The stability of protostellar disks with Hall effect and buoyancy

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Abstract. The stability properties of inviscid protostellar disks are examined taking into account the Hall effect and buoyancy. Depending on the parameters, different types of instabilities can exist in different regions of disks. In a very low ionized region, the instability associated with baroclinic effects of buoyancy is likely most efficient. The Hall-driven shear instability can lead to destabilization of regions with a higher ionization. The magnetorotational instability modified by buoyancy can only be a destabilizing factor in regions with strong magnetic field or a relatively high conductivity ($\sigma B^2/\rho > a_e\Omega$, with $a_e$ the magnetization parameter of electrons).

Key words. accretion: accretion disks - MHD - instabilities - turbulence - stars: formation

1. Introduction

The models of astrophysical disks require sufficiently strong turbulence to enhance the efficiency of angular momentum transport. At present, there is no commonly accepted view point as to how a laminar flow in disks is disrupted and turbulence generated. In general, turbulence may be generated due to various hydrodynamic and hydromagnetic instabilities which can arise in differentially rotating non-uniform gaseous disks but the exact origin of turbulence is still controversial.

In accretion disks, the origin of turbulence is often attributed to the magnetorotational instability since the necessary condition of instability, $\partial\Omega/\partial R < 0$ (i.e. a decrease of the angular velocity with cylindrical radius) is fulfilled (Velikov 1959; Kurzweg 1963; Balbus & Hawley 1991; Kaisig, Tajima & Lovelace 1992; Kumar, Coleman & Kley 1994; Zhang, Diamond & Vishniac 1994). This instability exists not only for short wavelength perturbations, but also for global modes with scales comparable to the disk height (Curry, Pudritz & Sutherland 1994; Curry & Pudritz 1995). Note that the magnetic shear instability can arise only if the field is not too strong, because this would suppress the instability (Urpin 1996; Kitchatinov & Rüdiger 1997). Simulations of the magnetorotational instability in disks (Hawley, Gammie & Balbus 1995; Matsumoto & Tajima 1995; Brandenburg et al. 1995; Torkelsson et al. 1996; Arlt & Rüdiger 2001) show that the generated turbulence may enhance the angular momentum transport.

Unlikely, that the magnetic shear instability is the only instability that can exist in such complex objects as astrophysical disks. A detailed analysis of MHD modes in stratified magnetic accretion disks demonstrates a much wider variety of instabilities than previously realized (Keppens, Casse & Goedbloed 2002). Therefore, the current view on the origin of turbulence can be very simplified. Even a pure hydrodynamic origin of turbulence cannot be excluded (see Richard & Zahn 1999).

The situation is particularly uncertain in cold and dense protostellar disks where the electrical conductivity is extremely small because of a low ionization degree. The magnetic Reynolds number is likely not very large in these disks, and the magnetic field cannot be considered as “frozen” into the gas (Gammie 1996). The behavior of the magnetic shear instability in the presence of ohmic dissipation has been considered in the linear (Jin 1996) and non-linear regimes (Sano, Inutsuka & Miyama 1998). As it was first pointed out by Wardle (1999), however, poorly conducting protostellar disks can be strongly magnetized if electrons are the main charge carriers. Magnetization leads to anisotropic electron transport with substantially different properties along and across the magnetic field (see, e.g., Spitzer 1978). If the field is sufficiently strong then the main contribution to the electric resistivity tensor is provided by the Hall component that produces the electric field perpendicular to both the magnetic field and electric current. This component is non-dissipative but it can change a geometry of the magnetic field. A linear stability analysis undertaken by Wardle (1999) shows that the Hall effect can provide an additional either stabilizing or destabilizing influence depending on a direction of the magnetic field. A more general consideration of the magnetic shear instability in the presence of Hall currents has been done by Balbus & Terquem (2001). They found that the Hall effect changes qualitatively the stability properties...
of rotating gas resulting in destabilization of even outwardly increasing differential rotation (see also Rüdiger & Shalybkov 2003). In their analysis, however, Balbus & Terquem (2001) neglected gravity which, in protostellar disks, is as important as rotation. Gravity influences the behavior of modes via buoyancy, and may lead to new Hall-driven instabilities missed in simplified considerations.

In the present paper we consider the linear stability properties of magnetic protostellar disks taking into account the Hall effect and gravity. We treat the behavior of different short wavelength magnetohydrodynamic modes which can exist in such objects and determine the parameter domain where these modes are unstable. The paper is organized as follows. In Section 2, we discuss the Hall effect in the conditions of protostellar clouds. In Section 2, the main equations are presented and a dispersion relation is derived that describes the behavior of short wavelength perturbations in the Boussinesq approximation. The stability criteria for different modes are discussed in Section 4, and the growth rates of instabilities are calculated in Section 5. Finally, our results are briefly summarized in Section 6.

