SUPER-HÖLDER VECTORS AND
THE FIELD OF NORMS

by

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Abstract. — Let $E$ be a field of characteristic $p$. In a previous paper of ours, we defined and studied super-Hölder vectors in certain $E$-linear representations of $\mathbb{Z}_p$. In the present paper, we define and study super-Hölder vectors in certain $E$-linear representations of a general $p$-adic Lie group. We then consider certain $p$-adic Lie extensions $K_\infty/K$ of a $p$-adic field $K$, and compute the super-Hölder vectors in the tilt of $K_\infty$. We show that these super-Hölder vectors are the perfection of the field of norms of $K_\infty/K$. By specializing to the case of a Lubin-Tate extension, we are able to recover $E[[Y]]$ inside the $Y$-adic completion of its perfection, seen as a valued $E$-vector space endowed with the action of $\mathcal{O}_K^\times$ given by the endomorphisms of the corresponding Lubin-Tate group.

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Introduction

Let $E$ be a field of characteristic $p$, for example a finite field. In our paper [BR22], we defined and studied super-Hölder vectors in certain $E$-linear representations of the $p$-adic Lie group $\mathbb{Z}_p$. These vectors are a characteristic $p$ analogue of locally analytic vectors. They allowed us to recover $E((X))$ inside the $X$-adic completion of its perfection, seen as a valued $E$-vector space endowed with the action of $\mathbb{Z}_p^\times$ given by $a \cdot f(X) = f((1+X)^a-1)$.

In the present paper, we define and study super-Hölder vectors in certain $E$-linear representations of a general $p$-adic Lie group. We then consider certain $p$-adic Lie extensions $K_\infty/K$ of a $p$-adic field $K$, and compute the super-Hölder vectors in the tilt of $K_\infty$. We show that these super-Hölder vectors are the perfection of the field of norms of $K_\infty/K$.

By specializing to the case of a Lubin-Tate extension, we are able to recover $E((Y))$ inside the $Y$-adic completion of its perfection, seen as a valued $E$-vector space endowed with the action of $O_K^\times$ given by the endomorphisms of the corresponding Lubin-Tate group.

We now give more details about the contents of our paper. Let $\Gamma$ be a $p$-adic Lie group. It is known that $\Gamma$ always has a uniform open pro-$p$ subgroup $G$. Let $G$ be such a subgroup, and let $G_i = G^{p^i}$ for $i \geq 0$. Let $M$ be an $E$-vector space, endowed with a valuation $\text{val}_M$ such that $\text{val}_M(xm) = \text{val}_M(m)$ if $x \in E^\times$. We assume that $M$ is separated and complete for the $\text{val}_M$-adic topology. We say that a function $f : G \to M$ is super-Hölder if there exist constants $e > 0$ and $\lambda, \mu \in \mathbb{R}$ such that $\text{val}_M(f(g) - f(h)) \geq p^\lambda \cdot p^{ei} + \mu$ whenever $gh^{-1} \in G_i$, for all $g, h \in G$ and $i \geq 0$. If $M$ is now endowed with an action of $G$ by isometries, and $m \in M$, we say that $m$ is a super-Hölder vector if the orbit map $g \mapsto g \cdot m$ is a super-Hölder function $G \to M$. We let $M^{G-e\text{-sh}, \lambda}$ denote the space of super-Hölder vectors for given constants $e$ and $\lambda$ as in the definition above. The space of vectors of $M$ that are super-Hölder for a given $e$ is independent of the choice of the uniform subgroup $G$, and denoted by $M^{e\text{-sh}}$. When $G = \mathbb{Z}_p$ and $e = 1$, we recover the definitions of [BR22].

If $\Gamma$ is a $p$-adic Lie group and $e = 1$, we get an analogue of locally $\mathbb{Q}_p$-analytic vectors. If $K$ is a finite extension of $\mathbb{Q}_p$, $\Gamma$ is the Galois group of a Lubin-Tate extension of $K$, and $e = [K : \mathbb{Q}_p]$, we seem to get an analogue of locally $K$-analytic vectors.

From now on, assume that $p \neq 2$. Let $K$ be a $p$-adic field and let $K_\infty/K$ be an almost totally ramified $p$-adic Lie extension, with Galois group $\Gamma$ of dimension $d \geq 1$. The tilt of $K_\infty$ is the fraction field $\mathcal{E}_{K_\infty}$ of $\varprojlim_{x \to x'} O_{K_\infty}/p$. It is a perfect complete valued field of characteristic $p$, endowed with an action of $\Gamma$ by isometries. The field $\mathcal{E}_{K_\infty}$ naturally contains the field of norms $X_K(K_\infty)$ of the extension $K_\infty/K$, and it is known that $\mathcal{E}_{K_\infty}$ is the completion of the perfection of $X_K(K_\infty)$. We have the following result (theorem 2.2.3).
Theorem A. — We have $\tilde{E}_{K_\infty}^{d-sh} = \bigcup_{n \geq 0} \varphi^{-n}(X_K(K_\infty))$. 

Assume now that $K$ is a finite extension of $\mathbb{Q}_p$, with residue field $k$, and let LT be a Lubin-Tate formal group attached to $K$. Let $K_\infty$ be the extension of $K$ generated by the torsion points of LT, so that $\text{Gal}(K_\infty/K)$ is isomorphic to $\mathcal{O}_K^\times$. The field of norms $X_K(K_\infty)$ is isomorphic to $k((Y))$, and $\mathcal{O}_K^\times$ acts on this field by the endomorphisms of the Lubin-Tate group: $a \cdot f(Y) = f([a](Y))$. Let $d = [K : \mathbb{Q}_p]$. The following (theorem 3.2.1) is a more precise version of theorem A in this situation.

Theorem B. — If $j \geq 1$, then $\tilde{E}_{K_\infty}^{1+p^j\mathcal{O}_K-d-sh,dj} = k((Y))$.

If $K = \mathbb{Q}_p$ and $K_\infty/K$ is the cyclotomic extension, theorem B was proved in [BR22]. A crucial ingredient of the proof of this theorem was Colmez’ analogue of Tate traces for $\tilde{E}_{K_\infty}$. If the Lubin-Tate group if of height $\geq 2$, there are no such traces (we state and prove a precise version of this assertion in §3.2). Instead of Tate traces, we a theorem of Ax and a precise characterization of the field of norms $X_K(K_\infty)$ inside $\tilde{E}_{K_\infty}$ in order to prove theorem A.

As an application of theorem B, we compute the perfectoid commutant of $\text{Aut}(LT)$. If $b \in \mathcal{O}_K^\times$ and $n \in \mathbb{Z}$, then $u(Y) = [b](Y^{p^n})$ is an element of $\tilde{E}_{K_\infty}^+$ that satisfies the functional equation $u \circ [g](Y) = [g] \circ u(Y)$ for all $g \in \mathcal{O}_K^\times$. Conversely, we prove the following (theorem 3.3.1).

Theorem C. — If $u \in \tilde{E}_{K_\infty}^+$ is such that $\text{val}_Y(u) > 0$ and $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_K^\times$, there exists $b \in \mathcal{O}_K^\times$ and $n \in \mathbb{Z}$ such that $u(Y) = [b](Y^{p^n})$.

In the last section, we give a characterization of super-Hölder functions on a uniform pro-$p$ group in terms of their Mahler expansions (theorem 4.3.4). In order to do so, we prove some results of independent interest on the space of continuous functions on $\mathcal{O}_K^d$ with values in a valued $E$-vector space $M$ as above.

At the end of [BR22], we suggested an application of super-Hölder vectors for the action of $\mathbb{Z}_p$ to the $p$-adic local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$. We hope that this general theory of super-Hölder vectors, especially in the Lubin-Tate case, will have applications to the $p$-adic local Langlands correspondence for other fields than $\mathbb{Q}_p$. 

1. Super-Hölder functions and vectors

In this section, we define Super-Hölder vectors inside a valued $E$-vector space $M$ endowed with an action of a $p$-adic Lie group $\Gamma$. The definition is very similar to the one
that we gave for $\Gamma = \mathbb{Z}_p$ in our paper [BR22]. The main new technical tool is the existence of uniform open subgroups of $\Gamma$. These uniform subgroups look very much like $\mathbb{Z}_p^d$ in a sense that we make precise.

1.1. Uniform pro-$p$ groups. — Uniform pro-$p$ groups are defined at the beginning of §4 of [DdSMS99]. We do not recall the definition, nor the notion of rank of a uniform pro-$p$ group, but rather point out the following properties of uniform pro-$p$ groups. A coordinate (below) is simply a homeomorphism.

**Proposition 1.1.1.** — If $G$ is a uniform pro-$p$ group of rank $d$, then
1. $G_i = \{g^{p^i}, g \in G\}$ is an open normal (and uniform) subgroup of $G$ for $i \geq 0$
2. We have $[G_i : G_{i+1}] = p^d$ for $i \geq 0$
3. There is a coordinate $c : G \to \mathbb{Z}_p^d$ such that $c(G_i) = (p^i\mathbb{Z}_p)^d$ for $i \geq 0$
4. If $g, h \in G$, then $gh^{-1} \in G_i$ if and only if $c(g) - c(h) \in (p^i\mathbb{Z}_p)^d$

**Proof.** — Properties (1-4) are proved in §4 of [DdSMS99]. Alternatively, a uniform pro-$p$ group $G$ has a natural integer valued $p$-valuation $\omega$ such that $(G, \omega)$ is saturated (remark 2.1 of [Klo05]). Properties (1-4) are then proved in §26 of [Sch11].

For example, the pro-$p$ group $\mathbb{Z}_p^d$ is uniform for all $d \geq 1$.

**Lemma 1.1.2.** — If $G$ is a uniform pro-$p$ group, and $H$ is a uniform open subgroup of $G$, there exists $j \geq 0$ such that $G_{i+j} \subset H_i$ for all $i \geq 0$.

