Shortened linear codes from APN and PN functions

Can Xiang, Chunming Tang and Cunsheng Ding

Abstract

Linear codes generated by component functions of perfect nonlinear (PN for short) and almost perfect nonlinear (APN for short) functions and first-order Reed-Muller codes have been an object of intensive study by many coding theorists. In this paper, we investigate some binary shortened code of two families of linear codes from APN functions and some $p$-ary shortened codes from PN functions. The weight distributions of these shortened codes and the parameters of their duals are determined. The parameters of these binary codes and $p$-ary codes are flexible. Many of the codes presented in this paper are optimal or almost optimal in the sense that they meet some bound on linear codes. These results show high potential for shortening to be used in designing good codes.

Index Terms

Linear code, shortened code, PN function, APN function, $t$-design

I. INTRODUCTION

Let $\mathbb{GF}(q)$ denote the finite field with $q = p^m$ elements, where $p$ is a prime and $m$ is a positive integer. An $[v, k, d]$ linear code $C$ over $\mathbb{GF}(q)$ is a $k$-dimensional subspace of $\mathbb{GF}(q)^v$ with minimum (Hamming) distance $d$. Let $A_i$ denote the number of codewords with Hamming weight $i$ in a code $C$ of length $v$. The weight enumerator of $C$ is defined by $1 + A_1 z + A_2 z^2 + \cdots + A_v z^v$. The sequence $(1, A_1, \ldots, A_v)$ is called the weight distribution of $C$ and it is an important research topic in coding theory, as it contains crucial information to estimate the error correcting capability. Thus the study of the weight distribution attracts much attention in coding theory and much work focuses on the determination of the weight distributions of linear codes (see, for examples, [32], [34], [14], [15], [16], [17], [36], [45], [46], [39]). Denote by $C^\perp$ and $(A_0^\perp, A_1^\perp, \ldots, A_v^\perp)$ the dual code of a linear code $C$ and its weight distribution, respectively. It is known that the Pless power moments [28]

$$\sum_{i=0}^{v} \binom{v}{i} A_i = \sum_{i=0}^{t} (-1)^i A_i^\perp \left[ \sum_{j=i}^{t} j! S(t, j) q^{k-j} (q-1)^{j-i} \left( \frac{v-i}{v-j} \right) \right],$$

(1)

play an important role in calculating the weight distributions of linear codes, where $A_0 = 1$, $0 \leq t \leq v$ and $S(t, j) = \frac{1}{t} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^t$. A code $C$ is said to be a $t$-weight code if the number of nonzero $A_i$ in the sequence $(A_1, A_2, \ldots, A_v)$ is equal to $t$. We call a $[v, k, d]$ code distance-optimal if no $[v, k, d+1]$ code exists and dimension-optimal if no $[v, k+1, d]$ code exists. An $[v, k, d]$ code is said to be length-optimal if there is no $[v', k, d]$ code exists with $v' < v$. A code is said to be optimal if it is distance-optimal, dimension-optimal and length-optimal.

Let $C$ be a $[v, k, d]$ linear code over $\mathbb{GF}(q)$ and $T$ a set of $t$ coordinate positions in $C$. We use $C^T$ to denote the code obtained by puncturing $C$ on $T$, which is called the punctured code of $C$ on $T$. Let

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Let \( C(T) \) be the set of codewords which are \( \mathbf{0} \) on \( T \). We now puncture \( C(T) \) on \( T \), and obtain a linear code \( C_T \), which is called the shortened code of \( C \) on \( T \). The following property plays an important role in determining the parameters of the punctured and shortened codes of \( C \) in [28, Theorem 1.5.7].

**Lemma 1.** [28] Let \( C \) be a \([v,k,d]\) linear code over \( \text{GF}(q) \) and \( d^\perp \) the minimum distance of \( C^\perp \). Let \( T \) be any set of \( t \) coordinate positions. Then

- \( (C_T)^\perp = (C^\perp)^T \) and \( (C_T)^\perp = (C^\perp)^T \).
- If \( t < \min\{d,d^\perp\} \), then the codes \( C_T \) and \( C_T^\perp \) have dimension \( k-t \) and \( k \), respectively.

It is worth noting that the shortening and puncturing technologies are two important approaches to constructing new linear codes. Very recently, Tang et al. obtained some ternary linear codes with few weights by shortening and puncturing a class of ternary codes in [37]. Afterwards, they presented a general theory for punctured and shortened codes of linear codes supporting t-designs and generalized Assmus-Mattson theorem in [38]. Some linear codes and t-designs can be obtained and their parameters can be also derived. However, till now few results on constructing punctured and shortened codes have been done and it is in general hard to determine their weight distributions. Motivated by this fact, we obtain some shortened codes of linear codes from almost perfect nonlinear (APN for short) and perfect nonlinear (PN for short) functions, and determine their parameters. Some of these codes are optimal or almost optimal.

The rest of this paper is arranged as follows. Section II introduces some notation and results related to group characters, Gauss sums, t-designs and linear codes from APN and PN functions. Section III gives some general results on shortened codes. Section IV investigates some shortened codes of binary linear codes from almost perfect nonlinear (APN for short) and perfect nonlinear (PN for short) functions. Section V studies some shortened codes of two classes of special linear codes from PN functions. Section VI concludes this paper and makes concluding remarks.

## II. Preliminaries

In this section, we briefly recall some results on group characters, Gauss sums, t-designs, and linear codes from APN and PN functions. These results will be used later in this paper. We begin this section by fixing some notations throughout this paper.

- \( p^* = (-1)^{(p-1)/2}p \).
- \( \zeta_p = e^{-2\pi i/p} \) is the primitive \( p \)-th root of unity.
- \( \text{GF}(q)^* = \text{GF}(q) \setminus \{0\} \).
- \( \text{Tr}_{q/p} \) is the trace function from \( \text{GF}(q) \) to \( \text{GF}(p) \).
- SQ and NSQ denote the set of all squares and nonsquares in \( \text{GF}(p)^* \), respectively.
- \( \eta \) and \( \bar{\eta} \) are the quadratic characters of \( \text{GF}(q)^* \) and \( \text{GF}(p)^* \), respectively. We extend these quadratic characters by letting \( \eta(0) = 0 \) and \( \bar{\eta}(0) = 0 \).

### A. Group characters and Gauss sums

An additive character of \( \text{GF}(q) \) is a nonzero function \( \chi \) from \( \text{GF}(q) \) to the set of nonzero complex numbers such that \( \chi(x+y) = \chi(x)\chi(y) \) for any pair \((x,y) \in \text{GF}(q)^2 \). For each \( b \in \text{GF}(q) \), the function

\[
\chi_b(c) = \zeta_p^{\text{Tr}(bc)} \quad \text{for all } c \in \text{GF}(q)
\]

defines an additive character of \( \text{GF}(q) \). When \( b = 0 \), \( \chi_0(c) = 1 \) for all \( c \in \text{GF}(q) \), and \( \chi_0 \) is called the trivial additive character of \( \text{GF}(q) \). The character \( \chi_1 \) in \[2\] is called the canonical additive character of \( \text{GF}(q) \). It is well known that every additive character of \( \text{GF}(q) \) can be written as \( \chi_b(x) = \chi_1(bx) \) [30, Theorem 5.7]. The orthogonality relation of additive characters is given by

\[
\sum_{x \in \text{GF}(q)} \chi_1(ax) = \begin{cases} q & \text{for } a = 0, \\ 0 & \text{for } a \in \text{GF}(q)^*. \end{cases}
\]
The Gauss sum $G(\eta, \chi)$ over $\text{GF}(q)$ is defined by
\[
G(\eta, \chi) = \sum_{c \in \text{GF}(q)^*} \eta(c)\chi_1(c) = \sum_{c \in \text{GF}(q)} \eta(c)\chi_1(c)
\tag{3}
\]
and the Gauss sum $G(\bar{\eta}, \bar{\chi}_1)$ over $\text{GF}(p)$ is defined by
\[
G(\bar{\eta}, \bar{\chi}_1) = \sum_{c \in \text{GF}(p)^*} \eta(c)\bar{\chi}_1(c) = \sum_{c \in \text{GF}(p)} \eta(c)\bar{\chi}_1(c),
\tag{4}
\]
where $\bar{\chi}_1$ is the canonical additive character of $\text{GF}(p)$.

The following four lemmas are proved in [30, Theorem 5.15, Theorem 5.33, Corollary 5.35] and [16, Lemma 7], respectively.

Lemma 2. [30] Let $q = p^m$ and $p$ be an odd prime. Then
\[
G(\eta, \chi_1) = (-1)^{m-1} (\sqrt{-1})^{(\frac{m-1}{2})^2m} \sqrt{q}
\]
\[
= \left\{ \begin{array}{ll}
(-1)^{m-1} \sqrt{q} & \text{for } p \equiv 1 \pmod{4}, \\
(-1)^{m-1} (\sqrt{-1})^m \sqrt{q} & \text{for } p \equiv 3 \pmod{4}.
\end{array} \right.
\]
and
\[
G(\bar{\eta}, \bar{\chi}_1) = \sqrt{-1}^{(\frac{m-1}{2})^2} \sqrt{p} = \sqrt{p^e}.
\]

Lemma 3. [30] Let $\chi$ be a nontrivial additive character of $\text{GF}(q)$ with $q$ odd, and let $f(x) = a_2x^2 + a_1x + a_0 \in \text{GF}(q)[x]$ with $a_2 \neq 0$. Then
\[
\sum_{c \in \text{GF}(q)} \chi(f(c)) = \chi(a_0 - a_1^2(4a_2)^{-1})\eta(a_2)G(\eta, \chi).
\]

Lemma 4. [30] Let $\chi_b$ be a nontrivial additive character of $\text{GF}(q)$ with $q$ even and $f(x) = a_2x^2 + a_1x + a_0 \in \text{GF}(q)[x]$, where $b \in \text{GF}(q)^*$. Then
\[
\sum_{c \in \text{GF}(q)} \chi_b(f(c)) = \left\{ \begin{array}{ll}
\chi_b(a_0)q & \text{if } a_2 = ba_1^2, \\
0 & \text{otherwise}.
\end{array} \right.
\]

Lemma 5. [16] If $m \geq 2$ is even, then $\eta(y) = 1$ for each $y \in \text{GF}(p)^*$. If $m \geq 1$ is odd, then $\eta(y) = \bar{\eta}(y)$ for each $y \in \text{GF}(p)$.

Let $e$ be a positive integer and $(a, b) \in \text{GF}(q)^2$, define the exponential sum
\[
S_e(a, b) = \sum_{x \in \text{GF}(q)} \chi_1 \left(ax^{e+1} + bx \right).
\tag{5}
\]

Then we have the following results in [12], [8], [9] and [43].

Lemma 6. [12] Let $e$ be a positive integer and $m$ be even with $\gcd(m, e) = 1$. Let $p = 2$, $q = 2^m$ and $a \in \text{GF}(q)^*$. Then
\[
S_e(a, 0) = \left\{ \begin{array}{ll}
(-1)^m 2^{\frac{m}{2}} & \text{if } a \neq \alpha^t \text{ for any } t, \\
-(-1)^m 2^{\frac{m}{2}+1} & \text{if } a = \alpha^t \text{ for some } t,
\end{array} \right.
\]
where $\alpha$ is a generator of $\text{GF}(q)^*$.

Lemma 7. [8] Let $e, h$ be positive integers and $m$ be even with $\gcd(m, e) = 1$. Let $p = 2$, $q = 2^m$ and $a \in \text{GF}(q)^*$. Then
\[
\sum_{b \in \text{GF}(q)^*} (S_e(a, b))^h = \left\{ \begin{array}{ll}
(2^m - 1)2^{\frac{m-h}{2}} & \text{if } h \text{ is even and } a \neq \alpha^t \text{ for any } t, \\
(2^{m-2} - 1)2^{\frac{m+1-h}{2}} & \text{if } h \text{ is even and } a = \alpha^t \text{ for some } t,
\end{array} \right.
\]
Then
\[ S_{e}(a, b) = \begin{cases} (-1)^{m-1} \sqrt{q} \eta(-a) \chi_{1}(-ax_{a, b}^{p^{e}+1}), & \text{if } p \equiv 1 \mod 4, \\ (-1)^{m-1} i^{3m} \sqrt{q} \eta(-a) \chi_{1}(-ax_{a, b}^{p^{e}+1}), & \text{if } p \equiv 3 \mod 4. \end{cases} \]

**Lemma 8.** \([9]\) Let \( p \) be an odd prime, \( q = p^{m} \), and \( e \) be any positive integer such that \( m/\gcd(m, e) \) is odd. Suppose \( a \in GF(q^{*}) \) and \( b \in GF(q^{*}) \). Let \( x_{a, b} \) be the unique solution of the equation
\[ a^{p^{e}} x^{p^{e}2^{e}} + ax + b^{p^{e}} = 0. \]
Then
\[ \Delta = \begin{cases} 0, & \text{if } \text{Tr}_{q/p}(a(x_{a, b})^{p^{e}+1}) = 0, \\ \eta(a) \sqrt{q}, & \text{if } p \equiv 1 \mod 4 \text{ and } \text{Tr}_{q/p}(a(x_{a, b})^{p^{e}+1}) \neq 0. \end{cases} \]

