THE MEAN CURVATURE FLOW FOR ISOPARAMETRIC SUBMANIFOLDS

XIAOBO LIU and CHUU-LIAN TERNG

Abstract
A submanifold in space forms is isoparametric if the normal bundle is flat and principal curvatures along any parallel normal fields are constant. We study the mean curvature flow with initial data an isoparametric submanifold in Euclidean space and sphere. We show that the mean curvature flow preserves the isoparametric condition, develops singularities in finite time, and converges in finite time to a smooth submanifold of lower dimension. We also give a precise description of the collapsing.

1. Introduction
The mean curvature flow (MCF) of a submanifold $M \subset \mathbb{R}^N$ over an interval $I$ is a map $f : I \times M \rightarrow \mathbb{R}^N$ such that for all $t \in I$ and $x \in M$, $\frac{\partial}{\partial t} f(t, x)$ is equal to the mean curvature vector of $M(t) = f(t, M)$ at the point $x(t) = f(t, x)$. Mean curvature flows of convex hypersurfaces have been extensively studied in the literature (see [GH], [Hu]). Comparatively, the behavior of mean curvature flows of submanifolds with higher codimension is less understood (see [W]). This is partly due to the lack of understanding of collapsing and the formation of singularities of the flow equations in the higher-codimensional case.

A submanifold $M$ of a Riemannian manifold is isoparametric if its normal bundle is flat and principal curvatures along any parallel normal vector field are constant. The codimension of $M$ is called the rank of $M$. An isoparametric submanifold $M$ in $\mathbb{R}^N$ is full if it is not contained in any proper hyperplane, and it is irreducible if it is not a product of two isoparametric submanifolds. We refer to [T] for the basic properties and structure theories for isoparametric submanifolds. Principal orbits of isotropy representations of symmetric spaces are isoparametric, and they are the only compact homogeneous isoparametric submanifolds in Euclidean space (see [PT1]). There are also infinite families of nonhomogeneous isoparametric submanifolds which arise from representations of Clifford algebras (see [FKM]). All these nonhomogeneous
examples have rank 2. A theorem of Thorbergsson [Th] asserts that compact full irreducible isoparametric submanifolds with rank bigger than 2 are always homogeneous.

A complete isoparametric submanifold of $\mathbb{R}^N$ can be decomposed as the product of a compact, irreducible, isoparametric submanifold and a subspace of $\mathbb{R}^N$. Since the MCF with an affine subspace of $\mathbb{R}^N$ as initial data is trivial and the MCF starting from a product submanifold stays a product, we consider only compact, full, irreducible isoparametric submanifolds.

Let $M$ be an isoparametric submanifold of $\mathbb{R}^N$, and let $\xi$ be a parallel normal vector field on $M$. Then $M_\xi = \{x + \xi(x) \mid x \in M\}$ is again a smooth submanifold (may have lower dimension), and the map $\pi : M \to M_\xi$ defined by $\pi(x) = x + \xi(x)$ is either a diffeomorphism or a fibration with a homogeneous isoparametric submanifold as fiber. The family of these parallel sets forms a singular foliation of the ambient Euclidean space $\mathbb{R}^N$. Top-dimensional leaves are all isoparametric in $\mathbb{R}^N$, and they are called parallel isoparametric submanifolds. Lower-dimensional leaves are no longer isoparametric, and they are called focal submanifolds.

We show that if $f : M \times [0, T) \to \mathbb{R}^N$ is a solution of the MCF in $\mathbb{R}^N$ with $f(\cdot, 0)$ isoparametric, then $f(\cdot, t)$ is isoparametric for all $t \in [0, T)$; that is, the MCF preserves isoparametric condition. This reduces the MCF to a system of ordinary differential equations (ODEs). There is a Weyl group $W$ associated to each isoparametric submanifold $M$ that acts on the normal plane $p + \nu_p M$. The ODE given by the MCF with initial data an isoparametric submanifold is given by a vector field $H$ smoothly defined on the interior of the Weyl chamber $C$ of $W$ but blows up at the boundary of $C$. However, we can use generators of $W$-invariant polynomials to change coordinates so that the vector field $H$ becomes a polynomial vector field, and its flows can be solved explicitly.

Every compact isoparametric submanifold is contained in a sphere. This sphere is also foliated by parallel isoparametric submanifolds and focal submanifolds. Each isoparametric foliation contains a unique isoparametric submanifold that is a minimal submanifold of this sphere. The MCF in $\mathbb{R}^N$ with initial data a minimal submanifold in $S^{N-1}$ behaves like the MCF of a sphere; that is, it just shrinks homothetically along the radial direction and collapses to a point in finite time (see Lemma 2.8). If $M$ is an isoparametric submanifold in $\mathbb{R}^N$ which is not minimal in the sphere, then its MCF converges to a focal submanifold $F$ of positive dimension (see Corollary 2.9). In fact, $M$ is a fibration over $F$ with each fiber a homogeneous isoparametric submanifold. Each fiber of this fibration collapses to a point under the MCF in a finite time.

We summarize some of the main results of this article in the following theorem (see Theorems 2.5, 2.6, Proposition 2.10).
THEOREM 1.1

The mean curvature flow in $\mathbb{R}^N$ with initial data a compact isoparametric submanifold

1. converges to a focal submanifold in finite time $T$;
2. if the fibration from the initial isoparametric submanifold to the limiting focal submanifold is a sphere fibration (this is the generic case), then the mean curvature flow $M(t)$ has type I singularity; that is, there is a constant $c_0$ such that $\|\Pi(t)\|_\infty^2(T-t) \leq c_0$ for all $t \in [0, T)$, where $\|\Pi(t)\|_\infty$ is the sup norm of the second fundamental form of $M(t)$;
3. every focal submanifold is the limit of the mean curvature flow with some parallel isoparametric submanifold as initial data;
4. if $M_1$ and $M_2$ are distinct parallel full isoparametric submanifolds in $\mathbb{R}^N$ which lie in the same sphere, then the mean curvature flows in $\mathbb{R}^N$ with initial data $M_1$ and $M_2$ collapse to two distinct focal submanifolds.

Remark. If the fibration from the initial isoparametric submanifold to the limiting focal submanifold is not spherical, we still expect that the mean curvature flow has type I singularity. However, the estimates for the second fundamental forms in those nongeneric cases are much harder to obtain.

The MCF in $S^{N-1}$ with initial data an isoparametric submanifold behaves very similarly to the Euclidean MCF. In particular, we have the following.

THEOREM 1.2

Let $M$ be an isoparametric submanifold of $S^{N-1}$. Then the mean curvature flow in $S^{N-1}$ with $M$ as initial data

1. is constant if $M$ is minimal in $S^{N-1}$ or
2. converges to a focal submanifold of positive dimension in finite time if $M$ is not minimal.

An isometric action of $G$ on a Riemannian manifold $N$ is polar if there exists a closed embedded submanifold $\Sigma$ of $N$ which meets all $G$-orbits and meets orthogonally. Such a $\Sigma$ is called a section of the polar action. Principal orbits of a polar action in $\mathbb{R}^n$ and $S^n$ are isoparametric (see [PT1]). We prove that if the $G$-action on $N$ is polar, then the mean curvature flow preserves $G$-orbits and the flow becomes an ODE on the section $\Sigma$. We expect that methods developed in this article can be applied to study mean curvature flows for orbits of polar actions with flat sections in symmetric spaces.

