A theory of induction and classification of tensor $C^*$–categories

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Abstract

We construct finite projective Hilbert bimodule representations over the $G$–ergodic $C^*$–algebra associated, in our previous work, with an inclusion of the representation category of the compact quantum group $G$ in an abstract tensor $C^*$–category $M$. In the commutative case this construction gives the bimodule of continuous sections of equivariant Hermitian bundles over compact homogeneous spaces and is closely related to Mackey’s induction.

If the inclusion of categories is surjective on objects, there is a natural Frobenius full tensor functor from $M$ to the category of bimodule representations of $G$.

We give two applications. The first constructs a full and faithful embedding of a tensor $C^*$–category with a generating object, which may or may not be selfconjugate, of intrinsic dimension $\geq 2$, into the category of bimodule representations of the quantum groups $A_n(F)$ or $A_u(F)$ for suitable matrices $F$. A large class of examples arise from finite index inclusions of factors.

The second shows that if the generating object is pseudoreal with intrinsic dimension 2, then $M$ may be identified with a full subcategory of the representation category of a closed subgroup of $SU(2)$. No permutation symmetry is assumed.
1 Introduction

In the late 80’s, S. Doplicher and the second-named author, motivated by algebraic QFT, proved a duality theory for compact groups asserting that if a tensor \( C^* \)-category with conjugates admits a permutation symmetry, it is equivalent, after completion under direct sums and subobjects to the tensor \( C^* \)-category of finite dimensional representations of a unique compact group with its natural permutation symmetry [10]. Changing the permutation symmetry may however change the group [18].

A tensor \( C^* \)-category with conjugates admitting an embedding into the tensor \( C^* \)-category of Hilbert spaces is, after completion under direct sums and subobjects, equivalent to the tensor \( C^* \)-category of finite dimensional representations of a compact quantum group by Woronowicz duality [52], that may be far from being unique.

The basic model of [10] is a fundamental representation of a compact Lie group.

Little seems to be known about the problem of classifying general non-embedable tensor \( C^* \)-categories, although there are plenty of examples arising in the theory of subfactors and in low dimensional QFT.

For instance, the tensor category generated by \( \mathcal{N} \mathcal{M} \mathcal{N} \), with \( \mathcal{N} \subset \mathcal{M} \) an irreducible \( II_1 \) inclusion with finite index, is a non-embedable tensor \( C^* \)-category whenever the index is not an integer and the inclusion is amenable in the sense
of Popa, see \[33\]. There is a similar result for an amenable object in a tensor C*-category with non-integral dimension \[22\].

As a trivial remark, if \( N \subset M \) is a finite index inclusion of \( II_1 \) factors, \( M \), as a bimodule over \( N \), has always a real structure given by the \( * \)-involution of \( M \), generalizing the reality property of the regular representation of a finite group. Hence not all tensor C*-categories with a single generating object are of this form.

It is an interesting problem to understand to what extent general tensor C*-categories with conjugates may be described as 'representation categories' of noncommutative group-like structures. We remark that other aspects of this problem involve the structure of the regular representation \[11, 1, 8, 12\]; and inclusions of factors with finite index and finite depth. More precisely, f.d. Kac algebras in their regular representations naturally arise from the tensor categories derived from irreducible depth 2 inclusions \[44, 17, 21\]. Tensor categories derived from reducible finite depth inclusions give rise to weak Hopf algebras (also named finite quantum groupoids) introduced in \[4, 5, 6\], see \[26, 27\].

In an algebraic framework, Hayashi and Ostrik proved that every semisimple rigid tensor category with finitely many irreducible objects is the representation category of a finite quantum groupoid \[15, 29\]. See also \[13\] and references therein.

We show that tensor C*-categories with conjugates (with possibly infinitely many irreducibles) admitting a generating object, that may or may not be self-conjugate, but are always of intrinsic dimension \( \geq 2 \) (or Jones index \( \geq 4 \), by \[20\], see also \[22\]), can be embedded faithfully into the tensor C*-category of Hilbert bimodules over a noncommutative C*-algebra, defined intrinsically by the category.

These bimodules are the noncommutative analogues of the bimodule of continuous sections of \( G \)-equivariant Hermitian bundles over compact homogeneous spaces. They carry actions of the universal orthogonal and unitary compact quantum groups \( A_o(F) \) or \( A_u(F) \) of Wang and van Daele and the embedding functor identifies the given category as the category of bimodule intertwiners of the quantum group. The quantum group is not uniquely determined but the representation category is always the same. The coefficient algebra of these bimodules, together with the ergodic C*-action of \( A_o(F) \) or \( A_u(F) \) describes the noncommutative homogeneous space.

Actually, ergodic C*-systems have been constructed in previous papers from an inclusion of the representation category of a compact quantum group \( G \) into an abstract tensor C*-category \( \mathcal{M} \) \[32, 34\].

We remark that, given an ergodic C*-action of a compact quantum group \( G \), the natural bimodule structure suggested by the commutative case, does not give rise to a bimodule \( G \)-representation structure in the noncommutative case. More precisely, if, e.g., the right module structure is given by right multiplication, the left module structure can not be given by left multiplication unless the Hopf algebra of \( G \) has enough mutually commuting elements (see Prop. 4.3).

Our construction provides an intrinsic left module structure making the natural right module representation of the quantum group into a bimodule representation. It reduces to the natural left module structure in the commutative case and it is not inner even for quantum quotient spaces. Furthermore, these bimodule representa-
tions are *full*, in a sense that we will describe more precisely later. This property implies that the ergodic $C^*$–system may be reconstructed from the category of bimodule representations.

Roughly speaking, the assumption on the intrinsic dimension allows us to realize solutions of the conjugate equations in the category of Hilbert spaces, hence representations of the universal compact quantum groups arise. The associated matrices $F$ then describe all possible realizations. These Hilbert space representations are then enlarged to Hilbert bimodule representations of the quantum group over the coefficient $C^*$–algebra by a method that we may term abstract induction.

Our ergodic actions do not, in general, arise from quantum subgroups but from what may be termed, following Mackey, a virtual quantum subgroup. Virtual subgroups are needed to describe cases where the category is non-embedable.

We construct a family of noncommutative compact $G$–spaces from a given category. In the commutative case it is well known that $G$–spaces are special cases of groupoids. Thus our result, although relying on different methods, should be compared with that of [26, 27]. It would be interesting to find an explicit connection when the category arises from a finite depth inclusion.

Our construction sheds light on the problem of recognizing which tensor $C^*$–categories with conjugates are embedable into the category of Hilbert spaces. More precisely, combining the bimodule construction with the imprimitivity theorem of Takesaki for dynamical systems on operator algebras [45] and the work of Høegh–Krohn, Landstad and E. Størmer [16] on ergodic actions, we show that if the category contains the representation category of a compact group and if the von Neumann completion of the coefficient algebra in the GNS representation of the unique invariant trace is of type $I$, then the given category may be identified with a full subcategory of the representation category of a closed subgroup.

The problem, posed in [16], of whether an ergodic action of a compact simple group is always of type $I$ is still open.

In particular, Wassermann’s result about the non-existence of ergodic actions of $SU(2)$ on the hyperfinite $II_1$ factor [50], allows one to conclude that a tensor $C^*$–category with a pseudoreal object of dimension 2 is always embedable.

Induced representations, introduced by Frobenius [14], have been studied by many authors in the course of developing the theory of representations of groups and algebras. In particular, Mackey, in the 50's, gave a general definition of induction for locally compact groups, emphasized their role in Quantum Mechanics and generalized and unified the work of his predecessors.

Notably, Mackey carried over concepts of group theory to measurable ergodic theory, and referred to this idea as the *virtual group* point of view in ergodic theory [24]. Introducing the concept of a *measurable groupoid*, as a generalization of the concept of a group, he unified measurable ergodic theory and group theory. Here Mackey was, in particular, motivated by wanting to give a conceptual explanation of certain previously known results in ergodic theory. Many authors then contributed to proving analogues in ergodic theory of results in group theory. In particular we recall the work of Ramsay on induction for virtual groups [38, 39].

As is well known, ergodic theory played an important role in deep developments in operator algebras and noncommutative geometry. Many von Neumann algebras...
and $C^*$–algebras were discovered to be associated with measurable or topological groupoids, for a beautiful recent survey see [25].

Induced actions of locally compact groups on noncommutative von Neumann algebras have been studied by Takesaki [45]. They played a role in his work on the structure of such algebras. In particular, Takesaki obtained an imprimitivity theorem reducing the study of ergodic actions of compact groups on finite von Neumann algebras to actions of closed subgroups on factors.

Høegh-Krohn, Laudstad and Størmer showed that a von Neumann algebra with an ergodic action of a compact group has a finite trace and is hyperfinite. Furthermore the multiplicity of an irreducible representation is always bounded by its dimension [10]. Combining this with Takesaki’s results, shows that any ergodic action of a compact group on a von Neumann algebra is always induced from an action on a full matrix algebra or the hyperfinite $II_1$ factor.

Wassermann developed a general method for studying ergodic actions of compact groups on operator algebras, characterized and classified ergodic actions with full multiplicity and proved that $SU(2)$ does not act ergodically on the hyperfinite $II_1$ factor [48, 49, 50].

Rieffel proposed a generalization of the theory of induced representations to $C^*$–algebras, where the inclusion of groups was replaced by an inclusion of $C^*$–algebras with a conditional expectation. He proved the theorem on induction in stages and an imprimitivity theorem. However, the structure in the group case is richer and parts of the theory of induction, such as Frobenius reciprocity and the tensor product theorem, did not seem to generalize to the $C^*$–algebra setting [40].

Induced representations have been further studied in the framework of quantum groups. In particular, versions of the Frobenius reciprocity theorem for a representation of a subgroup were proved see, e.g. [30, 46, 31]. However, our reference list is by no means complete.

We start with the following observation. Although tensor $C^*$–categories are generically non-embedable, they may contain an embedable subcategory $A$ with ‘few arrows’. More precisely, for every tensor $C^*$–category with conjugates $M$ generated by a single object, we may always find a universal tensor $C^*$–category $A$ generated by solutions of the conjugate equations (a Temperley-Lieb type category), embedable if the intrinsic dimension of the generating object is $\geq 2$. Universality gives a tensor $\ast$–functor $\mu : A \to M$.

Hence we may start with a pair of functors $\text{Hilb} \leftarrow A \xrightarrow{\tau} M$. The embedding $\tau$ is not unique, but a classification is known in certain cases. Different $\tau$ in general give rise to different compact quantum groups $G_\tau$.

The basic example is from a closed subgroup $K$ of a compact group $G$, with $\mu : \text{Rep}(G) \to \text{Rep}(K)$ the restriction functor. The duality theorem of [10] characterizes such inclusions: if $A \subset M$ is an inclusion of tensor $C^*$–categories, with conjugates and permutation symmetry and closed under direct sums and subobjects, then there is a unique inclusion $K \subset G$ of compact groups such that the given inclusion corresponds to $\mu : \text{Rep}(G) \to \text{Rep}(K)$ up to equivalence of categories.

When permutation symmetry is not assumed the framework becomes non-commutative. The special case of embedable inclusions $\mu : A \subset M$ describes
quantum subgroups of compact quantum groups in the sense of Woronowicz and Podleś.

Our theory is based on the concept of a \textit{quasitensor} functor \(\mu : \mathcal{A} \rightarrow \mathcal{M}\) between tensor \(C^\ast\)-categories, replacing the restriction functor arising from an inclusion of compact groups.

Quasitensor functors, first defined in [32], already arise in the case of groups. In fact, it was shown there that the spectral functors of ergodic actions of compact (quantum) groups on unital \(C^\ast\)-algebras are precisely the quasitensor functors from their representation categories to the category of Hilbert spaces. Ergodic actions on classical quotient spaces correspond to functors preserving the natural representation of the symmetric group. In this case, the spectral functor is determined by the restriction functor, which is tensorial. However, it is natural to replace the restriction functor by a quasitensorial functor in view of the role it plays in classifying ergodic actions.

Such functors have proved useful for connecting the theory of ergodic actions with Jones’s highly noncommutative theory of subfactors [19]. Their main feature is an analogue of Popa’s commuting squares [37], which play an important role in that theory.

In fact, the standard invariant of any \(II_1\) subfactor gives rise to such a functor, and hence to an ergodic action of \(S_\mu U(2)\) on a \(C^\ast\)-algebra for uniquely determined negative values of \(\mu\) [33].

Our results, described in the first part of the introduction, are obtained as applications of the following result.

If \(\mu : \mathcal{A} \rightarrow \mathcal{M}\) is a quasitensor functor and \(\tau : \mathcal{A} \rightarrow \text{Hilb}\) is a tensor functor, for each representation \(u\) of the Woronowicz dual \(G_\tau\) of \(\tau\) we construct an induced Hilbert \(G_\tau\)-bimodule representation \(\text{Ind}(\mu_u)\) over a noncommutative unital \(C^\ast\)-algebra, together with a Frobenius tensor \(\ast\)-isomorphism \(\mu_u \rightarrow \text{Ind}(\mu_u)\). \(\text{Ind}(\mu_u)\) turns out to be finite projective.

In the tensorial case, \(\text{Ind}(\mu_u)\) is free and finitely generated as a right module, in agreement with the subgroup case by Swan’s theorem [43].

These bimodule representations will be shown to be \textit{full} bimodule representations. The notion of full bimodule representation, introduced in Sect. 4, is central to this paper. It should not be confused with the usual notion of full \(C^\ast\)-module (density of the linear span of the inner products). It requires that every fixed point for the underlying module representation is central for the action of the coefficient algebra. In this way the left module structure is naturally compatible with the structure of right module representation and the obvious functor from the full subcategory of bimodule representations to that of module representations will be full.

This property allows us to classify the possible bimodule structures on type \(I\) von Neumann algebras, leading to the embedding results (Theorems 5.5, 5.7, Cor. 5.6). It also rules out certain ergodic actions arising from tensor \(C^\ast\)-categories.

If the induced modules are completed in the natural inner product defined by the unique invariant state, the functor \(\text{Ind}\) turns out to be a left adjoint of \(\mu\), essentially by construction.

The ergodic actions of compact quantum groups on noncommutative spaces of
of course correspond to taking \( u \) to be the tensor unit.

The paper is organized as follows. In Sect. 2 we explain the notion of bimodule representations for quantum groups, in Sect. 3 we review Mackey’s induced representation and Frobenius reciprocity from the viewpoint of bimodule representations. In the next section, we introduce the notion of full bimodule representation and discuss the example of quantum quotients. The induced bimodules constructed in the next sections will be shown to be full. In Sect. 5 we illustrate the main ideas and results of this paper. Sections 6-8 are dedicated to the construction of the induction functor to the tensor category of Hilbert \( C^* \)–bimodules. In sect. 6 we give the algebraic construction starting from a pair of quasitensor functors, \((\mu, \tau)\), in the next two sections we require one of them, say \( \tau \), to be tensorial. This restriction does not exclude any ergodic action of a compact quantum group and allows us to show positivity of the inner product. In Sect. 8 we extend the induction functor tensorially to the full tensor subcategory of \( \mathcal{M} \) generated by the image of \( \mu \). In Sect. 9 we show that \((\mu, \text{Ind})\) is an adjoint pair of functors. Sections 6–9 are summarized in Theorems 5.1–5.4. Sect. 10 and 11 are dedicated to the analysis of ergodic actions of compact groups. In Sect. 10 we classify full bimodule representations of compact groups on type \( I \) von Neumann algebras and use the classification in the next section to show the results on the embedding problem into Hilbert spaces (Theorems 5.5–5.7). A few computations in an appendix conclude the paper.

## 2 Bimodule representations of quantum groups

In this section we define unitary representations of compact quantum groups on Hilbert bimodules over unital \( C^* \)–algebras. These representations may be regarded as the noncommutative analogues of the \( G \)–equivariant Hermitian bundles over compact spaces introduced by Segal [42], where \( G \) is a compact group.

In the following general definition we shall not assume any finite projectivity property (a property corresponding to local triviality in the commutative case, by Swan’s theorem [43]), even though we shall eventually be interested in finite projective modules.

Let \( G = (\Omega, \Delta) \) be a compact quantum group [53] and let \( \alpha : \mathcal{C} \to \mathcal{C} \otimes \Omega \) be a fixed nondegenerate action of \( G \) on a unital \( C^* \)–algebra \( \mathcal{C} \). By a module representation of \( G \) we mean a \( \mathbb{C} \)–linear map

\[
v : X_v \to X_v \otimes \Omega,
\]

where \( X_v \) is a (right) Hilbert \( \mathcal{C} \)–module, \( \Omega \) is regarded as the trivial Hilbert \( \Omega \)–module and \( X_v \otimes \Omega \) denotes the exterior tensor product of Hilbert modules, such that

\[
<v(x), v(x')>_{\mathcal{C} \otimes \Omega} = \alpha(<x, x'>_{\mathcal{C}}), \quad x, x' \in X_v, \tag{2.1}
\]

\[
v(xc) = v(x)\alpha(c), \quad x \in X_v, c \in \mathcal{C}, \tag{2.2}
\]

\[
v \otimes 1_\Omega \circ v = 1_{X_v} \otimes \Delta \circ v, \tag{2.3}
\]
If $C = \mathbb{C}$, we recover the usual notion of a unitary representation on a Hilbert space.

The equivariance property is expressed by (2.2).

The action $\alpha$ is a module representation over $C$ regarded as a Hilbert module in the natural way. It will be called the trivial representation and will be denoted by $\iota$.

One can form the $C^*$–category $\text{Mod}_\alpha(G)$ with objects the module representations of $G$ and arrows the intertwiners:

$$ (v, v') := \{ T \in \mathcal{L}_C(X_v, X_{v'}) : v' \circ T = T \otimes 1 \circ v \}, $$

where $\mathcal{L}_C(X_v, X_{v'})$ is the space of bounded adjointable maps from $X_v$ to $X_{v'}$. $(\iota, \iota)$ can be identified with the fixed point algebra $C_\alpha := \{ c \in Z(C) : \alpha(c) = c \otimes I \}$.

We are interested in module representations of a compact quantum group where $X_v$ is a Hilbert bimodule, i.e. $X_v$ has in addition a left $C$–action given by a unital $^*$–homomorphism $\mathcal{C} \to \mathcal{L}_C(X_v)$. $v$ will be called a bimodule representation if in addition

$$ v(cx) = \alpha(c)v(x), \quad c \in \mathcal{C}, x \in X_v. \tag{2.5} $$

As an example, if we regard $\mathcal{C}$ as the trivial Hilbert bimodule, the trivial representation $\iota$ is a bimodule representation.

Bimodule representations form a tensor $C^*$–category, $\text{Bimod}_\alpha(G)$, where $(v, v')$ now denotes the space of bimodule intertwining operators. If $u, v$ are two objects of $\text{Bimod}_\alpha(G)$ we define, for $x \in X_u, y \in X_v$, the tensor product representation $u \otimes v$ by

$$ u \otimes v(x \otimes y) = u(x)_{13}v(y)_{23}, $$

an element of $X_u \otimes_C X_v \otimes \mathcal{Q}$. (2.2) and (2.5) show that $u \otimes v$ is well defined on the algebraic bimodule tensor product $X_u \otimes_C X_v$ and that (2.1), (2.2), (2.3) and (2.5) hold. The validity of (2.1) implies that $u \otimes v$ extends uniquely to a bounded $C$–linear map

$$ u \otimes v : X_u \otimes_C X_v \to X_u \otimes_C X_v \otimes \mathcal{Q}, $$

and the above equations still hold and (2.4) holds by construction. The tensor product of two intertwiners is now well defined and intertwines the tensor product representations. This leads to the following result.

2.1. **Theorem** The category $\text{Bimod}_\alpha(G)$ with objects Hilbert bimodule representations of $G$ and arrows the bimodule intertwining operators is a tensor $C^*$–category. The tensor unit is the trivial representation $\iota$ and $(\iota, \iota) = \{ c \in Z(\mathcal{C}) : \alpha(c) = c \otimes I \}$, the set of central fixed points. There is an obvious faithful $^*$–functor $\text{Bimod}_\alpha(G) \to \text{Mod}_\alpha(G)$.

We shall only consider ergodic actions: $\mathcal{C}^\alpha = \mathbb{C}$. Hence we only get categories with $(\iota, \iota) = \mathbb{C}$. 
The induced bimodules for compact groups

We recall Mackey’s definition of a representation induced from a closed subgroup of a compact group and the Frobenius reciprocity theorem in the form later generalized to tensor $C^*$–categories.

Our discussion has points in common with [41], in that we shall mainly regard the induced representation as acting on a Hilbert bimodule. The bimodule approach to induction is particularly convenient in the compact case as it provides finite dimensional objects by Swan’s theorem [43].

Let $K$ be a closed subgroup of a compact group $G$ and $v$ a (unitary, finite dimensional) representation of $K$ on the Hilbert space $H_v$. Mackey’s induced representation $\text{Ind}(v)$ is defined as right translation by elements of $G$ on the Hilbert space of $L^2$ functions $\zeta$ on $G$ with values in $H_v$ satisfying

$$\zeta(kg) = v(k)\zeta(g), \quad k \in K, g \in G,$$

where the inner product $\langle \zeta, \zeta' \rangle = \int_{K \setminus G} \langle \zeta(g), \zeta'(g) \rangle \, d\mu$ involves the unique normalized $G$–invariant measure $\mu$ on $K \setminus G$.

The main result is the Frobenius reciprocity theorem, asserting that there is an explicit linear isomorphism from the intertwining space $(u \upharpoonright K, v)$ to $(u, \text{Ind}(v))$, taking an intertwiner $S$ to the intertwiner $T$, where $T(\psi)(g) = S(u(g)\psi)$. The Frobenius isomorphism is natural in $u$ and $v$, and hence makes restriction and induction into a pair of adjoint functors.

