Intrinsic measurement errors for the speed of light in vacuum

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Received 20 February 2017, revised 22 June 2017
Accepted for publication 18 July 2017
Published 3 August 2017

Abstract
The speed of light in vacuum, one of the most important and precisely measured natural constants, is fixed by convention to \( c = 299 792 458 \) m s\(^{-1}\). Advanced theories predict possible deviations from this universal value, or even quantum fluctuations of \( c \). Combining arguments from quantum parameter estimation theory and classical general relativity, we here establish rigorously the existence of lower bounds on the uncertainty to which the speed of light in vacuum can be determined in a given region of space-time, subject to several reasonable restrictions. They provide a novel perspective on the experimental falsifiability of predictions for the quantum fluctuations of space-time.

Keywords: speed of light, quantum metrology, general relativity

(Some figures may appear in colour only in the online journal)

1. Introduction

It is generally accepted that the speed of light in vacuum \( c \) is a universal natural constant, isotropic, independent of frequency, and independent of the motion of the inertial frame with respect to which it is measured. These properties have been experimentally demonstrated with very high precision, e.g. isotropy up to a relative uncertainty of the order of \( \sim 10^{-9} \) [1], and lie at the basis of special relativity. By 1972, measurements of the speed of light became more precise than the definition of the meter [2], leading in 1983 to the definition of the speed of light in vacuum \( c = 299 792 458 \) m s\(^{-1}\). But attempts to quantize gravity have led to the concept of space-time as a fuzzy ‘quantum foam’ on the Planck length \( \ell_p = \sqrt{\hbar G/c^3} \approx 1.62 \times 10^{-35} \) m.
that implies an uncertainty or dispersion of $c$ [6–9]. Experimental data based on gamma-ray bursts, pulsars, and TeV-flares from active galaxies imply upper bounds on deviations of $c$ over cosmic distances [10–16]. Quantum fluctuations of $c$ were also proposed due to virtual fermion-anti-fermion pairs, leading to a scaling of the jitter of the arrival time of light pulses with propagation distance [17, 18]. Satellite experiments are being planned to verify fundamental space-time properties with unprecedented precision, such as the isotropy of $c$ and its independence from the laboratory frame velocity [1].

Here we establish how precisely $c$ in a given region of space–time may be determined in principle, i.e. independent of any technical challenges. Our approach is based on the firmly established quantum parameter estimation theory (q-pet) [19–26] and general relativity (GR) in semiclassical approximation [27]. Q-pet allows one to obtain a lower bound on the uncertainty with which a parameter $\theta$ may be estimated that parametrizes a quantum state specified by a density matrix $\rho(\theta)$. The power of q-pet is due to the facts that (i.) the bound is reachable in the limit of a large number of measurements, and (ii.) it is optimized over all possible quantum mechanical measurements (positive operator valued measures, POVM [28]) and all data-analysis schemes (unbiased estimator functions). This so-called quantum Cramér–Rao bound (QCRB) [19–22] becomes relevant once all technical noise problems have been solved, and only the fundamental quantum uncertainties remain. It is the ultimate achievable lower bound on the uncertainty with which any parameter can be measured. Recently, the q-pet formalism was applied to the measurement of parameters in relativistic quantum field theory such as proper times and accelerations, the Unruh effect, gravitation, or the estimation of the mass of a black hole [29–32]. In the present work we go a step further by examining the back-action of the quantum probe on the metric of space-time. Taking back-action into account was proposed before [33–37] but to the best of our knowledge we combine for the first time modern q-pet with a precise calculation of the back-action of the probe on the space-time metric. We show that there is an optimal photon number at which the perturbation of the space-time metric due to the probe equals the quantum uncertainty of the measurement itself, establishing thus an ultimate lower bound on the uncertainty with which $c$ can be determined.

## 2. Quantum parameter estimation

Any direct measurement of the speed of light has to use a light signal. Indirect measurements, e.g. through measuring the fine-structure constant, the electron charge and Planck’s constant, may need no light but do not reflect the definition of $c$ as a speed and need an elaborate theoretical framework for their interpretation. We consider definitions of $c$ through $c = \Delta x / \Delta t$ (i.e. runtime measurements of a light pulse) as well as through $c = \omega / k$ (where $\omega$ is $(2\pi$ times) the frequency and $k$ the wavevector of a monochromatic e.m. wave) as direct measurements, as these (i.) use a light signal; (ii.) correspond to how $c$ has actually been determined experimentally (in particular the most precise determinations of $c$ to date use $c = \omega / k$ [2]), and (iii.) are based on simple three-letter formulas that need no elaborate theoretical framework for extracting $c$. These two definitions give $c$ the meaning of a propagation speed or phase speed, respectively. Note that we only need $c = \omega / k$ at the frequency considered, not over all frequencies. For wave-lengths comparable to quantum-gravity length scales (assumed to be of order Planck-length), modifications of this linear dispersion relation have been proposed (see the discussion on rainbow gravity in section 5.2), but we restrict ourselves to frequencies where the linear dispersion is well verified experimentally. We emphasize that these definitions of speed are only needed to determine a systematic experimental error due to GR effects. The quantum-mechanical uncertainty of $c$ obtained from q-pet on the other hand is optimized
over all possible (POVM) measurements of the light signal and analysis schemes of the data, including those that measure the propagation distance $\Delta x$ of a light pulse over a time-interval $\Delta t$. We therefore do not have to worry about additional uncertainties of measurements of positions or times.

Any light signal can be decomposed in modes of the electromagnetic (e.m.) field which are the fundamental dynamical objects in quantum optics. Q-pet shows that with $m$ modes the sensitivity can be improved at most by a factor $1/m$ \[25\]. Below we find that with at most $n$ photons in a single mode the best sensitivity scales as $\propto 1/n$; one can thus achieve for given maximum photon number $nm$ the same sensitivity scaling as $\propto 1/(nm)$ as with $m$ modes (for a strict proof see appendix A). In \[38\] the problems of positioning and clock synchronization were analyzed. They were reduced to measuring a travel time of a light pulse with constant $c$, which is closely related to measuring $c$ for a known propagation distance. Also there it was shown that the best uncertainty in the arrival time of the pulse for a squeezed m-mode state scales as $1/(nm)$. Furthermore, using the Margolus–Levitin quantum speed limit theorem, it was argued in \[38\] that this is the optimal scaling possible for any state. The scaling $\propto 1/n$ for large average photon number $n$ was also obtained for phase estimation with two-mode squeezed light in \[39\]. As for relativistic effects, if we are interested in knowing $c$ in a given space-time region, they cannot be diluted by using several modes in parallel in different space-regions or sequentially. We can thus restrict ourselves to studying a single mode. For concreteness, we consider a cubic cavity with edges of length $L$, and perfectly reflecting walls or symmetric boundary conditions.

Maxwell’s equations in vacuum with appropriate boundary conditions impose quantized modes with wave vectors $k$ that are independent of $c$, whereas the frequency $\omega = c|k|$. Obtaining the best possible precision of $c$ is thus equivalent to the optimal frequency measurement of a harmonic oscillator, for which the quantum Cramér–Rao bound was calculated in \[40\]. The smallest $\delta \omega/\omega$, and hence smallest $\delta c/c$ for fixed maximum excitation $2n$ and for $\tau = \omega t \gg 1$, is achieved with the optimal state $|\psi_{\text{coh}}\rangle = (|0\rangle + |2n\rangle)/\sqrt{2}$. In a single measurement, it leads to a minimal $c$-uncertainty

$$\frac{\delta c}{c} \simeq \frac{1}{2\tau n}. \quad (1)$$

For existing measurements with large $n$, coherent states are more relevant than the optimal state. A coherent state with amplitude $\alpha$ at time $t = 0$, $|\psi_{\text{coh}}\rangle = |\alpha\rangle$, evolves according to $\alpha(t) = e^{-i\omega t} |\alpha\rangle$ and leads to

$$\frac{\delta c}{c} \simeq \frac{1}{2 \left( \sqrt{\frac{1}{2} n + \frac{1}{2}} \sin \frac{1}{2} \tau + \frac{1}{2} n \tau \sin \tau \right)} \simeq \frac{1}{2\tau \sqrt{n}}, \quad (2)$$

where the last equality is for large $\tau = \omega t$ and large average photon number $n = \alpha^2 (\tau^2 \alpha \gg 1)$ \[40\].