**Proof.** — This follows from the fact that $\{G_i\}_{i \geq 0}$ forms a basis of neighborhoods of the identity in $G$.

A $p$-adic Lie group is a $p$-adic manifold that has a compatible group structure. For example, $\text{GL}_n(\mathbb{Z}_p)$ and its closed subgroups are $p$-adic Lie groups. We refer to [Sch11] for a comprehensive treatment of the theory. Every uniform pro-$p$ group is a $p$-adic Lie group. Conversely, we have the following.

**Proposition 1.1.3.** — Every $p$-adic Lie group $\Gamma$ has a uniform open subgroup $G$, and the rank of $G$ is the dimension of $\Gamma$.

**Proof.** — See Interlude A (pages 97–98) of [DdSMS99].

**Proposition 1.1.4.** — Let $G$ be a pro-$p$ group of finite rank, and $N$ a closed normal subgroup of $G$. There exists an open subgroup $G'$ of $G$ such that $G', G' \cap N$ and $G'/G' \cap N$ are all uniform.

**Proof.** — This is stated and proved on page 64 of [DdSMS99] (their $H$ is our $G'$).
1.2. Super-Hölder functions and vectors. — Let $M$ be an $E$-vector space, endowed with a valuation $\text{val}_M$ such that $\text{val}_M(xm) = \text{val}_M(m)$ if $x \in E^\times$. We assume that $M$ is separated and complete for the $\text{val}_M$-adic topology. Throughout this §, $G$ denotes a uniform pro-$p$ group.

Definition 1.2.1. — We say that $f : G \to M$ is super-Hölder if there exist constants $\lambda, \mu \in \mathbb{R}$ and $e > 0$ such that $\text{val}_M(f(g) - f(h)) \geq p^\lambda \cdot p^{ei} + \mu$ whenever $gh^{-1} \in G$, for all $g, h \in G$ and $i \geq 0$.

Remark 1.2.2. — If $G = \mathbb{Z}_p$ and $e = 1$, we recover the functions defined in §1.1 (see also remark 1.12 of ibid).

In the above definition, $e$ will usually be equal to either 1 or $\text{dim}(G)$.

We let $\mathcal{H}_e^{\lambda,\mu}(G, M)$ denote the space of functions such that $\text{val}_M(f(g) - f(h)) \geq p^\lambda \cdot p^{ei} + \mu$ whenever $gh^{-1} \in G$, for all $g, h \in G$ and $i \geq 0$, and $\mathcal{H}_e^{\lambda}(G, M) = \cup_{\mu \in \mathbb{R}} \mathcal{H}_e^{\lambda,\mu}(G, M)$ and $\mathcal{H}_e(G, M) = \cup_{\lambda, \mu \in \mathbb{R}} \mathcal{H}_e^{\lambda,\mu}(G, M)$.

If $M, N$ are two valued $E$-vector spaces, and $f : M \to N$ is an $E$-linear map, we say that $f$ is Hölder-continuous if there exists $c > 0, d \in \mathbb{R}$ such that $\text{val}_N(f(x)) \geq c \cdot \text{val}_M(x) + d$ for all $x \in M$.

Proposition 1.2.3. — If $\pi : M \to N$ is a Hölder-continuous linear map, we get a map $\mathcal{H}_e(G, M) \to \mathcal{H}_e(G, N)$.

Proof. — Take $c, d \in \mathbb{R}$ of Hölder continuity for $\pi$, $f \in \mathcal{H}_e^{\lambda,\mu}(G, M)$, and $g, h \in G$ with $gh^{-1} \in G$. We have $\text{val}_N(\pi(f(g)) - \pi(f(h))) \geq c \cdot \text{val}_M(f(g) - f(h)) + d \geq cp^\lambda \cdot p^{ei} + (\mu + d)$, so that $\pi \circ f \in \mathcal{H}_e^{\lambda,\mu'}(G, N)$ with $p^\lambda' = cp^\lambda$, and $\mu' = \mu + d$. $\square$

Proposition 1.2.4. — If $\alpha : G \to H$ is a group homomorphism, we get a map $\alpha^* : \mathcal{H}_e(H, M) \to \mathcal{H}_e(G, M)$.

Proof. — By definition of the subgroups $G_i$ and $H_i$, we have $\alpha(G_i) \subset H_i$ for all $i$. Take $f \in \mathcal{H}_e^{\lambda,\mu}(H, M)$, and $g, h \in G$ with $gh^{-1} \in G_i$. We have $\text{val}_M(f(\alpha(g)) - f(\alpha(h))) \geq p^\lambda \cdot p^{ei} + \mu$ as $\alpha(g)\alpha(h)^{-1} \in H_i$, so that $\alpha^*(f) = f \circ \alpha \in \mathcal{H}_e^{\lambda,\mu}(G, M)$. $\square$

Proposition 1.2.5. — Suppose that $M$ is a ring, and that $\text{val}_M(mm') \geq \text{val}_M(m) + \text{val}_M(m')$ for all $m, m' \in M$. If $c \in \mathbb{R}$, let $M_c = M^{\text{val}_M \geq c}$.

1. If $f \in \mathcal{H}_e^{\lambda,\mu}(G, M_c)$ and $g \in \mathcal{H}_e^{\lambda,\nu}(G, M_d)$, and $\xi = \min(\mu + d, \nu + c)$, then $fg \in \mathcal{H}_e^{\lambda,\xi}(G, M_{c+d})$.

2. If $\lambda, \mu \in \mathbb{R}$, then $\mathcal{H}_e^{\lambda,\mu}(G, M_0)$ is a subring of $C^0(G, M)$.

3. If $\lambda \in \mathbb{R}$, then $\mathcal{H}_e^{\lambda}(G, M)$ is a subring of $C^0(G, M)$. 
Proof. — Items (2) and (3) follow from item (1), which we now prove. If \( x, y \in G \), then
\[
(fg)(x) - (fg)(y) = (f(x) - f(y))g(x) + (g(x) - g(y))f(y),
\]
which implies the claim. \( \square \)

We now assume that \( M \) is endowed with an \( E \)-linear action by isometries of \( G \). If \( m \in M \), let \( \text{orb}_m : G \to M \) denote the function defined by \( \text{orb}_m(g) = g \cdot m \).

**Definition 1.2.6.** — Let \( M^{G-\text{sh},\lambda,\mu} \) be those \( m \in M \) such that \( \text{orb}_m \in \mathcal{H}^{\lambda,\mu}(G, M) \), and let \( M^{G-\text{sh},\lambda} \) and \( M^{G-\text{sh}} \) be the corresponding sub-\( E \)-vector spaces of \( M \).

**Remark 1.2.7.** — We assume that \( G \) acts by isometries on \( M \), but not that \( G \) acts continuously on \( M \), namely that \( G \times M \to M \) is continuous. However, let \( M^{\text{cont}} \) denote the set of \( m \in M \) such that \( \text{orb}_m : G \to M \) is continuous. It is easy to see that \( M^{\text{cont}} \) is a closed sub-\( E \)-vector space of \( M \), and that \( G \times M^{\text{cont}} \to M^{\text{cont}} \) is continuous (compare with §3 of [Eme17]). We then have \( M^{\text{sh}} \subset M^{\text{cont}} \).

**Lemma 1.2.8.** — If \( m \in M \), then \( m \in M^{G-\text{sh},\lambda,\mu} \) if and only if for all \( i \geq 0 \), we have \( \text{val}_M(g \cdot m - m) \geq p^\lambda \cdot p^{ei} + \mu \) for all \( g \in G_i \).

**Proof.** — If \( m \in M \), then \( m \in M^{G-\text{sh},\lambda,\mu} \) if and only if the function \( \text{orb}_m \) is in \( \mathcal{H}^{\lambda,\mu}(G, M) \), that is, for all \( g, h \) with \( gh^{-1} \in G_i \), we have \( \text{val}_M(g \cdot m - h \cdot m) \geq p^\lambda \cdot p^{ei} + \mu \). As \( G \) acts by isometries, we have \( \text{val}_M(g \cdot m - h \cdot m) = \text{val}_M(h^{-1}g \cdot m - m) \). The result follows, as \( h^{-1}g = h^{-1} \cdot gh^{-1} \cdot h \in G_i \). \( \square \)

**Lemma 1.2.9.** — The space \( M^{G-\text{sh},\lambda,\mu} \) is a closed sub-\( E \)-vector space of \( M \).

**Lemma 1.2.10.** — If \( i_0 \geq 0 \), and \( m \in M \) is such that \( \text{val}_M(g \cdot m - m) \geq p^\lambda \cdot p^{ei} + \mu \) for all \( g \in G_i \) with \( i \geq i_0 \), then \( m \in M^{G-\text{sh},\lambda} \).

**Proof.** — Take \( i < i_0 \), and let \( R_i \) be a set of representatives of \( G_{i_0} \setminus G_i \). This is a finite set, so there exists \( \mu_i \in \mathbb{R} \) such that \( \text{val}_M(r \cdot m - m) \geq p^\lambda \cdot p^{ei} + \mu_i \) for all \( r \in R_i \). If \( g \in G_i \), it can be written as \( g = hr \) for some \( h \in G_{i_0} \) and \( r \in R_i \). We then have \( g \cdot m - m = hr \cdot m - h \cdot m + h \cdot m - m \), so that \( \text{val}_M(g \cdot m - m) \geq \min(\text{val}_M(r \cdot m - m), \text{val}_M(h \cdot m - m)) \) (recall that \( G \) acts by isometries), so \( \text{val}_M(g \cdot m - m) \geq \min(p^\lambda \cdot p^{ei} + \mu_i, p^\lambda \cdot p^{ei} + \mu) \geq p^\lambda \cdot p^{ei} + \min(\mu, \mu_i) \) as \( i_0 > i \). If \( \mu' \) is the min of \( \mu \) and the \( \mu_i \) for \( 0 \leq i < i_0 \), then \( m \in M^{G-\text{sh},\lambda,\mu'} \). \( \square \)

Recall that if \( k \geq 0 \), then \( G_k \) is also a uniform pro-\( p \) group.

