Holevo-ordering and the continuous-time limit for open Floquet dynamics

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Abstract
We consider an atomic beam reservoir as a source of quantum noise. The atoms are modelled as two-state systems and interact one-at-a-time with the system. The Floquet operators are described in terms of the Fermionic creation, annihilation and number operators associated with the two-state atom. In the limit where the time between interactions goes to zero and the interaction is suitably scaled, we show that we may obtain a causal (that is, adapted) quantum stochastic differential equation of Hudson-Parthasarathy type, driven by creation, annihilation and conservation processes. The effect of the Floquet operators in the continuous limit is exactly captured by the Holevo ordered form for the stochastic evolution.

1 Introduction
Periodically kicked quantum systems are described dynamically by applying a unitary $V$, called the Floquet operator, every $\tau$ seconds. In an open systems model, $V$ will be a unitary on a Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_R$ where $\mathcal{H}_S$ is the system’s state space and $\mathcal{H}_R$ is the state space for the environment at that particular time. Averaging over the environment leads to a dissipative reduced dynamics on $\mathcal{H}_S$. Here we address the question of an system being periodically kicked by independent environments and consider the continuous time limit $\tau \to 0$. To obtain an limit open dynamics we must re-scale the Floquet operators appropriately.

The approach which best captures the limit dynamics is that of the “time-ordered exponentials” introduced by Holevo [1]. This theory essentially deals with quantum stochastic Floquet operators and is equivalent to the usual Hudson-Parthasarathy approach [2]. We prefer the terminology Holevo ordered from for the former and Wick ordered form for the latter. This is because time ordered exponentials appear as Dyson series expansions: here the chronologically ordered terms lead to a Weyl-Stratonovich theory as opposed to the Wick-Itô of
Hudson and Parthasarathy [3]. Previously, we have established a quantum central limit for time-ordered exponentials involving reservoir fields $a_{+}^{\lambda}(t)$, $a_{-}^{\lambda}(t)$ satisfying commutation relations $[a_{-}^{\lambda}(t), a_{+}^{\lambda}(t)] = \frac{i}{\hbar} G(t)$, where $\lambda$ was a small parameter, and emission, absorption and scattering where present [4]. This is interpreted as Markovian limit where the auto-correlation time $\tau \propto \lambda^2$ vanishes. The fields $a_{+}^{\lambda}(t)$ are actually superpositions of creation/annihilation fields for fixed momenta states modulated by a $t$-dependent phase component: the Markovian approximation is then an infinite bandwidth limit.

In the present situation, the fields are in discrete time and are Kronecker-delta correlated. In this sense they are already discrete white noises. However we use the same strategy of anticipating the limit in terms of suitably scaled collective operators leading to a quantum central limit: see [5], [6] and chapter II of [7]. We have the advantage here that the dynamical updates involve operators independent of past state of the reservoir and so we avoid the finite-memory features appearing in Markovian approximations.

The limit we consider has been used to describe the open dynamics of a laser mode interacting with an atomic beam reservoir [8]. This problem has also been studied recently by Attal and Pautrat [9] and we obtain similar results to theirs. They, however, investigate the vacuum limit and use the Guichardet’s representation of Fock space processes and the toy-Fock approximation to Fock space [7]. For our purposes, we find the connection to Holevo’s formalism the most transparent. We conclude with a construction of non-vacuum limits.

2 Model For an Atomic Beam Reservoir

2.1 Open Floquet Dynamics

Let $V$ be a unitary operator on the Hilbert space $\mathfrak{h}_S \otimes \mathfrak{h}_R$ where $\mathfrak{h}_S$ is states space for a system of interest and $\mathfrak{h}_R$ the state space for its current environment. If we fix a reference density operator $\rho_R$ for the environment, then a completely positive map, $\Xi$, on the algebra $B(\mathfrak{h}_S)$ of bounded system operators is determined by

$$\text{tr}_{\mathfrak{h}_S}\{\rho_S \Xi(X)\} = \text{tr}_{\mathfrak{h}_S \otimes \mathfrak{h}_R}\{\rho_S \otimes \rho_R V^\dagger (X \otimes 1) V\}$$  (1)

If the environment was ignored, then $V$ would be referred to as a Floquet operator, particularly, when applied repeatedly. In such cases $\Xi(X) \equiv V^\dagger XV$ is a closed, and so non-dissipative, evolution.

Our aim is to study open Floquet systems and for simplicity we shall assume that the repeated applications of the Floquet operator involve copies of the same unitary however with different, independent environments.

We consider a repeated interaction strategy as a discrete-time open dynamics. At times $t = \tau, 2\tau, 3\tau, \ldots$ we have an application of a copy of the Floquet operator $V$. Let $\mathfrak{h}_{R,k}$ be the state space at time $t = k\tau$ - this will be a copy of
\[ \mathcal{H} = \bigotimes_{k=1}^{\lfloor t/\tau \rfloor} \mathfrak{h}_{R,k}, \quad \mathcal{H} = \bigotimes_{k=\lfloor t/\tau \rfloor + 1}^{\infty} \mathfrak{h}_{R,k}, \quad \mathcal{H} = \mathcal{H} \otimes \mathcal{H} \quad (2) \]

where \( \lfloor x \rfloor \) means the integer part of \( x \). (We fix a vector \( e_0 \in \mathfrak{h}_{R} \) and use this to stabilize the infinite direct product.) We refer to \( \mathcal{H} \) and \( \mathcal{H} \) as the past and future reservoir spaces respectively.

