TRAVELING WAVES IN AN SEIR EPIDEMIC MODEL WITH THE VARIABLE TOTAL POPULATION

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Abstract. In the present paper, we propose a simple diffusive SEIR epidemic model where the total population is variable. We first give the explicit formula of the basic reproduction number $R_0$ for the model. And hence, we show that if $R_0 > 1$, then there exists a constant $c^* > 0$ such that for any $c > c^*$, the model admits a nontrivial traveling wave solution, and if $R_0 < 1$ and $c > 0$ (or, $R_0 > 1$ and $c \in (0, c^*)$), then the model has no nontrivial traveling wave solution. Consequently, we obtain the full information about the existence and non-existence of traveling wave solutions of the model by determined by the constants $R_0$ and $c^*$. The proof of the main results is mainly based on Schauder fixed point theorem and Laplace transform.

1. Introduction. In 1927, Kermack and McKendrick [11] proposed the well-known deterministic susceptible-infected-removed (SIR) model describing the transmission of infectious diseases, and established a systematic “threshold theory” that determines whether a disease will become prevalent or not. These results gave information on prevalence of diseases, the transmission mechanisms and the effect of interventions [4, 8, 13]. Therefore, many researchers utilized ordinary differential equations to explore mechanisms and dynamical behaviors of communicable diseases qualitatively and quantitatively [4, 8]. When individuals move randomly, the corresponding reaction-diffusion model takes into the following diffusive SIR model

$$
\begin{align*}
\frac{\partial}{\partial t} S(t, x) &= d_1 \Delta S(t, x) - \beta S(t, x) I(x, t), \\
\frac{\partial}{\partial t} I(t, x) &= d_2 \Delta I(t, x) + \beta S(t, x) I(x, t) - \gamma I(t, x), \\
\frac{\partial}{\partial t} R(t, x) &= d_3 \Delta R(t, x) + \gamma I(t, x),
\end{align*}
$$

(1.1)

where $S(t, x)$, $I(t, x)$ and $R(t, x)$ denote respectively the number of the susceptible, infected and removed individuals at time $t$ and space $x \in \mathbb{R}$, $\Delta = \frac{\partial^2}{\partial x^2}$. The constant $\beta$ is the transmission coefficient, $\gamma$ is the recovery (or remove) rate, and $d_1$, $d_2$ and $d_3$ are the diffusion rates.

Note that $R$ does not appear in the first two equations of (1.1), it suffices to consider the two dimensional system for $(S, I)$ as following

$$
\begin{align*}
\frac{\partial}{\partial t} S(t, x) &= d_1 \Delta S(t, x) - \beta S(t, x) I(x, t), \\
\frac{\partial}{\partial t} I(t, x) &= d_2 \Delta I(t, x) + \beta S(t, x) I(x, t) - \gamma I(t, x).
\end{align*}
$$

(1.2)

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Hosono and Ilyas [9] showed that if \( S(0, x) = S^0 > 0 \) and \( \beta S^0 > \gamma \), then for each \( c \geq c^* = 2\sqrt{d_2(\beta S^0 - \gamma)} \) there exists a positive constant \( \varepsilon < S_0 \) such that model (1.2) has a traveling wave solution \((S(x + ct), I(x + ct))\) satisfying \( S(-\infty) = S^0, S(+\infty) = \varepsilon, I(\pm \infty) = 0 \). On the other hand, there is no traveling wave solution for (1.2) when \( \beta S^0 \leq \gamma \). There are substantial recent developments on the existence and non-existence of traveling wave solutions for Kermack-McKendrick SIR model, see [1, 2, 7, 12, 17, 19, 20, 24, 25, 27, 28] and references therein. In particular, Wang and Wang [17] have extended the results in [17] to a nonlocal dispersal SIR model.

In this paper, we will incorporate exposed individuals into model (1.3) and consider the following SEIR epidemic model with spatial diffusion

\[
\begin{align*}
\frac{\partial}{\partial t} S(t, x) &= d_1 \Delta S(t, x) - \frac{\beta S(t, x) I(t, x)}{N(t, x)}, \\
\frac{\partial}{\partial t} E(t, x) &= d_2 \Delta E(t, x) + \frac{\beta S(t, x) I(t, x)}{N(t, x)} - \kappa E(t, x) - \alpha I(t, x), \\
\frac{\partial}{\partial t} I(t, x) &= d_3 \Delta I(t, x) + \kappa E(t, x) - \alpha I(t, x), \\
\frac{\partial}{\partial t} R(t, x) &= d_4 \Delta R(t, x) + f \alpha I(t, x),
\end{align*}
\]

in which \( N = S + I + R \) is the total population, \( d_1, d_2 \) and \( d_3 \) are the diffusion rates of the susceptible (\( S \)), infective (\( I \)) and recovered (\( R \)) individuals, respectively. \( \gamma \geq 0 \) is the recovery rate and \( \delta \geq 0 \) is the death (or quarantine) rate of infective individuals. Considered the mobility of individuals, they assumed that the total population \( N(t, x) \) is not fixed. Based on Schauder fixed point theorem and Laplace transform, Wang and Wang [17] established some similar results as correspond to Hosono and Ilyas’ theorems for system (1.3). In a very recent work, Yang, Li and Wang [28] have extended the results in [17] to a nonlocal dispersal SIR model.

The purpose of the current paper is to study the existence and non-existence of traveling wave solutions of (1.4). To prove the existence theorem (Theorem 3.1), that is, there exists a constant \( c^* > 0 \) such that (1.4) has a traveling wave solution if \( c > c^* \) and \( \beta > \alpha \), we will employ Schauder fixed point theorem to the non-monotone operator used in a suitable invariant convex set. In order to construct the appropriate invariant convex set, we should use the ideas of the iteration process [3, 17, 19, 20] to construct the upper-lower solutions. One important feature of our method, which is different to [3, 17, 19, 20], is that we need to construct the vector-value upper-lower solutions for (2.1) (see, Section 2.2) developed by [21] since system (2.1) consists of four equations. Moreover, we establish that (1.4) has no traveling
wave for any $c > 0$ and $\beta < \alpha$ (Theorem 3.2). And also, we conclude the non-existence of traveling wave solutions for (2.1) for $c \in (0, c^*)$ and $\beta > \alpha$ (Theorem 3.3). Here the critical method is based on the two-side Laplace transform. As we know that the application of the Laplace transform requires the prior estimate of the exponential decay of the traveling wave solutions [5, 17, 19, 20]. However, it seems that the analytical method in [5, 17, 19, 20] cannot give the prior estimate due to the four dimensional system (2.1). Instead, we approve the Stable Manifold Theorem [14] to get the prior estimate. The approach in this paper provides a promising method to deal with high dimensional reaction-diffusion systems.

This paper is organized as follows. In Section 2, we establish some preliminary results. Sections 3 is devoted to the study of the existence and non-existence of traveling waves for system (1.4). Finally, we give a brief discussion in Section 4.

2. Preliminaries. In this section, we should give some preliminary results such as the basic reproduction number and the eigenvalue problems for the wave profile equation (2.1), constructing a pair of upper-lower solutions for system (2.1) and verifying the conditions of the Schauder fixed point theorem.

2.1. The basic reproduction number and eigenvalue problem. First of all, we give the basic reproduction number $R_0$ for system (1.4). By similar arguments to those in [18, Theorem 2.3], we can show that the basic reproduction number $R_0$ equals the spectral radius of the following $2 \times 2$ matrix

$$B := \begin{pmatrix} 0 & \beta \\ 1 & \frac{\alpha}{\beta} \end{pmatrix}.$$ 

Hence, $R_0 = \sqrt{\frac{\beta}{\alpha}}$. For the definition of the basic reproduction number $R_0$ for the reaction-diffusion models and its biological interpretation, we refer the readers to [18] for details.

Next we deal with the eigenvalue problems for the wave profile, which is obtained substituting $S(t, x) = \tilde{S}(x + ct)$, $E(t, x) = \tilde{E}(x + ct)$, $I(t, x) = \tilde{I}(x + ct)$, and $R(t, x) = \tilde{R}(x + ct)$ into (1.4). Here $(\tilde{S}, \tilde{E}, \tilde{I}, \tilde{R})$ is called the wave profile, $\xi := x + ct$ the wave coordinate and $c$ the speed. For the sake of convenience, we still use $S, E, I, R, t$ instead of $\tilde{S}, \tilde{E}, \tilde{I}, \tilde{R}, \xi$, and then get the following wave profile equation

$$
\begin{align*}
d_1 S''(t) - cS'(t) - \frac{\beta S(t)I(t)}{N(t)} &= 0, \\
d_2 E''(t) - cE'(t) + \frac{\beta S(t)I(t)}{N(t)} - \kappa E(t) &= 0, \\
d_3 I''(t) - cI'(t) + \kappa E(t) - \alpha I(t) &= 0, \\
d_4 R''(t) - cR'(t) + f\alpha I(t) &= 0,
\end{align*}
$$

(2.1)

here $N(t) = S(t) + E(t) + I(t) + R(t)$.