**Lemma 1.2.11.** — If \( k \geq 0 \) then \( M^{G-\text{sh},\lambda} = M^{G_{k-\text{sh},\lambda+k}} \).
Proof. — Note that \((G_k)_i = G_{i+k}\). The inclusion \(M_{G-e^s} \subset M_{G-e^s+k}\) is obvious, and the reverse inclusion follows from lemma \[1.2.10\].

**Proposition 1.2.12.** — The space \(M_{H-e^s}\) does not depend on the choice of a uniform open subgroup \(H \subset G\).

Proof. — Let \(H\) and \(H'\) be uniform open subgroups of \(G\). The group \(H \cap H'\) contains an open uniform subgroup by prop \[1.1.3\], so to prove the proposition, we can further assume that \(H' \subset H\). We then have \(H'_i \subset H_i\) for all \(i\), so that if \(m \in M_{H-e^s,\lambda,\mu}\), then \(m \in M_{H'-e^s,\lambda,\mu}\). This implies that \(M_{H-e^s,\lambda} \subset M_{H'-e^s,\lambda}\). Conversely, by lemma \[1.1.2\], there exists \(j\) such that \(H_j \subset H'\). The previous reasoning implies that \(M_{H'-e^s,\lambda} \subset M_{H_j-e^s,\lambda}\). Lemma \[1.2.11\] now implies that \(M_{H_j-e^s,\lambda} = M_{H-e^s,\lambda-j}\).

These inclusions imply the proposition. 

**Definition 1.2.13.** — If \(\Gamma\) is a \(p\)-adic Lie group that acts by isometries on \(M\), we let \(M_{e^s} = M_{G-e^s}\) where \(G\) is any uniform open subgroup of \(\Gamma\).

**Remark 1.2.14.** — If \(e \leq f\), then \(M_{I-e} \subset M_{e^s}\).

Recall that \(G\) is a uniform pro-\(p\) group. If a closed normal subgroup \(N\) of \(G\) acts trivially on \(M\), then \(G/N\) acts on \(M\).

**Proposition 1.2.15.** — If a closed normal subgroup \(N\) of \(G\) acts trivially on \(M\), then \(M_{G-e^s} = M_{G/N-e^s}\).

Proof. — By prop \[1.1.4\] \(G\) has an open subgroup \(G'\) such that \(G'\) and \(G'/N'\) are uniform (where \(N' = G' \cap N\)). By prop \[1.2.12\] we have \(M_{G-e^s} = M_{G'-e^s}\) and \(M_{G/N-e^s} = M_{G'/N'-e^s}\). Let \(\pi : G' \to G'/N'\) denote the projection. We have \(\pi(G'_i) = (G'/N')_i\), for all \(i\). Hence if \(m \in M\), then \(\val_M(g \cdot m - m) \geq p^\lambda \cdot p^{e_i + \mu}\) for all \(g \in G'_i\) if and only if \(\val_M(\pi(g) \cdot m - m) \geq p^\lambda \cdot p^{e_i + \mu}\) for all \(\pi(g) \in (G'/N')_i\).

**Proposition 1.2.16.** — Suppose that \(M\) is a ring, and that \(g(mm') = g(m)g(m')\) and \(\val_M(mm') \geq \val_M(m) + \val_M(m')\) for all \(m, m' \in M\) and \(g \in G\).

1. If \(v \in \mathbf{R}\) and \(m, m' \in M_{G-e^s,\lambda,\mu} \cap M_{\val_M \geq v}\), then \(m \cdot m' \in M_{G-e^s,\lambda,\mu+v}\).
2. If \(m \in M_{G-e^s,\lambda,\mu} \cap M^x\), then \(1/m \in M_{G-e^s,\lambda,\mu-2\val_M(m)}\).

Proof. — Item (1) follows from prop \[1.2.5\] and lemma \[1.2.8\] Item (2) follows from

\[g \left( \frac{1}{m} \right) - \frac{1}{m} = \frac{m - g(m)}{g(m)m} \]
2. The field of norms

Let $K$ be a $p$-adic field, and let $K_\infty$ be an algebraic Galois extension of $K$, whose Galois group $G$ is a $p$-adic Lie group of dimension $\geq 1$. We assume that $K_\infty/K$ is almost totally ramified, namely that the inertia subgroup of $G$ is open in $G$. Let $d = \dim(G)$ and let $\ell = p^d$. Let $\hat{E}^+_{K_\infty}$ denote the ring $\varprojlim_{x \rightarrow x^{\ell}} \mathcal{O}_{K_\infty}/p$. This is a perfect domain of characteristic $p$, which has a natural action of $G$. The map $(y_j)_{j \geq 0} \mapsto (y_{d^i})_{i \geq 0}$ gives an isomorphism between $\varprojlim_{x \rightarrow x^{\ell}} \mathcal{O}_{K_\infty}/p$ and $\hat{E}^+_{K_\infty}$, so that $\hat{E}^+_{K_\infty}$ is the ring of integers of the tilt of $\hat{K}_\infty$ (see §3 of [Sch12]).

If $x = (x_i)_{i \geq 0}$, and $\hat{x}_i$ is a lift of $x_i$ to $\mathcal{O}_{K_\infty}$, then $\ell^i \mathrm{val}_p(\hat{x}_i)$ is independent of $i \geq 0$ such that $x_i \neq 0$. We define a valuation on $\hat{E}^+_{K_\infty}$ by $\mathrm{val}_E(x) = \lim_{i \rightarrow +\infty} \ell^i \mathrm{val}_p(\hat{x}_i)$.

The aim of this section is to compute $(\hat{E}^+_{K_\infty})^{d\text{-sh}}$. Given definition 1.2.13, we assume from now on (replacing $K$ by a finite subextension if necessary) that $G$ is uniform and that $K_\infty/K$ is totally ramified. Let $k$ denote the common residue field of $K$ and $K_\infty$.

2.1. The field of norms. — Let $\mathcal{E}(K_\infty)$ denote the set of finite extensions $E$ of $K$ such that $E \subset K_\infty$. Let $X_K(K_\infty)$ denote the set of sequences $(x_E)_{E \in \mathcal{E}(K_\infty)}$ such that $x_E \in E$ for all $E \in \mathcal{E}(K_\infty)$, and $\mathcal{N}_{F/E}(x_F) = x_E$ whenever $E \subset F$ with $E, F \in \mathcal{E}(K_\infty)$.

If $n \geq 0$, let $K_n = K_\infty^n$ so that $[K_{n+1} : K_n] = \ell$, $\{K_n\}_{n \geq 0}$ is a cofinal subset of $\mathcal{E}(K_\infty)$, and $X_K(K_\infty) = \varprojlim_{n \geq 0} K_n$. If $x = (x_n)_{n \geq 0} \in X_K(K_\infty)$, let $\mathrm{val}_E(x) = \mathrm{val}_p(x_0)$.

**Theorem 2.1.1.** — Let $K$ and $K_\infty$ be as above.

1. If $x, y \in X_K(K_\infty)$, then $\{N_{K_{n+1}/K_n}(x_{n+j} + y_{n+j})\}_{j \geq 0}$ converges for all $n \geq 0$.
2. If we set $(x + y)_n = \lim_{j \rightarrow +\infty} N_{K_{n+j}/K_n}(x_{n+j} + y_{n+j})$, then $x + y \in X_K(K_\infty)$, and the set $X_K(K_\infty)$ with this addition law, and componentwise multiplication, is a field of characteristic $p$.
3. The function $\mathrm{val}_E$ is a valuation on $X_K(K_\infty)$, for which it is complete.
4. If $\varpi = (\varpi_n)_{n \geq 0}$ is a norm compatible sequence of uniformizers of $\mathcal{O}_K$, the valued field $X_K(K_\infty)$ is isomorphic to $k((\varpi))$ (with $\mathrm{val}(\varpi) = \mathrm{val}_p(\varpi_0)$).

**Proof.** — By a result of Sen [Sen72], $K_\infty/K$ is strictly APF in the terminology of §1.2 of [Win83] (see 1.2.2 of ibid). The theorem is then proved in §2 of ibid. \qed

Let $X^+_{K}(K_\infty) = \varprojlim_{n \geq 0} \mathcal{O}_K$ be the ring of integers of the valued field $X_K(K_\infty)$.

If $c > 0$, let $I^c_n = \{x \in \mathcal{O}_K$ such that $\mathrm{val}_p(x) \geq c\}$. If $m, n \geq 0$, the map $\mathcal{O}_K/I^c_n \rightarrow \mathcal{O}_{K_{m+n}}/I^c_{m+n}$ is well-defined and injective.
Proposition 2.1.2. — There exists \( c(K_\infty/K) \leq 1 \) such that if \( 0 < c \leq c(K_\infty/K) \), then \( \text{val}_p(N_{K^{n+k}/K_n}(x)/x^{[K_{n+k}:K_n]} - 1) \geq c \) for all \( n, k \geq 0 \) and \( x \in O_{K_{n+k}} \).

Proof. — See [Win83] as well as §4 of [CD15]. The result follows from the fact (see 1.2.2 of [Win83]) that the extension \( K_\infty/K \) is strictly APF. One can then apply 1.2.1, 4.2.2 and 1.2.3 of [Win83].

Using prop 2.1.2, we get a map \( \iota : X^+_K(K_\infty) \to \lim_{\leftarrow x \to x^t} O_{K_\infty}/I_{\infty}^c \) given by \( (x_n)_{n \geq 0} \in \lim_{\leftarrow_{N_{K_n}/K_{n-1}}} O_{K_n} \mapsto (\tau_n)_{n \geq 0} \). Let \( \lim_{\leftarrow x \to x^t} O_{K_n}/I_n^c \) denote the set of \( (x_n)_{n \geq 0} \in \lim_{\leftarrow_{x \to x^t}} O_{K_\infty}/I_{\infty}^c \) such that \( x_n \in O_{K_n}/I_n^c \) for all \( n \geq 0 \).

