Anytime Inference in Valuation Algebras

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Abstract. The novel contribution of this work is the construction of generic inference algorithms in a generic framework, which automatically gives us instantiations in many useful domains. We also show that semiring induced valuation algebras, an important subclass of valuation algebras are amenable to anytime inference. Anytime inference, and inference algorithms in general have been a well-researched area in the last few decades. Inference is an important component in most pattern recognition and machine learning algorithms; it also shares theoretical connections with other branches of computer science like theorem-proving. Anytime inference is important in applications with limited space, for efficiency reasons, such as in continuous learning and robotics. In this article we construct an anytime inference algorithm based on principles introduced in the theory of generic inference; and in particular, extending the work done on ordered valuation algebras \(^9\).

Keywords: Approximation; Anytime algorithms; Resource-bounded computation; Generic inference; Valuation algebras; Local computation; Binary join trees.

1 Introduction

The inference problem is one of the most-important and well-studied problems in the field of statistics and machine learning. Inference can be considered as the (1) combination of information from various sources, which could be in the form of probability distributions from a probabilistic graphical model \(^{12}\), belief functions in Dempster-Shafer theory \(^{3,8}\) or tables in a relational database; and (2) subsequent focusing or projection to variables of interest, which corresponds to projection for variables in probabilistic graphical models, or a query in the relational database. Our work is based on the theory of generic inference \(^{17}\) which abstracts and generalises the inference problem across these different areas.

The utility of generic inference can be understood as an analogue to sorting, which is agnostic to the specific data type, as long as there is a total order. Generic inference generalises inference algorithms by abstracting the essential components of information in an algebraic structure. In \(^{13}\), an algorithm was defined which solved the inference problem on Bayesian networks, using a technique called local computation. It was noted in \(^{23}\) that the same algorithm could be used to solve the inference problem on belief functions, and a sufficient set of axioms were proposed for an algebraic framework that is necessary for the generic inference algorithm. This was extended by Kohlas into a theory of valuation algebras, and a computer implementation of inference over valuation algebras along with concrete instantiations was developed in \(^{16}\).

Generic inference as formulated in \(^{17}\) solves the inference problem in the exact case. As exact inference is an \#P-hard problem \(^{25}\), in practice, we need frameworks for approximate inference. Approximation schemes exist for specific instances of valuation algebras (probability potentials \(^{4}\), belief potentials \(^{9}\)); as well as for the generic case \(^{5}\), but there is no such generic framework for anytime inference. In this paper, we extend the approximate inference framework in \(^{5}\) to support anytime inference.

In anytime algorithms, instead of an algorithm terminating after an unspecified amount of time with a specific accuracy, we are able to tune the accuracy via a parameter passed to the algorithm. The algorithm can also be designed to be interruptible, gradually improving its accuracy until terminated by the user. Such algorithms are important in online learning where new data is being streamed in \(^{24}\), in intelligent systems, decision making under uncertainty \(^{7}\) and robotics \(^{28}\) where due to the limitation of interacting in real-time there may not be sufficient time to compute an exact solution. We shall consider interruptible anytime algorithms which can be interrupted at any time and the approximation can be improved by resuming the algorithm. This affords the greatest flexibility from the user’s perspective, with applications of such algorithms to real-time systems such as sensor networks and path planning.

Table 1 notes the previous work done in the area of inference algorithms, in both the generic case and for the specific case of probability potentials, and situates our work in context.

| inference  | generic | probability potentials |
|-----------|---------|------------------------|
| exact     | \(^{17}\) | \(^{15}\) |
| approximate | \(^{5}\) | loopy belief propagation, \(^{2}\) |
| anytime   | [our work] | \(^{19}\) |

Table 1: Our work, in relation to various inference algorithms and frameworks.

We note that the successive rows in the above table refine upon the previous one, and include it; the approximate inference framework can also perform exact inference, and the anytime inference framework presented here gives an approximate solution which incrementally improves with time, converging on the solution obtained from exact inference given sufficient time.

This article is divided into the following sections. Section 2 reviews the framework of valuation algebras and ordered valuation algebras. Section 3 introduces our extension to ordered valuation algebras to support anytime inference, and proves soundness and completeness theorems for anytime inference. Section 4 describes instances of the framework, including its application to anytime inference in semiring-induced valuation algebras. Section 5 gives a complexity analysis of the algorithm. Section 6 shows implementation results of anytime inference on a Bayesian network. Section 7 concludes.

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2 Valuation Algebras

Valuation algebras are the core algebraic structure in the theory of
generic inference. In a valuation algebra, we consider the various
pieces of information in an inference problem (conditional probabil-
ity distributions, belief potentials, relational database tables, etc.) as
elements in an algebraic structure with a set of axioms. We review the
axioms of valuation algebra [17] below, preceded by some remarks
on notation.

All operations in the valuation algebra are defined on elements
denoted by lowercase Greek letters: \( \phi, \psi, \ldots \). We can think of a val-
uation as the information contained by the possible values of a set
of variables, which are denoted by Roman lowercase letters (with
possible subscripts): \( x, y, \ldots \) and denote sets of variables by uppercase
letters: \( S, T, \ldots \). Each valuation refers to the information contained in
a set of variables which we call the domain of a valuation, denoted by
\( d(\phi) \) for a valuation \( \phi \). For a finite set of variables \( D, \Phi_D \) denotes
the set of valuations \( \phi \) for which \( d(\phi) = D \). Thus, the set of all possible
valuations for a countable set of variables \( V \) is

\[
\Phi = \bigcup_{D \subseteq V} \Phi_D
\]

If \( \tilde{D} = \mathcal{P}_D(V) \) the finite powerset of \( V \), and \( \Phi \) the set of
valuations with domains in \( \tilde{D} \); we define the following oper-
ations on \( (\Phi, \tilde{D}) \): (i) labeling: \( \Phi \rightarrow \tilde{D}; \phi \rightarrow d(\phi) \)
(ii) combination: \( \Phi \times \Phi \rightarrow \Phi; (\phi, \psi) \rightarrow \phi \otimes \psi \) (iii) projection:
\( \Phi \times \tilde{D} \rightarrow \Phi; (\phi, X) \rightarrow \phi_X \) for \( X \subseteq d(\phi) \).

These are the basic operations of a valuation algebra. Using the
view of valuations as pieces of information which refer to questions
as valuations, the labelling operation tells us which set of variables
the valuation refers to; the combination operation aggregates the
information, and the projection operation focuses the information on a
particular question (query) of interest. Projection is also referred to as
focusing or marginalization. The following axioms are then imposed
on \( (\Phi, \tilde{D}) \):

(A1) Commutative semigroup: \( \Phi \) is associative and commutative
under \( \otimes \).

(A2) Labeling: For \( \phi, \psi \in \Phi \), \( d(\phi \otimes \psi) = d(\phi) \cup d(\psi) \).

