Q-ball dynamics from atomic Bose–Einstein condensates

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Abstract. Relativistic scalar field theories with a conserved global charge $Q$ often possess (meta)stable spherically symmetric soliton solutions, called Q-balls. We elaborate on the perfect formal analogy which exists between Q-balls and spherically symmetric solitons in certain non-relativistic atomic Bose–Einstein condensates, for which the dominant interatomic interaction can be tuned attractive. In a harmonic trap, present in existing experiments, the Q-ball solution is modified in an essential way. If the trap is significantly prolonged in one direction, however, then genuine solitons do appear, and actual experimental data can be obtained for some of the Q-ball properties studied numerically in the relativistic cosmological context, such as their formation and collisions. We also suggest conditions under which the same cosmologically relevant analogies could be extended to the fully three-dimensional case.

Keywords: dark matter, baryon asymmetry

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1. Introduction

Solitons, (meta)stable non-dispersive bound states, appear in all areas of physics described by non-linear field theories. They can be considered as lumps of spatially localized matter. As is well known, solitons can be topological or non-topological in nature. Here we shall be concerned with non-topological solitons, which are defined as field configurations having the same boundary conditions at asymptotic infinity as the vacuum state. For an extensive review on non-topological solitons in both relativistic and non-relativistic field theories, see [1].

Most soliton solutions exist in one space dimension only, as required by a theorem due to Derrick [2]. The theorem can, however, be circumvented, either by including gauge fields of non-zero spin or, as will be discussed here, by considering time-dependent solutions. The bottom line is that stable spherically symmetric non-topological solitons [3, 4], also called Q-balls [5], can exist if the theory contains scalar fields with suitable self-interactions, a conserved particle number, or ‘charge’ $Q$, and the charge is carried by massive particles.

In particle physics and cosmology, Q-balls have attracted a lot of attention recently. The reason is that the requirements for their existence are satisfied by (approximately) supersymmetric theories [6], considered among the best alternatives for physics beyond the standard model. Indeed, such theories do have new scalar fields, in the form e.g. of ‘squarks’. In this case the conserved charge is the baryon number $B$ (or some combination of the baryon and lepton numbers). The precise properties of Q-balls depend on the particular model of supersymmetry breaking, but for many conceivable alternatives, supersymmetry-based Q-balls may contribute significantly to the dark matter [7] and baryon contents [8]–[10] of the Universe (for recent reviews, see [11]). Stable Q-balls can also be directly searched for in existing and planned experiments [12].

In the present paper we transport the Q-ball formalism to non-relativistic atomic Bose–Einstein condensates (BECs) which, during the last few years, have been the subject
of exciting experimental developments. Our purpose is to elaborate on the formal analogy that exists between Q-balls and various solitons in BECs (see [1, 13] for reviews). We thus discuss, on one hand, whether it might be possible to observe spherically symmetric three-dimensional (3D) Q-balls in actual BEC experiments and, on the other hand, what kind of analogies can be drawn from the already existing experiments with essentially one-dimensional (1D) solitons, to support numerical studies of their dynamics. Our hope is that these analogies might allow us to obtain some insights on the behaviour of Q-balls in cosmology also.

2. Q-balls in relativistic field theory

In order to set up the relation to the non-relativistic case, let us start by briefly reiterating the properties of non-topological solitons, or Q-balls, at zero temperature in a relativistic field theory (for a recent review see, e.g., [14]). Consider a generic field theory containing a scalar field \( \phi \) and having a global U(1) symmetry. Let us denote the scalar potential by \( U(|\phi|) \). The Minkowskian action is then

\[
S_M = \int dt \, d^3x \left[ \left| \partial_t \phi \right|^2 - |\nabla \phi|^2 - U(|\phi|) \right].
\]  (2.1)

According to the Noether theorem, the system described by this action possesses a conserved ‘charge’, \( Q \),

\[
Q = \int d^3x \left[ i \left( \phi^* \partial_t \phi - \phi \partial_t \phi^* \right) \right].
\]  (2.2)

In the non-relativistic case \( Q \) could be the number of atoms, while in the relativistic case it could be the baryon number, as in supersymmetric theories.

