NULL CONTROLLABILITY FOR SINGULAR CASCADE SYSTEMS OF $n$-COUPLED DEGENERATE PARABOLIC EQUATIONS BY ONE CONTROL FORCE

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Abstract. In this paper, we consider a class of cascade systems of $n$-coupled degenerate parabolic equations with singular lower order terms. We assume that both degeneracy and singularity occur in the interior of the space domain and we focus on null controllability problem. To this aim, we prove first Carleman estimates for the associated adjoint problem, then, we infer from it an indirect observability inequality. As a consequence, we deduce null controllability result when a unique distributed control is exerted on the system.

1. Introduction and main result. In this paper we study the null controllability by one control force for a class of systems governed by $n$-coupled degenerate parabolic equations in presence of singular coupling terms. More precisely, for $n \geq 1$

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given, we consider the following linear parabolic system

\[
\begin{aligned}
\partial_t y_1 - d_1(a(x)y_{1x})_x - \sum_{j=1}^n \frac{\lambda_{kj}}{b_{kj}} y_j + \sum_{j=1}^n a_{1j} y_j &= v_1 \omega, \quad (t, x) \in Q, \\
\partial_t y_2 - d_2(a(x)y_{2x})_x - \sum_{j=1}^n \frac{\lambda_{kj}}{b_{kj}} y_j + \sum_{j=1}^n a_{2j} y_j &= 0, \quad (t, x) \in Q, \\
&\vdots \\
\partial_t y_n - d_n(a(x)y_{nx})_x - \sum_{j=1}^n \frac{\lambda_{kj}}{b_{kj}} y_j + \sum_{j=1}^n a_{nj} y_j &= 0, \quad (t, x) \in Q, \\
y_k(t, 0) &= y_k(t, 1) = 0, \quad 1 \leq k \leq n, \quad t \in (0, T), \\
y_k(0, x) &= y_k^0(x), \quad 1 \leq k \leq n, \quad x \in (0, 1),
\end{aligned}
\]

where \(\omega\) is a nonempty open subset of \((0, 1)\), \(d_k > 0, 1 \leq k \leq n\), \(T > 0\) fixed, \(Q := (0, T) \times (0, 1)\), \(1_\omega\) denotes the characteristic function of the set \(\omega\), \((y_1^0, \cdots, y_n^0) \in L^2(0, 1)^n\), is the initial condition, and \(v \in L^2(Q)\) is the control.

Moreover, we assume that the constants \(\lambda_{kj}, 1 \leq k, j \leq n\), satisfy suitable assumptions described below, and the functions \(a, b_{kj}, 1 \leq k, j \leq n\), degenerate at the same interior point \(x_0 \in (0, 1)\) of the spatial domain \((0, 1)\) that can belong to the control set \(\omega\) (for the precise assumptions we refer to section 2).

Equivalently, the previous system can be written as

\[
\begin{aligned}
\partial_t Y - KY - BY + CY &= e_1 v_1 \omega, \quad \text{in} \quad Q, \\
Y(t, 0) &= Y(t, 1) = 0, \quad t \in (0, T), \\
Y(0, x) &= Y^0(x), \quad x \in (0, 1),
\end{aligned}
\]

where \(Y = (y_k)_{1 \leq k \leq n}\), the operator \(K\) is given by

\[K = DK, \quad D = \text{diag}(d_1, \cdots, d_n),\]

and the differential operator \(K\) is defined by

\[Ky := (a(x)y_x)_x.\]

The matrix \(C = (a_{kj})_{1 \leq k, j \leq n}\) has its entries in \(L^\infty(Q)\), \(B\) is the singular matrix operator given by \(B = (B_{kj})_{1 \leq k, j \leq n}\), where \(B_{kj} := \frac{\lambda_{kj}}{b_{kj}}\), and finally \(e_1 = (1, 0, \cdots, 0)\) is the first element of the canonical basis of \(\mathbb{R}^n\).

The object of this paper is twofold: first we analyze the well-posedness of the evolution system (1); second, we investigate the effect of the singular coupling terms on observability/controllability aspects of such kind of systems. In particular, our main controllability result will be the following.

**Theorem 1.1.** Under the assumptions of Theorem 4.8, for any time \(T > 0\) and any initial datum \((Y^0)^* \in L^2(0, 1)^n\), there exists a control function \(v \in L^2(Q)\) such that the solution of (2) satisfies

\[y_k(T, \cdot) = 0 \quad \text{in} \quad (0, 1), \quad \forall k : 1 \leq k \leq n.\]
the authors obtain results concerning well-posedness, controllability and Carleman estimates. For inverse issues related to this type of equations we refer to [2].

Recently, in [29, 32] the authors considered a singular coupled system of degenerate parabolic equations (in divergence and nondivergence form) in the particular case of two equations (i.e., $n = 2$), and showed the null controllability of this system under some technical conditions on the coefficients.

In this work, as in [15, 16, 28], we want to generalize these results to the case of a general cascade system of $n$ linear degenerate and singular parabolic equations. To this end, we will suppose that $B$ and $C$ have the following structure

$$
B = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & 0 & \cdots & \cdots & 0 \\
\lambda_{12} & \lambda_{22} & \lambda_{23} & 0 & \cdots & 0 \\
0 & \lambda_{23} & \lambda_{33} & \lambda_{34} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & \lambda_{n-1,n} & \frac{b_{n-1,n}}{b_{n-1,n}} \\
\end{pmatrix}
$$

$$
C = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
0 & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{n,n-1} & a_{nn} \\
\end{pmatrix}
$$

In addition, to obtain our main null controllability result related to system (2), we assume that the singular matrix $B$ is symmetric, i.e.,

$$
\lambda_{kk-1} = \lambda_{k-1k} \quad \text{and} \quad b_{kk-1} = b_{k-1k}, \quad \forall k: 2 \leq k \leq n.
$$

**Remark 1.** As we shall see later, the above assumption is used to obtain the well-posedness result using semigroup theory, but the Galerkin method would prove that system (1) is wellposed without imposing the hypothesis that matrix $B$ is symmetric. However, this mentioned assumption is required to get the observability estimate.

It is well known that the main tools when dealing with null controllability properties of PDE are the so called Carleman inequalities. The main contributions are due to A. Fursikov and O. Yu. Imanuvilov, who developed the use of a Carleman inequality to the null controllability of classical (non degenerate) parabolic equations in [27].

To obtain Theorem 1.1, the first step relies on a Carleman estimate for the homogeneous dual problem corresponding to (1), which is proved in Theorem 4.8. With the Carleman estimate at hand, classical energy estimates (see Theorem 5.1) yield the observability estimate

$$
\|Z(0, \cdot)\|_{L^2(0,1)^n}^2 \leq CT \int_{(0,T) \times \omega} z^2(t,x) dx dt,
$$
for the solution $Z = (z_k)_{1 \leq k \leq n}$ of the dual homogeneous backward problem which, under assumption (3) (cascade system), has the form

\[
\begin{aligned}
\partial_t z_k + d_k(a(x)z_{kk})x + \sum_{j=k-1}^{k+1} \frac{\lambda_j}{b_{jk}} z_j - \sum_{j=1}^{k+1} a_{jk} z_j = 0, & \quad (t, x) \in Q, \\
z_k(t, 0) = 0, & \quad z_k(t, 1) = 0, \quad t \in (0, T), \\
z_k(T, x) = z_k^T(x), & \quad x \in (0, 1),
\end{aligned}
\]

where $z_k^T \in L^2(0, 1)$ and $1 \leq k \leq n$.

Let us remark that only one component of the unknown is observed. One calls this property indirect observability since by observing only one component of the solution on $\omega$, one can control all components of the state at the final time. Using the Hilbert uniqueness method, we then establish an indirect controllability result, which means that we drive back the full coupled system (1) to equilibrium at time $T$ by only controlling the first equation of the system. We refer to [1] for a discussion of various controllability and observability concepts.

The remainder of the paper is organized as follows. In Section 2 we introduce the functional analytic setting and recall some preliminary results, such as Hardy-Poincaré inequalities, that will be useful for the rest of the paper. In Section 3, we study well-posedness of the problem applying the previous inequalities. In Section 4, we prove Carleman estimates and we use them to prove observability inequality in Section 5.

2. Basic assumptions and preliminary results. In the following we will introduce the notions of weak and strong degeneracy for the real-valued functions $a$ and $b_{kj}$, $j = k - 1, k \quad \forall k : 1 \leq k \leq n$, defined on the interval $[0, 1]$. Accordingly, we will define suitable weighted spaces and recall some inequalities of Hardy-Poincaré type. These results will play a key role for the study of well-posedness of the system under analysis. The ways in which $a$ and $b_{kj}$, $\forall j = k - 1, k \quad \forall k : 1 \leq k \leq n$, degenerate at $x_0$ can be quite different, and for this reason we distinguish four different types of degeneracy. In particular, we consider the following cases (see, for instance [24]).

Hypothesis 2.1. Double weakly degenerate case (WWD). There exists $x_0 \in (0, 1)$ such that $a(x_0) = b_k \backslash \{x_0\} = 0$, $a, b_{kk} > 0$ in $[0, 1] \backslash \{x_0\}$, $a, b_{kk} \in C^1([0, 1] \backslash \{x_0\})$ and there exists $K, L_k \in (0, 1)$ such that $(x-x_0)a' \leq Ka$ and $(x-x_0)b_{kk}' \leq L_kb_{kk}$ a.e. in $[0, 1]$.

Hypothesis 2.2. Weakly strongly degenerate case (WSD). There exists $x_0 \in (0, 1)$ such that $a(x_0) = b_{kk}(x_0) = 0$, $a, b_{kk} > 0$ in $[0, 1] \backslash \{x_0\}$, $a \in C^1([0, 1] \backslash \{x_0\}), b_{kk} \in C^1([0, 1] \backslash \{x_0\}) \cap W^{1, \infty}(0, 1)$, there exists $K \in (0, 1), L_k \in (1, 2)$ such that $(x-x_0)a' \leq Ka$ and $(x-x_0)b_{kk}' \leq L_kb_{kk}$ a.e. in $[0, 1]$.

Hypothesis 2.3. Strongly weakly degenerate case (SWD). There exists $x_0 \in (0, 1)$ such that $a(x_0) = b_{kk}(x_0) = 0$, $a, b_{kk} > 0$ in $[0, 1] \backslash \{x_0\}$, $a \in C^1([0, 1] \backslash \{x_0\}) \cap W^{1, \infty}(0, 1), b_{kk} \in C^1([0, 1] \backslash \{x_0\})$, $\exists K \in [1, 2], L_k \in (0, 1)$ such that $(x-x_0)a' \leq Ka$ and $(x-x_0)b_{kk}' \leq L_kb_{kk}$ a.e. in $[0, 1]$.

Hypothesis 2.4. Double strongly degenerate case (SSD). There exists $x_0 \in (0, 1)$ such that $a(x_0) = b_{kk}(x_0) = 0$, $a, b_{kk} > 0$ in $[0, 1] \backslash \{x_0\}$, $a, b_{kk} \in C^1([0, 1] \backslash \{x_0\}) \cap W^{1, \infty}(0, 1)$, there exists $K, L_k \in [1, 2)$ such that $(x-x_0)a' \leq Ka$ and $(x-x_0)b_{kk}' \leq L_kb_{kk}$ a.e. in $[0, 1]$.

For the non diagonal terms we shall consider the following cases.
**Hypothesis 2.5.** The function $b_{kk-1}$ is weakly degenerate, that is, there exists $x_0 \in (0, 1)$ such that $b_{kk-1}(x_0) = 0$, $b_{kk-1} > 0$ on $[0, 1] \setminus \{x_0\}$, $b_{kk-1} \in C^1([0, 1] \setminus \{x_0\})$ and there exists $M_k \in (0, 1)$ such that $(x - x_0) b_{kk-1} \leq M_k b_{kk-1}$ a.e. in $[0, 1]$.

