Rational $p$-adic Hodge theory for $d$-de Rham-proper stacks

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with an appendix by Haoyang Guo

Abstract

In this follow-up paper we show that smooth Hodge-proper stacks over $\mathcal{O}_K$ are $\mathbb{Q}_p$-locally acyclic: namely the natural map between étale $\mathbb{Q}_p$-cohomology of the algebraic and Raynaud generic fibers is an equivalence. This establishes the $\mathbb{Q}_p$-case of general conjectures made in [KP21]. As a corollary, we get that if a smooth Artin stack over $K$ has a smooth Hodge-proper model over $\mathcal{O}_K$, its $\mathbb{Q}_p$-étale cohomology is a crystalline Galois representation. We then also establish a truncated version of the above results in more general setting of smooth $d$-de Rham-proper stacks over $\mathcal{O}_K$: here we only require first $d$ de Rham cohomology groups be finitely-generated over $\mathcal{O}_K$. As an application, we deduce a certain purity-type statement for étale $\mathbb{Q}_p$-cohomology of Raynaud generic fiber, as well as crystallinity of a first several étale cohomology groups in the presence of a Cohen–Macaulay model over $\mathcal{O}_K$ in the schematic setting.

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0 Introduction

0.1 $p$-adic Hodge theory for Hodge-proper stacks

Let $K$ be a complete discretely valued extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}_K$ and perfect residue field $k$. In [KP21] we developed some aspects of $p$-adic Hodge theory in the setting of Artin stacks over $\mathcal{O}_K$. The following concept played a key role:

**Definition 0.1 ([KP19])**. A smooth quasi-compact quasi-separated Artin stack $X$ over a Noetherian ring $R$ is called Hodge-proper if for any $i,j \in \mathbb{Z}_{\geq 0}$ the cohomology $H^j(X, \wedge^i L^\bullet_{X/R}) \in \text{Mod}_R$ is a finitely generated $R$-module.

Our observation, that we investigated in previous works [KP19], [KP21], was that Hodge-proper stacks\(^1\) are often as good as smooth proper schemes when considering certain cohomological features (like Hodge-to-de Rham degeneration in char 0 [KP19] or some aspects $p$-adic Hodge theory [KP21]). This work establishes more results supporting this idea in the context of rational $p$-adic Hodge theory (see Remark 0.4 below).

Let us start by reminding some examples of smooth Hodge-proper stacks:

**Example 0.2.**
1. A smooth proper stack $X$ over $R$ is Hodge-proper ([KP19, Corollary 2.2.13]).
2. Let $X$ be a smooth proper scheme over $R$ with an action of a reductive group $G$. Then the quotient stack $[X/G]$ is Hodge-proper ([KP21, Section 1.3]).
3. Let $X$ be a smooth scheme with an action $G$ and assume that $X$ has a $G$-invariant affine cover such that all intersections are affine. In this situation there is a well-defined “categorical quotient” scheme $X//G$. Then (see [KP21, Proposition 1.4.6]) if $X//G$ is proper, the stack $[X/G]$ is Hodge-proper. This includes many examples when $X$ is not necessary proper (see [KP19, Example 3.1.6]), e.g. $[\mathbb{A}^n/\text{SL}_n]$ with the tautological action.
4. Some smooth $\Theta$-stratified stacks (for the definition see [Hal18] and [Hal20]) are also Hodge-proper. Namely (see [KP21, Example 1.4.8]), if a smooth Artin stack $X$ with affine diagonal is endowed with a finite $\Theta$-stratification and all $\Theta$-strata are cohomologically proper (see [KP19, Definition 2.2.2]), then $X$ itself is Hodge-proper. The unstable strata in this criterion can also be replaced by their centra. The particular examples then include even more general quotient stacks $[X/G]$ where the action is only Kempf-Ness complete.

A particularly important aspect studied in [KP21] was the potential discrepancy between the étale cohomology of the algebraic and Raynaud generic fibers of a Hodge-proper stack which one doesn’t see in the usual smooth proper setting. Let us recall this in some detail. Namely, let $K$ be a finite extension of $\mathbb{Q}_p$ and let $C = \overline{K}$ be the completion of the algebraic closure of $K$. Let $\mathcal{O}_K \subset K$ and $\mathcal{O}_C \subset C$ be the corresponding rings of integers. In [KP21] to an Artin stack $X$ over $\mathcal{O}_C$ we associated three other stacks:

1. the algebraic generic fiber $X_C$, its analytification $X_C^{an}$ and the Raynaud generic fiber $\tilde{X}_C$ (the latter two are rigid-analytic stacks, see [KP21, Section 3.3]). One then has natural maps (see [KP21, Section 3.3, Remark 4.1.16])

\[
\begin{align*}
R\Gamma_\text{ét}(X_C, \mathbb{Z}_p) \xrightarrow{\varphi_X^{-1}} R\Gamma_\text{ét}(X_C^{an}, \mathbb{Z}_p) \xrightarrow{\psi_X^{-1}} R\Gamma_\text{ét}(\tilde{X}_C, \mathbb{Z}_p),
\end{align*}
\]

\(^{1}\text{Or slight variations like Hodge-properly spreadable stacks (see [KP19, Definition 1.4.1]).}\)
between the $\mathbb{Z}_p$-étale cohomology. Here the map $\varphi_X^{-1}$ is always an equivalence, while $\psi_X^{-1}$ is a priori an equivalence only in the smooth proper case. Still, in [KP21] we have shown that the map $\Upsilon_X$ is an equivalence when $X$ is as in part 2 of Example 0.2, but the situation in the general Hodge-proper case remained unclear.

In this work we study the map $\Upsilon_X$ rationally, namely when we replace $\mathbb{Z}_p$-coefficients with $\mathbb{Q}_p$ ones. We show that then the following general result holds:

**Theorem (3.8).** Let $X$ be a smooth Hodge-proper stack over $\mathcal{O}_K$. Then the map

$$\Upsilon_{X, \mathbb{Q}_p} : R\Gamma_{\text{ét}}(X_C, \mathbb{Q}_p) \longrightarrow R\Gamma_{\text{ét}}(\hat{X}_C, \mathbb{Q}_p)$$

is an equivalence.

**Remark 0.3.** This was stated as Conjecture 4.3.16 in [KP21]. By analogy with [KP21, Definition 4.1.18] stacks for which the map $\Upsilon_{X, \mathbb{Q}_p}$ is an equivalence could be called $\mathbb{Q}_p$-locally acyclic.

Let $G_K$ be the absolute Galois group of $K$. From Theorem 3.8, using the results in [KP21] one deduces that the Galois representation in étale cohomology of $X_C$ is crystalline:

**Corollary (3.9).** Let $X$ be a smooth Hodge-proper stack over $\mathcal{O}_K$. Then for any $i \geq 0$ the $G_K$-representation given by $H^i(X_C, \mathbb{Q}_p)$ is crystalline.

As a consequence we also obtain all the comparisons that one usually has for a smooth proper scheme (see Remark 3.10).

Let us try to motivate Theorem 3.8 and Corollary 3.9 by putting them into some broader context with remarks below.

**Remark 0.4** (Hodge-proper is (almost) as good as proper). Let $Y$ be a smooth scheme over $K$. Then using the work of de Jong on alterations [Jon96] Kisin showed in [Kis02] that $H^i(Y_C, \mathbb{Q}_p)$ is a potentially semi-stable $G_K$-representation. However, by the Fontaine’s $C_{\text{crys}}$-conjecture if $Y$ has a smooth proper model over $\mathcal{O}_K$ this representation has a finer structure: namely it is crystalline.

Let now $\mathcal{Y}$ be a smooth quasi-compact quasi-separated Artin stack over $K$. Resolving $\mathcal{Y}$ by a simplicial scheme and using smooth descent by a similar argument one can show that the étale cohomology $H^i(\mathcal{Y}_C, \mathbb{Q}_p)$ is also a potentially semistable representation of the Galois group $G_K$. However, Corollary 3.9 tells us that for the $G_K$-representation $H^i(\mathcal{Y}_C, \mathbb{Q}_p)$ to be crystalline it is in fact enough for $\mathcal{Y}$ to have a smooth Hodge-proper model over $\mathcal{O}_K$. This gives some evidence that from the point of view of $\mathbb{Q}_p$-étale cohomology smooth Hodge-proper stacks are as good as smooth proper schemes. Let us remark that there are some examples of smooth schemes that are Hodge-proper but not proper\(^2\) (see [KP19, Section 2.3.3]).

**Remark 0.5.** ($G$-equivariant $p$-adic Hodge theory). In [KP21] we proved a $G$-equivariant variant of $p$-adic Hodge theory for a smooth proper $\mathcal{O}_K$-scheme $X$ with a reductive group action (see [KP21, Section 0.4]). It is natural to ask whether the properness condition is necessary. Theorem 3.8 and Corollary 3.9 show that in fact a much milder condition is sufficient: $[X/G]$ should just be Hodge-proper (and smooth). This includes all the examples listed in 0.2.

To be more precise (see Remark 3.10 and apply to $X = [X/G]$) in this situation we have a $(\varphi, G_K)$-equivariant isomorphism

$$H^n_{\text{ét}}([X/G]_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{crys}} \simeq H^n_{\text{crys}}([X/G]_{k'/W(k)})[\frac{1}{p}] \otimes_{K_d} B_{\text{crys}},$$

a filtered $G_K$-equivariant isomorphism

$$H^n_{\text{ét}}([X/G]_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \simeq H^n_{\text{dR}}([X/G]/K) \otimes_K B_{\text{dR}}$$

and the Hodge-Tate decomposition

$$H^n_{\text{ét}}([X/G]_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \simeq \bigoplus_{i+j = n} H^j([X/G]_K, \wedge^i \Omega_{[X/G]/K}) \otimes_K C(-i).$$

\(^2\)Though we do not know such an example over $\mathcal{O}_K$ with $K/\mathbb{Q}_p$ finite, we do know some explicit examples over $W(k)$ with $k$ being the perfection of $\mathbb{F}_p(t)$ (e.g. adapting the construction in [KP19, Section 2.3.3]).
Moreover, if $K$ is a finite extension of $\mathbb{Q}_p$, after a choice of isomorphism $\iota: C \xrightarrow{\sim} \mathbb{C}$ by [KP21, Example 4.1.5 and Proposition 4.1.6] one can identify $H^i_{\text{ét}}([X/G]_C, \mathbb{Q}_p)$ with the $G(C)$-equivariant cohomology $H^i_{\text{sing},G(C)}(X(\mathbb{C}), \mathbb{Q}_p)$.

### 0.2 Generalization: $d$-de Rham proper stacks

In fact, in this work we show a more subtle statement in a more general setting. Here is our framework:

**Definition 0.6** ($d$-de Rham-proper stacks). Fix a positive integer $d \in \mathbb{Z}_{\geq 0}$. A smooth quasi-compact quasi-separated Artin stack $X$ over a Noetherian ring $R$ is called $d$-de Rham-proper if $H^i_{\text{dR}}(X/R)$ is a finitely generated $R$-module for $i \leq d$.

This is a $d$-truncated version of Definition 0.1, where Hodge cohomology is replaced by de Rham. This notion is much more general (in particular, a Hodge-proper stack is $d$-de Rham for any $d$), and the main motivation to consider it comes from the fact that there are many natural examples of $d$-de Rham-proper schemes that are not necessarily proper (see Lemma 1.4).

The possible expectation could be that a truncated variant of $p$-adic Hodge theory exists in this setting; namely, the standard properties of and comparisons between different cohomology should still be there, but maybe only up to a certain degree. This is exactly what we show.

**Theorem 0.7.** Let $X$ be a smooth $(d + 1)$-de Rham-proper stack over $\mathcal{O}_K$. Then

1. The natural map $\Upsilon_{X, \mathbb{Q}_p} : R\Gamma_{\text{ét}}(X_C, \mathbb{Q}_p) \to R\Gamma_{\text{ét}}(\widehat{X}_C, \mathbb{Q}_p)$ induces isomorphisms

$$H^i_{\text{ét}}(X_C, \mathbb{Q}_p) \simeq H^i_{\text{ét}}(\widehat{X}_C, \mathbb{Q}_p)$$

for $i \leq d$, and an embedding

$$H^{d+1}_{\text{ét}}(X_C, \mathbb{Q}_p) \hookrightarrow H^{d+1}_{\text{ét}}(\widehat{X}_C, \mathbb{Q}_p)$$

for $i = d + 1$ (Theorem 4.15).

2. The $G_K$-representation $H^i_{\text{ét}}(X_C, \mathbb{Q}_p) \simeq H^i_{\text{ét}}(\widehat{X}_C, \mathbb{Q}_p)$ is crystalline for $i \leq d$ with $D_{\text{crys}}(H^i_{\text{ét}}(X_C, \mathbb{Q}_p)) \simeq H^i_{\text{crys}}(X_{\overline{K}}/W(k))[[\frac{1}{p}]]$. Consequently, one has a $(G_K, \varphi)$-equivariant isomorphism

$$H^i_{\text{ét}}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{crys}} \xrightarrow{\sim} H^i_{\text{crys}}(X_{\overline{K}}/W(k)) \otimes_{W(k)} B_{\text{crys}}$$

in that range of degrees (Corollary 3.9).

3. Provided $H^i_{\text{crys}}(X/W(k))$ is $p$-torsion free and $i \leq d$, the Breuil-Kisin module $H^i_{\Delta}(X/\mathfrak{S})$ corresponds to the $G_K$-equivariant crystalline lattice $H^i_{\text{ét}}(\widehat{X}_C, \mathbb{Z}_p) \subset H^i_{\text{ét}}(X_C, \mathbb{Q}_p)$ under the Breuil-Kisin functor:

$$\text{BK}(H^i_{\text{ét}}(\widehat{X}_C, \mathbb{Z}_p)) \simeq H^i_{\Delta}(X/\mathfrak{S}),$$

(Remark 4.25).

We note right away that the proof of Theorem 0.7 essentially follows the same ideas as the proof of analogous results in [KP21]. However, to get the optimal degree range in the comparisons above one needs to analyze very carefully what happens with the completed tensor products in the boundary degrees (typically $d$ and $d - 1$). Condensed mathematics of Clausen-Scholze turned out to be a very convenient (and, in fact, so far the only suitable for us) framework to do so.

**Question 0.8.** Note that in part 3 of Theorem 0.7 the crystalline lattice corresponding to prismatic cohomology is given by the cohomology of the Raynaud generic fiber of $X$. This poses a natural question: do the lattices $H^i_{\text{ét}}(\widehat{X}_C, \mathbb{Z}_p)$ and $H^i_{\text{ét}}(X_C, \mathbb{Z}_p)$ agree inside $H^i_{\text{ét}}(X_C, \mathbb{Q}_p)$ (under the assumption that $X$ is $d$-de Rham proper and $i \leq d$)? We do not expect this to be true, but also don’t know a counterexample.
We then apply Theorem 0.7 to the schematic setting. A quite general example of a $d$-de Rham-proper scheme over $\mathcal{O}_K$ can be constructed as follows: take a proper Cohen-Macauley scheme $X$ over $\mathcal{O}_K$ and assume that the singularities $Z \to X$ have codimension $d + 1$ in $X$; then the complement $U := X \backslash Z$ is $(d - 1)$-de Rham proper over $\mathcal{O}_K$. Most interesting is the case when $Z$ is in fact contained in the closed fiber $X_k$ (and is of codimension $d$): then $U_C \cong X_C$, but the Raynaud generic fiber $\hat{U}_C$ is given by the complement in $X_C$ to an “open tube” around $Z \subset X$. Thus application of Theorem 0.7 in this situation leads to a purity-type result for erasing the latter, as well as the crystallinity of étale cohomology of $X_C$ in certain range.

**Theorem (5.1).** Let $X$ be a proper scheme over $\mathcal{O}_K$ that is Cohen-Macauley. Let $Z \to X_k$ be a codimension $d$ closed subscheme such that complement $X \backslash Z$ is smooth over $\mathcal{O}_K$. Then one has

1. (Purity) There are natural isomorphisms $H^i_{\text{ét}}(\hat{(X \backslash Z)}_C, \mathbb{Q}_p) \cong H^i_{\text{ét}}(\hat{X}_C, \mathbb{Q}_p)$ for $i \leq d - 2$ and an embedding $H^{d-1}_{\text{ét}}(\hat{X}_C, \mathbb{Q}_p) \to H^{d-1}_{\text{ét}}((X \backslash Z)_C, \mathbb{Q}_p)$;
2. (Crystalline) $H^i_{\text{ét}}(X_C, \mathbb{Q}_p)$ is a crystalline Galois representation for $i \leq d - 2$.

Another way to phrase Part 2 of Section 0.2 is that having a smooth proper scheme over $K$, if we found a Cohen-Macauley model over $\mathcal{O}_K$ such that singularities are in codimension $d$ in the closed fiber, then étale cohomology $H^i_{\text{ét}}(X_C, \mathbb{Q}_p)$ is still a crystalline $G_K$-representation at least up to degree $d - 2$.

**Question 0.9.** Is the bound in Theorem 5.1(2) sharp?

### 0.3 Plan of the proof

The original goal of this paper was to prove Theorem 3.8: namely that for Hodge-proper stack over $\mathcal{O}_K$ the étale $\mathbb{Q}_p$-cohomology of $\hat{X}_C$ and $X_C$ agree. The general strategy for our argument was inspired by the proof of Totaro’s inequality given in [BL21] by Bhatt and Li. Namely, by [KP21, Corollary 4.3.15] (at least in the case when $K/\mathbb{Q}_p$ is finite) we already knew that their dimensions agree, and so it remained to show that the map $H^i_{\text{ét}}(X_C, \mathbb{Q}_p) \to H^i_{\text{ét}}(\hat{X}_C, \mathbb{Q}_p)$ is injective for any $i \geq 0$. Our idea then was to replace “approximation by proper schemes” that had been used in [BL21] by taking values of some other cohomology theory. As a result we established injectivity of the above map in the more general $d$-de Rham-proper setting (where it holds in degrees up to $d$, see Proposition 3.6).

Let us briefly sketch our proof of injectivity. The key idea is to look at the “log-de Rham complex over $B_{dR}$” that was defined in [DLLZ18] and that we denote by $\Omega_{X,D,\log,dR} \otimes B_{dR}$. For $\Omega_{X,D,\log,dR} \otimes B_{dR}$ one has a natural “$B_{dR}$-comparison map” (see Construction 2.17): namely, given a smooth adic space $X$ over $K$ with a simple normal crossings divisor $D$ and the complement $U$ one has a natural map

$$\Theta_{X,D} : R\Gamma_{\text{ét}}(U, \mathbb{Q}_p) \otimes \mathbb{Q}_p \otimes B_{dR} \to R\Gamma(X, \Omega_{X,D,\log,dR}) \otimes B_{dR}.$$  

As shown in [DLLZ18], when $X$ is proper the map $\Theta_{X,D}$ is an equivalence. Now, given an affine smooth $\mathcal{O}_K$-scheme $U$ one can take the Raynaud generic fiber $\hat{U}_K$ with an empty divisor $D = \emptyset$, or take the algebraic generic fiber $U_K$ and consider the analytification of any compactification $X$ of $U_K$ by a normal crossings divisor $D$. For any choice of $(X,D)$ there is a natural map of pairs $(\hat{U}_K, \emptyset) \to (X^\text{an}, D^\text{an})$ and, consequently, a transformation between maps $\Theta_{\hat{U}_K, \emptyset}$ and $\Theta_{X^\text{an}, D^\text{an}}$.

We then notice two things: first, that the category $\text{Comp}(U_K)_{\text{nc}}$ of compactifications as above is weakly contractible and, second, that the functor $(X,D) \mapsto R\Gamma(X^\text{an}, \Omega_{X^\text{an}, D^\text{an}, \log,dR}) \otimes B_{dR}$ is identified with the constant functor $(X,D) \mapsto R\Gamma_{\text{dR}}(U_K/K) \otimes K \otimes B_{dR}$. From this we obtain a functorial commutative square (3.1) for any $U$, which after right Kan extension gives a commutative square

$$\xymatrix{ R\Gamma_{\text{ét}}(X \backslash Z, \mathbb{Q}_p) \ar[r]^{\pi} \ar[d] & R\Gamma_{\text{ét}}(\hat{X}_C, \mathbb{Q}_p) \ar[d] \\ R\Gamma_{\text{dR}}(X_K/K) \otimes K \otimes B_{dR} \ar[r] & R\Gamma_{\text{dR}}(\hat{X}_K/B_{dR}), }$$
(see Construction 3.3 and Proposition 3.5). The left vertical arrow here is an equivalence for any smooth quasi-compact quasi-separated Artin stack $X$, but when $X$ is $d$-de Rham proper over $O_K$, the bottom horizontal one gives an isomorphism in degrees up to $d - 1$ and an embedding in degree $d$ (Proposition 2.9). Consequently, the composition induces an embedding in cohomological degrees up to $d$, which forces $\mathbf{T}_{X,\mathbb{Q}_p}$ to be injective in the same range (Proposition 3.6).

To prove the $B_{\text{cris}}$-comparison in the case of $(d + 1)$-de Rham-proper stacks over $O_K$ we extend some relevant results of [KP21, Section 2 and 4] to this setting. This we do mainly in Sections 4.1 and 4.2. In this context it is only true that the truncation $\tau^{\leq d} \Gamma_{\Delta}^{\text{cris}}(X/\mathcal{S})$ is coherent (Corollary 4.4), and it causes a problem for establishing the usual comparisons with other cohomology theories (like $A_{\text{inf}}$ or étale). The problem is simple: the completed tensor products are not necessarily $t$-exact and if one blindly follows the strategy of loc.cit. one usually loses 1 or 2 last cohomological degrees in the comparisons. The problem is there even in the case when we tensor up with an $I$-completely free module: namely, if $I$ has at least two generators the derived $I$-completed direct sum functor can very well be not $t$-exact. At least in the latter case situation is better in the condensed world: namely, for prodiscrete $I$-complete solid modules derived $I$-completed direct sums are $t$-exact (see Proposition A.13). By an elaborate argument, that we perform in Section 4.3 we are able to deduce the $B_{\text{cris}}$-comparison from a condensed version of it. However to do so, we need to use a slightly unusual, however nicely behaved, period ring $B_{\max}$. The main property of the non-coherent part $\tau^{\leq d+1} \Gamma_{\Delta}^{\text{cris}}(X/\mathcal{S})$ of prismatic cohomology that allows to control the interplay between usual and condensed worlds is the following. Namely $d$-th and $(d - 1)$-st cohomology groups of the derived reduction $[\tau^{\leq d+1} \Gamma_{\Delta}^{\text{cris}}(X/\mathcal{S})/(p, u)]$ are finite dimensional $k$-vector spaces. As we show in Appendix A, this property implies a good behavior with respect to completed tensor products and, most importantly, that the derived $(p, u)$-completion $(\tau^{\leq d+1} \Gamma_{\Delta}^{\text{cris}}(X/\mathcal{S}))_{(p, u)}$ in solid modules is still concentrated in cohomological degrees $\geq d + 1$ (Lemma A.16).

Finally, in Section 5 we record the applications of the above results in the case of schemes (Theorem 5.1).

0.4 Notations and conventions

1. In this work by Artin stacks we always mean (higher) Artin stacks in the sense of [TV08, Section 1.3.3] or [GR17, Chapter 2.4]: these are sheaves in étale topology admitting a smooth $(n - 1)$-representable atlas for some $n \geq 0$. We stress that we (mostly) work with non-derived Artin stacks, i.e. they are defined on the category of ordinary commutative rings.

2. Let $R$ be a commutative ring equipped with an ideal $I$ and let $\mathcal{C}$ be a presentable stable $R$-linear $(\infty, 1)$-category. An object $X \in \mathcal{C}$ is called derived $I$-complete if for every $f \in I$ the limit

$$T(X, f) := \lim_{\leftarrow f} \left( \cdots \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} X \right)$$

vanishes. We denote the full subcategory of $\mathcal{C}$ spanned by derived $I$-complete objects by $\mathcal{C}_{I^{\text{comp}}}$. If $I$ is finitely generated then the inclusion functor $\mathcal{C}_{I^{\text{comp}}} \hookrightarrow \mathcal{C}$ admits a left adjoint $X \mapsto X^I$ that can be explicitly described (see formula (4.3)). Also see [SP20, Tag 091N] and [Lur18, Chapter 7] for more details.

3. Some arguments in the work rely on the theory of condensed mathematics developed by Clausen–Scholze in [CS19]. We closely follow the notations from loc. cit. In particular, for an $(\infty, 1)$-category $\mathcal{C}$ we denote the category of $\mathcal{C}$-valued sheaves on the pro-étale site of a point by $\text{Cond}(\mathcal{C})$. We also denote the natural functor $\text{Top} \to \text{Cond}(\text{Set})$ for [CS19, Lecture 1] by $X \mapsto X^\circ$. This functor induces a fully-faithful embedding $\text{D}(\text{Ab}) \hookrightarrow \text{D}(\text{Cond}(\text{Ab}))$ which by abuse of notation we will also denote by $M \\to M^\circ$.

Acknowledgments. This paper owes its existence to Sasha Petrov who pointed us to the log-$B_{\text{dR}}$-cohomology of Diao–Lan–Liu–Zhu and sketched how it could help to prove the injectivity of the map between étale cohomology of two generic fibers. The generalization to $d$-de Rham stacks was fundamentally inspired by conversations with Shizhong Li who pointed us to some potential applications in schematic setting (which are now Theorem 5.1). The optimal bounds for crystalline comparison would not be there
without the help of Peter Scholze, who showed to the first author how condensed mathematics could be of some help here and patiently explained some basic aspects of the theory. We are also grateful to Sasha Petrov, Shizhang Li and Haoyang Guo for the comments on different versions of the draft and, in the latter case, also writing the Appendix C that extends the generality of Theorem 5.1.

The first author would like to express his gratitude to Max Planck Institute for the excellent work conditions during his stay there. He is also grateful to Institut des Hautes Études Scientifiques where the last parts of this manuscript were written. The study has been funded within the framework of the HSE University Basic Research Program.

1 \ d-de Rham and \ d-Hodge-proper stacks

1.1 Definition and basic properties

One can naturally introduce the following truncated analogue of Hodge-properness condition.

Definition 1.1. A smooth quasi-compact quasi-separated Artin stack \( X \) over a Noetherian ring \( R \) is called \( d \)-Hodge-proper if the Hodge cohomology \( H^{i,j}(X/R) := H^i(X, \wedge^j \Omega^1_{X/R}) \) is finitely generated for all \( i + j \leq d \).

Remark 1.2. In other words, a smooth \( \mathrm{qcqs} \) Artin stack \( X \) is \( d \)-Hodge-proper if \( \tau^{\leq d-j}R\Gamma(X, \wedge^j \Omega^1_{X/R}) \in \mathrm{Coh}(R) \) for any \( j \geq 0 \).

Remark 1.3. Obviously, a stack \( X \) is Hodge-proper (see Definition 0.1) if and only if it is \( d \)-Hodge-proper for any \( d \geq 0 \).

The \( d \)-Hodge-properness condition is much more flexible than just Hodge-properness. In particular, there are many more schemes that are Hodge-proper up to some degree, but are not themselves proper:

Lemma 1.4. Let \( X \) be a Cohen–Macaulay scheme that is proper over \( \mathrm{Spec} \, R \). Assume \( Z \subseteq X \) is a closed \( R \)-subscheme that has codimension \( d + 2 \). Assume that the complement \( U := X \setminus Z \) is smooth over \( R \). Then \( U \) is Hodge-proper up to degree \( d \).

Proof. Let \( j : U \rightarrow X \) be the embedding. Since \( X \) is Cohen–Macaulay and \( Z \) has codimension \( d + 2 \) we have \( R^di_*\mathcal{O}_U = \mathcal{O}_X \) and \( R^dj_*\mathcal{O}_U = 0 \) for \( 0 < i \leq d \). From [SP20, Tag 0BLT] it follows that for \( 0 \leq i \leq d \) \( R^dj_*\mathcal{F} \) is a coherent sheaf when \( \mathcal{F} \) is finite locally free. In other words, \( \tau^{\leq d}\mathcal{F} \in \mathrm{Coh}(X) \). Taking \( \mathcal{F} = \Omega^k_{U/R} \) and applying Lemma 1.8 to the global sections functor \( R\Gamma : \mathrm{QCoh}(X) \rightarrow D(\mathrm{Mod}_R) \) (which is left \( t \)-exact) and \( M = Rj_*\Omega^k_{U/R} \) we get that for any \( k \geq 0 \)

\[
\tau^{\leq d}R\Gamma(X, \tau^{\leq d}Rj_*\Omega^k_{U/R}) \xrightarrow{\sim} \tau^{\leq d}R\Gamma(U, \Omega^k_{U/R}).
\]

Since \( X \) is proper over \( R \) we get \( \tau^{\leq d}R\Gamma(U, \Omega^k_{U/R}) \in \mathrm{Coh}(R) \) for any \( k \geq 0 \). We are done by Remark 1.2. \( \square \)

The property that will be actually used by us in the proofs will even be slightly weaker, though completely analogous in the spirit:

Definition 1.5. A smooth quasi-compact quasi-separated Artin stack \( X \) over a Noetherian ring \( R \) is called \( d \)-de Rham-proper if \( H^r_{\mathrm{dR}}(X/R) \) is finitely generated for \( i \leq d \).

Remark 1.6. In other words, a smooth \( \mathrm{qcqs} \) Artin stack \( X \) is \( d \)-de Rham-proper if \( \tau^{\leq d}R\Gamma_{\mathrm{dR}}(X/R) \in \mathrm{Coh}(R) \).

The following is immediate:

Lemma 1.7. If a smooth Artin stack \( X \) over \( R \) is \( d \)-Hodge-proper it is also \( d \)-de Rham-proper.

Proof. By [KP19, Corollary 1.1.6(2)] one has a convergent (Hodge-to-de Rham) spectral sequence

\[
E_1^{i,j} = H^j(X, \wedge^i \Omega^1_{X/R}) \Rightarrow H^{i+j}_{\mathrm{dR}}(X/R),
\]

from which we see that if \( X \) is \( d \)-Hodge-proper then \( H^{i+j}_{\mathrm{dR}}(X/R) \) is finitely generated over \( R \) for \( i \leq d \). \( \square \)
As we will see, the following simple lemma can be used to control the “degree of coherence” of complexes under certain operations:

**Lemma 1.8.** Let $\mathcal{C}$ and $\mathcal{D}$ be stable $\infty$-categories endowed with $t$-structures. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which is left $t$-exact up to a shift by $s$.\(^3\) Then for any $d \in \mathbb{Z}$ and any $M \in \mathcal{C}$ one has a natural equivalence

$$
\tau^d \sim F(\tau^d M) \xrightarrow{\sim} \tau^{d-s} F(M).
$$

**Proof.** Omitted. \qed

**Proposition 1.9.** Let $X$ be a smooth stack over $R$ that is $d$-Hodge-proper (resp. $d$-de Rham-proper).

1. Let $R \to R'$ be a faithfully flat map of Noetherian rings. Then $X$ is $d$-Hodge-proper (resp. $d$-de Rham-proper) if and only if the base change $X_{R'} := X \times_{\Spec R} \Spec R'$ is.

2. Let $R \to R'$ be a map of Noetherian rings that has Tor-amplitude $[-s,0]$. Then the base change $X_{R'}$ is $(d-s)$-Hodge proper (resp. $(d-s)$-de Rham-proper).

**Proof.** Consider the projection $q : X_{R'} \to X$. Since $X$ is smooth (and qcqs) and $R \to R'$ is of finite Tor-amplitude we can apply base change to get an equivalence

$$
R\Gamma(X, i^!L_{X/R}) \otimes_R R' \xrightarrow{\sim} R\Gamma(X_{R'}, q^!(i^!L_{X/R})),
$$

By base change for cotangent complex (and its exterior powers, see [KP21, Corollary A.2.48]) we also have $q^!(i^!L_{X/R}) \simeq i^!L_{X_{R'}/R'}$. Similarly, for de Rham cohomology we have

$$
R\Gamma_{dR}(X/R) \otimes_R R' \xrightarrow{\sim} R\Gamma_{dR}(X_{R'}/R').
$$

In point 1, by flatness of $R'$, $\tau^d \sim R\Gamma(X_{R'}, i^!L_{X_{R'}/R'}) \simeq R\Gamma(X, i^!L_{X/R}) \otimes_R R'$ for any $i \geq 0$ and we are done by Lemma 1.10 below. The proof for de Rham cohomology is analogous. For point 2, Lemma 1.8 (applied to the tensor product $- \otimes_R R'$ and $M = R\Gamma(X, i^!L_{X/R})$) gives us an equivalence

$$
\tau^{d-s-i} (\tau^{d-i} R\Gamma(X, i^!L_{X/R}) \otimes_R R') \xrightarrow{\sim} \tau^{d-s-i} R\Gamma(X_{R'}, i^!L_{X_{R'}/R'})
$$

for any $i \geq 0$ and, similarly, also

$$
\tau^{d-s} (\tau^{d} R\Gamma_{dR}(X/R)) \otimes_R R' \xrightarrow{\sim} \tau^{d-s} R\Gamma_{dR}(X_{R'}/R').
$$

Since $\tau^{d-i} R\Gamma(X, i^!L_{X/R}) \in \Coh(R)$ for any $i \geq 0$ by $d$-Hodge-properness of $X$ we have $\tau^{d-s} R\Gamma_{dR}(X, i^!L_{X/R}) \otimes_R R' \in \Coh(R')$ (see e.g. the argument in [KP21, Lemma 1.1.6]), which then also holds for the $(d-s-i)$-th truncation. This shows that $X_{R'}$ is $(d-s)$-Hodge-proper. The rest of the proof for the de Rham-proper condition is similar. \qed

**Lemma 1.10.** Let $R \to R'$ be a faithfully flat map. Then a complex $M \in D(\text{Mod}_R)$ lies in $\Coh(R)$ is and only if $M \otimes_R R'$ lies in $\Coh(R')$.

**Proof.** By flatness we have $H^i(M \otimes_R R') \simeq H^i(M) \otimes_R R'$. In particular, $M$ is bounded if and only if $M \otimes_R R'$ is. Finally, by [SP20, Tag 03C4(2)], $H^i(M) \otimes_R R'$ is finitely generated over $R'$ if and only if $H^i(M)$ is. \qed

\(^3\)More precisely, by this we mean that $F[-s]$ is left $t$-exact.
2 \( \mathbb{Q}_p \)-étale cohomology, de Rham cohomology over \( B_{dR} \) and log period sheaves

2.1 De Rham cohomology over \( B_{dR} \)

In this section we show that the “de Rham cohomology over \( B_{dR} \)” of the Raynaud generic fiber of a d-Hodge proper stack agrees with the algebraic de Rham cohomology up to degree \( d - 1 \). We describe in detail the notion of \((p, t)\)-completed tensor product \(-\widehat{\otimes}_K B_{dR}\) and show that it agrees with the one in [DLLZ18] in the cases of our interest (see Remark 2.8).

Recall the Fontaine’s map \( \theta: A_{\text{inf}} \to O_C \). Given a finite extension \( K \) of \( \mathbb{Q}_p \) we can consider the tensor product \( A_{\text{inf}, K} := O_K \otimes_{W(k)} A_{\text{inf}} \). Let \( e := [K : K_0] \) be the ramification index. Let also \( B_{dR, K}^+ := \left( A_{\text{inf}, K} \left( \frac{1}{p} \right) \right)^{\wedge}_{\ker \theta} \) be the corresponding analogue of \( B_{dR}^+ \). Note that the natural map \( O_K \otimes_{W(k)} B_{dR}^+ \to B_{dR, K}^+ \) is an equivalence. Indeed, both sides are \( t \)-adically complete \( B_{dR}^+ \)-modules and the map becomes identity modulo \( (t) \simeq \ker \theta \subset B_{dR} \).