2. Anisotropic electric resistivity in protostellar disks

The electrical conductivity is likely very low in protostellar disks because of a low temperature. The magnetic field cannot be considered as “frozen” into gas, and dissipative effects should be taken into account. However, despite low temperature and ionization, the electron gas can be magnetized as it was first pointed out by Wardle (1999). Magnetization of the electron gas is characterized by the product of electron gyroradius, \( \omega_B = eB/m_e c \) and the relaxation time of electrons, \( \tau \) (see, e.g., Spitzer 1978). In protostellar disks, \( \tau \) is likely determined by the scattering of electrons on neutrals, then \( \tau = 1/n(\sigma v) \) where \( \langle \sigma v \rangle \) is the average product of the cross-section and velocity and \( n \) is the number density of neutrals. Using the estimate of \( \langle \sigma v \rangle \) obtained by Draine, Roberge & Dalgarno (1983) for electron-neutral collisions, we can represent the magnetization parameter of electrons, \( a_e \), as

\[
a_e \equiv \omega_B \tau = 21 B n_{14}^{-1} T_2^{-1/2},
\]

where \( B \) is measured in Gauss, \( n_{14} \) has units \( 10^{14} \) cm\(^{-3} \), and \( T_2 \) has units 100 K. If this parameter is greater than 1, i.e.

\[
B > 0.048 n_{14} \sqrt{T_2} \text{ Gauss}
\]

(see Fig. 1), then electron transport is anisotropic, and we have to use a tensorial magnetic diffusivity instead of a scalar one. For more details concerning the generalized Ohm’s law in weakly ionized plasma we refer to the paper by Shalybkov & Urpin (1995) where this law has been considered using the relaxation time approximation for three components plasma with ions and neutrals of the same mass. In very weakly ionized plasma of protostellar disks, the difference between parallel and perpendicular resistivity is small (see, e.g., Balbus & Terquem 2001). The Hall-originated magnetic diffusivity is then given by

\[
a_e \eta = \beta B = \frac{eB}{4\pi e n_e^2},
\]

where \( \eta \) is the microscopic magnetic diffusivity and \( n_e \) is the number density of electrons. The magnetic diffusivity is

\[
\eta = 2.34 \times 10^3 f^{-1} T_2^{1/2} \text{ cm}^2 \text{ s}^{-1},
\]

where \( f = n_e/n \) is an ionization fraction.

The induction equation reads

\[
\frac{\partial B}{\partial t} = \text{rot}(U \times B) + \eta \Delta B - \beta \text{rot}(\text{rot} B \times B).
\]

(5)

The last term represents the Hall effect, and numerical evaluation done by Wardle & Ng (1999) indicate that this term can be of importance in some regions of protostellar disks. In (5), we neglect a non-uniformity of the resistivity and electron number density.

3. Basic equations and the dispersion relation

Consider the stability properties of a magnetized axisymmetric protostellar disk of a finite vertical extent. The unperturbed angular velocity can generally depend on both \( R \) and \( z \), so \( \Omega = \Omega(R, z) \), where \( (R, \phi, z) \) are cylindrical coordinates. The magnetic field, \( B = (B_R, B_\phi, B_z) \), is assumed to be weak in the sense that the Alfvén speed, \( c_A \), is small compared with the sound speed, \( c_s \). This enables us to employ the Boussinesq approximation.
In the unperturbed state, the disk is assumed to be in hydrostatic equilibrium in the $R$- and $z$-directions,

$$\frac{\nabla p}{\rho} = G + \frac{1}{4\pi\rho}\text{rot}\,B \times B, \quad G = g + \Omega^{2}R,$$ \hspace{1cm} (6)

where $g$ is the gravity. If $c_{s} > c_{A}$, the unperturbed Lorentz force is small compared with the pressure force, thus the disk structure is mainly determined by the balance between gravity, centrifugal force, and pressure.

We consider axisymmetric short wavelength perturbations with the space-time dependence $\exp(\gamma t - ik \cdot x)$ where $k = (k_{R}, 0, k_{z})$ is the wave vector, $|k \cdot x| \gg 1$. Small perturbations will be indicated by subscript 1, whilst unperturbed quantities will have no subscript, except for indicating vector components when necessary. The linearized momentum and continuity equations governing the behavior of such perturbations in the Boussinesq approximation read

$$\gamma U' + 2\Omega \times U' + e_{\phi}R(U' \cdot \nabla)\Omega = \frac{i k p'}{\rho} - \alpha G T' + \frac{i}{4\pi\rho}[k(B \cdot B') - B'(k \cdot B)],$$ \hspace{1cm} (7)

$$k \cdot U' = 0,$$ \hspace{1cm} (8)

where $U'$, $B'$, $p'$ and $T'$ are perturbations of the hydrodynamic velocity, magnetic field, pressure and temperature, respectively; $\alpha = - (\partial \ln \rho/\partial T)_{p}$ is the thermal expansion coefficient and $e_{\phi}$ is the unit vector in the azimuthal direction. In Eq. (6) it is assumed that the density perturbation in the buoyancy force is determined by the temperature perturbation, thus $p' = - \rho \alpha T'$, in accordance with the idea of the Boussinesq approximation.

