Uncertainty relation of Anandan-Aharonov and Intelligent states

Arun Kumar Pati
SEECS, Dean Street, University of Wales, Bangor LL 57 1UT, UK.
and
Theoretical Physics Division, 5th Floor, Central Complex
BARC, Mumbai-400 085, India

Abstract

The quantum states which satisfy the equality in the generalised uncertainty relation are called intelligent states. We prove the existence of intelligent states for the Anandan-Aharonov uncertainty relation based on the geometry of the quantum state space for arbitrary parametric evolutions of quantum states when the initial and final states are non-orthogonal.

email: akpati@sees.bangor.ac.uk

In recent years the study of geometry of the quantum state space and its implications have gained much importance. The introduction of Riemannian metric structure by Provost and Valle [1] and Fubini-Study metric by Anandan and Aharonov [2,3] into the projective Hilbert space of the quantum system has attracted a lot of attention. The relation between geometric distance function and geometric phase was studied and the equivalence of the above two metric structures (up to a scale factor) was pointed out [4]. The introduction of the length of the curve [4,5] has provided us a new way of understanding geometric phases in quantum systems. Subsequently, the metric structures were generalised to mixed states by Anandan [6] and the statistical distinguishability was used to define a metric structure by Braunstein and Caves [7]. Later, the Fubini-Study metric was generalised to non-unitary and non-linear quantum systems, and a metric approach to generalised geometric phase was proposed [8]. One of the outcome of the geometric approach is the parameter-based uncertainty relation (PBUR) in quantum theory. This is often useful when we do not have a Hermitian operator canonical conjugate to another operator which represent a physical quantity of our interest. The vivid example is the quest for time-energy uncertainty relation, when we do not have a Hermitian time operator canonical conjugate to energy.

In this letter we study the intelligent states (to be defined soon) for the Aharonov-Anandan uncertainty relation and prove the existence of such states when the initial and final states are non-orthogonal during an arbitrary parametric evolution of a quantum system.

To briefly recall the essential geometric ideas, let us consider a quantum system $S$ whose state vector $|\psi(t)\rangle \in \mathcal{H} = \mathbb{C}^N$ evolves in time from time $t_1$ to $t_2$. Geometrically the state is represented by a point in the projective Hilbert space $\mathcal{P} = \mathcal{H} - \{0\}/\mathbb{C}^*$, where $\mathbb{C}^*$ is a group of non-zero complex numbers. The time evolution of the system gives us a curve $C$ in $\mathcal{H}$, i.e. $C : t \to |\psi(t)\rangle, t_1 \leq t \leq t_2$. Since $\mathcal{H}$ is Riemannian this curve has a length which...
has been extensively studied by the present author \[4,5,9\]. The Hilbert space curve can be projected onto a curve \( \hat{C} = \Pi(C) \) via a projection map \( \Pi : \mathcal{H} \to \mathcal{P} \). The projected curve \( \hat{C} \) has a length which is induced from the inner product of the Hilbert space and is given by the Fubini-Study metric \[1–9\].

\[
S = \frac{2}{\hbar} \int_{t_1}^{t_2} \Delta H(t)dt, \tag{1}
\]

where \( \Delta H(t) = \left[ \langle \psi(t)|H(t)|\psi(t) \rangle - \langle \psi(t)|H(t)|\psi(t) \rangle^2 \right]^{1/2} \) is the usual uncertainty in the energy of the system. This distance is independent of a particular Hamiltonian used to evolve the quantum system and is invariant under a gauge transformation. On the other hand if \( S_0 \) is the shortest distance between the initial and final states joining the points \( \Pi(|\psi(t_1)\rangle) \) and \( \Pi(|\psi(t_2)\rangle) \) then the distance measured by Fubini-Study metric is always greater than the geodesic distance connecting the initial and final points, where \( S_0 \) is given by the Bargmann angle \[5\] formula \( \cos^2 S_0 = |\langle \psi(t_1)|\psi(t_2) \rangle|^2 \). This gives \( S \geq S_0 \) and the equality holds only for those states that evolve along a geodesic in \( \mathcal{P} \). The above geometric reasoning is the basis of the Anandan-Aharonov uncertainty relation for time and energy, which is given by