Recall that the embedding \( K \hookrightarrow C \simeq B_{dR}^+/(t) \) lifts canonically to a map \( K \to B_{dR}^+ \) since \( B_{dR}^+ \) is a strictly Henselian local ring. One then has a canonical map \( B_{dR, K}^+ \simeq O_K \otimes_{W(k)} B_{dR}^+ \to B_{dR, K}^+ \) which is the tensor product of the identity map \( B_{dR}^+ \to B_{dR}^+ \) and the restriction of \( K \to B_{dR}^+ \) to \( O_K \).

**Remark 2.1.** Note that \( B_{dR, K}^+ \) is in fact isomorphic to the product of \( e \) copies of \( B_{dR}^+ \). Indeed, since \( B_{dR}^+ \) is a \( K \)-algebra and \( O_K \otimes_{W(k)} K \simeq K^e \), one has isomorphisms

\[
B_{dR, K}^+ \xrightarrow{\sim} O_K \otimes_{W(k)} B_{dR}^+ \xrightarrow{\sim} (K)^{\otimes e} \otimes_K B_{dR}^+ \xrightarrow{\sim} (B_{dR}^+)^{\otimes e}.
\]

The map \( B_{dR, K}^+ \to B_{dR}^+ \) in these terms is just the projection on one of the components. Note that this map is flat.

The following construction plays a key role for this paper. Below, we also consider the composition \( B_{dR, K}^+ \to B_{dR}^+ \to B_{dR} \) (recall that \( B_{dR} := B_{dR}^+ \left( \frac{1}{t} \right) \)).

**Construction 2.2.** Let \( M \) be a complex of \( O_K \)-modules. We define the \((p, t)\)-completed tensor products \( M \otimes_{O_K} B_{dR}^+ \) and \( M \otimes_{O_K} B_{dR} \) as follows:

\[
M \otimes_{O_K} B_{dR}^+ := \left( (M \otimes_{W(k)} A_{\text{inf}}) \left( \frac{1}{p} \right) \right)^{\wedge}_{\ker \theta} \otimes_{B_{dR, K}^+} B_{dR}^+;
\]

\[
M \otimes_{O_K} B_{dR} := \left( (M \otimes_{W(k)} A_{\text{inf}}) \left( \frac{1}{p} \right) \right)^{\wedge}_{\ker \theta} \otimes_{B_{dR, K}^+} B_{dR}.
\]

**Remark 2.3.** By construction, \( O_K \otimes_{O_K} B_{dR} \simeq B_{dR}^+ \otimes_{B_{dR, K}} B_{dR} \simeq B_{dR} \). Also, for any \( M \) we have \( M \otimes_{O_K} B_{dR} \simeq \left( M \otimes_{O_K} B_{dR} \right) \left( \frac{1}{t} \right) \).

The following lemma will be useful later:

**Lemma 2.4.** The \((p, t)\)-completed tensor product functor \( - \otimes_{O_K} B_{dR} : D(\text{Mod}_{O_K}) \to D(\text{Mod}_{B_{dR}}) \) is left \( t \)-exact up to a shift by 1.

**Proof.** Note that the functor \( M \mapsto (M \otimes_{W(k)} A_{\text{inf}}) \left( \frac{1}{p} \right) / \xi^k \) is left \( t \)-exact up to a shift by 1. Indeed, for \( k = 1 \) we have \( M \mapsto (M \otimes_{W(k)} O_C) \left( \frac{1}{p} \right) / \xi \), which is the composition of tensor product \( - \otimes_{W(k)} O_C \) (which is \( t \)-exact since \( O_C \) is \( p \)-torsion free), derived \( p \)-adic completion (which is left \( t \)-exact up to a shift by 1) and localization (which is \( t \)-exact). To get the statement for a general \( k \) one can argue by induction using the fiber sequence

\[
(M \otimes_{W(k)} A_{\text{inf}}) \left( \frac{1}{p} \right) / \xi^{k-1} \twoheadrightarrow (M \otimes_{W(k)} A_{\text{inf}}) \left( \frac{1}{p} \right) / \xi^k \twoheadrightarrow (M \otimes_{W(k)} A_{\text{inf}}) \left( \frac{1}{p} \right) / \xi.
\]

\( M \otimes_{O_K} B_{dR} \) then is obtained by first taking limit of \( (M \otimes_{W(k)} A_{\text{inf}}) \left( \frac{1}{p} \right) / \xi^k \) over \( k \) and then the tensor product \( - \otimes_{B_{dR, K}} B_{dR} \) (which is \( t \)-exact by Remark 2.1). Since limits are left \( t \)-exact this gives the claim. \( \square \)
Remark 2.5. The map $\theta$ and the natural embedding $\mathcal{O}_K \hookrightarrow \mathcal{O}_C$ agree on $W(k)$ and produce a natural morphism

$$\theta_K: A_{\inf,K} \longrightarrow \mathcal{O}_C.$$ 

One can show that there is also another formula for $M \otimes_{\mathcal{O}_K} B_{dR}^+$ in terms of $\ker \theta_K$-adic completion

$$M \otimes_{\mathcal{O}_K} B_{dR}^+ := \left( (M \otimes_{W(k)} A_{\inf})_{[1]} \right)_{\ker \theta_K}.$$ 

Now we can define “Hodge” and “de Rham” cohomology of the Raynaud generic fiber “over $B_{dR}$”.

Construction 2.6. Let $U$ be a smooth affine scheme over $\mathcal{O}_K$. We define

$$R\Gamma(\check{U}_K, \Omega^i \otimes_{\mathcal{O}_K} B_{dR}) := \Omega^i_U \otimes_{\mathcal{O}_K} B_{dR} \quad \text{and} \quad R\Gamma_{dR}(\check{U}_K/B_{dR}) := \Omega^i_B \otimes_{\mathcal{O}_K} B_{dR}.$$ 

We call $R\Gamma_{dR}(\check{U}_K/B_{dR})$ the de Rham cohomology of $\check{U}_K$ over $B_{dR}$. We extend this definition to all smooth Artin $\mathcal{O}_K$-stacks via the right Kan extension. That is, for a smooth Artin stack $\mathcal{O}_K$-stack $X$ we have

$$R\Gamma(\check{X}_K, \Omega^i \otimes_{\mathcal{O}_K} B_{dR}) \simeq \lim_{\longleftarrow U \in \mathcal{A}_U} R\Gamma(\check{U}_K, \Omega^i \otimes_{\mathcal{O}_K} B_{dR}),$$

$$R\Gamma_{dR}(\check{X}_K/B_{dR}) \simeq \lim_{\longleftarrow U \in \mathcal{A}_U} R\Gamma_{dR}(\check{U}_K/B_{dR}).$$

One also has an obvious version with $B_{dR}^+$ instead of $B_{dR}$.

Remark 2.7. Note that for a smooth affine scheme $U$ the complex $R\Gamma_{dR}(\check{U}_K/B_{dR})$ admits a finite Hodge filtration with associated graded pieces $R\Gamma(\check{U}_K, \Omega^i \otimes_{\mathcal{O}_K} B_{dR})$. Passing to the limit we find a similar complete Hodge filtration on $R\Gamma_{dR}(\check{X}_K/B_{dR})$ for any smooth Artin $\mathcal{O}_K$-stack $X$.

Remark 2.8. We point out that the definitions in Construction 2.6 agree with the ones in [LZ17, Section 3.1] and [DLLZ18, Definition 3.1.1] in the case of Raynaud generic fiber of a smooth scheme. Indeed, let $X = \text{Spec}(R, R^+)$ be a smooth affinoid adic space over $\text{Spec}(K, \mathcal{O}_K)$. The complex $R\Gamma(X_{\text{an}}, \Omega_X \otimes_{\mathcal{O}_K} B_{dR})$ (in the notations of [LZ17, Section 3.1]) is by definition given by the limit over $n$ of the $p$-completed tensor products $\Omega_X \otimes_{\mathcal{O}_K} B_{dR}^+/t^n$. Let $X = \check{X}_K$ be the Raynaud generic fiber of a smooth affine $\mathcal{O}_K$-scheme $U := \text{Spec}(A)$; in this case $(R, R^+) \simeq (A_p^{\omega[1]}, A_p)$, and $\mathcal{O}_U \otimes_{\mathcal{O}_K} B_{dR}$ will then follow by inverting $t$ and considering the Hodge filtration correspondingly. Note that by smoothness, $\mathcal{O}_U$ is a perfect $A$-module, and since $\Omega_X \simeq \Omega_X \otimes_{A} R$, it is enough to consider the case $i = 0$; namely, of the structure sheaf. This reduces to the existence of a natural equivalence

$$\Omega^i_U \otimes_{\mathcal{O}_K} B_{dR}^+ \longrightarrow \Omega^i_B \otimes_{\mathcal{O}_K} B_{dR}^+,$$

the analogous isomorphisms $\Omega^i_U \otimes_{\mathcal{O}_K} B_{dR} \simeq \Omega^i_B \otimes_{\mathcal{O}_K} B_{dR}$ and $\Omega^i_{\mathcal{U}_dR} \otimes_{\mathcal{O}_K} B_{dR} \simeq \Omega^i_{\mathcal{X}_{dR}} \otimes_{\mathcal{O}_K} B_{dR}$ will then follow by inverting $t$ and considering the Hodge filtration correspondingly. Note that by smoothness, $\mathcal{O}_U$ is a perfect $A$-module, and since $\Omega_X \simeq \Omega_X \otimes_{A} R$, it is enough to consider the case $i = 0$; namely, of the structure sheaf. This reduces to the existence of a natural equivalence

$$\left( (A \otimes_{W(k)} A_{\inf})^{\omega[1]} \right)_{p^{1/1}} \otimes_{B_{dR,K}/t,n} B_{dR}^+/t \otimes_{B_{dR,K}/t^n} R\otimes_K B_{dR}^+/t^n$$

for any $n \geq 0$. For this recall how the completion in the tensor product $R\otimes_K (B_{dR}^+/t^n)$ can be defined: namely, the submodule $R^+ \otimes_{W(k)} (A_{\inf}/\xi^n) \subset R\otimes_K (B_{dR}^+/t^n)$ defines a $p$-adic lattice and $R\otimes_K (B_{dR}^+/t^n)$ is the completion of $R\otimes_K (B_{dR}^+/t^n)$ with respect to the corresponding topology. Consider another tensor product $R\otimes_{K_0} (B_{dR}^+/t^n)$ with the $p$-adic completion defined by $R^+ \otimes_{W(k)} (A_{\inf}/\xi^n) \subset R\otimes_{K_0} (B_{dR}^+/t^n)$; one has a formula

$$R\otimes_{K_0} (B_{dR}^+/t^n) \simeq (R^+ \otimes_{W(k)} (A_{\inf}/\xi^n))^{\omega[1]}.$$ 

Note that since $R^+ \simeq A_p$, we have an equivalence

$$\left( (A \otimes_{W(k)} (A_{\inf}/\xi^n))_{p^{1/1}} \right) \longrightarrow (R^+ \otimes_{W(k)} (A_{\inf}/\xi^n))_{p^{1/1}}.$$
Also, since \( A_{\text{inf}}/\xi^n \) is a perfect \( A_{\text{inf}} \)-module, one can commute it through the completion:
\[
(A \otimes_{W(k)} A_{\text{inf}})[1/p]//\xi^n \rightarrow A \otimes_{W(k)} (A_{\text{inf}}/\xi^n) [1/p].
\]
Finally, note that \( R\widehat{\otimes}_{K_0}(B^+_{\text{dr}}/t^n) \) is a \( K \otimes K_0 (B^+_{\text{dr}}/t^n) \simeq B^+_{\text{dr},K}/t^n \)-algebra and, moreover, that
\[
R\widehat{\otimes}_{K_0}B^+_{\text{dr}}/t^n \simeq R\widehat{\otimes}_{K_0}(B^+_{\text{dr}}/t^n) \otimes B^+_{\text{dr},K}/t^n (B^+_{\text{dr}}/t^n).
\]
Putting the last three isomorphisms together gives the desired isomorphism (2.1).

We will need the following basic result about \( B_{\text{dr}} \)-cohomology.

**Proposition 2.9.** Let \( X \) be a smooth Artin stack over \( O_K \). Then

1. The natural map
\[
R\Gamma(X, \wedge^n \mathcal{L}_X/O_K) \widehat{\otimes}_{O_K} B_{\text{dr}} \rightarrow R\Gamma(\hat{X}_K, \Omega^n \otimes_{O_K} B_{\text{dr}})
\]
is an equivalence.
2. So is the natural map \( R\Gamma_{\text{dr}}(X/O_K) \widehat{\otimes}_{O_K} B_{\text{dr}} \rightarrow R\Gamma_{\text{dr}}(\hat{X}_K/B_{\text{dr}}) \).
3. If additionally \( X \) is \( d \)-de Rham-proper over \( O_K \), then the natural map
\[
R\Gamma_{\text{dr}}(X_K/K) \otimes_K B_{\text{dr}} \rightarrow R\Gamma_{\text{dr}}(\hat{X}_K/B_{\text{dr}})
\]
induces an isomorphism \( H^i_{\text{dr}}(X_K/K) \otimes_K B_{\text{dr}} \simeq H^i_{\text{dr}}(\hat{X}_K/B_{\text{dr}}) \) for \( i \leq d - 1 \) and an embedding \( H^d_{\text{dr}}(X_K/K) \otimes_K B_{\text{dr}} \hookrightarrow H^d_{\text{dr}}(\hat{X}_K/B_{\text{dr}}) \).

**Proof.** For the first assertion note that by construction both sides coincide on smooth affine schemes. Hence to prove the claim it is enough to show that both sides satisfy smooth descent. For the left hand side this follows from the flat descent for cotangent complex (see e.g. [KP19, Proposition 1.1.5]) and Lemma 2.11 below. For the right hand side, note that by the same lemma the functor on smooth affine schemes that sends \( U \mapsto \Omega^i_{U,K} \otimes_{O_K} B_{\text{dr}} \) also satisfies smooth descent. It is then formal that the right Kan extension also satisfies smooth descent. The proof for the second part is completely analogous.

Finally, part (3) follows from (2) and applying Lemma 2.12 below to \( M = R\Gamma_{\text{dr}}(X/O_K) \).

**Remark 2.10.** If \( X \) is Hodge-proper over \( O_K \), then it is \( d \)-Hodge and, consequently, \( d \)-de Rham proper for any \( d \geq 0 \). Thus, from Proposition 2.9(3) we get that the map
\[
R\Gamma_{\text{dr}}(X_K/K) \otimes_K B_{\text{dr}} \rightarrow R\Gamma_{\text{dr}}(\hat{X}_K/B_{\text{dr}})
\]
is a quasi-isomorphism.

**Lemma 2.11.** Let \( M^\bullet \) be a co-simplicial diagram of uniformly bounded below complexes of \( O_K \)-modules. Then the natural map
\[
\text{Tot}(M^\bullet) \otimes_{O_K} B_{\text{dr}} \rightarrow \text{Tot}(M^\bullet \otimes_{O_K} B_{\text{dr}})
\]
is an equivalences.

**Proof.** This follows from [KP21, Corollary C.6], since the functor \( - \otimes_{O_K} B_{\text{dr}} \) is left \( t \)-exact up to a shift (Lemma 2.4).

**Lemma 2.12.** Let \( M \in D(\text{Mod}_{O_K}) \) be such that \( \tau^{\leq d} M \in \text{Coh}(O_K) \). Then the natural map
\[
M \otimes_{O_K} B_{\text{dr}} \rightarrow M \otimes_{O_K} B_{\text{dr}}
\]
induces an isomorphism \( H^i(M) \otimes_{O_K} B_{\text{dr}} \simeq H^i(M \otimes_{O_K} B_{\text{dr}}) \) for \( i \leq d - 1 \) and an embedding \( H^d(M) \otimes_{O_K} B_{\text{dr}} \hookrightarrow H^d(M \otimes_{O_K} B_{\text{dr}}) \).
Proof. The map in question is an equivalence for $M = \mathcal{O}_K$ (Remark 2.3), hence it is also so for all perfect (or equivalently, coherent) $\mathcal{O}_K$-modules. This applies in particular to $\tau^{\leq d}(M)$. Recall that $- \otimes_{\mathcal{O}_K} B_{\text{dR}}$ is left $t$-exact up to a shift by 1, while $- \otimes_{\mathcal{O}_K} B_{\text{dR}}$ is just left $t$-exact since $B_{\text{dR}}$ is $p$-torsion free. From Lemma 1.8 we get a commutative square

$$
\begin{array}{c}
\tau^{\leq d-1} (\tau^{\leq d}(M) \otimes_{\mathcal{O}_K} B_{\text{dR}}) \quad \longrightarrow \\
\tau^{\leq d-1} (\tau^{\leq d}(M) \otimes \mathcal{O}_K B_{\text{dR}})
\end{array}
$$

which shows that the low horizontal map is an equivalence. By passing to cohomology, this gives the statement for $i \leq d - 1$.

Note also that for any $i$ (from left $t$-exactness of $- \otimes_{\mathcal{O}_K} B_{\text{dR}}$ up to a shift by 1) we have a short exact sequence

$$
0 \to H^0(H^i(M) \otimes_{\mathcal{O}_K} B_{\text{dR}}) \to H^i(M \otimes_{\mathcal{O}_K} B_{\text{dR}}) \to H^{-1}(H^{i+1}(M) \otimes_{\mathcal{O}_K} B_{\text{dR}}) \to 0.
$$

By the assumption, for $i = d$ the module $H^d(M)$ is still finitely generated, so $H^d(M) \otimes_{\mathcal{O}_K} B_{\text{dR}} \simeq H^d(M) \otimes_{\mathcal{O}_K} B_{\text{dR}}$, and the short exact sequence gives that $H^d(M) \otimes_{\mathcal{O}_K} B_{\text{dR}} \to H^{d}(M \otimes_{\mathcal{O}_K} B_{\text{dR}})$.

\[ \square \]

2.2 Log period sheaves and $B_{\text{dR}}$-comparison map

Let $X$ be a smooth adic space over $\text{Spa}(K, \mathcal{O}_K)$. Let $i: D \to X$ be a normal crossings divisor (see [DLLZ19, Example 2.3.17] for the definition). The divisor $D$ endows $X$ with a natural structure of a (smooth) log adic space (see [DLLZ18, Example 2.1.2]). We will denote by $X_{D,\text{ké t}}$, $X_{D,\text{proké t}}$ the associated Kummer étale (see [DLLZ19, Section 4.1]) and pro-Kummer étale (see [DLLZ19, Definition 5.1.2]) sites correspondingly.

Example 2.13. When $D = \emptyset$ is empty, $X_{D,\text{ké t}}$ and $X_{D,\text{proké t}}$ are identified with the étale and pro-étale sites of $X$.

Remark 2.14. Let $U := X \setminus D$ be the complement to $D$ and let $j: U \to X$ be the natural embedding. Pull-back of $D$ to $U$ is empty, $U_{\emptyset, \text{ké t}} \simeq U_{\emptyset}$, and thus one has a natural (derived) push-forward functor $j_*: \mathcal{O}(U_{\emptyset, \text{ké t}}, \mathbb{Z}/n) \to \mathcal{O}(X_{D,\text{ké t}}, \mathbb{Z}/n)$ between the derived categories of sheaves of $\mathbb{Z}/n$-modules (for any $n \geq 0$). By [DLLZ19, Lemma 4.6.5] for the constant sheaf $\mathbb{Z}/n \in \mathcal{O}(U_{\emptyset, \text{ké t}}, \mathbb{Z}/n)$ one has an equivalence $j_* \mathbb{Z}/n \simeq \mathbb{Z}/n$; this then induces an equivalence

$$
R\Gamma(U_{\emptyset}, \mathbb{Z}/n) \sim R\Gamma(X_{D,\text{ké t}}, \mathbb{Z}/n)
$$

for the global sections. Taking $n = p^k$ and passing to the limit over $k$ one also gets an equivalence

$$
R\Gamma(U_{\emptyset}, \mathbb{Z}_p) \sim R\Gamma(X_{D,\text{ké t}}, \mathbb{Z}_p).
$$

(2.2)

In [DLLZ18] the period sheaves $\mathcal{B}_{\text{dR}}^+, \mathcal{B}_{\text{dR, log}}^+, \mathcal{O}_{\mathcal{B}_{\text{dR, log}}}^+$ on $X_{D,\text{proké t}}$ were introduced. While the definition of $\mathcal{B}_{\text{dR}}^+$, $\mathcal{B}_{\text{dR}}$ is essentially the same as in [Sch13] (see [DLLZ18, Definition 2.2.3]), the definition of $\mathcal{O}_{\mathcal{B}_{\text{dR, log}}}^+$, $\mathcal{O}_{\mathcal{B}_{\text{dR, log}}}$ is more involved and takes into account the log structure in a more significant way (see [DLLZ18, Definition 2.2.10]). Let also $\Omega_{X,\text{D, log}}^1$ be the sheaf of log differential 1-forms on $X$ (see [DLLZ19, Definition 3.2.25]). Similarly, one defines $\Omega_{X,\text{D, log}}^1 := \bigwedge^j \Omega_{X,\text{D, log}}^1$ for all $j \geq 0$ (see [DLLZ19, Definition 3.2.29]).

We denote by $\mu: X_{D,\text{proké t}} \to X_{\text{an}}$ the natural map of sites. We also consider the map $\mu': (X_C)_{D,\text{proké t}} \simeq (X_{D,\text{proké t}})/X_C \to X_{\text{an}}$ where $X_C := X \times_K C$ is the base change of $X$ to $C := \overline{K}$. For brevity we will continue to denote the pull-backs $\mu^*\Omega_{X,\text{D, log}}^1$ and $\mu'^*\Omega_{X,\text{D, log}}^1$ by $\Omega_{X,\text{D, log}}^1$. The sheaf $\mathcal{O}_{\mathcal{B}_{\text{dR, log}}}^+$ on $X_{D,\text{proké t}}$ has a natural log connection $\nabla: \mathcal{O}_{\mathcal{B}_{\text{dR}}^+} \to \mathcal{O}_{\mathcal{B}_{\text{dR, log}}}^+ \otimes \mathcal{O}_X \Omega_{X,\text{D, log}}^1$ (see [DLLZ18, p. 2.2.15]) which then extends to a log connection $\nabla: \mathcal{O}_{\mathcal{B}_{\text{dR, log}}}^+ \to \mathcal{O}_{\mathcal{B}_{\text{dR, log}}}^+ \otimes \mathcal{O}_X \Omega_{X,\text{D, log}}^1$ on $\mathcal{O}_{\mathcal{B}_{\text{dR, log}}}$ [DLLZ18, p. 2.2.17]. We then have:
Proposition (The Poincaré lemma, [DLLZ18, Corollary 2.4.2]). The complex

\[ 0 \to \mathbb{B}_{\text{DR}} \to \mathcal{O}_{\mathbb{B}_{\text{DR}} \otimes \mathcal{O}_X} \to \mathcal{O}_{\mathbb{B}_{\text{DR}} \otimes \mathcal{O}_X} \to \mathcal{O}_{\mathbb{B}_{\text{DR}} \otimes \mathcal{O}_X} \to \cdots \]

of sheaves on \( X_{\text{prokét}} \) is acyclic.

This gives a quasi-isomorphism \( \mathbb{B}_{\text{DR}} \sim \mathcal{O}_{\mathbb{B}_{\text{DR}} \otimes \mathcal{O}_X} \nabla \) in the derived categories of sheaves on \( X_{\text{prokét}} \) and \( (X_C)_{\text{prokét}} \) (here \( D_{\text{DR}} \) denotes the log de Rham complex).

One can also explicitly describe the pushforwards with respect to the map \( \mu' : (X_C)_{\text{prokét}} \to X_{\text{an}} \) of the individual terms of the de Rham complex \( D_{\text{DR}}(\mathcal{O}_{\mathbb{B}_{\text{DR}} \otimes \mathcal{O}_X} \nabla) \). Namely, recall the sheaf \( \mathcal{O}_X \otimes B_{\text{DR}} \) ([DLLZ18, Definition 3.1.1], see also the discussion in Remark 2.8) and consider \( \Omega^{j}_{X_{\text{log}}, D_{\text{DR}}} \otimes B_{\text{DR}} \) (these sheaves are called \( \Omega^{j}_{X_{\text{log}}, D_{\text{DR}}} \) in [DLLZ18, Definition 3.1.6]). Let \( D_{\text{DR}} : \mathcal{O}_X \otimes B_{\text{DR}} \to \Omega^{j}_{X_{\text{log}}, D_{\text{DR}}} \otimes B_{\text{DR}} \) be the map induced by the natural log connection \( d := \mathcal{O}_X \to \Omega^{j}_{X_{\text{log}}, D_{\text{DR}}} \) on \( \mathcal{O}_X \) and let

\[ \Omega^{j}_{X_{\text{log}}, D_{\text{DR}}} \otimes B_{\text{DR}} \to \Omega^{j}_{X_{\text{log}}, D_{\text{DR}}} \otimes B_{\text{DR}} \to \cdots \]

be the corresponding log de Rham complex.

Returning to the previous discussion, one has an equivalence

\[ \mathcal{O}_X \otimes B_{\text{DR}} \sim R\mu'_*(\mathbb{B}_{\text{DR}}) \]

(see [DLLZ18, Lemma 3.3.2]), which, by projection formula, also gives \( \Omega^{j}_{X_{\text{log}}, D_{\text{DR}}} \otimes B_{\text{DR}} \sim R\mu'_*(\mathbb{B}_{\text{DR}} \otimes \Omega^{j}_{X_{\text{log}}, D_{\text{DR}}}) \) for any \( j \geq 0 \). These isomorphisms then ultimately lead to equivalences

\[ R\mu'_*(\mathbb{B}_{\text{DR}}) \sim R\mu'_*(D_{\text{DR}}(\mathcal{O}_{\mathbb{B}_{\text{DR}} \otimes \mathcal{O}_X} \nabla)) \sim R\mu'_*(\mathbb{B}_{\text{DR}}) \]

(2.3)

Remark 2.15. It is noted in [DLLZ18, Remark 2.2.11] that when the log-structure is trivial the construction of the log-period ring \( \mathcal{O}_{\mathbb{B}_{\text{DR}} \otimes \mathcal{O}_X} \) still differs from the one of \( \mathbb{B}_{\text{DR}} \) in [Sch13, Section 6] by a certain completion procedure. However, as noted in loc.cit all features essential for the applications (namely the Poincaré lemma and the description of the pushforward to the analytic site) stay valid, thus in fact for most of the arguments one can freely choose between any of the two versions.

Remark 2.16. If \( D = \emptyset \) then the complex \( \Omega^{j}_{X_{\text{log}}, D_{\text{DR}}} \otimes B_{\text{DR}} \) on \( X \) agrees with \( \Omega^{j}_{X_{\text{log}}, D_{\text{DR}}} \otimes B_{\text{DR}} \) (see [LZ17, Section 3.1] or Remark 2.8 to recall the definition). By Remark 2.8, when \( X = \hat{U}_K \) is the Raynaud generic fiber of a smooth affine \( \mathcal{O}_K \)-scheme \( U \), this complex agrees with the “de Rham cohomology over \( B_{\text{DR}} \)” \( R\Gamma_{\text{DR}}(\hat{U}_K/B_{\text{DR}}) \) that we have defined in Construction 2.6.

We can now pass to the construction of the \( B_{\text{DR}} \)-comparison map.

Construction 2.17. Let \( X, i : D \hookrightarrow X \) be as in the beginning of this section and let \( U := X \setminus D \). Let \( R\Gamma_{\text{et}}(U_C, \mathbb{Q}_p) := R\Gamma_{\text{et}}(U_C, \mathbb{Z}_p)[\frac{1}{p}] \) and \( R\Gamma^{(X_C)}_{\text{D, két}, \mathbb{Q}_p} := R\Gamma^{(X_C)}_{\text{D, két}, \mathbb{Z}_p}[\frac{1}{p}] \). By (2.2) we have an equivalence

\[ R\Gamma_{\text{et}}(U_C, \mathbb{Q}_p) \sim R\Gamma^{(X_C)}_{\text{D, két}, \mathbb{Q}_p}. \]

Having this (with a further identification \( R\Gamma^{(X_C)}_{\text{D, két}, \mathbb{Z}_p} \sim R\Gamma^{(X_C)}_{\text{D, prokét}, \mathbb{Z}_p} \)) the map \( \mathbb{Z}_p \to \mathbb{B}_{\text{DR}} \) of sheaves on \( X_{\text{prokét}} \) induces a natural map

\[ R\Gamma_{\text{et}}(U_C, \mathbb{Q}_p) \otimes \mathbb{Z}_p \to R\Gamma^{(X_C)}_{\text{D, prokét}, \mathbb{Z}_p} \]

Composing further with the equivalence \( R\Gamma^{(X_C)}_{\text{D, prokét}, \mathbb{B}_{\text{DR}}} \sim R\Gamma^{(X_C)}_{\text{an}, \mathbb{Q}_p} \otimes \mathbb{B}_{\text{DR}} \) coming from (2.3) we get a natural map

\[ \Theta_{X,D} : R\Gamma_{\text{et}}(U_C, \mathbb{Q}_p) \otimes \mathbb{Z}_p \to R\Gamma^{(X_C)}_{\text{an}, \mathbb{Q}_p} \otimes \mathbb{B}_{\text{DR}} \]

which we will call the \( B_{\text{DR}} \)-comparison map.
Example 2.18. By [DLLZ18, Theorem 3.2.3(3)] if $X$ is proper the map $\Theta_{X,D}$ is an equivalence for any normal crossings divisor $D$. Moreover, by [DLLZ18, Lemma 3.6.2] one can make an identification

$$R\Gamma_{\log,dR}(X/K) \otimes_K B_{dR} \xrightarrow{\sim} R\Gamma(X,\Omega_{X,D,\log,dR}^\bullet \otimes B_{dR}).$$

In the case of the analytification $X^\text{an}, D^\text{an}$ of a smooth proper $K$-scheme $X$ with a normal crossings divisor $D$, one can make further natural identifications

$$R\Gamma_{\log,dR}(X^\text{an}/K) \xrightarrow{\text{GAGA}} R\Gamma_{\log,dR}(X/K) \xrightarrow{\sim} R\Gamma_{dR}(U/K),$$

where the first equivalence is induced by GAGA and the second one by the restriction of the algebraic log de Rham complex (as a complex of sheaves) to $U := X \setminus D$.

3 Proof of $\mathbb{Q}_p$-local acyclicity for Hodge-proper stacks

3.1 The key commutative square

Let $U$ be a scheme over $\mathcal{O}_K$. Let $U_K := U \times_{\mathcal{O}_K} K$ and $\hat{U}_K$ be the algebraic and Raynaud generic fibers of $U$. We have a natural map $\psi_U: \hat{U}_K \to U_K^\text{an}$, where $U_K^\text{an}$ is the analytification of $U_K$ (for a more detailed discussion of this situation see [KP21, Section 3.3]).

Below we will also need to consider the auxiliary data of a compactification $X$ of $U_K$ by a simple normal crossings divisor $D$. To clarify, by this we mean a smooth proper $K$-scheme $X$ with a simple normal crossings divisor $D \to X$ and an isomorphism $\varepsilon: U_K \xrightarrow{\sim} X \setminus D$. Let $\text{Comp}(U_K)_{\text{nc}}$ be the category of such compactifications $(X, D, \varepsilon)$. An important property of $\text{Comp}(U_K)_{\text{nc}}$ is that it is cofiltered:

Lemma 3.1. Let $S$ be a smooth finite type scheme over a characteristic 0 field. Then the category $\text{Comp}(S)_{\text{nc}}$ of compactifications of $S$ by a simple normal crossings divisor is co-filtered.

Proof. First of all $\text{Comp}(S)_{\text{nc}}$ is non-empty. Indeed, by Nagata’s compactification there exists a reduced proper scheme $X'$ containing $S$ as a dense open subscheme. By Hironaka’s resolution of singularities one then has a birational map $X \to X'$ which is trivial on $S$ and such that the complement $X \setminus S$ is a simple normal crossing divisor.

Let $X_1, X_2$ be a pair of compactifications of $S$ by normal crossings divisors. Let $Z'$ be a closure of $S$ in $X_1 \times X_2$. Resolving the singularities as above we find a $Z \to Z'$ which is trivial over $S$ and such that the complement $Z \setminus S$ is a simple normal crossings divisor. By construction such $Z$ maps both to $X_1$ and $X_2$ (using projections $X_1 \leftarrow Z' \to X_2$).

Let now $f, g: X_1 \xrightarrow{\sim} X_2$ be a pair of parallel arrows in $\text{Comp}(S)_{\text{nc}}$. Consider the closure $Z'$ of the diagonal embedding $S \hookrightarrow X_1 \times X_2 X_1$. Let $Z \to Z'$ be a resolution of singularities as above. By construction there is a map $e: Z \to Y$ such that $f \circ e = g \circ e$. \qed

Remark 3.2. In particular it follows that the nerve $N(\text{Comp}(S)_{\text{nc}})$ is weakly contractible.

In the next construction for an affine smooth $\mathcal{O}_K$-scheme $U$ we construct the following natural commutative square:

$$\begin{array}{ccc}
R\Gamma\text{et}(U_C, Q_p) \otimes Q_p B_{dR} & \xrightarrow{T_{U,C}} & R\Gamma\text{et}(\hat{U}_C, Q_p) \otimes Q_p B_{dR} \\
\downarrow{\sim} \downarrow{\Theta_{U_K}} & & \downarrow{\Theta_{\hat{U}_K}} \\
R\Gamma\text{dR}(U_K/K) \otimes_K B_{dR} & \xrightarrow{{\sim}} & R\Gamma\text{dR}(\hat{U}_K/B_{dR}).
\end{array}$$

(3.1)

This will be the key ingredient in the proof of local acyclicity.

Construction 3.3. 1. Let $S$ be a smooth affine $K$-scheme and let $S^\text{an}$ be its analytification. By associating to $(X, D, \varepsilon) \in \text{Comp}(S)_{\text{nc}}$ the pair $(X^\text{an}, D^\text{an})$ we can further consider a complex $R\Gamma(X^\text{an}, \Omega^\bullet_{X^\text{an}, D^\text{an}, \log,dR} \otimes B_{dR}) \in D(\text{Mod}_{B_{dR}})$. This extends to a functor

$$R\Gamma(-^\text{an}, \Omega^\bullet_{-, -, \log,dR} \otimes B_{dR}): (\text{Comp}(S)_{\text{nc}})^\text{op} \longrightarrow D(\text{Mod}_{B_{dR}}).$$

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Note that since for any \((X,D,\varepsilon) \in \text{Comp}(S)_{\text{nc}}\) the scheme \(X\) is smooth and proper, so by (the references in) Example 2.18 there is a sequence of natural identifications

\[
\begin{align*}
\text{RG}_{\text{dR}}(S/K) \otimes_K B_{\text{dR}} & \xrightarrow{\sim} \text{RG}_{\log \text{dR}}(X/K) \otimes_K B_{\text{dR}} \\
\text{RG}_{\log \text{dR}}(X^{\text{an}}/K) \otimes_K B_{\text{dR}} & \xrightarrow{\sim} \text{RG}(X^{\text{an}}, \Omega^{\text{an}}, D^{\text{an}}) \otimes B_{\text{dR}},
\end{align*}
\]

which ultimately identifies \(\text{RG}(-^{\text{an}}, \Omega^{\text{an}}, D^{\text{an}}) \otimes B_{\text{dR}}\) with the constant functor \(\text{RG}_{\text{dR}}(S/K) \xrightarrow{\sim} \text{RG}_{\text{dR}}(S/K) \otimes F B_{\text{dR}}\).

