ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF
AOKI-SHIDA-SHIGESADA MODEL IN BOUNDED DOMAINS

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Abstract. The beginning of the transition from a hunter-gatherer way of
life to a more settled, farming-based one in Europe is dated to the Neolithic
period. The spread of farming culture from the Middle East is associated,
among other things, with the transformation of landscape, cultivation of
domesticated plants, domestication of animals, as well as it is identified with the
distribution of certain human genetic lineages. Ecological models attribute the
Neolithic transition either to the spread of the initial farming populations or to
the dispersal of farming knowledge and ideas with the simultaneous conversion
of hunter-gatherers to farmers. A reaction-diffusion model proposed by Aoki,
Shida and Shigesada in 1996 is the first model that includes the populations of
initial farmers and converted farmers from hunter-gatherers. Both populations
compete for the same resources in this model, however, otherwise they evolve
independently of each other from a genetic point of view. We study the large
time behaviour of solutions to this model in bounded domains and we explain
which farmers under what conditions dominate over the other and eventually
occupy the whole habitat.

1. Introduction. The Neolithic transition in Europe, the shift from foraging (i.e.,
hunting and gathering) to agriculture and settlement, started around 10 000 years
ago and happened gradually at many different times in the whole Europe. Radio-
carbon dating has produced a large amount of evidence regarding this transition.
For example, it follows from the data that the Neolithic transition was steadily
shifted with an almost constant velocity, which is roughly estimated to be 0.8–1.2
km/year, despite the fact that Europe is geographically heterogeneous [1, 8].
In order to explain the Neolithic transition theoretically, Ammerman and Cavalli-Sforza proposed in [1] a model for the spread of farmers and hunter-gatherers, which is a natural extension of the Fisher-KPP equation, namely

\[ \begin{cases} \partial_t F = d_F \Delta F + r_F (1 - F/K_F) F + e F H, \\ \partial_t H = d_H \Delta H + r_H (1 - H/K_H) H - e F H, \end{cases} \]

where \( F(x,t) \) and \( H(x,t) \) are, respectively, the densities of farmers and hunter-gatherers at the time \( t > 0 \) and the position \( x \) in a space domain; \( d_F, r_F \) and \( K_F \) (resp. \( d_H, r_H \) and \( K_H \)) are the random dispersal rate, the intrinsic growth rate and the carrying capacity of the farming (resp. hunting-gathering) population. The acculturation (conversion) rate between hunter-gatherers and farmers is denoted by \( e \). All these rates are positive constants.

By writing

\[ F^* = \frac{F}{K_F}, \quad H^* = \frac{H}{K_H}, \quad d = \frac{d_H}{d_F}, \quad b = \frac{r_H}{r_F}, \quad s = \frac{e K_H}{r_F}, \quad g = \frac{e K_F}{r_F}, \]

\[ t^* = r_F t, \quad x^* = \sqrt{\frac{r_F}{d_F}} x, \]

and omitting the asterisks for notational simplicity, (1) is rescaled to

\[ \begin{cases} \partial_t F = \Delta F + (1 - F) F + s F H, \\ \partial_t H = d \Delta H + b(1 - H) H - g F H, \end{cases} \]

where \( d, b, s \) and \( g \) are positive constants. The existence, uniqueness and large time behavior of solutions to the model (2) in a bounded domain with the Neumann boundary conditions and nonnegative initial data are studied in [10]. In particular, it is proved in [10] that the solution \((F,H)\) converges to \((1,0)\) if \( g \geq b \) and \((F^*,H^*)\) if \( g < b \) as time tends to infinity, where \( F^* \) and \( H^* \) are uniquely determined by the equations

\[ 1 - F^* + s H^* = 0 \quad \text{and} \quad b(1 - H^*) - g F^* = 0. \]

Hence, the final state of \((F,H)\) of the model (2) depends on the growth rate of hunter-gatherers \( b \) and the conversion rate of hunter-gatherers to farmers \( g \).

A genetic study of mitochondrial DNA from late European hunter-gatherers, early farmers and modern Europeans revealed that the first farmers in Central Europe were not the descendants of local hunter-gatherers [3]. This indicates that the Neolithic transition in Europe was mainly driven by the reproduction and dispersal of initial farmers. In addition to the initial farmers, hunter-gatherers adapted to agriculture by acquiring knowledge from the farmers; hence, there are two genetically different types of farmers at the same time [5, 6]. The model (2) does not, however, explain which farming subpopulation eventually occupies the final state. Aoki, Shida and Shigesada proposed in [2] a model which takes into account the densities of initial farmers \( F \), converted farmers from hunter-gatherers \( C \) and hunter-gatherers \( H \), namely

\[ \begin{cases} \partial_t F = \Delta F + a(1 - (F + C)) F, \\ \partial_t C = \Delta C + (1 - (F + C)) C + s(F + C) H, \\ \partial_t H = d \Delta H + b(1 - H) H - g(F + C) H, \end{cases} \]
where $a$, $b$, $d$, $g$, and $s$ are positive constants. In the model (3), the initial and converted farmers compete for the same resources, however, it is assumed that they stay separated from each other due to missing intermarriage among the populations.

If $a = 1$, then the model (3) rewritten for the total farmers $F + C$ and hunter-gatherers $H$ becomes (2). Therefore, either $\lim_{t \to \infty} (F + C, H) = (1, 0)$ if $g \geq b$ or $\lim_{t \to \infty} (F + C, H) = (F^*, H^*)$ if $g < b$. Thus, as in the previous case, it is not clear from these limits which farmers occupy the equilibrium state of the model (3) even if $a = 1$.