(A3) Projection: For \( \phi \in \Phi, X \subseteq \tilde{D} \) and \( X \subseteq d(\phi) \), \( d(\phi_X) = X \).

Alternatively this is equivalent to the following elimination opera-
tion, \( \phi_X = \phi \setminus d(\phi_X) \) where all the variables except those in \( X \) are eliminated:

(A4) Transitivity: For \( \phi \in \Phi \) and \( X \subseteq Y \subseteq d(\phi) \), \( d(\phi_Y) \setminus X = \phi_X \).

(A5) Combination: For \( \phi, \psi \in \Phi \) with \( d(\phi) = X, d(\psi) = Y \) and
\( Z \subseteq D \) such that \( X \subseteq Z \subseteq X \cup Y \), \( d(\phi \otimes \psi) \setminus Z = \phi \otimes \psi \setminus Z \).

(A6) Domain: For \( \phi \in \Phi \) with \( d(\phi) = X \), \( \phi_X = \phi \).

For the intuitive reading of these axioms, we refer the reader to
[17][23].

Before proceeding to approximate inference, we formally define
the inference problem:

Definition 1. The inference problem is the task of computing
\[
\phi_X = (\phi_1 \otimes \cdots \otimes \phi_n)_X
\]

for a given knowledgebase \( \{ \phi_1, \ldots, \phi_n \} \subseteq \Phi \); domain \( X \) is the query
for the inference problem.

Next we consider approximate inference. Existing approximation
schemes, like the mini-bucket scheme [2] are either not general
enough or do not provide a reliable measure of the approximation and
how to improve the approximation in an anytime algorithm. In this
article, we have used the ordered valuation algebra framework defined
in [8] as a basis for constructing an anytime algorithm. We thus review
the extra axioms of the ordered valuation algebra framework, which
introduces the notion of a partial order into the valuation algebra, and
defines a partial combination operator \( \otimes \) to construct approximate
inference algorithms.

Firstly we define a relation \( \succeq \) which represents an information
ordering. If \( \phi, \phi' \) are two valuations, then \( \phi \succeq \phi' \) means that \( \phi \) is
more complete than \( \phi' \). Intuitively, the information contained in \( \phi \) is
more informative and a better approximation than the information
contained by \( \phi' \); generally this means \( \phi' \) has a more compact or sparse
representation than \( \phi \). Furthermore, we assume that this relation is a
partial order. It is also reasonable to assume that approximations
are only valid for valuations with equal domains; thus \( \phi \succeq \phi' \) implies
\( d(\phi) = d(\phi') \) for all \( \phi, \phi' \in \Phi \). Thus \( \succeq \) actually defines separate
completeness relations \( \succeq \) for each sub-semigroup \( \Phi_D \).

We also impose the condition of each sub-semigroup \( \Phi_D \) having a
zero element, denoted by \( n_D \), where \( \phi \otimes n_D = n_D \otimes \phi = n_D \). For
notational simplicity we shall also denote the neutral element by
\( \otimes \) (without a subscript), denoting the appropriate neutral element
corresponding to a particular domain.

An ordered valuation algebra is still a valuation algebra, so it retains
all the axioms (A1)-(A6) introduced previously. The additional axioms
are about how \( \succeq \) behaves under combination and marginalization:

(A7) Partial order: There is a partial order \( \succeq \) on \( \Phi \) such that \( \phi \succeq \phi' \)
implies \( d(\phi) = d(\phi') \) for all \( \phi, \phi' \in \Phi \).

(A8) Zero element: We assume that the zero element for the combi-
nation operation, \( n_D \) is the least element of the approximation order
\( \geq_D \) for all \( D \subseteq V \). Also, since zero elements for a particular domain
are unique, \( n_D \otimes n_D = n_D \cup n_D \) for \( D_1, D_2 \subseteq V \). Also, \( n_D \otimes D \subseteq D \).

(A9) Combination preserves partial order: If \( \phi_1, \phi'_1, \phi_2, \phi'_2 \in \Phi \) are
valuations such that \( \phi_1 \succeq \phi'_1 \) and \( \phi_2 \succeq \phi'_2 \), then \( \phi_1 \otimes \phi_2 \succeq \phi'_1 \otimes \phi'_2 \).

(A10) Marginalisation preserves partial order: If \( \phi, \phi' \in \Phi \) are valuations such that \( \phi \succeq \phi' \), then \( \phi_X \succeq \phi'_X \) for all \( D \subseteq d(\phi) = d(\phi') \).

Definition 2. the time-bounded combination operator [8] \( \otimes_t : \Phi \times \Phi \rightarrow \Phi \) is used to approximate the exact computation during
the propagation phase. \( \otimes_t \) performs a partial combination of two val-
uations within time \( t \) units, where \( t \in \mathbb{R}^+ \). The following properties are
satisfied by \( \otimes_t \):

(R1) \( \phi_1 \otimes \phi_2 \succeq \phi_1 \), \( \otimes_1 \phi_2 \).
(R2) \( \phi_1 \otimes_t \phi_2 \succeq \phi_1 \otimes \phi_2 \) for all \( t' > t \).
(R3) \( \phi_1 \otimes_0 \phi_2 = n_D(\phi_1) \cup n_D(\phi_2) \).
(R4) \( \phi_1 \otimes_{\omega} \phi_2 = \phi_1 \otimes \phi_2 \).

Definition 3. Such a system \( (\Phi, V, \geq, \otimes, \otimes_t) \) of valuations \( \Phi \),
variables \( V \), a completeness relation \( \geq \) and a time-bounded combination
operation \( \otimes_t \) is called an ordered valuation algebra, if the labeling
operations \( d, \otimes \), and marginalization \( \downarrow \) satisfy
(A1)-(A10).

Definition 4. A binary join tree (BJT) \( N = (V,E) \) corresponding to a knowledgebase \( \{ \phi_1, \ldots, \phi_n \} \) is a covering junction tree for the
we refer to as (P2): (A1-A10) for ordered valuation algebras, with the properties (P1) and

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3 Anytime Ordered Valuation Algebras

In this section, we augment ordered valuation algebras in a structure we refer to as *anytime ordered valuation algebras*. We introduce the extension, and in the following section give examples of anytime ordered valuation algebras. The primary purpose of introducing anytime ordered valuation algebras is to develop an anytime inference algorithm within the framework of generic inference. Such extensions preserve the generic structure of valuation algebras, but add restrictions to simplify or add features to the inference algorithm; in another instance, valuation algebras were extended to weighted valuation algebras to study communication complexity [18].

Before defining anytime ordered valuation algebras, we define a couple of prerequisites; the composition operation and a truncation function.

**Definition 5.** The composition operator, \( \odot : \Phi \times \Phi \rightarrow \Phi \); \( (\phi', \phi'') \mapsto \phi \) combines valuations \( \phi' \) and \( \phi'' \) into a valuation \( \phi \) more complete than either \( \phi \geq \phi', \phi \geq \phi'' \). This is not to be confused with the combination operation \( \odot \) which generally combines valuations from different domains. The valuations being composed belong to the same approximation order \( \succeq_D \), where \( D = d(\phi') = d(\phi'') = d(\phi) \). It is natural in this context to consider whether composition should be a supremum operation. However, this cannot be assumed in general.

