Observability inequalities for transport equations through Carleman estimates

Piermarco Cannarsa, Giuseppe Floridia and Masahiro Yamamoto

Abstract We consider the transport equation \( \partial_t u(x,t) + H(t) \cdot \nabla u(x,t) = 0 \) in \( \Omega \times (0,T) \), where \( T > 0 \) and \( \Omega \subset \mathbb{R}^d \) is a bounded domain with smooth boundary \( \partial \Omega \). First, we prove a Carleman estimate for solutions of finite energy with piecewise continuous weight functions. Then, under a further condition on \( H \) which guarantees that the orbit \( \{ H(t) \in \mathbb{R}^d, 0 \leq t \leq T \} \) intersects \( \partial \Omega \), we prove an energy estimate which in turn yields an observability inequality. Our results are motivated by applications to inverse problems.

1 Introduction

Let \( d \in \mathbb{N} \) and \( \Omega \subset \mathbb{R}^d \) be a bounded domain with smooth boundary \( \partial \Omega \), \( \nu = \nu(x) \) be the unit outward normal vector at \( x \) to \( \partial \Omega \), and let \( x \cdot y \) and \( |x| \) denote the scalar product of \( x, y \in \mathbb{R}^d \) and the norm of \( x \in \mathbb{R}^d \), respectively. We set \( Q := \Omega \times (0,T) \), and we consider

\[
Pu(x,t) := \partial_t u + H(t) \cdot \nabla u = 0 \quad \text{in } Q,
\]

where \( H(t) := (H_1(t), \ldots, H_d(t)) : [0,T] \to \mathbb{R}^d \), \( H \in C^1([0,T]; \mathbb{R}^d) \).

Equation (1) is called a transport equation and \( H(t) \) describes the velocity of the flow, which is here assumed to be independent of the spatial variable \( x \).

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Example 1 Setting
\[ v(x,t) = f(x - \alpha(t)) \]
with \( f \in C^1(\Omega; \mathbb{R}) \) and \( \alpha = (\alpha_1, \ldots, \alpha_d) \in C^2([0,T]; \mathbb{R}^d) \), we see that \( v = v(x,t) \) satisfies (7) where \( H(t) = \alpha'(t), \ 0 \leq t \leq T \). Thus (7) is related to an inverse problem of determining a moving source \( f \) with given \( \alpha \) in the diffusion equation
\[ \partial_t w = \Delta w + f(x - \alpha(t)), \ x \in \Omega, \ 0 < t < T. \]

Problem formulation

We assume
\[ H_0 := \min_{t \in [0,T]} |H(t)| > 0, \]
so that
\[ |H(t)| \geq H_0 > 0, \ \forall t \in [0,T]. \]

Without loss of generality let us suppose that \( \mathbf{0} = (0,\ldots,0) \in \Omega \). Otherwise we can translate \( \Omega \) suitably to reduce to the case of \( \mathbf{0} \in \overline{\Omega} \).

Let us recall the following definition.

Definition 1.1 A partition \( \{t_j\}^m_0 \) of \([0,T]\) is a strictly increasing finite sequence \( t_0, t_1, \ldots, t_m \nolimits \) (for some \( m \in \mathbb{N} \)) of real numbers starting from the initial point \( t_0 = 0 \) and arriving at the final point \( t_m = T \). In the following we will call it a uniform partition when the length of the intervals \([t_j, t_{j+1}]\) is constant for \( j = 0, \ldots, m-1 \), that is, \( t_j = \frac{T}{m}j, \ j = 0, \ldots, m \).

In the following Lemma 1.2 we show that the vector-valued function \( H(t) \), satisfying (3), admits a partition \( \{t_j\}^m_0 \) of \([0,T]\) such that the angles of oscillations of the vector \( H(t) \) are less than \( \frac{\pi}{2} \) in any time interval \([t_j, t_{j+1}]\), \( j = 0, \ldots, m-1 \) (see Figure 1).

Given a partition \( \{t_j\}^m_0 \) of \([0,T]\), let us set
\[ \eta_j := \frac{H(t_j)}{|H(t_j)|}, \ j = 0,\ldots,m-1. \]

Lemma 1.2 Let \( S_* \in \left( \frac{1}{\sqrt{2}}, 1 \right) \). For a given vector-valued function \( H \in Lip([0,T];\mathbb{R}^d) \) satisfying condition (3), there exist \( m \in \mathbb{N} \) and a partition \( \{t_j\}^m_0 \) of \([0,T]\) such that
\[ \frac{H(t)}{|H(t)|} \eta_j \geq S_*, \ \forall t \in [t_j, t_{j+1}], \ \forall j = 0, \ldots, m-1, \]
where \( \eta_j \) are defined in (4).
Lemma 1.2 is proved in Appendix.

Fig. 1 In this picture $S_* = \cos \frac{\pi}{6}$, $m = 6$ and $H_j := H(t_j)$, $j = 0, \ldots, 5$.

Condition (5) means that there exist $m$ cones in $\mathbb{R}^d$ such that the axis of every cone, that is, the straight line passing through the apex about which the whole cone has a circular symmetry, is the line between $O = (0, \ldots, 0)$ and $\eta_j$, $j = 0, \ldots, m - 1$. Moreover, a straight line passing through the apex is contained in the cone if the angle between this line and the axis of the cone is less than $\frac{\pi}{4}$. Indeed, the inequality (5), that is

$\frac{H(t)}{|H(t)|} \cdot \eta_j > S_* = \cos \vartheta^*$, for some $\vartheta^* \in (0, \frac{\pi}{4})$,

is equivalent to the fact that the angle between $H(t)$ and $\eta_j$ is less than $\frac{\pi}{4}$. Thus, the vector $H(t)$ is contained in the same cone $\forall t \in [t_j, t_{j+1}]$. Let us note that it can occur that $\eta_i = \eta_j$, for $i \neq j$.

