Donaldson invariants of product ruled surfaces 
and two-dimensional gauge theories

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Using the $u$-plane integral of Moore and Witten, we derive a simple expression for the Donaldson invariants of product ruled surfaces $\Sigma_g \times S^2$, where $\Sigma_g$ is a Riemann surface of genus $g$. This expression generalizes a theorem of Morgan and Szabó for $g = 1$ to any genus $g$. We give two applications of our results: (1) We derive Thaddeus' formulae for the intersection pairings on the moduli space of rank two stable bundles over a Riemann surface. (2) We derive the eigenvalue spectrum of the Fukaya-Floer cohomology of $\Sigma_g \times S^1$.

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1. Introduction

The Donaldson invariants of smooth four-manifolds have played a very important role in physics and mathematics in the last years. Since the reformulation of Donaldson theory by Witten in terms of twisted $\mathcal{N} = 2$ Yang-Mills theory [1], the physical approach to Donaldson theory has opened unsuspected perspectives. The main breakthrough, in this respect, was the introduction of Seiberg-Witten invariants in [2] and Witten’s “magic” formula relating the Donaldson and Seiberg-Witten invariants of simply-connected four-manifolds with $b_2^+ > 1$ and of simple type. This relation was fully explained in the fundamental paper of Moore and Witten [3], which also analyzed Donaldson-Witten theory for manifolds of $b_2^+ = 1$ using the formalism of the $u$-plane integral.

The $u$-plane integral of Moore and Witten has been studied during the last two years from many different points of view [4][5][6][7][8][9][10]. One of the most interesting outcomes of this approach has been a complete understanding of Donaldson invariants of non-simply connected manifolds. The study of this problem from the point of view of the $u$-plane integral was initiated in [3] and completed in [6], where (among other things) a general wall-crossing formula for non-simply connected manifolds was derived, generalizing the results obtained in [11] using algebro-geometric methods.

1.1. Product ruled surfaces

Among non-simply connected manifolds of $b_2^+ = 1$, product ruled surfaces play an important role in Donaldson theory. These are four-manifolds of the form $\Sigma_g \times S^2$, where $\Sigma_g$ is a Riemann surface of genus $g$. In [6], a direct application of the lattice reduction technique of [3] led to explicit expressions for the Donaldson invariants of these surfaces, in the chamber where the volume of $S^2$ is small. Using these expressions and summing up the infinite number of wall-crossing terms, one can derive in principle the Donaldson-Witten generating function of product ruled surfaces in the chamber where the volume of $\Sigma_g$ is small. This was the approach followed in [6], where explicit and compact formulae were written down using Kronecker’s double identity to sum up the wall-crossing terms.

However, formulae for the Donaldson invariants based on wall-crossing tend to be ineffective when the instanton number is large. Based on physical intuition, we would expect that some of the properties of the Donaldson-Witten series will not be apparent in these kinds of expressions, which are based on calculations made in the “electric” frame. The approach based on wall-crossing formulae makes it difficult to write generating formulae,
even for lower genus, and in fact one of the motivations of this paper was to reproduce the simple result of Morgan and Szabó for $g = 1$ [12] using the $u$-plane integral.

The approach that we follow in this paper is to perform a direct calculation of the $u$-plane integral in the chamber where $\Sigma_g$ is very small. This requires a slight generalization of the computation in [3] to allow for a non-zero Stiefel-Whitney class. In this chamber, there is a very important contribution coming from the Seiberg-Witten invariants. In this case, this contribution can be also computed from the $u$-plane integral via wall-crossing, with the important difference that the number of walls is always finite. This is the main reason for the relative simplicity of our final expression, which is expressed in terms of “magnetic” variables. Therefore, together with the results given in [3], one finds two different expressions for the Donaldson invariants of product ruled surfaces. This is somewhat similar to the case of $\mathbb{C}P^2$, whose invariants were computed in [13] by summing up an infinite number of wall-crossings, and in [3] by direct evaluation of the $u$-plane integral. There is, however, one important difference: in the case of $\mathbb{C}P^2$ both expressions are expressed in terms of “electric” variables, while in this case one of them is written in “electric” variables, and the other in “magnetic” variables. Depending on the problem we are interested on, we will find useful one expression or the other.

1.2. Relation to the moduli space of stable bundles on a curve

Apart from its intrinsic interest, the importance of having explicit expressions for the Donaldson invariants of product ruled surfaces comes from their relation to other interesting moduli problems. First of all, for zero instanton number, the moduli space of instantons on $\Sigma_g \times S^2$ is nothing but the moduli space of flat connections on the Riemann surface $\Sigma_g$. This space has been extensively studied by mathematicians, and the structure of its cohomology ring has been explored using gauge-theoretic techniques, starting with the seminal paper of Atiyah and Bott [14]. Using the connection to Verlinde formula [15], Thaddeus [16] was able to compute the intersection pairings for the generators of the cohomology ring. The moduli space of flat connections can be also described by a two-dimensional version of Donaldson-Witten theory. In [17], Witten gave a physical derivation of these intersection pairings by exploiting the relation of this two-dimensional topological field theory to physical 2d Yang-Mills theory [18]. He found in fact explicit formulae for higher rank gauge groups. In this paper, we will give another derivation of Thaddeus’ formulae using the Donaldson invariants of product ruled surfaces. In a sense, our derivation can be regarded as the dimensional reduction of Donaldson-Witten theory down to two dimensions. In this case, as we are considering zero instanton number, it is preferable to use the “electric” expressions given in [3].
1.3. Relation to Fukaya-Floer cohomology

Another important application of the Donaldson invariants of product ruled surfaces is to the (Fukaya)-Floer cohomology of \( \Sigma_g \times S^1 \). The Fukaya-Floer cohomology of \( \Sigma_g \times S^1 \) gives the gluing theory for Donaldson invariants along this three-manifold, and the gluing theory can be used in turn to derive important properties of Donaldson invariants of general four-manifolds [19][20][21]. The ring structure of the Floer cohomology of \( \Sigma_g \times S^1 \) has been studied from many different points of view. In [22], an explicit presentation was obtained under the assumption that the eigenvalues can be obtained from the Donaldson invariants of \( \Sigma_g \times \Sigma_1 \). This presentation was finally derived by V. Muñoz in [23], and a partial determination of the ring structure of the Fukaya-Floer cohomology was obtained in [19]. In a sense, the information contained in the Fukaya-Floer cohomology of \( \Sigma_g \times S^1 \) is equivalent to the information contained in the Donaldson invariants of product ruled surfaces. Using our “magnetic” expression for the Donaldson invariants, it is straightforward to find the eigenvalue spectrum of the Fukaya-Floer cohomology. It is interesting to notice that this spectrum is by no means obvious from the “electric” expressions, in other words, it can not be seen in a semiclassical instanton expansion.

1.4. Relation to the quantum cohomology of the moduli space of stable bundles over curves

We have seen that the Donaldson invariants of product ruled surfaces that correspond to zero Pontriagin number give the intersection pairings on the moduli space of stable bundles over a Riemann surface. It has been argued that the Donaldson invariants with non-zero Pontriagin numbers, when computed in the chamber where \( \Sigma_g \) is small and with Stiefel-Whitney classes satisfying \((w_2(E), [\Sigma_g]) \neq 0\), give essentially the Gromov-Witten invariants of this moduli space. This was shown in [22] using the dimensional reduction of topological Yang-Mills theory on \( \Sigma_g \times S^2 \) to a type A topological sigma model whose target space was the moduli space of flat connections on the Riemann surface [0]. The relation between the Gromov-Witten invariants and the Donaldson invariants is equivalent to the Atiyah-Floer conjecture, which says that the Floer cohomology of \( \Sigma_g \times S^1 \) is isomorphic as a ring to the quantum cohomology of the moduli space of stable bundles over \( \Sigma_g \). The

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\[1 \text{ In [4] it was argued, by performing the dimensional reduction in the low-energy action, that the effective two-dimensional theory can be formulated in terms of a topological Landau-Ginzburg model, which would then give an equivalent description of the quantum cohomology in a way reminiscent of mirror symmetry. It would be interesting to check this in some detail.} \]
isomorphism of vector spaces was proved in [24], and the ring isomorphism was proved in [23]. Using this isomorphism, one can interpret our formula (5.15) for the generating function of the Donaldson invariants as a generating function for the Gromov-Witten invariants.

1.5. Organization of the paper

The organization of this paper is as follows: in section 2, we give a brief summary of the results of [6] for the Donaldson invariants of non-simply connected manifolds. In section 3, we compute the Donaldson invariants of product ruled surfaces in the chambers of small volume for $\Sigma_g$ and for $S^2$ by direct evaluation. In section 4, we derive Thaddeus’ formulae for the intersection pairing and Verlinde’s formulae. In section 5, we explain the connection to Fukaya-Floer cohomology and derive the eigenvalue spectrum.

2. Donaldson-Witten theory on non-simply connected manifolds

The Donaldson invariants of smooth, compact, oriented four-manifolds $X$ [26] are defined by using intersection theory on the moduli space of anti-self-dual connections. The cohomology classes on this space are associated to homology classes of $X$ through the slant product [26] or, in the context of topological field theory, by using the descent procedure [1]. In this paper, we will restrict ourselves to the Donaldson invariants associated to zero, one and two-homology classes. The inclusion of three-classes has been considered in [3]. Define

$$A(X) = \text{Sym}(H_0(X) \oplus H_2(X)) \otimes \wedge^* H_1(X).$$

Then, the Donaldson invariants can be regarded as functionals

$$D_{X}^{w_2(E)} : A(X) \rightarrow \mathbb{Q},$$

where $w_2(E) \in H^2(X, \mathbb{Z})$ is the second Stiefel-Whitney class of the gauge bundle. It is convenient to organize these invariants as follows. Let $\{\delta_i\}_{i=1,\ldots,b_1}$ be a basis of one-cycles, $\{\beta_i\}_{i=1,\ldots,b_1}$ the corresponding dual basis of harmonic one-forms, and $\{S_i\}_{i=1,\ldots,b_2}$ a basis of two-cycles. We introduce the formal sums

$$\delta = \sum_{i=1}^{b_1} \zeta_i \delta_i, \quad S = \sum_{j=1}^{b_2} v_j S_j,$$
where \( v_i \) are complex numbers, and \( \zeta_i \) are Grassmann variables. The generator of the 0-class will be denoted by \( x \in H_0(X, \mathbb{Z}) \). We then define the Donaldson-Witten generating function:

\[
Z_{DW}(p, \zeta_i, v_i) = D_x^{w_2(E)}(e^{px+\delta+S})
\]  

(2.4)

so that the Donaldson invariants can be read off from the expansion of the left-hand side in powers of \( p, \zeta_i \) and \( v_i \). The main result in [1] is that \( Z_{DW} \) can be understood as the generating functional of a twisted version of the \( \mathcal{N} = 2 \) supersymmetric gauge theory – with gauge group \( SU(2) \) – in four dimensions – see [1][2][27] for details. In the twisted theory one can define observables \( O(x), I_1(\delta) = \int_\delta O_1, I_2(S) = \int_S O_2 \) (where \( O_i \) are functionals of the fields of the theory) in one to one correspondence with the homology classes of \( X \), and in such a way that the generating functional

\[
\langle e^{pO+I_1(\delta)+I_2(S)} \rangle
\]

is precisely \( Z_{DW}(p, \zeta_i, v_i) \).

Based on the low-energy effective descriptions of \( \mathcal{N} = 2 \) gauge theories obtained in [28][29], Witten obtained a explicit formula for (2.4) in terms of Seiberg-Witten invariants for manifolds of \( b_2^+ > 1 \) and simple type [2]. The general framework to give a complete evaluation of (2.4) was established in [3]. The main result of Moore and Witten is an explicit expression for the generating function \( Z_{DW} \):

\[
Z_{DW} = Z_u + Z_{SW}
\]  

(2.5)

which consists of two pieces. \( Z_{SW} \) is the contribution from the moduli space \( \mathcal{M}_{SW} \) of solutions of the Seiberg-Witten monopole equations. \( Z_u \) (the \( u \)-plane integral henceforth) is the integral of a certain modular form over the fundamental domain of the group \( \Gamma^0(4) \), that is, over the quotient \( \Gamma^0(4) \backslash \mathcal{H} \), where \( \mathcal{H} \) is the upper half-plane. The explicit form of \( Z_u \) was derived in [3] for simply connected four-manifolds, and extended to the non-simply connected case in [4]. \( Z_u \) is non-vanishing only for manifolds with \( b_2^+ = 1 \), and provides a simple physical explanation of the failure of topological invariance of the Donaldson invariants on those manifolds [3].
2.1. The u-plane integral

We will start by considering the u-plane piece. We will assume for simplicity that \( b_1 \) is even, although the general story is very similar. We can assume that \( X \) has \( b_2^+ = 1 \) (otherwise the u-plane integral is zero). In this case, there is a normalized self-dual two form or period point \( \omega \), with \( \omega^2 = 1 \), which generates \( H^2_+ (X, \mathbb{R}) \simeq \mathbb{R} \). The self-dual and anti-self-dual projections of a two-form \( \lambda \) are then given by \( \lambda_+ = (\lambda, \omega) \omega, \lambda_- = \lambda - \lambda_+ \).