The question is, what kind of solutions are there for the classical equations of motion derived from \( S_M \), given some fixed value of \( Q \)? In order to answer this question it is convenient to introduce a Lagrange multiplier \( \mu \) conjugate to \( Q \), and consider the expression for the energy of the system in the sector of a fixed \( \mu \) first (see, e.g., [1, 15]). Then, a Q-ball solution has the form [3, 4]

\[
\phi(x, t) = \exp(-i\mu t)\phi(x).
\]  (2.3)

The energy related to this configuration in the given ensemble is

\[
\Omega(\mu) = \int d^3x \left[ |\nabla \phi|^2 - \mu^2 |\phi|^2 + U(|\phi|) \right].
\]  (2.4)

The chemical potential \( \mu \) is related to the total charge of the solution \( Q \) through

\[
Q = -\frac{\partial \Omega(\mu)}{\partial \mu} = 2\mu \int d^3x |\phi|^2. \tag{2.5}
\]

Finally, the energy in the sector of a fixed charge is obtained by a Legendre transform,

\[
E(Q) = \Omega(\mu) + \mu Q, \tag{2.6}
\]

where \( \mu \) is expressed in terms of \( Q \) by inverting equation (2.5).

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3 In this section we employ the conventional natural units \( \hbar = c = 1 \).

4 Often the notation \( \omega \) is used instead of \( \mu \), but in a thermodynamic sense the quantity in question is precisely the chemical potential for \( Q \), which is why we prefer this notation.
To study whether $\Omega(\mu)$ has non-trivial extrema, one writes
\begin{equation}
\phi(x) = \frac{1}{\sqrt{2}} v(x) e^{i\alpha(x)}.
\end{equation}

A spatial variation in $\alpha(x)$ costs energy, so that we may assume it a constant, and without loss of generality, choose it to vanish. Moreover, energy is minimized by a spherically symmetric configuration [16]. The corresponding profile $v(r)$ is then determined by the classical equation of motion following from $\Omega(\mu)$,
\begin{equation}
\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr}\right)v(r) = -\mu^2 v + \frac{dU(v/\sqrt{2})}{dv}.
\end{equation}

By choosing a suitable $\mu$ this system does have a non-trivial localized solution, called a Q-ball, provided that $U(v/\sqrt{2})/v^2$ has a minimum for some $v > 0$ [3]–[5]. Physically, this implies the existence of an attractive interaction which can bind a collection of elementary quanta into a condensate, or ‘lump’, of semiclassical matter.

The fact that a solution exists for the classical equation of motion does not yet guarantee that it is stable with respect to quantum fluctuations. This is the case only if the energy of the solution, $E(Q)$, is smaller than the energy of an ensemble of free particles of mass $m$ carrying the same charge:
\begin{equation}
E(Q) < mQ.
\end{equation}

Under these conditions Q-balls are absolutely stable [4]. If equation (2.9) is not satisfied, then Q-balls can still be metastable [4] but possibly long-lived (see, e.g., [17]).

These basic considerations can be refined in a number of ways. For instance, finite temperature corrections can be addressed through the grand canonical potential, $\Omega(T, V, \mu)$. It is a standard procedure to derive a Euclidean (‘imaginary time’) path integral expression for $\Omega(T, V, \mu)$, and one can generically carry out also ‘dimensional reduction’ in this expression, integrating out the non-zero Fourier modes for the dependence of the fields on the time-like coordinate. The result is just equation (2.4), only with modified parameters, containing now all relevant dependence on the temperature $T$ [9]. One can also address a wide variety of different potentials: for instance, if the potential is ‘flat’ at large $|\phi|$, modulo possible logarithmic corrections, then the energy of the solution $E(Q)$ scales as $E(Q) \sim MQ^{3/4}$ [4, 18], allowing it to satisfy equation (2.9) for $Q \gg (M/m)^4$, and making Q-balls absolutely stable. With other potentials the growth may be slower than $|\phi|^2$ only by radiatively induced logarithmic corrections [19], but it is still possible to find regions in the parameter space where Q-balls are absolutely stable [20]. Finally, it is possible to address the formation of Q-balls from the fragmentation [7, 8, 21] of an essentially homogeneous initial condensate [22] as well as, in case Q-balls are only metastable, their decays and lifetime [17], particularly at finite temperatures [7]–[10].