The purpose of this paper is to study convergence to steady states for solutions of the system (3) in a bounded domain with homogeneous Neumann boundary conditions, that is

$$\begin{aligned}
\partial_t F &= \Delta F + a(1 - (F + C))F, \\
\partial_t C &= \Delta C + (1 - (F + C))C + s(F + C)H, \\
\partial_t H &= d\Delta H + b(1 - H)H - g(F + C)H, \\
\frac{\partial F}{\partial n} - \frac{\partial C}{\partial n} - \frac{\partial H}{\partial n} &= 0, \\
(F(0, x), C(0, x), H(0, x)) &= (F_0(x), C_0(x), H_0(x)),
\end{aligned} \quad (4)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial \Omega$, and $n = n(x)$ is the unit normal vector to $\partial \Omega$ at $x \in \partial \Omega$. We assume that the initial functions $(F_0(x), C_0(x), H_0(x))$ are nonnegative for $x \in \Omega$ and such that $F_0 + C_0 \neq 0$. \hfill (5)

It is well known, see for example [9], that this initial boundary value problem with nonnegative continuous initial data possesses a unique classical solution. The classical solution of (4) is uniformly bounded in $C^2(\Omega, \mathbb{R}^3)$ for $t \in [1, \infty)$ by standard parabolic estimates. Moreover, the Schauder estimates for parabolic equations imply that $\{(F(\cdot, t), C(\cdot, t), H(\cdot, t)) \mid t \geq 1\}$ is precompact in $C^2(\Omega, \mathbb{R}^3)$.

We prove that as time tends to infinity the solution $(F, C, H)$ converges to

(i) $(\lambda^*, 1 - \lambda^*, 0)$ if $g \geq b$, where $\lambda^*$ is a constant satisfying $0 \leq \lambda^* \leq 1$, 
(ii) $(0, C^*, H^*)$ if $g < b$, where $C^*$ and $H^*$ are uniquely determined by

$$1 - C^* + sH^* = 0 \quad \text{and} \quad b(1 - H^*) - gC^* = 0.$$  

These results anthropologically state that if the conversion rate of hunter-gatherers to farmers, $g$, is higher than or equal to the growth rate of hunter-gatherers, $b$, then hunter-gatherers fade away and that the initial and converted farmers eventually occupy the whole habitat. The coexistence of farmers and hunter-gatherers is only possible if the growth of hunter-gatherers is sufficient to compensate for the conversion to farmers. In this case, however, the final farming population is entirely composed of the converted farmers.

We end this introduction by a remark that other models explaining the Neolithic spread of farming in Europe exist in the literature. For more details we refer the reader to the reviews by Fort [7] and Steele [12].

The rest of the paper is organised in two sections. The convergence result for the case when $g \geq b$ is proved in Section 2 and the convergence result for the case when $g < b$ in Section 3.
2. High conversion rate case \((g \geq b)\). In this section we study convergence to steady states of solutions of (4)–(5) in the case when \(g \geq b\). Our main result is the following theorem.

**Theorem 2.1.** Assume that \(g \geq b\). Let \((F, C, H)\) be the solution of (4) with initial data \((F_0(x), C_0(x), H_0(x)) \geq 0\) for \(x \in \Omega\) satisfying (5). Then there exists \(\lambda^* \in [0, 1]\) such that

\[
\lim_{t \to \infty} \sup_{x \in \Omega} |(F(x, t), C(x, t), H(x, t)) - (\lambda^*, 1 - \lambda^*, 0)| = 0.
\]

Moreover, if \(F_0 \neq 0\) and \(g \neq b\), then

\[
\lambda_* = \lim_{t \to \infty} F(\cdot, t) \in (0, 1).
\]

Before we prove this theorem, we show uniform estimates for \(G = F + C\) and \(H\) in space and we prove convergence of \(F\) and \(C\) to their respective spatial averages as \(t \to \infty\).

**Lemma 2.2.** Let \(G(x, t) = F(x, t) + C(x, t)\) for \(x \in \Omega\) and \(t > 0\). Under the same assumptions and notations as in Theorem 2.1, there exist constants \(\varepsilon_k > 0\), \(C_k > 0\) for \(k = 1, 2, 3\) such that

\[
\begin{cases}
1 - C_1 e^{-\varepsilon_k t} \leq G(x, t) \leq 1 + C_2 e^{-\varepsilon_k t}, & x \in \Omega, \ t > 0 \quad \text{if} \quad g > b, \\
0 \leq H(x, t) \leq C_3 e^{-\varepsilon_k t}, & x \in \Omega, \ t > 0 \quad \text{if} \quad g = b.
\end{cases}
\]

**Proof.** The population densities of total farmers \(G = F + C\) and hunter-gatherers \(H\) satisfy the following inequalities:

\[
\begin{aligned}
\partial_t G - \Delta G &\geq \min\{1, a\} G(1 - G) \quad \text{in} \quad \{(x, t) \mid G(x, t) < 1\}, \quad (6) \\
\partial_t G - \Delta G &\leq \min\{1, a\} G(1 - G) + sHG \quad \text{in} \quad \{(x, t) \mid G(x, t) > 1\}, \quad (7) \\
\partial_t H - d\Delta H &= bH(1 - H) - gGH, \quad x \in \Omega, \ t > 0, \quad (8) \\
\frac{\partial G}{\partial n} &= \frac{\partial H}{\partial n} = 0, \quad x \in \partial \Omega, \ t > 0.
\end{aligned}
\]

Let \(\delta_0 = \min\left\{1/2, \inf_{x \in \Omega} G(x, 1)\right\} \in (0, 1/2]\) and

\[
G(t) = 1 - \frac{e^{-\min\{1, a\} t}}{1 - \lambda_0 + e^{-\min\{1, a\} t}} \quad (t \geq 0).
\]

Then \(G\) satisfies

\[
\begin{cases}
\partial_t G - \Delta G = \min\{1, a\} G(1 - G), & x \in \Omega, \ t > 0, \\
\frac{\partial G}{\partial n} = 0, & x \in \partial \Omega, \ t > 0, \\
G(0) = \delta_0 \leq G(x, 1), & x \in \Omega.
\end{cases}
\]

