac{\partial^3 \lambda}{\partial \theta_r \partial \theta_s \partial \theta_t} - \frac{1}{\lambda^2} \left[ \frac{\partial^2 \lambda}{\partial \theta_r \partial \theta_s} \frac{\partial \lambda}{\partial \theta_t} + \frac{\partial^2 \lambda}{\partial \theta_s \partial \theta_t} \frac{\partial \lambda}{\partial \theta_r} + \frac{\partial^2 \lambda}{\partial \theta_r \partial \theta_t} \frac{\partial \lambda}{\partial \theta_s} \right] + \frac{2}{\lambda^3} \frac{\partial \lambda}{\partial \theta_r} \frac{\partial \lambda}{\partial \theta_s} \frac{\partial \lambda}{\partial \theta_t}. \tag{3.8}
\]

Denoting \( \psi = (\mu_1, \ldots, \mu_M, \omega_1^2, \ldots, \omega_M^2) \), the absolute values of the terms on the right hand side of the above equation are bounded by a sum of terms of the forms

\[
\left| \frac{1}{\lambda} \frac{\partial f}{\partial \psi_r \partial \psi_s} \right| , \quad \left| \frac{1}{\lambda} \frac{\partial^2 f}{\partial \psi_r} \right| , \quad \left| \frac{1}{\lambda^2} \frac{\partial f}{\partial \psi_r} \right| \quad \text{and} \quad \left| \frac{\partial f}{\lambda} \right|.
\]

That expectation of the upper bound of each term is finite, is proved in Section S-2 of the supplement, which implies that (3.5) and (3.6) clearly hold.

Now, following Delattre et al. (2013), Maitra and Bhattacharya (2016), Maitra and Bhattacharya (2015), we assume:

(H4) The true value \( \theta_0 \in \text{int} (\Omega) \).

\[
\mathcal{I}(\theta) = E_{\theta} \left[ \frac{\partial \log \lambda}{\partial \theta_r} \frac{\partial \log \lambda}{\partial \theta_s} \right] = -E_{\theta} \left[ \frac{\partial^2 \log \lambda}{\partial \theta_r \partial \theta_s} \right]
\]

That \( \mathcal{I}(\theta) \) is well-defined is clear due to existence of derivatives up to the second order. Now, let \((b_1, b_2, \ldots, b_M)\) be a real row vector where \( \theta \) is \( M \)-dimensional. Then

\[
E \left[ b_1 \frac{\partial \log \lambda}{\partial \theta_1} + b_2 \frac{\partial \log \lambda}{\partial \theta_2} + \ldots + b_M \frac{\partial \log \lambda}{\partial \theta_M} \right]^2 \geq 0
\]

implying

\[
\sum_{r=1}^{M} \sum_{s=1}^{M} b_r b_s \mathcal{I}_{rs} \geq 0.
\]

This shows that \( \mathcal{I} \) is positive semi-definite. We assume

(H5) \( \mathcal{I}(\theta_0) \) is positive definite.

Hence, asymptotic normality of the MLE, of the form (3.7), holds in our case. Formally,

**Theorem 4** Assume the iid setup and conditions (H1) – (H5). Then the MLE is asymptotically normally distributed as (3.7).

### 4 Consistency and asymptotic normality of MLE in the non-iid set-up

In this section, as in Maitra and Bhattacharya (2016) and Maitra and Bhattacharya (2015) we allow \( T_i \neq T \) and \( x^i \neq x \) for each \( 1 \leq i \leq n \). Consequently, here we deal with the set-up where the processes \( X_i(t); i = 1, \ldots, n \) are independently, but not identically distributed. Following Maitra and Bhattacharya (2016) and Maitra and Bhattacharya (2015) we assume the following:
(H6) The sequences \( \{T_1, T_2, \ldots \} \) and \( \{x^1, x^2, \ldots \} \) are sequences in compact sets \( \Sigma \) and \( \mathcal{X} \), respectively, so that there exist convergent subsequences with limits in \( \Sigma \) and \( \mathcal{X} \). For notational convenience, we continue to denote the convergent subsequences as \( \{T_1, T_2, \ldots \} \) and \( \{x^1, x^2, \ldots \} \). Let us denote the limits by \( T^\infty \) and \( x^\infty \), where \( T^\infty \in \Sigma \) and \( x^\infty \in \mathcal{X} \).

Following Maitra and Bhattacharya (2016) and Maitra and Bhattacharya (2015) we denote the process associated with the initial value \( x \) and time point \( t \) as \( X(t, x) \), so that \( X(t, x^i) = X_i(t) \), and \( X_i = \{X_i(t); t \in [0, T_i]\} \). We also denote by \( \phi(x) \) the random effect parameter associated with the initial value \( x \) such that \( \phi(x^i) = \phi_i \). We assume

(H7) \( \phi(x) \) is a real-valued, continuous function of \( x \), and that for \( k \geq 1 \), \( \sup_{x \in \mathcal{X}} E |\phi(x)|^{2k} < \infty \).

As in Proposition 1 of Delattre et al. (2013), assumption (H7) implies that for any \( T > 0 \),

\[
\sup_{t \in [0, T], x \in \mathcal{X}} E |X(t, x)|^{2k} < \infty. \tag{4.1}
\]

For \( x \in \mathcal{X} \) and \( T \in \Sigma \), let

\[
U(x, T) = \int_0^T \frac{b(X(s, x))}{\sigma^2(X(s, x))} dX(s, x); \tag{4.2}
\]

\[
V(x, T) = \int_0^T \frac{b^2(X(s, x))}{\sigma^2(X(s, x))} ds. \tag{4.3}
\]

Clearly, \( U(x^i, T_i) = U_i \) and \( V(x^i, T_i) = V_i \), where \( U_i \) and \( V_i \) are given by (1.2).

Even in this non-iid case (H3) ensures that moments of all orders of \( V(x, T) \) are finite. Then, by Theorem 5 of Maitra and Bhattacharya (2016), the moments of uniformly integrable continuous functions of \( U(x, T) \), \( V(x, T) \) and \( \theta \) are continuous in \( x \), \( T \) and \( \theta \). In particular, the Kullback-Leibler distance and the information matrix, which we denote by \( \mathcal{K}_{x,T}(\theta_0, \theta) \) (or, \( \mathcal{K}_{x,T}(\theta_0, 0) \)) and \( \mathcal{I}_{x,T}(\theta) \) respectively to emphasize dependence on the initial values \( x \) and \( T \), are continuous in \( x \), \( T \) and \( \theta \). For \( x \equiv x^k \) and \( T = T_k \), if we denote the Kullback-Leibler distance and the Fisher’s information as \( \mathcal{K}_k(\theta_0, \theta) \) \( (\mathcal{K}_k(\theta_0, 0)) \) and \( \mathcal{I}_k(\theta) \), respectively, then continuity of \( \mathcal{K}_{x,T}(\theta_0, \theta) \) (or \( \mathcal{K}_{x,T}(\theta_0, 0) \)) and \( \mathcal{I}_{x,T}(\theta) \) with respect to \( x \) and \( T \) ensures that as \( x^k \rightarrow x^\infty \) and \( T_k \rightarrow T^\infty \), \( \mathcal{K}_{x^k,T_k}(\theta_0, \theta) \rightarrow \mathcal{K}_{x^\infty,T^\infty}(\theta_0, \theta) = \mathcal{K}(\theta_0, \theta) \), say. Similarly, \( \mathcal{K}_{x^k,T_k}(\theta_0, \theta_0) \rightarrow \mathcal{K}(\theta_0, \theta_0) \) and \( \mathcal{I}_{x^k,T_k}(\theta) \rightarrow \mathcal{I}_{x^\infty,T^\infty}(\theta) = \mathcal{I}(\theta) \), say. Thanks to compactness, the limits \( \mathcal{K}(\theta_0, \theta) \), \( \mathcal{K}(\theta_0, \theta_0) \) and \( \mathcal{I}(\theta) \) are well-defined Kullback-Leibler divergences and Fisher’s information, respectively. Consequently (see Maitra and Bhattacharya (2016), Maitra and Bhattacharya (2015)), the following hold for any \( \theta \in \Omega \),

