Berry phase and entanglement of 3 qubits in a new Yang-Baxter system

Taotao Hu
Department of Physics, Northeast Normal University, Changchun, Jilin 130024,
People’s Republic of China

Kang Xue
E-mail: Xuekang@nenu.edu.cn
Department of Physics, Northeast Normal University, Changchun, Jilin 130024,
People’s Republic of China

Chunfeng Wu
Department of Physics, National University of Singapore, 2 Science Drive 3,
Singapore 117542
Centre for Quantum Technologies, National University of Singapore, 3 Science
Drive 2, Singapore 117543

Abstract. In this paper we construct a new $8 \times 8$ matrix from the $4 \times 4$ matrix, where the $M$ matrix is the image of the braid group representation. The $8 \times 8$ matrix and the $4 \times 4$ matrix both satisfy extraspecial 2-groups algebra relations. By Yang-Baxteration approach, we derive a unitary $\hat{R}(\theta, \phi)$ matrix from the $M$ matrix with parameters $\varphi$ and $\theta$. Three-qubit entangled states can be generated by using the $\hat{R}(\theta, \varphi)$ matrix. A Hamiltonian for 3 qubits is constructed from the unitary $\hat{R}(\theta, \varphi)$ matrix. We then study the entanglement and Berry phase of the Yang-Baxter system.

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1. Introduction

Recently, it has been revealed that there are natural and profound connections between quantum information theory and braid group theory [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In a recent paper by Franko, Rowell and Wang [5], the images of the unitary braid group representations generated by the original 4 × 4 Bell matrix have been identified as extensions of extraspecial 2-groups [9, 10]. Extraspecial 2-groups are now known to play an interesting role in theory of quantum information, particularly in the theory of quantum error correction. They provide a bridge between quantum error correcting codes and binary orthogonal geometry [11]. Moreover, they form a subgroup of the pauli group, which is of importance in the theory of stabilized code [12, 13]. In Ref. [5], the authors utilized extraspecial 2-groups to study the images of braid group \( B_n \) under the representation associated with the 4 × 4 Bell matrix. It is found that the extraspecial 2-groups are the central link between almost-complex structures and unitary braid representations. New higher dimensional unitary braid group representations can be constructed by considering an extension of the extraspecial 2-groups [9].

Many efforts have been devoted to investigate the braiding operators and Yang-Baxter equation (YBE) to the field of quantum information and quantum computation. This provides a novel way to study the quantum entanglement [14, 15, 16, 17]. It is shown that braiding transformation is a natural approach describing quantum entanglement by applying the unitary braiding operators to realize entanglement swapping and generate the GHZ states as well as the linear cluster states. Unitary solutions of braid group relation [1] [2] and the YBE [3, 4] can be identified with universal quantum gates [18]. The investigation of many types of Yang-Baxter systems have attracted researchers’ attention. Usually, a Hamiltonian can be constructed from the unitary \( \hat{R}(\theta, \phi) \) matrix by Yang-Baxterization approach. Yang-Baxterization [19, 20] is exploited [3, 4] to derive the Hamiltonian for the unitary evolution of entangled states.

In this paper, we present a new 8 × 8 \( M \) matrix from the 4 × 4 \( M \) matrix, where \( M/M \) is the image of braid group representation. In Sec 2, we briefly review the Yang-Baxter systems of 4 × 4 \( M \) matrix. Then we construct a new 8 × 8 \( M \) matrix, which satisfies the extraspecial 2-groups algebra relation, from the 4 × 4 \( M \) matrix. By using Yang-Baxterization approach, we derive a unitary \( \hat{R}(\theta, \phi) \) matrix based on the 8 × 8 \( M \)-matrix. We analyze the effect of \( \hat{R}(\theta, \phi) \) matrix on the entanglement of three qubits. In Sec 3, we construct a Hamiltonian of 3 qubits from the unitary \( \hat{R}(\theta, \phi) \) matrix and investigate the entanglement and Berry phase of this system. The results are summarized in the last section.

