Local non-periodic order and diam-mean equicontinuity on cellular automata

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ABSTRACT
Diam-mean equicontinuity is a dynamical property that has been of use in the study of non-periodic order. Using some type of 'local' skew product between a shift and an odometer looking cellular automaton (CA), we will show that there exists an almost diam-mean equicontinuous CA that is not almost equicontinuous (and hence not almost locally periodic).

Previously, we constructed a CA that is almost mean equicontinuous [L.D.I.S. Baños and F. García-Ramos, Mean equicontinuity and mean sensitivity on cellular automata, Ergodic Theory Dynam. Systems 41 (12) (2021), pp. 3704–3721] but not almost diam-mean equicontinuous [L.D.I.S. Baños and F. García-Ramos, Diameter mean equicontinuity and cellular automata, Proceedings of the 27th International Workshop on Cellular Automata and Discrete Complex Systems, arXiv:2106.09641, 2021].

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1. Introduction

The study of non-periodic order, in the sense of finding and classifying order in non-periodic structures, has been a topic of interest among mathematicians, physicists and biologists for many years. For example, in 1940 Hedlund and Morse studied non-periodic sequences with minimal complexity which they called Sturmian sequences [19]. The role of dynamical systems in this line of research is natural, as it provides different formal ways to define order.

Cellular automata (CA) were introduced by Ulam and von Neumann to model the evolution of cells. CA can be studied in the context of topological dynamical systems (TDS), that is, the study of pairs $(X, T)$, where $X$ is a compact metric space and $T : X \to X$ a continuous function [4, 16]. CA are sometimes used as discrete models of differential equations [21, 22].

In the study of CA (or symbolic dynamics in general) periodicity is linked to the concept of equicontinuity. A TDS is equicontinuous if the family $\{T^n\}_{n\in\mathbb{N}}$ is equicontinuous. One may also study this notion locally. A point $x \in X$ is an equicontinuity point if the diameter of the images of a small ball around $x$ will always stay small, that is, if
for every $\epsilon > 0$ there exists $\delta > 0$ such that $\text{diam}(T_i^kB(x)) < \epsilon$ for every $i \in \mathbb{N}$. We say a TDS is almost equicontinuous if the equicontinuity points are residual. Given a CA it is not difficult to check that a point is equicontinuous if and only if it is locally eventually periodic; that is, if every column is eventually periodic (Proposition 2.6). For CA, almost equicontinuity is more natural than equicontinuity since it can be used to classify CA using sensitivity to initial conditions (Theorem 4.3). A weaker notion than equicontinuity, diam-mean equicontinuity, requires the diameter of small balls to stay small on average (see Definition 2.2). The notion of diam-mean equicontinuity has been used to characterize regularity properties of the maximal equicontinuous factor [9], and an even weaker property, mean equicontinuity (introduced in [5, 20]), has been shown to be connected with the concept of (measure-theoretic) discrete spectrum [6, 8, 14, 17] and almost periodic functions [10] (for a survey on mean equicontinuity see [18]). These two notions (mean equicontinuity and diam-mean equicontinuity) appear naturally in the study of aperiodic order; nonetheless, the viewpoint of this paper is not quite the same as in the study of quasicrystals and aperiodic order as in [1], since we are only studying the properties locally and the systems we study may exhibit chaotic properties like positive entropy.

We have the following relations:

\[
\text{equicontinuity} \rightarrow \text{diam-mean equicontinuity} \rightarrow \text{mean equicontinuity}.
\]

For subshifts, one can construct examples to show that these implications are strict. Actually, non-periodic regular Toeplitz subshifts are always diam-mean equicontinuous but not equicontinuous [9].

Dynamics on CA tend to be more restricted than on subshifts. Nonetheless, in [2], the authors constructed a CA (the Pacman CA) with a dense set of mean equicontinuity points, and no diam-mean equicontinuity points [3]. The question of whether there exists a CA with diam-mean equicontinuity points but no equicontinuity points remained open and is addressed in this paper.

The example we build here is very different to the Pacman CA and more in the spirit of a Toeplitz subshift. We construct a CA by taking some form of local skew-product between a very regular CA (similar to an odometer) and a very chaotic one (the shift map). It turns out that this CA is almost diam-mean equicontinuous but not almost equicontinuous (Theorem 3.22).

As an application of the previous construction, we deduce that there exist CA that are neither almost diam-mean equicontinuous nor diam-mean sensitive (Theorem 4.5). Note that CA (not necessarily transitive) are almost equicontinuous or sensitive (Kurka’s dichotomy [15]).

In summary, almost equicontinuous CA always exhibit eventually periodic behaviour. CA are either almost equicontinuous or sensitive. There exist very chaotic CA, like the shift, which satisfy all the sensitivity-type properties. Nonetheless, among the sensitive CA, there exist almost diam-mean equicontinuous CA, and CA that are neither almost diam-mean equicontinuous nor diam-mean sensitive. Among diam-mean sensitive CA, there exist almost mean equicontinuous, and CA that are neither mean sensitive nor almost mean equicontinuous.
2. Preliminaries

We say \((X, T)\) is a TDS if \(X\) is a compact metric space (with metric \(d\)) and \(T : X \to X\) is a continuous function. Given a metric space \(X\), we set \(B_\delta(x) = \{ y \in X : d(x, y) < \delta \}\), and we denote the diameter of a subset \(A\) with \(\text{diam}(A)\). A subset of a topological space is residual (or comeagre) if it includes the intersection of countably many dense open sets.

**Definition 2.1:** Let \((X, T)\) be a TDS and \(x \in X\).

1. The point \(x\) is an **equicontinuity point** if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that
   \[
   \text{diam}(T^iB_\delta(x)) < \varepsilon
   \]
   for every \(i \in \mathbb{N}\). The set of equicontinuity points of \((X, T)\) is denoted by \(\text{EQ}\).
2. \((X, T)\) is **equicontinuous** if \(\text{EQ} = X\).
3. \((X, T)\) is **almost equicontinuous** if \(\text{EQ}\) is residual.

Diam-mean equicontinuity was introduced in [8] and studied in [9].

**Definition 2.2:** Let \((X, T)\) be a TDS.

- We say \(x\) is a **diam-mean equicontinuity point** if for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that
  \[
  \limsup_{n \to \infty} \frac{\sum_{i=0}^{n} \text{diam}(T^iB_\delta(x))}{n+1} < \varepsilon
  \]
  We denote the set of diam-mean equicontinuity points by \(\text{EQ}^d\).
- \((X, T)\) is **diam-mean equicontinuous** if \(\text{EQ}^d = X\).
- \((X, T)\) **almost diam-mean equicontinuous** if \(\text{EQ}^d\) is residual.

It is trivial to see that every equicontinuity point is a diam-mean equicontinuity point.

**Proposition 2.3:** Let \((X, T)\) be a TDS and \(\varepsilon > 0\). We define
\[
\text{EQ}^d_\varepsilon = \left\{ x \in X : \exists \delta > 0, \limsup_{n \to \infty} \frac{\sum_{i=0}^{n} \text{diam}(T^iB_\delta(x))}{n+1} < \varepsilon \right\}.
\]
Then, \(\text{EQ}^d_\varepsilon\) is open and \(\text{EQ}^d = \bigcap_{m>0} \text{EQ}^d_\varepsilon\). Furthermore, \(\text{EQ}^d\) is dense if and only if it is a residual set.

