On finitely graded Iwanaga-Gorenstein algebras and the stable categories of their (graded) Cohen-Macaulay modules

to the memory of R.O. Buchweitz

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December 11, 2018

Abstract

We discuss finitely graded Iwanaga-Gorenstein (IG) algebras $A$ and representation theory of their (graded) Cohen-Macaulay (CM) modules. By quasi-Veronese algebra construction, in principle, we may reduce our study to the case where $A$ is a trivial extension algebra $A = \Lambda \oplus C$ with the grading $\deg \Lambda = 0$, $\deg C = 1$. In [15] we gave a necessary and sufficient condition that $A$ is IG in terms of $\Lambda$ and $C$ by using derived tensor products and derived Homs. For simplicity, we assume that $\Lambda$ is of finite global dimension in the sequel. In this paper, we show that the condition that $A$ is IG, has a triangulated categorical interpretation. We prove that if $A$ is IG, then the graded stable category $\mathcal{CM}^Z_A$ of CM-modules is realized as an admissible subcategory of the derived category $D^b(\text{mod } \Lambda)$. As a corollary, we deduce that the Grothendieck group $K_0(\mathcal{CM}^Z_A)$ is free of finite rank.

We give several applications. Among other things, for a path algebra $\Lambda = kQ$ of an $A_2$ or $A_3$ quiver $Q$, we give a complete list of $\Lambda$-$\Lambda$-bimodule $C$ such that $\Lambda \oplus C$ is IG (resp. of finite global dimension) by using the triangulated categorical interpretation mentioned above.

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1 Introduction

Representation theory of (graded) Iwanaga-Gorenstein (IG) algebra was initiated by Auslander-Reiten [1], Happel [9] and Buchweitz [4], has been studied by many researchers and is recently
getting interest from other areas. One source of interest on representation theory IG-algebra is theory of cluster categories, since higher cluster category is often realized as the stable category \( \text{CM} A \) of ungraded CM-modules over a finitely graded algebra \( A \). Another source is in the study of Mirror symmetry, as is mentioned below.

Recall that a graded algebra \( A = \bigoplus_{i=0}^{\infty} A_i \) is called IG if it is Noetherian on both sides and of finite self-injective dimension on both sides. The central object of representation theory of graded IG-algebra \( A \) is the category \( \text{CM}^Z A \) of graded Cohen-Macaulay (CM) modules and its stable categories \( \text{CM}^Z A \). The latter has a canonical structure of a triangulated category. Fundamental results are following equivalences of triangulated categories

\[
(1-1) \quad K^\text{ac}(\text{proj}^Z A) \overset{\mathbb{Z}/0}{\longrightarrow} \text{CM}^Z A \overset{\beta}{\longrightarrow} \text{Sing}^Z A
\]

where \( K^\text{ac}(\text{proj}^Z A) \) is the homotopy category of acyclic complexes of graded projective \( A \)-modules and \( \text{Sing}^Z A \) is the graded singularity category. It is defined by the Verdier quotient \( \text{Sing}^Z A := D^b(\text{mod}^Z A)/K^b(\text{proj}^Z A) \). If we perform the same construction to an algebraic variety \( X \), then we obtain a triangulated category \( \text{Sing} X \) which only related to the singular locus of \( X \). Hence, the name. Singularity categories play important roles in theory of Mirror symmetry. This is another reason that representation theory of IG-algebra have been becoming to have much attention. Story until now is the same with the ungraded situation. There is a special feature for the graded situation, the Orlov subcategory \( O \). Under the assumption that the degree 0-subalgebra \( A_0 \) coincides with the base field \( k \), Orlov [20] found another triangulated category \( O \) which is equivalent to \( \text{CM}^Z A \) in study of Mirror symmetry. The category \( O \) is a triangulated subcategory \( O \) of \( D^b(\text{mod}^Z A) \) such that the restriction functor \( \pi|_O \) of the canonical quotient functor \( \pi : D^b(\text{mod}^Z A) \to \text{Sing}^Z A \) gives an equivalence.

\[
(1-2) \quad \pi|_O : O \overset{\sim}{\longrightarrow} \text{Sing}^Z A.
\]

We postpone giving the definition of \( O \) under more general assumption that \( A_0 \) is IG until Section 3.3.

The aim of this paper is to study finitely graded IG-algebras \( A = \bigoplus_{i=0}^{\ell} A_i \) over a field and the stable categories of their graded Cohen-Macaulay (CM) modules by using results of [15]. One of our achievement is the following result.

**Theorem 1.1.** Let \( A = \bigoplus_{i=0}^{\ell} A_i \) be a finite dimensional graded IG-algebra. If the degree 0-part algebra \( A_0 \) is of finite global dimension, then the Gorthendieck group \( K_0(\text{CM}^Z A) \) is free and its rank is bounded by \( \ell|A_0| \) from above:

\[
\text{rank } K_0(\text{CM}^Z A) \leq \ell|A_0|
\]

where \( |A_0| \) denotes the number of non-isomorphic simple \( A_0 \)-modules.

One important trick here is the quasi-Veronese algebra construction: from a finitely graded algebra \( A = \bigoplus_{i=0}^{\ell} A_i \), we can construct the \( \ell \)-th quasi-Veronese algebra \( A^{[\ell]} \), whose important properties are that it is graded Morita equivalent to \( A \) and that \( (A^{[\ell]})_i = 0 \) for \( i \neq 0, 1 \). We denote \( \nabla A := (A^{[\ell]})_0, \Delta A := (A^{[\ell]})_1 \) and call \( \nabla A \) the Beilinson algebra.

\[
A^{[\ell]} = \nabla A \oplus \Delta A.
\]

Since they are graded Morita equivalent, \( A \) is IG if and only if so is \( A^{[\ell]} \) and if this is the case, we have an equivalence \( \text{CM}^Z A \simeq \text{CM}^Z A^{[\ell]} \). Therefore, we may and will concentrate in the case \( A = A_0 \oplus A_1 \). In this case \( A \) is regarded as the trivial extension algebra of \( A_0 \) by the bimodule \( A_1 \) over it. Hence, the basic set up of this paper is the followings: \( A \) is an algebra, \( C \) is a bimodule over it and \( A = A \oplus C \) is the trivial extension with the canonical grading \( \text{deg } A = 0, \text{deg } C = 1 \). This is precisely the situation studied in the former paper [15].
In that paper, we call a bimodule $C$ over $\Lambda$ *asid* (as an abbreviation of “attaching self-injective dimension”) if the trivial extension algebra $A = \Lambda \oplus C$ is IG. We gave a characterization of asid bimodule in terms of $- \otimes^L_A C^a$ and $\mathbb{R}\text{Hom}_A(C^a, -)$ where $C^a$ denotes the iterated derived tensor product of $C$, namely, for a natural number $a > 0$,

$$C^a := C \otimes^L_A C \otimes^L_A \cdots \otimes^L_A C \quad (a\text{-times})$$

and $C^0 := \Lambda$. One of our main theorem gives a categorical characterization of asid bimodules.

**Theorem 1.2.** Let $\Lambda$ be a finite dimensional algebra of finite global dimension and $C$ a bimodule over $\Lambda$. Then the trivial extension algebra $A = \Lambda \oplus C$ is IG if and only if there exists an admissible subcategory $T \subset \mathbb{D}^b(\text{mod } \Lambda)$ which satisfies the following conditions.

1. The functor $T = - \otimes^L_A C$ acts on $T$ as an equivalence, i.e., $T(T) \subset T$ and the restriction functor $T|_T$ is an autoequivalence.
2. The functor $T = - \otimes^L_A C$ nilpotently acts on $T^\perp$, i.e., $T(T^\perp) \subset T^\perp$ and there exists a natural number $a \in \mathbb{N}$ such that $T^a(T^\perp) = 0$.

It might be looked that the categorical characterization is only an abstract result. However, in Section 7 this characterization becomes an essential tool to solve a concrete problem that classifies asid bimodules over a path algebra of $A_2$ or $A_3$ quiver.

This theorem is proved in Theorem 4.10 for the more general case where $\Lambda$ is IG with a little modification that replace $\mathbb{D}^b(\text{mod } \Lambda)$ with $\mathbb{K}^b(\text{proj } \Lambda)$. We note that in Theorem 4.10 we give another categorical characterization in terms of $C$-duality functors

$$(-)^* := \mathbb{R}\text{Hom}_A(-, C) : \mathbb{D}^b(\text{mod } \Lambda) \rightleftarrows \mathbb{D}^b(\text{mod } \Lambda^{\text{op}}^{\text{op}}) : \mathbb{R}\text{Hom}_A^{\text{op}}(-, C) =: (-)^*.$$

We call an admissible subcategory $T \subset \mathbb{D}^b(\text{mod } \Lambda)$ an *asid subcategory* if it satisfies the conditions (1) and (2) of above theorem. The next main result, Theorem 1.3 asserts a uniqueness of asid subcategory by giving a description using an asid bimodule. For this, we use two invariants of an asid bimodule, the *right asid number* $\alpha_r$ and the *left asid number* $\alpha_\ell$ which are introduced in [15]. The original definitions is recalled in Definition 4.3. Here we give a formula for the right asid number $\alpha_r$ under the assumption that $\Lambda$ is finite dimensional.

$$\alpha_r = 1 - \min\{a \in \mathbb{Z} \mid \exists n \text{ s.t. } \text{soc}(\Omega^{-n}A)_a \neq 0\}.$$

This formula tells us that it essentially counts the minimal degree of the socle of the graded cosyzygies $\Omega^{-n}A$ as graded $A$-modules. The left asid number $\alpha_\ell$ is given by the same formula involving the left graded cosyzygies. Thus the first statement of Theorem 1.3 can be regarded as a result concerning on right-left symmetry of graded self-injective resolution of $A$.

**Theorem 1.3** (Theorem 4.12). Assume that $\text{gldim } \Lambda < \infty$. If $A = \Lambda \oplus C$ is IG, then the following assertions hold.

1. We have $\alpha_r = \alpha_\ell$. We put $\alpha := \alpha_r = \alpha_\ell$.
2. The subcategory $T$ of (2) of Theorem 1.2 is uniquely determined as $T = \text{thick } C^a$.
3. We have $T^\perp = \text{Ker}(- \otimes^L_A C^a)$. 

3
To state the next theorem, first we recall Happel’s result. Let $D(\Lambda) := \text{Hom}_k(\Lambda, k)$ be the $k$-dual of $\Lambda$ with the canonical bimodule structure. Then the trivial extension algebra $T(\Lambda) := \Lambda \oplus D(\Lambda)$ is self-injective and $\text{CM}^Z T(\Lambda) = \text{mod}^Z T(\Lambda)$. Happel [8] showed that there is a canonical embedding functor

$$\mathcal{H} : D^b(\text{mod} \Lambda) \hookrightarrow \text{mod}^Z A.$$  

He also showed that $\mathcal{H}$ gives an equivalence if and only if $\Lambda$ is of finite global dimension.

Observe that even in the case $C \neq D(\Lambda)$, if $A = \Lambda \oplus C$ is IG, then there are an analogue of Happel’s functor constructed as the following composite functor

$$\mathcal{H} : D^b(\text{mod} \Lambda) \hookrightarrow D^b(\text{mod}^Z A) \rightarrow \text{Sing}^Z A \xrightarrow{\beta^{-1}} \text{CM}^Z A$$

where the first two arrows are canonical functors and the third is the inverse of the functor $\beta$ in (1-1). We note that quasi-Veronese algebra construction yields the functor below which is also denoted by $\mathcal{H}$.

$$\mathcal{H} : D^b(\text{mod} \nabla A) \rightarrow \text{CM}^Z A \overset{[\ell]}{\rightarrow} \text{CM}^Z A$$

In [10] we study the conditions that generalized Happel’s functor $\mathcal{H}$ is fully faithful or gives an equivalence.

In the next result, the asid subcategory $T$ turns out to be equivalent to the stable category $\text{CM}^Z A$ via $\mathcal{H}$.

**Theorem 1.4** (Theorem 4.12, Theorem 5.6). Assume that $\text{gldim} \Lambda < \infty$ and $A = \Lambda \oplus C$ is IG. Let $T$ be the asid subcategory. Then the following assertions hold.

1. The functor $\mathcal{H}$ restricts to give an equivalence

$$\mathcal{H}|_T : T \hookrightarrow \text{CM}^Z A.$$

2. We have $T^\perp = \text{Ker} \mathcal{H}$.

What we actually prove is that the functor $\mathcal{H}|_T$ is an equivalence which fits into the following commutative diagram three arrows of which are equivalences of (1-1) and (1-2).

$$\begin{array}{ccc}
K^{\text{ac}}(\text{proj}^Z A) & \xrightarrow{Z^0} & \text{CM} A \\
\downarrow \mathcal{H}|_T & & \downarrow \beta \\
T & \xrightarrow{\mathcal{H}|_T} & \text{Sing}^Z A
\end{array}$$

The functor $\mathcal{H}|_T$ is the restriction of the canonical embedding $\mathcal{H} : D^b(\text{mod} \Lambda) \hookrightarrow D^b(\text{mod}^Z A)$ and $p_0$ is a functor which assign a projective complex $P$ with its “degree 0-generators”. These equivalence can be proved in the more general case where $\Lambda$ is IG. However for this we need to use “locally perfect” graded modules, which is introduced in Section 3.

As a summary, we give the following theorem which can be stated for a general finitely graded IG-algebra $A$ which is not necessary a trivial extension algebra. We mention that Theorem 1.1 is a consequence of this theorem.

**Theorem 1.5.** Let $A = \bigoplus_{i=0}^\ell A_i$ be a finite dimensional graded IG-algebra. Assume that $\text{gldim} \Lambda < \infty$. Then there exists the recollement of the following form

$$\text{CM}^Z A \xrightarrow{\mathcal{H}} D^b(\text{mod} \nabla A) \xrightarrow{\mathcal{H}} \text{Ker} \mathcal{H}$$

where in is a canonical inclusion.
This is the end of abstract results. We give three concrete applications. In the first two application we discuss a finite dimensional graded IG-algebra of finite CM-type. To obtain conclusions of ungraded CM-modules from the results of graded CM-modules obtained before, we use a CM-version of Gabriel theorem by the first author and M. Yoshiwaki [17]. This theorem asserts that a finite dimensional graded IG-algebra $A$ is of graded finite CM-type if and only if it is of (ungraded) CM-type.

It is known that a finite dimensional algebra $\Lambda$ is a iterated tilted algebra if and only if the trivial extension algebra $T(\Lambda) = \Lambda \oplus D(\Lambda)$ by $D(\Lambda)$ is of finite representation type (see Happel [8, Section V.2]. Since the algebra $T(\Lambda)$ is always self-injective, our first application can be looked as a CM-generalization of one implication of this result. A CM-generalization of the other implication is discussed in [17].

**Theorem 1.6** (Theorem 6.2). Let $\Lambda$ be an iterated tilted algebra of Dynkin type. If a trivial extension algebra $A = \Lambda \oplus C$ is IG, then it is of finite CM-type.

In this theorem, CM representation theory of $A$ is controlled by the degree 0-part $\Lambda$.

An easiest way to obtain a bimodule is taking a tensor product $C = N \otimes_k M$ of a left $\Lambda$-module $N$ and a right $\Lambda$-module $M$. In the second application, we study this case under the assumption $\text{gldim} \Lambda < \infty$. In Theorem 6.3 we determine the condition that $A = \Lambda \oplus C$ is IG (or $\text{gldim} A < \infty$) and give a description of $\mathcal{CM} A$. We see that if $A$ is IG, then it is always of finite CM type and the number of indecomposable CM-modules is given by $\# \text{ind} \mathcal{CM} A = \text{pd} M + 1$. Contrary to the first application, CM-representation theory is controlled by the degree 1-part $N \otimes_k M$.

As the third and final application, using Theorem 1.2 we give a complete list of $\Lambda$-$\Lambda$-bimodules $C$ such that $\Lambda \oplus C$ is IG in the case where $\Lambda = kQ$ is the path algebra of a quiver $Q$ of $A_2$ type or $A_3$ type.

The organization of the paper is the following. In Section 2 collects results which are need in the sequel. Among other things, in Section 2.1 we recall the quasi-Veronese algebra construction and the decomposition functor $p_i$ which plays a key role in the paper. In Section 3 we introduce the notion of locally perfect complexes and locally perfect CM-modules. They can be defined over arbitrary graded algebras. In Section 4 we prove categorical characterization of asid bimodules (Theorem 4.10) and a uniqueness of asid subcategories $T$ of an asid bimodule $C$ (Theorem 4.12). In Section 5 we show that the asid subcategory $T$ is equivalent to the stable category $\mathcal{CM}_{\Lambda}^Z C$ of locally perfect CM-modules. As a consequence, for a finitely graded algebra $A = \bigoplus_{i=0}^\ell A_i$ we see that $\mathcal{CM}_{\Lambda}^Z A$ is realized as an admissible subcategory of $K^b(\text{proj} \nabla A)$. In Section 6 we apply our result to study two particular classes of trivial extension algebras. In Section 7 we give a complete list of asid bimodules $C$ in the case where $\Lambda = kQ$ is the path algebra of a quiver $Q$ of $A_2$ type or $A_3$ type.

### 1.1 Notation and convention

Throughout this paper the symbol $k$ denotes a field and “algebra” means $k$-algebra. For generalization to the case where the base commutative ring $k$ is not a field, see remark 2.12. The symbol $D$ denotes the $k$-dual functor $D := \text{Hom}_k(\cdot, k)$.

Let $\Lambda$ be an algebra. Unless otherwise stated, the word “$\Lambda$-modules” means a right $\Lambda$-modules. We denote the opposite algebra by $\Lambda^{op}$. We identify left $\Lambda$-modules with (right) $\Lambda^{op}$-modules. A $\Lambda$-$\Lambda$-bimodule $D$ is always assumed to be $k$-central, i.e., $ad = da$ for $d \in D$, $a \in k$. Therefore we may identify $\Lambda$-$\Lambda$-bimodules with modules over the enveloping algebra $\Lambda^e := \Lambda^{op} \otimes_k \Lambda$. For a $\Lambda$-$\Lambda$-bimodule $D$, we denote by $D_\Lambda$ and $\Lambda D$ the underlying right and left $\Lambda$-modules respectively. So for example, $\text{id} D_\Lambda$ denotes the injective dimension of $D$ regarded a (right) $\Lambda$-module.
We note the obvious equality $\text{HOM}_A(M, N) = \bigoplus_{i \geq 0} A_i$. We denote by $\text{Mod}^Z_A$ the category of (right) $A$-modules $M = \bigoplus_{i \in \mathbb{Z}} M_i$ and graded $A$-module homomorphisms $f: M \to N$, which, by definition, preserve degree of $M$ and $N$, i.e., $f(M_i) \subseteq N_i$. We define the truncation $M_{\geq j}$ by $(M_{\geq j})_i = M_i$ $(i \geq j)$, $(M_{< j})_i = 0$ $(i < j)$. We set $M_{< j} := M/M_{\geq j}$ so that we have an exact sequence $0 \to M_{\geq j} \to M \to M_{< j} \to 0$.

For a graded $A$-module $M$ and an integer $j \in \mathbb{Z}$, we define the shift $M(j) \in \text{Mod}^Z_A$ by $(M(j))_i = M_{i+j}$. For $M, N \in \text{Mod}^Z_A$, $n \in \mathbb{Z}$ and $i \in \mathbb{Z}$, we set $\text{EXT}^n_A(M, N)_i := \text{Ext}^n_{\text{Mod}^Z_A}(M, N(i))$ and $\text{EXT}^n_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{EXT}^n_A(M, N)_i = \bigoplus_{i \in \mathbb{Z}} \text{Ext}^n_{\text{Mod}^Z_A}(M, N(i))$. We note the obvious equality $\text{HOM}_A(M, N)_0 = \text{Hom}_{\text{Mod}^Z_A}(M, N)$. We use the similar notation for $\text{RHOM}_A(M, N)$ where $M, N$ are objects of $\text{D}(\text{Mod}^Z_A)$.

We denote by $\text{mod}^Z_A \subset \text{Mod}^Z_A$ the full subcategory of finitely generated graded $A$-modules. For $i \in \mathbb{Z}$, we denote by $\text{Mod}^{Z\geq i}_A \subset \text{Mod}^Z_A$ the full subcategory consisting of $M \in \text{Mod}^Z_A$ such that $M_{< i} = 0$. We set $\text{mod}^{Z\geq i}_A := (\text{mod}^Z_A) \cap (\text{Mod}^{Z\geq i}_A)$. Similarly, we define the full subcategories $\text{mod}^{< i}_A$ and $\text{mod}^{\leq i}_A$.

### 2.1 Quasi-Veronese algebras

A (non-negatively) graded algebra $A = \bigoplus_{i \geq 0} A_i$ is called finitely graded if $A_i = 0$ for $i \gg 0$.

Let $A$ be a finitely graded algebra. We fix a natural number $\ell$ such that $A_i = 0$ for $i \geq \ell + 1$. (It is not necessary to assume that $A_{\ell} \neq 0$.) We recall that the Beilinson algebra $\nabla A$ of $A$ (which rigorously should be called the Beilinson algebra of the pair $(A, \ell)$) and its bimodule $\Delta A$ are defined to be

$$
\nabla A := \begin{pmatrix}
A_0 & A_1 & \cdots & A_{\ell-1} \\
0 & A_0 & \cdots & A_{\ell-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_0
\end{pmatrix},
$$

$$
\Delta A := \begin{pmatrix}
A_\ell & 0 & \cdots & 0 \\
A_{\ell-1} & A_\ell & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_1 & A_2 & \cdots & A_\ell
\end{pmatrix}
$$

where the algebra structure and the bimodule structure are given by the matrix multiplications. Then, the trivial extension algebra $\nabla A \oplus \Delta A$ with the grading $\text{deg} \nabla A = 0$, $\text{deg} \Delta A = 1$ is the $\ell$-th quasi-Veronese algebra $A^{[\ell]}$ of $A$ introduced by Mori [13, Definition 3.10].

$$
A^{[\ell]} = \nabla A \oplus \Delta A.
$$
By [18, Lemma 3.12] $A$ and $A^{[\ell]}$ are graded Morita equivalent to each other. More precisely, the functor $q^v := q^v_A$ below gives a $k$-linear equivalence.

$$q^v : \text{Mod}^Z A \xrightarrow{\cong} \text{Mod}^Z A^{[\ell]},$$

$$q^v(M) := \bigoplus_{i \in Z} q^v(M)_i,$$

$$q^v(M)_i = M_{i\ell} \oplus M_{i\ell+1} \oplus \cdots \oplus M_{(i+1)\ell-1}.$$

This equivalence is restricted to an equivalence $\text{Mod}^{\geq 0} A \xrightarrow{\sim} \text{Mod}^{\geq 0} A^{[\ell]}$.

