A DISCONTINUOUS CAPACITY

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Abstract. We introduce the spherical capacity and show that it is not continuous on a smoothly bounded smooth family of open sets in dimension four.

1. INTRODUCTION

In [2] K. Cieliebak, H. Hofer, J. Latschev, and F. Schlenk asked the following question:

Question ([2, Problem 7]). Are capacities continuous on all smooth families of domains bounded by smooth hypersurfaces? Here a family of domains is called smooth if their boundaries fit in a smooth isotopy of embeddings.

The answer is no in dimension 4. Below we define a capacity, which we call the spherical capacity and prove the following theorem.

Theorem 1.1. There is a smooth family $U_\varepsilon$, $\varepsilon \in (0,1)$, of ellipsoidal shells in $\mathbb{R}^4$ such that for the spherical capacity $s$ the function

$$\varepsilon \mapsto s(U_\varepsilon)$$

is not continuous.

The proof involves a non-embedding result for ellipsoids by F. Schlenk [11] and the symplectic 4-ball theorem by M. Gromov [6]. We begin with the definition of the spherical capacity:

Definition 1.2. For symplectic manifolds $(V, \omega)$ of dimension $\geq 4$ we call

$$s(V, \omega) := \sup\{\pi r^2 > 0 \mid \exists \text{ symplectic embedding } S^{2n-1}_r \hookrightarrow (V, \omega)\}$$

the spherical capacity. By a symplectic embedding of the sphere $S^{2n-1}_r$ we mean a symplectic embedding of a neighbourhood of $S^{2n-1}_r \subset \mathbb{R}^{2n}$.

In [6] Gromov proved the non-squeezing theorem which says that the open ball $B_r = B^{2n}_r$ of radius $r > 0$ embeds symplectically into the symplectic cylinder $Z_R = B^2_R \times \mathbb{R}^{2n-2}$ if and only if $r \leq R$. It is natural to ask whether a similar result holds for symplectic embeddings of the sphere $S^{2n-1}_r = \partial B_r$ into $Z_R$, i.e. embeddings of neighbourhoods of $S^{2n-1}_r \subset \mathbb{R}^{2n}$. If the dimension $2n \geq 4$ a positive answer was given in [12, 5], where it was shown, that such an embedding exists precisely if $r < R$. As Gromov’s non-squeezing leads to a symplectic capacity

$$w(V, \omega) = \sup\{\pi r^2 > 0 \mid \exists \text{ symplectic embedding } B_r \hookrightarrow (V, \omega)\},$$

the Gromov width, the spherical non-squeezing theorem from [12, 5] is related to the spherical capacity. That this is a normalized symplectic capacity in dimension $\geq 4$ follows from [12, 5]. Recall that this means the following.
A normalized symplectic capacity is an assignment of a real number \( c(V, \omega) \in [0, \infty] \) to a symplectic manifold \((V, \omega)\) of fixed dimension satisfying the following conditions:

**Monotonicity:** If there exists a symplectic embedding \((V, \omega) \hookrightarrow (V', \omega')\), then \( c(V, \omega) \leq c(V', \omega') \).

**Conformality:** For any \( a > 0 \) we have \( c(V, a\omega) = a c(V, \omega) \).

**Normalization:** \( c(B_1) = \pi = c(Z_1) \).

Given a symplectic manifold \((V, \omega)\) an extrinsic capacity on subsets \( U \subset V \) is a real number \( c(U, \omega) \in [0, \infty] \) satisfying the above conditions with monotonicity replaced by:

**Relative monotonicity:** If there exists a symplectomorphism of \((V, \omega)\) which maps \( U_1 \) into \( U_2 \), then \( c(U_1, \omega) \leq c(U_2, \omega) \).

We suppress the standard symplectic structure \( dx \wedge dy \) on \( \mathbb{R}^{2n} \) in the notation.

