Arbitrarily large violations of non-contextuality in single mode photon states with positive Wigner function

S M Roy

HBCSE, Tata Institute of Fundamental Research, Mumbai, Maharashtra, India

E-mail: smroy@hbcse.tifr.res.in

Received 3 December 2018, revised 28 March 2019
Accepted for publication 24 April 2019
Published 8 May 2019

Abstract

Banaszek, Wódkiewicz and others (Banaszek and Wódkiewicz 1998 Phys. Rev. A 58 4345; Chen et al 2002 Phys. Rev. Lett. 88 040406; Chen and Zhang 2002 Phys. Rev. A 65 044102) made the surprising discovery that Einstein–Bell locality inequalities can be violated by the two mode squeezed vacuum by a factor $\sqrt{2}$, in spite of the fact that the state has a positive Wigner function. I use here the more general Gleason–Kochen–Specker assumption of non-contextuality (Gleason 1957 J. Math. Mech. 6 885) to express classicality. I then derive non-contextuality Bell inequalities for correlations of $N$ pseudo-spins embedded in an infinite dimensional continuous variable Hilbert space, and show that their maximum possible quantum violation is by a factor $2^{(N-1)/2}$. I find quantum states for which this maximum violation is reached. I also show that the familiar displaced squeezed vacuum for a single optical mode, which has a positive Wigner function, can violate the inequality by a factor $0.842(\sqrt{2})^{N-1}$ for odd $N \geq 3$. The arbitrarily large non-classicality means that realizations of the pseudo-spin measurements even in a single mode photon state might afford similar opportunities in quantum information tasks as entangled $N$ qubit systems with large $N$.

Keywords: continuous variable quantum information, non-locality, non-contextuality, Bell inequalities, quantum optics

Introduction

Bell [5] pointed out that Gleason’s theorem [4] of impossibility of dispersion free quantum states is based on a fundamental ‘non-contextuality’ hypothesis: ‘that measurement of an observable must yield the same value independent of what other measurements may be made simultaneously’. When states are not dispersion free, this assumption can be changed into
the less restrictive ‘stochastic non-contextuality hypothesis’: that measured probability distribution of an observable must be independent of what other measurements may be made simultaneously. When the commuting observables are space-like separated, this is the same as Einstein’s local reality hypothesis [6] and leads to Bell’s theorem [7] that quantum mechanics violates Einstein locality. There is now a definite realisation that contextuality (including non-locality induced by entanglement) is a valuable resource [8] for quantum computation.

Bell correlations may be generally defined as linear combinations of correlations between jointly measurable (and hence commuting) observables, different terms of the linear combination being mutually non-commuting. Even when the commuting observables involved in Bell correlations are not spacelike separated, non-contextuality implies inequalities on these correlations. Quantum violation of non-contextuality is a decisive signal of non-classical behaviour.

Einstein locality violations concern two or more spacelike separated systems, the maximum violation being by a factor up to \( \sqrt{2} \) for two particles of spin 1/2 [7] or general spin \( S \) [9], and by a factor up to \( 2^{N-1}/2 \) for \( N \) qubits ([10–12]). In contrast, violations of non-contextuality can occur even for a single system with Hilbert space of dimension \( \geq 3 \) [4]; e.g. for a particle of spin \( S \) with \( 2S + 1 \geq 2^N \), and \( N \) odd, non-contextuality violations by a factor of \( \sqrt{(2S + 1)/2} \) has been demonstrated [13].

**Phas space Bell inequalities**

There has been considerable theoretical and experimental progress on quantum optical continuous variable EPR systems [14–16]. Many years ago we [17] derived phase space Bell inequalities for a four dimensional phase space. E.g. assuming existence of a positive phase space density \( \rho(\vec{x}, \vec{p}) \) reproducing the four experimental probability densities for \((q_1, q_2), (q_1, p_2), (p_1, q_2), (p_1, p_2)\) as marginals,

\[
| \int d\vec{x} d\vec{p} \rho(\vec{x}, \vec{p}) [(sqnF_1(q_1) - sqnF_2(q_2)) + sqnF_1(q_1) sqnG_2(p_2) + sqnG_1(p_1) sqnF_2(q_2) - sqnG_1(p_1) sqnG_2(p_2)] | \leq 2, \tag{1}
\]

where, \( F_1, F_2, G_1, G_2 \) are arbitrary non-vanishing functions, which need not be periodic. The corresponding quantum inequalities, with the phase space variables replaced by operators, and phase space averages replaced by quantum expectation values, are necessarily obeyed by quantum states with positive Wigner functions; but there exist states for which they are violated. Optimisation of these inequalities for experiments have been considered in [18].

Following a surge of interest in quantum information applications of modular observables, i.e. periodic functions of phase space variables ([19–23]), Arora and Asadian [22] obtained for a state with a positive Wigner function,

\[
| \text{Tr}\rho A_1(A_2 + A'_2) | + | \text{Tr}\rho A'_1(A_2 - A'_2) | \leq 2, \tag{2}
\]

where \( A_1, A'_1, A_2, A'_2 \) are observables whose Wigner transforms \( A_1(q_1, p_1), A'_1(q_1, p_1), A_2(q_2, p_2), A'_2(q_2, p_2) \) are of magnitude \( \leq 1 \). They discussed practical measurements on states violating this inequality using modular observables. Comparison with the similar inequality (1) suggests that their investigations may be extended to non-modular observables also.

\[1\] Einstein [6]. See esp. Einstein’s ‘autobiographical notes’ and replies by Bohr here and in Bohr (1935). See also, reprints of related articles in Wheeler and Zurek (1983).
EPR wave function

The above results correspond to Bell’s remarks on measurements of linear combinations of position and momentum using the EPR wave function. He concluded that the original (non-normalisable) EPR wave function leads to a (non-normalisable) positive Wigner function and therefore has no non-locality problem [24]. A significant achievement of Banaszek, Wódkiewicz and others ([1, 2]) was to show that this was incorrect. They demonstrated the non-locality of a normalisable EPR-like state, the two-mode squeezed vacuum or NOPA (non-degenerate optical parametric amplifier) state which has a positive Wigner function. They showed that locality inequalities on Bell correlations of phase-space displaced parity operators or of pseudo-spin observables for this system are violated by the quantum correlations by a factor $\sqrt{2}$. Thus, there exist observables for which locality need not be connected to the positivity of the Wigner function.