Notation

Let $\delta_\Omega := \text{diam}(\Omega) = \sup_{x, y \in \Omega} |x - y|$. Let us fix $S_* \in \left(\frac{1}{\sqrt{2}}, 1\right)$, $r > 0$ and define

$x_j := -R_j \eta_j$, $j = 0, \ldots, m - 1$,

(6)

where $\eta_j$ is defined in (4) and

$$\begin{cases}
R_j = 2/R_0 + (2^j - 1)(\delta_\Omega + r), \\
R_0 = \frac{1 + S_*}{1 - S_*} \delta_\Omega.
\end{cases}
$$

(7)

We note that from (7) it follows
$x_j \not\in \overline{\Omega}$, \hspace{1em} $j = 0, \ldots, m - 1$.

Fig. 2 In this picture $S_\ast = \cos \frac{\pi}{6}$, $m = 3$ and $H_j := H(t_j)$, $j = 0, 1, 2$.

For every $j = 0, \ldots, m - 1$, let us define

$$M_\Omega(x_j) := \max_{x \in \overline{\Omega}} |x - x_j| \quad \text{and} \quad d_\Omega(x_j) := \min_{x \in \overline{\Omega}} |x - x_j|,$$

$$\beta := (2S_\ast^2 - 1)H_0d_\Omega(x_0),$$

with $H_0$ and $d_\Omega(x_0)$ defined by (2) and (8), respectively.
Fig. 3 In this picture: \( \Omega := \{(x,y) \in \mathbb{R}^2 : |(x,y)-(1,0)| < 3\} \), \( C = (1,0) \), \( S_x = \cos \alpha \in \left(\frac{1}{\sqrt{2}},1\right) \), \( m = 1 \), \( H_j := H(t_j), j = 0,1 \), and \( \beta, \gamma > \alpha, \alpha_0 = \alpha, \delta \leq \alpha \). We note that \( \text{dist}(x_0,G) = d_{\Omega}(x_0) \) and \( M_{\Omega}(x_0) = d_{\Omega}(x_0) + 6 \).

**Main results**

In this article, under condition (3), we establish an observability inequality for (1) which estimates the \( L^2 \)-norm of \( u(x,0) \) by lateral boundary data \( u|_{\partial \Omega \times (0,T)} \) under some conditions on \( H(t) \) (see Theorem 1.4). This observability inequality is a consequence of the following Carleman estimate.

**Theorem 1.3** Let \( u \in H^1(Q) \) be a solution of equation (1), where \( H \in C^1([0,T];\mathbb{R}^d) \) satisfies (3). Let \( \{t_j\}_{j}^{m} \) be a partition of \([0,T]\) associated to \( H(t) \) satisfying (5). Then, there exist constants \( s_0, C_0, C > 0 \) such that for all \( s > s_0 \) we have

\[
\begin{align*}
    s^2 \int_Q |u|^2 e^{2s\varphi} dx dt + se^{-C_0 s} \sum_{j=0}^{m-1} \int_{\Omega} |u(x,t_j)|^2 dx \\
    \leq C \int_Q |Pu|^2 e^{2s\varphi} dx dt + Cse^{Cs} \int_{\Sigma} |u|^2 d\gamma dt + Cse^{Cs} \int_{\Omega} |u(x,T)|^2 dx,
\end{align*}
\]

where \( \varphi(x,t) : Q \rightarrow \mathbb{R} \) is the piecewise continuous weight function defined in (9) and

\[
\Sigma = \{(x,t) \in \partial \Omega \times (0,T) : H(t) \cdot \nu(x) \geq 0\}.
\]

We now give the observability inequality for the equation (1).

**Theorem 1.4** Let \( g \in L^2(\partial \Omega \times (0,T)) \) and let us consider the following problem

\[
\begin{align*}
    \frac{\partial}{\partial t} u + H(t) \cdot \nabla u &= 0 \quad \text{in } Q := \Omega \times (0,T), \\
    u|_{\partial \Omega \times (0,T)} &= g.
\end{align*}
\]
Let us suppose that there exists a partition \( \{ t_j \}_{j=0}^{m} \) of \([0,T]\) associated to \( H(t) \) satisfying (5) such that the following condition holds

\[
\max_{0 \leq j \leq m-1} \frac{(t_{j+1} - t_j) \delta \Omega(x_j)}{M_\Omega^2(x_j)} > \frac{1}{H_0(2S_\rho^2 - 1)}, \tag{12}
\]

where \( M_\Omega(x_j), \delta \Omega(x_j) \) and \( H_0 \) are defined in (8) and (9), respectively. Then, there exists a constant \( C > 0 \) such that the following inequality holds

\[
\| u(\cdot,t) \|_{L^2(\Omega)} \leq C \| g \|_{L^2(\partial \Omega \times (0,T))}, \quad 0 \leq t \leq T,
\]

for any \( u \in H^1(\Omega) \) satisfying (11).

In the following Counterexample 1 we show that it need on \( H(t) \) some further condition to obtain an observability inequality. The assumption (12) guarantees that the orbit \( \{ H(t) \in \mathbb{R}^d : t \in [0,T] \} \) intersects \( \partial \Omega \).

**Explanation about the assumption (12): the lack of observability**

Let us consider, for simplicity, the plane, that is the case \( \mathbb{R}^d \) with \( d = 2 \). For \( \rho > 0 \) we consider \( \Omega_\rho := \{ x \in \mathbb{R}^2 : |x| < \rho \} \). Let us start with a simple positive example.

