Eat AND Study but Wii OR Ski! Differentiating Between ‘Basic’ and ‘Non-basic’ Dimensions in a Multidimensional Index

Jaya Krishnakumar1 · Mario Biggeri2 · Mauro Vincenzo3

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Abstract
Measuring and monitoring multidimensional wellbeing have become central issues in international policy debates in recent years. Interpreting heterogeneity in achievement levels across dimensions in terms of individual freedom of choice, and arguing that the degree of substitutability should not only depend on this variation, but also on the typology of dimensions, we propose a Generalized Multidimensional Synthetic index that introduces several layers of flexibility in the aggregation methodology. We classify dimensions into different groups, for example basic and non-basic, and allow for varying substitutability rates within as well as between groups. After examining the theoretical properties of the new index, we perform simulation experiments which highlight the salient features of our index compared to other frequently used indices.

Keywords Capability approach · Composite indices · Human development · Multidimensional wellbeing · Substitutability

JEL Classifications C43 · I31

1 Introduction
Measuring and monitoring multidimensional poverty and wellbeing has been an important subject of debate and discussion in the international arena in recent years and in particular in the 2030 Agenda. Although there is a growing literature on this topic, it still deserves a careful attention from a capability approach perspective (Anand et al., 2011a, 2011b). The Multidimensional Synthesis of Indicators (MSI) introduced by (Mauro et al., 2018a, 2018b) presents a way to take into account the heterogeneity (variation) of individual achievements across dimensions by making a case that it can be interpreted in terms

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1 Institute of Economics and Econometrics, Geneva School of Economics and Management, University of Geneva, 40, Bd. du Pont d’Arve, 1211 Geneva 4, Switzerland
2 Department of Economics and Management, University of Florence, Florence, Italy
3 Department of political science, communication, and international relations, University of Macareta, Macareta, Italy

Jaya Krishnakumar
jaya.krishnakumar@unige.ch
of the freedom of choice of the individual. In other words, at low levels of achievement, a large variation across dimensions is normally a reflection of lack of choice, whereas the same could be due to deliberate decisions at higher levels. Hence the former situation is not a ‘good’ one in terms of overall wellbeing freedom whereas the latter is acceptable. The point that we would like to make here is that the decision on the degree of substitutability among the different dimensions in the aggregation procedure is theoretically weak and often relegated to an arbitrary weighting system.

The MSI provides a method to operationalise the idea of making the degree of substitutability among dimensions depend on the level of the general wellbeing. However, the original form of the MSI, does not make allowance for the fact that certain dimensions may be considered essential and hence might not be substitutable even at high levels of aggregated wellbeing. The innovation in MSI consisted in introducing a flexible substitution mechanism through a function that depends on the overall level of wellbeing. But it did not bring in the question of the nature of the dimensions considered while deciding on the degree of substitutability among them.

The aim of this paper is to go deeper into the question of substitutability by saying that one should have a mechanism within the aggregation procedure by which it should be possible to decide on the substitutability or not between two different dimensions depending on their nature. For this purpose, we extend the theoretical reasoning of MSI in two directions leading to a Generalized Multidimensional Synthetic (GeMS) index. First, we allow for the possibility to have different groups of dimensions depending on the degree of substitutability within and between them. Thus we would have some groups within which substitutability will be low and others within which substitutability will be high. The second generalization is to let the substitutability between groups also vary with the overall wellbeing level across key groups.

In the next section, the general principles underlying the GeMS index are introduced. In Sect. 3, Amartya Sen’s capability approach is taken as an illustrative framework for explaining the relevance of the extension proposed in the GeMS index. Section 4 is concerned with the technical details of the aggregation procedure. Section 5 discusses the theoretical properties of GeMS index. Section 6 reports results of some simulation experiments that support the theoretical claims. The concluding section contains a summary of the main findings and further challenges.

2 The Generalized Multidimensional Synthetic (GeMS) Index

The development of state-of-the-art synthetic measures is crucial for a serious study of such a complex phenomenon as well-being. A fast-changing world, with its deluge of information, calls for adequate statistical techniques to be developed at a rapid pace as well, providing synthetic measures that can deal with situations that are completely different from those of even a decade ago.

One of the main advancements towards an appropriate handling of this complex structure is the composite index methodology, of which a pioneer in the field of wellbeing has been UNDP with its launch of the Human Development Index (HDI) in 1990 as a credible alternative to go beyond GDP (UNDP-HDR, 1990) and many other subsequent indices.
Having received major criticisms on the use of arithmetic mean i.e. on the perfect substitutability among dimensions (Chowdhury & Squire, 2006; Desai, 1991), the developers of HDI changed the aggregating function from linear to geometric, with the aim of “...addressing one of the most serious criticisms of the linear aggregation formula, which allowed for perfect substitution across dimensions” (UNDP-HDR, 2010, p. 216). Although the geometric mean is only a partial solution (Zhou & Chen, 2010), it represents one of the first moves towards a flexible solution to the crucial issue of substitutability. The importance of this switch lies more in the fact of having defined a key area for research rather than in the formula itself. In fact, more flexible forms such as the generalized mean have been analysed in related research, for instance the multidimensional poverty measure proposed by Bourguignon and Chakravarty (2003).

Nonetheless, literature on the management of substitutability among a large number of dimensions is still scarce. The main stages of the construction of an index, like the choice of dimensions, the normalization of their indicator values, harmonization, weighting, and aggregation methodology, are strictly interconnected when it comes to taking into account substitutability rates and heterogeneity among achievement levels. For example, the weighting system, widely debated over decades, is a function of the trade-off between pairs of indices and therefore should be interpreted as such from the very beginning. Decancq and Lugo (2013) and Krishnakumar (2018) shed some light on this key issue, showing how the weights are closely linked to marginal rates of substitution among pairs of indices.

One of the important limitations of the aggregate indices currently proposed in the literature is that the expression of the marginal rate of substitution is the same for any pair of dimensions. Although the MSI proposed by (Mauro et al. 2018a, 2018b) tries to introduce some flexibility in the substitutability rate according to a ‘general’ level of wellbeing, it still does not allow variable substitutability rates (functions) between two dimensions, that is substitutability varying according to the dimensions chosen. The GeMS index of this paper precisely introduces this additional layer of flexibility, by classifying dimensions into different groups based on some well-defined criteria and allowing for varying substitutability rates within and between these dimension groups.

The GeMS index introduced in this paper is a general index that includes the MSI (and many other commonly used indices) as special cases depending on the values of a few parameters. The core feature of this new index is a finer management of the chosen dimensions in terms of their number as well as their nature. We divide the dimensions into different groups according to the degree of substitutability allowed within the groups as well as between groups.

The Mazziotta-Pareto Index (MPI, cf. De Muro et al. (2011), Mazziotta and Pareto (2016) is another index that adjusts for imbalance of achievement levels across indicators while summarising them into a composite index. However, MPI is different from our approach since “It is based on a non-linear function which, starting from the arithmetic mean, introduces a penalty for the units with unbalanced values of the indicators” (Mazziotta & Pareto, 2016, p. 987).

