Causal structure and diffeomorphisms in Ashtekar’s gravity

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Abstract

A manifestly diffeomorphism invariant extension of Einstein gravity is constructed, which includes singular metrics, and whose ADM formulation is Ashtekar’s gravity. The latter is shown to be locally equivalent to the covariant theory. It turns out that exactly those kinds of degenerate four dimensional metrics are allowed which do not destroy the causal structure of spacetime. It is also shown that Ashtekar’s gravity possesses an extension that provides a local $SO(3,\mathbb{C})$ invariance, without complexifying or changing the signature of the metric.

1 What is the problem?

The purpose of this paper is to reexamine a question considered by Bengtsson [1] some time ago, which also appears in the recent work of Reisenberger [2], and which has some interesting relations to the ideas of Dragon [3]. The question is whether there is a manifestly diffeomorphism invariant formulation of Ashtekar’s ‘polynomial’ gravity, where the metric is real Lorentzian but possibly degenerate. More precisely, the question is whether Ashtekar’s gravity is the ‘ADM’ [4] formulation of some covariant theory, obtained in the same way as the usual ADM formulation of gravity is obtained from Einstein’s theory. Bengtsson investigated this question and came to the conclusion that a theory exists which is equivalent to Ashtekar’s theory for complexified gravity and non-vanishing lapse function. Reisenberger has shown that (even without the last restriction) Ashtekar’s theory is invariant under infinitesimal diffeomorphisms, but he pointed out some problems concerning finite diffeomorphisms. He argued that these problems are related to the causal structure of spacetime. Finally, Dragon considered, quite generally, manifestly covariant extensions of Einstein gravity allowing degenerate metrics. In his results, the close relation to Ashtekar’s variables is not apparent. Here I will show that there is an improved version of Bengtsson’s theory, which is an extension of real Einstein gravity, and where the restriction ‘lapse $\neq 0$’ is replaced by a fully covariant condition, which implies that spacetime has a well defined causal structure, at least locally. The resulting theory can equivalently be considered as a restricted version of the vierbein formulation of Dragon’s theory.
Ashtekar’s theory

To set up the notation, let me briefly review what the problem actually is. As is well known, the Lagrange density of Ashtekar’s gravity can be written as

\[ \mathcal{L} = i\tilde{e}_a^m \partial_t A_{ma} + iA_{ta} D_m \tilde{e}_a^m - iN^m \tilde{e}_a^n F_{mna} + \frac{1}{2} N \varepsilon_{abc} \tilde{e}_a^m \tilde{e}_b^n F_{mnc}, \]  

(1.1)

where spacetime is split into a spatial hypersurface \( \mathcal{N} \) with local coordinates labeled by indices \( m, n, \ldots \) and a time coordinate \( t \). The basic fields are the \( \mathfrak{so}(3, \mathbb{C}) \) connection \( A_{\mu a} \) (where \( \mu, \nu, \ldots \) are spacetime indices taking the ‘values’ \( m, n, \ldots \) and \( t \)), the real densitized inverse dreibein \( \tilde{e}_a^m \) (where \( e \) is the determinant of the spatial dreibein \( e_{ma} \)) and the lapse and shift fields \( N \) and \( N^m \). As the lapse function represents a pure gauge degree of freedom, we can restrict it by \( N > 0 \). As mentioned above, \( N \neq 0 \) is a restriction already found in [1], for the transformation to the covariant theory to exist. The sign fixing does not impose any additional constraint, as long as \( N \) is continuous.

The flat indices \( a, b, \ldots \) take the values 1, 2, 3, and \( \varepsilon_{abc} \) are the structure constants of \( \mathfrak{so}(3) \), defining the covariant derivative and field strength

\[ D_m \tilde{e}_a^m = \partial_m \tilde{e}_a^m + \varepsilon_{abc} A_{mb} \tilde{e}_c^m, \]

\[ F_{mna} = \partial_m A_{na} - \partial_n A_{ma} + \varepsilon_{abc} A_{mb} A_{nc}. \]  

(1.2)

In the following, I will call this theory ‘Ashtekar’s gravity’. Its maybe most interesting feature is the polynomial form of the action in terms of the canonical variables, provided that \( t \) it used as the canonical time variable. This makes it possible to take it as an extension of Einstein gravity, allowing certain kinds of singular metrics, corresponding to non-invertible matrices \( \tilde{e}_a^m \). Thereby the densitized spatial metric \( \tilde{g}^{mn} = \tilde{e}_a^m \tilde{e}_a^n \) becomes degenerate as well, but remains positive semidefinite. Hence, for canonical formulation and quantization of gravity, the action (1.1) has turned out to be a promising starting point. Here I want to proceed into the opposite direction. The main question to be considered is whether there is a manifestly diffeomorphism invariant theory which, after introducing a spacetime slicing ‘a la ADM’, leads to the Lagrangian (1.1). Having such a theory, we do not only get an elegant proof for the diffeomorphism invariance of Ashtekar’s gravity, which will be a simple consequence of the manifest invariance of the ‘higher’ theory. It will also provide a deeper understanding of its geometrical structure, and it is this I want to focus on. In particular, we will see what the singular metrics ‘look like’ in four dimensions and how they affect the causal structure of spacetime.

Diffeomorphisms

The first important question is whether Ashtekar’s theory is really invariant under diffeomorphisms of spacetime. This is not obvious because to write down the action one has to introduce a particular coordinate \( t \) as a background structure. You can think of it as given by a scalar field \( T(\mathbf{x}) \) on spacetime, subject to the condition \( \partial_\mu T \neq 0 \), which provides the ADM slices as the \( T = \) constant hypersurfaces. This time coordinate not only appears explicitly in the action. It is also required to define the fields themselves. Whereas the connection can be combined into a four dimensional one-form, there is no covariant object which is linear in, or even a homogeneous polynomial of the dreibein and the lapse and shift fields. However, we know that the action reduces to the Einstein Hilbert action for invertible dreibeins (and after
Ashtekar’s theory is therefore invariant at least under ‘small’ diffeomorphisms, but from the order terms it is not guaranteed that they can be integrated to give finite diffeomorphisms. Hence, we have well defined infinitesimal diffeomorphisms, but because of the higher that the action is in fact invariant (up to a total time derivative) under these transform-
ations. A straightforward way to find out whether the transformation under diffeomorphisms is continuous in $\tilde{e}_a^m$, $N$, and $N^m$, is to compute the transformation explicitly. Let $\zeta^\mu = (\zeta^t, \zeta^m)$ be a generating vector field of an infinitesimal transformation. Then $A_{\mu a}$ should transform by its four dimensional Lie derivative. If one takes this as an ansatz and computes the transformation of the dreibein, one finds that it becomes complex. This is because the Lagrangian involves some kind of gauge fixing, which will be discussed in detail below. However, one can compensate for the imaginary contributions by introducing an additional \footnote{Unfortunately, a contradictory result obtained in a previous paper \cite{6} has turned out to be wrong. The correct result has been derived in \cite{2}, which focuses on gauge transformations generated by the canonical constraints instead of directly computing field transformations induced by four dimensional diffeomorphisms. This leads to different formulae for the canonical fields (which coincide with \cite{1} if the equations of motion are satisfied) but identical formulae for the multipliers.} SO(3, C) rotation, with field dependent parameter. As a consequence, all the transformations become highly non-linear, but polynomial. The explicit formulae are

$$
\delta N = \zeta^n \partial_n N - \partial_n \zeta^n N + \partial_t (\zeta^t N) - 2 \partial_n \zeta^t NN^m,
$$

$$
\delta N^m = \zeta^n \partial_n N^m - \partial_n \zeta^m N^m + \partial_t (\zeta^t N^m) - \partial_n \zeta^t (N^2 \tilde{g}^{mn} + N^m N^m) + \partial_t \zeta^m,
$$

$$
\delta \tilde{e}_a^m = \partial_n (\zeta^n \tilde{e}_a^m) - \partial_n \zeta^m \tilde{e}_a^m + \zeta^t \partial_n \tilde{e}_a^m + \partial_n \zeta^t (\tilde{e}_a^m N^m - \tilde{e}_a^m N^m),
$$

(1.3)

for the metric fields. In each expression, the first two terms are the transformations of the fields under spatial diffeomorphisms. The third terms describe the behaviour under pure rescaling of the time coordinate: the lapse and shift functions are densities of weight one with respect to these transformations, whereas $\tilde{e}_a^m$ has weight zero. The last term in $\delta N^m$ can be understood by interpreting $N^m$ as the gauge field associated with spatial diffeomorphisms. The remaining nonlinear terms appear whenever $\partial_n \zeta^t \neq 0$. This means that the time rescaling depends on the space point, so that the slicing itself is affected by the diffeomorphism. A similar term appears in the transformation of the connection, which explicitly shows the compensating gauge transformation:

$$
\delta A_{\mu a} = \zeta^\nu \partial_\nu A_{\mu a} + \partial_\nu \zeta^\nu A_{\nu a} + D_\mu (i \partial_n \zeta^t N \tilde{e}_a^n).
$$

(1.4)

Without the last term, we had to add the corresponding rotation to $\delta \tilde{e}_a^m$, which would produce an imaginary contribution. A cumbersome but straightforward computation now shows that the action is in fact invariant (up to a total time derivative) under these transformations. Hence, we have well defined infinitesimal diffeomorphisms, but because of the higher order terms it is not guaranteed that they can be integrated to give finite diffeomorphisms. Ashtekar’s theory is therefore invariant at least under ‘small’ diffeomorphisms, but from the
somewhat awkward formulae (1.3) we do not learn much about the geometrical nature of the
theory, especially it is hardly possible to see what ‘small’ means, i.e. whether (1.3) can be
integrated or not for a given generating vector field $\zeta^\mu$.