Another important aspect of non-simply connected four-manifolds of \( b_2^+ = 1 \) is that the image of the map

\[ \wedge : H^1 (X, \mathbb{Z}) \otimes H^1 (X, \mathbb{Z}) \rightarrow H^2 (X, \mathbb{Z}) \]  

(2.6)

is generated by a single rational cohomology class \( \Lambda \), so that for any two elements of the basis \( \{ \beta_i \}_{i=1, \ldots, b_1} \) of \( H^1 (X, \mathbb{Z}) \), \( \beta_i \wedge \beta_j = a_{ij} \Lambda \), where \( a_{ij} \) is an antisymmetric \( b_1 \times b_1 \) matrix.

As shown in [3][6], the new ingredient in the u-plane integral for non-simply connected manifolds is an integration over the Jacobian torus of \( X, T_{b_1} = H^1 (X, \mathbb{R}) / H^1 (X, \mathbb{Z}) \). There is a basis of one-forms on \( T_{b_1} \) that we will denote by \( \beta_i^\# \in H^1 (T_{b_1}, \mathbb{Z}) \), and which are dual to \( \beta_i \in H^1 (X, \mathbb{Z}) \). Notice that there is an isomorphism \( H_1 (X, \mathbb{Z}) \simeq H^1 (T_{b_1}, \mathbb{Z}) \), given by \( \delta \rightarrow \beta_i^\# \). We will then define \( \delta^\# = \sum_{i=1}^{b_1} \zeta_i \beta_i^\# \) as the image of \( \delta \) in (2.3) under this isomorphism. Finally, we introduce a symplectic two-form on \( T_{b_1} \) as

\[
\Omega = \sum_{i < j} a_{ij} \beta_i^\# \wedge \beta_j^\#
\]  

(2.7)

which does not depend on the choice of basis. This is a volume element for the torus, hence

\[
\text{vol}(T_{b_1}) = \int_{T_{b_1}} \frac{\Omega_{b_1/2}^{b_1/2}}{(b_1/2)!}.
\]  

(2.8)

We can now write the u-plane integral in the non-simply connected case:

\[
Z_u = \left\langle e^{\mu \mathcal{O}(P) + W_1 (\Gamma) + I (S)} \right\rangle_{u}^{w_2 (E)} = -4\pi i \int_{\Gamma_0 (4) \setminus \mathcal{H}} \frac{dx dy}{y^{1/2}} \int_{T_{b_1}} \tilde{f}_\infty (p, \delta, S, \tau, y) \Psi (\tilde{S}),
\]  

(2.9)

\[ ^2 \] The class \( \Lambda \) was denoted by \( \Sigma \) in [6][11].
where \( x = \text{Re}(\tau) \), \( y = \text{Im}(\tau) \). In this expression, the function \( \hat{f}_\infty(p, \delta, S, \tau, y) \) is an almost holomorphic modular form, as well as a differential form on \( T_{b1} \), given by:

\[
\hat{f}_\infty(p, \delta, S, \tau, y) = \sqrt{2} h_{b1}^{b1} q_{4}^{-3} f_{2\infty}^{-1} e^{2p u_{\infty} + S^{2} T_{\infty}} \exp \left[ 2 f_{1\infty}(S, \Lambda) \Omega + i h_{\infty}^{-1} \delta^{z} \right].
\]

(2.10)

We have denoted the intersection form in two-cohomology by \( (\ , \ ) \) and used Poincaré duality to convert cohomology classes in homology classes. \( \Psi(\tilde{S}) \) is a Siegel-Narain theta function given by:

\[
\Psi(\tilde{S}) = \exp(2\pi i \lambda_{0}^{2}) \exp \left[ -\frac{1}{8\pi y} h_{\infty}^{-2} \tilde{S}^{2} \right] \\
\cdot \sum_{\lambda \in H^{2} + \frac{1}{2} w_{2}(E)} \exp \left[ -i \pi \bar{\tau}(\lambda_{+})^{2} - i \pi \tau(\lambda_{-})^{2} + i \pi(\lambda - \lambda_{0}, w_{2}(X)) \right] \\
\cdot \exp \left[ -ih_{\infty}^{-1}(\tilde{S}_{-}, \lambda_{-}) \right] \left( \lambda_{+}, \omega \right) + \frac{i}{4\pi y} h_{\infty}^{-1}(\tilde{S}_{+}, \omega),
\]

(2.11)

where

\[
\tilde{S} = S - 16 f_{2\infty} h_{\infty}(\Lambda \otimes \Omega).
\]

(2.12)

so (2.11) is also a differential form on \( T_{b1} \), and \( 2\lambda_{0} \) is a integer lifting of \( w_{2}(E) \). Finally, in the above expressions \( u_{\infty}, T_{\infty}, h_{\infty}, f_{1\infty} \) and \( f_{2\infty} \) are modular forms defined as follows:

\[
u_{\infty} = \frac{1}{2} \vartheta_{2}^{4} + \vartheta_{3}^{4} = \frac{1}{8q^{1/4}}(1 + 20q^{1/2} - 62q + \cdots),
\]

\[
T_{\infty} = -\frac{1}{24} \left( \frac{E_{2}}{h_{\infty}^{2}} - 8u_{\infty} \right) = q^{1/4}(1 - 2q^{1/2} + 6q + \cdots),
\]

\[
h_{\infty}(\tau) = \frac{1}{2} \vartheta_{2}\vartheta_{3} = q^{1/8}(1 + 2q^{1/2} + q + \cdots),
\]

\[
f_{1\infty}(q) = \frac{2 E_{2} + \vartheta_{2}^{4} + \vartheta_{3}^{4}}{3\vartheta_{4}^{4}} = 1 + 24q^{1/2} + \cdots,
\]

\[
f_{2\infty}(q) = \frac{\vartheta_{2}\vartheta_{3}}{2\vartheta_{4}} = q^{1/8} + 18q^{5/8} + \cdots.
\]

(2.13)

In this formulae, \( q = e^{2\pi i \tau} \), and \( \vartheta_{i}, i = 2, 3, 4 \) are the Jacobi theta functions (we follow the notation in \[8\]). Notice that \( T_{\infty} \) does not transform well under modular transformations,

\[\footnote{\text{We have absorbed all the } b_{1} \text{ dependent factors in the } u \text{-plane integral of } \[6\] in the normalization of the differential forms on } T_{b1}, \text{ in order to get more compact expressions.} \]
due to the presence of the second Eisenstein series. In (2.10), we have used the related form
\[ \hat{T}_\infty = T_\infty + \frac{1}{8\pi y} h_\infty^2, \] (2.14)
which is not holomorphic but transforms well under modular transformations. We also define the related holomorphic function \( f_\infty(p, \delta, S, \tau) \) as in (2.10) but with \( T_\infty \) instead of \( \hat{T}_\infty \).

One immediate application of the \( u \)-plane integral formalism is the derivation of the wall-crossing behavior of Donaldson invariants. As explained in [3], the integral (2.9) has a discontinuous variation at the cusps of \( \Gamma^0(4) \backslash \mathcal{H} \) whenever the cohomology class \( \lambda \in H^2(X; \mathbb{Z}) \) is such the period \( \omega \cdot \lambda \) changes sign. We then say that \( \lambda \) defines a wall. The cusps are located at \( \tau = i\infty, \tau = 0, \text{ and } \tau = 2 \). The wall-crossing behavior associated to the cusp at infinity gives the wall-crossing properties of the Donaldson invariants, while the discontinuous variation of the integral at \( \tau = 0, 2 \) must cancel against the contribution to wall-crossing from the Seiberg-Witten piece \( Z_{SW} \). As shown in [3], this cancellation completely fixes the structure of \( Z_{SW} \).

The wall-crossing of the \( u \)-plane integral at \( \tau = i\infty \) can be easily derived by imitating the analysis in section 4 of [3]. The conditions for wall-crossing are \( \lambda^2 < 0 \) and \( \lambda_+ = 0 \), and one finds [3]:
\[
WC(\lambda) = -\frac{i}{2}(-1)^{(\lambda - \lambda_0, w_2(X))} e^{2\pi i \lambda^2_0} \\
\cdot \left[ q^{-\lambda^2/2 h_\infty(\tau)} b_1^{-2} \theta_4^2 f_2^{-1} \exp \left\{ 2pu_\infty + S^2 T_\infty - i(\lambda, S)/h_\infty \right\} \right. \\
\left. \cdot \int_{T^{b_1}} \exp \left( 2f_1(q)(S, \Lambda)\Omega + 16i f_2(q)(\lambda, \Lambda)\Omega + i \frac{\delta^\text{d}}{h_\infty} \right) \right] q^0. \tag{2.15}
\]

Using the \( q \)-expansion of the different modular forms, it is easy to check that the wall-crossing term is different from zero only if \( 0 > \lambda^2 \geq p_1/4 \), where \( p_1 \) is the Pontriagin number of the gauge bundle (and \( p_1 \equiv w_2(E)^2 \mod 4 \)). The expression (2.15) generalizes the wall-crossing formula of [13] to non-simply connected manifolds. Wall-crossing terms for non-simply connected manifolds were computed in [4] using algebro-geometric methods in some particular cases [4].

\footnote{In comparing to the expressions in [13] [4], one has to take into account that what they call \( \xi \) or \( \zeta \) is in fact our \( 2\lambda \).}
2.2. The Seiberg-Witten contribution

The structure of the Seiberg-Witten contribution \( Z_{SW} \) on non-simply connected four-manifolds has been studied in detail in [6], following the approach in [3]. We will not repeat the analysis here but we will write down several formulas which will be useful later.

The SW part \( Z_{SW} \) contains two pieces which correspond to the cusps at \( \tau = 0, 2 \), and is written in terms of universal modular forms and of the Seiberg-Witten invariants introduced in [2]. A crucial ingredient in the discussion of the Seiberg-Witten contribution for non-simply connected manifolds is that we have to consider generalized Seiberg-Witten invariants, which involve integration of differential forms on the moduli space of solutions to the monopole equations. These differential forms can be constructed, in the context of topological field theory, using the descent procedure, and they are associated to one-cycles in the four-manifold \( X \). Equivalently, to every element \( \beta_i \) in the basis of one-forms of \( X \) introduced above, with \( i = 1, \ldots, b_1 \), we associate a one-form \( \nu_i \) on \( M_\lambda \). The generalized Seiberg-Witten invariants are then introduced as follows. Let \( \lambda \in H^2(X, \mathbb{Z}) + w_2(X)/2 \) be a Spin\(^c\) structure on \( X \), and let \( M_\lambda \) be the corresponding Seiberg-Witten moduli space, with virtual dimension \( d_\lambda = \lambda^2 - (2\chi + 3\sigma)/4 \). We then define:

\[
SW(\lambda, \beta_1 \wedge \cdots \wedge \beta_r) = \int_{M_\lambda} \nu_1 \wedge \cdots \wedge \nu_r \wedge \frac{a_D^{d_\lambda - r}}{a_D^{2r}},
\]

where \( a_D \) is a two-form which represents the first Chern class of the universal line bundle on the moduli space. These generalized invariants (and their wall-crossing properties) have been considered in [30].

We can now write a general expression for \( Z_{SW} \) in the case of a four-manifold of \( b_2^+ = 1 \). To do this, we first introduce the following modular forms:

\[
\begin{align*}
  u_M(q_D) &= \frac{\vartheta_3^4 + \vartheta_4^4}{2(\vartheta_3 \vartheta_4)^2} = 1 + 32q_D + 256q_D^2 + \cdots, \\
  h_M(q_D) &= \frac{1}{2i} \vartheta_3 \vartheta_4 = \frac{1}{2i} (1 - 4q_D + 4q_D^2 + \cdots), \\
  f_{1M}(q_D) &= \frac{2E_2 - \vartheta_3^4 - \vartheta_4^4}{3\vartheta_2^8} = -\frac{1}{8} (1 - 6q_D + 24q_D^2 + \cdots), \\
  f_{2M}(q_D) &= \frac{\vartheta_3 \vartheta_4}{2i \vartheta_2^8} = \frac{1}{2 \vartheta_2^4} \left( \frac{1}{q_D} - 12 + 72q_D + \cdots \right), \\
  T_M(q_D) &= -\frac{1}{24} \left( \frac{E_2}{h_M^2} - 8u_M \right) = \frac{1}{2} + 8q_D + 48q_D^2 + \cdots.
\end{align*}
\]

\[\text{In the mathematical literature, Spin}^c\text{ structures are rather given by integral cohomology classes which reduce to } w_2(X) \text{ mod 2. They correspond to } 2\lambda \text{ in our notation.}\]
These modular forms are (up to the modular weight) the same modular forms as in (2.13) but evaluated at \( \tau_D = -1/\tau \), that is, \( \tau h_M(\tau) = h_\infty(-1/\tau) \), and so on. In the above expansions, we have used the dual variable \( q_D = e^{2\pi i \tau_D} \). We will denote by \( \delta_* = \sum_{i=1}^{b_1} \zeta_i \beta_i \) the formal combination of one-forms which is dual to \( \delta \) in (2.3). \( Z_{SW} \) can then be written as the sum of two terms. The first one (corresponding to the cusp at \( \tau = 0 \), or monopole cusp) is given by the following expression:

