The Schrodinger Cat Family in Attractive Bose Gases and Their Interference

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We show that the ground state of an attractive Bose gas in a double well evolves from a coherent state to a Schrodinger Cat like state as the tunneling barrier is decreased. The latter exhibits super-fragmentation as spin-1 Bose gas with antiferromagnetic interaction, which is caused by the same physics. We also show that the fragmented condensates of attractive and repulsive Bose gases in double wells lead to very different interference patterns.

Before proceeding, we shall comment on two recent related developments. There have been increasing efforts in recent years to create a Schrodinger cat state in condensed matter and atomic systems. The Schrodinger Cat state (or “Cat state” for short) is a superposition of two macroscopic quantum states. It is often used to illustrate the peculiarity of Quantum Theory, which admits such superpositions even though they have never been observed on the macroscopic scale. The usual explanation is that Cat states are highly unstable against entanglement with environment. Despite such difficulties, Cat states on the mesoscopic scale have recently been created. Since the discovery of BEC, there have been suggestions to produce “bigger” Cat states using atomic Bose condensates. These include using binary mixtures of Bose condensates, as well as performing projections on the coherent states. The attractive Bose gas in a double well is mathematically similar to the binary mixtures of Bose condensates, but is a much simpler system. As we shall see, it has Cat-like ground states over a wide range of parameters, a fact that does not seem to be generally appreciated.

In a separate development, recent studies of spin-1 Bose gases with antiferromagnetic interaction show that strict spin conservation will lead to fragmented ground states, which can be Fock like (with zero spin fluctuation) or superfragmented (with $N^2$ spin fluctuation), depending on the magnetization of the system. Although the superfragmented states of spin-1 Bose gas and attractive Bose gas in double wells assume very different forms, the origin of their formation and their fluctuations are identical. That it is caused by the existence of degenerate minima in the interaction energy in number space and the quantum fluctuations between them. In contrast, a Fock state is caused by a single deep minimum in the interaction energy which strongly suppresses number fluctuations. This mechanism can be seen in spin-1 Bose gas, and is best illustrated in the double well example below.

The Two Site Boson Hubbard Model: Consider the Hamiltonian

$$H = -t(a\dagger b + b\dagger a) + \frac{U}{2} [n_a(n_a - 1) + n_b(n_b - 1)], \tag{1}$$

with $n_a + n_b = N$, where $a$ and $b$ destroy Bosons at site (i.e. well) $a$ and $b$. $n_a = a\dagger a$, $n_b = b\dagger b$, $N$ is the total number of Bosons, $t > 0$ is the tunneling matrix element, and $U$ is the interaction between particles which has the same sign as the s-wave scattering length. Eq.(1) is meant to describe the actual system when reduced to the lowest doublet $(a \pm b)$. Although there are residual terms in the effective Hamiltonian in the reduction, and that $t$ and $U$ in eq.(1) have density dependence, we shall not consider such features because they do not affect the basic physics of the problem (i.e. the competition between tunneling and interaction). Instead, we shall focus on the simple model (eq.(1)) which deserves to be studied in its own right.
For a system with $N$ Bosons, the Hilbert space is \[ \{|\ell\rangle = (\frac{1}{\sqrt{N}} + \ell \frac{1}{\sqrt{N}}) \}, \] where $|N_a, N_b\rangle = a^{N_a} b^{N_b} |0\rangle/\sqrt{N_a! N_b!}$ is the state with $N_a$ and $N_b$ Bosons at site $a$ and $b$. We shall take $N$ even, a choice of convenience that has no effects on our results. If $|\Psi\rangle = \sum_{\ell} \Psi_{\ell} |\ell\rangle$ is an eigenstate with energy $E$, eq. (3) implies
\[ E\Psi_{\ell} = -(t_{\ell-1} \Psi_{\ell-1} + t_{\ell} \Psi_{\ell+1}) + U\ell^{2} \Psi_{\ell}, \tag{2} \]
where $t_{\ell} = \sqrt{\frac{N}{(\ell + 1)^{1/2}} - \ell}$. It is clear that $t_{\ell}$ strongly favors large amplitudes near $\ell = 0$. This is reflected in the non-interacting ground state, $|C\rangle = (2^{-N} N!)^{-1/2} (a^1 + b)^N |0\rangle$, which shows that $\Psi_{\ell}^C$ is a Gaussian centered at $\ell = 0$ with a width $\sigma_{\ell} = \sqrt{N/2}$ since $\Psi_{\ell}^C \approx (\frac{1}{\sqrt{\pi}})^{1/4} e^{-\ell^2/2 N}$ for $N \gg 1$. When $U > 0$, the potential $U\ell^2$ suppresses particle fluctuations between $a$ and $b$, narrowing the Gaussian toward the delta-function $\delta_{\ell=0}$, (corresponding to the Fock state $|N/2, N/2\rangle$) [10,1].