Hence in view of (6) and since \(G(t) < 1\) for \(t \geq 0\), it follows from the comparison principle that

\[
G(x, t + 1) \geq G(t) > 1 - \frac{1 - \delta_0}{\delta_0} e^{-\min\{1, a\} t}, \quad x \in \Omega, \ t \geq 0.
\]
Thus
\[ G(x, t) \geq G(t - 1) > 1 - \frac{1}{\delta_0} e^{-\min\{1, a\}(t-1)}, \quad x \in \Omega, \ t \geq 1. \]
Since \( \delta_0 \in (0, 1/2] \), then
\[ G(x, t) \geq 0 \geq 1 - \frac{1}{\delta_0} e^{-\min\{1, a\}(t-1)}, \quad x \in \Omega, \ t \in [0, 1]. \]
By writing \( \varepsilon_1 = \min\{1, a\} \) and \( C_1 = e^{\varepsilon_1}(1 - \delta_0)/\delta_0 \), we obtain
\[ G(x, t) \geq 1 - C_1 e^{-\varepsilon t}, \quad x \in \Omega, \ t \geq 0. \]
Next we put \( h_0 = \max \{1, \sup_{x \in \Omega} H_0(x)\} \), \( \psi_1(t) = -(g - b)t + \frac{C_1}{\varepsilon_1}(1 - e^{-\varepsilon t}) \) \((t \geq 0)\)
and
\[ \mathcal{H}(t) = \frac{e^{\psi_1(t)}}{h_0 + b \int_0^t e^{\psi_1(\tau)} d\tau} \quad (t \geq 0). \]
Then \( \mathcal{H} \) satisfies
\[
\begin{align*}
&\partial_t \mathcal{H} - d \Delta \mathcal{H} = b \mathcal{H}(1 - \mathcal{H}) - g \mathcal{H} \mathcal{H}, \\
&\quad \mathcal{H}(1 - \mathcal{H}) - g \mathcal{H} \mathcal{H}, \quad x \in \Omega, \ t > 0, \quad x \in \partial \Omega, \ t > 0, \quad x \in \Omega, \\
&\frac{\partial \mathcal{H}}{\partial n} = 0, \quad \mathcal{H}(0) = h_0 \geq H_0(x), \quad x \in \Omega,
\end{align*}
\]
where \( \widetilde{G}(t) = 1 - C_1 e^{-\varepsilon t} \leq G(x, t) \) \((x \in \Omega, \ t \geq 0)\). Hence it follows from (8) and the comparison principle that
\[ H(x, t) \leq \mathcal{H}(t), \quad x \in \Omega, \ t \geq 0. \]
Therefore if \( g > b \), then
\[ H(x, t) \leq \mathcal{H}(t) < h_0 e^{\frac{C_1}{\varepsilon_1}} e^{-(g-b)t}, \quad x \in \Omega, \ t \geq 0. \]
If \( g = b \), then
\[ H(x, t) \leq \mathcal{H}(t) < \frac{e^{\frac{C_1}{\varepsilon_1}}}{h_0 + bt}, \quad x \in \Omega, \ t \geq 0. \]
Thus if we put \( \varepsilon_3 = g - b \) and \( C_3 = h_0 e^{\frac{C_1}{\varepsilon_1}} \), then
\[ H(x, t) \leq \mathcal{H}(t) < \begin{cases} C_3 e^{-\varepsilon_3 t}, & x \in \Omega, \ t \geq 0 \quad \text{if} \quad g > b, \\ 1/C_3 + bt, & x \in \Omega, \ t \geq 0 \quad \text{if} \quad g = b. \end{cases} \tag{9} \]
Finally, by putting \( g_0 = \max \{1, \sup_{x \in \Omega} G_0(x)\} \), \( \psi_2(t) = \int_0^t (-\min\{1, a\} + s \mathcal{H}(s)) ds \) for \( t \geq 0 \) and
\[ \overline{G}(t) = 1 + e^{\psi_2(t)}(g_0 - 1) + s \int_0^t e^{\psi_2(t) - \psi_2(s)} \mathcal{H}(s) ds \quad (t \geq 0), \]
we obtain

\[
\begin{cases}
\partial_t \overline{G} - \Delta \overline{G} \geq \min\{1, a\} \overline{G}(1 - \overline{G}) + s \overline{H} \overline{G} \\
\geq \min\{1, a\} \overline{G}(1 - \overline{G}) + s \overline{H} \overline{G}, \quad x \in \Omega, \ t > 0,
\end{cases}
\]

\[
\frac{\partial \overline{G}}{\partial t} = 0, \quad x \in \partial \Omega, \ t > 0,
\]

\[
\overline{G}(0) = g_0 \geq G_0(x), \quad x \in \Omega.
\]

We deduce from (7) and the comparison principle that

\[
G(x, t) \leq \overline{G}(t) = 1 + I_1(t) + I_2(t) + I_3(t), \quad x \in \Omega, \ t \geq 0,
\]

where

\[
I_1(t) = e^{\psi_2(t)}(g_0 - 1),
\]

\[
I_2(t) = se^{\psi_2(t) - \psi_2(t/2)} \int_0^{t/2} e^{\psi_2(t/2) - \psi_2(\tau)} \overline{H}(\tau) d\tau,
\]

and

\[
I_3(t) = s \int_{t/2}^t e^{\psi_2(t) - \psi_2(\tau)} \overline{H}(\tau) d\tau.
\]

Since \( \lim_{t \to \infty} \overline{H}(t) = 0 \), there exists \( T_1 > 0 \) such that

\[
-\min\{1, a\} + s \overline{H}(t) \leq -\frac{\min\{1, a\}}{2}, \quad t \geq T_1,
\]

which implies for \( t' \in [T_1, t] \) that

\[
\psi_2(t) - \psi_2(t') = \int_{t'}^t (-\min\{1, a\} + s \overline{H}(\tau)) d\tau \leq -\frac{\min\{1, a\}}{2} (t - t').
\]

Thus for any \( t \geq 2T_1 \) we obtain

\[
\int_{T_1}^{t/2} e^{\psi_2(t/2) - \psi_2(\tau)} d\tau \leq \int_{T_1}^{t/2} e^{-\frac{\min\{1, a\}}{2} (t/2 - \tau)} d\tau
\]

\[
= \int_{0}^{t/2 - T_1} e^{-\frac{\min\{1, a\}}{2} \tau} d\tau < \int_{0}^{\infty} e^{-\frac{\min\{1, a\}}{2} \tau} d\tau,
\]

\[
\int_{t/2}^t e^{\psi_2(t) - \psi_2(\tau)} d\tau \leq \int_{t/2}^t e^{-\frac{\min\{1, a\}}{2} (t - \tau)} d\tau
\]

\[
= \int_{0}^{t/2 - T_1} e^{-\frac{\min\{1, a\}}{2} \tau} d\tau < \int_{0}^{\infty} e^{-\frac{\min\{1, a\}}{2} \tau} d\tau.
\]

