Finite Volume Dependence of the Quark-Antiquark Vacuum Expectation Value

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Abstract

A general formula is derived for the finite volume dependence of vacuum expectation values analogous to Lüscher’s formula for the masses. The result involves no integrals over kinematic quantities and depends only on the matrix element between pions at zero momentum transfer thus presenting a new way to calculate the latter, i.e. pion sigma terms.

The full order $p^6$ correction to the vacuum condensate $\langle \bar{q}q \rangle$ is evaluated and compared with the result from the Lüscher formula. Due to the size of the $p^6$ result no conclusion about the accuracy of the Lüscher formula can be drawn.
1 Introduction

Quantum Chromo Dynamics (QCD) at low energy remains a difficult problem. One of the ways to deal with this problem is to numerically evaluate the functional integral of QCD. This approach, known as lattice QCD, has now reached the stage where realistic calculations with fairly light quark masses are now possible. One side effect of this is that finite volume corrections are becoming more important. Luckily in many cases these corrections can be evaluated analytically using Chiral Perturbation Theory (ChPT) \[1, 2\]. The application of ChPT to finite volume was started by Gasser and Leutwyler \[3\]. A review of recent work in this area can be found in \[4\]. Note that ChPT is applicable to finite volume as soon as the typical momenta that are relevant are small enough. This imposes a size restriction on the volume as

\[ F_\pi L > 1. \] (1)

Here $F_\pi$ is the pion decay constant and $L$ is the linear size of the volume. This paper is concerned with the $p$-regime. This is the regime where the volume is large enough such that the zero momentum fluctuations of the meson fields can be treated perturbatively. This is the regime with in addition the requirement that

\[ m^2_\pi F_\pi^2 V >> 1. \] (2)

These finite volume corrections have been evaluated for many quantities up to one-loop order. This is the order where the first nontrivial dependence on the volume shows up. One purpose of this paper is to calculate the full two-loop finite volume corrections to the vacuum condensate $\langle \bar{q}q \rangle$. This is the one of the calculations of finite volume effects to this order\(^1\) The vacuum condensate at finite volume has been studied at one-loop in Ref. \[6\].

An alternative approach to finite volume corrections was introduced by Lüscher where the leading part of the finite volume corrections was derived to all orders in perturbation theory for the mass in terms of a scattering amplitude \[7, 8\]. This was extended to the finite volume corrections for the pion decay constant in \[9\]. The other purpose of this paper is to extend the Lüscher formula also to vacuum expectation values. This will in general connect the finite volume corrections of an operator to the zero-momentum transfer matrix element of that operator between pion states as shown in Eq. \[5\]. This allows for new ways to calculate sigma terms from the finite volume variation of vacuum condensates.

Note that all our formulas are for the case of an infinite extension in the time direction but a finite volume in the three spatial directions. The formulas can be easily extended to a small fourth direction as well by replacing the integrals with the expressions for that case.

\(^1\)Ref. \[5\] with another calculation appeared essentially simultaneously. There the pion mass at finite volume was studied at two-loop order.
2 A Lüscher Formula for the Vacuum Condensate

It turns out to be straightforward to extend Lüscher’s formula for the mass to the case of the vacuum condensate. In fact the formula has an even simpler structure than the one for the mass or the decay constant [9]. The underlying observation of Lüscher’s method is that the leading finite volume corrections come from one of the propagators feeling the finite volume effects only. We write the finite volume propagator \( G_V(q) \) as

\[
G_V(q) = \sum_{\vec{n}} e^{-iq \cdot \vec{n} L_i} G_\infty(q).
\]

(3)

Here the sum is over a vector of integers and \( L_i \) is the length of the volume in the \( i \)-th direction. The term with \( \vec{n} = \vec{o} \) gives the infinite volume case. Lüscher then kept only the term with one of the \( n_i = \pm 1 \) since this is parametrically leading. As suggested in [10] one can also easily keep the entire series, thus keeping part of the subleading corrections. The remainder of the Lüscher’s analysis goes by taking the component of \( \vec{q} \) parallel to \( \vec{n} \), \( q_n \) and distorting the integration contour along that direction to \( q_n \to q_n - is \). In the limit of \( s \to \infty \) the contribution from that integral vanishes and only the parts coming from the singularities encountered while deforming the contour remain. These happen when relevant propagators can go on-shell. A pedagogical introduction can be found in [8].

