Helical Versus Fundamental Solitons in Optical Fibers

Boris A. Malomed1*, G. D. Peng2 and P. L. Chu2

1Department of Interdisciplinary Studies, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel
2Optical Communications Group, School of Electrical Engineering and Telecommunications, University of New South Wales, Sydney 2052, Australia

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Abstract

We consider solitons in a nonlinear optical fiber with a single polarization in a region of parameters where it carries exactly two distinct modes, viz., the fundamental one and the first-order helical mode. From the viewpoint of applications to dense-WDM communication systems, this opens a way to double the number of channels carried by the fiber. Aside from that, experimental observation of helical (spinning) solitons (that can be launched and detected, using helicity-generating phase masks) and collisions between them and with fundamental solitons in (ordinary or hollow) optical fibers is an issue of fundamental interest, especially because it has been very recently found that spatiotemporal spinning solitons in bulk optical media with various nonlinearities are unstable. We introduce a system of coupled nonlinear Schrödinger equations for fundamental and helical modes, computing nonstandard values of the cross-phase-modulation coupling constants in it, and investigate, analytically and numerically, results of “complete” and “incomplete” collisions between solitons carried by the two modes. We conclude that the collision-induced crosstalk is partly attenuated in comparison with the usual WDM system, which sometimes may be crucially important, preventing merger of the colliding solitons into a breather. The interaction between the two modes is found to be additionally strongly suppressed in comparison with that in the WDM system in the case when a dispersion-shifted or dispersion-compensated fiber is used.

1. Introduction

A commonly adopted approach to the description of nonlinear light propagation in optical fibers is based on separation of the transverse modal structure, that may be described in the linear approximation, and slow longitudinal and temporal evolution of the signal’s envelope, which is essentially affected by the temporal dispersion and Kerr nonlinearity. The analysis following this way ends up with the derivation of the nonlinear Schrödinger (NLS) equation for the envelope [1,2].

Usually, both experimental and theoretical studies of the soliton propagation are confined to the case when parameters of the fiber admit a single (fundamental) transverse mode, because in a multimode fiber an initial pulse excites different modes in an uncontrollable fashion. However, using well-known data for fibers of the simplest step-index type [3], it is easy to find that a situation with exactly two modes takes place when the standard waveguide parameter

\[ V \equiv k \rho \sqrt{n_{co}^2 - n_{cl}^2}, \]  

(1)

where \( k, \rho \) and \( n_{co,cl} \) are, respectively, the carrier wave’s propagation constant, core radius, and the refractive index in the core and cladding, takes values

\[ 2.405 < V < 3.832. \]  

(2)

For instance, in the case of the standard carrier wavenumber \( \lambda = 1.54 \mu m \) admitting the soliton propagation in optical fibers, and taking the usual value \( n_{co} - n_{cl} = 0.01 \), the interval (2) implies \( 3 \mu m < \rho < 4.75 \mu m \), i.e., quite realistic values of the core’s radius.

Inside the interval (2), the fiber carries a fundamental mode (FM) and the first helical mode (HM). The transverse structure of the latter one is described by the expressions

\[ J_1(Ur) \exp(\pm i\theta) \]  

in the core and \( K_1(Wr) \exp(\pm i\theta) \) in the cladding, where \( J_1 \) and \( K_1 \) are the standard cylindrical functions, \( U \) and \( W \) are the associate waveguide parameters defined in the usual way [3], and \( (r, \theta) \) are the polar coordinates in the fiber’s cross section. In this case, one is actually dealing with the set of three coexisting modes, as there are two degenerate HMs with helicities \( S = \pm 1 \). Note also that, because the physical fields are proportional to the real part of the complex expressions, the presence of the multiplier \( \exp(\pm i\theta) \) means that HM solitons are spinning in the course of the propagation along the fiber.

