D-DIMENSIONAL RADIATIVE PLASMA: A KINETIC APPROACH

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Abstract

The covariant kinetic approach for the radiative plasma, a mixture of a relativistic moving gas plus radiation quanta (photons, neutrinos, or gravitons) is generalized to D spatial dimensions. The operational and physical meaning of Eckart’s temperature is reexamined and the D-dimensional expressions for the transport coefficients (heat conduction, bulk and shear viscosity) are explicitly evaluated to first order in the mean free time of the radiation quanta. Weinberg’s conclusion that the mixture behaves like a relativistic imperfect simple fluid (in Eckart’s formulation) depends neither on the number of spatial dimensions nor on the details of the collisional term. The case of Thomson scattering is studied in detail, and some consequences for higher dimensional cosmologies are also discussed.

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I. INTRODUCTION

In many astrophysical and cosmological problems, the matter-radiation content is modeled by a so-called radiative plasma, a two component fluid consisting of some material medium (perfect fluid, relativistic gas, ionized hydrogen atoms, etc.) plus radiation quanta (photons, neutrinos, or gravitons). In the simplest situation, the material medium is assumed to be locally in thermal equilibrium with itself (very short mean free times and mean free paths), whereas the massless component has a finite mean free time $\tau$. However, even in this case, an idealized perfect fluid description of this mixture is inappropriate, and a deeper insight is provided by a kinetic theory approach. In particular, since the radiation quanta are out of equilibrium with the material component, all transport properties of this system, to first order of approximation, depend exclusively on the mean free time of radiation.

The first self-consistent application of the relativistic kinetic theory to this system is due to Thomas [1], who computed the correct values of the heat and shear viscosity transport coefficients. Later on, by considering the concept of local equilibrium temperature, Weinberg [2] obtained the bulk viscosity coefficient, thereby showing that such a mixture behaves like an imperfect relativistic simple fluid in the framework of the hydrodynamical formulation for dissipative processes developed by Eckart [3]. Masaki [4], recovered the results of Thomas using a modern relativistic approach (manifestly covariant). This work was considerably simplified by Straumann [5], who also shown that if the Thomson scattering dominates, the bulk viscosity coefficient goes to zero. More recently, by considering the 9-moment Grad approximation method to solve the Boltzman equation, Schweizer [6] calculated the transient transport coefficients appearing in the linear, causal, hydrodynamical formulation proposed by Israel and Stewart [7]. The physical and operational meaning of local equilibrium temperature (Eckart’s temperature) has also been discussed by several authors [2], [1], [4].
On the other hand, since multidimensional models are presently believed to be the natural framework to unify all fundamental interactions, it is very common nowadays to study all phenomena and the underlying physics in an arbitrary number of dimensions. Historically, arbitrary dimensionality (including fractal dimension) has been widely considered in the context of Newtonian gravitational theory and general relativity, quantum field theory, cosmology, phase transitions, renormalization group, quantum mechanics, lattice models and random walks [10]- [21]. In domain of cosmology, for instance, several compactification mechanisms for extra dimensions, presumably taking place in the very early universe, have been suggested to explain the now observed 4-dimensional structure of spacetime [11]- [16].

Given the lack of knowledge about the initial state of the universe, the simplest dynamic approach to dimensional reduction is to construct a multidimensional anisotropic cosmological model based on some fundamental theory (superstrings, Kaluza-Klein or Einstein’s gravitational theory) where the extra spatial dimensions are collapsing while the “physical dimensions” are expanding, and eventually, leading to the present day universe. Isotropization and cosmological dimensional reduction has also been considered in the framework of nonequilibrium higher dimensional cosmologies. Concrete models have been proposed in the case of bulk and shear viscosities leading to the conclusion that viscosity may work efficiently to reduce the number of spatial dimensions. [14]- [16]. As a matter of fact, this is a new version of the earlier program of chaotic cosmology, for which the ordinary dissipative processes (viscosity, heat flow, ...etc.) were believed to be the mechanisms damping out any anisotropy and inhomogeneity existing in the primeval plasma [17]. However, even considering that the kinetic approach is more fundamental than its related thermodynamics, the analysis mentioned above are usually based only on macroscopic equations. In particular, to the best of our knowledge, the explicit form of the transport coefficients considered in such works have not been derived from kinetic theory in D spatial dimensions. This one of the main aims of this article.
It is also worth noticing that the definition of a radiative plasma, in principle, is not restricted to the class of systems mentioned above. Under certain conditions, more exotic system as, for instance, a mixture of massive and massless modes in superstrings models could, formally, be interpreted as a radiative plasma in D dimensions. The reason for this possibility may be easily understood. In the collisional limit, two different temperatures are naturally defined in an radiative plasma, namely: the matter temperature($T_m$) and the effective radiation kinetic temperature($T_r$). However, to first order in the mean free time, the latter is different from the former only if the expansion rate is different from zero. The expansion works to continuously pulling radiation and matter out of thermal equilibrium, because the components have different cooling rates. Basically, this is the mechanism accounting for the presence of the bulk viscosity in such a mixture as well as in a relativistic simple gas [8]. On the other hand, even in a thermodynamic setting, a gas of hot strings(superstrings, or Hagedorn’s fireballs) may also be described as a two temperature system when we have a mixture of heavy strings with a gas of light strings(see, for instance, [18]).

From the above considerations, it seems interesting to analyze the transport properties of a D-dimensional radiative plasma. In this work, by extending the covariant 4-dimensional kinetic approach developed by Straumann [3], we compute the transport coefficients for such a system. Hopefully, the expressions derived here may be useful to study the effects of dissipative processes in higher dimensional cosmologies as well as in modeling a mixture of massive and massless modes in the framework of superstring theory.

The paper is structured as follows: In the next section, the energy-momentum tensor for a D-dimensional radiative plasma is computed exactly by solving the D-dimensional Boltzmann transport equation to first order in the mean free time of the radiation quanta. In section 3 we discuss the physical meaning of Eckart’s temperature and obtain the transport coefficients of heat conduction, bulk and shear.
viscosities, and in section 4, the photon and total entropy production are discussed. In section 5, a specific model, namely, Thomson scattering, is studied. We conclude, in section 6, with a discussion of our main results for higher dimensional cosmology, and in the following two appendix we proof the basic mathematical properties used in the article. In our units the physical constants $\hbar = k_B = c = 1$, where $k_B$ is Boltzmann’s constant and the D+1-dimensional signature of the spacetime metric is $(+,-,-,-,...,-)$.

