Higher spin gauge theory on fuzzy $S^4_N$

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Abstract
We examine in detail the higher spin fields which arise on the basic fuzzy sphere $S^4_N$ in the semi-classical limit. The space of functions can be identified with functions on classical $S^4$ taking values in a higher spin algebra associated to $\mathfrak{so}(5)$. We derive an explicit and complete classification of the scalars and one-forms on the semi-classical limit of $S^4_N$. The resulting kinematics is reminiscent of Vasiliev theory. Yang–Mills matrix models naturally provide an action formulation for higher spin gauge theory on $S^4$, with four irreducible modes for each spin $s \geq 1$. We diagonalize the quadratic part of the effective action and exactly evaluate the quadratic part in the spin 2 sector. By identifying the linear perturbation of the effective metric, we obtain the exact kinetic term for all graviton candidates. At the classical level, matter $T_{\mu\nu}$ leads to three different contributions to the linearized metric: one consistent with linearized GR, one more rapidly decreasing contribution, and one non-propagating contribution localized at $T_{\mu\nu}$. The latter is too large to be physically acceptable, unless there is a significant induced quantum action. This issue should be resolved on generalized fuzzy spaces.

Keywords: fuzzy 4-sphere, higher spin, Yang–Mills matrix models

1. Introduction

The fuzzy 4-sphere $S^4_N$ [1, 2] is a noncommutative space which can be viewed as a quantization of the round 4-sphere. It is characterized by the radius $R$ and by an integer $N$, which sets the UV scale $L_{\text{NC}} \sim \frac{R}{\sqrt{N}}$ where noncommutativity becomes important. Functions on the sphere are replaced by finite-dimensional matrices, which act on a large irreducible representation (irrep) $\mathcal{H}_N$ of $\mathfrak{so}(5)$. This provides a finite 4D quantum geometry which is fully covariant.

\footnote{This is in contrast to e.g. the Moyal–Weyl plane $\mathbb{R}^4_\theta$, which is not compatible with rotations.}
under \( SO(5) \). The fuzzy 4-sphere has been considered in several different contexts, including string theory [2–5], matrix models [6, 7], and condensed matter theory [8, 9]. Geometrical and structural aspects were studied e.g. in [10–14].

Due to the presence of an intrinsic UV scale \( L_{NC} \) as well as an IR scale \( R \), the fuzzy 4-sphere is a very promising background for formulating fundamental physical models, and to realize ideas on emergent gravity [15] in a covariant (Euclidean) setting. However, the non-trivial internal structure of \( S^4_N \) leads to some unusual features. Most notably, its algebra of ‘functions’ \( \text{End}(\mathcal{H}_N) \) is much richer than the classical counterpart. Besides the usual scalar functions, it contains further modes, which can be interpreted as higher spin modes with \( s = 1, 2, \ldots, N \). This suggests that \( S^4_N \) should naturally lead to a higher spin theory, as observed by several authors [6, 9, 16] and further examined in [17].

A systematic study of the higher spin theories arising on fuzzy \( S^4 \) was recently initiated in [18], with focus on the gravity sector. The natural framework for realizing gauge theory on \( S^4_N \) is provided by matrix models, in particular the maximally supersymmetric IKKT model [19]. In [18], spin 2 modes on fuzzy \( S^4_N \) were identified which have the required features for gravitons including the appropriate coupling to matter, and the transformation under diffeomorphisms. However, it was found that the linearized Einstein equations arise only on certain \textit{generalized fuzzy} spheres \( S^4_N \), with some assumptions and caveats. The underlying issue is a constraint between the translational and rotational spin 2 modes on the basic \( S^4_N \). The analysis in [18] was, however, incomplete due to the mixing of several different modes, which were not fully disentangled.

In the present paper, we provide a complete and systematic classification of the higher spin fields which arise on the basic fuzzy sphere \( S^4_N \) in the semi-classical limit, completing the analysis in [18]. First, we realize in section 3 the space of functions in terms of suitable Young diagrams, or equivalently in terms of traceless rank \( s \) tensor field on \( S^4 \). There is one such mode for each spin \( s \). This can be captured succinctly in terms of function on \( S^4 \) taking values in an infinite-dimensional higher spin algebra \( \mathfrak{h} \) associated to \( \mathfrak{so}(5) \). Locally, \( \mathfrak{h} \) coincides with the semi-classical limit of Vasiliev’s higher spin algebra, but the global structure is more intricate. The fuzzy case provides a finite truncation of \( \mathfrak{h} \).

Second, we provide in section 4 a complete and explicit classification of all ‘vector’ fluctuation modes on \( S^4_N \) in the framework of Yang–Mills matrix models. It turns out that there are 4 distinct irreducible (off-shell, gauge-fixed) modes for each spin \( s \geq 1 \), which are explicitly realized in terms of suitable Young diagrams or, equivalently, in terms of rank \( s \) tensor fields. All these modes can be arranged in terms of a tangential 1-form on \( S^4 \) taking values in \( \mathfrak{h} \), and a (radial) function on \( S^4 \) taking values in \( \mathfrak{h} \). This provides a kinematical link with Vasiliev theory [20, 21]. The local representation of these modes involves a combinations of gauge fields and their field strength, which is explicitly worked out for spin 1 and 2.

The next step is the formulation of physically interesting higher spin theories. Matrix models provides a natural action for a interacting gauge theory on fuzzy \( S^4_N \), where all fields transform under \( \mathfrak{h} \)-valued local gauge transformations. This is a remarkable statement, given the notorious difficulty in finding an action formulation of higher spin theories. We explicitly diagonalize the quadratic part of this action in section 6. Focusing on the spin 2 sector, we identify the effective metric fluctuation (graviton) \( h_{\mu\nu} \), which is a linear combination of the basic spin 2 modes. We recover the appropriate transformation law under diffeomorphisms, which are part of the higher spin gauge invariance.

Given the full classification of the modes, we compute exactly the quadratic part of the effective action for these spin 2 modes coupled to matter via \( T_{\mu\nu} \). It turns out that the quadratic action for \( h_{\mu\nu} \) arises primarily via its spin connection, so that its dominant role is that of
a non-propagating auxiliary field, which is strongly localized at the matter source. However, there is a (sub-leading) mode which does mediate linearized Einstein gravity, and yet another mode which is more rapidly decaying, but nevertheless large. This means that the classical higher spin theory on the basic $S^4_N$ in Yang–Mills matrix models does not lead to realistic gravity, consistent with [18].

Nevertheless, we point out two possibilities which might lead to (more) realistic gravity in this context: First, the inclusion of one-loop quantum effects leads, as usual, to induced gravity terms in the effective action. If these induced terms are large, the above conclusion is reversed: the would-be auxiliary graviton is transmuted into a proper graviton governed by the appropriate linearized Einstein equations, while the other modes lead to sub-leading long-distance modifications, somewhat reminiscent of conformal gravity [22]. However, this scenario requires special parameter regimes.

The second, perhaps more interesting possibility to obtain (more) realistic gravity is to replace the basic fuzzy sphere $S^4_N$ by the generalized fuzzy sphere $S^4_\Lambda$, as suggested in [18]. The point is that $S^4_{\Lambda}$ admits translational modes which are independent of the rotational modes, unlike in the basic case. Consistent with the preliminary results in [18], we identify an appropriate graviton mode on $S^4_\Lambda$ which appears to avoid the undesired behavior found on $S^4_N$. This could be investigated along the same lines as in the present paper, but is postponed to future work.

Even if the model under consideration may not yet lead to the desired physics, the main message is, nonetheless, remarkable and very promising: Matrix models provide a natural and simple framework for actions for higher spin gauge theories on (fuzzy) $S^4$, which arise from the twisted bundle structure of a higher-dimensional noncommutative space over $S^4$. This provides a covariant quantization of a 4D space in a rigorous framework, and a simple geometric origin for higher spin theories. For example, the higher spin gauge transformations on $S^4$ are recognized as symplectomorphisms on $\mathbb{C}P^3$.

The present paper is restricted to the Euclidean case, having the advantage that the rich structure is completely under control. Of course one would like to move on to Lorentzian signature. There are also some candidates for analogous covariant fuzzy spaces with Minkowski signature [16, 23, 24]. An analogous study for such spaces is postponed to future work.

2. The basic fuzzy 4-sphere $S^4_N$

We are interested in covariant fuzzy 4-spheres, which are defined in terms of 5 hermitian matrices $X^a$, $a = 1, \ldots, 5$ acting on some finite-dimensional Hilbert space $\mathcal{H}$, and transforming as vectors under $SO(5)$, i.e.

\[
[M_{ab}, X_c] = i(\delta_{ac}X_b - \delta_{bc}X_a),
\]

\[
[M_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} - \delta_{bd}M_{ac} - \delta_{bc}M_{ad} + \delta_{ad}M_{bc}).
\]

(2.1)

Throughout this paper, indices are raised and lowered with $g_{ab} = \delta_{ab}$. The $M_{ab}$ for $a, b \in \{1, \ldots, 5\}$ generate a suitable representation of $\mathfrak{so}(5)$ on $\mathcal{H}$, and $X^a \in \text{End}(\mathcal{H})$ are operators interpreted as quantized embedding functions $X^a \sim x^a : S^4 \to \mathbb{R}^5$. Then the radius

\[
X^a X_a = R^2
\]

(2.2)

is a scalar operator of dimension $L^2$. The commutator of the $X^a$ will be denoted by

\[
[X^a, X^b] = :i\Theta^{ab}:
\]

(2.3)
Such relations constitute a **covariant quantum 4-sphere**. Particular realizations of such fuzzy 4-spheres are obtained from generators \( \mathcal{M}^{ab} \), \( a, b = 1, \ldots, 6 \) of \( \mathfrak{so}(6) \cong \mathfrak{su}(4) \) via
\[
X^a = r \mathcal{M}^{a6}, \quad a = 1, \ldots, 5, \quad \Theta^{ab} = r^2 \mathcal{M}^{ab}.
\] (2.4)
Here \( r \) is a scale parameter of dimension \( L \), and \( \mathcal{H} \) is some irreducible representation (irrep) of \( \mathfrak{so}(6) \). This class of quantum spheres was considered in [18] as a promising basis for a higher spin theory including gravity, and their geometry was studied further in [25].

These covariant quantum 4-spheres can be viewed as compact versions of Snyder space [26, 27]. The crucial feature is that the classical isometry group \( SO(5) \) is fully realized. This is in marked contrast to the basic quantum spaces such as the Moyal–Weyl quantum plane \( \mathbb{R}^6_q \), where the Poisson tensor \( \Theta^{ab} \) breaks this symmetry. The price to pay is that the algebra of ‘coordinates’ \( X^a \) does not close, because extra generators \( \Theta^{ab} \) are involved. Nevertheless, one can define physical theories on such spaces via matrix models, leading to fully covariant higher spin theories with large gauge symmetry, including a gauged version of \( SO(5) \).

In this paper we will focus on the simplest example of the above construction: the ‘basic’ fuzzy 4-sphere \( S^4_N \) [1, 2, 4]. This is obtained for the highest weight irrep \( \mathcal{H} = \mathcal{H}_\Lambda \) of \( \mathfrak{so}(6) \) with \( \Lambda = (N,0,0) \). Throughout this paper we denote highest weights by their Dynkin indices.

Then the following relations hold:
\[
X^a X_a = R^2 = r^2 R_N^2 \mathbb{1}, \quad R^2_N = \frac{1}{4} N(N + 4),
\]
\[
\{X_a, \Theta^{ab}\} = 0, \quad \frac{1}{2} \{\Theta^{ab}, \Theta^{cd}\} + \delta^{ac} \Theta^{bd} = r^2 (g^{bc} - \frac{1}{2 R^2} \{X^b, X^c\}) + 4(N + 2) r^3 X^e, \quad \epsilon_{abcd} \Theta^{ab} \Theta^{cd} = 4(N + 2) r^3 X^e, \quad (2.5)
\]
for indices \( a, b, \ldots = 1, \ldots, 5 \). Here \( \{\cdot, \cdot\} \) denotes the anti-commutator. The first relation expresses the fact that \( \mathcal{H}_\Lambda \) remains irreducible as representation of \( \mathfrak{so}(5) \subset \mathfrak{so}(6) \), which no longer holds for generic \( \Lambda \).

**Oscillator construction.** It is worthwhile recalling the following oscillator construction of fuzzy \( S^4_N \) [1, 2]. Consider four bosonic oscillators
\[
[a^\alpha, a^\beta_\dagger] = \delta^\alpha_\beta, \quad \alpha, \beta = 1, \ldots, 4, \quad (2.6)
\]
which transform in the spinorial representation of \( \mathfrak{so}(6) \) (and \( \mathfrak{so}(5) \)). Then define (for \( r = 1 \))
\[
X^c = \frac{1}{2} a^\dagger \gamma^c a, \quad \mathcal{M}^{ab} = a^\dagger \Sigma^{ab} a = -i[X^a, X^b] \quad (2.7)
\]
(suppressing spinorial indices), where \( \gamma^c, \ c = 1, \ldots, 5 \) are the gamma matrices associated to \( SO(5) \) acting on \( \mathbb{C}^4 \). It is then easy to check that (2.1) is satisfied, and (2.2), (2.5) hold on the \( N \)-particle Hilbert space \( \mathcal{H}_N = a_{1}^\dagger \cdots a_{N}^\dagger |0\rangle \cong (0, N)_{SO(5)} \).

2.1. **Semi-classical limit \( S^4 \)**

To understand the geometrical meaning of \( \Theta^{ab} \), it is best to view the fuzzy sphere as quantization of the 6D coadjoint orbit \( CP^3 \) of \( SO(6) \); this viewpoint naturally extends to the generalized spheres \( S^4_\Lambda \) of [18, 25]. The generic construction is as follows:\footnote{See e.g. [28] for a nice introduction to (quantized) coadjoint orbits.} For any given
(finite-dimensional) irrep $\mathcal{H}_\Lambda$ of $SO(6)$ with highest weight $\Lambda$, the generators $\mathcal{M}^{ab} \in \text{End}(\mathcal{H}_\Lambda)$ of its Lie algebra $\mathfrak{so}(6)$ are viewed as quantized embedding functions

$$\mathcal{M}^{ab} \sim m^{ab} : \quad \mathcal{O}_\Lambda \hookrightarrow \mathbb{R}^{15} \cong \mathfrak{so}(6)$$

of the homogeneous space (coadjoint\(^3\) orbit)

$$\mathcal{O}_\Lambda = \{ g \cdot \Lambda \cdot g^{-1} : g \in SO(6) \} \cong SO(6)/K \subset \mathbb{R}^{15},$$

with $K$ denoting the stabilizer of $\Lambda$ in $SO(6)$. As customary, one can identify $\Lambda$ with a Cartan generator of $\mathfrak{so}(6)$ via $\Lambda \in \mathfrak{b}^* \leftrightarrow H_\Lambda \in \mathfrak{b}$. For $\Lambda = (N, 0, 0)$, this gives $\mathcal{O}_\Lambda \cong CP^3$, which is naturally a $S^2$-bundle over $S^4$ via the Hopf map

$$x^a = r \ m^{ab}, \quad CP^3 \to S^4 \hookrightarrow \mathbb{R}^5.$$  

We denote this $SO(5)$-equivariant bundle with $S^4 \cong CP^3$ in this paper. Define $\theta^{ab} = r^2 m^{ab}$ for $a, b = 1, \ldots, 5$, then one can show the following semi-classical analogs of (2.5):

$$x^a x_a = R^2,$$

$$x_a \theta^{ab} = 0,$$

$$\theta^{ab} \theta^{cd} g_{au} = \frac{L_{NC}^4}{4} \left( g^{bc} - \frac{1}{R^2} x^b x^c \right) = \frac{L_{NC}^4}{4} P_T^{bc},$$

$$\epsilon_{abcd} \theta^{ab} \theta^{cd} = 2 L_{NC}^4 \frac{x_e}{R},$$

for $a, b = 1, \ldots, 5$, where

$$\theta = \frac{r^2}{R}, \quad L_{NC}^2 = 2rR$$

are parameters of dimension $L^2$. We refer to the tensor $P_T^{bc}$ as tangential projector because it satisfies $P_T^{bc} x^c = x^b$ and $P_T^{bc} P_T^{de} = P_T^{bc}$.

**Poisson structure.** Any such coadjoint orbit carries a (Kirillov–Kostant) Poisson structure,

$$\{ \theta^{ab}, \theta^{cd} \} = \theta \left( g^{au} \theta^{bd} - g^{ad} \theta^{bc} - g^{bc} \theta^{ad} + g^{bd} \theta^{ac} \right), \quad a, b, c, d = 1, \ldots, 5,$$

$$\{ \theta^{ab}, x^c \} = \theta \left( g^{ac} x^b - g^{bc} x^a \right), \quad a, b, c = 1, \ldots, 5,$$

$$\{ x^a, x^c \} = \theta \delta^{ac}, \quad a, c = 1, \ldots, 5,$$

which is $SO(5)$-invariant. This can also be obtained from an oscillator construction as in (2.7), replacing the creation- and annihilation generators $a_\alpha, a_\alpha^\dagger$ by holomorphic coordinate functions on $\mathbb{C}^4$ with Poisson structure $\{ x^a, a_\alpha \} = -i \delta^{a\alpha}$. In particular, we note

$$\{ \theta^{ab}, x^c \} = -4 \theta x^c,$$

which are the equations of motion of the ‘Poisson matrix model’ (5.2) introduced on section 5.2. Consequently, the $S^4$ generated by $x^a$ is a solution thereof.

For an arbitrary point $p \in S^4$ (e.g. the ‘north pole’ $p = R(0, 0, 0, 0, 1)$), we can decompose $\mathfrak{so}(5)$ into rotation generators $\mathcal{M}^{a \nu} (\mu, \nu = 1, \ldots, 4)$ and translation generators $P^\mu$, given by

$$P^\mu = \frac{1}{R} \theta^{\mu 5}, \quad \mu = 1, \ldots, 4.$$

\(^3\)For simplicity we identify the Lie algebra with its dual.
Although they vanish as classical functions due to $p^\mu \propto \{x^\mu, x^a\}x_a = 0$, see also [18], they do not vanish as generators, and cannot be dropped.

**Coherent states.** It is well-known that the quantized coadjoint orbits $O_\Lambda$ allow for the introduction of coherent states, which lie on the $SO(6)$ orbits of the highest weight state $|\Lambda\rangle \in \mathcal{H}_\Lambda$, i.e.

$$|x\rangle \equiv |x; \xi\rangle := g_x \cdot |\Lambda\rangle, \quad g_x \in SO(6).$$

(2.16)

Here, we labelled the points on $O_\Lambda \cong \mathbb{C}P^3$ by $x \in S^4$ and the fiber coordinate $\xi$, where the 'north pole' $p$ corresponds to $|\Lambda\rangle$. Coherent states are optimally localized, i.e. they minimize the uncertainty in position space. Using the defining relations (2.5), one computes in the large $N$ approximation

$$\Delta^2 := \sum_{a=1}^{5} (\langle X^a \rangle - \langle X^a \rangle^2) = \sum_{a=1}^{5} \langle (X^a)^2 \rangle - \langle (X^a) \rangle^2
\approx r^2 \left( \frac{R^2}{N} - \frac{N^2}{4} \right) \equiv \frac{4}{N} R^2 \approx 2rR \equiv L_{NC}^2.$$  

(2.17)

This then defines the length scale $L_{NC}$, which appeared in (2.12), and highlights its role as non-commutativity scale.

**Scales.** Before proceeding we emphasize that there are two scales involved: the IR or ‘cosmological’ scale $R$ giving the size of the sphere, and the UV scale $L_{NC} = \sqrt{\frac{r}{N}}$. This is the scale where non-commutative corrections in the star product become relevant, since

$$f \star g = f \cdot g + O\left( (L_{NC} \cdot \partial)^2 (f, g) \right).$$

(2.18)

Strictly speaking there is also a third scale $\frac{R}{\sqrt{N}}$, which is the UV cutoff on $S^4$.

### 2.2. Calculus and forms

We want to develop a differential calculus which allows to efficiently work with this semi-classical $S^4$. We introduce formal Grassmann variables $\xi_a$, which transform as a vector of $SO(5)$, and satisfy

$$\xi^a \xi^b = -\xi^b \xi^a \quad a, b = 1, \ldots, 5.$$  

(2.19)

The space $\Omega^* S^4$ is defined as the $S^4$-module of forms of degree $n$ in $\xi^a$, and the (exterior) algebra of forms on $S^4$ is

$$\Omega^* S^4 = \bigoplus_{n=0}^{5} \Omega^n S^4,$$

with $\Omega^0 S^4 \equiv \mathcal{C}$.

(2.20)

Here $\mathcal{C}$ is the space of functions on $S^4 \cong \mathbb{C}P^3$ with generators $x^a$ and $\theta^{ab}$. There are three special $SO(5)$-invariant forms

$$\xi := x_a \xi^a \in \Omega^1 S^4,$$

$$\omega := \theta_{ab} \xi^a \xi^b \in \Omega^2 S^4,$$

$$\Omega := \epsilon_{abcde} \xi^a \ldots \xi^e \in \Omega^5 S^4,$$

(2.21)

which play a special role. $\Omega$ is the 5D ‘volume’ form. Using the invariant metric, we can define the 4D cotangent space as orthogonal complement to $\xi$.  

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Consider the following $SO(5)$ intertwiners:

\[
Q : \Omega^0 S^4 \to \Omega^{+1} S^4, \quad Q(\alpha) = \{\xi, \alpha\}_\pm,
\]

\[
J : T^* S^4 \to T^* S^4, \quad J(\xi^a A_b) = \xi^b A_a,
\]

\[
I : \Omega^1 S^4 \to \Omega^1 S^4, \quad I(\xi^a A_b) = \xi^b (\theta^a, A_b),
\]

\[
N : \Omega^1 S^4 \to \mathcal{C}, \quad N(\xi^a A_b) = x_a A^b.
\]

where $\{\cdot, \cdot\}_\pm$ denotes the appropriately graded Poisson bracket. They satisfy

\[
Q^2 \alpha = \frac{1}{2}\{\omega, \alpha\}, \quad J^2 = -\theta R^2 p_f, \quad J(\xi) = 0.
\]

$J$ arises from the complex (Kähler) structure on the bundle space $\mathbb{C}P^3$. We will see in section 7.2 that $Q(A)$ is the infinitesimal gauge transformation of the $S^4$ background. Note that $Q$ and $J$ are tangential, which means they are annihilated by $\mathcal{N}$:

\[
\mathcal{N}(J(\alpha)) = 0, \quad \alpha \in \Omega^1 S^4,
\]

\[
\mathcal{N}(Q(f)) = 0, \quad \text{i.e. } Qf = \xi_a(x^a f) \in T^* S^4,
\]

because $x_a(x^a f) = \frac{1}{2}\{R^2, f\} = 0$ and $f \in \mathcal{C}$. Moreover, we can write

\[
Q^2(f) = \theta R^2 \xi^a \xi^c p_f, \quad f \in \mathcal{C},
\]

for instance at the north pole, since $\{\mathcal{M}^{\mu\nu}, \cdot\}$ vanishes for functions. This might seem reminiscent of supersymmetry, however $Q^2$ involves a commutator rather than an anti-commutator. Furthermore, using (2.14) one can show the following identities (see appendix A.1 for more details):

\[
I(\xi \phi) = -4\theta^b \xi_b \phi - J(Q(\phi)),
\]

\[
I \circ J(\xi^a A_b) = -4\theta^b J(A) + \theta^b \{x_a, A^b\} + \theta Q(\mathcal{N}(A)) - J \circ I(A).
\]

One can also define a Hodge star operator either on $\mathbb{R}^5$ or on $S^4$ as follows:

\[
*: \Omega^1 S^4 \to \Omega^2 S^4, \quad *(\xi^a) = c e^{abcd} \xi_d \xi_b \xi_c \xi_e \text{ etc}
\]

\[
*_4 : \Omega^1 S^4 \to \Omega^3 S^4, \quad *_4(\alpha) = *(\alpha \xi)
\]

normalized such that $** = 1$; however this will not be important the present paper. Now consider the functional

\[
\mathcal{G} : \Omega^1 S^4 \to \mathcal{C}, \quad \mathcal{G}(A) := \{x^a, A_a\}, \quad A \equiv \xi_a A^a \in \Omega^1 S^4,
\]

which will be used for gauge fixing in section 5.2. The kernel of $\mathcal{G}$ contains $J(Q(\phi))$, because

\[
\mathcal{G}[J(Q(\phi))] = \{x^a, \theta^{ab} \{x_b, \phi\}\}
\]

\[
= \{x^a, \theta^{ab} \{x_b, \phi\} + \theta^{ab} \{x^a, x_b, \phi\}\}
\]

\[
= \theta^{ab} \{x^a, x_b, \phi\} = 0,
\]

noting that

\[
\theta^{ab} \{x^a, x_b, \phi\} = -\theta^{ab} \{\phi, \{x^b, x^a\}\} + \{\phi, \{x^a, x^b\}\}
\]

\[
= -\theta^{ab} \{x^a, x_b, \phi\} - \theta^{ab} \{\phi, \{x^a, x^b\}\}.
\]
Hence
\[ 2\theta^{ab}\{x^a, \{x_b, \phi\}\} = -\theta^{ab}\{\phi, \theta^{ab}\} = 0, \tag{2.33} \]
which holds for any function \(\phi \in C\). Finally there is a natural ‘Poisson’ Laplacian, given by
\[ \Box f := \{x^a, \{x_a, f\}\}, \quad f \in C \tag{2.34} \]
or equivalently \(\Box f = c' \ast Q \ast Qf\), for some number \(c'\). The vector Laplacian is given by
\[ \Box^2 : \Omega^1 S^4 \rightarrow \Omega^1 S^4, \quad \Box^2 A_a = (\Box - 2I)A_a, \tag{2.35} \]
which will be discussed in section 5.2.