The ground state for general $U > 0$ (referred to as the “repulsive family”) is
\[ \Psi_{\ell}^{(-)} = G_{\sigma}(\ell), \quad \frac{2}{\sigma_{\ell}^2} = \left( \frac{1}{N^2} + \frac{U}{tN} \right)^{1/2} \tag{3} \]
where $G_{\sigma}(\ell) = \sum_{\ell} e^{-\ell^2/2 \sigma^2}/(\pi^{\sigma^2}/4)$. Eq. (3) follows from the fact that in the continuum limit eq. (2) near $\ell = 0$ is the Schrodinger equation of a simple harmonic oscillator. Its validity can also be verified numerically.

For $U < 0$, the potential $U\ell^2$ favors large amplitudes at $\ell = \pm N/2$. It competes strongly with tunneling and tends to split the Gaussian $\Psi_{\ell}^C$ at $U = 0$ into two. These features are found in the numerical solutions of eq. (2), which also show to a good approximation, the ground state (referred to as the “attractive family”) is
\[ \Psi_{\ell}^{(+)} = Q(\Psi_{\ell}^C + \Psi_{\ell}^R) \tag{4} \]
where $Q$ is a normalization constant, $\Psi_{\ell}^{(R)} = G_{\sigma}(\ell - A)$, $\Psi_{\ell}^{(L)} = (\frac{1}{\sqrt{N}} \Psi_{\ell}^C)$ (See figure 1). In other words, the ground state is a superposition of the coherent/Fock-like states $|L\rangle$ and $|R\rangle$, $|\Psi_{\ell}^{(-)}\rangle = Q(|L\rangle + |R\rangle)$,
\[ |L\rangle = \sum_q G_{\sigma}(q)|q\rangle N_{a,N_{-}}, \quad |R\rangle = \sum_q G_{\sigma}(q)|q\rangle N_{b,N_{+}}, \]
where $|q\rangle_{N_{a},N_{-}} \equiv |N_{a} + q, N_{-} - q\rangle$, $N_{k} = \frac{N}{2} \pm A$, and $N_{+}$ is the average number of $a$ (or $b$) particle in $|L\rangle$ (or $|R\rangle$). We also found numerically that the quantities $A^* = A/(N/2)$ and $\sigma^* = \sigma/\sigma_c$ ($\sigma_c = \sqrt{N/2}$) are functions of the combination $UN/t$ only [10] with a dividing behavior at $UN/t = -2$ as shown in figure 2.

As $UN/t$ increases below $-2$, we have $A \to N/2$ and $\sigma \to 0$. The system is driven towards the “extreme” Cat state $|\text{Cat}\rangle^* = (|N,0\rangle + |0,N\rangle)/\sqrt{2}$. In fact, the overlap between $|L\rangle$ and $|R\rangle$ vanishes rapidly when $A > \sigma$, or $A^*/\sigma^* > \sqrt{2/N}$. For a system with $N = 1000$ particles, we find that $|L\rangle$ and $|R\rangle$ cease to overlap when $UN/t < -2.1$. The system is essentially a superposition to two non overlapping mesoscopic condensates.