**Proof:** The fact that \(\text{EQ}^d_\varepsilon\) is open follows from [8, Lemma 46] and [9, Lemma 4.4].

It is easy to check that \(\text{EQ} = \bigcap_{m>0} \text{EQ}^d_\varepsilon\). Thus, by Baire’s theorem, \(\text{EQ}\) is residual if and only if it is dense.

Throughout this paper, given \(n \in \mathbb{N}\), we use \([0, n] = \{0, 1, \ldots, n\}\). Now, we will give the setup of basic symbolic dynamics.

1. Given a finite set \(A\) (called an alphabet), we define the **symbol space of \(A\)** as \(A^\mathbb{Z}\).
(2) Given $x \in A^\mathbb{Z}$, we represent the $i$th coordinate of $x$ as $x_i$. Also, given $i, j \in \mathbb{Z}$ with $i < j$, we define the finite word $x_{[i:j]} = x_i \ldots x_j$.

(3) We denote the set of all finite words as $A^+$. 

(4) We endow any symbol space with the metric

$$d(x, y) = \begin{cases} 2^{-i} & \text{if } x \neq y \text{ where } i = \min\{|j| : x_j \neq y_j|}; \\ 0 & \text{otherwise.} \end{cases}$$

This metric generates the prodiscrete topology $A^\mathbb{Z}$.

(5) For any symbol space $A^\mathbb{Z}$, we define the shift map $\sigma : A^\mathbb{Z} \to A^\mathbb{Z}$ by $\sigma(x)_i = x_{i+1}$. The shift map is continuous (with respect to the previously defined metric).

**Definition 2.4:** We say that $(X, T)$ is a cellular automaton (CA) if $X$ is a symbol space and $T : X \to X$ is continuous and commutes with $\sigma$, i.e. $\sigma \circ T = T \circ \sigma$.

**Remark 2.5:** Note that $T x_i$ represents the $i$th coordinate of the point $T x$, and $T x_{[0,n]}$ the word extracted from the $[0, n]$-coordinates of the point $T x$.

The following fact can be extracted from the proof of [15, Theorem 4].

**Proposition 2.6:** Let $(X, T)$ be a CA. If $x \in X$ is an equicontinuity point then $x$ is locally eventually periodic, i.e. for every $i \in \mathbb{Z}$ we have that $T^nx_i$ is an eventually periodic sequence (of $n$).

For CA, there is a simple combinatorial characterization of a diam-mean equicontinuity using upper density.

**Definition 2.7:** Let $S \subseteq \mathbb{N}$. We define the upper density of $S$ by

$$\overline{D}(S) = \limsup_{n \to \infty} \frac{|S \cap [0, n-1]|}{n}.$$

Let $n \in \mathbb{N}$. We will denote the balls of radius $2^{-n}$ with $B_n(x)$. That is,

$$B_n(x) = \{y \in A^\mathbb{Z} : x_i = y_i \forall i \in [-n, n]\}.$$ 

Now, we will define sensitivity sets on a set of columns.

**Definition 2.8:** Let $J \subset \mathbb{Z}$ be finite set and $n \in \mathbb{N}$. We define

$$S_J(x, n) = \{i \in \mathbb{N} : \exists y, z \in B_n(x), T^i y_j \neq T^i z_j\}.$$

**Proposition 2.9:** Let $(X, T)$ be a CA and $x \in X$. Then, $x$ is a diam-mean equicontinuity point if and only if for every $m \geq 0$ there exists $m' \geq 0$ such that

$$\overline{D}(S_{[-j, j]}(x, m')) \leq \frac{1}{2^{m+2}}$$

for all $0 \leq j \leq m + 1$. 
Proof: $\Rightarrow$: Suppose there exists $m \geq 0$ such that for all $m' \geq 0$ there exists $l \in [0, m + 1]$ such that

$$\overline{D}(S_{[-l,l]}(x, m')) > \frac{1}{2^{m+2}}.$$ 

This implies that

$$\limsup_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^{n} \text{diam}(T^i B_{m'}(x)) \geq \limsup_{n \to \infty} \frac{1}{n+1} \sum_{i \in S_{[-l,l]}(x,m')} \text{diam}(T^i B_{m'}(x))$$

$$\geq \frac{1}{2^l} \limsup_{n \to \infty} \frac{1}{n+1} \#(S_{[-l,l]}(x,m') \cap [0,n])$$

$$\geq \frac{1}{2^{m+1}} \overline{D}(S_{[-l,l]}(x, m')) > \frac{1}{2^{2m+3}}.$$ 

Therefore, $x$ is not a diam-mean equicontinuity point.

$\Leftarrow$: For every pair of integers $n,k \in \mathbb{N}$ and every $x \in X$ we define the set

$$S^n_k(x, m) = S_{[-k,k]}(x, m) \cap [0,n].$$

Note that for every $k$ we have that

$$S^n_k(x, m) \subseteq S^n_{k+1}(x, m)$$

and

$$S^n_{k+1}(x, m) \setminus S^n_k(x, m) = S^n_k(x, m) \cap [0,n].$$

Now, let us assume that for every $m \geq 0$ there exists $m' \geq 0$ such that

$$\overline{D}(S_{[-j,l]}(x, m')) \leq \frac{1}{2^m}$$

for every $0 \leq j \leq m + 1$. For sufficiently large $m$ we conclude that

$$\limsup_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^{n} \text{diam}(T^i B_{m'}(x)) \leq \frac{1}{2^m} + \sum_{i=1}^{m+1} \frac{1}{2^i} \cdot \frac{1}{2^m} + \sum_{i=m+2}^{\infty} \frac{1}{2^i}$$

$$\leq \frac{1}{2^{m-3}}.$$ 

This implies $x$ is a diam-mean equicontinuity point. $\blacksquare$
3. Almost diam-mean equicontinuity

As we mentioned in the previous section, every almost equicontinuous CA is almost diam-mean equicontinuous. In this section, we will construct an almost diam-mean equicontinuous CA that is not almost equicontinuous. For other examples of CA related to these concepts see [2, 23].