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^n \mathcal{K}_k(\theta_0, \theta)}{n} = \mathcal{K}(\theta_0, \theta); \tag{4.4}
\]

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^n \mathcal{K}_k(\theta, \theta_0)}{n} = \mathcal{K}(\theta, \theta_0); \tag{4.5}
\]

\[
\lim_{n \to \infty} \frac{\sum_{k=1}^n \mathcal{I}_k(\theta)}{n} = \mathcal{I}(\theta). \tag{4.6}
\]

We assume that

(H8) For any \( \theta \in \Omega \), \( \mathcal{I}(\theta) \) is positive definite.
4.1 Consistency of MLE in the non-iid set-up

Following Hoadley [1971] we define the following:

\[ R_i(\theta) = \log \frac{\lambda_i(X_i|\theta)}{\lambda_i(X_i|\theta_0)} \quad \text{if} \quad \lambda_i(X_i|\theta_0) > 0 \]
\[ = 0 \quad \text{otherwise}. \quad (4.7) \]

\[ R_i(\theta, \rho) = \sup \{ R_i(\xi) : \|\xi - \theta\| \leq \rho \} \quad (4.8) \]
\[ \mathcal{V}_i(r) = \sup \{ R_i(\theta) : \|\theta\| > r \}. \quad (4.9) \]

Following Hoadley [1971] we denote by \( r_i(\theta), r_i(\theta, \rho) \) and \( v_i(r) \) to be expectations of \( R_i(\theta), R_i(\theta, \rho) \) and \( \mathcal{V}_i(r) \) under \( \theta_0 \); for any sequence \( \{A_i; i = 1, 2, \ldots\} \) we denote \( \sum_{i=1}^{n} A_i/n \) by \( \hat{A}_n \).

Hoadley [1971] proved that if the following regularity conditions are satisfied, then the MLE \( \hat{\theta}_n \to \theta_0 \):

1. \( \Omega \) is a closed subset of \( \mathbb{R}^{3M} \).
2. \( \lambda_i(X_i|\theta) \) is an upper semicontinuous function of \( \theta \), uniformly in \( i \), a.s. \([P_{\theta_0}]\).
3. There exist \( \rho^* = \rho^*(\theta) > 0, r > 0 \) and \( 0 < K^* < \infty \) for which
   (i) \( E_{\theta_0} [R_i(\theta, \rho)]^2 \leq K^*, \quad 0 \leq \rho \leq \rho^* \);
   (ii) \( E_{\theta_0} [\mathcal{V}_i(r)]^2 \leq K^* \).
4. (i) \( \lim_{n \to \infty} \bar{r}_n(\theta) < 0, \quad \theta \neq \theta_0 \);
   (ii) \( \lim_{n \to \infty} \bar{v}_n(r) < 0 \).
5. \( R_i(\theta, \rho) \) and \( \mathcal{V}_i(r) \) are measurable functions of \( X_i \).

Actually, conditions (3) and (4) can be weakened but these are more easily applicable (see Hoadley [1971] for details).

4.1.1 Verification of the regularity conditions

Since \( \Omega \) is compact in our case, the first regularity condition clearly holds.

For the second regularity condition, note that given \( X_i \), \( \lambda_i(X_i|\theta) \) is continuous (as \( \lambda_i(X_i|\theta) = \sum_{k=1}^{M} a_k f_k(X_i|\beta_k) \) where each \( f_k(X_i|\beta_k) \) is continuous), in fact, uniformly continuous in \( \theta \) in our case, since \( \Omega \) is compact. Hence, for any given \( \epsilon > 0 \), there exists \( \delta_i(\epsilon) > 0 \), independent of \( \theta \), such that \( \|\theta_1 - \theta_2\| < \delta_i(\epsilon) \) implies \( |\lambda(X_i|\theta_1) - \lambda(X_i|\theta_2)| < \epsilon \). Now consider a strictly positive function \( \delta_{x,T}(\epsilon) \), continuous in \( x \in \mathcal{X} \) and \( T \in \mathcal{T} \), such that \( \delta_{x,T}(\epsilon) = \delta_i(\epsilon) \). Let \( \delta(\epsilon) = \inf_{x \in \mathcal{X}, T \in \mathcal{T}} \delta_{x,T}(\epsilon) \).

Since \( \mathcal{X} \) and \( \mathcal{T} \) are compact, it follows that \( \delta(\epsilon) > 0 \). Now it holds that \( \|\theta_1 - \theta_2\| < \delta(\epsilon) \) implies \( |\lambda(X_i|\theta_1) - \lambda(X_i|\theta_2)| < \epsilon \), for all \( i \). Hence, the second regularity condition is satisfied.

Let us now focus attention on condition (3)(i). It follows from (3.4) that

\[ R_i(\theta) \leq \sum_{k=1}^{M} C_1(U_i, V_i, \beta_1, \beta_{0,k}) + |\log a_1| \quad (4.10) \]

where \( C_1(U_i, V_i, \beta_1, \beta_{0,k}) \) is given by (3.1). Let us denote \( \{\xi \in \mathbb{R} \times \mathbb{R}^+ : \|\xi - \beta_1\| \leq \rho \} \) by \( B(\rho, \beta_1) \). Here \( 0 < \rho < \rho^*(\beta_1) \), and \( \rho^*(\beta_1) \) is so small that \( B(\rho, \beta_1) \subset \Omega_{\beta} \) for all \( \rho \in (0, \rho^*(\beta_1)) \). It then follows from (3.1) that
Theorem 5

Let \( \zeta(x, \theta) = \log \lambda_i(x|\theta) \); also, let \( \zeta_j(x, \theta) \) be the \( 3M \times 1 \) vector with \( j \)-th component \( \zeta_j(x, \theta) = \frac{\partial}{\partial \theta_j} \zeta_i(x, \theta) \), and let \( \zeta_{ij}(x, \theta) \) be the \( 3M \times 3M \) matrix with \( (j, k) \)-th element \( \zeta_{ij}(x, \theta) = \frac{\partial^2}{\partial \theta_j \partial \theta_k} \zeta_i(x, \theta) \).

For proving asymptotic normality in the non-iid framework, [Headley (1971)] assumed the following regularity conditions:
(1) $\Omega$ is an open subset of $\mathbb{R}^{3M}$.

(2) $\hat{\theta}_n \overset{P}{\to} \theta_0$.

(3) $\zeta'_i(X_i, \theta)$ and $\zeta''_i(X_i, \theta)$ exist a.s. $[P_{\theta_0}]$.

(4) $\zeta''_i(X_i, \theta)$ is a continuous function of $\theta$, uniformly in $i$, a.s. $[P_{\theta_0}]$, and is a measurable function of $X_i$.

(5) $E_{\theta} [\zeta'_i(X_i, \theta)] = 0$ for $i = 1, 2, \ldots$.

(6) $I_\theta = E_{\theta} [\zeta'_i(X_i, \theta)\zeta'_i(X_i, \theta)^T] = -E_{\theta} [\zeta''_i(X_i, \theta)]$, where for any vector $y$, $y^T$ denotes the transpose of $y$.

(7) $\hat{\theta}_n(\theta) \to \hat{\theta}(\theta)$ as $n \to \infty$ and $\hat{\theta}(\theta)$ is positive definite.

(8) $E_{\theta_0} \left| \zeta'_{i,j}(X_i, \theta_0) \right|^3 \leq K_2$, for some $0 < K_2 < \infty$.