**Lemma 9.** \([43]\) Let \( p \) be an odd prime, \( q = p^{m} \), and \( e \) be any positive integer such that \( m/\gcd(m, e) \) is odd. Suppose \( a \in GF(q^{*}) \) and \( b \in GF(q^{*}) \). Let \( x_{a, b} \) be the unique solution of the equation \( a^{p^{e}} x^{p^{e}2^{e}} + ax + b^{p^{e}} = 0 \) and \( \Delta = \sum_{c \in GF(p^{*})} S_{e}(ac, bc) \). Then we have the following results.
- If \( m \) is odd, then
  \[ \Delta = \begin{cases} 0, & \text{if } \text{Tr}_{q/p}(a(x_{a, b})^{p^{e}+1}) = 0, \\ \eta(a) \sqrt{q}, & \text{if } p \equiv 1 \mod 4 \text{ and } \text{Tr}_{q/p}(a(x_{a, b})^{p^{e}+1}) \neq 0. \end{cases} \]
- If \( m \) is even, then
  \[ \Delta = \begin{cases} -(p-1) \eta(a) \sqrt{q}, & \text{if } p \equiv 1 \mod 4 \text{ and } \text{Tr}_{q/p}(a(x_{a, b})^{p^{e}+1}) = 0, \\ \eta(a) \sqrt{q}, & \text{if } p \equiv 1 \mod 4 \text{ and } \text{Tr}_{q/p}(a(x_{a, b})^{p^{e}+1}) \neq 0, \\ -im(p-1) \eta(a) \sqrt{q}, & \text{if } p \equiv 3 \mod 4 \text{ and } \text{Tr}_{q/p}(a(x_{a, b})^{p^{e}+1}) = 0, \\ im \eta(a) \sqrt{q}, & \text{if } p \equiv 3 \mod 4 \text{ and } \text{Tr}_{q/p}(a(x_{a, b})^{p^{e}+1}) \neq 0. \end{cases} \]

**Lemma 10.** \([43]\) Let \( p \) be an odd prime, \( m \) and \( e \) be positive integers such that \( m/\gcd(m, e) \) is odd. Let \( q = p^{m} \). Define
\[ \hat{N}(a, b) = \{ x \in GF(q) : \text{Tr}_{q/p}(ax^{p^{e}+1} + bx) = 0 \}. \]
Then we have the following results.
- If \( a = 0 \) and \( b = 0 \), then \( \hat{N}(a, b) = q. \)
- If \( a = 0 \) and \( b \neq 0 \), then \( \hat{N}(a, b) = p^{m-1}. \)
- If \( a \neq 0 \) and \( b = 0 \), then
  \[ \hat{N}(a, b) = \begin{cases} p^{m-1}, & \text{if } m \text{ is odd}, \\ \frac{1}{q}(q-(p-1) \eta(a) \sqrt{q}), & \text{if } m \text{ is even and } p \equiv 1 \mod 4, \\ \frac{1}{q}(q-im(p-1) \eta(a) \sqrt{q}), & \text{if } m \text{ is even and } p \equiv 3 \mod 4. \end{cases} \]
- If \( a \neq 0 \) and \( b \neq 0 \), then \( \hat{N}(a, b) = \frac{1}{p}(q + \Delta), \) where \( R \) was given in Lemma \([9]\). 

**B. \( t \)-designs and related results**

Let \( k, t \) and \( v \) be positive integers with \( 1 \leq t \leq k \leq v \). Let \( \mathcal{P} \) be a set of \( v \geq 1 \) elements, and let \( \mathcal{B} \) be a set of \( k \)-subsets of \( \mathcal{P} \). The incidence structure \( \mathcal{D} = (\mathcal{P}, \mathcal{B}) \) is said to be a \( t \)-(\( v, k, \lambda \)) design if every \( t \)-subset of \( \mathcal{P} \) is contained in exactly \( \lambda \) elements of \( \mathcal{B} \). The elements of \( \mathcal{P} \) are called points, and those of \( \mathcal{B} \) are referred to as blocks. We usually use \( b \) to denote the number of blocks in \( \mathcal{B} \). A \( t \)-design is called simple if \( \mathcal{B} \) has no repeated blocks. A \( t \)-design is called symmetric if \( v = b \) and trivial if \( k = t \) or \( k = v \). When \( t \geq 2 \) and \( \lambda = 1 \), a \( t \)-design is called a Steiner system and traditionally denoted by \( S(t, k, v) \).
Linear codes and $t$-designs are companions. A $t$-design $\mathbb{D} = (\mathcal{P}, \mathcal{B})$ induces a linear code over $\text{GF}(p)$ for any prime $p$. Let $\mathcal{P} = \{p_1, \ldots, p_\nu\}$. For any block $B \in \mathcal{B}$, the characteristic vector of $B$ is defined by the vector $c_B = (c_1, \ldots, c_\nu) \in \{0, 1\}^\nu$, where
\[
c_i = \begin{cases} 1, & \text{if } p_i \in B, \\ 0, & \text{if } p_i \not\in B. \end{cases}
\]

For a prime $p$, a linear code $C_p(\mathbb{D})$ over the prime field $\text{GF}(p)$ from the design $\mathbb{D}$ is spanned by the characteristic vectors of blocks of $\mathcal{B}$, which is the subspace $\text{Span}\{\mathbf{v}_B : B \in \mathcal{B}\}$ of the vector space $\text{GF}(p)^{\nu}$. Linear codes $C_p(\mathbb{D})$ from designs $\mathbb{D}$ have been studied and documented in the literature (see, for examples, [1], [17], [40], [41]). A major approach to constructing $t$-designs from linear codes is the use of linear codes with $t$-homogeneous or $t$-transitive automorphism groups (see [14, Theorem 4.18]). Another major approach to constructing $t$-designs from codes is the use of the Assmus-Mattson Theorem [3], [28]. The following Assmus-Mattson Theorem for constructing simple $t$-designs is given in [2].

**Theorem 11.** Let $C$ be a linear code over $\text{GF}(q)$ with length $\nu$ and minimum weight $d$. Let $C^\perp$ with minimum weight $d^\perp$ denote the dual code of $C$. Let $t$ be an integer such that $1 \leq t < \min\{d, d^\perp\}$ and $\nu - t \leq d^\perp$. Then the following holds:

- $(\mathcal{P}(C), \mathcal{B}_w(C))$ is a simple $t$-design provided that $A_k \neq 0$ and $d \leq k \leq w$, where $w$ is defined to be the largest integer satisfying $w \leq \nu$ and
  \[w - \left\lceil \frac{w + q - 2}{q - 1} \right\rceil < d,\]

- $(\mathcal{P}(C^\perp), \mathcal{B}_w(C^\perp))$ is a simple $t$-design provided that $A_k^\perp \neq 0$ and $d^\perp \leq k \leq w^\perp$, where $w^\perp$ is defined to be the largest integer satisfying $w^\perp \leq \nu$ and
  \[w^\perp - \left\lceil \frac{w^\perp + q - 2}{q - 1} \right\rceil < d^\perp.\]

We will need the following results on the punctured and shortened codes of $C$ in [38, Lemma 3.1, Theorem 3.2].

**Lemma 12.** [38] Let $C$ be a linear code of length $\nu$ and minimum distance $d$ over $\text{GF}(q)$ and $d^\perp$ the minimum distance of $C^\perp$. Let $t$ and $k$ be two positive integers with $0 < t < \min\{d, d^\perp\}$ and $1 \leq k \leq \nu - t$. Let $T$ be a set of $t$ coordinate positions in $C$. Suppose that $(\mathcal{P}(C), \mathcal{B}_i(C))$ is a $t$-design for all $i$ with $k \leq i \leq k + t$. Then
\[
A_k(C^T) = \sum_{i=0}^{t} \frac{\binom{\nu - t}{i} \binom{k + i}{t} \binom{t}{i}}{\binom{\nu - t + i}{t}} A_{k+i}(C).
\]

**Theorem 13.** [38] Let $C$ be a $[\nu, k, d]$ linear code over $\text{GF}(q)$ and $d^\perp$ be the minimum distance of $C^\perp$. Let $t$ be a positive integer with $0 < t < \min\{d, d^\perp\}$. Let $T$ be a set of $t$ coordinate positions in $C$. Suppose that $(\mathcal{P}(C), \mathcal{B}_i(C))$ is a $t$-design for any $i$ with $d \leq i \leq \nu - t$. Then the shortened code $C^T$ is a linear
code of length \( v - t \) and dimension \( \tilde{k} - t \). The weight distribution \( (A_k(C_T))_{k=0}^{Y-t} \) of \( C_T \) is independent of the specific choice of the elements in \( T \). Specifically,

\[
A_k(C_T) = \binom{k}{t} \frac{(Y-t)}{(k)} \frac{k}{(k-t)} A_k(C).
\]

C. Linear codes from APN and PN functions

Let \( m, \tilde{m} \) be two positive integers with \( m \geq \tilde{m} \) and \( F \) be a mapping from \( GF(p^m) \) to \( GF(p^{\tilde{m}}) \). Define

\[
\delta_F = \max \{ \delta_F(a, b) : a \in GF(p^m)^*, b \in GF(p^{\tilde{m}}) \},
\]

where \( \delta_F(a, b) = \#\{ x \in GF(p^m) \setminus F(x+a) = F(x) = b, a \in GF(p^m) \} \) and \( \delta_F \) is a polynomial over \( GF(p^m) \). The function \( F(x) \) is called PN function if \( \delta_F = p^{m-\tilde{m}} \) and it is called APN function if \( m = \tilde{m} \) and \( \delta_F = 2 \). From the above definition one immediately has that \( F(x) \) is PN if and only if \( F(x+a) = F(x) \) is balanced for each \( a \in GF(p^m)^* \). Currently, all known PN and APN functions over \( GF(p^m) \) can be summarized in [14], [4], [10], [11], [13], [22], [44], [7], [6]. It is known that PN and APN functions are very important functions by constructing linear codes with good parameters (see, for examples, [43], [5], [33], [42]).

Let \( q = p^m \) and \( C \) denote the linear code of length \( q \) as follows:

\[
C = \{ ((Tr_{q/p}(af(x) + bx + c))_{x \in GF(q)} : a, b, c \in GF(q) \}, \tag{6}
\]

where \( f(x) \) is a polynomial over \( GF(q) \). Then we can regard \( GF(q) \) as the set of the coordinate positions \( \mathcal{P}(C) \) of \( C \). It is known that \( C \) has dimension \( 2m + 1 \) and the weight distribution in Table I when \( p = 2 \), \( m \geq 5 \) is odd and \( f(x) = x^t \) is an APN function, where \( s \) takes the following values [14].

• \( s = 2^e + 1 \), where \( gcd(e, m) = 1 \) and \( e \) is a positive integer.
• \( s = 2^e - 2^e + 1 \), where \( e \) is a positive integer and \( gcd(e, m) = 1 \).
• \( s = 2(3m-1)/2 + 3 \),
• \( s = 2(3m-1)/2 + 2(3m-1)/4 - 1 \), where \( m \equiv 1 \pmod{4} \).
• \( s = 2(3m-1)/2 + 2(3m-1)/4 - 1 \), where \( m \equiv 3 \pmod{4} \).

When \( f(x) = x^{2^e+1}, p = 2 \) and \( m \geq 4 \) is even with \( gcd(e, m) = 1 \), the code \( C \) defined in (6) has dimension \( 2m + 1 \) and the weight distribution in Table III [14].

| Weight | Multiplicity |
|--------|-------------|
| \( 2m-1 \) | \( 2^{m-1} \) |
| \( 2m-1 \) | \( 2^{m-1} \) |
| \( 2m-1 \) | \( 2^{m-1} \) |
| \( 2m \) | \( 1 \) |

It is known that the code \( C \) defined in (6) has dimension \( 2m + 1 \) and a few weights when \( p \) is an odd prime and \( f(x) = x^t \) is a PN function. If \( s \) takes the following values [14], [31]

• \( s = 2 \),
• \( s = p^e + 1 \), where \( m/gcd(m,e) \) is odd.
• \( s = (3^e + 1)/2 \), where \( p = 3 \), \( e \) is odd and \( gcd(m,e) = 1 \),

then \( f(x) = x^t \) is a PN and also planar function, \( Tr_{q/p}(\beta f(x)) \) is a weakly regular bent function [23], [27] for any \( \beta \in GF(q)^* \), and the code \( C \) defined in (6) has four or six weights [31].

Let \( f(x) \) be a function from \( GF(q) \) to \( GF(p) \), the Walsh transform of \( f \) at a point \( \beta \in GF(q) \) is defined by

\[
W_f(\beta) = \sum_{x \in GF(q)} \zeta_p^{f(x) - Tr_{q/p}(\beta x)}.
\]
The function $f(x)$ is said to be a $p$-ary bent function, if $|\mathcal{W}_f(\beta)| = p^m$ for any $\beta \in \mathbb{F}_q$. A bent function $f(x)$ is weakly regular if there exists a complex $u$ with unit magnitude satisfying $\mathcal{W}_f(\beta) = u^{\pm \beta} \mathcal{W}_f^*(\beta)$ for some function $f^*(x)$. Such function $f^*(x)$ is called the dual of $f(x)$. A weakly regular bent function $f(x)$ satisfies

$$\mathcal{W}_f(\beta) = \varepsilon \sqrt{p^m} \mathcal{W}_f^*(\beta),$$

where $\varepsilon = \pm 1$ is called the sign of the Walsh Transform of $f(x)$. Let $\mathcal{R} \mathcal{F}$ be the set of $p$-ary weakly regular bent functions with the following two properties:

- $f(0) = 0$; and
- $f(ax) = a^h f(x)$ for any $a \in \mathbb{GF}(p)^*$ and $x \in \mathbb{GF}(q)$, where $h$ is a positive even integer with $\gcd(h - 1, p - 1) = 1$.