In fact, for very special solutions of the field equations, it has been shown in [2] that there
are finite diffeomorphisms connected to the identity for which (1.3) can not be integrated.
It is shown that this is similar to a well known feature of the ADM formulation of metric
gravity. There, the slices have to be spacelike and a given solution in one slicing cannot be
transformed into another slicing unless the new slices are everywhere spacelike. In addition,
the pure existence of some slicing already restricts the possible filed configurations: spacetime
must be time orientable and closed causal (i.e. positive oriented timelike or lightlike) curves
are forbidden. By constructing the fully covariant theory associated with Ashtekar’s gravity
below, we will see that it is exactly this what happens here, too. There will be a well defined
‘causal structure’ on spacetime, and the restrictions we get are identical to those of ADM
gravity.

Instead of using infinite diffeomorphisms and integrating them, it might be more convenient
to consider finite diffeomorphisms right from the beginning. Moreover, instead of computing
the transformations of the fields under diffeomorphisms explicitly, it is more suitable to con-
struct covariant objects, i.e. proper four dimensional tensors, whose transformations laws then
become very simple. Let us see whether there is a tensor that carries the $\text{SO}(3)$ invariant
information about the dreibein, the lapse and the shift fields. Hence, it should depend on $\tilde{g}^{mn}$,
$N^m$ and $N$. These are $6 + 3 + 1 = 10$ independent components, so it is not surprising that the
tensor is symmetric of rank two [1]:

$$\tilde{G}^{\mu\nu} = \begin{pmatrix} \tilde{G}^{tt} & \tilde{G}^{tn} & \tilde{G}^{mt} & \tilde{G}^{mn} \\ N^{-1} & N^{-1}N^m & N^{-1}N^n \end{pmatrix} = \begin{pmatrix} -N^{-1} & N^{-1}N^n \\ -N^{-1}N^m & N\tilde{g}^{mn} - N^{-1}N^mN^n \end{pmatrix}. \quad (1.5)$$

For invertible metrics, this is nothing but the densitized inverse four dimensional metric
$\sqrt{-GG^{\mu\nu}}$, which appears in [3] as the basic field variable. Here, the restriction $N > 0$ is
essential to provide the correct sign.

What is the range of this tensor? First of all, we obviously have $\tilde{G}^{tt} < 0$. Moreover, because
of the positivity of $\tilde{g}^{mn}$, the signature of $\tilde{G}^{\mu\nu}$ is $(-, +, +, +)$, corresponding to invertible
metrics, or some (or all) of the ‘+’ signs may be replaced by 0, leading to degenerate metrics.
Under these conditions the transformation back to Ashtekar’s variables is unique up to $\text{SO}(3)$
rotations. The restriction $\tilde{G}^{tt} < 0$, however, is still non-covariant, because it refers to the
special coordinate $t$. But we already got some hint to how a covariant version could look like.
There must be exactly one ‘−’ in the signature of $\tilde{G}^{\mu\nu}$. Hence, a covector $\xi_\mu$ must exist such
that $\xi_\mu\xi_\nu\tilde{G}^{\mu\nu} < 0$, which in some sense (and in the usual sense for invertible metrics)
means that there is at least one ‘timelike direction’. I will not go into more details at this point,
but we can see that this will have something to do with timelike curves and causality, and a
restriction like this will be the central point of the covariant theory.

To restore the $\text{SO}(3)$ gauge freedom, one can go back from $\tilde{G}^{\mu\nu}$ to its square root, which
is linear instead of quadratic in $\tilde{e}_a^m$. There is an obvious square root of (1.3):

$$\tilde{E}_A^\mu = \begin{pmatrix} \tilde{E}_0^t \\ \tilde{E}_a^m \end{pmatrix} = \begin{pmatrix} \sqrt{N}^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{N}^{-1}N^m \\ -\sqrt{N}\tilde{e}_a^m \end{pmatrix}. \quad (1.6)$$

Here, $A$ is a flat Lorentz index taking the values 0, 1, 2, 3 (and hence $a$ refers to a subset of
these values), which is raised and lowered, and has to be contracted with the Lorentz metric.
\[ \eta^{AB} = \text{diag}(-1,1,1,1) \] to give \( \tilde{G}^{\mu\nu} \). The signs are chosen for later convenience. In fact, this is the basic variable used in [1] to define a covariant theory. It is a vector density of weight \( \frac{1}{2} \), and I will simply call it the vierbein. However, as defined in (1.4), the vierbein is not yet a covariant object, as it is subject to the conditions \( \tilde{E}_a^t = 0 \) and \( \tilde{E}_0^t > 0 \). They still refer to the coordinate \( t \). While the first condition can be treated as a gauge fixing, the second one is somehow more subtle. Moreover, they are not independent and the first one is not just a Lorentz gauge fixing: the transformation to the upper triangular form (1.6) is not always possible, as it already requires \( \tilde{E}_a^t \) to be timelike, or equivalently \( \tilde{G}^{tt} < 0 \). Hence, the gauge fixing already affects the diffeomorphism group.

One way out of this dilemma is to consider (1.6) as a gauge fixed version of an arbitrary complex vierbein (or alternatively switch to Euclidean gravity), thereby extending the gauge symmetry to \( \text{SO}(1,3,\mathbb{C}) = \text{SO}(4,\mathbb{C}) \) (or replacing it by \( \text{SO}(4,\mathbb{R}) \)). This is done in 1 and also assumed in 2. The transformation to the upper triangular form is then always possible by a pure \( \text{SO}(4) \) rotation. From the physical point of view, however, this is rather unsatisfactory, as it takes us away from real Lorentzian gravity, and moreover the complex theory is even different from complexified Einstein gravity: the equivalence of Ashtekar’s and Einstein gravity (for invertible metrics) is based on the fact that the Einstein Hilbert action is the real part of the complex action (1.1), which is holomorphic in all the complex fields, so the field equations are the same for both. However, this equivalence only holds if the antiself-dual part of the connection is the complex conjugate of the self-dual part, and this relation is lost when going over to complex gravity with gauge group \( \text{SO}(4,\mathbb{C}) \).

So the question of how to construct covariant objects out of the three dimensional fields naturally leads us to complexification. Moreover, it is not only the lack of a manifestly covariant formulation of Ashtekar’s gravity that is somehow unsatisfactory. Another weak point of (1.1) is that, though one of the fields is complex and acts as an \( \text{so}(3,\mathbb{C}) \) valued connection, there is no local \( \text{SO}(3,\mathbb{C}) \) invariance of the action. That is because the dreibein has to be real, so we only have an \( \text{SO}(3,\mathbb{R}) \) gauge invariance, and as is also well known this leads to various problems with reality conditions to be imposed on the canonical variables. Obviously, this can also be solved by complexifying the fields. But as just described, straightforwardly complexifying everything takes us to complex and non-Einstein gravity. Nevertheless complexification is not a bad idea, and before coming to the covariant formulation in section 3, I will show in the following section that there is a different extension of Ashtekar’s gravity, which has gauge group \( \text{SO}(3,\mathbb{C}) \) but does not take us away from positive semidefinite metrics.

## 2 Complexifying without complexifying

Instead of simply making \( \tilde{e}_a^m \) complex, a local \( \text{SO}(3,\mathbb{C}) \) invariance can be obtained in a slightly different way and without making the metric itself complex. The resulting action is a bit more complicated than (1.1), but it has some advantages when considering Ashtekar’s gravity as the ADM formulation of its covariant formulation, i.e. when we are going away from the canonical formalism back to a manifestly covariant theory. First of all, the connection field really becomes an \( \text{SO}(3,\mathbb{C}) \) gauge field, without extending the physical phase space of the theory, i.e. we only need to add extra gauge degrees of freedom. Secondly, there will be no gauge fixing necessary when going over from the full four dimensional theory in terms of the vierbein to the ADM formulation in terms of dreibein, lapse and shift. This sounds a bit mysterious first, but note that \( \text{SO}(3,\mathbb{C}) \simeq \text{SO}(1,3)_+ \) is nothing but the four dimensional
The prize we have to pay for this is that on some of the fields the gauge symmetry will be realized non-linearly. Hence, it is not clear whether this kind of complexification is really suitable for the canonical treatment or even quantization of Ashtekar’s gravity, I just want to present it to show that there is a formulation of real Lorentzian gravity in terms of Ashtekar’s variables which admits an SO(3, C) invariance, and that it can be obtained as the ADM formulation of a covariant theory without gauge fixing. So if you don’t mind gauge fixing you can skip this section, which is not necessary to understand the remainder of the article.