Consequently, the spectrum of the induced representation is the set of all irreducible $G$–representations $u$ for which $(u \upharpoonright K, v)$ is nonzero. Another consequence is that any irreducible representation $v$ of $K$, and hence any $v$, is a subrepresentation of some restriction to $K$ of a representation $u$ of $G$. Finally, the explicit form of the isomorphism shows that if $T \in (u, \text{Ind}(v))$ all the functions $T(\psi)$ are continuous. Hence we may pass from the Hilbert space of the induced representation to the space $C_v$ of continuous $H_v$–valued functions $\zeta$ as above, a module over the commutative $C^*$–algebra $C(K \setminus G)$ of continuous functions on the quotient space. $C_v$ has an inner product given by pointwise evaluation of the inner product of $H_v$. This inner product is constant on each left coset $Kg$ as $v$ is unitary, and $C_v$ becomes a Hilbert module over $C(K \setminus G)$. Hence $\text{Ind}(v)$ becomes a Hilbert $C(K \setminus G)$–module representation of $G$ in the sense of the previous section, where $\alpha$ is given by right translation by elements of $G$ on the quotient space. $C_v$ is the bimodule corresponding to the equivariant vector bundle induced from $v$. We thus have a $^*$–functor

$$\text{Ind} : \text{Rep}(K) \to \text{Mod}_\alpha(G),$$

taking $T \in (v, v')$ in $\text{Rep}(K)$ to the arrow $\text{Ind}(T) \in (\text{Ind}(v), \text{Ind}(v'))$ defined by $\text{Ind}(T)\xi(g) = T\xi(g)$.

If $u \upharpoonright K$ is the restriction to $K$ of a representation $u$ of $G$, there is a natural faithful module map

$$U : C_u \upharpoonright K \to H_u \otimes C(K \setminus G),$$

$$U\zeta(g) = u(g^{-1})\zeta(g).$$
Thus in this section we introduce a central notion for this paper, that of a full bimodule representations

tensor functor. In the sense recalled in Sect. 5. We summarise this discussion in

3.1. Theorem Mackey’s induction is a full and faithful *-functor \( \text{Ind} : \text{Rep}(K) \to \text{Bimod}_*(G) \) into the bimodule representation category of \( G \), where \( \alpha \) is given by right translation of \( G \) on \( C(K \setminus G) \). For any \( v \in \text{Rep}(K) \), the \( (K \setminus G) \)-bimodule \( C_v \) of \( \text{Ind}(v) \) is finite projective. In particular, if \( v \) is the restriction of a representation of \( G \), \( C_v \) is free. The natural inclusions \( C_u \otimes C_v \to C_{u \otimes v} \) make \( \text{Ind} \) into a relaxed tensor functor.

4 Full bimodule representations

In this section we introduce a central notion for this paper, that of a full bimodule representation of a compact quantum group. As we shall see in later sections, this property selects a proper subclass of noncommutative ergodic spaces still rich enough to include compact quantum quotient spaces but also noncommutative ergodic spaces arising from \( II_1 \) subfactors as in [33] or from tensor \( C^* \)-categories as in [34].

The noncommutative situation involves a unital \( C^* \)-algebra \( \mathcal{E} \) with an ergodic action \( \alpha \) of a compact quantum group \( G \). Following Mackey, one may regard the ergodic action \( (\mathcal{E}, \alpha) \) as arising from a virtual subgroup. Restricting or inducing
4 FULL BIMODULE REPRESENTATIONS

a representation now looses its strict meaning. What is left is the analogue of \(\text{Ind}(u_{|K})\), the representation induced by the restriction to a subgroup \(K\) of a representation \(u\) of \(G\). More precisely, if \(u\) is a unitary representation of \(G\), there is a right Hilbert module \(H_u \otimes \mathcal{C}\) with inner product:

\[
<\psi \otimes c, \psi' \otimes c'> = <\psi, \psi'> c'c,
\]

\(\psi, \psi' \in H_u, c, c' \in \mathcal{C}\).

It is easy to see that the map \(u \otimes \alpha\) defined by

\[
\psi \otimes \alpha(\psi) := u(\psi)\alpha(c)_{23} \in H_u \otimes \mathcal{C} \otimes \mathcal{C}, \quad \psi \in H_u, c \in \mathcal{C}.
\]

is a module \(G\)-representation on \(H_u \otimes \mathcal{C}\).

Given \(T \in (u, v)\), with \(u, v \in \text{Rep}(G)\), define \(T \otimes 1_\mathcal{C} : H_u \otimes \mathcal{C} \to H_v \otimes \mathcal{C}\) by \(T \otimes 1_\mathcal{C}(\psi \otimes c) = T\psi \otimes c\) then \(T \otimes 1_\mathcal{C} \in (u \otimes \alpha, v \otimes \alpha)\) in \(\text{Mod}_\alpha(G)\).

4.1. Proposition The map \(\text{Rep}(G) \to \text{Mod}_\alpha(G)\), taking \(u \to u \otimes \alpha\) and \(T \in (u, v) \to T \otimes 1_\mathcal{C}\) is a faithful \(^*\)-functor.

Our aim is to show that a large class of tensor \(C^*\)-categories give rise to ergodic actions of compact quantum groups where an analogue of Theorem 3.1 holds. In particular, the corresponding \(u \otimes \alpha\) become bimodule \(G\)-representations with a left module structure naturally compatible with the right module representation structure. In this section we introduce this compatibility condition.

We first motivate the need for a condition by taking \(G\) to be a compact group, but allowing \(\mathcal{C}\) to be noncommutative.

Note that the natural left \(\mathcal{C}\)-action on \(H_u \otimes \mathcal{C}\) can always be completed to a bimodule representation.

However, an intertwiner between two such bimodule representations, being a bimodule map, must lie in \(\mathcal{B}(H_u, H_{u'}) \otimes Z(\mathcal{C})\). Hence these intertwiners do not see the noncommutativity of \(\mathcal{C}\), in contrast to the module intertwining spaces, where \((\iota, u \otimes \alpha)\) is the space of multiplets \(\xi = (c_1, \ldots, c_d)\), with \(c_i \in \mathcal{C}\) and \(\alpha(c_i) = \sum_j c_j \otimes u_{ji}^*\), \(d\) being the dimension of \(u\). As \(u\) varies over the irreducible spectral representations of \(G\), the linear span of the corresponding \(c_i\)’s is a dense invariant \(^*\)-subalgebra.

Hence if \(u\) lies in the spectrum, the natural left action on \(H_u \otimes \mathcal{C}\) gives rise to a tensor category which does not allow one to reconstruct \(\mathcal{C}\). It would be desirable to use instead a left \(\mathcal{C}\)-action on \(H_u \otimes \mathcal{C}\) where all the module \(G\)-intertwiners become bimodule \(G\)-intertwiners. This leads us to the notion of full bimodule representation.

Definition Let \(G\) be a compact quantum group. A fixed vector \(\xi\) for a module representation \(v\) on \(X_v\) is an element \(\xi \in X_v\) such that \(v(\xi) = \xi \otimes I\). The set of fixed vectors for \(v\) is precisely the intertwining space \((\iota, v)\) in \(\text{Mod}_\alpha(G)\). A bimodule representation \(v\) will be called full if every fixed vector \(\xi\) for the underlying module \(G\)-representation is central: \(\xi c = c\xi\), for \(c \in \mathcal{C}\).

Note that the trivial representation is full since \(\alpha\) is ergodic. The next result shows that if \(G\) is a group, classical induction is characterized among the functors \(u \to u \otimes \alpha\) from \(\text{Rep}(G)\) to \(\text{Mod}_\alpha(G)\) by the property that under the natural left action each \(u \otimes \alpha\) becomes a full bimodule representation.
4.2. Proposition Let $G$ be a compact group and $\alpha$ an ergodic nondegenerate action of $G$ on a unital $C^*$–algebra $\mathcal{C}$. Then the natural left $\mathcal{C}$–action turns $u \otimes \alpha$ into a full representation for all $u \in \text{Rep}(G)$ if and only if $\mathcal{C}$ is commutative. In this case, $\mathcal{C} = C(K \backslash G)$ for a closed subgroup $K$, unique up to conjugation where $\alpha$ acts by right translation, $\alpha_g f(g') = f(g'g)$, $f \in C(K \backslash G)$. Hence $u \otimes \alpha$ corresponds to the classical induced representation $\text{Ind}(u |_K)$.

Proof If $\mathcal{C}$ is commutative, $c \eta = \eta c$ for $c \in \mathcal{C}$, $\eta \in H_u \otimes \mathcal{C}$ and $u \in \text{Rep}(G)$. Hence any module intertwiner between $u \otimes \alpha$ and $u' \otimes \alpha$ is a bimodule intertwiner. In particular, $u \otimes \alpha$ is full for all $u$. Conversely, assume that all the $u \otimes \alpha$ are full. We have already seen that a fixed vector $\xi$ for $u \otimes \alpha$ has the form $\xi = \sum_j \psi_j \otimes c_j$ for an orthonormal basis $(\psi_j)$ of $H_u$ where $c_j$ transforms like the complex conjugate representation $u^* = (u^*_{i,j})$ under $\alpha$. Since $\xi$ is supposed central, the elements $c_j$ are central in $\mathcal{C}$. If $u$ varies in the spectrum of $\alpha$, we get a dense commutative $*$–subalgebra of $\mathcal{C}$, hence $\mathcal{C}$ is commutative. As is well known, when $\alpha$ is an ergodic action on a unital commutative $C^*$–algebra, the action is right translation on $C(K \backslash G)$ by elements of $G$ for a closed subgroup $K$ of $G$, unique up to conjugation. Hence $u \otimes \alpha$ can be identified with $\text{Ind}(u |_K)$.

As we shall see from examples arising from group actions, discussed in Sect. 10, if $u$ is in the spectrum of $\alpha$, there may be no left $\mathcal{C}$–action on all of $H_u \otimes \mathcal{C}$ making $u \otimes \alpha$ into a full bimodule representation, but there always is a nonzero $G$–submodule representation $X_u$ with this property.

If $G$ is a quantum group, the natural left action on $H_u \otimes \mathcal{C}$ does not even lead to a bimodule representation.

4.3. Proposition Let $G$ be a compact quantum group, $u \in \text{Rep}(G)$ and $\alpha$ a nondegenerate ergodic action of $G$ on a unital $C^*$–algebra $\mathcal{C}$. Then $u \otimes \alpha$ is a bimodule representation for the natural left module structure if and only if all coefficients of the irreducibles in the spectrum of $\alpha$ commute with the coefficients of $u$.

Proof Since $\alpha$ is nondegenerate, $\mathcal{C}$ is generated as a Banach space, by the entries of rectangular matrices $(c^\alpha_{i,j})$ transforming like irreducible $G$–representations

$$\alpha(c^\alpha_{i,j}) = \sum_p c^\alpha_{i,p} \otimes v_{p,j}.$$ 

These entries are linearly independent \cite{36, 2, 32} and the conclusion follows from (2.5).

We conclude the section with examples of full bimodule representations arising from quantum quotients.

Examples from quantum quotients.

Let $G = (\Omega, \Delta)$ be a compact quantum group, and $K$ a quantum subgroup. $G$ acts on the quotient space $K \backslash G$ by right translation, given by restriction of the coproduct $\Delta$ of $G$. One can consider the left $C(K \backslash G)$–action on $H_u \otimes C(K \backslash G)$
defined as follows. For \( c \in C(K \setminus G) \), consider the element \( \lambda_u(c) \in \mathcal{L}(H_u) \otimes Q \) defined by

\[
\langle \psi_i \otimes I, \lambda_u(c) \psi_j \otimes I \rangle := \sum_h u_{hi}^* cu_{hj},
\]

where \((\psi_i)\) is an orthonormal basis of \( H_u \). It is easy to check that this element is independent of the choice of orthonormal basis. One could show directly that \( \lambda_u(c) \in \mathcal{L}(H_u) \otimes C(K \setminus G) \) and that \( \lambda_u \) makes \( u \otimes \alpha \) into a full bimodule representation. However, we refrain from giving complete details, as this will be proved in more generality in Sect. 7. We just verify that it is full. If \( \xi := \sum \psi_j \otimes c_j \in H_u \otimes \mathbb{C} \) is a fixed vector, i.e., \( \Delta(c_j) = \sum_j c_j \otimes u_{j_1}^* \), then

\[
c\xi = \sum_k \psi_k \otimes \langle \psi_k \otimes I, \lambda_u(c) \xi \rangle = \sum_{k,j,h} \psi_k \otimes u_{hk}^* cu_{hj} c_j
\]

whereas

\[
\xi c = \sum_k \psi_k \otimes c_k c,
\]

hence we need to show that for every \( c \in C(K \setminus G) \),

\[
\sum_{h,j} u_{hk}^* cu_{hj} c_j = c_k c
\]

for all \( k \). On the other hand, for a quotient space, we can find a \( K \)–fixed vector \( \eta \) of \( H_u \) such that \( c_j = u_{\eta,j}^* \), and the desired equality follows from the unitarity of \( u \).

In Sections 6 and 7 we discuss full bimodule representations of compact quantum groups arising from tensor \( C^* \)–categories with conjugates, whilst in Sect. 10 we shall classify full bimodule representations arising from ergodic actions of compact groups on type \( I \) von Neumann algebras. The class of examples of Cor. 10.10 or those following Theorem 10.9 for \( SU(2) \) shows that the condition that \( u \otimes \alpha \) has a full bimodule representation structure for all \( u \in \text{Rep}(G) \) rules out many ergodic actions.

5 Main results

In this section we illustrate the main ideas and results. Proofs will be given in later sections. As discussed in the introduction, a closed subgroup \( K \) of a compact group \( G \) admits a categorical counterpart, an inclusion \( \mu : \text{Rep}(G) \subset \text{Rep}(K) \) of tensor \( C^* \)–subcategories of the category Hilb of Hilbert spaces, where \( \mu \) is the functor restricting a representation to the subgroup.

Conversely, given an inclusion \( \mu : \mathcal{A} \subset \mathcal{M} \) of tensor \( C^* \)–categories with conjugates and permutation symmetry (and subobjects and direct sums) we may, by \[10\], find a unique pair of \( K \) and \( G \) such that, up to equivalence, \( \mathcal{A} = \text{Rep}(G) \), \( \mathcal{M} = \text{Rep}(K) \), with their natural permutation symmetries, the inclusion being given by the restriction functor.
5 MAIN RESULTS

We may thus generalize the above, dropping the existence of permutation symmetry and considering tensor $C^*$-categories $\mathcal{A}$ and $\mathcal{M}$ and an inclusion $\mathcal{A} \subset \mathcal{M}$. Consequently, $\mathcal{M}$ may not be embedable into Hilbert spaces. We will, however, assume that $\mathcal{A}$ has conjugates defined abstractly as in [22] (see Sect. 6).

The subcategory $\mathcal{A}$, to be regarded as containing few arrows, is usually embeddable and we then pick a tensor $^*$--functor $\tau: \mathcal{A} \to \text{Hilb}$. The inclusion $\mathcal{A} \subset \mathcal{M}$ is described by a by a quasitensor $^*$--functor $\mu: \mathcal{A} \to \mathcal{M}$, tensorial only in favourable circumstances. This functor generalizes the restriction functor of an inclusion of compact groups.

The study of quasitensor functors is motivated by the main result in [32] asserting that the spectral functors of ergodic actions of compact quantum groups $G$ on unital $C^*$--algebras are precisely the quasitensor functors from the representation category of $G$ to the category of Hilbert spaces. The definition will be recalled shortly.

All our tensor categories will be assumed to be strict. The tensor product between objects $u$ and $v$ will be denoted by $u \otimes v$ and between arrows $S$ and $T$ by $S \otimes T$. The tensor unit, denoted by $\iota$, will be assumed irreducible: $(\iota, \iota) = \mathbb{C}$.

The n-th tensor power of an object $u$ will be denoted $u^n$.

A $^*$--functor $\mu: \mathcal{A} \to \mathcal{M}$ together with a collection of isometries $\tilde{\mu}_{u,v} \in (\mu_u \otimes \mu_v, \mu_{u \otimes v})$, for objects $u, v \in \mathcal{A}$, is called quasitensor if

$$\mu_u = \iota,$$

$$\tilde{\mu}_{u,u} = \tilde{\mu}_{v,v} = 1_{\mu_u},$$

$$\tilde{\mu}_{u,v} \circ \tilde{\mu}_{u \otimes v, w} = 1_{\mu_u} \otimes \tilde{\mu}_{v,w} \circ \tilde{\mu}_{u,v} \otimes 1_{\mu_w}$$

and if

$$\mu(S \otimes T) \circ \tilde{\mu}_{u,v} = \tilde{\mu}_{u',v'} \circ \mu(S) \otimes \mu(T),$$

for any other pair of objects $u', v'$ and arrows $S \in (u, u')$, $T \in (v, v')$. In particular, a tensor functor $\mu$ is quasitensor with $\tilde{\mu}_{u,v} := 1_{\mu_u \otimes \mu_v}$. More generally, if all the isometries $\tilde{\mu}_{u,v}$ are unitary, we recover the known notion of a relaxed tensor functor.

We assume that $\mathcal{A}$ and $\mathcal{M}$ are tensor $C^*$--categories, that $\mathcal{A}$ has conjugates, that $\tau: \mathcal{A} \to \text{Hilb}$ is a tensor $^*$--functor to Hilbert spaces, and $\mu: \mathcal{A} \to \mathcal{M}$ is quasitensor functor. As we shall see in the next section, some of the results hold when $\tau$ is just quasitensor.

The embedding $\tau$ defines, by Woronowicz duality, a compact quantum group $G_\tau$ such that for every object $u \in \mathcal{A}$ there is an associated representation $\hat{u} \in \text{Rep}(G_\tau)$ on the Hilbert space $\tau_u$.

Our first result, proved in Sections 6 and 7, concerns the construction of the induced bimodules. Recall that $\mathcal{M}$ is a unital $C^*$--algebra, $\mathcal{E}$, and an ergodic action $\alpha$ of $G_\tau$ on $\mathcal{E}$.

5.1. Theorem Pick an object $u$ of $\mathcal{A}$ with $1_{\mu_u} \neq 0$. The linear space $^\circ \mathcal{H}_u$ obtained quotienting $\sum_v (\mu_v, \mu_u) \otimes \tau_v$ by the linear subspace generated by elements of the form $M \circ \mu(A) \otimes \psi - M \otimes \tau(A) \circ \psi$ can be naturally completed into a full Hilbert bimodule $\mathcal{H}_u$ over $\mathcal{E}$. 
There is a unique, full, bimodule representation, \( \text{Ind}(\mu_u) \), of \( G_\tau \) on \( \mathcal{H}_u \) with
\[
\text{Ind}(\mu_u)M \otimes \psi = M \otimes \hat{v}\psi,
\]
for \( M \in (\mu_v, \mu_u) \), \( \psi \in \tau_v \), \( \hat{v} \) being the representation of \( G_\tau \) on \( \tau_v \).

If \( \mu \) is relaxed tensor, \( \mathcal{H}_u \) is naturally isomorphic to \( \tau_u \otimes \mathcal{C} \) as a right module and \( \text{Ind}(\mu_u) \) becomes \( u \otimes \alpha \) under this isomorphism. Otherwise, as a module \( G_\tau \)-representation, \( \mathcal{H}_u \) is a projective subrepresentation of \( \tau_u \otimes \mathcal{C} \).

We shall see in Sect. 7 that \( \text{Ind} \) becomes a \( * \)-functor defined on the full \( C^* \)-subcategory of \( \mathcal{M}_u \) with objects the images of objects of \( A \) under \( \mu \). In the tensorial case, it takes an arrow \( \mu(A) \) to \( \tau(A) \otimes I \) under the isomorphism \( \mathcal{H}_u \cong \tau_v \otimes \mathcal{C} \).

Now, \( \mathcal{M}_u \) is not a tensor category if \( \mu \) is just quasitensor. We ask, however, whether \( \text{Ind} \) extends \textit{tensorially} to the smallest full tensor subcategory \( \mathcal{M}^\otimes_u \) of \( \mathcal{M} \) generated by \( \mathcal{M}_u \). Somewhat surprisingly, the answer is that it does. Sections 8 and 9 will be devoted to discussing the following result that may be regarded as the analogue of Theorem 3.1.

**5.2. Theorem** The induction functor \( \mu_u \rightarrow \text{Ind}(\mu_u) \) extends uniquely to a full and faithful relaxed tensor \( * \)-functor from \( \mathcal{M}^\otimes_u \) to \( \text{Bimod}_u(G_\tau) \). Furthermore, \( (\text{Ind}, \mu) \) gives rise to an adjoint pair of functors.

We next give two applications of our results, that originally motivated our work. The first concerns a tensor \( C^* \)-category with conjugates whose object set contains a distinguished generating element. We give two results, corresponding to the selfconjugate or non-selfconjugate case.

**5.3. Theorem** Let \( \mathcal{M} \) be a tensor \( C^* \)-category with objects \( \iota, x, x^2, \ldots \), where \( x \) is a real or pseudoreal object defined by a solution \( R \in (\iota, x^2) \) of the conjugate equations with \( \|R\|^2 \geq 2 \). Let \( F \in \text{Mat}_n(\mathbb{C}) \) be an invertible matrix with \( \text{Tr}(FF^*) = \text{Tr}(FF^*)^{-1} = \|R\|^2 \). Then there is a full and faithful tensor \( * \)-functor \( \mathcal{M} \rightarrow \text{Bimod}_u(A_\alpha(F)) \), where \( \alpha \) is the ergodic action of \( A_\alpha(F) \) on \( \mathcal{C} \) constructed in [22].

