Convexity Analysis of Optimization Framework of Attitude Determination from Vector Observations

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Abstract—In the past several years, there have been several representative attitude determination methods developed using derivative-based optimization algorithms. Optimization techniques e.g. gradient-descent algorithm (GDA), Gauss-Newton algorithm (GNA), Levenberg-Marquadt algorithm (LMA) suffer from local optimum in real engineering practices. A brief discussion on the convexity of this problem is presented recently [1] stating that the problem is neither convex nor concave. In this paper, we give analytic proofs on this problem. The results reveal that the target loss function is convex in the common practice of [1] stating that the problem is neither convex nor concave. In this paper, we give analytic proofs on this problem. The results reveal that the target loss function is convex in the common practice of

In consumer electronics, where the sensor precisions are relatively low, the vector sensors e.g. accelerometer and magnetometer are usually integrated with gyroscope for more smooth estimates [11]. For these estimators, there is always a need of measurement source for direct attitude reconstructions from sensors. According to bare computation resources, many algorithms are also developed to extract orientation by means of simple optimization methods. The Gaussian-Newton algorithm (GNA, [12]) is almost the first one doing this. Later, the gradient-descent algorithm (GDA, [13]) is applied to the same problem as well. The performances are improved by levenberg-marquardt algorithm (LMA, [14]) and improved

This paper is arranged as follows: Section II addresses the problem background and our main results. Section III contains numerical examples. In Section IV, we present discussion and concluding remarks.

II. PROBLEM BACKGROUND AND MAIN RESULTS

A direction cosine matrix (DCM) relates a vector observation pair with

$$\mathbf{D}^b = \mathbf{C}\mathbf{D}^r$$

where \(\mathbf{C}\) denotes the DCM; \(\mathbf{D}^b = (D^b_x, D^b_y, D^b_z)^T\) and \(\mathbf{D}^r = (D^r_x, D^r_y, D^r_z)^T\) are the normalized vector observations from one sensor in the body frame \(b\) and reference frame
One can rewrite this loss function into the system as follows:

\[
\frac{\partial^2 C}{\partial q_0^2} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \frac{\partial^2 C}{\partial q_0 \partial q_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad \frac{\partial^2 C}{\partial q_0 \partial q_2} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad \frac{\partial^2 C}{\partial q_0 \partial q_3} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
\frac{\partial^2 C}{\partial q_1^2} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{\partial^2 C}{\partial q_1 \partial q_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}, \quad \frac{\partial^2 C}{\partial q_1 \partial q_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
\frac{\partial^2 C}{\partial q_2^2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad \frac{\partial^2 C}{\partial q_2 \partial q_1} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{\partial^2 C}{\partial q_2 \partial q_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
\frac{\partial^2 C}{\partial q_3^2} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \frac{\partial^2 C}{\partial q_3 \partial q_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad \frac{\partial^2 C}{\partial q_3 \partial q_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

(17)

\[ r \text{ respectively. With several pairs of vector observations, the rotation matrix can be computed with the Wahba’s problem that employs the following loss function}
\]

\[
L(C) = \sum_{i=1}^{n} a_i \| D_i^b - CD_i^r \|^2
\]

(2)

where \(a_i\) is the positive weight of \(i\)-th sensor with \(\sum_{i=1}^{n} a_i = 1\).

One can re-write this loss function into the system as follows:

\[
\begin{cases}
\sqrt{a_1} (D_1^b - CD_1^r) = 0 \\
\sqrt{a_2} (D_2^b - CD_2^r) = 0 \\
\vdots \\
\sqrt{a_n} (D_n^b - CD_n^r) = 0
\end{cases}
\]

(3)

Optimization algorithms usually seek the minimum point of the Wahba’s loss function by parameterizing the DCM with quaternion \(q = (q_0, q_1, q_2, q_3)^T\), such that

\[
\arg \min_{\|q\|=1} \sum_{i=1}^{n} a_i \| D_i^b - CD_i^r \|^2
\]

(4)

A general solution i.e. the q-method solves the maximum eigenvalue and its associated eigenvector of the Davenport K matrix given as follows [7]

\[
K = \begin{bmatrix} B + B^T - tr(B)I & z \\ z^T & tr(B) \end{bmatrix}
\]

(5)

where

\[
B = \sum_{i=1}^{n} a_i (D_i^b)^T(D_i^b)^T
\]

\[
z = \sum_{i=1}^{n} a_i D_i^b \times D_i^r
\]

(6)

If the target multi-variate function is concave, there be local optimum setting up obstacles for global solving. In next subsection, we are going to investigate the convexity of this optimization problem.