The basic idea is quite simple: consider the real dreibein $\tilde{e}_a^m$, and act on it with an arbitrary SO(3, C) transformation $\lambda_{ab}$. The resulting dreibein $\tilde{e}_a^m = \lambda_{ab} \tilde{e}_b^m$ will in general be complex, but the metric $\tilde{g}_{mn} = \tilde{e}_a^m \tilde{e}_a^n = \tilde{e}_a^m \tilde{e}_a^n$ is still real and positive semidefinite. If we now replace $\tilde{e}_a^m$ by $\tilde{e}_a^m$ in $\text{(1.1)}$, then we obviously get an SO(3, C) invariant theory. All we have to do to keep the metric real is to restrict the range of $\tilde{e}_a^m$: only those values are allowed that admit a rotation $\lambda_{ab} \in \text{SO}(3, \mathbb{C})$ such that $\tilde{e}_a^m \lambda_{ab} \in \mathbb{R}$, or alternatively we must have $\tilde{g}_{mn}$ positive semidefinite. But this is a rather strange restriction, and the resulting range is not a proper vector space or submanifold of $\mathbb{C}^{3 \times 3}$, and so there is no well-defined action principle any more.

To see this, let us examine what the allowed range for $\tilde{e}_a^m$ looks like. We know that there is an over-complete coordinate system on this space which is given by specifying a real dreibein $\tilde{e}_a^m \in \mathbb{R}^{3 \times 3}$ and a complex rotation $\lambda_{ab} \in \text{SO}(3, \mathbb{C})$, which makes $9 + 6 = 15$ real coordinates. However, if we multiply $\lambda_{ab}$ by an element of the SO(3, $\mathbb{R}$) subgroup, we can compensate this by choosing a different real dreibein, so in fact there are three ambiguous coordinates corresponding to the SO(3, $\mathbb{R}$) subgroup, and the space of the $\tilde{e}_a^m$ has, at a generic point, 12 real dimensions only. Actually the complex dreibein is given by a real dreibein and an element of the coset space $\text{SO}(3, \mathbb{C})/\text{SO}(3, \mathbb{R})$. A suitable coordinate on the coset space is a real three-vector $v_a$, and a possible standard representative is given by

$$\lambda_{ab} = \alpha \delta_{ab} - (1 + \alpha)^{-1} v_a v_b + i \varepsilon_{abc} v_c, \quad \alpha = \sqrt{1 + v_a v_a}. \quad (2.1)$$

Let us call such an element of $\text{SO}(3, \mathbb{C})$ a ‘boost’ generated by $v_a$ (it is in fact the image of a boost in $\text{SO}(1, 3)_+$ under the group isomorphism (3.7)). It is straightforward to verify that $\lambda_{ab} \lambda_{ac} = \delta_{bc}$ for any $v_a \in \mathbb{R}^3$, and that every group element can be written uniquely as a product of some boost and a real SO(3) rotation. Hence, the coset is in fact (topologically) an $\mathbb{R}^3$. Together with the real dreibein we get 12 real coordinates $(\tilde{e}_a^m, v_a)$ for $\tilde{e}_a^m$, and explicitly we can write

$$\tilde{e}_a^m = \lambda_{ab} \tilde{e}_b^m = \alpha \tilde{e}_a^m - (1 + \alpha)^{-1} v_a v_b \tilde{e}_b^m + i \varepsilon_{abc} v_c \tilde{e}_b^m. \quad (2.2)$$

It is still not obvious what subset of $\mathbb{C}^{3 \times 3}$ is covered by this map. We can change coordinates from $(\tilde{e}_a^m, v_a)$ to $(\tilde{r}_a^m, w_a)$ by setting

$$\tilde{r}_a^m = \alpha \tilde{e}_a^m - (1 + \alpha)^{-1} v_a v_b \tilde{e}_b^m,$$
$$w_a = \alpha^{-1} v_a. \quad (2.3)$$

The range of these new coordinates is $\tilde{r}_a^m$ arbitrary, but for $w_a$ we have $w_a w_a < 1$, i.e. it takes values inside the unit ball in $\mathbb{R}^3$ only. We can check that (2.3) is a proper change of
coordinates, simply by giving the inverse explicitly:

\[ \tilde{e}_a^m = \alpha^{-1} \tilde{r}_a^m + \alpha (1 + \alpha)^{-1} w_a w_b \tilde{r}_b^m, \]
\[ v_a = \alpha w_a, \quad \alpha = 1/\sqrt{1 - w_a w_a}. \] (2.4)

Inserting the new coordinates into (2.2), we find

\[ \tilde{e}_a^m = \tilde{r}_a^m + i \varepsilon_{abc} w_c \tilde{r}_b^m, \quad w_a w_a < 1. \] (2.5)

Hence, we got a rather simple decomposition of \( \tilde{c}_a^m \) into its real and imaginary part. In contrast to this, (2.2) is rather a decomposition of the complex dreibein into its modulus and phase (where the phase is the boost which does not affect its modulus squared, i.e. the metric (2.7)). We can now see what the range of \( \tilde{c}_a^m \) is. It covers the whole real hyperplane in \( \mathbb{C}^{3 \times 3} \), and at each point there is a three dimensional ‘ball’ attached which extends into the imaginary region. The orientation and radius of the ball (or rather the axes of the ellipsoid) are linear functions of \( \tilde{r}_a^m \). The ball becomes degenerate if \( \tilde{r}_a^m \) does. You can think of the resulting range as some kind of angle, like, e.g., the spacelike region of a Minkowski space including the origin. The latter is also given by attaching a one dimensional ‘ball’ to each point in the ‘\( x_0 = 0 \)’ plane of Minkowski space, the (Euclidean) radius of that ball, i.e. its length, being the distance of its center from the origin.

The resulting space is neither an open subset of \( \mathbb{C}^{3 \times 3} \) nor an open subset of any submanifold, so we cannot choose \( \tilde{c}_a^m \) to be our basic field variable. However, what we can do is to take either \( (\tilde{r}_a^m, w_a) \) or \( (\tilde{c}_a^m, v_a) \) as the basic set of fields. Then \( \tilde{c}_a^m \) is given as a composite field by (2.3) or (2.2), respectively, and inserting this into (1.1) leads to an \( \text{SO}(3, \mathbb{C}) \) invariant Lagrangian

\[ \mathcal{L} = i \tilde{c}_a^m \partial_t A_{ma} + i A_{ta} D_m \tilde{c}_a^m - i N^m \tilde{c}_a^n F_{mna} + \frac{1}{2} N \varepsilon_{abc} \tilde{c}_a^m \tilde{c}_b^n F_{mnc}. \] (2.6)

Note that with \( \tilde{r}_a^m \) and \( w_a \) as the basic fields, this is still polynomial, but it does not share with (1.1) the property that \( \tilde{r}_a^m \) and \( A_{ma} \) can be treated as canonically conjugate variables in a straightforward Hamilton Jacobi formulation, due to the fact that now each component of \( A_{ma} \) carries more than one real degree of freedom. Note also that (2.3) is not one-to-one, as for singular values of \( \tilde{r}_a^m = \text{Re} \tilde{c}_a^m \) we cannot necessarily recover \( w_a \), so we cannot simply change to \( \tilde{c}_a^m \) as the basic field.

