ON A REDUCTION MAP FOR DRINFELD MODULES

WOJCIECH BONDAREWICZ, PIOTR KRASOŃ

Abstract. In this paper we investigate a local to global principle for Mordell-Weil group defined over a ring of integers $\mathcal{O}_K$ of $t$-modules that are products of the Drinfeld modules $\hat{\varphi} = \phi_1^{e_1} \times \cdots \times \phi_t^{e_t}$. Here $K$ is a finite extension of the field of fractions of $A = \mathbb{F}_q[t]$. We assume that the rank($\phi_i$) = $d_i$ and endomorphism rings of the involved Drinfeld modules of generic characteristic are the simplest possible, i.e. $\text{End}(\phi_i) = A$ for $i = 1, \ldots, t$. Our main result is the following numeric criterion. Let $N = N_1^{e_1} \times \cdots \times N_t^{e_t}$ be a finitely generated $A$-submodule of the Mordell-Weil group $\hat{\varphi}(\mathcal{O}_K) = \phi_1^{e_1}(\mathcal{O}_K)^{e_1} \times \cdots \times \phi_t^{e_t}(\mathcal{O}_K)^{e_t}$, and let $\Lambda \subset N$ be an $A$-submodule. If we assume $d_i \geq e_i$ and $P \in N$ such that $r_W(P) \in r_W(\Lambda)$ for almost all primes $W$ of $\mathcal{O}_K$, then $P \in \Lambda + N_{tor}$. We also build on the recent results of S.Baraniczuk [B17] concerning the dynamical local to global principle in Mordell-Weil type groups and the solvability of certain dynamical equations to the aforementioned $t$-modules.

1. Introduction

The local to global principle we investigate is of the following type. Assume we are given an object $A$ (of a small category $\mathcal{C}$) associated with a ring of integers $\mathcal{O}_R$ of some ring $R$. Assume that for any prime ideal $p \triangleleft R$ there exists a reduced object $A_p$. Assume further that we are given a property $PROP$ of $A$ such that the corresponding property $PROP_p$ for $A_p$ makes sense. Then one can raise the following question:

Question 1.1 (Local to global principle). Assume that for almost all prime ideals $p \triangleleft R$ properties $PROP_p$ hold. Does it follow that property $PROP$ hold for $A$?

In 1975 A. Schinzel [Sch75], generalized the work of Skolem [Sk37] from 1937 and proved the following theorem concerning exponential equations.

Theorem 1.2 (A.Schinzel). If $\alpha_1, \ldots, \alpha_k, \beta$ are non-zero elements of $K$ and the congruence $\alpha_1^{x_1} \alpha_2^{x_2} \cdots \alpha_k^{x_k} \equiv \beta \mod p$ is soluble for almost all prime ideals $p$ of a number field $K$ then the corresponding equation is soluble in rational integers. That is there exist $n_1 \ldots n_k \in \mathbb{Z}$ such that $\beta = \alpha_1^{n_1} \cdots \alpha_k^{n_k}$.

This theorem is in fact the detecting linear dependence problem for number fields and of the kind considered in Question 1.1. The reduction map is the usual reduction modulo a non Archimedean prime in a number field. It is well known that some questions concerning number fields can be translated to the context of abelian varieties. An analogous question

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for abelian varieties was raised by W. Gajda (see [We03]) and has been studied extensively [Ko03, We03, BGK05, GG09, Jo13]. A related problem, called the support problem, originated from a question by P. Erdős for integers and was solved in [CRS97] for number fields and elliptic curves. The support problem for some abelian varieties and intermediate Jacobians was treated in [BGK03] and solved for all abelian varieties by M. Larsen in [La03]. In the case of Drinfeld modules, the support problem was studied by A. Li in [Li06]. For the history of the detecting linear dependence problem for abelian varieties, as well as generalisation of it to linear algebraic groups see [FK17].

In [BK11] the second named author and G. Banaszak proved the following theorem:

**Theorem 1.3.** Let $\mathbb{A}/F$ be an abelian variety defined over a number field $F$. Assume that $\mathbb{A}$ is isogeneous to $\mathbb{A}_1^{e_1} \times \cdots \times \mathbb{A}_t^{e_t}$ with $\mathbb{A}_i$ simple, pairwise nonisogenous abelian varieties such that

$$\dim_{\text{End}_{F'}(\mathbb{A}_i)^0} H_1(\mathbb{A}_i(\mathbb{C}); \mathbb{Q}) \geq e_i$$

for each $1 \leq i \leq t$, where $\text{End}_{F'}(\mathbb{A}_i)^0 := \text{End}_{F'}(\mathbb{A}_i) \otimes \mathbb{Q}$ and $F'/F$ is a finite extension such that the isogeny is defined over $F'$. Let $P \in \mathbb{A}(F)$ and let $\Lambda$ be a subgroup of $\mathbb{A}(F)$. If $r_v(P) \in r_v(\Lambda)$ for almost all $v$ of $\mathcal{O}_F$ then $P \in \Lambda + \mathbb{A}(F)_{\text{tor}}$. Moreover if $\mathbb{A}(F)_{\text{tor}} \subset \Lambda$, then the following conditions are equivalent:

1. $P \in \Lambda$
2. $r_v(P) \in r_v(\Lambda)$ for almost all $v$ of $\mathcal{O}_F$.

Theorem 1.3 gives a numeric criterion needed for a local to global principle to hold (up to torsion). Let us make a few more comments on Theorem 1.3. This principle for abelian varieties with a commutative endomorphism ring was proven in [We03]. Notice that the reduction map makes sense, since for an abelian variety $\mathbb{A}$ over a number field one has the Néron model $\mathcal{A}$ such that the Mordell-Weil group $\mathcal{A}(\mathcal{O}_F) = \mathbb{A}(F)$ (cf. [BLR90]). Notice also that for abelian varieties over global fields, one has the Poincaré decomposition theorem, i.e. any abelian variety $\mathbb{A}/F$ can be decomposed over some field $F' \supset F$ uniquely up to an isogeny as a product $\mathbb{A} = \mathbb{A}_1^{e_1} \times \cdots \times \mathbb{A}_t^{e_t}$ where $\mathbb{A}_i$, $i = 1, \ldots, t$ are geometrically simple abelian varieties cf. [M70].

The main result of this paper is the result corresponding to Theorem 1.3 for Drinfeld modules, or rather Anderson $t$-modules [AS9, BP99] that are products of Drinfeld modules. The general framework of the proof is similar to that of [BK11] or [BK13]. (In [BK13] the local to global principle for étale $K$-theory of curves was treated.) As in previous papers, there are significant subtle differences between the proofs. Our main theorem is the following:

**Theorem 1.4.** Let $\tilde{\phi} = \phi_1^{e_1} \times \cdots \times \phi_t^{e_t}$ be a $t$-module where $\phi_i$, $1 \leq i \leq t$ are pairwise non-isogenous Drinfeld modules defined over $\mathcal{O}_K$ and such that $\text{End} \phi_i = \mathbb{A}$. Let $N_i \subset \phi_i(\mathcal{O}_K)$ be a finitely generated $\mathbb{A}$-submodule of the Mordell-Weil group. Pick $\Lambda \subset N = N_1^{e_1} \times \cdots \times N_t^{e_t}$ to be an $\mathbb{A}$-submodule. Let $d_i = \text{rank}(\phi_i) \geq e_i$ for each $1 \leq i \leq t$. Let $P \in N$ and assume that for almost all primes $\mathcal{P}$ of $\mathcal{O}_K$ we have $\text{red}_P(P) \in \text{red}_P(\Lambda)$. Then $P \in \Lambda + N_{\text{tor}}$. 
The basic definitions concerning Drinfeld modules and reduction maps are given in the following sections. Let us emphasize the major differences here between the situations described in Theorems 1.3 and 1.4. Firstly, the category of Drinfeld modules is not semisimple, therefore we have to state our theorem for $t$-modules that are products of Drinfeld modules. Secondly, there is no Néron model, but for every Drinfeld module defined over a field $K$ there exists a Drinfeld module isogenous to it defined over $\mathcal{O}_K$. Thirdly, the Mordell-Weil group of a Drinfeld module of generic characteristic is not finitely generated, thus we have to be careful to use a finitely generated $A$-submodule of the Mordell-Weil group. Finally, our proof relies on the reduction theorem below. In the proof of Theorem 1.5 we use the Ribet-Bashmakov method, developed for Drinfeld modules in [P16, H11]. This method works nicely for Drinfeld modules $\phi$ for which $\text{End } \phi = A$.