**Fragmentation and Super-fragmentation:** That interaction will lead to fragmentation can be seen from that the single particle density matrices of both $|\Psi^{(+)}\rangle$ and $|\Psi^{(-)}\rangle$, $\rho \equiv \left( \begin{array}{cc} (a^\dagger a) & (a^\dagger b) \\ (b^\dagger a) & (b^\dagger b) \end{array} \right)$, where $x = e^{-1/(4\sigma^2)}$ for $|\Psi^{(+)}\rangle$ with $\sigma$ given in eq. (3), and $x = (e^{-1/(4\sigma^2)}(1 - (2A)^2 + e^{-(A+1/2)/\sigma^2})/(1 + e^{-A^2/\sigma^2})$ for $|\Psi^{(-)}\rangle$ with $\sigma$ given in fig.2. The eigenvalues of $\rho$ are $\lambda_+ = (N/2)(1 \pm x)$, When $U > 0$, we have $x = 1$ for both signs of $U$, and $\rho$ has only one macroscopic eigenvalue ($\lambda_+ = N$ and $\lambda_- = 0$). This is expected since the ground state reduces to the coherent state $|C\rangle$. As $|U|$ increases, $x \to 0$ for both signs of $U$. The condensate becomes fragmented since both eigenvalues of $\rho$ becomes macroscopic, $\lambda_+, \lambda_- \to N/2$.

However, when examining the internal number fluctuations $\Delta N_{a}^2 \equiv \langle (N_{a} - \langle N_{a}\rangle)^2 \rangle$, one finds $\Delta N_{a}^2 = \sigma^2/2$ for $|\Psi^{(+)}\rangle$, which vanishes as $U/tN$ increases. In contrast, $\Delta N_{a}^2 = \frac{8}{N} \frac{1}{1 - e^{-x^2/2}}$ for $|\Psi^{(-)}\rangle$ which is of order $N^2$ since $A \sim N$ and $\sigma \sim \sqrt{N}$ for large $|U|/N/t$. The results of $\Delta N_{a}^2$ obtained from numerical solution of eq. (2) are shown in figure 2. We have plotted $\ln(4\Delta N_{a}^2)/\ln N$ instead of $\Delta N_{a}^2$ because the former assumes the simple value of 2 and 1 for the “extreme” Cat state $|\text{Cat}\rangle$ and the coherent state $|C\rangle$ respectively. Fig.2 shows that $\Delta N_{a}^2$ reaches a substantial fraction of its maximum value $N^2/4$ over the interval $\Delta(UN/t) = (-3, -2)$. Thus, from the viewpoint of achieving a superfragmented structure, it is not necessary to go deeply into the Cat regime, (i.e. $|U|/N/t > 1$).

At first sight, such an interval may seem physically irrelevant because for any finite interval $\Delta(UN/t)$, the corresponding range in $U/t$ vanishes as $1/N$ in the thermodynamic limit, and that the system is either deep in the Cat regime or is a coherent state depending on whether $U < 0$ or $U = 0$. This, however, is not true. The reason is that in general, the tunneling matrix element $t$ depends on an energy barrier $V_0$ in an exponential fashion, $\ln \propto -V_0$; (and $V_0$ is proportional to the intensity of the Laser producing the barrier). Thus, the range $\Delta V_0$ corresponding to the interval $\Delta(UN/t)$ is only proportional to $\ln N$, making the system highly tunable from one regime to another. Two other facts also make this transition region relevant. Firstly, quantum gases are mesoscopic instead of macroscopic systems, with $N < 10^6$ instead of $N \sim 10^{23}$. Secondly, recent experiments have shown that the scattering length of Rb$^{85}$ can be tuned to zero by varying the external magnetic field [13]. These show that it is possible to have a system with a small UN even for N as high as $10^9$. The fact that the ground state changes continuously from a coherent to a Cat-like structure over a wide range of parameter allows us to
explore how the phase coherence of the system as the crossover takes place.