Since the thermal conductivity of protostellar clouds is low because of a low temperature ($T \sim 10 - 10^{3}$ K), we adopt the adiabatic equation to describe the evolution of temperature perturbations,

$$\gamma T' + U' \cdot (\Delta \nabla T) = 0,$$ \hspace{1cm} (9)

where $(\Delta \nabla T) = \nabla T - \nabla_{ad} T$ is the difference between the actual and adiabatic temperature gradients.

The linearized induction equation and the divergence free condition read

$$(\gamma + \omega_{R})B' = -iU'(k \cdot B) + Re_{\phi}(B' \cdot \nabla)\Omega + \beta(k \cdot B)k \times B',$$ \hspace{1cm} (10)

$$k \cdot B' = 0,$$ \hspace{1cm} (11)

where $\omega_{R} = \eta k^{2}$ is the inverse timescale of the ohmic field decay.

The dispersion equation governing the behavior of perturbations may be obtained in the standard way. Equating the determinant of the set of Eqs. (7)–(11) to zero, we obtain

$$\gamma^{5} + a_{4}\gamma^{4} + a_{3}\gamma^{3} + a_{2}\gamma^{2} + a_{1}\gamma + a_{0} = 0,$$ \hspace{1cm} (12)

where

$$a_{4} = 2\omega_{R},$$
$$a_{3} = \omega_{R}^{2} + \omega_{H}(\omega_{H} + \omega_{sh}) + 2\omega_{A}^{2} + \omega_{g}^{2} + Q^{2},$$
$$a_{2} = 2\omega_{H}(\omega_{g}^{2} + \omega_{A}^{2} + Q^{2}),$$
$$a_{1} = [\omega_{R}^{2} + \omega_{H}(\omega_{H} + \omega_{sh})](\omega_{g}^{2} + Q^{2}) + 2\omega_{A}^{2} + 2\omega_{g}^{2} + 2\omega_{H}(\omega_{H} + \omega_{sh}) + 2\Omega^{2}k^{2},$$
$$a_{0} = \omega_{R}^{2}(\omega_{g}^{2} + \omega_{A}^{2}),$$

and

$$Q^{2} = 4\Omega^{2}k_{g}^{2} + 2\Omega^{2}k_{g}^{2}
= k_{g}^{2}(2\Omega \partial \frac{\partial \Omega}{\partial R} - k_{R} \partial \frac{\partial \Omega}{\partial z}),$$

$$\omega_{g}^{2} = - \alpha \Delta \nabla T \cdot \left[ G - \frac{k}{k_{g}}(k \cdot G) \right],$$
$$\omega_{H} = \beta k(k \cdot B),$$
$$\omega_{sh} = R\left(k_{g} \frac{\partial \Omega}{\partial R} - k_{R} \frac{\partial \Omega}{\partial z}\right),$$

$\omega_{g}$ is the frequency of buoyancy waves; $\omega_{A} = (k \cdot B)/\sqrt{4\pi\rho}$ is the Alfvén frequency, $Q^{2}$ represents the effects associated with differential rotation, $\omega_{H}$ and $\omega_{sh}$ are the characteristic frequencies of the Hall- and shear-driven processes, respectively.

If gravity is neglected ($\omega_{g} = 0$) then we recover the dispersion equation derived by Balbus & Terquem (2002)

$$\gamma^{4} + b_{3}\gamma^{3} + b_{2}\gamma^{2} + b_{1}\gamma + b_{0} = 0,$$ \hspace{1cm} (13)

where

$$b_{3} = 2\omega_{R},$$
$$b_{2} = \omega_{R}^{2} + \omega_{H}(\omega_{H} + \omega_{sh}) + 2\omega_{A}^{2} + Q^{2},$$
$$b_{1} = \omega_{R}(\omega_{A}^{2} + Q^{2}),$$
$$b_{0} = \omega_{R}^{2}Q^{2} + \left[ \omega_{A}^{2} + 2\Omega^{2}k_{g}(\omega_{H} + \omega_{sh}) \right] \times
$$
$$\left[ \omega_{A}^{2} + \omega_{H} \left( 2\Omega^{2}k_{g}^{2} + \omega_{sh} \right) \right].$$

Assuming that the condition of instability is given by

$$b_{0} < 0,$$ \hspace{1cm} (14)

Balbus & Terquem (2001) argued that the Hall effect can destabilize any differential rotation laws in protostellar disks, even those with angular velocity increasing outward.