The Floquet operator to be applied at time \( t = k\tau \) will be denoted \( V_k \) and acts on the joint space \( \mathfrak{h}_{S} \otimes \mathfrak{h}_{R,k} \) but has non-trivial action only on the factors \( \mathfrak{h}_{S} \) and \( \mathfrak{h}_{R,k} \). The unitary operator \( U^{(\tau)}_t \) describing the evolution from initial time to time \( t \) is therefore

\[ U^{(\tau)}_t = V_{\lfloor t/\tau \rfloor} \cdots V_2 V_1 \quad (3) \]

It acts on \( \mathfrak{h}_{S} \otimes \mathfrak{h}_{R} \) but, of course, has trivial action on the future reservoir space. The same is true of the discrete time dynamical evolution of observables \( X \in B(\mathfrak{h}_{S}) \) given by

\[ J^{(\tau)}_t (X) = U^{(\tau)}_t (X \otimes 1) U^{(\tau)}_t \quad (4) \]

We therefore have the difference equation

\[ \frac{1}{\tau} \left( U^{(\tau)}_{t/\tau + \tau} - U^{(\tau)}_{t/\tau} \right) = \left( V_{\lfloor t/\tau \rfloor + \tau} - 1 \right) U^{(\tau)}_{t/\tau}. \quad (5) \]

Our objective is to obtain a (quantum) stochastic differential equation for limiting situation \( \tau \to 0 \). To this end, we require a reference state for the reservoir and we choose the pur state determined by the vector \( \Phi^{\tau} \) on \( \mathcal{H} \) given by

\[ \Phi^{\tau} = e_0 \otimes e_0 \otimes e_0 \otimes e_0 \cdots \]

and, since \( e_0 \) will typically be identified as the ground state on \( \mathfrak{h}_{R} \), we shall call \( \Phi^{\tau} \) the vacuum vector for the reservoir.

The situation can be describe alternatively as follows. The Hamiltonian describing the combined system and reservoir is the formal operator on \( \mathfrak{h}_{S} \otimes \mathfrak{h}_{R,k} \) given by

\[ H^{(\tau)}_t = \sum_{k=1}^{\infty} \delta(t - k\tau) H^{(\tau)}_k \quad (6) \]

where \( H^{(\tau)}_k \) acts non-trivially only on the factors \( \mathfrak{h}_{S} \) and \( \mathfrak{h}_{R,k} \). The Floquet operators are then

\[ V_k = \exp \left\{ -i\tau H^{(\tau)}_k \right\}. \quad (7) \]

Note that we include a dependence on the time-scale parameter \( \tau \) in \( H^{(\tau)}_k \) as we shall require some control over the interaction in the limit \( \tau \to 0 \).
2.2 Atomic Beam Reservoirs

One situation that we can model in this way is when the reservoir \( R \) is a beam consisting of discrete atoms, each having state space \( \mathfrak{b}_R \). The system might be a photon mode inside a cavity. The atoms pass through the cavity in a regular sequence and it is assumed that the atom-mode interaction takes place over a time period \( \tau_{\text{int}} \) shorter than the time \( \tau \) taken for a single atom to pass through the cavity: therefore the atoms are independent and at any time at most one atom interacts with the mode. We shall therefore assume that the interaction is instantaneous - that is, the system receives a “kick” from each atom.

For simplicity, the atoms are taken to be just two-level atoms with ground state \( e_0 \) and excited state \( e_1 \). The transition operator from the ground state to the excited state of the \( k \)-th atom is \( \sigma_k^+ \), and is the ampliation of \( \sigma_1^+ = |e_1\rangle \langle e_0| \); its adjoint is denoted as \( \sigma_k^- = |e_0\rangle \langle e_1| \) and the ampliation by \( \sigma_k^- \). The operators \( \sigma_k^\pm \) are Fermionic variables and satisfy the anti-commutation relation

\[
\{\sigma_k^+, \sigma_k^-\} = 0, \quad \{\sigma_k^-, \sigma_k^+\} = 1. \tag{8}
\]

The operators commute for different atoms. The Hamiltonian for the beam is (formally) \( H_R = \sum_{k=1}^{\infty} \hbar \omega \sigma_k^+ \sigma_k^- \).

The preparation procedure is the same for each atom and corresponds to an ensemble state \( \varrho \) on \( \mathfrak{b}_R \). We take the general form

\[
\varrho = p_1 \sigma^+ \sigma^- + p_0 \sigma^- \sigma^+ \tag{9}
\]

where \( p_1, p_0 \) are the probabilities to be in the excited state and ground state respectively. The vacuum state is, of course, specified by \( p_0 = 1 \).

The Hamiltonian is specified by setting

\[
H_k := \frac{1}{\tau} H_{11} \otimes \sigma_k^+ \sigma_k^- + \frac{1}{\sqrt{\tau}} H_{10} \otimes \sigma_k^+ + \frac{1}{\sqrt{\tau}} H_{01} \otimes \sigma_k^- + H_{00}. \tag{10}
\]

where we take \( H_{11} \) and \( H_{00} \) to be self-adjoint and require that \( (H_{01})^\dagger = H_{10} \). We may identify \( H_{00} \) with the free system Hamiltonian \( H_S \), while \( H_{11} \) may be considered to contain the \( H_R \) as a component. We shall assume that the operators \( H_{\alpha\beta} \) are bounded with \( H_{11} \) also bounded away from zero.

We shall also employ the following summation convention: whenever a repeated raised and lowered Greek index appears we sum the index over the values zero and one. With this convention,

\[
H_k \equiv H_{\alpha\beta} \otimes \left[ \frac{\sigma_k^+}{\sqrt{\tau}} \right]^{\alpha} \left[ \frac{\sigma_k^-}{\sqrt{\tau}} \right]^{\beta} \tag{11}
\]

were we interpret the raised index as a power: that is, \([x]^0 = 1\), \([x]^1 = x\).
2.3 The Collective Operators

We define the collective operators $A^\pm (t; \tau), \Lambda (t; \tau)$ to be

\[
A^+ (t; \tau) := \sqrt{\tau} \sum_{k=1}^{\lfloor t/\tau \rfloor} \sigma_k^+; \quad A^- (t; \tau) := \sqrt{\tau} \sum_{k=1}^{\lfloor t/\tau \rfloor} \sigma_k^-;
\]
\[
\Lambda (t; \tau) := \sum_{k=1}^{\lfloor t/\tau \rfloor} \sigma_k^+ \sigma_k^-.
\]  

(12)

For times $t, s > 0$, we have the commutation relations

\[
\begin{align*}
[A^- (t; \tau), A^+ (s; \tau)] &= \tau \left( \frac{t \wedge s}{\tau} \right) - 2\tau \Lambda (t \wedge s, \tau), \\
[A (t; \tau), A^+ (s; \tau)] &= A^+ (t \wedge s; \tau), \\
[A^- (t; \tau), \Lambda (s; \tau)] &= A^- (t \wedge s; \tau),
\end{align*}
\]

where $s \wedge t$ denotes the minimum of $s$ and $t$. In the limit where $\tau$ goes to zero while $s$ and $t$ are held fixed, we have the approximation

\[
[A^- (t; \tau), A^+ (s; \tau)] \approx t \wedge s.
\]  

(13)

This suggest that the collective fields $A^\pm (t; \tau)$ converge to Bosonic quantum Brownian motions as $\tau \to 0$ and that $\Lambda (t; \tau)$ will converge to the Bosonic conservation process.

