MORI DREAM SPACES EXTREMAL CONTRACTIONS OF K3 SURFACES

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Abstract. We will give a criterion to assure that an extremal contraction of a K3 surface which is not a Mori Dream Space produces a singular surface which is a Mori Dream Spaces. We list the possible Néron-Severi groups of K3 surfaces with this property such that the Picard number is greater than or equal to 10. We give a detailed description of two geometric examples for which the Picard number of the K3 surface is 3, i.e. the minimal possible in order to have the required property. Moreover we observe that there are infinitely many examples of K3 surfaces with the required property and Picard number equal to 3.

1. Introduction

The Mori Dream Spaces are projective varieties for which the Minimal Model Program can be applied successfully, i.e. the necessary flops and contractions exist and the program terminates. They were introduced in [HK] where it is also proved that the property of being a Mori Dream Space is essentially equivalent to the finite generation of the Cox ring of the variety.

In this paper we discuss the following situation: let us consider a variety which is not a Mori Dream Space and let us contract exactly one extremal ray of its effective cone. Is it possible to obtain in such a way a Mori Dream Space? The answer to this question is known to be yes, so the property "to be a Mori Dream Space" is yet not preserved by resolution of simple singularities. The examples which allows one to give a positive answer to this question are provided in [O, Theorem 5.2]. These examples are constructed as follows: let $S$ be a K3 surface which is a Mori Dream Space. Then the singular variety $V := (S \times S)/\mathbb{S}_2$, where $\mathbb{S}_2$ is the symmetric group, is known to be a Mori Dream Space. The variety $V$ is obtained from $H := \text{Hilb}^2(S)$ by contracting the exceptional divisor over the diagonal, so $V$ is an extremal contraction of $H$. In [O] Oguiso gives examples of K3 surfaces $S$ such that $S$ and $V$ are Mori Dream Spaces but $H$ is not. The proof of the fact that $S$ and $V$ are a Mori Dream Space and $H$ is not is based on the computation of the automorphism groups of these varieties. Indeed, it is possible to construct K3 surfaces $S$ such that $S$ and $V$ have a finite automorphism groups, but the automorphism group of $H$ is not finite.

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The relation among the finiteness of the automorphism group and the finite generation of the Cox ring (so the property of being a Mori Dream Space), is deep as shown for example by the following result, by Artebani, Hausen and Laface, [AHL], which will be be fundamental in this paper: a K3 surface is a Mori Dream Space if and only if its automorphism group is finite.

In view of the result by Oguiso on the varieties $H$ and $V$, it is quite natural to ask if there exist similar results in lower dimension, in particular for K3 surfaces. So the aim of this note is to positively answer to the following question:

**Question 1.1.** Are there K3 surfaces $X$ which are not Mori Dream Spaces and such that the surface $X'$ obtained by contracting exactly one extremal ray of $X$ is a Mori Dream Space?

We will answer to the question 1.1 both by providing an abstract algorithm which produces the Néron Severi groups of admissible examples and describing the geometry of some of them.

In Section 2 we prove that the singular surface $X'$ is a Mori Dream Space if and only if the K3 surface $Y$ whose Néron–Severi group is isometric to the one of $X'$ is a Mori Dream Space. In Section 3 we give a criterion to find the Néron–Severi group of the K3 surfaces $X$ such that there exists an extremal contraction producing a singular surface $X'$ which is a Mori Dream Space. Here we use the results of [AHL] on K3 surfaces which are Mori Dream Spaces and the results of [Kon] on K3 surfaces with finite automorphism group. The main result of the Section 3 is Theorem 3.4 where we prove that if $X$ is a K3 surface which is not a Mori Dream Space, but which admits an extremal contraction producing a singular surface which is a Mori Dream Space, then $\rho(X) \geq 3$ and $\rho(X) \neq 19$. Moreover, we classify all the admissible Néron–Severi groups of $X$ if $\rho(X) \geq 10$ (the reason of this bound is only computational, and is essentially due to the fact that in [Kon] the same bound is considered). Section 4 is the geometric heart of this paper. Here we give two examples of K3 surfaces $X$ which are not Mori Dream Spaces, but which admit an extremal contraction producing a singular surface which is a Mori Dream Space. In both the cases $\rho(X) = 3$ and the Néron–Severi groups of the surfaces obtained after the contraction are the same. We will show that the automorphism group of the K3 surface $X$ is infinite and we show that some automorphisms do not descend to automorphisms of the singular model. In Section 5 we provide other examples, giving some geometric details. In particular we will show that there exists an infinite number of K3 surfaces of Picard number 3 which are not Mori Dream Spaces but such that the singular surface obtained by one extremal contraction is a Mori Dream Spaces, see Proposition 5.3 for a more precise statement.

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2. A CRITERION TO CONCLUDE THAT THE CONTRACTION OF AN EXTREMAL RAY OF A K3 SURFACE PRODUCES A MORI DREAM SPACE

2.1. **The K3 surfaces $X$ and $Y$ and the singular surface $X'$**.

**Definition 2.1.** A K3 surface is a complex compact surface with trivial canonical bundle and trivial irregularity.
We denote by $\Lambda_{K3}$ the unique even unimodular lattice with signature $(3, 19)$ and we observe that if $S$ is a K3 surface, then $H^2(S, \mathbb{Z}) \cong \Lambda_{K3}$.

In the following for each lattice $L$ and each element $l \in L$ we denote by $l^\perp$ the sublattice of $L$ whose elements are orthogonal to $l$, i.e. $l^\perp := \{k \in L \text{ such that } kl = 0\}$.

Let $X$ be an algebraic K3 surface (i.e. a K3 surface with an ample line bundle). The Néron–Severi group of $X$ is a sublattice of $\Lambda_{K3}$ and it is hyperbolic (i.e. its signature is $(1, \rho(X) - 1)$). We assume that $\rho(X) := \text{rank}(NS(X)) \geq 2$. We denote by $N$ the abstract lattice such that $L \cong NS(X)$. Let us assume that there exists a smooth rational curve $N$ on $X$. We denote by $\Lambda$ also the class of the curve $N$ in $NS(X)$ and we recall that $N^2 = -2$. We denote by $M$ the sublattice of $NS(X) \cong L$ which is orthogonal to $N$, i.e. $M := N^\perp$. Since $L$ is a hyperbolic lattice and $\langle N \rangle$ is a negative definite lattice, the lattice $M$ is a hyperbolic lattice. Moreover, the lattice $M$ is primitively embedded in $L$, by definition. Since there exists a primitive embedding of $L$ in $\Lambda_{K3}$, there exists a primitive embedding of $M$ in $\Lambda_{K3}$. As a consequence of the surjectivity of the period map for K3 surfaces, there exists at least a K3 surface (and in fact infinitely many) whose Néron–Severi group is isometric to $M$. Let $Y$ be a K3 surface such that $NS(Y) \cong M$. We observe that $\rho(Y) = \rho(X) - 1$.

Let us denote by $B_M = \{B_1, \ldots, B_{\text{rank}(M)}\}$ a $\mathbb{Z}$-basis of the abstract lattice $M$. The set $B_L = \{B_1, \ldots, B_{\text{rank}(M)}, N\}$ is a $\mathbb{Q}$-basis of the lattice $L$, i.e. every element in $L$ is a linear combination with rational coefficients of $\{B_1, \ldots, B_{\text{rank}(M)}, N\}$. For certain specific lattice $L$ the set $B_L$ is also a $\mathbb{Z}$-basis for $L$, but this can not be guaranteed in a general context. However, every element in $L$ which has a trivial intersection with $N$ is a linear combination with integer coefficients of $\{B_1, \ldots, B_{\text{rank}(M)}\}$.

Let $\phi : X \to X'$ be the map which contracts the curve $N$ to a point. Then $X'$ is a singular variety with exactly one singular point, of type $A_1$, which is the point $\phi(N)$.

The surface $X'$ is a normal surface and its Néron–Severi group is isometric to $M$. So, the Néron–Severi group $NS(X')$ and the Néron–Severi group $NS(Y)$ are isometric, and both isometric to $M$.

By construction, we fixed a specific primitive embedding of $M$ in $NS(X)$, which is in fact a marking. Let us denote this embedding by $f_X : M \to NS(X)$. So $NS(X')$ and $NS(Y)$ are identified with $N^\perp$, and the marking $f_X$ induces the isometries $f_{X'} : M \to NS(X')$ by $f_Y : M \to NS(Y)$. For each vector $v \in M$ we denote by $D_X := f_X(v)$, $D_{X'} := f_{X'}(v)$ and $D_Y := f_Y(v)$. We say that the divisors $D_X$, $D_{X'}$ and $D_Y$ are associated if they are the images (for the map $f_X$, $f_{X'}$ and $f_Y$ respectively) of the same vector $v \in M$. Let $D_X$ be a divisor on $X$ such that $D_X = f_X(v)$ for a certain $v \in M$. Then $D_XN = 0$ and so the associated vectors are $D_{X'} = f_{X'}(f_X^{-1}(D_X))$ and $D_Y = f_Y(f_X^{-1}(D_X))$.

### 2.2. The Nef cones of $Y$ and $X'$

**Lemma 2.2.** Let $D_X$ be a divisor on $X$ such that $D_X$ is nef and $D_XN = 0$. The divisor $D_{X'} \in NS(X')$ associated to $D_X$ is nef.

**Proof.** We denote by $v_D$ the vector in $M$ such that $f_X(v_D) = D_X$, and we recall that $D_{X'} = f_{X'}(v_D)$.
Since $D_X$ is nef, for every curve $C_X \subset X$, $D_X C_X \geq 0$. Let us now consider a curve $C_{X'} \subset X'$. The class of the curve $C_{X'} \in NS(X')$ is represented by $f_{X'}(v_C)$ for a certain vector $v_C \in M$. The strict transform of $C_{X'}$ on $X$ is a curve, $C_X$, whose class is $f_X(v_C) + \eta N$. So we have

$$D_X C_{X'} = v_D v_C = v_D(v_C + \eta N) = D_X C_X \geq 0,$$

where we used that $f_X$ and $f_{X'}$ are isometries and the fact that $v_D N = 0$. Hence $D_{X'}$ has a non-negative intersection with all the curves in $X'$. We conclude that $D_{X'}$ is nef.

\textbf{Remark 2.3.} Let us consider an ample divisor $H_Y \in NS(Y)$. It exists since $Y$ is a projective K3 surface. Let $X$ be its associated divisor on $X$, i.e. $X := f_X(f_Y^{-1}(H_Y))$. Since $H_Y^2 = H^2 > 0$, $H_X$ is a class in the positive cone $C_X$. The positive cone $C_X$ is divided in chambers by the walls (spaces orthogonal to the classes of the $(-2)$-curves). Each of these chambers can be chosen to be the ample cone. We know that $H_X N = 0$, so $H_X$ is on a wall of some chambers. We can chose the ample cone to be one of the chambers such that $H_X$ is in the closure of the ample cone and so $H_X$ is a pseudoample class on $X$. So we now fix the ample cone of both $Y$ and $X$ and we can assume that there is a divisor $H_Y$ which is ample on $Y$ such that the associated vector $H_X$ is pseudoample on $X$.

