FRACTIONAL KOLMOGOROV OPERATOR AND DESINGULARIZING WEIGHTS

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Abstract. We establish sharp upper and lower bounds on the heat kernel of the fractional Laplace operator perturbed by Hardy-type drift by transferring it to appropriate weighted space with singular weight.

1. Introduction

The fractional Kolmogorov operator \((-\Delta)^{\frac{\alpha}{2}} + f \cdot \nabla\), \(1 < \alpha < 2\) with a (locally unbounded) vector field \(f : \mathbb{R}^d \to \mathbb{R}^d, \, d \geq 3\), plays important role in probability theory where it arises as the generator of symmetric \(\alpha\)-stable process with a drift (in contrast to diffusion processes, \(\alpha\)-stable process has long range interactions). It has been the subject of intensive study over the past two decades. There is now a well developed theory of this operator with \(f\) belonging to the corresponding Kato class. This class, in particular, contains the vector fields \(f\) with \(|f| \in L^p, \, p > \frac{d}{\alpha-1}\) and is, indeed, responsible for existence of the standard (local in time) two-sided bound on the heat kernel \(e^{-t\Lambda(x,y)}, \, \Lambda \supset (-\Delta)^{\frac{\alpha}{2}} + f \cdot \nabla\), in terms of \(e^{-t(-\Delta)^{\frac{\alpha}{2}}} (x,y)\), see [BJ].

The authors in [KSS] studied the fractional Kolmogorov operator \(\Lambda = (-\Delta)^{\frac{\alpha}{2}} + b \cdot \nabla\), \(b(x) = \kappa |x|^{-\alpha} x, \, 0 < \kappa < \kappa_0\), where \(\kappa_0\) is the borderline constant for existence of \(e^{-t\Lambda(x,y)} \geq 0\). The model vector field \(b\) lies outside of the scope of the Kato class, and exhibits critical behaviour both at \(x = 0\) and at infinity making the standard upper bound on \(e^{-t\Lambda(x,y)}\) in terms of \(e^{-t(-\Delta)^{\frac{\alpha}{2}}} (x,y)\) invalid. Instead, the two-sided bounds \(e^{-t\Lambda(x,y)} \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}} (x,y) \varphi(t)(y) \) \((y \neq 0)\) hold for an appropriate weight \(\varphi(t) \geq \frac{1}{2}\) unbounded at \(y = 0\) [KSS Theorem 3].

The present paper continues [KSS]. We study the heat kernel \(e^{-t\Lambda(x,y)}\) of the fractional Kolmogorov operator with the drift of opposite sign (“repulsion case”)

\[\Lambda = (-\Delta)^{\frac{\alpha}{2}} - b \cdot \nabla, \quad b(x) = \kappa |x|^{-\alpha} x, \quad 0 < \kappa < \infty.\] (1)
Although the standard (global) upper bound in terms of $e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x, y)$ holds true for $e^{-t\Lambda}(x, y)$ (Theorem 3 below), the singularity of $b$ at $x = 0$ makes it off the mark. Namely, in Theorem 4 and Theorem 5 below we establish sharp upper and lower bounds

$$e^{-t\Lambda}(x, y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x, y)\psi_t(y), \quad x, y \in \mathbb{R}^d, \quad t > 0,$$

(UL$B_w$)

where the continuous weight $0 \leq \psi_t(y) \leq 2$ vanishes at $y = 0$ as $|y|^\beta$, $\beta > 0$ (Theorem 2). (Here notation $a(z) \approx b(z)$ means that $c^{-1}b(z) \leq a(z) \leq cb(z)$ for some constant $c > 1$ and all admissible $z$.) The order of vanishing $\beta$ ($< \alpha$) depends explicitly on the value of the multiple $\kappa > 0$ and tends to $\alpha$ as $\kappa \uparrow \infty$.

The key step in proving the upper and lower bound (UL$B_w$) is the weighted Nash initial estimate

$$0 \leq e^{-t\Lambda}(x, y) \leq C t^{-\frac{d}{\alpha}}\psi_t(y), \quad x, y \in \mathbb{R}^d, \quad t > 0.$$  

(NIE$w$)

The proof of (NIE$w$) uses the method of desingularizing weights [MS0, MS1, MS2] based on ideas set forth by J. Nash [N]: it depends on the “desingularizing” $(L^1, L^1)$ bound on the weighted semigroup $\psi_1e^{-t\Lambda}\psi_t^{-1}$.

The operator (1.44) in the local case $\alpha = 2$ has been studied in [MeSS, MeSS2] by considering it in the space $L^2(\mathbb{R}^d, |x|^{\gamma}dx)$ for appropriate $\gamma$ where the operator becomes symmetric. This approach, however, does not work for $\alpha < 2$.

Recently, the authors in [CKSV], [JW] considered the fractional Schrödinger operator $H_{\kappa} = (-\Delta)^{\frac{\alpha}{2}} + V$, $V(x) = \kappa|x|^{-\alpha}, 0 < \alpha < 2, \kappa > 0$, and established, using different methods, sharp two-sided bounds

$$e^{-tH_{\kappa}}(x, y) \approx e^{-t(-\Delta)^{\frac{\alpha}{2}}}(x, y)\psi_t(x)\psi_t(y)$$

for appropriate weights $\psi_t(x)$ vanishing at $x = 0$. We apply some ideas from [JW] (in the proof of Theorem 4).

In contrast to the cited papers, this work deals with purely non-local and non-symmetric situation. This leads to new difficulties, and requires new ideas. Even the proof of the standard upper bound $e^{-t\Lambda}(x, y) \leq Ce^{-t(-\Delta)^{\frac{\alpha}{2}}}(x, y)$ (Theorem 3), as well as the construction of semigroups $e^{-t\Lambda}, e^{-t\Lambda^*}$ (Sections 8 and 9) become non-trivial. The same applies to the Sobolev regularity of $e^{-t\Lambda}f, f \in C_0^\infty$ established in Section 8.2. We consider these results, along with Theorem 4 and Theorem 5, as the main results of this article.

Below we apply the scheme of the proof of the upper and lower bounds in [KSS], although with comprehensive modifications in the method, both at the level of the abstract desingularization theorem (Theorem 1) and in the proofs of (NIE$w$), (UL$B_w$) and of the standard upper bound.

We note that the heat kernel of the operator $(-\Delta)^{\frac{\alpha}{2}} + f \cdot \nabla$ with $\text{div } f = 0$ was studied in [MM, MM2]. For properties of the Feller process determined by (1.44) see [KM].

Let us mention that the vector field $b(x) = \kappa|x|^{-\alpha}x$ exhibits critical behaviour even if we remove the singularity of $b$ at the origin. Namely, if we consider $\Lambda$ with $b$ bounded in $B(0, 1)$ but having slower decay at infinity, $b(x) = \kappa|x|^{-\alpha+\varepsilon}x, \varepsilon > 0$ for $|x| \geq 1$, then the global in time upper bound $e^{-t\Lambda}(x, y) \leq Ce^{-t(-\Delta)^{\frac{\alpha}{2}}}(x, y)$ of Theorem 3 would no longer be valid.

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2. Desingularization in abstract setting

We first prove a general desingularization theorem in abstract setting, that we will apply in the next section to the fractional Kolmogorov operator.

Let $X$ be a locally compact topological space, and $\mu$ a $\sigma$-finite Borel measure on $X$. Set $L^p = L^p(X, \mu)$, $p \in [1, \infty]$, a (complex) Banach space. We use the notation

$$\langle u, v \rangle := \int_X \overline{u} v d\mu, \quad \| \cdot \|_p \rightarrow q = \| \cdot \|_{L^p} \rightarrow_{L^q}.$$ 

Let $-\Lambda$ be the generator of a contraction $C_0$ semigroup $e^{-t\Lambda}$, $t > 0$, in $L^2$.

Assume that, for some constants $M \geq 1$, $c_S > 0$, $j > 1$, $c$,

$$\|e^{-t\Lambda}f\|_1 \leq M\|f\|_1, \quad t \geq 0, \quad f \in L^1 \cap L^2. \quad (B_{11})$$

**Sobolev embedding property:**

$$\text{Re}(\langle Au, u \rangle) \geq c_S \|u\|_{2j}^2, \quad u \in D(\Lambda). \quad (B_{12})$$

$$\|e^{-t\Lambda}\|_{2 \rightarrow \infty} \leq ct^{-\frac{j}{j-1}}, \quad t > 0, \quad j' = \frac{j}{j-1}. \quad (B_{13})$$

Assume also that there exists a family of real valued weights $\psi = \{\psi_s\}_{s>0}$ on $X$ such that, for all $s > 0$,

$$0 \leq \psi_s, \psi_s^{-1} \in L^1_{\text{loc}}(X - N, \mu), \quad \text{where } N \text{ is a closed null set}, \quad (B_{21})$$

and there exist constants $\theta \in [0, 1[, \theta \neq \theta(s), c_i \neq c_i(s)$ $(i = 2, 3)$ and a measurable set $\Omega^s \subset X$ such that

$$\psi_s(x)^{-\theta} \leq c_2 \text{ for all } x \in X - \Omega^s, \quad (B_{22})$$

$$\|\psi_s^{-\theta}\|_{L^{q'}(\Omega^s)} \leq c_3 s^{j'/q'}, \quad \text{where } q' = \frac{2}{1-\theta}. \quad (B_{23})$$
Theorem 1. In addition to (B11) – (B23) assume that there exists a constant \( c_1 \neq c_1(s) \) such that, for all \( \frac{s}{2} \leq t \leq s \),
\[
\| \psi_s e^{-t \Lambda} \psi_s^{-1} f \|_1 \leq c_1 \| f \|_1, \quad f \in L^1.
\]
Then there is a constant \( C \) such that, for all \( t > 0 \) and \( \mu \) a.e. \( x, y \in X \),
\[
| e^{-t \Lambda}(x, y) | \leq C t^{-j} \psi_t(y).
\]

Remark 1. In application of Theorem 1 to concrete operators, the main difficulty is in verification of the assumption (B3).

Proof of Theorem 1. Set \( \psi \equiv \psi_s \) and put \( L^2_\psi := L^2(X, \psi^2 d\mu) \). Define a unitary map \( \Psi : L^2_\psi \to L^2 \) by \( \Psi f = \psi f \). Set \( \Lambda_{\psi} = \Psi^{-1} \Lambda \Psi \) of domain \( D(\Lambda_{\psi}) = \Psi^{-1} D(\Lambda) \). Then
\[
 e^{-t \Lambda_{\psi}} = \Psi^{-1} e^{-t \Lambda} \Psi, \quad \| e^{-t \Lambda_{\psi}} \|_{2, \psi \to 2, \psi} = \| e^{-t \Lambda} \|_{2 \to 2}, \quad t \geq 0.
\]
Here and below the subscript \( \psi \) indicates that the corresponding quantities are related to the measure \( \psi^2 d\mu \).

Set \( u_t = e^{-t \Lambda_{\psi}} f, \ f \in L^2_\psi \cap L^1_\psi \). Applying (B12), and then the Hölder inequality, we have
\[
-\frac{1}{2} \frac{d}{dt} (u_t, u_t)_\psi = \text{Re} (\Lambda u_t, u_t)_\psi \\
= \text{Re} (\Lambda \psi u_t, \psi u_t) \\
\geq c_S \| \psi u_t \|_{2j}^2 \\
\geq c_S \| \psi u_t \|_{q}^{2(r-1)},
\]
where \( q = \frac{2}{1+\theta}(< 2) \) and \( r = \frac{(1+\theta)j-1}{2j} \).

Noticing that (B11) + (B12) implies the bound \( \| e^{-t \Lambda} \|_{1 \to 2} \leq c t^{-j} \) (for details, if needed, see Remark 2 below), we have by the interpolation inequality
\[
\| e^{-t \Lambda} \|_{1 \to q} \leq c_4 t^{-\frac{L'}{q'}}, \quad q' = \frac{q}{q-1}, \quad c_4 = M^{\frac{2}{q}-1} e^\frac{2}{q};
\]
also, by (B11) and interpolation, \( \| e^{-t \Lambda} \|_{q \to q} \leq M^{\frac{2}{q}-1} \). Therefore,
\[
\| \psi u_t \|_q = \| e^{-t \Lambda \psi} f \|_q = \| e^{-t \Lambda} |\psi|^{-\theta} |\psi|^{\frac{2}{q}} f \|_q \\
(\text{we are applying (B22), (B23)}) \\
\leq c_2 \| e^{-t \Lambda} \|_{q \to q} \| f \|_q + \| e^{-t \Lambda} \|_{1 \to q} \| \psi|^{-\theta} \|_{L^{q'}(\Omega')} \| f \|_{q, \psi} \\
\leq (c_2 M^{\frac{2}{q}-1} + c_3 c_4 (s/t)^{\frac{j}{q'}}) \| f \|_{q, \psi}.
\]

Thus, setting \( w = (u_t, u_t)_\psi \), we obtain
\[
\frac{d}{dt} w^{1-r} \geq 2(r-1) c_S (c_2 M^{\frac{2}{q}-1} + c_3 c_4 (s/t)^{\frac{j}{q'}})^{-2(r-1)} \| f \|_{q, \psi}^{-2(r-1)}.
\]
Integrating this differential inequality yields
\[
\| u_t \|_{2, \psi s} \leq C_1 t^{-j'} \left( \frac{1}{q} - \frac{1}{2} \right) \| f \|_{q, \psi s}, \quad s/2 \leq t \leq s.
\]
The last inequality and (B₃) rewritten in the form \(\|u_t\|_{1,\psi} \leq c_1\|f\|_{1,\psi}\) yield according to the Coulhon-Raynaud Extrapolation Theorem (Theorem 13 in Appendix B)
\[
\|u_t\|_{2,\psi_s} \leq C_2t^{-\frac{\nu}{2}}\|f\|_{1,\psi_s}, \quad s/2 \leq t \leq s,
\]
or
\[
\|e^{-t\Lambda}h\|_2 \leq C_2t^{-\frac{\nu}{2}}\|h\|_{1,\sqrt{\psi}}, \quad h \in L^2 \cap L^1_{\sqrt{\psi}}, \quad s/2 \leq t \leq s,
\]
where \(L^1_{\sqrt{\psi}} := L^1(X, \psi_s d\mu)\).
Since \(\|e^{-2t\Lambda}h\|_{\infty} \leq \|e^{-t\Lambda\}|_{2 \to \infty}\|e^{-t\Lambda}h\|_2\), we have, employing (B₁₃),
\[
\|e^{-2t\Lambda}h\|_{\infty} \leq cC_2t^{-\frac{\nu}{2}}\|h\|_{1,\sqrt{\psi}},
\]
and so the assertion of Theorem 1 follows.

\[\square\]

Remark 2. The standard argument yields: \((B₁₁) + (B₁₂) \Rightarrow \|e^{-t\Lambda}\|_{1 \to 2} \leq \hat{c}t^{-\frac{\nu}{2}}, t > 0\). Indeed, setting \(u_t := e^{-t\Lambda}f, f \in L^2 \cap L^1\), we have applying (B₁₂), Hölder’s inequality and (B₁₁)
\[
-\frac{1}{2} \frac{d}{dt}\|u_t\|_2^2 = \text{Re}\langle \Lambda u_t, u_t \rangle \\
\geq cs\|u_t\|_2^2 \\
\geq cs\|u_t\|_2^{2+\frac{\nu}{2}}\|u_t\|_1^{-\frac{\nu}{2}} \\
\geq csM^{-\frac{\nu}{2}}\|u_t\|_2^{2+\frac{\nu}{2}}\|f\|_1^{-\frac{\nu}{2}}.
\]
Thus, \(w := \|u_t\|_2^2\) satisfies \(\frac{d}{dt}w^{\frac{1}{2}} \geq C\|f\|_1^{-\frac{\nu}{2}}, C = \frac{2csM^{-\frac{\nu}{2}}}{\frac{\nu}{2}}\), so integrating this inequality we obtain
\[
\|e^{-t\Lambda}\|_{1 \to 2} \leq C^{-\frac{\nu}{2}}t^{-\frac{\nu}{2}}.
\]

It is now seen that \((B₁) \equiv (B₁₁) + (B₁₂) + (B₁₃)\) implies the bound \(e^{-t\Lambda}(x, y) \leq \hat{c}t^{-\frac{\nu}{2}}\).

3. Heat kernel \(e^{-t\Lambda}(x, y)\) for \(\Lambda = (-\Delta)^{\frac{\nu}{2}} - \kappa|x|^{-\alpha}x \cdot \nabla\), \(1 < \alpha < 2, \kappa > 0\)

We now state in detail our main result concerning the fractional Kolmogorov operator \((-\Delta)^{\frac{\nu}{2}} - \kappa|x|^{-\alpha}x \cdot \nabla\) in \(L^r\), \(1 \leq r < \infty\).

1. Let us outline the construction of an appropriate operator realization \(\Lambda^r\) of \((-\Delta)^{\frac{\nu}{2}} - \kappa|x|^{-\alpha}x \cdot \nabla\) in \(L^r\), \(1 \leq r < \infty\). Set
\[
b_\varepsilon(x) := \kappa|x|^{-\alpha}_\varepsilon x, \quad |x|_\varepsilon := \sqrt{|x|^2 + \varepsilon}, \quad \varepsilon > 0,
\]
define the approximating operators in \(L^r\)
\[
\Lambda^\varepsilon \equiv \Lambda^\varepsilon := (-\Delta)^{\frac{\nu}{2}} - b_\varepsilon \cdot \nabla, \quad D(\Lambda^\varepsilon) = W^{\alpha,r} := (1 + (-\Delta)^{\frac{\nu}{2}})^{-1}L^r, \quad 1 \leq r < \infty,
\]
and in \(C_u\) (the space of uniformly continuous bounded functions with standard sup-norm),
\[
\Lambda^\varepsilon \equiv \Lambda^\varepsilon_C := (-\Delta)^{\frac{\nu}{2}} - b_\varepsilon \cdot \nabla, \quad D(\Lambda^\varepsilon_C) = D((-\Delta)^{\frac{\nu}{2}})_{C_u}.
\]
The operator \(-\Lambda^\varepsilon\) is the generator of a holomorphic semigroup in \(L^r\) and in \(C_u\). For details, if needed, see Section 3 below.

It is well known that
\[
e^{-t\Lambda^\varepsilon}L^r_+ \subset L^r_+ \text{ and } e^{-t\Lambda^\varepsilon}C^+_u \subset C^+_u
\]
where \(L^r_+ := \{ f \in L^r \mid f \geq 0 \}, \) \(C^+_u := \{ f \in C_u \mid f \geq 0 \}.\) Also
\[
\|e^{-tA}f\|_\infty \leq \|f\|_\infty, \quad f \in L^r \cap L^\infty, \text{ or } f \in C_u.
\]

In Proposition 10 below we show that, for every \(r \in [1, \infty[,\) the limit
\[
s-L^r \lim_{\varepsilon \downarrow 0} e^{-t\varepsilon A_r} \quad \text{(loc. uniformly in } t \geq 0)\]
exists and determines a positivity preserving, contraction \(C_0\) semigroup in \(L^r\), say \(e^{-tA_r}\); the (minus)
generator \(A_r\) is an appropriate operator realization of the fractional Kolmogorov operator \((-\Delta)^{\frac{\alpha}{2}} - \kappa |x|^{-\alpha} \cdot \nabla\) in \(L^r\); there exists a constant \(c\) such that
\[
\|e^{-tA_r}\|_{r \rightarrow q} \leq ct^{\frac{\alpha}{\alpha + (\frac{1}{r} - \frac{1}{q})}}, \quad t > 0,
\]
for all \(1 \leq r < q \leq \infty;\) by construction, the semigroups \(e^{-tA_r}\) are consistent:
\[
e^{-tA_r} \upharpoonright L^r \cap L^p = e^{-tA_p} \upharpoonright L^r \cap L^p.
\]
Using Proposition 10 we obtain
\[
\langle A_r u, h \rangle = \langle u, (-\Delta)^{\frac{\alpha}{2}} h \rangle + \langle u, b \cdot \nabla h \rangle + \langle u, (\text{div } b) h \rangle, \quad u \in D(A_r), \quad h \in C^\infty_c
\]
(cf. KSS Prop. 9).

2. We now introduce the desingularizing weights for \(e^{-tA}.\) Define \(\beta\) by
\[
\beta \frac{d + \beta - 2}{d + \beta - \alpha} \gamma(d + \beta - 2) = \kappa,
\]
where
\[
\gamma(\alpha) := \frac{2\alpha \pi \frac{d}{2} \Gamma(\frac{d}{2})}{\Gamma\left(\frac{d}{2} - \frac{\alpha}{2}\right)}.
\]
Direct calculations show that \(\beta \in ]0, \alpha[\) exists (see Figure 1), and that \(|x|^\beta\) is a Lyapunov’s function of the formal adjoint operator \(A^* = (-\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b,\) i.e. \(A^* |x|^{-\beta} = 0.\)
Set \( \psi(x) \equiv \psi_s(x) := \eta(s^{-\frac{1}{\alpha}} |x|) \), where \( \eta \) is given by

\[
\eta(t) = \begin{cases} 
  t^\beta, & 0 < t < 1, \\
  \beta t(2 - \frac{1}{2}) + 1 - \frac{3}{2}\beta, & 1 \leq t \leq 2, \\
  1 + \frac{\beta}{2}, & t \geq 2.
\end{cases}
\]

Applying Theorem 1 to the operator \( \Lambda_r \) and the weights \( \psi_s \), we obtain

**Theorem 2.** \( e^{-t\Lambda_r} \) is an integral operator for each \( t > 0 \) with integral kernel \( e^{-t\Lambda(x,y)} \geq 0 \). There exists a constant \( c_{N,w} \) such that the weighted Nash initial estimate

\[
e^{-t\Lambda(x,y)} \leq c_{N,w} t^{-\frac{\alpha}{2}} \psi_t(y).
\]

is valid for all \( x, y \in \mathbb{R}^d \) and \( t > 0 \).