\[
\langle \Delta H(t) \rangle \Delta t \geq \frac{\hbar}{4}, \tag{2}
\]

where \( \langle \Delta H(t) \rangle = \frac{1}{t_2-t_1} \int_{t_1}^{t_2} \Delta H(t)dt \) is the time average of the energy uncertainty and \( \Delta t = S_0/\pi(t_2-t_1) \) is the “uncertainty in time”. When the initial and final states are orthogonal (which are distinguishable by quantum mechanical tests) then the shortest distance is \( \pi \). In this case time-energy uncertainty relation takes a simple form (for a time-independent Hamiltonian) as

\[
\Delta H \Delta t \geq \frac{\hbar}{4}, \tag{3}
\]

where \( \Delta t = (t_2 - t_1) \) is the ordinary time difference that is required to make a transition to a orthogonal state. There are various approaches to time-energy uncertainty relation and to estimate the time required to make a transition to orthogonal states in the literature \[10–14\]. These estimations are not only of theoretical interest but have important implication on how fast one can do a computation in a quantum computer \[13,13\]. The advantage of the geometric uncertainty relation is that we do not have to look for a Hermitian operator for time. It can remain just as a parameter and uncertainty in the parameter would mean how good we can estimate it given a certain amount of uncertainty in the conjugate variable. This fact can be grounded by the observation that we can go beyond the time-energy uncertainty relation. If we have any continuous parameter \( \lambda \) and any Hermitian observable \( A(\lambda) \) which is the generator of the parametric evolution, then a similar geometric reasoning would give us

\[
\langle \Delta A(\lambda) \rangle \Delta \lambda \geq \frac{\hbar}{4}, \tag{4}
\]

where \( \langle \Delta A(\lambda) \rangle = \frac{1}{(\lambda_2-\lambda_1)} \int_{\lambda_1}^{\lambda_2} \Delta A(\lambda)d\lambda \) is the parameter average of the observable uncertainty and \( \Delta \lambda = \frac{S_0}{\pi_0}(\lambda_2 - \lambda_1) \) is the scaled displacement in the space of the conjugate variable of
A. This generalised uncertainty relation would hold for position-momentum, phase-number or any possible combinations. Recently, Yu [16] has discussed the PB UR for position-momentum case. We have also proposed a geometric uncertainty relation without reference to any Hermitian operator and shown that (2) and (4) follows as a special case of our generalised uncertainty relation [17].

It is known that given two non-commuting observables we can derive an uncertainty relation for them and the class of states that satisfy the equality sign in the inequality are called intelligent states [18,19]. So the natural question would be what are the intelligent states for geometric uncertainty relation? To characterise these states in general is quite difficult. But one pretty observation is that since the initial and final state can be connected by a geodesic, states which satisfy the equality have to be equivalent (i.e. belong to a ray) to a state that satisfies the geodesic equation. Since solution to geodesic equation lies in a two-dimensional (real) subspace [9,21], the intelligent states should also belong to a two-dimensional subspace \( H_s \). This will be clear as we proceed to our next discussion. Recently, Horesh and Mann [20] have constructed a state that satisfies the equality in PBUR when the initial and final states are orthogonal. These state indeed belong to a 2-dim. subspace of the full Hilbert space. However, their construction fails when the initial and final states are non-orthogonal. In this letter we find the intelligent states for PBUR or geometric uncertainty relation (GUR) when the initial and final states are non-orthogonal. We prove a theorem concerning the intelligent states for both orthogonal and non-orthogonal cases. This will provide an answer to the question what kind of states do evolve along a geodesic in the projective Hilbert space.