Present work

Here, I shall demonstrate that even for a continuous variable system with only one degree of freedom, and positive Wigner function, non-contextuality inequalities on Bell correlations may be violated by an arbitrarily large factor. The state may be as simple as a squeezed coherent state. Relevant correlation measurements and quantum information applications may be possible using simple techniques of continuous variable quantum computation such as balanced homodyne measurements and unitary phase space displacement operations on electromagnetic quadratures. In fact, it has been claimed that ‘this simplicity and the high efficiency when measuring and manipulating the continuous quadratures are the main reason why continuous-variable schemes appear more attractive than those based on discrete variables such as the photon number’. (Braunstein and van Loock in [15]).

My first step will be to define $N$ qubit pseudo-spin operators in a single continuous variable Hilbert space, namely, quantum optical quadratures for a single mode.

$N$ qubit pseudo-spin operators

A single electromagnetic mode of frequency $\omega$ corresponds to an oscillator Hamiltonian with ground state energy subtracted,

$$H = \hbar \omega \hat{a}^{\dagger} \hat{a},$$

where the annihilation operator $\hat{a}$ may be expressed in terms of dimensionless Hermitian quadrature operators $\hat{x}, \hat{p}$,

$$\hat{x} = (\hat{a} + \hat{a}^{\dagger})/\sqrt{2}; \quad \hat{p} = (\hat{a} - \hat{a}^{\dagger})/(i\sqrt{2})$$

$$\hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}; \quad [\hat{a}, \hat{a}^{\dagger}] = 1, \quad [\hat{x}, \hat{p}] = i.$$ (4)

I divide the configuration space $x \in \mathbb{R} = (-\infty, +\infty)$ of eigenvalues of the quadrature $\hat{x}$ into discrete intervals $(aL(s - \frac{1}{2}), aL(s + \frac{1}{2}))$, each of length $La$ and centred at $aLs$ where $s$ is an integer, zero, positive or negative; and $L = 2^N$. The arbitrary parameter ‘$a$’ is not to be confused with the operator $\hat{a}$. With a view to corresponding to $2^N$ basis states of $N$ qubits, I sub-divide the $s$th interval into $2^N$ sub-intervals labelled by $m$, each of length $a$ (see figure 1),
\[ I_{m,s} = (aL(s - \frac{1}{2}) + ma, aL(s - \frac{1}{2}) + (m + 1)a), \]
\[ s = \text{integer}, \ L = 2^N, \ m = 0, 1, \ldots L - 1. \]  

Then,
\[ \int_{x,s} dx|x| = \int_y^m dy(m,s,y)\langle m,s,y|, \]
\[ |m,s,y\rangle \equiv |aL(s - 1/2) + am + y|. \]

The completeness of the states \(|x\rangle\) and orthonormality relations become
\[ 1 = \sum_{j=-\infty}^{\infty} \sum_{m=0,1\ldots}^{2^N-1} \int_{y=0}^{\infty} dy(m,s,y)|m,s,y\rangle\langle m,s,y|, \]
\[ \langle m,s,y|m',s',y'\rangle = \delta_{s,s'}\delta_{m,m'}\delta(y - y'). \]

I want to define \(N\) pseudo-spin operators \(\sigma_z^{(j)}\) and the corresponding \(2^N\) eigenstates with eigenvalues \(m_j = \pm 1\), for \(j = 1, 2, \ldots N\). Like the set \((m_1, m_2, \ldots, m_N)\), the integer \(m\) takes \(2^N\) values. If I define,
\[ m(m_1, m_2, \ldots, m_N) = \sum_{j=1}^{N} 2^{(j-1)}(1 + m_j)/2, \]
then the resulting values of \(m\) are 0, 1, \ldots, \(2^N - 1\), with \(m = 0\) for all \(m_j = -1\) and \(m = 2^N - 1\) for all \(m_j = +1\). For each \(m\), the relation can be inverted to solve uniquely for \((m_1, m_2, \ldots, m_N)\), i.e. for \(m_j = \pm 1\) the correspondence
\[ m \leftrightarrow (m_1, m_2, \ldots, m_N) \]
is one-to-one. This is obvious from equation (10) because
\[ ((m_N + 1)/2)((m_{N-1} + 1)/2)\ldots((m_1 + 1)/2) \]
is just the binary representation of \(m\), each of the \(N\) digits being 0 or 1.

Due to the one-to-one correspondence (11), we may write the orthonormality relation (9) for the states as
\[ |m,s,y\rangle \equiv |m_1, m_2, \ldots, m_N; s, y\rangle, \]
\[ |m',s,y\rangle \equiv |m'_1, m'_2, \ldots, m'_N; s, y\rangle, \]
\[ \langle m,s,y|m',s',y'\rangle = \delta_{s,s'}\delta(y - y') \prod_{j=1}^{N} \delta_{m_j,m'_j}. \]

For a given \(m\), I now define the sub-intervals \(I_{m,s}\) for all \(s\) to correspond to an eigenstate of \(\sigma_z^{(1)}, \ldots, \sigma_z^{(N)}\), i.e.
\[ \sigma_z^{(j)}|m,s,y\rangle = m_j|m,s,y\rangle, \text{ for all } j \text{ and any } s, y, \]
\[ M \equiv \sum_{j=1}^{N} 2^{(j-1)}(1 + \sigma_z^{(j)})/2, \]
\[ M|m,s,y\rangle = m|m,s,y\rangle. \]
Equation (7) shows that the operator $M$ is periodic in $\hat{x}$ with period $a^2 N$. Next, the raising and lowering operators $\sigma^{(j)}_\pm$ should convert a state with $m_j = \mp 1$ into a state with $m_j = \pm 1$, leaving all other $m_{j' \neq j}$ unchanged, and should annihilate states with $m_j = \pm 1$. The definition of $m$, equation (10), shows that if all $m_{j' \neq j}$ are unchanged, the value of $m$ for $m_j = 1$ is greater than it is value for $m_j = -1$ by $2^{j-1}$. Hence, 

$$|m_1, \ldots, m_{j-1}, -m_j, \ldots, m_N; s, y\rangle = |m - 2^{(j-1)}m_j, s, y\rangle,$$  

and I stipulate that, 

$$(\sigma^{(j)}_x \pm i \sigma^{(j)}_y)|m, s, y\rangle = (1 \mp m_j)|m - 2^{(j-1)}m_j, s, y\rangle.$$  

