Cooperative Multi-Sender Index Coding

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Abstract

In this paper, we establish new capacity bounds for the multi-sender unicast index-coding problem. We first revisit existing outer and inner bounds proposed by Sadeghi et al. and identify the suboptimality of their inner bounds in general. We then present an alternative multi-sender maximal-acyclic-induced-subgraph outer bound that simplifies the existing one. For the inner bound, we identify shortcomings of the state-of-the-art partitioned Distributed Composite Coding (DCC) in the strategy of sender partitioning and in the implementation of multi-sender composite coding. We then modify the existing sender partitioning by a new joint link-and-sender partitioning technique, which allows each sender to split its link capacity so as to contribute to collaborative transmissions in multiple groups if necessary. This leads to a modified DCC (mDCC) scheme that is shown to outperform partitioned DCC and suffice to achieve optimality for some index-coding instances. We also propose cooperative compression of composite messages in composite coding to exploit the potential overlapping of messages at different senders to support larger composite rates than those by point-to-point compression in the existing DCC schemes. Combining joint partitioning and cooperative compression proposed, we develop a new multi-sender Cooperative Composite Coding (CCC) scheme for the problem. The CCC scheme improves upon partitioned DCC and mDCC in general, and is the key to achieve optimality for a number of index-coding instances. The usefulness of each scheme is illuminated via examples, and the capacity region is established for each example.

Index Terms

Composite coding, cooperative compression, index coding, random binning.

I. INTRODUCTION

Index coding addresses efficient broadcast problems where multiple receivers each wish to decode some messages from a common channel, and they each know a subset of messages a
priori. Originally motivated from satellite communications \[1\], \[2\], the index-coding problem is shown to have rich connections with network coding \[3\], \[4\], coded caching \[5\], distributed storage \[6\], \[7\] and topological interference management in wireless communications \[8\].

A. Background and Related Works

In the classic index-coding setup, one sender encodes a set of messages and broadcasts the codeword to multiple receivers through a noiseless channel. Each message is requested by only one receiver, and each receiver requests only one message. The aim is to find the optimal broadcast rate (i.e., the normalized codeword length) or the capacity region such that each receiver can correctly decode what it wants given what it knows. This problem is referred to as the single-sender unicast index-coding problem \[1\], \[2\], \[9\]–\[18\].

Most studies above cast the problem into side-information digraphs and derived bounds on the broadcast rate using graph-related quantities, such as the size of maximum acyclic induced subgraph (MAIS) for the lower bound \[2\], and the min-rank over digraph-induced matrices \[2\], the (partial) clique-covering number \[1\], the cycle-covering number \[10\], and the local chromatic number \[11\] for upper bounds. The optimality of each approach was proved for certain classes of graphs, but none of them is optimal in general. Recently, Thapa et al. \[16\] introduced the interlinked-cycle structure (a general form of overlapping cycles) and proposed an interlinked-cycle-cover (ICC) scheme using scalar linear index codes. It has been proved that the ICC scheme is optimal for a class of digraphs and can outperform all schemes above for certain digraphs.

Arbabjolfaei et al. \[12\], \[13\] instead viewed the problem information-theoretically and developed bounds using tools such as random coding (binning) arguments from network information theory. In particular, they devised a layered random coding scheme for the achievability, in which the sender first enumerates all possible combinations of the messages (i.e., “composite messages”) and encodes each composite message into a “composite” index at an appropriate rate by random binning, and then uses flat coding to encode the composite indices and broadcast to the receivers, while upon reception, each receiver leverages its side information, to first retrieve all composite indices, and then to decode the desired messages from the relevant composite indices. The scheme is termed as composite coding, and is shown to achieve the optimal broadcast rate \[13\] and the information capacity region \[12\] asymptotically (as the message size tends to infinity) for all
unicast index-coding problems with five or fewer receivers. However, this composite coding is still not optimal in general, as shown by a 6-receiver instance [16] and a 7-receiver instance [8].

Single-sender multicast index-coding problems (where each message may be requested by more than one receiver) have also been studied in the literature using both graph theory [10], [19]–[21] and rate distortion theory [22]. In particular, the optimal broadcast rate was established [22] for a number of multicast scenarios, including all instances with up to three receivers. But the general multicast index-coding problem has been shown to be NP-hard [10].

In the setups above, all messages are assumed to be stored and transmitted by a central sender. However, in various scenarios of interest, messages might be distributed across multiple senders, and users might be served jointly by these senders. For instance, in satellite communications, multiple satellites with local messages might jointly serve multiple clients on the downlink for better coverage. As another example, in the video-content-driven 5G heterogenous networks [23], caching parts of video content (i.e., messages) at distributed small-cell base stations and at user devices during off-peak hours has great potential in alleviating the load of the backhaul network and reducing end-to-end transmission delay during peak hours. Assume that video content is already properly cached. Unleashing the potential of cached video calls for design of efficient transmission schemes from multiple senders with cached-messages to multiple users each with some side-information, i.e., a general multi-sender index-coding problem.

Ong et al. [24] were the first to investigate an instance of such a setting. In particular, they proposed the joint use of information-flow graph and message graph to represent a multi-sender index-coding problem, and developed lower and upper bounds on the optimal index-code length for the multicast single-uniprior instances. It was shown that the bounds coincide for the special case where no two distinct senders have any messages in common. More recently, Thapa et al. [25] extended the single-sender version of the cycle-cover, clique-cover and local-chromatic number schemes to the two-sender unicast problem, and established the optimal broadcast rate under certain combinations of side-information graph and message graph.

Sadeghi et al. [26] considered a general multi-sender unicast setting, where there are $2^N - 1$ senders each containing a different subset of the $N$ messages in the system and each connected to all $N$ receivers by noiseless broadcast links of arbitrary finite capacity. Both inner and outer bounds on the capacity region have been proposed for the problem studied. In particular, the inner bound was attained by a distributed version of aforementioned composite coding [12], which
in its general form consists of partitioning senders into non-overlapping sender-groups, solving the composite coding problem for each sender-group, and then combining the corresponding achievable rates. This scheme was referred to as partitioned Distributed Composite Coding (DCC). It was indicated that partitioned DCC with all senders in the same group suffices to achieve the capacity region for all non-isomorphic index-coding instances with \( N = 3 \) messages for arbitrary link capacities \([26, \text{Section IV.B}]\). In addition, partitioned DCC with appropriate sender grouping was shown to be useful to achieve the sum-rate outer bound through an example with \( N = 4 \) messages and 15 senders each with unit link capacity \([26, \text{Section IV.C}]\). However, we show that for \( N = 4 \) messages, partitioned DCC can be sub-optimal even for an index-coding instance with only two senders (meaning that the link capacities associated with the remaining senders in the system are set to zero), see Example later in Section V.

B. Our Contributions

In this paper, we generalize partitioned DCC and develop new achievable schemes and new bounds on the capacity region for the multi-sender index-coding problem. We focus on the unicast setting with \( N \) messages and assume that there are \( K \) active senders in general, each connected to all receivers by noiseless broadcast links of arbitrary finite capacity.

We first present a multi-sender MAIS outer bound that simplifies the existing one \([26]\). Next, we revisit the partitioned DCC scheme \([26]\), and identify its limitations via several examples. We first observe that in partitioned DCC, each sender belongs exclusively to a sender-group and exhausts its link capacity for the transmission in that group. This strategy turns out to be sub-optimal in some instances, as it precludes senders carefully allocating their resources and contributing to collaborative transmissions in different groups.

To overcome this shortcoming, we propose a joint link-and-sender partitioning technique, in which each sender is allowed to split its link capacity appropriately so as to participate in multiple link-sender groups if necessary. This joint partitioning technique contains the existing sender partitioning approach as a special case. It also contains another extreme, link partitioning only, as a special case, where all senders are in the same group, but their link capacities can

\[1\]

We would like to point out that, in fact, partitioned DCC can achieve the entire capacity region for this particular example, if the second and the fifth groups \([25, \text{Table II}]\) are combined as a single group (see discussions in Example here). The converse proof needs customized Shannon-type inequalities as shown in Appendix in this paper.
still be split to accommodate different transmission strategies (e.g., different decoding strategies at some receivers). In the special case that all senders always allocate the same fraction of their link capacities to each transmission strategy, link partitioning in this circumstance is equivalent to a time-sharing technique among different transmission strategies. But this equivalence is not true in general. By the use of joint link-and-sender partitioning, we develop a modified DCC (mDCC) scheme, which is shown to strictly improve partitioned DCC and suffice to achieve capacity for a number of index-coding instances.

We also observe that in partitioned DCC, within each sender-group, a sender always treats its composite messages as independent source data and employs point-to-point compression (binning) to generate composite indices to broadcast to receivers, regardless of whether or not some messages might be common to two or more senders in the group. This strategy consequently fails to exploit the potential overlapping of messages at different senders for more efficient description of composite messages to each receiver.

To overcome this limitation, we propose a cooperative compression technique, where in the compression of composite messages in composite coding, senders that share a composite message will use the same composite index to represent this message. Each sender then will have a mixed set of private and common composite indices. In this way, the description of composite indices from multiple senders to an arbitrary receiver in the index-coding problem here can be viewed as a Slepian-Wolf-Cover like problem of transmitting multiple correlated sources through a multiple-access channel [27]–[29], with orthogonal links and some side-information at the receiver. In general, cooperative compression is more effective than point-to-point compression in the sense that it can support a larger composite rate region, which in turn leads to a larger message rate region.

We then develop a new achievable scheme for the multi-sender index-coding problem by the combined use of joint link-and-sender partitioning and cooperative compression. The scheme in its most general form consists of forming different link-sender groups by joint link-and-sender partitioning, splitting messages if they appear in different groups, implementing composite coding with cooperative compression of composite messages, solving the composite coding problem for each group and then combining the corresponding achievable rates. We term this new scheme as multi-sender Cooperative Composite Coding (CCC). The CCC scheme improves upon partitioned DCC and mDCC in general, and it is instrumental in achieving the capacity region for a number
TABLE I
A SUMMARY OF NEW mDCC AND CCC PROPOSED IN THIS PAPER AND EXISTING PARTITIONED DCC

| Rate-Region/Highlighted Techniques | Cooperative Compression | Link Partitioning | Sender Partitioning | Proposed by |
|-----------------------------------|-------------------------|-------------------|---------------------|-------------|
| \(\mathcal{R}_\text{DCC-a}\)     | \(\times\)              | \(\times\)       | \(\times\)         | Sec. IV.B, [26] |
| \(\mathcal{R}_\text{DCC}\)       |                        | \(\times\)       |                     | Sec. IV.C, [26] |
| \(\mathcal{R}_\text{mDCC-a}\)    | \(\times\)              | \(\checkmark\)   | \(\times\)         | this paper   |
| \(\mathcal{R}_\text{mDCC}\)      |                        | \(\checkmark\)   |                     |             |
| \(\mathcal{R}_\text{CCC-a}\)     | \(\checkmark\)         | \(\times\)       | \(\times\)         |             |
| \(\mathcal{R}_\text{CCC-b}\)     | \(\checkmark\)         | \(\times\)       | \(\checkmark\)     |             |
| \(\mathcal{R}_\text{CCC-c}\)     | \(\checkmark\)         | \(\checkmark\)   | \(\times\)         |             |
| \(\mathcal{R}_\text{CCC}\)       | \(\checkmark\)         | \(\checkmark\)   | \(\checkmark\)     |             |

Fig. 1. Performance comparison of different schemes: The arrow from \(\mathcal{R}_i\) to \(\mathcal{R}_j\) indicates that \(\mathcal{R}_i \subseteq \mathcal{R}_j\), and the inclusion can be strict for some index-coding instances, as shown by relevant examples. For notational convenience and if no ambiguity is caused, we simply use “\(\mathcal{R}_i \subset \mathcal{R}_j\)” to denote the relationship. Note that this relationship is transitive, i.e., whenever \(\mathcal{R}_i \subset \mathcal{R}_j\) and \(\mathcal{R}_j \subset \mathcal{R}_k\), then also \(\mathcal{R}_i \subset \mathcal{R}_k\).

Table I provides a summary of new mDCC and CCC proposed, existing partitioned DCC [26] and their special cases with the use of only some components in the full versions. The reason
why we also highlight the special cases is that they sometimes suffice to achieve the capacity region, as shown in Section [IV] and Section [V]. The performance comparison of different schemes is depicted in Fig. [1].

It is noted that very recently, Liu et al. proposed a new fractional DCC scheme in an independent and parallel work [30]. Fractional DCC includes the concept of joint link-and-sender partitioning proposed in this paper and also includes a new strategy to accommodate more flexible composite-rate allocation among different decoding choices. This new strategy subsumes the time-sharing strategy used in mDCC as a special case. Hence, fractional DCC improves upon mDCC in general, but a disadvantage of fractional DCC is its increase in the computational complexity to characterize a full achievable rate region [30]. In addition, we have verified that fractional DCC still fails to achieve the capacity region for the two-sender index-coding instance in Example [2] due to the lack of cooperative compression for composite coding, while CCC is able to achieve the capacity region for this instance. Conceptually, incorporating the new composite-rate allocation strategy [30] into CCC would lead to another new advanced scheme. However, given the significant increase in computational complexity for numerical evaluations, it remains unclear under what circumstance the advanced scheme as envisioned can outperform CCC. This is an interesting direction for future work.