**5.4. Theorem** If the set of objects of \( \mathcal{M} \) is generated, as a semigroup, by \( x \) and a conjugate \( \overline{x} \), with intrinsic dimension \( d(x) \geq 2 \), then conclusions analogous to Theorem 5.3 hold where the quantum group is now \( A_\alpha(F) \).

Examples Many examples of noncommutative ergodic spaces are known to arise from pairs \( (\mu_\tau) \). Compact quantum quotients \( C(K\backslash G) \) \cite{36} \cite{47} \cite{51} arise from the restriction functor \( \mu : \text{Rep}(G) \rightarrow \text{Rep}(K) \). The examples with high multiplicities of \( [2] \) are associated with the composition of a tensorial isomorphism with the embedding functor, \( \mu : \text{Rep}(G) \simeq \text{Rep}(G^\prime) \rightarrow \text{Hilb} \). Any ergodic action of a compact quantum group arises from a quasitensor \( * \)-functor, its spectral functor, \( \overline{\tau} : \text{Rep}(G) \rightarrow \text{Hilb} \) \cite{32}. In these cases \( \tau \) is the embedding functor of \( \text{Rep}(G) \).

Examples of categories of the type described in Theorems 5.3 and 5.4 arise from inclusions of \( II_1 \) factors \( N \subset M \) with finite Jones index. The ergodic action corresponding to the real object \( N \cdot M_N \) is made explicit in [33]. For any finite index inclusion of infinite factors described by an endomorphism \( \rho \) with \( d(\rho) \geq 2 \),
the tensor $C^*$–category generated by $\rho$ and $\mathcal{P}$ is of the form described in Theorem 5.4.

The proofs of Theorems 5.3 and 5.4 will be given at the end of Sect. 8. The next application concerns tensor categories $\mathcal{M}$ which admit an embedding of the representation category of a compact group. The following results, discussed in Sections 10 and 11, shed light on the problem of recognizing which tensor categories can be embedded into Hilbert spaces. They are obtained combining our bimodule construction with the work of Takesaki [45], Høegh–Krohn, Landstad and Størmer [16] and Wassermann [50].

In the following theorem $G$ is a compact group with a distinguished faithful representation $u$, in the sense that every irreducible of $G$ is a subrepresentation of a tensor power of $u$. We denote by $\mathcal{S}_G$ be the full subcategory of $\text{Rep}(G)$ with objects $\iota, u, u^2, \ldots$.

5.5. Theorem Let $G$ be a compact group with a distinguished faithful representation $u$, and let $\mu: \mathcal{S}_G \to \mathcal{M}$ be a tensor $^*$–functor. Let $\mathcal{E}$ be the ergodic $C^*$–algebra associated with $\mu$ and the embedding functor $\tau$ of $\mathcal{S}_G$ into the category of Hilbert spaces. Assume that the von Neumann algebra $\mathcal{E}''$ generated by $\mathcal{E}$ in the GNS representation of the unique $G$–invariant trace is of type $I$ and let $K$ be a closed subgroup of $G$ such that $L^\infty(K\backslash G) \cong Z(\mathcal{E}'')$ as ergodic $W^*$–systems. Then there is a full and faithful tensor $^*$–functor $\epsilon: \mathcal{M}_\mu \otimes \mu \to \text{Rep}(K)$.

Notice that in the above theorem $\mathcal{M}_\mu \otimes \mu$ is simply the full subcategory of $\mathcal{M}$ with objects the tensor powers of $\mu_u$.

Remarks

1) Recall that the representation category of a compact group does not determine the group uniquely as a tensor $^*$–category. In fact, Izumi and Kosaki have shown the existence of two different finite groups with tensor $^*$–isomorphic representation categories [13].

2) As we shall see in Sect. 11, the functor $\epsilon$ is naturally associated with $\mu$. However, the set of objects in the image of $\epsilon$ in general does not generate all the representations of $K$ under tensor products, subobjects and direct sums. In fact, in the particular case where each irreducible of $G$ has multiplicity in $\mathcal{E}$ equal to its dimension, then $\epsilon$ maps each object to the trivial representation of $K$. Hence $\mathcal{M}_\mu$ admits a tensor $^*$–functor to a full subcategory of the category of Hilbert spaces. Furthermore, any full multiplicity ergodic action of $G$ on a type $I$ von Neumann algebra arises from a relaxed tensor functor $\mu$, its spectral functor.

At the other extreme, if $\mathcal{E}$ is commutative, and this happens in particular if $\mathcal{M}$ has permutation symmetry, see [32], then we recover the following result, an important step used in [10] to prove the abstract duality theorem for compact groups.

5.6. Corollary If in particular $\mathcal{E}$ is commutative, and hence $\mathcal{E} = C(K\backslash G)$ as ergodic $C^*$–systems, then $\epsilon(\mu_u) = u|_K$. Hence the completion of the image of $\epsilon$ under subobjects contains any irreducible of $K$. 

It is still an open problem, posed in [16], whether ergodic actions of compact simple groups on von Neumann algebras are always of type $I$. Wassermann has shown this to be true for $SU(2)$ [50]. Consequently, we obtain the following embedding result for tensor $C^*$-categories containing a distinguished pseudoreal object of dimension 2. No permutation symmetry is assumed.

5.7. Corollary Let $\mathcal{M}$ be a tensor $C^*$–category whose object semigroup is generated by a pseudoreal object $x$ of dimension 2, i.e. with an intertwiner $R \in (\iota, x^2)$ such that
\[ R^* \otimes 1_x \circ 1_x \otimes R = -1_x, \]
\[ \|R\|^2 = 2. \]

Then there is a closed subgroup $K$ of $SU(2)$ and a full and faithful tensor $*$–functor $\mathcal{M} \rightarrow \text{Rep}(K)$.

Remark Our results suggest the problem of classifying the ergodic actions of compact quantum groups arising from tensor $*$–functors $\text{Rep}(G) \rightarrow \mathcal{M}$. (Of course, any ergodic action arises from a quasitensor functor into Hilbert spaces, its spectral functor). All the examples previously described are of this form. This problem is clearly related to that of classifying ergodic actions of $S_\mu U(2)$, or, more generally, of the Wang-van Daele quantum groups. It is not yet clear whether non-quotient Podles spheres [36] or Wang’s examples [47] can arise in this way. We notice, however, that already for ergodic $C^*$–actions of compact groups on type $I$ factors, not all ergodic actions can arise. For instance, neither the adjoint action by a non-trivial irreducible representation of $SU(2)$, nor those with full spectrum and an irreducible of low multiplicity can arise, see Sect. 10.

6 Algebraic bimodules from pairs of functors

Let $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{T}$ be tensor $C^*$–categories with irreducible tensor units $\iota$. We assume that $\mathcal{A}$ has conjugates. Let $(\mu, \bar{\mu}) : \mathcal{A} \rightarrow \mathcal{M}$ and $(\tau, \bar{\tau}) : \mathcal{A} \rightarrow \mathcal{T}$ be a pair of quasitensor functors. In [32] we have associated a unital $C^*$–algebra $\mathcal{C}$ completing a canonical dense $*$–subalgebra $\mathcal{C}$ if one of the functors, say $\tau$, is tensorial and embeds $\mathcal{A}$ into the category of Hilbert spaces. In this section we generalize that construction at the algebraic level to any pair of quasitensor $*$–functors, obtaining bimodules over a $*$–algebra.

Pick objects $t$ and $u$ of $\mathcal{A}$. Let $\mathcal{M}_{\mu}^{\mu}$ be the linear space $\sum_{(\mu, \mu)} (\tau_t, \tau_v)$, the sum being taken over the objects of $\mathcal{A}$, quotiented by the linear subspace generated by elements of the form
\[ M \circ \mu(A) \otimes T - M \otimes \tau(A) \circ T. \]

Notice that, as the objects involved have conjugates [32], [34], we are actually taking a sum of finite dimensional vector spaces [22]. We next define bilinear maps
\[ \mathcal{M}_{\mu}^{\mu} \times \mathcal{M}_{\mu}^{\mu} \rightarrow \mathcal{M}_{\mu}^{\mu} \otimes \mathcal{M}_{\mu}^{\mu}. \]
For $\xi = L \otimes S$, $\eta = M \otimes T$, where $L \in (\mu_w, \mu_u)$, $M \in (\mu_v, \mu_{w'})$, $S \in (\tau_t, \tau_u)$, $T \in (\tau_v, \tau_{w'})$, set:

$$\xi \cdot \eta := \tilde{\mu}_{u,w'} \circ (L \otimes M) \circ \tilde{\mu}^*_{w,v} \otimes \tilde{\tau}_{w,v} \circ (S \otimes T) \circ \tilde{\tau}^*_{t,t'}.$$  

It is easy to check that these maps are well defined and associative as a consequence of associativity of the functors $\mu$ and $\nu$. In particular, $\mathcal{H}_u$ is an algebra that we denote by $^0\mathcal{C}$ and $^\circ\mathcal{H}_u$ is a $^0\mathcal{C}$–bimodule.

Given $X \in (\tau_t, \tau_{w'})$ and $Y \in (\mu_u, \mu_{w'})$, we can define a functor of two variables

$$\nu(Y, X) := \lambda(Y) \rho(X) = \rho(X) \lambda(Y) : ^\circ\mathcal{H}_u \to ^0\mathcal{H}_{w'},$$

setting

$$\rho(X)(M \otimes T) := M \otimes (T \circ X),$$

$$\lambda(Y)(M \otimes T) := (Y \circ M) \otimes T.$$  

It is easily checked that $\nu(Y, X)$ is a bimodule map and hence a functor, covariant in the first variable and contravariant in the second, from the product of the full subcategories of $M$ and $\mathcal{T}$ whose objects are the images of objects of $\mathcal{A}$ into the category of $^0\mathcal{C}$–bimodules.

It should be noted that the bimodules $^\circ\mathcal{H}_u$ and the structure introduced above in fact depend only on $\mu_u$ and $\tau_u$. We next attempt to define an adjoint on these bimodules. Here matters are more complicated.

Recall [22] that an object $\bar{\tau}$ is called a conjugate of $u$ if there are intertwiners $R_u \in (\tau, \bar{\tau} \otimes u)$, $\overline{R}_u \in (\tau, u \otimes \bar{\tau})$ satisfying the conjugate equations

$$\overline{R}_u \otimes 1_\bar{\tau} \circ 1_u \otimes R_u = 1_u$$

$$R_u^* \otimes 1_\bar{\tau} \circ 1_\bar{\tau} \otimes \overline{R}_u = 1_\bar{\tau}.$$  

If $R_u$, $\overline{R}_u$ is a solution of the conjugate equations for $u$, any solution is of the form $X^{*\leftarrow} \otimes 1_u \circ R_u$ and $1_u \otimes X \circ \overline{R}_u$, where $X \in (\bar{\tau}, \bar{u})$ is an invertible intertwiner.

We pick a solution $R_u$, $\overline{R}_u$ or each object $u$ of $\mathcal{A}$ and define the associated conjugation,

$$A \in (v, u) \to A^\bullet := R_v^* \otimes 1_\bar{\tau} \circ 1_\bar{\tau} \otimes A^* \otimes 1_\bar{\tau} \circ 1_\bar{\tau} \otimes \overline{R}_u, \quad \in (\bar{\tau}, \bar{\tau}).$$

This conjugation depends on the choice of conjugates: changing the solution of the conjugate equations using invertibles $X \in (\bar{\tau}, \bar{u})$ and $Y \in (\bar{\tau}, \bar{v})$, $A^\bullet$ becomes $X \circ A^\bullet \circ Y^{-1}$.

$$R_{u \otimes v} := 1_\bar{\tau} \otimes R_u \otimes 1_v \circ R_v$$

and

$$\overline{R}_{u \otimes v} := 1_u \otimes \overline{R}_u \otimes 1_\bar{\tau} \circ \overline{R}_u$$

is a solution for $u \otimes v$, called a tensor product solution. Similarly, $R_{\bar{\tau}} := \overline{R}_u$ and $\overline{R}_{\bar{\tau}} := R_u$ the solution for $\bar{\tau}$, called a conjugate solution. A choice $u \mapsto R_u$ of solutions of the conjugate is said to be homomorphic, if $R_{u \otimes v}$ is always a product solution and $R_{\bar{\tau}}$ is always a conjugate solution.

The main properties of conjugation are the following:

$$(A \circ B)^\bullet = A^\bullet \circ B^\bullet,$$
and
\[(A \otimes B)^\bullet = B^\bullet \otimes A^\bullet,\]
for a tensor product solution.

If \((\mu, \hat{\mu})\) is a quasitensor functor then \(\hat{R}_u := \hat{\mu}_{\tau,u} \circ \mu(R_u)\) and \(\tilde{R}_u := \tilde{\mu}_{\tau,u} \circ \mu(R_u)\) is a solution of the conjugate equations for \(\mu_u\), called the image solution of \(R_u, \tilde{R}_u\) under \(\mu\) [32]. This solution defines a conjugation on the category whose objects are those of \(\mathcal{A}\) but where the set of arrows from \(u\) to \(v\) are \((\mu_u, \mu_v)\) and we have
\[\mu(A)^\bullet = \mu(A^\bullet).\]

Fixing solutions of the conjugate equations for objects \(t\) and \(u\) of \(\mathcal{A}\), we define an antilinear map
\[^* : \mathcal{H}_u \to \mathcal{H}_u^\tau\]
by setting
\[(M \otimes T)^* := M^\bullet \otimes T^\bullet,\]
where the conjugation is defined using image solutions of the conjugate equations under \(\mu\) and \(\tau\). Notice that \(^*\) is well defined and independent of the choice of solutions of the conjugate equations in \(\mathcal{A}\) for the running objects \(v\) appearing in the sum. However, if we change the solution of the conjugate equations for \(u\) and \(t\) using invertibles \(Y \in (\tilde{\tau}, \tilde{t})\) and \(X \in (\tau, \tilde{u})\), \((M \otimes T)^*\) becomes \((\mu(X)M^\bullet) \otimes (T^\bullet \tau(Y^{-1})) = \lambda_\tau(\mu(X))\rho_\tau(\tau(Y^{-1}))(M \otimes T)^* = \nu(\mu(X), \tau(Y^{-1}))(M \otimes T)^*\). Furthermore, the conjugation depends on \(u\) and not just on \(\mu_u\).

However, for \(u = t = \iota\) the \(^*\)–operation is independent and makes \(\mathcal{H}\) into a unital \(^*\)–algebra.

We have the following result. If \(u \mapsto R_u\) is a homomorphic choice of solutions of the conjugate equations, then \(^*\) has the properties expected of an adjoint.

6.1. Proposition Let \(u, u', t, t'\) be objects of \(\mathcal{A}\). If \(\xi \in \mathcal{H}_u\) and \(\xi' \in \mathcal{H}_{u'},\)
\[(\xi, \xi')^* = \xi^* \cdot \xi\quad \text{and} \quad \xi^{**} = \xi,\]
where we have used tensor product solutions of the conjugate equations for \(u \otimes u'\) and \(t \otimes t'\) and conjugate solutions for \(\tau\) and \(\tilde{\tau}\).

We next summarize the main properties of the functors \(\lambda\) and \(\rho\) with respect to the product \(\cdot\) and the adjoint \(^*\).

6.2. Proposition For \(Y \in (\mu_u, \mu_{u'}),\) \(\xi \in \mathcal{H}_u,\) \(\xi' \in \mathcal{H}_{u'},\) \(A \in (u, z),\) \(A' \in (u', z'),\)
\[(\lambda(\lambda(Y))\xi)^* = \lambda(Y^\bullet)\xi^*,\]
\[\lambda(\mu(A \otimes A'))\xi \cdot \xi' = (\lambda(\mu(A))\xi) \cdot (\lambda(\mu(A'))\xi').\]
Similar relations hold for \(\rho\).

We next define a \(\mathcal{H}\)–valued form on \(\mathcal{H}_u\) by
\[< \xi, \xi' > := \nu(\mu(R_u)^*, \tau(R_t))(\xi^* \cdot \xi').\]
The explicit formula, for \(\xi = M \otimes T,\) \(\xi' = M' \otimes T',\) with \(M \in (\mu_v, \mu_u),\) \(T \in (\tau_t, \tau_v),\) \(M' \in (\mu_{v'}, \mu_u),\) \(T' \in (\tau_{t'}, \tau_{v'})\) is
\[< \xi, \xi' > := (\tilde{R}_u^* \circ M^\bullet \otimes M' \otimes \tilde{\mu}_{\tau,v'}(\tilde{\tau}_{v'} \otimes T^\bullet \otimes T' \circ \tilde{R}_t)).\]
7 THE INDUCED $C^*$–BIMODULES

Hence if we change the solution of the conjugate equations for $u$ using an invertible $X$ then $T^\ast$ becomes $\mu(X) \circ T^\ast$, and this cancels the simultaneous change of $R_u$. However, the formula is less symmetric than it appears, as $\ast$ does not in general commute with the adjoint. If we change the conjugate of $t$ using an invertible $Y$, the tensor at the right hand side becomes $(\tilde{\tau}_{\tau,\nu'} \circ T^\ast \otimes T \circ \tau((Y^\ast Y)^{-1}) \otimes 1_{\tau} \circ \tilde{R}_t)$. (This problem can be overcome by choosing standard solution $s$ of the conjugate equations for $t$, since $Y$ is then unitary and the inner product does not depend on the choice of the standard solutions for $t$.) In the next section we anyway take $t = \iota$ and $\mathcal{T}$ to be the category of Hilbert spaces, hence the form becomes independent of the solutions of the conjugate equations. We shall show that the form is positive in this case.

We conclude this section with an explicit computation of the right hand side needed later.

\[
\tilde{R}_u^\ast \circ M^\ast \otimes M' = \tilde{R}_u^\ast \circ [(\tilde{R}_v^\ast \circ 1_{M'_{\tau}} \otimes 1_{M^\ast_{\tau}} \circ M^\ast \otimes 1_{M_{\tau}} \circ 1_{M'_{\tau}} \circ \tilde{R}_u) \otimes M'] = \\
\tilde{R}_v^\ast \circ \tilde{R}_u^\ast \circ 1_{M'_{\tau}} \otimes 1_{M^\ast_{\tau}} \circ 1_{M_{\tau}} \circ 1_{M'_{\tau}} \circ \tilde{R}_u \otimes 1_{M'_{\tau}} \circ 1_{M_{\tau}} \circ M' = \\
\tilde{R}_v^\ast \circ 1_{M_{\tau}} \otimes (M^\ast \circ M').
\]

Hence

\[
\langle \xi, \xi' \rangle = (\tilde{R}_u^\ast \circ 1_{M_{\tau}} \otimes (M^\ast \circ M') \circ \tilde{\mu}_{\tau,\nu'}^\ast \otimes (\tilde{\tau}_{\tau,\nu'} \circ T^\ast \otimes T' \circ \tilde{R}_t).
\]

Remark This formula shows that the form depends only on $\mu_u$ and $\tau_t$ and not on $u$ and $t$.

7 The induced $C^*$–bimodules

In this section we show that the form defined in the last section is positive when one of the two functors, say $\tau$, is a tensor $^\ast$–functor into a (strict) tensor $C^*$–category of Hilbert spaces and we consider the $^\ast C$–bimodule $^\circ \mathcal{H}_u$, simply denoted by $^\circ \mathcal{H}_u$. Hence we have $\tilde{\tau}_{\tau,\nu} = 1_{\tau_{\nu} \otimes \tau_{\tau}}$. This assumption on $\tau$, suggested by our main application, also helps to simplify formulae. Positivity of the form will be discussed in a different paper in more generality [35].

We let $G_{\tau}$ denote the compact quantum group derived from $\tau$ by Woronowicz duality.

Let $\alpha$ be the ergodic action of $G_{\tau}$ on $^\circ \mathcal{C}$ such that $\alpha(M \otimes T) = M \otimes \hat{\nu}(\psi)$ for $M \in (\mu_u, \iota), \psi \in (\iota, \tau_{\nu})$, where $\hat{\nu}$ denotes the representation of $G_{\tau}$ on $\tau_{\nu}$.

The main result of this section, generalizing [32], constructs the bimodule representations associated with ergodic actions of compact quantum groups.

7.1. Theorem If $1_{\mu_u} \neq 0$, $^\circ \mathcal{H}_u$ can be completed to give a full Hilbert $C^*$–module $\mathcal{H}_u$ over the maximal completion $\mathcal{C}$ of $^\circ \mathcal{C}$. $\mathcal{H}_u$ is finite projective. If $\mu$ is relaxed tensor, $\mathcal{H}_u$ is free. The left $\mathcal{C}$–action $\mathcal{H}_u$ is given by a faithful $^\ast$–homomorphism $\mathcal{C} \rightarrow \mathcal{L}(\mathcal{H}_u)$, the set of adjointable right module maps. There is a full bimodule representation $\text{Ind}(\mu_u)$ of $G_{\tau}$ on $\mathcal{H}_u$ such that $\text{Ind}(\mu_u)(M' \otimes \psi) = M' \otimes \hat{\nu}(\psi)$, $M' \in (\mu_v, \mu_u), \psi \in (\iota, \tau_{\nu})$. 
λ is now a covariant functor from $M_{\mu}$ to the category of $^\circ \mathcal{C}$–bimodules.

Any object $x$ of Hilb can be naturally identified with $(t, x)$. Tensor products of intertwiners go into tensor products of vectors. Hence we may think of $^\circ \mathcal{H}_u$ as being linearly spanned by simple tensors $M \otimes \psi$, with $M \in (\mu_v, \mu_u)$, $\psi \in \tau_u$.

