The distribution of quantum fidelities

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When applied to different input states, an imperfect quantum operation yields output states with varying fidelities, defined as the absolute square of their overlap with the desired states. We present an expression for the distribution of fidelities for a class of operations applied to a general qubit state, and we present general expressions for the variance and input-space averaged fidelities of arbitrary linear maps on finite dimensional Hilbert spaces.

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I. INTRODUCTION

In quantum control and quantum information theory one attempts to control the dynamics of a quantum system, such that the net mapping of a quantum state by the dynamics yields a specific output state. If the system is known to initially populate a specific state, the success of a given operation on that state is determined by the square of the overlap between the final state and the desired output state, while, more generally, the successful implementation of a certain map, e.g., a quantum gate in a quantum computer, should be judged by an evaluation of this overlap for all possible input states. Imperfections may be due to a variety of reasons, such as dissipative coupling to auxiliary degrees of freedom and imprecise knowledge of the physical parameters characterizing the system. A wide range of methods have thus been applied to counter such effects: composite pulses, quantum control, bang-bang control, error-correcting codes, and use of decoherence-free subspaces. To optimize these methods it is necessary to have definite functionals of the gate operation, that one can determine and, hopefully, improve by suitable variation of accessible control parameters.

In this paper, we address the distribution of fidelities obtained when the input states are taken uniformly from the full Hilbert space of the physical system or from a suitable subspace, to which physical circumstances may restrict the initial state. In Sec. II we present a derivation of the fidelity distribution of unitary gates applied to a qubit, i.e., a two-level quantum system, and an extension to the case of a unitarily diagonalizable linear map. In Sec. III, we review our recent derivation of the mean value of the fidelity of an arbitrary linear map, and in Sec. IV, we determine the variance of the fidelity distribution of such linear maps. Sec. V concludes the paper.

II. FIDELITY DISTRIBUTION FOR A ONE-QUBIT GATE

Consider a two-level system, a qubit, subjected to a unitary operator $U$, while the desired operation on the system is given by the unitary operator $U_0$. Under ideal circumstances, the unitary operator $M = U_0 U = U_0^{-1} U$ is the identity operator, and the fidelity of the quantum operation $\langle \psi | U_0^* U | \psi \rangle = 1$ for all normalized states $| \psi \rangle$. Under less ideal circumstances, however, $M$ is not the identity, but it can be diagonalized by a unitary transformation, and has two complex eigenvalues $e^{i\phi_0}$ and $e^{i\phi_1}$, with $\phi_0 \neq \phi_1$. We observe that the fidelity of the gate operation is unity when $\psi$ is equal to either of the eigenvectors $|0\rangle$, $|1\rangle$ of $M$, while expanding $|\psi\rangle$, we get the lowest fidelity, $f = |(e^{i\phi_0} + e^{i\phi_1})/2|^2 = \cos^2((\phi_1 - \phi_0)/2)$, for any equal weight superposition of these eigenvectors.

A Bloch sphere picture with the eigenvectors as the north and south pole, the fidelity is only a function of the polar angle $\theta$ of the input state, $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$, $f(\theta) = |(e^{i\phi_0} \cos^2(\theta/2) + e^{i\phi_1} \sin^2(\theta/2))|$, with a uniform distribution of states over the surface of the Bloch sphere, and the corresponding polar angle distribution $P_0 = \frac{1}{2} \sin \theta$ one readily determines the fidelity distribution, $P_f = \sum |d\theta|/d\theta |P_0|$, where the contributions to the sum come from symmetric polar angles above and below the equator with the same fidelity.

The expression for the fidelity distribution of a unitary, erroneous qubit gate thus reads,

$$P_f = \frac{1}{2 \sin \left(\frac{\phi_1 - \phi_0}{2}\right) \sqrt{f - \cos^2 \left(\frac{\phi_1 - \phi_0}{2}\right)}}, \quad (1)$$

for $\cos^2((\phi_1 - \phi_0)/2) \leq f \leq 1$.