(9) There exist $\epsilon > 0$ and random variables $B_{i,j,k}(X_i)$ such that

(i) $\sup \left\{ \left| \zeta''_{i,j}(X_i, \xi) \right| : \| \xi - \theta_0 \| \leq \epsilon \right\} \leq B_{i,j,k}(X_i)$.

(ii) $E_{\theta_0} \left| B_{i,j,k}(X_i) \right|^{1+\delta} \leq K_2$, for some $\delta > 0$.

Condition (8) can be weakened but is relatively easy to handle. Under the above regularity conditions, Hoadley (1971) prove that

$$\sqrt{n} \left( \hat{\theta}_n - \theta_0 \right) \overset{\mathcal{L}}{\to} N(0, \bar{I}(\theta_0)) \quad \text{(4.14)}$$

4.2.1 Validation of asymptotic normality of MLE in the non-iid SDE set-up

Note that although condition (1) requires the parameter space $\Omega$ to be an open subset, the proof of asymptotic normality presented in Hoadley (1971) continues to hold for compact $\Omega$; see Maitra and Bhattacharya (2016).

Conditions (2), (3), (5), (6) are clearly valid in our case. Condition (4) can be verified in exactly the same way as condition (2) of Section 4.1, measurability of $\zeta''_i(X_i, \theta)$ follows due to its continuity with respect to $X_i$. Condition (7) simply follows from (4.6).

For conditions (8), (9)(i) and (9)(ii) note that, by the same arguments as in Section 3.2.1, finiteness of moments of all orders of the derivatives are seen to hold for every $x \in \mathcal{X}, T \in \mathcal{T}$. Then compactness of $\mathcal{X}, \mathcal{T}$ and $\Omega$, ensures that the conditions (8), (9)(i) and (9)(ii) hold.

In other words, in our non-iid SDE case we have the following theorem on asymptotic normality.

**Theorem 6** Assume the non-iid SDE setup and conditions (H1) – (H8). Then (4.14) holds.

5 Consistency and asymptotic normality of the Bayesian posterior in the iid set-up

5.1 Consistency of the Bayesian posterior distribution

To verify posterior consistency we make use of Theorem 7.80 presented in Schervish (1995); below we state the general form of the theorem.

**Theorem 7** (Schervish (1995)) Let $\{X_n\}_{n=1}^\infty$ be conditionally iid given $\theta$ with density $\lambda_1(x|\theta)$ with respect to a measure $\nu$ on a space $(\mathcal{X}^1, \mathcal{B}^1)$. Fix $\theta_0 \in \Omega$, and define, for each $S \subseteq \Omega$ and $x \in \mathcal{X}^1$,

$$Z(S, x) = \inf_{\psi \in \mathcal{S}} \log \frac{\lambda_1(x|\theta_0)}{\lambda_1(x|\psi)}.$$
Assume that for each \( \theta \neq \theta_0 \), there is an open set \( N_0 \) such that \( \theta \in N_0 \) and that \( E_{\theta_0} Z(N_0, X_i) > -\infty \). Also assume that \( \lambda_1(x|\cdot) \) is continuous at \( \theta \) for every \( \theta \), a.s. \([P_{\theta_0}]\). For \( \epsilon > 0 \), define \( C_\epsilon = \{ \theta : K_1(\theta_0, \theta) < \epsilon \} \), where

\[
K_1(\theta_0, \theta) = E_{\theta_0} \left( \log \frac{\lambda_1(X_1|\theta_0)}{\lambda_1(X_1|\theta)} \right)
\]

(5.1)

is the Kullback-Leibler divergence measure associated with observation \( X_1 \). Let \( \pi \) be a prior distribution such that \( \pi(C_\epsilon) > 0 \), for every \( \epsilon > 0 \). Then, for every \( \epsilon > 0 \) and open set \( N_0 \) containing \( C_\epsilon \), the posterior satisfies

\[
\lim_{n \to \infty} \pi_n(N_0|X_1, \ldots, X_n) = 1, \quad \text{a.s.} \quad [P_{\theta_0}].
\]

(5.2)

### 5.1.1 Verification of posterior consistency

The condition \( E_{\theta_0} Z(N_\theta, X_i) > -\infty \) of the above theorem is verified in the context of Theorem 7.102 in Section 3.1.1.

Now, all we need to ensure is that there exists a prior \( \pi \) which gives positive probability to \( C_\epsilon \) for every \( \epsilon > 0 \). From the identifiability result given by Proposition 7 (i) of Delattre et al. (2013) it follows that \( K_1(\theta_0, \theta) = 0 \) if and only if \( \theta = \theta_0 \) (up to a label switching). Hence, for any \( \epsilon > 0 \), the set \( C_\epsilon \) is non-empty, since it contains at least \( \theta_0 \). In fact, continuity of \( K_1(\theta_0, \theta) \) in \( \theta \) follows from the fact that upper bound of \( \left| \log \frac{\lambda_1(x|\theta_0)}{\lambda_1(x|\theta)} \right| \) has finite \( E_{\theta_0} \) expectation as shown in Section 3.1.1 and since the parameter space \( \Omega \) is compact, it follows that \( K_1(\theta_0, \theta) \) is uniformly continuous on \( \Omega \). The rest of the verification remains the same as Section 2.1.1 of Maitra and Bhattacharya (2015).

In other words, the following result on posterior consistency holds.

**Theorem 8** Assume the iid set-up and conditions (H1), (H2) and (H4). For \( \epsilon > 0 \), define \( C_\epsilon = \{ \theta : K_1(\theta_0, \theta) < \epsilon \} \), where \( K_1(\theta_0, \theta) \) is the Kullback-Leibler divergence measure associated with observation \( X_1 \). Let the prior distribution \( \pi \) of the parameter \( \theta \) satisfy \( \frac{d\pi}{dv} = h \) almost everywhere on \( \Omega \), where \( h(\theta) \) is any positive, continuous density on \( \Omega \) with respect to the Lebesgue measure \( v \). Then the posterior \( \{P_n\} \) is consistent in the sense that for every \( \epsilon > 0 \) and open set \( N_0 \) containing \( C_\epsilon \), the posterior satisfies

\[
\lim_{n \to \infty} \pi_n(N_0|X_1, \ldots, X_n) = 1, \quad \text{a.s.} \quad [P_{\theta_0}].
\]

(5.3)

### 5.2 Asymptotic normality of the Bayesian posterior distribution

To investigate asymptotic normality of our SDE-based posterior distributions we exploit Theorem 7.102 in conjunction with Theorem 7.89 provided in Schervish (1995). Below we state the four requisite conditions for the iid set-up.

#### 5.2.1 Regularity conditions – iid case

(1) The parameter space is \( \Omega \subseteq \mathbb{R}^M \) for some finite \( M \).

(2) \( \theta_0 \) is a point interior to \( \Omega \).

(3) The prior distribution of \( \theta \) has a density with respect to Lebesgue measure that is positive and continuous at \( \theta_0 \).

(4) There exists a neighborhood \( \mathcal{N}_0 \subseteq \Omega \) of \( \theta_0 \) on which \( \ell_n(\theta) = \log \lambda(X_1, \ldots, X_n|\theta) \) is twice continuously differentiable with respect to all co-ordinates of \( \theta \), a.s. \([P_{\theta_0}]\).