We will need the following results on $p$-ary weakly regular bent functions in [39].

**Lemma 14.** [39] Let $\beta \in \mathbb{GF}(q)^*$ and $f(x) \in \mathcal{R} \mathcal{F}$ with $\mathcal{W}_f(0) = \varepsilon \sqrt{p^m}$. Define

$$N_{f,\beta} = \{x \in \mathbb{GF}(q) : f(x) = 0 \text{ and } \text{Tr}_{q/p}(\beta x) = 0\}.$$

If $f^*(\beta) = 0$, then

$$N_{f,\beta} = \begin{cases} p^{m-2} + \varepsilon \sqrt{p^m} / 2 (-1)(p - 1) p^{(m-2)/2}, & \text{if } m \text{ is even;} \\ p^{m-2}, & \text{if } m \text{ is odd.} \end{cases}$$

**Lemma 15.** [39] Let $\beta \in \mathbb{GF}(q)^*$ and $f(x) \in \mathcal{R} \mathcal{F}$ with $\mathcal{W}_f(0) = \varepsilon \sqrt{p^m}$. Let

$$N_{sq,\beta} = \{x \in \mathbb{GF}(q) : f(x) \in SQ \text{ and } \text{Tr}_{q/p}(\beta x) = 0\},$$

and

$$N_{nsq,\beta} = \{x \in \mathbb{GF}(q) : f(x) \in NSQ \text{ and } \text{Tr}_{q/p}(\beta x) = 0\}.$$

We have the following results.

- If $m$ is even and $f^*(\beta) = 0$, then

$$N_{sq,\beta} = N_{nsq,\beta} = \frac{p - 1}{2} \left( p^{m-2} - \varepsilon \sqrt{p^m} / 2 (-1)(p - 1) p^{(m-2)/2}\right).$$

- If $m$ is odd and $f^*(\beta) = 0$, then

$$N_{sq,\beta} = \frac{p - 1}{2} \left( p^{m-2} + \varepsilon \sqrt{p}^{m-1}\right)$$

and

$$N_{nsq,\beta} = \frac{p - 1}{2} \left( p^{m-2} - \varepsilon \sqrt{p}^{m-1}\right).$$

### Table II

| Weight       | Multiplicity |
|--------------|--------------|
| $2^{m-1}$    | 1            |
| $2^{m-1} - 2^{m/2}$ | $2^m - 1)^{2m-2}/3$ |
| $2^{m-1} - 2^{(m-2)/2}$ | $(2^m - 1)^{2m+1}/3$ |
| $2^{m-1}$    | $2(2^m - 1)^{2m-2} + 1$ |
| $2^{m-1} + 2^{m-2}/2$ | $(2^m - 1)^{2m+1}/3$ |
| $2^{m-1} + 2^{m/2}$ | $(2^m - 1)^{2m-2}/3$ |
| $2^m$        | 1            |
III. Shortened binary linear codes with special weight distributions

In this section, we give some general results on the shortened codes of linear codes with the weight distributions in Tables I and II. Let \( T \) be a \( t \)-subset of \( P \). Let \( \Lambda_{T,w}(C) = \{\text{Supp}(c) : \text{wt}(c) = w, \; c \in C \text{ and } T \subseteq \text{Supp}(c)\} \).

and \( \lambda_{T,w}(C) = \#\Lambda_{T,w}(C) \).

A. Shortened linear codes holding \( t \)-designs

Let \( p = 2 \) and \( q = 2^m \). Notice that if a binary code \( C \) has length \( 2^m \) and the weight distribution in Table II (resp. Table III), then the code \( C \) holds \( 3 \)-design (resp. \( 2 \)-design) in [14], [21]. The following two theorems are easily derived from Theorem [13], Tables I and II and we omit their proofs.

**Theorem 16.** Let \( m \geq 5 \) be odd, and \( C \) be a binary linear code with length \( 2^m \) and the weight distribution in Table I. Let \( T \) be a \( t \)-subset of \( P(C) \). We have the following results.

- If \( t = 1 \), then the shortened code \( C_T \) is a \([2^m - 1, 2m, 2^{m-1} - 2^{(m-1)/2}]\) binary linear code with the weight distribution in Table III.
- If \( t = 2 \), then the shortened code \( C_T \) is a \([2^m - 2, 2m - 1, 2^{m-1} - 2^{(m-1)/2}]\) binary linear code with the weight distribution in Table IV.
- If \( t = 3 \), then the shortened code \( C_T \) is a \([2^m - 3, 2m - 2, 2^{m-1} - 2^{(m-1)/2}]\) binary linear code with the weight distribution in Table V.

| Table III | The weight distribution of \( C_T \) for \( m \) odd and \( t = 1 \) |
|-----------|---------------------------------------------------------------|
| Weight \( 2^{m-1} - 2^{(m-1)/2} \) \( 2^{(m-5)/2}(2^m - 1)(2 + 2^{1+m}/2) \) | Multiplicity \( 1 \) |
| Weight \( 2^{m-1} \) \( -1 + 2^{m-1} + 2^{2m-1} \) | |
| Weight \( 2^{m-1} + 2^{(m-1)/2} \) \( 2^{(m-5)/2}(2^m - 1)(-2 + 2^{1+m}/2) \) | |

| Table IV | The weight distribution of \( C_T \) for \( m \) odd and \( t = 2 \) |
|-----------|---------------------------------------------------------------|
| Weight \( 2^{m-1} - 2^{(m-1)/2} \) \( 2^m - 7)(-4 + 2^{2+m} + 2^{1+3m}/2) \) | Multiplicity \( 1 \) |
| Weight \( 2^{m-1} \) \( -1 + 2^{2m-2} \) | |
| Weight \( 2^{m-1} + 2^{(m-1)/2} \) \( 2^{(m-7)/2}(4 - 2^{2+m} + 2^{1+3m}/2) \) | |

**Example 17.** Let \( m = 5 \) and \( T \) be a \( 1 \)-subset of \( P(C) \). Then the shortened code \( C_T \) in Theorem [16] is a \([31, 10, 12]\) binary linear code with the weight enumerator \( 1 + 310z^{12} + 527z^{16} + 186z^{20} \). The code \( C_T \) is optimal. The dual code \( C_T^* \) has parameters \([31, 21, 5]\) and is optimal according to the tables of best known codes maintained at [http://www.codetables.de](http://www.codetables.de).

**Example 18.** Let \( m = 5 \) and \( T \) be a \( 2 \)-subset of \( P(C) \). Then the shortened code \( C_T \) in Theorem [16] is a \([30, 9, 12]\) linear code with the weight enumerator \( 1 + 190z^{12} + 255z^{16} + 66z^{20} \). The code \( C_T \) is optimal.
The dual code of $C_T$ has parameters $[30, 21, 4]$ and is optimal according to the tables of best known codes maintained at [http://www.codetables.de](http://www.codetables.de).

**Example 19.** Let $m = 5$ and $T$ be a 3-subset of $\mathcal{P}(C)$. Then the shortened code $C_T$ in Theorem 16 is a $[29, 8, 12]$ binary linear code with the weight enumerator $1 + 114z^{12} + 119z^{16} + 22z^{20}$. The code $C_T$ is optimal. The dual code of $C_T$ has parameters $[29, 21, 3]$ and is almost optimal according to the tables of best known codes maintained at [http://www.codetables.de](http://www.codetables.de).

**Theorem 20.** Let $m \geq 4$ be even, and $C$ be a binary linear code with length $2^m$ and the weight distribution in Table VI. Let $T$ be a $t$-subset of $\mathcal{P}(C)$. We have the following results.

- If $t = 1$, then the shortened code $C_T$ is a $[2^m - 1, 2m, 2^{m-1} - 2^{m/2}]$ binary linear code with the weight distribution in Table VI.
- If $t = 2$, then the shortened code $C_T$ is a $[2^m - 2, 2m - 1, 2^{m-1} - 2^{m/2}]$ binary linear code with the weight distribution in Table VII.

**Table V**

| Weight                | Multiplicity                        |
|-----------------------|-------------------------------------|
| $2^{m-1} - 2^{(m-1)/2}$ | $-2^{(m-3)/2} + 3 \cdot 2^{(3m-7)/2} + 2^{m-3} + 2^{2m-4}$ |
| $2^{m-1}$             | $(-1 + 2^{m-2})(1 + 2^{m-1})$      |
| $2^{m-1} + 2^{(m-1)/2}$| $2^{(m-3)/2} - 3 \cdot 2^{(3m-7)/2} + 2^{m-3} + 2^{2m-4}$ |

**Table VI**

| Weight                | Multiplicity                        |
|-----------------------|-------------------------------------|
| $2^{m-1} - 2^{m/2}$   | $1/3 \cdot 2^{3+m/2}(2 + 2^{m/2})(-1 + 2^m)$ |
| $2^{m-1} - 2^{(m-2)/2}$ | $1/3 \cdot 2^{m/2}(-1 + 2^{m/2})(1 + 2^{m/2})^2$ |
| $2^{m-1}$             | $(2^m - 1)(1 + 2^{m-2})$            |
| $2^{m-1} + 2^{(m-2)/2}$ | $1/3 \cdot 2^{m/2}(-1 + 2^{m/2})^2(1 + 2^{m/2})$ |
| $2^{m-1} + 2^{m/2}$   | $1/3 \cdot 2^{3+m/2}(-2 + 2^{m/2})(-1 + 2^m)$ |

**Table VII**

| Weight                | Multiplicity                        |
|-----------------------|-------------------------------------|
| $2^{m-1} - 2^{m/2}$   | $1/3 \cdot 2^{m/2-4}(2 + 2^{m/2})(2^{m} + 2^{1+m/2} - 2)$ |
| $2^{m-1} - 2^{(m-2)/2}$ | $1/3 \cdot 2^{m/2-1}(1 + 2^{m/2})(2^{m} + 2^{m/2} - 2)$ |
| $2^{m-1}$             | $(2^{m-1} - 1)(1 + 2^{m-2})$      |
| $2^{m-1} + 2^{(m-2)/2}$ | $1/3 \cdot 2^{m/2-1}(-1 + 2^{m/2})(2^{m} - 2^{m/2} - 2)$ |
| $2^{m-1} + 2^{m/2}$   | $1/3 \cdot 2^{m/2-4}(4 + 2^{1+m/2} + 2^{m/2} - 2^{1+m})$ |
Example 21. Let \( m = 4 \) and \( T \) be a 1-subset of \( \mathcal{P}(C) \). Then the shortened code \( C_T \) in Theorem 20 is a \([15,8,4]\) linear code with the weight enumerator \( 1 + 15z^4 + 100z^6 + 75z^8 + 60z^{10} + 5z^{12} \). This code \( C_T \) is optimal. Its dual \( C_T^\perp \) has parameters \([15,7,5]\) and is optimal according to the tables of best known codes maintained at [http://www.codetables.de](http://www.codetables.de).

Example 22. Let \( m = 4 \) and \( T \) be a 2-subset of \( \mathcal{P}(C) \). Then the shortened code \( C_T \) in Theorem 20 is a \([14,7,4]\) binary linear code with the weight enumerator \( 1 + 11z^4 + 60z^6 + 35z^8 + 20z^{10} + z^{12} \). This code \( C_T \) is optimal. Its dual \( C_T^\perp \) has parameters \([14,7,4]\) and is optimal according to the tables of best known codes maintained at [http://www.codetables.de](http://www.codetables.de).

B. Several general results on shortened codes

Lemma 23. Let \( m \geq 5 \) be odd (resp. \( m \geq 4 \) be even), and \( C \) be a binary linear code with the length \( 2^m \) and the weight distribution in Table II (resp. Table II). Then the dual code \( C^\perp \) of \( C \) has parameters \([2^m,2^m - 2m - 1,6]\).

**Proof.** The weight distribution in Table II (or II) means that the dimension of \( C \) is \( 2m + 1 \). Thus, the dual code \( C^\perp \) of \( C \) has dimension \( 2^m - 2m - 1 \). Since the code length of \( C \) is \( 2^m \), from the weight distribution in Table II (or II) and the first seven Pless power moments in (II), it is easily obtain that \( A_6(C^\perp) = 0 \) and \( A_i(C^\perp) = 0 \) for any \( i \in \{1,2,3,4,5\} \). The desired conclusions then follow.

**Theorem 24.** Let \( m \geq 5 \) be odd (resp. \( m \geq 4 \) be even), and \( C \) be a binary linear code with length \( 2^m \) and the weight distribution in Table II (resp. Table II). Let \( T \) be a 4-subset of \( \mathcal{P}(C) \) and \( \lambda_{T,6}(C^\perp) = \lambda \), then \( \lambda = 0 \) or 1. Furthermore, we have the following results.

(I) If \( m \geq 5 \) is odd and \( \lambda = 0 \), then the shortened code \( C_T \) is a \([2^m - 4,2m - 3,2^m - 2,2^{(m-1)/2}]\) binary linear code with the weight distribution in Table VIII.