As neither higher-order radial (nonhedral) mode, nor any HM with a helicity \( > 1 \) exists in the interval (2), the two-mode situation is controllable; a light pulse with zero helicity can excite solely FM, while excitation of HM is possible by a pulse that carries the necessary helicity. A light beam can be lent helicity, passing it through a specially designed phase mask, which is quite feasible in a real experiment, see, e.g., Ref. [4]. Due to their distinct topological nature, FM and HM do not linearly mix, provided that the fiber remains circular. As is well known, it is easy to fabricate a long silica fiber whose deviation from circularity is negligible. Fiber bending will induce no linear mixing either, providing that the bending radius is much larger than the wavelength. In this work, however, we do not discuss exact limitations on the deviation of the fiber from circularity and similar details. Instead, we focus on principal issues, such as collisions between solitons carried by FM and HM.

It is necessary to distinguish between the fundamental and helical solitons at the receiver end of the fiber. In an experiment with single or few copropagating carrier frequencies, this is quite simple, as the fundamental and helical modes have an appreciable difference in their propagation constants (see below), thus the two types of solitons can be discriminated between by means of a simple wavelength filter. Besides that, there is a possibility to create a “helicity filter”, which would also work in the case of a multi-channel WDM system. Indeed, if it is known that two species of the solitons in the fiber have the helicities \( S = 0 \) and \( S = +1 \), at the receiver end the incoming signal can be passed through a phase mask that adds extra helicity.
\(\Delta S = -2\). Then, the arriving FM and HM solitons will change their helicities to \(-2\) and \(-1\), respectively, so that only the latter one will survive, as in the selected parametric region the modes with \(S = \pm 2\) do not propagate in the fiber. On the other hand, passing the incoming signal through the phase mask adding \(\Delta S = +1\) will transform the former \(S = 0\) soliton into a propagating \(S = +1\) plus, while the former \(S = 1\) soliton will have \(S = +2\), hence it will not be able to propagate.

Thus, launching solitons independently in each of the two modes, one can implement a two-channel system inside the core. Note that standard elements of the fiber communication systems, such as amplifiers and guiding filters [2], will act in essentially the same way on the solitons in both modes (although the gain coefficients of an Er-doped or Raman fiber amplifier may differ for the two modes, depending, e.g., on the density distribution of the doping atoms in the fiber’s transverse plane). Moreover, if one starts with a WDM (wavelength-division-multiplexed) multichannel system already implemented in the fiber, one can double the number of channels by means of this two-mode scheme (we demonstrate below that it is not really possible to triple the number of channels, using two HMs with opposite helicities). The feasibility of such a “mode-division” channel doubling may be quite important, as it has been demonstrated that doubling by means of polarization-division multiplexing is incompatible with WDM [5]. Indeed, while the polarization of the soliton can be easily changed by various imperfections of the system, the mode helicity is expected to be robust, as it is a topological invariant. Note that we do not consider the polarization structure of the modes, assuming that either they belong to one polarization, or (more realistically for the applications) the polarization can be effectively averaged out. Thus, our helical mode has nothing in common with the circular polarization.

Due to the Kerr nonlinearity, the linearly orthogonal solitons borne by the two modes interact via the cross-phase modulation (XPM). The main technical objective of this work is to study XPM-induced effects of collisions between solitons. As is well known, the collisional crossstalk is the most fundamental problem in the soliton-based multi-channel communication system, see e.g., Refs. [6–8]. It should be stressed that, while the application of the proposed mode-division doubling to WDM soliton communication systems is not straightforward, as some technical problems remain to be resolved, experimental observation of narrow subpicosecond helical (spinning) solitons and their collisions between themselves and with fundamental solitons in relatively short optical fibers is a problem of fundamental physical interest by itself. A combination of the above-mentioned helicity-generating phase masks with the well-developed experimental technique admitting, e.g., direct observation of the polarization structure of subpicosecond solitons in short fibers [9] should make the observation of the helical solitons and their collisions quite feasible. An additional interest to the latter problem is lent by the fact that a similar object in a bulk optical medium, viz., a spinning light bullet (spatiotemporal soliton), has been very recently found to be unstable in models with various nonlinearities [10]. Therefore, the optical fiber may be a unique medium in which the existence of a stable spinning temporal soliton may be possible.