**Hypothesis 2.6.** The function $b_{kk-1}$ is strongly degenerate, that is, there exists $x_0 \in (0, 1)$ such that $b_{kk-1}(x_0) = 0$, $b_{kk-1} > 0$ on $[0, 1] \setminus \{x_0\}$, $b_{kk-1} \in C^1([0, 1] \setminus \{x_0\}) \cap W^{1, \infty}(0, 1)$ and there exists $M_k \in [1, 2)$ such that $(x - x_0) b_{kk-1} \leq M_k b_{kk-1}$ a.e. in $[0, 1]$.

To prove well-posedness of (1), as in [24], we start by introducing the following weighted Hilbert spaces

\[
H^1_a(0, 1) := \left\{ y \in W^{1,1}_0(0, 1) : \sqrt{a} y_x \in L^2(0, 1) \right\},
\]

\[
H^1_{a,b_{kk}}(0, 1) := \left\{ y \in H^1_a(0, 1) : \frac{y}{\sqrt{b_{kk}}} \in L^2(0, 1) \right\},
\]

endowed with the respective norms defined by

\[
\|y\|_{H^1_a}^2 := \|y\|_{L^2(0, 1)}^2 + \|\sqrt{a} y_x\|_{L^2(0, 1)}^2,
\]

\[
\|y\|_{H^1_{a,b_{kk}}}^2 := \|y\|_{H^1_a}^2 + \left\| \frac{y}{\sqrt{b_{kk}}} \right\|_{L^2(0, 1)}^2.
\]

For our further results, it is important to remind the following fundamental Hardy-Poincaré inequality.

**Lemma 2.7.** ([24, Proposition 2.14]). If one of the Hypotheses 2.1, 2.2, 2.3 holds with $K + L_k \leq 2$, then there exists a constant $C^k > 0$ such that for all $y \in H^1_{a,b_{kk}}(0, 1)$, we have

\[
\int_0^1 \frac{y^2}{b_{kk}} \, dx \leq C^k \int_0^1 ay_x^2 \, dx.
\] (5)

In our situation, due to the presence of singular coupling terms, the functional setting must contain some information on the behaviour of the functions at the singularity. Thus, we introduce the weighted Hilbert space

\[
\mathcal{H}_k := \left\{ y \in H^1_{a,b_{kk}}(0, 1) : \frac{y}{\sqrt{b_{kk}}} \in L^2(0, 1) \right\},
\]

endowed with the associated norm

\[
\|y\|_{\mathcal{H}_k}^2 := \|y\|_{H^1_{a,b_{kk}}}^2 + \left\| \frac{y}{\sqrt{b_{kk}}} \right\|_{L^2(0, 1)}^2.
\]

Using the weighted space introduced above, one can prove the next result.

**Lemma 2.8.** Assume that one among the Hypothesis 2.5 or 2.6 holds and let $0 < K, M_k < 2$ be such that

\[
\begin{cases}
K \in [0, 2) \setminus \{1\}, & 0 < M_k \leq 2 - K, \\
K = 1, & 0 < M_k < 2 - K = 1.
\end{cases}
\]

Then there exists a constant $C^{kk-1} > 0$ such that for all $y \in \mathcal{H}_k$, we have

\[
\int_0^1 \frac{y^2}{b_{kk-1}} \, dx \leq C^{kk-1} \int_0^1 ay_x^2 \, dx.
\] (6)
Remark 2. It is well known that when $K = M_k = 1$, an inequality of the form (6) does not hold [24, Remark 2.15]. Being such an inequality fundamental not only for the well-posedness but also to obtain the observability inequality, it is not surprising if with our techniques we cannot handle this case.

3. Abstract setting and well-posedness. In order to study the well-posedness of problem (1), let us make precise our assumptions on the parameters.

Hypothesis 3.1. Throughout this section, we assume the following hypotheses:

1. Either of the following holds:
   - One among the Hypotheses 2.1, 2.2 or 2.3 holds with $K + L_k \leq 2$, $\forall k : 1 \leq k \leq n$ and we assume that
     \[
     \lambda_{kk} \in \left(0, \frac{d_k}{C^k}\right), \quad \forall k : 1 \leq k \leq n.
     \]
   - Hypotheses 2.1, 2.2, 2.3 or 2.4 hold with $\lambda_{kk} < 0$.

2. We shall admit Hypothesis 2.5 or 2.6 with $0 < K, M_k < 2$, satisfying
   \[
   \begin{cases}
   K \in [0, 2) \setminus \{1\}, & 0 < M_k \leq 2 - K, \\
   K = 1, & 0 < M_k < 2 - K = 1,
   \end{cases}
   \]
   and we assume that
   \[
   \begin{align*}
   \lambda_{21} & \in \left(0, \frac{\sqrt{\Lambda_1 \Lambda_2}}{\sqrt{2} C^{21}}\right), & \lambda_{n-1} & \in \left(0, \frac{\sqrt{\Lambda_n \Lambda_{n-1}}}{\sqrt{2} C^{n(n-1)}}\right), \\
   \lambda_{kk-1} & \in \left(0, \frac{\sqrt{\Lambda_k \Lambda_{k-1}}}{2 C^{kk-1}}\right), & 3 \leq k & \leq n - 1,
   \end{align*}
   \]
   where $\Lambda_k, k \in \{1, \cdots, n\}$, is given in (10).

Remark 3. The upper bound for the range of the coefficients of the singular terms considered in (9) is required for the well-posedness of the problem. Thus, to look for controllability properties, we will focus our study on this range of constants.

Using the lemmas given in the previous section one can prove the next inequality, which is crucial to obtain well-posedness and observability properties.

Proposition 1. Assume Hypothesis 3.1.1. Then there exists $\Lambda_k \in (0, d_k]$ such that

\[
\forall y \in H_k, \quad d_k \int_0^1 a y_x^2 \, dx - \lambda_{kk} \int_0^1 \frac{y^2}{b_{kk}} \, dx \geq \Lambda_k \int_0^1 a y_x^2 \, dx
\]

Proof. If $\lambda_{kk} < 0$, the result is obvious taking $\Lambda_k = d_k$. Now, assume that $\lambda_{kk} \in \left(0, \frac{d_k}{C^k}\right)$. Then, by Lemma 2.7, we have

\[
\begin{align*}
&d_k \int_0^1 a y_x^2 \, dx - \lambda_{kk} \int_0^1 \frac{y^2}{b_{kk}} \, dx \\
&\geq d_k \int_0^1 a y_x^2 \, dx - \lambda_{kk} C^k \int_0^1 a y_x^2 \, dx \\
&= (d_k - \lambda_{kk} C^k) \int_0^1 a y_x^2 \, dx \\
&\geq \Lambda_k \int_0^1 a y_x^2 \, dx.
\end{align*}
\]
Finally, we introduce the Hilbert space

\[ H_{a,b}^{2}(0,1) := \left\{ y \in H_{a}^{1} : ay_{x} \in H^{1}(0,1) \quad \text{and} \quad A_{k}y \in L^{2}(0,1) \right\}, \]

where

\[ A_{k}y := d_{k}(a(x)y_{k,x})x + \frac{\lambda_{k}}{b_{kk}}y_{k} \quad \text{with} \quad D(A_{k}) = H_{a,b}^{2}(0,1). \]

To study the well-posedness of the system (1) we write it as the first order evolution equation in the Hilbert space \( \mathbb{H} := L^{2}(0,1)^{n} \)

\[
\begin{cases}
    \dot{Y}(t) = KY(t) + BY(t) - CY(t) + F(t) \\
    Y(0) = (y_{1}^{0}, \cdots, y_{n}^{0})^{*},
\end{cases}
\]

where

\[ Y = (y_{1}, \cdots, y_{n})^{*}, \quad \text{and} \quad F(t) = e_{1}v(t)1_{\omega}. \]

Observe that, since \( C \) is a bounded perturbation, from now on we focus on the well-posedness of the following abstract inhomogeneous Cauchy problem

\[
\begin{cases}
    \dot{Y}(t) = KY(t) + BY(t) + F(t) \\
    Y(0) = (y_{1}^{0}, \cdots, y_{n}^{0})^{*}. \quad (11)
\end{cases}
\]

To show that \( \mathbb{A} := \mathbb{K} + \mathbb{B} \) generates a \( C_{0} \)-semigroup on \( \mathbb{H} \), we split it as

\[ \mathbb{A} := \mathbb{A} + \mathbb{B}_{0} \quad \text{with} \quad D(\mathbb{A}) = \left\{ Y^{*} = (y_{1}, \cdots, y_{n}) \in \mathcal{H} := \prod_{k=1}^{n} H_{k} : (AY)^{*} \in \mathbb{H} \right\}, \]

where the operator \( (\mathbb{A}, D(\mathbb{A})) \) is written in matrix form as

\[ \mathbb{A} = diag(A_{1}, \cdots, A_{n}) \quad \text{with} \quad D(\mathbb{A}) = \prod_{k=1}^{n} D(A_{k}), \]

and

\[
\mathbb{B}_{0} = \begin{pmatrix}
0 & \frac{\lambda_{12}}{b_{12}} & 0 & \cdots & \cdots & 0 \\
\frac{\lambda_{21}}{b_{21}} & 0 & \frac{\lambda_{32}}{b_{32}} & 0 & \cdots & 0 \\
0 & \frac{\lambda_{32}}{b_{32}} & 0 & \frac{\lambda_{43}}{b_{43}} & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \cdots & \frac{\lambda_{n-1,n}}{b_{n-1,n}} & 0 \\
0 & 0 & \cdots & 0 & \frac{\lambda_{n,n-1}}{b_{n,n-1}} & 0
\end{pmatrix}.
\]

Let us now show that the operator \( (\mathbb{A}, D(\mathbb{A})) \) generates an analytic semi-group in the pivot space \( \mathbb{H} \) for the equation (11). This aim relies on this fact.

**Lemma 3.2.** Assume that Hypothesis 3.1 is satisfied. Then, the operator \( \mathbb{A} \) with domain \( D(\mathbb{A}) \) is non-positive and self-adjoint on \( \mathbb{H} \).

**Proof.** Observe that \( D(\mathbb{A}) \) is dense in \( \mathbb{H} \).
(i) \( \mathbb{A} \) is non-positive. By Proposition 1 and integration by parts [23, Lemma 2.1], it follows that, for any \( Y^* = (y_1, \ldots, y_n) \in D(\mathbb{A}) \), we have
\[
-\langle AY, Y \rangle = -\langle (A + B_0)Y, Y \rangle \\
= \sum_{k=1}^{n} \left( d_k \int_0^1 ay_k^2 dx - \int_0^1 \frac{\lambda_{kk}}{b_{kk}} y_k^2 dx \right) \\
- 2 \sum_{k=2}^{n} \int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} y_k y_{k-1} dx \\
\geq \sum_{k=1}^{n} \Lambda_k \int_0^1 ay_k^2 dx - 2 \sum_{k=2}^{n} \int_0^1 \frac{\lambda_{kk-1}}{b_{kk-1}} y_k y_{k-1} dx.
\]

We now apply Young’s inequality and Lemma 2.8, to obtain
\[
\int_0^1 \frac{y_k y_{k-1}}{b_{kk-1}} dx \leq \delta_k \int_0^1 \frac{y_k^2}{b_{kk-1}} dx + \frac{1}{4\delta_k} \int_0^1 \frac{y_{k-1}^2}{b_{kk-1}} dx \\
\leq \delta_k C^{kk-1} \int_0^1 ay_k^2 dx + C^{kk-1} \int_0^1 ay_{k-1}^2 dx,
\]
where \( (\delta_k)_{k=2}^n \) is a sequence of positive constants that will be chosen later on. Hence,
\[
-\langle AY, Y \rangle \geq \sum_{k=1}^{n} \Lambda_k \int_0^1 ay_k^2 dx \\
- 2 \sum_{k=2}^{n} \lambda_{kk-1} \left( \delta_k C^{kk-1} \int_0^1 ay_k^2 dx + \frac{C^{kk-1}}{4\delta_k} \int_0^1 ay_{k-1}^2 dx \right) \\
= \left( \Lambda_1 - 2\frac{\lambda_{21} C^{21}}{4\delta_2} \right) \int_0^1 ay_1^2 dx + \left( \Lambda_n - 2\lambda_n \frac{C^{nn-1}}{4\delta_n} \right) \int_0^1 ay_n^2 dx \\
+ \sum_{k=2}^{n-1} \left( \Lambda_k - 2(\lambda_{kk-1} \delta_k C^{kk-1} + \lambda_{k+1} \frac{C^{k+1}}{4\delta_{k+1}}) \right) \int_0^1 ay_k^2 dx.
\]

Now, by hypothesis (9), one can find \( (\delta_k)_{k=2}^n \) such that
\[
\begin{align*}
\frac{\lambda_{21} C^{21}}{4\Lambda_1} &< \delta_2 < \frac{\Lambda_1}{4\lambda_{21} C^{21}}, \\
\frac{\lambda_{kk} C^{kk-1}}{4\Lambda_{k-1}} &< \delta_k < \frac{\Lambda_{k-1}}{4\lambda_{kk} C^{kk-1}}, \\
\frac{\lambda_n C^{nn-1}}{4\Lambda_{n-1}} &< \delta_n < \frac{\Lambda_{n-1}}{4\lambda_n C^{nn-1}}.
\end{align*}
\]

With this particular choice, we deduce that there is a constant \( C > 0 \) such that
\[
\forall Y \in D(\mathbb{A}), \quad -\langle AY, Y \rangle \geq C \|Y\|^2_H \geq 0, \quad (12)
\]
which proves the result.