The difficulty in proving this is that it needs to be done to all orders in perturbation theory, this was done in [7]. Here we only sketch the lines of reasoning. Of the three types of contributions shown in Fig. 4 of [7] only one is relevant here and is shown in Fig. (b). The case of operators with contributions of the type shown in diagram (a) is not relevant for ChPT, one can always add the parity-conjugate operator to remove the contribution from diagram (a).

One difference with the case of the mass or decay constant here is that when the singularity is encountered, there is no freedom left in the relevant matrix element and the integral over the momentum of the propagator can be fully done. In the case of the mass and decay constant there is an external momentum available, \( p_{ext} \), this leaves after putting the deformed \( q \) on-shell a freedom in \( q \cdot p_{ext} \) which results in the final integration over \( \nu \). Here there is no external momentum and hence there is no such freedom left.

The formula after putting everything on-shell becomes

\[
\langle O \rangle_V - \langle O \rangle_\infty = - \sum_{\vec{n} \neq \vec{o}} \frac{1}{16\pi^2} \int_0^\infty \frac{dq^2 q^2}{m_0^2 + q^2} e^{-\sqrt{\vec{n}^2 (m_0^2 + q^2)}L^2} \langle \phi | O | \phi \rangle ,
\]

(4)

for a real boson with infinite volume mass \( m_0 \) and the matrix element on the right hand side should be taken at zero momentum transfer. The integral can now be done explicitly in terms of the generalized Bessel function \( K_1 \). It also only depends on \( k = \vec{n}^2 \) which is also integer. The number of times in the sum that \( \vec{n}^2 = k \) we call \( x(k) \). We obtain

\[
\langle O \rangle_V - \langle O \rangle_\infty = - \sum_{k=1,\infty} x(k) \frac{m_0^2}{16\pi^2} \frac{K_1(\zeta(k))}{\sqrt{\zeta(k)}} \langle \phi | O | \phi \rangle ,
\]

(5)

for a real boson with infinite volume mass \( m_0 \) and the matrix element on the right hand side should be taken at zero momentum transfer.
with $\zeta(k) = \sqrt{k} m_0 L$. Note that the above formula is for the case of one real scalar. The multiplicity factors for complex scalars can be trivially taken into account. Note that we have left the sum over all modes in as suggested for the mass in [11].

In particular for the case of $\langle \bar{q}q \rangle$ the relevant matrix element is the sigma term. The finite volume corrections to the vacuum condensate thus are another option to calculate the pion sigma term. This can then be compared with the direct calculation of the sigma term via the matrix element $\langle \pi | \bar{q}q | \pi \rangle$ or via the Feynman-Hellman theorem from $\partial m_\pi^2 / \partial m_q$.

3 The finite volume vacuum condensate at two-loops

The vacuum condensate at two-loop was calculated in [12], here we repeat that calculation taking into account the finite volume effects. The calculation in terms of the lowest order meson masses is straightforward and proceeds exactly as in [12]. The details of calculation of two loops in ChPT can be found in [12]. The diagrams that contribute are shown in Fig. 2.

In infinite volume the loop diagrams contain the integrals

\[
A(m^2) = \frac{1}{i} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - m^2},
\]

\[
B(m^2) = \frac{1}{i} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)^2}.
\]

These are expanded in terms of $\epsilon = (4 - d)/2$ as

\[
A(m^2) = \frac{m^2}{16\pi^2} \left( \frac{1}{\epsilon} - \gamma_E + \log(4\pi) + 1 \right) + \bar{A}(m^2) + \epsilon A'(m^2) + \mathcal{O}(\epsilon^2),
\]
B(\(m^2\)) = \frac{1}{16\pi^2} \left( \frac{1}{\epsilon} - \gamma_E + \log(4\pi) + 1 \right) + \frac{1}{16\pi^2} \sum_{k=1,\infty} x(k) \frac{4m^2}{\lambda_k} K_1(\lambda_k)