First, we define a CA that resembles an odometer. Let $A_1 = \{0\} \cup Z_3 = \{0, [0], [1], [2]\}$. We define the CA $T_1 : A_1^\mathbb{Z} \to A_1^\mathbb{Z}$ locally as follows: $T_1x_i = \square$ if and only if $x_i = \square$, otherwise $T_1x_i \in \{x_i, (x_i + 1) \mod 3\}$, with $T_1x_i = (x_i + 1) \mod 3$ if and only if $x_{i+1} \in \{[2], [2]\}$. In other words,

$$T_1x_i = \begin{cases} \square & \text{if } x_i = \square \\ 0 & \text{if } (x_i = 2 \land x_{i+1} \in \{[0], [2]\}) \lor (x_i = 0 \land x_{i+1} \in A_1 \setminus \{[0], [2]\}); \\ 1 & \text{if } (x_i = 0 \land x_{i+1} \in \{[0], [2]\}) \lor (x_i = 1 \land x_{i+1} \in A_1 \setminus \{[0], [2]\}); \\ 2 & \text{if } x_i = 1 \land x_{i+1} \in \{[0], [2]\}) \lor (x_i = 2 \land x_{i+1} \in A_1 \setminus \{[0], [2]\}). \end{cases}$$

Example 3.1: Let $x \in A_1^\mathbb{Z}$ such that $x_{[0,5]} = \square[0][0][0][0][0]$. We have that

$$T_1^6x_{[0,5]} = \square[0][0][0][0][0]$$
$$T_1^7x_{[0,5]} = \square[0][0][0][1][0]$$
$$T_1^8x_{[0,5]} = \square[0][0][0][0][2]$$
$$T_1^9x_{[0,5]} = \square[0][0][0][1][2]$$
$$T_1^{10}x_{[0,5]} = \square[0][0][0][2][2]$$
$$T_1^{11}x_{[0,5]} = \square[0][0][0][2][1]$$
$$T_1^{12}x_{[0,5]} = \square[0][0][0][2][0]$$
$$T_1^{13}x_{[0,5]} = \square[0][0][0][1][0]$$

Remark 3.2: Let $x, y \in A_1^\mathbb{Z}$. Note that $T_1x_i$ only depends on $x_i$ and $x_{i+1}$. Hence, if there exist $M, N \in \mathbb{Z}$ such that $x_{M+i} = y_{N+i}$ for every $i \geq 0$, then $T_1x_{M+i} = T_1y_{N+i}$ for every $i \geq 0$.

Remark 3.3: It is not difficult to see that $T_1$ is almost equicontinuous. In fact, $\square$ is a blocking word; that is if $x, y \in A_1^\mathbb{Z}$ with $x_{[m',m]} = y_{[m',m]}$ and $x_m = \square$, then $T_1^n x_{[m',m]} = T_1^n y_{[m',m]}$ for every $n > 0$. Moreover, in this situation, one can check that there exist $M > 0$ and $p > 0$ such that $T_1^{M+ip} x_{[m',m]} = T_1^{M} x_{[m',m]}$ for all $i \geq 0$. 
Given a CA \((X, T)\), we say a point \(x \in X\) is periodic with period \(p\) if \(T^p x = x\). This should not be confused with the following definition. Let \(J \subset \mathbb{Z}\). We say \(x_j\) is periodic, if the sequence \(\{T^n x_j\}_{n \in \mathbb{N}}\) is periodic. If \(x\) is periodic then \(x_j\) is periodic for every \(J \subset \mathbb{Z}\). The converse may not hold (the period can increase).

We will now present some statements that will help us to understand the behaviour of \(T_1\).

**Lemma 3.4:** Let \(x \in A_1^\mathbb{Z}\) such that \(x_{[0,1]} = 0 [\square]\). Then, \(x_{[0,1]}\) is periodic with period 3.

**Proof:** Observe that \(T_1 x_i = \square\) if and only if \(x_i = \square\). Hence,

- \(T_1 x_0 = 0\) and \(T_1 x_1 = \square\);
- \(T_2^3 x_0 = 2\) and \(T_2^3 x_1 = \square\); and
- \(T_3^3 x_0 = 0\) and \(T_3^3 x_1 = \square\).

Therefore, \(x_0\) has period 3. \(\blacksquare\)

**Remark 3.5:** Using that \(T_1\) commutes with the shift we obtain that for every \(z \in \mathbb{Z}\) if \(x_{[z,z+1]} = 0 [\square]\), then \(x_{[z,z+1]}\) is periodic with period 3.

**Lemma 3.6:** Let \(x \in A_1^\mathbb{Z}\) such that \(x_{[0,2]} = 0 [\square][\square]\). Then, \(x_{[0,2]}\) has period 9.

**Proof:** Remark 3.3 and Lemma 3.4 implies that \(T_1^3 x_1 = 0\) and \(T_1^3 x_2 = \square\) for all \(k \in \mathbb{N}\). Thus, we have that

- \(T_1^3 x_{[0,1]} = 0 [\square][\square]\) and \(T_1^3 x_2 = \square\);
- \(T_1^6 x_{[0,1]} = 2 [\square][\square]\) and \(T_1^6 x_2 = \square\);
- \(T_1^9 x_{[0,1]} = 0 [\square][\square]\) and \(T_1^9 x_2 = \square\).

Therefore, \(x_{[0,2]}\) have period 9. \(\blacksquare\)

From the proof of Lemma 3.6, we can conclude the next result.

**Lemma 3.7:** If \(x \in A_1^\mathbb{Z}\) such that \(x_{[0,2]} = 0 [\square][\square]\), then

- (1) \(T_1^i x_0 = 0\) for all \(0 \leq i \leq 2\);
- (2) \(T_1^i x_0 = 0\) for all \(3 \leq i \leq 5\);
- (3) \(T_1^i x_0 = 2\) for all \(6 \leq i \leq 8\).

Something to be careful about is that the period does not necessarily increase if the amount of \(0\)s increases (one of the differences with an odometer). The following lemma is an evidence of this comment.

**Lemma 3.8:** Let \(x, y \in A_1^\mathbb{Z}\) such that \(x_{[0,2]} = 0 [\square][\square]\) and \(y_{[0,3]} = 0 [\square][\square][\square][\square]\). Then \(x_{[0,2]}\) and \(y_{[0,3]}\) have period 9.
Proof: By Lemmas 3.6 and 3.7 we have that
\[ T_1^i y_0 = [0], \]
for all \( 0 \leq i \leq 6 \). Observe that
\[ T_1^{6+i} y_0 = T_1^i y_2, \]
for all \( 0 \leq i \leq 3 \). Hence, \( T_1^9 y_{\lfloor 0,3 \rfloor} = [0] \). ■

To generalize Lemma 3.8 first we need the following statement.

Lemma 3.9: Let \( m, k > 0, x, y \in A_1^\mathbb{Z} \) such that \( x_{\lfloor 0,k \rfloor} = y_{\lfloor 0,k \rfloor} \), and \( \{T_1^i x_{k+1}, T_1^i y_{k+1}\} \subset \{[0,2] \} \) for every \( i \in [0, m] \). Then, \( T_1^i x_{\lfloor 0,k \rfloor} = T_1^i y_{\lfloor 0,k \rfloor} \) for every \( i \in [0, m] \).

Proof: The proof can be obtained using Remark 3.2, the shift commuting property of a CA, and the fact that if \( x', y' \) satisfies that \( x'_0 = y'_0 \) and \( \{x'_1, y'_1\} \subset \{[0,2] \} \), then \( T_1 x'_0 = T_1 y'_0 \). ■

Now, we will prove more important properties of \( T_1 \).

Proposition 3.10: Let \( l \in \mathbb{N}, j \in [0, 2^l - 1], \) and \( x, y \in A_1^\mathbb{Z} \) with \( x_{\lfloor 0,2^l \rfloor} = [0] \) and \( y_{\lfloor 0,2^l+j \rfloor} = [0] \). We have that \( x_{\lfloor 0,2^l \rfloor} \) and \( y_{\lfloor 0,2^l+j \rfloor} \) have period \( 3^{l+1} \). Furthermore,

- \( T_1^i x_0 = [0] \) for all \( 0 \leq i < 3^l \);
- \( T_1^i x_0 = [1] \) for all \( 3^l \leq i < 3^{l+1} \);
- \( T_1^i x_0 = [2] \) for all \( 2(3^l) \leq i < 3^l \).

