Periodic Instantons in $SU(2)$ Yang-Mills-Higgs Theory

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The properties of periodic instanton solutions of the classical $SU(2)$ gauge theory with a Higgs doublet field are described analytically at low energies, and found numerically for all energies up to and beyond the sphaleron energy. Interesting new classes of bifurcating complex periodic instanton solutions to the Yang-Mills-Higgs equations are described.
Introduction. Anomalous baryon and lepton number violating processes in the electroweak theory are dominated in the semiclassical weak coupling limit, $\alpha_W \to 0$, by field configurations which solve the classical Euclidean Euler-Lagrange equations. At zero temperature and energy the classical solutions that contribute to anomalous winding number transitions are the familiar BPST instantons/anti-instantons \([1]\). The rate of these vacuum tunneling transitions is exponentially suppressed $\exp(-2S_I)$, where $2S_I = 4\pi/\alpha_W$ is the Euclidean action of a widely separated instanton (I)/anti-instanton ($\bar{I}$) pair. At temperatures much higher than $M_W$, the transitions are classical thermal activation transitions over the potential barrier between vacua, with a Boltzmann rate $\sim \exp(-E_s/k_B T)$ controlled by the energy $E_s \sim 4M_W/\alpha_W$ of a certain unstable classical stationary field configuration called the sphaleron \([2]\).

At intermediate temperatures, or at finite energy (not necessarily arranged in thermal equilibrium), very little is known about the rate of anomalous transitions between states of different winding number. The first step in studying these transitions is to find the classical solutions which dominate the semiclassical rate, and calculate their action. This involves solving the classical nonlinear field equations in Euclidean time. At low energies one can construct solutions of the classical Euclidean Yang-Mills-Higgs equations consisting of periodic chains of $I\bar{I}$ pairs arrayed along the imaginary time axis. The action of these periodic instanton solutions can be expressed as a power series in $(E/E_s)^{2/3}$ for small $E/E_s$ \([3]\). This perturbative treatment of low energy periodic instanton solutions can be recast as an expansion in powers of $(M_W/\beta)^2$ where $\beta$ is the period of the solution. For larger energies or periods the solutions can be found numerically. In this Letter we describe the qualitative properties of and present numerical results for these periodic instanton solutions of the $SU(2)$ Yang-Mills-Higgs equations, i.e. the bosonic sector of the standard electroweak theory with $\theta_W = 0$. An unexpectedly rich structure of bifurcating periodic instanton solutions, both real and complex has been found, whose physical consequences for B and L violating transitions in the electroweak theory remains to be more fully investigated.

Periodic Instantons at Low Energy. As a simple example of a periodic potential with periodic instanton solutions consider the pendulum potential,

$$V(q) = \omega^2 (1 - \cos q) .$$

The zero energy instanton which interpolates between the vacuum states at $q = 0$ and $q = 2\pi$ is the kink configuration, $q_I(\tau)$, given by

$$\cos \left( \frac{q_I(\tau)}{2} \right) = -\tanh(\omega \tau)$$

which solves the classical Euclidean equations $\ddot{q}_I = V'(q_I)$ and has action $S_I = 8\omega$. The anti-instanton solution is $q_{\bar{I}}(\tau) = q_I(-\tau)$ with the same action. Consider now the widely separated $I - \bar{I}$ pair configuration,

$$q_{I\bar{I}}(\tau) = q_I(\tau) + q_{\bar{I}}(\tau - \bar{\tau}) - 2\pi$$

with $\bar{\tau} \gg 1/\omega$. The action of this configuration can be computed to first order in the interaction between the pair, with the result
\[ S[q_{II}] = 2S_I - 32 \omega e^{-\omega \bar{\tau}} + O\left(e^{-2\omega \bar{\tau}}\right). \]  \hspace{1cm} (4)

The negative sign reflects the attractive interaction between the \( I \) and \( \bar{I} \).