II. D-DIMENSIONAL KINETIC APPROACH

The basic object for computing the energy-momentum tensor(EMT) and other macroscopic quantities describing a D-dimensional radiative plasma is the radiation distribution function $F(k, x)$. As usual, it is defined in such a way that $F(k, x)d^Dx d^Dk$ gives the number of quanta in the D-dimensional volume element located at $x$, and whose D-momentum $k$ lie within $d^Dk$. Now, by assuming that the radiation quanta (photons for the sake of simplicity [22]) are out but close to equilibrium, we may expand $F(k, x)$, to first order in mean free time, as

$$F = F^{(0)} + F^{(1)},$$

(1)

where $F^{(0)}$ is the equilibrium distribution in the $D$-dimensional space

$$F^{(0)} = \frac{2}{(2\pi)^D} \frac{1}{e^{\frac{k U}{T}} - 1},$$

(2)

and the interaction part $F^{(1)}$ satisfies $|F^{(1)}| << F^{(0)}$. In the above expressions, $U^\mu$ is the normalized four-velocity field ($U^\mu U_\mu = 1$) of the material medium and $T \equiv T(x)$ its local temperature which, in equilibrium, coincides with the uniform temperature $T_M$ of the material medium.

The distribution function satisfies the Boltzmann transport equation which to first order may be written as
\[ k^\mu \frac{\partial F^{(0)}}{\partial x^\mu} = L[F^{(1)}], \quad (3) \]

where \( L \) is a linear operator which depends on the specific model (interaction of radiation and material fluid). Following Strauman [5], the solution of the above equation will be discussed, as far as possible, in rather general terms. An application of these results for a specific model will be presented in section 5.

As is widely known, the above Eq.(3) cannot, in general, be exactly solved so that some approximation scheme need to be implemented. One which is often considered is provided by the Grad’s relativistic 9-moments method applied in Refs. [5,6] to the 4-dimensional case. In the present D-dimensional framework it may be generalized as is explained below.

Let \( G \) be the \( D \)-dimensional Lorentz group of the Minkowski-like space-time \( M \) and \( G_x \) the little group associated to a timelike vector \( U(x) \). This subgroup of \( G \) leaves \( U(x) \) invariant. As a consequence \( G_x \) is isomorphic to \( SO(D) \), thereby implying that \( F(k, x) \) is a function of \( \omega = k_\mu U^\mu \), \( n^\mu \) and \( x \) where

\[ n^\mu = \frac{k^\mu}{\omega} - U^\mu \quad .\quad (4) \]

Note that \( n_\mu U^\mu = 0 \) and \( n_\mu n^\mu = -1 \). As we shall see next, the key ingredient to make the integrals on the hypersphere \( S_{D-1} \) (space-group where \( G_x \) acts upon) easier to compute comes from the fact that \( \omega = k_\mu U^\mu \) is obviously invariant under \( G_x \) at every point \( x \).

The calculational basis of Grad’s method applied to radiative problems is to expand the perturbative term \( F^{(1)}(\omega, n^\mu, x) \) with respect to \( n^\mu \) into irreducible polynomials under the action of the little group \( G_x \)

\[ F^{(1)}(\omega, n^\mu, x) = A(\omega, x) + B_\mu(\omega, x)n^\mu + C_{\mu\nu}(\omega, x)(n^\mu n^\nu + \frac{1}{D} h^{\mu\nu}) + ... \quad , \quad (5) \]

where

\[ h^{\mu\nu} = \eta^{\mu\nu} - U^\mu U^\nu \quad , \quad h^\mu = D \quad .\quad (6) \]
is the projector onto the hyperplane normal to $U^\mu$, and $\eta^{\mu\nu}$ is the Minkowski matrix. The former object also satisfies the identities: $h^\mu_\nu U^\nu = 0$ and $h^\mu_\nu n^\nu = n^\mu$. Naturally, the vectorial and tensorial parts in the decomposition (5) are defined in the rest space normal to $U^\mu$ i.e., $h^\mu_\nu B^\nu = B^\mu$ and $h^\sigma_\nu h^\lambda_\rho C^\lambda_\sigma = C^\mu_\nu$. In addition, since $n^\mu n_\nu + \frac{1}{D}h^\mu_\nu$ is symmetric and traceless, without loss of generality, one may assume that the same holds for $C^\mu_\nu$, that is:

$$C^\nu_\mu = C^\mu_\nu , \quad C^\lambda_\lambda = 0 \quad .$$

On the other hand, since $L$ behaves like a scalar under the action of the little group $G_x$, this means that it operates in the irreducible subspaces spanned by the irreducible polynomials in (5) as a multiple of the unit operator. It thus follows that

$$L[F^{(1)}] = -\omega [\kappa_0 A + \kappa_1 B^\mu n^\mu + \kappa_2 C^\mu_\nu (n^\mu n^\nu + \frac{1}{D}h^\mu_\nu) + ...] \quad ,$$

where $\kappa_i = \tau_i^{-1}$ ($i = 0, 1, 2$), is the inverse of the mean free time for the process in consideration (bulk viscosity, heat flux, shear viscosity). As usual, we assume that these quantities are functions solely of $\omega$ and $x$.

Using the transport equation (3) we can express the coefficients $A$, $B^\mu$ and $C^\mu_\nu$ in terms of $U^\mu$, $T$, and $\kappa_i$. To do that we introduce a measure $d\Omega_U$ on the $(D - 1)$-dimensional hypersurface

$$S_{D-1} = \{ k/ k^2 = 0, \; k^0 > 0, \; k.U = \text{const} \} \quad ,$$

for which $d\Omega_U$ is the unique $G_x$-invariant measure normalized as

$$\int_{S_{D-1}} d\Omega_U = \frac{2\pi^{D/2}}{\Gamma(D/2)} \equiv S_{D-1} \quad ,$$

where $\Gamma$ is the gamma function.

This normalization is easily understood in the comoving frame. Its nothing more than the “area” of unit $(D - 1)$-dimensional sphere. In the above formula we have denoted the area of $S_{D-1}$ with this same symbol, but this will not cause any misunderstanding to the reader.
Now, with respect to this measure, the irreducible polynomials are normalized by (see Appendix 1)

\[
\frac{1}{S_{D-1}} \int_{S_{D-1}} n^\mu n^\nu d\Omega_U = \frac{1}{D} h^{\mu\nu} ,
\]

\[
\frac{1}{S_{D-1}} \int_{S_{D-1}} (n^\mu n^\nu + \frac{1}{D} h^{\mu\nu})(n^\sigma n^\rho + \frac{1}{D} h^{\sigma\rho}) d\Omega_U =
\frac{1}{D(D+2)} (h^{\mu\nu} h^{\sigma\rho} + h^{\mu\sigma} h^{\nu\rho} + h^{\mu\rho} h^{\nu\sigma}) - \frac{1}{D^2} h^{\mu\nu} h^{\sigma\rho} ,
\]

Also we have the very useful result

\[
\int_{S_{D-1}} n^\mu n^\nu \ldots n^\sigma d\Omega_U = 0 ,
\]

for an odd number of components \(n^{(s)}\) of the vector \(n\). From a straightforward (although unduly long) calculation we can get the coefficients \(A, B_\mu, C_{\mu\nu}\) in (5) by evaluating the moments of the Boltzmann equation (3), and using the relations (11)-(13). The results are:

\[
A = \frac{1}{\kappa_0 T_M} \phi' \left( \frac{\omega}{T_M} \right) \left[ \frac{1}{T_M} U^\mu \partial_\mu T_M + \frac{1}{D} U_\mu \right] ,
\]

\[
B_\mu = \frac{1}{\kappa_1 T_M} \phi' \left( \frac{\omega}{T_M} \right) \left[ \frac{1}{T_M} h^\nu_\mu \partial_\nu T_M - U^\nu \partial_\nu U_\mu \right] ,
\]

\[
C_{\mu\nu} = -\frac{1}{2\kappa_2} \phi' \left( \frac{\omega}{T_M} \right) \left[ h^\nu_\mu \partial_\nu U_\mu + h^\lambda_\mu \partial_\lambda U_\mu - \frac{2}{D} h^{\mu\nu} U^\lambda_\lambda \right] ,
\]

where \(\phi(\frac{\omega}{T_M}) \equiv F^0(\omega, x)\), and a prime denotes its derivative.