The functor $q^v$ has the following compatibility with the degree shift functors over $A$ and $A^{[\ell]}$.

$$(1) \circ q^v \cong q^v \circ (\ell).$$

For simplicity we set $B = A^{[\ell]}$. We may identify $(A^{op})^{[\ell]}$ with $B^{op}$. The composite functor $q^v'$ gives an equivalence.

$$q^v' := q^v_{A^{op}} \circ (-\ell + 1) : \text{Mod}^Z A^{op} \rightarrow \text{Mod}^Z B^{op}.$$

It induces an equivalence $\text{Mod}^{> 0} A^{op} \xrightarrow{\sim} \text{Mod}^{> 0} B^{op}$ and fits into the following commutative diagram

$$
\begin{array}{ccc}
\text{Mod}^Z A^{op} & \xrightarrow{\text{HOM}_{A^{op}}(-, A)} & \text{Mod}^Z A \\
\downarrow{q^v'} & & \downarrow{q^v} \\
\text{Mod}^Z B^{op} & \xrightarrow{\text{HOM}_{B^{op}}(-, B)} & \text{Mod}^Z B
\end{array}
$$

### 2.1.2 Decomposition of a complex of graded projective $A$-modules

Let $A = \bigoplus_{i=0}^{\ell} A_i$ be a finitely graded algebra. We recall from [15] a decomposition of a complex of graded projective $A$-modules. For notational simplicity, we set $\Lambda := A_0$.

First we deal with a graded projective $A$-module. For an integer $i \in \mathbb{Z}$, we denote by $p_i : \text{Proj}^Z A \rightarrow \text{Proj} \Lambda$ the functor $p_i P := (P \otimes_A \Lambda)_i$. Then $t_i P := (p_i P) \otimes_A A(-i)$ is a graded projective $A$-module and there exists an isomorphism of graded $A$-modules, from which we deduce the following lemma.

$$P \cong \bigoplus_{i \in \mathbb{Z}} t_i P.$$

**Lemma 2.1.** A graded projective $A$-module $P \in \text{Proj}^Z A$ is finitely generated if and only if $p_i P$ is finitely generated and $p_i P = 0$ for $|i| \gg 0$.

We use the same symbol $p_i, t_i$ for graded $A^{op}$-modules. If $P \in \text{Proj}^Z A$ is such that $p_i P$ is finitely generated for $i \in \mathbb{Z}$, then $\text{HOM}_A(P, A)$ is a graded projective $A^{op}$-module and moreover we have an isomorphism of $\Lambda$-modules.

$$p_i \text{HOM}_A(P, A) \cong \text{Hom}_\Lambda(p_{-i} P, \Lambda).$$

We discuss compatibility of $p_i$ with the quasi-Veronese algebra construction. For $r = 0, \cdots, \ell$, we define a projective $\nabla A$-module $R_r$ to be

$$R_r = (0, \cdots, 0, A_0, A_1, \cdots, A_{\ell-1-r}).$$

It is clear that $\nabla A \cong \bigoplus_{r=0}^{\ell-1} R_r$ as $\nabla A$-modules. We note that $R_r$ has a canonical $A^{op}$-module structure.

We leave the verification of the following lemma to the readers.
Lemma 2.2. For \( P \in \text{Proj} \mathbb{Z} A \), we have the following isomorphism of \( \nabla A \)-modules.

\[
p_i q v P \cong \bigoplus_{r=0}^{\ell-1} (p_{i+r} P) \otimes_A R_r.
\]

From now we deal with a complex of graded projective \( A \)-modules. By abuse of notations, we denote the functors \( p_i : C(\text{Proj} \mathbb{Z} A) \to C(\text{Proj} \Lambda) \), \( p_i : K(\text{Proj} \mathbb{Z} A) \to K(\text{Proj} \Lambda) \) induced from the functor \( p_i : \text{Proj} \mathbb{Z} A \to \text{Proj} \Lambda \) by the same symbols. The symbol \( t_i \) is used in the same way. The following lemma plays an important role in the sequel.

Lemma 2.3 ([15, Lemma 4.2]). Assume that \( A = \Lambda \oplus C \). Let \( P \in C(\text{Proj} \mathbb{Z} A) \). Then there exists a morphism \( q_i : p_i P \to (p_{i-1} P) \otimes_A C \) in \( C(\text{Mod} \Lambda) \), which gives an exact triangle in \( K(\text{Mod} \Lambda) \).

\[(2-3) \quad p_{i-1} P \otimes_A C \to P_i \to p_i P \xrightarrow{q_i} p_{i-1} P \otimes_A C[1].\]

In particular \( H(P)_i = 0 \) if and only if the morphism \( q_i \) is an isomorphism in \( D(\text{Mod} \Lambda) \).

Assume that \( p_i P \) belongs to \( C^-(\text{Proj} \Lambda) \) for some \( i \in \mathbb{Z} \). Then the complex \( p_i P \otimes_A C \) is quasi-isomorphic to the derived tensor product \( p_i P \otimes^A_C C \). This observation yields the following corollary.

Corollary 2.4. Assume that \( A = \Lambda \oplus C \). Let \( P \in C^-(\text{Proj} A) \) such that \( p_i P \in C^- (\text{Proj} \Lambda) \) for \( i \in \mathbb{Z} \). Then the following assertions hold.

(1) If there exists \( j \in \mathbb{Z} \) such that \( H(P)_{>j} = 0 \), then for \( i \geq j \), we have an isomorphism \( p_i P \cong p_j P \otimes^A_C C^{i-j}[i-j] \) in \( D(\text{Mod} \Lambda) \).

(2) If there exists \( j \in \mathbb{Z} \) such that \( H(P)_{\leq j} = 0 \), then for \( i \leq j \), we have an isomorphism \( p_j P \cong p_i P \otimes^A_C C^{j-i}[j-i] \) in \( D(\text{Mod} \Lambda) \).

2.2 (Graded) Iwanaga-Gorenstein algebras

In this section, we collect basic facts about Iwanaga-Gorenstein (IG) algebra. In particular, we recall the constructions of equivalences in (1-1), since we use them in this paper. First we recall the definition of IG-algebras.

Definition 2.5. A (graded) algebra \( A \) is said to be Iwanaga-Gorenstein(IG) if it is (graded) Noetherian on both sides and (gr-) id \( A < \infty \), (gr-) id \( A < \infty \).

We recall the fundamental observation due to Iwanaga, which is frequently and tacitly used in the sequel.

Proposition 2.6 (Iwanaga [11]). Let \( A \) be a (graded) IG-algebra. For a (graded) \( A \)-module \( M \) the following conditions are equivalent

(1) (gr-) \( \text{pd} M < \infty \).

(2) (gr-) \( \text{id} M < \infty \).

Next we recall the definition of Cohen-Macaulay modules.

Definition 2.7. Let \( A \) be a (graded) IG-algebra. A (graded) \( A \)-module \( M \) is called Cohen-Macaulay (CM) if it is finitely generated and satisfies the condition

\[ \text{Ext}^n_A(M, A) = 0 \text{ for } n > 1. \]
We denote by $\text{CM}^\mathbb{Z} A \subset \text{mod}^\mathbb{Z} A$ the full subcategory of graded CM-modules. The ungraded version is denoted by $\text{CM} A \subset \text{mod} A$.

Let $A$ be a graded IG-algebra. The category $\text{CM}^\mathbb{Z} A$ is a Frobenius category, whose admissible projective-injective modules are precisely graded projective modules. Therefore, by [8], the stable category $\text{CM}^\mathbb{Z} A := \text{CM}^\mathbb{Z} A / \text{proj}^\mathbb{Z} A$ canonically has a structure of triangulated category.

Now we recall fundamental equivalences of triangulated categories which relate the stable category $\text{CM}^\mathbb{Z} A$ to other important triangulated categories.

The graded singularity category $\text{Sing}^\mathbb{Z} A$ is defined to be the Verdier quotient $\text{Sing}^\mathbb{Z} A := \text{D}^b(\text{mod}^\mathbb{Z} A) / \text{K}^b(\text{proj}^\mathbb{Z} A)$.

We consider the following diagram

\[
\begin{array}{ccc}
\text{CM}^\mathbb{Z} A & \xleftarrow{i} & \text{mod}^\mathbb{Z} A & \xleftarrow{j} & \text{D}^b(\text{mod}^\mathbb{Z} A) \\
\downarrow & & \downarrow & & \downarrow \\
\text{CM}^\mathbb{Z} A & \xrightarrow{\beta} & \text{Sing}^\mathbb{Z} A
\end{array}
\]

where the top arrows are canonical embeddings and the vertical arrows are canonical functors. Since $\pi ji(\text{proj}^\mathbb{Z} A) = 0$, there exists a unique functor $\beta$ which makes the above diagram commutative.

**Theorem 2.8** (Buchweitz [4], Happel [9]). The functor $\beta$ gives an equivalence of triangulated categories.

We denote by $\text{C}^{\text{ac}}(\text{proj}^\mathbb{Z} A) \subset \text{C}(\text{proj}^\mathbb{Z} A)$ the full subcategory of acyclic complexes of finitely generated graded projective modules. Let $Z^0 : \text{C}^{\text{ac}}(\text{proj}^\mathbb{Z} A) \to \text{mod}^\mathbb{Z} A$ be the functor which sends a complex $X$ to its 0-th cocycle group $Z^0(X)$. It can be shown that the functor $Z^0$ has its image in $\text{CM}^\mathbb{Z} A$ and descents to the functor $Z^0 : \text{K}^{\text{ac}}(\text{proj} A) \to \text{CM}^\mathbb{Z} A$.

\[
\begin{array}{ccc}
\text{C}^{\text{ac}}(\text{proj}^\mathbb{Z} A) & \xrightarrow{Z^0} & \text{CM}^\mathbb{Z} A \\
\downarrow & & \downarrow \\
\text{K}^{\text{ac}}(\text{proj}^\mathbb{Z} A) & \xrightarrow{Z^0} & \text{CM}^\mathbb{Z} A
\end{array}
\]

**Theorem 2.9** (Buchweitz [4]). The functor $Z^0$ gives an equivalence of triangulated categories.

### 2.3 Derived tensor products and derived Hom functors which involve bimodules

In this section 2.3 we discuss the compatibility between the derived functors involving bimodules and the restriction functors.

Let $\Lambda$ be an algebra over a field $k$. Recall that we identify ($k$-central) $\Lambda$-$\Lambda$-bimodules with module over the enveloping algebra $\Lambda^e = \Lambda^{\text{op}} \otimes_k \Lambda$ and regard the category $\text{Mod} \Lambda^e$ as the category of $\Lambda$-$\Lambda$-bimodules.

Let $D$ be a $\Lambda$-$\Lambda$-bimodules. We denote by $D_\Lambda$ and $\Lambda D$ the underlying right and left $\Lambda$-modules of $D$. The assignments $D \mapsto D_\Lambda$ and $D \mapsto \Lambda D$ extend to functors, which are called the restriction functors.

$\text{Mod} \Lambda^e \to \text{Mod} \Lambda$, $D \mapsto D_\Lambda$, $\text{Mod} \Lambda^e \to \text{Mod} \Lambda^{\text{op}}$, $D \mapsto \Lambda D$, 

9
We equip the Hom-space Hom\(_A(D, E)\) and the tensor product \((D) \otimes_A E\) with a canonical \(\Lambda\)-\(\Lambda\)-bimodule structures and denote them by Hom\(_\Lambda(D, E)\) and \(D \otimes \Lambda E\) respectively the bimodules so obtained. A \(\Lambda\)-\(\Lambda\)-bimodule \(D \in \text{Mod} \Lambda^e\) induces (covariant or contravariant) functors

\[
D \otimes \Lambda - , \ - \otimes \Lambda D, \ \text{Hom}\_\Lambda(D, -) : \text{Mod} \Lambda^e \to \text{Mod} \Lambda^{\text{op}},
\]

\[- \otimes \Lambda D, \ \text{Hom}\_\Lambda(\ - , D) : \text{Mod} \Lambda \to \text{Mod} \Lambda, \ D \otimes \ - : \text{Mod} \Lambda^{\text{op}} \to \text{Mod} \Lambda^{\text{op}},
\]

\[\text{Hom}\_\Lambda(\ - , D) : \text{Mod} \Lambda \to \text{Mod} \Lambda^{\text{op}}.\]

The aim of this section is to prove the following lemma. Since the restriction functors are exact, we denote the derived functors of them by the same symbol.

**Lemma 2.10.** For objects \(D, E \in \text{D}(\text{Mod} \Lambda^e)\), there exist natural isomorphisms

\[
(D \otimes^L \Lambda E)_\Lambda \cong (D) \otimes^L \Lambda E, \ \Lambda(D \otimes^L \Lambda E) \cong D \otimes^L \Lambda (\Lambda E),
\]

\[
\Lambda \mathbb{R}\text{Hom}\_\Lambda(D, E) \cong \mathbb{R}\text{Hom}(D, \Lambda E), \ \mathbb{R}\text{Hom}(D, E)_\Lambda \cong \mathbb{R}\text{Hom}\_\Lambda(D, E_\Lambda).
\]

For example, the first natural isomorphism says that the left diagram of (2-4) is commutative up to a natural isomorphism.

(2-4) \[
\begin{array}{ccc}
D(\text{Mod} \Lambda^e) & \xrightarrow{- \otimes^L \Lambda E} & D(\text{Mod} \Lambda^e) \\
(-)_\Lambda \downarrow & & \downarrow (-)_\Lambda \\
D(\text{Mod} \Lambda) & \xrightarrow{- \otimes^L \Lambda E} & D(\text{Mod} \Lambda), \quad \text{C}(\text{Mod} \Lambda^e) & \xrightarrow{- \otimes^L \Lambda E} & \text{C}(\text{Mod} \Lambda).
\end{array}
\]

The key is the following lemma for which we need to assume the base ring \(k\) is a field.

**Lemma 2.11.** The restriction functor \(D \mapsto D_\Lambda\) sends projective (resp. injective) \(\Lambda^e\)-modules to projective (resp. injective) \(\Lambda\)-modules.

Similar statements hold for the restriction functor \(D \mapsto \Lambda D\).

**Proof.** In this proof, we make use of the assumption that we are dealing with an algebra over a field.

Since \(\Lambda\) is free over the base field \(k\), the restriction \(\Lambda^e_\Lambda = (\Lambda^{\text{op}} \otimes_k \Lambda)_\Lambda\) is free \(\Lambda\)-module. Thus the assertion for projective modules holds.

To prove the assertion for injective modules, first note that the restriction functor is the restriction along the algebra homomorphisms \(\iota : \Lambda \to \Lambda^e = \Lambda^{\text{op}} \otimes_k \Lambda\) defined by \(\iota(a) := 1_{\Lambda^{\text{op}}} \otimes a \) (\(a \in \Lambda\)). Namely, \(\Lambda^e\) has a \(\Lambda\)-\(\Lambda^e\)-bimodule structure whose left module structure is induced from \(\iota\) and there exists a natural isomorphism \(D_\Lambda \cong \text{Hom}_{\Lambda^e}(\Lambda^e, D)\) of \(\Lambda\)-modules. Since \(\Lambda\) is free over the base field \(k\), \(\Lambda^e\) is flat as a left \(\Lambda\)-module when it is regarded as left \(\Lambda\)-module. Now it is easy to check that if \(D\) is an injective \(\Lambda^e\)-module, then the \(\Lambda\)-module \(D_\Lambda = \text{Hom}_{\Lambda^e}(\Lambda^e, D)\) is injective by using \(\otimes\)-Hom-adjunction.

**Proof of Lemma 2.10.** For simplicity, we only give a proof of the first natural isomorphism. The others can be proved in similar ways.

First we recall the construction of left derived functors from \([12]\). An object \(P\) of \(\text{C}(\text{Proj} \Lambda)\) is said to have property \((P)\) if it has an increasing filtration \(0 = P_{(0)} \subset P_{(1)} \subset \cdots\) such that each graded quotient \(P_{(i)}/P_{(i-1)}\) is an object of \(\text{C}(\text{Proj} \Lambda)\) with 0 differential and that \(\bigcup_{i \geq 0} P_{(i)} = P\). Every object \(M \in \text{C}(\text{Mod} \Lambda)\) is quasi-isomorphic to an object \(P \in \text{C}(\text{Proj} \Lambda)\) having property \((P)\) and the derived tensor product \(M \otimes^L_{\Lambda} E\) is defined to be the quasi-isomorphism class of \(P \otimes^L_{\Lambda} E\).

It follows from Lemma 2.11 that the restriction functor \((-)_\Lambda\) preserves the property \((P)\). Now the desired statement follows from the commutativity of the right diagram of (2-4).

Thanks to Lemma 2.10 in what follows we safely drop the sign of restriction functors.
Remark 2.12. We remark that if we deal with derived functors involving bimodules in a suitable way, then we can generalize the contents of the paper to the case where the base $k$ is not a field but a commutative ring.

As is pointed out in [23, Remark 1.12] in the case where the base commutative ring $k$ is not a field, a proper way to deal with bimodules is provided by theory of DG-algebras (see e.g. [21]).

First, we need to resolve $\Lambda$ as DG-algebras. Namely, we take a quasi-isomorphism $\tilde{\Lambda} \xrightarrow{\sim} \Lambda$ of DG-algebra with a cofibrant DG-algebra. It is known that the derived category $D(\tilde{\Lambda})$ of DG-$\tilde{\Lambda}$-modules is triangulated equivalent to $D(Mod \, \Lambda)$. Then, the derived category $D(\Lambda^{op} \otimes_k \Lambda)$ of DG-$\Lambda^{op} \otimes_k \Lambda$-modules is served as a proper derived category of $\Lambda$-$\Lambda$-bimodules. An analogue of Lemma 2.10 holds in this derived category even when $k$ is a commutative ring. Moreover, using the derived category $D(\Lambda^{op} \otimes_k \Lambda)$ we can prove all the results given in this paper.

2.4 Admissible subcategories and recollements

In this Section we recall the notion of an admissible subcategory and related results.

Let $D$ be a triangulated category. Recall that a full triangulated subcategory $E \subset D$ is called thick if it is closed under taking direct summands. For an object $d \in D$, we denote the smallest thick subcategory of $D$ which contains $d$ by thick $d$ and call it the thick hull of $d$. An object $e \in D$ belongs to thick $d$ if and only if it is obtained from $d$ by taking shifts, cones and direct summands finitely many times. For a thick subcategory $E$, we define its right orthogonal subcategory to be

$$E^\perp := \{ d \in D \mid \text{Hom}_D(e, d) = 0 \text{ for all } e \in E \}.$$ 

In the similar way, we define the left orthogonal subcategory $^\perp E$.

Recall that a thick subcategory $E \subset D$ is said to be right (resp. left) admissible if the inclusion functor $i_E : E \hookrightarrow D$ has a right (resp. left) adjoint functor, which will be often denoted by $\tau$ in the sequel. A thick subcategory is called admissible if it is both left and right admissible.

We say that a triangulated category $D$ has a semiorthogonal decomposition $D = E \perp F$ by thick subcategories $E, F \subset D$ if $E$ is right admissible and $E^\perp = F$. We will use the following characterization of right admissibility for thick subcategories in the proof of Lemma 4.19.

Lemma 2.13 ([2]). For a thick subcategory $E \subset D$, the following conditions are equivalent.

1. $E \subset D$ is right admissible.

2. Every $d \in D$ fits into an exact triangle $e \hookrightarrow d \twoheadrightarrow k \twoheadrightarrow e[1]$ such that $e \in E, \ k \in E^\perp$.

Moreover is one of the above conditions is satisfied, then the right adjoint functor $\tau$ sends objects $d \in D$ to $e \in E$ which is appeared as the left most term of the exact triangle in (2).

Assume that we have a semiorthogonal decomposition $D = E \perp F$. We have the following commutative diagram

$$
\begin{array}{cccccc}
E & \xrightarrow{i_E} & D & \xrightarrow{q_E} & D/E & \\
\downarrow{(q_E \circ i_E)_{|E}} & & \downarrow{(q_E \circ i_E)_{|E}}^{-1} & & \downarrow{(q_E \circ i_E)_{|E}}^{-1} & \\
D/F & \xrightarrow{q_F} & D & \xrightarrow{i_F} & F, & \\
\end{array}
$$

where $i_E, i_F$ are canonical inclusions and $q_E, q_F$ are canonical quotient functors. It is known that the restriction functor $(q_E \circ i_E)_{|E}$ is an equivalence and that the composite functor ${((q_E \circ i_E)_{|E})^{-1} \circ q_F}$ is naturally isomorphic to the right adjoint functor $\tau$ of $i_E$. We mention that the similar composite functor gives an equivalence $F \xrightarrow{\sim} D/E$. We can immediately check the following lemma.
Lemma 2.14. For \(d, d' \in D\), we have the following natural isomorphism
\[
\text{Hom}_D(\tau(d), d') \cong \text{Hom}_{D/F}(\text{qt}_F(d), \text{qt}_F(d'))
\]
where we suppress the inclusion functor \(\text{in}_E\) as custom and write the object \(\text{in}_E(\tau(d))\) of \(D\) as \(\tau(d)\).

A thick subcategory \(E \subset D\) is admissible if and only if it fits into the following recollement of the left of (2-5).

\[
(2-5) \quad E \xrightarrow{\text{in}_E} D \xrightarrow{\text{qt}_E} D/E, \quad D/F \xrightarrow{\text{qt}_F} D \xrightarrow{\text{in}_F} F.
\]

If an admissible subcategory \(E\) is a piece of a semi-orthogonal decomposition \(D = E \perp F\), then the recollement of the left of (2-5) become the recollement of the right of (2-5) via the equivalences \(E \cong D/F, D/E \cong F\) mentioned above.

3 Locally perfect complexes and locally perfect CM-modules

In Section 3, \(A = \bigoplus_{i=0}^\ell A_i\) denotes a finitely graded Noetherian algebra. We introduce notions of locally perfect complex and locally perfect CM-module over \(A\).

3.1 Locally perfect complexes

Definition 3.1. A complex \(P \in C(\text{proj}^Z A)\) is called locally perfect if the complex \(p_iP\) belongs to \(C^b(\text{proj} A)\) for any \(i \in \mathbb{Z}\). We denote by \(C_{lp}(\text{proj}^Z A)\) the full subcategory of \(C(\text{proj}^Z A)\) consisting of locally perfect complexes. We define
\[
C^\bullet_{lp}(\text{proj}^Z A) := C_{lp}(\text{proj}^Z A) \cap C^\bullet(\text{proj}^Z A).
\]

where \(\bullet = -, b, ac\), etc. We denote by \(K^\bullet_{lp}(\text{proj}^Z A)\) the homotopy category of \(C^\bullet_{lp}(\text{proj}^Z A)\).