With these estimates at hand, we obtain for any \( t \geq 2T_1 \) that

\[
I_1(t) \leq (g_0 - 1)e^{\psi_2(T_1)} e^{-\frac{\min\{1, a\}}{4} (t - T_1)},
\]

\[
I_2(t) \leq s e^{-\frac{\min\{1, a\}}{4} t} \sup_{\sigma \geq 0} \overline{H}(\sigma) \left\{ \int_0^{T_1} e^{\psi_2(T_1) - \psi_2(\tau)} d\tau + \int_{T_1}^{t/2} e^{-\frac{\min\{1, a\}}{2} (t/2 - \tau)} d\tau \right\}
\]

\[
\leq s e^{-\frac{\min\{1, a\}}{4} t} \sup_{\sigma \geq 0} \overline{H}(\sigma) \left\{ \int_0^{T_1} e^{\psi_2(T_1) - \psi_2(\tau)} d\tau + \int_{0}^{\infty} e^{-\frac{\min\{1, a\}}{2} \tau} d\tau \right\},
\]

\[
I_3(t) \leq s \int_{0}^{\infty} e^{-\frac{\min\{1, a\}}{2} \tau} d\tau \sup_{\sigma \in [t/2, t]} \overline{H}(\sigma) \leq \begin{cases} 
2s \min\{1, a\} C_3 e^{-\varepsilon s t} & \text{if } g > b, \\
2s \min\{1, a\} C_3 / C_3 + bt & \text{if } g = b,
\end{cases}
\]
where we use the upper bound (9) in the estimate for $I_3$. Since the integrals in the estimates exist and $\mathcal{H}$ is bounded uniformly in time, if we put $\varepsilon_2 = \min \{ \varepsilon_3, \frac{\min(1, \alpha)}{4} \}$, then there exists a positive constant $C_2$ such that

$$G(t) \leq \begin{cases} 1 + C_2 e^{-\varepsilon_2 t} & \text{if } g > b, \\ 1 + \frac{C_2}{1+t} & \text{if } g = b \end{cases}$$

for $t \geq 2T_1$. Therefore by putting

$$C_2 = \max \left\{ C_2, \max_{t \in [0,2T_1]} G(t)e^{2\varepsilon_2 T_1}, \max_{t \in [0,2T_1]} G(t)(1+2T_1) \right\},$$

we obtain

$$G(x,t) \leq \begin{cases} 1 + C_2 e^{-\varepsilon_2 t}, & x \in \Omega, \ t \geq 0 \quad \text{if } g > b, \\ 1 + \frac{C_2}{1+t}, & x \in \Omega, \ t \geq 0 \quad \text{if } g = b. \end{cases}$$

The proof is completed. \hfill \Box

**Remark 1.** The lower estimate

$$G(x,t) \geq 1 - C_1 e^{-\varepsilon_1 t}, \ x \in \Omega, \ t > 0$$

is still true in the case $g < b$.

**Lemma 2.3.** Under the same assumptions and notations as in Theorem 2.1, it holds that

$$\lim_{t \to \infty} \sup_{x \in \Omega} \left( \frac{1}{|\Omega|} \int_{\Omega} F(x,t) - \frac{1}{|\Omega|} \int_{\Omega} F(y,t)dy \right) + \left( \frac{1}{|\Omega|} \int_{\Omega} C(x,t) - \frac{1}{|\Omega|} \int_{\Omega} C(y,t)dy \right) = 0.$$

**Proof.** For the solution $(F, C, H)$ of (4) with the initial data $(F_0, C_0, H_0) \geq 0$ satisfying (5), the orbit $\{(F(\cdot, t), C(\cdot, t), H(\cdot, t)) | t \geq 1 \}$ is precompact in $C^2(\bar{\Omega}, \mathbb{R}^3)$.

By Lemma 2.2, the $\omega$-limit set

$$\omega = \bigcap_{t \geq 1} \{ (F(\cdot, \tau), C(\cdot, \tau), H(\cdot, \tau)) | \tau \geq t \} \in C^2(\bar{\Omega}, \mathbb{R}^3)$$

is contained in the set

$$\{ (u_0, v_0, 0) | u_0, v_0 \in C^2(\bar{\Omega}), \ u_0, v_0 \geq 0, \ u_0 + v_0 = 1, \ \partial u_0/\partial n = \partial v_0/\partial n = 0 \text{ on } \partial \Omega \},$$

where for a set $A \subset C^2(\bar{\Omega}, \mathbb{R}^3)$, the set $\overline{A}^{C^2(\bar{\Omega}, \mathbb{R}^3)}$ denotes the closure of $A$ in $C^2(\bar{\Omega}, \mathbb{R}^3)$. Moreover by [9, Theorem 4.3.3], the set $\omega$ is invariant with respect to the dynamical system defined by the equation (4), that is, for any $(u_0, v_0, w_0) \in \omega$, there is an entire orbit $(u, v, w) \in C^1(\mathbb{R}, \omega)$ satisfying (4). Thus, for any $(u_0, v_0, w_0) \in \omega$, $w_0 = 0$ holds and there exists a function $(u, v) \in C^{2,1}(\bar{\Omega} \times \mathbb{R}, \mathbb{R}^2)$ such that

$$\begin{cases} \partial_t u - \Delta u = 0, & x \in \Omega, \ t \in \mathbb{R}, \\
\partial_t v - \Delta v = 0, & x \in \mathbb{R}, \\
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial \Omega, \ t \in \mathbb{R}, \\
u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), & x \in \Omega, \end{cases}$$

since $u_0 + v_0 = 1$ and $w_0 = 0$ for $(u_0, v_0, w_0) \in \omega$. Thus $(u_0, v_0, w_0) = (u_0, v_0, 0)$ must be constant. Thus

$$\omega \subset \{ (\lambda, 1 - \lambda, 0) | \lambda \in [0,1] \}.$$
Hence
\[ \lim_{t \to \infty} \sup_{x \in \Omega} |\nabla F(x, t)| = 0, \quad \lim_{t \to \infty} \sup_{x \in \Omega} |\nabla C(x, t)| = 0. \]