We shall not attempt a derivation of the equivalent distribution of fidelities in higher Hilbert space dimensions. A related problem, dealing with the distribution of matrix elements of a Hermitian operator, was studied by von Neumann [2], and illustrates how the solution of such a problem breaks up in a large number of different cases. We shall, however, extend our analysis to the special situation where a quantum system is known to initially populate a two-dimensional subspace of a three-dimensional Hilbert space, including an excited state with suitable interaction properties, such that excitation from the lower qubit states to the excited states...
can be used to communicate between different quantum systems, e.g., by an excited state dipole-dipole interaction. In \cite{3} a robust one-qubit gate scheme was proposed, in which both qubit states are simultaneously coupled to the excited state, giving rise to a dark state superposition, $|0\rangle$, and a bright state superposition, $|1\rangle$, with destructive and constructive interference of the couplings. Transferring the bright state into the excited state and back with laser fields with different phases implements a controllable phase in the dark/bright basis, equivalent to an arbitrary qubit rotation in the computational qubit basis \cite{3}. If the coherent coupling of the bright and excited state is imprecise and does not return the population fully to the low lying states, we obtain $U = \left( \begin{array}{cc} 1 & 0 \\ 0 & \alpha \gamma \end{array} \right)$ in the basis $\{|0\rangle, |1\rangle, |e\rangle\}$. This implies that the effect on the qubit space of the application of $U$ can be written in terms of $U$ and the projection operator $P$ on that space, $PUP = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right)$ in the basis $\{|0\rangle, |1\rangle\}$. $\alpha$ is a complex number, and if $|\alpha| = 1$ the fidelity distribution is given by Eq. (1) with the difference between $arg(\alpha)$ and the desired phase shift by $U_0$ replacing $\phi_1 - \phi_0$.

An operation with $|\alpha| < 1$ yields a special example of the slightly more general case, where the state vector is mapped, $|\psi\rangle \rightarrow N|\psi\rangle$, by a normal $2 \times 2$ matrix $N$. The matrix $M = U_0^* N$ is also normal, i.e., it can be diagonalized by a unitary transform with eigenvalues $\lambda_0, \lambda_1$ that we can arrange such that $|\lambda_0| \leq |\lambda_1|$. Employing the expansion on the corresponding eigenstates, parametrized by $c_0 = \cos(\theta/2)$ and $c_1 = e^{i\phi} \sin(\theta/2)$, where $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$, we have $f = |\lambda_0|^2 f_\lambda$, where

$$f_\lambda = \cos^4 \frac{\theta}{2} + \sin^4 \frac{\theta}{2} |\lambda|^2 + \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} (\lambda + \lambda^*)$$

and $\lambda = \lambda_1/\lambda_0$. With the same argument as above we need

$$\frac{df}{d\theta} = \frac{1}{2} |\lambda_0|^2 \sin \theta (|\lambda|^2 - 1 - |1 - \lambda|^2 \cos \theta),$$

and we obtain

$$|\lambda_0|^2 P_f = \left| |\lambda|^2 - 1 - |1 - \lambda|^2 \cos \theta \right|^{-1},$$

where, from (2),

$$\cos \theta = \frac{|\lambda|^2 - 1 + 2 \sqrt{f_\lambda |1 - \lambda|^2 - \text{Im}(\lambda)^2} }{|1 - \lambda|^2}.$$  

The fidelity is a non-monotonic function of the polar angle $\theta$, and adding the contributions from different polar angles yielding the same fidelity, we finally obtain the probability distribution (recall $|\lambda_0|^2 < |\lambda_1|^2$). We have to specify the results for two different cases.

If $|\lambda_0 - \frac{1}{2} \lambda_1| < \frac{1}{2} |\lambda_1|$, then

$$P_f = \frac{1}{2 |\lambda_0 - \lambda_1|} \frac{1}{\sqrt{f - f_0}}, \quad |\lambda_0|^2 \leq f \leq |\lambda_1|^2, \quad (3)$$

and if $|\lambda_0 - \frac{1}{2} \lambda_1| \geq \frac{1}{2} |\lambda_1|$, then

$$P_f = \left\{ \begin{array}{ll} \frac{1}{2 |\lambda_0 - \lambda_1|} \frac{1}{\sqrt{f - f_0}}, & f_0 \leq f \leq |\lambda_0|^2, \\
\frac{1}{2 |\lambda_0 - \lambda_1|} \frac{1}{\sqrt{f - f_0}}, & |\lambda_0|^2 \leq f \leq |\lambda_1|^2, \end{array} \right. \quad (4)$$

where $f_0 = \text{Im}(\lambda_0^2 \lambda_1^2)^{1/2}/|\lambda_0 - \lambda_1|^2$.

