A note on latticeability in vector-valued sequence spaces

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Abstract

Adjusting a technique due to Jiménez-Rodríguez, we prove the complete latticeability of the set of disjoint non-norm null weakly null sequences and of the set of disjoint non-norm null regular-polynomially null sequences in Banach lattices.

1 Introduction

In this note we give a contribution to the fashionable subject of lineability, which is the search for linear structure inside nonlinear environments. The book [6] is a very good reference for the state of the art in lineability. Among tons of lineability-type results which have appeared in the last years, our focus in this note is the following result (actually its proof) proved by Jiménez-Rodríguez [20]: if $E$ is a Banach space failing the Schur property, then the set of non-norm null weakly null $E$-valued sequences contains, except for the origin, a closed infinite-dimensional subspace of $c_0^w(E)$, which is the closed subspace of $\ell_\infty(E)$ formed by weakly null sequences. We address in this note three questions that arise naturally from Jiménez-Rodríguez’ result. Some terminology is needed to state these questions precisely.

Polynomially Schur spaces were introduced by Carne, Cole and Gamelin [12] and have been developed by several authors (see, e.g. [5, 7, 18, 21]). A sequence $(x_j)_{j=1}^\infty$ in a Banach space $E$ is polynomially null if $P(x_j) \to 0$ for every scalar-valued continuous homogeneous polynomial $P$ on $E$. A Banach space $E$ is polynomially Schur if every polynomially null $E$-valued sequence is norm null.

A subset $A$ of a topological vector space $E$ is spaceable (see [6]) if there exists a closed infinite dimensional subspace of $E$ all of whose elements but the origin belong to $A$. Jiménez-Rodríguez’ result is a spaceability result in $c_0^w(E)$.

Question 1. If $E$ is a non-polynomially Schur Banach space, then the set of non-norm null polynomially null sequences is spaceable in $c_0^w(E)$?

In Remark 2.3 we shall explain why we do not go to a space smaller than $c_0^w(E)$.

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The *positive Schur property* in Banach lattices (positive – or, equivalently, disjoint – weakly null sequences are norm null) was introduced by Wnuk [29] and Räbiger [27] and has been extensively studied, for some recent developments see [1, 8, 9, 14, 28, 31, 32]. Oikhberg [25] coined the following terms: a subset $A$ of a Banach lattice is *latticeable* (completely latticeable) if there exists a (closed) infinite dimensional sublattice of $E$ all of whose elements but the origin belong to $A$ (see also [26]).

Question 2: If $E$ is a Banach lattice failing the positive Schur property, then the set of disjoint non-norm null weakly null $E$-valued sequences is latticeable in $\ell_\infty(E)$? Completely latticeable?

A sequence $(x_j)_{j=1}^\infty$ in a Banach lattice $E$ is *regular-polynomially null* if $P(x_j) \longrightarrow 0$ for every scalar-valued regular homogeneous polynomial $P$ on $E$. The following class of Banach lattices was studied in [10]: a Banach lattice $E$ is *positively polynomially Schur* if positive regular-polynomially null $E$-valued sequences are norm null.

Question 3: If $E$ is a non-positively polynomially Schur Banach lattice, then the set of disjoint non-norm null regular-polynomially null $E$-valued sequences is latticeable in $\ell_\infty(E)$? Completely latticeable?

In this note we show that the Jiménez-Rodríguez technique can be adjusted to solve affirmatively the three questions above. We will also justify why we have to work with $\ell_\infty(E)$ in Questions 2 and 3 (see Remark 2.3).

All sequence spaces in this note are considered as Banach lattices with the coordinatewise order. By $B_E$ we denote the closed unit ball of the Banach space $E$. For the general theory of Banach lattices we refer to [2, 24], for regular homogeneous polynomials in Banach lattices we refer to [11, 15, 23]. By a disjoint sequence in a Riesz space we mean a pairwise disjoint sequence.

### 2 Results

Recall that $\ell_\infty(E)$ is a Banach lattice with the coordinatewise order whenever $E$ is a Banach lattice [2, p. 177] and that $c^0_0(E)$ is a closed subspace of $\ell_\infty(E)$ whenever $E$ is a Banach space [16, p. 33].

