STABILITY OF THREE-DIMENSIONAL STOCHASTIC NAVIER-STOKES EQUATION WITH MARKOV SWITCHING

PO-HAN HSU

ABSTRACT. A right continuous Markov chain is introduced in the noise terms of the three-dimensional stochastic Navier-Stokes equation, and we call such stochastic system as stochastic Navier-Stokes equation with Markov switching. In the present article, we study the p-th moment exponential stability and the almost surely exponential stability of the solution to the equation.

1. INTRODUCTION

Let $G$ be an open bounded domain in $\mathbb{R}^3$ with a smooth boundary. Let the three-dimensional vector-valued function $u(x, t)$ and the real-valued function $p(x, t)$ denote the velocity and pressure of the fluid at each $x \in G$ and time $t \in [0, T]$. The motion of viscous incompressible flow on $G$ with no slip at the boundary is described by the Navier-Stokes system:

\begin{align}
\frac{\partial}{\partial t} u - \nu \Delta u + (u \cdot \nabla) u - \nabla p &= f(t) \quad \text{in } G \times [0, T], \\
\nabla \cdot u &= 0 \quad \text{in } G \times [0, T], \\
u u(x, t) &= 0 \quad \text{on } \partial G \times [0, T], \\
u u(x, 0) &= u_0(x) \quad \text{on } G \times \{t = 0\},
\end{align}

where $\nu > 0$ denotes the viscosity coefficient, and the function $f(t)$ is an external body force. Recalling the Helmholtz decomposition, which implies that $L^2(G)$ can be written as a direct sum of solenoidal part and irrotational part, and applying the Leray projector to equation (1.1), one may write equation (1.1) in the abstract evolution form on a suitable space as follows (see, e.g., [16, 24] for details):

\begin{equation}
\frac{du(t)}{dt} + [\nu A u(t) + B(u(t))]dt = f(t)dt + \sigma(t, u(t))dW(t),
\end{equation}

where $A$ is the Stokes operator and $B$ is the nonlinear inertial operator introduced in Section 2.

A random body force, in the form of a multiplicative noise driven by a Wiener process $W(t)$, is added to the model (see, e.g., [4]) so that one obtains

\begin{equation}
\frac{du(t)}{dt} + [\nu A u(t) + B(u(t))]dt = f(t)dt + \sigma(t, u(t))dW(t).
\end{equation}

Moreover, if the noise is allowed to be “discontinuous,” then a term driven by a Poisson random measure $N_1(dz, ds)$ (which is independent of $W(t)$) is added so that the equation becomes

\begin{equation}
\frac{du(t)}{dt} + [\nu A u(t) + B(u(t))]dt = f(t)dt + \sigma(t, u(t))dW(t) + \int_Z G(t, u(t-), z)\tilde{N}_1(dz, dt),
\end{equation}

\begin{date}March 31, 2022.\end{date}

2010 Mathematics Subject Classification. 35Q30, 60H15, 37L15.

Key words and phrases. Stochastic Navier-Stokes equations, hybrid switching diffusions, stability, jump processes.
where \( \tilde{N}_1(dz, dt) := N_1(dz, dt) - \nu_1(dz)dt \) and \( \nu_1(dz)dt \) is the intensity measure of \( N_1(dz, dt) \).

Let \( m > 0 \) be a fixed integer, \( S = \{1, 2, \ldots, m\} \), and \( \{r(t) : t \in \mathbb{R}^+\} \) be a right continuous Markov chain taking values in \( S \). The following equation whose noise terms depend on the Markov chain \( r(t) \) allows for transition in the type of random forces that perturb the Navier-Stokes equation:

\[
\frac{du(t)}{dt} + [\nu A u(t) + B(u(t))]dt = \sigma(s, u(t), r(t))dW(t) + \int_Z G(t, u(t), r(t), z)\tilde{N}_1(dz, dt),
\]

(1.3)

where \( u(0) = x \) in a specified space, and is called the stochastic Navier-Stokes equation with Markov switching. In particular, if there is no discontinuous noise, the equation appears to be

\[
\frac{du(t)}{dt} + [\nu A u(t) + B(u(t))]dt = \sigma(s, u(t), r(t))dW(t)
\]

(1.4)

with \( u(0) = x \).

The stochastic Navier-Stokes equations with Markov switching was introduced by the author and his adviser [11]; it is shown that the equation admits a weak (in the sense of stochastic analysis and partial differential equations) solution. Later, some asymptotic behaviors was further discussed in [12]. In the present article, we will study the stability of the solution.

Stability is a classical topic in the study of differential equations. For the (deterministic) Navier-Stokes equations, the interested reader may consult, e.g., [3, 7, 9] and references therein; for stability of the stochastic Navier-Stokes equations, the interested reader may consult, e.g., [5, 10, 29].

For a general class of stochastic differential equations, stability is also a topic that has been studied by a number of authors at various levels of generality; the interested reader is referred to the papers by Wu et al [25] and Zhu [28] and the books by Arnold [1], Khasminskii [14], and Mao [17]. If there is a Markov chain in the noise term, then such equations will be called equations with Markov/Markovian switching or hybrid diffusion in literature. For the equations with Markov switching, Mao and his collaborator studied the exponential stability and the asymptotic stability in distribution for a class of nonlinear stochastic differential equations with Markov switching in [18] and [26], respectively. Recently, Zhou et al studied the exponential stability for a delay hybrid system [30], and Deng et al studied the stability for hybrid differential equations by stochastic feedback controls [6]. For a more complete discussion on the stochastic differential equations with Markov switching, we refer the interested reader to the books by Mao and Yuan [19] and Yin and Zhu [27].

The aim of the present article is to investigate the \( p \)-th moment exponential stability and the almost surely exponential stability of both the equations (1.3) and (1.4).

We recall the definitions of the \( p \)-th moment exponential stability and almost surely exponential stability below for the benefit of the reader. Let \( u(t) \) denote the solution to equation (1.3) or (1.4). The equation is called

1. \( p \)-th moment exponentially stable if

\[
\limsup_{t \to \infty} \frac{\log(\mathbb{E}|u(t)|^p)}{t} < 0,
\]

and the case \( p = 2 \) is called exponential stability in mean square;
almost surely exponentially stable if

\[
\limsup_{t \to \infty} \frac{\log(|u(t)|)}{t} < 0
\]

almost surely.

For both equations (1.3) and (1.4), indeed, they are \( p \)-th moment exponentially stable for \( p \geq 2 \) (Theorem 3.1) if the noise terms satisfy a suitable growing and Lipschitz conditions (list as Hypotheses \( \mathbf{H} \) in Section 3).

Suppose, further, the noise terms satisfy a stronger condition (list as Hypotheses \( \mathbf{H}' \) in Section 4), then both equations (1.3) and (1.4) are indeed almost surely exponentially stable (Theorem 4.1).

The rest of the present article is organized as follows. The preliminaries and functional analytic setup are introduced in Section 2. In section 3, we introduce the Hypotheses \( \mathbf{H} \) and study the \( p \)-th moment exponential stability. In section 4, we introduce the Hypotheses \( \mathbf{H}' \) and study the almost surely exponential stability.

2. Preliminaries and Functional Analytic Setup

2.1. Function Space and Operators. Let \( G \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary, \( \mathcal{D}(G) \) be the space of \( C^\infty \)-functions with compact support contained in \( G \), and \( V := \{ u \in \mathcal{D}(G) : \nabla \cdot u = 0 \} \). Let \( \mathcal{H} \) and \( \mathcal{V} \) be the completion of \( V \) in \( L^2(G) \) and \( W_{0}^{1,2}(G) \), respectively. Then it can be shown that (see, e.g., [24, Sec. 1.4, Ch. II])

\[
\mathcal{H} = \{ u \in L^2(G) : \nabla \cdot u = 0, \ u \cdot n|_{\partial G} = 0 \}, \\
\mathcal{V} = \{ u \in W_{0}^{1,2}(G) : \nabla \cdot u = 0 \},
\]

and we denote the \( H \)-norm (\( V \)-norm, resp.) by \( | \cdot | \) (\( \| \cdot \| \), resp.) and the inner product on \( H \) (on \( V \), resp.) by \( \langle \cdot, \cdot \rangle \) (\( \langle \cdot, \cdot \rangle \), resp.). The duality pairing between \( V' \) and \( V \) is denoted by \( \langle \cdot, \cdot \rangle_{\mathcal{V}} \), or simply by \( \langle \cdot, \cdot \rangle \) when there is no ambiguity. In addition, we have the following inclusion between the spaces: \( V \hookrightarrow H \hookrightarrow V' \), and both of the inclusions \( V \hookrightarrow H \) and \( H \hookrightarrow V' \) are dense, compact embeddings (see, e.g., [23, Lemma 1.5.1 and 1.5.2, Ch. II]).

