Hilbert Polynomials of Kähler Differential Modules for Fat Point Schemes

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Abstract Given a fat point scheme $W = m_1P_1 + \cdots + m_sP_s$ in the projective $n$-space $\mathbb{P}^n$ over a field $K$ of characteristic zero, the modules of Kähler differential $k$-forms of its homogeneous coordinate ring contain useful information about algebraic and geometric properties of $W$ when $k \in \{1, \ldots, n+1\}$. In this paper we determine the value of its Hilbert polynomial explicitly for the case $k = n+1$, confirming an earlier conjecture. More precisely this value is given by the multiplicity of the fat point scheme $Y = (m_1-1)P_1 + \cdots + (m_s-1)P_s$. For $n = 2$, this allows us to determine the Hilbert polynomials of the modules of Kähler differential $k$-forms for $k = 1, 2, 3$, and to produce a sharp bound for the regularity index for $k = 2$.

1 Introduction

Let $X = \{P_1, \ldots, P_s\}$ be a set of points in projective $n$-space $\mathbb{P}^n$ over a field $K$ of characteristic zero, and let $I_X$ be the homogeneous vanishing ideal of $X$ in $S = K[X_0, \ldots, X_n]$. One important reason why the Hilbert function of $I_X$ has been studied extensively is that the elements of $(I_X)_d$ control the non-uniqueness of the solution of the homogeneous polynomial interpolation problem in degree $d$. When

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we switch to the Hermite interpolation problem, this non-uniqueness is controlled by the set of all polynomials having the property that not only their values, but also their derivatives up to some order \( m \), vanish at the point \( P_i \) for \( i = 1, \ldots, s \).

In the language of algebraic geometry, this means that we are interested in the Hilbert function of the homogeneous vanishing ideal \( I_\mathcal{W} = I_{P_1} \cap \cdots \cap I_{P_s} \) of the fat point scheme \( \mathcal{W} = m_1 P_1 + \cdots + m_s P_s \) in \( \mathbb{P}^n \). Such schemes have arisen in several contexts, for instance in the study of projective varieties which are obtained from blowing up sets of points in \( \mathbb{P}^n \) (see \[1\]).

The Hilbert function of the ideals \( I_\mathcal{W} \), or equivalently, of the homogeneous coordinate rings \( R_\mathcal{W} = S/I_\mathcal{W} \) have undergone intense scrutiny in the past. A major breakthrough was the paper \[1\] in which Alexander and Hirshowitz determined the Hilbert function of fat point schemes consisting of double points (i.e., with \( m_1 = \cdots = m_s = 2 \)) for the case of a generic support \( \mathcal{X} \). For this, they used a local differential method which they called \textit{La methode d’Horace}. It is therefore a natural approach to study also global differentials for fat point schemes. They are given by the module of Kähler differentials \( \Omega^k_{R_\mathcal{W}/K} \) and its exterior powers, the modules of Kähler differential \( k \)-forms \( \Omega^k_{R_\mathcal{W}/K} \) for \( k \geq 1 \). In \[3\], De Dominics and the first author initiated a careful examination of the structure and the Hilbert function of \( \Omega^1_{R_\mathcal{W}/K} \) for a set of points \( \mathcal{X} \) (i.e., for the case \( m_1 = \cdots = m_s = 1 \)). Then, in \[8\], the present authors started the study of the modules of Kähler differential \( k \)-forms for fat point schemes, and in \[9\] this work was continued.

A major open question in \[9\] is a formula for the Hilbert polynomial of \( \Omega^{n+1}_{R_\mathcal{W}/K} \), i.e., for the value of the Hilbert function in large degrees, which is given as a conjecture there. Our main result here is to prove this formula. More precisely, in Theorem 3.7 we show that, for a fat point scheme \( \mathcal{W} = m_1 P_1 + \cdots + m_s P_s \), the Hilbert polynomial of \( \Omega^{n+1}_{R_\mathcal{W}/K} \) is equal to the multiplicity of the slimming \( \mathcal{Y} = (m_1 - 1)P_1 + \cdots + (m_s - 1)P_s \) of \( \mathcal{X} \), and is therefore given by \( \sum_{i=1}^s \binom{m_i + n - 2}{n} \).

To achieve this goal, we proceed as follows. After recalling the definition and some basic results for fat point schemes in Section 2, we compare the Hilbert functions of a fat point scheme \( \mathcal{W} = m_1 P_1 + \cdots + m_s P_s \), its support \( \mathcal{X} = P_1 + \cdots + P_s \), and its slimming \( \mathcal{Y} = (m_1 - 1)P_1 + \cdots + (m_s - 1)P_s \). More precisely, we show in Proposition 2.6 that the Hilbert functions of \( I_\mathcal{W} \) and of \( I_\mathcal{Y} \) agree in sufficiently large degrees. In the last part of this section we recall the definition of the module of Kähler differential \( k \)-forms of \( R_\mathcal{W}/K \) and recall some of its basic properties.

Section 3 contains the main results of this paper. First we show that, for a set of points \( \mathcal{X} \) in \( \mathbb{P}^n \) and a subset \( Y \) of \( \mathcal{X} \), the vanishing ideals satisfy \( \binom{I_k^2 \cdot I_k^2}{i} \subseteq (\partial (I_k^{i+1} \cdot I_k^1)) \) for all \( k, \ell \geq 0 \) in sufficiently large degrees \( i \gg 0 \) (see Lemma 3.3). Then we use a substitute induction argument to prove that, for a sequence of point sets \( \mathcal{Y}_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_t \) in \( \mathbb{P}^n \) and for \( n_1, \ldots, n_t > 0 \), we have \( \binom{I_{n_1} \cdots I_{n_t}}{i} \subseteq (\partial (I_{n_{i+1}} \cdot I_{n_{i+1}} \cdots I_{n_{i+1}})) \) in sufficiently large degrees \( i \gg 0 \) (see Proposition 3.5). Using the presentation \( \Omega^{n+1}_{R_\mathcal{W}/K} \equiv (S/\partial I_\mathcal{W})/(n-1) \) and starting from the equimultiple case \( m_1 = \cdots = m_s \), we then prove inductively Theorem 3.7 which says that the Hilbert function of the module of Kähler \( (n+1) \)-forms of a fat point scheme \( \mathcal{W} = m_1 P_1 + \cdots + m_s P_s \) is given in large degrees by the Hilbert function of its slimming \( \mathcal{Y} \). On other words, we have \( \text{HP}(\Omega^{n+1}_{R_\mathcal{W}/K}) = \sum_{i=1}^s \binom{m_i + n - 2}{n} \).

The final section is devoted to applying this theorem in the case of fat point schemes in the plane \( \mathbb{P}^2 \). In this case we determine the Hilbert polynomials of \( \Omega^k_{R_\mathcal{W}/K} \) for all three relevant cases \( k = 1, 2, 3 \) (see Proposition 4.1) and provide a
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bound from where on the canonical exact sequence

\[ 0 \rightarrow (I_W^{(1)}/I_W^{(2)})_i \xrightarrow{\alpha} (I_W^1/K/I_W^{(1)})_i \xrightarrow{\beta} (I_W^2/K/I_W^{(2)})_i \xrightarrow{\gamma} (\Omega^2_{R_W}/K)_i \rightarrow 0. \]

is exact in degree \( i \) (see Proposition 4.3). Here \( W^{(1)} \) and \( W^{(2)} \) are the first and second fattening

\[ W^{(j)} = (m_1 + j)P_1 + \cdots + (m_s + j)P_s \]

of \( W \). Using an explicit example, we certify that this result yields a sharp bound for the regularity index

\[ \text{Reg}_W \]

of \( \Omega^2_{R_W}/K \).

The research underlying this paper and the calculation of the examples were greatly aided by an implementation of the relevant objects and functions in the computer algebra system ApCoCoA (see [2]). Unless explicitly stated otherwise, we adhere to the definitions and notation introduced in the books [10] and [11], as well as in our previous papers [8] and [9].