From these results one is tempted to conclude that $\delta c/c$ can be made arbitrarily small by increasing $n$. However, the energy-momentum tensor increases $\propto n$ for $n \gg 1$, and will at some point perturb itself the metric of space-time. We argue that the ultimate sensitivity is reached when the general relativistic modification of space-time becomes comparable to the minimal quantum uncertainty of the measurement. This leads to a finite optimal number of photons, and a finite optimal sensitivity. Increasing the photon number even more will modify space-time to a point where one cannot speak of light propagation in vacuum anymore. In principle one may re-calculate from the measured value using GR what the speed of light in vacuum would be, but this is a counterfactual reasoning and not a direct measurement of $c$. On the other hand,
reducing the photon number would increase the quantum noise. The situation is very similar to the optimization of the photon number in LIGO-type gravitational wave interferometers, where one balances photon-shot noise against radiation pressure noise [42–44]. However, whereas radiation pressure noise is specific to the measurement instrument, in our case the properties of space-time itself and thus the very meaning of light propagation in vacuum are affected when increasing the photon number further, and this effect is unavoidable.

The gravitational effects sought here are well in the regime where Einstein’s field equations are valid: Firstly, we consider light at wavelengths $\lambda$ and structures of the energy-momentum tensor on scales much larger than the Planck-length (e.g. $\lambda = 500$ nm and a standard (possibly lossy) cavity of size $L = 1$ km). Secondly, we consider light fields of very large intensity and effects linear in the perturbation of the metric, for which the energy-momentum tensor should be well approximated by its quantum mechanical expectation value [45]. It is the effect of this average energy-momentum tensor on space-time that we calculate and compare to the minimal uncertainty of $c$ obtained from q-pet, not the fluctuations of space-time themselves. The former is established on the solid ground of general relativity, whereas the latter would require a quantum gravity theory to make reliable predictions. The quantum fluctuations that we are interested in here are those of light probing the space-time, which are reliably described by quantum optics. Our results therefore rely only on well-tested theories, in distinction to predictions of the fluctuations of space-time obtained by various theories of quantum gravity.

3. Perturbation of metric due to light intensity

The modification of the metric of space-time is found from the weak field limit of the Einstein field equations, where the metric tensor is given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, i.e. the flat Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1,1,1,1)$ (in terms of $ct,x,y,z$) plus a small perturbation, $|h_{\mu\nu}| \ll 1$. Einstein’s equations yield a wave equation for the trace inverse, $\ddot{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} \eta^{\alpha\beta} h_{\alpha\beta}$,

$$\Box \bar{h}^{\mu\nu} = -16\pi \frac{G}{c^4} T^{\mu\nu},$$

(3)

where the (flat space-time) Lorenz gauge (FLG) condition $\bar{h}^{\mu\nu,\nu} = 0$ is used; see equation (18.8b) in [46]. The energy-momentum tensor $T^{\mu\nu}$ of the e.m. field reads [46]

$$T^{00} = \frac{1}{c} (\epsilon_0 E^2 + \mu_0 H^2), \quad T^{0i} = T^{i0} = \frac{1}{c} (E \times H),$$

$$T^{ij} = -\left( \epsilon_0 E_i E_j + \mu_0 H_i H_j \right) + T^{00} \delta_{ij},$$

(4)

where $i,j \in \{1,2,3\} = \{x,y,z\}$. We use the q.m. expectation value of $T^{\mu\nu}$ as source term in (3) for the (011) and the (01M) modes ($k_i = l_i \pi / L$, $l_x = 0$, $l_y = 1$, and $l_x = l_z = M$, respectively; $\Omega_i = c|k_i|$ and $V = L^3$). This ‘semiclassical approximation’ is justified if one is interested only in effects to first order in $h_{\mu\nu}$ [45]. Using the (0 1 1) mode is motivated by the fact that it has lowest frequency and hence expected lowest GR impact. This will be verified by comparing to the (01M) mode with large $M$. For $|\psi_{\text{opt}}\rangle$ with $n \gg 1$, the solution of (3) for the (0 1 1) mode reads ($\xi = \pi x / L$)

$$\bar{h}^{\mu\nu}(\xi) = \mathcal{P} \int_0^\pi \int_0^\pi d\eta' d\zeta' \mathcal{I}(\xi, \eta - \eta', \zeta - \zeta') \rho^{\mu\nu}(\eta', \zeta'),$$

where

$$\mathcal{P} = \frac{4\sqrt{2n}}{\pi \kappa}, \quad \kappa = \left( \frac{l_n}{L} \right)^2,$$

(5)
$$I(x, \eta, \zeta) = \ln \left( \frac{\xi + \sqrt{\xi^2 + \eta^2 + \zeta^2}}{\xi - \pi + \sqrt{(\xi - \pi)^2 + \eta^2 + \zeta^2}} \right),$$

(6)

with dimensionless trigonometric functions \( \mu^{\mu\nu} := T^{\mu\nu}/(nh\Omega_t/V) \) of order one inside the cavity, and zero outside (see appendix B). \( T_\mu^\nu = 0 \) for the e.m. field [47]; hence \( h_\mu^\nu = 0 \) and \( h^{\mu\nu} = h^{\mu\nu} \).

The deviations of \( h^{\mu\nu} \) in (5) from FLG are of second order in \( h \) and can be neglected [48]. For \( |\psi_{\text{coh}}\rangle \), \( h^{\mu\nu} \) is the same as for \( |\psi_{\text{opt}}\rangle \) plus retarded oscillation on top of it, with an amplitude of the same order. We therefore restrict the analysis to the time-independent part. For the \((01M)\) mode, and \( n, M \gg 1 \), only \( h^{\mu0} \) and \( h^{33} \) are non-negligible,

$$h^{00} = h^{33} \approx 4 \mathcal{P} M \int_0^\pi \int_0^\pi d\eta' d\zeta' I(\xi, \eta', \zeta - \zeta') \sin^2 \eta'.$$

From the geodesic condition \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu = 0 \), the local modification of the coordinate speed of light

$$\frac{\delta c(x)}{c} = \frac{1}{2} (h_{00} + h_{11})$$

(7)