We now consider a periodic arrangement of \( I \) and \( \bar{I} \) at equal intervals along the imaginary time axis, with period \( \beta \). This means that the separation between nearest neighbor \( I \) and \( \bar{I} \) is \( \bar{\tau} = \beta/2 \). The attractive force between neighbors can now exactly balance and yield an extremum of the action, the periodic instanton solution. Since there are two nearest neighbor \( II \) interactions per period we expect the action per period of this solution to be

\[ S(\beta) = 16 \omega - 64 \omega e^{-\omega \beta/2} + O\left(e^{-\omega \beta}\right), \]  \hspace{1cm} (5)

in the limit of large \( \omega \beta \). Since

\[ E(\beta) = \frac{dS(\beta)}{d\beta} = 32 \omega^2 e^{-\omega \beta/2} + \ldots \]  \hspace{1cm} (6)

large \( \beta \) corresponds to low energy. In the one dimensional pendulum example the exact periodic instanton solution with this action and energy are easily found explicitly in terms of elliptic functions by simple quadrature. Because of the attractive interaction between the \( I \) and \( \bar{I} \) along the chain it is clear that there is a single negative mode of the second order fluctuation operator, \(-\partial_\tau^2 + V''(q(\tau))\), around this periodic instanton solution, a fact that is also reflected by the second derivative of the action,

\[ \frac{d^2S(\beta)}{d\beta^2} = \frac{dE(\beta)}{d\beta} = -16 \omega^2 e^{-\omega \beta/2} + \ldots < 0. \]  \hspace{1cm} (7)

The monotonic decrease of period \( \beta \) with increasing energy persists up to \( E = E_s = 2\omega^2 \), where the curve of \( S(\beta) \) vs. \( \beta \) of the periodic instanton becomes tangent to the constant sphaleron solution, corresponding in this simple model with the unstable static configuration \( q_s = \pi \). This occurs at \( \beta = \beta_- = 2\pi/\omega \) equal to the period of oscillation in the inverted potential at \( q = q_s \). At this \( \beta \) the action of the periodic instanton is \( E_s \beta_- = 4\pi \omega < 16\omega \), reflecting the fact that the action is monotonically decreasing as \( \beta \) ranges from \( \infty \) down to \( \beta_- \), and as \( E \) increases from 0 to \( E_s \).

Beyond the point where the periodic instanton and sphaleron solutions merge, a complex solution bifurcates from the sphaleron and continues with real decreasing action. This may be understood by the amplitude of the zero mode at \( \beta = \beta_- \) turning from real to pure imaginary as \( \beta \) is decreased through the critical value. The generic behavior described here is what we call type (I) behavior of the periodic instanton solutions, for which the monotonic negative sign in the first derivative of the action in (7) and \( E_s \beta_- < 2S_I \) are characteristic.

A different pattern is possible when the instanton has additional zero modes, and therefore additional parameters enter the description. Such is the case in field theory models with exact or softly broken conformal invariance. We have studied this case in some detail in the \( O(3) \) nonlinear sigma model in two dimensions, softly broken by a mass term [4]. This model shares many features with the bosonic sector of the \( SU(2) \) electroweak theory. Although there is no isolated single \( I \) or \( \bar{I} \) solution in the broken theory, due to Derrick’s theorem (which tells us that zero scale size \( \rho \to 0 \) has minimum action), a periodic instanton solution does exist in which the scale size \( \rho \) is adjusted to a certain value as a function of
period $\beta$. At this value the attractive interaction between $I$ and $\bar{I}$ exactly balances the tendency of each individual $I$ or $\bar{I}$ to collapse to zero size. It is again possible to understand this at low energies by first finding the two-body interaction between well isolated $I$ and $\bar{I}$, and then arranging them periodically along the imaginary time axis, calculating the change in the action per period from that of a single $I\bar{I}$ pair due to the sum of the first order interactions between them. In this calculation the scale size $\rho$ can be treated as a variational parameter with the value on the solution $\rho(\beta)$ determined by extremizing $S(\beta, \rho)$ with respect to $\rho$. Substituting into $S$ then gives $S(\beta)$ on a low energy periodic instanton solution. The resultant behavior depends upon the curvature of $S(\beta)$, which is of opposite sign relative to the type (I) models, and we will consequently refer to this case as type (II), i.e.