2. a new 8 × 8 \( M \) matrix and its Yang-Baxterization in 3-qubits system

For 4 × 4 Yang-Baxter systems of 2 qubits, we know that the rational solution of the YBE can be expressed as \( \hat{R}_{i,i+1}(\theta, \phi) = \sin \theta I_i \otimes I_{i+1} + \cos \theta M_{i,i+1} \) [14], where \( \theta \) is the spectral parameter and \( M_{i,i+1} \) is an operator of braid group which can be expressed in the form of spin operators \( S_i^+ = S_i^1 + iS_i^2 \) and \( S_i^- = S_i^1 - iS_i^2 \) as follows

\[
M_{i,i+1} = e^{-i\phi}S_i^-S_{i+1}^+ - e^{i\phi}S_i^+S_{i+1}^- + S_i^+S_{i+1}^+ - S_i^-S_{i+1}^-.
\]  

(1)

The operator \( M_{i,i+1} \) satisfies the extraspecial 2-groups relations [10]:

\[
M_{i,i+1}^2 = \alpha I
\]
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\[ M_{i,i+1} M_{i+1,i+2} M_{i,i+1} = M_{i+1,i+2} \]
\[ M_{i+1,i+2} M_{i,i+1} M_{i+1,i+2} = M_{i,i+1}. \]

(2)

In terms of the standard basis \{\{00\}, \{01\}, \{10\}, \{11\}\} of 2 qubits, the \( M_{i,i+1} \) matrix is of the following form:

\[
M_{i,i+1} = \begin{pmatrix}
0 & 0 & 0 & e^{-i\varphi} \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-e^{i\varphi} & 0 & 0 & 1
\end{pmatrix}.
\]

(3)

When the unitary matrix \( \hat{R}_{i,i+1}(\theta, \varphi) = \sin \theta I + i \cos \theta M_{i,i+1} \) acts on the standard basis of 2 qubits, one gets four entangled states which possess the same entanglement degree of \(|\sin 2\theta| \ [14]\).

In the following, we construct a 8 × 8 matrix \( \mathcal{M} \) from this 4 × 4 matrix \( M \). In order to connect three qubits, we write \( \mathcal{M} \) by using \( M_{j,j+1} \) \((j = i, i + 1)\)

\[
\mathcal{M} = \frac{1}{\sqrt{3}}(M_{i,i+1} \otimes I_{i+2} + I_{i} \otimes M_{i+1,i+2} + M_{i+1,i+2} M_{i,i+1}).
\]

(4)

\( \mathcal{M} \) can be expressed by using spin operators:

\[
\mathcal{M} = \frac{1}{\sqrt{3}}[e^{-i\varphi}(S_{i}^{+} S_{i+1}^{-} + S_{i+1}^{+} S_{i}^{-}) - e^{i\varphi}(S_{i}^{-} S_{i+1}^{+} + S_{i+1}^{-} S_{i}^{+})]
\]

\[
+ (S_{i}^{+} S_{i+1}^{+} + S_{i+1}^{-} S_{i}^{-} - S_{i}^{-} S_{i+1}^{+} - S_{i+1}^{+} S_{i}^{-})
\]

\[
2S_{i+1}^{3}(e^{-i\varphi} S_{i}^{+} S_{i+1}^{-} - e^{i\varphi} S_{i}^{-} S_{i+1}^{+} + S_{i}^{+} S_{i+1}^{-} - S_{i}^{-} S_{i+1}^{+})
\]

(5)

Write in terms of the basis of three qubits \{\{000\}, \{001\}, \{010\}, \{011\}, \{100\}, \{101\}, \{110\}, \{111\}\}, the 8 × 8 matrix \( \mathcal{M} \) is

\[
\mathcal{M} = \frac{1}{\sqrt{3}}
\begin{pmatrix}
0 & 0 & 0 & e^{-i\varphi} & 0 & e^{-i\varphi} & 0 & e^{-i\varphi} \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & e^{-i\varphi} \\
0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\
-e^{i\varphi} & 0 & 0 & 0 & 0 & 1 & 0 & -e^{i\varphi} \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & e^{-i\varphi} \\
-e^{i\varphi} & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & -e^{i\varphi} & e^{i\varphi} & 0 & -e^{i\varphi} & 0 & 0 & 0
\end{pmatrix}.
\]

(6)

