1. INTRODUCTION.

Usual way to represent real number $x \in (0, 1)$ is via its infinite decimal fraction. More formally, consider the transformation $T_{10}(x) = \lfloor 10x \rfloor = 10x - \lfloor 10x \rfloor$ where $\lfloor x \rfloor$ and $\{x\}$ denote integer and fractional parts of $x$ correspondingly. Since transformation Dec is ten-folded, to be able to restore $x$ from its image, one has to get track of the lost information.

Partition the unit interval onto ten parts $(0, 1] = \bigcup_{j=1}^{10} \left( \frac{j-1}{10}, \frac{j}{10} \right]$ and denote $a_n = \lfloor 10T_{10}^{n-1}(x) \rfloor$. In other words, sequence $\{a_n\}$ encodes the trajectory of the point $x$ under the transformation Dec subject to the above partition. Then $0.a_1a_2\ldots$ is the decimal fraction representation of $x$. Convergent $x_n = 0.a_1a_2\ldots a_n$ corresponds to the whole segment of the initial points lying in the $10^{-n}$ vicinity since for all these points the first $n$ symbols in the encoding of the trajectory coincides. Thus to achieve the $\varepsilon$-accuracy of the approximation by the convergents one has to consider $\log \frac{1}{\varepsilon}$ iterations of the transformation $T_{10}$.

Another representation can be achieved via famous Gauss transformation: $Gx = \{\frac{1}{x}\}$. Again, one could encode the trajectory of any point $x \in [0, 1]$ by the information which is lost at each step. Partition the unit interval as $(0, 1] = \bigcup_{n=1}^{\infty} \left( \frac{1}{n+1}, \frac{1}{n} \right]$ and denote $a_k = \lfloor \frac{1}{(G^k)x} \rfloor$. Then the numbers $a_k$ will correspond to the the elements of continued fraction representation

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}.$$  

Convergents obtained by from the first digits of the continuous fraction provide the best rational approximation of the point $x$. However, as a payoff one has to use infinite set of symbols to encode the trajectory.

In the present note we consider the Finite Gauss Transformation. It uses the partition similar to those for the Gauß, but having only finite
number of the intervals. We present invariant density for such transformation and discuss several examples together with periodic orbits.

We start with the following transformation of the circle $S^1$. Fix $n \in \mathbb{N}$ and consider the regular $n$-gon $P = \{P_1, \ldots, P_n\}$ circumscribed around $S^1$. Denote by $A_j$ the point of tangency of $S^1$ with the $j$-th side $P_jP_{j+1}$. Points $A_j$ partition $S^1$ onto $n$ arcs. Transformation $T : S^1 \mapsto S^1$ is defined as follows (see Fig. 1):

\begin{equation}
Tz = S^1 \cap P_jz
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{finite_gauss_transformation}
\caption{Finite Gauss transformation.}
\end{figure}

Our main result concerns the invariant measure of the transformation (1). We claim that this invariant measure is absolutely continuous with respect to the Lebesgue measure with the infinite density having the form

\begin{equation}
\rho(\varphi) = \frac{\sin \frac{\pi}{n}}{\cos \frac{\pi}{n} - \cos \left\{ \frac{m\varphi}{2\pi} \right\}}.
\end{equation}

In the next section we present a proof of this result and discuss its connection to Gauss transformation.

For $n = 3$ transformation (1) was considered in [1], [2] as describing the mass evolution in the cosmological models of Bianchi type. Amazingly, this transformation for $n = 4$ has been considered only thirty years later describing the dynamics of Pythagorean triples (see [4]).

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2. Computations.

We start with parametrization of $\mathbb{S}^1$ by the stereographic projection from $\mathbb{R}$ to $\mathbb{S}^1$. The coordinates of a point $z \in \mathbb{S}^1$ are given by $z = \left( \frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right)$ and point $A_1$ correspond to the value $t_* = \tan \frac{\pi}{2n}$. Vertex $P_1$ has coordinates $\left( 0, \frac{1}{\cos \frac{\pi}{n}} \right)$.

