String Equation for String Theory on a Circle

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We derive a constraint (string equation) which together with the Toda Lattice hierarchy determines completely the integrable structure of the compactified 2D string theory. The form of the constraint depends on a continuous parameter, the compactification radius $R$. We show how to use the string equation to calculate the free energy and the correlation functions in the dispersionless limit. We sketch the phase diagram and the flow structure and point out that there are two UV critical points, one of which (the sine-Liouville string theory) describes infinitely strong vortex or tachyon perturbation.

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Introduction

Bosonic string theory with a two-dimensional target space, or \( c = 1 \) string theory (see, e.g., the review \([1]\)), is commonly considered as completely solved. In fact, this is not quite true, and some of the most intriguing questions here are still waiting to be answered. One of these questions is whether strong perturbations of the world sheet action can change the geometry of the target space. According to the FZZ conjecture \([2]\), the compactified \( c = 1 \) string theory in presence of a sufficiently strong vortex or tachyon source is related by S-duality to a 2d string theory in a Euclidean black hole background \([3,4]\). The string theory in a strong vortex background forms a new phase, which we call the sine-Liouville phase, because the role of the Liouville potential here is played by a sine-Liouville potential\(^1\).

A vortex perturbation of the 2d string theory is equivalent to a time-dependent tachyon perturbation in the T-dual theory. The simplest nontrivial example of such a perturbation is the sine-Gordon model coupled to quantum gravity, which was first studied by G. Moore \([6]\). Some interesting results followed for the general case, in particular the discovery of the integrable structure of the Toda Lattice hierarchy \([7]\).

Recently, this problem has been reconsidered in \([8]\), where the fact that the flows generated by allowing vortices on the world sheet are described by the Toda Lattice hierarchy was used. In this paper we complete the approach of \([8]\) by deriving a constraint (a ”string equation”), which, together with the Toda Lattice hierarchy, completely determines the theory. We consider in more detail the dispersionless (genus zero) limit and show how to use the string equation to perform calculations. We then check that the string equation correctly reproduces the free energy in the presence of a pair of vortex operators obtained in \([8]\), as well as the one- and two-point correlators in the Liouville phase, recently calculated in \([9]\). Finally, we discuss the phase structure of the theory and give a qualitative description of the sine-Liouville phase.

Tachyon and vortex sources in the compactified \( c = 1 \) string theory

Euclidean 2D string theory in a flat background is described on the string worldsheet by the conformal field theory of a massless scalar \( x \) coupled to a \( c = 25 \) Liouville field \( \phi \), with worldsheet Lagrangian

\[
\mathcal{L} = \frac{1}{4\pi}[((\partial x)^2 + (\partial \phi)^2 - 2\hat{R}\phi + \mu e^{-2\phi}] + \mathcal{L}_{\text{ghost}}. \tag{1}
\]

\(^1\) The relation between the topological version of the 2d Euclidean black hole and the \( c=1 \) string compactified at the self-dual radius has been previously pointed out by Mukhi and Vafa \([5]\).
If the $c = 1$ coordinate is compactified at radius $R$, then the theory has a discrete spectrum and the excitations carrying the momentum and the winding modes are represented by the tachyon operators $T_n$ and the vortex operators $V_n$ ($n = 0, \pm 1, \pm 2, ...$)

$$T_n \sim \int_{\text{world sheet}} e^{inx/R} e^{(|n|/R-2)\phi}$$

$$V_n \sim \int_{\text{world sheet}} e^{in\tilde{x}R} e^{(|n|R-2)\phi},$$

where the field $\tilde{x}$ is T-dual to $x$. The vortex operators can be created as topological defects describing localized winding modes: a vortex of charge $nR$ is located at the endpoint of a line along which the time coordinate has discontinuity $2\pi nR$.

- Matrix model formulation and Toda Lattice symmetry

The $c = 1$ string theory compactified at length $2\pi R$ can be constructed as a large $N$ matrix model (Matrix Quantum Mechanics), which can be viewed as a dimensional reduction of an $N$-color 2d Yang-Mills theory (for details see the review [1]). After being dimensionally reduced to the circle $x + 2\pi R \equiv x$, the YM theory is described in terms of two hermitian $N \times N$ matrix fields, the Higgs field $M = M_i^j(x)$ and the gauge field $A = A_i^j(x)$. It is more convenient to consider the grand canonical partition function, in which the chemical potential $\mu$ plays the role of a cosmological constant:

$$Z(\mu, R, \hbar) = \sum_{N=0}^{\infty} e^{-\frac{1}{\hbar} 2\pi R \mu N} \int_{A(x+2\pi R) = A(x)} DMDA e^{-\frac{1}{\hbar} Tr \int_0^{2\pi R} (-\frac{1}{2} [i\partial_x + A, M]^2 + \frac{1}{2} M^2) dx}.$$  

We have introduced explicitly the Planck constant $\hbar$, which is also the string interaction constant. The dominant contribution of the sum over $N$ comes from $N \sim 1/\hbar$. In order to make sense of the path integral, the measure $DM$ should be stabilized by an appropriate cutoff, for example an infinite potential wall placed far from the origin.

The tachyon operators are represented as the discrete momentum modes of the Higgs field $M(x)$:

$$T_n \sim \int_0^{2\pi R} dx \ e^{i\frac{\pi}{n} x} [M(x)]^{in\pi}$$

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2 We will use units in which the T-duality acts as $R \to 1/R$ and thus $R_{KT} = 2$.

3 This expression for the tachyon operators in the scaling limit is due to V. Kazakov.
while the vortex operators can be constructed as the moments of the holonomy factor of the gauge field $A(x)$ around the spacetime circle $[8]$

$$V_n \sim \text{Tr} \Omega^n, \quad \Omega = \hat{T} \exp \int_0^{2\pi R} dx A(x). \quad (6)$$

Note that, in spite of the asymmetry in the definition of the tachyon and vortex operators, their expectation values and correlation functions are related by the T-duality ($R \rightarrow 1/R$). 