The next step is to deduce the following global in time “standard” upper bound on \( e^{-t\Lambda(x,y)} \):

**Theorem 3.** (i) There is a constant \( C_1 \) such that, for all \( t > 0, \ x, y \in \mathbb{R}^d \),

\[
e^{-t\Lambda(x,y)} \leq C_1 e^{-t(-\Delta)^{\frac{\alpha}{2}}} (x, y).
\]

(ii) Moreover, for a given \( \delta \in ]0, 1[ \), there is a constant \( D = D_\delta > 0 \) such that

\[
e^{-t\Lambda(x,y)} \leq (1 + \delta)e^{-t(-\Delta)^{\frac{\alpha}{2}}} (x, y), \quad |x| > Dt^\frac{1}{\alpha}, \ y \in \mathbb{R}^d.
\]

Theorem 2 and Theorem 3 are the key tools which allow us to establish the upper bound on \( e^{-t\Lambda(x,y)} \):

**Theorem 4.** There is a constant \( C \) such that, for all \( t > 0, \ x, y \in \mathbb{R}^d \),

\[
e^{-t\Lambda(x,y)} \leq Ce^{-t(-\Delta)^{\frac{\alpha}{2}}} (x, y)\psi_t(y).
\]

Using Theorem 4, we prove the lower bound on \( e^{-t\Lambda(x,y)} \):

**Theorem 5.** There is a constant \( \bar{C} > 0 \) such that, for all \( t > 0, \ x, y \in \mathbb{R}^d \),

\[
e^{-t\Lambda(x,y)} \geq \bar{C} e^{-t(-\Delta)^{\frac{\alpha}{2}}} (x, y)\psi_t(y).
\]

### 4. Proof of Theorem 2 The weighted Nash initial estimate

The proof follows by applying Theorem 1 to \( e^{-t\Lambda_r} \).

The conditions \((B_{11})\) and \((B_{13})\) (with \( j' = \frac{d}{\alpha} \)) are satisfied by Proposition 10. Let us prove \((B_{12})\). By Proposition 8 (\( \Lambda^\varepsilon \equiv \Lambda^\varepsilon_2 \)),

\[
\text{Re}(\Lambda^\varepsilon(1 + \Lambda^\varepsilon)\overline{g}, (1 + \Lambda^\varepsilon)^{-1}g) \geq c_S\| (1 + \Lambda^\varepsilon)^{-1}g \|_{2j}^2, \quad g \in L^2, \quad j = \frac{d}{d - \alpha}, \quad c_S \neq c_S(\varepsilon),
\]

i.e.

\[
\text{Re}(g - (1 + \Lambda^\varepsilon)^{-1}g, (1 + \Lambda^\varepsilon)^{-1}g) \geq c_S\| (1 + \Lambda^\varepsilon)^{-1}g \|_{2j}^2.
\]

Using the convergence \((1 + \Lambda^\varepsilon)^{-1} \xrightarrow{\varepsilon \downarrow 0} (1 + \Lambda)^{-1}\) in \( L^2 \) as \( \varepsilon \downarrow 0 \) (Proposition 10), we pass to the limit \( \varepsilon \downarrow 0 \) in the last inequality to obtain \( \text{Re}(\Lambda(1 + \Lambda)^{-1}g, (1 + \Lambda)^{-1}g) \geq c_S\| (1 + \Lambda)^{-1}g \|_{2j}^2 \) for all \( g \in L^2 \), and so \((B_{12})\) is proven.
The condition \((B_{21})\) is evident from the definition of the weights \(\psi_s\). It is easily seen that \((B_{22}), (B_{23})\) hold with \(\Omega^s = B(0, s^{\frac{2}{d}})\) and \(\theta = \frac{(2-\alpha)d}{(2-\alpha)d + 8\beta}\). It remains to prove the desingularizing \((L^1, L^1)\) bound \((B_3)\), which presents the main difficulty.

**Proof of \((B_3)\).** We modify the proof of the analogous \((L^1, L^1)\) bound in [KSS] (see also Remark 6 below). We will appeal to the Lumer-Phillips Theorem applied to specially constructed \(C_0\) semigroups in \(L^1\), corresponding to operators with smooth coefficients and smooth weights, which approximate \(\psi_se^{-t\Lambda}\psi_s^{-1}\).

Recall that \(b_\varepsilon(x) := \kappa |x|^{-\alpha}x, \ |x|_\varepsilon := \sqrt{|x|^2 + \varepsilon}, \ \varepsilon > 0, \)

\[
\Lambda^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon \cdot \nabla, \ \ D(\Lambda^\varepsilon) = \mathcal{W}^0,1 := (1 + (-\Delta)^{\frac{\alpha}{2}})^{-1}L^1, \\
(\Lambda^\varepsilon)^* = (-\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b_\varepsilon, \ \ D(\Lambda^\varepsilon)^* = \mathcal{W}^0,1.
\]

By the Hille Perturbation Theorem, for each \(\varepsilon > 0\), we have \(e^{-t\Lambda^\varepsilon}, e^{-t(\Lambda^\varepsilon)^*}\) can be viewed as \(C_0\) semigroups in \(L^1\) and \(C_u\) (see Sections 8 and 9).

Define approximating weights

\[
\phi_{n,\varepsilon} := n^{-1} + e^{-\frac{(\Lambda^\varepsilon)^*}{\varepsilon}} \psi, \ \ \psi = \psi_s.
\]

**Remark 3.** This choice of the regularization of \(\psi\) is dictated by the method: \(e^{-\frac{(\Lambda^\varepsilon)^*}{\varepsilon}}\) will be needed below to control the auxiliary potential \(U_\varepsilon\). See also Remark 5 below.

In \(L^1\) define operators

\[
Q = \phi_{n,\varepsilon} \Lambda^\varepsilon \phi_{n,\varepsilon}^{-1}, \ \ D(Q) = \phi_{n,\varepsilon} D(\Lambda^\varepsilon),
\]

where \(\phi_{n,\varepsilon} D(\Lambda^\varepsilon) := \{\phi_{n,\varepsilon} u \mid u \in D(\Lambda^\varepsilon)\}\),

\[
P_{\varepsilon,n} = \phi_{n,\varepsilon} e^{-t\Lambda^\varepsilon} \phi_{n,\varepsilon}^{-1}.
\]

Since \(\phi_{n,\varepsilon}, \phi_{n,\varepsilon}^{-1} \in L^\infty\), these operators are well defined. In particular, \(P_{\varepsilon,n}\) are bounded \(C_0\) semigroups in \(L^1\), say \(P_{\varepsilon,n} = e^{-tG}\).

Set

\[
M := \phi_{n,\varepsilon} (1 + (-\Delta)^{\frac{\alpha}{2}})^{-1} [L^1 \cap C_u] \\
= \phi_{n,\varepsilon} (\lambda_{\varepsilon} + \Lambda^\varepsilon)^{-1} [L^1 \cap C_u], \ \ 0 < \lambda_{\varepsilon} \in \rho(-\Lambda^\varepsilon).
\]

Clearly, \(M\) is a dense subspace of \(L^1\), \(M \subset D(Q)\) and \(M \subset D(G)\). Moreover, \(Q \upharpoonright M \subset G\). Indeed, for \(f = \phi_{n,\varepsilon} u \in M\),

\[
Gf = sL^1 \lim_{t \downarrow 0} t^{-1}(1 - e^{-tG})f = \phi_{n,\varepsilon} sL^1 \lim_{t \downarrow 0} t^{-1}(1 - e^{-t\Lambda^\varepsilon})u = \phi_{n,\varepsilon} \Lambda^\varepsilon u = Qf.
\]

Thus \(Q \upharpoonright M\) is closable and \(\tilde{Q} := (Q \upharpoonright M)^{\text{clos}} \subset G\).

**Proposition 1.** The range \(R(\lambda_{\varepsilon} + \tilde{Q})\) is dense in \(L^1\).

**Proof of Proposition 7.** If \(\langle (\lambda_{\varepsilon} + \tilde{Q})h, v \rangle = 0\) for all \(h \in D(\tilde{Q})\) and some \(v \in L^\infty\), \(\|v\|_{L^\infty} = 1\), then taking \(h \in M\) we would have \(\langle (\lambda_{\varepsilon} + \tilde{Q}) \phi_{n,\varepsilon} (\lambda_{\varepsilon} + \Lambda^\varepsilon)^{-1} g, v \rangle = 0\), \(g \in L^1 \cap C_u\), or \(\langle \phi_{n,\varepsilon} g, v \rangle = 0\).

Choosing \(g = e^{\frac{\lambda_{\varepsilon}}{\varepsilon}} (\chi_m v)\), where \(\chi_m \in C_c^\infty\) with \(\chi_m(x) = 1\) when \(x \in B(0, m)\), we would have \(\lim_{\varepsilon \uparrow \infty} \langle \phi_{n,\varepsilon} g, v \rangle = \langle \phi_n \chi_m, |v|^2 \rangle = 0\), and so \(v = 0\). Thus, \(R(\lambda_{\varepsilon} + \tilde{Q})\) is dense in \(L^1\). □
Proposition 2. There are constants $\hat{c} > 0$ and $\varepsilon_n > 0$ such that, for every $n$ and all $0 < \varepsilon \leq \varepsilon_n$,

$$\lambda + \hat{Q}$$

is accretive whenever $\lambda \geq \hat{c}s^{-1} + n^{-1}$.

Proof of Proposition 2. Recall that both $e^{-t\Lambda^\varepsilon}$, $e^{-t(\Lambda^\varepsilon)^*}$ are holomorphic in $L^1$ and $C_u$ due to Hille’s Perturbation Theorem. We have

$$\psi = \psi_1 + \psi(u), \quad 0 \leq \psi_1 \in D((-\Delta)^{\frac{\alpha}{2}}), \quad 0 \leq \psi(u) \in D((-\Delta)^{\frac{\alpha}{2}} C_u).$$

For instance,

$$\psi(u) := 1 + \frac{\beta}{2}, \quad \psi_1 := \psi - 1 - \frac{\beta}{2} \quad \text{(so, sprt $\psi_1 \subset B(0, 2s^\frac{1}{2})$).}$$

In $B(0, s^\frac{1}{2})$, the weight $\psi$ coincides with $\tilde{\psi}(x) \equiv \tilde{\psi}_s(x) := s^{-\frac{\beta}{2}}|x|^{\beta}$, so $\psi_1 \in D((-\Delta)_{1})$. Thus, $\psi(1) \in D((-\Delta)^{\frac{\alpha}{2}})$ (see, e.g. [Ka] Ch.V, sect.3.11]). Therefore,

$$(\Lambda^\varepsilon)^*\psi \quad = \quad (\Lambda^\varepsilon)^*_{L^1} \psi_1 + (\Lambda^\varepsilon)^*_{C_u} \psi(u)$$

is well defined and belongs to $L^1 + C_u = \{w + v \mid w \in L^1, v \in C_u\}$.

We verify that $\text{Re}(\langle \lambda + \hat{Q} \rangle f, \frac{f}{|f|}) \geq 0$ for all $f \in D(\hat{Q})$. For $f = \phi_{n, \varepsilon} u \in M$, we have

$$\langle Qf, \frac{f}{|f|} \rangle = \langle \phi_{n, \varepsilon} \Lambda^\varepsilon u, \frac{f}{|f|} \rangle = \lim_{t \to 0} t^{-1} \langle \phi_{n, \varepsilon}(1 - e^{-t\Lambda^\varepsilon})u, \frac{f}{|f|} \rangle,$$

$$\text{Re}\langle Qf, \frac{f}{|f|} \rangle \geq \lim_{t \to 0} t^{-1} \langle (1 - e^{-t\Lambda^\varepsilon})|u|, \phi_{n, \varepsilon} \rangle$$

$$= \lim_{t \to 0} t^{-1} \langle (1 - e^{-t\Lambda^\varepsilon})|u|, n^{-1} \rangle + \lim_{t \to 0} t^{-1} \langle (1 - e^{-t(\Lambda^\varepsilon)^*})e^{-\frac{\Lambda^\varepsilon}{\pi}}|u|, \psi \rangle$$

$$= \lim_{t \to 0} t^{-1} \langle |u|, (1 - e^{-t(\Lambda^\varepsilon)^*})n^{-1} \rangle + \lim_{t \to 0} t^{-1} \langle e^{-\frac{\Lambda^\varepsilon}{\pi}}|u|, (1 - e^{-t(\Lambda^\varepsilon)^*})^* \psi \rangle$$

$$= \langle |u|, (\Lambda^\varepsilon)^*n^{-1} \rangle + \langle e^{-\frac{\Lambda^\varepsilon}{\pi}}|u|, (\Lambda^\varepsilon)^* \psi \rangle,$$

where the first term is positive since $(\Lambda^\varepsilon)^*n^{-1} = n^{-1}\text{div } b_\varepsilon = n^{-1}(d|x|^{-\alpha} - \alpha|x|^{-\alpha-2}|x|^2) \geq n^{-1}(d - \alpha)|x|^{-\alpha} \geq 0$. Thus,

$$\text{Re}\langle Qf, \frac{f}{|f|} \rangle \geq \langle e^{-\frac{\Lambda^\varepsilon}{\pi}}|u|, (\Lambda^\varepsilon)^* \psi \rangle,$$

so it remains to bound $J := \langle e^{-\frac{\Lambda^\varepsilon}{\pi}}|u|, (\Lambda^\varepsilon)^* \psi \rangle$ from below. For that, we estimate from below

$$(\Lambda^\varepsilon)^* \psi = (\Delta)^{\frac{\alpha}{2}} \psi + \text{div}(b_\varepsilon \psi).$$

Claim 1. $(-\Delta)^{\frac{\alpha}{2}} \psi \geq -\beta(d + \beta - 2)\frac{\gamma(d + \beta - 2)}{\gamma(d + \beta - \alpha)}|x|^{-\alpha} \tilde{\psi}.$

Proof of Claim 1. All identities are in the sense of distributions:

$$(-\Delta)^{\frac{\alpha}{2}} \psi = -I_{2-\alpha} \Delta \psi$$

$$= -I_{2-\alpha} \Delta \tilde{\psi} - I_{2-\alpha} \Delta(\psi - \tilde{\psi}),$$

where $I_\nu = (-\Delta)^{-\frac{\nu}{2}}$ is the Riesz potential, and we evaluate the first term

$$-I_{2-\alpha} \Delta \tilde{\psi} = -s^{-\frac{\beta}{2}} \beta(d + \beta - 2)I_{2-\alpha}|x|^{\beta-2}$$

$$= -s^{-\frac{\beta}{2}} \beta(d + \beta - 2)\frac{\gamma(d + \beta - 2)}{\gamma(d + \beta - \alpha)}|x|^{\beta-\alpha},$$
while the second term is positive and can be omitted: $-I_{2-\alpha}\Delta(\psi - \tilde{\psi}) \geq 0$ (see Remark [4] below for detailed calculation). The proof of Claim [1] is completed.

Claim 2. $\text{div} (b_x \psi) \geq \text{div} (b \tilde{\psi}) - U_{\tilde{\psi}} \psi - cs^{-1}\psi$ for a constant $\tilde{c} \neq \tilde{c}(\varepsilon,n)$, where $U_{\tilde{\psi}}(x) := \kappa(d + \beta - \alpha)(|x|^{-\alpha} - |x|_{\varepsilon}^{-\alpha}) > 0$.

Proof. We represent

$$\text{div} (b_x \psi) = \text{div} (b \tilde{\psi}) + \text{div} (b_x \psi) - \text{div} (b \tilde{\psi})$$

and estimate the difference $\text{div} (b_x \psi) - \text{div} (b \tilde{\psi})$:

$$\text{div} (b_x \psi) - \text{div} (b \tilde{\psi}) = \text{div} [b(\psi - \tilde{\psi})] + \text{div} [(b_x - b)\psi] = h_1 + \text{div} [(b_x - b)\psi],$$

where $h_1 \in C_{\infty}$ (continuous functions vanishing at infinity), $h_1 = 0$ in $B(0, s^{\frac{1}{2}})$. In turn,

$$\text{div} [(b_x - b)\psi] = (b_x - b) \cdot \nabla \psi + (\text{div} b_x - \text{div} b)\psi$$

$$\geq \kappa(|x|^{-\alpha} - |x|_{\varepsilon}^{-\alpha})x \cdot \nabla \tilde{\psi} + h_2 + \kappa [d|x|^{-\alpha} - \alpha |x|_{\varepsilon}^{-\alpha - 2}|x|^2 - (d - \alpha)|x|^{-\alpha}] \psi$$

(where $h_3 := \kappa(|x|^{-\alpha} - |x|_{\varepsilon}^{-\alpha})x \cdot \nabla (\psi - \tilde{\psi}) \in C_{\infty}$, $h_2 = 0$ in $B(0, s^{\frac{1}{2}})$)

$$\geq \kappa(|x|^{-\alpha} - |x|_{\varepsilon}^{-\alpha})\beta \tilde{\psi} + h_2 + \kappa [d|x|_{\varepsilon}^{-\alpha} - \alpha |x|_{\varepsilon}^{-\alpha - 2}|x|^2 - (d - \alpha)|x|^{-\alpha}] \psi$$

$$\geq \kappa(|x|^{-\alpha} - |x|_{\varepsilon}^{-\alpha})\beta \tilde{\psi} + h_2 + \kappa(d - \alpha)(|x|_{\varepsilon}^{-\alpha} - |x|^{-\alpha}) \psi.$$  

Thus,

$$\text{div} (b_x \psi) \geq \text{div} (b \tilde{\psi}) + \kappa(d + \beta - \alpha)(|x|_{\varepsilon}^{-\alpha} - |x|^{-\alpha}) \tilde{\psi} + h_1 + h_2 + h_3,$$

where $h_3 := \kappa(d - \alpha)(|x|_{\varepsilon}^{-\alpha} - |x|^{-\alpha})(\psi - \tilde{\psi}) \in C_{\infty}$, $h_3 = 0$ in $B(0, s^{\frac{1}{2}})$.

A straightforward calculation shows that $h_i \geq -c_i s^{-1}$ with $c_i \neq c_i(\varepsilon,n)$, $i = 1, 2, 3$ (we have used that $h_i = 0$ in $B(0, s^{\frac{1}{2}})$). The assertion of Claim 2 follows.

Now, we combine Claim [1] and Claim [2]. In view of the choice of $\beta$, $-\beta(d + \beta - 2)\frac{\gamma(d + \beta - 2)}{\gamma(d + \beta - \alpha)} |x|^{-\alpha} \tilde{\psi} + \text{div} (b \tilde{\psi}) = 0$ (that is, formally, $\Lambda^* \tilde{\psi} = 0$), and so

$$(\Lambda^*)^\ast \psi \geq -U_{\tilde{\psi}} \psi - \tilde{c}s^{-1} \psi.$$

It follows that

$$J \equiv \langle e^{-\frac{\Delta^*}{n}} |u|, (\Lambda^*)^\ast \psi \rangle \geq -\tilde{c}s^{-1} \langle e^{-\frac{\Delta^*}{n}} |u|, \psi \rangle - \langle e^{-\frac{\Delta^*}{n}} |u|, U_{\tilde{\psi}} \rangle$$

$$\geq -\tilde{c}s^{-1} (|u|, e^{-\frac{(\Lambda^*)^\ast}{n}} \psi) - \langle e^{-\frac{\Delta^*}{n}} |u|, U_{\tilde{\psi}} \rangle$$

$$\geq -\tilde{c}s^{-1} (|u|, n^{-1} + e^{-\frac{(\Lambda^*)^\ast}{n}} \psi) - \langle e^{-\frac{\Delta^*}{n}} |u|, U_{\tilde{\psi}} \rangle$$

(recall that $|u| = \phi^{-1}_n |f|$ and $\phi_n, \varepsilon = n^{-1} + e^{-\frac{(\Lambda^*)^\ast}{n}} \psi$)

$$= -\tilde{c}s^{-1} ||f||_1 - (|u|, e^{-\frac{(\Lambda^*)^\ast}{n}} (U_{\tilde{\psi}})).$$
Now, for every $n \geq 1$, we have
\[
\|e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(U_\varepsilon \tilde{\psi})\|_\infty \leq \|e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(1_{B^c(0,R)}U_\varepsilon \tilde{\psi})\|_\infty + \|e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(1_{B(0,R)}U_\varepsilon \tilde{\psi})\|_\infty
\]
(we are using that $e^{-t(\Lambda^\varepsilon)^*}$ is a $L^\infty$ contraction and ultra-contraction, see Proposition 11)
\[
\leq \|1_{B^c(0,R)}U_\varepsilon \tilde{\psi}\|_\infty + c_N n^\frac{d}{n} \|1_{B(0,R)}U_\varepsilon \tilde{\psi}\|_1
\]
(we fix $R = R_n$ such that $\|1_{B^c(0,R)}U_\varepsilon \tilde{\psi}\|_\infty \leq 2^{-1}n^{-2}$
and choose $\varepsilon_n > 0$ such that for all $\varepsilon \leq \varepsilon_n$ $\|1_{B(0,R)}U_\varepsilon \tilde{\psi}\|_1 \leq 2^{-1}n^{-2}(c_N n^\frac{d}{n})^{-1}$
\[
\leq n^{-2}.
\]
Therefore, since $\phi_{n,\varepsilon} \geq n^{-1}$, we have for every $n$ and all $\varepsilon \leq \varepsilon_n$ $\|\phi_{n,\varepsilon}^{-1}e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(U_\varepsilon \tilde{\psi})\|_\infty \leq n^{-1}$ and so
\[
\langle |u|, e^{-\frac{(\Lambda^\varepsilon)^*}{n}}(U_\varepsilon \tilde{\psi}) \rangle \leq n^{-1}\|f\|_1. \text{ Thus,}
\]
\[
J \geq -\left(\hat{c}s^{-1} + n^{-1}\right)\|f\|_1.
\]
Returning to (3), one can easily see that the latter yields the assertion of Proposition 2.

