Elements of the theory of induced representations
for quantum groups

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Abstract

We analyze the elements characterizing the theory of induced representations of Lie
groups, in order to generalize it to quantum groups. We emphasize the geometric and
algebraic aspects of the theory, because they are more suitable for generalization in the
framework of Hopf algebras. As an example, we present the induced representations of a
quantum deformation of the extended Galilei algebra in (1 + 1) dimensions.

1 Introduction

The theory of group representation plays a fundamental role in theoretical and mathematical
physics. Groups and physics are connected by the concept of symmetry, i.e., any transformation
on a system that keep it invariant in some sense to make explicit. Induced representations are
fundamental in theory of representations of Lie groups and in many of their physical applications.
If the configuration space, X, of a physical system is a homogeneous space of a certain Lie group
of transformations G, the (unitary) representations supported by this system can be obtained
inducing from the (unitary) representations of a Lie subgroup K of G, such that X ≃ G/K.
So, there exists a deep relation between geometry and representation theory of Lie groups via
induced representations.

One of the applications of quantum algebras and quantum groups is as generalized sym-
metries of the symmetries associated to Lie groups. So, it looks natural to study the induced
representations of quantum groups. Some attempts has been made up to now. In Ref. 3–4 the
induction procedure of corepresentations for quantum groups is analyzed in a general
way. Corepresentations for some deformations of the Galilei group can be found in Ref. 5. 6. 7. Induced representations of quantum groups has been studied in some particular cases: the one-
dimensional quantum Galilei group in Ref. 8, 9, slq(2) and e_q(2) in Ref. 10, 11, and the quantum
Heisenberg algebra in Ref. 12.

The aim of this paper is to analyze and give an algebraic formulation of the elements of
geometry as well as of the theory of induced representations suitable to be generalized to quantum

1Talk given by M.A.O. at the VIII Encuentro de Geometría y Física (Valencia, September 1998)
algebras and groups. In this way we obtain an induction method of representations for quantum groups \cite{10,11}, whose main advantage is that it is not necessary the use of corepresentations or of Kirillov’s theory \cite{13}.

It is worthy to mention that the framework of modules and comodules is the appropriate one for our procedure, since it allows to profit the algebraic character of the structures involved. The concept of pairing, which is the implementation of the idea of duality to Hopf algebras (the algebraic structures underlying quantum groups and quantum algebras) plays an important role, since it shows the equivalence of modules/comodules and spaces/co-spaces in the theory of induced representations. By means of the duality starting from regular and induced representations we get also coregular and coinduced ones.

The main conclusion of our analysis is that in the construction of induced representations is necessary to take in consideration two regular co-spaces: one of them determines the equivariance condition which characterized the carrier space of the induced representation, and a suitable restriction of the action on the other co-space gives the induced representation itself.

Finally, to mention that in the example studied in section 6, the quantum deformation of the extended Galilei (1 + 1) algebra $U_q(\mathfrak{g}(1,1))$, is fundamental to have dual bases of it and its dual $\text{Fun}_q(G(1,1))$ to obtain the coregular and coinduced representations of $U_q(\mathfrak{g}(1,1))$.

2 Algebraic preliminaries

2.1 Hopf algebras

We present a brief review of the main ideas about Hopf algebras and that are relevant for our purpose (for more details see, for instance, Ref. \cite{14}).

**Definition 1** An associative algebra with unit element over a commutative field, $\mathbb{K}$ (\(\mathbb{R}\) or $\mathbb{C}$), is a triad $A = (V, m, \eta)$, where $V$ is a linear vector space and the maps $m : V \otimes V \rightarrow V$ and $\eta : \mathbb{K} \rightarrow V$ are linear and verify:

$$m \circ (m \otimes id) = m \circ (id \otimes m), \quad m \circ (\eta \otimes id) = id = m \circ (id \otimes \eta).$$

**Definition 2** A coassociative coalgebra with counit element over a commutative field, $\mathbb{K}$, is a triad $C = (V, \Delta, \epsilon)$, where $V$ is a linear vector space and the maps coproduct, $\Delta : V \rightarrow V \otimes V$, and counit, $\epsilon : V \rightarrow \mathbb{K}$, are linear and verify

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta, \quad (\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta.$$

In general, coalgebras have a behaviour “dual” of that of algebras. The coproduct of an element $c \in V$ can be written as $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$.

**Definition 3** A bialgebra is a pair $B \equiv (A, C)$ made up of an algebra $A$ and a coalgebra $C$, both with the same underlying vector space $V$, such that $\Delta$ and $\epsilon$ are morphisms of $A$. 
Definition 4 A bialgebra \( H = (V, m, \eta, \Delta, \epsilon) \) is said to be a Hopf algebra if there exists an antipode, i.e., a bijective linear map \( S : V \to V \) verifying
\[
m \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\text{id} \otimes S) \circ \Delta.
\]
It can be proved that the antipode is an anti-automorphism of algebras and an anti-coautomorphism of coalgebras, and if it exists then is unique.

As examples of Hopf algebras we can mention: finite group algebras, the algebra of functions on a finite Lie group, and enveloping algebras of Lie algebras. All of them have the property of commutativity or cocommutativity, i.e., \( m \circ \tau = m \) or \( \tau \circ \Delta = \Delta \), with \( \tau \) the twist operator, i.e., \( \tau(a \otimes b) = b \otimes a \), \( a, b \in V \). All of them are non-deformed or “classical”.

2.2 Quantum algebras and quantum groups

Quantum groups and quantum algebras are Hopf algebras which are neither commutative nor cocommutative. The usual definition of quantum algebra (in the sense of Drinfel’d [15] and Jimbo [16]) is as follows.