What we have now is a complexified version of Ashtekar’s theory, which still describes real gravity. The densitized spatial metric is given by

\[ \tilde{g}^{mn} = \tilde{c}_a^m \tilde{c}_a^n = \tilde{c}_a^m \tilde{c}_a^n = (1 - w_b w_b) \tilde{r}_a^m \tilde{r}_a^n + w_a \tilde{r}_a^m w_b \tilde{r}_b^n. \] (2.7)

It is real and positive semidefinite, which follows from \( w_a w_a < 1 \) if we choose the last representation in terms of \( \tilde{r}_a^m \) and \( w_a \). Let us stick to them, together with \( A_{ma} \) and the multipliers \( N, N^m \) and \( A_{ta} \), as the basic fields, and give an explicit representation for an \( \text{SO}(3, \mathbb{C}) \) gauge transformation. In principle, we can start form \( \tilde{c}_a^m \mapsto \lambda_{ab} \tilde{c}_b^m \) and use the decomposition of \( \text{SO}(3, \mathbb{C}) \) elements into boosts and real rotation to find the transformation of \( \tilde{e}_a^m \) and \( v_a \), then using (2.3) to find those of \( \tilde{r}_a^m \) and \( w_a \). However, it is more convenient to obtain the infinitesimal transformations form (2.5) directly. What we must have for the action (2.6) to be invariant is

\[ \delta \tilde{c}_a^m = \varepsilon_{abc} (\lambda_c + i \sigma_c) \tilde{c}_b^m, \quad \delta A_{ma} = D_m (\lambda_a + i \sigma_a), \] (2.8)
where the infinitesimal generator $\lambda_a + i\sigma_a \in \mathfrak{so}(3, \mathbb{C}) \simeq \mathbb{C}^3$ has been split into its real and imaginary part. It is now straightforward to verify that the nonlinear transformations
\begin{align*}
\delta \tilde{r}^m_a &= (\varepsilon_{abc} \lambda^c + w_d \sigma_d \delta_{ab} - w_a \sigma_b) \tilde{r}^m_b, \\
\delta w_a &= \sigma_a + \varepsilon_{abc} \lambda^c w_b - \sigma_b w_b w_a,
\end{align*}
inserted into (2.3), lead to (2.8). We also see that $\delta (w_a w_a) = 2(1 - w_a w_a)w_b \sigma_b$ vanishes at the boundary of the unit ball so that we cannot get out of the range of $w_a$ by a gauge transformation.

By choosing $\sigma_a \propto -w_a$ and making a suitable finite gauge transformation, we can always achieve $w_a = 0$, which gives Ashtekar’s theory back as a gauge fixed version of (2.3). Hence, it is not so obvious what we actually got, beside a strange kind of complexification, which seems to be more complicated than the original theory. However, now $A_{ma}$ is really an $SO(3, \mathbb{C})$ gauge field, and the action is still polynomial in all the variables. When discussing the ADM formulation of the covariant theory in the next section, it will be possible to transform the real four dimensional vierbein directly into the real fields $\tilde{r}_a^m$ and $w_a$ by very simple relations, and without any gauge fixing. Hence, the four dimensional Lorentz group $SO(1,3)$ can be directly identified with the $SO(3, \mathbb{C})$ appearing here.

As a result, the three steps leading from ‘classical’ Einstein gravity (in vierbein formulation) to Ashtekar’s theory (1.1), namely extension to degenerate metrics, ADM spacetime decomposition, and Lorentz gauge fixing, are now completely separated. Moreover, we can perform these steps in any order, thereby building up a cube of theories, Einstein and gauge fixed Ashtekar gravity in two opposite corners, and the three space dimensions representing the three steps. Two of these steps are rather technical, whereas extension to degenerate metrics changes physics. We almost filled up all the corners of this cube. With the results of this section, we got a non-gauge fixed version of Ashtekar’s gravity, which allows degenerate metrics. What is still lacking is the theory that allows degenerate metrics but is manifestly diffeomorphism invariant and leads to (2.3) by ADM decomposition and then to Ashtekar’s theory by gauge fixing. As this is technically the analog of Einstein gravity, we expect it to provide the best insight into the ‘real physics’ of Ashtekar’s gravity, as seen from the spacetime point of view. Though ADM gravity is useful when considering questions like quantization or numerical computations, it is the covariant formulation in terms of the 4-metric that ‘explains’ the nature of gravity. Hence, to understand the difference between Einstein and Ashtekar gravity we need the last corner of the cube and it is this theory I want to present in the following section.

3 The covariant theory

Let us forget Ashtekar’s theory for a moment and start right from the beginning by defining an extension of (real, Lorentzian) general relativity. We choose the basic fields to be the $\mathfrak{so}(1,3)$ spin connection $\Omega_{\mu AB}$ (with field strength $R_{\mu \nu AB}$ defined in a straightforward way) and the vierbein $\tilde{E}_A^\mu$, which transforms as a density of weight one half under diffeomorphisms. As the action, we can take
\begin{equation}
\mathcal{L}' = \frac{1}{2} \tilde{E}_A^\mu \tilde{E}_B^\nu R_{\mu \nu AB}, \tag{3.1}
\end{equation}
which is formally the same as in [1], and which may also be considered as the vierbein version of the action used in [3]. It defines an extension of Einstein gravity for degenerate metrics, corresponding to singular matrices $\tilde{E}_A^\mu$.

### Self-dual representation

By introducing a complex basis of $\mathfrak{so}(1,3)$, we can expand the spin connection in terms of its self-dual and antiself-dual part (see the appendix and [3, 4] for the definition of the $J$ symbols):

$$\Omega_{\mu AB} = A_{\mu a} J_{aAB} + A_{\mu a}^* J_{aAB}^*, \quad a = 1, 2, 3. \quad (3.2)$$

The basis $J_{aAB} = -J_{aBA}$ is orthonormal in the sense of (A.10) and provides the natural map of $\mathfrak{so}(1,3)$ onto $\mathfrak{so}(3,\mathbb{C})$. The $\mathfrak{so}(3,\mathbb{C})$ field strength is

$$F_{\mu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + \varepsilon_{abc} A_{\mu b} A_{\nu c} = J_a^{AB} R_{\mu\nu AB}.$$  

(3.3)

Expanding the field strength like (3.2), $\mathcal{L}'$ becomes the real part of

$$\mathcal{L} = \tilde{E}_A^\mu \tilde{E}_B^\nu J_{aAB} F_{\mu\nu a}.$$ \quad (3.4)

This provides an extension of Einstein gravity for singular metrics as well, which is slightly different from $\mathcal{L}'$. It is $\mathcal{L}$ that will be equivalent to Ashtekar’s gravity. The difference between the two extensions is not important for our purpose here, but it is quite interesting. The equations of motion for the spin connection are identical, because $\mathcal{L}$ is holomorphic in $A_{\mu a}$ and therefore becomes stationary if and only if its real part is stationary. However, $\mathcal{L}$ provides an additional field equation for the vierbein. We find

$$\frac{\delta \mathcal{L}}{\delta \tilde{E}_A^\mu} = 2 \tilde{E}_B^\nu J_a^{AB} F_{\mu\nu a} = 0$$

$$\iff \tilde{E}_B^\nu R_{\mu\nu AB} = 0, \quad \tilde{E}_B^\nu \varepsilon^{ABCD} R_{\mu\nu CD} = 0. \quad (3.5)$$

The last two equations are obtained by taking the real and imaginary part of the first equation. The first one is the Einstein equation, which is also the equation of motion for $\tilde{E}_A^\mu$ in $\mathcal{L}'$. The second equation is not implied by $\mathcal{L}'$. For invertible metrics, it is the first Bianchi identity for the Riemann tensor $R_{\mu[\nu\rho\sigma]} = 0$, written in vierbein formulation. For singular vierbein fields this is in general not a consequence of the remaining equations of motion. So the self-dual formulation $\mathcal{L}$ additionally imposes the Bianchi identity for the curvature tensor as an equation of motion, whereas $\mathcal{L}'$ does not.

To see the relation between the Lorentz group $\text{SO}(1,3)_+$ and its self-dual representation $\text{SO}(3,\mathbb{C})$, let us consider gauge symmetries of the self-dual action. A finite Lorentz transformation is given by a real $4 \times 4$ matrix $\Lambda_A^B \in \text{SO}(1,3)_+$ (obeying $\Lambda_A^C \Lambda_B^D \eta_{CD} = \eta_{AB}$, det$(\Lambda) = 1$, and $\Lambda_0^0 > 0$), or equivalently by a complex $3 \times 3$ matrix $\lambda_{ab} \in \text{SO}(3,\mathbb{C})$ (obeying $\lambda_{ab} \lambda_{ac} = \delta_{bc}$ and det$(\lambda) = 1$). Thereby, the fields have to transform such that

$$\tilde{E}_A^\mu \mapsto \Lambda_A^B \tilde{E}_B^\mu, \quad F_{\mu\nu a} \mapsto \lambda_{ab} F_{\mu\nu b}. \quad (3.6)$$

To make this a symmetry of the action, we must have

$$\lambda_{ab} = J_a^{AB} J_b^{CD} A_A^C A_B^D = -A_{00} A_{ab} + A_{a0} A_{0b} + i \varepsilon_{bcd} A_{ac} A_{0d}, \quad (3.7)$$

9
which follows by direct computation from the properties of the $J$ symbols given in the appendix. This is the natural map $\text{SO}(1,3)_+ \to \text{SO}(3,\mathbb{C})$, which can be checked to be a one-to-one group homomorphism (there is no sign ambiguity due to the quadratic form, because $A_{00} < 0$). It obviously maps the $\text{SO}(3,\mathbb{R})$ subgroups identically onto each other. Hence, we already have the $\text{SO}(3,\mathbb{C})$ representation of the Lorentz group on the four dimensional level, $A_{\mu a}$ being the corresponding gauge field. To get Ashtekar’s gravity after a spacetime decomposition, all we have to do is to identify the vierbein components with the fields appearing in (2.6), or after a gauge fixing with those appearing in (1.1).

