Compact Relativistic Stars under Karmarkar Condition

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ABSTRACT

A class of new solutions for Einstein’s field equations, by choosing the ansatz $e^{\lambda(r)} = \frac{1 + ar^2}{1 + br^2}$ for metric potential, are obtained under Karmarkar condition. It is found that a number of pulsars like 4U 1820-30, PSR J1903+327, 4U 1608-52, Vela X-1, PSR J1614-2230, Cen X-3 can be accommodated in this model. We have displayed the nature of physical parameters and energy conditions throughout the distribution using numerical and graphical methods for a particular pulsar 4U 1820-30 and found that the solution satisfies all physical requirements.

Subject headings: General relativity; Exact solutions; Anisotropy; Relativistic compact stars; Charged distribution
1. Introduction

Eversince Schwarzschild (1916) obtained the first solution of Einstein’s field equations, a plethora of exact solutions are available at present, in literature. The interest in the study of anisotropic distributions has started with theoretical investigations of Ruderman (1972) and Canuto (1974) regarding the anisotropic nature of matter distribution in ultra-high densities. The impact of anisotropy on equilibrium of stellar configuration can be seen in the pioneering work of Bowers and Liang (1974). Herrera and Santos (1997) have studied matter distribution incorporating anisotropy in pressure. A class of anisotropic solutions of spherically symmetric distribution of matter have been studied by Mak and Harko (2003). Maharaj and Chaisi (2006) have shown a procedure to generate anisotropic solutions from the known isotropic solutions. The impact of shear and electromagnetic field on stellar configuration has been studied by Sharma and Maharaj (2007).

A number of researchers have worked on spacetimes whose physical space obtained by putting $t = 0$ has a definite geometry. Vaidya and Tikekar (1982) have studied spherical distributions of matter on spacetime whose physical space has 3-spheroidal geometry. Charged analogue of this metric has been studied by Patel and Kopper (1987). Tikekar and Patel (1988) have obtained models of non-adiabatic gravitationally collapsing models with radial heat flux on the background of spheroidal spacetime. The impact of anisotropy on Vaidya and Tikekar (1982) model has been studied by Karmarkar et. al. (2007).

Tikekar and Thomas (1998) have studied relativistic models of stars on the background of pseudo-spheroidal spacetime and have shown that it can be used to describe equilibrium models of superdense stars. It has further shown that these models are stable under radial modes of pulsation. Non-adiabatic gravitational collapse of spherical stars incorporating radial heat flux have been studied by Thomas et al. (2005) on the background of pseudo-spheroidal spacetime. Chattopadhyay and Paul (2010) have obtained the higher dimensional analogue
of pseudo-spheroidal stellar models of Tikekar and Thomas (1999). Ratanpal et. al. (2015) have studied spherical distribution of matter by choosing a specific form for radial pressure on pseudo-spheroidal spacetime. Ratanpal et. al. (2016) have studied anisotropic models of superdense stars on the background of pseudo-spheroidal spacetime.

Another useful and geometrically significant spacetime widely used by researchers is the paraboloidal spacetime studied by Tikekar and Jotania (2007). Tikekar and Jotania (2009) have used this spacetime to obtain core-envelope models of superdense stars. Anisotropic models of stars on paraboloidal spacetime admitting quadratic equation of state have been studied by Sharma and Ratanpal (2013). New anisotropic solutions of relativistic star on paraboloidal spacetime has been obtained by Ratanpal et. al. (2017). Thomas and Pandya (2017) have obtained anisotropic compact star models with linear equation of state on the background of paraboloidal spacetime.

The embedding problems are geometrically significant problems in general theory of relativity. It was first studied by Schlai (1871). Nash (1956) proposed first isometric embedding theorem. The condition for embedding 4-dimensional spacetime in 5-dimensional Euclidean space was derived by Karmarkar (1948). Such spacetimes are usually referred to a spacetimes of Class-I. The Karmarkar condition is given by

\[ R_{1414}R_{2323} = R_{1212}R_{3434} + R_{1224}R_{1334}, \]  

Pandey and Sharma (1981) have found that for spherically symmetric spacetime metric to be of Class-I, it is further required that \( R_{2323} \neq 0 \) in (1). Relativistic models of stars satisfying Karmarkar’s condition have been extensively studied by Maurya et. al. (2015[a]), Maurya et. al. (2015[b]), Maurya et. al. (2016[a]), Maurya et. al. (2016[b]), Maurya et. al. (2016[c]), Maurya et. al. (2017[a]), Maurya et. al. (2017[b]).