**Example 2** Let \( \Omega := \Omega_\rho, S_\rho = \left( \frac{1}{\sqrt{\rho}}, 1 \right) \) and \( H \in C^1([0,T]; \mathbb{R}^2) \) such that

\[
\frac{H(t)}{|H(t)|} \geq S_\rho \quad \forall t \in [0,T].
\]

Then, \( H(t) \) admits the trivial partition of \([0,T]\), that is the partition \( \{ t_j \}_{j=0}^{m} \) with \( m = 1 \). Let \( x_0 = -R_0 \eta_0, \) with \( \eta_0 = \frac{H(0)}{|H(0)|} \) and \( R_0 = \frac{1 + S_\rho}{\delta \Omega(\delta \Omega = diam(\Omega))} \), we write (12) as

\[
T > \frac{M_\Omega^2(x_0)}{H_0(2S_\rho^2 - 1) \delta \Omega(x_0)}.
\tag{13}
\]

By the particular geometry of \( \Omega \) we have

\[
\delta \Omega = 2 \rho, \quad \delta \Omega(x_0) = R_0 - \rho, \quad M_\Omega(x_0) = R_0 + \rho,
\]

so (13) becomes

\[
T > \frac{M_\Omega^2(x_0)}{H_0(2S_\rho^2 - 1) \delta \Omega(x_0)} = \frac{(R_0 + \rho)^2}{H_0(2S_\rho^2 - 1)(R_0 - \rho)} = C(S_\rho) \rho,
\]

for some positive constant \( C(S_\rho) \). Thus, if the time \( T \) is bigger than the threshold time \( T_0 := C(S_\rho) \rho > 0 \) we obtain the observability.

Now we give a negative example. We show that the observability inequality in Theorem 1.4 fails without some additional assumption on \( H \) such as (12).
Counterexample 1 Let $\sigma > 0$ and $\rho \in (0, \frac{2}{3}\sigma)$. Let $f \in C^1(\overline{\Omega}_\sigma; \mathbb{R})$ such that $\text{supp}(f) \subset \Omega_\sigma^2 \subseteq \overline{\Omega}_\sigma$ and let $\alpha(t) = (\rho \cos t, \rho \sin t)$, $t \in [0, 2\pi]$. As in Example 1 we set $v(x,y,t) = f(x - \rho \cos t, y - \rho \sin t)$, thus $v = v(x,y,t)$ satisfies (1), where $H(t) = \alpha'(t)$, $0 \leq t \leq T$, and $v$ vanishes at the boundary of $\Omega_\sigma$, so

$$
\begin{cases}
\partial_t v + \alpha'(t) \cdot \nabla v = 0 & \text{in } \Omega_\sigma \times (0,T), \\
v|_{\partial \Omega_\sigma \times (0,T)} = g,
\end{cases}
$$

with $g \equiv 0$. We note that $|\alpha'(t)| = \rho > 0$ and for $t \in [0,T]$ the support of $v(\cdot, \cdot, t)$ is

$$\text{supp}(v(\cdot, \cdot, t)) = \left\{ (x,y) \in \mathbb{R}^2 : |(x - \rho \cos t, y - \rho \sin t)| < \frac{\rho}{2} \right\}.$$  

Then, from (14) and (15) it follows that an observability inequality doesn’t hold.

We can establish an estimate similar to Theorem 1.4 with the maximum norm by the method of characteristics. Our proof is based on the Carleman estimate, which naturally provides an $L^2$-estimate. The $L^2$-estimate, not an estimate in the maximum norm, is related to the exact controllability and more flexibly applied to other problems such as inverse problems, although we do not here discuss details.

Main references and the plan of the paper

Carleman estimates for transport equations are proved in Gaitan and Ouzzane [4], Gölgeleyen and Yamamoto [5], Klibanov and Pamyatnykh [6]. For applications of the Carleman estimates to energy estimates and inverse problems, see in Beilina and Klibanov [1], Bellassoued and Yamamoto [3], Yamamoto [9]. Related references to inverse problems for transport equations are Belinskij [2], Gaitan and Ouzzane [4], Klibanov and Pamyatnykh [6], Machida and Yamamoto [7], Chapter 5 in Romanov [8].

The plan of the paper is the following. In Section 2 we prove the Carleman estimate (Theorem 1.3). In Section 3 we obtain the observability inequality (Theorem 1.4). Finally, in the Appendix we put the proof of Lemma 1.2.

2 Proof of the Carleman estimate

Let $S_s \in \left(\frac{1}{\sqrt{2}}, 1\right)$ and $\{t_j\}_{0}^{m}$ a partition of $[0, T]$ associated to $H(t)$ such that (5) is satisfied.
2.1 Some preliminary lemmas

**Lemma 2.1** Given $R_j$, $j = 0, \ldots, m - 1$, as in (7), then

\[ (x + R_j \eta_j) \cdot \eta_j \geq S_* |x + R_j \eta_j|, \quad \forall x \in \Omega, \]  

(16)

where $\eta_j$ are defined in (7).