(ii) \( \mathbb{A} \) is self-adjoint. Let \( T : H \to H \) be the mapping defined in the following usual way: to each \( F \in H \) associate the weak solution \( Y = T(F) \in H \) of
\[
-\langle AY, Z \rangle = \langle F, Z \rangle
\]
for every \( Z \in H \). Note that \( T \) is well defined by Lax-Milgram Lemma via the part (i), which also implies that \( T \) is continuous. Now, it is easy to see that \( T \) is injective and symmetric. Thus it is self adjoint. As a consequence, \( \mathbb{A} = T^{-1} : D(\mathbb{A}) \to H \) is self-adjoint (for example, see [14, Proposition X.2.4]). \( \square \)
As a consequence of the previous Lemma, the system (11) (and thus (1)) is well-posed in the sense of semigroup theory.

**Proposition 2.** Assume Hypothesis 3.1. Then, the operator \( A : D(A) \to \mathbb{H} \) generates an analytic contraction semigroup of angle \( \pi/2 \) on \( \mathbb{H} \). Moreover, for all \( F^* \in L^2(Q)^n \) and \((Y^0)^* \in \mathbb{H}\), there exists a unique weak solution of (11) that belongs to

\[
C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathcal{H}).
\]

In addition, if \((Y^0)^* \in D(A)\) and \(F^* \in W^{1,1}(0, T, \mathbb{H})\), then \(Y^* = (y_1, y_2, \ldots, y_n) \in C^1(0, T; \mathbb{H}) \cap C([0, T]; D(A))\).

**Proof.** Since \( A \) is a non-positive, self-adjoint operator on a Hilbert space, it is well known that \((A, D(A))\) generates a cosine family and an analytic contractive semigroup of angle \( \pi/2 \) on \( \mathbb{H} \) (see [4, Theorem 3.14.17]). Being \( A \) the generator of a strongly continuous semigroup on \( \mathbb{H} \), the assertion concerning the assumption \((Y^0)^* \in \mathbb{H}\) and the regularity of the solution \(Y\) when \((Y^0)^* \in D(A)\) is a consequence of the results in [5] and [13, Lemma 4.1.5 and Proposition 4.1.6].

4. **Carleman estimates.** The object of this section is to prove an interesting estimate of Carleman type that will be used to show the observability inequality which yields the controllability result. To this end, as in [8] or in [25, Chapter 4], we first define the following time and space weight functions. For \( x \in [D, 1] \), where \( D \) is chosen in such a way that \(-x_0 < D < 0\), let us introduce the function \( \psi(x) = \int_{x_0}^x \frac{y - x_0}{a(y)} dy - d \), where \( a(y) = \begin{cases} a(x), & x \in [0, 1], \\ a(-x), & x \in [-1, 0]. \end{cases} \) (14)

Further, we need to define the following weights function associated to nondegenerate Carleman estimates in a general interval \((A, B)\) which are suited to our purpose. For \( x \in [A, B] \), set

\[
\Phi(t, x) = \theta(t)\Psi(x), \quad \Psi(x) = e^{r\zeta(x)} - e^{2\rho},
\]

where

\[
\zeta(x) = \int_x^B \frac{dy}{\sqrt{a(y)}}, \quad \rho = 2r\zeta(A).
\]

Observe that the function \( \theta(t) \) satisfy

\[
\lim_{t \to 0^+} \theta(t) = \lim_{t \to T} \theta(t) = +\infty, \quad |\theta_1| \leq c\theta^{\frac{3}{2}}, \quad |\theta_2| \leq c\theta^{\frac{3}{2}}
\]

for some \( c > 0 \) depending on \( T \).
Here the parameters $\gamma, r$ and $d$ are chosen such that

\[
d > 4^\alpha \tilde{d}^*, \quad \rho > \ln \left( \frac{4^n (d - \tilde{d}^*)}{d - 4^n \tilde{d}^*} \right),
\]

\[
e^{2\rho} - 1 < \gamma < \frac{4^n}{(4^n - 1)d} (e^{2\rho} - e^{\rho}),
\]

where

\[
\tilde{d}^* := \sup_{[D,1]} \int_{x_0}^x y - x_0 \frac{a(y)}{d} dy,
\]

and $n$ is the size of the system (1).

For this choice of the parameters $\gamma, r$ and $d$ it is straightforward to check that the weight functions $\varphi$ and $\Phi$ satisfy the following inequalities which are needed in the sequel.

**Lemma 4.1.** One has:

1. For $(t, x) \in [0, T] \times [0, 1]$:
   \[
   \varphi(t, x) \leq \Phi(t, x).
   \]

2. For $(t, x) \in [0, T] \times [0, -D]$:
   \[
   \varphi(t, -x) \leq \Phi(t, x).
   \]

3. For $(t, x) \in [0, T] \times [0, 1]$:
   \[
   4^n (4^n - 1) \Phi(t, x) < \varphi(t, x).
   \]

**Proof.** Let us set

\[
d^* := \sup_{[D,1]} \int_{x_0}^x y - x_0 \frac{a(y)}{d} dy,
\]

1. From (16), we have
   \[
   \gamma > e^{2\rho} - 1 > \frac{e^{2\rho} - 1}{d - \tilde{d}^*}.
   \]
   Thus
   \[
   \max_{x \in [0,1]} \psi(x) = \gamma (d^* - d) = \min_{x \in [0,1]} \Psi(x)
   \]
   and the conclusion follows immediately.

2. With the aid of (16), one has
   \[
   \max_{x \in [0, -D]} \psi(-x) \leq \max_{x \in [0,1]} \psi(x) = \gamma (d^* - d) = \min_{x \in [0,1]} \Psi(x) \leq \min_{x \in [0, -D]} \Psi(x).
   \]
   Hence, $\psi(-x) \leq \Psi(x)$, which completes the proof of the desired result.

3. This follows easily by (16). Indeed
   \[
   \frac{4^n}{(4^n - 1)} \max_{x \in [0,1]} \Psi(x) = \frac{4^n}{(4^n - 1)} (e^{2\rho} - e^{\rho}) < -\gamma d = \min_{x \in [0,1]} \psi(x).
   \]

We also need the following result, whose proof is similar to the one given in [16, Lemma 3.3].

**Lemma 4.2.** Let the sequence $\Phi_k$ defined by

\[
\Phi_k = 4^{n-k} (\Phi - \varphi) + \varphi, \quad \forall k : 1 \leq k \leq n.
\]

Then, we have

- $\varphi < \Phi_k < 0$, $k = 1, \cdots, n$.
- $\Phi_n = \Phi < \Phi_{n-1} < \cdots < \Phi_1$. 

4.1. Carleman estimate with boundary observation for the inhomogeneous adjoint system. In this subsection we show Carleman estimates with boundary observation for solutions of the following nonhomogeneous adjoint system:

\[
\begin{aligned}
& z_{kt} + d_k(a(x)z_{kx})_x + \sum_{j=k-1}^{k+1} \frac{\lambda_{jk}}{\gamma_{jk}} z_j - \sum_{j=1}^{k+1} a_{jk} z_j = g_k, \\
&(t, x) \in Q, \quad 1 \leq k \leq n, \\
& z_k(t, 0) = z_k(t, 1) = 0, \quad 1 \leq k \leq n, \quad t \in (0, T), \\
& z_k(T, x) = z_k^T(x), \quad 1 \leq k \leq n, \quad x \in (0, 1),
\end{aligned}
\]  

(17)

which is derived taking inspiration from the works [24] and [29]. Here $z_k^T \in L^2(0, 1)$ and $g_k \in L^2(Q)$ ($1 \leq k \leq n$), while on the degenerate functions $a, b_{kk}, b_{kk-1}$ we make the following assumptions.

Hypothesis 4.3. From now on, we assume the following hypotheses:

1. Hypothesis 3.1 holds. Moreover, if $K > \frac{4}{7}$, then there exists a constant $\vartheta \in (0, K]$ such that the function

\[
x \mapsto \frac{a(x)}{|x - x_0|^{\vartheta}}
\]

is nonincreasing on the left of $x = x_0$, is nondecreasing on the right of $x = x_0$.

2. Moreover, we suppose that

\[
(x - x_0)b_{kk-1}^\prime(x) \geq 0 \quad \text{in} \quad [0, 1].
\]

3. Also, if $\lambda_{kk} < 0$, we require that

\[
(x - x_0)b_{kk}^\prime(x) \geq 0 \quad \text{in} \quad [0, 1].
\]

Our main result is the following.

Theorem 4.4. Assume Hypothesis 4.3. Then, there exist two positive constant $C$ and $s_0$ such that every solution solution $Z = (z_1, z_2, \cdots, z_n)^*$ of (17) in

\[
\mathcal{V} := L^2(0, T; D(A_h)) \cap H^1(0, T; \mathcal{H}),
\]

satisfies

\[
\begin{aligned}
& \sum_{k=1}^{n} \int_Q \left( s\theta a(x)z_{kx}^2 + s^3 \theta^3 \frac{(x - x_0)^2}{a(x)} z_k^2 \right) e^{2s\varphi} dx dt \\
& \leq C \sum_{k=1}^{n} \left( \int_Q g_k^2 e^{2s\varphi} dx dt + s\gamma \int_0^T \left[ \theta(x - x_0)az_{kx}^2 e^{2s\varphi} \right]_{x=0}^{x=1} dt \right),
\end{aligned}
\]

(19)

forall $s \geq s_0$, where $\gamma$ is the constant of (13).

Throughout this paper we will suppose that $z_0 \equiv z_{n+1} \equiv 0$.

Proof. First, observe that the system (17) can be rewritten in the following abstract form

\[
\begin{aligned}
& \partial_t Z + KZ + BZ - C^*Z = G, \quad (t, x) \in Q, \\
& Z(t, 0) = Z(t, 1) = (0, \cdots, 0)^*, \quad t \in (0, T), \\
& Z(T, x) = Z^T(x), \quad x \in (0, 1),
\end{aligned}
\]

(20)

where $G := (g_1, \cdots, g_n)^*$.