\textbf{Lemma 2.4.} Let $D_X$ be a divisor on $X$ such that $D_X$ is nef and $D_X N = 0$. The divisor $D_Y \in NS(Y)$ associated to $D_X$ is nef.

\textbf{Proof.} We consider a divisor $D_X$ which is nef (with respect to the choice of the ample cone mentioned in Remark 2.3). Let us consider the associated divisor $D_Y \in NS(Y)$. It suffices to show that for every effective $(-2)$-class $R_Y \in NS(Y)$, $R_Y D_Y \geq 0$. Let $R_Y$ be an effective $(-2)$-class in $NS(Y)$. It is associated to a divisor $R_Y \in NS(X)$. Clearly $R_Y^2 = R_Y^2 = -2$. Since $X$ is a K3 surface, by the Riemann-Roch theorem there are only the following two possibilities: either $R_X$ is effective or $-R_X$ is effective. If $R_X$ is effective, then $R_X D_X \geq 0$. By definition, $R_X N = 0$ and $D_X N = 0$ so $R_X D_X = R_Y D_Y$. Hence, $R_Y D_Y \geq 0$ and $D_Y$ is nef. So it now suffices to exclude that $-R_X$ is effective. Let us consider the intersection $R_X H_X$, since both $R_X$ and $H_X$ are contained in $M \subset NS(X)$, we have $R_X H_X = R_Y H_Y$, which is non-negative, since $H_Y$ is ample and $R_Y$ is effective. So $R_X H_X > 0$ and $H_X$ is pseudoample, this implies that $-R_X$ can not be effective.

\textbf{Lemma 2.5.} Let $D_{X'} \in NS(X')$ be a nef divisor. Let $D_X$ be the divisor associated to $D_{X'}$. Then $D_X \in NS(X)$ is nef (and $D_X N = 0$).

\textbf{Proof.} In order to show that $D_X$ is nef we show that for every curve $C_X \in NS(X)$, $D_X C_X \geq 0$. If $C_X = N$, then $D_X C_X = 0$. We now assume that $C_X \neq N$. So $\phi(C_X)$ is a curve in $X'$ and $C_X$ is the strict transform of $\phi(C_X)$ with respect to the blow up $\phi$. The class $\phi(C_X)$ is represented by an element in $NS(X')$, and in particular there exists a vector $v_C \in M$ such that $\phi(C_X) = f_{X'}(v_C)$. The curve $C_X$ is the strict transform of $\phi(C_X)$ and so it is represented in $NS(X)$ by the class $f_X(v_C) + \eta N$. Let us denote by $v_D \in M$ the vector such that $D_X = f_X(v_D)$. Thus,

$$D_X C_X = f_X(v_D)(f_X(v_C) + \eta N) = v_D v_C = f_{X'}(v_D) f_{X'}(v_C) = D_{X'} \phi(C_X)$$

where we used that $f_X$ and $f_{X'}$ are isometries. Since $D_{X'}$ is nef, $D_{X'} \phi(C_X) \geq 0$, so $D_X C_X \geq 0$ for every curve $C_X$ in $X$. \hfill \Box
Lemma 2.6. Let $D_Y \in NS(Y)$ be a nef divisor on $Y$. Let $D_X \in NS(X)$ be the associated divisor on $X$. Then $D_X$ is nef (and $D_X N = 0$).

Proof. We first prove that there is no a curve $C_X \subset X$ such that $C^2_X = -2$ and $D_X C_X < 0$. We already observed in proof of Lemma 2.4 that the class of every curve in $X$ is either the class of $N$ or $f_X(v_C) + \eta N$ for a certain vector $v_C \in M$. By definition $D_X = f_X(f_X^{-1}(D_Y))$ and so $D_X N = 0$. We can now assume that the class of $C_X$ is $f_X(v_C) + \eta N$. Then

$$-2 = (C_X)^2 = (f_X(v_C) + \eta N)^2 = (f_X(v_C))^2 - 2\eta^2.$$

So $(f_X(v_C))^2 = 2\eta^2 - 2 \geq -2$.

Let us denote by $C_Y \in NS(Y)$ the divisor $f_Y(v_C)$. Since $C^2_Y \geq -2$ either $C_Y$ or $-C_Y$ is effective.

Let $H_Y$ be the ample divisor of Remark 2.3 and $H_X$ its associated pseudoample divisor on $X$. By $C_X H_X = C_Y H_Y$ and by that fact that $C_X$ is effective, we deduce that $C_Y$ is effective.

If $C^2_Y > -2$, then the linear system $|C_Y|$ is sum of a movable part and a (possibly empty) fixed part. So let us write $C_Y = M + F$ where $|M|$ is a linear system without base points and $F$ is the fixed part of $|C_Y|$. The linear system $|M|$ contains an irreducible curve (by Bertini’s theorem) and thus $D_Y M \geq 0$. Similarly, $F$ is the union of the curves in the fixed part of $|C_Y|$ and thus its class consists of the sum (with positive coefficients) of the classes of these curves. Since $D_Y$ is nef, the intersection number $D_Y F$ is non negative. So $D_Y C_Y \geq 0$ and

$$C_X D_X = (f_X(v_C) + \eta N) f_X(v_D) = v_C v_D = C_Y D_Y \geq 0.$$

If $C^2_Y = -2$, then the linear system $|C_Y|$ does not have a movable part, but it coincides with its (non empty) fixed part and we conclude as before. □

Proposition 2.7. The Nef cones $\text{Nef}(X')$ and $\text{Nef}(Y)$ of $X'$ and $Y$ respectively are isomorphic.

Proof. We first recall that the lattices $NS(X')$, $NS(Y)$ and $M$ are all isometric.

Let $D_{X'}$ be a nef divisor on $X'$. Then the associated divisor $D_X$ is a nef divisor on $X$ and $D_X N = 0$, by Lemma 2.5. This implies, by Lemma 2.4, that the corresponding divisor $D_Y$ on $Y$ is nef.

Viceversa, let $D_{Y}$ be a nef divisor on $Y$. Then the associated divisor $D_X$ is nef on $X$ and $D_X N = 0$ by Lemma 2.4. This implies, by Lemma 2.2, that the corresponding divisor $D_{X'}$ on $X'$ is nef. □

Lemma 2.8. Let $D_X \in NS(X)$ be a semi-ample divisor (i.e. there exists a positive integer $m > 0$ such that $|mD_X|$ is without base points) such that $D_X N = 0$. Then the associated divisor $D_{X'} \in NS(X')$ is semi-ample.

Proof. Since $D_X$ is semi-ample, there exists an integer $m \in \mathbb{N}$ such that $|mD_X|$ is base points free. In particular $|mD_X|$ does not have fixed component. Since $D_X N = 0$, $mD_X N = 0$ and thus $H^0(X, mD_X)$ contains sections which do not pass through any point of $N$. The images of these sections under the map $\phi$ do not pass through the point $\phi(N) := P$.

The map $\phi : X \to X'$ is an isomorphism outside $N$. Let us consider the sections of $H^0(X, mD_X)$, we call them $s_j$, $j = 1, \ldots, 1 + (mD_X)^2/2$. The curves $\phi(s_j)$ are sections of $H^0(X', mD_{X'})$ and form a basis of this space. Viceversa, let us consider a section $s'$ in $H^0(X, mD_{X'})$. It is the image of a curve in $X$ which is
2.3. Mori Dream Spaces: the surfaces $Y$ and $X'$. We recall the definition of Mori Dream Space, in the context of the surfaces (and we recall that any small $\mathbb{Q}$-factorial modification is an isomorphism if we are considering surfaces).

**Definition 2.9.** ([HK]) We will call a normal projective surface $X$ a Mori Dream Space provided the following hold:

1. $X$ is $\mathbb{Q}$-factorial and $\text{Pic}(X) \otimes \mathbb{Q} \equiv \text{NS}(X) \otimes \mathbb{Q}$;
2. $\text{Nef}(X)$ is the affine hull of finitely many semi–ample line bundles.

By [HK] Proposition 2.9 a variety $X$ such that $\text{Pic}(X) \otimes \mathbb{Q} = \text{NS}(X) \otimes \mathbb{Q}$ is a Mori Dream Space if and only if its Cox ring $\mathcal{R}(X)$ is finitely generated.

**Proposition 2.10.** The surface $Y$ is a Mori Dream Space if and only if the surface $X'$ is a Mori Dream Space.

**Proof.** We recall that every semi-ample divisor is nef. On a K3 surface also the viceversa holds, i.e. a divisor on a K3 surface is semi-ample if and only if it is nef.

Let $Y$ be a Mori Dream Space. By definition $\text{Nef}(Y)$ is the convex hull of a finite set of semi-ample divisors, $D_Y^{(1)}, \ldots, D_Y^{(r)}$. Then $\text{Nef}(X')$ is the convex hull of a finite set of nef divisors, which are the divisors $D_X^{(1)}, \ldots, D_X^{(r)}$ associated to $D_Y^{(1)}, \ldots, D_Y^{(r)}$.

It remains to show that $D_X^{(1)}, \ldots, D_X^{(r)}$ are semi-ample.

The associated divisors on $X$, $D_X^{(1)}, \ldots, D_X^{(r)}$ are nef and such that $D_X^{(i)} \cdot N = 0$ for each $i = 1, \ldots, r$, by Lemma 2.6. So they are semi-ample divisors on $X$ (which is a K3 surface). ByLemma 2.8 the divisors $D_X^{(1)}, \ldots, D_X^{(r)}$ are semi-ample on $X'$. Thus $X'$ is a Mori Dream Space.

Let us now assume that $Y$ is not a Mori Dream Space, then $X'$ is not a Mori Dream Space. Indeed, since $Y$ is a K3 surface, it is $\mathbb{Q}$-factorial, $\text{Pic}(Y) \otimes \mathbb{Q} \equiv \text{NS}(Y) \otimes \mathbb{Q}$, and the nef divisors are semi-ample. So if $Y$ is not a Mori Dream Space, then $\text{Nef}(Y)$ is not the convex hull of finitely many nef divisors, which implies that also $\text{Nef}(X')$ is not the convex hull of finitely many nef divisors, and thus it can not be a Mori Dream Space. 

3. Admissible pairs $(X, X')$

**Definition 3.1.** An admissible pair is a pair of surfaces $(X, X')$ such that $X$ is a K3 surface that is not a Mori Dream Space and $X'$ is obtained by contracting exactly one extremal ray of $X$ and is a Mori Dream Space.