$$\frac{d^2 S(\beta)}{d\beta^2} = \frac{dE(\beta)}{d\beta} > 0, \quad \text{type (II)}. \quad (8)$$

The periodic instanton solutions in this case have two negative modes rather than just one, with the second negative mode corresponding to variation of the scale size $\rho$ away from its extremal value $\rho(\beta)$. Such periodic instanton solutions do not contribute to thermal winding number processes at finite temperature. However, they can contribute to anomalous finite-energy non-thermal transitions.

Because of the existence of the conformal mode in $SU(2)$ BPST instantons, we would expect the Yang-Mills-Higgs theory to behave qualitatively similar to the $O(3)$ sigma model, and to also be of type (II). Indeed for low energies we can show that this is exactly what happens. The action for an isolated pure $SU(2)$ $I\bar{I}$ pair with scale size $\rho$ separated by distance $\tau$ is

$$S_{II} = 2S_I - \frac{96\pi^2 \rho^4}{g^2 \tau^4} + \mathcal{O}\left(\frac{\rho^6}{g^2 \tau^6}\right), \quad (9)$$

with $S_I = \frac{8\pi^2}{g^2} = 2\pi/\alpha$ the single instanton action and the second term the well-known dipole-dipole attractive interaction between the $I$ and $\bar{I}$ aligned in an internal $SU(2)$ direction. When the periodic chain of $I$ and $\bar{I}$ separated by $(n + \frac{1}{2})\beta$ is constructed, this leads to the total interaction,

$$S_{\text{int}} = -\frac{96\pi^2 \rho^4}{g^2 \beta^4} \sum_{n=-\infty}^{\infty} \frac{1}{(n + \frac{1}{2})^4} = -\frac{4\pi}{\alpha} \left(\frac{2\pi^4 \rho^4}{\beta^4}\right). \quad (10)$$

When the $SU(2)$ doublet Higgs field is added to the action it can be solved for at leading order in the $I$ or $\bar{I}$ background and gives a contribution,

$$S_{\text{Higgs}} = \frac{4\pi}{\alpha} \left(\frac{M_\chi^2 \rho^2}{2}\right) + \mathcal{O}\left(\frac{M_\chi^2 \rho^4}{g^2 \beta^2}\right), \quad (11)$$

which was first calculated by ‘t Hooft [4].

This positive contribution expresses the fact that Derrick’s theorem drives the single isolated $I$ or $\bar{I}$ scale size to zero $\rho$; however, at finite $\beta$ this is opposed by the dipole-dipole interaction $S_{\text{int}}$, and the variational action
\[ S(\beta, \rho) = \frac{4\pi}{\alpha} \left[ 1 + \frac{M_W^2 \rho^2}{2} - \frac{2\pi^4 \rho^4}{\beta^4} + \mathcal{O}\left(\frac{M_W^4 \rho^4}{\beta^6}\right) \right] \] 

(12)

has an nontrivial extremum at \( \rho = \rho(\beta) \) when \( \partial S/\partial \rho = 0 \), or
\[
\rho = \frac{\sqrt{2}}{M_W} \left[ x^2 + \mathcal{O}\left(x^4\right) \right],
\]

(13)
denoting by \( x \) the expansion parameter \( M_W \beta/(2\pi) \). This stationarity condition implies that the various next-to-leading contributions to the action from both the gauge and Higgs fields are all of order \( x^6 \). Hence the periodic instanton action,
\[
S = \frac{4\pi}{\alpha} \left[ 1 + \frac{1}{2} x^4 + \mathcal{O}\left(x^6\right) \right]
\]

(14)
for small \( M_W \beta \), and
\[
E = \frac{dS(\beta)}{d\beta} = \frac{4M_W}{\alpha} x^3 \left[ 1 + \mathcal{O}\left(x^2\right) \right]
\]