Introduce \( \mathcal{M} = -i \mathcal{M} \), it is not difficult to find that \( \mathcal{M} \) satisfies the following extraspecial 2-groups relations [10]:

\[
\mathcal{M}_{2}^{2} = \alpha I
\]

\[
\mathcal{M}_{12}^{4} \mathcal{M}_{12}^{4} \mathcal{M}_{23}^{\frac{3}{2}} \mathcal{M}_{23}^{\frac{3}{2}} = \mathcal{M}_{12}^{\frac{3}{2}}
\]

\[
\mathcal{M}_{12}^{\frac{3}{2}} \mathcal{M}_{23}^{\frac{3}{2}} \mathcal{M}_{23}^{\frac{3}{2}} = \mathcal{M}_{23}^{\frac{3}{2}}
\]

(7)

It is known that a unitary solution of the YBE can be found via Yang-Baxterization on the solution of the extraspecial 2-groups relations. The Yang-Baxterization of the extraspecial 2-group operator \( \mathcal{M} \) is [14]:

\[
\hat{R}(x) = \rho(x)(I + G(x) \mathcal{M}).
\]

(8)

where \( \rho(x) \) and \( G(x) \) are some functions of \( x \) to be determined, \( I = I_{i} \otimes I_{i+1} \otimes I_{i+2} \) is identity matrix. One can choose appropriate \( \rho(x) \) and \( G(x) \) so that \( \hat{R}(x) \) is unitary. The unitary \( \hat{R} \)-matrix satisfies the YBE which is of the form,

\[
\hat{R}_{12}(x) \hat{R}_{23}(xy) \hat{R}_{12}(y) = \hat{R}_{23}(y) \hat{R}_{12}(xy) \hat{R}_{23}(x),
\]

(9)
where multiplicative parameters $x$ and $y$ are known as the spectral parameters. In order to make $\hat{R}(x)$ a unitary matrix, $\hat{R}(x)$ should be equal to the inverse $\hat{R}^{-1}(x)$. In this way, we obtain that $\hat{R}(x) = \frac{z^{-1} + 1}{2} (I + \frac{z^{-1} - 1}{2} M)$. By introducing a new variable parameter $\theta$ as $\frac{z^{-1} - 1}{2} = i \cos \theta$ and $\frac{z^{-1} + 1}{2} = \sin \theta$, the matrix $\hat{R}(x)$ can be rewritten as $\hat{R}(\theta, \varphi) = \sin \theta I + i \cos \theta M = \sin \theta I + \cos \theta M$. In the following, we express $\hat{R}(\theta, \varphi)$ in the form of spin operators

\[
\hat{R}(\theta, \varphi) = \sin \theta I_i \otimes I_{i+1} \otimes I_{i+2} + \frac{1}{\sqrt{3}} \cos \theta (e^{-i\varphi} (S_i^+ S_{i+1}^+ + S_{i+1}^+ S_{i+2}^+) \nonumber \\
- e^{i\varphi} (S_i^- S_{i+1}^- + S_{i+1}^- S_{i+2}^-) + (S_i^+ S_{i+1}^- + S_{i+1}^+ S_{i+2}^- - S_i^- S_{i+1}^+ - S_{i+1}^- S_{i+2}^-) \nonumber \\
+ 2S_{i+1}^3 (e^{-i\varphi} S_i^+ S_{i+1}^+ - e^{i\varphi} S_i^- S_{i+1}^- + S_i^+ S_{i+2}^+ - S_i^- S_{i+2}^-)).
\]

When the unitary matrix $\hat{R}(\theta, \varphi)$ acts on the direct product states $|klm\rangle \equiv |k\rangle_i \otimes |l\rangle_{i+1} \otimes |m\rangle_{i+2}$, the $\hat{R}(\theta, \varphi)$ matrix transfers product states to entangled states