Fig. 1, shows the fidelity distribution \cite{1} for the case of $\lambda_0 = 0.7 \cdot e^{i\pi/8}$, $\lambda_1 = 0.8 \cdot e^{i4\pi/5}$. The vertical bars in the figure indicate a histogram obtained by drawing states on the Bloch sphere at random and binning their individual fidelities. We observe that the results agree very well.

Having the full fidelity distribution at our disposal, we can see how changes in the map, reflected in the parameters $\{\lambda_0, \lambda_1\}$, change the fidelities, and we can optimize according to different criteria: minimum value, mean value or some higher moment, which can readily be obtained from the distribution.

In the following, we shall limit our analysis of the fidelity distribution to a characterization of its mean value and variance. Rather than obtaining these numbers from integrals over the distribution \cite{41}, \cite{42}, $(f) = \int_{f_{\text{min}}}^{f_{\text{max}}} P_f f df$ and $(f^2) = \int_{f_{\text{min}}}^{f_{\text{max}}} P_f f^2 df$, we shall provide a more general expression, valid for any map $|\psi\rangle \rightarrow N|\psi\rangle$, in any finite dimensional setting.

\section{The Average Fidelity Measure}

Derivations have been given in the literature for the average fidelity of qubit \cite{4} and qudit operations \cite{1, 5}, as well as for their evaluation as a sum over a properly chosen discrete set of states \cite{4, 6}. The latter approach is particularly relevant in connection with quantum process tomography \cite{3}, which provides a procedure for determining the quantum operation acting on a system from experimental observations.
For any linear operator $M$ acting on an $n$-dimensional complex Hilbert space, the uniform average of $|\langle \psi | M | \psi \rangle|^2$ over state vectors $|\psi\rangle$ on the unit sphere $S^{2n-1}$ in $\mathbb{C}^n$ is given by

$$\langle f \rangle = \frac{1}{n(n+1)} \int \sum_{n=1}^{2n-1} |\langle \psi | M | \psi \rangle|^2 dV$$

where $dV$ is the normalized measure on the sphere.

Recent detailed proofs of this result are given in [1, 9, 10], and it is also readily verified using a recent result for the averages of general polynomials of state vector amplitudes over the unit sphere [11].

The expression (5) is readily evaluated for any matrix $M$, and in particular, for the qubit example with eigenvalues $\lambda_0$ and $\lambda_1$, we get the simple result, $\langle f \rangle = (|\lambda_0|^2 + |\lambda_1|^2 + \text{Re}(\lambda_0\lambda_1^*))/3$. If $M$ is unitary, $MM^\dagger$ is the identity matrix with trace equal to the dimension $n$ in Eq. (5).

In the following we recall a few useful results, derived in [1].

### A. Subspace averaged fidelity of a unitary transformation

In a number of quantum information scenarios, auxiliary quantum levels are used to mediate the desired operations. In these protocols, the auxiliary levels of the quantum system may, with certainty, be unpopulated before the process and, consequently, we should only average the fidelity over the relevant input states. Since the final state is, ideally, also in the same subspace, we consider the matrix $M = (PU_q^\dagger P)(PU)$, where $P$ is the projection operator on the relevant, quantum information carrying subspace $S$, and $U_q$ represents the desired unitary evolution of the relevant subspace. In a matrix notation, the outermost applications of the projection operator $P$ amounts to the extraction of the square $n_{rel} \times n_{rel}$ submatrix $M_{rel}$ with the relevant columns and rows of the full matrix $M$, and to compute the mean fidelity over the subspace, we employ (5) for this reduced matrix:

$$F = \frac{1}{n_{rel}(n_{rel} + 1)} \left[ \text{Tr} (M_{rel}M_{rel}^\dagger) + |\text{Tr}(M_{rel})|^2 \right], \quad (6)$$

If population leaks to the auxiliary levels, $M_{rel}$ is not unitary, and hence both terms of (6) have nontrivial values.