With the above conditions, the relevant theorem (Theorem 7.102 of Schervish (1995)) is as follows:
Theorem 9 (Schervish (1995)) Let \( \{X_n\}_{n=1}^{\infty} \) be conditionally iid given \( \theta \). Assume the above four regularity conditions; also assume that there exists \( H_r(x, \theta) \) such that, for each \( \theta_0 \in \text{int}(\Omega) \) and each \( k, j \),

\[
\sup_{||\theta - \theta_0|| \leq r} \left| \frac{\partial^2}{\partial \theta_k \partial \theta_j} \log \lambda_1(x|\theta_0) - \frac{\partial^2}{\partial \theta_k \partial \theta_j} \log \lambda_1(x|\theta) \right| \leq H_r(x, \theta_0),
\]

(5.4)

with

\[
\lim_{r \to 0} E_{\theta_0} H_r(X, \theta_0) = 0.
\]

(5.5)

Further suppose that the conditions of Theorem 7 hold, and that the Fisher's information matrix \( I(\theta_0) \) is positive definite. Now denoting by \( \hat{\theta}_n \) the MLE associated with \( n \) observations, let

\[
\Sigma_n^{-1} = \begin{cases} -\ell''_n(\hat{\theta}_n) & \text{if the inverse and } \hat{\theta}_n \text{ exist} \\ \mathbb{I}_{3M} & \text{if not} \end{cases}
\]

(5.6)

where for any \( t \),

\[
\ell''_n(t) = \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} \ell_n(\theta) \right)_{\theta = t},
\]

(5.7)

and \( \mathbb{I}_{3M} \) is the identity matrix of order \( 3M \). Thus, \( \Sigma_n^{-1} \) is the observed Fisher's information matrix.

Letting \( \Psi_n = \Sigma_n^{-1/2}(\theta - \hat{\theta}_n) \), for each compact subset \( B \) of \( \mathbb{R}^{3M} \) and each \( \epsilon > 0 \), the following holds:

\[
\lim_{n \to \infty} P_{\theta_0} \left( \sup_{\Psi_n \in B} \left| \pi_n(\Psi_n|X_1, \ldots, X_n) - \xi(\Psi_n) \right| > \epsilon \right) = 0,
\]

(5.8)

where \( \xi(\cdot) \) denotes the density of the standard normal distribution.

5.2.2 Verification of posterior normality

Firstly, note that (H5) ensures positive definiteness of \( I(\theta_0) \). Now observe that the four regularity conditions in Section 5.2.1 trivially hold. The remaining conditions of Theorem 9 are verified in the context of Theorem 2 in Section 3.1. Briefly, \( \frac{\partial^2}{\partial \theta_k \partial \theta_j} \log \lambda_1(x|\theta) \) is differentiable in \( \theta \) and the derivative has finite expectation, which ensure (5.4) and (5.5). Hence, (5.8) holds in our SDE set-up. Thus, the following theorem holds:

**Theorem 10** Assume the iid set-up and conditions (H1) – (H5). Let the prior distribution \( \pi \) of the parameter \( \theta \) satisfy \( \frac{d\pi}{dn} = h \) almost everywhere on \( \Omega \), where \( h(\theta) \) is any density with respect to the Lebesgue measure \( \nu \) which is positive and continuous at \( \theta_0 \). Then, letting \( \Psi_n = \Sigma_n^{-1/2}(\theta - \hat{\theta}_n) \), for each compact subset \( B \) of \( \mathbb{R}^{3M} \) and each \( \epsilon > 0 \), the following holds:

\[
\lim_{n \to \infty} P_{\theta_0} \left( \sup_{\Psi_n \in B} \left| \pi_n(\Psi_n|X_1, \ldots, X_n) - \xi(\Psi_n) \right| > \epsilon \right) = 0.
\]

(5.9)

6 Consistency and asymptotic normality of the Bayesian posterior in the non-iid set-up

In this section, as in Section 4 we assume (H7) – (H9). The equations (4.4), (4.5) and (4.6) as described in Section 4 will also have important roles in our proceedings. For consistency in the Bayesian framework we utilize the theorem of Choi and Schervish (2007), and for asymptotic normality of the posterior we make use of Theorem 7.89 of Schervish (1995).
6.1 Posterior consistency in the non-iid set-up

Analogous to Section 3.1 of Maitra and Bhattacharya (2015), here we need to ensure existence of moments of the form

\[
\sup_{x \in X, T \in \mathbb{T}} E_{\theta} \left[ \exp \left\{ \alpha \left| \omega_{0,k}^2 - \omega_1^2 \right| \left( \frac{U(x,T)}{1 + \omega_{0,k}^2 V(x,T)} \right)^2 \left( 1 + \frac{\omega_{0,k}^2}{\omega_1^2} \right) \right\} \right],
\]

for some \(0 < \alpha < \infty\). Hence, we assume the following assumption analogous to assumption (H10') of Maitra and Bhattacharya (2015).

(H9) For \(k = 1, \ldots, M\) there exists a strictly positive function \(\alpha^*(x, T, \beta_1)\), continuous in \((x, T, \beta_1)\), such that for any \((x, T, \beta_1)\),

\[
E_{\theta} \left[ \exp \left\{ \alpha^*(x, T, \beta_1) K_1 U^2(x, T) \right\} \right] < \infty,
\]

where \(K_1 = \sup_{\omega_1: \beta_1 \in \Omega_\beta, 1 \leq k \leq M} \left| \omega_{0,k}^2 - \omega_1^2 \right| \left( 1 + \frac{\omega_{0,k}^2}{\omega_1^2} \right).

Now, let

\[
\alpha_{\min}^* = \inf_{x \in X, T \in \mathbb{T}, \beta_1 \in \Omega_\beta} \alpha^*(x, T, \beta_1),
\]

and

\[
\alpha = \min \{ \alpha_{\min}^*, c^* \},
\]

where \(0 < c^* < 1/16\).

Compactness ensures that \(\alpha_{\min}^* > 0\), so that \(0 < \alpha < 1/16\). It also holds due to compactness that for \(\beta_1 \in \Omega_\beta\),

\[
\sup_{x \in X, T \in \mathbb{T}} E_{\theta} \left[ \exp \left\{ \alpha K_1 U^2(x, T) \right\} \right] < \infty. \tag{6.3}
\]

This ensures that

\[
\sup_{x \in X, T \in \mathbb{T}} E_{\theta} \left[ \exp \left\{ \alpha \left| \omega_{0,k}^2 - \omega_1^2 \right| \left( \frac{U(x,T)}{1 + \omega_{0,k}^2 V(x,T)} \right)^2 \left( 1 + \frac{\omega_{0,k}^2}{\omega_1^2} \right) \right\} \right]
\leq \sup_{x \in X, T \in \mathbb{T}} E_{\theta} \left[ \exp \left\{ \alpha K_1 U^2(x, T) \right\} \right]
< \infty. \tag{6.4}
\]

This choice of \(\alpha\) ensuring (6.3) will be useful in verification of the conditions of Theorem 11 which we next state.

**Theorem 11** (Choi and Schervish (2007)) Let \(\{X_i\}_{i=1}^{\infty}\) be independently distributed with densities \(\{\lambda_i(\cdot|\theta)\}_{i=1}^{\infty}\), with respect to a common \(\sigma\)-finite measure, where \(\theta \in \Omega\), a measurable space. The densities \(\lambda_i(\cdot|\theta)\) are assumed to be jointly measurable. Let \( \theta_0 \in \Omega \) and let \( P_{\theta_0} \) be the joint distribution of \(\{X_i\}_{i=1}^{\infty}\) when \(\theta_0\) is the true value of \(\theta\). Let \(\{\Theta_n\}_{n=1}^{\infty}\) be a sequence of subsets of \(\Omega\). Let \(\theta\) have prior \(\pi\) on \(\Omega\). Define the following:

\[
\Lambda_i(\theta_0, \theta) = \log \frac{\lambda_i(X_i|\theta_0)}{\lambda_i(X_i|\theta)},
\]

\[
K_i(\theta_0, \theta) = E_{\theta_0}(\Lambda_i(\theta_0, \theta))
\]

\[
\theta_i(\theta_0, \theta) = \text{Var}_{\theta_0}(\Lambda_i(\theta_0, \theta)).
\]

Make the following assumptions:
(1) Suppose that there exists a set \( B \) with \( \pi(\mathcal{B}) > 0 \) such that

(i) \[ \sum_{n=1}^{\infty} \frac{\vartheta_n(\theta_n, \theta) - \vartheta_n(\theta_n, \bar{\theta}_n)}{\varepsilon^2} < \infty, \quad \forall \; \theta \in B, \]

(ii) For all \( \varepsilon > 0 \), \( \pi\{ B \cap \{ \theta : K_3(\theta_n, \theta) < \varepsilon, \; \forall \; i \} \} > 0 \).