The rest of the paper is organized as follows. Section [II] formalizes the multi-sender index coding problem considered. Section [III] revisits the key ideas and some related results in [26]. Sections [IV], [V] detail the development of new achievable schemes and the establishment of new bounds/capacity results in this work. Finally, concluding remarks are given in Section [VI].

Notation: For a pair of integers $z_1 \leq z_2$, we use notation $[z_1 : z_2]$ to denote the discrete interval \{ $z_1$, $z_1 + 1$, $\ldots$, $z_2$ \}. More generally, for any real number $c \geq 0$ and an integer $z_1 \leq 2^c$, we define $[z_1 : 2^c] = \{ z_1, z_1 + 1, \ldots, 2^\lceil c \rceil \}$, where $\lceil . \rceil$ is the conventional ceiling function. Notation $\prod_{l \in \mathcal{A}} M_l$ is used to denote the Cartesian product of sets $\{M_l, l \in \mathcal{A} \}$. Notation $|\mathcal{A}|$ denotes the cardinality of a set $\mathcal{A}$. Notation $\mathbb{R}_+^n$ denotes the set of nonnegative real vectors in $n$ dimensions. Finally, all the sets used in the paper are ordered sets.

II. Problem Setup and Definitions

We consider a multi-sender unicast index-coding problem that consists of the following:

- $N$ independent messages, the collection of which is denoted by $\mathcal{M} = \{M_1, M_2, \ldots, M_N\}$;
• $K$ senders: each sender is indexed by a scalar $k$, $k \in [1 : K]$, and the $k$-th sender having message indices $S_k \subseteq [1 : N]$ is also referred to as sender $S_k$. These two sender descriptions are used interchangeably in the paper, and each is useful in different derivations. Then the messages available at sender $S_k$ can now be represented by $M_{S_k} = \{M_j, j \in S_k\} \subseteq \mathcal{M}$. Without loss of generality, assume that $M_{S_{k_1}} \neq M_{S_{k_2}}$, $\forall k_1 \neq k_2$, and $\bigcup_{k=1}^{K} M_{S_k} = \mathcal{M}$;  
• $N$ receivers: each receiver $j$ ($j \in [1 : N]$) knows messages $\mathcal{M}_{A_j}$ a priori, i.e., a subset of $\mathcal{M}$ indexed by $A_j \subseteq [1 : N] \setminus \{j\}$, and requests message $M_j$;  
• $K$ broadcast links: Each sender $S_k$ is connected to all receivers via a noiseless broadcast link of an arbitrary link capacity $C_k > 0$ in bits/channel use (bcu).

Note that each sender contains a distinct subset of the messages $\mathcal{M}$. Therefore, with $N$ messages, the maximum number of admissible senders is $K_{\text{max}} = 2^N - 1$, and thus we have $1 \leq K \leq K_{\text{max}}$. Fig. 2(a) depicts an example of the multi-sender index-coding problem with $K = 3$ and $N = 4$. It is also noted that Sadeghi et al. [26] considered a model with $K = K_{\text{max}}$ but allowed link capacity $C_k = 0$, i.e., $K_{\text{max}}$ senders are all present but some are inactive. So the problem described above and the one studied by Sadeghi et al. are essentially equivalent.

Given a sender setting, similarly to the single-sender setup, a multi-sender index-coding problem can be described by a receiver side-information digraph, $G$, with $N$ vertices, in which each vertex represents a receiver and an arc exists from vertex $i$ to vertex $j$ if and only if receiver $i$ has message $M_j$ (requested by receiver $j$) as its side information. Fig. 2(b) depicts the side-information digraph for the index-coding instance in Fig. 2(a).

We now define a multi-sender index code for the setup above:

**Definition 1 (Multi-Sender Index Code):** Assume that each message $M_j$ is independently and uniformly distributed over the set $\mathbb{M}_j = [1 : 2^{nR_j}]$, where $n$ denotes the code block length and $R_j$ denotes the information bits per transmission, $j \in [1 : N]$. A $(2^{nR_1}, \ldots, 2^{nR_N}, n)$ multi-sender index code consists of

1) an encoder mapping $f_k$ at each sender $S_k$: 
$$\prod_{i \in S_k} \mathbb{M}_i \rightarrow [1 : 2^{nC_k}],$$  
which maps its messages to an index $L_k \in [1 : 2^{nC_k}]$ sent to all receivers;
2) and a decoder mapping \( g_j \) at each receiver \( j \):

\[
[1 : 2^nC_1] \times [1 : 2^nC_2] \times \cdots \times [1 : 2^nC_K] \times \prod_{i \in A_j} M_i \to \hat{M}_j,
\]

which maps its received indices and its side information to a message estimate \( \hat{M}_j \in M_j \).

The average probability of error is defined as \( P_e^{(n)} = \Pr[(\hat{M}_1, \cdots, \hat{M}_N) \neq (M_1, \cdots, M_N)] \). A rate tuple \((R_1, \cdots, R_N)\) is said to be achievable if there exists a sequence of \((2^{nR_1}, \cdots, 2^{nR_N}, n)\) multi-sender index codes defined as above such that \( P_e^{(n)} \to 0 \) as \( n \to \infty \). The capacity region \( \mathcal{C} \) is the closure of the set of achievable rate tuples.

The goal is to characterize the full capacity region or to establish bounds on the capacity region of this multi-sender index-coding problem.

### III. Preliminaries

For comparison, we revisit the key ideas and re-state related results by Sadeghi et al. \[26\] using our notation for consistency.

The following proposition re-states a simple yet useful outer bound \[26\].

**Proposition 1** (\[26\] Corollary 2): Given any fixed sender configuration and link capacities \( \{C_k, k \in [1 : K]\} \), and for a multi-sender index-coding problem represented by the side-information digraph \( G \), if a rate tuple \((R_1, \cdots, R_N)\) is achievable, then for any \( T \subseteq [1 : N] \), it
must satisfy
\[
\sum_{j \in S} R_j \leq \sum_{k \in [1:K]: S_k \cap T \neq \emptyset} C_k,
\]
for all \( S \subseteq T \) for which the subgraph of \( G \) induced by \( S \) is acyclic.

The outer bound in Proposition 1 was interpreted as a generalized version of the maximal-acyclic-induced subgraph (MAIS) bound from the single-sender problem. In the following, we present an equivalent but simplified version of the MAIS bound above with an alternative proof.

**Corollary 1 (An Equivalent but Simplified MAIS Bound):** Given any fixed sender configuration and link capacities \( \{C_k, k \in [1:K]\} \), and for a multi-sender index-coding problem represented by the side-information digraph \( G \), if a rate tuple \((R_1, \cdots, R_N)\) is achievable, it must satisfy
\[
\sum_{j \in S} R_j \leq \sum_{k \in [1:K]: S_k \cap S \neq \emptyset} C_k,
\]
for all \( S \subseteq [1:N] \) for which the subgraph of \( G \) induced by \( S \) is acyclic.

**Remark 1:** The outer bound in Corollary 1 simplifies that in Proposition 1 in the sense that the former preserves the same set of critical rate constraints induced by all acyclic-subgraph subsets \( S \) of \([1:N]\) as in the latter, but it does not include the additional constraint induced by all pairs of acyclic-subgraph subset \( S \) and subset \( T \subseteq [1:N] \) where \( S \subset T \) as in the latter. These additional constraints turn out to be redundant anyway, because for any subsets \( S \) and \( T \) such that \( S \subset T \), we must have
\[
\sum_{j \in S} R_j \leq \sum_{k \in [1:K]: S_k \cap S \neq \emptyset} C_k \leq \sum_{k \in [1:K]: S_k \cap T \neq \emptyset} C_k.
\]
The proof of Corollary 1 is deferred to Appendix A.

To define the inner bound, we use \( \Pi \) to denote a partition of the set of all senders into disjoint subsets, and \( P \) to denote a subset (a sender-group) in \( \Pi \). Therefore, we must have \( P \neq \emptyset \), \( \bigcup_{P \in \Pi} P = [1:K] \), and if \( P, Q \in \Pi \), then \( P \cap Q = \emptyset \). If \( |\Pi| = 1 \), all senders are in the same group and this is just a trivial partition. In addition, for any \( P \in \Pi \), we use \( S_P \) to denote the union of message indices available at the senders in \( P \), i.e., \( S_P = \bigcup_{k \in P} S_k \). For any \( j \in S_P \), let \( A_{j,P} = A_j \cap S_P \) denote the side information receiver \( j \) has in \( P \), and let \( D_{j,P} \subseteq S_P \), s.t. \( j \in D_{j,P} \), denote the index of messages that receiver \( j \) decodes from the senders in \( P \). Note that by having a receiver \( j \) decode more messages than it requires, that is, \( D_{j,P} \), we may increase the rate \( R_j \).
by decoding and canceling some “interfering” messages. The following proposition re-states an inner bound attained by partitioned DCC.

**Proposition 2 ([26, Section IV.C]):** Consider an arbitrary partition \( \Pi \) of all senders. For each sender-group \( P \in \Pi \), let \( R_{j,P} \) denote the message rate receiver \( j \) can obtain from the senders in \( P \), let \( R_P = (R_{j,P}, j \in S_P) \), and define the rate region \( \mathcal{R}_P \) as the collection of all admissible rate tuples \( R_P \). Under a fixed decoding choice \( \{D_{j,P}, j \in S_P\} \), the following polymatroidal rate region is attained by partitioned DCC for sender-group \( P \):

\[
\mathcal{R}_P = \left\{ R_P \in \mathbb{R}^{\left| S_P \right|} : \begin{align*}
& (a): \sum_{i \in T_j} R_{i,P} \leq \sum_{J_1 \subseteq D_{j,P} \cup A_{j,P}: J_1 \cap T_j \neq \emptyset} \sum_{k : k \in P, J_k \subseteq S_k} \gamma_{J_1,P}^{(k)}, \\
& \text{s.t.} \quad \forall T_j \subseteq D_{j,P} \setminus A_{j,P}, \forall j \in S_P, \\
& (b): \sum_{J_2 \subseteq S_k : J_2 \not\subseteq A_{j,P}} \gamma_{J_2,P}^{(k)} \leq C_k, \\
& \forall j \in S_P, \forall k \in P; \\
& \gamma_{J,P}^{(k)} \geq 0, \forall J \subseteq S_k, \forall k \in P. 
\end{align*} \right\}.
\]

Therefore, by considering all possible decoding combinations for the receivers involved, the following rate region \( \mathcal{R}_P \) is achieved for sender-group \( P \):

\[
\mathcal{R}_P = \bigcup_{\{D_{j,P}, j \in S_P\} \mid \forall j \in S_P} \mathcal{R}_P = \mathcal{R}_P \bigcup_{\{D_{j,P}, j \in S_P\} \mid \forall j \in S_P},
\]

where “\( \bigcup \)” denotes the union operation of multiple rate regions.

After finding the rate region \( \mathcal{R}_P \) for each sender-group \( P \in \Pi \), a combined achievable rate region can thus be obtained by applying the following constraints:

\[
R_j \leq \sum_{P \in \Pi : j \in S_P} R_{j,P}, \quad j \in [1 : N] 
\]

subject to

\[
R_P = (R_{j,P}, j \in S_P) \in \mathcal{R}_P, P \in \Pi
\]

and eliminating \( \{R_{j,P}, j \in S_P, P \in \Pi\} \) through Fourier-Motzkin elimination.

**Remark 2:** The basic idea of partitioned DCC consists of partitioning senders into non-overlapping sender-groups, splitting messages if they appear in different sender-groups, solving
the composite coding problem for each group, and then combining the corresponding achievable rates. Specifically, finding the rate region of (6) corresponds to solving the composite coding problem for a group $P$. In particular, as illustrated in Fig. 3, each sender $S_k$ individually uses point-to-point compression (binning) to map each of its composite messages (denoted by $\mathcal{M}_{J,P}$, $J \subseteq S_k$) to a composite index (denoted by $W_{J,P}^{(k)}$) at rate $\gamma_{J,P}^{(k)} \geq 0$ bau, and all composite indices will then be conveyed to receivers by a flat coding $X_n^k \left( W_{J,P}^{(k)}, J \subseteq S_k \right)$.

Upon reception, in the first step, each receiver $j$ first recovers each sender’s composite indices separately based on its side-information, which hence leads to $|P|$ constraints on link capacities (see (6).b.) by each receiver. In the second step, also based on its side-information, each receiver employs simultaneous nonunique decoding [12] to decode a set of messages $\mathcal{M}_{j,P}$ through a set of relevant composite indices. The resultant achievable rates are bounded by (6).a. The overall achievable rate region for each sender group is the union of rate regions evaluated for all possible decoding choices at the receivers involved (see (7)). A final combined rate region for the index-coding problem is then obtained via (8)–(9) by Fourier-Motzkin elimination [31].