If a measurement assures that the final state does not populate the complement to $S$, it is meaningful to define the average conditional fidelity $F_c$. If we only accept the state if it is in $S$, the squared overlap between the conditional, renormalized state $\frac{PUP|\psi\rangle}{||PUP|\psi\rangle||}$ and the ideal state $U_qP|\psi\rangle$ must be weighted with the acceptance probability $||PUP|\psi\rangle||^2$ and renormalized by the integrated acceptance probability over the input Hilbert space, $\int ||PU|\psi\rangle||^2 dV$. We hence obtain the average conditional fidelity [20]

$$F_c = \int \frac{|\langle \psi | PUP_0 |U_q |PUP|\psi\rangle|^2}{||PU|\psi\rangle||^2} dV$$

$$= \frac{1}{n_{rel} + 1} \left( \frac{\text{Tr}(U_q^\dagger PUPU_q)}{\text{Tr}(U_q^\dagger PUP)} \right)^2 + \frac{1}{n_{rel} + 1} \left( \text{Tr}(M_{rel}) \right)^2$$

where the numerator is evaluated using (5) and the denominator follows from $\int |\langle \psi | M | \psi \rangle|^2 dV = \text{Tr}(M)/n$.

### B. Average fidelity of a general quantum operation

Our quantum system may not be fully isolated from its surroundings, and ancillary quantum systems may play significant roles in various quantum information processing scenarios: quantum memory protocols in a very explicit manner involve an extra quantum system, quantum teleportation requires an extra entangled pair of systems, and in quantum computing with trapped ions, a motional degree of freedom is used to couple the particles. The ancillary systems are ideally disentangled from the qubits before and after the process, but in general they act as an environment and cause decoherence of the quantum system of interest. This forces us to generalize the formalism and take into account the general theory of quantum operations, according to which the mean dynamics is accounted for by density matrices which transform by completely positive maps. According to the Kraus representation theorem, any completely positive trace-preserving map $G$ admits the representation

$$G(\rho) = \sum_k G_k \rho G_k^\dagger, \quad (7)$$

where $\sum_k G_k G_k^\dagger = I_n$ is the $n \times n$ identity matrix [8].

If the pure input state $\rho = |\psi\rangle\langle\psi|$ is mapped to the output state $G(\rho)$ the mean fidelity with which our operation yields a unitary transformation $U_0$ is

$$F = \frac{1}{n(n+1)} \text{Tr} \left( \sum_k M_k^\dagger M_k + \sum_k |\text{Tr}(M_k)|^2 \right), \quad (8)$$

where $M_k = U_0^\dagger G_k$.

Equation (8) enables the calculation of the average fidelity of any quantum operation, as soon as it has been put in the Kraus form. For examples, see [1].
IV. THE VARIANCE OF THE FIDELITY DISTRIBUTION

To further characterize the fidelity distribution, we shall now obtain an explicit formula for the variance of the fidelity distribution. Note that it is not meaningful to define the variance of a general quantum transformation governed by the Kraus form, since the density matrix formulation already incorporates an averaging procedure, due to the trace over unobserved degrees of freedom of the surroundings of the quantum system. The density matrix evolution may be understood, and simulated, by an average over randomly evolving wave functions, but this unravelling is not unique, and only if measurements are actually carried out on the surroundings, a specific stochastic dynamics of the state vectors happens, which is invariant under the exchange of any two observables.

We shall proceed and derive a relation for the uniform average of $\left|\langle \psi | S | \psi \rangle\right|^4$ over state vectors $|\psi\rangle$ on the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ given by

\[
\int_{S^{2n-1}} \left|\langle \psi | S | \psi \rangle\right|^4 dV = \frac{1}{[6\text{Tr}(S^4) + 8\text{Tr}(S^3)\text{Tr}(S) + 3\text{Tr}(S^2)]^2 + 6\text{Tr}(S^2)\text{Tr}(S)^2 + \text{Tr}(S^4)].
\]

(9)

Our demonstration of this result proceeds along the lines of our demonstration of Eq. (5), given in more detail in [1]. First we note the invariance of both sides of (9) under conjugation by any unitary operator $U$, which allows us to restrict the analysis to a diagonal matrix $A$ with real eigenvalues $\lambda_1, \ldots, \lambda_n$. Letting $L$ denote the left-hand side of (9), we observe that $L(A)$ is a homogeneous polynomial of degree 4 in the real variables $\lambda_1, \ldots, \lambda_n$, which is invariant under the exchange of any two $\lambda_i$ and $\lambda_j$. It is easy to demonstrate that the set

\[
\{ \text{Tr}(A^4), \text{Tr}(A^3)\text{Tr}(A), \text{Tr}(A^2)^2, \text{Tr}(A^2)\text{Tr}(A)^2, \text{Tr}(A)^4 \}
\]