Let \( A : \mathcal{V} \to \mathcal{V}' \) be the Stokes operator and \( \lambda_1 \) be the first eigenvalue of \( A \). Then the Poincaré inequality in the context of the present article appears:

\[
\lambda_1 |u|^2 \leq \|u\|^2
\]

for all \( u \in V \) (see [8, Eq. (5.11), Ch. II]). In addition, for all \( u \in \mathcal{D}(A) \), one has (see, e.g., [8, Sec. 6, Ch. III])

\[
\langle Au, u \rangle_{\mathcal{V}} = \|u\|.
\]

Define \( b(\cdot, \cdot, \cdot) : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \to \mathbb{R} \) by

\[
b(u, v, w) := \sum_{i,j=1}^{3} \int_{G} u_{i} \frac{\partial v_{j}}{\partial x_{i}} w_{j} dx.
\]

Then \( b \) is a trilinear form which induces a bilinear form \( B(u, v) \) by \( b(u, v, w) = \langle B(u, v), w \rangle_{\mathcal{V}} \). In addition, \( b \) enjoys the following properties (see, e.g., [24, Lemma 1.3, Sec. 1, Ch. II]):

\[
b(u, v, v) = 0,
\]

\[
b(u, v, w) = -b(u, w, v).
\]
2.2. Noise Terms.

(i) Let \( Q \in \mathcal{L}(H) \) be a nonnegative, symmetric, trace-class operator. Define \( H_0 := Q^{\frac{1}{2}}(H) \) with the inner product given by \( (u,v)_0 := (Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v)_H \) for \( u,v \in H_0 \), where \( Q^{-\frac{1}{2}} \) is the inverse of \( Q \). Then it follows from [20] Proposition. C.0.3 (i)] that \((H_0, (\cdot, \cdot)_0)\) is again a separable Hilbert space. Let \( \mathcal{L}_2(H_0, H) \) denote the separable Hilbert space of the Hilbert-Schmidt operators from \( H_0 \) to \( H \). Then it can be shown that (see, e.g., [20, p. 27])

\[
\|Q\|_{\mathcal{L}_2(H_0, H)} = \|L \circ Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H_0, H)} \quad \text{for each} \quad L \in \mathcal{L}_2(H_0, H).
\]

Moreover, we write \( \|L\|_{L_Q} = \|L\|_{\mathcal{L}_2(H_0, H)} \) for simplicity.

Let \( T > 0 \) be a fixed real number and \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})\) be a filtered probability space. Let \( W \) be an \( H \)-valued Wiener process with covariance \( Q \).

Let \( \sigma : [0, T] \times \Omega \to \mathcal{L}_2(H_0, H) \) be jointly measurable and adapted. If we have \( \mathbb{E} \int_0^T \|\sigma(s)\|^2_{\mathcal{L}_2(H_0, H)} ds < \infty \), then for \( t \in [0, T] \), the stochastic integral \( \int_0^t \sigma(s)dW(s) \) is well-defined and is an \( H \)-valued continuous square integrable martingale.

(ii) Let \((Z, \mathcal{B}(Z))\) be a measurable space, \( M \) be the collection of all nonnegative integer-valued measures on \((Z, \mathcal{B}(Z))\), and \( B(M) \) be the smallest \( \sigma \)-field on \( M \) with respect to which all \( \eta \to \eta(B) \) are measurable, where \( \eta \in M \), \( \eta(B) \in \mathbb{Z}^+ \cup \{\infty\} \), and \( B \in \mathcal{B}(Z) \). Let \( \tilde{N} : \Omega \to M \) be a Poisson random measure with intensity measure \( \nu \).

For a Poisson random measure \( N(dz, ds), \tilde{N}(dz, ds) := N(dz, ds) - \nu(dz)ds \) defines its compensation. Then it can be shown that (see, e.g. [13] Sec. 3, Ch II.) \( \tilde{N}(dz, ds) \) is a square integrable martingale, and for predictable \( f \) such that

\[
\mathbb{E} \int_0^{t+} \int_Z |f(\cdot,z,s)|\nu(dz)ds < \infty, \quad \text{then}
\]

\[
\int_0^{t+} \int_Z f(\cdot,z,s)\tilde{N}(dz,ds)
\]

\[
= \int_0^{t+} \int_Z f(\cdot,z,s)N(dz,ds) - \int_0^t \int_Z f(\cdot,z,s)\nu(dz)ds
\]

is a well-defined \( \mathcal{F}_t \)-martingale.

(iii) Let \( m \in \mathbb{N} \). Let \( \{\tau(t) : t \in \mathbb{R}^+\} \) be a right continuous Markov chain with generator \( \Gamma = (\gamma_{ij})_{m \times m} \) taking values in \( S := \{1, 2, 3, \ldots, m\} \) such that

\[
\mathcal{R}_t(i,j) = \mathcal{R}(\tau(t+h) = j | \tau(t) = i)
\]

\[
= \begin{cases} 
\gamma_{ij}h + o(h) & \text{if } i \neq j, \\
1 + \gamma_{ii}h + o(h) & \text{if } i = j,
\end{cases}
\]

and \( \gamma_{ii} = -\sum_{i \neq j} \gamma_{ij} \).

In addition, \( \tau(t) \) admits the following stochastic integral representation (see, e.g. [22] Sec. 2.1, Ch. 2]): Let \( \Delta_{ij} \) be consecutive, left closed, right open intervals
of the real line each having length $\gamma_{ij}$ such that
\[ \Delta_{12} = [0, \gamma_{12}), \Delta_{13} = [\gamma_{12}, \gamma_{12} + \gamma_{13}), \cdots \]
\[ \Delta_{1m} = \left[ \sum_{j=2}^{m-1} \gamma_{1j}, \sum_{j=2}^{m} \gamma_{1j} \right), \cdots \]
\[ \Delta_{2m} = \left[ \sum_{j=2}^{m} \gamma_{1j} + \sum_{j=1,j \neq 2}^{m-1} \gamma_{2j}, \sum_{j=2}^{m} \gamma_{1j} + \sum_{j=1,j \neq 2}^{m} \gamma_{2j} \right) \]
and so on. Define a function $h : S \times \mathbb{R} \rightarrow \mathbb{R}$ by
\[ h(i, y) = \begin{cases} j - i & \text{if } y \in \Delta_{ij}, \\ 0 & \text{otherwise}. \end{cases} \]
(2.5)

Then
\[ d \tau(t) = \int_{\mathbb{R}} h(\tau(t-), y)N_2(dt, dy), \]
(2.6)

with initial condition $\tau(0) = \tau_0$, where $N_2(dt, dy)$ is a Poisson random measure with intensity measure $dt \times \mathcal{L}(dy)$, in which $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$.

We assume that such a Markov chain, Wiener process, and the Poisson random measure are independent.

2.3. Some Results from the Itô Formula. The Itô formula in the context of equation (1.3) was introduced in [11, Sec. 2.4]. Here, we collect certain results from the Itô formula for the benefit of the reader.

For equation (1.3), the Itô formula implies that
\[ |u(t)|^2 + 2\nu \int_0^t \|u(s)\|^2 ds \]
\[ = |x|^2 + \int_0^t \|\sigma(s, u(s), \tau(s))\|^2_{L_Q} ds + 2 \int_0^t \langle u(s), \sigma(s, u(s), \tau(s))dW(s) \rangle \]
\[ + \int_0^t \int_{\mathcal{Z}} \left( |u(s) + G(s, u(s), \tau(s), z)|^2 - |u(s)|^2 \right) \tilde{N}_1(dz, ds) \]
\[ + \int_0^t \int_{\mathcal{Z}} \left( |u(s) + G(s, u(s), \tau(s), z)|^2 - |u(s)|^2 \right. \]
\[ - 2 \langle u(s), G(s, u(s), \tau(s), z) \rangle_H \Big) \nu_1(dz)ds; \]
(2.7)

for equation (1.4), the Itô formula implies that
\[ |u(t)|^2 + 2\nu \int_0^t \|u(s)\|^2 ds \]
\[ = |x|^2 + \int_0^t \|\sigma(s, u(s), \tau(s))\|^2_{L_Q} ds + 2 \int_0^t \langle u(s), \sigma(s, u(s), \tau(s))dW(s) \rangle. \]
(2.8)