Casadio Tarabusi and Guarini (2013, 2016) also propose indices that take into account the disparity or the imbalance in achievement across different dimensions. In Casadio Tarabusi and Guarini (2013), the adjustment for imbalance is operationalised through two key parameters—called $\alpha$ and $\beta$ in their mean-min aggregation function. Whereas, in Casadio Tarabusi and Guarini (2016), the adjustment is obtained by setting different values for a pair of parameters named $(a, b)$ (for example $(-1, 1), (1, 1)$ and $(0, 1)$) in their Trichotomy mean index which is a particular case of a generalized mean using a very special transformation $f(\cdot)$ of the data. Various combinations of the two parameters yield varying degrees of
substitutability. In our proposal, we achieve flexibility by making the substitutability parameter depend on the general (mean) level of wellbeing. Thus, although Casadio Tarabusi and Guarini (2013, 2016) achieve the same goal as ours, namely varying degrees of substitutability at different levels of well-being, they do it through different combinations of some key parameters whereas we obtain the same by making the substitutability parameter itself a function of the observed level of wellbeing. Further, our approach allows for an easy classification of different dimensions into groups such as basic and non-basic (see Sect. 3 below).

Now, suppose we have $n$ indicators for each individual $i$ ($i = 1, ..., N$), normalised to the $[0, 1]$ scale, and these indicators are divided into $k$ groups of varying levels of substitutability, each group $j$ containing $n_j$ indicators, $j = 1, ..., k$, such that $\sum_{j=1}^k n_j = n$. One can then compute the aggregate GeMS index in two stages as follows.

First the aggregate for a given group (of indicators), say, the $j$-th group:

$$GeMS_{jd} = 1 - \left[ \frac{1}{n_j} \sum_{h=1}^{n_j} (1 - x_{h,j}) g_j(x_{j,h}) \right]^{\frac{1}{g_j(x_{j,h})}} \tag{1}$$

where the generic entry $x_{h,j}$ represents the $h$-th indicator in the $j$-th group for unit $i$, $x_{j,h}$ represents the $n_j$-vector of indicators of group $j$ for individual $i$, and the $k$ functions $g_j(\cdot), j = 1, ..., k$ are scalar real-valued continuous functions representing the theoretical link to the degree of substitutability between achievements within group $j$, with $g_j(x_{j,h}) \geq 1$ and decreasing in its arguments.

Let us denote as $\gamma_i$ the $k$-vector of group indices with the generic element $GeMS_{j,i}, j = 1, ..., k$, given in the previous formula. Then the aggregate over groups, for an individual $i$, denoted as $GeMS_i$, is given by:

$$GeMS_i = 1 - \left[ \frac{1}{k} \sum_{j=1}^{k} (1 - GeMS_{j,i}) g(\gamma_i) \right]^{\frac{1}{g(\gamma_i)}} \tag{2}$$

where the scalar-valued continuous function $g(\gamma_i)$ represents the degree of substitutability between groups, with $g(\gamma_i) \geq 1$ and decreasing in its arguments.

The idea of making the degree of substitutability depend on the level of wellbeing, either at the group level or at the aggregate level, is this. When the wellbeing is low at either of these levels, variation in achievement levels among the different indicators/groups is not ‘good’ as it implies that the individual is at an even lower level of achievement in some indicators/dimensions. Hence there should be very little or no substitutability at these (low) aggregate levels, whereas one could allow for substitutability at higher aggregate levels. So the degree of substitutability should be directly proportional to the overall level of achievement (note that a higher value of $g(\cdot)$ means lower substitutability, hence $g(\cdot)$ is inversely related to the substitutability).

One can visualise how the substitutability changes with the level of wellbeing by plotting the isoquants of our GeMS index function at various values of the function. As an illustration, the graph below (Fig. 1) plots the isoquants of our GeMS index for two dimensions$^1$. One can see that as the general level of well-being approaches zero, the isoquants reflect less and less substitutability between the two dimensions.

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$^1$ The isoquants are drawn for the $g(\cdot)$ function given in Sect. 4, i.e. for the index defined by Eqs. (4 and 5), with $n = 2, a = 4$, and $z = 0.5$. Here the authors would like to thank Rajiv Krishnakumar for assistance in programming the graphical representation of the GeMS function (Fig. 2) as well as its isoquants (Fig. 1) using Matlab.
Substituting (1) into (2), we can write

\[
GeMS_i = 1 - \left( \frac{1}{k} \sum_{j=1}^{k} \left( 1 - \left[ \frac{1}{n_j} \sum_{h=1}^{n_j} (1 - x_{h,j,i}) g_j(x_{h,j,i}) \right] \right) \right)^{g(r_j)^{\frac{1}{\sigma_{r_j}}}}
\]

or

\[
GeMS_i = 1 - \left[ \frac{1}{k} \sum_{j=1}^{k} \left( \frac{1}{n_j} \sum_{h=1}^{n_j} (1 - x_{h,j,i}) g_j(x_{h,j,i}) \right) \right]^{g(r_j)^{\frac{1}{\sigma_{r_j}}}}
\]

(3)

This general version of the index allows for an extreme flexibility. Nonetheless, it must be noted that the number of parameters to be set increases significantly with the number of groups, implying a trade-off between the flexibility of the index and its simplicity. This is mainly due to the rapid increase in the complexity of the theoretical formulation with the number of groups \(k\). Moreover, a high value of \(k\) requires additional assumptions on the procedure of aggregation over the \(k\) group indices that are neither straightforward nor easily justifiable.

Most of the common indices used in literature assume only one group (i.e. \(k = 1\)) and hence do not take into account different degrees of substitutability for different groups of dimensions. We suggest to assume either \(k = 2\) or \(k = 3\) depending on the number of total dimensions available and their nature, based on a simulation procedure (presented in Sect. 5). Although the GeMS index remains formally sound for any value of \(k\), the
simulations suggest that for values greater than three, the complexity induced is hardly compensated by the added flexibility and should therefore be considered with a caveat.

3 A Capability Approach Perspective as Illustrative Example

The GeMS index proposed in the previous section is suitable for the construction of any composite index for any concept that warrants the consideration of multiple aspects of different importance. In this section we would like to take the capability approach (CA) as an illustrative framework to capture the potential use of the GeMS index from a theoretical perspective.

The essential idea of the capability approach is that social arrangements should aim to expand people’s capabilities—their freedom to achieve valuable beings and doings (Sen, 1999). In particular, “the focus of the CA is not just on what a person actually ends up doing, but also on what she is capable of doing, whether or not she chooses to make use of that opportunity” (Sen, 2009, p. 17). In this process of choice, there are some capability domains/dimensions that are intrinsically and instrumentally essential or at least more important than others (Sen, 1999). Even within the same dimension of wellbeing, different components can have different relevance and can be more or less essential. This differentiated importance of dimensions is reflected in the distinction between basic and non-basic dimensions in the capability literature.

The idea of basic domains is that there are certain dimensions that are fundamental for human life, of primary importance for wellbeing and, hence they should be a top priority for social arrangements and public policies (Sen, 1999). Therefore, it should not be possible within these basic dimensions to substitute wellbeing in one dimension for that in another, at least up to a relatively high level of achievement in all of them (determined by a high level of the aggregate). The non-basic dimensions are by definition not essential although they may be relevant ‘beings’ and ‘doings’ for human flourishing, and hence, even if a person is at a low level in any of these it should not be of a serious concern from a policy perspective. Following Aristotle, the capabilities of a person have been associated with human flourishing, at the same time they can be realized in many different ways (Nussbaum, 2000). Another categorisation of dimensions could be: basic, quasi-basic and non-basic capabilities².