Let us consider an admissible pair $(X, X')$. By assumption the Picard number of $X$ is greater then or equal to 2. Under this condition, by [Kov] Theorem 2 the extremal rays of the effective cone of $X$ are curves with self intersection either 0 or $-2$. Since we are assuming that $X'$ is a surface, the contraction associated to the extremal ray can not be a fibration and so we are contracting a $-2$-curve.
In order to answer positively to the question 1.1 it suffices to find an admissible pair. In this section we describe how to find the Néron–Severi groups of the K3 surfaces $X$ in admissible pairs (see Condition 3.3) and we give a list of admissible pairs such that $\rho(X) \geq 10$ (cf. Theorem 3.4).

Let us assume that $X$ and $Y$ are K3 surfaces as in Section 2 (i.e. $X$ admits a $(-2)$ curve $N$ such that the lattice orthogonal to $N$ in $NS(X)$ is the Néron–Severi group of $Y$). We recall that $NS(X') \simeq NS(Y)$, where $X'$ is obtained by $X$ contracting $N$. By Proposition 2.10 the pair of surfaces $(X, X')$ is an admissible pair if and only if $X$ is not a Mori Dream Space and $Y$ is a Mori Dream Space. Both $X$ and $Y$ are K3 surfaces and since the K3 surfaces which are Mori Dream Spaces are classified, this provide a way to construct pairs $(X, X')$ as required.

Here we summarize some results on K3 surfaces, which will be used in the following.

**Theorem 3.2.** ([AHL, Theorems 2.7, 2.11, 2.12]) An algebraic K3 surface $S$ is a Mori Dream Space if and only if the automorphism group $\text{Aut}(S)$ is finite.

In particular, if $\rho(S) = 1$, then $S$ is a Mori Dream Space; if $\rho(S) = 2$, then $S$ is a Mori Dream Space if and only if $NS(S)$ contains at least an element with self intersection either 0 or $-2$; if $\rho(S) \geq 3$, then $S$ is a Mori Dream Space if and only if $NS(S)$ belongs to a finite known list of hyperbolic lattices.

The previous Theorem gives a constructive way to produce admissible pairs $(X, X')$, indeed it suffices to have a K3 surface $X$ with the following properties:

**Condition 3.3.**
- $|\text{Aut}(X)| = \infty$;
- there exists a rational curve $N$ on $X$;
- if $M := N^\perp_{NS(X)}$ and $Y$ is a K3 surface such that $NS(Y) \simeq M$, then $|\text{Aut}(Y)| < \infty$.

**Theorem 3.4.** Let $(X, X')$ be an admissible pair of surfaces. Then $\rho(X) \geq 3$ and $\rho(X) \neq 19$.

If $\rho(X) \geq 4$, then there are finitely many possible choices for the lattice $NS(X)$.

The complete list of the Néron–Severi groups $NS(X)$ of admissible pairs $(X, X')$ such that $\rho(X) \geq 10$ is given in the table (3.1).

If $L$ is as in Table (3.1) and $X$ is a K3 surface generic among the ones such that $NS(X) \simeq L$, then there exists a smooth irreductible rational curve $N \subset X$ such that, denoted by $X'$ the surface obtained contracting $N$, $NS(X') \simeq M$ and thus $(X, X')$ is an admissible pair.
Proof. If $X$ is a K3 surface and it is not a Mori Dream Space then $\text{Aut}(X)$ is not finite. In particular this implies that $\rho(X) \geq 2$ and that if $\rho(X) = 2$, then $X$ does not admit curves with self-intersection equal either to 0 or $-2$. Since we require that $X$ admits a $(-2)$-curve, $\rho(X) \geq 3$.

By hypothesis, $\text{NS}(X)$ is an overlattice of finite index of $M \oplus N$ where $N$ is the class of the $(-2)$-curve contracted and $M$ is primitively embedded in $\text{NS}(X)$. We denote by $r$ the index of the inclusions $M \oplus N \hookrightarrow \text{NS}(X)$. If $r \neq 1$, then there exists a class $(m + N)/r \in \text{NS}(X)$ such that $m \in M$, $(m + N)/r \notin M \oplus N$. Clearly this implies that $N((m + N)/r) = N^2/r = -2/r \in \mathbb{Z}$ and thus $r$ is either 1 or 2. So $\text{NS}(X)$ is either $M \oplus N$ or an overlattice of index 2 of $M \oplus N$.

The number of hyperbolic lattices $M$ with rank$(M) \geq 3$ such that if $Y$ is a K3 surface with $\text{NS}(Y) \simeq M$, then $|\text{Aut}(Y)| < \infty$ is finite. If $(X, X')$ is an admissible pair, then $\text{NS}(Y) \simeq \text{NS}(X') \simeq M$, and the admissible choices for $M$ are finite. The lattice $\text{NS}(X)$ is an overlattice of index 1 or 2 of $\text{NS}(X') \oplus N \simeq M \oplus A_1$. Since the number of overlattices of index two of $M \oplus A_1$ is finite (up to isometries), it follows that the possible choices for $\text{NS}(X)$ are finite if rank$(M) \geq 3$, i.e. if $\rho(\text{NS}(X)) \geq 4$.

In order to construct the list of the Néron–Severi groups of the admissible pairs $(X, X')$ (and to exclude the case $\rho(X) = 19$), we check the list of the Néron–Severi groups of the K3 surfaces with a finite group of automorphisms, given in [Ko]. We denote by $M$ a lattice in this list. If $\text{NS}(X') \simeq M$, then $X'$ is a Mori Dream Space: Indeed $X'$ is a Mori Dream Space if and only if $Y$ is a Mori Dream Space by Proposition 2.10 and $Y$ is a Mori Dream Space if and only if $Y$ is a K3 surface with a finite automorphism group, by Theorem 3.2. But $\text{NS}(Y) \simeq M$, so $Y$ is a Mori Dream Space.

\[
\begin{array}{c|c|c}
\rho(X) & \text{L} \simeq \text{NS}(X) & \text{M} \simeq \text{NS}(X') \\
\hline
20 & U \oplus E_8^2 \oplus A_1^2 & U \oplus E_8^2 \oplus A_1 \\
18 & U \oplus E_8 \oplus E_7 + A_1 & U \oplus E_8 \oplus E_7 \\
17 & U \oplus E_8 \oplus D_6 + A_1 & U \oplus E_8 \oplus D_6 \\
16 & U \oplus E_8 \oplus D_4 + A_1^2 & U \oplus E_8 \oplus D_4 + A_1 \\
15 & U \oplus E_8 \oplus A_1^2 & U \oplus D_8 + D_4 \\
14 & U \oplus E_7 \oplus A_1^2 & U \oplus E_7 \oplus A_1^2 \\
14 & U \oplus E_8 \oplus A_3 + A_1 & U \oplus E_8 \oplus A_3 \\
13 & U \oplus D_6 + A_1^2 & U \oplus D_6 + A_1 \\
12 & U \oplus E_8 + A_2 + A_1 & U \oplus E_8 + A_2 \\
12 & U \oplus D_4 + A_1^2 & U \oplus D_4 + A_1^2 \\
11 & U \oplus A_1^4 & U \oplus A_1^3 \\
10 & U \oplus A_1 \oplus A_1^3 & U \oplus A_1^2 \\
10 & U \oplus D_4 + A_3 + A_1 & U \oplus D_4 + A_3 \\
10 & U \oplus D_5 + A_2 + A_1 & U \oplus D_5 + A_2 \\
10 & U \oplus D_2 + A_1 & U \oplus D_2 \\
10 & U \oplus E_6 + A_1 + A_1 & U \oplus E_6 + A_1 \\
\end{array}
\]
So for each $M$ in the list given in \cite{Kon} we have to construct the lattice $M \oplus N$ and the overlattices of index 2 of $M \oplus N$. Each of these lattices is a good candidate to be the Néron–Severi group of $X$. Let us denote by $L$ one of these lattices and by $X$ a K3 surface such that $NS(X) \simeq L$. We now have to check that $X$ is not a Mori Dream Space, so we have to check that $|\text{Aut}(X)| = \infty$, i.e. that $L$ is not in the list given in \cite{Kon}. In this way one produces the list of the possible Néron–Severi groups of $X$ and $X'$. Then a geometric argument can be used in order to show that there exists a model of $X$ such that the class $N$ represents a smooth irreducible $(-2)$-curve and so an extremal ray.

Our first step is to construct the list of the Néron–Severi groups given in Table 3.1. We give all the details for the first lines of the Table, the other cases are very similar. Let us check the list given in \cite{Kon} of the lattices $M$ such that if $Y$ is a K3 surface with $NS(Y) \simeq M$, then $|\text{Aut}(Y)| < \infty$. If rank($M$) = 19, then $M \simeq U \oplus E_8^2 \oplus A_1$. So the lattice $L$ is either $M \oplus N \simeq U \oplus E_8^2 \oplus A_1^2$ or an overlattice of index 2 of $M \oplus N \simeq U \oplus E_8^2 \oplus A_1^2$. The discriminant group of the lattice $U \oplus E_8^2 \oplus A_1^2$ is $(\mathbb{Z}/2\mathbb{Z})^2$, so an overlattice of index two of this lattice is unimodular. But there exits no a hyperbolic even unimodular lattice of rank 20. So $L \simeq U \oplus E_8^2 \oplus A_1^2$. We now assume $NS(X) \simeq L$. Since $L$ is not contained in the list given in \cite{Kon} we conclude that $|\text{Aut}(X)| = \infty$ and thus $X$ is not a Mori Dream Space. This guarantee that if the pair $(X, X')$ is admissible and $\rho(X) = 20$, then $NS(X) \simeq U \oplus E_8^2 \oplus A_1^2$ and $NS(X') \simeq U \oplus E_8^2 \oplus A_1$.

Let us consider the lattice $M$ of the list in \cite{Kon} with rank($M$) = 18. In this case $M \simeq U \oplus E_8^2$, which is a unimodular lattice. A priori $L$ could be either $L \simeq M \oplus N \simeq U \oplus E_8^2 \oplus A_1$ or an overlattice of index 2 of $M \oplus N \simeq U \oplus E_8^2 \oplus A_1$. The discriminant group of $U \oplus E_8^2 \oplus A_1$ is $\mathbb{Z}/2\mathbb{Z}$, so there are no overlattices of index two of $U \oplus E_8^2 \oplus A_1$, and thus $L \simeq M \oplus N \simeq U \oplus E_8^2 \oplus A_1$. Hence, if there exists an admissible pair $(X, X')$ such that $\rho(X) = 19$, then $NS(X) \simeq U \oplus E_8^2 \oplus A_1$. But if $NS(X) \simeq U \oplus E_8^2 \oplus A_1$, then $|\text{Aut}(X)| < \infty$ by \cite{Kon}, and so $X$ is a Mori Dream Space, by Theorem 3.2. We conclude that there are no admissible pairs $(X, X')$ such that $\rho(X) = 19$. The other cases are similar. The other cases in the Table are similar.

