Green’s function and Bloch theory for the analysis of the dynamic response of a periodically supported beam to a moving load

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Abstract. This paper describes an analytical method for the wave field induced by a moving load on a periodically supported beam. The Green’s function for an Euler beam without support is evaluated by using the direct integration. Afterwards, it introduces the supports into the model established by using the superposition principle which states that the response from all the sleeper points and from the external point force add up linearly to give a total response. The periodicity of the supports is described by Bloch’s theorem. The homogeneous system thus obtained represents a linear differential equation which governs rail response. It is initially solved in the homogeneous case, and it admits a no null solution if its determinant is null, this permits the establishment the dispersion equation to Bloch waves and wave bands. The Bloch waves and dispersion curves contain all the physics of the dynamic problem and the wave field induced by a dynamic load applied to the system is finally obtained by decomposition into Bloch waves, similarly to the usual decomposition into dynamic modes on a finite structure. The method is applied to obtain the field induced by a load moving at constant velocity on a thin beam supported by periodic elastic supports.

1. Introduction
The dynamic response analysis of infinite periodic beams under moving loads has been one of the main of the research interests in railway engineering. Genin and Chung have proposed a numerical algorithm for calculation of the response of a moving vehicle on a continuous periodically supported beam [1]. Smith and Wormley [2] use the Floquet principle and the Fourier transform of the beam displacement and then numerically inverting it. The free vibration propagation and forced vibration induced by stationary harmonic loads is analyzed by [3]. A mathematical model is presented by [4] to predict the vibration of a rail excited by a rolling wheel. The track is modelled as an infinitely long beam supported over a finite section by discrete support systems for a non moving load. Vibration of

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an infinite periodic beam subject to a moving harmonic load has been investigated by [5]. In this
study, the author considered a single segment only by using boundary conditions derived according
to the Euler beam theory. The rail is modelled as an Euler beam with flexible supports in the work of [6].
The supports of the beam represent the pad/sleeper/ballast system of the railway track; they are spaced
regularly. The exact solution, in the frequency domain, of the linear differential equation is presented.
The method used to obtain the response to the dynamic load includes the computation of transfer
matrices [7, 8], the use of Bloch waves [9, 6] and finite element solutions [10, 11]. For a Timoshenko
beam, [12] produced the dispersion curves and showed that different modes appear, related to the
coupling between different degrees of freedom of the beam: flexion, torsion, shear.... The response of
a Timoshenko beam to a static (non moving) dynamic load was produced by [13]. A method is
proposed to calculate the response of periodic structures to subject to a moving load based on the
Floquet decomposition [14]. The aim of this paper describes the response of a periodically supported
beam using the Green function, the superposition principle and the Bloch transform [15,16]. The
model consists of a moving load on a discretly supported rail. The track is reduced to a rail modelled
as an infinite Euler beam.

2. Response to an harmonic load on the infinite beam :Green function

The vertical beam displacement is the solution of the dynamic beam equation Euler model is
considered in equation (1):

\[ \frac{EI}{\lambda^4} \frac{\partial^4 u(x,t)}{\partial x^4} + m \frac{\partial^2 u}{\partial t^2} = F(x,t) \]

where \(m\) is the mass of the beam by unit of length and \(E\) the Young’s modulus, \(I\) an inertia section
moment and \(F(x,t)\) a moving load.

If the harmonic concentrated load is applied at the point \(x_0\), in the Fourier space, the movement
equation becomes:

\[ \frac{EI}{\lambda^4} \frac{\partial^4 \bar{U}(x,\omega)}{\partial x^4} - m\omega^2 \bar{U}(x,\omega) - \delta(x-x_0) = 0 \]

Where \(\bar{U}\) is the Fourier transform of \(U\).

The solution of this equation is given by:

\[ G(x,x_0) = \frac{1}{4EI} \left[ e^{ik_b |x-x_0|} - e^{-ik_b |x-x_0|} \right] \]

Where:

\[ k_b = \frac{\sqrt{m\omega^2}}{EL} \]

3. Bloch wave function

We consider a beam periodically supported with \(N\) supports placed between \(-\infty\) and \(+\infty\). The period
is given by the regular spacing \(l\). The rigidity of the \(N\)th discrete elastic support, corresponding to the
deforation of the system (elastic support, mass support...), is \(Z_n\). It is considered that all the
discrete supports are characterized by the same rigidity \(Z_n = Z\). The global response of the
structure can be described by applying the superposition principle, which consists of the linear
summation of all the answers due to the various supports. Then, the displacement with the
position \(x\) is given by:

\[ \eta(x) = \sum F_n G(x,x_n) \]

Where: \(F_n = -Z_n U(n,l)\) is the force transmitted by the discrete support.

Introducing the value of \(G(x,x_n)\) is given by the equation (3) for the Euler beam.

The relation (6) is thus valid for any point \(x_m\).
The application of the Floquet theorem gives the relation between two adjacent supports:

\[ \eta(x_{m+1}) = \eta(x_m) e^{-\gamma l} \]  

(7)

With:

\[ \gamma = \alpha + ik \]  

(8)

\( \gamma \) is the Bloch constant propagation. It is generally complex. \( \alpha \) is the wave attenuation and \( k \) is the Bloch wave number.