Proposition 2.1.3. — Let \( 0 < c \leq c(K_\infty/K) \) be as in prop 2.1.2.

1. the natural map \( \tilde{E}_{K_\infty}^+ \to \lim_{\leftarrow x \to x^t} O_{K_\infty}/I_{\infty}^c \) is a bijection
2. the map \( \iota : X^+_K(K_\infty) \to \lim_{\leftarrow x \to x^t} O_{K_\infty}/I_{\infty}^c = \tilde{E}_{K_\infty}^+ \) is injective and isometric
3. the image of \( \iota \) is \( \lim_{\leftarrow x \to x^t} O_{K_n}/I_n^c \).

Proof. — See [Win83] and §4 of [CD15]. We give a few more details for the convenience of the reader. Item (1) is classical (see for instance prop 4.2 of [CD15]). The map \( \iota \) is obviously injective and isometric. For (3), choose \( x = (x_n)_{n \geq 0} \in \lim_{\leftarrow x \to x^t} O_{K_n}/I_n^c \), and choose a lift \( \hat{x}_n \in O_{K_n} \) of \( x_n \). One proves that \( \{N_{K_{n+j}/K_n}(\hat{x}_n+j)\}_{j \geq 0} \) converges to some \( y_n \in O_{K_n} \), and that \( (y_n)_{n \geq 0} \in X^+_K(K_\infty) \) is a lift of \( (x_n)_{n \geq 0} \). See §4 of [CD15] for details, for instance the proof of lemma 4.1. □

Prop 2.1.3 allows us to see \( X^+_K(K_\infty) \), and hence \( \varphi^{-n}(X^+_K(K_\infty)) \) for all \( n \geq 0 \), as a subring of \( \tilde{E}_{K_\infty}^+ \).

Proposition 2.1.4. — The ring \( \bigcup_{n \geq 0} \varphi^{-n}(X^+_K(K_\infty)) \) is dense in \( \tilde{E}_{K_\infty}^+ \).

Proof. — See §4.3 of [Win83]. □

2.2. Decompleting the tilt. — We now compute \( (\tilde{E}_{K_\infty}^+)_{d-sh} \). Since prop 2.2.1 below is vacuous if \( p = 2 \), we assume in this § that \( p \neq 2 \).

Proposition 2.2.1. — If \( 0 < c \leq 1 - 1/(p-1) \), and \( x \in O_{K_\infty} \) is such that \( \text{val}_p(g(x) - x) \geq 1 \) for all \( g \in G_n \), then the image of \( x \) in \( O_{K_\infty}/I_{\infty}^c \) belongs to \( O_{K_n}/I_n^c \).

Proof. — If \( \text{val}_p(g(x) - x) \geq 1 \) for all \( g \in \text{Gal}(K^{alg}/K_n) \), then by theorem 1.7 of [LB10] (an optimal version of a theorem of Ax), there exists \( y \in K_n \) such that \( \text{val}_p(x - y) \geq 1 - 1/(p-1) \). This implies the proposition. □

Proposition 2.2.2. — If \( c = p^\gamma \) is as above, then \( X^+_K(K_\infty) \subset (\tilde{E}_{K_\infty}^+)_{G-d-sh,\gamma,0} \).
In particular, $\pi$ power series giving the addition law in $L_T$ in that coordinate. Recall that Lemma 3.1.1

$\pi = q$ with ring of integers $O$. We see that $\text{val}_K(x) = \min\{e(K_{\infty}/K), 1 - 1/(p - 1)\}$. Take $x = (x_n)_{n \geq 0} \in \lim_{\longleftarrow x \to x^e} O_{K_{\infty}}/I_{n}^c$. If $n \geq 0$ and $x \in (\tilde{E}_{K_{\infty}}^{+})^{G_{dsh}, 0, 0}$, then $\text{val}_K(x) = p^d$.

**Theorem 2.2.3.** — We have

1. $(\tilde{E}_{K_{\infty}}^{+})^{G_{dsh}, 0, 0} \subset X_{K_{\infty}}^{+}(K_{\infty})$
2. $(\tilde{E}_{K_{\infty}}^{+})^{dsh} = \cup_{n \geq 0} \varphi^{-n}(X_{K_{\infty}}^{+}(K_{\infty}))$ and $\tilde{E}_{K_{\infty}}^{dsh} = \cup_{n \geq 0} \varphi^{-n}(X_{K_{\infty}}^{+}(K_{\infty}))$

**Proof.** — Take $c \leq \min\{e(K_{\infty}/K), 1 - 1/(p - 1)\}$. Take $x = (x_n)_{n \geq 0} \in \lim_{\longleftarrow x \to x^e} O_{K_{\infty}}/p$. If $n \geq 0$ and $x \in (\tilde{E}_{K_{\infty}}^{+})^{G_{dsh}, 0, 0}$, then $\text{val}_K(x) = p^d$.

Since $\text{val}_K(\varphi(x)) = p \cdot \text{val}_K(x)$, item (2) follows from (1) and props 2.2.2 and 1.2.16.

**Remark 2.2.4.** — We have $\tilde{E}_{K_{\infty}}^{1sh} \subset \tilde{E}_{K_{\infty}}^{1sh}$. The field $\tilde{E}_{K_{\infty}}^{1sh}$ contains the field of norms $X_{K_{\infty}}(L_{\infty})$ of any $p$-adic Lie extension $L_{\infty}/K$ contained in $K_{\infty}$. Indeed, $\tilde{E}_{L_{\infty}} \subset \tilde{E}_{K_{\infty}}$ and if $e = \dim \text{Gal}(L_{\infty}/K)$, then $X_{K_{\infty}}(L_{\infty}) \subset \tilde{E}_{L_{\infty}}^{1sh} \subset \tilde{E}_{K_{\infty}}^{1sh}$ (see prop 1.2.15).

Can one give a description of $\tilde{E}_{K_{\infty}}^{1sh}$, for example along the lines of §5 of [Ber16]?  

3. The Lubin-Tate case

We now specialize the constructions of the previous section to the case when $K_{\infty}$ is generated over $K$ by the torsion points of a Lubin-Tate formal group.

3.1. Lubin-Tate formal groups. — Let $K$ be a finite extension of $\mathbb{Q}_p$ of degree $d$, with ring of integers $O_K$, inertia index $f$, ramification index $e$, and residue field $k$. Let $q = p^f = \text{Card}(k)$ and let $\pi$ be a uniformizer of $O_K$. Let LT be the Lubin-Tate formal $O_K$-module attached to $\pi$ (see [LT65]). We choose a coordinate $Y$ on LT. For each $a \in O_K$ we get a power series $[a](Y) \in O_K[Y]$, that we now see as an element of $k[[Y]]$.

In particular, $[\pi](Y) = Y^{q}$. Let $S(T, U) \in k[T, U]$ denote the reduction mod $\pi$ of the power series giving the addition law in LT in that coordinate. Recall that $S(T, 0) = T$ and $S(0, U) = U$.

**Lemma 3.1.1.** — If $a, b \in O_K$ and $i \geq 0$, then $\text{val}_Y([a + p^i b](Y) - [a](Y)) \geq p^{di}$.

**Furthermore,** $[1 + \pi^i](Y) = Y + Y^{q^i} + O(Y^{q^i+1})$.

**Proof.** — We have $[\pi](Y) = Y^q$, so $\text{val}_Y([\pi](Y)) = p^f$. Writing $p = u \pi^e$ for a unit $u$, we see that $\text{val}_Y([p^f b](Y)) \geq p^{di}$ if $b \in O_K$. If $a, b \in O_K$ and $i \geq 0$, then $[a + b p^i](Y) = \cdots$.
$S([a](Y), [bp](Y))$. We have $S(T, U) = T + U + TU \cdot R(T, U)$, so that $[a + bp](Y) - [a](Y) = S([a](Y), [bp](Y)) - [a](Y) \in [bp](Y) \cdot k[Y]$. This implies the first result.

The second claim follows likewise from the fact that $[1 + \pi^n](Y) = S(Y, [\pi^n](Y)) = Y + [\pi^n](Y) + Y \cdot [\pi^n](Y) \cdot R(Y, [\pi^n](Y))$. □

Let $E = k((Y))$. Let $E_n = k((Y^{1/q^n}))$ and let $E_n = \cup_{n \geq 0} E_n$. These fields are endowed with the $Y$-adic valuation $\text{val}_Y$, and we let $E_n^+$ denote the ring of integers of $E_n$. The group $\mathcal{O}_K$ acts on $E_n$ by $a \cdot f(Y^{1/q^n}) = f([a](Y^{1/q^n}))$.

**Lemma 3.1.2.** — If $j \geq 1$ ($j \geq 2$ if $p = 2$), then $1 + p^j \mathcal{O}_K$ is uniform, and $(1 + p^j \mathcal{O}_K)_i = 1 + p^{j+i} \mathcal{O}_K$.

**Proof.** — The map $1 + p^j \mathcal{O}_K \rightarrow \mathcal{O}_K$, given by $x \mapsto p^{-j} \cdot \log_p (x - 1)$, is an isomorphism of pro-$p$ groups taking $1 + p^{j+i} \mathcal{O}_K$ to $p^j \mathcal{O}_K$.

Recall that $d = [K : \mathbb{Q}_p]$, that $f = [k : E]$, and that $q = p^f$.