Let $L$ be a lattice given in the second column of Table 3.1 and $M$ the corresponding lattice given in the third column of the same Table. Now we prove that the generic K3 surface $X$ such that $NS(X) \simeq L$ admits an extremal contraction such that the corresponding surface $X'$ has the property $NS(X') \simeq M$. This implies that $(X, X')$ is an admissible pair. It suffices to prove that there exists a smooth irreducible rational curve which is represented in $NS(X)$ by the class $N$ such that $N_{NS(X)} \simeq M \simeq NS(X')$. In all the cases but $\rho(X) = 15$ and $NS(X) \simeq U(2) \oplus A_1^8$, the lattice $L \simeq NS(X)$ is $L \simeq U \oplus R \oplus A_1$ for a certain root lattice $R$ and the lattice $M \simeq NS(X')$ is $M \simeq U \oplus R$.

Since $L \simeq U \oplus R \oplus A_1$, the surface $X$ admits an elliptic fibration such that the reducible components of the reducible fibers which do not intersect the zero section are represented by the lattice $R \oplus A_1$. In particular, the lattice $A_1$ which appears as direct summand in the decomposition $L \simeq U \oplus R \oplus A_1$, represents an irreducible component of a fiber of type $I_2$ (or $II$). Thus $A_1$ is generated by the class of a smooth irreducible rational curve. The contraction of this curve gives the singular surface $X'$, whose Néron–Severi group is naturally identified with $U \oplus R \simeq M$. 
In Section 5.1 an explicit equation of the elliptic fibration on $X$ associated to the decomposition $NS(X) \simeq U \oplus E_8^2 \oplus A_1^2$ is provided, as example.

The existence of the smooth irreducible rational curve $N$ in the remaining cases ($\rho(X) = 15$ and $NS(X) \simeq U(2) \oplus A_1^2$) is proved in Section 5 (more precisely in Example 5.1 and in Section 5.2 respectively).

**Remark 3.5.** In Theorem 3.3 we proved that if $(X, X')$ is an admissible pair and $\rho(X) \geq 4$, then the lattice $NS(X)$ is isometric to a lattice in a finite list of hyperbolic lattice. On the other hand, if $\rho(X) = 3$ then it is possible to construct infinitely many admissible pairs $(X_n, X'_n)$ such that $n \in \mathbb{N}$ and $NS(X_n) \not\simeq NS(X'_n)$ if $n \neq n'$. Examples are provided in Proposition 5.2.

**Remark 3.6.** There exist examples of both the following cases:

- $(X_1, X'_1)$ and $(X_2, X'_2)$ are two admissible pairs such that $NS(X_1) \simeq NS(X_2)$, but $NS(X'_1) \not\simeq NS(X'_2)$
- $(X_1, X'_1)$ and $(X_2, X'_2)$ are two admissible pairs such that $NS(X'_1) \simeq NS(X'_2)$, but $NS(X_1) \not\simeq NS(X_2)$

An example of the first case is given in the Table 5.1. $\rho(X) = 15$. We briefly describe the geometry of this case in Example 5.1.

An example of the second case is provided in Section 4 where the geometric details are given. Moreover, an infinite series of examples is presented in Proposition 5.2.

Since we described the deep relation between the automorphism group and the property of being a Mori Dream Space, we now consider the relation between the automorphism groups of the surfaces involved in our construction. Since $X'$ is obtained from $X$ by contracting a curves, $Aut(X') \subset Aut(X)$. More precisely, every automorphism $\alpha$ which does not preserve the curve $N$ does not descend to an automorphism of $X'$ and every automorphism $\alpha$ which preserves $N$ descend to an automorphism $\alpha'$ of $X'$. Every automorphism $\alpha'$ of $X'$ lifts to an automorphism $\alpha$ of $X$ which leaves invariant the rational curve $N$. Moreover there is also a strong relation among the automorphism group of $X'$ and $Y$:

**Corollary 3.7.** The group of automorphisms of $X'$ is contained in the group of automorphisms of $Y$.

**Proof.** Let $\alpha'$ be an automorphism of $X'$. It lifts to an automorphism $\alpha$ of $X$ which preserves $N$. So $\alpha$ induces an effective Hodge isometry of $H^2(X, \mathbb{Z})$. We denote by $T_X$ the transcendental lattice of $X$ and we observe that the isometry induced by $\alpha$ preserves the splitting $M \oplus N \oplus T_X$. Hence it is a Hodge isometry of $H^2(Y, \mathbb{Z})$, whose Néron–Severi group is $M$. Moreover, the Nef cone of $Y$ can be identified with the one of $X'$ by the Proposition 2.7. Since $\alpha'$ is an automorphism of $X'$, it preserves the nef cone of $X'$ and so $\alpha^*$ preserves the ample cone of $Y$. So $\alpha^*$ is an Hodge effective isometry for $Y$ and thus it is induced by an automorphism $\alpha_Y$ of $Y$. □

4. Two examples

This section is devoted to the geometric description of two examples. First we consider a lattice $M$ of rank 2 such that if $Y$ is a K3 surface with $NS(Y) \simeq M$, then $Y$ is a Mori Dream Space. We describe the geometry and the automorphism group of $Y$ in Section 4.2.1 $Y$ admits a model as quartic hypersurface with a node.
and a model as double cover of \( \mathbb{P}^2 \). For a generic choice of \( Y \) the automorphism group is generated by the cover involution.

In Section 4.1 we construct two different lattices \( L \) of rank 3: one of them is isometric to \( M \oplus A_1 \), the other one is the unique even overlattice of index 2 of \( M \oplus A_1 \). If the Néron–Severi group of a K3 surface is isometric to one of these two lattices, then the K3 surface is not a Mori Dream Space, but the contraction of a \((-2)\)-curve on it produces a Mori Dream Space whose Néron–Severi group is isometric to \( M \), so we construct the Néron–Severi groups of two admissible pairs.

We describe the geometry of the K3 surfaces of these two admissible pairs in Sections 4.3.1 and 4.3.3: one of them admits a model as quartic in \( \mathbb{P}^3 \) with two nodes (which clearly specializes the model of \( Y \)) and as double cover of \( \mathbb{P}^2 \), the other is an elliptic fibration and admits a model as double cover of the Hirzebruch surface \( \mathbb{F}_4 \). We show that the automorphism group of both these K3 surfaces are infinite, but that several automorphisms do not descend to automorphism of the contracted model, which is in fact a Mori Dream Space.

In the following proposition we summarize the results obtained in this section.

**Proposition 4.1.** Let \( M \) be the lattice \( \langle 4 \rangle \oplus \langle -2 \rangle \). Let \( L \) be an overlattice of \( M \) such that:

- There exists a vector \( n \in L \), such that \( n^2 = -2 \) and \( n^A \cong M \).
- \( L \) admits a primitive embedding in \( \Lambda_{K3} \).

Then \( L \) is a hyperbolic even lattice of rank 3 and it is isometric either to \( \langle 4 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle \) or to \( U \oplus \langle -4 \rangle \) (see Section 4.2.1). Moreover,

1. Let \( Y \) be a generic K3 surface such that \( \text{NS}(Y) \cong M \). Then \( Y \) admits a map \( \phi : Y \to \mathbb{P}^2 \) which is a \( 2:1 \) cover branched along a smooth sextic \( C_6 \subset \mathbb{P}^2 \). There exists a conic \( c_2 \) which is tangent to \( C_6 \) in their six intersection points. The automorphism group of \( Y \) is generated by the cover involution \( \iota \), in particular it is finite (see Section 4.2.1).

2. Let \( X \) be a generic K3 surface such that \( \text{NS}(X) \cong \langle 4 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle \). Then \( X \) admits two different maps \( \phi_i : X \to (\mathbb{P}^2)_i, \ i = 1, 2 \). Each of them is a \( 2:1 \) cover branched along a sextic \( (C_6)_i \subset (\mathbb{P}^2)_i \) with one node in the point \( (P)_i \in (C_6)_i \). There exists a conic \( (c_2)_i \) such that \( (P)_i \not\in (c_2)_i \) and \( (c_2)_i \) is tangent to \( (C_6)_i \) in their six intersection points. The automorphism group of \( X \) is infinite and contains the two (non commutative!) cover involutions \( \iota_i \).

The map \( \phi_1 \) (resp. \( \phi_2 \)) contracts exactly one rational curve, which is \( (c_2)_2 \) (resp. \( (c_2)_1 \)). Let \( X' \) be the double cover of \( \mathbb{P}^2 \) branched along \( (C_6)_1 \).

Then \( ((c_2)_2)^{-\text{NS}(X)} \cong \text{NS}(X') \) is isometric to \( M \) and \( (X, X') \) is an admissible pair. Moreover \( \text{Aut}(X') \) is generated by the cover involution \( \iota_1 \) (in particular it is finite) and the induced effective Hodge isometry coincides with the one induced by the involution \( \iota \) of \( Y \) (see Section 4.3.1).

3. Let \( S \) be a generic K3 surface such that \( \text{NS}(S) \cong U \oplus \langle -4 \rangle \). Then \( S \) admits an elliptic fibration \( \mathcal{E} : S \to \mathbb{P}^1 \) such that \( \text{MW}(\mathcal{E}) = \mathbb{Z} \) is generated by a section \( s_1 \) which has a trivial intersection with the zero section. The automorphism group of \( S \) is infinite and contains the translation by \( s_1 \) (which is an automorphism of infinite order).

There exists a \( 2:1 \) map \( \varphi : S \to \mathbb{P}^5 \) whose image is the cone, \( C \), over the twisted rational quartic in \( \mathbb{P}^4 \) and let \( B \subset C \) be the branch locus. The map \( \varphi \) contracts exactly one rational curve \( s \), which is a section of the fibration.
\( \mathcal{E} \) and whose image is the vertex of the cone. Let \( S' \) be the double cover of \( C \) branched along \( B \). Then \( s^{\text{NS}(S)} \simeq \text{NS}(S') \) is isometric to \( M \) and \( (S, S') \) is an admissible pair. Moreover \( \text{Aut}(S') \) is generated by the cover involution (in particular it is finite) and the induced effective Hodge isometry coincides with the one induced by the involution \( \iota \) of \( Y \) (see Section 4.3.3).

4.1. **Lattice enhancements.** Let \( M \) be the lattice \( \langle 4 \rangle \oplus \langle -2 \rangle \) and let us denote by \( \{n_1, n_2\} \) its basis. The generators of the discriminant group are \( M^\vee/M \simeq \langle n_1/4, n_2/2 \rangle \).

Let \( L \) be a lattice such that \( L \) admits a primitive embedding in \( \Lambda_{K3} \). Since \( \Lambda_{K3} \) is an even lattice, \( L \) is an even lattice.