\[
|000\rangle \rightarrow \sin \theta |000\rangle - \frac{1}{\sqrt{3}} \cos \theta e^{i\varphi} |011\rangle - \frac{1}{\sqrt{3}} \cos \theta e^{i\varphi} |101\rangle - \frac{1}{\sqrt{3}} \cos \theta e^{i\varphi} |110\rangle, \\
|001\rangle \rightarrow \sin \theta |001\rangle - \frac{1}{\sqrt{3}} \cos \theta |010\rangle - \frac{1}{\sqrt{3}} \cos \theta |100\rangle - \frac{1}{\sqrt{3}} \cos \theta e^{i\varphi} |111\rangle, \\
|010\rangle \rightarrow \sin \theta |010\rangle + \frac{1}{\sqrt{3}} \cos \theta |001\rangle - \frac{1}{\sqrt{3}} \cos \theta |100\rangle + \frac{1}{\sqrt{3}} \cos \theta e^{i\varphi} |111\rangle, \\
|011\rangle \rightarrow \sin \theta |011\rangle + \frac{1}{\sqrt{3}} \cos \theta e^{-i\varphi} |000\rangle - \frac{1}{\sqrt{3}} \cos \theta |101\rangle + \frac{1}{\sqrt{3}} \cos \theta |110\rangle, \\
|100\rangle \rightarrow \sin \theta |100\rangle + \frac{1}{\sqrt{3}} \cos \theta |001\rangle + \frac{1}{\sqrt{3}} \cos \theta |010\rangle - \frac{1}{\sqrt{3}} \cos \theta e^{i\varphi} |111\rangle, \\
|101\rangle \rightarrow \sin \theta |101\rangle + \frac{1}{\sqrt{3}} \cos \theta e^{-i\varphi} |000\rangle + \frac{1}{\sqrt{3}} \cos \theta |011\rangle - \frac{1}{\sqrt{3}} \cos \theta |110\rangle, \\
|110\rangle \rightarrow \sin \theta |110\rangle + \frac{1}{\sqrt{3}} \cos \theta e^{-i\varphi} |001\rangle - \frac{1}{\sqrt{3}} \cos \theta e^{-i\varphi} |010\rangle + \frac{1}{\sqrt{3}} \cos \theta e^{-i\varphi} |100\rangle,
\]

(11)

For convenience we label the $i$-th, $(i + 1)$-th, and $(i + 2)$-th qubits by A, B, and C respectively. Tripartite entangled states can be measured by three-tangle (or residual tangle) $\tau_{ABC}$ proposed by Coffman, Kud, and Wootters [21]. $\tau_{ABC}$ is expressed by using the composing coefficients $a_{ijk}$ corresponding to the basis state $|ijk\rangle$,

\[
\tau = 4|d_1 - 2d_2 + 4d_3| \\
d_1 = a^2_{000} a^2_{111} + a^2_{001} a^2_{110} + a^2_{010} a^2_{101} + a^2_{100} a^2_{011} \\
d_2 = a_{000} a_{111} a_{010} a_{100} + a_{000} a_{111} a_{010} a_{010} + a_{000} a_{111} a_{110} a_{001} + a_{011} a_{000} a_{100} a_{010} + a_{011} a_{100} a_{100} a_{001} + a_{100} a_{010} a_{100} a_{001} \\
d_2 = a_{000} a_{110} a_{011} a_{001} + a_{011} a_{000} a_{001} a_{010} + a_{111} a_{000} a_{001} a_{010}
\]

(12)

The above eight 3-qubit entangled states are found to have the same entanglement degree as

\[
\tau_{ABC} = \frac{16 \sqrt{3} |\sin \theta \cos^3 \theta|}{9}.
\]

(13)
The concurrence [22] measuring bipartite entangled states is defined as
\[ C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \] (14)
where \(\{\lambda_i\}\) are the eigenvalues of the matrix \(\rho_{12}(\sigma^1_y \otimes \sigma^2_y)\rho_{12}^\dagger(\sigma^1_y \otimes \sigma^2_y)\), here \(\rho_{12}\) is the density matrix of the pair and it can be either pure or mixed with \(\rho_{12}^\dagger\) denoting the complex conjugate of \(\rho\) and \(\sigma^1_y/2\) are the Pauli matrices for atoms 1 and 2. This quantity attains its maximum value of 1 for maximally entangled states and vanishes for separable states.