3. Q-balls in non-relativistic field theory

Let us now turn to the non-relativistic case appropriate for atomic BECs. As is conventional in this context, we denote the scalar field by $\psi$ instead of $\phi$, and reintroduce $\hbar$.

The ‘vacuum’ action describing the weakly interacting atoms can be written as
\begin{equation}
S_M = \int dt d^3x \left\{ i\hbar \psi^* \partial_t \psi - \frac{\hbar^2}{2m} |\nabla \psi|^2 - \left[V_0 + V(x)\right]|\psi|^2 - \frac{2\pi\hbar^2 a}{m} |\psi|^4 - \ldots \right\}.
\end{equation}
Here \( m \) is the atom mass, \( a \) is the s-wave scattering length, and \( V(\mathbf{x}) \) is a possible external potential. The conserved Noether charge corresponding to this theory is just the particle number,

\[
Q = \int d^3x \, |\psi|^2, \quad (3.2)
\]

and the energy for stationary configurations, obtained for an ensemble with a chemical potential \( \mu \) conjugate to \( Q \), is then

\[
\Omega(\mu) = \int d^3x \left\{ \frac{\hbar^2}{2m} |\nabla \psi|^2 + \left[ -\mu + V(\mathbf{x}) \right] |\psi|^2 + \frac{2\pi \hbar^2 a}{m} |\psi|^4 + \ldots \right\}. \quad (3.3)
\]

As is conventional, the chemical potential has been additively redefined here such that it contains the part \( V_0 \sim m \) in equation (3.1). The energy for a fixed charge is again obtained from

\[
E(Q) = \Omega(\mu) + \mu Q, \quad (3.4)
\]

where \( \mu \) is expressed in terms of \( Q \) by inverting equation (3.2) for a given solution \( \psi \) depending on \( \mu \). Let us remark that finite temperature effects could be taken into account in complete analogy with the relativistic case: one can again write down a Euclidean path integral expression for the grand canonical potential and carry out dimensional reduction, to arrive at an expression of precisely the form in equation (3.3), only with modified parameter values [23].

The equation of motion following from \( \Omega(\mu) \) is the (stationary) Gross–Pitaevskii equation [24],

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 - \mu + V(\mathbf{x}) + \frac{4\pi \hbar^2 a}{m} |\psi|^2 + \ldots \right] \psi = 0. \quad (3.5)
\]

We observe that (apart from trivial changes) equations (3.3), (3.5) describe precisely the same physics as equations (2.4), (2.8), if \( V(\mathbf{x}) = 0 \). Note that in the relativistic case, \( U(|\phi|) \) includes also a term quadratic in \( \phi \), \( U(|\phi|) = m^2 |\phi|^2 + \ldots \), so that \( m^2 - \mu_{\text{rel}}^2 \sim -2m \mu_{\text{non-rel}} \).

Since equations (2.4), (3.3) are equivalent for the homogeneous case \( V(\mathbf{x}) = 0 \), the conditions for the existence of (meta)stable Q-ball solutions are also equivalent: the potential needs to grow more slowly than \( |\psi|^2 \). This can be achieved if there is an attractive interaction between the atoms or, equivalently, if the s-wave scattering length is negative, \( a < 0 \). This is indeed the case for instance for the alkali vapour \(^7\text{Li} \) [25].

More generally, the magnitude of \( a \) in BECs can be tuned in a wide range, including both positive and negative values, using a magnetic field close to a so-called Feshbach resonance (see, e.g., [26]), as has been demonstrated also for \(^{23}\text{Na} \) [27], \(^{85}\text{Rb} \) [28] and \(^{133}\text{Cs} \) [29]. In the following, we thus assume that \( a < 0 \).