Transformation (1) is defined as

$$Tz = z + \tau(z - P_1), \quad \tau = \frac{2z \cdot (P_1 - z)}{|z - P_1|^2}.$$  

Since $t_*$ is the tangent of the half of $\frac{\pi}{n}$, we get the expressions $\cos \frac{\pi}{n} = \frac{1-t_*^2}{1+t_*^2}$ and $\sin \frac{\pi}{n} = \frac{2t_*}{1+t_*^2}$. Therefore $\frac{1}{\cos \frac{\pi}{n}} = \frac{1+t_*^2}{1-t_*^2}$ and so

$$\tau = \frac{(t_*^2 - 1)(t^2 - t_*^2)}{t^2 + t_*^4}.$$  

Immediate computation shows that $Tz = \left( \frac{2s}{1+s^2}, \frac{1-s^2}{1+s^2} \right)$ where

$$s = Q(t) = \frac{t_*^2}{t} : [-t_*, t_*] \mapsto [-t_*, t_*]^c.$$  

In order to iterate our transformation we have to rotate the image $Tz$ back to the original interval $[-t_*, t_*]$ by the rotation

$$R_k s = \frac{a_{k,n}s - b_{k,n}}{b_{k,n}s + a_{k,n}}, \quad \text{for } s \in \left[ \tan \frac{\pi(2k-1)}{2n}, \tan \frac{\pi(2k+1)}{2n} \right].$$  

**Figure 2.** Transformation on the segment $[-t_*, t_*]$ is the composition of the transformation $Q$ and rotation $R_k$. 

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Since Corollary 1. □ is proven.

Theorem 1. Transformation (6) has invariant measure with the density

\[ \rho(t) = \frac{2t_1}{(t^2 - t_1^2)}. \]

Proof. We have to show that function \( \rho \) satisfies the equation

\[ \rho(t) = \sum_{k=1}^{n-1} \rho\left(F_k^{-1}(t)\right) \left|\left(F_k^{-1}\right)'\right| \]

Since \( \rho(t) = \frac{1}{t - t_1} - \frac{1}{t + t_1} \), we have to check that

\[ \frac{1}{t - t_1} - \frac{1}{t + t_1} = \sum_{k=1}^{n-1} \left( \frac{\left|\left(F_k^{-1}\right)'\right|}{F_k^{-1}(t) - t_1} - \frac{\left|\left(F_k^{-1}\right)'\right|}{F_k^{-1}(t) + t_1} \right) = \left( \ln \prod_{k=1}^{n-1} \frac{F_k^{-1}(t) - t_1}{F_k^{-1}(t) + t_1} \right)' \]

Introducing the expression for the preimage

\[ F_k^{-1} = \frac{t_1^2(b_{k,n}t - a_{k,n})}{-a_{k,n}t - b_{k,n}} \]

and denoting \( \tan\frac{\pi k}{n} \) by \( t_{n,k} \), we get

\[ \prod_{k=1}^{n-1} \frac{F_k^{-1}(t) - t_1}{F_k^{-1}(t) + t_1} = \prod_{k=1}^{n-1} \frac{t_1(1 - tt_{n,k}) + (t + t_{n,k})}{t_1(1 - tt_{n,k}) - (t + t_{n,k})} = \prod_{k=1}^{n-1} \frac{t(1 - t_1t_{n,k}) + (t_{n,k} + t_1)}{t(1 + t_1t_{n,k}) + (t_{n,k} - t_1)}. \]

Since \( t_1 = \tan\frac{\pi}{2n} \) we can rewrite \( t_{n,k} \pm t_1 \) as \( t_{2n,2k\pm 1}(1 \mp t_1t_{n,k}) \) and so

\[ \prod_{k=1}^{n-1} \frac{F_k^{-1}(t) - t_1}{F_k^{-1}(t) + t_1} = \prod_{k=1}^{n-1} \frac{1 - t_1t_{n,k}}{1 + t_1t_{n,k}} \prod_{k=1}^{n-1} \frac{t + t_{2n,2k+1}}{t + t_{2n,2k-1}}. \]

The first product equals to one, since \( t_{n,k} = -t_{n,n-k} \). The last product is telescopic and so it is equal to \( \frac{t - t_1}{t + t_1} \). Since \( \left( \ln \frac{t - t_1}{t + t_1} \right)' = \rho(t) \) the theorem is proven. □

Corollary 1. Original density equals \( \rho(\varphi) = \frac{\sin \frac{\varphi}{n}}{\cos \frac{\varphi}{n} - \cos \varphi} \).
3. Discussion.