(II) If \( m \geq 5 \) is odd and \( \lambda = 1 \), then the shortened code \( C_T \) is a \([2^m - 4,2m - 3,2^m - 2,2^{(m-1)/2}]\) binary linear code with the weight distribution in Table IX.

| Weight | Multiplicity | Weight | Multiplicity |
|---|---|---|---|
| \( 2^m - 1 \) | 1 | \( 2^m - 1 \) | 1 |
| \( 2^m - 1 \) | 1 | \( 2^m - 1 \) | 1 |
| \( 2^m - 1 \) | 1 | \( 2^m - 1 \) | 1 |
| \( 2^m - 1 \) | 1 | \( 2^m - 1 \) | 1 |

| Weight | Multiplicity | Weight | Multiplicity |
|---|---|---|---|
| \( 2^m - 1 \) | 1 | \( 2^m - 1 \) | 1 |
| \( 2^m - 1 \) | 1 | \( 2^m - 1 \) | 1 |
| \( 2^m - 1 \) | 1 | \( 2^m - 1 \) | 1 |
| \( 2^m - 1 \) | 1 | \( 2^m - 1 \) | 1 |

**Proof.** By definition, we have

\[
\lambda = \lambda_{T,6}(C^\perp) = \#A_T(C^\perp) = \#\{\text{Supp}(e) : \text{wt}(e) = 6, \ e \in C^\perp \ and \ T \subseteq \text{Supp}(e)\}.
\]
Suppose that $\lambda \geq 2$. There exist $\text{Supp}(c_1), \text{Supp}(c_2) \in \Lambda_{T,6}(C^\perp)$. Then $c_1 + c_2 \in C^\perp$ and the weight $w_t(c_1 + c_2) \leq 4$. This is a contradiction to the minimal distance 6 of $C^\perp$ in Lemma 2\textsuperscript{3} Thus, $\lambda = 0$ or 1.

We now prove the two cases as follows.

(I) The case that $\lambda = 0$ and $m$ is odd.

By Lemma 2\textsuperscript{3} the minimal distance of $C^\perp$ is 6. Thus,

$$A_1 \left( \left( C^\perp \right)^T \right) = A_2 \left( \left( C^\perp \right)^T \right) = 0, \quad A_1 \left( \left( C^\perp \right)^\perp \right) = A_2 \left( \left( C^\perp \right)^\perp \right) = 0 \quad (7)$$

and the shortened code $C_T$ has length $n = 2^m - 4$ and dimension $k = 2m - 3$ from $\lambda_{T,6}(C^\perp) = 0$ and Lemma 1\textsuperscript{1} By the definition and Lemma 2\textsuperscript{3} we have $A_i \left( C_T \right) = 0$ for $i \not\in \{0, i_1, i_2, i_3\}$, where $i_1 = 2^{m-1} - 2^{(m-1)/2}$, $i_2 = 2^{m-1}$ and $i_3 = 2^{m-1} + 2^{(m-1)/2}$. Therefore, from (7) and (1), the first three Pless power moments

$$\begin{cases} \ A_{i_1} + A_{i_2} + A_{i_3} = 2^{2m-3} - 1, \\ i_1 A_{i_1} + i_2 A_{i_2} + i_3 A_{i_3} = 2^{2m-3-1} (2^m - 4), \\ i_1^2 A_{i_1} + i_2^2 A_{i_2} + i_3^2 A_{i_3} = 2^{2m-3-2} (2^m - 4) (2^m - 4 + 1). \end{cases}$$

yield the weight distribution in Table VIII. This completes the proof of (I).

(II) The case that $\lambda = 1$ and $m$ is odd.

The proof is similar to that of (I). Since $\lambda_{T,6}(C^\perp) = 1$ and the minimal distance of $C^\perp$ is 6, from Lemma 1\textsuperscript{1} we have

$$A_1 \left( \left( C^\perp \right)^T \right) = 0, \quad A_2 \left( \left( C^\perp \right)^T \right) = 1,$$

$$A_1 \left( \left( C^\perp \right)^\perp \right) = 0, \quad A_2 \left( \left( C^\perp \right)^\perp \right) = 1. \quad (8)$$

Then the desired conclusions follow from (8), the definitions and the first three Pless power moments of (1). This completes the proof. \hfill \square

Lemma 25. Let $m \geq 4$ be even, and $C$ be a binary linear code with length $2^m$ and the weight distribution in Table IV. Let $T$ be a 3-subset of $P(C)$. Suppose $\lambda_{T,6}(C^\perp) = \lambda$, then $A_1 \left( \left( C^\perp \right)^T \right) = A_2 \left( \left( C^\perp \right)^T \right) = 0$, $A_3 \left( \left( C^\perp \right)^T \right) = \lambda$ and $A_4 \left( \left( C^\perp \right)^T \right) = 2 \cdot (2^{m-2} - 1)^2 - 3\lambda$.

Proof: By Lemma 2\textsuperscript{3} the minimal distance of $C^\perp$ is 6. Thus, from $\#T = 3$ and the definitions, we have $A_1 \left( \left( C^\perp \right)^T \right) = A_2 \left( \left( C^\perp \right)^T \right) = 0$ and $A_3 \left( \left( C^\perp \right)^T \right) = \lambda$. Note that the code $C$ has length $2^m$ and dimension $2m+1$. By Lemma 2\textsuperscript{3} Table IV and the first seven Pless power moments of (1), we have

$$A_6(C^\perp) = \frac{1}{45} \cdot 2^{m-4} (2^m - 4)^2 (2^m - 1).$$

Further, from Theorem IV Lemmas 2\textsuperscript{3} and 1\textsuperscript{1} we have $(P(C^\perp), B_6(C^\perp))$ is a 2-design and

$$A_4 \left( \left( C^\perp \right)^{\{t_1,t_2\}} \right) = \binom{6}{2} \cdot A_6(C^\perp) = \frac{2}{3} \cdot (2^{m-2} - 1)^2$$

for any $\{t_1,t_2\} \subseteq P(C)$. Since $\#T = 3$ and the minimal distance of $C^\perp$ is 6,

$$A_4 \left( \left( C^\perp \right)^T \right) = \binom{3}{2} \left( A_4 \left( \left( C^\perp \right)^{\{t_1,t_2\}} \right) - \lambda \right).$$

Then the desired conclusions follow. \hfill \square

Theorem 26. Let $m \geq 4$ be even, and $C$ be a binary linear code with length $2^m$ and the weight distribution in Table IV. Let $T$ be a 3-subset of $P(C)$. Suppose $\lambda_{T,6}(C^\perp) = \lambda$, then the shortened code $C_T$ is a $[2^{m-3} - 3, 2m - 2, 2^{m-1} - 2^{m/2}]$ binary linear code with the weight distribution in Table X.
Proof. The proof is similar to that of Theorem [24] From Lemma 1 and \#T = 3, the shortened code \( C_T \) has length \( n = 2^m - 3 \) and dimension \( k = 2m - 2 \). By the definitions and the weight distribution in Table II, then \( A_i(C_T) = 0 \) for \( i \not\in \{0, i_1, i_2, i_3, i_4, i_5\} \), where \( i_1 = 2^{m-1} - 2^{m/2}, i_2 = 2^{m-1} - 2^{(m-2)/2}, i_3 = 2^{m-1}, i_4 = 2^{m-1} + 2^{(m-2)/2} \) and \( i_5 = 2^{m-1} + 2^{m/2} \). Moreover, from Lemmas 1 and 25 we have \( A_1 \left( (C_T) \perp \right) = A_2 \left( (C_T) \perp \right) = 0, A_3 \left( (C_T) \perp \right) = \lambda \) and \( A_4 \left( (C_T) \perp \right) = 2 \cdot (2^{m-2} - 1)^2 - 3\lambda \). Therefore, the first five Pless power moments of (I) yield the weight distribution in Table [X]. This completes the proof.  

| Weight             | Multiplicity |
|--------------------|-------------|
| \( 2^{m-1} - 2^{m/2} \) | \( 1/3 \cdot 2^{m/2 - 5}(8 + 2^3 + m/2 + 2^{3m/2} + 2^{2+m} + 12\lambda) \) |
| \( 2^{m-1} - 2^{(m-2)/2} \) | \( 1/3 \cdot 2^{m/2 - 3}((2 + 2^{m/2})(-8 + 3 \cdot 2^{m/2} + 2^{1+m} - 6\lambda) \) |
| \( 2^{m-1} \)       | \( -1 + 4^{m-2} \) |
| \( 2^{m-1} + 2^{(m-2)/2} \) | \( 1/3 \cdot 2^{m/2 - 3}((-2 + 2^{m/2})(-8 - 3 \cdot 2^{m/2} + 2^{1+m}) + 6\lambda) \) |
| \( 2^{m-1} + 2^{m/2} \) | \( 1/3 \cdot 2^{m/2 - 5}(-8 + 2^3 + m/2 + 2^{3m/2} - 2^{2+m} - 12\lambda) \) |

IV. SHORTENED LINEAR CODES FROM APN FUNCTIONS

Let \( p = 2 \) and \( q = 2^m \). In this section, we study some shortened codes \( C_T \) of linear codes \( C \) defined by (6) and determine their parameters, where \( f(x) = x^{2^e+1} \) is a special APN function, \( e \) is a positive integer, and \( \gcd(m, e) = 1 \). It is known that \( C \) has the weight distribution in Tables II (resp. Tables III) when \( m \) is odd (resp. \( m \) is even).

Let \( T \) be a \( t \)-subset of \( P(C) \). We will consider some shortened code \( C_T \) of \( C \) for the case \( m \geq 4 \) and \( t = 3 \) (or \( t = 4 \)).

A. Some shortened codes for the case \( t = 4 \) and \( m \) odd

We notice that it is difficult to determine the conditions satisfied by the four coordinate positions \( x_1, x_2, x_3, x_4 \in T \) such that \( \lambda_{T,6}(C) = 0 \) (or 1) in Theorem [24]. We start with the special code satisfying Theorem [24] and give the necessary and sufficient conditions in Theorem [28]. In order to prove Theorem [28], we need the result in the following lemma.

Lemma 27. Let \( e \) and \( m \geq 4 \) be positive integers with \( \gcd(m, e) = 1 \). Let \( q = 2^m \) and \( \{x_1, x_2, x_3, x_4\} \subseteq GF(q) \). Denote \( S_i = x_1^i + x_2^i + x_3^i + x_4^i \). Let \( N \) be the number of solutions \( (x, y) \in (GF(q))^2 \) of the system of equations

\[
\begin{align*}
&x_1 + x_2 + x_3 + x_4 + x + y = 0, \\
&x_1^{2e+1} + x_2^{2e+1} + x_3^{2e+1} + x_4^{2e+1} + x^{2e+1} + y^{2e+1} = 0 \tag{9}
\end{align*}
\]

where \( x_1, x_2, x_3, x_4, x, y \) are pairwise distinct. Then \( N = 4 \) if \( S_1 \not= 0 \) and \( \text{Tr}_{q/2}(S_1^{2e+1} + 1) = 0 \), and \( N = 0 \) otherwise.

Proof. By definition, the system (9) can be written as follows:

\[
\begin{align*}
x + y &= S_1, \\
x^{2e+1} + y^{2e+1} + S_2^{e+1} &= 0 \tag{10}
\end{align*}
\]
Substituting \( y = x + S_1 \), the second equation of (10) leads to
\[
x^{2^e + 1} + (x + S_1)^{2^e + 1} + S_{2^e + 1} \\
= x^{2^e + 1} + (x^2 + S_1^2)(x + S_1) + S_{2^e + 1} \\
= S_1 x^{2^e} + S_1^{2^e} \cdot x + S_1^{2^e + 1} + S_{2^e + 1} = 0.
\] (11)

Further, Equation (11) with \( S_1 \neq 0 \) is equivalent to
\[
\frac{S_1 x^{2^e} + S_1^{2^e} \cdot x + S_1^{2^e + 1} + S_{2^e + 1}}{S_1^{2^e + 1}} = \frac{(\frac{x}{S_1})^{2^e} + \frac{x}{S_1} + 1 + \frac{S_{2^e + 1}}{S_1^{2^e + 1}} = 0.}
\] (12)

This means that Equation (11) with \( S_1 \neq 0 \) has only two different solutions \( x = u, v \in \text{GF}(q) \) if and only if \( \text{Tr}_{q/2}(\frac{x^{2^e + 1}}{S_1^{2^e + 1}} + 1) = 0 \), and no solution otherwise. Note that if \( (x, y) \) is the solution of (9), then so is \( (y, x) \). Thus, \( N = 4 \) when Equation (11) with \( S_1 \neq 0 \) has solutions. Note that if (9) has nontrivial solution (i.e., not all being zero or pairwise equal), all elements \( x_1, x_2, x_3, x_4, y \) have to be distinct since the cyclic code with zero set \( \{1, 2^e + 1\} \) has minimum distance 5, where \( \gcd(m, e) = 1 \). Thus, \( x_1, x_2, x_3, x_4, u, v \) are pairwise distinct. Then the desired conclusions follow.

**Theorem 28.** Let \( e \) and \( m \geq 4 \) be positive integers with \( \gcd(m, e) = 1 \). Let \( q = 2^m \), \( f(x) = x^{2^e + 1} \) and \( C \) be defined in (6). Let \( \alpha \) be a primitive element of \( \text{GF}(q) \) and \( T = \{x_1, x_2, x_3, x_4\} \) be a 4-subset of \( \mathcal{P}(C) \). Then \( \lambda_{T,6}(C^\perp) = 1 \) iff \( \sum_{i=1}^{4} x_i \neq 0 \) and \( \text{Tr}_{q/2} \left( \sum_{i=1}^{4} x_i^{2^e + 1} / \sum_{i=1}^{4} x_i^{2^e + 1} + 1 \right) = 0 \), and \( \lambda_{T,6}(C^\perp) = 0 \) otherwise.