This article is organized as follows. We present proofs of results stated in Theorem 1.1 in Section 2, and we construct explicit solutions of the MCF in $\mathbb{R}^N$ with initial data an isoparametric submanifold in Section 3. Since focal submanifolds are smooth manifolds, we can consider their mean curvature flow. Most properties of the mean...
curvature flows for isoparametric submanifolds also hold for focal submanifolds. This is briefly discussed in Section 4. We describe MCF in spheres with initial data an isoparametric submanifold in Section 5. Combining results of Sections 2–5, we have a more complete picture of MCF for an isoparametric submanifold $M$. The MCF with initial data $M$ develops singularity in finite time $T_1$ and converges to a lower-dimensional focal submanifold $F_1$ as $t \to T_1$. The MCF with $F_1$ as initial data again develops singularity and converges to a focal submanifold $F_2$ with smaller dimension in finite time. After repeating this process finitely many times, the MCF in $\mathbb{R}^n$ ($S^{n-1}$, resp.) converges to a point (a focal minimal submanifold in $S^{n-1}$, resp.) in finite time. In Section 6, we discuss MCF in a Riemannian manifold with initial data a principal orbit of a polar action.

2. Mean curvature flows for general isoparametric submanifolds
Let $M$ be a full compact isoparametric submanifold of $\mathbb{R}^N$, fix $x_0 \in M$, and let $W$ be the Coxeter group associated to $M$. We prove that the MCF stays isoparametric and that the MCF equation becomes a flow equation of a vector field $H$ which is defined in the interior of the Weyl chamber of $W$ containing $x_0$ in $x_0 + v_{x_0}M$. The vector field $H$ tends to infinity at the boundary of the Weyl chamber. We prove that solutions of the ODE $x' = H(x)$ exist only for finite time. To see the finer structure of the behavior of the blow-up of MCF, we use $W$-invariant polynomials to construct a new coordinate system for the Weyl chamber so that the corresponding vector field $H$ becomes a polynomial vector field. We then analyze the behavior of flows of this polynomial vector field to obtain information on the collapsing of the MCF.

**Proposition 2.1**
If $f : M \times [0, T) \to \mathbb{R}^N$ satisfies the mean curvature flow in $\mathbb{R}^N$ and $f(\cdot, 0)$ is isoparametric, then $f(\cdot, t)$ is isoparametric for all $t \in [0, T)$.

**Proof**
The tangent bundle of an isoparametric submanifold $M$ can be decomposed into orthogonal sums of curvature distributions $\{E_i \mid i = 1, \ldots, g\}$ for some integer $g > 0$. At each point of $M$, $E_i$ is a common eigenspace of the shape operators of $M$ at that point. There are parallel normal vector fields $n_i$ such that the shape operator $A_{\xi}$ has the property

$$A_{\xi}|_{E_i} = \langle \xi, n_i \rangle \text{Id}_{E_i}$$

for all normal vectors $\xi$. Each vector field $n_i$ is called the curvature normal of $E_i$. The rank of $E_i$ is called the multiplicity of $n_i$, which is denoted by $m_i$. Each $E_i$ is an integrable distribution whose leaves are $m_i$-dimensional round spheres with radius $1/\|n_i\|$. Such spheres are called curvature spheres.
Fix \( x_0 \in M \). Since \( vM \) is globally flat, we can identify a vector \( v \in v_{x_0}M \) and the unique parallel normal field \( \hat{v} \) along \( M \) defined by \( \hat{v}(x_0) = v \). In particular, we may view each curvature normal \( n_i \) either as a global parallel normal vector field along \( M \) or an element in \( v_{x_0}M \). The precise meaning should be clear from the context.

The Coxeter group \( W \) of \( M \) acts as a reflection group on the affine normal space \( x_0 + v_{x_0}M \) generated by reflections along hyperplanes

\[
L_i := \{ x_0 + \xi \mid \xi \in v_{x_0}M, \ 1 - \langle \xi, n_i(x_0) \rangle = 0 \}
\]

for all \( i = 1, \ldots, g \). The intersection \( \bigcap_{i=1}^{g} L_i \) consists of a single constant point, which is denoted by \( a \). Then \( M \) is contained in a sphere that is centered at \( a \). Without loss of generality, we always assume that \( a = 0 \); that is, \( M \) is contained in a sphere centered at the origin of \( \mathbb{R}^N \). This condition is equivalent to

\[
\langle -x, n_i(x) \rangle = 1 \quad (2.1)
\]

for all \( x \in M \) and \( i = 1, \ldots, g \) (see [T, Corollary 1.17]).

For any parallel normal vector field \( \xi \) on \( M \), define

\[
M_{\xi} := \{ x + \xi(x) \mid x \in M \}.
\]

If

\[
1 - \langle \xi(x_0), n_i(x_0) \rangle \neq 0 \quad (2.2)
\]

for all \( i = 1, \ldots, g \), then \( M_{\xi} \) is again an isoparametric submanifold with the same dimension as \( M \). The curvature normals of \( M_{\xi} \) at the point \( x + \xi(x) \) are given by

\[
\frac{n_i(x)}{1 - \langle \xi(x), n_i(x) \rangle}
\]

with same multiplicities \( m_i \) for \( i = 1, \ldots, g \). The mean curvature vector of \( M_{\xi} \) at \( x + \xi(x) \) is given by

\[
H(x + \xi(x)) = \sum_{i=1}^{g} \frac{m_i n_i(x)}{1 - \langle \xi(x), n_i(x) \rangle}. \quad (2.3)
\]

Let \( \xi(t) \in v_{x_0}M \) be a one-parameter family of normal vectors satisfying the flow equation

\[
\dot{\xi}(t) = \sum_{i=1}^{g} \frac{m_i n_i}{1 - \langle \xi(t), n_i \rangle}, \quad \xi(0) = 0. \quad (2.4)
\]
It follows from (2.3) that $\xi$ is a solution of (2.4) if and only if the one-parameter family of parallel submanifolds $M(t) := M_{\xi(t)}$ satisfies the mean curvature flow equation with $M(0) = M$. In other words, the MCF preserves the isoparametric condition. The proposition is thus proved. \hfill \Box

Equation (2.4) does not make sense if $\langle \xi(t), n_i \rangle = 1$ for some $i$, which is equivalent to the condition that $M(t)$ is a focal submanifold. We study only the flow equation under the condition

$$\langle \xi(t), n_i \rangle < 1$$

for all $i = 1, \ldots, g$. In other words, we require that $x_0 + \xi(t)$ stay in the same Weyl chamber of $W$ as $x_0$ for all $t$. Under this condition, all $M(t)$ are still isoparametric.

Note that (2.4) is a system of nonlinear ODEs given by a vector field defined on the Weyl chamber

$$C := \{ x_0 + \xi \mid \xi \in \nu_{x_0}M, \langle \xi, n_i \rangle < 1 \}$$

and that the vector field blows up along the boundary of $C$. For any parallel normal vector field $\xi$ on $M$, the intersection of $M_{\xi}$ with $x_0 + \nu_{x_0}M$ is an orbit of $W$. In particular, if $M_{\xi}$ is a parallel isoparametric submanifold, then it intersects each open Weyl chamber of $W$ exactly once. Moreover, $M_{\xi}$ is a focal submanifold if and only if $x_0 + \xi(x_0)$ is contained in $\bigcup_{i=1}^g L_i$. The study of MCF with isoparametric submanifolds as initial data reduces to the study of solutions of the ODE system (2.4).

**Theorem 2.2**

Let $M \subset S^{N-1}(r_0)$ be an $n$-dimensional isoparametric submanifold in $\mathbb{R}^N$, and let $x_0 \in M$. If $\xi(t)$ satisfies the mean curvature flow equation (2.4), then $x(t) = x_0 + \xi(t)$ satisfies

$$x'(t) = -\sum_{i=1}^g \frac{m_i n_i}{\langle x(t), n_i \rangle}$$

with $x(0) = x_0$. Let $H(t)$ be the mean curvature vector of $M(t) = M_{\xi(t)}$ at the point $x(t)$. Then

(a) $\langle x(t), H(t) \rangle = -n$,

(b) $\|x(t)\|^2 = \|x(0)\|^2 - 2nt$.

