Complex structure and the construction of the $\phi^4_4$: quantum field theory in four-dimensional space-time *

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Abstract

We announce results about the nonperturbative mathematically rigorous construction of the $\phi^4_4$: quantum field theory in four-dimensional space-time. The complex structure of solutions of the classical nonlinear (real-valued) wave equation and quantization are closely connected among themselves and allow to construct non-perturbatively the quantum field theory with interaction $\phi^4_4$: in four-dimensional space-time. We consider vacuum averages, in particular, we construct Wightman functions and matrix elements of the scattering operator as generalized functions for finite energies. The constructed theory is obviously nontrivial.

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In this Letter we announce a part of results about the nonperturbative mathematically rigorous construction of the $\phi^4_4$: quantum field theory in four-dimensional space-time. The announced results are steps in realization of the project $\phi^4_4 \cap M$.

The construction of the quantum field is based on properties of dynamics of the classical system. Namely, there exists an intimate relationship between the dynamics of classical system, its complete integrability, its complex structure, and quantization. About the complete integrability of the classical $u^4_4$ interaction, see for instance, [1]. The complex structure for the classical nonlinear wave equation with interaction $u^4_4$ was constructed in [16], see also [16]. We formulate the statement about the complex structure in a convenient form.

We consider the solutions of the classical nonlinear wave equation in four-dimensional space-time

$$\frac{\partial^2}{\partial t^2} u - \Delta u + m^2 u + \lambda u^3 = 0, \quad m > 0, \quad \lambda > 0. \quad (1)$$

A solution is given uniquely by its initial data (by its canonical coordinate and canonical momentum), for instance, at time zero,

$$\varphi(x) = u(t, x)|_{t=0}, \quad \pi(x) = \frac{\partial}{\partial t} u(t, x)|_{t=0},$$

and these initial data belong to the space $H^1 \oplus L^2$, $(\varphi, \pi) \in H^1 \oplus L^2$. Let $U(t)$, $W$, and $S$ be, respectively, the operator of dynamics, the wave operator to the backward, and the scattering operator of the nonlinear wave equation (1). These nonlinear operators are correctly defined for initial data from $H^1 \oplus L^2$, are invertible and are canonical transforms (i.e. symplectomorphisms) (see, for instance, [8,9,1]).

Let $R$ be the map

$$R(\varphi, \pi) = \varphi + i\mu^{-1}\pi \equiv \varphi^+, \quad R^{-1}\varphi^+ = (\text{Re} \varphi^+, \mu \text{Im} \varphi^+),$$

where $\mu = (-\Delta + m^2)^{1/2}$. $R$ maps an initial data on the positive frequency part of the free solution with this initial data (at time zero).

The essential feature of the construction of the $\phi^4_4$: quantum field theory is the statement about the complex structure of the classical nonlinear wave equation (1).
Theorem 1 (complex structure) (see [15, Theorem 1.1]). The maps \(RU(t)R^{-1},\) \(RW R^{-1},\) \(RS R^{-1}\) are correctly defined and are complex holomorphic maps on the space \(H^1(\mathbb{R}^3, \mathbb{C})\) into itself. In particular, for

\[
z(\alpha) = \sum_{j=1}^{N} \alpha_j z_j, \quad z_j \in H^1(\mathbb{R}^3, \mathbb{C}), \quad \alpha_j \in \mathbb{C}, \quad h \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}),
\]

the functions \(\int d^3 x (RU(t)R^{-1}z(\alpha))(x)h(x),\)

\[
\int d^3 x (RW R^{-1}z(\alpha))(x)h(x), \quad \int d^3 x (RS R^{-1}z(\alpha))(x)h(x)
\]

are complex holomorphic functions on \((\alpha_1, ..., \alpha_N) \in \mathbb{C}^N.\)

The construction of the quantum field can be done by several different ways (which are in agreement with themselves). Here we consider the one of them.

To construct the \(\phi_4^4\) quantum field theory we use the existence of the wave operator of the classical nonlinear equation (1) and the statement of Theorem 1 about the complex structure. We can define the interacting field as

\[
\phi = \frac{1}{2} (:RW R^{-1}(\phi_{in}):+:R\Theta^T W \Theta^T R^{-1}(\phi_{in})::),
\]

(2)

where \(\Theta^T\) is the operator of time reflection for the classical system, \(\phi_{in}\) is the incoming quantum field (it is free), \(::\) means the Wick normal ordering with respect to the free quantum field \(\phi_{in}.\) Expression (2) gives the possibility to define the Wick kernel of the interacting quantum field (the Wick kernel is equal to matrix elements of (2) on coherent vectors). We note that on the diagonal the Wick symbol of the interacting quantum field (2) is a (real) solution of the classical nonlinear equation (1), and it is namely the same solution, in which the wave operator maps the free solution given by the diagonal of the Wick symbol of the free quantum field \(\phi_{in}.\)

Taking into account the holomorphity of the map \(RW R^{-1}\) and the \(T\)-symmetry of Eq. (1) the expression (2) has quite correct sense. In particular, it is a bilinear form defined on coherent vectors of the Fock space of the \(in\)-field (see [323][3][15][17]).
The other way of construction of the quantum field (and the construction of its Wick kernel) is the definition of nonlinear quantum equation in the integral form,

$$\phi(t, x) = \phi_{in}(t, x) - \lambda \int_{-\infty}^{t} d\tau d^3y R(t - \tau, x - y) :\phi^3(\tau, y):$$  \hspace{1cm} (3)

with normal ordering with respect to the $in$-field, see [6,13,14,23]. This way is consistent with the definition of quantum field given by (2).