We now require that there exists a vector \( n \in L \), such that \( n^2 = -2 \) and \( n^{\perp_L} \simeq M \). In particular this implies that \( L \) is an overlattice of finite index \( r \in \mathbb{N} \) of \( M \oplus \mathbb{Z}n \).

Since \( M \oplus \mathbb{Z}n \) is a hyperbolic lattice of rank 3, \( L \) is a hyperbolic lattice of rank 3.

We recall that every even hyperbolic lattice of rank less than 11 admits a primitive embedding in \( \Lambda_{K3} \). So its enough to construct all the non isometric even overlattices of \( M \oplus \mathbb{Z}n \) of index \( r \in \mathbb{N} \) in order to classify the admissible lattices \( L \).

We already proved that \( r \) is either 1 or 2 in proof of Theorem 3.4.

The first admissible choice for \( L \) is \( M \oplus \mathbb{Z}n \simeq \langle 4 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle \) (which clearly corresponds to \( r = 1 \)).

Now we look for an overlattice of index \( r = 2 \). A \( \mathbb{Q} \)-basis of \( L \) is given by \( n_1, n_2 \) and \( n \). If this is not a \( \mathbb{Z} \) basis, then there exists a vector \( w := (a_1 n_1 + a_2 n_2 + a_3 n)/2 \), \( a_i \in \mathbb{Z}/2\mathbb{Z} \), such that \( w \not\in (M \oplus \mathbb{Z}n) \), \( wn_1 \in \mathbb{Z} \), \( wn_2 \in \mathbb{Z} \), \( wn \in \mathbb{Z} \) and \( ww \in 2\mathbb{Z} \).

The condition \( ww \in 2\mathbb{Z} \) immediately implies \( a_2 \equiv a_3 \mod 2 \). If \( a_2 \) and \( a_3 \) are both even, then \( w \equiv n_1/2 \mod (M \oplus \mathbb{Z}n) \) and \( (n_1/2)^2 \equiv 1 \mod 2 \mathbb{Z} \). So \( a_2 \equiv a_3 \equiv 1 \mod 2 \). Again the condition \( ww \in 2\mathbb{Z} \) implies that \( a_1 \equiv 1 \mod 2 \). So there exists a unique overlattice of index \( r \neq 1 \) of \( M \oplus \mathbb{Z}n \) and it is the lattice obtained adding to the \( \mathbb{Q} \) basis \( \{n_1, n_2, n\} \) the vector \( (n_1 + n_2 + n)/2 \). We can now find a change of bases in order to give a better description of this overlattice: let us consider the \( \mathbb{Z} \)-basis \( \{(n_1 - n_2 - n)/2, (n_1 + n_2 - n)/2, -n_1 + 2n\} \). Computing the bilinear form on this basis we find \( U \oplus \langle -4 \rangle \), so the unique even hyperbolic overlattice of index 2 of \( M \oplus \mathbb{Z}n \) is isometric to \( U \oplus \langle -4 \rangle \).

4.1.1. **Remarks on the orthogonal to \( M \).** There exists a unique embedding (up to isometries) of \( M \) in \( \Lambda_{K3} \simeq U \oplus U \oplus U \oplus E_8 \oplus E_8 \) which is given by

\[
\begin{align*}
n_1 &:= \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ \mathbf{0} \oplus \mathbf{0} \end{pmatrix}, \\
n_2 &:= \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 0 \oplus 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \oplus 0 \end{pmatrix}.
\end{align*}
\]

Let us denote by \( T \) the lattice \( M^{\perp_{K3}} \). It is generated by the generators of the third copy of \( U \), by the generators of \( E_8 \oplus E_8 \) and by the two generators:

\[
\begin{align*}
t_1 &:= \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 0 \oplus 0 \end{pmatrix}, \\
t_2 &:= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \oplus 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \oplus 0 \end{pmatrix}.
\end{align*}
\]

Since \( \Lambda_{K3} \) is unimodular, it is an overlattice of index 8 of \( M \oplus T \) and indeed \( \Lambda_{K3}/(L \oplus M) \) is generated by \( (t_1 + n_1)/4 \) and \( (t_2 + n_2)/2 \).

In order to construct a lattice isometric to \( M \oplus \mathbb{Z}n \) we have to identify a vector \( n \in T \) with \( n^2 = -2 \). Let us consider the following two possibilities:

- \( n := \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \oplus 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \oplus 0 \end{pmatrix} \). In this case there is no overlattice of \( L \oplus \mathbb{Z}n \) contained in \( \Lambda_{K3} \) and so \( L \) is \( \langle 4 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle \). The orthogonal
the computation of the moduli of the family of the nodal quartics, implies that

\[ A \] in the point \((1 : 0 : 0 : 0)\).

4.2.1. Quartic with a node. Let us consider a K3 surface \(Y\) which admits a model as quartic with exactly one ordinary double point, which is a singularity of type \(A_1\). Then there exists a pseudo-ample divisor \(H\) in \(NS(Y)\) with self intersection 4 and which is orthogonal to a \((-2)\)-vector. Indeed \(\varphi|_H : Y \rightarrow \mathbb{P}^3\) contracts exactly one rational curve to the ordinary double point. The class of this curve, called \(N_1\), has self intersection \(-2\) (since the curve is a rational curve on a K3 surface) and is orthogonal to \(H\) (since the curve is contracted by \(\varphi|_H\)). Hence the lattice \(\langle H, N_1 \rangle \simeq \langle 4 \rangle \oplus \langle -2 \rangle\) is primitively embedded in the Néron–Severi group of a K3 surface which admits a model as quartic with a singularity of type \(A_1\). Moreover the computation of the moduli of the family of the nodal quartics, implies that generically \(NS(Y) \simeq \langle 4 \rangle \oplus \langle -2 \rangle\).

Up to projective transformations, we can assume that the node of the quartic is in the point \((1 : 0 : 0 : 0) \in \mathbb{P}^3_{(x_0:x_1:x_2:x_3)}\) and so an equation for \(Y\) is of the form

\[
(4.1) \quad x_0^2F_2(x_1 : x_2 : x_3) + x_0F_3(x_1 : x_2 : x_3) + F_4(x_1 : x_2 : x_3),
\]

where \(F_i\) are generic homogenous polynomials of degree \(i\).

Let us consider the projection of the quartic (4.1) from \((1 : 0 : 0 : 0)\) to \(\mathbb{P}^2_{(x_1:x_2:x_3)}\). It exhibits \(Y\) as double cover of \(\mathbb{P}^2_{(x_1:x_2:x_3)}\) branched along the sextic curve

\[
(4.2) \quad C_6 := V \left( F_3(x_1 : x_2 : x_3)^2 - 4F_2(x_1 : x_2 : x_3)F_4(x_1 : x_2 : x_3) \right) \subset \mathbb{P}^2_{(x_1:x_2:x_3)}
\]

The conic \(c_2 := V (F_2(x_1 : x_2 : x_3))\) intersects the branch locus in the six points \(F_2(x_1 : x_2 : x_3) = F_3(x_1 : x_2 : x_3) = 0\) and each of them has multiplicity 2.

The projection from the singular point of the quartic is associated to the divisor \(H - N_1\).

The cover involution of the \(2 : 1\) map \(\varphi|_{H - N_1} : Y \rightarrow \mathbb{P}^2\) is an involution, called \(\iota\), of \(Y\) and it does not preserve the symplectic structure of \(Y\) (indeed \(Y/\iota\) is birational to \(\mathbb{P}^2\)). The class of \(H - N_1\) is clearly preserved by the isometry \(\iota^*\), but the class \(N_1\) is not. Indeed, since \(c_2\) is everywhere tangent to the branch locus, its inverse image consists of two disjoint curves. One of them is \(N_1\) and the other is \(2H - 3N_1\).

So \(\iota^*\) acts as \(-1\) on the transcendental lattice and as

\[
(4.3) \quad \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}
\]

on the basis \(\{H, N_1\}\) of the Néron–Severi group.

Remark 4.2. The K3 surface \(Y\) admits two different (equivalent up to automorphism of the surface, but not up to projectivity of \(\mathbb{P}^3\)) models as singular quartic in \(\mathbb{P}^3\). One of them is associated to the divisor \(H\), the other one to \(\iota^*(H) = 3H - 4N_1\).
Since $Y$ has a smooth rational curve and its Picard number is 2, the automorphism group of $Y$ is finite. To be more precise it is known that the automorphism group of $Y$ generically coincide with $\mathbb{Z}/2\mathbb{Z} \simeq \langle \iota \rangle$ (see e.g. [GLP]).

4.2.2. *K3 surfaces with an elliptic fibration.* Let $V$ be a K3 surface and $\mathcal{E} : V \to \mathbb{P}^1$ be an elliptic fibration (i.e. a fibration in curves of genus 1 which admits at least one section). We will denote by $F$ the class of the fiber of $\mathcal{E}$ and by $s_0$ the class of a given section (called zero section) of $h$.

Let $t_{10}$ and is represented by the class 6.

Up to projective transformations we can assume that the singular point $P$ is (1 : 0 : 0 : 0) $\in \mathbb{P}^3$ where $H$ is the identity on $\mathbb{P}^3$ obtained contracting the exceptional curve.

The divisor $4F + 2s_0$ defines a $2 : 1$ map $\varphi_{[4F+2s_0]} : V \to \mathcal{C} \subset \mathbb{P}^3$ where $\mathcal{C}$ is a cone over the twisted quartic curve in $\mathbb{P}^4$ (see [SD]). We observe that $\mathcal{C}$ is a model of $V$ that is generic among the K3 surfaces admitting an elliptic fibration, then $NS(V) \cong U$.

On each elliptic curve (so on each smooth fiber of $\mathcal{E}$) the hyperelliptic involution is well defined. The hyperelliptic involutions on the fibers glue together giving an involution, called $h$, of $V$. The quotient for this an involution is the Hirzebruch surface $\mathbb{F}_4$ and the ramification locus consists of the zero section and of the trisection $t_{10}$ passing through the 2-torsion of each fiber. The trisection $t_{10}$ is a curve of genus 10 and is represented by the class $6F + 3s_0$.

Since both the class of the section $s_0$ and the class of the fiber $F$ are preserved by $h$, $h^*$ is the identity on $NS(V)$ and $-1$ on the transcendental lattice.

The divisor $4F + 2s_0$ defines a $2 : 1$ map $\varphi_{[4F+2s_0]} : V \to \mathcal{C} \subset \mathbb{P}^3$ where $\mathcal{C}$ is a cone over the twisted quartic curve in $\mathbb{P}^4$ (see [SD]). We observe that $\mathcal{C}$ is a model of $V$ obtained contracting the exceptional curve.

The zero section $s_0$ is contracted by $\varphi_{[4F+2s_0]}$ and its image is the vertex of the cone. So the double cover $V \to \mathcal{C}$ is branched along the image of the trisection, i.e. on $\varphi_{[4F+2s_0]}(t_{10})$. The involution $h$ is exactly the cover involution of this double cover.