The most well-known and diffused indicators drawing inspiration from the capability approach, are the HDI (UNDP-HDRO, 1990) and the Multidimensional Poverty Index (MPI) proposed by (Alkire & Foster, 2011) and UNDP-HDRO (2010)³, but few of them recall the freedom of choice underlying this approach.

From a philosophical perspective, the MSI (Mauro et al. 2018a, 2018b) is in better alignment with the capability approach by taking individual heterogeneity in achievements across dimensions as reflecting freedom of choice and making the substitutability

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² Nussbaum proposes a different categorization between simple and complex capabilities (2000). This categorization is not used here.
³ The MPI method of aggregation is based on two thresholds or cut-offs. A single cut-off (thresholds) is set for each dimension to define whether or not the person is deprived obtaining a binary definition of deprivation. The scores relating to each dimension are then aggregated and a second cut-off used (i.e. dual cut-off) to determine multidimensional poverty (Chakravarty & D’Ambrosio, 2006). This index, however, does not discuss the substitutability among dimensions as well as heterogeneity in achievement levels across dimensions, that are intrinsic to the value added of the capability approach.
parameter depend on it. However, it does not take into consideration the nature of the various dimensions included, in particular whether we are dealing with basic or non-basic capabilities. The GeMS index presents a neat extension that (a) enables to restrict substitutability within the group of basic dimensions, especially at low wellbeing levels, possibly releasing the restriction at high levels, and (b) allows for greater substitutability within non-basic dimensions at all wellbeing levels. Thus there is a two-layer flexibility in the treatment of dimensions—with respect to their nature, say basic or non-basic, as well as with respect to the degree of substitutability within and between groups. These theoretical considerations that are important for a fuller operationalisation of the capability approach (Sen, 1985, 1999; Comim et al., 2008) are easily generalised to the case when the number of groups is higher 4.

Let us illustrate our index in a CA setting with two groups of well-being dimensions. Let us say we have a first group of \( n_1 \) basic dimensions with a degree of substitutability that changes according to the level of a certain variable (usually the overall level of wellbeing, although Biggeri and Mauro (2018) suggest the possibility of using a single dimensional index such as the environmental group index as an alternative), and another group containing \( n_2 \) non-basic dimensions with a fixed (high) degree of substitutability.

Dimensions in the first group can be aggregated into a first sub-index using the MSI methodology which allows the substitutability parameter to be determined by a function \( g_b \) that will depend on the mean level of well-being and which also incorporates an adjustment for the variation in the \( n_1 \) dimensional achievements of this group. The function \( g_b \) will reflect the idea that, below a certain level of well-being, basic dimensions (such as health and education) should have a very low degree of substitutability, if not zero.

Dimensions in the second group are synthesized into a second sub-index through a parameter reflecting a high level of substitutability and therefore not necessarily dependent on the level of well-being 5.

The overall value of the index is then given by aggregating the two group indices using a substitutability parameter which can be made a function \( g(\cdot) \) of one or both of the group indices. This function takes into account the between-groups heterogeneity that cannot be captured by the group-indices.

The mathematical structure of the GeMS index allows a high flexibility in the management of both the within and the between heterogeneity induced by splitting the dimensions into different groups. For example, choosing the substitutability parameter \( g = 1 \) transforms the group indices into a simple arithmetic mean (i.e. perfect substitutability among dimensions within a group), and choosing \( g(\cdot) = 1 \) does the same between groups. Higher-order means, Leontief function and many other standard methods of aggregations, including the MSI of Mauro et al. (2018a, 2018b), can be easily obtained as special cases of the GeMS index.

Thus the GeMS index provides the researcher with a simple and flexible tool capable of taking into account any theoretical assumptions on the nature of the dimensions chosen, their classification into groups and the structure of substitutability within and between the groups. Thus this index can be implemented for all multidimensional concepts in any field such as health, justice, sustainability, biodiversity and so on. Moreover, these assumptions are incorporated in the GeMS index in a transparent way, so that they can be easily tested

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4 See last paragraph in Sect. 2.

5 It is also possible in this group too to further increase the substitutability parameter value at high mean wellbeing levels, however there is no theoretical justification for this.
through sensitivity analyses in order to check the robustness of the results to different specifications of the parameters of the substitutability or aggregation functions.

4 Construction of the Index in the Two Group Case

4.1 Basic Dimensions

The idea here is that there are certain dimensions which are basic for human life, and therefore wellbeing in all these dimensions is fundamental and essential. Thus the substitutability between the basic dimensions should be low, at least up to a relatively high level in all of them (determined by a high mean level of wellbeing). In addition, disparities in achievement levels among these dimensions is also not good at low levels (as more disparity implies even lower wellbeing than the mean in certain dimensions). One can imagine that once individuals have reached a relatively high level in all (as given by the mean), one could allow some trade-off. Suppose there are \( n_1 \) basic dimensions, let us denote the level of wellbeing in these \( n_1 \)-vector \( x_{hb,i} \), with the generic element \( x_{hb,i}^h \), \( h = 1, \ldots, n_1 \).

Denoting the aggregate index as \( GeMS_{b,i} \), we propose

\[
GeMS_{b,i} = \begin{cases} 
1 - \left[ \frac{1}{n_1} \sum_{h=1}^{n_1} (1 - x_{hb,i})^\alpha \right]^{\frac{1}{\alpha}} & \text{for } \mu_{b,i} \leq z_b \\
1 - \left[ \frac{1}{n_1} \sum_{h=1}^{n_1} (1 - x_{hb,i}) g_b(x_{b,i}) \right]^{\frac{1}{g_b(x_{b,i})}} & \text{for } \mu_{b,i} > z_b
\end{cases}
\]

where \( \mu_{b,i} \) denotes the mean achievement in the basic dimensions, that is the average over \( h \) of the elements of the \( n_1 \)-vector \( x_{b,i} \). So a constant and high \( \alpha \) say equal to 4 until the mean level of wellbeing reaches a certain level \( z_b \), and then \( \alpha = g_b(x_{b,i}) \), with \( g_b(x_{b,i}) > 1 \). For ensuring continuity of the aggregate index \( GeMS_{b,i} \) at the threshold, we should have

\[
g_b(x_{b,i}) = \frac{(1 - \alpha) \mu_{b,i}}{(1 - z_b)} + \frac{(\alpha - z_b)}{(1 - z_b)}
\]

4.2 Non-basic Dimensions

The non-basic dimensions are by definition not essential, and hence even if a person is at a low level in any of these it should not be of serious concern. Hence we allow for substitutability among them. Suppose there are \( n_2 \) non-basic dimensions.

\[
GeMS_{nb,i} = 1 - \left[ \frac{1}{n_2} \sum_{h=1}^{n_2} (1 - x_{hb,i})^\beta \right]^{\frac{1}{\beta}}
\]

with an \( \beta \) corresponding to maximum substitutability say \( \beta = 1 \).