The introduction of the quantum field in the form (2), or the construction of the bilinear form – solution of Eq. (3), allows to write out the explicit expression for the total Hamiltonian. The total Hamiltonian is the self-adjoint positive operator. It is correctly defined as a bilinear form on coherent vectors from $D_{coh}(H^1(\mathbb{R}^3, \mathcal{U}))$. The total Hamiltonian is equal to the following expressions

$$H = \frac{1}{2} \int d^3x \left( :\dot{\phi}^2(t, x): + :\nabla \phi^2(t, x): + m^2 :\phi^2(t, x): + \frac{\lambda}{2} :\phi^4(t, x): \right)$$

$$= \frac{1}{2} \int d^3x \left( :\dot{\phi}_{in}^2(t, x): + :\nabla \phi_{in}^2(t, x): + m^2 :\phi_{in}^2(t, x): \right)$$

$$= \frac{1}{2} \int d^3x \left( :\dot{\phi}_{out}^2(t, x): + :\nabla \phi_{out}^2(t, x): + m^2 :\phi_{out}^2(t, x): \right),$$

see [6,13,15,17].

The possibility to construct the Wick kernel, or the Wick symbol, of the interacting quantum field simplifies a consideration of its properties and the construction of vacuum averages.

The consideration of this Wick kernel and the construction of bilinear form is based on the Fock-Bargmann-Berezin-Segal integral representation over coherent states, see [22,20,10]. For this description we use initial data of the $in$-field. This choice of coordinates is natural from the physical point of view. It requires to consider integrals with Gaussian promeasure and simplifies estimates and the construction of operator–valued generalized function, corresponding to the Wick kernel of the interacting field, see [17,18]. Here we do not present the exposition of these results important for the formulation of locality condition, we only illustrate them by the consideration of smoothed quantum field for finite momentum and finite energy.
The consideration of finite energies (for our massive case!) is technically more simple and allows to construct vacuum averages.

To construct vacuum averages we consider the following smoothing of the quantum field

\[ \int dt_1 dt_2 d^3 x_1 d^3 x_2 e^{iHt_1+iPx_1} \phi(0, 0) e^{iHt_2+iPx_2} f_1(t_1, x_1) f_2(t_2, x_2). \] (4)

The expression \( \phi(t, x)|_{t=0,x=0} \) is correctly defined as a bilinear form and (4) corresponds to the following bilinear form

\[ f_1^\sim(H, P)\phi(0, 0)f_2^\sim(H, P), \]

where \( f_1^\sim, f_2^\sim \) is the Fourier transform of functions \( f_1 \), and, respectively, \( f_2, H \) is the total Hamiltonian and \( P \) is the operator of momentum. As test functions we shall take smooth functions with compact support in momentum space. Using holomorphy (Theorem 1) it is possible to write the Taylor expansion (at zero)

\[ \phi = \sum_{n=1}^{\infty} \phi_n :\phi^{in}_n\phi^{in}_n:, \]

coefficients \( \phi_n \) of this expansion are equal to the corresponding derivatives of the classical wave operator,

\[ \phi_n = \frac{1}{2^n n!} (d^n RWR^{-1}(0) + d^n R\Theta^T W\Theta^T R^{-1}(0)). \]

These Taylor coefficients \( \phi_n \) are correctly defined as Schwartz distributions, \( \phi_n \in S'(\mathbb{R}^3) \).

Since we use as test functions smooth functions with compact support in momentum space and, taking into account, that we consider the massive case, it is easy to see, that

\[ f_1^\sim(H, P)\phi(0, 0)f_2^\sim(H, P) = \sum_{1}^{N(f_1, f_2)} f_1^\sim(H, P)\phi_n (:\phi^{in}_n\phi^{in}_n:)f_2^\sim(H, P). \] (5)

Here \( N(f_1, f_2) < \infty \) for functions \( f_1, f_2 \) with compact support in momentum space. Thus, the bilinear form (5) is a Wick polynomial and it is easy to prove that it defines a unique bounded operator.