is obtained for the \((011)\) mode, with similar expressions for \( \delta c(y) \) and \( \delta c(z) \) (see also figure B1 in appendix B). For the \((01M)\) mode with \( n, M \gg 1 \), \( \delta c(x)/c = \delta c(y)/c = -\frac{1}{2} h_{00} \), \( \delta c(z)/c = 2 \delta c(x)/c \). One may object that according to the equivalence principle one could always find a coordinate system (CS) in which \( c(x) = c(y) = c(z) = c \), and that by the definition of \( c \) one should go to the free falling CS for measuring \( c \), where \( c \) is always the same. However, one has to distinguish between the universal constant \( c \) entering Lorentz-transformations, and the experimental value \( c_{\text{exp}} \) of the propagation speed of light obtained in measurements. The experimental definition of \( c \), \( c_{\text{exp}} = \Delta x/\Delta t \), where \( \Delta x \) is the distance that a light signal travels in time \( \Delta t \) implies that for any finite \( \Delta x \) the measurement is non-local, which precludes transforming the discussed GR effect away by a local transformation. It is to be expected that this non-local effect can be made arbitrarily small by moving the two points arbitrarily close to each other. More importantly, however, the measurement apparatus cannot be free falling in the gravitational field of the light it contains, as it carries that light with it. A time delay can be measured with a single clock by passing a short light pulse through a beam splitter (BS), reflecting it on a mirror and sending it back to the BS. The two passes through the BS trigger start/stop of the clock by light scattered into detectors adjacent to the BS. The clock measures its proper time, \( d\tau = \sqrt{-g_{00}} dt \). \( \Delta x \) has to be measured independently, i.e. with standard measurement rods. Hence, \( \Delta x \) corresponds to the ‘proper length’ of the apparatus (distance between BS and mirror for a runtime experiment or length of the cavity when using \( \omega = ck \)). ‘Proper length’ (not to be confused with ‘proper distance’) is defined as the length measured with standard measurement rods in the frame where the object is at rest [49]. We may assume the measurement rods as well as the measurement apparatus as sufficiently ‘rigid’ (gravitational forces and modification of the e.m. forces that determine the shapes of these objects much smaller than the e.m. forces that determine their shape and arrangement [50, 51]), which means that \( \Delta x \) remains unchanged when the light intensity is increased. In the limit \( R \gg L \) (\( R \) = typical radius of curvature of space time), the experimentally found value \( c_{\text{exp}}(x) = \Delta x/\Delta \tau \simeq dx/d\tau = c(x)/\sqrt{-g_{00}} \) is then directly related to the coordinate speed \( c(x) \) determined above. This gives \( \delta c_{\text{exp}}/c = -h_{11}/2 \) for the \((011)\) mode, where \( \delta c_{\text{exp}}(x) := c_{\text{exp}}(x) - c \) can be considered a systematic error in the determination of \( c \).
Since q-pet was based on the uncertainties $\delta \omega$, we also compare q-pet and GR based on the GR shift of the cavity resonance frequencies by solving the e.m. wave equation in the entire cavity with mirrors at $0, x_L$ and symmetric boundary conditions (SBC), $A^\mu(0, y, z) = A^\mu(x_L, y, z)$ (and correspondingly for the other directions). The unperturbed single modes are plane waves $A^\mu(t, x, y, z) = (h/(2\omega_0 V))^{1/2}(e^{i(x+ct)} a + h.c.)$, $A^\mu = 0$ for $\mu \in \{0, 1, 2\}$, and $k := k_0 = k_1 > 0$. This leads to $T^{\mu\nu} = -h\omega/(2\omega_0 V) [(a e^{i(x-c\tau)} + h.c.)^2]$ for $(\mu, \nu) \in \{00, 01, 10, 11\}$ inside the cavity, and $T^{\mu\nu} = 0$ else or outside. For $|\psi_{\text{coh}}\rangle$, $T^{\mu\nu}$ is time-independent, and for $|\psi_{\text{opt}}\rangle$ we once more consider only the time-independent part. Then, $h^{\mu\nu}(\xi) = \epsilon(\xi)$ for $(\mu, \nu) \in \{00, 01, 10, 11\}$ and $h^{\mu\nu} = 0$ else, where

$$
\epsilon(\xi) := \sqrt{2PM} \int_0^\pi d\eta' d\zeta' I(\xi, \eta' - \eta, \zeta - \zeta').
$$

(8)

The wave equation describing the propagation of light in curved space-time reads $\nabla^\beta F^{\alpha\beta} = 0$ (see (22.17a) in [46]), with $F^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}(A_{\nu,\mu} - A_{\mu,\nu})$. Using FLG for $A$ and $h$, and $h^\mu_\nu = 0$, we obtain to first order in $\epsilon$

$$
0 = -A^{\alpha,\nu}_{\mu} + (h^{\alpha}_{\mu,\nu} - h^{\nu}_{\nu,\mu})A^{\mu,\nu},
$$

(9)

where indices are pulled up or down by the full metric $g^{\mu\nu}$. Equation (9) is solved exactly by the original plane wave despite the changed metric, as the $\epsilon$-correction in $A^3$ is $\propto (\partial_x + \partial_t)^2$. This reflects the well-known result that two parallelly propagating beams of light do not affect each other gravitationally [52, 53]. The existence of a mode with unchanged dispersion relation suggests that judging whether the vacuum may still be considered as such based on the change of a single mode frequency can be insufficient. In such a case, the change of the metric can normally still be probed using other modes. In the example above the frequencies of modes propagating in different directions, e.g. $A^3 \propto \exp(ik(x + c\tau))$, are modified locally by a relative amount of order $\epsilon(x)$, as can be shown by solving (9) in eikonal approximation.

To summarize, up to numerical prefactors of order 1, both systematic errors $\delta c_{\text{exp}}$ obtained by measuring length over time or a shift of a cavity resonance, possibly in another mode, scale as

$$
\frac{\delta c_{\text{exp}}}{c} \sim -\kappa n M
$$

(10)

for $n, M \gg 1$. With this, we can now obtain the smallest possible uncertainty with which $c$ can be determined in a given region of space-time.

4. Minimal uncertainty of speed-of-light measurements

For $|\psi_{\text{opt}}\rangle$, equating (1) and the absolute value of (10) leads with $M \sim L/\lambda$ to an optimal photon number $n_{\text{opt}} \sim (\lambda/lp) \sqrt{L/(cT)}$, and a minimal

$$
\frac{\delta c}{c} \sim (cT L)^{1/2},
$$

(11)

independent of frequency: the gain in quantum mechanical sensitivity due to longer dimensionless evolution time for more energetic photons is exactly cancelled by the increased perturbation of the metric.

In an experiment, the measurement time is bounded from above by the finite photon-storage time of the photons in the cavity. While obtaining optimal bounds including photon loss requires mixed state q-pet [54, 55], the sensitivity cannot be better than that obtained from the
pure states from which the state is mixed [56]. For known dissipation and decoherence mechanisms one can try to find an adapted optimal state. However, the sensitivity cannot be better than if one had access to the full system and its environment. For photon loss the environment can be modelled by additional modes coupled to the central mode by beam-splitter couplings, and including such ancilla modes cannot improve the estimation of a parameter of the original system when optimized over all initial states [57, 58], if the ancillas are independent of the $c$ we are interested in (which is the case for the modes outside the cavity and hence outside the space-time region considered). Our q-pet bound calculated for the ideal situation without photon loss therefore remains valid, but can in general in the presence of dissipation or decoherence not be reached anymore. For a cavity of length $L$ and finesse $F$, the measurement time is bounded by $T = LF/(\pi c)$. This leads to an optimal number of photons independent of the length of the cavity, $n \sim \lambda/(\hbar F^{1/2})$. For numerical estimates we use in the following a standard situation: visible light with $\lambda = 500$ nm, a finesse $F = 10 000$, and $L = 1000$ m. The optimal $n$ for the optimal state is then $n \sim 10^{26}$, and the minimal uncertainty $\delta c/c \sim \lambda/(\hbar F^{3/2}) \sim 10^{-40}$.

For $|\psi_{\text{coh}}\rangle$, equating (2) and (10) leads to $n_{\text{opt}} \sim (L\lambda^2/(\hbar F c T)^2)^{1/3}$, and a minimal uncertainty

$$\delta c/c \sim \left( \frac{\hbar \lambda^2}{L(cT)^2} \right)^{1/3}.$$  

(12)

For a cavity with finesse $F$, the length of the cavity is again irrelevant for the optimal photon number, $n_{\text{opt}} \sim (\lambda/\hbar)^{4/3}/F^{2/3}$, and $\delta c/c \sim \hbar\lambda^{1/3}/(LF^{2/3})$. Contrary to $|\psi_{\text{eq}}\rangle$, the minimal uncertainty depends here on the wavelength. In principle, $\delta c/c$ could therefore be smaller for $|\psi_{\text{coh}}\rangle$ than for $|\psi_{\text{eq}}\rangle$, but only for wavelengths $\lambda < \hbar c T/L$ in lossless cavities, and for $\lambda < \hbar c T/F$ in cavities with finesse $F$, which are outside the validity of the theory. For the lossy cavity considered, the optimal coherent state photon number is $n \sim 10^{35}$ and $\delta c/c \gtrsim 10^{-31}$, demonstrating the superiority of $|\psi_{\text{eq}}\rangle$. We display the various $n$-scaling regimes and the optimal photon numbers located at the minima of the overall dependence of $\delta c/c$ on $n$ in figure 1.