**Theorem 1.5.** Let $A = \mathbb{F}_q[t]$, $\mathcal{P} \in M_A$ be a maximal ideal and $\pi_\mathcal{P}$ its generator. Let $L/K$ be a finite extension. Let $x_{i,j} \in \phi_i(\mathcal{O}_L)$ for $1 \leq j \leq s_i$ be linearly independent elements over $A$ for each $1 \leq i \leq t$. There is an infinite set of prime ideals $\mathcal{W}$ of $\mathcal{O}_L$ such that $\text{red}_\mathcal{W}(x_{i,j}) = 0$ in $\phi_i^w(\mathcal{O}_L/\mathcal{W})_{\pi_\mathcal{P}}$ for all $1 \leq j \leq s_i$ and $1 \leq i \leq t$.

As a corollary we obtain the following:

**Theorem 1.6.** Let $\mathcal{P} \in M_A$ and $m \in \mathbb{N} \cup \{0\}$ and $L/K$ be a finite extension. Let $x_{i,j} \in \phi_i(\mathcal{O}_L)$ for $1 \leq j \leq s_i$ be linearly independent elements over $A$ and let $T_{i,j} \in \phi_i([\mathcal{P}^m])$ be arbitrary torsion points for all $1 \leq j \leq s_i$ and $1 \leq i \leq t$. Then there is a set of prime ideals $\mathcal{W}$ of $\mathcal{O}_L$ of positive density such that

$$\text{red}_\mathcal{W}(T_{i,j}) = \text{red}_\mathcal{W}(x_{i,j})$$

in $\phi_i^w(\mathcal{O}_L/\mathcal{W})_{\pi_\mathcal{P}}$ for all $1 \leq j \leq s_i$ and $1 \leq i \leq t$, where $\mathcal{W}$ is a prime in $\mathcal{O}_L(\phi_i([\mathcal{P}^m]))$ over $\mathcal{W}$, $\text{red}_\mathcal{W} : \phi_i(\mathcal{O}_L(\phi_i([\mathcal{P}^m]))) \rightarrow \phi_i^w(k_{\mathcal{W}})$ and $k_{\mathcal{W}} = \mathcal{O}_L(\phi_i([\mathcal{P}^m]))/\mathcal{W}$.

Here $\phi_i^w(\mathcal{O}_L/\mathcal{W})_{\pi_\mathcal{P}} = \{ \alpha \in \phi_i^w(\mathcal{O}_L/\mathcal{W}) \mid \exists k \in \mathbb{N} \text{ such that } \pi_\mathcal{P}^k\alpha = 0 \}$. We view Theorems 1.5 and 1.6 as interesting on their own, not only as one of the key steps in proving Theorem 1.4. Some other applications are described in Section 6, where we show that the recent results of S.Barańczuk can be extended to the situation we consider. The content of the paper is as follows. In Section 2 we review some general definitions and facts concerning Drinfeld modules. The reader is advised to consult the general sources [G96, L96, BP09]. Section 3 is devoted to Kummer theory [P16, H11]. In Section 4 we prove the reduction theorems, i.e. Theorems 1.5 and 1.6. In Section 5 we give a proof of the local to global principle for $t$-modules that are products of Drinfeld modules. In Section 6 we state theorems analogous to S.Barańczuk’s and indicate how to prove them.

2. Preliminaries on Drinfeld modules

Let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements. Let $F$ be a field of transcendence degree 1 over $\mathbb{F}_q$, i.e. a function field of a smooth projective curve $X$ over $\mathbb{F}_q$ and let $A$ be the ring of elements of $F$ regular outside a fixed closed point $\infty$. Let $K$ be a finitely generated field over $\mathbb{F}_q$. The ring of $\mathbb{F}_q$-linear endomorphisms $\text{End}_{\mathbb{F}_q}(\mathcal{G}_{a,K})$ of the additive algebraic group over $K$ is the twisted (non-commutative) polynomial ring in one variable $K\{\tau\}$. The endomorphism $\tau$ corresponds to $u \rightarrow u^q$ and the commutation relation is $\tau u = u^q \tau$, $u \in K$. 

Definition 2.1. An ideal \( P \) of \( A \) is a prime ideal of \( A \). The kernel of \( \nu \) is a prime ideal \( \mathcal{P} \) of \( A \) called the characteristic. The characteristic of \( \nu \) is called finite if \( \mathcal{P} \neq 0 \), or generic (zero) if \( \mathcal{P} = 0 \).

Definition 2.2. A Drinfeld \( A \)-module is a homomorphism \( \phi : A \to K\{\tau\} \), \( a \to \phi_a \), of \( \mathbb{F}_q \)-algebras such that

1. \( D \circ \phi = \nu \),
2. for some \( a \in A \), \( \phi_a \neq \nu(a)\tau^0 \),

where \( D(\sum_{i=0}^{\nu} a_i\tau^i) = a_0 \). The characteristic of a Drinfeld module is the characteristic of \( \nu \).

Let \( \phi \) be a Drinfeld module over the \( A \)-field \( K \). Let

\[ \mu_\phi(a) := -\deg \phi_a(\tau), \quad \mu(0) = -\infty. \]

It is easy to prove (cf. [G96] Lemma 4.5.1) that \( \mu(a) = -d \deg(a) \). The integer \( d \) is called the rank of the Drinfeld module \( \phi \). Assume that \( K \) is an \( A \)-field with a non-trivial discrete valuation \( v \) and \( v(A) \geq 0 \). Let \( \mathcal{O}_v = \{ \gamma \in K \mid v(\gamma) \geq 0 \} \) be the valuation ring of \( K \). Let \( \phi \) be a Drinfeld module with integral coefficients, i.e. every \( \phi_a \in \mathcal{O}_K\{\tau\} \). There exists a reduction \( \phi^v \) of a Drinfeld module defined over \( k_v = \mathcal{O}_v/\mathfrak{m}_v \) (cf. [G96] Definition 4.10.1). By [G96] Lemma 4.10.2, for any Drinfeld module defined over \( K \) there exists a Drinfeld module with integral coefficients isogenous to it. For the definition of an isogeny of Drinfeld modules see [G96] Definition 4.4.3.

Definition 2.3. Let \( \phi \) be a Drinfeld module over \( K \), and \( L \) be an algebraic extension of \( K \). The Mordell-Weil group \( \phi(L) \) is the additive group of \( L \) viewed as an \( A \)-module via evaluation of the polynomials \( \phi_a \), \( a \in A \).

The rank of an \( A \)-module \( M \) is the dimension of the \( F \)-vector space \( M \otimes_A F \). An \( A \)-module is called tame if all its submodules of finite rank are finitely generated. B. Poonen ([P95] Theorem 1) proved that \( \phi(L) \) is the direct sum of a finite torsion submodule and a free \( A \)-module of rank \( \aleph_0 \). However, by [P95] Lemma 4, \( \phi(L) \) is a tame \( A \)-module.

Let \( I \subseteq A \) be an ideal. In general, for any \( A \), \( I \) is generated by two elements \( \{a_{i_1}, a_{i_2}\} \). Let \( \phi_I \) be a monic polynomial which is a right greatest common divisor of \( \phi_{a_{i_1}} \) and \( \phi_{a_{i_2}} \). It exists since in \( K\{\tau\} \) one has a right division algorithm (cf. [G96]). \( \phi_I \) is a generator of the left ideal (in \( K\{\tau\} \)) generated by \( \phi_{a_{i_1}} \) and \( \phi_{a_{i_2}} \) ([G96] Definition 4.4.4). Let \( \bar{K} \) denote an algebraic closure of \( K \).

Definition 2.4. For an ideal \( I \) let \( \phi[I] \subset \phi(\bar{K}) \) be the finite subgroup of roots of \( \phi_I \).

Notice that since \( I \) is an ideal of \( A \), \( \phi[I] \) is stable under \( \{\phi_a\} \), \( a \in A \) and the Galois group \( G_K = \text{Gal}(K^{\text{sep}}/K) \), of the separable closure \( K^{\text{sep}} \subset \bar{K} \), acts on \( \phi[I] \). For any ideal \( I \) prime to characteristic we have (cf. [Ro02] Corollary to Theorem 13.1, [G96])

\[ \phi[I] \cong (A/I)^d \]

and thus a representation

\[ \bar{\rho}_I : G_K \to \text{Aut}_A(\phi[I]) \cong \text{GL}_d(A/I). \]
Definition 2.5. Let $\phi$ be Drinfeld module and $\mathcal{P}$ be a maximal ideal different from the characteristic $\mathcal{P}_0$. The $\mathcal{P}$-adic Tate module is defined as

$$T_{\mathcal{P}}(\phi) = \text{Hom}_A(F_{\mathcal{P}}/A_{\mathcal{P}}, \phi[\mathcal{P}^\infty])$$

where $\phi[\mathcal{P}^\infty] = \bigcup_{m \geq 1} \phi[\mathcal{P}^m]$ and $F_{\mathcal{P}}$ (resp. $A_{\mathcal{P}}$) is the $\mathcal{P}$-adic completion of $F$ (resp. $A$).