Equivalently, 

$$\sigma^{(j)}_x|m, s, y\rangle = |m - 2^{(j-1)}m_j, s, y\rangle,$$

$$\sigma^{(j)}_y|m, s, y\rangle = im_j|m - 2^{(j-1)}m_j, s, y\rangle.$$ 

If $\vec{n}$ are unit vectors, these definitions may be summarised by 

$$\langle m', s', y' | \sigma^{(j)}_x \cdot \vec{n}|m, s, y\rangle = \langle m' | \vec{n} | m \rangle \delta_{s,s'} \delta(y - y') \prod_{l \neq j} \delta_{m_l, m_{l'}},$$  

where $\vec{\sigma}$ without the superscript $j$ denote the usual Pauli matrices, and $|\pm 1\rangle$ denote eigenvectors of $\sigma_z$. The standard commutation rules follow, 

$$\sigma^{(j)}_x \sigma^{(j)}_y = -\sigma^{(j)}_y \sigma^{(j)}_x = i \sigma^{(j)}_z,$$  

operators for different $j$ being mutually commuting on account of equation (17).
Using the completeness relations (8), these definitions are equivalent to,
\[
\sigma_x^{(j)} = \sum_{s,m} \int_{y=0}^{a} dy |m, s, y\rangle \langle m - 2^{(j-1)}m_j, s, y|,
\]
\[
\sigma_y^{(j)} = -i \sum_{s,m} m_j \int_{y=0}^{a} dy |m, s, y\rangle \langle m - 2^{(j-1)}m_j, s, y|,
\]
\[
\sigma_z^{(j)} = \sum_{s,m} \int_{y=0}^{a} dy m_j |m, s, y\rangle \langle m, s, y|.
\] (19)

Note that the operators $\sigma_x^{(j)}$, $\sigma_y^{(j)}$ are not diagonal in the quadrature basis.

**Measurement of the pseudo-spin operators**

Since the correspondence $m \leftrightarrow (m_1, m_2, \ldots, m_N)$ is unique, an experimental coarse-grained $x$-quadrature measurement finding $x \in I_{m,s}$ for some integer $s$ yields the eigen values of $\sigma_x^{(1)}, \ldots, \sigma_x^{(N)}$. These are homodyne measurements. Measurement of a single operator $\sigma_z^{(j)}$ is an even more coarse-grained quadrature measurement. For example, the projection to $\sigma_z^{(j)} = 1$ of a state $|\psi\rangle$ just removes those regions of $x = aL(s - 1/2) + am + y$ which correspond to $m_j = -1$, irrespective of the values of $m_k, k \neq j$,
\[
\frac{1 + \sigma_z^{(j)}}{2} |\psi\rangle = \sum_{s,m,m_j \neq -1} \int_{y=0}^{a} dy |m, s, y\rangle \langle m, s, y| |\psi\rangle.
\]

Towards the measurement of $\sigma_x^{(1)}$, $\ldots$, $\sigma_x^{(N)}$ and $\sigma_y^{(1)}$, $\ldots$, $\sigma_y^{(N)}$, it is sufficient to be able to realize their eigen states. The eigen value equations,
\[
(\sigma_y^{(j)} \pm 1)(|m, s, y\rangle \pm im_j |m - 2^{(j-1)}m_j, s, y\rangle) = 0,
\]
\[
(\sigma_y^{(j)} \pm 1)(|m, s, y\rangle \pm |m - 2^{(j-1)}m_j, s, y\rangle) = 0
\] (20)

show that the corresponding eigen states are just superpositions of a state with one value of quadrature and another state suitably phase-shifted and with the quadrature translated by $2^{(j-1)}m_j a$. As stated by Braunstein and van Loock [15], quadrature translations are easy and efficient using continuous variable techniques. So, a beam-splitter may split the beam into two beams, apply a quadrature translation on one of them and recombine the two beams with their paths being adjusted to obtain the required phase difference. Though less easy than a measurement of $\sigma_x^{(j)}$, measurements of $\sigma_y^{(j)}$, and $\sigma_y^{(j)}$ (and analogously a component of $\sigma_z^{(j)}$ along any direction) seem possible, and potentially rewarding.

**Post-measurement states**

We see from above that a pure state $|\psi\rangle$ corresponds to eigen value of $\sigma_x^{(1)} = \pm 1$ if, for all $m_2, \ldots, m_N, s, y$,
\[
\langle m_1 = 1, m_2, \ldots, m_N, s, y|\psi\rangle = \pm \langle m_1 = -1, m_2, \ldots, m_N, s, y|\psi\rangle.
\] (21)

This condition remains unaffected by projection to $\sigma_z^{(j)} = \pm 1$, if $j \neq 1$, because the projection just removes the $m_j = \mp 1$ component of the state; i.e. the eigen value of $\sigma_x^{(1)}$ is unaltered.
by a measurement of $\sigma(z)^{(j),j \neq 1}$. If $N = 2, j = 2$, the commutation of $\sigma(z)^{(2)}$ and $\sigma(z)^{(1)}$ is clearly exhibited by the equation,

$$\sigma(z)^{(2)} \sigma(z)^{(1)} = \sigma(z)^{(1)} \sigma(z)^{(2)}$$

$$= \sum_{s,m_1,m_2} \int_{-\hbar}^{\hbar} dy \, m_2 |m_1, m_2, s, y\rangle \langle -m_1, m_2, s, y|.$$  \hspace{1cm} (22)

This enables the proof that for any arbitrary state with density operator $\rho$, if $\sigma(z)^{(2)} = A$ is measured first to obtain the state $\rho'$, a subsequent measurement of $\sigma(x)^{(1)} = B$ on the post-measurement state must yield the same expectation value as in the initial state. If two self-adjoint operators $A$ and $B$ commute, so do their projectors $P_A$ and $P_B$. On measuring $A$,

$$\rho \rightarrow \rho' = \sum_l P_A \rho P_A.$$  \hspace{1cm} (23)

A measurement of $B$ on the post-measurement state yields the expectation value,

$$\text{Tr} \rho' B = \text{Tr} \sum_l P_A \rho P_A B = \text{Tr} \sum_l P_A \rho B P_A$$

$$= \text{Tr} \rho B \sum_l P_A P_A = \text{Tr} \rho B,$$  \hspace{1cm} (24)

which is the same as the expectation value in the initial state. Here we used the commutation of $P_A$ with $B$, the cyclicity of the Trace, the idempotence and the completeness of the projectors $P_A$. Though the pseudo-spin operators $\sigma(z)^{(1)}$ and $\sigma(z)^{(2)}$ refer to the same continuous variable system, their joint measurement characteristics are parallel to those of two independent spin-half systems.