Let us consider a quantum system represented by a state \( |\psi(\lambda)\rangle \). Let \( A \) be the Hermitian generator of the parameter translation. For simplicity we assume that \( A \) does not depend on \( \lambda \). The unitary evolution operator \( U = \exp(-iA\hbar) \) generates the evolution path in \( \mathcal{H} \) starting from a given state \( |\psi(0)\rangle \). The length of the actual evolution path in \( \mathcal{P} \) is measured by Fubini-Study metric. This can be defined via length of a parallel-transported vector \( |\bar{\psi}(\lambda)\rangle \), where \( |\bar{\psi}(\lambda)\rangle = \exp(i\int_0^\lambda \langle \psi(\lambda')|A|\psi(\lambda')\rangle d\lambda')|\psi(\lambda)\rangle \). This satisfies the parallel transport condition \( \langle \bar{\psi}(\lambda)|d/d\lambda\bar{\psi}(\lambda)\rangle = 0 \), which physically means the vector \( |\bar{\psi}(\lambda)\rangle \) does not rotate locally but undergoes a net rotation when comes back to its original point in the projective Hilbert space. The geodesic in \( \mathcal{P} \) can be defined as the one for which length of the path traced by the parallel-transported vector is minimum. The length of the curve traced by the parallel-transported state is given by

\[
l(\bar{\psi}) = \int_0^\lambda \left( \frac{d}{d\lambda'} \bar{\psi}(\lambda') \right)^2 \frac{d\lambda'}{2} = \int_0^\lambda \left( \frac{d}{d\lambda'} \bar{\psi}(\lambda') \right)^2 d\lambda'.
\]

The variational calculation gives an equation for geodesic (see for details [3] and [21])

\[
\frac{d^2|\bar{\psi}(\lambda)\rangle}{d\lambda^2} + v^2|\bar{\psi}(\lambda)\rangle = 0,
\]

where \( v = \frac{\Delta A}{\hbar} \). The geodesic equation has a general solution

\[
|\bar{\psi}(\lambda)\rangle = \cos(v\lambda)|\bar{\psi}(0)\rangle + \frac{1}{v} \sin(v\lambda)|\dot{\bar{\psi}}(0)\rangle,
\]

where \( \langle \bar{\psi}(0)|\bar{\psi}(0)\rangle = 1 \), \( \langle \dot{\bar{\psi}}(0)|\dot{\bar{\psi}}(0)\rangle = v^2 \), \( v \) being the speed of the system point in \( \mathcal{P} \) and \( |\dot{\bar{\psi}}(0)\rangle \) denotes derivative wrt the parameter. Though the geodesic on \( \mathcal{P} \) is obtained by
projecting a curve $C : \lambda \rightarrow |\tilde{\psi}(\lambda)\rangle$ in $\mathcal{H}$, we have expressed the equation of geodesic (6) in terms of the state $|\tilde{\psi}(\lambda)\rangle$. This is because, given a projected path $\Pi(C_{gs})$ in $\mathcal{P}$ there is a horizontal lift of the path in $\mathcal{H}$ such that the length of the path $\tilde{C}$ is same as that of the $\Pi(C_{gs})$.

Thus the states which satisfy geodesic equation are linear superposition of two mutually orthogonal vectors. The corresponding one-dimensional projector $\rho(\lambda) = |\tilde{\psi}(\lambda)\rangle\langle\tilde{\psi}(\lambda)|$ belongs to a two-dimensional real space. Note that $\rho(\lambda)$ can be expressed as

$$\rho(\lambda) = \cos^2(v\lambda)\rho_{11} + \sin^2(v\lambda)\rho_{22} + \cos(v\lambda)\sin(v\lambda)(\rho_{12} + \rho_{21}),$$

where $\rho_{ij} = |\tilde{\psi}_i\rangle\langle\tilde{\psi}_j|$, $|\tilde{\psi}_i\rangle = |\tilde{\psi}(0)\rangle$ and $|\tilde{\psi}_j\rangle = \frac{1}{i}\tilde{\psi}(0)\rangle$ ($ij = 1, 2$). This is a two-dimensional geometric quantity. This is consistent with the well-known fact [22] that the geodesics on the unit sphere are intersections of two-dimensional subspace $\mathcal{H}_s = \text{span}\{|\tilde{\psi}_i\rangle, |\tilde{\psi}_j\rangle\}$ with the sphere, i.e. they live in a two-dimensional manifold. Now we state the following theorem.

**Theorem:** If $|\psi(\lambda)\rangle$ is the intelligent state which satisfy the equality in PBUR and $|\tilde{\psi}(\lambda)\rangle$ is the parallel transported state obtained from $|\psi(\lambda)\rangle$ then $|\tilde{\psi}(\lambda)\rangle$ satisfy the equation of geodesic and can always be expressed in the form (7) for both the orthogonal and non-orthogonal initial and final states.