Equality (2.7) and (2.8) are also called the energy equality for equations (1.3) and (1.4), respectively.
Let $p > 2$. Then equation (1.3) admits the following equality (see, e.g., [2] Eq. (16), p. 65):

$$\|u(t)\|^p + p\nu \int_0^t |u(s)|^{p-2} \|u(s)\|^2 ds$$

$$= |x|^p + \frac{p(p-1)}{2} \int_0^t |u(s)|^{p-2} \|\sigma(s, u(s), r(s))\|_{L_q}^2 ds$$

$$+ p \int_0^t |u(s)|^{p-2} \langle u(s), \sigma(s, u(s), r(s)) \rangle dW(s)$$

(2.9)

$$+ \int_0^t \int_Z \left( |u(s) + G(s, u(s), r(s), z)|^p - |u(s)|^p \right) \tilde{N}_i(dz, ds)$$

$$+ \int_0^t \int_Z \left( |u(s) + G(s, u(s), r(s), z)|^p - |u(s)|^p - p|u(s)|^{p-2} \langle u(s), G(s, u(s), r(s), z) \rangle\right) \nu_1(dz) ds;$$

a similar equality holds for equation (1.4):

$$\|u(t)\|^p + p\nu \int_0^t |u(s)|^{p-2} \|u(s)\|^2 ds$$

$$= |x|^p + \frac{p(p-1)}{2} \int_0^t |u(s)|^{p-2} \|\sigma(s, u(s), r(s))\|_{L_q}^2 ds$$

$$+ p \int_0^t |u(s)|^{p-2} \langle u(s), \sigma(s, u(s), r(s)) \rangle dW(s).$$

(2.10)

3. P-TH MOMENT EXPONENTIAL STABILITY

In this section, the noise coefficients $\sigma: [0, T] \times H \times S \rightarrow L_2(H_0, H)$ and $G: [0, T] \times H \times S \times Z \rightarrow H$ are assumed to satisfy the following Hypotheses H:

**H1.** For all $t \in (0, T)$ and all $i \in S$, there exists a constant $K > 0$ such that

$$\| \sigma(t, u, i) \|_{L_q}^p \leq K(1 + |u|^p)$$

for all $p \geq 2$ (growth condition on $\sigma$).

**H2.** For all $t \in (0, T)$, there exists a constant $L > 0$ such that for all $u, v \in H$ and $i \in S$

$$\| \sigma(t, u, i) - \sigma(t, v, i) \|_{L_q}^p \leq L(|u - v|^p)$$

for all $p \geq 2$ (Lipschitz condition on $\sigma$).

**H3.** For all $t \in (0, T)$ and all $i \in S$, there exist a constant $K > 0$ such that

$$\int_Z |G(t, u, i, z)|^p \nu_1(dz) \leq K(1 + |u|^p)$$

for all $p \geq 1$ (growth condition on $G$).

**H4.** For all $t \in (0, T)$, there exists a constant $L > 0$ such that for all $u, v \in H$ and $i \in S$,

$$\int_Z |G(t, u, i, z) - G(t, v, i, z)|^p \nu_1(dz) \leq L(|u - v|^p)$$

for all $p \geq 2$ (Lipschitz condition on $G$).
The aim of this section is to prove the following $p$-th moments stability for both of the equations.

**Theorem 3.1** ($P$-th moment exponential stability).

1. Assume that $\mathbb{E}|x|^3 < \infty$. If
   a. $K < \frac{1}{2} \nu \lambda_1$, then the equation \((1.3)\) is exponentially stable in mean square;
   b. $K < 2 \nu \lambda_1$, then the equation \((1.4)\) is exponentially stable in mean square.

2. Let $p \geq 3$ be an integer and $\mathbb{E}|x|^p < \infty$. If
   a. $K$ satisfies \((3.1)\), then the equation \((1.3)\) is $p$-th moment exponentially stable;
   b. $K < \frac{2 \nu \lambda_1}{p-1}$, then the equation \((1.4)\) is $p$-th moment exponentially stable.

We start the discussion from equation \((1.4)\) with $u(0) = x$ being an $H$-valued random variable and $\tau(0) = i$.

Recall that $K$ is the constant appears in the Hypotheses $H$, $\nu$ is the viscosity constant, and $\lambda_1$ is the first eigenvalue of $A$.

**Lemma 3.2.** Assume that $\mathbb{E}|x|^3 < \infty$. If $K < 2 \nu \lambda_1$, then there exist a real number $t_0 > 0$ such that, for $t > t_0$,

$$
\mathbb{E}|u(t)|^2 \leq 2 \mathbb{E}|x|^2 e^{-\frac{2 \nu \lambda_1}{3} t}.
$$

**Remark.** For the proof of Lemma 3.2, we only need to assume that $\mathbb{E}|x|^2 < \infty$. The higher requirement of the initial data is for the existence of a solution (see, [11, Theorem 1.2]). Same reasoning apply to Lemma 3.4.

**Proof.** By the Hypotheses $H1$, we have

\[(3.1)\]

$$
\int_0^t \| \sigma(s, u(s), \tau(s)) \|_{L_Q}^2 ds \leq K t + K \int_0^t |u(s)|^2 ds.
$$

By the Poincaré inequality \((2.1)\), we have

\[(3.2)\]

$$
\lambda_1 \nu |u(s)|^2 \leq \nu \| u(s) \|^2.
$$

Plugging estimates \((3.1)\) and \((3.2)\) in \((2.8)\), and then taking expectation, we have

$$
\mathbb{E}|u(t)|^2 + 2 \nu \lambda_1 \mathbb{E} \int_0^t |u(s)|^2 ds \leq \mathbb{E}|x|^2 + K t + K \mathbb{E} \int_0^t |u(s)|^2 ds,
$$

which implies

$$
\mathbb{E}|u(t)|^2 \leq \mathbb{E}|x|^2 + K t + (K - 2 \nu \lambda_1) \mathbb{E} \int_0^t |u(s)|^2 ds.
$$

Employing the Gronwall inequality, one deduces that

$$
\mathbb{E}|u(t)|^2 \leq (\mathbb{E}|x|^2 + K t) e^{(K - 2 \nu \lambda_1) t}.
$$

The term

$$
K t e^{\frac{(K - 2 \nu \lambda_1) t}{2}} \to 0
$$

as $t \to \infty$; therefore, there exists $t_0 > 0$ such that

\[(3.3)\]

$$
K t e^{\frac{(K - 2 \nu \lambda_1) t}{2}} < \mathbb{E}|x|^2
$$

for $t > t_0$. Hence, for $t > t_0$, we conclude

$$
\mathbb{E}|u(t)|^2 \leq (\mathbb{E}|x|^2 + K t e^{K - 2 \nu \lambda_1) t}) \leq (2 \mathbb{E}|x|^2) e^{-\frac{2 \nu \lambda_1}{3} t}.
$$

□
Lemma 3.3. Let \( p \geq 3 \) be an integer and \( \mathbb{E}|x|^p < \infty \). If \( K < \frac{2\nu\lambda_1}{p-1} \), then there exist positive constants \( A^i \) and \( B^i \) and a real number \( t^i \) such that

\[
\mathbb{E}|u(t)|^p \leq A^i \mathbb{E}|x|^p e^{-B^i t}
\]

when \( t > t^i \).

Proof. We start with the case \( p = 3 \) and then use the strong induction to deal with the general \( p \geq 4 \).

It follows from the Hypothesis H1 that

\[
3 \int_0^t |u(s)||\sigma(s, u(s), \nu(s))|^2 \text{d}s \\
\leq 3 \int_0^t |u(s)|K(1 + |u(s)|^2) \text{d}s = 3K \int_0^t |u(s)| \text{d}s + 3K \int_0^t |u(s)|^3 \text{d}s.
\]

Then one uses the Poincaré inequality (2.1) in (2.10), and then taking expectation to obtain

\[
\mathbb{E}|u(t)|^3 + 3\nu\lambda_1 \mathbb{E} \int_0^t |u(s)|^3 \text{d}s \\
\leq \mathbb{E}|x|^3 + 3K \mathbb{E} \int_0^t |u(s)| \text{d}s + 3K \mathbb{E} \int_0^t |u(s)|^3 \text{d}s.
\]

Denote \( C_0 = \mathbb{E} \int_0^{t_0} |u(s)|^2 \text{d}s \). For \( t > t_0 \), we use Lemma 3.2 with \( A = 2 \) and \( B = \frac{2\nu\lambda_1 - K}{2} \) and Schwarz inequality to obtain

\[
\mathbb{E} \int_0^t |u(s)| \text{d}s \leq \left( \mathbb{E} \int_0^t |u(s)|^2 \text{d}s + \mathbb{E} \int_0^t |u(s)|^3 \text{d}s \right)^{\frac{1}{2}} \\
\leq \left( C_0 + \int_0^t A \mathbb{E}|x|^2 e^{-B s} \text{d}s \right)^{\frac{1}{2}} \leq \left( C_0 + \int_0^t A \mathbb{E}|x|^2 e^{-B s} \text{d}s \right)^{\frac{1}{2}} \\
\leq \left( C_0 + A \mathbb{E}|x|^2 \left( \frac{1}{B} - \frac{1}{B} e^{-B t} \right) \right)^{\frac{1}{2}} \leq \left( C_0 + \frac{A}{B} \mathbb{E}|x|^2 \right)^{\frac{1}{2}}.
\]