Next we define, for $s > 0$, the function

\[
W(t, x) = e^{2s\varphi} Z(t, x),
\]
where $Z$ is a solution of (20) in the class

$$V = H^1(0, T; H) \cap L^2(0, T; D(A)).$$

Then $W$ solves the following system

$$\begin{cases} 
\partial_t(e^{-s\varphi}W) + K(e^{-s\varphi}W) + B(e^{-s\varphi}W) = G + C^*(e^{-s\varphi}W), & (t, x) \in Q, \\
W(t, 0) = W(t, 1) = (0, \cdots, 0)^*, & t \in (0, T), \\
W(T, x) = W(0, x) = (0, \cdots, 0)^*, & x \in (0, 1). 
\end{cases}$$

Equivalently, the previous system can be written as

$$P_s^+W + P_s^-W = G_s,$$

with

$$P_s^+ = \text{diag}(P_{s1}^+, \cdots, P_{sn}^+), \quad B_0, \quad P_s^- = \text{diag}(P_{s1}^-, \cdots, P_{sn}^-) \quad \text{and} \quad G_s = Ge^{s\varphi} + C^*W.$$

The operators $P_{sk}^+$ and $P_{sk}^-$ are given by

$$P_{sk}^+w_k := d_k(aw_{kx})_x + \frac{\lambda_{kk}}{b_{kk}}w_k - s\varphi_1w_k + s^2a\varphi_x^2w_k,$$

and

$$P_{sk}^-w_k := w_{kt} - 2sa\varphi_xw_{kx} - s(a\varphi_x)_xw_k.$$

We have,

$$||P_s^+W||^2 + ||P_s^-W||^2 + 2\langle P_s^+W, P_s^-W \rangle = ||G_s||^2.$$

Then,

$$2\langle P_s^+W, P_s^-W \rangle \leq ||G_s||^2.$$

Here $|| \cdot ||$ stands for $L^2(Q)^n$ norm and $\langle \cdot, \cdot \rangle$ for the corresponding scalar product.

Now, we compute the inner product $\langle P_s^+W, P_s^-W \rangle$, to obtain

$$\langle P_s^+W, P_s^-W \rangle = \sum_{k=1}^n \langle P_{sk}^+w_k, P_{sk}^-w_k \rangle_{L^2(Q)} + \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \langle \frac{\lambda_{jk}}{b_{jk}}w_j, P_{sk}^-w_k \rangle_{L^2(Q)}.$$

Moreover, using the same computations of [24, Lemma 3.6], we find

$$\langle P_{sk}^+w_k, P_{sk}^-w_k \rangle_{L^2(Q)} \geq C \int_Q \left( s\theta a(x)w_{kx}^2 + s^3\theta^3(x-x_0)^2w_k^2 \right) dx dt - s^2\gamma \int_0^T \left[ \theta(x-x_0)aw_{kx}^2 \right]_{x=0}^{x=1} dt. \quad (21)$$

Furthermore, one has

$$\langle \frac{\lambda_{jk}}{b_{jk}}w_j, P_{sk}^-w_k \rangle_{L^2(Q)}$$

$$= \int_Q \frac{\lambda_{jk}}{b_{jk}}w_j w_k (w_{kt} - 2sa\varphi_xw_{kx} - s(a\varphi_x)_xw_k) dx dt. \quad (22)$$

Using the fact that $\lambda_{kj} = \lambda_{jk}$ and $b_{kj} = b_{jk}$, an integration by parts leads to

$$\sum_{k=1}^n \sum_{j=k+1}^{n-1} \int_Q \frac{\lambda_{jk}}{b_{jk}}w_j w_k dx dt = \sum_{k=1}^n \sum_{j=k+1}^{n-1} \lambda_{jk} \int_0^T \frac{w_j w_k |_{t=T}}{2\theta_{jk}} dx. \quad (23)$$
Similarly, one has
\[
-2s \sum_{k=1}^{n} \sum_{j=k}^{k+1} \int_{Q} \frac{\lambda_{jk}}{b_{jk}} a\varphi_{w} w_{j} dx \ dt \\
= -s \sum_{k=1}^{n} \sum_{j=k-1 \atop j \neq k}^{k+1} \lambda_{jk} \int_{0}^{T} \left[ \frac{a\varphi_{x} w_{j} w_{k}}{b_{jk}} \right]_{x=0}^{x=1} \ dt \\
+ s \sum_{k=1}^{n} \sum_{j=k-1 \atop j \neq k}^{k+1} \lambda_{jk} \int_{Q} \frac{(a\varphi_{x})_{x} b_{jk} - a\varphi_{x} b'_{jk}}{b_{jk}^{2}} w_{j} w_{k} \ dx \ dt.
\]

So, combining (22)-(24), we obtain
\[
\sum_{k=1}^{n} \sum_{j=k-1 \atop j \neq k}^{k+1} \langle \lambda_{jk} \frac{b_{jk}}{w_{j}}, P_{\lambda_{jk}} w_{k} \rangle_{L^{2}(Q)} = \sum_{k=1}^{n} \sum_{j=k-1 \atop j \neq k}^{k+1} \lambda_{jk} \int_{0}^{1} \frac{w_{j} w_{k}}{b_{jk}} \ dx \\
- s \sum_{k=1}^{n} \sum_{j=k-1 \atop j \neq k}^{k+1} \lambda_{jk} \int_{0}^{T} \left[ \frac{a\varphi_{x} w_{j} w_{k}}{b_{jk}} \right]_{x=0}^{x=1} \ dt \\
- s \sum_{k=1}^{n} \sum_{j=k-1 \atop j \neq k}^{k+1} \lambda_{jk} \int_{Q} \frac{a\varphi_{x} b'_{jk}}{b_{jk}^{2}} w_{j} w_{k} \ dx \ dt.
\]

As in \cite[Lemma 3.6]{24}, using the definition of \( \varphi \) and the boundary conditions on \( W \), the boundary terms appearing in the above identity reduce to 0.

On the other hand, applying Young’s inequality and using the assumptions on \( b_{jk} \), we obtain
\[
- s \sum_{k=1}^{n} \sum_{j=k-1 \atop j \neq k}^{k+1} \lambda_{jk} \int_{Q} \frac{a\varphi_{x} b'_{jk}}{b_{jk}^{2}} w_{j} w_{k} \ dx \ dt \\
= -s\gamma \sum_{k=1}^{n} \sum_{j=k-1 \atop j \neq k}^{k+1} \lambda_{jk} \int_{Q} \frac{\theta (x-x_{0}) b'_{jk}}{b_{jk}^{2}} w_{j} w_{k} \ dx \ dt \\
\leq \frac{s\gamma}{2} \sum_{k=1}^{n} \sum_{j=k-1 \atop j \neq k}^{k+1} \lambda_{jk} \left( \int_{Q} \frac{(x-x_{0}) b'_{jk}}{b_{jk}^{2}} \theta w_{j}^{2} \ dx \ dt + \int_{Q} \frac{\theta}{b_{jk}^{2}} w_{k}^{2} \ dx \ dt \right) \\
\leq \frac{s\gamma}{2} \sum_{k=1}^{n} \sum_{j=k-1 \atop j \neq k}^{k+1} \lambda_{jk} M(j,k) \left( \int_{Q} \frac{\theta}{b_{jk}} w_{j}^{2} \ dx \ dt + \int_{Q} \frac{\theta}{b_{jk}^{2}} w_{k}^{2} \ dx \ dt \right),
\]

with
\[
M(j,k) = \begin{cases} 
M_{k} & \text{if } j = k - 1, \\
M_{k+1} & \text{if } j = k + 1.
\end{cases}
\]
By the Hardy-Poincaré inequalities, one obtains
\[
\frac{s^\gamma}{2} \sum_{k=1}^{n} \sum_{j=k-1}^{k+1} \lambda_{jk} M(j,k) \left( \int_Q \frac{\theta}{b_{jk}} w_j^2 \, dx \, dt + \int_Q \frac{\theta}{b_{jk}} w_k^2 \, dx \, dt \right)
\]
\[
\leq \frac{s^\gamma}{2} \sum_{k=1}^{n} \sum_{j=k-1}^{k+1} \lambda_{jk} M(j,k) C(j,k) \left( \int_Q s^\gamma \theta a w_j^2 \, dx \, dt + \int_Q s^\gamma \theta a w_k^2 \, dx \, dt \right). \quad (27)
\]

Here, the constant \( C(j,k) > 0 \) is given by
\[
C(j,k) = \begin{cases} 
C_{kk} - 1 & \text{if } j = k - 1, \\
C_{k+1} & \text{if } j = k + 1,
\end{cases}
\]
where \( C_{kk} \) is the constant appearing in (6). By (25)-(27), we obtain
\[
\sum_{k=1}^{n} \sum_{j=k-1}^{k+1} \lambda_{jk} M(j,k) C(j,k) \left( \int_Q \theta a w_j^2 \, dx \, dt + \int_Q \theta a w_k^2 \, dx \, dt \right). \quad (28)
\]

Hence, from (21) and (28), we deduce
\[
\langle P_s W, P_s^{-} W \rangle \geq \sum_{k=1}^{n} \left[ C \int_Q \left( \frac{A_1}{s \theta a(x) w_k^2} + \frac{A_2}{s^3 \theta^3 (x - x_0)^2 a(x) w_k^2} \right) \, dx \, dt \\
- \frac{s^\gamma}{2} \int_0^T \left( \theta(x - x_0) a w_k^2 \right)_{x=0}^{x=1} \, dt \right] \quad (29)
\]
\[
- \frac{s^\gamma}{2} \sum_{k=1}^{n} \sum_{j=k-1}^{k+1} \lambda_{jk} M(j,k) C(j,k) \left( \int_Q \theta a w_j^2 \, dx \, dt + \int_Q \theta a w_k^2 \, dx \, dt \right). \quad (28)
\]

At this stage, let us remark that one can assume \( C \) as large as desired, provided that \( s_0 \) increases as well. Indeed, taken \( k > 0 \), from
\[
C(sA_1 + s^3 A_2) = kC(\frac{s}{k} A_1 + \frac{s^3}{k} A_2),
\]
we can choose \( s_0 = ks_0 \) and \( C = kC \) large as needed. As an immediate consequence, one can prove that the distributed terms in the right hand side of (29) can be estimated as
\[
\langle P_s^+ W, P_s^- W \rangle \geq \sum_{k=1}^{n} \left[ C \int_Q \left( s^\gamma \theta a(x) w_k^2 + s^3 \theta^3 (x - x_0)^2 a(x) w_k^2 \right) \, dx \, dt \\
- \frac{s^\gamma}{2} \int_0^T \left( \theta(x - x_0) a w_k^2 \right)_{x=0}^{x=1} \, dt \right], \quad (30)
\]
where $C$ is a positive constant. On the other hand, we have
\[
\|G_s\|^2 = \|Ge^{s\varphi} + C^*W\|^2
\]
\[
\leq 2 \sum_{k=1}^{n} \int_Q g_k^2 e^{2s\varphi} \, dx \, dt + C \sum_{k=1}^{n} \sum_{j=1}^{k+1} \int_Q a_{jk}^2 w_j^2 \, dx \, dt
\]
\[
\leq 2 \sum_{k=1}^{n} \int_Q g_k^2 e^{2s\varphi} \, dx \, dt + C \sum_{k=1}^{n} \int_Q w_k^2 \, dx \, dt. \tag{31}
\]
Then, by the Hardy-Poincaré inequality given in [26, Proposition 2.6], we get
\[
\sum_{k=1}^{n} \int_0^1 w_k^2 \, dx \leq C_0 \sum_{k=1}^{n} \int_0^1 \frac{a^4(x)}{(x-x_0)^{4/3}} w_k^2 \, dx
\]
\[
\leq C_0 \sum_{k=1}^{n} \int_0^1 \frac{p}{(x-x_0)^{2/3}} w_k^2 \, dx
\]
\[
\leq C_0 C_{HP} \sum_{k=1}^{n} \int_0^1 pw_k^2 \, dx
\]
\[
\leq C_0 \max \{C_1, C_2\} C_{HP} \sum_{k=1}^{n} \int_0^1 a(x)w_k^2 \, dx. \tag{32}
\]
Here $p(x) = (a(x)|x-x_0|^{4/3}$ if $K > \frac{4}{3}$ or $p(x) = \max_{[0,1]}a^{1/3}|x-x_0|^{4/3}$ otherwise,
\[
C_0 := \max \left\{ \left( \frac{x_0^2}{a(0)} \right)^{1/3}, \left( \frac{1-x_0^2}{a(1)} \right)^{1/3} \right\},
\]
\[
C_1 := \max \left\{ \left( \frac{x_0^2}{a(0)} \right)^{2/3}, \left( \frac{1-x_0^2}{a(1)} \right)^{2/3} \right\},
\]
\[
C_2 := \max_{[0,1]}q^{1/3} \times \max \left\{ \frac{x_0^{4/3}}{a(0)}, \frac{(1-x_0)^{4/3}}{a(1)} \right\}
\]
and $C_{HP}$ is the Hardy-Poincaré constant given in [26, Proposition 2.6]. Observe that the function $p$ satisfies the assumptions of [26, Proposition 2.6] thanks to [26, Lemma 2.1]. Substituting the inequality (32) in (31), it follows that
\[
\|G_s\|^2 \leq \sum_{k=1}^{n} \left( 2 \int_Q g_k^2 e^{2s\varphi} \, dx \, dt + \varepsilon \int_Q s\theta a(x)w_k^2 \, dx \, dt \right) \tag{33}
\]
for $\varepsilon > 0$. Thus, by choosing $\varepsilon$ small and $s$ large enough, (30) and (33) imply
\[
\sum_{k=1}^{n} \int_Q \left( s\theta a(x)w_k^2 + s^3\theta^4 \frac{(x-x_0)^2}{a(x)} w_k^2 \right) \, dx \, dt
\]
\[
\leq C \left( \sum_{k=1}^{n} \int_Q g_k^2 e^{2s\varphi} \, dx \, dt + s^4 \int_0^T \left[ \theta(x-x_0)aw_k^2 \right]_{x=0}^{x=1} \, dt \right),
\]
where $C$ is a positive constant. Recalling the definition of $w_k$, one thus obtains the asserted Carleman estimate for our original variables.