\[
Z_{SW,M} = \sum_{\lambda} \sum_{b\geq 0} \sum_{n=0}^{b} \frac{(-1)^n}{n!(b-n)!} 2^{-6n-5b+b_1/2} e^{2i\pi(\lambda_0 \cdot \lambda + \lambda_0^2)} \cdot \left[ q_D^{-\lambda^2/2} h_M^{b_1-3} \vartheta_2^{8+\sigma} (if_{1M}/f_{2M})^{(b+n-b_1)/2} \exp \left\{ 2pu_M - ih_M(S, \lambda) + S^2 T_M(u) \right\} \right. \\
\left. \cdot \left( 2f_{1M}(S, \Lambda) + 16i \left( \frac{1}{8} f_{1M} \right) f_{2M}(\lambda, \Lambda) \right) \right]^{\lambda_0} \cdot \sum_{i_p, j_p = 1}^{b_1} a_{i_1 j_1} \cdots a_{i_n j_n} \text{SW}(\lambda, \beta_{i_1} \wedge \beta_{j_1} \wedge \cdots \wedge \beta_{i_n} \wedge \beta_{j_n} \wedge \delta_{-n}^b),
\]

where the first sum is over all the Spin\(^c\)-structures on \( X \). As shown in [2], only a finite number of \( \lambda \) give non-zero Seiberg-Witten invariants (for a given metric). The second piece in \( Z_{SW} \) corresponds to the cusp at \( \tau = 2 \) (the dyon contribution) and it has exactly the same form, with the only difference that one has to use the modular forms

\[
u_{dy} = -u_M, \quad h_{dy} = ih_M, \quad f_{1dy} = f_{1M}, \quad f_{2dy} = if_{2M}, \quad T_{dy} = -T_M
\]

and include an extra factor \( \exp(-2\pi i \lambda_0^2) \). It is easy to check that

\[
Z_{SW,dy}(p, \zeta_i, v_i) = e^{-2\pi i \lambda_0^2} \int_{T^b_1} Z_{SW,M}(-p, i \zeta_i, -iv_i),
\]

in agreement with the arguments based on \( R \)-symmetry [2, 3, 27]. The structure of \( Z_{SW} \) is such that the wall-crossing behavior due to the Seiberg-Witten invariants exactly cancels the wall-crossing behavior of the \( u \)-plane integral at the cusps \( \tau = 0, 2 \). For example, at \( \tau = 0 \) one can see that (2.9) has a discontinuous behavior for \( \lambda \in H^2(X, \mathbb{Z}) + w_2(X)/2 \) (i.e. a Spin\(^c\) structure on \( X \)) such that \( \lambda^2 < 0 \), and \( \lambda \cdot \omega = 0 \). These are precisely the conditions for SW wall-crossing. The discontinuity is given by:

\[
\frac{i}{8} e^{2\pi i (\lambda_0 \cdot \lambda + \lambda_0^2)} \left[ q_D^{-\lambda^2/2} h_M^{b_1-3} \vartheta_2^{8+\sigma} \exp \left\{ 2pu_M - ih_M(S, \lambda) + S^2 T_M(u) \right\} \right. \\
\left. \cdot \int_{T^b_1} \exp \left( 2f_{1M}(S, \Lambda) \Omega + 16i f_{2M}(\lambda, \Lambda) \Omega + \frac{i}{h_M} \delta^2 \right) \right]^{\lambda_0}. \tag{2.21}
\]
If one compares this expression with (2.18), one actually recovers the general wall-crossing formulae of [30]:

$$WC(SW(\lambda, \beta_1 \wedge \cdots \wedge \beta_r)) = (-1)^{b_1 - r} \frac{b_1 - r}{(b_1 - 2)!} (\lambda, \Lambda)^{b_1 - r} \int_{T^{b_1}} \beta^\sharp_1 \wedge \cdots \wedge \beta^\sharp_r \wedge \Omega^{b_1 - r}. \quad (2.22)$$

3. Donaldson invariants of product ruled surfaces

In this section we will derive explicit results for the Donaldson invariants on product ruled surfaces, that is, on four-manifolds of the form $X = \Sigma_g \times S^2$, where $\Sigma_g$ is a Riemann surface of genus $g$. For these surfaces, $b_1 = 2g$, $b_2 = 2$, $b_2^+ = 1$, so $\sigma = 0$ and $\chi = 4 - 2b_1$. $H^2(X, \mathbb{Z})$ is generated by the cohomology classes $[S^2]$, $[\Sigma_g]$, with intersection form $H^{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (3.1)

These surfaces are Spin, therefore $w_2(X) = 0$. The basis of one forms on $\Sigma_g \times S^2$ is given by the duals to the usual symplectic basis of one cycles on $\Sigma_g$, $\delta_i$, $i = 1, \ldots, 2g$, with $\delta_i \cap \delta_{i+g} = 1$, $i = 1, \ldots, g$. The matrix $a_{ij}$ is then the symplectic matrix $J$, and $\Lambda = [S^2]$ (the Poincaré dual to the two-homology class of $S^2$). It follows that

$$\Omega = \sum_{i=1}^g \beta^\sharp_1 \wedge \beta^\sharp_{i+g}. \quad (3.2)$$

and $\text{vol}(T^{b_1}) = 1$. As it will become clear in the computation, all the Donaldson polynomials involving the cohomology classes associated to one-cycles can be expressed in terms of the $\text{Sp}(2g, \mathbb{Z})$-invariant element in $\wedge^{\text{even}}H_1(X, \mathbb{Z})$ given by $\iota = \sum_{i=1}^g \delta_i \delta_{i+g}$. This element of $A(X)$ corresponds to the degree 6 differential form on the moduli space of instantons given by:

$$\gamma = -2 \sum_{i=1}^g I(\delta_i)I(\delta_{i+g}). \quad (3.3)$$

If we write $S = s\Sigma_g + tS^2$, we see that the generating function that we want to compute is $Z_{DW}(p, r, s, t) = D_X^{w_2(E)}(e^{px+rt+s\Sigma_g+tS^2})$. In the previous section we have computed the generating functional including $\iota$. If we want to include $\iota$ in the $u$-plane integral, we just

\[\text{Up to a numerical constant that appears when one compares the normalization of the fermion fields in the twisted theory to the topological normalization of the one-forms } \nu_i. \text{ See [3] for details.}\]
take into account that the 3-class $I(\delta_1)$ on the moduli space gives $(i/h_\infty)\beta_1^\sharp$ in the $u$-plane integral. Therefore, using (3.2), we find the correspondence

$$\gamma \to \frac{2r}{h_\infty^2}\Omega,$$

(3.4)

and to obtain $Z_{DW}(p,r,s,t)$ from the above formulae we just have to change $(i/h_\infty)\delta^\sharp$ by (3.4) in the $u$-plane integral. For the Seiberg-Witten contribution, the modification is very similar.

As $b_2^+ = 1$, the generating function (or Donaldson series) for the Donaldson invariants will be given by the sum of the $u$-plane integral and the SW contributions. The resulting Donaldson polynomials are not topological invariants. In this case, it is interesting to compute the polynomials in limiting chambers, i.e., in chambers where one of the factors in the product is very small (and the other factor is then very big). Once the invariants are known in these chambers, we can compute the invariants in any other chamber by adding a sum of wall-crossings. But the most important reason to study the invariants in the limiting chambers is that, for special choices of the second Stiefel-Whitney classes, the Donaldson polynomials have a simple structure. Moreover, the connection to Fukaya-Floer theory involves the limiting chamber in which $\Sigma_g$ is small.

The limiting chambers can be analyzed in a fairly simple way using the general expression of the period point, as in [3]:

$$\omega(\theta) = \frac{1}{\sqrt{2}}(e^{\theta}[S^2] + e^{-\theta}[\Sigma_g]).$$

(3.5)

The limiting chambers are $\theta \to \pm\infty$, which correspond to the limit of small volume for $S^2$ and $\Sigma_g$, respectively. As explained in [3], in the chamber where $S^2$ is small and $\theta \to \infty$, the scalar curvature is positive and the Seiberg-Witten invariants vanish [2]. This has two important consequences: first, in this chamber the Donaldson invariants are given just by the $u$-plane integral. Second, the Seiberg-Witten invariants in any other chamber can be computed by wall-crossing and they will be given by the topological expression (2.22). In particular, the SW contribution to the Donaldson invariants will be given by wall-crossing of the $u$-plane integral at the cusps $\tau = 0, 2$, and we can use the simple expression (2.21).
3.1. Computing the u-plane integral

We start by computing the u-plane integral. We follow closely the analysis in section 8 of [3]. We first rewrite (2.9) as

\[
G(\rho) = \int_{\Gamma_0(4) \backslash \mathcal{H}} \frac{dx dy}{y^{3/2}} \int_{T^b_1} \hat{f}_\infty(p, S, \tau, y) \bar{\Theta}, \tag{3.6}
\]

where \( \hat{f}_\infty \) is given in (2.10) and \( \bar{\Theta} \) is a Siegel-Narain theta function introduced in [3]:

\[
\bar{\Theta} = \exp \left[ \frac{1}{2y} (\xi^2_+ - \xi^2_-) \right] \sum_{\lambda \in H^2 + \beta} \exp \left[ -i \pi \bar{\tau}(\lambda_+)^2 - i \pi \tau(\lambda_-)^2 - 2 \pi i (\xi, \lambda) + 2 \pi i (\lambda, \alpha) \right], \tag{3.7}
\]

with

\[
\bar{\xi} = \rho y h_{\infty} \omega + \frac{1}{2 \pi h_{\infty}} \tilde{S}_-, \quad \alpha = 0, \quad \beta = \frac{1}{2} w_2(E). \tag{3.8}
\]

\( Z_u \) is obtained from \( G \) by

\[
Z_u = (\tilde{S}, \omega) G(\rho) |_{\rho = 0} + 2 \frac{dG}{d\rho} |_{\rho = 0}. \tag{3.9}
\]

Next we bring the integral (2.9) over \( \Gamma_0(4) \backslash \mathcal{H} \) to an integral over the fundamental domain of \( SL(2, \mathbb{Z}) \). Recall that a fundamental domain for \( \Gamma_0(4) \) can be obtained from a fundamental domain \( F \) for \( SL(2, \mathbb{Z}) \) as follows

\[
\Gamma_0(4) \backslash \mathcal{H} \simeq F \cup (T \cdot F) \cup (T^2 \cdot F) \cup (S \cdot F) \cup (T^2 S \cdot F), \tag{3.10}
\]

where \( T \) and \( S \) are the standard generators of \( SL(2, \mathbb{Z}) \). The first four domains correspond to the cusp at \( \tau \to i \infty \) and will be referred to as the semiclassical cusp. The domain \( S \cdot F \) corresponds to the cusp at \( \tau = 0 \) (the monopole cusp.) The domain \( T^2 S \cdot F \) corresponds to the cusp at \( \tau = 2 \), or dyon cusp.

Taking this into account, we can bring the integral (3.6) to the form

\[
G(\rho) = \int_{\Gamma_0(4) \backslash \mathcal{H}} \frac{dx dy}{y^2} \int_{T^b_1} \hat{f} y^{1/2} \bar{\Theta} = \int_{\Gamma_0(4) \backslash \mathcal{H}} \frac{dx dy}{y^2} \int_{T^b_1} \mathcal{I}(\tau) = \int_{F} \frac{dx dy}{y^2} \sum_{I} \int_{T^b_1} \mathcal{I}_I(\tau), \tag{3.11}
\]

where \( \mathcal{I}_I = \hat{f}_I y^{1/2} \bar{\Theta}_I \) denotes the modular forms