Plugging (3.5) into (3.4), we have

\[
\mathbb{E}|u(t)|^3 \leq \mathbb{E}|x|^3 + 3K \left( C_0 + \frac{A}{B} \mathbb{E}|x|^2 \right)^{\frac{1}{2}} + 3(K - \nu\lambda_1) \mathbb{E} \int_0^t |u(s)|^3 \text{d}s.
\]

Thus, the Gronwall inequality implies that

\[
\mathbb{E}|u(t)|^3 \leq \left( \mathbb{E}|x|^3 + 3K \left( C_0 + \frac{A}{B} \mathbb{E}|x|^2 \right)^{\frac{1}{2}} \right) \exp \left( 3(K - \nu\lambda_1) t \right).
\]

Since \( K < \nu\lambda_1 \), there exists a \( t_1 > 0 \) such that

\[
K \left( C_0 + \frac{A}{B} \mathbb{E}|x|^2 \right)^{\frac{1}{2}} \exp \left( \frac{3}{2}(K - \nu\lambda_1) t \right) < \mathbb{E}|x|^3
\]

when \( t > t_1 \). Let \( t^i = \max\{t_0, t_1\} \). Then (3.6) implies that

\[
\mathbb{E}|u(t)|^3 \leq 2 \mathbb{E}|x|^3 \exp \left( -\frac{3}{2}(\nu\lambda_1 - K) t \right)
\]

when \( t > t^i \). Hence, the case of \( p = 3 \) is established.
Now fix $p \geq 4$, we assume such an assertion is valid for all positive integers $q < p$, i.e., there exist positive constants $A$ and $B$ and a real number $t_\ast > 0$ such that, for all $t > t_\ast$,

\begin{equation}
E|u(t)|^q \leq A E|x|^q e^{Bt}.
\end{equation}

It follows from the Hypothesis $\text{H1}$ that

\begin{align*}
\frac{p(p-1)}{2} \int_0^t |u(s)|^{p-2} \|\sigma(s, u(s), t(s))\|_{Lq}^2 ds &
\leq \frac{p(p-1)}{2} \int_0^t |u(s)|^{p-2} K (1 + |u(s)|^2) ds \\
&= \frac{p(p-1)}{2} K \int_0^t |u(s)|^{p-2} ds + \frac{p(p-1)}{2} K \int_0^t |u(s)|^p ds.
\end{align*}

Utilizing the estimates above and the Poincaré inequality (2.1) in (2.10), and then taking expectation, one reaches

\begin{align}
E|u(t)|^p + &\nu p \lambda_1 E \int_0^t |u(s)|^p ds \\
\leq E|x|^p + &\frac{p(p-1)}{2} K E \int_0^t |u(s)|^{p-2} ds + \frac{p(p-1)}{2} K E \int_0^t |u(s)|^p ds.
\end{align}

For $t > t_\ast$, we write $C_\ast = E \int_0^{t_\ast} |u(s)|^{p-2} ds$ and deduce from the induction hypothesis (3.7) that

\begin{align}
E \int_0^t |u(s)|^{p-2} ds = &E \int_0^{t_\ast} |u(s)|^{p-2} ds + E \int_{t_\ast}^t |u(s)|^{p-2} ds \\
\leq &C_\ast + \int_{t_\ast}^t A E|x|^{p-2} e^{-Bt} ds \leq C_\ast + \int_0^t A E|x|^{p-2} e^{-Bt} ds \\
= &C_\ast + A E|x|^{p-2} \left(1 - \frac{1}{B} e^{-Bt}\right) \leq C_\ast + \frac{A}{B} E|x|^{p-2}.
\end{align}

Plugging (3.9) in (3.8), we obtain, upon a simplification,

\begin{align}
E|u(t)|^p \leq &E|x|^p + \frac{p(p-1)}{2} K \left(C_\ast + \frac{A}{B} E|x|^{p-2}\right) + \left(\frac{p(p-1)}{2} K - \nu p \lambda_1 \right) E \int_0^t |u(s)|^p ds,
\end{align}

which further implies

\begin{equation}
E|u(t)|^p \leq \left(E|x|^p + \frac{p(p-1)}{2} K \left(C_\ast + \frac{A}{B} E|x|^{p-2}\right)\right) \exp \left(\left(\frac{p(p-1)}{2} K - \nu p \lambda_1 \right)t\right)
\end{equation}

by the Gronwall inequality.

Let

$$B' = \nu p \lambda_1 - \frac{p(p-1)}{2} K.$$

Then there exist a real number $t_2 > 0$ such that

$$\left(\frac{p(p-1)}{2} K \left(C_\ast + \frac{A}{B} E|x|^{p-2}\right)\right) e^{-B't} < E|x|^p$$

for $t > t_1$. Let $t^\dagger = \max\{t_\ast, t_2\}$. Then (3.10) implies that

$$E|u(t)|^p \leq 2E|x|^p e^{-\frac{B't^\dagger}{2}}.$$
for $t > t^\dagger$, which implies the required estimate. The proof is thus complete by the induction.

After the preparation of the lemmata above, we are in a position to prove the second part of Theorem 3.1.

**Proof of the second part of Theorem 3.1.** The case of $p = 2$ and 3 follows from Lemma 3.2 and (3.6). For general $p \geq 3$, it follows from (3.10) that
\[
\log \mathbb{E} |u(t)|^p \leq \log \left( \mathbb{E} |x|^p + \frac{p(p-1)}{2} K \left( C_\ast + \frac{A}{B} \mathbb{E} |x|^{p-2} \right) \right) + \left( \frac{p(p-1)}{2} K - \nu p \lambda_1 \right) t.
\]
Then the result follows from the assumption on $K$. □

Next, we consider equation (1.3) with initial condition $(u(0), r(0)) = (x, i)$. **Lemma 3.4.** Assume $\mathbb{E} |x|^3 < \infty$. If $K < \nu \lambda_1$, then there exist a real number $t_0 > 0$ such that,
\[
\mathbb{E} |u(t)|^2 \leq 2 \mathbb{E} |x|^2 e^{-\left( \nu \lambda_1 - K \right) t}.
\]

**Proof.** Taking expectation on the both side of (2.7), we have
\[
\mathbb{E} |u(t)|^2 + 2 \nu \mathbb{E} \int_0^t \|u(s)\|^2 ds = \mathbb{E} |x|^2 + \mathbb{E} \int_0^t \|\sigma(s, u(s), r(s))\|^2_{L_Q} ds + \mathbb{E} \int_0^t \int_Z |G(s, u(s), r(s), z)|^2 \nu_1(dz) ds.
\]
Employing the Hypotheses H1 and H3 and the Poicaré inequality (2.1), we obtain
\[
\mathbb{E} |u(t)|^2 \leq \left( \mathbb{E} |x|^2 + 2Kt \right) + 2(K - \nu \lambda_1) \mathbb{E} \int_0^t |u(s)|^2 ds,
\]
which together with the Gronwall inequality further imply
\[
\mathbb{E} |u(t)|^2 \leq \left( \mathbb{E} |x|^2 + 2Kt \right) e^{2(K - \nu \lambda_1) t}.
\]
Carrying out an analogous argument as in (3.3), we conclude that there exist a real number $t_0$ such that
\[
\mathbb{E} |u(t)|^2 \leq 2 \mathbb{E} |x|^2 e^{-\left( \nu \lambda_1 - K \right) t}
\]
for $t > t_0$. □

**Lemma 3.5.** Let $p \geq 3$ be an integer and $\mathbb{E} |x|^p < \infty$. If $K$ satisfies that
\[
(3.11) \quad K \left( \frac{p(p-1)}{2} + 2^p - 1 + p \right) - p p \lambda_1 < 0,
\]
then there exist positive constants $A^\dagger$ and $B^\dagger$ and a real number $t^\dagger$ such that
\[
\mathbb{E} |u(t)|^p \leq A^\dagger \mathbb{E} |x|^p e^{-B^\dagger t}
\]
when $t > t^\dagger$. □
**Proof.** We begin with the case \( p = 3 \) and then use the strong induction for general \( p \geq 4 \). Taking expectation on both sides of (2.9) and then use the Poincaré inequality (2.1), we have

\[
\begin{align*}
\mathbb{E}|u(t)|^3 + 3\nu \lambda_1 \mathbb{E} \int_0^t |u(s)|^3 ds & \\
& \leq \mathbb{E}|x|^3 + 3\mathbb{E} \int_0^t |u(s)||\sigma(s, u(s), r(s))|_{L_q}^2 ds \\
& \quad + \mathbb{E} \int_0^t \int_Z \left( |u(s) + G(s, u(s), r(s), z)|^3 - |u(s)|^3 \right. \\
& \quad \left. - 3|u(s)||u(s), G(s, u(s), r(s), z)|_H \right) \nu_1(\nu_1) d\nu_1(\nu_1) ds.
\end{align*}
\]