**Proof.** By definitions, the dual code \( C^\perp \) of \( C \) has minimum distance 6. For any \( \{x_1, x_2, x_3, x_4\} \subseteq \text{GF}(q) \), there exists a unique set \( \{x_1, x_2, x_3, x_4, y\} \) satisfying Equation (9) and \( 0 \in \{x_1, x_2, x_3, x_4, y\} \), since the code with zero set \( \{1, 2^e + 1\} \) has minimum distance 5, where \( \gcd(m, e) = 1 \). From the definitions we have \( \lambda_{T,6}(C^\perp) = \frac{N}{2^e} - 1 \), where \( N \) was defined in Lemma [27]. Then the desired conclusions follow from Lemma [27].

By Theorems 28 and 24, we have the following theorem, which is one of the main results in this paper.

**Theorem 29.** Let \( m \geq 5 \) be odd and \( e \) be a positive integer with \( \gcd(m, e) = 1 \). Let \( q = 2^m \), \( f(x) = x^{2^e + 1} \) and \( C \) be defined in (6). Let \( \alpha \) be a primitive element of \( \text{GF}(q) \) and \( T = \{x_1, x_2, x_3, x_4\} \) be a 4-subset of \( \mathcal{P}(C) \). Then \( \lambda_{T,6}(C^\perp) = 1 \) if \( \sum_{i=1}^{4} x_i \neq 0 \) and \( \text{Tr}_{q/2} \left( \sum_{i=1}^{4} x_i^{2^e + 1} / \sum_{i=1}^{4} x_i^{2^e + 1} + 1 \right) = 1 \), and \( \lambda_{T,6}(C^\perp) = 0 \) otherwise. The parameters of the shortened code \( C_T \) were given in Theorem 24.

**Example 30.** Let \( m = 5, e = 1 \) and \( T = \{\alpha^1, \alpha^2, \alpha^3, \alpha^5\} \). Then \( \gamma = \alpha^{17} \), \( \text{Tr}_{q/2} \left( \frac{y}{\gamma} \right) = 0 \) and \( \lambda_{T,6}(C^\perp) = 0 \), where \( \gamma = \alpha^1 + \alpha^2 + \alpha^3 + \alpha^5 \). The shortened code \( C_T \) in Theorem 29 is a \([28,7,12]\) binary linear code with the weight enumerator \( 1 + 66x^{12} + 55x^{16} + 6x^{20} \). The code \( C_T \) is optimal according to the tables of best known codes maintained at [http://www.codetables.de](http://www.codetables.de).

**Example 31.** Let \( m = 5, e = 1 \) and \( T = \{\alpha^1, \alpha^2, \alpha^3, \alpha^4\} \). Then \( \gamma = \alpha^{24} \), \( \text{Tr}_{q/2} \left( \frac{y}{\gamma} \right) = 1 \), and \( \lambda_{T,6}(C^\perp) = 1 \), where \( \gamma = \alpha^1 + \alpha^2 + \alpha^3 + \alpha^4 \). The shortened code \( C_T \) in Theorem 29 is a \([28,7,12]\) binary linear code with the weight enumerator \( 1 + 68x^{12} + 51x^{16} + 8x^{20} \). The code \( C_T \) is optimal according to the tables of best known codes maintained at [http://www.codetables.de](http://www.codetables.de).

**B. Some shortened codes for the case \( t = 3 \) and \( m \) even**

For any \( T = \{x_1, x_2, x_3\} \subseteq \mathcal{P}(C) \), we notice that it is difficult to determine the value of \( \lambda \) in Theorem 26. We will study a class of special linear codes \( C \) defined by (6), where \( f(x) = x^{2^e + 1} \) is an APN function, \( m \geq 4 \) is even and \( \gcd(e, m) = 1 \). Note that the code \( C \) satisfies Theorem 26. In the following, we will determine the value of \( \lambda \) in Theorem 33 and the parameters of the shortened code \( C_T \). We need the result in the following lemma.
Lemma 32. Let \( e \) be a positive integer, \( m \) be even with \( \gcd(m,e) = 1 \), and \( q = 2^m \). Let \( \hat{N} \) be the number of solutions \( (x,y,z) \in (\mathbb{GF}(q))^3 \) of the system of equations

\[
\begin{align*}
\begin{cases}
x + y + z = a, \\
x^{2^e+1} + y^{2^e+1} + z^{2^e+1} = b
\end{cases}
\end{align*}
\]  

(13)

where \( a, b \in \mathbb{GF}(q) \) and \( a^{2^e+1} \neq b \). Then \( \hat{N} = 2^m + (-2)^{m/2} - 2 \) if \( a^{2^e+1} + b \) is not a cubic residue, and \( \hat{N} = 2^m + (-2)^{m/2} + 2 \) if \( a^{2^e+1} + b \) is a cubic residue.

Proof. Replacing \( x \) with \( x + a \), \( y \) with \( y + a \) and \( z \) with \( z + a \), we have

\[
\begin{align*}
\begin{cases}
x + y + z = 0, \\
x^{2^e+1} + y^{2^e+1} + z^{2^e+1} = a^{2^e+1} + b
\end{cases}
\end{align*}
\]

(14)

By \( x + y + z = 0 \), we have the following equation

\[
x^{2^e} y + y^{2^e} x = a^{2^e+1} + b.
\]

(15)

Thus, \( \hat{N} \) equals the number of solutions \( (x,y) \in (\mathbb{GF}(q))^2 \) to Equation (15).

Replacing \( y \) with \( xy \) in Equation (15), we get

\[
x^{2^e} y + y^{2^e} x = x^{2^e+1} y + y^{2^e} x^{2^e+1} = x^{2^e+1}(y + y^{2^e}) = a^{2^e+1} + b.
\]

(16)

Since \( a^{2^e+1} + b \neq 0 \), a rearrangement of Equation (16) yields

\[
y + y^{2^e} = (a^{2^e+1} + b)x^{-(2^e+1)}.
\]

Then

\[
\text{Tr}(a^{2^e+1} + b)x^{-(2^e+1)} = 0.
\]

(17)

Further, from Lemma 5 we get

\[
\begin{align*}
\#\{x \in \mathbb{GF}(q)^* : \text{Tr}(a^{2^e+1} + b)x^{-(2^e+1)} = 0\} \\
= \#\{x \in \mathbb{GF}(q)^* : \text{Tr}(a^{2^e+1} + b)x^{(2^e+1)} = 0\} \\
= \frac{1}{2} \cdot \sum_{u \in \mathbb{GF}(2)} \sum_{x \in \mathbb{GF}(q)^*} (-1)^{\text{Tr}(u(a^{2^e+1} + b)x^{(2^e+1)})} \\
= \frac{1}{2} \cdot (q - 2 + \sum_{x \in \mathbb{GF}(q)} (-1)^{\text{Tr}((a^{2^e+1} + b)x^{(2^e+1)})}) \\
= \begin{cases}
\frac{1}{2} \cdot (q - 2 + (-1)^{\frac{m}{2^e}2^m}) & \text{if } a^{2^e+1} + b \text{ is not a cubic residue,} \\
\frac{1}{2} \cdot (q - 2 - (-1)^{\frac{m}{2^e}2^m+1}) & \text{if } a^{2^e+1} + b \text{ is a cubic residue.}
\end{cases}
\end{align*}
\]

(18)

Note that if \( (x,y) \) is the solution of Equation (15), then so is \( (y,x) \). This means that \( \hat{N} \) equals twice the number of solutions \( x \in \mathbb{GF}(q)^* \) to Equation (17). Then the desired conclusions follow from Equation (18). 

\( \square \)

Theorem 33. Let \( e \) be a positive integer and \( m \geq 4 \) be even with \( \gcd(m,e) = 1 \). Let \( q = 2^m \), \( f(x) = x^{2^{e+1}} \) and \( C \) be defined in \( 5 \). Let \( \alpha \) be a primitive element of \( \mathbb{GF}(q) \) and \( T = \{x_1,x_2,x_3\} \) be a 3-subset of \( \mathcal{P}(C) \). Then

\[
\lambda = \lambda_{T,6}(C_\perp) = \begin{cases}
\frac{1}{6} \cdot (q - 2 + (-1)^{\frac{m}{2^e}2^m}) - 1 & \text{if } a^{2^e+1} + b \text{ is not a cubic residue,} \\
\frac{1}{6} \cdot (q - 2 - (-1)^{\frac{m}{2^e}2^m+1}) - 1 & \text{if } a^{2^e+1} + b \text{ is a cubic residue,}
\end{cases}
\]

(19)

where \( a = \sum_{i=1}^3 x_i \) and \( b = \sum_{i=1}^3 x_i^{2^e+1} \). Moreover, the shortened code \( C_T \) is a \([2^m - 3, 2m - 2, 2^{m-1} - 2^{m/2}]\) binary linear code with the weight distribution in Table X.
Lemma 36. Let $m \geq 4$ be even and $q = 2^m$. Define

\[
R_{(3,i)}(x) = \# \{ x \in \mathbb{F}_q^* : \text{Tr}_{q/2}(x) = i \text{ and } x \text{ is a cubic residue} \}
\]

\[
\bar{R}_{(3,i)}(x) = \# \{ x \in \mathbb{F}_q^* : \text{Tr}_{q/2}(x) = i \text{ and } x \text{ is not a cubic residue} \}.
\]

where $i = 0$ or $i = 1$. Then

\[
R_{(3,0)} = \frac{1}{6} (2^m - 2 + (-2)^{m/2+1})
\]

\[
R_{(3,1)} = \frac{2^m - 1}{3} - \frac{1}{6} (2^m - 2 + (-2)^{m/2+1})
\]

\[
\bar{R}_{(3,0)} = (2^m - 1) - \frac{1}{6} (2^m - 2 + (-2)^{m/2+1})
\]

\[
\bar{R}_{(3,1)} = \frac{2^m}{3} - \frac{(-2)^{m/2}}{3}.
\]

Proof. By definitions, we get

\[
R_{(3,0)} = \frac{1}{3} \cdot \# \{ x \in \mathbb{F}_q^* : \text{Tr}_{q/2}(x^3) = 0 \}
\]

\[
= \frac{1}{6} \sum_{z \in \mathbb{F}_q^2} \sum_{x \in \mathbb{F}_q^*} (-1)^{\text{Tr}_{q/2}(x^3)}
\]

\[
= \frac{1}{6} \left( q - 2 + \sum_{x \in \mathbb{F}_q^*} (-1)^{\text{Tr}_{q/2}(x^3)} \right).
\]

Example 34. Let $m = 4$, $e = 1$ and $T = \{ \alpha^1, \alpha^2, \alpha^4 \}$. Then $\lambda = 0$ and the shortened code $C_T$ in Theorem 26 is a $[13, 6, 4]$ binary linear code with the weight enumerator $1 + 7z^4 + 36z^6 + 15z^8 + 4z^{10} + z^{12}$. This code $C_T$ is optimal. Its dual $C_T^\perp$ has parameters $[13, 7, 4]$ and is optimal according to the tables of best known codes maintained at http://www.codetables.de.

Example 35. Let $m = 4, e = 1$ and $T = \{ \alpha^2, \alpha^5, \alpha^7 \}$. Then $\lambda = 2$ and the shortened code $C_T$ in Theorem 26 is a $[13, 6, 4]$ binary linear code with the weight enumerator $1 + 8z^4 + 34z^6 + 15z^8 + 6z^{10}$. This code $C_T$ is optimal. Its dual $C_T^\perp$ has parameters $[13, 7, 3]$ and is almost optimal according to the tables of best known codes maintained at http://www.codetables.de.

C. Some shortened codes for the case $t = 4$ and $m$ even

For any $T = \{ x_1, x_2, x_3, x_4 \} \subseteq \mathcal{P}(C)$, Magma programs show that the weight distribution of $C_T$ for some code $C$ is very complex. Thus, it is difficult to determine their parameters in general. In this subsection, we will study a class of special linear codes $C$ with the weight distribution in Table 11 and determine parameters of $C_T$ with the special set $T$ and $t = \#T = 4$ in Theorem 38.

In order to determine the parameters of $C_T$, we need the next two lemmas.

Lemma 36. Let $m \geq 4$ be even and $q = 2^m$. Define

\[
R_{(3,i)}(x) = \# \{ x \in \mathbb{F}_q^* : \text{Tr}_{q/2}(x) = i \text{ and } x \text{ is a cubic residue} \}
\]

\[
\bar{R}_{(3,i)}(x) = \# \{ x \in \mathbb{F}_q^* : \text{Tr}_{q/2}(x) = i \text{ and } x \text{ is not a cubic residue} \}.
\]

where $i = 0$ or $i = 1$. Then

\[
R_{(3,0)} = \frac{1}{6} (2^m - 2 + (-2)^{m/2+1})
\]

\[
R_{(3,1)} = \frac{2^m - 1}{3} - \frac{1}{6} (2^m - 2 + (-2)^{m/2+1})
\]

\[
\bar{R}_{(3,0)} = (2^m - 1) - \frac{1}{6} (2^m - 2 + (-2)^{m/2+1})
\]

\[
\bar{R}_{(3,1)} = \frac{2^m}{3} - \frac{(-2)^{m/2}}{3}.
\]

Proof. By definitions, we get
Then the value of $R_{(3,0)}$ follows from Lemma [6] Note that it has $\frac{q-1}{3}$ cubic residues in $GF(q)^*$. Then the desired conclusions follow from Lemma 36.

Let $m \geq 4$ be even, $e$ be a positive integer with $\gcd(m,e) = 1$, and $q = 2^m$. Let $N_{(0,1)}$ be the number of solutions $(x, y, z, u) \in (GF(q))^4$ of the system of equations

\[
\begin{cases}
x + y + z + u = 0, \\
x^{2e+1} + y^{2e+1} + z^{2e+1} + u^{2e+1} = 1
\end{cases}
\]

where $x, y, z, u$ are pairwise distinct. Then $N_{(0,1)} = q(q - 2 - (-1)^{m/2}2^{m/2+1})$.