**Proof**

By equation (2.1),

$$\langle x(t), n_i \rangle = \langle x(0), n_i \rangle + \langle \xi(t), n_i \rangle = -1 + \langle \xi(t), n_i \rangle$$
for all $i = 1, \ldots, g$. Since

$$H(t) = -\sum_{i=1}^{g} \frac{m_i n_i}{\langle x(t), n_i \rangle},$$

we have $\langle x(t), H(t) \rangle = -\sum_{i=1}^{g} m_i = -n$. This proves part (a). Part (b) follows from integrating the following formula:

$$\frac{d}{dt} \|x(t)\|^2 = 2\langle x(t), x'(t) \rangle = 2\langle x(t), H(t) \rangle = -2n.$$ 

Hence we have the following.

**COROLLARY 2.3**

The mean curvature flow in $\mathbb{R}^N$ with initial data an isoparametric submanifold in $S^{N-1}(r_0)$ exists only for finite time with maximal interval $[0, T)$, where $0 < T \leq T_0 = r_0^2/(2n)$.

The following theorem is the key in proving that

1. the limits of two flows of (2.5) have two different limits on the boundary $\partial C$ of the Weyl chamber $C$;
2. every point of $\partial C$ is a limit of some flow of (2.5).

**THEOREM 2.4**

Let $M$ be a compact isoparametric submanifold in $\mathbb{R}^N$, let $W$ be the Weyl group associated to $M$, let $x_0 \in M$ be a fixed point, and let $V = x_0 + v_{x_0} M$. Let $P_1, \ldots, P_k$ be a set of generators of the ring $\mathbb{R}[V]^W$ of $W$-invariant polynomials on $V$ such that $P_i$ are homogeneous polynomials of degree $d_i$ with $P_1(x) = \|x\|^2$ and $d_1 \leq d_2 \leq \cdots \leq d_k$, and let $C$ be the Weyl chamber of $W$ containing $x_0$ in $V$. Let $P : \bar{C} \rightarrow \mathbb{R}^k$ be the map defined by $P(x) = (P_1(x), \ldots, P_k(x))$. Then $P$ is a homeomorphism from $\bar{C}$ to a closed subset $B = P(\bar{C})$. Moreover, there is a polynomial map

$$\eta = (\eta_1, \ldots, \eta_k) : \mathbb{R}^k \rightarrow \mathbb{R}^k$$

with $\eta_1 = -2n$, and $\eta_j(y)$ is a polynomial in $y_1, \ldots, y_{j-1}$ such that if $x : [0, T) \rightarrow C$ is a solution of (2.5), then $y(t) = P(x(t))$ is a solution of

$$y'(t) = \eta(y(t)) = (\eta_1(y(t)), \ldots, \eta_k(y(t))).$$
Proof
Since each orbit of $W$ intersects $\overline{C}$ exactly once and the algebra of invariant polynomials separates $W$-orbits, the map
\[
P : \overline{C} \longrightarrow \mathbb{R}^k, \quad x \mapsto (P_1(x), \ldots, P_k(x))
\]
is an injective continuous map. Since $P$ is injective and proper, $B = P(\overline{C})$ is a closed subset of $\mathbb{R}^k$ and $P$ is a homeomorphism from $\overline{C}$ to $B$.

The Coxeter group $W$ acts on $V$. If $f$ is a $W$-invariant homogeneous polynomial of degree $j$ on $V$, then by [T, Lemma 3.2],
\[
F(x) := \sum_{i=1}^{g} m_i \frac{\langle \nabla f(x), n_i \rangle}{\langle x, n_i \rangle}
\] (2.7)
is a $W$-invariant homogenous polynomial of degree $j - 2$. Let $x(t)$ be a solution of (2.5), and let $f(t) = f(x(t))$. By equation (2.6),
\[
f'(t) = \langle \nabla f(x(t)), x'(t) \rangle = \langle \nabla f(x(t)), H(t) \rangle = -F(x(t)).
\]
Hence $f'(t)$ is the value of a $W$-invariant polynomial of degree $k - 2$ evaluated at $x(t)$. In particular, $\frac{df}{dt}(t) = 0$ if $j > k/2$. Therefore $f(t)$ is a polynomial in $t$.

Let $y(t) = (y_1(t), \ldots, y_k(t)) = P(x(t))$, and let
\[
F_i(x) = \sum_{i=1}^{g} m_i \frac{\langle \nabla P_i(x), n_i \rangle}{\langle x, n_i \rangle}.
\]
Then $F_i$ is a $W$-invariant homogeneous polynomial on $V$ of degree $d_i - 2$. Since $\mathbb{R}[V]^W = \mathbb{R}[P_1, \ldots, P_k]$,
\[
F_i = -\eta_i(P_1, \ldots, P_{i-1})
\]
for some polynomial $\eta_i$. But we have shown above that $y'_i(t) = -F_i(x(t))$, so
\[
y'_i(t) = -F_i(x(t)) = \eta_i(y_1(t), \ldots, y_{i-1}(t)).
\]

This shows that $y(t)$ is an integral curve of the polynomial vector field $\eta$ on $\mathbb{R}^k$. Since $y_1(t) = \|x(0)\|^2 - 2nt$, solution $y$ can be solved explicitly by integrations. 

The MCF equation (2.5) is given by the vector field
\[
H(x) = -\sum_{i=1}^{g} \frac{m_i n_i}{\langle x, n_i \rangle},
\]
which is smoothly defined on the Weyl chamber $C$ of $x_0 + \nu x_0 M$ and blows up at the boundary $\partial C$. If we use generators of $W$-invariant polynomials on $x_0 + \nu x_0 M$ to change coordinates to $P$ as in Theorem 2.4, then the vector field $H$ becomes the polynomial vector field $\eta$ on $P(C)$. Moreover, the flow of $\eta$ can be solved explicitly and globally. Then apply $P^{-1}$ to flows of $\eta$ to get flows of (2.5).

THEOREM 2.5
For any compact isoparametric submanifold $M$ in $\mathbb{R}^N$, the mean curvature flow always converges to a focal submanifold at a finite time. Moreover, if $M_1$ and $M_2$ are parallel full isoparametric submanifolds that are contained in the same sphere, then mean curvature flows with initial data $M_1$ and $M_2$ never intersect and they converge to two distinct focal submanifolds.

Proof
We use the same notation as in Theorem 2.4. Let $x : [0, T) \rightarrow C$ be the maximal interval for a solution of the mean curvature flow equation (2.5). Note that $P_i(t) = P_i(x(t))$ are well defined since $P_i(t)$ are polynomials in $t$. Therefore the mean curvature flow of $x_0 \in M$ must converge to $P^{-1}(P_1(T), \ldots, P_k(T))$ which lies on the boundary of $\overline{C}$. The mean curvature flow of $M$ then converges to the focal submanifold passing through this point.

We may assume that $x_i(0)$ lies in the unit sphere. Let $T_i$ denote the maximum time for the solution $x_i(t)$. If $T_1 \neq T_2$, then $\|x_i(t)\|^2 = 1 - 2nt$, so $\lim_{t \rightarrow T_i^-} \|x_1(t)\|^2 \neq \lim_{t \rightarrow T_i^-} \|x_2(t)\|^2$. If $T_1 = T_2 = T$, then since $x_i(t)$ are solutions of (2.5) and $\langle x_i(t), n_j \rangle < 0$, we have

$$
\frac{1}{2} \frac{d}{dt} \|x_1(t) - x_2(t)\|^2 = \sum_{i=1}^{q} m_i \frac{(x_1(t) - x_2(t), n_i)^2}{\langle x_1(t), n_i \rangle \langle x_2(t), n_i \rangle} \geq 0. \quad (2.8)
$$

This implies that $\|x_1(t) - x_2(t)\|^2$ increases in $t \in [0, T)$; hence

$$
\lim_{t \rightarrow T^-} x_1(t) \neq \lim_{t \rightarrow T^-} x_2(t). \quad \square
$$

THEOREM 2.6
Every focal submanifold is a limit of the mean curvature flow of certain isoparametric submanifold.