### A. Single Vector Observation Pair

Starting from a single vector observation pair, one can define the scalar loss function as

\[
F_i(q) = e_i^T(q)e_i(q)
\]

(7)

where \(e_i(q) = D_i^b - CD_i^r\) is the error vector function. Minimizing this target function can be achieved by GDA, GNA, LMA and etc. For instance, the LMA conduct optimization iteration by [18]

\[
q_p = q_{p-1} - (J_i^TJ_i + \kappa I)J_i^Ter_i(q_{p-1})
\]

(8)

where \(p\) is the recursion index; \(\kappa\) denotes a tiny positive number ensuring invertibility of matrix \(J_i\) stand for the Jacobian of \(e_i(q_{p-1})\) with respect to \(q_{p-1}\). Such optimization relies on the Hessian that determines whether there is local optimum or not. To study the convexity of the function \(F_i(q)\), we can simplify it into

\[
F_i(q) = (CD_i^r - D_i^b)^T(\ CD_i^r - D_i^b)
\]

\[
= (D_i^r)^TC^TCD_i^r + (D_i^b)^TD_i^b - (D_i^b)^TCD_i^r - (D_i^r)^TC^TD_i^b
\]

\[
= 1 + (q_0^2 + q_1^2 + q_2^2 + q_3^2)^2 - (D_i^r)^TC^TCD_i^r - (D_i^r)^TC^TD_i^b
\]

\[
= 1 + (q_0^2 + q_1^2 + q_2^2 + q_3^2)^2 - 2A
\]

(9)

where we use

\[
(D_i^b)^TD_i^b = (D_i^r)^TD_i^r = 1
\]

\[
C^TC = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^2
\]

(10) and

\[
A = (D_i^r)^TC^TCD_i^r
\]

(11)

for simplification. The kernel problem is to deduce the convexity of function \(A\). Note that \((q_0^2 + q_1^2 + q_2^2 + q_3^2)^2 = \|q\|^4\) is not simplified to 1 because of possible loss of normalization between successive optimization updates. The main thought of the proof is to show that the Hessian of \(F_i(q)\) belongs to positive semidefinite matrices [19].

Taking the Hessian of \(F_i(q)\), we obtain

\[
H_{F_i} = \begin{pmatrix}
\frac{\partial^2 F_i}{\partial q_0^2} & \frac{\partial^2 F_i}{\partial q_0 q_1} & \frac{\partial^2 F_i}{\partial q_0 q_2} & \frac{\partial^2 F_i}{\partial q_0 q_3} \\
\frac{\partial^2 F_i}{\partial q_1 q_0} & \frac{\partial^2 F_i}{\partial q_1^2} & \frac{\partial^2 F_i}{\partial q_1 q_2} & \frac{\partial^2 F_i}{\partial q_1 q_3} \\
\frac{\partial^2 F_i}{\partial q_2 q_0} & \frac{\partial^2 F_i}{\partial q_2 q_1} & \frac{\partial^2 F_i}{\partial q_2^2} & \frac{\partial^2 F_i}{\partial q_2 q_3} \\
\frac{\partial^2 F_i}{\partial q_3 q_0} & \frac{\partial^2 F_i}{\partial q_3 q_1} & \frac{\partial^2 F_i}{\partial q_3 q_2} & \frac{\partial^2 F_i}{\partial q_3^2}
\end{pmatrix}
\]

(12)
\[
Q_A = \begin{bmatrix}
\frac{(D^k_x-D^k_y)(D^k_z-D^k_r)}{A^k - 1} & \frac{(D^k_x+D^k_y)(D^k_z+D^k_r)}{A^k - 1} \\
\frac{(D^k_x-D^k_y)(D^k_z+D^k_r)}{A^k - 1} & \frac{(D^k_x+D^k_y)(D^k_z-D^k_r)}{A^k - 1}
\end{bmatrix}
\]

in which
\[
\frac{\partial A}{\partial q_k} = (D^k)^T \frac{\partial C}{\partial q_k} D^k, \quad k = 0, 1, 2, 3
\]  

(13)

where \( k = 0, 1, 2, 3 \) are the quaternion indices and the derivatives of \( C \) can be computed by

\[
\begin{align*}
\frac{\partial C}{\partial q_0} &= 2 \begin{pmatrix}
q_0 & q_3 & -q_2 \\
-q_3 & q_0 & q_1 \\
q_2 & -q_1 & q_0
\end{pmatrix} \\
\frac{\partial C}{\partial q_1} &= 2 \begin{pmatrix}
q_1 & q_2 & q_3 \\
q_2 & q_1 & q_0 \\
q_3 & q_0 & q_1
\end{pmatrix} \\
\frac{\partial C}{\partial q_2} &= 2 \begin{pmatrix}
-q_2 & q_1 & -q_0 \\
q_1 & q_2 & q_3 \\
q_0 & q_3 & -q_2
\end{pmatrix} \\
\frac{\partial C}{\partial q_3} &= 2 \begin{pmatrix}
-q_3 & q_0 & q_1 \\
-q_0 & -q_3 & q_2 \\
q_1 & q_2 & q_3
\end{pmatrix}
\end{align*}
\]