Next, we discuss locally perfectness of objects of derived categories. For simplicity, we only deal with the bounded derived category \(D^b(\text{mod}^Z A)\). An object \(M \in D^b(\text{mod}^Z A)\) is called locally perfect if it is represented by a locally perfect complex \(P \in C_{lp}(\text{proj}^Z A)\). We note that \(M \in D^b(\text{mod}^Z A)\) is locally perfect if and only if it is represented by \(P \in C_{lp}^{-b}(\text{proj}^Z A)\). We denote by \(D_{lp}^b(\text{mod}^Z A)\) the full subcategory of \(D^b(\text{mod}^Z A)\) consisting of locally perfect objects. Obviously we have
\[
(3-6) \quad K^b(\text{proj}^Z A) \subset D_{lp}^b(\text{mod}^Z A) \subset D^b(\text{mod}^Z A).
\]

Since the functor \(p_i : K(\text{proj}^Z A) \to K(\text{proj} A_0)\) is exact, the first statement of the lemma below follows.

Lemma 3.2. (1) \(K^*_b(\text{proj}^Z A)\) is a thick subcategory of \(K^*_b(\text{proj}^Z A)\).

(2) \(D_{lp}^b(\text{mod}^Z A)\) is a thick subcategory of \(D^b(\text{mod}^Z A)\).

We can deduce the following lemma from Lemma 2.2.

Lemma 3.3. The equivalence \(qv\) induces an equivalence
\[
qv : D_{lp}^b(\text{mod}^Z A) \sim D_{lp}^b(\text{mod}^Z A^{[\ell]}).
\]
If \( \text{pd } A < \infty \), then the locally perfectness of \( M \in \mathcal{D}^b(\text{mod}^Z A) \) can be checked by looking graded pieces \( M_i \in \mathcal{D}^b(\text{mod } A_0) \).

**Proposition 3.4.** Assume that \( \text{pd } A < \infty \). Then, an object \( M \in \mathcal{D}^b(\text{mod}^Z A) \) is locally perfect if and only if \( M_i \) belongs to \( \mathbf{K}^b(\text{proj } A_0) \) for any \( i \in \mathbb{Z} \).

First we show that we can reduce the problem to the case of a trivial extension algebra by using the quasi-Veronese algebra construction.

**Lemma 3.5.** Assume that \( \text{pd } A < \infty \). Let \( M \) be an object of \( \mathcal{D}^b(\text{mod}^Z A) \). Then the \( i \)-th graded submodule \( M_i \) belongs to \( \mathbf{K}^b(\text{proj } A_0) \) for \( i \in \mathbb{Z} \) if and only if \( (\text{qv} M)_i \) belongs to \( \mathbf{K}^b(\text{proj } \nabla A) \) for \( i \in \mathbb{Z} \).

**Proof.** Let \( \Gamma := \begin{pmatrix} \Gamma_0 & D \\ 0 & \Gamma_1 \end{pmatrix} \) be an upper triangular matrix algebra and \( e_0 := \begin{pmatrix} 1_{\Gamma_0} & 0 \\ 0 & 0 \end{pmatrix}, e_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1_{\Gamma_1} \end{pmatrix} \). Assume that \( \Gamma \) is Noetherian. We note that it is equivalent to assume that \( \Gamma_0 \) and \( \Gamma_1 \) are Noetherian and the modules \( \Gamma_0 D \) and \( D \Gamma_1 \) are finitely generated over \( \Gamma_0 \) and \( \Gamma_1 \) respectively. Assume moreover that \( \text{pd } D < \infty \). Then, by a similar argument to the proof of [15, Proposition 6.1] we can prove that an object \( N \in \mathcal{D}^b(\text{mod } \Gamma) \) belongs to \( \mathbf{K}^b(\text{proj } \Gamma) \) if and only if \( Ne_i \) belongs to \( \mathbf{K}^b(\text{proj } \Gamma_i) \) for \( i = 0, 1 \).

Applying the last statement repeatedly to the quasi-Veronese algebra construction, we prove the desired statement.

For the proof, we introduce a notion, which is also used later. Let \( \Lambda \) be an algebra and \( Q \in \mathcal{C}(\text{proj } \Lambda) \) a complex which is homotopic to a bounded complex \( Q' \in \mathcal{C}^b(\text{proj } \Lambda) \). We define the upper bound \( \text{ub } Q \in \mathbb{Z} \) of \( Q \) by \( \text{ub } Q := \max\{n \in \mathbb{Z} \mid H^n(Q) \neq 0\} \).

**Proof of Proposition [8, 2].** By Lemma 3.5, we may assume \( A = \Lambda \oplus C \).

We prove “if” part. Let \( M \neq 0 \) be an object in \( \mathcal{D}^b(\text{mod}^Z A) \) such that \( M_i \in \mathbf{K}^b(\text{proj } \Lambda) \) for \( i \in \mathbb{Z} \). Let \( P \in \mathcal{C}^{-b}(\text{proj } \Lambda) \) be a projective resolution of \( M \). We note that \( M_i = P_i \) in \( \mathcal{D}^b(\text{mod } \Lambda) \). We set \( k := \min\{i \in \mathbb{Z} \mid P_i \neq 0\} \).

We claim that \( p_i P \) belongs to \( \mathbf{K}^b(\text{proj } \Lambda) \) for \( i \in \mathbb{Z} \). The case \( i < k \) is clear since \( p_i P = 0 \). The case \( i = k \) follows from \( P_k = p_k P \). The case \( i > k \) is shown by induction. Assume that \( p_{i-1} P \) belongs to \( \mathbf{K}^b(\text{proj } \Lambda) \). Since \( \text{pd } C_\Lambda < \infty \), the object \( p_{i-1} P \otimes_{\Lambda} C \) belongs to \( \mathbf{K}^b(\text{proj } \Lambda) \). By the assumption \( P_i \) belongs to \( \mathbf{K}^b(\text{proj } \Lambda) \). Using the exact triangle \([2, 3]\), we deduce that \( p_i P \) belongs to \( \mathbf{K}^b(\text{proj } \Lambda) \).

By the claim, the complex \( p_i P \in \mathcal{C}^{-b}(\text{proj } \Lambda) \) is homotopic to some complex \( Q_i \in \mathcal{C}^b(\text{proj } \Lambda) \) for \( i \in \mathbb{Z} \). We may take \( Q_i = 0 \) for \( i < k \) and assume that for \( i \geq k \) the following equality holds

\[
\text{(3-7)} \quad \max\{m \in \mathbb{Z} \mid (Q_i)^{m} \neq 0\} = \text{ub}(p_i P).
\]

By [15, Lemma 4.7], there is a complex \( P' \in \mathcal{C}(\text{Proj}^Z \Lambda) \) which is homotopic to \( P \) such that \( p_i P' = Q_i \) in \( \mathcal{C}(\text{proj } \Lambda) \) for \( i \in \mathbb{Z} \). Observe that \( P' \) belongs to \( \mathcal{C}^{-b}_{\text{ip}}(\text{Proj}^Z \Lambda) \).

To complete the proof, we only have to show that the graded projective \( A \)-module \((P')^n \) is finitely generated for \( n \in \mathbb{Z} \). We fix \( n \in \mathbb{Z} \) and use the criterion of Lemma [21]. By the construction each \( p_i (P')^n = Q_i^n \) is finitely generated for \( i \in \mathbb{Z} \) and \( p_i (P')^n = Q_i^n = 0 \) for \( i < k \). Therefore, it is enough to show that \( p_i (P')^n = Q_i^n = 0 \) for \( i > k \).

Since \( M \) is assumed to have bounded cohomology groups and \( A \) is finitely graded, there exists \( j \in \mathbb{Z} \) such that \( H(M)_i = 0 \) for \( i \geq j \). Thus, using Corollary [24](1), we deduce that \( \text{ub}(p_i P) < n \).
for \( i \gg 0 \). By (3-7), we conclude that \( Q_i^n = 0 \) for \( i \gg 0 \) as desired. This completes the proof of “if” part.

We prove “only if” part. Let \( M \in \mathcal{D}_{lp}^{b}(\mod \mathbb{Z} A) \). There exists a locally perfect complex \( P \in C_{lp}^{-,b}(\proj \mathbb{Z} A) \) which is quasi-isomorphic to \( M \). Since by the assumption \( \mathfrak{p}_i P \) and \( \mathfrak{p}_{i-1} P \otimes_{\Lambda} C \) belong to \( \mathcal{K}^{b}(\proj \Lambda) \), we conclude by the exact triangle \( 2 \) that \( M_i \) belongs to \( \mathcal{K}^{b}(\proj \Lambda) \).

From the above proposition, we deduce a condition that every object \( M \in \mathcal{D}^{b}(\mod \mathbb{Z} A) \) is locally perfect.

**Corollary 3.6.** The following conditions are equivalent.

(1) \( \mathcal{D}_{lp}^{b}(\mod \mathbb{Z} A) = \mathcal{D}^{b}(\mod \mathbb{Z} A) \).

(2) \( \mathcal{K}^{b}(\proj \mathfrak{A}_0) = \mathcal{D}^{b}(\mod \mathfrak{A}_0) \).

**Proof.** We may assume \( \mathfrak{A} = \Lambda \oplus C \). Assume that the condition (1) holds. Then \( \mathcal{D}^{b}(\mod \Lambda) \subset \mathcal{D}_{lp}^{b}(\mod \mathbb{Z} A) \). It follows from [15, Lemma 4.13] that \( D^{b}(\mod \Lambda) \subset \mathcal{K}^{b}(\proj \Lambda) \). This shows the implication \( (1) \Rightarrow (2) \).

Assume that the condition (2) holds. We remark that since \( \mathfrak{A}_i \) is finitely generated \( \mathfrak{A}_0 \)-module, it has finite projective dimension. Now, it is easy to deduce the condition (1) by using Proposition 3.4. \( \square \)

In the case where \( \mathfrak{A} \) is finite dimensional, the condition (2) is equivalent to the condition \( \text{gldim} \mathfrak{A}_0 < \infty \).

**Corollary 3.7.** Assume that \( \mathfrak{A} \) is a finite dimensional graded algebra. Then the equality \( \mathcal{D}_{lp}^{b}(\mod \mathbb{Z} A) = \mathcal{D}^{b}(\mod \mathbb{Z} A) \) holds if and only if \( \text{gldim} \mathfrak{A}_0 < \infty \).

The following proposition concerning on the \( \mathfrak{A} \)-dualities plays important roles in the sequel.

**Proposition 3.8.** Assume that \( \mathfrak{A} \) is IG and that it has finite projective dimension as a left and right \( \mathfrak{A}_0 \)-module. Then the \( \mathfrak{A} \)-duality \( (-)^* \) induces an equivalence

\[
(-)^* := \mathcal{R}\text{Hom}_{\mathfrak{A}} (-, \mathfrak{A}) : \mathcal{D}_{lp}^{b}(\mod \mathbb{Z} A) \xrightarrow{\sim} \mathcal{D}_{lp}^{b}(\mod \mathbb{Z} A^{\text{op}})^{\text{op}} : \mathcal{R}\text{Hom}_{\mathfrak{A}^{\text{op}}} (-, \mathfrak{A}) =: (-)^*. \]

**Proof.** By [15, Proposition 6.1], \( \Delta \mathfrak{A} \) has finite \( \nabla \mathfrak{A} \)-projective dimension on both sides. Hence, we may reduce the problem to the case where \( \mathfrak{A} = \Lambda \oplus C \) by Lemma 3.3.

It is enough to show that \( (-)^* \) preserves locally perfectness. Since the argument is left-right symmetric, it is enough to prove that if \( M \in \mathcal{D}^{b}(\mod \mathbb{Z} A) \) is locally perfect, then so is \( M^* \). Since \( M^* \) belongs to \( \mathcal{D}^{b}(\mod \mathbb{Z} A^{\text{op}}) \), it is enough to show that \( (M^*)_i \) belongs to \( \mathcal{K}^{b}(\proj \Lambda^{\text{op}}) \) by the left version of Proposition 3.4.

Let \( P \in C_{lp}^{-,b}(\proj \mathbb{Z} A) \) be a projective resolution of \( M \). Then, substituting \( P' = A(i) \) in the exact sequence (4-5) of [15], we obtain an exact triangle in \( \mathcal{D}(\mod \Lambda^{\text{op}}) \) for \( i \in \mathbb{Z} \)

\[
\text{Hom}_{\Lambda}(p_{i+1}P, C) \to (M^*)_i \to \text{Hom}_{\Lambda}(p_{i}P, \Lambda) \to .
\]

The right most term belongs to \( \mathcal{K}^{b}(\proj \Lambda^{\text{op}}) \). The assumption \( \text{pd} C < \infty \) implies that so does the left most term. Thus we conclude that so does \( (M^*)_i \). \( \square \)
3.2 Locally perfect graded CM-modules

Definition 3.9. For a graded IG-algebra $A$, we define

$$CM_{lp}^Z A = \{ Z^0(P) \mid P \in C_{lp}^b(\text{proj}^Z A) \}.$$ 

This is a Frobenius full subcategory of $CM^Z A$ containing $\text{proj}^Z A$. So the stable category $CM_{lp}^Z A$ is a triangulated full subcategory of $CM^Z A$.

By the definition, it is obvious that the equivalence $Z^0 : K^{ac}(\text{proj}^Z A) \xrightarrow{\sim} CM^Z A$ induces an equivalence

$$Z^0 : K^{ac}_{lp}(\text{proj}^Z A) \xrightarrow{\sim} CM_{lp}^Z A.$$ 

The equivalence $\beta : CM^Z A \xrightarrow{\sim} \text{Sing}^Z A$ can be restricted to the locally perfect subcategories.

Lemma 3.10. Let $A$ be a finitely graded IG-algebra. If $A$ has finite projective dimension as a left and right $A_0$-module, we have

$$CM_{lp}^Z A = (CM^Z A) \cap D_{lp}^b(\text{mod}^Z A) = \{ M \in CM^Z A \mid \text{pd}(M_{A_0}) < \infty \}.$$ 

Proof. By Proposition 3.4, we have the second equality. It is obvious that $CM_{lp}^Z A$ is contained in $(CM^Z A) \cap D_{lp}^b(\text{mod}^Z A)$. In the following, we prove the converse inclusion.

Let $M \in (CM^Z A) \cap D_{lp}^b(\text{mod}^Z A)$. By Proposition 3.8, $M^*$ belongs to $(CM_{lp}^Z A^{op}) \cap D_{lp}^b(\text{mod}^Z A^{op})$. Let $f : Q \xrightarrow{\sim} M^*$ be a quasi-isomorphism with $Q \in C_{lp}^{-b}(\text{proj}^Z A^{op})$. We may assume that $Q^{>0} = 0$.

We regard $Q$ as a projective resolution

$$\cdots \rightarrow Q^1 \rightarrow Q^0 \rightarrow M^* \rightarrow 0$$

of $M^*$ in $\text{mod}^Z A^{op}$. Applying $\text{HOM}_{A^{op}}(-, A)$, we have an exact sequence

(3-8) \hspace{1cm} 0 \rightarrow M^{**} \rightarrow P^1 \rightarrow P^2 \rightarrow \cdots 

in $\text{mod}^Z A$ where we set $P^i := \text{HOM}_{A^{op}}(Q^{-i+1}, A)$. Similarly, we take a projective resolution $P \xrightarrow{\sim} M$ with $P \in C_{lp}^{-b}(\text{proj}^Z A)$ satisfying $P^{>0} = 0$ and regard it as a projective resolution

(3-9) \hspace{1cm} \cdots \rightarrow P^{-1} \rightarrow P^0 \rightarrow M \rightarrow 0 

of $M$ in $\text{mod}^Z A$. Splicing the exact sequences (3-8) and (3-9), we have an acyclic locally perfect complex

$$P : \cdots \rightarrow P^{-1} \rightarrow P^0 \rightarrow P^1 \rightarrow P^2 \rightarrow \cdots$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow 

M \xrightarrow{\sim} M^{**}$$

such that $Z^0(P) \simeq M$. Thus $M$ belongs to $CM_{lp}^Z A$, and so the first equality holds. \hfill $\Box$

Definition 3.11. Let $A$ be a graded IG-algebra. Then by the observation (3-6), we can define the locally perfect singularity category as the Verdier quotient

$$\text{Sing}_{lp}^Z A := D_{lp}^b(\text{mod}^Z A)/K^b(\text{proj}^Z A),$$

which can be regarded as a triangulated full subcategory of $\text{Sing}^Z A$. 

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Lemma 3.12. Let $A$ be a finitely graded IG-algebra. We assume that $A$ has finite projective dimension as a left and right $A_0$-module. Then the equivalence $\beta : \text{CM}^Z A \xrightarrow{\simeq} \text{Sing}^Z A$ induces an equivalence
\[
\beta : \text{CM}^Z_{\text{lp}} A \xrightarrow{\simeq} \text{Sing}^Z_{\text{lp}} A.
\]

Proof. It follows from Lemma 3.10 that $\beta(M)$ belongs to $\text{Sing}^Z_{\text{lp}} A$ for $M \in \text{CM}^Z_{\text{lp}} A$. So we get a fully faithful functor $\beta : \text{CM}^Z_{\text{lp}} A \to \text{Sing}^Z_{\text{lp}} A$. We prove this functor is essentially surjective. Let $X$ be an object of $\text{Sing}^Z_{\text{lp}} A$ and $X' \in D^b_{\text{lp}}(\text{mod}^Z A)$ a representative of $X$. There exists $M \in \text{CM}^Z A$ such that $\beta(M) \cong X$ by Theorem 2.8. Then there exists a diagram inside $D^b(\text{mod}^Z A)$
\[
M \xleftarrow{f} N \xrightarrow{g} X'
\]
such that the cones $\text{cn}(f), \text{cn}(g)$ of $f, g$ belong to $K^b(\text{proj}^Z A)$. In other words there exist exact triangles
\[
N \xrightarrow{g} X' \rightarrow \text{cn}(g) \rightarrow, \quad N \xrightarrow{f} M \rightarrow \text{cn}(f) \rightarrow.
\]
From the left exact triangle of (3-10), we see that $N$ is locally perfect. Then from the right exact triangle of (3-10), we deduce $M \in D^b_{\text{lp}}(\text{mod} A)$. Therefore, $M$ belongs to $\text{CM}^Z A \cap D^b_{\text{lp}}(\text{mod} A) = \text{CM}^Z_{\text{lp}} A$. □

3.3 Orlov’s equivalence

Let $A$ be a finitely graded IG-algebra. Assume that $A$ has finite projective dimension as a left and right $A_0$-module.

Following Orlov [20], we set
\[
O := D^b_{\text{lp}}(\text{mod}^{\geq 0} A) \cap D^b_{\text{lp}}(\text{mod}^{> 0} A^{\text{op}})^*.
\]

Theorem 3.13. The canonical functor $\pi : D^b_{\text{lp}}(\text{mod}^Z A) \to \text{Sing}^Z_{\text{lp}} A$ induces an equivalence
\[
\pi|_O : O \to \text{Sing}^Z_{\text{lp}} A.
\]

Although we can prove this theorem by the same method with [20], we give a different proof in Section 5.

4 Categorical characterizations of an asid bimodule

Thanks to the quasi-Veronese algebra construction, representation theoretic problem of a finitely graded algebra $A = \bigoplus_{i=0}^t A_i$ can be reduced to that of a trivial extension algebra $A = \Lambda \oplus C$ with the canonical grading $\deg \Lambda = 0, \deg C = 1$.

It was obtained in [15] the condition that $A = \Lambda \oplus C$ is IG in terms of $\Lambda$ and $C$ by using derived tensor products and derived Hom. The aim of Section 4 is to prove two theorems. The first one, Theorem 4.10, gives two categorical characterizations that $A = \Lambda \oplus C$ is IG. The second, Theorem 4.12, verifies several properties of thick subcategories and invariants appearing in the categorical characterizations.
4.1 The asid conditions and the asid numbers

In Section 4.1, we recall the result of [15].

For a bimodule $C$ over $\Lambda$, we define a morphism $\tilde{\lambda}_r : \Lambda \to \text{Hom}_\Lambda(C, C)$ by the formula $\tilde{\lambda}_r(x)(c) := xc$ for $x \in \Lambda$ and $c \in C$. We denote the composite morphism $\lambda_r = \text{can} \circ \tilde{\lambda}_r$ in $D(\text{Mod} \Lambda)$ where $\text{can}$ is the canonical morphism $\text{Hom}_\Lambda(C, C) \to \mathbb{R}\text{Hom}_\Lambda(C, C)$.

$$\lambda_r : \Lambda \to \text{Hom}_\Lambda(C, C) \xrightarrow{\text{can}} \mathbb{R}\text{Hom}_\Lambda(C, C).$$

We denote by $\lambda_\ell$ the left version of $\lambda_r$.

$$\lambda_\ell : \Lambda \to \mathbb{R}\text{Hom}_{\Lambda^{\text{op}}}(C, C).$$

Using these morphisms, we give a condition for $C$ that the trivial extension algebra $A = \Lambda \oplus C$ is IG.

**Theorem 4.1 ([15, Theorem 5.14, Proposition 5.16])**. Let $\Lambda$ be an IG-algebra and $C$ a bimodule over $\Lambda$ which is finitely generated on both sides. Then the trivial extension algebra $A = \Lambda \oplus C$ is IG if and only if the following conditions are satisfied.

1. $C$ has finite projective dimensions on both sides.
2. “The right asid condition”. The morphism $\mathbb{R}\text{Hom}_\Lambda(C^a, \lambda_r)$ is an isomorphism for $a \gg 0$.
3. “The left asid condition”. The morphism $\mathbb{R}\text{Hom}_{\Lambda^{\text{op}}}(C^a, \lambda_\ell)$ is an isomorphism for $a \gg 0$.

**Remark 4.2.** In [15] the right and the left asid conditions are called “the right and the left ASID condition 3”. The first condition in the above theorem is called the “the right and the left ASID condition 1”.

**Definition 4.3.**

1. A $\Lambda$-$\Lambda$-bimodule $C$ is called a **asid (attaching self-injective dimension) bimodule** if the trivial extension algebra $A = \Lambda \oplus C$ is IG.

2. For an asid bimodule $C$, we define the **right asid number** $\alpha_r$ and the **left asid number** $\alpha_\ell$ to be

   $$\alpha_r := \min\{a \geq 0 \mid \mathbb{R}\text{Hom}_\Lambda(C^a, \lambda_r) \text{ is an isomorphism}\},$$

   $$\alpha_\ell := \min\{a \geq 0 \mid \mathbb{R}\text{Hom}_{\Lambda^{\text{op}}}(C^a, \lambda_\ell) \text{ is an isomorphism}\}.$$

Another description of the right and the left asid numbers is given in the following proposition.

**Proposition 4.4 ([15, Corollary 5.12])**. Let $C$ be a asid bimodule over $\Lambda$ and $A = \Lambda \oplus C$ the trivial extension algebra. We regard a minimal injective resolution $I$ of $A$ as a complex. Then,

$$\alpha_r = 1 - \min\{a \in \mathbb{Z} \mid \mathbb{R}\text{Hom}_A(\Lambda, A)^a \neq 0\} = 1 - \min\{a \in \mathbb{Z} \mid i_{a} I \neq 0 \text{ in } D(\text{Mod} \Lambda)\}.$$

**Remark 4.5.** For $a > 1$, we always have $\mathbb{R}\text{Hom}_A(\Lambda, A)^a = 0$ and $i_{a} I = 0$. Therefore the proposition above is the same with [15, Corollary 5.12].