2.4 Bosonic Noise

Let $\mathfrak{h}$ be a fixed Hilbert space. The $n$-particle Bose states take the basic form $\phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_n = \sum_{\sigma \in \mathfrak{S}_n} \phi_1 \otimes \cdots \otimes \phi_n$ where we sum over the permutation group $\mathfrak{S}_n$. The $n$-particle state space is denoted $\mathfrak{h}^{\otimes n}$ and the Bose Fock space, with one particle space $\mathfrak{h}$, is then $\Gamma^+ := \bigoplus_{n=0}^{\infty} \mathfrak{h}^{\otimes n}$ with vacuum space $\mathfrak{h}^{\otimes 0}$ spanned by a single vector $\Psi$.

The Bosonic creator, annihilator and differential second quantization fields are, respectively, the following operators on Fock space

\[
\begin{align*}
A^+ (\psi) \phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_n &= \sqrt{n+1} \psi \phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_n, \\
A^- (\psi) \phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_n &= \frac{1}{\sqrt{n+1}} \sum_j (\psi \phi_j) \phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_j \hat{\otimes} \cdots \hat{\otimes} \phi_n, \\
d\Gamma (T) \phi_1 \hat{\otimes} \cdots \hat{\otimes} \phi_n &= \sum_j \phi_1 \hat{\otimes} \cdots \hat{\otimes} (T \phi_j) \hat{\otimes} \cdots \hat{\otimes} \phi_n
\end{align*}
\]

where $\psi \in \mathfrak{h}$ and $T \in B (\mathfrak{h})$.

Now choose $\mathfrak{h} = L^2 (\mathbb{R}^+, dt)$ and set

\[
A^\pm_t := A^\pm (1_{[0, t]}); \quad \Lambda_t := d\Gamma (1_{[0, t]})
\]  

(14)
where $1_{[0,t]}$ is the characteristic function for the interval $[0,t]$ and $\hat{1}_{[0,t]}$ is the operator on $L^2(\mathbb{R}^+, dt)$ corresponding to multiplication by $1_{[0,t]}$.

An integral calculus can be built up around the processes $A_t^\pm, \Lambda_t$ and $t$ and is known as (Bosonic) quantum stochastic calculus. This allows us to consider quantum stochastic integrals of the type

$$\int_0^T F_{10}(t) \otimes dA_t^+ + F_{01}(t) \otimes dA_t^- + F_{11}(t) \otimes d\Lambda_t + F_{00}(t) \otimes dt$$

on $h_0 \otimes \Gamma_+ \left( L^2(\mathbb{R}^+, dt) \right)$ where $h_0$ is some fixed Hilbert space (termed the initial space).

We note the natural isomorphism $h_0 \otimes \Gamma_+ \left( L^2(\mathbb{R}^+, dt) \right) \cong h_0 \otimes h_t$ where $h_t = h_0 \otimes \Gamma_+ \left( L^2([0,t], dt) \right)$ and $h_t = \Gamma_+ \left( L^2((t,\infty), dt) \right)$. A family $(F_t)_t$ of operators on $h_0 \otimes \Gamma_+ \left( L^2(\mathbb{R}^+, dt) \right)$ is said to be adapted if $F_t$ acts trivially on the future space $h_t$ for each $t$.

The Leibniz rule however breaks down for this theory since products of stochastic integrals must be put to Wick order before they can be re-expressed again as stochastic integrals. The new situation is summarized by the quantum Itô rule

$$d(FG) = (dF)G + F(dG) + (dF)(dG)$$

and the quantum Itô table

$$\begin{array}{c|cccc}
\times & dA^+ & d\Lambda & dA^- & dt \\
\hline
dA^+ & 0 & 0 & 0 & 0 \\
d\Lambda & dA^+ & d\Lambda & 0 & 0 \\
dA^- & dt & dA^- & 0 & 0 \\
dt & 0 & 0 & 0 & 0 \\
\end{array}$$

It is convenient to denote the four basic processes as follows:

$$A_t^{\alpha\beta} = \begin{cases}
\Lambda_t, & (1,1); \\
A_t^+, & (1,0); \\
A_t^-, & (0,1); \\
t, & (0,0).
\end{cases}$$

The Itô table then simplifies to $dA_t^{\alpha\beta}dA_t^{\mu\nu} = 0$ except for the cases

$$dA_t^{11}dA_t^{13} = dA_t^{22}.$$ (15)

The next theorem is from [2].

**Theorem 2.1:** There exists an unique solution $U_t$ to the quantum stochastic differential equation (qsde)

$$dU_t = L_{\alpha\beta} \otimes dA_t^{\alpha\beta}, \quad U_0 = 1$$

whenever the coefficients $L_{\alpha\beta}$ are in $B(h_0)$. The solution is automatically adapted and, moreover, will be unitary provided that the coefficients take the form

$$L_{11} = W - 1; \quad L_{10} = L; \quad L_{10} = -L^\dagger W; \quad L_{00} = -iH - \frac{1}{2}L^\dagger L$$ (16)

with $W$ unitary and $H$ self-adjoint.
2.5 Convergence of the Collective Processes

Let $\phi, \psi \in L^2(\mathbb{R}^+, dt)$ be Riemann integrable and $T \in L^\infty(\mathbb{R}^+, dt)$ continuous. We define the pre-limit fields

$$A^+ (\phi, \tau) := \sqrt{\tau} \sum_k \phi(\tau k) \sigma_+^k, \quad A^- (\psi, \tau) := \sqrt{\tau} \sum_k \psi^*(\tau k) \sigma_-^k,$$

$$\Lambda (T, \tau) := \sum_{k=1}^{\lfloor t/\tau \rfloor} T(\tau k) \sigma_+^k \sigma_-^k.$$

then

$$[A^-(\psi, \tau), A^+(\phi, \tau)] = \tau \sum_k \psi^*(\tau k) \phi(\tau k) - 2\tau \sum_k \psi^*(\tau k) \phi(\tau k) \sigma_+^k \sigma_-^k.$$

which converges to $\langle \psi | \phi \rangle = \int_0^\infty \psi^*(t) \phi(t) dt$ as $\tau \to 0$ in the vacuum state $\Phi^\tau$.