The integral over the gauge field $A$ can be done explicitly and the resulting integral with respect to the eigenvalues of the Higgs field $M$ can be written as the partition function of free nonrelativistic fermions defined on the spacetime circle. This remains true also if a purely tachyonic source $\sum_n \lambda_n T_n$ is added to the exponent in (4). Using the properties of the amplitudes for tachyon scattering [10], Dijkgraaf, Moore and Plesser showed in [7] that the partition function (4) with purely tachyonic source added is a $\tau$-function of the Toda Lattice hierarchy $[11]$. 

On the other hand, if the theory is perturbed by a purely vortex source $\sum_{n \neq 0} t_n V_n$, then the integral with respect to the Higgs field $M$ can be done exactly and one obtains an effective one-plaquette gauge theory whose only variable is the unitary matrix $\Omega$, the holonomy factor around the spacetime circle $[8][5]$. The partition function of this theory can be expressed in terms of free fermions defined on the unit circle and is shown to be again a $\tau$-function of the Toda Lattice hierarchy.

In this way the $c = 1$ theory perturbed by a purely vortex or purely tachyonic source is completely integrable and its integrable structure is described by the Toda Lattice hierarchy. The Toda Lattice hierarchy is formulated in terms of the “time” variables $t_{\pm 1}, t_{\pm 2}, ...$ and a discrete “spatial” variable $s \in \mathbb{h} \mathbb{Z}$. It is natural to split the time variables into two sets $t = \{ t_n \}_{n=1}^{\infty}$ and $\tilde{t} = \{ \tilde{t}_n \}_{n=1}^{\infty}$ where $\tilde{t}_n \equiv t_{-n}$. The correspondence with 2D string

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4 The T-duality can be made explicit if the matrix model on a circle (3) is considered as the limit of Yang-Mills theory defined on a two-dimensional torus, when one of the periods tends to zero. Then the tachyon and vortex operators (3)-(4) are constructed as Polyakov loops winding $n$ times around one of the two periods of the torus.

5 This observation was first made by by Boulatov and Kazakov [12], who used it to study the non-singlet states of the $c = 1$ string theory.

6 This matrix model belongs to a class of integrable models [13,14,15] representing various generalizations of the Gaudin gas [14].

7 The integrability is however lost if the theory is perturbed by both tachyon and vortex sources.
theory requires to consider a complex variable $s$, which is related to the cosmological constant $\mu$ as $s = \frac{1}{2} - i\mu$. The Planck constant $\hbar$, which appears as the spacing unit of the difference operators in the Lax formalism, plays the role of string interaction constant.

The exact statement, made in [7] on the basis of the analysis of the tachyon S-matrix in [10], is that the free energy of the string theory

$$\mathcal{F}(\mu, t, \bar{t}) = \sum_{g \geq 0} \hbar^{2g} \langle \langle e^{\sum_{n \neq 0} t_n V_n} \rangle \rangle_g$$

where $\langle \langle \rangle \rangle_g$ means the connected expectation value on a genus-$g$ worldsheet, is related to the $\tau$-function of the Toda Lattice hierarchy as

$$\tau = e^{F/\hbar^2}.$$ (8)

Thus the free energy of the string theory satisfies an infinite chain of PDE, the first of which is the Toda equation

$$\frac{\partial}{\partial t_1} \frac{\partial}{\partial \bar{t}_1} \mathcal{F}(\mu) + \exp \left( \frac{1}{\hbar^2} [\mathcal{F}(\mu + i\hbar) + \mathcal{F}(\mu - i\hbar) - 2\mathcal{F}(\mu)] \right) = 0.$$ (9)

This equation determines the flow in the compact $c = 1$ string theory perturbed by the lowest vortex operators $V_{\pm 1}$. In [8], the free energy on the sphere and on the torus was obtained as the solution of (9) satisfying at $t = \bar{t} = 0$ the boundary condition

$$\mathcal{F}(\mu) = -\frac{1}{2} R \mu^2 \log \mu - \hbar^2 \frac{R + R^{-1}}{24} \log \mu + \mathcal{O}(\hbar^2).$$ (10)

The string susceptibility $\chi = \partial^2 \mathcal{F}$ at genus zero was obtained as a solution of a simple algebraic equation,

$$\mu e^{\chi/R} + (R - 1) t_1 \bar{t}_1 e^{(2-R)\chi/R} = 1,$$ (11)

which resums the perturbative expansion found in [6]. The same approach was subsequently applied to calculate the vacuum expectation values of the vortex operators and the two-point correlators [9].

The algebraic equation (11) suggests strongly that an operator constraint (a “string equation”) similar to those found for the $c < 1$ matrix models [18] might exist. Below we derive the string equation, which leads to (11), but before doing that we will briefly recall the Lax formulation of the Toda hierarchy [11].

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8 The derivation of [7] was critically reconsidered in [17].
Lax formalism

The Lax formalism is based on finite-difference operators where the Planck constant $\hbar$ (which in our interpretation plays the role of string interaction constant) emerges as a spacing unit \[19\]. The Lax operators $L$ and $\bar{L}$ are defined as series expansions in the shift operator

$$\hat{\omega} = e^{i\bar{\hbar}\partial/\partial \mu}$$

with coefficients depending on $\mu$ and the couplings $t$ and $\bar{t}$

$$L = r\hat{\omega} + \sum_{k=0}^{\infty} u_k \hat{\omega}^{-k}, \quad \bar{L} = \hat{\omega}^{-1} r + \sum_{k=0}^{\infty} \hat{\omega}^k \bar{u}_k.$$  \(13\)

The commuting flows along the “times” $t_n$ and $\bar{t}_n$ are generated by the operators

$$H_n = (L^n)_{>0} + \frac{1}{2}(L^n)_0, \quad \bar{H}_n = (\bar{L}^n)_{<0} + \frac{1}{2}(\bar{L}^n)_0, \quad (n = 1, 2, \ldots)$$ \(14\)

where the symbol $(\cdot)_>$ means the positive (negative) parts of the series in the shift operator $\hat{\omega}$ and $(\cdot)_0$ means the constant part. If we define the “covariant derivatives”

$$\nabla_n = \hbar \frac{\partial}{\partial t_n} - H_n \quad (n = \pm 1, \pm 2, \ldots),$$  \(15\)

then the Lax equations

$$[\nabla_n, L] = [\nabla_n, \bar{L}] = 0,$$  \(16\)

are equivalent to the zero-curvature conditions

$$[\nabla_m, \nabla_n] = 0.$$  \(17\)