For the last term in (3.12), we use the triangle inequality and the binomial theorem to obtain (we suppress all the variables for the sake of simplicity)

\[
|u + G|^3 \leq \left( |u| + |G| \right)^3 = |u|^3 + 3|u|^2|G| + 3|u||G|^2 + |G|^3;
\]

it is not hard to see that

\[
\left| - 3|u|(u, G)_H \right| \leq 3|u|^2|G|.
\]

Using these two observations in the last term of (3.12), we have

\[
\begin{align*}
& \left| \mathbb{E} \int_0^t \int_Z \left( |u(s) + G(s, u(s), t(s), z)|^3 - |u(s)|^3 \\
& \quad - 3|u(s)||u(s), G(s, u(s), t(s), z)|_H \right) \nu_1(\nu_1) d\nu_1(\nu_1) ds \right| \\
& \quad \leq 6\mathbb{E} \int_0^t \int_Z |u(s)|^2|G(s, u(s), t(s), z)| \nu_1(\nu_1) d\nu_1(\nu_1) ds \\
& \quad + 3\mathbb{E} \int_0^t \int_Z |u(s)||G(s, u(s), t(s), z)|^2 \nu_1(\nu_1) d\nu_1(\nu_1) ds + \mathbb{E} \int_0^t \int_Z |G(s, u(s), t(s), z)|^3 \nu_1(\nu_1) d\nu_1(\nu_1) ds;
\end{align*}
\]

then we employ the Hypothesis H3 to deduce

\[
\begin{align*}
\mathbb{E} \int_0^t \int_Z \left( |u(s) + G(s, u(s), t(s), z)|^3 - |u(s)|^3 \\
& \quad - 3|u(s)||u(s), G(s, u(s), t(s), z)|_H \right) \nu_1(\nu_1) d\nu_1(\nu_1) ds \\
& \quad \leq 10K\mathbb{E} \int_0^t |u(s)|^3 ds + 6K\mathbb{E} \int_0^t |u(s)|^2 ds + 3K\mathbb{E} \int_0^t |u(s)| ds + Kt.
\end{align*}
\]

Using the Hypothesis H2, we have

\[
3\mathbb{E} \int_0^t |u(s)||\sigma(s, u(s), t(s))|_{L_q}^2 ds \leq 3K\mathbb{E} \int_0^t |u(s)|^3 ds + 3K\mathbb{E} \int_0^t |u(s)| ds
\]

Plugging (3.13) and (3.14) into (3.12), we obtain

\[
\begin{align*}
\mathbb{E}|u(t)|^3 + 3\nu \lambda_1 \mathbb{E} \int_0^t |u(s)|^3 ds & \\
& \leq \mathbb{E}|x|^3 + Kt + 6K\mathbb{E} \int_0^t |u(s)| ds + 6K\mathbb{E} \int_0^t |u(s)|^2 ds + 13K\mathbb{E} \int_0^t |u(s)|^3 ds.
\end{align*}
\]
By (3.5) and Lemma 3.2, we see that
\[ 6K\mathbb{E} \int_0^t |u(s)|^2 ds \text{ and } 6K\mathbb{E} \int_0^t |u(s)|^3 ds \]
are bounded by a constant \( C(\nu, \lambda_1, K, \mathbb{E}|x|^2) \). Therefore, by the Gronwall inequality,
\[ \mathbb{E}|u(t)|^3 \leq \left( \mathbb{E}|x|^3 + Kt + C(\nu, \lambda_1, K, \mathbb{E}|x|^2) \right) \exp \left( (13K - 3\nu\lambda_1)t \right), \]
which implies that there exists sufficient large \( t^* \) such that
\[ \mathbb{E}|u(t)|^3 \leq 2\mathbb{E}|x|^3 \exp \left( (-3\nu\lambda_1 - 13K)\frac{t}{2} \right) \]
when \( t > t^* \). Thus, the case for \( p = 3 \) is established.

Now fix \( p \geq 4 \), we assume such an assertion is valid for all positive integers \( q < p \), i.e., there exist positive constants \( A \) and \( B \) and a real number \( t_* > 0 \) such that, for all \( t > t_* \),
\[ (3.15) \quad \mathbb{E}|u(t)|^q \leq A\mathbb{E}|x|^q e^{-Bt}. \]

By the Hypothesis H1, the second term of the right side of (2.9) implies
\[ (3.16) \quad \frac{p(p-1)}{2} \int_0^t |u(s)|^{p-2} \| \sigma(s, u(s), \tau(s)) \|^2_{L_\infty} ds \]
\[ \leq \frac{p(p-1)}{2} \int_0^t |u(s)|^{p-2} \left( K(1 + |u(s)|^2) \right) ds \]
\[ \leq \frac{p(p-1)}{2} K \int_0^t |u(s)|^{p-2} ds + \frac{p(p-1)}{2} K \int_0^t |u(s)|^p ds. \]

For the last term in the right side of (2.9) (suppress all the variables of functions for simplicity), using the binomial theorem, we obtain
\[ |u + G|^p \leq (|u| + |G|)^p = |u|^p + \sum_{n=1}^p \binom{p}{n} |u|^{p-n}|G|^n, \]
therefore,
\[ (3.17) \quad |u + G|^p - |u|^p \leq |u|^p + \sum_{n=1}^p \binom{p}{n} |u|^{p-n}|G|^n - |u|^p = \sum_{n=1}^p \binom{p}{n} |u|^{p-n}|G|^n. \]

On the other hand,
\[ (3.18) \quad \left| - p|u|^{p-2} (u, G) H \right| \leq p|u|^{p-1}|G| \]
Utilizing estimates (3.17) and (3.18) in
\[ \int_0^t \int_Z \left( |u(s) + G(s, u(s), \tau(s), z)|^p - |u(s)|^p \right) \nu_1(dz) ds, \]

we have

$$
\left| \int_0^t \int_Z \left( |u(s) + G(s, u(s), r(s), z)|^p - |u(s)|^p \\
- p |u(s)|^{p-2} (u(s), G(s, u(s), r(s), z)) \right) \nu_1(dz) ds \right|
\leq \int_0^t \int_Z \sum_{n=1}^p \binom{p}{n} |u(s)|^{p-n} |G(s, u(s), r(s), z)|^n \nu_1(dz) ds
\leq \int_0^t \int_Z p |u(s)|^{p-1} |G(s, u(s), r(s), z)| \nu_1(dz) ds.
$$

Therefore, by the Hypothesis H3, we obtain

$$
\left( 3.19 \right)
\left| \int_0^t \int_Z \left( |u(s) + G(s, u(s), r(s), z)|^p - |u(s)|^p \\
- p |u(s)|^{p-2} (u(s), G(s, u(s), r(s), z)) \right) \nu_1(dz) ds \right|
\leq K \int_0^t \sum_{n=1}^p \binom{p}{n} |u(s)|^{p-n} ds + K \int_0^t \sum_{n=1}^p \binom{p}{n} |u(s)|^p ds
\leq pK \int_0^t |u(s)|^{p-1} ds + pK \int_0^t |u(s)|^p ds.
$$

Using estimates \(3.16\) and \(3.19\) and then taking expectation on \(2.9\), we further use
the Poincaré inequality \(2.1\) to obtain

$$
\left( 3.20 \right)
\mathbb{E}|u(t)|^p + p \nu \lambda_1 \mathbb{E} \int_0^t |u(s)|^{p-2} |u(s)|^2 ds
\leq \mathbb{E}|x|^p + \frac{p(p-1)}{2} K \mathbb{E} \int_0^t |u(s)|^{p-2} ds + \frac{p(p-1)}{2} K \mathbb{E} \int_0^t |u(s)|^p ds
\leq \mathbb{E}|x|^p + K \mathbb{E} \int_0^t \sum_{n=1}^p \binom{p}{n} |u(s)|^{p-n} ds + K \mathbb{E} \int_0^t \sum_{n=1}^p \binom{p}{n} |u(s)|^p ds
\leq pK \mathbb{E} \int_0^t |u(s)|^{p-1} ds + pK \mathbb{E} \int_0^t |u(s)|^p ds.
$$

Let \(C\) denote the central binomial coefficient and notice that \(\sum_{n=1}^p \binom{p}{n} = 2^p - 1\). We simplify \(3.20\) to obtain

$$
\left( 3.21 \right)
\mathbb{E}|u(t)|^p \leq \left( \mathbb{E}|x|^p + C K \mathbb{E} \int_0^t \sum_{n=1}^p |u(s)|^{p-n} ds \right)
\leq \left( K \left( \frac{p(p-1)}{2} + 2^p - 1 + p \right) - p \nu \lambda_1 \right) \mathbb{E} \int_0^t |u(s)|^p ds.
$$