**Definition 6.** The truncation function \( \rho : \Phi \times \mathbb{R}^+ \rightarrow \Phi \) performs a truncation of the information contained in the valuation, according to the real valued parameter. Also, \( \rho \) is defined to be monotonically increasing with the real valued parameter, thus \( \rho(\phi, k) \geq \rho(\phi, k') \) whenever \( k \geq k' \).

The time-bounded combination operation \( \otimes_r \) can be recast such that truncation of the original pair of valuations followed by exact combination is equivalent to doing a time-bounded combination:

\[ \phi_1 \otimes_r \phi_2 = \rho(\phi_1, k_1) \otimes \rho(\phi_2, k_2) \]  

(3)

The parameters \( k_1, k_2 \) determining the truncated portions of \( \phi_1, \phi_2 \) will be important later in defining the partial valuations which will be used in the refinement algorithm for anytime inference. As \( k_1, k_2 \) are parameters that depend on the particular valuations \( \phi_1, \phi_2 \) and the time \( t \), this assumes a function \( K(\phi_1, \phi_2, t) = (k_1, k_2) \).

Following these two definitions, we extend the system of axioms (A1-A10) for ordered valuation algebras, with the properties (P1) and (P2):

(P1) The combination operation \( \odot \) distributes over \( \oplus \):

\[ (\phi_1' \oplus \phi_2') \oplus (\phi_1'' \oplus \phi_2'') = (\phi_1' \oplus \phi_1'') + (\phi_2' \oplus \phi_2'') \]

(4)

Here, \( \phi_1' \otimes \phi_2' = \rho(\phi_1, k_1) \otimes \rho(\phi_2, k_2) \) is a truncated valuation of the exact combined valuation \( \phi_1 \otimes \phi_2 \); **REFINE** is the part of the exact valuation that needs to be composed with the truncated valuation \( \phi_1' \otimes \phi_2' \) to complete the valuation. We also use the time-bounded operation **REFINE** for the same operation bounded by a time \( t \), with an analogous definition in terms of truncation functions as \( \otimes_r \) in equation [3]

\[ \text{REFINE}((\phi_1', \phi_2'), (\phi_1'', \phi_2'')) = \text{REFINE}'(\phi_1', \rho(\phi_1', k_1), \phi_2', \rho(\phi_2', k_2)) \]  

(5)

where the parameters \( k_1, k_2 \) are obtained from an assumed function \( K'(\phi_1, \phi_1', \phi_2', \phi_2') = (k_1, k_2) \).

(P2) The projection operation \( \downarrow \) distributes over \( \oplus \):

\[ (\phi' \oplus \phi'' \downarrow R) = \phi' \downarrow R \oplus \phi'' \downarrow R, D \subseteq d(\phi). \]  

(6)

We can now formally define the anytime ordered valuation algebra.

**Definition 7.** An *anytime ordered valuation algebra* \( \langle V, \Phi, d, \circ, \odot, \ominus, \downarrow R, \geq, \oplus, \downarrow D \rangle \) with the additional operations of composition \( \odot \) and the function \( \rho \), making the structure \( \langle V, \Phi, d, \odot, \ominus, \downarrow R, \geq, \oplus, \downarrow D \rangle \), which satisfies properties (P1) and (P2).

We show by construction that the composition operator \( \odot : \Phi \times \Phi \rightarrow \Phi \) with (P1), (P2) along with the truncation function \( \rho : \Phi \times \mathbb{R}^+ \rightarrow \Phi \) is sufficient to construct a refinement algorithm to improve the accuracy of a valuation.

To describe a refinement algorithm to improve upon the result provided by **INWARD**(N,t), we need to cache the partial valuations at each step so that we can use **REFINE** to improve upon them. We use a modified version of the propagation algorithm [22, 5], where \( \tau \) and \( \tau \) store the partial and complementary partial valuations respectively for a particular BJT node, where the complementary partial valuation \( \rho(\phi, k) \) is such that \( \rho(\phi, k) \oplus \rho(\phi, k) = \phi \). In the following procedures, \( \Delta(n) = d(n) \rightarrow d(P(n)) \) is the set of variables to be eliminated as we propagate messages to the parent node. To get the solution to the inference problem at the final step, we also define \( \Delta(n) = d(n) \rightarrow X \) where \( X \) is the query. There are \( r \) valuations in the knoweldbase resulting in \( r - 1 \) combination steps in the BJT. \( P(n) \) is the parent of \( n, \phi(n) \) is the valuation at node \( n, \phi(n) \) is the message from \( n \) to \( P(n); L(n), R(n) \) are the left and right nodes of \( n \) respectively and

\[ \text{next}(N) = \{ n \in N : \phi(n) = \text{nil}; \phi(L(n)) \neq \text{nil}, \phi(R(n)) \neq \text{nil} \} \]  

(7)

Both **INWARD**(N,t) and **REFINE**(N,t) return valuations which are the **(approximate)** solution to the inference problem.

1: procedure **INWARD**(N,t)
2: \( s \leftarrow r - 1; \)
3: initialise timer to \( t \) units.
4: for all \( n \in \text{leaves}(N) \) do \( \phi(n) \leftarrow \phi(n) \rightarrow \Delta(n) \)
5: while next(N) \( \neq \text{nil} \) do
6: select \( n \in \text{next}(N) \)
7: \( (k_1, k_2) \leftarrow K(\phi(L(n)), \phi(R(n))), t/s \)
8: \( \phi(n) \leftarrow \phi(L(n)) \ominus_{k_2} \phi(R(n)); D \subseteq d(\phi) \)
9: \( \tau(L(n)) \leftarrow \rho(\phi(L(n)), k_1) \)
10: \( \tau(R(n)) \leftarrow \rho(\phi(R(n)), k_2) \)
11: \( \tau(L(n)) \leftarrow \rho(\phi(L(n)), k_1) \)
12: \( \bar{\tau}(R(n)) \leftarrow \bar{\rho}(\phi(R(n)), k_2) \)
13: \( \phi(n) \leftarrow \phi(n)^{-\Delta(n)} \)
14: \( s \leftarrow s - 1 \)
15: \( t \leftarrow \operatorname{timer}() \)
16: end while
17: return \( \phi_n(\text{root}(N)) \)
18: end procedure

We can use the cached partial valuations in \( \tau \) and \( \bar{\tau} \) to define the refinement algorithm that follows in a similar manner to the algorithm in [9].