Proof: We will prove this result using induction on \( l \). From Lemma 3.6, we have the result for \( l = 0 \). Let us assume that the results hold for \( l = k \); that is

\begin{align*}
T_1^i x_0 &= [0] \text{ for all } 0 \leq i < 3^k; \\
T_1^i x_0 &= [1] \text{ for all } 3^k \leq i < 2(3^k); \\
T_1^i x_0 &= [2] \text{ for all } 2(3^k) \leq i < 3^k; \\
x_{\lfloor 0,2^k \rfloor} \text{ and } y_{\lfloor 0,2^k+j \rfloor} \text{ have period } 3^{k+1}. 
\end{align*}

(1)

Now, let \( l = k + 1 \). By (1) and Remark 3.3 we have that

- \( T_1^i x_{2^k} = [0] \) for all \( 0 \leq i < 3^k \);
- \( T_1^i x_{2^k} = [1] \) for all \( 3^k \leq i < 2(3^k) \);
- \( T_1^i x_{2^k} = [2] \) for all \( 2(3^k) \leq i < 3^{k+1} \).
Hence, we have that $T_1^i x_{[0,2^k]} = [0]^{2^k}$ for all $0 \leq i \leq 2(3^k)$. Observe that

$$T_1^{2(3^k)} x_{[0,2^k]} = [0]^{2^k} [2^{k+1-1}]$$

and

$$x_{[2^k,2^{k+1}]} = [0]^{2^k}.$$ 

Using Lemma 3.9 and the fact that $T_1$ commutes with the shift, we obtain that

$$T_1^{2(3^k)+i} x_{[0,2^k]} = T_1^i x_{[2^k,2^{k+1}]}$$

for all $0 \leq i < 3^k$. This implies that $T_1^i x_0 = [0]^{2^k}$ for all $0 \leq i < 3^k+1$.

By (1) we have that $y_{[0,2^{k+1}-1]} = [0]^{2^{k+1}-1}$ has period $3^{k+1}$. Since $x_{[0,2^k+1]} = y_{[0,2^{k+1}-1]}$, Remark 3.3 gives us that

$$T_1^{3^{k+1}} x_{[0,2^k+1]} = [0]^{2^{k+1}-1}.$$ 

Thus,

$$T_1^{3^{k+1}-1} x_{[0,2^k+1]} = [0]^{2^{k+1}-1} [2]$$

and

$$T_1^{3^{k+1}} x_{[0,2^k+1]} = [1]^{2^{k+1}-1} [0].$$

By this, and a similar use of (1), we obtain that

- $T_1^{3^{k+1}+i} x_{2^k} = [0]^{2^k}$ for all $0 \leq i < 3^k$;
- $T_1^{3^{k+1}+i} x_{2^k} = [0]^{2^k}$ for all $3^k \leq i < 2(3^k)$;
- $T_1^{3^{k+1}+i} x_{2^k} = [2]^{2^k}$ for all $2(3^k) \leq i < 3^k+1$.

Then, $T_1^{3^{k+1}+i} x_{[0,2^k]} = [0]^{2^k} [0]^{2^{k+1}-1}$ for all $0 \leq i \leq 2(3^k)$. So, using Lemma 3.9 and the fact that $T_1$ commutes with the shift, we have that

$$T_1^{3^{k+1}+i} x_{[0,2^k]} = T_1^i x_{[2^k,2^{k+1}]}$$

for all $2(3^k) \leq i < 3^{k+1}$. Therefore, $T_1^i x_0 = [0]^{2^k}$ for all $3^{k+1} \leq i < 2(3^{k+1})$. In a similar way, we have that $T_1^i x_0 = [2]^{2^k}$ for all $2(3^{k+1}) \leq i < 3^{k+2}$. Hence,

- $T_1^i x_0 = [0]^{2^k}$ for all $0 \leq i < 3^{k+1}$;
- $T_1^i x_0 = [0]^{2^k}$ for all $3^{k+1} \leq i < 2(3^{k+1})$; and
- $T_1^i x_0 = [2]^{2^k}$ for all $2(3^{k+1}) \leq i < 3^{k+2}$.

Therefore, $x_{[0,2^k+1]}$ has period $p = 3^{k+2}$. With this, we conclude part 1 of the proposition.
Now, let $0 \leq j < 2^{k+1}$. Using the induction hypothesis we have that $T_1^i y_{[0,j]} = [0]$ for all $0 \leq i \leq 2(3^{k+1})$. Using Lemma 3.9 and the fact that $T_1$ commutes with the shift, we obtain that

$$T_1^{2(3^{k+1})+i} y_{[0,j]} = T_1^i x_{[2^{k+1} - j, 2^{k+1}]}$$

for all $0 \leq i \leq 3^{k+1}$. Since $x$ has period $3^{k+1}$ we have that

$$T_1^{3^{k+1}} x_{[2^{k+1} - j, 2^{k+1}]} = [0]$$

Therefore, $y_{[0,2^{k+1}+j]}$ has period $2(3^{k+1}) + 3^{k+1} = 3^{k+2}$. □

**Proposition 3.11**: If $x \in A_1^Z$ is such that $x_j = \Box$ for some $j \in \mathbb{Z}$, then for all $i \in \mathbb{N}$ we have $T_1^n x_{j-i} \in \{\Box, 2\}$ for infinitely many $n > 0$.

**Proof**: We will prove this result by induction on $i$. The result follows for $i = 0$ using straightforward applications of the rules of the automaton (as in Lemma 3.4). Assume the result holds for $i > 0$. We may assume that $x_{j-i} \neq \Box$ and $x_{j-i-1} \neq \Box$ otherwise, the result is straightforward. Hence, we have that $\{n > 0 : T_1^n x_{j-i} = 2\}$ is infinite. Furthermore, by the rules of the automaton for every $n' \in \{n > 0 : T_1^n x_{j-i} = 2\}$ we have that $T_1^{n'+1} x_{j-i-1} = (T_1^{n'} x_{j-i-1} + 1) \mod 3$. With this we can conclude the result. □

Now, we will combine $T_1$ with the shift map. Let $A_2 = \{\Box, 2\}$, and $\sigma : A_2^Z \to A_2^Z$ the shift map.

Let $A = A_1 \times A_2$. At times we will identify $A$ with the following set:

$$A = \{\Box, 0, 1, 2, \Box, 0, 1, 2\}.$$ 

Note that with this notation, we identify the point $\Box \times \Box \in A$ with $2$. In general it will be clear which $\Box$ we are referring to. Let $\gamma_1 : A \to A_1$ and $\gamma_2 : A \to A_2$ be the projection functions. We also extend such functions to $A^Z \to A_1^Z$ and $A^Z \to A_2^Z$, respectively.