Obviously, setting just \( a < 0 \) in equation (3.3) is somewhat discomforting, because the theory is then not well-defined, being unbounded from below. This implies that the system tends to undergo a phase transition to the true ground state, possibly a Bose liquid [13, 30]. It is observed experimentally, however, that at least on short enough time scales a weakly interacting gaseous phase is still present, even when \( a < 0 \).
The theory in equation (3.3) can be explicitly stabilized, however, by adding higher order operators, for instance [31]

\[
\delta \Omega(\mu) = \int d^3 x \left\{ A |\nabla (\psi^* \psi)|^2 + B (\psi^* \psi)^3 \right\},
\]

(3.6)

where \( A \) parametrizes the effective range of the two-body scattering problem, and \( B \) the amplitude for three-body collisions. Relativistic Q-ball solutions in the case that \( a \) is negative but \( B \) is non-vanishing have been discussed in [32]. On the other hand, there is a range of chemical potentials where we are in the region of the ‘thick-wall approximation’, and any stabilizing terms, such as \( B \), can be neglected [15]. In the following we will for simplicity ignore \( A \) but keep \( B \), in order to understand when effects from operators such as those in equation (3.6) are important. Note that in the dilute and (almost) homogeneous limit the operator multiplied by \( B \) can be argued to be parametrically more important than that multiplied by \( A \) [31].

4. Solution in homogeneous space

Let us now consider in more detail the non-relativistic but homogeneous case, that is \( V(\mathbf{x}) \equiv 0 \) in equations (3.1), (3.3), (3.5), but \( B \neq 0 \) in equation (3.6). The solution resembles very much the relativistic one discussed in [32], the main difference being in the relation of \( \mu \) and \( Q \), but for completeness and since the solution does not appear to be widely appreciated in the atomic BEC literature, we briefly present some of its main features here, using the notation conventional in that context.

4.1. Equations of motion

As in equation (2.7), we can write the solution of equation (3.5) in the form

\[
\psi = \frac{1}{\sqrt{2}} v e^{i \alpha},
\]

(4.1)

where \( v \geq 0 \) and \( \alpha \) can be chosen to vanish. Equation (3.5) then takes the form corresponding to equation (2.8),

\[
\left[ -\frac{\hbar^2}{2m} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) - \mu + \frac{2\pi \hbar^2 a}{m} v^2 + \frac{3}{4} B v^4 \right] v = 0.
\]

(4.2)

The boundary conditions are that

\[
v'(0) = 0; \quad \lim_{r \to \infty} v(r) = \lim_{r \to \infty} v'(r) = 0.
\]

(4.3)

Since the system is over-constrained, solutions are only found for specific values of \( v(0) \).

The parametric dependences of all the properties of the solution of equations (4.2), (4.3) can easily be found out. A non-trivial solution exists for \( \mu < 0 \), and in the attractive case we are interested in here, \( a < 0 \). We can then rescale

\[
r \equiv \hat{r} a_r, \quad v \equiv \hat{v} a_v, \quad B \equiv \beta a_B,
\]

(4.4)

with

\[
a_r \equiv \sqrt{\frac{\hbar^2}{2m|\mu|}}, \quad a_v \equiv \sqrt{\frac{m|\mu|}{2\pi \hbar^2 |a|}}, \quad a_B \equiv \left( \frac{\pi \hbar^2 |a|}{m} \right)^2 \frac{1}{|\mu|},
\]

(4.5)
whereby a common factor $|\mu|$ can be dropped out from the equation. After this rescaling, equation (4.2) becomes

$$\left( \frac{d^2}{d\hat{r}^2} + \frac{2}{\hat{r}} \frac{d}{d\hat{r}} \right) \hat{\psi} = \hat{\psi} - \frac{3}{16} \beta \hat{\psi}^5. \tag{4.6}$$

