Mattila–Sjölin Type Functions: A Finite Field Model

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Abstract
Let \( \phi(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be a function. We say \( \phi \) is a Mattila–Sjölin type function of index \( \gamma \) if \( \gamma \) is the smallest number satisfying the property that for any compact set \( E \subset \mathbb{R}^d \), \( \phi(E, E) \) has a non-empty interior whenever \( \dim_H(E) > \gamma \). The usual distance function, \( \phi(x, y) = |x - y| \), is conjectured to be a Mattila–Sjölin type function of index \( \frac{d}{2} \). In the setting of finite fields \( \mathbb{F}_q \), this definition is equivalent to the statement that \( \phi(E, E) = \mathbb{F}_q \) whenever \( |E| \geq q^\gamma \). The main purpose of this paper is to prove the existence of such functions with index \( \frac{d}{2} \) in the vector space \( \mathbb{F}_q^d \).

Keywords Erdős–Falconer distance problem · Falconer distance conjecture · Mattila–Sjölin type functions · Finite fields

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1 Introduction
For \( E \subset \mathbb{R}^d \), we define its distance set by \( \Delta(E) := \{|x - y|: x, y \in E\} \). The classical Falconer distance conjecture [5] says that for any compact set \( E \subset \mathbb{R}^d \), if the Hausdorff dimension \( \dim_H(E) \) of \( E \) is greater than \( \frac{d}{2} \), then the Lebesgue measure \( \mathcal{L}(\Delta(E)) \) of \( \Delta(E) \) is positive.

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The first result was given by Falconer [5] in 1983, which says that if \( \dim_H(E) > \frac{d+1}{2} \), then \( \mathcal{L}^d(\Delta(E)) > 0 \). The threshold \( \frac{d+1}{2} \) was improved to \( \frac{d}{2} + \frac{1}{4} \) by Wolff [23] in 1999 for \( d = 2 \) and Erdős [4] in 2005 for \( d \geq 3 \). The followings are the current best results:

- \( d = 2 \), Guth, Iosevich, Ou, and Wang [6] (2019): \( 1 + \frac{1}{4} \).
- \( d = 3 \), Du, Guth, Ou, Wang, Wilson, and Zhang [1] (2017): \( \frac{3}{2} + \frac{3}{10} \).
- \( d \geq 4 \) even, Du, Iosevich, Ou, Wang, and Zhang [3] (2020): \( \frac{d}{2} + \frac{1}{4} \).
- \( d \geq 5 \) odd, Du and Zhang [2] (2018): \( \frac{d}{2} + \frac{d}{2d-2} \).

In another direction, Mattila and Sjölin [19] obtained a stronger conclusion under the same condition \( \dim_H(E) > \frac{d+1}{2} \), namely, one has that the distance set has non-empty interior. It has also been conjectured that the right dimension should be \( \frac{d}{2} \), see [18, Conjecture 4.4.]. Several extensions of this result for general functions and configurations have been obtained recently, for instance, see [7, 8]. In a very recent paper, the second, third, and fourth listed authors obtained an improvement for Cartesian product sets, namely, \( E = A^d \subset \mathbb{R}^d \). In particular, we have

**Theorem 1.1** ([15]) Let \( A \) be a compact set in \( \mathbb{R} \). Then we have \( \text{Int}(\Delta(A^d)) \neq \emptyset \) provided that

\[
\dim_H(A) > \begin{cases} 
\frac{d+1}{2d}, & \text{if } 2 \leq d \leq 4, \\
\frac{d+1}{2d} - \frac{d-1}{2d(2d-4)}, & \text{if } 5 \leq d \leq 26, \\
\frac{d+1}{2d} - \frac{2d(2d-228)}{114d(2d-4)}, & \text{if } 27 \leq d.
\end{cases}
\]

Let \( \phi(x, y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be a function in \( 2d \) variables. We say that \( \phi \) is a Mattila–Sjölin type function of index \( \gamma \) if \( \gamma \) is the smallest number satisfying the property that for any compact set \( E \subset \mathbb{R}^d \), \( \text{Int}(\phi(E, E)) \) contains an interval whenever the Hausdorff dimension of \( E \) is greater than \( \gamma \), where \( \phi(E, E) := \{\phi(x, y) : x, y \in E\} \). Therefore, it follows directly from Mattila and Sjölin’s result that the distance function is of index at most \( \frac{d+1}{2} \). It is conjectured that its index should be as small as \( \frac{d}{2} \).

Let \( \mathbb{F}_q \) be a finite field of order \( q \) which is a prime power. The distance function between two points \( x \) and \( y \) in the space \( \mathbb{F}_q^d \) is defined by \( \|x - y\| := (x_1 - y_1)^2 + \cdots + (x_d - y_d)^2 \).