By definition, the total energy-momentum tensor \(T^{\mu\nu}\) of the mixture (matter plus radiation) is given by

\[
T^{\mu\nu} = T_M^{\mu\nu} + T_R^{\mu\nu} + \int k^\mu k^\nu F^{(1)} \frac{d^Dk}{k^D} ,
\]

where \(T_M^{\mu\nu}\) is the EMT of the material component, which from now on, will be described by a perfect fluid
\[ T_{M}^{\mu \nu} = \rho_{M} U^{\mu} U^{\nu} - p_{M} h^{\mu \nu} \]  

and \( \rho_{M} \) and \( p_{M} \) are, respectively, the energy density and pressure. \( T_{R}^{\mu \nu} \) is the radiation EMT for the equilibrium state, which may be readily evaluated to be \[ T_{R}^{\mu \nu} = \int k^{\mu} k^{\nu} F_{0} \frac{d^{D}k}{k^{0}} = a_{D} T_{M}^{D+1} \left( U^{\mu} U^{\nu} - \frac{1}{D} h^{\mu \nu} \right) , \]

or equivalently, in the standard form \[ T_{R}^{\mu \nu} = \rho_{R} U^{\mu} U^{\nu} - p_{R} h^{\mu \nu} , \]

where

\[ \rho_{R} = a_{D} T_{M}^{D+1} , \quad p_{R} = \frac{1}{D} \rho_{R} , \]

and

\[ a_{D} = \frac{4}{(2\pi)^{D/2}} \frac{\pi^{D/2}}{\Gamma(D+1) \Gamma(D+1/2)} \zeta(D+1) \]

is the \( D \)-dimensional radiation constant, and \( \zeta \) denotes the Riemann zeta function.

The last term (the integral) in Eq.(17) is the only out of equilibrium contribution (due to interaction between matter and radiation), and can properly be interpreted as a dissipative term to the equilibrium energy-momentum tensor. The coefficients \( A, B_{\mu}, C_{\mu \nu} \) give rise to contributions of different tensorial ranks, which in the thermodynamic theory are usually named, respectively, as bulk viscosity, heat flux, and shear tensor. Substituting (14)-(16) in the last integral of (17), we get the terms:

a) Scalar (bulk viscosity) \[ F_{b_{e}}^{(1)} = A \]

\[ \int k^{\mu} k^{\nu} F_{b_{e}}^{(1)} \frac{d^{D}k}{k^{0}} = -\frac{(D+1)a_{D}T_{M}^{D+1}}{\kappa_{0}}(U^{\mu} U^{\nu} - \frac{1}{D} h^{\mu \nu}) \left( \frac{1}{T_{M}} U^{\lambda} \partial_{\lambda} T_{M} + \frac{1}{D} U^{\lambda} U^{,\lambda} \right) \]

(22)

b) Vectorial (heat flow) \[ F_{h_{f}}^{(1)} = B^{\mu} \eta_{\mu} \]

\[ \int k^{\mu} k^{\nu} F_{h_{f}}^{(1)} \frac{d^{D}k}{k^{0}} = \left( \frac{D+1}{D} \right) \frac{a_{D}T_{M}^{D}}{\kappa_{1}}(h^{\mu \lambda} U^{\nu} + h^{\nu \lambda} U^{\mu})(\partial_{\lambda} T_{M} - T_{M} U^{\sigma} U_{,\sigma}) \]

(23)
c) Tensorial (shear viscosity) 

\[ F_{sh}^{(1)} = \frac{C_{\mu\nu}}{D} h^{\mu\nu} \]

\[ \int k^\mu k^\nu F_{sh}^{(1)} \frac{d^D k}{k_0} = (D + 1) a_D T_M^{D+1} h^{\mu\nu} h^{\nu\rho} (U_{\sigma,\rho} - U_{\rho,\sigma} - \frac{2}{D} h_{\sigma\rho} U_{\lambda}) \] .

(24)

In the above equations, \( \bar{\kappa}_i \) \( (i = 0, 1, 2) \) are the \( D \)-dimensional Rosseland means for \( \kappa_i \). In general, for an arbitrary quantity \( \kappa_i \) we define its \( D \)-dimensional Rosseland mean \( \bar{\kappa}_i \) by

\[ \frac{1}{\bar{\kappa}_i} = \frac{\int_0^\infty \frac{1}{\kappa_i} \omega^{D+1} \phi'(\frac{\omega}{T_M}) d\omega}{\int_0^\infty \omega^{D+1} \phi'(\frac{\omega}{T_M}) d\omega} \] .

(25)

### III. ECKART TEMPERATURE AND THE TRANSPORT COEFFICIENTS

In the phenomenological theory of Eckart, the EMT of a relativistic dissipative simple fluid reads

\[ T^{\mu\nu} = \rho(T, n) U^{\mu} U^{\nu} - p(T, n) h^{\mu\nu} + \Delta T^{\mu\nu} \] ,

(26)

where the last term is the macroscopic description of the irreversible processes. Its \( D \)-dimensional canonical form is given by (the factor 1/D appears explicitly because the shear viscosity stress is traceless)

\[ \Delta T^{\mu\nu} = \zeta_D h^{\mu\nu} U^\lambda + \chi_D (h^{\mu\lambda} U^\nu + h^{\nu\lambda} U^\mu) (T_\lambda - T U^{\sigma} U_{\lambda,\sigma}) + \eta_D h^{\mu\sigma} h^{\nu\rho} (U_{\sigma,\rho} + U_{\rho,\sigma} - \frac{2}{D} g_{\sigma\rho} U^\lambda_\lambda) \] ,

(27)

where \( \zeta_D, \chi_D, \) and \( \eta_D \) are, respectively, the bulk viscosity, heat conducting, and the shear viscosity transport coefficients. In principle, the above expression should be justified by kinetic contributions like (22)-(24). It is worth noticing that all thermodynamic quantities (explicitly or implicitly) present in (26) are measured in the rest frame of \( U_\alpha \). However, for a radiative plasma there are, in the collisional limit, at least three concepts of temperature [9]. Since the physical features relating these temperatures concepts are independent (to first order in \( \bar{\kappa}_0^{-1} \)) of the shear and
heat conducting effects, for a moment, we will restrict our discussion to the isotropic case for which only bulk viscosity is relevant. To do that, let us first introduce (from the bulk viscosity contribution (22)) the following auxiliary quantity:

\[ \Pi_M = -(D + 1)a_D T_M^{D+1} \frac{1}{\bar{\kappa_0}} \left( \frac{1}{T_M} U^\lambda \partial_\lambda T_M + \frac{1}{D} U^\lambda,\lambda \right). \] (28)