2.3. Derivation and connection

**Derivation.** We can define the following \(SO(5)\)-covariant derivation on \(C\):
\[ \partial := -\frac{1}{\theta R^2} J \circ Q : C \rightarrow T^*S^4 \tag{2.36} \]
or more explicitly
\[ \partial^a \phi := -\frac{1}{\theta R^2} \theta^{ab}\{x_b, \phi\}, \quad \phi \in C, \tag{2.37} \]
which is indeed tangential and satisfies the Leibniz rule. The definition (2.36) is equivalent to
\[ \{x^a, \cdot\} = \theta^{ab} \partial_b. \tag{2.38} \]
In particular,
\[ \partial^a x^a = -\frac{1}{\theta R^2} \theta^{ab}\{x_b, x^a\} = P^a. \tag{2.39} \]
Therefore \(\partial\) reduces to the ordinary tangential derivative \(\partial^\mu f|_p\) for scalar functions \(f(x)\) at any given point \(p \in S^4\), e.g. the north pole. The derivation acts on the \(\theta^{ab}\) generators as
\[ \partial^a \theta^{cd} = -\frac{1}{\theta R^2} \theta^{ab}\{x_b, \theta^{cd}\} = \frac{1}{R^2} \left( -\theta^{ad} x^c + \theta^{ac} x^d \right), \tag{2.40} \]
which at the north pole reduces to
\[ \partial^\mu \theta^{\nu\eta} = 0, \quad \partial^\mu P^\nu = \frac{1}{R^2} \theta^{\mu \nu}, \quad P^\nu = \frac{1}{R} \theta^{\mu 5}. \tag{2.41} \]
Note that the second relation connects \(\theta^{\mu \nu}\) and \(P^\nu\). In particular, although the \(P^\mu\) vanish as functions on \(CP^3\), they do not vanish as generators, and cannot be dropped. As a consistency check, we note that
\[ 0 = \partial^\mu (\theta^{\nu a} x_a) = \frac{1}{R} \theta^{\mu \nu} R + \theta^{\nu a} \delta_\mu^a. \tag{2.42} \]
Moreover, the following intertwiner \(\text{div} : \Omega^1 S^4 \rightarrow C\) defined as:
\[ \text{div} A := \partial \cdot A = \partial^a A_a \]
\[ = -\frac{1}{\theta R^2} \theta^{ab}\{x^a, A_b\} = \frac{1}{\theta R^2} x^a \{\theta^{ab}, A_b\} = \frac{1}{\theta R^2} N(I(A)) \tag{2.43} \]
reduces to the divergence for tangential vector fields. It satisfies
\[
\text{div} J(A) = \frac{1}{\theta R^2}N(J(J(A))) = \{x^\mu, A_\mu\} = F(A)
\]
\[= \theta^{\mu\nu} \partial_\mu A_\nu = \frac{1}{2} \theta^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) \quad (2.44)
\]
using (2.28). In particular, \(\{x^\mu, A_\mu\}\) is some component of the field strength \(F(A)\) of.

Connection. Now consider tensor fields on \(S^4\) such as \(A_a, A_{ab}, \ldots\) which are tangential, i.e. \(A_{ab}x^a = 0\) etc. Then \(\partial\) does not respect these constraints: for example, if \(A_a\) is a tangential vector field, i.e. \(x^a A_a = 0\), then \(\partial_b A_a\) is not tangential in the index \(b\), since
\[
x^b \partial_a A_b = \partial_a (x^b A_b) - A_b \partial_a x^b = -A_b P_{T}^{b\mu} = -A_a \neq 0.
\]
To remedy this, we project on the tangential indices with \(P_T\), and define
\[
\nabla := P_T \circ \partial,
\]
where \(P_T\) acts on all components. For example if \(A_a\) is tangential, then
\[
\nabla_a A_b = \partial_a A_b + \frac{1}{R^2} x_b A_a
\]
is indeed tangential. \(\nabla\) is an \(SO(5)\)-equivariant connection on \(S^4\) which does not respect the sub-bundle corresponding to \(P^\mu\), due to the second relation in (2.41).

We conclude this section with two remarks. First, this calculus can be naturally extended to act on forms \(A \in \Omega^* S^4\) by defining
\[
\partial_a \xi_b = 0.
\]
This amounts to \(\{\theta^{ab}, \xi_c\} = 0 = \{x^a, \xi^b\}\). Second, there is another connection besides the above, which is the canonical \(SO(5)\)-equivariant connection given at the north pole by
\[
\partial'_\mu = \{P_\mu, \cdot\}.
\]
This derivation differs from \(\partial\) because \(\partial' \theta^{\alpha\beta} \sim P \neq \partial \theta^{\alpha\beta}\), while \(\partial' P = \partial P\).

3. Functions on \(S^4\) and higher spin

It is well-known that the algebra of functions on \(S^4\) decomposes into \(SO(5)\) harmonics as follows [4]:
\[
C \equiv C^\infty(CP^3) \cong \bigoplus_{s \geq 0} C^s,
\]
\[
\phi \mapsto \sum_{s \geq 0} \phi^{(s)}.
\]
Here
\[
C^s \cong \bigoplus_{n \geq 0} (n, 2s)_{so(5)}
\]
is a module over the algebra of scalar functions on \(S^4\), corresponding to certain spin \(s\) fields. The component of \(\phi \in C\) in the module \(C^s\) will be denoted by \(\phi^{(s)}\). We will provide a more explicit interpretation of these modules below. However, the full algebra respects this gradation.
in $s$ only modulo 2, because of the constraints (2.11). More details about the bundles $C^r$ and the corresponding field strength etc shall be discussed elsewhere.

**Averaging.** Consider the map [4]

$$ C = C^\infty(S^4) \to C^0 \quad \phi^{(s)} \mapsto [\phi^{(s)}]_0 := \delta^{(s)} \phi^{(s)} $$

which projects to the spin 0 scalar functions $C^0$. This amounts to averaging over the $S^2$ fiber at each point $x \in S^4$. Explicitly, this is given by (see [18])

$$ [\theta^{ab}]_0 = 0, \quad (3.4a) $$

$$ [\theta^{ab} \theta^{cd}]_0 = \frac{1}{3} \theta R^2 \left( P^a_P^b P_T^d - P^b_P^d P_T^a + \frac{1}{R} \mathcal{O}^{abcde} x^e \right), \quad (3.4b) $$

$$ [\theta^{ab} \theta^{cd} \theta^{ef}]_0 = 0, \quad (3.4c) $$

$$ [\theta^{ab} \theta^{cd} \theta^{ef} \theta^{gh}]_0 = \frac{3}{5} \left( [\theta^{ab} \theta^{cd}]_0 [\theta^{ef} \theta^{gh}]_0 + [\theta^{ab} \theta^{ef}]_0 [\theta^{cd} \theta^{gh}]_0 + [\theta^{ab} \theta^{gh}]_0 [\theta^{cd} \theta^{ef}]_0 \right), \quad (3.4d) $$

e etc. Similarly, there is a natural integral over $S^4$ defined by projecting $C$ to the unique trivial component. The normalization is given by the semi-classical limit of the trace:

$$ \text{Tr}_{\text{End}(\mathcal{H})} \sim \int d\Omega = \frac{\dim \mathcal{H}}{\text{vol}(S^4)} \int_{S^4} \frac{d\Omega}{\text{vol}(S^4)} \int [\cdot]_0. \quad (3.5) $$

This is an integral over $S^4 = \mathbb{C}P^3$ with the canonical symplectic measure $d\Omega$, which can be written as an integral over the 4-sphere $S^4$ after averaging $[\cdot]_0$ over the $S^2$ fiber. The appropriate factor $\text{vol}(S^4)$ or $\text{vol}(S^2)$ is understood in the following, and we will drop $d\Omega$ if no confusion can arise.

**$C^0$ as scalar fields on $S^4$.** For $n \geq 0$, $(n, 0)$ can be realized as space of symmetric traceless $SO(5)$ tensors $\phi^{(0)}_{a_1...a_n}$ corresponding to Young diagrams $\begin{ytableau} 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \end{ytableau}$ consisting of only one line of length $n$. These are in one-to-one correspondence to polynomial functions on $S^4$,

$$ \phi^{(0)} = \phi^{(0)}_{a_1...a_n} x^{a_1} ... x^{a_n} =: \phi^{(0)}(x) \in C^0. \quad (3.6) $$

**$C^1$ as vector fields on $S^4$.** Consider the space of $(n, 2)$ functions on $S^4$, for $n \geq 0$. These modes can be similarly characterized by Young diagrams $\begin{ytableau} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0; 1 & 1 \end{ytableau}$ with one row of length $n + 1$ and one column of length 2. This defines irreducible representations

$$ \phi^{(1)}_{a_1...a_n b c} := (P_S \circ P_A) \phi^{(1)}_{a_1...a_n b c} \subset (C^5)^{\otimes (n+2)}, \quad (3.7) $$

which are totally symmetric in $a_1, ... , a_n, b$, Here, we have chosen a basis of tensors exhibiting the symmetry of Young diagrams by first antisymmetrizing in columns by $P_A$ and subsequently symmetrizing in rows by $P_S$. By contracting these tensors with generators of $S^4$, they define $C^1$ modes via

$$ \phi^{(1)} = \phi^{(1)}_{a_1...a_n b c} x^{a_1} ... x^{a_n} \theta^{bc} =: \phi^{(1)}(x) \theta^{bc} \in (n, 2) \subset C^1. \quad (3.8) $$

There is a canonical vector field (or one-form) on $S^4$ associated to such a $\phi^{(1)} \in C^1$, with components given by

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4 Useful discussions and collaboration with Ramgoolam are gratefully acknowledged here.
\[
\phi^{(1)}_c(x) := \phi^{(1)}_{a_1 \ldots a_b c} x^{a_1} \ldots x^{a_b} x^c.
\]

We will denote this \(\phi^{(1)}_c(x)\) as symbol for \(\phi^{(1)} \in \mathcal{C}^1\). It amounts to a one-form
\[
\mathcal{A}^{(0)} := \phi^{(1)}_c(x) \xi^c,
\]
which is tangential and divergence-free
\[
\phi^{(1)}_c(x) \partial^c = 0 = \mathcal{N}(\mathcal{A}^{(0)}),
\]
\[
\partial^c \phi^{(1)}_a(x) = 0 = \text{div} \mathcal{A}^{(0)}.
\]

Hence \(\mathcal{C}^1\) can be identified with divergence-free rank 1 tensor fields on \(S^4\), via

\[
\Psi : \mathcal{C}^1 \to T^*S^4
\]
\[
\phi^{(1)} = \phi^{(1)}_{b c}(x) \theta^{b c} \mapsto \phi^{(1)}_c(x).
\]

We will see that \(\phi^{(1)}_c(x) \propto [\{x_a, \phi^{(1)}\}]_0 \) in (4.40), and the inverse of this map is given by
\[
\{x^c, \phi^{(1)}_c(x)\} = -(n + 1) \phi^{(1)}
\]
restricted to divergence-free \(\phi_a\). Hence \(\mathcal{C}^1\) can be identified with volume-preserving diffeomorphisms on \(S^4\).

\(\mathcal{C}^s\) as tensor fields on \(S^4\). For \(n \geq 0\), \((n, 2s)\) is the space of totally traceless \(SO(5)\) tensors \(\phi^{(s)}_{a_1 \ldots a_b d_1 \ldots d_s} \) corresponding to Young diagrams \((b, s)\) which are first antisymmetrized in each pair \(b(c)\), and then symmetrized in \((a_1 \ldots a_n b_1 \ldots b_s)\) and \(c_1 \ldots c_s\). Then define
\[
\phi^{(s)} := \phi^{(s)}_{a_1 \ldots a_b d_1 \ldots d_s} x^{a_1} \ldots x^{a_n} \theta^{b d_1} \ldots \theta^{b d_s} =: \phi^{(s)}_{b_1 \ldots b_s c_1 \ldots c_s}(x) \theta^{b d_1} \ldots \theta^{b d_s} \in \mathcal{C}^s.
\]

We associate\(^5\) to each such \(\phi^{(s)}\) a symmetric rank \(s\) tensor field on \(S^4\) via
\[
\phi^{(s)}_{c_1 \ldots c_s}(x) := \phi^{(s)}_{a_1 \ldots a_b d_1 \ldots d_s} x^{a_1} \ldots x^{a_n} x^{d_1} \ldots x^{d_s}
\]
denoted as symbol for \(\phi^{(s)} \in \mathcal{C}^s\). These are traceless, tangential and divergence-free,
\[
\phi_{c_1 \ldots c_s}(x) x^c = 0,
\]
\[
\phi_{c_1 \ldots c_s}(x) x^{c_{c_s}} = 0,
\]
\[
\partial^c \phi_{c_1 \ldots c_s}(x) = 0.
\]

Hence, \(\mathcal{C}^s\) can be identified\(^6\) with symmetric traceless divergence-free rank \(s\) tensor fields on \(S^4\).

\[
\Psi : \mathcal{C}^s \to T^*S^4
\]
\[
\phi^{(s)} = \phi^{(s)}_{b_1 \ldots b_s c_1 \ldots c_s}(x) \theta^{b d_1} \ldots \theta^{b d_s} \mapsto \phi^{(s)}_{c_1 \ldots c_s}(x) = \phi^{(s)}_{b_1 \ldots b_s c_1 \ldots c_s}(x) x^{b_1} \ldots x^{b_s}.
\]

The \(\phi^{(s)} \in \mathcal{C}^s\) are potentials of the tensor fields, in the sense that
\[
\phi_{a_1 \ldots a_n}(x) \propto [\{x_{a_1}, \ldots, \{x_{a_n}, \phi^{(s)}\}\}]_0.
\]

\(^5\) We will often drop the superscript \(^{(s)}\) if no confusion can arise.

\(^6\) It is important that the identification (3.17) only applies to those tensor fields which are obtained from irreducible Young diagrams as in (3.14).
Note that the projection on $\mathcal{C}^0$ entails symmetrization, as for instance
\[
\{x_{d1}, \{x_{d2}, \phi^{(2)} \} \} = \{x_{d2}, \{x_{d1}, \phi^{(2)} \} \} = \{\theta_{d1d2}, \phi^{(2)} \}
\tag{3.19}
\]
is in $\mathcal{C}^2$. Conversely, we have
\[
\{x^i, \ldots \{x^i, \phi^{(s)}_{\lambda_1\ldots\lambda_r} (x) \} \ldots \} = (-1)^s (n + s) \ldots (n + 1) \phi^{(s)}
\tag{3.20}
\]
because $\phi^{(s)}$ is defined in terms of traceless Young tensors. (3.17) constitutes the first important result of this paper.

**Spin 2 identities.** For $s = 2$ and $\phi^{(2)} \in (n, 4)$, we note the following identities:
\[
\{x^i, \{x^j, \phi^{(2)}_{ab} \} \} = (n + 2)(n + 1) \phi^{(2)}
\]
\[
\{x^i, \phi^{(2)}_{ab} (x) \} = -(n + 2) \phi^{(2)}_{ai\ldots a_b b_i a \ldots a_s} x^{a_i} x^{a_b} \theta^{b_i c}
\]
\[
\{x^i, \{x^j, \phi^{(2)} \} \} = -\theta (n + 2) (n + 5) - 6 \phi^{(2)}
\]
\[
\{\{x^i, \{x^j, \phi^{(2)} \} \} \} = c_n \phi^{(2)}
\]
\[
c_n = 2 \frac{\theta}{15} \frac{(n + 5)(n + 4)(n + 3)}{n + 1}
\tag{3.21}
\]
The details of the computation of the constant $c_n$ are in appendix A.1. We further need the following integral identities$^7$: 
\[
\int \{x^i, \phi^{(2)}_{ab} \} \{x^j, \phi^{(2)}_{cd} \} = \frac{1}{3} (n + 3)(n + 4) \theta \int \phi^{(2)}_{ab} \phi^{(2)}_{cd} \approx \frac{1}{3} \theta \int \phi^{(2)}_{ab} \phi^{(2)}_{cd}
\]
\[
\int \phi^{(2)}_{ab} \phi^{(2)} = \frac{2}{15} \frac{(n + 5)(n + 4)(n + 3)}{(n + 1)^2 (n + 2)} \theta^2 \int \phi^{(2)}_{ab} \phi^{(2)}
\tag{3.22}
\]
Here, $\approx$ indicates statements valid for large $n$. The second line of (3.22) is a consequence of (3.21), i.e.
\[
c_n \int \phi^{(2)}_{ab} \phi^{(2)} = \int \phi^{(2)}_{ab} \{x^i, \{x^j, \phi^{(2)} \} \} = \int \{x^i, \{x^j, \phi^{(2)} \} \} \phi^{(2)}
\]
\[
= (n + 2)(n + 1) \int \phi^{(2)} \phi^{(2)}
\tag{3.23}
\]
While the first line of (3.22) is derived exactly in appendix B; it can also be understood more intuitively using the following semi-classical leading-order computation: 
\[
\int \{x^i, \phi^{(2)}_{ab} \} \{x^j, \phi^{(2)}_{cd} \} = \int \theta^{\mu\nu} \theta^{\rho\sigma} \partial_\nu \phi^{(2)}_{ab} \partial_\rho \phi^{(2)}_{cd}
\]
\[
= \frac{1}{3} \theta R^2 \int (g^{\mu\nu} g^{\rho\sigma} - g^{\rho\sigma} g^{\mu\nu} + \epsilon (x)) \partial_\nu \phi^{(2)}_{ab} \partial_\rho \phi^{(2)}_{cd}
\]
\[
\approx \frac{1}{3} \theta R^2 \int \partial_\nu \phi^{(2)}_{ab} \phi^{(2)}_{cd} + O \left( \frac{1}{R} \right)
\]
\[
\approx \frac{1}{3} \theta R^2 \int \phi^{(2)}_{ab} \phi^{(2)}_{cd}
\tag{3.24}
\]
because $\phi_{ab}$ is divergence-free and radius $R$ is assumed to be sufficiently large. In the second line we used the averaging (3.4).

$^7$The integral here is simply the projector of $\mathcal{C}$ to the unique trivial mode. More details will be given in section 5.1.
3.1. Relation to higher spin algebras

According to (3.14), the module $C$ is described by functions on $S^4$ taking values in the direct sum of all rectangular Young diagrams with 2 rows. This provides the relation to the higher spin algebra of Vasiliev theory. To see this, we resort to the Lie algebra normalization $N^{ab} = \theta^{-1} g^{ab}$, and consider the subspace $\mathfrak{hs} \subset C$ with basis

$$\phi_{b_1\ldots b_i\ldots c_i\ldots} \mathcal{M}^{b_1c_1} \ldots \mathcal{M}^{b_ic_i} \in (0, 2s) \subset C^s, \ s = 1, 2, 3, \ldots$$

where the $\phi_{b_1\ldots b_i\ldots c_i\ldots}$ are constant, totally traceless and have the symmetries of a rectangular two-row Young diagram. Thus as a vector space,

$$\bigoplus \mathcal{Y} \cong \mathfrak{hs} := \bigoplus_{s=1}^{\infty} (0, 2s)$$

Moreover, $\mathfrak{hs}$ inherits a bracket from the Poisson structure (2.13) on $C$, given by

$$\{\mathcal{M}^{ab}, \mathcal{M}^{cd}\} = g^{ac} \mathcal{M}^{bd} - g^{ad} \mathcal{M}^{bc} - g^{bc} \mathcal{M}^{ad} + g^{bd} \mathcal{M}^{ac}, \quad (3.27a)$$

$$\{\mathcal{M}^{ab}, \phi_{b_1\ldots b_i\ldots c_i\ldots} \mathcal{M}^{b_1c_1} \ldots \mathcal{M}^{b_ic_i} \ldots \mathcal{M}^{b_i\ldots c_i\ldots}\} = s \ g^{ab} \phi_{b_1\ldots b_i\ldots c_i\ldots} \mathcal{M}^{b_1c_1} \ldots \mathcal{M}^{b_ic_i} \ldots \mathcal{M}^{b_i\ldots c_i\ldots} \pm \ldots \quad (3.27b)$$

which has the same form as in Vasiliev theory [29]. This can be truncated to $\mathfrak{so}(5)$ at $s = 1$, but not at any other finite $s$. Due to the tangential projector in (2.11), $\mathfrak{hs}$ does not close as (Poisson) algebra, which comes to no surprise given the origin of an $SO(5)$ covariant bundle, also known as Penrose twistor fibration. Nonetheless, as vector spaces $\mathfrak{hs}$ and the higher spin algebra $\mathfrak{hs}$ in Vasiliev theory coincide due to (3.26). By construction, the latter arises from relations imposed on the vector space of rectangular traceless Young diagrams, i.e. $\mathfrak{hs} = \mathcal{U}(\mathfrak{so}(5))/\mathfrak{J}$ were $\mathfrak{J}$ is the Joseph ideal (see [29] for a review).

To make the relation to the Joseph ideal manifest, recall that the definition of the fuzzy 4-sphere entails similar relations (2.11)

$$\mathcal{M}^{ab} \mathcal{M}^{ac} = \frac{R^2}{\theta} \ p^{bc}, \quad \varepsilon_{abcd} \mathcal{M}^{ab} \mathcal{M}^{cd} = \frac{R^2}{\theta} x^c,$$  

but besides the $\mathfrak{so}(5)$ generators $\mathcal{M}^{ab}$ also the coordinate functions $x^a$ on $S^4$ are involved. Now consider the quotient algebra $\mathfrak{ch} \mathfrak{s} := C/C^0$, where $C \equiv C^\infty(CP^3), C^0 \equiv C^\infty(S^4)$ from (3.1). Then the relations (3.28) on $C$ imply the following relations on $\mathfrak{ch}s$

$$\mathcal{M}^{ab} \mathcal{M}^{ac} = \frac{R^2}{\theta} \ g^{bc}, \quad \varepsilon_{abcd} \mathcal{M}^{ab} \mathcal{M}^{cd} = 0 \quad (3.29)$$

recovering the commutative limit of $\mathfrak{J}$. However, the quotient does not respect the Poisson brackets of $\mathfrak{hs}$. Equivalently, one can define $\mathfrak{chs} \cong C[[\mathcal{M}^{ab}]_{a, b = 1, \ldots, 5}]/((3.29))$ as a commutative quotient algebra. Consequently, $\mathfrak{ch}s$ can be understood as the Euclidean commutative or semi-classical vector space analog of the (Euclidean) higher spin Lie algebra of Vasiliev theory [21]. Thus locally, i.e. ‘forgetting’ the functions on $S^4$, $\mathfrak{ch}s$ coincides with the conventional $\mathfrak{hs}$, but not globally. The difference is tied to the presence of a scale $L_{NC}$. For finite $N$, the fuzzy case yields a truncation of our $\mathfrak{hs}$.

A somewhat related approach in (A)dS signature has been elaborated in [30], based on a Lorentz-covariant slicing, where the role of our $X$ is taken over by ‘momenta’ $P$.

Most importantly, we obtain a geometrical interpretation of $\mathfrak{hs}$ or $\mathfrak{ch}s$ as space of functions on $CP^3$ which are constant along $S^4$. More generally, $\mathfrak{hs}$-valued functions on $S^4$ are identified with the space of all functions on $CP^3$. 

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This leads to $\mathfrak{hs}$-valued gauge fields on $S^4$, as elaborated below. Hence, the semi-classical $S^4$ provides a natural realization of this higher spin algebra (and associated gauge theories as we will see). The $\theta^{ab}$ generators arise as functions on the $S^2$ fiber over $S^4$, and $\mathfrak{hs}$ is an $SO(5)$-invariant truncation of $\mathcal{C}$. However in the formulation of gauge theory discussed below, these $\theta^{ab}$ also act on the $x^a$, unlike in Vasiliev’s approach where classical space-time is added by hand. In the fuzzy case, these $\mathfrak{hs}$ generators are inseparably linked to the $x^a$, because the analog of $P^{\mu}$ of (3.28) becomes non-commutative.

3.2. Local representation and constraints

We have seen that functions on $S^4$ can be viewed as functions on ordinary $S^4$ taking values in $\mathfrak{hs}$. To understand better the meaning of this statement, we decompose $\phi^{(n)}$ into ordinary tensor fields on $S^4$. Fixing an arbitrary point $p \in S^4$ (e.g. the ‘north pole’ $p = R(0, 0, 0, 0, 1)$), we can decompose $\mathfrak{so}(5)$ into translation generators $P^\mu$, see (2.15), and rotation generators $M^{\mu\nu}$ for $\mu, \nu = 1, \ldots, 4$. This leads to an expansion of a spin $s$ mode of the form

$$\phi_{b_1 b_2 \ldots b_{2s}}^s(x) \theta^{b_1 c_1} \ldots \theta^{b_{2s} c_{2s}} \equiv \sum_{k=0}^{s} f_{n_1 \ldots n_k a_1 a_2 \ldots a_{2n_k}}(x) P^{n_1} \ldots P^{n_k} M^{\mu_1 \nu_1} \ldots M^{\mu_s \nu_s}$$

similar to [18]. The main consequence of the above classification of modes is that the coefficients $f_{n_1 \ldots n_k a_1 a_2 \ldots a_{2n_k}}$ of these generators are not independent; for example for $s = 1$, both are determined by a single underlying divergence-free vector field. This has important consequences for the resulting physics and we therefore elaborate these constraints explicitly.

3.2.1. $s = 1$ and field strength. With the above conventions, we can write

$$\phi^{(1)} = \phi_{a_1 a_2 b_1 c_1} x^{a_1} \ldots x^{a_2} \theta^{b_1 c_1} = A_\mu(x) P^\mu + \omega_{\mu\nu}(x) M^{\mu\nu} \in (n, 2) \subset \mathcal{C}^1.$$  

Here $\omega_{\mu\nu}(x)$ is naturally defined to be antisymmetric (unlike the underlying $\phi_{a_1 a_2 b_1 c_1}$). Carefully comparing the coefficients of $P^\mu$ using (A.13), we obtain

$$A_\mu(x) = -\frac{n + 2}{n + 1} \frac{1}{n + 2} \phi_{a_1 a_2 b_1 c_1} x^{a_1} \ldots x^{a_2} = -\frac{n + 2}{n + 1} \frac{1}{n + 2} \phi^{(1)}(x)$$

with $\partial^\mu A_\mu = 0$, (3.32)

is nothing but the canonical tangential divergence-free vector field (3.9) associated to $\phi^{(1)}$. Since this vector field uniquely determines $\phi^{(1)}$, it must also determine the tangential components $\omega_{\mu\nu}(x)$. Indeed, we find

$$\partial_\nu A_\mu = -(n + 2) \theta \phi_{a_1 a_2 b_1 c_1} x^{a_1} \ldots x^{a_s}.$$  

By further contracting $\partial_\mu A_\nu$ with $M^{\mu\nu}$ and comparing to (3.31), we conclude

$$\omega_{\mu\nu} = -\frac{1}{2(n + 2)} (\partial_\mu A_\nu - \partial_\nu A_\mu),$$

i.e. $\omega_{\mu\nu}$ is the field strength associated to the one-form $A_\mu$. Conversely, it follows that

\[\text{Note that one cannot simply read off } \omega_{\mu\nu} \text{ by comparing coefficients in (3.31), since } \phi_{a_1 a_2 b_1 c_1} \text{ is not anti-symmetric in } (bc), \text{ and } M^{\mu\nu} \text{ is self-dual. However, the coefficient of } P^\mu \text{ is uniquely specified. Although } P^\mu \text{ vanishes as function at } p \in S^4, \text{ it does not vanish as generator in the Poisson algebra.}
\]

\[\text{Here } \partial^\mu \text{ amounts to the Levi-Civita connection on } S^4.\]
because $A_\nu$ is divergence-free, and $\partial \cdot A_\nu = (n+1)(n+4)A_\nu$. Thus, $\phi^{(1)}$ encodes a multiplet consisting of a divergence-free vector field and its field strength.