\[
\begin{align*}
\mathcal{I}_{(\infty, 0)}(\tau) &= \mathcal{I}(\tau), & \mathcal{I}_{(\infty, 3)}(\tau) &= \mathcal{I}(\tau + 3), \\
\mathcal{I}_{(\infty, 1)}(\tau) &= \mathcal{I}(\tau + 1), & \mathcal{I}_M(\tau) &= \mathcal{I}(-1/\tau), \\
\mathcal{I}_{(\infty, 2)}(\tau) &= \mathcal{I}(\tau + 2), & \mathcal{I}_{dy}(\tau) &= \mathcal{I}(2 - 1/\tau). \tag{3.12}
\end{align*}
\]
Therefore, the integral splits into 6 integrals over the modular domain of \( SL(2, \mathbb{Z}) \). Now we can analyze each of these following the general procedure described in section 8 of [3]. From the modular properties of the theta function (3.7) – see for example appendix B of [3] – one notes that the corresponding theta functions have
\[
\begin{align*}
\alpha^I & \quad 0 \quad w_2(E)/2 \quad 0 \quad w_2(E)/2 \quad w_2(E)/2 \\
\beta^I & \quad w_2(E)/2 \quad w_2(E)/2 \quad w_2(E)/2 \quad w_2(E)/2 \quad 0 \quad 0
\end{align*}
\]
where we have taken into account that \( w_2(X) = 0 \) for product ruled surfaces. The computation of the \( u \)-plane integral proceeds as in [3], following a general strategy due to Borcherds [31], which is a generalization of standard techniques in one-loop threshold corrections in string theory – see for example [32]. This strategy is called lattice reduction. To perform the lattice reduction, one has to choose a reduction vector \( z \) in the cohomology lattice. \( z \) must be a primitive vector of zero norm, and the computation using lattice reduction will be valid in the chamber with \( (z, \omega)^2 \) very small. We have then two possible choices of \( z \) depending on the limiting chamber we choose: for small \( \Sigma_g \), one has \( z = [\Sigma_g] \), and for small \( S^2 \) one has \( z = [S^2] \). One also chooses another norm zero vector \( z' \) such that \( (z, z') = 1 \). The value of the \( u \)-plane integral depends of course on the value of the Stiefel-Whitney class, that we can write as
\[
w_2(E) = \epsilon z + \epsilon' z',
\]
where \( \epsilon, \epsilon' = 0, 1 \). We then write
\[
\beta^I = \frac{q^I}{2} z + \frac{r^I}{2} z'
\]
where \( q^I = \epsilon, r^I = \epsilon' \) for the cusps at infinity, and are zero otherwise. We write the elements in the lattice \( H^2(X, \mathbb{Z}) \) as \( cz' + nz \), where \( c, n \in \mathbb{Z} \). The contribution of the \( I \)-th cusp to the integral (3.6) reads, after a Poisson summation on \( n \),
\[
\begin{align*}
\frac{e^{2\pi i \lambda_0^2}}{2z_+^2} & \int_F \frac{dx dy}{y^2} \int_{U_+} \hat{f}_I \sum_{c,d \in \mathbb{Z}} \exp \left[ -\frac{\pi}{2yz_+^2} |(c + r^I/2) \tau + d|^2 - \frac{\pi}{yz_+^2} (\xi^I_+, z_+)(\xi^I_-, z_-) \right] \\
& \exp \left[ -\frac{\pi}{z_+^2} (\xi^I_+, z_+)(c + r^I/2) \tau + d \right] - \frac{\pi}{z_+^2} (\xi^I_-, z_-)(c + r^I/2) \tau + d \right] + \frac{\pi}{yz_+^2} (\xi^I, z)(\alpha^I, z) \\
& \exp \left[ -i\pi q^I d - \frac{\pi}{2yz_+^2} (\alpha^I, z)^2 + \frac{\pi}{yz_+^2} (\alpha^I_+, z_+)(c + r^I/2) \tau + d \right] + \frac{\pi}{yz_+^2} (\alpha^I_-, z_-)(c + r^I/2) \tau + d \right].
\end{align*}
\]
We will now analyze the $u$-plane integral in the two limiting chambers. We first consider
the case of $S^2$ small. In this case, $z = [S^2]$. The first thing to notice is that, if $\epsilon' \neq 0$,
then the cusps at infinity do not give any contribution. This is due to the non-zero $r^I$
in the exponentials in $\mathcal{I}$, and it can be proved using the analysis in section 5 of \cite{3}.
Moreover, the cusps at $\tau = 0,2$ do not contribute either, because the measure in the
$u$-plane integral goes like $q_D + \ldots$. Therefore, if $(w_2(E), [S^2]) \neq 0$, the $u$-plane integral
vanishes in the chamber where the volume of $[S^2]$ is very small. This is an example of the
vanishing theorem of \cite{3}.

We then have to consider only the case of $w_2(E) = \epsilon[S^2]$. In this case, the contribution
comes from the cusps at infinity, which have $q^I = \epsilon$. Notice that $\alpha^I = (\epsilon/2)z$ for $I = (\infty,1)$
and $(\infty,3)$. It is easy to see that the inclusion of $\alpha^I$ for these cusps is equivalent to an
extra phase $-\pi i \epsilon c$. Following \cite{3}, we now apply the unfolding technique to the integral
$\mathcal{I}$. The action of $SL(2,\mathbb{Z})$ on $c$ and $d$ has two classes of orbits: non-degenerate orbits
with $c, d$ not both zero, and the degenerate orbit with $c = d = 0$. Non-degenerate orbits
can be transformed by $SL(2,\mathbb{Z})$ to have $c = 0$, giving an integral over a strip $0 \leq x \leq 1$
in the upper half plane, together with a sum over $d \in \mathbb{Z} \setminus \{0\}$. In this case, the contribution of
the degenerate orbit can be combined with the contribution of the non-degenerate orbits.
As $c = 0$ in any case, the four cusps at infinity give the same contribution and they add to

$$-8\sqrt{2}e^{2\pi i \lambda_0^2} \left[ \int_{T^b_1} f_{\infty} h_{\infty} \sum_{d=-\infty}^{\infty} \frac{e^{-\pi i \epsilon d}}{d + A_{\infty}(z)} \right] q^0,$$  \hspace{1cm} (3.17)

where

$$A_I(z) \equiv \frac{\tilde{S}_z}{2\pi h_I}.$$  \hspace{1cm} (3.18)

In this case, as $z = [S^2]$ and therefore $(\Lambda, z) = 0$, one has

$$A_\infty(z) = \frac{s}{2\pi h_\infty}.$$  \hspace{1cm} (3.19)

Using now the identity,

$$\sum_{d=-\infty}^{\infty} \frac{e^{i \theta d}}{d + B} = 2\pi i e^{-i B \theta} \frac{e^{i B \theta}}{1 - e^{-2\pi i B}},$$  \hspace{1cm} (3.20)

which is valid for $0 \leq \theta < 2\pi$, and integrating over $T^b_1$, one finally obtains the expressions

$$Z_{g,S^2}^{w_2(E)=0} = -\frac{i}{4} \left[ (h_{\infty}^2 f_{2\infty})^{-1} e^{2p u_\infty + S^2 T_{\infty}} \left( 2 f_{1\infty} h_{\infty}^2 s + 2r \right)^g \coth \left( \frac{is}{2h_\infty} \right) \right] q^0,$$  \hspace{1cm} (3.21)
where we used (3.20) with $\epsilon = 0$, as in the original computation in [3], and

$$Z_{g,S^2}^{w_2(E)=S^2} = -\frac{1}{4} \left[ (h_{\infty}^2 f_{2\infty})^{-1} e^{2puu_{\infty}} + S^2 T_{\infty} \left( 2f_{1\infty} h_{\infty}^2 s + 2r \right)^9 \csc \left( \frac{s}{2h_{\infty}} \right) \right]_{q_0},$$

(3.22)

where we used again the identity (3.20) with $\theta = \pi$. This changes the coth in a csc. For $g = 0$, one recovers the expressions for $S^2 \times S^2$ which were obtained in [3][33].

Let us now consider the other limiting chamber, in which the volume of $\Sigma_g$ is small. We then have $z = [\Sigma_g]$. We will also restrict ourselves to the Stiefel-Whitney classes of the form $\omega_g(\Sigma^2) = [S^2] + \epsilon [\Sigma_g]$, i.e., $\epsilon' = 1$. The reason for this is simple: in this case, $r^I = 1$ for the cusps at infinity and the vanishing argument of [3] applies. Therefore, there is no contribution from these cusps. In other words, $(\beta^I, [\Sigma_g]) \neq 0$ and there is a non-zero magnetic flux through the vanishing fiber $\Sigma_g$. Notice that this does not imply that the whole $u$-plane integral is zero. As it has been already remarked at the end of section 5 in [3], the vanishing of the monopole and dyon cusps depends crucially on the behavior of the measure.

Let us then focus on the contribution from the monopole cusp (the contribution from the dyon cusp is related to that of the monopole cusp by (2.19)). It is easy to see that we can absorb the $\alpha^M$ dependence in (3.10) in a shift $d \rightarrow d - \frac{1}{2}$ plus an extra $\tau$-dependent phase $-i\pi \epsilon c$. We can now apply the unfolding procedure. The shift in $d$ is crucial to the final result. There is a subtlety here, since $c$ and $d$ come in the combination $c \tau + \tilde{d}$, where $\tilde{d} = d - \frac{1}{2}$. $SL(2,\mathbb{Z})$ elements \( \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) with $\gamma$ odd make $c$ half-integer (and hence not zero) and those with $\delta$ even result in $\tilde{d} \in \mathbb{Z}$. Both problems are related and can be solved as follows. If $\gamma$ is odd, $\epsilon' = \alpha c + \gamma(d - \frac{1}{2}) \in \mathbb{Z} - \frac{1}{2}$. Looking back to (3.10), we see that a half-integer $c$ corresponds to having a non-zero flux through the vanishing fiber, so the integral corresponding to those values of $c$ vanishes. If $\delta$ is even, then, as $\alpha \delta - \beta \gamma = 1$ (which means in particular that $\gamma$ and $\delta$ are coprime), $\gamma$ is necessarily odd, and the integral vanishes as well. So we do not get any contribution from orbits with $c \in \mathbb{Z} + \frac{1}{2}$ or $\tilde{d} \in \mathbb{Z}$. Therefore, the final result is the same as in equation (C.2) in [3] (with minor changes):

$$-2\sqrt{2} e^{2\pi i \lambda_0^2} \left[ \int_{T^4 \Sigma} f_M h_M \sum_{d=-\infty}^{\infty} \frac{1}{d - \frac{1}{2} + A_M(z)} \right]_{q_0 D},$$

(3.23)

where

$$A_M(z) = \frac{t}{2\pi h_M} - \frac{8}{\pi} f_{2M} \Omega.$$

(3.24)
and $f_M$ is the magnetic version of (2.10):

$$f_M h_M = \sqrt{\frac{2}{64}} h_M^{(g-1)} f_{2M}^{-1} e^{2\rho_M + S^2 T_M} \exp \left[ 2 \left( f_{1M} s + \frac{r}{h_M^2} \right) \Omega \right]. \quad (3.25)$$

Using the identity (3.20) with $B = A_M(z) - 1/2$, we finally obtain the final result for the $u$-plane integral in the limiting chamber where the volume of $\Sigma_g$ is small:

$$Z_{u,\Sigma_g}^{w_2(E)} = -4\sqrt{2}i e^{2\pi i \lambda_0^2} \left\{ \int_{T^{b_1}} \frac{f_M h_M}{1 + e^{-2\pi i A_M}} dq_D^0 + e^{-2\pi i \lambda_0^2} \left[ \int_{T^{b_1}} \frac{f_{dy} h_{dy}}{1 + e^{-2\pi i A_{dy}}} dq_{dy}^0 \right] \right\}. \quad (3.26)$$

To obtain an explicit expression for (3.26), we have to integrate over the Jacobian $T^{b_1}$. To do this, we have to expand the exponential involving $\Omega$ in the numerators of (3.26). Notice that

$$\frac{1}{1 + e^{t+x}} = \frac{1}{1 + e^t} + \sum_{m \geq 1} \text{Li}_{-m}(-e^t) \frac{x^m}{m!}, \quad (3.27)$$

where $\text{Li}_n$ is the polylogarithm of index $n$. We have taken into account the fact that, for negative index, the polylogarithm is given by

$$\text{Li}_{-m}(e^t) = \left( \frac{d}{dt} \right)^{|m|} \frac{1}{1 - e^t}, \quad (3.28)$$

and using this, the identity (3.27) follows immediately. One also has:

$$\text{Li}_{-m}(-1) = \frac{1}{2} E_m = -\frac{1}{m+1} (2^{m+1} - 1) B_{m+1}, \quad (3.29)$$

where $E_m$ are the Euler numbers and $B_m$ are the Bernoulli numbers. To write the final expression for the $u$-plane integral, we also define the normalized modular forms:

$$\tilde{h}_M = 2i h_M, \quad \tilde{f}_{1M} = -8 f_{1M}, \quad \tilde{f}_{2M} = 2^9 i f_{2M}. \quad (3.30)$$

The $u$-plane integral for the Stiefel-Whitney class $w_2(E) = [S^2] + \epsilon [\Sigma_g], \epsilon = 0, 1$, is then given by:

$$Z_{u,\Sigma_g}^{\epsilon} (p, r, s, t) = -2^8 (-1)^\epsilon \left[ e^{2\rho_M + 2sT_M} \sum_{m=1}^{g} \left( \frac{g}{m} \right) (-1)^m 2^{-6m} (\tilde{h}_M^2 \tilde{f}_{2M})^{m-1} \right. \left. \cdot \left( 2r + \frac{s}{16} \tilde{h}_M^2 \tilde{f}_{1M} \right) \right] q_D^0 \left[ e^{-2\rho_M - 2sT_M} \sum_{m=1}^{g} \left( \frac{g}{m} \right) (-1)^m 2^{-6m} (\tilde{h}_M^2 \tilde{f}_{2M})^{m-1} \right. \left. \cdot \left( 2ir - \frac{is}{16} \tilde{h}_M^2 \tilde{f}_{1M} \right) \right] q_D^0 \quad (3.31)$$
where we have used that $\lambda_0^2 = w_2(E)^2/4 \pmod{\mathbb{Z}} = \epsilon/2 \pmod{\mathbb{Z}}$. The first piece corresponds to the monopole cusp at $\tau = 0$, while the second one corresponds to the dyon cusp at $\tau = 2$. Notice that in the above expression we have not included the term $m = 0$ in the sum which comes from the expansion (3.27). This is due to the fact that this term has an overall $\tilde{f}_M^{-1} = q_D + \cdots$. As the rest of the modular forms are analytic in $q_D$, there can not be any $q_D^0$ term in the expansion, so this term does not contribute.