2. Let \(U\) be an affine smooth \(\mathcal{O}_K\)-scheme and \(U_K\) be the algebraic generic fiber. For any \((X,D,\varepsilon) \in \text{Comp}(U_K)_{\text{nc}}\) we have a natural map of pairs \((U_K^{\text{an}}, \emptyset) \to (X^{\text{an}}, D^{\text{an}})\). Precomposing it with \(\psi_U : \hat{U}_K \to \hat{U}_K\) we also get a map \((\hat{U}_K, \emptyset) \to (X^{\text{an}}, D^{\text{an}})\), which further induces a map

\[
\text{RG}(X^{\text{an}}, \Omega^{\text{an}}, D^{\text{an}}) \otimes B_{\text{dR}} \xrightarrow{\sim} \text{RG}_{\text{dR}}(\hat{U}_K / B_{\text{dR}});
\]

here for the equivalence on the right see Remark 2.16. Recall also the map \(\Theta\) defined in Construction 2.17. Applying it to pairs \((X^{\text{an}}, D^{\text{an}})\) and \((\hat{U}_K, \emptyset)\) we get a natural commutative square

\[
\begin{align*}
\text{RG}_{\text{et}}(U^{\text{an}}_C, \mathbb{Q}_p) \otimes \mathbb{Q}_p B_{\text{dR}} & \xrightarrow{\sim} \text{RG}_{\text{et}}(\hat{U}_C, \mathbb{Q}_p) \otimes \mathbb{Q}_p B_{\text{dR}} \\
\Theta_{X,D}^{-1} & \xrightarrow{\sim} \Theta_{\hat{U}_K, \emptyset}^{-1} \\
\text{RG}(X^{\text{an}}, \Omega^{\text{an}}, D^{\text{an}}) \otimes B_{\text{dR}} & \xrightarrow{\sim} \text{RG}_{\text{dR}}(\hat{U}_K / B_{\text{dR}}),
\end{align*}
\]

where the left vertical arrow is an equivalence. Passing to colimit over \((X,D) \in \text{Comp}(U_K)_{\text{nc}}\), using the equivalence \(\text{RG}(-^{\text{an}}, \Omega^{\text{an}}, D^{\text{an}}) \otimes B_{\text{dR}} \simeq \text{RG}_{\text{dR}}(U_K / K) \otimes K B_{\text{dR}}\) in (1) and the fact that \(N(\text{Comp}(U_K)_{\text{nc}})\) is weakly contractible (Remark 3.2) we obtain a natural commutative square

\[
\begin{align*}
\text{RG}_{\text{et}}(U^{\text{an}}_C, \mathbb{Q}_p) \otimes \mathbb{Q}_p B_{\text{dR}} & \xrightarrow{\sim} \text{RG}_{\text{et}}(\hat{U}_C, \mathbb{Q}_p) \otimes \mathbb{Q}_p B_{\text{dR}} \\
\Theta_{X,D}^{-1} & \xrightarrow{\sim} \Theta_{\hat{U}_K, \emptyset}^{-1} \\
\text{RG}_{\text{dR}}(U_K / K) \otimes K B_{\text{dR}} & \xrightarrow{\sim} \text{RG}_{\text{dR}}(\hat{U}_K / B_{\text{dR}}).
\end{align*}
\]

3. Finally, by [Hub96, Theorem 3.8.1] one has a natural equivalence

\[
\varphi_U^{-1} : \text{RG}_{\text{et}}(U_C, \mathbb{Q}_p) \xrightarrow{\sim} \text{RG}_{\text{et}}(U^{\text{an}}_C, \mathbb{Q}_p)
\]

(see also [KP21, Section 3.3 and Remark 4.1.16] to recall the precise setup). The composition

\[
\psi_U^{-1} \circ \varphi_U^{-1} : \text{RG}_{\text{et}}(\hat{U}_C, \mathbb{Q}_p) \xrightarrow{\sim} \text{RG}_{\text{et}}(U_C, \mathbb{Q}_p)
\]

is by definition the comparison map \(\Upsilon_{U, \mathbb{Q}_p}\) that appears in Remark 0.3 (see also [KP21, Construction 4.1.14] for more details). Precomposing left vertical and upper horizontal map in (3.2) with \(\varphi_U^{-1}\) we then get the natural commutative square (3.1).

Remark 3.4. It is not hard to see from the construction and identification in Remark 2.8 that the lower horizontal map \(\text{RG}_{\text{dR}}(U_K / K) \otimes K B_{\text{dR}} \to \text{RG}_{\text{dR}}(\hat{U}_K / B_{\text{dR}})\) in diagrams (3.2) and (3.1) is exactly the map that induces an equivalence in Proposition 2.9(2) in the Hodge-proper context.

\(^4\) Consisting of a smooth adic space and a normal crossings divisor.
3.2 The map $\Upsilon_{-,Q_p}$ for $d$-Hodge-proper stacks

In this section we establish $Q_p$-local acyclicity of a Hodge-proper stack over $\mathcal{O}_K$. First we extend the square (3.1) to smooth Artin $\mathcal{O}_K$-stacks.

**Proposition 3.5.** Let $X$ be a smooth quasi-compact quasi-separated Artin stack over $\mathcal{O}_K$. Then there is a commutative square of the form

$$
\begin{array}{ccc}
R\Gamma_{\text{et}}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} & \xrightarrow{\Upsilon_{X,Q_p}} & R\Gamma_{\text{et}}(\hat{X}_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \\
\sim & & \sim \\
R\Gamma_{\text{dR}}(X_K/K) \otimes K B_{\text{dR}} & \rightarrow & R\Gamma_{\text{dR}}(\hat{X}_K/B_{\text{dR}}),
\end{array}
$$

where the left vertical map is an equivalence.

**Proof.** Consider the right Kan extension of the commutative square in (3.1) along the embedding $(\text{Aff}_{\mathcal{O}_K}^{\text{sm}})^{\text{op}} \rightarrow (\text{PSh}_{K^{\text{op}}}^{\text{op}})$. We claim that the terms appearing in this Kan extension are the ones in the diagram (3.3). For the lower right term this holds by definition (Construction 2.6). For the others note that if a functor $F: (\text{Aff}_{\mathcal{O}_K}^{\text{sm}})^{\text{op}} \rightarrow D(\text{Mod}_{\mathbb{L}})^{\geq 0}$ where $L$ is either $K$ or $Q_p$ satisfies smooth descent then so does $F \otimes_{L} B_{\text{dR}}$; indeed, by [KP21, Corollary 3.1.13] $- \otimes K B_{\text{dR}}$ preserves totalizations of 0-cocncetive objects. The functors $R\Gamma_{\text{et}}((-)_C, \mathbb{Q}_p), R\Gamma_{\text{et}}((-)_C, Q_p)$ and $R\Gamma_{\text{dR}}((-)_K/K)$ satisfy smooth descent, thus picking a hypercover $|U_*| \rightarrow X$ (as in [Pri15, Theorem 4.7]) with $U_i$ being smooth affine schemes we reduce to the case of a smooth affine scheme, which is tautological.

**Proposition 3.6.** Let $X$ be a smooth $d$-de Rham-proper stack over $\mathcal{O}_K$. Then the map $\Upsilon_{X,Q_p}$ induces an embedding

$$
H^i_{\text{et}}(X_C, \mathbb{Q}_p) \rightarrow H^i_{\text{et}}(\hat{X}_C, \mathbb{Q}_p)
$$

for $i \leq d$.

**Proof.** From the commutative diagram (3.3) and Proposition 2.9(3) it follows that in the $d$-de Rham-proper case the composition of left vertical and low horizontal maps in (3.3) is injective on cohomology in the range $i \leq d$. Thus so is the vertical map; the claim of the proposition follows.

**Example 3.7.** Recall that any $d$-Hodge-proper stack is $d$-de Rham proper, so Proposition 3.6 holds for them as well. However, there are examples (see Section 3.3) of $d$-de Rham proper stacks that are not $d$-Hodge-proper, which motivates stating Proposition 3.6 in that generality.

We get the following result for Hodge-proper stacks as an immediate consequence.

**Theorem 3.8.** Let $X$ be a smooth Hodge-proper stack over $\mathcal{O}_K$. Then the natural map

$$
\Upsilon_{X,Q_p}: R\Gamma_{\text{et}}(X_C, \mathbb{Q}_p) \rightarrow R\Gamma_{\text{et}}(\hat{X}_C, \mathbb{Q}_p)
$$

is an equivalence.

**Proof.** Moreover, since the left vertical arrow in (3.3) is an isomorphism we have $\dim_{\mathbb{Q}_p} H^i_{\text{et}}(X_C, \mathbb{Q}_p) = \dim_K H^i_{\text{dR}}(X_K/K)$. Moreover, since $X$ is Hodge-proper, we have $\dim_K H^i_{\text{dR}}(X_K/K) = \dim_{\mathbb{Q}_p} H^i_{\text{et}}(\hat{X}_C, \mathbb{Q}_p)$ by [KP21, Proposition 4.3.14]. Thus: $\dim H^i_{\text{et}}(X_C, \mathbb{Q}_p) = \dim H^i_{\text{et}}(\hat{X}_C, \mathbb{Q}_p)$ for any $i \geq 0$. On the other hand a Hodge-proper stack is $d$-de Rham-proper for any $d$, and thus by Proposition 3.6 $\Upsilon_{X,Q_p}$ is injective on cohomology. So $\Upsilon_{X,Q_p}$ induces an isomorphism on cohomology, and thus is a quasi-isomorphism.

**Corollary 3.9.** Let $X$ be a smooth Hodge-proper stack over $\mathcal{O}_K$. Then for any $i \geq 0$ the Galois representation given by $H^i(X_C, \mathbb{Q}_p)$ is crystalline and $D_{\text{cryst}}(H^i(X_C, \mathbb{Q}_p)) = H^{i}_{\text{cryst}}(X_K/W(k))[[\mathbb{Q}_p]]$. 
Proof. From Theorem 3.8 for any $i \geq 0$ we get an isomorphism $H^i(X_C, \mathcal{Q}_p) \simeq H^i(\hat{X}_C, \mathcal{Q}_p)$, which is automatically $G_K$-equivariant. By [KP21, Theorem 4.3.25], $H^i(\hat{X}_C, \mathcal{Q}_p)$ is crystalline and $D_{cryst}(H^i(\hat{X}_C, \mathcal{Q}_p)) = H^i_{cryst}(X_k/W(k))[\frac{1}{p}]$.

Remark 3.10. More generally, using [KP21, Proposition 4.3.35 and Theorem 4.3.38], from Theorem 3.8 it follows that for a Hodge-proper stack over $\mathcal{O}_K$ one has all the comparisons that one usually has in $p$-adic Hodge theory. Namely, for any $n \geq 0$ one has a $(\varphi, G_K)$-equivariant isomorphism

$$H^n_{\text{et}}(X_C, \mathcal{Q}_p) \otimes \mathcal{Q}_p B_{\text{cryst}} \simeq H^n_{\text{cryst}}(X_k/W(k))[\frac{1}{p}] \otimes_{W(k)[\frac{1}{p}]} B_{\text{cryst}},$$

a filtered $G_K$-equivariant isomorphism

$$H^n_{\text{et}}(X_C, \mathcal{Q}_p) \otimes \mathcal{Q}_p B_{\text{dR}} \xrightarrow{\sim} H^n_{\text{dR}}(X/K) \otimes _K B_{\text{dR}}$$

and the Hodge-Tate decomposition

$$H^n_{\text{et}}(X_C, \mathcal{Q}_p) \otimes \mathcal{Q}_p C \simeq \bigoplus_{i+j=n} H^i(X_K, \wedge^j L_{X/K}) \otimes_K C(-i),$$

where $C(-i)$ denotes the $-i$-th Tate twist.

Remark 3.11. Note that from the proof of Theorem 3.8 it follows that all arrows in (3.3) except the right vertical one are in fact equivalences. As a result, it is also true for the right vertical one. Using the fact that all maps in the diagram are in fact filtered and $G_K$-equivariant, one can obtain from this another proof of the de Rham comparison for $H^n_{\text{et}}(\hat{X}_C, \mathcal{Q}_p)$ (for the previous one using prismatic cohomology see [KP21, Proposition 4.3.35]). However, it is not clear to us how to see from the same diagram that this representation is in fact crystalline.

Remark 3.12. In fact one could produce another proof of Theorem 3.8 by indentifying the right vertical map in (3.3) with the comparison map from [KP21] which was constructed via prismatic cohomology. By [KP21, Theorem 4.3.25], in Hodge-proper case the latter map is an equivalence. Consequently, for Hodge-proper $X$ all arrows in (3.3) except the top horizontal one are equivalences and thus so is the latter.

3.3 Example: the complement to a complete intersection in $\mathbb{P}^n$.

Here we would to illustrate a difference between $d$-de Rham properness and $d$-Hodge-properness on a particular example.

Namely, let $Z \hookrightarrow \mathbb{P}^n_{\mathcal{O}_K}$ be a complete intersection given by $f_1 = f_2 = \ldots = f_d = 0$, where each $f_i$ is a homogeneous polynomial of degree $d_i$. We will also assume that $Z$ is flat over $\mathcal{O}_K$. Let $X := \mathbb{P}^n_{\mathcal{O}_K} \setminus Z$. By Lemma 1.4, $X$ is $(d-2)$-Hodge-proper over $\mathcal{O}_K$. We claim that

- $X$ is not $(d-1)$-Hodge-proper,
- $X$ is still $(d-1)$-de Rham proper.

Denote by $j : X \to \mathbb{P}^n_{\mathcal{O}_K}$ the natural embedding. Also, let $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^n}$ be the sheaf of ideals defining $Z$. It is generated by the images of maps $\alpha_i : \mathcal{O}_{\mathbb{P}^n}(-d_i) \to \mathcal{O}_{\mathbb{P}^n}$ given by $f_i \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i))$. Since $Z$ is a complete intersection, the sheaf $\mathcal{O}_Z := \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_Z$ has a free Koszul-type resolution, namely

$$\text{Kos}^*_p (\alpha_1, \ldots, \alpha_d) := \bigotimes_{i=1}^d (\mathcal{O}_{\mathbb{P}^n}(-d_i) \xrightarrow{\alpha_i} \mathcal{O}_{\mathbb{P}^n}) \simeq \mathcal{O}_Z,$$

where $\mathcal{O}_{\mathbb{P}^n}(-d_i) \xrightarrow{\alpha_i} \mathcal{O}_{\mathbb{P}^n}$ is considered as a two-term complex concentrated in cohomological degrees 0 and -1. For a $d$-tuple $z := (s_1, \ldots, s_d) \in \mathbb{N}^d$ denote by $Z^z \subset \mathbb{P}^n_{\mathcal{O}_K}$ the subscheme defined by $f_1^{s_1} = f_2^{s_2} = \ldots = f_d^{s_d} = 0$. The subscheme $Z^z$ is still a complete intersection, and we have

$$\text{Kos}^*_p (\alpha_1^{s_1}, \ldots, \alpha_d^{s_d}) \simeq \mathcal{O}_{Z^z}.$$
where \( \alpha_i^s: \mathcal{O}_{\mathbb{P}^n}(-s_i d_i) \rightarrow \mathcal{O}_{\mathbb{P}^n} \) is the map induced by \( f_i^s \). Sheaves \( \mathcal{O}_{\mathbb{P}^n} \), as well as the Koszul complexes\(^5\) \( \text{Kos}^s_{\mathbb{P}^n}(\alpha_1^s, \ldots, \alpha_i^s, \ldots, \alpha_d^s) \) naturally form an inverse system (via the termwise partial ordering on \( \mathbb{N}^d \)). Also note that one has the following formula for the dual complex:

\[
\text{Kos}^s_{\mathbb{P}^n}(\alpha_1^s, \ldots, \alpha_d^s)^\vee \simeq \text{Kos}^s_{\mathbb{P}^n}(\alpha_1^s, \ldots, \alpha_d^s)(s_1d_1 + \ldots + s_dd_d)[-d]
\]

(this follows from the isomorphism of complexes \((\mathcal{O}_{\mathbb{P}^n}(-s_i d_i) \rightarrow \mathcal{O}_{\mathbb{P}^n})^\vee \simeq (\mathcal{O}_{\mathbb{P}^n}(-s_i d_i) \rightarrow \mathcal{O}_{\mathbb{P}^n})(s_i d_i)[-1] \)).

One has a fiber sequence

\[
\text{R}\Gamma_{\mathcal{Z}}(\mathcal{O}_{\mathbb{P}^n}) \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow Rj_* \mathcal{O}_X,
\]

where \( \text{R}\Gamma_{\mathcal{Z}}(\mathcal{O}_{\mathbb{P}^n}) \) is the (sheaf of) local cohomology. We get that \( H^i(X, \mathcal{O}_X) \) is finitely-generated if and only if \( H^{i-1}(\mathbb{P}^n, \text{R}\Gamma_{\mathcal{Z}}(\mathcal{O}_{\mathbb{P}^n})) \) is. Thus to show that \( X \) is not \((d-1)\)-Hodge-proper it is enough to show that \( H^d(\mathbb{P}^n, \text{R}\Gamma_{\mathcal{Z}}(\mathcal{O}_{\mathbb{P}^n})) \) is not finitely generated.

The local cohomology \( \text{R}\Gamma_{\mathcal{Z}}(\mathcal{O}_{\mathbb{P}^n}) \) is given explicitly by

\[
\text{R}\Gamma_{\mathcal{Z}}(\mathcal{O}_{\mathbb{P}^n}) \simeq \text{colim}_{s \in \mathbb{N}^d} \text{RHom}_{\mathcal{O}_\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n})
\]

(see e.g. [SP20, Tag 0956]). Naturally, one can use Koszul resolutions to compute \( \text{RHom} \): one sees that

\[
\text{RHom}_{\mathcal{O}_\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) \simeq \text{Kos}^s_{\mathbb{P}^n}(\alpha_1^s, \ldots, \alpha_d^s)^\vee \simeq \mathcal{O}_{\mathbb{P}^n}(s_1d_1 + \ldots + s_dd_d)[-d].
\]

This way

\[
\text{R}\Gamma_{\mathcal{Z}}(\mathcal{O}_{\mathbb{P}^n}) \simeq \text{colim}_{s \in \mathbb{N}^d} \text{RHom}_{\mathcal{O}_\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(s_1d_1 + \ldots + s_dd_d)[-d].
\]

In particular, \( \tau < d \text{R}\Gamma_{\mathcal{Z}}(\mathcal{O}_{\mathbb{P}^n}) \simeq 0 \)

Note that \( \mathcal{O}_{\mathbb{P}^n} \) has a finite filtration\(^6\) by a poset \( P^s \subseteq \{ t \in \mathbb{N}_{>0}^d \ | \ t \prec s \} \) with the associated graded given by \( \oplus_{t \in P^s} \mathcal{O}_{\mathbb{P}^n}(-t_1 d_1 - \ldots - t_d d_d) \). For the twist \( \mathcal{O}_{\mathbb{P}^n}(s_1d_1 + \ldots + s_dd_d) \) this gives a filtration with the associated graded \( \oplus_{t \in P^s} \mathcal{O}_{\mathbb{P}^n}((t_1+1)d_1 + \ldots + (t_d+1)d_d) \).

Moreover, for \( s' > s \) the corresponding map on the associated graded

\[
\oplus_{t \in P^s} \mathcal{O}_{\mathbb{P}^n}((t_1+1)d_1 + \ldots + (t_d+1)d_d) \rightarrow \oplus_{t \in P^{s'}} \mathcal{O}_{\mathbb{P}^n}((t_1+1)d_1 + \ldots + (t_d+1)d_d)
\]

is the embedding induced by \( P^s \rightarrow P^{s'} \). This way one sees that for the cokernel \( K_{s' s} \) of the above map has an associated graded given by direct sum of line bundles \( \mathcal{O}_{\mathbb{P}^n}(\ell) \) with \( \ell > \min_i (s_i d_i) \); moreover, as \( s' \rightarrow \infty \) the number of these line bundles grows to \( \infty \) as well. Thus, for a fixed \( s \gg 0 \) and \( s' \rightarrow \infty \) we have \( \text{rk}_K H^0(\mathbb{P}^n, K_{s' s}) \rightarrow \infty \). Using the exact sequence

\[
0 \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(\sum_i s_i d_i)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(\sum_i s_i' d_i)) \rightarrow H^0(K_{s' s}) \rightarrow H^1(\mathcal{O}_{\mathbb{P}^n}(\sum_i s_i d_i))
\]

one sees that

\[
H^d(\text{R}\Gamma_{\mathcal{Z}}(\mathcal{O}_{\mathbb{P}^n})) \simeq H^0(\text{colim}_{s \in \mathbb{N}^d} \mathcal{O}_{\mathbb{P}^n}(\sum_i s_i d_i)) \simeq \text{colim}_{s \in \mathbb{N}^d} H^0(\mathcal{O}_{\mathbb{P}^n}(\sum_i s_i d_i))
\]

has infinite rank over \( \mathcal{O}_K \), and so is not finitely generated.

We now consider the de Rham cohomology of \( X \). We need to show that \( H_{dR}^{d-1}(\mathcal{X}/\mathcal{O}_K) \) is finitely generated over \( \mathcal{O}_K \). For any \( k \geq 0 \) by projection formula we have

\[
Rj_* \Omega^k_X \simeq Rj_* j^* \Omega^k_{\mathbb{P}^n} \simeq Rj_* \mathcal{O}_X \otimes_{\mathcal{O}_\mathbb{P}^n} \Omega^k_{\mathbb{P}^n}.
\]

\(^5\)Here, the map \( \text{Kos}^s_{\mathbb{P}^n}(\alpha_1^s, \ldots, \alpha_i^s, \ldots, \alpha_d^s) \rightarrow \text{Kos}^s_{\mathbb{P}^n}(\alpha_1^s, \ldots, \alpha_i^{s_i-1}, \ldots, \alpha_d^s) \) is induced by the commutative square

\[
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^n}(-s_i d_i) & \xrightarrow{\alpha_i^{s_i}} & \mathcal{O}_{\mathbb{P}^n} \\
\alpha_i & & \downarrow{\text{id}} \\
\mathcal{O}_{\mathbb{P}^n}(-(s_i-1)d_i) & \xrightarrow{\alpha_i^{s_i-1}} & \mathcal{O}_{\mathbb{P}^n}
\end{array}
\]

with left vertical arrow induced by \( f_i \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d_i)) \), and identity on the other components of the tensor product.

\(^6\)The subsheaf corresponding to a given \( \ell \) is defined by the ideal spanned by \( f_i^{s_{i-1}} \cdot \cdot \cdot f_d^{s_d} \) with \( \ell \leq \ell \leq s \).
Lemma 4.1. Let $\tau < d \mathcal{R}_d^*(\mathcal{O}_\mathbb{P}^n) = 0$ we have that $\tau < d \mathcal{R}_d^*(\mathcal{O}_\mathbb{P}^n) \simeq \mathcal{O}_{\mathbb{P}^n}$ and, consequently $\tau < d \mathcal{R}_d^*(\mathcal{O}_\mathbb{P}^n) \simeq \mathcal{O}_{\mathbb{P}^n}$. In particular, for $i > 0$ and $j < d - 1$ we get that $H^{i,j}(X/\mathcal{O}_K) \simeq H^{i,j}(\mathbb{P}^n/\mathcal{O}_K)$. This way, on the first page of the Hodge-to-de Rham spectral sequence there is a single Hodge cohomology of total degree $d - 1$ that is infinitely-generated, namely $H^{d-1}(X/\mathcal{O}_K) := H^d(X, \mathcal{O}_X)$. Let $E_\infty^{i,d-1} \subset H^{i,d-1}(X/\mathcal{O}_K)$ be the corresponding term on the infinite page; it would be enough to show that $E_\infty^{i,d-1}$ is finitely generated. However, we identified $H^{0,d-1}(X/\mathcal{O}_K)$ with $H^d(R\mathcal{R}_d^*(\mathcal{O}_\mathbb{P}^n)) \simeq \text{colim}_{s \in \mathbb{N}} H^d(\mathcal{O}_{\mathbb{P}^n}(\sum_i s_i^d))$, so it is $p$-torsion free since we assumed $Z$ to be flat over $\mathcal{O}_K$. Consequently so is $E_\infty^{i,d-1}$ and thus it will be enough to show that it is finitely generated after inverting $p$. Note that the $p$-localization of the Hodge-to-de Rham spectral sequence for $X$ exactly gives the one for $X_K$. Since over a field of characteristic $0$ via excision we have $H^3_{\text{dR}}(X_K/K) \simeq H^i_{\text{dR}}(\mathbb{P}^n/K)$ for $i < 2d$, we get that $H^{d-1}_{\text{dR}}(X_K/K)$, and $E_\infty^{i,d-1}[\frac{1}{p}]$ in particular, are finite-dimensional.

Remark 3.13. In fact, in the proof of $(d-1)$-de Rham properness we never used that the ambient scheme is $\mathbb{P}^n$, only that it is smooth and proper. Indeed, given a codimension $d$ locally complete intersection $Z$ in an $\mathcal{O}_K$-scheme $X$ we have $\tau < d \mathcal{R}_d^*(\mathcal{O}_\mathbb{P}^n) \simeq 0$. Moreover, if $Z$ is flat over $\mathcal{O}_K$ then so is the $p$-th local cohomology sheaf $H^2_z(\mathcal{O}_X)$. Then, if $X$ is smooth and proper we already know that $X := X \setminus Z$ is $(d - 2)$-Hodge-proper, so it remains to show that $H^{d-1}_{\text{dR}}(X/\mathcal{O}_K)$ is finitely generated. Here, using the same argument as above, we can reduce to $H^{d-1}_{\text{dR}}(X_K/K)$, which is finitely-generated since $H^i_{\text{dR}}(X_K/K) \simeq H^i_{\text{dR}}(\mathcal{O}_{\mathbb{P}^n}/K)$ for $i < 2d$.

Remark 3.14. One can also see that the bound on de Rham-properness is sharp; namely in general (e.g. when $Z$ is a point) $X$ as above will not be $d$-de Rham proper.

4 A variant of $p$-adic Hodge theory for $d$-Hodge-proper stacks

4.1 Coherence properties of various cohomology

Let $R$ be a (classical) Noetherian ring, and let $I = (x_1, \ldots, x_n) \subset R$ be an ideal generated by a regular sequence (so that $R/I \simeq \otimes_{i=1}^n \text{cobar}(R \xrightarrow{z_i} R)$). Assume also that $R$ is $I$-adically complete (in the classical sense).

Lemma 4.1. Let $M \in D(\text{Mod}_R)$ be derived $I$-complete and assume that $\tau < d(M \otimes_R R/I) \in \text{Coh}(R/I)$. Then $\tau \mathcal{T}^d M \in \text{Coh}(R)$.

Proof. By induction on the number of generators we can assume that $I$ is principal and generated by a non-zero divisor $x \in R$. Note that $\tau \mathcal{T}^d M$ is also derived $I$-complete; thus, e.g. by [KP21, Proposition 1.1.7], it is enough to show that $\tau \mathcal{T}^d M \otimes_R R/I \in \text{Coh}(R)$. Since $\tau \mathcal{T}^d M \otimes_R R/I \simeq \tau \mathcal{T}^{d-1}(\tau \mathcal{T}^d M \otimes_R R/I)$, which is coherent, this reduces to showing that $H^d(\tau \mathcal{T}^d M \otimes_R R/I) \simeq H^d(M/I)$ is a finitely generated $R/I$-module. But by the universal coefficients formula $H^d(M/I)$ embeds into $H^d(M \otimes_R R/I)$ which is finitely generated by the assumption.

Here comes the first application.

Corollary 4.2. Let $X$ be a $d$-de Rham proper stack over a perfect field $k$ of characteristic $p$. Then $\tau \mathcal{T}^d(R\mathcal{E}_{\text{crys}}(X/W(k))) \in \text{Coh}(W(k))$.

Proof. By de Rham and crystalline comparisons [KP21, Propositions 2.3.20 and 2.5.7], we have

$$R\mathcal{E}_{\text{crys}}(X/W(k)) \otimes_{W(k)} k \simeq R\mathcal{E}_{\text{dR}}(X/k).$$

The rest then follows from Lemma 4.1, since $R\mathcal{E}_{\text{crys}}(X/W(k))$ is derived $p$-complete.

Remark 4.3. We note that $M := \tau \mathcal{T}^{d+1} R\mathcal{E}_{\text{crys}}(X/W(k))[d + 1]$ also satisfies the following finiteness condition:

$$\tau < 0([M/p]) \in \text{Coh}(k).$$
More generally, given a stack $X$ over $\mathcal{O}_K$ one can establish similar properties for the prismatic cohomology (over the Breuil-Kisin prism). Here, let $K$ be a discretely valued complete field extension of $\mathbb{Q}_p$ with a perfect residue field $k$. For a choice of uniformizer $\pi \in \mathcal{O}_K$ we have the corresponding Breuil-Kisin prism $(\mathfrak{S}, E(u)) = ((W(k))[u], E(u))$ with $\mathfrak{S}/E(u) = \mathcal{O}_K$. Recall ([KP21, Definition 2.2.1]) that for a prestack $X$ over $\mathcal{O}_K$ we have the prismatic cohomology complex $R\Gamma_{\Delta}(X/\mathfrak{S}) \in D(\text{Mod}_{\mathfrak{S}})$, as well as its twisted version $R\Gamma_{\Delta}(X/\mathfrak{S}) \in D(\text{Mod}_{\mathfrak{S}})$, which, in the case of the Breuil-Kisin prism is also identified with the “Frobenius twist” $\varphi^*\mathfrak{S}R\Gamma_{\Delta}(X/\mathfrak{S})$. The complexes $R\Gamma_{\Delta}(X/\mathfrak{S})$, $R\Gamma_{\Delta}(X/\mathfrak{S})$ are derived $(p, u)$-adically complete and one has a natural “Frobenius map”

$$\varphi_{\Delta}: R\Gamma_{\Delta}(X/\mathfrak{S}) \longrightarrow R\Gamma_{\Delta}(X/\mathfrak{S}),$$

which becomes an equivalence after inverting $E(u)$ in the case $X$ is a smooth Artin stack ([KP21, Remark 2.2.16]).

**Corollary 4.4.** Let $X$ be a smooth Artin stack over $\mathcal{O}_K$. If the reduction $X_k$ is $d$-de Rham proper over $k$, then both $\tau^{\leq d}R\Gamma_{\Delta}(X/\mathfrak{S}) \in \text{Coh}(\mathfrak{S})$ and $\tau^{\leq d}R\Gamma_{\Delta}(X/\mathfrak{S}) \in \text{Coh}(\mathfrak{S})$.

**Proof.** The map $\varphi_{\mathfrak{S}}: \mathfrak{S} \to \mathfrak{S}$ is faithfully flat, so by Lemma 1.10 it is enough to show that $\tau^{\leq d}R\Gamma_{\Delta}(X/\mathfrak{S}) \in \text{Coh}(\mathfrak{S})$. The claim then follows from crystalline comparison $R\Gamma_{\Delta}(X/\mathfrak{S}) \otimes_{\mathfrak{S}/u} \mathfrak{S}/u \simeq R\text{G}_{\text{cris}}(X_k/\mathfrak{S}(k))$ ([KP21, Remark 2.5.10]) and Corollary 4.2 above.

**Remark 4.5.** By Corollary 4.4, we get that if the reduction $X_k$ is $d$-de Rham proper, the $\mathfrak{S}$-modules $H^i_{\Delta}(X/\mathfrak{S})$ are finitely generated if $i \leq d$. The isomorphisms

$$\varphi_{\Delta}\left[\frac{1}{p}\right]: \varphi_{\mathfrak{S}}H^i_{\Delta}(X/\mathfrak{S})[\frac{1}{p}] \rightarrow H^i_{\Delta}(X/\mathfrak{S})[\frac{1}{p}]$$

then endow each of them with a Breuil-Kisin module structure. Just given the existence of this structure, we get that $H^i_{\Delta}(X/\mathfrak{S})[\frac{1}{p}]$ is a finite free $\mathfrak{S}[\frac{1}{p}]$-module for $i \leq d$ (see e.g. [BMS18, Proposition 4.3]). In fact, we will see that (under some torsion-freeness assumptions) the corresponding lattice in a crystalline Galois representation is described by étale cohomology of the Raynaud generic fiber (see Remark 4.25).

**Remark 4.6.** The $d$-de Rham properness of $X_k$ also implies some important properties of prismatic cohomology in degrees higher than $d$, that will turn out to be rather crucial later. Namely, even though $M := \tau^{\leq d+1}R\Gamma_{\Delta}(X/\mathfrak{S})[d + 1]$ is not an object of $\text{Coh}(\mathfrak{S})$, it is still derived $(p, u)$-complete and satisfies the following finiteness condition: namely, $\tau^{<0}[M/(p, u)] \in \text{Coh}(k)$. Indeed, let $C := R\Gamma_{\Delta}(X/\mathfrak{S})[d + 1]$: we have $M \simeq \tau^{<0}(C)$. Note that by our assumptions on $X_k$ and de Rham comparison we have $\tau^{<0}(C/(p, u)) \in \text{Coh}(k)$. We have a fiber sequence $\tau^{<0}C \to C \to M$ which gives a fiber sequence

$$[\tau^{<0}C/(p, u)] \to [C/(p, u)] \to [M/(p, u)]$$

We have that $\tau^{<0}C$ is coherent by Corollary 4.4, and so it follows that $\tau^{<0}([M/(p, u)]) \in \text{Coh}(k)$.

Consequently, applying $[-/(p, u)]$ to the fiber sequence $H^0(M)[0] \to M \to \tau^{<0}M$ and considering the corresponding long exact sequence of cohomology we get that $H^{-2}(H^0(M)/(p, u)) \simeq H^{-2}([M/(p, u)])$ while there is also an embedding $H^{-1}(H^0(M)/(p, u)) \hookrightarrow H^{-1}([M/(p, u)])$, and so both are finite-dimensional vector spaces over $k$. Recalling the definition of $M$ we get from this that the $(p, u)$-torsion $H^{d+1}_{\Delta}(X/\mathfrak{S})[p][u]$ is a finite dimensional $k$-vector space.

We record a part of the discussion in Remark 4.6 as Lemma 4.7 below for a future reference.

**Lemma 4.7.** Let $X$ be a smooth Artin stack over $\mathcal{O}_K$ such that the reduction $X_k$ is $d$-de Rham proper over $k$. Then $M := \tau^{\leq d+1}R\Gamma_{\Delta}(X/\mathfrak{S})[d + 1]$ satisfies

$$\tau^{<0}([M/(p, u)]) \in \text{Coh}(k).$$

Let $R$ be a $p$-bounded $p$-complete ring. The following notation will be convenient:
Definition 4.8. We define the $p$-completed Hodge cohomology $RH^i(\hat{X}/R)$ of an Artin $R$-stack $X$ as the derived $p$-completion $\Gamma_R^i(X/R)^\wedge_p$. We also denote

$$H^{j,i}(\hat{X}/R) := H^i(R\Gamma(\hat{X}, \wedge^j\Omega_{X/R})^\wedge).$$

Similarly, if $X$ is smooth over $R$ we define the $p$-completed de Rham cohomology

$$R\Gamma_{dR}(\hat{X}/R) := R\Gamma_{dR}(X/R)^\wedge.$$

Naturally, we have coherence results for $R\Gamma_H(\hat{X}/R)$ and $R\Gamma_{dR}(\hat{X}/R)$ implied by $d$-Hodge and $d$-de Rham-properness of the $mod \ p$ reduction $X_{R/p}$ of $X$.