For \(t > t_0\), by an analogous argument as in \(3.9\), we see that

$$
\mathbb{E} \int_0^t \sum_{n=1}^p |u(s)|^{p-n} ds \leq \sum_{n=0}^{p-1} C_n \mathbb{E}|x|^n
$$


where \( \{C_n\} \) are positive constants. Therefore, by Gronwall inequality, we deduce from (3.21) that

\[
E|u(t)|^p \leq \left( \sum_{n=0}^{p} C_n^p E|x|^n \right) \exp \left( \left( K \left( \frac{p(p-1)}{2} + 2^{p-1} + p \right) - p \nu \lambda_1 \right) t \right),
\]

where \( \{C_n^p\} \) are positive constants. By the assumption on \( K \), there exists a real number \( t_1 \) such that

\[
\left( \sum_{n=0}^{p-1} C_n^p E|x|^n \right) \exp \left( \left( K \left( \frac{p(p-1)}{2} + 2^{p-1} + p \right) - p \nu \lambda_1 \right) \frac{t}{2} \right) < E|x|^p
\]

for \( t > t_1 \). Now, taking \( t^1 = \max\{t_*, t_1\} \), we then infer from (3.22) that

\[
E|u(t)|^p \leq (1 + C_p^p) E|x|^p \exp \left( \left( K \left( \frac{p(p-1)}{2} + 2^{p-1} + p \right) - p \nu \lambda_1 \right) \frac{t}{2} \right).
\]

Hence, the proof is completed by induction. □

Now we are in a position to prove the first part of Theorem 3.1. By the assumption on \( K \), this theorem is a direct result of Lammata 3.4 and 3.5.

4. ALMOST SURELY EXPONENTIAL STABILITY

For the almost surely exponential stability, we require stronger hypotheses on the noise coefficients. The noise coefficients \( \sigma : [0, T] \times H \times S \rightarrow L_2(H_0, H) \) and \( G : [0, T] \times H \times S \times Z \rightarrow H \) are assumed to satisfy the following Hypotheses \( H' \):

**H1'**. For all \( t \in (0, T) \) and all \( i \in S \), there exists a function \( K \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \) such that

\[
\|\sigma(t, u, i)\|_{L_q}^p \leq K(t)(1 + |u|^p)
\]

for all \( p \geq 2 \) (growth condition on \( \sigma \)).

**H2'**. For all \( t \in (0, T) \), there exists a function \( L \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \) such that for all \( u, v \in H \) and \( i, j \in S \)

\[
\|\sigma(t, u, i) - \sigma(t, v, i)\|_{L_q}^p \leq L(t)(|u - v|^p)
\]

for all \( p \geq 2 \) (Lipschitz condition on \( \sigma \)).

**H3'**. For all \( t \in (0, T) \) and all \( i \in S \), there exist a function \( K \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \) such that

\[
\int_Z |G(t, u, i, z)|^p \nu_1(dz) \leq K(t)(1 + |u|^p)
\]

for all \( p \geq 1 \) (growth condition on \( G \)).

**H4'**. For all \( t \in (0, T) \), there exists a function \( L \in L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+) \) such that for all \( u, v \in H \) and \( i, j \in S \),

\[
\int_Z |G(t, u, i, z) - G(t, v, i, z)|^p \nu_1(dz) \leq L(t)(|u - v|^p)
\]

for all \( p \geq 2 \) (Lipschitz condition on \( G \)).

The aim of the section is to establish the following almost surely stability theorem for both of the equations.
Theorem 4.1.

(1) If \( \| K \|_\infty < \nu \lambda_1 \), then the equation (1.3) is almost surely exponentially stable.
(2) If \( \| K \|_\infty < 2 \nu \lambda_1 \), then the equation (1.4) is almost surely exponentially stable.

We begin the argument with the study of equation (1.4). First, we observe that under Hypotheses \( H' \), the solution obeys a different estimate (cf. [11, Proposition 3.1]).

Lemma 4.2. Under the Hypotheses \( H' \), the solution \( u \) to equation (1.4) satisfies the following estimates:

1. For all \( t > 0 \), one has

\[
\mathbb{E} |u(t)|^2 \leq (\mathbb{E}|x|^2 + \| K \|_1)e^{(\| K \|_\infty - 2\nu \lambda_1)t}.
\]

2. Suppose further, \( \| K \|_\infty < 2 \nu \lambda_1 \). Then one has

\[
\mathbb{E} \sup_{t \geq 0} |u(t)|^2 \leq 2(\mathbb{E}|x|^2 + 5\| K \|_1) + 10\| K \|_\infty \| K \|_1 - \| K \|_\infty.
\]

Proof. Making use of the Poincaré inequality (2.1) and the Hypothesis \( H' \), and then taking expectation on the both side of (2.8), one has

\[
\mathbb{E}|u(t)|^2 + 2\nu \lambda_1 \mathbb{E} \int_0^t |u(s)|^2 ds \\
\leq \mathbb{E}|u(t)|^2 + 2\nu \mathbb{E} \int_0^t ||u(s)||^2 ds \leq \mathbb{E}|x|^2 + ||K||_1 + ||K||_\infty \mathbb{E} \int_0^t |u(s)|^2 ds,
\]

which implies

\[
\mathbb{E}|u(t)|^2 \leq (\mathbb{E}|x|^2 + ||K||_1) + (||K||_\infty - 2\nu \lambda_1) \mathbb{E} \int_0^t |u(s)|^2 ds.
\]

Thus, (4.1) follows from the Gronwall inequality.

By the Davis inequality, the Hypothesis \( H' \), and the basic Young inequality, one has

\[
2 \mathbb{E} \sup_{t \geq 0} \left| \int_0^t \langle u(s), \sigma(s, u(s), r(s)) \rangle dW(s) \right| \\
\leq 2 \sqrt{2} \mathbb{E} \left\{ \left( \int_0^\infty \| \sigma^*(s, u(s), r(s)) \|_{\mathcal{L}_Q}^2 ds \right)^{\frac{1}{2}} \right\} \\
\leq 2 \sqrt{2} \mathbb{E} \left\{ \sup_{t \geq 0} |u(t)| \left( \int_0^\infty \| \sigma^*(s, u(s), r(s)) \|_{\mathcal{L}_Q}^2 ds \right)^{\frac{1}{2}} \right\} \\
\leq 2 \sqrt{2} \epsilon_1 \mathbb{E} \sup_{t \geq 0} |u(t)|^2 + 2 \sqrt{2} C_{\epsilon_1} \mathbb{E} \int_0^\infty \| \sigma^*(s, u(s), r(s)) \|_{\mathcal{L}_Q}^2 ds \\
\leq 2 \sqrt{2} \epsilon_1 \mathbb{E} \sup_{t \geq 0} |u(t)|^2 + 2 \sqrt{2} C_{\epsilon_1} \| K \|_1 + 2 \sqrt{2} C_{\epsilon_1} \| K \|_\infty \mathbb{E} \int_0^\infty |u(s)|^2 ds,
\]

where \( \epsilon_1 \) and \( C_{\epsilon_1} \) will be determined later.

Taking suprenum over \( t \geq 0 \) and then expectation on the both side of (2.8), we use the Hypothesis \( H' \), the Poincaré inequality (2.1), and (4.3) with \( \epsilon_1 = \frac{1}{\sqrt{2}} \) and \( C_{\epsilon_1} = \sqrt{2} \).
to obtain
\[
E \sup_{t \geq 0} |u(t)|^2 + 2\nu \lambda_1 E \int_0^\infty |u(s)|^2 ds \leq E \sup_{t \geq 0} |u(t)|^2 + 2\nu E \int_0^\infty \|u(s)\|^2 ds
\]
\[
\leq E|x|^2 + \|K\|_1 + \|K\|_\infty E \int_0^\infty |u(s)|^2 ds
\]
\[
+ \frac{1}{2} E \sup_{t \geq 0} |u(t)|^2 + \|K\|_1 + 4\|K\|_\infty E \int_0^\infty |u(s)|^2 ds,
\]
which implies
\[
E \int_0^\infty |u(s)|^2 ds \leq E |x|^2 + \|K\|_1 + 5\|K\|_\infty E \int_0^\infty |u(s)|^2 ds.
\]
If \(\|K\|_\infty < 2\nu \lambda_1\), then (4.1) implies
\[
E \int_0^\infty |u(s)|^2 ds \leq \frac{E|x|^2 + \|K\|_1}{2\nu \lambda_1 - \|K\|_\infty}.
\]
Finally, (4.2) follows from (4.4) and (4.5).