By substituting the expression of the Green function given by equation (3) in equation (5) and by multiplying this equation by \( e^{r ml} \), changing the index of summation \( m - n \) to \( n \) gives the following equation which defines the relation between \( \gamma \) and the displacement \( \eta_0 \).

\[ \eta_0 + Z \eta_0 \sum_{z = \pm} \left[ \frac{1}{4EIk_b^3} (e^{ik_b|z-ml|} - e^{ik_b|n-m|l}) e^{r ml} \right] = 0 \]  

(9)

In the following, only the waves not attenuated will be considered, which means that \( \alpha = 0 \) and \( \gamma = -ik \). The sum (9) can be evaluated by the usual formulas for the geometric series.

For the Euler model, we can have:

\[ 1 + \frac{Z}{4EIk_b^3} \left[ \frac{\sin(k_b l)}{\cos(k_b l) - \cos kl} - \frac{\sinh(k_b l)}{\cosh(k_b l) - \cos kl} \right] = 0 \]  

(10)

This is a second degree polynomial for \( \cos kl \). The solution produces the values of the wave number \( k \) which is between 0 and \( \pi \). It is a function of \( k_b \) which depends on the radial frequency equation (4). It leads therefore to the dispersion equation.

The shape of the Bloch wave can be obtained by using a similar process. Let us consider the free motion related to a couple \( k, k_b \) applying to equation (10).

The displacement at position \( x \) between two supports is related to the displacements of all supports by using Green’s function \( G \), leading to:

Introducing Green’s function and the propagating constant \( \gamma \) leads to:

\[ \eta(x) = -Z \eta_0 \frac{1}{4EIk_b^3} \sum_{m=\pm} \left[ (e^{ik_b|z-ml|} - e^{-k_b|x-x_m|}) e^{-r x m} \right] \]  

(11)

Taking into account that the Bloch function must be normalized, it can be defined as:

\[ \eta(x) = \left[ \frac{e^{ik_b z} \sin(k_b z) - \sin(k_b (z - l))}{\cos(k_b l) - \cos kl} - \frac{e^{ik_b z} \sin(k_b (l - z)) + \sin(k_b l)}{\cosh(k_b l) - \cos kl} \right]/C \]  

(12)

The value of \( \eta_0 \) and therefore the value of \( C \) must be chosen to build from the Bloch waves an orthonormal basis of \( L^2(Y) \) where \( Y \) is the periodic cell obtained after transforming the space variable \( x \) into the non dimensional variable \( X^* = x \frac{2\pi}{l} \). Through this process, the period \([0, l] \) is transformed into \([0, 2\pi] \).
The constant C is chosen to ensure that the $L^2$ norm of $\eta(X)$ (or $\eta_1(X)$) on $[0, 2\pi]$ is equal to 1, leading to:

$$C^{-1} = \left[ \frac{2 \pi}{l} \int_0^l \left| \frac{N_2}{D_1} + \frac{N_3}{D_2} \right|^2 \, dx \right]^{1/2}$$

(13)

Where:

$$N_1(x, k) = e^{ikx} \sin(k_b x) - \sin(k_b(x - l)) N_2(x, k) = -\left( e^{ikx} \sinh(k_b x) - \sinh(k_b(x - l)) \right)$$

$$D_1(k) = \cos(k_b l) - \cos( kl) \quad D_2(k) = \cosh(k_b l) - \cos( kl)$$

4. The response of the railway structure to a moving load

As for the space coordinate, it is convenient, to define the Bloch transform, by using a no-dimensional wave-number $K^*$ defined by: $K^* = \frac{k_1}{2\pi}$

The Bloch transform of a function $F(x)$ is then defined by:

$$\tilde{F}_n(K^*) = \int_0^{2\pi} F(x) e^{ik_x x} \, dx$$

(14)

Where $\eta_n(X, K^*)$ is the complex conjugate of the Bloch wave contained in the $n^{th}$ band obtained by equation (5). The Bloch decomposition theorem, whose proof and conditions of validity can be found in [15, 16], states that.

$$F(X^*) = \sum_{n=1}^{N} \frac{1}{2\pi} \tilde{F}_n(K^*) \eta_n(X^*, K^*) \, dK^*$$

(15)

The Bloch transform of equation (1), considering that the Bloch wave $\eta_n(x, k)$ is the solution of the homogeneous equation for an harmonic equation at radial frequency $\omega_n(k)$, leads to:

$$m\omega_n^2 \tilde{U}_n(K^*, t) + m \frac{\partial^2}{\partial t^2} \tilde{U}_n(K^*, t) = \tilde{F}_n(K^*, t)$$

(16)

This equation shows that each Bloch component is a solution of the dynamic equation for a ‘1 DOF’ system.

Let us use this result to obtain the response of the beam to a load having a constant intensity $F_0$ which is moving at the velocity $V$. It means that $F$ is given by:

$$F(x, t) = \delta(x - Vt)F_0$$

Where $\delta$ is the Dirac distribution.