Corollary 4.9. Let $R$ be a Noetherian $p$-torsion free $p$-complete ring and let $X$ be a smooth qcqs Artin stack over $R$.

1. If the reduction $X_{R/p}$ is $d$-Hodge-proper over $R/p$ we have
   $$\tau^{\leq d}R\Gamma_H(\hat{X}/R) \in \text{Coh}(R).$$
   Equivalently, $H^{j,i}(\hat{X}/R)$ is a finitely generated $R$-module for any $i + j \leq d$.

2. If the reduction $X_{R/p}$ is $d$-de Rham-proper over $R/p$ we have
   $$\tau^{\leq d}R\Gamma_{dR}(\hat{X}/R) \in \text{Coh}(R).$$

Proof. Since $p$ is a non-zero divisor in $R$, we have $R/p \simeq \text{cofib}(R \xrightarrow{p} R)$. Thus, for any $M \in D(\text{Mod}_R)$ and the definition of derived $p$ completion we have $M_R^\wedge \otimes_R R/p \simeq M \otimes_R R/p$. Also, $R/p$ is of Tor-amplitude $[-1,0]$ as an $R$-module. This, together with base change for Hodge and de Rham cohomology ([KP19, Proposition 1.1.8]), leads to equivalences

$$R\Gamma_H(\hat{X}/R) \otimes_R R/p \xrightarrow{\sim} R\Gamma_H(X_{R/p}/(R/p)) \quad R\Gamma_{dR}(\hat{X}/R) \otimes_R R/p \xrightarrow{\sim} R\Gamma_{dR}(X_{R/p}/(R/p)).$$

We then are done by Lemma 4.1. \qed

We also record a statement analogous to Corollary 4.4 but which applies to more general base prisms. We refer to [KP21, Definition 2.2.1] for the definitions of prismatic cohomology $R\Gamma_{\Delta}(X/A) \in D(\text{Mod}_A)$, as well as its twisted version $R\Gamma_{\Delta^{(1)}}(X/A) \in D(\text{Mod}_A)$.

Corollary 4.10. Let $(A, I)$ be a transversal prism such that the underlying ring $A$ is Noetherian. Let $X$ be a qcqs smooth Artin stack over $A/I$ such that the reduction $X_{A/(I,p)}$ is $d$-de Rham-proper over $A/(I,p)$.

Then

$$\tau^{\leq d}R\Gamma_{\Delta^{(1)}}(X/A) \in \text{Coh}(A).$$

If the reduction $X_{A/(I,p)}$ is $d$-Hodge-proper over $A/(I,p)$, then

$$\tau^{\leq d}R\Gamma_{\Delta}(X/A) \in \text{Coh}(A).$$

Proof. Note that we have isomorphisms

$$R\Gamma_{\Delta^{(1)}}(X/A) \otimes_A A/I \xrightarrow{\sim} R\Gamma_{dR}(X/(A/I)) \xrightarrow{\sim} R\Gamma_{dR}(\hat{X}/(A/I))^\wedge_p.$$

Here the first arrow is given by the de Rham comparison ([KP21, Proposition 2.3.20]), while the second equivalence holds for any smooth Artin $X$ (see [KP21, Proposition 2.3.18]). Since $(A, I)$ is transversal we can apply Corollary 4.9(2), getting that $\tau^{\leq d}R\Gamma_{dR}(\hat{X}/(A/I))$. The first part of the proposition then follows from Lemma 4.1.

\footnote{Recall that by definition this means that $A/I$ is $p$-torsion free.}
For the second part recall that the Hodge-Tate complex \( R\Gamma_{\Delta/I}(X/A) := R\Gamma(X/A) \otimes^L_A A/I \) has a natural conjugate filtration with the associated graded \( gr_t = R\Gamma(X, \wedge^t \mathcal{L}_X(A/I)) \) (see [KP21, Section 2.4]). Since \( X \) is smooth this filtration is exhaustive [KP21, Proposition 2.4.3] and induces a convergent spectral sequence \( E^{i,j} = H^j(X(A/I)) \Rightarrow H^{i+j}_{\Delta/I}(X/A) \) (see [KP21, Section 4.3.3]). Recall that the twist \( \{-i\} \) here is just tensoring with the invertible \( A/I \)-module \((I/I^2)^{\otimes i}\) and so doesn’t change coherence. It is then enough to show that \( H^{i,j}(X/(A/I)) \) is finitely generated for \( i + j \leq d \), which follows from Corollary 4.9(1).

4.2 Some integral \( p \)-adic Hodge theory in \( d \)-de Rham-proper setting

In this section we adapt some results of [KP21, Sections 2 and 4] about prismatic cohomology of Hodge-proper stacks to the setting of stacks that are only \( d \)-de Rham proper.

In this section we will mostly assume that the base prism \((A, I) = (\mathcal{O}, E(u))\) is a Breuil-Kisin prism.

Let \( \pi \in \mathcal{O}_K \) be the uniformizer to which the chosen Breuil-Kisin prism \((\mathcal{O}, E(u))\) corresponds (so \( \pi \) is the image of \( u \) under the identification \( \mathcal{O}/E(u) \isom \mathcal{O}_K \)). A compatible choice of \( p^n \)-roots of \( \pi \) in \( \mathcal{O}_C \) gives an element \( \pi^\flat := (\pi, \pi^{1/p}, \pi^{1/p^2}, \ldots) \in \mathcal{O}_C^\flat \) in the tilt \( \mathcal{O}_C^\flat := \lim_{\rightarrow, x \in \mathbb{P}} \mathcal{O}_C/p^x \). This then defines a homomorphism \( s_{\pi^\flat} : \mathcal{O} \to A_{\text{inf}} := W(\mathcal{O}_C^\flat) \) by sending \( u \mapsto [\pi^x] \). We can also compose it further with the natural map \( A_{\text{inf}} \to W(C^p) \) induced by \( \mathcal{O}_C^\flat \to C^p := \mathcal{O}_C^\flat[1/p] \). Recall that by the étale comparison ([KP21, Corollary 4.3.3]) and base change we have an equivalence

\[
R\Gamma_{\text{ét}}(\hat{X}_C, \mathbb{Z}_p) \isom \left( R\Gamma_{\Delta}^1(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^p) \right)^{\varphi_{\Delta} = 1} \tag{4.1}
\]

for any smooth qcqs Artin \( \mathcal{O}_K \)-stack \( X \).

Remark 4.11. We use the twisted version of the étale comparison here for a future convenience. The difference only affects the natural map

\[
R\Gamma_{\text{ét}}(\hat{X}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(C^p) \longrightarrow R\Gamma_{\Delta}^1(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^p),
\]

namely it becomes twisted by Frobenius on \( W(C^p) \) (see [KP21, Remark 4.3.7]; to apply the remark, note that by base change [KP21, Proposition 2.2.13] \( R\Gamma_{\Delta}^1(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^p) \isom R\Gamma_{\Delta}^1(X_{\text{ét}}/A_{\text{inf}}) \otimes_{A_{\text{inf}}} W(C^p) ) \).

Proposition 4.12. Let \( X \) be a smooth qcqs Artin stack over \( \mathcal{O}_K \) such that the reduction \( X_k \) is \( d \)-de Rham-proper. Then

1. There is a natural equivalence

\[
\tau^{\leq d} R\Gamma_{\Delta}^1(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^p) \isom \tau^{\leq d} (R\Gamma_{\Delta}^1(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^p)).
\]

2. The natural map

\[
\tau^{\leq d} R\Gamma_{\text{ét}}(\hat{X}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(C^p) \longrightarrow \tau^{\leq d} (R\Gamma_{\Delta}^1(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^p))
\]

is also an equivalence.

3. Consequently, one has isomorphisms

\[
H^i_{\Delta}(\hat{X}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(C^p) \isom H^i_{\Delta}^1(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^p)
\]

for all \( i \leq d \).
Proof. 1. First let us show that the natural map $\tau^{\leq d} R\Gamma_{\Delta^{(1)}}(X/\mathcal{O}) \to R\Gamma_{\Delta^{(1)}}(X/\mathcal{O})$ induces an equivalence

$$
\tau^{\leq d} R\Gamma_{\Delta^{(1)}}(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^0) \xrightarrow{\sim} \tau^{\leq d} (R\Gamma_{\Delta^{(1)}}(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^0)).
$$

Denote $M := R\Gamma_{\Delta^{(1)}}(X/\mathcal{O})$. Note that the $p$-completed tensor product functor $- \otimes_{\mathcal{O}} W(C^0)$ is $t$-exact up to a right shift by 1 since $W(C^0)$ is flat over $\mathcal{O}$. We have a fiber sequence

$$
\tau^{\leq d} M \otimes_{\mathcal{O}} W(C^0) \longrightarrow M \otimes_{\mathcal{O}} W(C^0) \longrightarrow \tau^{>d} M \otimes_{\mathcal{O}} W(C^0)
$$

and so to get the claim it is just enough to show that $H^d(\tau^{>d} M \otimes_{\mathcal{O}} W(C^0))$ is 0. Note that $H^d(\tau^{>d} M \otimes_{\mathcal{O}} W(C^0)) \cong H^d(H^{d+1}(M)[-d-1] \otimes_{\mathcal{O}} W(C^0)) \cong H^{-1}(H^{d+1}(M) \otimes_{\mathcal{O}} W(C^0))$.

By Remark 4.6 $H^{d+1}(M)$ satisfies the conditions of Corollary A.11, and so the latter group is 0. Since $\tau^{\leq d} R\Gamma_{\Delta^{(1)}}(X/\mathcal{O}) \notin \text{Coh}(\mathcal{O})$ by Corollary 4.4, the tensor product $\tau^{\leq d} R\Gamma_{\Delta^{(1)}}(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^0)$ is already $p$-complete. We get a natural equivalence

$$
\tau^{\leq d} R\Gamma_{\Delta^{(1)}}(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^0) \xrightarrow{\sim} \tau^{\leq d} (R\Gamma_{\Delta^{(1)}}(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^0)),
$$

as desired.

2. We keep the notation $M := R\Gamma_{\Delta^{(1)}}(X/\mathcal{O})$. We have a fiber sequence

$$(\tau^{\leq d} M \otimes_{\mathcal{O}} W(C^0))^{\varphi_{\Delta} = 1} \longrightarrow (M \otimes_{\mathcal{O}} W(C^0))^{\varphi_{\Delta} = 1} \longrightarrow (\tau^{>d} M \otimes_{\mathcal{O}} W(C^0))^{\varphi_{\Delta} = 1}.$$ 

As we saw in the proof of 1, $\tau^{>d} M \otimes_{\mathcal{O}} W(C^0) \cong \tau^{>d} (M \otimes_{\mathcal{O}} W(C^0))$ is $d$-coconnected, and thus so is $(\tau^{>d} M \otimes_{\mathcal{O}} W(C^0))^{\varphi_{\Delta} = 1}$. Moreover, $\tau^{\leq d} M \otimes_{\mathcal{O}} W(C^0) \cong \tau^{d} M \otimes_{\mathcal{O}} W(C^0)$ is coherent and so $(\tau^{\leq d} M \otimes_{\mathcal{O}} W(C^0))^{\varphi_{\Delta} = 1}$ lies in $D(\text{Mod}_{\mathcal{O}})^{\leq d}$. Together, this shows that

$$
\tau^{\leq d} (M \otimes_{\mathcal{O}} W(C^0))^{\varphi_{\Delta} = 1} \xrightarrow{\sim} (\tau^{\leq d} M \otimes_{\mathcal{O}} W(C^0))^{\varphi_{\Delta} = 1}.
$$

Moreover, by [Bha18, Lemma 8.5] (again crucially using that $\tau^{\leq d} M \otimes_{\mathcal{O}} W(C^0)$ is coherent) we get that the natural map

$$
\tau^{\leq d} (M \otimes_{\mathcal{O}} W(C^0))^{\varphi_{\Delta} = 1} \otimes_{\mathcal{O}} W(C^0) \longrightarrow \tau^{\leq d} M \otimes_{\mathcal{O}} W(C^0)
$$

is an equivalence. It remains to note that by (4.1) there is a natural equivalence

$$
\tau^{\leq d} R\Gamma_{\mathcal{O}}(\mathcal{X}_C, \mathcal{Z}_p) \xrightarrow{\sim} \tau^{\leq d} (M \otimes_{\mathcal{O}} W(C^0))^{\varphi_{\Delta} = 1}.
$$

3. From 1 and 2 we get an equivalence

$$
\tau^{\leq d} R\Gamma_{\mathcal{O}}(\mathcal{X}_C, \mathcal{Z}_p) \otimes_{\mathcal{O}} W(C^0) \xrightarrow{\sim} \tau^{\leq d} R\Gamma_{\Delta^{(1)}}(X/\mathcal{O}) \otimes_{\mathcal{O}} W(C^0).
$$

The statement of 3 then follows by passing to cohomology.

We will also need the following lemma which controls the (classical) base change of prismatic cohomology to $A_{\inf}$.

**Lemma 4.13.** Let $X$ be a stack over $\mathcal{O}_K$ such that its reduction $X_k$ is $d$-de Rham proper. Then we have a natural isomorphism

$$
\tau^{\leq d} R\Gamma_{\Delta^{(1)}}(X/\mathcal{O}) \otimes_{\mathcal{O}} A_{\inf} \cong \tau^{\leq d} R\Gamma_{\Delta^{(1)}}(X_{\mathcal{O}_C}/A_{\inf}).
$$

Indeed, we need to show that $\varphi_{\Delta} - 1$ is surjective on $H^d(M) \otimes_{\mathcal{O}} W(C^0)$. This is enough to show modulo $p$, where $H^d(M) \otimes_{\mathcal{O}} C^0$ is a finite-dimensional vector space over $C^0$ and the statement is standard (e.g. see [Cha98, Expose III, Lemma 3.3]).
Proof. By base change for prismatic cohomology ([KP21, Proposition 2.2.17]) we have
\[ R\Gamma_{\Delta (1)}(X_{O_C}/A_{\inf}) \simeq R\Gamma_{\Delta (1)}(X/\mathcal{S}) \otimes_{\mathcal{S}} A_{\inf}. \]
Note that \( A_{\inf} \) is a \((p, u)\)-completely free module over \( \mathcal{S} \). Indeed, picking a basis \( \{x_s\}_{s \in S} \) of \( \mathcal{O}_C/\pi^b \) over \( k \) and lifts \( \tilde{x}_i \in A_{\inf} \) (under the projection \( A_{\inf} \to \mathcal{O}_C/\pi^b \)) we get a map
\[
\bigoplus_{s \in S} \mathcal{S} \longrightarrow A_{\inf}
\]
which is an isomorphism modulo \((p, u)\), and thus is itself an isomorphism by Nakayama lemma. Moreover, by Remark 4.6 we have that \( M := \tau^{\leq d+1} R\Gamma_{\Delta (1)}(X/\mathcal{S})[d + 1] \) satisfies the condition of Proposition A.13, and thus \( M \otimes_{\mathcal{S}} A_{\inf} \in D(\text{Mod} A_{\inf}) \simeq d \). Since the completed tensor product is right \( t \)-exact we get from this that
\[
\tau^{\leq d} R\Gamma_{\Delta (1)}(X_{O_C}/A_{\inf}) \simeq \tau^{\leq d} (R\Gamma_{\Delta (1)}(X/\mathcal{S}) \otimes_{\mathcal{S}} A_{\inf}) \simeq \tau^{\leq d} (R\Gamma_{\Delta (1)}(X/\mathcal{S})) \otimes_{\mathcal{S}} A_{\inf}.
\]
By the assumption on \( X_k \) we have \( \tau^{\leq d} (R\Gamma_{\Delta (1)}(X/\mathcal{S})) \in \text{Coh}(\mathcal{S}) \), and so the classical tensor product \( \tau^{\leq d} (R\Gamma_{\Delta (1)}(X/\mathcal{S})) \otimes_{\mathcal{S}} A_{\inf} \) is already derived \((p, u)\)-complete. This way we get an isomorphism
\[
\tau^{\leq d} (R\Gamma_{\Delta (1)}(X/\mathcal{S}))(1) \otimes_{\mathcal{S}} A_{\inf} \longrightarrow \tau^{\leq d} R\Gamma_{\Delta (1)}(X_{O_C}/A_{\inf})
\]
as desired. \( \square \)

**Corollary 4.14.** Let \( X \) be a stack over \( O_K \) such that its reduction \( X_k \) is \( d \)-de Rham proper. Then for any \( 0 \leq i \leq d \) we have a \( G_K \)-equivariant isomorphism
\[
H^i_{\text{et}}(\hat{X}_C, Z_p) \otimes_{Z_p} A_{\inf}[\frac{1}{\mu}] \longrightarrow H^i_{\Delta (1)}(X_{O_C}/A_{\inf}) \otimes_{A_{\inf}} A_{\inf}[\frac{1}{\mu}]
\]
that extends to a \((G_K, \varphi)\)-equivariant isomorphism
\[
H^i_{\text{et}}(\hat{X}_C, Z_p) \otimes_{Z_p} B_{\text{crys}} \longrightarrow H^i_{\Delta (1)}(X_{O_C}/A_{\inf}) \otimes_{A_{\inf}} B_{\text{crys}}.
\]

**Proof.** Recall that the map \( \mathcal{S} \to A_{\inf} \) is flat ([BMS18, Lemma 4.30]). Thus from Lemma 4.13 we get that for any \( 0 \leq i \leq d \) there is an isomorphism \( H^i_{\Delta (1)}(X_{O_C}/A_{\inf}) \simeq H^i_{\Delta (1)}(X/\mathcal{S}) \otimes_{\mathcal{S}} A_{\inf} \). Applying Proposition 4.12(3) we then get a \( \varphi \)-equivariant isomorphism
\[
H^i_{\text{et}}(\hat{X}_C, Z_p) \otimes_{Z_p} W(C^\phi) \longrightarrow H^i_{\Delta (1)}(X_{O_C}/A_{\inf}) \otimes_{A_{\inf}} W(C^\phi), \tag{4.2}
\]
which is in fact induced by the étale comparison map
\[
R\Gamma_{\text{et}}(\hat{X}_C, Z_p) \longrightarrow R\Gamma_{\Delta (1)}(X_{O_C}/A_{\inf}) \otimes_{A_{\inf}} W(C^\phi)
\]
and thus is \( G_K \)-equivariant.

By Remark 4.5, \( H^i_{\Delta (1)}(X/\mathcal{S}) \) is a Breuil-Kisin module in the range \( 0 \leq i \leq d \) and this way \( H^i_{\Delta (1)}(X_{O_C}/A_{\inf}) \simeq H^i_{\Delta (1)}(X/\mathcal{S}) \otimes_{\mathcal{S}} A_{\inf} \) is a Breuil-Kisin-Fargues module. Then by [BMS18, Lemma 4.26] the isomorphism in (4.2) restricts to an isomorphism
\[
H^i_{\text{et}}(\hat{X}_C, Z_p) \otimes_{Z_p} A_{\inf}[\frac{1}{\mu}] \longrightarrow H^i_{\Delta (1)}(X_{O_C}/A_{\inf}) \otimes_{A_{\inf}} A_{\inf}[\frac{1}{\mu}],
\]
which is also \( G_K \)-equivariant, since \( G_K \)-action on \( W(C^\phi) \) preserves \( A_{\inf}[\frac{1}{\mu}] \subset W(C^\phi) \). This then automatically extends to a \((G_K, \varphi)\)-equivariant isomorphism
\[
H^i_{\text{et}}(\hat{X}_C, Z_p) \otimes_{Z_p} A_{\inf}[\frac{1}{\mu}, \frac{1}{\varphi(\mu)}, \frac{1}{\varphi^2(\mu)}, \ldots] \longrightarrow H^i_{\Delta (1)}(X_{O_C}/A_{\inf}) \otimes_{A_{\inf}} A_{\inf}[\frac{1}{\mu}, \frac{1}{\varphi(\mu)}, \frac{1}{\varphi^2(\mu)}, \ldots]
\]
Indeed, \( G_K \) acts on \( \mu := [\varepsilon] - 1 \) through the cyclotomic character and multiplies it by a unit (e.g. see the proof of [KP21, Lemma 6.2.15]).
where localization $A_{\text{inf}}[\frac{1}{\mu}, \frac{1}{\varphi(\mu)}], \frac{1}{\varphi(\mu)} \ldots ]$ is the minimal $\varphi$-invariant subring of $W(C^p)$ containing $A_{\text{inf}}[\frac{1}{\mu}]$. The $G_K$-equivariant embedding $A_{\text{inf}}[\frac{1}{\mu}] \to B_{\text{crys}}$ extends to a $(G_K, \varphi)$-equivariant embedding

$$A_{\text{inf}}[\frac{1}{\mu}, \frac{1}{\varphi(\mu)}], \frac{1}{\varphi(\mu)} \ldots ] \to B_{\text{crys}}.$$

Taking base change under this morphism we arrive at $(G_K, \varphi)$-equivariant isomorphisms

$$H^i_{\text{et}}(\hat{X}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{crys}} \sim H^i_{\Delta(i)}(X_C/\mathbb{A}_{\text{inf}}) \otimes_{A_{\text{inf}}} B_{\text{crys}}.$$

We can now prove an analogue of Theorem 3.8 in the de Rham proper context.

**Theorem 4.15.** Let $X$ be a $(d+1)$-de Rham proper stack over $\mathcal{O}_K$. Then the map $\Upsilon_{X, \mathbb{Q}_p} : R\Gamma_{\text{et}}(X_C, \mathbb{Q}_p) \to R\Gamma_{\text{et}}(\hat{X}_C, \mathbb{Q}_p)$ induces an isomorphism

$$H^i_{\text{et}}(X_C, \mathbb{Q}_p) \simeq H^i_{\text{et}}(\hat{X}_C, \mathbb{Q}_p)$$

for $i \leq d$ and an embedding

$$H^{d+1}_{\text{et}}(X_C, \mathbb{Q}_p) \hookrightarrow H^{d+1}_{\text{et}}(\hat{X}_C, \mathbb{Q}_p)$$

for $i = d + 1$.

**Proof.** With Proposition 3.6 in hand, it is enough to show that $\dim H^i_{\text{et}}(X_C, \mathbb{Q}_p) \geq \dim H^i_{\text{et}}(\hat{X}_C, \mathbb{Q}_p)$ for $i \leq d$. By Proposition 1.9(2) we have that the reduction $X_\kappa$ is $d$-de Rham proper. By Remark 4.5 we have that $H^i_{\Delta(i)}(X/\mathcal{O}_K)[\frac{1}{p}]$ is a free $\mathcal{O}_K[\frac{1}{p}]$-module for $i \leq d$. Inverting $p$ in Proposition 4.12(3) and comparing the dimensions of both sides over $W(C^p)[\frac{1}{p}]$ we get that $\dim_{\mathbb{Q}_p} H^i_{\text{et}}(\hat{X}_C, \mathbb{Q}_p) = \dim_{\mathbb{Q}_p} H^i_{\Delta(i)}(X/\mathcal{O}_K)[\frac{1}{p}]$ for $i \leq d$.

Denote $H^i_{\text{dR}}(\hat{X}_K/K) := H^i_{\text{dR}}(\hat{X}/\mathcal{O}_K)[\frac{1}{p}]$. By de Rham comparison [KP21] we have an equivalence $R\Gamma_{\text{dR}}(\hat{X}/\mathcal{O}_K) \simeq [R\Gamma_{\Delta(i)}(X/\mathcal{O}_K)/E(u)]$. This gives a short exact sequence

$$0 \longrightarrow H^i_{\Delta(i)}(X/\mathcal{O}_K)[\frac{1}{p}]/E \longrightarrow H^i_{\text{dR}}(\hat{X}_K/K) \longrightarrow (H^{i+1}_{\Delta(i)}(X/\mathcal{O}_K)[\frac{1}{p}]/E) \longrightarrow 0$$

for all $i$. In particular, for $i \leq d$ we get that

$$\dim_K H^i_{\text{dR}}(\hat{X}_K/K) \geq \dim_{\mathbb{Q}_p} H^i_{\Delta(i)}(X/\mathcal{O}_K)[\frac{1}{p}] = \dim_{\mathbb{Q}_p} H^i_{\text{et}}(\hat{X}_C, \mathbb{Q}_p).$$

On the other hand, we claim that for $i \leq d$ we have an isomorphism $H^i_{\text{dR}}(\hat{X}_K/K) \simeq H^i_{\text{dR}}(X_K/K)$. Indeed, derived $p$-completion is $t$-exact up to a right shift by 1 and so

$$\tau^{\leq d} R\Gamma_{\text{dR}}(\hat{X}/\mathcal{O}_K) \simeq \tau^{\leq d}((\tau^{\leq d+1} R\Gamma_{\text{dR}}(X/\mathcal{O}_K))^{\wedge}_p)$$

by Lemma 1.8. However, by our assumption $\tau^{\leq d+1} R\Gamma_{\text{dR}}(X/\mathcal{O}_K)$ is coherent and so automatically $p$-complete. It follows that $\tau^{\leq d} R\Gamma_{\text{dR}}(\hat{X}/\mathcal{O}_K) \simeq \tau^{\leq d} R\Gamma_{\text{dR}}(X/\mathcal{O}_K)$, and consequently $H^i_{\text{dR}}(\hat{X}_K/K) \simeq H^i_{\text{dR}}(X_K/K)$ for $i \leq d$. Finally, from the left vertical isomorphism in Proposition 3.5 we know that $\dim_{\mathbb{Q}_p} H^i_{\text{et}}(X_C, \mathbb{Q}_p) = \dim_K H^i_{\text{dR}}(X_K/K)$ for all $i$. We get that

$$\dim H^i_{\text{et}}(X_C, \mathbb{Q}_p) \geq \dim H^i_{\text{et}}(\hat{X}_C, \mathbb{Q}_p)$$

for $i \leq d$ as desired. □

**Remark 4.16.** Lest us record a fact with a similar flavor to Remark 4.6. By de Rham comparison for prismatic cohomology we have a short exact sequence

$$0 \longrightarrow H^i_{\Delta(i)}(X/\mathcal{O}_K)/E(u) \longrightarrow H^i_{\text{dR}}(\hat{X}/\mathcal{O}_K) \longrightarrow H^{i+1}_{\Delta(i)}(X/\mathcal{O}_K)[E(u)] \longrightarrow 0.$$
If $X$ is $(d+1)$-de Rham proper over $\mathcal{O}_K$, then $X_{\mathcal{O}_K/p}$ is $d$-de Rham proper over $\mathcal{O}_K/p$ and so $H^d_{\text{dR}}(\hat{X}/\mathcal{O}_K)$ is a finitely-generated $\mathcal{O}_K$-module. Consequently, we also get some partial information about $H^d_{\Delta(1)}(X/\mathcal{O}_\mathfrak{S})$: namely, its $E(u)$-torsion $M := H^d_{\Delta(1)}(X/\mathcal{O}_\mathfrak{S})[E(u)]$ is finitely-generated over $\mathcal{O}_K$. However, from the proof of Theorem 4.15 one can see more: namely $M$ dies after we invert $p$: $M[\frac{1}{p}] = 0$. Indeed, $M[\frac{1}{p}]$ is the last homomorphism in the short exact sequence

$$0 \longrightarrow H^d_{\Delta(1)}(X/\mathcal{O}_\mathfrak{S})[\frac{1}{p}] / E(u) \longrightarrow H^d_{\text{dR}}(\hat{X}_K/K) \longrightarrow (H^d_{\Delta(1)}(X/\mathcal{O}_\mathfrak{S})[\frac{1}{p}])[E(u)] \longrightarrow 0$$

and we saw that $\dim_K H^d_{\Delta(1)}(X/\mathcal{O}_\mathfrak{S})[\frac{1}{p}] / E(u) \simeq \dim_{\mathcal{O}_p} H^d_{\text{ét}}(\hat{X}_C, \mathbb{Q}_p)$, as well as the equalities

$$\dim_{\mathcal{O}_p} H^d_{\text{ét}}(\hat{X}_C, \mathbb{Q}_p) = \dim_{\mathcal{O}_p} H^d_{\text{ét}}(X_C, \mathbb{Q}_p) = \dim_K H^d_{\text{dR}}(X_K/K) = \dim_K H^d_{\text{dR}}(\hat{X}_K/K).$$

Thus $\dim_K M[\frac{1}{p}] = 0$, meaning that $M[\frac{1}{p}] = 0$.

Let us also record the following analogue of [KP21, Proposition 4.3.14] for $d$-Hodge-proper stacks. Let $K_0 := W(k)[\frac{1}{p}]$.

**Corollary 4.17.** Let $X$ be a smooth $(d+1)$-de Rham proper stack over $\mathcal{O}_K$. Then for any $0 \leq i \leq d$

$$\dim_{K_0} H^i_{\text{cris}}(X_k/W(k))[\frac{1}{p}] = \dim_K H^i_{\text{dR}}(X_K/K) = \dim_{\mathcal{O}_p} H^i_{\text{ét}}(\hat{X}_C, \mathbb{Q}_p) = \dim_{\mathcal{O}_p} H^i_{\text{ét}}(X_C, \mathbb{Q}_p).$$

**Proof.** All equalities except the first one follow from the proof of Theorem 4.15. Also, we showed that $H^d_{\text{dR}}(\hat{X}_K/K) \simeq H^d_{\text{dR}}(X_K/K)$ for $i \leq d$. Thus the first equality follows from the Berthelot-Ogus isomorphism (Proposition B.4), which is a consequence of the point $\ast \in (\ast)_{\text{proét}}$ gives an isomorphism $\Gamma_{\text{cris}}(X_k/W(k)) \otimes W(k) K \simeq \Gamma_{\text{dR}}(\hat{X}_K/K))$. \hfill \Box

### 4.3 Fontaine’s $C_{\text{cris}}$-conjecture for $d$-de Rham proper stacks

In this subsection we prove that for $i \leq d$ the $i$-th rational étale cohomology of the generic fiber of a $(d+1)$-de Rham proper stack over $\mathcal{O}_K$ is crystalline. We deduce it from a comparison between truncated prismatic and crystalline cohomology for which we will use the simplified period ring $B_{\text{max}}$.

**Construction 4.18** (Rings $A_{\text{max}}$ and $B_{\text{max}}$). The period ring $A_{\text{max}}$ is defined as the $p$-completion $(A_{\text{inf}}[\frac{1}{p}])_p$. It can be equivalently described as the completion of $A_{\text{inf}}[\frac{1}{p}] = A_{\text{inf}}[\frac{1}{p^\nu}]$ given by power series

$$A_{\text{max}} := \left\{ \sum_{i \in \mathbb{Z}} [x_i] \cdot p^i \mid x_i \in \mathcal{O}_C, \text{ s.t. } (\text{val}_K(x_i) + i) \geq 0 \text{ and } (\text{val}_K(x_i) + i) \to +\infty \text{ as } i \to -\infty \right\},$$

where $\text{val}_K$ is the natural valuation on $\mathcal{O}_C$ satisfying $\text{val}_K([p^\nu]) = 1$. We define the period ring $B_{\text{max}}$ as $A_{\text{max}}[\frac{1}{p}]$. Since for $x_i \in \mathcal{O}_C$ one has $\text{val}_K(x_i^p) = p \cdot \text{val}_K(x_i)$, Frobenius on $\mathcal{O}_C$ induces a map $\varphi: A_{\text{max}} \to A_{\text{max}}$ which is in fact an embedding and has the image

$$\varphi(A_{\text{max}}) \simeq \left\{ \sum_{i \in \mathbb{Z}} [x_i] \cdot p^i \mid (\text{val}_K(x_i + pi) \geq 0 \text{ and } (\text{val}_K(x_i + pi) \to +\infty \text{ as } i \to -\infty \right\}.$$

We also have natural $G_K$-actions on $A_{\text{max}}$ and $B_{\text{max}}$ induced by the one on $\mathcal{O}_C$. Note that it commutes with $\varphi$.

Recall (e.g. see [Car19, Section 3.2.2]) that any element $x \in A_{\text{cris}}$ has a (unique) expression as

$$x = \sum_{i \in \mathbb{Z}} [x_i] \cdot p^i \text{ with } x_i \in \mathcal{O}_C \text{ such that } (\text{val}_K(x_i - \nu(i)) \geq 0 \text{ and } (\text{val}_K(x_i - \nu(i)) \to +\infty \text{ as } i \to -\infty,$$
where \( \nu(i) = 0 \) if \( i \geq 0 \) and if \( i \leq 0 \) one puts \( \nu(i) \) to be the minimal integer \( n \) such that \( \text{val}_p(n!) + i \geq 0 \). It is not hard to see that for \( i \leq 0 \) one has

\[-ip \geq \nu(i) \geq -i\]

and this way comparing the convergence conditions one gets natural \( G_K \)-equivariant embeddings

\[\ldots \subset \varphi(A_{\text{crys}}) \subset \varphi(A_{\text{max}}) \subset A_{\text{crys}} \subset A_{\text{max}} \quad \text{and} \quad \ldots \subset \varphi(B_{\text{crys}}) \subset \varphi(B_{\text{max}}) \subset B_{\text{crys}} \subset B_{\text{max}}.\]

Consequently, we have embeddings \( \varphi(B_{\text{crys}})^{G_K} \subset \varphi(B_{\text{max}})^{G_K} \subset B_{\text{crys}}^{G_K} \subset B_{\text{max}}^{G_K} \) of \( G_K \)-invariants. One has \( B_{\text{crys}}^{G_K} \cong K_0 \) and \( \varphi: B_{\text{crys}} \to B_{\text{crys}} \) maps it to itself via Frobenius on \( K_0 \); in particular \( \varphi(B_{\text{crys}})^{G_K} \cong B_{\text{crys}}^{G_K} \), and consequently \( \varphi(B_{\text{max}})^{G_K} \cong K_0 \cong B_{\text{crys}}^{G_K} \). More generally, having a finite-dimensional \( G_K \)-representation \( V \) over \( \mathbb{Q}_p \) we get maps

\[(V \otimes_{\mathbb{Q}_p} \varphi(B_{\text{crys}}))^{G_K} \subset (V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_K} \subset (V \otimes_{\mathbb{Q}_p} B_{\text{max}})^{G_K}\]

and, since \( \varphi \) induces a \( G_K \)-equivariant isomorphism \( B_{\text{max}} \xrightarrow{\sim} \varphi(B_{\text{max}}) \), we get that

\[\dim_{K_0}(V \otimes_{\mathbb{Q}_p} \varphi(B_{\text{crys}}))^{G_K} = \dim_{K_0}(V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_K} = \dim_{K_0}(V \otimes_{\mathbb{Q}_p} B_{\text{max}})^{G_K}.\]

In particular, \( V \) crystalline if and only if it is \( B_{\text{max}} \)-admissible: namely,

\[\dim_{K_0}(V \otimes_{\mathbb{Q}_p} B_{\text{max}})^{G_K} = \dim_{\mathbb{Q}_p} V.\]

**Construction 4.19** (Ring \( \mathcal{S}_{\text{max}} \)). Let \( e \) be the ramification index of \( \mathcal{S}_{\text{max}} \). We define the ring \( \mathcal{S}_{\text{max}} \) as the \( p \)-adic completion

\[\mathcal{S}_{\text{max}} := \left( \mathcal{S}[\frac{u}{p}] \right)^{\wedge}_p.\]

Picking a uniformizer \( \pi \in \mathcal{O}_K \) with minimal polynomial \( E(u) \) we have \( E(u) \equiv u^e \mod p\mathcal{S} \) and so one also has \( \mathcal{S}_{\text{max}} \cong \mathcal{S}[\frac{E(u)}{p}]^{\wedge}_p \). Via the choice of \( \pi^e \in \mathcal{O}_C \) we have a map \( \mathcal{S} \to A_{\text{inf}} \) sending \( u \) to \( [\pi^e] \). Since \( \text{val}_p([\pi^e]) = 1 \), this map naturally extends to a map

\[\mathcal{S}_{\text{max}} \to A_{\text{max}}.\]

**Remark 4.20.** Note that \( u^e \in p \cdot \mathcal{S}_{\text{max}} \). Thus \( (p, u^e) \)-adic topology on \( \mathcal{S}_{\text{max}} \) is equivalent to \( p \)-adic one. Similarly, \( (p, [p^e]) \)-adic topology on \( A_{\text{max}} \) is equivalent to \( p \)-adic one.