The Lax operators $L, \bar{L}$ can be thought as the result of dressing transformations

$$L = W \hat{\omega} W^{-1}, \quad \bar{L} = \bar{W} \hat{\omega}^{-1} \bar{W}^{-1},$$  \(18\)

where the dressing operators $W$ and $\bar{W}$ have series expansions

$$W = \sum_{n \geq 0} w_n \hat{\omega}^{-n}, \quad \bar{W} = \sum_{n \geq 0} \bar{w}_n \hat{\omega}^n.$$  \(19\)

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9 We use slightly different notations than in \[11\] because we would like to preserve the symmetry between $L$ and $\bar{L}$, which is appropriate for the double scaling limit of the matrix model.
Lax-equations can be converted into Sato equations for the dressing operators

$$\nabla_n W = \nabla_n \bar{W} = 0, \quad n = \pm 1, \pm 2, \ldots$$

(20)

The dressing operators are related to the \( \tau \)-function (8) by

$$z^{-i\mu} W z^{i\mu} = \tau^{-1} e^{-\frac{i}{\hbar} D_t(z)} - \frac{i}{\hbar} \partial \tau$$

$$\bar{z}^{-i\mu} \bar{W} \bar{z}^{i\mu} = \tau^{-1} e^{-\frac{i}{\hbar} D_{\bar{t}}(\bar{z})} + \frac{i}{\hbar} \partial \bar{\tau}$$

(21)

where we introduced the spectral parameters \( z \) and \( \bar{z} \) and the symbols

$$D_t(z) = \sum_n \frac{1}{n} z^n \frac{\partial}{\partial t_n}, \quad D_{\bar{t}}(\bar{z}) = \sum_n \frac{1}{n} \bar{z}^n \frac{\partial}{\partial \bar{t}_n}. \quad (22)$$

The \( \tau \)-function contains all the data of the Toda system. For example, the first coefficients in the expansion of Lax operators (13) and the dressing operators are related to the \( \tau \)-function as

$$r\bar{r}(\mu) = \frac{\tau(\mu + i\hbar) \tau(\mu - i\hbar)}{\tau^2(\mu)}. \quad (23)$$

The dressing operators are determined by (18) up to a factor that commutes with the shift operator. In particular, eq. (18) is satisfied by the wave operators

$$\mathcal{W} = W \exp\left( \frac{1}{\hbar} \sum_{n \geq 1} t_n \hat{\omega}^n / n \right), \quad \bar{\mathcal{W}} = \bar{W} \exp\left( \frac{1}{\hbar} \sum_{n \geq 1} \bar{t}_n \hat{\omega}^{-n} / n \right) \quad (24)$$

which have the following important property. The operator

$$\mathcal{A} = \mathcal{W}^{-1} \bar{\mathcal{W}} \quad (25)$$

does not depend on \( t \) and \( \bar{t} \) (see Theorem 1.5 in ref. [11] and Proposition 1.2 in ref. [20]). The operator (25) characterizes the particular solution of the Toda Lattice hierarchy.

**Orlov-Shulman operators**

The Lax system can be extended by adding Orlov-Shulman operators [21]

$$M = W \left( \mu + i \sum_{n \geq 1} n t_n \hat{\omega}^n \right) W^{-1}, \quad (26)$$

$$\bar{M} = \bar{W} \left( \mu - i \sum_{n \geq 1} n \bar{t}_n \hat{\omega}^{-n} \right) \bar{W}^{-1}.$$
The Lax and Orlov-Shulman operators are expressed in terms of the wave operators (24) as
\[ L = \mathcal{W}_\hat{\omega} \mathcal{W}^{-1}, \quad M = \mathcal{W}_\mu \mathcal{W}^{-1} \]
\[ \bar{L} = \mathcal{W}_{\bar{\omega}} \mathcal{W}^{-1}, \quad \bar{M} = \mathcal{W}_{\bar{\mu}} \mathcal{W}^{-1}. \] (27)
Therefore the pairs \( L, M \) and \( \bar{L}, -\bar{M} \) satisfy the same commutation relations as the pair \((\hat{\omega}, \mu)\):
\[ [\hat{\omega}, \mu] = i\hbar \hat{\omega}, \quad [L, M] = i\hbar L, \quad [\bar{L}, \bar{M}] = -i\hbar \bar{L}. \] (28)
Since the “spatial” parameter of the Toda system \( s = \frac{1}{2} - i\mu \) is continuous in our consideration, one can also consider arbitrary powers \( \hat{\omega}^\alpha \) of the shift operator and generalize (28) to
\[ [\hat{\omega}^\alpha, \mu] = i\alpha \hbar \hat{\omega}^\alpha, \quad [L^\alpha, M] = i\alpha \hbar L^\alpha, \quad [\bar{L}^\alpha, \bar{M}] = -i\alpha \hbar \bar{L}^\alpha. \] (29)