**Interference of Attractive Bose Gas in the Superfragmented Regime:** As mentioned earlier, many authors have pointed out that there will be no distinctions between the interference patterns of a coherent state and a Fock state. The numerical evidence of this effect was given Javanainen and Yoo (JY) [4]. Later, using analogies with quantum optics, Castin and Dalibard (CD) simulate the particle collection process by the “beam splitter” operators $a \pm b$ and showed explicitly how the measurement process changes a Fock state into a coherent state. The exact spatial pattern, however, was not derived. In the following, we shall modify the calculation of CD to obtain the spatial interference pattern of the attractive family eq.(4). We shall see that the operators for particle collection are different from the “beam splitter” operators, and that our calculation when applied to the Fock state furnishes a derivation of JY’s result [4].

To illustrate the key features, it is sufficient to consider the one dimensional case. At time $t = 0$, the trap is turned off and the condensates at $a$ and $b$ begin to expand and to overlap. For simplicity, we shall assume the atoms expand as non-interacting particles for $t > 0$, which implies $\psi(x,t) = \sqrt{\frac{2\pi}{\sigma^2 t}} \int ds e^{im(x-s)^2/2\sigma^2} \psi(s)$. If $a$ and $b$ are Wannier states localized at $\pm x_o$, then we have $\psi(x) = \gamma (e^{-i\zeta(x)/2}a + e^{-i\zeta(x)/2}b)$, $\zeta(x) = (2\sigma_o M/\hbar t)x$, and $\gamma = \sqrt{\frac{\pi}{2\sigma^2 t}} e^{im(x^2 + s^2)/2\sigma^2}$. If the Bosons were photons, the operators $(a+b)^{k^+}(a-b)^{k^-}$ represent detecting different numbers of photons in different beam splitters. However, in an interference experiment, particle detections are products of $\psi(x,t)$, which are specific combinations of $a$ and $b$ rather than than products of $(a+b)^{k^+}(a-b)^{k^-}$.

Next we consider a series of particle detectors located at $x_i$, $i = 1, \ldots, D$. The joint probability of detecting a total of $k$ particles ($k << N$) with $k_i$ particles in the detector at $x_i$ is, (see Appendix),

$$\mathcal{P} \{ k_i \} = \frac{(N-k)!}{N!} \frac{k!}{k_1! \ldots k_D!} ||\hat{O}(\Psi)_N||^2, \quad (6)$$

where $\hat{O} = \prod_i \hat{\psi}^{k_i}(x_i)$ removes $k_i$ particles at $x_i$, $|\Psi>_N$ is a normalized state with $N$ particles, and $\sum_{i=1}^D k_i = k$. The measured density at $x_i$ is given by the most probable set $\{ \tilde{k}_i \}$ which optimizes $\mathcal{P} \{ k_i \}$, i.e. $n(x_i) = \tilde{k}_i$. In case there are many such sets, the measured density will change from experiment to experiment, as each experiment samples a different optimal set.

For CAT like states (eq.(3)), we have $\mathcal{P} \{ k_i \} = Q^2 [\mathcal{P}_L \{ k_i \} + \mathcal{P}_R \{ k_i \}] + \mathcal{P}_{LR} \{ k_i \}$, where $\mathcal{P}_L$ and $\mathcal{P}_R$ are eq.(3) evaluated at $|\Psi>_L$ and $|\Psi>_R$, and $\mathcal{P}_{LR}$ is eq.(3) with the norm replaced by $\langle L|\hat{O}|\hat{O}|R \rangle + c.c.$ Since $\mathcal{P}_{LR}$ depends on the overlap of $|L⟩$ and $|R⟩$, it is non-vanishing only within the range of $U/N$ such that $A < \sigma$. For $A > \sigma$, $\mathcal{P}$ is dominated by $\mathcal{P}_L$ and $\mathcal{P}_R$, and the interference pattern $n(x_j)$ is proportional to $\tilde{k}_j^L + \tilde{k}_j^R$, where $\{ \tilde{k}_j^L \}$ and $\{ \tilde{k}_j^R \}$ are the optimal set of $\mathcal{P}_L$ and $\mathcal{P}_R$ respectively.