Since $V$ contains a rational curve and $\rho(V) = 2$, $|\text{Aut}(V)| < \infty$. To be more precise, for a generic choice of $V$, $\text{Aut}(V) = \langle h \rangle$ (see e.g. [GLP]).

4.3. *K3 surfaces with Picard number 3.*

4.3.1. *Quartic with two nodes (lattice $L \simeq \langle 4 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle$).* Let us now consider a K3 surface $X$ such that $NS(X) \simeq \langle 4 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle$. We assume that $X$ is generic among the ones with this property and we denote by $\{H, N_1, N_2\}$ the basis of $NS(X)$. We can assume that $H$ is a pseudo ample divisor. The map $\varphi_{|H|} : X \to \mathbb{P}^3$ exhibits $X$ as quartic surface with two double points (the contraction of $N_1$ and $N_2$).

Up to projective transformations we can assume that the singular point $\varphi_{|H|}(N_1)$ is $(1 : 0 : 0 : 0) \in \mathbb{P}^3_{(x_0 : x_1 : x_2 : x_3)}$ (as in Section 4.2.1), and the singular point $\varphi_{|H|}(N_2)$ is $(0 : 0 : 0 : 1) \in \mathbb{P}^3_{(x_0 : x_1 : x_2 : x_3)}$. So any quartic in this family has the following equation:

\begin{equation}
4x_0^2F_2(x_1 : x_2 : x_3) + x_0(G_3(x_1 : x_2) + x_3G_2(x_1 : x_2) + x_2^2G_1(x_1 : x_2)) + \frac{1}{2}x_2^2H_2(x_1 : x_2) + x_3H_3(x_1 : x_2) + H_4(x_1 : x_2)
\end{equation}

where $F_i, G_i, H_i$ are homogenous polynomial of degree $i$.

As in Section 4.2.1, we consider the projection from $(1 : 0 : 0 : 0)$ which corresponds to the divisor $H - N_1$. This gives a $2 : 1$ map $\varphi_{|H-N_1|} : X \to \mathbb{P}^2_{(x_1 : x_2 : x_3)}$. 

which is a double cover branched along the sextic:

\[
C_6 := V \left( (G_3(x_1 : x_2) + x_3G_2(x_1 : x_2) + x_2^2G_1(x_1 : x_2))^2 + 
-4F_2(x_1 : x_2 : x_3) (x_2^2H_2(x_1 : x_2) + x_3H_3(x_1 : x_2) + H_4(x_1 : x_2)) \right).
\]

The sextic \(C_6\) has a unique singular point, which is an ordinary node, in \((0 : 0 : 1)\), i.e., in the point \(\varphi|_{H-N_1}|(N_2)\). Moreover (as in section 4.2.1), the conic \(V(F_2(x_1 : x_2 : x_3))\) intersect \(C_6\) in six smooth points and is tangent to \(C_6\) in each of these points. This conic is the image for \(\varphi|_{H-N_1}|\) of the rational curve \(N_1\).

The cover involution \(\iota_1\) preserves both the class \(H - N_1\) (which is the pull back of the hyperplane section of \(\mathbb{P}^2_{(x_1 : x_2 : x_3)}\)) and the class \(N_2\) (which is sent to the node of the sextic \(C_6\) and thus is preserved by the cover involution). Viceversa, the curve corresponding to \(N_1\) is not preserved by \(\iota_1\) and \(\iota_1^*(N_1) = 2H - 3N_1\) (see Section 4.2.1). So the involution \(\iota_1^*\) on \(NS(X)\) is represented, with respect to the basis \(\{H, N_1, N_2\}\), by the matrix:

\[
\iota_1^* := \begin{bmatrix}
3 & 2 & 0 \\
-4 & -3 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Since the quartic surface \(\mathbb{P}_2\) has two nodes, we can consider two different projections to \(\mathbb{P}^2\): the projection (already considered) from \((1 : 0 : 0 : 0)\) to \(\mathbb{P}^2_{(x_1 : x_2 : x_3)}\), associated to the linear system \(|H - N_1|\), which exhibits \(X\) as double cover of \(\mathbb{P}^2_{(x_1:x_2:x_3)}\) whose cover involution is \(\iota_1\); and the projection from \((0 : 0 : 0 : 1)\) to \(\mathbb{P}^2_{(x_0:x_1:x_3)}\), which is associated to the linear system \(|H - N_2|\). It exhibits \(X\) as double cover of \(\mathbb{P}^2_{(x_0:x_1:x_3)}\) and we call the cover involution \(\iota_2\). The involution \(\iota_2^*\) of \(NS(X)\) is represented with respect to the basis \(\{H, N_1, N_2\}\) by the matrix

\[
\iota_2^* := \begin{bmatrix}
3 & 0 & 2 \\
0 & 1 & 0 \\
-4 & 0 & -3
\end{bmatrix}.
\]

We observe that \(\iota_1\iota_2\) is an automorphism of infinite order of \(X\), indeed the associated isometry of \(NS(X)\) has infinite order and is represented on the basis \(\{H, N_1, N_2\}\) by the matrix

\[
(\iota_1\iota_2)^* := \begin{bmatrix}
9 & 2 & 6 \\
-12 & -3 & -8 \\
-4 & 0 & -3
\end{bmatrix}.
\]

So \(Aut(X)\) is infinite (indeed \(\iota_1\iota_2 \in Aut(X)\)).

The map \(\varphi|_{H-N_1}|\) gives a model of \(X\) which contracts exactly one rational curve, the curve corresponding to the class \(N_2\). Indeed \((H - N_1)^{+NS(X)} \simeq \langle -4 \rangle \oplus \langle -2 \rangle\) which clearly contains exactly two vectors with self intersection \(-2\) and only one of them is effective. We observe that \(\iota_2^*\) does not preserves the class \(H - N_1\), so it does not descend to an automorphism of \(\varphi|_{H-N_1}|(X)\). Vice versa \(\iota_1\) is (by definition) an automorphism of the model associated to \(\varphi|_{H-N_1}|\). The restriction of \(\iota_1^*\) to \(N_2^{+NS(X)}\) \(\simeq NS(Y)\) coincides with the action of \(\iota^*\) (where \(\iota\) is the unique non trivial automorphism of \(Y\)) on \(NS(Y)\), given in \[1.3\].

Denoted by \(X'\) the (singular) double cover of \(\mathbb{P}^2\) branched along \(C_6\), \((X, X')\) is an admissible pair and \(Aut(X')\) is generated by the cover involution. Indeed,
the Mordell–Weil group of \( \phi \) of the fibration

4.3.2. Elliptic fibrations with non trivial Mordell Weil group (lattices \( U \oplus \langle -2d \rangle \)). Here we consider K3 surfaces with Picard number 3 and an elliptic fibration. Since \( U \) is primitively embedded in the Néron–Severi group of any K3 surface admitting an elliptic fibration and it is a unimodular lattice, the Néron–Severi group of a K3 surface with an elliptic fibration and with Picard number 3 is isometric to \( U \oplus \langle -2d \rangle \), \( d \in \mathbb{N}_{>0} \).

We will denote by \( S_d \) a K3 surface such that \( NS(S_d) \simeq U \oplus \langle -2d \rangle \) and we will assume that \( S_d \) is generic among the K3 surfaces with this property. First we discuss the geometric properties of these surfaces \( S_d \) and then we focus on the case \( d = 2 \).

Let us denote by \( \{b_1, b_2, b_3\} \) the basis of \( NS(S_d) \) on which the bilinear form is represented by \( U \oplus \langle -2d \rangle \). Then we put \( F := b_1 \) and \( s_0 := b_2 - b_1 \). The lattice \( F^{1, NS(S_d)} \) consists of the vector \( w := (w_1, w_2, w_3) \) such that \( w_2 = 0 \). The bilinear form computed on \( w \) is \( 2w_1w_2 - 2dw_3^2 \), so \( F^{1, NS(S_d)} \) contains a \( (-2) \)-class if and only if \( d = 1 \).

So if \( d = 1 \), then the elliptic fibration \( \varphi_{|F|} : S_d \to \mathbb{P}^1 \) admits exactly one reducible fiber, which is necessarily of type \( I_2 \), since \( S_1 \) is generic.

If \( d > 1 \), then the elliptic fibration \( \varphi_{|F|} : S_d \to \mathbb{P}^1 \) has no reducible fibers and thus, by Shioda–Tate formula, the rank of the Mordell Weil group is 1. From now on we assume \( d > 1 \) and we denote by \( s_1 \) a section of \( \varphi_{|F|} : S_d \to \mathbb{P}^1 \) which generates the Mordell–Weil group. Hence \( \{F, s_0, s_1\} \) is a basis of \( NS(S_d) \). It is related to the basis \( \{b_1, b_2, b_3\} \) by the following: \( F = b_1, s_0 = b_2 - b_1, s_1 = (d - 1)b_1 + b_2 + b_3 \). It is immediate to check that the bilinear form computed on this basis is

\[
\begin{bmatrix}
0 & 1 & 1 \\
1 & -2 & d - 2 \\
1 & d - 2 & -2 \\
\end{bmatrix}
\]

So the K3 surfaces \( S_d, d > 1 \), can be geometrically described as the K3 surfaces admitting an elliptic fibration such that the rank of the Mordell–Weil group is 1 and the intersection among the zero section and a generator of the Mordell–Weil group is \( d - 2 \).

We observe that the classes \( s_n := (dn^2 - 1)b_1 + b_2 + nb_3, n \in \mathbb{Z} \) are the sections of the fibration \( \varphi_{|F|} : S_d \to \mathbb{P}^1 \) and there is an isomorphism of groups between the Mordell–Weil group of \( \varphi_{|F|} \) and \( \mathbb{Z} \) given by \( s_n \mapsto n \). Fixed a value \( d \), we have \( s_0s_n = dn^2 - 2 \). Moreover \( s_is_{i+n} = s_0s_n \), since \( s_i \) is the translation (by \( s_i \)) of \( s_0 \) and \( s_{i+n} \) is the translation (by \( s_i \)) of \( s_n \). It is immediate to check that \( s_0s_k > s_0s_h \) if \( |k| > |h| \), so \( s_0s_1 \) is the minimal possible intersection number among two sections of \( \varphi_{|F|} : S_d \to \mathbb{P}^1 \).

4.3.3. The K3 surface \( S := S_2 \) \( (NS(S) \simeq L \simeq U \oplus \langle -4 \rangle) \). In case \( d = 2 \), \( NS(S_2) \simeq U \oplus \langle -4 \rangle \). In the following we will denote by \( S \) the surface \( S_2 \), in order to simplify the notation. The K3 surface \( S \) is the generic K3 surface with an elliptic fibration, such that there is a section of infinite order which has a trivial intersection with the zero section.