In the sequel, we always assume that the initial free equilibrium is $(S^0, 0, 0, 0)$ with $S^0 > 0$. We now consider the eigenvalue problem at $(S^0, 0, 0, 0)$. Linearizing of the second and third equations of (2.1) at $(S^0, 0, 0, 0)$ gives

$$
\begin{align*}
d_2 E''(t) - cE'(t) - \kappa E(t) + \beta I(t) &= 0, \\
d_3 I''(t) - cI'(t) - \alpha I(t) + \kappa E(t) &= 0.
\end{align*}
$$

Plugging $E(t) = v_EE^{\lambda t}$ and $I(t) = v_IE^{\lambda t}$ into the above equations, we get the following eigenvalue problem

$$
\begin{pmatrix} v_E \\ v_I \end{pmatrix}, \quad A(\lambda) \begin{pmatrix} v_E \\ v_I \end{pmatrix} = 0,
$$

$$
det A(\lambda) = 0,$$
where

\[ A(\lambda) = \begin{pmatrix} h_E(\lambda) & \beta \\ \kappa & h_I(\lambda) \end{pmatrix} \]

with \( h_E(\lambda) = d_2 \lambda^2 - c \lambda - \kappa \) and \( h_I(\lambda) = d_3 \lambda^2 - c \lambda - \alpha \). Clearly, the fact \( R_0 > 1 \) implies

\[ \det A(0) = \alpha \kappa (1 - R_0^2) < 0. \]

Then the characteristic equation \( \det A(\lambda) = 0 \) has at least one positive root.

For convenience, we denote

\[ \lambda_2^\pm := \frac{c \pm \sqrt{c^2 + 4d_2 \kappa}}{2d_2}, \quad \lambda_3^\pm := \frac{c \pm \sqrt{c^2 + 4d_3 \alpha}}{2d_3}, \]

and

\[ \lambda_M^\pm := \max\{\lambda_2^\pm, \lambda_3^\pm\}, \quad \lambda_m^\pm := \min\{\lambda_2^\pm, \lambda_3^\pm\}. \]

Note that \( h_E(\lambda_2^\pm) = 0 \) and \( h_I(\lambda_3^\pm) = 0 \). Similar to [10, Lemma 2.1], see also [22, Lemma 2.1] and [23, Lemma 4.2], we can obtain the following result.

**Lemma 2.1.** Assume that \( R_0 > 1 \). There exists a positive number \( c^* \) such that the following holds.

1. If \( c > c^* \), then the characteristic equation \( \det A(\lambda) = 0 \) has three positive roots

\[ 0 < \lambda_1 < \lambda_2 < \lambda_3, \]

and a negative root \( \lambda_4 \) with

\[ \lambda_1, \lambda_2 \in (0, \lambda_m^\pm), \quad \lambda_3 \in (\lambda_M^\pm, +\infty), \quad \lambda_4 \in (-\infty, \lambda_m^\pm). \]

Furthermore,

\[ \det A(\lambda) > 0, \quad \lambda \in (\lambda_1, \lambda_2); \quad \det A(\lambda) < 0, \quad \lambda \in (0, \lambda_m^\pm) \setminus (\lambda_1, \lambda_2). \]

2. If \( c = c^* \), then the characteristic equation \( \det A(\lambda) = 0 \) has two different positive roots, and a negative root.

3. If \( 0 < c < c^* \), then the characteristic equation \( \det A(\lambda) = 0 \) has a positive root, a negative root, and a pair of complex roots with positive real parts.

### 2.2. Construction of the upper and lower solutions

To establish existence of traveling wave solutions of (1.4), we will construct a convex invariant. For this, noting that system (2.1) consists of four equations, we use the iteration process [3, 17, 19, 20] to construct a pair of vector-value upper-lower solutions for (2.1). The idea constructing a pair of vector-value upper-lower solutions is motivated by Weng and Zhao [21], see also Fang and Zhao [6], Wang [16] and Xu and Ai [26].

In the following, we always assume that \( R_0 > 1 \), i.e., \( \beta > \alpha \), and \( c > c^* \). Let \( \lambda_1 \) be the smallest eigenvalue defined as in Lemma 2.1(1) and \((v_E, v_I) \gg 0\) its associating eigenvector, which satisfies

\[ h_E(\lambda_1)v_E + \beta v_I = 0 \quad \text{and} \quad \kappa v_E + h_I(\lambda_1)v_I = 0. \tag{2.2} \]

Also by Lemma 2.1(1), for a sufficient small \( \epsilon \in (0, \lambda_2 - \lambda_1) \), we get

\[ h_E(\lambda_1 + \epsilon)h_I(\lambda_1 + \epsilon) - \beta \kappa > 0, \quad h_E(\lambda_1 + \epsilon) < 0, \quad h_I(\lambda_1 + \epsilon) < 0. \]

Then we can choose a constant \( h(\lambda_1 + \epsilon) > 0 \) such that

\[ -\frac{\kappa}{h_I(\lambda_1 + \epsilon)} < h(\lambda_1 + \epsilon) < -\frac{h_E(\lambda_1 + \epsilon)}{\beta}. \tag{2.3} \]
Now, for \( t \in \mathbb{R} \), we define eight continuous functions as follows

\[
\mathcal{S}(t) = S_0,
\]

\[
\mathcal{E}(t) = \min \left\{ v_E e^{\lambda_1 t}, \frac{\alpha}{\kappa} \left( \frac{\beta}{\alpha} - 1 \right) S^0 \right\},
\]

\[
\mathcal{I}(t) = \min \left\{ v_I e^{\lambda_1 t}, \left( \frac{\beta}{\alpha} - 1 \right) S^0 \right\},
\]

\[
\mathcal{R}(t) = q_1 e^{st},
\]

and

\[
\mathcal{S}(t) = \max \left\{ 0, S_0 - \frac{1}{\sigma} e^{\sigma t} \right\},
\]

\[
\mathcal{E}(t) = \max \left\{ 0, v_E e^{\lambda_1 t} - q_2 e^{(\lambda_1 + \epsilon) t} \right\},
\]

\[
\mathcal{I}(t) = \max \left\{ 0, v_I e^{\lambda_1 t} - q_2 h(\lambda_1 + \epsilon) e^{(\lambda_1 + \epsilon) t} \right\},
\]

\[
\mathcal{R}(t) = 0,
\]

in which \( \sigma, \epsilon, q_1, q_2 \) are positive constants determined in the following lemmas.

**Lemma 2.2.** The following inequalities hold.

\[
d_3 \mathcal{S}''(t) - c \mathcal{S}'(t) - \frac{\beta \mathcal{S}(t) \mathcal{I}(t)}{\mathcal{S}(t) + \mathcal{E}(t) + \mathcal{I}(t) + \mathcal{R}(t)} \leq 0, \quad \forall t \in \mathbb{R}; \tag{2.4}
\]

\[
d_2 \mathcal{E}''(t) - c \mathcal{E}'(t) + \frac{\beta \mathcal{S}(t) \mathcal{I}(t)}{\mathcal{S}(t) + \mathcal{E}(t) + \mathcal{I}(t) + \mathcal{R}(t)} - \kappa \mathcal{E}(t) \leq 0, \quad \forall t \neq t_1; \tag{2.5}
\]

\[
d_3 \mathcal{I}''(t) - c \mathcal{I}'(t) + \kappa \mathcal{E}(t) - \alpha \mathcal{I}(t) \leq 0, \quad \forall t \neq t_2; \tag{2.6}
\]

\[
d_4 \mathcal{R}''(t) - c \mathcal{R}'(t) + f \alpha \mathcal{I}(t) \leq 0, \quad \forall t \in \mathbb{R}, \tag{2.7}
\]

for \( q_1 > 0 \) large enough and \( \epsilon \in (0, \min\{ \frac{v_I}{c}, \lambda_1 \}) \) small sufficient, and

\[
t_1 := \frac{1}{\lambda_1} \ln \left( \frac{\alpha S^0}{\kappa v_E} \left( \frac{\beta}{\alpha} - 1 \right) \right), \quad t_2 := \frac{1}{\lambda_1} \ln \left( \frac{S^0}{v_I} \left( \frac{\beta}{\alpha} - 1 \right) \right).
\]

Here the function \((\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{R}(t))\) is called an upper solution of \((2.1)\).

**Proof.** Note that the functions \( \mathcal{S}(t) = S_0, \mathcal{E}(t) \) and \( \mathcal{R}(t) \) are positive, and \( \mathcal{I}(t) \) is nonnegative. Then (2.4) holds clearly.