2 Kähler Differentials for Fat Point Schemes

Throughout this paper we work over a field \( K \) of characteristic zero, and we let \( S = K[X_0, \ldots, X_n] \) be a standard graded polynomial ring over \( K \). By \( \mathfrak{M} \) we denote the homogeneous maximal ideal \( \langle X_0, \ldots, X_n \rangle \) of \( S \). The ring \( S \) is the homogeneous coordinate ring of the projective \( n \)-space \( \mathbb{P}^n \) over \( K \).

Let \( P_1, \ldots, P_s \) be distinct \( K \)-rational points in \( \mathbb{P}^n \). The prime ideals in \( S \) corresponding to the points \( P_1, \ldots, P_s \) will be denoted by \( I_{P_1}, \ldots, I_{P_s} \), respectively.

Definition 2.1 Let \( m_1, \ldots, m_s \) be positive integers.

(a) A zero-dimensional scheme \( W \) in \( \mathbb{P}^n \) is called a fat point scheme if it is defined by a saturated ideal of the form \( I_W = I_{P_1}^{m_1} \cap \cdots \cap I_{P_s}^{m_s} \). In this case, we also write

\[ W = m_1 P_1 + \cdots + m_s P_s. \]

(b) The number \( m_j \) is called the multiplicity of the point \( P_j \) for \( j = 1, \ldots, s \).

(c) If \( m_1 = \cdots = m_s = m \), we refer to \( W \) as an equimultiple fat point scheme and denote it also by \( mX \).

(d) The set of points \( \text{Supp}(W) := \{ P_1, \ldots, P_s \} \) is called the support of \( W \). Subsequently, we also write \( \text{Supp}(W) = P_1 + \cdots + P_s \).

It is well known that the homogeneous coordinate ring \( R_W := S/I_W \) of \( W \) is a one-dimensional, Cohen-Macaulay, standard graded \( K \)-algebra. The Hilbert function \( \text{HF}_{R_W}(i) := \dim_K(R_W)_i \) of \( R_W \) will be denoted by \( \text{HF}_W \). Note that \( \text{HF}_W \) is strictly increasing until it reaches the degree \( \deg(W) = \sum_{i=1}^s (m_i + n - 1) \) of \( W \) at which it stabilizes. The least integer \( i \) for which \( \text{HF}_W(i) = \deg(W) \) is called the regularity index of \( \text{HF}_W \) and is denoted by \( r_W \). In this setting, the Hilbert polynomial of \( \text{HF}_W \) is the constant polynomial given by \( \text{HP}_W(z) = \deg(W) \).
Definition 2.2 Let $\mathcal{W} = m_1P_1 + \cdots + m_sP_s$ be a fat point scheme in $\mathbb{P}^n$, let $j \in \{1, \ldots, s\}$, and let $\mathcal{W}_j \subseteq \mathcal{W}$ be the fat point scheme

\[
\mathcal{W}_j = m_1P_1 + \cdots + m_{j-1}P_{j-1} + (m_j - 1)P_j + m_{j+1}P_{j+1} + \cdots + m_sP_s
\]

obtained by reducing the multiplicity of $P_j$ by one. If $m_j = 1$, then $P_j$ does not appear in the support of $\mathcal{W}_j$. Further, let $\nu_j = \deg(\mathcal{W}) - \deg(\mathcal{W}_j)$.

(a) An element $F \in I_{\mathcal{W}_j} \setminus I_{\mathcal{W}}$ is called a separator of $\mathcal{W}_j$ in $\mathcal{W}$.

(b) A set of homogeneous polynomials $\{F_1, \ldots, F_t\}$ is called a minimal set of separators of $\mathcal{W}_j$ in $\mathcal{W}$ if $t = \nu_j$ and $I_{\mathcal{W}_j} = I_{\mathcal{W}} + \langle F_1, \ldots, F_t \rangle$.

According to [3, Theorem 3.3], a minimal set of separators of $\mathcal{W}_j$ in $\mathcal{W}$ always exists. Some basic properties of separators are described by the following lemma which follows from [6, Lemma 5.1].

Lemma 2.3 In the setting of Definition 2.2, let $\{F_1, \ldots, F_t\}$ be a minimal set of separators of $\mathcal{W}_j$ in $\mathcal{W}$, and suppose that $\deg(F_1) \leq \cdots \leq \deg(F_t)$.

(a) For $k = 1, \ldots, t$, we have $(I_{\mathcal{W}} + \langle F_1, \ldots, F_{k-1} \rangle) : \langle F_k \rangle = I_{F_k}$.

(b) For $k = 1, \ldots, t$, the ideal $I_{\mathcal{W}} + \langle F_1, \ldots, F_k \rangle$ is a saturated ideal.

By using separators, we want to figure out a connection between the homogeneous vanishing ideals of $\mathcal{W}$, of the support $X := \text{Supp}(\mathcal{W})$, and of the fat point scheme $Y := (m_1 - 1)P_1 + \cdots + (m_s - 1)P_s$ which we call the slimming of $\mathcal{W}$. First we have the following relation between the intersection and the product of two homogeneous ideals in $S$.

Lemma 2.4 Let $I, J$ be two homogeneous ideals of $S$. Then the following statements hold true.

(a) The graded module $M = (I \cap J)/(I \cdot J)$ is annihilated by $I + J$.

(b) If the Hilbert polynomial of $S/(I + J)$ satisfies $\text{HP}_{S/(I + J)}(z) = 0$, then we have the equality $\text{HP}_{S/(I \cap J)}(z) = \text{HP}_{S/I}(z)$.

In particular, we have $(I \cap J)_i = (I \cdot J)_i$ for $i \gg 0$.

Proof Claim (a) follows from the fact that for $f \in I$, $g \in J$ and $h \in I \cap J$ we have $(f + g)h = fh + gh \in I \cdot J$.

Now we prove (b). Since we have $\text{HP}_{S/(I + J)}(z) = 0$, there exists $i_0 \in \mathbb{N}$ such that $(I + J)^i \subseteq \mathfrak{M}_i$, for all $i \geq i_0$. It follows from (a) that $I + J \subseteq \text{Ann}_S(M)$, and hence $(\text{Ann}_S(M))_i = \mathfrak{M}_i$ for all $i \geq i_0$. This implies $\dim(S/\text{Ann}_S(M)) = 0$. Using [11, Theorem 5.4.10], we then get $\dim(M) = \dim(S/\text{Ann}(M)) = 0$. Consequently, by [11, Theorem 5.15], the Hilbert polynomial of $M$ is $\text{HP}_M(z) = 0$. Thus (b) follows from the homogeneous exact sequence

\[
0 \rightarrow M \rightarrow S/(I \cdot J) \rightarrow S/(I \cap J) \rightarrow 0 \square
\]

Remark 2.5 Let $\mathcal{W} = m_1P_1 + \cdots + m_sP_s$ and $\mathcal{V} = m'_1P'_1 + \cdots + m'_sP'_s$ be fat point schemes in $\mathbb{P}^n$ such that $\text{Supp}(\mathcal{W}) \cap \text{Supp}(\mathcal{V}) = \emptyset$. According to [11, Proposition 5.4.16], we have

\[
\text{HP}_{S/(I_{\mathcal{W}} \cap I_{\mathcal{V}})}(z) = \text{HP}_{S/I_{\mathcal{W}}}(z) + \text{HP}_{S/I_{\mathcal{V}}}(z) - \text{HP}_{S/(I_{\mathcal{W}} + I_{\mathcal{V}})}(z).
\]

The assumption yields that $\text{HP}_{S/(I_{\mathcal{W}} \cap I_{\mathcal{V}})}(z) = \text{HP}_{S/I_{\mathcal{W}}}(z) + \text{HP}_{S/I_{\mathcal{V}}}(z)$, and so we get $\text{HP}_{S/(I_{\mathcal{W}} + I_{\mathcal{V}})}(z) = 0$. In this case the lemma implies $\text{HP}_{S/(I_{\mathcal{W}} + I_{\mathcal{V}})}(z) = \text{HP}_{S/(I_{\mathcal{W}} + I_{\mathcal{V}})}(z)$. 

\[
\text{HP}_{S/(I_{\mathcal{W}} \cap I_{\mathcal{V}})}(z) = \text{HP}_{S/I_{\mathcal{W}}}(z) + \text{HP}_{S/I_{\mathcal{V}}}(z) - \text{HP}_{S/(I_{\mathcal{W}} + I_{\mathcal{V}})}(z).
\]
Proposition 2.6 Let $\mathcal{W} = m_1 P_1 + \cdots + m_s P_s$ be a fat point scheme supported at $X$ in $\mathbb{P}^n$, and let $\mathcal{Y}$ be the slimming $\mathcal{Y} = (m_1 - 1) P_1 + \cdots + (m_s - 1) P_s$ of $\mathcal{W}$.