**Phase space displaced pseudo-spin operators**

The unitary operator for phase space displacement is,

$$D(\alpha) = \exp (i\hat{x} \theta / \hbar), \quad \alpha \equiv (q + i \hat{p}) / \sqrt{2}.$$  \hspace{1cm} (25)

in units $\hbar = 1$, where the quadrature operators $\hat{x}$, $\hat{p}$ obey $[\hat{x}, \hat{p}] = i$. Then,

$$D(\alpha)|x\rangle = \exp \left( i \hat{x} \theta / 2 \right) |x + \theta\rangle$$

$$\langle x| D(\alpha)|\psi\rangle = \exp \left( i \hat{x} \theta / 2 \right) \langle x - \theta|\psi\rangle.$$  \hspace{1cm} (26)

For an arbitrary operator $A$, and state $|\psi\rangle$, I define the displaced operator $A_\alpha$ and displaced state $|\psi_\alpha\rangle$,

$$A_\alpha = D(\alpha) A D(\alpha)^\dagger, \quad |\psi_\alpha\rangle = D(\alpha)|\psi\rangle.$$  \hspace{1cm} (28)

Hence, the displaced pseudo-spin operators are given by,

$$(\sigma_{x,\alpha}^{(j)} \pm i \sigma_{y,\alpha}^{(j)}) = \exp \left( \pm i a 2^{j-1} \hat{p} \right) \sum_{l,m} (1 \mp m_j)$$

$$\times \int_{y=0}^{\hbar} dy |m \pm 2^{j-1}, s, y + \theta\rangle \langle m, s, y + \theta|.$$  \hspace{1cm} (29)
and
\[ \sigma_{\alpha}^{(j)} = \sum_{i,m} \int_0^\infty dy \, m_j |m, s, y + \tilde{q}\rangle \langle m, s, y + \tilde{q}|, \]  
\[ \text{where, } s \text{ is summed over all integers and } m \text{ over all integers } \in [0, 2^N - 1]. \]  

Bell correlations

I now define Bell correlations of the pseudo-spin operators and phase space displaced pseudo-spin operators. If \( \vec{a}^{(j)} \) are unit vectors,
\[ (\vec{x}^{(j)}, \vec{y}^{(j)})^2 = 1, \]
\[ [\vec{x}^{(j)}, \vec{y}^{(j)}, \vec{z}^{(j)}, \vec{z}^{(j)}] = 0, j' \neq j. \]  
The observables
\[ A^{(j)}(a^{(j)}) \equiv (\vec{x}^{(j)} \cdot \vec{a}^{(j)}) \equiv A^{(j)}; \]
\[ A^{(j)}(a^{(j)})' \equiv (\vec{x}^{(j)} \cdot (\vec{a}^{(j)})') \equiv (A^{(j)})'; \]  
have eigenvalues \( \pm 1 \). For brevity we may sometimes write \( A^{(j)} \) and \( (A^{(j)})' \) instead of \( A^{(j)}(a^{(j)}) \) and \( A^{(j)}(a^{(j)})' \), but it will be understood that \( A^{(j)} \) depends only on \( a^{(j)} \) and \( (A^{(j)})' \) depends only on \( (a^{(j)})' \). \( A^{(j)} \) commutes with \( (A^{(k)}) \) and \( (A^{(k)})' \) for \( j \neq k \). Consider the quantum operators,
\[ E^{(N)}(a^{(1)}, a^{(2)}, \ldots, a^{(N)}, (a^{(1)})', (a^{(2)})', \ldots, (a^{(N)})') \]
\[ = \prod_{j=1}^N (A^{(j)}(a^{(j)}) + iA^{(j)}(a^{(j)})'), \]  
or, suppressing dependences on the orientations \( a^{(j)}, (a^{(j)})' \),
\[ E^{(N)} \equiv \prod_{j=1}^N (A^{(j)} + i(A^{(j)})') \equiv E_1^{(N)} + iE_2^{(N)}; \]  
where the Hermitian operators \( E_1^{(N)}, E_2^{(N)} \)
\[ E_1^{(N)} \equiv (E^{(N)} + (E^{(N)})')/2; \]
\[ E_2^{(N)} \equiv (E^{(N)} - (E^{(N)})')/(2i), \]  
are linear combinations of \( 2^{(N-1)} \) terms, each term being a product of \( N \) commuting observables and hence experimentally measurable. Since,
\[ E^{(N+1)} = (E^{(N)} + iE_2^{(N)})A^{(N+1)} + i(A^{(N+1)})', \]
we can express higher order Bell operators in terms of lower order ones, as in \((3, 11, 26)\). I define conveniently normalised even and odd order Hermitian Bell operators,
\[ B_1^{(2r)} = \frac{E_1^{(2r)} + E_2^{(2r)}}{2r}, B_2^{(2r)} = \frac{E_1^{(2r)} - E_2^{(2r)}}{2r}, r = 1, 2, \ldots \]
\[ B_1^{(2r+1)} = \frac{E_1^{(2r+1)}}{2r}, B_2^{(2r+1)} = \frac{E_2^{(2r+1)}}{2r}, r = 0, 1, 2, \ldots \]
I then have the Bell operator recursion relations,

\[ B_{1(2r)} = B_{1(2r-1)} A^{(2r)} + \frac{(A^{(2r)})'}{2} \]

\[ + B_{2(2r-1)} A^{(2r)} - \frac{(A^{(2r)})'}{2}, \]  

or equivalently,

\[ B_{2(2r)} = -B_{2(2r-1)} A^{(2r)} + \frac{(A^{(2r)})'}{2} \]

\[ + B_{1(2r-1)} A^{(2r)} - \frac{(A^{(2r)})'}{2}, \]  

\[ B_{2(2r+1)} = -B_{1(2r+1)} A^{(2r+1)} + \frac{(A^{(2r+1)})'}{2} \]

\[- B_{2(2r+1)} A^{(2r+1)} - \frac{(A^{(2r+1)})'}{2}. \]  

Similarly, the corresponding displaced observables

\[ A_j^j(a_j^j)_{\alpha} = D(\alpha) A_j^j(a_j^j) D(\alpha)^\dagger, \]

have eigenvalues \( \pm 1 \) and are mutually commuting for different values of \( j \). Defining,

\[ E^{(N)}_{\alpha} \equiv \prod_{j=1}^{N} (A_j^j + i(A_j^j)') \equiv E^{(N)}_{1,\alpha} + iE^{(N)}_{2,\alpha}, \]  

displaced analogues of equations (36)–(42) are obtained by the replacements:

\[ A_j^j \rightarrow A_{\alpha}^j, (A_j^j)' \rightarrow (A_{\alpha}^j)', E^{(N)} \rightarrow E^{(N)}_{\alpha}, \]

\[ E^{(N)}_i \rightarrow E^{(N)}_{i,\alpha}, B_j^{(N)} \rightarrow B_{j,\alpha}^{(N)}. \]  