Definition 5 Let \( \mathcal{U}(\mathfrak{g}) \) be the universal enveloping algebra of a Lie algebra \( \mathfrak{g} \). It is a “classical” Hopf algebra with coproduct, counit and antipode defined by
\[
\Delta(X) = 1 \otimes X + X \otimes 1, \quad \Delta(1) = 1 \otimes 1; \quad \epsilon(X) = 0, \quad \epsilon(1) = 1; \quad S(X) = -X; \quad X \in \mathfrak{g}.
\]
The extension to the remaining elements of \( \mathcal{U}(\mathfrak{g}) \) is made by linearity.

A quantization or deformation of \( \mathcal{U}(\mathfrak{g}) \) is obtained by means of a deformed Hopf structure on the associative algebra of formal power series in \( z \) and coefficients in \( \mathcal{U}(\mathfrak{g}), \mathcal{U}_z(\mathfrak{g}) \equiv \mathcal{U}(\mathfrak{g}) \hat{\otimes} \mathbb{C}[[z]] \), such that \( \mathcal{U}_z(\mathfrak{g})/z\mathcal{U}_z(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{g}) \), i.e., \( \mathcal{U}_z(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \) when \( z \to 0 \) (the equivalence is in the sense of Hopf algebras).

On the other hand, there are several approaches for quantum groups [17, 18]. Let \( G \) be a finite dimensional Lie group and \( \mathfrak{g} \) its Lie algebra. Let us consider the commutative and associative algebra of smooth functions of \( G \) on \( \mathbb{C} \), \( \text{Fun}(G) \), with the usual product of functions (i.e., \( (fg)(x) = f(x)g(x), f, g \in \text{Fun}(G), x, y \in G \)). It has a Hopf structure (which is cocommutative if and only if \( G \) is abelian) given by
\[
(\Delta(f))(x, y) = f(xy), \quad \epsilon(f) = f(e), \quad (S(f))(x) = f(x^{-1}),
\]
where \( e \) is the identity of \( G \).

Note that, in general, \( \text{Fun}(G) \otimes \text{Fun}(G) \subseteq \text{Fun}(G \times G) \). When the group is finite the equality is strict, but if \( G \) is not a finite group \( \Delta(f) \) may not belong to \( \text{Fun}(G) \otimes \text{Fun}(G) \). This problem can be solved by an adequate restriction of the space \( \text{Fun}(G) \).

After deformation the above commutative Hopf algebra becomes non-commutative \( \text{Fun}_q(G) \). In section [3] we display the quantum Galilei algebra \( U_q(\mathfrak{g}(1, 1)) \) as well as of the quantum Galilei group \( F_q(G(1, 1)) \).
2.3 Pairing of Hopf algebras

In the category of linear vector spaces \((V)\) over \(\mathbb{K}\) the dual object of \(V\) is defined as the vector space of its linear forms, i.e., \(V^* = \mathcal{L}(V, \mathbb{K})\).

If \((V, m, \eta)\) is a finite algebra its dual object \((V^*, m^*, \eta^*)\) is defined to be a coalgebra. Reciprocally, the dual of a coalgebra is an algebra. However, the category of bialgebras and Hopf algebras is self-dual.

When \(V\) is infinite dimensional, \((V \otimes V)^*\) and \(V^* \otimes V^*\) are not isomorphic (in fact, there is only an inclusion of \(V^* \otimes V^*\) into \((V \otimes V)^*\)). Hence, the dual of a product map, \(m : V \otimes V \to V\), does not give rise to a coproduct in a natural way, but the concept of pairing \([14]\) solves this difficult.

**Definition 6** A pairing between two Hopf algebras, \(H\) and \(H'\), is a bilinear map \(\langle \cdot, \cdot \rangle : H \times H' \to \mathbb{K}\) that verifies the following properties:

\[
\begin{align*}
\langle h, m'(h' \otimes k') \rangle &= \langle \Delta(h), h' \otimes k' \rangle, \\
\langle h, 1_{H'} \rangle &= \epsilon(h), \\
\langle h \otimes k, \Delta'(h') \rangle &= \langle m(h \otimes k), h' \rangle, \\
\langle h, S'(h') \rangle &= \langle S(h), h' \rangle.
\end{align*}
\]

The pairing is left (right) non-degenerate if \([\langle h, h' \rangle = 0, \forall h' \in H'] \Rightarrow h = 0\) (\([\langle h, h' \rangle = 0, \forall h \in H \Rightarrow h' = 0\]). We say that a pairing is non-degenerate if it is simultaneously left and right non-degenerate. A non-degenerate pairing allows us to restrict (in the infinite dimensional case) the dual \(H^*\) to a more “manageable” subspace isomorphic to \(H'\) via \(h'(h) = \langle h, h' \rangle\).

Incidentally, \(\text{Fun}(G)\) \((\text{Fun}_q(G))\) is the Hopf algebra dual of \(\mathcal{U}(\mathfrak{g})\) \((\mathcal{U}_q(\mathfrak{g}))\).

2.4 Star structures on Hopf algebras

Complex numbers solve some problems that appear using real numbers, but the results of the physical measures are real numbers. So, it is pertinent to introduce “real forms” (star structures) on the complex spaces we are using.

**Definition 7** A star structure on a Hopf algebra, \(H\), is a map \(\ast : H \to H\) such that

\[
\begin{align*}
h^{**} &= h, \\
(\lambda h + \mu g)^\ast &= \bar{\lambda} h^\ast + \bar{\mu} g^\ast, \\
(hg)^\ast &= g^\ast h^\ast, \\
\Delta(h^\ast) &= \Delta(h)^\ast, \\
\epsilon(h^\ast) &= \epsilon(h), \\
1^\ast &= 1,
\end{align*}
\]

where \(h, g \in H\), \(\lambda, \mu \in \mathbb{C}\) and the bar denotes complex conjugate.

The axioms displayed in expression \([\text{I}]\) show that \(\ast\) is an antimorphism of algebras and a morphism of coalgebras, which is the usual choice \([14] [19]\).