The finite field model of the Falconer distance problem was studied by Iosevich and Rudnev in [13]. More precisely, they proved that for any \( E \subset \mathbb{F}_q^d \), if \( |E| \geq 4q^{\frac{d+1}{2}} \), then \( \mathcal{L}^d(\Delta(E)) = \mathbb{F}_q \), which is directly in line with the Mattila–Sjölin’s result. Notice that in the continuous setting, if \( \dim_H(E) > \alpha \) implies \( \mathcal{L}^d(\Delta(E)) > 0 \) or \( \text{Int}(\Delta(E)) \neq \emptyset \), then it is expected in finite fields that if \( |E| \gg q^{\alpha \gamma} \), then \( \Delta(E) \) covers a positive proportion of all distances, or \( \Delta(E) = \mathbb{F}_q \), respectively. Comparing the finite field and continuous settings, there is a crucial difference when \( q \equiv 1 \mod 4 \) or \( d - 1 = 4k \), \( k \in \mathbb{N} \), namely, under one of those conditions, one can easily construct null-subspaces \( V \) of reasonably large dimensions, i.e. subspaces \( V \) with \( \|x\| = 0, x \cdot y = 0 \) for all \( x, y \in V \), which of course does not exist in \( \mathbb{R}^d \). As a consequence, one can construct sets \( E \subset \mathbb{F}_q^d \) which have highly arithmetic structures, and for these sets, it has been indicated in [9] that the exponent \( \frac{d+1}{2} \) is best possible. However, in case \( q \equiv 3 \mod 4 \), if \( d \geq 2 \) is even, or \( d = 4k - 1 \), \( k \in \mathbb{N} \), the two settings follow in the same way, and it has been conjectured that the right exponent should be \( \frac{d}{2} \) for a positive proportion of all distances, or even the whole field. This conjecture is still wide open. In the setting of prime fields, the current best exponent in the plane is \( \frac{5}{4} \) due to Murphy, Petridis, Pham, Rudnev, and Stevens [20]. We refer the reader to [16, 20] for recent progress.
In the setting of finite fields, we say that a function \( \phi(x, y) : \mathbb{F}_q^d \times \mathbb{F}_q^d \rightarrow \mathbb{F}_q \) is a Mattila–Sjölin type function of index \( \gamma \) in \( \mathbb{F}_q^d \) if \( \gamma \) is the smallest number satisfying the property that for any \( E \subset \mathbb{F}_q^d \) with \( |E| \gg q^\gamma \), we have \( \phi(E, E) = \mathbb{F}_q \).

In both the finite field (with some conditions) and continuous settings, the distance function is conjectured to be of index \( \frac{d}{2} \). However, to the best knowledge of the authors, we are not aware of any function of index \( \frac{d}{2} \). The main purpose of this paper is to prove the existence of such a function in vector spaces over finite fields.

To state our main theorems, we need the following definition.

**Definition 1.2** Let \( d = 2n \) for a positive integer \( n \). For each \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{F}_q^d \), we denote \( x' = (x_1, x_2, \ldots, x_n) \) and \( x'' = (x_{n+1}, x_{n+2}, \ldots, x_d) \). For each \( x, y \in \mathbb{F}_q^d \), we define

\[
\phi(x, y) := \begin{cases} 
\frac{\|x' - y'\|}{\|x'' - y''\|} & \text{if } \|x'' - y''\| \neq 0, \\
0 & \text{if } \|x'' - y''\| = 0.
\end{cases}
\]

In the following theorems, we will see that the function \( \phi \) defined as in Definition 1.2 is a Mattila–Sjölin type function of index \( \frac{d}{2} \) if and only if \( 3 \mod 4 \) and \( 4 \). Unlike the case of usual distance function, Theorems 1.3 and 1.4 provide a complete description in the case of finite fields.

**Theorem 1.3** Let \( q \equiv 3 \mod 4 \). If \( E \subset \mathbb{F}_q^d \) and \( |E| > q^2 \), then

\[
\phi(E, E) = \mathbb{F}_q.
\]

The sharpness of this theorem can be checked easily. For example, if we take \( E = \mathbb{F}_q^2 \times \{(0, 0)\} \), then \( |E| = q^2 \) and \( \phi(E, E) = \{0\} \). Hence, when \( d = 4 \), the exponent \( d/2 \) cannot be improved. Moreover, the assumption that \( q \equiv 3 \mod 4 \) cannot be dropped, since otherwise there is an element \( i \in \mathbb{F}_q \) such that \( i^2 = -1 \) so that one can take \( E = \mathbb{F}_q^2 \times \{(t, it) : t \in \mathbb{F}_q \} \) for which \( |E| = q^3 \) and \( \phi(E, E) = \{0\} \).

For all cases other than those specified in Theorem 1.3, it turns out that the \( \frac{d}{2} \)-exponent cannot be attained. More precisely, we have the following theorem, which is also optimal up to a constant factor.

**Theorem 1.4** Let \( E \) be a subset of \( \mathbb{F}_q^d \).

1. If \( d = 4k \) for some \( k \in \mathbb{N} \), and \( |E| \geq Cq^{\frac{3d}{4}} \), then \( \phi(E, E) = \mathbb{F}_q \). Furthermore, if \( k \) is odd and \( q \equiv 3 \mod 4 \), then the condition \( |E| \geq Cq^{\frac{3d-4}{4}} \) is enough.
2. If \( d = 4k + 2 \) and for some \( k \in \mathbb{N} \), and \( |E| \geq Cq^{\frac{3d-2}{4}} \), then \( \phi(E, E) = \mathbb{F}_q \).

It follows from Theorems 1.3 and 1.4 that one can expect a function to be a Mattila–Sjölin type function of index \( \frac{d}{2} \) in \( \mathbb{R}^d \) only in some specific dimensions. This leads us to the following conjecture which will be addressed in a sequel paper.

**Conjecture 1.5** Let \( \phi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) be a function defined as in Definition 1.2. We have \( \phi \) is a Mattila–Sjölin type function of index \( \frac{d}{2} \) if and only if \( d = 4 \).

**Remark 1.1** We note here that when \( E = A \times A \) for \( A \subset \mathbb{F}_q^m \) or \( E = A^{2n} \) for \( A \subset \mathbb{F}_q \), the cardinality of \( \phi(E, E) \) has been studied in earlier papers by Iosevich, Koh, and...
Parshall [12], Pham and Suk [21], respectively, with different approaches. More precisely, the method in [12] only works for the case \( E = A \times A, A \subseteq \mathbb{F}_q^n \), and the key idea in the approach of [21] comes from arithmetic structures of sets. If the function \( \phi(x, y) \) is defined by \( \| x' - y' \| \cdot \| x'' - y'' \| \), then the authors of [11] proved that the index of this function is at most \( d^2 + \frac{1}{4} \) for only the family of sets \( E = A \times A, A \subseteq \mathbb{F}_q^n \).

**Remark 1.2** In a recent paper [8], Greenleaf, Iosevich, and Taylor studied several extensions of Mattila–Sjölin’s theorem. More precisely, they showed that for several families of functions \( \phi : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}^m \), for any \( E \subseteq \mathbb{R}^d \), if \( \dim_H(E) > \frac{d+1}{2} \), then the interior of \( \phi(E, \cdots, E) \) is non-empty. Our definition of Mattila–Sjölin type functions can also be extended in this general setting.