(2) Suppose that there exist test functions \( \{ \Phi_n \}_{n=1}^{\infty} \), sets \( \{ \Omega_n \}_{n=1}^{\infty} \) and constants \( C_1, C_2, c_1, c_2 > 0 \) such that

(i) \[ \sum_{n=1}^{\infty} E_\theta \Phi_n < \infty, \]

(ii) \[ \sup_{\theta \in \Theta_n \cap \Omega_n} E_\theta \left( 1 - \Phi_n \right) \leq C_1 e^{-c_1 n}, \]

(iii) \[ \pi(\Omega_n) \leq C_2 e^{-c_2 n}. \]

Then,

\[ \pi_n(\theta \in \Theta_n^c | x_1, \ldots, x_n) \to 0 \quad a.s. \; [\pi_0]. \quad (6.5) \]

### 6.1.1 Validation of posterior consistency

Recall that \( \lambda_i(X_i | \theta) \) in our case is given by (2.1). From Section 3.1.1 it follows that \( |\log \frac{\lambda_i(X_i | \theta_0)}{\lambda_i(X_i | \theta)}| \) has an upper bound which has finite expectation and square of expectation under \( \theta_0 \), and is uniform for all \( \theta \in B \), where \( B \) is an appropriate compact subset of the relevant parameter space. The rest of the verification of condition (1)(i) and the verification of (1)(ii) are similar to those of Maitra and Bhattacharya (2015).

In verification of (2)(iii), we let \( \Omega_n = \{ \Omega_1 \times \mathbb{R}^{3M-1} \} \) (since our parameter set \( \theta \) contains \( 3M \) parameters), where \( \Omega_1 = \{ \alpha_1 : |\alpha_1| < \bar{M}_1 \} \), where \( \bar{M}_n = O(e^n) \). Note that

\[ \pi(\Omega_n) = \pi(\Omega_1^n) = \pi(|\alpha_1| > \bar{M}_1) < E_\pi(|\alpha_1|) \bar{M}_1^{-1} \quad (6.6) \]

implies (2)(iii) holds, assuming that the prior \( \pi \) is such that the expectation \( E_\pi(|\alpha_1|) \) is finite (which holds for proper priors on \( \alpha_1 \)).

The verification of (2)(i) will follow in the same way as the verification in Maitra and Bhattacharya (2015) except the corresponding changes. Hence we will only mention the changes at which the verifications differ. Firstly, in our set-up \( L_n(\theta) = \prod_{i=1}^{n} \lambda_i(X_i | \theta) \) and \( \ell_n(\theta) = \sum_{i=1}^{n} \log \lambda_i(X_i | \theta) \), that is, \( f_i \) is now replaced with \( \lambda_i \). The existence of the third order derivative of \( \log \lambda_i \) is already established in Section 3.1.1.

Here also the continuity of the moments of \( V(x, T) \) and \( \frac{U(x, T)}{V(x, T)} \) with respect to \( x \) and \( T \) holds (which follows from Theorem 5 of Maitra and Bhattacharya (2016) where uniform integrability is ensured by finiteness of the moments of the aforementioned functions for every \( x, T \) belonging to compact sets \( \mathcal{X} \) and \( \mathcal{T} \)). Moreover, Kolmogorov’s strong law of large numbers for non-iid cases holds due to finiteness of the moments of \( V \) and \( \frac{U}{V} \) for every \( x \) and \( T \) belonging to the compact spaces \( \mathcal{X} \) and \( \mathcal{T} \).

Now, assuming \( \hat{\theta}_n = \zeta = (\gamma, \beta) \) we obtain from Section 3.1.1 the following:

\[ \log \lambda_i(x | \theta_0) - \log \lambda_i(x | \hat{\theta}_n) \geq - \sum_{k=1}^{p} C_1(U_i, V_i, \beta_1, \beta_0, k) - |\log a_1|, \]

where

\[ C_1(U_i, V_i, \beta_1, \beta_0, k) = \frac{1}{2} \left\{ \log \left( 1 + \frac{\omega_i^2}{\omega_{0,k}^2} \right) + \left| \frac{\omega_i^2 - \omega_{0,k}^2}{\omega_i^2} \right| \right\} + \frac{1}{2} \left| \frac{\omega_i^2}{\omega_{0,k}^2} - \omega_i^2 \right| \left( \frac{U_i}{1 + \omega_{0,k}^2 V_i} \right)^2 \left( 1 + \frac{\omega_{0,k}^2}{\omega_i^2} \right)

+ \left| \mu_1 \right| \left| \frac{U_i}{1 + \omega_{0,k}^2 V_i} \right| \left( 1 + \left| \frac{\omega_{0,k}^2 - \omega_i^2}{\omega_i^2} \right| \right) + \frac{\mu_0^2 V_i}{2(1 + \omega_{0,k}^2 V_i)} \left| \mu_0 k U_i \right| \left( 1 + \frac{\omega_{0,k}^2}{\omega_i^2} \right). \quad (6.7) \]
The rest of the verification of (2)(i) is the same as in Maitra and Bhattacharya (2015).

For the verification of (2)(ii), we define \( \Theta_n = \Theta_\delta = \{ (\gamma, \beta) : \mathcal{K}(\theta, \theta_0) < \delta \} \) where \( \mathcal{K}(\theta, \theta_0) \), defined as in (4.5), is the proper Kullback-Leibler divergence and the verification will be in a similar manner as in Maitra and Bhattacharya (2015). Hence, posterior consistency (6.5) holds in our non-iid SDE set-up. The result can be summarized in the form of the following theorem.

**Theorem 12** Assume the non-iid SDE set-up. Also assume conditions (H1) – (H9). For any \( \delta > 0 \), let \( \Theta_\delta = \{ (\gamma, \beta) : \mathcal{K}(\theta, \theta_0) < \delta \} \), where \( \mathcal{K}(\theta, \theta_0) \), defined as in (4.5), is the proper Kullback-Leibler divergence. Let the prior distribution \( \pi \) of the parameter \( \theta \) satisfy \( \frac{d\pi}{dv} = h \) almost everywhere on \( \Omega \), where \( h(\theta) \) is any positive, continuous density on \( \Omega \) with respect to the Lebesgue measure \( \nu \). Then,

\[
\pi_n (\theta \in \Theta_\delta | X_1, \ldots, X_n) \to 0 \quad \text{a.s.} \quad [P_{\theta_0}].
\]

(6.8)

### 6.2 Asymptotic normality of the posterior distribution in the non-iid set-up

For asymptotic normality of the posterior in the iid situation, four regularity conditions, stated in Section 5.2.1, were necessary. In the non-iid framework, three more are necessary, in addition to the already presented four conditions. They are as follows (see Schervish (1995) for details).

#### 6.2.1 Extra regularity conditions in the non-iid set-up

(5) The largest eigenvalue of \( \Sigma_n \) goes to zero in probability.

