PHASE DEPENDENT CURRENT STATISTICS IN SHORT-ARM ANDREEV INTERFEROMETER

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Abstract

We calculate analytically the current statistics for a short diffusive wire between the normal reservoir and a short superconductor-normal metal-superconductor (SNS) junction, at arbitrary applied voltages and temperatures. The cumulant-generating function oscillates with the phase difference $\phi$ across the junction, approaching the normal-state value at $\phi = \pi$. At $T = 0$ and at the applied voltage much smaller than the proximity gap $\Delta_0$, the current noise $P_I$ doubles and the third current cumulant $C_3$ is 4 times larger with respect to their values in the normal state; at $eV \gg \Delta_0$ they acquire large excess components. At the gap edge, $eV = \Delta_0$, the effective transferred charge defined through $dP_I/dI$ and $dP_I/dV$ approaches $0e$ and $3e$, respectively, which makes doubtful the interpretation of these quantities as physical elementary transferred charge. At $T \neq 0$, $C_3$ shows a non-monotonous voltage dependence with a dip near $eV = \Delta_0$.

During last years, statistics of quantum and thermal fluctuations of the electric current in mesoscopic systems attracted rapidly growing attention. It was established both experimentally and theoretically that the fluctuation properties of mesoscopic conductors provide important information about correlations and statistics of charge carriers which are not accessible through conductance measurements. A powerful theoretical approach to the fluctuation problem has been developed by using the concept of full counting statistics (FCS), i.e., statis-
tics of number of particles transferred through the conductor. The concept of FCS, which first appeared in quantum optics, was extended to normal electron systems in Ref. [1], and then applied to superconducting structures in Ref. [2].

The basic problem of the FCS is to calculate a probability \( P_{t_0}(N) \) for \( N \) particles to pass a system during an observation time \( t_0 \). Equivalently, one can find a cumulant generating function (CGF) \( S(\chi) \),

\[
\exp[-S(\chi)] = \sum_N P_{t_0}(N) \exp(iN\chi),
\]

which determines the irreducible current correlation functions (cumulants) \( C_n \),

\[
C_n = - \left( \frac{\partial}{\partial \chi} \right)^n S(\chi) \bigg|_{\chi=0}.
\]

The first two cumulants, \( C_1 = \langle N \rangle \equiv \sum_N NP_{t_0}(N) \) and \( C_2 = \langle (N - \langle N \rangle)^2 \rangle \), correspond to the average current \( I = (e/t_0)C_1 \) and noise power \( P_I = (2e^2/t_0)C_2 \). Intensive studies of the current noise have led to a number of interesting results concerning statistical correlations in the current transport (for a review, see Ref. [3]), and an effective charge \( q_{\text{eff}} \) transferred during an elementary transport event. The third cumulant \( C_3 = \langle (N - \langle N \rangle)^3 \rangle \) has recently attracted a special interest as the lowest-order correlation function which is not masked by equilibrium fluctuations [4]. First measurements of \( C_3(V) \) in the tunnel junction [5] have revealed its high sensitivity to an electromagnetic environment [6].

In normal metal/superconducting (NS) hybrid structures, the basic mechanism of charge transport at subgap energies, \( E < \Delta \), is due to Andreev reflection of quasiparticles at the NS boundary [7], i.e., conversion of electrons incident from the normal metal to retroreflected holes, accompanied by escape of Cooper pairs into the superconductor. During an elementary Andreev reflection event, the effective charge transferred through the NS interface is twice the electron charge, \( q_{\text{eff}} = 2e \). This charge doubling strongly affects the current statistics in NS junctions; in particular, it is known as the reason for the doubling, compared to normal junctions, of a zero-bias shot noise [2, 8]. At finite bias, the effective charge becomes dependent on applied voltage [9, 10], due to variations of the size of the proximity region near the NS boundary, where the quantum coherence holds between the electrons and retroreflected holes.

In the Andreev interferometers (see Fig. 1), the phase relations between the electron and hole wavefunctions in the normal wire can be controlled by magnetic flux enclosed by a superconducting loop, which results in a periodical dependence of the transport characteristics of the interferometer on the superconducting phase difference \( \phi \) across the SNS junction. First, the oscillations of the conductance have been investigated experimentally (see a review in Ref. [11]) and theoretically [12], and, recently, the oscillations of the current noise were reported [10].

Motivated by growing interest in higher correlation functions, we discuss in this paper the full statistics of charge transport in Andreev interferometers.
We will adopt several simplifying assumptions, which enables us to present an analytical solution for the CGF and thus to clearly demonstrate essential features of coherent effects in the current statistics in NS structures. Our approach is based on the extended Keldysh-Green technique [13] (see also [14]) in which the CGF is determined by the equation

\[
(-ie/t_0) \frac{\partial S}{\partial \chi} = I(\chi), \quad I(\chi) = \frac{1}{8e} \int dE \text{Tr} \tau K \tilde{I}, \quad \tilde{I} = \sigma \tau.
\]  

(3)