\[\square\]
4.2. \(\omega\)-Carleman estimate for the homogeneous adjoint system. In this subsection we consider the following homogeneous parabolic system
\[
\begin{aligned}
\partial_t Z + KZ + BZ - C^* Z = (0, \cdots, 0)^* & \quad (t, x) \in Q, \\
Z(T, x) = Z^T(x) & \in D(\mathcal{A}^2).
\end{aligned}
\]
(34)

Let us recall that \(D(\mathcal{A}^2) = \{Z^\star \in H; (\mathcal{A}Z)^\star \in D(\mathcal{A})\}\). Notice that \(D(\mathcal{A}^2)\) is densely defined in \(D(\mathcal{A})\) for the graph norm (see, for instance [9, Theorem 2.7]) and hence in \(H\). As in [26], we define the following class of functions
\[
W = \{Z \text{ is a solution of (34)}\}.
\]
From [9, Theorem 7.5], we have
\[
W \subset C^1([0, T]; D(\mathcal{A})) \subset V \subset U,
\]
where \(V\) is defined in (18) and
\[
U := C([0, T]; H) \cap L^2(0, T; H).
\]
To obtain Carleman estimates for the system (34), we assume that the control region \(\omega\) satisfies the following assumption:

**Hypothesis 4.5.** The control set \(\omega\) is such that
\[
\omega = \omega_1 \cup \omega_2,
\]
where \(\omega_i (i = 1, 2)\) are intervals with \(\omega_1 := (\alpha_1, \beta_1) \subset (0, x_0), \omega_2 := (\alpha_2, \beta_2) \subset (x_0, 1),\) and \(x_0 \notin \bar{\omega}\).

**Remark 4.** In fact, it is proved in [11] that null controllability of the parabolic operator
\[
Pu = u_t - (|x - x_0|^K u_x)_x, \quad x \in (0, 1),
\]
which degenerates at the interior point \(x_0 \in (0, 1)\), under the action of a locally distributed control supported only on one side of the domain with respect to the point of degeneracy
1. fails for \(K \in [1, 2)\),
2. holds true when \(K \in [0, 1)\).

In particular, in order that (1) to be null controllable, the control support must lie on both sides of the degeneracy point.

Next, we introduce
\[
\omega' = \omega_1' \cup \omega_2'
\]
where \(\omega_i' := (\alpha_i', \beta_i') \subset \omega_i\) and \(\omega_2' := (\alpha_2', \beta_2') \subset \omega_2\).

We claim the following.

**Theorem 4.6.** Assume Hypotheses 4.3 and 4.5. Then, there exist two positive constant \(C_1\) and \(s_0\) such that, every solution \((z_1, \cdots, z_n) \in W\) of (34) satisfies, \(\forall s \geq s_0\),
\[
\sum_{k=1}^{n} \int_Q \left( s\theta u(x)z_k^2 + s^3\theta^3 \frac{(x - x_0)^2}{a(x)} z_k^2 \right) e^{2s\varphi} \, dx \, dt \leq C_1 \sum_{k=1}^{n} \int_{Q_{\omega'}} s^2\theta^2 z_k^2 e^{2s\varphi} \, dx \, dt.
\]
(35)

Here, \(Q_{\omega'} := (0, T) \times \omega'\).
For the proof of the above result, we shall use the following non degenerate non singular classical Carleman estimate in a suitable interval \((A, B)\) (see [26] or [24, Proposition 4.8]), which will be used far away from \(x_0\) within a localization procedure via cut-off functions.

**Proposition 3.** Let \(z\) be the solution of
\[
\begin{align*}
  z_t + (az_x)_x + \frac{\lambda}{b(x)} z &= h, & (t, x) \in (0, T) \times (A, B), \\
  z(t, A) &= z(t, B) = 0, & t \in (0, T),
\end{align*}
\]
where \(h \in L^2((0, T) \times (A, B)), a \in C^1([a, b]), \) and \(b \in C([a, b])\) is a strictly positive function and \(b \geq b_0 > 0 \) in \([a, b]\). Then, there exist two positive constants \(C\) and \(s_0\) such that for any \(s \geq s_0\),
\[
\begin{align*}
  \int_0^T \int_A^B s\theta e^{\epsilon \zeta} z^2 e^{2s\phi} dx dt + \int_0^T \int_A^B \frac{3\theta^2 e^{3\epsilon \zeta}}{s} z^2 e^{2s\phi} dx dt 
  \leq C \left( \int_0^T \int_A^B f^2 e^{2s\phi} dx dt - \int_0^T \left[ r_s\theta e^{\epsilon \zeta} z^2 e^{2s\phi} \right]_{x=A}^{x=B} dt \right)
\end{align*}
\]
for some positive constant \(C\).

**Proof of Theorem 4.6.** To prove the statement we use a technique based on cut-off functions. For this purpose, we define five points \(\delta_1, \delta_2, \alpha_1'', \alpha_2', \beta_2''\) such that
\[
\alpha_1' < \alpha_1'' < \delta_1 < \beta_1' \quad \text{and} \quad \alpha_2' < \alpha_2'' < \delta_2 < \beta_2'' < \beta_2.
\]
From now on, the point \(D\) will be fixed such that \(-x_0 < D < -\beta_1'.\)

Now, let us consider a smooth function \(\tau: [0, 1] \to [0, 1]\) such that
\[
\tau(x) = \begin{cases} 
1, & x \in [\delta_2, 1], \\
0, & x \in [0, \alpha_2'].
\end{cases}
\]
Define \(W = \tau Z\), where \(Z\) is the solution of (34). Then, \(W_k\) \((1 \leq k \leq n)\), satisfies
\[
\begin{align*}
  \partial_t w_k + d_k(a(x)w_{kx})_x - \sum_{j=1}^{k+1} a_{jk} w_j + \sum_{j=k-1}^{k+1} \frac{\lambda_{jk}}{b_{jk}} w_j 
  = (a\tau_x z_k)_x + a\tau_x z_{kx}, & (t, x) \in (0, T) \times (\alpha_2', 1), \\
  w_k(t, \alpha_2') = w_k(t, 1) = 0, & t \in (0, T).
\end{align*}
\]
Since \(x \in (\alpha_2', 1)\), observe that the system above is a nondegenerate nonsingular problem. Thus, we can apply the analogue of Proposition 3 for the component \(w_k\) in \((\alpha_2', 1)\) in place of \((A, B)\), obtaining that there exist two positive constants \(C\) and \(s_0\) \((s_0\) sufficiently large)\), such that \(w_k\) satisfies, for all \(s \geq s_0\),
\[
\begin{align*}
  \int_0^T \int_{\alpha_2'}^1 \left( s\theta e^{\epsilon \zeta} w_{kx}^2 + \frac{3\theta^2 e^{3\epsilon \zeta}}{s} w_k^2 \right) e^{2s\phi} dx dt 
  \leq C \int_0^T \int_{\alpha_2'}^1 \sum_{j=1}^{k+1} a_{jk} w_j - \sum_{j=k-1}^{k+1} \frac{\lambda_{jk}}{b_{jk}} w_j + (a\tau_x z_k)_x + a\tau_x z_{kx} \right)^2 e^{2s\phi} dx dt.
\end{align*}
\]
Let us remark that the boundary term in \(x = 1\) is nonpositive, while the one in \(x = \alpha_2'\) is 0, so that they can be neglected in the classical Carleman estimate.
Moreover, taking into account the fact that \( a_{jk} \in L^\infty(Q) \) and the coefficients \( \frac{1}{b_{jk}} \) are bounded in \([\alpha_2', 1]\), we find
\[
\sum_{k=1}^{n} \int_0^T \int_{\alpha_2'} \left( s\theta e^{-r} w_{kx}^2 + s^3 \theta^3 e^{-3r} w_k^2 \right) e^{2\Phi} \, dx \, dt \\
\leq C \sum_{k=1}^{n} \int_0^T \int_{\alpha_2'} w_k^2 e^{2\Phi} \, dx \, dt + C \sum_{k=1}^{n} \int_0^T \int_{\alpha_2'} ((a\tau_x z_k)_{x} + a\tau_x z_{kx}) e^{2\Phi} \, dx \, dt.
\]

Using the fact that \( \tau_x \) and \( \tau_{xx} \) are supported in \([\alpha_2'', \delta_2]\), we obtain
\[
\sum_{k=1}^{n} \int_0^T \int_{\alpha_2'} \left( s\theta e^{-r} w_{kx}^2 + s^3 \theta^3 e^{-3r} w_k^2 \right) e^{2\Phi} \, dx \, dt \\
\leq C \sum_{k=1}^{n} \int_0^T \int_{\alpha_2'} w_k^2 e^{2\Phi} \, dx \, dt + C \sum_{k=1}^{n} \int_0^T \int_{\alpha_2'} (z_k^2 + z_{kx}^2) e^{2\Phi} \, dx \, dt.
\]

For \( s \) large enough, we have
\[
C \sum_{k=1}^{n} \int_0^T \int_{\alpha_2'} w_k^2 e^{2\Phi} \, dx \, dt \leq \frac{1}{2} \sum_{k=1}^{n} \int_0^T \int_{\alpha_2'} s\theta e^{-r} w_k^2 e^{2\Phi} \, dx \, dt,
\]
and then
\[
\sum_{k=1}^{n} \int_0^T \int_{\alpha_2'} \left( s\theta e^{-r} w_{kx}^2 + s^3 \theta^3 e^{-3r} w_k^2 \right) e^{2\Phi} \, dx \, dt \\
\leq C \sum_{k=1}^{n} \int_0^T \int_{\alpha_2''} (z_k^2 + z_{kx}^2) e^{2\Phi} \, dx \, dt.
\]  
\[
\tag{39}
\]

At this level, we shall also use the following Caccioppoli’s inequality.

**Lemma 4.7.** [7, Proposition 5.1] Let \( \omega' \) and \( \omega'' \) be two non empty open subsets of \((0, 1)\) such that \( \omega'' \subset \omega' \subset (0, 1) \) and \( x_0 \notin \omega' \). Then, there exist two positive constant \( C > 0 \) and \( s_0 > 0 \) such that any solution \( z \) of the equation (36) satisfies, \( \forall s \geq s_0, \)
\[
\int_{Q_{\omega''}} z_k^2 e^{2\Phi} \, dx \, dt \leq C \int_{Q_{\omega'}} (f^2 + s^2 \theta^4 z^2) e^{2\Phi} \, dx \, dt,
\]
with \( \phi(t, x) := \theta(t) \rho(x) \), where \( \rho \in C^2(\overline{\omega'}, \mathbb{R}) \).