**Proof.** By definitions, we have

\[
N_{(0,1)} = \frac{1}{q^2} \sum_{x,y,z,u \in GF(q)} \sum_{a,b \in GF(q)} \chi_1(b(x+y+z+u))\chi_1(a(x^{2e+1} + y^{2e+1} + z^{2e+1} + u^{2e+1} - 1))
\]

\[
= \frac{1}{q^2} \sum_{a,b \in GF(q)} \chi_1(-a) \left( \sum_{x \in GF(q)} \chi_1(ax^{2e+1} + bx) \right)^4
\]

\[
= \frac{1}{q^2} \left( \sum_{a \in GF(q)} \chi_1(-a) \left( \sum_{x \in GF(q)} \chi_1(ax^{2e+1}) \right)^4 + \sum_{a \in GF(q)^*} \sum_{b \in GF(q)^*} \chi_1(-a) \left( \sum_{x \in GF(q)} \chi_1(ax^{2e+1} + bx) \right)^4 \right)
\]

\[
= \frac{1}{q^2} \left( (q^4 + (R_{(3,0)} - R_{(3,1)}) \cdot 2^{m+4} + (\tilde{R}_{(3,0)} - \tilde{R}_{(3,1)}) \cdot 2^m + \right.
\]

\[
(R_{(3,0)} - R_{(3,1)}) \cdot (2^{3m+2} - 2^{2m+4}) + (\tilde{R}_{(3,0)} - \tilde{R}_{(3,1)}) \cdot (2^m - 1) 2^m),
\]

where $R_{(3,0)}, R_{(3,1)}, \tilde{R}_{(3,0)}$ and $\tilde{R}_{(3,1)}$ were defined in (21) and the last equality holds from Lemmas [6] and [7]. Then the desired conclusions follow from Lemma [36].

**Theorem 38.** Let $m \geq 4$ be even and $e$ be a positive integer with $\gcd(m,e) = 1$. Let $q = 2^m$, $f(x) = x^{2e+1}$ and $C$ be defined in (6). Let $T = GF(4) = \{0, 1, w, w^2\} \subseteq GF(q)$, where $w$ is a generator of $GF(4)^*$. Then the shortened code $C_T$ is a $[2^m - 4, 2m - 3, 2^{m-1} - 2^{m/2}]$ binary linear code with the weight distribution in Table [XI].

**Proof.** By definitions and Lemma [23] $C$ is a $[2^m, 2m + 1, 2^{m-1} - 2^{m/2}]$ with the weight distribution in Table [II] and the minimal distance of $C^{\perp}$ is 6. Note that $\lambda_{T,6}(C^{\perp}) = 0$ from Theorem [28] and the definition of $T$. Thus, the minimal distance of $(C^{\perp})^T$ is at least 3. This means that

\[
A_1 \left( \left( C^{\perp} \right)^T \right) = A_2 \left( \left( C^{\perp} \right)^T \right) = 0.
\]
Note that the solutions of the system (22) have symmetrical property and we obtain

\[ \text{Combining Equations (24), (26) and (27), Equation (28) yields} \]

\[ N \]

the weight distribution in Table XI. This completes the proof.

Hence, from Lemma 1 and Equations (23), (25) and (29), the first five Pless power moments of (1) yield

\[ \text{Let } m \]

Further, from Theorem 33 and the definition of \( T \), we have

\[ \lambda_{\hat{T},6}(C^\perp) = \frac{1}{6} \cdot (q - 2 - (-1)^m 2^m + 1) - 1 \]

for any \( \hat{T} = \{\hat{x}_1, \hat{x}_2, \hat{x}_3\} \subseteq T \). Therefore,

\[ A_3 \left( \left( C^\perp \right)^T \right) = \left( \frac{4}{3} \right) \cdot \left( \frac{1}{6} \cdot (q - 2 - (-1)^m 2^m + 1) - 1 \right). \]

Note that the solutions of the system (22) have symmetrical property and \( (x, y, z, u) = (0, 1, w, w^2) \) is a solution of the system (22). Thus, from the definitions and Lemma 37 we get

\[ \lambda_{T,8}(C^\perp) = \frac{N_{(0,1)}}{4!} - 1 - \left( \frac{4}{3} \right) \cdot \lambda_{\hat{T},6}(C^\perp), \]

where \( N_{(0,1)} \) was defined in Lemma 37 and \( \lambda_{\hat{T},6}(C^\perp) \) was defined in (24). By the proof of Lemma 25 we obtain

\[ A_4 \left( \left( C^\perp \right)^{\{\hat{x}_1, \hat{x}_2\}} \right) = \frac{2}{3} \cdot (2^{m-2} - 1)^2 \]

for any \( \{\hat{x}_1, \hat{x}_2\} \subseteq T \). It is obvious that there exists only two 3-subsets \( \hat{T} \) of \( T \) with \( \{\hat{x}_1, \hat{x}_2\} \subseteq \hat{T} \subseteq T \) for any \( \{\hat{x}_1, \hat{x}_2\} \subseteq T \). Thus, from the definitions we have

\[ A_4 \left( \left( C^\perp \right)^T \right) = \left( \frac{4}{2} \right) \cdot \left( A_4 \left( \left( C^\perp \right)^{\{i_1, i_2\}} \right) - 2 \cdot \lambda_{T,6}(C^\perp) \right) + \lambda_{T,8}(C^\perp). \]

Combining Equations (24), (26) and (27), Equation (28) yields

\[ A_4 \left( \left( C^\perp \right)^T \right) = 4 \cdot (2^{m-2} - 1)^2 - \frac{8}{3} \cdot (2^{m-2} - (-1)^m 2^m + 1) + \frac{N_{(0,1)}}{24} + 15. \]

Note that the shortened code \( C_T \) has length \( 2^m - 4 \) and dimension \( 2m - 3 \), since \#\( T = 4 \) and Lemma 1. By the definitions and the weight distribution in Table XI then \( A_i(C_T) = 0 \) for \( i \not\in \{0, i_1, i_2, i_3, i_4, i_5\} \), where \( i_1 = 2^{m-1} - 2^{m/2} \), \( i_2 = 2^{m-1} - 2^{(m-2)/2} \), \( i_3 = 2^{m-1} \), \( i_4 = 2^{m-1} + 2^{(m-2)/2} \) and \( i_5 = 2^{m-1} + 2^{m/2} \). Hence, from Lemma 1 and Equations (23), (25) and (29), the first five Pless power moments of (1) yield the weight distribution in Table XI. This completes the proof.

**Example 39.** Let \( m = 4 \) and \( e = 1 \) (or 3). Then the shortened code \( C_T \) in Theorem 38 is a [12,5,4] binary linear code with the weight enumerator \( 1 + 3z^4 + 24z^6 + 3z^8 + z^{12} \). This code \( C_T \) is optimal. Its dual \( \overline{C_T} \) has parameters [12,7,4] and is optimal according to the tables of best known codes maintained at [http://www.codetables.de](http://www.codetables.de).
V. SHORTENED LINEAR CODES FROM PN FUNCTIONS

In this section, we study some shortened codes of linear codes \( C \) from the special PN functions and determine their parameters.

Let \( p \) be odd prime and \( q = p^m \). Let \( f(x) = x^2 \) and \( C \) defined by (6). It is known that the code \( C \) is a \([q, 2m+1]\) linear code with the weight distribution in \([31]\). Note that the code \( C \) is affine invariant, and thus holds 2-designs. Then the following theorem is easily derived from the parameters of the code \( C \) in \([31]\) and Theorem 13 and we omit its proof.

**Theorem 40.** Let \( p \) be an odd prime, \( m \) and \( t \) be positive integers. Let \( q = p^m \), \( f(x) = x^2 \) and \( C \) be defined in (6). Suppose \( T \) is a \( t \)-subset of \( \mathcal{P}(C) \). We have the following results:

- If \( t = 1 \), then the shortened code \( \tilde{C}_T \) is a \([2^m - 1, 2m]\) linear code with the weight distribution in Table XII (resp. Table XIV) when \( m \) is odd (resp. even).
- If \( t = 2 \), then the shortened code \( \tilde{C}_T \) is a \([2^m - 2, 2m - 1]\) linear code with the weight distribution in Table XIII (resp. Table XV) when \( m \) is odd (resp. even).

| TABLE XII |
|-----------|
| **The weight distribution of \( \tilde{C}_T \) for \( m \) odd and \( t = 1 \)** |
| Weight | Multiplicity |
|--------|--------------|
| \( 0 \) | \( 1 \) |
| \( p^{m-1}(p-1) \) | \( (p^m-1)(1+p^{m-1}) \) |
| \( p^{m-1}(p-1) - p^{m+1} \) | \( 1/2 \cdot (p-1)p^{(m-3)/2}(p^m-1)(p+p(1+m)/2) \) |
| \( p^{m-1}(p-1) + p^{m+1} \) | \( 1/2 \cdot (p-1)p^{(m-3)/2}(p^m-1)(-p+p(1+m)/2) \) |

| TABLE XIII |
|-----------|
| **The weight distribution of \( \tilde{C}_T \) for \( m \) odd and \( t = 2 \)** |
| Weight | Multiplicity |
|--------|--------------|
| \( 0 \) | \( 1 \) |
| \( p^{m-1}(p-1) \) | \( p^{2m-2} - 1 \) |
| \( p^{m-1}(p-1) - p^{m+1} \) | \( (p-1)(-p^{(m-1)/2} + p^{2m-2} + 2p^{(3m-3)/2})/2 \) |
| \( p^{m-1}(p-1) + p^{m+1} \) | \( (p-1)(p^{(m-1)/2} + p^{2m-2} - 2p^{(3m-3)/2})/2 \) |

| TABLE XIV |
|-----------|
| **The weight distribution of \( \tilde{C}_T \) for \( m \) even and \( t = 1 \)** |
| Weight | Multiplicity |
|--------|--------------|
| \( 0 \) | \( 1 \) |
| \( (p-1)(p^{m-1} - p^{m+1}) \) | \( p^{m/2-1}(p+t^{m/2} - 1)(p^m - 1)/2 \) |
| \( (p-1)p^{m-1} - p^{m+1} \) | \( (p-1)p^{m/2-1}(p^{(m/2) - 1})/(1+p^{m/2})/2 \) |
| \( (p-1)p^{m-1} \) | \( p^{m-1} \) |
| \( (p-1)p^{m-1} + p^{m+1} \) | \( (p-1)p^{m/2-1}(p^{m/2} - 1)^2(1+p^{m/2})/2 \) |
| \( (p-1)(p^{m-1} + p^{m+1}) \) | \( p^{m/2-1}(-p+p^{m/2} + 1)(p^m - 1)/2 \) |

**Example 41.** Let \( m = 3 \), \( p = 3 \) and \( T \) be a \( 1 \)-subset of \( \mathcal{P}(C) \). Then the shortened code \( \tilde{C}_T \) in Theorem 40 is a \([26, 6, 15]\) linear code with the weight enumerator \( 1 + 312z^{15} + 260z^{18} + 156z^{21} \). This code \( \tilde{C}_T \) is
Then the following results follow.

| Weight | Multiplicity |
|--------|--------------|
| 0      | \((p-1)p^{m-1}\) |
| 1      | \(p^m/2-2(p^m/2-1)(-1+p+p^m/2)(p+p^m/2)/2\) |
| \((p-1)p^{m-1}p^{m-2}\) | \((p-1)p^{m-2}(1+p^m/2)(p+p^m/2)/2\) |
| \((p-1)p^{m-1}p^{m-2}\) | \(p^{m-1}-1\) |
| \((p-1)p^{m-1}p^{m-2}\) | \((p-1)p^{m-2}(1+p^m/2)(-p-p^m/2+p^m)/2\) |
| \((p-1)p^{m-1}p^{m-2}\) | \(p^{m-2}(p^m/2+1)(1-p+p^m/2)(p^m/2-p)/2\) |

Example 42. Let \(m = 4\), \(p = 3\) and \(T\) be a 1-subset of \(\mathcal{P}(C)\). Then the shortened code \(C_T\) in Theorem 40 is a \([80, 8, 48]\) linear code with the weight enumerator \(1 + 1320z^{46} + 2400z^{51} + 80z^{54} + 1920z^{57} + 840z^{60}\). This code \(C_T\) is optimal. Its dual \(C_T^\perp\) has parameters \([26, 20, 4]\) and is optimal according to the tables of best known codes maintained at \(\text{http://www.codetables.de}\).

Example 43. Let \(m = 5\), \(p = 3\) and \(T\) be a 2-subset of \(\mathcal{P}(C)\). Then the shortened code \(C_T\) in Theorem 40 is a \([241, 9, 153]\) linear code with the weight enumerator \(1 + 8010z^{153} + 6560z^{162} + 5112z^{171}\). This code \(C_T\) is optimal. Its dual \(C_T^\perp\) has parameters \([241, 232, 3]\) and is optimal according to the tables of best known codes maintained at \(\text{http://www.codetables.de}\).

Example 44. Let \(m = 4\), \(p = 3\) and \(T\) be a 2-subset of \(\mathcal{P}(C)\). Then the shortened code \(C_T\) in Theorem 40 is a \([79, 7, 48]\) linear code with the weight enumerator \(1 + 528z^{48} + 870z^{51} + 26z^{54} + 552z^{57} + 210z^{60}\). This code \(C_T\) is almost optimal. Its dual \(C_T^\perp\) has parameters \([79, 72, 3]\) and is almost optimal according to the tables of best known codes maintained at \(\text{http://www.codetables.de}\).