We need a couple of lemmas to prove Theorem 2.6. First, a simple lemma on scaling and the proof is obvious.
LEMMA 2.7
If \( f : M \times [0, T) \to \mathbb{R}^N \) is a solution to the mean curvature flow with \( f(x, 0) = f_0(x) \), then given any \( r \neq 0 \), \( \tilde{f}(x, t) = rf(x, r^{-2}t) \) is a solution with \( \tilde{f}(x, 0) = rf_0(x) \).

LEMMA 2.8
Let \( f : M^n \to S^{N-1}(r_0) \) be an immersed minimal submanifold of a sphere with radius \( r_0 \). For any \( x \in M \), the solution to the mean curvature flow equation in \( \mathbb{R}^N \) with initial data \( M \) is given by
\[
F(x, t) = \sqrt{1 - \frac{2nt}{r_0^2}} f(x).
\]
In particular, the mean curvature flow of \( M \) shrinks to a point homothetically in finite time \( T_0 = r_0^2 / (2n) \).

Proof
For a minimal submanifold of the sphere \( S^{N-1}(r) \) with radius \( r \), the mean curvature vector (as submanifold in \( \mathbb{R}^N \)) at a point \( x \) is \( -nx/r^2 \). Let \( F(x, t) = rf(x) \) for \( x \in M \) with \( r(t) \geq 0 \). Then the mean curvature vector field of \( F(\cdot, t) \) at point \( x \) is given by \( -(n/(r_0^2 r(t))) f(x) \). So \( F(x, t) \) satisfies the mean curvature flow equation for \( f \) if and only if
\[
r'(t) = -\frac{n}{r_0^2 r(t)} \quad \text{and} \quad r(0) = 1.
\]
It follows that \( r(t) = \sqrt{1 - (2nt/r_0^2)} \).

Proof of Theorem 2.6
In each isoparametric family, there exists a unique isoparametric submanifold \( M \subset S^{N-1}(1) \), which is minimal in \( S^{N-1}(1) \). Let \( x_0 \in M \). The mean curvature flow for minimal submanifolds in spheres can be solved explicitly as in Lemma 2.8; that is, \( x(t) = \sqrt{1 - 2nt} x_0 \) is a solution of (2.5), and \( x(t) \in C \) for all \( t \in [0, 1/(2n)) \).

Recall that integral curves of \( H(x) = -\sum_{i=1}^g \frac{m_i \cdot n}{\langle x, n \rangle} \) map to integral curves of the polynomial vector field \( \eta \) under the homeomorphism \( P \) defined in Theorem 2.4. Since the integral curve starting from \( x_0 \) lies in \( C \), the flow of \( \eta \) starting at \( P(x_0) \) lies in \( P(C) \). But \( -\eta \) is a polynomial vector field, and the one-parameter subgroup \( \phi_t \) generated by \( -\eta \) is a globally defined polynomial map. So there exists \( \delta > 0 \) and an open subset \( \mathcal{U} \) of \( P(C) \) such that \( \mathcal{U} \) contains the origin and \( \phi_t(z) \in P(C) \) for \( t \in (0, \delta) \) and \( z \in \mathcal{U} \). This shows that the flow of \( -\eta \) starting at the boundary of \( \mathcal{U} \) points inward in \( P(C) \). Hence any boundary point of \( P^{-1}(\mathcal{U}) \) is a limit of some MCF with initial data in \( C \).
It follows from Lemma 2.7 that any focal submanifold can be a limit of some MCF with some initial isoparametric submanifold.

As consequence of Theorem 2.5 and Lemma 2.8, we have the following.

**COROLLARY 2.9**

Let $M$ be an isoparametric submanifold. The mean curvature flow of $M$ converges to a point if and only if it is minimal in the sphere containing it.

Below, we describe the rate of collapsing of the MCF for isoparametric submanifolds. Recall that a MCF, $M_t$, which collapses at time $T < \infty$, is said to have type I singularity (see [W]) if there is a constant $c_0$ such that

$$\|\Pi(t)\|_\infty^2 (T - t) \leq c_0$$

for all $t \in [0, T)$, where $\|\Pi(t)\|_\infty$ is the sup norm of the second fundamental form for $M_t$.

**PROPOSITION 2.10**

Let $x(t)$ be a solution of the MCF (2.5), and let $T$ be the maximal time. Then

1. $x(T) := \lim_{t \to T^-} x(t)$ exists and belongs to the boundary $\partial C$ of the Weyl chamber $C$;
2. $\lim_{t \to T^-} \frac{\|x(t) - x(T)\|^2}{T - t} = 2m$, where $m = \dim(M_{x(0)}) - \dim(M_{x(T)})$;
3. if $x(T)$ lies in a highest-dimensional stratum of $\partial C$, then the MCF has type I singularity.

**Proof**

We have already proved (1) in Theorem 2.5. Statement (2) follows from l’Hôpital’s rule:

$$\lim_{t \to T^-} \frac{\|x(t) - x(T)\|^2}{T - t} = \lim_{t \to T^-} \frac{2\langle x(t) - x(T), x'(t) \rangle}{T - t} - 1$$

$$= 2 \lim_{t \to T^-} \sum_{i=1}^{g} \left( x(t) - x(T), \frac{m_i n_i}{\langle x(t), n_i \rangle} \right)$$

$$= 2 \lim_{t \to T^-} \sum_{i \in I} m_i \frac{\langle x(t) - x(T), n_i \rangle}{\langle x(t), n_i \rangle} + 2 \sum_{i \in I} m_i \frac{\langle x(T), n_i \rangle}{\langle x(t), n_i \rangle} = 2 \sum_{i \in I} m_i,$$

which is the dimension of the fiber of $M_{x(0)} \to M_{x(T)}$. Here

$$I = \{ 1 \leq i \leq g \mid \langle x(T), n_i \rangle = 0 \}.$$
We now prove statement (3). If \( x(T) \) lies in a highest-dimensional stratum of \( \partial C \), then there exists a unique \( i \) such that \( x(T) \) lies in the hyperplane defined by \( n_i \); that is, \( \langle x(T), n_i \rangle = 0 \). We may assume that \( i = 1 \). Note that the norm square of the second fundamental form of \( M_{x(t)} \) satisfies
\[
\left\| \Pi(x(t)) \right\|^2 (T - t) \\
\leq \sum_{i=1}^{g} m_i \| n_i \|^2 \frac{(T - t)}{\langle x(t), n_i \rangle^2} \\
= \frac{m_1 \| n_1 \|^2 (T - t)}{\langle x(t), n_1 \rangle^2} + \sum_{i=2}^{g} \frac{m_i \| n_i \|^2}{\langle x(t), n_i \rangle^2} (T - t).
\]
As \( t \to T^− \), the second term tends to zero because \( \langle x(T), n_i \rangle \neq 0 \) for all \( i \geq 2 \), and by l'Hôpital's rule, the first term has the same limit as
\[
\frac{-m_1 \| n_1 \|^2}{-2\langle x(t), n_1 \rangle \sum_{i=1}^{g} m_i (\langle n_i, n_1 \rangle / \langle x(t), n_i \rangle)}.
\]
But the denominator tends to \(-2m_1 \| n_1 \|^2\), so the limit is \( 1/2 \).

We remark that there is an open dense subset \( \mathcal{O} \) of the Weyl chamber \( C \) such that the solution \( x(t) \) of (2.5) with \( x(0) \in \mathcal{O} \) converges to a point in a highest-dimensional stratum of \( \partial C \).

### 3. Solutions to the mean curvature flow equation

In this section, we use Theorem 2.4 to construct explicit solutions of the MCF (2.5) by selecting a set of generators \( P_1, \ldots, P_k \) for the \( W \)-invariant polynomials and calculating flows of the polynomial vector field \( \eta \).