(15)
is an increasing function of period \( \beta \) for small \( M_W \beta \). The second derivative of \( S(\beta) \) is also clearly positive. After a rather elaborate calculation \[5\], one can evaluate the coefficients of the next to leading terms in the perturbative expansion. One thus finds
\[
S = \frac{4\pi}{\alpha} \left[ 1 + \frac{1}{2} x^4 + \frac{4}{3} x^6 + \mathcal{O}\left(x^8\right) \right]
\]

(16)
\[
E = \frac{4M_W}{\alpha} x^3 \left[ 1 + 4 x^2 + \mathcal{O}\left(x^4\right) \right].
\]

(17)

Although the \( SU(2) \) Higgs theory starts out at low energy and small \( \beta \) behaving like the type (II) case, there is an additional independent parameter in the 4D gauge theory, namely the quartic Higgs self-coupling \( \lambda \), or equivalently the Higgs mass, \( M_H \). Thus, we cannot preclude a more complicated behavior at larger energies, depending on the value of \( M_H/M_W \). This is indeed what we have found in our numerical study.

**Periodic Instantons at Finite Energy in the \( SU(2) \)-Higgs Theory.** We consider the \( SU(2) \) gauge theory with a doublet Higgs field in 4D Euclidean space with the action,
\[
S = \frac{1}{g^2} \int d^4x \left\{ \frac{1}{2} \text{Tr}(F_{\mu\nu}F_{\mu\nu}) + (D_{\mu}\Phi)^\dagger(D_{\mu}\Phi) + \frac{\lambda}{g^2} \left( \Phi^\dagger \Phi - \frac{g^2 v^2}{2} \right)^2 \right\},
\]
\[
F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - i[A_{\mu}, A_{\nu}], \quad D_{\mu}\Phi = (\partial_{\mu} - iA_{\mu})\Phi,
\]

(18)
and \( A_{\mu} = A_{\mu}^a \sigma^a/2 \). The corresponding classical Euler-Lagrange equations are
\[
D_{\mu}F_{\mu\nu} + i(D_{\nu}\Phi^\dagger) \times \Phi - i\Phi^\dagger \times (D_{\nu}\Phi) = 0
\]
\[
\left[ -D^2 + \frac{2\lambda}{g^2} \left( \Phi^\dagger \Phi - \frac{g^2 v^2}{2} \right) \right] \Phi = 0,
\]

(19)
where the covariant derivative acting on the $A$-field in the adjoint representation is $D_\mu A_\nu = \partial_\mu A_\nu - i [A_\mu, A_\nu]$, and the $\times$ denotes the outer product of the two spinors. We use the standard conventions for the Higgs vacuum expectation value $v$ and the self-coupling $\lambda$ in which the $W$- and and Higgs-masses are $M_W = \frac{1}{2} g v$ and $M_H = \sqrt{2} \lambda v$ respectively.

The spherical Ansatz is given by expressing the gauge field $A_\mu$ and the Higgs field $\Phi$ in terms of six real functions $a_0, a_1, \alpha, \gamma, u$ and $w$ of $r$ and $\tau$:

\[
A_0(x, \tau) = \frac{1}{2} a_0(r, \tau) \bar{\sigma} \cdot \hat{x}
\]
\[
A_i(x, \tau) = \frac{1}{2} [a_1(r, \tau) \bar{\sigma} \cdot \hat{x} i + \alpha(r, \tau) (\sigma_i - \bar{\sigma} \cdot \hat{x} i) + \frac{\gamma(r, \tau)}{r} \epsilon_{ijk} \hat{x} j \sigma_k]
\]
\[
\Phi(x, \tau) = \sqrt{2} M_W [u(r, \tau) + i w(r, \tau) \bar{\sigma} \cdot \hat{x}] \hat{\zeta}
\]

where $\hat{\zeta}$ is an arbitrary unit two-component spinor, $\hat{x}$ is the unit three-vector in the radial spatial direction, and $\bar{\sigma}$ is a three-vector of Pauli matrices.