In view of Lemma 4.7, one can estimate the right-hand side of (39) as follows
\[
\int_0^T \int_{\alpha_2''} (z_k^2 + z_{kx}^2) e^{2\Phi} \, dx \, dt \\
\leq C \int_{Q_{\omega'}} \left( z_k^2 + \left( \sum_{j=1}^{k+1} a_{jk} z_j - \sum_{j=k+1}^{k+1} \frac{L_{jk}}{b_{jk}} z_j \right)^2 + s^2 \theta^2 z_k^2 \right) e^{2\Phi} \, dx \, dt \\
\leq C \sum_{j=1}^{k+1} \int_{Q_{\omega'}} s^2 \theta^2 z_j^2 e^{2\Phi} \, dx \, dt.
\]  
\[
\tag{40}
\]
Combining (39) and (40), we get

\[
\sum_{k=1}^{n} \int_{0}^{T} \int_{\alpha_{2}}^{1} \left( s\theta e^{-\frac{x\theta}{w_{kx}}} + s^{3}\theta^{3} e^{3\theta\frac{x}{w_{k}}} \right) e^{2s\varphi} \, dx \, dt \\
\leq C \sum_{k=1}^{n} \int_{Q_{\omega}} s^{2}\theta^{2} z_{k}^{2} e^{2s\varphi} \, dx \, dt.
\] (41)

Having in mind the fact that \(e^{2s\varphi} \leq e^{2s\Phi}\), one can easily check that there exists a positive constant \(C\) such that for every \((t, x) \in [0, T] \times [\alpha_{2}, 1]\), we have

\[
a(x)e^{2s\varphi} \leq Ce^{-\rho(x)} \quad \text{and} \quad \frac{(x - x_{0})^{2}}{a(x)}e^{2s\varphi} \leq Ce^{3\rho(x)}e^{2s\Phi}.\] (42)

By (41) and (42), using the definition of \(z_{k}\), we deduce

\[
\sum_{k=1}^{n} \int_{0}^{T} \int_{\delta_{2}}^{1} \left( s\theta a(x)z_{k}^{2} + s^{3}\theta^{3}\frac{(x - x_{0})^{2}}{a(x)}z_{k}^{2} \right) e^{2s\varphi} \, dx \, dt \\
= \sum_{k=1}^{n} \int_{0}^{T} \int_{\delta_{2}}^{1} \left( s\theta a(x)w_{kx}^{2} + s^{3}\theta^{3}\frac{(x - x_{0})^{2}}{a(x)}w_{k}^{2} \right) e^{2s\varphi} \, dx \, dt \\
\leq C \sum_{k=1}^{n} \int_{0}^{T} \int_{\delta_{2}}^{1} \left( s\theta e^{-\frac{x\theta}{w_{kx}}} + s^{3}\theta^{3} e^{3\theta\frac{x}{w_{k}}} \right) e^{2s\varphi} \, dx \, dt \\
\leq C \sum_{k=1}^{n} \int_{Q_{\omega}} s^{2}\theta^{2} z_{k}^{2} e^{2s\varphi} \, dx \, dt.
\] (43)

To complete the proof, it is sufficient to prove a similar inequality on the interval \([0, \delta_{2}]\). To this aim, recalling that \(D\) is chosen in such a way that \(-x_{0} < D < -\beta_{1} < 0\), we follow a reflection procedure already introduced in [26], considering the function

\[
\tilde{z}_{k}(t, x) = \begin{cases} 
  z_{k}(t, x), & x \in [0, 1], \\
  -z_{k}(t, -x), & x \in [-1, 0],
\end{cases}
\]

where \(z_{k}\) solution of the \(k\)-th equation of (34). Let us define

\[
\tilde{\alpha}_{jk}(t, x) = \begin{cases} 
  \alpha_{jk}(t, x), & x \in [0, 1], \\
  \alpha_{jk}(t, -x), & x \in [-1, 0],
\end{cases} \quad \tilde{\beta}_{jk}(x) = \begin{cases} 
  \beta_{jk}(x), & x \in [0, 1], \\
  \beta_{jk}(-x), & x \in [-1, 0],
\end{cases}
\]

\[
\tilde{Z} = (\tilde{z}_{1}, \cdots, \tilde{z}_{n}) \quad \text{and} \quad \tilde{K}\tilde{Z} = \text{diag}\left( (\tilde{a}(x)\tilde{z}_{1x})_{x}, (\tilde{a}(x)\tilde{z}_{2x})_{x}, \cdots, (\tilde{a}(x)\tilde{z}_{nx})_{x} \right).
\]

Therefore, \(\tilde{Z}\) solves the system

\[
\begin{aligned}
\partial_{t}\tilde{Z} + \tilde{K}\tilde{Z} + \tilde{B}\tilde{Z} - \tilde{C}^{*}\tilde{Z} &= (0, \cdots, 0)^{*}, & (t, x) &\in (0, T) \times (-1, 1), \\
\tilde{Z}(t, -1) &= \tilde{Z}(t, 1) = (0, \cdots, 0)^{*}, & t &\in (0, T),
\end{aligned}
\] (44)

where, \(\tilde{C}\) is the analogue of \(C\) with \(a_{jk}\) replaced by \(\tilde{a}_{jk}\) and \(\tilde{B}\) is defined as \(\tilde{C}\). Now, consider a smooth function \(\rho : [-1, 1] \rightarrow [0, 1]\) such that

\[
\rho(x) = \begin{cases} 
  1, & x \in [-\alpha_{2}, \delta_{2}], \\
  0, & x \in [-1, -\delta_{1}] \cup [\delta_{2}, 1],
\end{cases}
\]

and define the function \(\tilde{V} = \rho\tilde{Z}\), where \(\tilde{Z}\) is the solution of (44). Then \(\tilde{V}\) solves

\[
\begin{aligned}
\partial_{t}\tilde{V} + \tilde{K}\tilde{V} + \tilde{B}\tilde{V} - \tilde{C}^{*}\tilde{V} &= \tilde{G}, & (t, x) &\in (0, T) \times (-\beta_{1}, 1), \\
\tilde{V}(t, -\beta_{1}) &= \tilde{V}(t, 1) = (0, \cdots, 0)^{*}, & t &\in (0, T),
\end{aligned}
\] (45)
where $\tilde{G} := (\tilde{a} \rho_z \tilde{Z})_x + \tilde{a} \rho_z \tilde{Z}_x$.

Applying the Carleman estimate (19) to (45) (which still holds true, in $(-\beta_1', 1)$ in place of $(0, 1)$ and $\tilde{a}$ instead of $a$, since $\tilde{a} \in W^{1,1}(-\beta_1', 1)$ in weakly degenerate case and $\tilde{a} \in W^{1,\infty}(-\beta_1', 1)$ in strongly degenerate one), it follows that

$$
\sum_{k=1}^n \int_0^T \int_{-\beta_1'}^{1} \left( s \tilde{a}(x) \tilde{v}_k^2 + s^3 \theta^3 \frac{(x - x_0)^2}{\tilde{a}(x)} \tilde{v}_k^2 \right)e^{2s\varphi} \, dx \, dt
\leq C \sum_{k=1}^n \int_0^T \int_{-\beta_1'}^{1} \left( (\tilde{a} \rho_x \tilde{z}_k)_x + \tilde{a} \rho_x \tilde{z}_k \right)^2 e^{2s\varphi} \, dx \, dt
\leq C \sum_{k=1}^n \int_0^T \int_{-\delta_1}^{1} \left( z_k^2 + z_k^2 \right)e^{2s\varphi} \, dx \, dt + C \sum_{k=1}^n \int_0^T \int_{\delta_2}^{1} \left( z_k^2 + z_k^2 \right)e^{2s\varphi} \, dx \, dt.
$$

Furthermore, using the definition of $\tilde{z}_k$ and thanks to the oddness of the involved functions, one can write

$$
\int_0^T \int_{-\delta_1}^{1} \left( z_k^2 + z_k^2 \right)e^{2s\varphi} \, dx \, dt = \int_0^T \int_{-\alpha''_1}^{1} \left( z_k^2(-x) + z_k^2(-x) \right)e^{2s\varphi(x)} \, dx \, dt
= \int_0^T \int_{\alpha''_1}^{1} \left( z_k^2(x) + z_k^2(x) \right)e^{2s\varphi(-x)} \, dx \, dt.
$$

At this point, by Lemma 4.1, for $(t, x) \in [0, T] \times [\alpha''_1, \delta_1]$, one has

$$
\varphi(t, -x) \leq \Phi(t, x).
$$

It follows from this last inequality and (47) that

$$
\int_0^T \int_{-\delta_1}^{1} \left( z_k^2 + z_k^2 \right)e^{2s\varphi} \, dx \, dt \leq \int_0^T \int_{\alpha''_1}^{1} \left( z_k^2 + z_k^2 \right)e^{2s\Phi} \, dx \, dt.
$$

Putting the above inequality in (46) and using the fact that $\varphi \leq \Phi$, we get

$$
\sum_{k=1}^n \int_0^T \int_{-\beta_1'}^{1} \left( s \tilde{a}(x) \tilde{v}_k^2 + s^3 \theta^3 \frac{(x - x_0)^2}{\tilde{a}(x)} \tilde{v}_k^2 \right)e^{2s\varphi} \, dx \, dt
\leq C \sum_{k=1}^n \int_0^T \int_{\alpha''_1}^{1} \left( z_k^2 + z_k^2 \right)e^{2s\Phi} \, dx \, dt + C \sum_{k=1}^n \int_0^T \int_{\delta_2}^{1} \left( z_k^2 + z_k^2 \right)e^{2s\Phi} \, dx \, dt.
$$

Now, proceeding as in (40), it is not difficult to see that

$$
\int_0^T \int_{\alpha''_1}^{1} \left( z_k^2 + z_k^2 \right)e^{2s\Phi} \, dx \, dt + \int_0^T \int_{\delta_2}^{1} \left( z_k^2 + z_k^2 \right)e^{2s\Phi} \, dx \, dt
\leq C \int_{Q_{\omega'}} s^2 \theta^2 z_k^2 e^{2s\Phi} \, dx \, dt.
$$

Thus, it appears that

$$
\sum_{k=1}^n \int_0^T \int_{-\beta_1'}^{1} \left( s \tilde{a}(x) \tilde{v}_k^2 + s^3 \theta^3 \frac{(x - x_0)^2}{\tilde{a}(x)} \tilde{v}_k^2 \right)e^{2s\varphi} \, dx \, dt
\leq C \sum_{k=1}^n \int_{Q_{\omega'}} s^2 \theta^2 z_k^2 e^{2s\Phi} \, dx \, dt.
$$
Using the definitions of $\tilde{z}_k, \tilde{v}_k$ and $\rho$, we obtain

$$
\sum_{k=1}^{n} \int_0^T \int_0^{\delta_2} \left( s\theta a(x)z_{k,x}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)}z_k^2 \right) e^{2s\rho} \, dx \, dt
$$

$$
= \sum_{k=1}^{n} \int_0^T \int_0^{\delta_2} \left( s\theta a(x)v_{k,x}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)}v_k^2 \right) e^{2s\rho} \, dx \, dt
$$

$$
\leq \sum_{k=1}^{n} \int_0^1 \int_0^{\delta_1} \left( s\theta a(x)v_{k,x}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)}v_k^2 \right) e^{2s\rho} \, dx \, dt
$$

$$
\leq C \sum_{k=1}^{n} \int_{Q_{\omega}} s^2\theta^2 z_k^2 e^{2s\rho} \, dx \, dt. \quad (49)
$$

Adding (43) and (49), we finally obtain Theorem 4.6. □

To establish the null controllability of the parabolic system (1) with one control force, we need the following crucial Carleman estimate with one observation.

**Theorem 4.8.** Assume Hypotheses 4.3 and 4.5. Moreover, we suppose that for some open subset $\hat{\omega} \subset \subset \omega$, we have

$$
-a_{kk-1} + \frac{\lambda_{kk-1}}{b_{kk-1}} \geq 0, \quad \text{in } (0, T) \times \hat{\omega}, \quad \forall k : 2 \leq k \leq n. \quad (50)
$$

Then, there exist two positive constant $C$ and $s_0$, such that every solution of (34) satisfies, $\forall s > s_0$,

$$
\sum_{k=1}^{n} \int_{Q} \left( s\theta a(x)z_{k,x}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)}z_k^2 \right) e^{2s\rho} \, dx \, dt \leq C \int_{Q_{\omega}} z_k^2 \, dx \, dt. \quad (51)
$$

**Remark 5.** We point out that the proof of Theorem 4.8 is still valid if we assume (50) or

$$
-a_{kk-1} + \frac{\lambda_{kk-1}}{b_{kk-1}} \leq -b_0 < 0, \quad \text{in } (0, T) \times \hat{\omega}, \quad \forall k : 2 \leq k \leq n.
$$

To prove the above Theorem, we will need the following lemma.