## 2. Motivation

The spherical capacity is a variant of the regular coisotropic capacity of hypersurfaces introduced in \([12]\). Consider a hypersurface \( M \) in a symplectic manifold \((V, \omega)\) such that the characteristics are all closed, form a smooth fibration over the leaf space with fibre \( S^1 \), and are contractible in \( V \). Let \( \inf(M) \) denote the least positive symplectic area a smooth disc in \( V \) with boundary on a closed characteristic of \( M \) can have and set

\[
a(V, \omega) := \sup \{ \inf(M) \mid M \subset (V, \omega) \},
\]

where the supremum runs over all hypersurfaces as described. This defines a normalized capacity on all symplectically aspherical symplectic manifolds, see \([12]\), the regular coisotropic hypersurface capacity. The restriction to spheres is denoted by \( a_S \) and we get

\[
w \leq s \leq a_S \leq a.
\]

The contact type embedding capacity

\[
c(V, \omega) := \sup \{ \inf(\alpha) \mid \exists \text{ contact type embedding } (M, \alpha) \hookrightarrow (V, \omega) \},
\]

see \([3, 5]\), yields a second approach to the spherical capacity. Here the supremum is taken over all closed contact manifolds \((M, \alpha)\) of dimension \((2n - 1)\), where \( \inf(\alpha) \) is the infimum of all positive periods of closed Reeb orbits w.r.t. the contact from \( \alpha \). By a contact type embedding \( j : (M, \alpha) \hookrightarrow (V, \omega) \) we mean that there is a Liouville vector field \( Y \) for \( \omega \) defined near \( j(M) \) such that \( j^*(i_Y \omega) = \alpha \). If one restricts in the definition of \( c \) to contact manifolds diffeomorphic to the \((2n - 1)\)-sphere, one obtains a normalized capacity \( c_S \) as well. These capacities yield a second proof of the spherical non-squeezing theorem and we have

\[
w \leq s \leq c_S \leq c.
\]

**Definition 2.1.** For symplectic manifolds \((V, \omega)\) of dimension \( \geq 4 \) we call

\[
e(V, \omega) := \sup \{ \pi r_1^2 > 0 \mid \exists \text{ symplectic embedding } \partial E \hookrightarrow (V, \omega) \}.
\]

the ellipsoidal capacity. By a symplectic embedding of the boundary of \( E \) we mean a symplectic embedding of a neighbourhood of

\[
\partial E := \left\{ \frac{x_1^2 + y_1^2}{r_1^2} + \ldots + \frac{x_n^2 + y_n^2}{r_n^2} = 1 \right\} \subset \mathbb{R}^{2n}
\]
with positive symplectic half axes $r_1 \leq \ldots \leq r_n$.

Notice that

$$s \leq e \leq c_S.$$  

The question which now appears is whether the two capacities $s$ and $e$ coincide.

**Theorem 2.2.** The boundary of each 4-dimensional ellipsoid with different symplectic main axes has a tubular neighbourhood $U$, such that $s(U) < e(U)$.

**Remark 2.3.** Both quantities $s$ and $c_S$ do not define capacities in dimension 2. Because they satisfy the monotonicity axiom, they would otherwise measure the area of the annuli $B_1^+ \setminus B_{1-\varepsilon}$ in $\mathbb{R}^2$, see [7]. Alternatively, for the first observe that $(r, \theta) \mapsto (\sqrt{r^2 + a}, \theta)$ maps $S^1 = \partial B$ symplectically to the circle of radius $\sqrt{1 + a}$ for all $a \in (-1, \infty)$. For the second, consider the contact form $\frac{1}{2}(r^2 + a)d\theta$ on $S^1$. Its smallest action equals $(1 + a)\pi$.

In contrary, if one measures the largest minimal action an embedding of restricted contact type (with image in a certain open subset) has, this results in an extrinsic normalized capacities also in dimension 2, cf. [4]. In this case the monotonicity axiom is only valid in the weaker sense requiring all symplectomorphisms defined on the ambient space, cf. [9, p. 375].