Next, we show the inequality (2.5) holds. Indeed, note that \( \mathcal{I}(t) \leq v_I e^{\lambda_1 t} \) for all \( t \in \mathbb{R} \), and \( \mathcal{E}(t) = v_E e^{\lambda_1 t} \) when \( t < t_1 \). Then, by (2.2), for \( t < t_1 \),

\[
d_2 \mathcal{E}''(t) - c \mathcal{E}'(t) + \frac{\beta \mathcal{S}(t) \mathcal{I}(t)}{\mathcal{S}(t) + \mathcal{E}(t) + \mathcal{I}(t) + \mathcal{R}(t)} - \kappa \mathcal{E}(t)
\]

\[
\leq d_2 \mathcal{E}''(t) - c \mathcal{E}'(t) + \beta \mathcal{I}(t) - \kappa \mathcal{E}(t)
\]

\[
= (h_E(\lambda_1)v_E + \beta v_I)e^{\lambda_1 t}
\]

\[
= 0.
\]

When \( t > t_1 \), \( \mathcal{E}(t) = \frac{c}{\alpha} \left( \frac{\beta}{\alpha} - 1 \right) S^0 \), and it follows from the facts \( \mathcal{S}(t) = S^0 \) and \( 0 < \mathcal{I}(t) \leq \left( \frac{\beta}{\alpha} - 1 \right) S^0 \) for all \( t \in \mathbb{R} \) that

\[
d_2 \mathcal{E}''(t) - c \mathcal{E}'(t) + \frac{\beta \mathcal{S}(t) \mathcal{I}(t)}{\mathcal{S}(t) + \mathcal{E}(t) + \mathcal{I}(t) + \mathcal{R}(t)} - \kappa \mathcal{E}(t)
\]
Proof. If \( t \geq t_3 \), then the inequality (2.8) holds immediately since \( \mathcal{S}(t) = 0 \) on \([t_3, \infty)\). If \( t < t_3 \), then \( \mathcal{S}(t) = \mathcal{S}_0 = \frac{\sigma}{\epsilon} \). Noting that \( \mathcal{T}(t) \leq v_1 e^{\lambda_1 t} \) for all \( t \in \mathbb{R} \), we see

\[
\begin{align*}
d_1 \mathcal{S}''(t) - c \mathcal{S}'(t) - \frac{\beta \mathcal{S}(t) \mathcal{T}(t)}{\mathcal{S}(t) + \mathcal{E}(t) + \mathcal{T}(t) + \mathcal{R}(t)} & \geq 0, \quad \forall t \neq t_3; \\
d_1 \mathcal{E}''(t) - c \mathcal{E}'(t) + \frac{\beta \mathcal{S}(t) \mathcal{T}(t)}{\mathcal{S}(t) + \mathcal{E}(t) + \mathcal{T}(t) + \mathcal{R}(t)} - \kappa \mathcal{E}(t) & \geq 0, \quad \forall t \neq t_4; \\
d_1 \mathcal{I}''(t) - c \mathcal{I}'(t) + \alpha \mathcal{E}(t) - \beta \mathcal{I}(t) & \geq 0, \quad \forall t \neq t_5; \\
d_1 \mathcal{R}''(t) - c \mathcal{R}'(t) + f \alpha \mathcal{I}(t) & \geq 0,
\end{align*}
\]

where \( t_3 := \frac{1}{\epsilon} \ln(\sigma \mathcal{S}_0) \), and \( q_2 > v_1 \) is a sufficiently large constant such that

\[
\begin{align*}
t_4 := \frac{1}{\epsilon} \ln \frac{v_1}{q_2} & < 0, \quad t_5 := \frac{1}{\epsilon} \ln \frac{v_1}{q_2 \epsilon \mathcal{I} + \alpha} & < 0.
\end{align*}
\]

Here the function \((\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{R}(t))\) is called a lower solution of (2.1).

Proof. If \( t > t_3 \), then the inequality (2.8) holds immediately since \( \mathcal{S}(t) = 0 \) on \([t_3, \infty)\). If \( t < t_3 \), then \( \mathcal{S}(t) = \mathcal{S}_0 = \frac{\sigma}{\epsilon} \). Noting that \( \mathcal{T}(t) \leq v_1 e^{\lambda_1 t} \) for all \( t \in \mathbb{R} \), we see

\[
\begin{align*}
d_1 \mathcal{S}''(t) - c \mathcal{S}'(t) - \frac{\beta \mathcal{S}(t) \mathcal{T}(t)}{\mathcal{S}(t) + \mathcal{E}(t) + \mathcal{T}(t) + \mathcal{R}(t)} & \geq d_1 \mathcal{S}''(t) - c \mathcal{S}'(t) - \beta \mathcal{T}(t) \\
& \geq -d_1 \sigma e^{\sigma t} + c e^{\sigma t} - \beta v_1 e^{\lambda_1 t} \\
& = e^{\sigma t} (-d_1 \sigma + c - \beta v_1 e^{(\lambda_1 - \sigma) t})
\end{align*}
\]
Note that $e^{(\lambda_1-\epsilon)t} < (\sigma S^0)^{(\lambda_1-\epsilon)} < \sigma S^0$ for $t < t_3$. Then, for $t < t_3$,
\[ d_1 \frac{S''(t)}{S(t)} - e^{(\lambda_1-\epsilon)t} \frac{\beta S(t)I(t)}{S(t) + E(t) + I(t) + R(t)} \geq e^{\sigma t}(-d_1 \sigma + c - \beta v_t S^0 \sigma) > 0, \]
i.e., (2.8) holds for all $t \neq t_4$.

Next, we show that (2.9) holds. In fact, for $t > t_4$, the inequality (2.9) holds immediately since $E(t) = 0$ on $[t_4, \infty)$. Note that $S(t) \geq \frac{1}{2} S^0$ for $t \leq \frac{1}{\sigma} \ln(\frac{S_0}{2S^0})$ since the inequality $q_2 \geq \frac{1}{2} S^0$ implies that $t_4 \leq \frac{1}{\sigma} \ln(\frac{S_0}{2S^0})$. In view of the fact that $v_t e^{\lambda_1 t} - q_2 h(\lambda_1 + \epsilon)e^{(\lambda_1+\epsilon)t} \leq I(t) \leq v_t e^{\lambda_1 t}$ and $R(t) = q_1 e^{\epsilon t}$ for all $t \in \mathbb{R}$, and $E(t) = v_t e^{\lambda_1 t} - q_2 e^{(\lambda_1+\epsilon)t}$ for $t < t_3$. Therefore, by (2.2), for $t < t_4$,
\[ d_3 E''(t) - e^{(\lambda_1-\epsilon)t} \frac{\beta S(t)I(t)}{S(t) + E(t) + I(t) + R(t)} - \kappa E(t) \]
\[ = d_2 E''(t) - e^{(\lambda_1-\epsilon)t} - \kappa E(t) + \beta I(t) - \frac{\beta I(t)(E(t) + I(t) + R(t))}{S(t) + E(t) + I(t) + R(t)} \]
\[ \geq d_2 E''(t) - e^{(\lambda_1-\epsilon)t} - \kappa E(t) + \beta I(t) - \frac{\beta I(t)(E(t) + I(t) + R(t))}{S(t)} \]
\[ \geq -e^{(\lambda_1+\epsilon)t} \left( C_2 (h_E(\lambda_1 + \epsilon) + \beta h(\lambda_1 + \epsilon)) + \frac{2b}{S^0} ((v_E + v_I)e^{(\lambda_1-\epsilon)t} + q_1) \right) \]
\[ \geq -e^{(\lambda_1+\epsilon)t} \left( C_2 (h_E(\lambda_1 + \epsilon) + \beta h(\lambda_1 + \epsilon)) + \frac{2b}{S^0} (v_E + v_I + q_1) \right), \]
since $e^{(\lambda_1-\epsilon)t} < 1$ for $t < t_4 < 0$. By (2.3), we see $h_E(\lambda_1 + \epsilon) + \beta h(\lambda_1 + \epsilon) < 0$. Thus, taking $q_2$ large enough, we get (2.9) holds.

Finally, we show (2.10) holds. Clearly, (2.10) holds since $I(t) = 0$ for $t > t_5$. Note that $E(t) \geq v_t e^{\lambda_1 t} - q_2 e^{(\lambda_1+\epsilon)t}$ for all $t \in \mathbb{R}$, and $I(t) = v_t e^{\lambda_1 t} - q_2 h(\lambda_1 + \epsilon)e^{(\lambda_1+\epsilon)t}$ for $t < t_5$. Hence, by (2.3), for $t < t_5$,
\[ d_3 I''(t) - c E'(t) + \alpha I(t) \geq -q_2 e^{(\lambda_1+\epsilon)t}(h_I(\lambda_1 + \epsilon)h(\lambda_1 + \epsilon) + \kappa) > 0. \]
Thus, (2.10) holds.

Clearly, (2.11) holds since $R(t) = 0$ for $t \in \mathbb{R}$. This completes the proof. \hfill \Box

2.3. The verification of Schauder fixed point theorem. In this subsection, we will use the upper and lower solutions $(\bar{S}(t), \bar{E}(t), \bar{I}(t), \bar{R}(t))$ and $(\underline{S}(t), \underline{E}(t), \underline{I}(t), \underline{R}(t))$ constructed in section 2.2 to verify that the conditions of Schauder fixed point theorem hold.