4. Criteria of instability of protostellar disks

The equation (12) describes five low-frequency modes which can exist in protostellar disks. The condition that at least one of the roots of equation (12) has a positive real part (unstable mode) is equivalent to one of the following inequalities

$$a_{0} < 0,$$
$$A_{1} \equiv a_{4}a_{3} - a_{2} < 0,$$
$$A_{2} \equiv a_{2}(a_{4}a_{3} - a_{2}) - a_{4}(a_{4}a_{3} - a_{2}) < 0,$$
$$A_{3} \equiv (a_{4}a_{3} - a_{2})[a_{2}(a_{4}a_{3} - a_{2}) - a_{4}(a_{4}a_{3} - a_{2})] -$$
$$- a_{0}(a_{4}a_{3} - a_{2})^{2} < 0,$$ \hspace{1cm} (15)

being fulfilled (see, e.g., Aleksandrov, Kolmogorov & Laurentiev 1985). Since $\omega_{g}$ is positive defined quantity, the first condition ($a_{4} < 0$) will never apply, and only the four given conditions determine the instability in disks.
4.1. The condition $a_0 < 0$

Since $\omega_R^2 > 0$, the condition $a_0 < 0$ is equivalent to

$$\omega_k^2 < 0,$$  \hspace{1cm} (16)

or

$$k^2 (G \cdot \nabla T) - (k \cdot G) (k \cdot \Delta \nabla T) > 0.$$  \hspace{1cm} (17)

Generally, $\omega_k^2 < 0$ if the temperature gradient is superadiabatic, i.e. it exceeds its adiabatic value. In this case, the standard convective instability arises. However, $\omega_k^2$ may also be negative if the temperature gradient is subadiabatic but $\Delta \nabla T$ is not parallel to the “effective gravity”, $G$ (see Urpin & Brandenburg 1998). This obliqueness can be caused, in principle, either by the dependence of $\Omega$ on $z$ or by radiative heat transport in the radial direction. Introducing the angle $\psi$ between the vectors $G$ and $k$ and representing $\Delta \nabla T$ as a sum of components parallel and perpendicular to $G$, $\Delta \nabla T = (\Delta \nabla T)_\parallel + (\Delta \nabla T)_\perp$, the inequality (17) can be rewritten in the form

$$G(\Delta \nabla T)_\parallel [\sin^2 \psi - \sin \psi \cos \psi (\Delta \nabla T)_\perp/(\Delta \nabla T)_\parallel] > 0.$$  \hspace{1cm} (18)

If stratification is stable according to the standard Schwarzschild criterion of convection, $G(\Delta \nabla T)_\parallel < 0$, then the instability arises at

$$\sin^2 \psi - \sin \psi \cos \psi (\Delta \nabla T)_\perp/(\Delta \nabla T)_\parallel < 0.$$  \hspace{1cm} (19)

This condition can be fulfilled due to the obliqueness of $G$ and $\Delta \nabla T$ for perturbations with a small (but non-zero) angle $\psi$. Estimating $(\Delta \nabla T)_\perp$ as $(z/R)(\Delta \nabla T)_\parallel$, we obtain that $\omega_k^2$ for unstable perturbations is small,

$$\omega_k^2 \sim -\Omega^2 (H/R)^2.$$  \hspace{1cm} (20)

Therefore, convection caused by obliqueness of $G$ and $\Delta \nabla T$ is probably relatively slow.

4.2. The condition $A_1 < 0$

This condition reads

$$\omega_A^2 + \omega_R^2 + \omega_H^2 + \omega_{H\omega_{sh}} < 0,$$  \hspace{1cm} (21)

or, substituting the frequencies,

$$\beta R (k \cdot B) \left( k_z \frac{\partial \Omega}{\partial R} - k_R \frac{\partial \Omega}{\partial z} \right) < -\omega_A^2 - \omega_R^2 - \beta^2 k^2 (k \cdot B)^2.$$  \hspace{1cm} (22)

Despite this inequality is the only condition (15) that does not depend on gravity, it differs from the criterion of instability derived by Balbus & Terquem (2001) (see Eq. (85) of their paper). The condition (22) describes a new instability that appears due to combined influence of shear and the Hall effect. This instability differs from the magnetic shear instability because the only term that can provide a destabilizing influence is proportional to the Hall frequency and shear stresses, and this term is vanishing if $\omega_H \to 0$. To satisfy the inequality (22) the sign of the left hand side should be negative since all three terms on the right hand side are negative. Obviously, for any dependence of $\Omega$ on $R$ and $z$ and for any direction of $B$, there exist wave vectors that satisfies the inequality

$$(k \cdot B) \left( k_z \frac{\partial \Omega}{\partial R} - k_R \frac{\partial \Omega}{\partial z} \right) < 0,$$  \hspace{1cm} (23)

and makes the left hand side of Eq. (22) negative. Therefore, any differential rotation can generally be unstable if the Hall effect is sufficiently strong.