(a) We have $I_{\mathcal{W}} : S I_{\mathcal{Y}} = I_X$.
(b) There exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$ we have $(I_{\mathcal{W}})_i = (I_X \cdot I_{\mathcal{Y}})_i$.

Proof (a) Clearly, we have $I_X \cdot I_{\mathcal{Y}} \subseteq I_{\mathcal{W}}$, and so $I_X \subseteq I_{\mathcal{W}} : S I_{\mathcal{Y}}$. For the other inclusion, let $\nu_j = \binom{m_j + n - 2}{n-1}$ and let $\{F_j, \ldots, F_{\nu_j}\}$ be a minimal set of separators of $\mathcal{W}$ in $\mathcal{Y}$ such that $\deg(F_{j1}) \leq \cdots \leq \deg(F_{\nu_j})$, where $\mathcal{W}_j = m_1 P_1 + \cdots + m_{j-1} P_{j-1} + (m_j - 1) P_j + m_{j+1} P_{j+1} + \cdots + m_s P_s$. Then, by Lemma 2.3, we have

$$I_{\mathcal{Y}} = I_{\mathcal{W}} + \langle F_{11}, \ldots, F_{1\nu_1}, \ldots, F_{s1}, \ldots, F_{s\nu_s} \rangle.$$ 

Suppose for a contradiction that there exists a homogeneous element $F \in (I_{\mathcal{W}} : S I_{\mathcal{Y}}) \setminus I_X$. Since $F \notin I_X$, there exists $j \in \{1, \ldots, s\}$ such that $F \notin I_{P_j}$. On the other hand, we have $F \cdot F_j \in I_{\mathcal{W}}$ for all $j = 1, \ldots, s$ and $k = 1, \ldots, \nu_j$. In particular, we get $F \in I_{\mathcal{W}} : S (F_j)$. Also, by Lemma 2.3, the separator $F_j$ satisfies $I_{\mathcal{W}} : S (F_j) \subseteq I_{P_j}$. Hence we obtain $F \in I_{P_j}$, a contradiction.

(b) Note that $(I_X \cdot I_{\mathcal{Y}})_i \subseteq (I_{\mathcal{W}})_i$ for all $i \in \mathbb{N}$. Set $V := m_1 P_1 + \cdots + m_{s-1} P_{s-1}$. It follows from Remark 2.4 that

$$\text{HP}_{S/(I_{\mathcal{W}}+I_{m_1 P_1})}(z) = 0.$$ 

An application of Lemma 2.3 yields that there exists $t \in \mathbb{N}$ such that for all $i \geq t$ we have $(I_{\mathcal{W}})_i = (I_X \cdot I_{m_1 P_1})_i$. By induction on $s$ we find $i_0 \in \mathbb{N}$ such that

$$(I_{\mathcal{W}})_i = (I_X \cdot I_{m_1 P_1})_i = (I_{m_1 P_1} \cdots I_{m_{s-1} P_{s-1}} \cdot I_{m_s P_s})_i$$

$$= (I_{m_1 P_1} \cdots I_{m_s P_s})_i$$

for all $i \geq i_0$, where the equality (*) follows from the fact that $I_{P_j}$ is a complete intersection ideal for $j = 1, \ldots, s$. Moreover, observe that $I_{P_1} \cdots I_{P_s} \subseteq I_X$ and $I_{P_1}^{m_1-1} \cdots I_{P_s}^{m_s-1} \subseteq I_X$. This implies $(I_{\mathcal{W}})_i \subseteq (I_X \cdot I_{\mathcal{Y}})_i$, and therefore we get the equality $(I_{\mathcal{W}})_i = (I_X \cdot I_{\mathcal{Y}})_i$.

Notice that if $\mathcal{W}$ is an equimultiple fat point scheme in $\mathbb{P}^n$ such that its support $X$ is a complete intersection, then claim (b) of Proposition 2.6 holds true for $i_0 = 0$. However, when $\mathcal{W}$ is not an equimultiple fat point scheme and $X$ is a complete intersection, the following example shows that the number $i_0$ in Proposition 2.6 should be chosen large enough.

Example 2.7 Let $X \subseteq \mathbb{P}^2$ be the scheme $X = P_1 + P_2 + \cdots + P_8$ consisting of eight points given by $P_1 = (1 : 0 : 0)$, $P_2 = (1 : 0 : 1)$, $P_3 = (1 : 1 : 0)$, $P_4 = (1 : 1 : 1)$, $P_5 = (1 : 2 : 0)$, $P_6 = (1 : 2 : 1)$, $P_7 = (1 : 3 : 0)$ and $P_8 = (1 : 3 : 1)$. The homogeneous vanishing ideal of $X$ is given by

$$I_X = \langle X_0 X_2 - X_2^2, 6X_0^3 X_1 - 11X_0^2 X_1^2 + 6X_0 X_1^3 - X_1^4 \rangle \subseteq S = K[X_0, X_1, X_2],$$

and so $X$ is a complete intersection of type $(2, 4)$. 
Now we consider the fat point scheme $\mathcal{W} = P_3 + 2P_2 + P_3 + 2P_4 + 2P_5 + P_6 + 5P_7 + P_8$ supported at $X$. Let the subscheme $\mathcal{Y}$ of $\mathcal{W}$ be given by $\mathcal{Y} = P_2 + P_3 + P_5 + 4P_7$.

A calculation using ApCoCoA (see [2]) yields the Hilbert functions

$$HF_X : 1 \ 3 \ 5 \ 7 \ 8 \ 8 \ \cdots,$$
$$HF_Y : 1 \ 3 \ 6 \ 10 \ 13 \ 13 \ \cdots,$$
$$HF_{\mathcal{W}} : 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 26 \ 27 \ 28 \ 28 \ \cdots.$$ 

This implies that $HF_{\mathcal{W}}(7) = 27 < 28 = HF_{S/(I_X I_Y)}(7)$. Hence we get $I_{\mathcal{W}} \neq I_X \cdot I_Y$.

But we may check that $(I_{\mathcal{W}})_i = (I_X \cdot I_Y)_i$ for all $i \geq 8$.

The next example shows that, if the support of an equimultiple fat point scheme $\mathcal{W}$ is not a complete intersection, then Proposition 2.7(b) does not always hold for all $i \in \mathbb{N}$.

**Example 2.8** Let us consider the set $X' = X \setminus \{P_4\} \subseteq \mathbb{P}^2$ where $X$ is the complete intersection given in Example 2.7. Then $X'$ is an almost complete intersection with

$$I_{X'} = I_X + \langle X_3^2 X_2 - 5 X_1^2 X_2^2 + 6 X_1 X_2^3 \rangle \subseteq S = K[X_0, X_1, X_2].$$

The double point scheme $\mathcal{W}' = 2P_1 + 2P_2 + 2P_3 + 2P_5 + 3P_6 + 3P_7 + 3P_8 \subseteq \mathbb{P}^2$ supported at $X'$ has the Hilbert function $HF_{\mathcal{W}'} : 1 \ 3 \ 6 \ 10 \ 14 \ 18 \ 20 \ 21 \ \cdots$. In this case the Hilbert function of $S/I_{X'}^2$ is given by $HF_{S/I_{X'}^2} : 1 \ 3 \ 6 \ 10 \ 14 \ 18 \ 20 \ 21 \ \cdots$. Therefore we obtain $HF_{\mathcal{W}'}(7) = 21 < 22 = HF_{S/I_{X'}^2}(7)$, and hence $I_{\mathcal{W}'} \neq I_{X'} \cdot I_{X'}$.