More generally, we have convergence of the type

$$\lim_{\tau \to 0} \left\langle \Phi^\tau | \prod_{j=1,\ldots,n} \exp \{ i\Lambda (T_j, \tau) + iA^+ (\phi_j, \tau) + iA^- (\psi_j, \tau) \} \Phi^\tau \right\rangle$$

$$= \left\langle \Psi | \prod_{j=1,\ldots,n} \exp \{ i\Lambda (T_j) + iA^+ (\phi_j) + iA^- (\psi_j) \} \Psi \right\rangle \quad (17)$$

where $\prod_{j=1,\ldots,n} X_j = X_1 X_2 \cdots X_n$ denotes an ordered product of operators.

3 Decomposition of the Floquet operators

Let $\sigma^\pm$ be the two-level transition operators. We set

$$\mathcal{H} = H_{\alpha\beta} \otimes \left[ \frac{\sigma^+}{\sqrt{\tau}} \right]^\alpha \left[ \frac{\sigma^-}{\sqrt{\tau}} \right]^\beta$$

and assume that $\| H_{\alpha\beta} \| \leq C$. Using the anti-commutation relations, we have that

$$(\tau \mathcal{H})^2 = \left[ H_{11} \mathcal{H}_{11} + \tau H_{11} H_{00} H_{01} - \tau H_{01} H_{10} + H_{00} H_{11} \right] \sigma^+ \sigma^-$$

$$+ \sqrt{\tau} \left[ H_{11} H_{10} + \tau H_{10} H_{00} \right] \sigma^+$$

$$+ \sqrt{\tau} \left[ H_{01} H_{11} \right] \sigma^-$$

$$+ \tau \left[ H_{01} H_{10} + \tau H_{00} H_{01} \right].$$

More generally, for $m \geq 2$, we have that (recall our summation convention!)

$$(\tau \mathcal{H})^m = \tau^m H_{\alpha_n\beta_n} \cdots H_{\alpha_1\beta_1} \otimes \left[ \frac{\sigma^+}{\sqrt{\tau}} \right]^\alpha_n \left[ \frac{\sigma^-}{\sqrt{\tau}} \right]^\beta_n \cdots \left[ \frac{\sigma^+}{\sqrt{\tau}} \right]^\alpha_1 \left[ \frac{\sigma^-}{\sqrt{\tau}} \right]^\beta_1 \quad (18)$$
and, again by repeated use of the anti-commutation relations, we are lead to the form

\[(\tau \mathcal{H})^n = \tau K^{(n)}_{\alpha\beta}(\tau) \otimes \left[ \frac{\sigma^+}{\sqrt{\tau}} \right]^\alpha \left[ \frac{\sigma^-}{\sqrt{\tau}} \right]^\beta \]  \hspace{1cm} (19)

where

\[K^{(n)}_{\alpha\beta}(\tau) := H_{\alpha 1} (H_{11})^{n-2} H_{1\beta} + O(\tau).\]  \hspace{1cm} (20)

The remainder is a polynomial of degree \(m\) in \(\tau\), whose coefficients are sums of \(n\)-fold products of the \(H_{\alpha\beta}\)'s, having no constant term. Here \(O(\tau)\) means a term going to zero in operator-norm faster than \(\tau\) as \(\tau \to 0\).

We next of all compute the Floquet operator:

\[V = \exp\{-i\tau \mathcal{H}\} = 1 + \tau \{L_{\alpha\beta} + R_{\alpha\beta}(\tau)\} \otimes \left[ \frac{\sigma^+}{\sqrt{\tau}} \right]^\alpha \left[ \frac{\sigma^-}{\sqrt{\tau}} \right]^\beta.\]  \hspace{1cm} (21)

Here the \(L_{\alpha\beta}\) and \(R_{\alpha\beta}(\tau)\) are bounded operators on the system space with \(R_{\alpha\beta}(\tau) = O(\tau)\). Explicitly, the coefficients \(L_{\alpha\beta}\) are given by

\[
L_{11} = e^{-iH_{11}} - 1; \\
L_{10} = \frac{e^{-iH_{11}} - 1}{H_{11}} H_{10}; \\
L_{01} = \frac{e^{-iH_{11}} - 1}{H_{11}} H_{01}; \\
L_{00} = -iH_{00} + H_{01} \frac{e^{-iH_{11}} - 1 + iH_{11} H_{10}}{(H_{11})^2}.\]  \hspace{1cm} (22)

We remark that these coefficients take the form (16) where \(W := \exp\{-iH_{11}\}\) is unitary, \(H := H_{00} - H_{01} \frac{e^{-iH_{11}} - \sin(H_{11})}{(H_{11})^2} H_{10}\) is self-adjoint and \(L\) is bounded but otherwise arbitrary. (Note that \(\frac{e^{-\sin x} - x}{x^2} > 0\) for \(x > 0\).)

### 4 Limit For the Vacuum State

We begin by assuming that \(p_1 = 0\) and that therefore the reservoir is in the vacuum state \(\Phi^\tau\).

