We investigate the possible formation of a molecular condensate, which might be, for instance, the analogue of the alpha condensate of nuclear physics, in the context of multicomponent cold atoms fermionic systems. A simple paradigmatic model of $N$-component fermions with contact interactions loaded into a one-dimensional optical lattice is studied by means of low-energy and numerical approaches. For attractive interaction, a quasi-long-range molecular superfluid phase, formed from bound-states made of $N$ fermions, emerges at low density. We show that trionic and quartetting phases, respectively for $N = 3, 4$, extend in a large domain of the phase diagram and are robust against small symmetry-breaking perturbations.

1. Introduction

Since the discovery of Bose-Einstein condensation, cold atom systems have become a major field of research for investigating the physics of strong correlations in a widely tunable
range and in unprecedentedly clean systems. In particular, loading cold atomic gases into an optical lattice allows for the realization of bosonic and fermionic (depending on the number of neutrons of the underlying atom) lattice models and the experimental study of exotic quantum phases in a new context\textsuperscript{[1]}. A prominent example is the observation of the Mott insulator-superfluid quantum phase transition with cold bosonic atoms in an optical lattice\textsuperscript{[2]}. Optical traps offer the opportunity to investigate the main features of the hyperfine spin ($F$) degeneracy on the properties of ultracold fermionic quantum gases. For instance, three component Fermi gases may be created experimentally by trapping the lowest three hyperfine states of $^6\text{Li}$ ($F = 1/2$) atoms in a magnetic field or by considering $^{40}\text{K}$ ($F = 9/2$) atoms. In addition, the magnetic field dependence of the three scattering lengths of $^6\text{Li}$ are known experimentally and can be tuned via Feshbach resonance\textsuperscript{[3]} which opens the experimental realization of a three-component fermionic lattice model. In fact, such a degenerate Fermi gas has been created experimentally very recently\textsuperscript{[4]}. The existence of these internal degrees of freedom in fermionic atoms is expected to give rise to some exotic superfluid phases. In this respect, a molecular superfluid (MS) phase might be stabilized where more than two fermions form a bound state. Such a non-trivial superfluid behavior has already been found in different contexts. In nuclear physics, a four-particle condensate—the $\alpha$ particle—is known to be favored over deuteron condensation at low densities\textsuperscript{[5]}. This quartet condensation can also occur in semiconductors with the formation of biexcitons\textsuperscript{[6]}. A quartetting phase, which stems from the pairing of Cooper pairs, has also been found in a model of one-dimensional (1D) Josephson junctions\textsuperscript{[7]} and in four-leg Hubbard ladders\textsuperscript{[8]}. A possible experimental observation of quartets might be found in superconducting quantum interference devices with (100)/(110) interfaces of two d-wave superconductors\textsuperscript{[9]}. The $hc/4e$ periodicity of the critical current with applied magnetic flux has been interpreted as the formation of quartets with total charge $4e$\textsuperscript{[10]}. More recently, the emergence of trions (i.e. three-fermion bound states) and quartets has been proposed to occur in the context of ultracold fermionic atoms\textsuperscript{[11], [12], [13], [14], [15]}. A simple paradigmatic model to investigate the formation of this exotic physics is the 1D $N$-component fermionic Hubbard model with attractive (s-wave) contact interaction. Such a model is defined by the following Hamiltonian:

$$
\mathcal{H} = -t \sum_{i,\alpha} [c_{\alpha,i}^\dagger c_{\alpha,i+1} + \text{H.c.}] + \frac{U}{2} \sum_i n_i^2,
$$

where $c_{\alpha,i}^\dagger$ is the fermion creation operator corresponding to the $N$ hyperfine states $\alpha = 1, \ldots, N$ and $n_i = \sum_\alpha c_{\alpha,i}^\dagger c_{\alpha,i}$ is the density at site $i$. Model (1) displays an extended $U(N) = U(1) \times SU(N)$ symmetry where the $SU(N)$ hyperfine spin symmetry is defined by: $c_{\alpha,i} \to \sum_\beta U_{\alpha\beta} c_{\beta,i}$ ($U_{\alpha\beta}$ being a $SU(N)$ matrix). The MS instability is built from the $N$ fermions: $M_i^I = c_{1,i}^\dagger c_{2,i}^\dagger \cdots c_{N,i}^\dagger$ which is a singlet under the $SU(N)$ symmetry. In the following, we will show, by means of a combination of analytical and numerical results obtained by the density-matrix renormalization group (DMRG) technique\textsuperscript{[16]} that this MS phase emerges at small enough density $n$ ($n = \sum_i n_i$) for an attractive interaction ($U < 0$). This work is a brief summary of the results obtained in Ref. [14] It includes also new results
concerning the stability of the MS phase upon switching on SU($N$) symmetry-breaking terms for $N = 3, 4$.