Now we introduce the algebraic object associated to $\mathcal{W}$ which we are most interested in. The enveloping algebra of $R_{\mathcal{W}}$ is the graded algebra $R_{\mathcal{W}} \otimes_K R_{\mathcal{W}} = \bigoplus_{j \geq 0} (\bigoplus_{k \geq 0} (R_{\mathcal{W}})_j \otimes (R_{\mathcal{W}})_k)$. Let $\mu : R_{\mathcal{W}} \otimes_K R_{\mathcal{W}} \to R_{\mathcal{W}}$ be the canonical multiplication map given by $\mu(f \otimes g) = fg$ for all $f, g \in R_{\mathcal{W}}$. This map is homogeneous of degree zero and its kernel $\mathcal{J} := \text{Ker}(\mu)$ is a homogeneous ideal of $R_{\mathcal{W}} \otimes_K R_{\mathcal{W}}$.

**Definition 2.9** Let $k$ be a positive integer.

(a) The graded $R_{\mathcal{W}}$-module $\Omega^1_{R_{\mathcal{W}}/K} : = \mathcal{J}/\mathcal{J}^2$ is called the module of Kähler differentials of $R_{\mathcal{W}}/K$. The homogeneous $K$-linear map $d : R_{\mathcal{W}} \to \Omega^1_{R_{\mathcal{W}}/K}$ given by $f \mapsto f \otimes 1 - 1 \otimes f + \mathcal{J}^2$ is called the universal derivation of $R_{\mathcal{W}}/K$.

(b) The exterior power $\Omega^k_{R_{\mathcal{W}}/K} := \bigwedge^k_{R_{\mathcal{W}}} \Omega^1_{R_{\mathcal{W}}/K}$ is called the module of Kähler differential $k$-forms of $R_{\mathcal{W}}/K$.

For $i = 0, \ldots, n$, we denote the image of $X_i$ in $R_{\mathcal{W}}$ by $x_i$. Then we have $\deg(dx_i) = \deg(x_i) = 1$ and $\Omega^k_{R_{\mathcal{W}}/K} = R_{\mathcal{W}} dx_0 + \cdots + R_{\mathcal{W}} dx_n$. Hence we see that $\Omega^k_{R_{\mathcal{W}}/K} = 0$ for all $k \geq n + 2$, and $\Omega^k_{R_{\mathcal{W}}/K}$ is a finitely generated $R$-module for all $k \geq 1$. Moreover, from [20] Proposition 4.12 or [29] Proposition 3.2.11 we get the following presentation for $\Omega^k_{R_{\mathcal{W}}/K}$.

**Proposition 2.10** Let $1 \leq k \leq n + 1$ and let $dI_{\mathcal{W}} = \langle dF \mid F \in I_{\mathcal{W}} \rangle$. Then the graded $R_{\mathcal{W}}$-module $\Omega^k_{R_{\mathcal{W}}/K}$ has a presentation

$$\Omega^k_{R_{\mathcal{W}}/K} \cong \Omega^k_{S/K}/(I_{\mathcal{W}} \Omega^k_{S/K} + dI_{\mathcal{W}} \Omega^{k-1}_{S/K}).$$
Given a non-zero homogeneous ideal $I$ of $S$, we denote the homogeneous ideal $(\frac{\partial I}{\partial X^i})_{0 \leq i \leq n} \subseteq S$ by $\partial I$. The ideal $\partial I$ is also known as the $n$-th Jacobian ideal (or the $n$-th Kähler different) of the $K$-algebra $S/I$ (see [12, Section 10]). If \{$F_1, \ldots, F_s$\} is a set of generators of $I$, then $\partial I$ is generated by all 1-minors of the Jacobian matrix $(\frac{\partial F_i}{\partial X^j})_{0 \leq j \leq n, 1 \leq k \leq n}$, i.e., $\partial I = (\frac{\partial F_i}{\partial X^j} | 0 \leq j \leq n, 1 \leq k \leq r)$. Furthermore, by Euler’s relation [12, Section 1], we always have $I \subseteq \partial I$.

Lemma 2.11 Let $I$ and $J$ be two proper homogeneous ideal of $S$ such that $I_i = J_i$ for $i \gg 0$. Then we have $(\partial I)_i = (\partial J)_i$ for $i \gg 0$.

Proof It suffices to prove the inclusion $(\partial I)_i \subseteq (\partial J)_i$ for $i \gg 0$, since $I$ and $J$ may be interchanged. Let $i_0 \in \mathbb{N}$ be a number such that $I_i = J_i$ for all $i \geq i_0$. For $i \geq i_0$ let $F \in (\partial I)_i$. There are homogeneous polynomials $G_{jk} \in S$ and $H_{jk} \in I$ such that $F = \sum_{j=0}^n \sum_{k=1}^n G_{jk} \frac{\partial H_{jk}}{\partial X^j}$, Then, for $j \in \{0, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, we have $G_{jk} \frac{\partial H_{jk}}{\partial X^j} = G_{jk} \frac{\partial G_{jk}}{\partial X^j} = H_{jk} \frac{\partial G_{jk}}{\partial X^j}$. Clearly, $G_{jk} H_{jk} \in I_i+1 = J_i+1$, and hence $\frac{\partial G_{jk}}{\partial X^j} \in (\partial J)_i$. Also, we have $J \subseteq \partial J$ and $H_{jk} \frac{\partial G_{jk}}{\partial X^j} \in J_i$. Thus $G_{jk} \frac{\partial H_{jk}}{\partial X^j} \in (\partial J)_i$. Consequently, we get $F \in (\partial J)_i$, as we wanted to show. 

As a consequence of Proposition 2.10 we obtain the following explicit description for the module of Kähler differential $(n+1)$-forms of $S/K$ (see also [9, Corollary 2.3]).

Corollary 2.12 There is an isomorphism of graded $R_{\mathbb{W}}$-modules

$$\Omega^{n+1}_{R_{\mathbb{W}}/K} \cong (S/\partial I_{\mathbb{W}})(-n-1)$$

In particular, we have $HF_{R_{\mathbb{W}}/K}^{n+1}(i) = HF_{S/\partial I_{\mathbb{W}}}(i-n-1)$ for all $i \in \mathbb{Z}$.

3 Hilbert Polynomials of Kähler Differential Modules

In this section we look at the Hilbert function of the module of Kähler differential $k$-forms of a fat point scheme $\mathbb{W} = m_1 P_1 + \cdots + m_n P_n$ in $\mathbb{P}^n$, where $1 \leq k \leq n+1$. Especially, for the case $k = n+1$, we determine a formula for the Hilbert polynomial of the module of Kähler differential $k$-forms of $\mathbb{W}$.

Clearly, the ring $R_{\mathbb{W}}$ is Noetherian and the module of Kähler differential $k$-forms $\Omega^k_{R_{\mathbb{W}}/K}$ is a finitely generated graded $R_{\mathbb{W}}$-module. Thus the Hilbert polynomial $HP_{\Omega^k_{R_{\mathbb{W}}/K}}(z) \in \mathbb{Q}[z]$ of $\Omega^k_{R_{\mathbb{W}}/K}$ exists (see e.g. [3, Theorem 4.1.3]). The (Hilbert) regularity index of $\Omega^k_{R_{\mathbb{W}}/K}$ is defined by $ri(\Omega^k_{R_{\mathbb{W}}/K}) := \min\{i \in \mathbb{Z} | AH_{\Omega^k_{R_{\mathbb{W}}/K}}(j) = HP_{\Omega^k_{R_{\mathbb{W}}/K}}(j) \text{ for all } j \geq i\}$.