We define the CA $T : A^Z \to A^Z$ locally (and coordinate-wise) as the only CA that satisfying

$$(\gamma_1(Tx))_i = (T_1 \gamma_1(x))_i$$

and

$$(\gamma_2(Tx))_i = \begin{cases} (\sigma \gamma_2(x))_i & \text{if } \{\Box, 2\} \cap \{x_i, x_{i+1}\} \neq \emptyset, \\ (\gamma_2(x))_i & \text{otherwise.} \end{cases}$$

In other words, on the first coordinate $T$ acts exactly as $T_1$; on the second coordinate, an arrow advances to the left if and only if the first coordinate is a $\Box$ or a $2$. In case two arrows overlap they superimpose each other (i.e. the CA is not conservative).

In the introduction, we said this construction was a ‘local’ skew product. On a skew product, the phase space is the product, one coordinate acts normally and the second acts only if the first coordinate is in a certain position. The main difference is that here the shift only acts locally.
Though the next table is not needed for the proofs we provide it in case it is of assistance to the reader.

\[
Tx_i = \begin{cases} 
\Box & \text{if } x_i \in \{ \Box, \blacksquare \} \land x_{i+1} \in A \setminus \{ \Box, \blacksquare \}; \\
\blacksquare & \text{if } x_i \in \{ \Box, \blacksquare \} \land x_{i+1} \in \{ \Box, \blacksquare \}; \\
0 & \text{if } (x_i \in \{ 2, \blacksquare \} \land x_{i+1} \in \{ \Box, \blacksquare \}) \lor \\
& (x_i = 0 \land x_{i+1} \in A \setminus \{ \Box, \blacksquare, \blacksquare \}); \\
1 & \text{if } (x_i = 0 \land x_{i+1} \in \{ \Box, \blacksquare \}) \lor \\
& (x_i = 1 \land x_{i+1} \in A \setminus \{ \Box, \blacksquare, \blacksquare \}); \\
2 & \text{if } (x_i = 1 \land x_{i+1} \in \{ \Box, \blacksquare \}) \lor \\
& (x_i = 0 \land x_{i+1} \in \{ \Box, \blacksquare, \blacksquare \}) \lor \\
& (x_i = 1 \land x_{i+1} \in A \setminus \{ \Box, \blacksquare, \blacksquare \}); \\
0 & \text{if } (x_i \in \{ 2, \blacksquare \} \land x_{i+1} \in \{ \Box, \blacksquare \}) \lor \\
& (x_i = 0 \land x_{i+1} \in A \setminus \{ \Box, \blacksquare, \blacksquare \}); \\
1 & \text{if } (x_i = 0 \land x_{i+1} \in \{ \Box, \blacksquare \}) \lor \\
& (x_i = 1 \land x_{i+1} \in \{ \Box, \blacksquare, \blacksquare \}) \lor \\
& (x_i = 1 \land x_{i+1} \in A \setminus \{ \Box, \blacksquare, \blacksquare \}); \\
2 & \text{if } (x_i = 1 \land x_{i+1} \in \{ \Box, \blacksquare \}) \lor \\
& (x_i = 0 \land x_{i+1} \in \{ \Box, \blacksquare, \blacksquare \}) \lor \\
& (x_i = 1 \land x_{i+1} \in A \setminus \{ \Box, \blacksquare, \blacksquare \}).
\end{cases}
\]

When we write \( x := \infty \Box 0 \Box \infty \), we mean the point \( x \) such that \( x_0 = \Box, x_{-i} = \Box \) and \( x_i = \blacksquare \) for every \( i \neq 0 \).

**Example 3.12:** Let \( x \in A^\infty \) such that \( x = \infty \Box 0000 \Box \infty \). We have that

\[
\begin{array}{cccccccc}
T^0x_{[0,6]} &=& \Box & 0 & 0 & 0 & 0 & 0 \\
T^1x_{[0,6]} &=& \Box & 0 & 0 & 0 & 1 & 0 \\
T^2x_{[0,6]} &=& \Box & 0 & 0 & 0 & 2 & 0 \\
T^3x_{[0,6]} &=& \Box & 0 & 0 & 1 & 0 & 0 \\
T^4x_{[0,6]} &=& \Box & 0 & 0 & 1 & 0 & 0 \\
T^5x_{[0,6]} &=& \Box & 0 & 0 & 1 & 0 & 0 \\
T^6x_{[0,6]} &=& \Box & 0 & 0 & 1 & 0 & 0 \\
T^7x_{[0,6]} &=& \Box & 0 & 0 & 1 & 0 & 0 \\
T^8x_{[0,6]} &=& \Box & 0 & 0 & 1 & 0 & 0 \\
T^9x_{[0,6]} &=& \Box & 0 & 0 & 1 & 0 & 0 \\
T^{10}x_{[0,6]} &=& \Box & 0 & 0 & 1 & 0 & 0 \\
T^{11}x_{[0,6]} &=& \Box & 0 & 0 & 1 & 0 & 0 \\
T^{12}x_{[0,6]} &=& \Box & 0 & 0 & 1 & 0 & 0 \\
T^{13}x_{[0,6]} &=& \Box & 0 & 0 & 1 & 0 & 0 \\
\end{array}
\]
Example 3.13: Let \( x \in \mathbb{A}^\mathbb{Z} \) such that \( x = \infty \square 0 0 0 \square \infty \). We have that

\[
\begin{align*}
T^0 x_{[0,6]} &= 0 0 0 0 0 0 \quad T^{14} x_{[0,6]} &= 2 2 2 2 2 2 \\
T^1 x_{[0,6]} &= 1 0 0 1 0 1 \quad T^{15} x_{[0,6]} &= 0 0 0 0 2 0 \quad T^{16} x_{[0,6]} &= 1 1 0 0 0 1 \\
T^2 x_{[0,6]} &= 2 0 0 0 1 0 \quad T^{17} x_{[0,6]} &= 0 0 0 0 0 2 \\
T^3 x_{[0,6]} &= 0 0 1 0 0 0 \quad T^{18} x_{[0,6]} &= 0 0 0 0 0 1 \\
T^4 x_{[0,6]} &= 1 0 1 0 1 0 \quad T^{19} x_{[0,6]} &= 0 0 0 0 0 1 \quad T^{20} x_{[0,6]} &= 2 2 0 0 0 1 \\
T^5 x_{[0,6]} &= 2 0 2 0 2 0 \quad T^{21} x_{[0,6]} &= 0 0 0 0 0 1 \\
T^6 x_{[0,6]} &= 2 0 0 0 2 0 \quad T^{22} x_{[0,6]} &= 0 0 0 0 0 1 \\
T^7 x_{[0,6]} &= 0 0 1 0 1 0 \quad T^{23} x_{[0,6]} &= 2 2 0 0 0 1 \\
T^8 x_{[0,6]} &= 2 0 2 0 2 0 \quad T^{24} x_{[0,6]} &= 0 0 0 0 0 1 \\
T^9 x_{[0,6]} &= 0 0 0 0 0 2 \quad T^{25} x_{[0,6]} &= 0 0 0 0 0 1 \\
T^{10} x_{[0,6]} &= 1 0 1 0 1 0 \quad T^{26} x_{[0,6]} &= 2 2 0 0 0 1 \\
T^{11} x_{[0,6]} &= 0 0 1 0 1 0 \\
T^{12} x_{[0,6]} &= 0 0 1 0 1 0 \\
T^{13} x_{[0,6]} &= 0 0 1 0 1 0
\end{align*}
\]

Lemma 3.14: Let \( x \in \mathbb{A}^\mathbb{Z} \) such that \( x_{[0,2]} = \square 0 0 \square \). Then, \( T^n x_0 = \square \) for all \( n \neq 3k \), where \( k \geq 1 \).

