Percolation of randomly distributed growing clusters: Finite Size Scaling and Critical Exponents

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We study the percolation properties of the growing clusters model. In this model, a number of seeds placed on random locations on a lattice are allowed to grow with a constant velocity to form clusters. When two or more clusters eventually touch each other they immediately stop their growth. The model exhibits a discontinuous transition for very low values of the seed concentration \( p \) and a second, non-trivial continuous phase transition for intermediate \( p \) values. Here we study in detail this continuous transition that separates a phase of finite clusters from a phase characterized by the presence of a giant component. Using finite size scaling and large scale Monte Carlo simulations we determine the value of the percolation threshold where the giant component first appears, and the critical exponents that characterize the transition. We find that the transition belongs to a different universality class from the standard percolation transition.

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I. INTRODUCTION

Percolation represents a paradigmatic model of a geometric phase transition \cite{1,2}. It has been widely studied and has numerous applications in many areas of physics \cite{3-17}. Its importance and practical applications are described in detail elsewhere, see for example \cite{2}. Here, we will present for the sake of clarity and completeness, some necessary definitions of important quantities that commonly appear in the literature. In the classical site percolation model, the sites of a square lattice are randomly occupied with particles with probability \( p \), or remain empty with probability \( 1-p \). Neighboring occupied sites are considered to belong to the same cluster and percolation theory simply deals with the number and properties of these clusters. When the occupation probability \( p \) is small, the occupied sites are either isolated or form very small clusters. On the other hand, for large \( p \) there are a lot of occupied sites that have formed one large cluster and it is possible to find several paths of occupied sites which a walker can use to move from one side of the lattice to the other. In this latter case, it is said that a giant component of connected sites exists in the lattice. This component does not appear in a gradual “linear” way with increasing \( p \). It appears suddenly at a critical occupation probability \( p_c \). Below \( p_c \) there are only small clusters and even if we increase the lattice size considerably, these clusters remain small, i.e. the largest cluster does not depend on the system size. Above \( p_c \), suddenly, small clusters join together to form a large cluster whose size scales with system size. Hence, the term giant component or infinite cluster which is very common in the literature \cite{1}.

In percolation, \( p \) plays the same role as the temperature in thermal phase transitions, i.e. that of the control parameter, while the order parameter is the probability \( P_\infty \) that a site belongs to the infinite cluster. For \( p > p_c \), \( P_\infty \) increases with \( p \) by a power law

\[
P_\infty \sim (p - p_c)^\beta
\]

Other important quantities are the correlation length \( \xi \) which is defined as the mean distance between two sites on the same finite cluster and the mean number of sites \( S \) of a finite cluster. When \( p \) approaches \( p_c \), \( \xi \) increases as

\[
\xi \sim (p - p_c)^{-\nu}
\]

The mean number of sites \( S \) of a finite cluster also diverges at \( p_c \)

\[
S \sim (p - p_c)^{-\gamma}
\]

The critical exponents \( \beta, \nu \) and \( \gamma \) describe the critical behavior associated with the percolation transition and are universal. They do not depend on the structure of the lattice (e.g., square or triangular) or on the type of percolation (site, bond or even continuum) \cite{2}.

In this paper we study numerically the percolation properties of the growing clusters model \cite{18} which we describe in Sec. II. We focus on the intermediate concentration regime and find that the model exhibits a non-trivial percolation transition which belongs in a different universality class from standard percolation. We determine quite accurately the position of \( p_c \) and the values of the critical exponents associated with this transition.

II. THE GROWING CLUSTERS MODEL

The growing clusters model is presented in detail in \cite{18}. Here, we provide a brief description of it. A square lattice of size \( L \) is randomly populated with “seeds” with probability \( p \). These seeds are allowed to grow to clusters isotropically and stop when they touch another growing aggregate, see fig. I for a schematic of the system evolution. At every time step, i.e. one Monte Carlo Step.
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Monte Carlo simulations as described in [18] and we mon-
stable cluster. To study the model we perform large scale

Neighboring sites are considered to belong in the same
growing seeds. Thus, each seed becomes an evolv-
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Investigation sequence is random in order. Each seed is
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III. FINITE SIZE SCALING

Equations [15] are valid for infinite systems close to the
critical threshold. In practice, however, it is not possible
to use them to calculate the critical exponents with con-
siderable accuracy due to finite size effects. Thus, one
has to resort to finite size scaling [2] techniques. Due to
the finite size of the lattices that can be simulated the or-
der parameter is expected to depend on the system size.
Assuming that we are close to the critical threshold so
that the correlation length $\xi$ is comparable to the system
size $L$ it can be shown that the probability $P_{\text{max}}$ that
a site belongs to the largest cluster follows the scaling
relation:

$$P_{\text{max}} = L^{-\beta/\nu} F[L^{1/\nu}(p - p_c)]$$  \(4\)

where we have deliberately used the notation $P_{\text{max}}$ in-
stead of $P_\infty$ in order to emphasize the finiteness of the
systems. Here $F$ is a suitable scaling function. Simi-
larly, any other quantity varying as $|p - p_c|^x$ is expected
to scale similar to eq. 4 with $\beta$ replaced by the appro-
propriate exponent $x$ and with a different scaling function
$F$. At $p = p_c$, these quantities are expected to scale as
power law since the scaling function reduces to a simple
proportionality constant. This result gives a way to
determine the critical exponents. One important char-
acteristic of the standard percolation transition is that
exactly at the critical point, the largest cluster has a
fractal geometry meaning that the mass of the largest
cluster $S_1$ scales with the system size as $S_1 \sim L^{d_f}$, where
the fractal dimension $d_f$ is known to be equal to 91/48.
Since $S_1 = P_{\text{max}} L^d$, where $d$ is the dimensionality of the
space ($d = 2$ in the lattice case) one can easily derive a
scaling law relating $d_f, \beta$ and $\nu$, namely $d_f = d - \beta/\nu$.