2. Low-energy approach

The low-energy effective field theory corresponding to the SU($N$) model (1) for attractive interactions has been discussed at length in Refs. 13, 14. The starting point of this approach is the linearization at the two Fermi points ($\pm k_F = \pm \pi n/N a_0$, $a_0$ being the lattice spacing) of the dispersion relation of free $N$-component fermions\textsuperscript{17}. For incommensurate filling, the resulting low-energy Hamiltonian separates into two commuting U(1) density and SU($N$) hyperfine spin parts: $H = \mathcal{H}_d + \mathcal{H}_s$. This results is the famous “spin-charge” separation which is the hallmark of 1D electronic quantum systems\textsuperscript{17}. Within this low-energy approach, the U(1) density excitations are critical and display metallic properties in the so-called Luttinger liquid universality class.\textsuperscript{17} In contrast, for attractive interaction ($U < 0$), a spectral gap opens in the hyperfine spin sector. The dominant instability which governs the physics of model (1) is the one with the slowest decaying correlations at zero temperature. In Ref. 14, we have found that the equal-time density correlation $N(x) = \langle n_i n_{i+x} \rangle$ associated to an atomic-density wave (ADW) and the equal-time MS correlations $M(x) = \langle M_i M_{i+x}^\dagger \rangle$ display the following power-law decay at long distance:

\[
N(x) \sim \cos(2k_F x) x^{-2K/N} \quad (2)
\]

\[
M(x) \sim x^{-N/(2K)} \quad \text{for } N \text{ even} \quad (3)
\]

\[
M(x) \sim \sin(k_F x) x^{-(K+N^2/K)/(2N)} \quad \text{for } N \text{ odd}, \quad (4)
\]

where $K$ is the Luttinger parameter which depends on the interaction $U$ and density $n$. It is a non-universal parameter which controls the power-law decay of correlation functions and is central to the Luttinger-liquid paradigm.\textsuperscript{17} A perturbative estimate of $K$ can be determined: $K = [1 + U(N - 1)/(\pi v_F)]^{-1/2}$ ($v_F = 2ta_0 \sin(\pi n/N)$ being the Fermi velocity). We thus observe that ADW and MS instabilities compete and the key point of the analysis is the one which dominates. In particular, we deduce from Eqs. (2, 3, 4) that a dominant MS instability requires $K > N/2$ ($K > N/\sqrt{3}$) for $N$ even (odd respectively). The existence and stability of this MS phase stem from the knowledge of the full non-perturbative behavior of the Luttinger parameter $K$ as a function of the density $n$ and the interaction $U$. This parameter can be numerically determined in the simplest cases $N = 3, 4$ by computing dominant correlations with the DMRG technique.

3. DMRG results

We have performed extensive DMRG calculations with open-boundary conditions for both the $N = 3$ and $N = 4$ cases and for a wide range of densities $n$ and interactions $U$. We see in Fig. 1(a) and Fig. 2(a) that the density and MS correlations $N(x)$ and $M(x)$ display a power-law behavior for $N = 3$ and $N = 4$ respectively at typical values of $n$ and $U = -4t$. The phase diagrams for SU(3) and SU(4) models are presented in Figs. 1(b), 2(b) which give a map of $K$ vs interaction and density. The values of $K$ were obtained from the power-law behavior of the molecular correlation $M(x)$ using Eqs. (3, 4). We find that trions and
quartet superfluid phases emerge in a wide portion of the phase diagrams (grey area) at low density separated from a ADW phase by a cross-over line $n_c(U)$. Interestingly enough, the MS phase extends to small values of $U$ at sufficiently small densities. In absence of the optical lattice, i.e. $n = 0$, we recover the known results that trionic and quartetting phases are stabilized respectively for $N = 3$ and $N = 4$. Indeed, the continuum SU($N$) fermionic model with a delta interaction is integrable by means of the Bethe ansatz and a phase with the formation of bound-state of $N$ fermions is found for an attractive interaction $^{18}$. 

