The Painlevé analysis for $N=2$ super KdV equations

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Abstract

The Painlevé analysis of a generic multiparameter $N=2$ extension of the Korteweg-de Vries equation is presented. Unusual aspects of the analysis, pertaining to the presence of two fermionic fields, are emphasized. For the general class of models considered, we find that the only ones which manifestly pass the test are precisely the four known integrable supersymmetric KdV equations, including the SKdV$_1$ case.
1. Motivation and summary of the results

Even in its simplest form, the Painlevé analysis [1], as applied to a multiparameter class of equations, is a powerful tool for identifying those special values of the parameters for which the equations are potentially integrable.

However, for fermionic – and in particular, supersymmetric – extensions of known integrable systems, the application of the test is made a little tricky by the presence of fermionic fields [2, 3] and actually very few systems have been fully studied so far (see also [4]). For a single fermionic extension (with a fermionic field of degree 3/2 in the normalization where the degree of $\partial_x$ is 1) of the Korteweg-de Vries (KdV) equation, which contains 3 free parameters, the test has selected the very two integrable nontrivial extensions of the KdV equation, namely the Kuper-KdV [5] (which is not invariant under supersymmetry) and the supersymmetric KdV (sKdV - the small s refers to $N = 1$) equation [6, 7, 8] - the latter being called here the sKdV$_3$ equation for reasons explained below. There is an additional integrable supersymmetric system [6,9] which will be referred to, in the following, as the sKdV$_0$ equation. Although the latter is somewhat trivial in that the fermionic fields do not appear in the bosonic evolution equations (and for this reason it was excluded from the generic family considered in [2]), this will not be an issue here [10]. That the integrability of these models had already been established by other means supports the validity of the application of the test, or more precisely, its reliability as an integrability indicator, in the presence of fermionic fields.

No similar studies have been performed for the extension of the KdV equation with two fermions and an additional bosonic field. Six such systems are known to be integrable: the three usual $N = 2$ supersymmetric KdV (SKdV - the capital S is used for $N = 2$) equations, i.e., the SKdV$_a$ equation (where $a$ is a free parameter in a second hamiltonian formulation) for $a = -2, 1, 4$ [11, 12, 13], the SKdV$_O$ (where the subscript stands for ‘odd’) equation [14], which has an odd Poisson bracket formulation, the SKdV-B equation [15] and the $osp(2, 2)$ KdV equation, the direct extension of the Kuper-KdV equation (which is thus not invariant under $N = 2$ supersymmetry) [11, 16].

The details of the Painlevé analysis of these systems has never been presented in the literature. Actually, it has been claimed that for the SKdV$_1$ equation, the test is failed (see in particular the concluding remarks in [13]). The particular interest for this case, at the time, was due to its conjectural integrability status for some years before the discovery of
its Lax formulation in [13]. But given that this system is now known to be integrable [17],
that it does not have the Painlevé property sounds as an extremely surprising statement.
Clearly, the failure of the Painlevé test is not by itself a clear indication of nonintegrability.
For instance, the equation might have to be somewhat transformed in order to successfully
pass the test. However, in a multiparameter deformation of an equation, we definitely ex-
pect that if the test is satisfied for some values of the parameters (corresponding to a known
integrable system), it should be equally satisfied for all other values of the parameters for
which the equations are known to be integrable. But to rule on the SKdV1 equation, we
need to perform the test for the other cases too in order to see if, in the presence of two
fermionic fields, it is again a reliable integrability indicator.

The natural expectation is that all six extended KdV equations known to be integrable
should have the Painlevé property. However, precisely because there are two fermions, the
test displays unusual features. This point in itself is certainly not surprising given some of
the odd technical aspects of the test as applied to a single fermionic extension of the KdV
equation [2, 3]. Clearing up the status of the Painlevé property for the SKdV1 equation
was our first motivation for this work.

We present here the result of a ‘complete’ Painlevé analysis for four supersymmetric
integrable systems (excluding the SKdV-B equation by requiring an $O(2)$ invariance –
see below). More precisely, we perform a simplified analysis, in which, in addition to
verifying the plain properties of a genuine pole behavior of the leading singularities and
the integrality of the resonance positions, we only check the compatibility conditions at
the nonnegative resonances. The qualitative ‘complete’ refers to the fact that we consider
the full set of four evolution equations in each case. In addition to be rather complicated,
even though the analysis is done with the simplified Kruskal ansatz [18], it reveals an
unusual feature: in two cases out of four (and this includes the SKdV1 equation), in order
to verify the last resonance conditions – whenever this resonance is bosonic – say, at level
$n$, we need to solve the set of recursion equations at level $n + 1$. In other words, at
first sight the compatibility conditions are not satisfied. However, they involved some field
components that get determined only at the next recursion level. But when this is done and
the solutions are substituted back into the level $n$ resonance relations, the compatibility
conditions are found to be satisfied. We thus conclude that, in this context, the Painlevé
test is still in par with the other integrability indicators.
A second motivation for this work was to initiate the search for new integrable $N = 2$ extensions of the KdV equations by using the Painlevé property as a probing tool to test generic deformations of the known SKdV equations. In the present work we treat the most general deformation (which contains 4 free parameters) compatible with a natural $O(2)$ invariance.

Instead of starting with a brute force analysis of this four-parameter equation, we use a simple observation in order to constrain these parameters, which is that the reduction (by which we mean setting some fields equal to zero) of an integrable system has to be integrable. For instance, a clear signal of this integrability persistence is that, after the reduction of an integrable system, there remains an infinite number of conservation laws. In particular, the $N = 1$ reduction of an integrable SKdV equation has to be either the sKdV$_3$ or sKdV$_0$ equations. This fixes two parameters and selects two classes of two-parameter equations. Another simplifying feature of the above observation is that the bosonic core of the full set of equations (obtained by setting the two fermionic fields equal to zero) must also be integrable. The analysis of such bosonic systems (here a system of two coupled evolution equations) is much easier and puts severe constraints on the remaining parameters. In fact, the bosonic core of the test is satisfied (modulo a technical restriction discussed below) for only four cases, which are precisely the four known integrable supersymmetric systems.

Our search for new systems is thus unsuccessful. The results suggest in particular, that (most probably) there are no integrable deformations of the SKdV$_{\varnothing}$ equation.

The analysis of the complete fermionic systems is then performed case by case and the Painlevé property is verified in all four cases, as already mentioned.

We should point out a technical limitation of the present analysis, which is restricted to the study of the so-called principal family – in the terminology of [19]. That means that we only look for nonnegative resonances, in addition to the resonance at level $n = -1$. For a complete analysis, solutions with negative resonances must also be considered. The perturbative Painlevé test [19] provides a method for investigating such solutions. However, there is no finite algorithm ensuring the absence of movable logarithms (only for the principal families one can guarantee that the system has the Painlevé property). As a result, the computations are much more involved. We intend to return to this question elsewhere. Here we only indicate the cases where the negative resonances occur but without further analysis. Our statements concerning the non-existence of new integrable systems must thereby be tinged by this technical restriction.
The article is organized as follows. In section 2, we present the general class of \( N = 2 \) supersymmetric equations to be studied and discuss the constraints resulting from integrability under truncation to \( N = 1 \) supersymmetric equations. The general structure of the recursion relations is displayed in section 3. The delicate question of fixing the dominant resonance of the fermionic fields is discussed in full detail in appendix B. In the following section, we present the essential results of the bosonic-core analysis, relegating the details to appendix A. This analysis turns out to be rather involved, necessitating the consideration of a large number of special cases. Finally, section 5 presents a brief discussion of the study of the equations incorporating the fermionic fields. Here we only present the salient features of the \( \text{SKdV}_1 \) case and briefly comment on the differences that occur in the other cases. Our conclusions are reported in section 6.