**Lemma 4.9.** Under the assumptions of Theorem 4.8 and given $l \in \mathbb{N}$, $\varepsilon > 0$, $k = 2, \ldots, n$ and two open sets $\omega_0$ and $\omega'$ such that $\omega' \subset \subset \omega_0 \subset \subset \omega$, there exist $C_k = C_k(\omega_0, \omega', b_0) > 0$ and $l_j \in \mathbb{N}$, $1 \leq j \leq k-1$ such that if $Z = (z_1, \ldots, z_n)^*$ is the solution to (34), one has

$$
J_{\omega_0}(l, \Phi_k, z_k) \leq \varepsilon \sum_{j=k}^{k+1} \int_{Q} \left( s\theta a(x)z_{j,x}^2 + s^3\theta^3 \frac{(x-x_0)^2}{a(x)}z_j^2 \right) e^{2s\rho} \, dx \, dt
$$

$$
+ C_k \left( 1 + \frac{1}{\varepsilon} \right) \sum_{j=1}^{k-1} J_{\omega_0}(l_j, \Phi_{k-1}, z_j), \quad (52)
$$

with $J_{\omega}(d, \phi, z) := s^d \int_{Q_{\omega}} \theta^d z^2 e^{2s\phi} \, dx \, dt$ and $l_j = \max(3, 2l + 1)$.

**Proof of Theorem 4.8.** In order to eliminate the local terms

$$
\int_{Q_{\omega}} s^2\theta^2 z_k^2 e^{2s\rho} \, dx \, dt \quad \forall k = 2, \ldots, n,
$$

...
in the right hand side of (35), we proceed as in [28]. Thus, we consider a sequence of open sets \((O_k)_{k=2}^{n}\) such that \(\omega' \subset \subset O_n \subset \subset O_{n-1} \subset \subset \cdots \subset \subset O_2 \subset \subset \omega\).

At first, to absorb the term that depend on the component \(z_n\), we apply the formula (52) for

\[
k = n, \quad l = 2, \quad \omega_0 = \omega', \quad \omega_0 = O_n \quad \text{and} \quad \varepsilon = \frac{1}{2C_1},
\]

with \(C_1\) is the constant appearing in (35). Therefore, from (35), one has

\[
\sum_{k=1}^{n} \int_Q \left( s \theta a(x) z_{kx}^2 + s^3 \theta^3 \frac{(x-x_0)^2}{a(x)} z_k^2 \right) e^{2s\varphi} \, dx \, dt \\
\leq \tilde{C}_n \sum_{j=1}^{n-1} J_{O_n}(l_j, \Phi_{n-1}, z_j),
\]

(53)

where \(\tilde{C}_n = 2 \max\left( C_1, C_n (1 + 2C_1) \right)\). We can go on applying (52) for

\[
k = n-1, \quad l = l_{n-1}, \quad \omega_0 = O_n, \quad \omega_0 = O_{n-1}, \quad \text{and} \quad \varepsilon = \frac{1}{2C_n},
\]

and eliminate in (53) the local term \(J_{O_n}(l_{n-1}, \Phi_{n-1}, z_{n-1})\), obtaining

\[
\sum_{k=1}^{n} \int_Q \left( s \theta a(x) z_{kx}^2 + s^3 \theta^3 \frac{(x-x_0)^2}{a(x)} z_k^2 \right) e^{2s\varphi} \, dx \, dt \\
\leq \tilde{C}_{n-1} \sum_{j=1}^{n-2} J_{O_{n-1}}(l_j, \Phi_{n-2}, z_j).
\]

By (a finite) iteration of this argument, we obtain

\[
\sum_{k=1}^{n} \int_Q \left( s \theta a(x) z_{kx}^2 + s^3 \theta^3 \frac{(x-x_0)^2}{a(x)} z_k^2 \right) e^{2s\varphi} \, dx \, dt \\
\leq \tilde{C}_2 J_{O_2}(l_1, \Phi_1, z_1) \\
\leq C \int_{Q_\omega} s \theta l_1 z_1 e^{2s\Phi_1} \, dx \, dt,
\]

with \(C\) is a positive constant. Finally, since \(\sup_{(t,x) \in Q} s \theta l_1 e^{2s\Phi_1} < +\infty\), we readily deduce (51), which concludes the proof of Theorem 4.8.

\[\Box\]

**Proof of Lemma 4.9.** Let us consider a smooth cut-off function \(\xi \in C^\infty(0,1)\) such that

\[
\xi(x) = \begin{cases} 
1, & x \in \omega', \\
0, & x \in (0,1) \setminus \omega_0,
\end{cases}
\]

(54)
On the other hand, thanks to Lemma 4.2, one can easily check that
\[
\xi_x, \xi_{xx} \in L^\infty(0,1).
\]
Multiplying the equation satisfied by \( z_{k-1} \) in (34) by \( s^t \theta^j \xi e^{2s \Phi_k} z_k \) and integrating over \( Q \), we get
\[
\begin{align*}
b_0 J_{(\cdot)}(l, \Phi_k, z_k) & \leq \iint_Q \left( - a_{kk-1} + \frac{\lambda_{kk-1}}{b_{kk-1}} \right) s^t \theta^j \xi e^{2s \Phi_k} z_k^2 \, dx \, dt \\
& \leq \left\{ \begin{array}{c}
\int_Q z_{k-1,t} s^t \theta^j \xi e^{2s \Phi_k} z_k \, dx \, dt \\
\int_Q z_k \, dx \, dt \\
\int_Q a(x) z_{k-1,x} s^t \theta^j \xi e^{2s \Phi_k} z_k \, dx \, dt \\
\int_Q a_{j-1} z_j s^t \theta^j \xi e^{2s \Phi_k} z_k \, dx \, dt
\end{array} \right\} + \left\{ \begin{array}{c}
\sum_{j=1}^{k-1} \int_Q a_{j-1} z_j s^t \theta^j \xi e^{2s \Phi_k} z_k \, dx \, dt \\
\sum_{j=k-1}^{k-2} \int_Q a_{j-1} z_j s^t \theta^j \xi e^{2s \Phi_k} z_k \, dx \, dt
\end{array} \right\}.
\end{align*}
\]

Using the same computations as in [16, Lemma 3.7], one can prove that
\[
K_1 + K_2 + K_3 \leq \frac{\varepsilon}{2} \sum_{j=k}^{k+1} \iint_Q \left( s^t \theta^j \xi e^{2s \Phi_k} z_k \, dx \, dt \right)
\]
\[
+ C_k \left( 1 + \frac{1}{\varepsilon} \right) \sum_{j=1}^{k-1} \int_{Q_{x_0}} s^t \theta^j \xi e^{2s \Phi_k-1} \, dx \, dt,
\]
with \( l_1 = \max(3, 2l+1) \) and \( C_k \) is a positive constant depending on \( k \). In addition, the term \( K_4 \), can be estimated using Young’s inequality, in the following way
\[
K_4 = \left| \sum_{j=k-1}^{k-2} \int_Q \frac{\lambda_{jk-1}}{b_{jk-1}} z_j s^t \theta^j \xi e^{2s \Phi_k} z_k \, dx \, dt \right|
\]
\[
\leq \sum_{j=k-2}^{k-1} \left| \int_Q \left( \frac{\lambda_{jk-1}}{b_{jk-1}} \right) s^t \theta^j \xi e^{2s \Phi_k} z_k \, dx \, dt \right|
\]
\[
\leq \frac{1}{\varepsilon} \sum_{j=k-2}^{k-1} \int_Q \left( \frac{\lambda_{jk-1}}{b_{jk-1}} \right) s^t \theta^j \xi e^{2s \Phi_k} \, dx \, dt
\]
\[
+ \frac{\varepsilon}{2} \int_Q s^t \theta^j \xi e^{2s \Phi_k} \, dx \, dt.
\]

Moreover, since \( \text{supp} (\xi) \subset \omega_0 \) and \( x_0 \notin \overline{\omega_0} \), using the fact that the functions \( \frac{1}{b_{jk}} \) and \( \frac{a}{(x-x_0)^2} \) are bounded in \( \overline{\omega_0} \), one has
\[
K_4 \leq \frac{C}{\varepsilon} \sum_{j=k-2}^{k-1} \int_{Q_{x_0}} (s \theta)^{2l-3} \xi e^{2s (2\Phi_k-\varphi)} \, dx \, dt
\]
\[
+ \frac{\varepsilon}{2} \int_Q s^3 \theta^3 \frac{(x-x_0)^2}{a} e^{2s \Phi_k} \, dx \, dt.
\]

On the other hand, thanks to Lemma 4.2, one can easily check that
\[
e^{2s (2\Phi_k-\varphi)} \leq e^{2s \Phi_{k-1}}.
\]
Adding (55) and (56), by (57), we can deduce
\[
K_1 + K_2 + K_3 + K_4 \leq \varepsilon \sum_{j=k}^{k+1} \int_{Q} \left( s \partial a(x) \frac{z^2_j}{x} + \frac{s^3 \partial^3 (x-x_0)^2}{a(x)^2} \right) e^{2s\varphi} \, dx \, dt
\]
\[ + C_k \left( 1 + \frac{1}{\varepsilon} \right) \sum_{j=1}^{k-1} \int_{Q_0} s^{l_j} \theta \frac{z^2_j}{x} e^{2s\varphi k-1} \, dx \, dt,
\]
where \( l_j = \max(3, 2l + 1) \). Finally,
\[
J_{\omega_0}(l, \Phi_k, z_k) \leq \varepsilon \sum_{j=k}^{k+1} \int_{Q} \left( s \partial a(x) \frac{z^2_j}{x} + \frac{s^3 \partial^3 (x-x_0)^2}{a(x)^2} \right) e^{2s\varphi} \, dx \, dt
\]
\[ + C_k \left( 1 + \frac{1}{\varepsilon} \right) \sum_{j=1}^{k-1} J_{\omega_0}(l_j, \Phi_{k-1}, z_j).
\]
This concludes the proof of Lemma 4.9.

5. Observability and null controllability. In this section, we establish an indirect observability estimate using certain ideas from [8] and [29]. For this, on the coefficients \( a \) and \( b_{jk} \) we essentially start with the assumptions made so far, with the exception of Hypothesis 2.4. More precisely, we shall apply the just established Carleman inequalities to prove observability inequality for the homogeneous adjoint problem (4) and deduce the null controllability for the system (1). In particular, our main observability result is the following.

**Theorem 5.1.** Under the assumptions of Theorem 4.8, there exists a positive constant \( C_T \) such that for every \((z^T_1, \cdots, z^T_n) \in \mathbb{H}, \) the corresponding solution \( Z \) to (4) satisfies
\[
\|Z(0, \cdot)\|^2_{\mathbb{H}} \leq C_T \int_{Q_\omega} |z_1(t, x)|^2 \, dx \, dt.
\]

In order to prove the previous theorem, we need to prove the following observability inequality in the case of a regular final-time datum.