4. Effect of symmetry breaking perturbations

At this point, the natural question is whether the MS phases survives to the breaking of the artificial SU($N$) symmetry of model $^{1}$. In the $N = 3$ case, the basic assumption of the SU(3) symmetry is that the three scattering lengths of the underlying cold atoms problem are the same. For the $^6$Li atoms, these scattering lengths are different and can be varied with an external magnetic field $^{3}$. The minimal model to investigate the low-energy properties of $^6$Li fermionic atoms in 1D reads as follows:

$$
\mathcal{H} = -t \sum_{i,\alpha} [c_{\alpha,i}^\dagger c_{\alpha,i+1} + \text{H.c.}] + U_{12} \sum_i n_{i,1} n_{i,2} + U_{13} \sum_i n_{i,1} n_{i,3} + U_{23} \sum_i n_{i,2} n_{i,3},
$$

(5)
where $n_{i,\alpha} = c_{\alpha,i}^\dagger c_{\alpha,i}$ is the density of the fermion on site $i$ with hyperfine spin component $\alpha = 1, 2, 3$. We observe that the SU(3) symmetry of model (1) is broken down to $U(1)^3$. In Fig. 3 (a), we show for an anisotropic situation close to the SU(3) point, which is relevant to the $^6$Li problem, that the trionic phase survives the breaking of the artificial SU(3) symmetry. The investigation of the full phase diagram of model (5) at zero temperature will be discussed in a forthcoming paper. For the $N = 4$ case, we consider the model for spin-$3/2$ (i.e. four-component) fermionic cold atoms with s-wave contact interaction derived in Ref. [19]. The interacting part depends on $U_J$ parameters corresponding to the total spin $J$ of two spin-$3/2$ atoms which takes only even integers value due to Pauli’s principle (the hyperfine spin part of the wave function should be antisymmetric): $J = 0, 2$. The resulting model reads as follows:

$$\mathcal{H} = - t \sum_{i,\alpha} [c_{\alpha,i}^\dagger c_{\alpha,i+1} + \text{H.c.}] + \frac{U}{2} \sum_i n_i^2 + V \sum_i P_{00,i}^\dagger P_{00,i},$$

where $c_{\alpha,i}^\dagger$ is the fermion creation operator corresponding to the spin-$3/2$ states $\alpha = \pm 3/2, \pm 1/2$ and $P_{00,i}^\dagger = c_{3/2,i}^\dagger c_{-3/2,i}^\dagger - c_{1/2,i}^\dagger c_{-1/2,i}^\dagger$ is the singlet-pairing operator with $V = U_0 - U_2$. As shown in Ref. [20] the presence of this BCS singlet-pairing term reduces the SU(4) symmetry down to SO(5) when $V \neq 0$. We show typical data for $U/t = -4$ and $V/t = -2$ at the density $n = 1/2$ in Fig. 3(b). Quartet correlations are (quasi) long ranged and dominate over ADW ones. For large negative $V$, a BCS pairing phase emerges [21] but
Fig. 3. (a): Trions and density correlations for the anisotropic three-components fermionic model \((5)\) at density \(n = 1/3\). The value of the Luttinger parameter is \(K = 2.65\) which signals the onset of the trionic phase. (b): Quartet and density correlations for the SO(5) model with \(U/t = -4\) and \(V/t = -2\) at the density \(n=0.5\). The value of the Luttinger parameter is \(K = 2.73\) which shows that the quartetting phase survives the breaking of the SU(4) symmetry.

the main point here is to show that the quartetting phase is not an artifact of the SU(4) symmetry and does exist in more realistic models at low density.

5. Concluding remarks

We have shown that a quasi-long-range MS phase can emerge in 1D multicomponent fermionic model with attractive contact interaction at sufficiently low density. This phase, characterized by the formation of a bound-state made of \(N\) fermions, is not an artifact of the artificial SU\((N)\) symmetry of model \((1)\). In this respect, the trionic and quartetting phases in the simplest \(N = 3, 4\) cases might be explored experimentally in the context of spinor ultracold fermionic atoms. As a first step, we have assumed here a homogeneous optical lattice and have neglected the parabolic confining potential of the atomic trap. We expect that this potential will not affect the main qualitative properties of this MS phase at low density. The effect of a harmonic trap could be investigated by means of DMRG calculations for quantitative comparisons as it has been done for 1D two-component fermionic systems \([22]\). Such a study is currently under progress in the \(N = 3, 4\) cases. We hope that future experiments in ultracold spinor fermionic atoms will reveal the existence of these trionic and quartetting phases.
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