Remark 3: Note that the region $\mathcal{R}_P$ of (7) is not necessarily convex. An additional convex hull operation can be used in (7) after the union operation to convexify the region. This modification can be viewed as the incorporation of a time-sharing strategy [31] among different decoding choices into the scheme and can be beneficial in enlarging the rate region, see Example 1 later.

Remark 4: The achievable rate region under an arbitrary feasible partition as defined above [26]...
Section IV.C is denoted by $\mathcal{R}_{\text{DCC}}$ in this work, see Table I. Specially, when only a trivial partition with all senders in the same group is considered, the resultant rate region [26, Section IV.B] is denoted by $\mathcal{R}_{\text{DCC-a}}$ in Table I. It was indicated [26] that $\mathcal{R}_{\text{DCC-a}}$ corresponds to the capacity region for all non-isomorphic index-coding instances with $N = 3$ receivers (messages) and with arbitrary link capacities for senders. In addition, it was shown that $\mathcal{R}_{\text{DCC}}$ with appropriate sender grouping strictly improves upon $\mathcal{R}_{\text{DCC-a}}$ for an index-coding instance with $N = 4$ receivers and $K = 15$ senders each with unit link capacity (see Example 3 later). However, as we will show in Section IV, partitioned DCC is suboptimal in general. □

IV. A NEW ACHIEVABLE SCHEME: INTRODUCING LINK-PARTITIONING

In this section, we will describe a limitation of the state-of-the-art partitioned DCC scheme in the strategy of sender partitioning and develop a new achievable scheme that can strictly improve partitioned DCC.

A. A Motivating Example

We first examine a 4-sender index-coding instance with $N = 4$ messages defined below.

Example 1: In this example, there are

- $K = 4$ senders, with indices of messages $S_1 = \{1, 3\}$, $S_2 = \{3, 4\}$, $S_3 = \{1, 4\}$, and $S_4 = \{2, 3, 4\}$, and with link capacities $C_1 = 2$, $C_2 = 1$, $C_3 = 2$, $C_4 = 1$ bcu, respectively;
- $N = 4$ receivers, with side-information $A_1 = \{4\}$, $A_2 = \{1, 3\}$, $A_3 = \{1, 2\}$, and $A_4 = \{2, 3\}$, respectively. This corresponds to the side-information digraph depicted in Fig. 2(b).

For this example, we note that the MAIS outer bound of (4) evaluates to

$$\mathcal{R}_{\text{out}}^{\text{MAIS}} = \left\{ \left( R_1, R_2, R_3, R_4 \right) \in \mathbb{R}_+^4 : \begin{array}{l} R_1 \leq 4, \quad R_2 \leq 1, \quad R_3 \leq 4, \\ R_1 + R_3 \leq 6, \quad R_1 + R_4 \leq 6, \\ R_2 + R_4 \leq 4, \quad R_3 + R_4 \leq 6. \end{array} \right\}. \quad (10)$$

To evaluate the inner bound by partitioned DCC in Proposition 2, we consider all possible partitions (15 in total) of the four senders, compute the best possible achievable rate region by considering all possible decoding combinations for the receivers under each partition, and obtain the following:
(i) when \( \Pi = \{\{1, 2, 3\}, \{4\}\} \), i.e., all senders are in the same group, the following rate region is achievable under decoding choice \( D_{1, \Pi} = \{1\} \) and \( D_{j, \Pi} = [1 : 4] \setminus A_j \) for \( j = 2, 3, 4 \):

\[
\mathcal{R}_{\text{DCC}, 1a}^{\Pi} = \left\{ (R_1, R_2, R_3, R_4) \in \mathbb{R}_+^4 : \\
R_1 \leq 4, \quad R_2 \leq 1, \quad R_3 \leq 4, \\
R_1 + R_3 \leq 6, \quad R_1 + R_4 \leq 6, \\
R_2 + R_4 \leq 4, \quad R_3 + R_4 \leq 6, \\
R_1 + R_3 + R_4 \leq 8 \right\} \tag{11}
\]

while the following rate region is achievable under decoding choice \( D_{1, \Pi} = \{1, 3\} \) and \( D_{j, \Pi} = [1 : 4] \setminus A_j \) for \( j = 2, 3, 4 \):

\[
\mathcal{R}_{\text{DCC}, 1b}^{\Pi} = \left\{ (R_1, R_2, R_3, R_4) \in \mathbb{R}_+^4 : \\
R_1 \leq 4, \quad R_2 \leq 1, \\
R_1 + R_4 \leq 6, \quad R_2 + R_3 \leq 4, \\
R_2 + R_4 \leq 4, \quad R_3 + R_4 \leq 6, \\
R_1 + R_2 + R_3 \leq 6 \right\} \tag{12}
\]

Note that none of these two regions strictly contains the other. Hence, by time-sharing these two decoding choices (recall Remark 3), the following region is further obtained:

\[
\mathcal{R}_{\text{DCC}, 1}^{\Pi} = \left\{ (R_1, R_2, R_3, R_4) \in \mathbb{R}_+^4 : \\
R_1 \leq 4, \quad R_2 \leq 1, \quad R_3 \leq 4, \\
R_1 + R_3 \leq 6, \quad R_1 + R_4 \leq 6, \\
R_2 + R_4 \leq 4, \quad R_3 + R_4 \leq 6, \\
R_1 + 2R_2 + R_3 + R_4 \leq 10, \\
3R_1 + 2R_2 + 3R_3 + 2R_4 \leq 24 \right\} \tag{13}
\]

(ii) when \( \Pi = \{\{1, 2, 3\}, \{4\}\} = \{P_1, P_2\} \) with two sender-groups, the following best possible rate region is obtained:

\[
\mathcal{R}_{\text{DCC}, 2}^{\Pi} = \left\{ (R_1, R_2, R_3, R_4) \in \mathbb{R}_+^4 : \\
R_1 \leq 4, \quad R_2 \leq 1, \quad R_3 \leq 4, \\
R_1 + R_3 \leq 6, \quad R_2 + R_4 \leq 4, \\
R_3 + R_4 \leq 6, \quad R_1 + R_2 + R_4 \leq 6, \\
2R_1 + R_3 + R_4 \leq 11 \right\} \tag{14}
\]
where relevant decoding sets are $D_{j,P_1} = S_{P_1} \setminus A_{j,P_1}$ for $j = 1, 3, 4$, and $D_{j,P_2} = S_{P_2} \setminus A_{j,P_2}$ for $j = 2, 3, 4$;

(iii) when $\Pi$ takes other forms of partition, the rate region is always smaller than $R_{\text{DCC},1}^{\text{el}}$.

Since none of $R_{\text{DCC},1}^{\text{el}}$ and $R_{\text{DCC},2}^{\text{el}}$ strictly contains the other, they are the best possible achievable rate regions among different partitions considered. However, there is clearly a gap between these inner bounds and the outer bound \[10\]. In particular, it can be checked that the rate tuple $(R_1 = 3, R_2 = 1, R_3 = 3, R_4 = 3)$ lies in $R_{\text{out}}^{\text{el}}$, but it does not lie in $R_{\text{DCC},1}^{\text{el}}, R_{\text{DCC},2}^{\text{el}}$ or the convex union of these two regions. This hence hints at some potential room left to improve upon the existing partitioned DCC scheme \[16\].

Towards this end, we first digress a little from the current composite coding approach and make some observations from a linear index-coding strategy \[24\], \[25\]. Consider messages $M_j$ ($j = 1, 3, 4$) each split into three parts ($M_j^{(i)}, i = 1, 2, 3$), each of 1 bcu, and message $M_2$ that is also of 1 bcu. Subject to the message availability and link capacity constraints at senders, the following XOR packets can be transmitted: $i)$ $c_1 = [c_{11} \ c_{12}] = [M_1^{(1)} \oplus M_3^{(1)} \ M_1^{(2)} \oplus M_3^{(2)}]$ from sender $S_1$; $ii)$ $c_2 = M_3^{(1)} \oplus M_4^{(1)}$ from sender $S_2$; $iii)$ $c_3 = [c_{31} \ c_{32}] = [M_1^{(2)} \oplus M_4^{(2)} \ M_1^{(3)} \oplus M_4^{(3)}]$ from sender $S_3$; and $iv)$ $c_4 = M_2 \oplus M_3^{(3)} \oplus M_4^{(3)}$ from sender $S_4$. Given these constructions, it can be checked that both pairs of $(c_{11}, c_2)$ and $(c_{12}, c_{31})$ are valid cyclic index codes that exploit the cycle “$1 \rightarrow 4 \rightarrow 3 \rightarrow 1$” in the side-information graph $G$. Thus each receiver $j$ ($j = 1, 3, 4$) can recover its message components $(M_j^{(1)}, M_j^{(2)})$ by combining packets from sender-group $\{S_1, S_2, S_3\}$. In addition, $(c_{32}, c_4)$ is a valid interlinked-cycle-covering (ICC) index code \[16\], which allows each receiver $j$ ($j = 1, 2, 3, 4$) to decode its message component by exploiting its side information upon reception from sender-group $\{S_3, S_4\}$. Therefore, the rate tuple $(R_1 = 3, R_2 = 1, R_3 = 3, R_4 = 3)$ is even achievable with this linear index-coding strategy.

Note that in the linear index-coding strategy described, the senders within each group collaboratively form some index code for the relevant receivers, and more importantly, sender $S_3$ splits its resource and participates in two different groups. Inspired by this observation and returning to the random coding approach focused here, we would argue that, instead of allocating each sender to a unique group as in the existing partitioned DCC, it can be very beneficial to allow some sender to split its link capacity appropriately and collaborate in transmission in more than one sender-group if necessary. We term this strategy as joint link-and-sender partitioning.
Indeed, due to this new modification, we manage to establish the capacity region for the problem in Example 1. For the achievability, we propose to split $C_3 = 2$ into two equal parts as $C_{31} = C_{32} = 1$, form two link-sender groups (say \{S_1, S_2; S_3; C_1 = 2, C_2 = 1, C_{31} = 1\} and \{S_3, S_4; C_{32} = 1, C_4 = 1\}), implement composite coding and solve the composite coding problem for each group as in partitioned DCC, and then combine the corresponding achievable rates. Table II lists the rate region attained by each group and also the combined rate region that matches $R_{\text{out}}^{\text{eq}}$ of (10).

### B. A New Achievable Scheme and A New Inner Bound

In general, a modified DCC (mDCC) scheme is proposed by the use of joint link-and-sender partitioning here in partitioned DCC. To characterize the inner bound attained by the new mDCC scheme, we use $\Pi_L$ to denote a joint partition of all senders and their link capacities \{$C_k, k \in [1 : K]\}$, and use $P_L$ to denote an element (termed as a link-sender group) of $\Pi_L$. Each group $P_L$ is jointly determined by a set of senders’ indices for all senders in $P_L$ and a set of link capacities allocated to transmission in $P_L$. Mathematically, each $P_L$ can thus be represented by $P_L = \{ \tilde{P} \subseteq [1 : K]; C_{\tilde{P}} = \{C_{k,\tilde{P}}, k \in \tilde{P}\}\}$, where $\tilde{P} \neq \emptyset$ is the sender component and $C_{\tilde{P}}$ is the link capacity component with each $C_{k,\tilde{P}} \geq 0$. Note that a joint partition $\Pi_L$ is said to be admissible if i) $\bigcup_{\tilde{P}|P_L \in \Pi_L} \tilde{P} = [1 : K]$; and ii) the following sum link-capacity constraints are

| Link-Sender Groups | Achievable Rate regions | Relevant Decoding Sets |
|--------------------|------------------------|------------------------|
| \{S_1, S_2; S_3; C_1 = 2, C_2 = 1, C_{31} = 1\} \(\tilde{P}_1 = \{1, 2, 3\}\) | $R_1, \tilde{p}_1 \leq 3, R_3, \tilde{p}_1 \leq 3, R_4, \tilde{p}_1 \leq 2,$ $R_1, \tilde{p}_1 + R_3, \tilde{p}_1 \leq 4, R_1, \tilde{p}_1 + R_4, \tilde{p}_1 \leq 4,$ $R_3, \tilde{p}_1 + R_4, \tilde{p}_1 \leq 4.$ | $\mathcal{D}_j, \tilde{p}_1 = S_{\tilde{p}_1} \setminus \mathcal{A}_{j, \tilde{p}_1}$, $j = 1, 3, 4.$ |
| \{S_3, S_4; C_{32} = 1, C_4 = 1\} \(\tilde{P}_2 = \{3, 4\}\) | $R_1, \tilde{p}_2 \leq 1, R_2, \tilde{p}_2 \leq 1, R_3, \tilde{p}_2 \leq 1,$ $R_1, \tilde{p}_2 + R_4, \tilde{p}_2 \leq 2, R_2, \tilde{p}_2 + R_4, \tilde{p}_2 \leq 2,$ $R_3, \tilde{p}_2 + R_4, \tilde{p}_2 \leq 2.$ | $\mathcal{D}_j, \tilde{p}_2 = \{1\};$ $\mathcal{D}_j, \tilde{p}_2 = S_{\tilde{p}_2} \setminus \mathcal{A}_{j, \tilde{p}_2}, j = 2, 3, 4.$ |
| Combined \(R_2 = R_2, \tilde{p}_2, R_j = R_j, \tilde{p}_1 + R_j, \tilde{p}_2, j = 2, 3, 4\) | $R_1 \leq 4, R_2 \leq 1, R_3 \leq 4,$ $R_1 + R_3 \leq 6, R_1 + R_4 \leq 6,$ $R_2 + R_4 \leq 4, R_3 + R_4 \leq 6.$ |
satisfied:

\[ \sum_{P \mid \bar{P} \in \Pi_L, k \in \bar{P}} C_{k, \bar{P}} \leq C_k, \ \forall k \in [1 : K]. \]  

(15)

We also use notation consistent with sender-partitioning in partitioned DCC. In particular, given any \( P_L \) and \( \bar{P} \) of \( P_L \), let \( S_{\bar{P}} = \bigcup_{k \in \bar{P}} S_k \) be the union of message indices at the senders in \( P_L \), let \( A_{j, \bar{P}} = A_j \cap S_{\bar{P}} \) be the side information receiver \( j \) has in \( P_L \), and let \( D_{j, \bar{P}} \subseteq S_{\bar{P}} \), s.t. \( j \in D_{j, \bar{P}} \), be the index of messages that receiver \( j \) decodes from the senders in \( P_L \).