In this article we have studied solutions of Einstein’s field equations satisfying Karmarkar
condition (1) by choosing the metric potential the ansatz $e^{\lambda(r)} = \frac{1+ar^2}{1+br^2}$. If $a = -\frac{k}{R^2}$ and $b = -\frac{1}{R^2}$, the metric in Schwarzschild coordinates represents spheroidal spacetime metric proposed by Vaidya and Tikekar (1982). If $a = \frac{k}{R^2}$ and $b = \frac{1}{R^2}$, the spacetime metric reduces to pseudo-spheroidal spacetime metric considered by Tikekar and Thomas (1998). If we take $b = 0$ and $a = \frac{1}{R^2}$, the spacetime metric reduces to paraboloidal spacetime metric discussed by Tikekar and Jotania (2007).

We have organized the article as follows: In section 2, we have given the Einstein’s field equations and Karmarkar condition. The solution of Einstein field equations under Karmarkar condition is obtained in section 3. In section 4, physical plausibility conditions are described. The nature of various physical quantities throughout the distribution has been examined by taking a particular pulsar 4U 1820-30. In section 5, It has been concluded that a large variety of pulsars can be accommodated in this model incorporating Karmarkar condition.

2. Einstein’s field equations and Karmarkar condition

We consider the interior spacetime metric for static spherically symmetric fluid distribution as

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

with energy-momentum tensor

$$T_{ij} = (\rho + p) u_i u_j - pg_{ij} + \pi_{ij}, \quad u^i u_i = 1,$$

where $\rho$ and $p$ represent density and isotropic fluid pressure respectively, $u^i$ is the unit four velocity and anisotropic stress tensor $\pi_{ij}$ is given by (Maharaj and Maartens (1989))

$$\pi_{ij} = \sqrt{3} S [c_i c_j - \frac{1}{3} (u_i u_j - g_{ij})],$$
where $S = S(r)$ denotes the magnitude of anisotropy and $c^i = (0, -e^{\lambda/2}, 0, 0)$ denotes radially directed vector. The non-vanishing components of energy-momentum tensor are given by

$$T^0_0 = \rho, \quad T^1_1 = -\left(P + \frac{2S}{\sqrt{3}}\right), \quad T^2_2 = T^3_3 = -\left(P - \frac{S}{\sqrt{3}}\right).$$

(5)

We shall denote

$$p_r = P + \frac{2S}{\sqrt{3}}, \quad p_\perp = P - \frac{S}{\sqrt{3}},$$

(6)

and hence magnitude of anisotropy is given by

$$S = \frac{p_r - p_\perp}{\sqrt{3}}.$$  

(7)

The Einstein’s field equations, for spacetime metric (2) with energy-momentum tensor (3), are given by

$$8\pi\rho = \frac{e^{-\lambda} \lambda'}{r} + \frac{1 - e^{-\lambda}}{r^2},$$

(8)

$$8\pi p_r = \frac{e^{-\lambda} \nu'}{r} + \frac{e^{-\lambda} - 1}{r^2},$$

(9)

$$8\pi p_\perp = e^{-\lambda} \left(\frac{\nu''}{2} + \frac{\nu^2}{4} - \frac{\nu' \lambda'}{4} + \frac{\nu' - \lambda'}{2r}\right).$$

(10)

The spacetime metric (2) is said to be of class-I type if it satisfies the Karmarkar condition (1).