For the 2-qubit subsystem of three qubits, we need consider the reduced density. i.e. \(\rho_{AB} = Tr_C\rho\), here \(\rho\) is the density of 3 qubits. It is pure and has the form \(\rho = |\psi\rangle\langle\psi|\). Though calculating the reduced density of the eight 3 qubits states we get in Eq (11), it is not difficult to obtain the corresponding concurrences as follows
\[ C_{AB} = C_{BC} = C_{AC} = \frac{1}{\sqrt{3}}|\sin 2\theta| - \frac{2}{3}\cos^2 \theta. \] (15)
It is worth noting that the eight 3 qubits state we get in Eq (11) all have the same pairwise concurrence as Eq (15). On the other hand, the entanglement between qubit A and the pair BC can be calculated [21] as
\[ C_{A(BC)}^2 = C_{BC}^2 + C_{AC}^2 + \tau_{ABC}. \] (16)
Thus we have
\[ C_{A(BC)}^2 = \frac{8}{9}\cos^2 \theta(1 + 2\sin^2 \theta). \] (17)
One can easily verify that \(C_{B(AC)}^2 = C_{(AB)}^2 = C_{A(BC)}^2\). So it is worth noting that by this \(8 \times 8\) \(\hat{R}(\theta, \varphi)\) acting on the product states of 3 qubits, one can also get eight tripartite entangled states with the same degree of entanglement. While for the \(4 \times 4\) Yang-Baxter system, they get the same entanglement degree of 2 qubits \(C_{t+1} = |\sin 2\theta|\). From Eqs (13), (15) and (17), one can see that when \(\theta = \pi/6\), \(C_{AB} = C_{BC} = C_{AC} = 0\) and \(C_{A(BC)}^2 = C_{C(AB)}^2 = C_{A(BC)}^2 = \tau_{ABC} = 1\). This case is just the GHZ state [21]. According to Eq. (13), when \(\cos \theta = 0\), the three-tangle \(\tau_{ABC}\) is equal to 0 and \(C_{AB} = C_{BC} = C_{AC} = 0\). This tells us that the states in Eq. (11) are separable in this case. When \(\sin \theta = 0\), \(C_{A(BC)}^2 = C_{BC}^2 + C_{AC}^2\) while \(\tau_{ABC} = 0\). This corresponds to the W state [21]. So one can see the \(8 \times 8\) \(\hat{R}(\theta, \varphi)\) generates arbitrary tripartite entangled states which can be achieved depending on the parameters \(\theta\).

3. Hamiltonian, Entanglement and Berry phase

Generally, multi-spin interaction Hamiltonians can be constructed based on the YBE. As \(\hat{R}\) is unitary, it can define the evolution of a state \(|\Psi(0)\rangle\)
\[ |\Psi(t)\rangle = \hat{R}_t(t)|\Psi(0)\rangle, \] (18)
here \(\hat{R}_t(t)\) is time-dependent, which can be realized by specifying corresponding time-dependent parameter of \(\hat{R}_t\). By taking partial derivative of the state \(|\Psi(t)\rangle\) with respect to time \(t\), we have an equation
\[ \frac{i\hbar}{\partial t}|\Psi(t)\rangle = i\hbar \left[ \frac{\partial}{\partial t}\hat{R}_t(t) \right]|\Psi(t)\rangle = H(t)|\Psi(t)\rangle, \] (19)
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where $H(t) = i\hbar \frac{\partial}{\partial t} \hat{R}(t)$ is the Hamiltonian governing the evolution of the state $|\Psi(t)\rangle$. Thus, the Hamiltonian $H(t)$ for the Yang-Baxter system is derived through the Yang-Baxterization approach.