(6) For \( \delta > 0 \), define \( \mathcal{N}_0(\delta) \) to be the open ball of radius \( \delta \) around \( \theta_0 \). Let \( \rho_n \) be the smallest eigenvalue of \( \Sigma_n \). If \( \mathcal{N}_0(\delta) \subseteq \Omega \), there exists \( K(\delta) > 0 \) such that

\[
\lim_{n \to \infty} P_{\theta_0} \left( \sup_{\theta \in \Omega \setminus \mathcal{N}_0(\delta)} \rho_n [\ell_n(\theta) - \ell_n(\theta_0)] < -K(\delta) \right) = 1. \tag{6.9}
\]

(7) For each \( \epsilon > 0 \), there exists \( \delta(\epsilon) > 0 \) such that

\[
\lim_{n \to \infty} P_{\theta_0} \left( \sup_{\theta \in \mathcal{N}_0(\delta(\epsilon)), \|\eta\| = 1} \left| 1 + \eta^T \Sigma_n^{1/2} \ell_n^\prime(\theta) \Sigma_n^{1/2} \eta \right| < \epsilon \right) = 1. \tag{6.10}
\]

In the non-iid case, the four regularity conditions presented in Section 5.2.1 and additional three provided above, are sufficient to guarantee (5.8).

#### 6.2.2 Verification of the regularity conditions

The verification follows from the verification presented in Section 3.2.2 of Maitra and Bhattacharya (2015), as finiteness of the expectation of \( \log \lambda \) up to the third order derivative is already justified in Section 3.2.1. So, we have our result in the form of the following theorem.

**Theorem 13** Assume the non-iid set-up and conditions (H1) – (H8). Let the prior distribution \( \pi \) of the parameter \( \theta \) satisfy \( \frac{d\pi}{dv} = h \) almost everywhere on \( \Omega \), where \( h(\theta) \) is any density with respect to the Lebesgue measure \( \nu \) which is positive and continuous at \( \theta_0 \). Then, letting \( \Psi_n = \Sigma_n^{-1/2} (\theta - \hat{\theta}_n) \), for each compact subset \( B \) of \( \mathbb{R}^{3M} \) and each \( \epsilon > 0 \), the following holds:

\[
\lim_{n \to \infty} P_{\theta_0} \left( \sup_{\Psi_n \in B} |\pi_n(\Psi_n | X_1, \ldots, X_n) - \xi(\Psi_n)| > \epsilon \right) = 0. \tag{6.11}
\]
7 Summary and discussion

Considering the linearity assumption in the drift term involving the random effect parameter in the SDE model [Delattre et al. (2013)] have proved convergence in probability and asymptotic normality of the MLE of the parameter in the iid situation when the random effect parameters have the normal distribution. In a similar set up as in Delattre et al. (2013), Maitra and Bhattacharya (2016) established strong consistency and asymptotic normality of the MLE under both iid and non-iid set-ups, with weaker assumptions compared to Delattre et al. (2013). Maitra and Bhattacharya (2015) extended the classical set-up to the Bayesian framework, establishing consistency and asymptotic normality of the posterior in both iid and non-iid situations.

In this work, we consider the random effect parameter having finite mixture of normal distributions. Inherent flexibility of the mixture framework offers far greater generality compared to normal distributions considered so far in this direction of research. Even in this set up we are able to prove strong consistency and asymptotic normality of the MLE and in the corresponding Bayesian framework we establish posterior consistency and asymptotic normality of the posterior distribution, our results encompassing both iid and non-iid situations, for both the paradigms. It is important to note that no extra assumptions are needed besides the assumptions of Maitra and Bhattacharya (2016) and Maitra and Bhattacharya (2015) to conclude the corresponding results under the consideration of finite normal mixture. In other words, without any extra assumption we could achieve asymptotic results regarding the random effect parameters having a much larger class of distributions, which seems to hold importance from theoretical and practical perspectives.

Although Delattre et al. (2016) also considered the SDE mixture model, they required an extra, strong assumption to prove even weak consistency of MLE in the iid set-up, compared to our stronger, almost sure convergence result, without the assumption. Moreover, the weak consistency result in the iid situation is the only asymptotic result they provided, while we delved into consistency and asymptotic normality of both MLE and the Bayesian posterior distribution, in both iid and non-iid set-ups.

However, in this work we have considered one dimensional SDE’s. The generalization of our asymptotic theories to high dimensions will be considered in our future endeavors.

Acknowledgments

The first author gratefully acknowledges her CSIR Fellowship, Govt. of India.
**Supplementary Material**

Throughout, we refer to our main manuscript Maitra and Bhattacharya (2017) as MB.

### S-8 Proof that differentiation can be passed under the integral sign

Let us denote the marginal distribution of \( \{X_i(t) : t \in [0, T]\} \) by \( Q_{\theta}^1 \) on \((C_T, C_T)\), where \( C_T \) is the space of real continuous functions on \([0, T]\) and \( C_T \) is the corresponding \( \sigma \)-algebra. As in MB here \( \theta = (\gamma, \beta) = (a_1, \ldots, a_M, \mu_1, \ldots, \mu_M, \omega_2, \ldots, \omega_M) \).

Let \( \tau = (1, 0, \ldots, 0, \mu_1, \ldots, \mu_M, \omega_1, \ldots, \omega_M) \) and set

\[
p_1(\theta) = \frac{\lambda_1(X_1, \theta)}{\lambda_1(X_1, \tau)} = \frac{\sum_{k=1}^M \frac{a_k}{(1+\omega_k^2 V_1)} \exp \left( -\frac{V_1}{2(1+\omega_k^2 V_1)} \left( \mu_k - \frac{U_1}{V_1} \right)^2 \right) \exp \left( \frac{U_1^2}{2V_1} \right)}{\frac{1}{(1+\omega_1^2 V_1)} \exp \left( -\frac{V_1}{2(1+\omega_1^2 V_1)} \left( \mu_1^* - \frac{U_1}{V_1} \right)^2 \right) \exp \left( \frac{U_1^2}{2V_1} \right)} = \frac{\sum_{k=1}^M a_k (1+\omega_k^2 V_1)^{\frac{1}{2}} \exp \left( -\frac{V_1}{2(1+\omega_k^2 V_1)} \left( \mu_k - \frac{U_1}{V_1} \right)^2 + \frac{V_1}{2(1+\omega_1^2 V_1)} \left( \mu_1^* - \frac{U_1}{V_1} \right)^2 \right)}{(1+\omega_1^2 V_1)^{\frac{1}{2}}} \]

(S-8.1)

so that \( \int_{C_T} p_1(\theta) dQ_{\tau}^1 = 1 \). The assurance of interchange of integration with respect to \( Q_{\theta}^1 \) and differentiation with respect to \( \theta \) implies interchange of integration with respect to \( Q_{\theta}^1 \) and differentiation with respect to \( \theta \). So, here we will justify interchange of integration with respect to \( Q_{\theta}^1 \) and differentiation with respect to \( \theta \).