**Proof.** For every $x \in \Omega$, we have $|x| = |x - 0| \leq \delta_{\Omega}$ since $0 \in \Omega$, and

\[ S_* |x + R_j \eta_j| \leq S_* (|x| + |R_j\eta_j|) = S_* (|x| + R_j) \leq S_* (\delta_{\Omega} + R_j), \]  

(17)

and, since $-x \cdot \eta_j \leq |x \cdot \eta_j| \leq |x||\eta_j| = |x| \leq \delta_{\Omega}$,

\[ (x + R_j \eta_j) \cdot \eta_j = x \cdot \eta_j + R_j \eta_j \cdot \eta_j = x \cdot \eta_j + R_j \geq R_j - |x| \geq R_j - \delta_{\Omega}. \]  

(18)

From (17) and (18) it follows that a sufficient condition for the inequality (16) is the following

\[ R_j - \delta_{\Omega} \geq S_* (\delta_{\Omega} + R_j), \]

that is, $R_j \geq \frac{1 + S_*}{2} \delta_{\Omega}$. For every $j = 1, \ldots, m - 1$, the last condition is verified by $R_j$ defined as in (7). \hfill \square

By the definition (7) of the sequence $\{R_j\}$ the following Lemma 2.2 follows.

**Lemma 2.2** Let $x_j = -R_j \eta_j$, $j = 0, \ldots, m - 1$, with $R_j$ defined as in (7). Then

\[ M_\Omega(x_j) = \max_{x \in \Omega} |x - x_j| < \min_{x \in \Omega} |x - x_{j+1}| = d_\Omega(x_{j+1}), \quad j = 0, \ldots, m - 2. \]  

(19)

By Lemma 2.2 (see also Figure 2) we deduce

\[ \max_{j=0,\ldots,m-1} M_\Omega(x_j) = M_\Omega(x_{m-1}) \quad \text{and} \quad \min_{j=0,\ldots,m-1} d_\Omega(x_j) = d_\Omega(x_0). \]  

(20)

**Lemma 2.3** Let $x_j = -R_j \eta_j$, $j = 0, \ldots, m - 1$, with $R_j$ defined as in (7). Then,

\[ H(t) \cdot (x - x_j) \geq C_* H_0 d_\Omega(x_0), \quad t_j \leq t \leq t_{j+1}, \quad j = 0, \ldots, m - 1, \quad x \in \Omega, \]

where $C_* = 2S_*^2 - 1 > 0$ and $H_0 = \min_{t \in [0,T]} |H(t)| > 0$.

**Proof.** Let $\vartheta^* \in (0, \frac{\pi}{2})$ such that $\cos \vartheta^* = S_*$. For $t \in [t_j, t_{j+1}], \quad j = 0, \ldots, m - 1$, from (20) we deduce

\[ H(t) \cdot (x - x_j) \geq \cos 2\vartheta^* H_0 d_\Omega(x_j) \geq (2S_*^2 - 1) H_0 d_\Omega(x_0), \quad x \in \Omega. \]

(21)
2.2 Derivation of the Carleman estimate

After introducing the previous lemmas in Section 2.1, we are able to prove Theorem 1.3. In this section, for simplicity of notation, for \( j = 0, \ldots, m - 1 \) let us set
\[
M_j := M_\Omega (x_j) \quad \text{and} \quad \mu_j := d_\Omega (x_j),
\]
see (8) for the definitions of \( M_\Omega (x_j) \) and \( d_\Omega (x_j) \).

**Proof.** (of Theorem 1.3). We derive a Carleman estimate on
\[
Q_j := \Omega \times (t_j, t_{j+1}), \quad 0 \leq j \leq m - 1.
\]

Let
\[
w_j := e^{\phi_j u}, \quad \text{where} \quad \phi_j \text{ is defined in (9), and}
\]
\[
L_j w_j := e^{\phi_j} P (e^{-\phi_j} w_j).
\]

By direct calculations, we obtain
\[
L_j w_j = \partial_t w_j + H(t) \cdot \nabla w_j - s (P \phi_j) w_j \quad \text{in} \quad Q_j,
\]
where, keeping in mind (9) and the definition of the operator \( P \) contained in (1),
\[
P \phi_j (x, t) = \partial_t \phi_j + H(t) \cdot \nabla \phi_j = -\beta + 2H(t) \cdot (x - x_j), \quad 0 \leq j \leq m - 1.
\]

By Lemma 2.3 and (10), since \( \beta = (2S^2_\kappa - 1) H_0 \mu_0 \in ((0, 2(2S^2_\kappa - 1) H_0 \mu_0)) \) we have
\[
P \phi_j = -\beta + 2H(t) \cdot (x - x_j) \geq C_{\kappa} H_0 \mu_0,
\]
where \( C_{\kappa} = 2S^2_\kappa - 1 \) Therefore, by (24) we obtain
\[
\begin{align*}
\int_{Q_j} |L_j w_j|^2 dx dt & \geq -2s \int_{Q_j} (P \phi_j) w_j (\partial_t w_j + H(t) \cdot \nabla w_j) dx dt \\
& \quad + s^2 \int_{Q_j} |2H(t) \cdot (x - x_j) - \beta|^2 |w_j|^2 dx dt \\
& \geq I_1 + I_2 + C_{\kappa}^2 H_0^2 \mu_0^2 s^2 \int_{Q_j} |w_j|^2 dx dt,
\end{align*}
\]
where
\[
I_1 := -2s \int_{Q_j} (P \phi_j) w_j \partial_t w_j dx dt \quad \text{and} \quad I_2 := -2s \int_{Q_j} (P \phi_j) H(t) \cdot (w_j \nabla w_j) dx dt.
\]

We have
\[
I_1 = -2s \int_{Q_j} (P \phi_j) w_j \partial_t w_j dx dt = -s \int_{t_j}^{t_{j+1}} \int_{\Omega} (P \phi_j) \partial_t (w_j^2) dx dt
\]
\[
\begin{align*}
= s \int_{Q_j} \left[ P \phi_j(x,t) |w_j(x,t)|^2 \right]_{t=t_j}^{t=t_{j+1}} dx + s \int_{Q_j} \partial_t (P \phi_j(x,t)) |w_j|^2 dx dt.
\end{align*}
\]