If $R_u, \overline{R}_u$ is a solution of the conjugate equations for an object $u$ of $A$, the conjugation $\star$ defined by the image solution $\hat{R}_u, \overline{R}_u$ when restricted to $\tau_u$ is an antilinear invertible $j_u : \tau_u \to \tau_u^\ast$, satisfying

$$\hat{R}_u = \sum_i \phi_i \otimes j_i^{-1} \phi_i, \quad \overline{R}_u = \sum_k \psi_k \otimes j_k \psi_k,$$

where $(\psi_k)$ and $(\phi_i)$ are orthonormal bases of $\tau_u$ and $\tau_u^\ast$ respectively.

Now define $L_{t, u}(\xi)$ to be the operator of left multiplication by $\xi$ on $^\circ \mathcal{H}_u$, hence obviously a right module morphism. The following relations hold for $\xi \in ^\circ \mathcal{H}_v, \xi' \in ^\circ \mathcal{H}, Y \in (\tau_s, \tau_t), Y \in (\mu_v, \mu_w)$:

$$L_{s \otimes t, v \otimes u}(\eta)L_{t, u}(\xi) = L_{t, u}(\eta \cdot \xi),$$

$$\nu(\hat{\mu}_{t, v} \circ 1_{\mu_v} \otimes Y \circ \hat{\mu}_{t, u}^\ast \hat{\tau}_{x, t} \circ 1_{\tau_u} \otimes X \circ \hat{\tau}_{x, s})L_{t, u}(\xi') = L_{s, w}(\xi')\nu(Y, X).$$

Now set

$$L_{t, u}(\xi)^\ast := \nu(\mu(R_v^\ast \otimes 1_u), \tau(R_v \otimes 1_t))L_{s \otimes t, v \otimes u}(\xi^\ast),$$

for $\xi \in ^\circ \mathcal{H}_v$. If we change solutions of the conjugate equations using invertibles $Y \in (\overline{\tau}, \overline{s}), X \in (\overline{\tau}, \overline{v})$, $\xi^\ast$ becomes $\nu(\mu(X), \tau(Y^{-1}))\xi^\ast$, hence $L_{t, u}(\xi)^\ast$ does not change.

**Lemma 7.2.** $\langle \eta, L_{t, u}(\xi)^\ast \hat{\xi} \rangle \equiv \langle L_{t, u}(\eta \cdot \xi) \rangle$

**Proof** We may suppose that we have chosen product solutions of the conjugate equations for $v \otimes u$ and $s \otimes t$. Then

$$\langle \eta, L_{t, u}(\xi)^\ast \hat{\xi} \rangle = \nu(\mu(R_v^\ast \otimes 1_u), \tau(1_{\tau_v} \otimes R_v \otimes 1_t))L_{s \otimes t, v \otimes u}(\xi^\ast \cdot \hat{\xi})$$

$$= \nu(\mu(R_v^\ast \otimes 1_{\tau_v} \otimes R_v \otimes 1_t), \tau(1_{\tau_v} \otimes R_v \otimes 1_t \circ R_v))\eta^\ast \cdot \xi^\ast \cdot \hat{\xi} = \langle L_{t, u}(\eta \cdot \xi) \rangle.$$

We shall later see that the $\mathbb{C}$–valued form $\langle \cdot, \cdot \rangle$ is positive so that $L_{t, u}(\xi)^\ast$ is the adjoint of $L_{t, u}(\xi)$ as the notation suggests.

Given $\phi \in (\iota, \mu_z)$, it proves convenient to write for brevity

$$L_u(\phi) := L_{t, u}(1_{\mu_z} \otimes \phi).$$

Since $1_{\mu_z} \otimes \phi \in ^\circ \mathcal{H}_z$,

$$L_u(\phi) : ^\circ \mathcal{H}_v \to ^\circ \mathcal{H}_z \otimes u.$$

Thus

$$L_u(\phi)^\ast = \lambda \mu(R_v^\ast \otimes 1_u)L_{z \otimes u}(\phi^\ast).$$

These maps satisfy the following properties.

**7.3. Lemma**
a) \( \lambda(\tilde{\mu}_{z,w} \circ 1_{\mu_z} \otimes X \circ \tilde{\mu}^*_{z,w}) \circ L_u(\phi) = L_w(\phi) \circ \lambda(X) \),

b) \( \lambda \mu(A \otimes 1_u) L_u(\phi) = L_u(\tau(A) \phi) \),

c) \( \sum_i L_u(\phi_i) L_u(\phi_i)^* = \lambda(\tilde{\mu}_{z,u} \circ \tilde{\mu}^*_{z,u}) \), with \( \phi_i \) an orthonormal basis of \( \tau_z \).

If \( \mu \) is tensor we also have:

c) \( L_u(\phi \otimes \phi') = L_{u' \otimes u}(\phi) \circ L_u(\phi') \),

d) \( L_u(\eta)^* L_u(\phi) = < \eta, \phi > \),

for \( X \in (\mu_u, \mu_{u'}) \), \( \phi, \eta \in \tau_z \), \( \phi' \in \tau_{z'} \), \( \xi \in \mathcal{O} \mathcal{H}_u \), \( \zeta', \xi' \in \mathcal{O} \mathcal{H}_{u'} \), \( \zeta \in \mathcal{O} \mathcal{H}_{z \otimes u} \).

For \( u = t \) the corresponding maps \( \mathcal{O} \mathcal{C} \to \mathcal{O} \mathcal{H}_z \), \( \mathcal{O} \mathcal{H}_z \to \mathcal{O} \mathcal{C} \) will simply be denoted \( L(\phi) \) and \( L(\phi)^* \). By c), for any orthonormal basis \( (\phi_i) \) of \( \tau_z \) we have:

\[ \sum L(\phi_i) L(\phi_i)^* = 1. \]

This relation can be used to define a faithful right module map \( S_z : \mathcal{O} \mathcal{H}_z \to \tau_z \otimes \mathcal{O} \mathcal{C} \) by

\[ S_z \xi := \sum_i \phi_i \otimes L(\phi_i)^* \xi, \]

clearly independent of the choice of the orthonormal basis.

Property b) of Lemma 7.3 easily shows that, for \( A \in (u, u') \), the following diagrams are commutative:

\[ \begin{array}{ccc}
\mathcal{O} \mathcal{H}_u & \xrightarrow{\lambda \mu(A)} & \mathcal{O} \mathcal{H}_{u'} \\
S_u \downarrow & & \downarrow S_{u'} \\
\tau_u \otimes \mathcal{O} \mathcal{C} & \xrightarrow{\tau(A) \otimes I} & \tau_{u'} \otimes \mathcal{O} \mathcal{C} 
\end{array} \]

In the previous section we have defined a \( \mathcal{O} \mathcal{C} \)-valued form on \( \mathcal{O} \mathcal{H}_u \) that we now show to be faithful and positive in the special case considered here. For later use, we recall that

\[ < \xi, \xi' > = (\tilde{R}_\psi^* \circ 1_{\mu_{\psi'}} \otimes (M^* \circ M') \circ \tilde{\mu}_{\mu_{\psi'}, \mu_{\psi'}}) \otimes (j_{\psi} \psi \otimes \psi'), \quad (7.1) \]

for \( \xi = M \otimes \psi \), \( \xi' = M' \otimes \psi' \), \( M \in (\mu_{\psi'}, \mu_{\psi}) \), \( M' \in (\mu_{\psi'}, \mu_{\psi}) \), \( \psi \in \tau_{\psi}, \psi' \in \tau_{\psi'} \).

7.4. Proposition If \( \tau_z \otimes \mathcal{O} \mathcal{C} \) is considered as a free right prehilbertian \( \mathcal{O} \mathcal{C} \)-module, the map \( S_z \) satisfies:

\[ < S_z \xi, S_z \xi' > = < \xi, \xi' >, \quad \xi, \xi' \in \mathcal{O} \mathcal{H}_z. \]

Hence \( \mathcal{O} \mathcal{H}_z \) is a finite projective, right prehilbertian module over \( \mathcal{O} \mathcal{C} \) and \( S_z \) is an isometric right \( \mathcal{O} \mathcal{C} \)-module map with adjoint \( S_z^* : \tau_z \otimes \mathcal{O} \mathcal{C} \to \mathcal{O} \mathcal{H}_z \) given by

\[ S_z^*(\psi \otimes I) = 1_{\mu_z} \otimes \psi \]
for $\psi \in \tau_z$.

**Proof**

$$< S_z \xi, S_z \xi' > = \sum_i < L(\phi_i)^* \xi, L(\phi_i)^* \xi' > = \sum_i < \xi, L(\phi_i) L(\phi_i)^* \xi' > = < \xi, \xi' > .$$

Hence $< \cdot, \cdot >$ is a faithful, positive, $^{0}\mathcal{C}$–valued inner product on $^{0}\mathcal{H}_z$ and $S_z$ an isometry. We next compute the adjoint of $S_z$. If $\xi \in ^{0}\mathcal{H}_z$, $\psi \in \tau_z$ then

$$< \xi, S_z^* \psi \otimes I > = < S_z \xi, \psi \otimes I > = \sum_i < \phi_i \otimes L(\phi_i)^* \xi, \psi \otimes I > = \sum_i (\phi_i, \psi) < \xi, 1_z \otimes \phi_i > = < \xi, 1_z \otimes \psi > ,$$

as required.

We next decide when $S_z$ is unitary by computing its range projection $P_z = S_z S_z^*$. If $\psi \in \tau_z$,

$$P_z (\psi \otimes I) = S_z (1_{\mu_z} \otimes \psi) = \sum_i \phi_i \otimes L(\phi_i)^* (1_{\mu_z} \otimes \psi) = \sum_i \phi_i \otimes ((\mu(R_z^*) \circ \tilde{\mu}_{\tau_z} \circ \tilde{\mu}_{\tau_z}^*) \otimes (j_z \phi_i \otimes \psi)).$$

**7.5. Corollary** If $\tilde{\mu}_{\tau_z} \circ \tilde{\mu}_{\tau_z} \circ \mu(R_z) = \mu(R_z)$ (e.g. when $\mu$ is relaxed tensor), $S_z$ is a unitary, and hence $^{0}\mathcal{H}_z$ is a free right $^{0}\mathcal{C}$–module.

**Proof** We now have

$$\mu(R_z^*) \circ \tilde{\mu}_{\tau_z} \circ \tilde{\mu}_{\tau_z}^* \otimes (j_z \phi_i \otimes \psi) = \mu(R_z^*) \otimes (j_z \phi_i \otimes \psi) = 1_i \otimes \tau(R_z^*) j_z \phi_i \otimes \psi = < \phi_i, \psi > .$$

Hence $P_z$ is the identity projection.

We next show that the inner product of $^{0}\mathcal{H}_u$ is algebraically full.

**7.6. Proposition** Let $u$ be an object of $A$ for which $1_{\mu_u} \neq 0$, then the coefficients $< \xi, \xi' >$, $\xi, \xi' \in ^{0}\mathcal{H}_u$, span $^{0}\mathcal{C}$.

**Proof** Choose $v = u$, $v' = u \otimes v''$, $M = 1_{\mu_u}$, $M' = M'' \circ \tilde{\mu}_{u,v''}$ with $M'' \in (\mu_u \otimes \mu_{v''}, \mu_u)$, $\psi = j_u^{-1} \phi_i$, $\psi' = j_u^{-1} \phi_i \otimes \psi''$ in (7.1) where $(\phi_i)$ is an orthonormal basis of $\tau_u$ and $\psi'' \in \tau_{v''}$. Summing over $i$ gives

$$\left( \hat{R}_u \circ 1_{\mu_u} \otimes (M'' \circ \tilde{\mu}_{u,v''}) \circ \tilde{\mu}_{\tau_u \otimes \tau_{v''}} \right) \otimes (\tau(R_u) \otimes \psi''''') =$$

$$\left( \hat{R}_u \circ 1_{\mu_u} \otimes (M'' \circ \tilde{\mu}_{u,v''}) \circ \tilde{\mu}_{\tau_u \otimes \tau_{v''}} \circ \mu(R_u \otimes 1_{v''}) \right) \otimes \psi''' =$$

$$\left( \hat{R}_u \circ 1_{\mu_u} \otimes (M'' \circ \tilde{\mu}_{u,v''}) \circ \tilde{\mu}_{\tau_u \otimes \tau_{v''}} \circ \mu(R_u) \otimes 1_{\mu_{v''}} \right) \otimes \psi''' = (\hat{R}_u \circ 1_{\mu_u} \otimes M'' \circ \hat{R}_u \otimes 1_{\mu_{v''}}) \otimes \psi''''.$$
Now we recall, see e.g. [22] that if $\rho$, $\sigma$, $\tau$ are objects of a tensor $C^*$–category with conjugates, the map

$$T \in (\rho \otimes \sigma, \tau) \rightarrow 1_R \otimes T \circ R \in (\sigma, \tau)$$

is a linear isomorphism. Hence $X := 1_{\mu^u} \otimes M'' \circ R \otimes 1_{\mu^u}$ is a generic element of $(\mu^u, \mu^u \otimes \mu^u)$ and can, in particular, be any element of the form $X = R \circ Y$ where $Y \in (\mu^u, \iota)$. Hence the linear span of the coefficients of the inner product of $\mathcal{H}_u$ is $C^*$ as it contains any element of the form $Y \otimes \psi''$.

We next give a useful formula for the left $C^*$–action on $\mathcal{H}_z$. As $x_i := S_i(\psi_i \otimes I) = 1_{\mu^u} \otimes \psi_i$, where $(\psi_i)$ is an orthonormal basis of $\tau_z$, is a Hilbert module basis, we need only specify $< x_i, c \cdot x_j >$ for $c \in C^*$.

**7.7. Proposition** If $c = T \otimes \phi \in C^*$, with $T \in (\mu^u, \iota)$, $\phi \in \tau_v$,

$$< x_i, c \cdot x_j > = (R^* \circ 1_{\mu^u} \otimes T \circ 1_{\mu^u} \circ \mu^u \otimes \iota) \otimes (j_{z_\psi} \psi_i \otimes \phi \otimes \psi_j).$$

**Example** Let $K \subset G$ be an inclusion of a compact quantum groups. Then we have a tensor $*$–functor $\mu : \text{Rep}(G) \rightarrow \text{Rep}(K)$ taking a representation $u$ of $G$ to its restriction $u |_{K}$ to the subgroup. Hence $\hat{\mu} = 1$ and $\mathcal{H}_z$ is a free module. $C^*$ is the canonical dense $*$–subalgebra of $C(K\setminus G)$ [32]. The above formula then gives:

$$< x_i, c \cdot x_j > = \sum_{r} z_{ri} c z_{rj}, \quad c \in C^*(K\setminus G).$$

This example was discussed at the end of Sect. 4.

**Example** Let $G$ be a compact quantum group acting ergodically on a unital $C^*$–algebra $\mathcal{E}$, and let $\mathcal{L} : \text{Rep}(G) \rightarrow \text{Hilb}$ be the spectral functor of the action as in [32], and shown there to be a quasitensor functor. Then $C^*$ is the $*$–algebra spanned by the elements of $\mathcal{E}$ transforming under the action like unitary irreducible $G$–representations. $\mathcal{H}_u \neq 0$ precisely when $\mathcal{L}_u \neq 0$ and, if $u$ is irreducible, this is equivalent to requiring $u$ to lie in the spectrum of the action. We thus get a finite projective $C^*$–bimodule $\mathcal{H}_u$, with left $C^*$–action given by

$$< x_i, c \cdot x_j > = \sum_{r} c_{r,s}^u x_r c_{r,s}^u,$$

where $c_{r,s}^u$ is a multiplet transforming under the action like the contragredient $u^*_s = (u^*_r)$ and an orthonormal basis of $\mathcal{L}_u$.

**Remark** The restriction functor and the spectral functor of a quantum quotient give rise to the same algebra $\mathcal{E}$. However, as expected, the associated bimodules are different.

Consider $C^*$ with its maximal $C^*$–norm, which is finite by [32]. Completing $\mathcal{H}_z$ in the norm $\|\xi\| := \|< \xi, \xi >\|^{1/2}$, gives a right Hilbert module $\mathcal{H}_z$ over the completion $C^*$ of $C^*$. There is an isometry $\mathcal{H}_z \rightarrow \tau_z \otimes C^*$ extending the algebraic
isometry $S_z$. Hence $\mathcal{H}_z$ is a finite projective right Hilbert $\mathcal{C}$–module. As a consequence, every right module map on $^o\mathcal{H}_z$ extends to an adjointable bounded map on $\mathcal{H}_z$. Hence the left $^o\mathcal{C}$–action extends to a unital $^*$–homomorphism $\mathcal{C} \to \mathcal{L}_c(\mathcal{H}_z)$ thus making $\mathcal{H}_z$ into a Hilbert $\mathcal{C}$–bimodule.

To show that the left action is faithful we need norm continuity of the product of bimodules.

7.8. Proposition The map

$$\xi \otimes \xi' \in {}^o\mathcal{H}_u \otimes {}^o\mathcal{H}_w \rightarrow \xi \cdot \xi' \in {}^o\mathcal{H}_{u \otimes w}$$

is an isometry of prehilbertian $^o\mathcal{C}$–bimodules. Hence $\|\xi \cdot \xi'\| \leq \|\xi\|\|\xi'\|$.

Proof Using a product solution of the conjugate equations,

$$<\xi \xi', \eta \eta'> = \lambda(\mu(R_u^*))\xi^* \xi^* \eta \eta' = \lambda(\mu(R_w^*))\lambda(\mu(1_u \otimes R_u^* \otimes 1_w'))\xi^* \xi^* \eta \eta' =

(\lambda(\mu(R_u^*))\xi^*) \cdot (\lambda(\mu(R_u^*))\xi^* \cdot \eta) \cdot \eta' =<\xi',<\xi, \eta \cdot \eta'> =<\xi \otimes \xi',\eta \otimes \eta'>,$$

as required.

Consequently, $\cdot$ extends to an associative product $\xi \cdot \xi'$ on the completed bimodules $\mathcal{H}_u$ and $\mathcal{H}_w$.

7.9. Proposition The extended left action of $\mathcal{C}$ on $\mathcal{H}_u$ is faithful whenever $1_{\mu_u} \neq 0$.

Proof If $c \cdot \xi = 0$ for all $\xi \in \mathcal{H}_u$, then

$$c \cdot (\lambda(\mu(R_u))\xi \cdot \eta) = \lambda(\mu(R_u))c \cdot \xi \cdot \eta = 0,$$

for all $\eta \in \mathcal{H}_u$. On the other hand, $\lambda(\mu(R_u))\xi \cdot \eta =<\xi^*, \eta>$, and these coefficients span $^o\mathcal{C}$ if $1_{\mu_u} \neq 0$ i.e. if $1_{\mu_u} \neq 0$.

We next construct quantum group representations on the bimodules $\mathcal{H}_u$. Let $G_\tau$ denote, as before, the compact quantum group that is the Woronowicz dual of $\tau$.

7.10. Proposition Given an object $u$ of $A$, there is a unique bimodule representation $Ind(\mu_u)$ of $G_\tau$ on $\mathcal{H}_u$ such that $Ind(\mu_u)(M' \otimes \psi) = M' \otimes \hat{\psi}(\psi)$, $M' \in (\mu_v, \mu_u)$, $\psi \in (i, \tau_v)$. $Ind(\mu_u)$ is a full bimodule representation.

Proof The relation between the invertible antilinear maps $j_v : \tau_v \rightarrow \tau_{\tau v}$ and the coefficients of the corresponding representations of $G_\tau$ is given by $\hat{\psi}j_v \psi = \sum \phi_i \otimes \hat{\phi}_i^v$, where $\phi_i$ is an orthonormal basis of $\tau_{\tau v}$. This relation together with (7.1) allows us to verify (2.1), (2.2), (2.3) and (2.5) follow from straightforward computations, whilst (2.4) is a consequence of the corresponding relation for the Hilbert space representation $\hat{\psi}$. We show that $Ind(\mu_u)$ is a full representation. A $G$–fixed vector $\xi$ in $\mathcal{H}_u$ for the underlying module representation is a simple tensor of the form $\xi = T \otimes 1_\tau$, $T \in (i, \mu_u)$. For any element in $\mathcal{C}$ of the total subset of elements of the form $c = Y \otimes \psi$, $Y \in (\mu_\tau, i)$, $\psi \in \tau_v$, we have

$$\xi \cdot c = (T \otimes Y) \otimes \psi = (T \otimes 1_\tau \circ 1_\tau \otimes Y) \otimes \psi =$$
\[(T \circ Y) \otimes \psi = (1_i \otimes T \circ Y \otimes 1_i) \otimes \psi = (Y \otimes T) \otimes \psi = c \cdot \xi.\]

The next task of this section is to extend the functor \( \lambda \) to the completed bimodules. (7.1) shows that \( \lambda \) is a \( ^* \)-functor from the \( C^* \)-category \( \mathcal{M}_\mu \) to the category of prehilbertian \( \mathbb{C} \)-bimodules. Thus \( \lambda(X) \) is bounded and hence extends uniquely to a bimodule map between the completed Hilbert bimodules. On the other hand, the obvious commutation relations between the action of \( G \) and that of the \( \lambda(X) \) imply that \( \lambda(X) \) is an intertwining operator between the corresponding bimodule representations of \( G \).

We summarize this discussion in the following result.

**7.11. Theorem** The algebraic functor \( \lambda \) gives rise to a \( ^* \)-functor from the \( C^* \)-category \( \mathcal{M}_\mu \) to the \( C^* \)-category of \( \mathbb{C} \)-bimodule representations of \( G_r \) : \( \lambda : \mathcal{M}_\mu \to \text{Bimod}_\alpha(G_r) \).