5. Comparison with similar bounds

The minimal uncertainties of $c$ and hence the metric of flat space-time that we have derived are reminiscent of ideas about the fuzziness of space-time on the Planck scale, their different physical meaning not withstanding. The minimal uncertainty of $\delta c$ that we have derived here translates, in experiments where a length $L$ is measured through $L = cT$, to fluctuations $\delta L$ of $L$. There has been a vast amount of work aiming at demonstrating a minimal length scale in physics and working out its consequences, see [59] for an excellent review. The majority of these works has tried to establish smallest uncertainties of positions or length measurements, but there have also been attempts to find minimal uncertainties of volumes, areas, gravitational fields, event horizons, and others. Here we focus on previous predictions of minimal uncertainties of lengths or positions. For simplicity we set $\hbar = c = 1$ in the rest of this section and neglect factors of order 1, unless otherwise noted.

5.1. Previous thought experiments

Closest to our analysis are previous thought experiments that one way or another use classical gravity effects to bound quantum uncertainties from below. An illustrative example is the Heisenberg microscope with gravity [60]. In addition to the familiar Heisenberg microscope,
where attempts to resolve the position of a particle by scattering light from it result in an unknown momentum kick of order $\omega$ onto the particle, while limiting the spatial resolution to roughly the wavelength of the light $\delta x_{QM} \sim 1/\omega$, one also considers the gravitational interaction of the photon with the particle. This leads to an acceleration of the particle of at least $G\omega/\mathcal{R}^2$ if the photon is detected at distance $\mathcal{R}$, and a corresponding displacement between the photon-particle interaction and the photon detection of order $\delta x_{GR} \sim G\omega$. Taking the geometric mean of the two uncertainties gives immediately $\delta x \sim \sqrt{G} = l_{Pl}$. Alternatively, we can take the sum of the two uncertainties and minimize it over $\omega$. This gives $\omega \sim 1/\sqrt{G} = m_{Pl}$, the Planck mass, and, up to a factor 2, again $\delta x \sim l_{Pl}$.

Another popular argument goes back at least to Bronstein in 1936 [61], who, in the context of investigating how precisely a gravitational field might possibly be measured, came up with the request that the test particle should not collapse to a black hole. Later, Wigner and Salecker introduced a similar limitation to length measurements with light pulses [33, 34], where the clock should not become a black hole. The idea was refined for the measurement of lengths based on ‘material reference systems’ (MRS) [36], consisting of reference points of size $s$ and mass $M$ that contain a clock, light-gun and detector, arranged in space. The request that no event-horizon should form around the reference points beyond $s$ implies $M < s/l_{Pl}^2$.

We can apply the black-hole argument to the Heisenberg-microscope, requesting that the photon’s event horizon should be at least smaller than the distance $\mathcal{R}$, i.e. $\omega < R/l_{Pl}$. Then $\delta x_{QM} \gtrsim l_{Pl}/R$, a bound obviously much weaker than the previous one for $R \gg l_{Pl}$. On the other hand, for the MRS the black-hole criterion leads again to $\delta L \gtrsim l_{Pl}$ if we assume $s \sim L$ and argue that the quantum mechanical uncertainty for a material particle scales as $\delta L \gtrsim \sqrt{L/M}$.

![Figure 1. Minimal uncertainty $\delta c/c$ as a function of the number of photons $n$: The dashed red/blue line shows the minimal uncertainty obtained from the quantum Cramér–Rao bound for the optimal and coherent states given in equations (1) and (2), respectively. The dashed green line corresponds to the unavoidable systematic error in the measurement of $c$ due to the light’s own gravitational effect. The sum of the minimal uncertainty given by the quantum Cramér–Rao bound and the systematic error for the optimal/coherent state is shown by the solid orange/light blue lines. The optimal number of photons minimizing $\delta c/c$ for either optimal or coherent states lies at the minima of the solid orange/light blue lines. Parameters are $\lambda = 500$ nm, $\tau = 1$, $L = 1$ km und $M = L/\lambda$.](image)
This latter scaling is based on a semi-classical picture [36] with an initial width of a wave-package leading to a minimal width in momentum space, that is interpreted as particles spreading out with a corresponding momentum distribution, giving a correspondingly larger uncertainty for the position measurement at a later time $T$. The argument can be made more rigorous by minimizing the quantum-mechanically calculated expectation value $\langle \delta x(0) \delta x(t) \rangle$ of a particle by minimizing over its mass [62]. One also recognizes in $\delta L \gtrsim \sqrt{L/M}$ the standard quantum limit (SQL), and in particular for $M = N \omega$ for a device dominated by the mass of $N$ photons a scaling with $1/\sqrt{N}$.

5.2. Quantum gravity theories and phenomenological models

For most microscopic theories of quantum gravity it is difficult to extract bounds on minimal uncertainties of lengths. In [59], a generalized uncertainty principle (GUP) of the form $\delta x' \delta p' \gtrsim 1 + l_{Pl} E$ is given as a prediction of string theory, as well as a space-time uncertainty $\delta x \delta T \gtrsim l_{Pl}$, where $l_{Pl}$ is a (yet unknown) string scale that might be of the order of $l_{Pl}$, and $E$ the energy with which the string is tested. In [15] it was stated that Lie-algebra non-commutative space-times with non-commuting position coordinates, $[x_\alpha, x_\beta] = i R_{\alpha \beta} x_\gamma / m_{Pl}$, lead to a $\delta T$ of the form $\delta T \sim L^{n} E^{m} / m_{Pl}^{1+m-n}$ where $m, n$ are some model-dependent powers with $1 + m - n > 0$. The lowest-order non-trivial case $n = m = 1$ that gives an energy dependence, corresponds to $\delta T \sim L E / m_{Pl}$. Considering $T$ as the travel time of a particle from source to detector, $\delta T$ implies an uncertainty of the radar length. Combining this $\delta T$ with the standard contribution from the Heisenberg uncertainty principle and minimizing over the energy gives a minimal length uncertainty that can be written in the form

$$\delta L \gtrsim l_{Pl} L^{1-\alpha}$$

with some real value $\alpha \in [0,1]$ [15].

Given the mentioned difficulty to extract predictions of fluctuations of positions or lengths from microscopic quantum gravity theories, mostly phenomenological GUPs have been used to generalize lower bounds based on the standard uncertainty principle. It is clear from dimensional grounds that (13) is the generic form of a power law scaling with $l_{Pl}$ if only $l_{Pl}$ and $L$ exist as length scales. Such a form is therefore also obtained in many other phenomenological theories, notably models that assume fluctuations on the scale of the Planck length and then ask how these accumulate during the propagation of a light signal. The simplest case are random walk models, which lead to $\alpha = 1/2$ [63, 64]; $\alpha = 2/3$ is known as the holographic model. If one assumes a fluctuation $\delta \lambda$ of the wavelength $\lambda$ of the light used to measure distances with $\alpha = 1/2$, $\delta \lambda \gtrsim l_{Pl} (\lambda / l_{Pl})^{1/2}$, the fluctuations of the total length are given in the random walk model by $\delta L \gtrsim \delta \lambda (L / \lambda)^{1/2} = \sqrt{\delta \lambda L^{1/2}}$, i.e. the new length-scale $\lambda$ drops out. However, if the fluctuations $\delta \lambda$ are added up coherently, i.e. all with the same sign, a much larger value results,