In terms of \( \Pi_M \), the isotropic part of the total EMT (17) can be written as (see (22))

\[ T^{\mu\nu} = \left[ \rho_M(T_M, n) + a_D T_M^{D+1} + \Pi_M \right] U^\mu U^\nu - \left[ p_M(T_M, n) + \frac{1}{D} a_D T_M^{D+1} + \frac{1}{D} \Pi_M \right] h^{\mu\nu}. \] (29)

Note that by defining the effective radiation kinetic temperature:

\[ T_R = T_M \left[ 1 - \bar{\kappa_0}^{-1} \left( \frac{1}{T_M} U^\lambda \partial_\lambda T_M + \frac{1}{D} U^\lambda,\lambda \right) \right], \] (30)

the above EMT assume the form

\[ T^{\mu\nu} = \left( \rho_M(T_M, n) + a_D T_M^{D+1} \right) U^\mu U^\nu - \left( p_M(T_M, n) + \frac{1}{D} a_D T_M^{D+1} \right) h^{\mu\nu}, \] (31)

which has only two standard terms, which are characteristic of the perfect fluid description. In particular, the radiation quanta behaves as if they were in equilibrium at temperature \( T_R \) (compare (19)). However, the difference in temperature between the material and radiative components give rise to an irreversible heat transference between them. This is the physical mechanism accounting for the bulk viscosity process in this mixture. In order to extract its analytical expression, we recall that the Eckart temperature \( T \) is defined as being the local equilibrium temperature, which is implicitly fixed by (see Ref. [2])

\[ U_\mu U_\nu T^{\mu\nu} = \rho(T, n) = \rho_M(T_M, n) + a_D T_M^{D+1} + \Pi_M = \rho_M(T, n) + a_D T^{D+1}. \] (32)

Since the equilibrium temperature \( T \in [T_M, T_R] \), whose length \( |T_R - T_M| \) is of first order in \( \bar{\kappa_0}^{-1} \), we can expand \( \rho(T, n) \) as a power series

\[ \rho(T, n) = \rho(T_M, n) + \left( \frac{\partial \rho}{\partial T} \right)_n (T - T_M) + \cdots, \] (33)
where

\[ \rho(T_m, n) = \rho_M(T_m, n) + a_D T_M^{D+1} \quad . \]  

(34)

Inserting (34) into (33) and comparing with (32) we get

\[ T = T_M + \left( \frac{\partial \rho}{\partial T} \right)_n^{-1} \Pi_M \quad . \]  

(35)

Analogously, the total pressure \( p(T, n) \) can be expanded, to first order, as

\[ p(T, n) = p(T_M, n) + \left( \frac{\partial p}{\partial T} \right)_n (T - T_M) + \cdots \quad , \]  

(36)

or still, inserting \( p(T_M, n) \) from (31), and using (35)

\[ p(T, n) = p_M(T_M, n) + \frac{1}{D} a_D T_M^{D+1} + \left( \frac{\partial p}{\partial \rho} \right)_n \Pi_M \quad . \]  

(37)

With the help of (28), (32), and (37), the total energy-momentum tensor (29) takes the form

\[ T^{\mu \nu} = \rho(T, n) U^\mu U^\nu - \left\{ p(T, n) - \frac{D + 1}{\bar{\kappa}_0} a_D T_M^{D+1} \left[ \frac{1}{D} - \left( \frac{\partial p}{\partial \rho} \right)_n \right]^2 U^\lambda_{,\lambda} \right\} h^{\mu \nu} \quad . \]  

(38)

where we have used the thermodynamic relation \[ \] and the fact that, to first order, \( T_M \) can be replaced by \( T \) when multiplied by \( \bar{\kappa}_0^{-1} \).

The above EMT has exactly the Eckart’s canonical isotropic form, namely:

\[ T^{\mu \nu} = \rho(T, n) U^\mu U^\nu - [p(T, n) - \zeta_D \theta] h^{\mu \nu} \quad , \]  

(40)

where \( \theta = U^\lambda_{,\lambda} \) is the expansion parameter, and from (38) and (40) we read

\[ \zeta_D = \frac{D + 1}{\bar{\kappa}_0} a_D T_M^{D+1} \left[ \frac{1}{D} - \left( \frac{\partial p}{\partial \rho} \right)_n \right]^2 \quad . \]  

(41)

To obtain the remaining coefficients it is necessary only to observe that the vectorial and tensorial contributions (23) and (24) are not modified (to the first order)
when rewritten in terms of the Eckart temperature. Hence, a direct comparison with the canonical form (27) give us

\[ \chi_D = \frac{D + 1}{D} \frac{a_D T^D}{\bar{\kappa}_1}, \]

and

\[ \eta_D = \frac{(D + 1)a_D T^{D+1}}{D(D + 2)\bar{\kappa}_2}. \]

Expressions (41), (42), and (43) are the transport coefficients for a D-dimensional radiative plasma. In particular, for \( D = 3 \), the standard results are recovered [2,5]. From (41) we see that structureless point particles in D-dimensions have negligible bulk viscosity in the extreme relativistic regime \( (p \approx 1/D \rho) \), as should be expected in physical grounds. This is also a consequence of the fact that the D-dimensional equilibrium distribution of photons is preserved under expansion, or equivalently, that the ratio \( T/\omega \) is an adiabatic invariant in the sense of Ehrenfest, regardless of the number of spatial dimensions. Hence, when the universe is radiation dominated, there is no radiative bulk viscosity in the D-dimensional Friedmann-Robertson-Walker (FRW) spacetime. This generalize the widely known 3-dimensional result [2].

**IV. ENTROPY PRODUCTION FOR A D-DIMENSIONAL RADIATIVE PLASMA**

In Eckart's frame, the current of entropy and the local entropy production for a D-dimensional relativistic simple fluid are given by

\[ S^\mu = s(T, n)U^\mu + \frac{q^\mu}{T}, \]

and

\[ \partial_\mu S^\mu = \frac{\Pi^2}{\zeta_D T} + \frac{q_\mu q^\mu}{\chi_D T^2} + \frac{\Pi_{\mu\nu}\Pi^{\mu\nu}}{2\eta_D T}, \]
where $s$ is the entropy density, and $\Pi$, $q^\mu$, and $\Pi^{\mu\nu}$ are the classical dissipative stresses (bulk viscosity, heat flow and shear viscosity):

$$\Pi = \zeta_D \theta \quad ,$$

$$q^\mu = \chi_D h^{\mu\lambda}(T,\lambda - TU^\sigma U_{\lambda,\sigma}) \quad ,$$

$$\Pi^{\mu\nu} = \eta_D h^{\mu\sigma} h^{\nu\rho}(U_{\sigma,\rho} + U_{\rho,\sigma} - \frac{2}{D} h_{\sigma\rho} U^\lambda_{,\lambda}) \quad .$$