We use the root system of the Coxeter group given in [GB]. Let \( M \) be a compact, irreducible isoparametric submanifold in \( \mathbb{R}^N \), let \( W \) be its Weyl group, and let \( n_i \) be its curvature normals. Let \( \Pi \) denote a set of simple roots of \( W \), and let \( \Delta^+ \) be the set of positive roots defined by \( \Pi \). Then \( \{ \mathbb{R} n_i \mid 1 \leq i \leq g \} \) is equal to \( \{ \mathbb{R} \alpha \mid \alpha \in \Delta^+ \} \). So the Weyl chamber \( C \) containing \( x_0 \) is precisely given by
\[
C = \{ x \in V \mid -\langle x, \alpha \rangle > 0 \text{ for all } \alpha \in \Pi \}.
\]
The closure of \( C \) is
\[
\overline{C} = \{ x \in V \mid -\langle x, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Pi \},
\]
and the MCF (2.5) becomes

\[ x'(t) = -\sum_{\alpha \in \Delta^+} \frac{m_\alpha}{\langle x(t), \alpha \rangle} \alpha, \]  

(3.1)

where \( m_\alpha \) is the multiplicity of the curvature normal which is parallel to \( \alpha \). Since (3.1) is invariant under rescaling of each \( \alpha \), we may normalize roots of the Coxeter group to be of unit length.

If \( M = M_1 \times M_2 \) with \( M_i \) an isoparametric submanifold of \( \mathbb{R}^{N_i} \) for \( i = 1, 2 \), then the Weyl group of \( M \) is the product of the Weyl groups of \( M_1 \) and \( M_2 \) and the mean curvature flow of \( M \) is the product of the mean curvature flows of \( M_1 \) and \( M_2 \). So, without loss of generality, we may assume that \( M \) is an irreducible isoparametric submanifold. In the rest of this section, we work out explicit solutions for mean curvature flow equations for compact isoparametric submanifolds whose Coxeter groups are \( A_k, B_k, D_k \), and \( G_2 \).

Example 3.1: The \( A_k \)-case

Suppose that \( k \geq 2 \). Let \( \{e_1, \ldots, e_{k+1}\} \) be the standard orthonormal basis of \( \mathbb{R}^{k+1} \), and let \( (x_1, \ldots, x_{k+1}) \) be the corresponding coordinate. The set \( (1/\sqrt{2})(e_i - e_{i+1}) \) with \( 1 \leq i \leq k \) is a simple root system of \( A_k \), and the set of positive roots is \( (1/\sqrt{2})(e_i - e_j) \) with \( 1 \leq i < j \leq k + 1 \). Let

\[ V := \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} \mid x_1 + \cdots + x_{k+1} = 0\}. \]

The normal space of an isoparametric submanifold of type \( A_k \) can be identified with \( V \). The Coxeter group acts on \( V \) and is generated by all permutations of \( \{e_1, \ldots, e_{k+1}\} \). The open positive Weyl chamber \( C \) containing \( x_0 \) is

\[ C = \{(x_1, \ldots, x_{k+1}) \in V \mid x_1 < x_2 < \cdots < x_{k+1}\}. \]

Since the multiplicities of curvature spheres are invariant under the action of the Coxeter group, there is only one possible multiplicity, which we denote by \( m \). The MCF equation (3.1) is equivalent to

\[ \frac{1}{m} \frac{d}{dt} x_i = \frac{1}{x_j - x_i}, \quad 1 \leq i \leq k + 1. \]

(3.2)

If \( x(0) \in V \), then \( x(t) \in V \) for all \( t \). This follows from the simple fact that \( \frac{d}{dt}(x_1 + \cdots + x_{k+1}) = 0 \).

Let \( \sigma_r \) be the \( r \)th elementary symmetric polynomial in \( x_1, \ldots, x_{k+1} \),

\[ \sigma_r = \sum_{1 \leq i_1 < \cdots < i_r \leq k+1} x_{i_1} \cdots x_{i_r}. \]
and let $\sigma_0 = 1$. Let $x(t)$ be a solution of (3.2), and let

$$y_r(t) = \sigma_r(x(t)).$$

A straightforward computation shows that

$$\begin{cases}
y_2' = n, \\
y_r' = \frac{1}{2} m(k - r + 3)(k - r + 2)y_{r-2}.
\end{cases} \quad (3.3)$$

The explicit formula for $y_r(t)$ can be obtained from this equation recursively, and it is a polynomial in $t$ with initial conditions $x_1(0), \ldots, x_{k+1}(0)$. For each $t$, we can obtain $x_1(t), \ldots, x_{k+1}(t)$ as the $k + 1$ solutions of the following polynomial equation in $z$:

$$\sum_{r=0}^{k+1} (-1)^{k+1-r} y_{k+1-r}(t) z^r = 0 \quad (3.4)$$

with the property

$$x_1(t) < x_2(t) < \ldots < x_{k+1}(t).$$

**Example 3.2: The $B_k$-case**

Suppose that $k \geq 2$. Let $\{e_1, \ldots, e_k\}$ be the standard orthonormal basis of $\mathbb{R}^k$, and let $(x_1, \ldots, x_k)$ be the corresponding coordinate. We identify $\mathbb{R}^k$ with a normal space of an isoparametric submanifold of type $B_k$. The set $e_k$ and $(1/\sqrt{2})(e_i - e_{i+1})$ with $1 \leq i \leq k - 1$ is a simple root system of $B_k$, and the set of positive roots is $e_i$ with $1 \leq i \leq k$ and $(1/\sqrt{2})(e_i \pm e_j)$ with $1 \leq i < j \leq k$. The Coxeter group is generated by all permutations and sign changes of $\{e_1, \ldots, e_k\}$. Since the multiplicities of curvature spheres are invariant under the action of the Coxeter group, there are only two possible multiplicities. Let $m_1$ be the multiplicities of curvature spheres corresponding to $(1/\sqrt{2})(e_i \pm e_j)$, and let $m_2$ be the multiplicities of curvature spheres corresponding to $e_i$. So the MCF (3.1) is equivalent to

$$-x_i' = \frac{m_2}{x_i} + m_1\left(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j}\right).$$

Set $y_i = x_i^2$. Then we have

$$\frac{1}{2} \frac{d}{dt} y_i = -m_2 - m_1 \sum_{j \neq i} \frac{2y_i}{y_i - y_j}. \quad (3.5)$$

Let $s_0 = 1$, let $s_i$ be the $i$th elementary symmetric polynomial of $y_1, \ldots, y_k$, and let

$$\zeta_r(t) = s_r(y(t)).$$
A straightforward computation shows that
\[
\zeta_j' = -2(k - j + 1)(m_2 + m_1(k - j))\zeta_{j-1}, \quad 1 \leq j \leq k.
\] (3.6)

Note that when \( j = 1 \), the right-hand side is \(-2k(m_2 + m_1(k - 1))\), which is equal to \(-2n\). An explicit formula for \( \zeta_j(t) \) can be obtained from (3.6) recursively. Hence we get the solution to the MCF equation for the \( B_k \)-cases.

**Example 3.3: The \( D_k \)-case**

Suppose that \( k \geq 4 \). Let \( \{e_1, \ldots, e_k\} \) be the standard orthonormal basis of \( \mathbb{R}^k \), and let \( (x_1, \ldots, x_k) \) be the corresponding coordinate. We identify \( \mathbb{R}^k \) with a normal space of an isoparametric submanifold of type \( D_k \). The set of simple roots is \((1/\sqrt{2})(e_i - e_i)\) with \( 1 \leq i \leq k - 1 \), and the set of positive roots is \((1/\sqrt{2})(e_i \pm e_j) \mid 1 \leq i < j \leq k\). The Coxeter group is generated by all permutations and even numbers of sign changes of \( \{e_1, \ldots, e_k\} \). The open positive Weyl chamber \( C \) is

\[
C := \{x \in \mathbb{R}^k \mid x_1 < x_2 < \cdots < x_k \text{ and } x_{k-1} + x_k < 0\}.
\]

All multiplicities for curvature spheres are equal and are denoted by \( m \). The mean curvature flow equation (3.1) becomes

\[
\frac{1}{m} \frac{d}{dt} x_i = \sum_{q \neq i} \frac{2x_i}{x_q^2 - x_i^2}.
\]

Multiply both sides by \( x_i/2 \) to obtain

\[
\frac{1}{4m} \frac{d}{dt} y_i = \sum_{q \neq i} \frac{y_i}{y_q - y_i},
\] (3.7)

where \( y_i := x_i^2 \) for all \( i = 1, \ldots, k \).