\( o \) Gaussian field representation

The Orlov-Shulman operators can be represented as the currents of the gaussian field \( \Phi(z, \bar{z}) = \Phi(z) + \bar{\Phi}(\bar{z}) \) whose left and right chiral components are defined by
\[ \Phi(z) = \mu \log z - \frac{1}{2} i\hbar^2 \frac{\partial}{\partial \mu} + i \sum_{n=1}^{\infty} z^n t_n - i\hbar^2 \frac{z^{-n}}{n} \frac{\partial}{\partial t_n} \]
\[ \bar{\Phi}(\bar{z}) = \mu \log \bar{z} + \frac{1}{2} i\hbar^2 \frac{\partial}{\partial \mu} + i \sum_{n=1}^{\infty} \bar{z}^n \bar{t}_n - i\hbar^2 \frac{\bar{z}^{-n}}{n} \frac{\partial}{\partial \bar{t}_n}. \] (30)
Indeed, introducing the spectral parameters \( z \) and \( \bar{z} \), we can write (21) in the form
\[ \mathcal{W}_z^{i\mu} = \left\langle e^{\Phi(z)/\hbar} \right\rangle \]
\[ \bar{\mathcal{W}}_{\bar{z}}^{i\mu} = \left\langle e^{\bar{\Phi}(\bar{z})/\hbar} \right\rangle \] (31)
where for any differential operator \( O \) in \( t, \bar{t} \) and \( \mu \) we denote \( \left\langle O \right\rangle = \tau^{-1} O \tau \). From (31) and (27) we find
\[ z^{-i\mu} M z^{i\mu} = \left\langle z \partial_z \Phi(z) \right\rangle \]
\[ \bar{z}^{-i\mu} \bar{M} \bar{z}^{i\mu} = \left\langle \bar{z} \partial_{\bar{z}} \bar{\Phi}(\bar{z}) \right\rangle. \] (32)
Eq. (32) yields, together with (14), the following expansions for the Orlov-Shulman operators as Laurent series in $L$ and $\bar{L}$

$$M = \mu + i \sum_{n=1}^{\infty} (nt_n L^n + v_n L^{-n})$$

$$\bar{M} = \mu - i \sum_{n=1}^{\infty} (n\bar{t}_n \bar{L}^n + \bar{v}_n \bar{L}^{-n})$$

(33)

with coefficients $v_n$ and $\bar{v}_n$ equal to the expectation values of the vortex operators

$$v_n = \frac{\partial F}{\partial t_n} = \langle V_n \rangle, \quad \bar{v}_n = \frac{\partial F}{\partial \bar{t}_n} = \langle \bar{V}_{-n} \rangle.$$  

(34)

The operators $\Phi(z)$ and $\bar{\Phi}(\bar{z})$ can interpreted as creating and annihilating world-sheet boundaries and the spectral parameters $z$ and $\bar{z}$ play the role of boundary cosmological constants. Similarly, the operators $M$ and $\bar{M}$ create and annihilate boundaries with marked points because of the derivatives in $z$ and $\bar{z}$. In this way Orlov-Shulman operators allow the spectrum of local scaling operators on the world sheet to be studied.

For sufficiently weak perturbations, these are the vortex and anti-vortex operators associated with the expansion (33) in integer powers of $z = L$ and $\bar{z} = \bar{L}$. For strong perturbations, where the series (33) do not converge, the operators $M$ and $\bar{M}$ can be given meaning via analytic continuation. We will return to this point at the end of the paper.

○ The “string equation”

Now we are ready to proceed with the derivation of the constraint which, together with the Toda Lattice hierarchy, defines the integrable structure of the compactified string theory. Such constraints are commonly called “string equations”.

Let us first consider the case of the unperturbed theory ($t = \bar{t} = 0$). In this case the all-genus free energy of the 2d string theory compactified at radius $R$ is given by the integral

$$F(\mu) = h^2 \log \tau(\mu) = \frac{\hbar^2}{4} \operatorname{Im} \int_{-\infty}^{\infty} \frac{dy}{y} \frac{e^{-2iRy\mu/\hbar}}{\sinh(yR) \sinh(y)}$$

(35)

and therefore satisfies the functional equation

$$4 \sin \left( \frac{\hbar \partial \mu}{2R} \right) \sin \left( \frac{\hbar \partial \mu}{2} \right) \log \tau = \log \left( \frac{1}{\mu} \right).$$

(36)
Now we would like to rewrite (36) as an algebraic relation for the Lax operators. In the case of zero potential, only the first term in the expansions (13) survives
\[ L = r \hat{\omega} = W \hat{\omega} W^{-1}, \quad \bar{L} = \hat{\omega}^{-1} \bar{r} = \bar{W} \hat{\omega}^{-1} \bar{W}^{-1} \] (37)
where \( W(\mu) \) and \( \bar{W}(\mu) \) are ordinary functions. By (21) we find for their ratio
\[ \frac{W(\mu)}{W(\mu)} = \frac{\tau(\mu - i\hbar/2)}{\tau(\mu + i\hbar/2)}, \] (38)
and therefore the functional equation (36) is equivalent to
\[ \frac{\bar{W}(\mu + i\hbar/2R)W(\mu - i\hbar/2R)}{W(\mu + i\hbar/2R)\bar{W}(\mu - i\hbar/2R)} = \mu, \] (39)
which means that the operator \( A = W^{-1} \bar{W} \) satisfies the following identities
\[ A \hat{\omega}^{-1/R} A^{-1} \hat{\omega}^{1/R} = \mu - i\hbar/2R, \] (40)
\[ A^{-1} \hat{\omega}^{1/R} A \hat{\omega}^{-1/R} = \mu + i\hbar/2R. \]
A third identity follows from the fact that for \( t = \bar{t} = 0 \) the dressing operators do not contain shifts and therefore commute with \( \mu \):
\[ A \mu A^{-1} = \mu. \] (41)
Further, at \( t = \bar{t} = 0 \) the dressing operators \( W \) and \( \bar{W} \) coincide with the wave operators \( \mathcal{W} \) and \( \bar{\mathcal{W}} \) defined by (24) and we can replace the operator \( A \) in (40)-(41) by \( A = \mathcal{W}^{-1} \bar{\mathcal{W}} \). Therefore the identities (40)-(41) actually hold for arbitrary couplings \( t \) and \( \bar{t} \).