For the definition of $i_{-a} I$, we refer [15, Section 2.3]. Roughly speaking, it is a complex of injective $\Lambda$-modules formed by the cogenerating module of each term $I^n$ in the graded degree $-a$.

In the case where $\Lambda$ is a finite dimensional algebra, the above formula turns out to be written down by graded cosyzygies of $A$. 

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Corollary 4.6. Let $\Lambda, A$ and $C$ be as in the above proposition. Assume moreover that $\Lambda$ is a finite dimensional algebra. We consider the graded cosyzygies $\Omega^{-n}A$ of $A$. Then,

$$\alpha_r = 1 - \min\{a \in \mathbb{Z} | \exists n \text{ s.t. } \text{soc}(\Omega^{-n}A)_a \neq 0\}.$$ 

Remark 4.7. If we assume $C \neq 0$, then $\max\{a \in \mathbb{Z} | \exists n \text{ s.t. } \text{soc}(\Omega^{-n}A)_a \neq 0\} = 1$. Hence we obtain the equation below which tells that $\alpha_r$ is the amplitude of the degrees of the socles of graded cosyzygies $\Omega^{-n}A$.

\begin{equation}
\alpha_r = \max\{a \in \mathbb{Z} | \exists n \text{ s.t. } \text{soc}(\Omega^{-n}A)_a \neq 0\} - \min\{a \in \mathbb{Z} | \exists n \text{ s.t. } \text{soc}(\Omega^{-n}A)_a \neq 0\}.
\end{equation}

4.1.1 Happel’s functor $\varpi$

Let $A = \Lambda \oplus C$ be a trivial extension algebra. We may regard an $\Lambda$-modules $M$ as a graded $A$-module $M$ such that $M_i = 0$ for $i \neq 0$ and embed the category $\text{Mod} \Lambda$ as the full subcategory of $\text{Mod}^{\mathbb{Z}}A$ consisting of graded $A$-modules $M$ such that $M_i = 0$ for $i \neq 0$. We also embed the derived category $D(\text{Mod} \Lambda)$ as a full subcategory of $D(\text{Mod}^{\mathbb{Z}}A)$ consisting of objects $M$ such that $H(M)_i = 0$ for $i \neq 0$.

We assume that $A$ is Noetherian or equivalently $\Lambda$ is Noetherian and $C$ is finitely generated over $\Lambda$. We define a functor $\varpi : D^b(\text{mod} \Lambda) \to \text{Sing}^{\mathbb{Z}}A$ which plays a key role in the paper as the following composite functor

$$\varpi : D^b(\text{mod} \Lambda) \hookrightarrow D^b(\text{mod}^{\mathbb{Z}}A) \xrightarrow{\pi} \text{Sing}^{\mathbb{Z}}A$$

where $\pi$ is the canonical quotient functor. As is mentioned in the introduction, in the case where $A$ is IG, the functor $\mathcal{H} := \beta^{-1} \varpi : D^b(\text{mod} \Lambda) \to \text{CM}^{\mathbb{Z}}A$ is a generalization of the functor $\mathcal{H} : D^b(\text{mod} \Lambda) \to \text{mod} T(\Lambda)$ constructed by Happel. Thus, we call $\varpi$ Happel’s functor.

Assume moreover that $\text{pd} C_{\Lambda} < \infty$. Then for $a \geq 0$, the functor $(- \otimes_{\Lambda}^L C^a)$ is restricted to the endofunctor $(- \otimes_{\Lambda}^L C^a)|_K$ of $\text{K}^b(\text{proj} \Lambda)$. Observe that there exists the following increasing sequence of thick subcategories of $\text{K}^b(\text{proj} \Lambda)$.

\begin{equation}
\text{Ker}(- \otimes_{\Lambda}^L C)|_K \subset \text{Ker}(- \otimes_{\Lambda}^L C^2)|_K \subset \cdots \subset \text{Ker}(- \otimes_{\Lambda}^L C^a)|_K \subset \cdots.
\end{equation}

It was shown that the union of this sequence is the kernel of $\varpi$.

Proposition 4.8 ([13, Corollary 4.18]). Under the above situation, we have

$$\text{Ker} \varpi = \bigcup_{a \geq 0} \text{Ker}(- \otimes_{\Lambda}^L C^a)|_K.$$ 

We end this section by pointing out a relationship between the functor $- \otimes_{\Lambda}^L C$ and the degree shift functor $(1)$ on $\text{Sing}^{\mathbb{Z}}A$. We denote by $\varpi|_K$ the restriction to the homotopy category $\text{K}^b(\text{proj} \Lambda)$.

Lemma 4.9. Under the above situation, we have the following natural isomorphism between the functors from $\text{K}^b(\text{proj} \Lambda)$ to $\text{Sing}^{\mathbb{Z}}A$.

$$\varpi|_K \circ (- \otimes_{\Lambda}^L C[1])|_K \cong (1) \circ \varpi|_K.$$ 

We note that since $\Lambda$ is the degree 0-subalgebra of $A$, we may regard a graded $A$-$A$-bimodule $A X_A$ as a graded $\Lambda$-$A$-bimodule $\Lambda X_A$.

Proof. Let $M \in \text{K}^b(\text{proj} \Lambda)$. Applying $M \otimes_{\Lambda}^L -$ to the standard exact sequence

$$0 \to C \to A(1) \to \Lambda(1) \to 0,$$

we obtain a morphism $f : M(1) \to M \otimes_{\Lambda}^L C[1]$ in $D^b(\text{mod}^{\mathbb{Z}}A)$ which is natural in $M$. It is easy to see that the cone $\text{cn}(f) = M \otimes_{\Lambda}^L A[1]$ belongs to $\text{K}^b(\text{proj}^{\mathbb{Z}}A)$. Therefore $f$ becomes an isomorphism after passing to $\text{Sing}^{\mathbb{Z}}A$. 

\end{proof}
4.2 Statements of Theorems

In Section 4.2, we give statements of main theorem of this section.

Assume that \( \Lambda \) is Noetherian and \( C \) has finite projective dimension on both sides. Then the \( C \)-dual functors \((-)^* := \mathcal{R} \text{Hom}(-, C) \) can be restricted to the perfect derived categories

\[
(-)^* := \mathcal{R} \text{Hom}_\Lambda(-, C) : \mathcal{K}^b(\text{proj} \Lambda) \cong \mathcal{K}^b(\text{proj} \Lambda^\text{op})^\text{op} : \mathcal{R} \text{Hom}_{\Lambda^\text{op}}(-, C) =: (-)^*.
\]

We also use the \( \Lambda \)-dual functors. Since it is necessary to distinguish the right \( \Lambda \)-duality and the left \( \Lambda \)-duality, we denote by \((-)^\dagger := \mathcal{R} \text{Hom}_\Lambda(-, \Lambda) \) the right \( \Lambda \)-dual functor and by \((-)^\ddagger := \mathcal{R} \text{Hom}_{\Lambda^\text{op}}(-, \Lambda) \) the left \( \Lambda \)-dual functor.

\[
(-)^\dagger := \mathcal{R} \text{Hom}_\Lambda(-, \Lambda) : \mathcal{D}(\text{mod} \Lambda) \cong \mathcal{D}(\text{mod} \Lambda^\text{op})^\text{op} : \mathcal{R} \text{Hom}_{\Lambda^\text{op}}(-, \Lambda) =: (-)^\ddagger.
\]

We remark that these functors induce contravariant equivalences between the homotopy categories \( \mathcal{K}^b(\text{proj} \Lambda) \) and \( \mathcal{K}^b(\text{proj} \Lambda^\text{op}) \).

The aim of this section is to prove the following two theorems. The first one gives categorical characterizations of asid bimodules.

**Theorem 4.10.** Let \( \Lambda \) be an IG-algebra and \( C \) a \( \Lambda \)-\( \Lambda \)-bimodules which is finitely generated on both sides and has finite projective dimension on both sides. The following conditions are equivalent

1. \( C \) is an asid bimodule.
2. \( \mathcal{K}^b(\text{proj} \Lambda) \) has an admissible subcategory \( T \) such that
   2-a) the functor \( T = - \otimes^\mathbb{L}_{\Lambda} C \) acts on \( T \) as an equivalence, i.e., \( T(T) \subset T \) and the restriction functor \( T|_T \) is an autoequivalence.
   2-b) the functor \( T = - \otimes^\mathbb{L}_{\Lambda} C \) nilpotently acts on \( T^\perp \), i.e., \( T(T^\perp) \subset T^\perp \) and there exists a natural number \( a \in \mathbb{N} \) such that \( T^a(T^\perp) = 0 \).
3. \( \mathcal{K}^b(\text{proj} \Lambda) \) has a thick subcategory \( T \) such that
   3-a) the \( C \)-dual functors induce equivalences
       \[ (-)^* : T \cong T^\triangleright : (-)^* \]

   where \( T^\triangleright \) denotes the image of \( T \) by the functor \((-)^\triangleright \).
   3-b) there exists a natural number \( a \geq 0 \) such that \( C^a, (C^a)^\dagger \in T \).

**Remark 4.11.** First we remark that what we actually show in the proof provided in the next section is that if \( \Lambda \) is Noetherian and a bimodule \( C \) over it has finite projective dimension from both sides, then the right and the left asid conditions are equivalent to the conditions (2) or (3) of the above theorem.

The second main result of this section shows that the left and right asid numbers coincide with each other and that the triangulated subcategories \( T \) and \( T^\perp \) appeared in the above theorem are uniquely determined by an asid bimodule \( C \).

**Theorem 4.12.** Let \( \Lambda \) be an IG-algebra and \( C \) an asid bimodule. Then the following assertions hold.
(1) We have $\alpha_r = \alpha_{\ell}$. We put $\alpha := \alpha_r = \alpha_{\ell}$.

(2) The subcategory $T$ of (2) or (3) of Theorem 4.10 is uniquely determined as $T = \text{thick } C^\alpha$.

(3) We have $T^\perp = \text{Ker } \varpi = \text{Ker}(\omega \otimes^L_C C^\alpha)|_K$.

We introduce a terminology.

**Definition 4.13.** Let $\Lambda$ be an IG-algebra, $C$ an asid bimodule and $\alpha := \alpha_r = \alpha_{\ell}$. We call the subcategory $\text{thick } C^\alpha$ the *asid* subcategory of $C$.

### 4.3 Proof of Theorem 4.10 and Theorem 4.12

In the rest of Section 4, we keep the notation and the assumption of Theorem 4.10.

#### 4.3.1 Interpretations of the asid conditions

First, we give interpretations of the asid conditions. We only discuss the right asid condition, since the left version follows from the dual argument.

For the purpose, we collect natural isomorphisms which relate to the morphism $\lambda_r : \Lambda \to \mathbb{R}\text{Hom}_\Lambda(C, C)$.

We will show up natural isomorphisms which are also used in the proofs of Theorem 4.10 and Theorem 4.12 so that the readers can easily consult their definitions.

- The natural isomorphism $\mathcal{F}$: We denote by $\mathcal{F}_{M,N} : \mathbb{R}\text{Hom}_\Lambda(M, N) \to \mathbb{R}\text{Hom}_\Lambda(M \otimes^L_A C, N \otimes^L_A C)$ the morphism induced by the functor $T = - \otimes^L_A C$.

- The natural isomorphism $\mathcal{E}$: Let $D$ be a $\Lambda\Lambda$-bimodule and $M \in \text{Mod } \Lambda$. We define a morphism $\mathcal{E}_{D,M} : \text{Hom}_\Lambda(D, M) \otimes^L_A D \to M$ by the formula $\mathcal{E}_{D,M}(f \otimes d) = f(d)$ for $f \in \text{Hom}_\Lambda(D, M)$ and $d \in D$. We note that $\mathcal{E}_{D,M}$ is natural in $D$ and $M$ and that it is the counit morphism of the adjoint pair $- \otimes^L_A D : \text{Mod } \Lambda \rightleftarrows \text{Mod } \Lambda : \text{Hom}_\Lambda(D, -)$.

  Let $D$ be a complex of $\Lambda\Lambda$-bimodules and $M \in \text{D(Mod } \Lambda)$. We denote by $\mathcal{E}_{D,M} : \mathbb{R}\text{Hom}_\Lambda(D, M) \otimes^L_A D \to M$ the morphism which is derived from the morphism $\mathcal{E}$. We note that $\mathcal{E}_{D,M}$ is the counit morphism of the adjoint pair $- \otimes^L_A D : \text{D(Mod } \Lambda) \rightleftarrows \text{D(Mod } \Lambda) : \mathbb{R}\text{Hom}_\Lambda(D, -)$.

- The natural isomorphisms $\epsilon_r, \epsilon_{\ell}$: We set $\epsilon_r := \mathcal{E}_{C,\Lambda}$ and $\epsilon_{\ell}$ to be the left version of $\epsilon_r$.

  $\epsilon_r : C^r \otimes^L_A C = \mathbb{R}\text{Hom}_\Lambda(C, \Lambda) \otimes^L_A C \to \Lambda$, \hspace{1em} $\epsilon_{\ell} : C \otimes^L_A C^{\alpha} = C \otimes^L_A \mathbb{R}\text{Hom}_{\Lambda^{op}}(C, \Lambda) \to \Lambda$.

**Proposition 4.14.** For a natural number $a \in \mathbb{N}$, the following conditions are equivalent.

(1) The morphism $\mathbb{R}\text{Hom}_\Lambda(C^a, \lambda_r)$ is an isomorphism.

(2) The morphism $\mathcal{F}_{C^a,\Lambda} : \mathbb{R}\text{Hom}_\Lambda(C^a, \Lambda) \to \mathbb{R}\text{Hom}_\Lambda(C^{a+1}, C)$ is an isomorphism.

(3) The morphism $\mathcal{F}_{M \otimes^L_C C^a, N} : \text{Hom}_\Lambda(M \otimes^L_A C^a, N) \to \text{Hom}_\Lambda(M \otimes^L_A C^{a+1}, N \otimes^L_A C)$ is an isomorphism for $M, N \in \mathcal{K}^b(\text{proj } \Lambda)$ where $\mathcal{F}_{M \otimes^L_C C^a, N}$ is the morphism of Hom-spaces associated to the functor $T = - \otimes^L_A C$.

(4) $C^a \otimes^L_A \epsilon_{\ell}$ is an isomorphism.
Before starting the proof, we need to introduce a natural isomorphism \( \mathcal{F} \).

- The natural isomorphism \( \mathcal{F} \): For \( \Lambda \)-modules \( M, N \) and \( \Lambda \)-\( \Lambda \)-bimodules \( D, E \), we define a morphism \( \mathcal{F}_{M,N,D,E} \) to be

\[
\mathcal{F}_{M,N,D,E} : N \otimes_{\Lambda} \text{Hom}_{\Lambda}(D, E) \otimes_{\Lambda} \text{Hom}_{\Lambda}(M, \Lambda) \to \text{Hom}_{\Lambda}(M \otimes_{\Lambda} D, N \otimes_{\Lambda} E),
\]

for \( n \in N, f \in \text{Hom}_{\Lambda}(D, E), \phi \in \text{Hom}_{\Lambda}(M, \Lambda), m \in M, d \in D \). For \( M, N \in \text{D}(\text{Mod} \, \Lambda) \) and complexes \( D, E \) of \( \Lambda \)-\( \Lambda \)-bimodules, we define \( \mathcal{F}_{M,N,D,E} \) to be the morphism derived from \( \mathcal{F} \).

We note that if \( M, N \) belong to \( K^b(\text{proj} \, \Lambda) \), then \( \mathcal{F}_{M,N,D,E} \) is an isomorphism.

In the proof and thereafter, we (tacitly) use the following natural isomorphisms.

- The natural isomorphism \( \mathcal{F} \): By \( \mathcal{F}_M : M \to M^{\text{op}} \) we denote the unit map of the adjoint pair \((-)^{\text{op}} \dashv (-)^{\text{op}}\). We note that it is an isomorphism for \( M \in K^b(\text{proj} \, \Lambda) \).

**Proof of Proposition 4.14** Let \( \text{adj}_{C^a,C} : \text{RHom}_{\Lambda}(C^a, \text{RHom}_{\Lambda}(C, C)) \cong \text{RHom}_{\Lambda}(C^{a+1}, C) \) be the isomorphism induced from the adjoint pair \(- \otimes_{\Lambda} C \dashv \text{RHom}_{\Lambda}(C, -)\). Then, we can check the equation \( \mathcal{F}_{C^a,C} = \text{adj}_{C^a,C} \circ \text{RHom}_{\Lambda}(C^a, \lambda_C) \), which implies the equivalence \( (1) \Leftrightarrow (2) \).

\[
\begin{array}{ccc}
\text{RHom}_{\Lambda}(C^a, \Lambda) & \xrightarrow{\text{RHom}(C^a, \lambda_C)} & \text{RHom}_{\Lambda}(C^a, \text{RHom}_{\Lambda}(C, C)) \\
\downarrow & & \downarrow \text{adj}_{C^a,C} \\
\text{RHom}_{\Lambda}(C^{a+1}, C) & & \\
\end{array}
\]

We prove the implication \( (3) \Rightarrow (2) \). First observe that the condition \( (3) \) is satisfied if and only if \( (3') \) the morphism \( \mathcal{F}_{M \otimes_{\Lambda} C^a,N} : \text{RHom}_{\Lambda}(M \otimes_{\Lambda} C^a, N) \to \text{RHom}_{\Lambda}(M \otimes_{\Lambda} C^{a+1}, N \otimes_{\Lambda} C) \) is an isomorphism for \( M, N \in K^b(\text{proj} \, \Lambda) \). If we substitute both \( M \) and \( N \) with \( \Lambda \) in the condition \( (3') \), then we see that \( (3') \) implies \( (2) \).

We prove the implication \( (2) \Rightarrow (3) \). We can check that the following diagram is commutative.

\[
\begin{array}{ccc}
N \otimes_{\Lambda} \text{RHom}_{\Lambda}(C^a, \Lambda) & \otimes_{\Lambda} M^{\text{op}} & N \otimes_{\Lambda} \text{RHom}_{\Lambda}(C^{a+1}, C) \otimes_{\Lambda} M^{\text{op}} \\
\downarrow \mathcal{F}_{M,N,C^a,N} \quad \cong & \downarrow \mathcal{F}_{M,N,C^{a+1},N} \\
\text{RHom}_{\Lambda}(M \otimes_{\Lambda} C^a, N) & & \text{RHom}_{\Lambda}(M \otimes_{\Lambda} C^{a+1}, N \otimes_{\Lambda} C).
\end{array}
\]

Since we are assuming that \( M, N \) belongs to \( K^b(\text{proj} \, \Lambda) \), the vertical arrows are isomorphism. This shows that the condition \( (2) \) implies the condition \( (3') \) and hence the condition \( (3) \).

To prove the equivalence \( (1) \Leftrightarrow (4) \), we need to introduce one more natural morphism, which is denoted by \( \mathcal{H} \). First for \( \Lambda \)-\( \Lambda \)-bimodules \( D, E \), we define a morphism \( \mathcal{H}_{D,E} \) to be

\[
\mathcal{H}_{D,E} : D \otimes_{\Lambda} E \otimes_{\Lambda} \text{Hom}_{\Lambda\text{op}}(E, \Lambda) \to \text{Hom}_{\Lambda\text{op}}(\text{Hom}_{\Lambda}(D, \text{Hom}_{\Lambda}(E, E)), \Lambda),
\]

\[
\mathcal{H}_{D,E}(d \otimes e \otimes f)(\phi) := f(\phi(d)(e))
\]

for \( d \in D, e \in E, f \in \text{Hom}_{\Lambda\text{op}}(E, \Lambda), \phi \in \text{Hom}_{\Lambda}(D, \text{Hom}_{\Lambda}(E, E)) \). For complexes \( D, E \) of \( \Lambda \)-\( \Lambda \)-bimodules, we define a morphism \( \mathcal{H}_{D,E} \) to be the morphism derived from \( \mathcal{H} \).

\[
\mathcal{H}_{D,E} : D \otimes_{\Lambda} E \otimes_{\Lambda} E_{\text{op}} \to \text{RHom}_{\Lambda}(D, \text{RHom}_{\Lambda}(E, E))_{\text{op}}.
\]
We can check that the following diagram is commutative and that the left vertical arrow \( H_{C^a,C} \) is an isomorphism

\[
\begin{array}{c}
C^a \otimes^\Lambda_C C \otimes^\Lambda_C C^{<} \xrightarrow{H_{C^a,C}} C^a \otimes^\Lambda_C \Lambda \\
\mathbb{R}\text{Hom}_A(C^a, \mathbb{R}\text{Hom}_A(C, C))^{<} \xrightarrow{\mathbb{R}\text{Hom}(C^a, \Lambda^{<})} \mathbb{R}\text{Hom}_A(C^a, \Lambda)^{<}
\end{array}
\]

where the right vertical arrow is a canonical isomorphism. Now it is clear that (1) ⇔ (4).

Remark 4.15. The equivalence (1) ⇔ (2) ⇔ (3) of Proposition 4.14 is true without the assumption that \( C \) belongs to \( \mathcal{K}_b^{b}(\text{proj} \Lambda) \) and \( C \) belongs to \( \mathcal{K}_b^{b}(\text{proj} \Lambda^{op}) \).

4.3.2 The thick hull \( \text{thick} C^a \)

We remind the following fact which will be frequently used in the sequel. Let \( M, N \) be objects of \( \mathcal{K}_b^{b}(\text{proj} \Lambda) \) and \( D \) a complex of \( \Lambda-\Lambda \)-bimodules. Then \( \text{thick} N \subset \text{thick} M \) if and only if \( N \in \text{thick} M \). Moreover if \( N \in \text{thick} M \), then \( N \otimes^\Lambda_C D \in \text{thick} M \otimes^\Lambda_C D \).

By the assumption \( C \) belongs to \( \text{thick} \Lambda = \mathcal{K}_b^{b}(\text{proj} \Lambda) \). Thus taking \( M = \Lambda, N = C, D = C^a \) in the above consideration, we see that \( \text{thick} C^{a+1} \subset \text{thick} C^a \) and obtain the following descending chain of thick subcategories

\[
\mathcal{K}_b^{b}(\text{proj} \Lambda) \supset \text{thick} C \supset \text{thick} C^2 \supset \cdots \supset \text{thick} C^a \supset \cdots
\]

We show that if the left and right asid conditions are satisfied, then this chain terminates. But actually, we prove more properties of these and related thick subcategories in the following lemma.

Lemma 4.16. We assume that \( C \) satisfies the left and right asid conditions. Let \( \alpha_r \) and \( \alpha_\ell \) be right and left asid number. Then we have the following equalities in \( \mathcal{K}_b^{b}(\text{proj} \Lambda) \).