3.2. The Seiberg-Witten contribution

Let us now turn to the Seiberg-Witten contribution. As we have already remarked, the Seiberg-Witten invariants vanish in the chamber of small volume for $S^2$, and the Seiberg-Witten invariants in the chamber where $\text{vol}(\Sigma_g) \to 0$ are given by the sum of wall-crossing terms of the form (2.22). But these terms match the wall-crossings at the cusps $\tau = 0, 2$ of the $u$-plane integral. Therefore, the SW contribution to the Donaldson invariants is given by the sum of wall-crossings (2.21). Which walls do we cross in going from the limiting chamber where $\text{vol}(S^2) \to 0$ to the chamber where $\text{vol}(\Sigma_g) \to 0$? Any $\lambda$ with $\lambda^2 < 0$ and $d_\lambda = \lambda^2 - (2\chi + 3\sigma)/4 = \lambda^2 + 2(g - 1) \geq 0$ defines a wall. If we set $\lambda = -b[\Sigma_g] + a[S^2]$, $a, b \in \mathbb{Z}$, these conditions give:

$$0 < ab \leq g - 1.$$  \hfill (3.32)

We then see that, for a fixed genus $g$, there is only a finite number of walls for the Seiberg-Witten contribution. This is in sharp contrast with the wall-crossing terms coming from the cusps at infinity, analyzed in [6], where there is an infinite number of walls. Notice that $\lambda$ and $-\lambda$ define the same wall, but the wall-crossing term has opposite sign. We then obtain the following expression for the Seiberg-Witten contribution:

$$Z_{SW,\Sigma_g}^c(p, r, s, t) = 2^8(-1)^c \sum_{0 < ab \leq g - 1} \left[ \text{sgn}(a)q_D^{ab}(-1)^{ae+b}(\tilde{h}_M^{-2}\tilde{f}_M^{-1})^{-1}e^{2pu_M+2stT_M} \right. $$

$$\cdot \left( \left( \frac{s}{16}\tilde{h}_M^{2}\tilde{f}_M^{1} + 2^{-7}b\tilde{h}_M^{2}\tilde{f}_M^{2} + 2r \right) e^{\frac{2\alpha+2\lambda}{\tilde{h}_M}} \right) q_D^0 $$

$$+ 2^8i^{1-g} \sum_{0 < ab \leq g - 1} \left[ \text{sgn}(a)q_D^{ab}(-1)^{ae+b}(\tilde{h}_M^{-2}\tilde{f}_M^{-1})^{-1}e^{-2pu_M-2stT_M} \right. $$

$$\cdot \left( \left( -\frac{is}{16}\tilde{h}_M^{2}\tilde{f}_M^{1} + 2^{-7}b\tilde{h}_M^{2}\tilde{f}_M^{2} + 2ir \right) e^{\frac{2\alpha+2\lambda}{\tilde{h}_M}} \right) q_D^0. $$  \hfill (3.33)
where again the first piece corresponds to the monopole cusp, and the second piece to the dyon cusp. The $\text{sgn}(a)$ appears because, as we explained before, the wall-crossing terms corresponding to $\lambda$ and $-\lambda$ have opposite sign. The overall sign can be fixed by looking at the sign of the Seiberg-Witten invariants in the wall-crossing formula of [30]. Notice that the constraint $g - 1 \geq ab$ is in fact redundant, as the whole expression above vanishes if this constraint is not fulfilled. This is due to the fact that the most negative power of $q_D$ in the above expression comes from $\tilde{f}_{2M}^{g-1} = q_D^{1-g}(1 + O(q_D))$. Therefore, if $ab > g - 1$, there are no terms in $q_0^D$.

The Donaldson-Witten generating function in the chamber where $\Sigma_g$ is small, and for $w_2(E) = [S^2] + \epsilon[\Sigma_g]$, is then given by the sum of (3.33) and (3.31), and we will denote it by $Z_{\Sigma_g}^\xi(p, r, s, t)$. Notice that $Z_{\Sigma_g}^\xi(p, r, s, t)$ can be also computed by adding the generating function in the chamber of small volume for $S^2$ and the infinite sum of wall-crossings. The resulting expressions were given in [4], in terms of Weierstrass $\sigma$ functions. The fact that the generating functions in [4] are equal to $Z_{\Sigma_g}^\xi(p, r, s, t)$ gives a remarkable identity. They encode the information about the Donaldson invariants in two different ways, that we can call “magnetic” and “electric.” We have checked that they give indeed the same invariants in many cases, but an analytic proof would be rather formidable.

It is interesting to notice that the $u$-plane integral can be written as

$$-2^8(-1)^e e^{2pu_M+2stT_M(\tilde{h}_M^2\tilde{f}_{2M})^{-1}}$$

$$\cdot \left(2r + \frac{s}{16}\tilde{h}_M^2\tilde{f}_{1M} - 2^{-8}\tilde{f}_{2M}\tilde{h}_M^3\frac{d}{dt}\right) \sum_{n=0}^\infty (-1)^n e^{\frac{2nu}{q_M}} \int_{q_D^0}^{\infty}$$

where we have only written the monopole contribution. This piece has the same structure of the Seiberg-Witten contribution (3.33), but where the sum is now over an infinite number of basic classes of the form $\lambda = n[\Sigma_g]$, i.e. with $b = -n \leq 0$ and $a = 0$. Similar considerations have been made in [33] for simply-connected manifolds.

### 3.3. Some properties and examples

An interesting corollary of our computation is that the manifold $\Sigma_g \times S^2$ is of $g^{th}$ finite type, when one considers the chamber of small volume for $\Sigma_g$ and a Stiefel-Whitney class such that $(w_2(E), \Sigma_g) = 0$. This is an immediate consequence of our expressions. To see it, notice that both for the $u$-plane and the SW contributions, the only source of a negative
power of $q_D$ is the term $\tilde{f}_{2M}$. The rest of the modular forms involved in our formulae are analytic in $q_D$. On the other hand, the maximum possible power of $\tilde{f}_{2M}$ is precisely $g - 1$. As $u_M = 1 + 32q_D + \ldots$, an insertion of $(u_M \pm 1)^g$ in the monopole (respectively, dyon) contribution to (3.34) or (B.33), will make the generating function vanish. But an insertion of $u_M \pm 1$ is equivalent to acting with

$$\frac{\partial}{\partial p} \pm 2$$

on the Donaldson-Witten generating function. We then find that

$$\left(\frac{\partial^2}{\partial p^2} - 4\right)^g Z_g^\epsilon(p, r, s, t) = 0.$$  

(3.36)

$g$ is in fact the minimum power we need to kill the generating function, since

$$\left(\frac{\partial^2}{\partial p^2} - 4\right)^{g-1} Z_g^\epsilon(p, r, s, t) = (-1)^{g-1}2^g \left(\text{Li}_{-g}(-e^{2it})e^{2p+st} + (-1)^{g-1}\text{Li}_{-g}(-e^{-2it})e^{-2p-st}\right).$$

(3.37)

We conclude that $\Sigma_g \times S^2$ is of $g^{th}$ finite type for the chamber and the Stiefel-Whitney class under consideration. This was proved in [12] for $g = 1$.

We will now give explicit expressions for the Donaldson-Witten generating function at low genus. For $g = 1$, the Seiberg-Witten contribution vanishes as there are no walls (i.e., the conditions (B.32) have no solution). In this case, only the $u$-plane contributes. The only polylogarithm involved here is

$$\text{Li}_{-1}(-e^t) = -\frac{e^t}{(1 + e^t)^2} = -\frac{1}{4(\cosh(t/2))^2}.$$  

(3.38)

It is clear that for $g = 1$ we only need the first term in the expansion of the modular forms. We then find,

$$Z_{1,\Sigma_1}^\epsilon(p, r, s, t) = Z_{u,\Sigma_1}^\epsilon(p, r, s, t) = -\frac{1}{2}(-1)^\epsilon \left[ \frac{e^{2p+st}}{\cosh^2(t)} + (-1)^\epsilon \frac{e^{-2p-st}}{\cosh^2(-it)} \right].$$

(3.39)

As in this case the manifold has a simple type behavior, we can define the Donaldson series

$$\mathbb{D}^\epsilon = Z_{DW}^\epsilon|_{p=0} + \frac{1}{2} \frac{\partial}{\partial p} Z_{DW}^\epsilon|_{p=0}. \text{ If we write it as a functional on } \text{Sym}(H^2(X, \mathbb{Z})), \text{ we find from (B.33):}$$

$$\mathbb{D}^\epsilon = (-1)^{(e^2 - 2e_F)} \frac{e^{Q/2}}{\cosh^2 F},$$

(3.40)
where \( e = w_2(E) \), \( F = [\Sigma_1] \), and \( Q \) is the intersection form. This is in perfect agreement with the Theorem 1.3 in [12].

It is clear that, as we consider larger values of \( g \), the expression for the Donaldson-Witten generating function becomes more and more complicated. The main source of this complexity is the \( t \)-dependence in the \( u \)-plane integral. For example, for \( g = 2 \) the monopole contribution to the \( u \)-plane integral is given by:

\[
Z^\varepsilon_{u,M} = -(-1)^\varepsilon \frac{e^{2p+st+2t}}{16(1+e^{2t})^4} (5 - 5e^{4t} + 16(1 + e^{4t})p + 128(1 + e^{2t})^2r + 4s) \\
+ 8e^{2t}s + 4e^{4t}s - 2t + 8e^{2t}t - 2e^{4t}t - 4st + 4e^{4t}s),
\]

while the Seiberg-Witten contribution is

\[
Z^\varepsilon_{SW,M} = (-1)^\varepsilon \frac{e^{2p+st}}{64} (e^{2t-2s} - e^{2s-2t}).
\]

4. Application 1: intersection pairings on the moduli space of stable bundles

4.1. The moduli space of stable bundles

Let \( \Sigma_g \) be a Riemann surface of genus \( g \). The moduli space of flat \( SO(3) \) connections on \( \Sigma_g \), with Stiefel-Whitney class \( w_2 \neq 0 \) turns out to be a very rich and interesting space. One of the reasons for this richness is the fact that this moduli space can be understood in many different ways: using the Hitchin-Kobayashi correspondence, we can think about this space as the moduli space of rank two, odd degree stable bundles over \( \Sigma_g \) with fixed determinant. On the other hand, due to the classical theorem of Narashiman and Seshadri, we can identify this moduli space with the representations in \( SU(2) \) of the fundamental group of the punctured Riemann surface \( \Sigma_g \setminus D_p \), where \( D_p \) is a small disk around the puncture \( p \), and with holonomy \( -1 \) around \( p \) (the fact that we require a non-trivial holonomy is due precisely to the non-zero Stiefel-Whitney class). In any case, this moduli space, that we will denote by \( \mathcal{M}_g \), is a smooth projective variety of (real) dimension \( 6g - 6 \). Similarly, we can consider the moduli space of flat \( SU(2) \) connections, i.e. with \( w_2 = 0 \). This moduli space can be identified with the moduli space of stable rank two vector bundles of even degree and it is singular. We will denote it by \( \mathcal{M}^+_g \).

The cohomology ring of \( \mathcal{M}_g \) can be studied by using a two-dimensional version of the \( \mu \) map which arises in Donaldson theory. This map sends homology classes of \( \Sigma_g \) to
cohomology classes of $\mathcal{M}_g$. The generators of $H_*(\Sigma_g)$ give in fact a set of generators in $H^{4-*}(\mathcal{M}_g)$ that are usually taken as follows [16][25]:

$$\alpha = 2\mu(\Sigma_g) \in H^2(\mathcal{M}_g),$$
$$\psi_i = \mu(\gamma_i) \in H^3(\mathcal{M}_g),$$
$$\beta = -4\mu(x) \in H^4(\mathcal{M}_g),$$

where $x$ is the class of the point in $H_0(\Sigma_g)$. We also define the $\text{Sp}(2g, \mathbb{Z})$-invariant cohomology class in $H^6(\mathcal{M}_g)$,

$$\gamma = -2\sum_{i=1}^{g} \psi_i \psi_{i+g}.$$  

One can show that the moduli space of anti-self-dual connections on $\Sigma_g \times S^2$ with instanton number zero is isomorphic to the moduli space of flat connections on $\Sigma_g$. In particular, the generators of the cohomology in (4.1) correspond precisely to the Donaldson cohomology classes, and we have that

$$\alpha = 2I(\Sigma_g), \quad \psi_i = I(\gamma_i), \quad \beta = -4\mathcal{O},$$

while the invariant form $\gamma$ corresponds to (3.3).