Let
\[
M_1(t) := \int_0^t \langle u(s), \sigma(s, u(s), r(s))dW(s) \rangle.
\]

**Lemma 4.3.** Suppose that \(\|K\|_\infty < 2\nu \lambda_1\). Then there exists random variable \(M_1(\infty) \in L^1(\Omega)\) such that
\[
\lim_{t \to \infty} M_1(t) = M_1(\infty)
\]
almost surely.

**Proof.** The proof is based on the Doob Martingale Convergence Theorem, therefore, it suffices to verify that \(\sup_t E|M_1(t)| < \infty\).

By the Davis inequality and the Hypothesis H1', we have
\[
\sup_t E|M_1(t)| \leq E \sup_t |M_1(t)| \leq \sqrt{2} E \left\{ \left( \|s^*(s, u(s), r(s)u(s))\|_{L^2}^2 ds \right) ^{\frac{1}{2}} \right\}
\]
\[
\leq \sqrt{\frac{\nu}{2}} E \sup_t |u(t)|^2 + \frac{\sqrt{\nu}}{2} \|K\|_1 + \frac{\sqrt{\nu}}{2} \|K\|_\infty E \int_0^\infty |u(s)|^2 ds.
\]
Then the lemma follows from (4.2) and (4.5).

Now we are in a position to prove the second part of Theorem 4.1.

**Proof of the second part of Theorem 4.1** Utilizing the Poincaré inequality (2.1) and the Hypothesis H1' in (2.8), we have
\[
|u(t)|^2 + 2\nu \lambda_1 \int_0^t |u(s)|^2 ds \leq |u(t)|^2 + 2\nu \int_0^t \|u(s)\|^2 ds
\]
\[
\leq |x|^2 + \|K\|_1 + \|K\|_\infty \int_0^t |u(s)|^2 ds + 2M_1(t),
\]
where \(M_1\) is defined in (4.6). Thus, we have
\[
|u(t)|^2 \leq (|x|^2 + \|K\|_1 + 2M_1(t)) + (\|K\|_\infty - 2\nu \lambda_1) \int_0^t |u(s)|^2 ds,
\]
which further implies that (by the Gronwall inequality)

\[ |u(t)|^2 \leq (|x|^2 + \| K \|_1 + 2M_1(t))e^{(\| K \|_{\infty} - 2\nu \lambda_1)t} \]

Taking logarithm on both sides and then dividing by \( t \), we have

\[
\frac{2 \log |u(t)|}{t} \leq \frac{\log(|x|^2 + \| K \|_1 + 2M_1(t))}{t} + (\| K \|_{\infty} - 2\nu \lambda_1).
\]

Then the theorem follows from Lemma 4.3 and the hypothesis on \( K \).

Next, we consider the equation (1.3).

**Lemma 4.4.** Under the Hypotheses \( H' \), the solution \( u \) to equation (1.3) satisfies the following estimate:

\[ E|u(t)|^2 \leq (E|x|^2 + 2\| K \|_1) e^{2(\| K \|_{\infty} - \nu \lambda_1)t} \]  \hspace{1cm} (4.7)

for all \( t > 0 \). Moreover, if \( \| K \|_{\infty} < \nu \lambda_1 \), then we have

\[ E \sup_{t \geq 0} |u(t)|^2 \leq C(x, K, \nu, \lambda_1). \]  \hspace{1cm} (4.8)

**Proof.** It follows from the Hypothesis \( H3' \) that

\[
\int_0^t \int_Z \left( |u(s) + G(s, u(s), r(s), z)|^2 - |u(s)|^2 - 2(u(s), G(s, u(s), r(s), z))_H \right) \nu_1(dz)ds
\]

\[ = \int_0^t \int_Z |G(s, u(s), r(s), z)|^2 \nu_1(dz)ds \leq \| K \|_1 + \| K \|_{\infty} \int_0^t |u(s)|^2 ds.
\]

Employing the Poincaré inequality (2.1), the estimate above, and the Hypothesis \( H1' \) to \( \sigma \) in (2.7), we have

\[
E|u(t)|^2 + 2\nu \lambda_1 E \int_0^t |u(s)|^2 ds \leq E|u(t)|^2 + 2\nu E \int_0^t |u(s)|^2 ds
\]

\[ \leq E|x|^2 + 2\| K \|_1 + 2\| K \|_{\infty} E \int_0^t |u(s)|^2 ds,
\]

which implies

\[ E|u(t)|^2 \leq (E|x|^2 + 2\| K \|_1) + 2(|| K ||_{\infty} - \nu \lambda_1) E \int_0^t |u(s)|^2 ds.
\]

Then the Gronwall inequality implies (4.7).

It is not hard to see that

\[ \int_0^t \int_Z \left( |u(s) + G(s, u(s), r(s), z)|^2 - |u(s)|^2 \right) \tilde{N}_1(dz, ds)
\]

\[ = 2 \int_0^t \int_Z (u(s), G(s, u(s), r(s), z))_H \nu_1(dz)ds.
\]
Therefore, the Davis inequality, the Hypothesis H3’, and the basic Young inequality imply
\begin{align}
\mathbb{E} \sup_{t \geq 0} \left| \int_0^t \int_Z \left( |u(s) + G(s, u(s), r(s), z)|^2 - |u(s)|^2 \right) \tilde{N}_1(dz, ds) \right| \\
\leq 2 \sqrt{10} \mathbb{E} \left\{ \left( \int_0^\infty \int_Z \left( |u(s), G(s, u(s), r(s), z)| \right) \nu_1(dz) ds \right)^{\frac{1}{2}} \right\} \\
\leq \frac{1}{4} \mathbb{E} \sup_{t \geq 0} |u(t)|^2 + 40 \| K \|_1 + 40 \| K \|_\infty \mathbb{E} \int_0^\infty |u(s)|^2 ds.
\end{align}

Taking supremum over \( t \geq 0 \) on the both side of (2.7) and then expectation, using the Poincaré inequality (2.1), (4.3) with \( \epsilon_1 = \frac{1}{8\sqrt{2}} \) and \( C_{\epsilon_1} = 2\sqrt{2} \), and (4.9), we obtain
\begin{align}
\mathbb{E} \sup_{t \geq 0} |u(t)|^2 + 2\nu \lambda_1 \mathbb{E} \int_0^\infty |u(s)|^2 ds \leq \mathbb{E} \sup_{t \geq 0} |u(t)|^2 + 2\nu \mathbb{E} \int_0^\infty \| u(s) \|^2 ds \\
\leq \mathbb{E}|x|^2 + 50\| K \|_1 + 50\| K \|_\infty \mathbb{E} \int_0^\infty |u(s)|^2 ds + \frac{1}{2} \mathbb{E} \sup_{t \geq 0} |u(t)|^2,
\end{align}
which implies
\begin{align}
\frac{1}{2} \mathbb{E} \sup_{t \geq 0} |u(t)|^2 \leq (\mathbb{E}|x|^2 + 50\| K \|_1) + (50\| K \|_\infty - 2\nu \lambda_1) \mathbb{E} \int_0^\infty |u(s)|^2 ds.
\end{align}

Now, if \( 50\| K \|_\infty - 2\nu \lambda_1 \leq 0 \), then (4.10) implies
\begin{align}
\frac{1}{2} \mathbb{E} \sup_{t \geq 0} |u(t)|^2 \leq \mathbb{E}|x|^2 + 50\| K \|_1.
\end{align}

Take \( C(x, K, \nu, \lambda_1) = 2(\mathbb{E}|x|^2 + 50\| K \|_1) \). Then the estimate above implies (4.8); if \( 50\| K \|_\infty - 2\nu \lambda_1 > 0 \), then using (4.7), we have
\begin{align}
\int_0^\infty \mathbb{E}|u(s)|^2 ds \leq \mathbb{E} \int_0^\infty |u(s)|^2 ds \leq \frac{\mathbb{E}|x|^2 + 2\| K \|_1}{2(\nu \lambda_2 - \| K \|_\infty)}.
\end{align}

Plugging (4.11) into (4.10), we have
\begin{align}
\mathbb{E} \sup_{t \geq 0} |u(t)|^2 \leq 2(\mathbb{E}|x|^2 + 50\| K \|_1) + 2(50\| K \|_\infty - 2\nu \lambda_1) \frac{\mathbb{E}|x|^2 + 2\| K \|_1}{2(\nu \lambda_2 - \| K \|_\infty)}
\end{align}

Take
\begin{align}
C(x, m, K, \nu, \lambda_1) \\
= 2(\mathbb{E}|x|^2 + 50\| K \|_1) + 2(50\| K \|_\infty - 2\nu \lambda_1) \frac{\mathbb{E}|x|^2 + 2\| K \|_1}{2(\nu \lambda_2 - \| K \|_\infty)}.
\end{align}
The proof is therefore completed. \( \square \)