Compare characteristic frequencies in Eq. (21). If the Hall parameter is large, $a_e \gg 1$, then the “ohmic frequency” is negligible in Eq. (21). The characteristic value of the Hall frequency, $\omega_H$, is

$$\omega_H \sim 2 \times 10^{-4} B n_e^{-1} \lambda_{11}^{-2} \frac{\lambda}{11} s^{-1},$$  \hspace{1cm} (24)

where $\lambda = 2\pi/k$ is the wavelength, and $\lambda_{11}$ has units $10^{11}$ cm; $n_e$ are in units $100$ cm$^{-3}$. Assuming that gas is weakly ionized and $n \gg n_e$, we can estimate

$$|\omega_{sh}| > |\omega_H|.$$  \hspace{1cm} (25)

Assuming $|\partial \Omega/\partial R| \sim \Omega/R$ as is usual in astrophysical disks, we can represent the necessary condition (26) as

$$B < \frac{\Omega}{\beta k^2} < 10^{-3} P^{-1} n_e \lambda_{11}^2$$  \hspace{1cm} (27)

where $P = 2\pi/\Omega$ is the rotation period, $P$ has units 1 yr. The condition that the Hall effect dominates ohmic dissipation, $a_e > 1$, yields

$$B > 0.048 \ n_{14} T_{2}^{1/2}$$  \hspace{1cm} (28)

The conditions (27) and (28) are consistent only if

$$P_{yr} < 2.2 \times 10^{-2} f_{-12} \lambda_{11}^{-2} T_{2}^{-1/2}.$$  \hspace{1cm} (29)

where $f_{-12} = f/10^{-12}$. If we estimate the thickness of a disk, $H$, at the distance $R$ as $H \sim 0.1 R$ and assume that $\lambda \sim 0.1 H$ to justify a short wavelength approximation, then we have at $R = 1$ AU,

$$P_{yr} < 2.2 \times 10^{-2} f_{-12} T_{2}^{-1/2}.$$  \hspace{1cm} (30)

To be fulfilled this condition requires the temperature lower than $100$ K and ionization higher than $10^{-12}$. Therefore, the condition (29) seems to be very restricting and hardly to be fulfilled in the conditions of protostellar disks.
4.3. The condition $A_2 < 0$

Substituting the expressions for coefficients, we can represent the condition $A_2 < 0$ as

$$\omega_g^2 + 2\omega_R^2 + 2 \left( 2\Omega \frac{k_z}{k} - \omega_H \right)^2 < 0.$$  \hfill (31)

All terms on the left hand side are positive except $\omega_g^2$ and, hence, the criterion (31) can be satisfied only if $\omega_g^2 > 0$. However, this criterion requires larger negative $\omega_g^2$ than the condition (10). Therefore, the criterion $A_2 < 0$ can be fulfilled only if the condition $a_0 < 0$ is already fulfilled.

4.4. The condition $A_3 < 0$

This condition generalizes the criterion obtained by Balbus and Terquem (2001) for the protostellar disk with no gravity. If we assume $q = 0$ (and, hence, $\omega_g^2 = a_0 = 0$) then $A_3 = a_3 a_1 A_2$. Since $a_3$ is always positive and $A_2$ is positive at $g = 0$ (see Eq. (31)), the criterion $A_3 < 0$ reduces to

$$a_1|_{g=0} \equiv b_0 < 0,$$ \hfill (32)

that is the condition derived by Balbus & Terquem (2001).

In the general case, the criterion $A_3 < 0$ yields

$$D \equiv b_0 + q \left\{ \left( \omega_R^2 + \omega_H^2 + \omega_H \omega_{sh} \right) \omega_g^2 + Q^2 - \omega_A^2 \right\}$$

$$+ \omega_R^2 + 2 \left( 2\Omega \frac{k_z}{k} - \omega_H \right)^2 - \omega_A^2 - \omega_H \omega_{sh}$$

$$+ \omega_A^2 \left( \frac{1}{2} \omega_g^2 + Q^2 \right) \right\} < 0,$$ \hfill (33)

where

$$q = \frac{1}{2} \frac{\omega_g^2}{\omega_R^2 + \left( 2\Omega \frac{k_z}{k} - \omega_H \right)^2}.$$ \hfill (34)

Gravity can provide either positive or negative contribution to the left hand side of Eq. (33) and, hence, can be either a stabilizing or destabilizing factor depending on the characteristic frequencies. The condition (33) is generally rather cumbersome, and we consider only the particular cases of astrophysical interest.