The maps \( S_u \) extend uniquely to isometries \( S_u : \mathcal{H}_u \to \tau_u \otimes \mathcal{C} \) making the following diagrams commute for \( A \in \langle u, u' \rangle \):

\[
\begin{array}{c}
\mathcal{H}_u \xrightarrow{\lambda \mu(A)} \mathcal{H}_{u'} \\
S_u \downarrow \quad \downarrow S_{u'} \quad \tau_u \otimes \mathcal{C} \xrightarrow{\tau(A) \otimes I} \tau_{u'} \otimes \mathcal{C}
\end{array}
\]

**7.12. Proposition** \( S_u \in (\text{Ind}(\mu_u), \hat{\pi} \otimes \alpha) \) in the category \( \text{Mod}_\alpha(G_r) \).

**Proof** For \( \xi = M \otimes \psi_i \), with \( M \in (\mu_v, \mu_u) \), \( (\psi_j) \) an orthonormal basis of \( \tau_v \), and orthonormal bases \( (\phi_r) \) and \( (\eta_p) \) of \( \tau_u \) and \( \tau_r \) respectively,

\[
\hat{u} \otimes \alpha \circ S_u(M \otimes \psi_i) = \sum_r \hat{u} \otimes \alpha(\phi_r \otimes L(\phi_r)^*(M \otimes \psi_i)) =
\]

\[
\sum_r \hat{u} \otimes \alpha(\phi_r \otimes (\mu(R_u^*) \circ \hat{\mu}_{r,u} \otimes 1_{\mathcal{M}_\mu} \otimes M \circ \hat{\mu}_v^*)) \otimes (j_u \phi_r \otimes \psi_i)) =
\]

\[
\sum_{r,s,p,j} \phi_s \otimes (\mu(R_u^*) \circ \hat{\mu}_{r,u} \otimes 1_{\mathcal{M}_\mu} \otimes M \circ \hat{\mu}_v^*) \otimes (\eta_p \otimes \psi_j)) \otimes \hat{u}_{s,r} (\hat{u}_{j_u \cdot \eta_p, \phi_r} \cdot \hat{v}_{ji}) =
\]

\[
\sum_{r,s,p,j,h} \phi_s \otimes ((\mu(R_u^*) \circ \hat{\mu}_{r,u} \otimes 1_{\mathcal{M}_\mu} \otimes M \circ \hat{\mu}_v^*) \otimes (\eta_p \otimes \psi_j)) \otimes \hat{u}_{s,r} (\hat{u}_{j_u \cdot \eta_p, \phi_r} \cdot \hat{v}_{ji}) < \eta_p, j_u \phi_h > =
\]

\[
\sum_{j,h} S_u(M \otimes \psi_j) \otimes \hat{v}_{ji} = S_u \otimes 1_Q \circ \text{Ind}(\mu_u)(M \otimes \psi_i).
\]

**Remark** \( u \to S_u \) is a natural transformation from \( \tau \otimes 1 \) to \( \lambda \mu \) taking values in \( \text{Bimod}_\alpha(G_r) \).
8 Extending Ind to a full tensor functor

Now $\mathcal{M}_\mu$ is a $C^*$–category, but not a tensor $C^*$–category in general. This suggests looking for an extension of $\lambda$ to the full tensor subcategory $\mathcal{M}_\mu^0$ of $\mathcal{M}$ generated by the images under $\mu$ of the objects of $\mathcal{A}$. Here we construct new bimodule representations $\mathcal{H}_u$ over $\mathcal{C}$ associated with finite sequences $u = (u_1, \ldots, u_n)$ of objects of $\mathcal{A}$.

We assume that $\mathcal{A}$ has conjugates and make a choice $u \to R_u$ of solutions of the conjugate equations. We define a $C^*$–category $A^d$ by adding formal conjugates where the arrows are given by replacing formal conjugates by conjugates. In particular, the conjugate and the formal conjugate are isomorphic and the embedding functor of $\mathcal{A}$ into $A^d$ is an equivalence of $C^*$–categories. For the formal conjugate of $u$ we pick $R_u$ as a solution of the conjugate equations. With this choice, the corresponding conjugation of arrows $\cdot$ is involutive.

For book-keeping purposes we introduce a new tensor $C^*$–category $A^\otimes$ whose objects are finite sequences of objects of $A^d$ with a sequence of length 0 corresponding to the tensor unit $\underline{1}$. The arrows of $A^\otimes$ connect only objects of the same length, $(\underline{u}, \underline{1}) = \underline{C}$, and the remaining arrows are the tensor product spaces of arrows of $A$: for $u := (u_1, u_2, \ldots, u_n)$ and $v := (v_1, \ldots, v_n)$, $(u, v) := (u_1 \otimes \cdots \otimes u_n, v_n)$, where formal conjugates are to be replaced by conjugates. The tensor product in $A^\otimes$ is defined by juxtaposition, it being understood that $\underline{1}$ acts as a tensor unit. $A$ may be identified in an obvious way with a $C^*$–subcategory of $A^\otimes$. There is an involutive covariant functor $\cdot^*$. It takes an object $(u_1, u_2, \ldots, u_n)$ into $(\overline{u}_n, \ldots, \overline{u}_2, \overline{u}_1)$, where the conjugate of a formal conjugate of $u$ must obviously be interpreted as $u$. On arrows we set $(A_1 \otimes A_2 \otimes \cdots \otimes A_n)^* := A_n^* \otimes \cdots \otimes A_2^* \otimes A_1^*$. We have: $(A \otimes B)^* = B^* \otimes A^*$. Despite this conjugation, $A^\otimes$ does not of course have conjugates.

If $\mu$ is a $\cdot^*$–functor from $A$ to a tensor $C^*$–category $M$ then $\mu$ extends to a $\cdot^*$–functor from $A^d$ sending formal conjugates into images of conjugates and acting on the arrows as the ‘identity’. After this, $\mu$ extends uniquely to a tensor $\cdot^*$–functor $\mu$ from $A^\otimes$ to $M$. But if $\mu$ is the identity functor on $A$, we write simply $f$.

If $u$ is an object of $A^\otimes$ then we write $\underline{R}_u$ to indicate the solution of the conjugate equations for $f(\underline{u})$ given by the product formula. A quasitensor functor $(\mu, \tilde{\mu})$ gives rise to an isometric natural transformation from $\mu$ to $\mu f\cdot$, also denoted by $\tilde{\mu}$. $\tilde{\mu}_t := 1_t$, if $u$ has length 1 then $\tilde{\mu}_u := 1_{\mu_u}$, whilst, if $\underline{u} := (u_1, u_2, \ldots, u_n)$, we set $\tilde{\mu}_u := \tilde{\mu}_{u_1, u_2, \ldots, u_n}$, where on the right hand side formal conjugates are to be replaced by conjugates and $\tilde{\mu}_{u_1, u_2, \ldots, u_n}$ denotes the natural transformation

$$\mu(S_1 \otimes S_2 \otimes \cdots \otimes S_n) \circ \tilde{\mu}_{u_1, u_2, \ldots, u_n} = \tilde{\mu}_{u_1, u_2, \ldots, u_n} \circ \mu(S_1) \otimes \mu(S_2) \otimes \cdots \otimes \mu(S_n),$$

derived from $\tilde{\mu}$ using associativity. The two defining equations for the natural transformation in $(\mu, \tilde{\mu})$ yield

$$\tilde{\mu}_{u \cdot t} = \tilde{\mu}_{t \cdot u} = 1_{\mu_u},$$

$$\tilde{\mu}_{u \cdot f(v \otimes w)} \circ \mu_{f(v \otimes w) \cdot u} = 1_{\mu_u} \otimes \tilde{\mu}_{v \otimes w} \circ \tilde{\mu}_{u \otimes v} \otimes 1_{\mu_w}.$$
The associative law for $\tilde{\mu}$ reads
\[
\tilde{\mu}_{y,z} \otimes 1_{\mu_v} = \tilde{\mu}_y \otimes f(y) \circ 1_{\mu_v} \otimes \tilde{\mu}_z.
\]
This provides a way of computing $\mu_z$ by induction. Thus given objects $v$ and $w$ of $A$ then
\[
\tilde{\mu}_{v \otimes z \otimes w} = \tilde{\mu}_{v \otimes f(z \otimes w)} \circ 1_{\mu_v} \otimes \tilde{\mu}_{f(z \otimes w)} \circ 1_{\mu_w}
\]
\[
\tilde{\mu}_{v \otimes f(z) \otimes w} \circ 1_{\mu_v} \otimes \tilde{\mu}_{f(z) \otimes w} \otimes 1_{\mu_w}.
\]
We have made a choice $u \mapsto R_u$ of solutions of the conjugate equations in $A$ for objects of $A^\otimes$. These induce solutions $\hat{R}_u := \tilde{\mu}_{z,u} \circ \mu(R_u)$ in $\mathcal{M}$. Finally, we extend this to a map $\hat{u} \mapsto \hat{R}_u$ derived from the $\hat{R}_u$, using the product formula. $u \mapsto \hat{R}_u$ is a homomorphic choice of solutions of the conjugate equations.

**8.1. Lemma** $\hat{R}_w = \tilde{\mu}_{z \otimes w} \circ \mu(R_u)$.

**Proof** We prove the lemma by induction on the length of $u$. The result holds by definition if this length is one. Suppose $u = v \otimes w$. Then
\[
\hat{R}_w = 1_{\mu_v} \otimes \hat{R}_w \otimes 1_{\mu_w} \circ \hat{R}_w.
\]
Hence by the induction hypothesis,
\[
\hat{R}_w = 1_{\mu_v} \otimes \tilde{\mu}_{z \otimes w} \circ 1_{\mu_v} \otimes \mu(R_u) \otimes 1_{\mu_w} \circ \tilde{\mu}_{z \otimes w} \circ \mu(R_u).
\]
But
\[
1_{\mu_v} \otimes \mu(R_u) \otimes 1_{\mu_w} \circ \tilde{\mu}_{z \otimes w} = \tilde{\mu}_{z \otimes f(v \otimes w) \otimes w} \circ \mu f(1_{\mu} \otimes R_u \otimes 1_w),
\]
by naturality. As we have seen above
\[
1_{\mu_v} \otimes \tilde{\mu}_{z \otimes w} \circ 1_{\mu_v} \circ \tilde{\mu}_{z \otimes f(v \otimes w) \otimes w} = \tilde{\mu}_{z \otimes w},
\]
completing the proof.

We next aim to define a Hilbert $C^*$–bimodule $\mathcal{H}_u$ associated with an object $u$ of $A^\otimes$. As for the previous bimodules $\mathcal{H}_u$ associated with objects of $A$, we start by considering the more symmetric case of a pair of quasi-tensor functors $(\mu, \tilde{\mu}) : A \rightarrow M$ and $(\tau, \tilde{\tau}) : A \rightarrow \mathcal{S}$.

Given objects $t$ and $u$ of $A^\otimes$, let $\mathcal{H}_{t \otimes u}$ be the linear space $\sum_v (\mu_{tv}, \mu_{uv}) \otimes (\tau_{tv}, \tau_{uv})$, the sum being taken over the objects of $A$, quotiented by the linear subspace generated by elements of the form $M \circ \mu(A) \otimes T - M \otimes \tau(A) \circ T$.

We next define bilinear maps
\[
\tilde{\mathcal{H}}_{t \otimes u} \times \tilde{\mathcal{H}}_{t \otimes u} \rightarrow \mathcal{H}_{t \otimes u}.
\]
For $\xi = L \otimes S$, $\eta = M \otimes T$, where $L \in (\mu_w, \mu_w)$, $M \in (\mu_v, \mu_w)$, $S \in (\tau_L, \tau_v)$, $T \in (\tau_L, \tau_v)$, set:
\[
\xi \cdot \eta := (L \otimes M) \circ \tilde{\mu}_{w \otimes v} \circ \tilde{\tau}_{w \otimes v} \circ (S \otimes T).
\]
It is easy to check that these maps are well defined and associative.

Remark In the above construction, we can envisage taking \( v \) in place of \( v \) using the relation \( M \circ \mu(A) \otimes T = M \otimes \tau(A) \circ T \) for \( M \in (\mu_w, \mu_w), \ T \in (\tau_L, \tau_L) \ A \in (w, w) \).

We define an antilinear map
\[
* : \mathcal{H}_m \to \mathcal{H}_m^* 
\]
by setting
\[
(M \otimes T)^* := M^* \otimes T^*.
\]
Notice that \( * \) is well defined and independent of the choice of solutions of the conjugate equations in \( A \) for the running objects \( v \) appearing in the sum. However, if we change the solution of the conjugate equations for \( u \) and \( t \) using invertibles \( u' \in (\check{L}, \check{L}) \) and \( t' \in (\check{M}, \check{M}) \), \( M \otimes T \) becomes \((\mu(X) \circ M^*) \otimes (T^* \circ \tau(Y^{-1}))\).

Nevertheless since \( u \to \check{R}_u \) is homorphic, \( * \) is an antilinear map satisfying the following properties.

8.2. Proposition Let \( u, u', t, t' \) be objects of \( A^\circ \). If \( \xi \in \mathcal{H}_m \) and \( \xi' \in \mathcal{H}_m^* \),
\[
(\xi \cdot \xi')^* = \xi^* \cdot \xi^* \text{ and } \xi^* = \xi.
\]

Thus \( \mathcal{H}_m \) is again the \( * \)–algebra \( \circ \mathcal{C} \) and \( \mathcal{H}_m \) is a \( \circ \mathcal{C} \)–bimodule.

Given \( X \in (\tau_L, \tau_L) \) and \( Y \in (\mu_w, \mu_w') \) we define
\[
\nu(Y, X) := \lambda(Y) \rho(X) = \rho(X) \lambda(Y) : \mathcal{H}_m \to \mathcal{H}_m^* \to \mathcal{H}_m
\]
by setting
\[
\lambda(X) (M \otimes T) := M \otimes (T \circ X), \quad \rho(Y) (M \otimes T) := (Y \circ M) \otimes T.
\]
\( \lambda(X) \) and \( \rho(Y) \) are bimodule maps.

Hence \( \lambda \) and \( \rho \) are covariant and contravariant functors from \( \mathcal{M}_w^\circ \) and \( \mathcal{T}_v^\circ \), respectively, into the category of \( \circ \mathcal{C} \)–bimodules. We next define a \( \circ \mathcal{C} \)–valued form on \( \mathcal{H}_m^* \) by
\[
\langle \xi, \xi' \rangle := \nu(\check{R}_u, \check{R}_u^*)(\xi^* \cdot \xi').
\]
The explicit formula, for \( \xi = M \otimes T, \xi' = M' \otimes T' \), with \( M \in (\mu_w, \mu_w), \ T \in (\tau_L, \tau_L), \ M' \in (\mu_w', \mu_w'), \ T' \in (\tau_L, \tau_L) \) is
\[
\langle \xi, \xi' \rangle := (\hat{R}_u \circ M^* \otimes M' \circ \check{R}_u) \otimes (\tau_{\tau_{u'}} \circ T^* \otimes T' \circ \hat{R}_u).
\]
One can easily check that
\[
\hat{R}_u^* \circ M^* \otimes M' = \hat{R}_u^* \circ 1_{\mu_w} \otimes (M^* \circ M').
\]
Hence
\[
\langle \xi, \xi' \rangle = (\hat{R}_u^* \circ 1_{\mu_w} \otimes (M^* \circ M') \circ \check{R}_{u'}^*) \otimes (\tau_{u' \tau} \circ T^* \otimes T' \circ \hat{R}_u).
\]
From this last expression, it is obvious that if \( Y \in (u, u') \), then \( \lambda(Y)^* = \lambda(Y^*) \). Thus \( \lambda \) is a \( * \)-functor. The same is true of \( \rho \), bearing in mind that \( X^* \circ 1_{\mu} \circ \hat{R} = 1_{\mu} \circ X^* \circ \hat{R} \).

As before, this form is independent of the choice of the conjugate of \( u \) but depends on that of \( t \), hence, as before, we break the symmetry assuming that \( \tau \) is a tensor functor into the category of Hilbert spaces and we choose \( t = \iota \). The corresponding bimodules will be denoted by \( H_u \).

We next regard the associative product \( \xi, \xi' \to \xi \cdot \xi' \) as a bimodule map defined on the bimodule tensor product \( H_u \otimes C \otimes H_{u'} \): \( H_u \otimes C \otimes H_{u'} \to H_u \otimes H_{u'} \).

We also have linear maps: \( \lambda(\hat{\mu}_u) : H_u \to H_u \otimes \cdots \otimes u_n \).

Since \( \hat{\mu}_u \) is an isometry, the previous computation of the inner product shows that \( \lambda(\hat{\mu}_u) \) preserves the corresponding forms: \(<\lambda(\hat{\mu}_u)\xi, \lambda(\hat{\mu}_u)\xi'> = <\xi, \xi'>, \xi, \xi' \in \hat{\mathcal{H}}_u \> \).

On the other hand, \( \hat{\mathcal{H}}_{u_1 \cdots u_n} \) is a finite projective prehilbertian bimodule, so \( \hat{\mathcal{H}}_u \) becomes a finite projective prehilbertian bimodule.

8.3. Proposition \( \lambda \) is a \( * \)-functor from \( \mathcal{M}_\mu^\otimes \) to the category of prehilbertian \( \hat{\mathcal{C}} \)-bimodules.

Proof For \( X \in (\mu_u, \mu_{u'}) \), \( \xi \in \hat{\mathcal{H}}_u \), \( \xi' \in \hat{\mathcal{H}}_{u'} \),

\[<\lambda(X)\xi, \lambda(X)\xi'> = <\lambda(\hat{\mu}_u)\lambda(X)\xi, \lambda(\hat{\mu}_{u'})\lambda(X)\xi'> = <\lambda(\hat{\mu}_u \circ X \circ \hat{\mu}_{u'})\lambda(\hat{\mu}_u)\xi, \lambda(\hat{\mu}_{u'})\lambda(\hat{\mu}_u)\xi'> \]

since \( \lambda \) is a functor. As \( \lambda \) is a \( * \)-functor on \( \mathcal{M}_\mu \) we conclude that it is a \( * \)-functor on \( \hat{\mathcal{C}}^\otimes \).

Completing \( \hat{\mathcal{H}}_u \) with respect to the norm derived from the maximal \( \hat{\mathcal{C}}^\otimes \)-norm of \( \hat{\mathcal{C}} \) yields a Hilbert \( \hat{\mathcal{C}} \)-bimodule \( \hat{\mathcal{H}}_u \). \( \lambda \) extends to a \( * \)-functor from \( \hat{\mathcal{C}}^\otimes \) to the \( \hat{\mathcal{C}}^\otimes \)-category of Hilbert \( \hat{\mathcal{C}} \)-bimodules. In this category \( \hat{\mathcal{H}}_u \) is a subobject of \( \hat{\mathcal{H}}_{u_1 \cdots u_n} \).

8.4. Lemma \( \hat{\lambda}_{\mu_u, \mu_{u'}} : \hat{\mathcal{H}}_u \otimes \hat{\mathcal{C}} \otimes \hat{\mathcal{H}}_{u'} \to \hat{\mathcal{H}}_{u_\mu, u'_\mu} \) is a unitary bimodule map.

Proof The products of the new and old bimodules are related by

\[\lambda(\hat{\mu}(u, u'))(\xi \cdot \eta) = (\lambda(\hat{\mu}_u)\xi) \cdot (\lambda(\hat{\mu}_{u'})\eta) \]

Using successively that \( \lambda(\hat{\mu}_u) \) is isometric, Proposition 7.8, and the above relation, we conclude that the \( \hat{\lambda}_{\mu_u, \mu_{u'}} \) are isomeries. We next show that they are surjective, and hence unitary. Since the products between the bimodules \( \hat{\mathcal{H}}_u \) are associative, it suffices to choose \( u \) to be a sequence \((u)\) consisting of a single element. Consider
an element of $^*\mathcal{H}_{(u,w)}$ of the form $M \otimes \psi$, where $M \in (\mu_v, \mu_u \otimes \mu_{w'})$ and $\psi \in \tau_v$.

Again using the explicit linear isomorphism $(\mu_v, \mu_u \otimes \mu_{w'}) \simeq (\mu_{\overline{w}}, \mu_v, \mu_{w'})$, we may write

$$M = 1_{\mu_u} \otimes T \circ \overline{\mathcal{R}}_u \otimes 1_{\mu_v},$$

where $T \in (\mu_{\overline{w}} \otimes \mu_v, \mu_{w'})$. We may also write

$$T = T' \circ \overline{\mu}_{\overline{w}, v},$$

with

$$M = 1_{\mu_u} \otimes (T' \circ \overline{\mu}_{\overline{w}, v}) \circ \overline{\mathcal{R}}_u \otimes 1_{\mu_v} = 1_{\mu_u} \otimes T' \circ 1_{\mu_u} \otimes \overline{\mu}_{\overline{w}, v} \circ \overline{\mu}_{\overline{w}, \overline{v}} \circ 1_{\mu_v} \circ \mu(\overline{\mathcal{R}}_u) \otimes 1_{\mu_v} = 1_{\mu_u} \otimes T' \circ \overline{\mu}_{\overline{w}, \overline{v}} \circ \mu(\overline{\mathcal{R}}_u) \otimes 1_{\mu_v}.$$

Substituting this into $M \otimes \psi$ gives

$$M \otimes \psi = (1_{\mu_u} \otimes T' \circ \overline{\mu}_{\overline{w}, \overline{v}}) \circ (\tau(\overline{\mathcal{R}}_u) \otimes \psi).$$

Writing $\tau(\overline{\mathcal{R}}_u) = \sum_j \phi_j \otimes j_u \phi_j$, for an orthonormal basis $(\phi_j)$ of $\tau_u$, gives $M \otimes \psi = \sum_j \xi_j \cdot \eta_j$ with $\xi_j = 1_{\mu_u} \otimes \phi_j \in ^*\mathcal{H}_u$, $\eta_j = T' \otimes (j_u \phi_j \otimes \psi) \in ^*\mathcal{H}_{\overline{w}}$, completing the proof.