Let $\alpha_1 > \beta, \alpha_2 > \kappa, \alpha_3 > \alpha$ and $\alpha_4 > 0$ be four constants, and let
\[ \lambda_{1i} = \frac{c - \sqrt{c^2 + 4d_i \alpha_i}}{2d_i}, \quad \lambda_{2i} = \frac{c + \sqrt{c^2 + 4d_i \alpha_i}}{2d_i}, \quad i = 1, 2, 3, 4. \]

Now we choose $\alpha_i > 0$ (i.e., $i = 1, 2, 3, 4$) sufficiently large such that $-\lambda_{1i} > \lambda_i$ (i.e., $i = 1, 2, 3, 4$). For any given constant $\mu \in (\lambda_1, \min \{-\lambda_{1i}\})$, we define the Banach space
\[ B_{\mu}(\mathbb{R}, \mathbb{R}^4) = \{ u = (u_1, u_2, u_3, u_4) \in C(\mathbb{R}, \mathbb{R}^4) : \sup_{t \in \mathbb{R}} |u_i(t)| e^{-\mu|t|} < +\infty, i = 1, 2, 3, 4 \} \]
equipped with norm
\[ |u|_{\mu} = \max_{1 \leq i \leq 4} \{ \sup_{t \in \mathbb{R}} |u_i(t)| e^{-\mu|t|} \}. \]
With aid of the upper and lower solutions, we define a convex set $\Gamma$ as
\[
\Gamma = \left\{ (S, E, I, R) \in C(\mathbb{R}, \mathbb{R}^2) : \begin{array}{l}
S(t) \leq S(t) \leq S^0, \quad E(t) \leq E(t) \leq E(t), \\
I(t) \leq I(t) \leq I(t), \quad R(t) \leq R(t) \leq R(t)
\end{array} \right\}
\]
Since $\mu > \lambda_1$, it is easily seen that $\Gamma$ is uniformly bounded with respect to the norm $| \cdot |_\mu$. Furthermore, we define an operator $F : \Gamma \to C(\mathbb{R}, \mathbb{R}^4)$. And for a given $u = (u_1, u_2, u_3, u_4) \in B_\mu(\mathbb{R}, \mathbb{R}^4)$, let
\[
F(u)(t) = (F_1(u), F_2(u), F_3(u), F_4(u))(t),
\]
where
\[
F_i(u)(t) = \frac{1}{\rho_i} \left( \int_{-\infty}^{t} e^{\lambda_1(t-s)} + \int_{t}^{+\infty} e^{\lambda_2(t-s)} \right) H_i(u)(s) ds, \quad i = 1, 2, 3, 4,
\]
in which
\[
\rho_i = d_i(\lambda_{i2} - \lambda_{i1}) = \sqrt{c^2 + 2d_i \alpha_i}, \quad i = 1, 2, 3, 4,
\]
and
\[
H_1(u)(t) := \alpha_1 u_1(t) - \frac{\beta u_1(t)u_3(t)}{u_1(t) + u_2(t) + u_3(t) + u_4(t)},
\]
\[
H_2(u)(t) := (\alpha_2 - \kappa) u_2(t) + \frac{\beta u_1(t)u_3(t)}{u_1(t) + u_2(t) + u_3(t) + u_4(t)},
\]
\[
H_3(u)(t) := (\alpha_3 - \alpha) u_3(t) + \kappa u_2(t),
\]
\[
H_4(u)(t) := \alpha_4 u_4(t) + f_\alpha u_3(t).
\]

**Lemma 2.4.** The operator $F$ maps $\Gamma$ into $\Gamma$.

**Proof.** For any $u = (S, E, I, R) \in \Gamma$, it is obvious that we only need to show that the following inequalities
\[
S(t) \leq F_1(u)(t) \leq S^0, \quad E(t) \leq F_2(u)(t) \leq E(t),
\]
\[
I(t) \leq F_3(u)(t) \leq I(t), \quad R(t) \leq F_4(u)(t) \leq R(t)
\]
hold for all $t \in \mathbb{R}$.

We now consider $F_1(u)(t)$. Note that $H_1(u)(t) \leq \alpha_1 S(t) \leq \alpha_1 S^0$ for all $t \in \mathbb{R}$. Thus
\[
F_1(u)(t) \leq \frac{\alpha_1 S^0}{\rho_1} \left( \int_{-\infty}^{t} e^{\lambda_1(t-s)} + \int_{t}^{+\infty} e^{\lambda_2(t-s)} \right) ds = S^0.
\]
By (2.8), for $t \neq t_3$,
\[
H_1(S, E, I, R)(t) \geq -d_1 S''(t) + c_s S'(t) + \alpha_1 S(t).
\]
Then, when $t > t_3$,
\[
F_1(u)(t) \geq F_1(S, E, I, R)(t)
\]
\[
\geq \frac{1}{\rho_1} \left( \int_{-\infty}^{t} e^{\lambda_1(t-s)} + \int_{t}^{+\infty} e^{\lambda_2(t-s)} \right) (-d_1 S''(s) + c_s S'(s) + \alpha_1 S(s)) ds
\]
\[
= S(t) + \frac{d_1}{\rho_1} e^{\lambda_1(t-t_3)} (S'(t_3 + 0) - S'(t_3 - 0))
\]
\[
\geq S(t).
\]
Similarly, when \( t < t_3 \), by (2.8), we also show \( F_1(u)(t) \geq S(t) \) holds. Thus, we have shown that \( S(t) \leq F_1(u)(t) \leq S^0 \) holds for all \( t \in \mathbb{R} \). The proofs of the following inequalities

\[
E(t) \leq F_2(\mathcal{S}, E, I, R)(t) \leq F_2(u)(t) \leq F_2(\mathcal{S}, E, I, R)(t) \leq E(t),
\]

\[
I(t) \leq F_3(\mathcal{S}, E, I, R)(t) \leq F_3(u)(t) \leq F_3(\mathcal{S}, E, I, R)(t) \leq I(t),
\]

\[
R(t) \leq F_4(\mathcal{S}, E, I, R)(t) \leq F_4(u)(t) \leq F_4(\mathcal{S}, E, I, R)(t) \leq R(t)
\]

are similar to that of \( S(t) \leq F_1(u)(t) \leq S^0 \) for all \( t \in \mathbb{R} \), and the proof is completed.

\[ \Box \]

**Lemma 2.5.** The operator \( F = (F_1, F_2, F_3, F_4) : \Gamma \to \Gamma \) is continuous and compact with respect to the norm \( | \cdot |_{\mu} \).

**Proof.** We first show that \( F = (F_1, F_2, F_3, F_4) : \Gamma \to \Gamma \) is continuous with respect to the norm \( | \cdot |_{\mu} \). Indeed, for any \( u_i = (S_i, E_i, I_i, R_i) \in \Gamma, \ i = 1, 2 \), it is easy to see that there exists a constant \( L_1 > 0 \) such that

\[
|H_1(u_1)(t) - H_1(u_2)(t)| \\
\leq L_1(|S_1 - S_2| + |E_1 - E_2| + |I_1 - I_2| + |R_1 - R_2|).
\]

Therefore,

\[
|F_1(u_1)(t) - F_1(u_2)(t)| e^{-\mu|t|} \\
\leq \frac{L_1}{\rho_1}(|S_1 - S_2| + |E_1 - E_2| + |I_1 - I_2| + |R_1 - R_2|) C(t), \tag{2.12}
\]

where

\[
C(t) = e^{-\mu|t|} \left( \int_{-\infty}^{t} e^{\lambda_1(t-s)+|\mu|s} ds + \int_{t}^{+\infty} e^{\lambda_2(t-s)+|\mu|s} ds \right).
\]

The direct calculations show that

\[
C(t) \leq \int_{-\infty}^{t} e^{\lambda_1(t-s)+|\mu|s} ds + \int_{t}^{+\infty} e^{\lambda_2(t-s)+|\mu|s} ds = \frac{1}{\lambda_{12} - \mu} - \frac{1}{\lambda_{11} + \mu},
\]

that is, \( C(t) \) is uniformly bounded on \( \mathbb{R} \), which follows from (2.12) that the operator \( F_1 \) is continuous with respect to the norm \( | \cdot |_{\mu} \). Similarly, we also can show that operator \( F_i : \Gamma \to \Gamma, \ i = 2, 3, 4 \), is continuous with respect to the norm \( | \cdot |_{\mu} \). Consequently, \( F \) is a continuous operator on \( \Gamma \) with respect to the norm \( | \cdot |_{\mu} \).

Next, we use the similar arguments as in [17] to prove the compactness of \( F \), that is, we shall make use of Arzalà-Ascoli theorem and a standard diagonal process. Let \( I_k = [-k, k] \) with \( k \in \mathbb{N} \) be a compact interval on \( \mathbb{R} \) and temporarily we regard \( \Gamma \) as bounded subset of \( C(I_k, \mathbb{R}^4) \) equipped with the norm \( | \cdot |_{\mu} \). Since \( F \) maps \( \Gamma \) into \( \Gamma \), it is obvious that \( F \) is uniformly bounded on \( I_k \). In the following, we shall show that \( F \) is equi-continuous on \( I_k \). To this end, we first establish four inequalities for the derivative of \( F \). In fact, note that \( u = (S, E, I, R) \in \Gamma \), it is easy to see that there is a constant \( H_0 > 0 \) such that

\[
|H_i(u)(t)| \leq H_0, \ \ i = 1, 2, 3, \ |H_4(u)| \leq H_0 e^{\lambda_1 t}, \ \forall t \in \mathbb{R}.
\]