4.2. The intersection pairings

To determine the ring structure of the cohomology of $\mathcal{M}_g$, once a set of generators has been found, one only has to find a set of relations. Due to Poincaré duality, the intersection pairings of the generators in (4.1) give all the information needed to find the relations. In other words, to find the structure of the cohomology ring it is enough to evaluate the intersection pairings

$$\langle \alpha^m \beta^n \gamma^p \rangle_{\mathcal{M}_g} = \int_{\mathcal{M}_g} \alpha^m \wedge \beta^n \wedge \gamma^p,$$

as all the intersection pairings involving the $\psi_i$’s can be reduced to (4.4) by $\text{Sp}(2r, \mathbb{Z})$ symmetry. These numbers can be considered as the two-dimensional analogs of Donaldson invariants, which in fact can be formulated as correlation functions of a two-dimensional topological gauge theory [17][34]. In [16], Thaddeus computed (4.4) in two steps: first, he obtained a recursive relation which allows to eliminate the $\gamma$ classes. Second, he computed the pairings $\langle \alpha^m \beta^n \rangle$ using Verlinde’s formula [15]. The same steps are followed by Witten in [17]. We will also prove first the recursion relation, and then compute the remaining
pairings. To do this, we use the Donaldson invariants of product ruled surfaces. The relation between the pairings in (4.4) and the Donaldson invariants are as follows (see [25], Remark 13):

\[ \langle \alpha^m \beta^n \gamma^p \rangle_{\mathcal{M}_g} = \epsilon([S^2]) D_{\Sigma_g \times S^2}^{w_2 = [S^2]} \left( (2 \Sigma_g)^m (-4x)^n \nu^p \right). \quad (4.5) \]

In this equation, \( \epsilon(w) = (-1)^{K_w + w^2} \), where \( w \) is an integer lift of the second Stiefel-Whitney class. This sign appears due to the following reason. The Donaldson invariants that appear in the right hand side are defined using the natural orientation of the moduli space of anti-self-dual connections. For algebraic surfaces, this moduli space can be realized as the Gieseker compactification of the moduli space of rank two stable bundles \( \mathcal{M}(c_1, c_2) \), where \( c_1 = w \). This complex space has a natural orientation induced by its complex structure, and the difference between these orientations in the computation of the invariants is given by \( \epsilon(w) \). Now, when \( c_2 = 0 \) and \( c_1 = [S^2] \) (i.e., there is a non-zero flux through the Riemann surface \( \Sigma_g \)), then \( \mathcal{M}(0, c_1) = \mathcal{M}_g \), and the intersection pairings in the left hand side are in fact computed with the complex orientation. In this case, \( \epsilon([S^2]) = -1 \).

Notice that the pairing above is only different from zero when \( 2m + 4n + 6p = 6g - 6 \). The Donaldson invariants in (4.5) can be computed in any chamber, since for \( c_2 = 0 \) and \( w_2(E) = [S^2] \) one has \( p_1 = 0 \) and therefore there are no walls. The computation turns out to be much simpler in the chamber of small volume for \( S^2 \). One can make in principle the computation in the other limiting chamber, using \( Z_{\Sigma_g}^{\epsilon = 0} \). This turns out to be very complicated analytically, although we have explicitly checked that the answers agree in many cases. Physically, the computation of Thaddeus’ intersection pairings (as well as the computation of Donaldson invariants involving low instanton numbers) is rather semiclassical and is best performed in the electric frame, while the “magnetic” expression \( Z_{\Sigma_g}^{\epsilon = 0} \) gives information about global aspects of the generating function which are useful, for example, to compute the eigenvalue spectrum of Fukaya-Floer cohomology. It has also been noticed in [22] that in fact, to compute the intersection pairings on \( \mathcal{M}_g \), the chamber of small volume for \( S^2 \) is more natural, as the topological reduction in this chamber gives the twisted \( \mathcal{N} = 2 \) Yang-Mills in two dimensions in a direct way. We will then extract these pairings from the Donaldson invariants given by (3.22).

The first thing that we can prove is the recursive relation of Thaddeus. Using the explicit formula (3.22), one easily sees that

\[ \frac{\partial}{\partial r} Z^{w_2(E) = [S^2]}_{g, S^2} = 2g Z^{w_2(E) = [S^2]}_{g-1, S^2}, \quad (4.6) \]
and this implies, using (4.3), that
\[ \langle \alpha^m \beta^n \rangle_{\mathcal{M}_g} = 2g \langle \alpha^m \beta^n \gamma^{p-1} \rangle_{\mathcal{M}_{g-1}}, \] (4.7)
which is precisely Thaddeus’ recursive relation.

We now compute the intersection pairings \( \langle \alpha^m \beta^n \rangle \). To do this, we use the expansion:
\[ \csc z = \sum_{k=0}^{\infty} (-1)^{k+1} (2^{2k} - 2) B_{2k} \frac{z^{2k-1}}{(2k)!}, \] (4.8)
where \( B_{2k} \) are the Bernoulli numbers. We have to extract now the powers \( s^m p^n \) from
the generating function (3.22). Notice that a power \( s^g \) comes already from the overall \( g \)-dependent factor in (3.22). We then have to extract the power \( s^{m-g} \) from the series expansion in \( s/2h \). Taking now into account the comparison factors from (4.5), and the dimensional constraint \( 2m + 4n = 6g - 6 \), one finds
\[ \langle \alpha^m \beta^n \rangle = \frac{1}{4} 2^m (-4)^n \epsilon^{m-g+1} 2^{2g+n-m} m! \frac{(2^{m-g+1} - 2)}{(m-g+1)!} B_{m-g+1} [h^{3g-m-2} u_\infty f_{1\infty} f_{2\infty}^{-1}] q^n. \] (4.9)
Fortunately, only the leading term contributes in the \( q \)-expansion involved in (4.9). One finally obtains,
\[ \langle \alpha^m \beta^n \rangle = (-1)^g \frac{m!}{(m-g+1)!} 2^{2g-2} (2^{m-g+1} - 2) B_{m-g+1}, \] (4.10)
which is exactly Thaddeus’ formula for the intersection pairings. This expression, as well as the recursive relation (4.7), has been obtained by Witten in [17] by exploiting the relation to physical two-dimensional Yang-Mills theory. A derivation of (4.10) based on gluing techniques has been worked out in [35].

One can formally consider the intersection pairings in the case of even degree, and
extract them from the Donaldson invariants for vanishing Stiefel-Whitney class. These invariants are given by (3.21). Using now the expansion
\[ \coth z = \sum_{k=0}^{\infty} 2^{2k} B_{2k} \frac{z^{2k-1}}{(2k)!}, \] (4.11)
and taking into account that \( \epsilon(0) = 1 \), one finds
\[ \langle \alpha^m \beta^n \rangle = (-1)^g \frac{m!}{(m-g+1)!} 2^{m+g-1} B_{m-g+1}. \] (4.12)
These are also the pairings that one obtains using two-dimensional gauge theory \[17\][34]. However, one has to be extremely careful about the mathematical interpretation of \((4.12)\), since the space \(\mathcal{M}_g^+\) is singular for \(g \geq 3\). To define the intersection pairings one has then to use intersection cohomology \[36\] or consider a partial desingularization of \(\mathcal{M}_g\) that has only orbifold singularities \[37\]. The computation of the intersection pairings for the partial desingularization was performed in \[37\] using the strategy of \[16\]. The result agrees with \((4.12)\) for \(m \geq g\), but for \(m < g\) there are correction terms \[7\]. It would be interesting to see if these corrections can be obtained using physical methods. In this sense, the approach in \[17\] seems more appropriate, as the singular character of the moduli space shows up as a non-regular term in the partition function.

4.3. Relation to Verlinde’s formula

The derivation of the intersection pairings \((4.10)\) in \[16\] was based in the \(SU(2)\) Verlinde’s formula for the WZW model \[15\]. In fact, one can reverse the logic in \[16\] and give a derivation of Verlinde’s formula from the intersection pairings. In this section, we will closely follow the arguments given in \[16\] for \(\mathcal{M}_g\), and we will also show that they can be formally extended to \(\mathcal{M}_g^+\).

Verlinde’s formula gives an explicit expression for the number of conformal blocks in CFT. In the case of the \(SU(2)\) WZW model, the space of conformal blocks at level \(k\) (where \(k\) is a positive integer) can be identified with the space of sections of the line bundle \(L^{k/2}\), where \(L\) is a fixed line bundle over \(\mathcal{M}_g\) which generates \(\text{Pic}(\mathcal{M}_g) \simeq \mathbb{Z}\). The canonical bundle of \(\mathcal{M}_g\) is given by \(L^{-2}\). We are interested in computing \(\dim H^0(L^{k/2}, \mathcal{M}_g)\). As explained in \[16\], this can be done using Hirzebruch-Riemann-Roch. The canonical bundle of \(\mathcal{M}_g\) is negative, and by the Kodaira vanishing theorem one has that \(H^i(L^{k/2}, \mathcal{M}_g) = 0\) for \(i > 0\). We then have,

\[
\dim H^0(L^{k/2}, \mathcal{M}_g) = \chi(L^{k/2}, \mathcal{M}_g) = \int_{\mathcal{M}_g} \text{ch} \ L^{k/2} \text{td} \mathcal{M}_g. \tag{4.13}
\]

Notice that the cohomology classes involved in Hirzebruch-Riemann-Roch can be expressed in terms of the generators of the cohomology ring \((4.1)\), and therefore \((4.13)\) can be computed in principle once the intersection pairings are known. Explicit expressions for the characteristic classes of the tangent bundle to \(\mathcal{M}_g\) have been obtained by Newstead \[38\].

\[7\] We are grateful to Y.-H. Kiem for explaining these issues to us.
(see also [33]) and read \( c_1(\mathcal{M}_g) = 2\alpha, \ p(\mathcal{M}_g) = ((1 + \beta)^{2g-2}. \) One also has \( c_1(L) = \alpha, \) and this gives:

\[
\dim H^0(L^{k/2}, \mathcal{M}_g) = \int_{\mathcal{M}_g} \exp\left(\frac{k+2}{2}\alpha\right)\left(\frac{\sqrt{\beta}/2}{\sinh \sqrt{\beta}/2}\right)^{2g-2}, \tag{4.14}
\]

If we expand the characteristic classes in the right hand side of (4.14), we get a polynomial in \( k + 2 \) of the form

\[
\sum_{m=0}^{3g-3} \frac{P_{3g-3-m}(k+2)^m}{2^{3g-3} m!} \int_{\mathcal{M}_g} \alpha^m \beta^{(3g-3-m)/2}, \tag{4.15}
\]

where \( P_n \) is the coefficient of \( x^n \) in the series expansion of \( (x/\sinh x)^{2g-2}. \) Using now (4.10), it is easy to see that this is the coefficient of \( x^{3g-3} \) in

\[
\left(\frac{-k+2}{2}x\right)^{g-1}\left(\frac{x}{\sinh x}\right)^{2g-2}\frac{(k+2)x}{\sinh(k+2)x}. \tag{4.16}
\]

An argument due to Zagier and explained in [16], Proposition (19), shows that this coefficient is given by

\[
\dim H^0(L^{k/2}, \mathcal{M}_g) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{n=1}^{k+1} \frac{(-1)^{n+1}}{(\sin \frac{n\pi}{k+2})^{2g-2}}, \tag{4.17}
\]

which is precisely Verlinde’s formula in the case of odd degree.

As we pointed out before, the moduli space of rank two stable bundles of even degree is a singular space, and in principle the intersection numbers are not well-defined. The answer (4.12) should be considered as a regularization of these pairings in the context of the \( u \)-plane integral. If we assume that the Riemann-Roch formula is still valid, one obtains in fact the usual Verlinde’s formula for \( SU(2) \) [8]. If we consider the expression (4.13) with the intersection pairings given in (4.12), one finds that the dimension of \( H^0(L^{k/2}, \mathcal{M}_g^+) \) is now given by the coefficient of \( x^{3g-3} \) in

\[
-\left(\frac{-k+2}{2}x\right)^{g-1}\left(\frac{x}{\sinh x}\right)^{2g-2}(k+2)x \coth(k+2)x. \tag{4.18}
\]

Going through the argument in [16], Proposition (19), one easily obtains:

\[
\dim H^0(L^{k/2}, \mathcal{M}_g^+) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{n=1}^{k+1} \frac{1}{(\sin \frac{n\pi}{k+2})^{2g-2}}, \tag{4.19}
\]

8 This has also been observed in [10] [11].
which gives the right formula for the number of conformal blocks for the (untwisted) $SU(2)$ case. This computation is, however, formal, as there is no suitable Riemann-Roch formula for a singular space like $\mathcal{M}_g^+$. Using the orbifold desingularization of $\mathcal{M}_g^+$, one can apply the Kawasaki-Riemann-Roch formula and relate the intersection pairings to Verlinde’s formula [4.19]. In fact, this is how the corrections to the pairings (4.12) are obtained in [37].

5. Application 2: Fukaya-Floer cohomology

5.1. Floer cohomology and gluing rules

The Floer (co)homology groups of three-manifolds and their relations to Donaldson invariants can be understood in a simple way using the axiomatic approach introduced by Atiyah [42], which in fact is a formalization of heuristic considerations involving path integrals. According to the axiomatic approach, a topological field theory in 3+1-dimensions is essentially a functor $\Phi$ from the category of three-dimensional manifolds to the category of complex vector spaces, $\Phi : \text{Man}(3) \to \text{Vect}$, and satisfying certain properties. In the case of Donaldson-Witten theory, this functor associates to any compact, oriented three-manifold $Y$ the graded vector space given by the Floer homology groups $HF_*(Y)$. These homology groups can be defined by using Morse theory with the Chern-Simons functional on the moduli space of $SO(3)$ connections on $Y$ with second Stiefel-Whitney class $w_2 \in H^2(Y, \mathbb{Z})$.