**Proposition 3 (mDCC Rate Region):** Consider an arbitrary admissible joint link-and-sender partition \( \Pi_L \). For each link-sender group \( P_L \in \Pi_L \), let \( R_{P, \bar{P}} \) denote the message rate receiver \( j \) can obtain from the senders in \( P_L \), let \( \mathcal{R}_{P_L} = (R_{j, \bar{P}}, j \in S_{\bar{P}} | P_L) \), and define the rate region \( \mathcal{R}_{P_L} \) as the collection of all admissible rate tuples \( \mathcal{R}_{P_L} \). Under a fixed decoding choice \( \{D_{j, \bar{P}}, j \in S_{\bar{P}}\} \), the following polymatroidal rate region is attained by mDCC for link-sender group \( P_L \):

\[
\mathcal{R} \left( \left\{ D_{j, \bar{P}}, j \in S_{\bar{P}} \right\} \mid \left\{ A_{j, \bar{P}}, j \in S_{\bar{P}} \right\}, \{C_{k, \bar{P}}, k \in \bar{P}\}, P_L \right) = \\
\left\{ \begin{array}{l}
R_{P, \bar{P}} \in \mathbb{R}^{\mid S_{\bar{P}} \mid} : \\
\sum_{i \in T_j} R_{i, \bar{P}} \leq \sum_{J_i \subseteq D_{j, \bar{P}} \cup A_{j, \bar{P}}} \sum_{k, k \in \bar{P}} \gamma^{(k)}_{J_i, \bar{P}}, \\
\forall T_j \subseteq D_{j, \bar{P}} \setminus A_{j, \bar{P}}, \forall j \in S_{\bar{P}}, \\
\text{s.t.} \\
\sum_{J_k \subseteq S_k, J_k \not\subseteq A_{j, \bar{P}}} \gamma^{(k)}_{J_k, \bar{P}} \leq C_{k, \bar{P}}, \\
\forall j \in S_{\bar{P}}, \forall k \in \bar{P}, \\
C_{k, \bar{P}} \geq 0, \ \gamma^{(k)}_{J_k, \bar{P}} \geq 0, \ \forall J_k \subseteq S_k, \forall k \in \bar{P}.
\end{array} \right.
\]  

(16)

Therefore, by considering all possible decoding combinations for the receivers involved, the following rate region \( \mathcal{R}_{P_L} \) is achieved for link-sender group \( P_L \):

\[ \mathcal{R}_{P_L} = \text{conv} \bigcup_{\{D_{j, \bar{P}} \subseteq S_{\bar{P}} : j \in D_{j, \bar{P}} \}} \mathcal{R} \left( \left\{ D_{j, \bar{P}}, j \in S_{\bar{P}} \right\} \mid \left\{ A_{j, \bar{P}}, j \in S_{\bar{P}} \right\}, \{C_{k, \bar{P}}, k \in \bar{P}\}, P_L \right). \]  

(17)

where “\( \text{conv} \bigcup \)” denotes the convex hull of the union of multiple rate regions.

After finding the rate region \( \mathcal{R}_{P_L} \) for each link-sender group \( P_L \in \Pi_L \), a combined achievable rate region can thus be obtained by applying the following constraints:

\[ R_j \leq \sum_{P \mid \bar{P} \in \Pi_L, j \in S_{\bar{P}}} R_{j, \bar{P}}, \ j \in [1 : N] \]  

(18)
\[ \mathbf{R}_{P_L} = (R_{j,\hat{P}, j} \in S_{\hat{P}} | P_L) \in \mathcal{R}_{P_L}, \quad P_L \in \Pi_L \]  
\text{s.t.} \quad \mathcal{R}_{P_L} \quad (19) 
and \quad \sum_{P_L \in \Pi_L} C_{k,\hat{P}} \leq C_k, \quad k \in [1:K], \quad (20)

and eliminating \( \{R_{j,\hat{P}, j} \in S_{\hat{P}} | P_L \in \Pi_L\} \) and \( \{C_{k,\hat{P}}, k \in \hat{P} | P_L \in \Pi_L\} \) through Fourier-Motzkin elimination.

\textit{Proof:} The proof is similar to that for Proposition 2 by replacing the sender-group therein with a link-sender group and details are thus omitted here. \qed

The final combined rate region attained by mDCC is denoted by \( \mathcal{R}_{mDCC} \) here, see Table I.

\textit{Corollary 2:} \( \mathcal{R}_{DCC} \subseteq \mathcal{R}_{mDCC} \) in general, and the inclusion can be strict.

\textit{Proof:} \( \mathcal{R}_{DCC} \) of Proposition 2 can be specialized from \( \mathcal{R}_{mDCC} \), by restricting \( \Pi_L \) to be a \( \Pi_{(sender-partitioning\ only)} \) as defined previously in partitioned DCC, and setting \( C_{k,\hat{P}} = C_k \), i.e., forcing each sender to exhaust its link capacity for transmission within one group. Therefore, \( \mathcal{R}_{DCC} \) is dominated by \( \mathcal{R}_{mDCC} \) in general, and \( \mathcal{R}_{DCC} \subset \mathcal{R}_{mDCC} \) as shown by Example 7. \qed

\textit{Remark 5:} In (17), a time-sharing strategy among different decoding combinations is incorporated by the use of the convex union operation. It is noted that the same region can also be obtained by a link-partitioning argument. For exposition, we introduce \( D_P = \{D_{j,\hat{P}} \subseteq S_{\hat{P}}; j \in D_{j,\hat{P}}, \forall j \in S_{\hat{P}}\} \) as the set of all possible decoding combinations for the receivers within \( \hat{P} \), and its size is \( N_D \). Each decoding combination is now compactly represented by \( D_{n_D} = \{D_{j,\hat{P}}, j \in S_{\hat{P}}\} \in D_P, n_D = 1, \ldots , N_D \). In general, each sender can split its link capacity \( C_{k,\hat{P}} \) as \( C_{k,\hat{P}} = \alpha_1 C_{k,\hat{P}} + \alpha_2 C_{k,\hat{P}} + \cdots + \alpha_{N_D} C_{k,\hat{P}} \), subject to \( \alpha_{n_D} \geq 0 \) and \( \sum_{n_D=1}^{N_D} \alpha_{n_D} = 1 \). Each sender then allocates \( \alpha_{n_D} C_{k,\hat{P}} \) resource to the transmission with decoding combination \( D_{n_D} \). By exhausting different splitting strategies, we can obtain the following rate region \( \mathcal{R}_{P_L} \):

\[ \bigcup_{\{\alpha_1, \ldots , \alpha_{N_D}\} \in \mathbb{R}^N_{+}, \sum_{n_D=1}^{N_D} \alpha_{n_D} \leq 1} \bigg( \bigcup_{D_{n_D} \in D_P} \mathcal{R} \left( D_{n_D} \mid \{A_{j,\hat{P}}, j \in S_{\hat{P}}\}, \{\alpha_{n_D} C_{k,\hat{P}}, k \in \hat{P}\}, P_L \right) \right) \]  
(21)

which is equivalent to the time-sharing induced region of (17). \qed

\textit{Remark 6:} Based on the previous remark, the time-sharing among different decoding choices within a group can be realized in a form of link partitioning. Therefore, even without sender partitioning, the concept of link partitioning alone can also be useful in enlarging the achievable rate region for the multi-sender index-coding problem. Denote the rate region attained by mDCC
with link partitioning only by $R_{\text{mDCC-a}}$, see Table I. It is straightforward to have $R_{\text{mDCC-a}} \subseteq R_{\text{mDCC}}$, and the inclusion can be strict, i.e., $R_{\text{mDCC-a}} \subset R_{\text{mDCC}}$, as shown by Example 7 before.

V. A NEW ACHIEVABLE SCHEME: INTRODUCING COOPERATIVE COMPRESSION

In the previous section, we have improved the state-of-the-art partitioned DCC scheme by the use of joint link-and-sender partitioning. The resultant mDCC scheme has been verified to be optimal for Example 7 as detailed above, and for other instances in our computer-aided studies (details are not shown here). However, it still fails to achieve optimality in some cases. In this section, we will describe a limitation of partitioned DCC and mDCC in the implementation of composite coding and develop a new achievable scheme.

A. A Motivating Example

We first examine a simple 2-sender index-coding example as defined below.

Example 2: In this example, there are

- $K = 2$ senders, with indices of messages $S_1 = \{1, 2, 3\}$, $S_2 = \{2, 3, 4\}$, and with link capacities $C_1 = C_2 = 1$ bcu;
- $N = 4$ receivers, with side-information $A_1 = \{4\}$, $A_2 = \{1, 3\}$, $A_3 = \{1, 2\}$, and $A_4 = \{2, 3\}$, respectively. This again corresponds to the same digraph $G$ as depicted in Fig. 2(b).

For this example, note that the MAIS outer bound of (4) is specialized to

$$R_{\text{eg2 out}} = \left\{ (R_1, R_2, R_3, R_4) \in \mathbb{R}_+^4 : \right\}
\begin{align*}
R_1 &\leq 1, \quad R_4 \leq 1, \\
R_1 + R_2 &\leq 2, \quad R_1 + R_3 \leq 2, \\
R_2 + R_4 &\leq 2, \quad R_3 + R_4 \leq 2.
\end{align*}
$$

(22)

To derive the inner bound, we use Proposition 3 and observe that the best possible achievable rate region is attained by grouping these two senders together and time-sharing between the following two critical decoding choices (a form of link-partitioning only, see Remarks 5–6):

(i) with $D_1 = \{1\}$ and $D_j = [1 : 4] \setminus A_j$ for $j = 2, 3, 4$, the following rate region is achieved:

$$R_{\text{eg2 mDCC}, 1} = \left\{ (R_1, R_2, R_3, R_4) \in \mathbb{R}_+^4 : \right\}
\begin{align*}
R_1 &\leq 1, \quad R_4 \leq 1, \\
R_1 + R_3 + R_4 &\leq 2, \\
R_1 + R_2 + R_4 &\leq 2
\end{align*}
$$

(23)
(ii) with $D_j = [1 : 4] \setminus A_j$ for $j = 1, 2, 3, 4$, the following rate region is achieved:

$$R_{\text{mDCC},2} = \left\{ (R_1, R_2, R_3, R_4) \in \mathbb{R}^4_+ : \right\}
\begin{align*}
R_1 &\leq 1, \quad R_4 \leq 1, \\
R_3 + R_4 &\leq 2, \quad R_2 + R_4 \leq 2, \\
R_1 + R_2 + R_3 &\leq 2
\end{align*}$$

Since none of $R_{\text{mDCC},1}$ and $R_{\text{mDCC},2}$ strictly contains the other, a convex hull of the union of these two regions according to (17) yields the best possible achievable rate region $R_{\text{mDCC}}$ by mDCC for the 2-sender index-coding problem considered:

$$R_{\text{mDCC}} = \left\{ (R_1, R_2, R_3, R_4) \in \mathbb{R}^4_+ : \right\}
\begin{align*}
R_1 &\leq 1, \quad R_4 \leq 1, \\
R_1 + R_2 &\leq 2, \quad R_1 + R_3 \leq 2, \\
R_2 + R_4 &\leq 2, \quad R_3 + R_4 \leq 2, \\
R_1 + R_2 + R_3 + 2R_4 &\leq 4, \\
2R_1 + R_2 + R_3 + R_4 &\leq 4
\end{align*}$$

It can be seen that there is a gap between this inner bound and the MAIS outer bound (22). In particular, a symmetric-rate tuple $(R_j = 1, j \in [1 : 4])$ is in $R_{\text{out}}$, but it is not in $R_{\text{mDCC}}$ above.