The quantum Bell correlations are given by the expectation values,

\[ \langle B_{1(2r)}^{(N)} \rangle_{Q} = \text{Tr} \rho B_{1(2r)}^{(N)}, \]

\[ \langle B_{2(2r)}^{(N)} \rangle_{Q} = \text{Tr} \rho B_{2(2r)}^{(N)}, \]  

\[ \langle B_{2(2r+1)}^{(N)} \rangle_{Q} = \text{Tr} \rho B_{2(2r+1)}^{(N)}, \]  

where \( i = 1, 2 \), and \( \rho \) is the density operator for the state.

We want to compare the quantum Bell correlations with the predictions of a non-contextual hidden variable theory.

**Non-contextual hidden variables (NCHV)**

In a non-contextual stochastic hidden variable theory, the state with hidden variables \( \lambda \) with probability distribution \( \mu(\lambda) \), specifies the values or at least expectation values corresponding to the \( j \)th observables (32) as \( A_j^j(\lambda, a_j^j) \) which must lie in the interval \([-1, +1]\) and be
independent of the orientations of \( \vec{a}^{(j)} \) for \( j \neq j' \). I denote the NCHV expectation value of a quantum operator \( A \) by \( \langle A \rangle = \langle A \rangle_{\text{NCHV}} \), and the corresponding quantum expectation value by \( \langle A \rangle_{\text{QM}} \). Hence the NCHV expectation value corresponding to the operator \( E^{(N)} \) is given by

\[
\langle E^{(N)} \rangle = \langle E^{(N)}_{1} \rangle + i \langle E^{(N)}_{2} \rangle \\
= \int d\lambda \mu(\lambda) \prod_{j=1}^{N} (A^{(j)}(\lambda) + iA^{(j)(j)'}(\lambda)) \\
= \int d\lambda \mu(\lambda) E^{(N)}(\lambda),
\]

(46)

where \( E^{(N)}_{1}(\lambda), E^{(N)}_{2}(\lambda) \) are the real and imaginary parts of \( E^{(N)}(\lambda) \). It will be understood that \( A^{(j)}(\lambda) = A^{(j)}(\lambda, a^{(j)}) \) also depends on \( a^{(j)} \), and \( (A^{(j)(j)'}(\lambda) = (A^{(j)(j)'}(\lambda, (a^{(j)j})')) \) depends also on \( (a^{(j)j})' \). The normalisation conditions are,

\[
\int d\lambda \mu(\lambda) = 1, \mu(\lambda) \geq 0, \\
|A^{(j)}(\lambda)| \leq 1, |(A^{(j)(j)'}(\lambda)| \leq 1.
\]

(48)

They imply already that,

\[
|E^{(1)}_{1}(\lambda)| \leq 1, |\langle E^{(1)}_{1} \rangle| \leq 1; i = 1, 2; \\
|E^{(N)}(\lambda)| \leq 2^{N/2}, |\langle E^{(N)} \rangle| \leq 2^{N/2},
\]

(49) (50)

but the \( N \) 1 results can be improved.

The NCHV expectation value \( \langle E^{(N)}_{\alpha} \rangle \) is

\[
\langle E^{(N)}_{\alpha} \rangle = \langle E^{(N)}_{1,\alpha} \rangle + i \langle E^{(N)}_{2,\alpha} \rangle \\
= \int d\lambda \mu(\lambda) \prod_{j=1}^{N} (A^{(j)}_{\alpha}(\lambda) + iA^{(j)(j)'}_{\alpha}(\lambda)) \\
= \int d\lambda \mu(\lambda) E^{(N)}_{\alpha}(\lambda),
\]

(51)

\[
E^{(N)}_{\alpha}(\lambda) = E^{(N)}_{1,\alpha}(\lambda) + iE^{(N)}_{2,\alpha}(\lambda),
\]

(52)

where \( A^{(j)}_{\alpha}(\lambda) = A^{(j)}_{\alpha}(\lambda, a^{(j)}) \) depends also on \( a^{(j)} \), and \( (A^{(j)(j)'}_{\alpha}(\lambda) = (A^{(j)(j)'}_{\alpha}(\lambda, (a^{(j)j})')) \) depends also on \( (a^{(j)j})' \). Further,

\[
|A^{(j)}_{\alpha}(\lambda)| \leq 1, |(A^{(j)(j)'}_{\alpha}(\lambda)| \leq 1,
\]

(53)

and hence,

\[
|E^{(1)}_{1,\alpha}(\lambda)| \leq 1, |\langle E^{(1)}_{1,\alpha} \rangle| \leq 1; i = 1, 2; \\
|E^{(N)}_{\alpha}(\lambda)| \leq 2^{N/2}, |\langle E^{(N)}_{\alpha} \rangle| \leq 2^{N/2},
\]

(54) (55)

which too can be improved for \( N > 1 \).
Writing the NCHV value of each operator $A$ in the hidden state $\lambda$ as $A(\lambda)$, the NCHV correlations corresponding to the Bell operators $B_i^{(j)}$, $B_{i,\alpha}^{(j)}$ are given by,

$$
\langle B_i^{(j)} \rangle = \int d\lambda \lambda B_i^{(j)}(\lambda);
$$

$$
\langle B_{i,\alpha}^{(j)} \rangle = \int d\lambda \lambda B_{i,\alpha}^{(j)}(\lambda); i = 1, 2.
$$

Thus, the operator relations (37)–(42) yield corresponding recursion relations between the hidden variable values. E.g.