Note that

\[
\frac{\partial p_1(\theta)}{\partial a_k} \leq (1+\omega_1^2 V_1)^{\frac{1}{2}} \exp \left( -\frac{V_1}{2(1+\omega_k^2 V_1)} \left( \mu_k - \frac{U_1}{V_1} \right)^2 + \frac{V_1}{2(1+\omega_1^2 V_1)} \left( \mu_1^* - \frac{U_1}{V_1} \right)^2 \right). 
\]

(S-8.2)

Let

\[
\pi^2 = \max \{\mu_k^2, \mu_1^*^2\}; \quad \mu^2 = \min \{\mu_k^2, \mu_1^*^2\}; \quad \omega = \min \{\omega_k, \omega_1\}. 
\]

Now

\[
\mu_k^2 - 2\mu_k \frac{U_1}{V_1} + \frac{U_1^2}{V_1^2} \geq \mu^2 - 2\mu_k \frac{U_1}{V_1} + \frac{U_1^2}{V_1^2} \implies -\left( \mu_k^2 - 2\mu_k \frac{U_1}{V_1} + \frac{U_1^2}{V_1^2} \right) \leq -\left( \mu^2 - 2\mu_k \frac{U_1}{V_1} + \frac{U_1^2}{V_1^2} \right). 
\]

(S-8.3)

Also,

\[
\mu_1^*^2 - 2\mu_1^* \frac{U_1}{V_1} + \frac{U_1^2}{V_1^2} \leq \pi^2 - 2\mu_1^* \frac{U_1}{V_1} + \frac{U_1^2}{V_1^2}, 
\]

(S-8.4)

\[
\frac{V_1}{2(1+\omega_k^2 V_1)} \leq \frac{V_1}{2(1+\omega_1^2 V_1)} \quad \text{and} \quad \frac{V_1}{2(1+\omega_1^2 V_1)} \leq \frac{V_1}{2(1+\omega_1^2 V_1)}. 
\]

(S-8.5)
Due to (S-8.3), (S-8.4) and (S-8.5) it follows that

$$
\frac{\partial p_1(\theta)}{\partial \alpha_k} \leq (1 + \omega_1^2 V_1)^{\frac{1}{2}} \exp \left[ -\frac{V_1}{2 (1 + \omega_1^2 V_1)} \left( \mu_1^2 - 2 \mu_k \frac{U_1}{V_1} \right) + \frac{V_1}{2 (1 + \omega_1^2 V_1)} \left( \mu_k^2 - 2 \mu_k \frac{U_1}{V_1} \right) \right] = (1 + \omega_1^2 V_1)^{\frac{1}{2}} \exp \left[ \frac{V_1}{2 (1 + \omega_1^2 V_1)} (\mu_1^2 - \mu_k^2 + \frac{U_1}{V_1} (\mu_k - \mu_1^4)) \right] \leq (1 + \omega_1^2 V_1)^{\frac{1}{2}} \exp \left[ \frac{1}{2 \omega_1^2} (\mu_1^2 - \mu_k^2) + \frac{U_1}{1 + \omega_1^2 V_1} (\mu_k - \mu_1^4) \right] = K_1(U_1, V_1) \tag{S-8.6}
$$

where the last inequality follows as $\frac{V_1}{2 (1 + \omega_1^2 V_1)} < \frac{1}{2 \omega_1^2}$. Note that $K_1(U_1, V_1)$ has finite expectation, that is, integrable with respect to $Q_1^i$, thanks to existence of all order moments of $V_1$, Lemma 1 of Delattre et al. (2013), and the Cauchy-Schwartz inequality.

Further, with $\mu_{\text{max}} = \max \{|\mu_1|, \ldots, |\mu_M|\}$, note that,

$$
\begin{align*}
\frac{\partial p_1}{\partial \omega_k} &\leq K_1(U_1, V_1) \left[ \mu_{\text{max}} + \frac{U_1}{V_1} \right] ; 
\frac{\partial^2 p_1}{\partial \omega_k^2} &\leq K_1(U_1, V_1) \left[ V_1 + V_1^2 \left( \mu_{\text{max}} + \left| \frac{U_1}{V_1} \right| \right)^2 \right] ; 
\frac{\partial^2 p_1}{\partial \mu_k \partial \omega_k} &\leq K_1(U_1, V_1) \left[ V_1 + V_1^2 \left( \mu_{\text{max}} + \left| \frac{U_1}{V_1} \right| \right)^2 + V_1^3 \left( \mu_{\text{max}} + \left| \frac{U_1}{V_1} \right| \right)^4 \right] ; 
\frac{\partial^2 p_1}{\partial \mu_k \partial \omega_k} &\leq K_1(U_1, V_1) \left[ 2V_1^2 \left( \mu_{\text{max}} + \left| \frac{U_1}{V_1} \right| \right) + V_1^3 \left( \mu_{\text{max}} + \left| \frac{U_1}{V_1} \right| \right)^3 \right] ,
\end{align*}
$$

where each upper bound has finite expectation. This easily follows due to existence of all order moments of $V_1$ and $\frac{U_1}{V_1}$, compactness of the parameter space, the fact that the expectation of $K_1(U_1, V_1)$ is finite, and the Cauchy-Schwartz inequality. Therefore, the interchange is justified.

**S-9 Upper bounds of the third order partial derivatives of loglikelihood**

Likelihood corresponding to the $i$-th individual is

$$
\lambda_i(X_i, \theta) = \sum_{k=1}^{M} a_k f(X_i|\beta_k),
$$

where

$$
f(X_i|\beta_k) = \frac{1}{(1 + \omega_k^2 V_i)^{1/2}} \exp \left[ -\frac{V_i}{2 (1 + \omega_k^2 V_i)} \left( \mu_k - \frac{U_i}{V_i} \right)^2 \right] \exp \left( \frac{U_i^2}{2V_i} \right) .
$$
Hence we obtain,

\[
\frac{|f|}{\lambda} \leq 1; \tag{S-9.1}
\]
\[
\frac{1}{\lambda} \frac{\partial f}{\partial \mu} \leq V \left[ \mu_{\text{max}} + \left| \frac{U}{V} \right| \right]; \tag{S-9.2}
\]
\[
\frac{1}{\lambda} \frac{\partial f}{\partial \omega} \leq \left[ V + V^2 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right)^2 \right]; \tag{S-9.3}
\]
\[
\frac{1}{\lambda} \frac{\partial^2 f}{\partial \mu^2} \leq \left[ V + V^2 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right)^2 \right]; \tag{S-9.4}
\]
\[
\left| \frac{1}{\lambda} \frac{\partial^2 f}{\partial \omega^2} \right| \leq \left[ V^2 + 3V^3 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right)^2 + V^4 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right) \right]; \tag{S-9.5}
\]
\[
\left| \frac{1}{\lambda} \frac{\partial^2 f}{\partial \mu \partial \omega} \right| \leq \left[ 2V^2 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right) + V^3 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right) \right]; \tag{S-9.6}
\]
\[
\left| \frac{1}{\lambda} \frac{\partial^3 f}{\partial \mu^3} \right| \leq \left[ 3V^2 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right) + V^3 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right) \right]; \tag{S-9.7}
\]
\[
\left| \frac{1}{\lambda} \frac{\partial^3 f}{\partial \mu \partial \omega^2} \right| \leq \left[ 2V^2 + 3V^3 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right)^2 + V^4 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right) \right]; \tag{S-9.8}
\]
\[
\left| \frac{1}{\lambda} \frac{\partial^3 f}{\partial \mu \partial \omega^2} \right| \leq \left[ 4V^3 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right) + 4V^4 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right)^3 + V^4 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right)^5 \right]; \tag{S-9.9}
\]
\[
\left| \frac{1}{\lambda} \frac{\partial^3 f}{\partial \omega^2 \partial \omega^2} \right| \leq \left[ 2V^3 + 11V^4 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right)^2 + 5V^5 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right)^4 + V^6 \left( \mu_{\text{max}} + \left| \frac{U}{V} \right| \right)^6 \right]; \tag{S-9.10}
\]

where each of the upper bounds has finite expectation. This can be seen from the existence of all order moments of \( V \) and \( \left| \frac{U}{V} \right| \), along with the compactness of the parameter space, and using the Cauchy-Schwartz inequality.