**Proposition 3.1.3.** — We have $E_n^+ = (E_n^+)^{1 + p^j \mathcal{O}_{K-d-sh,dj} - fn,0}$.

**Proof.** — If $b \in \mathcal{O}_K$ and $i, j \geq 0$, then by lemma 3.1.1, we have

$$\text{val}_Y((1 + p^{i+j}b)(Y^{1/q^n}) - Y^{1/q^n}) \geq 1/q^n \cdot p^{d(i+j)} = p^{dj-fn} \cdot p^{dj}.$$  

Lemma 3.1.2 then implies that $Y^{1/q^n} \in (E_n^+)^{1 + p^j \mathcal{O}_{K-d-sh,dj} - fn,0}$. The lemma now follows from prop 1.2.16 and lemma 1.2.9. □

**Corollary 3.1.4.** — We have $E = E^{1 + p^j \mathcal{O}_{K-d-sh,dj}}$.

**Proof.** — This follows from prop 3.1.3 with $n = 0$, and prop 1.2.16. □

**Proposition 3.1.5.** — If $\varepsilon > 0$, then $k[[Y]]^{1 + p^j \mathcal{O}_{K-d-sh,dj+\varepsilon}} \subset k[[Y^p]]$.

**Proof.** — Take $f(Y) \in k[Y]$. There is a power series $h(T, U) \in k[[T, U]]$ such that

$$f(T + U) = f(T) + U \cdot f'(T) + U^2 \cdot h(T, U).$$

If $m \geq 0$, lemma 3.1.1 implies that $[1 + \pi^m](Y) = Y + Y^{q^m} + O(Y^{q^m+1})$. Therefore,

$$f([1 + \pi^m](Y)) = f(Y) + (Y^{q^m} + O(Y^{q^m+1})) \cdot f'(Y) + O(Y^{2q^m}).$$

If $f(Y) \notin k[[Y^p]]$, then $f'(Y) \neq 0$. Let $\mu = \text{val}_Y(f'(Y))$. The above computations imply that $\text{val}_Y(f([1 + \pi^m + \varepsilon](Y)) - f(Y)) = p^d \cdot p^{dj} + \mu$ for $i \geq 0$.

This implies the claim, since $\pi^\varepsilon \mathcal{O}_K = p \mathcal{O}_K$. □

**Corollary 3.1.6.** — We have $E_n^{1 + p^j \mathcal{O}_{K-d-sh,dj} - fn} = E_n$. ■
Proof. — We prove that, more generally, \( 
abla_1^{+1 + p^iO_{K^{dsh, dj} - \ell}} = k((Y^{1/p^f})) \). Take \( f(Y^{1/p^m}) \in (\nabla_1^{+1 + p^iO_{K^{dsh, dj} - \ell}} \cap \nabla_1^{+1 + p^iO_{K^{dsh, dj} - \ell}}) \). Since \( \text{valy}(h^p) = p \cdot \text{valy}(h) \) for all \( h \in \nabla_1^+ \), we have \( f^{p^m}(Y) \in (\nabla_1^{+1 + p^iO_{K^{dsh, dj} - \ell + m}} \cap \nabla_1^{+1 + p^iO_{K^{dsh, dj} - \ell + m}}) \). Let \( m \) denote the \( m \) - torsion points of \( \nabla_1^{+1 + p^iO_{K^{dsh, dj} - \ell + m}} \). This implies the claim.

3.2. Decompletion of \( \nabla_1^{+} \). — Since we use the results of §2.2, we once more assume that \( p \neq 2 \). Let \( \nabla_1^{+} \) denote the \( Y \)-adic completion of \( \nabla_1^{+} \).

Theorem 3.2.1. — We have \( \nabla_1^{+1 + p^iO_{K^{dsh, dj}}} = \nabla_1^{+} \), and \( \nabla_1^{+dsh} = \nabla_1^{+} \).

Proof. — Let \( K_\infty = K(LT[\pi_\infty]) \) denote the extension of \( K \) generated by the torsion points of \( LT \), and let \( \Gamma = \text{Gal}(K_\infty/K) \). The Lubin-Tate character \( \chi_\pi \) gives rise to an isomorphism \( \chi : \Gamma \to \nabla_1^{+} \). For \( n \geq 1 \), let \( K_n = K(LT[\pi^n]) \). If \( (\pi_n)_{n \geq 1} \) is a compatible sequence of primitive \( \pi^n \)-torsion points of \( LT \), then \( \pi_n \) is a uniformizer of \( \nabla_1^{+} \), \( \omega = (\pi_n)_{n \geq 0} \) belongs to \( \lim_{\to \infty} N_{K_n/K_{n-1}} \nabla_1^{+} \), and \( X_K(K_\infty) = k((\omega)) \) by theorem 2.1.1.

If \( g \in \Gamma \), then \( g(\omega) = [\chi_\pi(g)](\omega) \), so that if we identify \( \Gamma \) and \( \nabla_1^{+} \), then \( X_K(K_\infty) = \nabla_1^{+} \) with its action of \( \nabla_1^{+} \). Prop 2.1.3 implies that \( \nabla_1^{+} = \nabla_1^{+} \) as valued fields with an action of \( \Gamma \). We can therefore apply theorem 2.2.3 and get \( \nabla_1^{+dsh} = \nabla_1^{+} \).

This implies the second statement. The first one then follows from coro 3.1.6. \( \square \)

Remark 3.2.2. — In the above proof, note that \( K_\infty^{+1 + p^iO_K} = K_{ne} \), so that the numbering is not the same as in §2.1.

Remark 3.2.3. — We can define Lubin-Tate \( \Gamma \)-modules over \( \nabla_1^{+} \) as in §3.2 of [BR22]. The results proved in that section carry over to the Lubin-Tate setting without difficulty.

In theorem 2.9 of [BR22], we proved theorem 3.2.1 above in the cyclotomic case, using Tate traces. There are no such Tate traces in the Lubin-Tate case if \( K \neq Q_p \). We now explain why this is so. More precisely, we prove that there is no \( \Gamma \)-equivariant \( k \)-linear projector \( \nabla_1^{+} \to \nabla_1^{+} \) if \( K \neq Q_p \). Choose a coordinate \( T \) on \( LT \) such that \( \log_{LT}(T) = \sum_{n \geq 0} T^n/\pi^n \), so that \( \log_{LT}'(T) \equiv 1 \mod \pi \). Let \( \partial = 1/\log_{LT}'(T) \cdot d/dT \) be the invariant derivative on \( LT \). Let \( \varphi = \varphi' \) where \( q = p^f \).

Lemma 3.2.4. — We have \( d_\gamma(Y)/dY \equiv \chi_\pi(\gamma) \) in \( \nabla_1^{+} \) for all \( \gamma \in \Gamma \).

Proof. — Since \( \log_{LT}' \equiv 1 \mod \pi \), we have \( \partial = d/dY \) in \( \nabla_1^{+} \). Applying \( \partial \circ \gamma = \chi_\pi(\gamma) \gamma \circ \partial \) to \( Y \), we get the claim. \( \square \)

Lemma 3.2.5. — If \( \gamma \in \Gamma \) is nontorsion, then \( \nabla_1^{+} = k \).
**Proposition 3.2.6.** — If \( K \neq \mathbb{Q}_p \), there is no \( \Gamma \)-equivariant map \( R : E \to E \) such that \( R(\varphi_q(f)) = f \) for all \( f \in E \).

**Proof.** — Suppose that such a map exists, and take \( \gamma \in \Gamma \) nontorsion and such that \( \chi_\pi(\gamma) \equiv 1 \pmod{\pi} \). We first show that if \( f \in E \) is such that \((1 - \gamma)f \in \varphi_q(E)\), then \( f \in \varphi_q(E) \). Write \( f = f_0 + \varphi_q(R(f)) \) where \( f_0 = f - \varphi_q(R(f)) \), so that \( R(f_0) = 0 \) and \((1 - \gamma)f_0 = \varphi_q(g) \in \varphi_q(E)\). Applying \( R \), we get \( 0 = (1 - \gamma)R(f_0) = g \). Hence \( g = 0 \) so that \((1 - \gamma)f_0 = 0 \). Since \( E^{\gamma = 1} = k \) by lemma 3.2.5, this implies \( f_0 \in k \), so that \( f \in \varphi_q(E) \).

However, lemma 3.2.4 and the fact that \( \chi_\pi(\gamma) \equiv 1 \pmod{\pi} \) imply that \( \gamma(Y) = Y + f_\gamma(Y^p) \) for some \( f_\gamma \in E \), so that \( \gamma(Y^{q/p}) = Y^{q/p} + \varphi_q(g_\gamma) \). Hence \((1 - \gamma)(Y^{q/p}) \in \varphi_q(E) \) even though \( Y^{q/p} \) does not belong to \( \varphi_q(E) \). Therefore, no such map \( R \) can exist.

**Corollary 3.2.7.** — If \( K \neq \mathbb{Q}_p \), there is no \( \Gamma \)-equivariant \( k \)-linear projector \( \varphi_q^{-1}(E) \to E \). A fortiori, there is no \( \Gamma \)-equivariant \( k \)-linear projector \( E \to E \).

**Proof.** — Given such a projector \( \Pi \), we could define \( R \) as in prop 3.2.6 by \( R = \Pi \circ \varphi_q^{-1} \).

### 3.3. The perfectoid commutant of \( \text{Aut}(LT) \).

In §3.1 of \([BR22]\), we computed the perfectoid commutant of \( \text{Aut}(G_m) \). We now use theorem 3.2.1 to do the same for \( \text{Aut}(LT) \). We still assume that \( p \neq 2 \).

**Theorem 3.3.1.** — If \( u \in \hat{E}^+ \) is such that \( \text{val}_Y(u) > 0 \) and \( u \circ [g] = [g] \circ u \) for all \( g \in \mathcal{O}_K^\times \), there exists \( b \in \mathcal{O}_K^\times \) and \( n \in \mathbb{Z} \) such that \( u(Y) = [b](Y^n) \).