spans all such polynomials $L(A)$, and by evaluating the integrals

\[
\int_{S^{2n-1}} |c_i|^4 dV = \frac{24}{n(n+1)(n+2)(n+3)}
\]

\[
\int_{S^{2n-1}} |c_i|^4 |c_j|^4 dV = \frac{4}{n(n+1)(n+2)(n+3)}
\]

\[
\int_{S^{2n-1}} |c_i|^6 |c_j|^2 dV = \frac{6}{n(n+1)(n+2)(n+3)}
\]

\[
\int_{S^{2n-1}} |c_i|^4 |c_j|^2 |c_k|^2 dV = \frac{2}{n(n+1)(n+2)(n+3)}
\]

\[
\int_{S^{2n-1}} |c_i|^2 |c_j|^2 |c_k|^2 |c_l|^2 dV = \frac{1}{n(n+1)(n+2)(n+3)}.
\]

(10)

for different $i, j, k, l$, and by choosing five appropriate matrices for which the integrals are readily obtained, we finally obtain the coefficients in (9) by solving a linear system of equations.

Note that (9) also holds for an anti-Hermitian matrix $A$, since $L(A) = L(iA) = R(iA) = R(A)$, where $L$ and $R$ denote the left- and right-hand sides of (9), respectively.

Having obtained the uniform average of $\left|\langle \psi | S | \psi \rangle\right|^4$, where $S$ is Hermitian, we now proceed to the general case. We decompose the general matrix $M = S + A$ as a sum of a Hermitian and an anti-Hermitian matrix, denoted by $S$ and $A$, respectively, and we note that

\[
\int_{S^{2n-1}} |\langle \psi | M | \psi \rangle|^4 dV = \int_{S^{2n-1}} |\langle \psi | S | \psi \rangle|^4 + 2|\langle \psi | A | \psi \rangle|^4 + |\langle \psi | A | \psi \rangle|^2 |\langle \psi | S | \psi \rangle|^2 dV.
\]

(11)

The first two terms on the right-hand side of (11) are readily evaluated using (9) and the explicit expressions $S = \frac{1}{2} (M + M^\dagger)$ and $A = \frac{1}{2} (M - M^\dagger)$.

To calculate the third term, we use the conjugation invariance to diagonalize the Hermitian part $S$, and thus

\[
\int_{S^{2n-1}} |\langle \psi | S | \psi \rangle|^2 |\langle \psi | A | \psi \rangle|^2 dV = \int_{S^{2n-1}} |\langle \psi | A | \psi \rangle|^2 |\langle \psi | \tilde{A} | \psi \rangle|^2 dV,
\]

(12)

where $A = U S U^{-1}$ is a diagonal matrix with elements $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and $\tilde{A} = U A U^{-1}$ is an anti-Hermitian matrix, which is not necessarily diagonal. Expanding the state vectors in the eigenbasis of $S$, $|\psi\rangle = \sum c_j |j\rangle$ and employing the invariance of the integral under the transformation $c_j \rightarrow \exp(i\theta_j)c_j$, it follows that

\[
\int_{S^{2n-1}} |\langle \psi | \tilde{A} | \psi \rangle|^2 |\langle \psi | \tilde{A} | \psi \rangle|^2 dV = \sum_{i,j,k,l} [\lambda_i^2 |\tilde{A}_{ii}|^2 + \lambda_j^2 |\tilde{A}_{jj}|^2 + |\tilde{A}_{ij}|^2 + 2|\tilde{A}_{ij}|^2 + 2|\tilde{A}_{ij}|^2 + 2|\tilde{A}_{ij}|^2]
\]

(13)

where the last step follows upon insertion of the expressions obtained in (12). The notation $\sum'$ indicates that all indices must be different. After a lengthy, but straightforward calculation, (13) can be rewritten in
terms of traces of products of powers of $\Lambda$ and $\hat{\Lambda}$. Invoking trace invariance, we can replace $\Lambda$ and $\hat{\Lambda}$ with $S$ and $A$, respectively, and insert the explicit expressions for $S$ and $A$ in terms of $M$ and $M^\dagger$. Collecting terms in (11), we finally obtain the mean value of the squared fidelity for a linear transformation $M$, 

$$
\langle f^2 \rangle = \frac{1}{n(n+1)(n+2)(n+3)} \left[ 4\text{Tr}(M^2M^\dagger) + 2\text{Tr}(MM^\dagger M^\dagger M^\dagger) 
+ 4\text{Tr}(M)\text{Tr}(MM^\dagger) + 4\text{Tr}(M^\dagger)\text{Tr}(M^2M^\dagger) 
+ \text{Tr}(M^2\text{Tr}(M^\dagger)^2) + 2\text{Tr}(MM^\dagger)^2 
+ \text{Tr}(M^\dagger)^2\text{Tr}(M)M^\dagger 
+ 4\text{Tr}(M(M^\dagger)^2)\text{Tr}(M^2M^\dagger) + \text{Tr}(M^2\text{Tr}(M^\dagger)^2) \right].
$$