Therefore
\[
\begin{align*}
\sup_{x \in \Omega} \left| F(x, t) - \frac{1}{|\Omega|} \int_{\Omega} F(y, t)dy \right| & \leq \sup_{x, y, z \in \Omega} |\nabla F(x, t)||z - y| \to 0, \\
\sup_{x \in \Omega} \left| C(x, t) - \frac{1}{|\Omega|} \int_{\Omega} C(y, t)dy \right| & \leq \sup_{x, y, z \in \Omega} |\nabla C(x, t)||z - y| \to 0,
\end{align*}
\]
as \( t \to \infty \). The proof is completed. \( \square \)

**Proof of Theorem 2.1.** If we put \( G(x, t) = F(x, t) + C(x, t) \) for \( x \in \Omega \) and \( t > 0 \), then by Lemma 2.2 the following estimates hold in the case \( g > b \):
\[
\begin{align*}
\int_0^\infty \int_{\Omega} |F(y, t)(1 - G(y, t))|dydt & < \infty, \\
\int_0^\infty \int_{\Omega} |C(y, t)(1 - G(y, t)) + sH(y, t)G(y, t)|dydt & < \infty.
\end{align*}
\]

Thus it follows from the equation (4) that
\[
0 \leq |\Omega| \lambda_F = \lim_{t \to \infty} \int_{\Omega} F(y, t)dy = \int_{\Omega} F_0(y)dy + \int_0^\infty \int_{\Omega} \partial_t F(y, t)dydt
\]
\[ = \int_{\Omega} F_0(y)dy + a \int_0^\infty \int_{\Omega} F(y, t)(1 - G(y, t))dydt, \]
\[ 0 \leq |\Omega| \lambda_C = \lim_{t \to \infty} \int_{\Omega} C(y, t)dy = \int_{\Omega} C_0(y)dy + \int_0^\infty \int_{\Omega} \partial_t C(y, t)dydt
\]
\[ = \int_{\Omega} C_0(y)dy + \int_0^\infty \int_{\Omega} (C(y, t)(1 - G(y, t)) + sH(y, t)G(y, t))dydt. \]

Hence by Lemma 2.3,
\[ \lim_{t \to \infty} \sup_{x \in \Omega} (|F(x, t) - \lambda_F| + |C(x, t) - \lambda_C|) = 0 \]
and by Lemma 2.2,
\[ \lambda_F + \lambda_C = \lim_{t \to \infty} \frac{1}{|\Omega|} \int_{\Omega} G(y, t)dy = 1. \]

In the case \( g = b \),
\[
\frac{d}{dt} \int_{\Omega} F(y, t)dy = a \int_{\Omega} F(1 - G)dy \leq aC_1|\Omega| \max_{x \in \Omega, \tau \geq 0} F(x, \tau)e^{-\varepsilon_1\tau} = Me^{-\varepsilon_1t}, \quad (10)
\]
where \( M = aC_1|\Omega| \max_{x \in \Omega, \tau \geq 0} F(x, \tau) \). This implies that there exists \( \lambda_F \geq 0 \) such that
\[
\lim_{t \to \infty} \frac{1}{|\Omega|} \int_{\Omega} F(y, t)dy = \lambda_F. \quad (11)
\]

In fact, if we assume that \( \int_{\Omega} F(y, t)dy \) does not converge as \( t \to \infty \), then
\[
\int_0^\infty \left| \frac{d}{dt} \int_{\Omega} F(y, t)dy \right| dt = \infty.
\]

On the other hand, by (10),
\[
\int_0^\infty \max \left\{ 0, \frac{d}{dt} \int_{\Omega} F(y, t)dy \right\} dt \leq \int_0^\infty Me^{-\varepsilon_1t}dt < \infty.
\]
Thus
\[
0 \leq \int_{\Omega} F(y,t)dy = \int_{\Omega} F_0(y)dy + \int_0^t \frac{d}{dt} \int_{\Omega} F(y,t)dydt \to -\infty \quad \text{as} \quad t \to \infty.
\]
This is a contradiction. By Lemma 2.2 and (11), we obtain
\[
0 \leq \frac{1}{|\Omega|} \int_{\Omega} C(y,t)dy = \frac{1}{|\Omega|} \int_{\Omega} G(y,t)dy - \frac{1}{|\Omega|} \int_{\Omega} F(y,t)dy \to 1 - \lambda_F \quad \text{as} \quad t \to \infty.
\]
By Lemma 2.3, if we put \( \lambda^* = \lambda_F \in [0,1] \), then
\[
\lim_{t \to \infty} F(\cdot,t) = \lim_{t \to \infty} \frac{1}{|\Omega|} \int_{\Omega} F(y,t)dy = \lambda^*,
\]
\[
\lim_{t \to \infty} C(\cdot,t) = \lim_{t \to \infty} \frac{1}{|\Omega|} \int_{\Omega} C(y,t)dy = 1 - \lambda^*.
\]
To prove the rest of the conclusion of Theorem 2.1, we assume that \( F_0 \neq 0 \) and \( g > b \). Then it follows from (4) and the strong maximum principle that
\[
F(\cdot,t) > 0, \ H(\cdot,t) > 0, \ t > 0.
\]
By Lemma 2.2, \( C \) satisfies
\[
\partial_t C - \Delta C = C(1 - G) + sH(F + C) \geq -C_2e^{-\varepsilon_2t}C + sHF, \ t > 0
\]
and we deduce from the strong maximum principle that
\[
C(\cdot,t) > 0, \ t > 0.
\]
By Lemma 2.2,
\[
\frac{d}{dt} \int_{\Omega} Fdy = a \int_{\Omega} (1 - G)Fdy \geq -aC_2e^{-\varepsilon_2t} \int_{\Omega} Fdy,
\]
\[
\frac{d}{dt} \int_{\Omega} Cdy \geq \int_{\Omega} (1 - G)Cdy \geq -C_2e^{-\varepsilon_2t} \int_{\Omega} Cdy.
\]
Thus if we fix \( t_0 > 0 \) arbitrarily, then
\[
\left\{ \begin{array}{l}
\int_{\Omega} F(y,t)dy \geq e^{-\frac{C_2}{\varepsilon_2}(1-e^{-\varepsilon_2(t-t_0)})} \int_{\Omega} F(y,t_0)dy, \\
\int_{\Omega} C(y,t)dy \geq e^{-\frac{C_2}{\varepsilon_2}(1-e^{-\varepsilon_2(t-t_0)})} \int_{\Omega} C(y,t_0)dy,
\end{array} \right. \quad t \geq t_0.
\]
Therefore
\[
|\Omega|\lambda^* = \lim_{t \to \infty} \int_{\Omega} F(y,t)dy \geq e^{-\frac{C_2}{\varepsilon_2}} \int_{\Omega} F(y,t_0)dy > 0,
\]
\[
|\Omega|(1 - \lambda^*) = \lim_{t \to \infty} \int_{\Omega} C(y,t)dy \geq e^{-\frac{C_2}{\varepsilon_2}} \int_{\Omega} C(y,t_0)dy > 0
\]
and the proof is completed. \( \square \)