1. \( \text{thick} C^a = \text{thick} C^{a+1} \) for \( a \geq \alpha_\ell \).
2. \( \text{thick} C^a = \text{thick}(C^a)^{<} \) for \( a \geq \max\{\alpha_r, \alpha_\ell\} \).
3. \( \text{thick}(C^a)^{<} = \text{thick}(C^{a+1})^{<} \) for \( a \geq \alpha_r \).
4. \( \text{Ker}(- \otimes^\Lambda_C C^a)|^\mathcal{K} = \text{Ker}(- \otimes^\Lambda_C C^{a+1})|_\mathcal{K} \) for \( a \geq \alpha_r \).

To prove this, we need to verify a compatibility between the iterated derived tensor products \( C^a \) and the \( \Lambda \)-duality. For this, we introduce the following natural morphism.

- The natural isomorphism \( \mathcal{I} \): Let \( D \) be a complex of \( \Lambda-\Lambda \)-bimodules and \( M \in D(\text{Mod} \Lambda) \). For simplicity, we set \( \mathcal{I}_{D,M} := \mathcal{I}_{M,\Lambda,D,\Lambda} \) and regard this as a morphism \( \mathcal{I}_{D,M} : D^\circ \otimes^\Lambda \Lambda^{\circ} \to (M \otimes^\Lambda \Lambda)^{\circ} \) via canonical isomorphisms as below

\[
\mathcal{I}_{D,M} : D^\circ \otimes^\Lambda \Lambda^{\circ} \cong \Lambda \otimes^\Lambda \mathbb{R}\text{Hom}_A(D, \Lambda)^{\circ} \xrightarrow{\mathcal{I}_{M,\Lambda,D,\Lambda}} \mathbb{R}\text{Hom}_A(M \otimes^\Lambda \Lambda, \Lambda)^{\circ} \cong (M \otimes^\Lambda \Lambda)^{\circ}
\]

We leave the proof of the following lemma to the readers.
Lemma 4.17. Let $D_1, D_2$ be $\Lambda\Lambda$-bimodule complexes which are perfect as left and right $\Lambda$-complexes. Then, we have the following commutative diagram

\[
\begin{array}{c}
D_2^\ast \otimes^L_A D_1^\ast \otimes^L_A D_1 \otimes^L_A D_2 \\
\downarrow \quad \downarrow \quad \downarrow \\
(D_1 \otimes^L_A D_2)^\ast \otimes^L_A D_1 \otimes^L_A D_2 \\
\end{array}
\]

where $\mathcal{E}^{(i)} = \mathcal{E}_{D_i, \Lambda}$ for $i = 1, 2$ and $\mathcal{E}^{(12)} = \mathcal{E}_{D_1 \otimes^L_A D_2, \Lambda}$.

The thick hull of an object $L$ of $K^b(\text{proj } \Lambda^{\text{op}})$ is denoted by $\text{thick}_{\Lambda^{\text{op}}} L$.

**Proof of Lemma 4.17.** (1) By the left version of Proposition 4.14, we have an isomorphism $C^a \cong C^a \otimes^L_A C^{a+1}$ for $a \geq \alpha_\ell$.

For an object $M \in \text{D}(\text{Mod } \Lambda)$, we denote by $\text{Loc } M$ the localizing subcategory generated by $M$, i.e., the smallest triangulated subcategory of $\text{D}(\text{Mod } \Lambda)$ containing $M$ which is closed under taking arbitrarily coproducts. Since $\text{D}(\text{Mod } \Lambda) = \text{Loc } \Lambda$, we see that $C^a \cong C^a \otimes^L_A C^{a+1}$ belongs to Loc $C^{a+1}$. Therefore $C^a$ belongs to $K^b(\text{proj } \Lambda) \cap \text{Loc } C^{a+1}$.

For a triangulated category $\text{D}$, we denote by $\text{D}^{\text{cpt}}$ the full subcategory of compact objects. It is well-known that $\text{D}(\text{Mod } \Lambda)^{\text{cpt}} = K^b(\text{proj } \Lambda)$. Therefore by [19, Theorem 2.1, Lemma 2.2], we have

$$K^b(\text{proj } \Lambda) \cap \text{Loc } C^{a+1} = (\text{Loc } C^{a+1})^{\text{cpt}} = \text{thick } C^{a+1}.$$  

Thus, we conclude $C^a \in \text{thick } C^{a+1}$ as desired.

(2) Let $a \geq \alpha_\ell$. Since $C^a \cong C^b \otimes^L_A C^{a+1} \cong (C^b)^\ast \otimes^L_A C^{a+b} \cong (C^b)^\ast \otimes^L_A C^{a+b}$ for $b \geq 1$, we have $\Lambda C^a \in \text{thick}_{\Lambda^{\text{op}}}(C^b)^\ast \subset \text{D}^b(\text{mod } \Lambda^{\text{op}})$. Thus $(C^a)^\ast \subset \text{thick } C^b$ for $b \geq 1$.

In the same way, for $a \geq \alpha_r$, we deduce $C^a \in \text{thick } (C^b)^\ast$ for $b \geq 1$. Hence we have $\text{thick } C^a = \text{thick } (C^a)^\ast$.

(3) and (4) follows from the left version of (1).

By Lemma 4.16 (4), if $C$ satisfies the right and the left asid conditions, then the increasing sequence (4-13) terminates at $a = \max\{\alpha_r, \alpha_\ell\}$. From Corollary 4.8 we deduce the following description of $\text{Ker } \varpi$.

**Corollary 4.18.** Assume that $C$ satisfies the right and the left asid conditions. Then we have

$$\text{Ker } \varpi = \text{Ker }((- \otimes^L_A C^a)|_K)$$

for $a \geq \max\{\alpha_r, \alpha_\ell\}$.

By Lemma 4.16 (1), if $C$ satisfies the right and the left asid conditions, then the decreasing sequence (4-14) terminates at $a = \alpha_\ell$. In the next lemma which is a key for the main theorems we study the subcategory $\text{thick } C^{\max(\alpha_r, \alpha_\ell)}$.

**Lemma 4.19.** Assume that $C$ satisfies the right and the left asid conditions. We set $\alpha = \max\{\alpha_r, \alpha_\ell\}$, $T := \text{thick } C^\alpha$. Then the following assertions hold.

(1) The functor $T := - \otimes^L_A C$ acts $T$ as an equivalence.

(2) We have a semi-orthogonal decomposition

$$K^b(\text{proj } \Lambda) = T \perp \text{Ker } \varpi.$$
In the proof and thereafter, we (tacitly) use the following natural isomorphisms.

- The natural isomorphism $\mathcal{J}: \text{Let } M, N \text{ be complex of A-modules. For simplicity, we set } \mathcal{J}_{M,N} := \mathcal{J}_{M,N,A,A} \text{ and regard this as a morphism } \mathcal{J}_{M,N} : N \otimes^L_A M^\circ \to \mathbb{R} \text{Hom}_A(M, N). \text{ We note that } \mathcal{J}_{M,N} \text{ is an isomorphism if one of } M, N \text{ belongs to } \mathcal{K}^b(\text{proj } A).

Proof. (1) It is clear that $\mathcal{T}(T) \subset T$ by Lemma 4.16 (1).

Applying Proposition 4.17, we see that the morphism $\mathcal{T}_{C^\alpha,C^\alpha[n]}$ is an isomorphism for $n \in \mathbb{Z}$.

$$\mathcal{T}_{C^\alpha,C^\alpha[n]} : \text{Hom}_A(C^\alpha, C^\alpha[n]) \to \text{Hom}_A(C^{\alpha+1}, C^{\alpha+1}[n])$$

Since $\mathcal{T} = \text{thick } C^\alpha$, we conclude that $\mathcal{T}|_\mathcal{T}$ is fully faithful by standard argument.

Since $C^{\alpha+1} \in \text{Im } \mathcal{T}|_\mathcal{T}$ and $\mathcal{T}|_\mathcal{T}$ is fully faithful, we see that $\text{thick } C^{\alpha+1} \subset \text{Im } \mathcal{T}|_\mathcal{T}$ by standard argument. Thus by Lemma 4.16, we conclude that $\mathcal{T}|_\mathcal{T}$ is essentially surjective.

(2) First we claim that $\text{Hom}_A(X, Y) = 0$ for $X \in \mathcal{T}, Y \in \text{Ker } \varpi$. Indeed, since $\mathcal{T} = \text{thick } (C^\alpha)^{\circ}$, it is enough to check the case $X = (C^\alpha)^{\circ}$. It follows from $\text{Ker } \varpi = \text{Ker } (- \otimes^L_A C^\alpha)$ that

$$\mathbb{R} \text{Hom}_A((C^\alpha)^{\circ}, Y) \cong Y \otimes^L_A (C^\alpha)^{\circ \circ} \cong Y \otimes^L_A C^\alpha = 0.$$ 

Now it is enough to prove that every $M \in \mathcal{K}^b(\text{proj } A)$ fits into an exact triangle $X \to M \to Y \to$ with $X \in \mathcal{T}, Y \in \text{Ker } \varpi$. Since $\mathbb{R} \text{Hom}_A(C^\alpha, M) \otimes^L_A C^\alpha$ belongs to $\mathcal{T} = \text{thick } C^\alpha$ it is enough to show that the cone of the derived evaluation map $\varepsilon_{C^\alpha,M} : \mathbb{R} \text{Hom}_A(C^\alpha, M) \otimes^L_A C^\alpha \to M$ belongs to $\text{Ker } (- \otimes^L_A C^\alpha)$. In other words, if we set $\varepsilon'_M := \varepsilon_{C^\alpha,M}$, we only have to show that $\varepsilon'_M \otimes^L_A C^\alpha$ is an isomorphism.

First observe that we have the following commutative diagram where the bottom arrow is the canonical isomorphism.

$$\begin{array}{ccc}
M \otimes^L_A (C^\alpha)^{\circ} \otimes^L_A C^\alpha & \xrightarrow{\mathcal{J} \otimes^L_A C^\alpha} & \mathbb{R} \text{Hom}_A(C^\alpha, M) \otimes^L_A C^\alpha \\
M \otimes^L_A L & \cong & M
\end{array}$$

Therefore the problem is reduced to the case where $M = L$.

Let $\mathcal{J}' : (C^\circ)^{\alpha} \to (C^\alpha)^{\circ}$ be the isomorphism which is obtained from the morphism $\mathcal{J}$. For simplicity we set $\varepsilon_{(\alpha)} := (C^\circ)^{\alpha} \otimes^L \varepsilon_r \otimes^L C^\alpha$. We identify $(C^\circ)^{\alpha-1} \otimes^L_A \Lambda \otimes^L_A C^{\alpha-1}$ with $(C^\circ)^{\alpha-1} \otimes^L_A C^{\alpha-1}$ via the canonical isomorphism and regard $\varepsilon_{(\alpha)}$ as the morphism from $(C^\circ)^{\alpha} \otimes^L_A C^\alpha$ to $(C^\circ)^{\alpha-1} \otimes^L_A C^{\alpha-1}$. Then by Lemma 4.17 we have $\varepsilon'_\Lambda \circ (\mathcal{J}' \otimes^L C^\alpha) = \varepsilon_r \circ \varepsilon_{(1)} \circ \cdots \circ \varepsilon_{(\alpha-1)}$.

$$(C^\circ)^{\alpha} \otimes^L_A C^\alpha \xrightarrow{\varepsilon_{(\alpha-1)}} (C^\circ)^{\alpha-1} \otimes^L_A C^{\alpha-1} \xrightarrow{\varepsilon_{(\alpha-2)}} \cdots \to C^\circ \otimes^L_A C \xrightarrow{\varepsilon_r} \Lambda$$

Since $\varepsilon_r \otimes^L C^\alpha$ is an isomorphism by the left version of Proposition 4.14, we conclude that $\varepsilon'_\Lambda \otimes^L C^\alpha$ is an isomorphism as desired. \hfill \Box

In the proof of Proposition 4.19 we obtained the following result.

**Corollary 4.20.** The functor $\tau : \mathcal{K}^b(\text{proj } A) \to \mathcal{T}$, $\tau(M) := \mathbb{R} \text{Hom}_A(C^\alpha, M) \otimes^L_A C^\alpha$ is a right adjoint functor of the inclusion functor $i : \mathcal{T} \to \mathcal{K}^b(\text{proj } A)$. 

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4.3.3 Lemmas

The following abstract lemma clarify the situation.

Lemma 4.21. Let $D$ be a triangulated subcategory, $E \subset D$ an admissible subcategory and $G : D \rightleftarrows D : F$ an adjoint pair of exact endofunctors. Assume that $F$ acts on $E$ as an equivalence and nilpotently acts on $E^\perp$. Then for a natural number $a \geq 0$, the following conditions are equivalent.

1. $F^a(D) \subset E$.
2. $F^a(E^\perp) = 0$.
3. The morphism $F_{F^a(d),d'} : \text{Hom}_D(F^a(d), d') \to \text{Hom}_D(F^{a+1}(d), F(d'))$ is an isomorphism for $d, d' \in D$.
4. $G^a(D) \subset E$.
5. $G^a(E^\perp) = 0$.
6. The morphism $G_{d,G^a(d')} : \text{Hom}_D(d, G^a(d')) \to \text{Hom}_D(G(d), G^{a+1}(d'))$ is an isomorphism for $d, d' \in D$.

Proof. (2) $\Rightarrow$ (1). Let $d$ be an object of $D$ and $e' \to d \to k \to$ an exact triangle such that $e' \in E, k \in E^\perp$. Since $F^a(k) = 0$, we have $F^a(d) \cong F^a(e')$.

(3) $\Rightarrow$ (2). Let $k \in E^\perp$. Then the morphism

$$F_{F^a(k),F^a(k)}^b : \text{Hom}_D(F^a(k), F^a(k)) \to \text{Hom}_D(F^{a+b}(k), F^{a+b}(k))$$

is an isomorphism for $b \geq 0$. Since $F^{a+b}(k) = 0$ for $b \gg 0$, we conclude $F^a(k) = 0$.

For the implication (1)$\Rightarrow$(3), it is enough to show that the map $F_{e,d} : \text{Hom}_D(e, d) \to \text{Hom}_D(F(e), F(d))$ is an isomorphism for $e \in E, d \in D$.

Let $e' \xrightarrow{\varphi} d \to k$ be the exact triangle such that $e' \in E, k \in E^\perp$. Since $\text{Hom}_D(e,k[n]) = 0$ for $n = 0, -1$, the induced map $\varphi_* : \text{Hom}_D(e, e') \to \text{Hom}_D(e, d)$ is an isomorphism. By the assumptions, we have $F(E) \subset E$ and $F(E^\perp) \subset E^\perp$. Thus, $\text{Hom}_D(F(e), F(k)[n]) = 0$ for $n = 0, -1$. Hence, the induced map $F(\varphi)_* : \text{Hom}_D(F(e), F(e')) \to \text{Hom}_D(F(e), F(d))$ is an isomorphism. Since $F_{e,d} \circ \varphi_* = F(\varphi)_* \circ F_{e,e'}$ and $F_{e,e'}$ is an isomorphism, we conclude that $F_{e,d}$ is an isomorphism.

We prove (1) $\Rightarrow$ (5). Let $\ell \in E^\perp, d \in D$. We have $\text{Hom}_D(G^a(\ell), d) \cong \text{Hom}_D(\ell, F^a(d)) = 0$. Therefore $G^a(\ell) = 0$.

We have proved the equivalence (1) $\iff$ (2) $\iff$ (3) and the implication (1) $\Rightarrow$ (5).

Next we claim that

Claim 4.22. (a) $G$ acts $E$ as an equivalence.

(b) $G$ nilpotently acts on $E^\perp$.

Proof of Claim. (a) For $e \in E$, we have $\text{Hom}_D(G(e), k) = \text{Hom}_D(e, F(k)) = 0$ for any $k \in E^\perp$. Hence $G(e)$ belongs to $\underleftarrow{\text{Hom}}(E^\perp) = E$. This shows that $G(E) \subset E$. Now $G|_E$ can be regarded as an endofunctor of $E$. Then it is a left adjoint of the equivalence $F|_E$. Hence $G|_E = (F|_E)^{-1}$ and in particular $G|_E$ is an equivalence.

(b) We claim that $G(E^\perp) \subset E^\perp$. Indeed, for $\ell \in E^\perp, t \in E$, we have the equality $\text{Hom}_D(G(\ell), e) = \text{Hom}_D(\ell, F(e)) = 0$. Thus $G(\ell) \in E^\perp$.

By the assumption $F$ nilpotently acts on $E^\perp$. Therefore by the implication (2) $\Rightarrow$ (5), we check that $G^b(E^\perp) = 0$ for $b \gg 0$. \qed
Since $G$ acts $\mathcal{E}$ as an equivalence and nilpotently acts on $\mathcal{E}$, we can apply dual argument to $G$ to prove the equivalence (4) $\iff$ (5) $\iff$ (6) and the implication (4) $\Rightarrow$ (2).

To apply above lemma to the proofs of Theorem 4.10 and Theorem 4.12, we need to show that the functor $- \otimes^L_A \mathcal{C}$ has a left adjoint functor.

**Lemma 4.23.** We have the following adjoint pair

$$- \otimes^L_A \mathcal{C} : K^b(\text{proj } \mathcal{A}) \rightleftarrows K^b(\text{proj } \mathcal{A}) : - \otimes^L_A \mathcal{C}. $$

**Proof.** By the assumption that the complex $\mathcal{C}$ is perfect as left modules and right modules, we see that the adjoint pair $- \otimes^L \mathcal{C} \dashv \mathbb{R}\text{Hom}_{\mathcal{A}}(\mathcal{C}^a, -)$ of endofunctors on $\mathcal{D}(\text{Mod } \mathcal{A})$ can be restricted $K^b(\text{proj } \mathcal{A})$.

$$- \otimes^L_A \mathcal{C} : K^b(\text{proj } \mathcal{A}) \rightleftarrows K^b(\text{proj } \mathcal{A}) : \mathbb{R}\text{Hom}_{\mathcal{A}}(\mathcal{C}^a, -).$$

On the other hands by the assumption, we have isomorphism of functors below induced by the natural morphisms $\mathcal{J}$ and the left version $\mathcal{G}_t$ of $\mathcal{G}$

$$- \otimes^L \mathcal{C} \xrightarrow{\cong \mathcal{G}_t} - \otimes^L \mathcal{C}^a \xrightarrow{\cong \mathcal{J}} \mathbb{R}\text{Hom}_{\mathcal{A}}(\mathcal{C}^a, -).$$

Combining these observations, we obtain the desired adjoint pair.

### 4.3.4 Proof of Theorem 4.10

We proceed a proof of Theorem 4.10.

**Proof of Theorem 4.10.** We prove the implication (1) $\Rightarrow$ (2). We set $T = \text{thick } C^\alpha_{\text{max} \{\alpha_r, \alpha_l\}}$. By Lemma 4.19 the functor $T$ acts on $T$ as an equivalence and the subcategory $T$ is right admissible. It follows from Lemma 4.19 and Corollary 4.18 that the functor $T$ nilpotently acts on $T^\perp$.

By the left version of Lemma 4.19 the subcategory $\text{thick}_{\mathcal{A}} C^\alpha$ is a right admissible subcategory of $K^b(\text{proj } \mathcal{A}^{\text{op}})$. Since the $\mathcal{A}$-dual functor $(-)^{\wedge} : K^b(\text{proj } \mathcal{A}^{\text{op}}) \xrightarrow{\sim} K^b(\text{proj } \mathcal{A})$ gives a contravariant equivalence, the subcategory $T = \text{thick}(C^\alpha)^{\wedge} = (\text{thick}_{\mathcal{A}^{\text{op}}} C^\alpha)^{\wedge}$ is left admissible. This finishes the proof (1) $\Rightarrow$ (2).

We prove the implication (2) $\Rightarrow$ (3). First note that by the assumption, Proposition 4.14 and Lemma 4.23 we can apply Lemma 4.21 and Claim 4.22 to the adjoint pair $- \otimes^L_A \mathcal{C}^a \dashv - \otimes^L_A \mathcal{C}$.

By Claim 4.22 the functor $- \otimes^L_A \mathcal{C}$ equivalently acts on $T$. Therefore the functor $C \otimes^L_A = - \otimes^L_A \mathcal{C}^{\alpha}$ equivalently acts on the full subcategory $T^{\text{op}}$ of $K^b(\text{proj } \mathcal{A}^{\text{op}})$.

Recall that we have an isomorphism $\mathcal{J}_{C,M} : C \otimes^L_A M^{\text{op}} \cong M^{*}$ which is natural in $M \in K^b(\text{proj } \mathcal{A})$. We also have an isomorphism $N^{\alpha} \otimes^L_A C \cong N^{*}$ as the left version of the above isomorphism which is natural in $N \in K^b(\text{proj } \mathcal{A}^{\text{op}})$. Therefore, we see that the $\mathcal{C}$-dual functors $(\alpha)^{*}$ induce an equivalence

$$(-)^{*} : T \xrightarrow{\sim} T^{\text{op}} : (-)^{*}. $$

By (1) of Lemma 4.21 we conclude that $C^\alpha \in T$. By (4) of Lemma 4.21 we conclude that $(C^\alpha)^{\wedge} \cong (C^{\wedge})^\alpha \in T$. This finishes the proof of the implication (2) $\Rightarrow$ (3).

Finally, we prove the implication (3) $\Rightarrow$ (1). Let $\mathcal{X}_N : N \rightarrow N^{**}$ be the evaluation morphism for $N \in \mathcal{D}(\text{Mod } \mathcal{A}^{\text{op}})$. We remark that the assumption (3-a) implies that if $N \in T$, then $\mathcal{X}_N$ is an isomorphism. For $M \in \mathcal{D}(\text{Mod } \mathcal{A})$, we denote by $\mathcal{L}_M : M \otimes^L_A \mathcal{C} \rightarrow M^{\text{op}}$ the composite morphism

$$\mathcal{L}_M : M \otimes^L_A C \xrightarrow{\mathcal{G}\otimes^{\text{L}}_A C} M^{\alpha} \otimes^L_A C \xrightarrow{\mathcal{J}} \mathbb{R}\text{Hom}_{\mathcal{A}^{\text{op}}}(M^{\alpha}, C) = M^{\text{op}}.$$
Then we can check that the following diagram is commutative for $M \in \text{D(Mod } \Lambda)$

$$
\begin{array}{ccc}
M^\triangleright & \xrightarrow{\mathcal{X}_{M^\triangleright}} & M^{\triangleright\triangleright} \\
\| & & \| \\
\mathbb{R}\text{Hom}_{\Lambda}(M, \Lambda) & \xrightarrow{\mathcal{I}_{M, \Lambda}} & \mathbb{R}\text{Hom}_{\Lambda}(M \otimes^L_{\Lambda} C, C).
\end{array}
$$

Since the object $C^\alpha$ belongs to $T$ by the assumption (3-b), the morphism $\mathcal{X}_{(C^\alpha)^\triangleright}$ is an isomorphism. Since $C^\alpha$ belongs to $K^b(\text{proj } \Lambda)$, the morphism $\mathcal{L}_{C^\alpha}$ is an isomorphism and hence so is $\mathcal{I}_{C^\alpha, \Lambda}$. By Proposition 4.14 we check the right asid condition. Since the condition (3) is right-left symmetric, in the same way, we can check the left asid condition. This finishes the proof of $(3) \Rightarrow (1)$.