### $3.2.2. \ s = 2$ and curvature

Similarly, for a mode $\phi^{(2)}$ the decomposition around the north pole yields

$$\phi^{(2)} = \phi_{a_1\ldots a_d b c d e} x^{a_1} \ldots x^{a_d} \theta^{b} \theta^{c} \theta^{d} \theta^{e} =: h_{\mu \nu}(x) P^{\mu} P^{\nu} + \omega_{\mu \alpha \beta}(x) P^{\mu} M^{\alpha \beta} + \Omega_{\alpha \beta \mu \nu}(x) M^{\alpha \beta} M^{\mu \nu} \in (n, 4) \subset C^4.$$  

(3.36)

Here $\Omega_{\alpha \beta \mu \nu}(x)$ is naturally antisymmetric in $(\alpha \beta)$ and $(\mu \nu)$ separately (in contrast to the underlying $\phi$). Carefully comparing coefficients of $P^{\mu} P^{\nu}$ at the north pole using (A.21), we obtain

$$h_{\alpha \beta} = \frac{n + 3}{n + 1} h_{\alpha \beta} = \frac{(n + 2)(n + 3)}{n + 1} \theta^{a_1} \ldots \theta^{a_d} x^{a_1} \ldots x^{a_d} x^\alpha = \theta^2 \phi^{(2)}(x) \quad \text{with} \quad \partial^\mu h_{\mu \nu} = 0, $$

(3.37)

which is proportional to the symmetric rank 2 tensor $\phi^{(2)}_{\mu \nu}(x)$ of (3.15) associated to $\phi^{(2)}$. Since this tensor field uniquely determines $\phi^{(2)}$, it also determines $\omega_{\mu \alpha \beta}$ and $\Omega_{\alpha \beta \mu \nu}$. Indeed, consider derivatives of the above

$$\partial_\mu h_{\alpha \beta} = \frac{(n + 2)(n + 3)}{n + 1} \theta^2 \phi_{a_1 \ldots a_d \mu \alpha \beta} x^{a_1} \ldots x^{a_d} x^\alpha,$$

$$\partial_\mu \partial_\nu h_{\alpha \beta} = \frac{(n + 2)(n + 3)}{n + 1} \theta^2 \phi_{a_1 \ldots a_d \mu \nu \alpha \beta} x^{a_1} \ldots x^{a_d}.$$

(3.38)

Contracting these expressions with $M^{\mu \nu}$ and/or $P^\mu$ and comparing with (3.36), we conclude

$$-\omega_{\mu \alpha \beta}(x) P^\mu M^{\alpha \beta} = 2 \theta^2 \phi_{a_1 \ldots a_d \mu \alpha \beta} x^{a_1} \ldots x^{a_d} x^\alpha P^\alpha M^{\mu \beta} = \frac{n + 1}{(n + 2)(n + 3)} \partial_\mu h_{\alpha \beta} P^\alpha M^{\mu \beta},$$

$$\omega_{\mu \alpha \beta} = -\frac{n + 1}{(n + 2)(n + 3)} (\partial_\mu h_{\beta \alpha} - \partial_\beta h_{\alpha \mu}).$$

(3.39)

Hence, up to a factor, $\omega_{\mu \alpha \beta}$ is the spin connection defined by $h_{\alpha \beta}$. Similarly,

$$(n + 2)(n + 3) \Omega_{\mu \nu \alpha \beta}(x) M^{\mu \nu} M^{\alpha \beta} = (n + 2)(n + 3) \theta^2 \phi_{a_1 \ldots a_d \mu \nu \alpha \beta} x^{a_1} \ldots x^{a_d} M^{\mu \alpha} M^{\nu \beta}
= \partial_\mu \partial_\nu h_{\alpha \beta} M^{\mu \alpha} M^{\nu \beta},$$

(3.40)

To understand the RHS, consider the linearized Riemann tensor associated to $h_{\mu \nu}$

$$R_{\alpha \beta \mu \nu} = \frac{1}{2} (\partial_{\alpha \mu} h_{\beta \nu} + \partial_{\beta \mu} h_{\alpha \nu} - \partial_{\alpha \nu} h_{\beta \mu} - \partial_{\beta \nu} h_{\alpha \mu}).$$

(3.41)

Contracting with $M^{\alpha \beta} M^{\mu \nu}$ gives

$$R_{\alpha \beta \mu \nu} M^{\alpha \beta} M^{\mu \nu} = \frac{1}{2} (\partial_{\alpha \mu} h_{\beta \nu} + \partial_{\beta \mu} h_{\alpha \nu} - \partial_{\alpha \nu} h_{\beta \mu} - \partial_{\beta \nu} h_{\alpha \mu}) M^{\alpha \beta} M^{\mu \nu}$$

$$= \partial_{\alpha \mu} h_{\beta \nu} M^{\alpha \beta} M^{\mu \nu} = -\partial_{\mu} \partial_{\nu} h_{\alpha \beta} M^{\mu \alpha} M^{\nu \beta}. $$

(3.42)
Comparing with (3.40), we conclude
\[
\Omega_{\alpha\beta\mu\nu}(x) = -\frac{1}{(n+2)(n+3)}R_{\alpha\beta\mu\nu}.
\]
(3.43)
Thus, \(\phi^{(2)}\) encodes a multiplet consisting of a divergence-free symmetric traceless tensor field (graviton), its spin connection, and the (linearized) curvature tensor. Similar relations hold between the tangential and radial components of general \(\phi^{(s)}\).

4. Vector harmonics on \(S^4\) and higher spin fields on \(S^4\)

In this section, we derive the complete classification of one-forms (i.e. vector modes) on \(S^4\). These are the basic degrees of freedom which arise in the semi-classical limit of matrix models on \(S^4\). We will first obtain the abstract classification from \(\mathfrak{so}(5)\) representation theory, which for generic \(s\) leads to five modes for each spin \(s\). In a second step, we will provide an explicit realization of these modes in terms of five ‘Ansätze’ involving \(\mathfrak{so}(5)\) tensors and Young diagrams. By elaborating their properties and comparing with the group-theoretical results, we show that this provides the complete set of modes. This explicit realization will be the basis for the further analysis.

The tensor product decomposition of \(\Omega^0S^4\) is given by [31]
\[
A = \xi^aA_a \in (1,0) \otimes (n,2s) = (n+1,2s) \oplus (n-1,2s+2) \oplus (n,2s) \oplus (n+1,2s-2) \oplus (n-1,2s)
\]
(4.1)
for generic \((n,2s)\). For \(s = 0\) the decomposition truncates as follows:
\[
(1,0) \oplus (n,0) \equiv (n+1,0) \oplus (n-1,2) \oplus (n-1,0), \quad n \geq 1,
\]
(4.2)
and for \(n = 0\)
\[
(1,0) \otimes (0,2s) = (1,2s) \oplus (0,2s) \oplus (1,2s-2), \quad s \geq 1.
\]
(4.3)
The irreducible components are characterized by their eigenvalues of the intertwiner \(\mathcal{I}\) of (2.23), which commutes\(^{10}\) with the Laplacian \(\Box\) defined in (2.34). This fact can be seen by expressing \(\mathcal{I}\) as follows:
\[
-\theta(M_{\text{ad}}^{(5)} \otimes M_{\text{ad}}^{(5)} A)_a = -(M_{\text{ad}}^{(5)} \cdot \theta^{(d)\cdot})A_a = 2\{\theta^{ab},A_b\} = 2\mathcal{I}(A_a).
\]
(4.4)
Here
\[
(M_{\text{ad}}^{(5)})_{\bar{a}}^{\bar{b}} = \delta_{\bar{b}}^{\bar{a}}\delta_{\alpha\beta}^{\alpha\beta} - \delta_{\bar{a}}^{\bar{b}}\delta_{\alpha\beta}^{\alpha\beta}
\]
(4.5)
is the vector generator of \(\mathfrak{so}(5)\), and \(M_{\text{ad}}^{(5)} = \{M_{\text{bc}},\cdot\}\) denotes the representation of \(\mathfrak{so}(5)\) induced by the Poisson structure on \(S^4\) (see (2.13)). Therefore \(\mathcal{I}\) measures the product of internal and space-time (angular) momentum, analogous to the spin-orbit coupling of the modes. Now, we find
\[
-M_{\text{bc}}^{(ad)} \otimes M_{\text{bc}}^{(5)} = -C^2[\mathfrak{so}(5)]^{(5)\otimes(\cdot)} + C^2[\mathfrak{so}(5)]^{(ad)\cdot} + C^2[\mathfrak{so}(5)]^{(\cdot)}
\]
(4.6)
\(^{10}\) Because \(\{\theta^{ab},\cdot\}\) is the adjoint action of a \(\mathfrak{so}(5)\) generator on \(C\), which commutes with the Casimir \(\Box\).
i.e. \( \mathcal{I} \) is the difference between the total Casimir and the orbital and spin Casimirs. This yields

\[
\mathcal{I}(\mathcal{A}^{(n+1,2)}) = \theta(n+s)\mathcal{A}^{(n+1,2)}, \\
\mathcal{I}(\mathcal{A}^{(n-2,2)}) = \theta(s-1)\mathcal{A}^{(n-2,2)}, \\
\mathcal{I}(\mathcal{A}^{(n,2)}) = -2\theta\mathcal{A}^{(n,2)}, \\
\mathcal{I}(\mathcal{A}^{(n+1,2-2)}) = -\theta(s+2)\mathcal{A}^{(n+1,2-2)}, \\
\mathcal{I}(\mathcal{A}^{(n-1,2)}) = -\theta((n+s)+3)\mathcal{A}^{(n-1,2)},
\]

(4.7)

for \( \mathcal{I} \) acting on \( \mathcal{A} \in (n,2s) \otimes (1,0) \). To identify these modes explicitly, we will define five intertwiners

\[
\mathcal{A}^{(0)}[\cdot] : \quad \mathcal{C}' \cong \bigoplus_{i=0}^{2} \{1, 2, 3\} \\
\phi^{(s)} \mapsto \phi^{(s)}_{a_1\ldots a_s} \mapsto \mathcal{A}^{(0)}[\phi^{(s)}]
\]

(4.8)

for each \( s = 0, 1, 2, \ldots \) (except for \( s = 0 \) where only \( i = 2,3,R \) arise). These five modes \( \mathcal{A}^{(i)} \) provide a one-to-one realization of all vector fluctuations (4.7) on \( S^4 \). This will be established by diagonalizing \( \mathcal{I} \), which will recover precisely the above eigenvalues. In the following paragraphs we discuss separately the cases \( s = 0,1,2 \) and higher \( s \).

**Notation.** As noted at the beginning of this section, we refer to one-forms on \( S^4 \) also as vector fields because they are tangential fluctuations around a background and are the degrees of freedom in the matrix model. The complementary group theoretical content of the one-forms/vector fields is captured by their spin. Hence, there are tangential fluctuations of arbitrary spin \( s \), which are referred to as spin \( s \) vector fields.

### 4.1. Spin 0 vector fields \( \mathcal{A} \)

First consider the three spin 0 vector modes \( \mathcal{A} \). They correspond to a Young diagram with one line, or equivalently \( (n,0) \).

There are three such vector modes for each \( n \), two from \((n+1,0) \oplus (n-1,0) \subset (n,0) \otimes (1,0) \) and one from \((n+1,0) \oplus (n,0) \subset (n-1,2) \otimes (1,0) \). Explicitly, they are given as follows:

\[
\mathcal{A}^{(2)} := \xi_c\mathcal{A}^{(2)} = \xi_c\phi^{(0)}_{a_0\ldots a_n} \ldots x^{a_n} \in (n+1,0) \subset (n,0) \otimes (1,0),
\]

(4.9a)

\[
\mathcal{A}^{(3)} := \mathcal{J}(\mathcal{A}^{(2)}) = \xi_c\theta^{(0)}_{a_0\ldots a_n} x^{a_0} \ldots x^{a_n} \in (n+1,0) \subset (n,2) \otimes (1,0),
\]

(4.9b)

\[
\mathcal{A}^{(R)} := \xi_c x^{a}\phi^{(0)} \in (n,0) \subset (n+1,0) \otimes (1,0).
\]

(4.9c)

The notation will become clear later when considering spin \( s \) fields in (4.68) and their transformation into \( \mathcal{I} \)-eigenmodes (4.69) and (4.71).

#### 4.1.1. Properties of spin 0 fields

Consider first the vector field \( \mathcal{A}^{(2)} \), which satisfies

\[
\mathcal{N}(\mathcal{A}^{(2)}) = x^c\phi^{(0)}_{a_0\ldots a_n} x^{a_0} \ldots x^{a_n} = \phi^{(0)},
\]

(4.10)

\[
\mathcal{G}(\mathcal{A}^{(2)}) = \{x^c, \mathcal{A}^{(2)}\} = 0,
\]

(4.11)
\( \mathcal{I}(A^{(2)}) = \zeta_{\nu} \{ \theta^{\alpha\nu}, A^{(2)}_\nu \} \)
\( = (n - 1) \xi_{\sigma} \delta^{(0)}_{\nu \cdots \mu} \cdots \, A^{(2)}_\nu \theta^{\alpha\nu}, x^{\alpha\nu} \}
\( = (n - 1) \theta A^{(2)} \),
\( (4.12) \)
using (2.31) and tracelessness of \( \delta^{(0)} \). The last relation implies that \( A^{(2)} \) is an example of the first line in (4.7) for the case \( s = 0 \). We observe from (4.10) that \( A^{(2)} \) is not tangential, but its tangential projection can be straightforwardly worked out to read
\[ P_T A^{(2)} = \frac{1}{n} \partial \phi^{(0)} . \]  
(4.13)
Hence \( P_T A^{(2)} \) is essentially the differential of a function on \( S^4 \). Next, \( A^{(3)} \) satisfies
\[ \mathcal{N}(A^{(3)}) = 0, \]
\[ \mathcal{G}(A^{(3)}) = \{ x^{\nu}, A^{(3)}_\nu \} = 0, \]
\[ \mathcal{I}(A^{(3)}) = \mathcal{I}(J(A^{(2)})) = -4 \theta J(A^{(2)}) + \theta \mathcal{Q}(\mathcal{N}(A^{(2)})) - \mathcal{J} \circ \mathcal{I}(A^{(2)}) \]
\[ = -3 \theta A^{(3)} , \]  
(4.14)
using (2.28) and (4.15) for the last relation, where we also need the following expression for the spin 0 pure gauge modes
\[ \mathcal{Q}(\phi^{(0)}) = \mathcal{Q}(\phi^{(0)}) x^{\nu} \cdots x^{\nu} = n A^{(3)}. \]  
(4.15)
Hence, one recognizes \( A^{(3)} \) as an example of the fourth line in (4.7) for the case \( s = 1 \), and, most notably, \( A^{(3)} \) equals a gauge transformation generated by \( \frac{1}{n} \phi^{(0)} \). Finally, for the radial mode \( A^{(R)} \) we find
\[ \mathcal{I}(A^{(R)}) = -4 \theta \xi \phi^{(0)} - \mathcal{J}(\mathcal{Q}(\phi^{(0)})) \]
\[ = -(4 + n) \theta A^{(R)} + n \theta R^2 A^{(2)} , \]  
(4.16)
using (2.27). Thus, we have explicitly realized all three \((n, 0)\) vector modes \( A \) in terms of irreducible Young tableaux or tensors with one line. In particular, \( A^{(3)} \) is recognized as pure gauge field in noncommutative \( U(1) \) Yang–Mills gauge theory\(^{11}\), and \( A^{(2)} \) is completely determined by a function and its differential.

### 4.1.2. Diagonalization of \( \mathcal{I} \)

Collecting the results of the \( \mathcal{I} \) action on the modes \( A^{(R)} , R^2 A^{(2)} \), and \( A^{(3)} \) we find\(^{12}\)
\[ \mathcal{I} \left( A^{(R)} R^2 A^{(2)} \right) = \theta \begin{pmatrix} -(n + 4) & n \\ 0 & (n - 1) \end{pmatrix} \left( \begin{array}{c} A^{(R)} \\ R^2 A^{(2)} \end{array} \right) , \]  
(4.17a)
\[ \mathcal{I}(A^{(3)}) = -3 \theta A^{(3)} . \]  
(4.17b)

The eigenvalues of \( \mathcal{I} \) are \((n - 1)\), \(-(n + 4)\), and \(-3\) with corresponding eigenmodes given by
\[ C[\phi^{(0)}] := A^{(R)} - \frac{n}{2n + 3} (R^2 A^{(2)}), \]  
(4.18a)
\(^{11}\) In the semi-classical limit the \( U(1) \) gauge field is a Maxwell field, but it becomes necessarily Yang–Mills in the non-commutative setting.

\(^{12}\) The clumsy-looking organization will become more transparent once we proceed to higher spin fields.
\[ D[\phi^{(0)}] := R^2 A^{(2)}, \quad (4.18b) \]
\[ F[\phi^{(0)}] := A^{(3)}. \quad (4.18c) \]

We observe that \( C \) exemplifies the fifth line in (4.7) for the case of \( s = 0 \).

### 4.2. Spin 1 vector fields \( A \)

Now we account for all five \( s = 1 \) modes \( A \) in \((n, 2)\) in a systematic fashion. They are determined in different ways in terms of mixed Young diagrams or corresponding tensors \( \phi_{a_1 \ldots a_n b_c} \).

We have the following spin 1 vector modes:

\[
A^{(0)} := \xi_a A^{(0)} = \phi_{a_1 \ldots a_n b_c}^1 \xi_c x^{a_1} \ldots x^{a_n} x^b \in (n, 2) \subset (n + 1, 0) \otimes (1, 0),
\]
\[
A^{(1)} := J(A^{(0)}) = \xi_a \phi_{a_1 \ldots a_n b_c}^1 \theta^b x^{a_1} \ldots x^{a_n} x^b \in (n, 2) \subset (1, 2) \otimes (1, 0),
\]
\[
A^{(2)} := \xi^k \phi_{a_1 \ldots a_n b_c}^1 x^a \ldots x^a \mathcal{A}^{bc} \in (n, 2) \subset (1, 2) \otimes (1, 0),
\]
\[
A^{(3)} := J(A^{(2)}) = \xi^k \theta^b \phi_{a_1 \ldots a_n b_c}^1 x^a \ldots x^a \mathcal{A}^{bc} \in (n, 2) \subset (2, 4) \otimes (1, 0),
\]
\[
A^{(4)} := \frac{\xi \phi^{(1)}}{\theta} = \xi_a \phi_{a_1 \ldots a_n b_c}^1 x^a \ldots x^a \mathcal{A}^{bc} \in (n, 2) \subset (2, 4) \otimes (1, 0). \quad (4.19)
\]

#### 4.2.1. Properties of spin 1 fields

Start with the divergence-free tangential vector field \( A^{(0)} \), see also (3.9). We can identify this with the second line in (4.7) for \( s = 0 \), because

\[
N(A^{(0)}) = 0, \quad (4.20)
\]
\[
G(A^{(0)}) = -(n + 1) \phi^{(1)}, \quad (4.21)
\]
\[
div A^{(0)} = 0, \quad (4.22)
\]
\[
I(A^{(0)}) = \phi_{a_1 \ldots a_n b_c}^1 \xi_a \{ \theta^b x^{a_1} \ldots x^{a_n} x^b \}
= \theta(n + 1) \xi_a \phi_{a_1 \ldots a_n b_c}^1 x^a \ldots x^a x^b
= -\theta A^{(0)}, \quad (4.23)
\]

using (3.13) and (A.13). Furthermore, we find that \( A^{(1)} \) satisfies

\[
N(A^{(1)}) = 0, \quad (4.24)
\]
\[
G(A^{(1)}) = \phi_{a_1 \ldots a_n b_c}^1 \{ \theta^b x^{a_1} \ldots x^{a_n} x^b \}
= \phi_{a_1 \ldots a_n b_c}^1 \left( (n + 1) \theta R^2 x^{a_1} \ldots x^{a_n} \theta_{\mu}^{\nu} \right) = 0, \quad (4.25)
\]
\[
I(A^{(1)}) = I(J(A^{(0)})) = -4 \theta J(A^{(0)}) + \theta \{ x^c, A^{(0)} \} + \theta Q(N(A^{(0)})) - J \circ I(A^{(0)})
= -3 \theta A^{(1)} - (n + 1) \theta \xi \phi^{(1)}, \quad (4.26)
\]

using (2.28). Thus, \( A^{(1)} \) is not an eigenvector of \( I \). For the mode \( A^{(2)} \), we obtain
\( \mathcal{N}(A^{(2)}) = \frac{\phi^{(1)}}{\theta}, \) \hspace{1cm} (4.27)

\( \mathcal{G}(A^{(2)}) = \phi^{(1)}_{aa_2...a_nbc} \{ x^{a_1} x^{a_2} \ldots x^{a_n} \mathcal{M}^{bc} \} = 0, \) \hspace{1cm} (4.28)

\( \mathcal{I}(A^{(2)}) = \xi^a \phi^{(1)}_{aa_2...a_nbc} \{ \theta^{da} x^{a_2} \ldots x^{a_n} \mathcal{M}^{bc} \} \)
\[= (n - 1)\theta A^{(2)} + \theta \xi^a \phi^{(1)}_{aa_2...a_nbc} x^{a_2} \ldots x^{a_n} \mathcal{M}^{bc} \]
\[= n\theta A^{(2)}. \] \hspace{1cm} (4.29)

This is consistent with the first line in (4.7). Next, the vector mode \( A^{(3)} \) satisfies

\( \mathcal{G}(A^{(3)}) = \{ x_d, \theta^d \phi^{(1)}_{a_2...a_nbc} x^{a_2} \ldots x^{a_n} \mathcal{M}^{bc} \} \)
\[= -(n + 4)\phi^{(1)}, \] \hspace{1cm} (4.30)

\( \mathcal{I}(A^{(3)}) = \mathcal{I}(\mathcal{J}(A^{(2)})) = -4\theta \mathcal{J}(A^{(2)}) + \theta \xi \{ x^c, A^{(2)}_c \} + \theta Q(\mathcal{N}(A^{(2)})) - \mathcal{J} \circ \mathcal{I}(A^{(2)}) \)
\[= -4\theta \mathcal{J}(A^{(2)}) + Q(\phi^{(1)}) - n\theta \mathcal{J}(A^{(2)}) \]
\[= -(4 + n)\theta \mathcal{J}(A^{(2)}) + Q(\phi^{(1)}) \]
\[= -4\theta A^{(3)} + \frac{n + 2}{n + 1} \theta A^{(0)}, \] \hspace{1cm} (4.31)

using (2.28). Finally, for the radial mode \( A^{(6)} \) we compute

\( \mathcal{G}(A^{(6)}) = x_a \{ x^a, \phi^{(1)} \} = 0, \) \hspace{1cm} (4.32)

\( \mathcal{I}(A^{(6)}) = -4\xi \phi^{(1)} - \frac{1}{\theta} \mathcal{J}(Q(\phi^{(1)})) \)
\[= -(4 + n)\xi \phi^{(1)} + n\theta R^2 A^{(2)} - \frac{n + 2}{n + 1} A^{(1)}, \] \hspace{1cm} (4.33)

using (2.27), (4.29), and \( A^{(3)} = \mathcal{J} A^{(2)} \). Again \( \mathcal{I} \) does not diagonalize on the radial mode, but decomposes into radial and tangential components.

4.2.2. Diagonalization of \( \mathcal{I} \). Collecting the results, \( \mathcal{I} \) acts on the modes \( A^{(1)}/\theta, A^{(6)}, \) and \( R^2 A^{(2)} \) as follows:

\[ \mathcal{I} \begin{pmatrix} A^{(1)}/\theta \\ A^{(6)} \\ R^2 A^{(2)} \end{pmatrix} = \theta \begin{pmatrix} -3 & -(n + 1) & 0 \\ \frac{-n + 2}{n + 1} & -(4 + n) & n \\ 0 & 0 & n \end{pmatrix} \begin{pmatrix} A^{(1)}/\theta \\ A^{(6)} \\ R^2 A^{(2)} \end{pmatrix}. \] \hspace{1cm} (4.34)

This matrix has indeed eigenvalues \(-2, -(n + 5), n\), in complete agreement with lines three, five and one of (4.7). The corresponding eigenvectors are

\( B[\phi^{(1)}] := \frac{A^{(1)}}{\theta} - \frac{n + 1}{n + 2} A^{(6)} + \frac{n(n + 1)}{(n + 2)^2} R^2 A^{(2)}, \) \hspace{1cm} (4.35a)

\( C[\phi^{(1)}] := \frac{1}{n + 1} \frac{A^{(1)}}{\theta} + A^{(6)} - \frac{n}{2n + 5} R^2 A^{(2)}, \) \hspace{1cm} (4.35b)
\[ D[\phi(1)] := R^2 A(2) \]  
(4.35c)

All of these are physical, i.e. they are annihilated by \( G \) which means they are gauge fixed, see section 5.2. Similarly, the action of \( I \) on \( A(0) \), \( A(3) \) can be diagonalized as follows:

\[ I \left( \begin{array}{c} A(0) \\ A(3) \end{array} \right) = \theta \left( \begin{array}{cc} -1 & 0 \\ \frac{n+2}{n+1} & -4 \end{array} \right) \left( \begin{array}{c} A(0) \\ A(3) \end{array} \right), \]

(4.36)

which gives rise to the eigenvectors

\[ E[\phi(1)] := A(0), \]

(4.37a)

\[ F[\phi(1)] := A(3) - \frac{1}{3} \frac{n+2}{n+1} A(0), \]

(4.37b)

with eigenvalues \(-1\) and \(-4\), respectively. This corresponds to line two and four in (4.7). Hence, we have a complete description of all spin 1 modes. In particular, this means that the \( F \) modes live in \( (1, 0) \otimes C^2 \).