To calculate $\langle \tilde{k}_j^L \rangle$, we consider the coherent state

$$|\alpha,\beta>_N = \frac{1}{\sqrt{N!}} (ua^0 + vb^0)^N |0>_N \quad (7)$$

where $u \equiv e^{-\alpha^2/2 \sigma^2}, v = e^{i\alpha^2/2 \sigma^2} \cos \frac{\beta^2}{2 \sigma^2} \equiv N+/N$. It follows from eq.(8) that $\langle a^0|a^0⟩ = N_+, \langle b^0|b^0⟩ = N_-, N_+ = N/2 \pm A$. Eq.(8) has the expansion

$$|\alpha,\beta>_N = \sum_q \Psi_q^{(\alpha)} e^{-i(A+q)\alpha} |\tilde{q}>_{N_+,N_-} \quad (8)$$

where $\Psi_q^{(\alpha)} = G_{\sigma_o}(q)$ with $\sigma_q^2 = 2N, N_-/N$ up to $1/N_+, 1/N_-$ corrections. Inverting eq.(8), we can express $\langle \tilde{q}|_{N_+,N_-}$ and hence $|L⟩ [\text{eq.(3)}]$ in terms of $|\alpha,\beta>_N$. We then have

$$\hat{O}|L⟩ = \Gamma \sum_q f(q) \int_{-\pi}^{\pi} \frac{d\alpha}{2\pi} e^{i\alpha(A+q)} \prod_{i=1}^D |W_{x_i}(\alpha)|^{k_i} |\alpha,\beta>_{N-k} \quad (9)$$

where $f(q) = G_{\sigma_o}(q)/G_{\sigma_o}(q), \ W_{x}(\alpha) = e^{i\zeta(x)/2} + e^{-i\zeta(x)/2}v$, and $\Gamma = \sqrt{N!} (N-k)$. Since $|\alpha,\beta>_{N-k}$ is close to the Fock state $e^{i\alpha_0 N_{+}}|N_+,N_->$, and since the range of $q$ is restricted to $1/\sigma$, the combination $e^{i\alpha(A+q)}|\alpha,\beta>_{N-k}$ varies slowly with $\alpha$. Since $k_i >> 1$, one can use the method of steepest descents to determine the phase angle $\alpha^*$ that maximizes the magnitude $\prod_{i=1}^D |W_{x_i}(\alpha)|^{k_i}$, and we have $\hat{O}|L⟩ \propto |\alpha^*⟩$. This shows that the measurement process (specified by the set $\{ \tilde{k}_i \}$ projects the state $|L⟩$ into the coherent state $|\alpha^*,\beta⟩$.

To calculate $\mathcal{P}_L$, we note that $|\alpha',\beta,\alpha,\beta⟩ \equiv \exp[-i(N-k)\sin^2(\beta)|\alpha^* - \alpha|^2 + i\zeta(N-k)\sin^2(\alpha)]$ for $N-k >> 1$. We then have

$$\mathcal{P}_L(\{ k_i \}) = \eta^k \int \frac{d\alpha}{2\pi} |\tilde{f}(\alpha)|^2 \prod_{i=1}^D \frac{|W_{x_i}(\alpha)|^{2k_i}}{k_i!} \quad (10)$$

where $\tilde{f}(\alpha) = \sum_q f(q) e^{i\alpha q} = \sqrt{\sigma_o/\sigma} \sum_q e^{-q^2/2\sigma^2 + i\alpha q}$, where $\sigma_o^2 = \sigma^2 - 2\sigma^2$, and $\eta = \sqrt{\frac{8\pi}{(N-k)\sin^2(\beta)}}$. As the ground state becomes more Cat-like, $\sigma \rightarrow 0$, $\tilde{f}(\alpha)$ has a weak $\alpha$ dependence. To find the optimal set $\{ \tilde{k}_i \}$, we rewrite the $k_i$ in eq.(11) using Stirling formula, and optimize the product in eq.(10) using method of steepest descent subject to the constraint $\sum_{i=1}^D k_i = k$. One then obtains $\tilde{k}_i\approx |W_{x_i}(\alpha^*)|^2$, or