3. **Low conversion rate case** \((g < b)\). In this section we consider (4) for the case when \( g < b \). The main result of this section is the following theorem.

**Theorem 3.1.** Assume that \( g < b \). Let \((F, C, H)\) be the solution of (4) with the initial data \((F_0(x), C_0(x), H_0(x)) \geq 0 \quad (x \in \Omega)\) satisfying (5) and \( H_0 \neq 0 \). Then it holds that
\[
\lim_{t \to \infty} \sup_{x \in \Omega} |F(x,t), C(x,t), H(x,t)| = 0,
\]
where \( C^* = \frac{(1+s)b}{b+sg} \) and \( H^* = \frac{b-g}{b+sg} \).
The idea of the proof of this statement comes from that of [11, Theorem 2.10]. First we prove auxiliary lemmas which are key for the proof of Theorem 3.1.

**Lemma 3.2.** Assume that $g < b$. Let

$$
\Phi(v, w) = g \int_{v^{*}}^{w} \frac{\eta - C^{*}}{\eta} d\eta + s \int_{H^{*}}^{w} \frac{\xi - H^{*}}{\xi} d\xi
$$
and let $(v, w)$ be the solution of the following equation

$$
\begin{cases}
\frac{\partial v}{\partial t} - \Delta v = v(1 - v) + swv, \\
\frac{\partial w}{\partial t} - d\Delta w = bw(1 - w) - gwv,
\end{cases}
$$

for the initial data $v_0(x) > 0$, $w_0(x) > 0$ for $x \in \Omega$. Then the functional $L(t) = \int_{\Omega} \Phi(v(y, t), w(y, t)) dy \geq 0$ satisfies

$$
\frac{dL}{dt} = -\int_{\Omega} \left( \frac{gC^{*}}{v^{2}}|\nabla v|^{2} + \frac{dH^{*}}{w^{2}}|\nabla w|^{2} \right) dy - \int_{\Omega} \left( g(v - C^{*})^{2} + sb(w - H^{*})^{2} \right) dy.
$$

**Proof.** The proof is straightforward and relies on the fact that $C^{*}$ and $H^{*}$ satisfy the identities $C^{*} - sH^{*} = 1$ and $gC^{*} + bH^{*} = b$ (see the proof of [4, Lemma 3.5] for the detail).

**Lemma 3.3.** There exists $\varepsilon_0 > 0$ such that for any $(F_0, C_0, H_0) \geq 0$ satisfying (5) and $H_0 \neq 0$, the solution $(F, C, H)$ of (4) with initial data $(F_0, C_0, H_0)$ satisfies

$$
\limsup_{t \to \infty} \sup_{x \in \Omega} H(x, t) \geq \varepsilon_0.
$$

**Proof.** We prove the conclusion by contradiction. Let us assume that for every $\varepsilon > 0$ there are $(F_{0, \varepsilon}, C_{0, \varepsilon}, H_{0, \varepsilon}) \geq 0$ satisfying (5) and $H_{0, \varepsilon} \neq 0$ and $T_\varepsilon > 0$ such that for the solution $(F_\varepsilon, C_\varepsilon, H_\varepsilon)$ of (4) with initial data $(F_{0, \varepsilon}, C_{0, \varepsilon}, H_{0, \varepsilon})$, it holds that

$$
H_\varepsilon(x, t) \leq \varepsilon, \quad x \in \Omega, \quad t \geq T_\varepsilon.
$$

Then the population density of total farmers $G_\varepsilon(x, t) = F_\varepsilon(x, t) + C_\varepsilon(x, t)$ satisfies

$$
\frac{\partial G_\varepsilon}{\partial t} - \Delta G_\varepsilon \leq \min\{1, a\} G_\varepsilon(1 - G_\varepsilon) + s\varepsilon G_\varepsilon
$$

in the set \{(x, t) \mid G_\varepsilon(x, t) > 1, \ t \geq T_\varepsilon\}. By using the comparison principle, we obtain

$$
G_\varepsilon(x, t) \leq \overline{G}_\varepsilon(t), \quad t \geq T_\varepsilon,
$$

where \overline{G}_\varepsilon(t)$ for $t \geq T_\varepsilon$ is the solution of ODE

$$
\begin{cases}
\frac{dG_\varepsilon}{dt} = \overline{G}_\varepsilon \left( \min\{1, a\} + s\varepsilon - \min\{1, a\}\overline{G}_\varepsilon \right), \quad t > T_\varepsilon, \\
\overline{G}_\varepsilon(T_\varepsilon) = \sup_{x \in \Omega} G_\varepsilon(x, T_\varepsilon).
\end{cases}
$$

Thus

$$
\limsup_{t \to \infty} \sup_{x \in \Omega} G_\varepsilon(x, t) \leq \lim_{t \to \infty} \overline{G}_\varepsilon(t) = 1 + \frac{s\varepsilon}{\min\{1, a\}}.
$$