(14)

Combining (5) and (14), we obtain the variance of the fidelity distribution for a linear transformation $|\psi\rangle \mapsto N|\psi\rangle$, using 

$$
\sigma_f^2 = \langle f^2 \rangle - \langle f \rangle^2 = \int_{S^{2n-1}} \langle f^2 \rangle dV - \left( \int_{S^{2n-1}} \langle f \rangle dV \right)^2,
$$

with $M = U_0^d N$, where $U_0$ and $N$ are the desired and actual evolution operators of the system. As in Sec. IIIA this result is easily extended to the case where the average is solely performed over a subset of input states by simply replacing $n$ with $n_{rel}$, and $M$ with $M_{rel} = PU_0^d PNP$, where $P$ is the corresponding projection operator on the relevant subspace.

V. SUMMARY

We have in this paper derived an expression for the fidelity distribution, applicable to a normal linear state vector transformation in a two dimensional complex Hilbert space. We have also presented simple and compact expressions for the average fidelity of a general quantum operation, and for the variance of the fidelity of a linear state vector map. Such simple expressions constitute a good starting point for further analysis, e.g., for the achievements of error correcting codes [15], decoherence free subspaces [16, 17], and protection of quantum information by dynamical decoupling [18]. Our expression can also be handled and generalized analytically, as illustrated by our study in [1] of a $K$-qudit register, which provides insight into the scaling of errors. This may have applications in quantum error correction, the capacity of quantum channels, and the way that, e.g., communication with quantum repeaters [19] and entanglement distillation should optimally be carried out. Although we did not consider that possibility here, we note that the ability to restrict averages to subspaces may also enable generalization of our formalism to deal with non-uniform averages, assuming nontrivial prior probability distributions on the Hilbert space.

[1] L. H. Pedersen, N. M. Møller and K. Mølmer, Phys. Lett. A 367, 47 (2007)
[2] J. von Neumann, Ann. Math. Stat. 12, 367 (1941)
[3] I. Roos and K. Mølmer, Phys. Rev. A 69, 022321 (2004)
[4] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. A 60, 1888 (1999)
[5] M. A. Nielsen, Phys. Lett. A 303, 249 (2002)
[6] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. A 60, 1888 (1999)
[7] E. Bagan, M. Baig and R. Muñoz-Tapia, Phys. Rev. A 67, 014303 (2003)
[8] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge Univ. Press, 2000)
[9] P. Zanardi and D. A. Lidar, Phys. Rev. A 70, 012315 (2004)
[10] C. Dankert, Efficient simulation of random quantum states and operators, Master’s thesis, University of Waterloo (2005), quant-ph/0512217
[11] A. Ambainis and J. Emerson, Proceedings from IEEE Conference on Computational Complexity ’07; quant-ph/0701126
[12] J. Dalibard, Y. Castin and K. Mølmer, Phys. Rev. Lett. 68, 580 (1992)
[13] K. Mølmer and Y. Castin, Quantum and Semiclass. Opt. 8, 49 (1996).
[14] P. Treutlein, P. Hommelhoff, T. Steinmetz, T. W. Hänsch, and J. Reichel, Phys. Rev. Lett. 92, 203005 (2004).
[15] A. M. Steane, Nature 399, 124 (1999)
[16] P. Zanardi and M. Rasetti, Phys. Rev. Lett. 79, 3306 (1997)
[17] D. A. Lidar, I. L. Chuang and K. B. Whaley, Phys. Rev. Lett. 81, 2594 (1998)
[18] L. Viola, E. Knill and S. Lloyd, Phys. Rev. Lett. 82, 2417 (1999)
[19] L. M. Duan, M. D. Lukin, J. I. Cirac and P. Zoller, Nature 414, 413 (2001)
[20] This expression replaces an incorrect expression in [1].