1: procedure \texttt{REFINE}(N, t)
2: initialise timer \( t \) units
3: \( s \leftarrow t - 1 \)
4: while next(N) \( \neq \emptyset \) do
5: select \( n \in \text{next}(N) \)
6: \( (k_1, k_2) \leftarrow K'(\tau(L(n)), \bar{\tau}(L(n)), \tau(R(n)), \bar{\tau}(R(n)), t/s) \)
7: \( v \leftarrow \texttt{REFINE}_i^j(t(L(n)), \bar{\tau}(L(n)), \tau(R(n)), \bar{\tau}(R(n))) \)
8: \( t \leftarrow \operatorname{timer}() \)
9: \( \bar{\tau}(L(n)) \leftarrow \bar{\rho}(v, \bar{\tau}(L(n)), k_1) \)
10: \( \tau(R(n)) \leftarrow \tau(R(n)) \oplus \rho(v, \tau(R(n)), k_2) \)
11: \( \bar{\tau}(L(n)) \leftarrow \bar{\rho}(v, \bar{\tau}(L(n)), k_1) \)
12: \( \tau(R(n)) \leftarrow \rho(v, \tau(R(n)), k_2) \)
13: \( \phi(n) \leftarrow \phi(n) \oplus v \)
14: \( \bar{\tau}(n) \leftarrow \bar{\tau}(n) \oplus \bar{\nu}^{-\Delta(n)} \)
15: \( s \leftarrow s - 1 \)
16: end while
17: return \( \phi_n(\text{root}(N)) \)
18: end procedure

This procedure refines the existing valuations in the binary join tree \( N \), taking at most \( t \) units. We ensure that the algorithm is interruptible in lines 9–12 using appropriate caching of partial valuations. A diagram of the truncation of a valuation is shown below to illustrate anytime refinement.

Here, and in the following proof, the notation \( \phi^k := \rho(\phi, k) \) and \( \phi_n := \bar{\rho}(\phi, k) \). We shall also abbreviate the notation \( \tau(L(n)) \) as \( \bar{\tau}_L \) and \( \bar{\tau}(L(n)) \) as \( \bar{\tau}_L \) (accordingly for \( R(n) \)), and \( \bar{\tau}(n) \) as \( \bar{\tau} \). The shaded region \( \phi^k \) is the part that has already been combined, while \( \phi_n \) represents the cached part that has not been combined yet. The dotted region represents the part of \( \phi \) that is yet uncomputed, due to truncated messages from child nodes; line 14 in \texttt{REFINE}(N, t) shrinks the uncomputed portion by extending \( \bar{\tau} \).

**Theorem 1** (Soundness of anytime inference). If \( \phi_{[0:k-1]} \) is the valuation returned after the following invocations: \texttt{INWARD}(N_0, t_0), \texttt{REFINE}(N_1, t_1), \ldots, \texttt{REFINE}(N_j, t_j) \), where \( N_{k+1} \) is the modified BJT with the cached valuations after step \( k \), then \( \phi_{[0:k-1]} \leq \cdots \leq \phi_{[0:k-2]} \leq \cdots \leq \phi \) where \( \phi \) is the exact valuation. The sequence becomes strictly increasing (upto the exact valuation) if \( t_i > t_k \) for all \( i > 0 \) where \( t_k \) is the minimum time required for the refinement to update one valuation.

**Proof.** We split the proof into two parts: (S1) proving that the sequence of valuations returned from successive calls to \texttt{REFINE} are partially ordered and (S2) showing the upper bound is the exact valuation, to which the partial valuations converge after a finite time.

Proving (S1) is trivial; for each node, \( \phi \) is updated once (line 13), thus \( \phi' = (\phi \oplus \nu) \geq \phi \), where \( \phi' \) is the valuation at node \( n \) after a call to \texttt{REFINE}. Using transitivity of the partial order, we obtain (S1). In the case when \( t_i > t_k \), at least one valuation is updated, resulting in \( \nu \gg \phi' \), which gives \( \phi' \geq \phi \).

To prove (2) we shall note the following statements

\( (T1) \quad (\phi_k)_m = (\phi^{k+m})_k \)
\( (T2) \quad (\phi_k \oplus \phi_l) = \phi \)
\( (T3) \quad \phi^k \oplus (\phi_l)_m = \phi^{k+m} \)
\( (T4) \quad (\phi_k)_m = \phi_k + m \)

For notational simplicity, only for the following proof, we denote \( \phi \psi := \phi \otimes \psi \) and \( + := \oplus \).

Since each node is only updated once, we can consider a particular node; let’s denote by \( \phi \) the valuation at node \( n \) after \texttt{INWARD}(N_0, t_0). If \( (k_1, k_2) \) are the parameters obtained from \( k' \) in \texttt{REFINE}(N_1, t_1) then the updated valuation \( \phi' = \phi + \bar{\tau}_L \phi_k + \bar{\tau}_R \phi_k + \bar{\tau}_L \phi_k + \bar{\tau}_R \phi_k \), where \( \phi = \tau_\text{INWARD} \).

Here we note that we can replace \((\bar{\tau}_L \phi)_k^m \) with their exact counterpart \((\bar{\tau}_L \phi)_k^m \), where we use the \( \phi \) to denote the exact valuation. This can be done as the truncation function is invariant under extension of the valuation to incorporate previously uncomputed information. Following this, we drop the superscript and use \( \bar{\tau}_L \) to denote \((\bar{\tau}_L \phi)_k^m \).

Then we consider a subsequent call, \texttt{REFINE}(N_2, t_2), \( \phi'' = \phi + \bar{\tau}_L \phi_k^m + \bar{\tau}_R \phi_k^m + \bar{\tau}_L \phi_k^m + \bar{\tau}_R \phi_k^m \), where the additional prime indicates the value for this iteration and \( (m_1, m_2) \) are the parameters obtained from \( k'' \).

From lines 9–12 in \texttt{REFINE} we get:

\[ \tau_L = \tau_L + \phi_k^m, \quad \tau_R = \tau_R + \phi_k^m, \]
\[ \bar{\tau}_L = (\bar{\tau}_L)_{k_1}, \quad \bar{\tau}_R = (\bar{\tau}_R)_{k_2} \]

Expanding \( \phi'' \) we get:

\[ \phi'' = \tau_L + \tau_R + \phi_k^m + \bar{\tau}_L \phi_k^m + \bar{\tau}_R \phi_k^m + \bar{\tau}_L \phi_k^m + \bar{\tau}_R \phi_k^m \]

Here we use (T1,T3) to simplify the expression. Note that this is the same form as \( \phi' = \phi + \bar{\tau}_L \phi_k^m + \bar{\tau}_R \phi_k^m + \bar{\tau}_L \phi_k^m + \bar{\tau}_R \phi_k^m \), with \( k_1 \to k_1 + m_1, k_2 \to k_2 + m_2 \). Thus, subsequent calls to \texttt{REFINE} will always result in \( \phi \) having the same form by induction. From the definition of the truncation function, \( \phi^k \geq \phi^{k+1} \) for \( k \geq k' \), from which (S1) follows as well, by preservation of partial order under combination and composition. To show (S2) we note that for finite valuations, there exists \( k \), such that \( \phi^k = \phi \). As the exponent is monotonically increasing with subsequent calls to \texttt{REFINE}, we shall eventually get \( \phi_{[0:k]} = \tau_L + \tau_R + \bar{\tau}_L + \bar{\tau}_R = (\tau_L + \bar{\tau}_L)(\tau_R + \bar{\tau}_R), \) the
exact valuation at node \( n \). Thus, we shall eventually get the exact valuation at the root after finite invocations of \textsc{Refine}.