4.3.5 Proof of Theorem 4.12

We proceed a proof of Theorem 4.12.

Proof of Theorem 4.12 (1) For a triangulated category $D$ and its exact endofunctor $F$, we set

$$
a(F, D) := \min\{a \geq 0 \mid F_{F^d(d), d'} \text{ is an isomorphism for all } d, d' \in D\}
$$

where we are assuming that the set of which we take the minimal value is not empty.

By Proposition 4.19 we can apply Lemma 4.21 to the case where $D = K^b(\text{proj } \Lambda)$, $E = \text{thick } C^\alpha$, $F = T = - \otimes^L_{\Lambda} C$ and $G = - \otimes^L_{C} C^\alpha$. Set $T_\ell := C \otimes^L_{\Lambda} -$. Then we have the following equalities

$$
\alpha_\ell \overset{(a)\text{ of }}{=} a(T, K^b(\text{proj } \Lambda)) \overset{(b)}{=} \min\{a \geq 0 \mid G_{d, G^d(d)} \text{ is an isomorphism for all } d, d' \in D\}
$$

where the equality (a) follows from Proposition 4.14, the equality (b) follows from Lemma 4.21, the equality (d) follows from the left version of Proposition 4.14 and the equality (c) follows from the observation that the endofunctor $G$ of $K^b(\text{proj } \Lambda)$ corresponds to the endofunctor $T_\ell = C \otimes^L_{\Lambda} -$ of $K^b(\text{proj } \Lambda^\alpha)$ via the contravariant equivalence $(-)^\triangleright$.

(2) Assume that an admissible subcategory $T \subset K^b(\text{proj } \Lambda)$ satisfies the conditions (2-a) and (2-b) of Theorem 4.10. Then, by Lemma 4.21 $C^\alpha = T^\alpha(\Lambda)$ belongs to $T$. Hence $\text{thick } C^\alpha \subset T$. On the other hand, since $T \subset K^b(\text{proj } \Lambda)$, we have $T^\alpha(T) \subset \text{thick } C^\alpha$. Moreover, the conditions (2-a) implies $T = T^\alpha(T)$ and hence $T = T^\alpha(T) \subset \text{thick } C^\alpha$. Thus, $T = \text{thick } C^\alpha$.

Assume that a thick subcategory $T \subset K^b(\text{proj } \Lambda)$ satisfies the conditions (3-a) and (3-b) of Theorem 4.10. By (3-b), thick $C^\alpha \subset T$. Therefore, we have a semi-orthogonal decomposition $T = (\text{thick } C^\alpha) \perp (T \cap \text{Ker } \varpi)$ induced from that of $K^b(\text{proj } \Lambda)$ in Lemma 4.19. Since $M \otimes^L_{\Lambda} C \cong M^{\triangleright\triangleright}$ for $M \in K^b(\text{proj } \Lambda)$, the functor $- \otimes^L_{\Lambda} C$ acts $T$ as an equivalence. It follows that $T \cap \text{Ker } \varpi = 0$ by Lemma 4.18. Thus we conclude that $T = \text{thick } C^\alpha$.

(3) follows from Corollary 4.18 and Lemma 4.19.

We can verify the following assertion whose proof is left to the readers.

Proposition 4.24. Let $\Lambda$ and $C$ as in Theorem 4.10. Then we have

$$
\alpha_\ell = \min\{a \geq 0 \mid (\text{Ker } \varpi) \otimes^L_{\Lambda} C^\alpha\}.
$$

5 On an asid subcategory

In Section 5 we discuss an asid subcategory $T = \text{thick } C^\alpha$ for an asid bimodule $C$ over an IG-algebra $\Lambda$. The main result given in Theorem 5.6 proves that $T$ is equivalent to the stable category $\text{CM}_{lp} A$ of locally perfect graded CM-modules over $A = \Lambda \oplus C$.  

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5.1 A-duality and C-duality

The aim of Section 5.1 is to prove Theorem 5.4 which gives conditions that $M \in \mathcal{K}^b(\text{proj} \Lambda)$ belongs to $T$ in terms of A-duality $(-)^* = \mathcal{R}\text{Hom}_A(-, A)$.

We need a preparation. Let $\Lambda$ be a Netherian algebra and $C$ a bimodule over $\Lambda$ finitely generated on both sides. Then the trivial extension algebra $A = \Lambda \oplus C$ is Noetherian. Let $M$ be an object of $\mathcal{D}^b(\text{mod} \Lambda)$. and $\gamma_M$ the morphism induced from the canonical morphism $C \to A(1)$.

$$
\gamma_M : M^* = \mathcal{R}\text{Hom}_A(M, C) \to \mathcal{R}\text{Hom}_A(M, A)(1) = M^*(1).
$$

We give a description of $(M^*)_i$ by using $\gamma_M$ and $\mathcal{T}$.

**Lemma 5.1.** Let $M$ be an object in $\mathcal{D}^b(\text{mod} \Lambda)$. Then, there exist isomorphisms below in $\mathcal{D}(\text{mod} \Lambda^{\text{op}})$

$$
\mathcal{R}\text{Hom}_A(M, A)_i \cong \begin{cases} 
0 & (i > 1) \\
\mathcal{R}\text{Hom}_A(M, C) & (i = 1) \\
\text{cn} \left( \mathcal{T}_M \otimes^\Lambda_C [-i, \Lambda] \right) [-1] & (i < 1) 
\end{cases}
$$

Moreover, the isomorphism of the case $i = 1$ is the degree 0-part of $\gamma_M$.

Before giving a proof, we collect two immediate consequences of Lemma 5.1.

**Corollary 5.2.** The following conditions are equivalent for $M \in \mathcal{D}^b(\text{mod} \Lambda)$.

1. The morphism

$$
\mathcal{T}_M \otimes^\Lambda_C : \mathcal{R}\text{Hom}_A(M \otimes^\Lambda_C \Lambda, \Lambda) \to \mathcal{R}\text{Hom}_A(M \otimes^\Lambda_C \Lambda, \Lambda)
$$

is an isomorphism in $\mathcal{D}(\text{mod} \Lambda^{\text{op}})$ for any $i \geq 0$.

2. The morphism $\gamma_M$ is an isomorphism in $\mathcal{D}(\text{mod}^{\mathbb{Z}} A^{\text{op}})$.

3. The object $(M^*)_{\leq 0}$ is isomorphic to 0 in $\mathcal{D}(\text{mod}^{\mathbb{Z}} A^{\text{op}})$.

**Corollary 5.3.** We assume that $\Lambda$ is IG and $C$ has finite injective dimension on both sides. Then the following conditions are equivalent.

1. $C$ is a right asid bimodule.

2. The morphism $\gamma_C$ is an isomorphism in $\mathcal{D}(\text{mod}^{\mathbb{Z}} A)$ for some integer $a \geq 0$.

Moreover we have $\alpha_r = \min \{ a \geq 0 \mid \gamma_C$ is an isomorphism $\}$.

**Proof of Lemma 5.1.** Let $P \in C^-(\text{proj}^{\mathbb{Z}} A)$ be a projective resolution of $M$. We may assume that $P_{<0} = 0$ and hence $p_i P = 0$ for $i < 0$. Thus, for $i > 1$,

$$
\mathcal{R}\text{Hom}_A(M, A)_i = \text{Hom}_{\text{Mod}^{\mathbb{Z}} A}(P, A(i)) = 0.
$$

Since $p_0 P = P_0$ is a projective resolution of $M$ in $C(\text{proj} \Lambda)$ by Lemma [15, Lemma 4.13], we have isomorphisms

$$
\mathcal{R}\text{Hom}_A(M, A)_1 \cong \text{Hom}_{\text{Mod}^{\mathbb{Z}} A}(P, A(1)) \cong \text{Hom}_A(p_0 P, C) \cong \mathcal{R}\text{Hom}_A(M, C)
$$

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We consider the remaining case \( i < 1 \). Substituting \( P' = A(i) \) in the exact sequence (4-5) of [15] we obtain the following exact triangle in \( \mathcal{D}(\text{mod} \Lambda^{op}) \)

\[
\mathbb{R}\text{Hom}_{\Lambda}(p_{-i+1}P, C) \rightarrow \mathbb{R}\text{Hom}_{\text{Mod}^Z \Lambda}(P, A(i)) \rightarrow \mathbb{R}\text{Hom}_{\Lambda}(p_{-i}P, \Lambda) \xrightarrow{F_i} \mathbb{R}\text{Hom}_{\Lambda}(p_{-i+1}P, C[1]).
\]

We can check that \( F_i \) to be the composite morphism

\[
\mathbb{R}\text{Hom}_{\Lambda}(p_{-i}P, \Lambda) \xrightarrow{\mathcal{S}_{p_{-i}P, \Lambda}} \mathbb{R}\text{Hom}_{\Lambda}(p_{-i}P \otimes_{\Lambda} C, C) \xrightarrow{\Sigma} \mathbb{R}\text{Hom}_{\Lambda}(p_{-i}P \otimes_{\Lambda} C[1], C[1]) \xrightarrow{\text{Hom}(q_{-i+1}C[1])} \mathbb{R}\text{Hom}_{\Lambda}(p_{-i+1}P, C[1])
\]

where \( \Sigma \) is the map induced from the shift functor [1].

By Lemma 2.3, \( q_{-i+1} \) is isomorphism in \( \mathcal{D}(\text{mod} \Lambda) \) for \( i < 1 \). Since the morphism \( \Sigma \) is also isomorphism, we see that \( \mathbb{R}\text{Hom}_{\text{Mod}^Z \Lambda}(M, A(i)) \) is a co-cone of \( \mathcal{S}_{p_{-i}P, \Lambda} \). On the other hand, since \( p_{-i}P \cong M \otimes_{\Lambda} C^{-i}[-i] \) for \( i \leq 0 \) by Corollary 2.4(1), we finish the proof.

We give the main result of Section 5.1.

**Theorem 5.4.** Assume that \( \Lambda \) and \( A = \Lambda \oplus C \) is IG. We set \( \alpha := \alpha_{\tau} = \alpha_{\ell} \) and \( T := \text{thick} C^\alpha \). Then for an object \( M \in \mathcal{K}(\text{proj} \Lambda) \) the following conditions are equivalent.

1. \( M \in \mathcal{T} \).
2. \( \gamma_M \) is an isomorphism.
3. \( (M^*)_{\leq 0} = 0 \) in \( \mathcal{D}(\text{mod}^Z \Lambda^{op}) \).

**Proof.** The equivalence (2) \( \iff \) (3) follows from Lemma 5.1.

We prove the implication (1) \( \Rightarrow \) (2). By Corollary 5.3 \( \gamma_{C^\alpha} \) is an isomorphism. Since \( T \) is isomorphism, we conclude that \( \gamma_M \) is an isomorphism for \( M \in \mathcal{T} \).

We prove the implication (3) \( \Rightarrow \) (1). By the assumption \( M^* \cong M^*(-1) \) belongs to \( \mathcal{K}(\text{proj} \Lambda^{op})(-1) \). We remark that this implies that \( M^* \) is locally perfect by Proposition 3.3.

Let \( Q \in \mathcal{C}(\text{proj}^Z \Lambda^{op}) \) be a projective resolution of \( M^* \). We may assume that \( Q_{\leq 0} = 0 \) and \( p_i Q \in \mathcal{C}(\text{proj} \Lambda^{op}) \) for \( i \geq 0 \). Set \( P := \text{HOM}^*(Q, A) \). Then since \( p_i P = \text{Hom}_{\Lambda}(p_{-i}Q, \Lambda) \), we see that \( p_i P \in \mathcal{K}(\text{proj} \Lambda) \) for \( i < 0 \) and \( t_{\geq 0} P = 0 \). The latter property implies that \( P_0 = p_{-1}P \otimes_{\Lambda} C \). Since \( P \) is isomorphic to \( M^* = M \in \mathcal{D}(\text{mod}^Z \Lambda) \), \( P_0 \cong M \in \mathcal{D}(\text{mod} \Lambda) \) and the complex \( P_{<0} \) is acyclic. By Corollary 2.4 we obtain an isomorphism \( p_{-1}P \cong p_{-1}P \otimes_{\Lambda} C^\alpha[-1][i-1] \) for \( i > 0 \). Therefore, \( M \cong p_{-1}P \otimes_{\Lambda} C^\alpha[\alpha-1] \).

### 5.2 The equivalence \( \mathcal{T} \simeq \text{CM}_\Lambda^{Z, A} \)

In this section 5.2 \( \Lambda \) is an IG-algebra and \( C \) is a \( \Lambda \)-\( \Lambda \)-bimodule such that the trivial extension algebra \( A = \Lambda \oplus C \) is IG. We set \( \alpha := \alpha_{\tau} = \alpha_{\ell} \) and \( T := \text{thick} C^\alpha \). To state the main theorem of this section, we point out the following observation.

**Lemma 5.5.** (1) The essential image of the functor \( p_0 : \mathcal{K}^{Z, \Lambda}_{\text{ip}}(\text{proj}^Z \Lambda) \rightarrow \mathcal{K}(\text{proj} \Lambda) \) is contained in \( \mathcal{T} \).

2. Let \( \text{in} : \mathcal{D}(\text{mod} \Lambda) \hookrightarrow \mathcal{D}(\text{mod}^Z \Lambda) \) be the canonical inclusion. Then the essential image of \( \mathcal{T} \) by \( \text{in} \) is contained in the Orlov subcategory \( \mathcal{O} = \mathcal{D}(\text{mod}^{Z, 0} \Lambda) \cap \mathcal{D}(\text{mod}^{Z, >0} \Lambda^{op})^* \).
Proof. (1) Let \( P \) be an object of \( \underline{C}^{ac}_{\mathfrak{lp}}(\text{proj}^{Z} A) \). Then, by Corollary 2.4, we have isomorphisms \( \mathfrak{p}_{0}P \cong \mathfrak{p}_{-\mathfrak{a}}P \otimes_{\Lambda}^{L} C^{\mathfrak{a}}[\mathfrak{a}] \). Thus \( \mathfrak{p}_{0}P \in \text{thick} C^{\mathfrak{a}} = T \).

(2) Let \( M \) be an object of \( T \). It is clear that \( M \) belongs to \( D^{b}(\text{mod}^{\geq 0} A) \). We set \( T_{\Lambda^{op}} := \text{thick}_{\Lambda^{op}} C^{\mathfrak{a}} \). Then, \( T^{*} = T_{\Lambda^{op}} \) by Theorem 4.10 and the left version of Lemma 4.16. In particular \( M^{*} \) belongs to \( T_{\Lambda^{op}} \subset D^{b}(\text{mod} A^{op}) \). It follows from the isomorphism \( M^{*} = M^{*}(-1) \) obtained in Theorem 5.4 that \( M^{*} \in D^{b}(\text{mod}^{>0} A^{op}) \). This shows that \( M \) belongs to \( O \). \( \square \)

Now we can state the main theorem of this section.

**Theorem 5.6.** (1) The following diagram is commutative up to natural isomorphism.

\[
\begin{array}{ccc}
K^{ac}_{\mathfrak{lp}}(\text{proj}^{Z} A) & \xrightarrow{\mathfrak{p}_{0}} & \underline{C}M^{Z}_{\mathfrak{lp}} A \\
\downarrow \mathfrak{p} & & \downarrow \beta \\
T & \xrightarrow{\mathfrak{i}_{\mathfrak{p}T}} & O & \xrightarrow{\pi_{0}} & \text{Sing}_{\mathfrak{lp}}^{Z} A
\end{array}
\]

(2) All the functors appeared in the above diagram are equivalences.

We use the upper bound \( \text{ub} Q \) which was used in the proof of Proposition 5.4. Let \( Q \in C(\text{proj} \Lambda) \) be a complex which is homotopic to a bounded complex \( Q' \in D^{b}(\text{proj} \Lambda) \). We define the lower bound \( \text{lb} Q \) of \( Q \) by \( \text{lb} Q := -\text{ub}(Q^{op}) \). A complex \( Q \) is said to be of minimal amplitude if \( Q^{n} = 0 \) for \( n < \text{lb} Q \) or \( n > \text{ub} Q \). We remark that a complex \( Q \in C(\text{proj} \Lambda) \) which is homotopic to a bounded complex \( Q' \in D^{b}(\text{proj} \Lambda) \) is homotopic to a complex of minimal amplitude.

**Lemma 5.7.** Let \( P \in C^{ac}(\text{proj}^{Z} A) \). Assume that for \( i \in \mathbb{Z} \), the complex \( \mathfrak{p}_{i}P \) belongs to \( C(\text{proj} \Lambda) \) and is homotopic to an object of \( D^{b}(\text{proj} \Lambda) \) for all \( i \in \mathbb{Z} \). Then the following assertions hold.

(1) \( \text{ub}(\mathfrak{p}_{i}P) \leq \text{ub}(\mathfrak{p}_{0}P) - i \) for \( i \geq 1 \).

(2) \( \text{lb}(\mathfrak{p}_{-i}P) \geq \text{lb}(\mathfrak{p}_{0}P) + i \) for \( i \geq 1 \).

(3) Assume moreover that \( \mathfrak{p}_{i}P \) is of minimal amplitude for \( i \in \mathbb{Z} \). Then \( P \) belongs to \( C^{ac}_{\mathfrak{lp}}(\text{proj}^{Z} A) \).

*Proof. (1) follows from Corollary 2.3.(1).

(2) Since \( A \) is IG, the \( A \)-dual complex \( P^{*} \) is acyclic. Since \( \mathfrak{p}_{i}(P^{*}) \cong (\mathfrak{p}_{-i}P)^{op} \), the complex \( P^{*} \in C^{ac}(\text{proj}^{Z} A^{op}) \) satisfies the assumption of (1). Thus by (1) for \( i \geq 1 \), we have the inequality \( \text{ub} \mathfrak{p}_{i}(P^{*}) \leq \text{ub} \mathfrak{p}_{0}(P^{*}) - i \), from which we deduce the inequality of (2).

(3) For a fixed integer \( n \), we have \( (\mathfrak{p}_{i}P)^{n} = 0 \) for \( |i| \gg 0 \) by (1) and (2). Since each term \( P^{n} \) is finitely generated, hence \( P \) belongs to \( C^{ac}_{\mathfrak{lp}}(\text{proj}^{Z} A) \) by Lemma 2.1. \( \square \)

Let \( n \) be an integer. We denote by \( \sigma^{\leq n} P \) the brutal truncation of \( P \in C(\text{proj}^{Z} A) \).

\[
\sigma^{\leq n} P : \cdots \rightarrow P^{n-2} \rightarrow P^{n-1} \rightarrow P^{n} \rightarrow 0 \rightarrow 0 \rightarrow \cdots
\]

*Proof of Theorem 5.6.* (1) Let \( P \) be an object of \( K^{ac}_{\mathfrak{lp}}(\text{proj}^{Z} A) \). By Lemma 15, Lemma 4.7 and Lemma 5.4.(3), we may assume that \( P \) belongs to \( C^{ac}_{\mathfrak{lp}}(\text{proj}^{Z} A) \) such that \( \mathfrak{p}_{i}P \) is of minimal amplitude for \( i \in \mathbb{Z} \). We construct an isomorphism \( \beta \circ Z^{b}P \cong \mathfrak{w} \circ \mathfrak{p}_{0}(P) \). The canonical map \( \sigma^{\leq 0} P \rightarrow Z^{0}P \) is an isomorphism in \( D^{b}(\text{mod}^{\geq 0} A) \). Let \( n > \max\{\text{ub}(\mathfrak{p}_{0}P), 0\} \) be an integer. Since the kernel \( \text{Ker} \mathfrak{cs} \) of the canonical surjection \( \mathfrak{cs} : \sigma^{\leq n} P \rightarrow \sigma^{\leq 0} P \) is of the following form

\[
\text{Ker} \mathfrak{cs} : \cdots \rightarrow 0 \rightarrow 0 \rightarrow P^{1} \rightarrow P^{2} \rightarrow \cdots \rightarrow P^{n-1} \rightarrow P^{n} \rightarrow 0 \rightarrow 0 \rightarrow \cdots
\]
it belongs to $C^b(\text{proj} Z A)$. Therefore the morphism $c$s becomes an isomorphism in $\text{Sing}^Z_{\text{fp}} A$. Since $\sigma^{\leq n} P$ is bounded above, the complex $t_{< 0} \sigma^{\leq n} P$ belongs to $C^b(\text{proj} Z A)$ by Lemma 5.7(2). Therefore the canonical map $\sigma^{\leq n} P \rightarrow t_{\geq 0} \sigma^{\leq n} P$ become an isomorphism in $\text{Sing}^Z_{\text{fp}} A$. By Lemma 5.7(1) and the definition of $n$, the canonical map $t_{\geq 0} P \rightarrow t_{\geq 0} \sigma^{\leq n} P$ is an isomorphism in $C(\text{proj} Z A)$. Since $P$ is acyclic, $t_{\geq 0} P$ is acyclic in degree greater than 0, i.e., $H(t_{\geq 0} P)_{> 0} = 0$. Thus the canonical map $t_{\geq 0} P \rightarrow (t_{\geq 0} P)_{\leq 0} = p_0 P$ is an isomorphism in $D^b(\text{mod}^Z A)$. Combining all, we obtain the following diagram which give an isomorphism $\beta \circ Z^0(P) \cong \varpi \circ p_0(P)$ in $\text{Sing}^Z_{\text{fp}} A$.

$$Z^0(P) \leftarrow \sigma^{\leq 0} P \leftarrow \sigma^{\leq n} P \rightarrow t_{\geq 0} \sigma^{\leq n} P \rightarrow t_{\geq 0} P \rightarrow p_0 P.$$

It can be easily checked that this isomorphism does not depend on the choice of $n$ and is natural in $P$. Therefore, this diagram gives the desired natural isomorphism $\beta \circ Z^0 \cong \varpi \circ p_0$.

(2) We have already proved that $\beta, Z^0$ are equivalence.

We prove that the functor $p_0$ is an equivariant.

Since the functors $\beta, Z^0$ gives equivalences, it follows from (1) that the functor $p_0$ is faithful. Thus, it is enough to prove that $p_0 : K^a_{\text{fp}}(\text{proj} Z A) \rightarrow T$ is essentially surjective and full.

We prove $p_0$ is essentially surjective. Let $M \in T$. It is enough to give a way to construct $P \in K^a_{\text{fp}}(\text{proj} Z A)$ such that $p_0 P = M$. We denote by $T'$ the autoequivalence $- \otimes_C^L C[1]$ of $T$. For $i \in \mathbb{Z}$, we take $Q_i \in C^b(\text{proj} \Lambda)$ be a projective representative of $(T')^i(M)$ having minimal amplitude. Then there exists a quasi-isomorphism $\alpha_i : Q_i \rightarrow Q_{i-1} \otimes C[1]$ in $C(\text{mod} \Lambda)$. By the argument of beginning of [13], there exists an object $P \in C(\text{proj} Z A)$ such that $p_0 P = Q_i$ in $C(\text{proj} \Lambda)$ and that the morphisms $\alpha_i$ equals to $\omega_i$ in the exact triangle $(2-3)$. Thus the complex $P$ is acyclic by Lemma 2.3. By Lemma 5.7(3), $P$ belongs to $C^a_{\text{fp}}(\text{proj} Z A)$.