In line with our reasoning about basic and non-basic dimensions in Sect. 3, we argue that there is no need to take variation of achievements into account for non-basic dimensions, as these dimensions are, by definition, not indispensable for a decent quality of life. Otherwise the distinction between basic and non-basic groups will vanish.
4.3 Combination of the Two

Here again, following the same logic as in the distinction between basic and non-basic dimensions, the substitutability factor should only depend on the the aggregate wellbeing in the basic dimensions.

\[ GeMS_i = 1 - \left[ \frac{1}{2} (1 - GeMS_{b,i})^{g(GeMS_{b,i})} + \frac{1}{2} (1 - GeMS_{nb,i})^{g(GeMS_{b,i})} \right]^{\frac{1}{g(GeMS_{b,i})}} \]  (7)

4.4 Aggregation Over Individuals

Our idea of introducing flexibility in substitutability decisions can be applied in the case of vertical aggregation too, i.e. for aggregating GeMS indices of \( N \) individuals in a population. For this, we propose:

\[ GeMS = 1 - \left[ \frac{1}{N} \sum_i (1 - GeMS_i)^{g(m)} \right]^{\frac{1}{f(m)}} \]  (8)

where \( m \) is the \( N \)-vector of GeMS indices of all individuals in the population i.e. \( m = [GeMS_1, ..., GeMS_N]' \) and the substitutability parameter is made to depend on the general wellbeing level in the population (say the mean wellbeing for example) through \( f(m) \) with \( f(m) > 1 \). Note that we use a different \( g(\cdot) \) for the aggregation over individuals, calling it \( f(\cdot) \).

In this case, the fact that the substitutability parameter depends on the variability of wellbeing among individuals implies that inequality of distribution of wellbeing in the population is ‘penalised’. We do not elaborate further on this point of aggregation across individuals, in order to keep the length of our paper reasonable and retain our focus on the aggregation across dimensions.

5 Properties of GeMS Index

5.1 At the Individual Level i.e. Aggregation Over Dimensions

We would like to state at the outset that we prove the properties of the GeMS index as defined in formulas (4) and (5) for the basic dimensions, the index for non-basic dimensions being a standard generalised mean whose properties are well-known. As we have two stages in our procedure—aggregation at the group level and then aggregation across groups, with the same generic formula at both levels, we will show the properties at only one level, say the second level of aggregation across groups. The same properties hold for the within group aggregation. Note that, even if the formula is in two parts for above and below a threshold within a group, the derivation of properties can be repeated for each part, as we have ensured continuity at the threshold.

The formula for \( GeMS \) for individual \( i \) is given by:

\[ GeMS_i = 1 - \left[ \frac{1}{2} (1 - GeMS_{b,i})^{g(GeMS_{b,i})} + \frac{1}{2} (1 - GeMS_{nb,i})^{g(GeMS_{b,i})} \right]^{\frac{1}{g(GeMS_{b,i})}} \]
Or with $k$ groups:

$$\text{GeMS}_i = 1 - \left[ \frac{1}{k} \sum_{j=1}^{k} (1 - \text{GeMS}_{j,i})^{g(\gamma_i)} \right]^{\frac{1}{g(\gamma_i)}}$$

with $g(\gamma_i) > 1$.

1.1 **Continuity**: It is easily seen that our index is continuous in all its arguments $\text{GeMS}_{j,i}$, $j = 1, \ldots, k$.

1.2 **Normalisation**: $\text{GeMS}_i = z$ when $\text{GeMS}_{j,i} = z \forall j$. It is satisfied. Proof in the Appendix at the end.

1.3 **Linear homogeneity** (homogeneity of degree 1): This property states that when all dimension indices are multiplied by $\lambda$, i.e. all $\text{GeMS}_{j,i}$ become $\lambda \ast \text{GeMS}_{j,i}$, then the aggregate index is also multiplied by $\lambda$. It is not satisfied. This is due to the fact that this index aggregates shortfalls and then subtracts the aggregate shortfall from one to obtain the wellbeing index value. Proof (in one line) is in the Appendix at the end.

1.4 **Symmetry in dimensions**: It is easy to verify that permuting dimensions ($j$’s) does not change $\text{GeMS}_i$.

1.5 **Monotonicity**: This property states that when the wellbeing increases in a particular dimension, with all other dimensions unchanged, then the overall wellbeing increases. It is satisfied. Proof in the Appendix at the end. In addition to the analytical proof in the Appendix, one can also visually see the monotonic nature of GeMS index in the graph of $\text{GeMS}_i$ as a function of one of its elements say $\text{GeMS}_{j,1}$ (see Fig. 2 above), the curve being exactly the same for all elements of $\text{GeMS}_i$. 

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{Fig2.png}
  \caption{Graph of $\text{GeMS}_i$ as a function of one of its elements $\text{GeMS}_{i,1}$.}
\end{figure}
5.2 At the Population Level i.e. Aggregation Over Individuals

Let us recall the population index.

\[ GeMS = 1 - \left( \frac{1}{N} \sum_{i} (1 - GeMS_i)^{f(m)} \right) \]

with \( f(m) > 1 \).

P.1 Continuity: It is once again easily verified that the index is continuous for all \( GeMS_j \).

P.2 Normalisation: \( GeMS = z \) when \( GeMS_i = z \) \( \forall \ i \). It is satisfied. Proof in the Appendix at the end.

P.3 Linear homogeneity (homogeneity of degree 1): As for the individual index, this property does not hold for the population aggregate index.

P.4 Symmetry in individuals: It is easy to confirm that permuting individuals does not change \( GeMS \).

P.5 Monotonicity: The proof of monotonicity is exactly similar to the one given for the individual index, and hence not repeated.

P.6 Population replication invariance: Let us say we have a population distribution of \( GeMS \) given by the following vector

\[ y = (GeMS_1 GeMS_2 \ldots GeMS_M) \]

Now let us consider a replication of this population \( \ell \) times (i.e. \( \ell N \) individuals).

\[ y^* = (GeMS_1 GeMS_2 \ldots GeMS_N GeMS_1 GeMS_2 \ldots GeMS_N \ldots (\ell \ times) \]

Then the property states that \( GeMS(y^*) = GeMS(y) \). It is satisfied. Proof in the Appendix at the end.

P.7 Subgroup consistency: This property states that if wellbeing of one group increases, everything else remaining the same, then overall wellbeing increases. This result follows directly from monotonicity.

P.8 Subgroup decomposability: Dividing the population into different groups, this property states that the aggregate wellbeing can be decomposed into a sum of group aggregates. This property is satisfied by construction.

P.9 Path independence: This property states that the aggregate index will be the same whether we aggregate first over dimensions and then over individuals or vice versa. It is rather easily seen that this property is not satisfied

6 Simulations and Discussion of Results

This paragraph introduces a simulation study performed in order to test the behaviour of our GeMS index in different scenarios, compared with other common indices\(^6\).