Eqs. (40) and (41) can be formulated as algebraic relations between \( L, \bar{L}, M \) and \( \bar{M} \), namely
\[ \bar{L}^{1/R} L^{1/R} = \bar{M} - \frac{i\hbar}{2R}, \] (42)
\[ L^{1/R} \bar{L}^{1/R} = M + \frac{i\hbar}{2R}, \]
\[ M = \bar{M}. \] (43)
Thus we arrive at the constraint, which allows to express the Orlov-Shulman operators as bilinears of Lax operators
\[ M = \bar{M} = \frac{1}{2} \left( L^{1/R} \bar{L}^{1/R} + \bar{L}^{1/R} L^{1/R} \right). \] (44)
Given the "string equation" (44), the canonical commutation relations (29) are equivalent to the following constraint for the Lax operators

\[
[L^\alpha, \bar{L}^{1/R}] = i\alpha \hbar L^{\alpha-1/R}, \quad [\bar{L}^\alpha, L^{1/R}] = -i\alpha \hbar \bar{L}^{\alpha-1/R}.
\] (45)

◦ The dispersionless limit \( \hbar \to 0 \)

In the sequel we will concentrate on the genus zero case and consider the dispersionless limit \( \hbar \to 0 \) of Toda hierarchy, which is a particular case of the universal Whitham hierarchy introduced by Krichever [22]. The solutions of the dispersionless hierarchy can be parametrized by canonical transformations in a 2D phase space so that any solution is determined by the choice of a canonical pair of variables [19].

In the limit \( \hbar \to 0 \), the operator \( \hat{\omega} \) can be considered as a classical variable \( \omega \), conjugate to \( \mu \)

\[ \{\omega, \mu\} = \omega \] (46)

where the Poisson bracket is defined by

\[ \{f, g\} = i\omega(\partial_\omega f \partial_\mu g - \partial_\omega g \partial_\mu f) \], (47)

and the Lax operators \( L, \bar{L} \) are considered as c-functions of the phase space coordinates \( \omega \) and \( \mu \). The Hamiltonians as functions of \( \omega \) and \( \mu \) give the expectation values of the vortex operators, \( H_n = \partial_n \mathcal{F} = \langle \mathcal{V}_n \rangle \).

The Lax equations are equivalent to the exterior differential relations

\[ d \log L \wedge dM = d \log \bar{L} \wedge d\bar{M} = d \log \omega \wedge d\mu + i \sum_{n \neq 0} dH_n \wedge dt_n \] (48)

which imply the existence of functions \( S(L, \mu, t, \tilde{t}) \) and \( \bar{S}(\bar{L}, \mu, t, \tilde{t}) \) whose differentials are given by

\[ dS = M d \log L + \log \omega d\mu + i \sum_{n \neq 0} H_n dt_n \]
\[ d\bar{S} = \bar{M} d \log \bar{L} + \log \omega d\mu + i \sum_{n \neq 0} H_n dt_n. \] (49)

The functions \( S \) and \( \bar{S} \) can be thought of as the generating functions for the canonical transformations \( (\omega, \mu) \to (L, M) \) and \( (\omega, \mu) \to (\bar{L}, \bar{M}) \). The new coordinates \( z = L(\omega, \mu) \) and \( \bar{z} = \bar{L}(\omega, \mu) \) are analytic functions of \( \omega \), given by the \( \hbar \to 0 \) limit of the expansions [13].
The classical Hamiltonians $H_n(\omega, \mu)$ associated with the “times” $t_n$ are related $z = L(\omega)$ and $\bar{z} = \bar{L}(\omega)$ by (13).

Conversely, the coordinate $\omega$ can be considered as a meromorphic function of either $z$ or $\bar{z}$

$$\omega = e^{\partial_\mu S(z)} = e^{\partial_\mu \bar{S}(\bar{z})}. \quad (50)$$

Eq. (50) defines the canonical transformation between $(L, M)$ and $(\bar{L}, \bar{M})$.

Further, the functions $S(z)$ and $S(\bar{z})$ are equal to the expectation values of the chiral components of the gaussian field (30)

$$S(z) = \langle \Phi(z) \rangle$$
$$\bar{S}(\bar{z}) = \langle \bar{\Phi}(\bar{z}) \rangle \quad (51)$$

and the functions representing Orlov-Shulman operators are given by the derivatives

$$M(z) = z \partial_z S(z), \quad \bar{M}(\bar{z}) = \bar{z} \partial_{\bar{z}} \bar{S}(\bar{z}). \quad (52)$$

In the dispersionless limit we can write (44) and (45) as

$$M = \bar{M} = L^{1/R} L^{1/R} \quad (53)$$

and

$$\{L^{1/R}, L^{1/R}\} = 1/R. \quad (54)$$

The second constraint is a consequence of the first one and the canonical commutation relations

$$\{L, M\} = L, \quad \{\bar{L}, \bar{M}\} = -\bar{L}. \quad (55)$$

The dispersionless “string equation” (53) generalizes the constraint found for the self-dual radius $(R = 1)$ by Eguchi and Kanno [23], and for all integer values of $\beta = 1/R$ by Nakatsu [24]. In this last case one can rewrite the “string equation” as a linear constraint ($W_{1+\infty}$ constraint) on the $\tau$-function [25].

○ Two-point correlators in the dispersionless limit

Knowing the functions $z(\omega)$ and $\bar{z}(\omega)$ or their inverse functions $\omega(z)$ and $\omega(\bar{z})$, we can evaluate, using (50) and (51), the two-point functions $\partial_\mu \partial_n F$:

$$\partial_\mu \partial_n F = \frac{1}{2\pi i} \int_\infty \omega d\omega(z), \quad \partial_\mu \partial_{-n} F = \frac{1}{2\pi i} \int_0 \bar{\omega} d\omega(\bar{z}) \quad (n \geq 1). \quad (56)$$
The generating functions for the two-point correlators $\partial_m \partial_n F$ and $\partial_m \partial_{-n} F$ ($m, n > 0$) are given by
\[ \frac{1}{2} \partial^2_{\mu} F + D_t(z_1) D_t(z_2) F = \log \frac{\omega(z_1) - \omega(z_2)}{z_1 - z_2} \] (57)
\[ D_t(z_1) D_t(\bar{z}_2) F = -\log \left( 1 - \frac{1}{\omega(z_1)\omega(\bar{z}_2)} \right) \] (58)
where the operators $D_t(z)$ and $D_t(\bar{z})$ are defined by (22) and the functions $\omega(z)$ and $\omega(\bar{z})$ are defined by (54). Eqs. (57) and (58) are obtained directly from the dispersionless Hirota equations for Toda hierarchy [19] and hold independently of the constraint (54).