2. The general equations and the \( N = 1 \) constraints

The \( N = 1 \) supersymmetrization of the KdV equation

\[ u_t = -u_{xxx} + 6uu_x \]  

is obtained by extending the \( u \) field to a fermionic superfield as

\[ u(x) \to \phi(x, \theta) = \theta u(x) + \xi(x). \]

Here \( \theta \) is a grassmannian variable (\( \theta^2 = 0 \)) and \( \xi \) is a fermionic field: \( \xi(x)\xi(x') = -\xi(x')\xi(x) \). The direct supersymmetrization reads [6,20]

\[ \phi_t = -\phi_{xxx} + c(\phi D\phi)_x + (6 - 2c)\phi_x(D\phi), \]

where \( c \) is a free parameter and \( D \) is the superderivative: \( D = \theta\partial_x + \partial_\theta \) so that \( D^2 = \partial_x \). It turns out that this equation is integrable only if \( c = 0 \) or 3 \([6]\). We call the resulting equation the sKdV_\( c \) equation. Its component version reads

\[ u_t = -u_{xxx} + 6uu_x - c\xi_{xx} \]  
\[ \xi_t = -\xi_{xxx} + (6 - c)u\xi_x + c u_x \xi. \]  

For \( c = 0 \) we see that \( \xi \) decouples from the first equation \([21]\).
The $N = 2$ super-extension is obtained by lifting $u$ to a bosonic superfield defined as follows (with the time dependence being implicit):

$$\Phi(x, \theta_1, \theta_2) = \theta_2 \theta_1 u(x) + \theta_1 \xi^{(2)}(x) + \theta_2 \xi^{(1)}(x) + w(x), \quad (2.5)$$

$\xi^{(1)}$ and $\xi^{(2)}$ are two fermionic fields and $w$ is a new bosonic field. Using the superderivatives

$$D_i = \theta_1 \partial_x + \partial_{\theta_i} \quad \Rightarrow \quad D_i^2 = \partial_x \quad (i = 1, 2) \quad D_1 D_2 = -D_2 D_1 \quad (2.6)$$

the most general version (subject to some restrictions to be specified shortly) of the $N = 2$ extension of the KdV equation reads

$$\Phi_t = -\Phi_{xxx} + \alpha_1 \Phi D_1 D_2 \Phi_x + \alpha_2 \Phi_x D_1 D_2 \Phi + \frac{\alpha_3}{2} (D_1 D_2 \Phi^2)_x + \frac{\alpha_4}{3} (\Phi^3)_x. \quad (2.7)$$

This equation contains all possible terms that are compatible with an homogeneity requirement under a gradation defined by $\text{deg } \Phi = 1$, $\text{deg } D_i = 1/2$ and the $O(2)$ invariance, that is, invariance under the transformation $\Phi \rightarrow -\Phi$ and $D_1 \leftrightarrow D_2$. (For instance, terms like $[D_1(\Phi_x D_2 \Phi) - D_2(\Phi_x D_1 \Phi)]$ or $[D_1(\Phi D_2 \Phi_x) - D_2(\Phi D_1 \Phi_x)]$ or even $[(D_1 \Phi)(D_2 \Phi_x) - (D_2 \Phi)(D_1 \Phi_x)]$ are not independent – i.e., they are linear combinations of those already given).

The $N = 1$ reduction is obtained by setting

$$\Phi(x, \theta_1, \theta_2) = \theta_2 \phi(x, \theta_1) + F(x, \theta_1) \quad (2.8)$$

and keeping only the linear terms in $\theta_2$ with $F = 0$. All integrable versions of this four-parameter equation must reduce to the sKdV$_c$ equation for either $c = 0$ or 3. This fixes two parameters:

$$\alpha_1 = c, \quad \alpha_2 = 6 - c. \quad (2.9)$$

The other two are redefined as follows:

$$\alpha_3 = \alpha - 1, \quad \alpha_4 = \beta \quad (2.10)$$

and we are left with two distinct two-parameter equations:

$$\Phi_t = -\Phi_{xxx} + c \Phi D_1 D_2 \Phi_x + (6 - c) \Phi_x D_1 D_2 \Phi + \frac{\alpha - 1}{2} (D_1 D_2 \Phi^2)_x$$

$$+ \beta \Phi^2 \Phi_x \quad (2.11)$$
(c = 0, 3). In terms of component fields, it leads to four coupled equations:

\[
\begin{align*}
    u_t &= -u_{xxx} + 6u_x - c\xi^{(i)}_x\xi^{(i)}_x - cww_{xxx} - (6 - c)w_xw_x \\
    &\quad - \frac{\alpha - 1}{2}(w^2)_{xxx} + \beta (uw^2)_x + 2\beta(\xi^{(2)}_x\xi^{(1)}_x)_x \\
    \xi^{(i)}_t &= -\xi^{(i)}_x + c u_x\xi^{(i)}_x + (6 - c)u\xi^{(i)}_x - c\epsilon_{ij}\xi^{(j)}_xw \\
    &\quad - (6 - c)\epsilon_{ij}\xi^{(j)}_xw_x - (\alpha - 1)\epsilon_{ij}(\xi^{(j)}_xw)_x + \beta(\xi^{(i)}_xw^2)_x \\
    w_t &= -w_{xxx} + c u_xw + (6 - c)uw_x + (\alpha - 1)(uw + \xi^{(2)}_x\xi^{(1)}_x)_x \\
    &\quad + \beta w^2w_x
\end{align*}
\]  

(2.12)

with \( i, j = 1, 2 \) and \( \epsilon_{12} = -\epsilon_{21} = 1 \).

3. The Painlevé analysis: recursion relations

We next proceed with the Painlevé analysis by solving the recursion equations in order to find those values of the parameters \( \alpha \) and \( \beta \) for which the test is satisfied. In the present work, we content ourself with a minimal version of the test, which consists in verifying:

1- that the leading singularity is integer (i.e., pole-like),

2- the resonances occur at integer levels,

3- the compatibility conditions are satisfied at the nonnegative resonances.

We will further give all possible solutions with integer resonances but without further analysis of these last cases.

The expansion of the component fields about a movable singular manifold \( \varphi(x, t) \) reads

\[
\begin{align*}
    u &= \sum_{n=0}^{\infty} u_n\varphi^{n-p} , \\
    \xi^{(i)} &= \sum_{n=0}^{\infty} \xi^{(i)}_n\varphi^{n-r} , \\
    w &= \sum_{n=0}^{\infty} w_n\varphi^{n-q}
\end{align*}
\]  

(3.1)

\( i = 1, 2 \). By symmetry, the value of the leading singularity must be the same for the two fermionic fields. To simplify the analysis, we will use the Kruskal’s ansatz:

\[
\begin{align*}
    \varphi(x, t) &= x - f(t) , \\
    u_n &= u_n(t) , \\
    \xi^{(i)}_n &= \xi^{(i)}_n(t) , \\
    w_n &= w_n(t).
\end{align*}
\]  

(3.2)

The first step amounts to fix the leading singularity: we easily find that \( p = 2, q = 1 \). Note that this is a consequence of the SKdV degree-homogeneity already mentioned: setting
deg(\partial_x) = 1, it follows that deg(u) = 2 and deg(w) = 1. Now since deg(\varphi) = deg(x) = \(-1\) and \(u_0\) and \(w_0\) are constants, hence of degree zero, we thus conclude that deg(u) = 2 and deg(w) = 1 only if \(p = 2\) and \(q = 1\).

The determination of the leading singularity for the fermionic fields is a bit tricky (see for instance \[3\]); it is shown in appendix B that \(r = 2\) is a solution and all other possible solutions do not pertain to a principal family. However, for the rest of this section, we leave \(r\) unspecified since the recursion relations themselves are needed to fix it – cf. appendix B. Moreover, the precise value found for \(r\) depends explicitly on the first bosonic terms and these are fixed from the bosonic-core analysis.