The Pauli matrices \( \sigma (\tau) \) operate in the Nambu (Keldysh) space. The counting current \( I(\chi) \) is to be found from the quantum kinetic equations [15] for the \( 4 \times 4 \) matrix Keldysh-Green function \( \tilde{G} \) in the mesoscopic normal region of the interferometer confined between the reservoirs,

\[
\sigma_N [\sigma; E, \tilde{G}] = i\hbar \mathcal{D} \partial \tilde{I}, \quad \tilde{I} = \sigma_N \tilde{G} \partial \tilde{G}, \quad \tilde{G}^2 = 1.
\]  

(4)

In this equation, \( \mathcal{D} \) is the diffusion coefficient, \( \partial \) denotes spatial derivative, and \( \sigma_N \) is the normal conductivity per unit length. The counting field \( \chi \) is introduced via a modified boundary condition involving the gauge transformation of the local-equilibrium function \( \tilde{G}_R \), e.g., in the right \( (R) \) normal reservoir,

\[
\tilde{G}_R(\chi) = \exp(i\chi \tilde{\tau}_K/2) \tilde{G}_R \exp(-i\chi \tilde{\tau}_K/2).
\]  

(5)

A brief overview of this technique in the particular case of normal structures can be found in the Appendix.

For a multi-terminal structure in Fig. 1, the solution of Eq. (4) has to be found separately in each arm of the interferometer, with the matching condition following from the Kirchhoff’s rule for partial counting currents at the node [16]. The problem simplifies when the junction length \( d \) is much smaller than the length \( L \) of the interferometer wire (or, more precisely, the wire resistance dominates the net interferometer resistance). In this case, the interferometer wire weakly affects the spectrum of the junction [17], which thus can be considered as an effective left \( (L) \) reservoir. Correspondingly, the function \( \tilde{G}_L \) which imposes the boundary condition to Eq. (4) at the junction node, is to be constructed from the Green’s and distribution functions at the middle of a closed
equilibrium SNS junction. Furthermore, if $d$ is much smaller than the coherence length $\xi_0 = \sqrt{\hbar D/\Delta}$, these Green’s functions take the BCS form, with the phase-dependent proximity gap $\Delta_\phi = \Delta |\cos(\phi/2)|$ [18]. Within such model, the problem of current statistics in the Andreev interferometer reduces to the calculation of the CGF for an NS junction with the effective order parameter $\Delta_\phi$ in the superconducting reservoir.

Proceeding to this calculation, we encounter a common difficulty, violation of the standard triangle form of $\tilde{G}$ in the Keldysh space resulting from the gauge transformation in Eq. (5). In such situation, Eq. (4) cannot be decomposed into the Usadel equation for the Green’s functions and the kinetic equation for the distribution functions, which makes well developed methods for solving Keldysh-Green’s equations quite unusable. For this reason, the FCS problem in NS structures generally requires a numerical analysis of the whole $4 \times 4$ matrix boundary problem which has been so far performed only in the limit of small characteristic energies $\{eV, T\} \ll \Delta$ [9, 10].

In some particular cases, an analytical solution of this problem can be obtained by means of the generalized circuit theory [19, 20]. Within this approach, the CGF for a mesoscopic connector between two reservoirs is expressed in terms of the distribution $\rho(T)$ of the transparencies of the conduction channels,

$$S(\chi) = \frac{g t_0}{4e^2} \int dE \int_0^1 dT \text{Tr} \ln \tilde{W}(E, T, \chi),$$  \hfill (6)

$$\tilde{W} = 1 + (T/4)(\{\tilde{G}_L, \tilde{G}_R(\chi)\} - 2),$$  \hfill (7)

where $g$ is the connector conductivity. Equation (6) generally applies to the normally conducting structures with arbitrary $\rho(T)$. It was also applied to the superconducting tunnel junctions [21] and point contacts [20, 22] with a singular transparency distribution localized at the junction transparency. In general NS structures, due to dephasing between the electron and hole wavefunctions described by the commutator term in Eq. (4), statistics of conducting modes, in contrast to normal structures [23, 24], does not reduce to statistics of transparencies but concerns full scattering matrices. However, if the characteristic energies are much smaller than the Thouless energy, $\{eV, T\} \ll E_{Th} = \hbar D/L^2$, the dephasing term in Eq. (4) can be neglected, which makes it possible to apply the transparency statistics for a normal wire [23] to the superconducting structure. In long diffusive junctions, $L \gg \xi_0$, where the Thouless energy is small, $E_{Th} \ll \Delta$, the quasiparticle spectrum is structureless at small energies, $E \ll E_{Th}$, which results in linear voltage dependence of the CGF and, correspondingly, of all cumulants at $eV \ll E_{Th}$ [14]. In the opposite limit, $eV \gg E_{Th}$, the CGF for a long junction can be found within so-called “incoherent” approximation [25], by neglecting the contribution of the coherent proximity region. The calculations in [14, 25] lead to the conclusion that the FCS exhibits the reentrance effect: In both limits, $eV \ll E_{Th}$ and $eV \gg E_{Th}$, it is described by the same ex-
pression for \( S(\chi) \). An interesting situation occurs in NS junctions with opaque interfaces dominating the net resistance of the junction \[26\]. In this case, the crossover between the coherent and incoherent transport regimes occurs at very small voltage of the order of the inverse dwell time of quasiparticles confined between the interface barriers.