○ Landau-Ginsburg description

The topological description of the $c = 1$ string theory proposed for the self-dual radius in [26] and [27] can be readily generalized for any value of the compactification radius. It was shown in [28] that the WDVV equations are a consequence of the dispersionless Hirota equations.

○ Relation to conformal maps

The solutions of the dispersionless Toda hierarchy can be given a geometrical interpretation as conformal maps from simply connected domains in the complex plane to the unit disk. This correspondence was first established for a particular solution of the hierarchy [29,30] and then for any generic solution in [31].

The general form of a conformal map from the exterior of the unit disc to a simply connected domain $D$ containing the point $z = \infty$ is given by the first series expansion in [13]. The parameter $r$, which we will assume to be real and positive, is called the conformal radius of the domain $D$. Let us further assume that the couplings $t_n$ and $\bar{t}_n$ are complex conjugate. Then, along the boundary of the unit disc $\tilde{\omega} = 1/\omega$, $\tilde{\bar{z}} = \bar{L}(\omega)$ and $z = L(\omega)$ are complex conjugates and define a smooth curve $\gamma$ in the complex $z$-plane, which is the boundary of the domain $D$. If $t = \bar{t} = 0$, then $\gamma$ is a circle of radius $r = \mu R/2$; in the general case the form of the contour depends on the values of the couplings $t_n$. The couplings $t_n$ ($n \neq 0$) and $\mu$ can be thought of as a set coordinates in the space of closed curves. The coordinates of the curve $\gamma = \partial D$ are given by the moments of the domain $D$ with respect to the measure $\sigma = \sigma(z, \bar{z}) dz \wedge d\bar{z}$
\[ t_n = -\frac{1}{\pi n} \int_D z^{-n} \sigma(z, \bar{z}) d^2z, \quad \bar{t}_n = -\frac{1}{\pi n} \int_D \bar{z}^{-n} \sigma(z, \bar{z}) d^2\bar{z}, \quad \mu = \frac{1}{\pi R} \int_D \sigma(z, \bar{z}) d^2z \] (59)
and the free energy is

\[ \mathcal{F} = \frac{1}{2\pi^2} \int_{\mathcal{D}} d^2z \int_{\mathcal{D}} d^2\zeta \sigma(z, \bar{z}) \log|z^{-1} - \zeta^{-1}|^2 \sigma(\zeta, \bar{\zeta}). \]

In [29] it is assumed that the density is homogeneous, which in our case is so only at the selfdual point \( R = 1 \). The case of generic density \( \sigma(z, \bar{z}) \) was worked out in [31].

The density function is fixed by the constraint imposed on the Toda system. In our case, the string equation (54) means that if we parametrize the phase space by \( z = L \) and \( \bar{z} = \bar{L} \), then the volume form is \( \sigma(z, \bar{z}) dz \wedge d\bar{z} = R dz^{1/R} \wedge d\bar{z}^{1/R} \). This means that the density function is

\[ \sigma(z, \bar{z}) = \frac{1}{R} \left( z\bar{z}\right)^{\frac{1}{1-R}}. \]  

(60)

If we introduce the potential \( U(z, \bar{z}) \) such that \( \sigma(z, \bar{z}) = \partial \bar{\partial} U(z, \bar{z}) \), the couplings \( t, \bar{t} \) and the cosmological constant \( \mu \) are given by contour integrals along the boundary \( \gamma = \partial \mathcal{D} \)

\[ t_n = \frac{1}{2\pi i n} \oint_{\gamma} z^{-n} \partial_z U dz, \quad \bar{t}_n = \frac{1}{2\pi i n} \oint_{\gamma} \bar{z}^{-n} \partial_{\bar{z}} U d\bar{z}, \quad \mu = \frac{1}{2\pi R} \oint_{\gamma} \partial_z U dz. \]  

(61)

In our case \( U(z, \bar{z}) = R \left( z\bar{z}\right)^{1/R} \). The derivative of the potential can be considered as a meromorphic function in the vicinity of the curve \( \gamma \) and is related to the Orlov-Shulman operator by \( z \partial_z U(z, \bar{z}) = M(z) \). Similarly, \( \bar{z} \partial_{\bar{z}} U(z, \bar{z}) = M(\bar{z}) \). The dispersionless string equation (54) is equivalent to \( \{L, \bar{L}\} = R(L\bar{L})^{-\frac{1}{1-R}} = \sigma^{-1}(L, \bar{L}) \).

The formulation of the dispersionless limit of the theory as a conformal map problem is analogous to the now standard geometrical description of the tachyon excitations in the 2D string theory within MQM. The tachyons there appear as small fluctuations of the profile of the Fermi surface [32]. In our case, similarly, a small change of the couplings \( t_n \) leads to a small deformation of the boundary of the domain \( \mathcal{D} \). Typically the deformations of the domain satisfy a \( W_\infty \) symmetry, but a local realization of this symmetry exists only for \( R = 1 \). In order to have the standard realization of the \( W_\infty \) symmetry, we have to go to coordinates \( y = z^{1/R}, \bar{y} = \bar{z}^{1/R} \). In these coordinates the density is homogenous, but the conformal map has a conical singularity in the origin.