Proof: From Lemma 3.4 we have that \( \gamma_1 (T^n x_1) = \square 2 \) if and only if \( n = 3k-1 \), where \( k \geq 1 \). Therefore, \( T^n x_0 = \square \) for all \( n \neq 3k \), where \( k \geq 1 \).

Lemma 3.15: Let \( k, l \geq 0 \), and \( x \in \mathbb{A}^\mathbb{Z} \) such that \( x_{[0,2^{l+1}]} = \square 0 0 \square \). If \( n \neq k(3^{l+1}) + 2(3^l) + 1 \) then \( T^n x_0 = \square \).

Proof: We will use the following two properties that are easily checked for any \( y \in X \) with \( y_0 = \square \).

- If \( \gamma_1 (y_1) \notin \{\square, \square 2\} \) then \( \gamma_2 (Ty_0) = \square \).
- If \( \gamma_1 (y_2) \notin \{\square, \square 2\} \) and \( \gamma_1 (y_1) = \square 2 \), then \( \gamma_2 (T^2 y_0) = \square \).

Now, by Proposition 3.10 we have that \( \gamma_1 (x_1) \) has period \( 3^{l+1} \) (for \( T^1 \)) and

- \( \gamma_1 (T^i x_1) = \square 0 \) for all \( 0 \leq i < 3^l \);
- \( \gamma_1 (T^i x_1) = \square 0 \) for all \( 3^l \leq i < 2(3^l) \);
- \( \gamma_1 (T^i x_1) = \square 2 \) for all \( 2(3^l) \leq i < 3^{l+1} \).

This implies that \( \gamma_1 (T^i x_2) \neq \square 2 \) for all \( 2(3^l) \leq i < 3^{l+1} \) (otherwise value on coordinate 1 would change).

Using the previous properties, we conclude that \( T^i x_0 = \square \) for all \( 2(3^l) + 2 \leq i \leq 3^{l+1} \).

Actually, using the periodicity of \( \gamma_1 (x_{[0,2^{l+1}]}) \) (Proposition 3.10) we conclude that \( T^n x_0 = \square \) for all \( n \neq k(3^{l+1}) + 2(3^l) + 1 \), for some \( k \geq 0 \).
Lemma 3.16: Let \( x \in A^\mathbb{Z} \) and \( j \in \mathbb{Z} \) such that \( \gamma_2(x_i) = \square \) for all \( i \geq j \) and \( \gamma_1(x_j) = \square \). For every \( k \geq j \) there exists \( N > 0 \) such that \( \gamma_2(T^n x_k) = \square \) for every \( n \geq N \).

Proof: Using Proposition 3.11, we can see that every arrow on a position \( k \geq j \) will eventually move to the left (since they move when \( \gamma_1 \) is \( \square \) or \( \[\) ). For every \( k \geq j \) we have that \( \gamma_2(x_n) = \square \) for only finitely many \( n > k \). Since arrows only move to the left one can conclude the result. ■

The following statement will be used to show that \((A^\mathbb{Z}, T)\) is not almost equicontinuous.

Proposition 3.17: Let \( m > 0 \) and \( w \in A^{2m+1} \). There exists \( x, y \in A^\mathbb{Z} \) with \( x|_{[-m, m]} = w = y|_{[-m, m]} \) such that \( T^n x_0 \neq T^n y_0 \) for some \( n > 0 \).

Proof: Let \( x, y \) as in the hypothesis and with \( x_i = \square \) and \( y_i = \square \), for every \( i \in \mathbb{Z} \) such that \( |i| > m \). By Lemma 3.16 we have that there exists \( N > 0 \) such that for all \( n \geq N \) we have that \( \gamma_2(T^n x_i) = \square \) for all \( i \in [-m, m] \). Using that \( \gamma_i = \square \) for all \( i > m \) and Proposition 3.11 we obtain that \( \{n \in \mathbb{N} : \gamma_2(T^n y_0) = \square \} \) is infinite. Hence, there exists \( N' \in \{n \in \mathbb{N} : \gamma_2(T^n y_0) = \square \} \) such that \( N' > N \) and \( T^{N'} x_0 \neq T^{N'} y_0 \). ■

In the following lemma, we will use the sets \( S_{[-i,i]}(x, m) \) from Definition 2.8.

Remark 3.18: Let \( m > 0 \) and \( x \in A^\mathbb{Z} \) so that \( \gamma_2(x_i) = \square \) for all \( i \in \mathbb{Z} \) and \( \gamma_1(x_m) = \square \). Then,

\[
S_{[i]}(x, m) = \{i \in \mathbb{N} : \exists y \in B_m(x), \ \gamma_2(T^j y_j) = \square \}
\]

for all \( |j| \leq m \).

Lemma 3.19: Let \( l \geq 0 \) and \( x \in A^\mathbb{Z} \) such that \( x|_{[0, 2^l+1]} = \square \ 0 \ 0 \ 0 \ 0 \) Then

\[
\overline{D}(S_{[0]}(x, 2^l + 1)) \leq \frac{1}{3^{l+1}}.
\]

Proof: By Lemma 3.15 we have that

\[
S_{[0]}(x, 2^l + 1) \subseteq \{i \in \mathbb{N} : i = 3^{l+1}k + 2(3^l) + 1 \ \forall \ k \geq 0 \}.
\]

Therefore,

\[
\overline{D}(S_{[0]}(x, 2^l + 1)) \leq \overline{D}(\{i \in \mathbb{N} : i = 3^{l+1}k + 2(3^l) + 1 \ \forall \ k \geq 0 \})
\]

\[
= \limsup_{k \to \infty} \frac{k + 1}{3^{l+1}k + 2(3^l) + 1} = \frac{1}{3^{l+1}}.
\]

Lemma 3.20: Let \( l \geq 0, m = 2^l + 2^{l-1} + 2 \) and \( x \in A^\mathbb{Z} \) with \( x|_{[0, m]} = \square \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \) Then

\[
\overline{D}(S_{[i]}(x, m)) \leq (2(3^{l-1}) + 1)\overline{D}(S_{[2^{l-1}+1]}(x, m))
\]

and

\[
\overline{D}(S_{[0]}(x, m)) = \overline{D}(S_{[2^{l-1}+1]}(x, m))
\]

for all \( i \in [1, 2^{l-1}] \).
**Proof:** By Lemma 3.10 we have that

- \( \gamma_1(T^n x_1) = 0 \) for all \( 0 \leq n < 3^{l-1} \);
- \( \gamma_1(T^n x_1) = 0 \) for all \( 3^{l-1} \leq n < 2(3^{l-1}) \);
- \( \gamma_1(T^n x_1) = 2 \) for all \( 2(3^{l-1}) \leq n < 3^l \).