In the set of \(\mathbb{C}\)-valued functions on the space \(X\) the star structure is defined by \(f^\ast(x) = \bar{f(x)}\). If \(G\) is a finite group, the star structure to be considered on the algebra \(\mathbb{C}[G]\) is given by \(g^\ast = g^{-1}\), \(\forall g \in G\). When \(G\) is a Lie group, the canonical star structure on the complexification of the enveloping algebra \(\mathcal{U}(\mathfrak{g}) \otimes \mathbb{C}\) is defined for the generators \(X\) of \(\mathfrak{g}\) as \(X^\ast = -X\), and it is extended to all the remaining elements taking into account that \(\ast\) is a semilinear antimorphism of algebras. Note that with Def. \([\text{I}]\), \(S \circ \ast = S^{-1} \circ \ast\).
Given a pair of Hopf algebras equipped with a non-degenerate pairing \((H, H', (\cdot, \cdot))\), the star structure on \(H\) can be translated to \(H'\) by means of \(\langle \varphi^*, h \rangle = \langle \varphi, S(h)^* \rangle\).

### 2.5 Modules and comodules

The concept of module appears from that of linear vector space substituting the field of scalars by a ring, and dualizing modules we get comodules.

**Definition 8** Let \((V, \alpha, A)\) a triad made of an associative \(\mathbb{K}\)-algebra, \(A\), with unit element, a linear vector \(\mathbb{K}\)-space, \(V\), and a linear map \(\alpha : A \otimes \mathbb{K} V \to V\), that we call action, and denote by \(a \triangleright v = \alpha(a \otimes v)\) over the decomposable elements of the tensor product. We will say that \((V, \alpha, A)\) (or \((V, \triangleright, A)\)) is a left \(A\)-module if verifies: \(a \triangleright (b \triangleright v) = (ab) \triangleright v, \ \forall a, b \in A, \forall v \in V\).

We only consider “left objects” but, obviously, right versions can be defined in a similar way. In the regular \(A\)-modules \((A, \triangleright, A)\) and \((A, \triangleleft, A)\) the actions are obtained via the product of the algebra by \(a' \triangleright a = a'a\) and \(a \triangleleft a' = aa'\), respectively. The morphisms between two \(A\)-modules \((V, \alpha, A)\) and \((V', \alpha, A)\) are the linear maps \(f : V \to V'\) such that are equivariant respect to the action, i.e., \(f(a' \triangleright v) = a' \triangleright f(v), \ \forall a \in A, \forall v \in V\).

**Definition 9** Let \((V, \beta, C)\) be a triad made of a coassociative \(\mathbb{K}\)-coalgebra with counit, \(C\), a linear vector \(\mathbb{K}\)-space, \(V\), and a linear map \(\beta : V \to C \otimes \mathbb{K} V\), that we call coaction, and denote by \(v = \beta(v) = v^{(1)} \otimes v^{(2)}\). We will say that \((V, \beta, C)\) (or \((V, \triangleleft, C)\)) is a left \(C\)-comodule if:

\[
v^{(1)} \otimes v^{(2)} = v^{(1)} \otimes v^{(2)}(1) \otimes v^{(2)}(2), \quad \epsilon(v^{(1)})v^{(2)} = v, \quad \forall v \in V.
\]

The regular right comodule \((C, \triangleleft, C)\) (left comodule \((C, \triangleright, C)\)) has defined its coaction in terms of the coproduct on \(C\): \(c = c^{(1)} \otimes c^{(2)}\). The morphisms between two \(C\)-comodules, \((V, \triangleright, C)\) and \((V', \triangleright', C)\), are linear maps \(f : V \to V'\) verifying \(v^{(1)} \otimes f(v^{(2)}) = f(v^{(1)}) \otimes f(v^{(2)})\), \(\forall v \in V\).

The definition of dual modules and dual comodules is possible when they are building up over a Hopf algebra \(H\). Note that if \((V, \triangleright, H)\) is a left \(H\)-module then the dual space \(V^*\) is equipped in a natural way with a structure of right \(H\)-module by \(f \triangleright h, v) = (f, h \triangleright v), \ f \in V^*, \ v \in V, \ h \in H\). When \((V, \triangleleft, H)\) is a left \(H\)-comodule then \(V^*\) is a right \(H\)-comodule by means of \(\langle \varphi, f \triangleright v \rangle \triangleleft \varphi = \langle \varphi \otimes f, v \rangle\), \(\varphi \in V^*, \ v \in V, \varphi \in H^*\). The module dual of \((V, \triangleright, H)\) is obtained making the pull–back with respect to the antipode \(S\) of \(H\), i.e., \(v \otimes h = S(h) \triangleright v\). The comodule dual of \((V, \triangleleft, H)\) is the push–out with respect to \(S\), i.e., \(v = S(v^{(1)}) \otimes v^{(2)} \triangleright v\).

A representation of the algebra \(A\) on a linear vector space \(V\) is an algebra morphism \(\rho : A \to \text{End}(V)\). The expression \(\rho(a)(v) = a \triangleright v\) establishes a one-to-one correspondence between representations of \(A\) and left \(A\)-modules, in such a way that any concept defined for representations has its analogue in the language of modules.

When a bialgebra acts or coacts in a linear vector space equipped with another structure like algebra, coalgebra or bialgebra some compatibility relations are necessary [14, 21]. In the following definitions \(B\) and \(B'\) denote bialgebras, \(H\) and \(H'\) Hopf algebras, \(A\) an algebra and \(C\) a coalgebra.
Definition 10 The left $B$–module $(A, \triangleright, B)$ is an algebra-module if $m_A$ and $\eta_A$ are morphisms of $B$–modules, i.e., if $b \triangleright (aa') = (b_{(1)} \triangleright a)(b_{(2)} \triangleright a')$, and $b \triangleright 1 = \epsilon(b)1$, $\forall b \in B$, $\forall a, a' \in A$.