A direct substitution of (3.1), (3.2) with \(p = 2\), \(q = 1\) into (2.12) leads to the general recursion formulae:

\[
\begin{align*}
    u_{n-3,t} + (n-4)u_{n-2}\varphi_t &= -(n-2)(n-3)(n-4)u_n + 3(n-4) \sum_{m=0}^{n} u_{n-m}w_m \\
    &\quad - (\alpha - 1 + c) \sum_{m=0}^{n} (m-1)(m-2)(m-3) w_{n-m}w_m \\
    &\quad - \frac{1}{2}(3\alpha + 3 - c)(n-4) \sum_{m=0}^{n} (n-m-1)(m-1) w_{n-m}w_m \\
    &\quad + \beta (n-4) \sum_{m=0}^{n} \sum_{l=0}^{m} u_{n-m}w_{m-l}w_l \\
    &\quad + \frac{1}{2}c(n-4) \sum_{m=0}^{n+2r-3} (n+2r-3-2m) \xi^{(i)}_{n+2r-3-m} \xi^{(i)}_m \\
    &\quad + 2\beta (n-4) \sum_{m=0}^{n+2r-3} \sum_{l=0}^{m} \xi^{(2)}_{n+2r-3-m} \xi^{(1)}_{m-l}w_l \\
    \xi^{(i)}_{n-3,t} + (n-r-2)\xi^{(i)}_{n-2}\varphi_t &= -(n-r)(n-r-1)(n-r-2) \xi^{(i)}_n \\
    &\quad + c \sum_{m=0}^{n} (n-m-2) u_{n-m} \xi^{(i)}_m \\
    &\quad + (6 - c) \sum_{m=0}^{n} (m-r) u_{n-m} \xi^{(i)}_m \\
    &\quad + \beta (n-r-2) \sum_{m=0}^{n} \sum_{l=0}^{m} \xi^{(i)}_{n-m}w_{m-l}w_l
\end{align*}
\]
Here $n$ takes any integer value from $\min(3 - 2r, 0)$ to $\infty$ (sticking to the principal family). It is understood that every field-component with a negative index is zero. For the system of equations to be integrable, the solution needs to contain a sufficient number of arbitrary functions. For the case under study, the system being composed of four coupled third order equations, there should be twelve arbitrary functions, six bosonic and six fermionic. With the leading singularities fixed, we need to determine those (recursion) levels $n$ – the resonances – in (3.3) for which there are arbitrary functions. That clearly requires $n$ to be an integer. At each such level, the equation must vanish identically without enforcing any constraints on the lower-order arbitrary functions. These are the compatibility conditions at the resonances. We then proceed in two steps. We first find all the possible values of the free parameters for which the bosonic-core system has the Painlevé property. Then, for those special parameters, we complete the analysis for the full system with the fermionic fields reintroduced.
4. The Painlevé analysis of the bosonic core

The bosonic-core analysis is the most important and also the most involved part of this work. It amounts to consider one-by-one a long sequence of special cases. Although the analysis is straightforward for most of them, there is a number of cases (that include cases in which the test is satisfied) for which this is not so. For this reason, a somewhat detailed presentation of all the possibilities is required. It is reported in appendix A. For the ease of reading, we collect in this section the final results of this appendix.

The only cases for which the Painlevé property of the bosonic core is fully satisfied are listed below. Note that the ‘body’ (i.e., without the nilpotent part) values of \( u_0 \) and \( w_0 \) represent an important part of the data since it is necessary to fix uniquely the leading singularity of the fermionic fields. Here, \( k = \pm i \).

(I) \( SKdV_{-2} \)
\[
c = 3, \quad \alpha = -2, \quad \beta = -6, \quad u_0 = -1, \quad w_0 = k.
\]

(II) \( SKdV_1 \)
\[
c = 3, \quad \alpha = 1, \quad \beta = 3, \quad u_0 = 1, \quad w_0 = k.
\]

(III) \( SKdV_4 \)
\[
c = 3, \quad \alpha = 4, \quad \beta = 12, \quad u_0 = \frac{1}{2}, \quad w_0 = \frac{1}{2}k.
\]

(IV) \( SKdV_0 \)
\[
c = 0, \quad \alpha = 1, \quad \beta = 0, \quad u_0 = 1, \quad w_0 = k.
\]

(V) \( SKdV_{-2} \) (‘degenerate’ case)
\[
c = 3, \quad \alpha = -2, \quad \beta = -6, \quad u_0 = 2, \quad w_0 = 0.
\]

Cases (I)-(V) are those with nonnegative resonances. For completeness, we also present all other possible solutions that are not in principal families. These are listed below. It is always understood that \( j_1, k_1 \) and \( k_2 \) are integers.

(VI) \( c = 3, \quad \alpha = \frac{1}{2} j_1 (j_1 - 3) - 1, \quad \beta = \frac{3}{2} j_1 (j_1 - 3) - 1, \quad u_0 = 2, \quad w_0 = 0 \)

\( (j_1 \geq 4) \)

(VII) \( c = 3, \quad \alpha = \frac{6k_1}{k_1 (3 - k_2) + k_2 (k_2 + 1) - 6} - 2, \quad \beta = 108 \frac{k_1 - 2}{(k_1 (3 - k_2) + k_2 (k_2 + 1) - 6)^2}, \quad u_0 = \frac{1}{6} (k_1 (3 - k_2) + k_2 (k_2 + 1) - 6), \quad w_0 = k u_0 \)
\[(k_1 \geq \max (5, 2k_2 + 1), \quad k_2 \geq -1).\]

(VIII) \[c = 3, \quad \alpha = \frac{6k_1}{k_1(3-k_2)+k_2(k_2+1)-6} - 2, \quad \beta = 108 \frac{k_1-2}{(k_1(3-k_2)+k_2(k_2+1)-6)^2}, \quad u_0 = \frac{1}{6}(k_1(3-k_2)+k_2(k_2+1)-6), \quad w_0 = k u_0 \]
\[(k_1 \geq 5, \quad k_2 \leq -4).\]

(IX) \[c = 3, \quad \alpha = 2 \frac{7-k_2^2}{2+k_1}, \quad \beta = \frac{108}{(2+k_1)^2}, \quad u_0 = \frac{1}{6}(2+k_1^2), \quad w_0 = k u_0 \]
\[(k_1 \geq 5).\]

(X) \[c = 3, \quad \alpha = -4 \frac{k_1}{5k_1+6}, \quad \beta = 108 \frac{k_1}{(5k_1+6)^2}, \quad u_0 = \frac{5}{6}k_1 + 1, \quad w_0 = k u_0 \]
\[(k_1 \leq -7).\]

(XI) \[c = 3, \quad \alpha = -4 \frac{k_1}{5k_1+6}, \quad \beta = 108 \frac{k_1}{(5k_1+6)^2}, \quad u_0 = \frac{5}{6}k_1 + 1, \quad w_0 = k u_0 \]
\[(k_1 \geq 3).\]

(XII) \[c = 0, \quad \alpha = 7, \quad u_0 = 2, \quad w_0 = 0.\]

(XIII) \[c = 0, \quad \alpha = 1, \quad u_0 = 2, \quad w_0 = 0.\]

(XIV) \[c = 0, \quad \alpha = -\frac{7}{31}, \quad \beta = -\frac{78}{961}, \quad u_0 = -31, \quad w_0 = k u_0.\]

(XV) \[c = 0, \quad \alpha = -\frac{5}{21}, \quad \beta = -\frac{6}{49}, \quad u_0 = -21, \quad w_0 = k u_0.\]

(XVI) \[c = 0, \quad \alpha = -\frac{1}{7}, \quad \beta = \frac{15}{98}, \quad u_0 = 14, \quad w_0 = k u_0.\]
\( c = 0, \quad \alpha = -\frac{7}{5}, \quad \beta = 0, \quad u_0 = -5, \quad w_0 = k u_0. \)

5. The Painlevé analysis of the full fermionic systems of the four integrable \( N = 2 \) supersymmetric systems

In a second step, the Painlevé analysis is completed for the fermionic extension of the successful bosonic systems. We omit the details of the SKdV\(_{-2,4,\sigma}\) analysis and sketch some aspects of the analysis of the SKdV\(_1\) equation.

5.1. Analysis of the SKdV\(_1\) equation

In appendix B, it is shown that the leading singularity of fermionic fields must be \( r = 2 \) and that the following condition must hold:

\[
\xi_0^{(2)} = k_0 \xi_0^{(1)} \quad (5.1)
\]

where \( k_0^2 = -1 \) [22]. With this condition, (3.3) for \( n = -1 \), which reads

\[
-30 \alpha \xi_0^{(1)} \xi_0^{(2)} w_0 = 0 \quad -4(\alpha - 1) \xi_0^{(1)} \xi_0^{(2)} = 0, \quad (5.2)
\]

is automatically satisfied.

From the resonance equations obtained in appendices A and B, the bosonic resonances must occurs at the roots of

\[
(n + 1)(n - 1)(n - 2)(n - 3)(n - 4)(n - 6) = 0 \quad (5.3)
\]

corresponding to the arbitrariness of \( \varphi, w_1, w_2, w_3, u_4 \) and \( w_6 \), whereas the fermionic ones are determined by the roots of

\[
n(n - 2)^2(n - 4)^2(n - 6) = 0 \quad (5.4)
\]

corresponding to the arbitrariness of \( \xi_0^{(1)}, \xi_2^{(1)}, \xi_2^{(2)}, \xi_4^{(1)}, \xi_4^{(2)} \) and \( \xi_6^{(1)} \).