In the following, we will consider the shortened code \(C_T\) of \(C\) for the case \(#T = p\). In order to determine the parameters of \(C_T\), we need the next lemmas.

Lemma 45. Let \(q = p^m\) with \(p\) an odd prime. Then

\[
\sharp\{a \in \text{GF}(q)^* : \eta(a) = 1 \text{ and } \text{Tr}_{q/p}(a) = 0\} = \begin{cases} 
\frac{p^{m-1}-(p-1)p^{m-2}m}{2} & \text{for even } m, \\
\frac{p^{m-1}-1}{2} & \text{for odd } m,
\end{cases}
\]

and

\[
\sharp\{a \in \text{GF}(q)^* : \eta(a) = -1 \text{ and } \text{Tr}_{q/p}(a) = 0\} = \begin{cases} 
\frac{p^{m-1}+(p-1)p^{m-2}m}{2} & \text{for even } m, \\
\frac{p^{m-1}-1}{2} & \text{for odd } m.
\end{cases}
\]

Lemma 46. Let \(q = p^m\) with \(p\) an odd prime and \((a, b) \in \text{GF}(q)^2\). Denote

\[
N_0(a, b) = \sharp\{x \in \text{GF}(q) : \text{Tr}_{q/p}(ax^2 + bx) = 0\}.
\]

Then the following results follow.
• If $m$ is odd, then

$$N_0(a, b) = \begin{cases} 
  p^m & \text{if } (a, b) = (0, 0), \\
  p^{m-1} & \text{if } a = 0, b \neq 0, \text{ or } a \neq 0, 0 = \operatorname{Tr}_{q/p}(\frac{b^2}{4a}); \\
  p^{m-1} + p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)+4}{4}} & \text{if } a \neq 0, 0 \neq \operatorname{Tr}_{q/p}(\frac{b^2}{4a}), \eta(a) = 1, \\
  p^{m-1} + p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)+4}{4}} & \text{if } a \neq 0, 0 \neq \operatorname{Tr}_{q/p}(\frac{b^2}{4a}), \eta(a) = -1. 
\end{cases}$$

• If $m$ is even, then

$$N_0(a, b) = \begin{cases} 
  p^m & \text{if } (a, b) = (0, 0), \\
  p^{m-1} & \text{if } a = 0, b \neq 0, \\
  p^{m-1} + (p-1)p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)+4}{4}} & \text{if } a \neq 0, 0 = \operatorname{Tr}_{q/p}(\frac{b^2}{4a}), \eta(a) = 1, \\
  p^{m-1} + (p-1)p^{\frac{m-2}{2}}(-1)^{\frac{m(p-1)+4}{4}} & \text{if } a \neq 0, 0 \neq \operatorname{Tr}_{q/p}(\frac{b^2}{4a}), \eta(a) = -1. 
\end{cases}$$

Lemma 47. Let $q = p^m$ with $p$ an odd prime and $a \in \operatorname{GF}(q)^*$. Define $f(x) = \operatorname{Tr}_{q/p}(\frac{-1}{4a}x^2)$. Then the dual of $f(x)$ is $f^*(x) = \operatorname{Tr}_{q/p}(ax^{p-1})$ and the sign of the Walsh transform of $f(x)$ is

$$\varepsilon = \eta\left(-\frac{1}{4a}\right)(-1)^{\frac{m(p-1)(p-3)}{8} + m - 1} = \eta(a)(-1)^{\frac{m(p-1)(p-3)}{8} + m - 1 + \frac{p-1}{2}}.$$

Proof. Note that $f(x)$ is a weakly regular bent function. Then the desired conclusions follow from the definitions and Lemmas 2 and 3.

Lemma 48. Let $q = p^m$ with $p$ an odd prime, $a \in \operatorname{GF}(q)^*$ and $e$ be any positive integer such that $m/\gcd(m, e)$ is odd. Define $f(x) = \operatorname{Tr}_{q/p}(ax^{p^{e}+1})$. Then the dual of $f(x)$ is $f^*(\beta) = \operatorname{Tr}_{q/p}(-a(x_{a, -\beta})^{p^{e}+1})$ for any $\beta \in \operatorname{GF}(q)$ and the sign of the Walsh transform of $f(x)$ is

$$\varepsilon = \eta(a)(-1)^{\frac{m(p-1)(p-3)}{8} + m - 1 + \frac{p^e-1}{2}},$$

where $x_{a, -\beta}$ is the unique solution of the equation

$$a^{p^e}x^{p^e} + ax + (-\beta)^{p^e} = 0.$$

Proof. Note that $f(x)$ is a weakly regular bent function. By definitions, we have

$$f^*(\beta) = \operatorname{Tr}_{q/p}(-a(x_{a, -\beta})^{p^{e}+1}).$$

From the definitions and Lemma 3, we have

$$\varepsilon = \begin{cases} 
  (-1)^{m-1 + \frac{p-1}{2} + \frac{m(p-1)}{4}}\eta(a), & \text{if } p \equiv 1 \text{ mod } 4, \\
  (-1)^{m-1 + \frac{p-1}{2} + \frac{m(p-3)}{4}}\eta(a), & \text{if } p \equiv 3 \text{ mod } 4. 
\end{cases}$$

Then the desired conclusions follow.

Lemma 49. Let $q = p^m$ with $p$ an odd prime, $a \in \operatorname{GF}(q)^*$, $\gamma \in \operatorname{GF}(p)$ and $e$ be any positive integer such that $m/\gcd(m, e)$ is odd. Define

$$\tilde{N}_b = \#\{b \in \operatorname{GF}(q) : \operatorname{Tr}_{q/p}(a(x_{a, b})^{p^{e}+1}) = \gamma \text{ and } \operatorname{Tr}_{q/p}(b) = 0\}$$
where \( x_{a,b} \) is the unique solution of the equation \( a^{p^x}x^{p^x} + ax + (b)^{p^x} = 0 \). If \( \text{Tr}_{q/p}(a) = 0 \), then

\[
\hat{N}_b = \begin{cases} 
p^{m-2} + \varepsilon \eta^{m/2}(-1)(p-1)p^{(m-2)/2}, & \text{if } \gamma = 0 \text{ and } m \text{ is even}; \\
p^{m-2}, & \text{if } \gamma = 0 \text{ and } m \text{ is odd}; \\
p^2 \left( p^{m-2} - \varepsilon \eta^{m/2}(-1)p^{(m-2)/2} \right), & \text{if } \gamma \neq 0 \text{ and } m \text{ is even}; \\
p^2 \left( p^2 - \varepsilon \sqrt{p^{m-1}} \right), & \text{if } \gamma \in \text{SQ and } m \text{ is odd}; \\
p^2 \left( p^2 - \varepsilon \sqrt{p^{m-1}} \right), & \text{if } \gamma \in \text{NSQ and } m \text{ is odd}; 
\end{cases}
\]

where \( \varepsilon \) is given in Lemma 48.

**Proof.** By definitions, we have

\[
a^{p^x}(x_{a,b})^{p^x} + ax_{a,b} + (b)^{p^x} = (ax_{a,b})^{p^x} + ax_{a,b} + (b)^{p^x} = 0,
\]

which deduces

\[
\text{Tr}_{q/p}(ax_{a,b}^{p^x} + ax_{a,b} + (b)^{p^x}) = \text{Tr}_{q/p}(ax_{a,b}^{p^x} + ax_{a,b}) = \text{Tr}_{q/p}(a^{p^x} + a)x_{a,b} = 0.
\]

When \( \text{Tr}_{q/p}(b) = 0 \), we have

\[
\hat{N}_b = \#\{x \in GF(q) : \text{Tr}_{q/p}(ax_{a,b}^{p^x+1}) = \gamma \text{ and } \text{Tr}_{q/p}(a^{p^x} + a)x = 0\}.
\]

Note that \( x = 1 \) is the unique solution of the equation

\[
a^{p^x}(x)^{p^x} + ax + (-a^{p^x} - a)^{p^x} = 0.
\]

By Lemma 48 we get

\[
f^*(a^{p^x} + a) = \text{Tr}_{q/p}(-a) = 0
\]

when \( \text{Tr}_{q/p}(a) = 0 \), where \( f^* \) is the dual of \( \text{Tr}_{q/p}(ax_{a,b}^{p^x+1}) \). By Equations (33) and (34), the desired conclusions follow from Lemmas 14, 15 and 48.

**Theorem 50.** Let \( p \) be an odd prime and \( m \) be a positive integer. Let \( q = p^m \), \( f(x) = x^2 \) and \( C \) be defined in (6). Let \( T = GF(p) \). Then the shortened code \( C_T \) is a \([2^m - p, 2m - 2] \) linear code. If \( m \) is odd, the weight distribution of \( C_T \) is given in Table XVI if \( m \geq 2 \) is even, the weight distribution of \( C_T \) is given in Table XVI where \( B = (-1)^{\frac{m(p-1)(p-3)}{8} + m - 1 + \frac{q-1}{2}} \sqrt{p^{m-1}} \), \( B_1 = \frac{p^{m-1} - 1 - (p-1)p^{m-2}}{2} \) and \( B_2 = (-1)^{\frac{m(p-1)(p-3)}{8} + m - 1 + \frac{q-1}{2}} \eta^{m/2}(-1) \cdot (p-1)p^{(m-2)/2} \).

**Proof.** By definitions and Lemma 1, the shortened code \( C_T \) has length \( 2^m - p \) and dimension \( 2m - 2 \). Since \( T = GF(p) \), the weight distribution of \( C_T \) is the same as the code

\[
\hat{C} = \left\{ (\text{Tr}_{q/p}(ax + bx)_{x \in GF(q)} : a, b \in GF(q) \text{ and } \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0 \right\}.
\]
Then the Hamming weight of \(a\) and \(b\) runs through \(GF(q)\times GF(q)\) with \(\text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0\), define the corresponding codeword \(c(a, b) = (\langle \text{Tr}_{q/p}(ax^2 + bx) \rangle_{x \in GF(q)} \rangle) \in \hat{C}\).

Then the Hamming weight of \(c(a, b)\) is

\[
\text{wt}(c(a, b)) = q - N_0(a, b)
\]

with \(\text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0\), where \(N_0(a, b)\) was defined in Equation (30). We discuss the value of \(\text{wt}(c(a, b))\) in the following two cases.

1. The case that \(m\) is odd.

From Equation (31), we get

\[
\text{wt}(c(a, b)) = q - N_0(a, b)
\]

\[
= \begin{cases} 
0 & \text{if } (a, b) = (0, 0), \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0, \\
p^{m-1}(p-1) & \text{if } a = 0, b \neq 0, \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0, \\
p^{m-1}(p-1) - p^{m-1}(p-1)q - m \cdot \left(1 - \frac{(p-1)(m+1)}{4} \right) & \text{if } a \neq 0, \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0 \\
& \text{or } a \neq 0, \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0, \\
p^{m-1}(p-1) + p^{m-1}(p-1)q - m \cdot \left(1 - \frac{(p-1)(m+1)}{4} \right) & \text{if } \eta(a) \eta\left(-\text{Tr}_{q/p} \left(\frac{b^2}{4a}\right)\right) = 1, \\
& \text{or } a \neq 0, \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0, \\
& \text{with } 1 \text{ time}, \\
& \text{with } (p^{m-1} - 1)(p^{m-2} + 1) \text{ time}, \\
\end{cases}
\]

when \((a, b)\) runs through \(GF(q) \times GF(q)\), where \(B = \left(-1\right)^{m\cdot(m-1)(m-2)/8 + m - 1} + \frac{2}{2} \cdot \sqrt{p^{m-1}}\) and the frequency is obtained based on Lemmas [45] [47] [14] and [15]. We first compute the frequency \(A_w\) of the nonzero weight \(w\), where

\[
w = p^{m-1}(p-1) - p^{m-1}(p-1)q - m \cdot \left(1 - \frac{(p-1)(m+1)}{4} \right),
\]

and

\[
A_w = \mathcal{Z}\{(a, b) : a \neq 0, \text{Tr}_{q/p} \left(\frac{b^2}{4a}\right) \neq 0, \eta(a) \eta\left(-\text{Tr}_{q/p} \left(\frac{b^2}{4a}\right)\right) = 1, \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0\}.
\]

Clearly, the number of \(a \in GF(q)^*\) such that \(\text{Tr}_{q/p}(a) = 0\) and \(\eta(a) = 1\), is \(n_a = \frac{p^{m-1} - 1}{2}\) by Lemma [45]. If we fix \(a\) with \(\text{Tr}_{q/p}(a) = 0\) and \(\eta(a) = 1\), the number of \(b\) such that \(\text{Tr}_{q/p}(b) = 0\) and \(\bar{\eta}\left(-\text{Tr}_{q/p} \left(\frac{b^2}{4a}\right)\right) = 1\),

| Weight                  | Multiplicity |
|-------------------------|--------------|
| 0                       | \(p^{m-1}(p-1)\) |
| \(p^{m-1}(p-1)\)       | \(1\) |
| \(p^{m-1}(p-1) - p^{m-1}(p-1)q - m \cdot \left(1 - \frac{(p-1)(m+1)}{4} \right)\) | \(B_1 \cdot (p^{m-2} + B_2)\) |
| \(p^{m-1}(p-1) - p^{m-1}(p-1)q - m \cdot \left(1 - \frac{(p-1)(m+1)}{4} \right)\) | \((p^{m-1} - 1 - B_1) \cdot (p^{m-2} - B_2)\) |
| \(p^{m-1}(p-1) - p^{m-1}(p-1)q - m \cdot \left(1 - \frac{(p-1)(m+1)}{4} \right)\) | \((p^{m-1} - 1 - B_1) \cdot (p^{m-1} - p^{m-2} + B_2)\) |
is \( \tilde{n}_b = \frac{p-1}{2}(p^{m-2} + (-1)^{m(p-1)(p-3) + m - 1 + \frac{q-1}{2}} \cdot \sqrt{p^m}) \) by Lemmas \[47\] and \[15\]. Meanwhile, the number of \( a \in \text{GF}(q)^* \) such that \( \text{Tr}_{q/p}(a) = 0 \) and \( \eta(a) = -1 \), is \( \tilde{n}_a = \frac{p^{m-1}}{2} \) by Lemma \[45\]. If we fix \( a \) with \( \text{Tr}_{q/p}(a) = 0 \) and \( \eta(a) = -1 \), the number of \( b \) such that \( \text{Tr}_{q/p}(b) = 0 \) and \( \bar{\eta}(-\text{Tr}_{q/p}(b^2)) = -1 \), is \( \tilde{n}_b = \frac{p-1}{2}(p^{m-2} + (-1)^{m(p-1)(p-3) + m - 1 + \frac{q-1}{2}} \cdot \sqrt{p^m}) \) by Lemmas \[47\] and \[15\]. Hence,

\[
A_w = \tilde{n}_a\tilde{n}_b + \tilde{n}_a\tilde{n}_b = \frac{p^{m-1} - 1}{2} \cdot (p - 1)(p^{m-2} + (-1)^{m(p-1)(p-3) + m - 1 + \frac{q-1}{2}} \cdot \sqrt{p^m}).
\]

The frequencies of other nonzero weights can be similarly derived.