Given the solution with the boundary conditions corresponding to equation (4.3), we can compute the dimensionless equivalents of equations (3.2), (3.3), (3.4):

$$Q_\beta \equiv \beta^{-1/2} \int_0^\infty d\hat{r} \ 4\pi \hat{r}^2 \cdot \frac{1}{2} \hat{\psi}^2, \tag{4.7}$$

$$\Omega_\beta \equiv \beta^{1/2} \int_0^\infty d\hat{r} \ 4\pi \hat{r}^2 \cdot \left[ \frac{1}{2} |\nabla \hat{\psi}|^2 + \frac{1}{2} \hat{\psi}^2 - \frac{1}{4} \hat{\psi}^4 + \frac{\beta}{32} \hat{\psi}^6 \right], \tag{4.8}$$

$$E_\beta \equiv \Omega_\beta - \beta Q_\beta. \tag{4.9}$$

Factors of $\beta$ have been chosen such that rescalings back to physical units contain no $|\mu|$s, other than implicitly inside the $\beta$ in $Q_\beta, E_\beta$:

$$Q(|\mu|) = \left( \frac{mB}{2\hbar^2} \right)^{1/2} \left( \frac{1}{2\pi|a|} \right)^2 Q_\beta, \tag{4.10}$$

$$E(Q) = \left( \frac{1}{2mB} \right)^{1/2} \hbar^3 \frac{1}{4m} E_\beta. \tag{4.11}$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Left: the Q-ball solution for the normalized (inverted) potential $P(\hat{\psi}) = -\hat{\psi}^2/2 + \hat{\psi}^4/4$, corresponding to equation (4.6) with $\beta = 0$. The dots are equidistant in $\hat{r}$, corresponding to uniform time intervals in the classical mechanics analogue. Right: the corresponding profiles $\hat{\psi}(\hat{r})$, for $\beta = 0.0, 0.5, 0.825$.}
\end{figure}
4.2. Exact numerical solution

It is well known (cf [4]) that a non-trivial solution exists for equation (4.6) with the boundary conditions of equation (4.3), for \( \beta \leq 1 \). (Because we want the theory to be stable, we also assume \( \beta \geq 0 \).) We may recall that the simplest way to understand this is to think of \( \hat{r} \) in equation (4.6) as a time variable, \( \hat{v} \) as a position, and to note that equation (4.6) then corresponds to the movement of a classical particle in a potential \( P(\hat{v}) = -\frac{1}{2} \hat{v}^2 + \frac{1}{4} \hat{v}^4 - \frac{\beta}{32} \hat{v}^6 \), under the influence of some friction. The situation is illustrated in figure 1 (left), for \( \beta = 0 \). A solution can be found by a simple overshooting–undershooting algorithm, and is also illustrated in figure 1. We find the range

\[
\hat{v}(0) = 4.3374 \ldots 2.0,
\]

for \( \beta = 0.0 \ldots 1.0 \), respectively. The corresponding \( Q_\beta, E_\beta \) are shown in figure 2.

4.3. Analytic considerations

Let us next discuss some analytic estimates in the regimes of small and large \( \beta \). The regime of small \( \beta \), or small \( B \), is called the ‘thick-wall’ regime: there the Q-ball resembles a lump without a separate interior and a boundary; in other words, the boundary (or ‘wall’) is as thick as the radius. Some properties of the solution, such as its behaviour at large \( \hat{r} \), have previously been discussed, e.g., in \([1, 33]\). The quantity \( Q_\beta \) behaves as

**Figure 2.** Left: the rescaled charge, \( Q_\beta \) (equation (4.7)), as a function of \( \beta \) (equation (4.4)). The two branches of solutions, obtained with small and large \( \beta \), are well visible. Right: the rescaled energy of the Q-ball, \( E_\beta \) (equation (4.9)), as a function of \( Q_\beta \) (equation (4.7)). The upper branch (corresponding to \( \beta < 0.14 \)) leads to unstable solitons, while the lower branch (\( \beta > 0.14 \)) leads to metastable (\( E_\beta > 0 \)) or stable (\( E_\beta < 0 \)) ones.
\[ Q_\beta = \hat{Q}/\beta^{1/2}, \] where \( \hat{Q} \approx 9.4486. \) From equations (4.4), (4.5), (4.10), this gives the relation

\[ Q(\mu) = \left( \frac{\hbar^2}{2m|\mu|} \right)^{1/2} \frac{\hat{Q}}{4\pi|a|}. \] (4.13)