## 2 Preliminaries and Key Lemmas

We first start by recalling notations from Fourier analysis over finite fields from [14, 17]. The Fourier transform of \( f \), denoted by \( \widehat{f} \), is defined by

\[
\widehat{f}(m) := q^{-d} \sum_{x \in \mathbb{F}_q^n} \chi(-m \cdot x) f(x).
\]

Let \( f \) be a complex valued function on \( \mathbb{F}_q^n \). Here, and throughout this paper, \( \chi \) denotes the canonical additive character of \( \mathbb{F}_q \).

The Fourier inversion theorem is given by

\[
f(x) = \sum_{m \in \mathbb{F}_q^n} \chi(m \cdot x) \widehat{f}(m).
\]

The orthogonality of the additive character \( \chi \) says that

\[
\sum_{\alpha \in \mathbb{F}_q^n} \chi(\beta \cdot \alpha) = \begin{cases} 0 & \text{if } \beta \neq (0, \ldots, 0), \\ q^d & \text{if } \beta = (0, \ldots, 0). \end{cases}
\]

By the orthogonality of \( \chi \), it is not hard to prove that

\[
\sum_{m \in \mathbb{F}_q^n} |\widehat{f}(m)|^2 = q^{-d} \sum_{x \in \mathbb{F}_q^n} |f(x)|^2.
\]

This formula is referred to as the Plancherel theorem. For example, it is a direct consequence of the Plancherel theorem that for any set \( E \) in \( \mathbb{F}_q^n \),

\[
\sum_{m \in \mathbb{F}_q^n} |\widehat{E}(m)|^2 = q^{-n} |E|.
\]

Here, and throughout this paper, we identify a set \( E \) with the indicator function \( 1_E \) on \( E \).

Throughout this paper, we denote by \( \eta \) the quadratic character of \( \mathbb{F}_q \) with the convention that \( \eta(0) = 0 \). More precisely, we take \( \eta(t) = 1 \) for a square number \( t \) in \( \mathbb{F}_q^* \), and \( -1 \) otherwise. For a non-zero element \( a \in \mathbb{F}_q^* \), the Gauss sum \( G_a \) is defined by

\[
G_a := \sum_{x \in \mathbb{F}_q^*} \eta(s) \chi(as).
\]
The Gauss sum $G_a$ is also written as follows: for $a \in \mathbb{F}_q^*$,

$$G_a = \sum_{s \in \mathbb{F}_q} \chi(as^2) = \eta(a)G_1.$$  

The explicit form of $G_a$ is presented in the following lemma.

**Lemma 2.1** ([17, Theorem 5.15]) Let $\mathbb{F}_q$ be a finite field with $q = p^\ell$, where $p$ is an odd prime and $\ell \in \mathbb{N}$. Then we have

$$G_1 = \begin{cases} (-1)^{\ell-1}q^{\frac{1}{2}} & \text{if } p \equiv 1 \mod 4, \\ (-1)^{\ell-1}i^{\ell}q^{\frac{1}{2}} & \text{if } p \equiv 3 \mod 4. \end{cases}$$

There are several consequences of this lemma, which will be used throughout the paper. More precisely, one can prove the following estimate by completing the square and using a change of variables:

$$\sum_{s \in \mathbb{F}_q} \chi(as^2 + bs) = \eta(a)G_1 \chi(\frac{b^2}{-4a}).$$  \hspace{1cm} (1)

We recall the fact that $q \equiv 3 \mod 4$ if and only if $l$ is odd and $p \equiv 3 \mod 4$, so

$$G_1^2 = \eta(-1)q.$$  \hspace{1cm} (2)

### 2.1 Key Lemmas

In this section, we present key lemmas in the proofs of Theorems 1.3 and 1.4.

Given an even integer $d \geq 4$, we define

$$S_0 := \left\{ x' \in \mathbb{F}_q^{d/2} : \|x'\| = 0 \right\},$$

which is called the zero sphere in $\mathbb{F}_q^{d/2}$. We shall invoke the well-known Fourier transform on the zero sphere $S_0$ in $\mathbb{F}_q^{d/2}$, which is closely related to the Gauss sum.

**Lemma 2.2** ([10, Lemma 4]) For $d \geq 4$ even, let $S_0$ be the zero sphere in $\mathbb{F}_q^{d/2}$. Then, for $m' \in \mathbb{F}_q^{d/2}$, we have

$$\widehat{S}_0(m') = \frac{\delta_0(m')}{q^{d/2}} + q^{-d+4} \eta^{d/2}(-1)G_1^{d/2} \sum_{r \in \mathbb{F}_q^*} \eta^{d/2}(r) \chi(r\|m'\|),$$

where $\delta_0(x) = 1$ if $x = 0$, and 0 otherwise.

Using the explicit value of the Gauss sum $G_1$, Lemma 2.2 yields the following corollaries.

**Corollary 2.3** For $d/2 \geq 2$ even, let $S_0$ be the zero sphere in $\mathbb{F}_q^{d/2}$ and let $m' \in \mathbb{F}_q^{d/2}$. Then the following statements hold.