\(^6\) The Mazziotta Pareto index (MPI, Mazziotta & Pareto, 2016) is an ideal benchmark to compare with GeMS. However, the MPI relies on a different method of normalisation with mean 100 and standard deviation 10, that makes the comparison impossible, as it is hard to disentangle the effect of the aggregating function from that of the normalisation approach while comparing the adjustments for heterogeneity. Nonetheless, the reader should consider that the behaviour of the MPI is similar to that of the geometric mean when both are normalised in the MPI way.
The data used for the test are initially obtained from a simulated dataset of \( n = 2,000,000 \) observations measured on \( k = 7 \) variables. The variables are divided in two groups, with \( k_1 = 4 \) basic (such as nourishment, education, health, living conditions) and \( k_2 = 3 \) non-basic dimensions (such as electronic entertainment, luxury sports). All the variables are generated using a uniform distribution bounded between 0 and 1, mimicking the process of normalisation usually performed in presence of dimensions measured with different magnitude or unit of measurement. The variables are generated as independent and with positive polarity, so that higher values correspond to better scenarios.

The mean of the \( k \) variables, as a function of the sum of a number of independent uniform random variables follows a Irwin-Hall distribution, as confirmed by Fig. 3a. As a result, most of the observations present an overall mean close to the central value of 0.5. The goal of the simulation is to monitor the behaviour of the index in different situations (i.e. at different levels of mean and variance of the achievements of a unit), so that a Irwin-Hall distribution could bias the results by implicitly assigning higher weights to units around the centre. In order to avoid this issue, the observations are trimmed, obtaining an (approximately) uniform distribution of the means of the units, as shown in Fig. 1B. The final number of observations after the trimming is 2000, for which Table 1 reports some basic descriptive statistics. It is important to underline that the trimming

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Alternative distributions like Normal, Gamma, and Beta were tested, and the results did not differ significantly. As a result, we decided to propose the Uniform distribution for sake of simplicity.
procedure induces a positive correlation between the variables that is sometimes significant. Nonetheless, some degree of correlation among the variable is a very common scenario, so we decided to keep it. A more detailed analysis about the structure of the correlation could be an interesting extension of the research that remains beyond the aims of this paper.

Once the 2000 units are generated, the first 4 basic variables are aggregated using the formula for basic dimensions (see Eq. 4) of the GeMS, setting a threshold of $z = 0.4$. When the arithmetic mean of the basic achievements of a unit is smaller than $z$, $\alpha$ is set to 4 ($\alpha \geq 1$ is the power of the function, the greater its value, the more severe is the heterogeneity adjustment); otherwise, the value of $\alpha$ is a linear function $\alpha(\cdot)$ identified by the two constraints i) $\alpha(z) = 4$ (for the function to be continuous) and ii) $\alpha(1) = 1$ (so that the upper limit of the achievements corresponds to a situation with perfect substitutability among achievements). The non-basic variables are aggregated using a simple arithmetic mean.

The final index (GeMS) aggregates the synthetic measures of the two groups of basic and non-basic dimensions into a single score. In this phase of aggregation, the level of $\alpha$ for a unit is the same used for the basic group aggregation step, in order to stress the higher importance given to (possible) imbalance in the basic dimensions.

Figure 4 shows the performances of the GeMS and the Geometric mean calculated on the group of 2000 units. They are both compared with the arithmetic mean, that assumes perfect substitutability among achievements. The left hand side panel shows the differences for the overall index and the right panel only for the aggregate of basic dimensions.

Both the GeMS (green dots) and the Geometric mean (orange dots) lie below the arithmetic mean (red line), as the heterogeneity of the achievements negatively affects their overall score. The Geometric mean presents the well-known problem of an excessive adjustment for units that present one or more zeros in their achievements. As a result, the orange straight line at the bottom represents the points affected by this issue, especially severe for units on the left, that are more likely to present small values. Comparing the two panels, one can that these differences are more pronounced for the basic dimensions as the green dots go much lower than the orange ones here. We observe the same pattern if we use relative differences instead of the absolute differences shown in Fig. 4. The graphs are not reproduced for saving space and are available with the authors upon request.

Table 1 Summary of the 7 variables after the trimming (4 basic and 3 non basic)

| Var | Obs. | Mean  | St.Dev. | Min | Max |
|-----|------|-------|---------|-----|-----|
| $V_1$ | 2000 | 0.505 | 0.337   | 0   | 1   |
| $V_2$ | 2000 | 0.496 | 0.297   | 0   | 1   |
| $V_3$ | 2000 | 0.506 | 0.316   | 0   | 1   |
| $V_4$ | 2000 | 0.498 | 0.326   | 0   | 1   |
| $V_5$ | 2000 | 0.502 | 0.299   | 0   | 1   |
| $V_6$ | 2000 | 0.501 | 0.333   | 0   | 1   |
| $V_7$ | 2000 | 0.493 | 0.307   | 0   | 1   |
Table 2 reports the average levels of reduction of the two indices analysed with respect to the arithmetic mean, considering all the units. The Geometric mean seems to be associated on average with higher reductions (-25.5%) than the GeMS (-17.4%). This result could be biased by units that are set to zero, so the table also reports the results obtained after dropping these units. As expected, the adjustment induced by the Geometric mean is still higher that of GeMS, but the two indices are now closer. The GeMS seems to induce a higher adjustment for units with a low arithmetic mean (deprived8 units, bottom left). This is more evident in Fig. 4, where the green dots are mostly lower than the orange ones (excluding the zeroes) even at low values of the arithmetic mean.

This first comparison shows how GeMS discriminates more between “deprived” and “non-deprived” units. Its distribution, in fact, appears positively skewed, while the

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Table 2  Performances of the GeMS and Geometric Mean compared to the Arithmetic mean

| Index    | Mean  | St.Dev. | Min   | Max   |
|----------|-------|---------|-------|-------|
| GeMS     | -0.146| 0.109   | -0.723| 0     |
| Geom. mean| -0.255| 0.278   | -1    | -0.001|
| GeMS*    | -0.136| 0.101   | -0.581| 0     |
| Geom mean*| -0.174| 0.137   | -0.617| -0.001|

*Only units without 0s

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8 We refer to units with a low level of multidimensional wellbeing as deprived and those with a relatively high wellbeing level as non-deprived units.
Geometric mean seems to have a negative skewness (i.e. contrary to expectations, the adjustment seems slightly higher for non-deprived units).

In order to capture these dynamics in more detail, the analysis has been repeated comparing the performances in 6 different situations. The units are first divided into
Fig. 7 Performances of the GeMS and the Geometric mean compared to the arithmetic mean using absolute differences—High Variability

Table 4 Performances of the GeMS and Geometric Mean for different groups of units compared to the Arithmetic mean in case of Medium Variability

| Status          | Index | Mean | Mean* | St.Dev. | Min  | Max  |
|-----------------|-------|------|-------|---------|------|------|
| Not deprived    | GeMS  | −0.159 | 0.057 | −0.398  | −0.060 |      |
|                 | Gmean | −0.151 | 0.101 | −1      | −0.034 |      |
| Deprived        | GeMS  | −0.194 | 0.087 | −0.623  | −0.005 |      |
|                 | Gmean | −0.269 | 0.239 | −1      | −0.044 |      |
| Total           | GeMS  | −0.176 | 0.075 | −0.623  | −0.005 |      |
|                 | Gmean | −0.209 | 0.190 | −1      | −0.034 |      |