ADM formulation

In Einstein gravity, the ADM formulation is possible only for special spacetime manifolds (those that admit a slicing $\mathcal{M} = \mathcal{N} \times \mathbb{R}$), and a given slicing also restricts the space of field configurations, because the slices have to be spacelike everywhere. Expressed in terms of the scalar function $T(\mathbf{x})$ defining the slicing, this means that the covector orthogonal to the slices $(\partial_\mu T)$ has to be timelike $(\partial_\mu T \partial_\nu T G^{\mu \nu} < 0)$, which imposes a restriction on the (invertible) metric $G_{\mu \nu}$. As a consequence, the ADM formulation is no longer manifestly covariant under four dimensional diffeomorphisms, and it also excludes some field configurations, e.g. those with closed timelike curves. However, for any given field configuration such that the slices are spacelike, we can always change the slicing ‘slightly’ in any direction, such that the new slices are still spacelike. So the ADM formulation does allow ‘small’ diffeomorphisms, which can also be checked by computing the transformations of the fields under infinitesimal diffeomorphisms.

When the diffeomorphisms become ‘too big’, then these infinitesimal transformations can no longer be integrated. In Einstein gravity we can easily say when this is going to happen. Namely, when the slices become lightlike at some point. The aim of this section is now to make an analogous and straightforward construction for the covariant theory defined by (3.4), leading to Ashtekar’s gravity as its ADM formulation.

An important notion in ADM formulation of Einstein gravity is that of a spacelike hypersurface. It has a straightforward generalization for degenerate metrics. Let us call a surface spacelike if its normal covector $\zeta_\mu$ is timelike in the sense that $\zeta_\mu \tilde{E}_\mu^A$ is a timelike vector in Minkowski space, or $\zeta_\mu \zeta_\nu \tilde{G}^{\mu \nu} < 0$. An important feature of this definition is that it allows ‘small’ deformations, as described above for invertible metrics. Consider a given metric $\tilde{G}^{\mu \nu}$ at some fixed point in spacetime. Then, the set of all normal vectors $\zeta_\mu$ satisfying $\zeta_\mu \zeta_\nu \tilde{G}^{\mu \nu} < 0$ is obviously a (possibly empty but) open subset of $\mathbb{R}^4$. Therefore, a spacelike surface can be deformed slightly in any direction, thereby remaining spacelike. Anticipating the result that Ashtekar’s theory is the ADM formulation of our covariant theory, this is in agreement with the fact that there are well defined infinitesimal transformation (1.3) for Ashtekar’s theory. The fact that these might not be integrable for ‘too big’ diffeomorphisms means that the slices become ‘non-spacelike’ (there is no straightforward generalization of ‘lightlike’ or ‘timelike’ surfaces).

For degenerate vierbeins $\tilde{E}_\mu^A$ strange things can happen: it might be that at a given point there is no spacelike hypersurface at all. This happens when the image of $\tilde{E}_\mu^A$, viewed as a linear map from the cotangent space of spacetime into the four dimensional Minkowski space, does not contain any timelike vector, so for example in the trivial case $\tilde{E}_\mu^A = 0$. Hence, there are different kinds of degeneracy, and we will see that due to this there is a crucial difference between the relation of Einstein gravity to its ADM formulation and the relation between our
covariant theory and Ashtekar’s gravity. In this sense (3.4) is not yet what we should call the covariant version of Ashtekar’s gravity. I will come back to this problem below. Here, this ‘worst’ kind of degeneracy will be ruled out simply by imposing the usual ADM restriction: for a given slicing \( T(x) \) of \( M \), we restrict the range of the vierbein such that for all \( x \in M \)

\[
\tilde{E}_\mu(x) \partial_\mu T(x) \quad \text{is negative timelike.} \tag{3.8}
\]

A vector \( \zeta^A \) is called negative timelike if \( \zeta^A \zeta_A < 0 \) and \( \zeta^0 < 0 \). This means that in addition to requiring the slicing to be spacelike we also fix the sign of \( \tilde{E}_\mu \), which in some sense means that \( T \) increases when going towards the physical time direction. It does not restrict the set of solutions, because with \( \tilde{E}_\mu, -\tilde{E}_\mu \) solves the field equations as well, and we just have to replace \( T \mapsto -T \) to find the solutions excluded by the word ‘negative’ in (3.3). In the next section I will give a more precise definition of how the physical arrow of time is assumed to be included in the the vierbein field, and you can check that ‘negative’ in (3.8) implies that \( T \) increases when going towards the future. Of course, (3.8) also excludes the case that there are points where no spacelike hypersurface exists.

Introducing the coordinate \( t = T(x) \) and three more (possibly local) coordinates \( x^m \), the action (3.4) splits into

\[
\mathcal{L} = 2 \tilde{E}_A^t \tilde{E}_B^m J_{aB} F_{tma} + \tilde{E}_A^m \tilde{E}_B^n J_{aB} F_{mn}. \tag{3.9}
\]

Now there are several possible ways to proceed. Because of (3.8), we now have \( \tilde{E}_A^t \) negative timelike, so we can find a boost \( A_{AB} \in SO(3, \mathbb{C}) \) such that the rotated vierbein is of the form (1.4), with positive \( \tilde{E}_0^t \) (note the position of the 0 index). Hence, we can impose a gauge fixing and it is straightforward to verify that (3.9) becomes the same as (1.1).

If we want to avoid any kind of gauge fixing, we must find a transformation from (3.9) to (2.4), i.e. we have to define the fields \( N, N^m, \tilde{r}_a^m \) and \( w_a \) appearing therein as a function of \( \tilde{E}_\mu \) such that

\[
2 \tilde{E}_A^t \tilde{E}_B^m J_{aB} = i \tilde{c}_c^m, \\
\tilde{E}_A^m \tilde{E}_B^n J_{aB} = \frac{1}{2} N \varepsilon_{abc} \tilde{c}_b^m \tilde{c}_c^n - i N^{[m} \tilde{c}_a^{n]}.
\tag{3.10}
\]

By writing out real and imaginary parts of these equations and using the explicit representation (A.3) for the \( J \) symbols one finds the very simple relations

\[
N^{-1} = -\tilde{E}_a^t \tilde{E}_a^t + \tilde{E}_0^t \tilde{E}_0^t, \quad \tilde{r}_a^m = \tilde{E}_a^t \tilde{E}_0^m - \tilde{E}_0^t \tilde{E}_a^m, \\
N^{-1} N^m = \tilde{E}_a^t \tilde{E}_a^m - \tilde{E}_0^t \tilde{E}_0^m, \quad w_a = \tilde{E}_a^t / \tilde{E}_0^t. \tag{3.11}
\]

Of course, \( N^{-1} = -\tilde{G}^{rt} \) and \( N^{-1} N^m = \tilde{G}^{tm} \), I just wrote out the components of the vierbein to show the similarity of these formulae. Note that (3.8) ensures that \( w_a w_a < 1 \) and \( N > 0 \). The relations are invertible, as for a given set \( N, N^m, \tilde{r}_a^m, w_a \) we can recover \( \tilde{E}_A^t \) from \( N \) and \( w_a \). The remaining equations are linear in \( \tilde{E}_A^m \), and it is easy to check that they have a unique solution. Hence, the somewhat unmotivated introduction of the ‘canonical’ variables \( \tilde{r}_a^m \) and \( w_a \) in (2.3) not only leads to the decomposition (2.4) of the complex dreibein. It also gives this surprisingly simply transformation from the vierbein to the three dimensional variables. If \( \tilde{E}_0^t = 0 \), i.e. if we impose a gauge fixing, we have \( w_a = 0, \tilde{r}_a^m = \tilde{c}_a^m \), and the three dimensional fields are given by (1.4).

If you prefer the variables \( \tilde{c}_a^m \) and \( w_a \) to parameterize the complex dreibein \( \tilde{c}_a^m \) in (2.4), you can proceed as follows. Given the vierbein \( \tilde{E}_A^\mu \), subject to (3.8), find the (unique) boost...
$A_A^B$ that brings the vierbein into the upper triangular form. Then you get $\tilde{e}_a^m$ by (1.1), and $v_a$ is obtained by mapping the boost into $SO(3, \mathbb{C})$, which is then of the form (2.1).

As a result, we found a manifest covariant theory of gravity (3.4), whose ADM formulation is Ashtekar’s gravity. However, there remains one crucial point to be considered which makes this ADM formulation different from that of Einstein gravity. It is the fact that in our theory the metric might be degenerate in such a way that at some point no spacelike hypersurface exists. Such a field configuration has no representation in the ADM formulation, and therefore no representation in Ashtekar’s theory. This is a well known problem in Einstein gravity too, as there are many field configuration which do not have representations in ADM gravity, e.g. those with closed timelike curves etc. But there is a difference: in our case, the problem is local, whereas the obstacles arising in the ADM formulation of Einstein gravity are always of a global type. In some sense, Einstein gravity is locally equivalent to its ADM formulation, whereas our covariant theory is not yet locally equivalent to Ashtekar’s theory.