In principle, the Eckart description for a D-dimensional radiative plasma will be considered complete only if the kinetic equations lead to (44) and (45) in Eckart’s frame. In particular, one may argue that the above defined entropy density should be introduced, in a consistent way, by the same kind of algorithm used to fix the Eckart temperature. Indeed, this is exactly what happens. To show that let us consider the kinetic expression for the entropy flux

$$S^\mu_R = -\frac{2}{(2\pi)^D} \int \frac{d^D k}{k^0} [F \ln F - (1 + F) \ln(1 + F)] \quad ,$$

where from (2), (14), (15), and (16) the distribution function $F$ given in (5) may be written as

$$F = \phi \left( \frac{\omega}{T_M} \right) + \kappa_0^{-1} \frac{\omega}{T_M} \phi' \left( \frac{\omega}{T_M} \right) \Pi + \kappa_1^{-1} \frac{\omega}{T_M} \phi' \left( \frac{\omega}{T_M} \right) \tilde{\Pi} n^\mu$$

$$-\kappa_2^{-1} \frac{\omega}{T_M} \phi' \left( \frac{\omega}{T_M} \right) \tilde{\Pi}^{\mu\nu}(n^\mu n^\nu + \frac{1}{D} h^{\mu\nu}) \quad .$$

where

$$\tilde{\Pi} = \frac{1}{T_M} U^\mu \partial_\mu T_M + \frac{1}{D} U^\mu$$

$$\tilde{q}_\mu = \frac{1}{T_M} h^\nu_{,\mu} \partial_\nu T_M - U^\nu \partial_\nu U_\mu$$

$$\Pi^{\mu\nu} = h^\lambda_{\mu} U_{\nu,\lambda} + h^\lambda_{\nu} U_{\mu,\lambda} - \frac{2}{D} h_{\mu\nu} U^\lambda_{,\lambda}$$

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Inserting the above expression into (49) a straightforward, although lengthy, calculation yields
\[
S^\mu_R = \left( \frac{D+1}{D} a_D T_M^D + \frac{\Pi_M}{T_M} \right) U^\mu + \frac{q^\mu_M}{T_M} ,
\]
where \( q^\mu \) has been indexed to recall that it is in the matter temperature. However, since \( q^\mu_M \) is already of first order in \( \bar{\kappa}_1^{-1} \), this means that \( \frac{q^\mu}{T_M} \approx \frac{q^\mu}{T} \), and using the definition of \( T_R \) given by (30), the above equation becomes
\[
S^\mu_R = \frac{D+1}{D} a_D T_R^D U^\mu + \frac{q^\mu}{T} .
\]
Note that the radiation entropy density is \( U^\mu S^\mu_R = \frac{D+1}{D} a_D T_R^D \), as should be expected in physical grounds. In particular, for \( D = 3 \), one has \( s_R \sim T_R^3 \), which, in the absence of heat flow, corresponds to the usual radiation fluid with \( \rho_R \sim T_R^4 \). These results are in agreement with our discussion in the earlier section. The quanta behave (in the isotropic case) as a radiation perfect fluid at temperature \( T_R \) regardless of the number of dimensions.

Now, as a self-consistency check of the algorithm used to introduce the temperature of Eckart, we need only to show that the total entropy density of the mixture reduces to the equilibrium expression at the temperature \( T \).

The total entropy of matter plus radiation is given by \( S^\mu = S^\mu_M + S^\mu_R \) where \( S^\mu_R \) is given by (55) and \( S^\mu_M = s_M(T_M, n) U^\mu \) is the entropy flux of matter. Since the material component is in equilibrium at temperature \( T_M \), the total entropy density can be written as (see (54))
\[
U^\mu S^\mu = s_M(T_M, n) + \frac{D+1}{D} a_D T_M^D + \frac{\Pi_M}{T_M} ,
\]
where \( s_M(T_M, n) \) is the entropy density of the material medium. By definition of Eckart’s frame, the right hand side of the above equation should be \( s(T, n) \). In fact, expanding \( s(T, n) \) in power series
\[
s(T, n) = s(T_M, n) + \left( \frac{\partial s}{\partial T} \right)_n (T - T_M) + o(\bar{\kappa}_0^{-2}) ,
\]

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and using (35), the above expression (to first order) becomes

\[ s(T_M, n) = s(T, n) - \left( \frac{\partial s}{\partial \rho} \right)_n \Pi_M, \tag{58} \]

where

\[ s(T_M, n) = s_M(T_M, n) + \frac{D + 1}{D} \alpha \nu T^D_M. \tag{59} \]

Inserting (58) into (56), we can rewrite the entropy density as

\[ U_\mu S^\mu = s(T, n) - \left( \frac{\partial s}{\partial \rho} \right)_n \Pi_M + \frac{\Pi_M}{T}, \tag{60} \]

where we have used that \( \frac{\Pi_M}{T} \approx \frac{\Pi}{T} \) to first order in \( \bar{\tau}_0 = \bar{\kappa}^{-1}_0 \). Finally, by considering the standard relation \( \left( \frac{\partial s}{\partial \rho} \right)_n = \frac{1}{T} \), we obtain

\[ U_\mu S^\mu = s(T, n). \tag{61} \]

Thus, in Eckart’s temperature, the total entropy four-vector of a radiative plasma, assume the same form usually applied for a relativistic simple fluid (see (44)).

Now, we proceed to calculate the photon entropy production, namely, \( \partial_\mu S^\mu_R \). From (49) we have

\[ \partial_\mu S^\mu_R = -\frac{2}{(2\pi)^D} \int \frac{d^D k}{k^0} k^\mu \partial_\mu [F \ln F - (1 + F) \ln(1 + F)] \] . \tag{62} \]

Using the Boltzmann equation (3) a straightforward calculation leads to

\[ \partial_\mu S^\mu_R = \frac{2}{(2\pi)^D} \int \frac{d^D k}{k^0} \frac{\omega}{T} L[F^{(1)}] + \frac{2}{(2\pi)^D} \int \frac{d^D k}{k^0} \frac{F^{(1)}}{\Delta^{(0)} F^{(0)}} L[F^{(1)}], \tag{63} \]

where \( \Delta^{(0)} = 1 + F^{(0)} \). Using (50) these integrals are readily evaluated

\[ \partial_\mu S^\mu_R = (D + 1) \alpha \nu T^D \left[ \frac{\bar{\Pi}}{\bar{\kappa}_0} + \frac{\bar{\Pi}^2}{D \bar{\kappa}_1} + \frac{1}{2D(D + 2)} \frac{\bar{\Pi}_\mu \bar{\Pi}^{\mu \nu}}{\bar{\kappa}_2} \right], \tag{64} \]

where the linear term came from the first integral in (63). As it will be seen, if we include the contribution from the material medium to the total entropy production, we render the total linear term zero as a consequence of the conservation of the total
energy-momentum tensor (such approach has been applied by Schweizer for the 3-dimensional case). In order to do this we first introduce the notation

\[ \frac{\omega}{T} = k^\nu \frac{U_\nu}{T} \equiv k^\nu \beta_\nu \]  \hspace{1cm} (65)

and using the Boltzmann equation (3), (65), and (19), the first integral in (63) can be evaluated leading to

\[ \partial_\mu S^\mu_R = \beta_\nu T^\mu_{R,\nu} + QP \]  \hspace{1cm} (66)

where QP stands for the second integral in (63) which gives rise to the quadratic part in (64). Similarly, for the material medium one has

\[ \partial_\mu S^\mu_M = \beta_\nu T^\mu_{M,\nu} \]  \hspace{1cm} (67)