Inserting into equation (4.4), the physical distance is obtained from \( \hat{r} \) as

\[ r = 4\pi|a|\hat{r} \cdot \frac{Q}{\hat{Q}}, \] (4.14)

and the particle number density at a given distance is given by

\[ n(r) = |\psi|^2(r) = \frac{1}{2} \bar{\psi}^2(r) \cdot \left( \frac{\hat{Q}}{Q} \right)^2 \left( \frac{1}{4\pi|a|} \right)^3. \] (4.15)

It is observed that the central density is smaller for larger particle numbers, but the size of the solution is larger. The solution remains weakly interacting even in the centre,

\[ n(r)|a|^3 \ll 1, \] (4.16)

for large values of \( Q \). Note that all dependence on \( B \) has cancelled in equations (4.13)–(4.16).

These thick-wall solutions of the equations of motion are not stable, however. Using \( Q = -\partial\Omega(\mu)/\partial\mu \) and equation (3.4), one can derive \( E(Q) \) from the \( Q(\mu) \) in equation (4.13), to obtain

\[ E(Q) = \left( \frac{\hbar\hat{Q}}{4\pi|a|} \right)^2 \frac{1}{2mQ}. \] (4.17)

Note that in contrast to the relativistic case, this energy does not contain the particle rest masses. The fact that the total binding energy \( E(Q) \) is positive, implies that this solution is an excited state. In fact, it is not even metastable: following [4], one can inspect \( Q \)-conserving field variations around the solution \( \hat{v}(\hat{r}) \), and find that there is a direction in the field space where even a small variation leads to a decrease of \( E(Q) \). Therefore, the thick-wall solutions correspond to an unstable branch, as shown in figure 2. There is, however, another solution with the same charge but a lower energy, to which we now turn.

The so-called thin-wall regime is obtained as \( \beta = |\mu|B(m/(\pi\hbar^2|a|))^2 \) approaches unity. At the same time, \( \hat{\sigma}(0) \) approaches 2.0. In this limit the core of the soliton is essentially in a homogeneous phase, corresponding to the global minimum of the theory, and has a well defined boundary, or ‘wall’, which is thin compared with the radius. Following [1, 13], this solution could also be called a droplet of Bose liquid. In this limit, clearly, the physics depends in an essential way on the value of the stabilizing terms, in our case the coefficient \( B \).

The properties of the thin-wall solution can again be found in the standard way. One may first compute the interface tension of a planar wall at \( \beta = 1, \sigma = \int_0^\hat{r} d\hat{\sigma} \sqrt{2[-\hat{P}(\hat{\sigma})]} = 2/3 \). Then \( \Omega_\beta \) may be approximated as a sum of a surface term, \( 4\pi\hat{R}^2\hat{\sigma} \), and a volume term, and extremizing this expression allows us to solve for the radius \( \hat{R} \). Consequently, \( Q_\beta \) and \( E_\beta \) are easily obtained, to leading order in \( 1 - \beta \). We find the charge

\[ Q_\beta \approx 4\pi \left( \frac{2}{3} \right)^4 \frac{1}{(1 - \beta)^3}, \] (4.18)
and the radius

$$R \approx \left( \frac{mB}{2} \right)^{1/2} \frac{1}{\pi |a|} \left( \frac{3Q\beta}{8\pi} \right)^{1/3}. \quad (4.19)$$

The binding energy becomes

$$E(Q) = -\left( \frac{\pi h^2|a|}{m} \right)^2 \frac{Q}{B} + O(Q^{2/3}), \quad (4.20)$$

and, being negative for large $Q$, the solitons are absolutely stable. At the same time, the central density becomes

$$n|a|^3 \approx \frac{\pi h^2|a|^4}{mB}, \quad (4.21)$$

independent of $Q$. Thus the interactions are no longer weak, for a small $B$.