Let $\phi^n : \phi[\mathcal{P}^{n+1}] \to \phi[\mathcal{P}^n]$ be the multiplication by $\pi_{\mathcal{P}}$ map. Then (2.1) can be written in the following way

$$T_{\mathcal{P}}(\phi) \cong \bigcup \phi[\mathcal{P}^m],$$

Notice that (2.2) becomes a free $\mathcal{P}$-module of rank $d$ and $G_K$ acts on $T_{\mathcal{P}}(\phi)$ continuously. Since the action of $G_K$ commutes with the multiplication by elements of $A_{\mathcal{P}}$ we obtain a $\mathcal{P}$-adic representation

$$\rho_{\mathcal{P}} : G_K \to \text{Aut}_{A_{\mathcal{P}}}(T_{\mathcal{P}}(\phi)) \cong \text{GL}_d(A_{\mathcal{P}}).$$

Let $r_{\mathcal{P}} : \text{GL}_d(A_{\mathcal{P}}) \to \text{GL}_d(k_{\mathcal{P}})$ be the projection map. Then we have $\bar{r}_{\mathcal{P}} = r_{\mathcal{P}} \circ \rho_{\mathcal{P}}$.

Let $M_A$ be the set of all maximal ideals of $A$ and let $\hat{A} = \prod_{\mathcal{P} \in M_A} A_{\mathcal{P}}$. Under the assumption that $\text{End}_K(\phi) = A$ in [PR109], the authors proved that the adelic representation

$$\rho_{\text{ad}} : G_K \to \text{GL}_d(\hat{A})$$

has open image.

For our purposes, especially for the proof of Theorem 1.5, we need the following general result ([Wa01] Proposition 6) concerning Mordell-Weil groups of Drinfeld modules defined over finitely generated (over the field of fractions $F$ of $A$) fields $L$.

Proposition 2.1 ([Wa01]). Let $\bar{L}$ (resp. $L^{\text{sep}}$) be an algebraic (resp. separable) closure of $L$. Each of the $A$-modules $\phi(\bar{L})$ and $\phi(L^{\text{sep}})$ is the direct sum of a $F$-vector space of dimension $\aleph_0$ with a torsion submodule. Furthermore, when the $A$-characteristic is generic, the torsion submodule of each case is isomorphic to $(F/A)^d$. When the $A$-characteristic of $L$ is $\mathcal{P}$, the torsion submodule of $\phi(\bar{L})$ (resp. $\phi(L^{\text{sep}})$) is isomorphic to $\bigoplus_{\beta \neq \mathcal{P}} \phi[F_{\beta}/A_{\beta}]^d \oplus (F_{\mathcal{P}}/A_{\mathcal{P}})^{d-h}$ (resp. a submodule between $\bigoplus_{\beta \neq \mathcal{P}} \phi[F_{\beta}/A_{\beta}]^d$ and $\bigoplus_{\beta \neq \mathcal{P}} \phi[F_{\beta}/A_{\beta}]^d \oplus (F_{\mathcal{P}}/A_{\mathcal{P}})^{d-h}$) where $\beta$ is taken in Spec$A$, $d$ and $h$ are the rank and the height of $\phi$ respectively.

3. KUMMER THEORY

From now on we assume that $A = F_q[t]$ is the ring of polynomials in one variable and $F = F_q(t)$ is the field of rational functions over $F_q$. In order to prove Theorem 1.5 we need Kummer theory in the context of Drinfeld modules, which corresponds to one constructed by Ribet [R79] for the extensions of abelian varieties by tori. Such an extension was developed in [H11] and [P16]. Now we recall from these references the relevant facts below. For a detailed exposition see the original sources.

Consider a Drinfeld module $\phi : A \to K\{\tau\}$ of generic characteristic. We will consider $K^{\text{sep}}$ as an $A$-module via $\phi$. We assume that $\text{End}_K(\phi) = A$. Let $s \geq 1$ and $\Lambda$ be an $A$-submodule of $K$ generated by $s$ $A$-linearly independent elements $x_1, \ldots, x_s$. Let $y_i \in \phi^{-1}(\{x_i\}) \subset K^{\text{sep}}$. This is possible since $\phi_1 - x_i$ is a separable polynomial. Define

$$\eta : G_K \to \phi[I]^s \cong \text{Mat}_{d \times s}(A/I), \quad \eta(\sigma) = (\sigma(y_1) - y_1, \ldots, \sigma(y_s) - y_s)$$
and

\[(3.2) \quad \Phi_I : G_K \to \text{Mat}_{d \times s}(A/I) \times \text{GL}_d(A/I), \quad \Phi_I(\sigma) = (\eta(\sigma), \bar{\rho}_I(\sigma)).\]

The action of \(\text{GL}_d\) on \(\text{Mat}_{d \times s}(A/I)\times \text{GL}_d(A/I)\) is given by matrix multiplication. Now, following [H11], we describe the map analogous to (3.2) for \(A_P, P \in M_A\).

Let \(\phi : \mathcal{P}^\infty = \bigcup_{n \geq 0} \phi[\mathcal{P}^n] \subset K^{sep}, \phi[\mathcal{P}^0] := 0, K^{sep} = K(\phi[\mathcal{P}^\infty]), K_{ad} = \prod_{P \in M_A} K_P^{\infty}\) and \(\pi^n_P\) be a generator of \(\mathcal{P}^n\). For an \(A\)-submodule \(M\) of \(K^{sep}\) and \(n \geq 0\) denote as \(\phi_{\pi^n_P}^{-1}(M) \subset K^{sep}\) the inverse image of \(M\) under the endomorphism \(\phi_{\pi^n_P}\) (cf. Definition 2.2). By a slight abuse of notation, denote by \(\phi^n\) (cf. (2.2)) the map \(\phi^n : \phi_{\pi^n_P}^{-1}(M) \to \phi_{\pi^n_P}^{-1}(M)\) as well. The extended \(\mathcal{P}\)-adic Tate module is defined as \(T_P[M] = \varprojlim \phi_{\pi^n_P}^{-1}(M).\) Let \(K \subset L \subset K^{sep},\) then the absolute Galois group \(G_L\) acts continuously on \(T_P[M]\) by the following formula:

\[\sigma(t_n) = (\sigma(t_n)), (t_n) \in T_P[M].\]

One has an exact sequence of \(G_L\-A\)-modules

\[0 \to T_P(\phi) \to T_P[M] \to M \to 0\]

where the surjection in (3.3) is given by the map \(\text{pr}_M : T_P[M] \to M, \quad \text{pr}_M((t_n)_{n \geq 0}) = t_0\) and \(G_L\) acts trivially on \(M\). For any \(\sigma \in G_L\) and \((t_n) \in T_P[M]\) one has \(\sigma(t_n) - t_n \in \ker(\phi_{\pi^n_P})\) and therefore

\[(3.4) \quad \phi^n(\sigma(t_{n+1}) - t_{n+1}) = \sigma(\phi^n(t_{n+1})) - \phi^n(t_{n+1}) = \sigma(t_n) - t_n.\]

Formulas (3.4) show that the map

\[\xi_{L,M} : T_P[M] \to \text{Map}(G_L, T_P(\phi)) \quad (t_n) \to [\sigma \to (\sigma(t_n) - t_n)_{n \geq 0}]\]

is well defined. One can specialize this construction to \(M = \Lambda\) and \(K = L\). For each \(1 \leq i \leq s\) choose \((t_{i,n}) \in \text{pr}_{\Lambda}^{-1}(x_i)\) and construct the map \(\eta_P : G_K \to (T_P(\phi))^s \cong \text{Mat}_{d \times s}(A_P)\) by the formula:

\[\eta_P(\sigma) = (\xi_{K,\Lambda}(t_{i,n})_{n \geq 0}(\sigma), \ldots, \xi_{K,\Lambda}(t_{s,n})_{n \geq 0}(\sigma)).\]

Using [H11] Lemma 4.3 one obtains the \(A\)-module homomorphism

\[\xi_{K_{ad},\Lambda} : \Lambda \to \text{Hom}(G_{K_{ad}}, T_P(\phi)) \quad (t_{n \geq 0}) \to [\sigma \to (\sigma(t_n) - t_n)_{n \geq 0}].\]

Let \(\psi_i = \xi_{K_{ad},\Lambda}(x_i) \in \text{Hom}(G_{K_{ad}}, T_P(\phi))\) for the fixed generators \(x_1, \ldots, x_s\) of \(\Lambda\) and let

\[(3.5) \quad \Psi_P : G_{K_{ad}} \to (T_P(\phi))^s \cong \text{Mat}_{d \times s}(A_P), \quad \sigma \to (\psi_1(\sigma), \ldots, \psi_s(\sigma)).\]

For \(T \subset M_A\) there is the homomorphism

\[\Psi_T : G_{K_{ad}} \to \prod_{P \in T} \text{Mat}_{d \times s}(A_P), \quad \sigma \to (\Psi_P(\sigma))_{P \in T}.\]

Denote \(\Psi_{ad} := \Psi_{M_A}\). The main results in [H11] are the following:

**Proposition 3.1** ([H11] Proposition 4.6). The image of \(\Psi_P\) is equal to \(\text{Mat}_{d \times s}(A_P)\) for almost all \(P \in M_A\) and is open for all \(P \in M_A\).