We will be interested here in the gluing rules that relate the ring structure of the Floer homology to the Donaldson invariants of four-manifolds. Let us consider a four-manifold $X$ with boundary $\partial X = Y$, together with an element $z$ in $A(X)$. According to the axiomatic approach, the functor $\Phi$ also assigns to the pair $(X, z)$ a “relative invariant” of $X$, which is an element in the Floer homology of $Y$ (i.e., $\Phi(X, z) \in \Phi(\partial X)$). This relative invariant can be understood in a simple way, as explained in [4], in terms of path-integrals. Let $z = S_{i_1} \ldots S_{i_p} x^n \gamma_{j_1} \ldots \gamma_{i_q}$ be an element of $A(X)$, where $S_{i_\mu}$, $\gamma_{j_\mu}$ are two and one-homology classes, respectively, and $x$ is the class of the point. This determines a BRST-invariant operator given by the $\mu$-map, namely

$$A_z = I(S_{i_1}) \ldots I(S_{i_p}) O^n I(\gamma_{j_1}) \ldots I(\gamma_{i_q}). \quad (5.1)$$

We then define the relative invariant through the usual correspondence operators/states in quantum field theory:

$$\left( \Phi(X, z) \right)_X^Y (\phi_Y) = \int_X [D\phi] |_{\phi|_Y = \phi_Y} e^{-S_{\text{TYM}}} A_z, \quad (5.2)$$
which is a functional of the fields restricted to the boundary. In this path integral, one integrates over all the gauge fields with Stiefel-Whitney class $w_X \in H^2(X, \mathbb{Z})$, where $w_X$ restricts to $w$ on $Y$.

There are some extra structures in $HF_*(Y)$ that will be important to our analysis. Our presentation will closely follow the excellent surveys in [20][23]; for more details, one can see [43][44]. First of all, there is an associative and graded commutative ring structure $HF_*(Y) \otimes HF_*(Y) \to HF_*(Y)$. Second, as in ordinary homology, one can define the dual of $HF_*(Y)$ to obtain the Floer cohomology of $Y$. Moreover, if $-Y$ denotes the manifold with opposite orientation, one has $HF^*(Y) \simeq HF^*(-Y)$. Finally, there is a natural, non-degenerate pairing $\langle , \rangle : HF^*(Y) \otimes HF^*(Y) \to \mathbb{C}$. When the states in the Floer cohomology are given by relative invariants, this pairing can be understood heuristically from path integral arguments. Consider two manifolds with boundary, $X_1, X_2$, such that $\partial X_1 = Y$ and $\partial X_2 = -Y$ (i.e., $Y$ with the opposite orientation). We can glue the manifolds together to obtain a closed four-manifold $X$. There is then a pairing between $HF_*(Y)$ and $HF_*(-Y)$ which is given by

$$\langle \Phi(X_1, z_1)^{w_{X_1}}, \Phi(X_2, z_2)^{w_{X_2}} \rangle = \int_X [D\phi] e^{-S_{\text{TYM}}} A_{z_1} A_{z_2}.$$  (5.3)

In other words, the pairing is essentially given by Donaldson invariants of the four-manifold $X$.

In order to give a precise gluing result, we have to be careful with two things: first of all, which is the Stiefel-Whitney class that one has to pick in order to define the Donaldson invariant on the right hand side of (5.3)? and, in case $b^+_2(X) = 1$, in which chamber should we compute the Donaldson invariant? These issues are discussed in [19][13] in detail. To answer the first question, consider a $w \in H^2(X, \mathbb{Z})$ such that $w|_Y = w_2$. Also consider a cohomology class $[\Sigma] \in H^2(X, \mathbb{Z})$, given as the Poincaré dual of a two-class $\Sigma$ which lies in the image of $i_* : H_2(Y, \mathbb{Z}) \to H_2(X, \mathbb{Z})$, and satisfying $w \cdot [\Sigma] = 1 \pmod{2}$. The pair $(w, \Sigma)$ is called in [19] an allowable pair. One then defines

$$D^{(w, \Sigma)}_X = D^w_X + D^{w+[\Sigma]}_X.$$  (5.4)

The gluing theorem of [23][13] is then

$$\langle \Phi(X_1, z_1)^{w_{X_1}}, \Phi(X_2, z_2)^{w_{X_2}} \rangle = D^{(w, \Sigma)}_X(z_1, z_2).$$  (5.5)
If $b^n_2(X) = 1$, then one considers the metric giving a long neck, i.e., one takes $X = X_1 \cup (Y \times [0, R]) \cup X_2$, with $R$ very large.

We are interested in the Floer (co)homology of $Y = \Sigma_g \times S^1$, with Stiefel-Whitney class $w_2 = [S^1]$. This manifold has an orientation-reversing diffeomorphism given by conjugation on $S^1$, and therefore there is a natural isomorphism $HF^*(Y) \simeq HF_*(Y)$. We will work with the ring cohomology from now on. The first thing to do is to find the generators of this ring. Consider the four-manifold with boundary $X^0 = \Sigma_g \times D^2$, where $D^2$ is the two-disk. Then, $\partial X^0 = \Sigma_g \times S^1$. We can define relative invariants of $X$ associated to elements in $A(X^0) = A(\Sigma_g)$. Clearly, one has to take $w = [D^2] \in H^2(X^0, \mathbb{Z})$, which restricts to $[S^1]$ at the boundary. The generators of $HF^*(Y)$ are then given by [23]:

$$
\begin{align*}
\alpha &= 2\Phi^w(X^0, \Sigma_g) \in HF^2(Y), \\
\psi_i &= \Phi^w(X^0, \gamma_i) \in HF^3(Y), \\
\beta &= -4\Phi^w(X^0, x) \in HF^4(Y),
\end{align*}
$$

where $\Sigma_g$, $\gamma_i$ and $x$ are the generators of $H_*(\Sigma_g)$. Notice that this basis is very similar to the basis of $\mathcal{M}_g$ presented in (4.1). In fact, $HF^*(Y)$ and $H^*(\mathcal{M}_g)$ are isomorphic as vector spaces [24]. The product structure in the Floer cohomology is given, for these relative invariants, by $\Phi^w(X^0, z)\Phi^w(X^0, z') = \Phi^w(X^0, zz')$. We will restrict ourselves to the invariant part of $HF^*(Y)$ (as in the analysis of the cohomology of $\mathcal{M}_g$), which is generated by $\alpha$, $\beta$ and

$$
\gamma = -2 \sum_{i=1}^g \Phi^w(X^0, \gamma_i \gamma_{i+g}).
$$

The last ingredient we need is the gluing rule. If we consider the pairing of two relative invariants constructed from $X^0$, we will have to glue two copies of $X^0$ along their boundaries. Clearly, this gives the closed four-manifold $X = \Sigma_g \times S^2$ (where $S^2$ comes from gluing the two disks along their boundaries $S^1$). The long neck metric is the one that makes $S^2$ very big, and then corresponds to the chamber where $\Sigma_g$ is small. Finally, we have to specify the allowable pair. In $X$, $w = [S^2]$ restricts to $w_2 = [S^1]$ on $Y$. On the other hand, the image of $H_2(Y, \mathbb{Z})$ in $H^2(X, \mathbb{Z})$ is generated by $\Sigma_g$. This means that the gluing rule for the relative invariants is

$$
\langle \Phi^{w_0}(X^0, z_1), \Phi^{w_0}(X^0, z_2) \rangle = D^{w_0, \Sigma_g}_X(z_1 z_2) = D^{w_2 = [S^2]}_X(z_1 z_2) + D^{w_2 = [S^2] + [\Sigma_g]}_X(z_1 z_2).
$$

(5.8)
The Fukaya-Floer cohomology $HFF^*(Y)$ of an oriented three-manifold $Y$ needs the extra input of a loop $\delta \simeq S^1$ in $Y$. A review of this construction can be found in [19]. Here we will consider that $\delta$ is the $S^1$ factor in $Y = \Sigma_g \times S^1$. In this case, one has that $HFF^*(Y) = HF^*(Y) \times \mathbb{C}[[t]]$. A basis of generators can be also constructed using relative invariants of the manifold $X^0$, with the insertion of the operator $\exp tI(D^2)$ in the path integral. In this way, we obtain the generators

$$\hat{\alpha} = 2\Phi^w_0(X^0, \Sigma_t e^{tD^2}) \in HFF^2(Y),$$
$$\hat{\psi}_i = \Phi^w_0(X^0, \gamma_i e^{tD^2}) \in HFF^3(Y),$$
$$\hat{\beta} = -4\Phi^w_0(X^0, x e^{tD^2}) \in HFF^4(Y).$$

(5.9)

The gluing rule is now

$$\langle \Phi^w_0(X^0, z_1 e^{tD^2}), \Phi^w_0(X^0, z_2 e^{tD^2}) \rangle = D_X^{(w, \Sigma_g)}(z_1 z_2 e^{tS^2}),$$

(5.10)

and therefore the Donaldson invariants involved in the Fukaya-Floer cohomology include the cohomology class associated to $S^2$. This makes the determination of this cohomology more difficult.

5.2. Eigenvalue spectrum of the Fukaya-Floer cohomology

As we have seen, the intersection pairings in Floer and Fukaya-Floer cohomology are given by Donaldson invariants of $\Sigma_g \times S^2$ in the chamber where $\Sigma_g$ is small. These invariants completely determine, in principle, the ring structure of the (Fukaya)-Floer cohomology, but this does not mean that we are able to give an explicit presentation of the relations of the ring. Already in the comparatively simpler case of the classical cohomology of $M_g$, to obtain the explicit relations at genus $g$ starting from the intersection pairings (4.10) turns out to be a very complicated combinatorial problem (solved in [39]). In this section, we want to show that an important aspect of the ring structure, namely the eigenvalue spectrum, can be deduced in a simple way from the generating function that we found in section 2. In the case of Floer cohomology, the spectrum was obtained in [22] under some extra assumptions, and finally derived in [23] from an explicit presentation of the relations. The spectrum of the Fukaya-Floer cohomology was conjectured in [19], based on the computation of the spectrum for a submodule. Our calculation confirms this conjecture. Our strategy will be in a way the reverse to that in [20]. In these papers, the information about Fukaya-Floer cohomology obtained in [19] is used to understand the
structure of Donaldson invariants. Here, we will use the Donaldson invariants of $\Sigma_g \times S^2$ to deduce results about the Fukaya-Floer cohomology of $\Sigma_g \times S^1$.

The basic procedure to obtain the eigenvalue spectrum is to find elements in the ideal of relations of the Fukaya-Floer cohomology, i.e., to find vanishing polynomials in the generators $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$:

$$P(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) = 0. \quad (5.11)$$

We can easily translate this identity in terms of the generating function for the Donaldson invariants of $\Sigma_g \times S^2$: as the pairing (5.3) is non-degenerate, to prove the identity (5.11) it is enough to prove that

$$\langle P(\hat{\alpha}, \hat{\beta}, \hat{\gamma}), \Phi(X^0, ze^{tD^2}) \rangle = 0, \quad (5.12)$$

for any $z \in A(\Sigma_g)$. Due to the gluing rule (5.10), the above pairing is nothing but $D_X^{(w, \Sigma_g)}(P(2\Sigma_g, -4x, i)e^{tS^3})$. The vanishing of (5.12) for any $z$ is then equivalent to the following differential equation,

$$P\left(2\frac{\partial}{\partial s}, -4\frac{\partial}{\partial p}, \frac{\partial}{\partial r}\right) Z_{g, \Sigma_g}(p, r, s, t) = 0, \quad (5.13)$$

where we have defined the generating functional corresponding to the invariants (5.4):

$$Z_{g, \Sigma_g}(p, r, s, t) = Z_{g}^{\epsilon=0}(p, r, s, t) + Z_{g}^{\epsilon=1}(p, r, s, t). \quad (5.14)$$

What we have computed in section 3 are precisely the generating functions involved in (5.14). We then have to study the differential equations satisfied by our function. First of all, using (3.31) and (3.33), one immediately finds:

$$Z_{g, \Sigma_g} = -2^{g-1} \left[ e^{-2pu_M - 2stT_M} \sum_{m=1}^{g} (\frac{2}{m}) (-1)^{m} 2^{-6m} (\tilde{h}_{M}^{2} \tilde{f}_{2M})^{m-1} \cdot \left( 2ir - \frac{is}{16} \tilde{h}_{M}^{2} \tilde{f}_{1M} \right)^{g-m} \text{Li}_{-m}(-e^{-2ith_{M}^{-1}}) \right] q_D^0$$

$$+ 2^g \sum_{a \text{ odd}}^{a<ab \leq g-1} \left[ \text{sgn}(a)q_D^a(-1)^b e^{2pu_M + 2stT_M} (\tilde{h}_{M}^{2} \tilde{f}_{2M})^{-1} \cdot \left( \frac{s}{16} \tilde{h}_{M}^{2} \tilde{f}_{1M} + 2^{-7} b \tilde{h}_{M}^{2} \tilde{f}_{2M} + 2r \right) e^{\frac{2a}{h_{M}}} \right] q_D$$

$$+ 2^g \sum_{a \text{ even}}^{a<ab \leq g-1} \left[ \text{sgn}(a)q_D^a(-1)^b e^{-2pu_M - 2stT_M} (\tilde{h}_{M}^{2} \tilde{f}_{2M})^{-1} \cdot \left( -\frac{is}{16} \tilde{h}_{M}^{2} \tilde{f}_{1M} + 2^{-7} b \tilde{h}_{M}^{2} \tilde{f}_{2M} + 2ir \right) e^{-\frac{2a}{h_{M}}} \right] q_D. \quad (5.15)$$