In this paper, we focus on the case of short NS junctions with the length smaller than \( \xi_0 \) and, correspondingly, with large Thouless energy, \( E_{Th} \gg \Delta_0 \).

In such situation, the energy region of negligibly small dephasing, \( E_0 \ll E_{Th} \), overlaps with the region \( E_0 \gg \Delta_0 \), in which the NS junction behaves as the normal system. This enables us to apply Eq. (6) and the transparency statistics for diffusive normal conductor at arbitrary voltages and temperatures, and present analytical solution of the problem.

The calculation of the integrand in Eq. (6) is briefly discussed below. The Keldysh-Green’s function \( G_R(\chi) \) in the normal reservoir is traceless in the Keldysh space and therefore it can be expanded over the Pauli matrices \( \tau \) as

\[
\tilde{G}_R(\chi) = \tilde{\tau} (\tilde{g}_1 + \sigma_z \tilde{g}_z), \quad \tilde{g}_1 \tilde{g}_z = 0, \quad \tilde{g}_1^2 + \tilde{g}_z^2 = 1, \quad \tilde{\tau} = (\tau_x, \tau_y, \tau_z),
\]

where the vectors \( \tilde{g}_1(\chi) \) are expressed through local-equilibrium distribution functions in the voltage biased electrode. In the subgap energy region, \( E_0 < \Delta_0 \), the function \( \tilde{G}_L \) at the junction node is the unity matrix in the Keldysh space proportional to the Nambu matrix Green’s function \( \hat{g} \) in the superconductor,

\[
\tilde{G}_L = \tilde{g} = \sigma_z \exp(\sigma_x \theta_0), \quad \tilde{g}_1^2 = 1, \quad \theta_0 = \arctanh (E/\Delta_0).
\]

Then the calculation of the trace in the Nambu space in Eq. (7) is reduced to the summation over the eigenvalues \( \sigma = \pm 1 \) of the matrix \( \tilde{g} \),

\[
\text{Tr} \ln \tilde{W} = \sum_\sigma \text{Tr} \ln \tilde{W}_\sigma, \quad \tilde{W}_\sigma = a + \tilde{\tau} \tilde{b},
\]

\[
a = 1 - T/2, \quad \tilde{b} = (T/2)(\sigma_1/2 - i \sigma_z \sinh \theta_0).
\]

Noticing that any \( 2 \times 2 \) matrix can be presented in exponential form as

\[
\tilde{W}_\sigma = \exp(\ln w + \varphi \tilde{p}),
\]

\[
w^2 = a^2 - \tilde{b}^2, \quad \cosh \varphi = a/w, \quad \tilde{p} = \varphi \tilde{b}/w \sinh \varphi, \quad \text{Tr} \tilde{p} = 0,
\]

where \( w \) is independent of \( \sigma \) due to orthogonality of the vectors \( \tilde{g}_1 \) and \( \tilde{g}_z \), one easily obtains \( \text{Tr} \ln \tilde{W}_\sigma = \ln w^2 \) and \( \text{Tr} \ln \tilde{W} = 2 \ln w^2 \).

At \( E > \Delta_0 \), the function \( G_L \) is traceless in the Keldysh space,

\[
\tilde{G}_L = \tilde{g}(\tilde{\tau} \tilde{g}_L), \quad \tilde{g} = \sigma_z \exp(\sigma_x \theta_0), \quad \theta_0 = \arctanh (\Delta_0/E),
\]

where the vector \( \tilde{g}_L \) is constructed from the equilibrium distribution function at zero potential. In this case, the \( 4 \times 4 \) matrix \( \tilde{W} \) has the form \( \tilde{W} = a + \sigma \tilde{b} \), where
arrive at the final expression for the CGF, obtain Tr ln ˇW and (13), with the traceless matrix ˇW therefore it can also be transformed to the matrix exponent similar to Eqs. (12) and (13), with the traceless matrix ˇW = ⃗σb/w sinh φ. Following this line, we obtain Tr ln ˇW = 2 ln w², and then, performing integration over T in Eq. (6), we arrive at the final expression for the CGF,

\[
S(\chi) = \frac{g} {4e^2} \int_0^\infty dE S(E, \chi), \quad S(E, \chi) = \begin{cases} 2\theta_0^2, & E < \Delta_0, \\ \theta_0^2 + \theta_0^- E, & E > \Delta_0, \end{cases}
\]

(17)

where the quantities θ and θ± are given by explicit relations,

\[
Z(0) \cosh^2 \theta = Z(2\chi) \cosh^2 \theta_0,
\]

(18)

\[
Z(0) \cosh \theta_\pm = |Z(\chi) + \cos \chi - 1| \cosh \theta_0 \pm \tanh \frac{\epsilon}{2} \left[ \sinh \theta \pm \frac{\cosh \theta + 1}{\cosh \theta - 1} \sinh^2 \theta_0 \right],
\]