\[\square\]

**Remark 4.** Let us show that $-\Delta(\psi - \tilde{\psi}) \geq 0$. Without loss of generality, $s = 1$. The inequality is evidently true on $\{0 < |x| \leq 1\} \cup \{|x| \geq 2\}$. Now, let $1 < |x| < 2$. Then
\[
\Delta(\tilde{\psi} - \psi) = \beta(d + \beta - 2)|x|^{\beta - 2} - \eta''(|x|)|x|^{-2} - \eta'(|x|)(d - 1)|x|^{-1}
\]
\[
= \beta(d + \beta - 2)|x|^{\beta - 2} + \beta|x|^{-2} - \beta(2 - |x|)(d - 1)|x|^{-1}
\]
\[
= \beta|x|^{-2}(d + \beta - 2)|x|^{\beta + 1 - (d - 1)(2 - |x|)|x|)
\]
\[
\geq \beta|x|^{-2}(d + \beta - 2) + 1 - (d - 1) \geq 0.
\]

[\square]

The fact that $\tilde{Q}$ is closed together with Proposition 1 and Proposition 2 imply $R(\lambda_\varepsilon + \tilde{Q}) = L^1$ (Appendix C). Then, by the Lumer-Phillips Theorem, $\lambda + \tilde{Q}$ is the (minus) generator of a contraction semigroup, and $\tilde{Q} = Q$ due to $\tilde{Q} \subset G$. Thus, it follows that, for all $n$ and all $\varepsilon \leq \varepsilon_n$
\[
\|e^{-t\tilde{Q}}\|_{1 \rightarrow 1} \equiv \|\phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}\phi_{n,\varepsilon}^{-1}\|_{1 \rightarrow 1} \leq e^{\omega t}, \quad \omega = \hat{c}s^{-1} + n^{-1}. \quad \text{(\star)}
\]

To obtain (B3), it remains to pass to the limit in (\star): first in $\varepsilon \downarrow 0$ and then in $n \rightarrow \infty$. It suffices to prove (B3) on positive functions. By (\star),
\[
\|\phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}f\|_1 \leq e^{\omega t}\|f\|_1, \quad 0 \leq f \in L^1,
\]
or taking $f = \phi_{n,\varepsilon}h$, $0 \leq h \in L^1$,
\[
\|\phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}h\|_1 \leq e^{\omega t}\|\phi_{n,\varepsilon}h\|_1.
\]

Using Proposition 10, we have
\[
\|\phi_{n,\varepsilon}e^{-t\Lambda^\varepsilon}h\|_1 = \langle n^{-1}e^{-t\Lambda^\varepsilon}h \rangle + \langle \psi, e^{--(t+\frac{1}{n})\Lambda^\varepsilon}h \rangle \rightarrow \langle n^{-1}e^{-t\Lambda^\varepsilon}h \rangle + \langle \psi, e^{--(t+\frac{1}{n})\Lambda^\varepsilon}h \rangle \quad \text{as } \varepsilon \downarrow 0,
\]
and
\[
\|\phi_{n,\varepsilon}h\|_1 = n^{-1}\langle h \rangle + \langle \psi, e^{-\frac{\Lambda^\varepsilon}{n}h} \rangle \rightarrow n^{-1}\langle h \rangle + \langle \psi, e^{-\frac{\Lambda}{n}h} \rangle \quad \text{as } \varepsilon \downarrow 0.
\]
Thus,
\[
\langle n^{-1}e^{-tA}h \rangle + \langle \psi, e^{-(t+\frac{1}{2})A}h \rangle \leq e^{\omega t}(n^{-1}\langle h \rangle + \langle \psi, e^{-\frac{1}{2}A}h \rangle).
\]
Taking \(n \to \infty\), we obtain \(\langle \psi e^{-tA}h \rangle \leq e^{\epsilon x^{-1}t}\langle \psi h \rangle\). (B3) now follows.

The proof of Theorem 2 is completed. \(\square\)

**Remark 5.** (On the choice of the regularization \(\phi_{n, \epsilon}\) of the weight \(\psi\)). In [KSS], we construct the regularization of the weight in the same way as above, although there the factor \(e^{-\frac{1}{2}(\Lambda^*)^*}\) serves a different purpose (in [KSS] the drift term \(b \cdot \nabla\) has the opposite sign, and so the corresponding weight is unbounded). (As a by-product, this allows us to consider \((-x, y)\) for all \(\langle\rangle\).

**Remark 6.** In the proof of the analogous \((L^1, L^1)\) bound in [KSS] proof of Theorem 2], where we consider the vector field \(b\) of the opposite sign, we first pass to the limit in \(n \to \infty\), and then in \(\epsilon \downarrow 0\). In the proof of Theorem 2 above this order is naturally reversed.

As a consequence of the \((L^1, L^1)\) bound \((B3)\), we obtain

**Corollary 1.** \(\langle e^{-tA}(-, x)\psi_t(\cdot) \rangle \leq c_1\psi_t(x)\) for all \(x \in \mathbb{R}^d, x \neq 0, t > 0\).

As a consequence of Corollary 1 and \((NIE_w)\), we obtain

**Corollary 2.** \(\langle e^{-tA}(-, x) \rangle \leq C_2\psi_t(x)\) for all \(x \in \mathbb{R}^d, x \neq 0, t > 0\).

**Proof.** We have
\[
\langle e^{-tA}(-, x) \rangle \leq \langle 1_{B(0,t^{\frac{1}{2}})} \rangle e^{-tA}(-, x) \rangle + \langle 1_{B^c(0,t^{\frac{1}{2}})} \rangle e^{-tA}(-, x) \rangle \psi_t(\cdot) \rangle \leq I_1 + I_2.
\]
By \((NIE_w)\), \(I_1 \leq c\psi_t(x)\), and by Corollary 1 \(I_2 \leq c''\psi_t(x)\), for appropriate constants \(c', c'' < \infty\). Set \(C_2 := c' + c''\). \(\square\)

5. **Proof of Theorem 3** The standard upper bounds

(i) For brevity, put \(A := (-\Delta)^{\frac{1}{2}}\). Recall that
\[
k_0^{-1}t((-x, y) \leq e^{-tA}([x - y]^{-d-\alpha} \wedge t^{-d+\alpha}) \leq k_0t([x - y]^{-d-\alpha} \wedge t^{-d+\alpha})
\]
for all \(x, y \in \mathbb{R}^d, x \neq y, t > 0\), for a constant \(k_0 = k_0(d, \alpha) > 1\).

In view of Proposition 10, it suffices to prove the a priori bound
\[
e^{-tA}(-, x) \leq C_1e^{-tA}(-, x), \quad x, y \in \mathbb{R}^d, \quad t > 0, \quad C_1 \neq C_1(\epsilon).
\]

By duality, it suffices to prove
\[
e^{-tA^*}(-, x) \leq C_1e^{-tA}(-, x), \quad x, y \in \mathbb{R}^d, \quad t > 0, \quad C_1 \neq C_1(\epsilon).
\]

**Step 1:** For every \(D > 1\) and all \(t > 0\), \(|x| \leq Dt^{\frac{1}{2}}, |y| \leq Dt^{\frac{1}{2}}\) the following bound
\[
e^{-tA^*}(-, x) \leq k_0c_N(2D)^{d+\alpha}e^{-tA}(-, x) \leq k_0c_N(2D)^{d+\alpha}e^{-tA}(-, x) \leq k_0c_N(2D)^{d+\alpha}e^{-tA}(-, x) \text{ for all } x, y \in \mathbb{R}^d, t > 0.
\]
is valid.

In fact, we will prove

Lemma 6. Let \( t > 0 \) and \( D > 1 \). Then

(i) \( e^{-t\langle \Lambda^\varepsilon \rangle^*}(x, y) \leq k_0c_N(2D)^{d+\alpha}e^{-tA}(x, y), \quad |x| \leqDt^{\frac{1}{\alpha}}, \ |y| \leq Dt^{\frac{1}{\alpha}}. \)

(ii) \( e^{-t\Lambda^\varepsilon}(x, y) \leq k_0c_{N,w}(1+D)^{d+\alpha}e^{-tA}(x, y)\psi_t(x), \quad |x| \leq t^{\frac{1}{\alpha}}, \ |y| \leq Dt^{\frac{1}{\alpha}}. \)

Proof. (i) Note that \( (|x| \leq Dt^{\frac{1}{\alpha}}, \ |y| \leq Dt^{\frac{1}{\alpha}}) \Rightarrow t^{-\frac{2}{\alpha}} \leq (2D)^{d+\alpha}t|x-y|^{-d-\alpha} \). The latter means that \( t^{-\frac{2}{\alpha}} \leq k_0(2D)^{d+\alpha}e^{-tA}(x, y) \) is valid. By \( (NIE_w) \) (Theorem 12), the Nash initial estimate

\[
e^{-t\langle \Lambda^\varepsilon \rangle^*}(x, y) \leq c_Nt^{-\frac{d}{\alpha}}, \ x, y \in \mathbb{R}^d, \ t > 0 \tag{NIE}\]

is proved. Therefore,

\[
e^{-t\langle \Lambda^\varepsilon \rangle^*}(x, y) \leq c_Nt^{-\frac{d}{\alpha}} \leq k_0c_N(2D)^{d+\alpha}e^{-tA}(x, y).
\]

(ii) Clearly, \( (|x| \leq Dt^{\frac{1}{\alpha}}, \ |y| \leq t^{\frac{1}{\alpha}}) \Rightarrow t^{-\frac{2}{\alpha}} \leq (1+D)^{d+\alpha}t|x-y|^{-d-\alpha} \), and so the inequality \( t^{-\frac{2}{\alpha}} \leq k_0(1+D)^{d+\alpha}e^{-tA}(x, y) \) is valid. By \( (NIE_w) \) (Theorem 2), \( e^{-t\Lambda^\varepsilon}(x, y) \leq c_{N,w}t^{-\frac{d}{\alpha}}\psi_t(x) \) for all \( t > 0, \ x, y \in \mathbb{R}^d \). Therefore,

\[
e^{-t\Lambda^\varepsilon}(x, y) \leq k_0c_{N,w}(1+D)^{d+\alpha}e^{-tA}(x, y)\psi_t(x).
\]

\( \square \)

In what follows, we will need the following estimates.

Lemma 7. Set \( E^t(x, y) = t(|x-y|^{-d-\alpha} \land t^{-\frac{d+\alpha+1}{\alpha}}), \ E^t f(x) := \langle E^t(x, \cdot), f(\cdot) \rangle, \ t > 0. \)

Then there exist constants \( k_i \ (i = 1, 2, 3) \) such that for all \( 0 < t < \infty, \ x, y \in \mathbb{R}^d \)

(i) \( |\nabla(xe^{-tA}(x, y))| \leq k_1E^t(x, y); \)

(ii) \[ \int_0^t e^{-(t-\tau)A}(x, \cdot)E^\tau(\cdot, \cdot)d\tau \leq k_2t^{-\frac{\alpha-1}{\alpha}}e^{-tA}(x, y); \]

(iii) \[ \int_0^t (E^{t-\tau}(x, \cdot)E^\tau(\cdot, \cdot))d\tau \leq k_3t^{-\frac{\alpha-1}{\alpha}}E^t(x, y). \]

Proof. For the proof of (i), (ii) see e.g. \([10]\). Essentially the same argument yields (iii), see e.g. \([26]\) sect. 5 for details. \( \square \)

Step 2: Fix \( \delta \in ]0, 2^{-1}[. \ Set C_g := \kappa k_1(2k_2 + k_3), \ R := (C_g\delta^{-1})^{\frac{1}{\alpha-1}} \) and \( m = 1 + 2k_0k_1. \)

If \( D \geq Rm, \) then the following bound

\[
e^{-t\langle \Lambda^\varepsilon \rangle^*}(x, y) \leq (1+\delta)e^{-tA}(x, y), \ x \in \mathbb{R}^d, \ |y| > Dt^{\frac{1}{\alpha}}, \ t > 0 \tag{4}\]

is valid.

We use the Duhamel formula

\[
e^{-t\langle \Lambda^\varepsilon \rangle^*} = e^{-tA} + \int_0^t e^{-(t-\tau)A}(B^{t}_{\varepsilon,R} + B^{t,c}_{\varepsilon,R})e^{-(t-\tau)A}d\tau
\]

\[ = e^{-tA} + K^t_R + K^{t,c}_R, \quad R := (C_g\delta^{-1})^{\frac{1}{\alpha-1}}, \tag{5}\]

where

\[ B^{t}_{\varepsilon,R} := \mathbf{1}_{B(0, R t^{\frac{1}{\alpha}})}B_\varepsilon, \quad B^{t,c}_{\varepsilon,R} := \mathbf{1}_{B(0, R t^{\frac{1}{\alpha}})}B_\varepsilon, \quad B_\varepsilon := -b_\varepsilon \cdot \nabla - W_\varepsilon, \]
where \( W_\varepsilon(x) := \kappa(d|x|_\varepsilon^{-\alpha} - \alpha|x|_\varepsilon^{-\alpha-2}|x|^2) \).

Set
\[
M^t_R(x, y) := (d - \alpha)\kappa \int_0^t \left( e^{-\tau(\Lambda^\varepsilon)^*}(x, \cdot) \mathbf{1}_{B(0, R^t \pi)}(\cdot) \right) \cdot |\varepsilon^{-\alpha} e^{-(t-\tau)A}(\cdot, y))d\tau.
\]

**Claim 3.** For every \( D \geq Rm \) and all \( |y| > Dt^{\frac{1}{\alpha}} \), \( x \in \mathbb{R}^d \), we have
\[
K^t_R(x, y) \leq -\frac{1}{2} M^t_R(x, y).
\]

**Proof of Claim 3.** Using Lemma \( \ref{lem:interpolation} \)(i), we obtain
\[
K^t_R(x, y) \equiv \int_0^t \left( e^{-\tau(\Lambda^\varepsilon)^*}(x, \cdot) B^t\varepsilon, R(\cdot)e^{-(t-\tau)A}(\cdot, y))d\tau
\]
\[
\leq k_1 \int_0^t \left( e^{-\tau(\Lambda^\varepsilon)^*}(x, \cdot) \mathbf{1}_{B(0, R^t \pi)}(\cdot) |\varepsilon^{-\alpha} e^{-(t-\tau)A}(\cdot, y))d\tau
\]
\[
- \int_0^t \left( e^{-\tau(\Lambda^\varepsilon)^*}(x, \cdot) \mathbf{1}_{B(0, R^t \pi)}(\cdot) W_\varepsilon(\cdot)e^{-(t-\tau)A}(\cdot, y))d\tau =: I_1 + I_2,
\]
where \( |b_\varepsilon(x)| = \kappa|x|_\varepsilon^{-\alpha}|x| \).

Using \( E^{t-\tau}(z, y) \leq k_0 e^{-(t-\tau)A}(z, y)|z - y|^{-1} \), we obtain
\[
I_1 \leq k_0 k_1 \int_0^t \left( e^{-\tau(\Lambda^\varepsilon)^*}(x, \cdot) \mathbf{1}_{B(0, R^t \pi)}(\cdot) |b_\varepsilon(\cdot)| |e^{-(t-\tau)A}(\cdot, y)| \cdot |y|^{-1} d\tau
\]
\[
(\text{we are using } \mathbf{1}_{B(0, R^t \pi)}(\cdot) |b_\varepsilon(\cdot)| \cdot |y|^{-1} \leq \mathbf{1}_{B(0, R^t \pi)}(\cdot) R(D - R)^{-1} \kappa |\varepsilon^{-\alpha})
\]
\[
\leq k_0 k_1 R(D - R)^{-1} \kappa \int_0^t \left( e^{-\tau(\Lambda^\varepsilon)^*}(x, \cdot) \mathbf{1}_{B(0, R^t \pi)}(\cdot) |\varepsilon^{-\alpha} e^{-(t-\tau)A}(\cdot, y))d\tau
\]
\[
= k_0 k_1 R(D - R)^{-1} (d - \alpha)^{-1} M^t_R(x, y).
\]

We now compare the RHS of the last estimate with \( I_2 \). Since \( W_\varepsilon(\cdot) \geq \kappa(d - \alpha) |\varepsilon^{-\alpha} \), we have
\[
K^t_R(x, y) \leq (k_0 k_1 R(D - R)^{-1} (d - \alpha)^{-1} - 1) M^t_R(x, y).
\]

Since \( k_0 k_1 R(D - R)^{-1} \leq \frac{k_0 k_1}{m - 1} \leq \frac{1}{2} \) and \( d - \alpha > 1 \) by our assumptions, we end the proof of Claim 3.

**Claim 4.** For every \( D \geq Rm \) and all \( |y| > Dt^{\frac{1}{\alpha}} \), \( x \in \mathbb{R}^d \), we have
\[
K^t_R(x, y) \leq \delta(M^t_R(x, y) + e^{-\tau A}(x, y)).
\]

**Proof of Claim 4.** Recall that
\[
K^t_{R}^{t,c}(x, y) \equiv \int_0^t \left( e^{-\tau(\Lambda^\varepsilon)^*}(x, \cdot) B^t\varepsilon, R(\cdot)e^{-(t-\tau)A}(\cdot, y))d\tau,
\]
where \( B^t\varepsilon, R = \mathbf{1}_{B(0, R^t \pi)} (-b_\varepsilon \cdot \nabla - W_\varepsilon) \). Thus, discarding in \( K^t_{R}^{t,c} \) the term containing \(-W_\varepsilon\) and using Lemma \( \ref{lem:interpolation} \)(i), we obtain
\[
K^t_{R}^{t,c}(x, y) \leq k_1 R^{1-\alpha} \int_0^t \left( e^{-\tau(\Lambda^\varepsilon)^*}(x, \cdot) E^{t-\tau}(\cdot, y))d\tau.
\]

We will have to estimate the integral in the RHS of (*).
By the Duhamel formula
\[
\int_0^t (e^{-\tau(\Lambda^*)} E^{t-\tau}) (x, y) d\tau
\]
\[
= \int_0^t (e^{-\tau A} E^{t-\tau}) (x, y) d\tau + \int_0^t \int_0^t (e^{-\tau(\Lambda^*)} (B_{\varepsilon,R}^t + B_{\varepsilon,R}^c) e^{-(\tau-\tau') A} d\tau' E^{t-\tau}) (x, y) d\tau
\]
\[
= \int_0^t (e^{-\tau A} E^{t-\tau}) (x, y) d\tau + J_R(x, y) + J_R^c(x, y),
\]
where, by Lemma 7(ii), \( \int_0^t (e^{-\tau A} (x, \cdot) E^{t-\tau} (\cdot, y)) (x, y) d\tau \leq k_2 t^{\frac{\alpha-1}{\alpha}} e^{-tA} (x, y) \). Let us estimate \( J_R(x, y) \) and \( J_R^c(x, y) \).

In \( J_R(x, y) \), discarding the term containing \(-W_{\varepsilon}\) and applying Lemma 7(i), we obtain
\[
J_R(x, y) \leq k_1 \int_0^t \int_0^t (e^{-\tau(\Lambda^*)} \mathbf{1}_{B(0,R)^{\frac{1}{m}}} |b_\varepsilon| E^{t-\tau'} d\tau' E^{t-\tau}) (x, y) d\tau
\]
(we are changing the order of integration and applying Lemma 7(iii))
\[
\leq k_1 k_3 \int_0^t \int_0^t (e^{-\tau(\Lambda^*)} \mathbf{1}_{B(0,R)^{\frac{1}{m}}} |b_\varepsilon| (t - \tau')^{\frac{\alpha-1}{\alpha}} E^{t-\tau'}) (x, y) d\tau'
\]
\[
\leq k_1 k_3 t^{\frac{\alpha-1}{\alpha}} \int_0^t (e^{-\tau(\Lambda^*)} \mathbf{1}_{B(0,R)^{\frac{1}{m}}} |b_\varepsilon| E^{t-\tau'}) (x, y) d\tau'.
\]
Now, repeating the corresponding argument in the proof of Claim 3 we obtain
\[
J_R(x, y) \leq C_2 t^{\frac{\alpha-1}{\alpha}} M_R(x, y), \quad C_2 = k_0 k_1 k_3 R(D - R)^{-1} (d - \alpha)^{-1} \leq \frac{k_3}{2}.
\]
(C_2 \leq \frac{k_0 k_1 k_3}{m-1} (d - \alpha)^{-1} \leq \frac{k_3}{2} (d - \alpha)^{-1} \leq \frac{k_3}{2}.)

In turn, \( J_R^c = \int_0^t (J_R^c)^{\tau} E^{t-\tau} d\tau \), where
\[
(J_R^c)^{\tau} := \int_0^\tau e^{-\tau(\Lambda^*)} B_{\varepsilon,R}^c e^{-(\tau-\tau') A} d\tau'.
\]
Again, discarding the \(-W_{\varepsilon}\) term in \( B_{\varepsilon,R}^c \) and applying Lemma 7(i), we obtain
\[
| (J_R^c)^{\tau} (x, y) | \leq \kappa k_1 R^{1-\alpha} t^{\frac{\alpha-1}{\alpha}} \int_0^\tau (e^{-\tau(\Lambda^*)} E^{t-\tau'}) (x, y) d\tau'.
\]
Due to Lemma 7(iii),
\[
| J_R^c (x, y) | \leq \kappa k_1 k_3 R^{1-\alpha} t^{\frac{\alpha-1}{\alpha}} \int_0^t (e^{-\tau(\Lambda^*)} (x, \cdot) (t - \tau')^{\frac{\alpha-1}{\alpha}} E^{t-\tau'} (\cdot, y)) d\tau'
\]
\[
\leq \kappa k_1 k_3 R^{1-\alpha} \int_0^t (e^{-\tau(\Lambda^*)} (x, \cdot) E^{t-\tau'} (\cdot, y)) d\tau'.
\]
Thus, due to \( \kappa k_1 k_3 R^{1-\alpha} \leq \delta < \frac{1}{2} \),
\[
\int_0^t (e^{-\tau(\Lambda^*)} (x, \cdot) E^{t-\tau'} (\cdot, y)) d\tau
\]
\[
\leq k_2 t^{\frac{\alpha-1}{\alpha}} e^{-tA} (x, y) + \frac{k_3}{2} t^{\frac{\alpha-1}{\alpha}} M_R(x, y) + \frac{1}{2} \int_0^t (e^{-\tau(\Lambda^*)} (x, \cdot) E^{t-\tau'} (\cdot, y)) d\tau.
\]
Thus, we obtain \( \int_0^t (e^{-\tau (\Lambda^\varepsilon)^*}(x,\cdot)E^{t-\tau}(\cdot,y))d\tau \leq 2k_2 t^{\frac{\alpha-1}{\alpha}} e^{-tA}(x,y) + k_3 t^{\frac{\alpha-1}{\alpha}}M^\varepsilon_K(x,y) \). Substituting the latter in (3), we obtain Claim 4.