Upon substituting (20) into the action (18) one finds [7]

\[
S = \frac{4 \pi}{g^2} \int d\tau \int_0^\infty dr \left[ \frac{1}{4} r^2 f_{\mu \nu} f_{\mu \nu} + (\bar{D}_\mu \bar{\chi}) D_\mu \chi + r^2 (\bar{D}_\mu \bar{\phi}) D_\mu \phi + \frac{1}{2 r^2} (\bar{\chi} \chi - 1)^2 + \frac{1}{2} (\bar{\chi} \chi + 1) \bar{\phi} \phi + \text{Re}(i \bar{\chi} \phi^2) + \frac{\lambda}{g^2} r^2 \left( \bar{\phi} \phi - 2 M_W^2 \right)^2 \right],
\]

where the indices now run over 0 and 1 and

\[
f_{\mu \nu} = \partial_\mu a_\nu - \partial_\nu a_\mu ,
\]
\[
\chi = \alpha + i \left( \gamma - 1 \right), \quad \bar{\chi} = \alpha - i \left( \gamma - 1 \right),
\]
\[
\phi = \sqrt{2} M_W (u + i w) , \quad \bar{\phi} = \sqrt{2} M_W (u - i w),
\]
\[
D_\mu \chi = (\partial_\mu - i a_\mu) \chi , \quad \bar{D}_\mu \bar{\chi} = (\partial_\mu + i a_\mu) \bar{\chi},
\]
\[
D_\mu \phi = (\partial_\mu - i a_\mu) \phi , \quad \bar{D}_\mu \bar{\phi} = (\partial_\mu + i a_\mu) \bar{\phi} .
\]

The equations of motion for the reduced theory are

\[
-\partial_\mu (r^2 f_{\mu \nu}) = i \left[ (\bar{D}_\nu \bar{\chi}) \chi - \bar{\chi} D_\nu \chi \right] + \frac{i}{2} r^2 \left( (\bar{D}_\nu \bar{\phi}) \phi - \bar{\phi} D_\nu \phi \right),
\]
\[
-\partial_\mu D_\mu + \frac{1}{r^2} (\bar{\chi} \chi - 1) + \frac{i}{2} \bar{\phi} \phi \chi = -\frac{i}{2} \phi^2 ,
\]
\[
-\partial_\mu (r^2 D_\mu) + \frac{1}{2} (\bar{\chi} \chi + 1) + \frac{2 \lambda}{g^2} r^2 \left( \bar{\phi} \phi - 2 M_W^2 \right) \phi = i \chi \bar{\phi} .
\]

Note that the overbar on $\phi, \chi$ and $D_\mu$ denotes changing $i \to -i$ in the definitions (22) above, which is the same as complex conjugation only if the six fields $a_\mu, \alpha, \gamma, u$ and $w$ are real. These equations can be obtained by either imposing the the spherical Ansatz (20) on the four dimensional equations (19), or by varying the action (21) directly.
The spherical Ansatz (20) has a residual $U(1)$ gauge invariance under the $U(1)$ gauge transformation,

$$
a_\mu \rightarrow a_\mu + \partial_\mu \Omega,
\chi \rightarrow e^{i\Omega} \chi,
\phi \rightarrow e^{i\Omega/2} \phi,
$$

The complex scalar fields $\chi$ and $\phi$ have $U(1)$ charges of 1 and 1/2 respectively, $a_\mu$ is the $U(1)$ gauge field, $f_{\mu\nu}$ is the field strength, and $D_\mu$ is the covariant derivative. The residual $U(1)$ gauge invariance must be fixed for numerical solution of the equations. In our numerical work we chose the temporal gauge $a_0 = 0$. The remaining time independent gauge freedom is fixed by a boundary condition at the quarter period time slice $\tau = \beta/4$ to be specified below. In the $a_0 = 0$ gauge the $\nu = 0$ component of the first of Eqs. (23), i.e. the Gauss law constraint, must be imposed on the initial $\tau = 0$ surface, whereupon it will be satisfied for all $\tau$.