First we prove it for the case when initial and final states are orthogonal. From the work of Horesh and Mann [20] we know that all states of the form

$$|\psi(\lambda)\rangle = \frac{1}{\sqrt{2}} \left( \exp(-\frac{i}{\hbar}a_1\lambda)|\psi_i\rangle + \exp(-\frac{i}{\hbar}a_2\lambda)|\psi_j\rangle \right), \quad i \neq j$$

are the only intelligent states which satisfy the equality in (4). It is assumed that $A$ has a complete basis of normalised state $\{|\psi_i\rangle\}_{i \in I}$ with non-degenerate spectrum $\{a_i\}_{i \in I}$, with $I$ a set of quantum numbers. This state at parameter value $\lambda = \frac{\pi\hbar}{(a_j - a_i)}$ becomes orthogonal to the initial state. The parallel transported state $|\tilde{\psi}(\lambda)\rangle = e^{\frac{i}{\hbar}(A)\lambda}|\psi(\lambda)\rangle$ is given by

$$|\tilde{\psi}(\lambda)\rangle = \frac{1}{\sqrt{2}} \left( \exp(-\frac{i}{2\hbar}(a_i - a_j)\lambda)|\psi_i\rangle + \exp\left(\frac{i}{2\hbar}(a_i - a_j)\lambda\right)|\psi_j\rangle \right), \quad i \neq j.$$  

(10)

It is easy to check that this state satisfies geodesic equation (6). Further, by noting that $v = \Delta A = \frac{1}{2}(a_i - a_j)$ we can reexpress the above state as

$$|\tilde{\psi}(\lambda)\rangle = \cos(v\lambda)|\tilde{\psi}(0)\rangle + \frac{i}{v}\sin(v\lambda)|\dot{\tilde{\psi}}(0)\rangle,$$

where $|\tilde{\psi}(0)\rangle = \frac{1}{\sqrt{2}}(|\psi_i\rangle + |\psi_j\rangle)$ and $\frac{1}{v}|\dot{\tilde{\psi}}(0)\rangle = \frac{i}{\sqrt{2}}(|\psi_i\rangle - |\psi_j\rangle)$. This completes the proof when the initial and final states are orthogonal.

However, the states given in (9) do not satisfy the equality sign in (4) when the initial and final states are non-orthogonal. In quantum theory the study of non-orthogonal states are crucial because they display variety of non-classical features. Moreover, non-orthogonal states cannot be distinguished by ordinary measurement processes without error. Therefore, it is important to know what kind of states can evolve along a geodesic connecting two non-orthogonal states. Now we seek those states which satisfy the equality in PBUR or GUR of Anandan-Aharonov for non-orthogonal initial and final states. In general for arbitrary
generator $A$ it is not possible to find a state which will be an intelligent one. So, the question is what condition should we impose on the Hermitian operator $A$, such that we will be able to find intelligent states for non-orthogonal initial and final states. The problem is more acute when we do not know the complete set of eigenstates of the operator $A$. But suppose we know the spectrum of a part of the operator $A$. Then we propose the following:

**Proposition:** *If the Hermitian generator $A$ of the parametric evolution can be split into two parts $A_0 + A_1$ such that $A_0$ has a complete basis of normalised eigenstates $\{|\psi_i\rangle\}_{i \in I}$ with degenerate spectrum $\{a_0\}$, with $I$ a set of quantum numbers and $A_1$ has matrix elements $(A_1)_{ii} = 0 = (A_1)_{jj}$ and $(A_1)_{ij} = (A_1)_{ji} = a_1$, then all states of the form*

$$\psi(\lambda) = e^{i a_0 \lambda}(\cos(a_0 \frac{\lambda}{\hbar})|\psi_i\rangle - i \sin(a_0 \frac{\lambda}{\hbar})|\psi_j\rangle), \quad i \neq j$$