The components of Riemann curvature tensor $R_{ijkl}$ for spacetime metric (2) are given by

$$R_{2323} = r^2 \sin^2 \theta \left(1 - e^{-\lambda}\right),$$

$$R_{1212} = \frac{1}{2} r \lambda',$$

$$R_{2424} = \frac{1}{2} r \nu' e^\nu e^{-\lambda},$$

$$R_{1224} = 0,$$

$$R_{1414} = e^\nu \left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda' \nu'}{4}\right),$$
\[ R_{3434} = R_{2424} \sin^2 \theta. \]

The Karmarkar condition (1) now takes the form

\[ \frac{v''}{v'} + \frac{v'}{2} = \frac{\lambda' e^\lambda}{2(e^\lambda - 1)}. \] (11)

The general solution of equation (11) is given by

\[ e^v = \left[ A + B \int \sqrt{(e^\lambda(r) - 1)} dr \right]^2, \] (12)

where \( A \) and \( B \) are constants of integration and \( e^\lambda(r) \neq 1 \). Using (9), (10), (12) in (7), the magnitude of anisotropy can be expressed in the form

\[ 8\pi \sqrt{3} S = -\frac{v' e^{-\lambda}}{4} \left[ \frac{2}{r} - \frac{\lambda'}{e^\lambda - 1} \right] \left[ \frac{v' e^v}{2rB^2} - 1 \right]. \] (13)

In the case of isotropic distribution of matter, we have \( S = 0 \) which leads to either \( \frac{2}{r} - \frac{\lambda'}{e^\lambda - 1} = 0 \) or \( \frac{v' e^v}{2rB^2} - 1 = 0 \). The former case leads to Schwarzschild (1916) exterior solution and the latter gives the solution given by Kohler and Chao (1965).

3. Anisotropic solution under Karmarkar condition

The explicit expression for the potential \( v \) can be obtained by choosing appropriate form for \( \lambda \).

We choose \( e^\lambda \) in the form

\[ e^\lambda = \frac{1 + ar^2}{1 + br^2}, \] (14)

where \( a \) and \( b \) are constants. If \( a = \frac{K}{R} \) and \( b = 1 \), metric (2) represents the pseudo-spheroidal spacetime discussed by Tikekar and Thomas (1998). If \( a = -\frac{K}{R^2} \) and \( b = -1 \) gives the Vaidya and Tikekar (1982) spacetime and \( a = 1, b = 0 \) represent the paraboloidal spacetime studied by Tikekar and Jotania (2007).
We shall assume here, that both $a$ and $b$ are not equal to zero. Substituting (14) in (12), gives $e^\nu$ in the form
\[
e^\nu = \left( A + B \sqrt{\frac{a-b}{b}} \right)^2.
\] (15)

The spacetime metric (2) now takes the explicit form
\[
ds^2 = \left[ A + B \sqrt{\frac{a-b}{b}} \right]^2 dt^2 - \left( \frac{1+ar^2}{1+br^2} \right) dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right).
\] (16)

The expressions of matter density, radial pressure and tangential pressure are given by
\[
8\pi \rho = \frac{(a-b)(3+ar^2)}{(1+ar^2)^2},
\] (17)
\[
8\pi p_r = \frac{Ab(b-a) + B\sqrt{a-b}\sqrt{1+br^2}(3b-a)}{(1+ar^2)(Ab + B\sqrt{a-b}\sqrt{1+br^2})},
\] (18)
\[
8\pi p_\perp = \frac{\sqrt{a-b}}{(1+ar^2)^2(Ab + B\sqrt{a-b}\sqrt{1+br^2})} [\sqrt{a-b}[-Ab\sqrt{a-b} + B\sqrt{1+br^2}(3b-a + abr^2)]].
\] (19)

The spacetime metric (16) should match continuously with schwarzschild exterior metric
\[
ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 - \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\] (20)
at the boundary of the star $r = R$. It leads to the following equations
\[
1 - \frac{2M}{R} = \frac{1+bR^2}{1+aR^2},
\] (21)
\[
\sqrt{1 - \frac{2M}{R}} = A + B \sqrt{a-b}\sqrt{1+bR^2}.
\] (22)

Further, the boundary condition $P_r (r = R) = 0$ gives
\[
Ab\sqrt{a-b} = B\sqrt{1+bR^2} (3b-a).
\] (23)