The introduction of the fermionic fields brings a little complexity right at the beginning of the analysis in that it is necessary to use both the \( n = 0 \) and \( n = 1 \) conditions in order to
fix $u_0$, $\xi^{(i)}_0$ and $w_0$ unambiguously. Once this is settled, the remaining part of the analysis is straightforward, apart from the plain fact that the equations are rather complicated.

The most general solution to the recursion formulae at level $n = 0$, for which the bosonic part reduces to the one found in the bosonic-core analysis (with constant $k$ appearing in $w_0$ fixed to $k = -k_0$, as shown in appendix B), is:

$$\begin{align*}
u_0 &= 1 - \frac{2}{3} \lambda_0 \xi^{(1)}_0, \quad \xi^{(2)}_0 = k_0 \xi^{(1)}_0, \\
w_0 &= -k_0 + \lambda_0 \xi^{(2)}_0, \quad \xi^{(2)}_1 = k_0 \xi^{(1)}_1 + \frac{4}{3} k_0 \lambda_0 + k_1 \xi^{(2)}_0, \tag{5.5}
\end{align*}$$

where $k_0^2 = -1$, $k_1$ is a (even) constant, $\lambda_0$ is a fermionic constant and $\xi^{(1)}_0$ is an arbitrary fermionic function.

In order to fix uniquely $u_0$ and $w_0$, we need to consider the equations for $n = 1$. At this level, a substitution of (5.5) into the recursion equations leads to ($k_0^2 = -1$):

$$\begin{align*}
u_0 &= 1, \quad \xi^{(1)}_0 \text{ arbitrary}, \quad \xi^{(2)}_0 = k_0 \xi^{(1)}_0, \quad w_0 = -k_0, \\
u_1 &= 0, \quad \xi^{(1)}_1 = 0, \quad \xi^{(2)}_1 = 0, \quad w_1 = 0. \tag{5.6}
\end{align*}$$

Pursuing the analysis of (3.3), one can verify that all the compatibility conditions at the various resonances are satisfied. Since the resulting equations are very long, this part of the analysis will be omitted. Notice that since the equation for $w_6$ depends upon the value of $\xi^{(i)}_7$ ($i = 1, 2$), we have to go up to level $n = 7$ to fix completely the different non-arbitrary functions needed to verify this particular compatibility condition. The analysis for levels $n = 5$ to $n = 7$ is actually very complicated; the computations have been made with Maple (with the package Grassmann).

5.2. Comments on the other three cases

The analysis for the other three cases singled out by the bosonic-core analysis has also been performed successfully.

For the SKdV$_{-2}$ equation, only one of the two possible cases identified by the bosonic-core analysis is found to have the Painlevé property: this is case (I). The analysis for this case is not too difficult since the last resonance is fermionic, occurring at level $n = 5$, so that we only have to push the analysis up to this level. The arbitrary functions are: $\varphi$, $\xi^{(1)}_0$, $u_1$, $\xi^{(1)}_2$, $u_3$, $\xi^{(1)}_3$, $w_3$, $u_4$, $\xi^{(1)}_4$, $\xi^{(2)}_4$, $w_4$ and $\xi^{(1)}_5$. 

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For the so-called ‘degenerate’ SKdV−2 case (V), the Painlevé test immediately fails at the first level since both \( \xi^{(1)}_0 \) and \( \xi^{(2)}_0 \) need to be arbitrary and, at the same time, satisfy 
\[ \xi^{(1)}_0 \xi^{(2)}_0 = 0 \] (cf. (5.2)). This condition is completely independent of the value of \( r \).

For the SKdV4 equation, the compatibility conditions are all verified and \( \varphi, \xi^{(1)}_0, w_1, \xi^{(1)}_2, u_3, \xi^{(1)}_3, w_3, u_4, \xi^{(2)}_4, u_5 \) and \( \xi^{(1)}_5 \) are found to be arbitrary. Notice that there are two resonances at level \( n = 5 \), one of which being bosonic; the analysis must then be extended up to level \( n = 6 \).

For the SKdV\( \mathcal{O} \) equation, the bosonic evolution equations decouple from the fermionic ones; this eliminates the necessity of extending the analysis to a higher level in order to check the compatibility condition at the highest resonance. With \( r = 2 \) we find that 
\[ \varphi, \xi^{(1)}_0, w_1, \xi^{(1)}_2, \xi^{(2)}_2, w_2, \xi^{(1)}_3, w_3, u_4, \xi^{(1)}_4, u_6 \text{ and } \xi^{(1)}_7 \] are arbitrary hence it is not necessary to perform the test for other values of \( r \) (since we already know that the system SKdV\( \mathcal{O} \) has the Painlevé property).

6. Conclusion

In this work, we have presented the Painlevé analysis (at least, a reduced form of it) for the complete set of bosonic and fermionic evolution equations pertaining to a general multiparameter family of \( N = 2 \) supersymmetric equations. Such an analysis is interesting for a number of reasons. First, very few fermionic extensions of integrable systems have been analyzed from that point of view. The detailed analysis of specific examples is of a clear interest in view of confirming (or limiting or even, in principle, invalidating) the direct extension of the test to fermionic systems. The successful analysis presented here for four \( N = 2 \) supersymmetric extensions of the KdV equation, known to be integrable from other methods, indeed confirms the validity of the naive extension of the test. This, in turn, gives credit to the test when viewed as an exploratory tool in the search for new integrable systems among a multiparameter class of equations. In that respect, the present results have not signaled the existence of a single new integrable equation (although new integrable systems could still in principle be revealed by an analysis that goes beyond the principal families).

Manifestly, that only a rather limited number of examples have been studied so far is partly due to the intrinsic complications of such computations (involving here four coupled
nonlinear equations); however, it is also due to the special complications brought by the fermionic fields themselves. In particular, even the determination of the leading singularity is somewhat problematic (and a simple degree-homogeneity requirement cannot be put forward). This particular question has been treated in great detail here. Another aspect of the present analysis is to have put in light unusual features of the verification of the compatibility conditions for systems involving fermions, unusual in that these conditions can be satisfied in some case only if higher-order recursion relations are solved.

For completeness, we have also checked the Painlevé property for the \( osp(2,2) \) KdV equation, which is \( O(2) \) symmetric but not supersymmetric, and found that it successfully passes the test. The equations take the form

\[
\begin{align*}
  u_t &= \partial_x [-u_{xx} + 3u^2 - 12\xi^{(i)} \xi_x^{(i)} + 24\xi^{(2)} \xi^{(1)} w + 2(w_x)^2 + 2ww_{xx} - 6w^2 + 3w^4] \\
  \xi_t^{(i)} &= -4\xi_{xx}^{(i)} + 3u_x \xi^{(i)} + 6u \xi_x^{(i)} + 6w^2 \xi^{(i)} + 6ww_x \xi^{(i)} \\
  &\quad - 12\epsilon_{ij} \xi_x^{(j)} w - 12\epsilon_{ij} \xi_x^{(j)} w_x - 4\epsilon_{ij} \xi^{(j)} w_x w + 6u \epsilon_{ij} \xi^{(j)} w - 2\epsilon_{ij} \xi^{(j)} w^3
\end{align*}
\]

with \( w_t = 0 \). Since the \( w \)-evolution equation is trivial this field has no singularity; however, relying on the degree-homogeneity property, we have set \( w = \sum_{n=0}^{\infty} w_n \varphi^{n-1} \) and \( w_0 = 0 \). Moreover, the fermionic fields are also singularity-free at the leading order; therefore, \( u \) is the only field having a leading singularity, which in itself is a rather uncommon feature.

We stress finally that the analysis has been presented here in terms of the component fields. Hence, we have not taken advantage of the economical superfield formalism. In fact, the Painlevé test has never been formulated in superspace. That would be a definite progress since it is only in such a case that we could face a more refined analysis that does not rely upon the simplified Kruskal ansatz. The benefit of such a generalization is the ability to make contact, in the early steps of the analysis, with Backlund transformations and Lax pairs (see e.g., [23]). We hope to report elsewhere on this topic.