\[ \square \]

Theorem 2 (Completeness of anytime inference). If \( \Phi_{t_0} \) is the valuation returned after the following invocations: \([\textsc{INWARD}(N,t_0), \textsc{Refine}(N',t_1)]\), where \( N' \) is the modified BJT with the cached valuations after the call to \textsc{INWARD}(N,t_0), then there exists a \( T \) such that for all \( t \geq T \) \( \Phi_{t_0} = \Phi_{t_0} = \Phi = (\otimes_{\Phi,\Psi} \Psi)^{|X|} \), the exact solution to the inference problem.

\[ \textbf{Proof.} \] We consider two cases:

\textbf{Case 1:} \textsc{INWARD}(N,t_0) has performed exact inference.

We shall show that \textsc{Refine}(N,t) is a null operation which does not change \( \Phi, \tau, \tau \); then the statement of the theorem follows if we set \( T = t_0 \).

\( \Phi' = \Phi \oplus \nu \) (line 13), so if we show \( \nu \equiv \emptyset \), we are done.

\[ \nu = \textsc{Refine}^{t_0}_{\ell_0}(\ell_0, \tau_0, \tau_0, \tau_0) \text{, but } \tau_0 = \tau_0 = \emptyset \text{ as } \tau \text{ represents the partial valuation that has not been combined, which is null for the exact inference case. Thus } \nu = \emptyset. \]

\textbf{Case 2:} \textsc{INWARD}(N,t_0) gives a partial result.

In general, \( \nu \) is also a partial valuation due to the time restriction. Since we are operating on finite datasets, the combination operation at a particular node in \textsc{Refine} takes a finite amount of time, say \( t_0 \).

Thus \textsc{Refine} \( t_0 \) at a node \( n \) is the exact refinement, making \( \Phi(n) \) exact after line 13, and thus \( m(\text{root}(N)) \) is exact after completion of the propagation. So we set \( T = \sum_{t \in \mathbb{N}} t_0 \) to get the time bound, such that for all \( t \geq T \) we get the exact result. \[ \square \]

4 Instances of anytime ordered valuation algebras

In the following sections, we describe instances of anytime ordered valuation algebras. Specifically we show that the important class of semiring induced valuation algebras, \([3]\), can be considered as anytime ordered valuation algebras. We also remark on the application of our framework to belief potentials.

4.1 Semiring induced valuation algebras

Semiring induced valuation algebras are a subclass of valuation algebras with several useful instances like probability potentials and disjunctive normal forms. We use the definition of semiring induced valuation algebras from \([3]\) and review the following standard notation. The semiring is denoted by \( \mathcal{S} = (\mathbb{A}, +, \times) \) with the semiring operations \(+, \times\) on a set \( \mathbb{A} \), where \( +, \times \) are assumed to be commutative and associative, with \( \times \) distributing over \(+\). Lowercase letters like \( x \) are variables, with a corresponding finite set of values for \( x \), called the \textit{frame} of \( x \) and denoted by \( \Omega_x \). Each \( \Omega_x \) also has an associated total order on its elements. If the frame has two elements, then it is the frame of a \textit{binary variable}. If the binary elements represent true or false, then we call the variable \textit{propositional}. For a domain \( D \subseteq V \) where \( V \) is the set of all variables in the system, the corresponding set of possible values becomes the Cartesian product \( \Omega = \prod \{ \Omega_x : x \in D \} \), whose elements \( x \in \Omega \) are called \textit{D-configurations} or \textit{D-tuples}. For a subset \( D' \subseteq D \), \( x^{D'} \in \Omega_{D'} \) is the projection of \( x \) to \( D' \). Where \( D \) is empty, we use the convention that the frame is a singleton set: \( \Omega_x = \{ \emptyset \} \). Any set of \( D \)-configurations can be ordered using a lexicographical order.

\textbf{Definition 8.} An \( \mathcal{S} \)-valuation \( \phi \) with domain \( D \) associates a value in \( A \) with each configuration \( x \in \Omega_D \), i.e. \( \phi \) is a function \( \phi : \Omega_D \rightarrow \mathbb{A} \).

The set of all such \( \mathcal{S} \)-valuations with a domain \( D \) is denoted by \( \Phi_D \), and the union of all such sets with \( D \subseteq V \) is the set of all \( \mathcal{S} \)-valuations \( \Phi \). The operations \( +, \times \) on \( A \) then induce a valuation algebra structure on \( (\Phi, \mathcal{P}(V)) \) where \( \mathcal{P}(V) \) is the finite powerset of the set of variables \( V \) \([3]\) Theorem 2], using the following definitions of combination and projection:

1. **Combination:** \( \otimes : \Phi \times \Phi \rightarrow \Phi \) defined for \( x \in \Omega_{D(\phi,\psi)} \) by \[ \Phi = \Phi \otimes \psi(x) = \Phi(\chi^{D(\phi)}) \times \psi(\chi^{D(\psi)}) \] \[ (8) \]
2. **Projection:** \( \setminus : \Phi \rightarrow \Phi \) defined for all \( \phi \in \Phi \) and \( T \subseteq d(\phi) \) by \[ \phi(T) = \sum_{x \in \Omega_T} \phi(x) \] \[ (9) \]

\textbf{Theorem 3.} Semiring induced valuation algebras, provided the underlying semiring has a zero element, form an ordered valuation algebra.

\[ \textbf{Proof.} \] To show semiring induced valuation algebras are an ordered valuation algebra, we have to show (A7-A10):

(A7) The preorder \( \geq \) is defined by \( \Phi \geq \Phi' \) iff \( \Phi(x) \geq \Phi'(x) \) for all \( x \in \Omega_{\phi(\psi)} \), where \( \geq_{\Omega} \) is the preorder on the semiring \([3] \text{ Prop. 1, p1362} \) defined as \( b \geq_{\Omega} a \) if \( a - b \) or there exists \( c \) such that \( a + c = b \), with \( d(\phi) = d(\phi') \) as it only makes sense to compare valuations on the same domain. However we need a \textit{partial order} for this axiom, which is possible if the additive monoid is positive, has a zero element and is cancellative:

\textbf{Lemma 4.} The preorder \( \preceq \) defined on a positive, cancellative, commutative monoid, \( (\mathbb{A}, +) \) with a zero element, is a partial order.

\[ \textbf{Proof.} \] A preorder implies \( a \leq b \iff a + c = b \). For a partial order, we need asymmetry: if \( a \leq b \) and \( b \leq a \), then \( a = b \).

\[ a \leq b \] implies there exists \( c \) such that \( a + c = b \); similarly there exists \( d \) such that \( b + d = a \); substituting gives us \( b + d + c = b + 0 \), the cancellative property implies \( d + c = 0 \) and the positivity property implies \( c = 0 \), implying \( a = b \), and we have a partial order. \[ \square \]

(A8) Zero element: Most common instances of semiring induced valuation algebras have a zero element. Specifically semirings with zero elements induce valuation algebras with the zero element \( n_0 \) such that \( n_0(x) = 0 \) for all \( x \in \Omega_D \).