The Hilbert polynomial of $\Omega^k_{R_{\mathbb{W}}/K}$ is easily shown to be a constant polynomial. However, except the case $n = 1$ (see e.g. [14]), to determine the Hilbert polynomial of $\Omega^k_{R_{\mathbb{W}}/K}$ is an interesting non-trivial task. In [9, Sections 4 and 5], the authors gave the following bounds for the Hilbert polynomial $HP_{\Omega^k_{R_{\mathbb{W}}/K}}(z)$ and its regularity index.
Using Corollary 3.2, we find also be generated by homogeneous polynomials of degrees $\leq r - j$ for all $i$ and $r$. The claim is equivalent to proving the inclusion

$$\min\{\max\{r_{\mathbf{w}} + k, r_{\mathbf{v}} + k - 1\}, \max\{r_{\mathbf{w}} + n, r_{\mathbf{v}} + n - 1\}\}.$$  

In particular, for $\nu \geq 1$, we have $\mathrm{HP}_{\nu}^{(r_{\mathbf{w}} + 1)\mathbf{v}, \mathbf{w}}(z) = 0$ and

$$\mathrm{HP}_{\nu}^{(r_{\mathbf{w}} + 1)\mathbf{v}, \mathbf{w}}(z) = \mathrm{HP}_{\nu \mathbf{x}}(z) = s^{\nu + n - 1}.$$  

Also, the above lower bound for $\mathrm{HP}_{\nu}^{(r_{\mathbf{w}} + 1)\mathbf{v}, \mathbf{w}}(z)$ is attained for a fat point scheme whose support is contained in a hyperplane (see [9, Proposition 5.1]) and the upper bound for the regularity index of $\Omega_{R_{\mathbf{w}}/K}$ is sharp as well (see [9, Example 4.3]). Based on the isomorphism of graded $R_{\mathbf{w}}$-modules $\Omega_{R_{\mathbf{w}}/K}^{\nu + 1} \cong (S/\partial I_{\mathbf{w}})(-n - 1)$, we obtain from Propositions 2.6 and 3.1 the following consequence.

**Corollary 3.2** Let $X = P_1 + \cdots + P_s$ be a set of $s$ distinct points in $\mathbb{P}^n$, and let $\nu \geq 1$. There exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$ we have $(\partial I_X)_{i} = \mathfrak{M}_i$ and

$$(\partial I_X^\nu + 1)_{i} = (\partial I_{(\nu + 1)\mathbf{v}}\mathbf{x})_{i} = (I_{\mathbf{x}}^\nu)_{i}.$$  

For a set of distinct points $X$ and a subset $Y$ of $X$, the vanishing ideals $I_X$ and $I_Y$ satisfy the following relation.

**Lemma 3.3** Let $X = P_1 + \cdots + P_s$ be a set of $s \geq 2$ distinct points in $\mathbb{P}^n$, let $Y$ be a non-empty subset of $X$, and let $k, \ell \geq 0$. Then, for $i \gg 0$, we have

$$(I_X^k \cdot I_Y^\ell)_{i} \subseteq (\partial(I_X^{k + 1} \cdot I_Y^\ell))_{i}.$$  

**Proof** Let $\nu := k + \ell$. The claim is equivalent to proving the inclusion

$$(I_X^{\nu - j} \cdot I_Y^\ell)_{i} \subseteq (\partial(I_X^{\nu - j + 1} \cdot I_Y^\ell))_{i}$$  

for $i \gg 0$ and $j \in \{0, 1, \ldots, \nu\}$. We proceed by induction on $j$. By Corollary 3.2 we have the equalities

$$(I_X^\nu)_{i} = (I_{\mathbf{x}})_{i} = (\partial I_X^{\nu + 1})_{i}$$

for $i \gg 0$, and hence (1) holds true for $j = 0$. Suppose that (1) holds true for $j - 1 \geq 0$, i.e., there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$ we have

$$(I_X^{\nu - j + 1} \cdot I_Y^\ell)_{i} \subseteq (\partial(I_X^{\nu - j + 2} \cdot I_Y^\ell - 1))_{i}.$$  

Using Corollary 3.2 we find $i_1 \in \mathbb{N}$ such that

$$(I_X^{\nu - j})_{i} = (I_{(\nu - j)\mathbf{x}})_{i} = (\partial I_X^{\nu - j + 1})_{i} \quad \text{and} \quad (I_Y^\ell)_{i} = (I_{\mathbf{y}})_{i},$$

for all $i \geq i_1$. Further, let $r_{\mathbf{w}}$ be the regularity index of $HF_{\mathbf{w}}$. Then the homogeneous ideal $I_Y$ can be generated by homogeneous polynomials of degrees $\leq r_{\mathbf{w}} + 1$ by [7] Proposition 1.1. Set $r := \max\{i_0, i_1, r_{\mathbf{w}} + 1\}$. The ideal $I_Y^\ell$ can also be generated by homogeneous polynomials of degrees $\leq r$. Let $i \geq 2r$, and
Thus the formula of the lemma does not hold in general, when
$Y_k$ for
Since $HF_{S/I}$ to hold, as the following example shows.
Hence, from (2) and (3) we get
Clearly, we have
Moreover, from the inclusion $I_X \subseteq I_Y$ yields
Hence, from (2) and (3) we get
for all $p, q.$ This implies $F \cdot G \in (\partial(I_X^{\nu-j} \cdot I_Y))$. Since $(I_X^{\nu-j} \cdot I_Y)_i$ is a $K$-vector space generated by elements of the form $F \cdot G$, we obtain the desired inclusion
Notice that the assumption that $Y$ is a subset of $X$ is essential for this lemma to hold, as the following example shows.

**Example 3.4** Let $K = \mathbb{Q}$ and let $X, Y$ be two complete intersections in $\mathbb{P}^2$ given by $X = P_1 + P_2 + P_3 + P_4$ and $Y = P_1 + P_2 + P_3 + P_4$, where $P_1 = (1 : 0 : 0)$, $P_2 = (1 : 0 : 1)$, $P_3 = (1 : 1 : 0)$, $P_4 = (1 : 1 : 1)$, $P_5 = (1 : 2 : 0)$ and $P_6 = (1 : 2 : 1)$. Then, in $S = K[X_0, X_1, X_2]$, we have
\[
I_X = \langle X_0X_1 - X_2^2, X_0X_2 - X_2^2 \rangle \quad \text{and} \quad I_Y = \langle 2X_0X_1 - X_1^2, X_0X_2 - X_2^2 \rangle.
\]
For $k = \ell = 1$, we see that
\[
HF_{S/(I_X, I_Y)} : 1 \ 3 \ 6 \ 10 \ 11 \ 10 \ 10 \cdots \quad \text{and} \quad HF_{S/\partial(I_X^{i+1} - I_Y)} : 1 \ 3 \ 6 \ 10 \ 15 \ 8 \ 8 \cdots.
\]
Since $HF_{S/(I_X, I_Y)}(4) = 11 < 15 = HF_{S/\partial(I_X^{i+1} - I_Y)}(4)$, we see that $I_X \cdot I_Y \nsubseteq \partial(I_X^{i+1} - I_Y)$. Thus the formula of the lemma does not hold in general, when $Y$ is not a subset of $X$. 