By (5), (16) and (24), similarly to Lemma 2.3, we can obtain
\[
\partial_t (P \phi_j(x,t)) = 2(x - x_j) \cdot H'(t) \geq 2|x - x_j| |H'(t)| C_s \geq 2 C_s \mu_0 H_0',
\]
where \(C_s = 2 S_2^2 - 1\) and we set \(H_0' = \min_{t \in [0,T]} \left| H'(t) \right| \geq 0\). Consequently, from (26) we deduce
\[
I_1 \geq s \int_{Q_j} \left[ P \phi_j(x,t) |w_j(x,t)|^2 \right]_{t=t_j}^{t=t_{j+1}} dx + s 2 C_s \mu_0 H_0' \int_{Q_j} |w_j|^2 dx dt.
\]

Then, for \(I_2\) we deduce
\[
I_2 = -2s \int_{Q_j} (P \phi_j) H(t) \cdot (w_j \nabla w_j) dx dt = -s \int_{t_j}^{t_{j+1}} \int_{Q_j} P \phi_j \sum_{k=1}^d H_k(t) \partial_k (w_j^2) dx dt
\]
\[
= s \int_{t_j}^{t_{j+1}} \int_{Q_j} (\partial_t (P \phi_j)) H(t) |w_j|^2 dx dt - s \int_{t_j}^{t_{j+1}} \int_{\partial \Omega} P \phi_j (H(t) \cdot \nu(x)) |w_j|^2 d\gamma dt.
\]

We note that
\[
H(t) \cdot (x - x_j) \leq |H(t)||x - x_j| \leq H_s M_s,
\]
where we set (see (20))
\[
M_s = M_m - 1 \quad \text{and} \quad H_s := \max_{t \in [0,T]} \left| H(t) \right| > 0.
\]

Therefore, since \(P \phi_j > 0\) by (24) and \(\partial_t (P \phi_j) = 2 H_k(t)\), we estimate \(I_2\) in the following way:
\[
I_2 \geq 2s \int_{t_j}^{t_{j+1}} \int_{Q_j} (\partial_t (P \phi_j)) H(t) |w_j|^2 dx dt - s \int_{t_j}^{t_{j+1}} \int_{\partial \Omega} P \phi_j (H(t) \cdot \nu(x)) |w_j|^2 d\gamma dt
\]
\[
\geq 2s \int_{t_j}^{t_{j+1}} \int_{Q_j} |H(t)|^2 |w_j|^2 dx dt
\]
\[
- s \int_{\Sigma_j} (-\beta + 2H(t) \cdot (x - x_j)) (H(t) \cdot \nu(x)) |w_j|^2 d\gamma dt
\]
\[
\geq 2 H_0^2 s \int_{t_j}^{t_{j+1}} \int_{Q_j} |w_j|^2 dx dt - 2 s \int_{\Sigma_j} (H(t) \cdot (x - x_j))(H(t) \cdot \nu(x)) |w_j|^2 d\gamma dt
\]
\[
\geq 2 H_0^2 s \int_{Q_j} |w_j|^2 dx dt - 2H_s M_s s \int_{\Sigma_j} |H(t)||\nu(x)||w_j|^2 d\gamma dt
\]
\[
\geq 2 H_0^2 s \int_{Q_j} |w_j|^2 dx dt - 2H_0^2 M_s s \int_{\Sigma_j} |w_j|^2 d\gamma dt,
\]
\[ \Sigma_j = \{(x,t) \in \partial \Omega \times (t_j,t_{j+1}) : H(t) \cdot \nu(x) \geq 0 \}. \]

Hence, by (25), (27) and (29), we obtain
\[
\int_{Q_j} [L_j w_j]^2 dx dt \geq s \int_{\Omega} \left[ P(\psi_j(x,t)|w_j(x,t)|)^2 \right]_{t=t_{j+1}}^{t=t_j} dx + C_1 s \int_{Q_j} |w_j|^2 dx dt + C_1 s^2 \int_{Q_j} |w_j|^2 dx dt + 2H_j^2 s \int_{\Sigma_j} |w_j|^2 d\nu dt,
\]
for some positive constant \( C_1 \). Since \( w_j := e^{2\psi}u \), from the previous inequality, for \( j = 0, \ldots, m-1 \), by (22) we deduce that there exists also a positive constant \( C_2 \) such that
\[
\int_{t_j}^{t_{j+1}} \int_{\Omega} |P(u) e^{2\psi} u|^2 dx dt \geq s \int_{\Omega} \psi_j(x) dx + C_1 (s + s^2) \int_{Q_j} e^{2\psi} |u|^2 dx dt - C_2 s e^{C_2 t} \int_{\Sigma_j} |u|^2 d\nu dt,
\]
where \( C_1, C_2 \) are positive constants and
\[
\psi_j(x) := \left[ P(\psi_j(x,t) e^{2\psi_j(x,t)} |u(x,t)|)^2 \right]_{t=t_{j+1}}^{t=t_j} .
\]
By (9) and (24) we obtain
\[
\psi_j(x) = \left[ (2H(t) \cdot (x-x_j) - \beta) e^{2\psi_j} |u(x,t_j)|^2 \right]_{t=t_{j+1}}^{t=t_j} = (2H(t_j) \cdot (x-x_j) - \beta) e^{2\psi_j} |u(x,t_j)|^2 - (2H(t_{j+1}) \cdot (x-x_j) - \beta) e^{2\psi_j} |u(x,t_{j+1})|^2 .
\]
Therefore, summing in \( j \) from 0 to \( m-1 \) and keeping in mind that \( t_0 = 0 \) and \( t_m = T \) by (10) and (28) we have
\[
\sum_{j=0}^{m-1} \psi_j(x) \geq (2H(t_0) \cdot (x-x_0) - \beta) e^{2\psi_0} |u(x,0)|^2 + \sum_{j=1}^{m-1} q_j(x) |u(x,t_j)|^2 - (2H(t) \cdot (x-x_{m-1}) - \beta) e^{2\psi_M} |u(x,t)|^2 \geq \mu_0 H_0 e^{2\mu_0^2} |u(x,0)|^2 - 2M_0 e^{2\mu_0^2} |u(x,T)|^2 + \sum_{j=1}^{m-1} q_j(x) |u(x,t_j)|^2 ,
\]
where, for \( j = 1, \ldots, m-1 \), we set
$q_j(x) := (2H(t_j) \cdot (x-x_j) - \beta) e^{2s|x-x_j|^2} - (2H(t_j) \cdot (x-x_{j-1}) - \beta) e^{2s|x-x_{j-1}|^2}$.