Table 5 Performances of the GeMS and Geometric Mean for different groups of units compared to the Arithmetic mean in case of High Variability

| Status          | Index | Mean | Mean* | St.Dev. | Min  | Max  |
|-----------------|-------|------|-------|---------|------|------|
| Not deprived    | GeMS  | −0.212 | 0.051 | −0.419  | −0.202 |      |
|                 | Gmean | −0.355 | 0.223 | −1      | −0.115 |      |
| Deprived        | GeMS  | −0.332 | 0.106 | −0.723  | −0.032 |      |
|                 | Gmean | −0.493 | 0.306 | −1      | −0.112 |      |
| Total           | GeMS  | −0.248 | 0.104 | −0.723  | −0.032 |      |
|                 | Gmean | −0.396 | 0.302 | −1      | −0.112 |      |

two groups, with a mean of the basic variables below or above the average value of 0.5, and then divided into three groups according to their coefficient of variation\(^{9}\) (low,

\(^{9}\) Since units with extreme values of the mean are less likely to present a high variability, there is a significant correlation between the mean of a unit and its standard deviation. Units with values close to 0.5 tend to have higher variability. The correlation between mean and CV is lower so we decided to use the latter.
medium, and high). As a result, the performances of the index are analysed in six different scenarios.

The results are reported in Tables 3, 4, and 5, and the analysis is completed with Figures 5, 6, and 7 that show the behaviour of both the GeMS and the Geometric mean for the three subgroups with different CVs. The benchmark is, as usual, the arithmetic mean.

In case of low variability (Fig. 5 and Table 3), the behaviour of the two indices seem different: GeMS appears more balanced, providing mild adjustments for both the deprived and non-deprived, with the deprived units getting more adjustments as expected. The same pattern applies to the Geometric mean, but the adjustment appears quite severe for both groups (−14.6% on average). When we drop units with excessive adjustments due to the presence of 0s, the Geometric mean does not seem to discriminate between groups, while GeMS index remains balanced.

For medium variability levels, as shown in Fig. 6 and Table 4, the behaviour is approximately the same, especially when the analysis is limited to units without 0s to avoid the problem of the Geometric mean.

In case of high variability (Fig. 7 and Table 5) both indices induce a heavy adjustment when compared with the arithmetic mean. Since the high variability is correlated with the presence of outliers, the Geometric mean is severely affected by the issue of the zero values (even more than in the case of low and medium variability), and the orange line at the bottom appears thicker than in the other figures. The behaviour of the GeMS is more robust: the pattern of the green dots is wider than the Geometric mean, since the index discriminates more accurately between units. If we focus on the subgroup of units without 0s, the two indices are not very different. Nonetheless, the GeMS index seems to discriminate more between the two groups, thus presenting a higher degree of adjustment (12% approximately) than the Geometric Mean (3% approximately).

To sum up, the GeMS index seems to perform better than the Geometric Mean for at least two reasons:

1. It completely avoids the collapse to zero that affects the geometric mean, thus allowing the analysis of a higher number of units.
2. It still remains more flexible in assigning a different degree of adjustment to non-deprived and deprived units, even for units not having any zeros, especially when the heterogeneity is very low or very high.

Before ending this section, it is important to underline that the GeMS index is a very flexible index that can be tailored to many different situations. Consequently, the simulations presented in this section only represent one possible real-life scenario among a large number of potential situations that can be investigated. In particular, one could change the following parameters:

1. \( n = 2,000 \), the total number of observations
2. \( k = 7 \), the total number of dimensions
3. Uniform, the data generating distribution
4. \( k_1 = 4 \) and \( k_2 = 3 \), the number of dimensions in the basic and non-basic groups
5. \( z = 0.4 \), the threshold of the basic wellbeing level for altering the substitutability rate

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10 And possibly many others, in case one wants to extend the simulation experiment.
(6) \( \alpha = 4 \), the substitutability rate parameter when the mean wellbeing is smaller than \( z \)
(7) \( \text{Linear} \), the function used to set \( \alpha \) when the arithmetic mean is greater than \( z \)
(8) \( CV \), the index used to disentangle the measured effect
(9) \( 3 \) (low/med/high variability), the number of subgroups to test different scenarios
(10) \( \text{Deprived/non-deprived} \), the threshold for classifying units into these two categories (above or below 0.5)
(11) \( \text{The indices} \) that we compare GeMS with (for example, we could also make a comparison with the standard MSI, in addition to the arithmetic and geometric means)

Although a study of all possible combinations of the above parameters is infeasible and beyond the scope of this paper, the results presented in this section were tested in a few other settings, and the conclusions stated above seem to hold. This sensitivity analysis suggested that the final results seem to be ‘robust’ to the choice of the parameters listed above.

7 Concluding Remarks

The main message of this paper is that in the context of multidimensional wellbeing and deprivation, the degree of substitutability between dimensions should be distinguished according to the typology of the dimensions under study. To this effect, we introduce a new index called the Generalized Multidimensional Synthetic (GeMS) Index. The GeMS index introduces several layers of flexibility in the aggregation methodology by classifying dimensions into different groups, for example basic and non-basic, and allowing for varying substitutability rates within and between groups (this is the first generalization). The second generalization is to let the substitutability between groups also vary with the overall wellbeing level across key groups.

Our GeMS index presents a neat extension that a) enables to restrict substitutability within the group of basic dimensions, especially at low wellbeing levels, releasing the restriction at high levels, and b) allows for greater substitutability within non-basic dimensions at all wellbeing levels. Thus there is a two-layer flexibility in the treatment of dimensions—with respect to their nature, say basic or non-basic, as well as with respect to the degree of substitutability within and between groups.

These theoretical considerations can suit many conceptual frameworks and empirical approaches. For instance, they enable a fuller operationalization of the capability approach, in particular they allow for an explicit characterisation of certain dimensions as basic, which is often mentioned in the theoretical literature (e.g. Sen, 1999) but seldom implemented in practical settings. Moreover, as shown in the theoretical part, the methodology is easily generalised to the case when the number of groups is more than two.

Our simulations show that GeMS is able to reflect the above theoretical reasoning in an adequate manner, and in a better way than frequently used indices such as the arithmetic mean, geometric mean as well as the MSI (which is a special case of GeMS). The simulations bring out the particular feature that distinguishes GeMS index from the earlier ones, namely its flexibility in the choice of the degree of substitutability between dimensions depending on the nature of dimensions concerned, thus pushing the group aggregate value even lower for a ‘basic’ group than for a ‘non-basic’ group, in presence of within-group heterogeneity of achievement levels across dimensions.

Thus the GeMS index provides the researcher with a transparent and flexible tool capable of taking into account any theoretical structure of substitutability among the
dimensions chosen. The theoretical assumptions are incorporated in the GeMS index in a clear way, and can be tested through sensitivity analysis in order to check the robustness of the results to different specifications of the parameters of substitutability or different aggregation functions.

The strength of GeMS is its flexibility. However, if we increase the parameters in GeMS we are going to add more subjectivity to the analysis. This also means that systematic simulation packages need to be built using statistical programs. Therefore, future direction or extensions should go in the direction of empirical applications and systematic sensitivity analysis.