To summarize, soliton solutions exist independent of the value of $B$. They come, however, in two branches, and the branch which remains there in the limit $B \to 0$ (thick-wall, or small $\beta$) corresponds to unstable Q-balls. Therefore stabilizing terms, for instance of the form in equation (3.6), are essential for the properties of stable atomic Q-balls, just as they are in the relativistic case. The stabilizing terms tend to lead, however, to strong interactions in the interior of the soliton, which may in fact resemble a Bose liquid rather than a dilute gas.

Finally, let us mention that according to figure 2, the Q-ball solutions have a minimum charge, $Q\beta \approx 41$, corresponding in physical units to $Q_{\text{min}} \approx 41(mB/(2\hbar^2))^{1/2}(2\pi|a|)^{-2}$. For a vanishing stabilizing term, therefore, $Q_{\text{min}} \to 0$, and the Q-ball could have any charge. Note, however, that quantum corrections become important for small $Q$ [34], and our classical analysis is no longer trustworthy. On the side of large $Q$, it has been suggested [35] that there can also exist a maximal charge, $Q_{\text{max}}$, beyond which the system undergoes a phase transition to the stable (‘Bose liquid’) phase. Whether this can happen depends on the ensemble: in our case (large volume, fixed $Q$) it cannot, because the charge density in the stable phase is so large (cf equation (4.21)) that there are simply not enough particles present to fill the whole volume with this phase: the Q-ball solution is in fact the optimal configuration, and absolutely stable.

5. Solutions in harmonic traps, and experimental data

In experiments where atomic BECs are studied, space is not homogeneous, but there is a harmonic trap, characterized by the potential $V(x)$ in equation (3.5). This modifies the solution in a qualitative way. We reiterate here the situation for spherically (‘3D’) and axially (‘1D’) symmetric potentials\(^5\). We do not present any new solutions but simply point out how the Q-ball picture fits the qualitative pattern of the condensate behaviour in these traps.

\(^5\) We follow standard terminology although it is not without the danger of some confusion: In the ‘1D’ case the trap is narrow in two directions, while in the ‘3D’ case it is narrow in three.
3D case. Condensate solutions for a (nearly) spherical trap, \( V(\mathbf{x}) \equiv m\omega_r^2 r^2 / 2 \), were obtained in \[36\]. Their essential properties are as follows. The trap has a finite width, characterized by \( l_r = \sqrt{\hbar / (m\omega_r)} \), where \( \omega_r \) could typically be in the range of 100 Hz or so. Because of a finite \( l_r \), the radius of the soliton cannot grow freely, but is restricted. Therefore, as more particles condense, the only way to accommodate them is to increase the central density. This behaviour is opposite to either of the branches discussed in section 4.3. Therefore the trapped ‘soliton’ belongs to yet a different ‘branch’ of solutions than the genuine Q-balls. Another way to express the issue is that in the genuine Q-ball solutions \( \mu < 0 \) (cf section 4), while in the trap solution \( \mu > 0 \) \[36\].

Because of the growth of the central density, trapped solutions with large charges are unstable. Indeed, once the central density increases beyond a certain limit, various losses become overwhelming \[37\], and the condensate collapses, as is also observed experimentally \[38\]. After the collapse, the condensate may start to grow again, only to experience yet another collapse later on \[39\]. The collapse happens when \( Q \sim O(l_r/|a|) \), imposing an upper limit on the charge, or particle number, in the condensate.

1D case. In the 1D case the trap has a small finite width only in two directions, but is very long in one direction. It turns out that in this case genuine solitons can be observed. The chemical potential corresponding to them is negative, as in section 4. This solution, called a ‘bright soliton’, was discussed in detail already in \[40\] \[6\].