$$
E^{(N+1)}(\lambda) = (E_1^{(N)} + iE_2^{(N)})\lambda + (A^{(N+1)} + i(A^{(N+1)})')(\lambda),
$$

$$
B_1^{(2r)}(\lambda) = \frac{E_1^{(2r)}(\lambda) + E_2^{(2r)}(\lambda)}{2^r}; r = 1, 2,..
$$

$$
B_2^{(2r)}(\lambda) = \frac{E_1^{(2r)}(\lambda) - E_2^{(2r)}(\lambda)}{2^r}, r = 1, 2,..
$$

$$
B_1^{(2r+1)}(\lambda) = \frac{E_1^{(2r+1)}(\lambda)}{2^r}; r = 0, 1, 2,..
$$

$$
B_2^{(2r+1)}(\lambda) = \frac{E_2^{(2r+1)}(\lambda)}{2^r}, r = 0, 1, 2,..
$$

These lead to recursion relations for hidden variable values of the Bell operators,

$$
B_1^{(2r)}(\lambda) = B_1^{(2r-1)}(\lambda) \frac{(A^{(2r)} + (A^{(2r)})')(\lambda)}{2} + B_2^{(2r-1)}(\lambda) \frac{(A^{(2r)} - (A^{(2r)})')(\lambda)}{2},
$$

$$
B_2^{(2r)}(\lambda) = -B_2^{(2r-1)}(\lambda) \frac{(A^{(2r)} + (A^{(2r)})')(\lambda)}{2} + B_1^{(2r-1)}(\lambda) \frac{(A^{(2r)} - (A^{(2r)})')(\lambda)}{2},
$$

or equivalently,

$$
B_1^{(2r+1)}(\lambda) = B_2^{(2r)}(\lambda) \frac{(A^{(2r+1)} + (A^{(2r+1)})')(\lambda)}{2} + B_1^{(2r)}(\lambda) \frac{(A^{(2r+1)} - (A^{(2r+1)})')(\lambda)}{2},
$$

$$
B_2^{(2r+1)}(\lambda) = B_1^{(2r)}(\lambda) \frac{(A^{(2r+1)} + (A^{(2r+1)})')(\lambda)}{2} - B_2^{(2r)}(\lambda) \frac{(A^{(2r+1)} - (A^{(2r+1)})')(\lambda)}{2}.
$$

These relations enable a recursive proof of $N$-qubit NCHV inequalities similar to the original Bell-CHSH locality inequalities [7], and their $N$-party generalisations ([10–12]). Thus, from equations (59) and (60),
\[ |B_i^{(2r)}(\lambda)| \leq \left( |(A^{(2r)})(\lambda) + (A^{(2r)})'(\lambda)| + |(A^{(2r)} - (A^{(2r)})')'(\lambda)| \right) / 2 \times \max \{|B_i^{(2r-1)}(\lambda)|, |B_j^{(2r-1)}(\lambda)|\}, \quad i, j = 1, 2. \]  

(63)

Using the normalisation conditions (48) we have,

\[ |(A^{(2r)} + (A^{(2r)})')(\lambda)| + |(A^{(2r)} - (A^{(2r)})')'(\lambda)| \leq 2. \]  

(64)

On multiplying equation (63) by \(\mu(\lambda)\) and integrating over \(\lambda\) we obtain,

\[ \max(|\langle B_i^{(2r)} \rangle|, |\langle B_j^{(2r)} \rangle|) \leq \max(|\langle B_i^{(2r-1)} \rangle|, |\langle B_j^{(2r-1)} \rangle|), \]  

(65)

for any positive integer \(r\). This proof is analogous to the original two party Bell-CHSH proof [7] and an alternative to the variational \(N\) party proofs ([10–12]). Similarly, equations (61) and (62) yield, for any positive integer \(r\),

\[ \max(|\langle B_i^{(2r+1)} \rangle|, |\langle B_j^{(2r+1)} \rangle|) \leq \max(|\langle B_i^{(2r)} \rangle|, |\langle B_j^{(2r)} \rangle|). \]  

(66)

From \(B_i^{(1)} = E_i^{(1)}\) and \(|\langle E_i^{(1)} \rangle| \leq 1\) for \(i = 1, 2\), we obtain recursively the NCHV Bell inequalities in a single continuous variable system,

\[ \max(|\langle B_i^{(j)} \rangle|, |\langle B_j^{(j)} \rangle|) \leq 1; j = 1, 2, \ldots N. \]  

(67)

Exactly the same procedure yields the NCHV Bell inequalities for the displaced operators in a single continuous variable system,

\[ \max(|\langle B_i^{(j)} \rangle|, |\langle B_j^{(j)} \rangle|) \leq 1; j = 1, 2, \ldots N. \]  

(68)

Our normalisations (38) of the Bell operators have ensured that their NCHV expectation values are bounded by unity. What are the upper limits on the quantum expectation values?

**Maximum possible quantum violations of NCHV inequalities**

The Cirel’son theorem in the two qubit case [25] and generalised Cirel’son theorems in \(N\)-qubit case [26] and \(N\) continuous variable systems ([2, 3]) set limits on maximum possible quantum violations of local hidden variable inequalities. I derive analogous limits on quantum violations of NCHV inequalities using pseudo-spin observables for a single continuous variable system.

From equation (39), setting \(M = 2r\) and writing the pseudo-spin operators explicitly, I obtain,

\[ B_1^{(M)} = B_1^{(M-1)} \frac{\tilde{\sigma}^{(M)}}{2} + \frac{\tilde{\sigma}^{(M)} + (\tilde{\sigma}^{(M)})'}{2}, \]  

(69)

\[ + B_2^{(M-1)} \frac{\tilde{\sigma}^{(M)} - (\tilde{\sigma}^{(M)})'}{2}, \]
we get,
\[
(B_1^{(M)})^2 = \left(B_1^{(M-1)}\right)^2 \frac{\left(1 + \vec{a}^{(M)} \cdot (\vec{a}^{(M)})'\right)}{2} \\
+ \left(B_2^{(M-1)}\right)^2 \frac{\left(1 - \vec{a}^{(M)} \cdot (\vec{a}^{(M)})'\right)}{2} \\
+ \frac{i}{2} [B_1^{(M-1)}, B_1^{(M-1)}] \vec{a}^{(M)} \cdot (\vec{a}^{(M)})'.
\]
(70)

Using $||A^2|| = ||A||^2$ for a Hermitian $A$, I have, for $M$ even,
\[
||B_1^{(M)}||^2 \leq 2 \text{Max} (||B_1^{(M-1)}||^2, ||B_2^{(M-1)}||^2).
\]
(71)

Similarly, equation (40) yields, for even $M$,
\[
||B_1^{(M)}||^2 \leq 2 \text{Max} (||B_1^{(M-1)}||^2, ||B_2^{(M-1)}||^2).
\]
(72)

Similarly, equations (41) and (42) imply that the relations (71) and (72) also hold for odd $M$. Using $(B_1^{(1)})^2 = 1, (B_2^{(1)})^2 = 1$, I now obtain recursively, both for $N$ even, and for $N$ odd,
\[
||B_1^{(N)}||^2 \leq 2^{N-1}; ||B_2^{(N)}||^2 \leq 2^{N-1}.
\]
(73)