(A9, A10) \textbf{Combination and marginalisation preserve partial order.} This follows from the fact that \( \times \) and \( + \) preserve partial order in the underlying semiring structure.

\[ \square \]

Having shown that semiring induced valuation algebras satisfy the ordered valuation algebra axioms (A7-A10) provided the underlying semiring has a zero element and the additive commutative monoid is
1. We denote the composition operator on semiring induced valuation algebras as $\phi \otimes \phi'\langle x \rangle = \phi(x) + \phi'(x)$, where $\phi$ is the configuration in $\Omega(\phi)$.

2. The function $\rho$ is defined on the semiring induced valuation algebra as $\rho(\phi, k) = \phi$ for the first $k$ (lexicographically ordered on $x$) elements of graph $\phi$; where $\text{graph}(\phi) = \{(x, \phi(x)) | x \in \Omega(\phi)\}$. For efficient implementation, we only store $(x, \phi(x))$ where $\phi(x) \neq 0$.

We also define the time-bounded combination operation $\phi_1 \otimes \phi_2$, where $L_{\phi_1} = ([x_1, \phi_1(x_1)], \ldots)$, and $L_{\phi_2} = ([y_1, \phi_2(y_1)], \ldots)$. $xy$ denotes the configuration in $\Omega(\phi_1 \cdot \phi_2)$ such that $(xy)^{\rho(\phi_1)} = x$ and $(xy)^{\rho(\phi_2)} = y$.

We define helper functions INSERT, which inserts a combination into the configuration space provided there is a common support and COMBINE-EXTEND which incrementally adds combinations into the configuration and updates the state, going from the state $\rho(\phi_1, i) \otimes \rho(\phi_2, j)$ to $\rho(\phi_1, i + i') \otimes \rho(\phi_2, j + j')$. Finally, we define COMBINE which performs the combination operation within the allocated time constraint.

---

**Theorem 5.** Semiring induced valuation algebras, provided the underlying semiring has a zero element, along with the composition operator and the truncation function defined above form an anytime ordered valuation algebra.

**Proof.** Semiring induced valuation algebras form an ordered valuation algebra as shown in Theorem III. To show that they also constitute an anytime ordered valuation algebra, we have to show properties (P1), (P2), i.e. combination and projection distribute over $\otimes$.

(P1) If $p_1 = p_1' \otimes p_1''$ and $p_2 = p_2' \otimes p_2''$, then we have to show that:

\[
p_1 \otimes p_2 = (p_1' \otimes p_2') \oplus (p_1' \otimes p_2') \oplus (p_1'' \otimes p_2') \oplus (p_1' \otimes p_2'').
\]

LHS applied to $x$ is $p_1(x) \times p_2(x^T)$, where $d(p_1) = S$ and $d(p_2) = T$.

RHS is $p_1(x) \times p_2(x) + p_1' (x') \times p_2(x) + p_1'' (x') \times p_2(x) + p_1' (x') \times p_2' (x') = LHS$ using distributivity of $\times$ over $\oplus$.

(P2) We have to show that if $p = p' \otimes p''$ that $p^{1D} = p'^{1D} \oplus p''^{1D}$, where $D \subseteq d(p)$. The LHS applied to $x$ is $p^{1D}(x) = \sum_{x^{1D}=x} p(x) = \sum_{x^{1D}=x} (p' \oplus p'')(x)$, and the RHS is

\[
(p'^{1D} \oplus p''^{1D})(x) = \sum_{x^{1D}=x} p'(x) + \sum_{x^{1D}=x} p''(x) = \sum_{x^{1D}=x} (p' \oplus p'')(x)
\]

where we use the associativity and commutativity of $\oplus$. □

As stated earlier, several common instances of valuation algebra can be considered as semiring induced. We present a couple of important examples below:

**Example 1.** Probability potentials are semiring induced valuation algebras on $\mathbb{R}^n$ with the semiring operations being the arithmetic addition and multiplication. Also known as arithmetic potentials, these describe (unnormalised) probability distributions, and thus inference in probabilistic graphical models.

**Example 2.** Disjunctive normal forms (abbreviated as DNF) are of the form $\alpha_1 \vee \alpha_2 \cdots \vee \alpha_m$ where $\alpha_i$ is of the form $x_1 \wedge x_2 \wedge \cdots \wedge x_k$ and $x_j$ is a literal; either a logical variable or its negation. All frames are binary reflecting true and false values respectively. DNF potentials
are induced by the semiring with $+$ and $\times$ being defined as $a + b = \max(a, b)$ and $a \times b = \min(a, b)$, which are equivalent to the definition of logical-or and logical-and.

There are many other examples of semiring induced valuation algebras, a detailed introduction to which can be found in [2]. In certain cases, the valuation algebra induced by the semiring has the idempotent property, i.e. $\rho \otimes \rho = \rho$; then we may use more efficient architectures for local computation such as the Lauritzen-Spiegelhalter architecture [10].

It is also pertinent to mention that for DNF potentials, one can alternately consider the valuation algebra over the formulae itself instead of the models [11], which simplifies computation extensively. This alternative representation is also an anytime ordered valuation algebra, but we have omitted the proof for the purposes of brevity.

4.2 Belief functions

Belief potentials are a generalisation of probability potentials to subsets of the configuration space in Dempster-Shafer’s theory of evidence [21]. The advantage of belief potentials over standard probability theory is in their ability to express partially available information in a manner not possible in probability theory. This is the reason for the usage of belief functions in sensor network literature, which involves fusion of information from various sources [14, 4, 20, 27].

For the instance of belief functions, with the composition operator defined as $[\phi \otimes \phi']_m(A) = [\phi]_m(A) + [\phi']_m(A)$, where $[\phi]_m$ is the mass function associated with the belief function $\phi$, our framework specialises to anytime inference in belief potentials as described in [6].

Theorem 6. Belief functions, along with the composition operator defined above, and the truncation operation $\rho(\phi, k)$ as the potential that contains the $k$ focal sets of $\phi$ with the highest masses, form an anytime ordered valuation algebra.