(19)

\[
\theta_0 = \text{arctanh} \left[ \left( \frac{\Delta_0}{E} \right)^{\text{sgn}(E-\Delta_0)} \right],
\]

(20)

\[
Z(\chi) = \cosh(\epsilon) + \cosh(p-i\chi), \quad \epsilon = E/T, \quad p = eV/T.
\]

(21)

By using Eqs. (2) and (17)-(21), one can obtain analytical expressions for all cumulants. At zero temperature, the calculation essentially simplifies. Indeed, at T → 0 and E > eV, the dominating terms in Eqs. (18)-(21) are proportional to \exp(\epsilon), and therefore θ and θ± are equal to θφ. This implies that the CGF is independent of the counting field at these energies, and all cumulants turn to zero. At E < eV, the terms with \exp(p - i\chi) dominate, and we obtain

\[
\cosh \theta = e^{-i\chi} \cosh \theta_0, \quad \cosh \theta_\pm = e^{-i\chi} \cosh \theta_0 \pm (e^{-i\chi} - 1).
\]

(22)

At subgap voltage, eV < \Delta_0, when the charge transport at T = 0 is only due to the Andreev reflection, the current I, the shot noise power P_t, and the third cumulant C_3 read

\[
I = I_\Delta q(z), \quad q(z) = \int_0^z \frac{dx}{x} \text{arctanh} x, \quad P_t = 2e [I - I_\Delta f(z^{-1})],
\]

(23)

\[
C_3 = \mathcal{N} - \frac{N_\Lambda}{2z^2} \left[ (5z^2 - 3) f(z^{-1}) + z \right], \quad I_\Delta = \frac{g\Delta_0}{e}, \quad \mathcal{N}_\Lambda = \frac{I_\Delta \theta_0}{e},
\]

(24)

\[
f(z) = (1/2) (z - (z^2 - 1) \text{arctanh} z^{-1}), \quad z = eV/\Delta_0.
\]

(25)
Phase dependent current statistics in short-arm Andreev interferometer

Figure 2. Shot noise power and third cumulant vs superconducting phase (a,b) at different voltages and $T = 0$, and vs voltage at different temperatures (c,d). Dashed lines denote voltage dependencies in the normal state at $T = 0$. In the panel (d), zero-bias slopes of the normalized $C_3(V)$ are indicated.

At small voltage, $eV \ll \Delta_0$, the shot noise doubles, $P = (4/3)eI$, and $C_3 = 4N/15$ is four times larger compared to the normal case [8, 2, 27–1]. When the voltage increases and exceeds the gap edge, $eV > \Delta_0$, the normal electron processes at the energies $E > \Delta_0$ begin to contribute to the charge transport, providing the normal-state voltage dependencies of the cumulants at $eV \gg \Delta_0$.

At large voltage, the Andreev reflected particles produce voltage-independent excess components of the cumulants (in ballistic NS junctions, existence of the excess noise was first predicted in Ref. [28]),

$$I = I_N - I_{\Delta}f(z) + I^{ex}, \quad P_I = 2eI_{\Delta}(z^2 - 1)f(z) + P_I^{ex}, \quad I_N = gV,$$

$$C_3 = \frac{N_{\Delta}}{2}(z + 1)\{(z - 1)[8z/3 - (8z^2 - 3)f(z)] - 1/3\} + C_3^{ex},$$

$$I^{ex} = \frac{I_{\Delta}}{2}\left(\frac{\pi^2}{4} - 1\right), \quad P_I^{ex} = 2eI^{ex}, \quad C_3^{ex} = \frac{N_{\Delta}}{2}\left(\frac{\pi^2}{4} - \frac{4}{3}\right).$$

At nonzero temperatures, $T \neq 0$, we calculate the cumulant spectral densities $I(E), P(E)$ and $C(E)$ defined as

$$I = I_N\int_0^\infty dE I(E), \quad P_I = 2eI_{\Delta}\int_0^\infty dE P(E), \quad C_3 = N_{\Delta}\int_0^\infty dE C(E).$$
Here \( I(E) = f_1 \sinh p / Z(0) \), and the functions \( P(E) \) and \( C(E) \) at \( E < \Delta_\phi \) read

\[
P(E) = \frac{2}{Z^2(0)} \left[ 2Q f_1 + (1 - f_2) \sinh^2 p \right], \quad Q = 1 + \cosh \varepsilon \cosh p, \quad (30)
\]

\[
C(E) = \frac{\sinh p}{Z^3(0)} \left[ 4 f_1 \sinh^2 \varepsilon + (2 f_2 + 3 f_3) \sinh^2 p + 2Q \left( 3(1 - f_2) - 2 f_1 \right) \right], \quad (31)
\]

whereas at \( E > \Delta_\phi \) they are given by equations,

\[
P(E) = \frac{2}{Z^2(0)} \left[ Q \left( 1 + 2 f_1 - 2 f_2 \frac{\cosh p - 1}{\cosh \varepsilon + 1} \right) + \sinh^2 p - Z(0) \right], \quad (32)
\]