Let \( 1 \leq i \leq 2^{l-1} \). Any arrow on this region will not move for at most \( 2(3^{l-1}) \) iterations, that is for any \( y \in B_{2^{-m}}(x) \) we have that \( \gamma_2(T^n y_i) = \square \) for at most \( 2(3^{l-1}) \) consecutive \( n \). Hence, by Remark 3.18, we can conclude that

\[
\sharp(S_{[i]}(x, m) \cap [0, (k + 1)3^{l+1}]) \leq (2(3^{l}) + 1)\sharp(S_{[2^{l-1}+1]}(x, m) \cap [0, (k + 1)3^{l+1}])
\]

for all \( k \geq 0 \). Therefore,

\[
\overline{D}(S_{[i]}(x, m)) \leq (2(3^{l-1}) + 1)\overline{D}(S_{[2^{l-1}+1]}(x, m)).
\]

Now, for \( y \in B_{2^{-m}}(x) \), one can check that \( \gamma_2(T^n x_{2^{l-1}+1}) = \square \) if and only if \( \gamma_2(T^n x_{2^{l-1}+1}) = \square \) (see Proposition 3.10 and Lemma 3.14). Using this and Remark 3.18, we obtain that

\[
S_{[0]}(x, m) = S_{[2^{l-1}+1]}(x, m) + 2(3^{l-1}).
\]

Therefore,

\[
\overline{D}(S_{[0]}(x, m)) = \overline{D}(S_{[2^{l-1}+1]}(x, m)).
\]

**Proposition 3.21:** Let \( k > 0, w \in A^k \) and

\[
x := \infty \square w \square 0 \square 0 \square 0 \square 2^2 \square 0 \square 2^n \cdots.
\]

We have that \( x \) is a diam-mean equicontinuity point.

**Proof:** We will prove that \( x \) is a diam-mean equicontinuity point with the use of Proposition 2.9. Let \( m \geq 0 \). First notice that, without loss of generality, we may assume that \( \gamma_2(w_i) = \square \) for every \( 1 \leq i \leq k \) (from Lemma 3.16 there exists \( M > 0 \) such that \( \gamma_2(T^M x_i) = \square \) for all \( 0 \leq i < k \)). Let \( l > 0 \) so that \( k < 2^l \) and

\[
\frac{2(3^{l-1} + 1)}{3^{l+1}} \leq \frac{1}{2^{m+2}}.
\]

Let \( k \leq j \leq k + l + \sum_{i=0}^{l-1} 2^i \). By applying Lemma 3.20 recurrently (and using that \( k < 2^l \)), we have that

\[
\overline{D}\left(S_{[j]}(x, k + l + \sum_{i=0}^{l-1} 2^i)\right) \leq 2(3^{l-1} + 1)\overline{D}\left(S_{[k+l+\sum_{i=0}^{l-1} 2^i]}(x, k + l + \sum_{i=0}^{l-1} 2^i)\right).
\]

Recall, by Remark 3.3, that \( \gamma_1(T^n x_{[0,k]}) = \gamma_1(T^n y_{[0,k]}) \) for all \( y \in B_{k+l+\sum_{i=0}^{l-1} 2^i}(x) \) and all \( n \geq 0 \). Hence, since \( k < 2^l \) we can conclude all \( j \leq k + l + \sum_{i=0}^{l-1} 2^i \) satisfies Equation (2).
By Lemma 3.19 and the choice of \( l \) we obtain that

\[
\overline{D} \left( S_{\lfloor j \rfloor} \left( x, k + l + \sum_{i=0}^{l} 2^i \right) \right) \leq \frac{2(3^{l-1} + 1)}{3^{l+1}} \leq \frac{1}{2m+2}
\]

for all \( j \leq k + l + \sum_{i=0}^{l-1} 2^i \).

It is not hard to check that

\[
\overline{D} \left( S_{\lfloor -j \rfloor} \left( x, k + l + \sum_{i=0}^{l} 2^i \right) \right) \leq \overline{D} \left( S_{\lfloor 0 \rfloor} \left( x, k + l + \sum_{i=0}^{l} 2^i \right) \right)
\]

for every \( j > 0 \).

Therefore, by Proposition 2.9, we have that \( x \) is a diam-mean equicontinuity point.

**Theorem 3.22:** The CA \((A^Z, T)\) is almost diam-mean equicontinuous but not almost equicontinuous.

**Proof:** From Proposition 3.21, we have \( EQ^d \) is dense. Hence, by Proposition 2.3, \( EQ^d \) is residual. So, \((A^Z, T)\) is almost diam-mean equicontinuous. By Proposition 3.17, there are no equicontinuity points. Therefore, \((A^Z, T)\) is not almost equicontinuous.

**4. Diam-mean sensitivity**

Actually the previous result holds as long as a TDS, \((X, T)\), is sensitive if there exists \( \varepsilon > 0 \) such that for every non-empty open set \( U \subseteq X \) there exist \( x, y \in U \) and \( n > 0 \) such that

\[
d(T^n x, T^n y) > \varepsilon.
\]

Diam-mean sensitivity was introduced in [9].

**Definition 4.1:** Let \((X, T)\) be a TDS. We say that \((X, T)\) is diam-mean sensitive if there exists \( \varepsilon > 0 \) such that for every open set \( U \) we have

\[
\limsup_{n \to \infty} \frac{\sum_{i=1}^{n} \diam(T^i U)}{n} > \varepsilon.
\]

We say a TDS is transitive if for every pair of non-empty open sets \( U \) and \( V \) there exists \( n > 0 \) such that \( T^{-n} U \cap V \neq \emptyset \).

**Theorem 4.2:** Let \((X, T)\) be a transitive TDS.

- \((X, T)\) is either almost equicontinuous or sensitive [15]; and
- \((X, T)\) is either almost diam-mean equicontinuous or diam-mean sensitive [8].

For other related dichotomies see [13].
The first part of the previous theorem holds for non-necessarily transitive CA.

**Theorem 4.3 ([15]):** Let \((X, T)\) be a CA. Then, \((X, T)\) is either almost equicontinuous or sensitive.

Sensitivity on CA has been studied in several papers (e.g. [7, 11, 12]). We find it natural to ask if the previous dichotomy holds for the diam-mean notions on CA. We will show this has a false answer.

Let \(T : A^\mathbb{Z} \to A^\mathbb{Z}\) the CA from the previous subsection and \(A_3 = \{a, b, c\}\). We define \(T_3 : A_3^\mathbb{Z} \to A_3^\mathbb{Z}\) as

\[
T_3x_i = \begin{cases} 
  a & \text{if } x_i = a; \\
  b & \text{if } x_i = c; \\
  c & \text{if } x_i = b.
\end{cases}
\]

We set \(A_S := A \times A_3\). For every \(x \in A_S\), we have that \(x^1\) is the component on \(A\) and \(x^2\) is the component on \(A_3\). Let \(id : A_3^\mathbb{Z} \to A_3^\mathbb{Z}\) be the identity function and \(T_S : A_S^\mathbb{Z} \to A_S^\mathbb{Z}\) a CA defined locally with

\[
T_Sx_i = \begin{cases} 
  (Tx_i^1, id(x_i^2)) & \text{if } \gamma_2(x_i^1) = \square; \\
  (Tx_i^1, T_3x_i^2) & \text{otherwise.}
\end{cases}
\]

Thus, on \(A\), \(T_S\) behaves exactly as \(T\), and on \(A_3\), as \(T_3\) except if there is an arrow on the first coordinate. When this happens, the periodicity on \(b\) and \(c\) changes phase.