Let \( s_i \) be the \( i \)th elementary symmetric polynomial of \( y_1, \ldots, y_k \). We set \( s_0 = 1 \). Then \( s_1, \ldots, s_{k-1} \) and \( \sqrt{s_k} \) generate the algebra of polynomials invariant under the action of the Coxeter group. Note that equation (3.7) is a special case of equation (3.5) with \( m_2 = 0 \) and \( m_1 = m \). Set \( \zeta_j(t) = s_j(x(t)) \). From equation (3.6), we obtain

\[
\zeta_r' = -2m(k - r + 1)(k - r)\zeta_{r-1}, \quad 1 \leq r \leq k.
\]

This gives a recursive solution to \( \zeta_r \) and hence produces a solution to the MCF equation.

**Example 3.4: The \( G_k \)-case**

In rank 2 cases, the Weyl group is the dihedral group with \( 2g \) elements (\( g = 3 \) for \( A_2 \), \( g = 4 \) for \( B_2 \), and \( g = 6 \) for \( G_2 \)). We identify the normal space of a rank 2
isoparametric submanifold with $\mathbb{R}^2 = \mathcal{G}$, and we use $e^{ik\pi/g}$ with $0 \leq k < g$ as positive roots. Then

$$P_1(x) = x_1^2 + x_2^2, \quad P_2(x) = \text{Re}((x_1 + ix_2)^g)$$

form a set of generators for the ring of invariant polynomials. Since $A_2$- and $B_2$-cases are covered in the above examples, we need only discuss the $G_2$-case. In this case, all multiplicities are equal and are either 1 or 2. Let $F_i = \sum_{i=1}^{6} m_i \langle \nabla P_i, n_i \rangle$ for $i = 1, 2$ be as in Theorem 2.4. Then $F_1(x) = 2n$ and $F_2(x) = 0$ (see [PT2]). If $x(t)$ is a solution of the MCF, then it follows from Theorem 2.4 that $y(t) = (P_1(x(t)), P_2(x(t)))$ satisfies $y'(t) = (-2n, 0)$. By Lemma 2.7, it suffices to consider initial data $x_0 = e^{i\theta_0}$ for the MCF. Then $y(t) = \left(1 - 2nt, \cos(6\theta_0)\right)$. Set $x(t) = r(t)e^{i\theta(t)}$. Since $y_1(t) = r^2(t)$ and $y_2(t) = r^6(t) \cos 6\theta(t)$, the solution of the MCF for the $G_2$-case with initial data $e^{i\theta_0}$ is

$$r(t) = (1 - 2nt)^{1/2} \quad \text{and} \quad \phi(t) = \frac{1}{6} \cos^{-1}\left(\frac{\cos 6\theta_0}{(1 - 2nt)^3}\right).$$

4. Mean curvature flows of focal submanifolds

We consider the mean curvature flows of focal submanifolds of isoparametric submanifolds. They behave very similarly to the mean curvature flows of isoparametric submanifolds. With slight modification, most results in Section 2 also hold for mean curvature flows of focal submanifolds.

Let $M_0$ be an isoparametric submanifold, and let $p \in M_0$ be a fixed point. As before, we assume that $M_0$ is contained in a sphere centered at the origin. Let $C \subset p + v_p M_0$ be the Weyl chamber containing $p$, and let $\Delta^+$ be the set of positive roots of the Weyl group $W$ associated to $M_0$. Then

1. $\tilde{C}$ is a stratified space;
2. the isotropy subgroup of any two points in the same stratum $\sigma$ is the same, and it is denoted by $W_\sigma$;
3. for $x \in \partial C$, let

$$\Delta^+(x) = \{\alpha \in \Delta^+ \mid \langle x, \alpha \rangle > 0\};$$

then $\Delta^+(x_1) = \Delta^+(x_2)$ if and only if $x_1, x_2$ lie in the same stratum $\sigma$, and it is denoted by $\Delta^+(\sigma)$;
4. $\sigma$ is the Weyl chamber of $W_x$ for $x \in \sigma$, and $\sigma$ is an open simplicial cone in the following linear subspace:

$$V(\sigma) = \{x \in p + v_p M_0 \mid \langle x, \alpha \rangle = 0 \text{ for all } \alpha \in \Delta^+ \setminus \Delta^+(\sigma)\}. $$

Let $\sigma$ be a stratum in $\partial C$, let $x_0 \in \sigma$, and let $M$ be the focal submanifold of $M_0$ through $x_0$. By [T, Theorem 4.1], the mean curvature vector field of $M$ at $x_0$ is given...
by

\[ H(x_0) = - \sum_{\alpha \in \Delta^+(\sigma)} \frac{m_\alpha}{\langle x_0, \alpha \rangle} \alpha, \]

(4.1)

where \( m_\alpha \) are multiplicities of curvature spheres of \( M_0 \). Moreover,

\[ \langle x_0, \alpha \rangle = \langle H(x_0), \alpha \rangle = 0 \]

(4.2)

for all \( \alpha \in \Delta^+ \setminus \Delta^+ (\sigma) \). The MCF equation of \( M \) is the following ODE on \( \sigma \):

\[ \frac{dx}{dt} = - \sum_{\alpha \in \Delta^+(\sigma)} \frac{m_\alpha}{\langle x, \alpha \rangle} \alpha. \]

(4.3)

The analogue of Theorem 2.2 also holds for this case. In particular, if \( x(t) \) satisfies the flow equation (4.3), then

\[ \|x(t)\|^2 = \|x(0)\|^2 - 2nt, \]

(4.4)

where \( n = \sum_{\alpha \in \Delta^+(\sigma)} m_\alpha \) is the dimension of \( M \). Therefore we have the following.

**Theorem 4.1**

The maximal interval for the solution of the MCF equation for any focal submanifold is finite.

Suppose that \( x(t) \) and \( y(t) \) satisfy equation (4.3) on \( \sigma \), and suppose that \( x(0) \neq y(0) \). Use the same computation for (2.8) to get

\[
\frac{1}{2} \frac{d}{dt} \|x(t) - y(t)\|^2 = \langle x(t) - y(t), x'(t) - y'(t) \rangle
\]

\[ = \sum_{\alpha \in \Delta^+(\sigma)} m_\alpha \frac{\langle x(t) - y(t), \alpha \rangle^2}{\langle x(t), \alpha \rangle \langle y(t), \alpha \rangle} > 0. \]

(4.5)

Then the proofs given in Section 2 work, so we have the following.

**Theorem 4.2**

Let \( M^n \subset S^{n+k-1} \) be a compact isoparametric submanifold in \( \mathbb{R}^{n+k} \), let \( W \) be its Weyl group, let \( C \) be the Weyl chamber in \( x_0 + \nu(M)_{x_0} \) containing \( x_0 \in M \), and let \( M_\gamma \) be the submanifold parallel to \( M \) through \( y \). If \( \sigma \subset \overline{C} \) is a stratum, then

(1) there is a unique \( y_\sigma \in \sigma \) such that the focal submanifold \( M_{y_\sigma} \) is minimal in \( S^{n+k-1} \), and the MCF in \( \mathbb{R}^{n+k} \) with initial data \( M_{y_\sigma} \) homothetically shrinks to a point.
(2) if \( y_0 \in \sigma \cap S^{n+k-1} - \{ y_\sigma \} \), then the MCF in \( \mathbb{R}^{n+k} \) with \( M_{y_0} \) as initial data blows up in finite time \( T < 1/(2n) \), \( x(t) \in \sigma \) for all \( t \in [0, T) \), and \( \lim_{t \to T^-} x(t) \in \partial \sigma \); in particular, the limit is a focal submanifold with lower dimension; 

(3) if \( y_1, y_2 \in \sigma \cap S^{n+k-1} \) are distinct, then the MCF in \( \mathbb{R}^{n+k} \) with initial data \( M_{y_1} \) and \( M_{y_2} \) converge to distinct focal submanifolds of lower dimensions.