**Appendix A. Analysis of the bosonic core**

In this appendix, we analyze the bosonic core of the generic supersymmetric KdV equations. This boils down to the study of the recursion formulae (3.3) in which we set all fermionic fields equal to zero: \( \xi_n^{(i)} = 0 \) for \( i = 1, 2 \). Further references to (3.3) in this appendix are to be understood with this restriction, which transforms this system into a set of two coupled bosonic equations.
Before considering the general recursion equations and determining its resonances, we first analyze the recursion formulae at levels \( n = 0, 1 \) for the cases \( c = 0, 3 \), in order to impose as much constraints as possible in the very early steps of the analysis. For every solution found at those levels, we need to write the resonance equation in order to identify those cases that are potentially Painlevé admissible. Note however that a single solution to the \( n = 0, 1 \) relations can lead to more than one resonance equation; the different possibilities must then be analyzed one by one. The cases \( c = 0, 3 \) are studied separately.

### A.1. The \( c = 3 \) case

The recursion formulae (3.3), with \( c = 3 \), can be written under the form

\[
\begin{align*}
A^1_1 u_n + A^1_2 w_n &= F^1_n \\
A^2_1 u_n + A^2_2 w_n &= F^2_n
\end{align*}
\]

(A.1)

where

\[
\begin{align*}
A^1_1(n) &= -(n - 2)(n - 3) + 6 u_0 + \beta w_0^2(n - 4) \\
A^1_2(n) &= -(\alpha + 2)n^2 + (5\alpha + 4)n - 6(\alpha + 1) + 2\beta u_0(n - 4) w_0 \\
A^2_1(n) &= (\alpha + 2)(n - 3) w_0 \\
A^2_2(n) &= -(n - 1)(n - 2) + (\alpha + 2) u_0 + \beta w_0^3(n - 3)
\end{align*}
\]

(A.2)

and \( F^1_n \) and \( F^2_n \) are functions of \( u_0, u_1, ... u_{n-1} \) and \( w_0, w_1, ... w_{n-1} \). There is a resonance when this system is not defined, that is, when

\[
A(n) = \det |A^i_j(n)| = 0 \quad (i, j = 1, 2).
\]

(A.3)

The substitution of \( \alpha, \beta, u_0 \) and \( w_0 \) (whenever they are known) for each case identified will then yield the values of the resonance levels \( n \). It should be stressed that we are particularly interested in cases in which there is a resonance at level \( n = -1 \) (corresponding to the arbitrariness of the singular manifold \( \varphi \)) with the other ones being integers \( \geq 1 \), unless either \( u_0 \) or \( w_0 \) is arbitrary, in which case we also need a resonance at level zero. The cases for which negative resonances appear will be given, but the search for movable logarithms will be omitted.

Here are the solutions of the recursion equations for levels \( n = 0, 1 \), the results of the resonance analysis and their compatibility conditions. When the test is not satisfied, we simply indicate the reason (and avoid repeating; therefore the test is not satisfied).
(i) \( u_0 = 0, \quad u_1 = 0, \)
\( w_0 = 0, \quad w_1 \) arbitrary.

This case can readily be eliminated given the absence of singularities. Similarly, cases (ii) and (iii) below could have been eliminated from the start since there are no singularities for the field \( w \); however, being interested in a supersymmetric extension for the field \( u \) – which is thus the ‘leading’ field – this restriction will not be imposed.

(ii) \( u_0 = 2, \quad u_1 = 0, \)
\( w_0 = 0, \quad w_1 = 0, \)
\[ A(n) = (n + 1)(n - 3)(n - 4)(n - 6)(n^2 - 3n - 2(\alpha + 1)). \]

Given that the two roots of the second order polynomial are \( j_1 \) and \( j_2 \), we thus have
\[ j_2 = 3 - j_1, \quad \alpha = \frac{1}{2}j_1(j_1 - 3) - 1 \]
and we can choose \( j_1 \geq j_2 \). Now since the coefficients at level 0 and 1 are fixed, a resonance at one of those levels would signal the presence of a movable logarithm. In consequence, there is no solution in a principal family (with both \( j_1 \) and \( j_2 \geq 0 \)) free from movable logarithms. The only other cases left are those for which \( j_1 \geq 4 \). For those cases, the compatibility conditions at level 6 are satisfied only for \( \beta = 3\alpha \).

(iii) \( \alpha = -2, \)
\( u_0 = 2, \quad u_1 = 0, \)
\( w_0 = 0, \quad w_1 \) arbitrary,
\[ A(n) = (n + 1)(n - 1)(n - 2)(n - 3)(n - 4)(n - 6). \]

The resonances correspond to the arbitrariness of \( \varphi, w_1 w_2, w_3, u_4, u_6 \). All compatibility conditions are verified without constraints on the parameters, except the one at level 6 which forces \( \beta = 3\alpha = -6 \). This will turn out to correspond to a non-integrable solution of the recursion relations associated to the SKdV\(_{\alpha=-2} \) equation.

(iv) \( \alpha = -1, \)
\( u_0 \) not fixed yet or arbitrary, \( u_1 = 0, \)
\( w_0 = k \sqrt{\frac{3}{2}(u_0 - 2)}, \quad w_1 = 0, \)
\[ A(n) = (n + 1)n(n - 3)(n - 4)(n^2 - 9n - (u_0 - 20) - (u_0 - 2)) \]
where (here and below) \( k^2 = -1 \). The resonance at level 0 would signal a movable logarithm if \( u_0 \) would have to be fixed. Moreover, \( A(n) \) would need to be independent of \( u_0 \) for this coefficient to be arbitrary. This leads to \( \beta = -3 \) and
\[ A(n) = (n + 1)n(n - 3)^2(n - 4)(n - 6). \]

(A.5)
However, the compatibility conditions at level 3 are not satisfied.

(v) \[ \alpha = -1, \quad \beta = 3\left[\frac{u_0^2}{P}\right], \]
\[ u_0 \text{ fixed by value of } \beta, \quad u_1 = 0, \]
\[ w_0 = k\sqrt{P}, \quad w_1 = 0, \]
\[ A(n) = (n + 1)n(n - 3)(n - 4)[(n - 1)(n - 8) - 3u_0(u_0 - 2)] \]

with \( P = 3(u_0)^2 - 7u_0 + 12 \). The resonance at level 0 signals the presence of a movable logarithm.

(vi) \[ \alpha = -1, \quad \beta = -\frac{8}{3}, \]
\[ u_0 = -\frac{4}{3}, \quad u_1 = 0, \]
\[ w_0 = \frac{1}{2}k\sqrt{15}, \quad w_1 = 0, \]
\[ A(n) = (n + 1)n(n - 3)(n - 4)[n^2 - 9n + \frac{211}{12}] \]

\( A(n) \) has non-integer roots.

(vii) \[ \beta = 3\left[\frac{\alpha + 2}{u_0}u_0 - 2\right], \]
\[ u_0 \text{ fixed by the value of } \beta, \quad u_1 = 0, \]
\[ w_0 = ku_0, \quad w_1 = 0, \]
\[ A(n) = (n + 1)(n - 3)(n - 4)(n - m)[n^2 + (m - 9)n + 6(4 - u_0 - m)] \]

where \( m = 4 - (\alpha + 2)u_0 \). Writing the two roots of the second order polynomial as \( j_1 \) and \( j_2 \), the general solution can be written (with \( k_1 \) and \( k_2 \) integers)

\[ m = 4 - k_1, \quad j_1 = 3 + k_2, \quad j_2 = 2 + k_1 - k_2, \quad k_1 \geq 2k_2 + 1 \quad (A.6) \]

with

\[ u_0 = \frac{1}{6}(k_1(3 - k_2) + k_2(k_2 + 1) - 6), \]
\[ \alpha = \frac{6k_1}{k_1(3 - k_2) + k_2(k_2 + 1) - 6} - 2, \quad (A.7) \]
\[ \beta = 108\frac{k_1 - 2}{(k_1(3 - k_2) + k_2(k_2 + 1) - 6)^2}. \]

The principal families are characterized by

\[ -1 \leq k_2 \leq 2k_2 + 1 \leq k_1 \leq 2 \quad (A.8) \]

for which the possible solutions are
Case vii.b can be eliminated since there are no singularities (and moreover $\alpha, \beta \to \infty$). For cases vii.a, c, d and f, the compatibility conditions at level $n = 2, 3, 3, 2$ respectively are not satisfied. For vii.e all the conditions are satisfied so that this system passes the test ($\varphi, u_1, u_3, w_3, u_4$ and $w_4$ are all arbitrary functions). It corresponds to the bosonic core of the SKdV$_{\alpha=-2}$ equation.