### 3. The Boundary Gromov width

For open subsets $U$ of a symplectic manifold $(V, \omega)$ there is version of the Gromov width which interpolates between $w$ and $s$. Consider symplectic embeddings of the closed ball $B_r$ into $(V, \omega)$ which map the boundary sphere $\partial B_r$ into $U$. The **boundary Gromov width** $w_\partial(U)$ is then defined to be the supremum of $\pi r^2 > 0$ taken over all such embeddings and is a normalized extrinsic capacity.

As a first step in the proofs of Theorems 1.1 and 2.2 we estimate $w_\partial(U_\varepsilon)$ for open subsets $U_\varepsilon$ of $\mathbb{R}^{2n}$. For that we consider an ellipsoid

$$E := E(r_1, \ldots, r_n) = \left\{ \frac{x_1^2 + y_1^2}{r_1^2} + \ldots + \frac{x_n^2 + y_n^2}{r_n^2} < 1 \right\}$$

with positive symplectic half axes $r_1 \leq \ldots \leq r_n$. We define an ellipsoidal shell via

$$U_\varepsilon := (1 + \varepsilon)E \setminus (1 - \varepsilon)E$$

provided $\varepsilon > 0$ is sufficiently small.

**Lemma 3.1.** If two of the symplectic radii of $E$ are different we have

$$w_\partial(U_\varepsilon) \to 0$$

as $\varepsilon$ tends to 0.

**Proof.** We consider a symplectic embedding $\varphi$ of $B_r$ into $\mathbb{R}^{2n}$ such that $\varphi(S^{2n-1})$ is contained in $U_\varepsilon$. Then there are two cases which we need to consider: either $\varphi(B_r)$ is contained in $U_\varepsilon$ or not. The latter implies that the bounded component of the complement of $\varphi(S^{2n-1})$ contains $(1 - \varepsilon)\partial E$ and, hence, $(1 - \varepsilon)E \subset \varphi(B_r)$. We claim that for $\varepsilon > 0$ small enough the second case can not appear, so that necessarily $\varphi(B_r) \subset U_\varepsilon$. The lemma follows then by comparing the volume.

Assume now that $\varphi(B_r) \not\subset U_\varepsilon$ for some $\varepsilon > 0$. Then $(1 - \varepsilon)E \subset \varphi(B_r)$, as we remarked above. Again comparing the volume we get a lower bound

$$(1 - \varepsilon)^n r_1 \cdot \ldots \cdot r_n < r^n.$$
For an upper bound observe that \( \varphi(B_r) \subset (1 + \varepsilon)Z_{r_1} \). Invoking Gromov’s non-squeezing theorem we get
\[
r < (1 + \varepsilon)r_1.
\]
Combining both inequalities yields
\[
\left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^n < \frac{r^n}{r_1 \cdot \ldots \cdot r_n}.
\]
Because the \( r_j \) are not all the same the right hand side is < 1. Therefore, there exists a positive number \( \varepsilon_0 \), which only depends on the \( r_j \), such that \( \varepsilon > \varepsilon_0 \). The lemma follows now by taking \( \varepsilon \leq \varepsilon_0 \) which excludes the second case. \( \square \)

On the other hand:

**Lemma 3.2.** We have \( e(U_\varepsilon) \longrightarrow \pi r_1^2 \) and \( c(U_\varepsilon) \longrightarrow \pi r_1^2 \) as \( \varepsilon \) tends to 0.

**Proof.** Indeed,
\[
\pi r_1^2 = \inf(\lambda_{st} | \gamma \partial T) \leq e(U_\varepsilon) \leq e((1 + \varepsilon)E) = (1 + \varepsilon)^2 \pi r_1^2,
\]
where \( \lambda_{st} \) denotes the radial Liouville form \( \frac{1}{2}(x\,dy - y\,dx) \) on \( \mathbb{R}^{2n} \). For \( c \) the argument is the same. \( \square \)

### 4. DISCONTINUITY

A refinement of the proof of Lemma 3.1 shows that the boundary Gromov width \( w_\partial \) is discontinuous. We consider the ellipsoid \( E \) with radii \( r_1 = \ldots = r_{n-1} = 1 \) and \( r_n = R \) for a real number \( R \in (1, \sqrt{2}) \). The corresponding ellipsoidal shell is again denoted by \( U_\varepsilon \).