The Bloch transform of $F(t)$ is:

$$\tilde{F}_n(K^*, t) = \tilde{\eta}_n \left( X^* = Vt \frac{2\pi}{l}, K^* \right) F_0 \frac{2\pi}{l}$$

(17)

The Bloch components of the displacement induced by the moving load are therefore solutions of:

$$m\omega_n^2 \tilde{U}_n(K^*, t) + m \frac{\partial^2}{\partial t^2} \tilde{U}_n(K^*, t) = \tilde{F}_n \left( X^* = Vt \frac{2\pi}{l}, K^* \right) F_0 \frac{2\pi}{l}$$

(18)

Its solution is given by:

$$\tilde{U}_n(K^*, t) = \frac{2F_0 C^2}{l} \left[ \frac{e^{-2iK^* \sin(k_b Vt)} - \sin(k_b(Vt - l))}{1 - \cos(k_b l) - \cos(2K^*)} - \frac{e^{-2iK^* \sinh(k_b Vt)} - \sinh(k_b(Vt - l))}{2 - \cosh(k_b l) - \cos(2K^*)} \right]$$

(19)

Where $\gamma_1$ and $\gamma_2$ are given by:

$$\gamma_1 = m \left( \omega_n^2 - k_b^2 V^2 \right) \quad \gamma_2 = m \left( \omega_n^2 + k_b^2 V^2 \right)$$
The displacement induced by the moving force is finally obtained by a composition of all the Bloch components, leading to:

\[ U(X^*, t) = \sum_{n} \int_{K=0}^{1} \hat{U}_{n}(K^*, t) \eta_{n}(X^*, K^*) dK^* \]  

(20)

Coming back to \( x, k \) and taking into account that \( k \) is involved in \( \eta \) only by \( \cos(kl) \), leads to:

\[ U(x, t) = \sum_{n} \int_{k=0}^{\pi/l} V_{n}(k, x, t) \, dk \]  

(21)

The function \( V_{n}(k, x, t) = \frac{1}{\pi} \hat{U}_{n}(k, t) \eta_{n}(x, k) \) is given by:

\[ V_{n}(k, x, t) = 2F_0 C^2 \left[ \frac{N_{1}(v_x, k)}{\gamma_{1} D_{1}} + \frac{N_{2}(v_x, k)}{\gamma_{2} D_{2}} \right] \left[ \frac{N_{1}(x, k)}{D_{1}} + \frac{N_{2}(x, k)}{D_{2}} \right] \]  

(22)

This solution can be transformed by using new dimensionless variables \( T, X, B, K \) defined by \( x = Xl, \ t = Tl/V, \ k = Kl, \ k_{b} = Bl \).

Using these variables \( \hat{U} \) is given by:

\[ \hat{U}(X, T) = \frac{F_0}{\pi \beta l} \sum_{n} \int_{K=0}^{\pi} \frac{P}{\alpha_{1} \alpha_{2} Q} dK \]  

(23)

where: \( \alpha_{1} = B^{4} - B^{2} \phi \, ; \, \alpha_{2} = B^{4} + B^{2} \phi \, ; \, \phi = \frac{mv^{2} l^{2}}{EI} \, ; \, \beta = \frac{EI}{l^{4}} \)

From the previous results, it can be seen that the displacement depends on the non dimensional position and time and also on the three following parameters:

4.1 Example of application

Let us consider a rail whose properties are given in (table 1).

**Table 1.** Physical properties of the beam and support

|                    | Young’s modulus | Section’s inertia | Mass per un. length | Support stiffness | Spacing |
|--------------------|-----------------|------------------|----------------------|------------------|---------|
|                    | GPa             | m^2              | kg/m                 | GN/m             | m      |
|                    | 200             | 8.10-5           | 158                  | 30 and 0.02      | 0.6    |

The dispersion curves are show on figure 1 for the first three passing bands for the stiffer support (Z=30 GN/m). Similar results are obtained for the softer support at lower frequencies.
The displacement induced by a unit moving load at the center of a given period is next computed when the load is moving over that period with two speeds: a low speed 10m/s and a high speed 140m/s. Results are given in the case of a stiff support in figure 2 and in the case of a soft support in figure 3. It appears that the displacement is the same for low speed and high speed, in the case of the stiffer support, while the displacement depends significantly of the speed for the softer support figure 3 where the speeds for the second case are: 10m/s; 60m/s; 140 m/s; 200m/s.

This result is due to the fact that the passing bands are at higher frequencies for the stiffer supports. The figure 4 show the analysis displacement of the beam with various positions ( z=0; z=0.25; z=0.5), for this case the speed is constant v=140 m/s.

5. Conclusions

The Green function and Bloch theory were used for analysis of dynamic response of periodically supported beam to moving load. The proposed method uses the superposition principle with these two theories. It allows the computation of the response of the beam to any dynamic loading by computing the Bloch transform of the displacement induced by the
loading. It only involves the solution of a (continuous) set of decoupled one degree of freedom
dynamic equations. The method is applied to the dynamic response of a thin beam.

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