\( \circ \) How to use the string equation

13
It is convenient to rewrite the expansions \( (13) \) as

\[
L^{1/R} = (r \omega)^{\frac{1}{\Pi}} (1 + a_1/\omega + a_2/\omega^2 + \ldots), \\
\bar{L}^{1/R} = (\bar{r} / \omega)^{\frac{1}{\Pi}} (1 + \bar{a}_1 \omega + \bar{a}_2 \omega^2 + \ldots),
\]

where \( a_k, \bar{a}_k \) are some functions of the couplings \( t_k, \bar{t}_k \) \((k = 1, 2, \ldots)\) and \( \mu \). The constraint (53) then gives

\[
(r \bar{r})^{\frac{1}{\Pi}} (1 + a_1 \omega^{-1} + a_2 \omega^{-2} + \ldots) (1 + \bar{a}_1 \omega + \bar{a}_2 \omega^2 + \ldots) = M(\omega). \tag{63}
\]

In order to use (63) we need to express the coefficients of the Laurent series \( M(\omega) \) in terms of the couplings \( t_{\pm 1}, \ldots, t_{\pm n} \) and \( \mu \). This can be done using the expansions (33) as follows. It is easy to see that if we consider a perturbation by \( t_{\pm 1}, \ldots, t_{\pm n} \), then all the coefficients but \( a_1, \ldots, a_n \) and \( \bar{a}_1, \ldots, \bar{a}_n \) in (62) are zero. Then the coefficients of the positive [negative] powers of \( \omega \) are obtained by plugging \( M(\omega) = M(L(\omega)) \) \([M(\omega) = \bar{M}(\bar{L}(\omega))\)]\) into (63). Comparing the coefficients of \( \omega^{\pm 1}, \ldots, \omega^{\pm n} \) on both sides of (63), we obtain \( 2n \) conditions, which determine \( \bar{a}_1, \ldots, \bar{a}_n \).

\( \circ \) \text{ Example: sine-Gordon model coupled to gravity } (t_k = \delta_{k,1} t_n + \delta_{k,-1} \bar{t}_n) \)

Let us solve, as an example, the string equation for the case of only one pair of nonzero vortex couplings, \( t = t_1 \) and \( \bar{t} = t_{-1} \), which corresponds to a sine-Liouville term

\[
(t e^{i \bar{r} R} + \bar{t} e^{-i \bar{r} R}) e^{(R-2) \phi}
\]

added to the world-sheet action (1). In this case one can retain only \( a = a_1 \) and \( \bar{a} = \bar{a}_1 \) in (62)

\[
L = r \omega (1 + a/\omega)^R \\
\bar{L} = r \omega^{-1} (1 + \bar{a} \omega)^R
\]

and eq. (63) yields

\[
(r \bar{r})^{\frac{1}{\Pi}} (1 + a/\omega) (1 + \bar{a} \omega) = \begin{cases} 
rt(\omega + Ra) + \mu + \ldots & \text{for large } \omega, \\
\bar{r} \bar{t}(\omega^{-1} + R \bar{a}) + \mu + \ldots & \text{for small } \omega.
\end{cases}
\]

(66)

Comparing the coefficients in front of \( \omega^{0, \pm 1} \) we get three identities

\[
(r \bar{r})^{\frac{1}{\Pi}} \bar{a} = rt, \quad (r \bar{r})^{\frac{1}{\Pi}} a = \bar{r} \bar{t}, \quad (R - 1)a \bar{a} + \mu (r \bar{r})^{\frac{1}{\Pi}} = 1.
\]

(67)
Taking the limit $\hbar \to 0$ limit of (23), we can express the product $r\bar{r}$ through the susceptibility $\chi \equiv \partial^2_{\mu^2} F$ by
\[ r\bar{r} = e^{-\chi}. \]  

(68)

From eqs. (67) and (68) we get an algebraic equation for the susceptibility
\[ \mu e^{2\pi \chi} + tt(R - 1)e^{\frac{2\pi R}{\chi}} = 1 \]  

(69)

and, choosing the gauge $r = \bar{r}$, the explicit form of the functions $z = L(\omega)$, $\bar{z} = \bar{L}(\omega)$:
\[ z = e^{-\frac{1}{2}\chi} \omega (1 + \bar{t} e^{\frac{2\pi R}{\chi}} \omega^{-1})^R \]
\[ \bar{z} = e^{-\frac{1}{2}\chi} \omega^{-1} (1 + t e^{\frac{2\pi R}{\chi}} \omega)^R. \]  

(70)

The two-point correlators can now be obtained by plugging $\omega(z)$ and $\omega(\bar{z})$ in (57) and (58) and expanding in $z$ and $\bar{z}$. To compute the one-point correlators, one should integrate (56) with respect to $\mu$. We will not do this here, since the calculation is presented in detail in [9].

In a similar way, if all couplings but $t_n$ and $t_{-n}$ are zero, the string equation is solved by
\[ \mu e^{\frac{n}{R} \chi} + (nR - 1) t_n \bar{t}_n e^{\frac{2-n R}{\chi}} = 1 \]  

(71)

and
\[ z = e^{-\frac{1}{2}\chi} \omega (1 + \bar{t}_n e^{\frac{2-n R}{\chi}} \omega^{-n})^R \]
\[ \bar{z} = e^{-\frac{1}{2}\chi} \omega^{-1} (1 + t_n e^{\frac{2-n R}{\chi}} \omega^n)^R. \]  

(72)

As expected, the theory compactified at radius $R$ and perturbed by $t_1$ and $t_{-1}$ is equivalent to the theory compactified at radius $R/n$ and perturbed by $t_n$ and $t_{-n}$.

- $UV \to IR$ flows in the compactified the $c = 1$ string theory in a vortex background

The critical points in the space of couplings describe conformal matter coupled to gravity. The typical size of the world sheet diverges when one approaches the critical point and is inversely proportional to the cosmological constant, defined as the distance to this point. At a critical point the third derivative of the free energy in the cosmological constant diverges.
Let us analyze the solution of the string equation in the example considered above, where only the three couplings, \( \mu, t = t_1 \) and \( \bar{t} = \bar{t}_1 \) are nonzero. In this case the theory contains a single dimensionless parameter

\[ \lambda \equiv t\bar{t}\mu^{R-2} \tag{73} \]

which measures the strength of the sine-Liouville perturbation. The algebraic equation for the susceptibility (69) has three singular points

\[ \lambda = 0, \quad \lambda = \lambda^*, \quad \lambda = \infty \tag{74} \]

where we denoted \( \lambda^* = \frac{(1-R)(1-R)}{(2-R)^2} \). In the vicinity of these points the third derivative of the free energy behaves as

\[ \partial_3^3 \lambda F \sim \lambda^{-1}, \quad \partial_3^3 \lambda F \sim (\lambda - \lambda^*)^{-1/2}, \quad \partial_{\lambda^*}^3 F \sim (\lambda^{-1})^{-1}. \tag{75} \]

The critical point \( \lambda = 0 \) corresponds to the unperturbed string theory (\( c = 1 \) compactified boson coupled to gravity). It was argued in [8] that if \( 1 < R < 2 \), the critical point \( \lambda = \infty \) describes another \( c = 1 \) string theory, in which the role of the cosmological constant is played by the sine-Liouville coupling. (The latter theory, which can be called sine-Liouville or SL string theory is expected to be dual, according to the FZZ conjecture\[10\], to the 2D string theory in Euclidean black hole background.) Finally, \( \lambda = \lambda^* \) describes the trivial \( c = 0 \) string theory in which the matter field is the vortex-antivortex plasma with finite correlation length.