This lemma allows us to extend the product maps to bimodule unitaries between the completions

$$\hat{\lambda}_{\mu_u, \mu_{w'}} : \mathcal{H}_u \otimes \mathcal{H}_{\overline{w}} \to \mathcal{H}_{\overline{w} \otimes \overline{w}},$$

and we may regard these products as strictly associative realizations of the bimodule tensor products. In particular,

$$\mathcal{H}_u = \mathcal{H}_{u_1} \otimes \mathcal{H}_{u_2} \otimes \cdots \otimes \mathcal{H}_{u_n},$$

as vector spaces.

Thus if $1_{\mu_u} \neq 0$ for all $i$, $\mathcal{H}_u$ is a full right Hilbert module with a faithful right $\mathcal{C}$-action.

The unitaries $\hat{\lambda}_{\mu_u, \mu_{w'}}$ clearly satisfy the associativity property (5.3), leading to the following result.

**8.5. Theorem** The pair $(\lambda, \hat{\lambda})$ is a relaxed tensor functor from the tensor $C^*$–category $\mathcal{M}_{\mathcal{C}}^\mathcal{G}$ to the tensor $C^*$–category of Hilbert $\mathcal{C}$–bimodules.

We next define $G_\tau$–representations on the bimodules $\mathcal{H}_u$.

**8.6. Proposition** For any finite sequence $u = (u_1, \ldots, u_n)$ of objects of $\mathcal{A}$, there is a unique bimodule representation $\text{Ind}(\mu_u)$ of $G_\tau$ on $\mathcal{H}_u$ such that the bimodule unitary maps $\mathcal{H}_{u_1} \otimes \cdots \otimes \mathcal{H}_{u_n} \to \mathcal{H}_u$ intertwine $\text{Ind}(\mu_u_1) \otimes \cdots \otimes \text{Ind}(\mu_{u_n})$ and $\text{Ind}(\mu_u)$. $\text{Ind}(\mu_u)$ is a full bimodule representation.

**Proof** As for the old bimodules, we first define $\text{Ind}(\mu_u)$ on $^*\mathcal{H}_u$ setting

$$\text{Ind}(\mu_u)(M \otimes \psi) = M \otimes \hat{\psi}$$
for \( M \in (\mu, \mu') \), \( \psi \in \tau_v \). This map splits as the tensor product of the algebraic representations \( \text{Ind}(\mu u_1) \otimes \cdots \otimes \text{Ind}(\mu u_n) \), hence extends to a bimodule representation of \( G \) on \( \mathcal{K}_u \) making the product maps into intertwiners. Clearly this intertwining property determines \( \text{Ind}(\mu) \) uniquely. We are left to show that \( \text{Ind}(\mu) \) is full. A \( G \)-fixed vector \( \mathcal{K}_u \) is of the form \( M \otimes 1 \), with \( M \in (\iota, \mu_u) \). As for the old modules, Prop. 7.10, one shows this element is central.

Now if \( X \in (\mu_u, \mu_u') \) in \( M_{\mu}^\otimes \), \( \lambda(X) \in \langle \text{Ind}(\mu), \text{Ind}(\mu') \rangle \), so that

\[
\lambda : M_{\mu}^\otimes \to \text{Bimod}_{\mu}(G)
\]

is a functor into the category of bimodule representations of \( G \). The bimodule unitaries \( \tilde{\lambda} \) are also \( G \)-intertwiners. We are now ready to state a central result of this paper, a version of Frobenius reciprocity theorem for quasitensor functors.

**8.7. Theorem** The pair \((\lambda, \tilde{\lambda})\) is a full and faithful relaxed tensor functor from \( M_{\mu}^\otimes \) to the category of bimodule representations of \( G \).

**Proof** \( M_{\mu}^\otimes \) is a tensor \( C^* \)-category with conjugates and \( \lambda \) a relaxed tensor functor, hence automatically faithful [34]. It remains to show that \( \lambda \) is full. This follows from the linear isomorphisms \( \gamma : (\mu_u, \mu_u') \to (\iota, \mu_u\overline{u} \overline{u}') \), \( T \to X = T \otimes 1_{\mu_u} \circ \overline{R} \), and \( \delta : (\text{Ind}(\mu_u), \text{Ind}(\mu_u')) \to (\iota, \text{Ind}(\mu_u\overline{u} \overline{u}')) \), defined similarly, where \( \mu \) is replaced by the quasitensor \( * \)-functor \( \lambda \circ \mu \). Hence any intertwiner in \( (\text{Ind}(\mu_u), \text{Ind}(\mu_u')) \) is determined by a fixed vector in \( \mathcal{K}(\mu_u \overline{u} \overline{u}') \), which we already know to arise from an intertwiner in \( (\iota, \mu_u \overline{u} \overline{u}') \), hence lying in the image of \( \lambda \).

**Remark** The last proof uses only the functor of tensoring on the right by an identity arrow. This also makes sense for module intertwiners and hence shows the following result.

**8.8. Theorem** Any module intertwiner from \( \text{Ind}(\mu_u) \) to \( \text{Ind}(\mu_u') \), namely an intertwiner in the \( C^* \)-category \( \text{Mod}_u(G) \), is automatically a bimodule intertwiner.

We next prove Theorems 5.3 and 5.4.

**Proof of Theorems 5.3 and 5.4.** We briefly recall from [34] how to get a pair of functors \( \mu \) and \( \tau \). Assume that \( \mathcal{M} \) has conjugates and irreducible tensor unit \( \iota \), and fix an object \( x \) in \( \mathcal{M} \) with intrinsic dimension \( > 1 \) and a normalized solution \( R, \overline{R} \) of the conjugate equations for \( x \). By Jones’s result [19] the intrinsic dimension of \( x \) can only take the values \( d = 2 \cos \frac{\pi}{\ell} \), for \( \ell = 3, 4, \ldots \) or \( d \geq 2 \). Consider the universal tensor \( * \)-category \( \mathcal{T}_d \) with objects the finite words in \( u \) and \( \overline{u} \) and whose arrows are generated by two arrows \( S \in (\iota, \overline{u}u) \) and \( \overline{S} \in (\iota, u\overline{u}) \) subject to the relations expressing \((S, \overline{S})\) as a normalized solution of the conjugate equations for \( u \). \( \iota \) is the empty word and functions as a tensor unit. \( \mathcal{T}_d \) is a tensor \( C^* \)-category for the allowed values of \( d \). Furthermore there is a tensor \( * \)-functor \( \tau \) from \( \mathcal{T}_d \) to the category of Hilbert spaces if and only if \( d \geq 2 \), and all such functors can be easily classified. Picking an embedding \( \tau \), we get an associated compact quantum group \( G = A_u(F) \), where \( F \) is an invertible matrix determined by \( \tau \), such that
\[ \text{Tr}(FF^*) = \text{Tr}((FF^*)^{-1}) = R^*R. \]

Furthermore we have a canonical tensor \(^*\)-functor \( \mu : \mathcal{T}_d \to \mathcal{M} \) such that \( \mu(u) = x \), \( \mu(\psi) = \pi \), \( \mu(S) = R \), \( \mu(\mathcal{S}) = \mathcal{R} \), and we may now apply our main result.

Similarly, given a real or pseudoreal solution of the conjugate equations in \( \mathcal{M} \), namely \( R \in (\ell, x^2) \) with \( R^* \otimes 1_x \circ 1_x \otimes R = \pm 1_x \), we consider the associated universal Temperley–Lieb categories \( \mathcal{T}_{rd} \) and \( \mathcal{T}_{pd} \) with generating arrow \( S \in (\ell, uu) \).

If \( R^* \circ R \geq 2 \), a choice of an embedding of \( \mathcal{T}_{pd} \) into the Hilbert spaces provides a quantum group \( \mathcal{A}_o(F) \) with \( F \) an invertible matrix satisfying \( FF^* = \pm I \), \( \text{Tr}(FF^*) = \text{Tr}((FF^*)^{-1}) = R^* \circ R \).

9 An adjoint pair of functors

Recall that a pair of functors \( F : \Phi \to \Phi' \) and \( F' : \Phi' \to \Phi \) between linear categories is an adjoint pair if, for any pair of objects \( \phi \in \Phi \), \( \phi' \in \Phi' \), there is a linear isomorphism \( \beta_{\phi, \phi'} : (\Phi', F\phi) \to (F'\phi', \phi) \) natural in \( \phi \) and \( \phi' \).

In this section we show that, essentially by construction, the induction functor \( \text{Ind} \) is an adjoint of \( \mu \).

To this end, we assume that \( \tau : \mathcal{A} \to \text{Hilb} \) is a tensor \(^*\)-functor to the category of Hilbert spaces so that \( \mathcal{A} \) is a category of representations of a compact quantum group \( G_\tau \). We assume, as before, that \( \mu : \mathcal{A} \to \mathcal{M} \) is a quasitensor functor between strict tensor \( C^*\)-categories with irreducible tensor units and construct the corresponding Hilbert \( C^*\)-bimodules.

Following Mackey’s construction of the induced representation for locally compact groups, we consider the scalar-valued inner product on \( \mathcal{H}_\mu \) given by composing the \( \mathcal{C} \)-valued inner product with the unique \( G_\tau \)-invariant faithful state. We thus get a Hilbert space, \( \mathcal{H}_\mu \) and the bimodule representation \( \text{Ind}(\mu_\mu) \) of \( G_\tau \) defines a densely defined representation of \( G_\tau \) on \( \mathcal{H}_\mu \) with dense range. This representation is isometric as the state is invariant and hence extends uniquely to a unitary representation of \( G_\tau \) again denoted by \( \text{Ind}(\mu_\mu) \). However, although we start with a finite dimensional representation, the Hilbert space of the induced representation in general fails to be finite dimensional, hence we need to work with the category of not necessarily finite dimensional unitary representations of \( G_\tau \), denoted by \( \text{Rep}(G_\tau) \). We thus have a functor, \( \text{Ind} : \mathcal{M}_\mu^\otimes \to \text{Rep}(G_\tau) \). We also need to assume that \( \mathcal{M} \) is a tensor \( W^*\)-category with infinite direct sums and let \( \mathcal{M}_\mu^\otimes \) denote the full subcategory of \( \mathcal{M} \) whose objects are infinite direct sums of objects of \( \mathcal{M}_\mu^\otimes \). Hence \( \mu \) and \( \text{Ind} \) extend uniquely to \(^*\)-functors on \( \text{Rep}(G_\tau) \) and \( \mathcal{M}_\mu^\otimes \) respectively.

9.1. Theorem The pair of functors \( \text{Ind} : \mathcal{M}_\mu^\otimes \to \text{Rep}(G_\tau) \) and \( \mu : \text{Rep}(G_\tau) \to \mathcal{M}_\mu^\otimes \) is an adjoint pair.

Proof Note that the linear span of the images of elements of the form \( T \otimes \psi \), where \( T \in (\mu_\psi, \mu_\mu^\otimes) \) and \( \psi \in \tau_v \), where \( v \) runs over the irreducible representations of \( G_\tau \), is dense in \( \mathcal{H}_\mu \). If we fix an irreducible \( \psi \), the space of intertwiners \( (v, \text{Ind}(\mu_\mu)) \) is given precisely by the set of maps \( \tilde{T} : \psi \in \tau_v \to T \otimes \psi \in \mathcal{H}_\mu \) with \( T \in (\mu_\psi, \mu_\mu^\otimes) \).
Hence there is a linear isomorphism \((\mu_v, \mu_u) \rightarrow (v, \text{Ind}(\mu_u))\), natural in \(\mu_u\). This isomorphism extends uniquely to a linear isomorphism natural in \(v\).

10 Full bimodule representations from group actions

The aim of this section is to study the full \(G\)–bimodule representation structures on the projective \(G\)–module subrepresentations of \(u \otimes \alpha\), where \(G\) is a compact group acting ergodically on a \(C^*\)–algebra, classifying them if the group acts on a type \(I\) von Neumann algebra.

More precisely, a general imprimitivity result due to Takesaki [45], combined with the work of Høegh-Krohn, Landstad and Størmer [16], allows one on the one hand to reduce the study of ergodic actions of compact groups to those of closed subgroups on finite factors. Correspondingly, we shall see that any full bimodule representation over an algebra with an ergodic action is induced by such an action on a factor. On the other hand, we shall show that there is always a full \(G\)–module subrepresentation \(X_u\) of \(u \otimes \alpha\) arising from Wasserman’s eigenmatrices for the action, that can be made uniquely into a full bimodule representation. Now, the functor \(u \rightarrow X_u\) is naturally isomorphic to the spectral functor associated with the ergodic action, and is, in general, quasitensor rather than tensor [32].

Thus, we may describe the extensions of \(X_u\) to full \(G\)–bimodule subrepresentations of \(u \otimes \alpha\) in terms of the extensions of the corresponding bimodule for a closed subgroup acting on a finite factor. In the type \(I\) case, we shall give a classification in terms of certain representations of the subgroup.

These results show that in certain cases of low multiplicity, \(X_u\) is the maximal full \(G\)–module subrepresentation. Hence, only a restricted subclass of ergodic actions beyond the commutative ones will have a tensorial induction functor, among them, those whose irreducibles have full multiplicity.

Let \(\alpha : G \rightarrow \text{Aut}(\mathcal{C})\) be a continuous ergodic action of a compact group \(G\) on a unital \(C^*\)–algebra \(\mathcal{C}\). The finiteness theorem for the noncommutative ergodic space \(\mathcal{C}\) and the multiplicity bound theorem assert respectively that the unique \(G\)–invariant state of \(\mathcal{C}\) is a trace, and that the multiplicity of an irreducible representation of \(G\) in \(\alpha\) is bounded above by its dimension. Furthermore, any von Neumann algebra with an ergodic action of a compact group is necessarily hyperfinite [16]. (For completeness, we recall that the finiteness theorem fails for compact quantum groups, as the Haar measure is not a trace in general [51], see also [47], whilst the multiplicity bound theorem holds, provided multiplicity and dimension are replaced by their noncommutative analogues [3, 2].)

Recall that if \(\beta\) is an automorphic action of a closed subgroup \(K\) of \(G\) on a von Neumann algebra \(\mathcal{F}\), the induced von Neumann algebra is defined by:

\[
\text{Ind}(\mathcal{F}) := \{ f \in L^\infty(G, \mathcal{F}) : f(kg) = \beta_k(f(g)), k \in K, g \in G \} = (L^\infty(G \otimes \mathcal{F}))^\lambda \otimes \beta,
\]
where $\lambda$ is left translation of $K$ on $L^\infty(G)$. If $\mathcal{F}$ is a $C^*$-algebra, the von Neumann tensor product $\otimes$ is replaced by the minimal one, and $L^\infty$-functions by continuous ones. The induced algebra carries the induced action $\rho$ of $G$ given by right translation.

As recalled in [48], combining the above results with an imprimitivity theorem of Takesaki [45] for locally compact group actions on von Neumann algebras leads to the following result, reducing the study of ergodic actions on von Neumann algebras to those on finite factors.

**10.1. Theorem** Any ergodic action of a compact group $G$ on a von Neumann algebra $\mathcal{C}$ is induced by an action of a closed subgroup $K$ on a full matrix algebra or on the hyperfinite $II_1$ factor $R$.

Wassermann has shown the important result that $G = SU(2)$ does not act ergodically on $R$ [50]. For more results in this direction see also [28]. In general, it is not yet known whether ergodic actions of compact simple groups on $R$ exist at all, a problem raised in [10].

The above theorem allows us to regard ergodic actions of closed subgroups on finite factors as virtual subgroups. If a closed subgroup $K$ of $G$ acts ergodically on a factor $\mathcal{F}$ via $\beta$, the role of a representation of $(K, \beta, \mathcal{F})$ is naturally taken by the $\mathcal{F}$–module $K$–representations $u \otimes \beta$ on $H_u \otimes \mathcal{F}$, where $u$ is a f.d. representation of $K$. The representation induced by $u \otimes \beta$ may be defined in the obvious way on the module of $K$–equivariant functions from $G$ to $H_u \otimes \mathcal{F}$. It is an $\text{Ind}(\mathcal{F})$–module representation of $G$. Similarly, the restriction of a f.d. representation $v$ of $G$ corresponds to the $\mathcal{F}$–module representation $v \mid_K \otimes \beta$ of $(K, \beta, \mathcal{F})$. However, without bimodule structures, one cannot relate these constructions to tensor $C^*$–categories.

We start with $(K, \beta, \mathcal{F})$, where $\beta$ is an action of a compact group $K$ on a unital $C^*$–algebra $\mathcal{F}$. Given a representation $\nu$ of $K$, recall that the spectral space $\mathcal{L}_v$ is the complex conjugate of the set $L_v$ of all linear maps $T : H_v \rightarrow \mathcal{F}$ intertwining $v$ with $\beta$. The multiplicity bound theorem shows that $\dim(\mathcal{L}_v) \leq \dim(\nu)$. If $v$ is irreducible, $\dim(\mathcal{L}_v)$ is usually called the multiplicity of $v$ in $\beta$, and denoted by $\text{mult}(\nu)$. $\mathcal{L}_v$ is a Hilbert space with inner product $<\mathcal{S}, \mathcal{T}> := \sum_i S(\psi_i)\mathcal{T}(\psi_i)^*$, where $(\psi_i)$ is an orthonormal basis of $H_v$. Clearly, $\mathcal{L}_v$ can and will be identified as a Hilbert space with the subspace of $K$–fixed vectors in $H_v \otimes \mathcal{F}$ for the action $v \otimes \beta$ and inner product inherited from the $\mathcal{F}$–valued inner product on $H_v \otimes \mathcal{F}$ (which, indeed, takes values in $\mathcal{C}$ on that subspace).

If $\mathcal{L}_v \neq 0$, we construct a natural nonzero $K$–module subrepresentation $X_v$ of $H_v \otimes \mathcal{F}$ with a full bimodule structure.

Consider the inclusion map $\mathcal{L}_v \rightarrow H_v \otimes \mathcal{F}$ as an isometry $S_v$ in the space $\mathcal{L}(\mathcal{L}_v, H_v) \otimes \mathcal{F}$. The (adjoint of the) map $S_v$ is referred to as the eigenmatrix of $\beta$ in [18], as clearly $\nu \otimes \beta(S_v) = \nu(k)^* \otimes IS_v$. If $v$ is a unitary f.d. representation of $K$, the orthogonal projection $E_v := S_v S_v^* \in \mathcal{L}(H_v) \otimes \mathcal{F}$ defines a projective right $\mathcal{F}$–submodule of $H_v \otimes \mathcal{F}$

$$X_v := E_v(H_v \otimes \mathcal{F}).$$

Since $E_v$ commutes with all $(v \otimes \beta)(k)$, $X_v$ is $K$–invariant. There is a faithful
unital \textasteriskcentered\textendash homomorphism
\[ \zeta : \mathcal{F} \to E_v \mathcal{L}(H_v) \otimes \mathcal{F} E_v \]
defined by
\[ \zeta(f) = S_v I \otimes f S_v^* , \]
or, in coordinates,
\[ \zeta(f)_{hr} = \sum_j \xi_j(h) f \xi_j(r)^* , \]
where the set of \( \xi_h := \sum_i \psi_i \otimes \xi_h(i) \) is an orthonormal basis of \( \mathcal{T}_v \) and \( (\psi_i) \) an orthonormal basis of \( H_v \).

10.2. Proposition For any representation \( v \) of \( K \) with \( \mathcal{T}_v \neq 0 \), \( \zeta \) makes \( X_v \) into a full bimodule \( K \)–representation, naturally isomorphic to \( \mathcal{T}_v \otimes \mathcal{F} \) with trivial left and right \( \mathcal{F} \)–actions, where \( K \) acts as \( \mathcal{T}_v \otimes \beta \), and \( \mathcal{T}_v \) is the trivial representation of \( K \) on \( \mathcal{T}_v \).

Proof Since \( E_v \xi_i = \xi_i \) for all \( i \), all the \( K \)–fixed vectors of \( H_v \otimes \mathcal{F} \) lie in \( X_v \). On the other hand, \( X_v \) must be generated, as a right \( \mathcal{F} \)–module, by the \( x_r := E(\psi_r \otimes I) = \sum_{h,j} \psi_h \otimes \xi_j(h) \xi_j(r)^* = \sum_j \xi_j \xi_j(r)^* \), and hence by the \( (\xi_j) \)’s.

Therefore \( X_v \), as a right module, is isomorphic to \( \mathcal{T}_v \otimes \mathcal{F} \). It becomes a bimodule representation of \( K \) with left action defined by \( \zeta \), with \( K \) acting trivially on the first factor and as \( \beta \) on the second.