**Proposition 4.1.** For \( U_\varepsilon \subset \mathbb{R}^{2n}, \ n \geq 2 \), as described above the function
\[
\varepsilon \longmapsto w_\partial(U_\varepsilon)
\]
is not continuous.

**Proof.** We will show that the function jumps at
\[
\varepsilon_0 := \frac{R - 1}{R + 1}.
\]
For this we consider two cases.

We claim that for all \( \varepsilon \in (0, \varepsilon_0] \) we have \( w_\partial(U_\varepsilon) = w(U_\varepsilon) \). For this we need to exclude symplectic embeddings \( \varphi \) of \( B_r \) into \( \mathbb{R}^{2n} \), such that \( (1 - \varepsilon)E \subset \varphi(B_r) \), similarly to Lemma 3.1. We use a result of F. Schlenk [11, Theorem 1], which is based on the Ekeland-Hofer capacities [3]. By this result, since \( R \in (1, \sqrt{2}) \), there exists a symplectic embedding of \( (1 - \varepsilon)E \) into \( B_r \) only if \( (1 - \varepsilon)R \leq r \). Moreover, Gromov’s non-squeezing yields the inequality \( r < (1 + \varepsilon) \). Combining both we get
\[
\frac{1 - \varepsilon}{1 + \varepsilon} < \frac{1}{R},
\]
so that the symplectic embeddings under considerations can be excluded by our choice of \( \varepsilon_0 \). Hence \( w_\partial(U_\varepsilon) = w(U_\varepsilon) \). Because the Gromov width of \( U_\varepsilon \) is bounded in terms of its volume from above, we obtain
\[
w_\partial(U_\varepsilon) < \sqrt{(1 + \varepsilon)^{2n} - (1 - \varepsilon)^{2n}} \pi R^2.
\]
On the other hand, for \( \varepsilon > \varepsilon_0 \), where the spheres \( S^{2n-1} \subset \mathcal{U}_{\varepsilon_0} \) start to appear, we have the lower bound
\[
\frac{4\pi R^2}{(R+1)^2} \leq w_{\partial} (U_{\varepsilon}).
\]
Consequently, we get for all \( R \in (1, \sqrt{2}) \) and for all \( \varepsilon > \varepsilon_0 \) the following estimate:
\[
w_{\partial} (U_{\varepsilon_0}) < \sqrt{R^{2n-1} - \frac{4\pi R^2}{(R+1)^2}} < w_{\partial} (U_{\varepsilon}).
\]
In other words the function \( \varepsilon \mapsto w_{\partial} (U_{\varepsilon}) \) is not continuous at \( \varepsilon_0 \). \( \square \)

5. Proof of the Theorems

The second ingredient of the proofs of Theorems 1.1 and 2.2 are the following considerations:

Proposition 5.1. The capacities \( w \) and \( s \) coincide on closed minimal symplectic 4-manifolds.

Proof. As the Gromov width is the smallest capacity we get \( w \leq s \) on all symplectic manifolds. For the converse consider a symplectic embedding of \( S^3 \) into a closed minimal symplectic 4-manifold \((V, \omega)\). By [1] or [5, Proposition 4.10] its image \( S_r \) separates, so that \( S_r \) cuts out a strong symplectic filling \((W, \omega)\). Since \((W, \omega)\) is minimal it is symplectomorphic to the standard 4-ball \( B_4 \) by a theorem of Gromov [6, p. 311], c.f. [4, Remark 2.3] or [10, Theorem 9.4.2]. Hence, \( s \leq w \). \( \square \)

Notice that the minimality assumption is essential. The volume (and hence the Gromov width) of the symplectic blow up of \( \mathbb{C}P^2 \) obtained by cutting out a ball of radius \( 1 - \varepsilon \) in \( B_1 \subset \mathbb{C}P^2 \) can be made arbitrary small. In contrast, the spherical capacity stays \( \geq \pi \).