There are two parameters $\theta$ and $\varphi$ in the unitary matrix $\hat{R}(\theta, \varphi)$. Consider that $\theta$ is time-independent and $\varphi$ is time-dependent, a Hamiltonian can be constructed from the matrix $\hat{R}(\theta, \varphi)$ as done in Refs. [14, 16, 17]. The Hamiltonian is

$$
\hat{H}(\theta, \varphi) = \frac{1}{\sqrt{3}} \hbar \dot{\varphi} \sin \theta \cos \theta + 2S_{i+1}^z (e^{-i\varphi} S_i^+ S_{i+2}^- + e^{i\varphi} S_i^- S_{i+2}^+) 
+ e^{-i\varphi} (S_i^+ S_{i+1}^+ S_{i+2}^- S_{i+2}^- + e^{i\varphi} S_i^- S_{i+1}^- S_{i+2}^+ S_{i+2}^+) 
+ \frac{1}{3} \hbar \dot{\varphi} \cos^2 \theta (2S_{i+1}^z S_{i+2}^+ + S_{i+2}^z S_{i+1}^+) 
+ 2S_{i+1}^z (S_i^+ S_{i+2}^- S_{i+1}^+ + S_i^- S_{i+2}^+ S_{i+1}^- + S_{i+1}^+ S_{i+2}^+) 
- (S_i^+ S_{i+1}^+ + S_{i+2}^- S_{i+1}^+ S_{i+2}^+ + S_i^- S_{i+1}^+ + S_{i+1}^+ S_{i+2}^+)
$$

(20)

One can find the eigenvalues and corresponding eigenstates as follows,

$$
E_1 = E_2 = E_3 = E_4 = 0 \\
E_5 = E_6 = -\hbar \dot{\varphi} \cos \theta, \\
E_7 = E_8 = \hbar \dot{\varphi} \cos \theta
$$

$$
|\chi_1\rangle = \frac{1}{\sqrt{2}} (-|011\rangle + |110\rangle) \\
|\chi_2\rangle = \frac{1}{\sqrt{2}} (-|001\rangle + |100\rangle) \\
|\chi_3\rangle = \frac{1}{\sqrt{2}} (-|011\rangle + |101\rangle) \\
|\chi_4\rangle = \frac{1}{\sqrt{2}} (|001\rangle + |010\rangle) \\
|\chi_5\rangle = -\frac{1}{\sqrt{3}} e^{-i\varphi} \frac{\theta}{2} |001\rangle + \frac{1}{\sqrt{3}} e^{-i\varphi} \frac{\theta}{2} |010\rangle - \frac{1}{\sqrt{3}} e^{-i\varphi} \frac{\theta}{2} |100\rangle + \cos \theta \frac{\theta}{2} |111\rangle \\
|\chi_6\rangle = \frac{1}{\sqrt{3}} \cos \frac{\theta}{2} |001\rangle - \frac{1}{\sqrt{3}} \cos \frac{\theta}{2} |010\rangle + \frac{1}{\sqrt{3}} \cos \frac{\theta}{2} |100\rangle + \frac{\dot{\varphi} \sin \theta}{2} |111\rangle \\
|\chi_7\rangle = -\frac{1}{\sqrt{3}} e^{-i\varphi} \sin \frac{\theta}{2} |000\rangle + \frac{1}{\sqrt{3}} e^{-i\varphi} \sin \frac{\theta}{2} |011\rangle + \frac{1}{\sqrt{3}} \sin \theta \frac{\theta}{2} |101\rangle + \frac{1}{\sqrt{3}} \cos \theta \frac{\theta}{2} |110\rangle \\
|\chi_8\rangle = \cos \frac{\theta}{2} |000\rangle + \frac{1}{\sqrt{3}} e^{i\varphi} \sin \frac{\theta}{2} |011\rangle + \frac{1}{\sqrt{3}} e^{i\varphi} \sin \frac{\theta}{2} |101\rangle + \frac{1}{\sqrt{3}} e^{i\varphi} \sin \frac{\theta}{2} |110\rangle
$$

(21)

It is clear that four of the eigenstates $|\chi_i\rangle$ ($i = 1, 2, 3, 4$) can be decomposed to two-qubit entangled states and single-qubit states. We get $\tau_{1234}^{ABC} = 0$ since there exists no 3-qubit entanglement for the four states. One can easily understand that when any pair of qubit in a 3-qubit system has maximal entanglement, 3-qubit entanglement will vanish. According to Eq. (14), by calculating the reduced density for the first four eigenstates, we have $C_{AC}^1 = 1$ for the state $|\chi_1\rangle$, $C_{AC}^2 = 1$ for the state $|\chi_2\rangle$, $C_{AC}^3 = 1$ for the state $|\chi_3\rangle$ and $C_{AC}^4 = 1$ for the state $|\chi_4\rangle$. One can easily understand that when any pair of qubit in a 3-qubit system has maximal entanglement, 3-qubit entanglement will vanish. The other four eigenstates $|\chi_i\rangle$ ($i = 5, 6, 7, 8$) are 3-qubit entangled states. These eigenstates have the same 3-tangle $\tau_{ABC} = \frac{\sin \frac{\theta}{2} \cos^3 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2} \cos \frac{\theta}{2}}$.