We proceed a proof that $p_0$ is full. Let $g : M \rightarrow N$ be a morphism in $T$. We construct a morphism $f : P^M \rightarrow P^N$ such that $p_0(f) = g$ where $P^M$ and $P^N$ are objects of $C^a_{\text{fp}}(\text{proj} Z A)$ such that $p_0 P^M = M, p_0 P^N = N$ which are constructed in the previous paragraph. Let $Q^M_i$ and $Q^N_i$ be the objects of $C^b(\text{proj} \Lambda)$ which are constructed as in the previous paragraph for $M$ and $N$.

For $i \in \mathbb{Z}$, we take $g_i : Q^M_i \rightarrow Q^N_i$ to be a representative of $(T')^i(g) : (T')^i(M) \rightarrow (T')^i(N)$. First note that since the complex $Q^M_i$ is bounded above complex of projective $\Lambda$-modules, we have an isomorphism $\text{Hom}_{\text{proj}(\Lambda)}(Q^M_i, Q^N_{i+1} \otimes C[1]) \cong \text{Hom}_{\text{proj}(\Lambda)}(Q^M_i, Q^N_i \otimes C[1])$. For $i \in \mathbb{Z}$, the morphisms $(g_{i-1} \otimes C[1]) \circ q^M_i$ and $q^N_i \circ g_i$ from $Q^M_i$ to $Q^N_{i-1} \otimes C[1]$ in $C(\text{mod} \Lambda)$ represent the same morphism $(T')^i(g) : (T')^i(M) \rightarrow (T')^i(N)$ in $D(\text{mod} \Lambda)$. Therefore these two morphisms are homotopic. Thus by [13] Lemma 4.6, there exists a morphism $f : P^M \rightarrow P^N$ in $C(\text{proj} Z A)$ such that $p_0(f) = g$. In particular we have $p_0(f) = g$. This completes the proof that $p_0$ is an equivalent.

We prove that the functor $\text{in} T$ is equivalent.

Since in is fully faithful, so is $\text{in} T$. It only remains to prove that in $T \Lambda \rightarrow O$ is essentially surjective. We remark that since $A_i = 0$ for $i \neq 0, 1$, we have $D^b_{\text{fp}}(\text{mod}^{\geq 1} A^{op}) \subset D^b_{\text{fp}}(\text{mod}^{\leq 0} A^{op})$. Therefore $O$ is contained in the intersection $D^b_{\text{fp}}(\text{mod}^{\geq 0} A) \cap D^b_{\text{fp}}(\text{mod}^{\leq 0} A)$, which is identified with $K^b(\text{proj} \Lambda)$ via $\text{in}$. Let $M$ be an object in $O$. We regard $M$ as an object of $K^b(\text{proj} \Lambda)$. Then since $M^* \in D^b(\text{mod}^{\geq 1} A^{op})$, we have $(M^*)_{\leq 0} = 0$. Therefore by Theorem 5.4, $M$ belongs to $T$. This completes the proof that in $T$ is an equivalence.

Finally, $\pi | O$ is an equivalence, since the other functors in the diagram of (1) are equivalences.

As a corollary, we obtain a description of the image of the Happel’s embedding functor in the case where $\Lambda$ is IG, which is obtained by Chen-Zhang [6] Theorem 3.1].

**Corollary 5.8.** Let $\Lambda$ is IG finite dimensional algebra and $T(\Lambda) := \Lambda \oplus D(\Lambda)$ the trivial extension algebra. Then the Happel embedding functor $\varpi : D^b(\text{mod} \Lambda) \rightarrow \text{mod}^Z T(\Lambda)$ induces an equivalence

$$\varpi : K^b(\text{proj} \Lambda) \Rightarrow \text{mod}^Z T(\Lambda).$$
We give our proof of Orlov’s equivalence.

Proof of Theorem 5.13. From the discussion in Section 2.1.1 we see that the Orlov subcategory $O$ is invariant under quasi-Veronese algebra construction.

$$D^b_{lp}(\text{mod}^{\geq 0} A) \cap D^b_{lp}(\text{mod}^{> 0} A^\text{op})^* \cong D^b_{lp}(\text{mod}^{\geq 0} A^{[\ell]}) \cap D^b_{lp}(\text{mod}^{> 0} (A^{[\ell]})^{\text{op}})^*.$$  

Therefore, we may assume that $A = \Lambda \oplus C$. In that case $\pi|_O$ gives an equivalence by Theorem 5.6. □

5.3 A recollement involving the canonical functor $\varpi|_K$

Combining results which we have obtained, we see that the canonical functor $\varpi|_K : K^b(\text{proj} \Lambda) \to \text{Sing}^{\mathbb{Z}}_{lp} A$ fits into a recollement as follows. We also obtain a description of Hom-space of the singularity category $\text{Sing}^{\mathbb{Z}}_{lp} A$.

**Theorem 5.9.** Let $\Lambda$ be an IG-algebra and $C$ a $\Lambda$-$\Lambda$-bimodule such that the trivial extension algebra $A = \Lambda \oplus C$ is IG. Then the following assertions hold.

1. There exists the recollement of the following form

$$\text{Sing}^{\mathbb{Z}}_{lp} A \xrightarrow{\varpi} K^b(\text{proj} \Lambda) \xrightarrow{\text{in}} \ker \varpi$$

where in is a canonical embedding.

2. For $M, N \in K^b(\text{proj} \Lambda)$, we have

$$\text{Hom}_{\text{Sing}^{\mathbb{Z}}_{lp} A} (\varpi M, \varpi N) \cong \text{Hom}_{D^b_{\text{mod} \Lambda}}(R\text{Hom}(C^\alpha, M), R\text{Hom}(C^\alpha, N))$$

**Proof.** (1) follows from Theorem 4.10, Lemma 4.19, Theorem 5.6 and the relationship between an admissible subcategory and a recollement which is recalled in Section 2.4.

(2) is proved by the following string of isomorphisms

$$(\text{LHS}) \cong \text{Hom}_{K^b(\text{proj} \Lambda)}(\tau(M), N)$$

$$(b) \cong \text{Hom}_{K^b(\text{proj} \Lambda)}(R\text{Hom}_\Lambda(C^\alpha, M) \otimes^L_\Lambda C^\alpha, N) \cong (\text{RHS}).$$

where the isomorphism (a) can be obtained from Lemma 2.14 the isomorphism (b) is a consequence of Corollary 4.20 and the isomorphism (c) follows from the $- \otimes^L C^\alpha - R\text{Hom}(C^\alpha, -)$-adjunction. □

Recalling the quasi-Veronese algebra construction, we obtain a result which can be applied to any finite dimensional graded IG-algebra $A$ which is not necessary a trivial extension algebra. For a finite dimensional algebra $\Lambda$, we denote by $|\Lambda|$ the number of pairwise non-isomorphic indecomposable projective modules.

**Theorem 5.10.** Let $A = \bigoplus_{i=0}^\ell A_i$ be a finite dimensional (over some field $K$ which is not necessary the base field $k$) graded IG-algebra. If the degree $0$-part algebra $A_0$ is of finite global dimension, then the Gorthendieck group $K_0(\text{CM}^{\mathbb{Z}} A)$ is free and its rank is bounded by $\ell |A_0|$ from above.

$$\text{rank } K_0(\text{CM}^{\mathbb{Z}} A) \leq \ell |A_0|.$$  

**Proof.** Since $\nabla A$ is defined as an $\ell \times \ell$-upper triangular matrix all of whose diagonals are $A_0$, the assumption $\text{gldim } A_0 < \infty$ implies $\text{gldim } \nabla A < \infty$ (see e.g. [15, Corollary 6.3]). Therefore $K^b(\text{proj} \nabla A) = D^b(\text{mod} \nabla A)$. By Corollary 3.7 and Lemma 3.10 $\text{CM}^{\mathbb{Z}}_{lp} A = \text{CM}^{\mathbb{Z}} A$. Hence by Theorem 5.9, $\text{CM}^{\mathbb{Z}} A$ is realized as an admissible subcategory of $D^b(\text{mod} \nabla A)$. In particular, $K_0(\text{CM}^{\mathbb{Z}} A)$ is a direct summand of $K_0(D^b(\text{mod} \nabla A))$, which is known to be free of rank $|\nabla A|$. Now the result follows from the equation $|\nabla A| = \ell |A_0|$. □
6 Application I: two classes of IG algebra of finite CM-type

In this Section 6.1 we deal with two particular classes of trivial extension algebras. We show that if the classes of algebras are IG, then they are of finite (graded) Cohen-Macaulay (CM) type. For this, we use the following result which is a CM-version of Gabriel’s theorem in covering theory [7].

For a Krull-Schmidt category $A$, we denote by $\text{ind} A$ the set of indecomposable objects of $A$ up to isomorphisms.

Recall that an IG (resp. graded IG) algebra $A$ is said to be of finite CM (resp. graded CM) type, if the number of isomorphisms classes of CM modules (resp. graded CM-module) is finite (resp. finite up to degree shift) $\# \text{ind CM} A < \infty$ (resp. $\# \text{ind CM}^Z A/(1) < \infty$).

**Theorem 6.1** ([7]). Let $A$ be a finite dimensional graded IG-algebra. Then, $A$ is of finite CM type if and only if it is of graded CM-type. Moreover, if this is the case, the functor $\text{mod}^Z A \to \text{mod} A$ which forgets the grading of graded modules induces the equality $\text{ind CM}^Z A/(1) = \text{ind CM} A$.

6.1 The case where $\Lambda$ is iterated tilted of Dynkin type

The result of Section 6.1 is the following.

**Theorem 6.2.** Let $\Lambda$ be an iterated tilted algebra of Dynkin type and $C$ is a finite dimensional bimodule over $\Lambda$. If a trivial extension algebra $A = \Lambda \oplus C$ is IG, then it is of finite CM type.

We need a preparation. Let $S$ be a set and $F : S \xrightarrow{\cong} S$ a bijection. We denote by $S/F$ the quotient set $S/\mathbb{Z}$ by the action of $\mathbb{Z}$ on $S$ which is defined to be $n \cdot F := F^n(s)$ for $n \in \mathbb{Z}$ and $s \in S$. We note that if we take a complete set of representatives $S_0 \subset S$ of $S/F$, then any element $s$ is of the form $s = F^q(s_0)$ for some $q \in \mathbb{Z}$ and $s_0 \in S_0$.

**Lemma 6.3.** Let $S$ be a set and $F, G : S \xrightarrow{\cong} S$ bijections such that $FG = GF$. Assume that $\#S/F = r < \infty$ and that there exists a set of representatives $\{s_1, s_2, \ldots, s_r\}$ such that for all $i = 1, \ldots, r$ there exists a positive integer $p_i > 0$ such that $G(s_i) = F^{p_i}(s_{\sigma(i)})$ for some $\sigma(i) = 1, \ldots, r$. Then $\#S/F < \infty$.

**Proof.** First we claim that if $i \neq j$, then $\sigma(i) \neq \sigma(j)$. Indeed if $\sigma(i) = \sigma(j)$, then $F^{-p_i}G(s_i) = F^{-p_j}G(s_j)$. Thus we have the equation $s_i = F^{-p_i}(s_j)$ which contradicts to the assumption.

By the claim we may regard $\sigma$ as a permutation. We have $G^{p_i}(s_i) = F^{p_i}(s_i)$ where $P_i = \sum_{i=0}^{p_i-1} p_{\sigma(i)}$. Note that $P_i > 0$. We claim that every $\Gamma_i$-orbit contains one of the elements of the set $\{F^q(s_i) \mid 0 \leq q < P_i\}$. Indeed any element $s$ is of the form $s = F^q(s_i)$ for some $q \in \mathbb{Z}$ and $i = 1, \ldots, r$. Let $q_1, q_2 \in \mathbb{Z}$ be such that $q = q_1 P_i + q_2$, $0 \leq q_2 < P_i$. Then we have $G^{-q_2}f(s) = F^{q_2}(s_i)$.

Now by the second claim $\#S/F < \#S/G \leq \sum_{i=1}^{r} P_i < \infty$. □

**Proof of Theorem 6.2.** Let $C$ be an asid $\Lambda$-$\Lambda$-bimodule and $\alpha$ the asid number of $C$. We set $T := \text{thick } C^\alpha$. We claim that $\# \text{ind } T/[1] < \infty$. Indeed, let $Q$ be a Dynkin quiver such that $\Lambda$ is derived equivalent to $kQ$. Since all indecomposable object $M \in D^b(\text{mod } kQ)$ are of forms $M = M'[a]$ for some $M' \in \text{ind } M_\alpha$ and $a \in \mathbb{Z}$, we have $\text{ind } D^b(\text{mod } kQ)/[1] = \text{ind } \text{mod } kQ$. Hence $\# T/[1] \leq \# \text{ind } D^b(\text{mod } \Lambda)/[1] < \infty$.

Let $\{M_1, \ldots, M_r\}$ be a set of representatives of $\text{ind } T/[1]$. We may assume that $H^0(M_i) \neq 0$ and $H^2(M_i) = 0$. For $i = 1, \ldots, r$, we have $M_i \otimes^L_\Lambda C[1] = M_j[p_i]$ for some $p_i \in \mathbb{Z}$ and $j = 1, \ldots, r$. Then we have $H^{1-p_i}(M_i \otimes^L_\Lambda C) = H^0(M_j) \neq 0$. On the other hand, since $C$ is a $\Lambda$-module, we have $H^{2}(M_i \otimes^L_\Lambda C) = 0$. Thus we conclude that $p_i > 0$. Now by Lemma 6.3 we conclude that $\# \text{ind } T/(- \otimes^L_\Lambda C[1]) < \infty$. □
6.2 The case where \( C = \Lambda N \otimes_k M \)

A straightforward way to obtain a bimodule over an algebra \( \Lambda \) is to take a tensor product \( C = N \otimes_k M \) of a left \( \Lambda \)-module \( N \) and a right \( \Lambda \)-module \( M \) both of which are finite dimensional over \( k \). In Section 6.2 we investigate the case \( A = \Lambda \oplus (N \otimes_k M) \).

6.2.1 Results and examples

In this Section 6.2, \( \Lambda \) denotes a finite dimensional algebra of finite global dimension and \( N \) denotes a nonzero left \( \Lambda \)-module and \( M \) denotes a nonzero right \( \Lambda \)-module. We set \( C := N \otimes_k M \) and \( A := \Lambda \oplus C \).

For simplicity we restrict ourselves to deal with this case. We note that some of the statements can be verified in the more general case where \( \Lambda \) is IG and \( \text{id}_M < \infty \), \( \text{id}_N < \infty \).

**Theorem 6.4.** Let \( \Lambda \) be a finite dimensional algebra of finite global dimension, \( (1) \) \( \text{gldim} \ A < \infty \) if and only if \( M \otimes^\Lambda_k N = 0 \).
\( (2) \) \( A \) is IG and \( \text{gldim} \ A = \infty \) if and only if \( M \) is exceptional (i.e., \( \mathbb{R}\text{Hom}_\Lambda(M, M) \cong k \)) and \( \mathbb{R}\text{Hom}_\Lambda(M, \Lambda) = N[-p] \) for some \( p \in \mathbb{N} \).
\( (3) \) Assume that \( A \) is IG and \( \text{gldim} \ A = \infty \). Then the following assertions hold.

\( (3-a) \) Let \( p \) be the integer in \( (2) \). Then \( p = \text{pd}^/_A \Lambda M = \text{pd}^/_A \Lambda \text{op} \Lambda N \).

\( (3-b) \) There are equivalences \( \text{CM}^Z A \cong \text{D}^b(\text{mod} \ k) \) under which the graded degree shift functors \( (1) \) corresponds to the complex degree shift functor \( [p + 1] \).

\( (3-c) \) \( \text{CM} A \cong \text{D}^b(\text{mod} \ k)/[p + 1] \cong (\text{mod} \ k)^{\oplus p+1} \) and the syzygy functor \( \Omega \) is \((p + 1)\)-periodic.

\( (3-d) \) \( \text{ind} \ \text{CM} A = \{ M, \Omega M, \cdots, \Omega^p M \} \).

Combining Theorem 6.4 and a result by X-W. Chen, we can provide an example of a non-IG finite dimensional algebra \( A \) such that the singularity category \( \text{Sing} A \) is Hom-finite.

**Example 6.5.** Let \( \Lambda \) be a basic finite dimensional algebra of finite global dimension and \( e, f \in \Lambda \) idempotent elements. Then the algebra \( A = \Lambda \oplus (\Lambda e \otimes_k f \Lambda) \) is of finite global dimension if and only if \( f \Lambda e = 0 \). The algebra \( A \) is an IG-algebra of infinite global dimension if and only if \( e = f \) and \( \text{dim} e \Lambda e = 1 \).

On the other hands, X-W. Chen \[5\] showed that \( \text{Sing} A \) is Hom-finite if and only if \( \text{dim} f \Lambda e \leq 1 \). Thus we conclude that there are finite dimensional algebras \( A \) which is not IG but whose singularity category \( \text{Sing} A \) is Hom-finite.

**Example 6.6.** Let \( \Lambda \) be a \( d \)-representation infinite algebra and \( e \in \Lambda \) an idempotent such that \( \text{dim} e \Lambda e = 1 \). Set \( \theta = \text{Ext}^d_\Lambda(D(\Lambda), \Lambda) \).

Then \( A = \Lambda \oplus D(e \Lambda) \otimes_k e \theta \) is an IG-algebra such that \( \text{CM} A \cong (\text{mod} \ k)^{\oplus d+1}, \text{CM}^\text{op} A \cong (\text{mod} \ k)^{\oplus d+1} \).

6.2.2 Proof of Theorem 6.4.

First observe that \( C^a = N \otimes_k (M \otimes^L_\Lambda N)^{\otimes k a-1} \otimes_k M \) for \( a \geq 1 \). On the other hand, under the assumption that \( \text{gldim} \Lambda < \infty \), we have \( \text{gldim} A < \infty \) if and only if \( C^a = 0 \) for some \( a \geq 0 \) by \[15\] Corollary 4.15. Hence we deduce the following lemma.

**Lemma 6.7.** We have \( M \otimes^L_\Lambda N = 0 \) if and only if \( \text{gldim} A < \infty \).
Since \( \text{id}_C = \text{id} M < \infty \), \( \text{id}_C C = \text{id} N < \infty \), the trivial extension algebra \( A \) is IG if and only if \( C \) satisfies the right and left asid conditions by Theorem \[4.10]\.

We concentrate on the right asid condition. By Lemma \[4.14\] the right asid condition is satisfied if and only if the morphism \( \mathcal{T}_{C^a, A} : \mathbb{R} \text{Hom}_A(C^a, A) \to \mathbb{R} \text{Hom}_A(C^{a+1}, C) \) is an isomorphism. Moreover, if this is the case we have \( \alpha_r = \min\{a \geq 0 | \mathcal{T}_{C^a, A} \text{ is an isomorphism}\} \).

We give a description of the morphism \( \mathcal{T}_{C^a, A} \) in terms of more concrete morphisms, which are introduced now.

By \( \text{lm} \) we denote the left multiplication map \( \text{lm} : \Lambda \to \text{Hom}_k(N, N) \), i.e., \( \text{lm}(r)(n) = rn \) for \( r \in \Lambda \) and \( n \in N \).

We define a morphism \( h_0 : k \to \mathbb{R} \text{Hom}_A(M, M) \) in the following way. By \( \mathcal{h} \), we denote the homothety map \( \mathcal{h} : k \to \text{Hom}_A(M, M) \), i.e., \( \mathcal{h}(a)(m) := am \) for \( a \in k \) and \( m \in M \). Then, we set \( h := \text{can} \circ \mathcal{h} \) where \( \text{can} : \text{Hom}_A(M, M) \to \mathbb{R} \text{Hom}_A(M, M) \) is the canonical morphism.

By \( \mathcal{T} \), we denote the morphism \( \mathcal{T} : \mathbb{R} \text{Hom}_A(M, A) \to \text{Hom}_k(M \otimes^A N, N) \) in \( D^b(\text{mod } \Lambda) \) induced from the functor \( - \otimes^N \Lambda \cap \text{Hom}_k(M, N) \otimes^N \Lambda \) for some finite dimensional vector space \( \Lambda \).

\[ \mathcal{T} : \mathbb{R} \text{Hom}_A(M, A) \to \mathbb{R} \text{Hom}_k(M \otimes^A N, N) \cong \text{Hom}_k(M \otimes^A N, N) \]

We leave the verification of the following lemma to the readers.

**Lemma 6.8.** (1) Under the isomorphisms
\[
\mathbb{R} \text{Hom}_A(\Lambda, \Lambda) \cong \Lambda \otimes_k k \text{ and } \mathbb{R} \text{Hom}_A(N \otimes_k M, N \otimes_k M) \cong \text{Hom}_k(N, N) \otimes_k \mathbb{R} \text{Hom}_A(M, M),
\]
the morphism \( \mathcal{T}_{A, A} : \mathbb{R} \text{Hom}_A(\Lambda, \Lambda) \to \mathbb{R} \text{Hom}_A(N \otimes_k M, N \otimes_k M) \) corresponds to \( \text{lm} \otimes h \).

\[
\text{lm} \otimes h : \Lambda \otimes_k k \to \text{Hom}_k(N, N) \otimes_k \mathbb{R} \text{Hom}_A(M, M)
\]

(2) Let \( a \geq 1 \). Then under the isomorphisms
\[
\mathbb{R} \text{Hom}_A(C^a, A) \cong D(N \otimes_k (M \otimes^A N) \otimes_k a^{-1}) \otimes_k \mathbb{R} \text{Hom}_A(M, A) \otimes_k k \text{ and } \mathbb{R} \text{Hom}_A(C^{a+1}, C) \cong D(N \otimes_k (M \otimes^A N) \otimes_k a^{-1}) \otimes_k \text{Hom}_k(M \otimes^A N, N) \otimes_k \mathbb{R} \text{Hom}_A(M, M)
\]
the morphism \( \mathcal{T}_{C^a, A} \) corresponds to \( \text{id} \otimes \mathcal{T} \otimes h \).

\[
\text{id}_{L_0} \otimes \mathcal{T} \otimes h : L_0 \otimes_k \mathbb{R} \text{Hom}_A(M, A) \otimes_k k \to L_0 \otimes_k \text{Hom}_k(M \otimes^A N, N) \otimes_k \mathbb{R} \text{Hom}_A(M, M).
\]

where we set \( L_0 := D(N \otimes_k (M \otimes^A N) \otimes_k a^{-1}) \).

In the next proposition, we study when \( C = N \otimes_k N \) satisfies the right asid condition.