Definition 11 The left $B$–module $(C, \triangleright, B)$ is a coalgebra–module if $\Delta_C$ and $\epsilon_C$ are $B$–module morphisms, i.e., if for any $b, c \in B$ it is accomplished that $(b \triangleright c)_{(1)} \otimes (b \triangleright c)_{(2)} = (b_{(1)} \triangleright c_{(1)}) \otimes (b_{(2)} \triangleright c_{(2)})$ and $\epsilon_C(b \triangleright c) = \epsilon_B(b)\epsilon_C(c)$.

By dualizing the actions two new structures over a left $B$–comodule $(C, \blacklozenge, B)$ are obtained: coalgebra–comodule and algebra–comodule. When the object where one acts is a bialgebra other two structures are obtained.

Definition 12 It is said that $(B', \triangleright, B)$ is a left bialgebra–[co]module if is simultaneously an algebra–[co]module and a coalgebra–[co]module.

3 Spaces and co-spaces

The formalization of the idea of geometry is based on two objects: an space, $X$, and a group of transformations, $G$, acting on it. We understand for (right) action of $G$ on $X$ an external composition law $\alpha : X \times G \to X$, $(\alpha(x, g) = x \triangleleft g)$, verifying $(x \triangleleft (g \triangleleft g') = x \triangleleft (gg')$ and $x \triangleleft 1 = x$.

So, we consider a geometry as a triad $(X, \triangleleft, G)$ constituted by a space, $X$, a group, $G$, and a (right) action, $\triangleleft$, of $G$ on $X$. It is said that $X$ is a $G$–space or also a $G$–module. For instance, the group $G$ with the regular action, $g' \triangleleft g = gg'$, $g, g \in G$, is a $G$–module. The morphisms between the objects $(X, \triangleleft, G)$ and $(X', \triangleleft', G)$ are $G$–equivariant maps $f : X \to X'$, i.e., $f(x \triangleleft g) = f(x) \triangleleft' g$, $\forall x \in X$, $\forall g \in G$.

The action of $G$ can be extended in a natural way to other objects defined in terms of $X$. For instance, the set $F(X, X')$ of maps of $X$ on $X'$ can be equipped with a (left) action defined by $(g \triangleright f)(x) = f(x \triangleleft g)$.

The fact to study the geometry in terms of other triads associated to the original one is interesting for two reasons: some questions about $(X, \triangleleft, G)$ can be solved easier in the new object, and the characteristics of the new triad can allow a natural generalization of the original geometric concepts.

In this way, we replace the manifold $X$ by the algebra of $C^\infty \mathbb{C}$–valued functions on $X$, as well as the Lie group $G$ by the enveloping algebra $U(g)$ of its Lie algebra $g$. Since $(F(X), \triangleright, U(g))$ is an algebra–module over the Hopf algebra $U(g)$, we can generalize the concept of $G$–space in algebraic terms.

Definition 13 Let $H$ be a Hopf algebra. A left [right] $H$–co-space is an algebra module $(A, \triangleright, H)$ $[(A, \triangleright, H)]$.

The morphisms among $H$–co-spaces are the morphisms of $H$–modules and the concepts of subco-space or quotient co-space are equivalent to subalgebra–module or quotient algebra–module, respectively. The term co-space is not usual, but we have adopted this word instead of space to stress the dual character of $A$ as way of describing the initial “geometric object”. The
crucial point in the above definition is that no condition about the commutativity of the product in $A$ is required. Hence, it allows to write in a geometric language properties of algebras which are not the ring of coordinates over a space.

As examples we mention the following ones. Let $(X, \triangleleft, G)$ be a $G$–space, with $G$ a finite group. The action of $G$ on the $\mathbb{K}$–valued functions on $X$ by $(g \triangleright f)(x) = f(x \triangleleft g)$, determines the $\mathbb{K}[G]$–module $(F(X), \triangleright, \mathbb{K}[G])$, where $\mathbb{K}[G]$ is the group algebra. The right $G$–space $(M, \triangleleft, G)$, with $M$ a smooth manifold and $G$ a Lie group, has associated the $U(\mathfrak{g})$–co-space $(F(M), \triangleright, U(\mathfrak{g}))$, where the action is defined by $1 \triangleright f = f$, $(X \triangleright f)(p) = \frac{d}{dt} |_{t=0} f(p \triangleleft e^{tX})$, with $X \in \mathfrak{g}$. If $H$ is a finite Hopf algebra by dualizing the regular action we get the module $(H^*, \triangleright, H)$ with the action given by $h \triangleright \varphi = (h, \varphi_2)\varphi_1$, $h \in H$, $\varphi \in H^*$. In this example the module is also a $H$–co-space.

In the non-finite case it is necessary to consider a pair of algebras with a non-degenerate pairing $(H, H', \langle \cdot, \cdot \rangle)$. It allows, via dualization of the regular actions, to obtain the regular $H$–co-spaces $(H', \triangleright, H)$ and $(H', \triangleleft, H)$.

In the context of algebra of functions a subspace will be a quotient co-space and a quotient space a subco-space.

In the co-spaces $(F(X), \triangleright, \mathbb{C}[G])$ and $(F(M), \triangleright, U(\mathfrak{g}))$, the invariance of a function $f$ under the action means that $g \triangleright f = f$, $\forall g \in G$, and $X \triangleright f = 0$, $\forall X \in \mathfrak{g}$, respectively. Both expressions are summarized in the general case $(A, \triangleright, H)$ saying that $a \in A$ is invariant under the action of $H$ if $h \triangleright a = \epsilon(h)a$, $\forall h \in H$.