Now, applying Claim 3 and Claim 4 in (5), we have

\[
e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq e^{-tA}(x,y) - \frac{1}{2} M^\varepsilon_K(x,y) + \delta (M^\varepsilon_K(x,y) + e^{-tA}(x,y))
\]

\[
\leq (1 + \delta) e^{-tA}(x,y),
\]

thus ending the proof of Step 2.

**Step 3:** Set \( R = 1 \lor (2k_3)^{\frac{1}{\alpha-1}} \) and let \( D \geq 2R \). Then there is a constant \( C = C(d,\alpha,\kappa,R) \) such that the following bound

\[
e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq Ce^{-tA}(x,y), \quad |x| > 2Dt^\frac{1}{\alpha}, \quad |y| \leq Dt^\frac{1}{\alpha}, \quad t > 0.
\]

is valid

(See the proof below for explicit formula for \( C(d,\alpha,\kappa,R) \))

Using the Duhamel formula and applying Lemma 7(1), we have

\[
e^{-t(\Lambda^\varepsilon)^*}(x,y) \leq e^{-tA}(x,y) + k_1 \int_0^t (E^\tau |b_\varepsilon|e^{-(t-\tau)(\Lambda^\varepsilon)^*})(x,y)d\tau
\]

\[
\leq e^{-tA}(x,y) + k_1 L^t_{\varepsilon,R}(x,y) + k_1 L^{t,c}_{\varepsilon,R}(x,y).
\]

where

\[
L^t_{\varepsilon,R}(x,y) := \int_0^t (E^{\tau} 1_{B(0,Rt^\frac{1}{\alpha})} |b_\varepsilon|e^{-(t-\tau)(\Lambda^\varepsilon)^*})(x,y)d\tau,
\]

\[
L^{t,c}_{\varepsilon,R}(x,y) := \int_0^t (E^{\tau} 1_{B^c(0,Rt^\frac{1}{\alpha})} |b_\varepsilon|e^{-(t-\tau)(\Lambda^\varepsilon)^*})(x,y)d\tau.
\]

Let us estimate \( L^t_{\varepsilon,R}(x,y) \). Recalling that \( E^t(x,z) = t(|x-z|^{-d-\alpha-1} \land t^{-d+\alpha+1}) \) and taking into account that \( |x| \geq 2Dt^\frac{1}{\alpha}, |z| \leq Rt^\frac{1}{\alpha} \), we obtain \( E^t(x,z) \leq t|x-z|^{-d-\alpha-1} \leq t|x-z|^{-d-\alpha}(3R)^{-1}t^{-\frac{\alpha}{d}} \).
Therefore,

\[ L_{t,\varepsilon,R}^t(x,y) \leq (3R)^{-1} t^{-\frac{1}{2}} \int_0^t \left( t|x - \cdot|^{-\alpha - d} \mathbf{1}_{B(0,Rt^\frac{1}{\alpha})}(\cdot) |b_{\varepsilon}(\cdot)| e^{-(t-\tau)(\Lambda^\varepsilon)^*}(\cdot, y) \right) d\tau \]

(we are using that \(|x| > 2Dt^{\frac{1}{\alpha}}, |\cdot| \leq Rt^{\frac{1}{\alpha}}\))

\[ \leq (3R)^{-1} (4/3)^{d+\alpha} t^{-\frac{1}{3}} t|x|^{-\alpha - d} \int_0^t \left( \mathbf{1}_{B(0,Rt^\frac{1}{\alpha})}(\cdot) |b_{\varepsilon}(\cdot)| e^{-(t-\tau)(\Lambda^\varepsilon)^*}(\cdot, y) \right) d\tau \]

(we are using that \(|y| \leq Dt^{\frac{1}{\alpha}}, D \geq 2R\) and setting \(c = 3^{-1}(16/9)^{d+\alpha}\))

\[ \leq cR^{-1} t^{-\frac{1}{3}} t|x - y|^{-\alpha - d} \int_0^t \left( \mathbf{1}_{B(0,Rt^\frac{1}{\alpha})}(\cdot) |b_{\varepsilon}(\cdot)| e^{-(t-\tau)(\Lambda^\varepsilon)^*}(\cdot, y) \right) d\tau \]

(we are using \(t|x - y|^{-\alpha - d} = t(|x - y|^{-\alpha - d} \wedge t^{-\frac{d+\alpha}{\alpha}})\)

since \(|x - y|^{-\alpha - d} \leq (2R)^{-d-\alpha} t^{-\frac{d+\alpha}{\alpha}} < t^{-\frac{d+\alpha}{\alpha}}\), and are re-denoting \(t - \tau\) by \(\tau\))

\[ \leq k_0 c R^{-1} t^{-\frac{1}{3}} e^{-tA}(x,y) \int_0^t \left( t^{d-\alpha} \right) \mathbf{1}_{B(0,Rt^\frac{1}{\alpha})}(\cdot) |b| \|1\|_\infty d\tau \]

(we are applying Proposition 8)

\[ \leq k_0 c R^{-1} t^{-\frac{1}{3}} e^{-tA}(x,y) c_\infty \int_0^t \tau^{-\frac{d}{\alpha}} \mathbf{1}_{B(0,Rt^\frac{1}{\alpha})}(\cdot) |b| \|1\|_p \quad (p = \frac{d}{\alpha - 2}). \]

Since \(\int_0^t \tau^{-\frac{d}{\alpha}} d\tau = 2\alpha t^{\frac{1}{2}}\) and \(\|1\|_p = \kappa R^{\frac{1}{2}} t^{\frac{1}{2\alpha}} c\), \(\tilde{c} = \tilde{c}(d) < \infty\), we have

\[ L_{t,\varepsilon,R}^t(x,y) \leq C'R^{-\frac{1}{2}} e^{-tA}(x,y), \quad C' = 2\kappa \alpha k_0 c c_\infty \tilde{c} \]

or, for convenience,

\[ L_{t,\varepsilon,R}^t(x,y) \leq C' e^{-tA}(x,y). \quad (7) \]

In turn, clearly,

\[ L_{t,\varepsilon,R}^t(x,y) \leq \kappa R^{1-\alpha} t^{-\frac{a-1}{\alpha}} \int_0^t E^{\tau} e^{-(t-\tau)(\Lambda^\varepsilon)^*} d\tau. \]
Let us estimate the integral in the RHS. Using the Duhamel formula, we obtain
\[
\int_0^t \left( E^* e^{-(t-\tau)(\Lambda^*)} \right) (x, y) d\tau
\]
\[
\leq \int_0^t \left( E^* e^{-(t-\tau)A} \right) (x, y) d\tau + \int_0^t \left( E^* \int_0^{t-\tau} E^{t-\tau-s} |b_\varepsilon| e^{-s(\Lambda^*)} ds \right) (x, y) d\tau.
\]
(we are applying Lemma 7(ii) and changing the order of integration)
\[
\leq k_2 t \frac{\alpha - 1}{\alpha} e^{-t A} (x, y) + \int_0^t t - s \frac{\alpha - 1}{\alpha} \left( E^{t-s} |b_\varepsilon| e^{-s(\Lambda^*)} \right) (x, y) ds
\]
\[
\leq k_2 t \frac{\alpha - 1}{\alpha} e^{-t A} (x, y) + k_3 t \frac{\alpha - 1}{\alpha} \int_0^t \left( E^{t-s} 1_{B(0, R t^{\frac{1}{2}})} |b_\varepsilon| e^{-s(\Lambda^*)} \right) (x, y) d\tau ds
\]
\[
+ k_3 t \frac{\alpha - 1}{\alpha} \int_0^t \left( E^{t-s} 1_{B(0, R t^{\frac{1}{2}})} |b_\varepsilon| e^{-s(\Lambda^*)} \right) (x, y) ds.
\]
(we are applying Lemma 7 to the second term, and note that \(k_3 \kappa R^{1-\alpha} \leq \frac{1}{2}\))
\[
\leq (k_2 + k_3 C') t \frac{\alpha - 1}{\alpha} e^{-t A} (x, y) + \frac{1}{2} \int_0^t \left( E^{t-s} e^{-s(\Lambda^*)} \right) (x, y) ds.
\]
Therefore,
\[
\int_0^t E^* \left( e^{-(t-\tau)(\Lambda^*)} \right) (x, y) d\tau \leq 2(k_2 + k_3 C') t \frac{\alpha - 1}{\alpha} e^{-t A} (x, y),
\]
and so
\[
I_{\varepsilon,R}^t (x, y) \leq 2\kappa (k_2 + k_3 C') R^{1-\alpha} e^{-t A} (x, y). \quad (8)
\]
Applying (7) and (8) in (6), we obtain the desired bound
\[
e^{-t(\Lambda^*)} (x, y) \leq Ce^{-t A} (x, y), \quad |x| > 2Dt^{\frac{1}{2}}, \quad |y| \leq Dt^{\frac{1}{2}},
\]
for all \(R > 1\) such that \(k_3 \kappa R^{1-\alpha} \leq \frac{1}{2}\); \(D \geq 2R\), where \(C := 1 + k_1 C' + k_1 2\kappa (k_2 + k_3 C') R^{1-\alpha}\). The assertion of Step 3 follows.

We are in position to complete the proof of Theorem 3(i), i.e. to prove the bound
\[
e^{-t(\Lambda^*)} (x, y) \leq C_1 e^{-t A} (x, y), \quad x, y \in \mathbb{R}^d, \quad t > 0,
\]
for appropriate constant \(C_1 = C_1(d, \alpha, \kappa)\).

To prove (9), we combine Steps 1-3 as follows. Fix \(D\) large enough so that the assertions of both Step 2 and Step 3 hold.

Without loss of generality, the assertion of Step 3 holds for all \(|x| > Dt^{\frac{1}{2}}\), \(|y| \leq Dt^{\frac{1}{2}}\) (indeed, by Step 1, (9) is true for all \(|x| \leq 2Dt^{\frac{1}{2}}\), \(|y| \leq 2Dt^{\frac{1}{2}}\) (with \(C_1 = C_0'(4D)^{d+\alpha}\) and so, in particular, for all \(Dt^{\frac{1}{2}} < |x| \leq 2Dt^{\frac{1}{2}}\), \(|y| \leq Dt^{\frac{1}{2}}\); the rest follows from the assertion of Step 3 as stated). Thus, the desired bound (9) is true for all \(|x| > Dt^{\frac{1}{2}}\), \(|y| \leq Dt^{\frac{1}{2}}\) and, by Step 2, for all \(x \in \mathbb{R}^d\), \(|y| > Dt^{\frac{1}{2}}\).
It remains to prove (12) in the case $|x| \leq Dt^{\frac{1}{2}}$, $|y| \leq Dt^{\frac{1}{2}}$. But this is the assertion of Step 1.
Thus, (12) is true, with constant $C_1$ equal to the maximum of the constants in Step 1 (with $2D$ in place of $D$) and in Steps 2, 3.

(ii) The result follows immediately from Step 2 in the proof of (i) upon taking $\varepsilon \downarrow 0$ (cf. Proposition 12).

The proof of Theorem 3 is completed. \hfill $\square$

6. Proof of Theorem 4: The weighted upper bound
Recall $A \equiv (-\Delta)^{\alpha}$. We are going to prove that there is a constant $C < \infty$ such that
\[ e^{-tA}(x, y) \leq Ce^{-tA}(x, y)\psi_t(y), \quad t > 0, \quad x, y \in \mathbb{R}^d. \quad (10) \]
Clearly, Theorem 2 and Theorem 3(i) combined, yield
\[ e^{-tA}(x, y) \leq C_1c_{N,w}\left(e^{-tA}(x, y) \wedge (t^{-\frac{d}{\alpha}}\psi_t(y))\right), \quad t > 0, \quad x, y \in \mathbb{R}^d. \quad (11) \]

1. If $|y| \geq t^{\frac{1}{2}}$, then $\psi_t(y) \geq 1$. Then, by (11),
\[ e^{-tA}(x, y) \leq C_1c_{N,w}e^{-tA}(x, y) \leq C_1c_{N,w}e^{-tA}(x, y)\psi_t(y), \]
i.e. (10) holds.

2. If $|x| \leq Dt^{\frac{1}{2}}$, $|y| < t^{\frac{1}{2}}$ for some constant $D > 1$, then by (11) (cf. Lemma 6(ii))
\[ e^{-tA}(x, y) \leq C_1c_{N,w}t^{-\frac{d}{\alpha}}\psi_t(y) \leq C_1c_{N,w}k_0^{-1}(D + 1)^{d+\alpha}e^{-tA}(x, y)\psi_t(y), \]
i.e. (10) holds.

3. It remains therefore to consider the case $|x| > Dt^{\frac{1}{2}}$, $|y| < t^{\frac{1}{2}}$.

By duality (cf. Proposition 12), it suffices to prove the estimate
\[ e^{-tA^*}(x, y) \leq Ce^{-tA}(x, y)\psi_t(x) \]
for all $|x| < t^{\frac{1}{2}}$, $|y| > Dt^{\frac{1}{2}}$, $t > 0$, for some $D > 1$.

We will use Corollary 2
\[ \langle e^{-tA^*}(x, \cdot) \rangle \leq C_2\psi_t(x) \quad \text{for all } x \in \mathbb{R}^d, \quad t > 0, \]
the “standard” upper bound (Theorem 3(i))
\[ e^{-tA^*}(x, y) \leq C_1e^{-tA}(x, y), \quad \text{for all } x, y \in \mathbb{R}^d, \quad t > 0, \]
and its partial improvement (Theorem 3(ii)): For every $\delta > 0$ there exists a sufficiently large $D$
such that for all $|x| < t^{\frac{1}{2}}$, $|y| > Dt^{\frac{1}{2}}$ and all $z \in B(y, \frac{|y-x|}{2})$
\[ e^{-tA^*}(x, z) \leq C_\delta e^{-tA}(x, z), \quad e^{-tA^*}(z, y) \leq C_\delta e^{-tA}(z, y), \quad C_\delta := 1 + \delta. \quad (13) \]

We will need the following elementary inequality:
\[ 2\left(1_{B(y, \frac{|y-x|}{2})}(\cdot)e^{-\frac{\alpha}{2}A}(x, \cdot)e^{-\frac{\alpha}{2}A}(\cdot, y)\right) \leq e^{-tA}(x, y). \quad (14) \]
Indeed, by symmetry, the LHS of (14) coincides with
\[
\langle 1_{B(y, |x-y|^2)}(\cdot) \rangle e^{-\frac{t}{2}A}(x, \cdot) + \langle 1_{B(x, |x-y|^2)}(\cdot) \rangle e^{-\frac{t}{2}A}(x, \cdot) e^{-\frac{t}{2}A}(y, \cdot) \rangle \\
\leq \langle e^{-\frac{t}{2}A}(x, \cdot) e^{-\frac{t}{2}A}(\cdot, y) \rangle = e^{-tA}(x, y),
\]
i.e. (14) follows.

**Proposition 3.** (i) There exists a constant $c_5$ such that
\[
e^{-tA^\ast}(x, y) \leq \langle 1_{B(y, |x-y|^2)}(\cdot) \rangle e^{-\frac{t}{2}A^\ast}(x, \cdot) e^{-\frac{t}{2}A^\ast}(\cdot, y) \rangle + c_5 e^{-tA}(x, y) \psi_1(x)
\]
(ii) If $|x| < t^\frac{1}{2}$, $|y| > D t^\frac{1}{2}$ with $D > 1$ sufficiently large, then
\[
e^{-tA^\ast}(x, y) \leq \left( \frac{C_5^2}{2} + c_5 \psi_1(x) \right) e^{-tA}(x, y).
\]

**Proof.** We have
\[
e^{-tA^\ast}(x, y) = \langle 1_{B(y, |x-y|^2)}(\cdot) \rangle e^{-\frac{t}{2}A^\ast}(x, \cdot) e^{-\frac{t}{2}A^\ast}(\cdot, y) \rangle + \langle 1_{B(y, |x-y|^2)}(\cdot) \rangle e^{-\frac{t}{2}A^\ast}(x, \cdot) e^{-\frac{t}{2}A^\ast}(\cdot, y) \rangle
\]
\[=: J_1 + J_2.
\]

(i) For $z \in B^c(y, |x-y|^2)$, $e^{-\frac{t}{2}A^\ast}(z, y) \leq C_1 e^{-\frac{t}{2}A}(z, y) \leq k_1 e^{-tA}(x, y)$. Thus,
\[
J_2 \leq k_1 e^{-tA}(x, y) \langle 1_{B(y, |x-y|^2)}(\cdot) \rangle e^{-\frac{t}{2}A^\ast}(x, \cdot) \rangle
\]
\[
(we \ are \ applying \ Corollary \ 2)
\]
\[\leq k_1 C_2 e^{-tA}(x, y) \psi_1(x) \leq c_5 e^{-tA}(x, y) \psi_1(x),
\]
and so (i) follows.

(ii) Using (i), it remains to estimate $J_1$. Applying (13), we have
\[
J_1 \leq C_5^2 \langle 1_{B(y, |x-y|^2)}(\cdot) \rangle e^{-\frac{t}{2}A}(x, \cdot) e^{-\frac{t}{2}A}(\cdot, y) \rangle
\]

Finally, we use (14). \qed

Let us complete the proof of Theorem 4.

By Proposition 3(ii),
\[
e^{-tA^\ast}(x, y) \leq \left( \frac{C_5^2}{2} + c_5 \psi_1(x) \right) e^{-tA}(x, y).
\]

Set $\nu := \frac{C_5^eta 2^n}{2}$, so that $\frac{C_5^2}{2} \psi_1/2 = \nu \psi_1$. Fix $\delta \in ]0, (\sqrt{2} - 1) \wedge (2^{1-\frac{\nu}{2}} - 1)]$. Then $\frac{C_5^2}{2} < 1$ and $\nu < 1$. Now, suppose that, for $n = 2, 3, \ldots$,
\[
e^{-tA^\ast}(x, y) \leq \left( \frac{C_5^\delta + 1}{2^n} + c_5(1 + \nu + \cdots + \nu^{n-1}) \psi_1(x) \right) e^{-tA}(x, y),
\]

(15)
Then, using Proposition [3](i), we have
\[
e^{-t\Lambda^*}(x, y) \leq \langle 1_{B(y, |x-y|)}(\cdot) e^{-\frac{t}{2}\Lambda^*}(x, \cdot) C_\delta e^{-\frac{t}{2}A}(\cdot, y) \rangle + c_5 e^{-tA}(x, y) \psi_I(x)
\]
\[
\leq \langle 1_{B(y, |x-y|)}(\cdot) C_\delta \left( \frac{C_\delta^{n+1}}{2^n} + c_5 (1 + \nu + \cdots + \nu^{n-1}) \psi_I(x) \right) e^{-\frac{t}{2}A}(x, \cdot) e^{-\frac{t}{2}A}(\cdot, y) \rangle + c_5 e^{-tA}(x, y) \psi_I(x)
\]
\[
(\text{we are applying (14)})
\]
\[
\leq \left( \frac{C_\delta^{n+2}}{2^{n+1}} + c_5 (1 + \nu + \cdots + \nu^n) \psi_I(x) \right) e^{-tA}(x, y) + c_5 e^{-tA}(x, y) \psi_I(x)
\]
\[
= \left( \frac{C_\delta^{n+2}}{2^{n+1}} + c_5 (1 + \nu + \cdots + \nu^n) \psi_I(x) \right) e^{-tA}(x, y).
\]
Thus by induction, (15) holds for \( n + 1 \). Sending \( n \to \infty \) there, we obtain
\[
e^{-t\Lambda^*}(x, y) \leq c_5 (1 - \nu)^{-1} e^{-tA}(x, y) \psi_I(x),
\]
as needed. The proof of (12) is completed. The proof of Theorem 4 is completed.

7. Proof of Theorem 5: The weighted lower bound

Recall that
\[
k_0^{-1} t(|x-y|^{d-\alpha} \wedge t^{-\frac{d+\alpha}{\alpha}}) \leq e^{-tA}(x, y) \leq k_0 t(|x-y|^{d-\alpha} \wedge t^{-\frac{d+\alpha}{\alpha}})
\]
for all \( x, y \in \mathbb{R}^d, x \neq y, t > 0 \), for a constant \( k_0 = k_0(d, \alpha) > 1 \).

1. First, we prove the “standard” lower bound away from the origin.

**Lemma 8.** There exists a generic constant \( 0 < \gamma < \frac{1}{2} \) such that, for all \( r \geq \gamma^{-2} \) and \( t > 0 \),
\[
e^{-t\Lambda^*}(x, y) \geq \frac{1}{2} e^{-tA}(x, y)
\]
whenever \( |x| \geq rt^\frac{1}{\alpha}, |y| \geq rt^\frac{1}{\alpha} \).

**Proof.** In view of Proposition [10] it suffices to prove the inequality \( e^{-t(\Lambda^*)^*}(x, y) \geq \frac{1}{2} e^{-tA}(x, y) \).