The only other solutions of interest (with negative resonances) are given by

\[
\begin{array}{cccccc}
\text{vii.g} & k_1 \geq \max (5, 2k_2 + 1) \\
\text{vii.h} & k_1 \geq 5 & k_2 \geq -1 \\
\end{array}
\]

\(\beta = \frac{1}{3}(\alpha + 2)^2,\)

\(u_0 = \frac{3}{\alpha + 2},\)

\(w_0 = k u_0,\)

\(u_1 = 0,\)

\(w_1 \text{ arbitrary},\)

\(A(n) = (n + 1)(n - 1)(n - 3)(n - 4)(n^2 - 8n - 6(u_0 - 3)).\)

We write the two roots of the quadratic term as

\[
j_1 = 4 - k_1, \quad j_2 = 4 + k_1, \quad k_1 \geq 1
\]

with $k_1$ an integer and

\[
u_0 = \frac{1}{6} (2 + k_1^2), \quad \alpha = \frac{7 - k_1^2}{2 + k_1^2}.
\]

The principal families (free from movable logarithms) are characterized by $k_1 = 1, 2$ so that we have

\[
\begin{array}{cccccc}
j_1 & j_2 & \alpha & \beta & u_0 \\
vii.a & 2 & 6 & 1 & 3 & 1 \\
vii.b & 3 & 5 & 4 & 12 & \frac{1}{2} \\
\end{array}
\]
For viii.a and b, all the compatibility conditions are satisfied. Those systems describe the bosonic core of the SKdV$_{\alpha=1}$ (with $\varphi, w_1, w_2, w_3, u_4$ and $w_6$ arbitrary) and SKdV$_{\alpha=4}$ (with $\varphi, w_1, u_3, w_3, u_4$ and $u_5$ arbitrary) equations respectively.

The other possible cases are those for which $k_1 \geq 5$.

(ix) $\beta = -\frac{9}{5} \alpha (5\alpha + 4)$, 
\[ u_0 = \frac{4}{5\alpha+4}, \quad u_1 = \frac{1}{2} \frac{(11\alpha + 4)}{\alpha + 2} k w_1, \]
\[ w_0 = k u_0, \quad w_1 \text{ arbitrary.} \]

Note however that in the singular case where $\alpha = -2$, $w_1 = 0$ and $u_1$ is arbitrary.

\[ A(n) = (n + 1)(n - 1)(n - 3)(n - 4)(n + \frac{6}{5} u_0 - \frac{16}{5})(n - \frac{6}{5} u_0 - \frac{24}{5}). \]

We can write the two roots of the last two factors as

\[ j_1 = 2 - k_1, \quad j_2 = 6 + k_1 \]  
(A.11)

with

\[ u_0 = \frac{5}{6} k_1 + 1, \quad \alpha = -4 \frac{k_1}{5k_1 + 6}. \]  
(A.12)

The principal families (with $-4 \leq k_1 \leq 0$) are thus

|   | $j_1$ | $j_2$ | $\alpha$ | $\beta$ | $u_0$ |
|---|---|---|---|---|---|
| ix.a | 2 | 6 | 0 | 0 | 1 |
| ix.b | 3 | 5 | 4 | -108 | $\frac{1}{6}$ |
| ix.c | 4 | 4 | -2 | $-\frac{27}{2}$ | $-\frac{3}{2}$ |
| ix.d | 5 | 3 | $-\frac{4}{3}$ | -4 | $-\frac{3}{2}$ |
| ix.e | 6 | 2 | $-\frac{8}{7}$ | $-\frac{108}{49}$ | $-\frac{7}{3}$ |

The resonance conditions are not met at level $n = 2, 3, 4, 3, 2$ respectively. The other solutions are

|   | $k_1 \leq -7$ |
|---|---|
| ix.f | ix.g | $k_1 \geq 3$ |

When $c = 3$, there are thus only 4 cases in principal families for which the Painlevé test is satisfied for the bosonic core of our multiparameter version of the SKdV equation. Those cases correspond to (I), (II), (III) and (V) in the list of section 4. There is also some other cases with negative resonances. Those cases will be classified as (VI), (VII), (VIII), (IX), (X) and (XI).
A.2. The $c = 0$ case

The recursion formulae (3.3) with $c = 0$ take the form (A.1) with

\begin{align*}
A_1^1(n) &= \left[-(n - 2)(n - 3) + 6u_0 + \beta w_0^2\right](n - 4) \\
A_2^1(n) &= \left[-(\alpha - 1)n(n + 1) + 6\alpha(n - 1) + 2\beta u_0\right](n - 4)w_0 \\
A_1^2(n) &= \left[(\alpha - 1)(n - 3) - 6\right]w_0 \\
A_2^2(n) &= -(n - 1)(n - 2)(n - 3) + (\alpha - 1)(n - 3)u_0 \\
&\quad + 6(n - 1)u_0 + \beta (n - 3)w_0^2. (A.13)
\end{align*}

The possible solutions of the resonance conditions are now listed in turn.

(i) $u_0 = 0$, $u_1 = 0$, $w_0 = 0$, $w_1$ arbitrary.

Again, this case is eliminated due to the absence of singularity but, as before, we will keep the cases (ii) and (iii) below even if $w$ is not singular.

(ii) $u_0 = 2$, $u_1 = 0$, $w_0 = 0$, $w_1 = 0$, $A(n) = (n + 1)(n - 4)(n - 6)(n^3 - 6n^2 + n - 2\alpha(n - 3))$.

Writing the last factor under the form $(n - j_1)(n - j_2)(n - j_3)$, the constants $j_1, j_2, j_3$ must satisfy

\begin{align*}
&j_1 + j_2 + j_3 = 6, \quad j_1j_2 + (j_1 + j_2)j_3 = 1 - 2\alpha, \quad j_1j_2j_3 = -6\alpha \quad (A.14)
\end{align*}

and we can choose $j_2 \leq j_3$. The second condition requires that $\alpha$ be an integer or half-integer so that the third condition allow us to choose $j_1 = 3m$ where $m$ is an integer. This leads to the equation

\begin{align*}
9m(m - 1) - 9(m - 1) + (m - 1)j_2j_3 = 8. \quad (A.15)
\end{align*}

$m - 1$ must thus be a divisor of 8. With this last condition, a case-by-case analysis leads to the only two possible solutions (which are not in principal families):

|      | $j_1$ | $j_2$ | $j_3$ | $\alpha$ |
|------|-------|-------|-------|----------|
| ii.a | -1    | 1     | 6     | 1        |
| ii.b | -3    | 2     | 7     | 7        |
Case (ii.a) can be eliminated since the resonance at level \( n = 1 \) signals a movable logarithm.

(iii) \( \alpha = 1, \quad u_0 = 2, \quad u_1 = 0, \quad w_0 = 0, \quad w_1 \text{ arbitrary}, \quad A(n) = (n + 1)^2(n - 1)(n - 4)(n - 6)^2. \)

This case is not a principal family solution but all positive resonances are verified.

(iv) \( \alpha = 0, \quad u_0 \text{ not fixed yet}, \quad u_1 = 0, \quad w_0 = k \sqrt{\frac{3}{\beta}(u_0 - 2)}, \quad w_1 = 0, \quad A(n) = (n + 1)n(n - 1)(n - 4)(n^2 - 12n^2 - \frac{1}{\beta}((3 + 5\beta)u_0 - 47\beta - 6)n - 9(1 - 3\beta)u_0 + 6\frac{3 - 10\beta}{\beta}) \)

(Recall that \( k = \pm i \)). To fix the three roots of the cubic polynomial, \( u_0 \) must be fixed so that the resonance at level \( n = 0 \) signals the presence of a movable logarithm.

(v) \( \alpha = 0, \quad u_0 = 12\left(\frac{1 - \beta}{6 - 11\beta}\right), \quad u_1 = -(3\beta + 2)\sqrt{\frac{3}{10}\left(\frac{1}{6 - 11\beta}\right)} k w_1, \quad w_0 = \sqrt{\frac{30}{6 - 11\beta}} k, \quad w_1 \text{ arbitrary}, \quad A(n) = (n + 1)n(n - 1)(n - 4)(n^2 - 11n + \frac{126 - 336\beta}{6 - 11\beta}). \)

There is a resonance at level \( n = 0 \) but \( u_0 \) and \( w_0 \) are both fixed.