As example of algebraic description we will analyze the compatibility between the action on the co-space $(A, \triangleright, H)$ and the star structures defined on $A$ and $H$. From $(F(M), \triangleright, U(\mathfrak{g}) \otimes \mathbb{C})$, now $F(M)$ denotes the space of $C^\infty \mathbb{C}$–valued functions on $M$, equipped with star structures defined in section 2.4, one gets: $(X \triangleright f)^* (p) = \frac{d}{dt} |_{t=0} f(p \triangleleft e^{tX}) = \frac{d}{dt} |_{t=0} f^*(p \triangleleft e^{tX}) = (X^* \triangleright f^*) (p)$, with $X \in \mathfrak{g}$, $f \in F(M)$ and $p \in M$. When $Z = X + iY \in \mathfrak{g} \otimes \mathbb{C}$ we have $(Z \triangleright f)^* = (X \triangleright f)^* - i(Y \triangleright f)^* = X^* \triangleright f^* - iY^* \triangleright f^* = -Z^* \triangleright f^*$. In the co-space $(F(X), \triangleright, \mathbb{C}[G])$ with the star structures displayed in section 2.4 we obtain the identity $(g \triangleright f)^* = g \triangleright f^*$. The last two expressions can be rewritten using the antipode as $(Z \triangleright f)^* = S(Z)^* \triangleright f^*$ and $(g \triangleright f)^* = S(g)^* \triangleright f^*$, respectively, which give a compatibility relation for the general case: $(h \triangleright a)^* = S(h)^* \triangleright a^*$.

Other important concept to be generalized is the measure on a space. Its algebraic expression is immediate, just like to note that in a space $X$ with a measure $\mu$ a functional on the set of measurable functions is defined by

$$I_\mu(f) = \int_X f(x) \, d\mu(x).$$

In fact Riesz’s theorem guarantees, under hypotheses very few restrictive, the existence of a one-to-one correspondence between measures on $X$ and positive linear functionals.

If the $H$–module $(A, \triangleleft, H)$ is finite we can consider the dual module, $(A^*, \triangleright, H)$, where the integral $I \in A^*$ is left invariant if $h \triangleright I = \epsilon(h)I$.

**Definition 14** A left invariant integral on the co-space $(A, \triangleleft, H)$ is a linear form, $I : A \to \mathbb{K}$, that verifies $\langle I, a \triangleleft h \rangle = \epsilon(h)\langle I, a \rangle$, $\forall h \in H$, $\forall a \in A$. The integral $I$ is normalized if $\langle I, 1_A \rangle = 1$.

For the regular co-space $(H', \triangleleft, H)$, presented below, the condition that determines the invariance of the integral $I : H' \to \mathbb{K}$ can be reformulated using the coproduct of $H'$ as $\langle h \otimes I, \Delta h' \rangle = \epsilon(h)\langle I, h' \rangle$. 

\[ \langle \Delta h', h' \rangle = \langle \Delta h, 1_A \rangle. \]
\[ \epsilon(h)(I, h'), \ \forall h \in H, \ \forall h' \in H'. \] In this case it is possible to demonstrate that such integral is unique up to a constant.

The definition of homogeneous space is not easy to generalize and in the literature there are some generalizations non-equivalent \(^{21, 22}\).

It is worthy to mention that the use of the duality supplies several alternatives for the same object. So, associated to the \(H\)-co-space \((A, \triangleright, H)\) there is another module, \((A^*, \triangleleft, H)\), and two comodules, \((A, \blacktriangleleft, H^*)\) and \((A^*, \blacktriangledown, H^*)\).

## 4 Induced representations

Let us start rewriting the method of induced representations from an algebraic point of view in terms of modules \(^{23}\) and giving its dual version.

**Definition 15** Let \(A'\) be a subalgebra of an algebra \(A\), which can be considered simultaneously as a right \(A'\)-module and a left \(A\)-module for the regular action, and \((V, \triangleright, A')\) a left \(A'\)-module. Then, \(A \otimes_A V\) equipped with a left action of \(A\) is a left \(A\)-module \((V^\uparrow, \triangleright, A)\) called \(A\)-module induced by \((V, \triangleright, A')\). If \(\rho\) and \(\pi\) are the representations associated to \((V, \triangleright, A')\) and \((V^\uparrow, \triangleright, A)\), respectively, it is said that \(\pi\) is the representation of \(A\) induced by \(\rho\).

**Definition 16** Let \(A'\) be a subalgebra of an algebra \(A\), which can be considered simultaneously as a left \(A'\)-module and a right \(A\)-module for the regular action, and \((V, \triangleright, A')\) a left \(A'\)-module. Then the space \(V^\downarrow = \text{Hom}_A(A, V)\) equipped with the action given by \((a \triangleright f)(b) = f(ba), \ \forall a, b \in A, \ \forall f \in \text{Hom}_A(A, V)\), is a left \(A\)-module \((V^\downarrow, \triangleright, A)\), called \(A\)-module coinduced by \((V, \triangleright, A')\). If \(\rho\) and \(\pi\) are the representations associated to \((V, \triangleright, A')\) and \((V^\downarrow, \triangleright, A)\), respectively, it is said that \(\pi\) is the representation of \(A\) coinduced by \(\rho\).

The above procedures of induction and coinduction appear in a natural way in this context since they are extensions of scalars in a module \(^{24}\). For infinite dimensional algebras the coinduced representations have non-countable dimension, this difficult is avoided using non-degenerate dual forms.

Let \(\langle \cdot, \cdot \rangle\) be a non-degenerate pairing between the Hopf algebras \(H\) and \(H'\). If \(K\) is a subalgebra of \(H\) and \((V, \triangleright, K)\) a \(K\)-module, the carrier space \(V^\uparrow\) of the induced module is the subspace of \(H' \otimes V\) with elements \(f\) verifying

\[ \langle f, kh \rangle = k \triangleright \langle f, h \rangle, \ \forall k \in K, \ \forall h \in H. \] (2)

The pairing of expression (2) is \(V\)-valued and given by \(\langle \varphi \otimes v, h \rangle = \langle \varphi, h \rangle v, \ \text{with} \ h \in H, \varphi \in H', v \in V\). Finally, the action \(h \triangleright f\) in the coinduced module is defined by

\[ \langle h \triangleright f, h' \rangle = \langle f, h'h \rangle, \ \forall h' \in H. \]

Let us suppose a real Lie group \(G\) and its Lie algebra \(\mathfrak{g}\). For any unitary representation of \(G\) on a complex Hilbert space \(\mathcal{H}\) there is a representation of \(U(\mathfrak{g} \otimes \mathbb{C})\) on the space of \(C^\infty\) vector fields of \(\mathcal{H}\), and other of \(U(\mathfrak{g} \otimes \mathbb{C})\) on the space \(\mathcal{H}^\infty\) of distribution vectors of \(\mathcal{H}\) \(^{23}\).
4.1 Group representations

In this section we show the importance of the associative algebras in the construction of group representations and, in particular, induced representations.