By the Duhamel formula,
\[
e^{-t(\Lambda^*)^*}(x, y) \geq e^{-tA}(x, y) - |M_t(x, y)|,
\]
\[
M_t(x, y) := \int_0^t e^{-(t-\tau)A} \nabla \cdot b \, e^{-\tau(\Lambda^*)^*} \, d\tau.
\]
Using Lemma [7](i), we have
\[
|M_t(x, y)| \leq k_1 \int_0^t (E^{t-\tau}(x, \cdot)) \cdot |e^{-\alpha+1} e^{-\tau(\Lambda^*)^*}(\cdot, y))| d\tau
\]
(we are using Theorem [3](i) – the standard upper bound)
\[
\leq k_1 C_1 \int_0^t (E^{t-\tau}(x, \cdot)) \cdot |e^{-\alpha+1} e^{-\tau A}(\cdot, y))| d\tau.
\]
Set
\[
J(1_{B(0, \gamma r^t \pi)} \cdot | \cdot |^{1-\alpha}) := \int_0^t \langle 1_{B(0, \gamma r^t \pi)}(\cdot)e^{-(t-\tau)A}(\cdot, \cdot) \rangle d\tau,
\]
\[
J(1_{B^c(0, \gamma r^t \pi)} \cdot | \cdot |^{1-\alpha}) := \int_0^t \langle 1_{B^c(0, \gamma r^t \pi)}(\cdot)e^{-(t-\tau)A}(\cdot, \cdot) \rangle d\tau,
\]
where \(0 < \gamma < 2^{-1}\).

Note that if \(|x| \geq r t^1_\pi\), then
\[
E^{t-\tau}(x, z) \leq C_5 e^{-(t-\tau)A}(x, z)|x - z|^{-1} \leq C_5 2r^{-1} t^{-\frac{1}{\alpha}} e^{-(t-\tau)A}(x, z) \quad z \in B(0, \gamma r^t \pi).
\]

Thus, using the inequality
\[
e^{-tA}(x, z)e^{-sA}(y, z) \leq K e^{-(t+s)A}(x, y)(e^{-tA}(x, z) + e^{-sA}(z, y)),
\]
which holds for a constant \(K = K(d, \alpha)\), all \(x, z, y \in \mathbb{R}^d\) and \(t, s > 0\) (see e.g. [BJ]), we have
\[
J(1_{B(0, \gamma r^t \pi)} \cdot | \cdot |^{1-\alpha}) \leq C_6 2r^{-1} t^{-\frac{1}{\alpha}} K e^{-tA}(x, y) \int_0^t \langle 1_{B(0, \gamma r^t \pi)}(\cdot) \cdot | \cdot |^{1-\alpha}(e^{-(t-\tau)A}(x, \cdot) + e^{-\tau A}(\cdot, \cdot)) \rangle d\tau.
\]

Next, for all \(0 < \tau < t\), \(|x| \geq r t^1_\pi\), \(|y| \geq r t^1_\pi\),
\[
1_{B(0, \gamma r^t \pi)}(\cdot)e^{-\tau A}(\cdot, y) \leq C_7 t^{-\frac{d}{\alpha} r^{-d-\alpha}} \quad \text{if } (1 - \gamma)r > 1,
\]
\[
1_{B(0, \gamma r^t \pi)}(\cdot)e^{-(t-\tau)A}(x, \cdot) \leq C_7 t^{-\frac{d}{\alpha} r^{-d-\alpha}} \quad \text{if } (1 - \gamma)r > 1,
\]
and so
\[
J(1_{B(0, \gamma r^t \pi)} \cdot | \cdot |^{1-\alpha}) \leq C_8 t^{-\frac{d}{\alpha} r^{-d-\alpha} - 1} e^{-tA}(x, y) \int_0^t \langle 1_{B(0, \gamma r^t \pi)}(\cdot) \cdot | \cdot |^{1-\alpha} \rangle d\tau
\]
\[
\leq C_9 t^{-2\alpha r^{-d-\alpha} + 1} e^{-tA}(x, y)
\]
\[
\leq C_9 2\alpha t^{-d-\alpha} + 1 e^{-tA}(x, y) \quad \text{if } r > (1 - \gamma)^{-1}.
\]

Therefore,
\[
J(1_{B(0, \gamma r^t \pi)} \cdot | \cdot |^{1-\alpha}) \leq C_{10} \gamma^{d-\alpha} + 1 e^{-tA}(x, y) \quad \text{if } r > (1 - \gamma)^{-1}, \quad 0 < \gamma < 2^{-1}.
\] (\(*\))

In turn,
\[
J(1_{B^c(0, \gamma r^t \pi)} \cdot | \cdot |^{1-\alpha}) \leq \frac{c_1 C}{2} C_0 (\gamma r t^\pi)^{1-\alpha} t^{-\frac{1}{\alpha}} e^{-tA}(x, y) = C_{11} (\gamma r)^{1-\alpha} e^{-tA}(x, y)
\]
as follows immediately from Lemma [7 ii):
\[
\int_0^t \langle e^{-(t-\tau)A}(x, \cdot) E^\tau(\cdot, y) \rangle d\tau \leq C_0 t^{1-\frac{1}{\alpha}} e^{-tA}(x, y).
\]

Thus, if \(r \geq \gamma^{-2}\), then
\[
J(1_{B^c(0, \gamma r^t \pi)} \cdot | \cdot |^{1-\alpha}) \leq C_{11} \gamma^{1-\alpha} e^{-tA}(x, y).
\] (\(**\))

Finally, selecting \(\gamma > 0\) sufficiently small: \(k_1 \kappa C(C_{10} \lor C_{11}) \gamma^{\alpha-1} \leq \frac{1}{4}\), and using (\(*\), (\(**\)), we have
\[
|M_t(x, y)| \leq \frac{1}{2} e^{-tA}(x, y),
\]
which ends the proof. \(\square\)
Corollary 3. For every $r > 0$, there is a constant $c(r) > 0$ such that

$$e^{-t\Lambda^*}(x, y) \geq c(r)e^{-tA}(x, y)$$

whenever $|x| \geq rt^\frac{1}{\alpha}$, $|y| \geq rt^\frac{1}{\alpha}$, $t > 0$.

Proof. In Lemma 8, fix some $r \geq \gamma^{-2}$, so that

$$e^{-t\Lambda^*}(x, y) \geq 2^{-1}e^{-tA}(x, y), \quad |x| \geq rt^\frac{1}{\alpha}, \quad |y| \geq rt^\frac{1}{\alpha}, \quad (18)$$

$$e^{-t\frac{1}{2}\Lambda^*}(x, y) \geq 2^{-1}e^{-\frac{1}{2}A}(x, y), \quad |x| \geq r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}, \quad |y| \geq r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}. \quad (19)$$

We now extend (18), by proving existence of a constant $0 < c_1 < 2^{-1}$ such that

$$e^{-t\Lambda^*}(x, y) \geq c_1e^{-tA}(x, y), \quad |x| \geq r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}, \quad |y| \geq r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}. \quad (18)$$

Clearly, we need to consider only the case $rt^\frac{1}{\alpha} \geq |x| \geq r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}, r \geq |y| \geq r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}}$. By the reproduction property,

$$e^{-t\Lambda^*}(x, y) \geq \langle e^{-\frac{1}{2}t\Lambda^*}(x, \cdot) \mathbf{1}_{B^c(0,r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}})}(\cdot)e^{-\frac{1}{2}tA^*}(\cdot, y)\rangle$$

(we are applying (19))

$$\geq 2^{-2}\langle e^{-\frac{1}{2}tA}(x, \cdot) \mathbf{1}_{B^c(0,r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}})}(\cdot)e^{-\frac{1}{2}tA}(\cdot, y)\rangle$$

$$\geq 2^{-2}\langle e^{-\frac{1}{2}tA}(x, \cdot) \mathbf{1}_{B(0,(r+1)\left(\frac{t}{2}\right)^{\frac{1}{\alpha}})-B(0,r\left(\frac{t}{2}\right)^{\frac{1}{\alpha}})}(\cdot)e^{-\frac{1}{2}tA}(\cdot, y)\rangle$$

(we are using the lower bound in (16))

$$\geq 2^{-2}\tilde{c}t^{-\frac{d}{\alpha}} \quad (\tilde{c} = \tilde{c}(r) > 0)$$

(we are using the upper bound in (16))

$$\geq c_1e^{-tA}(x, y) \quad \text{for appropriate } 0 < c_1 = c_1(r) < 2^{-1},$$

i.e. we have proved (18).

The same argument yields

$$e^{-\frac{1}{2}t\Lambda^*}(x, y) \geq c_1e^{-\frac{1}{2}tA}(x, y), \quad |x| \geq r\left(\frac{t}{2^2}\right)^{\frac{1}{\alpha}}, \quad |y| \geq r\left(\frac{t}{2^2}\right)^{\frac{1}{\alpha}}. \quad (19)$$

Thus, we can repeat the above procedure $m - 1$ times obtaining

$$e^{-t\Lambda^*}(x, y) \geq c_me^{-tA}(x, y), \quad |x| \geq r\left(\frac{t}{2^m}\right)^{\frac{1}{\alpha}}, \quad |y| \geq r\left(\frac{t}{2^m}\right)^{\frac{1}{\alpha}}$$

for appropriate $c_m > 0$, from which the assertion of Corollary 3 follows. □

2. Next, in Proposition 4 we will prove an “integral lower bound”. We need
Theorem. We have
\[ \psi \]

Proof. Recall that both
\[ \int \]
Since \( c \neq 1 \), \( \psi \leq c \psi_t \), \( c > 1 \), it suffices to prove Lemma 9 for weight \( \psi_{0,t} \).

For brevity, write \( \psi_t := \psi_{0,t} \). We have
\[ \| \psi_t h \|_1 = \langle 1_{B(0,r^{\frac{1}{\alpha}})}(\tau^{-\frac{\alpha}{\beta}}|x|)^\beta h \rangle + \langle 1_{B^c(0,r^{\frac{1}{\alpha}})}h \rangle, \]
and so
\[ \int_0^t \| \psi_t h \|_1 d\tau = \left( \int_0^t 1_{B(0,r^{\frac{1}{\alpha}})}(\tau^{-\frac{\alpha}{\beta}}|x|)^\beta d\tau \right) + \left( \int_0^t 1_{B^c(0,r^{\frac{1}{\alpha}})} d\tau \right) \]
If \( |x| \leq r^{\frac{1}{\alpha}} \), then
\[ \int_0^t 1_{B(0,r^{\frac{1}{\alpha}})}(x)\tau^{-\frac{\alpha}{\beta}} d\tau = \int_{|x|^\alpha}^t \tau^{-\frac{\alpha}{\beta}} d\tau = \frac{1}{1-\beta} (t^{1-\beta} - |x|^{-\beta+\alpha}) \]
and
\[ \int_0^t 1_{B^c(0,r^{\frac{1}{\alpha}})}(x) d\tau = \int_0^{|x|^\alpha} d\tau = |x|^\alpha. \]
If \( |x| > r^{\frac{1}{\alpha}} \), then
\[ \int_0^t 1_{B(0,r^{\frac{1}{\alpha}})}(x)\tau^{-\frac{\alpha}{\beta}} d\tau = 0, \quad \int_0^t 1_{B^c(0,r^{\frac{1}{\alpha}})} d\tau = t. \]
Thus,
\[ \int_0^t \| \psi_t h \|_1 d\tau = \frac{\alpha}{\alpha-\beta} \left( t^{\frac{\alpha}{\beta}+1} - |x|^{-\beta+\alpha} \right) |x|^{\beta h} + \langle 1_{B(0,r^{\frac{1}{\alpha}})} |x|^\alpha h \rangle + t \langle 1_{B^c(0,r^{\frac{1}{\alpha}})} \psi_t h \rangle \]
\[ = t \frac{\alpha}{\alpha-\beta} \langle 1_{B(0,r^{\frac{1}{\alpha}})} \psi_t h \rangle - \frac{\beta}{\alpha-\beta} \langle 1_{B(0,r^{\frac{1}{\alpha}})} |x|^\alpha h \rangle + t \langle 1_{B^c(0,r^{\frac{1}{\alpha}})} \psi_t h \rangle \]
\[ \leq t \frac{2\alpha-\beta}{\alpha-\beta} \langle \psi_t h \rangle. \]

\[ \square \]

Lemma 9. For every \( 0 \leq h \in L^1 \), \( t > 0 \)
\[ t^{-1} \int_0^t \| \psi_t h \|_1 d\tau \leq \hat{C} \| \psi_t h \|_1 \]
for a constant \( \hat{C} = \hat{C}(\alpha, \beta) \).

Proof. Define \( \psi_0, t(y) = \eta_0(t^{-\frac{1}{\alpha}} |y|) \), where
\[ \eta_0(u) = \begin{cases} u^\beta, & 0 < u < 1, \\ 1, & u \geq 1. \end{cases} \]
Since \( c^{-1} \psi_t \leq \psi_{0,t} \leq c \psi_t \), \( c > 1 \), it suffices to prove Lemma 9 for weight \( \psi_{0,t} \).

Proposition 4. Define \( g_t = \psi_t h \), \( 0 \leq h \in \mathcal{S} \)-the L. Schwartz space of test functions. Then, there exists generic constant \( \nu > 0 \) such that, for all \( t > 0 \),
\[ \langle \psi_t e^{-t\Lambda} \psi_t^{-1} g_t \rangle \geq \nu \langle g_t \rangle. \]

Proof. Recall that both \( e^{-t\Lambda^c} \), \( e^{-t(\Lambda^c)^*} \) are holomorphic in \( L^1 \) and \( C_u \) due to Hille’s Perturbation Theorem. We have \( \psi = \psi_{(1)} + \psi_{(u)} \), where
\[ \psi_{(1)} \in D((-\Delta)^{\frac{\alpha}{1}}) \quad (= D((\Lambda^*)^1) = D(\Lambda^*_1)), \]
\[ \psi_{(u)} \in D((-\Delta)^{\frac{\alpha}{C_u}}) \quad (= D((\Lambda^*)^C_u) = D(\Lambda^C_u)). \]
(see the proof of Proposition 2 for details), so \((\Lambda^\varepsilon)^* \psi = (\Lambda^\varepsilon)^*_{L^1} \psi(1) + (\Lambda^\varepsilon)^*_{C_u} \psi(u)\) and belongs to \(L^1 + C_u\).

Now, set \(g_{s,n} = \phi_{s,n} h, \phi_{s,n}(x) = e^{-(\frac{(\Lambda^\varepsilon)^*}{n}) \psi_s(x)}\). We have, for \(s > t > 0\),

\[
(g_{s,n}) - (\phi_{s,n} e^{-t \Lambda^\varepsilon}) = \int_0^t \langle \psi_s, \Lambda^\varepsilon e^{-\tau \Lambda^\varepsilon} e^{-\frac{x}{\varepsilon} h} \rangle d\tau
\]

\[
= \lim_{r \to 0} r^{-1} \int_0^t \langle \psi_s, (1 - e^{-r \Lambda^\varepsilon}) e^{-\tau \Lambda^\varepsilon} e^{-\frac{x}{\varepsilon} h} \rangle d\tau
\]

\[
= \lim_{r \to 0} r^{-1} \int_0^t \langle (1 - e^{-r(\Lambda^\varepsilon)^*}) \psi_s, e^{-\tau \Lambda^\varepsilon} e^{-\frac{x}{\varepsilon} h} \rangle d\tau
\]

\[
= \int_0^t \langle (\Lambda^\varepsilon)^* \psi_s, e^{-\tau \Lambda^\varepsilon} e^{-\frac{x}{\varepsilon} h} \rangle d\tau.
\]

Arguing as in the proof of Proposition 2, we represent

\[
(\Lambda^\varepsilon)^* \psi_s = 1_{B(0,s \frac{1}{\varepsilon})} W_\varepsilon \psi_s + v_\varepsilon,
\]

where \(W_\varepsilon(x) = \kappa(|x|^{-\alpha} - |x|^{-\alpha}) + \kappa d|x|^{-\alpha} - \alpha |x|^{-2}|x| - (d - \alpha)|x|^{-\alpha}\) and \(0 \leq \varepsilon \in L^\infty\).

Then

\[
(g_{s,n}) - (\phi_{s,n} e^{-t \Lambda^\varepsilon}) \leq \int_0^t \langle 1_{B(0,s \frac{1}{\varepsilon})} W_\varepsilon \psi_s, e^{-(\tau + \frac{1}{\varepsilon}) \Lambda^\varepsilon} \rangle d\tau + \int_0^t \langle v_\varepsilon, e^{-\tau \Lambda^\varepsilon} e^{-\frac{x}{\varepsilon} h} \rangle d\tau
\]

or, sending \(n \to \infty\),

\[
(g_s) - (\psi_s e^{-t \Lambda^\varepsilon}) \leq \int_0^t \langle 1_{B(0,s \frac{1}{\varepsilon})} W_\varepsilon \psi_s, e^{-\tau \Lambda^\varepsilon} \rangle d\tau + \int_0^t \langle v_\varepsilon, e^{-\tau \Lambda^\varepsilon} \rangle d\tau
\]

\[
\leq \int_0^t \langle 1_{B(0,s \frac{1}{\varepsilon})} W_\varepsilon \psi_s, e^{-\tau \Lambda^\varepsilon} \rangle d\tau + c' \int_0^t \|e^{-\tau \Lambda^\varepsilon} \|_1 d\tau.
\]

Next, we pass to the limit \(\varepsilon \downarrow 0\):

\[
(g_s) - (\psi_t e^{-t \Lambda^\varepsilon}) \leq c' \int_0^t \|e^{-\tau \Lambda^\varepsilon} \|_1 d\tau.
\]

We estimate the RHS of (*) using the upper bound:

\[
c' \int_0^t \|e^{-\tau \Lambda^\varepsilon} \|_1 d\tau \leq c' \int_0^t \|e^{-\tau \Lambda^\varepsilon} \psi_t \|_1 d\tau \leq c' \int_0^t \|\psi_t \|_1 d\tau
\]

(we are applying Lemma 9)

\[
\leq c' C \hat{C} \frac{t}{s} \|\psi_t \|_1,
\]

Therefore, using \(\psi_s \geq (\frac{t}{s})^\frac{\alpha}{\varepsilon} \psi_t\), we obtain

\[
c' \int_0^t \|e^{-\tau \Lambda^\varepsilon} \|_1 d\tau \leq c' C \hat{C} \frac{t}{s} \left( \frac{t}{s} \right)^{-\frac{\alpha}{\varepsilon}} \|g_s\|.
\]
Thus, by (\(*\)), \((1 - c'\hat{C}(\frac{t}{s})^{\alpha - \beta})\langle g_s \rangle \leq \langle \psi_se^{-t\Lambda}h \rangle\). Since \(\beta < \alpha\), we can select \(s > t\) such that \(c'\hat{C}(\frac{t}{s})^{\alpha - \beta} = \frac{1}{2}\), which yields the bound
\[
\langle \psi_se^{-t\Lambda}\psi_s^{-1}g_s \rangle \geq \frac{1}{2}\langle g_s \rangle.
\]
Finally, using \(\psi_t \geq \psi_s \geq (\frac{t}{s})^\alpha \psi_t\) and setting \(2\nu := (\frac{t}{s})^\alpha = (2c'\hat{C})^{-\alpha - \beta}\), we have
\[
\langle \psi_t e^{-t\Lambda} \psi_t^{-1}g_t \rangle = \langle \psi_t e^{-t\Lambda} \psi_s^{-1}g_s \rangle \geq \langle \psi_s e^{-t\Lambda} \psi_s^{-1}g_s \rangle \geq \frac{1}{2}\langle g_s \rangle \geq \frac{1}{2} \left(\frac{t}{s}\right)^\alpha \langle g_t \rangle = \nu \langle g_t \rangle.
\]
\[\square\]

**Remark 7.** In the proof of Proposition 4 we calculate \((\Lambda^\beta)^*\psi_s\) arguing as in the proof of Proposition 2
\[
\Lambda^\beta = (-\Delta)^{\beta/2} + \text{div } (b_\psi)\quad \psi = \psi_s,
\]
where
\[
(-\Delta)^{\beta/2} \psi = -s^{-\beta/2}\beta(d + \beta - 2)\frac{\gamma(d + \beta - 2)}{\gamma(d + \beta - \alpha)}|x|^{\beta - \alpha} + h_0
\]
for \(h_0 := -I_{2-\alpha}\Delta(\psi - \psi) \in L^\infty, \|h_0\|_\infty \leq c_0s^{-1}\). In turn,
\[
\text{div } (b_\psi) = \text{div } (b\tilde{\psi}) + W_\epsilon + h_1 + h_2 + h_3
\]
where \(\|h_i\|_\infty \leq c_is^{-1}, i = 1, 2, 3\). Since, by the choice of \(\beta\), \(-\beta(d + \beta - 2)\frac{\gamma(d + \beta - 2)}{\gamma(d + \beta - \alpha)}|x|^{\beta - \alpha} + \text{div } (b\tilde{\psi}) = 0\), we have
\[
\Lambda^\beta = \mathbf{1}_{B(0,s^{1/\beta})}W_\epsilon + v_\epsilon, \quad v_\epsilon := \mathbf{1}_{B(0,s^{1/\beta})}W_\epsilon + h_0 + h_1 + h_2 + h_3,
\]
where, it easily seen, \(\|v_\epsilon\|_\infty \leq c's^{-1}, \) as claimed.

**Proposition 5.** For every \(R_0 > 0\) there exist constants \(0 < r < R_0 < R\) such that for all \(t > 0\)
\[
\frac{\nu}{2} \langle \psi_t(x) \rangle \leq e^{-t\Lambda^\beta} \psi_t \mathbf{1}_{R_t, r_t}(x) \quad \text{for all } x \in B(0, R_{0,t}), \quad x \neq 0.
\]
where \(r_t := rt^{\frac{1}{\beta}}, R_{0,t} := R_0 t^{\frac{1}{\beta}}, R_t := Rt^{\frac{1}{\beta}}, \mathbf{1}_{R_t, r_t} := \mathbf{1}_{B(0, R_t)} - \mathbf{1}_{B(0, r_t)}\).