(II) The case that \( m \) is even.

From Equation \((32)\), we get

\[
\text{wt}((a,b)) = q - N_0(a,b)
\]

\[
= \begin{cases} 
0 & \text{if } (a,b) = (0,0), \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0, \\
\frac{p^{m-1}}{2}(p - 1) & \text{if } a = 0, b \neq 0, \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0, \\
\frac{p^{m-1}}{2}(p - 1) - (p - 1)p^{m-2}(-1)^{m(p-1)/4} & \text{if } a \neq 0, 0 = \text{Tr}_{q/p}(b^2), \eta(a) = 1, \\
\frac{p^{m-1}}{2}(p - 1) - (p - 1)p^{m-2}(-1)^{m(p-1)/4} & \text{if } a \neq 0, 0 = \text{Tr}_{q/p}(b^2), \eta(a) = -1, \\
\frac{p^{m-1}}{2}(p - 1) - p^{m-2}(-1)^{m(p-1)/4} & \text{if } a \neq 0, 0 \neq \text{Tr}_{q/p}(b^2), \eta(a) = 1, \\
\frac{p^{m-1}}{2}(p - 1) - p^{m-2}(-1)^{m(p-1)/4} & \text{if } a \neq 0, 0 \neq \text{Tr}_{q/p}(b^2), \eta(a) = -1, \\
\frac{p^{m-1}}{2}(p - 1) + p^{m-2}(-1)^{m(p-1)/4} & \text{with } 1 \text{ time}, \\
\frac{p^{m-1}}{2}(p - 1) + p^{m-2}(-1)^{m(p-1)/4} & \text{with } p^{m-1} - 1 \text{ time}, \\
\frac{p^{m-1}}{2}(p - 1) + p^{m-2}(-1)^{m(p-1)/4} & \text{with } B_1 \cdot (p^{m-2} + B_2) \text{ time}, \\
p^{m-1} - 1 - B_1 \cdot (p^{m-2} - B_2) & \text{time}, \\
p^{m-1} - 1 - B_1 \cdot (p^{m-2} - B_2) & \text{time}, \\
p^{m-1} - 1 - B_1 \cdot (p^{m-2} + B_2) & \text{time}, \\
p^{m-1} - 1 - B_1 \cdot (p^{m-2} + B_2) & \text{time},
\end{cases}
\]

with 1 time,

with \( p^{m-1} - 1 \) time,

with \( B_1 \cdot (p^{m-2} + B_2) \) time,

with \( (p^{m-1} - 1 - B_1) \cdot (p^{m-2} - B_2) \) time,

with \( (p^{m-1} - 1 - B_1) \cdot (p^{m-2} - B_2) \) time,

with \( (p^{m-1} - 1 - B_1) \cdot (p^{m-2} + B_2) \) time,

and the frequency is easy to obtain based on Lemmas \[45\] \[47\] \[14\] and \[15\].

By the above two cases, the weight distributions in Tables \[XVI\] and \[XVII\] follow. This completes the proof. \[\square\]

**Example 51.** Let \( m = 3 \) and \( p = 3 \). Then the shortened code \( C_T \) in Theorem \[50\] is a \([24,4,15]\) linear code with the weight enumerator \( 1 + 48z^{15} + 32z^{18} \). This code \( C_T \) is optimal. Its dual \( \tilde{C}_T \) has parameters \([24,20,3]\) and is optimal according to the tables of best known codes maintained at \[http://www.codetables.de\].

**Example 52.** Let \( m = 4 \) and \( p = 3 \). Then the shortened code \( C_T \) in Theorem \[50\] is a \([78,6,48]\) linear code with the weight enumerator \( 1 + 240z^{48} + 240z^{51} + 26z^{54} + 192z^{57} + 30z^{60} \). This code \( C_T \) is almost optimal. Its dual \( \tilde{C}_T \) has parameters \([78,72,2]\) and is almost optimal according to the tables of best known codes maintained at \[http://www.codetables.de\].

**Theorem 53.** Let \( p \) be an odd prime, \( m \) and \( e \) be positive integers such that \( m/gcd(m,e) \) is odd. Let \( q = p^m \), \( f(x) = x^{p^e} + 1 \) and \( C \) be defined in \[6\]. Let \( T = \text{GF}(p) \). Then the parameters of the shortened code \( C_T \) are the same as that of \( C_T \) in Theorem \[50\].
Proof. The proof is similar to that of Theorem 50. Recall that the code \( C \) has length \( q \) and dimension \( 2m+1 \). By Lemma 1, the shortened code \( C_T \) has length \( 2^m - p \) and dimension \( 2m - 2 \). Since \( T = \text{GF}(p) \), the weight distribution of \( C_T \) is the same as the code

\[
\hat{C} = \left\{ \left( \text{Tr}_{q/p}(ax^{p^r+1} + bx) \right)_{x \in \text{GF}(q)} : a, b \in \text{GF}(q) \text{ and } \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0 \right\}.
\]

(36)

For each \((a, b) \in \text{GF}(q) \times \text{GF}(q)\) with \(\text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0\), define the corresponding codeword

\[
c(a, b) = \left( \left( \text{Tr}_{q/p}(ax^{p^r+1} + bx) \right)_{x \in \text{GF}(q)} \right) \in \hat{C}.
\]

Then the Hamming weight of \(c(a, b)\) is

\[
\omega_t(c(a, b)) = q - \hat{N}_0(a, b),
\]

(37)

where \(\text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0\) and \(\hat{N}_0(a, b)\) was defined in Lemma 10.

We need prove the value of \(\omega_t(c(a, b))\) and its frequencies in the following four cases.

- (I) \(m\) is odd and \(p \equiv 1 \mod 4\).
- (II) \(m\) is odd and \(p \equiv 3 \mod 4\).
- (III) \(m\) is even and \(p \equiv 1 \mod 4\).
- (IV) \(m\) is even and \(p \equiv 3 \mod 4\).

Next we only give the proof for the case (I) and omit the proofs for the other three cases whose proofs are similar. Suppose that \(m\) is odd and \(p \equiv 1 \mod 4\). From Equation (37) and Lemma 10 we get

\[
\omega_t(c(a, b)) = q - \hat{N}_0(a, b)
\]

\[
= \begin{cases} 
0 & \text{if } (a, b) = (0, 0), \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0, \\
p^{m-1}(p-1) & \text{if } a = 0, b \neq 0, \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0, \\
p^{m-1}(p-1) - p^{m/2-1} \sqrt{p^s} & \text{if } a \neq 0, b = 0, \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0, \\
p^{m-1}(p-1) - p^{m/2-1} \sqrt{p^s} & \text{if } a \neq 0, b \neq 0, \text{Tr}_{q/p}(a(x,a,b)^{p^r+1}) \neq 0, \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0, \\
p^{m-1}(p-1) + p^{m/2-1} \sqrt{p^s} & \text{if } a \neq 0, b \neq 0, \text{Tr}_{q/p}(a(x,a,b)^{p^r+1}) \neq 0, \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0, \\
0 & \text{with 1 time,} \\
p^{m-1}(p-1) & \text{with } (p^{m-1} - 1)(p^{m-2} + 1) \text{ time,} \\
p^{m-1}(p-1) - p^{m-2} & \text{with } p^{m-1}-1 \cdot (p-1)(p^{m-2} + B) \text{ time,} \\
p^{m-1}(p-1) + p^{m-2} & \text{with } p^{m-1}-1 \cdot (p-1)(p^{m-2} - B) \text{ time,} 
\end{cases}
\]

when \((a, b)\) runs through \(\text{GF}(q) \times \text{GF}(q)\), where \(B = (-1)^{\frac{(p-1)(p-3)}{8} + m-1 + \frac{q-1}{2}} \sqrt{p^{m-1}}\) and the frequency is obtained by Lemmas 48 and 45. As an example, we just compute the frequency \(A_w\) of the nonzero weight \(w\), where

\[
w = p^{m-1}(p-1) - p^{m-2}
\]

and

\[
A_w = \{ (a,b) : a \neq 0, b \neq 0, \text{Tr}_{q/p}(a(x,a,b)^{p^r+1}) \neq 0, \text{Tr}_{q/p}(a) = \text{Tr}_{q/p}(b) = 0, \eta(a) \eta(\text{Tr}_{q/p}(a(x,a,b)^{p^r+1})) = 1 \}.
\]

Clearly, the number of \(a \in \text{GF}(q)^*\) such that \(\text{Tr}_{q/p}(a) = 0\) and \(\eta(a) = 1\), is \(\tilde{n}_a = \frac{p^{m-1}-1}{2}\) by Lemma 45. If we fix \(a\) with \(\text{Tr}_{q/p}(a) = 0\) and \(\eta(a) = 1\), the number of \(b\) such that \(\text{Tr}_{q/p}(b) = 0\) and \(\eta(\text{Tr}_{q/p}(a(x,a,b)^{p^r+1})) = 1\)
, is $\tilde{n}_b = \frac{p-1}{2} \left( p^{m-2} + (-1)^\frac{m(p-1)(p-3)}{8} + m-1 + \frac{1}{2} \cdot \sqrt{p^m-1} \right)$ by Lemma 48. Meanwhile, the number of $a \in \text{GF}(q)^*$ such that $\text{Tr}_{q/p}(a) = 0$ and $\eta(a) = -1$, is $\tilde{n}_a = \frac{\eta^{m-1}-1}{2}$ by Lemma 45 if we fix $a$ with $\text{Tr}_{q/p}(a) = 0$ and $\eta(a) = -1$, the number of $b$ such that $\text{Tr}_{q/p}(b) = 0$ with $b \neq 0$ and $\eta(\text{Tr}_{q/p}(a(x,a,b))^p+1)) = -1$, is $\frac{p-1}{2} \left( p^{m-2} + (-1)^\frac{m(p-1)(p-3)}{8} + m-1 + \frac{1}{2} \cdot \sqrt{p^m-1} \right)$ by Lemma 48. Hence,

$$A_w = \tilde{n}_a \tilde{n}_b + \tilde{n}_a \tilde{n}_b = \frac{p^{m-1}-1}{2} \cdot (p-1) \left( p^{m-2} + (-1)^\frac{m(p-1)(p-3)}{8} + m-1 + \frac{1}{2} \cdot \sqrt{p^m-1} \right).$$

The frequencies of other nonzero weights can be similarly derived. This completes the proof of the weight distribution of Table [XVI] for the case $m$ odd and $p \equiv 1 \mod 4$.

The proofs of the other three cases are similar. The desired conclusions follow from Equation (37), Lemmas 10, 45 and 48. This completes the proof.

VI. CONCLUDING REMARKS

In this paper, we mainly investigated some shortened codes of linear codes from PN and APN functions and determined their parameters. The obtained codes have a few weights and some of these codes are optimal or almost optimal. Specifically, the main results are summarized as follows:

- For any binary linear code $C$ with length $q = 2^m$ and the weight distribution in Table II we gave a general result on their shortened code $C_T$ with $\#T = 4$ in Theorem 24. Meanwhile, when $m$ is odd, the parameters of the shortened code $C_T$ of a class of binary linear codes from APN functions was determined in Theorem 28.

- For any binary linear code $C$ with length $q = 2^m$ and the weight distribution in Table II, we gave a general result on their shortened code $C_T$ with $\#T = 3$ in Theorem 26. Further, the parameters of the shortened code $C_T$ of a special class of linear codes from APN functions were determined for $\#T = 3$ and $\#T = 4$ in Theorems 33 and 38.

- Two classes of $p$-ary shortened code $C_T$ from PN functions were presented and their parameters were also determined in Theorems 50 and 53 where $p$ is an odd prime.

Some of these shortened codes in this paper are optimal or almost optimal. Many codes with good parameters can be produced by using shortening and puncturing technologies. It seems hard to determine their weight distributions. The reader is cordially invited to settle these problems.

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