B'(m^2) = \frac{1}{16\pi^2} \log \left( \frac{m^2}{\mu^2} \right) - \frac{1}{16\pi^2} + \frac{1}{16\pi^2} \sum_{k=1,\infty} x(k)2K_0(\lambda_k)

\begin{align*}
\mathcal{A}(m^2) &= - \frac{m^2}{16\pi^2} \log \left( \frac{m^2}{\mu^2} \right) - \frac{1}{16\pi^2} \sum_{k=1,\infty} x(k) \frac{4m^2}{\lambda_k} K_1(\lambda_k) \\
\mathcal{B}(m^2) &= - \frac{1}{16\pi^2} \log \left( \frac{m^2}{\mu^2} \right) - \frac{1}{16\pi^2} + \frac{1}{16\pi^2} \sum_{k=1,\infty} x(k)2K_0(\lambda_k)
\end{align*}

with \(\lambda_k = \sqrt{k} mL\). The functions \(K_1\) and \(K_2\) are modified Bessel functions and the integer quantity \(x(k)\) is the number of times the sum of squares \(\vec{k}^2 = k_1^2 + k_2^2 + k_3^2\) is equal to \(k\) when \(k_1, k_2, k_3\) are varied over all positive and negative integers. For large \(L\) the Bessel functions lead to an exponential fall-off with the finite size.

We have checked explicitly that at two-loop order the contributions containing \(A^\epsilon\) and \(B^\epsilon\) cancel, so we do not need to evaluate these at finite volume. That this cancellation happens follows from the cancellations of nonlocal divergences but in the calculation of [12] this calculation was not explicitly checked. We have also reperformed the full calculation since in [12] the relation \(\mathcal{A}(m^2) = m^2 \left( \mathcal{B}(m^2) + 1/(16\pi^2) \right)\) was used, which is no longer

Figure 2: The diagrams up to order \(p^6\) for \(\langle \bar{q} q \rangle\). The lines are meson propagators and the vertices are: \(\circ\) a \(p^2\) insertion of \(\bar{q} q\), \(\otimes\) a \(p^4\) insertion of \(\bar{q} q\), \(\odot\) a \(p^6\) insertion of \(\bar{q} q\), \(\bullet\) a \(p^2\) vertex and \(\times\) a \(p^4\) vertex.

\begin{align*}
B(m^2) &= \frac{1}{16\pi^2} \left( \frac{1}{\epsilon} - \gamma_E + \log(4\pi) + 1 \right) + \frac{1}{16\pi^2} \left( \mathcal{B}(m^2) + \epsilon B'(m^2) + O(\epsilon^2) \right),
\end{align*}

with a similar expansion for \(B(m^2)\). At finite volume the integrals over momenta get replaced by sums over the finite possible momenta. In this paper we only keep three of the four dimensions at the same finite length \(L\). Their principle of evaluation can be found in Refs. [3, 13] and explicit expressions are at finite volume. After renormalization the divergent parts cancel and the finite parts become subtraction-scale \(\mu\) dependent. The needed explicit expressions are

\begin{align*}
\mathcal{A}(m^2) &= - \frac{m^2}{16\pi^2} \log \left( \frac{m^2}{\mu^2} \right) - \frac{1}{16\pi^2} \sum_{k=1,\infty} x(k) \frac{4m^2}{\lambda_k} K_1(\lambda_k) \\
\mathcal{B}(m^2) &= - \frac{1}{16\pi^2} \log \left( \frac{m^2}{\mu^2} \right) - \frac{1}{16\pi^2} + \frac{1}{16\pi^2} \sum_{k=1,\infty} x(k)2K_0(\lambda_k)
\end{align*}
true at finite volume. In Ref. [12] the result was expressed in terms of the physical mass and decay constant. This can be done here as well but since we must then distinguish between the integrals at finite volume and those at infinite volume the formula is shorter if we leave meson masses and decay constants in their unrenormalized form.

The result, in three flavour ChPT, uses the $p^4$ low-energy constants (LECs) $L_i^r$ from Ref. [2] and the $C_i^r$ from [14]. The latter drop out when comparing finite and infinite volume corrections and the values of the $L_i^r$ only contribute to that difference at NNLO. We also suppress the $\mu$ dependence of all quantities. The lowest order masses for the pions, kaons and eta we denote by $\chi_\pi$, $\chi_K$ and $\chi_\eta$ respectively and in term of the average up and down quark-mass $\bar{m}$ and strange quark-mass $m_s$ they are

$$\chi_\pi = 2 B_0 \bar{m}, \quad \chi_K = B_0 (\bar{m} + m_s), \quad \chi_\eta = 2 B_0 (\bar{m} + 2 m_s)/3.$$  

(9)