Recall that a power series \( f(Y) \in k[[Y]] \) is separable if \( f'(Y) \neq 0 \). If \( f(Y) \in Y \cdot k[[Y]] \), we say that \( f \) is invertible if \( f'(0) \in k^\times \), which is equivalent to \( f \) being invertible for composition (denoted by \( \circ \)). We say that \( w(Y) \in Y \cdot k[[Y]] \) is nontorsion if \( w^{(n)}(Y) \neq Y \) for all \( n \geq 1 \). If \( w(Y) = \sum_{i \geq 0} w_i Y^i \in k[[Y]] \) and \( m \in \mathbb{Z} \), let \( w^{(m)}(Y) = \sum_{i \geq 0} w_i^{(m)} Y^i \). Note that \( (w \circ v)^{(m)} = w^{(m)} \circ v^{(m)} \).

**Proposition 3.3.2.** — Let \( w(Y) \in Y + Y^2 \cdot k[[Y]] \) be a nontorsion series, and let \( f(Y) \in Y \cdot k[[Y]] \) be a separable power series. If \( w^{(m)} \circ f = f \circ w \) for some \( m \in \mathbb{Z} \), then \( f \) is invertible.

**Proof.** — This is a slight generalization of lemma 6.2 of \([Lub94]\). Write

\[
  f(Y) = f_n Y^n + O(Y^{n+1})
\]

\[
  f'(Y) = g_j Y^j + O(Y^{j+1})
\]

\[
  w(Y) = Y + w_r Y^r + O(Y^{r+1}),
\]

where \( f_n \neq 0 \) and \( g_j \neq 0 \). Since \( w(Y) \in Y \cdot k[[Y]] \), we get

\[
  f_n w_r Y^{n+r} = f_n Y^n + O(Y^{n+1}) + g_j Y^j + O(Y^{j+1}) + w_r Y^r + O(Y^{r+1}) = f_n Y^n + O(Y^{n+1})
\]

Hence \( w_r Y^{n+r} = f_n Y^n \). This implies that \( w_r = f_n Y^{n-r} \), which is separable since \( f(Y) \) is.

Therefore, \( f \) is invertible.
with \( f_n, g_j, w_r \neq 0 \). Since \( w \) is nontorsion, we can replace \( w \) by \( w^{op} \) for \( \ell \gg 0 \) and assume that \( r \geq j + 1 \). We have
\[
        w^{(m)} \circ f = f(Y) + w^{(m)} f(Y)^r + O(Y^{nr+1}) \\
        = f(Y) + w^{(m)} f_n Y^{nr} + O(Y^{nr+1}).
\]
If \( j = 0 \), then \( n = 1 \) and we are done, so assume that \( j \geq 1 \). We have
\[
        f \circ w = f(Y + w r Y^r + O(Y^{r+1})) \\
        = f(Y) + w r f(Y) + O(Y^{2r}) \\
        = f(Y) + w r g[Y] f(Y + j + O(Y^{r+j+1}).
\]
This implies that \( nr = r + j \), hence \( (n - 1)r = j \), which is impossible if \( r > j \) unless \( n = 1 \). Hence \( n = 1 \) and \( f \) is invertible.

\[ \square \]

**Lemma 3.3.3.** — If \( u \in \mathbb{E}^+ \) is such that \( \text{val}_X(u) > 0 \) and \( u \circ [g] = [g] \circ u \) for all \( g \in \mathcal{O}_K^x \), then \( u \in (\mathbb{E}^+)^{\text{sh}} \).

**Proof.** — The group \( \mathcal{O}_K^x \) acts on \( \mathbb{E}^+ \) by \( g \cdot u = u \circ [g] \). By lemmas 3.1.1 and 3.1.2, the function \( g \mapsto [g] \circ u \) is in \( H^X_1(1 + p \mathcal{O}_K, \mathcal{E}^+) \), where \( p^X = \text{val}_y(u) \).

**Proof of theorem 3.3.1.** — Take \( u \in \mathbb{E} \) such that \( \text{val}_y(u) > 0 \) and \( u \circ [g] = [g] \circ u \) for all \( g \in \mathcal{O}_K^x \). By lemma 3.3.3 and theorem 3.2.1, there is an \( m \in \mathbb{Z} \) such that \( f(Y) = u(Y)^m \) belongs to \( Y \cdot k[Y] \) and is separable. Take \( g \in 1 + \pi \mathcal{O}_K \) such that \( g \) is nontorsion, and let \( w(Y) = [g](Y) \) so that \( u \circ w = w \circ u \). We have \( f \circ w = w^{(m)} \circ f \). By prop 3.3.2, \( f \) is invertible. In addition, \( f \circ w = w^{(m)} \circ f \) if \( w(Y) = [g](Y) \) for all \( g \in \mathcal{O}_K^x \). Hence \( f_0 \cdot \mathcal{G} = \mathcal{G}^{p^m} \cdot f_0 \), so that \( \mathcal{G}^{p^m} = a \) for all \( a = \mathcal{G} \in k \). This implies that \( F_q \subset F_{p^m} \), so that \( m = fn \) for some \( n \in \mathbb{Z} \). Hence \( w^{(m)} = w \), and \( f \circ [g] = [g] \circ f \) for all \( g \in \mathcal{O}_K^x \). Theorem 6 of [LS07] implies that \( f \in \text{Aut}(LT) \). Hence there exists \( b \in \mathcal{O}_K^x \) such that \( u(Y) = [b](Y^{q^n}) \).

\[ \square \]

4. **Mahler expansions and super-Hölder functions**

In §1.3 of [BR22], we proved an analogue of Mahler’s theorem for continuous functions \( \mathbb{Z}_p \to M \), and then gave a characterization of super-Hölder functions in terms of their Mahler expansions. We now indicate how these results generalize to functions \( G \to M \) for a uniform pro-\( p \) group \( G \). Given the definition of super-Hölder functions and the existence of a coordinate \( c : G \to \mathbb{Z}_p^d \) as in prop 1.1.1, it is enough to study functions \( \mathbb{Z}_p^d \to M \). We generalize the setting a little bit, and study functions \( \mathcal{O}_K^d \to M \) where \( K \)
is a finite extension of \( \mathbb{Q}_p \). Let \( K \) be such a field, fix a uniformizer \( \pi \) of \( \mathcal{O}_K \) and let \( k \) be the residue field of \( K \). Let \( q = \mathrm{Card}(k) \).

### 4.1. Good bases and wavelets.

Let \( X = \mathcal{O}_K^d \), which we endow with the valuation \( \mathrm{val}_X(x_1, \ldots, x_d) = \min_i \mathrm{val}_x(x_i) \). For \( n \geq 0 \), let \( X_n = \pi^n X = \{ x \in X, \mathrm{val}_X(x) \geq n \} \).

We endow \( X \) with the \( \mathrm{val}_X \)-adic topology. For any set \( Y \), we denote by \( \mathrm{LC}(X, Y) \) the set of locally constant functions \( X \to Y \). For \( n \geq 0 \) we denote by \( \mathrm{LC}_n(X, Y) \) the subset of elements of \( \mathrm{LC}(X, Y) \) that factor through \( X/X_n \). Let \( I = \bigcup_{n \geq 0} I_n \) be a set of indices, where \( I_n \subset I_{n+1} \) for all \( n \geq 0 \), and \( \mathrm{Card}(I_n) = \mathrm{Card}(X/X_n) = q^n d \). Let \( E \) be a field of characteristic \( p \).

**Definition 4.1.1.** — A family \( \{ h_i \}_{i \in I} \) is a good basis of \( \mathrm{LC}(X, E) \) if it is a basis of the \( E \)-vector space \( \mathrm{LC}(X, E) \) such that for all \( n \geq 0 \), \( \{ h_i \}_{i \in I_n} \) is a basis of \( \mathrm{LC}_n(X, E) \).

Let \( M \) be (as usual) an \( E \)-vector space with a valuation \( \mathrm{val}_M \), such that \( \mathrm{val}_M(ax) = \mathrm{val}_M(x) \) for all \( a \in E^\times \) and \( x \in M \). We assume that \( M \) is separated and complete for the \( \mathrm{val}_M \)-adic topology.

**Proposition 4.1.2.** — Every \( f \in \mathrm{LC}_n(X, M) \) can be written uniquely as \( \sum_{i \in I_n} h_i \cdot m_i \) for some elements \( m_i \in M \). Moreover, \( \inf_{x \in X} \mathrm{val}_M(f(x)) = \inf_{i \in I_n} \mathrm{val}_M(m_i) \).

**Proof.** — Let \( \{ h_i \}_{i \in I_n} \) be the basis of \( \mathrm{LC}_n(X, E) \) defined as follows: \( \delta_x \) is the characteristic function of \( x + X_n \). Then \( f \in \mathrm{LC}_n(X, M) \) is equal to \( \sum_{x \in X/X_n} \delta_x \cdot f(x) \).

As \( \{ h_i \}_{i \in I_n} \) is also a basis of \( \mathrm{LC}_n(X, E) \), we can write \( \delta_x = \sum_{i \in I_n} a_{i,x} h_i \) for some elements \( a_{i,x} \in E \). We now have \( f = \sum_{i \in I_n} h_i \cdot m_i \) where \( m_i = \sum_{x \in X/X_n} a_{i,x} f(x) \). This formula implies that \( \inf_{i \in I_n} \mathrm{val}_M(m_i) \geq \inf_{x \in X} \mathrm{val}_M(f(x)) \).