Of course these \( B, C, D, E, F \) eigenmodes are mutually orthogonal. This is one of the main points of going to the above basis, and it will be elaborated in detail in the spin 2 case.

**Pure gauge spin 1 vector fields.** Consider the pure gauge modes \( Q(\phi(1)) \) for \( \phi(1) \in (n, 2) \)

\[ Q(\phi(1)) = Q(\phi(1)_{, a}, a_{b c} x^{b c} \ldots x^{a b c}), \]

\[ = \theta n A(3) + \theta \frac{n+2}{n+1} A(0) \in C^2 \oplus C^0 \]

(4.38)

using (A.13). As a check, we compute

\[ \{ x^a, Q(\phi(1)) \}_a = \square \phi(1) = -\theta (n^2 + 5n + 2) \phi(1), \]

(4.39)

which has the correct eigenvalue for \((n, 2)\). Note that the \( A(3) \) term contains some \( C^0 \) components, and after a projection \( [\theta^a \theta^b \theta^c]_a \), using (4.37), one obtains

\[ [Q(\phi(1))]_0 = \frac{\theta}{3} \frac{1}{n+1}(n+3) A(0). \]

(4.40)

We will see in section 7 that this \( C^0 \) contribution corresponds to volume-preserving diffeomorphisms, while the contribution in \( C^2 \) leads to the corresponding gauge transformation of the graviton.

**Gauge fixing.** Imposing the gauge fixing condition \( G(A) = \{ x^a, A_a \} = 0 \) in the \( E, F \) sector leaves one physical mode

\[ G \left( A(3) + \alpha A(0) \right) = 0 \quad \text{for} \quad \alpha = -\frac{n+4}{n+1}. \]

(4.41)

Note that the \( Q[\phi] \) are exact zero modes before gauge fixing, and imposing \( G(A) = 0 \) removes these modes. In the Euclidean case, there is no need to further factor out pure gauge modes, because

\[ \{ x^a, Q(\phi) \} = \{ x^a, \{ x_\alpha, \phi \} \} = \square \phi \]

(4.42)

is positive definite and invertible on \( S^4 \). Therefore any \( A^a \) can indeed be gauge fixed uniquely via \( A^a \rightarrow A^a + Q(\phi) \), and gauge fixing removes only one mode in the Euclidean case. In the Minkowski case, this story would be somewhat different.
4.3. Spin 2 vector fields $\mathcal{A}$

We have the following spin 2 vector modes:

\[
\mathcal{A}^{(0)} := \xi a_{a_1 \ldots a_{bd}ce} x^{a_1} \ldots x^{a_6} x^{b} \mathcal{M}^{de} \in (n, 4) \subset (n + 1, 2) \otimes (1, 0),
\]

\[
\mathcal{A}^{(1)} := \mathcal{J}(\mathcal{A}^{(0)}) = \xi a_{a_1 \ldots a_{bd}ce} x^{a_1} \ldots x^{a_6} x^{b} \mathcal{M}^{de} \in (n, 4) \subset (n, 4) \otimes (1, 0),
\]

\[
\mathcal{A}^{(2)} := \xi a_{a_1 \ldots a_{bd}ce} x^{a_1} \ldots x^{a_6} \mathcal{M}^{bc} \mathcal{M}^{de} \in (n, 4) \subset (n, 0) \otimes (1, 0),
\]

\[
\mathcal{A}^{(3)} := \mathcal{J}(\mathcal{A}^{(2)}) = \xi a_{a_1 \ldots a_{bd}ce} x^{a_1} \ldots x^{a_6} \mathcal{M}^{bc} \mathcal{M}^{de} \in (n, 4) \subset (n, 4) \otimes (1, 0),
\]

\[
\mathcal{A}^{(R)} := \frac{1}{\Omega^2} \xi a_{a_1 \ldots a_{bd}ce} x^{a_1} \ldots x^{a_6} x^{b} \mathcal{M}^{de} \in (n, 4) \subset (n, 4) \otimes (1, 0).
\]

(4.43)

We will again compute $\mathcal{I}$ on these modes and diagonalize it. This will result in the eigenmodes of the vector Laplacian (2.35).

### 4.3.1 Properties of spin 2 fields

First we note the representation of $\mathcal{A}^{(0)}$ which follows from (3.21)

\[
A_a^{(0)} = -\frac{1}{\Omega^2} \frac{1}{n + 2} \{x^b, \phi^{(2)}_{ab} \}.
\]

(4.44)

We have

\[
\mathcal{N}(\mathcal{A}^{(0)}) = x^c \phi^{(2)}_{a_1 \ldots a_{bd}ce} x^{a_1} \ldots x^{a_6} x^{b} \mathcal{M}^{de} = 0,
\]

(4.45)

\[
\mathcal{G}(\mathcal{A}^{(0)}) = \phi^{(2)}_{a_1 \ldots a_{bd}ce} \{x^a, x^{a_1} \ldots x^{a_6} x^{b} \mathcal{M}^{de} \} = -\frac{1}{\Omega^2} (n + 1) \phi^{(2)}.
\]

(4.46)

\[
\mathcal{I}(\mathcal{A}^{(0)}) = \phi^{(2)}_{a_1 \ldots a_{bd}ce} \{x^a, x^{a_1} \ldots x^{a_6} x^{b} \mathcal{M}^{de} \} = 0.
\]

(4.47)

using (A.13) and the symmetry in (ce) in the last step. We can identify this with the second line of (4.7) for $s = 1$. For $\mathcal{A}^{(1)}$, we compute

\[
\mathcal{N}(\mathcal{A}^{(1)}) = 0,
\]

(4.48)

\[
\mathcal{G}(\mathcal{A}^{(1)}) = \phi^{(2)}_{a_1 \ldots a_{bd}ce} \{x^f, \theta^{ac} x^{a_1} \ldots x^{a_6} x^{b} \mathcal{M}^{de} \} = 0,
\]

(4.49)

\[
\mathcal{I}(\mathcal{A}^{(1)}) = \mathcal{I}(\mathcal{J}(\mathcal{A}^{(0)})) = -4\Omega \mathcal{J}(\mathcal{A}^{(0)}) - (n + 1) \xi \phi^{(2)} + \theta \mathcal{Q}(\mathcal{N}(\mathcal{A}^{(0)})) - \mathcal{J} \circ \mathcal{I}(\mathcal{A}^{(0)})
\]

\[
= -4\theta \mathcal{A}^{(1)} - (n + 1) \xi \phi^{(2)}.
\]

(4.50)

using (2.28). This is not an eigenvector, but this is addressed below. Furthermore,

\[
\mathcal{N}(\mathcal{A}^{(2)}) = \phi^{(2)}_{a_1 \ldots a_{bd}ce} x^f x^{a_2} \ldots x^{a_6} \mathcal{M}^{bc} \mathcal{M}^{de} = \frac{1}{\Omega^2} \phi^{(2)},
\]

(4.51)

\[
\mathcal{G}(\mathcal{A}^{(2)}) = \phi^{(2)}_{a_1 \ldots a_{bd}ce} \{x^a, x^{a_2} \ldots x^{a_6} \mathcal{M}^{bc} \mathcal{M}^{de} \} = 0,
\]

(4.52)

\[
\mathcal{I}(\mathcal{A}^{(2)}) = \xi a_{a_1 \ldots a_{bd}ce} \{\theta^{ac}, x^{a_2} \ldots x^{a_6} \mathcal{M}^{bc} \mathcal{M}^{de} \} = (n + 1) \theta \mathcal{A}^{(2)},
\]

(4.53)

because $\phi^{(2)}$ is traceless. Similarly, we can identify this with line one of (4.7) for $s = 2$. Next, for $\mathcal{A}^{(3)}$ we obtain
\[ \mathcal{N}(\mathcal{A}^{(3)}) = 0, \quad (4.54) \]

\[ \mathcal{G}(\mathcal{A}^{(3)}) = \{ x^\mu \partial^\mu \phi(a_1^{(2)} \ldots a_{n+4} \ldots x^a \ldots x^e \mathcal{M}^{ac} \mathcal{M}^{de} \} = -\frac{(n+5)}{\theta} \phi^{(2)}, \quad (4.55) \]

\[ \mathcal{I}(\mathcal{A}^{(3)}) = \mathcal{I}(\mathcal{J}(\mathcal{A}^{(2)})) = -4\theta \mathcal{J}(\mathcal{A}^{(2)}) + \theta \mathcal{Q}(\mathcal{N}(\mathcal{A}^{(2)})) - \mathcal{J} \circ \mathcal{I}(\mathcal{A}^{(2)}) = -4\theta \mathcal{A}^{(3)} + \frac{1}{\theta} \mathcal{Q}(\phi^{(2)}) - (n+1)\theta \mathcal{A}^{(3)} = -5\theta \mathcal{A}^{(3)} + 2\theta^2 \frac{n+2}{n+1} \mathcal{A}^{(0)}, \quad (4.56) \]

using (2.28) (for the basic \( S^2 \)) and (4.63). Finally, for the radial mode \( \mathcal{I} \) does not diagonalize, but decomposes into radial and tangential components. In detail, we obtain

\[ \mathcal{G}(\mathcal{A}^{(3)}) = \frac{1}{\theta^2} x^\mu \{ x^\mu, \phi^{(2)} \} = 0, \quad (4.57) \]

\[ \mathcal{I}(\mathcal{A}^{(3)}) = -4 \frac{1}{\theta} \phi^{(2)} - \frac{1}{\theta} \mathcal{J}(\mathcal{Q}(\phi^{(2)})) = -\theta(n+4)A^{(3)} + n\theta R^2 A^{(2)} - 2\frac{n+2}{n+1} A^{(1)}, \quad (4.58) \]

using (2.27), (4.29), and \( \mathcal{A}^{(3)} = \mathcal{J} \mathcal{A}^{(2)} \).

### 4.3.2. Diagonalization of \( \mathcal{I} \).

Now we can diagonalize \( \mathcal{I} \) as in the spin 1 case. Collecting the above results, we obtain for the modes \( \mathcal{A}^{(1)}/\theta, \mathcal{A}^{(2)}, \) and \( R^2 \mathcal{A}^{(2)} \) the following:

\[ \mathcal{I} \left( \begin{array}{c} \mathcal{A}^{(1)}/\theta \\ \mathcal{A}^{(2)} \\ R^2 \mathcal{A}^{(2)} \end{array} \right) = \theta \left( \begin{array}{ccc} -4 & -(n+1) & 0 \\ -\frac{n+2}{n+1} & -(n+4) & n \\ 0 & 0 & (n+1) \end{array} \right) \left( \begin{array}{c} \mathcal{A}^{(1)/\theta} \\ \mathcal{A}^{(2)} \\ R^2 \mathcal{A}^{(2)} \end{array} \right). \quad (4.59) \]

This matrix has eigenvalues \(-2, -(n+6), 1+n\), in complete agreement with lines three, five and one of (4.7). The corresponding eigenvectors are

\[ B[\phi^{(2)}] := \frac{A^{(1)}}{\theta} - \frac{n+1}{n+2} \mathcal{A}^{(2)} + \frac{n(n+1)}{(n+2)(n+3)} R^2 \mathcal{A}^{(2)}, \quad (4.60a) \]

\[ C[\phi^{(2)}] := \frac{2}{n+1} \frac{A^{(1)}}{\theta} + \mathcal{A}^{(2)} - \frac{n}{2n+7} R^2 \mathcal{A}^{(2)}, \quad (4.60b) \]

\[ D[\phi^{(2)}] := R^2 \mathcal{A}^{(2)}. \quad (4.60c) \]

They all satisfy the gauge fixing condition \( \mathcal{G}(\cdot) = 0 \). We have therefore obtained a complete basis of spin 2 eigenmodes of \( \mathcal{I} \). Similarly, we can compute the \( \mathcal{I} \) action on \( \mathcal{A}^{(0)} \) and \( \mathcal{A}^{(3)} \)

\[ \mathcal{I} \left( \begin{array}{c} \mathcal{A}^{(0)} \\ \mathcal{A}^{(3)} \end{array} \right) = \theta \left( \begin{array}{cc} 0 & 0 \\ \frac{n+2}{n+1} & -5 \end{array} \right) \left( \begin{array}{c} \mathcal{A}^{(0)} \\ \mathcal{A}^{(3)} \end{array} \right), \quad (4.61) \]

which has eigenvalues 0 and \(-5\). The corresponding eigenvectors are

\[ E[\phi^{(2)}] := \mathcal{A}^{(0)}, \quad (4.62a) \]
\[ F[\phi^{(2)}] := A^{(3)} - \frac{2}{5} n + 2 \frac{A^{(1)}}{n + 1}. \]  

(4.62b)

We can identify this with line 2 and 4 of (4.7). In particular, this means that the \( F \) modes live in \( C^2 \otimes (1, 0) \).

Pure gauge spin 2 vector modes. The gauge transformations generated by \( \phi^{(2)} \) read

\[
Q(\phi^{(2)}) = Q(\phi^{(2)}_{a_1...a_5b_1...b_5c_1...c_5}x^{a_1}...x^{a_5} \theta^{b_1} \theta^{b_2} \theta^{b_3}) \\
= \theta^2 n A^{(3)} + \xi_f \phi^{(2)}_{a_1...a_5b_1...b_5c_1...c_5} x^{a_1}...x^{a_5} \{ x^{b_1}, \theta^{b_2} \theta^{b_3} \} \\
= \theta^2 n A^{(3)} + 2 \theta^2 \frac{B + 2}{n + 1} A^{(0)}
\]

(4.63)

using (A.13), see (4.38). Hence, these are not new modes.

4.3.3. Inner product matrix. Later, when computing the kinetic terms in the action (5.5), we will need the following expressions:

\[
\int A^{(i)[\phi^{(2)}]} A^{(j)[\phi^{(2)}]} = K^{ij} \int \phi^{(2)}_{ab} \phi^{(2)}_{ab}, \quad i, j \in \{0, 1, 2, 3, R\}.
\]

(4.64)

We only present the results here, and delegate the derivation to appendix B:

\[
\int A^{(0)[\phi]} A^{(0)[\phi]} = \frac{(n + 3)(n + 4)}{3(n + 2)^2} \theta \int \phi_{ab} \phi_{ab}.
\]

(4.65a)

\[
\int A^{(0)[\phi]} A^{(3)[\phi]} = \frac{2}{15} \frac{(n + 4)(n + 3)}{(n + 1)(n + 2)} \theta \int \phi_{ab} \phi_{ab}.
\]

(4.65b)

\[
\int A^{(3)[\phi]} A^{(3)[\phi]} = \frac{2}{15} \frac{(n + 3)(n + 4)(n^2 + 8n + 21)}{n(n + 1)^2(n + 2)} \theta \int \phi_{ab} \phi_{ab}.
\]

(4.65c)

\[
\int A^{(1)[\phi]} A^{(1)[\phi]} = \frac{(n + 3)(n + 4)}{3(n + 2)^2} R^2 \theta \int \phi_{ab} \phi_{ab}.
\]

(4.65d)

\[
\int A^{(1)[\phi]} A^{(2)[\phi]} = \frac{2}{15} \frac{(n + 4)(n + 3)}{(n + 1)(n + 2)} \theta \int \phi_{ab} \phi_{ab}.
\]

(4.65e)

\[
\int A^{(1)[\phi]} A^{(R)[\phi]} = 0.
\]

(4.65f)

\[
\int A^{(2)[\phi]} A^{(2)[\phi]} = \frac{1}{\theta^2} \frac{2(n + 3)^2(n + 4)(2n + 7)}{15n(n + 1)^2(n + 2)R^2} \theta \int \phi_{ab} \phi_{ab}.
\]

(4.65g)

\[
\int A^{(2)[\phi]} A^{(R)[\phi]} = \frac{2}{15} \frac{(n + 5)(n + 4)(n + 3)}{(n + 1)^2(n + 2)} \theta \int \phi^{(2)}_{ab} \phi^{(2)}_{ab}.
\]

(4.65h)

\[
\int A^{(R)[\phi]} A^{(R)[\phi]} = \frac{2}{15} \frac{(n + 5)(n + 4)(n + 3)}{(n + 1)^2(n + 2)} R^2 \theta \int \phi^{(2)}_{ab} \phi^{(2)}_{ab}.
\]

(4.65i)
where $\phi \equiv \phi^{(2)}$. All other inner products vanish. To gain some insights, we will give a more transparent derivation of e.g. (4.65a) in (4.86). In the basis of the $I$-eigenmodes (4.60) and (4.62) we find the inner product matrix $K^U$ for $I, J \in \{B, C, D, E, F\}$, defined via

$$
\int B_I [\phi^{(2)}] B_J [\phi^{(2)}] = K^U \int \phi^{(2)}_{ab} \phi^{(2)}_{ab}, \quad I, J \in \{B, C, D, E, F\},
$$

(4.66)

to be diagonal $K^U = \delta^U K^U$ with coefficients

$$
K^B = \frac{(n+4)^2(n+5)}{5(n+2)^3} \frac{R^2}{\theta^2} \approx \frac{R^2}{\theta^2} \frac{5}{2},
$$

(4.67a)

$$
K^C = \frac{2(n+3)(n+4)^2(n+5)(n+7)}{15(n+1)^2(n+2)^2(2n+7)} \frac{R^2}{\theta^2} \approx \frac{R^2}{\theta^2} \frac{1}{15},
$$

(4.67b)

$$
K^D = \frac{2(n+3)(n+4)(2n+7)}{15n(n+1)^2(n+2)} \frac{R^2}{\theta^2} \approx \frac{R^2}{\theta^2} \frac{4}{15},
$$

(4.67c)

$$
K^E = \frac{(n+3)(n+4)}{3(n+2)^2} \frac{1}{\theta} \approx \frac{1}{\theta} \frac{1}{3},
$$

(4.67d)

$$
K^F = \frac{2(n+3)(n+4)}{25n(n+1)^2(n+2)} \frac{n^2 + 12n + 35}{\theta} \approx \frac{1}{\theta} \frac{1}{25}.
$$

(4.67e)

Here, $\approx$ indicates the leading contributions for large $n$. This calculation provides a non-trivial consistency check. Note that $\phi^{(2)}_{ab}$ is defined in terms of the same linear combination of $\phi_{ab}$ as the mode $B_I$ in terms of $A_i$.

### 4.4. Higher spin vector fields $A$

Now we briefly discuss the general structure of the fluctuations with generic spin $s$. From the examples above, it is clear that there are five vector modes for each spin $s$, realized by

$$
A^{(0)} = \xi_n \phi_{a_i a_{i+1} \ldots a_{i+s}} M^{bcde} \in (n, 2s) \subset (\ast, 2s) \otimes (1, 0),
$$

$$
A^{(1)} = J(A^{(0)}) = \xi \theta^b \phi_{a_{i+1} \ldots a_{i+s}} M \otimes M^{bcde} \in (n, 2s) \subset (\ast, 2s) \otimes (1, 0),
$$

$$
A^{(2)} = \xi \theta^b \theta^c \phi_{a_{i+1} \ldots a_{i+s}} M^{bcde} \otimes M^{cdfe} \in (n, 2s) \subset (\ast, 2s) \otimes (1, 0),
$$

$$
A^{(3)} = J(A^{(2)}) = \xi \theta^b \theta^c \phi_{a_{i+1} \ldots a_{i+s}} M^{bcde} \otimes M^{cdfe} \in (n, 2s) \subset (\ast, 2s+2) \otimes (1, 0),
$$

$$
A^{(4)} = \frac{1}{\theta^2} \xi \theta^b \phi_{a_{i+1} \ldots a_{i+s}} M^{bcde} \otimes M^{cdfe} \in (n, 2s) \subset (\ast, 2s) \otimes (1, 0).
$$

(4.68)

#### Properties spin $s$ fields.

Analogous to the previous calculations, one can evaluate $\mathcal{I}$ on the five generic spin $s$ modes (some details are in appendix D). One obtains

$$
\mathcal{I} \left( \begin{array}{c} A^{(1)}_\theta \\ A^{(4)}_\theta \\ R^2 A^{(2)} \end{array} \right) = \theta \left( \begin{array}{ccc} -(s+2) & -(n+1) & 0 \\ -s^2 & -(n+4) & n \\ 0 & 0 & n + s - 1 \end{array} \right) \left( \begin{array}{c} A^{(1)}_\theta \\ A^{(4)}_\theta \\ R^2 A^{(2)} \end{array} \right),
$$

(4.69)

which has eigenvalues $-2, -(n+s+4), n+s-1$. This agrees with table D1 of appendix D. The corresponding eigenvectors are defined as
\[ B[\phi^{(s)}] := \frac{A^{(s)}}{\theta} - \frac{n+1}{n+2} A^{(R)} + \frac{n(n+1)}{(n+2)(n+s+1)} (R^2 A^{(2)}), \tag{4.70a} \]

\[ C[\phi^{(r)}] := \frac{2}{n+1} A^{(1)} - \frac{n}{2(n+s+3)} (R^2 A^{(2)}), \tag{4.70b} \]

\[ D[\phi^{(1)}] := R^2 A^{(2)}. \tag{4.70c} \]

Likewise we find

\[ I \left( \begin{array}{cc} A^{(0)} \\ A^{(3)} \end{array} \right) = \theta \left( \begin{array}{cc} s - 2 & 0 \\ \frac{s+2}{n+1} & -(s+3) \end{array} \right) \left( \begin{array}{c} A^{(0)} \\ A^{(3)} \end{array} \right), \tag{4.71} \]

having eigenvalues \( s - 2 \) and \(- (s+3)\), see again table D1. The corresponding eigenvectors are given by

\[ E[\phi^{(x)}] := A^{(0)}, \tag{4.72a} \]

\[ F[\phi^{(s)}] := A^{(3)} - \frac{s(n+2)}{(2s+1)(n+1)} A^{(0)}. \tag{4.72b} \]

4.5. Recombination and relation with Vasiliev theory

To make contact with the standard formalism of Vasiliev theory and with the previous work [18], we observe that \( A^{(1)} \) can be absorbed in a trace part of \( A^{(2)} \), and the tangential part of \( A^{(2)} \) can be absorbed in a trace part of \( A^{(3)} \). To start with we rewrite

\[ A^{(1)}_a = P_{t^b}^a A^{(0)}_b = - \frac{1}{\theta R^2} \theta^{ac} \theta^{bd} A^{(2)}_d, \]

\[ P_{t^b}^a A^{(2)}_b = - \frac{1}{\theta R^2} \theta^{ac} \theta^{bd} A^{(2)}_d. \tag{4.73} \]

For example, the spin 1 mode of type \( A^{(0)} \) can be written as

\[ A^{(0)}[\phi^{(1)}] = - \frac{1}{\theta R^2} \xi_{\alpha} \theta^{\alpha \beta} \left( \phi^{(1)}_{\alpha_1 \ldots \alpha_{n-1} \alpha_{n+1}} \gamma^{b_1 b_2} \ldots \gamma^{b_{n-1} b_{n+1}} \theta^{b_{n+1}} \right) \]

\[ = - \frac{1}{\theta R^2} \xi_{\alpha} \theta^{\alpha \beta} \left( g^{b_{n+1} c} \phi^{(1)}_{\alpha_1 \ldots \alpha_{n-1} \alpha_{n+1}} \gamma^{b_1 b_2} \ldots \gamma^{b_{n-1} b_{n+1}} \theta^{b_{n+1}} \right) \]

\[ = \xi_{\alpha} \theta^{\alpha \beta} \left( \phi^{(1)}_{\alpha_1 \ldots \alpha_{n-1} \alpha_{n+1}} \gamma^{b_1 b_2} \ldots \gamma^{b_{n-1} b_{n+1}} \theta^{b_{n+1}} \right) \]

\[ = A^{(1)}[\phi^{(1)}], \tag{4.74} \]

where

\[ \phi^{(1)}_{\alpha_1 \ldots \alpha_{n-1} \alpha_{n+1}} = - \frac{1}{\theta R^2} P^{b_1 b_2 \ldots b_{n+1}} \phi^{(1)}_{\alpha_1 \ldots \alpha_{n-1} \alpha_{n+1}} + (h \leftrightarrow \alpha_{n+1}) \]

\[ g^{b_{n+1} c} \phi^{(1)}_{\alpha_1 \ldots \alpha_{n-1} \alpha_{n+1}} = \phi^{(1)}_{\alpha_1 \ldots \alpha_{n-1} \alpha_{n+1}}. \tag{4.75} \]

This is associated via \( \begin{array}{c} \hline \hline \end{array} \) of (4.8) to a Young diagram which is no longer traceless, but double traceless (but traceless within the horizontal lines).

Similarly, the tangential part of \( A^{(2)} \) can be absorbed in a trace contribution to \( A^{(3)} \), which we illustrate again for the spin 1 case
\[
\mathcal{P}_T \mathcal{A}^{(2)}[\phi^{(1)}] = -\frac{1}{\theta R^2} \mathcal{E}_{\theta} \theta^{\theta \mu} (\bar{\phi}_{\mu \nu}^{(1)} x^{\nu} + \ldots + \phi^{(1) \mu} \theta^{\nu}) \\
= \frac{1}{\theta R^2} \mathcal{E}_{\theta} \theta^{\theta \mu} (g_{\theta \nu} \bar{\phi}_{\mu \nu}^{(1)} x^{\nu} + \ldots + \phi^{(1) \mu} \theta^{\nu}) \\
= \mathcal{E}_{\theta} \theta^{\theta \mu} (\bar{\phi}_{\mu \nu}^{(1)} x^{\nu} + \ldots + \phi^{(1) \mu} \theta^{\nu}) \\
= A^{(1)}[\phi^{(1)}], \tag{4.76}
\]

where \(\bar{\phi}_{\mu \nu}^{(1)}\) coincides with \(4.75\). Likewise, the radial part of \(\mathcal{A}^{(2)}\) can be absorbed in the radial mode \(A^{(R)}\).