Our purpose is to describe a map (a geometric model) of \( S \) which contracts exactly one rational curve (indeed we already know that there exists a rational
curve on $S$ such that the orthogonal to the class of this curve in $NS(S)$ is isometric to the lattice $L \cong \langle 4 \rangle \oplus \langle -2 \rangle$, by Section 4.1.

We consider the map $4F + 2s_0$. It exhibits $S$ as double cover of a cone $C \subset \mathbb{P}^5$ over the twisted rational curve of degree 4 in $\mathbb{P}^4$, as in Section 4.2.2. The curve $s_0$ is contracted and it is the unique rational curve which is contracted by this map. Indeed the orthogonal to $4F + 2s_0 = 2b_1 + 2b_2$ in $NS(S_2)$ is generated by $b_2 - b_1 = s_0$ and $b_3$. So, if $r$ is the class of a rational curve in $S_2$ contracted by $4F + 2s_0$, then $r = z_1(b_2 - b_1) + z_2b_3$, $z_1, z_2 \in \mathbb{Z}$; $r^2 = -2$; $rF = rb_1 \geq 0$. This implies that $z_1 = 1$ and $z_2 = 0$, so $r = s_0$.

The family of K3 surfaces which are double covers of $C \subset \mathbb{P}^5$ is 18-dimensional (the K3 surface $V$ described in Section 4.2.2 is a general member of such a family), but $S$ is the general member of a 17-dimensional subfamily. Indeed there is a peculiarity in the branch locus, $\varphi|_{4F + 2s_0}(t_{10})$, of the double cover $\varphi|_{4F + 2s_0} : S \rightarrow C$: there exists a curve which is tangent to $\varphi|_{4F + 2s_0}(t_{10}) \subset C$ in each of their intersection points. Indeed the class $4F + 2s_0$ is equivalent to $s_1 + s_{-1}$ in $NS(S)$. This means that $\varphi|_{4F + 2s_0}(s_1) = \varphi|_{4F + 2s_0}(s_{-1})$ and thus the inverse image of $\varphi|_{4F + 2s_0}(s_1)$ splits in the double cover. So $\varphi|_{4F + 2s_0}(s_1)$ is tangent to $\varphi|_{4F + 2s_0}(t_{10})$ in all their intersection points, which are 6 = $s_1 t_{10}$. We observe that since $s_1 s_0 = 0$, the curve $\varphi|_{4F + 2s_0}(s_1)$ does not pass through the vertex of $C$.

The automorphism group of $S$ is surely infinite and contains at least the following automorphisms: $h$ which is the cover involution of the double cover $S \rightarrow C$ (i.e. it is the hyperelliptic involution of the elliptic fibration $\varphi|_F : S \rightarrow \mathbb{P}^1$) and the automorphism $T_{s_1}$ which is the translation by the section $s_1$. We observe that: $h(F) = F$, $h(s_0) = s_0$, $h(s_n) = s_{s-n}$ (indeed $h$ switches for example the sections $s_1$ and $s_{-1}$) and that $T_{s_1}(F) = F$ and $T_{s_1}(s_n) = s_{n+1}$, $n \in \mathbb{Z}$. With respect to the basis $\{b_1, b_2, b_3\}$ these automorphisms are represented by

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad T_{s_1} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$  

It is immediate to check that the class $4F + 2s_0 = 2b_1 + 2b_2$ is preserved by $h$ but not by $T_{s_1}$, so $T_{s_1}$ does not descend to an automorphism of the singular model $\varphi|_{4F + 2s_0} : S \rightarrow C$. Indeed the class $s_0$, which is the class of the unique curve contracted by $\varphi|_{4F + 2s_0}$, is not preserved by $T_{s_1}$.

The map $\varphi|_{4F + 2s_0}$ contracts $s_0$ and we observe that the lattice $s_0^{\perp_{NS(V^2)}} \simeq M = \langle 4 \rangle \oplus \langle -2 \rangle$ is generated by $2b_1 + 2b_2 + b_3$ and $b_1 + b_2 + b_3$. The involution $h^*$ restricted to the lattice $\langle (2b_1 + 2b_2 + b_3), (b_1 + b_2 + b_3) \rangle$ is represented by the matrix:

$$\begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix},$$

which in fact coincides with the involution $\nu^*$ defined on $M$ by the automorphism $\nu$ of $Y$.

Denoted by $S'$ the (singular) double cover of $C$ branched along $\varphi|_{4F + 2s_0}(t_{10})$, $(S, S')$ is an admissible pair and $\text{Aut}(S')$ is generated by the cover involution. Indeed, $h$ induces the cover involution on $S'$, so $\text{Aut}(S') \supset \mathbb{Z}/2\mathbb{Z}$. By Corollary 3.7 $\text{Aut}(S') \subset \text{Aut}(Y)$ and since $\text{Aut}(Y) = \mathbb{Z}/2\mathbb{Z}$, we deduce that $\text{Aut}(S') = \mathbb{Z}/2\mathbb{Z}$.
5. Other Examples

In this section we briefly describe other geometric examples: We conclude the proof of the Theorem \([\text{\ref{thm:NS2}}]\) in cases \(\rho(X) = 15\) and \(NS(X) \simeq U(2) \oplus A_1^2\), showing the existence of the smooth irreducible rational curve \(N\) which has to be contracted in order to obtain the Mori Dream Space \(X'\). We also give an explicit equation of an elliptic fibration mentioned in the proof of the same theorem.

The principal results of this section are geometric descriptions of the following situations:

- there exists two admissible pairs \((X_1, X_1')\) and \((X_2, X_2')\) such that \(X_1 \simeq X_2\) but \(X_1' \neq X_2'\) (see Example \([\text{\ref{ex:admissible}}]\);
- there exists two infinite series of admissible pairs \((S_d, S_d')\) and \((Q_d, Q_d')\) such that \(\rho(Q_d) = \rho(S_d) = 3\), (the minimal possible), and moreover \(NS(S_d) \neq NS(Q_d)\) but \(NS(S_d') \simeq NS(Q_d')\), see Proposition \([\text{\ref{prop:infseries}}]\).

5.1. The K3 surface with \(NS(X) \simeq U \oplus E_8^2 \oplus A_1^7\). Let \(X\) be the K3 surface such that \(NS(X) \simeq U \oplus E_8^2 \oplus A_1^7\). In the proof of Theorem \([\text{\ref{thm:NS2}}]\) we showed that \(X\) admits a smooth irreducible rational curve \(N\) which can be contracted in order to obtain a Mori Dream Space \(X'\) and that this curve is one of the two components of one of the two fibers of type \(I_2\) of a certain elliptic fibration \(E : X \to \mathbb{P}^1\). The existence of this fibration immediately follows from the decomposition of \(NS(X)\) in the direct sum \(U \oplus E_8^2 \oplus A_1^7\). Here we give also an explicit equation of this fibration.

Let us consider the pencil of plane cubics

\[
V((x_1^2 + x_0^2 x_2 + x_1^2 x_2) + t x_2^3) \subset \mathbb{P}^2_{(x_0:x_1:x_2)}
\]

generated by a triple line \(l\) and a smooth cubic \(C_3\) which admits the line \(l\) as inflectional tangent. It is well known (and easy to check) that this pencil induces an elliptic fibration on the rational surface which is the blow up of \(\mathbb{P}^2\) nine times in the intersection point between \(l\) and \(C_3\). This elliptic fibration has one fiber of type \(II^*\) (which is the pull back of \(l\)) and two fibers of type \(I_1\). With the coordinates \(x_2 = 1\), \(y = x_0\) and \(x = x_1 - 1/3\) we immediately obtain the Weierstrass form

\[
y^2 = x^3 - \frac{1}{3} x + t
\]

which has a fiber of type \(II^*\) at infinity and two fibers of type \(I_1\) in \(t = \pm 2/27\). So the double cover \(\mathbb{P}^1_{(T,S)} \to \mathbb{P}^1_{(t,s)}\) which is branched in \(t = \pm \frac{1}{27}\), induces the base change \(s := \frac{27}{4}(S^2 - T^2)\) and \(t := \frac{1}{2}(S^2 + T^2)\) which gives the elliptic fibration

\[
y^2 = x^3 - \frac{1}{3} x \left( \frac{27}{4} S^2 - \frac{27}{4} T^2 \right)^4 + \left( \frac{1}{2} S^2 + \frac{1}{2} T^2 \right) \left( \frac{27}{4} S^2 - \frac{27}{4} T^2 \right)^5.
\]

This is in fact the equation of the unique (up to projective transformation) elliptic fibration over \(\mathbb{P}^1_{(T,S)}\) with reducible fibers \(2II^* + 2I_2\) and so it is an equation of (a singular model of) the unique K3 surface \(X\) with \(NS(X) \simeq U \oplus E_8^2 \oplus A_1^7\). The fibers over \((1:0)\) and \((0:1)\) are of type \(I_2\), so each of them consists of two smooth irreducible rational curves meeting in 2 points. One of these rational curves meets the zero section of the fibration. The other can be chosen to be the curve \(N\).

5.2. K3 surfaces with \(NS(X) \simeq U(2) \oplus A_1^7\). We give a geometric description of the K3 surface \(X\), which is generic among the K3 surfaces such that \(NS(X) \simeq U(2) \oplus A_1^8\). We will show that it surely contains a smooth irreducible rational curve
which can be contracted in order to obtain a singular surface $X'$ whose Néron–Severi group is isometric to $U(2) \oplus A_1^7$. This concludes the proof of Theorem 3.4 in case $NS(X) \simeq U(2) \oplus A_1^8$.

Since there exists a unique even hyperbolic lattice $L$ such that: $\text{rank}(L) = 10$, the discriminant group is $(Z/2Z)^{10}$ and the discriminant form takes values in $\frac{1}{2}Z$ (and not in $Z$), we find $U(2) \oplus A_1^8 \simeq (2) \oplus (-2)^9$. So $NS(X) \simeq (2) \oplus (-2)^9$. We can assume that one of the primitive generators of the sublattice $(2) \hookrightarrow NS(X)$ is pseudo ample. We denote this divisor by $A$ and we observe that $\phi_{|A|} : X \to \mathbb{P}^2$ exhibits $X$ as double cover of $\mathbb{P}^2$ branched along an irreducible sextic $C_6$ with 9 nodes $P_1, \ldots, P_9$. The double cover of the blow up of $\mathbb{P}^2$ in these 9 points is a smooth minimal model of $X$ and it contains 9 smooth rational curves which are the double cover of the 9 exceptional divisors. The classes of these rational curves are represented by $(-2)$-classes orthogonal to $A$ and mutually orthogonal. So $X$ admits at least one smooth rational curve (indeed at least 9) such that the contraction of this curve produces a singular surface $X'$ whose Néron–Severi group is isometric to $(2) \oplus (-2)^8 \simeq U(2) \oplus A_1^7$. A geometric construction of $X'$ is the following: let us consider the irreducible sextic curve $C_6$ with 9 nodes $P_1, \ldots, P_9$ such that $\phi_{|A|} : X \to \mathbb{P}^2$ is branched along $C_6$. Let us blow up $\mathbb{P}^2$ in the eight points $P_1, \ldots, P_8$ and let us denote by $\widetilde{\mathbb{P}^2}$ the surface obtained by this blow up. Let us denote by $C_6$ the strict transform of $C_6$ for this blow up. We observe that $C_6$ has exactly one singular point. The double cover of $\widetilde{\mathbb{P}^2}$ branched along $C_6$ is $X'$, indeed the following diagram commute:

\[
\begin{array}{ccc}
X' & \xrightarrow{2:1} & \mathbb{P}^2 \\
\phi \downarrow & & \downarrow \phi \\
X & \xrightarrow{2:1} & \mathbb{P}^2
\end{array}
\]

where $\widetilde{\mathbb{P}^2}$ is the blow up of $\mathbb{P}^2$ in the unique singular point of $C_6$ and it coincides with the blow up of $\mathbb{P}^2$ in the nine points $P_1, \ldots, P_9$ and $\phi : X \to X'$ is the contraction of the smooth rational curve of $X$ which is the double cover of the exceptional divisor of the blow up $\widetilde{\mathbb{P}^2} \to \mathbb{P}^2$.