Proposition 5.2. For all open subsets \( U \) in \( \mathbb{R}^4 \) we have \( s(U) = w_{\partial} (U) \).

Proof. The argument is almost the one from Proposition 5.1. Just observe that any Liouville vector field defined near and determined by the symplectic embedding \( S_r \) of \( S^3 \) points out of the interior of \( S_r \). Otherwise, we could use a sufficiently large ball to cut out a connected symplectic manifold with convex boundary consisting of two standard spheres. This would violate [8, Theorem 1.2] or [5, Theorem 3.4]. \( \square \)

Question 5.3. The proceeding proposition is valid in greater generality, e.g. for all subcritical Stein surfaces, see [5, Theorem 3.4]. The critical case seems to be not known. Therefore, we ask:

Does the Liouville vector field defined by a closed hypersurface of contact type \((M, \alpha)\) in a critical Stein manifold of dimension \( \geq 4 \) point out of the interior of \( M \), if \((M, \alpha)\) is not of restricted contact type?

Proof of Theorem 2.2. By choosing \( \varepsilon > 0 \) small enough we can make \( e(U_{\varepsilon}) \) as close to \( \pi r^2 \) and \( w_{\partial} (U_{\varepsilon}) \) as small as we wish, see Lemmata 3.1 and 3.2. Moreover, by the proceeding proposition \( s(U_{\varepsilon}) \) and \( w_{\partial} (U_{\varepsilon}) \) are equal. \( \square \)

Proof of Theorem 1.1. By Proposition 5.2 \( w_{\partial} \) equals the spherical capacity on \( \mathbb{R}^4 \) - defines there an intrinsic capacity. The claim follows from Proposition 4.1. \( \square \)
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References

[1] P. Albers, B. Bramham, C. Wendl, On nonseparating contact hypersurfaces in symplectic 4-manifolds, *Algebr. Geom. Topol.*, 10 (2010), 697–737.

[2] K. Cieliebak, H. Hofer, J. Latschev, F. Schlenk, *Quantitative symplectic geometry*, In: Dynamics, ergodic theory, and geometry, Math. Sci. Res. Inst. Publ., 54, 1–44, Cambridge Univ. Press (2007).

[3] I. Ekeland, H. Hofer, Symplectic topology and Hamiltonian dynamics. II, *Math. Z.*, 203 (1990), 553–567.

[4] H. Geiges, K. Zehmisch, How to recognise a 4-ball when you see one, (2011), preprint, arXiv:1104.1549.

[5] H. Geiges, K. Zehmisch, Symplectic cobordisms and the strong Weinstein conjecture, *Math. Proc. Cambridge Philos. Soc.* (2012), to appear.

[6] M. Gromov, Pseudoholomorphic curves in symplectic manifolds, *Invent. Math.* 82 (1985), 307–347.

[7] H. Hofer, E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser Verlag, Basel, (1994).

[8] D. McDuff, Symplectic manifolds with contact type boundaries, *Invent. Math.* 103 (1991), 651–671.

[9] D. McDuff, D. Salamon, *Introduction to symplectic topology*, 2nd edition, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, (1998).

[10] D. McDuff, D. Salamon, *J-holomorphic Curves and Symplectic Topology*, Amer. Math. Soc. Colloq. Publ. 52, American Mathematical Society, Providence, RI (2004).

[11] F. Schlenk, Symplectic embeddings of ellipsoids, *Israel J. Math.*, 138 (2003), 215–252.

[12] J. Swoboda, F. Ziltener, Coisotropic displacement and small subsets of a symplectic manifold, *Math. Z.* 271 (2012), 415–445.

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