Let
\begin{align}
M_2(t) := \int_0^t \int_Z \left( |u(s) + G(s, u(s), r(s), z)|^2 - |u(s)|^2 \right) \tilde{N}_1(dz, ds)
\end{align}

**Lemma 4.5.** Suppose that \( \| K \|_\infty < \nu \lambda_1 \). Then there exists random variable \( M_2(\infty) \in L^1(\Omega) \) such that
\begin{align}
\lim_{t \to \infty} M_2(t) = M_2(\infty)
\end{align}
almost surely.
Proof. Similar to Lemma 4.3, we prove this lemma by the Doob Martingale Convergence Theorem. Consider
\[ \sup_{t \geq 0} \mathbb{E}[|M_2(t)|] \leq \mathbb{E} \sup_{t \geq 0} |M_2(t)|. \]
\[ \leq \frac{1}{4} \mathbb{E} \sup_{t \geq 0} |u(t)|^2 + 40 \|K\|_1 + 40 \|K\|_{\infty} \int_0^\infty |u(s)|^2 ds \]
by (4.9). Then we conclude that \( \sup_{t \geq 0} \mathbb{E}[|M_2(t)|] \leq C \) for some constant \( C \) by Lemma 4.4. Hence, we complete the proof. □

Finally, we are in a position to prove the first part of Theorem 4.1. Proof of the first part of Theorem 4.1. Making use of the Poincaré inequality (2.1) and the Hypotheses \( H_1' \) and \( H_3' \) in (2.7), we obtain
\[ |u(t)|^2 + 2 \nu \lambda_1 \int_0^t |u(s)|^2 ds \leq |u(t)|^2 + 2 \nu \int_0^t \|u(s)\|^2 ds \]
\[ \leq |x|^2 + 2 \|K\|_1 + 2M_1(t) + M_2(t) + 2\|K\|_{\infty} \int_0^t |u(s)|^2 ds, \]
where \( M_1(t) \) and \( M_2(t) \) are defined in (4.6) and (4.12), respectively. The inequality above implies that
\[ |u(t)|^2 \leq (|x|^2 + 2\|K\|_1 + 2M_1(t) + M_2(t)) + 2(\|K\|_{\infty} - \nu \lambda_1) \int_0^t |u(s)|^2 ds, \]
which further implies (via an application of the Gronwall inequality)
\[ |u(t)|^2 \leq (|x|^2 + 2\|K\|_1 + 2M_1(t) + M_2(t))e^{2(\|K\|_{\infty} - \nu \lambda_1)t}. \]

Taking logarithm on the both side above and then dividing by \( t \), we have
\[ \frac{2 \log |u(t)|}{t} \leq \frac{|x|^2 + 2\|K\|_1 + 2M_1(t) + M_2(t)}{t} + 2(\|K\|_{\infty} - \nu \lambda_1), \]
which implies the result by Lemmata 4.3 and 4.5 and the assumption on \( K \). □

Acknowledgements

The author would like to thank Professor Padmanabhan Sundar and Ju-Yi Yen for their kind encouragement on this project.

References

[1] L. Arnold: Stochastic differential equations: theory and applications. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.
[2] S. Albeverio, F. Flandoli, and Y. G. Sinai: SPDE in Hydrodynamic: recent progress and prospects. Lecture Notes in Mathematics, 1942. Springer-Verlag, Berlin; Fondazione C.I.M.E., Florence, 2008.
[3] J. Avrin: Exponential asymptotic stability of a class of dynamical systems with applications to models of turbulent flow in two and three dimensions. Proc. Roy. Soc. Edinburgh Sect. A 142 (2012) 225–238.
[4] B. Birnir: The Kolmogorov-Obukhov Theory of Turbulence: A Mathematical Theory of Turbulence. SpringerBriefs in Mathematics, Springer, New York, 2013.
[5] T. Caraballo, J. A. Langa, and T. Taniguchi: The exponential behaviour and stabilizability of stochastic 2D-Navier-Stokes equations. J. Differential Equations 179 (2002), 714–737.
[6] F. Deng, Q. Luo, and X. Mao: Stochastic stabilization of hybrid differential equations. Automatica J. IFAC 48 (2012), 2321–2328.
[7] B.-Q. Dong and Y. Jia: Stability behaviors of Leray weak solutions to the three-dimensional Navier-Stokes equations. Nonlinear Anal. Real World Appl. 30 (2016), 41–58.
[8] C. Foias, O. Manley, R. Rosa, and R. Temam: *Navier-Stokes Equations and Turbulence*. Encyclopedia of Mathematics and its Applications, 83. Cambridge University Press, Cambridge, 2001.
[9] I. Gallagher, D. Iftimie, and F. Planchon: Asymptotics and stability for global solutions to the Navier-Stokes equations. *Ann. Inst. Fourier* **53** (2003), 1387–1424.
[10] H. Liu, L. Lin, C. Sun, Q. Xiao: The exponential behavior and stabilizability of the stochastic 3D Navier-Stokes equations with damping. *Rev. Math. Phys.* **31** (2019), 1950023, 15 pp.
[11] P.-H. Hsu and P. Sundar: Three-dimensional stochastic Navier-Stokes equations with Markov switching. https://arxiv.org/abs/2203.14442
[12] P.-H. Hsu and P. Sundar: Ergodicity for three-dimensional stochastic Navier-Stokes Equations with Markov Switching. https://arxiv.org/abs/2203.15749
[13] N. Ikeda and S. Watanabe: *Stochastic Differential Equations and Diffusion Processes*. North-Holland Mathematical Library, Second edition, North-Holland Publishing Co., Amsterdam, 1989.
[14] R. Khasminskii: *Stochastic Stability of Differential Equations*. Second edition, Stochastic Modelling and Applied Probability, 66 Springer, Heidelberg, 2012.
[15] O. A. Ladyzhenskaya: *The Mathematical Theory of Viscous Incompressible Flow*. Second English edition, Mathematics and its Applications, Vol. 2 Gordon and Breach, Science Publishers, New York-London-Paris 1969.
[16] J. Leray: Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta math.* **63** (1934), 193-248.
[17] X. Mao: *Exponential Stability of Stochastic Differential equations*. Monographs and Textbooks in Pure and Applied Mathematics, 182. Marcel Dekker, Inc., New York, 1994.
[18] X. Mao: Stability of stochastic differential equations with Markovian switching. *Stochastic Process. Appl.* **79** (1999), 45–67.
[19] X. Mao and C. Yuan: *Stochastic Differential Equations with Markovian Switching*. Imperial College Press, London, 2006.
[20] C. Prévôt and M. Röckner: *A Concise Course on Stochastic Partial Differential Equations*. Lecture Notes in Mathematics, 1905, Springer, Berlin, 2007.
[21] W. S. Ożarowski and B. C. Pooley: Leray’s fundamental work on the Navier-Stokes equations: a modern review of “Sur le mouvement d’un liquide visqueux emplissant l’espace”. *Partial Differential Equations in Fluid Dynamics*, London Math. Soc. Lecture Note Ser., 452, Cambridge Univ. Press, 2018.
[22] A. V. Skorohod: *Asymptotic Methods in the Theory of Stochastic Differential Equations*. Translations of Mathematical Monographs, 78. American Mathematical Society, Providence, RI, 1989.
[23] H. Sohr: *The Navier-Stokes Equations. An Elementary Functional Analytic Approach*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2001.
[24] R. Temam: *Navier-Stokes Equations. Theory and numerical analysis*. North-Holland Publishing Co., Amsterdam, 1984.
[25] A. Wu, S. You, W. Mao, X. Mao, and L. Hu: On exponential stability of hybrid neutral stochastic differential delay equations with different structures. *Nonlinear Anal. Hybrid Syst.* **39** (2021), 100971, 17 pp.
[26] C. Yuan and X. Mao: Asymptotic stability in distribution of stochastic differential equations with Markovian switching. *Stochastic Process. Appl.* **103** (2003), 277–291.
[27] G. Yin and C. Zhu: *Hybrid Switching Diffusions: Properties and Applications*. Stochastic Modelling and Applied Probability, 63. Springer, New York, 2010.
[28] Q. Zhu: Asymptotic stability in the $p$-th moment for stochastic differential equations with Lévy noise. *J. Math. Anal. Appl.* **416** (2014), 126–142.
[29] Y. Zhou: Asymptotic stability for the 3D Navier-Stokes equations. *Comm. Partial Differential Equations* **30** (2005), 323–333.
[30] W. Zhou, J. Yang, X. Yang, A. Dai, H. Liu, and J. Fang: $p$ th moment exponential stability of stochastic delayed hybrid systems with Lévy noise. *Appl. Math. Model.* **39** (2015), 5650–5658.

4415 FRENCH HALL-WEST, UNIVERSITY OF CINCINNATI, CINCINNATI, OH, 45221-0025, USA.
*Email address: hsupa@ucmail.uc.edu*