According to Berry’s theory [24], when the parameter $\varphi$ evolves adiabatically
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from 0 to 2\(\pi\), the Berry phase accumulated is,

\[ \gamma = i \int_0^{2\pi} \langle \chi | \frac{d}{d\varphi} | \chi \rangle d\varphi \] (22)

From Eq. (21), one can see that the Berry phase of the states \(|\chi_1\rangle, |\chi_2\rangle, |\chi_3\rangle, |\chi_4\rangle\) are zero. While for the other four states, we obtain the Berry phases in the following,

\[ \gamma_5 = \gamma_7 = \pi (1 - \cos \theta) = \frac{\Omega}{2} \]

\[ \gamma_6 = \gamma_8 = - \pi (1 - \cos \theta) = - \frac{\Omega}{2} \] (23)

where \(\Omega = 2\pi (1 - \cos \theta)\) is the solid angle enclosed by the loop on the Bloch sphere.

Introduce three operators

\[
I_\pm = S^+_1 S^+_2 S^+_3 + S^-_2 S^-_3 + 2 S^3_1 S^3_2 S^3_3 \\
I_- = -2 S^-_1 S^-_2 S^-_3 + 2 S^3_1 S^3_2 S^3_3 \\
I_3 = S^+_1 S^-_2 S^-_3 + S^-_1 S^+_2 S^-_3 + S^-_1 S^-_2 S^+_3 \]

\[ -\frac{1}{2} (S^+_1 S^-_2 S^-_3 + S^-_1 S^+_2 S^-_3 + S^-_1 S^-_2 S^+_3) \] (24)

it is not difficult to find that \((I_\pm)^2 = 0\) and \(I_3^2 = \frac{1}{4}\). We thus have a SU(2) group formed by the three operators, fulfilling conditions \([I_+, I_-] = 2I_3\) and \([I_3, I_\pm] = \pm I_\pm\).

The Hamiltonian (20) can be rewritten by using the operators

\[ H(\theta, \varphi) = B_+ I_+ + B_- I_- + B_3 I_3 = B \cdot J \] (25)

where the parameters \(B_+ = \frac{1}{\sqrt{3}} \hbar \dot{\varphi} \sin \theta \cos \theta e^{-i\varphi}\), \(B_- = \frac{1}{\sqrt{3}} \hbar \dot{\varphi} \sin \theta \cos \theta e^{i\varphi}\), and \(B_3 = \frac{2}{3} \hbar \dot{\varphi} \cos^2 \theta\). This is the reason that Berry phase of the 3-qubit Yang-Baxter system is the consistence with that of the 2-qubit Yang-Baxter system [13]. We also get the same eigenvalues of Hamiltonian (20) as those of the 2-qubit system [13].

4. Summary

In summary, we have discussed a new 8 \(\times\) 8 extraspecial 2-groups operator \(M\) extending from the 4 \(\times\) 4 extraspecial 2-groups operator \(M\). A unitary \(\hat{R}(\theta, \varphi)\) matrix is constructed by the Yang-Baxterization approach. 3-qubit entangled states can be achieved by acting the unitary \(\hat{R}(\theta, \varphi)\) matrix on the product states of 3-qubit. Remarkably the eight states have the same degree of entanglement, and arbitrary tripartite entangled states can be achieved depending on the parameters \(\theta\). We have constructed a Hamiltonian of 3 qubits from the unitary matrix \(\hat{R}(\theta, \varphi)\) and investigated the Berry phase of the system. The Berry phase is found to be consistence with that of the Yang-Baxter system of 2 qubits. The reason is that the Hamiltonian can be rewritten by using SU(2) operators.

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