Thus, by (8), (21), (24) and (28), we obtain the following estimate

$$q_j(x) \geq \tilde{C}_0 H_0 e^{2s\mu_j^2} - C_{H_0} e^{2sM_{j-1}^2} = \tilde{C}_0 H_0 e^{2s\mu_j^2} \left(1 - \frac{M_{H_0}}{C_{H_0} H_0} e^{-2s(\mu_j^2 - M_{j-1}^2)}\right).$$

Thanks to (19) (see Lemma 2.2), the choice of the points $x_j$ permits to have $\mu_j - M_{j-1} > 0$, then we deduce that there exist $s_j > 0$ enough large, that is $s_j > \frac{1}{2(\mu_j^2 - M_{j-1}^2)} \log \left(\frac{2H_0 M_{j-1}}{C_0 M_0 H_0}\right)$, $j = 1, \ldots, m - 1$, such that, for every $s > s_0 := \max_{j=1,\ldots,m-1} s_j$, we have

$$q_j(x) \geq \frac{\mu_j H_0}{2} e^{2s\mu_j^2} \geq \frac{\mu_0 H_0}{2} e^{2s\mu_0^2} \geq C_0 e^{C_0 s},$$

for some positive constant $C_0 = C_0(s)$. Thus, by (30), (32) and (33) we have

$$\int_Q |Pu|^2 e^{2s\varphi} dx dt = \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \int_{\Omega} |Pu|^2 e^{2s\varphi} dx dt$$

$$\geq s \sum_{j=0}^{m-1} \int_{\Omega} \psi_j(x) dx + C_1 (s + s^2) \sum_{j=0}^{m-1} \int_Q e^{2s\varphi_j} |u|^2 dx dt$$

$$- C_2 s e^{C_2 s} \sum_{j=0}^{m-1} \int_{\Sigma_j} |u|^2 d\gamma dt$$

$$\geq C_1 (s + s^2) \int_Q e^{2s\varphi} |u|^2 dx dt - C_2 s e^{C_2 s} \sum_{j=0}^{m-1} \int_{\Sigma_j} |u|^2 d\gamma dt$$

$$+ C_0 s e^{C_0 s} \sum_{j=0}^{m-1} \int_{\Omega} |u(x,t_j)|^2 dx - C_2 s e^{C_2 s} \int_{\Omega} |u(x,T)|^2 dx.$$

The last estimate completes the proof of Theorem 1.3.

\[\square\]

### 3 Proof of the observability inequality

Let us give in Section 3.1 two lemmas and in Section 3.2 the proof of Theorem 1.4.
3.1 Energy estimates

Let us give the following energy estimates.

**Lemma 3.1** Let $g \in L^2(\partial \Omega \times (0,T))$ and let us consider the problem

\[
\begin{cases}
\partial_t u + H(t) \cdot \nabla u = 0 & \text{in } Q := \Omega \times (0,T), \\
\left. u \right|_{\partial \Omega \times (0,T)} = g.
\end{cases}
\]  

(11)

Then, for every $t \in [0,T]$, the following energy estimates hold

\[
\|u(\cdot,t)\|_{L^2(\Omega)}^2 \leq \|u(\cdot,0)\|_{L^2(\Omega)}^2 + H_* \|g\|_{L^2(\partial \Omega \times (0,T))}^2,
\]

(34)

\[
\|u(\cdot,0)\|_{L^2(\Omega)}^2 \leq \|u(\cdot,t)\|_{L^2(\Omega)}^2 + H_* \|g\|_{L^2(\partial \Omega \times (0,T))}^2,
\]

(35)

for any $u \in H^1(Q)$ satisfying (11), where $H_* := \max_{\xi \in [0,T]} |H(\xi)|$.