In this way the string theory with a vortex source exists in two phases, the ordinary Liouville phase (\( 0 < \lambda < \lambda^* \)) and the sine-Liouville phase (\( \lambda^* < \lambda < \infty \)). In both phases the theory flows to the same \( c = 0 \) IR fixed point \( \lambda = \lambda^* \).

Each of the two phases is characterized by its spectrum of local operators (the operators creating “microscopic loops” in the world sheet). In the Liouville phase these are the vortex operators of all possible vorticities, which appear in the Laurent expansion of the loop operator \( M(z) \). The corresponding coupling constants \( t_{\pm n} \) are the “times” of the Toda system.

If we are interested in the sine-Liouville phase, we should be more careful, because the loop operator has two branches for large \( z \). Indeed, let us examine the function

\[10\] The FZZ conjecture [4] actually concerns only the point \( R = \frac{3}{2} \), which can be considered as a \( c = 1 \) string theory compactified at \( R = \frac{3}{2} \) and perturbed by \( t_1 \mathcal{V}_1 + \bar{t}_1 \mathcal{V}_{-1} \).
\[ \log \omega(z) = \partial_\mu S(z), \] which can be thought of as the partition function of a string having a punctured disk as world sheet, with boundary cosmological constant \( z \). The Riemann surfaces of this function is shown in Fig.1. When \( 1 < R < 2 \), it follows from (65) that the functions \( \log \omega(z) \) has two branches at large \( z \), which are associated with the two UV fixed points \( \lambda = 0 \) and \( \lambda = \infty \)

\[
\omega(z) \sim \begin{cases} 
z, & \text{if } \lambda \to 0 \quad \text{(branch I)} \\
z^{1/R}, & \text{if } \lambda \to \infty \quad \text{(branch II).} 
\end{cases}
\] (76)

In the SL phase (\( \lambda^* < \lambda < \infty \)) it is the second branch which is the relevant one. In this branch the partition function on a punctured disk is expanded as

\[
\log \omega(z) = \log \tilde{z} + \sum_{k \geq 0} c_k \tilde{z}^k, \quad \tilde{z} = z^{1/R}.
\] (77)

If we want to calculate the correlation functions of the microscopic loop operators in the SL phase (which are no vortex operators), we can still use the general formulas (57) and (58), but we should expand \( \omega(z) \) in fractional powers of \( z \). Near the SL critical point the theory can be again described in terms of a gaussian field \( \Phi(\tilde{z}) \) with local expansion parameter \( \tilde{z} = z^{1/R} \).

Fig.1. The Riemann surface of \( \omega(z) \).
IR critical point occurs when the contour hits the branch point. If we are in the SL phase, then the contour $\gamma$ wraps the cylinder II and hits the branch point when $\lambda$ approaches $\lambda^*$ from above.

We have to show that the contour $\gamma$ hits the branch point when the coupling $\lambda$ reaches the critical value $\lambda^*$. At the branch point $dz/d\omega = 0$ we have $\omega = (R - 1)a$. On the other hand, the contour $\gamma$ corresponds to the unit circle $\omega = 1$. The condition $(R - 1)a = 1$ is satisfied exactly at the critical coupling $\lambda = \lambda^*$ defined by (74).

° A description of the IR critical point in terms of the dual sine-Gordon theory

An intuitive physical picture of the flows in the perturbed theory can be made in terms of the T-dual field $\tilde{x}$. Below we sketch the scenario proposed by Kutasov in [33] (see also [34]). If the compactification radius is in the interval $\frac{2}{m+1} < R < \frac{2}{m}$, then only the vortex operators $V_{\pm 1}, \ldots, V_{\pm m}$ are relevant. At large distances on the world sheet the effective couplings grow, the barriers between the minima of the effective potential $V(\tilde{x}) = t_{-m}e^{-m\tilde{x}/R} + \ldots + t_mE^{m\tilde{x}/R}$ become infinite and the original $c = 1$ theory splits up into $m$ theories with $c = 0$.

In general, the IR matter fields need not be massive. By tuning the coefficients of the potential appropriately one can achieve multicritical behavior $V(\tilde{x}) - V(\tilde{x}_0) = C(\tilde{x} - \tilde{x}_0)^{2k} + \ldots$, which gives, as was argued in [33], a string theory with central charge $c = 1 - \frac{6}{(k+1)(k+2)}$. Therefore one can expect that there can be multicritical points that correspond to a direct product of rational $c < 1$ conformal theories coupled to gravity. The general form of the solution (62) indeed confirms this physical picture. For example, if we switch on all $2m$ couplings $t_{\pm 1}, \ldots, t_{\pm m}$ for which the perturbation is relevant, then we can tune the coefficients in (62) and the cosmological constant $\mu$ so that the first $m$ derivatives of $z(\omega)$ vanish at some point $z_c$ such that $|\omega(z_c)| = 1$. Then the IR central charge of the matter field will be $c_{\text{IR}} = 1 - \frac{6}{(m+1)(m+2)}$. In this way all unitary $c < 1$ string theories can be obtained by turning on various perturbations of the compactified $c = 1$ string theory, as it was suggested in [33]. It would be interesting to exhibit the flow of the constraint (53) in Toda hierarchy to the Douglas equations [18] for generalized KdV hierarchies.

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