(14)

The above equations lead to further computations in (17). Then we obtain

\[
\frac{\partial^2 A}{\partial q_k \partial q_j} = (D^k)^T \frac{\partial^2 C}{\partial q_k \partial q_j} D^k, \quad k = 0, 1, 2, 3, \quad j = 0, 1, 2, 3
\]  

(15)

From the results of second-order derivative of \( C \) we can observe that

\[
\frac{\partial C}{\partial q_k \partial q_j} = \frac{\partial C}{\partial q_j \partial q_k}, \quad k = 0, 1, 2, 3, \quad j = 0, 1, 2, 3
\]  

(16)

This leads to the Hessian of \( A \) i.e. \( H_A \) being a symmetric matrix, such that

\[
\begin{align*}
H_{A,11} &= 2(D^k_x D^k_z + D^k_y D^k_y + D^k_b D^k_r) \\
H_{A,12} &= 2(D^k_x D^k_y - D^k_y D^k_x) \\
H_{A,13} &= 2(D^k_y D^k_z - D^k_b D^k_x) \\
H_{A,14} &= 2(D^k_y D^k_z + D^k_b D^k_x) \\
H_{A,22} &= 2(-D^k_x D^k_z + D^k_y D^k_y + D^k_b D^k_r) \\
H_{A,23} &= -2(D^k_y D^k_z - D^k_b D^k_x) \\
H_{A,24} &= -2(D^k_y D^k_z + D^k_b D^k_x) \\
H_{A,33} &= 2(D^k_y D^k_z - D^k_b D^k_x) \\
H_{A,34} &= -2(D^k_y D^k_z + D^k_b D^k_x) \\
H_{A,44} &= 2(D^k_x D^k_z - D^k_y D^k_y + D^k_b D^k_r)
\end{align*}
\]

(19)

where \( H_{A,jk} \) is the entry of \( H_A \) in the \( j \)-th row and \( k \)-th column. The Hessian of \((q_0^2 + q_1^2 + q_2^2 + q_3^2)^2\) can easily be computed by

\[
H_{\|q\|^4} = 4(q_0^2 + q_1^2 + q_2^2 + q_3^2)^2 \mathbf{I} + 8qq^T
\]  

(20)

Hence the final Hessian of \( F_i(q) \) takes the following form

\[
H_{F_i} = 4\|q\|^2 \mathbf{I} + 8qq^T - 2H_A
\]  

(21)

Notice that \( H_A \) has the eigenvalue decomposition of

\[
H_A = 2Q_A \Sigma_A Q_A^{-1}
\]  

(22)

where

\[
\Sigma_A = \text{diag}(1, 1, -1, -1)
\]  

(23)

while \( Q_A \) is given in (18) in which

\[
N = (D^k_x)^2 + (D^k_y)^2 - (D^k_z)^2 - (D^k_r)^2
\]  

(24)

Therefore we can see that \( 4\|q\|^2 \mathbf{I} - 2H_A \) has the eigenvalues of

\[
\lambda_{4\|q\|^2\mathbf{I} - 2H_A} = \begin{pmatrix}
4\|q\|^2 + 4 & 4\|q\|^2 + 4 & 4\|q\|^2 - 4 & 4\|q\|^2 - 4
\end{pmatrix}
\]

(25)

From another aspect, \( qq^T \) is a matrix with rank 1 and all non-negative eigenvalues, such that

\[
qq^T = Q_{qq^T} \Sigma_{qq^T} Q_{qq^T}^{-1}
\]  

(26)

where

\[
\Sigma_{qq^T} = \text{diag}(0, 0, 0, q_0^2 + q_1^2 + q_2^2 + q_3^2)
\]  

(27)

\[
Q_{qq^T} = \begin{pmatrix}
\frac{q_0}{q_0} & 0 & 0 & 1 \\
0 & \frac{q_0}{q_0} & 0 & 1 \\
0 & 0 & \frac{q_0}{q_0} & 1 \\
\frac{q_0}{q_3} & \frac{q_1}{q_3} & \frac{q_2}{q_3} & \frac{q_3}{q_3}
\end{pmatrix}
\]

(28)

Therefore we have

\[
\text{rank}(H_{F_i}) = \text{rank}(4\|q\|^2 \mathbf{I} - 2H_A + 8qq^T) \leq \text{rank}(4\|q\|^2 \mathbf{I} - 2H_A) \leq 8qq^T
\]

(29)