Consequently, for \( u = (S, E, I, R) \in \Gamma \),

\[
|F_i'(u)(t)| \leq \frac{-\lambda_{11} H_0}{\rho_1} \int_{-\infty}^{t} e^{\lambda_{11}(t-s)} ds + \frac{\lambda_{12} H_0}{\rho_1} \int_{t}^{+\infty} e^{\lambda_{12}(t-s)} ds = \frac{2H_0}{\rho_1}. \]
Similarly, we have, for all \( t \in \mathbb{R} \),
\[
|F'_2(u)(t)| \leq \frac{2H_0}{\rho_2}, \quad |F'_3(u)(t)| \leq \frac{2H_0}{\rho_3},
\]
and
\[
|F'_4(u)(t)| \leq -\frac{\lambda_{41}H_0}{\rho_4} \int_{-\infty}^{t} e^{\lambda_{41}(t-s)+\lambda_1 t} s ds + \frac{\lambda_{42}H_0}{\rho_4} \int_{t}^{+\infty} e^{\lambda_{42}(s-t)+\lambda_1 t} s ds
\]
\[
= \frac{H_0}{\rho_4} \left( -\frac{\lambda_{41}}{\lambda_1 - \lambda_{41}} + \frac{\lambda_{42}}{\lambda_{42} - \lambda_1} \right) e^{\lambda_1 t}.
\]

Let \( \{v_n\} \) be a sequence of \( \Gamma \), which can be also viewed as a bounded subset of \( C(I_k) \). Since \( F \) is uniformly bounded and equi-continuous on \( I_k \), by the Arzelà-Ascoli theorem and the standard diagonal process, we can extract a subsequence \( \{v_{n_k}\} \) such that \( v_{n_k} := F v_{n_k} \) converges in \( C(I_k) \) for any \( k \in \mathbb{N} \). Let \( v := \lim_{k \to \infty} v_{n_k} \). It is readily seen that \( v \in C(\mathbb{R}, \mathbb{R}^4) \). Furthermore, since \( F(\Gamma) \subset \Gamma \) (by Lemma 2.4 and \( \Gamma \) is closed), it follows that \( v \in \Gamma \). Note that \( \mu > \lambda_1 \), it follows from the definition of \( \bar{R}(t) \) that \( e^{-\mu |t|} \bar{R}(t) \) are uniformly bounded on \( \mathbb{R} \). Thus, \( \Gamma \) is uniformly bounded with respect to the norm \( | \cdot |_{\mu} \). Consequently, the norm \( |v_{n_k} - v|_{\mu} \) is uniformly bounded for all \( k \in \mathbb{N} \). For any \( \varepsilon > 0 \), we can find an integer \( N_0 > 0 \) independent of \( v_{n_k} \) such that
\[
|v_{n_k}(t) - v(t)| e^{-\mu t} < \varepsilon
\]
for any \( |t| > N_0 \) and \( k \in \mathbb{N} \). Since \( v_{n_k} \) converges to \( v \) on the compact interval \([-N_0, N_0]\) with respect to the maximum norm, there exists \( K \in \mathbb{N} \) such that
\[
|v_{n_k}(t) - v(t)| e^{-\mu t} < \varepsilon
\]
holds for any \( |t| \leq N_0 \) and \( k > K \). The above two inequalities imply that \( v_{n_k} \) converges to \( v \) with respect to the norm \( | \cdot |_{\mu} \). This proves the compactness of the map \( F \). Therefore, we complete the proof. \( \square \)

3. Existence and non-existence of traveling wave solutions.

3.1. Existence of traveling wave solutions. In this subsection, we establish the existence of traveling waves for system (1.4). To begin with, we first give two propositions.

Proposition 3.1. Assume that \( R_0 > 1 \). Then for any \( c > c^* \), system (1.4) admits a nontrivial traveling wave solution \((S(x+ct), E(x+ct), I(x+ct), R(x+ct))\) satisfying the following.

(1) \( \lim_{t \to -\infty} (S(t), E(t), I(t), R(t)) = (S^0, 0, 0, 0), \quad \lim_{t \to -\infty} (S'(t), E'(t), I'(t), R'(t)) = (0, 0, 0, 0) \).

(2) \( \lim_{t \to -\infty} e^{-\lambda_1 t} E(t) = v_E, \quad \lim_{t \to -\infty} e^{-\lambda_1 t} I(t) = v_I, \quad \lim_{t \to -\infty} e^{-\lambda_1 t} E'(t) = \lambda_1 v_E, \quad \lim_{t \to -\infty} e^{-\lambda_1 t} I'(t) = \lambda_1 v_I. \)

Proof. In view of Lemmas 2.4 and 2.5, it follows from Schauder fixed point theorem that there exists a pair of \( u = (S, E, I, R) \in \Gamma \), which is a fixed point of the operator \( F \). Consequently, the solution \((S(x+ct), E(x+ct), I(x+ct), R(x+ct))\) is a traveling wave solution of system (1.4), and for any \( t \in \mathbb{R} \),
\[
S^0 - \frac{1}{\sigma} e^{\alpha t} \leq S(t) \leq S^0, \quad v_E e^{\lambda_1 t} - q_2 e^{(\alpha+c)t} \leq E(t) \leq v_E e^{\lambda_1 t},
\]
\[ v_I e^{\lambda_1 t} - q_2 h(\lambda + \epsilon) e^{(\epsilon + t) t} \leq I(t) \leq v_I e^{\lambda_1 t}, \quad 0 \leq R(t) \leq q_1 e^{\epsilon t}, \]

which follow

\[ \lim_{t \to -\infty} u_1(t) = S^0, \quad \lim_{t \to -\infty} E(t) = 0, \quad \lim_{t \to -\infty} I(t) = 0, \quad \lim_{t \to -\infty} R(t) = 0, \]

and

\[ \lim_{t \to -\infty} e^{-\lambda_1 t} E(t) = v_E, \quad \lim_{t \to -\infty} e^{-\lambda_1 t} I(t) = v_I. \quad (3.1) \]

Note that \( u = (S, E, I, R) \in \Gamma \) is a fixed point of the operator of \( F \). Applying L'Hôpital rule to the operator \( F_i, \quad i = 1, 2, 3, 4 \), it is easy to see that \( \lim_{t \to -\infty} (S'(t), E'(t), I'(t), R'(t)) = 0 \). Hence, we have shown that the conclusion (1) holds.

Now, integrating both sides of the second equation of (2.1) from \(-\infty\) to \( t \) gives

\[ d_2 E'(t) = cE(t) - \beta \int_{-\infty}^{t} \frac{S(s)I(s)}{N(s)} ds + \kappa \int_{-\infty}^{t} E(s) ds. \quad (3.2) \]

Thus, recall that (3.1), by L'Hôpital rule again,

\[ \lim_{t \to -\infty} e^{-\lambda_1 t} E'(t) = \lim_{t \to -\infty} \left( \frac{c}{d_2} e^{-\lambda E(t)} - \frac{\beta}{d_2} e^{-\lambda_1 t} \int_{-\infty}^{t} \frac{S(s)I(s)}{N(s)} ds + \frac{\kappa}{d_2} e^{-\lambda_1 t} \int_{-\infty}^{t} E(s) ds \right) \]

\[ = \left( \frac{c}{d_2} - \frac{\beta}{d_2 \lambda_1} + \frac{\kappa}{d_2 \lambda_1} \right) v_E \]

\[ = \lambda_1 v_E. \]

Similarly, we also show \( \lim_{t \to -\infty} e^{-\lambda_1 t} I'(t) = \lambda_1 v_I \) holds. Hence, we complete the proof. \( \square \)

**Proposition 3.2.** Assume that \( R_0 > 1 \). For any \( c > c^* \), let \((S(x + ct), E(x + ct), I(x + ct), R(x + ct))\) be a nontrivial traveling wave solution of system (1.4) satisfying Proposition 3.1(1). Then the following holds.

1. \( S(t), E(t) \) and \(-R(t)\) are monotonically decreasing in \( t \in \mathbb{R} \).
2. \( \lim_{t \to +\infty} (S(t), E(t), I(t), R(t)) = (S_0, 0, 0, \frac{\kappa}{\alpha} (S^0 - S_0)), \quad \lim_{t \to -\infty} (S'(t), E'(t), I'(t), R'(t)) = (0, 0, 0, 0). \)
3. It holds.

\[ \int_{-\infty}^{+\infty} \frac{S(s)I(s)}{N(s)} ds = \frac{c}{\beta} (S^0 - S_0), \]

and

\[ \int_{-\infty}^{+\infty} E(t) dt = \frac{c}{\kappa} (S^0 - S_0), \quad \int_{-\infty}^{+\infty} I(t) dt = \frac{c}{\alpha} (S^0 - S_0). \]

**Proof.** We first show the conclusion (1) holds. Indeed, in view of the facts, \( \lim_{t \to -\infty} S(t) = S^0 \) and \( \lim_{t \to -\infty} S'(t) = 0 \), integrating the two sides of the first equation of (2.1) from \(-\infty\) to \( t \) follows

\[ d_1 S'(t) = c(S(t) - S^0) + \beta \int_{-\infty}^{t} \frac{S(s)I(s)}{N(s)} ds. \quad (3.3) \]
We now claim the integral
\[ \int_{-\infty}^{+\infty} S(s)I(s) \frac{ds}{N(s)} < +\infty. \] (3.4)

If not, note that the fact \(0 \leq S(t) \leq S^0\) for all \(t \in \mathbb{R}\), by (3.3), we then conclude that there exists \(\delta_0 > 0\) such that \(S'(t) > \delta_0\) for all large \(t > 0\), which implies that \(\lim_{t \to +\infty} S(t) = +\infty\), this is contradiction. Hence, the improper integral
\[ \int_{-\infty}^{+\infty} S(s)I(s) \frac{ds}{N(s)} \]
diverges, i.e., (3.4) holds. As a result, it follows \(S'(t)\) is uniformly bounded for all \(t \in \mathbb{R}\). Here it is clear from the first equation of (2.1) that
\[ (e^{-\frac{c}{R}t} S(t))' = \frac{\beta}{d_1} e^{-\frac{c}{R}t} S(t) \frac{I(t)}{N(t)}, \quad \forall t \in \mathbb{R}. \]