Adding (66) and (67), and using the conservation of the total energy-momentum tensor, \( T^{\mu\nu} = T^{\mu\nu}_R + T^{\mu\nu}_M \), we get

\[ \partial_\mu S^\mu = QP = (D + 1) a_D T^D \left[ \frac{\Pi^2}{\kappa_0} + \frac{\bar{q}_\mu \bar{q}^\mu}{D \bar{\kappa}_1} + \frac{1}{2D(D + 2)} \bar{\Pi}_{\mu\nu} \bar{\Pi}^{\mu\nu} \right] \]  \hspace{1cm} (68)

With a bit more algebra one can prove that

\[ (D + 1) a_D T^D \frac{\Pi^2}{\kappa_0} = \frac{1}{T} \frac{\Pi^2_M}{\zeta_D} \]  \hspace{1cm} (69)

\[ \frac{D + 1}{D} a_D T^D \frac{\bar{q}_\mu \bar{q}^\mu}{\bar{\kappa}_1} = \frac{1}{\chi_D T^2} q_\mu q^\mu \]  \hspace{1cm} (70)

\[ \frac{(D + 1)a_D T^D}{2D(D + 2)\bar{\kappa}_2} \bar{\Pi}_{\mu\nu} \bar{\Pi}^{\mu\nu} = \frac{\Pi_{\mu\nu} \bar{\Pi}^{\mu\nu}}{2\eta_D T} \]  \hspace{1cm} (71)

where \( \Pi_{\mu\nu} = \eta_D \bar{\Pi}_{\mu\nu} \). Finally, using (69)-(71) we can write (68) in the expected final form

\[ \partial_\mu S^\mu = \frac{1}{T} \frac{\Pi^2}{\zeta_D} + \frac{1}{\chi_D T^2} q_\mu q^\mu + \frac{1}{2\eta_D T} \Pi_{\mu\nu} \bar{\Pi}^{\mu\nu} \]  \hspace{1cm} (72)
V. MODEL WITH THOMSON SCATTERING

In D dimensions, the transport equation for the case of a nondegenerate material medium and temperature not too high, may be written in the form (see, for example, references [5],[6], and [14])

\[ k^\mu \partial_\mu F = -\omega N \sigma_a(1 - e^{-\omega/T})(F - F^{(0)}) \]
\[ -\omega N \sigma_s \left[ F - \int_{S_{D-1}} p(n, n') F(\omega, n') d\Omega'_U \right] . \]  

(73)

In this expression, the quantities \( \sigma_a(\omega) \) and \( \sigma_s(\omega) \) appearing in the linearized collisional term are, respectively, the absorption and scattering cross sections in D spatial dimensions, \( N \) is the number of scatters and we have considered coherent scattering only. We do not evaluate them here, since for our purposes this is not relevant. The phase function \( p(n, n') \) is given by relation

\[ d\sigma_s(n.n') = p(n, n') \sigma_s d\Omega'_U . \]  

(74)

If we consider only Thomson scattering, \( (\sigma_s = \sigma_{Th}) \), we have in D-dimensions (see Appendix B)

\[ p(n, n') = \frac{1}{S_{D-1}} \left[ 1 + \frac{D}{(D - 1)^2} (n^\mu n^\nu + \frac{1}{D} h^{\mu \nu}) (n'_\mu n'_\nu + \frac{1}{D} h_{\mu \nu}) \right] , \]  

(75)

which reduces to the usual Thomson formula for \( D = 3 \) (see [24],[25]).

In order to compute the integral in (73), we consider the total distribution function as given by (1), (2), and (5), namely

\[ F = F^{(0)} + A + B_\mu n^\mu + C_{\mu \nu}(n^\mu n^\nu + \frac{1}{D} h^{\mu \nu}) . \]  

(76)

Inserting (75) and (76) into (73), and evaluating the integral, we get

\[ k^\mu \partial_\mu F = -\omega \left\{ N \sigma_a(\omega)(1 - e^{-\omega/T}) A + N \left[ \sigma_a(\omega)(1 - e^{-\omega/T}) + \sigma_{Th} \right] B_\mu n^\mu \right. \]
\[ + N \left[ \sigma_a(\omega)(1 - e^{-\omega/T}) + \frac{D(D^2 - 3)}{(D - 1)^2(D + 2)} \sigma_{Th} \right] C_{\mu \nu}(n^\mu n^\nu + \frac{1}{D} h^{\mu \nu}) \} , \]  

(77)
Comparing the above expression with equation (8) we obtain the transport coefficients

\[ \kappa_0(\omega) = N\sigma_a(\omega)(1 - e^{-\omega/T}) \]  

\[ \kappa_1(\omega) = N\left[\sigma_a(\omega)(1 - e^{-\omega/T}) + \sigma_{Th}\right] \]  

\[ \kappa_2(\omega) = N[\sigma_a(\omega)(1 - e^{-\omega/T}) + \frac{D(D^2 - 3)}{(D - 1)^2(D + 2)}\sigma_{Th}] \]  

Note that when the Thomson scattering dominates, \( \kappa_0(\omega) \) is negligible, thereby implying that there is no radiative bulk viscosity in this case (see (25) and (41)). Under such a condition an homogeneous and isotropic mixture of matter and radiation could expand in equilibrium. However, for an inhomogeneous and anisotropic medium, heat conduction and shear viscosity are always present. The Rosseland means for \( \kappa_1 \) and \( \kappa_2 \) reads

\[ \frac{1}{\overline{\kappa}_1} = \frac{1}{N\sigma_{Th}} \]  

\[ \frac{1}{\overline{\kappa}_2} = \frac{1}{N}\frac{(D - 1)^2(D + 2)}{D(D - 3)\sigma_{Th}} \]  

It thus follows that the transport coefficients (41), (42), and (43) are given by:

\[ \chi_D = \frac{D + 1}{D} \frac{a_D T^D}{N\sigma_{Th}} \]  

\[ \eta_D = \frac{(D + 1)(D - 1)^2 a_D T^{D+1}}{D^2(D^2 - 3)} \frac{1}{N\sigma_{Th}} \]  

\[ \zeta_D = 0 \]  

In particular, for \( D = 3 \), the coefficients first computed by Straumann [5] are recovered.
VI. CONCLUSION

As we have seen, kinetic theory is indeed a very powerful approach to describe radiative plasmas irrespective of the number of spatial dimensions. It provides the transport coefficients and the standard entropy production rate, which is obtained from non-equilibrium thermodynamics. In addition, through the same algorithm used to define the Eckart temperature, the main macroscopic quantities are also consistently settled to have the same form of equilibrium, as required by the first order nonequilibrium thermodynamics.