5. Mean curvature flows for isoparametric submanifolds in spheres

If \( M^n \subset S^{n+k-1} \) is an isoparametric submanifold in \( \mathbb{R}^{n+k} \), then \( M \) is also isoparametric in \( \mathbb{R}^{n+k} \). So basic structure theory for isoparametric submanifolds in Euclidean spaces also applies to \( M \). For \( x \in M \), let \( H(x) \) and \( H_E(x) \) be the mean curvature vector fields of \( M \) at \( x \) as a submanifold of \( S^{n+k-1} \) and \( \mathbb{R}^{n+k} \), respectively. Then \( H(x) \) is the orthogonal projection of \( H_E(x) \) to \( T_x S^{n+k-1} \). More precisely,

\[
H(x) = H_E(x) + nx
\]

for all \( x \in M \). In particular, \( H \) is again a parallel normal vector field along \( M \). The mean curvature flow of \( M \) as a submanifold of \( S^{n+k-1} \) behaves similarly to its flow as a submanifold of \( \mathbb{R}^{n+k} \). With slight modifications, most results for mean curvature flows for isoparametric submanifolds in the Euclidean spaces also hold for isoparametric submanifolds in spheres. We need only explain how to deal with the arguments in the Euclidean case which cannot be applied directly to the spherical case.

**Proof of Theorem 1.2**

Fix \( x_0 \in M \), and let \( V = x_0 + v_{x_0} M \) be the normal space of \( M \) as a submanifold of \( \mathbb{R}^N \) at the point \( x_0 \), let \( W \) be its Coxeter group, and let \( C \subset V \) be the Weyl chamber containing \( x_0 \). The mean curvature flow of \( M \) in \( S^{n+k-1} \) is uniquely determined by the flow of \( x_0 \) in \( S := C \cap S^{k-1} \):

\[
x'(t) = - \sum_{\alpha \in \Delta} \frac{m_\alpha \alpha}{\langle x(t), \alpha \rangle} + nx(t).
\]

(5.1)

The set \( S \) is a geodesic \((k-1)\)-simplex on \( S^{k-1} \). Let \( x(t) \in S \) be a solution to equation (5.1) with initial condition \( x_0 \). Then

\[
y(t) = \sqrt{1 - 2nt} \cdot x \left( - \frac{1}{2n} \log(1 - 2nt) \right)
\]

satisfies the Euclidean mean curvature flow equation (3.1) with initial condition \( y(0) = x_0 \). Let \([0, T_x)\) and \([0, T_y)\) be the maximal intervals for the domains of \( x(t) \) and \( y(t) \), respectively. Then

\[
T_x = - \frac{1}{2n} \log(1 - 2nT_y)
\]
and

\[
\lim_{t \to T_y} y(t) = \sqrt{1 - 2nT_y} \lim_{t \to T_x} x(t).
\]

Note that by Theorem 2.2 and Corollary 2.9, \( T_y \leq 1/(2n) \) and the equality holds if and only if the isoparametric submanifold \( M_0 \) passing \( x_0 \) is minimal in the sphere \( S^{n+k-1} \). So by Theorem 2.5, if \( M_0 \) is not minimal in the sphere, then \( x(t) \) converges to a focal submanifold at a finite time \( T_x \). This proves Theorem 1.2.

If \( x_1(t) \in S \) and \( x_2(t) \in S \) satisfy the spherical mean curvature flow equation (5.1), then

\[
\frac{1}{2} \frac{d}{dt} \|x_1(t) - x_2(t)\|^2 = \|x_1 - x_2, (H_E(x_1) + nx_1) - (H_E(x_2) + nx_2)\|
\]

\[= n\|x_1 - x_2\|^2 + \langle x_1 - x_2, H_E(x_1) - H_E(x_2) \rangle.
\]

By (2.8), \( \langle x_1 - x_2, H_E(x_1) - H_E(x_2) \rangle \geq 0 \). Therefore

\[
\frac{d}{dt} \|x_1(t) - x_2(t)\|^2 \geq 2n\|x_1(t) - x_2(t)\|^2.
\]

(5.2)

We use (5.2) to give an estimate of the maximal interval \([0, T)\) for the spherical mean curvature flow \( x(t) \). Let \( p_0 \) be the unique point in \( S \) such that the isoparametric submanifold passing \( p_0 \) is minimal in the sphere \( S^{n+k-1} \). Set \( x_1(t) = x(t) \) and \( x_2(t) = p_0 \) in equation (5.2). Since \( x_2(t) \) exists for all \( t > 0 \), we obtain

\[
\|x(t) - p_0\| \geq e^{nt}\|x(0) - p_0\|
\]

for all \( t \) as long as \( x(t) \in S \). Let \( D \) be the diameter of \( S \); then \( D \leq 2 \) and

\[
T \leq \frac{1}{n} \log \frac{D}{\|x(0) - p_0\|}.
\]

Now, we discuss the behavior of invariant polynomials under the spherical mean curvature flow. Let \( x(t) \in S \) be the mean curvature flow of \( x_0 \). For any function \( f \) on \( V \), let \( f(t) = f(x(t)) \). Then

\[
f'(t) = \langle \nabla f(x(t)), H(x(t)) \rangle = \langle \nabla f(x(t)), H_E(x(t)) \rangle + n\langle \nabla f(x(t)), x(t) \rangle.
\]

If \( f \) is a homogenous polynomial of degree \( k \) which is invariant under the action of the Coxeter group \( W \), then as in the proof of Theorem 2.5,

\[
f'(t) = -F(x(t)) + nkf(t),
\]

(5.3)
where \( F \) is defined by equation (2.7) and it is an invariant polynomial of degree \( k - 2 \). If we have computed \( F(t) := F(x(t)) \), then we can solve \( f(t) \) from equation (5.3) and obtain

\[
f(t) = -e^{knt} \int e^{-knt} F(t) \, dt.
\] (5.4)

Note that there is no homogeneous invariant polynomial of degree 1. By induction on the degree, we obtain the following.

**THEOREM 5.1**

If \( x(t) \) satisfies the spherical mean curvature flow equation (5.1) and \( f \) is a \( W \)-invariant polynomial, then

\[
f(t) = f(x(t)) = c_1 e^{knt} + h(t)
\]

for some constant \( c_1 \) and polynomial \( h \).

In particular, \( f(t) \) is well defined for all \( t \in \mathbb{R} \). In Section 3, we have given explicit formulas for \( F_i \) for invariant homogeneous polynomials \( P_i \) for isoparametric submanifolds. We can use these formulas and (5.4) to construct explicit solutions to the spherical mean curvature flow equation for isoparametric submanifolds in spheres.

**Example 5.2: Phase portrait for rank 2 cases**

Let \( M^n \subset S^{n+1} \subset \mathbb{R}^{n+2} \) be an isoparametric hypersurface with \( g \) distinct principal curvatures. Then the Weyl group associated to \( M \) as a rank 2 isoparametric submanifold in \( \mathbb{R}^{n+2} \) is the dihedral group of \( 2g \) elements. Let \( C \) denote the Weyl chamber containing \( x_0 \in M \), and let \( D \) be the intersection of \( C \) and the normal circle at \( x_0 \) in \( S^{n+1} \). Let \( p_1, p_2 \) denote the end points of \( D \). The arc \( D = \overline{p_1p_2} \) has length \( \pi/g \). For \( y \in \bar{C} \), let \( M_y \) denote the submanifold through \( y \) which is parallel to \( M \) (a leaf of the isoparametric foliation). There exists a unique \( p_0 \in D \) such that \( M_{p_0} \) is minimal in \( S^{n+1} \).