Integrating the last equality from \(t\) to \(+\infty\) yields
\[ S'(t) = -\frac{\beta}{d_1} e^{-\frac{c}{R}t} \int_t^{+\infty} e^{-\frac{c}{R}s} S(s) \frac{I(s)}{N(s)} ds, \quad \forall t \in \mathbb{R}, \]
which, together with the fact \(S(t) \geq 0\) and \(I(t) \geq 0\) are continuous and not identically zero in \(t \in \mathbb{R}\), implies \(S'(t) < 0\) for all \(t \in \mathbb{R}\). Thus, \(S(t)\) is monotonically decreasing in \(t \in \mathbb{R}\), and let \(S_0 := \lim_{t \to +\infty} S(t)\), consequently, \(S^0 > S_0 \geq 0\). Note that \(E(t)\) satisfies the second equation of (2.1). Then
\[ E(t) = \frac{\beta}{\rho_2} \left( \int_{-\infty}^{t} e^{\lambda_2(t-s)} + \int_t^{+\infty} e^{\lambda_2(t-s)} \right) S(s) \frac{I(s)}{N(s)} ds, \quad \forall t \in \mathbb{R}, \] (3.5)

here,
\[ \lambda_2 = \frac{c - \sqrt{c^2 + 4d_2\kappa}}{2d_2}, \quad \lambda_2 = \frac{c + \sqrt{c^2 + 4d_2\kappa}}{2d_2}, \quad \rho_2 = d_2(\lambda_2 - \lambda_2). \]

By (3.4), let \(\int_{-\infty}^{+\infty} S(t)I(t) \frac{ds}{N(t)} = A_0 < \infty\), it follows from Fubini’s theorem that
\[ \int_{-\infty}^{+\infty} E(t) dt = \frac{\beta}{\rho_2} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{t} e^{\lambda_2(t-s)} + \int_t^{+\infty} e^{\lambda_2(t-s)} \right) S(s) \frac{I(s)}{N(s)} ds dt \]
\[ = \frac{\beta}{\rho_2} \left( -\frac{1}{\lambda_2} + \frac{1}{\lambda_2} \right) A_0 = \frac{\beta}{\kappa} A_0. \] (3.6)

Note that for \((S, E, I, R) \in \Gamma\), it is clear that
\[ 0 \leq \frac{S(t)I(t)}{N(t)} \leq S^0, \quad \forall t \in \mathbb{R}. \]

Then
\[ |E'(t)| \leq \frac{\beta}{\rho_2} \left( |\lambda_2| \int_{-\infty}^{t} e^{\lambda_2(t-s)} + \lambda_2 \int_t^{+\infty} e^{\lambda_2(t-s)} \right) S(s) \frac{I(s)}{N(s)} ds \]
\[ \leq \frac{\beta S^0}{\rho_2} \left( |\lambda_2| \int_{-\infty}^{t} e^{\lambda_2(t-s)} ds + \lambda_2 \int_t^{+\infty} e^{\lambda_2(t-s)} ds \right) \]
\[ = \frac{2\beta S^0}{\rho_2}, \quad \forall t \in \mathbb{R}, \]

Thus, \(E'(t)\) is uniformly bounded, which, together with that \(E(t) \geq 0\) is integrable on \(\mathbb{R}\) (by (3.6)), implies \(\lim_{t \to +\infty} E(t) = 0\). Furthermore, recall that \(\lim_{t \to +\infty} S(t) = S_0\) exists, letting \(t \to +\infty\) in (3.3), and note that (3.4), we know \(\lim_{t \to +\infty} S'(t)\) exists. It
follows from the fact $S'(t) < 0$ for all $t \in \mathbb{R}$ that $\lim_{t \to +\infty} S'(t) = 0$. Consequently, $\lim_{t \to +\infty} S'(t) = 0$. Otherwise, $\lim_{t \to +\infty} S'(t) < 0$, which implies $\lim_{t \to +\infty} S(t) = -\infty$, a contradiction to the fact $S(t) > 0$ for all $t \in \mathbb{R}$. Recall that $\lim_{t \to +\infty} S'(t) = 0$ and $\lim_{t \to +\infty} S(t) = S_0$. Letting $t \to +\infty$ in (3.3), we have

$$\int_{-\infty}^{+\infty} S(t)I(t)\frac{dt}{N(t)} = \frac{c}{\beta} (S_0 - S_0).$$

(3.7)

Combining (3.6) with (3.7), we derive that

$$\int_{-\infty}^{+\infty} E(t)dt = \frac{c}{\kappa} (S_0 - S_0).$$

(3.8)

Letting $t \to +\infty$ in (3.2), by (3.7) and (3.8), and recall that $\lim_{t \to +\infty} E(t) = 0$, we get $\lim_{t \to +\infty} E'(t) = 0$.

On the other hand, note that $I(t)$ satisfies the third equation of (2.1). Then

$$I(t) = \frac{\kappa}{\rho'_3} \left( \int_{-\infty}^{t} e^{\lambda_{31}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{32}(t-s)} \right) E(s) ds, \quad \forall t \in \mathbb{R},$$

(3.9)

here,

$$\lambda_{31} = \frac{c - \sqrt{c^2 + 4d_3 \alpha}}{2d_3}, \quad \lambda_{32} = \frac{c + \sqrt{c^2 + 4d_3 \alpha}}{2d_3}, \quad \rho'_3 = d_3 (\lambda'_{32} - \lambda'_{31}) \).$$

Then, by Fubini’s theorem again and (3.9),

$$\int_{-\infty}^{+\infty} I(t)dt = \frac{\kappa}{\rho'_3} \left( \int_{-\infty}^{t} e^{\lambda_{31}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{32}(t-s)} \right) E(s) ds dt$$

$$= \frac{\kappa}{\rho'_3} \left( - \frac{1}{\lambda_{31}} + \frac{1}{\lambda_{32}} \right) \frac{c}{\kappa} (S_0 - S_0)$$

$$= \frac{c}{\alpha} (S_0 - S_0).$$

(3.10)

As the same as the proof of the before, we also get $I'(t)$ is uniformly bounded, and then, together with (3.10), we get $\lim_{t \to +\infty} I(t) = 0$. Integrating the third equation of (2.1) from $-\infty$ to $t$ follows

$$d_3 I'(t) = cI(t) + \alpha \int_{-\infty}^{t} I(s) ds - \kappa \int_{-\infty}^{t} E(s) ds.$$

Letting $t \to +\infty$ in the above, which, together with (3.8) and (3.10), implies $\lim_{t \to +\infty} I'(t) = 0$. Noting that $\lim_{t \to +\infty} R(t) = 0$, and solving the fourth equation of (2.1) gives

$$R(t) = \frac{f_\alpha}{c} \int_{-\infty}^{t} I(s) ds + \frac{f_\alpha}{c} \int_{t}^{+\infty} e^{\frac{f_\alpha}{c}(t-s)} I(s) ds.$$

(3.11)

It follows from (3.11), and applying L’Hôpital’s rule, that

$$\lim_{t \to +\infty} R(t) = \frac{f_\alpha}{c} \int_{-\infty}^{+\infty} I(s) ds = \frac{f_\alpha}{c} (S_0 - S_0).$$

Moreover, differentiating (3.11) once yields

$$R'(t) = \frac{f_\alpha}{d_4} \int_{t}^{+\infty} e^{\frac{f_\alpha}{c}(t-s)} I(s) ds > 0,$$
which follows that $R(t)$ is monotonically increasing. Using L'Hôpital's rule again, we get $\lim_{t \to +\infty} R'(t) = 0$. Obviously, it follows from (3.7), (3.8) and (3.10) that the conclusion (3) holds, which completes the proof. \hfill \Box

In view of Propositions 3.1 and 3.2, we are ready to state and prove the existence of traveling wave solutions of system (1.4).

**Theorem 3.1.** Assume that $R_0 > 1$. Then for any $c > c^*$, system (1.4) admits a nontrivial traveling wave solution $(S(x+ct), E(x+ct), I(x+ct), R(x+ct))$ satisfying the following.

1. The following boundary conditions

   \[
   \lim_{t \to -\infty} (S(t), E(t), I(t), R(t))) = (S^0, 0, 0, 0), \quad \lim_{t \to +\infty} (S(t), E(t), I(t), R(t))) = \left( S_0, 0, 0, \frac{\alpha}{c} (S^0 - S_0) \right)
   \]

   hold.

