A NEW UPPER BOUND ON THE MINIMUM DEGREE OF MINIMAL RAMSEY GRAPHS

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Abstract. We prove that \(s_r(K_{k+1}) = O(k^3 r^3 \ln^3 k)\), where \(s_r(K_k)\) is the Ramsey parameter introduced by Burr, Erdős and Lovász in 1976, which is defined as the smallest minimum degree of a graph \(G\) such that any \(r\)-colouring of the edges of \(G\) contains a monochromatic \(K_k\), whereas no proper subgraph of \(G\) has this property.

1. Introduction

A graph \(G\) is called \(r\)-Ramsey for another graph \(H\), denoted by \(G \rightarrow (H)_r\), if every \(r\)-colouring of the edges of \(G\) contains a monochromatic copy of \(H\). Observe that if \(G \rightarrow (H)_r\), then every graph containing \(G\) as a subgraph is also \(r\)-Ramsey for \(H\). Some very interesting questions arise when we study graphs \(G\) which are minimal with respect to \(G \rightarrow (H)_r\), that is, \(G \rightarrow (H)_r\) but there is no proper subgraph \(G'\) of \(G\) such that \(G' \rightarrow (H)_r\). We call such graphs \(r\)-Ramsey minimal for \(H\) and we denote the set of all \(r\)-Ramsey minimal graphs for \(H\) by \(\mathcal{M}_r(H)\). It follows from the classical result of Ramsey [12] that \(\mathcal{M}_r(H)\) is non-empty for any choices of graph \(H\) and positive integer \(r\).

Many questions on \(\mathcal{M}_r(H)\) have been explored; for example, the Ramsey number \(R_r(H)\) denotes the smallest number of vertices of any graph in \(\mathcal{M}_r(H)\) and the size Ramsey number \(\hat{R}_r(H)\) denotes the smallest number of edges. We refer the reader to \[2, 4, 10, 13\] for various results on Ramsey minimal problems. In this paper, we will be interested in the smallest minimum degree of an \(r\)-Ramsey minimal graph, defined by

\[ s_r(H) := \min_{G \in \mathcal{M}_r(H)} \delta(G), \]

for a finite graph \(H\) and positive integer \(r\), where \(\delta(G)\) denotes the minimum degree of \(G\). Trivially, we have \(s_r(H) \leq R_r(H) - 1\), since the complete graph on \(R_r(H)\) vertices is \(r\)-Ramsey for \(H\) and is \((R_r(H) - 1)\)-regular (taking minimal Ramsey subgraphs of this graph cannot increase the minimum degree). This parameter was introduced by Burr, Erdős and Lovász [3] in 1976. They were able to show the rather surprising exact

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result, $s_2(K_{k+1}) = k^2$, where $K_{k+1}$ is the complete graph on $k + 1$ vertices, which is far away from the trivial exponential bound of $s_2(K_{k+1}) \leq R_2(k + 1) - 1$.

While no precise values are known for $s_r(K_{k+1})$ for $r > 2$, Fox, Grinshpun, Liebenau, Person, and Szabó [6] showed that $s_r(K_{k+1})$ is quadratic in $r$, up to a polylogarithmic factor, when the size of the clique is fixed. Formally, they showed that for all $k \geq 2$ there exist constants $c_k, C_k > 0$ such that for all $r \geq 3$, we have

$$c_k r^2 \frac{\ln r}{\ln \ln r} \leq s_r(K_{k+1}) \leq C_k r^2 (\ln r)^{8k^2}. \quad (1.1)$$

When $k = 2$, Guo and Warnke [7] settled the exact polylogarithmic factor, following earlier work in [6]. In the other regime, when the number of colours is fixed, Hán, Rödl, and Szabó [8] showed that $s_r(K_{k+1})$ is quadratic in the clique size $k$, up to a polylogarithmic factor. They proved that, for every integer $r \geq 2$ there exists a constant $C'$ such that for every integer $k \geq 3$

$$s_r(K_{k+1}) \leq C'(k \ln k)^2. \quad (1.2)$$

The constant in the upper bound of (1.1) is rather large ($C_k \sim k^2 2^{8k^2}$), and in particular not polynomial in $k$. To remedy this, Fox, Grinshpun, Liebenau, Person, and Szabó [6] also proved an upper bound which is polynomial in both $k$ and $r$ and is applicable for small values of $r$ and $k$.