\[
C(E) = \frac{\sinh p}{Z^3(0)(1 + \cosh \varepsilon)} \left\{ 4 f_1 (1 + \cosh \varepsilon)(Q + \sinh^2 \varepsilon) + 3 \left[ Z(0)(1 - 2 f_3) + Q \left( 4(1 - f_2 + f_3 \cosh \varepsilon) + 3 \cosh \varepsilon - 2 f_3 \right) \right.ight.
\]

\[
\left. \left. \quad \sinh^2 \varepsilon \left( 2 f_3 - \cosh \varepsilon + (3 - 5 \cosh \varepsilon) f_2 \right) \right] + f_2(5 \cosh \varepsilon - 1) \sinh^2 p \right\}, \quad (33)
\]

\[
f_1 = \theta_\phi \coth \theta_\phi, \quad f_2 = (f_1 - 1) / \sinh^2 \theta_\phi, \quad f_3 = (f_2 - 1/3) / \sinh^2 \theta_\phi. \quad (34)
\]

In Eqs. (30)-(34), the functions \( f_i \) describe energy variation of quasiparticle spectrum which is most essential in the vicinity of the gap edge \( \Delta_\phi \).

As shown in Fig. 2.(a,b), the cumulants oscillate with the phase and exhibit deep minima at \( \phi \mod 2\pi = \pi \), when the gap closes and the cumulants approach their normal values. When the proximity gap \( \Delta_\phi \) approaches \( eV \), \( P_1(\phi) \) exhibits a peak, while \( C_3(\phi) \) shows a step-like structure. The voltage dependence of the cumulants for different temperatures is plotted in Fig. 2.(c,d) in specifically normalized variables, which provides universality of the curves for any \( \phi \). As the temperature increases, the current noise approaches finite value at \( eV = 0 \) due to thermal fluctuations, and exhibits quadratic dependence on the applied voltage at \( eV \ll T \). Within the intermediate voltage region, \( T < eV < \Delta_\phi \), \( P_2(V) \) becomes linear with doubled slope produced by the Andreev reflected particles, and at \( eV > \Delta_\phi \), the slope turns to its normal-metal value. A considerable excess noise at large voltages is contributed by both the thermal fluctuations and Andreev reflection. A more interesting behavior is discovered for the third cumulant. As the temperature departs from zero, the zero-bias slope of the normalized \( C_3(V) \) increases by factor of 5 compared to the zero temperature (which is similar to the normal structure [4]), and approaches the value \( 4/3 \).

Then, at \( T < eV < \Delta_\phi \), the slope of the normalized \( C_3(V) \) returns to the subgap value \( 4/15 \) for \( T = 0 \). At \( eV \sim \Delta_\phi \), the curve \( C_3(V) \) shows \( \mathcal{N} \)-like feature, and finally, at \( eV > \Delta_\phi \), it approaches a straight line with the (normal-state) slope \( 1/15 \). This implies that \( C_3 \) acquires anomalously large thermal component at voltage \( eV \sim \Delta_\phi \), which, however, rapidly decreases at \( eV > \Delta_\phi \) and/or \( T > \Delta_\phi \) towards the normal metal level.
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The voltage dependence of the effective transferred charge defined as $q_{\text{eff}} = (3/2)dP_I/dI$ [10] and the normalized differential noise $(3R/2) dP_I/dV$ is shown in Fig. 3 for $T = 0$. At zero voltage, both quantities approach $2e$, whereas at $eV \gg \Delta$ they turn to $e$. However, the behavior of these quantities becomes very different when the voltage approaches the proximity gap edge $\Delta$: the normalized differential noise increases to $3e$, whereas the effective charge $q_{\text{eff}} \sim (dP_I/dV)(dV/dI)$ rapidly turns to zero, along with the differential resistance $dV/dI$ (see inset in Fig. 3). We note that the effect of zero differential resistance giving $q_{\text{eff}} = 0$ results from the resonant transparency enhancement at the proximity gap edge, due to the singularity in the density of states. Such an “exotic” behavior of these quantities at the gap edge leads us to the conclusion that none of them can be unambiguously associated with the physical elementary transferred charge, in contrast to what is commonly suggested. Similar effects have been predicted in Ref. [26] for an NS structure with opaque interfaces; however, in such case, a considerable enhancement of $dP_I/dV$ and suppression of $dP_I/dI$ occur at small applied voltage of the order of the inverse dwell time of quasiparticles.

It is instructive to compare our analytical results for short-arm interferometers with that obtained numerically for long NS junctions with a small minigap $E_g \sim E_{Th} \ll \Delta$. The results are qualitatively similar: in long junctions, $q_{\text{eff}}$ is equal to $2e$ at $eV \ll E_g$ and has a minimum at $eV \approx E_g(\phi)$, which moves towards small voltage when $\phi \to \pi$ [10]; the differential noise is also non-monotonous and approaches maximum at $eV \approx 5E_{Th}$ [9]. After this comparison we see that the proximity gap $\Delta$ in short junctions plays the role of the minigap $E_g$ in long junctions and determines the feature in the effective charge, though this feature at $eV \sim E_g$ in long junctions is much less pronounced. However, as noted above, a qualitative difference of long junctions is the existence of an intermediate incoherent voltage region $E_g \ll eV \ll \Delta$, where both the effective charge and the normalized differential noise have the value $2e$, and their crossover to $e$ occurs only at $eV \geq \Delta$ [29].