let $F \cdot G \in (I_X^{\nu-j} \cdot I_Y)$, with homogeneous polynomials $F \in I_X^{\nu-j}$ and $G \in I_Y$. Without loss of generality, we may assume that $F \in (I_X^{\nu-j})_{i-r}$ and $G \in (I_Y)_r$. Since $i - r \geq 2r - r = r \geq i_1$, this implies $F \in (\partial(I_X^{\nu-j+1})_{i-r}$. We may write
\[
F = \sum_{p=0}^{n} \sum_{q=0}^{u} H_{pq} \frac{\partial F_{pq}}{\partial X_p}
\]
where $F_{pq} \in (I_X^{\nu-j+1})_{\deg(F_{pq})}$ and $H_{pq} \in S_{i-r-\deg(F_{pq})+1}$ for all $0 \leq p \leq n$ and $1 \leq q \leq u$. Hence we have
\[
F \cdot G = \sum_{p=0}^{n} \sum_{q=0}^{u} H_{pq} G \frac{\partial F_{pq}}{\partial X_p} = \sum_{p=0}^{n} \sum_{q=0}^{u} H_{pq} (\partial(F_{pq}G) - F_{pq} \frac{\partial G}{\partial X_p})
\]
\[
= \sum_{p=0}^{n} \sum_{q=0}^{u} H_{pq} \frac{\partial(F_{pq}G)}{\partial X_p} - \sum_{p=0}^{n} \sum_{q=0}^{u} H_{pq} F_{pq} \frac{\partial G}{\partial X_p}.
\]
Clearly, we have $\frac{\partial(F_{pq}G)}{\partial X_p} \in (I_X^{\nu-j+1} \cdot I_Y)$ and thus $H_{pq} \frac{\partial(F_{pq}G)}{\partial X_p} \in (\partial(I_X^{\nu-j+1} \cdot I_Y))$. Moreover, from the inclusion $I_X \subseteq I_Y$ we deduce $F_{pq} \frac{\partial G}{\partial X_p} \in I_X^{\nu-j+1} \cdot I_Y$. Also, the inclusion $I_X \subseteq I_Y$ yields
\[
I_X^{\nu-j+2} \cdot I_Y^{-1} \subseteq I_X^{\nu-j+1} \cdot I_Y.
\]
Hence, from (2) and (3) we get
\[
H_{pq} F_{pq} \frac{\partial G}{\partial X_p} \in (I_X^{\nu-j+1} \cdot I_Y^{-1}) \subseteq (\partial(I_X^{\nu-j+2} \cdot I_Y^{-1})) \subseteq (\partial(I_X^{\nu-j+1} \cdot I_Y)),
\]
for all $p, q$. This implies $F \cdot G \in (\partial(I_X^{\nu-j+1} \cdot I_Y))$. Since $(I_X^{\nu-j} \cdot I_Y)_i$ is a $K$-vector space generated by elements of the form $F \cdot G$, we obtain the desired inclusion
\[
(I_X^{\nu-j} \cdot I_Y)_i \subseteq (\partial(I_X^{\nu-j+1} \cdot I_Y))_i.
\]
The preceding lemma can be generalized as follows.

**Proposition 3.5** Let \( t \geq 2 \), let \( \nu_1, \ldots, \nu_t \geq 1 \), and let \( Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_t \) be a descending chain of finite sets of distinct points in \( \mathbb{P}^n \). Then, for \( i \gg 0 \) we have

\[
(I_{Y_1}^{\nu_1} \cdot I_{Y_2}^{\nu_2} \cdots I_{Y_t}^{\nu_t})_i \subseteq (\partial(I_{Y_1}^{\nu_1+1} \cdot I_{Y_2}^{\nu_2} \cdots I_{Y_t}^{\nu_t}))_i.
\]

In the proof of this proposition we use the following lemma.

**Lemma 3.6** Let \( t \geq 2 \), let \( \nu_1, \ldots, \nu_t \geq 1 \), and let \( Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_t \) be a descending chain of finite sets of distinct points in \( \mathbb{P}^n \). If

\[
(I_{Y_1}^{\nu_1+1} \cdot I_{Y_2}^{\nu_2} \cdots I_{Y_t}^{\nu_t})_i \subseteq (\partial(I_{Y_1}^{\nu_1+2} \cdot I_{Y_2}^{\nu_2} \cdots I_{Y_t}^{\nu_t}))_i
\]

and

\[
(I_{Y_1}^{\nu_1} \cdot I_{Y_2}^{\nu_2} \cdots I_{Y_t}^{\nu_t})_i \subseteq (\partial(I_{Y_1}^{\nu_1+1} \cdot I_{Y_2}^{\nu_2} \cdots I_{Y_t}^{\nu_t}))_i
\]

for \( i \gg 0 \), then

\[
(I_{Y_1}^{\nu_1} \cdot I_{Y_2}^{\nu_2} \cdots I_{Y_t}^{\nu_t+1})_i \subseteq (\partial(I_{Y_1}^{\nu_1+1} \cdot I_{Y_2}^{\nu_2} \cdots I_{Y_t}^{\nu_t+1}))_i
\]

for \( i \gg 0 \).

**Proof** Note that the ideal \( I_{Y_i} \) can be generated by homogeneous polynomials of degree \( \leq \nu_i + 1 \), where \( \nu_i \) is the regularity index of \( H(I_{Y_i}) \). Set \( J := I_{Y_1}^{\nu_1} \cdot I_{Y_2}^{\nu_2} \cdots I_{Y_t}^{\nu_t} \).

By assumption, there exists an integer \( i_0 \geq \nu_i + 1 \) such that

\[
(I_{Y_1} \cdot J)_i \subseteq (\partial(I_{Y_1}^2 \cdot J))_i \quad \text{and} \quad J_i \subseteq (\partial(I_{Y_1} \cdot J))_i
\]

for every \( i \geq i_0 \). Now we let \( i \geq 2i_0 \) and consider \( F \in (J \cdot I_{Y_1})_i \) of the form

\[
F = G \cdot H \text{ with homogeneous polynomials } G \in J \text{ and } H \in I_{Y_1}. \]

By the choice of \( i_0 \), we may assume \( H \in (I_{Y_1})_{i_0} \) and \( G \in J_{i-i_0} \). Since \( i - i_0 \geq i_0 \), we get \( G \in J_{i-i_0} \subseteq (\partial(I_{Y_1} \cdot J))_{i-i_0} \). There are homogeneous polynomials \( G_{01}, G_{02}, \ldots, G_{nm} \in I_{Y_1} \cdot J \)

and \( H_{01}, H_{02}, \ldots, H_{nm} \in S \) such that

\[
G = \sum_{k=0}^{n} \sum_{\ell=1}^{m} H_{k\ell} \frac{\partial G_{k\ell}}{\partial X_k}
\]

For any \( k, \ell \geq 0 \), we observe that \( \frac{\partial(HG_{k\ell})}{\partial X_k} \in \partial(I_{Y_1} \cdot J \cdot I_{Y_1}) \). Moreover, we have

\[
H_{k\ell}G_{k\ell} \frac{\partial H}{\partial X_k} \in (I_{Y_1} \cdot J)_i \subseteq (\partial(I_{Y_1}^2 \cdot J))_i \subseteq (\partial(I_{Y_1} \cdot J \cdot I_{Y_1}))_i.
\]

Here the last inclusion follows from the fact that \( I_{Y_1} \subseteq I_{Y_1} \). So, we obtain

\[
F = \sum_{k=0}^{n} \sum_{\ell=1}^{m} H_{k\ell} \frac{\partial G_{k\ell}}{\partial X_k} = \sum_{k=0}^{n} \sum_{\ell=1}^{m} H_{k\ell}(\frac{\partial(HG_{k\ell})}{\partial X_k} - G_{k\ell} \frac{\partial H}{\partial X_k}) \in (\partial(I_{Y_1} \cdot J \cdot I_{Y_1}))_i.
\]

Since \( (J \cdot I_{Y_1})_i \) is a \( K \)-vector space generated by such elements \( F \), the inclusion \( (J \cdot I_{Y_1})_i \supseteq (\partial(I_{Y_1} \cdot J \cdot I_{Y_1}))_i \) is completely proved.
Proof (of Proposition 3.5) Let us prove the claim by induction on $t$. For $t = 2$, the claim follows from Lemma 3.3. Now we consider the case $t > 2$ and assume that the claim holds true for $t - 1$. We let $\nu_1, \ldots, \nu_t \geq 0$ and set $J := I^2_{\nu_2} \cdots I^{t-1}_{\nu_{t-1}}$. By the induction hypothesis, $(I^2_{\nu_2} \cdot J)_i \subseteq (\partial(I^3_{\nu_1} \cdot J))_i$ holds true for $i \gg 0$ and for all $k \geq 0$. Thus Lemma 3.6 yields that $(I^3_{\nu_1} \cdot J \cdot I_{\nu_k})_i \subseteq (\partial(I^4_{\nu_1} \cdot J \cdot I_{\nu_k}))_i$ holds true for $i \gg 0$ and for all $k \geq 0$. By applying Lemma 3.6 again, and by induction on the power of the ideal $I_{\nu_k}$, we obtain the inclusion $(I^\nu_{\nu_1} \cdot J \cdot I_{\nu_k})_i \subseteq (\partial(I^\nu_{\nu_1} \cdot J \cdot I_{\nu_k}))_i$ for $i \gg 0$, and the claim follows. \hfill $\Box$

Now we are ready to state and prove a formula for the Hilbert polynomial of the module of Kähler differential $(n + 1)$-forms for an arbitrary fat point scheme $\mathbb{W}$ in $\mathbb{P}^n$. This gives an affirmative answer to the conjecture made in [4, Conjecture 5.7].