Hence there exists $T_0 \geq T_\varepsilon$ such that

$$
G_\varepsilon(x, t) \leq 1 + \varepsilon + \frac{s\varepsilon}{\min\{1, a\}}, \quad x \in \Omega, \quad t \geq T_0.
$$
and hence
\[ \partial_t H_\varepsilon - d\Delta H_\varepsilon = bH_\varepsilon (1 - H_\varepsilon) - gG_\varepsilon H_\varepsilon \]
\[ \geq bH_\varepsilon (1 - H_\varepsilon) - g \left( 1 + \varepsilon + \frac{s\varepsilon}{\min\{1, a\}} \right) H_\varepsilon \]
\[ = bH_\varepsilon \left( 1 - \frac{g}{b} - \frac{g}{b} \left( \varepsilon + \frac{s\varepsilon}{\min\{1, a\}} \right) - H_\varepsilon \right), \quad x \in \Omega, \ t \geq T_0. \]

Since \( H_{\varepsilon,0} \neq 0 \), then
\[ \inf_{x \in \Omega} H_\varepsilon(x, T_0) > 0 \]
holds. Thus the solution \( H_\varepsilon \) of ODE
\[
\begin{cases}
\frac{dH_\varepsilon}{dt} = bH_\varepsilon \left( 1 - \frac{g}{b} - \frac{g}{b} \left( \varepsilon + \frac{s\varepsilon}{\min\{1, a\}} \right) - H_\varepsilon \right), & t \geq T_0, \\
H_\varepsilon(T_0) = \inf_{x \in \Omega} H_\varepsilon(x, T_0) > 0
\end{cases}
\]
satisfies
\[ H_\varepsilon(x, t) \geq H_\varepsilon(t), \quad x \in \Omega, \ t \geq T_0, \]
\[ \lim_{t \to \infty} H_\varepsilon(t) = 1 - \frac{g}{b} - \frac{g}{b} \left( \varepsilon + \frac{s\varepsilon}{\min\{1, a\}} \right). \]

Therefore by assumption (12),
\[ \varepsilon \geq \liminf_{t \to \infty} \inf_{x \in \Omega} H_\varepsilon(x, t) \geq \lim_{t \to \infty} H_\varepsilon(t) = 1 - \frac{g}{b} - \frac{g}{b} \left( \varepsilon + \frac{s\varepsilon}{\min\{1, a\}} \right). \]

By letting \( \varepsilon \to 0 \) we get a contradiction
\[ 0 \geq 1 - g/b > 0 \]
and the proof is completed. \( \square \)

**Lemma 3.4.** Under the same assumptions and notations as in Theorem 3.1, it holds that
\[ \lim_{t \to \infty} \sup_{x \in \Omega} F(x, t) = 0, \ \lim_{t \to \infty} \inf_{x \in \Omega} C(x, t) > 0, \ \lim_{t \to \infty} \inf_{x \in \Omega} H(x, t) > 0. \]

**Proof.** We prove the conclusion by contradiction. Let \( \lim_{t \to \infty} \inf_{x \in \Omega} H(x, t) = 0 \). Then there exists a sequence \( \tau_1 < \tau_2 < \tau_3 < \cdots \to \infty \) such that
\[ \lim_{n \to \infty} \inf_{x \in \Omega} H(x, \tau_n) = 0. \]

For any
\[ (u_0, v_0, w_0) \in S = \cap_{n \in \mathbb{N}} \{(F(\cdot, \tau_k), C(\cdot, \tau_k), H(\cdot, \tau_k)) | k \geq n \}^{C^2(\overline{\Omega}, \mathbb{R}^3)}, \]
\[ \inf_{x \in \Omega} w_0(x) = \lim_{n \to \infty} \inf_{x \in \Omega} H(x, \tau_n) = 0 \]
and there exists a function \((u, v, w) \in C^{2, 1}(\overline{\Omega} \times \mathbb{R}, \mathbb{R}^3) \) such that \((u, v, w) \geq 0 \) and
\[
\begin{align*}
\partial_t u - \Delta u &= au(1 - u - v), \\
\partial_t v - \Delta v &= v(1 - u - v) + sw(u + v), \quad x \in \Omega, \ t > 0, \\
\partial_t w - d\Delta w &= bw(1 - w) - gw(u + v), \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, \quad x \in \partial \Omega, \ t > 0, \\
u(x, 0) &= u_0(x), \ v(x, 0) = v_0(x), \ w(x, 0) = w_0(x), \quad x \in \Omega,
\end{align*}
\]
since the set of all cluster points
\[ S = \bigcap_{n \in \mathbb{N}} \big\{ (F(\cdot, \tau_k), C(\cdot, \tau_k), H(\cdot, \tau_k)) \mid k \geq n \big\}^{C^2(\Omega, \mathbb{R}^3)} \]
is contained in the \( \omega \)-limit set
\[ \bigcap_{t \geq 1} \big\{ (F(\cdot, \tau), C(\cdot, \tau), H(\cdot, \tau)) \mid \tau \geq t \big\}^{C^2(\Omega, \mathbb{R}^3)}. \]
Since \( w(x, t) \geq 0 \ (x \in \Omega, \ t \in \mathbb{R}) \) and \( \min_{x \in \Omega} w(x, 0) = 0 \), it follows from the strong maximum principle that
\[ w(x, t) = 0, \quad x \in \Omega, \ t \in \mathbb{R} \]
and hence \( w_0(x) = w(x, 0) = 0 \ (x \in \Omega) \). Thus
\[ S \subset \{ (u_0, v_0) \mid u_0, v_0 \in C^2(\overline{\Omega}) \}. \]
Hence
\[ \liminf_{n \to \infty} \sup_{x \in \Omega} H(x, \tau_n) = 0. \]
Therefore
\[ \liminf_{t \to \infty} \sup_{x \in \Omega} H(x, t) = 0. \]
This and Lemma 3.3 imply that there exist sequences \( t_1 < t_2 < t_3 < \cdots \to \infty \) and \( s_1 < s_2 < s_3 < \cdots \to \infty \) such that for \( t_n < s_n \)
\[ \sup_{x \in \Omega} H(x, t_n) = \varepsilon_0/2, \quad \sup_{x \in \Omega} H(x, s_n) = \frac{\varepsilon_0}{n + 2}, \]
\[ H(x, t) \leq \varepsilon_0/2, \quad x \in \Omega, \ t \in [t_n, s_n], \]
where \( \varepsilon_0 > 0 \) is the constant which appears in Lemma 3.3.
From the boundedness of the solution \((F, C, H)\) and a standard parabolic estimate, if we put \((F_n(x, t), C_n(x, t), H_n(x, t)) = (F(x, t+t_n), C(x, t+t_n), H(x, t+t_n))\)
for \( x \in \Omega \) and \( t \in [0, \infty) \), then up to extraction of subsequence, \((F_n, C_n, H_n)\) converges to \((u, v, w) \in C^{2,1}(\Omega \times [0, \infty), \mathbb{R}^3)\) locally uniformly in \( \overline{\Omega} \times [0, \infty) \) as \( n \to \infty \) and \((u, v, w)\) satisfies
\[
\begin{align*}
\partial_t u - \Delta u &= au(1 - u - v), \\
\partial_t v - \Delta v &= v(1 - u - v) + sw(u + v), \\
\partial_t w - d\Delta w &= bw(1 - w) - gw(u + v), \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, \quad x \in \partial \Omega, \ t > 0, \\
\sup_{x \in \Omega} w(x, 0) &= \varepsilon_0/2, \quad \sup_{x \in \Omega} w(x, t) \leq \varepsilon_0/2, \quad t > 0.
\end{align*}
\]
Here we remark that \( H_n(\cdot, t) \leq \varepsilon_0 / 2 \ (t \in [0, s_n - t_n]) \) and that \( s_n - t_n \to \infty \) as \( n \to \infty \), since
\[ H_n(\cdot, 0) \to w(\cdot, 0) \neq 0 \quad \text{and} \quad H_n(\cdot, s_n - t_n) \leq 1/n \to 0 \quad \text{as} \quad n \to \infty. \]
By Remark 1,
\[ 1 = \lim_{n \to \infty} (1 - C_1 e^{-\varepsilon_1 t_n}) \leq \lim_{n \to \infty} (F_n(\cdot, 0) + C_n(\cdot, 0)) = u(\cdot, 0) + v(\cdot, 0) \neq 0 \]
and hence by Lemma 3.3,
\[ \limsup_{t \to \infty} \sup_{x \in \Omega} w(x, t) \geq \varepsilon_0. \]
This contradicts that 
\[ \sup_{x \in \Omega} w(x, t) \leq \varepsilon_0/2, \quad t > 0. \]
Thus it holds that 
\[ \liminf_{t \to \infty} \inf_{x \in \Omega} H(x, t) > 0. \]
Therefore there exist \( \varepsilon_* > 0 \) and \( T > 0 \) such that 
\[ H(x, t) > \varepsilon_*, \quad x \in \Omega, \quad t \geq T. \]