In the context of section 3, it is well known that any finite group $G$ has associated in a natural way the group algebra $\mathbb{K}[G]$, whose elements can be interpreted either as formal linear combinations of elements of $G$ with coefficients on $\mathbb{K}$ or as $\mathbb{K}$-valued functions on $G$. Moreover, the category of representations on vector space over $\mathbb{K}$ is equivalent to that of the modules over $\mathbb{K}[G]$ [13].

If $K$ is a subgroup of a finite group $G$ and $(V, \rho)$ a representation of $K$ one can consider the algebras $\mathbb{K}[K]$ and $\mathbb{K}[G]$ together with the module $(V, \triangleright, \mathbb{K}[K])$ defined by $k \triangleright v = \rho(k)(v)$, $k \in K$, $v \in V$. The carrier space of the induced representation, $\text{Hom}_{\mathbb{K}[K]}(\mathbb{K}[G], V)$, is the set of linear maps $F$ defined in $\mathbb{K}[G]$ verifying $F(kg) = k \triangleright F(g)$, $\forall k \in \mathbb{K}[K]$, $\forall g \in \mathbb{K}[G]$. Since $G$ is a basis of $\mathbb{K}[G]$ only it is necessary to know the values of $F$ on the elements of $G$. This allows to characterize the carrier space, $F_\rho(G, V)$, of the induced representation as the subspace of $V$–valued functions on $G$ verifying the “equivariance condition”

$$f(kg) = \rho(k)(f(g)), \quad f = F|_G, \forall k \in K, \forall g \in G.$$ 

Note that the equivariance condition is completely determined by $K$, $G$ and $(V, \rho)$. The induced representation $\rho^\uparrow$ is given by

$$[\rho^\uparrow(g)f](g') = (g \triangleright f)(g') = f(g'g).$$

When the group $G$ is not finite, the equivalence between representations and $\mathbb{K}[G]$–modules is maintained, but the algebra is too complicated and usually other algebras are introduced [13].

Let us suppose now that $G$ and $K$ are connected Lie groups and, for the sake of simplicity, $V$ is finite dimensional. Then, the algebras to be consider are $U(\mathfrak{g})$ and $U(\mathfrak{g})$ together with the module $(V, \triangleright, U(\mathfrak{g}))$ defined by $X \triangleright v = \frac{d}{dt} \big|_{t=0} \rho(e^{tX})(v)$, $\forall X \in \mathfrak{g}$. The carrier space of the induced representation is the set of the linear $V$–valued functions on $U(\mathfrak{g})$ verifying

$$F(XY) = X \triangleright F(Y), \quad \forall X \in U(\mathfrak{g}), Y \in U(\mathfrak{g}).$$

4.2 Induced corepresentations

Taking into account the concept of duality the knowledge of the category of representations of the Hopf algebra $H$ is equivalent to that of the category of corepresentations of the dual of $H$ (at least if dim $H < \infty$). So, it looks natural to ask ourselves for the analogue of the induction algorithm in the category of comodules. In the following we discuss the finite dimensional case.

Let us consider the subalgebra $A'$ of the algebra $A$ with canonical injection $i : A' \rightarrow A$, and the module $(V, \triangleright, A')$. As we seen below the induction method originates the $A$–module $(V^\uparrow, \triangleright, A)$ with action $(a \triangleright F)(b) = F(ba)$ and carrier space $V^\uparrow = \text{Hom}_A(A, V) \subset \text{Hom}_\mathbb{K}(A, V)$, whose elements $F$ verify $F(a'a) = a' \triangleright F(a)$, $\forall (a', a) \in A' \times A$. The dualization gives the coalgebras $C' = A'^\ast$ and $C = A^\ast$ with canonical subjection $\pi = i^\ast : C \rightarrow C'$ and the comodules $(V, \triangleright, C)$, $(V^\uparrow, \triangleright, C)$. The last comodule is said to be the comodule induced by the former one. Note that the carrier space is the same in every corresponding pair (module/comodule).
The above procedure can be reformulated in terms of comodules. For that it is necessary to rewrite the condition that characterizes the carrier space using coactions and determine the coaction on the induced comodule.

The elements of $V^\dagger$ are of the form $F = \sum_{i\in I} v_i \otimes \varphi_i$, with $I$ a set of indices, $v_i \in V$ and $\varphi_i \in C$. For all $f \in V^*$, $a' \in A'$ and $a \in A$ we have that $\langle F(a'a), f \rangle = \langle (\id \otimes \Delta) F, f \otimes a' \otimes a \rangle$. Using the equivariance condition we get, for all $f \in V^*$, $a' \in A'$, $a \in A$, that $\langle F(a'a), f \rangle = \langle (\Box \otimes \id) F, f \otimes a' \otimes a \rangle$. Comparing both expressions the equivariance condition is equivalent to

$$(\id \otimes L) F = (\beta \otimes \id) F,$$

where $L = (\pi \otimes \id) \circ \Delta$ and $\beta$ is the coaction on the $C'$–comodule.

Noting that $\Box F \in (V \otimes C) \otimes C$ we can obtain the coaction on $F$. So, $\langle (f \otimes a) \otimes b, \Box F \rangle = \langle f \otimes a \otimes b, (\id \otimes \Delta) F \rangle$. From this result we deduce that the induced coaction is given by

$$\Box (\id \otimes \Delta) F.$$

Hence we have obtained a procedure to construct coinduced comodules.