1. If $d/2 = 4k$, $k \in \mathbb{N}$, then

$$\widehat{S}_0(m') = \begin{cases} q^{-d/4} - q^{-(d+4)/4} & \text{if } m' = (0, \ldots, 0), \\ q^{-d/4} - q^{-(d+4)/4} & \text{if } \|m'\| = 0, m' \neq (0, \ldots, 0), \\ -q^{-(d+4)/4} & \text{if } \|m'\| \neq 0. \end{cases}$$
2. If \( d/2 = 4k - 2, k \in \mathbb{N} \), then
\[
\hat{S}_0(m') = \begin{cases} 
q^{-1} + \eta(-1)q^{-d/4} - \eta(-1)q^{-(d+4)/4} & \text{if } m' = (0, \ldots, 0), \\
\eta(-1)q^{-d/4} - \eta(-1)q^{-(d+4)/4} & \text{if } \|m'\| = 0, m' \neq (0, \ldots, 0), \\
-\eta(-1)q^{-(d+4)/4} & \text{if } \|m'\| \neq 0.
\end{cases}
\]

3. If \( d/2 = 4k - 2, k \in \mathbb{N} \), and \( q \equiv 3 \mod 4 \), then
\[
\hat{S}_0(m') = \begin{cases} 
q^{-1} - q^{-d/4} + q^{-(d+4)/4} & \text{if } m' = (0, \ldots, 0), \\
-q^{-d/4} + q^{-(d+4)/4} & \text{if } \|m'\| = 0, m' \neq (0, \ldots, 0), \\
q^{-(d+4)/4} & \text{if } \|m'\| \neq 0.
\end{cases}
\]

Proof Since \( d/2 \) is even, we have \( \eta^{d/2} = 1 \). By Lemma 2.2,
\[
\hat{S}_0(m') = \frac{\delta_0(m')}{q} + q^{-(d+2)/2}G_1^{d/2} \sum_{r \in \mathbb{F}_q^d} \chi(r\|m'\|).
\]

By the orthogonality of \( \chi \), we notice that
\[
\sum_{r \in \mathbb{F}_q^d} \chi(r\|m'\|) = q\delta_0(\|m'\|) - 1.
\]

So we have
\[
\hat{S}_0(m') = \frac{\delta_0(m')}{q} + q^{-(d+2)/2}G_1^{d/2} \delta_0(\|m'\|) - q^{-(d+2)/2}G_2^{d/2}.
\]

By using (2), one has
\[
G_1^{d/2} = (\eta(-1))^{d/4}q^{d/4}.
\]

In Case (1), since \( d/4 \) is even, \( G_1^{d/2} = q^{d/4} \). In Case (2), since \( d/4 \) is odd, we have \( G_1^{d/2} = \eta(-1)q^{d/4} \). In Case (3), since \( d/4 \) is odd and \( q \equiv 3 \mod 4 \) (namely, \( \eta(-1) = -1 \)), we have \( G_1^{d/2} = -q^{d/4} \). Hence, the proof is complete.

Corollary 2.4 For \( d/2 \geq 3 \) odd, let \( S_0 \) be the zero sphere in \( \mathbb{F}_q^{d/2} \) and let \( m' \in \mathbb{F}_q^{d/2} \). Then, the following statements hold.

1. If \( d/2 = 4k - 1, k \in \mathbb{N} \), then
\[
\hat{S}_0(m') = \begin{cases} 
q^{-1} & \text{if } m' = (0, \ldots, 0), \\
0 & \text{if } \|m'\| = 0, m' \neq (0, \ldots, 0), \\
q^{-(d+2)/4}\eta(-\|m'\|) & \text{if } \|m'\| \neq 0.
\end{cases}
\]

2. If \( d/2 = 4k + 1, k \in \mathbb{N} \), then
\[
\hat{S}_0(m') = \begin{cases} 
q^{-1} & \text{if } m' = (0, \ldots, 0), \\
0 & \text{if } \|m'\| = 0, m' \neq (0, \ldots, 0), \\
q^{-(d+2)/4}\eta(\|m'\|) & \text{if } \|m'\| \neq 0.
\end{cases}
\]

Proof Since \( d/2 \) is odd, \( \eta^{d/2} = \eta \). By Lemma 2.2,
\[
\hat{S}_0(m') = \frac{\delta_0(m')}{q} + q^{-(d+2)/2}\eta(-1)G_1^{d/2} \sum_{r \in \mathbb{F}_q^d} \eta(r)\chi(r\|m'\|).
\]
Since $\sum_{r \in \mathbb{F}_q^n} \eta(r) \chi(r \parallel m') = \eta(\parallel m'\parallel)G_1$, we have
\[
\widehat{S}_0(m') = \frac{\delta_0(m')}{q} + q^{-(d+2)/2} \eta(-1) \eta(\parallel m'\parallel)G_1^{(d+2)/2}.
\]
By using (2),
\[
G_1^{(d+2)/2} = (\eta(-1))^{(d+2)/4} q^{(d+2)/4}.
\]
In Case (1), $G_1^{(d+2)/2} = q^{(d+2)/4}$, since $(d + 2)/4$ is even. On the other hand, in Case (2), since $(d + 2)/4$ is odd, we have $G_1^{(d+2)/2} = \eta(-1)q^{(d+2)/4}$. Hence, the proof follows.

For $t \in \mathbb{F}_q$, we define
\[
R_t := \left\{ x \in \mathbb{F}_q^d : \phi(x, 0) = t \right\},
\]
which can be viewed as the “sphere” centered at the origin of radius $t$ with respect to the function $\phi$.

The Fourier transform on the set $R_t$, $t \neq 0$, takes the following form, which plays a crucial role in proving main theorems.

**Lemma 2.5** For $t \neq 0$, we have
\[
\widehat{R}_t(m) = q^{-d} \delta_0(m) - \widehat{S}_0(m')\widehat{S}_0(m'') + G_1^{d}q^{-d-1}\eta^{d/2}(-t) \left( q\delta_0(t \parallel m'\parallel - \parallel m''\parallel) - 1 \right).
\]