In protostellar disks, we have typically $\omega_k \sim \Omega$ and, hence, $q \sim 1$. As it has been adopted in previous studies, we also assume that the Hall effect dominates ohmic dissipation, and $\omega_H > \omega_R$ (or $a_e > 1$),

$$\omega_R \approx 9.2 \times 10^{-6} f^{-1} T_{12}^{1/2} \lambda_{11}^{1/2} P_{yr}^{-1}.$$ \hfill (35)

Stability properties are sensitive to the relationship between the Hall frequency, $\omega_H$, and the angular velocity, $\Omega$. The inequality $\omega_H > \Omega$ is fulfilled only if the field is strong enough,

$$B \gg 10^{-3} n_{e2} \lambda_{11}^{3/2} P_{yr}^{-1}.$$ \hfill (36)

This field seems to be rather strong for the conditions of protostellar disks and, most likely, $\Omega > \omega_H$. Note also that at $\omega_H > \Omega$ we have $D > 0$ for both $\omega_H > \omega_A$ and $\omega_A > \omega_H$.

Hence, the instability does not appear, and the field, satisfying Eq. (36), is stabilizing for instability given by the criterion $A_3 < 0$. Therefore, we consider Eq. (33) only at $\Omega > \omega_H$.

If this is the case, the criterion of instability reads

$$\omega_H (\omega_H + \omega_{sh}) \left[ (1 + q)(\omega_g^2 + Q^2) - \omega_A^2 \right] + \omega_A^2 \left[ 2 \Omega \frac{k_z}{k} + \omega_{sh} \right] +$$

$$+ 2 \left( 2\Omega \frac{k_z}{k} + \omega_{sh} \right) + 2 \Omega \frac{k_z}{k} (\omega_H + \omega_{sh}) +$$

$$+ q \omega_A^2 \left( \frac{1}{2} \omega_g^2 + Q^2 \right) < 0.$$ \hfill (37)

As it was mentioned (see Eq. (26)), the Hall frequency can exceed the Alfvén frequency in protostellar disks, $\omega_H > \omega_A$, and we treat this case first. Under this assumption, the criterion (37) yields

$$\omega_H (\omega_H + \omega_{sh}) (1 + q)(\omega_g^2 + Q^2) < 0.$$ \hfill (38)

In a convectively stable disk with $\omega_g^2 > 0$ and $Q^2 > 0$, the shear-driven instability arises only if

$$\omega_H (\omega_H + \omega_{sh}) < 0.$$ \hfill (39)

To be fulfilled, this inequality requires a strong shear, satisfying Eq. (26), and a wave vector directed in accordance with Eq. (23). There is no instability due to the Hall effect without shear. It appears that the condition $\omega_{sh} > \omega_H$ is general for instabilities caused by a joint influence of the Hall effect and shear but, as we already mentioned, it is very restricting in protostellar disks. The sign of $\omega_{sh}$ plays no crucial role in the condition (39), and instability can appear for any $\partial \Omega / \partial R$ if Eq. (26) is fulfilled. Note that in the case $g = 0$, the instability can arise for any sign of shear as well (Balbus & Terquem 2001).

Consider now the criterion of instability (37) in the case when the Hall frequency is small compared to the Alfvén frequency, $\omega_A > \omega_H$. We again assume $\Omega > \omega_H$ but the Alfvén frequency can generally be larger or smaller than $\Omega$. If $\omega_A > \Omega$, or

$$B > 0.15 n_{i1}^{1/2} \lambda_{11} P_{yr}^{-1},$$ \hfill (40)

then $D \approx \omega_A^2 > 0$, and instability is not possible. The field, satisfying this condition, is typically even stronger than that given by (36). Unlike that such a strong field can exist in protostellar disks, and we consider instability under a more realistic condition $\Omega > \omega_A$.

If $\Omega > \omega_A > \omega_H$, then we have from the condition (37)

$$(1 + q) \left( \omega_g^2 + 4 \Omega \frac{k_z}{k} \omega_{sh} \right) < 0.$$ \hfill (41)

In a convectively stable disk, the instability appears if

$$4 \Omega \frac{k_z}{k} \omega_{sh} + \omega_g^2 < 0.$$ \hfill (42)

At $\omega_k \sim \Omega > \omega_A$, this condition is far more restricting than the criterion of the magnetorotational instability obtained by Balbus & Terquem (2001) for protostellar disks.

If $\Omega > \omega_H / \omega_A$, then we recover the instability criterion (38).
5. The growth rate of instability

A general analysis of the roots of Eq. (12) is very complicated. However, simple expressions for the roots can be obtained in some cases of astrophysical interest. Consider initially Eq. (12) in the case when the ohmic dissipative “frequency”, $\omega_R$, is small compared to other characteristic frequencies. Then, with accuracy in terms of the zeroth order in $\omega_R$, four roots corresponding to rapidly varying modes (either growing, or decaying, or oscillating) can be obtained from Eq. (12) with neglected ohmic dissipation. These roots satisfy the approximate equation

$$\gamma^4 + c_2 \gamma^2 + c_0 = 0,$$

where

$$c_2 \approx \omega_R^2 + Q^2$$

$$c_0 = \omega_A^2 \left( \omega_R^2 + 2\Omega \omega_{sh} \right) + \omega_H \left( \omega_H + \omega_{sh} \right) \left( \omega_R^2 + Q^2 \right).$$