**Proof.** Let $H(t) = (H_1(t), \ldots, H_d(t))$, $t \in [0,T]$. Multiplying the equation in (11) by $2u$ and integrating over $\Omega$, we have

\[
\int_{\Omega} 2u \partial_t u \, dx + \sum_{k=1}^d \int_{\Omega} H_k(t) 2u \partial_k u \, dx = 0,
\]

then,

\[
\partial_t \left( \int_{\Omega} |u(x,t)|^2 \, dx \right) + \sum_{k=1}^d \int_{\Omega} H_k(t) \partial_k (|u(x,t)|^2) \, dx = 0.
\]

So, integrating by parts, for every $t \in [0,T]$, we obtain

\[
\partial_t \left( \int_{\Omega} |u(x,t)|^2 \, dx \right) = - \sum_{k=1}^d \int_{\partial \Omega} H_k u^2 v_k d\gamma = - \int_{\partial \Omega} (H \cdot \nu) |g|^2 d\gamma,
\]

(36)

where $\nu = (v_1, \ldots, v_d)$ is the unit normal vector outward to the boundary $\partial \Omega$. Setting

\[
E(t) := \int_{\Omega} |u(x,t)|^2 \, dx, \quad t \in [0,T],
\]

by (36), integrating on $[0,t]$ we deduce

\[
|E(t) - E(0)| \leq - \int_{\partial \Omega} (H(\xi) \cdot \nu(x)) |g(x,\xi)|^2 d\gamma d\xi \leq H_* \|g\|_{L^2(\partial \Omega \times (0,T))}^2,
\]

where $H_* = \max_{\xi \in [0,T]} |H(\xi)|$. Thus, for all $t \in [0,T]$, we have

\[
E(t) \leq E(0) + H_* \|g\|_{L^2(\partial \Omega \times (0,T))}^2,
\]

and
\[ E(0) \leq E(t) + H_s \|g\|_{L^2(\partial\Omega \times (0,T))}^2. \]

\[ \square \]

**Lemma 3.2** Let \( 0 \leq s_1 < s_2 \leq T \), \( g \in L^2(\partial\Omega \times (0,T)) \). Let us assume that there exists a positive constant \( C = C(s_1, s_2) \) such that for every \( t \in [s_1, s_2] \) the following observability inequality holds

\[ \|u(\cdot, t)\|_{L^2(\Omega)} \leq C \|g\|_{L^2(\partial\Omega \times (0,T))}, \quad \text{for all } u \in H^1(Q) \text{ solution to (11)}. \] (37)

Then, there exists a positive constant \( C = C(s_1, s_2, T) \) such that the inequality (37) holds for every \( t \in [0, T] \).

**Proof.** Let \( E(t) = \|u(\cdot, t)\|_{L^2(\Omega)}^2 \), \( t \in [0, T] \). For every \( t \in [0, s_1] \), keeping in mind Lemma 3.1 by (34), (35) and (37) we obtain

\[ \|u(\cdot, t)\|_{L^2(\Omega)}^2 = E(t) \leq E(0) + H_s \|g\|_{L^2(\partial\Omega \times (0,T))}^2 \leq E(s_1) + 2H_s \|g\|_{L^2(\partial\Omega \times (0,T))}^2 \]
\[ \leq (C^2 + 2H_s) \|g\|_{L^2(\partial\Omega \times (0,T))}^2. \] (38)

For every \( t \in [s_2, T] \), using again Lemma 3.1 by (34) and (37) we deduce

\[ \|u(\cdot, t)\|_{L^2(\Omega)}^2 = E(t) \leq E(s_2) + H_s \|g\|_{L^2(\partial\Omega \times (0,T))}^2 \leq (C^2 + H_s) \|g\|_{L^2(\partial\Omega \times (0,T))}^2. \] (39)

From (38) and (39) the conclusion follows. \( \square \)

### 3.2 The proof

**Proof.** (of Theorem 1.4).

Let \( \phi \) be the weight function given in (9). By the assumption (12) it follows that there exists \( j^* \in \{0, \ldots, m-1\} \) such that

\[ \frac{(t_{j^*+1} - t_{j^*})d_\Omega(x_{j^*})}{M^2_{\Omega}(x_{j^*})} > \frac{1}{H_0(2S^2_{\Omega} - 1)}. \] (40)

By the definition of the weight function \( \phi(x, t) \) (see (9)), it follows that, for every \( x \in \overline{\Omega} \), we have

\[ \phi(x, t_{j^*}) = \phi_{j^*}(x, t_{j^*}) = |x - x_{j^*}|^2 > 0 \]
and, since (40) holds, keeping in mind that \( \beta = (2S^2_\Omega - 1)H_0d_\Omega(x_0) \),

\[ \lim_{t \to t_{j^*+1}^-} \phi_{j^*}(x, t) = |x - x_{j^*}|^2 - \beta(t_{j^*+1} - t_{j^*}) < 0. \]

Therefore, there exist \( \varepsilon \in \left(0, \frac{t_{j^*+1} - t_{j^*}}{2}\right) \) and \( \delta > 0 \) such that
Let \( u \in H^1(Q) \), satisfying (11) on \( Q = \Omega \times (0, T) \). Let us consider \( Q^* := \Omega \times (t_f, t_{f+1}] \subseteq Q \). Now we define a cut-off function \( \chi \in C^\infty_0([t_f, t_{f+1}]) \) such that

\[
0 \leq \chi \leq 1
\]

and, since \( s \to 1 \) and for some positive constant \( C \),

\[ \chi(t) = \begin{cases} 
1, & t \in [t_f, t_{f+1} - 2\epsilon], \\
0, & t \in [t_f + \epsilon, t_{f+1}]. 
\end{cases} \]

We set \( v(x,t) = \chi(t)u(x,t) \), \( (x,t) \in Q^* \), then, keeping in mind (11) and (42), we deduce

\[
\begin{align*}
\frac{\partial}{\partial t} v + H(t) \cdot \nabla v &= u(\partial_s \chi) \quad \text{in } Q^*, \\
v|_{\partial \Omega \times (t_f, t_{f+1})} &= \chi g, \\
v(x, t_{f+1}) &= 0, & x \in \Omega.
\end{align*}
\]

Applying Theorem 1.3 to the problem (43), since \( v(x,t) \leq u(x,t) \) for every \( (x,t) \in Q^* \) (see 42), we obtain

\[
s^2 \int_Q v^2 e^{2s\eta} \, dx \, dt \leq C \int_Q |u|^2 |\partial_s \chi|^2 e^{2s\eta} \, dx \, dt + C e^{Cs^2} \int_{\Sigma} |u|^2 \, d\gamma \, dt,
\]

for all large \( s > 0 \) and for some positive constant \( C \).