The Eckart approach considered here is plagued with serious undesirable features, like instabilities of the equilibrium states and superluminal velocities of the thermal and viscous signals [23]-[28]. Such shortcomings can be circumvented by the so-called second order thermodynamic theories as developed by Muller [24] and Israel [27]. In this way, it is interesting to reconsider the kinetic treatment developed here in the framework of the causal or transient relativistic kinetic theory [7]. The corrections of the extended theories, however, will be important only in regimes where the mean free paths are comparable to the macroscopic length scales [29].

Finally, we remark that it is not difficult to widen the scope of the results derived here to include general relativistic effects. As usual, one must assume that the mean free time of each process is much smaller than any typical scale of the gravitational field, and at the level of the computations, to implement the minimal coupling, viz.: to replace the ordinary derivatives by covariant derivatives and the (D+1)-Minkowski matrix $\eta^{\mu\nu}$ by $g^{\mu\nu}$ [2]. Naturally, D-gravity does not change the form of the D-dimensional transport coefficients, which should be considered in the analysis of the entropy production in high dimensional cosmologies, as well as in the role played by dissipative processes on the problem of dimensional reduction. In particular, as shown in section 3, the radiative bulk viscosity does not play any role in a D-dimensional radiation-dominated FRW universe. This generalizes the well
known results obtained by Weinberg in the three dimensional case [2]. Qualitatively, significant bulk viscosity may take place only if the plasma does a transition, for instance, from radiation to a matter-dominated phase. However, if only the Thomson scattering is considered, the bulk viscosity also vanishes, regardless of the number of spatial dimensions. Further considerations in these topics will be presented in a forthcoming communication.

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APPENDIX A: IRREDUCIBLE POLYNOMIALS NORMALIZATION

For completeness, we now outline the proofs of formulas (11)-(13), which are the key ingredients to perform integrals defining the usual thermodynamic quantities. These formulae on the normalization of the irreducible polynomials are frequently cited in the case $D = 3$ (see, for instance, ref(s) [4],[5] and [7]). Nevertheless, as far as we know, there is no simple proof of them in the general case. As mentioned earlier (see section 2) we make all calculations in the comoving frame.

The hypersphere parametrization is given by

$$k_1^2 + k_2^2 + \cdots + k_D^2 = \omega^2$$

where
\[ k_1 = \omega \cos \theta_1 \]
\[ k_2 = \omega \sin \theta_1 \cos \theta_2 \]
\[ k_3 = \omega \sin \theta_1 \sin \theta_2 \cos \theta_3 \]
\[ \vdots \]
\[ k_{D-1} = \omega \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{D-2} \cos \theta_{D-1} \]
\[ k_D = \omega \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{D-2} \sin \theta_{D-1} \]

with \( 0 \leq \theta_j \leq \pi \) for \( j = 1, 2, \ldots, D - 2 \) and \( 0 \leq \theta_{D-1} \leq 2\pi \). The element of volume on \( S_{D-1} \) is given by
\[
d\Omega_U = \sin^{D-2} \theta_1 \sin^{D-3} \theta_2 \ldots \sin^1 \theta_{D-2} \sin^0 \theta_{D-1} \, d\theta_1 \, d\theta_2 \ldots \, d\theta_{D-1} \]
and the volume of the \((D-1)\)-sphere is given by equation (10).

**a) Proof of equation (11)**

In the comoving frame \( \omega = k^0 \) and \( n^0 = 0 \), thereby implying that (11) is valid for \( \mu = 0 \) and \( \nu = 0, 1, 2, 3 \) (both sides are identically zero). Now consider \( i \neq j \).

Since in this case \( h^{ij} = 0 \), we need only to prove that \( \int_{S_{D-1}} n^i n^j d\Omega_U = 0 \), or the equivalent formula \( \int_{S_{D-1}} k^i k^j d\Omega_U = 0 \) (note that \( k^i = \omega n^i \)). Assuming, without lost of generality, that \( i < j \), and using the above parametrization of the \((D-1)\)-sphere, it follows that
\[
\int_{S_{D-1}} n^i n^j d\Omega_U = A \int_0^\pi d\theta_i \sin^{D-1} \theta_i \cos \theta_i = 0 \quad ,
\]
where \( A \) is a constant which appears from a multiple integral involving all angular variables but \( \theta_i \).

For \( i = j \) we have from spherical symmetry
\[
D \int_{S_{D-1}} n_i^2 d\Omega_U = \sum_{j=1}^D \int_{S_{D-1}} n_j^2 d\Omega_U =
\int_{S_{D-1}} d\Omega_U \sum_{j=1}^D n_j^2 = \int_{S_{D-1}} d\Omega_U = S_{D-1} \quad ,
\]

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and since $h^{ii} = -1$, $i = 1, 2, \ldots, D$ we can write
\[
\int_{S_{D-1}} n_i^2 d\Omega_U = -\frac{1}{D} h^{ii} S_{D-1} ,
\tag{A3}
\]
which means that (11) is valid for $\mu, \nu = 0, 1, 2, 3$.

**Proof of equation (12)**

By using equation (11) we get from the left hand side of (12)
\[
\frac{1}{S_{D-1}} \int_{S_{D-1}} (n^\mu n^\nu + \frac{1}{D} h^{\mu\nu})(n^\sigma n^\rho + \frac{1}{D} h^{\sigma\rho}) d\Omega_U = \\
\frac{1}{S_{D-1}} \int_{S_{D-1}} n^\mu n^\nu n^\sigma n^\rho d\Omega_U - \frac{1}{D^2} h^{\mu\nu} h^{\sigma\rho} .
\tag{A4}
\]
Let us now consider the integral
\[
\frac{1}{S_{D-1}} \int_{S_{D-1}} n^\mu n^\nu n^\sigma n^\rho d\Omega_U .
\tag{A5}
\]
To evaluate it we observe that $h^{\mu\nu}$ is the only second rank $D$-dimensional tensor at our disposal which does not depends on the vector $n^\mu$, and that all permutations of $(\mu\nu\sigma\rho)$ leave us with the same expression. The only 4-rank tensor matching these requirements is given by $h^{\mu\nu} h^{\sigma\rho} + h^{\mu\sigma} h^{\nu\rho} + h^{\mu\rho} h^{\nu\sigma}$. The integral must be proportional to this tensor, and the proportionality constant should depend only on the dimension $D$. Therefore we can write
\[
\frac{1}{S_{D-1}} \int_{S_{D-1}} n^\mu n^\nu n^\sigma n^\rho d\Omega_U = f(D)(h^{\mu\nu} h^{\sigma\rho} + h^{\mu\sigma} h^{\nu\rho} + h^{\mu\rho} h^{\nu\sigma}) ,
\tag{A6}
\]
where $f(D)$ is the constant to be determined. Since we have tensorial equation, we can evaluate the scalar $f(D)$ in the comoving frame. By specializing the integral for $n^\mu = n^\nu = n^\sigma = n^\rho = n^1 = \cos \theta_1$, we obtain
\[
f(D) = \frac{1}{3 S_{D-1}} \int_{S_{D-1}} n_1^4 d\Omega_U .
\tag{A7}
\]
A straightforward (although a bit lengthy) calculation leads to
\[
f(D) = \frac{1}{D(D+2)} ,
\tag{A8}
\]
and using (A.8), (A.6) and (A.4) one immediately obtains (12).