**Theorem 1.1** (Fox, Grinshpun, Liebenau, Person, Szabó). For all $k \geq 2$, $r \geq 3$, $s_r(K_{k+1}) \leq 8k^6 r^3$.

Hán, Rödl, and Szabó [8] further proved that the constant $C'$ of (1.2) is polynomial in $r$ when $r < k^2$ and $k$ is large enough. They showed that there exists a constant $k_0$ such that for every $k > k_0$ and $r < k^2$, we have $s_r(K_{k+1}) \leq 80^3 (r \ln r)^3 (k \ln k)^2$. Combining with (1.1), this result implies the existence of a large absolute constant $C$ and a polynomial upper bound for $s_r(K_{k+1})$.

**Theorem 1.2** (Hán, Rödl, Szabó). There exists an absolute constant $C$ such that for every $k \geq 2$ and $r < k^2$,

$$s_r(K_{k+1}) \leq C (r \ln r)^3 (k \ln k)^2.$$

Finally, using a group theoretic model of generalised quadrangles introduced by Kantor in 1980 [9], Bamberg and the authors [1] proved another polynomial bound, reducing the dependency in $r$, and improving on Theorem 1.1 for any $k, r$ and on Theorem 1.2 when $r > k^6$.

**Theorem 1.3** (Bamberg, Bishnoi, Lesgourgues). There exists an absolute constant $C$ such that for all $k \geq 2$, $r \geq 3$, $s_r(K_{k+1}) \leq C k^5 r^{5/2}$.

These theorems all use the equivalence between $s_r(K_k)$ and another extremal function, called the $r$-colour $k$-clique packing number [6]. Theorems 1.1 and 1.3 further use some ‘triangle-free’ point-line geometries, for which, under certain conditions on their parameters, any packing of these geometries implies an upper bound on the $r$-colour $k$-clique packing number. This argumentation, initially developed by Dudek and Rödl [5] and then by Fox et al. in [6], has been further optimized by Bamberg et al. in [1, Lemma 3.1]. Using this optimized argumentation from [1] and the finite geometric
construction of Fox et al. from [6], we show the following general upper bound that improves on the best known bounds for \( k \geq 8 \) and \( r \) in the range \( k^2 \leq r \leq O(k^4/\ln^6 k) \).

**Theorem 1.4.** For all \( k \geq 2, r \geq 3 \), \( s_r(K_{k+1}) \leq (8kr \ln k)^3 \).

In Section 3, we then proceed to show that this upper bound for \( s_r(K_{k+1}) \) is in some sense the ‘best possible’ bound one can obtain using triangle-free partial linear spaces and Lemma 2.1.

## 2. Packing partial linear spaces

A partial linear space is an incidence structure of points \( P \) and lines \( L \), with an incidence relation such that there is at most one line through every pair of distinct points. If every line is incident with exactly \( s + 1 \) points and every point is incident with exactly \( t + 1 \) lines, then the partial linear space has order \( (s, t) \). If there are no three distinct lines pairwise meeting each other in three distinct points, then the partial linear space is triangle-free. Generalised quadrangles are standard examples of triangle-free partial linear spaces, with the additional property that for every non-incident point-line pair \( x, \ell \) there exists a unique point \( x' \) incident to \( \ell \) such that \( x \) and \( x' \) are collinear (see the book by Payne and Thas [11] for a standard reference on finite generalised quadrangles).

The next lemma can be found in [1, Lemma 3.1]. Its proof follows a methodology initially developed by Dudek and Rödl [5], using the \( r \)-colour \( k \)-clique packing number developed in [6].

**Lemma 2.1** (Bamberg, Bishnoi, Lesgourgues). Let \( r, k, s, t \) be positive integers. Say there exists a family \( (I_i)_{i=1}^r \) of triangle-free partial linear spaces of order \( (s, t) \), on the same point set \( P \) and with pairwise disjoint line sets \( L_1, \ldots, L_r \), such that the point-line geometry \( (P, \bigcup_{i=1}^r L_i) \) is also a partial linear space. If \( s \geq 3rk\ln k \) and \( t \geq 3k(1+\ln r) \), then \( s_r(K_{k+1}) \leq |P| \).