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This generating function can be explicitly evaluated for low genus. The results are specially simple if we put $t = 0$. We obtain, for example:

$$Z_1^{(w, \Sigma_1)} = -e^{-2p},$$

$$Z_2^{(w, \Sigma_2)}(p, r, s) = -\frac{1}{8}e^{-2p}(32r - s) - \frac{1}{32}e^{2p}(e^{2s} - e^{-2s}),$$

$$Z_3^{(w, \Sigma_3)}(p, r, s) = -\frac{1}{4096}e^{-2p}(98 + 256p + 256p^2 + 49152r^2 - 3072rs + 48s^2)$$
$$+ \frac{1}{256}e^{2p+2s}(3 - 4p - 48r - 2s) + \frac{1}{256}e^{2p-2s}(3 - 4p + 48r + 2s)$$
$$+ \frac{1}{4096}e^{-2p}(e^{4is} + e^{-4is}).$$  \hspace{1cm} (5.16)

Let us now concentrate on the eigenvalue spectrum of the Fukaya-Floer cohomology. The first eigenvalue equation we can write is the one that corresponds to the finite-type condition that we obtained in section 3. It reads now,

$$\left(\hat{\beta}^2 - 64\right)^g = 0,$$  \hspace{1cm} (5.17)

therefore the eigenvalues of $\hat{\beta}$ must be $\pm 8$. To understand the eigenvalue spectrum of the remaining operators, it is useful first to be more precise about the structure of $Z_g^{(w, \Sigma_g)}$. If we look at (5.15), it is easy to see that it can be written as,

$$Z_g^{(w, \Sigma_g)} = \sum_{|a| \leq g-1} Z_a(p, r, s, t).$$  \hspace{1cm} (5.18)

Notice that the $u$-plane integral contribution corresponds to $a = 0$. The structure of $Z_a(p, r, s, t)$ is immediate from (5.15):

$$Z_a(p, r, s, t) = \begin{cases} f_a(p, r, s, t)e^{2p+st+2as}, & \text{for } a \text{ odd}, \\ f_a(p, r, s, t)e^{-2p-st-2ias}, & \text{for } a \text{ even}, \end{cases}$$  \hspace{1cm} (5.19)

where $f_a(p, r, s, t)$ is a polynomial in $p$, $r$ and $s$ and a power series in $t$ (similar remarks about the structure of the Donaldson invariants have been made in [20]). We are interested in the degree of the polynomial in $p$, $r$, $s$. This is again easy to see if we look at the modular

\footnote{For $t = 0$, the generating function (5.15) can be computed in principle using the Artinian decomposition of the Floer cohomology [23]. This procedure, however, does not give a general result for any genus and has to be worked out case by case. V. Muñoz has informed us that the above expressions for $g = 2, 3$ coincide with the results that can be obtained from this decomposition.}
forms involved in \((5.13)\). Assume \(g \geq 2\) (for \(g = 1\), \(f_0\) only depends on \(t\)). We know that the maximum power we can find in \(p\) is precisely \(g - 1\) (a simple consequence of the finite type condition). As one can see in \((5.37)\), this power appears in the \(u\)-plane integral contribution, and for \(a \neq 0\), the maximum power of \(p\) is in fact \(g - 2\).

Let us now find which is the maximum power of \(s\) in \(f_a\). If we group the powers of \(s\) in the modular form that gives \(f_a\), we easily see that the leading term in \(q_D\) has the form

\[
q_D^{ab+1+n-m} s^{g+n-m} t^n,
\]

up to numerical constants. It is clear that the maximum possible power of \(s\) which can appear in \(f_a\) is \(g - |a| - 1\), and occurs for \(|b| = 1\). This power actually appears in \(f_a\): using the above expression, it is easy to see that

\[
f_a(p, r, s, t) = -2^{6-4g-3|a|} \sum_{n=|a|+1}^{g} \left( \frac{g}{n} \right) (\sgn(a))^{n+1} \frac{(a+2t)^{n-|a|-1}}{(n-|a|-1)!} e^{-2(\sgn(a)) t} s^{g-|a|-1} + \ldots,
\]

for \(a\) odd, while for \(a\) even one obtains:

\[
f_a(p, r, s, t) =
(-1)^{g-|a|} i^{|a|} 2^{6-4g-3|a|} \sum_{n=|a|+1}^{g} \left( \frac{g}{n} \right) (\sgn(a))^{n+1} \frac{(a-2it)^{n-|a|-1}}{(n-|a|-1)!} e^{2i(\sgn(a)) t} s^{g-|a|-1} + \ldots.
\]

Finally, for \(a = 0\) (the \(u\)-plane contribution), one has:

\[
f_a(p, r, s, t) = (-1)^g 2^{6-4g} \sum_{n=1}^{g} \left( \frac{g}{n} \right) (-1)^{n} (-2it)^{n-1} \frac{(-1)^n (-2it)^{n-1}}{(n-1)!} \left( \frac{g}{n} \right) \Li_{-n}(-e^{-2it}) s^{g-1} + \ldots
\]

Notice that, for \(a \neq 0\), \(f_a(p, r, s, t)\) is a polynomial in \(e^{\pm t}\), while for \(a = 0\) it is a rational function in these variables.

By similar arguments, one finds that the maximum power of \(r\) appears for \(a = 0\) and is \(g - 1\). We then have the following differential equations for the \(Z_a\):

\[
(-4 \frac{\partial}{\partial p} + 8)^g Z_a = (2 \frac{\partial}{\partial s} - 4a - 2t)^g Z_a = 0, \quad a \text{ odd}
\]

\[
(-4 \frac{\partial}{\partial p} - 8)^g Z_a = (2 \frac{\partial}{\partial s} + 4ia + 2t)^g Z_a = 0, \quad a \text{ even, } a \neq 0,
\]

\[
(-4 \frac{\partial}{\partial p} - 8)^g Z_0 = (2 \frac{\partial}{\partial s} + 2t)^g Z_0 = 0, \quad a = 0
\]

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Notice that the powers that appear in these equations are in fact the minimum powers that are needed to kill the \( Z_a \), as it can be easily seen from (5.21)(5.22) and (5.23). We also have

\[
\frac{\partial^g}{\partial r^g} Z_g^{(w, \Sigma_a)} = 0,
\]

which is also the minimum power we need (\( r^g - 1 \) appears for \( a = 0 \), while for \( a \neq 0 \) the maximum power is \( r^g - 2 \)). Notice, in particular, that \( (\hat{\beta} \pm 8)^g \) kills all the \( Z_a \) with \( a \) odd (even, respectively). We can now deduce the eigenvalue spectrum of the Fukaya-Floer cohomology. From (5.24) and (5.25) we find the following operator equations:

\[
\hat{\gamma}^g = 0,
\]

\[
\prod_{a \text{ odd}} (\hat{\alpha} - 4a - 2t)^g - |a| \prod_{a \text{ even}} (\hat{\alpha} + 4ia + 2t)^g - |a| = 0.
\]

(5.26)

Therefore, the only eigenvalue of \( \hat{\gamma} \) is 0, and for \( \hat{\alpha} \) we find the eigenvalue spectrum \((0, \pm 4 + 2t, \pm 8i - 2t, \cdots)\). Notice that the eigenvalue 8 of \( \hat{\beta} \) only occurs for \( \hat{\alpha} = -4ia - 2t \), with \( a \) even. In the same way, we find that \(-8\) only occurs for \( \hat{\alpha} = 4a + 2t \), for \( a \) odd. This is due to the equation

\[
(\hat{\beta} + 8)^g \prod_{a \text{ even}} (\hat{\alpha} + 4ia + 2t)^g - |a| = 0,
\]

(5.27)

and a similar equation involving \( (\hat{\beta} - 8)^g \). Recalling now that \(-\( g - 1 \) \leq a \leq g - 1 \), our main conclusion is that the eigenvalue spectrum of \( (\hat{\alpha}, \hat{\beta}, \hat{\gamma}) \) is given by

\[
(0, 8, 0), \quad (\pm 4 + 2t, -8, 0), \quad \ldots \quad (\pm 4 (g - 1)i^g + (-1)^g 2t, (-1)^{g-1} 8, 0).
\]

(5.28)

This generalizes Proposition 20 in [23], and confirms the conjecture in [19] (see Theorem 5.13 and Remark 5.14 in that paper). It is easy to give an explicit construction of the eigenvectors corresponding to these eigenvalues. They are given by:

\[
v_a = \begin{cases} 
(\hat{\beta} + 8)^g (\hat{\alpha} + 4ia + 2t)^g - |a| \prod_{a' \text{ even}, a' \neq a} (\hat{\alpha} + 4ia' + 2t)^g - |a'|, & a \text{ even}, \\
(\hat{\beta} - 8)^g (\hat{\alpha} - 4a - 2t)^g - |a| \prod_{a' \text{ odd}, a' \neq a} (\hat{\alpha} - 4a' - 2t)^g - |a'|, & a \text{ odd}.
\end{cases}
\]

(5.29)

To see that these vectors are in fact not zero, one can easily prove that \( \langle v_a, \Phi(X^0, ze^{tS^1}) \rangle \neq 0 \) for any \( z \in A(X) \), using the explicit results for the generating function (5.15) in (5.21), (5.22) and (5.23).

Using now arguments from [22] [23] [45], together with our results, it is easy to rederive the presentation of the Floer cohomology of \( \Sigma_g \times S^1 \) given in [22] [23]. We will give some
brief indications on this respect. Let \( J_g \) be the ideal of relations at genus \( g \) (\( J_g \) is then generated by all the polynomials in \( \alpha, \beta, \gamma \) that vanish as elements of the invariant part of \( HF^* \)). First of all, notice that \( Z_{g}^{w, \Sigma_g} \) satisfies

\[
\frac{\partial}{\partial r} Z_{g}^{w, \Sigma_g} = 2gZ_{g-1}^{w, \Sigma_g},
\]

(5.30)

This implies immediately the following inclusion relation:

\[
\gamma J_g \subset J_{g+1} \subset J_g.
\]

(5.31)

Now one can use the fact that the Floer cohomology of \( Y = \Sigma_g \times S^1 \) is a deformation of the cohomology of \( M_g \) (this is rather elementary and does not assume the existence of a ring isomorphism between \( HF^*(Y) \) and the quantum cohomology of \( M_g \)). Using the explicit recursive presentation of the ring cohomology of \( M_g \) given in [39][46], and adapting the arguments of Proposition 3.2 in [45], one obtains the following result: the ideal of relations is given by \( J_g = (q_1^g, q_2^g, q_3^g) \), where the \( q_i^g \), \( i = 1, 2, 3 \), are given by the following recursive relations:

\[
\begin{align*}
q_{g+1}^1 &= \alpha q_g^1 + g^2 q_g^2, \\
q_{g+1}^2 &= (\beta + c_{g+1})q_g^1 + \frac{2g}{g+1} q_g^3, \\
q_{g+1}^3 &= \gamma q_g^1.
\end{align*}
\]

(5.32)

When \( c_{g+1} = 0 \), one recovers the classical cohomology of \( M_g \). The deformation is then encoded in the coefficient \( c_{g+1} \). Notice that the key fact which is used to derive (5.32) is the inclusion of ideals (5.31), which is in turn a consequence of (5.30). This recursion relation was conjectured in [45] for the generating function of the Gromov-Witten invariants of \( M_g \), and provides in that context a generalization of Thaddeus’ recursion relation (4.7) from the classical to the quantum pairings.

The last ingredient is then to compute the value of the coefficient \( c_{g+1} \). It was shown in [23] that this value can be easily deduced by induction using the eigenvector that corresponds to the maximum \( \alpha \)-eigenvalue. For \( g = 1 \), one immediately finds from (5.13) that \( Z_1^{w, \Sigma_1} = -e^{-2p} \) (we are putting \( t = 0 \), as we are considering the Floer rather than the Fukaya-Floer cohomology). It then follows that \( \beta = 8 \), so \( c_1 = -8 \). The argument in [23], Theorem 14, gives then \( c_g = (-1)^g 8 \).

It would be interesting to use the information contained in (5.15) to give a recursive presentation like (5.32) but for the Fukaya extension. Some steps in this direction have been
taken in [19]. In principle, all the information that one needs is contained in the generating function (5.15), but it is still a non-trivial problem to extract it in the particular form of an explicit presentation of the relations.

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