**are intelligent states for non-orthogonal initial and final states.**

To prove this, we focus on the parametric evolution of the state $|\psi(\lambda)\rangle$ in the two-dimensional subspace spanned by vectors $\{|\psi_i\rangle, |\psi_j\rangle\}$. Since $\lambda$ is a continuous parameter we can solve the equation of motion for $|\psi(\lambda)\rangle$, i.e. $\hbar \frac{d}{d\lambda}|\psi(\lambda)\rangle = A|\psi(\lambda)\rangle$ with a given initial condition. Without loss of generality we assume that initially the state is in $|\psi_i\rangle$. The state at any other parameter value can be written as $|\psi(\lambda)\rangle = c_i(\lambda)|\psi_i\rangle + c_j(\lambda)|\psi_j\rangle$. Then the solution to evolution equation gives

$$c_i(\lambda) = e^{-\frac{i a_0 \lambda}{\hbar}} \cos(a_0 \frac{\lambda}{\hbar}),$$

$$c_j(\lambda) = -i e^{-\frac{i a_0 \lambda}{\hbar}} \sin(a_0 \frac{\lambda}{\hbar}).$$

With the help of these amplitudes we find the uncertainty in the operator $A$. It is given by

$$\Delta A^2 = a_0^2 + a_1^2 + 4 a_0 a_1 \text{Re}(c_i^* c_j) - (a_0 + 2 a_1 \text{Re}(c_i^* c_j))^2 = a_1^2,$$

where we have used the fact that $c_i^* c_j$ is purely imaginary quantity. Next we calculate the shortest distance along the geodesic using $S_0 = 2 \cos^{-1}(\langle \psi(\lambda_1)|\psi(\lambda_2)\rangle)$, which is $\frac{\pi}{2} a_1 \lambda_2$. We have taken initial parameter value $\lambda_1 = 0$. Therefore, the “uncertainty in the parameter” is $\Delta \lambda = \frac{\pi}{S_0} \lambda_2 = \frac{\pi}{2 a_1 \hbar}$. Thus the lhs of the PBUR of Anandan-Aharonov becomes

$$\Delta A \Delta \lambda = a_1 \frac{\pi}{2 a_1} \hbar = \frac{\hbar}{4}.$$  

This proves that the states (11) in our proposition are indeed intelligent states for arbitrary parametric evolution of quantum systems, when the initial and final points are non-orthogonal.

Now we prove our theorem for non-orthogonal cases. It is easy to see that the parallel-transported state in this case is given by $|\tilde{\psi}(\lambda)\rangle = (\cos(a_1 \frac{\lambda}{\hbar})|\psi_i\rangle - i \sin(a_1 \frac{\lambda}{\hbar})|\psi_j\rangle)$. This satisfies the geodesic equation (6). Also, the above state can be expressed as $|\tilde{\psi}(\lambda)\rangle = \cos(v \lambda)|\psi(0)\rangle + \frac{i}{v} \sin(v \lambda)|\dot{\psi}(0)\rangle$, where one can check that $|\tilde{\psi}(0)\rangle = |\psi_i\rangle$ and $\frac{i}{v}|\dot{\psi}(0)\rangle = -i|\psi_j\rangle$. This ends the proof.
We remark that any other state which is intelligent has to be equivalent (where an equivalence relation \( \sim \) means \(|\psi(\lambda)\rangle \sim |\psi(\lambda)\rangle'\) if \(|\psi(\lambda)\rangle' = c|\psi(\lambda)\rangle\), where \(c\) is a complex number of unit modulus, i.e. \(c \in U(1)\) group) to the state given in (11). Any state of the form \(|\psi\rangle = c_i|\psi_i\rangle + c_j|\psi_j\rangle + c_k|\psi_k\rangle\), \((i \neq j \neq k)\) will not be an intelligent state because it will not satisfy geodesic equation.

To conclude this letter, we have proved a theorem which says that in general when the intelligent states satisfy geometric uncertainty relation, the corresponding parallel transported states satisfy geodesic equation. This guarantees that intelligent states belong to a two-dimensional manifold. For arbitrary parametric evolution of a quantum system we have found the explicit form of intelligent states by imposing certain condition on the generator when the initial and final states are non-orthogonal. The intelligent states for orthogonal case found in [20] are distinct from that of the non-orthogonal case found here.

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