With this notations the key result of this section is the following

**Theorem 4.21.** Let \( X \) be a smooth \((d + 1)\)-de Rham proper stack over \( \mathcal{O}_K \). Then there is a \((G_K, \varphi)\)-equivariant equivalence

\[(\tau^{\leq d} R\Gamma_{\Delta(i)}(X_{\mathcal{O}_C}/A_{\text{inf}})) \otimes_{A_{\text{inf}}} B_{\text{max}} \simeq (\tau^{\leq d} R\Gamma_{\text{crys}}(X_k/W(k))) \otimes_{W(k)} B_{\text{max}}.\]

We will prove this theorem in the remaining sections using a bit of condensed mathematics. Before introducing some necessary notation let us deduce from Theorem 4.21 the crystalliness of the \( G_K \)-representation given by the étale\(^{\text{10}} \) cohomology \( H^i_{\text{ét}}(X_C, \mathbb{Q}_p) \).

Recall that a finite-dimensional \( \mathbb{Q}_p \) representation \( V \) is called crystalline if the dimension of \( D_{\text{crys}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\text{crys}})^{G_K} \) over \( K_0 \) is equal to the dimension of \( V \).

**Corollary 4.22** (Fontaine’s \( C_{\text{crys}} \)-conjecture for \( d \)-de Rham proper stacks). Let \( X \) be a smooth \((d + 1)\)-de Rham proper stack over \( \mathcal{O}_K \). Then \( H^i_{\text{ét}}(X_C, \mathbb{Q}_p) \cong H^i_{\text{ét}}(X_C, \mathbb{Q}_p) \) is a crystalline Galois representation for \( i \leq d \). Moreover, \( D_{\text{crys}}(H^i_{\text{ét}}(X_C, \mathbb{Q}_p)) \cong H^i_{\text{crys}}(X_k/W(k))(\frac{1}{p}) \).

\(^{\text{10}}\)Note that from Theorem 4.15 it follows that in the range \( i \leq d + 1 \) it doesn’t matter whether we consider the Raynaud or the algebraic generic fiber here.
Proof. Recall that for $0 \leq i \leq d$ we have $H^i_{\Delta}(X_{\mathcal{O}_C}/A_{inf}) \simeq H^i_{\Delta}(X/\mathcal{S}) \otimes_{\mathcal{O}_C} A_{inf}$ by Lemma 4.13, and so by Remark 4.5 we have that $H^i_{\Delta}(X_{\mathcal{O}_C}/A_{inf})[\frac{1}{p}]$ are free modules over $A_{inf}[\frac{1}{p}]$. Thus applying cohomology to both sides of the isomorphism in Theorem 4.21 we get $(G_K, \varphi)$-equivariant isomorphisms

$$H^i_{\Delta}(X_{\mathcal{O}_C}/A_{inf}) \otimes_{A_{inf}} B_{max} \simeq H^i_{\text{crys}}(X_k/W(k)) \otimes_{W(k)} B_{crys}.$$  

Recall that by Corollary 4.14 we have a $(G_K, \varphi)$-equivariant isomorphisms

$$H^i_{et}(\mathcal{X}_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{crys} \cong H^i_{\Delta}(X_{\mathcal{O}_C}/A_{inf}) \otimes_{A_{inf}} B_{crys}.$$  

Tensoring it further by $B_{max}$ we then get $(G_K, \varphi)$-equivariant isomorphisms

$$H^i_{et}(\mathcal{X}_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{max} \cong H^i_{\Delta}(X_{\mathcal{O}_C}/A_{inf}) \otimes_{A_{inf}} B_{max} \cong H^i_{\text{crys}}(X_k/W(k)) \otimes_{W(k)} B_{max}.$$  

Put $V := H^i_{et}(\mathcal{X}_C, \mathbb{Q}_p)$. By the discussion in the end of Construction 4.18 we then have

$$D_{\text{crys}}(V) \cong (V \otimes_{\mathbb{Q}_p} B_{max})^{G_K} \simeq (H^i_{\text{crys}}(X_k/W(k)) \otimes_{W(k)} B_{max})^{G_K} \simeq H^i_{\text{crys}}(X_k/W(k) \otimes_{W(k)} K_0),$$

since the $G_K$-action on $H^i_{\text{crys}}(X_k/W(k))$ is trivial and $B_{max}^{G_K} \simeq K_0$. Since $H^i_{\text{crys}}(X_k/W(k)) \otimes_{W(k)} K_0 \simeq H^i_{\text{crys}}(X_k/W(k)[\frac{1}{p}])$, from Corollary 4.17 we get

$$\dim_{\mathbb{Q}_p} V = \dim_{K_0} H^i_{\text{crys}}(X_k/W(k)[\frac{1}{p}]) = \dim_{K_0} D_{\text{crys}}(V),$$

and so $V$ is crystalline.

Remark 4.23. By general theory of admissible rings of Fontaine, from Corollary 3.9 we get that for any $i \leq d$ there is a natural $(G_K, \varphi)$-equivariant isomorphism

$$H^i_{et}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{crys} \cong H^i_{\text{crys}}(X_k/W(k)) \otimes_{W(k)} B_{crys}.$$  

Remark 4.24. By a method similar to one in [KP21, Section 4.3.3] it should be possible to show that for $i \leq d$ the filtered $K$-vector space $D_{\text{dR}}(H^i_{et}(X_C, \mathbb{Q}_p))$ is given by $H^i_{\text{dR}}(X_K/K)$ together with Hodge filtration. This then would formally imply the Hodge-Tate decomposition in the same range of degrees.

Remark 4.25. Let $X$ be a smooth $(d+1)$-de Rham-proper stack over $\mathcal{O}_K$ and let

$$H^i_{et}(\mathcal{X}_C, \mathbb{Z}_p)_{\text{free}} := H^i_{et}(\mathcal{X}_C, \mathbb{Z}_p)/H^i_{et}(\mathcal{X}_C, \mathbb{Z}_p)_{\text{tors}} \subset H^i_{et}(\mathcal{X}_C, \mathbb{Z}_p).$$  

By Corollary 4.22, for $i \leq d$ this is a lattice in a crystalline $G_K$-representation. Following the same argument as in [KP21, Remark 4.3.27] one can identify the corresponding Breuil-Kisin module $\text{BK}(H^i_{et}(\mathcal{X}_C, \mathbb{Z}_p)_{\text{free}})$ with the free Breuil-Kisin module $H^i_{\Delta}(X/\mathcal{S})_{\text{free}}$ associated to $H^i_{\Delta}(X/\mathcal{S})$ (see [BMS18, Proposition 4.3] for the definition). If $H^i_{\Delta}(X/\mathcal{S})$ is free over $\mathcal{S}$, then by Proposition 4.12 $H^i_{et}(\mathcal{X}_C, \mathbb{Z}_p)$ is $p$-torsion free and the statement becomes:

$$\text{BK}(H^i_{et}(\mathcal{X}_C, \mathbb{Z}_p)) \simeq H^i_{\Delta}(X/\mathcal{S}).$$

The following lemma gives a sufficient condition for $H^i_{\Delta}(X/\mathcal{S})$ to be a free $\mathcal{S}$-module.

Lemma 4.26. Let $X$ be a $(d+1)$-de Rham-proper stack over $\mathcal{O}_K$. If $i \leq d$ and $H^i_{\text{crys}}(X_k/W(k))$ is $p$-torsion free, then $H^i_{\Delta}(X/\mathcal{S})$ is free as an $\mathcal{S}$-module.

Proof. In the proof all tensor products are assumed to be non-derived (unless noted otherwise). First note that $H^i_{\Delta}(X/\mathcal{S})$ is free over $\mathcal{S}$ if and only if $H^i_{\Delta}(X/\mathcal{S})$ is. Let more generally $C$ be a complex of $\mathcal{S}$-modules such that $H^i(C)$ is finitely generated over $\mathcal{S}$ and $H^i(C)[1/p]$ is free over $\mathcal{S}[1/p]$ (e.g. $C = R\Gamma_{\Delta}(X/\mathcal{S})$). We claim that if $H^i(C \otimes_{\mathcal{S}} W(k))$ is $p$-torsion free then $H^i(C)$ is free. Indeed, since $H^i(C \otimes_{\mathcal{S}} W(k))$ is a submodule of $H^i(C \otimes_{\mathcal{S}} W(k))$ it follows that $H^i(C) \otimes_{\mathcal{S}} W(k)$ is $p$-torsion free (equivalently free as a $W(k)$-module) as well. Hence it is enough to prove the following assertion: let $M$ be a finitely generated $\mathcal{S}$-module such that

The following lemma gives a sufficient condition for $H^i_{\Delta}(X/\mathcal{S})$ to be a free $\mathcal{S}$-module.
• $M[1/p]$ is free over $\mathcal{G}[1/p]$.
• $M \otimes_{\mathcal{G}} W(k)$ is free over $W(k)$.

Then $M$ is free over $\mathcal{G}$.

But under these assumptions

$$\dim_k M \otimes_{\mathcal{G}} k = \dim_{W(k)} M \otimes_{\mathcal{G}} W(k) = \dim_{W(k)[1/p]} M \otimes_{\mathcal{G}} W(k)[1/p] =$$

$$= \dim_{\mathcal{G}[1/p]} M \otimes_{\mathcal{G}} \mathcal{G}[1/p] = \dim_{\text{Frac} \mathcal{G}} M \otimes_{\mathcal{G}} \text{Frac} \mathcal{G}.$$ and thus $M$ is free by the semicontinuity of stalks.

\[\square\]

**Condensed prismatic and crystalline cohomology.** First, we set up some notation. We denote by $X \rightarrow \mathcal{X}$ the natural functor $\text{Top} \rightarrow \text{Cond}$. It commutes with products and so sends topological groups/rings to group/ring objects in condensed sets. We consider the full subcategory $\text{Solid} \subset \text{Cond}(\text{Ab})$ of solid abelian groups; recall that it is closed under all limits and colimits. The restriction of the functor $A \rightarrow A$ to discrete topological groups factors through Solid, inducing a fully faithful embedding $\text{Ab} \rightarrow \text{Solid}$. It then also extends to a fully faithful embedding $D(\text{Ab}) \rightarrow D(\text{Solid})$ between derived categories which by slight abuse of notation we will still denote by $M \rightarrow M$. Having an $\mathcal{E}$-ring $A$ in $D(\text{Solid})$ we will sometimes denote by $D(A) \simeq D(\text{Mod}_{\text{solid}}(D(\text{Solid})))$ the category of $A$-modules. It is endowed with the symmetric monoidal structure given by solid tensor tensor product $- \otimes_{\mathcal{A}} -$.

For a set $\{f_i\}_{i \in I}$ of elements of $R$ and an object $M \in D(R)$ we denote

$$\text{Kos}(M; \{f_i\}_{i \in I}) := M \otimes_{\mathbb{Z}[\{x_i\}_{i \in I}]} \mathbb{Z}$$

where $M$ is considered as a $\mathbb{Z}[\{x_i\}_{i \in I}]$-module via the homomorphism $\mathbb{Z}[\{x_i\}_{i \in I}] \rightarrow R$ sending $x_i$ to $f_i$ and all $x_i$’s act on $\mathbb{Z}$ by 0. Note that $\text{Kos}(M; \{f_i\}_{i \in I})$ is naturally a module over the derived ring $\text{Kos}(R; \{f_i\}_{i \in I})$.

In the case $M \in \text{Mod}_R$, the complex $\text{Kos}(M; \{f_i\}_{i \in I})$ can be explicitly computed by the “Koszul complex”

$$\cdots \longrightarrow \bigoplus_{i<j} M \longrightarrow \bigoplus_i M \xrightarrow{(f_i)_{i \in I}} M$$

where we fix some auxiliary total ordering on $I$ (see [SP20, Tag 0621]). Recall that given a finitely generated ideal $J = (x_1, \ldots, x_s) \subset R$ one defines a functor $M \mapsto M_J^\wedge$ as

$$M_J^\wedge := \lim_{\leftarrow n} \text{Kos}(M; x_1^n, \ldots, x_s^n).$$ (4.3)

The functor $(-)^\wedge_J$ is a left adjoint to the embedding $D(\text{Mod}_R)_{J-\text{comp}} \rightarrow D(\text{Mod}_R)$ (see [SP20, Tag 091N]) and so computes the “derived $J$-completion”. In particular, it doesn’t depend on the choice of generators $(x_1, \ldots, x_s)$.

**Construction 4.27.** 1. Let $R$ be a classical ring with a finitely generated ideal $J = (x_1, \ldots, x_s) \subset R$ (with a fixed set of generators). We define

$$R_J^\wedge := \lim_{\leftarrow n} \text{Kos}(R; x_1^n, \ldots, x_s^n) \in D(\text{Solid}),$$

to be the derived $J$-completion of the discrete ring $R$. The solid group $R_J^\wedge$ has the natural structure of a derived solid ring. In the case $(x_1, \ldots, x_s)$ is a regular sequence, $\text{Kos}(R; x_1^n, \ldots, x_s^n) \simeq R/(x_1^n, \ldots, x_s^n)$ and $R_J^\wedge \simeq R_J^\wedge \in \text{Solid}$ where $R_J^\wedge$ is considered as a topological ring via $J$-adic topology.

2. Let $M \in D(\text{Mod}_R)$ be a complex. We define

$$M_J^\wedge := \lim_{\leftarrow n} \text{Kos}(M; x_1^n, \ldots, x_s^n)$$
where \( \text{Kos}(M; x^n_1, \ldots, x^n_n) \) is the image of \( \text{Kos}(M; x^n_1, \ldots, x^n_n) \in D(\text{Mod}_R) \) under the natural functor \( D(\text{Ab}) \to D(\text{Solid}) \). The functor \( M \to M^\wedge \) defines a functor

\[
(-)^\wedge: D(\text{Mod}_R) \to D(R^\wedge).
\]

By construction it factors through the subcategory \( D(R^\wedge)_{J-\text{comp}} \subset D(R^\wedge) \) of derived \( J \)-complete solid \( R^\wedge \)-modules.

3. Let \((R, J)\) be as in part 1 and let \( A \simeq R[\frac{1}{r_s}]_{s \in S} \) be a localization of \( R \) with respect to a set of elements \( \{r_s\}_{s \in S} \). Then we define

\[
A^\wedge_J := R^\wedge[\frac{1}{r_s}]_{s \in S}.
\]

4. When it is clear from the context what \( J \) is, we will sometimes omit it to lighten the notation. We will occasionally call the operation \((-)^J \) as \((-)^J \) solid \( J \)-completion.

**Remark 4.28.** We note that since evaluation on the singleton \(* \in (\_)^{\text{pro}\text{-}et}\) commutes will all limits and colimits, one has isomorphisms \( R^\wedge_J(*) \simeq R^\wedge_J \) and \( M^\wedge_J(*) \simeq M^\wedge_J \) where \((-)^J \) is the derived \( J \)-completion. Similarly, \( A^\wedge_J(*) \simeq R^\wedge_J[\frac{1}{r_s}]_{s \in S} \).

**Remark 4.29.** Note that the solid module \( M^\wedge \) only depends on the pro-system \( \{(M/(x^n_1, \ldots, x^n_n))\}_n \) and not \( M \) itself. Consequently, the functor \((-)^\wedge: D(\text{Mod}_R) \to D(\text{Mod}_R)_{J-\text{comp}} \) factors through the usual derived completion \((-)^J: D(\text{Mod}_R) \to D(\text{Mod}_R)_{J-\text{comp}} \).

**Lemma 4.30.** The functor \((-)^\wedge: D(\text{Mod}_R) \to D(\text{Mod}_R)_{J-\text{comp}} \) is right \( t \)-exact. If \( J = (x_1, \ldots, x_s) \subset R \) is generated by \( s \) elements it is also left \( t \)-exact up to a shift by \( s \).

**Proof.** Indeed, \((-)^\wedge \) is the composition of fully faithful embedding \((-): D(\text{Mod}_R) \to D(\text{Ab}) \) (where \( R \) is endowed with discrete topology) which is \( t \)-exact and the derived \( J \)-adic completion in \( \mathcal{D}(\text{Mod}_R) \) which is right \( t \)-exact. Indeed, \( M^\wedge \simeq (((M\wedge_{x_1})\wedge_{x_2})\ldots \wedge_{x_s}) \). Now, having \( M \in \text{Mod}_R(\text{Solid}) \subset D(\text{Ab}) \) and \( x \in R \) the complex \( [M/(x^n)] \) has two cohomology modules: \( H^0([M/(x^n)]) = M/x \) and \( H^1([M/(x^n)]) = M\wedge x \) \((x\text{-tor})\). Since in the derived limit \( \lim_{n \in \mathbb{N}} \) (applied to objects in the heart) we only have two terms \( \lim^0 \) and \( \lim^1 \), we get the only non-zero term in \( H^{>0}(M\wedge x) \) could be given by \( \lim^1 M/x^n \), which is zero by Mittag-Leffler since maps \( M/x^n \to M/x^{n-1} \) are surjective.

For the second statement note that the functor \( M \mapsto [M/(x^n_1, \ldots, x^n_n)] \) is left \( t \)-exact up to a shift by \( s \), while the homotopy limit \( \lim\) is left \( t \)-exact.

**Warning 4.31.** We warn the reader that the functor \((-)^\wedge \) doesn’t need to be left \( t \)-exact even if we restrict to the subcategory of derived \( J \)-complete modules \( D(\text{Mod}_R)_{J-\text{comp}} \subset D(\text{Mod}_R) \). Nevertheless, it still satisfies some “left \( t \)-exactness” properties if we restrict to objects satisfying special finiteness conditions (see Lemma A.16).

We now give a variant of the prismatic and crystalline cohomology in this setting.

**Definition 4.32** (Solid \((p, I)\)-completed prismatic cohomology). Let \((A, I)\) be a bounded prism. Then we can consider the corresponding solid ring \( A^\wedge := A^\wedge_{(p, I)} \). We define functors

\[
\begin{align*}
R\Gamma^\wedge_{\Delta}(\_): \mathbb{P}\text{Stk}^{\text{op}}_{A/I} & \longrightarrow \text{Mod}_A(\text{Solid})
\end{align*}
\]

as the application of Construction 4.27(2) to functors

\[
\begin{align*}
R\Gamma^\wedge_{\Delta}(\_, I): \mathbb{P}\text{Stk}^{\text{op}}_{A/I} & \longrightarrow \text{Mod}_A(D(\text{Ab}))
\end{align*}
\]

(see [KP21, Definition 2.2.1]). Namely,

\[
R\Gamma^\wedge^J_{\Delta}(X) := R\Gamma^\wedge_{\Delta}(X/A)^\wedge, \quad R\Gamma^\wedge^J_{\Delta}(I)(X) := R\Gamma^\wedge_{\Delta}(I)(X/A)^\wedge_{(p, I)}.
\]

\(^{11}\)Since pro-systems \( \{(M/(x^n_1, \ldots, x^n_n))\}_n \) and \( \{(M^\wedge_J/(x^n_1, \ldots, x^n_n))\}_n \) are naturally equivalent.
Remark 4.33. Since $A$ is bounded, the derived $(p,I)$-completion of $A$ is identified with the classical one. So in this case $A^\wedge$ in fact lands in Solid $\subset D($Solid$)$, and is the solid ring $A$ represented by $A$ with $(p,I)$-adic topology.

Definition 4.34 (Solid PD-completed crystalline cohomology). Let $S \twoheadrightarrow T$ be a PD-thickening, where $T = \colim T_n$ is a formal scheme with $T_n$ being the $n$-th PD-neighborhood of $S$. Let $A := \mathcal{O}(T) := \lim_n \mathcal{O}(T_n)$ be the global functions on $T$ and $A^\wedge := \lim_n \mathcal{O}(T_n) \in \text{Solid}$ the corresponding completion in solid rings, where $\mathcal{O}(T_n)$ is considered as discrete topological group. We define the functor

$$R\Gamma_{\text{crys}}(-/T) : \mathcal{PStk}_{S}^{\text{op}} \longrightarrow \text{Mod}_{A^\wedge}(D($$Solid$))$$

as the limit in $D($Solid$)$

$$R\Gamma_{\text{crys}}(-/T) := \lim_{\leftarrow n} R\Gamma_{\text{crys}}(-/T_n),$$

where $R\Gamma_{\text{crys}}(-/T_n)$ is the right Kan extension of the crystalline cohomology functor from smooth $S$-schemes to prestacks.

Remark 4.35. In the following we will only consider a special case when $T = \text{Spf} A$ is a $p$-adic formal scheme. In this case there is a natural equivalence $R\Gamma_{\text{crys}}(X/T) \simeq R\Gamma_{\text{crys}}(X/T)(^\wedge) /\p \in D(\text{Mod}_{A^\wedge}($Solid$))$.

With this definition one can formally deduce the base change and the comparison between condensation of the prismatic and crystalline cohomology from the “discrete” version:

Proposition 4.36 (Base change). Let $(A,I) \rightarrow (B,IB)$ be a morphism of bounded prisms such that the $(p,IB)$-completed tensor product functor $\otimes_A B$ is left $t$-exact up to a shift. Also assume that $X$ is quasi-compact quasi-separated and syntomic Artin stack. Then the natural maps

$$R\Gamma_{\Delta}(X/A) \otimes_{A^\wedge} B^\wedge \longrightarrow R\Gamma_{\Delta}(X_B/IB/B) \quad R\Gamma_{\Delta(1)}(X/A) \otimes_{A^\wedge} B^\wedge \longrightarrow R\Gamma_{\Delta(1)}(X_B/IB/B)$$

are equivalences.

Proof. By Corollary A.21, the solid tensor product $R\Gamma_{\Delta}(X/A) \otimes_{A^\wedge} B^\wedge$ is naturally isomorphic to the solid $(p,I)$-completion $(R\Gamma_{\Delta}(X/A) \otimes_A B)^{\wedge}_{(p,I)}$, where the tensor product inside the brackets is also $(p,I)$-completed. By [KP21, Proposition 2.2.17], $R\Gamma_{\Delta}(X/A) \otimes_A B \simeq R\Gamma_{\Delta}(X_B/IB/B)$ and we get the comparison. The proof for twisted cohomology is completely analogous, using the same reference. □

Proposition 4.37 (Crystalline comparison). If $X$ is smooth and $I = (p)$ then there is a natural equivalence

$$R\Gamma_{\Delta(1)}(X/A) \simeq R\Gamma_{\text{crys}}(X/A).$$

Proof. This follows from the crystalline comparison $R\Gamma_{\Delta(1)}(X/A) \simeq R\Gamma_{\text{crys}}(X/A)(^\wedge)$ ([KP21, Proposition 2.5.7]) by applying $(^\wedge)$. □

Remark 4.38. One can also “topologize” the étale cohomology of a stack and (formally) upgrade the étale comparison of prismatic and étale cohomology to condensed setting, but we won’t need and so don’t discuss this here.

Notation 4.39. Below, we will consider instances of Construction 4.27 in the following situations:

- $W(k), \mathcal{O}_{\text{max}}, A_{\text{crys}}, A_{\text{max}}$ with $J = (p)$, and the corresponding $p$-localizations $K_0, \mathcal{O}_{\text{max}}[\frac{1}{J}]$, $B_{\text{crys}}, B_{\text{max}}$. We will denote the corresponding solid rings simply by $W(k)^\wedge, \mathcal{O}_{\text{max}}^{\wedge}[^\wedge], A_{\text{crys}}^\wedge, A_{\text{max}}^\wedge$ and $K_0^\wedge, \mathcal{O}_{\text{max}}[^\wedge], B_{\text{crys}}^\wedge, B_{\text{max}}^\wedge$.

- $\mathcal{G}, \mathcal{O}(T) := \mathcal{O}[T]^{\wedge}_{(p,u)}, A_{\text{inf}}$ with $J = (p,u)$ and $(p,[\pi^\wedge])$ correspondingly. Similarly, we denote the corresponding solid rings by $\mathcal{G}^\wedge, \mathcal{O}(T)^\wedge, A_{\text{inf}}^\wedge$.

Below, we will also need the following remarks:
Remark 4.40. \((- \otimes_{\mathcal{G}^\bullet} \mathcal{S}(T)^\bullet\) as completed direct sum). Note that by construction, the underlying \(\mathcal{G}^\bullet\)-module of \(\mathcal{S}(T)^\bullet\) is given by the \((p, u)\)-completed direct sum \(\bigoplus_{i \in \mathbb{N}} \mathcal{S}^\bullet \cdot T^i\). So, for any derived \((p, u)\)-complete \(\mathcal{G}^\bullet\)-module \(M \in D(\mathcal{G}^\bullet)_{(p, u)\text{--comp}}\) one has

\[ M \otimes_{\mathcal{G}^\bullet} \mathcal{S}(T)^\bullet \cong \bigoplus_{i \in \mathbb{N}} M \cdot T^i \]

as \(\mathcal{G}^\bullet\)-modules. Indeed, there is a natural map \(\bigoplus_{i \in \mathbb{N}} M \cdot T^i \to M \otimes_{\mathcal{G}^\bullet} \mathcal{S}(T)^\bullet\), then by Corollary A.21 the right hand side is \((p, u)\)-complete and so the map factors through the completion \(\bigoplus_{i \in \mathbb{N}} M \cdot T^i\). Moreover it is an isomorphism modulo \((p, I)\) since tensor product commutes with direct sums.

Remark 4.41. Also, one has a strict exact sequence

\[ 0 \to \mathcal{S}(T)^\bullet \overset{(pT-u^c)}{\to} \mathcal{S}(T)^\bullet \to \mathcal{S}_{\text{max}}^\bullet \to 0 \]

of topological \(\mathcal{S}\)-modules where the topology is \((p, u)\)-adic on \(\mathcal{S}(T)^\bullet\) and \(\mathcal{S}\), \(p\)-adic on \(\mathcal{S}_{\text{max}}^\bullet\) and the map on the left is given by multiplication by \(pT-u^c\). This then gives a short exact sequence

\[ 0 \to \mathcal{S}(T)^\bullet \overset{(pT-u^c)}{\to} \mathcal{S}(T)^\bullet \to \mathcal{S}_{\text{max}}^\bullet \to 0 \]

of solid \(\mathcal{G}^\bullet\)-modules. Indeed, terms in the sequence are represented by the corresponding topological groups. Then the first arrow is an embedding since the functor \(V \to \overline{V}\) is left exact and to see that the second arrow is a surjection one can use [And21, Lemma 3.1].

Proof of the comparison. Having introduced the necessary notations we can now formulate a more precise form of Theorem 4.21 which we actually prove:

Proposition 4.42. Let \(X\) be a smooth \((d+1)\)-de Rham proper stack over \(\mathcal{O}_K\). Then there is a \((G_K, \varphi)\)-equivariant equivalence

\[ \left( \tau^{\leq d} R\Gamma^{\mathcal{G}^\bullet}_{\mathcal{A}^\bullet} (X_{\mathcal{O}_C}/A_{\text{inf}}) \right) \otimes_{A_{\text{inf}}}^{\mathcal{G}^\bullet} B_{\text{max}}^\bullet \cong \left( \tau^{\leq d} R\Gamma_{\text{crys}}^{\mathcal{G}^\bullet} (X_k/W(k)) \right) \otimes_{W(k)}^{\mathcal{G}^\bullet} B_{\text{crys}}^\bullet. \]

Proof. Recall that \(B_{\text{crys}}^\bullet := A_{\text{crys}}^\bullet[\frac{1}{p}]\) as solid ring. By base change for the map of prisms \((A_{\text{inf}}, \xi) \to (A_{\text{crys}}, p)\) and crystalline comparison (Propositions 4.36 and 4.37), after inverting \(p\) we get a \((G_K, \varphi)\)-equivariant equivalence

\[ R\Gamma^{\mathcal{G}^\bullet}_{\mathcal{A}^\bullet} (X_{\mathcal{O}_C}/A_{\text{inf}}) \otimes_{A_{\text{inf}}}^{\mathcal{G}^\bullet} B_{\text{crys}}^\bullet \cong R\Gamma_{\text{crys}}^{\mathcal{G}^\bullet} (X_{k}/W(k)) \otimes_{W(k)}^{\mathcal{G}^\bullet} B_{\text{crys}}^\bullet. \]

Combining this with the Berthelot-Ogus isomorphism (Proposition B.6) we obtain a \((G_K, \varphi)\)-equivariant equivalence

\[ R\Gamma^{\mathcal{G}^\bullet}_{\mathcal{A}^\bullet} (X_{\mathcal{O}_C}/A_{\text{inf}}) \otimes_{A_{\text{inf}}}^{\mathcal{G}^\bullet} B_{\text{max}}^\bullet \cong R\Gamma_{\text{crys}}^{\mathcal{G}^\bullet} (X_{k}/W(k)) \otimes_{W(k)}^{\mathcal{G}^\bullet} B_{\text{crys}}^\bullet. \]

Finally, by tensoring it further with \(B_{\text{max}}^\bullet\) over \(B_{\text{crys}}^\bullet\) we obtain a \((G_K, \varphi)\)-equivariant equivalence

\[ R\Gamma^{\mathcal{G}^\bullet}_{\mathcal{A}^\bullet} (X_{\mathcal{O}_C}/A_{\text{inf}}) \otimes_{A_{\text{inf}}}^{\mathcal{G}^\bullet} B_{\text{max}}^\bullet \cong R\Gamma_{\text{crys}}^{\mathcal{G}^\bullet} (X_{k}/W(k)) \otimes_{W(k)}^{\mathcal{G}^\bullet} B_{\text{max}}^\bullet. \quad (4.4) \]

Thus to deduce the assertion it would be enough to show that the natural maps

\[
\left( \tau^{\leq d} R\Gamma_{\text{crys}}^{\mathcal{G}^\bullet} (X_{k}/W(k)) \right) \otimes_{W(k)}^{\mathcal{G}^\bullet} B_{\text{max}}^\bullet \to \tau^{\leq d} \left( R\Gamma_{\text{crys}}^{\mathcal{G}^\bullet} (X_{k}/W(k)) \otimes_{W(k)}^{\mathcal{G}^\bullet} B_{\text{max}}^\bullet \right),
\]

\[
\left( \tau^{\leq d} R\Gamma^{\mathcal{G}^\bullet}_{\mathcal{A}^\bullet} (X_{\mathcal{O}_C}/A_{\text{inf}}) \right) \otimes_{A_{\text{inf}}}^{\mathcal{G}^\bullet} B_{\text{max}}^\bullet \to \tau^{\leq d} \left( R\Gamma^{\mathcal{G}^\bullet}_{\mathcal{A}^\bullet} (X_{\mathcal{O}_C}/A_{\text{inf}}) \otimes_{A_{\text{inf}}}^{\mathcal{G}^\bullet} B_{\text{max}}^\bullet \right)
\]

are equivalences.

To see that the first arrow is an equivalence note that \(A_{\text{max}}\) is \(p\)-completely free as a \(W(k)\)-module. Indeed, pick a \(k\)-basis \(\{x_s\}_{s \in S}\) of \(A_{\text{max}}/p\), lifting it to \(A_{\text{max}}\) we get a map \(\oplus_{s \in S} W(k) \cdot x_s \to A_{\text{max}}\)
which becomes an isomorphism after $p$-completion. Consequently, $A_{\text{max}}^\bullet \simeq \oplus_{S \in S} W(k)^{\bullet}$ as a $W(k)^{\bullet}$-module. Then (by a similar argument to Remark 4.40) the tensor product functor $- \otimes_{W(k)^{\bullet}} A_{\text{max}}^\bullet$ is given by $p$-completed direct sum when restricted to $p$-complete derived category $D(W(k)^{\bullet})_{\text{p-comp}}$. Thus it is $t$-exact when restricted further to the essential image of the functor $(-)^\bullet := (-)^\bullet_p : D(\text{Mod}_{W(k)}) \to D(W(k)^{\bullet})_{\text{p-comp}}$ by Proposition A.12. Since $B_{\text{max}}^\bullet := A_{\text{max}}^\bullet[p,1]^\bullet$ is a filtered colimit of free $A_{\text{max}}^\bullet$-modules, it is flat over $A_{\text{max}}^\bullet$, hence the composite functor $- \otimes_{W(k)^{\bullet}} A_{\text{max}}^\bullet = (- \otimes_{W(k)^{\bullet}} A_{\text{max}}^\bullet) \otimes_{A_{\text{max}}^\bullet} B_{\text{max}}^\bullet$ is $t$-exact on the essential image of $(-)^\bullet$. Consequently, first arrow is an equivalence.

The second arrow requires a more involved analysis. Noting that $A_{\inf}$ is a $(p,u)$-completely free $G$-module (again, by picking a basis of $A_{\inf}/(p,u)$ over $k \simeq G/(p,u)$), similarly to the discussion above we get that the tensor product $- \otimes_{G^{\bullet}} A_{\inf}^\bullet$ is $t$-exact when restricted to the essential image of solid $(p,u)$-completion functor $(-)^\bullet : D(\text{Mod}_G) \to D(G^{\bullet})_{(p,u)\text{-comp}}$. Thus, factoring $G^{\bullet} \to B_{\text{max}}^\bullet$ as $G^{\bullet} \to A_{\inf}^\bullet \to B_{\text{max}}^\bullet$, using the above $t$-exactness and prismatic base change for $(G, E(u))$ we can identify the second arrow with

$$
(\tau^{\leq d} R\Gamma_{\Delta(1)}^G(X/\mathcal{S})) \otimes_{G^{\bullet}} B_{\text{max}} \to \tau^{\leq d} (R\Gamma_{\Delta}^G(X/\mathcal{S})) \otimes_{G^{\bullet}} B_{\text{max}}
$$

Finally, note that $A_{\text{max}}$ is $p$-completely free as a $G_{\text{max}}^{\bullet}$-module. Indeed, $G_{\text{max}}^{\bullet}/p \simeq k[u, t]/u^e$, where $t := \frac{u^e}{p}$ mod $p$. Similarly, $A_{\text{max}}/p \simeq G_{\text{max}}^{\bullet}/[\pi^e]$, and the map $G_{\text{max}}^{\bullet}/p \to A_{\text{max}}/p$ sends $u$ to $\pi^{e}$. A basis of $G_{\text{max}}^{\bullet}/[\pi^e]$ over $k$ then gives a basis of $A_{\text{max}}/p$ as a $G_{\text{max}}/p$-module; its lift to $A_{\text{max}}$ then provides a $p$-complete basis over $G_{\text{max}}$. Hence, as above, we deduce that the functor $- \otimes_{G_{\text{max}}^{\bullet}} A_{\text{max}}^\bullet$ is $t$-exact when restricted to the essential image of solid $p$-completion functor $(-)^\bullet_p : D(\text{Mod}_{G_{\text{max}}}) \to D(G_{\text{max}}^{\bullet})_{\text{p-comp}}$. Thus, in the end it just remains to show that the natural map

$$
(\tau^{\leq d} R\Gamma_{\Delta}^G(X/\mathcal{S})) \otimes_{G^{\bullet}} G_{\text{max}}^{\bullet}[\frac{1}{p}] \to \tau^{\leq d} (R\Gamma_{\Delta}^G(X/\mathcal{S})) \otimes_{G^{\bullet}} G_{\text{max}}^{\bullet}[\frac{1}{p}]
$$

is an equivalence.