It is also noteworthy that the helical soliton, whose local intensity vanishes at the central point of the fiber’s cross section, may be a natural object to exist in hollow nonlinear optical fibers, which have recently attracted a lot of attention (and where, incidentally, very narrow solitary pulses are quite possible), see, e.g., the works [11] and references therein. An interesting issue is a possibility to select parameters of the hollow fiber so that it would support solely a helical mode, which would then play the role of the fundamental one.

The paper is organized as follows. In Section 2, a system of coupled NLS equations for the three modes, FM with \(S = 0\) and two HMs with \(S = \pm 1\), is obtained. It has nonstandard values of the XPM coupling constants. In Section 3, collisions between solitons carried by the fundamental and helical modes is studied analytically, by means of the perturbation theory. The perturbative treatment applies to the case when the colliding solitons pass through each other quickly enough. In Section 4, an example of direct numerical simulations of the collision between the FM and HM solitons, illustrating a difference from the collision between FM solitons in the usual WDM system, is displayed. The difference may be crucial: the colliding solitons merge into a breather in the usual system, but survive the collision when they belong to the different modes. The paper is concluded by Section 5.

2. The model

A normalized system of coupled nonlinear Schrödinger (NLS) equations for the interacting modes can be derived by means of a standard asymptotic procedure [1].

\[
i(u_\alpha)_t + i\epsilon_\alpha(u_\alpha)_x + k_\alpha u_\alpha - \frac{1}{2}\beta_\alpha(u_\alpha)_x = (|u_\alpha|^2 + 2|u_\alpha|^2 + 2|u_\alpha|^2)u_\alpha = 0.
\]

(3)

\[
i(u_+)_t + i\epsilon_+(u_+)_x + k_+ u_+ - \frac{1}{2}\beta_+(u_+)_x = \left(|u_+|^2 + 2|u_+|^2 + 2|u_+|^2\right)u_+ = 0,
\]

(4)

\[
i(u_-)_t + i\epsilon_-(u_-)_x + k_- u_- - \frac{1}{2}\beta_-(u_-)_x = \left(|u_-|^2 + 2|u_-|^2 + 2|u_-|^2\right)u_- = 0
\]

(5)

(in the case of extremely narrow solitons, well-known higher-order terms [1,2] should be added to the system). We here consider the most general case, when two HMs with helicities \(\pm 1\), represented by the envelopes \(u_\pm\), interact with the zero-helicity FM \(u_0\); \(\beta_0\) and \(\beta_\pm\) are the corresponding mode-dependent dispersion coefficients (see below), \(c_{\alpha,1}\) and \(k_{\alpha,1}\) are the group-velocity and propagation-constant shifts of the two modes (these characteristics are also mode-dependent [3]), and the effective XPM coefficients \(\gamma_0\) and \(\gamma_\pm\) are given by the properly normalized overlapping integrals between FM and HM. Using known expressions for the transverse modal functions of the step-index fiber [3], we have calculated them numerically. In Fig. 1a, we display \(\gamma_0\) and \(\gamma_\pm\) vs. the waveguide parameter (1).

Note that Eqs. (3)–(5) do not contain four-wave mixing (FWM) terms. Some of them might originate from the terms \(\sim u_0^\star(u_\pm^\star)^2\) and its complex conjugate in the model’s Hamiltonian density. However, the full expressions to be inserted into the Hamiltonian are multiplied by the modal
angular dependences $\exp(\pm 2i\theta)$, hence they will give zero upon angular integration. Another possible source of FWM terms in Eqs. (3)-(5) could be the term $\sim u_0^2 u_1^* u_2^*$ and its complex conjugate in the Hamiltonian density. In these expressions, the angular dependence cancels out, hence the angular integration will not nullify them. However, the corresponding terms in Eqs. (3)-(5) will be rapidly oscillating in $\theta$ because of the difference between the propagation constants $k_0$ and $k_1$. A straightforward consideration yields an estimate for the relative wavenumber mismatch between FM and HM in the region of interest, $k_1/k \approx 0.4(n_{co} - n_{cl})/k_0$. Taking the same estimate for the refractive index difference, $n_{co} - n_{cl} \approx 0.01$, as above, we conclude that $k_1/k_0 \approx 0.005$, which corresponds to a beat length $\approx 200$ wavelengths. As it is disparately small in comparison with any propagation distance relevant to the solitons, all the FWM terms may be neglected.