Proof. Belief functions already form an ordered valuation algebra [5], as well as permit anytime inference [6]. The anytime inference algorithm in [6] turns out to be a specific case of the generic anytime inference framework presented in this article. In particular if we denote $\oplus := +$ in their notation, and the truncation function $\rho(\phi, k) := \rho_k(\phi)$ then [6] Theorem 9.10] shows that belief functions also form an anytime ordered valuation algebra according to the axioms in Section 3. □

5 Complexity Analysis

The anytime inference algorithm presented in Section 3 hides the time complexity of approximate inference by restricting the accuracy of the valuations. While we don’t have an explicit control over the accuracy, we can improve it by allocating more time to the refinement algorithm. In this section, we take an alternative approach of focussing on accuracy and estimating the time complexity, which also allows us to use a tuning parameter which scales from zero accuracy (null valuations) to the valuation obtained after exact inference.

Since complexity of exact (and approximate) inference depends upon the complexity of the combination operation (usually the more time-consuming operation among combination and focussing), we consider the specific instance of semiring-induced valuation algebras. As there are $n$ valuations, $\phi_1, \ldots, \phi_n$, the resulting BJT $N$ will have $2n - 1$ nodes, $n$ of which are the valuations themselves at the leaves of the tree. We denote the maximum frame size of a variable in the semiring induced valuation algebra as $m := \max\{[\Omega_x], x \in V\}$. As we are representing semiring induced valuations in memory in terms of a tuple of the configuration and its associated value, the number of words required to represent the configuration is a key component in the time and communication complexity. The upper bound on the size of the configuration space for a valuation is thus $m^{|\Delta(\phi)|}$.

Definition 9. The approximation parameter $k$ is a tunable parameter that goes from 0 to $m^\omega$, where $\omega$ is the treewidth of the binary join tree $N$.

$m^\omega$ is the maximum size of the configuration space that we have to process during the inward or outward propagation phase of the Shenoy-Shafer algorithm. Now we can define the following.

Definition 10. The approximate combination operation $\otimes^k : \Phi \times \Phi \rightarrow \Phi$ is defined as combining the elements of the configuration space of the valuations in a semiring-induced valuation algebra, until we get $k$ resultant elements.

Lemma 7. The complexity of the approximate combination operator $\otimes^k$ is $O(k)$.

Proof. The worst-case scenario is when the configuration spaces are independent (no variables in common). Then there is no requirement for common support and we can take the pairwise multiplication of the elements of the configuration space, till we get $k$ elements, giving us $O(k)$ complexity. □

The INWARD-APPROX($N, k$) algorithm is defined similarly to the INWARD algorithm, with the instances of the time-bound combination operator $\otimes$, replaced by the approximate combination operator $\otimes^k$. In the following, $K(\phi, \psi, k)$ returns $(k_1, k_2)$ such that $\rho(\phi, k_1) \otimes \rho(\psi, k_2)$ has at most $k$ elements.

function INWARD-APPROX($N, k$)
for all $n \in \text{leaves}(N)$ do
    $\Phi_i(n) \leftarrow \Phi_i(n)^{-\Delta(n)}$
while next($N$) $\neq$ do
    select $n$ from next($N$)
    $(k_1, k_2) \leftarrow K(\Phi_i(L(n)), \Phi_i(R(n)), k)$
    $\Phi_i(n) \leftarrow \Phi_i(L(n)) \otimes^k \Phi_i(R(n))$;
    $\Phi_i(n) \leftarrow \Phi_i(n)^{-\Delta(n)}$
    $\tau(L(n)) \leftarrow \rho(\Phi_i(L(n)), k_1)$
    $\tau(R(n)) \leftarrow \rho(\Phi_i(R(n)), k_2)$
    $\tau(L(n)) \leftarrow \rho(\Phi_i(L(n)), k_1)$
    $\tau(R(n)) \leftarrow \rho(\Phi_i(R(n)), k_2)$
    $\Phi_i(n) \leftarrow \Phi_i(n)^{-\Delta(n)}$
    $s \leftarrow s - 1$
end while
end function

Theorem 8. The time complexity of INWARD-APPROX($N, k$) in the Shenoy-Shafer architecture, with the approximation parameter of $k$, given that there are $n$ valuations in the knowledgebase is $O((n - 1)k)$.  

Proof. There are \( n - 1 \) combinations as the number of combinations in the binary join tree is the same as the number of non-leaf nodes. As each combination has a complexity of \( O(k) \), we get a complexity of \( O((n - 1)k) \). Projection has a complexity of \( O(k) \) as there are \( k \) elements in the configuration space, so at most \( k - 1 \) summations, which is the case when we are marginalising to the null set (equivalent to eliminating all the variables), thus it does not change the asymptotic complexity.

We get the same time complexity for an analogous \textsc{refine-approx} algorithm, with a modification to lines 6–7 of \textsc{refine} to combine at most \( k \) elements.

Matching in the exact inference case. In the exact inference case, the complexity is known to be in the class \#P-hard. In the discussion on complexity [17], Kohlas and Pouly derive the estimate \( O(|V|, f(\omega)) \) where \( \omega \) is the treewidth, with \( f(x) = m^x \) for the case of semiring induced valuation algebras with variables having a upper bound frame size of \( m \). \( |V| \) is the number of vertices in the join tree. Substituting \( |V| = n, k = m^\omega \) in the time complexity \( O(n - 1)k \) and taking \( k = n^\omega \), we get the same time complexity as the exact inference case; thus the approximate time complexity obtained in terms of the approximation parameter \( k \) gives us a transition from \( k = 0 \) (null valuations, obtained when we set the \( t = 0 \) in \textsc{inward}(N, t)) to \( k = n^\omega \), the exact inference case.

Estimation of accuracy from elapsed time. It can be useful to derive an estimate of the accuracy of a valuation given the elapsed time of the algorithm in specific cases. Here, we shall consider the example of probability potentials. The time-bound combination operator combines the configurations with the largest weight first so that we get diminishing returns; the accuracy also depends on the sparsity of the probability potential. For simplicity we consider uniform distributions, where the weights are uniformly distributed in the configuration space. Then we can state the following:

\textbf{Lemma 9.} The fractional error estimate compared to the exact probability potential is

\[
\varepsilon(t) = 1 - \max \left(1, \frac{t}{m^\omega c(n - 1)}\right)
\]

where \( \omega \) is the treewidth, \( c \) is the constant time required to combine two elements in the configuration space, and \( n \) is the number of valuations in the knowledgebase.

\textbf{Proof.} As each configuration has an uniform weight, the accuracy of combination at the root node (which is the solution to the inference problem obtained from the inward propagation algorithm) is directly proportional to the allocated time which is on average \( t/(n - 1) \) as there are \( n - 1 \) combinations. Considering that each combination takes \( c \) units, and in the worst-case each configuration has weight \( 1/m^\omega \) (for a normalised potential; for unnormalised, this introduces a constant factor which is cancelled out by considering a fractional error estimate), we get the fractional error estimate as above.

As can be easily seen, \( \varepsilon(0) = 1 \), and \( \varepsilon(\phi((n - 1)m^\omega)) = 0 \) where \( \phi((n - 1)m^\omega) \) is the exact inference time complexity.