The fifth mode describes a secular instability and varies on a long time scale proportional to $\omega_R$. For this mode, we have with the accuracy in terms of the lowest order in $\omega_R$,

$$\gamma_5 \approx -\frac{a_0}{a_1} \approx -\frac{1}{c_0} \omega_R \omega_A^2 \omega_R^2.$$

If $\omega_A^2 \gg |\omega_H (\omega_H + \omega_{sh})|$, then Eq. (46) yields a simple estimate for $\gamma_5$,

$$\left| \frac{\gamma_5}{\Omega} \right| \sim 44 f_{-12} T_2^{-1/2} \lambda_{11}^{-2} P_{yr}.$$ (47)

The solution of Eq. (43) is given by

$$\gamma^2 = -\frac{c_2}{2} \pm \sqrt{\frac{c_2^2}{4} - c_0}.$$

Note that the true expressions for $\gamma$ have to contain also small corrections, $\Delta \gamma$, proportional to $\omega_R$ which are neglected in Eq. (48). If $\Omega > \max (\omega_H, \omega_A)$ as the criteria of instability requires then $c_2^2 \gg 4c_0$, and we obtain

$$\gamma_{1,2}^2 = \frac{c_0}{c_2} \quad \gamma_{3,4}^2 \approx -c_2.$$

(49)

The modes 1 and 2 are relevant to the Hall-driven instability whereas the modes 3 and 4 describe convection and are stable in convectively stable disks. One of the modes $\gamma_{1,2}$ is unstable if $c_0 < 0$. Note that the secular mode $\gamma_{5}$ is unstable as well under this condition, and there exist two qualitatively different instabilities represented by the same criterion.

In the case $\Omega \sqrt{\omega_R / \Omega} > \omega_A$, we have for the Hall-driven modes

$$\gamma_{1,2}^2 \approx -\omega_H (\omega_H + \omega_{sh}).$$

If the condition (49) is fulfilled then one of these modes is unstable. Since $\omega_{sh} \sim \Omega$, we can estimate

$$\left| \frac{\gamma_{1,2}}{\Omega} \right| \sim \sqrt{\frac{\omega_H}{\Omega}} \sim 31 B^{1/2} n_{e2}^{-1/2} \lambda_{11}^{-1} P_{yr}^{1/2}.$$ (51)

If $\Omega > \omega_A > \Omega \sqrt{\omega_H / \Omega}$ then we obtain

$$\gamma_{1,2}^2 \approx -\frac{\omega_A^2}{\omega_R^2 + Q^2} \left( \omega_R^2 + 2\Omega \frac{k_z}{k} \omega_{sh} \right).$$ (52)

This is the known dispersion equation for magnetic shear-driven modes (see, e.g., Urpin 1996) in the limit $\omega_k \gg \omega_A$, and one of these modes is unstable if

$$2\Omega \frac{k_z}{k} \omega_{sh} < -\omega_R^2.$$ (53)

The growth rate is of the order of $\omega_A$. Note that the condition (53) differs from the more general condition (49) that predicts instability also within the range

$$-\omega_R^2 < 2\Omega \frac{k_z}{k} \omega_{sh} < -\frac{1}{2} \omega_R^2.$$ (54)

The difference originates from the approximate expressions (49) neglecting terms proportional $\omega_R$. If the inequality (53) is not fulfilled, we have $c_0 > 0$, and the roots $\gamma_1$ and $\gamma_2$ become imaginary. Therefore, their stability at $c_0 > 0$ is determined by small real corrections $\propto \omega_R$. These corrections can be calculated by making use of the standard perturbation procedure. For the modes 1 and 2, corrections are same and given by

$$\Delta \gamma_{1,2} \approx \frac{1}{2c_0} (a_0 - 2a_4 c_0) = -\frac{\omega_H \omega_A^2}{2c_0} \left( 4\Omega \frac{k_z}{k} \omega_{sh} + \omega_R^2 \right).$$ (55)

These corrections are positive under the condition (54) and lead to instability in complete agreement with the criterion (49). However, the instability caused by small resistive corrections is oscillatory and qualitatively different from the magnetic shear instability. Note that within the range (53) the secular mode (46) is stable.