**Proof** By definition, we have
\[
\widehat{R}_t(m) = q^{-d} \sum_{x \in \mathbb{F}_q^d : \phi(x, 0) = t} \chi(-m \cdot x).
\]
Since $t \neq 0$, we see
\[
\widehat{R}_t(m) = q^{-d} \sum_{x \in \mathbb{F}_q^d : \|x''\| \neq 0, \phi(x, 0) = t} \chi(-m \cdot x) - q^{-d} \sum_{x \in \mathbb{F}_q^d : \|x''\| = 0} \chi(-m \cdot x).
\]
By the definition of the Fourier transform of the indicator function on the zero sphere $S_0$ in $\mathbb{F}_q^d$, the last term above equals $-\widehat{S}_0(m')\widehat{S}_0(m'')$. Thus to complete the proof, it suffices to prove that
\[
M := q^{-d} \sum_{x \in \mathbb{F}_q^d : \|x''\| = 0} \chi(-m \cdot x)
\]
\[
= q^{-1}\delta_0(m) + G_1^{d}q^{-d-1}\eta^{d/2}(-t) \left( q\delta_0(t \parallel m'\parallel - \parallel m''\parallel) - 1 \right).
\]
To prove this, we apply the orthogonality of $\chi$ and the Gauss sum estimate. It follows that
\[
M = q^{-d} \sum_{x \in \mathbb{F}_q^d} q^{-1} \sum_{s \in \mathbb{F}_q} \chi(s(\|x'\|-t\|x''\|)) \chi(-m \cdot x).
\]
Considering the cases of $s = 0$ and $s \neq 0$, we have
\[
M = q^{-1}\delta_0(m) + q^{-d-1}\sum_{s \neq 0}\sum_{x \in \mathbb{F}_q^d} \chi(s\|x\| - m' \cdot x') \chi(-st\|x''\| - m'' \cdot x'').
\]

Applying the formula (1), we obtain
\[
M = q^{-1}\delta_0(m) + q^{-d-1} G_1^d \eta^{d/2}(-t) \sum_{s \neq 0} \left( \frac{t\|m'\| - \|m''\|}{-4st} \right).
\]

By the orthogonality of $\chi$, it is not hard to check that
\[
\sum_{s \neq 0} \left( \frac{t\|m'\| - \|m''\|}{-4st} \right) = q\delta_0(t\|m'\| - \|m''\|) - 1.
\]

Therefore, the proof is complete. \qed

3 Proofs of Main Results

We proceed with the counting function argument. For $t \in \mathbb{F}_q$, let $\nu(t)$ be the number of pairs $(x, y)$ in $E \times E$ such that $\phi(x, y) = t$. Since $0 \in \phi(E, E)$ for any $E \subset \mathbb{F}_q^d$, our task is to find size condition of a set $E$ in $\mathbb{F}_q^d$ such that $\nu(t) > 0$ for any nonzero $t$ in $\mathbb{F}_q$. Fix a nonzero $t \in \mathbb{F}_q^*$. Then, it is immediate that
\[
\nu(t) = \sum_{x, y \in \mathbb{F}_q^d} E(x)E(y)R_t(x - y).
\]

Applying the Fourier inversion theorem to the function $R_t(x - y)$, we have
\[
\nu(t) = q^{2d} \sum_{m \in \mathbb{F}_q^d} \widehat{R_t}(m)|\widehat{E}(m)|^2.
\]

Combining with Lemma 2.5, we see from a direct computation that
\[
\nu(t) = q^{-1}|E|^2 - q^{2d} \sum_{m \in \mathbb{F}_q^d} \widehat{S_0}(m')\widehat{S_0}(m'')|\widehat{E}(m)|^2
\]
\[
+ q^d G_1^d \eta^{d/2}(-t) \sum_{t\|m'\| - \|m''\| = 0} |\widehat{E}(m)|^2 - q^{-1} G_1^d \eta^{d/2}(-t)|E|.
\]

3.1 Proof of Theorem 1.3

Since $d = 4$, it is clear that $G_1^d = q^{d/2} = q^2$ and $\eta^{d/2} \equiv 1$. Furthermore, since $-1$ is not a square number, we have $S_0 = \{(0, 0)\}$ and so $\widehat{S_0}(m') = q^{-2} = \widehat{S_0}(m'')$ for all $m \in \mathbb{F}_q^d$. Hence, it follows from (3) that
\[
\nu(t) = q^{-1}|E|^2 - q^8 q^{-4} \sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^2 + q^4 q^2 \sum_{t\|m'\| - \|m''\| = 0} |\widehat{E}(m)|^2 - q^{-1} q^2 |E|.
\]

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To compute the above second term, we recall from the Plancherel theorem that \( \sum_{m \in \mathbb{F}_q} |\hat{E}(m)|^2 = q^{-1}|E| \). The above third term is estimated as follows:

\[
q^4q^2 \sum_{t \mid m'' - |m''| = 0} |\hat{E}(m)|^2 \geq q^4 q^2 |\hat{E}(0, 0, 0, 0)|^2 = q^{-2}|E|^2.
\]

Therefore, we have

\[
v(t) \geq q^{-1}|E|^2 - |E| + q^{-2}|E|^2 - q|E| = (q^{-1} + q^{-2})|E|(|E| - q^2).
\]

This implies that if \( |E| > q^2 \), then \( v(t) > 0 \). This completes the proof of Theorem 1.3.

### 3.2 Proof of Theorem 1.4(1)

It follows from the statement of Theorem 1.4(1) that we need to prove two separated cases:

- **A.** \( d = 8k + 4, k \in \mathbb{N}, \) and \( q \equiv 3 \) \( \text{mod} \ 4 \).
- **B.** \( d = 4k, k \in \mathbb{N} \).

We remark here that the case **A** is an extension of Theorem 1.3.