The same bounds follow for the displaced operators,
\[
||B_1^{(N)}_{\alpha}||^2 \leq 2^{N-1}; ||B_2^{(N)}_{\alpha}||^2 \leq 2^{N-1}.
\]
(74)

Hence, for an arbitrary normalised quantum state,
\[
|\langle B_i^{(N)} \rangle_{\text{QM}}| \leq ||B_i^{(N)}|| \leq 2^{(N-1)/2}, i = 1, 2
\]
(75)

and,
\[
|\langle B_i^{(N)}_{\alpha} \rangle_{\text{QM}}| \leq ||B_i^{(N)}_{\alpha}|| \leq 2^{(N-1)/2}, i = 1, 2.
\]
(76)

We see that just as for the Mermin–Roy–Singh multiparty locality inequalities ([10–12]), quantum correlations can violate the NCHV limits for $N$ pseudo spins in a continuous variable system by at most a factor $2^{(N-1)/2}$. Thus the generalised Cirel’son theorem for the NCHV single system case is similar to that for locality inequalities for the $N$ system case ([2, 3, 25, 26]). Can this maximal violation be reached?

**Quantum state showing maximal violation of NCHV inequality for a given $N$**

Consider the special choice,
\[
E^{(N)} \equiv \prod_{j=1}^{N} (\sigma_x^{(j)} - i \sigma_y^{(j)}),
\]
(77)

where, $\sigma_x^{(j)}, \sigma_y^{(j)}$ are the pseudo-spin operators ; the Hermitian operators $E_i^{(N)}, B_i^{(N)}$ for $i = 1, 2$ are derived from it using equations (36) and (38). I evaluate the quantum Bell correlations for the particular state,
\[ |\psi_0\rangle = \sum_{s=-\infty}^{\infty} \sum_{m=0,1,...}^{(2^N-1)} \int_0^a dy \]
\[ |m, s, y\rangle \psi_0(aL(s - \frac{1}{2}) + am + y) \]  
(78)

where \( \psi_0(aL(s - \frac{1}{2}) + am + y) = 0 \), unless \( m = 0 \) or \( L - 1 \), i.e.
\[ |\psi_0\rangle = \sum_s \int_0^a dy \left( |0, s, y\rangle \psi_0(aL(s - \frac{1}{2}) + y) + |L - 1, s, y\rangle \psi_0(aL(s - \frac{1}{2}) + a(L - 1) + y) \right) \]  
(79)

with the normalisation condition, \( \langle \psi_0 | \psi_0 \rangle = 1 \). E.g. the particular choice
\[ \psi_0(aL(s - \frac{1}{2}) + y) = (2/(\pi(2s - 1))) \chi(y), \]
\[ \int_0^a dy |\chi(y)|^2 = 1/2, \]
obeys the normalisation condition because (see [27])
\[ \sum_{k=1}^{\infty} (2k - 1)^{-2} = \pi^2/8. \]

The definition (77) of \( E^{(N)} \) and of the \( \sigma \) matrices yield,
\[ \langle \psi_0 | E^{(N)} | \psi_0 \rangle = 2^N \sum_s \int_0^a dy \psi_0^* (aL(s - 1/2) + y) \]
\[ \times \psi_0(aL(s - 1/2) + a(L - 1) + y). \]

I now choose,
\[ \psi_0(aL(s - 1/2) + a(L - 1) + y) = \exp(i\theta) \psi_0(aL(s - 1/2) + y), \]  
(80)

and use the normalisation condition to obtain,
\[ \langle \psi_0 | E^{(N)} | \psi_0 \rangle = 2^{N-1} \exp(i\theta). \]  
(81)

This gives the quantum correlations,
\[ N \text{ odd : } |\langle B_1^{(N)} \rangle_{\text{QM}}| = 2^{(N-1)/2}, \theta = 0; \]
\[ |\langle B_2^{(N)} \rangle_{\text{QM}}| = 2^{(N-1)/2}, \theta = \pi/2; \]  
(82)

\[ N \text{ even : } |\langle B_1^{(N)} \rangle_{\text{QM}}| = 2^{(N-1)/2}, \theta = \pi/4; \]
\[ |\langle B_2^{(N)} \rangle_{\text{QM}}| = 2^{(N-1)/2}, \theta = -\pi/4. \]  
(83)

Choosing \( \theta = 0 \) or \( \pi/2 \) for \( N \) odd, and \( \theta = \pm \pi/4 \) for \( N \) even, we see that the NCHV inequalities (67) are violated by quantum mechanics by the maximal factor \( 2^{(N-1)/2} \) which grows exponentially with the chosen \( N \). This concludes the formal proof, but the state may not be easily realizable. I now show that nearly maximal violation is possible using very familiar states.
Non-contextuality violation by the single mode squeezed vacuum (SMSV) state

I now consider the SMSV wave function, which is easy to realise quantum optically, and has a positive definite Wigner function,

\[ \psi(x) = C \exp \left( -x^2 / (4\sigma^2) \right); \quad C = (2\pi\sigma^2)^{-1/4}. \]  

(84)

The relation with the squeezing parameter \( r' \) is given by \( \sigma^2 = 1/2 \exp (-2r') \), where \( r' \) corresponds to position squeezing, and \( r' \) corresponds to momentum squeezing. Rewriting the states \( \left| x \right> \) in terms of the states \( \left| m, s, y \right> \) with definite eigenvalues of \( \sigma_x^{(j)} \), we have,

\[ \left| \psi \right> = \sum_{s=-\infty}^{\infty} \left( \sum_{m=0,1,..}^{2^N-1} \int_{y=0}^{a} dy \right) \left| m, s, y \right> \times C \exp \left( - (aL(s -1/2) + am + y)^2 / (4\sigma^2) \right). \]  

(85)

Note that, \( m = \frac{L}{2} - 1 \leftrightarrow x - Las \in (-a, 0) \) and \( m = \frac{L}{2} \leftrightarrow x - Las \in (0, a) \). Further, \( m = \frac{L}{2} - 1 \leftrightarrow (m_1,..,m_N) = (1,..,1,-1) \), and \( m = \frac{L}{2} \leftrightarrow (m_1,..,m_N) = (-1,..,-1,1) \).