Consider now the roots of Eq. (12) assuming that the dissipative “frequency”, $\omega_R$, is large compared to other characteristic frequencies including $\omega_H$. Most likely, that the angular velocity, $\Omega$, is largest among other frequencies, and the assumption $\omega_H > \Omega$ implies that

$$P_{yr} > 2.2 \times 10^{-2} f_{-12} \lambda_{11}^2 T_2^{-1/2}.$$ (56)

Probably, such inequality can be fulfilled in some regions of protostellar disks if ionization is low. If $\Omega > \omega_A$, then the coefficients of Eq. (12) are given by

$$a_1 \approx 2\omega_R, \quad a_3 \approx \omega_R^2, \quad a_2 \approx a_4 (\omega_R^2 + Q^2),$$

$$a_1 \approx \omega_R a_2 / 2, \quad a_0 \approx \omega_R^2 \omega_A^2 \omega_R^2.$$

Four roots of the Eq. (12) with such coefficients have negative real parts, and only one root can correspond to instability. With the accuracy in terms of the lowest order in $\omega_R^{-1}$ this root is given by

$$\gamma \approx -\omega_A^2 \omega_R^2 \left( \frac{\omega_A^2}{\omega_R^2 + Q^2} \right).$$ (57)

The growth rate is positive if $\omega_R^2 < 0$. As it was argued above (see Eq. (16)), the instability can appear under this condition if $G$ and $\Delta \nabla T$ are not parallel. In protostellar disks, the growth rate of this instability is probably relative small, $\gamma \sim \omega_A (\omega_A / \omega_R) (H / R)^2$. This instability, however, can exist if ionization is extremely low, and the magnetic Reynolds number is small even for rotation.
6. Conclusions

We have considered the stability properties of protostellar disks. Compared to other studies we took into account consistently gravity and buoyancy forces which play an important role in dynamics of disks. It has been shown that, under certain conditions, magnetic protostellar disks are unstable to different kinds of shear-driven instabilities. Some of these instabilities are relevant to the Hall effect but some can manifest themselves even if the Hall currents are negligible. All considered instabilities are influenced by the magnetic field, and a relatively strong field can suppress instabilities. The strength of the field which can stabilize the flow depends generally on the conditions in disks and on the wavelength of the perturbation. The stabilizing field is determined by the conditions (36) and (40) and is typically relatively strong.

The type of instability that is more efficient depends very much on the conditions in protostellar disks and the wavelength of perturbations, and can be different in different regions of the disk. The instability associated with the criterion (16) seems to be most general in protostellar disks. It can occur both in magnetized ($a_e > 1$) and non-magnetized ($a_e < 1$) regions and requires only non-parallel $G$ and $\Delta \nabla T$. The growth rate of this instability is given by equations (46) and (57) in the limiting cases of small and large $\omega_R$, respectively. Note that the baroclinic instability appears if dissipative processes are taken into account and, therefore, it can also exist in accretion disks. In the present study, the baroclinic convection is caused by the electrical resistivity and magnetic field but it can also exist, for example, in non-magnetic disks if one takes account of the thermal conductivity (see Urpin & Brandenburg 1998). The baroclinic convection, however, can be not the most efficient instability operating in protostellar disks since the inequality (16) is fulfilled only for a very narrow range of $k$ almost parallel to $G$. In magnetized regions with $a_e > 1$, other instabilities can manifest themselves as well if the wavelength of perturbations is sufficiently long. For the purpose of illustration, we consider the domains of different instabilities in the case when the wave vector is not parallel or perpendicular to $\Omega$ and $B$ and, hence, $k_z \sim k$ and $(k \cdot B) \sim kB$. Then, the regions of instability are determined by two characteristic wavelength, $\lambda_H$ and $\lambda_A$, corresponding to the conditions $\omega_H = \Omega$ and $\omega_A = \Omega$, respectively. These wavelengths are

$$\lambda_H = 3.1 \times 10^{12} B^{1/2} n_e^{-1/2} P_{yr}^{1/2} \text{ cm},$$

$$\lambda_A = 6.6 \times 10^{11} B n_{14}^{-1/2} P_{yr} \text{ cm},$$

and their ratio is given by

$$\xi = \frac{\lambda_A}{\lambda_H} \approx 0.21 B^{1/2} f^{-1/2} P_{yr}^{1/2}.$$  (60)

If $\xi \gg 1$ then for $\lambda \gg \lambda_A$ (that is equivalent to $\Omega \gg \omega_A \gg \Omega/\sqrt{\omega_H/\Omega}$) a particular case of the magnetic shear instability may occur with the growth rate given by Eq. (52). If $\xi < 1$ and $\lambda \gg \lambda_H$ (that is equivalent to $\Omega \gg \Omega/\sqrt{\omega_H/\Omega} \gg \omega_A$) then the Hall-driven shear instability can arise with the growth rate (50). Note that the both these instabilities co-exist with the baroclinic convection in their domains of instability but the latter can also occur for shorter $\lambda$.

In the present paper, we have addressed the behavior of only axisymmetric perturbations. It is clear, however, that the obtained results can apply to nonaxisymmetric perturbations with the azimuthal wavelength much longer than the vertical or radial ones, $\min(k_R, k_z) \gg k_\phi$. The turbulence that could be generated by the considered instabilities may be strongly anisotropic in the $(R, z)$-plane, because the instability criteria are very sensitive to the direction of the wave vector. However, the generated turbulence may be efficient in the radial transport of angular momentum.

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Fig. 2. The same as in Fig. 1 but only for $n_{14} = 1$....
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