Therefore, we can consider the expressions

\[ (\Omega, \prod_i(f_j^\sim(H, P)\phi(0, 0)f_{j+1}^\sim(H, P))\Omega). \]
Due to translation invariance and the Schwartz nuclear theorem the expressions (6) allow to define Wightman functions \((\Omega, \prod \phi(t_j, x_j)\Omega)\) as generalized functions. The direct consequences of these considerations are the following statements:

**Theorem 2 (operator-valued generalized function).** The interacting field \(\phi\) is a correctly defined operator-valued generalized function on the space \(\mathcal{F}(D(\mathbb{R}^4))\) in the Fock space of the free quantum in-field \(\phi_{\text{in}}\). Here \(\mathcal{F}\) the Fourier transform and \(D(\mathbb{R}^4)\) is the Schwartz space of smoothed test functions with compact support in momentum space. The analogous statement is valid for the quantum outgoing field \(\phi_{\text{out}}\).

**Theorem 3 (Wightman functions, [14]).** The Wightman functions are correctly defined as generalized functions on the space \(\mathcal{F}(D(\mathbb{R}^n))\). They satisfy the positivity condition, the spectrum condition, \(T\)- and \(P\)-symmetry.

The analogous assertion is valid for expressions \((\Omega, \prod \phi_{\#}(t_j, x_j)\Omega)\) also. Here \(\phi_{\#}\) is either the in-field, or the interacting field, or the out-field. To construct the quantum scattering operator (the \(S\) matrix) and to prove its unitarity [19,7] we need to consider these vacuum averages, too.

The next important properties of the quantum field are locality, unitarity [17,18], and nontriviality [13,21,18].

Here we shall formulate the assertion about nontriviality and we shall describe required steps for the proof of locality and unitarity.

**Theorem 4 (nontriviality, [13]).** The constructed \(\phi^4\) quantum field theory is nontrivial. The coupling constant \(\lambda\) is uniquely defined by matrix elements of the interpolating quantum field.

The consideration of locality requires a further progress. An appropriate and important step is the consideration of a quantum field–bilinear form with the help of the Fock–Bargmann–Berezin–Segal integral and the integral representation in the in-field coordinates [22,21,2,18]. This integral representation uses the Gaussian promeasure, Wick kernel (=...
Wick symbol), coherent states, holomorphic wave representation of the Fock space and allows to write an integral representation for singular operator and bilinear forms \[20,18\].

We note, that in coordinates of the \(in\)-field the vacuum is described by Gaussian promeasure. The possibility to use coordinates of the \(in\)-field is connected with the existence of wave operator. The existence of wave operator simplifies consideration essentially. A key moment in a capability to apply this integral representation is the existence of the complex structure (Theorem 1 and Theorem 2), see \[17,18,20\].

A bilinear form written with the help of the Fock–Bargmann–Berezin–Segal integral representation has the following form

\[
\phi(\chi_1, \chi_2) = \int_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} \int_{H^{1/2}(\mathbb{R}^3, \mathcal{L})} d\nu(z_1)d\nu(z_2) \phi(z_1, z_2) \chi_1(z_1)\chi_2(z_2).
\]

Here \(\phi(z_1, z_2)\) is the Wick kernel of the interacting quantum field, \(d\nu(z_1)\) is the promeasure corresponding to the vacuum (in the \(in\)-field coordinates the promeasure is Gaussian), see \[20,18\]. About promeasures, i.e. about a consistent set of finite–dimensional measures, or about a close analogue of measures on cylindrical sets, see, for instance, \[20\], \[4, v. 4, ch. 4\].

This integral representation (7), estimates, and the Bogoliubov “edge of the wedge” theorem \[26\] allow to extend the quantum field on much more wide class of test functions, including functions with compact support in coordinate space \[18\]. This fact gives a possibility to consider locality. In addition, locality requires the consideration of some suitable finite–dimensional approximations and the using of von Neumann’s theorem about unitary equivalence of finite-dimensional representations of algebra of canonical commutation relations \[19\].

Furthermore, the proof of unitarity is based on absence of bounded states (for the classical wave equation).

**Conclusion.** The presented scheme of construction of the \(:\phi^4_4:\) quantum field theory follows to the scheme, stated in \[3\], partially it is reflected in \[12,23\] (however, we note, that in \[23\] the iterations for the quantum field converge and they allow to reconstruct the quantum field, but this reconstruction requires the existence of complex structure for solutions of the
classical nonlinear equation). Long time a part of this scheme were propagandized by Segal (see, for example, [24,25]). This scheme is closely connected with the scheme of quantization that have been proposed by Kostant [11].

It is interesting to compare the stated exposition of the construction of the quantum field in four-dimensional space-time with the known construction in low dimension [5]. We think that our scheme of exposition is more appropriate from the physical point of view. It is interesting also to make comparison with the perturbation theory of the $\phi^4$ model. Note that the perturbation theory for the $\phi^4$ model is renormalizable.

There is wide possibilities to extend this scheme on higher dimensions, on higher degree of nonlinearities (for which the Sobolev inequalities have the other form), on Yang–Mills–Higgs fields and similar questions. A proper discussion of this issue is beyond the scope of this Letter.

This is the second announce paper of the project $\phi^4 \cap M$. The one of the goal of this project is to support partly the Russian Fundamental Researches.

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