**Proof of Case A** From our assumption that \( d = 8k + 4, k \in \mathbb{N} \), it is clear that \( G_1^d = q^{d/2} \) and \( q^{d/2} \equiv 1 \). Therefore, the identity (3) becomes

\[
v(t) = q^{-1}|E|^2 - q^{2d} \sum_{m \in \mathbb{F}_q^d} \hat{S}_0(m') \hat{S}_0(m''') |\hat{E}(m)|^2
\]

\[
+ q^{3d/2} \sum_{t \mid m'' - |m''| = 0} |\hat{E}(m)|^2 - q^{(d-2)/2}|E|.
\]

First notice from our hypothesis that we can apply the third part of Corollary 2.3, that is

\[
\hat{S}_0(m') = \begin{cases} 
q^{-1} - q^{-d/4} + q^{-(d+4)/4} & \text{if } m' = (0, \ldots, 0), \\
-q^{-d/4} + q^{-(d+4)/4} & \text{if } ||m'|| = 0, m' \neq (0, \ldots, 0), \\
q^{-(d+4)/4} & \text{if } ||m'|| \neq 0.
\end{cases}
\]

Since we aim to find a lower bound of \( v(t) \), we may ignore some positive terms appearing when we estimate \( v(t) \). Hence, we break down the sum \( \sum_{m \in \mathbb{F}_q^d} \) in (4) into 9 subsummands:

\[
\sum_{m', m'' = 0} + \sum_{m', m'' = 0, ||m'''|| = 0, m' \neq 0} + \sum_{m' = 0, ||m'''|| = 0} + \sum_{m' = 0, m'' \neq 0, m''' = 0} + \sum_{||m'''|| = 0, m' \neq 0, m'' = 0} + \sum_{||m'''|| = 0, m'' \neq 0, m' = 0} + \sum_{||m'''|| = 0, m'' \neq 0, m' \neq 0} + \sum_{||m'''|| = 0, m' \neq 0, m'' \neq 0} + \sum_{||m'''|| = 0, m' \neq 0, m'' = 0, m' \neq 0},
\]

and then we only compute such sums for which \( \hat{S}_0(m') \hat{S}_0(m'') \) takes a positive value, which can be easily evaluated by using the above explicit value of \( \hat{S}_0 \). Notice that it will be enough to consider the dominant main term of \( \hat{S}_0(m') \). Namely, we can use the following approximate value of \( \hat{S}_0 \):

\[
\hat{S}_0(m') \approx \begin{cases} 
q^{-1} & \text{if } m' = (0, \ldots, 0), \\
-q^{-d/4} & \text{if } ||m'|| = 0, m' \neq (0, \ldots, 0), \\
q^{-(d+4)/4} & \text{if } ||m'|| \neq 0.
\end{cases}
\]
In addition, notice that the third term in (4) is greater than or equal to the value
\[ q^{3d/2} \sum_{\|m'\|=0, \|m''\|=0} |\widehat{E}(m)|^2. \]

Then, it is not hard to obtain that
\[
v(t) \geq q^{-1}|E|^2 - q^{2d-2} \sum_{m', m''=0} |\widehat{E}(m)|^2 - q^{(7d-8)/4} \sum_{m'=0, \|m''\|\neq 0} |\widehat{E}(m)|^2
+ q^{3d/2} \sum_{\|m'\|=0, m'' \neq 0, \|m''\|=0} |\widehat{E}(m)|^2 - q^{(7d-8)/4} \sum_{m''=0, \|m''\|\neq 0} |\widehat{E}(m)|^2
-q^{(3d-4)/2} \sum_{\|m'\|=0, \|m''\|=0} |\widehat{E}(m)|^2 + q^{3d/2} \sum_{\|m''\|=0} |\widehat{E}(m)|^2 - q^{(d-2)/2} |E|.
\]

To find a concrete lower bound of the above terms, we first note that the second term above equals \(-q^{2d-2} |\widehat{E}(0, \ldots, 0)|^2\), which is \(q^{-2}|E|^2\). We also observe that the fourth and seventh terms above can be ignored since they are positive, the sum of the third and fifth terms is greater than or equal to the value \(-q^{(7d-8)/4} \sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^2\). For other summations, we dominate them by using the quantity \(\sum_{m \in \mathbb{F}_q^d} |\widehat{E}(m)|^2 = q^{-d}|E|\). In other words,
\[
v(t) \geq q^{-1}|E|^2 - q^{-2}|E|^2 - q^{(3d-8)/4}|E| - q^{(d-6)/2}|E| - q^{(d-2)/2}|E|.
\]

By a direct computation, we conclude that if \(|E| \geq Cq^{(3d-4)/4}\), then \(v(t) > 0\), as required.

**Proof of Case B** Since \(d = 4k, k \in \mathbb{N}\), it is not hard to see that \(G_d^\sigma = q^{d/2}\) and \(\eta^{d/2} \equiv 1\). Therefore, the equation (3) can be rewritten by
\[
v(t) = q^{-1}|E|^2 - q^{2d} \sum_{m \in \mathbb{F}_q^d} |\widehat{S}_0(m')| |\widehat{S}_0(m'')| |\widehat{E}(m)|^2
+ q^{d} q^{d/2} \sum_{\|m''\|=0} |\widehat{E}(m)|^2 - q^{-1} q^{d/2} |E|.
\]

Since the third term is positive, we obtain that
\[
v(t) > q^{-1}|E|^2 - q^{2d} \sum_{m \in \mathbb{F}_q^d} |\widehat{S}_0(m')| |\widehat{S}_0(m'')| |\widehat{E}(m)|^2 - q^{-1} q^{d/2} |E|.
\]

We estimate the sum in \(m \in \mathbb{F}_q^d\) by finding an upper bound of each of four subsummands:
\[
\sum_{m', m''=(0, \ldots, 0)} + \sum_{m'=(0, \ldots, 0), m'' \neq (0, \ldots, 0)} + \sum_{m''=(0, \ldots, 0), m' \neq (0, \ldots, 0)} + \sum_{m', m'' \neq (0, \ldots, 0)}.
\]

To bound each of them, we apply the following facts which are direct consequences of Corollary 2.3(1)-(2):
\[ |\widehat{S}_0(0, \ldots, 0)| \leq 2q^{-1} \quad \text{and} \quad \max_{m' \neq (0, \ldots, 0)} |\widehat{S}_0(m')| \leq q^{-d/4}. \]
By a simple algebra, we then obtain that

\[
\nu(t) > q^{-1}|E|^2 - 4q^{2d-2}|\hat{E}(0, 0)|^2 - 2q^{(7d-4)/4} \left[ \sum_{m'' \neq 0} |\hat{E}(0, m'')|^2 + \sum_{m' \neq 0} |\hat{E}(m', 0)|^2 \right] \\
- q^{3d/2} \sum_{m', m'' \neq 0} |\hat{E}(m', m'')|^2 - q^{-1}q^{d/2}|E|.
\]