In general, \(\mathcal{A}^{(0)}[\cdot]\) is absorbed in the trace part of the \(\mathcal{A}^{(1)}[\cdot]\), and \(\mathcal{P}_T \mathcal{A}^{(2)}[\cdot]\) is absorbed in the trace part of \(\mathcal{A}^{(3)}[\cdot]\). However, as exemplified in \(4.75\) the underlying tensor of the vector fields is different. In the classification of fluctuation modes \((4.7)\) and \((4.68)\) we used irreducible Young diagrams, whereas the class of tensors for the recombinations needs to be generalized. These tensors (or Young diagrams) do not correspond to irreducible representations any more, but have to advantage to repackage the tangential modes into only two objects.

Thus, we can collect all vector modes into a single form

\[
\mathcal{A}^\mu = P_{\tau}^\mu A^\tau + \frac{1}{R^2} x^\mu (\kappa_b A^b) \equiv \theta^{\mu \nu} A^\nu + x^\mu \phi, \tag{4.77}
\]

separated into tangential gauge fields and a transversal scalar field

\[
A^\nu = P_{\tau}^\nu A^\tau, \quad x^\mu \phi = \phi^\mu \phi^\nu = \phi^\mu (x) \Xi^\nu, \quad \Xi^\nu \in \mathfrak{h}^\ast, \tag{4.78}
\]

using \((3.26)\), where \(A^\mu\) is a \(\mathfrak{h}^\ast\)-valued one-form on \(S^4\) corresponding to double-traceless rectangular Young diagrams. \(A^\mu\) encodes all \(A^{(0)}\), \(A^{(1)}\), \(P_T A^{(2)}\), \(A^{(3)}\) associated to irreducible Young diagrams via the recombination into \(\mathcal{A}^{(1)}\) and \(\mathcal{A}^{(3)}\) build from more general diagrams. The only difference between the \(A^{(1)\mu}\) and the \(A^{(3)\mu}\) modes is that the external vector index \(a\) is linked via \(\theta^{\mu b}\) either to the second line or the first line of the Young diagram, leading to a different number of generators \(\mathcal{M}\) in \(\Xi^\mu\). In other words, there are two traceless contributions of the same form \(A^\mu = \theta^{\mu b} A^b\), describing an irreducible spin \(s\) gauge field, and one spin \(s - 1\) contribution which will be recognized as pure gauge sector.

In addition, the transversal degrees of freedom \((1 - \mathcal{P}_T) A^{(2)}\) and \(A^{(R)}\) are encoded in the \(\mathfrak{h}^\ast\)-valued scalar \(\phi\) on \(S^4\).

The 1-form \(A^\mu\) provides the kinematical link to Vasiliev theory, see \[18\].

### 4.6. Local representation and constraints

Now we will express the spin 1 and spin 2 modes \((4.19)\) and \((4.43)\) in terms of ordinary tensor fields near the north pole as in section \[3.2\]. As for the scalar fields, we will find that the \(P^\mu\) and the \(\mathcal{M}^{\mu \nu}\) components of the \(\mathfrak{so}(5)\)-valued fields are not independent. This leads again to constraints, which are illustrated in some examples.

**Spin 1 mode \(\mathcal{A}^{(2)}\).** The decomposition for \(\mathcal{A}^{(0)}\) and \(\mathcal{A}^{(1)}\) involves only a vector field and has already been discussed. Thus, consider the spin 1 mode \(\mathcal{A}^{(2)}\). Decomposed near the north pole of \(S^4\) into tangential and radial components, it reads
\[ A_{\nu}^{(2)} = \phi_{\mu_0 \ldots \mu_{n}} \cdot \omega_{
abla \mu_0 \ldots \mu_{n}} + \cdots = A_{\mu}^a P_{\mu}^a + F_{\mu \nu}^{a} M_{\mu \nu}. \] (4.79)

For any fixed \( a \), \( A_{\nu}^{(2)} \) can be viewed as an element in \( C^1 \), which using the results of section 3.2.1 can be written as

\[ F_{\mu \nu}^{a} = - \frac{1}{2(n+1)} \left( \partial_{\mu} A_{\nu}^{a} - \partial_{\nu} A_{\mu}^{a} \right). \] (4.80)

On the other hand, \( A_{\nu}^{(2)} \) is fully determined by its radial component

\[ \phi_{\mu}^{(1)} := x^{a} A_{\mu}^{(2)} = : A_{\mu} (x) P_{\mu}^{a} + F_{\mu \nu} (x) M_{\mu \nu}^{a}, \] (4.81)

which, by using (2.41), yields at the north pole

\[ A_{\mu}^{\alpha} = \frac{1}{n} \partial^{\alpha} A_{\mu}, \quad F_{\mu \nu}^{a} = - \frac{1}{n(n+2)} \partial^{\alpha} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}). \] (4.82)

Hence, these spin 1 modes also reduce to a vector field and its field strength tensor.

**Spin 1 mode \( A_{(3)} \).** The above result implies immediately

\[ A_{(3)}^{\mu} = \mathcal{J} (A_{(2)}^{a}) = \theta^{\rho \sigma} \left( A_{\rho \mu} P_{\mu}^{a} + F_{\rho \mu \nu} M_{\mu \nu}^{a} \right) \] (4.83)

at the north pole, where \( A_{\rho \mu} \) and \( F_{\rho \mu \nu} \) are as in (4.82). Note that \( A_{\rho \mu} \) is not necessarily symmetric.

**Spin 2 modes \( A_{(0)} \).** Now consider the spin 2 (graviton) modes \( A_{(0)} \), decomposed into tangential and radial components at the north pole of \( S^4 \)

\[ A_{\mu}^{(0)} = \phi_{\mu_0 \ldots \mu_{n}} \cdot \omega_{
abla \mu_0 \ldots \mu_{n}} + \cdots = h_{\mu \nu} P_{\mu \nu} + \omega_{\mu \nu \rho \sigma} (x) M_{\mu \nu \rho \sigma}, \] (4.84)

recall that \( A_{(0)} \) is tangential. For fixed \( \mu \), \( A_{\mu}^{(0)} \) can be viewed as an element in \( C^1 \), and applying the results of section 3.2 we obtain

\[ h_{\mu \nu} = - \frac{n + 2}{n(n+1)} \phi_{\mu \nu}^{(2)} = h_{\nu \mu}, \]

\[ \omega_{\mu \nu \rho \sigma} = \frac{1}{2(n+2)} (\partial_{\mu} h_{\nu \rho \sigma} - \partial_{\nu} h_{\mu \rho \sigma}). \] (4.85)

Hence \( \omega_{\mu \nu \rho \sigma} \) is proportional to the spin connection defined by the linearized metric mode \( h_{\mu \nu} \). As a check, we asymptotically recover the relation (4.65) for \( A_{(0)} \)

\[ \int A_{\mu}^{(0)} A_{\nu}^{(0)} \sim \frac{2 R^2}{3 \rho} \int \omega_{\mu \nu \rho \sigma} \omega_{\mu \nu \rho \sigma} \sim \frac{1}{3} \frac{R^2}{\rho} \int \phi_{\mu \nu}^{(2)} \phi_{\mu \nu}^{(2)} \sim \frac{1}{3 \rho} \int \phi_{\mu \nu}^{(2)} \phi_{\mu \nu}^{(2)}, \] (4.86)

dropping \( O \left( \frac{1}{\rho} \right) \) curvature terms, using \( [P_{\mu} P_{\nu}]_{0} = 0 \), see (3.4), and \( \partial \cdot \partial \sim - \frac{(n+1)^2}{R^4} \). Note that only the \( \omega_{\mu \nu \rho \sigma} \) contributes. For the generalized \( S^4 \), there are different modes where the \( h_{\mu \nu} \) term provides the dominant contribution. This leads to a very different behavior, which will be elaborated elsewhere.

**Spin 2 modes \( A_{(1)} \).** Similarly, we can specialize \( A_{(1)} = \mathcal{J} (A_{(0)}) \) at the north pole to

\[ A_{(1)}^{\mu} = \theta^{\mu \nu} \left( \phi_{\mu_0 \ldots \mu_{n}} \cdot \omega_{
abla \mu_0 \ldots \mu_{n}} + \cdots \right) = \theta^{\mu \nu} \left( h_{\nu \rho \sigma} P_{\rho \sigma} + \omega_{\nu \rho \sigma} (x) M_{\rho \sigma} \right), \] (4.87)
where $\omega_{\mu\alpha\beta}$ is proportional to the spin connection defined by the linearized metric $h_{\mu\nu}$. Hence $A^{(0)}$ and $A^{(1)}$ both encode some ‘metric’ tensor and the associated spin connection.

**Spin 2 modes** $A^{(2)}$. Now consider the $A^{(2)}$ spin 2 (graviton) modes, decomposed into tangential and radial components at the north pole of $S^4$ as in (3.36)

\[
A^{(2)} = \phi^{(2)}_{\alpha_1\ldots\alpha_{2s}} \cdots \times_{\alpha_{2s}} M^{\alpha_{2s}} M^{\alpha_{2s}}
\]

This is determined by its radial component

\[
\phi^{(2)} := \phi^{(2)}_{\alpha_1\ldots\alpha_{2s}} = h_{\mu\nu} P^\mu P^\nu + \omega_{\mu\alpha\beta} P^\mu M^{\alpha\beta} + \Omega_{\alpha\beta;\mu\nu} M^{\alpha\beta} M^{\mu\nu}.
\]

(4.88)

This is determined by its radial component

\[
\phi^{(2)} := x_\mu A^{(2)} = h_{\mu\nu} P^\mu P^\nu + \omega_{\mu\alpha\beta} P^\mu M^{\alpha\beta} + \Omega_{\alpha\beta;\mu\nu} M^{\alpha\beta} M^{\mu\nu}
\]

(4.89)

via

\[
h_{\mu\nu} = \partial_\mu h_{\mu\nu}.
\]

(4.90)

Applying the results of section 3.2.2, we obtain for example

\[
\omega_{\mu\alpha\beta} = \partial_\mu \omega_{\nu\alpha\beta},
\]

\[
\Omega_{\alpha\beta;\mu\nu} = \partial_\mu \Omega_{\alpha\beta;\mu\nu}
\]

(4.91)

up to curvature contributions of order $\frac{1}{R}$. The analysis of $A^{(3)}$ is analogous.

### 5. Action and equations of motion

So far we have obtained the higher spin fluctuation modes on $S^4$. The next step is the formulation of physical higher spin theories. There is a simple and natural framework to establish such higher spin actions on $S^4$, given by matrix models. In the semi-classical limit, this leads to higher spin gauge theories on $S^4$.

#### 5.1. Scalar theory on $S^4$

As a warm-up, consider first a ‘scalar field theory’ on $S^4$, with an action given by the semi-classical limit of a matrix model

\[
S = \text{Tr}(\Phi \Box \Phi + V(\Phi)) \sim \frac{\dim H}{\text{Vol} S^4} \int_{S^4} \phi(-\Box) \phi + V(\phi), \quad \Phi \in \text{End}(H) \sim \phi \in \mathcal{C},
\]

(5.1)

recalling the relation (3.5) between trace and integral. The spin 0 sector $\phi^{(0)} \in \mathcal{C}^0$ leads to a scalar field theory on $S^4$, deformed by the non-associativity of the commutative product [4] which leads to slightly non-local interactions. This is supplemented by a tower of spin $s$ fields $\phi^{(s)} \in \mathcal{C}^s$. Similar models have been considered for instance in [7, 32]. However this is not a gauge theory, and we will not consider it any further here.

#### 5.2. Vector theory on $S^4$ and vector Laplacian

Now consider a gauge theory for fields $A = \xi_\mu A^\mu \in \Omega^1 S^4$. Such a theory arises naturally as Poisson limit of Yang–Mills matrix models, such as the IKKT model. The action of the ‘Poisson matrix model’ reads as follows:
\[ S = \frac{1}{g^2} \int d\Omega \left( \{y_a, y_b\} \{y^a, y^b\} + \mu^2 y^a y_a \right), \quad (5.2) \]

where

\[ y^a = x^a + \mathcal{A}^a \]

are functions on \( S^4 \cong \mathbb{CP}^3 \), and \( x^a \) is the background which defines \( S^4 \). As above in (3.5) and (5.1), the integral is defined by the symplectic volume form on \( \mathbb{CP}^3 \). Collecting the variables in the formal one-forms \( Y = \xi_a x^a \), \( X = \xi_a A^a \) and expanding the action up to second order in \( \mathcal{A}^a \), one obtains

\[ S[Y] = S[X] + \frac{2}{g^2} \int d\Omega \left( 2A^a (-\Box + \frac{1}{2} \mu^2) x_a + \mathcal{A}_a (-\Box + \frac{1}{2} \mu^2) A_a + 2\{\mathcal{A}_a, \mathcal{A}_b\} \{x^a, x^b\} + G(A)^2 + O(A^3) \right). \quad (5.4) \]

Here \( \Box = \{x^a, \{x^a, \cdot\}\} \) is the Poisson–Laplacian defined in (2.34), and, recalling (2.30), \( G(A) = \{\mathcal{A}^a, x_a\} \) can be viewed as gauge fixing function, which transforms as \( \mathcal{G} \to \mathcal{G} + \Box \mathcal{A} \) under gauge transformations. Hence, the quadratic fluctuations \( \mathcal{A}^a \) are governed by the quadratic form

\[ \int d\Omega \mathcal{A}_a \left( \Box^2 + \frac{1}{2} \mu^2 \right) \mathcal{A}_a, \quad (5.5) \]

where the contribution from \( G(A)^2 \) has been canceled by adding a suitable Faddeev–Popov gauge-fixing term. Therefore, we recover that the relevant operator for the vector fluctuations is the ‘vector’ Laplacian (2.35), i.e.

\[ \Box^2 A = \left( -\Box - 2\mathcal{I} \right) A \quad (5.6) \]

where \( \mathcal{I} \) is the intertwiner defined in (2.23). Its eigenvalues can be determined by relating \( \Box^2 \) to basic group-theoretical operators (4.6) and (4.4),

\[ \Box = \{x_a, \{x^a, \cdot\}\} = \theta \sum_{a=1}^{5} \mathcal{M}_{ab}^{(ad)} \mathcal{M}_{bc}^{(ad)} = \theta(C^2[\mathfrak{so}(6)]^{(ad)} - C^2[\mathfrak{so}(5)]^{(ad)}), \]

\[ 2\mathcal{I} = \theta(-C^2[\mathfrak{so}(5)]^{(5)\oplus(ad)} + C^2[\mathfrak{so}(5)]^{(ad)} + C^2[\mathfrak{so}(5)]^{(5)}). \quad (5.7) \]

Here \( \mathcal{M}_{ab}^{(5)} \) is the vector generator of \( \mathfrak{so}(5) \), and \( \mathcal{M}_{bc}^{(ad)} = \{\mathcal{M}_{bc}, \cdot\} \) denotes the representation of \( \mathfrak{so}(5) \) on \( S^4 \) induced by the Poisson structure on \( S^4 \), see (2.13). This gives

\[ -\mathcal{D}^2 = \theta(C^2[\mathfrak{so}(6)]^{(ad)} - C^2[\mathfrak{so}(5)]^{(5)\oplus(ad)} + 4) \quad (5.8) \]

with \( C^2[\mathfrak{so}(5)]^{(5)} = 4 \). Now \( S^4 \) decomposes under \( \mathfrak{so}(5) \subset \mathfrak{so}(6) \) as follows:

\[ S^4 = \bigoplus_{n=0}^{\infty} (n, 0, 0)_{\mathfrak{so}(6)}, \quad (n, 0, 0) = \bigoplus_{s=0}^{n} (n - s, 2s)_{\mathfrak{so}(5)}. \quad (5.9) \]

13 To stabilize \( S^4 \) with the classical action, one needs a negative mass; however taking into account quantum corrections, a positive bare mass term suffices at one loop [31]. Alternatively one may add other terms such as \( \int e^{ab\mu}(y^a y^b) \{y^a, y^b\} \), see [6].
Therefore the eigenvalues of $C^2[\mathfrak{so}(6)]^{(ad)}$ acting on $(n', 2s) \subset (n, 0, n)$ as

$$C^2[\mathfrak{so}(6)]^{(ad)}\phi(n', 2s) = 2(n' + s)(n' + s + 3)\phi(n', 2s)$$  \hfill (5.10)

depend only on the combination $n' + s$. Thus, the three modes $A_{(n, 2s)} \subset ((n, 2s) \oplus (n - 1, 2s + 2) \oplus (n + 1, 2s - 2)) \otimes (1, 0)$ are degenerate under $D^2$, and their ‘wavefunctions’ $\phi(n', 2s)$ are related by $SO(6)$; these are the $B,E,F$-type modes discussed above.

The explicit eigenvalues are given in appendix D and in [31]. $D^2$ turns out to be positive except for some zero modes, given by the $(1, 2s)$ modes of type $D$. These are the $A^{(5)}$ modes without any explicit $x$ factors.

As a remark, the fluctuations or gauge fields $A$ take values in $\Omega^1 S^4$. One might be tempted to restrict $A$ to $T^* S^4$, i.e. purely tangential fluctuations, but that is inconsistent as one has to take all possible (matrix) fluctuations into account. In other words, one cannot eliminate the radial fluctuations in the (Poisson) matrix model (5.2).

### 6. Metric and graviton

Assuming an action of the above type (5.2), we can identify the effective metric and its linearized fluctuation along the lines of [15], and decompose it into the above spin modes. As always, the metric is encoded in the kinematics of the fluctuation modes on the background $v^a$ in the action. Their kinetic term arises from the bi-vector field

$$\gamma = \{y^c, \cdot\} \{y_c, \cdot\} = \gamma^{\mu\nu} \partial_\mu \partial_\nu,$$  \hfill (6.1)

due to the radial constraint (2.11). For the unperturbed $S^4$,  \hfill (6.3)

$$\gamma^{ab} = \{y^a, x^b\}.$$  \hfill (6.2)

This is indeed a tangential tensor field on $S^4$, since  \hfill (6.4)

$$\gamma^{ab} x_a = 0.$$  \hfill (6.5)

Explicitly, the corresponding metric fluctuation tensor is  \hfill (6.6)

$$H^{ab}[A] := \{x^a, x^b\} \{A_a, x^b\} + (a \leftrightarrow b) =: H[A].$$  \hfill (6.7)

which is tangential  \hfill (6.8)

$$H^{ab} x_b = 0.$$

Hence, $H^{ab}[A]$ defines an $SO(5)$ intertwiner from $\Omega^1 S^4$ to tangential symmetric 2-tensor fields. At low energies, only the $C^0$ modes of the energy-momentum tensor $T_{\mu\nu}$ are relevant, so that only the average metric is important, i.e. the projection

$$H^{ab} := \frac{4}{L_A^4} [H^{ab}]_0 \in (5) \otimes (5) \otimes C^0$$

(6.9)

see [18]. This will be called graviton in this paper. The normalization factor is chosen consistent with (6.4), such that the full effective metric is

$$\gamma^{\mu\nu} = g^{\mu\nu} + h^{\mu\nu}.$$ (6.10)

Then by construction, $h^{\mu\nu}$ couples to matter via its energy-momentum tensor$^{14}$,

$$\delta h_{\text{matter}} = \frac{1}{2} \int_{S^4} d^4 x h^{\mu\nu} T_{\mu\nu}.$$ (6.11)

6.1. Gravitons and eigenmodes

We can rewrite $H_{ab}[A]$ by means of

$$\theta^{ca}(A_c, x_b) = \{\theta^{ca} A_c, x_b\} - \theta(A_b x^a - g^{ab} A_c x^c)$$ (6.12)

as follows:

$$H_{ab}[A] = \{\theta^{ca} A_c, x_b\} + \theta(A_b x^a + A_a x_b) - 2\theta g_{ab}(A_c x^c).$$ (6.13)

In particular, the radial fluctuations $A^{(R)} c\phi$ give rise to a conformal metric fluctuation

$$H_{ab}[A^{(R)}] = 2\theta R^2 \phi \left( g_{ab} - R_{\phi}^{-2} A_a A_b \right) = 2\theta R^2 \phi P^{ab}_I.$$ (6.14)

while for tangential $A^\mu$, (6.13) simplifies to

$$H_{ab}[A] = \{\theta^{ca} A_c, x_b\} + \theta(A_b x^a + A_a x_b) .$$ (6.15)

We can now elaborate the graviton contributions of the different vector fluctuation modes considered in section 4:

Gravitons for spin 2 fields. To gain some intuition, consider first the graviton associated with the spin 2 mode $A^{(1)}$ using the local representation (4.87), i.e.

$$H_{\mu\nu}[A^{(1)}] = \frac{L_A^4}{4} \left( \theta^{\mu\alpha} \partial_\alpha A_\nu + (\mu \leftrightarrow \nu) \right)$$

$$= -R^2 \theta \theta^{\mu\alpha} \partial_\alpha \left( h_{\mu\alpha} P^\alpha_\nu + \omega_{\mu\alpha,\nu} A^{\alpha\nu} \right) + (\mu \leftrightarrow \nu).$$ (6.16)

Upon averaging (3.4), the $P^\alpha$ term drops out$^{15}$, and the leading term is

$$h_{\mu\nu}[A^{(1)}] = \frac{2R^2}{3} \partial_\mu \omega_{\nu,\mu} + (\mu \leftrightarrow \nu) \sim \frac{2}{3} \theta \phi_{\mu\nu}.$$ (6.17)

using (4.85), see [18].

$^{14}$This defines the normalization of $T_{\mu\nu}$. The conformal factor is a bit tricky, for a discussion see [18].

$^{15}$This is precisely the problem with the basic $S^4_A$. There are extra modes on generalized $S^4_A$ which survive this step.
We can derive exact expressions for all five modes $H_{ab}[A^{(j)}]$ using the Young diagram representations, which are given in appendix C. The physical gravitons are then obtained by averaging these via (3.4). This leads to

\[
H_{ab}[A^{(0)}] = 0, \tag{6.18}
\]

\[
H_{ab}[A^{(1)}] = \frac{-2}{3} \frac{(n+3)(n+4)}{n+2} \phi_{ab}^{(2)}(x), \tag{6.19}
\]

\[
H_{ab}[A^{(2)}] = \frac{4}{15} \frac{(n+3)(n+4)}{n+1} \frac{1}{\kappa^2} \phi_{ab}^{(2)}(x), \tag{6.20}
\]

\[
H_{ab}[A^{(3)}] = 0, \tag{6.21}
\]

\[
H_{ab}[A^{(4)}] = 0, \tag{6.22}
\]

which for large $n$ consistently reduces to the local derivation (6.17). The main feature is the factor $n$ in $H_{ab} \sim n \phi_{ab}$, which arises from the derivative contributions $\partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma} \phi_{\mu \nu \rho \sigma}$ in (6.17). This will imply that the quadratic action translates into $H_{ab}H_{ab}$, rather than $H_{ab} \Box H_{ab}$. As explained in section 8.1, we expect that this problem does not arise for the generalized fuzzy sphere, due to extra momentum-type generators $\mathfrak{r}^i$ which are non-vanishing as functions. This would lead to $H_{ab} \sim \phi_{ab}$ without factor $n$, hence to gravity at the classical level.

The gravitons of the $\mathcal{I}$ eigenmodes (4.60) then read

\[
H_{ab}^{(p)}[\phi^B] = -\frac{2}{5} \frac{(n+4)(n+5)}{n+2} \theta \phi_{ab}^{(1)}(x) \approx -\frac{2}{5} \theta n \phi_{ab}^{(2)}(x), \tag{6.23a}
\]

\[
H_{ab}^{(c)}[\phi^C] = -\frac{4}{5} \frac{(n+3)(n+4)(n+5)(n+7)}{(n+1)(n+2)(2n+7)} \frac{1}{\theta} \phi_{ab}^{(2)}(x) \approx -\frac{4}{5} \frac{1}{\theta} n \phi_{ab}^{(2)}(x), \tag{6.23b}
\]

\[
H_{ab}^{(d)}[\phi^D] = -\frac{4}{15} \frac{(n+3)(n+4)}{n+1} \frac{1}{\theta} \phi_{ab}^{(2)}(x) \approx -\frac{4}{15} \frac{1}{\theta} n \phi_{ab}^{(2)}(x), \tag{6.23c}
\]

\[
H_{ab}^{(e)}[\phi^E] = 0, \tag{6.23d}
\]

\[
H_{ab}^{(f)}[\phi^F] = 0, \tag{6.23e}
\]

and the approximations are valid for large $n$. Also, observe that all terms have a similar structure, including an explicit factor $n$.

**Gravitons for spin 1 fields.** The gravitons for the spin 1 modes (4.19) read as follows:

\[
H_{ab}[A^{(0)}] = -\frac{(n+1)}{3} (\phi_{a_1...a_n,b} + \phi_{a_1...a_n,b}) x^{a_1} ... x^{a_n} + \frac{n}{3 \kappa^2} \left( x^{a} A_{b}^{(0)} + x^{b} A_{a}^{(0)} \right), \tag{6.24a}
\]

\[
H_{ab}[A^{(1)}] = 0, \tag{6.24b}
\]

\[
H_{ab}[A^{(2)}] = 0, \tag{6.24c}
\]

\[
H_{ab}[A^{(3)}] = -\frac{(n+2)^2}{3n} (\phi_{a_1...a_n,b} + \phi_{a_1...a_n,b}) x^{a_1} ... x^{a_n} + \frac{(n+2)^2}{3(n+1) \kappa^2} \left( x^{a} A_{b}^{(0)} + x^{b} A_{a}^{(0)} \right), \tag{6.24d}
\]

\[
H_{ab}[A^{(4)}] = 0. \tag{6.24e}
\]
We observe that all spin 1 gravitons are traceless and tangential. One can rewrite the non-trivial modes as follows:

\[ h_{ab}[A^{(1)}] = -\frac{1}{3} \left( \nabla_a \phi_b^{(1)} + \nabla_b \phi_a^{(1)} \right), \]

\[ h_{ab}[A^{(2)}] = -\frac{1}{3} \frac{(n+2)^2}{n(n+1)} \left( \nabla_a \phi_b^{(1)} + \nabla_b \phi_a^{(1)} \right), \]

which is recognized as pure gauge contribution to the graviton. Here \( \phi_a^{(1)} \) is the canonical vector field associated to the Young diagram, and \( \nabla \) is defined in (2.46). In particular, the graviton contribution of the spin 1 pure gauge modes \( Q(\phi_a^{(1)}) \), see (4.38), is

\[ h_{ab}[Q(\phi_a^{(1)})] = -\frac{\theta}{3} \frac{(n+2)(n+3)}{n+1} \left( \nabla_a \phi_b^{(1)} + \nabla_b \phi_a^{(1)} \right). \]

### Gravitons for spin 0 fields.

Finally for the spin 0 modes (4.9), we obtain the graviton contributions

\[ h_{ab}[A^{(2)}] = -2 \left( \phi_{ab} \phi_{cd} \phi_{de} \phi_{ef} - \frac{1}{R^2} \left( \chi^a A^0_b + \chi^b A^0_a \right) + \phi^{ab}(0) \phi^{cd}(0) \phi^{ef}(0) \right), \]

\[ h_{ab}[J A^{(2)}] = 0, \]

\[ h_{ab}[A^{(0)}] = 2 P^{ab}_F \phi^{(0)}. \]

The first mode can be written as

\[ h_{ab}[A^{(2)}] = -\frac{2}{3n} \left( \nabla_a \nabla_b \phi(0) + \frac{n^2}{R^2} P^{ab} \phi(0) \right), \]

which encodes the pure gauge graviton associated to the vector field \( \partial_a \phi(0) \). Similarly, \( h_{ab}[A^{(R)}] \) is the conformal metric contribution. Note that in the Einstein–Hilbert action, the conformal mode suffers from an instability, see for instance [33]. There is no such instability in the present action.