In Ref. [2] it is discussed this construction, called induced representation, starting from a Hopf algebra $H$ and a $H'$–comodule for a quotient Hopf algebra of the first one. The results obtained correspond to unitary induced corepresentations of Hopf algebras.

5 Spaces and representations

In this section we present some situations where geometry and representation theory interact directly in order to stress the deep relationship between them. We emphasize either those aspects relevant on group theory that can be extended to quantum Hopf algebras or the problems that appear.

5.1 Invariant integrals and unitary representations

It is well known that the use of sums (or integrals) and averages over a group is very useful in group theory. Thus, now we will try to construct an invariant integral, since it allows to construct a unitary module.

Let us consider the simple case of a homogeneous $G$–space $(X, \trianglelefteq, G)$, where $X$ as well $G$ are finite, and the co–space $(F(X), \triangleright, \mathbb{C}[G])$. The dual of $F(X)$ can be identified with the space of formal linear combinations $\mathbb{C}[X]$ through the bilinear form determined by $\langle f, x \rangle = f(x), f \in F(X), x \in X$. So, a linear form on $F(X)$ will be

$$I = \sum_{x \in X} \alpha_x x.$$

If $I$ is an invariant integral then it satisfies $I \triangleleft h = \epsilon(h) I, \forall h \in \mathbb{C}[G]$. Taking $h = g^{-1}$ the l.h.s. of the invariance condition is $I \triangleleft g^{-1} = \sum_{x \in X} \alpha_x (x \triangleleft g^{-1}) = \sum_{x' \in X} \alpha_{x' \triangleleft g} x'$, and the r.h.s. $\epsilon(g^{-1}) I = I = \sum_{x' \in X} \alpha_{x' x}$. Both expressions give that $\alpha_{x' \triangleleft g} = \alpha_{x'}$. Since $X$ is a homogeneous space all the coefficients $\alpha_x$ are equal to a constant $\lambda \in \mathbb{C}$, hence $I = \lambda \sum_{x \in X} x$. Let $|X|$ be
the cardinal of $X$. The integral is normalized choosing $\lambda = |X|^{-1}$, and in this case $I(f) = |X|^{-1} \sum_{x \in X} f(x)$, with $f$ an arbitrary function. The integral $I$ combined with the star structure on the space of functions, i.e. $f^*(x) = \overline{f(x)}$, allows to define an inner product $\langle f_1, f_2 \rangle = I(f_1^* f_2)$. If the star structure on $\mathbb{C}[G]$ is given by $g^* = g^{-1}$ we get that $\langle g^* \circ f_1, f_2 \rangle = \langle f_1, g \circ f_2 \rangle$. This result can be extended by linearity to any element of $\mathbb{C}[G]$. So, we have proved that the module $(F(X), \triangleright, \mathbb{C}[G])$ becomes unitary.

The generalization of this result is provided by the following proposition.

**Proposition 1** Let $(A, \triangleright, H)$ be a co-space, such that $A$ and $H$ have star structures compatible in the sense of $(h \triangleright a)^* = S(h)^* \triangleright a^*, \forall a \in A, \forall h \in H$. If $I$ is a $H$–invariant integral on $A$, then the star structure defined on $H$ is compatible with the bilinear form $(a, b) = I(a^* b)$.

Note that if $(A, \langle \cdot, \cdot \rangle)$ is a Hilbert space then the representation associated to the $H$–module $(A, \triangleright, H)$ is unitary.

### 5.2 Co-spaces and induction

In practice the induction algorithm has two steps: the characterization of the carrier space $V^\uparrow = \text{Hom}_A(A, V)$ as subspace of $\text{Hom}_G(A, V)$ and the description of the action. When a Hopf algebra is involved in this procedure both steps can be formulated in terms of co-spaces. Effectively, let us consider a subalgebra $L$ (non necessarily a Hopf subalgebra) of a Hopf algebra $H$ and a $L$–module $(V, \triangleright, L)$, and suppose that $H$ as well as $V$ are finite dimensional for the sake of simplicity. The carrier space of the induced representation $V^\uparrow = \text{Hom}_L(H, V)$ is the subspace of $V \otimes H^*$, whose elements $\sum_i v_i \otimes \varphi_i$ verify $\sum_i (\varphi_i, lh) v_i = \sum_i (\varphi_i, h)(l \triangleright v_i)$, $\forall h \in H, \forall l \in L$. This condition can be expressed using the regular action of $H$ on its dual seen in section 3 (and no reference to the elements of $H$ is made)

\[
\sum_i v_i \otimes (\varphi_i \triangleright l) = \sum_i (l \triangleright v_i) \otimes \varphi_i, \quad \forall l \in L \subset H.
\]

Hence, the equivariance condition can be described in terms of the regular co-space $(H^*, \triangleleft, H)$. The connection between the action on the induced module and the other regular co-space is summarized in the following proposition.

**Proposition 2** The module $(V^\uparrow, \triangleright, H)$ is a submodule of the module tensor product $(V, \triangleright, H) \otimes (H^*, \triangleright, H)$, whose second factor is the left regular $H$–co-space and the action on the first factor is given by the counit by means of

\[
h \triangleright v = \epsilon(h)v, \quad h \in H, v \in V.
\]

The above considerations can be formulated in a simple way when the module from where one induces is unidimensional, since in this case the carrier space $V^\uparrow$ is the subspace of $H^*$ determined by the condition $\varphi \triangleleft l = \varphi, \forall l \in L$. Moreover, the induced module is simply a submodule of the regular module $(H^*, \triangleright, H)$. It suffices to consider Proposition 2 and to take into account that the character $\epsilon$ of $H$ is the unit element of the tensor product.