**Theorem 3.2** ([H11] Theorem 4.4). The image of \(\Psi_{ad}\) is open.

In the sequel for prime ideals \(P \in M_A\) we shall use the following modules:

\[V_P(\phi) = T_P(\phi) \otimes A_P F_P.\]

**Remark 3.1.** Notice that in view of (3.5) we have \(T_P(\phi) \cong \text{Mat}_{d \times 1}(A_P)\) and \(\dim_{F_P} V_P(\phi) = d.\)
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4. Proof of the reduction theorem

In what follows we assume that the fields of definitions for the Drinfeld modules involved are finite extension of $F = \mathbb{F}_q(t)$.

Let $\phi : A \to K\{\tau\}$ be a Drinfeld module of generic characteristic. Any Drinfeld module defined over $K$ is isogenous to a Drinfeld module defined over $\mathcal{O}_K$ - the ring of $A$-integers of $K$. For a maximal ideal $\mathcal{P} \subset \mathcal{O}_K$ we can reduce coefficients of $\phi$, for every $a \in A$, modulo $\mathcal{P}$ and obtain a Drinfeld module over a finite field $\mathcal{O}_K/\mathcal{P}$. This Drinfeld module has the special characteristic $\mathcal{P}$. We will denote it by $\phi^\mathcal{P}$.

Definition 4.1. Let $\phi$ be a fixed Drinfeld module of rank $d$ defined over $K$. A prime $\mathcal{P} \in \text{Spec}\mathcal{O}_K$ is called good if

1. there exists $\alpha \in K^\times$ such that $\phi_1 = \alpha \phi_1^{-1}$ has $\mathcal{P}$-integral coefficients.
2. the reduced map $\phi^\mathcal{P}_1 : A \to \mathcal{O}_K/\mathcal{P}\{\tau\}$ is a Drinfeld module of rank $d$.

Primes that are not good are called bad.

It is well known (because $A$ is a finitely generated ring over $\mathbb{F}_q$) that almost all primes $\mathcal{P} \in \text{Spec}\mathcal{O}_K$ are good. Moreover, for almost all primes one can take $\alpha = 1$ (cf. [G94], p.320, [Th04]). There also exists an analogue of classical Néron - Ogg - Šafarevič criterion for abelian varieties [Ta82], [G94].

Definition 4.2. By reduction map mod $\mathcal{P}$ for Mordell-Weil groups we will mean the $A$-module homomorphism

$$\text{red}_\mathcal{P} : \phi(\mathcal{O}_K) \to \phi^\mathcal{P}(\mathcal{O}_K/\mathcal{P}).$$

Let $L$ be a finite extension of $K$.

The following Proposition holds true ( [BKo95] Proposition 1.3):

Proposition 4.1. Let $\phi$ be a Drinfeld module over $\mathcal{O}_L$ of rank $d$, $\mathcal{P}$ be a maximal ideal of $\mathcal{O}_L$ and let $\mathfrak{p} = \text{Ker}(A \to \mathcal{O}_L \to \mathcal{O}/\mathcal{P})$ be the special characteristic. Let $I$ be prime to $\mathfrak{p}$. Then

1. $\text{Tor}_\phi(\mathcal{P}) = \{ x \in \mathcal{P} : \phi_a(x) = 0 \text{ for some } a \in A \}$ has no nontrivial $I$-torsion.
2. the reduction map is an injection on the $I$-torsion between $\phi(\mathcal{O}_L)[I]$ and $\phi^\mathcal{P}(\mathcal{O}_L/\mathcal{P})[I]$.

Remark 4.1. In [BKo95] the authors consider Drinfeld modules of rank 2 but their proof works for arbitrary rank.

Theorem 4.2. Let $x_{i,j} \in \phi_i(\mathcal{O}_L)$ for $1 \leq j \leq s_i$ be linearly independent elements over $A$ for each $1 \leq i \leq t$. There is an infinite set of prime ideals $\mathcal{W}$ of $\mathcal{O}_L$ such that $\text{red}_\mathcal{W}(x_{i,j}) = 0$ in $\phi_i^\mathcal{W}(\mathcal{O}_L/\mathcal{W})_{\pi_\mathcal{P}}$ for all $1 \leq j \leq s_i$ and $1 \leq i \leq t$.

Proof. By Proposition 3.1 there exists $m \in \mathbb{N}$ such that for any $\mathcal{P}$ one has

$$\mathcal{P}^m \prod_{i=1}^t T_{\mathcal{P}}(\phi_i)^{s_i} \subset \Psi_\mathcal{P}(G_{K_\mathcal{P}}) \subset \prod_{i=1}^t T_{\mathcal{P}}(\phi_i)^{s_i}.$$ 

Let $\Gamma = \sum_{i=1}^t \sum_{j=1}^{s_i} A x_{i,j}$, denote $L_{\mathcal{P}}^\infty := L(\hat{\phi}[\mathcal{P}^\infty])$, where $\hat{\phi}[\mathcal{P}^\infty] = \bigcup_{i=1}^t \phi_i[\mathcal{P}^\infty]$ and let

$$\frac{1}{\pi_\mathcal{P}} \Gamma := \{ x \in K^{\text{sep}} \mid \exists m \text{ such that } \pi_\mathcal{P}^m x \in \Gamma \}.$$
Similarly, define $L_{P^k} := L(\tilde{\varphi}[P^k])$ and
\[
\frac{1}{\pi^k_P} \Gamma := \{ x \in K^{\text{sep}} \mid \text{such that } \pi^k_P x \in \Gamma \}.
\]

We have the following commutative diagram:

\[
\begin{array}{ccc}
G(L_{P^\infty}(\frac{1}{\pi^\infty_P} \Gamma)/L_{P^\infty}) & \xrightarrow{\Psi_P} & \bigoplus_{i=1}^t T_P(\phi_i)^s_i / P^m \bigoplus_{i=1}^t T_P(\phi_i)^{s_i} \\
\downarrow & & \cong \\
G(L_{P^{k+1}}(\frac{1}{\pi^k_P} \Gamma)/L_{P^{k+1}}) & \xrightarrow{\Psi_{P^{k+1}}} & \bigoplus_{i=1}^t (\phi_i[\mathcal{P}^{k+1}])^{s_i} / P^m \bigoplus_{i=1}^t (\phi_i[\mathcal{P}^{k+1}])^{s_i} \\
\downarrow & & \cong \\
G(L_{P^k}(\frac{1}{\pi^k_P} \Gamma)/L_{P^k}) & \xrightarrow{\Psi_P} & \bigoplus_{i=1}^t (\phi_i[\mathcal{P}^k])^{s_i} / P^m \bigoplus_{i=1}^t (\phi_i[\mathcal{P}^k])^{s_i}
\end{array}
\]

The maps $\Psi_P$ and $\Psi_{P^{k+1}}$ for $k \geq 1$ are subquotients of the map (3.5). For large enough values of $k$ (i.e. $k > m$ ) the images of $\Psi_{P^{k+1}}$ and $\Psi_P$ in the diagram (4.1) are isomorphic. This is due to the fact that the Galois groups in the second and third rows are finite and the maps $\Psi_P$ and $\Psi_{P^{k+1}}$ are injections. In fact multiplication by $\pi_P$ gives us this isomorphism. Therefore the left vertical arrow in the diagram (4.1) is a surjection. Let us consider the following diagram of fields

\[
\begin{array}{ccc}
L_{P^{k+1}}(\frac{1}{\pi^k_P} \Gamma) & \xrightarrow{id} & L_{P^{k+1}}(\frac{1}{\pi^k_P} \Gamma) \\
\downarrow & & \downarrow \\
L_{P^{k+1}}(\frac{1}{\pi^k_P} \Gamma) & \xrightarrow{h} & L_{P^{k+1}} \\
\downarrow & & \downarrow \\
L_{P^k} & \xrightarrow{id} & L_{P^{k+1}} \\
\end{array}
\]