**Theorem 3.7** Let $\mathbb{W} = m_1 \mathbb{P}_1 + \cdots + m_s \mathbb{P}_s$ be a fat point scheme in $\mathbb{P}^n$, and let $\mathbb{Y}$ be the subscheme $\mathbb{Y} = (m_1 - 1) \mathbb{P}_1 + \cdots + (m_s - 1) \mathbb{P}_s$ of $\mathbb{W}$. Then the Hilbert polynomial of $\mathcal{O}_{\mathbb{W}/\mathbb{K}}^{n+1}$ is given by

$$\text{HP}_{\mathcal{O}_{\mathbb{W}/\mathbb{K}}^{n+1}}(z) = \text{HP}_{\mathbb{Y}}(z) = \sum_{j=1}^{s} \binom{m_j + n - 2}{n}.$$  

Proof Note that if $\mathbb{W}$ is an equimultiple fat point scheme in $\mathbb{P}^n$, i.e., if $m_1 = \cdots = m_s = \nu$ for some $\nu \geq 1$, the claim was proved in [4, Corollary 5.3]. Now we prove the claim for the general case. By reordering the indices of the points, we may write $\mathbb{W}$ as

$$\mathbb{W} = \nu_1 \mathbb{X}_1 + \nu_2 \mathbb{X}_2 + \cdots + \nu_t \mathbb{X}_t,$$

where $\mathbb{X}_1, \ldots, \mathbb{X}_t$ are disjoint subsets of points in $\mathbb{X}$ with $\mathbb{X} = \mathbb{X}_1 + \cdots + \mathbb{X}_t$, and where $1 \leq \nu_1 < \nu_2 < \cdots < \nu_t$ for some $t \geq 2$. For $k = 1, \ldots, t$, we set $\mathbb{Y}_k = \mathbb{X}_{k+1} + \cdots + \mathbb{X}_t$. Then we get a descending chain $\mathbb{X} = \mathbb{Y}_1 \supseteq \mathbb{Y}_2 \supseteq \cdots \supseteq \mathbb{Y}_t$ of finite sets of distinct points in $\mathbb{P}^n$ such that

$$\mathbb{W} = \nu_1 \mathbb{Y}_1 + \sum_{k=2}^{t} (\nu_k - \nu_{k-1}) \mathbb{Y}_k.$$

Also, it is obviously true that $\mathbb{Y} = (\nu_1 - 1) \mathbb{Y}_1 + \sum_{k=2}^{t} (\nu_k - \nu_{k-1}) \mathbb{Y}_k$. An application of Proposition 2.3 shows that, for $i \gg 0$, we have the equalities

$$(I_{\nu_1})_i = (I^\nu_{\nu_1} \cdot I^2_{\nu_2} \cdots I^{t-1}_{\nu_{t-1}})_i \quad \text{and} \quad (I_{\nu_1})_i = (I^\nu_{\nu_1} \cdot I^2_{\nu_2} \cdots I^{t-1}_{\nu_{t-1}})_i \quad (4)$$

It follows from Lemma 2.11 that

$$(\partial I_{\mathbb{W}})_i = (\partial(I^\nu_{\nu_1} \cdot I^2_{\nu_2} \cdots I^{t-1}_{\nu_{t-1}}))_i \quad (5)$$

for $i \gg 0$. Furthermore, Proposition 3.6 implies the inclusion

$$(I^\nu_{\nu_1} \cdot I^2_{\nu_2} \cdots I^{t-1}_{\nu_{t-1}})_i \subseteq (\partial(I^\nu_{\nu_1} \cdot I^2_{\nu_2} \cdots I^{t-1}_{\nu_{t-1}}))_i \quad (6)$$

for $i \gg 0$. From (4), (5) and (6) we get

$$(I_{\mathbb{Y}})_i = (I^\nu_{\nu_1} \cdot I^2_{\nu_2} \cdots I^{t-1}_{\nu_{t-1}})_i \subseteq (\partial I_{\mathbb{W}})_i$$
for $i \gg 0$. Since $\Omega_{R_W/K}^{n+1} \cong (S/\partial I_W)(-n - 1)$, the Hilbert polynomial of $\Omega_{R_W/K}^{n+1}$ satisfies

$$HP_{\Omega_{R_W/K}^{n+1}}(z) = HP_{S/\partial I_W}(z) \leq HP_{S/I_Y}(z) = HP_{Y}(z).$$

Moreover, Proposition 3.1 yields that

$$HP_{\Omega_{R_W/K}^{n+1}}(z) \geq \sum_{j=1}^{s} \left( \frac{m_j + n - 2}{n} \right) = HP_{Y}(z).$$

Therefore the desired equality for $HP_{\Omega_{R_W/K}^{n+1}}(z)$ follows.

4 Kähler Differential Modules of Fat Point Schemes in $\mathbb{P}^2$

In this section we apply the previous results to examine the case of fat point schemes in the projective plane $\mathbb{P}^2$. When a fat point scheme $W$ of $\mathbb{P}^2$ is equimultiple and is supported on a non-singular conic, the Hilbert function of the module of Kähler differential $k$-forms of $W$ was computed in [9, Section 6]). However, in the general case no such detailed information is available. In this section we use our preceding results to supplement our knowledge about $\Omega_{R_W/K}^n$ with some new information.

Proposition 4.1 Let $W = m_1 P_1 + \cdots + m_s P_s$ be a fat point scheme in $\mathbb{P}^2$. The Hilbert polynomials of the modules of Kähler differentials of $W$ are given by

$$HP_{\Omega_{R_W/K}^1}(z) = \sum_{j=1}^{s} \frac{1}{2} (3m_j - 2)(m_j + 1),$$

$$HP_{\Omega_{R_W/K}^2}(z) = \sum_{j=1}^{s} \frac{1}{2} (3m_j + 2)(m_j - 1),$$

$$HP_{\Omega_{R_W/K}^3}(z) = \sum_{j=1}^{s} \frac{1}{2} m_j(m_j - 1).$$

Proof Recall that, the first fattening of $W = m_1 P_1 + \cdots + m_s P_s$ is the fat point scheme $W^{(1)} := (m_1 + 1)P_1 + \cdots + (m_s + 1)P_s$. By [3] Theorem 1.7, there is a short exact sequence of graded $R_W$-modules

$$0 \longrightarrow I_W/I_W^{(1)} \longrightarrow R_W^3(-1) \longrightarrow \Omega_{R_W/K}^1 \longrightarrow 0,$$

and hence we get

$$HP_{\Omega_{R_W/K}^1}(z) = 4 \cdot \sum_{j=1}^{s} \binom{m_j + 1}{2} - \sum_{j=1}^{s} \binom{m_j + 2}{2}$$

$$= \sum_{j=1}^{s} \frac{1}{2} (4m_j(m_j + 1) - (m_j + 1)(m_j + 2))$$

$$= \sum_{j=1}^{s} \frac{1}{2} (3m_j - 2)(m_j + 1).$$
Also, Theorem 3.7 yields $\text{HP} \Omega^2_{R_w/K}(z) = \sum_{j=1}^s z \cdot m_j(m_j - 1)$. On the other hand, by [9, Proposition 2.4], we have an exact sequence of graded $R_w$-modules

$$0 \rightarrow \Omega^1_{R_w/K} \rightarrow \Omega^2_{R_w/K} \rightarrow \Omega^1_{R_w/K} \rightarrow 0$$

where $m_w$ is the homogeneous maximal ideal of $R_w$. Thus it follows that

$$\text{HP} \Omega^2_{R_w/K}(z) = \text{HP} \Omega^2_{R_w/K}(z) + \text{HP} \Omega^1_{R_w/K}(z) - \text{HP} m_w(z)$$

$$= \sum_{j=1}^s \frac{1}{2} m_j(m_j - 1) + \sum_{j=1}^s \frac{1}{2} (3m_j - 2)(m_j + 1) - \sum_{j=1}^s \frac{1}{2} m_j(m_j + 1)$$

$$= \sum_{j=1}^s \frac{1}{2} (3m_j + 2)(m_j - 1).$$

The next remark recalls some information about the degree from where on we now know the Hilbert function of $\Omega_w$.