$$\delta L \gtrsim (l_{Pl} L)^{1/2} (L / \lambda)^{1/2}.$$  

The choice of model has therefore important implications for the falsifiability of the predicted minimal fluctuations. E.g. in [65] the coherence of Hubble-space telescope images of distant galaxies was used to bound possible quantum fluctuations of space-time from below. No fluctuations were found, but the coherent addition of the fluctuations was subsequently questioned [66].
Modified commutation relations lead in general to a generalized uncertainty principle. In as much as this implies a fluctuating speed of light, Lorentz invariance can be violated, but need not (see e.g. the model of discrete space-time with modified commutation relations without violation of Lorentz invariance due to Snyder in 1947 [3]). In the same way, the (deterministic) dispersion relation of e.m. waves can be modified; such theories have become known as ‘rainbow gravity’. This class of theories contains doubly (or deformed) special relativity (DSR), with a kappa-deformed Poincaré group [67–72]. DSR is based on the idea that not only the speed of light is independent of the reference-frame, but also the small length-scale \( l_{\text{QG}} \) on which quantum-gravity effects become important, identified typically with the Planck-length. DSR has recently been elaborated further into ‘relative locality’ [73], a theory that emphasizes the importance of phase-space and suggests that momentum-space might be curved, which would imply non-linear conservation laws of energy and momentum, and a relativity of ‘locality’. Another formulation of DSR considered an energy-dependence of space-time [74, 67]. Earlier theories also proposed a time-dependent speed of light as solution to cosmological problems [75, 76].

In [77, 78] it was proposed that a non-linear dispersion relation might arise from averaging a quantum-fluctuating metric over a relevant length scale of a test particle. Considering a ‘measurement process’ in relativistic rather than quantum terms, it was suggested that the metric relevant for a measurement process of the momentum \( p_\alpha \) of a particle with energy \( E \) is the ‘classical’ metric of GR plus an averaged perturbation of quantum-gravitational origin, assumed non-vanishing when averaging over the de Broglie wavelength \( \lambda = 1/E \) of a deeply relativistic particle, thus introducing an extra energy-dependence into the (inverse) dispersion relation \( p_\alpha(E) \).

In [79] a modified dispersion relation was found in the context of a non-critical-string approach to quantum gravity. It leads to a minimal total uncertainty of a length measurement based on the propagation of massless particles

\[
\delta L \gtrsim \sqrt{\eta L l_{\text{Pl}} + l_{\text{Pl}}},
\]

where \( \eta \) is a dimensionless parameter of order one, and clearly the first term dominates for \( L \ll l_{\text{Pl}} \), giving (13) with \( \alpha = 1/2 \), but \( \alpha = 1 \) for \( L \simeq l_{\text{Pl}} \). Underlying (15) is an assumption about the form of a decoherence-term in the modified quantum Liouville equation that arises from coupling matter to the degrees of freedom of space-time fluctuations that scales as \( E^2/m_{\text{Pl}} \) with Planck-mass \( m_{\text{Pl}} \) and energy \( E \) of a particle. When generalizing this to a scaling \( E^n/m_{\text{Pl}}^{n-1} \), a dependence

\[
\delta L \gtrsim L^{1/n} l_{\text{Pl}}^{1-1/n}
\]

was predicted, which is again of the form (13).

In [80], it was argued that a finite minimal uncertainty of time measurements is linked to the perturbative approach to quantization, whereas in a non-perturbative approach in principle infinite resolution could be achieved, as long as particle energies are not bound from above (as might happen with a modified dispersion relation). On the other hand, the authors find a finite minimum resolution both in perturbative and non-perturbative approaches, with a minimum length uncertainty

\[
\delta L \gtrsim l_{\text{Pl}},
\]

whereas for large background times \( \bar{T} \)

\[
\delta L \gtrsim \sqrt{l_{\text{Pl}} c \bar{T}},
\]
as in the Wigner–Salecker case [33, 34]. In [64], other estimates of length fluctuations were discussed, one of them scaling as \( \delta L \gtrsim (l_{QG} c T)^{1/2} \), where \( l_{QG} \) is expected to be \( l_{QG} \gtrsim l_{Pl} \), which for \( L = cT \) is again in line with (13) with \( \alpha = 1/2 \).

5.3. Comparison with our bounds

When trying to compare these previously found bounds with ours, the first thing to keep in mind, is that our bounds are fundamentally for \( \delta c/c \), not \( \delta L/L \). This is important as there is no quantum mechanical operator for the speed of light, hence one cannot apply directly the standard Heisenberg uncertainty principle. Rather, we resorted to q-pet, which gives generalized uncertainty relations [22]. Secondly, our bounds are based directly on the light field itself, not the quantum mechanical uncertainty in the position of a clock, an MRS point, or a test-particle. We have furthermore the choice of the state of the probe, notably it can be a multi-photon state, whereas previous derivations typically considered single-particle uncertainty relations, with a state that saturates Heisenberg’s uncertainty relation. Moreover, since the QCRB is optimized over all possible measurements of the light field and has a clear interpretation in terms of the minimal uncertainty of an estimator of \( c \), there are no conceptual issues with the meaning of the measurement on very small length scales. Questions on how fluctuations at smaller length-scale add up do not arise. In random-walk models one might wonder why one should add up fluctuations of the wavelength, as no measurements are made at that length scale. In the q-pet approach, measurements on the length-scale of the wavelength are included just as any other measurement of the light field, and the uncertainty is the one of the best possible estimator of \( c \), rather than fluctuations of a measured observable (whose existence at a very small length scale might be questionable; this issue was indeed recognized as one of the most important ones in the field, see section 4.2.5 in [59]).

By using a light signal, another length-scale comes into play, namely the wavelength \( \lambda \) of the light, as well as the propagation time, which in a cavity can be much larger than the length of the cavity. Depending on the quantum state used, \( \lambda \) is still present in the final result for the lower bound.

If we do translate our bounds for \( \delta c/c \) into a bound for fluctuations of length estimations \( \delta L \) by assuming \( \delta L = T \delta c \) with fixed \( T \), we see from (11) that for the optimal state we get back \( \delta L \gtrsim l_{Pl} \) for \( L = cT \), i.e. this corresponds to \( \alpha = 1 \) in (13). However, for \( T \gg L/c \), one can get uncertainties much smaller than the Planck length, a fact that was not reflected by previous bounds. This insight results naturally from the use of q-pet, where time appears as a resource for more precise measurements, in sync with experimentalists’ habit to provide uncertainties per square root of Hz for fair comparison.

For a coherent state in a lossless cavity, the lower bound of \( \delta L \) implied by (12) reads \( \delta L \gtrsim l_{Pl}^{1/3} \lambda^{1/3} (L^2/(c T^2))^{1/3} \). If \( L = cT \), this is as (13) for \( \alpha = 2/3 \), but with \( L \) replaced by \( \lambda \).

One might wonder if there is a deeper reason behind the fact that a classical light signal reproduces the holographic model concerning the scaling of the smallest \( \delta L \) with \( l_{Pl} \). Compared to the coherently added up fluctuations equation (14), this is, in the optical domain, still a much smaller value for any \( L \) larger than about \( 10^{-12} \) m.

Given their fundamental measurement-based nature, our bounds can serve for judging the falsifiability of quantum gravity theories and phenomenological models: predictions of fluctuations in a given space-time region that are smaller than those given by our bounds can never be falsified through direct measurement as a matter of principle (subject to the made assumptions). While the prefactors depending on \( L, \lambda, T \) for the coherent state matter, as a rule of thumb, predictions of fluctuations with \( \alpha > 2/3 \) could not be measured with light in a
coherent state, as the measurements own smallest possible uncertainty $\propto l_{Pl}^{2/3}$ is larger. Length uncertainties $\propto \sqrt{t_{Pl}L}$ of Wigner–Salecka-type theories as well as the bound in (15) are at least in principle falsifiable with light in a coherent state. The fluctuations (16) cannot be measured with light in a coherent state as soon as $n > 3$, but they would be accessible at least in principle to ‘quantum enhanced measurements’ using the optimal quantum state of light. However, it is unlikely that an optimal state of light with a sufficiently large photon number can ever be built, given the experimental difficulties of producing superpositions of Fock states with even a few photons. The fluctuations predicted in [17] are well above our bounds for any cavity of realistic size.