The eigenvalues of \( H_{F_i} \) satisfies

\[
\min(\lambda_{4\|q\|^2\mathbf{I} - 2H_A}) + \min(\lambda_{8qq^T}) \leq \lambda_H \leq \max(\lambda_{4\|q\|^2\mathbf{I} - 2H_A}) + \max(\lambda_{8qq^T})
\]

yielding

\[
4\|q\|^2 - 4 \leq \lambda_H \leq 12\|q\|^2 + 4
\]

(30)

That is to say, \( H_{F_i} \) is currently indefinite. However, when conducting in the optimization ensuring that the quaternion is always normalized in last step, we would obtain \( \|q\| = 1 \). In fact, for all \( q \) owning \( \|q\| \geq 1 \), \( H_{F_i} \) is a positive semidefinite symmetric matrix with rank 3 or 4. As in all literatures, normalization of quaternion always takes place, then it is ensured that the optimization is convex [19].
B. Multi-Vector Case

Defining
\[
e(q) = \begin{bmatrix}
\sqrt{a_1} (D^b_1 - CD^r_1) \\
\sqrt{a_2} (D^b_2 - CD^r_2) \\
\vdots \\
\sqrt{a_n} (D^b_n - CD^r_n)
\end{bmatrix}
\] (31)

, one can easily find out that the target loss function defined by
\[
F(q) = e^T(q)e(q)
\] (32)
meets
\[
F(q) = \sum_{i=1}^{n} a_i F_i(q) = \sum_{i=1}^{n} a_i \|D^b_i - CD^r_i\|^2
\] (33)

Its Hessian \(H\)
\[
H = \sum_{i=1}^{n} a_i H_{F_i}
\] (34)
has the eigenvalue inequality of
\[
0 \leq \lambda_H \leq \sum_{i=1}^{n} a_i \left[ 4 + 12 \left( q_i^2 + q_j^2 + q_k^2 + q_l^2 \right) \right]
\] (35)

which proves the convexity of the attitude optimization from multi-vector observations.

III. Numerical Example

Assume that we obtain the following single vector observation pair from a vector sensor
\[
D^b = \begin{pmatrix}
-0.71282482753344 \\
-0.22577381096068 \\
0.664008732763561
\end{pmatrix}
\] (36)

\[
D^r = \begin{pmatrix}
-0.037453665434217 \\
0.500499809534146 \\
-0.864926102971707
\end{pmatrix}
\]

When we perform optimization based on last unnormalized updated quaternion such as
\[
q = \begin{pmatrix}
0.420683700201250 \\
0.40073998146962 \\
0.095142157864169 \\
0.496084391636530
\end{pmatrix}
\] (37)

with \(\|q\| = 0.77026822031943 < 1\), the Hessian’s eigenvalues can be computed by
\[
\lambda_H = \begin{pmatrix}
9.761874553883407 \\
6.373252553488999 \\
-0.268864411927401 \\
-1.626747464510996
\end{pmatrix}
\] (38)

indicating that the optimization is non-convex nor concave. However, with normalized quaternion of
\[
q = \begin{pmatrix}
0.118759061535262 \\
-0.346543560044311 \\
-0.817997262250335 \\
0.443491065576337
\end{pmatrix}
\] (39)

we can compute the Hessian’s eigenvalues by
\[
\lambda_H = \begin{pmatrix}
14.677631236006699 \\
7.99999999999996 \\
1.322368763993306 \\
0.00000000000000
\end{pmatrix}
\] (40)

which reflects \(H\) here is positive semidefinite. While with
\[
q = \begin{pmatrix}
-0.353622599929341 \\
0.0464354268087823 \\
0.0464354268087823 \\
-1.55051447747561
\end{pmatrix}
\] (41)
in which \(\|q\| = 1.777657443309303 > 1\), we have
\[
\lambda_H = \begin{pmatrix}
39.12760929877825 \\
16.640263943011881 \\
11.433446472169676 \\
8.640263943011886
\end{pmatrix}
\] (42)

which verifies the former derived results that the current \(H\) owns positive definiteness. Therefore, we verify that that the attitude determination problem in [4] is always convex provided that the last-step quaternion is normalized.

IV. Conclusion

In this paper, the optimization framework of the attitude determination from vector observations is revisited. Some closed-form results are derived showing the identities of the Hessian to the optimization. By eigenvalue analysis, it is found out that the original problem is sometimes concave but is rigorously convex after adding a quaternion normalization in advance. As such commitment is very common in real engineering practice ensuring unitary quaternion norm, the target optimization can be regarded as a fully convex one. Numerical examples containing simulated cases verify the derived results. It is fully proven in this paper that the previous derivative-based optimization techniques are robust in the case of convexity. And we hope that this contribution would benefit related research in the future.

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