Hence \( G = F + C \) satisfies 
\[ \partial_t G - \Delta G \geq \min\{1, a\}G(1 - G) + s\varepsilon_* G \quad \text{in} \quad \{(x, t) \mid G(x, t) < 1\}, \]
\[ \partial_t G - \Delta G \geq \max\{1, a\}G(1 - G) + s\varepsilon_* G \quad \text{in} \quad \{(x, t) \mid G(x, t) > 1\}. \]

By a similar argument to that in the proof of Lemma 3.3, these inequalities imply that 
\[ \liminf_{t \to \infty} \inf_{x \in \Omega} G(x, t) \geq 1 + \frac{s\varepsilon_*}{\max\{1, a\}}. \]
Hence there exists \( T_* > T \) such that 
\[ \partial_t F - \Delta F = aF(1 - G) \leq -\frac{a s\varepsilon_*}{2 \max\{1, a\}} F, \quad x \in \Omega, \quad t \geq T_. \]
This implies that 
\[ \limsup_{t \to \infty} \sup_{x \in \Omega} F(x, t) = 0 \]
and hence 
\[ \liminf_{t \to \infty} \inf_{x \in \Omega} C(x, t) = \liminf_{t \to \infty} \inf_{x \in \Omega} G(x, t) > 0. \]

The proof is completed. \( \square \)

**Proof of Theorem 3.1.** For the solution \((F, C, H)\) of (4) with the nonnegative initial data \((F_0, C_0, H_0)\) satisfying (5) and \(H_0 \neq 0\), the orbit \(\{(F(\cdot, t), C(\cdot, t), H(\cdot, t)) \mid t \geq 1\}\) is precompact in \(C^2(\overline{\Omega}, \mathbb{R}^3)\).

By Lemma 3.4, the \(\omega\)-limit set 
\[ \omega = \bigcap_{t \geq 1} \{(F(\cdot, \tau), C(\cdot, \tau), H(\cdot, \tau)) \mid \tau \geq t\} \]
\(C^2(\overline{\Omega}, \mathbb{R}^3)\) is contained in the set 
\[ \{(0, v_0, w_0) \mid v_0 > 0, \quad w_0 > 0, \quad \partial v_0/\partial n = \partial w_0/\partial n = 0 \quad (x \in \partial \Omega), \quad v_0, w_0 \in C^2(\overline{\Omega})\}. \]
Moreover by the same argument as in the proof of Lemma 2.3, for any \((u_0, v_0, w_0) \in \omega\), it holds that \(u_0 = 0\) and there exists a function \((v, w) \in C^{2,1}(\overline{\Omega} \times \mathbb{R}, \mathbb{R}^2)\) such that 
\[ \begin{cases} 
\partial_t v - \Delta v = v(1 - v) + swv, 
\quad x \in \Omega, \quad t \in \mathbb{R}, \\
\partial_t w - d\Delta w = bw(1 - w) - gwv, 
\quad x \in \Omega, \quad t \in \mathbb{R}, \\
\partial v/\partial n = \partial w/\partial n = 0, \quad x \in \partial \Omega, \quad t \in \mathbb{R}, \\
v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), \quad x \in \Omega, \\
\inf_{(x, t) \in \Omega \times \mathbb{R}} v(x, t) > 0, \quad \inf_{(x, t) \in \Omega \times \mathbb{R}} w(x, t) > 0.
\end{cases} \]
Thus by Lemma 3.2, \((u_0, v_0, w_0)\) must be equal to \((0, C^*, H^*)\), that is, 
\[ \omega = \{(0, C^*, H^*)\}. \]
The proof is completed. \(\square\)
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Received for publication July 2019.

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