Several works discussed the possibility to measure fluctuations of space-time created on the Planck-scale with gravitational wave interferometers such as LIGO [37, 64, 81]. Bounds on $l_{QG}$ were obtained from experimental data from Caltech’s 40 m interferometer [82]. In [81] it was argued that the stated displacement noise level of that interferometer of order $3 \cdot 10^{-19}$ m $\sqrt{\text{Hz}}^{-1}$ in the neighborhood of 450 Hz already rules out length fluctuations of the interferometer arms of order $l_{Pl}$ per Planck-time interval for the random-walk accumulation of individual Planck-cell fluctuations to a total uncertainty. References [10–16] attempted to bound the supposed quantum fluctuations of space-time using the broadening of light pulses from far-away astronomical sources, but so far the uncertainty in the emission time of the light pulses as well as other sources of spreading the pulse are too large to say much about quantum fluctuations of the metric [13].

6. Concluding discussion

Our results imply that one should not think of quantum fluctuations of space-time as existing independently of the measurement devices that probe them, but rather as something that can only be defined in conjunction with them. This is in line with the modern theory of quantum measurement, where the possible measurement results do not only depend on the quantum system, but also on the quantum probe and its initial quantum state.

Accordingly, we find different lower bounds for $\delta c/c$ for the optimal state and a coherent state. The former reproduces $\delta L \gtrsim l_{Pl}$ when translated to the uncertainty of a length and assuming a measurement time $T \simeq L/c$, whereas the latter is substantially enhanced and still depends on the wavelength, scaling only as $l_{Pl}^{2/3}$. Their derivation from standard quantum optics and GR is similar in nature to those of previous bounds based on Gedanken-experiments (see section 5.1) within QM and GR, but provides a conceptual advance by the use of q-pet, which includes the optimization over all possible measurements, and precise calculations rather than orders of magnitude arguments. Simple scaling arguments can be insufficient, as the discussions in the literature about how fluctuations on small scales add up on long distances have shown. Another example: in the Heisenberg microscope including gravity, one might arrange the particle half way between light source and detector. In that case the acceleration due to the gravitational pull will average to zero and it is not clear why the quantum uncertainty should be bounded from below by a gravitational effect—not to talk about questions of how the photon is supposed to be localized in space-time, when only its wavelength is specified. Such questions on how exactly the measurement is done, and whether a different setup might not avoid the limitations, do not arise in our q-pet approach.

Nevertheless, our bounds are of course subject to several (reasonable) restrictions as well: We consider direct measurements of the propagation speed or phase speed of an e.m. wave. Note, however, that the QCRB bounds the uncertainty for any measurement and estimation scheme,
as long as \( c \) is imprinted on the quantum state through the standard time evolution in quantum optics with (A.1) as Hamiltonian. Ambiguities arising from a proper definition of arrival time of the pulse pertain to the level of different data analysis schemes and are fully covered by the QCRB.

We want to know the value of \( c \) in a given region of space-time, and we assume a sufficiently rigid measurement apparatus whose length remains unchanged when the photon number is increased. Apparatuses with finite rigidity could deform under the influence of the gravity of the light signal and the modification of Coulomb’s law. For any realistic material that deformation should be negligible, however, compared to the one due to the light pressure; this will be examined in more detail in another publication [83]. The gravitational effect of the elastic energy was already shown in [50] to be smaller than the one of the e.m. field by a factor \( c_s/c \), where \( c_s \) is the speed of sound in the cavity walls. We rely on the validity of quantum mechanics (more precisely quantum optics and q-pet) and GR in semiclassical approximation (i.e. \( T^{\mu\nu} \) calculated as q.m. expectation value), and the validity of the linear dispersion relation \( \omega = ck \) for wavelengths well above the quantum-gravity/Planck length. For finding the optimal state, we assume a maximum possible photon number in the state. We neglect uncertainties in \( c \) due to the expansion of the Universe [51], non–inertial observers, local gravitation potentials e.g. from Earth or a (stochastic) gravitational-wave (GW) background [84], and quantum fluctuations of the mirror positions. In the quantum foam picture, also the latter should depend on the way they are measured, but in any case can only lead to reduced precision. The GW background at optical frequencies is expected to be extremely small, but might dominate at frequencies around 100–1000 Hz, where a large number of gravitational sources is expected to exist, see [85]. However, to cavities much shorter than the GW wavelength (300–3000 km for the above frequencies), the modified metric due to the GW appears as uniform, and the GW effect can hence in principle be eliminated by a cavity in free fall, in contrast to the GR effect of the light inside the cavity. More generally, any additional source of modification of the speed of light may lead to tighter lower bounds on the uncertainty of \( \delta c/c \) than ours, but will not invalidate them.

Acknowledgments

We thank Kostas Kokkotas, Nils Schopohl, Claus Zimmermann, Julien Fraisse, Dennis Rätzel and Friedrich Wilhelm Hehl for discussions and a critical reading of the manuscript. The research of URF was supported by the NRF of Korea, grant Nos. 2014R1A2A2A01006535 and 2017R1A2A2A05001422.

Appendix A. Single mode reduction of q-pet

We here prove that very generally for a given maximum amount of energy the optimal quantum measurement of \( c \) can be reduced to measuring a single mode of fixed frequency put into the optimal state \( |\psi_{\text{opt}}\rangle = (|0\rangle + |2n\rangle)/\sqrt{2} \). Starting point is the Hamiltonian \( H \) for the e.m. field, decomposed into modes labelled by a mode-index \( k \), consisting of wave-vector \( k \) and polarization \( \varepsilon \). Then

\[
H = \sum_k \hbar \omega_k n_k = \hbar c \sum_k k n_k, \tag{A.1}
\]

with angular frequency \( \omega_k = ck \) and \( k = |k| \). The Hamiltonian has the general form \( H = cG \) with a Hermitian generator \( G = \hbar \sum_k kn_k \). It leads in a given state \( |\psi\rangle \) and propagation over total time \( T \) to QFI [22].
\[ I_c = 4\Delta G^2 T^2 \equiv 4((G^2) - \langle G^2 \rangle)T^2. \]  

(A.2)

Let \( G = \sum_i e_i |i\rangle\langle i| \) be the spectral decomposition of \( G \), and \( |\psi\rangle = \sum_{i=1}^N c_i |i\rangle \), where we assume that \(|1\rangle (|N\rangle)\) are the states of lowest (largest) energy available. Then 
\[
\Delta G^2 = \sum_{i=1}^N p_i e_i^2 - \left( \sum_{i=1}^N p_i e_i \right)^2
\]
with \( p_i = |c_i|^2 \) and \( \sum_{i=1}^N p_i = 1 \). The Popoviciu inequality [86] states \( \Delta G^2 \leq (e_N - e_1)^2/4 \). It is saturated for \( p_1 = p_N = 1/2 \), \( p_i = 0 \) else. The state \(|\psi\rangle = (|1\rangle + e^{i\phi}|N\rangle)/\sqrt{2} \) with an arbitrary phase \( \phi \) saturates the inequality and thus maximizes \( I_c \). If \( e_0 \) or \( e_1 \) is degenerate, only the total probability for the degenerate energy levels is fixed to 1/2, and arbitrary linear combinations in the degenerate subspace are allowed. But the value of \( \Delta G^2 \) remains unchanged under such redistributions, and we may still choose just two non-vanishing probabilities \( p_1 = p_N = 1/2 \). The derivation did not make use of the multi-mode structure of the energy eigenstates. Hence, exactly the same minimal uncertainty of \( c \) can be obtained by superposing the ground state of a single mode with a Fock state of given maximum allowed energy as with an arbitrarily entangled multi-mode state containing components of up to the same maximum energy. Setting \( N = 2n \) leads to the announced optimal single-mode state.