**Proposition 6.9.** The right asid condition is satisfied if and only if one of the following conditions is satisfied.

(0) \( \Lambda = \text{End}_k(V) \), \( N = V \) and \( M = D(V) \) for some finite dimensional vector space \( V \).

(1) The module \( M \) is exceptional and \( \mathbb{R} \text{Hom}_A(M, A) \cong N[-p] \) for some \( p \in \mathbb{Z} \). The condition (0) does not hold.

(2) \( M \otimes^A N = 0 \)

Moreover, if one of conditions (a) is satisfied for \( a = 0, 1, 2 \), then we have \( \alpha_r = a \).
We prepare the following lemma

**Lemma 6.10.** (1) The morphisms \( lm \) and \( h \) are isomorphisms if and only if \( \Lambda = \text{End}_k(V) \), \( N = V \) and \( M = D(V) \) for some finite dimensional vector space \( V \).

(2) The morphism \( \mathcal{T} \) and \( h \) are isomorphisms if and only if \( M \) is exceptional and \( \mathbb{R}\text{Hom}_\Lambda(M, \Lambda) \cong N[-p] \) for some \( p \in \mathbb{Z} \).

Moreover, in either cases, we have \( M \otimes^L_\Lambda N \neq 0 \).

**Proof.** (1) Since the map \( lm \) is an algebra homomorphism, it is an isomorphism if and only if the algebra \( \Lambda \) is the endomorphism \( \text{End}_k(V) \) of the \( k \)-vector space \( V = N_k \). If this is the case, then \( \Lambda \) is a full matrix algebra. Thus it is easy to see that the homothety map \( h \) is an isomorphism if and only if \( M = D(V) \) as left modules and \( N \otimes_k M \cong \text{End}_k(V) \) as bimodules. In this case it is clear that \( M \otimes^L_\Lambda N \neq 0 \).

(2) First assume that the morphism \( \mathcal{T} \) and \( h \) are isomorphisms. Applying \( M \otimes^L_\Lambda - \) to the isomorphism \( \mathcal{T} \), we obtain the isomorphism \( M \otimes^L_\Lambda \mathcal{T} : \mathbb{R}\text{Hom}_\Lambda(M, M) \xrightarrow{\approx} \text{Hom}_k(M \otimes^L_\Lambda N, M \otimes^L_\Lambda N) \).

Since \( \mathbb{R}\text{Hom}_\Lambda(M, M) \cong k \), we must have \( M \otimes^L_\Lambda N \cong k[p] \) for some integer \( p \geq 0 \). In particular, we have \( M \otimes^L_\Lambda N \neq 0 \). Now \( \mathcal{T} \) become an an isomorphism \( \mathbb{R}\text{Hom}_\Lambda(M, \Lambda) \cong N[-p] \).

The converse implication can easily be proved and left to the readers. \( \Box \)

**Proof of Proposition 6.7.** We assume that the right asid condition is satisfied. By Lemma 6.8 if \( \alpha_r = 0 \), then \( lm \) and \( h \) are isomorphisms. Therefore by Lemma 6.10 the condition (0) holds. In the case \( \alpha_r = 1 \), \( \mathcal{T} \) and \( h \) are isomorphisms by Lemma 6.8. Therefore by Lemma 6.10 the condition (1) holds. Finally, we assume that \( \alpha_r \geq 2 \). If \( M \otimes^L_\Lambda N \neq 0 \), then we have \( L_a \neq 0 \). Therefore, since \( \mathcal{T}_c, \Lambda \) is an isomorphism for some \( a \geq 1 \), \( \mathcal{T} \) and \( h \) are isomorphisms by Lemma 6.8. Hence, again by Lemma 6.8 we have \( \alpha_r \leq 1 \). This contradicts to the assumption. Thus, if \( \alpha_r \geq 2 \), we must have \( M \otimes^L_\Lambda N = 0 \). Moreover, by Lemma 6.8 the last condition implies that \( \alpha_r = 2 \).

By Lemma 6.8 and Lemma 6.10 it is easy to see that if one of the conditions (0), (1), (2) is satisfied, then \( C \) satisfies the right asid condition and \( \alpha_r \) has desired value. \( \Box \)

We proceed a proof of Theorem 6.4.

**Proof of Theorem 6.4.** Combining Lemma 6.7, Proposition 6.9, we deduce (1) and (2).

(3-a) Since \( \text{pd} M < \infty \), the isomorphism \( \mathbb{R}\text{Hom}_\Lambda(M, \Lambda) \cong N[-p] \) implies that \( p = \text{pd} M \). Since derived the A-dual of the above isomorphism yields an isomorphism \( \mathbb{R}\text{Hom}_{\Lambda^{op}}(N, \Lambda) \cong M[-p] \), we conclude that \( p = \text{pd} N \).

(3-b) By Lemma 4.9 and Theorem 5.6, we have an equivalence \( \mathcal{CM}^Z \Lambda \cong T \) under which the autoequivalence (1) of \( \mathcal{CM}^Z \Lambda \) corresponds to \( - \otimes^L_\Lambda C[1] \).

Observe that \( T = \text{thick} M \). Since \( M \) is exceptional, we have an equivalence \( F : \text{thick} M \cong D^b(\text{mod} \Lambda) \) which sends \( M \) to \( \Lambda \). Using the isomorphism \( N \cong \mathbb{R}\text{Hom}_\Lambda(M, \Lambda)[p] \) we obtain the isomorphisms

\[
M \otimes^L_\Lambda C \cong M \otimes^L_\Lambda N \otimes_k M \cong \mathbb{R}\text{Hom}_\Lambda(M, M) \otimes_k M[p] \cong M[p].
\]

Therefore, under above equivalence \( F \), the autoequivalence \( - \otimes^L_\Lambda C[1] \) corresponds to \( [p + 1] \). Hence we obtain the desired equivalence.

(3-c) is obtained by combining (3-b) and Theorem 6.1.

(3-d) By (3-c), it is enough to show that \( M \) is an indecomposable CM-\( \Lambda \)-module. Since \( M \) is exceptional, it is indecomposable. To show that \( M \) is CM, we construct a complete resolution of \( M \) as \( \Lambda \)-module. Let \( 0 \rightarrow Q^p \xrightarrow{\partial_p} \cdots \xrightarrow{\partial_1} Q^0 \rightarrow M \rightarrow 0 \) be a minimal projective resolution as
Λ-modules. Since $M \otimes_{\Lambda} C \cong M \otimes_{\Lambda} N \otimes_k M \cong M[p]$, applying $- \otimes_{\Lambda} C$ to the minimal projective resolution, we obtain the following exact sequence of Λ-modules

$$0 \rightarrow M \rightarrow Q^{-p} \otimes_{\Lambda} C \xrightarrow{\partial^{-p} \otimes C} \cdots \xrightarrow{\partial^{-1} \otimes C} Q^{0} \otimes_{\Lambda} C \rightarrow 0.$$  

Therefore applying $- \otimes_{\Lambda} A$ to the minimal projective resolution, we obtain the following exact sequence of Λ-modules

$$0 \rightarrow M \rightarrow Q^{-p} \otimes_{\Lambda} A \xrightarrow{\partial^{-p} \otimes A} \cdots \xrightarrow{\partial^{-1} \otimes A} Q^{0} \otimes_{\Lambda} A \rightarrow M \rightarrow 0.$$  

Splicing the copies of this sequence we obtain a complete resolution $P$ of $M$ as an $A$-module as desired.

$$P : \cdots \rightarrow Q^{0} \otimes_{\Lambda} A \rightarrow Q^{-p} \otimes_{\Lambda} A \xrightarrow{\partial^{-p} \otimes A} \cdots \xrightarrow{\partial^{-1} \otimes A} Q^{0} \otimes_{\Lambda} A \rightarrow Q^{-p} \otimes_{\Lambda} A \xrightarrow{\partial^{-p} \otimes A} \cdots \rightarrow Q^{-p} \otimes_{\Lambda} C \rightarrow 0.$$  

\[\square\]

**Remark 6.11.** We remark that we can prove the Gorenstein symmetric conjecture for a trivial extension algebra $A$ of the form $A = \Lambda \oplus (N \otimes_k M)$. Namely, $A$ is IG if and only if $\text{id}_{A}A < \infty$ if and only if $\text{id}_{A}A < \infty$.

## 7 Application II: classification of asid bimodules

**Theorem 4.10** gives us a way to classify asid bimodules over an IG-algebra Λ as follows:

**Step 1.** Classify admissible subcategories $T$ of $K^b(\text{proj } \Lambda)$.

**Step 2.** For an admissible subcategory $T$, classify bimodules $C$ such that the functor $- \otimes_{\Lambda} C$ acts $T$ as an equivalence and nilpotently acts on $T^\perp$.

In Section 7.1, we deal with the path algebra $\Lambda = kQ$ of $A_2$ quiver $Q = 1 \leftarrow 2$. We demonstrate how to use the strategy and give the complete list of the asid bimodules over $\Lambda$.

In Section 7.2, we deal with the path algebra $\Lambda = kQ$ of $A_3$ quiver $Q = 1 \leftarrow 2 \rightarrow 3$. We give the complete list of the asid bimodules $C$ such that $\Lambda \oplus C$ has infinite global dimension. The restriction is put for the sake of space.

**Remark 7.1.** In this section, suffixes are used in a different way from other sections. For a Λ-module $M$ and $n \in \mathbb{N}$, we denote by $M^n$ the direct sum $M \oplus n$.

### 7.1 Classification of asid bimodules over the path algebra of $A_2$ quiver

Let $\Lambda = k[1 \leftarrow 2]$ be the path algebra, $e_1, e_2$ the idempotent elements corresponding to the vertex 1, 2. We denote by $S^1_1$ the simple left Λ-module which corresponds to the vertex 1. We denote by $S^r_2$ the simple right Λ-module which corresponds to the vertex 2.

In the list below, we give an admissible subcategory $T$ of $K^b(\text{proj } \Lambda) = D^b(\text{mod } \Lambda)$ and an asid bimodule $C$ such that $T = \text{thick } C^n$.

**Classification 7.2.**

1. $T_1 = D^b(\text{mod } \Lambda)$. $C = \Lambda, D(\Lambda)$.
2. $T_2 = \text{thick } e_1 \Lambda$. $C = \Lambda e_1 \otimes_k e_1 \Lambda$. 

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\( T_3 = \text{thick } e_2 \Lambda \). \( C = \Lambda e_2 \otimes_k e_2 \Lambda \).

\( T_4 = \text{thick } S'_2 \). \( C = S'_2 \otimes_k S'_2 \).

\( T_5 = 0 \). \( C = (\Lambda e_2 \otimes_k e_1 \Lambda)^n, (S'_2 \otimes_k e_2 \Lambda)^n, (\Lambda e_1 \otimes_k S'_2)^n \) for \( n \in \mathbb{N} \).

We note that the case \( \langle 1 \rangle \) is precisely the case \( \alpha = 0 \) and that the case \( \langle 5 \rangle \) is precisely the case \( \text{gldim } A < \infty \).

Before starting the classification, we recall basic facts about \( \Lambda = k[1 \leftarrow 2] \) (for the details we refer \[8\]). The algebra \( \Lambda \) has three indecomposable modules up to isomorphisms, whose quiver representations are given by

\[
P_1 = e_1 \Lambda = (k \leftarrow \cdots \leftarrow 0), \quad P_2 = e_2 \Lambda = (k \leftarrow \cdots \leftarrow k), \quad I_2 = S'_2 = (0 \leftarrow \cdots \leftarrow k),
\]

where solid arrow is identity map and dotted arrows are zero maps. The complete set of pairwise non-isomorphic indecomposable objects of \( \text{D}^b(\text{mod } \Lambda) \) is given by

\[
\{X[i] \mid X : \text{indecomposable } \Lambda\text{-module}, \ i \in \mathbb{Z}\}
\]

and the Auslander-Reiten quiver of \( \text{D}^b(\text{mod } \Lambda) \) is the following.

\[
\cdots \arrow{P_1[-1]} \arrow{I_2[-1]} \arrow{P_2[-1]} \arrow{I_2} \arrow{P_1} \arrow{I_2} \arrow{P_2[1]} \arrow{I_2} \arrow{P_1[1]} \arrow{I_2} \arrow{P_2[2]} \arrow{I_2} \cdots
\]

Now we give sketches of Step 1 and Step 2 for the algebra \( \Lambda \).

**Step 1.** We classify admissible subcategories of \( \text{K}^b(\text{proj } \Lambda) = \text{D}^b(\text{mod } \Lambda) \). Since \( \Lambda \) is a path algebra of a Dynkin quiver, we can classify thick subcategories of \( \text{D}^b(\text{mod } \Lambda) \) by applying \[3\] Theorem 5.1 and \[10\] Theorem 1.1. One can check that all of them are admissible subcategories. Thus we obtain the complete list of admissible subcategories, which is shown in the table below.

| \( \langle 1 \rangle \) | \( \langle 2 \rangle \) | \( \langle 3 \rangle \) | \( \langle 4 \rangle \) | \( \langle 5 \rangle \) |
|---|---|---|---|---|
| \( T_1 = \text{D}^b(\text{mod } \Lambda) \) | \( T_2 = \text{thick } P_1 \) | \( T_3 = \text{thick } P_2 \) | \( T_4 = \text{thick } I_2 \) | \( T_5 = 0 \) |

**Table 1:** List of admissible subcategories of \( \text{D}^b(\text{mod } \Lambda) \)

**Step 2.** We represent a \( \Lambda\)-\( \Lambda \)-bimodule \( C \) as a quiver representation

\[
\begin{array}{c}
e_1Ce_1 \ar^\alpha \ar_\alpha \e_1Ce_2 \\
e_2Ce_1 \ar^\alpha \ar_\alpha \e_2Ce_2
\end{array}
\]

which makes the square commutative. This is nothing but a quiver representation of \( C \) as a \( \Lambda^e \)-module.

By Morita Theory, we can recover a \( \Lambda\)-\( \Lambda \)-bimodule \( C \) from the functor \( \mathcal{T} = - \otimes^L_{\Lambda} C \) in the following way. The right \( \Lambda \)-module structure of \( C = e_1C \oplus e_2C \) can be read off from a canonical
isomorphism $\mathcal{T}(P_1) \cong e_1 C$. The left $\Lambda$-module structure of $C$ can be read off from the commutative diagram

$$
\begin{array}{ccc}
\mathcal{T}(P_1) & \xrightarrow{\mathcal{T}(\alpha)} & \mathcal{T}(P_2) \\
\cong & & \cong \\
e_1 C & \xrightarrow{\alpha} & e_2 C
\end{array}
$$

where the vertical maps are canonical isomorphisms. From these datum, we may represent $C$ as a quiver representation.

Thanks to the above observation, we can calculate quiver representations of asid bimodules corresponding to each admissible subcategory $\mathcal{T}$ by referring possible equivalences on $\mathcal{T}$ and nilpotent endofunctors on $\mathcal{T}^\perp$. In the following example, we explain how to do it for a concrete admissible subcategory.

**Example 7.3.** We deal with the admissible subcategory $\mathcal{T}_2$. First observe that $\mathcal{T}_2 = \text{add}\{P_1[i] \mid i \in \mathbb{Z}\}$ and $\mathcal{T}_2^\perp = \text{add}\{I_2[i] \mid i \in \mathbb{Z}\}$.

Let $C$ be a bimodule such that $\mathcal{T} = - \otimes_{\Lambda}^L C$ acts on $\mathcal{T}_2$ as an equivalence and nilpotently acts on $\mathcal{T}_2^\perp$. Since $\mathcal{T}$ acts on $\mathcal{T}_2$ as an equivalence, $\mathcal{T}(P_1) \cong P_1[n]$ for some $n \in \mathbb{Z}$. On the other hand $\mathcal{T}(P_1) = e_1 C$ belongs to mod $\Lambda$. Therefore we conclude that $e_1 C = \mathcal{T}(P_1) \cong P_1$.

We claim that $\mathcal{T}(I_2) = 0$. Indeed, by the assumption $\mathcal{T}(\mathcal{T}_2^\perp) \subset \mathcal{T}_2^\perp$, we have $\mathcal{T}(I_2) \cong I_m^m \oplus I_2[1]^n$ for some non-negative integers $m, n$. Since $\mathcal{T}^a(I_2) = 0$ for $a \gg 0$, $m$ and $n$ must be zero, namely $\mathcal{T}(I_2) = 0$.

By applying $\mathcal{T}$ to the triangle (7-15) below, we see that $\mathcal{T}(\alpha) : \mathcal{T}(P_1) \to \mathcal{T}(P_2)$ is an isomorphism.

(7-15)

$$I_2[-1] \to P_1 \xrightarrow{\alpha} P_2 \to I_1$$

We conclude that the right module structure of $C$ is

$$e_1 C = P_1 = (\begin{array}{c} k & \cdots & 0 \\ \end{array}), \quad e_2 C = P_1 = (\begin{array}{c} 0 \\ \end{array})$$

and the left module structure of $C$ is given by a map

$$
\begin{array}{ccc}
e_1 C & = & \begin{array}{c} k & \cdots & 0 \\ \end{array} \\
\alpha & \downarrow & \begin{array}{c} a \\ \end{array} \\
e_2 C & = & \begin{array}{c} 0 \\ \end{array}
\end{array}
$$

for some $a \in k \setminus \{0\}$. Thus the representation of $C$ is given by

$$
\begin{array}{c}
\begin{array}{c} k & \cdots & 0 \\ \end{array} \\
\alpha \\
\begin{array}{c} 0 \\ \end{array}
\end{array}
$$

It is easy to check that this bimodule is isomorphic to $\Lambda e_1 \otimes_k e_1 \Lambda$.

**Example 7.4.** We deal with the case $\mathcal{T}_5 = 0$. We note that this is precisely the case where $A = \Lambda \oplus C$ has finite global dimension.

Let $C$ be a bimodule. Then $\mathcal{T} = - \otimes_{\Lambda}^L C$ always acts on $\mathcal{T}_5 = 0$ as an equivalence. Thus we only have to study the condition that $\mathcal{T}$ nilpotently acts on $\mathcal{T}^\perp = D^b(\text{mod} \Lambda)$. We assume that $C$ satisfies this condition. Observes that at least one of $\mathcal{T}(P_1) = 0$, $\mathcal{T}(P_2) = 0$ and $\mathcal{T}(I_2) = 0$ holds.

We discuss the case $\mathcal{T}(P_1) = 0$. Since $\mathcal{T}(P_2) = e_2 C$ belongs to mod $\Lambda$, we have $\mathcal{T}(P_2) \cong P_1^m \oplus P_2^m \oplus I_2^n$ for some $l, m, n \in \mathbb{N}$. By applying $\mathcal{T}$ to the triangle (7-15), we have an isomorphism $\mathcal{T}(P_2) \cong \mathcal{T}(I_2)$. Since $\mathcal{T}^a(P_2) = 0$ for $a \gg 0$, we see that $m$ and $n$ must be zero, namely $\mathcal{T}(P_2) \cong P_1^n$. 39
We conclude that the right module structure of $C$ is
\[ e_1C = (0 \rightarrow \cdots \rightarrow 0), \quad e_2C = P^n_1 = (k^n \rightarrow \cdots \rightarrow 0). \]
Thus the representation of $C$ is given by
\[
\begin{array}{c}
0 \rightarrow \cdots \rightarrow 0 \\
\downarrow \\
k^n \rightarrow \cdots \rightarrow 0 \\
\end{array}
\]
It is clear that this bimodule is isomorphic to $(\Lambda e_2 \otimes_k e_1 \Lambda)^n$.

By the similar arguments, in the case $T(P_2) = 0$ or $T(I_2) = 0$, we can show that $C$ is isomorphic to one of the bimodules listed in (5). Consequently we have three asid bimodules corresponding to $T_5 = 0$.

### 7.2 Classification of asid bimodules over the path algebra of $A_3$ quiver

Let $\Lambda = kQ$ be the path algebra of $A_3$ quiver $Q$.

\[ Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3. \]

Below, we give the list of asid bimodules $C$ over $\Lambda$ such that $A$ has infinite global dimension. The restriction $\text{gl dim} A = \infty$ is put only for the sake of space.

We use the quiver representation as below to exhibit a bimodule $C$.

\[
\begin{array}{c}
e_1Ce_1 \xrightarrow{\alpha} e_1Ce_2 \xrightarrow{\beta} e_1Ce_3 \\
\alpha \downarrow \alpha \downarrow \alpha \\
e_2Ce_1 \xrightarrow{\alpha} e_2Ce_2 \xrightarrow{\beta} e_2Ce_3 \\
\beta \downarrow \beta \downarrow \beta \\
e_3Ce_1 \xrightarrow{\alpha} e_3Ce_2 \xrightarrow{\beta} e_3Ce_3 \\
\end{array}
\]

In the list, the asid bimodules exhibited in (i-s) and (i-t) have the same admissible subcategory as asid subcategories.

Table 2: List of asid bimodules over 1 ← 2 → 3

| (1-1) | 1 ← 2 → 3 | (1-2) | 1 ← 2 → 3 | (1-3) | 1 ← 2 → 3 | (1-4) | 1 ← 2 → 3 |
|-------|------------|-------|------------|-------|------------|-------|------------|
| 1     | 0 ← 0 → 0  | 2     | 0 ← 0 → k  | 3     | 0 ← k → k  | 1     | 0 ← k → 0  |
| k     | k → k → k  | k     | k → k → k  | 0     | k → 0 → 0  | 0     | k → 0 → k  |
| k     | 0 → 0 → k  | k     | 0 → 0 → 0  | k     | 0 → k → 0  | k     | 0 → k → k  |

| (2-1) | 1 ← 2 → 3 | (2-2) | 1 ← 2 → 3 | (3-1) | 1 ← 2 → 3 | (3-2) | 1 ← 2 → 3 |
|-------|------------|-------|------------|-------|------------|-------|------------|
| 2     | 0 ← 0 → 0  | 3     | 0 ← 0 → k  | 1     | 0 ← k → k  | 2     | 0 ← k → k  |
| k     | k → 0 → 0  | k     | k → 0 → 0  | k     | k → k → k  | k     | k → k → k  |
| k     | 0 → 0 → 0  | k     | 0 → k → k  | k     | 0 → k → k  | k     | 0 → k → k  |
Remark 7.5. In these examples, for every admissible subcategory $T$, there exists at least one asid bimodule $C$ having $T$ as the asid subcategory. However this is not the case for general algebras. There exists an algebra $\Lambda$ such that $K^b(\text{proj } \Lambda)$ has an admissible subcategory $T$ which is not the asid subcategory for any asid bimodule.

Let $\Lambda$ be the path algebra $k[1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3]$ with the relation $\beta \alpha$. We denote by $P_1$ the indecomposable projective $\Lambda$-module corresponds to the vertex 1 and by $S_3$ the simple $\Lambda$-module corresponds to the vertex 3. Then, the admissible subcategory $\text{thick}(P_1 \oplus S_3) \subset D^b(\text{mod } \Lambda)$ is not the asid subcategory for any asid bimodule.

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