Let \(t > 0\), then we are interested in the unitary \(U_{t}^{(\tau)} = V_{\lfloor t/\tau \rfloor} \cdots V_2 V_1\) in the limit \(\tau \to 0\). By virtue of an uniform estimate established in the next section, we have that the components \(R_{\alpha\beta}(\tau)\) make negligible contribution in the limit \(\tau \to 0\). It is easy to see that, if \(k = \lfloor t/\tau \rfloor\) and if we ignore the negligible component, then

\[
\frac{1}{\tau} \left( U_{t+\tau}^{(\tau)} - U_{t}^{(\tau)} \right) = \frac{1}{\tau} (V_{k+1} - 1) U_{t}^{(\tau)} \\
= L_{\alpha\beta} \otimes \left[ \frac{\sigma^{+}_{k+1}}{\sqrt{\tau}} \right]^\alpha \left[ \frac{\sigma^{-}_{k+1}}{\sqrt{\tau}} \right]^\beta U_{t}^{(\tau)} + \cdots.
\]
We shall replace this by a quantum stochastic differential equation shortly. The operator $U^{(\tau)}_t$ is then represented as

$$U^{(\tau)}_t = \prod_{j=[t/\tau],\ldots,1} \left( 1 + \tau \left[ L_{\alpha\beta} + R_{\alpha\beta}(\tau) \right] \otimes \left[ \frac{\sigma^+}{\sqrt{\tau}} \right]^{\alpha(j)} \left[ \frac{\sigma^-}{\sqrt{\tau}} \right]^{\beta(j)} \right)$$  \hspace{1cm} (23)$$

**Theorem 4.1:** In the above notations, the discrete time family $\{U^{(\tau)}_t\}$ converges to quantum stochastic process $U_t$ on $\mathfrak{h}_S \otimes \Gamma_+ \left( L^2(\mathbb{R}^+, dt) \right)$ in the sense that, for all $u, v \in \mathfrak{h}_S$, integers $n, m$ and for all $\phi_j, \psi_j \in L^2(\mathbb{R}^+, dt)$ Riemann integrable, we have the uniform convergence

$$\left\langle A^+ (\phi_m, \tau) A^+ (\phi_1, \tau) u \otimes \Phi^+ | U^{(\tau)}_t A^+ (\psi_n, \tau) A^+ (\psi_1, \tau) v \otimes \Phi^\tau \right\rangle \rightarrow \left\langle A^+ (\phi_n) A^+ (\phi_1) u \otimes \Phi | U_t A^+ (\psi_n) A^+ (\psi_1) v \otimes \Phi \right\rangle$$  \hspace{1cm} (24)$$

The process $U_t$ is moreover unitary, adapted and satisfies the (quantum) stochastic differential equation

$$dU_t = L_{\alpha\beta} \otimes dA_t^{\alpha\beta} U_t, \quad U_0 = 1.$$  \hspace{1cm} (25)$$

**Remark 1:** The solution to (25) can be written as

$$U_t = \tilde{N} \exp \left\{ \int_0^t ds L_{\alpha\beta} \left[ a^+_s \right]^{\alpha} \left[ a^-_s \right]^{\beta} \right\}$$

where $a^+_s$ are quantum white noises. Here the symbol $\tilde{N}$ stands for normal ordering and we understand the formal development

$$U_t = 1 + \sum_{n=1}^{\infty} \int_{t>t_1,\ldots,\, t_n>0} dt_n \ldots dt_1 L_{\alpha_n\beta_n} \ldots L_{\alpha_1\beta_1} \times \left[ a^+_t \right]^{\alpha_n} \ldots \left[ a^+_1 \right]^{\alpha_1} \left[ a^-_t \right]^{\beta_n} \ldots \left[ a^-_1 \right]^{\beta_1}.$$ 

The connection with the Hudson-Parthasarathy notation is made by the replacements $[a^+]^\alpha K_t [a^-]^\beta$ $dt \mapsto K_t dA_t^{\alpha\beta}$ for adapted $K_t$.

**Remark 2:** There is an alternative presentation $\tilde{H}$ which we refer to as the Holevo ordered form. We write

$$U_t = \tilde{H} \exp \left\{ \int_0^t ds H_{\alpha\beta} \left[ a^+_s \right]^{\alpha} \left[ a^-_s \right]^{\beta} \right\}$$

and understand this to be the Itô qsd e

$$dU_t = \left( e^{H_{\alpha\beta} dA_t^{\alpha\beta}} - 1 \right) U_t; \quad U_0 = 1.$$
From the quantum Itô table we have that
\[
e^{H_{\alpha_1 \beta_1} \cdots A_{\alpha_n \beta_n}} - 1 = \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} L_{\alpha_1 \beta_1} \cdots L_{\alpha_n \beta_n} \cdots L_{\alpha_1} \cdots L_{\alpha_n}.
\]

We see that the Itô coefficients, \(L_{\alpha \beta}\), and Holevo coefficients, \(H_{\alpha \beta}\), are connected according to the same relations as (22).

**Remark 3:** The theorem may be restated in a more elegant fashion. The discrete unitary process \(U_t(\tau) = \tilde{T} \exp \left\{ -i \int_0^\tau H_s(\tau) \, ds \right\}\), where \(H_t(\tau) = \sum_k \delta(t - k\tau) H_k\) with \(H_k\) given by (11), converges to the continuous-time unitary process \(U_t = \tilde{H} \exp \left\{ \int_0^t ds H_{\alpha \beta} [a_+^{\alpha} a_-^{\beta}] \right\}\).

**Remark 4:** The basic estimates in section 4 of [1] serve to show the convergence of (23) to the Holevo ordered form.

Next of all we turn our attention to the Heisenberg evolution. We begin by noting that if we set \(\Delta A^{\alpha \beta} = \tau \left[ \frac{\sigma_+}{\sqrt{\tau}} \right]^{\alpha} \left[ \frac{\sigma_-}{\sqrt{\tau}} \right]^{\beta}\) then
\[
\Delta A^{\alpha 1} A^{1 \beta} = \tau \left[ \frac{\sigma_+}{\sqrt{\tau}} \right]^{\alpha} \left[ \frac{\sigma_-}{\sqrt{\tau}} \right] \times \tau \left[ \frac{\sigma_+}{\sqrt{\tau}} \right]^{1} \left[ \frac{\sigma_-}{\sqrt{\tau}} \right]^{\beta} = \Delta A^{\alpha \beta}
\]
and that otherwise \(\Delta A^{\alpha \beta} \Delta A^{\mu \nu} = O(\tau) \Delta A^{\alpha \epsilon}\). This is the discrete form of the quantum Itô table (10).