1. The spherical MCF (5.1) has three orbits: a stationary point \( p_0 \), the orbit \( \overline{p_0p_1} \) with one end tending to \( p_0 \) and the other end tending to \( p_1 \), and the orbit \( \overline{p_0p_2} \) with one end tending to \( p_0 \) and the other end tending to \( p_2 \).
2. The MCF (2.5) in \( \mathbb{R}^{n+2} \) starting at \( M_y \) degenerates homothetically to one point (the origin) if \( y = p_0 \), to \( M_{rp_2} \) for some \( 0 < r < 1 \) if \( y = p_0p_2 \), and to \( M_{rp_1} \) for some \( 1 < r < 1 \) if \( y = \overline{p_1p_0} \).

**Example 5.3: Phase portrait for rank \( A_3 \) cases**

Let \( M^n \subset S^{n+2} \) be an isoparametric submanifold with Weyl group \( A_3 \) and uniform multiplicity \( m \), and let \( x_0 \in M \). Let \( C \) denote the Weyl chamber containing \( x_0 \), and let \( D \) be the intersection of \( C \) and the normal sphere at \( x_0 \). Then \( D \) is a geodesic
triangle with vertices $p_1, p_2, p_3$ and interior angles $\pi/3, \pi/3, \pi/2$. The phase spaces of spherical MCF (5.1) and Euclidean MCF (2.5) are $D$ and $C$, respectively. We describe the phase portraits.

(1) There exists a unique $p_0$ in the interior of $D$ such that $p_0$ is the fixed point of the ODE (5.1). This implies that the spherical MCF starting at the parallel submanifold $M_{p_0}$ is stationary and $M_{p_0}$ is minimal in $S^{n+k-1}$.

(2) For each $1 \leq i \leq 3$, there is a unique flow $\ell_i$ that at one end approaches $p_0$ and at the other end approaches $p_i$. This implies that for $y \in \ell_i$, the spherical MCF starting at $M_y$ collapses in finite time to the focal submanifold $M_{p_i}$ by collapsing fibers of the fibration $M_y \to M_{p_i}$. The fibers are isoparametric submanifolds with Weyl group $A_2$ for $i = 1, 2$ and $A_1 \times A_1$ for $i = 3$. In particular, when $m = 2$, $M_y$ is diffeomorphic to the manifold of flags in $\mathbb{C}^4$ and collapsing is along complex flag manifolds of $\mathbb{C}^3$ in $M_y$ for $i = 1, 2$ and $S^2 \times S^2$ in $M_y$ for $i = 3$.

(3) For distinct $i, j, k$, let $D_k$ denote the triangle with vertices $p_i, p_j, p_0$, edges $\ell_i, \ell_j,$ and geodesic segment $\hat{\ell}_i p_j$ in the sphere. The flow for (5.1) starting at a point in the interior $D_k^0$ of $D_k$ exists for finite time and converges to a point on the interior of $\hat{\ell}_i p_j$. This implies that for $y \in D_k^0$, the spherical MCF starting at $M_y$ converges in finite time to a focal submanifold $M_q$ with $q \in \hat{\ell}_i p_j \setminus \{p_i, p_j\}$ by collapsing one family of curvature spheres.

(4) The flow of (2.5) on $C$ starting at $p_0$ is the straight line joining the origin to $p_0$, the flow starting from a point in $D_k^0$ converges to a point on the wall containing $p_i, p_j$ (i, j, k distinct), and the flow starting at a point on $\ell_i$ converges to a point on the line segment $\overline{Op_i}$. This implies that the Euclidean MCF with initial data $M_y$

(i) shrinks homothetically to the origin if $y = p_0$;

(ii) converges to a focal submanifold $M_q$ for some $q$ which lies in the open cone spanned by $\hat{\ell}_i p_j$, and the collapsing is along a curvature $m$-sphere if $y \in D_k^0$;

(iii) converges to a focal submanifold $M_q$ with $q \in \overline{Op_i}$ for $y \in \ell_i$; moreover, the collapsing is along fibers of the fibration $M_y \to M_q$, and the fibers are isoparametric submanifolds with Weyl group $A_2, A_2, A_1 \times A_1$, respectively, for $i = 1, 2, 3$.

6. Mean curvature flow for polar action orbits

Let $G$ act on a Riemannian manifold $N$ isometrically, and let $G \cdot p$ be a principal orbit through $p$. If $v \in \nu(G \cdot p)_p$, then $\hat{v}(g \cdot p) = dg_p(v)$ is a globally defined normal vector field on $G \cdot p$ and is called a $G$-equivariant normal field. An isometric action of a compact Lie group $G$ on a Riemannian manifold $N$ is called polar if there is a totally geodesic submanifold $\Sigma$ that meets all $G$-orbits and meets orthogonally. Such a $\Sigma$ is called a section. We list some properties of polar actions (see [PT1]).
The action $G$ is polar if and only if every $G$-equivariant normal field is parallel with respect to the induced normal connection on $G \cdot p$ as a submanifold of $N$.

Let $N(\Sigma) = \{ g \in G \mid g \cdot \Sigma = \Sigma \}$ and $Z(\Sigma) = \{ g \in G \mid g \cdot x = x \ \forall \ x \in \Sigma \}$ denote the normalizer and centralizer of $\Sigma$, respectively. Then the quotient group $W(\Sigma) = N(\Sigma)/Z(\Sigma)$ is a finite group acting on $\Sigma$, and it is called the generalized Weyl group associated to the polar action.

The orbit space $\Sigma/W$ is isomorphic to $N/G$, and the ring of smooth $G$-invariant functions on $N$ is isomorphic to the ring of $W$-invariant functions on $\Sigma$ under the restriction map.

If $p_0 \in \Sigma$ is a singular point, that is, $G \cdot p_0$ is a singular orbit, then the slice representation of $G_{p_0}$ on the normal space of the orbit at $p_0$ is a polar representation.

**THEOREM 6.1**

Suppose that the isometric action $G$ on $N$ is polar, and suppose that $\Sigma$ is a section. Then

(i) if $x \in \Sigma$, the mean curvature vector $\xi(x)$ of $G \cdot x$ at $x$ is tangent to $\Sigma$ at $x$;

(ii) if $x'(t) = \xi(x(t))$ with $x(t)$ regular (i.e., $G \cdot x(t)$ is a principal orbit), $G \cdot x(t)$ satisfies the MCF in $N$; in other words, the MCF in $N$ with a principal $G$-orbit as initial data flows among principal $G$-orbits.

For general polar action, $W$ need not be a Coxeter group and the orbit space of the $W$-action on the section can be complicated. In fact, given any finite group $W$ and any compact $W$-manifold, there exist a Riemannian manifold $N$, a compact group $G$, and an isometric polar $G$-action on $N$ such that the induced action on the section is the given $W$-action (see [PT1]). Hence the behavior of the MCF for general polar actions is not as clear as in the sphere and Euclidean cases.

A polar action on a symmetric space is hyperpolar if the sections are flat. In this case, the fundamental domain of the $W$-action on a section is a geodesic simplex. A submanifold in a symmetric space is called equifocal if its normal bundle is flat, the exponential of each normal space is flat, and the focal radii along a parallel normal field are constant. It was proved in [TT] that principal orbits of a hyperpolar action on a symmetric space are equifocal, and parallel foliation of an equifocal submanifold is an orbitlike foliation. Moreover, generators of the ring of smooth functions which are constant along parallel leaves were constructed in [HLO]. Hence we believe that methods developed in this article can be used to solve the MCF starting with an equifocal submanifold in symmetric spaces.

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Liu
Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46566, USA; xliu3@nd.edu

Terng
Department of Mathematics, University of California at Irvine, Irvine, California 92697-3875, USA; cterng@math.uci.edu