Consider the fiber sequence

$$
\tau^{\leq d} R\Gamma_{\Delta(1)}^G(X/\mathcal{S}) \to \tau^{\leq d} R\Gamma_{\Delta}^G(X/\mathcal{S}) \to \tau^{\geq d+1} R\Gamma_{\Delta(1)}^G(X/\mathcal{S}).
$$

Applying the functor $M \mapsto M^{\bullet} \otimes_{G^{\bullet}} G_{\text{max}}^{\bullet}[\frac{1}{p}]$, and noting that $(\tau^{\leq d} R\Gamma_{\Delta(1)}^G(X/\mathcal{S}))^{\bullet} \simeq \tau^{\leq d} R\Gamma_{\Delta(1)}^G(X/\mathcal{S})^{\bullet}$ by Lemma 4.30 we get that it’s enough to show that

$$
\tau^{\leq d} (\tau^{\geq d+1} R\Gamma_{\Delta(1)}^G(X/\mathcal{S}))^{\bullet} \otimes_{G^{\bullet}} G_{\text{max}}^{\bullet}[\frac{1}{p}] \simeq 0.
$$

This follows from Lemma 4.43 below applied to $M := \tau^{\geq d+1} R\Gamma_{\Delta(1)}^G(X/\mathcal{S})[d+1]$. It satisfies the conditions of the lemma by 4.47 and Remark 4.16.

**Lemma 4.43.** Let $M \in D^{\geq 0}(\text{Mod}_G)_{(p,u)\text{-comp}}$ be a derived $(p,u)$-complete $G$-module and let $M^{\bullet} := M^{\bullet}_{(p,u)} \in D(G^{\bullet})$ denote its solid $(p,u)$-completion. Then, if $\tau^{< 0}[M/(p,u)] \in \text{Coh}(k)$ and $E(u)$-torsion $(H^0(M)[\frac{1}{p}])|E(u)| = 0$, one has

$$
\tau^{< 0} \left( M^{\bullet} \otimes_{G^{\bullet}} G_{\text{max}}^{\bullet}[\frac{1}{p}] \right) \simeq 0.
$$

**Proof.** First, note that by our assumption on $M$ and Lemma A.16 we have that $M^{\bullet} \in D(\text{Solid})^{\geq 0}$, or, in other words, $\tau^{< 0}(M^{\bullet}) \simeq 0$. Thus, it remains to show that the same holds after tensoring with $G_{\text{max}}^{\bullet}[\frac{1}{p}]$. By Remark 4.41 there is a short exact sequence of condensed $G$-modules

$$
0 \longrightarrow \mathcal{S}(T)^{\bullet} \longrightarrow \mathcal{S}(T)^{\bullet} \longrightarrow \mathcal{S}_{\text{max}}^{\bullet} \longrightarrow 0.
$$

By Remark 4.40 and Proposition A.13 the functor $- \otimes_{G^{\bullet}} \mathcal{S}(T)$ is $t$-exact on the essential image of $(-)^{\bullet}$. Thus from the above short exact sequence we get that

$$
\tau^{< -1} \left( M^{\bullet} \otimes_{G^{\bullet}} \mathcal{S}_{\text{max}}^{\bullet} \right) \simeq 0.
$$

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It remains to show that $H^{-1}(M \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}})[\frac{1}{p}] \simeq 0$.

Let $N := H^{-1}(M \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}})$. We claim that $N$ is a finitely presented $\mathfrak{S}_{\text{max}}$-module. Indeed, first note that $N$ is derived $p$-complete by Corollary A.21. Thus, by Lemma A.15 it is enough to check that the cohomology of $[N/p]$ are finitely generated modules over $\mathfrak{S}_{\text{max}}/p \simeq k[u,t]/u^e$ (here $t$ is the class of $\frac{u}{p}$ modulo $p$, and the topology on $k[u,t]/u^e$ is trivial). We have a fiber sequence

$$N[1] \longrightarrow M \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}} \longrightarrow \tau^{-\leq 0}(M \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}})$$

(4.5)

in $\text{Mod}_{\mathfrak{S}}(D(\text{Solid}))$-modules, which gives a fiber sequence

$$[N/p][1] \longrightarrow [(M \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}})/p] \longrightarrow [\tau^{-\leq 0}(M \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}})/p]$$

The complex $[N/p]$ has at most two cohomology: $H^0$ and $H^{-1}$. From the fiber sequence above, using that $[\tau^{-\leq 0}(M \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}})/p] \in D^{\geq -1}(\text{Solid})$, we see that

$$H^{-1}(N/p) \simeq H^{-2}([(M \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}})/p]) \quad \text{and} \quad H^0(N) \hookrightarrow H^{-1}([(M \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}})/p]).$$

Note that the map $\mathfrak{S} \to \mathfrak{S}_{\text{max}}/p$ is given by the composition of the projection $W(k[[u]]) \to k[[u]] \to k[u]/u^e$ and the unique $k[u]/u^e$-algebra map $k[u]/u^e \to k[t,u]/u^e$. Thus we can identify

$$[(M \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}})/p] \simeq [M/(p,u^e)] \otimes_{k[u]/u^e} k[u,t]/u^e$$

and, consequently,

$$\tau^{<0}([(M \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}})/p]) \simeq \tau^{<0}([M/(p,u^e)] \otimes_{k[u]/u^e} k[u,t]/u^e).$$

Since $\tau^{<0}[M/(p,u)] \in \text{Coh}(k)$ by our assumptions on $M$ and $[M/(p,u^e)]$ has a $k$-step filtration with associated graded pieces given by $[M/(p,u)]$, one sees that $\tau^{<0}[M/(p,u^e)] \in \text{Coh}(k[u]/u^e)$. Since $k[u,t]/u^e$ is flat over $k[u]/u^e$ from this we get that both $H^{-1}(N/p)$ and $H^0(N/p)$ are finitely generated $k[t,u]$-modules, and so $N$ itself is finitely presented.

By Lemma A.15, we then see that $N$ comes as the image of the functor $\text{Mod}_{\mathfrak{S}_{\text{max}}}^{fg} \to \text{Mod}_{\mathfrak{S}_{\text{max}}}^{\mathfrak{S}}(\text{Solid})$ given by $L \mapsto L \otimes_{\mathfrak{S}_{\text{max}}} \mathfrak{S}_{\text{max}}$ (where here $\mathfrak{S}_{\text{max}}$ and $L$ are considered with discrete topology). More explicitly, we have $N \simeq N(*) \otimes_{\mathfrak{S}_{\text{max}}} \mathfrak{S}_{\text{max}}$ where $* \in \text{Cond}$ is the singleton. In particular, we get that to show that $N[\frac{1}{p}] = 0$ it is enough to show that the discrete $\mathfrak{S}_{\text{max}}[\frac{1}{p}]$-module $N(*)[\frac{1}{p}]$ is 0.

First, let us show that $N(*)[\frac{1}{p}]$ is locally free. Note that $\mathfrak{S}_{\text{max}}[\frac{1}{p}] \simeq K_0(\mathcal{O}_L)$ is a ring of functions on an affine scheme of dimension 1 and so is a principal ideal domain. Thus it will be enough to show that the $\mathfrak{S}_{\text{max}}[\frac{1}{p}]$-module $N(*)[\frac{1}{p}]$ is $I$-torsion free for any maximal ideal $I \subset \mathfrak{S}_{\text{max}}[\frac{1}{p}]$. Since $I$ is automatically principal, the $I$-torsion $N(*)[\frac{1}{p}]/I$ is given by $H^{-1}([N(*)[\frac{1}{p}]/I])$ and so it will be enough to show that the latter group is 0. Any maximal ideal of $\mathfrak{S}_{\text{max}}[\frac{1}{p}]$ is given by the kernel of a surjective homomorphism $\mathfrak{S}_{\text{max}} \rightarrow L$ onto some finite extension $L/K_0$ that sends $u \in \mathfrak{S}_{\text{max}}$ to some element $x \in L$ such that $x^e/p \in \mathcal{O}_L$. In particular, $x$ itself should lie in $\mathcal{O}_L$ and so $I$ is generated by the minimal polynomial $f_x \in W(k)[u]$ of $x$ over $W(k)$. Note that this way $f_x$ in fact lies in $\mathfrak{S}$ and that also $\mathfrak{S}[\frac{1}{p}]/(f_x) \simeq L$. Thus the functor $- \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}}[\frac{1}{p}]/(f_x)$ can be rewritten as $- \otimes_{\mathfrak{S}_*} \mathfrak{S}[\frac{1}{p}]/(f_x)$; this way from the fiber sequence (4.5) we get a fiber sequence

$$[N[\frac{1}{p}]/f_x][1] \longrightarrow [M[\frac{1}{p}]/f_x] \longrightarrow [(\tau^{\leq 0}M \otimes_{\mathfrak{S}_*} \mathfrak{S}_{\text{max}}[\frac{1}{p}])/f_x].$$

(4.6)

Neither of the last two terms have $H^{-2}$, and so we get that $H^{-1}([N[\frac{1}{p}]/I]) = 0$. Consequently, $H^{-1}([N(*)[\frac{1}{p}]/I]) = H^{-1}([N[\frac{1}{p}]/I])(*) = 0$.

Thus, $N(*)[\frac{1}{p}]$ is locally free and (since Spec $\mathfrak{S}_{\text{max}}[\frac{1}{p}]$ is obviously connected) to show that it is 0 it is enough to show that so is its reduction $N(*)[\frac{1}{p}]/I$ modulo any ideal $I \not\subseteq R$ is 0. Note that $\mathfrak{S}_{\text{max}}/E(u) \simeq \mathcal{O}_K$ via the map sending $u$ to $\pi$; since $e$ is the ramification index of $K$ the element $\pi^e/p \in \mathcal{O}_K$, and so the
map above is well defined. Using the fiber sequence (4.6) for $f_x = E(u)$ we also get that $N[\frac{1}{p}]|E(u)$ embeds into $H^{-1}(M^\bullet[N[\frac{1}{p}]]|E(u))$. Thus, since evaluation at $* \in (s)_{prot}$ is t-exact, to check that $N(*)[\frac{1}{p}]|E(u)$ is 0 it is enough to show that $H^{-1}(M^\bullet[\frac{1}{p}]|E(u))(*)$ is. But since the original $M$ was $(p, u)$-complete by Remark 4.28 we have

$$M^\bullet[\frac{1}{p}]|E(u)(*) \simeq M[\frac{1}{p}]|E(u),$$

and so $H^{-1}$ in question is exactly given by $(H^0(M)[\frac{1}{p}])|E(u)]$, which is 0 by our assumptions on $M$. □

**Proof of Theorem 4.21:** We will obtain the isomorphism in Theorem 4.21 from the isomorphism in Proposition 4.42 by evaluating the latter on the point. By Remark 4.3 we have that $M \simeq \tau^{-d+1}R\Gamma_{\text{cryst}}(X_k/W(k))[d+1]$ satisfies $\tau^{-0}(\cdot|E) \in \text{Coh}(k)$, and so $(\tau^{-d+1}R\Gamma_{\text{cryst}}(X_k/W(k)))_p \in D^{\geq d+1}(W(k)^\bullet)$ by Lemma A.16. Since $(-)^\bullet$ is right t-exact, from this we get that $\tau^{-d}R\Gamma_{\text{cryst}}(X_k/W(k)) \simeq (\tau^{-d}R\Gamma_{\text{cryst}}(X_k/W(k)))^\bullet$. Thus, by Corollary A.21 we have

$$\tau^{-d}(R\Gamma_{\text{crys}}^\bullet(X_k/W(k)) \otimes_{W(k)}^\bullet B_{\text{max}} \cong (\tau^{-d}R\Gamma_{\text{crys}}(X_k/W(k)) \otimes_{W(k)}^\bullet A_{\text{max}})^\bullet[\frac{1}{p}],$$

where the tensor product on the right is $p$-completed. Consequently, we have a natural equivalence

$$(\tau^{-d}(R\Gamma_{\text{crys}}^\bullet(X_k/W(k)) \otimes_{W(k)}^\bullet B_{\text{max}})(*) \cong (\tau^{-d}R\Gamma_{\text{crys}}(X_k/W(k)) \otimes_{W(k)}^\bullet A_{\text{max}})[\frac{1}{p}]$$

for the value on the point $* \in (s)_{prot}$. Since by Corollary 4.2 we have $R\Gamma_{\text{crys}}(X_k/W(k)) \in \text{Coh}(k)$, the tensor product $\tau^{-d}R\Gamma_{\text{crys}}(X_k/W(k)) \otimes_{W(k)}^\bullet A_{\text{max}}$ is already $p$-complete and we get a natural equivalence

$$(\tau^{-d}(R\Gamma_{\text{crys}}^\bullet(X_k/W(k)) \otimes_{W(k)}^\bullet B_{\text{max}})(*) \cong (\tau^{-d}R\Gamma_{\text{crys}}(X_k/W(k)) \otimes_{W(k)}^\bullet B_{\text{max}}).$$

Also, from the proof of Proposition 4.42 we have an equivalence

$$(\tau^{-d}R\Gamma_{\text{crys}}^\bullet(X/\mathcal{S})) \otimes_{\mathcal{S}} B_{\text{max}} \cong (\tau^{-d}R\Gamma_{\text{crys}}^\bullet(X_{O_C}/A_{\text{inf}})) \otimes_{A_{\text{inf}}}^\bullet B_{\text{max}}$$

and $\tau^{-d}R\Gamma_{\Delta}^\bullet(X/\mathcal{S}) \simeq (\tau^{-d}R\Gamma_{\Delta}^\bullet(X/\mathcal{S}))^\bullet$. Thus, by Corollary A.21 we can rewrite left hand side as $(\tau^{-d}R\Gamma_{\Delta}^\bullet(X/\mathcal{S}) \otimes_{\mathcal{S}} A_{\text{max}})^\bullet[\frac{1}{p}]$, where the tensor product is $(p, u)$-$(or, equivalently, just (p), since u$\in (p)$ in $A_{\text{max}})$-completed. By Corollary 4.4 we have $\tau^{-d}R\Gamma_{\Delta}^\bullet(\mathcal{S}) \in \text{Coh}(\mathcal{S})$ and so the tensor product $\tau^{-d}R\Gamma_{\Delta}^\bullet(X/\mathcal{S}) \otimes_{\mathcal{S}} A_{\text{max}}$ is already $p$-complete. This way (by Lemma A.13) we can rewrite the left hand side further as $\tau^{-d}R\Gamma_{\Delta}^\bullet(X_{O_C}/A_{\text{inf}}) \otimes_{A_{\text{inf}}} A_{\text{max}})^\bullet[\frac{1}{p}]$. Thus, restricting to the point, we get that the natural map

$$\tau^{-d}R\Gamma_{\Delta}^\bullet(X_{O_C}/A_{\text{inf}}) \otimes_{A_{\text{inf}}} B_{\text{max}} \longrightarrow ((\tau^{-d}R\Gamma_{\Delta}^\bullet(X_{O_C}/A_{\text{inf}})) \otimes_{A_{\text{inf}}}^\bullet B_{\text{max}})(*)$$

is an equivalence. Finally, this way we get a commutative diagram

$$
\begin{array}{ccc}
\tau^{-d}R\Gamma_{\Delta}^\bullet(X_{O_C}/A_{\text{inf}}) \otimes_{A_{\text{inf}}}^\bullet B_{\text{max}}(*)& \longrightarrow & \tau^{-d}(R\Gamma_{\text{crys}}^\bullet(X_k/W(k)) \otimes_{W(k)}^\bullet B_{\text{max}})(*) \\
\downarrow & \cong & \downarrow \\
\tau^{-d}R\Gamma_{\Delta}^\bullet(X_{O_C}/A_{\text{inf}}) \otimes_{A_{\text{inf}}} B_{\text{max}} & \longrightarrow & \tau^{-d}R\Gamma_{\text{crys}}(X_k/W(k)) \otimes_{W(k)}^\bullet B_{\text{max}}
\end{array}
$$

that provides the desired isomorphism in Theorem 4.21.
5 Applications for schemes

Here we record a couple of applications of the above results about étale cohomology of $d$-de Rham proper stacks that lead to some new results in $p$-adic Hodge theory for schemes.

Let $X$ be a scheme over $\mathcal{O}_K$ and let $Z \hookrightarrow X_k$ be a closed subscheme of the special fiber. Then the geometric Raynaud generic fiber $(X/Z)_C \subset \hat{X}_C$ can be viewed as a complement (of a sort) in $\hat{X}_C$ to an open tubular neighborhood around $Z$ (meaning in particular that on the level of classical points of the corresponding rigid-analytic spaces we throw out the “residue discs” around all points of $Z$).

**Theorem 5.1.** Let $X$ be a proper scheme over $\mathcal{O}_K$ that is Cohen-Macauley. Let $Z \hookrightarrow X_k$ be a codimension $d$ closed subscheme of the special fiber that contains the singularities of $X$ (meaning that the complement $X \setminus Z$ is smooth over $\mathcal{O}_K$). Then

1. (Purity) There are natural isomorphisms $H^i_{\text{ét}}((X/Z)_C, \mathbb{Q}_p) \simeq H^i_{\text{ét}}(\hat{X}_C, \mathbb{Q}_p)$ for $i \leq d - 2$ and an embedding $H^{d-1}_{\text{ét}}(\hat{X}_C, \mathbb{Q}_p) \hookrightarrow H^{d-1}_{\text{ ét}}((X/Z)_C, \mathbb{Q}_p)$;
2. (Crystallinity) $H^i_{\text{ét}}(X_C, \mathbb{Q}_p)$ is a crystalline Galois representation for $i \leq d - 2$.

**Proof.** By Lemma 1.4 $X \setminus Z$ is $(d - 1)$-Hodge-proper, and so also $(d - 1)$-de Rham proper over $\mathcal{O}_K$. Note that $H^i_{\text{ ét}}(\hat{X}_C, \mathbb{Q}_p) \simeq H^i_{\text{ ét}}(X_C, \mathbb{Q}_p)$ since $X$ is proper. Also note that $(X/Z)_C \simeq X_C$, since $Z$ lies in the closed fiber, and so $H^i_{\text{ ét}}(\hat{X}_C, \mathbb{Q}_p) \simeq H^i_{\text{ ét}}((X/Z)_C, \mathbb{Q}_p)$. This way part (1) is a corollary of Theorem 4.15 while part (2) is a corollary of Corollary 3.9, both applied to $X \setminus Z$.

**Remark 5.2.** The part 2 of Theorem 5.1 can be reformulated as follows. Let $X$ be a smooth proper scheme over $K$. Assume that it has an integral model $\overline{X}$ over $\mathcal{O}_K$ that is Cohen-Macauley and such that the singularities of $\overline{X}$ are of codimension $d$ in the closed fiber. Then $H^i_{\text{ ét}}(X_C, \mathbb{Q}_p)$ is a crystalline $G_K$-representation for $i \leq d - 2$. In other words, we see that the existence of an integral Cohen-Macauley model such that singularities are of large codimension forces the étale cohomology be crystalline up to some degree depending on this codimension.

A Some homological algebra of complete modules

Here we prove some results on $t$-exactness of completed tensor products in certain situations. They will ultimately lead us to the correct bounds on the cohomological degrees in Sections 4.2 and 4.3.

Recall that for a ring $R$ we denote by $\text{Mod}_R$ the abelian category of $R$-modules, while by $D(\text{Mod}_R)$ we denote the unbounded derived category of $\text{Mod}_R$ (considered as an $\infty$-category). For a finitely generated ideal $I$ we denote by $D(\text{Mod}_{R/I}) \subseteq D(\text{Mod}_R)$ the full subcategory of derived $I$-complete modules.

**Lemma A.1.** Let $R$ be a ring and let $x \in R$ be an element. Let $\{M_\alpha\}$ be a set of derived $x$-complete $R$-modules. Then the derived completed direct sum $\bigoplus_\alpha M_\alpha$ is still concentrated in cohomological degree 0.

**Proof.** We need to show that $H^{-1} \left( \bigoplus_\alpha M_\alpha \right) \simeq 0$. By Milnor’s exact sequence

$$H^{-1} \left( \bigoplus_\alpha M_\alpha \right) \simeq \lim \limits_\leftarrow H^{-1} \left( \bigoplus_\alpha \left[ M_\alpha/x^n \right] \right) \simeq \lim \limits_\leftarrow \bigoplus_\alpha H^{-1} \left( \left[ M_\alpha/x^n \right] \right).$$

Since direct sums embed into direct products and since limits preserve injections we have

$$\lim \limits_\leftarrow \bigoplus_\alpha H^{-1} \left( \left[ M_\alpha/x^n \right] \right) \hookrightarrow \lim \limits_\leftarrow \prod_\alpha H^{-1} \left( \left[ M_\alpha/x^n \right] \right) \simeq \prod_\alpha \lim \limits_\leftarrow \left[ M_\alpha/x^n \right] \simeq 0,$$

where the right hand side vanishes by derived $x$-completeness of $M$.
Corollary A.2. The \( x \)-completed direct sum functor
\[
-\hat{\bigoplus}_S : D(\text{Mod}_R)_x^\wedge \longrightarrow D(\text{Mod}_R)_x^\wedge
\]
is \( t \)-exact.

Corollary A.3. Let \( k \) be a perfect field of characteristic \( p \). Then the \( p \)-completed tensor product functor
\[
-\hat{\otimes}_{Z_p} W(k) : D(\text{Mod}_{Z_p})_p^\wedge \longrightarrow D(\text{Mod}_{W(k)})_p^\wedge
\]
is \( t \)-exact.

Proof. Note that the forgetful functor \( \text{obl} : D(\text{Mod}_{W(k)})_p^\wedge \rightarrow D(\text{Mod}_{Z_p})_p^\wedge \) is conservative and \( t \)-exact. Thus it is enough to show the claim for the composition \( \text{obl} \circ (-\hat{\otimes}_{Z_p} W(k)) \). Pick a basis \( \{v_s\}_{s \in S} \) of \( k \) over \( \mathbb{F}_p \), where \( S \) is some indexing set. Consider a \( Z_p \)-submodule \( F_A := \oplus_{s \in S} \mathbb{Z}_p \cdot [v_s] \subset W(k) \), where \( [\cdot] : k \rightarrow W(k) \) is the Teichmüller lift; it is easy to see that \( W(k) \) is the derived \( p \)-completion of \( F_A \). Thus, composition of \( -\hat{\otimes}_{Z_p} W(k) \) with \( \text{obl} \) can be identified with the \( p \)-completed direct sum \(-\hat{\bigoplus}_S\). We are done by the corollary above.

We will now consider a slightly more complicated context. Namely, let \( \mathcal{S} := W(k)[[u]] \) for some perfect field \( k \) of characteristic \( p \). We endow \( \mathcal{S} \) with the \((p,u)\)-adic topology. We would like to have an analogue of Lemma A.1 where we take \( R = \mathcal{S} \) and replace the ideal \((x)\) by \((p,u)\). However, unfortunately it seems that the analogous statement just doesn’t hold. Nevertheless, it does hold if we make some further assumptions on \( M \).

Proposition A.4. Let \( \{M_\alpha\} \) be a set of derived \((p,u)\)-complete \( \mathcal{S} \)-modules. Assume that \( \tau^{<0}(M_\alpha \otimes_{\mathcal{S}} \mathcal{S}/(p,u)) \in \text{Coh}(k) \) for all \( \alpha \). Then the \((p,u)\)-completed direct sum \( \bigoplus_\alpha M_\alpha \) is concentrated in degree 0.

Proof. We need to show that
\[
H^{-1} \left( \lim_{\leftarrow n,m} \bigoplus_{\alpha} [M_\alpha/(p^n, u^m)] \right) \cong H^{-2} \left( \lim_{\leftarrow n,m} \bigoplus_{\alpha} M_\alpha/(p^n, u^m) \right) \cong 0.
\]

By Milnor’s exact sequence this is amount to the following vanishings:
\[
H^{-1} \cong 0 \iff \lim_{\leftarrow n,m} H^{-1} \left( \bigoplus_{\alpha} [M_\alpha/(p^n, u^m)] \right) \cong 0 \text{ and } \lim_{\leftarrow n,m} H^{-2} \left( \bigoplus_{\alpha} [M_\alpha/(p^n, u^m)] \right) \cong 0,
\]
\[
H^{-2} \cong 0 \iff \lim_{\leftarrow n,m} H^{-2} \left( \bigoplus_{\alpha} [M_\alpha/(p^n, u^m)] \right) \cong 0.
\]

Since direct sums embed into direct products and since limits preserve injections we have for \( i \in \{-1,-2\} \)
\[
\lim_{\leftarrow n,m} H^i \left( \bigoplus_{\alpha} [M_\alpha/(p^n, u^m)] \right) \cong \lim_{\leftarrow n,m} \bigoplus_{\alpha} H^i ([M_\alpha/(p^n, u^m)]) \]
\[
\cong \lim_{\leftarrow n,m} \prod_{\alpha} H^i ([M_\alpha/(p^n, u^m)]) \cong \lim_{\leftarrow n,m} \prod_{\alpha} H^i ([M_\alpha/(p^n, u^m)]) \cong 0,
\]
where the right hand side vanishes by the derived \((p,u)\)-completeness of \( M \). Hence it is left to prove the vanishing of \( \lim_{\leftarrow n,m} H^{-2} \). Note that
\[
H^{-2} \left( \bigoplus_{\alpha} [M_\alpha/(p^n, u^m)] \right) \cong \bigoplus_{\alpha} M_\alpha[p^n, u^m] \cong 0.
\]

But by Lemma A.6 below the diagram \( \{\bigoplus_{\alpha} M[p^n, u^m]\}_{n,m} \) is pro-zero (since the composition of \((N+1)\) successive vertical or horizontal maps is given by multiplication with \( p^{N+1} \) or \( u^{N+1} \), hence vanishes), so its \( \lim^1 \) is zero. \( \square \)
Definition A.5. Given a ring $R$ with an element $x \in R$, for a classical module $M \in \text{Mod}_\mathfrak{S}$ we denote by $T_x(M)$ the $x$-adic Tate module of $M$:

$$T_x(M) := \lim_{\leftarrow}^0 \left( \cdots \xrightarrow{x^n} M[x^n] \xrightarrow{x} M[x^{n-1}] \xrightarrow{x} \cdots \xrightarrow{x} M[x] \right).$$

Note that $T_x(M) \simeq H^{-1}(M_x^\mathfrak{S})$, where $M_x^\mathfrak{S}$ is the derived $x$-adic completion. In particular, if $M \in \text{Mod}_\mathfrak{S}$ is derived $x$-adically complete, then $T_x(M) = 0$.

**Lemma A.6.** Let $M \in \text{Mod}_\mathfrak{S}$ be a derived $(p, u)$-complete module such that $\tau < 0(M \otimes^\mathfrak{S}_\mathfrak{S} \mathfrak{S}/(p, u)) \in \text{Coh}(k)$. Then $M[p^\infty, u^\infty] = M[p^N, u^N]$ for some $N \geq 0$.

**Proof.** Note that $H^{-2}(M \otimes^\mathfrak{S}_\mathfrak{S} \mathfrak{S}/(p, u)) \simeq M[p][u]$ and so by our assumption on $M$, $\dim_k M[p, u] = d < \infty$. Also, $M \otimes^\mathfrak{S}_\mathfrak{S} \mathfrak{S}/p$ is derived $u$-complete, and thus so is $M[p]$. Since $T_u(M[p]) \simeq T_u(M[p][u^\infty])$, by Lemma A.7 we get that the $k[[u]]$-module $M[p][u^\infty]$ is of finite length.

Let $M_n := M[p^n][u^\infty]$ and $M_n := M_n/M_{n-1}$. We have $M_1 = M_1 = M[p][u^\infty]$. Multiplication by $p$ induces an embedding $N_1 \hookrightarrow N_1$, providing a chain of inclusions $\cdots \hookrightarrow N_i \hookrightarrow N_{i-1} \hookrightarrow \cdots \hookrightarrow N_1$. Since $N_1$ is of finite length, this chain stabilizes starting from some $n$. Also, each $M_i$ is of finite length, in particular $M_1 = M_1[p^N][u^N]$ for some $N \gg 0$. If $N = 0$, $M[p^\infty][u^\infty] = M_n$, we are done. If $N \neq 0$, take the quotient $M_i' = M_i/M_{i-1}$. Module $M_{n-1}$ is coherent, in particular derived $(p, u)$-complete; consequently, so is $M_i'$. Let $M_i' := M_i'[p]$ and $N_i' := N_i'/M_i'-1$; note that $N_i' \simeq N_{i+n-1}$. Since $N_i' \simeq N_{i+n-1}$ we get that $M_i' \simeq N_{i+n-1}$ is of finite length, contradicting our assumptions on $M$. If $N_1 = 0$, consider $M_i := M/M_{i-1}$. We have a fiber sequence $(M_{i-1})_i^\mathfrak{S} \rightarrow M_i^\mathfrak{S} \rightarrow (M_i')^\mathfrak{S}_i$, which gives a left-exact sequence

$$0 \longrightarrow T_u(M) \longrightarrow T_u(M_i') \longrightarrow T_u(M_{i-1}) \longrightarrow 0$$

since $M_{i-1}$ is finitely generated, and so is derived $u$-complete. Let $M_i' := M_i'[u]$ and $N_i' := M_i'/M_i'-1; note that $N_i' \simeq N_{i+n-1}$. We now have that multiplication by $u$ gives isomorphisms $N_i' \simeq N_{i+n-1}$ for all $i$. It follows that the map $M_i' \xrightarrow{M_i'} M_i'-1$ is surjective for all $i$, and since $M_i' \simeq N_n$ is non-zero we get $T_u(M_i') \neq 0$. Moreover, if we put $d := \dim_k M_i'$, then one has a non-canonical isomorphism $T_u(M_i') \simeq k[[u]]^\otimes d$ and, since $M_{i-1}$ is finite, this forces $T_u(M)$ to be isomorphic to $k[[u]]^\otimes d$ as well, in particular to be non-zero. \[\square\]

Consider the $p$-adic completion $\mathfrak{S}(\frac{1}{p}) := \mathfrak{S}(\frac{1}{p})^\mathfrak{S}$. We would also like to have an analogue of Lemma A.1 for the $p$-completed tensor product functor $-\otimes^\mathfrak{S} \mathfrak{S}/(p)$. Similarly to the completed direct sum, this functor is a composition of a filtered colimit of $(p, u)$-complete modules and derived completion. However, the colimit here is more complicated and this reduces the generality in which $-\otimes^\mathfrak{S} \mathfrak{S}/(p)$ is $t$-exact, as the examples below show.

**Example A.8.** Consider an $\mathfrak{S}$-module $N = \mathfrak{S}/(p)/\mathfrak{S}$. Explicitly, it is generated by elements $x_i := \frac{x_i}{p^i}$ for $i \geq 0$ and with relations given by $x_0 = 0$ and $p \cdot x_i = u \cdot x_{i-1}$. It is easy to see that $N$ is $u$-torsion free and that $p^N = u^n N$. Also, $p^N$-torsion $N[p^n]$ is generated by $x_1, \ldots, x_n$, so it is finitely-generated and in particular derived $u$-complete. It is also not hard to see that it is $u$-torsion free. We claim that the derived $(p, u)$-completion $M := N(p, u)$ coincides with the classical $p$-completion of $N$ which can be identified with
Indeed, the derived $p$-completion $N_p^\wedge$ maps to the classical $p$-adic completion $\mathcal{G}(\frac{u}{p})/\mathcal{G}$ with the fiber given by the two-term Milnor complex

$$\prod_n N[p^n] / \prod_n N[p^n]$$

where $(p)$ maps $(n_1, n_2, n_3, \ldots)$ to $(pn_2, pn_3, \ldots)$. Both source and target are derived $u$-complete and $u$-torsion free. Moreover $(p)$ is 0 modulo $u$, and so the induced map $\prod_n N[p^n]/u \to \prod_n N[p^n]/u$ is simply the identity. By derived Nakayama lemma we get that the above two-term complex is quasi-isomorphic to $\mathcal{G}(\frac{u}{p})/\mathcal{G}$.

We now claim that $\mathcal{G}(\frac{1}{u})$ is no longer concentrated in degree 0. Indeed, $H^{-1}(\mathcal{G}(\frac{1}{u}))$ is given by the $p$-adic Tate module $\lim_0 (M \otimes \mathcal{G}(\frac{1}{u}))[p^n]$, which is no longer zero after we have inverted $u$. Indeed, we have $\frac{1}{p^n} = x_n \cdot \frac{1}{u^n}$ and so $(\ldots, x_3/u^3, x_2/u^2, x_1/u)$ gives a non-zero element of $H^{-1}(\mathcal{G}(\frac{1}{u}))$.

**Remark A.9.** One can also show that the derived solid $(p, u)$-adic completion $M^\bullet\!/(p, u) \in D(\text{Solid})$ taken in (derived) condensed abelian groups lies in the heart Solid $\subset D(\text{Solid})$, but the derived $p$-completed tensor product $M^\bullet\!/(p, u) \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))} \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))}$ does not (since by Example A.8 $H^{-1} \neq 0$ for the value on the point $\mathcal{G}$). Thus, it is not true in the condensed setting that the $p$-completed tensor product $-\otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))}$ is $t$-exact as functor on derived $(p, u)$-complete solid $\mathcal{G}(\mathcal{G}(\frac{1}{u})))$-modules. In particular, the condition on $M$ that $M^\bullet\! \in D(\text{Solid})$ in this case is not sufficient.

Nevertheless, the finiteness assumptions on $M$ that we had in Proposition A.4 still turn out to be enough:

**Proposition A.10.** Let $M \in \text{Mod}_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))}$ be a derived $(p, u)$-complete module such that $\tau^\leq 0(M \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))} \mathcal{G}(\mathcal{G}(\frac{1}{u}))) \in \text{Coh}(k)$. Then the $p$-completed tensor product $M \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))} \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))}$ is concentrated in cohomological degree 0.

**Proof.** Note that if we have a map $f : M \to M'$ with $M'$ satisfying the same conditions as $M$, such that the fiber $\text{fib}(f) \in \text{Coh}(\mathcal{G}(\mathcal{G}(\frac{1}{u})))$ (equivalently, kernel and cokernel of $f$ are finitely generated) then the conclusion of the proposition holds for $M$ if it holds for $M'$. Indeed, by coherence $\text{fib}(f) \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))} \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))}$ and the latter is concentrated in cohomological degrees 0 and 1, since $M$ and $M'$ lie in the heart and $\mathcal{G}(\mathcal{G}(\frac{1}{u})))$ is flat over $\mathcal{G}(\mathcal{G}(\frac{1}{u})))$. The claim then follows by considering the long exact sequence of cohomology.