On the other hand we can also write \( h_i = \sum_{x \in X/X_n} b_{x,i} \delta_x \) for some elements \( b_{x,i} \in E \), so that \( f(x) = \sum_{i \in I_n} b_{x,i} m_i \). This implies that \( \inf_{i \in I_n} \mathrm{val}_M(m_i) \leq \inf_{x \in X} \mathrm{val}_M(f(x)) \). \( \square \)

We now give an example of a particularly nice good basis of \( \mathrm{LC}(X, E) \), the basis of wavelets (see §1.3 of [Col10] and §2.1 of [dS16]). Let \( \mathcal{T} \) be a set of representatives of \( X/X_1 \) in \( X \), chosen so that the representative of 0 is 0. For each \( n \geq 0 \), let \( \mathcal{R}_n \) be the set of representatives of \( X/X_n \) defined as follows: \( \mathcal{R}_0 = \{ 0 \} \), and for \( n \geq 1 \), \( \mathcal{R}_n = \{ \sum_{i=0}^{n-1} \pi^i x_i, \ x_i \in \mathcal{T} \text{ for all } i \} \). We have \( \mathcal{R}_1 = \mathcal{T} \), and \( \mathcal{R}_n \subset \mathcal{R}_{n+1} \) for all \( n \). Let \( \mathcal{R} = \bigcup_{n \geq 0} \mathcal{R}_n \). If \( r \in \mathcal{R} \) let \( \ell(r) \) be the smallest \( n \) such that \( r \in \mathcal{R}_n \). For \( r \in \mathcal{R} \), let \( \chi_r \) be the characteristic function of the closed disc \( r + X_{\ell(r)} = \{ x \in X, \mathrm{val}_X(x-r) \geq \ell(r) \} \).

**Proposition 4.1.3.** — The set \( \{ \chi_r \}_{r \in \mathcal{R}} \) is a good basis of \( \mathrm{LC}(X, E) \).
Proof. — We prove that for all \( n \geq 0 \), the set \( \{ \chi_r \}_{r \in \mathcal{R}_n} \) is a basis of \( \text{LC}_n(X, E) \). Consider the basis \( \{ \delta_r \}_{r \in \mathcal{R}_n} \) of \( \text{LC}_n(X, E) \), where \( \delta_r \) is the characteristic function of \( r + X_n \). We have
\[
\chi_r = \sum_{r' \in \mathcal{R}_{n-l(r)}} \delta_{r+\pi_l(r)r'}.
\]
This implies that if we write \( \mathcal{R}_n = (\mathcal{R}_n \setminus \mathcal{R}_{n-1}) \cup \ldots \cup (\mathcal{R}_1 \setminus \mathcal{R}_0) \cup \mathcal{R}_0 \) and we express the family \( \{ \chi_r \}_{r \in \mathcal{R}_n} \) in terms of the basis \( \{ \delta_r \}_{r \in \mathcal{R}_n} \), we get a unipotent matrix. This shows that \( \{ \chi_r \}_{r \in \mathcal{R}_n} \) is also a basis of \( \text{LC}_n(X, E) \). \( \square \)

4.2. Expansions of continuous functions. — We show that every continuous function \( X \to M \) has a convergent expansion along a good basis of \( X \), and prove some continuity estimates in terms of the coefficients of the expansion. If \( \{ m_i \}_{i \in I} \) is a family of \( M \), we say that \( m_i \to 0 \) if \( \inf_{i \in I} \text{val}_M(m_i) \to +\infty \) as \( n \to +\infty \).

Theorem 4.2.1. — Let \( \{ h_i \}_{i \in I} \) be a good basis of \( \text{LC}(X, E) \).

If \( \{ m_i \}_{i \in I} \) is a family of \( M \) such that \( m_i \to 0 \), the function \( f : X \to M \) given by \( f = \sum_{i \in I} h_i \cdot m_i \) belongs to \( C^0(X, M) \), and \( \inf_{x \in X} \text{val}_M(f(x)) = \inf_{i \in I} \text{val}_M(m_i) \).

Conversely, if \( f \in C^0(X, M) \), there exists a unique family \( \{ m_i(f) \}_{i \in I} \) of elements of \( M \) such that \( m_i(f) \to 0 \) and such that \( f = \sum_{i \in I} h_i \cdot m_i(f) \).

Proof. — Let \( \{ m_i \}_{i \in I} \) be a family of \( M \) such that \( m_i \to 0 \). If \( f_n = \sum_{i \in I} h_i \cdot m_i \), then \( f_n \in C^0(X, M) \), and \( f \) is the uniform limit of the \( f_n \). We have \( \inf_{x \in X} \text{val}_M(f_n(x)) = \inf_{i \in I} \text{val}_M(m_i) \) by prop 4.1.2. Since \( m_i \to 0 \), we have \( \inf_{i \in I} \text{val}_M(m_i) = \inf_{i \in I} \text{val}_M(m_i) \) for \( n \gg 0 \). Hence \( \inf_{x \in X} \text{val}_M(f_n(x)) = \inf_{i \in I} \text{val}_M(m_i) \) for \( n \gg 0 \). Since \( \inf_{x \in X} \text{val}_M(f(x)) = \lim_n \inf_{x} \text{val}_M(f_n(x)) \), we have \( \inf_{x \in X} \text{val}_M(f(x)) = \inf_{i \in I} \text{val}_M(m_i) \).

We now prove the converse. Let \( M_n = \{ m \in M, \text{val}_M(m) \geq n \} \), let \( \pi_n : M \to M/M_n \) be the projection, and for each \( n \), fix a lift \( \psi_n : M/M_n \to M \). Take \( f \in C^0(X, M) \), and let \( f_n = \psi_n \circ \pi_n \circ f \). As \( f \) and \( f_n \) coincide modulo \( M_n \), \( f \) is the uniform limit of the \( f_n \). On the other hand, \( \pi_n \circ f \) is locally constant, and therefore so is \( f_n \). As \( X \) is compact, there exists some \( k(n) \geq 0 \) such that \( f_n \in \text{LC}_{k(n)}(X, M) \). By prop 4.1.2, we can write \( f_n = \sum_{i \in I} h_i \cdot m_{i,n} \), where \( m_{i,n} = 0 \) if \( i \notin I_{k(n)} \). We have \( \text{val}_M(m_{i,n} - m_{i,n'}) \geq \min(n, n') \) by construction, so that for each \( i \), the sequence \( \{ m_{i,n} \} \) converges to some \( m_i \in M \).

Moreover, if \( i \notin I_{k(n)} \), then \( \text{val}_M(m_i) \geq n \), so that \( m_i \to 0 \). The continuous function \( \sum_{i \in I} h_i \cdot m_i \) is the uniform limit of the \( f_n \), so that finally \( f = \sum_{i \in I} h_i \cdot m_i \). \( \square \)

Proposition 4.2.2. — Take \( f \in C^0(X, M) \) and \( t \in \mathbb{Z}_{\geq 0} \). If \( \{ h_i \}_{i \in I} \) is a good basis of \( \text{LC}(X, E) \), and we write \( f = \sum_i h_i \cdot m_i \) with \( m_i \to 0 \), then \( \inf_{i \in I} \text{val}_M(m_i) \) depends only on \( f \) and not on the choice of the good basis.
Proof. — Fix two good bases \( \{ h_i \}_{i \in I} \) and \( \{ h'_i \}_{i \in I} \) of \( \mathbb{LC}(X, E) \). There exists a family \( \{ \lambda_{i,j} \}_{(i,j) \in I \times I} \) of elements of \( E \) such that \( h_i = \sum_j \lambda_{i,j} h'_j \) for all \( i \). Moreover, if \( i \in I_t \) then \( \lambda_{i,j} = 0 \) for all \( j \not\in I_t \). Now write \( f = \sum_{i \in I} h_i \cdot m_i(f) = \sum_{i \in I} h'_i \cdot m'_i(f) \). We also have
\[
f = \sum_i \left( \sum_j \lambda_{i,j} h'_j \right) \cdot m_i(f) = \sum_j h'_j \cdot \left( \sum_i \lambda_{i,j} m_i(f) \right),
\]
so that \( m'_i(f) = \sum_i \lambda_{i,j} m_i(f) \). If \( j \not\in I_t \), then \( m'_i(f) = \sum_{i \in I_t} \lambda_{i,j} m_i(f) \), as \( \lambda_{i,j} = 0 \) if \( i \in I_t \) and \( j \not\in I_t \). This implies that \( \inf_{j \not\in I_t} \text{val}_M(m'_j(f)) \geq \inf_{i \in I_t} \text{val}_M(m_i(f)) \).

By symmetry, we get that \( \inf_{j \not\in I_t} \text{val}_M(m'_j(f)) = \inf_{i \not\in I_t} \text{val}_M(m_i(f)) \).

\( \square \)

**Theorem 4.2.3.** — Take \( f \in \mathbb{C}^0(X, M) \) and \( t \in \mathbb{Z}_{\geq 0} \).

If \( \{ h_i \}_{i \in I} \) is a good basis of \( \mathbb{LC}(X, E) \), and we write \( f = \sum_i h_i \cdot m_i \) with \( m_i \to 0 \), then
\[
\inf_{i \not\in I_t} \text{val}_M(m_i) = \inf_{x, y \in X} \text{val}_M(f(x) - f(y))
\]

**Proof.** — Let \( C_t(f) = \inf_{x, y \in X, \text{val}_X(x - y) \geq t} \text{val}_M(f(x) - f(y)) \) and \( B_t(f) = \inf_{i \not\in I_t} \text{val}_M(m_i) \).

If \( x \in X \) and \( z \in X_t \), then \( f(x + z) - f(x) = \sum_{i \in I} \left( h_i(x + z) - h_i(z) \right) \cdot m_i(f) \). As \( h_i \in \mathbb{LC}_t(X, E) \) for \( i \in I_t \), the above equality gives us
\[
f(x + z) - f(x) = \sum_{i \not\in I_t} \left( h_i(x + z) - h_i(z) \right) \cdot m_i(f).
\]

This implies that \( C_t(f) \geq B_t(f) \).