**Theorem 5.2.** Under the assumptions of Theorem 4.8, there exists a positive constant \( C_T \) such that for every \((z^T_1, \cdots, z^T_n) \in D(k^2), \) the corresponding solution \( Z \) to (34) satisfies
\[
\|Z(0, \cdot)\|^2_{\mathbb{H}} \leq C_T \int_{Q_\omega} |z_1(t, x)|^2 \, dx \, dt.
\]

**Proof.** Multiplying the \( k \)-th equation of the adjoint system (4) by \( z_{kt} \) and then integrating over \((0, 1)\) with respect to \( x, \) one gets that
\[
0 = \sum_{k=1}^{n} \int_0^1 z^2_{kt} \, dx + \sum_{k=1}^{n} \left[ d_k a(x) z_{kt} z_{kx} \right]_{x=0}^{x=1} - \frac{1}{2} \frac{d}{dt} \sum_{k=1}^{n} \int_0^1 d_k a(x) z^2_{kx} \, dx
\]
\[ + \sum_{k=1}^{n} \sum_{j=k-1}^{k+1} \int_0^1 \lambda_{jk} \frac{z_{j} z_{kt}}{b_{jk}} \, dx - \sum_{k=1}^{n} \sum_{j=1}^{k-1} \int_0^1 a_{jk} z_j z_{kt} \, dx.
\]

(60)
On the other hand, using the fact that the matrix $B$ is symmetric, we obtain
\[
\sum_{k=1}^{n} \sum_{j=k+1}^{n} \int_{0}^{1} \lambda_{jj} z_{j} z_{k} \, dx = \frac{1}{2} \sum_{k=1}^{n} \frac{d}{dt} \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} z_{k}^{2} \, dx + \frac{d}{dt} \int_{0}^{1} \lambda_{k}^{1} \frac{b_{kk}}{b_{kk}^{1}} z_{k} z_{k-1} \, dx.
\]
Furthermore, since $a_{jk} \in L^{\infty}(Q)$, by Young’s inequality it follows that
\[
\sum_{k=1}^{n} \sum_{j=1}^{k} \int_{0}^{1} a_{jk} z_{j} z_{k} \, dx \leq C \sum_{k=1}^{n} \int_{0}^{1} z_{k}^{2} \, dx + \sum_{k=1}^{n} \int_{0}^{1} z_{k}^{2} \, dx
\]
for some positive constant $C$. Thus, (60) becomes
\[
- \frac{d}{dt} \sum_{k=1}^{n} \int_{0}^{1} d_{k} a(x) z_{k}^{2} \, dx + \frac{d}{dt} \sum_{k=1}^{n} \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} z_{k}^{2} \, dx + \frac{d}{dt} \sum_{k=1}^{n} \int_{0}^{1} \frac{\lambda_{kk}^{1}}{b_{kk}^{1}} z_{k} z_{k-1} \, dx \leq C \sum_{k=1}^{n} \int_{0}^{1} z_{k}^{2} \, dx.
\]
(61)
Then, by the same technique used in (32), we have
\[
\sum_{k=1}^{n} \int_{0}^{1} z_{k}^{2} \, dx \leq C \sum_{k=1}^{n} \int_{0}^{1} a(x) z_{k}^{2} \, dx.
\]
Substituting the above inequality in (61), we obtain
\[
- \frac{d}{dt} \sum_{k=1}^{n} \left( \int_{0}^{1} d_{k} a(x) z_{k}^{2} \, dx - \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} z_{k}^{2} \, dx - 2 \int_{0}^{1} \frac{\lambda_{kk}^{1}}{b_{kk}^{1}} z_{k} z_{k-1} \, dx \right) \leq C \sum_{k=1}^{n} \int_{0}^{1} a(x) z_{k}^{2} \, dx
\]
(62)
for a positive constant $C$.
At this stage, observe that from (12), one can find $C > 0$ such that
\[
-\langle \dot{A}Z, Z \rangle = \sum_{k=1}^{n} \left( \int_{0}^{1} d_{k} a(x) z_{k}^{2} \, dx - \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} z_{k}^{2} \, dx - 2 \int_{0}^{1} \frac{\lambda_{kk}^{1}}{b_{kk}^{1}} z_{k} z_{k-1} \, dx \right) \geq C \sum_{k=1}^{n} \int_{0}^{1} a z_{k}^{2} \, dx.
\]
(63)
Combining (62) and (63), we obtain
\[
- \frac{d}{dt} \sum_{k=1}^{n} \left( \int_{0}^{1} d_{k} a(x) z_{k}^{2} \, dx - \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} z_{k}^{2} \, dx - 2 \int_{0}^{1} \frac{\lambda_{kk}^{1}}{b_{kk}^{1}} z_{k} z_{k-1} \, dx \right) \leq C \sum_{k=1}^{n} \left( \int_{0}^{1} d_{k} a y_{k}^{2} \, dx - \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} y_{k}^{2} \, dx - 2 \int_{0}^{1} \frac{\lambda_{kk}^{1}}{b_{kk}^{1}} y_{k} y_{k-1} \, dx \right)
\]
for a positive constant $\hat{C}$. Hence
\[
\frac{d}{dt} \left\{ e^{\hat{C}t} \sum_{k=1}^{n} \left( \int_{0}^{1} d_{k} a(x) z_{k}^{2} \, dx - \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} z_{k}^{2} \, dx - 2 \int_{0}^{1} \frac{\lambda_{kk}^{1}}{b_{kk}^{1}} z_{k} z_{k-1} \, dx \right) \right\} \geq 0.
\]
Consequently, the function

\[ t \mapsto e^{\hat{C}T} \sum_{k=1}^{n} \left( \int_{0}^{1} d_k a(x) z_{k,x}^2 \, dx - \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} z_{k}^2(0, x) \, dx \right) \]

is increasing for all \( t \in [0, T] \). Thus,

\[
\sum_{k=1}^{n} \left( \int_{0}^{1} d_k a(x) z_{k,x}^2 (0, x) \, dx - \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} z_{k}^2(0, x) \, dx \right) \\
- 2 \int_{0}^{1} \frac{\lambda_{kk-1}}{b_{kk-1}} z_k(0, x) z_{k-1}(0, x) \, dx \leq e^{\hat{C}T} \sum_{k=1}^{n} \left( \int_{0}^{1} d_k a(x) z_{k,x}^2(t, x) \, dx \\
- \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} z_{k}^2(t, x) \, dx - 2 \int_{0}^{1} \frac{\lambda_{kk-1}}{b_{kk-1}} z_k(t, x) z_{k-1}(t, x) \, dx \right).
\]

Next, using Young’s inequality and applying the Hardy-Poincaré inequalities (5) and (6), it results:

\[
\sum_{k=1}^{n} \left( \int_{0}^{1} d_k a(x) z_{k,x}^2 (0, x) \, dx - \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} z_{k}^2(0, x) \, dx \right) \\
- 2 \int_{0}^{1} \frac{\lambda_{kk-1}}{b_{kk-1}} z_k(0, x) z_{k-1}(0, x) \, dx \leq e^{\hat{C}T} \sum_{k=1}^{n} \left( d_k + \lambda_{kk} C_k + \lambda_{kk-1} C_{kk-1} + \lambda_{k+1} C_{k+1,k} \right) \int_{0}^{1} a(x) z_{k,x}^2(t, x) \, dx.
\]

Integrating the previous inequality over \([\frac{T}{4}, \frac{3T}{4}]\), \( \theta \) being bounded therein, we find

\[
\sum_{k=1}^{n} \left( \int_{0}^{1} d_k a(x) z_{k,x}^2 (0, x) \, dx - \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} z_{k}^2(0, x) \, dx \right) \\
- 2 \int_{0}^{1} \frac{\lambda_{kk-1}}{b_{kk-1}} z_k(0, x) z_{k-1}(0, x) \, dx \leq 2 T e^{\hat{C}T} \sum_{k=1}^{n} \left( d_k + \lambda_{kk} C_k + \lambda_{kk-1} C_{kk-1} + \lambda_{k+1} C_{k+1,k} \right) \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{0}^{1} a(x) z_{k,x}^2 \, dx \, dt \\
\leq C_T \sum_{k=1}^{n} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{0}^{1} \theta a(x) z_{k,x}^2 e^{2s \theta} \, dx \, dt.
\]

Now, using the Carleman estimate (51), we readily deduce

\[
\sum_{k=1}^{n} \left( \int_{0}^{1} d_k a(x) z_{k,x}^2 (0, x) \, dx - \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}} z_{k}^2(0, x) \, dx \right) \\
- 2 \int_{0}^{1} \frac{\lambda_{kk-1}}{b_{kk-1}} z_k(0, x) z_{k-1}(0, x) \, dx \leq C \int_{0}^{T} z_{k}^2 \, dx \, dt,
\]

where \( C \) is a positive constant.
Then (63) implies that there exists a constant $C > 0$ such that
\[
\sum_{k=1}^{n} \int_{0}^{1} a(x)z_{k}^{2}(0, x) \, dx \leq C \sum_{k=1}^{n} \left( \int_{0}^{1} d_{k}a(x)z_{k}^{2}(0, x) \, dx \right)
\]
\[
- \int_{0}^{1} \frac{\lambda_{kk}}{b_{kk}}z_{k}^{2}(0, x) \, dx - 2 \int_{0}^{1} \frac{\lambda_{kk-1}}{b_{kk-1}}z_{k}(0, x)z_{k-1}(0, x) \, dx \right)
\]
\[
\leq C \int_{0}^{T} \int_{\omega} z_{1}^{2} \, dx \, dt.
\]
Finally, we proceed as in (32), to obtain
\[
\sum_{k=1}^{n} \int_{0}^{1} z_{k}^{2}(0, x) \, dx \leq C_{0} \max\{C_1, C_2\} C_{HP} \sum_{k=1}^{n} \int_{0}^{1} a(x)z_{k}^{2}(0, x) \, dx
\]
\[
\leq C \int_{0}^{T} \int_{\omega} z_{1}^{2} \, dx \, dt,
\]
for a positive constant $C$. Hence, the conclusion follows.

By Theorem 5.2 and using a density argument, as in [26, Proposition 4.1], one can prove Theorem 5.1. As an immediate consequence, we can prove, using a standard technique (e.g., see [31, Section 7.4]), the null controllability result for the linear degenerate/singular problem (1): if (58) holds, then for every $(y_{0}^{1}, \ldots, y_{n}^{0}) \in \mathbb{H}$, there exists a control $v \in L^{2}(Q)$ such that the solution of the parabolic system (1) satisfies
\[
y_{k}(T, \cdot) = 0, \quad \text{in } (0, 1), \quad \forall k : 1 \leq k \leq n.
\]
Moreover, there exists $C_{T} > 0$ such that
\[
\|v\|_{L^{2}(Q)}^{2} \leq C_{T}\|Y^{0}\|_{\mathbb{H}}^{2}.
\]

6. Conclusion and perspectives. In the present paper we treated the interior controllability for a coupled system of degenerate parabolic equations with singular potentials. The main particularity is the fact that the coupling is also done in the singular terms. By means of a Carleman inequality with only one observation for the problem under analysis, we obtained the null controllability employing one single distributed control supported in a suitable open subset of the domain.

We present here some remarks and perspective related to our work.

1. First of all, as in [28], we believe that the results of the present paper could be extended to more general cascade systems by introducing first order coupling terms. However, the employment of a weight for the Carleman inequality for the degenerate/singular part and a classical weight for the classical parabolic equation is not of use in this situation. We think that weight introduced in [12] (see also [18]) could be used for analyzing control properties of (1) involving coupling terms of first order.

2. Let us consider the following degenerate/singular operator:
\[
P_{\alpha, \mu}y := y_{t} - (x^{\alpha}y_{x})_{x} - \frac{\mu}{x^{2-\alpha}}y^{2}.
\]
It has been proved in [6] that, provided that the parameters $\alpha$ and $\mu$ satisfy some precise conditions, the mixed degenerate/singular operator $P_{\alpha, \mu}$ associated to homogeneous boundary conditions of Dirichlet type is boundary null.
controllable. Following [17], since we have an explicit knowledge of the spectrum of the operator $P_{\alpha,\mu}$, we believe that the moment method could be used for analyzing boundary controllability for degenerate/singular parabolic coupled systems. This and other related questions are being considered and will be addressed elsewhere.

3. The focus of this work was the controllability of a $n \times n$ degenerate/singular system by a locally distributed control that enters the model as an additive term describing the effect of some external force on the process at hand. However this is not always realistic to act on the system in such a way. In the spirit of the works [19, 30, 33], it would be interesting to study the problem of bilinear or multiplicative controllability for this class of systems.

4. Finally, in view of the new Carleman estimates introduced in [10], it will be important to analyse controllability properties for the hybrid systems, between transport and degenerate/singular equations. This is an issue for future research to explore.

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