Recall that, by Lemma A.6 we have $M[p^\infty]/[u^N] = M[p^N]/[u^N]$ for some $N \geq 0$. Thus, by the above we can replace $M$ by $M/(M[p^N]/[u^N])$ and assume that $M[p^\infty]$ is $u$-torsion free.

Let $M_n := M[p^n]$. We need to show that $H^{-1}(M \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))})$ vanishes, where

$$H^{-1}(M \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))}) \xrightarrow{n} \lim_{n} \cdots \xrightarrow{p} M_2 \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))} \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))} M_1 \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))} \otimes_{\mathcal{G}(\mathcal{G}(\frac{1}{u})))} \cdots).$$

Define $N_n := M_n/M_{n-1}$; these are $k[[u]]$-modules. By our assumptions on $M$, $N_1 = M_1 = M[p]$ is finitely generated and $u$-torsion free, and so is isomorphic to $k[[u]]^{\oplus s}$ for some $s$. Multiplication by $p$ induces an embedding $N_n \hookrightarrow N_{n-1}$ for all $n$, ultimately giving a sequence of embeddings $\cdots \hookrightarrow N_n \hookrightarrow N_{n-1} \hookrightarrow \cdots \hookrightarrow N_1$. Since $N_1 \cong k[[u]]^{\oplus s}$, all $N_i$ are necessarily free; starting from some $k \gg 0$ the rank $\text{rk } N_k$ stabilizes. Factoring $M$ by $M_k$ we can assume that all $N_i$ are isomorphic to $k[[u]]^{\oplus s}$ for a given $s$. Note that $s = 0$ we are done (because then $p$-torsion in $M$ is in fact bounded).

Our goal will now be to show that if $s \neq 0$ then either $M$ is not derived $p$-complete, or $M/p$ is too big: namely, either the $u$-torsion $(M/p)[u] \cong \text{Tor}_1^*(M,\mathcal{G}(\mathcal{G}(\frac{1}{u})))$ is finitely generated, or there is a $u$-divisible element in $M$ killed by $p$ (and thus $M/p$ is not derived $u$-complete). Note that we have an embedding $M[p^\infty]/pM[p^\infty] \to M/pM$. Moreover, for each $n$ we have an embedding $\iota_{n-1} : M_{n-1}/pM_{n-1} \to M_n/pM_{n-1}$, and $M[p^\infty]/pM[p^\infty] = \text{colim}_n (M_{n-1}/pM_n)$. If $\iota_{n}$ are isomorphisms starting from some $k$, then replacing $M$ by $M/M_k$ we can assume that the maps $M_n / p^{n-1} M_{n-1}$ are surjective for all $n$ and thus the $(p)$-adic Tate module of $M$ is non-zero. It follows that $H^{-1}(M_p^\wedge) \neq 0$ and so $M$ is not derived $p$-complete — a contradiction. We get that $\iota_{n}$ is a strict embedding for infinitely many $k$. 

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Moreover, we claim that each $M_i/pM_{i+1}$ differs from $M_{n-1}/pM_n$ by a (finitely generated) $u$-torsion module. Indeed, we need to understand $(M_i/pM_{i+1})/(M_{n-1}/pM_n) \simeq M_{n-1}/(M_{n-1} + pM_{n+1})$. By our assumptions, each $M_{n+1}$ is an extension of $k[[u]]^{\oplus s}$ by $M_n$. Using the resolution

$$0 \longrightarrow \mathcal{S} \xrightarrow{p} \mathcal{S} \longrightarrow k[[u]] \longrightarrow 0,$$

for any $L \in \text{Mod}_{\mathcal{S}}$ one can identify $\text{Ext}^1_{\mathcal{S}}(k[[u]]^{\oplus s}, L)$ with $\text{Hom}_{\mathcal{S}}(k[[u]]^{\oplus s}, L/p)$. For each $n$, let $e_n \in \text{Hom}_{\mathcal{S}}(k[[u]]^{\oplus s}, k[[u]]^{\oplus s})$ be the image of the class $[M_{n+1}] \in \text{Ext}^1_{\mathcal{S}}(k[[u]]^{\oplus s}, M_n)$ under the natural map $\text{Ext}^1_{\mathcal{S}}(k[[u]]^{\oplus s}, M_n) \to \text{Ext}_1(k[[u]]^{\oplus s}, k[[u]]^{\oplus s})$ (where we identify $M_n/M_{n-1} \simeq k[[u]]^{\oplus s}$ and the isomorphism $\text{Ext}^1_{\mathcal{S}}(k[[u]]^{\oplus s}, k[[u]]^{\oplus s}) \simeq \text{Hom}_{\mathcal{S}}(k[[u]]^{\oplus s}, k[[u]]^{\oplus s})$). Note that the image of $pM_{n+1}$ in $M_{n}/M_{n-1}$ is exactly the image of $e_n$. In particular, $M_n/(M_{n-1} + pM_{n+1}) \simeq \text{coker}(e_n)$. We claim that $e_n$ is an embedding: indeed, if $e_n(x) = 0$ for some $x \in k[[u]]^{\oplus s} \simeq M_{n+1}/M_n$, then the extension $[M_{n+1}] \in \text{Ext}^1_{\mathcal{S}}(k[[u]]^{\oplus s}, M_n)$ restricted to $k[[u]] \cdot x \hookrightarrow k[[u]]^{\oplus s}$ reduces to $M_{n-1}$ and this way any lift of $x$ to $M_{n+1}$ is in fact killed by $p^n$, and so belongs to $M_n$. We get that $x \in M_{n+1}/M_n$ is actually 0. It remains to note that $e_n$ is an embedding, its cokernel is a necessarily a finite $u$-torsion module.

This way we get that $L := M[p^\infty]/pM[p^\infty]$ has an infinite filtration $0 \to L_1 \to L_2 \to \ldots \subseteq L$ such that $L_i/L_{i-1}$ are non-zero finitely-generated $u^\infty$-torsion modules. Consider $u$-torsion $L_i[u]$; starting from some $i$, $L_i[u] \simeq L_{i+1}[u]$, otherwise $\dim_k L_i[u] = \infty$, and we are done (since $L_i[u] \to M/p[u]$, but by assumption $\text{Tors}_{I}(M,k) \simeq M/p[u]$ should be finite-dimensional). But then by Lemma A.7 the $u$-adic Tate module $T_u(L)$ is non-zero. Since $T_u$ is a left-exact functor (in the sense of abelian categories) it follows that $T_u(M/p)$ is non-zero, which is a contradiction, since $M \otimes^\mathbb{L} \mathcal{S}/p$ and, consequently, $M/p$ should be derived $u$-adically complete.

We will apply the above proposition in the following context. Namely, let $K/\mathbb{Q}_p$ be a complete discretely valued extension of $\mathbb{Q}_p$, with a perfect residue field $k$. Consider the $p$-completed algebraic closure $C = \overline{K}$ and its tilt $C^o$. Choice of a uniformizer $\pi$ and a collection $\pi^\nu := (\ldots, \pi^{1/p^2}, \pi^{1/p}, \pi) \in \mathcal{O}_C^*$ of compatible $p^\nu$-roots of $\pi$ gives a unique $W(k)$-linear map $\mathcal{S} \to W(C^o)$ which sends $u$ to the Teichmüller lift $[\pi^\nu]$.

**Corollary A.11.** Let $M \in \text{Mod}_{\mathcal{S}}$ be a derived $(p,u)$-complete module such that $\tau^{<0}(M \otimes^\mathbb{L}_{\mathcal{S}} \mathcal{S}/(p,u)) \in \text{Coh}(k)$. Then the $p$-completed tensor product $M \otimes_{\mathcal{S}} W(C^o)$ is concentrated in cohomological degree 0.

**Proof.** Note that the $p$-completed tensor product functor $-\otimes_{\mathcal{S}} W(C^o)$ decomposes as the composition of $-\otimes_{\mathcal{S}} \mathcal{S}(\frac{1}{p})$ and $-\otimes_{\mathcal{S}} \mathcal{S}(\frac{1}{p^2}) W(C^o)$. Then by Proposition A.10 $M \otimes_{\mathcal{S}} \mathcal{S}(\frac{1}{p})$ is concentrated in degree 0 (and is derived $p$-complete), while $-\otimes_{\mathcal{S}} \mathcal{S}(\frac{1}{p^2}) W(C^o)$ (considered as a functor to $\mathcal{S}(\frac{1}{p})$-modules) can be rewritten as a $p$-completed direct sum (indeed, pick a basis $\{x_s\}, s \in S$, of $C^o$ over $k((u)) \simeq \mathcal{S}(\frac{1}{p})/p$; then $W(C^o)$ as a $\mathcal{S}(\frac{1}{p})$-module is identified with the $p$-adic completion of $\oplus_{s \in S} \mathcal{S}(\frac{1}{p})_p \cdot [x_s] \subseteq W(C^o)$). Thus we are done by Lemma A.1. \[\Box\]

**Solid I-completions and direct sums** We also include another proof of (a weaker form of) Proposition A.4 which uses condensed mathematics and which was suggested to us by Peter Scholze (see Lemma A.16). Some results below that precede the proof are then also used in the proof of $B_{\text{crys}}$-comparison in Section 4.3. Recall our notations from Construction 4.27: for a ring $R$ equipped with a finitely generated ideal $I$ and $M \in \text{D}(\text{Mod}_R)$ we denote

$$M^I := (M)_I \in \text{D}(\text{Solid})$$

the derived $I$-completion of $M$ in $\text{D}(\text{Solid}) \subseteq \text{D}(\text{Cond}(\text{Ab}))$. $M^I$ is naturally an $R^I$-module and naturally lands in the subcategory $\text{D}(R^I)^{\text{-comp}}$ of derived $I$-complete solid $R^I$-modules.

Till the end of this section we will denote by $\gamma : \text{D}(\text{Mod}_R) \to \text{D}(\text{Solid})$ the natural functor extending the functor $- : \text{Mod}_R \to \text{Solid}$, associating to an $R$-module $M$ the solid group $M$ represented by $M$ with discrete topology. We will completely ignore any other natural topologies that $M$ by accident can have.

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Proposition A.12. Let $R$ be a ring equipped with a finitely generated ideal $I$. Consider the full subcategory $(D(\text{Mod}_R))^\wedge$ of $D(R^\dagger_I)_{\text{I-comp}}$ given by the essential image of the (solid) $I$-completion functor $(-)^\dagger_I: D(\text{Mod}_R) \to D(R_I^\dagger)_{\text{I-comp}}$. Then for any set $S$ the $I$-completed direct sum functor

$$\bigoplus_S: (D(\text{Mod}_R))^\wedge \longrightarrow (D(\text{Mod}_R))^\wedge$$

is $t$-exact.

Proof. For $M \in D(\text{Mod}_R)$ we have

$$\text{Hom}_{D(\text{Solid})} \left( \prod_S \mathbb{Z}, M^\dagger \right) \simeq \lim_n \text{Hom}_{D(\text{Solid})} \left( \prod_S \mathbb{Z}, [M/I^n] \right) \simeq \lim_n \bigoplus_S [M/I^n] \simeq \bigoplus_S M \simeq \bigoplus_S M^\dagger,$$

where in the second equivalence we have used the fact that for any complex of abelian groups $N$ the natural map

$$\bigoplus_S N \longrightarrow \text{Hom}_{D(\text{Solid})} \left( \prod_S \mathbb{Z}, N \right)$$

is an equivalence. This follows from the fact that by [CS19, Corollary 6.1] the product $\prod_S \mathbb{Z}$ is compact in $D(\text{Solid})$, hence the functor $\text{Hom}_{D(\text{Solid})} (\prod_S \mathbb{Z}, -)$ preserves small colimits, and the basic computation [CS19, Proof of Proposition 5.7]

$$\bigoplus_S \mathbb{Z} \longrightarrow \text{Hom}_{D(\text{Solid})} \left( \prod_S \mathbb{Z}, \mathbb{Z} \right).$$

Since by [CS19, Corollary 6.1] the object $\prod_S \mathbb{Z}$ is projective, we deduce the desired $t$-exactness of the $I$-completed direct sum functor.

Under additional finiteness assumptions on $M \in D(\text{Mod}_R)$ we can also deduce the (left) $t$-exactness of the completed direct sum $\bigoplus_S M$ in $D(\text{Mod}_R)$ (as opposed to $(D(\text{Mod}_R))^\wedge \subset D(R_I^\dagger)_{I-\text{comp}}$).

Proposition A.13. Let $R$ be a Noetherian ring equipped with an ideal $I$ generated by a regular sequence. Let $M \in D^{\geq 0}(\text{Mod}_R)$ be a derived $I$-complete complex of $R$-modules such that $\tau^{<0}([M/I])$ is coherent over $R/I$. Then for any set $S$ the $I$-completed direct sum $\bigoplus_S M$ is still concentrated in degrees $\geq 0$.

Proof. Since the evaluation on a point functor preserves limit and colimits we see that the $I$-completed direct sum functor $M \mapsto \bigoplus_S M$ factors as the composition

$$D(\text{Mod}_R) \xrightarrow{(-)^\dagger_I} D(R_I^\dagger)_{I-\text{comp}} \xrightarrow{\bigoplus_S} D(R_I^\dagger)_{I-\text{comp}} \xrightarrow{(-)^\dagger} D(\text{Mod}_R)_{I-\text{comp}}.$$  

(A.2)

Since by Proposition A.12 the completed direct sum functor (A.1) is $t$-exact we deduce that if $M^\dagger_I \in D^{\geq 0}(\text{Solid})$ then so is the $I$-completed direct sum $\bigoplus_S M^\dagger_I$. Moreover, since the evaluation at a point is $t$-exact too, by factorization (A.2) for any $M$ as above the $I$-completed direct sum $\bigoplus_S M$ is also concentrated in non-negative cohomological degrees. So it is left to prove that if $M \in D^{\geq 0}(\text{Mod}_R)$ is derived $I$-complete complex of $R$-modules such that $\tau^{<0}([M/I])$ is coherent over $R/I$ then $M^\dagger_I$ lies in $D^{\geq 0}(\text{Solid})$. This is the content of Lemma A.16 below.

Before proving Lemma A.16 we first establish a few auxiliary results useful in their own rights.

Proposition A.14. Any subgroup and any quotient group of a discrete condensed abelian group is discrete.
Proof. Let $M$ be an abelian group and let $\frac{M}{I} \rightarrow Q$ be a surjective map of condensed abelian groups. We claim that $Q$ is also discrete. To see this note that the counit of adjunction gives rise to a commutative square

$$
\begin{array}{ccc}
M & \longrightarrow & Q(*), \\
\sim & & q \\
\downarrow & & \downarrow q \\
\frac{M}{I} & \longrightarrow & Q.
\end{array}
$$

Since the left vertical and the bottom horizontal arrows are surjective, the right vertical map $q$ is also surjective.

It is left to prove that $q$ is injective. In fact, we claim that for any condensed set $X$ the natural map $q: X(*) \rightarrow X$ is injective. To see this let $S = \lim_{\alpha} S_\alpha$ be a pro-finite set. Unwinding the definitions we see that $q(S)$ is given by the natural map

$$
\lim_{\alpha} X(S_\alpha) \longrightarrow X(S).
$$

But note that for each $\alpha$ the projection $S \rightarrow S_\alpha$ admits a section (since the target is a disjoint union of points), hence the induced map $X(S_\alpha) \rightarrow X(S)$ is injective. The assertion then follows from the fact that a filtered colimit of monomorphisms of sets is a monomorphism.

Finally, let $N \rightarrow \frac{M}{I}$ be an injection of condensed abelian groups. Then $N \simeq \ker(\frac{M}{I} \rightarrow \frac{M}{I}/N)$. By the previous we know that $\frac{M}{I}/N$ is discrete. Since the condensation functor $L \mapsto L$ is fully faithful and exact we deduce that $N$ is also discrete. \hfill $\square$

Lemma A.15. Let $R$ be a Noetherian ring equipped with an ideal $I$ generated by a regular sequence. Let $L \in D^{<\infty}(\text{Mod}_R(\text{Solid}))$ be an eventually connective $I$-complete complex of $R$-modules such that all cohomology of $[L/I]$ are finitely generated over $R/I$. Then all cohomology of $L$ are finitely presentable $R^\wedge$-modules. In particular, for each $m \in \mathbb{Z}$ the natural map

$$
\frac{H^m(L(*)) \otimes_{R^\wedge} R^\wedge}{I^m} \longrightarrow H^m(L)
$$

is an isomorphism.

Proof. By shifting if necessary we can assume without loss of generality that $H^{>0}(L) \simeq 0$. By induction it is enough to prove that $H^0(L)$ is finitely presentable. Before proceed further note that by Mittag–Leffler (which applies in the setting of condensed abelian groups e.g. by [BS13, Proposition 4.2.8, Proposition 3.2.3(1), and Proposition 3.1.9]) $R^\wedge$ is concentrated in degree 0. Let now $\alpha: (R^\wedge)^{\otimes m} \rightarrow L$ be a map inducing a surjection $(R/I)^{\otimes m} \rightarrow H^0([L/I])$. We claim that $\alpha$ induces surjection on $H^0$. To see this it is enough to prove that $H^0(C) \simeq 0$, where $C$ denotes the cofiber of $\alpha$. By $I$-completeness of $C$ and Milnor’s exact sequence we have

$$
\begin{array}{c}
0 \longrightarrow \lim_{\leftarrow n} H^{-1}([C/I^n]) \longrightarrow H^0(C) \longrightarrow \lim_{\leftarrow n} H^0([C/I^n]) \longrightarrow 0.
\end{array}
$$

By induction on $n$ all $H^0([C/I^n]) \simeq 0$. Moreover, from $H^0([C/I]) \simeq 0$ it follows that for all $n$ the map $H^{-1}([C/I^n]) \rightarrow H^{-1}([C/I^{n-1}])$ is surjective. By Mittag–Leffler it follows that the limit $\lim_{\leftarrow n}$-term vanishes as well. This shows that $H^0(L)$ is finitely generated $R^\wedge$-module. To prove that $H^0(L)$ is finitely presented it is left to show that the kernel of $H^0(\alpha): (R^\wedge)^{\otimes m} \rightarrow H^0(L)$ is also finitely generated. But $H^0(\alpha)$ is covered by $H^{-1}(C)$, which is finitely generated by the previous argument (applied to $L' = C[-1]$). Hence so is the kernel of $H^0(\alpha)$, and hence $H^0(L)$ is finitely presented.

The last assertion follows from the fact that the functor

$$
\text{Mod}_{R^\wedge}(\text{Solid})^{fg} \longrightarrow \text{Mod}_{R^\wedge}^{fg}, \quad N \mapsto N(*)
$$

right adjoint to $\sim \otimes_{R^\wedge} R^\wedge$ is fully faithful. This reduces to the fact that the evaluation at a point functor preserves finite colimits and that

$$
R^\wedge \simeq \text{End}_{\text{Mod}_{R^\wedge}(\text{Solid})}(R^\wedge) \simeq \text{End}_{\text{Mod}_{R^\wedge}(\text{Solid})}(R^\wedge). \quad \square
$$
Lemma A.16. Let $R$ be a Noetherian ring equipped with an ideal $I$ generated by a regular sequence. Let $M \in D^{\geq 0}(\text{Mod}_R)$ be a derived $I$-compact complex of $R$-modules such that $\tau^{<0}([M/I])$ is coherent over $R/I$. Then $M^\wedge_i$ lies in $D^{\geq 0}(\text{Solid})$.

Proof. First we claim that $[\tau^{<0}(M^\wedge_i)/I]$ is coherent over $R/I$. Indeed, from the fiber sequence

$$[\tau^{<0}(M^\wedge_i)/I] \longrightarrow [M^\wedge_i/I] \cong [M/I] \longrightarrow [\tau^{\geq 0}(M^\wedge_i)/I]$$

for each $m < 0$ we obtain the 5-term exact sequence

$$H^{m-1}([M/I]) \xrightarrow{f_{m-1}} H^{m-1}([\tau^{\geq 0}(M^\wedge_i)/I]) \longrightarrow H^m([\tau^{<0}(M^\wedge_i)/I]) \longrightarrow H^m([M/I]) \xrightarrow{f_m} H^m([\tau^{\geq 0}(M^\wedge_i)/I]).$$

It follows that $H^m([\tau^{<0}(M^\wedge_i)/I])$ is an extension of the $\ker(f_m)$ by $\Im(f_{m-1})$. Since by assumption both $H^{m-1}([M/I])$ and $H^m([M/I])$ are discrete finitely generated $R/I$-modules so are $\Im(f_{m-1})$ and $\ker(f_m)$ (here we use the fact that a subgroup and a quotient of a discrete condensed abelian group is discrete, see Proposition A.14) and hence also so is $H^m([\tau^{<0}(M^\wedge_i)/I])$.

By applying Lemma A.15 to $L = \tau^{<0}(M^\wedge_i)$ we deduce that for each $m < 0$ the natural map

$$H^m(M^\wedge_i)(* \otimes_{R^i}^{\mathbb{L}} R^\wedge_i) \longrightarrow H^m(M^\wedge_i)$$

is an isomorphism. But $H^m(M^\wedge_i)(* \otimes_{R^i}^{\mathbb{L}} R^\wedge_i) \cong H^m(M^\wedge_i) \cong 0$, hence $\tau^{<0}(M^\wedge_i) \cong 0$. □

Complete tensor product. In this paragraph we show that for a ring $R$ equipped with a finitely generated ideal $I$ the $I$-completed tensor product functor $\hat{\otimes}_{I}^{\mathbb{L}}$ is closely related to the solid tensor product functor $\hat{\otimes}_{\text{Solid}}^{\mathbb{L}}$ in $\text{Mod}_{R^\wedge_i}(D(\text{Solid}))$. We would like to mention that most proofs here were explained to us by Peter Scholze (and in particular due to him).

Proposition A.17. Let $R$ be a ring with a finitely generated ideal $I \subseteq R$. Then for a pair of bounded above derived $I$-complete complexes of $R^\wedge_i$-modules $X,Y \in D(R^\wedge_i)^{<\infty}_{\text{comp}}$ their tensor product $X \hat{\otimes}_{R^\wedge_i}^{\mathbb{L}} Y$ is also derived $I$-complete.

Proof. Note that for any $f \in I$ the natural map $\mathbb{Z}[x] \to R$ sending $x$ to $f$ extends to a map $\mathbb{Z}[[x]] \to R^\wedge_i$ (since $\mathbb{Z}[[x]] \cong \mathbb{Z}[x]^{\wedge_i}$), where $\mathbb{Z}[[x]]$ is considered with $x$-adic topology. Note that a complex of $R^\wedge_i$-modules is $I$-complete if and only if it is $(x)$-complete as a $\mathbb{Z}[[x]]$-module for each such ring morphism $\mathbb{Z}[x] \to R^\wedge_i$ (that maps $x$ to an element of $I$). So it is enough to prove that $X \hat{\otimes}_{R^\wedge_i}^{\mathbb{L}} Y$ is $(x)$-complete for each such homomorphism $\mathbb{Z}[[x]] \to R^\wedge_i$.

To see the latter note that the tensor product $X \hat{\otimes}_{R^\wedge_i}^{\mathbb{L}} Y$ is equivalent to the geometric realization of the simplicial complex of solid $\mathbb{Z}[[x]]$-modules with terms given by

$$X \otimes_{\mathbb{Z}[[x]]}^{\mathbb{L}} R^\wedge_i \otimes_{\mathbb{Z}[[x]]}^{\mathbb{L}} \cdots \otimes_{\mathbb{Z}[[x]]}^{\mathbb{L}} \hat{\otimes}^{\mathbb{L}} \mathbb{Z}[[x]] Y.$$

Note that by the boundness assumption on $X$ and $Y$ for each $i \in \mathbb{Z}$ the $i$-cohomology module of the complex $X \hat{\otimes}_{R^\wedge_i}^{\mathbb{L}} Y$ receives contribution only from finitely many terms of (A.3). Since derived complete modules are closed under (co)kernels and extensions and since $X \hat{\otimes}_{R^\wedge_i}^{\mathbb{L}} Y$ is $(x)$-complete if and only if all of its cohomology $H^i(X \hat{\otimes}_{R^\wedge_i}^{\mathbb{L}} Y)$ are, it is enough to prove that all $X \otimes_{\mathbb{Z}[[x]]}^{\mathbb{L}} R^\wedge_i \otimes_{\mathbb{Z}[[x]]}^{\mathbb{L}} \cdots \otimes_{\mathbb{Z}[[x]]}^{\mathbb{L}} \hat{\otimes}^{\mathbb{L}} \mathbb{Z}[[x]] Y$ are $(x)$-complete. This reduces the assertion to the special case $R = \mathbb{Z}[x], I = (x)$.

To treat this case let $\omega_1$ be the smallest uncountable cardinal. Since the category $D(\text{Mod}_{\mathbb{Z}[[x]]}(\text{Solid}))$ is compactly generated by objects of the form $\prod I \mathbb{Z}[[x]]$ it is also $\omega_1$-compactly generated by objects of the form $\bigoplus \prod I \mathbb{Z}[[x]]$, where the direct sum is at most countable. In particular, both $X,Y$ can be written as a completion of an $\omega_1$-filtered colimit of objects of the form $\bigoplus \prod I \mathbb{Z}[[x]]$. But note that the $(x)$-completion functor, being a countable limit (hence $\omega_1$-small), commutes with $\omega_1$-filtered colimits. Hence both sides of the comparison map

$$X \otimes_{\mathbb{Z}[[x]]}^{\mathbb{L}} Y \longrightarrow X \hat{\otimes}_{\mathbb{Z}[[x]]}^{\mathbb{L}} Y$$


preserve $\omega_1$-filtered colimits in both variables. It follows that it is enough to prove that $X \otimes_{\mathbb{Z}[x]} Y$ is $(x)$-complete for $X = Y = \bigoplus_I \prod_I \mathbb{Z}[x]$.

Concretely, the completed direct sum $\bigoplus_{j} \prod_I \mathbb{Z}[x]$ can be described as

$$\colim_{f: N \to \mathbb{N}, f(n) \to \infty} \prod_{n \in \mathbb{N}} x^{f(n)} \prod_I \mathbb{Z}[x] \longrightarrow \prod_N \prod_I \mathbb{Z}[x],$$

where the colimit is taken over the diagram of functions $f: \mathbb{N} \to \mathbb{N}$ tending to $\infty$ when $n \to \infty$ partially ordered by the pointwise inequality. Indeed, the module above is classically $x$-adically complete and $x$-torsion free, hence is derived $x$-complete. Moreover, the natural embedding $\bigoplus_{j} \prod_I \mathbb{Z}[x] \to \prod_N \prod_I \mathbb{Z}[x]$ factors through a map

$$\alpha: \bigoplus_{j} \prod_I \mathbb{Z}[x] \longrightarrow \colim_{f: N \to \mathbb{N}, f(n) \to \infty} \prod_{n \in \mathbb{N}} x^{f(n)} \prod_I \mathbb{Z}[x].$$

So it is enough to see that $\alpha / (x)$ is an isomorphism, which is clear. Under this identification we have

$$\bigoplus_{j} \prod_I \mathbb{Z}[x] \otimes_{\mathbb{Z}[x]} \bigoplus_{j} \prod_I \mathbb{Z}[x] \simeq \colim_{f,g: N \to \mathbb{N}, f(n) \to \infty} \prod_{n \in \mathbb{N}} x^{f(n) + g(m)} \prod_I \mathbb{Z}[x],$$

$$\bigoplus_{j} \prod_I \mathbb{Z}[x] \otimes_{\mathbb{Z}[x]} \bigoplus_{j} \prod_I \mathbb{Z}[x] \simeq \colim_{h: N \times N \to \mathbb{N}, h(n,m) \to \infty} \prod_{n \in \mathbb{N}} x^{h(n,m)} \prod_I \mathbb{Z}[x].$$

So it is enough to prove that among all functions $h: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ tending to $\infty$ the functions of the form $f(n) + g(m)$ are cofinal. Concretely, for each such $h$ we need to construct a pair of functions $f, g$ such that $h(n,m) \leq f(n) + g(m)$ for all $n, m \in \mathbb{N}$. This inequality is satisfied e.g. by

$$f(n) = g(n) := \max_{i,j \leq n} h(i, j).$$

Note that the category $D(\mathbb{M}od_R)_{I-\text{comp}}$ is endowed with the natural symmetric monoidal structure given by the $I$-completed tensor product $- \otimes_R -$.

**Corollary A.18.** Let $R$ be a ring and let $\mathbb{I} \trianglelefteq R$ be a finitely generated ideal. Then the functor

$$D^{<\infty}(\mathbb{M}od_R)_{I-\text{comp}} \longrightarrow D(\mathbb{M}od_{R^I}^{\text{Solid}}), \quad M \mapsto M^I$$

is symmetric monoidal.

*Proof. For $M, N \in D(\mathbb{M}od_R)^I_I$ there is a natural comparison map $M^I \otimes_{R^I} N^I \to (M \otimes_{R^I} N)^I$ which becomes an equivalence modulo $\mathbb{I}$. Hence it is enough to prove that $M^I \otimes_{R^I} N^I$ is derived $I$-complete in $D(\mathbb{M}od_{R^I}^{\text{Solid}})$, which is a content of the previous proposition.*

**Corollary A.19.** Let $R$ be a ring equipped with a finitely generated ideal $\mathbb{I} \trianglelefteq R$ and assume that $R$ is derived $I$-complete. Let $S$ be a localization of $R$. Then for each $M, N \in D^{<\infty}(\mathbb{M}od_S^{\text{Solid}}))$ with condensed structure coming from derived $I$-complete $R$-lattices (i.e. $M \simeq (M_0)^I \otimes_{R^I} S^I$ for some derived $I$-complete $R$-module $M_0$ and similarly for $N$) their solid tensor product $M \otimes_{S^I}^\bullet N$ is equivalent to $(M_0 \otimes_{R^I} N_0)^I \otimes_{R^I} S^I$.

*Proof. This follows from the associativity of the tensor product and the previous corollary:

$$M \otimes_{S^I}^\bullet N \simeq ((M_0)^I \otimes_{R^I}^\bullet S) \otimes_{S^I}^\bullet ((N_0)^I \otimes_{R^I}^\bullet S^I) \simeq ((M_0)^I \otimes_{R^I}^\bullet (N_0)^I) \otimes_{R^I}^\bullet S^I \simeq (M_0 \otimes_{R^I} N_0)^I \otimes_{R^I} S^I.$$

**Example A.20.** Take $R = \mathbb{Z}_p$, $I = (p)$, $S = \mathbb{Q}_p$. Then for a pair of $\mathbb{Q}_p$-vector spaces $V, W$ with topology coming from $(p)$-complete $\mathbb{Z}_p$-lattices $V_0, W_0$ the solid tensor product of their condensation $V_p^\bullet \otimes_{\mathbb{Q}_p} W_p^\bullet$ is equivalent to the “continuous” tensor product $(V_0 \otimes_{\mathbb{Z}_p} W_0)_p^\bullet \otimes_{\mathbb{Q}_p} \mathbb{Q}_p$.

We will also need the following variant of Corollary A.18.
Corollary A.21. Let \( R \) be a ring and let \( I \subseteq R \) be a finitely generated ideal. Let \( N \in D^{\infty}(\text{Mod}_R)_{I-\text{comp}} \) be a complex of \( R \)-modules such that the derived \( I \)-completed tensor product functor \( - \hat{\otimes}_R N \) has bounded cohomological amplitude. Then for each \( M \in D(\text{Mod}_R)_{I-\text{comp}} \) the natural map

\[
M_I^\wedge \otimes^R_{I^\wedge} N_I^\wedge \longrightarrow (M \hat{\otimes}_R N)_I^\wedge
\]

is an equivalence.

Proof. It is enough to show that \( M_I^\wedge \otimes^R_{I^\wedge} N_I^\wedge \) is \( I \)-complete. Write \( M \) as a colimit of its Whitehead tower \( M \simeq \lim \tau^{\leq n}M \). By Corollary A.19 for each \( n \) the tensor product \( (\tau^{\leq n}M)_I^\wedge \otimes^R_{I^\wedge} N_I^\wedge \) is equivalent to \( (\tau^{\leq n}(M) \hat{\otimes}_R N)_I^\wedge \), in particular it is \( I \)-complete. Moreover, since by assumption both functors \( - \hat{\otimes}_R N \) and \( (-)_I^\wedge \) have bounded cohomological amplitude, for each \( i \in \mathbb{Z} \) the cohomology \( H^i((\tau^{\leq n}(M) \hat{\otimes}_R N)_I^\wedge) \) is isomorphic to \( H^i((\tau^{\leq n}(M) \hat{\otimes}_R N)_I^\wedge) \) for some \( n \gg 0 \). It follows that all cohomology of \( M_I^\wedge \otimes^R_{I^\wedge} N_I^\wedge \) are derived \( I \)-complete, hence so is \( M_I^\wedge \otimes^R_{I^\wedge} N_I^\wedge \).

\[ \square \]

B Berthelot–Ogus isomorphism

In this section we prove the Berthelot–Ogus isomorphism between rational crystalline and de Rham cohomology of smooth Artin stacks in a possibly ramified base field case. In the main part of the text we need this isomorphism to take into account the natural \( p \)-adic topology present on both sides, so we prove the equivalence in question between the crystalline and de Rham cohomology considered as objects of \( D(\text{Cond}(\text{Ab})) \). The argument is a more or less straightforward adaptation of the one from [BO'83, Section 2]. As above let \( K \) be a complete discretely valued field of characteristic 0 with the ring of integers \( \mathcal{O}_K \) and perfect residue field \( k \) of characteristic \( p \).

Proposition B.1. Let \( S \) be a base scheme of characteristic \( p \) and let \( S \hookrightarrow T \) be a \( p \)-thickening. Then for each smooth quasi-compact quasi-separated Artin \( S \)-stack \( X \) and non-negative integer \( n \) there exists a functorial morphism

\[
V_{\leq n}: \tau^{\leq n}\Gamma_{\text{crys}}(X/T) \longrightarrow \tau^{\leq n}\Gamma_{\text{crys}}(X^{(1)}/T)
\]

such that the composition of \( V_n \) with the \( n \)-th truncation of the pullback along the relative Frobenius \( \varphi_X: X \rightarrow X^{(1)} \) (in both orders) is equivalent to the multiplication by \( p^n \):

\[
\tau^{\leq n}(\varphi_X^n) \circ V_{\leq n} \simeq V_{\leq n} \circ \tau^{\leq n}(\varphi_X^n) \simeq p^n.
\]

Proof. Since homotopy limits are left \( t \)-exact one has

\[
\tau^{\leq n}\Gamma_{\text{crys}}(-/T) \simeq \tau^{\leq n}(\text{Ran}_{\mathcal{A}b_{/S}^{\emph{subfl}} \tau^{\leq n}\Gamma_{\text{crys}}(-/T)}),
\]

and similarly for \( \tau^{\leq n}\Gamma_{\text{crys}}(-^{(1)}/T) \). It follows that it is enough to construct a functorial map \( V_n \) for affine schemes over \( S \). This is a content of [BO'78, Theorem 8.20].