However, we again notice that this rate tuple is even achievable with a simple linear index-coding strategy. To see how it works, consider all messages $M_j$ are of 1 bcu. Subject to the message availability and link capacity constraints, sender $S_1$ simply transmits a XOR packet $c_1 = M_1 \oplus M_2 \oplus M_3$, while sender $S_2$ simply transmits a XOR packet $c_2 = M_2 \oplus M_3 \oplus M_4$. Given these constructions, upon reception from $\{S_1, S_2\}$, each receiver can decode its requested message by the use of its side information. Specifically, receiver 1 knows $M_4$, thus it can recover $W_{2,3} = M_2 \oplus M_3$ from $c_2$ and then cancel out $W_{2,3}$ in $c_1$ to decode message $M_1$; receiver 2 knows $\{M_1, M_3\}$ and can decode $M_2$ directly from $c_1$ (then also $M_4$ from $c_2$ as byproduct); similarly, receiver 3 knows $\{M_1, M_2\}$ and thus can decode $M_3$ from $c_1$ (then also $M_4$ from $c_2$ as byproduct), while receiver 4 knows $\{M_2, M_3\}$ and thus can decode $M_4$ from $c_2$ (also $M_1$ from $c_1$ as byproduct). Therefore, the symmetric-rate point $(R_j = 1, j \in [1 : 4])$ is indeed achievable by a linear index code $(c_1, c_2)$ collectively generated by the two senders.

Note that in the linear index code above, it is crucial to group senders $S_1$ and $S_2$ together to achieve the symmetric-rate point. The same grouping strategy is already used in the evaluation
of $\mathcal{R}_{mDCC}^{\mathcal{E}}$, but it is not yet capacity-achieving. We thus suspect that there might be certain limitations rooted in the implementation of composite coding in partitioned DCC and mDCC.

To corroborate this, we now probe into the composite-coding formulation of (16), examine the key composite indices to achieve $\mathcal{R}_{mDCC,1}^{\mathcal{E}}$ or $\mathcal{R}_{mDCC,2}^{\mathcal{E}}$ and uncover why some critical composite rates that would lead to the symmetric-rate point are not supported. In particular, consider the first decoding choice where $D_1 = \{ 1 \}$ and $D_j = [1 : 4] \setminus A_j$ for $j = 2, 3, 4$ in (16). To obtain $\mathcal{R}_{mDCC,1}^{\mathcal{E}}$, sender $S_1$ compresses composite messages $\mathcal{M}_1$ and $\mathcal{M}_{2,3}$ into composite indices $W_{1}^{(1)}$ and $W_{2,3}^{(1)}$ at rates $\gamma_{1}^{(1)}$ and $\gamma_{2,3}^{(1)}$, respectively, and sets the rates of the remaining indices to zero, while sender $S_2$ compresses $\mathcal{M}_4$ and $\mathcal{M}_{2,3}$ to composite indices $W_{4}^{(2)}$ and $W_{2,3}^{(2)}$ at rates $\gamma_{4}^{(2)}$ and $\gamma_{2,3}^{(2)}$, respectively, and sets the rates of the remaining indices to zero in (16). With these choices, constraints (redundant ones are discarded) in (16) read

$$R_1 \leq \gamma_{1}^{(1)}, \quad R_2 \leq \gamma_{2,3}^{(1)} + \gamma_{2,3}^{(2)}, \quad \gamma_{1}^{(1)} + \gamma_{2,3}^{(2)} \leq 1, \quad \gamma_{2,3}^{(1)} + \gamma_{2,3}^{(2)} \leq 1.$$  \hfill (26)

$$R_3 \leq \gamma_{2,3}^{(1)} + \gamma_{2,3}^{(2)}, \quad R_4 \leq \gamma_{4}^{(2)}, \quad \gamma_{2,3}^{(1)} + \gamma_{2,3}^{(2)} \leq 1.$$  \hfill (27)

It can be checked that eliminating $\{ \gamma_{1}^{(1)}, \gamma_{2,3}^{(1)}, \gamma_{4}^{(2)}, \gamma_{2,3}^{(2)} \}$ in the above bounds by Fourier-Motzkin elimination indeed leads to the rate region $\mathcal{R}_{mDCC,1}^{\mathcal{E}}$. Moreover, it can be seen that the sum of composite rates $\gamma_{1}^{(1)}$ and $\gamma_{2,3}^{(1)}$ is constrained by the link capacity of sender $S_1$, while the sum of composite rates $\gamma_{2,3}^{(2)}$ and $\gamma_{4}^{(2)}$ is constrained by the link capacity of sender $S_2$, recalling that each sender individually uses point-to-point compression to compress its composite messages. Therefore, to achieve $R_1 = R_4 = 1$, both $\gamma_{1}^{(1)}$ and $\gamma_{4}^{(2)}$ must be set to one, which in turn forces both $\gamma_{2,3}^{(1)}$ and $\gamma_{2,3}^{(2)}$ to be zero and leads to $R_2 = R_3 = 0$.

However, one may notice that composite message $\mathcal{M}_{2,3}$ is available at both senders. Therefore, as inspired by the linear index-coding strategy described above, instead of compressing $\mathcal{M}_{2,3}$ to different composite indices (i.e., $W_{2,3}^{(1)}$ and $W_{2,3}^{(2)}$), both senders can use the same composite index $W_{2,3}$. In this way, senders $S_1$ and $S_2$ share a common composite index $W_{2,3}$ at rate $\gamma_{2,3}$, and each has a private index, namely, $W_1$ at rate $\gamma_1$ and $W_4$ at rate $\gamma_4$, respectively. Then the description of these indices from the two senders to each receiver can now be viewed as a Slepian-Wolf-Cover like problem of transmitting multiple correlated sources through a multiple-access channel [27]–[29], with orthogonal links and some side-information at the receiver. According to Cover [28, Theorem 2] (also, see Han [29, Section I]), without receiver side-information, if the following
conditions on the composite rates are satisfied:

\[
\gamma_1 \leq 1, \quad \gamma_4 \leq 1, \quad (29)
\]
\[
\gamma_1 + \gamma_4 + \gamma_{2,3} \leq 2, \quad (30)
\]

then all the composite indices can be reliably reproduced at a receiver. Due to the side information at each receiver, the above conditions can be further relaxed to

\[
\gamma_1 \leq 1, \quad \gamma_4 \leq 1, \quad (31)
\]
\[
\gamma_1 + \gamma_{2,3} \leq 2, \quad \gamma_4 + \gamma_{2,3} \leq 2, \quad \gamma_1 + \gamma_4 \leq 2, \quad (32)
\]

for the index-coding instance studied. After recovering all composite indices, each receiver can decode its requested message by nonunique simultaneous decoding, with decoding choice \( D_1 = \{1\} \) and \( D_j = [1:4] \setminus A_j \) for \( j = 2, 3, 4 \) as before. The resultant rate region is characterized by

\[
R_1 \leq \gamma_1, \quad R_2 \leq \gamma_{2,3}, \quad (33)
\]
\[
R_3 \leq \gamma_{2,3}, \quad R_4 \leq \gamma_4, \quad (34)
\]
\[
\gamma_1 \leq 1, \quad \gamma_4 \leq 1, \quad (35)
\]
\[
\gamma_1 + \gamma_{2,3} \leq 2, \quad \gamma_4 + \gamma_{2,3} \leq 2, \quad \gamma_1 + \gamma_4 \leq 2. \quad (36)
\]

Interestingly, it can be seen that all composite rates can now be set to one simultaneously, the symmetric-rate tuple \((R_j, j \in [1:4])\) is therefore achievable. Moreover, it can be checked that eliminating \( \{\gamma_1, \gamma_{2,3}, \gamma_4\} \) in the above bounds by Fourier-Motzkin elimination (all variables involved are non-negative) leads to a new achievable rate region, which agrees with the outer bound \( R_{\text{out}}^{\text{nat}} \) of (22), thus establishing the capacity region.

B. A New Achievable Scheme and A New Inner Bound

The idea above can be applied in any multi-sender index-coding problem. Specifically, in the compression of composite messages, the senders that share a composite message will use the same composite index to represent this message. In this way, each sender will have a mixed set of private and common composite indices. The senders then collaboratively describe these indices to each receiver at appropriate composite rates by Slepian-Wolf-Cover binning [27]–[29]. We term this compression strategy as *cooperative compression* for multi-sender composite coding.
Along with the link-partitioning technique introduced earlier, we now propose a new achievable scheme for the multi-sender index-coding problem. The scheme proposed in its most general form consists of forming different groups by joint link-and-sender partitioning, splitting messages if they appear in different link-sender groups, implementing composite coding with cooperative compression of composite messages, solving the composite coding problem for each group and then combining the corresponding achievable rates. This scheme is termed as multi-sender Cooperative Composite Coding (CCC).

The following proposition states the new inner bound attained by CCC.

**Proposition 4 (CCC Rate Region):** Consider arbitrary admissible joint link-and-sender partition $\Pi_L$. For each link-sender group $P_L \in \Pi_L$, let $R_{P_L}$ denote the message rate receiver $j$ can obtain from the senders in $P_L$, let $R_{P_L} = (R_{j,\tilde{P}}, j \in S_P | P_L)$, and define the rate region $R_{P_L}$ as the collection of all admissible rate tuples $R_{P_L}$. For any $\tilde{K} \subseteq \tilde{P}$, let $\bar{\tilde{K}} = \bar{\tilde{P}} \setminus \tilde{K}$, and define $I_{\tilde{K}} = \bigcup_{k \in \tilde{K}} \{J : J \subseteq S_k \}$. Under a fixed decoding choice $\{D_j, \tilde{P}, j \in S_P \}$, the following polymatroidal rate region is attained by CCC for group $P_L$:

$$ R \left( \left\{ D_{j,\tilde{P}}, j \in S_P \right\} \mid \left\{ A_{j,\tilde{P}}, j \in S_P \right\}, \left\{ C_{k,\tilde{P}}, k \in \tilde{P} \right\}, P_L \right) = \left\{ \begin{array}{l}
R_{P_L} \in \mathbb{R}^{|S_P|} : \\
(a): \sum_{i \in T_j} R_{i,\tilde{P}} \leq \sum_{J_1 \subseteq D_{j,\tilde{P}} \cup A_{j,\tilde{P}} : J_1 \cap T_j \neq \emptyset} \gamma_{J_1,\tilde{P}}, \\
\forall T_j \subseteq D_{j,\tilde{P}} \setminus A_{j,\tilde{P}}, \forall j \in S_{\tilde{P}}, \\
\text{s.t.} \\
(b): \sum_{J_2 \in I_{\tilde{K}} : J_2 \not\subseteq I_{\tilde{K}}, J_2 \not\subseteq A_{j,\tilde{P}}} \gamma_{J_2,\tilde{P}} \leq \sum_{k \in \tilde{K}} C_{k,\tilde{P}}, \\
\forall j \in S_{\tilde{P}}, \forall \tilde{K} \subseteq \tilde{P} ; \\
C_{k,\tilde{P}} \geq 0, \gamma_{J,\tilde{P}} \geq 0, \\
\forall J \subseteq S_k, \forall k \in \tilde{P} .
\end{array} \right\} . \tag{37} $$

Therefore, by considering all possible decoding combinations for the receivers involved, the following rate region $\mathcal{R}_{P_L}$ is achieved for link-sender group $P_L$:

$$ \mathcal{R}_{P_L} = \text{conv} \bigcup_{\{D_{j,\tilde{P}}, j \in S_P \} \mid \forall j \in S_{\tilde{P}}, \forall j \in D_{j,\tilde{P}}} \mathcal{R} \left( \left\{ D_{j,\tilde{P}}, j \in S_P \right\} \mid \left\{ A_{j,\tilde{P}}, j \in S_P \right\}, \left\{ C_{k,\tilde{P}}, k \in \tilde{P} \right\}, P_L \right) . \tag{38} $$
where “conv U” denotes the convex hull of the union of multiple rate regions.

After finding the rate region \( R_{\mathcal{P}_L} \) for each link-sender group \( P_L \in \Pi_L \), a combined achievable rate region can thus be obtained by applying the following constraints:

\[
R_j \leq \sum_{\hat{P}|P_L \in \Pi_L : j \in S_{\hat{P}}} R_{j, \hat{P}}, \quad j \in [1 : N] \quad (39)
\]

subject to:

\[
\mathcal{R}_{\mathcal{P}_L} = (R_{j, \hat{P}}, j \in S_{\hat{P}} | P_L) \in \mathcal{R}_{\mathcal{P}_L}, \quad P_L \in \Pi_L \quad (40)
\]

and

\[
\sum_{\hat{P}|P_L \in \Pi_L : k \in \hat{P}} C_{k, \hat{P}} \leq C_k, \quad k \in [1 : K], \quad (41)
\]

and eliminating \( \{R_{j, \hat{P}}, j \in S_{\hat{P}} | P_L \in \Pi_L\} \) and \( \{C_{k, \hat{P}}, k \in \hat{P} | P_L \in \Pi_L\} \) through Fourier-Motzkin elimination.