Let \(J_t^{(\tau)}(X) = U_t^{(\tau)}(X \otimes 1) U_t(\tau)\). If \(k = [t/\tau]\), then
\[
\frac{1}{\tau} \left( J_{t+\tau}^{(\tau)}(X) - J_t^{(\tau)}(X) \right) = \left\{ J_t^{(\tau)} \left( L_{\alpha \beta}^1 X \right) + J_t^{(\tau)}(X L_{\alpha \beta}) + J_t^{(\tau)} \left( L_{1 \alpha}^1 XL_{1 \beta} \right) + S_{\alpha \beta}(\tau) \right\} \otimes \left[ \frac{\sigma_+^{k+1}}{\sqrt{\tau}} \right]^{\alpha} \left[ \frac{\sigma_-^{k+1}}{\sqrt{\tau}} \right]^{\beta}
\]
where \(S_{\alpha \beta}(\tau) = O(\tau)\). Again the terms \(S_{\alpha \beta}(\tau)\) will have negligible contribution in the \(\tau \to 0\) limit. We establish the appropriate uniform estimate in the next section.

**Theorem 4.2:** In the above notations, the discrete time family \(\left\{ J_t^{(\tau)}(X) \right\} \) converges to quantum stochastic process \(J_t(X) = U_t^{(\tau)}(X \otimes 1) U_t\) on \(\mathfrak{h}_S \otimes \Gamma^+ \left( L^2(\mathbb{R}^+, dt) \right)\) in the sense that, for all \(u, v \in \mathfrak{h}_S\), integers \(n, m\) and for all \(\phi_j, \psi_j \in L^2(\mathbb{R}^+, dt)\) Riemann integrable, we have the uniform convergence
\[
\left\langle A^+(\phi_m, \tau) \cdots A^+(\phi_1, \tau) \, u \otimes \Phi^+ \mid J_t^{(\tau)}(X) \, A^+(\psi_n, \tau) \cdots A^+(\psi_1, \tau) \, v \otimes \Phi^+ \right\rangle \to \left\langle A^+(\phi_m) \cdots A^+(\phi_1) \, u \otimes \Psi \mid J_t(X) \, A^+(\psi_n) \cdots A^+(\psi_1) \, v \otimes \Psi \right\rangle. \quad (26)
\]
The process $U_t$ is moreover unitary, adapted and satisfies the (quantum) stochastic differential equation

$$dJ_t (X) = J_t (\mathcal{L}_{\alpha\beta} X) \otimes dA_{\alpha\beta}, \quad J_0 (X) = X \otimes 1$$  \hspace{1cm} (27)

where

$$\mathcal{L}_{\alpha\beta} X := L_{\alpha\beta}^\dagger X + XL_{\alpha\beta} + L_{1\alpha} X L_{1\beta}.$$  \hspace{1cm} (28)

**Remark 5** A completely positive semigroup $\{\Xi_t : t \geq 0\}$ is then defined on $B (\mathcal{H})$ by $\langle u \otimes \Psi | J_t (X) v \otimes \Psi \rangle := \langle u | \Xi_t (X) v \rangle$ and we have $\Xi_t = \exp \{ t\mathcal{L}_0 \}$ where the Lindblad generator is

$$\mathcal{L}_0 (X) = \frac{1}{2} [L^\dagger, X] L + \frac{1}{2} L^\dagger [X, L] - i [X, H]$$

where $L = \frac{e^{-iH}}{H} H_{11} - H_{10}$ and $H = H_{00} - H_{01} H_{11} H_{10}^{-1} H_{11}^\dagger.$

## 5 Uniform Estimates

We now want to obtain a norm-estimate for the series (21) based on the expansion appearing in (18).

**Lemma 5.1:** In the notations of the previous section

$$\sum_{n \geq 0} \frac{\tau^n}{n!} \sum_{\alpha, \beta \in \{0, 1\}^n} \left\| H_{\alpha_n \beta_n} \cdots H_{\alpha_1 \beta_1} \otimes \begin{bmatrix} \sigma^+ & \gamma_n \alpha_n \\ \sqrt{\gamma} & \sqrt{\gamma} \end{bmatrix} \cdots \begin{bmatrix} \sigma^+ & \gamma_1 \alpha_1 \\ \sqrt{\gamma} & \sqrt{\gamma} \end{bmatrix} \right\|$$

$$\leq \exp \{ \tau (e^C - 1) \}.$$  \hspace{1cm} (29)

(Recall that $C = \max_{\alpha, \beta} \| H_{\alpha\beta} \|$, so $e^C > 1$, and note that we have broken from our summation convention to show explicitly that we have a sum of norms.)

**Proof.** Evidently we have the bound

$$\sum_{\alpha, \beta \in \{0, 1\}^n} \left\| H_{\alpha_n \beta_n} \cdots H_{\alpha_1 \beta_1} \otimes \begin{bmatrix} \sigma^+ & \gamma_n \alpha_n \\ \sqrt{\gamma} & \sqrt{\gamma} \end{bmatrix} \cdots \begin{bmatrix} \sigma^+ & \gamma_1 \alpha_1 \\ \sqrt{\gamma} & \sqrt{\gamma} \end{bmatrix} \right\|$$

$$\leq \tau^n C^n \sum_{\alpha, \beta \in \{0, 1\}^n} \left\| \begin{bmatrix} \sigma^+ & \gamma_n \alpha_n \\ \sqrt{\gamma} & \sqrt{\gamma} \end{bmatrix} \cdots \begin{bmatrix} \sigma^+ & \gamma_1 \alpha_1 \\ \sqrt{\gamma} & \sqrt{\gamma} \end{bmatrix} \right\|.$$  \hspace{1cm} (29)

Now $\tau^n \begin{bmatrix} \frac{\sigma^+}{\sqrt{\gamma}} & \gamma_n \alpha_n \\ \frac{\sqrt{\gamma}}{\sqrt{\gamma}} & \sqrt{\gamma} \end{bmatrix} \cdots \begin{bmatrix} \frac{\sigma^+}{\sqrt{\gamma}} & \gamma_1 \alpha_1 \\ \frac{\sqrt{\gamma}}{\sqrt{\gamma}} & \sqrt{\gamma} \end{bmatrix} \begin{bmatrix} \frac{\sigma^+}{\sqrt{\gamma}} \\ \frac{\sqrt{\gamma}}{\sqrt{\gamma}} \end{bmatrix}$ can be described as follows: we have $n$ ordered vertices labelled $j = 1, \ldots, n$ and at the $k$-th vertex will be either $\sigma^+, \sqrt{\gamma} \sigma^+, \sqrt{\gamma} \sigma^-$ or $\tau$ depending on whether $(\alpha_j, \beta_j) = (1, 1), (1, 0), (0, 1)$ or $(0, 0)$ respectively. Our first objective is to put this expression to Wick order. In placing the $\sigma^+$'s to the left of the $\sigma^-$'s, we must repeatedly use the anticommutation relations. This introduces, in the usual way, the notion of pair...
contractions between a $\sigma^+$ and a $\sigma^-$ at different vertices— that is, we replace $A\sigma^+ B\sigma^- C$ with $ABC$. (We ignore the possible minus sign occurring as we want a norm estimate.)