The above mentioned surjection of Galois groups yields the following equality for large enough values of $k$.

\[
L_{P^k}(\frac{1}{\pi^k_P} \Gamma) \cap L_{P^{k+1}} = L_{P^k}.
\]

Let $\tilde{h} \in G(L_{P^\infty}/L_{P^k})$ be the homothety $1 + \pi^k_P u, u \in \mathbb{F}_q^*$, acting on $T_P(\phi)$. For $k >> 0$ such a homothety exists according to the open image theorem of Pink and Rütsche [PR09] (cf. (2.4)) and the fact that $L_{P^k}$ is a finite extension of $K$. Let $h \in G(L_{P^{k+1}}/L_{P^k})$ be a projection of $\tilde{h}$. By (4.3) there exists $\sigma \in G(L_{P^{k+1}}(\frac{1}{\pi^k_P} \Gamma))$ such that $\sigma|_{L_{P^k}(\frac{1}{\pi^k_P} \Gamma)} = \text{id}$ and $\sigma|_{L_{P^{k+1}}} = h$. 
By the Chebotarev density theorem for global fields (cf. [FJ08], Theorem 6.3.1) there is a set of primes $\mathcal{W} \in \mathcal{O}_L$ of positive density such that there exists a prime $\mathcal{W}_1 \in \mathcal{O}_{L_{\text{p}^{k+1}}} \Gamma$ with the Frobenius in $G(L_{\text{p}^{k+1}} \Gamma)$ equal to $\sigma$. Let $\mathcal{W}$ and $\mathcal{W}_1$ be such primes and let $\mathcal{W}_2 \in \mathcal{O}_{L_{\text{p}^{k}}} \Gamma$ be the prime below $\mathcal{W}_1$. Consider the following commutative diagram:

\[\begin{array}{ccc}
\phi_i(\mathcal{O}_L) & \xrightarrow{\text{red}_\mathcal{W}} & \phi_i^\mathcal{W}(k_\mathcal{W})_{\pi_\mathcal{P}} \\
\downarrow & & \downarrow \\
\phi_i(\mathcal{O}_{L_{\text{p}^{k}}} \Gamma) & \xrightarrow{\text{red}_\mathcal{W}^2} & \phi_i^\mathcal{W}_2(k_\mathcal{W}_2)_{\pi_\mathcal{P}} \\
\downarrow & & \downarrow \\
\phi_i(\mathcal{O}_{L_{\text{p}^{k+1}}} \Gamma) & \xrightarrow{\text{red}_\mathcal{W}_1} & \phi_i^\mathcal{W}_1(k_\mathcal{W}_1)_{\pi_\mathcal{P}}
\end{array}\]

where for brevity we denoted $k_\mathcal{W} := \mathcal{O}_L/\mathcal{W}$ and similarly for $k_\mathcal{W}_1$ and $k_\mathcal{W}_2$. The subscript $\pi_\mathcal{P}$ for the Drinfeld modules with finite coefficients denotes the $\pi_\mathcal{P}$-torsion e.g. $\phi_i^\mathcal{W}(k_\mathcal{W})_{\pi_\mathcal{P}} = \{\alpha \in \phi_i^\mathcal{W}(k_\mathcal{W}) \mid \exists k \in \mathbb{N} \text{ such that } \pi_\mathcal{P}^k \alpha = 0\}$.

**Definition 4.3.** By the $\pi_\mathcal{P}$-order of a point $x \in \phi_i^\mathcal{W}(k_\mathcal{W})_{\pi_\mathcal{P}}$ we mean the least positive integer $m$ such that $\pi_\mathcal{P}^m x = 0$.

**Remark 4.2.** Notice that by our choice of $\mathcal{W}, \mathcal{W}_1$ and $\mathcal{W}_2$ we have $k_\mathcal{W}_2 = k_\mathcal{W}$ and therefore $\phi_i^\mathcal{W}_2(k_\mathcal{W}_2)_{\pi_\mathcal{P}} = \phi_i^\mathcal{W}(k_\mathcal{W})_{\pi_\mathcal{P}}$.

Let $c_{i,j}$ be the $\pi_\mathcal{P}$-order of $\text{red}_\mathcal{W}(x_{i,j}) \in \phi_i^\mathcal{W}(k_\mathcal{W})_{\pi_\mathcal{P}}$. All the vertical arrows in the diagram (4.4) are injections. Let $y_{i,j} = \frac{1}{\pi_\mathcal{P}} x_{i,j} \in \phi_i(\mathcal{O}_{L_{\text{p}^{k}}} \Gamma) \subseteq \phi_i(\mathcal{O}_{L_{\text{p}^{k+1}}} \Gamma)$. One readily verifies that the $\pi_\mathcal{P}$-order of $\text{red}_\mathcal{W}_1(y_{i,j})$ equals $k + c_{i,j}$. By Remark 4.2 the element $\text{red}_\mathcal{W}_1(y_{i,j})$ comes from an element of $\phi_i^\mathcal{W}(k_\mathcal{W})_{\pi_\mathcal{P}}$. Assume $c_{i,j} \geq 1$. We have

\[h(\pi_\mathcal{P}^{c_{i,j}-1} \text{red}_\mathcal{W}_1(y_{i,j})) = (1 + \pi_\mathcal{P}^k u) \pi_\mathcal{P}^{c_{i,j}-1} \text{red}_\mathcal{W}_1(y_{i,j}).\]

This is because the $\pi_\mathcal{P}$-order of $\pi_\mathcal{P}^{c_{i,j}-1} \text{red}_\mathcal{W}_1(y_{i,j})$ in $\phi_i^\mathcal{W}(k_\mathcal{W})_{\pi_\mathcal{P}}$ is equal to $k + 1$.

On the other hand by our choice of $\mathcal{W}$ the Frobenius at $\mathcal{W}_1$ acts on $\pi_\mathcal{P}^{c_{i,j}-1} \text{red}_\mathcal{W}_1(y_{i,j})$ via $h$. Thus since $\text{red}_\mathcal{W}_1(y_{i,j}) \in \phi_i^\mathcal{W}_2(k_\mathcal{W}_2)_{\pi_\mathcal{P}}$ we have

\[h(\pi_\mathcal{P}^{c_{i,j}-1} \text{red}_\mathcal{W}_1(y_{i,j})) = \pi_\mathcal{P}^{c_{i,j}-1} \text{red}_\mathcal{W}_1(y_{i,j}).\]

Comparing (4.5) and (4.6) we obtain

\[\pi_\mathcal{P}^{c_{i,j}-1+k} u \text{red}_\mathcal{W}_1(y_{i,j}) = \pi_\mathcal{P}^{c_{i,j}-1} u \text{red}_\mathcal{W}_1(x_{i,j}) = 0.\]

But this contradicts the assumption that the $\pi_\mathcal{P}$-order equals $c_{i,j}$. \qed

We also have the following
Theorem 4.3. Let \( P \in M_A \) and \( m \in \mathbb{N} \cup \{0\} \) and \( L/K \) be a finite extension. Let \( x_{i,j} \in \phi_i(\mathcal{O}_L) \) for \( 1 \leq j \leq s_i \) be linearly independent elements over \( A \) and let \( T_{i,j} \in \phi_i([P^m]) \) be arbitrary torsion points for all \( 1 \leq j \leq s_i \) and \( 1 \leq i \leq t \). Then there is a set of prime ideals \( \mathcal{W} \) of \( \mathcal{O}_L \) of positive density such that

\[
\text{red}_{\mathcal{W}}(T_{i,j}) = \text{red}_{\mathcal{W}}(x_{i,j})
\]

in \( \phi_i^m(\mathcal{O}_L/\mathcal{W})_{\pi_P} \) for all \( 1 \leq j \leq s_i \) and \( 1 \leq i \leq t \), where \( \mathcal{W} \) is a prime in \( \mathcal{O}_L(\phi_i[\mathcal{P}^m]) \) over \( \mathcal{W} \) and \( \text{red}_{\mathcal{W}} : \phi_i(\mathcal{O}_L(\phi_i[\mathcal{P}^m])) \to \phi_i^m(\mathcal{O}_L(\phi_i[\mathcal{P}^m])) \) and \( k_{\mathcal{W}} = \mathcal{O}_L(\phi_i([P^m]))/\mathcal{W}' \).