The action of the various discrete symmetries, C, P, and T on the two dimensional fields follows directly from the spherical Ansatz (20). In addition to these symmetries, we may consider the two dimensional reflection symmetry $R : \phi \rightarrow -\phi$. We can employ these discrete symmetries to help us select the appropriate boundary conditions for the periodic instanton solution. The $CPR$ even fields are $\gamma$ and $w$, whereas the other four fields are $CPR$ odd. Since we are searching for a periodic solution which returns to itself with period $\beta$, the time derivatives must reverse sign in the second half period relative to the first. This means that we should require the boundary condition that the $\tau$ derivatives of all remaining five functions in $a_0$ gauge vanish at $\tau = 0$ and $\tau = \beta/2$. This corresponds to an instanton at $\beta/4$ and an anti-instanton at $3\beta/4$ in the low energy limit. At the time slice $\tau = \beta/4$ the fields are sphaleron-like. In a gauge where the four dimensional fields are regular at the origin, the sphaleron is a $CPR$ even configuration, and therefore the $CPR$ odd fields change sign while the $CPR$ even fields reach a maximum at $\tau = \beta/4$. Hence we actually require the solution only on the quarter interval $[0, \beta/4]$, if we specify the boundary conditions,

$$
\dot{a}_1 = \dot{\alpha} = \dot{\gamma} = \dot{u} = \dot{w} = 0, \quad \tau = 0;
\quad a_1 = \alpha = u = 0 = \dot{\gamma} = \dot{w}, \quad \tau = \frac{\beta}{4}.
$$

These boundary conditions eliminate the time translational zero mode.

The Euler-Lagrange Eqs. (23) are also invariant under the two additional complex discrete transformations:

$$
C_1 : \quad \phi \rightarrow -\bar{\phi}, \quad \bar{\phi} \rightarrow -\phi, \quad \chi \rightarrow -\bar{\chi}, \quad \bar{\chi} \rightarrow -\chi, \quad a \rightarrow -a
$$

$$
C_2 : \quad \phi \rightarrow \bar{\phi}^*, \quad \bar{\phi} \rightarrow \phi^*, \quad \chi \rightarrow \bar{\chi}^*, \quad \bar{\chi} \rightarrow \chi^*, \quad a \rightarrow a^*.
$$

Under $C_1$: $S \rightarrow S$, under $C_2$: $S \rightarrow S^*$ (and similarly for $E$).

For the four dimensional fields to be regular at the origin, and to approach the vacuum at $r = \infty$ we require the boundary conditions,

$$
\alpha = \gamma = w = 0 = a'_1 = u', \quad r = 0;
\quad a_1 = \alpha = u = 0, \quad \text{but} \quad \gamma = 2, \quad w = 1, \quad r = \infty.
$$
where the last condition is necessary for a nonzero winding number, and agrees with the sphaleron boundary condition at \( r = \infty \) on the \( \tau = \beta/4 \) slice. The boundary conditions on \( a_1 \) at the origin and infinity are gauge choices, which completely eliminate the time independent gauge freedom in temporal gauge \( a_0 = 0 \).

**Numerical Results.** With a well-defined elliptic boundary value problem standard numerical methods may be applied. Here we only outline our computational procedure, the full details of which will be presented in a separate publication [5]. We discretized the Euler-Lagrange equations in space and time by using link variables for the gauge degrees of freedom, thus preserving exact gauge invariance under space dependent gauge transformations (consistent with our choice of the \( a_0 = 0 \) gauge). The code allows for non-uniform grids in both space and time to better approximate the fields in the regions of fastest variation. We also allowed for the analytic continuation of the solutions into complex valued functions by using complex variables to represent the fields \( a_1, \alpha, \gamma, u, w \) and by discretizing the equations in a manner compatible with analytic continuation. The coordinates \((r, \tau)\) are maintained real.