(vi) \( \alpha = \frac{(\beta u_0 - 3)u_0 + 6}{3u_0}, \quad u_0 \text{ fixed by the value of } \alpha, \quad u_1 = 0, \quad w_0 = k u_0, \quad w_1 = 0, \quad A(n) = (n + 1)(n - 1)(n - 4)(n^2 - (\frac{4}{3}\beta u_0^2 - 2u_0 - 5)n - \beta u_0^2 + 6) \]
\[ n^2 - (\frac{4}{3}\beta u_0^2 - 2u_0 + 7)n - 6u_0 + 2\beta u_0^2 + 12. \]

Writing
\[ \beta u_0^2 = k_1, \quad u_0 = \frac{1}{2} k_2 + \frac{1}{6} k_1 \]
the roots of the two quadratic polynomials \( j_1, j_2, j_3 \) and \( j_4 \) are the integers satisfying
\[ j_1 + j_2 = 5 + k_2, \quad j_1j_2 = 6 - k_1, \]
\[ j_3 + j_4 = 7 - k_2, \quad j_3j_4 = 12 + k_1 - 3k_2, \]
and we choose \( j_1 \leq j_2 \) and \( j_3 \leq j_4 \). Introducing the auxiliary integers \( k_3 \) and \( k_4 \) such that
\[ j_1 = 3 + k_2 - k_3, \quad j_2 = 2 + k_3, \]
\[ j_3 = 3 + k_4 - k_2, \quad j_4 = 4 - k_4, \]
(A.17)
with $2k_4 + 1 \leq k_2 \leq 2k_3 - 1$. The constraints can be written as

$$
k_2(k_3 + k_4 + 1) = 0
$$

$$
k_3 + k_4 = k_3^2 + k_4^2
$$

$$
k_4(k_4 + k_3 - k_3^2 - 1) = k_1(1 + k_3 + k_4)
$$

so that the only possible solution is $(k_1, k_2, k_3, k_4) = (0, 0, 1, 0)$ which correspond to

$$
(j_1, j_2, j_3, j_4, u_0, \alpha, \beta) = (2, 3, 3, 4, 0, \sim \frac{1}{u_0}, \sim \frac{1}{u_0})
$$

However, the compatibility conditions at level $n = 3$ are not satisfied.

(vii) \hspace{1cm} \alpha = \frac{3-2u_0}{u_0}, \quad \beta = -3\left(\frac{u_0-1}{u_0}\right),

$u_0$ fixed by \(\alpha\) and \(\beta\), \quad u_1 = 0,

$w_0 = k u_0$, \quad w_1 = 0,

$$
A(n) = (n + 1)(n - 1)(n - 4)(n - m)[n^2 + (m - 11)n - 2(2m - 15)]
$$

with \(m = 3(u_0 + 1)\). The two roots of the quadratic polynomial are

$$
j_1, j_2 = \frac{8 - 3u_0}{2} \mp \frac{1}{2} \sqrt{9u_0^2 - 8}
$$

(A.19)

Since \(j_1, j_2\) and \(m\) are integers, we must have \(u_0 = \frac{1}{3} k_1\) with \(k_1\) an integer. This leads to the condition

$$
9u_0^2 - 8 = k_1^2 - 8 = k_2^2
$$

(A.20)

where \(k_2\) is also an integer but this equation has a solution only when \(k_1 = \pm 3\). With \(k_1 = -3\), there should be a resonance at level \(n = 0\) so that there is a movable logarithm. The only solution is thus \(u_0 = 1\), which yields \(m = 6, j_1 = 2, j_2 = 3\) and \(\alpha = 1, \beta = 0\).

All the resonance conditions are verified: \(\varphi, w_1, w_2, w_3, u_4\) and \(u_6\) are genuine arbitrary functions. Actually, this system is the bosonic core of the \(SKdVO\) equation.

(viii) \hspace{1cm} \alpha = \frac{1}{5}(\frac{4-u_0}{u_0}), \quad \beta = \frac{6}{5}(\frac{2u_0-3}{u_0}),

$u_0$ fixed by \(\alpha\) and \(\beta\), \quad u_1 = -2\frac{9u_0-11}{9u_0+4} k w_1,

$w_0 = k u_0$, \quad w_1$ arbitrary,

$$
A(n) = (n + 1)(n - 1)(n - 4)(n - m)[n^2 - (11 - m)n + 2m]
$$

with \(m = \frac{6}{5}(4 - u_0)\). Writing the roots of the second order polynomial as \(j_1\) and \(j_2\), we thus have

$$
j_1 + j_2 = m - 11, \quad j_1j_2 = 2m
$$

(A.21)
and we choose \( j_1 \leq j_2 \). Elimination of \( m \) leads to the formula

\[
j_2 = \frac{26}{j_1 + 2} - 2 \quad \text{(A.22)}
\]

so that \( 2 + j_1 \) must be a divisor of 26. We thus find that the only possible solutions are

|   | \( m \) | \( j_1 \) | \( j_2 \) | \( u_0 \) | \( \alpha \) | \( \beta \) |
|---|---|---|---|---|---|---|
| viii.a | 42 | -28 | -3 | -31 | -\frac{7}{31} | -\frac{78}{961} |
| viii.b | 30 | -15 | -4 | -21 | -\frac{5}{21} | -\frac{6}{49} |
| viii.c | 0 | 0 | 11 | 4 | 0 | \frac{3}{8} |
| viii.d | -12 | -1 | 24 | 14 | -\frac{1}{7} | 15 \frac{48}{98} |

Case (viii.c) can be eliminated since there are movable logarithms at level \( n = 0 \).

(ix) \( \alpha = \frac{1}{3} \), \( \beta = 0 \),
\( u_0 = \frac{3}{2} \), \( u_1 = -\frac{2}{7} k w_1 \),
\( w_0 = k u_0 \), \( w_1 \) arbitrary,
\( A(n) = (n + 1)(n - 1)(n - 3)(n - 4)(n^2 - 8n + 6) \).

\( A(n) \) has non-integer roots.

(x) \( \beta = 0 \),
\( u_0 = \frac{2}{\alpha + 1} \), \( u_1 = 0 \),
\( w_0 = k u_0 \), \( w_1 = 0 \),
\( A(n) = (n + 1)(n - 3)(n - 4)(n - m)(n^2 + (m - 9)n + 6) \)

with \( m = 2(2 - u_0) \). The roots of the quadratic piece are

\[
j_1, j_2 = \frac{9 - m}{2} \pm \frac{1}{2} \sqrt{m^2 - 18m + 57} \quad \text{(A.23)}
\]

with \( j_1 \leq j_2 \). We can write the quantity inside the square root as

\[
m^2 - 18m + 57 = (m - 9)^2 - 24 = k_1^2 \quad \text{(A.24)}
\]

where \( k_1 \) must be an integer. In consequence, we must have \( m = 14 \) or \( m = 4 \). The choice \( m = 4, j_1 = 2, j_2 = 3 \) leads to \( \alpha \sim \infty \) and \( u_0 = w_0 = u_1 = 0 \) so that there are no singularities at all. The only possible case is thus

\[
(m, j_1, j_2, u_0, \alpha, \beta) = (14, -3, -2, -5, -\frac{7}{5}, 0) \quad \text{(A.25)}
\]

(xi) \( \alpha = 0 \), \( \beta = 0 \),
There is a resonance at level $n = 0$ but $u_0$ is already fixed and $w_0$ cannot be arbitrary since it enters in the expression of the other resonances.

\[(xii) \quad \alpha = 0, \quad \beta = 0, \quad u_0 = 2, \quad u_1 \text{ arbitrary}, \quad w_0 = \sqrt{5} k, \quad w_1 = \sqrt{5} k u_1, \quad A(n) = (n + 1)n(n - 1)(n - 4)(n^2 - 11n + 21).\]

There is a resonance at level $n = 0$ while $u_0$ and $w_0$ are both fixed and moreover $A(n)$ has non-integer roots.

For $c = 0$, we have thus found only one case in a principal family for which the bosonic core passes the test: this is case (IV) of section 4. Some other possibilities can be identified as cases (XII) through (XVII).