### 6.2. Spin 2 action and equations of motion

We want to understand the physics of the spin 2 modes in the presence of matter. Before embarking on the detailed computation, we should have some idea of what to expect. For example, the quadratic action for \( h_{\mu\nu}[A^{(1)}] \) is obtained from (6.17) and (4.86) approximately as

\[ \frac{1}{g^2} \int d^4 x A^{(1)} \nabla^2 A^{(1)} \approx \frac{L^2_{\text{NC}}}{4} \int d^4 x \phi^{(2)}(0) \phi^{(2)}(0) \approx \frac{3 L^2_{\text{NC}}}{4 g^2} \frac{\dim H}{\text{vol}(S^4)} \int_{S^4} h^{\mu\nu} h_{\mu\nu}. \]

Combined with the coupling to matter (6.11)

\[ \delta_S h_{\mu\nu} = \frac{1}{2} \int_{S^4} \delta_S h^{\mu\nu} T_{\mu\nu}, \]

we arrive at an equation of motion of the form
\[ h_{\mu\nu}[A^{(1)}] \approx \frac{3g^2}{4\mu^2} \frac{\text{vol}(S^4)}{\text{dim} H} T_{\mu\nu}. \]  

(6.30)

This means that \( h_{\mu\nu} \) behaves like a non-propagating auxiliary field, rather than a graviton. However, we need to take the mixing between the different modes \( A^{(1)} \) into account; this is taken care of by using the eigenbasis \( B_j^{(s)} \). We can then solve exactly the quadratic action governing the spin 2 sector. This will exhibit an interesting sub-leading behavior, and by, taking account of possible induced gravity terms from quantum corrections, it might even acquire the appropriate behavior of gravity.

Now we derive the precise equations of motion. Consider the action of the vector fluctuations \( A \) in the original matrix model in the semi-classical limit,

\[ S = \frac{1}{g^2} \int_{S^4} \text{d}\Omega \left( A D^2 A \right) \]  

(6.31)

where \( D^2 \) has been defined in (5.6). The fluctuation \( A \) can be expanded in the five (or three) independent spin \( s \) fields for \( s \geq 1 \) (or \( s = 0 \)) as follows:

\[ A = \sum_{i \geq 0} \left( A^{(0)_i} + \frac{1}{\theta} A^{(1)_i} + R^2 A^{(2)_i} + A^{(3)_i} + A^{(R)_i} \right) \equiv \sum_{i \geq 0} \sum_{j} \tilde{A}^{(i)_j}, \]  

(6.32)

where \( \{\tilde{A}^{(s)_j}\} = \{\frac{1}{\theta} A^{(1)_i}, A^{(R)_i}, R^2 A^{(2)_i}, A^{(3)_i}, A^{(R)_i}\} \). Recall that for \( s = 0 \) the modes \( A^{(0,x=0)} \) and \( A^{(1,x=0)} \) are absent, see (4.18). The modes \( \tilde{A}^{(i,j)} \) have a uniform dimension, unlike the modes \( A^{(i)} \) of (4.68). At each spin \( s \), the transformation matrix (4.70) and (4.72) for the basis change into the \( I \)-eigenmodes \( \{B_j^{(s)}\}_I \equiv \{B^{(i)}, C^{(i)}, D^{(i)}, E^{(i)}, F^{(i)}\} \) can be cast into the form

\[ B_j^{(i)} = \sum_i M_{ji}^{(i)} \tilde{A}^{(i)_j} \quad \text{with} \quad D^2 B_j^{(s)} = \theta \lambda_j^{(s)} B_j^{(s)}. \]  

(6.33)

Again, \( s = 0 \) has only a rank 3 transformations matrix, and the modes \( B^{(0)}, E^{(0)} \) are absent. Inserting this into the action (6.31), we obtain

\[ S = \frac{1}{g^2} \sum_{s' s} \sum_{ij} \int_{S^4} \text{d}\Omega \left( \tilde{A}^{(i,j)} D^2 \tilde{A}^{(i',j')} \right) \]

\[ = \frac{1}{g^2} \sum_{s' s} \sum_{ij} \int_{S^4} \text{d}\Omega \left( B_j^{(s)} D^2 B_j^{(r)} \right) \sum_{i} (M^{-1})^{(i)}_{j} (M^{-1})^{(r)}_{j} \]

\[ \times \theta \lambda_j^{(s)} \delta_{s,s'} \delta_{r,r'} \]

\[ = \frac{1}{g^2} \sum_{s' s} \left( \sum_{i} (M^{-1})^{(s)}_{i} \right)^2 \int_{S^4} \text{d}\Omega \left( B_j^{(s)} D^2 B_j^{(s)} \right). \]

(6.34)

The \( I \)-eigenmodes \( B_j^{(s)} \) can be canonically normalized by absorbing the normalizations \( N_j^{(s)} \) into the fluctuations via

\[ B_j^{(s)} \mapsto \tilde{B}_j^{(s)} := N_j^{(s)} B_j^{(s)}. \]

(6.35)
Then the action reads

\[ S = \frac{1}{g^2} \sum_{s} \int d\Omega \left( \bar{B}^{(i)}_I D^2 \bar{B}^{(i)}_I \right). \]  

(6.36)

Focusing on the spin \( s = 2 \) sector, we can evaluate the action in the semi-classical limit as follows:

\[ S_{s=2} = \frac{1}{g^2} \sum_i \int d\Omega \left( \bar{B}^{(2)}_I D^2 \bar{B}^{(2)}_I \right) = \frac{1}{g^2} \sum_i \theta \lambda_i^{(2)} \int_{S^4} d\Omega \left( \bar{B}^{(2)}_I \bar{B}^{(2)}_I \right) \]

\[
\sim \frac{\dim(H)}{g^2 \text{vol}(S^4)} \sum_i \theta \lambda_i^{(2)} K' \int \bar{\phi}_{ab} \phi_{ab},
\]

(6.37)

We used the inner product (4.67) and absorbed the normalizations \( N_i^{(2)} \) of the \( \bar{B}^{(2)}_I \) into \( \bar{\phi}_{ab} := N_i^{(2)} \phi_{ab} \), so that (4.66) turns into

\[ \int \bar{B}^{(i)}_I \bar{B}^{(i)}_J = K' \delta_{IJ} \int \bar{\phi}_{ab} \phi_{ab}, \]

(6.38)

where all \( K' \) are order one. The eigenvalues \( \lambda_i^{(2)} \) are given in appendix D and read

\[ \lambda_i^{(2)} = \lambda_i^{(2)} = \lambda_i^{(2)} = n(n+3) + 4(n+2), \]

\[ \lambda_i^{(2)} = \lambda_i^{(2)} = (n+3)(n+8). \]

(6.39)

The appearing degeneracy has been explained in section 5.2.

Now consider the coupling (6.11) of the spin 2 modes to matter

\[ \delta b S_{\text{matter}} = \frac{1}{2} \int_{S^4} h^{\mu\nu} [A] T_{\mu\nu}. \]

(6.40a)

Since the gravitons depend linearly on the modes (6.9), we can write

\[ \delta b S_{\text{matter}} = \sum_i \frac{1}{2} \int_{S^4} h^{\mu\nu} [\bar{A}^{(i)}(\bar{x})] T_{\mu\nu} = \sum_i \frac{1}{2} \int_{S^4} h^{\mu\nu} [\bar{B}^{(i)}(\bar{x})] T_{\mu\nu} \]

\[ = \sum_i \frac{1}{2} \int_{S^4} h^{\mu\nu} [\bar{B}^{(i)}(\bar{x})] T_{\mu\nu}. \]

(6.40b)

Restricting to \( s = 2 \) we observe that the form of (6.23) remains untouched, i.e.

\[ h_{ab}^{(B)}[\bar{\phi}] = -\frac{2}{5} \frac{(n+4)(n+5)}{\theta(n+2)} \bar{\phi}_{ab}(x), \]

(6.41a)

\[ h_{ab}^{(C)}[\bar{\phi}] = -\frac{4}{5} \frac{(n+3)(n+4)(n+5)(n+7)}{\theta(n+1)(n+2)(2n+7)} \bar{\phi}_{ab}(x), \]

(6.41b)

\[ h_{ab}^{(D)}[\bar{\phi}] = \frac{4}{15} \frac{(n+3)(n+4)}{\theta(n+1)} \bar{\phi}_{ab}(x). \]

(6.41c)

Using the short hand notation \( h_{ab}[\bar{B}^{(2)}] = \Xi^i \bar{\phi}_{ab} \), the equations of motion can be compactly written as
The coefficient in front of the energy momentum tensor is non-negative. Since \( \lambda \approx n^2 \) this leads to \( \phi_{\mu\nu} \sim \frac{1}{n} T_{\mu\nu} \), which is somewhat strange and non-local. The induced metric fluctuations are explicitly

\[
\begin{align*}
    h^{(b)}_{\mu\nu}[\tilde{\phi}^b] &= -\frac{4}{L^4_{NC} \dim(H)} \frac{g^2 \text{vol}(S^4)}{5} \left( 1 + \frac{2}{n^2 + 7n + 8} \right) T_{\mu\nu}, \\
    h^{(c)}_{\mu\nu}[\tilde{\phi}^c] &= -\frac{4}{L^4_{NC} \dim(H)} \frac{g^2 \text{vol}(S^4)}{5} \left( 1 + \frac{7}{3(2n + 7)} - \frac{2}{3(n + 8)} \right) T_{\mu\nu}, \\
    h^{(d)}_{\mu\nu}[\tilde{\phi}^d] &= -\frac{4}{L^4_{NC} \dim(H)} \frac{g^2 \text{vol}(S^4)}{15} \left( 1 + \frac{2}{3(n - 1)} - \frac{7}{3(2n + 7)} \right) T_{\mu\nu},
\end{align*}
\]

which results in the total metric fluctuation

\[
h_{\mu\nu} = \sum_{I=B,C,D} h^{(I)}_{\mu\nu}[\tilde{\phi}^I]
\]

\[
= -\frac{4}{L^4_{NC} \dim(H)} \frac{g^2 \text{vol}(S^4)}{45} \left( 39 - \frac{18}{n + 8} + \frac{56}{2n + 7} + \frac{2}{n - 1} + \frac{18}{n(n + 3) + 4(n + 2)} \right) T_{\mu\nu}.
\]

(6.44)

The leading contribution to this equation of motion is\(^{16}\)

\[
h^0_{\mu\nu} = -\frac{13}{15} \frac{4}{L^4_{NC}} \frac{g^2 \text{vol}(S^4)}{ \dim(H)} T_{\mu\nu},
\]

(6.45)

which agrees with (6.30). This is a non-propagating metric perturbation localized at the matter source, consistent with [18].

6.3. Flat limit

To understand the meaning of the above results (6.43), we focus on a region near the north pole of \( S^4 \), and assume that the radius \( R \) is much larger than any other relevant length scale. We can then relate the kinetic parameters on \( S^4 \) to ordinary momenta on the tangential \( R^4 \), using the tangential coordinates \( x^\mu \), \( \mu = 1, \ldots, 4 \). It is easy to see e.g. from (6.4) that the Laplace operator becomes [18]

\[
\Box \sim \frac{L^4_{NC}}{4} g^{\mu\nu} \partial_\mu \partial_\nu = -\frac{L^4_{NC}}{4} \Box_g
\]

(6.46)

neglecting curvature contributions \( \sim \frac{1}{R} \). Recalling \( \frac{L^4_{NC}}{4} = R^2 r^2 \) and \( \Box \approx -\theta n^2 \) on \( (n, 0) \) modes, we can identify

\[
\frac{L^4_{NC}}{4} \Box_g \approx -\theta n^2, \quad -n^2 \approx R^2 \Box_g
\]

(6.47)

for modes with \( n \gg 1 \). Now consider first the \( B \) mode of (6.43), which satisfies

\[\text{Since we focus on the spin 2 sector, only the traceless part of } T^{\mu\nu} \text{ enters here.}\]
\[ h^{(B)}_{\mu\nu} [\varphi^B] = \kappa \left( -1 + \frac{1}{R^2 \Box g} \right) T_{\mu\nu}, \]  

(6.48)

for some constant \( \kappa = \frac{4}{3\sqrt{\pi}} \frac{r^2 \text{vol}(S^4)}{\text{dim}(H)} \). Clearly the two source terms lead to two contributions

\[ h^{(B)}_{\mu\nu} [\varphi^B] = h^{(B,\text{loc})}_{\mu\nu} + h^{(B,\text{grav})}_{\mu\nu}, \]  

(6.49)

where

\[ \Box g h^{(B,\text{grav})}_{\mu\nu} = \kappa \frac{R^2}{2} T_{\mu\nu} \equiv G_N T_{\mu\nu}, \]

\[ h^{(B,\text{loc})}_{\mu\nu} = -\kappa T_{\mu\nu} \equiv -G_N R^2 T_{\mu\nu}. \]  

(6.50)

The first (sub-leading) term \( h^{(B,\text{grav})}_{\mu\nu} \) has indeed the structure of linearized gravity, and we have tentatively identified \( \kappa = 16\pi G_N R^2 \), such that \( G_N = \frac{4}{3\sqrt{\pi}} \frac{r^2 \text{vol}(S^4)}{\text{dim}(H)} \cong L_p^2 \). However, the local contribution \( h^{(B,\text{loc})}_{\mu\nu} \) is then too large: assuming \( R \approx 10^{27} \text{m} \) (\( \approx 10^{11} \text{ly} \)) we would have \( L_p^2 R^2 \approx 10^{-16} \text{m}^4 \), which results in \( h^{(B,\text{loc})}_{\mu\nu} \gg 1 \) even in the presence of modest energy densities, which is clearly unacceptable. Hence, although the present mechanism does lead to a propagating graviton contribution, it is not realistic in the current form. The most promising way to obtain more realistic gravity is by replacing \( S^4 \) with the generalized fuzzy sphere \( S^4_{\Lambda} \) [18, 25]. Then extra spin 2 contributions arise, which avoid the dominant derivative contributions. This will be discussed briefly in section 8. In the presence of an induced gravity term, this problem is also ameliorated, as we will see.

Now consider the \( C \) and \( D \) modes in (6.43), which lead to

\[ h^{(I)}_{\mu\nu} [\varphi^I] = -\kappa \left( 1 + \frac{1}{R \sqrt{\Box g}} \right) T_{\mu\nu}, \quad \text{for } I = C, D. \]  

(6.51)

This leads again to two contributions

\[ h^{(I)}_{\mu\nu} [\varphi^I] = h^{(I,\text{loc})}_{\mu\nu} + h^{(I,\text{nonloc})}_{\mu\nu}, \]  

(6.52)

where

\[ h^{(I,\text{loc})}_{\mu\nu} = -\kappa T_{\mu\nu} \equiv -G_N R^2 T_{\mu\nu}, \]

\[ h^{(I,\text{nonloc})}_{\mu\nu} = -\kappa T_{\mu\nu} \equiv -G_N R^2 T_{\mu\nu}. \]  

(6.53)

The relevant Green’s functions are computed in appendix E.1 in the flat limit \( R \to \infty \). To have a meaningful comparison between the various contribution, we assume a Gaussian mass distribution with variance \( a^2 \). The results are summarized as follows:

1. The Green’s function for the \( B \) mode \( h^{(B,\text{grav})}_{\mu\nu} \) (6.50) is \( \sim 1/|k|^2 \) in momentum space, which leads to the standard \( 1/p \) behavior in 4D space. In the presence of the Gaussian source term, this gives

\[ h^{(B,\text{grav})}_{\mu\nu} = G_N \frac{1 - e^{-\frac{a^2}{2r^2}}}{4\pi^2 r^2}, \]  

(6.54)

\[ \text{Note that } g^2 \text{ has dimension } L^4. \]
for any components $\mu \nu$. In the limit $a \to 0$, the source reduces to a 4D Delta-distribution, and we recover the usual $\frac{1}{4\pi r^2}$ Green’s function.

2. The Green’s function for the $C,D$ modes $h_{\mu \nu}^{(l, \text{nonloc})}$, in (6.53), is $\sim \frac{1}{|\vec{k}|}$ in momentum space, which for the same Gaussian source gives

$$h_{\mu \nu}^{(l, \text{nonloc})} = \frac{G_N R}{8 \sqrt{2\pi} 3^3 a^3} e^{-\frac{r^2}{2a^2}} \left( I_0 \left( \frac{r^2}{4a^2} \right) - I_1 \left( \frac{r^2}{4a^2} \right) \right),$$

for any $\mu \nu$, and $I_a(z)$ denotes the modified Bessel functions of first kind. The asymptotic behavior for small and large radial distances $r$ is

$$h_{\mu \nu}^{(l, \text{nonloc})} = \frac{G_N R}{8 \sqrt{2\pi} 3^3 a^3} e^{-\frac{r^2}{2a^2}} \left( 1 - \frac{r^2}{8a^2} \right), \quad 0 < r \ll 2a,$$

$$h_{\mu \nu}^{(l, \text{nonloc})} = \frac{G_N R}{3\pi^2 a^3} \left( 1 + \frac{3r^2}{a^2} + \frac{9r^4}{16a^4} + \ldots \right), \quad \frac{r}{2a} \to \infty. \quad (6.56)$$

3. All the fluctuations exhibit one mode $h_{\mu \nu}^{(l, \text{loc})}$, for $I = B,C,D$, which has an algebraic equation of motion. By the same analysis as in the previous cases, one finds

$$h_{\mu \nu}^{(l, \text{loc})} = \frac{G_N R^2}{4\pi^2 a^4} e^{-\frac{r^2}{2a^2}}, \quad (6.57)$$

for the Gaussian source, for any $\mu \nu$.

Equipped with these results, we can consider two regimes:

**Small $a$.** This is the regime where the observer is far from the source. Then the mode $h_{\mu \nu}^{(B, \text{grav})}$ behaves indeed like a graviton. However, $h_{\mu \nu}^{(l, \text{nonloc})}$ has a leading contribution $\sim \frac{\frac{G_N R}{8\sqrt{2\pi} 3^3 a^3}}{\frac{1}{4\pi r^2}}$, which by far dominates over the graviton $h_{\mu \nu}^{(B, \text{grav})}$. Sufficients far from the source, the ‘local’ term $h_{\mu \nu}^{(l, \text{loc})}$ is exponentially suppressed and therefore irrelevant.

In summary, the basic fuzzy 4-sphere does not lead to a physical gravity for localized matter distributions, but it does lead to a long-range $\sim \frac{R}{4\pi r^2}$ ‘gravity’.

**Large $a$.** This regime applies to observers within mass distributions that are spread out over large scales $a$ (such as dust), but still smaller than the extend of the entire space, $a \ll R$. Now the local contribution $h_{\mu \nu}^{(l, \text{loc})}$ for $I = B,C,D$ is enhanced by $\frac{R^2}{a^2}$ compared to $h_{\mu \nu}^{(B, \text{grav})}$. Similarly, $h_{\mu \nu}^{(l, \text{nonloc})}$, for $I = C, D$ is enhanced by $\frac{R}{a}$. On the other hand, $h_{\mu \nu}^{(B, \text{grav})}$ is of order $\frac{G_N}{8\sqrt{2\pi} 3^3 a^3}$, but otherwise still exhibits a graviton-like behavior. Nevertheless, the ‘local’ and ‘non-local’ modes dominate the graviton-like mode, which renders the scenario unphysical.

6.3.1. *Induced gravity effects.* Now we want to take the leading quantum effects into account. Integrating out any fields that couple to a background metric, one generically obtains induced gravity terms in the effective action, such as $\int d^4 x \Lambda^2 \mathcal{R} \mathcal{R}$. Here $\Lambda^2$ is the effective cutoff, presumably determined by the SUSY breaking scale. This would lead to a quadratic term $\int h_{ab} \Box h_{ab}$ in the action (there cannot be any linear contribution for the traceless modes). There may be a linear contribution to the conformal factor, which we ignore here; this is basically the question whether the background is stable, which relies on quantum effects anyway [31]. Hence the cosmological constant problem is tantamount to the issue of stability of the background.
Let us therefore add such an induced kinetic term to the action. For the $B$ mode the equation of motion becomes

$$
(1 + \sigma R^2 \Box_g) (h^{(B)}_{\mu\nu} \tilde{\phi}^{(B)})_{\mu\nu} = -\kappa \left( 1 + \frac{1}{R^2 \Box_g} \right) T_{\mu\nu},
$$

(6.58)

where $R$ is the effective cutoff scale for induced gravity\(^{18}\), and $\sigma = \pm 1$ depending on the precise field content of the model. Now the two source terms lead to two contributions

$$
h^{(B)}_{\mu\nu} \tilde{\phi}^{(B)} = h^{(B, \text{loc})}_{\mu\nu} + h^{(B, \text{grav})}_{\mu\nu},
$$

(6.59)

where

$$
\left( \sigma \Box_g + \frac{1}{R^2} \right) h^{(B, \text{loc})}_{\mu\nu} = -G_N \frac{R^2}{R^2} T_{\mu\nu},
$$

(6.60)

For $\sigma = -1$, $h^{(B, \text{loc})}_{\mu\nu}$ behaves like a massive spin 2 graviton with mass $1/R^2$, with appropriate coupling to matter by $\frac{R^2}{G_N} T_{\mu\nu}$. For $\sigma = 1$, it would be a tachyonic mode, which we discard.

For the $C, D$ modes the equation of motion becomes

$$
(1 + \sigma R^2 \Box_g) (h^{(I)}_{\mu\nu} \tilde{\phi}^{(I)})_{\mu\nu} = -\kappa \left( 1 + \frac{1}{R \sqrt{\Box_g}} \right) T_{\mu\nu}, \quad \text{for } I = C, D.
$$

(6.61)

This leads to

$$
h^{(I)}_{\mu\nu} \tilde{\phi}^{(I)} = h^{(I, \text{loc})}_{\mu\nu} + h^{(I, \text{nonloc})}_{\mu\nu},
$$

(6.62)

where

$$
\left( \sigma \Box_g + \frac{1}{R^2} \right) h^{(I, \text{loc})}_{\mu\nu} = -\frac{1}{R^2} \kappa T_{\mu\nu} = -G_N \frac{R^2}{R^2} T_{\mu\nu},
$$

(6.63)

Again for $\sigma = -1$, the first term $h^{(I, \text{loc})}_{\mu\nu}$ behaves like a massive graviton with mass $1/R^2$. We therefore assume $\sigma = -1$ from now on.

The induced gravity terms can have quite different implications depending on the scale of $R$. For small $R$, one recovers effectively the same physics as described by (6.50) and (6.53). However for large $R$, the quantum corrections changes the nature of $h^{(B, \text{loc})}_{\mu\nu}$, $h^{(B, \text{nonloc})}_{\mu\nu}$, $h^{(B, \text{grav})}_{\mu\nu}$, such that $h^{(B, \text{loc})}_{\mu\nu}$ behaves like a graviton. In any case, we note that the UV contribution is suppressed, as the mass term acts as a UV regulator. Hence we no longer need to assume a Gaussian source as above. In appendix E.2 we compute the Green’s functions for the PDEs in (6.60) and (6.63). With these results we find the following behavior for the two scaling regimes:

**R small.** In this regime, $R$ plays a similar role as $a$ in the previous section. $h^{(B, \text{grav})}_{\mu\nu}$ behaves like gravity for large distances $r \gg R$, while $h^{(B, \text{loc})}_{\mu\nu}$ (for $I = B, C, D$) is exponentially decaying at a scale $R^{-1}$ away from matter, see (E.9) for $r/R \to \infty$. However within uniformly

18 $R$ is some combination of the scale $A$, which could e.g. be the scale of SUSY breaking, and the other scales in the model such as $R$ and $L_{\text{NC}}$. 

40
distributed matter, it behaves the same as before, i.e., $\hat{h}^{(\text{loc})}_{\mu\nu} \sim -G_N R^2 T_{\mu\nu}$ for $I = B, C, D$. This is consistent with the behavior (6.57), but it is again too large and the situation is not improved.

The remaining mode $\hat{h}^{(\text{nonloc})}_{\mu\nu}$ exhibits essentially the same long-distance behavior as in (6.56) (see (E.14) for $r/R \to \infty$), with leading long-distance term $\hat{h}^{(\text{nonloc})}_{\mu\nu} \sim -\frac{1}{r}$ and higher-order terms suppressed by the scale factor $R$ (replacing $a$). Again due to the explicit $R$ in the source, this is too large to be physically acceptable for $\kappa = G_N R$.

$R$ large. Now consider the opposite limit where the induced gravity term $R$ is so large that the bare ‘mass’ term in (6.60) can be neglected. Then the modes $\hat{h}^{(\text{loc})}_{\mu\nu}$ for any $I = B, C, D$ can indeed play the role of a physical graviton, see (E.9) for $0 < r/R \ll 1$. In this case, $\hat{h}^{(\text{nonloc})}_{\mu\nu}$ for $I = C, D$ and $\hat{h}^{(B, grav)}_{\mu\nu}$ (being now misnomers) would lead to a sub-leading very-long-range interactions somewhat reminiscent of conformal gravity, as apparent from (E.14) and (E.11) for $0 < r/R \ll 1$. Hence this scaling regime might indeed lead to interesting gravitational physics, as long as the graviton mass is sufficiently small and $\sigma = -1$.