Appendix: Proofs of Properties

We will only prove the properties when the substitutability parameter is a function of the mean level of wellbeing i.e. when a $g(\cdot)$ function is used in the formula. It is straightforward to see that the properties are satisfied for a fixed $\alpha$ or $\beta$.

- **Proof of I.2**
  Indeed we have:
  \[
  GeMS_i = 1 - \left[ \frac{1}{k} \sum_j (1 - z)^{g(z)} \right]^{\frac{1}{g(z)}} \\
  = 1 - \left[ \frac{1}{k} k(1 - z)^{g(z)} \right]^{\frac{1}{g(z)}} \\
  = 1 - (1 - z) \\
  = z
  \]

- **Proof of I.3**
  When all dimension indices are multiplied by $\lambda$, i.e. all $GeMS_{j,i}$ become $\lambda \times GeMS_{j,i}$, then the aggregate index is given by:
  \[
  GeMS(\lambda \times GeMS_{j,i}, \forall j) = 1 - \left[ \frac{1}{k} \sum_j (1 - GeMS_{j,i})^{g(\lambda)} \right]^{\frac{1}{g(\lambda)}} \\
  \neq \lambda GeMS_i
  \]
  Hence linear homogeneity is NOT satisfied.

- **Proof of I.5**
  Let the wellbeing increase in the $\ell$-th dimension, everything else remaining the same, i.e. $GeMS_{\ell}$ increases. let us calculate $\frac{\partial GeMS}{\partial GeMS_{\ell}}$. We have :
  \[
  GeMS_i = 1 - \left[ \frac{1}{k} \sum_j (1 - GeMS_{j,i})^{g(\lambda)} \right]^{\frac{1}{g(\lambda)}} \\
  = 1 - B(\gamma_i) \left( \frac{1}{g(\lambda)} \right)
  \]
with $g(\gamma_i) > 1$. Hence

\[
\frac{\partial \text{GeMS}_{\ell}}{\partial \text{GeMS}_{i\ell}} = - \frac{\partial B(\gamma_i)}{\partial \text{GeMS}_{i\ell}} \frac{1}{g(\gamma_i)} \quad (11)
\]

Let us derive $\frac{\partial B(\gamma_i)}{\partial \text{GeMS}_{i\ell}}$. We first write:

\[
\ln B(\gamma_i) = \frac{1}{g(\gamma_i)} \ln B(\gamma_i) \quad (12)
\]

Deriving both sides, we get

\[
\frac{1}{B(\gamma_i)} \frac{\partial B(\gamma_i)}{\partial \text{GeMS}_{i\ell}} = \frac{1}{g(\gamma_i)} \frac{\partial \ln B(\gamma_i)}{\partial \text{GeMS}_{i\ell}} - \frac{1}{g(\gamma_i)^2} \frac{\partial g(\gamma_i)}{\partial \text{GeMS}_{i\ell}} \ln B(\gamma_i)
\]

\[
\Rightarrow \quad \frac{\partial B(\gamma_i)}{\partial \text{GeMS}_{i\ell}} = B(\gamma_i) \frac{1}{g(\gamma_i)} \left[ \frac{1}{B(\gamma_i)} \frac{1}{g(\gamma_i)} \frac{\partial B(\gamma_i)}{\partial \text{GeMS}_{i\ell}} - \frac{1}{g(\gamma_i)^2} \frac{\partial g(\gamma_i)}{\partial \text{GeMS}_{i\ell}} \ln B(\gamma_i) \right] \quad (13)
\]

Since $B(\gamma_i) \frac{1}{g(\gamma_i)}$ is always positive, the sign of $\frac{\partial B(\gamma_i)}{\partial \text{GeMS}_{i\ell}}$ is that of the expression in the square brackets in (13). Let us now examine the bracketed expression in greater depth. We have:

\[
\frac{1}{g(\gamma_i)} \frac{1}{B(\gamma_i)} \frac{\partial B(\gamma_i)}{\partial \text{GeMS}_{i\ell}} - \frac{1}{g(\gamma_i)^2} \frac{\partial g(\gamma_i)}{\partial \text{GeMS}_{i\ell}} \ln B(\gamma_i) \quad (14)
\]

Once again, noting that $g(\cdot)$ is always positive ($> 1$), we can ignore $\frac{1}{g}$ in both terms, and we are left with the following expression to sign:

\[
\frac{1}{B(\gamma_i)} \frac{\partial B(\gamma_i)}{\partial \text{GeMS}_{i\ell}} - \frac{1}{g(\gamma_i)} \frac{\partial g(\gamma_i)}{\partial \text{GeMS}_{i\ell}} \ln B(\gamma_i) \quad (15)
\]

We have

\[
\frac{\partial B(\gamma_i)}{\partial \text{GeMS}_{i\ell}} = \frac{1}{k} \sum_j \frac{\partial}{\partial \text{GeMS}_{i\ell}} \left[ (1 - \text{GeMS}_{j,i})^{g(\gamma_i)} \right] = \frac{1}{k} \sum_j \left[ (1 - \text{GeMS}_{j,i})^{g(\gamma_i)} \ln (1 - \text{GeMS}_{j,i}) \frac{\partial g(\gamma_i)}{\partial \text{GeMS}_{i\ell}} + \delta_{i\ell} g(\gamma_i)(1 - \text{GeMS}_{i\ell}^{-1}) \frac{\partial (1 - \text{GeMS}_{i\ell})}{\partial \text{GeMS}_{i\ell}} \right] \quad (16)
\]
\[
\delta_{j\ell} = \begin{cases} 
1 & \text{if } j = \ell \\
0 & \text{if } j \neq \ell
\end{cases}
\]

Substituting (16) into (15), and noting that \(\frac{\partial (1 - GeMS_{i\ell})}{\partial GeMS_{i\ell}}\) is \(-1\), we get

\[
\frac{1}{B} \left[ \frac{1}{k} \sum_j (1 - GeMS_{j,i}) g(\gamma_i) \ln (1 - GeMS_{j,i}) \frac{\partial g(\gamma_i)}{\partial GeMS_{i\ell}} \right] - \frac{1}{B} \left[ \frac{1}{k} \sum_j \delta_{j\ell} g(\gamma_i) (1 - GeMS_{i\ell})^{-1} \right] - \frac{1}{g(\gamma_i)} \frac{\partial g(\gamma_i)}{\partial GeMS_{i\ell}} \ln B(\gamma_i)
\]

(17)

Let us first note that (17) has three terms and that the second term is always negative

\[-\frac{1}{B} \left[ \frac{1}{k} \sum_j \delta_{j\ell} g(\gamma_i) (1 - GeMS_{i\ell})^{-1} \right] = - \frac{1}{B} \left\{ \frac{1}{k} \sum_{j > 0} g(\gamma_i) (1 - GeMS_{i\ell})^{-1} \right\} < 0 \]

Now, let us consider the first and the third terms together:

\[\frac{1}{B} \left[ \frac{1}{k} \sum_j (1 - GeMS_{j,i}) g(\gamma_i) \ln (1 - GeMS_{j,i}) \frac{\partial g(\gamma_i)}{\partial GeMS_{i\ell}} \right] - \frac{1}{g(\gamma_i)} \frac{\partial g(\gamma_i)}{\partial GeMS_{i\ell}} \ln B(\gamma_i)\]