We then have to contend with a sum over all possible pair contractions. We have at most one creator and one annihilator at each vertex. Therefore, in a typical term we shall have several vertices connected through pair contractions and these vertices form disjoint subsets of all the $n$ vertices. We also take each $(0,0)$ vertex to be a singleton set. In this way each term corresponds to a partition on the $n$ vertices into subsets. Now recall that the number of ways to partition $n$ objects into $m$ subsets is given by Stirling’s number, $S(n, m)$, of the second kind \[10\]. Each subset of the partition contributes a factor $\tau$: this is obvious for the singletons and more generally we have a subset with $\sqrt{\tau}$ for both of the terminal vertices and unity for the internal scattering vertices. We have in addition a product of the uncontracted $\sigma^\pm$ but this will have norm bounded by unity.

This leads to the following bound for (29): \[
\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=1}^{n} C^n \tau^m S(n, m) = \exp \left\{ \tau \left( e^C - 1 \right) \right\}
\] (30)

where we use the well known generating series \[11\] for the Stirling numbers. (We remark that the integer $S(n, m)$ gives the coefficient of $\lambda^m$ in the $n$-th moment of a Poisson distributed random variable with intensity $\lambda$.)

**Corollary 5.2:**

\[
\sum_{k=1}^{[t/\tau]} \sum_{n_1, \ldots, n_k \geq 0} \frac{\tau^{n_1 + \cdots + n_k}}{n_1! \cdots n_k!} \prod_{j} \sum_{\alpha, \beta \in \{0, 1\}^{n_j}} H_{\alpha n_j} H_{\beta n_j} \cdots H_{\alpha_1 n_1} H_{\beta_1 n_1} \leq \exp \left\{ t \left( e^C - 1 \right) \right\}.
\]

This establishes a uniform estimate for the series expansion of $U_t = V_{[t/\tau]} \cdots V_2 V_1$ based on the development \[21\]. We now do the same for the Heisenberg evolution. For $X \in B(\mathfrak{h}_S)$ we have

\[
V^\dagger (X \otimes 1) V = \sum_{n, n'} (-i)^{n-n'} \frac{\tau^{n+n'}}{n! n'} H_{\alpha n} H_{\alpha_1 n_1} X H_{\mu_2 n_2} H_{\nu_2 n_2} \cdots H_{\mu_1 n_1} H_{\nu_1 n_1} \otimes \left[ \sigma^+ \right]^{\alpha n} \left[ \sigma^- \right]^{\beta n} \cdots \left[ \sigma^+ \right]^{\alpha_1} \left[ \sigma^- \right]^{\beta_1} \times \left[ \sigma^+ \right]^{\mu_2 n_2} \left[ \sigma^- \right]^{\nu_2 n_2} \cdots \left[ \sigma^+ \right]^{\mu_1 n_1} \left[ \sigma^- \right]^{\nu_1 n_1}
\]
and by the previous arguments we see that the sum of the norms of these summands can be bounded by

$$\|X\| \sum_{n,n',m,m'} C_{n,m}^n C_{m,n'}^m S(n,m) S(n',m') = \|X\| \exp \left\{ 2\tau (e^C - 1) \right\}.$$  

Likewise, the series expansion of $J_t^{(\tau)}(X) = U_t^{(\tau)\dagger}(X \otimes 1) U_t^{(\tau)}$ in terms of the fundamental Hamiltonian components will be bounded by $\|X\| \exp \left\{ 2t (e^C - 1) \right\}$ uniformly.

6 Non-Vacuum State

6.1 Gaussian Case

We consider the situation where $H_{11} = 0$. Let $\langle \cdot \rangle$ be the state determined by the density matrix $\rho$ in (9). With the convention that $\sigma^z = 2 \sigma^+ \sigma^- - 1$, we set

$$\hat{\sigma}^\pm_k = \sqrt{p_0} \sigma^\pm_k \otimes 1 + \sqrt{p_1} \sigma^\mp_k \otimes \sigma^\mp_k.$$  

These operators commute for different labels $k$ and we have $\{\hat{\sigma}^+_k, \hat{\sigma}^-_k\} = 1$, $\left(\hat{\sigma}^+_k\right)^2 = 0$. Moreover, algebra generated by the $\hat{\sigma}^\pm_k$ with the pure state $e_0 \otimes e_0$ is isomorphic to the one generated by the $\sigma^\pm_k$ with mixed state $\langle \cdot \rangle$.

If we adopt the new representation then we can consider collective fields

$$\hat{A}^\pm (\phi, \tau) = \sqrt{p_0} B^\pm (\phi, \tau) + \sqrt{p_1} C^\mp (j\phi, \tau)$$

where

$$B^+ (\phi, \tau) : = \sqrt{\tau} \sum_k \phi(k\tau) \sigma^+_k \otimes 1;$$

$$C^- (\phi, \tau) : = \sqrt{\tau} \sum_k \phi(k\tau) \sigma^+_k \otimes \sigma^-_k.$$  

Here $j: \phi \mapsto \phi^*$ is the complex conjugation. We note that

$$[B^- (\phi, \tau), C^+ (\psi, \tau)] = 2\tau \sum_k \phi(k\tau)^* \psi(k\tau) \sigma^+_k \otimes \sigma^-_k$$

and that this and similar terms are negligible in the limit $\tau \to 0$ for the state $\Phi^\tau \otimes \Phi^\tau$.