**Proof:** Consider an arbitrary \( P_L \in \Pi_L \) with \( S_{\hat{P}} = \{S_k, k \in \hat{P} | P_L\} \) and each message \( M_{j, \hat{P}} \in [1 : 2^{nR_{j, \hat{P}}}], j \in S_{\hat{P}} \). The encoding operations at each sender and the decoding operations at any receiver \( j \) are illustrated in Fig. 4.

Specifically, for the encoding, in the first step, each sender \( S_k \) generates a list of its composite messages \( \{M_{J, \hat{P}} \in S_k, J \subseteq S_k\} \) and encodes (compresses) each of the composite message into a composite index \( W_{J, \hat{P}} \) at composite rate \( \gamma_{J, \hat{P}} \geq 0 \) bcu via a standard random binning [31]. Denote the collection of all composite indices at sender \( S_k \) by \( \mathcal{W}_{k, \hat{P}} = \{W_{J, \hat{P}} : J \subseteq S_k\} \).

Whenever two or more senders have a composite message in common, they will use the same composite index to represent this message (i.e., a cooperative compression strategy). Hence, \( \mathcal{W}_{k, \hat{P}} \) and \( \mathcal{W}_{\tilde{k}, \hat{P}} \) can be correlated if senders \( S_k \) and \( S_{\tilde{k}} \) have some common message. In this way, in the next step, the description of \( \mathcal{W}_{k, \hat{P}} \) from all senders to any receiver can now be viewed as a Slepian-Wolf-Cover like problem of transmitting multiple correlated sources through a multiple-access channel [27]–[29], with orthogonal links \( \{C_{k, \hat{P}}, k \in \hat{P}\} \) and some side-information at each receiver. Therefore, each sender \( S_k \) uses Slepian-Wolf-Cover random binning to encode \( \mathcal{W}_{k, \hat{P}} \) into a bin index \( L_k \) at rate \( C_{k, \hat{P}} \geq 0 \) bcu and broadcasts the index to each receiver via a codeword \( X_k^n(L_k) \).

Upon reception, receiver \( j \) (\( j \in S_{\hat{P}} \)) retrieves all bin indices \( \{L_1, \cdots, L_K\} \) and attempts to recover all composite indices from the senders by leveraging its side-information. As the number of channel uses \( n \to \infty \), receiver \( j \) can reliably recover all composite indices within the group,
as long as \[ [28, \text{Theorem 2}][29, \text{Section I}] \]

\[
H \left( W_{\tilde{K},\tilde{P}} \mid W_{\tilde{K}^c,\tilde{P}}, M_{A_j,\tilde{P}} \right) \leq \sum_{k \in \tilde{K}} C_{k,\tilde{P}}, \quad \forall \tilde{K} \subseteq \tilde{P}, \quad (42)
\]

where \( W_{\tilde{K},\tilde{P}} = \bigcup_{k \in \tilde{K}} W_{k,\tilde{P}} \) and \( \tilde{K}^c \) denotes the complement of \( \tilde{K} \) with respect to \( \tilde{P} \). Recall the notation \( \mathcal{I}_{\tilde{K}} = \bigcup_{k \in \tilde{K}} \{ J : J \subseteq S_k \} \) we have defined. With this notation, the above conditions are evaluated as

\[
H \left( W_{\tilde{K},\tilde{P}} \mid W_{\tilde{K}^c,\tilde{P}}, M_{A_j,\tilde{P}} \right) = \sum_{\begin{subarray}{c} J_2 \in I_{\tilde{K}}; \\ J_2 \notin \tilde{K}^c, J_2 \not\subseteq A_j,\tilde{P} \end{subarray}} \gamma_{J_2,\tilde{P}} \leq \sum_{k \in \tilde{K}} C_{k,\tilde{P}}, \quad (43)
\]

\[
\forall j \in S_{\tilde{P}}, \quad \forall \tilde{K} \subseteq \tilde{P}, \quad (44)
\]

which lead to constraints (b) of (37).

Given all composite indices recovered, receiver \( j \) then chooses a proper decoding set \( D_{j,\tilde{P}} \) such that \( j \in D_{j,\tilde{P}} \) and employs simultaneous nonunique decoding \[12\] to decode messages \( M_{D_{j,\tilde{P}}} \) by utilizing its side information. As \( n \to \infty \), the messages can be reliably decoded if the following bounds are satisfied \[12\]:

\[
\sum_{i \in T_j} R_{i,\tilde{P}} \leq \sum_{\begin{subarray}{c} J_1 \subseteq D_{j,\tilde{P}} \cup A_j,\tilde{P} \\ J_1 \cap T_j \neq \emptyset \end{subarray}} \gamma_{J_1,\tilde{P}}, \quad (45)
\]

\[
\forall T_j \subseteq D_{j,\tilde{P}} \setminus A_j,\tilde{P}, \quad \forall j \in S_{\tilde{P}}, \quad (46)
\]
which lead to bounds (a) of (37). This hence proves the achievable rate region attained by the CCC proposed under a fixed decoding choice.

The rate region \( R_{PL} \) for group \( PL \) is then obtained by time-sharing over all possible decoding choices (see (38)) and the final rate region is combined through all \( R_{PL} \)'s (see (39)–(41)).

The final combined rate region attained by multi-sender CCC is denoted by \( R_{CCC} \), see Table I.

**Corollary 3:** \( R_{mdCC} \subseteq R_{CCC} \) in general, and the inclusion can be strict. \( \square \)

**Proof:** The proof is referred to Appendix B. \( \square \)

**Remark 7:** Note that the general multi-sender CCC scheme contains the following three special cases: i) “CCC-a” with cooperative compression but without link-and-sender partitioning; ii) “CCC-b” with cooperative compression and sender-partitioning only; iii) “CCC-c” with cooperative compression and link-partitioning only, see Table I, whose corresponding achievable rate regions are denoted by \( R_{CCC-a} \), \( R_{CCC-b} \) and \( R_{CCC-c} \), respectively. By definition, we thus have that \( R_{CCC-a} \subseteq \{R_{CCC-b}, R_{CCC-c}\} \subseteq R_{CCC} \), and both the inclusions can be strict as shown by examples shortly. It is noted that depending on the index-coding setup, a special case of CCC sometimes suffices to achieve the capacity region. \( \square \)

**C. More Examples**

We now demonstrate the usefulness of general multi-sender CCC and its special cases.

First, for the 3-sender 4-message problem in Example 1 interestingly, we note that CCC-a (which is with cooperative compression alone and without link-and-sender partitioning) is also able to achieve the capacity region (which was attained by mDCC before). We also revisit a 15-sender 4-message index-coding instance in Sadeghi et al. [26].

**Example 3:** In this example, there are

- \( K = 15 \) senders, with indices of messages \( S_1 = \{1\}, S_2 = \{2\}, S_3 = \{3\}, S_4 = \{4\}, S_5 = \{1, 2\}, S_6 = \{1, 3\}, S_7 = \{3, 4\}, S_8 = \{2, 3\}, S_9 = \{2, 4\}, S_{10} = \{1, 4\}, S_{11} = \{1, 2, 3\}, S_{12} = \{1, 2, 4\}, S_{13} = \{1, 3, 4\}, S_{14} = \{2, 3, 4\}, S_{15} = \{1, 2, 3, 4\}, \) and each with link capacity \( C_k = 1 \) bcu, \( k \in [1:15] \).
- \( N = 4 \) receivers, with side-information \( A_1 = \{4\}, A_2 = \{3, 4\}, A_3 = \{1, 2\}, \) and \( A_4 = \{2, 3\}, \) respectively.
For this example, with decoding choice \( D_1 = \{1\} \) and \( D_j = [1 : 4] \setminus A_j \) for \( j = 2, 3, 4 \), we verify that CCC-a leads to the following rate region

\[
\mathcal{R}_{\text{CCC-a}}^{\text{eg3}} = \left\{ (R_1, R_2, R_3, R_4) \in \mathbb{R}^4_+ : \\
R_1 \leq 8, \ R_2 \leq 8, \ R_3 \leq 8, \ R_4 \leq 8, \\
R_1 + R_2 \leq 12, \ R_1 + R_3 \leq 12, \\
R_1 + R_4 \leq 12, \ R_3 + R_4 \leq 12, \\
R_1 + R_2 + R_3 \leq 18 \right\}, \tag{47}
\]

which strictly enlarges the previously reported region by partitioned DCC [26]. Note that the same region here can also be attained by the use of a better sender-grouping (e.g., the second and fifth groups in [26 Table II] are combined as a single group) in partitioned DCC. We further prove that the new achievable rate region \( \mathcal{R}_{\text{CCC-a}}^{\text{eg3}} \) is in fact the capacity region. The converse proof requires a set of customized Shannon-type inequalities, since the existing outer bounds (the MAIS bound of (3) or (4), and the polymatroidal bound [26]) are both loose for this example. Details on the converse proof are deferred to Appendix C.

We next consider a 4-sender 5-message index-coding example as defined below.

**Example 4:** In this example, there are

- \( K = 4 \) senders, with indices of messages \( S_1 = \{1, 2, 3\} \), \( S_2 = \{2, 3, 4\} \), \( S_3 = \{1, 2\} \), and \( S_4 = \{2, 4, 5\} \), and each with link capacity \( C_k = 1 \) bcu, \( k \in [1 : 4] \);
- \( N = 5 \) receivers, with side-information \( A_1 = \{4, 5\} \), \( A_2 = \{1, 3, 5\} \), \( A_3 = \{1, 2\} \), \( A_4 = \{2, 3, 5\} \) and \( A_5 = \{3\} \), respectively.

The above sender and receiver settings are depicted in Figs. 5 (a) and (c), respectively.

In this instance, note that if we only consider the first two senders and the messages/receivers involved, the resulting index-coding subproblem coincides with the problem in **Example 2**. As discussed earlier, cooperative compression is the key to achieve the capacity region for the problem. Moreover, in terms of optimal decoding choice, receiver 1 is restricted to decode only \( M_1 \). On the other hand, if we only consider the last two senders and the messages/receivers involved, we observe that DCC-a (thus also CCC-a) suffices to attain the capacity region for the corresponding subproblem, but it requires receiver 1 to at least decode both \( M_1 \) and \( M_2 \). There is clearly a conflict of decoding requirements for receiver 1 between these two subproblems (sender groups) said. Given these observations, we thus anticipate that CCC-b (i.e., with sender partitioning) is crucial to achieve the capacity region for the whole problem.
Indeed, by applying Proposition 4 to this index-coding instance, if all senders are restricted to be in the same group, we note that the best possible achievable rate region is achieved by CCC-c with link-partitioning (or time-sharing) between two different decoding choices: i) $D_1 = \{1\}$, $D_j = [1 : 5] \setminus A_j$, $j \in [2 : 5]$; and ii) $D_j = [1 : 5] \setminus A_j$, $j \in [1 : 4]$, $D_5 = \{5\}$. However, it is still not capacity-achieving. Instead, CCC-b with senders $\{S_1, S_2\}$ as a group and $\{S_3, S_4\}$ as another group leads to the following achievable rate region:

$$
R_{\text{CCC-b}}^{\text{mais}} = \left\{ (R_1, R_2, R_3, R_4, R_5) \in \mathbb{R}_+^5 : 
\begin{array}{l}
R_1 \leq 2, \ R_3 \leq 2, \ R_5 \leq 1, \\
R_1 + R_3 \leq 3, \ R_4 + R_5 \leq 2, \\
R_1 + R_2 + R_5 \leq 4, \\
R_2 + R_4 + R_5 \leq 4, \\
R_3 + R_4 + R_5 \leq 3.
\end{array}
\right\},
$$

which agrees with the MAIS outer bound (4) specialized to this index-coding problem, thus establishing the capacity region. Note that this example also confirms that $R_{\text{mDCC-a}} \subset R_{\text{CCC-c}}$ and $R_{\text{DCC}} \subset R_{\text{CCC-b}}$ as depicted in Fig. 1 due to the necessity of cooperative compression.

Finally, we study an example with the same receiver side-information graph as that in Example 4, but with only 3 senders as defined below.

**Example 5:** In this example, there are
• $K = 3$ senders, with indices of messages $S_1 = \{1, 2, 3\}, S_2 = \{2, 3, 4\},$ and $S_3 = \{1, 2, 4, 5\}$, and each with link capacity $C_k = 1$ bcu, $k \in [1 : 3]$;

• $N = 5$ receivers, with side-information $A_1 = \{4, 5\}, A_2 = \{1, 3, 5\}, A_3 = \{1, 2\}, A_4 = \{2, 3, 5\}$ and $A_5 = \{3\}$, respectively.