**Theorem 2.1.**

(a) Let $E$ be a non-polynomially Schur Banach space. Then the set of $E$-valued non-norm null polynomially null sequences is spaceable in $c^0_0(E)$.

(b) Let $E$ be a Banach lattice failing the positive Schur property. Then the set of $E$-valued disjoint non-norm null sequences is completely latticeable in $\ell_\infty(E)$.

(c) Let $E$ be a non-positively polynomially Schur Banach lattice. Then the set of $E$-valued disjoint non-norm null regular-polynomially null sequences is completely latticeable in $\ell_\infty(E)$.

**Proof.** We start with the construction due to Jiménez-Rodríguez [20] which will be used in the three proofs. Let $E$ be a Banach space, $\varepsilon > 0$ and $(x_j)_{j=1}^\infty \subset B_E$ be a sequence such that $\|x_j\| \geq \varepsilon$ for every $j \in \mathbb{N}$. Consider the set of prime numbers $\mathfrak{P} = \{p_k : k \in \mathbb{N}\}$ increasingly ordered, the surjective function

$$F: \mathbb{N} \setminus \{1\} \longrightarrow \mathbb{N}, \quad f(m) = k, \text{ where } p_k = \min\{p \in \mathfrak{P} : p \mid m\},$$
and the map

\[ T: \ell_\infty \rightarrow \ell_\infty(E), \quad T((a_n)_{n=1}^\infty) = (a_F(j+1)x_j)_{j=1}^\infty, \]

that is, \( T((a_n)_{n=1}^\infty)_j = a_F(j+1)x_j \) for every \( j \in \mathbb{N} \). An easy adaptation of the arguments of \cite{20} Theorem 2.1 yield that \( T \) is a well defined into isomorphism and that the nonzero elements of its range are non-norm null sequences. The range of \( T \) will be the space/lattice we are looking for.

(a) We can start with a non-norm null polynomially null sequence \( (x_j)_{j=1}^\infty \) in \( E \). Passing to a subsequence and normalizing if necessary, we can suppose that \( (x_j)_{j=1}^\infty \subset B_E \) and that there is \( \varepsilon > 0 \) such that \( ||x_j|| \geq \varepsilon \) for every \( j \in \mathbb{N} \). All that is left to prove is that the elements of the range of \( T \) are polynomially null \( E \)-valued sequences. This is true because, given \( (a_F(j+1)x_j)_{j=1}^\infty \in T(\ell_\infty) \) with \( (a_j)_{j=1}^\infty \in \ell_\infty, \ n \in \mathbb{N} \) and \( P \in \mathcal{P}^\infty(E) \), since \( P(x_j) \rightarrow 0 \) and \( \{a_F(j+1) : j \in \mathbb{N}\} = \{a_n : n \in \mathbb{N}\} \) is a bounded set, we have

\[ |P(a_F(j+1)x_j)| = |a_F(j+1)|^n \cdot |P(x_j)| \rightarrow 0. \]

In particular, the range of \( T \) lies in \( c_0^\infty(E) \).

(b) In this case we can start with a positive weakly null non-norm null sequence \( (x_j)_{j=1}^\infty \) in \( E \). By \cite{30} p.16 we can suppose that this sequence is disjoint and, as in the proof of (a), that \( (x_j)_{j=1}^\infty \subset B_E \) and that there is \( \varepsilon > 0 \) such that \( ||x_j|| \geq \varepsilon \) for every \( j \in \mathbb{N} \). From the proof of \cite{20} Theorem 2.1 we know that the elements of the range of \( T \) are weakly null sequences. As these elements are of the form \( (a_F(j+1)x_j)_{j=1}^\infty \) for some \( (a_n)_{n=1}^\infty \in \ell_\infty \) and the sequence \( (x_j)_{j=1}^\infty \) is disjoint, from \cite{3} Lemma 1.9(1)] we conclude that the range of \( T \) is formed by disjoint sequences. To finish the proof of this case, let us see that \( T \) is Riesz homomorphism: given \( (a_n)_{n=1}^\infty \) and \( (b_n)_{n=1}^\infty \) in \( \ell_\infty \), it holds \( (a_n)_{n=1}^\infty \wedge (b_n)_{n=1}^\infty = (a_n \wedge b_n)_{n=1}^\infty \) and since \( x_j \geq 0 \) for every \( j \), we have