3.1. Fixed points. Transformation (6) has $n - 1$ fixed points. If $s_k$ is the $k$-th fixed point of (6) then $T s_k = e^{2\pi ik/n} s_k$. Therefore $s_k$ correspond to the periodic point of the transformation (1) of period $\gcd(k, n)$ (see Fig.3).

If one orients the polygon $P$ in such a way that the point $A_n$ will belong to the vertical axis, then the arc $A_n A_1$ will correspond to the interval $[0, \tan \frac{\pi}{n}]$ in $t$ variable. Then points $Q^{-1}(A_k)$ will correspond to the values $t_k = \frac{1}{k} \tan \frac{\pi}{n}$. Thus, for example for $n = 4$ points of tangency will split the interval $[0, 1]$ on three intervals $[0, \frac{1}{3}] \cup [\frac{1}{3}, \frac{1}{2}] \cup [\frac{1}{2}, 1]$. And the corresponding transformation will have the form

$$T(t) = \begin{cases} \frac{3t-1}{t-1}, & t \in [0, \frac{1}{3}] \\ \frac{1}{t-2}, & t \in [\frac{1}{3}, \frac{1}{2}] \\ \frac{t-1}{1-3t}, & t \in [\frac{1}{2}, 1] \end{cases}$$
Obviously, in this orientation any periodic point of the transformation $F$ is the periodic point of some Moebius transformation with integer coefficients. Therefore, every periodic point of $F$ is a root of some quadratic polynomial. However, for the moment we do not know if any quadratic irrationality can be achieved as a periodic point of transformation $F$.

3.2. **Piecewise Moebius on the interval.** Transformation (3) for the case $n = 3$ can be considered not only for regular triangle. Projectively mapping generic triangle to the regular one, we will obtain the transformation (6) of the form

$$T_a t = \begin{cases} \frac{at+1}{(a-2)t-1}, & t \in [-1, 0] \\ \frac{at-1}{-(a-2)t-1}, & t \in [0, 1] \end{cases}$$

It is easy to see that such transformation has invariant density $\rho(t) = \frac{1}{t^2-1}$.

3.3. **Higher dimensions.** It would be interesting to look at the 3-dimensional analog of transformation $F$. Given vertices of the tetrahedron, provide the partition of the sphere on the four regions, defined by the perpendicular bisector planes to the edges of the tetrahedron. Each region correspond to the nearest vertex. Thus one can define the transformation via formulas (1).

![Figure 5. 3d Finite Gauss map.](image-url)
For the sphere we have the following parametrization:

\[ z = \left( \frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{1 - u^2 - v^2}{1 + u^2 + v^2} \right) \]

and \( P_1 = (0, 0, 3) \). Therefore from (3) we compute for \( z \) from the top part of the sphere (see Fig. 5) \( Tz = Q(u, v) \) where

\[ Q(u, v) = \left( \frac{u}{2(u^2 + v^2)}, \frac{v}{2(u^2 + v^2)} \right). \]

Our experiments shows that the invariant density is infinite at the points of tangency of the tetrahedron and the sphere and is smooth everywhere else. However, we still do not know the closed formula for it.

REFERENCES

[1] O. I. Bogojavlenskii. Qualitative theory of homogeneous cosmological models II. *Trudy Sem. Petrovsk.*, (Vyp. 2):67–112, 1976.

[2] O. I. Bogoyavlenskii. *Metody kachestvennoi teorii dinamicheskikh sistem v astrotfizike i gazovoi dinamike*. “Nauka”, Moscow, 1980.

[3] Karlheinz Gröchenig and Andrew Haas. Invariant measures for certain linear fractional transformations mod 1. *Proc. Amer. Math. Soc.*, 127(12):3439–3444, 1999.

[4] Dan Romik. The dynamics of Pythagorean triples. *Trans. Amer. Math. Soc.*, 360(11):6045–6064, 2008.

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