### Appendix B. Leading singularity and resonance equation for fermionic fields

The general recursion equations for the fermionic evolution equations can be written as (cf. (3.3)):

\[
B_1^1(n)\xi_n^{(1)} + B_2^1(n)\xi_n^{(2)} = G_n^1
\]

\[
B_1^2(n)\xi_n^{(1)} + B_2^2(n)\xi_n^{(2)} = G_n^2
\]

where

\[
B_1^1(n) = B_2^2(n) = -(n - r)(n - r - 1)(n - r - 2) - 2c u_0 + (6 - c)(n - r) u_0 + \beta(n - r - 2) w_0^2
\]

\[
B_2^1(n) = -B_2^1(n) = [c(n - r)(n - r - 1) - (6 - c)(n - r) + (\alpha - 1)(n - r - 1)(n - r - 2)] w_0
\]

$r$ is the leading singularity exponent (so that $r \leq 0$ corresponds to no singularity) and $G_n^i$ ($i = 1, 2$) are functions of $\varphi, u_0, \ldots u_n, \xi_0^{(1,2)}, \ldots \xi_{n-1}^{(1,2)}, w_0, \ldots, w_n$.

With $n = 0$, we have

\[G_0^i = 0\]

\[B_1^1(0) = r(r + 1)(r + 2) - 2c u_0 - (6 - c) r u_0 - \beta (r + 2) w_0^2\]

\[B_2^2(0) = [c r(r + 1) + (6 - c) r + (\alpha - 1)(r + 1)(r + 2)] w_0.\]
Multiplying the first equation in (B.1) with \( n = 0 \) by \( \xi_0^{(1)} \) and the second by \( \xi_0^{(2)} \) yields (using \( G_0^i = 0 \))

\[
B_1^1(0)\xi_0^{(1)}\xi_0^{(2)} = B_1^2(0)\xi_0^{(1)}\xi_0^{(2)} = 0 \tag{B.4}
\]

There are thus two possible types of solutions: either the two \( B_1^i \) coefficients vanish or \( \xi_0^{(1)}\xi_0^{(2)} = 0 \), that is:

1. \( B_1^1(0) = B_1^2(0) = 0 \) (no relation between \( \xi_0^{(1)} \) and \( \xi_0^{(2)} \)),
2. \( \xi_0^{(2)} = k_0 \xi_0^{(1)} \) (with \( k_0 \) a bosonic constant).

The leading singularity is fixed by introducing the values found in the bosonic-core analysis (corresponding to the ‘body piece’, i.e., without the nilpotent part, of the bosonic components) and verify the possible solutions for \( r \).

Before pursuing, the exact meaning of this computation should be clarified. The goal is to fix the leading singularity of the fermionic field for those 5 particular cases for which the bosonic-core analysis manifestly shows the Painlevé property. We thus look for the solutions (type-(1) or (2)) of (B.1) for the special values of the parameters \( c, \alpha, \beta, u_0 \) and \( w_0 \) given in section 4, appropriate to each possibility. The solutions with negative resonances will not be considered. Now, let us eliminate a possible source of ambiguity in our procedure: \textit{a priori}, the values of \( u_0 \) and \( w_0 \) entering in (B.1) should be those pertaining to the complete system, incorporating the fermions. However, as mentioned above, only the non-nilpotent parts are considered. The reason for this is that since the nilpotent piece can be eliminated by an appropriate multiplication, the bosonic core must also satisfy (B.4).

The solutions to case (1) are:

(I) \( r = -2 \),

(II) \( r = 0, -2 \),

(III) \( r = -2 \),

(IV) \( r = 0 \),

(V) \( r = 2, -2, -3 \).

For case (2), equations (B.1) for \( n = 0 \) can be written

\[
B_1^1(0) - k_0 B_1^2(0) = 0, \quad k_0^2 B_1^1(0) + k_0 B_1^2(0) = 0. \tag{B.5}
\]
The compatibility of these equations forces $k_0^2 = -1$ or $k_0 = \pm i$. The constant $k = \pm i$ that appear in the expression of the bosonic component $w_0$ (cf. section 4) can thus be either $k = \pm k_0$; both cases need thus to be considered (the precise relation being fixed by the resonance equations). We then find the following possible solutions for $r$:

(I) $k = +k_0$: $r = 2, -2, -3$
    $k = -k_0$: $r = 0, -1, -2$

(II) $k = +k_0$: $r = 0, -2, -4$
     $k = -k_0$: $r = 2, 0, -2$

(III) $k = +k_0$: $r = -1, -2, -3$
      $k = -k_0$: $r = 2, 0, -2$

(IV) $k = +k_0$: $r = 0, -1, -2$
      $k = -k_0$: $r = 2, 0, -5$

(V) $r = 2, -2, -3$.

Observe that the type-(1) solutions for $r$ are recovered as the intersection of the two set of solutions in each case: this is clear since in case (1) we do not assume any special relation between $k$ and $k_0$; it should then hold for all possibilities, in particular when $k = k_0$ and $-k_0$. In the following, we can thus restrict ourself to type-(2) solutions.

In order to uniquely fix the value of $r$ (and, thereby, the value of $k$ appropriate to each case), we must consider the resonance equations. Since the bosonic resonances are solutions of $A(n) = \det |A_{ij}(n)| = 0$, the fermionic resonances are necessarily given by

$$B(n) = \det |B_{ij}(n)| = 0 \quad (i, j = 1, 2). \quad (B.6)$$

Inserting the values already found for $r$, $u_0$ and $w_0$, the roots of $B(n)$ should then lead to the resonance levels for the fermionic fields. The idea is to select $r$ by requiring the corresponding polynomial $B(n)$ to have only integer roots. The explicit form of these polynomials is

(I) $r = 2$: $B(n) = n(n - 2)(n - 3)(n - 4)^2(n - 5)$
    $r = 0$: $B(n) = (n^5 - 4n^4 - n^3 + 16n^2 - 12n + 36)(n - 2)$
    $r = -1$: $B(n) = (n^5 + n^4 - 7n^3 - n^2 + 6n + 54)(n - 1)$
    $r = -2$: $B(n) = (n^5 + 6n^4 + 7n^3 - 6n^2 - 8n + 72)n$
    $r = -3$: $B(n) = (n^5 + 11n^4 + 41n^3 + 61n^2 + 30n + 90)(n + 1)$

(II) $r = 2$: $B(n) = n(n - 2)^2(n - 4)^2(n - 6)$
     $r = 0$: $B(n) = (n^4 - 4n^3 - 4n^2 + 52n - 54)n(n - 2)$

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We can already eliminate all cases for which there are non-integer roots. This leaves us with \( r = 2 \) as the only possibility for cases (I), (II) and (III) while we have some other possibilities for cases (IV) and (V). However, we argue in section 5 that for the other possibilities we can restrict to \( r = 2 \) (moreover, this amounts to restrict the study to the principal families).

The situation concerning the leading fermionic singularity is thus somewhat peculiar: we essentially keep track of all possibilities and determine the particular values which ensure integer-valued resonances. Quite interestingly, the same value for \( r \) is singled out in all cases when we restrict to principal family solutions. Actually, this value corresponds precisely to the one that follows from a naive consideration where the fermionic terms, in the bosonic evolution equations, have a dominant singular behavior comparable to that of the leading bosonic terms.

**ACKNOWLEDGEMENTS**

The work of S.B. was supported by NSERC (Canada), through an Undergraduate Research Student Award and that of P.M. by NSERC (Canada) and FCAR (Québec).

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$u_t = -u_{xxx} + 6uu_x - 3\xi \xi_{xx} \quad \xi_t = -\xi_{xxx} + 6(u\xi)_x$.

However, the change of variable $v = u - \xi \partial_x^{-1} \xi$ transforms it into (cf. D. Depireux and P. Mathieu, Phys. Lett. B 308 (1993) 272):

$v_t = -v_{xxx} + 6vv_x \quad \xi_t = -\xi_{xxx} + 6(v\xi)_x$

which is somewhat trivial (but clearly integrable) and manifestly not supersymmetric invariant.
22. Notice that even if the Painlevé analysis breaks the $O(2)$ invariance, we are still free to choose the sign of $k_0$. Clearly, in a formulation in terms of the redefined fields $\xi^{(\pm)} = \xi^{(1)} \pm i\xi^{(2)}$, we would be free to take either $\xi_0^{(+)}$ or $\xi_0^{(-)}$ as an arbitrary function, the other being null.

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