As for the dispersion coefficients in Eqs. (1)-(3), their parts accounted for by the waveguide geometry can also be calculated for the two modes on the basis of the data available from the linear-propagation theory [3]. The result of the calculation is shown in Fig. 1b. It is noteworthy that the waveguide-geometry part of $\beta_0$ changes its sign. One should, however, keep in mind that the full dispersion also contains a material (bulk) contribution, that may be essentially larger than that displayed in Fig. 1b.

Thus, the analysis of the interaction between solitons must admit different (but both negative, i.e., anomalous [1] effective dispersions $\beta_0$ and $\beta_1$ in Eqs. (3)-(5). Together with the nonstandard values of the XPM coefficients $\gamma_0$ and $\gamma_1$, these features constitute an essential mathematical difference of the present model from the usual three-channel WDM one (see, e.g., Ref. [7]).

The fundamental and helical modes are also characterized by a difference in their group velocities, which plays a crucially important role in the analysis of soliton–soliton collisions. Continuing the above estimates of the physical parameters for the present case, we obtain

$$ (\delta v_{gr})_{\text{mod/e}/v_{gr}} \sim 5 \cdot 10^{-6} $$

(6)

for the relative group-velocity difference between the modes. As concerns the possibility to use the mode-division doubling of the channels in the WDM system, it is relevant to mention that, in the WDM system implemented in the standard telecommunications fiber with the dispersion $\beta \sim -20 \text{ ps}^2/\text{km}$ at 1.54 $\mu$m, the relative group-velocity mismatch between the adjacent channels is estimated to be

$$ (\delta u_{gr})_{WDM}/v_{gr} \sim 10^{-2} \cdot (\delta \lambda/\lambda), $$

(7)

$\delta \lambda$ being the wavelength separation between the channels. The case of practical interest is $\Delta \lambda \sim 1 \text{ nm}$, hence we conclude that the relative group-velocity differences (6) and (7) are of the same order of magnitude. On the other hand, in dispersion-shifted (DS) or dispersion-compensated (DC) fibers, the effective value of the dispersion is much smaller than the above-mentioned value $-20 \text{ ps}^2/\text{km}$, hence in these cases the inference is that the corresponding WDM relative difference is negligible as compared to that between the fundamental and helical modes,

$$ (\delta v_{gr})_{\text{DS/DC}/v_{gr}} \ll (\delta v_{gr})_{\text{mod/e}/v_{gr}} $$

(8)

For typical solitons to be used in telecommunications, with the temporal width $T \sim 10 \text{ ps}$, the above estimate $(\delta v_{gr})_{\text{mod/e}/v_{gr}} \sim 5 \cdot 10^{-6}$ implies that a collision between the FM and HM solitons takes place at a propagation distance $z_{\text{coll}} \sim T(\delta v_{gr})_{\text{mod/e}/v_{gr}} \sim 500 \text{ m}$, which is much shorter than any soliton’s length scale. This circumstance allows us to treat the XPM-mediated interaction as a small perturbation in the course of the fast passage of one soliton through the other, as it was done in other contexts in Refs. [7,8]. Note that in laboratory experiments with subpicosecond solitons, the collision length may be $\leq 50$ m, implying that the experimental study of the collisions should be possible.