The final conclusion, as we mention in the introduction, is that in the induction procedure both regular co-spaces are used: $(H^*, \triangleleft, H)$ characterizes the equivariance condition, i.e., the
carrier space of the induced representation, and by restricting the action on \((H^*, \triangleright, H)\) the induced representation is obtained explicitly.

# 6 Induced representations of \(U_q(\mathfrak{g}(1, 1))\)

The quantum extended Galilei group is the Hopf algebra \(F_q(G(1, 1))\), generated by \(\mu, x, t\) and \(v\) with nonvanishing commutation relations

\[
[\mu, x] = -2a\mu, \quad [\mu, v] = av^2, \quad [x, v] = 2av.
\]

The coproduct, counit and antipode are

\[
\Delta \mu = \mu \otimes 1 + 1 \otimes \mu + v \otimes x + \frac{1}{2}v^2 \otimes t, \quad \Delta x = x \otimes 1 + 1 \otimes x + v \otimes t, \quad \Delta v = v \otimes 1 + 1 \otimes v;
\]
\[
\epsilon(\mu) = \epsilon(x) = \epsilon(t) = \epsilon(v) = 0;
\]
\[
S(\mu) = -\mu + x - \frac{1}{2}v^2t, \quad S(x) = -x + tv, \quad S(t) = -t, \quad S(v) = -v.
\]

The dual (enveloping) algebra \(U_q(\mathfrak{g}(1, 1))\) is generated by \(I, P, H\) and \(N\), in such a way that its pairing with \(F_q(G(1, 1))\) is given by

\[
\langle P^\mu H^\nu N^\alpha, \mu' x^\rho t^\tau v^\sigma \rangle = p!q!s! \delta^\mu_{\mu'} \delta^\nu_{\nu'} \delta^\alpha_{\alpha'}.
\]

The duality relations fix the Hopf algebra structure in \(U_q(\mathfrak{g}(1, 1))\). The nonvanishing commuting relations are

\[
[I, N] = -ae^{-2aP} I^2, \quad [P, N] = -e^{-2aP} I, \quad [H, N] = -\frac{1 - e^{-2aP}}{2a}.
\]

The coproduct, counit and antipode can be written as

\[
\Delta M = M \otimes e^{-aP} + e^{aP} \otimes M, \quad \Delta P = P \otimes 1 + 1 \otimes P, \quad \Delta H = H \otimes 1 + 1 \otimes H, \quad \Delta K = K \otimes e^{-aP} + e^{aP} \otimes K;
\]
\[
\epsilon(M) = \epsilon(P) = \epsilon(H) = \epsilon(K);
\]
\[
S(M) = -M, \quad S(P) = -P, \quad S(H) = -H, \quad S(K) = -K - aM;
\]

where \(M = e^{-aP}\) and \(K = e^{aP} N\). However, the pairing is simpler in terms of \((I, P, H, N)\) than instead of \((M, P, H, K)\).

After some computations (see [14]) one finds that:

1).- The action of the generators of \(U_q(\mathfrak{g}(1, 1))\) on the left regular module \((F_q(G(1, 1)), \triangleright, U_q(\mathfrak{g}(1, 1)))\) is given by

\[
I \triangleright f = \left(1 + a\bar{v} \frac{\partial}{\partial \mu} e^{-2a\frac{\partial}{\partial x}}\right) \frac{\partial}{\partial \mu} f,
\]
\[
P \triangleright f = \left[\frac{\partial}{\partial x} + \frac{1}{2a} \ln \left(1 + a\bar{v} \frac{\partial}{\partial \mu} e^{-2a\frac{\partial}{\partial x}}\right)\right] f,
\]
\[
H \triangleright f = \left[\frac{\partial}{\partial \mu} + \frac{1}{2a} \bar{v} \left(1 - \frac{e^{-2a\frac{\partial}{\partial x}}}{1 + a\bar{v} \frac{\partial}{\partial \mu} e^{-2a\frac{\partial}{\partial x}}}\right)\right] f;
\]
\[
N \triangleright f = \frac{\partial}{\partial v} f \quad f \in F_q(G(1, 1)).
\]
2).- The action on the right regular module \((F_q(G(1,1)), \prec, U_q(\mathfrak{g}(1,1)))\) is

\[
    f \prec I = \frac{\partial}{\partial \mu} f, \quad f \prec P = \frac{\partial}{\partial x} f, \quad f \prec H = \frac{\partial}{\partial t} f, \\
    f \prec N = \left[ \frac{\partial}{\partial v} + a\bar{\mu}e^{-2a\beta}e^{2\mu} + \bar{x}e^{-2a\beta}e^{4\mu} + \bar{t}e^{-2a\beta}e^{2\mu} \right] f.
\]

The above results allow us to construct a family of representations of \(U_q(\mathfrak{g}(1,1))\) coinduced by the character \((I^pP^qH^r) \vdash 1 = \alpha^p\beta^q\gamma^r\) of the Abelian subalgebra generated by \(I, P\) and \(H\). The carrier space \(C^\uparrow\) is the subspace of elements of \(F_q(G(1,1))\) of the form \(e^{\alpha\mu}e^{\beta x}e^{\gamma t}\phi(v)\), with \(\phi\) an arbitrary function. The action can be transferred to the space of formal power series \(C[[v]]\), where the action of the generators of \(U_q(\mathfrak{g}(1,1))\) is

\[
    I \vdash \phi(v) = \alpha(1 + aae^{-2a\beta}v)\phi(v), \\
    P \vdash \phi(v) = [\beta + \frac{1}{a} \ln(1 + aae^{-2a\beta}v)]\phi(v), \\
    H \vdash \phi(v) = [\gamma + \frac{1}{2a}(1 - \frac{e^{-2a\beta}}{1+aee^{-2a\beta}v})v]\phi(v), \\
    N \vdash \phi(v) = \phi'(v).
\]

Induced representations of \(U_q(\mathfrak{g}(1,1))\) were previously obtained by Bonechi et al [7]. We have deduced the coregular representations and constructed from them a family of coinduced representations including those presented in [7].

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