**S-10  Multidimensional linear random effects**

Here we consider \( d \)-dimensional random effect, that is, we consider SDE’s of the following form:

\[
dX_i(t) = \phi^T_i b(X_i(t)) dt + \sigma(X_i(t)) dW_i(t), \quad \text{with} \quad X_i(0) = x^i, \quad i = 1, \ldots, n. \tag{S-10.1}
\]

where \( \phi_i = (\phi_{i1}, \phi_{i2}, \ldots, \phi_{id})^T \) is a \( d \)-dimensional random vector and \( b(x) = (b^1(x), b^2(x), \ldots, b^d(x))^T \) is a function from \( \mathbb{R} \) to \( \mathbb{R}^d \). Here \( b(x, \varphi) = \sum_{i=1}^{d} \varphi^i b^i(x) \) satisfies (H1) of MB. We consider \( \phi_i \) having finite (say, \( M \)) mixture of normal distributions having expectation vectors \( \mu_k \) and covariance matrices \( \Sigma_k \) for \( k = 1, \ldots, M \), with density

\[
g(\varphi, \theta) d\nu(\varphi) \equiv \sum_{k=1}^{M} a_k N(\mu_k, \Sigma_k)
\]

such that \( a_k \geq 0 \) for \( k = 1, \ldots, M \) and \( \sum_{k=1}^{M} a_k = 1 \). Here the parameter set is

\[
\theta = (a_1, \mu_1, \Sigma_1, \ldots, a_M, \mu_M, \Sigma_M) = (\gamma, \beta),
\]

20
where, \( \gamma = (a_1, \ldots, a_M) \), and \( \beta = (\beta_1, \ldots, \beta_M) \), where, for \( k = 1, \ldots, M \), \( \beta_k = (\mu_k, \Sigma_k) \).

The sufficient statistics for \( i = 1, \ldots, n \) are

\[
U_i = \int_0^{T_i} \frac{b(X_i(s))}{\sigma^2(X_i(s))} dX_i(s)
\]

and

\[
V_i = \int_0^{T_i} \frac{b(X_i(s))b^T(X_i(s))}{\sigma^2(X_i(s))} ds.
\]

Note that \( U_i \) are \( d \)-dimensional random vectors and \( V_i \) are random matrices of order \( d \times d \). As in Delattre et al. (2013) we need to assume that \( V_i \) is positive definite for each \( i \geq 1 \) and for all \( \theta \).

By Lemma 2 of Delattre et al. (2013) it follows, almost surely, for all \( i \geq 1 \), for \( 1 \leq k \leq M \) and for all \( \theta \), that \( V_i + \Sigma_k^{-1}, I_d + V_i\Sigma_k, I_d + \Sigma_k V_i \) are invertible.

Setting \( R_i^{-1} = (I_d + V_i\Sigma_k)^{-1} V_i \) we have

\[
\lambda_i(X_i, \theta) = \sum_{i=1}^M a_k f(X_i|\beta_k), \tag{S-10.2}
\]

where

\[
f(X_i|\beta_k) = \frac{1}{\sqrt{\det(I_d + V_i\Sigma_k)}} \exp \left( -\frac{1}{2} (\mu_k - V_i^{-1}U_i)^T R_i^{-1} (\mu_k - V_i^{-1}U_i) \right) \exp \left( \frac{1}{2} U_i^T V_i^{-1} U_i \right) \tag{S-10.3}
\]

The asymptotic investigation for both classical and Bayesian paradigms in this multidimensional case can be carried out in the same way as the one-dimensional problem, using Theorem 5 of Maitra and Bhattacharya (2016) with relevant modifications and Proposition 10 (i) of Delattre et al. (2013) which is valid here for each \((\mu_k, \Sigma_k)\).

References

Atienza, N., J., J. G.-H., Munoz-Pichardo, J., and Villa, R. (2005). Some Integrability Results on Exponential Families. Technical report. Available at “http://www.personal.us.es/natienza/inves/docu/technical1.pdf”.

Banfield, J. and Raftery, A. (1993). Model Based Gaussian and non-Gaussian Clustering. *Biometrics, 49*, 803–821.

Bishop, C. M. (1995). *Neural Networks for Pattern Recognition*. Oxford University Press.

Brooks, S. (2001). On Bayesian Analyses and Finite Mixtures for Proportions. *Statistics and Computing, 11*, 179–190.

Choi, T. and Schervish, M. J. (2007). On Posterior Consistency in Nonparametric Regression Problems. *Journal of Multivariate Analysis, 98*, 1969–1987.

Das, M. and Bhattacharya, S. (2017). Transdimensional Transformation based Markov Chain Monte Carlo. Available at “https://arxiv.org/abs/1403.5207”.

Delattre, M., Genon-Catalot, V., and Samson, A. (2013). Maximum Likelihood Estimation for Stochastic Differential Equations with Random Effects. *Scandinavian Journal of Statistics, 40*, 322–343.

Delattre, M., Genon-Catalot, V., and Samson, A. (2016). Mixtures of Stochastic Differential Equations With Random Effects: Application to Data Clustering. *Journal of Statistical Planning and Inference, 173*, 109–124.
Dempster, A., Laird, N., and Rubin, D. (1977). Maximum Likelihood from Incomplete Data via the EM Algorithm (with discussion). *Journal of the Royal Statistical Society. Series B, 39*, 1–38.

Green, P. and Richardson, S. (2002). Hidden Markov Models and Disease Mapping. *Journal of the American Statistical Association, 97*, 1055–1070.

Green, P. J. (1995). Reversible jump Markov chain Monte Carlo computation and Bayesian model determination. *Biometrika, 82*, 711–732.

Hoadley, B. (1971). Asymptotic Properties of Maximum Likelihood Estimators for the Independent not Identically Distributed Case. *The Annals of Mathematical Statistics, 42*, 1977–1991.

Kass, R. E. and Raftery, R. E. (1995). Bayes factors. *Journal of the American Statistical Association, 90*(430), 773–795.

Keribin, C. (2000). Consistent Estimation of the Order of Mixture Models Mixture Models. *Sankhya, 62*, 49–66.

Leroux, B. (1992). Maximum Penalized Likelihood Estimation for Independent and Markov-Dependent Mixture Models. *Biometrics, 48*, 545–558.

Lindsay, B. (1995). *Mixture Models: Theory Geometry and Applications*. IMS, Hayward, CA.

Louzada-Neto, F., Mazucheli, J., and Achcar, J. (2002). Mixture Hazard Models for Lifetime Data. *Biometrical Journal, 44*, 3–14.

Maitra, T. and Bhattacharya, S. (2015). On Bayesian Asymptotics in Stochastic Differential Equations with Random Effects. *Statistics and Probability Letters, 103*, 148–159. Also available at “http://arxiv.org/abs/1407.3971”.

Maitra, T. and Bhattacharya, S. (2016). On Asymptotics Related to Classical Inference in Stochastic Differential Equations with Random Effects. *Statistics and Probability Letters, 110*, 278–288. Also available at “http://arxiv.org/abs/1407.3968”.

Maitra, T. and Bhattacharya, S. (2017). On Classical and Bayesian Asymptotics in Stochastic Differential Equations with Random Effects having Mixture Normal Distributions. submitted.

Norets, A. and Pelenis, J. (2012). Bayesian Modeling of Joint and Conditional Distributions. *Journal of Econometrics, 168*, 332–346.

Redner, R. (1981). Note on the Consistency of the Maximum Likelihood Estimate for Nonidentifiable Distributions. *The Annals of Statistics, 9*, 225–228.

Richardson, S. and Green, P. J. (1997). On Bayesian Analysis of Mixtures with an Unknown Number of Components (with discussion). *Journal of the Royal Statistical Society. Series B, 59*, 731–792.

Roeder, K. and Wasserman, L. (1997). Practical Bayesian Density Estimation using Mixtures of Normals. *Journal of the American Statistical Association, 92*, 894–902.

Schervish, M. J. (1995). *Theory of Statistics*. Springer-Verlag, New York.

Wirjanto, T. S. and Xu, D. (2009). The Applications of Mixture of Normal Distributions in Empirical Finance: A Selected Survey. Available at “http://economics.uwaterloo.ca/documents/mn-review-paper-CES.pdf”.

Zhu, H. T. and Lee, S. Y. (2001). A Bayesian Analysis of Finite Mixtures in the LISREL Model. *Psychometrika, 66*, 133–152.