![Figure 3](image-url)
Appendix

For reference purposes, in this Appendix we outline the procedure and summarize the results of calculation of the CGF for a diffusive connector between normal reservoirs, by using the extended Keldysh-Green’s technique. For generality, we consider a diffusive wire interrupted by tunnel barriers, which enables us to present several original results and to examine various limiting situations.

In normal systems, the matrices $\mathbf{G}$ and $\mathbf{I}$ are traceless in the Keldysh space and therefore they can be expressed through 3-vectors with the components diagonal in the Nambu space, $\mathbf{G} = \bar{g} \mathbf{G}$, $\mathbf{I} = \bar{p} \mathbf{I}$, where $\bar{r}$ is the vector of the matrices $\tau$, and $\bar{g}^2 = 1$. Since the left-hand side of Eq. (4) turns to zero in normal systems, the formal solution of Eq. (4) for the matrix current density $\mathbf{I}_N$ in each segment of the wire can be easily obtained,

$$\mathbf{I}_N = g_N \ln \mathbf{G}_1 \mathbf{G}_2 = g_N \ln [\bar{g}_1 \bar{g}_2 + \bar{r}(\bar{g}_1 \times \bar{g}_2)] = \bar{u}_N, \quad \mathbf{I}_N = i g_N \mathbf{p} \phi_N,$$  \hspace{1cm} (A.1)

where $g_N$ is the conductivity of the wire segment, $\mathbf{G}_1, \mathbf{G}_2$ are the Green’s functions at the left and right segment edges, respectively, $\phi_N = \arccos \bar{g}_1 \bar{g}_2$ is the angle between the (complex) unit vectors $\bar{g}_1$ and $\bar{g}_2$, and $\bar{r} = (\bar{g}_L \times \bar{g}_R)/\sin\phi_N$ is the unit vector perpendicular to $\bar{g}_1$ and $\bar{g}_2$.

The matrix current $\mathbf{I}_B$ through the tunnel barrier can be expressed through the Green’s functions $\mathbf{G}_-$ and $\mathbf{G}_+$ at the left and right sides of the barrier by using the boundary condition [30],

$$\mathbf{I}_B = \mathbf{I}_B = \frac{g_B}{2} [\mathbf{G}_- \mathbf{G}_+] = \bar{u}_B, \quad \mathbf{I}_B = i g_B \bar{p} \sin\phi_B, \quad \phi_B = \arccos \bar{g}_- \bar{g}_+,$$  \hspace{1cm} (A.2)

where $\bar{p} = (\bar{g}_- \times \bar{g}_+)/\sin\phi_B$ and $g_B$ is the barrier conductivity.

The conservation of the matrix current along the connector, $\bar{I} = \text{const}$, following from Eq. (4) and the boundary condition in Eq. (A.2), results in conservation of the vector current, $\bar{I} = \bar{I}_N = \bar{I}_B = \text{const}$. This implies that for all elements of the connector, the unit vectors $\bar{p}$ coincide, therefore the Green’s vectors $\bar{g}$ lie within the same plane and the vector $\bar{p}$ can be constructed from known Green’s vectors $\bar{g}_L$, and $\bar{g}_R$ in the reservoirs, $\bar{p} = (\bar{g}_L \times \bar{g}_R)/\sin\phi$, where $\phi$ is the angle between $\bar{g}_L$ and $\bar{g}_R$. From the current conservation, we also conclude that all barriers and wires are characterized by a single variable $\eta$,

$$g_B \sin\phi_B = g_N \phi_N = g\eta = \text{const},$$  \hspace{1cm} (A.3)

where the normalization constant $g$ is chosen to be equal to the conductance of the whole connector. Thus, the vector current is given by equation,

$$\bar{I} = \frac{i g\eta}{\sin\phi} (\bar{g}_L \times \bar{g}_R).$$  \hspace{1cm} (A.4)

The planar rotation of the Green’s vector along the connector results in the additivity of the angles between all consecutive vectors $\bar{g}$, therefore the sum of these angles is equal to $\phi$,

$$\sum_{\text{wires}} \phi_N + \sum_{\text{barriers}} \phi_B = \phi = \arccos (\bar{g}_L \bar{g}_R),$$  \hspace{1cm} (A.5)

$$\bar{g}_L \bar{g}_R = 1 + P_+ (e^{\gamma} - 1) + P_- (e^{-\gamma} - 1),$$  \hspace{1cm} (A.6)

$$P_{\alpha\beta} = n_{\alpha}(1 - n_{\beta}), \quad n_- = n_F(E), \quad n_+ = n_F(E + eV),$$  \hspace{1cm} (A.7)

which leads to the following equation for the parameter $\eta(\phi)$,

$$\gamma_N \eta + \sum_k \arcsin(\gamma_k \eta) = \phi, \quad \gamma_N = R_N/R, \quad \gamma_k = R_k/R, \quad \gamma_N + \sum_k \gamma_k = 1,$$  \hspace{1cm} (A.8)
where $R_k$ is the net resistance of all wires, $R_k$ is the resistance of the $k$-th barrier, and $R = e^{-1}$.