IV. RESULTS AND DISCUSSION

To obtain an indication of the system critical behav-
or, we start by monitoring the size of the largest cluster
$S_1$ as a function of the initial “seed” probability $p$, see
[18] and fig. 2 therein. There, it is evident that the
system exhibits two phase transitions: One very sharp
transition at $p = 0$ and a second, smoother, transition
around $p \approx 0.5$. The first transition is discussed in de-
tail at [15] and characterized by the fact that the size of the
largest cluster $S_1$, which is the order parameter of the
system, has a discontinuity for $p = 0$ and, thus, the
system exhibits a sharp, albeit artificial, first order phase
transition.

The second phase transition turns out to be rather in-
teresting. It is reminiscent of the classical percolation transi-
tion and it is important to clarify the extent of this
similarity. Thus, we simulate systems of several different
sizes and for several different initial seed concentrations.
After allowing the growth process to complete and the
systems to reach their final states, we study the proba-
bility $P_{\text{max}}$ that a randomly chosen site belongs to the
largest cluster. When there are only few initial seeds, af-
after the evolution of the system is completed, the clusters
formed are small and isolated. However, we expect that
with increasing $p$ when a large number of seeds is intro-
duced, the growth process will lead to the formation of a
large spanning cluster. This is the classic behavior seen
in a system which undergoes a continuous phase transi-
tion. In such cases, we can use finite size scaling to
determine the position of the critical concentration, $p_c$,
and the critical exponents. Initially, we are interested in
the geometry of the largest cluster at criticality and its
fractal dimension $d_f$. In fig. 2 we plot the size of the
largest cluster $S_1 = P_{\text{max}} L^2$ as a function of the lattice

FIG. 1: Top Left: At time $t = 0$ Monte Carlo Steps (MCS)
6 seeds are randomly placed on a $40 \times 40$ lattice. Top Right:
After $t = 3$ MCS there are 6 evolving clusters. Bottom Left:
After $t = 6$ MCS there are 4 evolving clusters, as 2 clusters
have touched each other and stopped growing. Bottom Right:
Final state of the system. There are no evolving clusters and
3 stable clusters have been formed.
FIG. 2: Size of the largest cluster, $S_1 = P_{\text{max}}L^2$, as a function of the lattice size $L$ for three different initial concentrations, $p = 0.48, 0.496$ and 0.52 (black squares, red dots and green triangles respectively). Points are Monte Carlo Simulation results and the straight line is a power law fit with slope 1.79.

FIG. 3: Snapshot of the final state of a system with $L = 100$ with initial seed concentration $p = 0.496$ which is equal to the critical concentration ($p = p_c$). Different colors correspond to different clusters. The final coverage is $p_f = 0.538$. The largest cluster is shown in black color.

FIG. 4: $P_{\text{max}}L^{0.206}$ vs $p$ for seven different system sizes, namely $L = 100, 200, 400, 600, 800, 1000, 1200$. The curves cross at $p_c = 0.496$ in agreement with the scaling relation eq. 4.
$L = 100, 200, 400, 600, 800, 1000, 1200$, and we vary $\beta/\nu$ until all curves cross at one single point only. This is done for $\beta/\nu = 0.206$ and for $p = p_c \simeq 0.496$. This result is in excellent agreement with our previous estimation for the $d_f$ and the scaling relation $d_f = d - \beta/\nu$. It also allows to determine the exact location of the critical point with more accuracy than the previous method.

Fig. 5 shows a plot of the mean mass $S$ of all the finite clusters as a function of $L$ for $p = p_c$. At criticality, this quantity is expected to scale as $S \sim L^{\gamma/\nu}$ (see section III). From the slope of the straight line we estimate $\gamma/\nu = 1.63$. This, as well as the previous result, are in very good agreement with our previous estimation for the classical percolation transition.

Finally, we calculate the critical exponent $\nu$. In fig. 6 we plot $P_{\text{max}} L^{0.206} \ (p - p_c) L^{1/\nu}$ for seven different system sizes $L = 100, 200, 400, 600, 800, 1000, 1200$, and we vary the exponent $1/\nu$ until all data collapse on one single curve. The data collapse, in agreement with eq. 4 enables us to determine the critical exponent $\nu$ with considerable accuracy. We find that $1/\nu = 0.85$.

Table I shows a comparison of the critical exponents between the classical percolation transition and the transition of the growing clusters model. The exponents $\tau$ and $\sigma$ are associated with the cluster size distribution [11].

| Exponent | Class. Percolation | Growing Clusters |
|----------|--------------------|------------------|
| $\beta$  | 0.138              | 0.24             |
| $\gamma$ | 2.38               | 1.91             |
| $\nu$    | 1.33               | 1.17             |
| $\sigma = 1/(\beta + \gamma)$ | 0.395 | 0.46 |
| $\tau = 1 + \sigma d$ | 2.05 | 2.08 |
| $d_f$    | 1.89               | 1.79             |

**V. CONCLUSIONS**

We have studied the growing clusters model and found that it exhibits two phase transitions with increasing seed concentration. A first order transition at $p = 0$ and a continuous transition at $p_c = 0.496$, separating a phase of isolated clusters for $p < p_c$ from a phase where a giant component is present for $p > p_c$. Using finite size scaling we have calculated the position of the phase transition and the critical exponents with considerable accuracy to establish that this transition belongs to a different universality class from the standard percolation transition.
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