5.3. Two admissible pairs $(X, X'_1)$ and $(X, X'_2)$ with $X'_1 \not\cong X'_2$ and $\rho(X) = 15$.

We consider a K3 surface $X$ whose Néron–Severi group is isometric to $U \oplus D_8 \oplus D_4 \oplus A_1$, so it is not a Mori Dream Space (cf. Proposition 3.4 and [Kon]). We show that it admits two different rational curves $N_1$ and $N_2$ such that, denoted by $X'_i$ the singular surface contraction of the curve $N_i$, the two pairs $(X, X'_1)$ and $(X, X'_2)$ are both admissible and $X'_1 \not\cong X'_2$. This gives geometric examples of the cases with $\rho(X) = 15$ of the Table 3.1 and concludes the proof of Theorem 3.4.

Example 5.1. Let $X$ be a K3 surface admitting an elliptic fibration $\mathcal{E} : X \to \mathbb{P}^1$ whose singular fibers are $I_4 + I_6 + I_2 + 6I_1$ and whose Mordell–Weil group is trivial. The Néron–Severi group of the surface is isomorphic to $U \oplus D_8 \oplus D_4 \oplus A_1$ and is generated by the classes $F, s_0, \Theta_i^{(1)}$, $i = 1, \ldots, 8$, $\Theta_j^{(2)}$, $j = 1, 2, 3, 4$, $\Theta_i^{(3)}$, where $F$ is the class of the fiber of the fibration $\mathcal{E}$, $s_0$ is the class of the unique section of the fibrations, $\Theta_i^{(1)}$ are the eight components of the fibers of type $I_4$ which do not intersect the zero section, $\Theta_j^{(2)}$ are the 4 components of the fiber of type $I_6$ which
Let $E$ be an elliptic fibration such that $NS(E) \simeq U \oplus D_8 \oplus D_4$. In the following we will assume that the intersection properties of the components of the fibers of type $I_2^*$ are numbered as follow:

Let us consider the divisor $D := 3\Theta_1 + 2\Theta_2 + 4\Theta_3 + 5\Theta_4 + 6\Theta_5 + 4\Theta_0 + 2\Theta_2 + 3\Theta_3$ on $X$.

We observe that $D^2 = 0$ and $D$ is an effective divisor. Moreover the curve $\Theta_6$ is a rational curve such that $D\Theta_6 = 1$. Thus, the linear system $|D|$ defines an elliptic fibration $\phi_1 : X \to \mathbb{P}^1$. The class of $D$ is the class of the fiber of the fibration and the divisor $D$ exhibits a reducible fiber of type $II^*$ of this fibration. Let us denote by $R$ the lattice $(\Theta_0, \Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, s_0, \Theta_6, 2\Theta_2, 3\Theta_3, 4\Theta_0)$. Then $R \simeq U \oplus E_8$ and the orthogonal complement of $R$ in $NS(X)$ is isometric to $A_1^5$ (since the discriminant group of $NS(X)$ is $(\mathbb{Z}/2\mathbb{Z})^5$). So the reducible fibers of the elliptic fibration induced by $|D|$ are $II^* + 5I_2$. In particular the class $\Theta_0^{(1)} \Theta_0^{(1)} + 2s_0 + 2\Theta_0^{(2)} + 2\Theta_3^{(2)} + \Theta_0^{(3)}$ is the class of an effective $(-2)$-curve orthogonal to $R$ (and in fact a bisection of the elliptic fibration $E$) and so it is the class of one of the components of one of the fibers of type $I_2$. The map which contracts exactly this curve produces a surface $X_2$ which is singular and whose Néron–Severi group is $U \oplus E_8 \oplus A_1^1$.

5.4. Infinite admissible pairs $(S_d, S'_d)$ with $\rho(S_d) = 3$. Let $S_d$ denote a generic K3 surface admitting an elliptic fibration $E_d : S_d \to \mathbb{P}^1$ such that $MW(E_d) = (s_1)$ and $s_0s_1 = d - 2$ (described in Section 4.3.2).

**Proposition 5.2.** Let $\phi : S_d \to S'_d$ be the contraction of the curve $s_1$. Then $NS(S'_d) \simeq (2) \oplus (-2d)$.

If $d$ is a square, then $(S_d, S'_d)$ is an admissible pair. In particular this gives an infinite number of admissible pairs such that the Picard number of the K3 surface is 3 (the minimal possible).

Moreover, for almost all the $d$ such that $d$ is a square, the pair $(Q_d, Q'_d)$ is also an admissible if $NS(Q'_d) \simeq NS(S'_d)$ and $NS(Q_d) \simeq (2) \oplus (-2d) \oplus (-2)$. So we have an infinite number of admissible pairs $(S_d, S'_d)$ and $(Q_d, Q'_d)$ such that $NS(S_d) \not\simeq NS(Q_d)$ and $NS(S'_d) \simeq NS(Q'_d)$.

If $d$ is not a square and $d \equiv 3 \pmod{4}$ or $d \equiv 0 \pmod{4}$, then $S'_d$ is not a Mori Dream Space, so $(S_d, S'_d)$ is not an admissible pair for infinitely many values of $d$.

**Proof.** We already observed that $S_d$ is not a Mori Dream Space, since the translation by the section $s_1$ is an automorphism of infinite order of $S_d$. 

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Let us denote by $\Theta_1, \Theta_2, \ldots, \Theta_n$ the points of $X$. Let $\Theta_1 := \Theta_{1}^{(3)}$ be the component of the fiber of type $I_2$ which does not intersect the zero section. The map $\phi : X \to X'_1$ which contracts the curve $N_1 := \Theta_{1}^{(3)}$ produces the singular surface $X'_1$ such that $NS(X'_1) \simeq U \oplus D_8 \oplus D_4$.

Furthermore, for almost all the $d$ such that $d$ is a square, the pair $(Q_d, Q'_d)$ is also an admissible if $NS(Q'_d) \simeq NS(S'_d)$ and $NS(Q_d) \simeq (2) \oplus (-2d) \oplus (-2)$. So we have an infinite number of admissible pairs $(S_d, S'_d)$ and $(Q_d, Q'_d)$ such that $NS(S_d) \not\simeq NS(Q_d)$ and $NS(S'_d) \simeq NS(Q'_d)$.

If $d$ is not a square and $d \equiv 3 \pmod{4}$ or $d \equiv 0 \pmod{4}$, then $S'_d$ is not a Mori Dream Space, so $(S_d, S'_d)$ is not an admissible pair for infinitely many values of $d$. 

**Proof.** We already observed that $S_d$ is not a Mori Dream Space, since the translation by the section $s_1$ is an automorphism of infinite order of $S_d$. 

The lattice \((s_1^{\perp}N S_{d})\) is generated by \(\langle 2F + s_1, -dF + s_0 - s_1 \rangle \simeq \langle 2 \rangle \oplus \langle -2d \rangle\). The surface \(S'_d\) is a Mori Dream Space if and only if the K3 surface \(Y_d\) with Néron–Severi group isometric to \(\langle 2 \rangle \oplus \langle -2d \rangle\) is a Mori Dream Space. Since the rank of the lattice is 2, we know that the K3 surface \(Y_d\) is a Mori Dream Space if and only if the lattice \(\langle 2 \rangle \oplus \langle -2d \rangle\) represents 0 or \(-2\).

The quadratic form associated to \(\langle 2 \rangle \oplus \langle -2d \rangle\) is \(2x^2 - 2dy^2\). The form represents the zero if and only if there exists \((x, y) \in \mathbb{Z}^2\) such that \(2x^2 - 2dy^2 = 0\) and represents \(-2\) if and only if \(2x^2 - 2dy^2 = -2\).

If \(d\) is a square, then there exists \(b \in \mathbb{Z}\) such that \(d = b^2\) and it suffices to chose \((x, y) = (b, 1)\). So if \(d\) is a square, then the quadratic form represents zero which implies that \(S'_d\) is a Mori Dream Space.

Viceversa if \(d\) is not a square, then the quadratic form does not represent 0. Let us assume that the quadratic form represents \(-2\). So there exists \((x, y) \in \mathbb{Z}^2, x^2 - dy^2 = -1\). Let us consider this equation modulo 4 (where the square are either 0 or 1). We have the following possible values for \((x^2, y^2)\) modulo 4, \((0, 0)\), \((0, 1)\), \((1, 0)\), \((1, 1)\). The choices \((0, 0)\) and \((1, 0)\) give a contradiction. So either \((x^2, y^2) \equiv (0, 1) \mod 4\) and in this case \(d \equiv 1 \mod 4\) or \((x^2, y^2) \equiv (1, 1) \mod 4\) and in this case \(d \equiv 2 \mod 2\). Therefore, if \(d \equiv 0 \mod 4\) or \(d \equiv 3 \mod 4\), then the quadratic form does not represent \(-2\). If moreover \(d\) is not a square, the quadratic form does not represent also 0. Thus \(S'_d\) is not a Mori Dream Space.

If \(d\) is a square, then the K3 surface \(Y_d\) is a Mori Dream Space. As a consequence both \(S'_d\) and \(Q'_d\) are Mori Dream Space. Hence the pair \((Q_d, Q'_d)\) is an admissible pair if and only if \(Q_d\) is not a Mori Dream Space. Since there are exactly 27 hyperbolic lattices \(L\) of rank 3 such that if the Néron–Severi group of a K3 surface is isometric to one of these lattices, the K3 surface has a finite automorphism group and so is a Mori Dream Space, we conclude that for almost all squares \(d\), \(Q_d\) is not a Mori Dream Space and \((Q_d, Q'_d)\) is an admissible pair.

\[\square\]

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