We now prove the converse inequality. By prop \( 4.2.2 \), \( B_t(f) \) is independent of the choice of a good basis, and we choose the wavelet basis of prop \( 4.1.3 \). Write \( f = \sum_{r \in \mathcal{R}} \chi_r \cdot m_r(f) \), so that we want to show that \( \inf_{s \in \mathcal{R}} \text{val}_M(m_s(f)) \geq C_t(f) \) for all \( s \not\in \mathcal{R}_t \). If \( x \in X \), define \( g_x : X \to M \) by \( g_x(z) = f(x + \pi^t z) - f(x) \), and write \( g_x = \sum_{r \in \mathcal{R}} \chi_r \cdot m_r(g_x) \). For each \( r \in \mathcal{R}_t \), we can write uniquely \( r = r_t + \pi^t s \) with \( r_t \in \mathcal{R}_t \), where \( s = 0 \) if \( r \in \mathcal{R}_t \), and \( s \neq 0 \in \mathcal{R}_{(r) - t} \) if \( r \not\in \mathcal{R}_t \). For \( x \in \mathcal{R}_t \) and \( r \not\in \mathcal{R}_t \), the map \( z \mapsto \chi_r(x + \pi^t z) - \chi_r(x) \) is the zero function if \( r_t \neq x \), and is \( \chi_s \) if \( r_t = x \). This implies that if \( x \in \mathcal{R}_t \), then
\[
g_x(z) = \sum_{r \in \mathcal{R}_t} \left( \chi_r(x + \pi^t z) - \chi_r(x) \right) \cdot m_r(f) = \sum_{r \not\in \mathcal{R}_t} \left( \chi_r(x + \pi^t z) - \chi_r(x) \right) \cdot m_r(f) = \sum_{s \not\in \mathcal{R}_0} \chi_s(z) \cdot m_{x + \pi^t s}(f).
\]
Therefore if \( x \in \mathcal{R}_t \), then \( m_0(g_x) = 0 \) and \( m_s(g_x) = m_{x + \pi^t s}(f) \) if \( s \neq 0 \). We have \( \inf_{s \in \mathcal{R}} \text{val}_M(m_s(g_x)) = \inf_{z \in X} \text{val}_M(g_x(z)) \geq C_t(f) \), so that \( \text{val}_M(m_s(g_x)) \geq C_t(f) \) for all \( x \in X \) and \( s \in \mathcal{R} \). This implies that for all \( x \in \mathcal{R}_t \) and \( s \neq 0 \), \( \text{val}_M(m_{x + \pi^t s}(f)) \geq C_t(f) \). Hence for all \( r \not\in \mathcal{R}_t \), we have \( \text{val}_M(m_r(f)) \geq C_t(f) \). \( \square \)
4.3. Mahler bases. — We now construct some other examples of good bases. For $n \geq 0$, let $\text{Int}_n(\mathcal{O}_K)$ denote the set of polynomials $f(T) \in K[T]$ such that $\deg(P) \leq n$ and $f(\mathcal{O}_K) \subset \mathcal{O}_K$. Recall (see for instance §1.2 of [dS16]) that a Mahler basis for $\mathcal{O}_K$ is a sequence $\{h_n\}_{n \geq 0}$ with $h_n(T) \in K[T]$ of degree $n$, and such that $\{h_0, \ldots, h_n\}$ is a basis of the free $\mathcal{O}_K$-module $\text{Int}_n(\mathcal{O}_K)$ for all $n \geq 0$. For example, if $K = \mathbb{Q}_p$, we can take $h_n(T) = \binom{T}{n}$. Let $\{h_n\}_{n \geq 0}$ be a Mahler basis for $\mathcal{O}_K$. Each $h_n$ defines a function $\mathcal{O}_K \to \mathcal{O}_K$ and hence $\mathcal{O}_K \to k$. Let $I = \mathbb{Z}_{\geq 0}$ and let $I_n = \{0, \ldots, q^n - 1\}$ for $n \geq 0$.

**Proposition 4.3.1.** — If $\{h_n\}_{n \geq 0}$ is a Mahler basis for $\mathcal{O}_K$, then $\{h_i\}_{i \in I}$ is a good basis of $\text{LC}(\mathcal{O}_K; k)$.

**Proof.** — By theorem 1.2 of [dS16], $\{h_0, \ldots, h_{q^n - 1}\}$ is a basis of the $k$-vector space $\text{LC}_n(\mathcal{O}_K; k)$ for all $m \geq 0$. This implies the claim. \hfill \Box

We now specialize to $K = \mathbb{Q}_p$. Write $N$ for $\mathbb{Z}_{\geq 0}$ and $n$ for an element $(n_1, \ldots, n_d) \in \mathbb{N}^d$. For each $n \in \mathbb{N}^d$, we denote by $h_n$ the function $\mathbb{Z}_d^+ \to \mathbb{E}$ given by $(x_1, \ldots, x_d) \mapsto (x_{n_1}) \cdots (x_{n_d})$. For $m \in \mathbb{Z}_{\geq 0}$, let $I_m = \{n \in \mathbb{N}^d \mid \max(n_1, \ldots, n_d) \leq m - 1\}$.

**Proposition 4.3.2.** — The functions $\{h_n\}_{n \in \mathbb{N}^d}$ form a good basis of $\text{LC}(\mathbb{Z}_d^+, \mathbb{F}_p)$.

**Proof.** — The claim follows from prop4.3.1 for $K = \mathbb{Q}_p$, and lemma 4.3.3 below. \hfill \Box

**Lemma 4.3.3.** — If $X$ and $X'$ are as in §4.1 and $\{h_i\}_{i \in I}$ and $\{h'_j\}_{j \in J}$ are good bases of $\text{LC}(X, E)$ and $\text{LC}(X', E)$, then $\{h_i \otimes h'_j\}_{(i,j) \in I \times J}$ is a good basis of $\text{LC}(X \times X', E)$, with $(I \times J)_n = I_n \times J_n$.

Let $G$ be a uniform pro-$p$ group, and let $c : G \to \mathbb{Z}_d^+$ be a coordinate as in prop4.3.1. The theorem below follows from prop4.3.2, theorem 4.2.1 and theorem 4.2.3.

**Theorem 4.3.4.** — If $\{m_n\}_{n \in \mathbb{N}^d}$ is a sequence of $M$ such that $m_n \to 0$, the function $f : G \to M$ given by $f(g) = \sum_{n \in \mathbb{N}^d} \left( c_{n_1}(g) \cdots c_{n_d}(g) \right) m_n$ belongs to $C^0(G, M)$. We have $\inf_{g \in G} \text{val}_M(f(g)) = \inf_{n \in \mathbb{N}^d} \text{val}_M(m_n)$.

Conversely, if $f \in C^0(G, M)$, there exists a unique sequence $\{m_n(f)\}_{n \in \mathbb{N}^d}$ such that $m_n(f) \to 0$ and such that $f(g) = \sum_{n \in \mathbb{N}^d} \left( c_{n_1}(g) \cdots c_{n_d}(g) \right) m_n(f)$.

We have $f \in \mathcal{H}_c^{\alpha, \mu}(G, M)$ if and only if for all $i \geq 0$, we have $\text{val}_M(m_n(f)) \geq p^\alpha \cdot p^{d_1} + \mu$ whenever $\max(n_1, \ldots, n_d) \geq p^i$.

**Remark 4.3.5.** — The first two assertions in the above theorem also follow from theorem 1.2.4 in §III of [Laz65] (we thank Konstantin Ardakov for pointing this out).
We finish by considering the case $G = \mathcal{O}_K$ for $K$ a finite extension of $\mathbb{Q}_p$, and working with a Mahler basis for $\mathcal{O}_K$. Let $K$ be a finite extension of $\mathbb{Q}_p$ as before. Assume that $E$ is an extension of $k$. Let $\{h_n\}_{n \geq 0}$ be a Mahler basis for $\mathcal{O}_K$. If $f \in C^0(\mathcal{O}_K, M)$, write $f = \sum_{n \geq 0} h_n m_n(f)$ with $m_n(f) \to 0$. Let $e$ denote the ramification index of $K$.

**Proposition 4.3.6.** — If $f = \sum_{n \geq 0} h_n m_n(f)$ as above, then $f \in H^\lambda_{te}(\mathcal{O}_K, M)$ if and only if $\text{val}_M(m_n(f)) \geq p^\lambda \cdot p^\mu i + \mu$ whenever $n \geq p^\mu i$.

**Proof.** — This follows from theorem 4.2.3, since $\text{val}_\pi(x - y) \geq i$ if and only if $\text{val}_{\pi}(x - y) \geq e_i$, and since $q^i = p^d$.

In this situation we can also define a slightly different version of super-Hölder functions. We say that a function $f : \mathcal{O}_K \to M$ is in $H^\lambda_{K,t}(\mathcal{O}_K, M)$ if $\text{val}_M(f(x) - f(y)) \geq p^\lambda \cdot p^\mu i + \mu$ whenever $\text{val}_\pi(x - y) \geq i$. We then have

$$H^\lambda_{te}(\mathcal{O}_K, M) \subset H^\lambda_{K,t}(\mathcal{O}_K, M) \subset H^\lambda_{te}(\mathcal{O}_K, M).$$

In particular, $H_{K,t}(\mathcal{O}_K, M) = H_{te}(\mathcal{O}_K, M)$. If $K/\mathbb{Q}_p$ is unramified then $H^\lambda_{K,t}(\mathcal{O}_K, M) = H^\lambda_{te}(\mathcal{O}_K, M)$. Moreover we have the following criterion:

**Proposition 4.3.7.** — If $f = \sum_{n \geq 0} h_n m_n(f)$ as above, then $f \in H^\lambda_{K,t}(\mathcal{O}_K, M)$ if and only if $\text{val}_M(m_n(f)) \geq p^\lambda \cdot p^\mu i + \mu$ whenever $n \geq p^\mu i$.

**Example 4.3.8.** — For all $n \geq 0$, there exists $c_n(T) \in \text{Int}_n(\mathcal{O}_K)$ such that $[a](Y) = \sum_{n \geq 0} c_n(a) Y^n$. This implies that $\text{val}_Y(m_n(a \mapsto [a](Y))) \geq n$, so that the function $a \mapsto [a](Y)$ is in $H^0_d(\mathcal{O}_K, E[Y])$, and in $H^0_{K,J}(\mathcal{O}_K, E[Y])$ where $q = p^d$.

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