The essential properties of the 1D solitons can easily be deduced from the Q-ball results in section 4, in the thick-wall limit \( B \rightarrow 0 \) (see also \[42, 43\] and references therein). Let us now denote by \( l_r \) the transverse width of the trap, and by \( a_r, a_v \) the scaling factors in equation (4.4). In the expression for \( Q \), then, the homogeneous 3D relation \( Q \sim \hat{Q} a^3 r a^2 v \) gets replaced with \( Q \sim \hat{Q} a_r l_r^2 a_v^2 \). Therefore, \( Q \propto |\mu|^{1/2} \). Consequently, the radius now scales as \( r \propto a_r \propto |\mu|^{-1/2} \propto 1/Q \), the central density as \( v^2 \propto a_v^2 \propto |\mu| \propto Q^2 \), the grand canonical potential as \( \Omega(\mu) \propto a_r l_r^2 a_v^2 |\mu| \propto |\mu|^{3/2} \), and the energy as \( E(Q) = \Omega(\mu) - \mu \delta \Omega(\mu)/\delta \mu \propto -\frac{1}{2} |\mu|^{3/2} \propto -Q^3 \). Because the binding energy is negative, these Q-balls are absolutely stable compared with the gaseous phase, even for \( B = 0 \). The central density grows with particle number, however, which may still cause an instability related to the practical experimental setup for large particle numbers, like for trapped 3D solitons.

Let us now note that once they have formed \[44, 45\], it is possible to study experimentally the collisions of such 1D solitons \[45\]. It is very interesting that the collision results are qualitatively similar to what has been found in numerical simulations of the relativistic case \[46\], namely that Q-balls with opposite phases repel each other \[45, 47\]. Such similarities may provide an exciting opportunity for studying supersymmetry-based post-inflationary cosmology in the laboratory.

6. Conclusions

We have emphasized in this paper that a rigorous formal analogy exists between the non-topological solitons, or Q-balls, of relativistic field theories, and three-dimensional solitons
that could be found in atomic BECs with a negative s-wave scattering length.

In order to be stable, the three-dimensional BEC soliton requires an additional stabilizing term, beyond the usual four-point interaction. The precise form of the stabilizing term is not essential, however; as an example, we have considered three-body scattering [31], the strength of which we denoted by $B$. Whether a stabilizing interaction of precisely this type could be obtained in atomic BECs with a negative scattering length, either directly or effectively as a consequence of some coupling of the atoms to external fields, remains at present an open issue. If it exists, as could be argued from general principles following [31], or if some other stabilizing mechanism takes over [13, 30] (ultimately even the hard core atomic repulsion should be sufficient), then the Q-ball is stable against decay into its quanta, the free atoms.

Without any stabilizing term, the stationary three-dimensional soliton still exists, but it has a finite lifetime. The lifetime is currently unknown, but it could be determined by solving the time-dependent Gross–Pitaevskii equation around the solution we have presented here.

These considerations have implications on both contexts in which Q-balls may appear. In the BEC case, it is an interesting question whether the spherical Q-ball solutions in a homogeneous space, which are quite different from the traditional 3D trapped ones discussed in section 5, could also be observed experimentally. This is no doubt a challenging task. In principle one could attempt to tune the trap frequency to as small a value as possible while $a > 0$, and then tune $a$ negative, to collapse an almost homogeneous BEC into a genuine Q-ball soliton. Alternatively one could start with a significantly prolonged trap holding a genuine 1D soliton, and then slowly decrease the trap frequency in the transverse directions, to try to restore spherical symmetry.

On the side of cosmology, where most of the interest in Q-balls has been in recent years, the current understanding of their dynamics is based on solving classical equations of motion. In the actual BEC experiments, of course, the system contains also quantum and thermal fluctuations, modifying the dynamics. Thus experimental results from BECs can to some extent test which features of the dynamics are robust with respect to these fluctuations. The existing 1D experiments [45, 47] are very encouraging in this respect, but it would of course be even more remarkable if they could be extended to the (almost) homogeneous 3D case, as outlined above.

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