The following lemma, that will imply Theorem 1.4, is a reformulation in the language of (triangle-free) partial linear space of a construction that can be found in [6, Proof of Lemma 4.4]. We include the proof for completeness.

**Lemma 2.2.** Let \( q \) be any prime power. There exists a family \( (I_i)_{i=1}^{q-1} \) of triangle-free partial linear spaces of order \( (q-1, q-2) \), on the same point set \( P \) and with pairwise disjoint line-sets \( L_1, \ldots, L_{q-1} \), such that the point-line geometry \( (P, \bigcup_{i=1}^{q-1} L_i) \) is also a partial linear space.

**Proof.** Let \( \mathbb{F}_q \) be the finite field of order \( q \) and for \( \lambda \in \mathbb{F}_q \setminus \{0\} \) let \( M_\lambda \) be the \( \lambda \)-moment curve,

\[
M_\lambda = \{(1, \lambda \alpha, \lambda \alpha^2) : \alpha \in \mathbb{F}_q \setminus \{0\}\}.
\]

Note that for non-zero \( \lambda_1 \neq \lambda_2 \) the two curves \( M_{\lambda_1} \) and \( M_{\lambda_2} \) do not intersect. A line in \( \mathbb{F}_q^3 \) is a set of the form \( \ell_{s,v} = \{\beta s + v : \beta \in \mathbb{F}_q\} \), where \( s \in \mathbb{F}_q^3 \setminus \{0\} \) is the slope of the curve. For \( \lambda \in \mathbb{F}_q \setminus \{0\} \), we define the incidence structure \( I_\lambda = (\mathbb{F}_q^3, L_\lambda) \) where \( L_\lambda \) is the set of lines with slope from the \( \lambda \)-moment curves, i.e.

\[
L_\lambda := \{\ell_{s,v} : s \in M_\lambda, v \in \mathbb{F}_q^3\}.
\]
Fox et al. [6] established the following properties about each structure \( I_\lambda, \lambda \in \mathbb{F}_q \setminus \{0\} \).

1. \( I_\lambda \) is a partial linear space (any two lines meet in at most one point).
2. Every line \( \ell \in L_\lambda \) contains \( q \) points and every point \( v \in \mathbb{F}_q^3 \) is contained in \( q - 1 \) lines.
3. \( I_\lambda \) is triangle-free. No three lines in \( L_\lambda \) intersect pairwise in three distinct points.

Further, they proved that for \( \lambda_1 \neq \lambda_2 \),

4. \( L_{\lambda_1} \cap L_{\lambda_2} = \emptyset \).

Given that any line in \( \bigcup_{i=1}^{q^2-1} L_i \) is a line of the affine space \( \mathbb{F}_q^3 \), Property (4) is sufficient to deduce that any two lines in \( \bigcup_{i=1}^{q^2-1} L_i \) meet in at most one point. Therefore the point-line geometries \( I_\lambda \) are such that \( (P, \bigcup_{i=1}^{q^2-1} L_i) \) is also a partial linear space. \( \square \)

Theorem 1.4 is a direct consequence of Lemma 2.2.

**Proof of Theorem 1.4.** Let \( k \geq 2, r \geq 3 \), and let \( q \) be the smallest prime such that \( q \geq 4kr \ln k \). By Bertrand’s postulate, \( q \leq 8kr \ln k \). By Lemma 2.2, there exists a family of \( r < q \) triangle-free partial linear spaces of order \( (q-1, q-2) \), on the same point set \( P \) and pairwise disjoint line-sets \( L_1, \ldots, L_r \), such that the point-line geometry \( (P, \bigcup_{i=1}^{q^2-1} L_i) \) is also a partial linear space. Note that with \( k \geq 2 \) and \( r \geq 3 \), we have \( q - 1 \geq 3rk \ln k \) and \( q - 2 \geq 3k(1 + \ln r) \). By Lemma 2.1, \( s_r(K_{k+1}) \leq |P| \), and then \( |P| = q^3 \) yields the desired bound. \( \square \)

**Remark 2.3.** Each point-line geometry \( (P, L_\lambda) \) in the construction above is a subgeometry of a \( T_2(O) \) generalized quadrangle (see Section 3.1.2 in [11]).