Appendix B. Calculation of the metric perturbation

The vector potential of the e.m. field in the cavity in Coulomb gauge \( A(r, t) = T q(t) \mathbf{v}(r) \), where \( T \) is a constant, \( q(t) \) the time dependent amplitude, and \( \mathbf{v}(r) \) the mode function, with components
\[
\mathbf{v}_l = N e_l \cos k_x x \sin k_y y \sin k_z z, \\
\mathbf{v}_y = N e_l \sin k_x x \cos k_y y \sin k_z z, \\
\mathbf{v}_z = N e_l \cos k_x x \sin k_y y \cos k_z z.
\]  

(B.1)

The polarization vector \( e = (e_x, e_y, e_z) \) is normalized to length one, and is orthogonal to the \( k \)-vector \( k = (k_x, k_y, k_z) \), where \( k_z = l_z \pi / L \), and \( l_z \in \mathbb{N}_0 \) and at most one of three given \( l_z \) can be zero. Therefore, there are two polarization directions (transverse modes) for each \( k \) vector, with the exception of cases where one of the \( l_z = 0 \), where only one polarization is possible. The request that the modes be orthonormal,
\[
\int d^3 r \mathbf{v}_l(r) \cdot \mathbf{v}_l'(r) = \delta_{ll'}
\]  

(B.2)

leads to \( N = \sqrt{8/\mathcal{V}} \), and we can define the mode-volume \( \mathcal{V}_l = \mathcal{V}/8 \). Note that the index \( l \) stands here for both the discrete \( k \) vector and the polarization direction \( (l_1, l_2) \). Finally, we choose \( T = 1/\sqrt{\epsilon_0} \), such that
\[
A(r, t) = \sum_l \frac{1}{\sqrt{\epsilon_0}} q_l(t) \mathbf{v}_l(r), \\
E(r, t) = -\sum_l \frac{1}{\sqrt{\epsilon_0}} \mathbf{q}_l(t) \mathbf{v}_l(r), \\
H(r, t) = \sum_l \frac{1}{\mu_0 \sqrt{\epsilon_0}} \mathbf{q}_l(t) \nabla \times \mathbf{v}_l(r).
\]  

(B.3)
After quantization, the amplitudes $q_l$ become the quadrature operators of a harmonic oscillator, 
\[
\hat{q}_l = \sqrt{\frac{\hbar}{2\pi}}(\hat{a}_l + \hat{a}^\dagger_l), \quad \hat{p}_l = \frac{\hbar}{2\sqrt{2\pi}}(\hat{a}_l - \hat{a}^\dagger_l),
\]
where $\Omega_l = |k_l|c$. In the semiclassical approach the energy-momentum tensor for a single mode with mode function $v$ is given by the quantum mechanical expectation value [46, 87],
\[
T^{00} = \frac{\hbar \Omega}{4} \left( -\langle (\hat{a} - \hat{a}^\dagger)^2 \rangle v^2 + \langle (\hat{a} + \hat{a}^\dagger)^2 \rangle (\nabla \times v)^2 / k^2 \right),
\]
\[
T^{0i} = \frac{i\hbar \Omega}{2k} \left( \langle (\hat{a}^\dagger)^2 \rangle - \langle \hat{a}^\dagger v \rangle \nabla \times v \right),
\]
\[
T^{ij} = \frac{\hbar \Omega}{2k} \left( \langle (\hat{a} - \hat{a}^\dagger)^2 \rangle v_i v_j, \right.
\]
\[
- \langle (\hat{a} + \hat{a}^\dagger)^2 \rangle (\nabla \times v)_i (\nabla \times v)_j / k^2 \right) + T^{00}\delta_{ij},
\]
where $k^2 = k^2$, and we have used the symmetrized form $(\hat{q}^\dagger + \hat{p}^\dagger)/2$ of the quantum mechanical operators for the $T^{0i}$ components.

For a $|10M\rangle$ mode, $l_x = 0, l_y = 1, l_z = M$ dictates $e = (1,0,0)$ as unique possible polarizer. For $M = 1$, the frequency $\Omega_l = \sqrt{2\pi c}/L$, and
\[
v = \sqrt{\frac{8}{V}} \sin(\pi y/L) \sin(\pi z/L)e_x,
\]
\[
\nabla \times v = \sqrt{\frac{8}{V \sqrt{L}}} \sin(\pi y/L) \cos(\pi z/L)e_x,
\]
\[
- \sqrt{\frac{8}{V \sqrt{L}}} \cos(\pi y/L) \sin(\pi z/L)e_z.
\]

For $|\psi_{opt}\rangle$ with $n \gg 1$, and neglecting terms of order $O(n^0)$ (all other terms are of order $n$), we find that for the fundamental $|011\rangle$ mode the only non-vanishing components of $T^{\mu\nu}$ can be expressed in terms of four functions,
\[
T^{\mu\nu} = n \frac{\hbar \Omega}{V} \mu^{\mu\nu}
\]
with the dimensionless tensor components $\mu^{00}(\eta, \zeta) = f_1(\eta, \zeta), \quad \mu^{11}(\eta, \zeta) = f_2(\eta, \zeta), \quad \mu^{22}(\eta, \zeta) = f_3(\eta, \zeta), \quad \mu^{12}(\eta, \zeta) = f_4(\eta, \zeta)$, and
\[
\begin{align*}
f_1(\eta, \zeta) &= 2 - \cos(2\eta) - \cos(2\zeta), \\
f_2(\eta, \zeta) &= \cos(2\eta) + \cos(2\zeta) - 2 \cos(2\eta) \cos(2\zeta), \\
f_3(\eta, \zeta) &= \frac{1}{2}(2 - 4 \cos(2\zeta) + 2 \cos(2\zeta) \cos(2\eta)) \\
f_4(\eta, \zeta) &= \sin(2\eta) \sin(2\zeta),
\end{align*}
\]
where we write $x, y, z$ in units of $L/\pi, \xi = x\pi/L, \eta = y\pi/L, \zeta = z\pi/L$, and thus $\xi, \eta, \zeta \in [0, \pi]$. Outside the cavity $T^{\mu\nu}$ vanishes. For this state the field equations are solved with a time-independent metric. The wave equation reduces to the Poisson equation,
\[
\Delta \Phi^{\mu\nu} = -16\pi G c^2 T^{\mu\nu}.
\]

The solution is obtained by integrating the inhomogeneity $T^{\mu\nu}$ over with the Green’s function of the Poisson equation, i.e.
\( \bar{h}^{\mu\nu} = \frac{4G}{c^4} \int \frac{T^{\mu\nu}(x')}{|x-x'|} d^3x' \)

\( = \mathcal{P} \int_0^\pi d\eta' d\zeta' I(\xi, \eta - \eta', \zeta - \zeta') t^{\mu\nu}(\eta', \zeta'), \quad (B.9) \)

where the parameter \( \mathcal{P} \) is given by

\( \mathcal{P} = 4\sqrt{2} \frac{\hbar G}{\pi c^3 L} = \frac{4\sqrt{2}}{\pi} \kappa, \quad \kappa = \left( \frac{l_{\text{Pl}}}{L} \right)^2. \quad (B.10) \)

The integral kernel reads

\( I(\xi, \eta, \zeta) = \ln \left( \frac{\xi + \sqrt{\xi^2 + \eta^2 + \zeta^2}}{\xi - \pi + \sqrt{(\xi - \pi)^2 + \eta^2 + \zeta^2}} \right). \quad (B.11) \)

Numerical evaluation of the two remaining integrals in equation (B.9) shows that they are of order one inside the cavity, and decay rapidly outside, as is required by the boundary conditions of a flat metric far from the cavity.