**Local slicing**

To make this more precise, let us define what is meant by ‘local equivalence’ of Einstein gravity and its ADM formulation. Let $G_{\mu \nu}$ be a field configuration of Einstein gravity, i.e. an invertible, differentiable, Lorentzian metric on some spacetime $\mathcal{M}$, and let $x_0 \in \mathcal{M}$. Choose any timelike covector $\xi_\mu(x_0)$ at $x_0$, and define a scalar field $T(x)$ such that $\partial_\mu T(x_0) = \xi_\mu(x_0)$. As the metric is continuous, there will be a neighbourhood of $x_0$ where $\partial_\mu T$ is timelike, and thus also $\partial_\mu T \neq 0$. Inside this neighbourhood $T$ defines a ‘local slicing’. One can introduce the ADM variables, and express the contribution of this volume element to the action in terms of the ADM action. The field equations derived from this action are equivalent to those from the Einstein Hilbert action, and hence, at least locally, one can always go over to the ADM formulation of gravity. Only when requiring that a global spacetime slicing exists, we get a restriction on the field configurations.

This is not true for our theory. If there is no spacelike hypersurface at $x_0$, then there is no local slicing $T(x)$, and the transition to Ashtekar’s variables and finally the action (1.1) is not possible. We somehow have to restrict the range of $\tilde{E}_A^\mu$, but in a fully covariant way, i.e. not referring to any coordinate system, to get a theory which is really the covariant version of Ashtekar’s theory. It is not so difficult to guess how this restriction has to look. For each $x \in \mathcal{M}$ we must have:

$$\exists \xi_\mu(x) \text{ such that } \xi_\mu(x)\tilde{E}_A^\mu(x) \text{ is negative timelike.}$$

This $\xi_\mu$ is the normal covector of some hypersurface, which is spacelike by definition. So we can equivalently require that

$$\text{there is a spacelike hypersurface at each point in spacetime.} \quad (3.13)$$

In contrast to (3.8), this does no longer refer to any global object like the scalar field $T$ or any coordinate. With (3.13), our theory is still covariant and an extension of Einstein gravity, as it is certainly fulfilled by invertible vierbeins. Another question is whether the action principle is still well defined, which requires that the set of allowed values for $\tilde{E}_A^\mu$ does not have boundaries. It is in fact open (as a subset of $\mathbb{R}^{4 \times 4}$). For every $\tilde{E}_A^\mu$ with $G^{\mu \nu} \xi_\mu \xi_\nu < 0$ for some $\xi_\mu$, all vierbeins solving this inequality (for the same $\xi_\mu$) have the required property, and they obviously form an open neighbourhood of $\tilde{E}_A^\mu$. 


With the restriction (3.13), we can now show that our covariant theory is locally equivalent to Ashtekar’s theory, by the same arguments as before for Einstein gravity. Starting at some point $x_0$, there is a negative timelike covector $\xi_\mu(x_0)$. Define $T(x)$ such that $\partial_\mu T(x_0) = \xi_\mu(x_0)$, then there is a neighbourhood where (3.8) holds, the transition to the variables (3.11) can be performed, and the contribution to the action from this volume element reads (2.6).

To summarize, the action (3.4) together with the restriction (3.13) provides a covariant version of Ashtekar’s gravity, allowing exactly the right kinds of singular metrics. There are two crucial points where the theory is different from that in [1]: first of all, here we are dealing with real gravity, there is no need to complexify the metric, and the gauge group is the four dimensional Lorentz group $SO(1,3)_+ \simeq SO(3,\mathbb{C})$. No gauge fixing is necessary to transform to the three dimensional variables in the complexified version of Ashtekar’s gravity, except for the ‘global’ ADM formulation where the same restriction (‘spacelike slices’) as in Einstein gravity is required, and which can (partly) be understood as a gauge fixing. The second difference is that it was possible to render the condition $\mathcal{N} \neq 0$ covariant, which in [1] somehow remains as a non-covariant relic. What remains to be done now is to consider the covariant theory from a physical point of view.

### 4 The causal structure

In this section I want to analyse the geometrical properties of the covariant theory, at a kinematical level, i.e. only considering the field configurations themselves but not the dynamics. This will give us a deeper understanding of what the singular metrics in Ashtekar’s theory actually are, in particular we will see how spacetime looks like ‘at the origin’ $\hat{e}_a \cdot n = 0$. We will find that, even in this highly degenerate case, there is (at least locally) a well defined causal structure. This is in contrast to a comment made by Bengtsson [3], who assumes that degenerate metrics describe spacetimes without causal structure. So it is quite remarkable that Ashtekar’s singular metrics are exactly those which do not destroy the causal structure.

On the other hand, this is not really surprising because otherwise the Hamilton-Jacobi formalism would not work. Nevertheless it is interesting to be considered from the four dimensional point of view. In contrast to Einstein gravity, where in some sense the ‘origin’ of the fields is flat spacetime, here this origin turns out to be a space consisting of a continuum of completely disconnected points, but with a well defined ‘time’ everywhere. The same kind of origin is considered by Dragon [3], but there the time direction becomes degenerate, too: not only the space points but also the spacetime events are completely disconnected. Hence, our theory is a little bit more ‘physical’ than Dragon’s, and an interesting fact is that a distinction between ‘time’ and ‘space’ is made without destroying the manifest invariance under diffeomorphisms, which is due to the notion ‘spacelike’ appearing in (3.13).

#### Future, past, and spacelike hypersurfaces

Given an arbitrary vierbein, not necessarily subject to (3.13), you may consider $\hat{E}_A^\mu(x) : \mathbb{M}^4 \to T_x \mathcal{M}$, $\zeta^A \mapsto \zeta^\mu(x) = \zeta^A \hat{E}_A^\mu(x)$ as a map from four dimensional Minkowski space into the tangent bundle of the spacetime $\mathcal{M}$ (actually from an $\mathbb{M}^4$ bundle over $\mathcal{M}$, which only has to be time orientable for our purpose, but let us for simplicity assume that it is trivial). Let us define the ‘future’ of a point $x \in \mathcal{M}$ as the set of all tangent vectors at $x$ which are images
of positive timelike or lightlike vectors in $\mathbb{M}^4$, i.e.

$$\mathcal{F}(x) = \{ \zeta^\mu(x) = \zeta^A \tilde{E}^\mu_A(x) | \zeta^A \zeta_A \leq 0, \zeta^0 > 0 \}.$$  \hfill (4.1)

The past is defined similarly as $-\mathcal{F}(x)$, and a causal curve is straightforwardly defined as a curve whose tangent vector is contained in the future at every point the curve passes through.  The time orientation is assumed to be something real physical, so that the gauge group is really $SO(1,3)_+$ and changing the sign of $\tilde{E}^\mu_A$ is not considered as a gauge transformation, because it changes the time direction.  If you like you can choose $O(1,3)_+ \simeq O(3,\mathbb{C})$ to be the gauge group, allowing spacelike parity transformations.  Hence, in addition to the metric, the vierbein is assumed to carry information about the physical arrow of time as well.

The shape of the future can be very different for different values of the vierbein.  If $\tilde{E}^\mu_A(x)$ is invertible, it is obviously the usual future lightcone together with its interior at $x$ (but $\zeta^\mu = 0$ excluded).  Let us call this a hypercone, as it has one dimension more than a usual cone.  If the rank of $\tilde{E}^\mu_A(x)$ is less than 4, the future becomes degenerate, and the question to be considered here is whether such a degenerate future still has the physical features it should have.  There are three possible ways in which a hypercone can become degenerate under a rank 3 linear map.  Depending on which direction it is projected into, it either becomes a (three-dimensional) cone, with the peak excluded, a hyperplane, or a half-hyperplane including the boundary.  The latter happens if it is projected along a lightlike direction.  From a physical point of view, the last two cases are worse than the first.  In the first case, we have one space direction in which light does not propagate (i.e. its velocity vanishes), but it propagates normally into two other directions, and the future somehow still looks like a lightcone, except that it is ‘a bit’ flattened.  However, if the future becomes a hyperplane, there is a direction in which light propagates infinitely fast, and, moreover, the future will intersect with, or even becomes equal to the past.  Hence, the causal structure of spacetime is lost.  A similar situation occurs for smaller rank of $\tilde{E}^\mu_A$: for rank 2 we get an angle, the peak excluded, a half-plane or a plane, and for rank 1 the future becomes a half line with or without end point, or a full line.  For $\tilde{E}^\mu_A = 0$, finally, the future is $\{0\}$.