Let us take out \(\frac{\partial g(\gamma_i)}{\partial GeMS_{i\ell}} \) as a common factor as it does not depend on the index of the sum, and omit it in the comparison. However, noting that it is \(< 0\), we have to remember that whatever sign we get at the end will have to be reversed when we add this factor again. This gives

\[\frac{1}{B} \left[ \frac{1}{k} \sum_j (1 - GeMS_{j,i}) g(\gamma_i) \ln (1 - GeMS_{j,i}) \right] - \frac{1}{g(\gamma_i)} \ln B(\gamma_i)\]

Let us denote \(a_j \equiv (1 - GeMS_j) g(\gamma_i)\). Then \(\ln a_j = g(\gamma_i) \ln (1 - GeMS_j)\), implying that \(\ln (1 - GeMS_j) = \frac{1}{g(\gamma_i)} \ln a_j\). Thus the expression becomes

\[\frac{1}{B} \left[ \frac{1}{k} \sum_j a_j \frac{1}{g(\gamma_i)} \ln a_j \right] - \frac{1}{g(\gamma_i)} \ln B(\gamma_i)\]

Since we are only determining the sign of the expression, let us multiply both terms by \(Bk\) (which is positive) and noting that \(Bk = \sum_j a_j\) and \(\ln B = \ln \sum_j a_j / k\), we get

\[\frac{1}{g(\gamma_i)} \left[ \sum_j a_j \ln a_j \right] - \left( \frac{\sum_j a_j}{k} \right) \left( \ln \frac{\sum_j a_j}{k} \right) \]

Ignoring the common factor \(\frac{1}{g(\gamma_i)}\), which is positive, we get

\[\left[ \sum_j a_j \ln a_j \right] - \left( \sum_j a_j \right) \left( \ln \frac{\sum_j a_j}{k} \right) \]
Using the log sum inequality i.e. \(\sum_i a_i \ln \frac{a_i}{b_i} \geq \left( \sum_i a_i \right) \ln \frac{\sum_i a_i}{\sum_i b_i} \), and setting \(a_j = a_j \) defined above, \(b_j \equiv 1\) such that \(\sum_j b_j = k\), we get
\[
\left[ \sum_i a_i \ln a_i \right] - \left( \sum_i a_i \right) \left( \ln \frac{\sum_i a_i}{k} \right) \geq 0
\]

Now, going backwards, we can get
\[
\frac{1}{B} \left[ \frac{1}{k} \sum_j \left( 1 - GeMS_{ji} \right)^{g(\gamma_i)} \ln \left( 1 - GeMS_{ji} \right) \right] - \frac{1}{g(\gamma_i)} \frac{\partial g(\gamma_i)}{\partial GeMS_{i\alpha}} \ln B(\gamma_i) \geq 0
\]
and hence the first and third terms combined of (17) is negative:
\[
\frac{1}{B} \left[ \frac{1}{k} \sum_j \left( 1 - GeMS_{ji} \right)^{g(\gamma_i)} \ln \left( 1 - GeMS_{ji} \right) \right] - \frac{1}{g(\gamma_i)} \frac{\partial g(\gamma_i)}{\partial GeMS_{i\alpha}} \ln B(\gamma_i) \leq 0
\]

Now adding the second term of (17) which was already shown to be also negative, we get
\[
\frac{1}{B} \left[ \frac{1}{k} \sum_j \left( 1 - GeMS_{ji} \right)^{g(\gamma_i)} \ln \left( 1 - GeMS_{ji} \right) \right] \cdot \frac{\partial g(\gamma_i)}{\partial GeMS_{i\alpha}} - \frac{1}{g(\gamma_i)} \frac{\partial g(\gamma_i)}{\partial GeMS_{i\alpha}} \ln B(\gamma_i) \leq 0
\]
Thus we have shown that (17) < 0. In other words, (15) < 0 as it is the same expression, and therefore (14) < 0 as (14) = (15) \(\frac{1}{g}\). This in turn gives (13) < 0 as (13) = (14) \(B(\gamma_i)^{\frac{1}{g}}\). Finally, we have \(\frac{\partial g(\gamma_i)}{\partial GeMS_{i\alpha}} > 0\) as it is -(13). Thus we see that GeMS\(_i\) is monotonic increasing in GeMS\(_{i\alpha}\).

An important remark: Note that the above derivation assumes differentiability of \(g(\cdot)\) with respect to each element of \(\gamma_i\). But we know that there is one point at which \(g(\cdot)\) is not differentiable i.e. at \(\mu_{b, i} \equiv i' \gamma_i = z_b\). So at this point, instead of calculating the change in GeMS\(_i\) as a differential, we should calculate it as a discrete change. That is, go through \(\Delta GeMS_i\). If all derivatives exist, then we have
\[
\Delta GeMS_i = \frac{\partial GeMS_i}{\partial GeMS_{i\alpha}} \Delta GeMS_{i\alpha}
\]

Now, for all terms in \(\frac{\partial GeMS_i}{\partial GeMS_{i\alpha}}\) that do not contain the derivative \(\frac{\partial g(\gamma_i)}{\partial GeMS_{i\alpha}}\), the derivation remains the same with the additional multiplication by \(\Delta GeMS_{i\alpha}\). For the terms containing \(\frac{\partial g(\gamma_i)}{\partial GeMS_{i\alpha}}\), one should replace, \(\frac{\partial g(\gamma_i)}{\partial GeMS_{i\alpha}} \cdot \Delta GeMS_{i\alpha}\) by \(\Delta g(z_b)\). At \(z_b\), when GeMS\(_{i\alpha}\) becomes GeMS\(_{i\alpha} + \Delta GeMS_{i\alpha}\), \(z_b\) becomes \(z_b + \Delta GeMS_{i\alpha}\).

From the definition of \(g(\cdot)\) given in Eq. (5), we can see that \(g(z_b + \varepsilon) - g(z_b) < 0\) if \(\varepsilon > 0\) and vice versa. Thus, the sign of \(\Delta g(z_b)\) is the same as that of \(\frac{\partial g(\gamma_i)}{\partial GeMS_{i\alpha}} \cdot \Delta GeMS_{i\alpha}\) when the latter exists, and everything else follows.

Hence \(\Delta GeMS_i > 0\) for a positive change in any of its components, and therefore monotonicity holds.
• Proof of P.2
  We have:

  \[ GeMS = 1 - \left[ \frac{1}{N} \sum_{i}(1 - z)^{f(i)} \right]^{\frac{1}{f(\tilde{m})}} \]
  \[ = 1 - \left[ \frac{1}{N} (1 - z)^{f(\tilde{m})} \right]^{\frac{1}{f(\tilde{m})}} \]
  \[ = 1 - (1 - z) \]
  \[ = z \]

• Proof of P.6

  \[ GeMS(y^*) = 1 - \left[ \frac{1}{\ell N} \sum_{i} \ell (1 - GeMS) f^{(\tilde{m})} \right]^{\frac{1}{f(\tilde{m})}} = GeMS(y) \]

  where \( \tilde{m} \) denotes the \( \ell N \)-vector of GeMS indices for the \( \ell N \) individuals in \( y^* \).

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