Proof. Put \( x_{i,j} = P_{i,j} - T_{i,j} \), take \( L(\phi_i[\mathcal{P}^m]) \) instead of \( L \) and apply Theorem 1.2. \( \square \)

5. Local to global principle for \( t \)-modules associated to direct sums of Drinfeld modules

In what follows we assume to be working in the category of Anderson \( t \)-modules that are direct products of Drinfeld modules. Such situation was considered in \([PT06]\). So, we assume that \( t \)-module \( \hat{\varphi} = \phi_1^{e_1} \times \cdots \times \phi_t^{e_t} \), where \( \phi_i \) is a Drinfeld module of generic characteristic of rank \( d_i \). We assume that for every \( 1 \leq i \leq t \), \( \text{End}(\phi_i) = A \) and all modules are defined over the same ring of integers \( \mathcal{O}_K \). We further assume that we are given a finitely generated \( A \)-submodule \( N = N_1^{e_1} \times \cdots \times N_t^{e_t} \) of the Mordell-Weil group \( \varphi(\mathcal{O}_K) = \phi_1(\mathcal{O}_K)^{e_1} \times \cdots \times \phi_t(\mathcal{O}_K)^{e_t} \), where \( N_i \subset \phi_i(\mathcal{O}_K) \).

Remark 5.1. According to the result of B.Poonen \([P95]\) the Mordell-Weil group is a direct sum of a free \( A \)-module on \( \mathbb{R}_0 \) generators and a finite torsion module.

In view of Remark 5.1 the theorem analogous to Theorem 4.1 of \([BK11]\) asserts the following:

Theorem 5.1. Let \( \hat{\varphi} = \phi_1^{e_1} \times \cdots \times \phi_t^{e_t} \) be a \( t \)-module where \( \phi_i, 1 \leq i \leq t \) are pairwise non-isogenous Drinfeld modules defined over \( \mathcal{O}_K \) such that \( \text{End}(\phi_i) = A \). Pick \( \Lambda \subset N \) to be an \( A \)-submodule. Let \( d_i = \text{rank}(\phi_i) \geq e_i \) for each \( 1 \leq i \leq t \). Let \( P \in N \) and assume that for almost all primes \( \mathcal{P} \) of \( \mathcal{O}_K \) we have \( \text{red}_\mathcal{P}(P) \in \text{red}_\mathcal{P}(\Lambda) \). Then \( P \in \Lambda + N_{\text{tor}} \).

Let \( F = \mathbb{F}_p(t) \) be the field of fractions of \( A \). Notice that \( d_i = \dim_{\mathbb{F}_p} V_{\mathcal{P}}(\phi_i) \) for \( \mathcal{P} \in M_A \) (cf. Remark 3.1).

Corollary 5.2. If \( N_{\text{tor}} \subset \Lambda \) then the following conditions are equivalent

1. \( P \in \Lambda \)
2. \( \text{red}_\mathcal{P}(P) \in \text{red}_\mathcal{P}(\Lambda) \).

Recall some facts from \([BK11]\) section 3 concerning modules over division algebras adapted for our situation. Let \( D_i = F = \text{End}(\phi_i) \otimes_A F \). Then \( \text{End}(\hat{\varphi}) = M_{e_1}(A) \times \cdots \times M_{e_t}(A) \) and \( \text{End}(\varphi) \otimes_A F = M_{e_1}(F) \times \cdots \times M_{e_t}(F) \).

Definition 5.1. Let \( K_{1}(j) \) be the left ideal of the matrix algebra \( M_{e_j}(F) \) consisting of matrices \( \alpha(j)_1 = (a(j)_{l,m})_1 \leq l, m \leq e_j \) such that \( a(j)_{l,m} = 0 \) if \( m \neq 1 \).

Definition 5.2. For a \( D_i \)-vector space \( W_i \) let \( \omega(i) = (\omega(i)_1, 0, \ldots, 0)^T \in W_i^{e_i}, \; \omega(i)_1 \in W_i \). Let \( D = \prod_{i=1}^t D_i, \; M_{e}(D) = \prod_{i=1}^t M_{e_i}(D_i) \), where \( e = (e_1, \ldots, e_t) \).
Remark 5.2. Let $W_i$ be a finite dimensional $F$-vector space over $D_i$, $1 \leq i \leq t$. Then $W = \bigoplus_{i=1}^t W_i^{e_i}$ has an obvious $\mathcal{M}_e(\mathbb{D})$-module structure.

The following lemma is essentially a specialisation of Corollary 3.2 of [BK11].

**Lemma 5.3.** Every nonzero simple $\mathcal{M}_e(\mathbb{D})$-submodule of $W = \bigoplus_{i=1}^t W_i^{e_i}$ is of the form

$$K(j)\omega(j) = \{ (a_{1,1}\omega(j), \ldots, a_{e,j}\omega(j))^T : a_{k,1} \in D_j = F, 1 \leq k \leq e_j \}$$

where $1 \leq j \leq t$ and $\omega(j) \in W_j$.

The trace homomorphism $\text{tr} : \mathcal{M}_e(\mathbb{D}) \rightarrow F$ is defined as $\text{tr} = \sum_{i=1}^t \text{tr}_i$ where $\text{tr}_i : M_e(D_i) \rightarrow F$ is the usual trace homomorphism. The following lemma corresponds to Lemma 3.3 of [BK11].

**Lemma 5.4.** The induced map $\text{tr} : \text{Hom}_{\mathcal{M}_e(\mathbb{D})}(W, \mathcal{M}_e(\mathbb{D})) \rightarrow \text{Hom}_F(W, F)$ is an isomorphism.

**Proof.** Replace $\mathbb{Q}$ by $F$ in the proof of Lemma 3.3 of [BK11].

Semiaisimple of $\mathcal{M}_e(\mathbb{D})$ implies that the module $W$ is semisimple and therefore for any $\pi \in \text{Hom}_{\mathcal{M}_e(\mathbb{D})}(W, \mathcal{M}_e(\mathbb{D}))$ there exists $s : \text{Im}\pi \rightarrow W$ such that $\pi \circ s = \text{Id}$. One has of course the following splittings $\pi = \prod_{i=1}^t \text{Im}\pi(i)$ and $s = \bigoplus_{i=1}^t s(i)$ where $\pi(i) \in \text{Hom}_{M_e(D_i)}(W_i^{e_i}, M_e(D_i))$, $s(i) \in \text{Hom}_{M_e(D_i)}(\text{Im}\pi(i), W_i^{e_i})$ and $\pi(i) \circ s(i) = \text{Id}$.

### 5.1. Proof of Theorem 5.1

**Proof.** Since the $A$-torsion of $N$ is finite (cf. Remark 5.1) we can consider a torsion free $A$-module $\Omega := cN$, where $c = g_1 \cdots g_k$ is the product of generators of the $A$-anihilator of $N_{\text{tor}}$, and replace $N$ by $\Omega$. We can also assume that $\Lambda \subset \Omega$ and $P \in \Lambda$. Let $P_1, \ldots, P_s$ be an $A$-basis of $\Omega$ such that

$$\Lambda = Av_1 P_1 + \cdots + Av_s P_s, \quad P = n_1 P_1 + \cdots + n_s P_s$$

where $n_i \in A$. Assume that $P \notin \Lambda$. This is equivalent to $P \otimes 1 \notin \Lambda \otimes_A \mathcal{O}_U$ for some $\widetilde{\mathcal{U}}/\mathcal{U}$, $\mathcal{U} \in A$ where $\mathcal{O}_U$ is the completion of $\mathcal{O}_L$ with respect to $\mathcal{U}$. Thus there exists $1 \leq j_0 \leq s$ and a natural number $m_1$ such that $\mathcal{U}^{m_1} |n_{j_0}$ and $\mathcal{U}^{m_1+1} |v_{j_0}$. Define the following map of $A$-modules

$$\pi : \Omega \rightarrow A, \quad \pi(R) = a_{j_0}, \quad R = \sum_{i=1}^s a_i P_i, \quad a_i \in A.$$  

We shall use the same letter $\pi$ as a short abbreviation for the map $\pi \otimes_A F : \Omega \otimes_A F \rightarrow F$. By Lemma 5.4, we obtain a map $\pi \in \text{Hom}_{\mathcal{M}_e(\mathbb{D})}(\Omega \otimes_A F, \mathcal{M}_e(\mathbb{D}))$ such that $\text{tr}\pi = \pi$. By the above discussion we also have $s$ such that $\pi \circ s = \text{Id}$. We have

$$\Omega \otimes_A F \cong \text{Im}s \oplus \ker\pi \quad \text{and} \quad \Omega^{e_i} \otimes_A F \cong \text{Im}s(i) \oplus \ker\pi(i), \quad 1 \leq i \leq t.$$  

By Lemma 5.3 we have the following decompositions

$$\text{Im}s(i) = \bigoplus_{k=1}^{k_i} K(i)\omega_k(i) \quad \text{and} \quad \ker\pi(i) = \bigoplus_{k=k_i+1}^{u_i} K(i)\omega_k(i).$$  