Now, which of these situations are physically reasonable and which are not?  What we want is that the future somehow ‘points into a direction’ and the past points into the opposite direction.  In particular, this means that the future is non-trivial and does not intersect with the past.  So let us impose the following condition on the vierbein

$$-\mathcal{F}(x) \cap \mathcal{F}(x) = \emptyset \iff 0 \notin \mathcal{F}(x).$$  \hfill (4.2)

The equivalence of the two conditions is easy to see.  $\Rightarrow$: if $0 \in \mathcal{F}$, then clearly $0 \in -\mathcal{F}$.  $\Leftarrow$: if there is some $\zeta^\mu \in -\mathcal{F} \cap \mathcal{F}$, then we have $\zeta^\mu \in \mathcal{F}$ and $-\zeta^\mu \in \mathcal{F}$, but with two vectors their sum is an element of $\mathcal{F}$ as well, which follows immediately from (1.11) and the fact that the sum of two positive timelike or lightlike vectors in Minkowski space is such a vector again.  So $0 \in \mathcal{F}$.

Condition (4.2) also implies that the future is non-trivial: by definition, it cannot be empty, so there is at least one non-zero tangent vector in $\mathcal{F}(x)$.  If (1.12) holds, the future is either a hypercone, a cone, an angle, or a half-line, depending on the rank of the vierbein.  In all these cases there is a at least one causal curve through $x$.  It is only the number of the degrees of freedom of this curve which is affected by the rank of $\tilde{E}^\mu_A$.  Moreover, the fact that future and past do not overlap means that there is space in between for, e.g., a hyperplane that does not intersect with either of them.  For invertible metrics such a surface is spacelike.  It is
reasonable to call a surface spacelike if it intersects neither with the future nor with the past, as it then contains causally disconnected points only. Let us check whether this definition of a spacelike surface is equivalent to that given in the last section, and whether the condition \((4.2)\) coincides with \((3.8)\). If this is the case, we have a new definition for the range of the vierbein in our theory, which is somehow more physical. To prove the equivalence, we can use a simple Lemma: the following two statements are equivalent:

(a) \(\xi_A \in \mathbb{M}^4\) is negative timelike, and

(b) \(\xi_A \zeta^A > 0\) for all positive timelike or lightlike \(\zeta^A \in \mathbb{M}^4\).

To proof this, you just have to write out the scalar product in components: for \((a) \Rightarrow (b)\) use that \(\xi^0 < -|\xi_a|, \zeta^0 \geq |\zeta_a|\), and the Schwarz inequality; for \((b) \Rightarrow (a)\) choose \(\zeta^A = (|\xi_a|, -\xi_a)\), which is positive lightlike and gives \(\xi_A \zeta^A > 0 \Rightarrow \xi^0 < -|\xi_a|\).

If we now consider the two definitions for spacelike surfaces, it is easy to proof their equivalence: a surface is spacelike if

(a) the surface does not intersect with \(\mathcal{F}\), or

(b) the normal covector \(\xi_\mu\) can be chosen such that \(\tilde{E}_A^\mu \xi_\mu\) is negative timelike.

Let \(\xi_\mu\) be the normal covector of the surface, then \((a)\) means that \(\xi_\mu \xi^\mu \neq 0\) for all \(\zeta^\mu \in \mathcal{F}\). As \(\mathcal{F}\) is connected, we can choose \(\xi_\mu\) such that the sign is positive, i.e. \(\xi_\mu \zeta^\mu = \xi_\mu \tilde{E}_A^\mu \zeta^A > 0\) for all positive timelike or lightlike \(\zeta^A\) (this is the definition of \(\mathcal{F}\)). Using the Lemma, this is equivalent to \((b)\), stating that \(\xi_\mu \tilde{E}_A^\mu\) is negative timelike.

What remains to be shown is that \((4.2)\) is equivalent to requiring that there is a spacelike hypersurface at any point in spacetime. Assume that there is such a surface, then certainly \(0 \notin \mathcal{F}\), because it lies inside the surface. On the other hand, if no spacelike hypersurface exists, then for every covector \(\xi_\mu\) we have \(E_A^\mu \xi_\mu\) spacelike or lightlike (otherwise this would be a normal covector of some spacelike hypersurface). So all these vectors lie on a non-timelike hyperplane in Minkowski space, having a positive timelike or lightlike normal vector \(\zeta^A\). Hence, we have \(\zeta^A \tilde{E}_A^\mu \xi_\mu = 0\) for all \(\xi_\mu\), i.e. \(\zeta^A \tilde{E}_A^\mu = 0\). But by definition this is an element of the future, so \(0 \in \mathcal{F}\).

We now have a ‘physical’ condition \((4.2)\), which defines the kinematics, i.e the set of allowed vierbein fields, of the covariant version of Ashtekar’s gravity. The result is that only those degenerate metrics are allowed which provide a local causal structure, in the sense that there is a subset of the tangent space at each point which denotes the future. This is separate from the past, pointing towards the opposite direction, and in between there is place to put hypersurfaces which do not intersect with the future (nor the past), and those are called spacelike. Another equivalent restriction on the vierbein is, as already guessed in the first section, that the metric \(\tilde{G}^{\mu\nu}\) must have signature \((-,+,+,+), (-,+,+,0), (-,+0,0)\) or \((-,0,0,0)\). I will not give an explicit proof for this, but you can infer from \((3.12)\) that there is at least one ‘-’ sign, and the signature of \(\eta_{AB}\) implies that there cannot be more ‘-’ signs in the signature of \(\tilde{G}^{\mu\nu}\). In principle, what all these equivalent conditions say is that space may become degenerate but not time. To illustrate the situation of a degenerate future, let us consider some simple examples.
Scalar fields and black holes

As we are only interested in the kinematics of degenerate metrics, let us consider a scalar field in a given background, i.e. with fixed values for the vierbein $\tilde{E}_A^\mu$ or the metric $\tilde{G}^{\mu\nu}$. Its action reads

$$I_{\text{mat}} = -\frac{1}{2} \int d^4x \tilde{G}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi.$$  \hfill (4.3)

Note that the simple structure of this action was one of the motivations in [3] to use $\tilde{G}^{\mu\nu}$ as the basic gravitational field (beside some regularity requirements which are not so different from Ashtekar’s ‘polynomiality’). As mentioned there, degeneracy of the metric means that there are directions in spacetime in which $\varphi$ can fluctuate without giving contributions to the energy, or equivalently without influencing the equations of motion. For a completely degenerate metric $\tilde{G}^{\mu\nu} = 0$, the field $\varphi$ drops out from the action and evolves arbitrarily. In our case, however, there is always at least one direction where the metric is not degenerate, and this direction is timelike. Given a causal curve through some point in spacetime, then there is always a positive contribution to the ‘kinetic’ energy of $\varphi$ in the action above, which is quadratic in $\dot{\varphi}$, where the dot denotes the derivative with respect to the curve’s parameter. As a consequence, the equation of motion can always be solved for $\ddot{\varphi}$, and we always have a unique time evolution if we follow a causal curve through spacetime.

In the ‘most degenerate’ case, the future consists of a half line only at every point in spacetime. The metric takes the form $\tilde{G}^{\mu\nu} = -\zeta^\mu \zeta^\nu$, where $\zeta^\mu$ is some vector (of weight one half) in $\mathcal{F}$, which itself consists of the positive multiples of $\zeta^\mu$ only. This corresponds to the ‘origin’ $\tilde{e}_a^m = 0$ of Ashtekar’s canonical variables. Let us assume that spacetime consists, at least locally, of a bundle of non-intersecting causal curves, such that exactly one of them passes through each point (you can find some strange vector fields $\zeta^\mu$ without this feature, but those do not admit a local slicing such that $\tilde{e}_a^m = 0$). Then, we can find a coordinate (or slicing) $t$ such that $\zeta^\mu = (1, 0, 0, 0)$, and the action for the scalar field becomes

$$I_{\text{mat}} = -\frac{1}{2} \int d^3x \int dt (\dot{\varphi})^2.$$  \hfill (4.4)

Obviously, this describes a continuum of completely decoupled fields $\varphi(x)$ for each ‘space’ point $x$ (if the slicing is global). Physically, space consists of a set of completely disconnected points, each having its own value of $\varphi$ evolving in time, thereby ignoring what is going on in the neighbourhood. Another way to see this, which will be useful below, is to consider a field that has a ‘step’, and to check whether this step contributes to the energy or not. So we set $\varphi = \varphi_0 + \alpha \theta$, where $\varphi_0$ is continuous, $\alpha$ some number, and $\theta$ the characteristic function of some region of space, which is one inside and zero outside that region. If we plug this into the action, we find that, whatever the region is, the action does not depend on $\alpha$ and therefore there is no contribution to the energy coming from the step along the boundary of the specified region. This is what is meant by ‘a fluctuation does not contribute to the energy’ above, and tells us that any given region and its complement are physically not connected, i.e. no information (about $\varphi$) can be passed from one point to another.