Notice that \(|\hat{E}(0, 0)| = q^{-d}|E| = \sum_{m \in \mathbb{F}_q} |\hat{E}(m)|^2\). Also observe that both the sums in the above bracket and the above sum over \(m', m'' \neq 0\) are dominated by the sum \(\sum_{m \in \mathbb{F}_q} |\hat{E}(m)|^2\). From these observations, we have

\[
\nu(t) > q^{-1}|E|^2 - 4q^{-2}|E|^2 - 2q^{(3d-4)/4}|E| - q^{d/2}|E| - q^{(d-2)/2}|E|
\geq q^{-1}|E|^2 - 4q^{-2}|E|^2 - 4q^{(3d-4)/4}|E|.
\]

We may assume that \(q\) is sufficiently large. Otherwise, we can take \(C > 0\) such that \(q^d = Cq^{3d/4}\) for which \(|E| = q^d\) and \(\phi(E, E) = \mathbb{F}_q\). Hence, we conclude that if \(|E| \geq Cq^{3d/4}\) for a sufficiently large constant \(C > 0\), then \(\nu(t) > 0\). This completes the proof.

### 3.3 Proof of Theorem 1.4(2)

The proof is almost identical with that of Theorem 1.4(1). The main difference is that since \(d = 4k + 2, k \in \mathbb{N}\), we can invoke Corollary 2.4, which gives much better Fourier decay on the zero sphere \(S_0\) than Corollary 2.3. For the sake of completeness, we now give a detailed proof. Since \(|G_1^d| = q^{d/2}\), the equation (3) implies that

\[
\nu(t) \geq q^{-1}|E|^2 - q^{2d} \sum_{m \in \mathbb{F}_q^d} |\hat{S}_0(m')||\hat{S}_0(m'')||\hat{E}(m)|^2 \\
- q^{3d/2} \sum_{m \in \mathbb{F}_q^d} |\hat{E}(m)|^2 - q^{(d-2)/2}|E|.
\]

We estimate the second term above by decomposing it into four subsummands as in (5). In addition, we use the following Fourier decay estimates on \(S_0\), which follow immediately from Corollary 2.4:

\[
|\hat{S}_0(0, \ldots, 0)| = q^{-1} \quad \text{and} \quad \max_{m' \neq (0, \ldots, 0)} |\hat{S}_0(m')| \leq q^{-(d+2)/4}.
\]

We then have

\[
\nu(t) > q^{-1}|E|^2 - q^{2d-2}|\hat{E}(0, 0)|^2 - q^{(7d-6)/4} \left[ \sum_{m'' \neq 0} |\hat{E}(0, m'')|^2 + \sum_{m' \neq 0} |\hat{E}(m', 0)|^2 \right] \\
- q^{(3d-2)/2} \sum_{m', m'' \neq 0} |\hat{E}(m', m'')|^2 - q^{(d-2)/2}|E|.
\]

As mentioned before, both the value in the above bracket and the sum over \(m', m'' \neq 0\) are less than \(\sum_{m \in \mathbb{F}_q^d} |\hat{E}(m)|^2\), which equals \(q^{-d}|E|\). Also recall that \(|\hat{E}(0, 0)| = q^{-d}|E|\). Then we see that

\[
\nu(t) > q^{-1}|E|^2 - q^{-2}|E|^2 - q^{(3d-6)/4}|E| - q^{(d-2)/2}|E| - q^{(d-2)/2}|E|
\geq q^{-1}|E|^2 - q^{-2}|E|^2 - 3q^{(3d-6)/4}|E|.
\]
This clearly implies that \( v(t) > 0 \) if \( |E| \geq Cq^{(3d-2)/4} \) for some large constant \( C > 0 \). This completes the proof.

Remark 3.1 It is quite natural to ask whether our main theorems can be extended for the case of two sets \( \phi(E, F) \). However, it seems that we need to find a new approach for this question. The main reason is that, for instance, in the proof of Theorem 1.3, the sign of the term \( \sum_{m \in \mathbb{N}_q} |\hat{E}(m)|^2 \) is negative, but for two different sets, one might get \( \sum_{m \in \mathbb{N}_q} |\hat{E}(m)|^2 \) which can be positive and quite large. This question will be addressed in a sequel paper.

4 Sharpness of Theorem 1.4

To construct desired sets, let us first recall the following lemma in a paper due to Vinh [22].

Lemma 4.1 ([22, Lemma 2.1]) Let \( S_0 = \{(x_1, \ldots, x_n) : x_1^2 + \cdots + x_n^2 = 0\} \) be a variety in \( \mathbb{F}_q^n \) with \( n \geq 2 \). If \( H \) is a subspace of maximal dimension contained in \( S_0 \), then we have the following facts:

1. If \( n \) is odd, then \( |H| = q^{n-1} \).
2. If \( n \) is even and \( (n(-1))^{\frac{n}{2}} = 1 \), then \( |H| = q^n \).
3. If \( n \) is even and \( (n(-1))^{\frac{n}{2}} = -1 \), then \( |H| = q^{n-2} \).

By using this lemma, we are able to construct sets that meet the exponents of Theorem 1.4.

Lemma 4.2 (Sharpness of Theorem 1.4(1)–Case A) Suppose \( d = 8k + 4 \) for some \( k \in \mathbb{N} \) and \( q \equiv 3 \mod 4 \). Then there is a set \( E \) in \( \mathbb{F}_q^d \) such that \( |E| = q^{(3d-4)/4} \) and \( \phi(E, E) = \{0\} \).