\[ (a_F(j+1) \wedge b_F(j+1))x_j = (a_F(j+1)x_j) \wedge (b_F(j+1)x_j) \]

for every \( j \in \mathbb{N} \). It follows that \( T((a_n)_{n=1}^\infty \wedge (b_n)_{n=1}^\infty) = T((a_n)_{n=1}^\infty) \wedge T((b_n)_{n=1}^\infty) \), proving that \( T \) is a Riesz homomorphism, hence its range is a closed sublattice of \( \ell_\infty(E) \) lattice isomorphic to \( \ell_\infty \).

(c) According to \cite{10} Proposition 2.4] we can start with a positive disjoint non-norm null regular-polynomially null sequence \( (x_j)_{j=1}^\infty \) in \( E \). Like we have done above, we can suppose that \( (x_j)_{j=1}^\infty \subset B_E \) and that there is \( \varepsilon > 0 \) such that \( ||x_j|| \geq \varepsilon \) for every \( j \in \mathbb{N} \). As we did in the proof of (b), the fact that the sequence \( (x_j)_{j=1}^\infty \) is positive and disjoint guarantees that \( T \) is a Riesz homomorphism, therefore its range is a closed sublattice of \( \ell_\infty(E) \), and that the elements of the range are disjoint sequences. As we did in the proof of (a), the fact that the sequence \( (x_j)_{j=1}^\infty \) is regular-polynomially null implies that the sequences in the range of \( T \) are regular-polynomially null as well.

It is natural to wonder if the closed sublattices of \( \ell_\infty(E) \) obtained in the proofs of (b) and (c) above are ideals in \( \ell_\infty(E) \). Next example shows that this is not the case in general, making clear that this is a direction that cannot be pursed using the Jiménez-Rodríguez technique.
Example 2.2. The most favorable situation we can imagine for \( T(\ell_\infty) \) to be an ideal of \( \ell_\infty(E) \) occurs when the starting sequence \((x_j)_{j=1}^\infty\) is formed by atoms of \( E \). In this example we show that, even in this case, \( T(\ell_\infty) \) may fail to be an ideal of \( \ell_\infty(E) \). We start with the sequence \((e_j)_{j=1}^\infty\) of canonical unit vectors in \( c_0 \), which is a positive disjoint non-norm null weakly null sequence formed by atoms. By \([17, Proposition 1.59]\) this sequence is also regular-polynomially null. Consider the corresponding operator
\[
T: \ell_\infty \longrightarrow \ell_\infty(c_0), \quad T((a_n)_{n=1}^\infty) = (a_{F(j+1)}e_j)_{j=1}^\infty,
\]
the positive vector \((e_1,0,0,\ldots) \in \ell_\infty(c_0)\) and the sequence \( e_1 \in \ell_\infty \). On the one hand,
\[
0 \leq (e_1,0,0,\ldots) \leq (e_1,0,e_3,0,e_5,0,\ldots) = T(e_1)
\]
in \( \ell_\infty(c_0) \). On the other hand, there is no element \((b_n)_{n=1}^\infty \in \ell_\infty \) such that \( T((b_n)_{n=1}^\infty) = (e_1,0,0,\ldots) \). Indeed, supposing that such a sequence \((b_n)_{n=1}^\infty\) exists, by the definition of \( T \) we would have
\[
(e_1,0,0,\ldots) = T((b_n)_{n=1}^\infty) = (b_1 e_1, b_2 e_2, b_1 e_3, \ldots),
\]
which gives \( 1 = b_1 = 0 \). This contradiction proves that \( T(\ell_\infty) \) is not an ideal in \( \ell_\infty(E) \).

Remark 2.3. (i) We cannot use \( c_0^u(E) \) instead of \( \ell_\infty(E) \) in Theorem 2.1(b) and (c) because \( c_0^u(E) \) is not always a sublattice of \( \ell_\infty(E) \). For instance, for \( 1 \leq p < \infty \), \( c_0^u(L_p[0,1]) \) is not a Riesz space due to the fact that the lattice operations are not weakly sequentially continuous \([24, Example, p. 114]\). But it is clear that the sublattices of \( \ell_\infty(E) \) created there are contained in \( c_0^u(E) \). Sometimes \( c_0^u(E) \) is a Banach lattice, for instance when \( E \) is either an AM-space or an atomic Banach lattice with order continuous norm (see \([2, Theorem 12.30]\) and \([24, Proposition 2.5.23]\)). In these cases, \( \ell_\infty(E) \) can be replaced with \( c_0^u(E) \) in Theorem 2.1(b) and (c).