Now recall the definition of condensation of crystalline cohomology Definition 4.34. With these notions we have:

Corollary B.2. Let \( S \) be a base scheme of characteristic \( p \) and let \( S \hookrightarrow T \) be a \( p \)-thickening, where \( T \) is a \( p \)-adic formal scheme. Then for a smooth quasi-compact quasi-separated Artin \( S \)-stack \( X \) the pullback along the relative Frobenius \( \varphi_X: X \rightarrow X^{(1)} \) induces an equivalence of rational crystalline cohomology

\[
\Gamma_{\text{crys}}^\wedge(X^{(1)}/T) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \Gamma_{\text{crys}}^\wedge(X/T) \otimes_{\mathbb{Z}} \mathbb{Q}.
\]

Proof. Since the embedding \( D(\text{Ab}) \hookrightarrow D(\text{Solid}) \) is \( t \)-exact, since the derived \( p \)-completion functor

\[
(-)_p^\wedge: D(\text{Solid}) \longrightarrow D(\text{Solid})
\]
is $t$-exact up to a shift, and since $\mathbb{Q}$ is flat in solid abelian groups we have that

$$RT^\Lambda_{\text{cris}}(-/T) \otimes \mathbb{Z} \mathbb{Q} \simeq \lim_{\leftarrow} \left( (\tau^{\le n} \Gamma_{\text{cris}}(-/T))^\Lambda_p \otimes \mathbb{Z} \mathbb{Q} \right).$$

So it is enough to show that for all $n \in \mathbb{Z}_{\ge 0}$ the map

$$(\tau^{\le n}(\varphi_X^*))_p^\Lambda \otimes \mathbb{Z} \mathbb{Q}: (\tau^{\le n} \Gamma_{\text{cris}}(X^{(1)}/T))_p^\Lambda \otimes \mathbb{Z} \mathbb{Q} \longrightarrow (\tau^{\le n} \Gamma_{\text{cris}}(X/T))_p^\Lambda \otimes \mathbb{Z} \mathbb{Q}$$

is an equivalence. But by applying the $p$-completion functor $(-)^\Lambda_p: D(\text{Ab}) \to D(\text{Solid})$ to the map $V_{\le n}$ from the previous proposition we see that $(\tau^{\le n}(\varphi_X^*))_p^\Lambda$ admits an inverse up to multiplication by $p^n$, hence rationally becomes an equivalence.

**Proposition B.3** (Nil-functoriality of the rational crystalline cohomology). *Let $i: S_0 \to S$ be a pro-nilpotent thickening of characteristic $p$ schemes and let $S \rightsquigarrow T$ be a pd-thickening, where $T$ is a $p$-adic formal scheme. Then there exists a functor

$$RT^\Lambda_{\text{cris}}(-/T) \otimes \mathbb{Q}: \text{Stk}_{/S_0}^{\text{sm,op}} \longrightarrow \text{Mod}_\mathbb{Q}(D(\text{Solid}))$$

making the following diagram commutative

$$\begin{array}{ccc}
\text{Stk}_{/S}^{\text{sm,op}} & \xrightarrow{RT^\Lambda_{\text{cris}}(-/T) \otimes \mathbb{Q}} & \text{Mod}_\mathbb{Q}(D(\text{Solid})).
\end{array}$$

Proof. By assumption $S \simeq \lim_{\leftarrow} S_\alpha$, where $S_\alpha$ are nilpotent thickenings of $S_0$. First we claim that for each $\alpha$ there is a functor

$$RT^\Lambda_{\text{cris,}\alpha}(-/T) \otimes \mathbb{Q}: \text{Stk}_{/S_0}^{\text{sm,op}} \longrightarrow \text{Mod}_\mathbb{Q}(D(\text{Solid}))$$

making the following diagram commutative

$$\begin{array}{ccc}
\text{Stk}_{/S_\alpha}^{\text{sm,op}} & \xrightarrow{- \times S_\alpha} & \text{Stk}_{/S}^{\text{sm,op}} \xrightarrow{RT^\Lambda_{\text{cris}}(-/T) \otimes \mathbb{Q}} \text{Mod}_\mathbb{Q}(D(\text{Solid})).
\end{array}$$

To construct it let $\mathcal{I}$ be the sheaf of ideals of the embedding $S_0 \to S_\alpha$. Since $\mathcal{I}$ is nilpotent there exists an integer $n$ large enough so that $\mathcal{I}^n = 0$. In particular, there exists a dashed arrow $\rho_{\alpha,n}: S_\alpha \to S_0$ making the diagram below commutative

$$\begin{array}{ccc}
S_0 & \xrightarrow{F_{S_0}} & S_0 \\
\downarrow & \searrow & \downarrow \\
S_\alpha & \xrightarrow{F_{S_\alpha}} & S_\alpha.
\end{array}$$

where $F_{S_0}$ and $F_{S_\alpha}$ are absolute Frobenii of $S_0$ and $S_\alpha$ respectively. We then define $RT^\Lambda_{\text{cris,}\alpha}(-/T) \otimes \mathbb{Q}$ as the composition

$$\begin{array}{ccc}
\text{Stk}_{/S_0}^{\text{sm,op}} & \xrightarrow{- \times S_0 \rho_{\alpha,n}} & \text{Stk}_{/S_\alpha}^{\text{sm,op}} \xrightarrow{- \times S_\alpha} \text{Stk}_{/S}^{\text{sm,op}} \xrightarrow{RT^\Lambda_{\text{cris}}(-/T)} \text{Mod}_\mathbb{Q}(D(\text{Solid})).
\end{array}$$

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We claim that there is a natural equivalence

$$R\Gamma_{\text{crys},a,n}(\times S_0/T) \otimes \mathbb{Q} \simeq R\Gamma_{\text{crys}}(\times S/T) \otimes \mathbb{Q}.$$ 

In order to construct it note that by construction the composition of the pullbacks along the embedding $S_0 \hookrightarrow S$, $\rho_{a,n}$ and the projection $S \to S_0$ sends an $S_0$-stack $X$ to the $n$-th relative (over $S$) Frobenius twist of $X \times S_0 S$. The $n$-th relative Frobenius then induces a natural morphism

$$\varphi_{a,n}(X \times S_0 S) \longrightarrow (X \times S_0 S)^{(n)} \simeq ((X \times S_0 S_0) \times S_0, \rho_{a,n}, S_0) \times S_0 S.$$ 

By applying the pullback in crystalline cohomology $R\Gamma_{\text{crys}}(\times T) \otimes \mathbb{Q}$ to $\varphi_{a,n}$ we then obtain a natural transformation

$$R\Gamma_{\text{crys},a,n}(\times S_0/T) \otimes \mathbb{Q} \longrightarrow R\Gamma_{\text{crys},a,n}(\times S_0 S_0/T) \otimes \mathbb{Q}$$

which is an equivalence by Corollary B.2.

Moreover, since for any integer $m \geq n$ we have $\rho_{a,m} \simeq \rho_{a,n} \circ F_{S_0}^{m-n}$ it follows that

$$(- \times S_0, S_0, S_0) \times S_0 S \simeq ((- \times S_0, S_0, S_0) \times S_0 S)^{(m-n)},$$

and the $(m-n)$-th relative Frobenius and pullback in crystalline cohomology induce a natural transformation of functors

$$R\Gamma_{\text{crys},a,m}(\times T) \otimes \mathbb{Q} \longrightarrow R\Gamma_{\text{crys},a,n}(\times T) \otimes \mathbb{Q}$$

which is an equivalence again by Corollary B.2. We then define

$$R\Gamma_{\text{crys},a}(\times T) \otimes \mathbb{Q} := \lim_{\rightarrow} R\Gamma_{\text{crys},a,n}(\times T) \otimes \mathbb{Q}$$

which is naturally equivalent to $R\Gamma_{\text{crys},a,n}(\times T) \otimes \mathbb{Q}$ for any $n$.

Finally, since for a morphism $f: S_0 \to S_0$ and large enough $n$ we have $\rho_{a,n} \simeq \rho_{a,n} \circ f$ there is a natural equivalence $R\Gamma_{\text{crys},a}(\times T) \otimes \mathbb{Q} \simeq R\Gamma_{\text{crys},a}(\times T) \otimes \mathbb{Q}$ under which the equivalence

$$R\Gamma_{\text{crys},a}(\times S_0 S_0/T) \otimes \mathbb{Q} \simeq R\Gamma_{\text{crys}}(\times S_0 S/T) \otimes \mathbb{Q}$$

identifies with the restriction along the pullback $\text{Stk}_{\text{op}}^{\text{sm},a} \to \text{Stk}_{\text{op}}^{\text{sm},a}$ of the corresponding equivalence for $R\Gamma_{\text{crys},a}(\times S_0 S_0/T)$. We conclude passing to the colimit over $\alpha$ and using the fact that by [KP19, Theorem 2.1.13] the pullbacks induce an equivalence $\lim_{\rightarrow} \text{Stk}_{\text{op}}^{\text{sm},a} \simeq \text{Stk}_{\text{op}}^{\text{sm},a}$.

**Proposition B.4** (Berthelot–Ogus comparison). *Let $X$ be a smooth quasi-compact quasi-separated Artin stack over $\mathcal{O}_K$. Then there exists a functorial equivalence*

$$R\Gamma_{\text{crys}}(\mathcal{X}_k/W(k)) \otimes W(k) \otimes K \simeq R\Gamma_{\text{crys}}(\hat{X}/\mathcal{O}_K) \otimes \mathcal{O}_K K.$$

**Proof.** The ideal $(p)$ in $\mathcal{O}_K$ admits a (unique) pd-structure and the inclusion $W(k) \hookrightarrow \mathcal{O}_K$ is a pdmorphism. It follows from the base change for crystalline cohomology Proposition B.5 below that we have a natural equivalence

$$R\Gamma_{\text{crys}}(\mathcal{X}_k/W(k)) \otimes W(k) \mathcal{O}_K \simeq R\Gamma_{\text{crys}}((\mathcal{X}_k \otimes_k \mathcal{O}_k/p)/\mathcal{O}_K).$$

On the other hand, since $X$ is a smooth lifting of $\mathcal{X}_k/p$ to $\mathcal{O}_K$ by smooth descent for crystalline and de Rham cohomology and [BO78, Corollary 7.4] in the affine case we have

$$R\Gamma_{\text{crys}}(\mathcal{X}_k/p/\mathcal{O}_K) \simeq R\Gamma_{\text{crys}}(\hat{X}/\mathcal{O}_K).$$

Moreover, since both $\mathcal{X}_k \otimes_k \mathcal{O}_k/p$ and $\mathcal{X}_k/p$ are nil-extensions of $\mathcal{X}_k$ we have

$$R\Gamma_{\text{crys}}((\mathcal{X}_k \otimes_k \mathcal{O}_k/p)/\mathcal{O}_K) \otimes \mathcal{O}_K K \simeq R\Gamma_{\text{crys}}((\mathcal{X}_k/p/\mathcal{O}_K) \otimes \mathcal{O}_K K$$

by Proposition B.3. Combining the last equivalence with the rationalizations of equivalences (B.1) and (B.2) we deduce the desired comparison. 

\[\Box\]
**Proposition B.5.** Let \( A \to A' \) be a ring morphism of finite \( p \)-complete Tor amplitude. Then for a smooth quasi-compact quasi-separated Artin stack \( X \) over \( A/(p) \) the natural maps

\[
R\Gamma_{crys}(X/A) \otimes_A A' \longrightarrow R\Gamma_{crys}(X'/A'),
\]

\[
R\Gamma_{crys}(X/A) \otimes_{A'} A' \longrightarrow R\Gamma_{crys}(X'/A')
\]

are equivalences, where \( X' \) denotes the base change \( X \times_{Spec A/(p)} Spec A'/(p) \).

**Proof.** Both sides of the first arrow are derived \( p \)-complete, hence it is enough to prove the base change modulo \( p \). In this case both parts identify with de Rham cohomology for which the base change is proved e.g. in [KP19, Proposition 1.1.8]. By construction and Corollary A.21 the second map is a condensation of the first one, hence is also an equivalence. \( \square \)

In the main part of the text we also use the following comparison of crystalline cohomology:

**Proposition B.6.** Let \( X \) be a smooth quasi-compact quasi-separated Artin stack over \( O_K \). There is a natural equivalence

\[
R\Gamma_{crys}(X_k/W(k)) \otimes_{W(k)} B_{crys} \simeq R\Gamma_{crys}(X_{O_{cp}/p}/A_{crys})[\frac{1}{p}].
\]

**Proof.** By the base change for crystalline cohomology we have the natural equivalence

\[
R\Gamma_{crys}(X_k/W(k)) \otimes_{W(k)} W(\kappa)^A \simeq R\Gamma_{crys}(X_{\kappa}/W(\kappa)).
\]

Hence it is enough to prove a more general statement: let \( Y \) be a smooth quasi-compact quasi-separated Artin stack over \( O_{C_p}/p \). Then there is a functorial equivalence

\[
R\Gamma_{crys}(Y_{\kappa}/W(\kappa)) \otimes_{W(\kappa)} B_{crys} \simeq R\Gamma_{crys}(Y/A_{crys})[\frac{1}{p}]. \tag{B.3}
\]

Note that the morphism \( O_{C_p}/(p) \to \kappa \) is an ind-nilpotent extension: indeed, \( O_{C_p}/(p) \simeq O_{\Gamma p}/(p) \) and \( O_{\Gamma p} \simeq \lim O_L \), where \( L \) runs over the poset of finite extensions of the maximal unramified extension \( K_{nr} \) of \( K \), and for each such \( L \) the ring \( O_L/(p) \) is a nilpotent extension of \( O_{K_{nr}}/(p) \simeq \kappa \). It follows from Proposition B.3 that there is a natural equivalence

\[
R\Gamma_{crys}(Y_{\kappa} \otimes_{\kappa} O_{C_p}/p/A_{crys})[\frac{1}{p}]) \simeq R\Gamma_{crys}(Y/A_{crys})[\frac{1}{p}]. \tag{B.4}
\]

Moreover, by the base change for crystalline cohomology again we have

\[
R\Gamma_{crys}(Y_{\kappa} \otimes_{\kappa} A_{crys}) \simeq R\Gamma_{crys}(Y_{\kappa} \otimes_{\kappa} O_{C_p}/p/A_{crys}). \tag{B.5}
\]

By combining (B.4) with (B.5) we deduce (B.3). \( \square \)

### C Cohomological descent, de Rham comparison, and local acyclicity for some singular schemes (by Haoyang Guo)

In this appendix, we recall cohomological descent for étale cohomology of an algebraic variety or a rigid space, and generalize the de Rham comparison in [DLLZ18] to singular varieties. As an application, we give a small extension of Theorem 4.15 for some schemes over \( O_K \) that are singular within the generic fiber in Theorem C.16.

We fix a complete discretely valued extension \( K \) of \( \mathbb{Q}_p \), and its complete algebraic closure \( C \) throughout the section.
C.1 Cohomological descent for étale cohomology

Denote \( \text{Var}_K \) to be the category of finite type schemes over \( K \), and \( \text{Rig}_K \) to be the category of qcqs rigid spaces over \( K \). Let \( X \) be either in \( \text{Var}_K \) or \( \text{Rig}_K \).

**Definition C.1.** Let \( a: X_\bullet \rightarrow X \) be a simplicial object with an augmentation over \( X \), in either \( \text{Var}_K \) or \( \text{Rig}_K \). It is called a simplicial resolution of singularities of \( X \) if it is a proper hypercovering in the sense of [SP20, Tag 0DHI], such that each \( X_{n+1} \rightarrow (\cosk_n sk_n X_\bullet)_{n+1} \) is a resolution of singularity as in [BM08, Theorem 1.1].

**Remark C.2.** By definition, a simplicial resolution of singularities for an algebraic variety is preserved under a field extension.

**Remark C.3.** When \( X \) is an open subspace inside of a proper algebraic/rigid variety \( X \), with a Zariski closed complement \( D \), one can apply the embedded resolution of singularities on the pair \((X,D)\) to get a pair of simplicial spaces \((\overline{X}_\bullet, D_\bullet) \rightarrow (\overline{X}, D)\), so that each \( D_n \) is a simple normal crossing divisor in the smooth variety \( \overline{X}_n \), and both \( \overline{X}_\bullet \rightarrow X \) and the simplicial open subset \( \overline{X}_\bullet \setminus D_\bullet \rightarrow X \) are simplicial resolution of singularities.

A simplicial resolution of singularities provides a tool of computing cohomology of singular spaces using that of smooth ones, by the following result.

**Theorem C.4.** Let \( a: X_\bullet \rightarrow X \) be a simplicial resolution of singularities in either \( \text{Var}_K \) or \( \text{Rig}_K \). Then we have
\[
R\Gamma_\text{ét}(X_C, \mathbb{Q}_p) \simeq R\Gamma_\text{ét}(X_\bullet C, \mathbb{Q}_p),
\]
where the latter is computed as the homotopy limit \( R\lim_{[n] \in \Delta^{op}} R\Gamma_\text{ét}(X_n C, \mathbb{Q}_p) \) over the simplicial diagram \( \Delta^{op} \).

**Proof.** For algebraic varieties, this is proved for example in [Con], where the only non-formal input is the proper base change theorem for a pullback from a point. For the case of rigid space, we apply the same proof and the proper base change theorem for rigid spaces as in [Hub96, Theorem 4.4.1.(b)].

C.2 Du Bois complex

We then recall the notion of the (Deligne)-Du Bois complex, introduced by Deligne and studied in Du Bois’s thesis. The idea is to generalize the de Rham complex to the non-smooth setting, using cohomological descent and resolution of singularities.

**Definition C.5.** Let \( X \) be either in \( \text{Var}_K \) or \( \text{Rig}_K \), and let \( a: X_\bullet \rightarrow X \) be a simplicial resolution of singularities. The Du Bois complex of \( X \), denoted as \( \Omega^\bullet_{X/K} \) is an object in the \( \mathbb{N}^{op} \)-filtered derived category of sheaves of \( K \)-modules over \( X \), defined as the homotopy limit of the Hodge-filtered de Rham complexes as below
\[
\Omega^\bullet_{X/K} := R\lim_{[n] \in \Delta^{op}} Ra_n\Omega^\bullet_{X_n/K}.
\]

**Remark C.6.** It can shown that the construction is independent of the choice of the simplicial resolution of singularities \( a: X_\bullet \rightarrow X \). See for example [PS08, Theorem 7.22] for algebraic varieties, and [Guo22, Section 5] for rigid spaces (with a slightly different notation as \( \Omega^\bullet_{\text{ch}} \)). In particular, when \( X \) is smooth itself, the Du Bois complex is filtered isomorphic to its de Rham complex with Hodge filtration.

**Remark C.7.** When \( K \) is replaced by the field of complex numbers, it can be shown that cohomology of the Du Bois complex for a complex algebraic variety is isomorphic to singular cohomology of \( X \bar{\otimes} \), and the induced filtration from \( \Omega^\bullet_{X/K} \) is the Hodge filtration on each singular cohomology group. See for example [PS08, Section 7.3].
Remark C.8. By construction, the $i$-th graded piece of the Du Bois complex, denoted as $\Omega^i_{X/K}[-i]$, is a bounded below complex with coherent cohomology over $O_X$. Using either classical Hodge theory ([PS08, Section 7.3]) or $p$-adic Hodge theory ([Guo22, Theorem 1.2.2], or an upcoming work of Bhatt-Lurie on $p$-adic Riemann-Hilbert correspondence), it can be shown that the coherent complex $\Omega^i_{X/K}$ lives in cohomological degree $[0, \dim(X) - i]$.

We also provide a useful computation tool for Du Bois complex using resolution of singularities.

Proposition C.9. Let $X$ be a finite type variety over $K$, and let $i: Z \to X$ be a Zariski closed subspace of $X$. Denote by $\pi: X' \to X$ the blowup of $X$ at $Z$, and let $Z'$ be the preimage of $Z$. Then for each $i \in \mathbb{N}$, there is a natural distinguished triangle as below

$$\Omega^i_{X/K} \to R\pi_* \Omega^i_{X'/K} \oplus i_* \Omega^i_{Z/K} \to R\pi_* \Omega^i_{Z/K}.$$ 

Proof. One can prove this for example following a filtered version of [PS08, Example 7.25], or [Guo22, Proposition 5.0.4].

C.3 de Rham comparison for singular varieties

In this subsection, we state the de Rham comparison for non-smooth algebraic varieties.

Theorem C.10. Let $X$ be in $\text{Var}_K$. Then there exists a natural isomorphism of complexes of $B_{\text{dR}}$-modules

$$R\Gamma_{\text{ét}}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \simeq R\Gamma(X, \Omega^\bullet_{X/K}) \otimes_K B_{\text{dR}}.$$ 

Proof. Let $X \to \overline{X}$ be a compatification with the complement $D$, and let $a: X_s \to X$ be a simplicial resolution of singularities, extended to an embedded resolution for $\overline{a}: \overline{X}_s \to \overline{X}$ as in Remark C.3. By the result of [DLLZ18] in Construction 2.17 and Example 2.18, for each $n \in \mathbb{N}$ we have a natural isomorphism

$$R\Gamma_{\text{ét}}(X_{nC}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \simeq R\Gamma_{\text{dR}}(X_n/K) \otimes_K B_{\text{dR}}.$$ 

On the other hand, by cohomological descent as in Theorem C.4, we have

$$R\Gamma_{\text{ét}}(X_C, \mathbb{Q}_p) \simeq R \lim_{[n] \in \Delta^{op}} R\Gamma_{\text{ét}}(X_{nC}, \mathbb{Q}_p).$$ 

Combine the above two isomorphisms, we obtain the following formula

$$R\Gamma_{\text{ét}}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \simeq (R \lim_{[n] \in \Delta^{op}} R\Gamma_{\text{dR}}(X_n/K)) \otimes_K B_{\text{dR}},$$ 

where we implicitly uses the convergence of the spectral sequence associated to this simplicial diagram ([SP20, Tag 0DHP]) to commute the homotopy limits with the tensor product with $B_{\text{dR}}$. Finally, using the definition of the Du Bois complex, the last term above can be rewritten as $R\Gamma(X, \Omega^\bullet_{X/K}) \otimes_K B_{\text{dR}}$, thus we get the formula as below

$$R\Gamma_{\text{ét}}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \simeq R\Gamma(X, \Omega^\bullet_{X/K}) \otimes_K B_{\text{dR}}.$$ 

As a consequence, we obtain the following result, which was first proved by Kisin using de Jong’s alteration.

Corollary C.11. Let $X$ be in $\text{Var}_K$. Then for each $i \in \mathbb{N}$, its étale cohomology $H^i_{\text{ét}}(X_C, \mathbb{Q}_p)$ is a de Rham Galois representation of $G_K$. 

C.4 Hodge properness and Hodge proper resolutions

We consider the $d$-Hodge properness for spaces over $K$ that are not necessarily smooth in this subsection.

**Definition C.12.** Let $X$ be either in $\text{Var}_K$ or $\text{Rig}_K$. It is called $d$-**Hodge proper** if for each integer $0 \leq i \leq d$, its cohomology of (shifted) graded piece of Du Bois complex $R\Gamma(X, \Omega^i_{X/K})$ is a perfect $K$-linear complex.

**Remark C.13.** When $X$ is a finite type algebraic variety that is smooth over $K$, the notion coincides with the definition as in Definition 1.1, following Remark C.6.

Our goal is to prove the following result on $d$-Hodge properness.

**Theorem C.14.** Let $X$ be in $\text{Var}_K$ or $\text{Rig}_K$, and let $X_{\text{sing}}$ be its singular locus. Assume $X$ admits a Cohen-Macaulay $^{13}$ compactification $X \to \overline{X}$ with the complement $D$ of codimension $d + 2$, such that the closure of $X_{\text{sing}}$ in $\overline{X}$ is $Z$ itself. Then $X$ is $d$-Hodge proper.

**Proof.** We first notice that when $X$ is smooth admitting a compactification as in the hypothesis, this is the commutative algebra statement as in Lemma 1.4 (together with Remark C.6).

In general, we reduce to the smooth case as follows. Let $Z$ be a Zariski closed subspace of $X$, inside of $X_{\text{sing}}$. By assumption, its closure in $\overline{X}$ is equal to itself, and we can form the following diagram of varieties, with each square being cartesian

$$
\begin{array}{cccc}
Z' & \longrightarrow & X' = \text{Bl}_X(Z) & \longrightarrow & \overline{X'} = \text{Bl}_X'(Z) \leftarrow D \\
\downarrow & & & & \downarrow \\
Z & \longrightarrow & X & \longrightarrow & \overline{X} \leftarrow D
\end{array}
$$

In particular, as $Z$ is a closed subscheme of $\overline{X}$, its preimage $Z' = Z \times_X X'$ in $X'$ is also a closed subscheme of $\overline{X}'$. Moreover, both $Z$ and $Z'$ are proper and hence Hodge proper over $K$.

To proceed, we use the resolution of singularities in [BM08, Theorem 1.1.(2')], which says that there exists a finite sequence of blowups

$$X = X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_t,$$

such that each $X_{i+1} \to X_i$ is a blowup at a Zariski closed subspace $C_i$ of $X_i$ for $C_i \subseteq X_i_{\text{sing}}$, and $X_t$ is smooth over $K$. In particular, inductively as in the paragraph above, the closure of $C_{i+1}$ in $\overline{X}_{i+1} = \text{Bl}_X(C_i)$ is $C_{i+1}$ itself, and thus $C_i$ and $C_i \times_X X_{i+1} = C_i \times_{\overline{X}_i} \overline{X}_{i+1}$ are proper over $K$ for each $i$.

At last, we apply Proposition C.9 to get the following distinguished triangle

$$\Omega^{i}_{C_i/K} \rightarrow R\pi_! \Omega^i_{X_{i+1}/K} \oplus \Omega^j_{C_i/K} \rightarrow R\pi_! \Omega^j_{C_i \times_X X_{i+1}/K}.$$

In this way, by the properness of $C_i$ and $C_i \times_X X_{i+1}$ and the descending induction from $X_t$, we get the $d$-Hodge properness of $X = X_0$ using long exact sequences applying at their derived global sections.

**Proposition C.15.** Let $X$ be in $\text{Var}_K$. Assume $X$ admits a Cohen-Macaulay compactification $X \to \overline{X}$ with the complement $D$ of codimension $d + 2$. Then there is a proper hypercovering $(X_\bullet, \overline{X}_\bullet)$ of the pair $(X, \overline{X})$, such that

- the map $X_\bullet \to X$ is a simplicial resolution of singularities;
- each $X_n \to \overline{X}_n$ is a compactification with its complement isomorphic to $D = \overline{X} \setminus X$.

In particular, each $X_n$ is smooth and $d$-Hodge proper.

---

$^{13}$The Cohen-Macaulay condition for rigid spaces can be found for example in [Ber93], and is defined as local rings being Cohen-Macaulay.
We give the construction inductively, using the same idea of proof as in Theorem C.14. For \( n = 0 \), we apply the resolution of singularity of \([BM08, \text{Theorem 1.1}]\) at \( X \), and extends to its compactification \( \overline{X} \) via blowing up at the same centers to get the pair \((X_0, \overline{X}_0)\). By the second and the third paragraph of the proof of Theorem C.14, the map \( X_0 \to X \) (and similarly for \( \overline{X}_0 \to \overline{X} \)) is a finite composition of blowups \( X_{0,i+1} = \text{Bl}_{X_{0,i}}(C_{0,i}) \to X_{0,i} \to \cdots \to X_{0,0} = X \), such that the closure of the blowup center \( C_{0,i} \) within the associated compactification \( \overline{X}_{0,i} \) is itself. In particular, the complement \( \overline{X}_0 \setminus X_0 \) is naturally isomorphic to \( D := \overline{X} \setminus X \), which is Cohen-Macaulay of codimension \( d + 2 \). So the pair \((X_0, \overline{X}_0)\) satisfies the requirement. Moreover, as each blowup center \( C_{0,i} \) is within the singular loci of \( X_{0,i} \), the natural map of pairs \((X_0, \overline{X}_0) \to (X, \overline{X})\) is an isomorphism away from \( X_{\text{sing}} \).

To get the construction for \( n + 1 \), by Definition C.1, it suffices to notice that the singular locus of \((\cosk_n \overline{X}_n, \overline{X}_n)_{n+1}\) is within the preimage of \( X_{\text{sing}} \) in \((\cosk_n \overline{X}_n, \overline{X}_n)_{n+1}\). The latter is because by induction the map of pairs \((X_m, \overline{X}_m) \to (X, \overline{X})\) are isomorphisms away from \( X_{\text{sing}} \) for all \( m \leq n \). In particular, the pair of truncated simplicial diagram \(((\cosk_n \overline{X}_n)_{n+1}, (\cosk_n \overline{X}_n)_{n+1})\) is isomorphic to the constant diagram when restricted to \((X \setminus X_{\text{sing}}, \overline{X} \setminus X_{\text{sing}})\). Thus the singular locus of \((\cosk_n \overline{X}_n, \overline{X}_n)_{n+1}\) is contained in \( X_{\text{sing}} \times X (\cosk_n \overline{X}_n, \overline{X}_n)_{n+1}\), whose closure in \((\cosk_n \overline{X}_n, \overline{X}_n)_{n+1}\) is disjoint with \( D \). So we are done. \( \square \)

### C.5 Local acyclicity in the singular case

In this subsection, we extend the local acyclicity in Theorem 4.15 to a finite type scheme \( X \) over \( \mathcal{O}_K \) that has singularity away from the special fiber.

Precisely, we prove the following.

**Theorem C.16.** Let \( X \) be a finite type scheme over \( \mathcal{O}_K \). Assume the following conditions hold for \( X \):

1. there is an open subscheme \( U \) of \( X \) containing the special fiber \( X_k \), such that \( U \) is smooth over \( \mathcal{O}_K \);
2. there is an open immersion \( X \subset \overline{X} \) into a proper and Cohen-Macaulay \( \mathcal{O}_K \)-scheme \( \overline{X} \), such that the closure of \( X \setminus U \) in \( \overline{X} \) is \( X \setminus U \) itself, and the complement \( D = \overline{X} \setminus X \) has codimension \( d + 2 \).

Then the natural map below is an isomorphism for \( 0 \leq i \leq d \) and an injection for \( i = d + 1 \)

\[
Y_{X, \mathbb{Q}_p} : H^i_{\text{ét}}(X_C, \mathbb{Q}_p) \to H^i_{\text{ét}}(\overline{X}_C, \mathbb{Q}_p).
\]

Here we note that by assumption, Raynaud’s generic fiber \( \overline{X}_K \), which is equal to \( \overline{X}_K \), is smooth over \( K \).

**Corollary C.17.** Assume \( X \) is as in Theorem C.16. Then for each \( i \leq d \), the étale cohomology \( H^i_{\text{ét}}(X_C, \mathbb{Q}_p) \) is a crystalline representation.

**Proof.** We first notice that if \( X \) is smooth over \( \mathcal{O}_K \) satisfying the condition (ii), then \( X \) is \( d \)-Hodge proper over \( \mathcal{O}_K \) (Lemma 1.4), and the statement is proved in Theorem 4.15. In general, by assumption (i) the \( \mathcal{O}_K \)-non-smooth locus \( X_{\text{sing}} \subset X \setminus U \) is supported within the generic fiber \( X_K \). Moreover, the closure of \( X_{\text{sing}} \) in \( \overline{X} \) is \( X_{\text{sing}} \) itself. As a consequence, by blowing up at subschemes within the generic fibers, we can apply the construction of Proposition C.15 to get a simplicial diagram of \( \mathcal{O}_K \)-schemes \((X_\bullet, \overline{X}_\bullet)\) over \((X, \overline{X})\), satisfying

1. the generic fiber \( X_\bullet \to X_K \) is a simplicial resolution of singularities;
2. each \( X_n \to \overline{X}_n \) is a Cohen-Macaulay compactification;
3. the restriction of \( \overline{X}_\bullet \to \overline{X} \) on the open subset below is isomorphic to the constant augmented diagram

\[
\overline{X}_\bullet \times_X (\overline{X} \setminus X_{\text{sing}}) \to \overline{X} \setminus X_{\text{sing}}.
\]

Notice that by assumption (a) and (c), each \( X_n \) is in particular smooth over \( \mathcal{O}_K \), and the compactification \( X_n \to \overline{X}_n \leftarrow D \) satisfies the assumption in Lemma 1.4. Here to check \( \overline{X}_n \) is Cohen-Macaulay, it suffices
to do so after by restricting at the preimage of the covering $\overline{X} = (X \setminus X_{\text{sing}}) \cup X_K$, where the claim then follows from the assumption (ii) of $\overline{X}$, property (c), and the smoothness in property (a) above.

Now let us consider the following diagram extending the derived version of $Y_{X,C}$:

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
R\Gamma_{\text{et}}(X_C, \mathbb{Q}_p) \\
\sim
\end{array}

\begin{array}{c}
\begin{array}{c}
R\Gamma_{\text{et}}(\hat{X}_C, \mathbb{Q}_p)
\end{array}
\end{array}

\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
R\Gamma_{\text{et}}(X^* C, \mathbb{Q}_p)
\end{array}

\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
R\Gamma_{\text{et}}(\hat{X}^* C, \mathbb{Q}_p).
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

Here the two vertical arrows are isomorphisms by cohomological descent at $X_K \to X_K$ and $\hat{X}_K \to \hat{X}_K$ (Theorem C.4). For each $[n] \in \Delta^p$, by applying Theorem 4.15 at the compactification $X_n \to \overline{X}_n \leftarrow D$, the map below is an isomorphism at degree $[0,d]$ and induces an injection on $H^{d+1}$

$$
R\Gamma_{\text{et}}(X_{nC}, \mathbb{Q}_p) \longrightarrow R\Gamma_{\text{et}}(\hat{X}_{nC}, \mathbb{Q}_p).
$$

On the other hand, we can consider the spectral sequence associated to the simplicial diagram as in [SP20, Tag 0DHP] (similarly for $\hat{X}_C$)

$$
E_1^{i,j}(X^* C) = H_{\text{et}}^i(X_{iC}, \mathbb{Q}_p) \Rightarrow H_{\text{et}}^{i+j}(X_C, \mathbb{Q}_p).
$$

Then the map of spectral sequences $E_1^{i,j}(X^* C) \to E_1^{i,j}(\hat{X}^* C)$ is an equivalence for $j \leq d$ and is an injection for $i = d + 1$. Notice that since the differential map $d_{r}^{i,j}$ sends $E_r^{i,j}$ to $E_r^{i+r,j-r+1}$, the total degree goes from $(i+j)$ to $(i+r) + (j-r+1) = (i+j+1)$. In particular, to calculate $E^\infty_{i,j}$, by the vanishing of negative terms in the spectral sequence, it only involves the following finite amount of degrees in the spectral sequence

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\{(0,i+j-1),(1,i+j-2),\ldots,(i-1,j),(i,j),(i+1,j),\ldots,(i+j+1,0)\}, \text{ if } i > 0;
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\{(0,j),(1,j),\ldots,(j+1,0)\}, \text{ if } i = 0.
\end{array}
\end{array}
\end{array}
$$

As a consequence, we get

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
E^\infty_{i,j}(X^* C) = E^\infty_{i,j}(\hat{X}^* C), \text{ for } i + j \leq d + 1, \text{ } i > 0;
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
E^\infty_{0,d+1}(X^* C) \hookrightarrow E^\infty_{0,d+1}(\hat{X}^* C),
\end{array}
\end{array}
\end{array}
\end{array}
$$

which implies the statement by the general fact about computing spectral sequences. \qed

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