We define the quantities

$$\langle \bar{u}u \rangle = -B_0 F_0^2 \left(1 + \frac{v_4^u}{F_0^2} + \frac{v_6^u}{F_0^3}\right), \quad \langle \bar{s}u \rangle = -B_0 F_0^2 \left(1 + \frac{v_4^s}{F_0^2} + \frac{v_6^s}{F_0^3}\right).$$  

(10)

The calculation gives

$$v_4^u = A(\chi_\pi) \left(\frac{3}{2}\right) + A(\chi_K) + A(\chi_\eta) \left(\frac{1}{6}\right) + 16 \chi_\pi L_6^r + 8 \chi_\pi L_8^r + 4 \chi_\pi H_2^r + 32 \chi_K L_6^r,$$

$$v_4^s = A(\chi_\pi) \left(2\right) + A(\chi_K) \left(2/3\right) + 16 \chi_\pi L_6^r + 4 (2 \chi_K - \chi_\pi) (2 L_8^r + H_2^r) + 32 \chi_K L_6^r,$$

$$v_6^u = A(\chi_\pi)^2 \left(-3/8\right) + A(\chi_K) + A(\chi_\eta) \left(1/4\right) + A(\chi_\pi) B(\chi_\pi) \left(-3/4 \chi_\pi\right) + A(\chi_K) + A(\chi_\eta) B(\chi_\eta) \left(-2/9 \chi_K\right) + A(\chi_\pi)^2 \left(1/72\right) + A(\chi_K) B(\chi_K) \left(-1/3 \chi_K\right) + A(\chi_\pi) B(\chi_\pi) \left(-7/108 \chi_\pi + 4/27 \chi_K\right) + A(\chi_K) \left(-36 \chi_\pi L_4^r - 24 \chi_\pi L_5^r + 72 \chi_\pi L_6^r + 48 \chi_\pi L_8^r - 24 \chi_K L_4^r + 48 \chi_K L_6^r\right) + A(\chi_\pi) \left(-8 \chi_\pi L_4^r + 16 \chi_\pi L_6^r - 48 \chi_K L_4^r - 16 \chi_K L_5^r + 96 \chi_K L_6^r + 32 \chi_K L_8^r\right) + A(\chi_\pi) \left(4/3 \chi_\pi L_4^r + 8/9 \chi_\pi L_5^r - 8/3 \chi_\pi L_6^r + 64/3 \chi_\pi L_7^r + 16/3 \chi_\pi L_8^r\right) + 40/3 \chi_K L_4^r - 32/9 \chi_K L_5^r + 80/3 \chi_K L_6^r - 64/3 \chi_K L_7^r,$$