7. Symmetries and gauge transformations

7.1. Global symmetries

The 5-dimensional matrix model (5.2) has a global ISO(5) symmetry. Consider first the SO(5) symmetry. Since the $X^a$ and $\Theta^{ab}$ are tensor operators, the SO(5) action on them

$$X^a \rightarrow \Lambda^a_b X^b = U^\Lambda_{\mu\nu} X^\mu U^{\Lambda\nu}_\mu^{-1}$$  (7.1)

(and similarly for $\Theta^{ab}$) is equivalent to a gauge transformation. In this sense, the background is ‘covariant’. This implies that the vector modes $A_a$ (4.77) transform indeed as vectors,

$$A_a \rightarrow \Lambda^A_a U^\Lambda_{\mu\nu} A_{\mu} U^{\Lambda\nu}_\mu^{-1}.$$  (7.2)

It also implies that the corresponding rotational zero modes (Goldstone bosons) are unphysical. However there is a tower of non-trivial zero modes, which arises from the $(1, 2s) \subset (1, 0) \otimes (0, 2s)$ modes in (4.1), see [31]. For $s = 0$ these are the translations $X^a \rightarrow X^a + e^a$ corresponding to the 5 zero modes in $(1, 0) \otimes (0, 0)$ in (4.2). For $s \geq 1$ they are associated with higher spin symmetries.

7.2. Gauge transformations of functions on $S^4$

Now suppose $\phi \in C$. Gauge transformations act on $\phi$ as

$$\phi \rightarrow U^{-1} \phi U, \quad \delta_{\Lambda} \phi = \{ \phi, \Lambda \}.$$  (7.3)

This is simply the action of symplectomorphisms on $CP^3$. We can make this more transparent by viewing $\Lambda$ as $\mathfrak{iso}(5)$-valued function on $S^4$,

$$\Lambda = \Lambda^a_a(s) \Xi_a.$$  (7.4)
Hence, gauge transformations are local (i.e. $x$-dependent) versions of the $\mathfrak{h}_s$ algebra acting on fields on $S^4$. We discuss a few aspects of these transformations.

**$C^0$ gauge transformations.** Gauge transformations generated by $\Lambda(x) \in C^0$ act on functions on $S^4$ via

$$\delta_\Lambda x^a = \{x^a, \Lambda\} = \mathcal{Q}(\Lambda^{(1)}) = \theta^{\mu\nu}\partial_\mu\Lambda(x) \in C^1. \quad (7.5)$$

This is no longer a function, but a spin 1 field, and it has the typical form of gauge transformation in noncommutative Yang–Mills gauge fields. Hence these transformations are naturally interpreted as local $U(1)$ gauge transformations\(^{19}\) of noncommutative gauge theory on $S^4_N$. There is no constant counterpart of that symmetry.

**$C^1$ gauge transformations.** Now consider gauge transformations generated by $\Lambda = \Lambda_{\mu\nu}^{(1)}(x)\theta^{\mu\nu} \in C^1$:

$$\delta_\Lambda x^a = \{x^a, \Lambda^{(1)}\} = \mathcal{A}_{bc}^{(1)}(x)\{x^b, \theta^{bc}\} + \{x^a, \mathcal{A}_{bc}^{(1)}(x)\}\theta^{bc}
= \mathcal{Q}(\Lambda^{(1)}) = \theta^{\mu\nu}\partial_\mu\mathcal{A}^{(0)} + \theta^{\mu\nu}\mathcal{A}^{(3)} \in C^0 \oplus C^2 \quad (7.6)$$

using (4.38). This contains in particular the global $SO(5)$ symmetry. We recall from section 3 that $\Lambda \in C^1$ is characterized by a divergence-free vector field, which is explicit in the local representation (3.31)

$$\Lambda = \Lambda_{\mu\nu}^{(1)}(x)\theta^{\mu\nu} = v_\mu(x)P^\mu + \omega_\mu\nu(x)\mathcal{M}^{\mu\nu}. \quad (7.7)$$

Here $\omega^{\mu\nu}$ is the field strength of the divergence-free vector field $v_\mu$. Then (2.41) gives\(^{20}\)

$$\delta_\Lambda x^a = -\nabla^a + (\partial^\mu v_\mu + \partial^\nu \omega_\nu\sigma \mathcal{M}^{\sigma\nu})\theta^{\mu\nu} \in C^0 \oplus C^2 \quad (7.8)$$

noting that $\{P^\mu, x^a\} = \delta^a_\mu$ and $\{\mathcal{M}^{\mu\nu}, x^a\} = 0$ at the north pole. Hence the non-derivative contribution in $C^0$ is a vector field $v$, which can be interpreted as diffeomorphism on $S^4$. This in turn determines the $C^2$ contribution, which involves derivatives of $v$, and will be recognized as gauge transformation of spin 2 gauge fields (gravitons) under diffeomorphisms. Thus the $C^1$ gauge transformations amount to (volume-preserving) diffeomorphisms, which act both on the $S^4$ space and the gauge fields on it.

Note that the $C^1$ gauge transformations are only a subset of the local $SO(5)$ gauge transformations, given by the group of area-preserving diffeomorphisms. The full local $SO(5)$ arises only on the generalized $S^4_N$, which was one reason for introducing it in [18].

**$C^s$ gauge transformations.** Gauge transformations generated by $C^s$ act similarly as

$$\delta_\Lambda x^a = \{x^a, \Lambda^{(s)}\} = \mathcal{Q}(\Lambda^{(s)}) \in C^{s-1} \oplus C^{s+1} \quad (7.9)$$

This includes global transformations with generators in the higher spin algebra $\mathfrak{h}_s$ (3.26). As before, we expect that the non-derivative $C^{s-1}$ contribution can be given a geometrical meaning on $S^4$ corresponding to some global symmetry, while the $C^{s+1}$ terms provide the associated (derivative) gauge transformations of spin $s + 1$ gauge fields.

---

19 Since $[\theta^{\mu\nu}]_0 = 0$ here, these are not related to diffeomorphisms on $S^4$, in contrast to e.g. gauge theory on the Moyal–Weyl quantum plane $\mathbb{R}^4_\hbar$ [15].

20 This is strictly speaking valid only at the north pole $p$, since (2.41) holds only at $p$. However since $p$ is arbitrary, there is no loss of generality.
73. Gauge transformations of gauge fields on $S^4$

Now consider an $S^4$ background with a generic perturbation $y^a = x^a + \mathcal{A}^a$, as in (6.5). Gauge transformations act on the background as

$$\delta_x y^a := \{y^a, \Lambda\} = \{x^a + \mathcal{A}^a, \Lambda\} =: D^a \Lambda, \quad \Lambda \in \mathcal{C},$$

(7.10)

which can be absorbed by a gauge transformation of the fluctuation by defining

$$\delta_x \mathcal{A}^a = D^a \Lambda = Q(\Lambda) + \{\mathcal{A}^a, \Lambda\}.$$  

(7.11)

The inhomogeneous contribution

$$Q(\Lambda)^a = \theta^{ab} \partial_b \Lambda =: \theta^{ab} \delta_x \mathcal{A}_b$$

$$\delta_x \mathcal{A}_a = \partial_a \Lambda = \partial_a (\Lambda_{ab}(x) \Xi^b),$$

(7.12)

is a gauge transformation of the embedding function $x^a \in \mathcal{C}$ as discussed above. We have seen that this has the general form $Q(\Lambda) \sim \mathcal{A}^{(0)} + \mathcal{A}^{(3)}$, where the ‘non-derivative’ contribution $\mathcal{A}^{(0)}$ can be interpreted as geometric transformation of $S^4$, and the ‘derivative’ contribution $\mathcal{A}^{(3)}$ can be interpreted as pure gauge contribution to $\mathcal{A}$. E.g. for the spin 1 gauge transformations (7.6), the $\mathcal{A}^{(0)}$ contribution is the vector field $\mathcal{A}_\mu$ corresponding to a (volume-preserving) diffeomorphism, while the $\mathcal{A}^{(3)}$ contribution is combined with the spin 2 gauge field $\mathcal{A}^{(1)}$, see (4.87), and leads to the pure gauge part of a spin 2 gravitons $h_{\mu \nu} = \partial_\mu v_\nu + \partial_\nu v_\mu$, see (6.25). The $\mathcal{A}^{(0)}$ contribution is absorbed in the trace part of $\mathcal{A}$. A similar discussion should apply for the higher spin case, leading to the Fronsdal form of gauge transformations of rank $s$ tensor fields via the appropriate identifications\footnote{One can associate a symmetric rank $s$ tensor field on $S^4$ to $\mathcal{A}$ generalizing $h_{\mu \nu}$ which then transforms as in the Fronsdal form. An explicit elaboration is postponed to future work.}\footnote{Another approach to gravity with similar non-standard actions of the generators on fields was recently discussed in [34].}.

Now separate $\mathcal{A}^a = \theta^{ab} \mathcal{A}_b + x^a \phi$ into tangential gauge fields and a transversal scalar field as in (4.77), and consider the homogeneous contribution to its gauge transformation

$$\{\mathcal{A}^a, \Lambda\} = \theta^{ab} \{\mathcal{A}_b, \Lambda\} + \mathcal{A}_b \{\theta^{ab}, \Lambda\} + \{x^a, \Lambda\} \phi + x^a \{\phi, \Lambda\}.$$  

(7.13)

Re-grouping and combining with $Q(\Lambda)$, we can write the non-linear gauge transformation as

$$\delta_x \mathcal{A}^a = D^a \Lambda = \theta^{ab} D_b \Lambda + x^a \delta_x \phi + \mathcal{A}_b \delta_x \theta^{ab}$$

(7.14)

where

$$\delta_x (\cdot) = \{\cdot, \Lambda\}, \quad D_a = (1 + \phi) \partial_a + \{A_a, \cdot\}.$$  

(7.15)

We can organize the perturbations and the gauge parameter in terms of $\mathfrak{h}$ generators (7.4):

$$\Lambda = \Lambda_{ab}(x) \Xi^a, \quad \mathcal{A}_b = A_{b \beta}(x) \Xi^\beta, \quad \phi = \phi_\beta(x) \Xi^\beta.$$  

(7.16)

This suggests to view $\mathcal{A}_b$ as $\mathfrak{h}$-valued gauge field on $S^4$, with Yang–Mills-type gauge transformations $\delta \mathcal{A}_b = D_b \Lambda$,

$$\{A_{\mu \beta}, \Lambda\} = A_{\mu \beta}(x) \Lambda_{\beta}(\Xi^\beta, \Xi^\beta) + \Lambda_{\beta}(x) \Xi^\beta + \ldots.$$  

(7.17)

The first term has the usual form of a nonabelian gauge theory; however the remaining terms are non-standard. Similar non-standard terms arise\footnote{One can associate a symmetric rank $s$ tensor field on $S^4$ to $\mathcal{A}$ generalizing $h_{\mu \nu}$ which then transforms as in the Fronsdal form. An explicit elaboration is postponed to future work.} in (7.14). These originate from the action $\delta_x (\cdot)$ of a local $\mathfrak{h}$ transformation on $x^a$ and on $\theta^{ab}$, which is precisely what allowed us to understand the inhomogeneous contributions from $Q(\Lambda)$ discussed above. Hence the present model does not coincide with a standard Yang–Mills formulation of $\mathfrak{h}$ gravity (and with
Vasiliev theory, to our understanding). However, the above features are crucial in the matrix model realization of \( \mathfrak{h} \mathfrak{s} \) gauge theory.

**Field strength.** As in noncommutative gauge theory, the gauge-covariant field strength arise from commutators or Poisson brackets

\[
\{ y^a, y^b \} = \{ x^a + A^a, x^b + A^b \} = \theta^{ab} + F^{ab},
\]

which transforms in the adjoint of the gauge group. Dropping the radial fluctuations for simplicity, i.e. \( A^a x_a = 0 \), we obtain

\[
F^{ab} = \theta^{ac} \partial_c A^b - \theta^{bc} \partial_c A^a + \{ A^a, A^b \}
\]

(7.18)

and, by further employing that also \( A_a x_b = 0 \) holds, we find

\[
F^{ab} = \theta^{ab} \theta^{ac} \partial_c A^b - \theta^{bc} \partial_c A^a + \{ A^a, A^b \}
\]

(7.19)

By evaluating \( F^{ab} \) at a point on the 4-sphere we can shed some light on its structure and make contact to conventional noncommutative Yang–Mills theory:

\[
F_{\mu\nu} = \theta_{\mu\rho} \theta_{\nu\sigma} \left( F_{\rho\sigma} + \frac{4\theta}{L_{NC}^2} (\theta^{\rho\sigma} A_\gamma A^\gamma + \theta^{\gamma\rho} A_{\sigma} A_\gamma - \theta^{\sigma\gamma} A_{\rho} A_\gamma) \right)
\]

(7.20)

by means of \( A_{\mu} = P_{\mu}^\alpha A_\alpha = \theta^{\mu\sigma} \frac{4}{L_{NC}^2} \theta^{\sigma\rho} A_\rho \) and \( \theta^{\mu\nu} = \theta^{\mu\rho} P_{\nu}^\rho = \theta^{\mu\rho} \theta^{\rho\sigma} \frac{4}{L_{NC}^2} \theta^{\sigma\mu} \). We observe that the corrections to the ‘usual’ field strength \( F_{\rho\sigma} \) are suppressed by the factor \( 4\theta/L_{NC}^2 = r/R \), hence the set-up resembles a Yang–Mills field strength. However, despite the conventional appearance, some unusual terms arise in \( F_{\mu\nu} \), because the \( \mathfrak{so}(5) \) generators do act on the functions and derivatives. For example, \( \partial^\mu P^\nu = \theta^{\mu\nu} \) arises from \( P \)-valued gauge fields. These extra terms are responsible for the present mechanism for gravity, see [18].

In any case, the quantities \( \{ y^a, y^b \} \) or \( F^{ab} \) transform in the adjoint under gauge transformations, and are therefore natural building blocks for a higher spin gauge theory. The action (5.2) under consideration in the present paper is of that type. Of course other, more complicated terms are conceivable, such as \( \int \{ y^a, y^b \} \{ y^c, y^d \} \varepsilon^\varepsilon_{abcde} \) see [6].

### 8. Remarks and outlook

We briefly comment on some interesting aspects and open questions which we have to set aside for the moment, to keep the paper within bounds.

#### 8.1. The generalized fuzzy sphere

As remarked earlier, the shortcomings of the gravitons (6.23) on the basic fuzzy 4-sphere may be overcome by considering the generalized fuzzy 4-sphere \( S_4^\Lambda \) as background instead of \( S_4^V \). Following the geometric description presented in [18, 25], the extended bundle structure leads to additional generators \( P^\mu \) in the algebra functions such that new fluctuation modes arise. A particular promising mode is given by

\[ \theta^{\mu\nu} \]

23The fully non-linear case can probably be made more transparent if some of the fluctuations are absorbed in \( \theta^{\mu\nu} \). This will be discussed elsewhere.
\[ A_a = \phi_1 \ldots \phi_\text{c} x^1 \ldots x^n = h_{ab}(x) \eta^b. \]  
(8.1)

The new feature is that derivative contributions to \( h_{ab} \) are suppressed, in contrast to (6.16) and (6.17) for the basic 4-sphere. More specifically, there is no accompanying ‘spin connection’ term \( \omega_{\mu\nu} M^{\mu\nu} \) which would spoil its contribution. There is no such mode on \( S_N^4 \).

Moreover, there should be similar Goldstone bosons arising from the generalized background

\[ Y^a = \begin{pmatrix} X^a \\ T^a \end{pmatrix} \]  
(8.2)

found in [25], i.e. \( X^a \rightarrow X^a + \Lambda_{ab} T^b \). These are physical (in contrast to the \( SO(5) \) would-be Goldstone bosons), massless, and are closely related to the above gravitons.

### 8.2. Decoupling and interactions of higher spin fields

For the intents and purposes of this paper, we restricted the analysis of fluctuation modes to the quadratic order. Nevertheless, the higher order terms in the action will of course lead to interactions between the various higher spin modes, and should be considered in more detail.

We focused on the spin 2 modes in this paper, which couple to other fields via the energy-momentum tensor (6.11). This involves two derivatives such as in \( \{ A, \phi(x) \} \{ X, \phi(x) \} \), which is suppressed by some dimensionful scale parameter (identified with the Newton constant). It is crucial that there is indeed a natural UV scale \( L_{NC} \) on the fuzzy sphere which can serve this purpose; that seems to be a major advantage over Vasiliev’s higher spin theory.

Similarly, all spin \( s \) fluctuation modes can interact with scalar fields only via \( s \) derivatives, or with spin \( l \) fields via \( s-l \) derivatives. Therefore the interactions of all higher spin fields should be suppressed via appropriate powers of that UV scale parameter, see (2.18). In contrast, the scale of higher derivative interactions in Vasiliev theory is given by the cosmological constant, which is the only available scale there. However, here we have two scales at our disposal: the UV and the cosmological scale. It is therefore quite plausible that these higher spin fields decouple at low energies, as they should.

Nonetheless, we cannot make any definite statements about the higher spin fields at this stage. The framework presented has the intriguing feature of formulating a higher spin gauge theory in the presence of a UV scale \( L_{NC} \), which naturally defines energies below \( L_{NC} \) as ‘low energies’. However, it is not at all clear whether the higher spin fields are massive or massless, i.e. whether they are part of the propagating low energy degrees of freedom or not. We have exemplified this in the spin 2 sector, whose physical behavior is more intricate than naively expected. To address this question more systematically one might need to refine the current framework.

There are also non-derivative interactions among the higher spin fields, which arise from their structure as \( hs \)-valued nonabelian Yang–Mills theory, see (7.21). It would be desirable to gain some insights into their significance and their relation to Vasiliev’s formulation.

### 9. Conclusion

We have shown that the fuzzy 4-sphere \( S_N^4 \) and its semi-classical limit \( S^4 \) naturally carry an tower of higher spin fields, which is finite or infinite respectively. We have provided an explicit classification of the scalar and vector fluctuation modes in sections 3 and 4, respectively. The resulting kinematics is very similar to that of Vasiliev theory, including a structure which plays
the role of a local higher spin algebra $\mathfrak{hs}$, as discussed in section 2.1. However, there are some differences compared to Vasiliev theory. The most distinctive new feature is the presence of an intrinsic UV scale $L_{\text{NC}}$, which is the scale where the underlying noncommutativity becomes significant. This scale plays a crucial role throughout.

Moreover, it turns out that matrix models and their semi-classical limit as ‘Poisson models’ provide a natural formulation of higher spin gauge theory on $S^4$. In particular, the Yang–Mills matrix model action (5.2) provides an action for interacting higher spin gauge fields, which can be arranged in terms of a tangential 1-form and a scalar field taking values in $\mathfrak{hs}$, as in Vasiliev theory. However, the resulting dynamics appears to be different. This is found by explicitly diagonalizing the quadratic part (6.36) of the resulting action.

It is natural to expect that the spin 2 sector of a higher spin theory contains gravity. We have checked this explicitly using the classification of fluctuation modes, which allowed for the identification of the physical gravitons (6.23) together with their effective action (6.29) including matter coupling. We found that although one of the three graviton modes mediates linearized Einstein gravity, the remaining modes are dominant, and behave as auxiliary nonpropagating or short-range fields. This confirms the preliminary result in [18] that Yang–Mills matrix models on the basic fuzzy 4-sphere do not provide a relativistic version of gravity, at least at the classical level. However, induced gravity due to one-loop effects might reverse this conclusion, by transmuting the auxiliary spin 2 field into a realistic graviton, as discussed in section 6.3.1. In that case, the remaining modes become, for a suitable parameter range, subdominant long-range phenomena.

In order to judge the physical relevance of the higher spin theory formulated by the Poisson matrix model, one would have to perform a full fledged analysis of all higher spin modes analogously to our considerations of the spin 2 modes. Only this would allow conclusive statements about the on-shell degrees of freedom and their relation to the low energy region set by $L_{\text{NC}}$. We have to leave this to future research.

In any case, the present paper provides the necessary tools to address the more promising generalized fuzzy sphere $S^4_{\Lambda}$. Following the preliminary analysis in [18] and supported by the geometric results in [25], we expect that this background will lead to the linearized Einstein equations, due to extra modes equation (8.1). These should provide the required properties for physical gravity as sketched in section 8.1, along with an interesting non-abelian Yang–Mills gauge theory. We hope to report on this in the near future.

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Appendix A. Details

A.1. Explicit derivations

Equation (2.28) is seen as follows:
\begin{align}
\mathcal{I} \circ \mathcal{J}(\xi^a A_a) &= \xi^a \{\theta^{ab}, \theta_{bc} A^c\} \\
&= \xi^a \left(- 3 \theta \theta^{ab} A^c + \{\theta^{ab}, \theta_{bc} A^c\} - \theta^{ab} \{\theta_{bc}, A^c\} \right) \\
&= \xi^a \left(- 3 \theta \theta^{ab} A^c - \theta \theta^{ab} \{\theta_{bc}, A^c\}\right) - \mathcal{J} \circ \mathcal{I}(\xi^a A_a) \\
&= -3 \theta \mathcal{J}(\xi^a A_a) + \theta \xi^a \{\theta_{bc}, A^c\} - \mathcal{J} \circ \mathcal{I}(\mathcal{A}) \\
&= -3 \theta \mathcal{J}(\mathcal{A}) + \theta \xi^a \{\theta_{bc}, A^c\} + \theta \mathcal{Q}(\mathcal{N}(\mathcal{A})) - \mathcal{J} \circ \mathcal{I}(\mathcal{A}). \quad (A.1)
\end{align}

A.1.1 Details spin 2 fields. To derive the first identity in (3.22), consider

$$\int \{x^a, \{x^b, \phi^{(2)}_{ab}\}\} = (n + 2)\phi^{(2)}_{ab} - (n + 2)(n + 1)\phi^{(2)}_{ab}$$

Hence,

$$\int \{x^a, \{x^b, \phi^{(2)}_{ab}\}\} = \left(- (n + 3) - \frac{1}{3}(n + 3)(n + 1)\right)\theta \phi^{(2)}_{ab}$$

for any symmetric matrix \(k^{ab}\). Using (A.16). Thus,

$$\int \{x^a, \phi^{(2)}_{ab}\} = - \int \phi^{(2)}_{ab} \{\{x^a, \{x^b, \phi^{(2)}_{ab}\}\}\} = \frac{1}{3}(n + 3)(n + 4)\theta \phi^{(2)}_{ab}.$$

In order to compute the coefficient \(c_n\) in (3.21), we need to compute \(\{x^a, \{x^b, \phi^{(2)}_{ab}\}\}\). To start with

\begin{align}
\{x^a, \phi^{(2)}_{ab}\} &= \phi^{(2)}_{ab} \left( n x^{a_1} \ldots x^{a_{n-1}} g^{b_1 a_1} g^{d^1 a_2} \ldots g^{d^{n-1} a_n} x^{b_n} g^{b n} g^{d^0} + g^{b n} x^n g^{d^0} g^{b^0} g^{d^0} \right) \\
&= \begin{cases} 
2 \theta x^n \ldots x^n \left( x^{a_1} \ldots x^{a_n} g^{b n} g^{b^0} g^{d^0} \right), \\
\end{cases} \quad (A.5)
\end{align}

then we find

\begin{align}
\{x^a, \{x^b, \phi^{(2)}_{ab}\}\} &= (n - 1)\phi^{(2)}_{ab} \left( n x^{a_1} \ldots x^{a_{n-2}} g^{b_1 a_1} g^{d^1 a_2} \ldots g^{d^{n-2} a_{n-1}} g^{b_{n-1} a_n} \ldots g^{b n} g^{b^0} g^{d^0} \right) \\
&= - n \theta g^{b n} \phi^{(2)}_{ab} + n \theta g^{b n} \phi^{(2)}_{ab} x^{a_1} \ldots x^{a_{n-2}} g^{b_1 a_1} g^{d^1 a_2} \ldots g^{d^{n-2} a_{n-1}} g^{b_{n-1} a_n} \ldots g^{b n} g^{b^0} g^{d^0} \\
&+ 2n \theta \phi^{(2)}_{ab} \left( \phi^{(2)}_{ab} - \phi^{(2)}_{ab} \right) x^{a_1} \ldots x^{a_{n-2}} g^{b_1 a_1} g^{d^1 a_2} \ldots g^{d^{n-2} a_{n-1}} g^{b_{n-1} a_n} \ldots g^{b n} g^{b^0} g^{d^0} \\
&+ 2n \theta \phi^{(2)}_{ab} \left( \phi^{(2)}_{ab} - \phi^{(2)}_{ab} \right) x^{a_1} \ldots x^{a_{n-2}} \ldots x^{a_n} g^{b n} g^{b^0} g^{d^0} \\
&+ 2 \theta \phi^{(2)}_{ab} \left( \phi^{(2)}_{ab} + \phi^{(2)}_{ab} \right) x^{a_1} \ldots x^{a_{n-2}} \ldots x^{a_n} g^{b n} g^{b^0} g^{d^0} \right). \quad (A.6)
\end{align}
As consistence check we perform
\[ g_{ab}\{x^a, \{x^b, \phi^{(2)}\}\} = -\theta(n^2 + 7n + 4)\phi^{(2)}. \] (A.7)
which is the correct result. Finally, we employ the averaging expressions (3.4) and obtain
\[ \{x^a, \{x^b, \phi^{(2)}\}\}\_0 = \frac{2}{15}\theta^2\frac{(n + 3)(n + 4)(n + 5)}{n + 1}\phi^{(2)} \] (A.8)

A.1.2. Details spin s fields. In order to compute the action of \( I \) on the spin s fields one needs the following results, obtained using (A.26):

\[ Q(\phi^{(s)}) = \theta^n A^{(3)} + \theta^n (n + 2) A^{(0)} \] (A.9)
\[ \mathcal{N}(A^{(0)}) = 0 \quad \{x^a, A^{(0)}_a\} = -\frac{n + 1}{\theta^{n-1}}\phi^{(s)} \] (A.10)
\[ \mathcal{N}(A^{(2)}) = \frac{1}{\theta^n} \phi^{(s)} \quad \{x^a, A^{(2)}_a\} = 0. \] (A.11)

A.2. Young tableaux and tensor fields

We discuss some properties of Young diagrams and projectors. For a useful resource in this context we refer to [35, appendix E].