\section{Implementation}

We implemented the anytime inference algorithm using the Python programming language, on a Core i5 CPU with 4GB RAM. While we have shown anytime inference in a Bayesian network here, the framework, being generic, can be applied to other valuation algebras which satisfy the necessary axioms.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure1}
\caption{Anytime inference progress in the CHILD dataset}
\end{figure}

The figure shows progress of anytime inference on the CHILD dataset, which was used as a case study for exact inference in [1]. The progress is shown as a function of the fractional error estimate with time units (the actual total time for the series of successive refinements, up to the exact valuation is < 10s):

\[
\varepsilon(t) = 1 - \frac{\sum L_0}{\sum L_\phi}
\]

Here the sum is over the weights of the configurations \( L_\phi \) of a valuation \( \phi; L_0 \) is the valuation obtained at the root after time \( t \), and \( \phi \) is the exact valuation. As expected, the fractional error estimate converges to zero as we obtain the exact valuation.

\section{Conclusion}

In this work, we have shown that we can construct anytime algorithms for generic classes of valuation algebras, provided certain conditions are satisfied. We have also shown that the important subclass of semiring induced valuation algebras admit an anytime inference algorithm as they meet the aforementioned conditions. This is useful as semiring induced valuation algebras include important valuation algebra instances like probability potentials, DNF potentials and relational algebras, among others.

From a broader perspective, the advantage of operating in the generic framework of valuation algebras has been addressed before [17]; we can target a large class of problems using a unified framework; the inference or projection problem can be found in various forms: Fourier transforms, linear programming and constraint satisfaction problems. Enriching the valuation algebra structure through extensions is thus useful. Anytime inference in particular has a wide spectrum of applications. We also plan to study the applicability of our framework across these various domains in future work.

We are currently working on implementation of other instances of anytime ordered valuation algebras, as well as conducting a complexity analysis of the algorithm in a distributed setting using the Bulk Synchronous Parallel [26] model.
REFERENCES

[1] Robert G Cowell, A Philip Dawid, Steffen L Lauritzen, and David J Spiegelhalter. Probabilistic networks and expert systems, exact computational methods for Bayesian networks series: information science and statistics, 2007.

[2] Rina Dechter, ‘Bucket elimination: A unifying framework for probabilistic inference’, in Learning in graphical models, 75–104, Springer, (1998).

[3] Arthur P Dempster, ‘A generalization of Bayesian inference’, in Learning in graphical models, 75–104, Springer, (1998).

[4] Thierry Denoeux, ‘A neural network classifier based on Dempster-Shafer theory’, Systems, Man and Cybernetics, Part A: Systems and Humans, IEEE Transactions on, 30(2), 131–150, (2000).

[5] Rolf Haenni, ‘Ordered valuation algebras: a generic framework for approximating inference’, International Journal of Approximate Reasoning, 37(1), 1–41, (2004).

[6] Rolf Haenni and Norbert Lehmann, ‘Resource bounded and anytime approximation of belief function computations’, International Journal of Approximate Reasoning, 31(12), 103 – 154, (2002).

[7] Michael C Horsch and David Poole, ‘An anytime algorithm for decision making under uncertainty’, in Proceedings of the Fourteenth conference on Uncertainty in artificial intelligence, pp. 246–255. Morgan Kaufmann Publishers Inc., (1998).

[8] Audun Jessang and Simon Pope, ‘Dempster’s rule as seen by little colored balls’, Computational Intelligence, 28(4), 453–474, (2012).

[9] Juerg Kohlas and Nic Wilson, ‘Semiring induced valuation algebras: Exact and approximate local computation algorithms’, Artificial Intelligence, 172(11), 1360–1399, (2008).

[10] Jürg Kohlas, Information algebras: Generic structures for inference, Springer Science & Business Media, 2003.

[11] Jürg Kohlas, Rolf Haenni, and Serafín Moral, ‘Propositional information systems’, Journal of Logic and Computation, 9(5), 651–681, (1999).

[12] Daphne Koller and Nir Friedman, Probabilistic Graphical Models: Principles and Techniques, Adaptive Computation and Machine Learning Series, MIT Press, 2009.

[13] S. L. Lauritzen and D. J. Spiegelhalter, ‘Readings in uncertain reasoning’, chapter Local Computations with Probabilities on Graphical Structures and Their Application to Expert Systems, 415–448, Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, (1990).

[14] Robin R Murphy, ‘Dempster-Shafer theory for sensor fusion in autonomous mobile robots’, Robotics and Automation, IEEE Transactions on, 14(2), 197–206, (1998).

[15] J. Pearl, Reverend Bayes on inference engines: a distributed hierarchical approach, Cognitive Systems Laboratory, School of Engineering and Applied Science, University of California, Los Angeles, 1982.

[16] Marc Poully, ‘NENOK – A software architecture for generic inference’, International Journal on Artificial Intelligence Tools, 19(01), 65–99, (2010).

[17] Marc Poully and Juerg Kohlas, Generic Inference: A Unifying Theory for Automated Reasoning, Wiley-Blackwell, May 2011.

[18] Marc Poully and Jürg Kohlas, ‘Minimizing communication costs of distributed local computation’, Technical report, (2005).

[19] Fabio Tozeto Ramos and Fabio Gagliardi Cozman, ‘Anytime anyspace probabilistic inference’, International Journal of Approximate Reasoning, 38(1), 53–80, (2005).

[20] Kari Sentz and Scott Ferson, Combination of evidence in Dempster-Shafer theory, volume 4015, Citeseer, 2002.

[21] Glenn Shafer et al., A mathematical theory of evidence, volume 1, Princeton university press Princeton, 1976.

[22] Prakash P Shenoy, ‘Binary join trees for computing marginals in the Shenoy-Shafer architecture’, International Journal of approximate reasoning, 17(2), 239–263, (1997).

[23] Prakash P Shenoy and Glenn Shafer, ‘Axioms for probability and belief-function propagation’, in Classic Works of the Dempster-Shafer Theory of Belief Functions, 499–528, Springer, (2008).

[24] Ken Ueno, Xiaopeng Xi, Eamonn Keogh, and Dah-kye Lee, ‘Anytime classification using the nearest neighbor algorithm with applications to stream mining’, in Data Mining, 2006. ICDM’06. Sixth International Conference on, pp. 623–632. IEEE, (2006).

[25] Leslie G Valiant, ‘The complexity of enumeration and reliability problems’, SIAM Journal on Computing, 8(3), 410–421, (1979).

[26] Leslie G. Valiant, ‘A bridging model for parallel computation’, Commun. ACM, 33(8), 103–111, (August 1990).

[27] Dong Yu and Deborah Frincke, ‘Alert confidence fusion in intrusion detection systems with extended Dempster-Shafer theory’, in Proceedings of the 43rd annual Southeast regional conference-Volume 2, pp. 142–147. ACM, (2005).

[28] Shlomo Zilberstein, ‘Using anytime algorithms in intelligent systems’, AI magazine, 17(3), 73, (1996).