$$+ A(\chi_\pi) \left(-24 \chi_\pi \chi_K L_4^r + 48 \chi_\pi \chi_K L_6^r - 12 \chi_\pi^2 L_4^r - 12 \chi_\pi^2 L_5^r + 24 \chi_\pi^2 L_6^r + 24 \chi_\pi^2 L_8^r\right) + A(\chi_\pi) \left(-8 \chi_\pi \chi_K L_4^r + 16 \chi_\pi \chi_K L_5^r - 16 \chi_\pi \chi_K L_6^r - 8 \chi_\pi \chi_K L_7^r + 32 \chi_\pi \chi_K L_8^r + 16 \chi_\pi \chi_K L_8^r\right) + A(\chi_\eta) \left(-8/9 \chi_\pi \chi_K L_4^r + 32/27 \chi_\pi \chi_K L_5^r + 16/9 \chi_\pi \chi_K L_6^r - 128/9 \chi_\pi \chi_K L_7^r - 64/9 \chi_\pi \chi_K L_8^r + 4/9 \chi_\pi^2 L_4^r - 4/27 \chi_\pi^2 L_5^r - 8/9 \chi_\pi^2 L_6^r + 64/9 \chi_\pi^2 L_7^r + 8/3 \chi_\pi^2 L_8^r - 32/9 \chi_\pi \chi_K L_4^r - 64/27 \chi_\pi \chi_K L_5^r + 64/9 \chi_\pi \chi_K L_6^r + 64/9 \chi_\pi \chi_K L_7^r + 64/9 \chi_\pi \chi_K L_8^r\right) + 192 \chi_\pi \chi_K C_{r1}^{\pi} + 8 \chi_\pi \chi_K C_{r4}^{\pi} + 48 \chi_\pi^2 C_{r9}^{\pi} + 80 \chi_\pi^2 C_{r19}^{\pi} + 48 \chi_\pi^2 C_{r20}^{\pi} + 48 \chi_\pi^2 C_{r21}^{\pi} - 4 \chi_\pi^2 C_{r4}^{\pi} + 64 \chi_\pi^2 C_{r19}^{\pi} + 192 \chi_\pi^2 C_{r21}^{\pi}.$
\[ v_6^s = + \overline{A}(\chi_\pi) \overline{B}(\chi_\eta) \left( \frac{1}{3} \chi_\pi \right) + \overline{A}(\chi_K) \overline{A}(\chi_0) \left( - \frac{2}{3} \right) + \overline{A}(\chi_K) \overline{B}(\chi_\eta) \left( - \frac{8}{9} \chi_K \right) \]
\[ + \overline{A}(\chi_\eta)^2 \left( \frac{2}{9} \right) + \overline{A}(\chi_\eta) \overline{B}(\chi_K) \left( - \frac{2}{3} \chi_K \right) + \overline{A}(\chi_\eta) \overline{B}(\chi_\eta) \left( - \frac{7}{27} \chi_\pi + \frac{16}{27} \chi_K \right) \]
\[ + \overline{A}(\chi_\eta) \left( - \frac{24}{9} \chi_\pi L_4^r + \frac{48}{9} \chi_\pi L_6^r \right) \]
\[ + \overline{A}(\chi_K) \left( -16 \chi_\pi L_4^r + 32 \chi_\pi L_6^r - 64 \chi_K L_4^r - 32 \chi_K L_6^r + 128 \chi_K L_6^r + 64 \chi_K L_8^r \right) \]
\[ + \overline{A}(\chi_\eta) \left( - \frac{8}{3} \chi_\pi L_4^r + 32/9 \chi_\pi L_5^r + 16/3 \chi_\pi L_6^r - 128/3 \chi_\pi L_5^r - 64/3 \chi_\pi L_8^r \right) \]
\[ - \frac{64}{3} \chi_K L_4^r - 128/9 \chi_K L_6^r + 128/3 \chi_K L_5^r + 128/3 \chi_K L_7^r + 128/3 \chi_K L_8^r \]
\[ + \overline{B}(\chi_K) \left( -16 \chi_\pi \chi_K L_4^r + 32 \chi_\pi \chi_K L_6^r - 32 \chi_R^2 L_4^r - 16 \chi_K^2 L_5^r + 64 \chi_K^2 L_6^r \right. \]
\[ \left. + 32 \chi_K^2 L_8^r \right) \]
\[ + \overline{B}(\chi_\eta) \left( - \frac{32}{9} \chi_\pi \chi_K L_4^r + 128/27 \chi_\pi \chi_K L_5^r + 64/9 \chi_\pi \chi_K L_6^r - 512/9 \chi_\pi \chi_K L_7^r \right. \]
\[ - 256/9 \chi_\pi \chi_K L_6^r + 16/9 \chi_\pi^2 L_4^r - 16/27 \chi_\pi^2 L_5^r - 32/9 \chi_\pi^2 L_6^r + 256/9 \chi_\pi^2 L_7^r \]
\[ + 32/3 \chi_\pi^2 L_8^r - 128/9 \chi_R^2 L_4^r - 256/27 \chi_R^2 L_5^r + 256/9 \chi_R^2 L_6^r + 256/9 \chi_R^2 L_7^r \]
\[ + 256/9 \chi_R^2 L_8^r \]
\[ - 192 \chi_\pi \chi_K C_{19}^r - 64 \chi_\pi \chi_K C_{20}^r + 192 \chi_\pi \chi_K C_{21}^r + 48 \chi_R^2 C_{19}^r + 16 \chi_R^2 C_{20}^r \]
\[ + 48 \chi_R^2 C_{21}^r + 4 \chi_R^2 C_{94}^r + 192 \chi_R^2 C_{19}^r + 192 \chi_R^2 C_{20}^r + 192 \chi_R^2 C_{21}^r. \]

These results agree analytically with those of Ref. [12]. In Ref. [12] numerical results were presented. Using the formulas above one obtains much smaller numerical corrections at NNLO then were obtained there. This effect is mainly due to the rewriting of the $1/F_0^2$ into $1/F_0^2$ and to a lesser extent of rewriting the masses in terms of the physical masses.