A.2.1. Spin 1. Let \( \phi^{(1)}_{a_1 \ldots a_{n_d}} \) have symmetry corresponding to the diagram \( \begin{array}{c} \text{E} \\ \text{E} \end{array} \). Then its total symmetrization in \( (a_1 \ldots a_{n_d}) \) vanishes. Since the symmetry in \( (a_1 \ldots a_{n_d}) \) is manifest, this means that
\[ 0 = P_s^{(n+2)} \phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} = \frac{1}{n + 2} \left( \phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} + \phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} + \sum_n \phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} \right) \]
\[ = \frac{1}{n + 2} \left( \phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} + \phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} + n\phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} \right). \]

Now contracting this with \( x^{a_1} \ldots x^{a_{n_d}} \xi^d \xi^e \) yields
\[ 0 = \frac{1}{n + 2} \left( \phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} \xi^d \xi^e + \phi^{(2)}_{a_1 \ldots a_{n_d} \sigma} \xi^d \xi^e + n\phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} \xi^d \xi^e \right)x^{a_1} \ldots x^{a_{n_d}} \]
\[ = \frac{1}{n + 2} \left( (n + 1)\phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} \xi^d \xi^e + \phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} \xi^d \xi^e \right)x^{a_1} \ldots x^{a_{n_d}}. \] (A.12)

Hence,
\[ (n + 1)\phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} x^{a_1} \ldots x^{a_{n_d}} \xi^d \xi^e + \phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} x^{a_1} \ldots x^{a_{n_d}} \xi^d \xi^e = 0. \] (A.13)

Similarly, we can deduce by contracting with \( x^{a_1} \ldots x^{a_{n_d}} \) that
\[ \phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} x^{a_1} \ldots x^{a_{n_d}} = \frac{1}{n} \left( \phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} + \phi^{(1)}_{a_1 \ldots a_{n_d} \sigma} \right)x^{a_1} \ldots x^{a_{n_d}}. \] (A.14)
A.2.2. Spin 2. Similarly, let $\phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}}$ have symmetries corresponding to the diagram \( \Box \Box \). Then its total symmetrization in $(a_1\ldots a_5)$ vanishes$^{24}$. Since the symmetry in $(a_1\ldots a_5)$ is manifest, this means that

$$0 = \tilde{p}^{(n+3)} S \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} = \frac{1}{n+3} \left( \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} + \phi^{(2)}_{a_1\ldots a_5}^{\text{cde bd}} + \phi^{(2)}_{a_1\ldots a_5}^{\text{bed ac}} + \sum_n \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} \right)$$

$$0 = \frac{1}{n+3} \left( \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} + \phi^{(2)}_{a_1\ldots a_5}^{\text{cde bd}} + \phi^{(2)}_{a_1\ldots a_5}^{\text{bed ac}} + n\phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} \right).$$

Now contracting this with $x^{a_1} \ldots x^{a_5} c^b \theta^{de}$ gives

$$0 = \left( \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} (n + 1) \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} c^b \theta^{de} \right) x^{a_1} \ldots x^{a_5}, \quad (A.15)$$

which is the spin 2 version of (A.13). Similarly, contracting this with $x^{a_1} \ldots x^{a_5} c^b \theta^{de}$ with some symmetric tensor $k^{ce}$ gives

$$0 = \left( \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} (n + 2) \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} c^b \theta^{de} \right) x^{a_1} \ldots x^{a_5}. \quad (A.16)$$

Finally, contracting with $\theta^{ba} \theta^{de}$ gives

$$0 = \left( \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} \theta^{bd} + \phi^{(2)}_{a_1\ldots a_5}^{\text{cde bd}} \theta^{de} + \phi^{(2)}_{a_1\ldots a_5}^{\text{bed ac}} \theta^{de} + n\phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} \theta^{de} \right) \theta^{ba}$$

$$0 = \left( \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} \theta^{bd} + \phi^{(2)}_{a_1\ldots a_5}^{\text{cde bd}} \theta^{de} + \sum_n \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} \theta^{de} \right) \theta^{ba} \theta^{de}. \quad (A.17)$$

Thus,

$$0 = \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} x^{a_1} \ldots x^{a_5} \theta^{bd} + \phi^{(2)}_{a_1\ldots a_5}^{\text{cde bd}} x^{a_1} \ldots x^{a_5} \theta^{de}$$

$$0 = \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} (x) \theta^{bd} + \phi^{(2)}_{a_1\ldots a_5}^{\text{cde bd}} (x) \theta^{de}. \quad (A.18)$$

This also implies

$$\phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} (x) \theta^{bd} \theta^{de} = \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} (x) \theta^{bd} \theta^{de} = -n \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} (x) \theta^{bd} \theta^{de} = -n \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} (x) \theta^{bd} \theta^{de}. \quad (A.19)$$

Similarly, one deduces the following identities:

$$\phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} x^{a_1} \ldots x^{a_5} c^b \theta^{de} = -\frac{1}{n+2} \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} x^{a_1} \ldots x^{a_5} c^b \theta^{de}, \quad (A.20)$$

$$\phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} x^{a_1} \ldots x^{a_5} c^b \theta^{de} = -\frac{1}{n+2} \left( \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} x^{a_1} \ldots x^{a_5} c^b \theta^{de} \right) x^{a_1} \ldots x^{a_5} c^b \theta^{de}$$

$$= \frac{2}{(n+2)(n+3)} \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} x^{a_1} \ldots x^{a_5} c^b \theta^{de}, \quad (A.21)$$

$$\{x^d, \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} x^{a_1} \ldots x^{a_5+1} \} = -\theta(n+2)(n+3) \phi^{(2)}_{a_1\ldots a_5}^{\text{bd cde}} x^{a_1} \ldots x^{a_5+1}. \quad (A.22)$$

$^{24}$Because that would correspond to a Young diagram with a row of length $n+3$ i.e. an irrep $(n+3, 0)$, which upon tensoring with $(1, 0)$ cannot give $(n+2, 4)$. 


A.2.3. Spin s. We can derive a spin \( s \) identity. Consider the symmetrization \( P_s^{p+1} \) of

\[
\begin{array}{cccccccc}
1 & 2 & \ldots & 1 & 2 & \ldots & n
\end{array}
\]

(A.23)
in \( a_1 \ldots a_nb_1 \ldots b_s; c_1 \ldots c_s \), which vanishes. Thus, we obtain

\[
0 = (x - 1)\phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \ldots + \phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \ldots
\]

\[
\sum_{j=1}^{n} \phi_{c_1 \ldots c_s} + \sum_{j=1}^{n} \phi_{c_2 \ldots c_s} + \ldots + \sum_{j=1}^{n} \phi_{c_1 \ldots c_s}
\]

(A.24)

Contracting (A.24) with \( \xi^{b_1}x^{a_1} \ldots x^{a_n}x^{c_1}M^{b_2c_2} \ldots M^{b_sc_s} \) yields

\[
0 = (n + s - 1 - (s - 1))\phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \ldots + \phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \ldots
\]

\[
\sum_{j=1}^{n} \phi_{c_1 \ldots c_s} + \sum_{j=1}^{n} \phi_{c_2 \ldots c_s} + \ldots + \sum_{j=1}^{n} \phi_{c_1 \ldots c_s}
\]

(A.25)

i.e. it yields the generalization of (A.13) and (A.15):

\[
0 = (n + 1)\phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \ldots + \phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \phi_{a_1 \ldots a_nb_1 \ldots b_s;c_1 \ldots c_s} + \ldots
\]

\[
\sum_{j=1}^{n} \phi_{c_1 \ldots c_s} + \sum_{j=1}^{n} \phi_{c_2 \ldots c_s} + \ldots + \sum_{j=1}^{n} \phi_{c_1 \ldots c_s}
\]

(A.26)

Appendix B. Inner product matrix

First, (4.44) together with (3.22) gives

\[
\int \mathcal{A}^{(0)}[\phi^{(2)}] \mathcal{A}^{(0)}[\phi^{(2)}] = \frac{1}{\theta^2(n + 2)^2} \int \{x^a, \phi^{(2)}_{ba}\} \{x^a, \phi^{(2)}_{ca}\} = \frac{(n + 3)(n + 4)}{3\theta(n + 2)^2} \int \phi^{(2)}_{ab} \phi^{(2)}_{ab}.
\]

Similarly, (4.63) allows to compute

\[
\int \mathcal{A}^{(3)}[\phi^{(2)}] \mathcal{A}^{(3)}[\phi^{(2)}] = \frac{1}{n \int} \left( \frac{1}{\theta^2} Q(\phi^{(2)}) - \frac{2n + 2}{n + 1} \mathcal{A}^{(0)} \right) \left( \frac{1}{\theta^2} Q(\phi^{(2)}) - \frac{2n + 2}{n + 1} \mathcal{A}^{(0)} \right)
\]

\[
= \frac{1}{n \int} \left( \phi^{(2)} - \frac{1}{\theta^2} \frac{4 n + 2}{n + 2} \mathcal{A}^{(0)} \right) \phi^{(2)} + \frac{4}{3 (n + 1)^2 \theta} \phi^{(2)}_{ab} \phi^{(2)}_{ab}
\]

\[
= \frac{2}{15} \frac{(n + 3)(n + 4)(n^2 + 8n + 21)}{n(n + 1)^2 (n + 2)} \int \phi^{(2)}_{ab} \phi^{(2)}_{ab},
\]

(B.1)
where we used
\[
\int Q(\phi^{(2)})A^{(0)} = \int \{x^a, \phi^{(2)}\}A_a^{(0)} = - \int \phi^{(2)}\{x^a, A_a^{(0)}\} = \frac{1}{\theta} (n + 1) \int \phi^{(2)}\phi^{(2)}
\]  
(B.2)

with (4.46), and
\[
\int Q(\phi^{(2)})Q(\phi^{(2)}) = \int \{x^a, \phi^{(2)}\}\{x^b, \phi^{(2)}\} = - \int \phi^{(2)}\square\phi^{(2)} = \theta((n + 2)(n + 5) - 6)\int \phi^{(2)}\phi^{(2)}.
\]  
(B.3)

Finally, separating \(A^{(2)}\) into normal and tangential parts, we have
\[
\int A^{(2)}[\phi]A^{(2)}[\phi] = \frac{1}{R^2} \int x^a A^a_{\phi} A^a_{\phi} + \frac{1}{\theta R^2} \int J\mathcal{A}^{(2)}[\phi]\mathcal{J}\mathcal{A}^{(2)}[\phi] = \frac{1}{R^2} \frac{1}{\theta^2} \int \phi + \frac{1}{\theta R^2} \int A^{(3)}[\phi]A^{(3)}[\phi].
\]  
(B.4)

Furthermore,
\[
n \int A^{(3)}_b[\phi]A^{(0)}_b[\phi] = \int \left(\frac{1}{\theta R^2} Q(\phi^{(2)}) - \frac{2n + 2}{n + 1} A^{(0)}_b A^{(0)}_b\right) = \frac{1}{\theta} (n + 1) \int \phi^{(2)}\phi^{(2)} - \frac{2n + 2}{n + 1} \int A^{(0)}_b A^{(0)}_b
\]
\[
\int A^{(3)}_b[\phi]A^{(0)}_b[\phi] = \frac{2}{15} \frac{(n + 4)(n + 3)}{(n + 1)(n + 2)} \frac{1}{\theta} \int \phi_{ab}\phi_{ab}.
\]  
(B.5)

Moreover, (4.51) gives
\[
\int A^{(R)}_b[\phi]A^{(2)}_b[\phi] = \frac{1}{\theta R^2} \int \phi^{(2)}\phi^{(2)}
\]  
(B.6)

and
\[
\int A^{(1)}_b[\phi]A^{(2)}_b[\phi] = \int J A^{(0)}_b[\phi]A^{(0)}_b[\phi] = - \int A^{(0)}_b[\phi]J A^{(2)}_b[\phi] = - \int A^{(0)}_b[\phi]A^{(3)}_b[\phi].
\]  
(B.7)

In addition,
\[
\int A^{(R)}_b[\phi]A^{(R)}_b[\phi] = \frac{1}{\theta R^2} \int R^2 \phi^{(2)}\phi^{(2)}
\]  
(B.8)

and
\[
\int A^{(1)}_b[\phi]A^{(1)}_b[\phi] = \theta R^2 \int P_T A^{(0)}_b A^{(0)}_b = \theta R^2 \int A^{(0)}_b[\phi]A^{(0)}_b[\phi].
\]  
(B.9)
Appendix C. Metric fluctuations

The full metric fluctuations $H_{ab}$ (6.15) for the modes $A^{(i)}$ are given by

$$H_{ab}[A^{(0)}] = (n+1)(\theta a^a \theta^b + \theta^b \theta^a) \phi_{a_1 \ldots a_d b}^{(2)} x^{a_1} \ldots x^{a_d} M^{de}$$

$$+ \phi_{a_1 \ldots a_b b}^{(2)} \theta^a_a x^{a_1} \ldots x^{a_d} x^b x^d - \phi_{a_1 \ldots a_b b}^{(2)} x^{a_1} \ldots x^{a_d} x^b x^d,$$

$$+ \phi_{a_1 \ldots a_b c} \theta^b_a x^{a_1} \ldots x^{a_d} x^b x^c - \phi_{a_1 \ldots a_b c}^{(2)} x^{a_1} \ldots x^{a_d} x^b x^c,$$  \hspace{1cm} (C.1)

$$H_{ab}[A^{(1)}] = (n+1) \theta R^2 \left( \phi_{a_1 \ldots a_{b'}} a_{b'} b_{b'} \theta^b + \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \theta^b \right) x^{a_1} \ldots x^{a_d} M^{de}$$

$$+ \theta R^2 \left( \phi_{a_1 \ldots a_{b'} b_{b'}} \theta^a + \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \theta^a \right) x^{a_1} \ldots x^{a_d} x^b x^d,$$

$$+ 2 \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \delta_{a_{b'}} x^{a_1} \ldots x^{a_d} x^b x^d,$$ \hspace{1cm} (C.2)

$$H_{ab}[A^{(2)}] = 2(n+1) \theta R^2 \left( \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \theta^a + \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \theta^a \right) x^{a_1} \ldots x^{a_d} M^{de}$$

$$+ \theta R^2 \left( \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \theta^a + \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \theta^a \right) x^{a_1} \ldots x^{a_d} x^b x^d,$$

$$+ 2 \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \delta_{a_{b'}} x^{a_1} \ldots x^{a_d} x^b x^d,$$ \hspace{1cm} (C.3)

$$H_{ab}[A^{(3)}] = (n+1) \theta R^2 \left( \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \theta^a + \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \theta^a \right) x^{a_1} \ldots x^{a_d} M^{de}$$

$$+ \theta R^2 \left( \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \theta^a + \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \theta^a \right) x^{a_1} \ldots x^{a_d} x^b x^d,$$

$$+ 2 \phi_{a_1 \ldots a_{b'} b_{b'}}^{(2)} \delta_{a_{b'}} x^{a_1} \ldots x^{a_d} x^b x^d,$$ \hspace{1cm} (C.4)

$$H_{ab}[A^{(R)}] = 2 \theta R^2 P_{ab}^{(2)} \phi_{a_{b'}}^{(2)}.$$ \hspace{1cm} (C.5)

Appendix D. Eigenvalues of $\mathcal{I}$ modes

According to [18, equation (2.42)], the eigenvalues of the Poisson Laplacian are given by

$$\Box(n - m, 2m) = \theta (-n(n + 3) + m(m + 1))$$ \hspace{1cm} (D.1)

and the vector mode Laplacian (2.35) is $D^2 = -\Box - 2\mathcal{I}$. The properties of the modes $B^i$ are summarized in table D1.

Appendix E. Green’s functions

E.1. Semi-classical

We consider the solutions to the equations (6.50) and (6.53) with a Gaussian source as inhomogeneity.
Table D1. The eigenvalues of the $I$-modes $B,C,D,E,F$ for any spin level $s$. (All $I$, $\square_s$, and $D^2$ eigenvalues are modulo $\theta$.) Note that the notation for the representations follows that of [18].

| Mode | Identification | $I$ eigenvalue | $\square_s$ eigenvalue | $D^2$ eigenvalue |
|------|----------------|----------------|------------------------|------------------|
| $B^{(s)}$ | $\tilde{n} = n + s$ | $-2$ | $-\tilde{n}(\tilde{n} + 3) + m(m + 1)$ | |
| | $(n, 2s)$ | $m = s$ | $-2$ | $-n^2 - n(2s + 3)$ | $n(n + 3) + 2s(n + 1) + 4$ |
| $C^{(s)}$ | $n = n + s + 1$ | $-\tilde{n} - 3$ | $-n^2 - n(3 + 5(2s) - 4(n + 3)(n + 2s + 4) s + 1)$ | |
| | $(n, 2s)$ | $m = s$ | $-n - s - 4$ | $n(n + 3) + 2s(n + 1) + 4$ | |
| $D^{(s)}$ | $\tilde{n} = n + s - 1$ | $\tilde{n}$ | $-n(\tilde{n} + 3) + m(m + 1)$ | |
| | $(n, 2s)$ | $m = s$ | $n + s + 1$ | $-n^2 - n(1 + 2\xi) + 2(n + 1)(n + 2\xi)$ | |
| $E^{(s)}$ | $\tilde{n} = m - 1$ | $m - 1$ | $-n(\tilde{n} + 3) + m(m + 1)$ | |
| | $(n, 2s)$ | $m = s - 1$ | $s - 2$ | $n^2 - n(2\xi + 3) - 4\xi$ | $n(n + 3) + 2s(n + 1) + 4$ |
| $F^{(s)}$ | $\tilde{n} = n + s$ | $-m - 2$ | $-n(\tilde{n} + 3) + m(m + 1)$ | |
| | $(n, 2s)$ | $m = s + 1$ | $-s - 3$ | $-n^2 - n(2\xi + 3) + 2(n + 3) + 2s(n + 1) + 4$ | |

E.1.1 Preliminaries. Define Fourier transforms and their inverse as (in 4d)

$$
\tilde{f}(\vec{k}) := \frac{1}{4\pi^2} \int d^4 x f(\vec{x}) e^{i\vec{k} \cdot \vec{x}}, \quad f(\vec{x}) := \frac{1}{4\pi^2} \int d^4 k \tilde{f}(\vec{k}) e^{-i\vec{k} \cdot \vec{x}}.
$$

(E.1)

We can then solve inhomogeneous differential equation with differential operator $D$ as usual

$$
D\psi(\vec{x}) = f(\vec{x}) \Leftrightarrow \tilde{D}\tilde{\psi}(\vec{k}) = \tilde{f}(\vec{k}) \Leftrightarrow \psi(\vec{x}) = \frac{1}{4\pi^2} \int d^4 k \tilde{D}^{-1}\tilde{f}(\vec{k}) e^{-i\vec{k} \cdot \vec{x}},
$$

(E.2)

where $\tilde{D}$ is the algebraic operator representing $D$. We consider a Gaussian source $f$ with

$$
f(\vec{x}) = \frac{1}{4\pi^2 a^4} e^{-\frac{x^2}{a^2}} \Rightarrow \tilde{f}(\vec{k}) = \frac{1}{4\pi^2} e^{-\frac{\vec{k}^2}{a^2}}
$$

(E.3)

such that $\int d^4 x f(\vec{x}) = 1$.

E.1.2. Computation. We compute the integral in 4d spherical coordinates: (i) radial coordinate $r$ and (ii) three angles $\psi_1, \psi_2 \in [0, \pi]$, and $\psi_3 \in [0, 2\pi]$. Then the volume element becomes $d^4 x = r^3 dr (\sin \psi_1)^2 \sin \psi_2 d\psi_2 d\psi_3$, and we can choose a direction such that $\vec{x} \cdot \vec{k} = kr \cos \psi_1$.

$$
\psi(\vec{x}) = \frac{1}{4\pi^2} \int d^4 k \tilde{D}^{-1}\tilde{f}(\vec{k}) e^{-i\vec{k} \cdot \vec{x}}
$$

$$
= \frac{1}{4\pi^2} \int k^3 dk (\sin \psi_1)^2 \sin \psi_2 d\psi_2 \sin \psi_3 \tilde{D}^{-1}\tilde{f}(\vec{k}) e^{-ikr \cos \psi_1}
$$

$$
= \frac{2}{(2\pi)^3} \int k^3 dk \tilde{D}^{-1} e^{-\frac{k^2}{a^2}} \int_0^1 d\xi \sqrt{1 - \xi^2} e^{-ikr \xi}
$$

$$
= \frac{1}{(2\pi)^2} \int_0^\infty dk k^3 \tilde{D}^{-1} J_1(kr) e^{-\frac{k^2}{a^2}},
$$

(E.4)

where $J_1(z)$ denotes the Bessel functions of first kind. We consider two examples

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\( \mathcal{D} \) corresponds to the 4d Laplacian, then \( \mathcal{D}^{-1} \sim \frac{1}{k^2} \). Then we obtain

\[
\psi(\vec{x}) = \frac{1}{(2\pi)^2} \int_0^\infty dk \frac{J_1(kr)}{r} e^{-\frac{kr}{r}} = \frac{1 - e^{-\frac{kr}{r}}}{4\pi^2 r^2} \tag{E.5}
\]

\( \mathcal{D} \) corresponds to some (first order) operator such that \( \mathcal{D}^{-1} \sim \frac{1}{k} \), then

\[
\psi(\vec{x}) = \frac{1}{(2\pi)^2} \int_0^\infty dk \frac{J_1(kr)}{r} k e^{-\frac{kr}{r}} = \frac{1}{8\sqrt{2\pi^3/2}} \int_0^\infty \frac{J_0(r^2/4a^2) - J_1(r^2/4a^2)}{r^3} e^{-r^2/4a^2} dr, \tag{E.6}
\]

where \( J_n(z) \) denotes the modified Bessel functions of first kind.

We can use the known asymptotic behavior, see for instance [36], and obtain the two regimes for \( \psi \) as follows:

\[
\psi(\vec{x}) = \begin{cases} 
  e^{-\frac{r}{2a}} \frac{2 - \frac{r^2}{4a^2}}{16\sqrt{2\pi^3/2}} & , 0 < r < 2a, \\
  \frac{1}{4\pi^2 r^2} + \frac{3r^2}{8\pi^2 r} + \frac{9r^4}{64\pi^2 r^2} + \ldots & , r \to \infty.
\end{cases} \tag{E.7}
\]

E.2. Induced gravity

The scale \( \bar{R} \) arising in induced gravity, see section 6.3.1, acts as regulator and one can readily compute the Green’s function by considering Delta-distribution sources.

E.2.1. Mode B. Considering the equations of motion (6.58) for the splitting (6.59), we derive the Green’s functions for (6.60).

- For the PDE for \( h^{(B,loc)}_{\mu\nu} \) we obtain

\[
G^{(B,loc)}(r, 0) = \frac{K_1(\frac{r}{\bar{R}})}{4\pi^2 r \bar{R}} \tag{E.8}
\]

where \( K_n(z) \) denotes the modified Bessel function of second kind. From the known asymptotic behavior [36] we arrive at

\[
G^{(B,loc)}(r, 0) = \frac{\frac{1}{4\pi^2 r^2} e^{-\frac{r}{\sqrt{2\pi} \bar{R}}}}{\frac{1}{4\pi^2 r^2} e^{-\frac{r}{\sqrt{2\pi} \bar{R}}} \left( 1 + \frac{3\bar{R}}{8r} - \frac{15\bar{R}^2}{128r^2} + \ldots \right)} , 0 < \frac{r}{\bar{R}} \ll \sqrt{2} \tag{E.9}
\]

- For the PDE for \( h^{(B,grav)}_{\mu\nu} \) we compute

\[
G^{(B,grav)}(r, 0) = \frac{1 - \frac{r}{\bar{R}} K_1(\frac{r}{\bar{R}})}{4\pi^2 r^2 \bar{R}}. \tag{E.10}
\]

Again, the known asymptotics [36] reveals
\[
G^{(B,\text{grav})}(r, 0) = \begin{cases} 
1 - 2\gamma_E \frac{2\ln(\frac{r}{\bar{R}})}{16\pi} + \frac{3 - 4\gamma_E + 4\ln(\frac{2}{\bar{R}}) \frac{r^2}{\bar{R}^2}}{(16\pi)^2} + \cdots & , 0 < \frac{r}{\bar{R}} \ll \sqrt{2} \\
\frac{1}{4\pi^2 \bar{R}^2} \left(1 - e^{-\frac{r}{\bar{R}}} \sqrt{\frac{2}{\bar{R}} \left(\frac{r}{\bar{R}} + \frac{3}{8} \frac{\bar{R}}{125 \bar{R}^2} + \cdots\right)}\right) & , \frac{r}{\bar{R}} \to \infty .
\end{cases}
\] (E.11)

where \(\gamma_E\) denotes the Euler–Mascheroni constant.

E.2.2. Mode C and D. Considering the equations of motion (6.61) for the splitting (6.62), we derive the Green’s functions for (6.63).

- The behavior of the \(h^{(I,\text{loc})}_{\mu\nu}\) part is identical to the previous one of \(h^{(B,\text{loc})}_{\mu\nu}\). Therefore,

\[
G^{(I,\text{loc})}(r, 0) = G^{(B,\text{loc})}(r, 0).
\] (E.12)

- All left to check is the Green’s function for \(h^{(I,\text{nonloc})}_{\mu\nu}\). We find

\[
G^{(I,\text{nonloc})}(r, 0) = \frac{1}{8\pi \bar{R}^2} \left( L_{-1} \left(\frac{r}{\bar{R}}\right) - I_1 \left(\frac{r}{\bar{R}}\right) \right).
\] (E.13)

where \(L_{n}(z)\) denotes the modified Struve function. Employing the tabulated expansion of [36], one can deduce the asymptotic behavior as follows:

\[
G^{(I,\text{nonloc})}(r, 0) = \begin{cases} 
\frac{1}{8\pi \bar{R}^2} \left(\frac{2}{\bar{R}^2} - \frac{r^2}{3\bar{R}^2} + \frac{2\bar{R}^2}{3\pi^2 \bar{R}^4} + \frac{2r^4}{3\bar{R}^4} + \cdots\right) & , 0 < \frac{r}{\bar{R}} \ll 1 , \\
\frac{1}{8\pi \bar{R}^2} \left(\frac{3r^2}{4\pi^2 \bar{R}^4} + \frac{4\bar{R}^2}{4\pi^2 \bar{R}^4} + \cdots\right) & , \frac{r}{\bar{R}} \to \infty.
\end{cases}
\] (E.14)

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