In the next lemma, we will establish that the set of diam-mean equicontinuity points is not dense.

**Lemma 4.4:** Let \(m > 0\) and \(w \in A_S^m\) such that \(w_0 = (\square, b)\). Then, there exist \(x, y \in A_S^\mathbb{Z}\) such that

\[
x[0,|w|−1] = y[0,|w|−1] = w
\]

and the set

\[
\mathbb{Z}_{n \geq 0} \setminus \{n \in \mathbb{Z}_{n \geq 0} : T^n_Sx_0 \neq T^nSy_0\}
\]

is finite.

**Proof:** Let \(w \in A_S^m\) such that \(w_0 = (\square, b)\). Let us define

\[
x = \infty (\square, a).w(\square, a)(\square, a)\infty
\]

and

\[
y = \infty (\square, a).w(\square, a)\infty.
\]

Using Lemma 3.16 we can assume, without loss of generality (by waiting until the arrows are gone), that

\[
w_i \in \{(p, q) : p \in (\square, 0, 1, 2) \land q \in A_3\}.
\]

Now, there exists \(N > 0\) such that \(T^n_Sx_0 = (\square, q)\), where \(q \in \{b, c\}\). We have two cases to prove.
Case 1: $T^N_S x_0 = (b, b)$.

This implies that $T^{N+1}_S x_0 = (b, b)$. Meanwhile, $T^{N+1}_S y_0 = (c, c)$. Therefore, we can easily see that $T^{N+i}_S x_0 \neq T^{N+i}_S y_0$, for all $i > 0$.

Case 2: $T^N_S x_0 = (b, c)$.

Again we have that $T^{N+1}_S x_0 = (b, c)$, so $T^{N+i}_S x_0 \neq T^{N+i}_S y_0$ for all $i \geq 0$.

Notice that for all $\varepsilon > 0$, any $y \in B_\varepsilon(x)$, where $x_0 = (b, b)$, is not a diam-mean equicontinuity point.

**Theorem 4.5:** $(A^Z_S, T_S)$ is neither diam-mean sensitive and nor almost diam-mean equicontinuous.

**Proof:** From Lemma 4.4, we conclude that for every $x \in A^Z_S$ such that $x_0 = (b, b)$, we have that $x$ is not a diam-mean equicontinuity point.

From Proposition 3.21, we have that

$$x := \infty(\square, a)(\square, a)(\square, a)(\square, a)(\square, a)^2(\square, a)(\square, a)^2 \cdots (\square, a)(\square, a)^2 \cdots$$

is a diam-mean equicontinuity point. Hence, for all $\varepsilon > 0$ there exists $l \geq 0$ such that

$$\limsup_{n \to \infty} \frac{\sum_{j=0}^{n} \text{diam}(T^j_S(B_{l+\sum_{i=0}^{l} 2^i(x)}))}{n+1} < \varepsilon.$$

Therefore, $(A^Z_S, T_S)$ is not diam-mean sensitive.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

**References**

[1] M. Baake and U. Grimm, *Aperiodic Order*, Vol. 1, Cambridge University Press, Cambridge, 2013.

[2] L.D.I.S. Baños and F. García-Ramos, Mean equicontinuity and mean sensitivity on cellular automata, Ergodic Theory Dynam. Systems 41(12) (2021), pp. 3704–3721.

[3] L.D.I.S. Baños and F. García-Ramos, Diameter mean equicontinuity and cellular automata. Proceedings of the 27th International Workshop on Cellular Automata and Discrete Complex Systems, arXiv:2106.09641, 2021.

[4] T. Ceccherini-Silberstein and M. Coornaert, *Cellular Automata and Groups*, Springer Science & Business Media, New York City, 2010.

[5] S. Fomin, On dynamical systems with pure point spectrum, Dokl. Akad. Nauk SSSR 77(4) (1951), pp. 29–32 (in Russian).

[6] G. Fuhrmann, M. Gröger, and D. Lenz, The structure of mean equicontinuous group actions, Israel J. Math. 247(1) (2022), pp. 75–123.

[7] F. García-Ramos, Limit behaviour of $\mu$-equicontinuous cellular automata, Theor. Comput. Sci. 623 (2016), pp. 2–14.
[8] F. García-Ramos, Weak forms of topological and measure-theoretical equicontinuity: Relationships with discrete spectrum and sequence entropy, Ergodic Theory Dynam. Systems 37(4) (2017), pp. 1211–1237.

[9] F. García-Ramos, T. Jäger, and X. Ye, Mean equicontinuity, almost automorphy and regularity, Israel J. Math. 243(1) (2021), pp. 155–183.

[10] F. García-Ramos and B. Marcus, Mean sensitive, mean equicontinuous and almost periodic functions for dynamical systems, Discrete Contin. Dyn. Syst.-A 39(2) (2019), pp. 729.

[11] R.H. Gilman, Classes of linear automata, Ergodic Theory Dynam. Systems 7(1) (1987), pp. 105–118.

[12] J. Guckenheimer, Sensitive dependence to initial conditions for one dimensional maps, Comm. Math. Phys. 70(2) (1979), pp. 133–160.

[13] W. Huang, S. Kolyada, and G. Zhang, Analogues of Auslander–Yorke theorems for multi-sensitivity, Ergodic Theory Dynam. Systems 38(2) (2018), pp. 651–665.

[14] W. Huang, J. Li, J.P. Thouvenot, L. Xu, and X. Ye, Bounded complexity, mean equicontinuity and discrete spectrum, Ergodic Theory Dynam. Systems 41(2) (2021), pp. 494–533.

[15] P. Kůrka, Languages, equicontinuity and attractors in cellular automata, Ergodic Theory Dynam. Systems 17(2) (1997), pp. 417–433.

[16] P. Kůrka, Topological and Symbolic Dynamics, Vol. Cours Specials 11, SMF, Paris, 2003.

[17] J. Li, S. Tu, and X. Ye, Mean equicontinuity and mean sensitivity, Ergodic Theory Dynam. Systems 35(8) (2015), pp. 2587–2612.

[18] J. Li, X. Ye, and T. Yu, Mean equicontinuity, complexity and applications, Discrete Contin. Dyn. Syst. 41(1) (2021), pp. 359–393.

[19] M. Morse and G.A. Hedlund, Symbolic dynamics II. Sturmian trajectories, Amer. J. Math. 62(1) (1940), pp. 1–42.

[20] J.C. Oxtoby, Ergodic sets, Bull. Amer. Math. Soc. 58(2) (1952), pp. 116–136.

[21] T. Toffoli, Cellular automata as an alternative to (rather than an approximation of) differential equations in modeling physics, Phys. D 10(1–2) (1984), pp. 117–127.

[22] T. Toffoli and N. Margolus, Cellular Automata Machines: A New Environment for Modeling, MIT Press, Cambridge, 1987.

[23] I. Törmä, A uniquely ergodic cellular automaton, J. Comput. Syst. Sci. 81(2) (2015), pp. 415–442.