2. $S(t)$ and $E(t)$ are monotonically decreasing in $t \in \mathbb{R}$, and

   \[0 < S(t) < S^0, \quad 0 < E(t) < S^0 - S_0, \quad 0 < I(t) < S^0 - S_0, \quad \forall t \in \mathbb{R},\]

   and

   \[ \lim_{t \to +\infty} \frac{E'(t)}{E(t)} = \lim_{t \to +\infty} \frac{I'(t)}{I(t)} = \lambda_1. \]

3. $R(t)$ is monotonically increasing in $t \in \mathbb{R}$, and

   \[0 < R(t) < \frac{f \alpha}{c} (S^0 - S_0), \quad \forall t \in \mathbb{R}.\]

**Proof.** To prove Theorem 3.1, by Propositions 3.1 and 3.2, it suffices to show that

\[0 < S(t) < S^0, \quad 0 < E(t) < S^0 - S_0, \quad 0 < I(t) < S^0 - S_0, \quad R(t) > 0, \quad \forall t \in \mathbb{R}\]

holds. To this end, we first show $0 < S(t) < S^0$ for all $t \in \mathbb{R}$. In fact, in view of Lemmas 2.4 and 2.5, it follows from Schauder fixed point theorem that there exists a pair of $u = (S, E, I, R) \in \Gamma$, which is a fixed point of the operator $F$. As a result, the solution $(S(x+ct), E(x+ct), I(x+ct), R(x+ct))$ is a traveling wave solution of system (1.4), and $0 \leq S(t) \leq S^0$. We next show that the strict inequalities hold. Indeed, note that $u = (S, E, I, R) \in \Gamma$ is a fixed point of the operator $F$, then $S(t) = F_1(S, E, I, R)(t)$. Consequently,

\[S(t) = F_1(S, E, I, R)(t) \geq F_1(S, E, I, R)(t)\]

\[= \frac{1}{\rho_1} \left( \int_{-\infty}^{t} e^{\lambda_1 (t-s)} + \int_{t}^{+\infty} e^{\lambda_2 (t-s)} \right) H_1(S, E, I, R)(s)ds\]

\[\geq \frac{\alpha_1 - \beta}{\rho_1} \left( \int_{-\infty}^{t} e^{\lambda_1 (t-s)} + \int_{t}^{+\infty} e^{\lambda_2 (t-s)} \right) \bar{S}(s)ds\]

\[> 0,
\]

since $S(t)$ is continuous and is not identically zero, and

\[H_1(S, E, I, R) = \alpha_1 S - \frac{\beta ST}{S + E + I + R} \geq (\alpha_1 - \beta) S.\]

Similarly, we can show that the inequalities $S(t) < S^0$, $E(t) > 0$, $I(t) > 0$ and $R(t) > 0$ hold for all $t \in \mathbb{R}$. 

Next, we show $E(t) < S^0 - S_0$ for any $t \in \mathbb{R}$. In fact, define
\[
G(t) := E(t) + \frac{k}{c} \int_{-\infty}^{t} E(s)ds + \frac{k}{c} \int_{t}^{+\infty} e^{\frac{c}{\sigma_2}(s-t)} E(s)ds, \quad \forall t \in \mathbb{R},
\]
(3.13
which, by L’Hôpital’s rule, together with (3.12), implies that
\[
\lim_{t \to -\infty} G(t) = 0, \quad \lim_{t \to +\infty} G(t) = S^0 - S_0.
\]
(3.14
By differentiating (3.13) once,
\[
G'(t) = E'(t) + \frac{k}{d_2} \int_{-\infty}^{+\infty} e^{\frac{c}{\sigma_2}(s-t)} E(s)ds,
\]
(3.15
which yields
\[
\lim_{t \to -\infty} G'(t) = 0, \quad \lim_{t \to +\infty} G'(t) = 0.
\]
Furthermore, by differentiating (3.15) once, and noting that $E(t)$ satisfies the second equation of (2.1), one gets
\[
e G'(t) = d_2 G''(t) + \frac{\beta S(t)I(t)}{N(t)}, \quad \forall t \in \mathbb{R},
\]
which follows that
\[
G'(t) = \frac{\beta}{d_2} e^{-\frac{c}{\sigma_2}t} \int_{t}^{+\infty} e^{\frac{c}{\sigma_2}s} \frac{S(s)I(s)}{N(s)} ds > 0, \quad \forall t \in \mathbb{R}.
\]
Consequently, $G(t)$ is increasing in $t \in \mathbb{R}$. Further, by (3.14),
\[
0 < E(t) < G(t) \leq \lim_{t \to +\infty} G(t) = S^0 - S_0, \quad \forall t \in \mathbb{R}.
\]
Similarly, we also get
\[
0 < I(t) < S^0 - S_0, \quad \forall t \in \mathbb{R}.
\]
Hence, we complete the proof.

3.2. Non-existence of traveling wave solutions. In this subsection, we will establish the non-existence of traveling waves for system (1.4) either $\mathcal{R}_0 < 1$ and $c > 0$, or $\mathcal{R}_0 > 1$ and $c \in (0, c^*)$.

Clearly, if the traveling wave solution $(S(x + ct), E(x + ct), I(x + ct), R(x + ct))$ of (1.4) satisfies (3.12), then $E(t)$ and $I(t)$ satisfies (3.5) and (3.9), respectively. Furthermore, applications of L’Hôpital rule to $E'(t)$ and $I'(t)$, we can easily get the following asymptotic behaviors of $E(t)$ and $I(t)$ as follows.
\[
\lim_{t \to \pm \infty} E(t) = 0, \quad \lim_{t \to \pm \infty} I(t) = 0, \quad \lim_{t \to \pm \infty} E'(t) = 0, \quad \lim_{t \to \pm \infty} I'(t) = 0.
\]
(3.16

**Theorem 3.2.** Assume that $\mathcal{R}_0 < 1$. Then for any $c > 0$, system (1.4) has no nontrivial and nonnegative traveling wave $(S(x + ct), E(x + ct), I(x + ct), R(x + ct))$ satisfying (3.12).

**Proof.** Suppose that there exists a nontrivial and nonnegative traveling wave solution $(S(x + ct), E(x + ct), I(x + ct), R(x + ct))$ of system (1.4) satisfying (3.12). Then $(S(x + ct), E(x + ct), I(x + ct), R(x + ct))$ satisfies (3.5) and (3.9). Note
that \((S(t), E(t), I(t), R(t)) \in \Gamma\), it is easy to see that \(\int_{-\infty}^{+\infty} E(t) dt < +\infty\) and \(\int_{-\infty}^{+\infty} I(t) dt < +\infty\). Then, by Fubini’s theorem and (3.9),

\[
\int_{-\infty}^{+\infty} I(t) dt = \frac{\kappa}{\rho_3'} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{t} e^{\lambda_{11}'(t-s)} + \int_{t}^{+\infty} e^{\lambda_{22}'(t-s)} \right) E(s) ds dt
\]

\[
= \frac{\kappa}{\rho_3'} \left( -\frac{1}{\lambda_{31}'} + \frac{1}{\lambda_{32}'} \right) \int_{-\infty}^{+\infty} E(s) ds
\]

\[
= \frac{\kappa}{\alpha} \int_{-\infty}^{+\infty} E(s) ds,
\]

(3.17)

and, by (3.5), we have

\[
\int_{-\infty}^{+\infty} E(t) dt \leq \frac{\beta}{\rho_2'} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{t} e^{\lambda_{11}'(t-s)} + \int_{t}^{+\infty} e^{\lambda_{22}'(t-s)} \right) I(s) ds dt
\]

\[
= \frac{\beta}{\rho_2'} \left( -\frac{1}{\lambda_{21}'} + \frac{1}{\lambda_{22}'} \right) \int_{-\infty}^{+\infty} I(s) ds
\]

\[
= \frac{\beta}{\kappa} \int_{-\infty}^{+\infty} I(s) ds.
\]

(3.18)

Then, combining (3.17) and (3.18), and noting that \(R_0 < 1\), i.e., \(\beta < \alpha\), we get

\[
\int_{-\infty}^{+\infty} I(t) dt \leq \frac{\kappa}{\alpha} \int_{-\infty}^{+\infty} E(t) dt \leq \frac{\beta}{\alpha} \int_{-\infty}^{+\infty} I(t) dt < \int_{-\infty}^{+\infty} I(t) dt,
\]

which is a contradiction. This ends the proof.


\[\begin{align*}
\textbf{Theorem 3.3.} & \text{ Assume that } R_0 > 1. \text{ Then for any } c \in (0, c^*), \text{ system } (1.4) \text{ has no nontrivial and nonnegative traveling wave } (S(x+ct), E(x+ct), I(x+ct), R(x+ct)) \text{ satisfying (3.12).} \\
\textbf{Proof.} & \text{ As the proof of Theorem 3.2, suppose that there exists a nontrivial and nonnegative traveling wave solution } (S(x+ct), E(x+ct), I(x+ct), R(x+ct)) \text{ of system (1.4) satisfying (3.12). Then } (S(x+ct), E(x+ct), I(x+ct), R(x+ct)) \text{ satisfies the following system.} \\
& d_2 E''(t) - cE'(t) - \kappa E(t) + \frac{\beta S(t) I(t)}{N(t)} = 0, \\
& d_3 I''(t) - cI'(t) + kE(t) - \alpha I(t) = 0.
\end{align*}\]