**Remark 4.2** Set $t := \max\{r_W + 1, r_{W^{(1)}}\}$. The regularity indices of the modules of Kähler differentials of $\mathbb{W} \subseteq \mathbb{P}^2$ are bounded by

$$\text{ri}(\Omega^1_{R_w/K}) \leq t \quad \text{and} \quad \text{ri}(\Omega^2_{R_w/K}) \leq t + 1 \quad \text{and} \quad \text{ri}(\Omega^3_{R_w/K}) \leq t + 1.$$

Moreover, if the support of $\mathbb{W}$ lies on a non-singular conic, we have $\text{ri}(\Omega^1_{R_w/K}) = t = r_W$ (see [9, Theorem 6.2]), and the bounds for $\text{ri}(\Omega^2_{R_w/K})$ and $\text{ri}(\Omega^3_{R_w/K})$ are sharp. For instance, the scheme $\mathbb{W} = P_1 + P_2 + P_3$ consisting of three non-collinear points in $\mathbb{P}^2$ satisfies $\text{ri}(\Omega^1_{R_w/K}) = t = 3$ and $\text{ri}(\Omega^2_{R_w/K}) = \text{ri}(\Omega^3_{R_w/K}) = t + 1 = 4$.

Next, let us look more closely at the module of Kähler differential 2-forms of fat point schemes $\mathbb{W} \subseteq \mathbb{P}^2$. In general, for a fat point scheme $\mathbb{W} = m_1 P_1 + \cdots + m_s P_s$ in $\mathbb{P}^n$ with the $i$-th fattening $\mathbb{W}^{(i)} = (m_1 + i)P_1 + \cdots + (m_s + i)P_s$ for $i \geq 1$, [9, Proposition 5.4] implies that the sequence of graded $R_w$-modules

$$0 \rightarrow I_{\mathbb{W}^{(1)}}/I_{\mathbb{W}^{(2)}} \rightarrow I_{\mathbb{W}^{(1)}}\Omega^1_{S/K} / I_{\mathbb{W}^{(2)}}\Omega^1_{S/K} \rightarrow I_{\mathbb{W}^{(1)}}\Omega^2_{S/K} / I_{\mathbb{W}^{(2)}}\Omega^2_{S/K} \rightarrow I_{\mathbb{W}^{(1)}}\Omega^3_{S/K} / I_{\mathbb{W}^{(2)}}\Omega^3_{S/K} \rightarrow 0 \quad (\text{7})$$

is a complex. Here the map $\alpha$ is given by $\alpha(F + I_{\mathbb{W}^{(2)}}) = dF + I_{\mathbb{W}^{(1)}}\Omega^1_{S/K}$, the map $\beta$ is given by $\beta(GdX_i + I_{\mathbb{W}^{(1)}}\Omega^1_{S/K}) = d(GdX_i) + I_{\mathbb{W}^{(1)}}\Omega^2_{S/K}$, and the map $\gamma$ is given by $\gamma(H + I_{\mathbb{W}^{(1)}}\Omega^2_{S/K}) = H + I_{\mathbb{W}^{(1)}}\Omega^3_{S/K}$. In addition, we have $\text{Im}(\beta) = \text{Ker}(\gamma)$. For a fat point scheme $\mathbb{W}$ in $\mathbb{P}^2$, the complex (7) satisfies the following exactness property which generalizes the case of an equimultiple fat point scheme studied in [9, Proposition 5.5].

**Proposition 4.3** Let $\mathbb{W} = m_1 P_1 + \cdots + m_s P_s$ be a fat point scheme in $\mathbb{P}^2$, let $t := \max\{r_{W^{(2)}}, r_{W^{(1)}} + 1, r_W + 2\}$, and let $\alpha, \beta$ and $\gamma$ be the maps defined above. Then, for all $i \geq t$, we have the exact sequence of $K$-vector spaces

$$0 \rightarrow (I_{\mathbb{W}^{(1)}}/I_{\mathbb{W}^{(2)}})_i \xrightarrow{\alpha} (I_{\mathbb{W}^{(1)}}\Omega^1_{S/K} / I_{\mathbb{W}^{(2)}}\Omega^1_{S/K})_i \xrightarrow{\beta} (I_{\mathbb{W}^{(1)}}\Omega^2_{S/K} / I_{\mathbb{W}^{(2)}}\Omega^2_{S/K})_i \xrightarrow{\gamma} (I_{\mathbb{W}^{(1)}}\Omega^3_{S/K} / I_{\mathbb{W}^{(2)}}\Omega^3_{S/K})_i \rightarrow 0.$$
Proof It suffices to show that \( \text{Im}(\alpha) = \text{Ker}(\beta) \). Equivalently, it suffices to prove the equality of Hilbert functions
\[
\text{HF}_{\Omega^2_{R/W}/K}(i) + \text{HF}_{I_{w(1)}/I_{w(2)}}(i) + \text{HF}_{\Omega^2_{R/W}/K}(i) = \text{HF}_{I_{w(1)}/I_{w(2)}}(i)
\]
for all \( i \geq t \). Since \( \Omega^1_{S/K} \) is a free \( S \)-module with basis \( \{dX_0, dX_1, dX_2\} \) and \( \Omega^2_{S/K} \) is a free \( S \)-module with basis \( \{dX_0dX_1, dX_0dX_2, dX_1dX_2\} \), for \( i \in \mathbb{Z} \) we have
\[
\text{HF}_{I_{w(1)}/I_{w(2)}}(i) = 3 \text{HF}_{W}(i-1) - 3 \text{HF}_{W}(i-1)
\]
and
\[
\text{HF}_{\Omega^2_{R/W}/K}(i) = 3 \text{HF}_{W}(i-2).
\]
So, the equality (8) can be written as
\[
\text{HF}_{\Omega^2_{R/W}/K}(i) + 3 \text{HF}_{W}(i-1) - 3 \text{HF}_{W}(i-1) = \text{HF}_{W}(i-2).
\]
By Proposition 4.1 and Remark 4.2, \( \text{HF}_{\Omega^2_{R/W}/K}(i) = \sum_{j=1}^{s} \left( \frac{1}{2} (3m_j + 2)(m_j - 1) + 4(m_j + 1)^2 - 6(m_j + 1) - (m_j + 3) \right) \)
for all \( i \geq t \). Therefore the claim follows.

Notice that the exactness of the sequence in this proposition allows us to compute values of the Hilbert function of \( \Omega^2_{R/W}/K \) in the corresponding degrees. Unfortunately, our final example indicates that this exactness property does not hold for all \( i \in \mathbb{Z} \), even when the support of \( W \) is a complete intersection in \( \mathbb{P}^2 \).

Example 4.4 Let \( W \) be the fat point scheme given in Example 2.7. A calculation using ApCoCoA yields
\[
\text{HF}_{I_{w(1)}/I_{w(2)}} : 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 2 \ 9 \ 15 \ 19 \ 23 \ 26 \ 29 \ 30 \ 31 \ 31 \ldots,
\]
\[
\text{HF}_{I_{w(1)}/I_{w(2)}} : 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 2 \ 9 \ 15 \ 18 \ 21 \ 22 \ 23 \ 23 \ 23 \ldots,
\]
\[
\text{HF}_{I_{w(1)}/I_{w(2)}} : 0 \ 0 \ 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 26 \ 27 \ 28 \ 28 \ 28 \ 28 \ 28 \ldots,
\]
\[
\text{HF}_{I_{w(1)}/I_{w(2)}} : 0 \ 0 \ 3 \ 9 \ 18 \ 30 \ 45 \ 57 \ 53 \ 51 \ 48 \ 47 \ 46 \ 46 \ 46 \ldots.
\]
Thus the proposition holds true for \( i = 0 \), for \( i = 1 \), and for \( i \geq 15 \). However, the sequence is not exact for \( 2 \leq i \leq 14 \).
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