By construction every \( K \)–fixed vector is \( \mathcal{F} \)–central in \( X_v \), as for all \( i \)
\[ \zeta(f) \xi_i = \sum_{j,h,q} \psi_h \otimes \xi_j(h) f \xi_j(q)^* \xi_i(q) = \sum_{j,h} \psi_h \otimes \xi_j(h) f \delta_{j,i} = \sum_h \psi_h \otimes \xi_i(h) f = \xi_i f, \]
where \( \psi_h \) is an orthonormal basis of \( H_v \). Property (2.5) follows easily: for all \( i, f, f' \in \mathcal{F} 
\]
\[ \begin{array}{l}
  v(k) \otimes \beta_k(\zeta(f) \xi_i f') = v(k) \otimes \beta_k(\xi_i f f') = v(k) \otimes \beta_k(\xi_i f f') = v(k) \otimes \beta_k(\xi_i) \beta_k(f) \beta_k(f') = \zeta(\beta_k(f)) \xi_i \beta_k(f') = \zeta(\beta_k(f)) v_k \otimes \beta_k(\xi_i f'). 
\end{array} \]

Given \( (\mathcal{F}, K, \beta) \), we look for extensions of \( X_v \) to full bimodule structures on intermediate projective \( K \)–module subrepresentations
\[ X_v \subset Y \subset H_v \otimes \mathcal{F}. \]

Clearly, such submodules are the ranges of projections \( E \in \mathcal{L}(H_v) \otimes \mathcal{F} \) satisfying
\[ E \geq E_v, \quad (10.1) \]
and the $K$–invariance condition
\[ \text{Ad}v(k) \otimes \beta_k(E) = E, \quad k \in K. \tag{10.2} \]

In what follows, we set $S_v = 0$ and $X_v = \{0\}$ if $\mathcal{T}_v = \{0\}$.

10.3. Lemma Given $v \in \text{Rep}(K)$ and a projection $E \in \mathcal{L}(H_v) \otimes \mathcal{F}$ satisfying (10.1) and (10.2), a unital $*$–homomorphism $\eta : \mathcal{F} \to \mathcal{E}(H_v) \otimes \mathcal{F}E$ defines a full $K$–bimodule representation on $Y = EH_v \otimes \mathcal{F}$ if and only if
\[ \eta(\beta_k(f)) = \text{Ad}v(k) \otimes \beta_k(\eta(f)), \quad k \in K, \tag{10.3} \]
\[ \eta(f)S_v = S_v I \otimes f, \quad f \in \mathcal{F}. \tag{10.4} \]

In particular, if every irreducible subrepresentation of $v$ has full multiplicity in $\beta$, $X_v = H_v \otimes \mathcal{F}$, hence $v \otimes \beta$ becomes a full bimodule $K$–representation in a unique way.

Proof The proof is straightforward. We just note that (10.3) corresponds to left $K$–equivariance in the sense of (2.5), whilst the property of being a full representation is expressed by $\sum_{\psi} \eta(f)\omega_\psi(\zeta(p)) = \zeta(i)f$ for all $f \in \mathcal{F}$ and $\xi = \sum \omega_\psi \otimes \xi(p) \in H_v \otimes \mathcal{F}$ a fixed point. Replacing $\xi$ by an orthonormal basis, leads to (10.4).

In particular, if $v$ has full multiplicity, $S_v$ is an isometry in $\mathcal{L}(H_v) \otimes \mathcal{F}$ and hence a unitary since that algebra has a faithful positive trace, and $X_v = H_v \otimes \mathcal{F}$. Hence $\xi$ is a unital $*$–homomorphism of $H_v \otimes \mathcal{F}$ into a full bimodule representation.

Now assume that $K$ is a closed subgroup of a compact group $G$ acting on $\mathcal{F}$, which may be either a $C^*$–algebra or a von Neumann algebra. This action, $\beta$, is supposed to be continuous in the appropriate topology. Let $v$ be a f.d. unitary representation of $G$. In the next, known, proposition we determine the spectral spaces for the induced action $\rho$ in terms of those of the original action $\beta$.

10.4. Proposition The map $T \in L^\beta_{v|\mathcal{K}} \to T' \in L^v_\beta$, with $T' : H_v \to \text{Ind}(\mathcal{F})$ defined by $T'(\psi)(g) := T(v(g)\psi)$, is unitary. As a consequence,
\[ S^\beta_v(g) = v(g)^* \otimes I S^\beta_{v|\mathcal{K}}, \]
hence
\[ E^\beta_v(g) = v(g)^* \otimes IE^\beta_{v|\mathcal{K}} v(g) \otimes I. \]

Proof Let us extend $\beta$ and $\rho$ to unitary representations of $K$ and $G$ respectively on the $L^2$–completions of $\mathcal{F}$ and $\text{Ind}(\mathcal{F})$ for the unique invariant traces. The extension of $\rho$ is clearly the representation induced from the extension of $\beta$ in the sense of Mackey. Extending in this way does not increment the spectra. Hence $L^\rho_v$ may be determined by the classical Frobenius reciprocity theorem, showing that $T \to T'$ is a linear isomorphism. It is easily checked to be an isometry. Therefore
\[ S^\rho_v(T')(g) = \sum_i \psi_i \otimes T(v(g)\psi_i)^*, \]
for $T \in T_{v[I]}^\beta$ and $(\psi_i)$ an orthonormal basis of $H_v$, showing that if $(T_j)$ is an orthonormal basis of $T_v$ then $(\xi'_j)$, with $\xi'_j(g) := \sum j_k \otimes T_j(v(g)\psi_k)^*$, is an orthonormal basis of the Hilbert space of $G$-fixed vectors in $H_v \otimes \text{Ind}(\mathcal{F})$, hence the $jr$– entry of $S_{v,I}$ is the function

$$T_r(v(g)\psi_j)^* = (v(g)^* \otimes IS_{v[I],I}^\beta)_{jr}.$$  

If $z$ and $v$ are representations of $K$, we may identify the space of $\mathcal{F}$–module maps $L_{\mathcal{F}}(H_z \otimes H_v \otimes \mathcal{F})$ with $L(H_z, H_v) \otimes \mathcal{F}$ in an obvious way. Hence

$$(z \otimes \beta, v \otimes \beta) = \{ T \in L(H_z, H_v) \otimes \mathcal{F} : \iota \otimes \beta_k(T) = v(k)^* \otimes IT_z(k) \otimes \iota \},$$

leading to an identification of $C^*$–categories. As a $K$–space, this space is linearly isomorphic to

$$(H_v \otimes \mathcal{F})^* \otimes \mathcal{F} \cong T_{v[I]}^\beta$$

and therefore finite dimensional. This remark, combined with the previous proposition, shows the following result, needed later. A module map $T \in L_{\text{Ind}(\mathcal{F})}(H \otimes \text{Ind}(\mathcal{F}), H' \otimes \text{Ind}(\mathcal{F}))$ will be regarded as a function $T : G \to L(H, H') \otimes \mathcal{F}$.

**10.5. Corollary** There is a full and faithful $*$–functor from the full subcategory of $\text{Mod}_\rho(G)$ with objects $v \otimes \rho$, $v \in \text{Rep}(G)$, to the category $\text{Mod}_\beta(K)$, given by

$$v \otimes \rho \mapsto v \big|_K \otimes \beta, \quad T \in (v \otimes \rho, v' \otimes \rho) \mapsto T(1) \in (v \big|_K \otimes \beta, v' \big|_K \otimes \beta).$$

The inverse map on arrows is given by $A \mapsto A'$ with $A'(g) := v'(g)^* \otimes IAv(g) \otimes I$.

The functor $T \mapsto T(1)$ defined in the above corollary will be referred to as the *evaluation functor*.

Given a projection $E \in L(H_v) \otimes \mathcal{F}$ and a unital $*$–homomorphism $\eta : \mathcal{F} \to \mathcal{E}L(H_v) \otimes \mathcal{F}E$ defining a full bimodule structure on the intermediate $K$–module $Y = EH_v \otimes \mathcal{F}$, i.e. satisfying conditions (10.1)–(10.4), we may consider the projection $\tilde{E} \in C(G, L(H_v) \otimes \mathcal{F}) \simeq L(H_v) \otimes C(G, \mathcal{F})$,

$$\tilde{E}(g) := v(g)^* \otimes IEv(g) \otimes I,$$

which clearly satisfies

$$\iota \otimes \beta_k(\tilde{E}(g)) = v(g)^* \otimes Iv(\iota \otimes \beta_k(E)v(g) \otimes I = v(kg)^* \otimes IEv(kg) \otimes I = \tilde{E}(kg),$$

hence $\tilde{E} \in L(H_v) \otimes \text{Ind}(\mathcal{F})$. We may also consider the map taking a continuous function $f$ on $G$ with values in $\mathcal{F}$ to the function

$$\tilde{\eta}(f)(g) := v(g)^* \otimes I\eta(f(g))v(g) \otimes I.$$  

Similar computations and (10.3) show that if $f \in \text{Ind}(\mathcal{F})$ then $\tilde{\eta}(f) \in L(H_v) \otimes \text{Ind}(\mathcal{F})$ and $\tilde{E} \tilde{\eta}(f) = \tilde{\eta}(f) = \tilde{\eta}(f)\tilde{E}$, hence $\tilde{\eta}$ is in fact a unital $*$–homomorphism between

$$\tilde{\eta} : \text{Ind}(\mathcal{F}) \to \tilde{E}L(H_v) \otimes \text{Ind}(\mathcal{F})\tilde{E},$$
hence \((\tilde{E}, \tilde{\eta})\) defines a bimodule over the induced algebra \(\text{Ind}(\mathcal{F})\). We shall refer to it as the induced bimodule.

**10.6. Theorem** The induced bimodule \((\tilde{E}, \tilde{\eta})\) satisfies (10.1)–(10.4) if \((E, \eta)\) does. Furthermore, if \(\mathcal{F}\) is the completion of the dense \(*\)–subalgebra of \(K\)–finite elements in the maximal \(C^*\)–norm, any intermediate projective \(G\)–module \(X_\xi \subset Y \subset H_v \otimes \text{Ind}(\mathcal{F})\) with a full bimodule structure is defined by such a pair \((E, \eta)\).

**Proof** The validity of (10.1)–(10.4) for a bimodule induced from one with analogous properties follows easily from the previous proposition. Conversely, let \((E', \eta')\) satisfy (10.1)–(10.4) with respect to the automorphism group \(\rho\) of the induced algebra. By (10.2), \(v(g) \otimes \rho(g)(E')v(g)^* \otimes I = E'\). Evaluating in \(g'\) gives \(v(g) \otimes \text{IE}(g')v(g')^* \otimes I = E'(g')\), hence \(E'(g) = v(g)^* \otimes \text{IE}v(g) \otimes I\), where \(E := E'(1)\).

It is now clear that \(E\) satisfies (10.1). Moreover, for \(k \in K\),

\[v(k)^* \otimes \text{IE}v(k) \otimes I = E'(k) = \iota \otimes \beta_k(E'(1)),\]

hence \(E\) satisfies (10.2).

On the other hand, \(E' \mathcal{L}(H_v) \otimes \text{Ind}(\mathcal{F})E', \) with \(G\)–action \(\text{Adv} \otimes \rho\), is isomorphic to the \(C^*\)–system induced by \(E \mathcal{L}(H_v) \otimes \mathcal{F}E\) with \(K\)–action \(\text{Adv} \upharpoonright K \otimes \beta\). An explicit \(G\)–equivariant isomorphism takes \(f \in E' \mathcal{L}(H_v) \otimes \text{Ind}(\mathcal{F})E'\) to the element of \(C(G, E \mathcal{L}(H_v) \otimes \mathcal{F}E)\) defined by \(g \in G \rightarrow \text{Adv}(g) \otimes \text{I}_f(g)\). Therefore condition (10.3) can be regarded as an intertwining relation between induced group representations. Hence, by Frobenius reciprocity, there is a map, a priori just linear, and densely defined on the \(*\)–subalgebra of \(K\)–finite elements, \(\eta : \mathcal{F} \rightarrow E \mathcal{L}(H_v) \otimes \mathcal{F}E\) satisfying the intertwining relation

\[\eta(\beta_k(f)) = \text{Adv}_k \otimes \beta_k(\zeta(f)),\]

and hence (10.3), for \(f \in \mathcal{F}, k \in K\), inducing \(\eta'\) via

\[\eta'(f)(g) = \text{Adv}(g)^* \otimes \text{I}_\eta(f(g)).\]

We show that \(\eta\) is a unital \(*\)–homomorphism. It is well known that for any \(K\)–finite element \(f_1 \in \mathcal{F}\), there is an element \(f \in \text{Ind}(\mathcal{F})\) such that \(f_1 = f(1)\). Thus, (10.4) follows. On the other hand, since \(\eta'\) is a unital \(*\)–homomorphism, the above formula, evaluated in 1, shows that \(\eta\) is a unital \(*\)–homomorphism on the dense \(*\)–subalgebra of \(K\)–finite elements. Since \(\mathcal{F}\) is the completion in the maximal \(C^*\)–norm, we may conclude that \(\eta\) extends uniquely to \(\mathcal{F}\) with the required properties.

We next give a simple method for constructing extensions of \(X_v\) to full bimodule representations on projective submodules of \(H_v \otimes \mathcal{F}\), even providing a complete list if \(\mathcal{F}\) is a full matrix algebra.

**10.7. Proposition** Pick a representation \(v\) of \(K\).

a) If there is a unitary representation \(z\) of \(K\) with

\[\dim(z) \leq \dim(v) - \dim(T_v)\]

(10.5)
and an isometry

$$W \in (z \otimes \beta, v \otimes \beta) \text{ such that } W^*S_v = 0,$$  \hspace{1cm} (10.6)\

then the intermediate K–module subrepresentation $X_v \subset Y \subset H_v \otimes \mathcal{F}$ defined by the projection $E := S_v S_v^* + W W^* \in \mathcal{L}(H_v) \otimes \mathcal{F}$ becomes a full bimodule K–representation with left action $\eta(f) := S_v I \otimes f S_v^* + W I \otimes f W^*$, if we can choose $z$ with $\dim(z) = \dim(v) - \dim(\mathcal{T}_v)$ then we get a full $K$–bimodule representation for $v \otimes \beta$.

**Proof** If $E$ and $\eta$ are defined as in a) they certainly satisfy the assumptions in Lemma 10.3, hence we obtain a full bimodule representation $Y$, and clearly b) follows.

**10.8. Theorem** Let $\beta$ be an ergodic action of a compact group $K$ on a factor $\mathcal{F}$ and $v$ a representation of $K$.

a) Two pairs $(z,W), (z',W')$ satisfying (10.5) and (10.6) define the same intermediate $K$–bimodule representation $Y$ if and only if there is a unitary intertwiner $U \in (z, z')$ such that $W = W' U \otimes I$.

b) if $\mathcal{F}$ is a full matrix algebra, then any full intermediate bimodule representation $X_v \subset Y \subset H_v \otimes \mathcal{F}$ arises from a pair $(z,W)$. In particular, structures of full $K$–bimodule representations on $H_v \otimes \mathcal{F}$ correspond to pairs $(z,W)$ satisfying (10.5) and (10.6) where the inequality of (10.5) is strengthened to an equality.

**Proof**

a) Obviously two equivalent pairs $(z,W), (z',W')$, as in a), give rise to the same intermediate $K$–module $Y$ with the same left action $\eta$. Conversely, suppose $(z,W)$ and $(z',W')$ define the same $K$–bimodule representation $Y$. Then clearly $W W^* = W' W'^*$. Since the two left actions coincide, $W I \otimes f W^* = W' I \otimes f W'^*$. Hence the unitary $W' W \in (z \otimes \beta, z' \otimes \beta)$ is of the form $U \otimes I$, with $U : H_z \to H_{z'}$, as $\mathcal{F}$ is a factor. Thus $W = W' U \otimes I$. Making the intertwinning property of $W$ and $W'$ explicit shows that $U \in (z, z')$.

b) Assume that $\mathcal{F} = \text{Mat}_r(\mathbb{C})$. If $Y$ is defined by $E$ and $\eta$, then $E$ needs to be of rank $qr$ with $q$ integer, as $\eta$ is unital, and $q \geq \dim(\mathcal{T}_v)$ as $E \geq E_v$. Set $\eta_1(f) := \eta(f)(E - E_v)$. We can write $\eta_1$ in the form $\eta_1(f) = W I \otimes f W^*$ with $W$ a partial isometry such that $W W^* = E - E_v =: E_1$ and $W^* W \in \mathcal{L}(H_v) \otimes \mathbb{C}$. The relation $W^* S_v = 0$ implies $\dim(W^* W H_v) + \dim(\mathcal{T}_v) \leq \dim(v)$. The covariance condition (2.5) for $Y$ becomes

$$\text{Ad}_v(k) \otimes \beta_k \eta(f) = \eta(\beta_k(f)), \hspace{1cm} f \in \text{Mat}_r(\mathbb{C}),$$

and is equivalent to requiring an analogous relation for $\eta_1$:

$$v(k) \otimes I \otimes \beta_k(W) I \otimes \beta_k(f) I \otimes I \otimes \beta_k(W^*) v(k)^* \otimes I = WI \otimes \beta_k(f) W^*,$$

or

$$W^* v(k) \otimes I \otimes \beta_k(W) \in \mathcal{L}(H_v) \otimes \mathbb{C}.$$
On the other hand, the map \( k \to z(k) \) with \( z(k) \) defined by
\[
z(k) \otimes I := W^* v(k) \otimes I_k \otimes \beta_k(W)
\]
is a unitary representation of \( K \) on the subspace \( W^* WH_v \), completing the proof of c).

**Remark** Given representations \( z \) and \( v \) of \( K \), express elements of \( \mathcal{L}(H_z \oplus H_v) \otimes \mathcal{F} \) as \( 2 \times 2 \) matrices. Connes’ well known argument shows that an isometry \( W \) satisfying the conditions in a) exists if and only if the projection \( 1_{H_z} \otimes I \oplus 0 \) is Murray-von Neumann equivalent to a subprojection of \( 0 \oplus (I - E_v) \), in the f.d. fixed point algebra \( \mathcal{L}(H_z \oplus H_v) \otimes \mathcal{F} \text{Ad}(z \otimes v) \otimes \beta \).

The following result summarizes the classification of full bimodule representations for type I ergodic actions achieved here.

**10.9. Theorem** Let \( F \) be a full matrix algebra, and let \( \beta \) be an ergodic action of a closed subgroup \( K \) of a compact group \( G \) on \( F \). Pick a unitary f.d. representation \( v \) of \( G \). Then

a) the full bimodule \( G \text{-} \text{representations} \) over intermediate projective \( G \text{-} \text{module subrepresentations} \), \( X_v \subset Y \subset H_v \otimes \text{Ind}(\mathcal{F}) \), are classified by equivalence classes of pairs \( (z, W) \), where \( z \) is a unitary f.d. representation of \( K \) and \( W \in (z \otimes \beta, v \upharpoonright_K \otimes \beta) \), an isometry satisfying
\[
W^* S_{v \upharpoonright_K}^\beta = 0.
\]

\((W, z)\) and \((W', z')\) are equivalent if there is a unitary intertwiner \( U \in (z, z') \) with \( W = W' U \otimes I \). In particular, the structures of full \( G \text{-} \text{bimodule representation} \) on \( H_v \otimes \text{Ind}(\mathcal{F}) \) correspond to the possibility of choosing \( W \) s.t.
\[
WW^* + S_{v \upharpoonright_K}^\beta S_{v \upharpoonright_K}^{\beta*} = I.
\]

b) The left module structure \( \tilde{\eta} : \text{Ind}(\mathcal{F}) \to \mathcal{L}(H_v) \otimes \text{Ind}(\mathcal{F}) \) corresponding to \((W, z)\) is given by
\[
\tilde{\eta}(f)(g) = v(g)^* \otimes (S_{v \upharpoonright_K}^\beta I \otimes f(g)S_{v \upharpoonright_K}^{\beta*} + WI \otimes f(g)W^*)v(g) \otimes I.
\]

**Remark** Hence, if the module \( G \text{-} \text{representation} \) \( v \otimes \rho \) over \( \text{Ind}(\mathcal{F}) \) can be made into a full bimodule \( G \text{-} \text{representation} \), and if \((z, W)\) induces this structure, we may form the \( K \text{-} \text{representation} \) \( z' := \nu_{T_v} \oplus z \) of the same dimension as \( v \). Then the original full bimodule structure for \( v \upharpoonright_K \otimes \beta \) inducing the given full bimodule structure for \( v \otimes \rho \), in the sense of Theorem 10.6, is in fact unitarily equivalent as a bimodule representation to \( z' \otimes \beta \) with trivial left module structure via the unitary of \( K \text{-} \text{bimodule representations} \) \( S_{v \upharpoonright_K}^\beta \oplus W \).

As a consequence of b) of Theorem 10.8, we notice that in some cases, \( v \otimes \beta \), for \( v \) in the spectrum, does not extend to a full bimodule \( K \text{-} \text{representation} \) unless \( v \) has full multiplicity. We discuss a class of examples.
Example Consider the adjoint action $\beta_r$ of the unique $r + 1$-dimensional irreducible representation $v_r$ of the group $K = SU(2)$ acting on the full matrix algebra $\text{Mat}_{r+1}(\mathbb{C})$. We show that $v \otimes \beta_r$ becomes a full bimodule representation only for certain $v$. Hence none of the actions $\beta_r$ arises from a relaxed tensor $^*$-functor $\text{Rep}(SU(2)) \to \mathcal{M}$ to a tensor $C^*$-category, as this functor would make all $\beta_r \otimes v$ into full bimodule representations by Theorem 5.1.

The spectrum of $\beta_r$ may be determined by the Clebsch–Gordan rule

$$v_r \otimes v_s \simeq v_{r-s} \oplus v_{r-s+2} \oplus \cdots \oplus v_{r+s}, \quad r \geq s,$$

after regarding $\beta_r$ as a Hilbert space representation with respect to the inner product defined by the ($K$–invariant) trace of $\text{Mat}_{r+1}(\mathbb{C})$. $v_r$ being selfconjugate, we have $\beta_r \simeq v_r \otimes v_r \simeq v_0 \oplus v_2 \oplus \cdots \oplus v_{2r}$. Hence any spectral representation has multiplicity 1.