By assumptions $k_i \leq e_i \leq p_i$, for every $1 \leq i \leq t$. The elements $\omega_1(i), \ldots, \omega_{k_i}(i), \ldots, \omega_{u_i}(i)$ constitute a basis for the $F$-vector space $\Omega_i \otimes_A F$. (Notice that $D_i = F$ by assumption.) Without loss of generality one can assume that $\omega_{k_i+1}(i), \ldots, \omega_{u_i}(i) \in \Omega_i$. $\Omega_i \otimes_A F$ is a free
F-module. We have \(\mathcal{R} = \text{End}_A \Omega \subset M_e(D) = \mathcal{R} \otimes_A F\). Since \(\Omega\) is a finitely generated A-module there exists a polynomial \(M_0 \in A\) such that the following homomorphisms are well defined \(M_0 \pi : \Omega \to \mathcal{R}\) and \(s : M_0 \pi \Omega \to \Omega\). Define the \(M_0(A)\)-module \(\Gamma = \sum_{k=1}^{k_i} K(i) M_0 \omega_k(i) + \sum_{k=k_i+1}^{u_k} K(i) \omega_k(i) \subset \Omega^e\). Let \(\Gamma = \bigoplus \Gamma(i) \subset \Omega\), \(M_2, M_3 \in A\) be polynomials of minimal degrees such that \(M_2 \Omega \subset \Gamma\) and \(M_3 \Omega \subset M_2 \Omega\). The choice of \(j_0\) implies \(\pi(P) \notin \pi(\Lambda \otimes_A O_{\tilde{\U}}) + \check{U}^m \pi(\Omega \otimes_A O_{\tilde{\U}})\) for every \(m \geq m_2\). Choose such an \(m\). Then since \(\text{tr} M_0 \pi = M_0 \pi\) we have

\[
M_0 \pi(P) \notin M_0 \pi(\Lambda \otimes_A O_{\tilde{\U}}) + M_0 \check{U}^m \pi(\Omega \otimes_A O_{\tilde{\U}}).
\]

Let \(K(i)_{1,\tilde{U}} = K(i)_{1,\tilde{U}} \subset M_e(O_{\tilde{U}})\) and \(Q \in \mathcal{A}\). By the definition of \(M_2 \in A\) we have

\[
M_2 \pi(P - Q \otimes 1) = M_0^2 \sum_{i=1}^{t} \sum_{k=1}^{k_i} (\alpha_k(i)_1 - \beta_k(i)_1) \pi(\omega_k(i))
\]

where \(\alpha_k(i)_1, \beta_k(i)_1 \in K(i)_{1,\tilde{U}}\), \(1 \leq k \leq u_i, 1 \leq i \leq t\). By (5.1) and (5.2) and the choice of \(M_3\) we have

\[
M_0 \sum_{i=1}^{t} \sum_{k=1}^{k_i} (\alpha_k(i)_1 - \beta_k(i)_1) \pi(\omega_k(i)) \notin \check{U}^m M_0 \pi(M_3 \Gamma).
\]

This implies that for some \(1 \leq i \leq t\) and \(1 \leq k \leq k_i\) we have

\[
\alpha_k(i)_1 - \beta_k(i)_1 \notin \check{U}^m M_3 K(i)_{1,\tilde{U}}.
\]

Notice that for all \(n' \in \mathbb{N}\) there is an isomorphism \(\phi_i[U^{n'}] \cong T_{\tilde{U}}(\phi_i)/v^{n'} T_{\tilde{U}}(\phi_i)\) where \(v\) is a generator of the ideal \(U\). Let \(L_{\tilde{U}} = \bigoplus_{i=1}^{t} T_{\tilde{U}}(\phi_i)\) and \(\eta_1(i), \ldots, \eta_k(i)\) be a basis of \(T_{\tilde{U}}(\phi_i)\) over \(A_{\tilde{U}}\) (cf. Remark 3.1). Let \(m_0\) and \(m_3\) be the natural numbers such that \(v^{m_0} | M_0\) and \(v^{m_3} | M_3\). Let \(V_{\tilde{U}, i} = T_{\tilde{U}}(\phi_i) \otimes_{A_{\tilde{U}}} F_{\tilde{U}}\). Then \(\dim F_{\tilde{U}} V_{\tilde{U}, i} = d_i\). The quotient \(T_{\tilde{U}}(\phi_i)/v^{n'} T_{\tilde{U}}(\phi_i)\) of \(A_{\tilde{U}}\)-modules is a free \(A_{\tilde{U}}/(v^{n'} A_{\tilde{U}})\)-module with the basis \(T_1(i), \ldots, T_{d_i}(i)\) where \(T_k(i)\) is the image of \(\eta_k(i)\) in \(\phi_i[U^{n'}]\). Take \(n' > m + m_0 + m_3\). By Theorem 4.1 there exists a set of primes \(\mathcal{W} \in \mathcal{O}_L\) of positive density such that

\[
\text{red}_{\mathcal{W}}(\omega_k(i)) = 0, \quad \text{for} \quad 1 \leq i \leq t, \quad k_i + 1 \leq k \leq u_i
\]

and

\[
\text{red}_{\mathcal{W}}(\omega_k(i)) = \text{red}_{\mathcal{W}}(T_k(i)), \quad \text{for} \quad 1 \leq i \leq t, \quad 1 \leq k \leq k_i.
\]

Pick such a prime \(\mathcal{W}\). Since by assumption \(\text{red}_{\mathcal{W}}(P) \in \text{red}_{\mathcal{W}}(A)\) we can choose \(Q \in \mathcal{A}\) such that \(\text{red}_{\mathcal{W}}(P) = \text{red}_{\mathcal{W}}(Q)\). Now we apply the reduction map \(\text{red}_{\mathcal{W}}\) to the equation

\[
M_2(P - Q) = \sum_{i=1}^{t} \sum_{k=1}^{k_i} (\alpha_k(i)_1 - \beta_k(i)_1) M_0 \omega_k(i) + \sum_{i=1}^{t} \sum_{k=k_i+1}^{u_k} (\alpha_k(i)_1 - \beta_k(i)_1) \omega_k(i).
\]

Thus we obtain \(0 = \sum_{i=1}^{t} \sum_{k=1}^{k_i} (\alpha_k(i)_1 - \beta_k(i)_1) M_0 T_k(i)\). Since the reduction map \(\text{red}_{\mathcal{W}}\) is injective on a torsion prime to a characteristic of a Drinfeld module, cf. Proposition 4.1, we have \(0 = \sum_{i=1}^{t} \sum_{k=1}^{k_i} (\alpha_k(i)_1 - \beta_k(i)_1) M_0 T_k(i)\). Therefore the element \(v^{m_0} \sum_{i=1}^{t} \sum_{k=1}^{k_i} (\alpha_k(i)_1 - \beta_k(i)_1) \eta_k(i)\) maps to zero in \(A_{\tilde{U}}/v^{n'} A_{\tilde{U}}\)-module \(L_{\tilde{U}}/v^{m_0} L_{\tilde{U}}\). This yields

\[
\sum_{i=1}^{t} \sum_{k=1}^{k_i} (\alpha_k(i)_1 - \beta_k(i)_1) \eta_k(i) \in v^{n'-m_0} L_{\tilde{U}}.
\]
But since $\eta_k(i), 1 \leq k \leq d_i$, constitute a basis of $T_d(\phi_i)$ over $A_d$ we obtain the following:

\begin{equation}
\alpha_k(i)_1 - \beta_k(i)_1 \in v'^{-m_0}K(i)_1.
\end{equation}

But (5.6) contradicts (5.3). \hfill \Box

We also have the following theorem:

**Theorem 5.5.** Let $\phi$ be a Drinfeld module of rank $d$ defined over $O_K$ with $\text{End}(\phi) = A$. Then the numerical bound in Theorem 5.1 is the best possible. That is, for the $t$-module $\phi^{d+1}$ the local to global principle of Theorem 5.7 does not hold.