The sender and receiver setting are illustrated in Figs. 5(b) and (c), respectively.

For this example, the MAIS bound of (4) is specialized to

\[
\mathcal{R}^{5\{5\}}_{\text{out}} = \left\{ (R_1, R_2, R_3, R_4, R_5) \in \mathbb{R}_+^5 : \begin{align*}
R_3 &\leq 2, \quad R_5 \leq 1, \\
R_1 + R_3 &\leq 3, \quad R_1 + R_5 \leq 2, \quad R_4 + R_5 \leq 2, \\
R_1 + R_2 + R_3 &\leq 3, \quad R_1 + R_4 + R_5 \leq 3, \\
R_2 + R_4 + R_5 &\leq 3, \\
2R_1 + R_2 + R_3 + R_4 + 2R_5 &\leq 7, \\
4R_1 + 3R_2 + R_3 + 3R_4 + 4R_5 &\leq 15, \\
3R_1 + 3R_2 + R_3 + 4R_4 + 4R_5 &\leq 15.
\end{align*} \right\}
\]

(49)

while by Proposition 4, CCC with only link-partitioning (i.e., CCC-c) between two different decoding choices: i) $\mathcal{D}_1 = \{1\}, \mathcal{D}_5 = \{5\}, \mathcal{D}_j = [1 : 5]\backslash A_j, j \in [2 : 4]$; and ii) $\mathcal{D}_j = [1 : 5]\backslash A_j, j \in [1 : 4], \mathcal{D}_5 = \{5\}$ leads to the following achievable rate region:

\[
\mathcal{R}^{5\{5\}}_{\text{CCC-c}} = \left\{ (R_1, R_2, R_3, R_4, R_5) \in \mathbb{R}_+^5 : \begin{align*}
R_3 &\leq 2, \quad R_5 \leq 1, \\
R_1 + R_3 &\leq 3, \quad R_1 + R_5 \leq 2, \quad R_4 + R_5 \leq 2, \\
R_1 + R_2 + R_3 &\leq 3, \quad R_1 + R_4 + R_5 \leq 3, \\
R_2 + R_4 + R_5 &\leq 3, \\
R_1 + 2R_3 + R_4 + R_5 &\leq 5.
\end{align*} \right\}
\]

(50)

and CCC-b with the best possible sender-partitioning $\bar{P}_1 = \{1, 2\}$ and $\bar{P}_2 = \{3\}$ leads to the following rate region:

\[
\mathcal{R}^{5\{5\}}_{\text{CCC-b}} = \left\{ (R_1, R_2, R_3, R_4, R_5) \in \mathbb{R}_+^5 : \begin{align*}
R_3 &\leq 2, \quad R_5 \leq 1, \\
R_1 + R_3 &\leq 3, \quad R_1 + R_5 \leq 2, \quad R_4 + R_5 \leq 2, \\
R_1 + R_2 + R_3 &\leq 3, \quad R_1 + R_4 + R_5 \leq 3, \\
R_2 + R_4 + R_5 &\leq 3, \\
R_1 + 2R_3 + R_4 + R_5 &\leq 5.
\end{align*} \right\}
\]

(51)
where relevant decoding sets are: i) \( \mathcal{D}_{1,\tilde{P}_1} = \{1\} \), \( \mathcal{D}_{j,\tilde{P}_1} = \mathcal{S}_{\tilde{P}_1} \setminus \mathcal{A}_{j,\tilde{P}_1} \) for \( j = 2, 3, 4 \); ii) \( \mathcal{D}_{j,\tilde{P}_2} = \mathcal{S}_{\tilde{P}_2} \setminus \mathcal{A}_{j,\tilde{P}_2} \) for \( j = 1, 2, 4 \), and \( \mathcal{D}_{5,\tilde{P}_2} = \{5\} \).

It can be checked that none of \( \mathcal{R}_{\text{CCC-c}}^{\mathcal{S}} \) and \( \mathcal{R}_{\text{CCC-b}}^{\mathcal{S}} \) strictly contains the other, and there is still a gap between these inner bounds and the outer bound \( \mathcal{R}_{\text{out}}^{\mathcal{S}} \) of (49). In particular, the rate tuple \( \mathbf{R}_c = (R_1 = 1.5, R_2 = 1, R_3 = 1.5, R_4 = 1, R_5 = 0.5) \) is in \( \mathcal{R}_{\text{out}}^{\mathcal{S}} \), but it is not in \( \mathcal{R}_{\text{CCC-c}}^{\mathcal{S}} \), \( \mathcal{R}_{\text{CCC-b}}^{\mathcal{S}} \), or the convex union of these two regions. To close the gap, we may resort to the general CCC scheme. But what would be the optimal link-sender groups we shall use in CCC?

As before, we might find some clues from the linear index-coding approach. Consider that message \( M_j(j = 1, 3) \) is split into three parts, each of 0.5 bcu and denoted by \( M_j^{(i)}(i = 1, 2, 3) \); message \( M_j(j = 2, 4) \) is split into two parts, each of 0.5 bcu and denoted by \( M_j^{(i)}(i = 1, 2) \); and message \( M_5 \) is of 0.5 bcu. Subject to the message availability and link capacity constraints at senders, the following XOR packets can be transmitted:

(i) \( c_1 = [c_{11} c_{12}] = [M_1^{(1)} \oplus M_2^{(1)} \oplus M_3^{(1)} \quad M_1^{(3)} \oplus M_3^{(3)}] \) from sender \( S_1 \);
(ii) \( c_2 = [c_{21} c_{22}] = [M_2^{(1)} \oplus M_3^{(1)} \oplus M_4^{(1)} \quad M_2^{(2)} \oplus M_3^{(2)} \oplus M_4^{(2)}] \) from sender \( S_2 \);
(iii) \( c_3 = [c_{31} c_{32}] = [M_2^{(1)} \oplus M_4^{(2)} \quad M_1^{(3)} \oplus M_5] \) from sender \( S_3 \).

Given these constructions, it can be checked that \((c_{11}, c_{21})\) collaboratively formed by senders \( \{S_1, S_2\} \) is a valid index code that exploits the subgraph \( \tilde{G} \) induced by the first four receivers, thus each receiver \( j \) can decode its relevant message component \( M_j^{(1)}, j \in [1 : 4] \). Similarly, \((c_{22}, c_{31})\) jointly formed by senders \( \{S_2, S_3\} \) is also a valid index code that exploits the same subgraph \( \tilde{G} \), thus each receiver \( j \) can decode its relevant message component \( M_j^{(2)}, j \in [1 : 4] \).

Lastly, \((c_{12}, c_{32})\) jointly formed by senders \( \{S_1, S_3\} \) is a cyclic index code that exploits the cycle “1 → 5 → 3 → 1”, thus receiver \( j \) (\( j = 1, 3 \)) decode its relevant message component \( M_j^{(3)} \) and receiver 5 can decode message \( M_5 \). In this way, each receiver reliably recovers all its message components. The rate tuple \( \mathbf{R}_c \) above is hence achieved by this linear index-coding scheme.

The scheme above indicates that pair-wise sender-groups are the critical ones. Inspired by this observation, in the evaluation of inner bound attained by CCC, we propose that each sender splits its link capacity \( C_k \) into two non-negative parts as \( C_k = C_{k,\tilde{P}_1} + C_{k,\tilde{P}_2} \) and devotes each to the transmission for a different pair-wise sender-group. Formally, the following three link-sender groups are formed for the index-coding example studied: \( P_{L,1} = \{\tilde{P}_1 = \{1, 2\}; C_{\tilde{P}_1} = \{C_{1,\tilde{P}_1}, C_{2,\tilde{P}_1}\}\), \( P_{L,2} = \{\tilde{P}_2 = \{2, 3\}; C_{\tilde{P}_2} = \{C_{2,\tilde{P}_2}, C_{3,\tilde{P}_2}\}\) and \( P_{L,3} = \{\tilde{P}_3 = \{1, 3\}; C_{\tilde{P}_3} = \{C_{1,\tilde{P}_3}, C_{3,\tilde{P}_3}\}\}, \) subject to \( C_{1,\tilde{P}_1} + C_{1,\tilde{P}_3} \leq C_1 = 1, C_{2,\tilde{P}_1} + C_{2,\tilde{P}_2} \leq C_2 = 1 \) and \( C_{3,\tilde{P}_2} + C_{3,\tilde{P}_3} \leq
TABLE III
THE RATE REGIONS ATTAINED BY INDIVIDUAL GROUPS AND THE FINAL COMBINED RATE REGION FOR EXAMPLE 5

| Link-Sender Groups | Achievable Rate regions                                                                 | Relevant Decoding Sets                                      |
|--------------------|-----------------------------------------------------------------------------------------|-------------------------------------------------------------|
| \( \tilde{P}_1 = \{1, 2\}; \tilde{C}_\rho_1 = \{C_1, \rho_1, C_2, \rho_1\} \)       | \( R_1, \rho_1 \leq C_1, \rho_1, \quad R_4, \rho_1 \leq C_2, \rho_1, \)  
\( R_1, \rho_1 + R_2, \rho_1 \leq C_1, \rho_1 + C_2, \rho_1, \)  
\( R_1, \rho_1 + R_3, \rho_1 \leq C_1, \rho_1 + C_2, \rho_1, \)  
\( R_2, \rho_1 + R_4, \rho_1 \leq C_1, \rho_1 + C_2, \rho_1, \)  
\( R_3, \rho_1 + R_4, \rho_1 \leq C_1, \rho_1 + C_2, \rho_1 \). | \( D_{1, \rho_1} = \{1\}; \)  
\( D_{1, \rho_1} = S_{\rho_1} \setminus A_{\rho_1}, \)  
\( j = 2, 3, 4. \) |
| \( \tilde{P}_2 = \{2, 3\}; \tilde{C}_\rho_2 = \{C_2, \rho_2, C_3, \rho_2\} \)       | \( R_3, \rho_2 \leq C_2, \rho_2, \quad R_1, \rho_2 + R_5, \rho_2 \leq C_3, \rho_2, \)  
\( R_1, \rho_2 + R_2, \rho_2 + R_5, \rho_2 \leq C_2, \rho_2 + C_3, \rho_2, \)  
\( R_1, \rho_2 + R_4, \rho_2 + R_5, \rho_2 \leq C_2, \rho_2 + C_3, \rho_2, \)  
\( R_2, \rho_2 + R_4, \rho_2 + R_5, \rho_2 \leq C_2, \rho_2 + C_3, \rho_2, \)  
\( R_3, \rho_2 + R_4, \rho_2 + R_5, \rho_2 \leq C_2, \rho_2 + C_3, \rho_2, \). | \( D_{1, \rho_2} = \{1\}, D_{3, \rho_2} = \{5\}; \)  
\( D_{1, \rho_2} = S_{\rho_2} \setminus A_{\rho_2}, \)  
\( j = 2, 3, 4. \) |
| \( \tilde{P}_3 = \{1, 3\}; \tilde{C}_\rho_3 = \{C_1, \rho_3, C_3, \rho_3\} \)       | \( R_3, \rho_3 \leq C_1, \rho_3, \)  
\( R_1, \rho_3 + R_3, \rho_3 \leq C_1, \rho_3 + C_3, \rho_3, \)  
\( R_1, \rho_3 + R_5, \rho_3 \leq C_3, \rho_3, \)  
\( R_1, \rho_3 + R_2, \rho_3 + R_5, \rho_3 \leq C_1, \rho_3 + C_3, \rho_3, \)  
\( R_1, \rho_3 + R_4, \rho_3 + R_5, \rho_3 \leq C_1, \rho_3 + C_3, \rho_3, \)  
\( R_2, \rho_3 + R_4, \rho_3 + R_5, \rho_3 \leq C_1, \rho_3 + C_3, \rho_3. \) | \( D_{1, \rho_3} = \{1\}, D_{3, \rho_3} = \{5\}; \)  
\( D_{1, \rho_3} = S_{\rho_3} \setminus A_{\rho_3}, \)  
\( j = 2, 3, 4. \) |
| Combined Region \( \mathcal{R}_{\text{CCC}}^{55} \) | \( R_3 \leq 2, \quad R_5 \leq 1, \)  
\( R_1 + R_3 \leq 3, \quad R_1 + R_5 \leq 2, \quad R_4 + R_5 \leq 2, \)  
\( R_1 + R_2 + R_5 \leq 3, \quad R_1 + R_4 + R_5 \leq 3, \)  
\( R_2 + R_4 + R_5 \leq 3, \quad R_3 + R_4 + R_5 \leq 3. \) | \( D_{1, \rho_1} = \{1\}, D_{3, \rho_1} = \{5\}; \)  
\( D_{1, \rho_1} = S_{\rho_1} \setminus A_{\rho_1}, \)  
\( j = 2, 3, 4. \) |

\( C_3 = 1. \) By these link-sender group construction and invoking Proposition 4, we thus can first obtain the individual rate region under each link-sender group through (37)–(38), and then obtain the final combined rate region \( \mathcal{R}_{\text{CCC}}^{55} \) via (39)–(41). Details on the optimal decoding sets and the individual rate regions are provided in Table III. We note that \( \mathcal{R}_{\text{CCC}}^{55} \) agrees with the MAIS outer bound \( \mathcal{R}_{\text{out}}^{55} \) in form of (49), thus establishing the capacity region.