**Proof.** It suffices to prove that, for all \(g := \psi_t h, 0 \leq h \in S\) with sprt \(h \subset B(0, R_{0,t})\),
\[
\frac{\nu}{2} \langle g \rangle \leq \langle \mathbf{1}_{R_t, r_t} \psi_t e^{-t\Lambda^\beta} \psi_t^{-1}g \rangle.
\]
By the upper bound,
\[
\langle \mathbf{1}_{B(0, r_t)} \psi_t e^{-t\Lambda^\beta} \psi_t^{-1}g \rangle \leq C \langle \mathbf{1}_{B(0, r_t)} \psi_t, e^{-t\Lambda}g \rangle
\]
\[
\leq CC_1t^{-\frac{d}{\alpha}}\mathbf{1}_{B(0, r_t)}\psi_t \|1\|g\|_1
\]
\[
= CC_1\mathbf{1}_{B(0, r_t)}\psi_1 \|1\|g\|_1, \quad \mathbf{1}_{B(0, r_t)} \|1\| \rightarrow 0 \text{ as } r \downarrow 0.
\]
Proof.

where at the last step we have used, for \( x \in B(0, R_0, t) \), \( y \in B^c(0, R_t) \) and \( \tilde{x} = R_0^{-1} t^{-\frac{1}{\alpha}} x \in B(0, 1) \),
\[
\tilde{y} = R^{-1} t^{-\frac{1}{\alpha}} y \in B^c(0, 1),
\]
\[
e^{-tA}(x, y) \leq k_0 |x - y|^{-d-\alpha} \leq k_0 |R_0 t^{\frac{1}{\alpha}} \tilde{x} - R t^{\frac{1}{\alpha}} \tilde{y}|^{-d-\alpha} < 2k_0 t^{-\frac{d}{\alpha}} (R - R_0)^{-d-\alpha} |\tilde{y}|^{-d-\alpha}.
\]

It remains to apply Proposition 4 to obtain \( \langle \psi_t e^{-tA} \psi_t^{-1} g \rangle \). \[\square\]

Proposition 6. \( \langle h \rangle = \langle e^{-tA^* h} \rangle \) for every \( h \in L^1, t > 0 \).

Proof. Proposition 6 follows from \( \langle h \rangle = \langle e^{-tA^*} h \rangle \) and Proposition 10. \[\square\]

Proposition 7. For every \( R_0 > 0 \) there exist constants \( 0 < r < R_0 < R \) such that for all \( t > 0 \)
\[
\frac{1}{2} \leq e^{-tA} 1_{B(t, r)}(x) \quad \text{for all } x \in B(0, R_0, t),
\]
where \( r_t := rt^{\frac{1}{\alpha}}, R_0, t := R_0 t^{\frac{1}{\alpha}}, R_t := R t^{\frac{1}{\alpha}}, 1_{B(0, R)} := 1_{B(0, R)} - 1_{B(0, r)} \).

Proof. We essentially repeat the proof of Proposition 5. It suffices to prove that, for all \( 0 \leq h \in S \) with \( \text{sprt} \, h \subset B(0, R_0, t) \),
\[
\frac{1}{2} \langle h \rangle \leq \langle 1_{B(0, r)} e^{-tA^*} h \rangle.
\]

By the upper bound,
\[
\langle 1_{B(0, r)} e^{-tA^*} h \rangle \leq C \langle 1_{B(0, R)} \psi_t e^{-tA} h \rangle \leq C C_t t^{-\frac{d}{\alpha}} 1 \langle 1_{B(0, r)} \psi_t 1 \rangle \| h \|_1 = o(r) \| h \|_1, \quad o(r) \rightarrow 0 \text{ as } r \downarrow 0;
\]
\[
\langle 1_{B^c(0, R)} e^{-tA^*} h \rangle \leq C \langle 1_{B^c(0, R)} \psi_t e^{-tA} h \rangle \leq C \langle e^{-tA} 1_{B^c(0, R)} h 1_{B(0, R_0, t)} \rangle \leq C \sup_{x \in B(0, R_0, t)} e^{-tA} 1_{B^c(0, R)}(x) \| h \|_1 = C(R_0, R) \| h \|_1, \quad C(R_0, R) \rightarrow 0 \text{ as } R - R_0 \uparrow \infty.
\]

The last two estimates and Proposition 6 yield \( \frac{1}{2} \langle h \rangle \leq \langle 1_{B(0, r)} e^{-tA^*} h \rangle \). \[\square\]

3. We are in position to complete the proof of the lower bound using the so-called 3q argument.
Set \( q_t(x, y) := \psi_t^{-1}(x) e^{-tA^*} (x, y) \), \( x \neq 0 \).
(a) Let \( x, y \in B^c(0, t^{\frac{1}{\alpha}}) \), \( x \neq y \). Then, using that \( \psi_{3t}^{-1} \geq 1 \), we have by Corollary 3
\[
q_{3t}(x, y) \geq e^{-3tA^*} (x, y) \geq ce^{-3tA}(x, y).
\]
Let $r_t = rt^\frac{1}{2}$, $R_t = Rt^\frac{1}{2}$ be as in Proposition 5 and Proposition 7 where we fix $R_0 = 1$ (hence $r < 1$).

(b) Let $x \in B(0, t^\frac{1}{2}), |y| \geq rt^\frac{1}{2}, x \neq y$. By the reproduction property,

\[ q_{2t}(x, y) \geq \psi_{2t}^{-1}(x)(e^{-tA^*}(x, \cdot)\psi_{2t}^{-1}(\cdot)e^{-tA^*}(\cdot, y)1_{R_t,r_t}(\cdot)) \]

\[ = \psi_{2t}^{-1}(x)(e^{-tA^*}(x, \cdot)\psi_{2t}^{-1}(\cdot)e^{-tA^*}(\cdot, y)1_{R_t,r_t}(\cdot)) \]

\[ \geq \psi_{2t}^{-1}(x)(e^{-tA^*}(\cdot, y)1_{R_t,r_t}(\cdot)) \]

\[ \geq \psi_{2t}^{-1}(x)\psi_{2t}^{-1}(R_t)(e^{-tA^*}1_{R_t,r_t}(\cdot))(x) \inf_{r_t \leq |z| \leq R_t} e^{-tA^*}(z, y) \]

(we are applying Corollary 3, Proposition 5 and using $\psi_{2t}^{-1}(R_t) = 1$)

\[ \geq \frac{\nu}{2}\psi_{2t}^{-1}(x)\psi_{t}(x)c(r) \inf_{r_t \leq |z| \leq R_t} e^{-tA}(z, y) \]

(we are using $\psi_t \geq \psi_{2t}$)

\[ \geq C_1e^{-2tA}(x, y). \]

(b') Let $x \in B(0, t^\frac{1}{2}), |y| \geq t^\frac{1}{2}, x \neq y$. Arguing as in (b), we obtain

\[ q_{3t}(x, y) \geq C_2e^{-3tA}(x, y). \]

(c) Let $|x| \geq rt^\frac{1}{2}, y \in B(0, t^\frac{1}{2}), x \neq y$. We have

\[ q_{2t}(x, y) \geq \psi_{2t}^{-1}(x)(e^{-tA^*}(x, \cdot)e^{-tA^*}(\cdot, y)1_{R_t,r_t}(\cdot)) \]

\[ = \psi_{2t}^{-1}(x)(e^{-tA^*}(x, \cdot)e^{-tA^*}(y, \cdot)1_{R_t,r_t}(\cdot)) \]

(we are using $\psi_{2t}^{-1} \geq 1$ and applying Corollary 3)

\[ \geq c(r)(e^{-tA}(x, \cdot)e^{-tA}(y, \cdot)1_{R_t,r_t}(\cdot)) \]

(we are applying (16))

\[ \geq C_3(r)(Rt^\frac{1}{2} + |x|)^{-\alpha}e^{-A}(y, \cdot)1_{R_t,r_t}(\cdot)) \]

(we are applying Proposition 7)

\[ \geq C_3(r)^{-1}(Rt^\frac{1}{2} + |x|)^{-\alpha} \geq C_4(r)e^{-2tA}(x, y). \]

(c') Let $|x| \geq t^\frac{1}{2}, y \in B(0, t^\frac{1}{2}), x \neq y$. Arguing as in (c), we obtain

\[ q_{3t}(x, y) \geq C_5(r)e^{-3tA}(x, y). \]
(d) Let \( x, y \in B(0, t^{\frac{1}{4}}) \), \( x \neq y \). By the reproduction property,

\[
q_{3t}(x, y) \geq \psi_{3t}^{-1}(x)(e^{-t\Lambda^*}(x, \cdot) e^{-2t\Lambda^*}(\cdot, y) 1_{R_t, r_t}(\cdot))
\]

(we are using (c))

\[
\geq C_4(r)\psi_{3t}^{-1}(x)(e^{-t\Lambda^*}(x, \cdot) \psi_{2t}(\cdot) e^{-2t\Lambda^*}(\cdot, y) 1_{R_t, r_t}(\cdot))
\]

(we are using \( \psi_{2t} \geq 2^2 \psi_t \)) and \( e^{-2t\Lambda^*}(z, y) \geq c(r, R)t^{-\frac{d}{\alpha}} > 0 \) for \( r_t \leq |z| \leq R_t \), \( |y| \leq t^{\frac{1}{4}} \)

\[
\geq c(r, R)C_42^\frac{\alpha}{\alpha} \psi_{3t}^{-1}(x) t^{-\frac{d}{\alpha}} (e^{-t\Lambda^*}(x, \cdot) 1_{R_t, r_t}(\cdot) \psi_t(\cdot))
\]

(we are applying Proposition \([\ref{prop}])\) and using \( \psi_t \geq \psi_{3t} \)

\[
\geq C_5(r, R)e^{-3tA^*}(x, y).
\]

By (a), (b'), (c'), (d), \( q_{3t}(x, y) \geq Ce^{-3tA^*}(x, y) \) for all \( x, y \in \mathbb{R}^d \), \( x \neq y \), \( x \neq 0 \), and so

\[
e^{-3tA^*}(x, y) \geq Ce^{-3tA^*}(x, y)\psi_{3t}(x), \quad t > 0.
\]

The lower bound is proved.

---

8. Construction of the semigroup \( e^{-t\Lambda^\varepsilon} \), \( \Lambda^\varepsilon = (-\Delta)^{\frac{\alpha}{2}} - b \cdot \nabla \) in \( L^r \), \( 1 \leq r < \infty \)

Set \( b_\varepsilon(x) := \kappa|x|^{-\alpha}x \), \( \kappa > 0 \), \( |x| := \sqrt{|x|^2 + \varepsilon} \), \( \varepsilon > 0 \),

\[
\Lambda^\varepsilon = (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon \cdot \nabla, \quad D(\Lambda^\varepsilon) = \mathcal{W}^{\alpha, r} := (1 + (-\Delta)^{\frac{\alpha}{2}})^{-1}L^r.
\]

To prove that \( -\Lambda^\varepsilon \equiv -\Lambda^\varepsilon \) is the generator of a holomorphic semigroup in \( L^r \), \( 1 \leq r < \infty \), we appeal to the Hille Perturbation Theorem \([Ka, \text{Ch. IX, sect. 2.2}] \). To verify its assumptions, we use a well known estimate

\[
|\nabla(\zeta + A)^{-1}(x, y)| \leq C(\text{Re}\zeta + A)^{-\frac{\alpha-1}{\alpha}}(x, y), \quad \text{Re}\zeta > 0, \quad C = C(d, \alpha), \quad A \equiv (-\Delta)^{\frac{\alpha}{2}}.
\]

Then for \( Y = L^p \)

\[
\|b_\varepsilon \cdot \nabla(\zeta + A)^{-1}\|_{Y \rightarrow Y} \leq C\|b_\varepsilon\|_\infty\|((\text{Re}\zeta + A)^{-\frac{\alpha-1}{\alpha}})\|_{Y \rightarrow Y} \leq C\|b_\varepsilon\|_\infty(\text{Re}\zeta)^{-\frac{\alpha-1}{\alpha}},
\]

and so \( \|b_\varepsilon \cdot \nabla(\zeta + A)^{-1}\|_{Y \rightarrow Y} \), \( \text{Re}\zeta \geq c_\varepsilon \), can be made arbitrarily small by selecting \( c_\varepsilon \) sufficiently large. It follows that the Neumann series for

\[
(\zeta + \Lambda^\varepsilon)^{-1} = (\zeta + A)^{-1}(1 + T)^{-1}, \quad T := -b_\varepsilon \cdot \nabla(\zeta + A)^{-1},
\]

converges in \( L^p \) and \( C_u \) and satisfies \( \|((\zeta + \Lambda^\varepsilon)^{-1}\|_{Y \rightarrow Y} \leq C_\varepsilon|\zeta|^{-1}, \text{Re}\zeta \geq c_\varepsilon \), i.e. \( -\Lambda^\varepsilon \) is the generator of a holomorphic semigroup.

The same argument (with \( Y = C_u \)) shows that \( \Lambda^\varepsilon := (-\Delta)^{\frac{\alpha}{2}} - b_\varepsilon \cdot \nabla \) with \( D(\Lambda^\varepsilon) := D((-\Delta)^{\frac{\alpha}{2}}) \) generates a holomorphic semigroup in \( C_u \).
Proposition 8. For every \( r \in [1, \infty) \) and \( \varepsilon > 0 \), \( e^{-t\Lambda_\varepsilon^r} \) is a contraction \( C_0 \) semigroup in \( L^r \). There exists a constant \( c \neq c(\varepsilon) \) such that

\[
\|e^{-t\Lambda_\varepsilon^r}\|_{r \to q} \leq c_N t^{-\frac{d}{2} \left( \frac{1}{r} - \frac{1}{q} \right)}, \quad t > 0,
\]

for all \( 1 \leq r < q \leq \infty \).

In particular, there is a constant \( c_S > 0 \), \( c_S \neq c_S(\varepsilon) \) such that \( (\Lambda_\varepsilon^r \equiv \Lambda_\varepsilon^r \mid \epsilon_1) \)

\[
\text{Re}(\Lambda_\varepsilon^r u, u) \geq c_S \|u\|_{2j}^2, \quad u \in D(\Lambda_\varepsilon^r).
\]

Proof. First, let \( 1 < r < \infty \). Set \( u \equiv u(t) := e^{-t\Lambda_\varepsilon^r} f, f \in L^1 \cap L^\infty \), and write \( A := (-\Delta)^{\frac{\alpha}{2}} \).

Multiplying the equation \( \partial_t u + \Lambda_\varepsilon^r u = 0 \) by \( u|u|^{-2} \) and integrating over the spatial variables we obtain (taking into account that \( D(\Lambda_\varepsilon^r) = D(A_r) \subset W^{1,r} \))

\[
\frac{1}{r} \partial_t \|u\|_r^r + \text{Re}(Au, u|u|^{-2}) - \text{Re}(b_e \cdot \nabla u, u|u|^{-2}) = 0.
\]

Note that, since \( -A \) is a Markov generator,

\[
\text{Re}(Au, u|u|^{-2}) \geq \frac{4}{rr'} \|A^{\frac{1}{2}} |u|^{\frac{r}{2}}\|^2_2
\]

(indeed, by [LS] Theorem 2.1) or by Theorem [10] in Appendix A \( \text{Re}(Au, u|u|^{-2}) \geq \frac{4}{rr'} \|A^{\frac{1}{2}} |u|^{\frac{r}{2}}\|^2_2 \), \( u^{\frac{r}{2}} := u|u|^{-\frac{r-1}{2}} \), and by the Beurling-Deny theory \( \|A^{\frac{1}{2}} u^{\frac{r}{2}}\|^2_2 \geq \|A^{\frac{1}{2}} |u|^{\frac{r}{2}}\|^2_2 \). Integration by parts yields

\[
-\text{Re}(b_e \cdot \nabla u, u|u|^{-2}) = \kappa \frac{d}{r} (d|x|^{-\alpha} - \alpha|x|^{-\alpha-2}|x|^2)|u|^r \geq \kappa \frac{d}{r} \langle |x|^{-\alpha}|u|^r \rangle.
\]

Thus,

\[
-\partial_t \|u\|_r^r \geq \frac{4}{rr'} \|A^{\frac{1}{2}} |u|^{\frac{r}{2}}\|^2_2 \tag{20}
\]

From (20) we obtain \( \|u(t)\|_r \leq \|f\|_r \), \( t \geq 0 \) and since \( L^1 \cap L^\infty \) is dense in \( L^r \), \( ||e^{-t\Lambda_\varepsilon^r}||_{r \to r} \leq 1 \) as needed.

Since \( e^{-t\Lambda_\varepsilon^r} \upharpoonright L^1 \cap L^\infty = e^{-t\Lambda_\varepsilon^r} \upharpoonright L^1 \cap L^\infty \), the latter clearly yields

\[
||e^{-t\Lambda_\varepsilon^r} f||_r \leq ||f||_r, \quad f \in L^1 \cap L^\infty.
\]

Sending \( r \uparrow \infty \), we have \( ||e^{-t\Lambda_\varepsilon^r} f||_\infty \leq ||f||_\infty \), and sending \( r \downarrow 1 \), we have \( ||e^{-t\Lambda_\varepsilon^r}||_{1 \to 1} \leq 1 \).

Let us prove the ultracontractivity of \( e^{-t\Lambda_\varepsilon^r} \). By (20),

\[
-\partial_t \|u\|^2_{2r} \geq \frac{4}{(2r)^2} \|A^{\frac{1}{2}} |u|^r\|^2_2, \quad 1 \leq r < \infty.
\]

Using the Nash inequality \( \|A^{\frac{1}{2}} h\|_2^2 \geq C_N \|h\|_2^{2+\frac{2\alpha}{d}} \|h\|_1^{2\frac{2\alpha}{d}} \) and \( ||u(t)||_r \leq ||f||_r \), we have, setting \( v := ||u||_{2r}^2 \),

\[
\partial_t v^{-\frac{2\alpha}{d}} \geq c_1 ||f||_r^{-\frac{2\alpha}{d}},
\]

where \( c_1 = C_N \frac{\alpha}{d} \frac{4}{(2r)^2} \). Integrating this inequality yields

\[
||e^{-t\Lambda_\varepsilon^r}||_{r \to 2r} \leq c_1 \frac{d}{2\alpha} t^{-\frac{d}{2}(\frac{1}{r} - \frac{1}{2})}, \quad t > 0,
\]

and so, by semigroup property,

\[
||e^{-t\Lambda_\varepsilon^r}||_{1 \to 2m} \leq c_N t^{-\frac{d}{2}(1 - \frac{1}{2m})}, \quad t > 0, \quad m \geq 1,
\]
where the constant $c_N \neq c_N(m)$. Thus, sending $m$ to infinity we arrive at $\|e^{-t\Lambda^\varepsilon}\|_{1 \to \infty} \leq c_N t^{-\frac{d}{4}}, \quad t > 0$. The latter and the contractivity of $e^{-t\Lambda^\varepsilon}$ in all $L^q$, $1 \leq q \leq \infty$ yield via interpolation the desired bound $\|e^{-t\Lambda^\varepsilon}\|_{p \to q} \leq c_N t^{-\frac{d}{4}(\frac{1}{p} - \frac{1}{q})}, \quad t > 0$, for all $1 \leq p < q \leq \infty$.

Finally, since $D(\Lambda^\varepsilon) = D(A)$, we have, for $u \in D(A)$, $\text{Re}(\Lambda^\varepsilon u, u) \geq \|A^\frac{1}{2} u\|_2^2 \geq c_S \|u\|_{2j}^2$. □

8.1. **Case $d \geq 4$.** We will first provide an elementary argument that allows to treat all $d = 4, 5, \ldots$ but the main case $d = 3$.

**Proposition 9.** For every $r \in [1, \infty[\text{ the limit}

$$s\cdot L^r \cdot \lim_{\varepsilon \downarrow 0} e^{-t\Lambda^\varepsilon} \quad (\text{loc. uniformly in } t \geq 0)$$

exists and determines a contraction $C_0$ semigroup on $L^r$, say $e^{-t\Lambda^r}$.

For all $1 \leq r < q \leq \infty$,

$$\|e^{-t\Lambda^r}\|_{r \to q} \leq c_N t^{-\frac{d}{4}(\frac{1}{r} - \frac{1}{q})}, \quad t > 0$$

with $c_N$ from Proposition 8.

**Proof of Proposition 9.** First, let $r = 2$. Set $u^\varepsilon(t) := e^{-t\Lambda^\varepsilon} f$, $f \in C_c^\infty$.