**Proof.** Our proof is modelled on the counterexample to the local to global principle for abelian varieties constructed by P. Jossen and A. Perucca in [JP10]. Let $e = d + 1$ and $P_1, \ldots, P_e \in \phi(O_K)$ be points linearly independent over $A$. Let

\[ P := \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_e \end{bmatrix} \quad \Lambda := \{ MP : M \in \text{Mat}_{e \times e}(A), \tr M = 0 \}. \]

Denote for shorthand $\kappa = O_K/P$. Notice that $P \notin \Lambda$ since $P_1, \ldots, P_e$ are $A$-linearly independent. Let $W \in O_K$ be a prime of good reduction for $\phi$. We will find a matrix $M \in \text{Mat}_{e \times e}(A)$ such that $P = M\overline{P}$ where $\overline{P} = [\overline{P}_1, \ldots, \overline{P}_e]^T$ is the reduction mod $W$ of the point $P$. This will show that $\text{red}_W P \in \text{red}_W \Lambda$. Since the Mordell-Weil group $\phi(\kappa)[P]$ is finite there exist polynomials $\alpha_1, \ldots, \alpha_e \in A$ of minimal degrees such that

\[ \alpha_1 \overline{P}_1 + m_{1,2} \overline{P}_2 + \cdots + m_{1,e} \overline{P}_e = 0 \\
\alpha_2 \overline{P}_1 + m_{2,1} \overline{P}_1 + \alpha_2 \overline{P}_2 + \cdots + m_{2,e} \overline{P}_e = 0 \\
\ldots \\
m_{e,1} \overline{P}_1 + m_{e,2} \overline{P}_2 + \cdots + \alpha_e \overline{P}_e = 0. \]

We will show that $D = \gcd(\alpha_1, \ldots, \alpha_e) = 1$. It is enough to show that for any prime ideal $P \triangleleft A$ the polynomial $D$ is not divisible by $P$. Assume opposite and let $\overline{P}$ be a prime that divides $D$. This means, by our choice of $\alpha_1, \ldots, \alpha_e$ that $P$ divides coefficients of any linear combination of points $\bar{P}_1, \ldots, \bar{P}_e \in \phi^{\overline{P}}$. By Proposition 2.1 the group $\phi^{\overline{P}}(\kappa)[P]$ is isomorphic to $(A/P)^{d-k}$. Therefore the group $Y = (\bar{P}_1, \ldots, \bar{P}_e) \cap \phi^{\overline{P}}(\kappa)[P]$ is generated by fewer than $e$ elements. We may assume without loss of generality that $Y = (\bar{P}_2, \ldots, \bar{P}_e) \cap \phi^{\overline{P}}(\kappa)[P]$. Let

\begin{equation}
\alpha_1 \bar{P}_1 + x_2 \bar{P}_2 + \cdots + x_e \bar{P}_e = 0
\end{equation}

be a linear relation. Then since the left hand side of (5.7) is in $Y$ we obtain a contradiction with the minimality of $\alpha_1$. Thus $D = 1$. Hence there exist $a_1, \ldots, a_e \in A$ such that

\[ e = a_1 \alpha_1 + \cdots + a_e \alpha_e. \]

Put $m_{i,i} = 1 - a_i \alpha_i$. Then $m_{1,1} + \cdots + m_{e,e} = 0$ and
\[
\begin{bmatrix}
m_{1,1} & \ldots & m_{1,e} \\
\vdots & \ddots & \vdots \\
m_{e,1} & \ldots & m_{e,e}
\end{bmatrix}
\begin{bmatrix}
\bar{P}_1 \\
\ldots \\
\bar{P}_e
\end{bmatrix}
= \begin{bmatrix}
\bar{P}_1 \\
\ldots \\
\bar{P}_e
\end{bmatrix}.
\]

Therefore \( \bar{P} \in \bar{A} \).

Essentially the same proof - with \( \mathbb{Z} \) substituted for \( A \) - works for abelian varieties with \( \text{End} A = \mathbb{Z} \) and we obtain the following:

**Theorem 5.6.** Let \( \mathbb{A}/F \) be an abelian variety defined over a number field \( F \) with \( \text{End} \mathbb{A} = \mathbb{Z} \) and \( d = \dim_{\text{End}(\mathbb{A})} H_1(\mathbb{A}(\mathbb{C}); \mathbb{Q}) = 2g \), where \( g = \dim \mathbb{A} \). Then the numerical bound in Theorem 1.3 is the best possible. That is, for \( \mathbb{A}^{d+1} \) the local to global principle of Theorem 7.3 does not hold.

6. Some other consequences of the reduction theorem

In [B17] S.Barańczuk introduced a dynamical version of the local to global principle [1.1]. In [B17] the author considers the case of abelian groups (modules over \( \mathbb{Z} \)) satisfying the following two axioms:

**Assumptions 6.1.** Let \( B \) be an abelian group such that there are homomorphisms \( r_v: B \to B_v \) for an infinite family \( v \in F \), whose targets \( B_v \) are finite abelian groups.

(1) Let \( l \) be a prime number, and \((k_1, \ldots, k_m)\) be a sequence of nonnegative integers. If \( P_1, \ldots, P_m \in B \) are points linearly independent over \( \mathbb{Z} \), then there is a family of primes \( v \in F \) such that \( l^{k_i} \mid \ord_v P_i \) if \( k_i > 0 \) and \( l \nmid \ord_v P_i \) if \( k_i = 0 \).

(2) For almost all \( v \) the map \( B_{\text{tors}} \to B_v \) is injective.

Here \( \ord_v P \) is the order of a reduced point \( P \mod v \). In [B17] and [Ba17] finitely generated abelian groups are considered. In our case we modify Assumption 6.1 (2) so that we deal appropriately with infinite torsion in Drinfeld modules. Instead of Assumption 6.1 (1) we assume the following:

**Assumptions 6.2.** Let \( B \) be an \( A \)-module such that there are homomorphisms \( \text{red}_U : B \to B_U \) for an infinite family \( U \in \text{Spec} \mathcal{O}_K \) such that \( B_U \) is a torsion \( A \)-module. For \( P \in B \) let \( \text{ord}_U P \) be the polynomial of minimal degree in \( \mathcal{O}_K \) such that it annihilates \( \text{red}_U P \).

(1) For \( U \) and \((k_1, \ldots, k_m)\), a sequence of nonnegative integers, the following holds true: If \( P_1, \ldots, P_m \in B \) are points linearly independent over \( A \), then there is a family of primes \( W \in \mathcal{O}_K \) such that \( U^{k_i} \mid \text{ord}_WP_i \) if \( k_i > 0 \) and \( U \nmid \text{ord}_WP_i \) if \( k_i = 0 \).

(2) For any \( U \) there are infinitely many primes \( W \in \mathcal{O}_K \) such that the reduction map is an injection on \( U \)-torsion, i.e. \( \text{red}_W : B[U] \to B_W[U] \).

The main theorem of [B17] can be extended to the case of \( t \)-modules considered in this paper in the following way.

**Theorem 6.3.** Let \( \tilde{\phi} = \phi_1^t \times \cdots \times \phi_t^t \) be a \( t \)-module where \( \phi_i \), \( 1 \leq i \leq t \) are pairwise non-isogenous Drinfeld modules defined over \( \mathcal{O}_K \) such that \( \text{End} \phi_i = A \). Let \( \Lambda \subset \phi(\mathcal{O}_K) \) be a finitely generated \( A \)-submodule. For \( x \in \phi(\mathcal{O}_K) \) and \( w(t) \in A \) let \( O_{w(t)}(x) = \{ w^n(x) : n \geq 0 \} \)
be an orbit of the point $x$ under the iterations of multiplication by $w(t) \in A$. Then the following are equivalent

1. For almost every $U \in \text{Spec}(O_K)$
   
   $O_{w(t)}(\text{red}_U(P)) \cap \text{red}_U(\Lambda) \neq \emptyset$

2. $O_{w(t)}(P) \cap \Lambda \neq \emptyset$

whereas the main theorem of [Ba17] extended to $t$-modules reads as follows:

**Theorem 6.4.** Let $\mathcal{F} = \phi_1^{e_1} \times \cdots \times \phi_t^{e_t}$ be a $t$-module where $\phi_i$, $1 \leq i \leq t$ are pairwise non-isogenous Drinfeld modules defined over $O_K$ such that $\text{End} \phi_i = A$. Let $P, Q \in \phi(O_K)$ and $w_1(t), w_2(t) \in A$. Suppose that for almost all $U \in \text{Spec}O_K$ there exists a natural number $n_U$ such that

$$\text{red}_U(w_1^{n_U}P - w_2^{n_U}Q) = 0$$

Then there exists a natural number $n$ and a torsion point $T \in \phi(O_K)$ of an order that divides some power of $\gcd(w_1, w_2)$ such that $w_1^nP - w_2^nQ = T$.

Proofs of Theorems 6.3 and 6.4 follow the lines of S. Barańczuk’s original proofs. In appropriate places one has to replace multiplication by a natural number (viewed as an element of $\text{End} \mathcal{B}$) by the $A$-module action of an element $w(t) \in A$ and Assumption 6.1 by 6.2. Additionally one has to check that Assumptions 6.2 is fulfilled. Assumption 6.1 (2) is fulfilled by $B = \phi(O_K)$ cf. (2) of Proposition 4.1. As for Assumption 6.1 (1) we readily see that it follows from Theorem 1.6.

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