Proof First, we have \( (n(-1))^{\frac{n}{2}} = -1 \) since \( q \equiv 3 \mod 4 \). Let \( n := d/2 = 4k + 2, k \in \mathbb{N} \). We apply the third part of Lemma 4.1 so that we can choose a subspace \( H \) in \( S_0 \subset \mathbb{F}_q^{d/2} \) with \( |H| = q^{(n-1)/2} = q^{(3d-4)/4} \). Moreover, since \( H - H = H \), we have \( \|a - b\| = 0 \) for all \( a, b \in H \). We now define \( E = \mathbb{F}_q^{d/2} \times H \). It is clear that \( |E| = q^{(3d-4)/4} \) and \( \phi(E, E) = \{0\} \), as required.

Lemma 4.3 (Sharpness of Theorem 1.4(1)–Case B) Suppose \( d = 4k \) for some \( k \in \mathbb{N} \). If \( k \) and \( q \) satisfy one of the following two conditions:

- \( k \) is odd and \( q \equiv 1 \mod 4 \),
- \( k \) is even,

then there exists a set \( E \) in \( \mathbb{F}_q^d \) such that \( |E| = q^{3d/4} \) and \( \phi(E, E) = \{0\} \).

Proof We proceed with the same argument as in the proof of Lemma 4.2. Recall that \( q \equiv 1 \mod 4 \) if and only if \( (n(-1))^{\frac{n}{2}} = 1 \). So, one can apply Lemma 4.1 to obtain a subspace \( H \) in \( S_0 \subset \mathbb{F}_q^{d/2} \) with \( |H| = q^{n/2} = q^{d/4} \). Setting \( E = \mathbb{F}_q^{d/2} \times H \), the proof follows.

\( \square \)

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Lemma 4.4 (Sharpness of Theorem 1.4(2)) Suppose that \( d = 4k + 2 \) for some \( k \in \mathbb{N} \). Then there is a set \( E \) in \( \mathbb{R}^d_q \) such that \( |E| = q^{(3d-2)/4} \), and \( \phi(E, E) = \{0\} \).

\[ \text{Proof} \] Let \( n = d/2 \). Since \( n \) is odd, we can use the first part of Lemma 4.1 with \( n = d/2 \) so that we can find a subspace \( H \) in \( S_0 \subset \mathbb{R}^{d/2}_q \) with \( |H| = q^{(n-1)/2} = q^{(d-2)/4} \). Since \( H - H = H \), we have \( \|a - b\| = 0 \) for any \( a, b \in H \). Hence, the set \( E = \mathbb{R}^{d/2}_q \times H \) is what we need.

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References

1. Du, X., Guth, L., Ou, Y., Wang, H., Wilson, B., Zhang, R.: Weighted restriction estimates and application to Falconer distance set problem. Amer. J. Math. 143, 175–211 (2021)
2. Du, X., Zhang, R.: Sharp \( L^2 \) estimates of the Schrödinger maximal function in higher dimensions. Ann. Math. 189, 837–861 (2019)
3. Du, X., Iosevich, A., Ou, Y., Wang, H., Zhang, R.: An improved result for Falconer’s distance set problem in even dimensions. arXiv:2006.06833 (2020)
4. Erdőgan, M.B.: A bilinear Fourier extension theorem and applications to the distance set problem. Int. Math. Res. Not. 2005, 1411–1425 (2005)
5. Falconer, K.J.: On the Hausdorff dimensions of distance sets. Mathematika 32, 206–212 (1985)
6. Guth, L., Iosevich, A., Ou, Y., Wang, H.: On Falconer’s distance set problem in the plane. Invent. math. 219, 779–830 (2020)
7. Greenleaf, A., Iosevich, A., Taylor, K.: Configuration sets with nonempty interior. J. Geom. Anal. https://doi.org/10.1007/s12220-019-00288-y (2019)
8. Greenleaf, A., Iosevich, A., Taylor, K.: On \( k \)-point configuration sets with nonempty interior. arXiv:2005.10796 (2020)
9. Hart, D., Iosevich, A., Koh, D., Rudnev, M.: Averages over hyperplanes, sum-product theory in vector spaces over finite fields and the Erdős-Falconer distance conjecture. Trans. Amer. Math. Soc. 363, 3255–3275 (2011)
10. Iosevich, A., Koh, D.: Extension theorems for spheres in the finite field setting. Forum Math. 22, 457–483 (2010)
11. Iosevich, A., Koh, D.: On the product set of the distance set. Preprint (2017)
12. Iosevich, A., Koh, D., Parshall, H.: On the quotient set of the distance set. Mosc. J. Comb. Number Theory 8, 103–115 (2019)
13. Iosevich, A., Rudnev, M.: Erdős distance problem in vector spaces over finite fields. Trans. Amer. Math. Soc. 359, 6127–6142 (2007)
14. Iwaniec, H., Kowalski, E.: Analytic Number Theory. Colloquium Publications, vol. 53. American Mathematical Society, Providence, RI (2004)
15. Koh, D., Pham, T., Shen, C.-Y.: On the Mattila-Sjölin distance theorem for product sets (submitted)
16. Koh, D., Pham, T., Vinh, L.A.: Extension theorems and Distance problems over finite fields. arXiv:1809.08699 (2018)
17. Lidl, R., Niederreiter, H.: Finite Fields. Encyclopedia of Mathematics and its Applications, vol. 20. Cambridge University Press, Cambridge (1997)
18. Mattila, P.: Fourier Analysis and Hausdorff Dimension, vol. 150. Cambridge University Press, Cambridge (2015)
19. Mattila, P., Sjölin, P.: Regularity of distance measures and sets. Math. Nachr. 204, 157–162 (1999)
20. Murphy, B., Petridis, G., Pham, T., Rudnev, M., Stevens, S.: On the pinned distances problem over finite fields. arXiv:2003.00510 (2020)
21. Pham, T., Suk, A.: On the structure of distance sets over prime fields. Proc. Amer. Math. Soc. 148, 3209–3215 (2020)
22. Vinh, L.A.: Maximal sets of pairwise orthogonal vectors in finite fields. Can. Math. Bull. 55, 418–423 (2012)
23. Wolff, T.: Decay of circular means of Fourier transforms of measures. Int. Math. Res. Not. 1999, 547–567 (1999)

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