A careful review of the arguments in [1, Lemma 3.1 and 5.2] would allow a small optimisation on the multiplicative constant of this corollary. However in light of the conjectured quadratic upper bound [1, Conjecture 5.2], we did not push this further.

### 3. Best possible total degree

We now prove that the upper bound from Theorem 1.4 is the best possible general polynomial bounds using triangle-free partial linear spaces and Lemma 2.1.

**Theorem 3.1.** For some \( \alpha \geq 1 \), assume that for any positive integers \( k, r \) there exist a family \( (P, L_i) \) of triangle-free partial linear spaces of order \( (q, q^\alpha) \), satisfying the assumptions of Lemma 2.1. Then \( |P| = \Omega(k^{2+\alpha}r^{2+\alpha}) \). Similarly if the partial linear spaces are of order \( (q^\alpha, q) \) then \( |P| = \Omega(k^{2\alpha+1}r^{2+1/\alpha}) \).

Note that Theorem 1.4 corresponds to the case \( \alpha = 1 \). It follows from this proposition that, ignoring polylogarithmic factors, any polynomial upper bound for \( s_r(K_k) \) achieved through Lemma 2.1 must have total degree at least 6, with \( k^3r^3 \) being the unique polynomial of degree 6 possible (up to polylogarithmic factors).

The proposition is a direct consequence of the following lemma, that yields a lower bound on the number of points in any triangle-free partial linear space.

**Lemma 3.2.** Let \( (P, L) \) be a triangle-free partial linear space of order \( (s, t) \). Then \( |P| \geq (st + 1)(s + 1) \).
Proof. For any point \(x \in \mathcal{P}\), let \(N(x)\) be the *neighbourhood* of \(x\), the set of points \(v\) such that \(x\) and \(v\) are collinear. Any point \(x \in \mathcal{P}\) is incident with \(t+1\) lines, each of which contains \(s+1\) points including \(x\), therefore \(|N(x)| = s(t+1)|.

Let \(\ell \in \mathcal{L}\). Given that the partial linear space is triangle-free, the set of points not incident to \(\ell\) contains the disjoint union of \(N(x) \setminus \ell\), for \(x \in \ell\), so,

\[
|\mathcal{P}| \geq (s+1) + \sum_{x \in \ell}|N(x) \setminus \ell| = (s+1) + (s+1)(s(t+1) - s) = (s+1)(st+1).
\]

Equality is attained if and only if any point not incident to \(\ell\) is in \(N(x)\) for some \(x \in \ell\), meaning that the partial linear space is a generalized quadrangle. \(\square\)

**Proof of 3.1.** Assume first that for any positive integers \(k, r\), the partial linear spaces have order \((q, q^\alpha)\) for \(q = q(k, r)\). Then given the condition of Lemma 2.1, \(q(k, r) \geq 3rk \ln k\) and \(q(k, r)^\alpha \geq 3k(1 + \ln r)\). As these inequalities have to be verified for any \(k, r\), then \(q(k, r)\) has to be at least linear in \(r\) and linear in \(k\), i.e. \(q(k, r) = \Omega(rk)\). Then Lemma 3.2 yields \(|\mathcal{P}| > s^2t = q^{2+\alpha} = \Omega(k^{2+\alpha}r^{2+\alpha}).\)

Similarly, if the partial linear spaces have order \((q^\alpha, q)\) for some \(\alpha \geq 1\), the condition of Lemma 2.1 implies \(q^\alpha \geq 3rk \ln k\) and \(q \geq 3k(1 + \ln r)\) and then \(q = \Omega(kr^{1/\alpha})\). Lemma 3.2 yields \(|\mathcal{P}| > s^2t = q^{2+1} = \Omega(k^{2+1}r^{2+1/\alpha}).\) \(\square\)

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