Defining this situation as the natural origin of the gravitational fields, which one usually does in most of the applications of Ashtekar’s variables, gravity is no longer the ‘deviation’ from flat spacetime. Instead, switching on the fields somehow switches on the communication between adjacent points in space, so that they know of each other and information can be
passed from one to another. Flat spacetime is just a very special configuration. In a perturbative formulation this leads to a new understanding of what a graviton is. Instead of being a perturbation of flat space, you should consider it as a deviation from the completely ‘discretized’ space described above. Flat space would no longer be the vacuum but some kind of condensate in perturbative quantum gravity, and because we are expanding around a very different classical field configuration, this might have some influence on renormalizability and related problems in perturbative quantum gravity, which so far have not been investigated in the context of Ashtekar’s variables.

Let us also consider a special field configuration where the metric is ‘slightly’ degenerate only. With global coordinates $t, x, y, z$, we take

$$
\tilde{G}^{\mu\nu} = \text{diag}(-1, 1, 1, f^2(z)),
$$

where $f$ is some function with $f(0) = 0$, $f'(0) = 1$ and $f(z) \to 1$ for $z \to \pm\infty$, so that spacetime is asymptotically flat for large $z$ and the metric is singular on the $z = 0$ plane. A possible choice is $f(z) = z/\sqrt{1 + z^2}$, but we won’t need $f$ explicitly. The action for the scalar field becomes

$$
I_{\text{mat}} = -\frac{1}{2} \int d^4x (\nabla \varphi \cdot \nabla \varphi + f^2(z) \partial_z \varphi \partial_z \varphi),
$$

where $\nabla$ denotes the derivative operator $(\partial_t, \partial_x, \partial_y)$ with respect to the ‘flat’ coordinates. To see what happens at the ‘wall’, we again consider a field which has a finite step, i.e. which is of the form $\varphi = \varphi_0 + \alpha \theta(z)$, where $\varphi_0$ is continuous, and $\theta(z)$ is the step function taking the values 0, 1 for $z < 0, z \geq 0$, respectively. As $\partial_z \theta(z) = \delta(z)$, this unfortunately produces a $\delta^2(z)$ term in the action, so we have to regularize somehow. We do this by defining a regularized step function

$$
\theta(z) = \begin{cases} 
0 & \text{for } z \leq -\epsilon/2, \\
\frac{z}{\epsilon} + \frac{1}{2} & \text{for } -\epsilon/2 \leq z \leq \epsilon/2, \\
1 & \text{for } z \geq \epsilon/2,
\end{cases}
$$

and taking the limit $\epsilon \to 0$. The contribution of the step to the kinetic energy then becomes

$$-rac{\alpha}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} dz f^2(z) \partial_z \varphi_0 - \frac{\alpha^2}{2\epsilon^2} \int_{-\epsilon/2}^{\epsilon/2} dz f^2(z).$$

Using $|f(z)| \leq |z|$ and the fact that $\varphi_0$ is continuous it is easy to show that this vanishes in the limit $\epsilon \to 0$. Hence, we see that the fields on both sides of the wall decouple completely and there is no correlation.

Another way to see this is to look at light rays approaching the wall. For $z > 0$, the metric is invertible and we have $G_{\mu\nu} = \text{diag}(-f, f, f, f^{-1})$. Hence, for a light ray in the $t, z$ plane we have $dt/dz = \pm f^{-1}(z)$, so that $t$ diverges logarithmically for $z \to 0$, and the light never reaches the wall, showing again that there is no communication between the two parts of spacetime. Of course, all these arguments depend crucially on the behaviour of $f$ near $z = 0$. Here we assumed that $f(z) \approx z$. If $f$ behaves differently, we get very different kind of ‘walls’. Without giving proofs (which are absolutely straightforward), let me just list a few ‘physical’ properties such a wall can have:

(1) if $zf^{-2}(z)$ is bounded for $z \to 0$, then the scalar fields on both sides do not decouple (in this case (4.8) does not to vanish);
(2) if $zf^{-1}(z) \to 0$ for $z \to 0$, light rays from outside can touch the wall (because $f^{-1}(z)$ is integrable at $z = 0$);

(3) if $zf^{-1/2}(z) \to 0$ for $z \to 0$, the spacelike distance between the wall and some point outside is finite.

We see that (1) implies (2) and (2) implies (3), which is reasonable from the physical point of view, too. But note that they are not equivalent. So already with this rather simple degenerate metric (4.5) we can describe very different singular structures in spacetime, in this case some kind of ‘domain walls’. Moreover, the singular metrics somehow enable us to glue together parts of spacetime which do not communicate with each other, i.e. they naturally describe horizons, which are however stronger than usual horizons which only exist with respect to specific observers (remain ‘outside’ for all times), and where information can be passed through in one direction. Remember, however, that we only considered the kinematics here, i.e. it is not clear whether all these structures may appear dynamically, i.e. as solutions of the full set of field equations.

A typical situation where such a wall in fact occurs dynamically is the Schwarzschild geometry, when simple polar coordinates $t, r, \theta, \varphi$ are used. There we have a coordinate singularity at the horizon $r = r_0$, which is similar to the $z = 0$ wall above. It has been shown in [4, 5] that Ashtekar’s variables can be used to avoid this singularity, i.e. one can choose coordinates such that all the fields are finite at the horizon. Those are quite different from Kruscal or related coordinates, which eliminate the singularity by a non-regular coordinate transformation. Instead, the transformation here is invertible, and there is still a degenerate metric at the horizon. Moreover, one finds very peculiar solutions of the field equations, called ‘empty black holes’, i.e. the metric inside the black hole is not of the Schwarzschild type but simply flat spacetime, with only a coordinate singularity at the origin.

With our results from above these solutions are no longer mysterious, as the ‘wall’ keeps the part of spacetime behind the horizon separate from that outside the black hole, so there is no relation between the ‘physics’ inside and outside the $r = r_0$ sphere. In fact, the region $r < r_0$ is not really the interior of the black hole. The latter can be found somewhere else, namely by transforming to, say, Kruscal coordinates and extending spacetime behind the true horizon. Bengtsson’s ‘empty black hole’ is therefore rather a combination of two different, and separated, spacetimes, glued together along a wall similar to the $z = 0$ wall above, but this time the wall has the shape of a sphere.

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I would like to thank Michael Reisenberger for drawing my attention to a mistake made in [6], which led to the wrong conclusion that diffeomorphisms are not well defined for singular values of Ashtekar’s variables. This paper was intended to correct the mistake and to clarify the problem of finite diffeomorphisms in Ashtekar’s gravity, at the classical level.
Appendix

This is a collection of formulae for the $J$ symbols, for more detailed information see [1, 7]. Explicit representation $A, B, \ldots = 0, 1, 2, 3, a, b, \ldots = 1, 2, 3$:

$$J_{aAB} = \frac{1}{2} \eta_{aA} \delta_B^0 - \frac{1}{2} \eta_{aB} \delta_A^0 - \frac{1}{2} \varepsilon^0_{aAB}, \quad (A.9)$$

Orthonormality and completeness:

$$J^a_{aAB} J^b_{bAB} = \eta_{ab}, \quad J^a_{aAB} J^*_{bAB} = \eta_{ab}, \quad J^a_{aAB} J^*_{bAB} = 0,$$

$$J^a_{aAB} J^b_{aCD} + J^a_{aAB} J^*_{aCD} = \delta_A^A \delta_B^C \delta_D^D. \quad (A.10)$$

Commuting representations of $\mathfrak{so}(3)$:

$$J^a_{aAB} J^b_{bBC} = -\frac{1}{3} \eta_{ab} \delta_C^C + \frac{1}{2} \varepsilon_{abc} J^c_{cCA},$$

$$J^*_{aAB} J^b_{bBC} = -\frac{1}{3} \eta_{ab} \delta_C^C + \frac{1}{2} \varepsilon_{abc} J^*_{cCA},$$

$$J^a_{aAB} J^*_{bBC} = J^*_{bAB} J^a_{aBC}. \quad (A.11)$$

Self-duality:

$$\varepsilon_{AB}^{CD} J_{aCD} = 2i J_{aAB}, \quad \varepsilon_{AB}^{CD} J^*_{aCD} = -2i J^*_{aAB}. \quad (A.12)$$

Other useful formulae:

$$\varepsilon_{abc} J_{aAB} J_{cCD} = \frac{1}{2} (\eta_{AD} J_{cBC} + \eta_{BC} J_{cAD} - \eta_{AC} J_{cBD} - \eta_{BD} J_{cAC}),$$

$$J_{aAB} J_{aCD} = \frac{1}{2} \eta_{A[C} \eta_{D]B} - \frac{1}{4} \varepsilon_{ABCD}. \quad (A.13)$$

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