Equations (21), (22) and (23) determine the constants $A$, $B$ and the total mass enclosed inside the radius $R$ as
\[
A = \frac{(3b-a)\sqrt{1+bR^2}}{2b\sqrt{1+aR^2}},
\] (24)
\[ B = \frac{\sqrt{a-b}}{2\sqrt{1+ar^2}}, \quad (25) \]

\[ M = \frac{(a-b)R^3}{2(1+ar^2)}. \quad (26) \]

Equations (17) through (19) now take the form

\[ 8\pi \rho = \frac{(a-b)(3+ar^2)}{(1+ar^2)^2}, \quad (27) \]

\[ p_r = \frac{(3b-a)(b-a)}{1+ar^2} \left[ \frac{\sqrt{1+bR^2} - \sqrt{1+br^2}}{(3b-a)\sqrt{1+bR^2} - (b-a)\sqrt{1+br^2}} \right], \quad (28) \]

\[ p_\perp = -\frac{(a-b)[3b(1+br^2-\sqrt{1+br^2}\sqrt{1+bR^2})+a(-1+b^2r^4+\sqrt{1+br^2}\sqrt{1+bR^2})]}{(1+ar^2)^2\sqrt{1+br^2}[b(\sqrt{1+br^2}-3\sqrt{1+bR^2})+a(-\sqrt{1+br^2}+\sqrt{1+bR^2})]} \quad (29) \]

The expression for anisotropy (13) can be explicitly written as

\[ 8\pi \sqrt{3}S = \frac{a(a-b)r^2[a(1+br^2-\sqrt{1+br^2}\sqrt{1+bR^2})+b(-2-2br^2+3\sqrt{1+br^2}\sqrt{1+bR^2})]}{(1+ar^2)^2\sqrt{1+br^2}[b(\sqrt{1+br^2}-3\sqrt{1+bR^2})+a(-\sqrt{1+br^2}+\sqrt{1+bR^2})]} \quad (30) \]

It can be noticed that anisotropy of the distribution is zero at the centre of the star.

### 4. Physical Plausibility Conditions

A physically acceptable stellar model should comply with the following conditions throughout its region of validity.

(i) \( \rho(r) \geq 0, \quad p_r(r) \geq 0, \quad p_\perp(r) \geq 0 \) for \( 0 \leq r \leq R \)

(ii) \( \frac{dp}{dr} \leq 0, \quad \frac{dp_r}{dr} \leq 0, \quad \frac{dp_\perp}{dr} \leq 0 \) for \( 0 \leq r \leq R \)

(iii) \( 0 < \frac{dp_r}{dp} < 1, \quad 0 < \frac{dp_\perp}{dp} < 1 \) for \( 0 \leq r \leq R \)

(iv) \( \rho - p_r - 2p_\perp \geq 0 \) for \( 0 \leq r \leq R \)

(v) \( \Gamma > \frac{4}{3} \), \quad for \( 0 \leq r \leq R \)
Table 1: Estimated physical values of parameters based on the observational data

| STAR          | M (M☉) | R (Km) | ρc (MeV fm⁻³) | ρR (MeV fm⁻³) | u(= M/R) |
|---------------|--------|--------|----------------|---------------|----------|
| 4U 1820-30    | 1.25   | 9.1    | 804.032        | 309.128       | 0.137    |
| PSR J1903+327 | 1.35   | 9.438  | 804.032        | 293.779       | 0.142    |
| 4U 1608-52    | 1.31   | 9.31   | 804.032        | 299.495       | 0.140    |
| Vela X-1      | 1.38   | 9.56   | 804.032        | 288.439       | 0.144    |
| PSR J1614-2230| 1.42   | 9.69   | 804.032        | 282.863       | 0.146    |
| Cen X-3       | 1.27   | 9.178  | 804.032        | 305.513       | 0.138    |