We therefore find that the fields $B^\pm (\cdot, \tau)$ and $C^\mp (\cdot, \tau)$ in the state $\Phi^\tau \otimes \Phi^\tau$ converge in distribution to independent (that is, commuting) Bose fields $B^\pm (\cdot)$ and $C^\mp (\cdot)$ with the double Fock vacuum state $\Psi \otimes \Psi$. We set $\hat{A} (\cdot) = \sqrt{p_0} B^\pm (\cdot) + \sqrt{p_1} C^\mp (j\cdot)$ and find the modified Itô table

$$d\hat{A}_t^- d\hat{A}_t^+ = p_0 dB_t^- dB_t^+ = p_0 \, dt,$$

$$d\hat{A}_t^+ d\hat{A}_t^- = p_1 dC_t^- dC_t^+ = p_1 \, dt.$$  

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It is relatively easy to see that there again exists a limit process $U_t$, this time driven by the fields $\hat{A}_t^\pm$. The limit qsde will be
\[
dU_t = \left\{ H_{10} \otimes d\hat{A}_t^+ + H_{01} \otimes d\hat{A}_t^- - \left(iH_{00} + \frac{1}{2}p_0H_{01}H_{10} + \frac{1}{2}p_1H_{10}H_{01}\right) \right\} U_t
\]
which is again unitary and adapted to the noise fields $\hat{A}_t^\pm$.

### 6.2 Asymptotic Case

We consider a new orthonormal basis for $\mathfrak{h}_R$
\[
|\tilde{e}_0\rangle = \theta \sqrt{p_0}|e_0\rangle + \theta^* \sqrt{p_1}|e_1\rangle; \\
|\tilde{e}_1\rangle = -\theta^* \sqrt{p_1}|e_0\rangle + \theta \sqrt{p_0}|e_1\rangle.
\]
where $\theta$ is a complex number of unit modulus and $p_0 + p_1 = 0$. Introduce new transition operators
\[
\tilde{\sigma}^+ = |\tilde{e}_1\rangle \langle \tilde{e}_0|, \tilde{\sigma}^- = |\tilde{e}_0\rangle \langle \tilde{e}_1|
\]
The new variables $\tilde{\sigma}^\pm$ again satisfy the proper anti-commutation relations however $\tilde{\sigma}^-$ annihilates the state $\tilde{e}_0$.

We note that $\langle \tilde{e}_0|\sigma^+\sigma^-|\tilde{e}_0\rangle = p_1, \langle \tilde{e}_0|\sigma^+|\tilde{e}_0\rangle = p_0, \langle \tilde{e}_0|\sigma^-|\tilde{e}_0\rangle = \sqrt{p_1p_0}\theta^2$ and $\langle \tilde{e}_0|\sigma^-|\tilde{e}_0\rangle = \sqrt{p_1p_0}\theta^2$. We find that
\[
\sigma^+ = p_0\tilde{\sigma}^+ - p_1\tilde{\sigma}^- - \sqrt{p_1p_0}(\theta^2 + \theta^{*2}) \tilde{\sigma}^+\tilde{\sigma}^- + \sqrt{p_1p_0}\theta^2; \\
\sigma^+\sigma^- = (p_0 - p_1)\tilde{\sigma}^+\tilde{\sigma}^- - \sqrt{p_1p_0}(\theta^2 + \theta^{*2})(\tilde{\sigma}^+ + \tilde{\sigma}^-) + p_1.
\]
After some rearrangement, we obtain
\[
H_{\alpha\beta} \otimes \begin{bmatrix} \sigma^+ & \sigma^- \end{bmatrix} \begin{bmatrix} \alpha & \beta \end{bmatrix} = \tilde{H}_{\alpha\beta} \otimes \begin{bmatrix} \tilde{\sigma}^+ & \tilde{\sigma}^- \end{bmatrix} \begin{bmatrix} \alpha & \beta \end{bmatrix}
\]
where
\[
\tilde{H}_{00} = H_{00} + \sqrt{\frac{p_0p_1}{\tau}}\theta^2H_{10} + \sqrt{\frac{p_0p_1}{\tau}}\theta^{*2}H_{01} + \frac{p_1}{\tau}H_{11}; \\
\tilde{H}_{10} = p_0H_{10} - p_1H_{01} + \sqrt{\frac{p_0p_1}{\tau}}\theta^2H_{11}; \\
\tilde{H}_{01} = p_0H_{01} - p_1H_{10} + \sqrt{\frac{p_0p_1}{\tau}}\theta^{*2}H_{11}; \\
\tilde{H}_{11} = (p_0 - p_1)H_{11} - \sqrt{\frac{p_0p_1}{\tau}}(\theta^2 + \theta^{*2})H_{01} - \sqrt{\frac{p_0p_1}{\tau}}(\theta^2 + \theta^{*2})H_{10}.
\]
Let us now adopt $\tilde{e}_0$ as the new ground state and stabilizing vector. We set $\tilde{\Phi} = \tilde{e}_0 \otimes \tilde{e}_0 \otimes \cdots$. Clearly we would like to use the tilded variables in place of the un-tilded ones however there is an explosion problem associated with the
coefficients $\tilde{H}_{\alpha\beta}$. One way to resolve this is to allow the state to depend on $\tau$ by taking

$$p_1 = \gamma^2 \tau + O(\tau^2).$$  \hfill (32)

In this case we have the asymptotic behaviour

$$\tilde{H}_{00} = H_{00} + \gamma \theta^2 H_{10} + \gamma \theta^2 H_{01} + \gamma^2 H_{11} + O(\tau);$$
$$\tilde{H}_{10} = H_{10} + \gamma \theta^2 H_{11} + O(\tau);$$
$$\tilde{H}_{01} = H_{01} + \gamma \theta^2 H_{11} + O(\tau);$$
$$\tilde{H}_{11} = H_{11} + O(\tau).$$

Here the continuous limit follows by using the above forms for the $\tilde{H}_{\alpha\beta}$ and ignoring the $O(\tau)$ terms. The scaling used in (32) is necessary if we wish to obtain a Gaussian limit for the collective operators: this is related to the notion of macroscopic states in statistical mechanics [11].

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