**Claim 5.** $\|\nabla u^\varepsilon(t)\|_2 \leq \|\nabla f\|_2$, $t \geq 0$.

**Proof of Claim 5.** Denote $u := u^\varepsilon$, $w := \nabla u$, $w_i := \nabla_i u$. Due to $f \in C_c^\infty$ and $\nabla_i^\varepsilon b_i \in C_c^\infty \cap L^\infty$, $i = 1, \ldots, d$, $n \geq 1$ we can and will differentiate the equation $\partial_t u + \Lambda^\varepsilon u = 0$ in $x$, obtaining

$$\partial_t w_i + (-\Delta)^\frac{n}{4} w_i - b_i \cdot \nabla w_i + (\nabla_i b_i) \cdot w = 0.$$

Multiplying the latter by $\bar{w}_i$, integrating by parts and summing up in $i = 1, \ldots, d$ we have

$$\frac{1}{2} \partial_t \|w\|_2^2 + \sum_{i=1}^d \|(-\Delta)^\frac{n}{4} w_i\|_2^2 - \text{Re} \sum_{i=1}^d \langle b_i \cdot \nabla w_i, w_i \rangle - \text{Re} \sum_{i=1}^d \langle (\nabla_i b_i) \cdot w, w_i \rangle = 0,$$

$$- \text{Re} \langle b_i \cdot \nabla w_i, w_i \rangle = \frac{\kappa}{2} \langle (|x|^\alpha - \alpha |x|^{-\alpha} - \alpha |x|^{-\alpha-2}|x|^2) w_i, w_i \rangle,$$

$$- \langle (\nabla_i b_i) \cdot w, w_i \rangle = -\kappa \langle |x|^{-\alpha} w_i, w_i \rangle + \kappa \alpha \langle |x|^{-\alpha-2} x_i \bar{w}_i(x \cdot w) \rangle.$$

Thus,

$$\frac{1}{2} \partial_t \|w\|_2^2 + \sum_{i=1}^d \|(-\Delta)^\frac{n}{4} w_i\|_2^2 + \kappa \frac{d - \alpha - 2}{2} \langle |x|^{-\alpha} |w|^2 \rangle + \frac{\kappa \alpha \varepsilon}{2} \langle |x|^{-\alpha-2} |x \cdot w|^2 \rangle - \kappa \langle |x|^{-\alpha} |w|^2 \rangle + \kappa \alpha \langle |x|^{-\alpha-2} |x \cdot w|^2 \rangle = 0,$$

and so, since $\kappa > 0$,

$$\frac{1}{2} \partial_t \|w\|_2^2 + \sum_{i=1}^d \|(-\Delta)^\frac{n}{4} w_i\|_2^2 + \kappa \frac{d - \alpha - 2}{2} \langle |x|^{-\alpha} |w|^2 \rangle + \kappa \alpha \langle |x|^{-\alpha-2} |x \cdot w|^2 \rangle \leq 0.$$

Since $d \geq 4$, $\alpha < 2$, we have $d - \alpha - 2 > 0$. Thus, integrating in $t$, we obtain $\|w(t)\|_2^2 \leq \|\nabla f\|_2^2$, $t \geq 0$, as needed. □

Next, set $u_n := u^{\varepsilon_n}$, $u_m := u^{\varepsilon_m}$ and $g(t) := u_n(t) - u_m(t), \quad t \geq 0$. 
Claim 6. \(\|g(t)\|_2 \to 0\) uniformly in \(t \in [0, 1]\) as \(n, m \to \infty\).

Proof of Claim 6. We subtract the equations for \(u_n\) and \(u_m\) and obtain
\[
\partial_t u + (-\Delta)^{\frac{1}{2}} u - b_n \cdot \nabla g - (b_n - b_m) \cdot \nabla u_m = 0,
\]
\[
\partial_t g + (-\Delta)^{\frac{1}{2}} g - b_n \cdot \nabla g - (b_n - b_m) \cdot \nabla u_m = 0.
\]
Concerning the last two terms, we have:
\[
-\text{Re}(b_n \cdot \nabla g, g) = \frac{\kappa}{2} \langle |x|^{-\alpha} - \alpha |x|^{-\alpha - 2} |x|^2 g, g \rangle \geq \frac{\kappa}{2} \frac{d - \alpha}{\alpha} \langle |x|^{-\alpha}, |g|^2 \rangle,
\]
\[
|\langle (b_n - b_m) \cdot \nabla u_m, g \rangle| \leq |\langle 1_{B(0,1)}(b_n - b_m) \cdot \nabla u_m, g \rangle| + |\langle 1_{B(0,1)}^c(b_n - b_m) \cdot \nabla u_m, g \rangle|\\text{(we are using } \|g\|_\infty \leq 2 \|f\|_\infty, \|g\|_2 \leq 2 \|f\|_2)\\text{(we are using Claim 5)}\\text{(we are using Claim 6)}
\]
\[
|\langle (b_n - b_m) \cdot \nabla u_m, g \rangle| \leq 2 \|\nabla f\|_2 \|f\|_\infty + \|1_{B(0,1)}^c(b_n - b_m)\|_\infty \|\nabla u_m\|_2 \|f\|_2
\]
\[
\to 0 \quad \text{as } n, m \to \infty.
\]
Thus, integrating (21) in \(t\) and using the last two observations, we end the proof of Claim 6. \(\square\)

By Claim 6, \(\{e^{-t\Lambda^{\varepsilon_n}} f\}_{\varepsilon_n=1}^\infty, f \in C_c^\infty\) is a Cauchy sequence in \(L^\infty([0, 1], L^2)\). Set
\[
T_2^t f := s \cdot L^2 - \lim_n e^{-t\Lambda^{\varepsilon_n}} f \text{ uniformly in } 0 \leq t \leq 1.
\]
(Clearly, the limit does not depend on the choice of \(\{\varepsilon_n\} \downarrow 0\).) Since \(e^{-t\Lambda^{\varepsilon_n}}\) are contractions in \(L^2\), we have \(\|T_2^t f\|_2 \leq \|f\|_2, t \in [0, 1]\). Extending \(T_2^t\) by continuity to \(L^2\), we obtain that \(T_2^t\) is strongly continuous. Furthermore,
\[
T_2^t f = \lim_n e^{-t\Lambda^{\varepsilon_n}} f \text{ in } L^2 \text{ for all } f \in L^2, \quad 0 \leq t \leq 1.
\]
Finally, extending \(T_2^t\) to all \(t \geq 0\) using the reproduction property, we obtain a contraction \(C_0\) semigroup \(T_2^t := e^{-t\Lambda}, t \geq 0\).

Now, let \(1 \leq r < \infty\). Since \(e^{-t\Lambda^{\varepsilon}}\) is a contraction in \(L^r\), we obtain, by construction (22) of \(e^{-t\Lambda f}, f \in C_c^\infty\), appealing e.g. to Fatou's Lemma, that
\[
\|e^{-t\Lambda f}\|_r \leq \|f\|_r, \quad t \geq 0.
\]
Thus, extending \(e^{-t\Lambda}\) by continuity to \(L^r\), we can define contraction semigroups \(T_r^t := [e^{-t\Lambda}]_{L^r \to L^r}^{\text{th}}\), \(t \geq 0\). The strong continuity of \(T_r^t\) in \(L^r\) is a consequence of strong continuity of \(e^{-t\Lambda}\), contractivity of \(T_r^t\) and Fatou’s Lemma. Write \(T_r^t := e^{-t\Lambda_r}\). Clearly,
\[
e^{-t\Lambda_r} = s \cdot L_r - \lim_n e^{-t\Lambda^{\varepsilon_n}}, \quad t \geq 0.
\]
The latter and Proposition 3 complete the proof of Proposition 9. \(\square\)
8.2. Case \( d = 3 \). The proof of the next proposition works in all dimensions \( d \geq 3 \).

**Proposition 10.** For every \( r \in [1, \infty[ \) the limit
\[
sL' \lim_{\varepsilon \downarrow 0} e^{-t\Lambda^\varepsilon_r} \quad (\text{loc. uniformly in } t \geq 0)
\]
exists and determines a contraction \( C_0 \) semigroup on \( L' \), say, \( e^{-t\Lambda_r} \). There exists a constant \( c_N \neq c_N(\varepsilon) \) such that
\[
\|e^{-t\Lambda_r}\|_{r \to q} \leq c_N t^{-\frac{d}{2}(1-\frac{2}{d})}, \quad t > 0,
\]
for all \( 1 \leq r \leq q \leq \infty \).

**Proof of Proposition 10.** Denote \( u^\varepsilon(t) := e^{-t\Lambda^\varepsilon_r} f, \ f \in C^\infty_c \). For brevity, write \( u \equiv u^\varepsilon \) and \( w := \nabla u \).

**Claim 7.** For every \( r \in ]1, \infty[ \),
\[
\frac{1}{r} \|w(t_1)\|_r^r + \frac{4}{rr'} \int_0^{t_1} \left( \frac{\alpha}{r} \|\nabla w\|_r\|w\|_r \right) dt
\]
\[
+ \frac{\alpha - \alpha - r}{r} \int_0^{t_1} \left( \frac{\alpha - 2}{r} \|w\|_r \right) dt + \alpha \kappa \int_0^{t_1} \langle |x|^{\alpha-2} |x \cdot w|^2 |w|^{r-2} \rangle dt \leq \frac{1}{r} \|\nabla f\|_r^r, \quad t_1 > 0.
\]

In particular, for \( 1 < r < d - \alpha \),
\[
\|w(t_1)\|_r^r + \frac{4}{rr'} \int_0^{t_1} \|w\|_r^r dt \leq \|\nabla f\|_r^r, \quad t_1 > 0, \quad j := \frac{d}{d - \alpha}.
\]

**Proof of Claim 7.** Set \( w_i := \nabla_i u \). We differentiate \( \partial_i u + \Lambda^\varepsilon_r u = 0 \) in \( x_i \), obtaining identity
\[
\partial_i w_i + (-\Delta)^{\frac{\alpha}{2}} w_i - b_\varepsilon \cdot \nabla w_i - (\nabla_i b_\varepsilon) \cdot w = 0,
\]
which we multiply by \( \tilde{w}_i |w|^{r-2} \), integrate over the spatial variables and then sum in \( 1 \leq i \leq d \) to obtain
\[
\frac{1}{r} \partial_t \|w\|_r^r + \text{Re}((-\Delta)^{\frac{\alpha}{2}} w, w|w|^{r-2}) - \text{Re} \sum_{i=1}^d \langle b_\varepsilon \cdot \nabla w_i, w_i |w|^{r-2} \rangle - \text{Re} \sum_{i=1}^d \langle (\nabla_i b_\varepsilon) \cdot w, w_i |w|^{r-2} \rangle = 0.
\]

By Theorem 11 (Appendix A),
\[
\text{Re}((-\Delta)^{\frac{\alpha}{2}} w, w|w|^{r-2}) \geq \frac{4}{rr'} \left((-\Delta)^{\frac{\alpha}{2}} (w|w|^{\frac{r-2}{2}}), (-\Delta)^{\frac{\alpha}{2}} (w|w|^{\frac{r-2}{2}})\right) \equiv \frac{4}{rr'} \sum_{i=1}^d \|(-\Delta)^{\frac{\alpha}{2}} (w_i |w|^{\frac{r-2}{2}})\|_2^2.
\]
Next, integrating by parts, we obtain
\[
-\text{Re} \sum_{i=1}^d \langle b_\varepsilon \cdot \nabla w_i, w_i |w|^{r-2} \rangle = \frac{\kappa}{r} (\|d|x|^{-\alpha} - \alpha |x|^{-\alpha-2} |x|^2 |w|^{r}) \geq \frac{\kappa - \alpha}{r} (|x|^{-\alpha} |w|^{r}),
\]
and
\[
\text{Re} \sum_{i=1}^d \langle (\nabla_i b_\varepsilon) \cdot w, w_i |w|^{r-2} \rangle = \kappa (|x|^{-\alpha} |w|^{r}) - \alpha \kappa (|x|^{-\alpha-2} (x \cdot w)^2 |w|^{r-2}).
\]
The first required inequality follows.
Now, let $1 < r < d - \alpha$. Note that
\[
\sum_{i=1}^{d} \|\langle -\Delta \rangle^{\frac{r}{2}} (w_i |w|^{\frac{r-2}{2}})\|_{2}^{2} \geq c_{S} \sum_{i=1}^{d} \|w_i |w|^{\frac{r-2}{2}}\|_{2j}^{2} = c_{S} \sum_{i=1}^{d} \langle |w_i|^{2j} |w|^{(r-2)j} \rangle^{\frac{1}{2}}
\]
\[
\geq c_{S} \left( \langle |w|^{(r-2)j} \sum_{i=1}^{d} |w_i|^{2j} \rangle \right)^{\frac{1}{2}}
\]
\[
\left( \text{we use } \left( \sum_{i=1}^{d} |w_i|^{2j} \right)^{1/j} \geq \left( \sum_{i=1}^{d} |w_i|^{2} \right)^{d-1/j'} = |w|^{2d-1/j'} \right)
\]
\[
\geq c_{S} d^{-1/j'} \langle |w|^{rj} \rangle^{\frac{1}{2}} = c_{S} d^{-\frac{2}{d}} \|w\|_{rj}^{\frac{1}{2}}.
\]

The second required inequality follows. \(\square\)

Next, set \(u_{n} = u^{\varepsilon_{n}}\), \(u_{m} = u^{\varepsilon_{m}}\). Let \(g(t) := u_{n}(t) - u_{m}(t)\), \(t \geq 0\).

**Claim 8.** \(\|g(t)\|_{2} \rightarrow 0\) uniformly in \(t \in [0, 1]\) as \(n, m \rightarrow \infty\).

**Proof of Claim 8.** We subtract the equations for \(u_{n}\) and \(u_{m}\):
\[
\partial_{t} g + (-\Delta)^{\frac{r}{2}} g - b_{n} \cdot \nabla g - (b_{n} - b_{m}) \cdot \nabla u_{m} = 0.
\]

Multiplying the latter by \(g\) and integrating, we obtain
\[
\|g(t_{1})\|_{2}^{2} + \int_{0}^{t_{1}} \|(-\Delta)^{\frac{r}{2}} g\|_{2}^{2} dt - \text{Re} \int_{0}^{t_{1}} \langle b_{n} \cdot \nabla g, g \rangle dt - \text{Re} \int_{0}^{t_{1}} \langle (b_{n} - b_{m}) \cdot \nabla u_{m}, g \rangle dt = 0
\]
for every \(t_{1} > 0\). Since
\[
-\text{Re} \langle b_{n} \cdot \nabla g, g \rangle = \frac{\kappa}{2} \langle \langle d|x\rangle^{-\alpha} - \alpha|x\rangle^{-\alpha-2} |x|^{2} g, g \rangle \geq \frac{\kappa}{2} \langle |x\rangle^{-\alpha}, |g|^{2} \rangle,
\]
we have
\[
\|g(t_{1})\|_{2}^{2} + \int_{0}^{t_{1}} \|(-\Delta)^{\frac{r}{2}} g\|_{2}^{2} dt + \frac{\kappa}{2} \int_{0}^{t_{1}} \langle |x|^{-\alpha}, |g|^{2} \rangle dt \leq \int_{0}^{t_{1}} \langle (b_{n} - b_{m}) \cdot \nabla u_{m}, g \rangle dt.
\]

(23)

Let us estimate the RHS of (10). Fix \(1 < r < d - \alpha\) (as in the second assertion of Claim 7). Then
\[
|\langle (b_{n} - b_{m}) \cdot \nabla u_{m}, g \rangle| \leq |\langle 1_{B(0,1)}(b_{n} - b_{m}) \cdot \nabla u_{m}, g \rangle| + |\langle 1_{B^{c}(0,1)}(b_{n} - b_{m}) \cdot \nabla u_{m}, g \rangle|
\]

(we apply estimates \(\|g\|_{\infty} \leq 2\|f\|_{\infty}, \|g\|_{(rj)'} \leq 2\|f\|_{(rj)'})
\]
\[
\leq \|1_{B(0,1)}(b_{n} - b_{m})\|_{(rj)'} \|\nabla u_{m}\|_{rj} \|f\|_{(rj)'} + \|1_{B^{c}(0,1)}(b_{n} - b_{m})\|_{\infty} \|\nabla u_{m}\|_{rj} \|f\|_{(rj)'}.
\]

Clearly \(\|1_{B^{c}(0,1)}(b_{n} - b_{m})\|_{\infty} \rightarrow 0\) as \(n, m \rightarrow \infty\). The same is true for \(\|1_{B(0,1)}(b_{n} - b_{m})\|_{(rj)'\infty}\) since \((rj)' = \frac{rd}{rd-d-\alpha} < \frac{d}{\alpha-1}\). Thus, in view of Claim 7
\[
\int_{0}^{t_{1}} |\langle (b_{n} - b_{m}) \cdot \nabla u_{m}, g \rangle| dt
\]
\[
\leq \left( \|1_{B(0,1)}(b_{n} - b_{m})\|_{(rj)'\infty} + \|1_{B^{c}(0,1)}(b_{n} - b_{m})\|_{\infty} \|f\|_{(rj)'\infty} \right) 2 \int_{0}^{t_{1}} \|\nabla u_{m}\|_{rj} dt \rightarrow 0
\]
as \(n, m \rightarrow \infty\). \(\square\)
Now, we argue as in the proof of Proposition \[9\] to obtain that for every \( r \in [1, \infty] \) the limit
\[
s-L^r\lim_{\varepsilon \downarrow 0} e^{-t\varepsilon_r^*}, \quad t \geq 0
\]
determines a contraction \( C_0 \) semigroup on \( L^r \). It is easily seen that the limit does not depend on the choice of \( \varepsilon_n \).

The last assertion follows now from Proposition \[8\]. The proof of Proposition \[10\] is completed.

\[\square\]

9. Construction of the semigroup \( e^{-t\Lambda^r}, \Lambda^r = (\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b \) in \( L^r, 1 \leq r < \infty \)

Set \( (\Lambda^\varepsilon)^* := (\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b_{\varepsilon} \), \( D((\Lambda^\varepsilon)^*) = \mathcal{W}^{\alpha,r} \). By the Hille Perturbation Theorem, \(-(\Lambda^\varepsilon)^*_r\) is the generator of a holomorphic \( C_0 \) semigroup in \( L^r \) (arguing as in Section \[8\] the argument there also shows that \((\Lambda^\varepsilon)^*: (\Delta)^{\frac{\alpha}{2}} + \nabla \cdot b_{\varepsilon}, D((\Lambda^\varepsilon)^*) = D((-\Delta)^{\frac{\alpha}{2}}) \) is the generator of a holomorphic semigroup in \( C_u \)).

**Proposition 11.** For every \( r \in [1, \infty[ \) and \( \varepsilon > 0 \), \( e^{-t(\Lambda^\varepsilon)^*_r} \) is a contraction \( C_0 \) semigroup. There exists a constant \( c_N \neq c_N(\varepsilon) \) such that
\[
\|e^{-t(\Lambda^\varepsilon)^*_r}\|_{r \to q} \leq c_N t^{-\frac{\alpha}{\frac{\alpha}{2}(1 \frac{1}{q})}, \quad t > 0},
\]
for all \( 1 \leq r \leq q \leq \infty \).

**Proof.** The semigroup \( e^{-t(\Lambda^\varepsilon)^*_r} \) is constructed in \( L^r \) repeating the argument in Section \[8\]. The ultra contractivity estimate for \( 1 < r \leq q < \infty \) follows from Proposition \[8\] by duality, and for all \( 1 \leq r \leq q \leq \infty \) upon taking limits \( r \downarrow 1, q \uparrow \infty \). \[\square\]

**Proposition 12.** For every \( r \in [1, \infty[ \) the limit
\[
s-L^r\lim_{\varepsilon \downarrow 0} e^{-t(\Lambda^\varepsilon)^*_r} \quad (\text{loc. uniformly in } t \geq 0)
\]
exists and determines a contraction \( C_0 \) semigroup in \( L^r \), say, \( e^{-t\Lambda^r} \). There exists a constant \( c_N \) such that
\[
\|e^{-t\Lambda^r}\|_{r \to q} \leq c_N t^{-\frac{\alpha}{\frac{\alpha}{2}(1 \frac{1}{q})}, \quad t > 0},
\]
for all \( 1 \leq r \leq q \leq \infty \).

We have for \( 1 < r < \infty \)
\[
\langle e^{-t\Lambda^r}(b)f, g \rangle = \langle f, e^{-t(\Lambda^\varepsilon)^*_r}g \rangle, \quad t > 0, \quad f \in L^r', \quad r' = \frac{r}{r-1}, \quad g \in L^r.
\]

**Proof.** First, let \( r = 2 \). In view of Proposition \[11\] we can argue as in the proof of \[KSS\] Prop.10, appealing to the Rellich-Kondrashov Theorem, to obtain: For every sequence \( \varepsilon_n \downarrow 0 \) there exists a subsequence \( \varepsilon_{n_m} \) such that the limit
\[
s-L^2\lim_{m} e^{-t(\Lambda^{\varepsilon_{n_m}})^*} \quad (\text{loc. uniformly in } t \geq 0)
\]
exists and determines a \( C_0 \) semigroup in \( L^2 \).

On the other hand, since
\[
\langle e^{-t\Lambda^\varepsilon}f, g \rangle = \langle f, e^{-t(\Lambda^\varepsilon)^*}g \rangle, \quad t > 0, \quad f, g \in L^2,
\]
it follows from Proposition \[10\] that for every \( g \in L^2 \) \( e^{-t(\Lambda^\varepsilon)^*}g \) converge weakly in \( L^2 \) as \( \varepsilon \downarrow 0 \). Thus, the limit in \[24\] does not depend on the choice of \( \varepsilon_{n_m} \) and \( \varepsilon_n \).
For $1 \leq r < \infty$, we repeat the argument in the end of the proof of Proposition 9 appealing to Proposition 11.

The last assertion follows from the analogous property of $e^{-tA_r^\varepsilon}$, $e^{-t(A_r^\varepsilon)^*}$, $\varepsilon > 0$ and Propositions 10, 12.

**Appendix A.** $L^r$ (vector) inequalities for symmetric Markov generators

Let $X$ be a set and $\mu$ a $\sigma$-finite measure on $X$. Let $T^t = e^{-tA}$, $t \geq 0$, be a symmetric Markov semigroup in $L^2(X, \mu)$. Let

$$T^t_r := [T^t \mid L^2 \cap L^r]_{L^r \to L^r}, \quad t \geq 0,$$

a contraction $C_0$ semigroup on $L^r$, $r \in [1, \infty]$. Put $T^t_r := e^{-tA_r}$.