**Proof of equation (13)**

In the comoving frame \((n^0 = 0)\), we need to proof (13) only for the spatial components. Suppose that we have a odd number, say \(p\), of components of the vector \(n\). Since \(n^i = k^i/k^0\) we get

\[
\int_{S_{D-1}} n^i n^j \ldots n^r d\Omega_U = \frac{1}{k_0^p} \int_{S_{D-1}} k^i k^j \ldots k^r d\Omega_U . \tag{A9}
\]

The vectors \(k = (k^0, k^1, k^2, \ldots, k^D)\) and \(\overline{k} = (k^0, -k^1, -k^2, \ldots, -k^D)\) cover the same \(D\)-dimensional sphere \(S_{D-1}\). It thus follows that

\[
\int_{S_{D-1}} k^i k^j \ldots k^r d\Omega_U = (-1)^p \int_{S_{D-1}} k^i k^j \ldots k^r d\Omega_U . \tag{A10}
\]

Since \(p\) is odd we get \(\int_{S_{D-1}} k^i k^j \ldots k^l k^r d\Omega_U = 0\) and so the proof is completed.

**APPENDIX B: \(D\)-DIMENSIONAL THOMSON CROSS SECTION**

In order to evaluate the phase function \(p(n, n')\) one must impose some hypothesis reflecting the symmetry of the specific scattering problem. For Thomson scattering, the results of references [5] and [19] are easily generalized as follows:

**H1** \(p(n, n')\) is written in terms of irreducible polynomials.

**H2** \(p(n, n') = p(n', n)\)

These two hypothesis imply that \(p(n, n')\) is constructed out from products of an even number of irreducible polynomials. Thus, we can write

\[
p(n, n') = A + Bn_\sigma n'^\sigma + C(n^\mu n'^\nu + \frac{1}{D} h^{\mu\nu})(n'_\mu n'_\nu + \frac{1}{D} h^{\mu\nu}) , \tag{B1}
\]

where \(A, B,\) and \(C\) are constants to be determined.

**H3** \(p(n, n')\) is normalized such that:

\[
\int_{S_{D-1}} p(n, n') d\Omega_U = 1 . \tag{B2}
\]
The mean value of $n^\mu$ on the sphere $S_{D-1}$ is zero, that is:

$$\int_{S_{D-1}} p(n, n') n^\mu d\Omega_U = 0 \quad .$$

(B3)

Substituting (B.1) into (B.2) we get $A = \frac{1}{S_{D-1}}$, and from (B.3) we get $B = 0$. Hence, $p(n, n')$ reduces to

$$p(n, n') = \frac{1}{S_{D-1}} + C(n^\mu n^\nu + \frac{1}{D} h^{\mu\nu})(n'_\mu n'_\nu + \frac{1}{D} h_{\mu\nu}) \quad .$$

(B4)

To compute the coefficient $C$, we consider the comoving frame where $n = (0, \vec{n}) = (0, n_1, n_2, \ldots, n_D)$, $n' = (0, \vec{n}') = (0, n'_1, n'_2, \ldots, n'_D)$, and define the angle $\theta$ by

$$n^\mu n'_\mu = -n^i n'_i = \cos \theta \quad .$$

(B5)

Inserting (B.5) into (B.4) we get

$$p(n, n') = \frac{1}{S_{D-1}}[1 + \alpha(\cos^2 \theta - \frac{1}{D})] \quad ,$$

(B6)

where $\alpha = CS_{D-1}$. The differential scattering cross section is defined by (see Fig 1)

$$d\sigma = \sin^2 \phi d\Omega_U = p(n, n')\sigma_s d\Omega_U \quad ,$$

(B7)

while the total scattering cross section $\sigma_s$ reads

$$\sigma_s = \int_{S_{D-1}} d\sigma = \frac{D-1}{D} S_{D-1} \quad .$$

(B8)

It thus follows from (B.6), (B.7), and (B.8) that

$$\sin^2 \phi = \frac{D-1}{D}[1 + \alpha(\cos^2 \theta - \frac{1}{D})] \quad .$$

(B9)

From Fig. 1 we get the relation between the angles (see, for example, reference[20] for a 3-dimensional version of our argument)

$$\cos \phi = \sin \theta \cos \psi \quad .$$

(B10)

Now, consider the sphere $S_{D-2} \subset S_{D-1}$, such that $\vec{n} \perp S_{D-2}$, that is, $\vec{n}$ is orthogonal to all vectors describing $S_{D-2}$. By denoting $< \mathcal{A} >$ as the mean on $S_{D-2}$ of an arbitrary quantity $\mathcal{A}$, namely

H4
\[ <A> = \frac{1}{S_{D-2}} \int_{S_{D-2}} A d\Omega_{D-2}, \quad (B11) \]

we obtain (note that only the angle \( \psi \) relies on \( S_{D-2} \))

\[ <\cos^2 \psi> = \frac{1}{D-1}, \quad (B12) \]

and from (B.10) and (B.12)

\[ <\sin^2 \phi> = 1 - \frac{1}{D-1} \sin^2 \theta. \quad (B13) \]

Now, taking the average on \( S_{D-2} \) of equation (B.9) and using (B.13) we get

\[ C = \frac{\alpha}{S_{D-1}} = \frac{D}{(D-1)^2 S_{D-1}}, \quad (B14) \]

and inserting (B.14) into (B.4) we find the D-dimensional Thomson cross section

\[ p(n, n') = \frac{1}{S_{D-1}} [1 + \frac{D}{(D-1)^2} (n^\mu n'^\nu + \frac{1}{D} h^{\mu\nu} (n'_\mu n'_\nu + \frac{1}{D} h^{\mu\nu}))]. \quad (B15) \]

As expected, for \( D = 3 \) this expression reproduces the usual Thomson formula \cite{24,25}

\[ p_{Th}(n, n') = \frac{3}{16\pi} (1 + \cos^2 \theta). \quad (B16) \]
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[22] The case of neutrinos can be easily covered by the same formalism if one defines

$$F^{(0)} = \frac{2}{(2\pi)^D} \left[ e^{\frac{k_0}{\varepsilon} U - \varepsilon} \right]^{-1},$$

where the number $\varepsilon$ is $+1$ for bosons and $-1$ for fermions.

[23] To get the above result we made the calculations in the comoving matter referential in which we have $\frac{d^Dk}{k^D} = \omega^{D-2} d\omega d\Omega_U$.

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FIG. 1. Thomson Scattering in $D$ dimensions. The geometry of the scattering is the same as in 3 dimensions but the mean of the angle $\psi$ has been done on the $(D - 2)$-dimensional sphere $S_{D-2}$. 