This example confirms that \( \mathcal{R}_{\text{CCC-b}} \subset \mathcal{R}_{\text{CCC}} \) and \( \mathcal{R}_{\text{CCC-c}} \subset \mathcal{R}_{\text{CCC}} \) as in Fig. 1. It also shows that \( \mathcal{R}_{\text{mDCC}} \subset \mathcal{R}_{\text{CCC}} \) due to the necessity of cooperative compression in composite coding.
VI. Conclusions

In this paper, we have developed new achievable schemes via a random coding approach and established new capacity bounds for the multi-sender unicast index-coding problem. In particular, we have revisited partitioned DCC \cite{26} and have identified its limitations in the sender grouping strategy and in the implementation of composite coding in a multi-sender setup. We have introduced a joint link-and-sender partitioning strategy and developed a modified DCC (mDCC) scheme, which strictly improves upon partitioned DCC and suffices to achieve optimality for a number of index-coding instances. We have also introduced a cooperative compression of composite messages in multi-sender composite coding to leverage potential overlapping of messages at different senders to support larger composite rates. By the combined use of the new techniques proposed, we have devised a new multi-sender cooperative composite coding (CCC) scheme, which improves upon both partitioned DCC and mDCC in general and hence leads to the best achievable rate region known to date for the problem studied. Overall, it can be seen that compared with single-sender index coding, the multi-sender index-coding problem is richer and more difficult to solve, due to the varying availability of messages and the nature of distributed encoding at each sender. The current work hence serves as an intermediate step towards a full understanding of the problem.

For future work, we will investigate whether or not CCC proposed here suffices to achieve the capacity region for all non-isomorphic 4- or 5-message index-coding instances with arbitrary admissible sender setting and with arbitrary link capacities, in the same spirit of the study to all 3-message index-coding instances by Sadeghi et al. \cite{26}. For the 4-message case, our preliminary studies indicate that there are 103 out of 218 non-isomorphic side-information graphs, for which the CCC-induced inner bound agrees with the MAIS outer bound, thus establishing the capacity region under any sender setting and with arbitrary link capacities. However, for the remaining 115 side-information graphs, no firm conclusion is reached yet. To completely solve them, we might need either customized link-sender partitioning optimization for each instance in CCC, or even some new advanced schemes, or new customized outer bounds as that derived for Example 3, in which case it is highly non-trivial and requires significant extra efforts.
APPENDIX A

PROOF OF THE SIMPLIFIED MULTI-SENDER MAIS OUTER BOUND

Recall that in the code Definition 1, $L_k$ denotes the output of each sender $S_k$. Let $L = \{L_1, L_2, \cdots, L_K\}$ denote the collection of outputs at all senders, let $L_{M_S} \subseteq L$ denote the collection of outputs that are completely determined by messages only in $M_S$, and let $S^c$ denote the complement of $S$ with respect to $[1:N]$. For any $S \subseteq [1:N]$ such that the subgraph of $G$ induced by $S$ is acyclic, we have:

$$\sum_{j \in S} R_j = H(M_S)$$

$$= H(M_S | M_{S^c})$$

$$= H(L, M_S | M_{S^c})$$

$$= H(L | M_{S^c})$$

$$\leq H(L \backslash L_{M_{S^c}})$$

$$\leq \sum_{k \in [1:K]} C_k,$$

where equality (53) holds due to the independence of messages; equality (54) follows from the encoding functions of (1), while equality (55) holds because $H(M_S | L, M_{S^c}) = 0$, which is deduced by the facts that the subgraph of $G$ induced by $S$ is acyclic and thus each message $M_j, j \in S$ can be always decoded in a certain order according to the decoding functions of (2), given $L, M_{S^c}$ and the subset of messages recovered before $M_j$; (56) holds because $H(L | M_{S^c}) = H(L \backslash L_{M_{S^c}} | M_{S^c})$ and conditioning reduces entropy; (57) holds because the output of sender $S_k$ is at most $C_k$ bcu constrained by its link capacity.

APPENDIX B

PROOF OF COROLLARY 3

We first prove that $R_{mdcc} \subseteq R_{ccc}$ in general.

Assume that the same arbitrary admissible joint link-and-sender partition $\Pi_L$ is used in Proposition 3 and Proposition 4. Consider any $\tilde{P} \in \Pi_L$ and any subset $\tilde{K} \subseteq \tilde{P}$. For convenience, define $f(\tilde{K}) = \{J_2 \in I_{\tilde{K}} : J_2 \notin I_{\tilde{K}^c}, J_2 \notin A_{j,\tilde{P}}\}$ as a set of indices of composite messages that appear in (37)(b) of $R_{ccc}$. Similarly, define $h(k) = \{J_2 \subseteq S_k : J_2 \notin A_{j,\tilde{P}}\}$ as a set of indices
of composite messages that appear in (16).(b) of $R_mDCC$. In addition, given any fixed admissible $J \in f(\tilde{K})$, define $g(J) = \{k : k \in \tilde{P}, \bar{J} \subseteq S_k\}$, and let $\gamma_{\bar{J}, \tilde{P}} = \sum_{k \in g(J)} \gamma_{\bar{J}, \tilde{P}}^{(k)}$.

For the chosen $\tilde{K}$ and $\bar{J}$, if $k \in g(J)$, we must have $k \in \tilde{K}$. We can prove this by contradiction. Specifically, if $k \in g(J)$, then $J \subseteq S_k$ by the definition of $g(J)$. This hence implies that $J \in I_k$ by the definition of $I_k$. Now, suppose that $k \in \tilde{K}$, then $J \subseteq I_k \subseteq I_{\tilde{K}}$, which contradicts the assumption that $J \in f(\tilde{K})$.

We now inspect the constraints imposed on composite rates in (37) of $R_{CCC}$ and (16) of $R_mDCC$, respectively. For any fixed $\tilde{K}$, constraint (37).(b) reads as

$$
\sum_{J \in f(\tilde{K})} \sum_{k \in g(J)} \gamma_{\bar{J}, \tilde{P}}^{(k)} \leq \sum_{k \in \tilde{K}} C_{k, \tilde{P}},
$$

(58)

while constraint (16).(b) implies that

$$
\sum_{k \in \tilde{K}} \sum_{J' \in h(k)} \gamma_{\bar{J}, \tilde{P}}^{(k)} \leq \sum_{k \in \tilde{K}} C_{k, \tilde{P}}.
$$

(59)

For each $(J, k)$ in the left-hand side of (58), the corresponding $\gamma_{\bar{J}, \tilde{P}}^{(k)}$ must appear in the left-hand side of (59), because: i) for any $J \in f(\tilde{K})$, if $k \in g(J)$, then $k \in \tilde{K}$ as proved before; and ii) if $J \in f(\tilde{K})$, then $J \in h(k)$ for some $k \in \tilde{K}$, as $f(\tilde{K}) \subseteq \bigcup_{k \in \tilde{K}} h(k)$. Since we only count distinct $(J, k)$’s, constraint (59) must be more restrictive than constraint (58). We therefore conclude that $R_mDCC \subseteq R_{CCC}$ in general.

The inclusion can be strict as shown by Example 2, Example 4 and Example 5 in Section V.

APPENDIX C

CONVERSE PROOF OF THE CAPACITY REGION FOR EXAMPLE 3

Recall from the multi-sender index code Definition 7, each message $M_j$ is independently and uniformly distributed over the set $[1 : 2^{nR_j}]$, where $n$ is the number of channel uses. The output $L_k$ of each sender $S_k$ is a function of the messages available at the sender. Thus, we have

$$
H(L_k | M_{S_k}) = 0, \quad k \in [1 : 15].
$$

(60)

Let $\mathbf{L} = \{L_1, L_2, \cdots, L_{15}\}$ denote the collection of all outputs at the 15 senders. By the definition of decoders, we have the following decodability conditions at receivers:

$$
H(M_1 | \mathbf{L}, M_4) = 0,
$$

(61)
\[ H(M_2 | L, M_3, M_4) = 0, \quad (62) \]
\[ H(M_3 | L, M_1, M_2) = 0, \quad (63) \]
\[ H(M_4 | L, M_2, M_3) = 0. \quad (64) \]

With these conditions, we now proceed with the converse proof.

1) The outer bounds on each individual rate \( R_j \leq 8 \) and on each rate pair concerned are included in the MAIS outer bound of (4). The proof is hence omitted here.

2) We now prove that \( R_1 + R_2 + R_3 \leq 18 \). We first note that

\[ n(R_1 + R_2 + R_3) = H(M_1, M_2, M_3) \quad (65) \]
\[ = H(L, M_1, M_2, M_3 | M_4) \quad (66) \]
\[ = H(L | M_4) + H(M_1, M_2, M_3 | M_4, L) \quad (67) \]
\[ \leq H(L | L_4) + H(M_1, M_2, M_3 | M_4, L) \quad (68) \]
\[ \leq 14n + H(M_1, M_2 | M_4, L) + H(M_3 | M_1, M_2, M_4, L) \quad (69) \]
\[ = 14n + H(M_1, M_2 | M_4, L) \quad (70) \]
\[ = 14n + H(M_1 | M_4, L) + H(M_2 | L_1, L_4, L) \quad (71) \]
\[ = 14n + H(M_2 | M_1, M_4, L), \quad (72) \]

where (66) follows the independence of messages and encoding functions of (60), (67) follows the chain rule of entropy, (68) holds as \( L_4 \) is a function of \( M_4 \) and conditioning reduces entropy, (70) holds due to the decodability condition (63) for \( M_3 \), and (72) holds due to the decodability condition (61) for \( M_1 \).

Now, suppose we can prove that

\[ H(M_2 | M_1, M_4, L) \leq 4n, \quad (73) \]

then we are done.

Towards this end, we examine the following two facts:

- \( H(M_2 | M_1, M_4) \leq H(L_2, L_5, L_8, L_9, L_{11}, L_{12}, L_{14}, L_{15} | M_1, M_4) \):

\[ H(M_2 | M_1, M_4) = H(M_2 | M_1, M_4, M_3) \quad (74) \]
\[ I(M_2; L_2, L_5, L_8, L_9, L_{11}, L_{12}, L_{14}, L_{15}|M_1, M_4) = H(M_2) - H(M_2|M_1, M_4, L_{12}, L_{14}, L_{15}) \]

where (75) follows the definition of encoding functions, (76) uses the chain rule of entropy, (77) holds as \( H(M_2|M_1, M_4, L) = 0 \) due to the decodability condition (62) for \( M_2 \), and (78) holds because conditioning reduces entropy.

- \( I(M_2; L_2, L_5, L_8, L_9, L_{11}, L_{12}, L_{14}, L_{15}|M_1, M_4) \) can be expanded in two ways:

\[ I(M_2; L_2, L_5, L_8, L_9, L_{11}, L_{12}, L_{14}, L_{15}|M_1, M_4) = H(M_2|M_1, M_4) - H(M_2|L_2, L_5, L_8, L_9, L_{11}, L_{12}, L_{14}, L_{15}) \]

\[ = H(L_2, L_5, L_8, L_9, L_{11}, L_{12}, L_{14}, L_{15}|M_1, M_4) - H(L_2, L_5, L_8, L_9, L_{11}, L_{12}, L_{14}, L_{15}|M_1, M_2, M_4) \]

\[ = H(L_8, L_{11}, L_{14}, L_{15}|M_1, M_2, M_4) \]

(79)

(80)

(81)

where (81) holds because \( L_2, L_5, L_9, L_{12} \) are deterministic functions of \( \{M_1, M_2, M_4\} \).

Now, noting that \( H(M_2|M_1, M_4) \leq H(L_2, L_5, L_8, L_9, L_{11}, L_{12}, L_{14}, L_{15}|M_1, M_4) \) from (78), and by comparing (79) and (81), we can prove that the following inequality holds:

\[ H(M_2|L_2, L_5, L_8, L_9, L_{11}, L_{12}, L_{14}, L_{15}) \leq H(L_8, L_{11}, L_{14}, L_{15}|M_1, M_2, M_4). \]

(82)

Given this inequality, we thus have

\[ H(M_2|M_1, M_4, L) \leq H(M_2|M_1, M_4, L_2, L_5, L_8, L_9, L_{11}, L_{12}, L_{14}, L_{15}) \]

\[ \leq H(L_8, L_{11}, L_{14}, L_{15}|M_1, M_2, M_4) \]

\[ \leq H(L_8, L_{11}, L_{14}, L_{15}), \]
Finally, combining bounds (72) and (86), we conclude that

\[ R_1 + R_2 + R_3 \leq 18, \]  

(87)

which completes the proof for the outer bound on the capacity region.
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