**Theorem 10.** Let $f_i \in D(A_r)$ ($1 \leq i \leq m$), $r \in [1, \infty]$. Set $f := (f_i)_{i=1}^m$, $f(r) := f|f|^{\frac{r-2}{2}}$. Then $f_i|f|^{\frac{r-2}{2}} \in D(A_r^\frac{1}{2})$ ($1 \leq i \leq m$) and, applying the operators coordinate-wise, we have

$$\frac{4}{r'r'}\langle A_r^\frac{1}{2}f(r), A_r^\frac{1}{2}f(r) \rangle \leq \text{Re}\langle A_r f, f|f|^{r-2}\rangle \leq \mathcal{K}(r)\langle A_r^\frac{1}{2}f(r), A_r^\frac{1}{2}f(r) \rangle,$$

where $\mathcal{K}(r) := \sup_{s \in [0, \frac{1}{2}] \left[ \frac{1}{(1 + s^2)(1 + s^2)^2} \right], r' = \frac{r}{r-1}}$, $|\text{Im}\langle A_r f, f|f|^{r-2}\rangle| \leq \frac{|r-2|}{2\sqrt{r-1}} \text{Re}\langle A_r f, f|f|^{r-2}\rangle$,

where

$$\langle A_r^\frac{1}{2}f(r), A_r^\frac{1}{2}f(r) \rangle = \sum_{i=1}^m \|A_r^\frac{1}{2}(f_i)|f_i|^{\frac{r-2}{2}}\|^2_2, \quad \langle A_r f, f|f|^{r-2}\rangle = \sum_{i=1}^m \langle A_r f_i, f_i|f_i|^{r-2}\rangle.$$

Theorem 10 is a prompt but useful modification of [LS, Theorem 2.1] (corresponding to the case $m = 1$): it allows us to control higher-order derivatives of $u(t) = e^{-tA}f$, $A \supset (-\Delta)^2 + b \cdot \nabla$, $f \in C^\infty_c$ in the proof of Proposition 10 (see Claim 7 there).

For the sake of completeness, we included the detailed proof below.

1. We will need

**Claim 9.** There exists a finitely additive measure $\mu_\pi$ on $X \times X$, symmetric in the sense that $\mu_\pi(A \times B) = \mu_\pi(B \times A)$ on any $\mu$-measurable sets of finite measure $A$ and $B$, and satisfying

$$\langle T^t f, g \rangle = \int_{X \times X} f(x)g(y)d\mu_\pi(x, y) \quad (f, g \in L^1 \cap L^\infty).$$

In order to justify the claim, let us introduce the Banach space $L^\infty = L^\infty(X, \mathcal{M}_\mu)$, the Banach space of all bounded $\mu$-measurable functions, endowed with the norm $\|f\| := \sup\{|f(x)| : x \in X\}$.

Let $N^\infty \equiv N^\infty(X, \mathcal{M}_\mu)$ be the set of all $\mu$-negligible functions, so that $L^\infty = L^\infty/N^\infty$. Denoting by $\pi : f \mapsto \tilde{f}$ the canonical mapping of $L^\infty$ onto $L^\infty$, we can identify $L^\infty$ with $\pi(L^\infty)$. Since $\mu$ is $\sigma$-finite, there exists a lifting $\rho : L^\infty \to L^\infty$, a linear multiplicative positivity preserving map such that

$$\rho(1_G) = 1_G \text{ for all } G \in \mathcal{M}_\mu \text{ with } \mu(G) < \infty.$$

Given $t > 0$ define $T^t_\rho : L^\infty \to L^\infty$ by

$$T^t_\rho f := \rho(T^t f),$$
and so \( T^t_\rho \) is a positivity preserving semigroup, and
\[
\langle T^t_\rho f, g \rangle = \langle T^t \tilde{f}, \tilde{g} \rangle \quad (\tilde{f}, \tilde{g} \in L^\infty \cap L^1).
\]
The following set function is associated with the semigroup \( T^t_\rho \):
\[
P(t, x, G) := (T^t_\rho 1_G)(x) \quad (t > 0, x \in X, G \in \mathcal{M}_\mu).
\]
This function satisfies the following evident properties:

1. \( P(t, x, G) (G \in \mathcal{M}_\mu) \) is finitely additive.
2. \( P(t, x, X) \leq 1 \).
3. \( \int f(y) P(t, \cdot, dy) \) exists and equals to \( T^t_\rho f(\cdot) (f \in \mathcal{L}^\infty) \).

Set by definition
\[
\mu_t(A \times B) = \int_A P(t, x, B) d\mu(x) \quad (A, B \in \mathcal{M}_\mu).
\]
The claimed symmetry of \( \mu_t \) is a direct consequence of the self-adjointness of \( T^t \) and the fact that we can identify \( T^t_\rho 1_G \) and \( T^t 1_G \) for every \( G \in \mathcal{M}_\mu \) of finite measure.

2. We are in position to complete the proof of Theorem 10.

**Proof of Theorem 10.** We will need the following elementary estimates: for all \( s, t \in [0, \infty[, \ r \in [1, \infty[, \)
\[
\frac{4}{rr'}(s^r + t^r - 2b(st)^{\frac{r}{r'}}) \\
\leq s^r + t^r - b(st^{r-1} + ts^{r-1}) \\
\leq \kappa(r)(s^r + t^r - 2b(st)^{\frac{r}{r'}}), \quad b \in [-1, 1]
\]
(Lemma 12(3), (5) below)
\[
|a||s^{r-1} - ts^{r-1}| \leq \frac{|r-2|}{2\sqrt{r-1}}[s^r + t^r - \sqrt{1-a^2}(st^{r-1} + ts^{r-1})], \quad a \in [-1, 1]
\]
(\(**\))
(Lemma 12(4) below).

We are going to establish the following inequalities: for all \( f \in L^r \)
\[
\frac{4}{rr'}\langle (1 - T^t_2)f, (r), f(r) \rangle \leq \text{Re}\langle (1 - T^t_1)f, f, f|f|^{r-2} \rangle \leq \kappa(r)\langle (1 - T^t_2)f, f(r), f(r) \rangle, 
\]
(25)
\[
|\text{Im}\langle (1 - T^t_1)f, f, f|f|^{r-2} \rangle | \leq \frac{|r-2|}{2\sqrt{r-1}}\text{Re}\langle (1 - T^t_1)f, f, f|f|^{r-2} \rangle.
\]
(26)
The the required estimates would follow from the definitions of \( A_r \) and \( A^\frac{3}{2} \). Indeed, for \( f \in D(A_r) \),
\[
s-L^p-\lim_{t \downarrow 0} \frac{1}{t}(1 - T^t_1)f \text{ exists and equals to } A_r f.
\]
Combining the LHS of (25) and Fatou’s Lemma, it is seen that \( J := \lim_{t \downarrow 0} \frac{1}{t}\langle (1 - T^t)f(r), f(r) \rangle \) exists and is finite. By the spectral theorem for self-adjoint operators, the latter means that \( f(r) \in D(A^\frac{3}{2}) \) and \( J = \|A^\frac{3}{2} f(r)\|^2_2 \).
First, let \( f \in L^1 \cap L^\infty \) with \( \text{sprt} \, f \subset G, \; G \in \mathcal{M}_\mu, \; \mu(G) < \infty \). Using Claim 9, we have

\[
\langle T^t f, f |f|^{-2} \rangle = \frac{1}{2} (T^t f, f |f|^{-2}) + \frac{1}{2} (f, T^t (f |f|^{-2})) = \frac{1}{2} \int \langle f(x) \cdot \bar{f}(y), f(y) |f(\bar{y})|^{-2} \rangle \mu(x, y),
\]

\[
\langle T^t f, f \rangle = \frac{1}{2} \int \bar{f}(x) \cdot f(x) \mu(x, y) + \frac{1}{2} \int f(x) \cdot \bar{f}(x) \mu(x, y),
\]

\[
\langle T^t 1_G, |f| \rangle = \langle 1_G, T^t |f| \rangle = \frac{1}{2} (P(t, \cdot, G)|f(\cdot)|^r) + \frac{1}{2} \int |f(y)|^r P(t, \cdot, dy)
\]

\[
\|f\|^r = \langle T^t 1_G, |f| \rangle + \| (1 - T^t 1_G), |f| \rangle.
\]

Setting \( s = |f(x)|, \; l = |f(y)|, \; \beta := \frac{f(x) \cdot f(y)}{|f(x)| \cdot |f(y)|}, \; b := \text{Re} \beta, \; a := \text{Im} \beta \), we obtain

\[
\langle (1 - T^t) f, f |f|^{-2} \rangle = \langle (1 - T^t 1_G), |f|^{-2} \rangle + \frac{1}{2} \int [s^r + l^r - \beta s^{r-1} - \beta \bar{l} s] \mu(x, y),
\]

\[
\text{Re} \langle (1 - T^t) f, f |f|^{-2} \rangle = \langle (1 - T^t 1_G), |f|^{-2} \rangle + \frac{1}{2} \int [s^r + l^r - b(s^{r-1} + \bar{l} s^{r-1})] \mu(x, y),
\]

\[
\langle (1 - T^t) f, f \rangle = \langle (1 - T^t 1_G), |f| \rangle + \frac{1}{2} \int [s^r + l^r + b(\bar{l} s^{r-1} - l s^{r-1})] \mu(x, y),
\]

\[
\text{Im} \langle (1 - T^t) f, f |f|^{-2} \rangle = \frac{1}{2} \int a(s^{r-1} - l s^{r-1}) \mu(x, y).
\]

Next, employing \([*, **] \), we obtain \([28], [29] \) but for \( f \in L^1 \cap L^\infty \) with \( \text{sprt} \, f \subset G, \; \mu(G) < \infty \).

To end the proof, we note that \( \mu \) is a \( \sigma \)-finite measure, and so we can first get rid of the condition “\( \text{sprt} \, f \subset G, \; \mu(G) < \infty \)”, and then, using the truncated functions

\[
g_n = \begin{cases} 
g, & \text{if } |g| \leq n, \\
0, & \text{if } |g| > n, \end{cases} \quad n = 1, 2, \ldots
\]

and the Dominated Convergence Theorem, to get rid of “\( f \in L^1 \cap L^\infty \)”. \( \square \)

For the sake of completeness, we also include the following result concerning the scalar case.

**Theorem 11.** If \( 0 \leq f \in D(A_r) \), then

\[
\frac{4}{r r'} \| A^{\frac{1}{r}} f^{\frac{1}{r}} \|^2 \leq \langle A_r f, f^{r-1} \rangle \leq \| A^{\frac{1}{r}} f^{\frac{1}{r}} \|^2, \quad (iii)
\]

Moreover, if \( r \in [2, \infty) \) and \( f \in D(A) \cap L^\infty \), then \( f_{(r)} := |f|^{\frac{1}{r}} \text{sgn} \, f \in D(A^{\frac{1}{2}}) \) and

\[
\frac{4}{r r'} \| A^{\frac{1}{2}} f_{(r)} \|^2 \leq \text{Re} \langle A f, f^{r-1} \text{sgn} \, f \rangle \leq \text{Re} \langle A f, f^{r-1} \text{sgn} \, f \rangle \| A^{\frac{1}{2}} f_{(r)} \|^2, \quad \text{sgn} \, f := \frac{f}{|f|} \quad (i')
\]
If \( r \in [2, \infty[ \) and \( 0 \leq f \in D(A) \cap L^\infty \), then \( f^\sharp \in D(A^\frac{1}{2}) \) and

\[
\frac{4}{rt^r} \| A^\frac{1}{2} f^\sharp \|_2^2 \leq \langle Af, f^{r-1} \rangle \leq \| A^\frac{1}{2} f^\sharp \|_2^2. \tag{i3}'
\]

**Proof.** Follows closely the proof of Theorem 10 where, instead of inequalities (25), (26), we use

\[
\frac{4}{rt^r} \langle (1 - T^r) f^\sharp, f^\sharp \rangle \leq \langle (1 - T^r) f, f^{r-1} \rangle \leq \langle (1 - T^r) f^\sharp, f^\sharp \rangle \quad (f \in L^r_+). \]

\[\Box\]

In the proof of Theorem 10 we use

**Lemma 12.** Let \( s, t \in [0, \infty[ \), \( r \in [1, \infty[ \) and \( b \in [-1, 1] \). Then

\[
\frac{4}{rt^r} (s^\sharp - t^\sharp)^2 \leq (s - t)(s^{r-1} - t^{r-1}) \leq (s^\sharp - t^\sharp)^2. \tag{l1}
\]

\[
(s^\sharp + t^\sharp)^2 \leq (s + t)(s^{r-1} + t^{r-1}) \leq \kappa (s^\sharp + t^\sharp)^2. \tag{l2}
\]

\[
\frac{4}{rt^r} (s^\sharp + t^\sharp + 2b(st)^\sharp) \leq s^r + t^r + b(st^{r-1} + ts^{r-1}). \tag{l3}
\]

\[
|b||st^{r-1} - ts^{r-1}| \leq \frac{|r - 2|}{2 \sqrt{r - 1}} [s^r + t^r - \sqrt{1 - b^2}(st^{r-1} + ts^{r-1})]. \tag{l4}
\]

\[
s^r + t^r + b(st^{r-1} + ts^{r-1}) \leq \kappa (s^r + t^r + 2b(st)^\sharp). \tag{l5}
\]

**Proof.** The RHS of (l1) and the LHS of (l2) are consequences of the inequality \( 2|\alpha||\beta| \leq \alpha^2 + \beta^2 \).

The RHS of (l2) follows from the definition of \( \kappa (r) \).

The LHS of (l1) follows from

\[
\frac{4}{r} (s^\sharp - t^\sharp)^2 = (\int_t^s z^\sharp - 1 dz)^2 \leq \int_t^s dz \cdot \int_t^s z^r - 2 dz.
\]

(l3) is a consequence of the LHS of (l1).

To derive (l4) set

\[
A = st^{r-1} - ts^{r-1}, \quad B = \frac{|r - 2|}{2 \sqrt{r - 1}} (st^{r-1} + ts^{r-1}), \quad C = \frac{|r - 2|}{2 \sqrt{r - 1}} (s^r + t^r),
\]

and note that \( A^2 + B^2 \leq C^2 \Rightarrow |A \sin \theta| + |B \cos \theta| \leq C. \)

The inequality \( A^2 + B^2 \leq C^2 \) follows from

\[
(st^{r-1} - ts^{r-1})^2 \leq \left( \frac{r - 2}{r} \right)^2 (s^r - t^r)^2 \quad (*)
\]

and the LHS of (l1) and (l2).

Setting \( v = s/t \), (*) takes the form

\[
|v^{r-1} - v| \leq \frac{|r - 2|}{r} |v^r - 1|.
\]

All possible cases are reduced to the case where \( v > 1 \) and \( r > 2 \).

If \( \frac{r - 2}{r} v \geq 1 \), then the inequality \( v^{r-1} - v \leq \frac{r - 2}{r} v^r - \frac{r - 2}{r} v \) is selfevident. If \( 1 < v < \frac{r}{r - 2} \), we set \( \psi(v) = \frac{r - 2}{r} v^r - v^{r-1} + v - \frac{r - 2}{r} v \) and note that \( \frac{d}{dv} \psi(v) \geq 0 \) by Young’s inequality.

Finally, (l5) follows from the RHS of (l2) and the following elementary inequality:

\[
\frac{A + BB}{A + bC} \leq \frac{A + B}{A + C} \quad (b \in [-1, 1]),
\]

provided that \( A > C \) and \( B \geq C > 0 \).
Let $R \subseteq L^1 \cap L^\infty$, where $\beta$ are valid for all $(t, s)$ and $f \in L^1 \cap L^\infty$. Then
\[
\|U^{t,s}f\|_r \leq M(t-s)^{-\nu/(1-\beta)}\|f\|_p,
\]
where $\beta = \frac{r q-p}{q-r-p}$ and $M = 2\nu/(1-\beta)^2 M_1 M_2^{1/(1-\beta)}$.

**Proof.** Set $2t_s = t + s$. The hypotheses and Hölder’s inequality imply
\[
\|U^{t,s}f\|_r \leq M_2(t - t_s)^{-\nu/2} \|U^{t_s,s}f\|_q \\
\leq M_2(t - t_s)^{-\nu} \|U^{t_s,s}f\|_q^2 \|U^{t_s,s}f\|_r^{1-\beta} \\
\leq M_2 M_1^{-\beta}(t - t_s)^{-\nu} \|U^{t_s,s}f\|_r \|f\|_p^{1-\beta},
\]
and hence
\[
(t - s)^{\nu/(1-\beta)} \|U^{t,s}f\|_r \leq M_2 M_1^{-\beta} 2^{\nu/(1-\beta)} \left((t_s - s)^{\nu/(1-\beta)} \|U^{t_s,s}f\|_r / \|f\|_p\right)^{\beta}.
\]
Setting $R_{2T} := \text{sup}_{t-s \in [0, T]} \left((t - s)^{\nu/(1-\beta)} \|U^{t,s}f\|_r / \|f\|_p\right)$, we obtain from the last inequality that $R_{2T} \leq M^{1-\beta}(R_T)^\beta$. But $R_T \leq R_{2T}$, and so $R_{2T} \leq M$. \(\square\)

**Corollary 4.** Let $U^{t,s} : L^1 \cap L^\infty \to L^1 + L^\infty$ be an evolution family of operators. Suppose that, for some $1 < p < q < r \leq \infty$, $\nu > 0$, $M_1$ and $M_2$, the inequalities
\[
\|U^{t,s}f\|_r \leq M_1 \|f\|_r \quad \text{and} \quad \|U^{t,s}f\|_q \leq M_2(t-s)^{-\nu} \|f\|_p
\]
are valid for all $(t, s)$ and $f \in L^1 \cap L^\infty$. Then
\[
\|U^{t,s}f\|_r \leq M(t-s)^{-\nu/(1-\beta)} \|f\|_p,
\]
where $\beta = \frac{r q-p}{q-r-p}$ and $M = 2\nu/(1-\beta)^2 M_1 M_2^{1/(1-\beta)}$.

**Appendix C. The range of an accretive operator**

In the proof of Theorem 2 we use the following well known result.

Let $P$ be a closed operator on $L^1$ such that $\text{Re}((\lambda + P)f, \frac{f}{|f|}) \geq 0$ for all $f \in D(P)$, and $R(\mu + P)$ is dense in $L^1$ for a $\mu > \lambda$.

Then $R(\mu + P) = L^1$. 

Indeed, let \( y_n \in R(\mu + P) \), \( n = 1, 2, \ldots \), be a Cauchy sequence in \( L^1 \); \( y_n = (\mu + P)x_n, x_n \in D(P) \). Write \([f, g] := \langle f, \frac{g'}{g} \rangle\). Then

\[
(\mu - \lambda)\|x_n - x_m\|_1 = (\mu - \lambda)[x_n - x_m, x_n - x_m] \\
\leq (\mu - \lambda)[x_n - x_m, x_n - x_m] + [(\lambda + P)(x_n - x_m), x_n - x_m] \\
= [(\mu + P)(x_n - x_m), x_n - x_m] \leq \|y_n - y_m\|_1.
\]

Thus, \( \{x_n\} \) is itself a Cauchy sequence in \( L^1 \). Since \( P \) is closed, the result follows.

References

[BJ] K. Bogdan and T. Jakubowski, Estimates of heat kernel of fractional Laplacian perturbed by gradient operators, Comm. Math. Phys., 271 (2007), p. 179-198.

[CKSV] S. Cho, P. Kim, R. Song and Z. Vondraček, Factorization and estimates of Dirichlet heat kernels for non-local operators with critical killings, arXiv:1809.01782 (2018).

[JW] T. Jakubowski and J. Wang, Heat kernel estimates for fractional Schrödinger operators with negative Hardy potential, arXiv:1809.02425 (2018).

[Ka] T. Kato. Perturbation Theory for Linear Operators. Springer-Verlag Berlin Heidelberg, 1995.

[KM] D. Kinzebulatov and K.R. Madou, On admissible singular drifts of symmetric \( \alpha \)-stable process, Preprint, arXiv:2002.07001 (2020).

[KS] D. Kinzebulatov and Yu. A. Semënov, On the theory of the Kolmogorov operator in the spaces \( L^p \) and \( C_\infty \). Ann. Sc. Norm. Sup. Pisa (5), to appear.

[KSS] D. Kinzebulatov, Yu. A. Semënov and K. Szczypkowski, Heat kernel of fractional Laplacian with Hardy drift via desingularizing weights, arXiv:1904.07363 (2019).

[LS] V. A. Liskevich, Yu. A. Semënov, Some problems on Markov semigroups, In: “Schrödinger Operators, Markov Semigroups, Wavelet Analysis, Operator Algebras” M. Demuth et al. (eds.), Mathematical Topics: Advances in Partial Differential Equations, 11, Akademie Verlag, Berlin (1996), 163-217.

[MM] Y. Maekawa, H. Miura, Upper bounds for fundamental solutions to non-local diffusion equations with divergence free drift, J. Funct. Anal., 264 (2013), p. 2245-2268.

[MM2] Y. Maekawa, H. Miura, On fundamental solutions for non-local parabolic equations with divergence free drift, Adv. Math., 247 (2013), p. 123-191.

[MeSS] G. Metafune, M. Sobajima and C. Spina, Kernel estimates for elliptic operators with second order discontinuous coefficients, J. Evol. Equ. 17 (2017), p. 485-522.

[MeSS2] G. Metafune, L. Negro and C. Spina, Sharp kernel estimates for elliptic operators with second-order discontinuous coefficients, J. Evol. Equ. 18 (2018), p. 467-514.

[MS0] P. D. Milman and Yu. A. Semënov, Desingularizing weights and heat kernel bounds, Preprint (1998).

[MS1] P. D. Milman and Yu. A. Semënov, Heat kernel bounds and desingularizing weights, J. Funct. Anal., 202 (2003), p. 1-24.

[MS2] P. D. Milman and Yu. A. Semënov, Global heat kernel bounds via desingularizing weights, J. Funct. Anal., 212 (2004), p. 373-398.

[N] J. Nash. Continuity of solutions of parabolic and elliptic equations, Amer. Math. J., 80 (1) (1958), p. 931-954.

[VSC] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon. “Analysis and Geometry on Groups”, Cambridge Univ. Press, 1992.