The equations were solved by the Newton-Raphson technique. Denoting a definite trial configuration of the fields by \( f_i \), where \( i \) stands for the discretized two dimensional lattice point \((r, \tau)\) as well as the various field components themselves, we may calculate the gauge invariant discretized action functional \( S[f] \), the discretized first variation, \( \partial S/\partial f_i \), and the second order fluctuation operator \( \partial^2 S/\partial f_i \partial f_i' \) on the trial configuration. Since the first variation must vanish on the solution, one can find the first order correction to the configuration, \( \delta f \), by solving the linear equations,

\[
\sum_{i'} \frac{\partial^2 S}{\partial f_i \partial f_{i'}} \delta f_{i'} + \frac{\partial S}{\partial f_i} = 0 .
\]

(29)

Adding \( \delta f \) to \( f \) yields a corrected trial configuration, and the process may now be iterated. Clearly, all gauge and translational zero modes must be removed from the second order variation by the gauge fixing and boundary conditions in order for the inverse of \( \partial^2 S/\partial f_i \partial f_i' \) to exist and the procedure to be well defined.

The algorithm converges quite rapidly, the error decreasing quadratically with the number of complete Newton-Raphson iterations. The most time and memory consuming step is the inversion of the second order fluctuation operator. With grids consisting of as many as \( 128 \times 128 \) points (or more) and 5 complex valued fields per point, a direct solution of the above system of linear equations is prohibitive. Because the equations of motion are local, the matrix to be inverted is a sparse banded matrix, and it is much more efficient to use the method of eliminating alternate time slices instead of a direct inversion. This effectively reduces the dimensionality of the linear system one must invert to the size of the space grid only, times the 5 components of the fields. Quite modest lattices \((64 \times 64)\) are sufficient for accuracy of order one percent, except in the small \( \beta \), low energy region, where the larger, adaptive lattices \((128 \times 128 \) or greater\) were used. From this solution at a given \( \beta \) and \( \lambda \) other solutions were found by using the previous one as a trial configuration for the new values of the parameters, changing the values of period and \( \lambda \) in small increments.

Our results for the dependence of the action on the period are shown in Figs. 1 for two different values of the quartic Higgs coupling, \( \lambda = 0.7g^2 \) and \( \lambda = 3.6g^2 \), respectively. They clearly exhibit a pattern of bifurcations.
FIG. 1. The action of the periodic instanton as a function of period along the four distinct branches of solutions. The action of the sphaleron is shown by the solid diagonal lines. In (a), $\lambda = 0.7g^2 < \lambda_{\text{crit}}$. The only real solutions lie along branch 1, which joins onto the perturbative solutions at small period. Branch 1 merges with the sphaleron at $s$, and may be analytically continued onto branch 2. Branches 3 and 2 have real action and energy and form a cusp at point $c$, beyond which along branch 4 the action becomes complex (only the real part of the action is shown). The cusp $c$ always lies above the sphaleron line in this regime of Higgs coupling. In (b), $\lambda = 3.6g^2 > \lambda_{\text{crit}}$ and the situation is similar, except that it is now the complex branch 3 that merges with the sphaleron. Branches 1 and 2 have real action and energy and form a cusp $c$ that lies below the sphaleron line. Again, the action and energy along branch 4 are complex.

Since the numerical method and our code works equally well for real or complex solutions, we were able to follow the action and energy of the latter as well. We observe an interesting
pattern of bifurcating solutions depending on the Higgs self-coupling $\lambda$, as anticipated in ref. 8.