By using the definitions in Eq. (3), we obtain the counting electric current $I(\chi)$ and the CGF,

$$I(\chi) = \frac{1}{2\epsilon} \int_0^\infty dE \operatorname{Tr} \sigma I = \frac{i g_0}{2\epsilon} \int_0^\infty dE \operatorname{Tr} \frac{\sigma}{\sin \phi} (\hat{G}_L \times \hat{G}_R)_x,$$

(A.9)

$$S(\chi) = \frac{g_0}{4\epsilon^2} \int dE \operatorname{Tr} \left[ r \eta^2 / 2 + \sum_k \left( 1 - \sqrt{1 - r_k^2 \eta^2} \right) / r_k \right],$$

(A.10)

We note that the statistics is insensitive to the position of the barriers, and depends only on the barrier resistances and the net resistance of the diffusive part of the connector. In the absence of barriers, $r_k \to 0$, the CGF has the form,

$$S(\chi) = \frac{g_0}{4\epsilon^2} \int dE \phi^2 = \frac{g_0}{4\epsilon^2} \int dE \arccos^2 \left[ 1 + P_+ \left( e^{2\chi} - 1 \right) + P_- \left( e^{-2\chi} - 1 \right) \right].$$

(A.11)

At zero temperature, the integration over energy in Eq. (A.10) can be explicitly performed,

$$S(\chi) = \frac{\mathcal{N}}{2} \left[ r \eta^2 / 2 + \sum_k \left( 1 - \sqrt{1 - r_k^2 \eta^2} \right) / r_k \right],$$

(A.12)

where $\mathcal{N} = gV_0/e$. From Eq. (A.12) we find the Fano factor $F$ in the shot noise power $P_T = eF$, $F = (2/3)(1 + 2B_3)$, $B_n = \sum_k r_k^n$,

(A.13)

which varies between the Poissonian value $F = 2$ for the tunnel connector and 1/3-suppressed value, $F = 2/3$, in the absence of barriers. The third cumulant $C_3$ varies between $\mathcal{N}$ for Poissonian statistics in the single barrier case and $\mathcal{N}/15$ for a diffusive conductor,

$$C_3(V, 0) = \left[ (\mathcal{N}/15) \left[ 1 + 10B_3 \left( 1 + 4B_3 \right) - 36B_3 \right] \right].$$

(A.14)

It is interesting to note that Eq. (A.8) can be easily transformed into equation for the transparency distribution $\rho(T)$, by making use of the relation of the generalized circuit theory between the counting current $I(\chi)$ and the matrix current $I$ following from Eqs. (6) and (3),

$$I(\chi) = \frac{1}{4\epsilon} \int_0^\infty dE \operatorname{Tr} \tau_x \sigma I, \quad I = \frac{g}{2} \int_0^1 dT \rho(T) |\tilde{G}_L, \tilde{G}_R(\chi)| W^{-1}.$$  

(A.15)

Rewriting this equation in the vector representation, comparing it with Eq. (A.9), and introducing the variable $z = (1/2)(1 - e^{i\phi})$, we obtain the integral equation for $\rho(T)$,

$$\int_0^1 \frac{T \rho(T)}{1 - z T} = \frac{\eta}{2\sqrt{z(1-z)}},$$

(A.16)

where $\eta$ obeys Eq. (A.8) with the function $\phi = 2\arcsin \sqrt{z}$ in the right-hand side (rhs). The solution of Eq. (A.16) has the form $\rho(T) = \operatorname{Re} \eta/2\pi T \sqrt{1-T}$, where $\eta(\chi)$ is the solution of Eq. (A.8) with the function $\pi + 2i\arccosh(1/\sqrt{T})$ in the rhs [24].

In some limiting cases, one can obtain an analytical solution of Eq. (A.8). In particular, if the number $M$ of the barriers is large, $M \gg 1$, then the resistance of each barrier is small compared to the net resistance, $R_k \ll R$. In this case, the approximate solution of Eq. (A.8) is $\eta = \phi$, and the CGF coincides with that for diffusive wire, $S(\chi) = \mathcal{N} \arccos^2 e^{\chi/2}$. In the tunnel limit, when the resistance of each barrier much exceeds the net resistance of diffusive segments, $R_k \gg \mathcal{N}$, the first term in Eq. (A.8) can be neglected. Then an analytical expression for the parameter $\eta$ and the CGF at arbitrary $M$ can be obtained in the case of equivalent barriers, $r_k = 1/M$,

$$\eta = M \sin \phi / \mathcal{M}, \quad S(\chi) = \mathcal{N} M^2 \sin^2 \phi \arccos e^{\chi/2} / M,$$

(A.17)
when the Fano factor is given by \( F = (2/3) \left( 1 + 2/M^2 \right) \). In the limit of large number of the barriers, \( M \gg 1 \), we return to the “diffusive” statistics, while for single-barrier structure, \( M = 1 \), we obtain pure Poissonian statistics, \( S(\chi) = \mathcal{N}(e^\chi - 1) \).