For \( |\psi_{\text{coh}}\rangle \), we have to consider the full retarded solution of the wave equation according to

\( \bar{h}^{\mu\nu} = \frac{4G}{c^4} \int \frac{T^{\mu\nu}(t-|x-x'|/c, x')}{|x-x'|} d^3x'. \quad (B.12) \)

For example, the \( yz \) component reads

\( \bar{h}^{yz} = \bar{h}_{\text{opt}}^{yz} + \frac{4\hbar G}{\pi c^3 L} \int d\zeta'd\eta' d\zeta'' \sin(2\omega t - |x-x'|/c) \sin(2\zeta') \sin(2\zeta''). \)

This metric element is thus the solution of \( |\psi_{\text{opt}}\rangle \) (B.9) plus some retarded oscillation on top of it, which is of the same order. In the following we will therefore restrict our analysis to the time-independent part given by \( |\psi_{\text{opt}}\rangle \).

Figure B1. Relative change of the local coordinate speed of light in \( x \)-direction as function of dimensionless coordinates \( \eta, \zeta \) at \( \xi = 1.5 \) in units of \( \mathcal{P} = (4\pi/\pi)\kappa \) with \( \kappa = (l_{\text{Pl}}/L)^2 \) (see equation (B.10)) for the (011) mode. The cavity extends from 0 to \( \pi \) in these units.
For the \((01M)\) mode, with \(M > 1\), \(l_v = 0, l_i = 1, l_k = M\), the general expressions for \(T^{\mu\nu}\) are more complicated, but for \(|\psi_{\text{type}}|\) with \(n \gg 2\), and in the limit of \(M \gg 1\), we have \(T^{00} = T^{33} = 4n(h\Omega/V)\sin^2 \eta, T^{11} = -T^{22} = 4n(h\Omega/V)\sin^2 \eta \cos(2M\zeta)\). Corrections are of order \(1/M\). All other tensor elements of \(T\) vanish to order \(M^0\). The rapidly oscillating term \(\cos(2M\zeta)\) in \(T^{11}, T^{22}\) leads to a rapid decay of \(\tilde{h}^{11}\) and \(\tilde{h}^{22}\) as function of \(M\). Numerics indicates that the decay is roughly as \(1/M\) for fixed \((\xi, \eta, \zeta)\), including the factor \(M\) that is gained due to the prefactor \(\Omega \propto M\) for large \(M\). This means that for large \(n\) and \(M\), only \(T^{00} = T^{33}\) are non-negligible, with

\[
\tilde{h}^{00} = \tilde{h}^{33} \simeq \mathcal{P} m_{\text{e}} e^2 / (\hbar c) \sin^2 \eta, \\
\tilde{h}(\xi, \eta, \zeta) := 4 \int_0^\pi \int_0^{2\pi} \ld d\eta' d\zeta' L(\xi, \eta - \eta', \zeta - \zeta') \sin^2 \eta',
\]

where the dimensionless function \(\tilde{h}(\xi, \eta, \zeta)\) is once more of order 1 inside the cavity and falls off rapidly outside. So using a higher mode has the effect of reducing the perturbation of the metric essentially to two diagonal elements of the metric tensors, but increases the perturbation by a factor equal to the mode-index \(M\).

In all cases, the amplitude of the space-time perturbation due to the e.m. field in the cavity scales as

\[
\tilde{h}^{\mu\nu} \sim \left( \frac{L}{L_0} \right)^2 n M,
\]

proportional to the number of photons \(n\) in the cavity, the mode index \(M\), and the squared ratio \(L_0/L\) of Planck length \(L_0 \simeq 1.62 \times 10^{-35}\) m and size \(L\) of the cavity. The expression remains valid for the fundamental mode with \(M = 1\).

We note that throughout our analysis we tacitly assume that the photon densities in the cavity are small enough and the cavity sufficiently large, such that we stay well below the critical (electric) field strength \(E_c = m_\text{e} c^3 / (\hbar e) = 1.3 \times 10^{18}\) V m\(^{-1}\), where \(m_\text{e}\) is the mass of the electron, beyond which nonlinear corrections to Maxwellian electrodynamics due to polarization of the quantum vacuum become important [88]. This condition may be translated into a minimal cavity size \(L\) using an energy density \(O(\hbar n M/L^3)\) and a critical energy density \(O(0.1 \mathcal{E}_0^2)\). We obtain that \(L \gg (\hbar c^3/e^2 \epsilon_0^{-1/4} M^{-1} e^{-5/4})(n M)^{1/4} = (2.1 \times 10^{-13}\) m \((n M)^{1/4}\)

for linear electrodynamics in the cavity to hold. For the two types of cavities considered and all combinations of \(n_{\text{opt}}\) and \(M\), the lower bound on \(L\) is satisfied by the cavity sizes considered.

From \(h_{\mu\nu}\) we now calculate a local measure of the modification of the coordinate speed through the geodesics of the modified metric.

A finite \(h_{\mu\nu}\) leads to a new line element

\[
dx^2 = -(1 - h_{00})c^2 dt^2 + (1 + h_0)(dx')^2 + 2h_{23}dydz
\]

where the metric elements are, for the \((011)\) mode,

\[
h_{00} = \frac{1}{2} P(g_1 + g_2 + g_3 + \tilde{g}_3), \\
h_{11} = \frac{1}{2} P(g_1 + g_2 - g_3 - \tilde{g}_3), \\
h_{22} = \frac{1}{2} P(g_1 - g_2 + g_3 - \tilde{g}_3), \\
h_{33} = \frac{1}{2} P(g_1 - g_2 + \tilde{g}_3 - g_3), \\
h_{23} = \mathcal{P} g_4,
\]
with the definitions, see (B.9),
\[ g_i = \int_0^{\pi} \int_0^{\pi} d\eta' d\xi' I(\xi, \eta - \eta', \zeta - \zeta') f_i(\eta', \xi'). \]

(B.17)

For the (01M) mode we have \( h_{00} = h_{33} \) with \( h_{00} \) given by (B.16) whereas \( h_{\mu\nu} \) vanishes for all other values of \( \mu, \nu \). The light ray trajectories are determined through the geodesic condi-
tion \( d\xi^2 = 0 \). The speed of light in \( x^1 \)-direction (meaning all other \( dx^j = 0, j \neq 1 \), i.e. locally straight paths along \( x^1 = x \)) is then \( c(x) = c \sqrt{(1 - h_{00})/(1 + h_{11})} \), and correspondingly for the other directions. The relative change of the coordinate speed of light in \( x^1 \)-direction then reads, for the (011) mode with \( n \gg 1 \),
\begin{align*}
\delta c(x)/c &= -\frac{1}{2} h_{00} + h_{11} = -\frac{1}{2} \mathcal{P}(g_1 + g_2), \\
\delta c(y)/c &= -\frac{1}{2} (h_{00} + h_{22}) = -\frac{1}{2} \mathcal{P}(g_1 + g_3), \\
\delta c(z)/c &= -\frac{1}{2} (h_{00} + h_{33}) = -\frac{1}{2} \mathcal{P}(g_1 + \bar{g}_3). \tag{B.18}
\end{align*}

For the (01M) mode with \( n, M \gg 1 \),
\begin{align*}
\delta c(x)/c &= \delta c(y)/c = -\frac{1}{2} h_{00} = -\frac{\mathcal{P} M}{4} (g_1 + g_2 + g_3 + \bar{g}_3), \\
\delta c(z)/c &= 2 \delta c(x)/c. \tag{B.19}
\end{align*}

where the equalities in terms of the \( g_i, \bar{g}_i \) are for \( |\psi_{\text{opt}}\rangle \).

In figure B1, we plot the relative change of the coordinate speed of light in \( x \)-direction for the (011) mode. We see that up to position dependent functions of order 1 the relative change of speed of light is given by equation (5) in the main text. Very similar plots are obtained for other directions.

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