However, there is no group-velocity difference between the two HMs $u_{\pm}$, hence the collision distance for the corresponding solitons may be very large, giving rise to a strong crosstalk between them. Moreover, the collision between two solitons with the helicities $S = \pm 1$ may result in their annihilation or transformation into a pair of $S = 0$ solitons, while, due to the conservation of the topological invariant, the collision between solitons with $S = 0$ and $S = 1$ is expected to be much closer to an elastic one. In view of this, it makes sense to assume only the doubling of the number of channels by means of the “mode-division multiplexing”
(i.e., to use only one HM) in the context of the WDM systems, but not tripling, that might seem possible due to the existence of two HMs with $S = \pm 1$. Irrespective of that, a study of collisions between solitons with $S = +1$ and $S = -1$ is a challenge for experiments with narrow solitons in optical fibers.

3. Analytical treatment of soliton collisions

Proceeding to a perturbative analysis of the collision between solitons carried by FM and HM, we should take into consideration that, in view of the asymmetry between Eqs. (3) and (4), (5), the FM and HM solitons may have different widths, $T_0$ and $T_1$. This circumstance makes it technically impossible to base the perturbative treatment of the collision on the exact unperturbed soliton waveforms of the sech type, as the corresponding overlapping integrals will be intractable. The only possibility to develop an efficient perturbation theory is to use, as the zero-order approximation, the Gaussian ansatz for the unperturbed soliton solutions to the uncoupled equations (3) and (4),

$$u_t^{(0)}(z, \tau) = A_t \exp \left( \frac{(\tau - t)^2}{2T_t^2} \right), \quad \frac{dA_t}{dz} = c_t; \quad l = 0, 1, \quad (9)$$

where a relation between the amplitude and width of the soliton can be found by means of the variational approximation [12],

$$A_t^2 = \sqrt{2} |\beta_l| / T_l^2$$

(10)

(the propagation constants $K_{\gamma}$ will not be needed here). In fact, the difference between the approximate soliton shape given by Eq. (9) and the exact sech shape is fairly small, see, e.g., Fig. 5 in Ref. [12].

A soliton moving in the given reference frame is obtained from Eq. (9) as its Galilean transform,

$$u(z, \tau) = u_t^{(0)}(z - t(z)) \exp(-i\omega_t \tau + iqz), \quad (11)$$

where $\omega_t$, is an arbitrary transform-generating frequency shift, the propagation-constant shift $q$ is not essential, and (cf. Eq. (9))

$$\frac{dA_t}{dz} = c_t - |\beta_l| q, \quad (12)$$

If the XPM term in Eqs. (1)-(3) is, effectively, a small perturbation (in the case of a fast collision, see above), the collision between the solitons may be described as that between two quasiparticles interacting through an effective potential. Following the lines of the analysis developed for similar problems earlier [7,8], it is straightforward to derive the following perturbation-induced evolution equations for the solitons' frequency shifts:

$$\frac{d\gamma_t}{dz} + \frac{4 |\beta_l| \gamma_t}{T_t - t(z) \sqrt{T_0^2 + T_1^2}} \cdot \frac{d \gamma_t}{dz} \cdot \exp \left[ - \frac{(t(z) - t_0)^2}{2(T_0^2 + T_1^2)} \right] = 0, \quad (13)$$

where Eq. (10) was used to eliminate the amplitudes in favor of the widths $T_l$ (recall that $\gamma_t$ are the relative XPM coupling constants in Eqs. (3)-(5)). Combining Eqs. (13) with Eqs. (12) and assuming, in the first approximation, $T_t = \text{const.}$ furnish a closed dynamical system governing the evolution of the temporal positions $t_l$ of the two solitons.

To further apply perturbation theory to Eq. (13), we recall that, according to the estimates obtained above, the difference of the inverse group velocities, $c_1 - c_0$, is, effectively, a large parameter. Hence, in the lowest-order approximation, one may set $t_1 - t_0 \approx c_2$ in the argument of the exponential in Eq. (13), thus strongly simplifying the equation:

$$\frac{d\gamma_t}{dz} = \frac{4(-1)^l |\beta_l| \gamma_t}{cT_l - t(z) \sqrt{T_0^2 + T_1^2}} \cdot \frac{d \gamma_t}{dz} \cdot \exp \left[ - \frac{(c_2)^2}{2(T_0^2 + T_1^2)} \right]. \quad (14)$$