$$\tilde{k}_i = \lambda \left[ 1 + \sin(2\beta) \cos \left( \frac{2Mx_i\alpha}{\hbar t} - \alpha^* \right) \right], \quad (11)$$

where $\lambda$ is a constant. The stationary condition for $\alpha^*$ can be derived in a straightforward manner and will not
be presented here. Repeating the calculation for \( \mathcal{P}_R \) we find \( \bar{\mathcal{P}}_k = \bar{\mathcal{P}}_b^L \). The measured density \( n(x_j) = \bar{\mathcal{P}}_j^L + \bar{\mathcal{P}}_b^R \) therefore consists of a uniform background and a sinusoidal oscillation with wavevector \( \hbar^{-1}M(2x_0/t) \). The amplitude of oscillation \( \sin \beta \) vanishes as the systems becomes more Cat-like, since \( \beta \to 0 \) as \( A \to N/2 \). This is in contrast to the repulsive case, where the interference pattern is independent of the barrier height which causes the system to fragment into a Fock state.

Appendix. Eq. (3) was derived in the Appendix in \[5\] using continuous quantum measurement theory. It can also be derived from the following elementary considerations: For a complete set of states (labelled by \( p \)) with creation operator \( \{ A_p^\dagger \} \), the probability of detecting a particle in state \( i \) in an \( N \)-particle system \( |\Psi\rangle \) is
\[
\mathcal{P}^{(1)} = \frac{\langle A_i^\dagger A_i \rangle_\Psi}{N(N-1)_\Psi} = \frac{|\langle A_i |\Psi\rangle|^2}{(N||\Psi||)^2}.
\]
The probability of detecting a second particle in state \( i \) after the first particle is detected (provided that the state evolves very little between detections) is
\[
\mathcal{P}^{(2)} = \frac{|\langle A_i^\dagger A_i |\Psi\rangle|^2}{(N-1)||\Psi||^2}.
\]
The joint probability of detecting \( k \) particles in state \( i \) is
\[
\bar{\mathcal{P}}_k = \frac{\bar{\mathcal{P}}^{(k-1)}_i}{N^k - \mathcal{P}^{(k-1)}_i} = \frac{\bar{\mathcal{P}}^{(k-1)}_i}{N^k - \mathcal{P}^{(k-1)}_i}.
\]
Eq. (3) is obtained by generalizing the detection to more than one states and with \( A_i \) replaced by \( \psi(x_i) \). The combinatoric factors in \( k_i \) accounts for the arbitrary ordering in the detection of different \( i \) states.

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\[13\] For attractive Bose gas, we shall consider the density range where the condensate in each well remains stable.

Caption

Figure 1. Numerical solution of \( \Psi_y(x) \) calculated from eq. (3) for different \( UN/t \) for a system with \( N = 1000 \) particles. The results for \( U < 0 \) can be well fitted by the functional form in eq. (4) with parameters \( A \) and \( \sigma \) shown in figure 2. For \( UN/t < -2 \), \( \Psi_y(x) \) begins to split up into two Gaussians. The split-up is complete for \( UN/t \approx -2.1 \).

Figure 2. With the solutions of eq. (4) for \( U < 0 \) well fitted by eq. (4), we find that for different \( (N, U, \text{ and } t) \), each of the parameters \( A^*, A/(N/2), \sigma/\sigma_c, \) and \( \ln(4\Delta N_0^2)/\ln N \) when plotted against \( UN/t \) falls into a single curve. Note that \( \ln(4\Delta N_0^2)/\ln N = 2 \) and 1 for the Cat state \( |N, 0\rangle \pm |0, N\rangle \sqrt{2} \) and the coherent state \( |C\rangle \) respectively. At \( UN/t = -2, \ln(4\Delta N_0^2)/\ln N = 1.32 \).
\[ \Psi_\ell \]

\[ \ell / N \]

-4.0
-2.5
-2.1
0.0

\( UN/t = -2 \)
\[ \log\left( \frac{4\Delta N^2}{\Delta} \right) / \log(N) \]

\[ A^\ast = \frac{2A}{N} \]

\[ \sigma^\ast = \frac{\sigma}{\sigma_c} \]