Numerical results are presented in terms of the ratio

\[ R_q = \frac{\langle \overline{q}q \rangle_0 - \langle \overline{q}q \rangle_\infty}{\langle \overline{q}q \rangle_\infty} \]  

(12)

where we calculate both numerator and denominator to NLO or NNLO in ChPT. As input parameters for $F_0$ and the $L_i^r$ we use the values obtained in fit 10 of Ref. [15]. In addition, we have set $H_2^r = 0$. The results for both $R_u$ and $R_s$ are shown in Fig. 3 for $\chi_K = (450 \text{ MeV})^2$ and three values of the lowest order pion mass $\chi_\pi = (100 \text{ MeV})^2$, $(250 \text{ MeV})^2$ and $(450 \text{ MeV})^2$. The finite volume corrections to the strange quark vacuum expectation value are always small. The light quark vacuum expectation value can have sizable finite volume corrections for the smaller pion mass.

### 4 Comparison and conclusions

The sigma terms in infinite volume are known to two-loop order in ChPT. Either directly [16] or via the derivative of the meson mass to the quark mass [12, 17]. The sigma terms are also known for two-flavour case [18, 19, 20].
Figure 3: (a) The ratio $R_u$ for three values of the input lowest order pion mass and $\chi_K = (450 \text{ MeV})^2$ (b) The same for $R_s$, with $\chi_K = (450 \text{ MeV})^2$ and $\chi_\pi = (100 \text{ MeV})^2$, and $(450 \text{ MeV})^2$.

A major motivation for this work was to test the accuracy of the Lüscher type of finite volume formulas. So, how well do the two approaches compare.

At one-loop the comparison is rather trivial since at most one-propagator can show up in the relevant one-loop diagrams. We can also apply the formula (5 ) for the different species of meson separately and thus construct the exact one loop ChPT formula. At one loop the only test possible is thus how fast the sum over $k$ converges and how quickly it converges to an exponential. This was already studied in [11] and the convergence to the leading exponential is rather slow while the sum over $k$ converges faster. We find the same conclusions. The extended Lüscher formula agrees analytically with the full one-loop expressions and is thus fully accurate.

At two loop level the two formulas have a different behaviour. There are diagrams now allowing two propagators simultaneously to feel the effect of the finite volume. A small complication that needs to be taken into account here is that the Lüscher formula is with the infinite volume mass at one loop. Thus when changing from the lowest order mass to the physical mass in the one loop formulas this needs to be done with the infinite volume expressions. But, even after doing this, the corrections are very small. The two-loop calculation as plotted in Fig. 3 is obviously very small. In addition, it is dependent on precisely how one defines the one-loop order. E.g., there are ambiguities in using the the physical pion decay constant or the lowest order one, and how much one uses the Gell-Mann-Okubo relation in the one-loop expression. With these changes the two-loop calculation can be changed significantly but remains mostly small. The Lüscher formula at this order has also small corrections since the sigma terms have small corrections at
one-loop. We have not plotted it since it will essentially be on top of the other curves in Fig. 3.

Both the extended Lüscher formula and the full two-loop calculations thus indicate small two-loop order corrections. The actual numerical results of the two-loop expressions depends strongly on the inherent ambiguities in defining it. So, both approaches give comparable results at this order, but we cannot draw conclusions on the accuracy of the extended Lüscher formula.

The extended Lüscher formula allows also to include effects from even higher orders by using the sigma terms at higher orders. This quantity is known to NNLO but its numerical value depends strongly on the input parameters chosen [16]. There might thus be sizable effects at higher orders but there need not be.

In conclusion, we have derived an extended Lüscher formula for the finite volume effects on the quark vacuum condensate. We have also calculated these effects to two-loop order in ChPT. At one-loop order the extended Lüscher formula is exactly equal to the full ChPT calculation. At two-loop order, the latter includes extra effects, but both approaches indicate very small corrections. The difference is within the inherent uncertainty of the full ChPT calculation at that order and we thus cannot conclude if the difference is due to the inherent uncertainty in the two-loop order result or to the effects not captured by the Lüscher formula.

Acknowledgements

We thank Gilberto Colangelo and Christoph Haefeli for discussions. This work is supported by the European Union TMR network, Contract No. HPRN-CT-2002-00311 (EURIDICE) and by the European Community-Research Infrastructure Activity Contract No. RII3-CT-2004-506078 (HadronPhysics). KG acknowledges a fellowship from the Iranian Ministry of Science.

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