To proceed, it is necessary to specify the type of collision to be considered. One should distinguish between “complete” and “incomplete” collisions [8]. In the former case, the solitons are, originally, far separated; in the course of the interaction, the faster soliton catches up with the slower one and passes it. In the first approximation, the complete interaction does not give rise to a net frequency shift (a change of the frequency would be tantamount to a change of the soliton’s velocity, according to Eq. (12)), as the integration of the right-hand side of Eq. (14) from $z = -\infty$ to $z = +\infty$ yields zero. However, finding a nonzero instantaneous frequency shift from Eq. (14), inserting it into Eq. (12), and integrating the latter equation yield a nonzero collision-induced position shift $\delta t_l$ of the soliton’s center which is the main effect of the complete collision. A final result can be conveniently written as a relative position shift, normalized to the soliton’s temporal width:

$$\frac{\delta t_l}{T_l} = (-1)^{l-1} \cdot 4\sqrt{2\pi} \cdot \frac{\beta_0 \beta_1}{c^2 T_0 T_1} \gamma_t, \quad (15)$$

An “incomplete” collision takes place if the solitons are essentially overlapped at the initial point, $z = 0$. This kind of collision is more significant, as it gives rise to a nonzero net frequency shift $\delta \omega_t$ (hence, to a velocity shift too). The most important (dangerous) case is that when centers of the colliding solitons exactly coincide at $z = 0$. In this case, $\delta \omega_t$ is found by straightforward integration of Eq. (14) from $z = 0$ to $z = +\infty$. The result can be presented in a more natural form, multiplying the net frequency shift by the soliton’s temporal width (i.e., normalizing the frequency shift to the soliton’s spectral width):

$$T_l \delta \omega_t = 4(-1)^{l-1} \cdot \frac{|\beta_1| T_l}{cT_l - t(z) \sqrt{T_0^2 + T_1^2}} \gamma_t, \quad (16)$$

The only difference of Eqs. (15) and (16) from similar results for the usual WDM system are the specific XPM coefficients $\gamma_t$, which are $\equiv 1$ in the usual case. The most promising range for the applications is around $V = 3.6$ (Fig. 1a), which gives

$$\gamma_0 \approx 0.98 \quad \text{and} \quad \gamma_1 \approx 0.62 \quad (17)$$

This implies that the crosstalk between the FM and HM solitons is attenuated by the factor 0.62 for the HM soliton, as compared to the usual WDM system, while for the FM soliton the crosstalk strength is not different from that in the usual system (taking, instead, the values around

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Although the above-mentioned frequency-shift-attenuation factor 0.62 is not really small, sometimes it may be important. In Fig. 2a, we display an example of a disastrous incomplete collision in the usual WDM system, which leads to a merger of the solitons into a breather (simulations of the complete collision at the same values of the parameters show that it is fairly mild, producing only small position shifts of the solitons). Replacing the usual XPM coefficients $\gamma = 1$ by those for the FM–HM collision, given by Eq. (17), we find that the same solitons survive the incomplete collision (Fig. 2b).

5. Conclusion

We have considered solitons in a nonlinear optical fiber in a parametric region where the fiber supports exactly two distinct modes, the fundamental one and the first helical mode, which allows one to double the number of soliton channels in the fiber. We have introduced a system of coupled NLS equations for the two modes and computed nonstandard values of the relative XPM coupling constants in it. Then, we investigated analytically and numerically results of both “complete” and “incomplete” collisions between solitons carried by the helical and fundamental modes, concluding that the crosstalk effects are partly attenuated for the helical solitons. The crosstalk between the two modes is found to be additionally suppressed in comparison with that in the WDM system in the case when the latter system is realized in a dispersion-shifted or dispersion-compensated fiber. Irrespective of the possible applications, experimental observation of narrow helical (spinning) temporal solitons in ordinary or hollow optical fibers is a challenge, especially because it has been recently demonstrated that spinning spatiotemporal solitons (“light bullets”) are unstable in bulk media.

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