There are two regimes, separated by a critical value $\lambda_{cr}$. As illustrated in Fig. 1a, for $\lambda < \lambda_{cr}$ the perturbative solutions (branch 1) merge with the sphaleron at some value $\beta = \beta_1$, which is similar in behavior to the $O(3)$ sigma model. For $\beta > \beta_1$, branch 1 can be analytically continued up to a second bifurcation point $\beta_2$ to yield complex solutions whose energy and action however remain real (branch 2). At $\beta_2$, branch 2 merges with yet another branch of complex solutions (branch 3), and the action and energy of these solutions are also real. The numerical results for this second complex branch suggest that its action decreases monotonically with decreasing $\beta$, and that at a sufficiently low (but positive) $\beta$ its action vanishes, with potentially interesting physical consequences (however, as indicated below, the energy along this branch becomes arbitrarily large). Branches 2 and 3 form a cusp $c$ at $\beta_2$, beyond which the solutions may be analytically continued onto a fourth branch. The action and the energy on branch 4 ($\beta > \beta_2$) are complex, and the cusp $c$ always lies above the sphaleron line (only the real part of the action has been graphed).

Figure 1b illustrates the situation for $\lambda > \lambda_{cr}$. The pattern is similar, except in this case it is the complex solutions along branch 3, and not the real solutions along branch 1, that merge with the sphaleron, and whose extension beyond $\beta_1$ becomes branch 2. Branch 1 (the perturbative one) and branch 2 are real up to $\beta_2$, where they form a cusp $c$ beyond which the solutions are complex with complex energy and action (branch 4). In this case, the cusp always lies below the sphaleron line, except at a critical coupling $\lambda_{crit}$ where the cusp intersects the sphaleron. We found $\lambda_{cr} \approx 1.198 g^2$, corresponding to $M_H \approx 3.096 M_W$, in good agreement with $3.091 M_W$, obtained in ref. 9 by a careful treatment of perturbations away from the static sphaleron solution.

![FIG. 2. The energy as a function of period near the bifurcation points for two values of $\lambda$ on either side of the critical Higgs coupling and for the critical coupling $\lambda_{crit}$ itself. The solid horizontal lines labeled by $E_s$ denote the constant sphaleron energy for each value of $\lambda$, while the dashed, dotted, and solid curves are the energies of the corresponding solutions in Figs. 1.](image)
The bifurcation structure for the energy of the complex solutions is shown in Fig. 2 for two values of $\lambda$ on either side of the critical value and for the critical coupling $\lambda_{\text{crit}}$ itself. The associated sphaleron energies $E_s(\lambda)$ are illustrated by the short horizontal lines. The real solutions are indicated by the bold solid lines, the complex solutions with real energy by the dashed lines, and the complex solutions with complex energy by the dotted lines extending down from the cusps $c$ (only the real part of the energy has been graphed). The critical coupling $\lambda_{\text{crit}}$ occurs when the energy $E_0(\lambda)$ of the cusp crosses the sphaleron line, i.e. when $E_0(\lambda_{\text{crit}}) = E_s(\lambda_{\text{crit}})$. Finally, it should be noted that the equations are analytic (both in $\lambda$ and in $\beta$), and therefore the total number of solutions cannot change at the bifurcation points. Indeed, branches 1, 2 and 3 consist of two independent solutions each (with the same energy and action), whereas branch 4 consists of 4 solutions (in two conjugate pairs). By continuity in $\lambda$, these (unsuspected) new complex instanton solutions continue to exist even at lower $\lambda$. The physical consequences of these new solutions for rates of anomalous processes at finite energy in the electroweak theory are currently under investigation.

FIG. 3. The action (a) and energy (b) of the periodic instanton as a function of $\beta$ in the perturbative regime (small $\beta$). The numerical data points are denoted by the triangles, the leading order calculations (14)-(15) by the dashed lines, and the next to leading order calculations (16)-(17) by the solid lines.
To conclude, let us remark that the accuracy of our numerical calculations is high enough that they can provide a verification of the perturbative results for low energy, low period. In Fig. 3 we plot our numerical results for action and energy together with the results of the perturbative expansions to leading order (14)-(15) and next to leading order (16)-(17). The agreement with the next to leading order perturbative results is quite good.

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Note Added in Proof: Yaffe and Frost have reported similar findings in ref. [10].