(ii) Castillo, García and Gonzalo in \([13, Theorem 5.5]\) proved that the sum of two polynomially null sequences is not necessarily polynomially null. This is why we cannot pass to a space smaller than \( c_0^u(E) \) in Theorem 2.1(a).

(iii) We have already explained why \( c_0^u(E) \) cannot be used in general in Theorem 2.1(c). But one might wonder if we could have gone to a smaller space, formed by regular-polynomially null sequences. In order to see that we cannot, next we show that the counterexample given in \([13, Theorem 5.5]\) is good enough to show that the sum of two regular-polynomially null sequences may fail to be regular-polynomially null.

Theorem 2.4. The sum of two regular-polynomially null sequences in a Banach lattice is not necessarily regular-polynomially null.

Proof. Let \( d(w;1) \) be the Lorentz space of \([13, Theorem 5.4]\) and denote by \( d_*(w;1) \) its predual. The sequence of canonical unit vectors \((e_j)_{j=1}^\infty\) is a 1-unconditional basis for \( d(w;1) \) (see \([1]\) and the sequence of coordinate functionals \((e_j^*)_{j=1}^\infty\) is an unconditional basis for \( d_*(w;1) \) (see \([19]\)), hence it is a 1-unconditional basis (see \([22, p. 19]\)). We consider \( d_*(w;1) \) as a Banach lattice with the order given by its 1-unconditional basis and \( d(w;1) \) with its dual structure (which coincides, by the way, with the order given by the 1-unconditional basis \((e_j^* )_{j=1}^\infty\)). Thus, \( d_*(w;1) \times d(w;1) \) is a Banach lattice with the coordinatewise order, in which we can consider, without loss of generality, the norm \( \| \cdot \|_1 \). According to
Theorem 5.5, the sequences \(((e_j^*, 0))_{j=1}^{\infty}\) and \(((0, e_j))_{j=1}^{\infty}\) are polynomially null, hence regular-polynomially null, in \(d_*(w; 1) \times d(w; 1)\). Let us see that their sum \(((e_j^*, e_j))_{j=1}^{\infty}\) is not regular-polynomially null. To do so, consider the symmetric bilinear form \(A\) on \((d_*(w; 1) \times d(w; 1)) \times (d_*(w; 1) \times d(w; 1))\) given by

\[
A((x^*, x), (y^*, y)) = 1/2(x(y^*) + (y(x^*))
\]

(see [17 Example 1.18]). It is easy to check that \(A\) is positive, from which it follows that its associated 2-homogeneous polynomial \(\hat{A}\) is positive, hence regular. Since \(\hat{A}((e_j^*, e_j)) = 1\) for every \(j \in \mathbb{N}\), we conclude that \(((e_j^*, e_j))_{j=1}^{\infty}\) is not regular-polynomially null.

Now it is easy to see that, for every Banach space \(E\), the set \(PN\) of polynomially null \(E\)-valued sequences is spaceable in \(c_0^w(E)\): if \(E\) is not polynomially Schur, in Theorem 2.1(a) we proved that a set much smaller than \(PN\) is spaceable; if \(E\) is polynomially Schur, it is easy to check that \(PN = c_0(E)\), the closed subspace of \(c_0^w(E)\) formed by norm null sequences. Theorem 2.4 rises the question of the complete latticeability (or not) of the sets of regular-polynomially null sequences and of disjoint regular-polynomially null sequences in a Banach lattice \(E\). In Theorem 2.1(c) we proved that a set much smaller than these is completely latticeable whenever the Banach lattice \(E\) is not positively polynomially Schur. But we do not know what happens in the general case.

Open question. In an arbitrary Banach lattice \(E\), are the sets of regular-polynomially null sequences and of disjoint regular-polynomially null sequences (completely) latticeable in \(\ell_\infty(E)\)?

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