Set \(u_1 = E, \ u_2 = u_1' = E', \ u_3 = I \) and \(u_4 = u_3' = I'\). Then (3.19) can be rewritten as

\[
y' = By + f(t, y),
\]

where

\[
y = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{\beta}{d_2} & \frac{c}{d_2} & -\frac{\beta}{d_2} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\beta}{d_2} & 0 & \frac{c}{d_2} & \frac{\beta}{d_2} \end{pmatrix}, \quad f(t, y) = \begin{pmatrix} 0 \\ \frac{\beta}{d_2} u_3 - \frac{\beta}{d_2} S u_3 \\ 0 \\ 0 \end{pmatrix}.
\]

Since \((S(t), E(t), I(t), R(t)) \text{ satisfies (3.12), by (3.16), we have } \lim_{t \to +\infty} y(t) = 0.\) It is easy to show that the characteristic equation for \(B\) is given by \(\det A(\lambda)\). It follows from Lemma 2.1(3) that the equation \(\det A(\lambda) = 0\) have one positive eigenvalue, one negative eigenvalue and a pair of complex conjugate eigenvalue with positive real parts. Hence the initial equilibrium \((S^0, 0, 0, 0)\) is hyperbolic. Then it follows
from Stable Manifold Theorem [14, p107] (see also the proof of [29, Lemma 3.1]) that there exists a positive constant \( \alpha_0 \) such that
\[
\sup_{t \in \mathbb{R}} E(t)e^{-\alpha_0 t} < +\infty, \quad \sup_{t \in \mathbb{R}} I(t)e^{-\alpha_0 t} < +\infty, \quad \sup_{t \in \mathbb{R}} |E'(t)|e^{-\alpha_0 t} < +\infty, \quad \sup_{t \in \mathbb{R}} |I'(t)|e^{-\alpha_0 t} < +\infty.
\] (3.20)

Then, by (3.19) and (3.20), we get
\[
\sup_{t \in \mathbb{R}} \{E''(t)e^{-\alpha_0 t}\} < +\infty, \quad \sup_{t \in \mathbb{R}} \{|I''(t)|e^{-\alpha_0 t}\} < +\infty.
\]

Moreover, note that \( \lim_{t \to -\infty} N(t) = S^0 \), and \( \frac{E(t)}{N(t)} \leq 1, \frac{I(t)}{N(t)} \leq 1 \), then
\[
\sup_{t \in \mathbb{R}} \left\{ e^{-\alpha_0 t} \frac{E(t)}{N(t)} \right\} < +\infty, \quad \sup_{t \in \mathbb{R}} \left\{ e^{-\alpha_0 t} \frac{I(t)}{N(t)} \right\} < +\infty.
\] (3.21)

Noting that \( \lim_{t \to -\infty} R(t) = 0 \), we solve the fourth equation of (2.1) and obtain
\[
R(t) = \frac{f \alpha}{c} \int_{-\infty}^{t} I(s) ds + \frac{f \alpha}{c} \int_{t}^{0} e^{\frac{c}{\beta}(t-s)} I(s) ds + C_1 e^{\frac{c}{\beta}t},
\]
where \( C_1 \) is a constant of integration. In view of the fact \( e^{-\alpha_0 t} I(t) \) is uniformly bounded as \( t \to -\infty \). Choosing \( \alpha_1 \in (0, \min\{\alpha_0, \frac{f \alpha}{c\beta}\}) \), we have, for \( t < 0 \),
\[
e^{-\alpha_1 t} R(t) = \frac{f \alpha}{c} \int_{-\infty}^{t} e^{-\alpha_1(s-t)} e^{-\alpha_1 s} I(s) ds
+ \frac{f \alpha}{c} \int_{t}^{0} e^{\left(\frac{c}{\beta} - \alpha_1\right)(t-s)} e^{-\alpha_1 s} I(s) ds + C_1 e^{\left(\frac{c}{\beta} - \alpha_1\right)t}
\leq \frac{f \alpha}{c} \int_{-\infty}^{t} e^{-\alpha_1 s} I(s) ds + \frac{f \alpha}{c} \int_{t}^{0} e^{-\alpha_1 s} I(s) ds + C_1
= \frac{f \alpha}{c} \int_{-\infty}^{0} e^{-\alpha_1 s} I(s) ds + C_1.
\]

Since \( e^{-\alpha_1 t} I(t) \) is uniformly bounded as \( t \to -\infty \) and \( \alpha_1 < \alpha_0 \), it follows from the above inequality that \( e^{-\alpha_1 t} R(t) \) is uniformly bounded as \( t \to -\infty \). Therefore,
\[
\sup_{t \in \mathbb{R}} \left\{ e^{-\alpha_1 t} \frac{R(t)}{N(t)} \right\} < +\infty.
\] (3.22)

Now we rewrite system (3.19) as follows
\[
d_2 E''(t) - cE'(t) - kE(t) + \beta I(t) = \beta I(t) \frac{E(t) + I(t) + R(t)}{N(t)},
\]
\[
d_3 I''(t) - cI'(t) - \alpha I(t) + kE(t) = 0.
\]

Then, we introduce two-side Laplace transform on the above system and get
\[
h_E(\lambda)L_E(\lambda) + \beta L_I(\lambda) = \beta \int_{-\infty}^{+\infty} e^{-\lambda t} I(t) \frac{E(t) + I(t) + R(t)}{N(t)} dt,
\]
\[
h_I(\lambda)L_I(\lambda) + kL_E(\lambda) = 0,
\]
where \( L_E(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda t} E(t) dt \) and \( L_I(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda t} I(t) dt, \lambda > 0 \). Multiplying the first equation of the above system by \( k \) and substituting the second one into the
obtained equation yields
\[
\det A(\lambda) \int_{-\infty}^{+\infty} e^{-\lambda t} I(t) dt = -\kappa\beta \int_{-\infty}^{+\infty} e^{-\lambda t} \frac{E(t) + I(t) + R(t)}{N(t)} dt,
\tag{3.23}
\]
where \( \det A(\lambda) \) is the characteristic function. The integrals on both side of the above equality are well defined for \( \lambda \in (0, \alpha_0) \). Furthermore, the two Laplace integrals of (3.23) can be analytically continued to the whole right half plane; otherwise the integral on the left has a singularity at \( \lambda = \lambda^* \) and it is analytic for \( \lambda < \lambda^* < \alpha_0 \) (cf, [5, 17, 19, 20]). However, note that (3.21) and (3.22), the integral on the right is actually analytic for all \( \lambda < \lambda^* + \alpha_1 \), a contradiction. Thus, (3.23) holds for \( \lambda > 0 \) and can be rewritten as
\[
\int_{-\infty}^{+\infty} e^{-\lambda t} I(t) \left( \det A(\lambda) + \frac{\kappa\beta(E(t) + I(t) + R(t))}{N(t)} \right) dt = 0.
\]
This again leads to a contradiction because
\[
\det A(\lambda) + \frac{\kappa\beta(E(t) + I(t) + R(t))}{N(t)} \to +\infty \text{ as } \lambda \to +\infty,
\]
and \( e^{-\lambda t} I(t) \) is always nonnegative for all \( t \in \mathbb{R} \). Thus we conclude the proof. \( \square \)

4. Discussion. In this paper, we present a diffusive SEIR epidemic model (1.4) with standard incidence rate where the total population is variable. Applying [18, Theorem 2.3], we give the explicit formula of the basic reproduction number \( R_0 \) for system (1.4).

For the model under consideration, the traveling wave solutions describes the disease propagation into the susceptible individuals from an initial disease-free equilibrium to the final, also disease-free equilibrium. In this paper, we first use the iteration process to construct the vector-value upper-lower solutions for (2.1). Together with the Schauder fixed point theorem, we can establish existence of such a traveling wave solution. Second, we use the two-sides Laplace transform to establish the non-existence of such a traveling wave solution. These results could formulate the possible propagation models of the disease.

Theorem 3.1 gives some asymptotic behaviors of the traveling wave solution \((S(x+ct), E(x+ct), I(x+ct), R(x+ct))\) of system (1.4) with speed \( c > c^* \). Note that, in the condition, \( S^0 > 0 \) is a constant representing the number of the susceptible individuals before being infected. Clearly, at any fixed \( x \in \mathbb{R} \), Theorem 3.1(1) and (2) describe that all the individuals were susceptible a long time ago \((t \to -\infty)\) and all the susceptible individuals will be decreasing to \( S_0 \) after a long time \((t \to +\infty)\). In particular, if \( S_0 = 0 \) (due to \( S_0 \) may be zero), then all the susceptible individuals will become the removed individuals also after a long time \((t \to +\infty)\). Hence, the natural question arises. Can we know the value of \( S(+\infty) = S_0 \)? As pointed in [19], for general system such as (1.4) with nonzero diffusion terms, it seems impossible to obtain the value of \( S_0 \), see [19] for a brief discussion on related works for this problem. We conjecture \( S_0 = 0 \), but unfortunately, we do not know to prove it. How to overcome technical problems that prevent a full analysis of relations of them should be a challenging work and leave it as our further project.

As a final remark, we would like to comment Theorems 3.1 and 3.3 by stressing that no existence or non-existence of wave has been derived for the wave speed \( c^* \) (in the case \( R_0 > 1 \)). Let us mention that a limiting argument [15] (looking at the convergence of a sequence of a traveling wave with speed \( \{c_n\} \) such that
\[ \lim_{n \to \infty} c_n = c^* \] is complicated because in this paper we cannot show whether the waves are monotone. We expect that the constant \( c^* \) is the minimum wave speed but this question remains an open problem.

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