At arbitrary temperature, the cumulants can be found analytically by asymptotic expansion in Eqs. (A.8), (A.10) over small \( \eta \) and \( \chi \). In particular, the noise power,

\[
P_L(V,T) = \frac{4T}{3R} \left[ (1 + 2B_3) \frac{p}{2} \coth \frac{p}{2} + 2(1 - B_3) \right],
\]

(A.18)

exhibits crossover between the shot noise at \( T \ll eV \) and the Johnson thermal noise \( P_T = 4T/R \) at large temperature, \( T \gg eV \). The voltage dependence of the third cumulant,

\[
C_3(V,T) = C_3(V,0) + \frac{2}{5} N \sum_{V} (1 - 10B_3^2 + 9B_3) \frac{\sinh p - p}{p \sinh^2(p/2)}.
\]

(A.19)

is linear in both limits and approaches \((\mathcal{N}/3)(1 + 2B_3)\) at high temperatures. In the limit of tunnel connector, \( B_n = 1 \), the second term in Eq. (A.19) vanishes, and \( C_3 \) becomes temperature independent. In the absence of barriers, \( B_n = 0 \), Eq. (A.19) reproduces the result of a modified kinetic theory of fluctuations for a diffusive wire[4].

In order to access FCS in multi-terminal structures, which consist of a set of connectors attached between several normal electrodes and a diffusive island (node) with negligibly small resistance, separate counting fields \( \chi \alpha \) and parameters \( \eta \alpha \) are to be introduced in each arm[16],

\[
\tilde{I}_\alpha = i \tilde{\xi}_\alpha (\tilde{g}_\alpha \times \tilde{g}_c), \quad \tilde{\xi}_\alpha = g_\alpha \eta_\alpha / \sin \phi_\alpha.
\]

(A.20)

The quantities \( \eta_\alpha \) obey the equations similar to Eq. (A.8), with the angles \( \phi_\alpha = \arccos(\tilde{g}_\alpha \cdot \tilde{g}_c) \) in the rhs, where the Green’s vector \( \tilde{g}_c \) at the node can be found from the current conservation law, \( \sum_\alpha \tilde{I}_\alpha = 0 \),

\[
\tilde{g}_c = \tilde{G} / \sqrt{\sum_\alpha \tilde{G}_\alpha} = \tilde{g}_c / \sqrt{\sum_\alpha \tilde{G}_\alpha}.
\]

(A.21)

According to Eq. (A.21), the vector \( \tilde{g}_c \) depends on all counting fields \( \chi_\alpha \), which reflects cross-correlations between the currents in different connectors. For the system of tunnel connectors, where the quantities \( \tilde{\xi}_\alpha \) are equal to the conductances \( g_\alpha \) and therefore become independent of \( \chi \), the CGF at zero temperature can be explicitly evaluated [21],

\[
S(\chi) = \frac{V_{10}}{2e} G \sqrt{1 + 4 \sum_\alpha \tilde{G}_\alpha (e^{\tilde{\chi}_\alpha} - 1)}, \quad \tilde{G}_\alpha = g_\alpha / G, \quad G = \sum_\beta \tilde{G}_\beta.
\]

(A.22)

where the index \( V \) denotes the voltage biased electrode.

For arbitrary connectors, the cumulants can be found from asymptotic solutions of the equations for \( \eta_\alpha \) and \( \tilde{g}_c \) at small \( \chi_\alpha \). For instance, the partial current through \( \alpha \)-th connector is \( I_\alpha = V \tilde{G}_\alpha \tilde{g}_c \), and the Fano factors defined as \( F_{\alpha\beta} = \langle 2eI_\alpha \rangle / \langle \partial H_\alpha \chi \rangle / \partial \chi_\beta \chi=0 \) read

\[
F_{\alpha\beta} = \left( 2 - \frac{4}{3} \tilde{G}_\alpha \right) \delta_{\alpha\beta} - \frac{4}{3} \tilde{G}_\beta \left[ 1 + \tilde{G}_\gamma (B_{3a} + B_{3\beta}) - B_{3\gamma} (1 - \tilde{G}_\gamma)^2 - \tilde{G}_\gamma \sum_{\gamma^2} \tilde{G}_\gamma B_{3\gamma} \right].
\]

(A.23)

The diagonal elements \( F_{\alpha\alpha} \) of the matrix \( F_{\alpha\beta} \) have the meaning of the Fano factors for the shot noise in \( \alpha \)-th connector and may vary between \( 2/3 \) and \( 2 \), approaching Poissonian value \( F_{\alpha\alpha} = 2 \) for a large number of the terminals, when the normalized conductances \( \tilde{G}_\alpha \) become small. The cross-correlators \( F_{\alpha\beta} (\alpha \neq \beta) \) between the currents in different terminals are negative [29]. In a particular case of diffusive connectors (\( B_n = 0 \)), Eq. (A.23) reproduces the result of Ref. [31] for a so-called star-shaped geometry.
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