This time we choose the operator \( E^{(N)} \) defined by

\[ E^{(N)} = (\sigma_x^{(N)} - i\sigma_y^{(N)}) \prod_{j=1}^{(N-1)} \left( \sigma_x^{(j)} + i\sigma_y^{(j)} \right) \]  

(86)

which takes states with \( m = L/2, \) to states with \( m = L/2 - 1, \)

\[ E^{(N)} \left| m = L/2, s, y \right> = 2^N \left| m = L/2 - 1, s, y \right> \]  

(87)

and annihilates all states with \( m \neq \frac{L}{2} \). Using this, and the orthonormality of the states \( \left| m, s, y \right> \), I obtain, for odd \( N, \)

\[ \langle \psi | E^{(N)} | \psi \rangle = \sum_{s=-\infty}^{\infty} \int_{y=0}^{a} dy \times C^2 \exp \left( - \frac{(aLs - a + y)^2 + (aLs + y)^2}{4\sigma^2} \right), \]  

(88)

and

\[ (B^{(N)}_1)_{QM} = \text{Re} \frac{\langle \psi | E^{(N)} | \psi \rangle}{2^{(N-1)/2}}. \]  

(89)

Since each \( s \) gives a positive contribution to \( (B^{(N)}_1)_{QM} \), I obtain for odd \( N, \) a lower bound on this Bell correlation by keeping only \( s = 0, \)
\[ \langle B_1^{(N)} \rangle_{QM} \geq 2^{(N+1)/2} \exp \left( -\frac{\mu^2}{2} \right) A(\mu), \]
\[ \mu \equiv \frac{a}{2\sigma}, \quad A(\mu) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\mu}^{\mu} dt \exp \left( -\frac{t^2}{2} \right). \]  

(90)

The choice \( a = 1.8\sigma \) yields,
\[ \langle B_1^{(N)} \rangle_{QM} \geq 0.842 \times 2^{(N-1)/2}; N \text{ odd.} \]

(91)

This contradicts the NCHV bound \(|\langle B_1^{(N)} \rangle| \leq 1\) by a factor \(0.842(\sqrt{2})^{N-1}\) for odd \( N \geq 3 \). This violation holds for the vacuum state \((\sigma^2 = 1/2)\) too, with the choice \( a = 1.8/\sqrt{2} \).

**Non-contextuality violation by a single mode squeezed coherent state**

Consider a squeezed coherent state \(|\psi_\alpha\rangle\) that is obtained by displacing the squeezed vacuum state \(|\psi\rangle\) of the last section,
\[ |\psi_\alpha\rangle = D(\alpha)|\psi\rangle. \]

It has a positive Wigner function,
\[ W_\alpha(x,p) = \frac{1}{\pi} \exp \left( -\frac{(x - \bar{x})^2}{2\sigma^2} - 2\sigma^2(p - \bar{p})^2 \right). \]

(92)

Without any extra work, the quantum prediction for expectation value of \( E_\alpha^{(N)} \) obtained by displacement of the operator \( E^{(N)} \) (equation (86)) in the displaced state \(|\psi_\alpha\rangle\) is obtained simply by using,
\[ \langle \psi_\alpha | E_\alpha^{(N)} | \psi_\alpha \rangle = \langle \psi | E^{(N)} | \psi \rangle. \]

(93)

Hence using the result of the last section, for \( N \) odd,
\[ \langle \psi_\alpha | B_1^{(N)} | \psi_\alpha \rangle = \langle \psi | B_1^{(N)} | \psi \rangle \geq 0.842 \times 2^{(N-1)/2}, \]

(94)

which again contradicts the NCHV bound on \(|\langle B_1^{(N)} \rangle|\) by a factor \(0.842(\sqrt{2})^{N-1}\) for \( a = 1.8\sigma \) and odd \( N \geq 3 \). This violation is obtained for the usual coherent state (no squeezing) too, if we choose \( a = 1.8/\sqrt{2} \), showing the inequivalence of non-contextuality with the alternate classicality criterion of ‘coherence’. The non-contextuality violations can be tested experimentally.

**Conclusions and comparisons with earlier work**

Non-contextuality is a hallmark of classical behaviour, and contextuality a valuable quantum resource [8]. It is shown that arbitrarily large violation of non-contextuality is possible even in states of a single continuous variable system with positive Wigner function. Although the pseudo-spin observables are defined here in \( x \)-space, they are non-diagonal. It is known that a classical probability description for any diagonal operator such as \(|x\rangle\langle x|\) or \(|p\rangle\langle p|\) (or \(|c\rangle\langle c|\) where \( c \) is an eigenvalue of any linear combination of the two quadrature operators), is possible for a quantum state with a positive Wigner function. The present work which extends to ‘contextuality’ the earlier work ([1–3]) on ‘non-locality’ shows very clearly that the ‘classicality’
of states with positive Wigner function does not hold for pseudo-spin operators which are non-diagonal in x-space; the non-classicality can be arbitrarily large.

There is at least another class of non-contextuality inequalities violated by quantum states irrespective of the positivity of the Wigner function. Plastino and Cabello, and others [20] obtained state-independent non-contextuality inequalities for continuous (and discrete) variables. In particular, state independent inequalities involving 18 modular observables for a two dimensional configuration space were shown to be violated by any quantum state by a factor $2/\sqrt{3}$.

In contrast with all earlier work, the present non-contextuality inequalities for a one-dimensional configuration space can be violated by some quantum states by arbitrarily large factors (when $N$ is chosen large enough). Practical demonstrations of ‘contextuality’ using the inequalities presented here will involve measurement of Bell correlations of displaced pseudo-spin observables in quantum optical states. The squeezed coherent state seems particularly promising. Efficient techniques have been developed for manipulation and measurement of continuous quadratures and their phase space displacements [15]. If practical techniques to measure the pseudo-spin observables non-diagonal in quadrature space are developed, applications of large classicality violation to quantum information tasks would be possible. It will be interesting to know if decoherence effects may be less severe in a single system used here than in multiparty systems used before.

Acknowledgments

I thank Sibasish Ghosh, Aditi Sen De, Saptarshi Roy, N Mukunda and Anupam Garg for intense discussions during the QFF2018 conference at the Raman Research Institute, Bangalore, April 30 to May 4, 2018, Guy Auberson for the reference to phase space Bell inequalities optimal for experimental tests [18], and A Asadian for references to their work [22] on applications of modular observables. I thank the organisers Urbasi Sinha, Dipankar Home and Alexandre Matzkin for invitation to this conference, and the Indian National Science Academy for the INSA honorary scientist position.

ORCID iDs

S M Roy https://orcid.org/0000-0002-0743-5416

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