We shall use the above conditions to find the bounds on the model parameters a and b. Density ρ is positive and decreasing throughout the distribution if \( a > b \). The radial pressure \( p_r \) is positive and decreasing throughout the distribution if \( a \leq \frac{4}{R^2} \) and \( a > 3b \). The transverse pressure \( p_\perp \) is positive and decreasing throughout the distribution if \( 0 < \frac{dp_\perp}{d\rho} < 1 \) and \( 0 < \frac{dp_\perp}{d\rho} < 1 \) impose the restrictions \( a \leq \frac{2.7847}{R^2} \) and \( 0 < \frac{0.4384}{R} < a \leq \frac{2.7025}{R^2} \). The adiabatic index \( \Gamma > \frac{4}{3} \) if \( a \leq \frac{0.8202}{R^2} \). Thus the conditions (i) through (v) are satisfied if

\[
0 < a \leq \frac{2}{R^2}, \quad a > b.
\]

We shall examine the viability of the present model to represent some well-known pulsars whose mass and size are known.

5. Discussion

We have used the present model to a large variety of compact stars like 4U 1820-30, PSR J1903+327, 4U 1608-52, Vela X-1, PSR J1614-2230, Cen X-3, whose masses and radii are known (Gangopadhyay et al. (2013)). The central and surface densities are calculated and displayed.
in Table 1 along with the compactification factor. We have shown that energy and stability conditions are satisfied for all pulsars listed in Table 1 for suitable bounds for the parameters $a$ and $b$ depending on the radii of different pulsars under consideration. Due to the complexity of expressions involved, it is difficult to examine the physical acceptability conditions analytically. Hence we have adopted graphical method.

In order to examine the nature of physical quantities throughout the distribution, we have considered the pulsar 4U 1820-30 whose estimated mass is $M = 1.25 M_{\odot}$ and radius $R = 9.1 \text{ km}$. The expressions (31) now take the form

$$0.0053 < a \leq 0.0099, \quad a > 3b.$$  \hspace{1cm} (32)

We have taken the value of $a$ as the upper bound 0.0099, $b = 0.001$ and examined the physical, energy and stability conditions of the pulsar throughout its region of validity.

In Fig. 1 we have shown the variation of density for $0 \leq r \leq 9.1$. It is clear from the graph that the density is a decreasing function of $r$. In Fig. 2 and Fig. 3 we have shown the variation of radial and transverse pressure throughout the star. It can be seen that both pressures are decreasing.
radially outward. In Fig. 4 and Fig. 5 we have displayed the variation of $\frac{dp_r}{d\rho}$ and $\frac{dp_\perp}{d\rho}$ against $r$. Both quantities satisfy the restriction $0 < \frac{dp_r}{d\rho} < 1$ and $0 < \frac{dp_\perp}{d\rho} < 1$ indicating that the sound speed throughout the star is less than the speed of light.

The variation of anisotropy is shown in Fig. 6. It can be noticed that anisotropy vanishes at the centre and decreases towards the boundary. Fig. 7 indicates that the strong energy condition $\rho - p_r - 2p_\perp > 0$ is satisfied throughout the distribution. In order that a relativistic equilibrium model of a compact star is stable model, the adiabatic index $\Gamma = \frac{\rho + p_r}{p_r} \frac{dp_r}{d\rho} > \frac{4}{3}$ throughout the distribution. Fig. 8 indicates that the condition $\Gamma > \frac{4}{3}$ is satisfied in the region $0 \leq r \leq 9.1$. In Fig. 9, we have shown that redshift $z$ is less than 1 and decreases radially outward.

It has been concluded that a large number of pulsars with known masses and radii can be accommodated in the present model satisfying Karmarkar condition.
Fig. 3.— Variation of transverse pressures against radial variable $r$

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Fig. 4.— Variation of $\frac{dp_r}{d\rho}$ against radial variable $r$.

Fig. 5.— Variation of $\frac{dp_\perp}{d\rho}$ against radial variable $r$. 
Fig. 6.— Variation of anisotropies against radial variable $r$.

Fig. 7.— Variation of strong energy condition against radial variable $r$. 
Fig. 8.— Variation of adiabatic Index against radial variable $r$.

Fig. 9.— Variation of surface redshift against radial variable $r$. 

$\Gamma$ (MeV Fm$^{-3}$)  

$Z$
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