In particular, $v_1$ is never in the spectrum of $\beta_r$, and the full bimodule structures on $H_{v_1} \otimes \text{Mat}_{r+1}(\mathbb{C})$ are described by pairs $(z, W)$ with $\dim(z) = \dim(v_1) = 2$. Since $z$ can never contain the trivial representation, we necessarily have $z = v_1$. Hence we need to specify a unitary

$$W \in (v_1 \otimes \beta_r, v_1 \otimes \beta_r) \simeq (v_1 \otimes v_r, v_1 \otimes v_r) \simeq (v_{r-1} \oplus v_{r+1}, v_{r-1} \oplus v_{r+1}) \simeq \mathbb{C} \oplus \mathbb{C}.$$ 

Hence $v_1 \otimes \beta_r$ admits full bimodule structures, and they are classified by $T$.

On the other hand, low multiplicity of a representation in the spectrum in general rules out full bimodule structures on $v \otimes \beta_r$ as the following simple argument shows. If there were a structure of a full $K$–bimodule representation on $v_2 \otimes \beta_r$ defined by $(z, W)$, then we must have $\dim(z) = \dim(v_2) - \text{mult}(v_2) = 2$. Since $z$ cannot contain the trivial representation, $z = v_1$. On the other hand the space of module intertwiners $(v_1 \otimes \beta_r, v_2 \otimes \beta_r)$ is isomorphic to $(v_1 \otimes v_r, v_2 \otimes v_r)$ which is trivial, again by the Clebsch–Gordan rule. Hence $v_2 \otimes \beta_r$ admits no full bimodule structure, and actually $X_{v_2}$ admits no proper extension to a full bimodule representation.

10.10. Corollary Let $K$ act ergodically on a full matrix algebra.

a) Let $v$ be a representation with $\mathcal{L}_v \neq \{0\}$ and assume that any irreducible of smaller dimension has full multiplicity. If $\dim(\mathcal{L}_v) < \dim(v)$ then $X_v$ does not admit any proper extension to a full bimodule $K$–representation.

b) If $\beta$ has full spectrum (hence $K$ is finite) and each $v \otimes \beta$ can be made into a full bimodule representation then each irreducible is of full multiplicity in $\beta$.

Proof Let $z$ and $W$ be as required in a) of Prop. 10.7. Since $\dim(z) < \dim(v)$, any irreducible subrepresentation of $z$ has full multiplicity in $\beta$. Hence there is a unitary $U \in \mathcal{L}(H_z) \otimes \text{Mat}_r(\mathbb{C})$ with $v \otimes \beta_k(U) = z(k)^* \otimes IU$. Hence every column of $WU$ gives an element of $\mathcal{L}_v$ orthogonal to $\mathcal{L}_v$ itself, as $W^*S_v = 0$. So $WU = 0$ and $W = 0$. This completes the proof of a) and b) follows easily.
11 Tensorial properties of the evaluation functor

In this section we use the classification of full Hilbert bimodule structures on type I von Neumann algebras obtained in the previous section to prove Theorem 5.5, and Corollaries 5.6 and 5.7.

We need a few simple lemmas that clarify the tensorial properties of the evaluation functor defined in the previous section. We thus assume that we are given an action $\beta$ of a closed subgroup $K$ of a compact group $G$ on a $C^*$-algebra $\mathcal{F}$ and that for each $v \in \text{Rep}(G)$ we have a full bimodule structure for $v \mid_K \otimes \beta$ defined by the $^*-$homomorphism $\eta_v : \mathcal{F} \to \mathcal{L}(H_v) \otimes \mathcal{F}$. We consider the full bimodule structure $\tilde{\eta}_v$ for $v \otimes \rho$ induced by $\eta_v$.

11.1. Lemma If $T \in (v \otimes \rho, v' \otimes \rho)$ is a bimodule map then $T(1) \in (v \mid_K \otimes \beta, v' \mid_K \otimes \beta)$ is a bimodule map as well.

Proof The proof is straightforward. By Cor. 10.5, we may write $T$ in the form $T(g) = v'(g)^* \otimes IT(1)v(g) \otimes I$, with $T(1) \in (v \mid_K \otimes \beta, v' \mid_K \otimes \beta)$. The intertwining relation for $T$ evaluated at 1 gives the intertwining relation for $T(1)$.

Let us now consider a unital $C^*$-algebra $\mathcal{C}$ and two f.d. Hilbert spaces $H$ and $L$. Consider the right $C^*$-modules $H \otimes \mathcal{C}$ and $L \otimes \mathcal{C}$. If $L \otimes \mathcal{C}$ also has a left $\mathcal{C}$-module structure defined by a unital $^*-$homomorphism $\eta : \mathcal{C} \to \mathcal{L}(L) \otimes \mathcal{C}$ then we may form the tensor product right Hilbert $C^*$-module $(H \otimes \mathcal{C}) \otimes_{\mathcal{C}} (L \otimes \mathcal{C})$, to be identified with $(H \otimes L) \otimes \mathcal{C}$. We may thus form tensor products $T \otimes S$ of a module intertwiner $T \in \mathcal{L}(H, H') \otimes \mathcal{C}$ with a bimodule intertwiner $S \in \mathcal{C}\mathcal{L}(L \otimes \mathcal{C}, L' \otimes \mathcal{C})$ giving an element of $\mathcal{L}(H \otimes L, H' \otimes L') \otimes \mathcal{C}$.

11.2. Lemma Let us consider $H_v \otimes \text{Ind}(\mathcal{F})$ and $H'_v \otimes \text{Ind}(\mathcal{F})$ as right Ind($\mathcal{F}$)-modules. Let $\tilde{\eta}_v$, $\tilde{\eta}_v'$ make $H_v \otimes \text{Ind}(\mathcal{F})$ and $H'_v \otimes \text{Ind}(\mathcal{F})$ into Ind($\mathcal{F}$)-bimodules. For a module intertwiner $T \in (v \otimes \rho, v' \otimes \rho)$ and a bimodule intertwiner $S \in (u \otimes \rho, u' \otimes \rho)$, we have

$$(T \otimes S)(1) = T(1) \otimes S(1).$$

Proof Notice that $S(1)$ is a bimodule intertwiner by the previous lemma, hence the right hand side makes sense. Let $H, H', L, L'$ be f.d. Hilbert spaces and $\eta, \eta'$ left $\mathcal{C}$-module structures on $L \otimes \mathcal{C}$ and $L' \otimes \mathcal{C}$ respectively. Given a module intertwiner $T \in \mathcal{L}(H, H') \otimes \mathcal{C}$, and a bimodule intertwiner $S \in \mathcal{C}\mathcal{L}(L \otimes \mathcal{C}, L' \otimes \mathcal{C})$, a simple computation shows that if $T$ is represented by the $\mathcal{C}$-valued matrix $(t_{rs})$, in the sense that $T = \sum_{rs} t_{rs} \otimes t_{rs}$, where $t_{rs}$ are matrix units, and if $S$ is represented by $(s_{pq})$ then the module intertwiner $T \otimes S$ regarded as an element of $\mathcal{L}(H \otimes L, H' \otimes L') \otimes \mathcal{C}$ is represented by the matrix whose $(rp)(sq)$-entry is $\sum_{h} \eta_r(h) t_{rs} \eta_s(h)$. We apply this to $H_v, H'_v, H, H'$ and Ind($\mathcal{F}$) respectively. By Cor. 10.5, we may write $(t_{rs})(g) = \eta_r(h) t_{rs}(g) (s_{pq})(g) = \eta_s(h) t_{rs}(g)$, where $T(1)$ and $S(1)$ are now represented as $\mathcal{C}$-valued matrices. Recalling how $\tilde{\eta}_v'$ was defined before Theorem 10.6, the $(rp)(sq)$-entry of $T \otimes S$ is the function

$$\sum_h \eta_r(h) t_{rs}(g) s_{pq}(h) = \sum_{h,l,m} \eta_r(h) t_{rs}(g) (s_{pq})(h) = \sum_{h,l,m} \eta_r(h) t_{rs}(g) (s_{pq})(h).$$
\[
\sum_{h,l,m,i,j,k,t} \overline{u}(g)_{lp}u'(g)_{ir}\eta'_{h}(T(1)_{ij})lmv(g)_{js}u'(g)_{mh}\overline{u}(g)_{kh}S(1)_{kl}u(g)_{tq} = \\
\sum_{i,j,k,t} v' \otimes u'(g)_{(r_p)(ij)}\eta'_{i}(T(1)_{ij})lkv(g)_{js}S(1)_{kt}v \otimes u(g)_{(jt)(sq)} = \\
\sum_{i,j,k,t} v' \otimes u'(g)_{(r_p)(ij)}(T(1) \otimes S(1))_{(ij)(jt)}v \otimes u(g)_{(jt)(sq)}.
\]

Hence \((T \otimes S)(1) = T(1) \otimes S(1)\).

Notice that if \(H \otimes \mathcal{C}\) and \(L \otimes \mathcal{C}\) have left bimodule structures defined by \(\eta : \mathcal{C} \rightarrow \mathcal{L}(H) \otimes \mathcal{C}\) and \(\zeta : \mathcal{C} \rightarrow \mathcal{L}(L) \otimes \mathcal{C}\) then under the unitary module map \((H \otimes \mathcal{C}) \otimes \mathcal{C} \simeq (H \otimes L) \otimes \mathcal{C}\) the left module structure \(\mathcal{C} \rightarrow \mathcal{L}(H \otimes L) \otimes \mathcal{C}\) corresponding to the tensor product bimodule is given by \(i_{\mathcal{L}(H)} \otimes \zeta \circ \eta, i_{\mathcal{L}(H)}\) being the identity map on \(\mathcal{L}(H)\). This tensor product left action will be denoted by \(\eta \otimes \zeta\).

11.3. Lemma If the induced set of left actions \(\{\tilde{\eta}_u, u \in \text{Rep}(G)\}\) on the \(C^*\)-modules \(H_u \otimes \text{Ind}(\mathcal{F})\) is tensorial, i.e. \(\tilde{\eta}_u \otimes v = \tilde{\eta}_u \otimes \tilde{\eta}_v\) for \(u, v \in \text{Rep}(G)\) then the original set \(\{\eta_u, u \in \text{Rep}(G)\}\) is tensorial too.

Proof It suffices to evaluate the tensorial relation for the \(\tilde{\eta}_u\)'s at 1.

We summarize the above lemmas as follows.

11.4. Theorem Let \(\beta\) be an action of a closed subgroup \(K\) of a compact group \(G\) on a \(C^*\)-algebra \(\mathcal{F}\). Assume that for each \(v \in \text{Rep}(G)\) we have a full bimodule structure for \(v \mid_K \otimes \beta\) defined by the \(*\)-homomorphism \(\eta_v : \mathcal{F} \rightarrow \mathcal{L}(H_v) \otimes \mathcal{F}\). If the set of induced bimodule structures \(\tilde{\eta}_v\) for \(v \otimes \rho\) is tensorial then the evaluation functor \(T \rightarrow T(1)\) restricts to a faithful tensor \(*\)-functor from the full tensor \(C^*\)-subcategory of \(\text{Bimod}_\rho(G)\) with objects \(v \otimes \rho\) to \(\text{Bimod}_\beta(K)\).

Proof of Theorem 5.5. and Cor. 5.6. Theorem 5.1, applied to the given tensor functor \(\mu : \mathcal{F}G \rightarrow \mathcal{M}\) and to the embedding functor \(\tau : \mathcal{F}G \rightarrow \text{Hilb}\), allows us to identify \(\mathcal{M}_\mu^r\) with the full subcategory of \(\text{Bimod}_\rho(G)\) with objects \(u^r \otimes \alpha\), \(r = 0, 1, 2, \ldots\), where \(u\) is the distinguished representation of \(G\) and, as before, \(\alpha\) is the ergodic action of \(G\) on the associated \(C^*\)-algebra \(\mathcal{C}\). That theorem provides us with a full \(G\)-bimodule representation for each \(u^r \otimes \alpha\) and the collection of these left module structures is tensorial. Since the von Neumann completion of \(\mathcal{C}\) in the GNS representation of the \(G\)-invariant trace state is of type I, we may identify the completed ergodic system with a von Neumann ergodic system \((\text{Ind}(\mathcal{F}), \rho)\) induced from a closed subgroup \(K\), unique up to conjugation, where \(\mathcal{F}\) is a matrix algebra with an ergodic action \(\beta\) of \(K\). The left \(\mathcal{C}\)-action of \(H_u^r \otimes \mathcal{C}\) is defined by a unital \(*\)-homomorphism \(\eta : \mathcal{C} \rightarrow \mathcal{L}(H_u) \otimes \mathcal{C}\) intertwining \(\alpha\) with \(\text{Ad}(u) \otimes \alpha\) by Lemma 10.3. Hence, if \(\text{tr}\) and \(\tau\) are the normalized \(G\)-invariant traces on \(\mathcal{L}(H_u)\) and \(\mathcal{C}\) respectively, \((\text{tr} \otimes \tau) \circ \eta\) is a \(G\)-invariant trace on \(\mathcal{C}\). Such a trace
is unique so \((\text{tr} \otimes \tau) \circ \eta = \tau\). Thus \(\eta\) induces a normal \({}^*\)-homomorphism from \(\text{Ind}(\mathcal{F})\) to \(\mathcal{L}(H_u) \otimes \text{Ind}(\mathcal{F})\). Correspondingly, we get a set of tensorial full bimodule structures for \(u^* \otimes \rho\). Thus by Theorem 11.4 there is a faithful tensor \({}^*\)-functor from the full subcategory of \(\text{Bimod}_\rho(G)\) with objects \(u^* \otimes \rho\) to the full subcategory \(\mathcal{T}\) of \(\text{Bimod}_\rho(K)\) with objects \(u^* |_K \otimes \beta\). We next apply Theorem 10.9 to the full bimodule \(K\)-representation \(u |_K \otimes \beta\) fixing a pair \((z, W)\). We set \(z' := i_{\mathcal{T}(u)}^* u |_K \oplus z\) and \(U := S_{u |_K} \oplus W\), a \(K\)-bimodule unitary in \((z' \otimes \beta, u |_K \otimes \beta)\) if \(z'\) has the trivial left \(\mathcal{C}\)-action. We define a \({}^*\)-functor \(\mathcal{T} \rightarrow \text{Rep}(K)\) taking \(u^* |_K \otimes \beta\) to \((z'' |_K \otimes \beta)\) and a bimodule intertwiner \(T \in (u^* |_K \otimes \beta, u^* |_{\beta \otimes \beta})\) to \(U^{* \otimes * TU^{\otimes r}}\). We need to show that \(U^{* \otimes * T U^{\otimes r}}\) lies in the subspace \(\mathcal{L}(H_{v^*}, H_{v^*}) \otimes \mathcal{C}\) of \(\mathcal{L}(H_{v^*}, H_{v^*}) \otimes \mathcal{F}\). To this end, recall that Theorem 5.1 ensures that any module \(G\)-intertwiner is in fact a bimodule intertwiner, see Theorem 8.8. The same property holds for the bimodule structures of the \(u^* |_K \otimes \beta\)'s and hence for the bimodule structures of the \(z'' |_K \otimes \beta\), unitarily related to them, since the evaluation functor is full and faithful, see Cor. 10.5. But now each \(z'' |_K \otimes \beta\) has the trivial left module structure over a f.d. factor, hence a bimodule intertwiner lies in \(\mathcal{L}(H_{v^*}, H_{v^*}) \otimes \mathcal{C}\), see the discussion following Prop. 4.1. This argument completes the proof of Theorem 5.5. If in particular \(\mathcal{C}\) is commutative then \(\mathcal{F} = \mathcal{C}\), and \(z' = u |_K\), completing the proof of Cor. 5.6.

**Proof of Cor. 5.7** The condition on \(R\) allows us to define a tensor \({}^*\)-functor from \(S_{\mathcal{U}(2)}\) to \(\mathcal{M}\) taking the defining representation \(u\) to \(x\) and the determinant element to \(R\), see \([3]\). We may now apply Theorem 5.5.

### 12 Appendix

In this appendix we collect some computations with quasitensor functors that we have used throughout the paper.

**12.1. Proposition** If we take \(1_{\mu \otimes \mu} \otimes \tilde{R}_u \otimes 1_{\mu \otimes \mu} \circ \tilde{R}_v\) as a solution of the conjugate equations for \(\mu \otimes \mu\) and \(\tilde{R}_u \otimes \tilde{R}_v\) as the solution for \(\mu \otimes \mu\), where \(\tilde{R}_u \otimes \tilde{R}_v\) is the image solution of the tensor product solution for \(u \otimes v\), then

\[
\tilde{\mu}_{u,v}^* = \tilde{\mu}_{\pi, \pi}^*, \quad \tilde{\mu}_{u,v}^{**} = \tilde{\mu}_{\pi, \pi}^*.
\]
\(\circ 1_{\mu_T} \otimes \tilde{\mu}_{\pi,u} \otimes 1_{\mu_T} \circ 1_{\mu_T} \otimes 1_{\mu_T} \otimes \tilde{R}_u = \)
\(\tilde{R}_u^* \circ 1_{\mu_T} \otimes \tilde{\mu}_{\pi,u,\pi} \circ 1_{\mu_T} \otimes \mu(\tilde{R}_u \otimes 1_{\pi}) \circ 1_{\mu_T} \otimes \mu(R_u^*) \circ 1_{\mu_T} \circ 1_{\mu_T} \circ \tilde{\mu}_{\pi,u} \otimes 1_{\mu_T} \circ 1_{\mu_T} \otimes \tilde{R}_u.\)

Now
\[\mu(R_u^*) \otimes 1_{\mu_T} \circ \tilde{\mu}_{\pi,u} \otimes 1_{\mu_T} \circ 1_{\mu_T} \otimes \tilde{R}_u = \]
\[\mu(R_u^*) \otimes 1_{\mu_T} \circ \tilde{\mu}_{\pi,u} \otimes 1_{\mu_T} \circ 1_{\mu_T} \circ \tilde{\mu}_{\pi,u,\pi} \otimes 1_{\mu_T} \otimes \mu(\tilde{R}_u) = \]
\[\mu(R_u^*) \otimes 1_{\mu_T} \circ \tilde{\mu}_{\pi,u,\pi} \circ \mu(\tilde{R}_u \otimes 1_{\pi}) = \]
\[\mu(R_u^*) \circ 1_{\mu_T} \otimes \tilde{\mu}_{\pi,u,\pi} \circ \mu(\tilde{R}_u \otimes 1_{\pi}) = \]
\[\mu(R_u^* \circ 1_{\mu_T} \otimes \tilde{\mu}_{\pi,u,\pi} \circ 1_{\mu_T} \otimes \mu(\tilde{R}_u) = 1_{\mu_T}. \]

Substituting this into our calculation gives
\[\tilde{\mu}_{\mu,u,v}^* = \tilde{R}_u^* \circ 1_{\mu_T} \circ 1_{\mu_T} \otimes \tilde{\mu}_{\mu,v,\pi} \otimes \mu(\tilde{R}_u \otimes 1_{\pi}) = \]
\[\mu(R_u^*) \circ 1_{\mu_T} \otimes \tilde{\mu}_{\mu,v,\pi} \circ \mu(\tilde{R}_u \otimes 1_{\pi}) = \]
\[\mu(R_u^*) \circ 1_{\mu_T} \circ \tilde{\mu}_{\mu,v,\pi} \circ 1_{\mu_T} \otimes \mu(\tilde{R}_u \otimes 1_{\pi}) = \]
\[\mu(R_u^* \circ 1_{\mu_T} \otimes \tilde{\mu}_{\mu,v,\pi} \circ 1_{\mu_T} \otimes \mu(\tilde{R}_u) = \tilde{\mu}_{\mu,u,\pi}. \]

Dualizing with respect to \(\otimes\) yields \(\tilde{\mu}_{\mu,u,v}^* = \tilde{\mu}_{\mu,u,\pi}\) and taking adjoints completes the proof.

12.2. Corollary For \(M \in (\mu_u, \mu_v), N \in (\mu_v, \mu_v'),\)
\[(\tilde{\mu}_{u,v'} \circ M \circ N \circ \tilde{\mu}_{u,v'})^* = \tilde{\mu}_{u,v'} \circ (M \circ N) \circ \tilde{\mu}_{u,v'}^* = \tilde{\mu}_{u,v'} \circ M \circ N \circ \tilde{\mu}_{u,v'}.\]

with respect to the image of a tensor product solution of the conjugate equations.

Proof By the previous proposition,
\[(\tilde{\mu}_{u,v'} \circ M \circ N \circ \tilde{\mu}_{u,v'})^* = \tilde{\mu}_{u,v'} \circ (M \circ N) \circ \tilde{\mu}_{u,v}^* = \tilde{\mu}_{v', u,v} \circ N \circ M \circ \tilde{\mu}_{v', u,v}. \]

12.3. Proposition If we take the conjugate solution \(R_{\pi} = \tilde{R}_u\) as a solution of the conjugate equations for \(\pi\) and the tensor product solution \(R_{\pi \otimes u} = 1_{\pi} \circ R_{\pi} \otimes 1_u \circ R_u\) for \(\pi \otimes u\) then \(R_{u}^* = R_u\) for \(R_u\).

Proof
\[R_u^* = R_u^* \circ 1_{\pi \otimes u} \circ \tilde{R}_{\pi \otimes u} = \]
\[R_u^* \circ 1_{\pi \otimes u} \circ 1_{\pi} \circ \tilde{R}_u \circ 1_u \circ R_u = R_u. \]

Dualizing again with respect to \(\otimes\) gives \(R_u^{**(u)} = R_u^*\).

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