Therefore, by (42) and (11) we have

\[
s^2 \int_Q v^2 e^{2s\eta} \, dx \, dt \geq s^2 \int_{t_f}^{t_f+\epsilon} \int_{\Omega} |u|^2 e^{2s\eta} \, dx \, dt \geq s^2 e^{2s\delta} \int_{t_f}^{t_f+\epsilon} \int_{\Omega} |u|^2 \, dx \, dt (45)
\]

and, since \( \chi \in C^\infty_0([t_f, t_{f+1}]) \), we also deduce

\[
\int_{Q^*} |u|^2 |\partial_s \chi|^2 e^{2s\eta} \, dx \, dt = \int_{t_{f+1} - 2\epsilon}^{t_{f+1} - \epsilon} \int_{\Omega} |u|^2 |\partial_s \chi|^2 e^{2s\eta} \, dx \, dt \\
\leq K_1 e^{-2s\delta} \int_{t_{f+1} - 2\epsilon}^{t_{f+1} - \epsilon} \int_{\Omega} |u|^2 \, dx \, dt \leq K_1 \|u\|_{L^2(Q^*)}^2 e^{-2s\delta},
\]

for all large \( s > 0 \) and for some positive constant \( K_1 \).

From (44), by (45) and (46) we obtain

\[
s^2 e^{2s\delta} \int_{t_f}^{t_f+\epsilon} \int_{\Omega} |u|^2 \, dx \, dt \leq C_1 \|u\|_{L^2(Q^*)}^2 e^{-2s\delta} + C_1 e^{Cs^2} \|g\|_{L^2(\partial \Omega \times (0, T))}^2
\]

for all large \( s > 0 \) and for some positive constant \( C_1 \).

Setting

\[ E(t) := \int_{\Omega} |u(x,t)|^2 \, dx, \quad t \in [t_f, t_{f+1}], \]

by the energy estimate (35) of Lemma 3.1 we deduce
\[ \int_{t_j}^{t_{j+1}} \int_\Omega |u|^2 \, dx \, dt = \int_{t_j}^{t_{j+1}} |u|^2 \, dx \, dt \leq \int_{t_j}^{t_{j+1}} (E(t) - H_\varepsilon \|g\|_{L^2(\partial \Omega \times (0,T))}) \, dt \]

and, by the energy estimate (34) of Lemma 3.1 we obtain

\[ \|u\|_{L^2(Q^*)}^2 = \int_{t_j}^{t_{j+1}} E(t) \, dt = \int_{t_j}^{t_{j+1}} (E(t) + H_\varepsilon \|g\|_{L^2(\partial \Omega \times (0,T))}) \, dt \]

Substituting (48) and (49) into (47), we have

\[ (s^2 e^{2\delta} - C^2 T e^{-2\delta}) E(t_j) \leq (C e^{C^1 \varepsilon} + s^2 e^{2\delta} \varepsilon H_\varepsilon + C_1 e^{-2\delta} H_\varepsilon T) \|g\|_{L^2(\partial \Omega \times (0,T))} \]

for all large \( s > 0 \). Hence,

\[ s^2 e^{2\delta} - C_1 T e^{-2\delta} > 0. \]

Thus, using again (34), for every \( t \in [t_j, t_{j+1}] \), we obtain

\[ \|u(\cdot, t)\|_{L^2(\Omega)} = E(t) \leq E(t_j) + H_\varepsilon \|g\|_{L^2(\partial \Omega \times (0,T))} \leq C_2 \|g\|_{L^2(\partial \Omega \times (0,T))}, \]

for some positive constant \( C_2 \). The conclusion of the proof of Theorem 1.4 follows from the above inequality, namely using Lemma 3.2 we can extend the above observability inequality from \([t_j, t_{j+1}]\) to \([0, T]\). \( \square \)

**Appendix**

In this appendix we prove Lemma 1.2.

**Proof.** (of Lemma 1.2). Since \( H \in \text{Lip}([0,T]; \mathbb{R}^d) \) there exists \( L > 0 \) such that

\[ |H(t) - H(s)| \leq L|t - s|, \forall t, s \in [0,T]. \]
Let us consider, for simplicity, a uniform partition \( \{ t_j \}_{0}^{m} \) of \([0, T]\). Let us set
\[
\eta_j := \frac{H(t_j)}{|H(t_j)|}, \quad j = 0 \ldots, m - 1.
\]
For \( t \in [t_j, t_{j+1}] \), \( j = 0 \ldots, m - 1 \), we deduce
\[
H(t) \cdot \eta_j = (H(t) - H(t_j)) \cdot \eta_j + H(t_j) \cdot \eta_j \geq -|H(t) - H(t_j)| + |H(t_j)|
\]
\[
\geq -L|t - t_j| + |H(t_j)| \geq -L \frac{T}{m} + |H(t_j)|,
\]
and, since \( |H(t)| \leq |H(t) - H(t_j)| + |H(t_j)| \),
\[
|H(t_j)| \geq |H(t)| - |H(t) - H(t_j)| \geq |H(t)| - L|t - t_j| \geq |H(t)| - L \frac{T}{m}.
\]
From (50) and (51), if we choose the uniform partition with \( m \geq \frac{2LT}{H_0(1 - S^*)} \), where we recall that \( H_0 = \min_{t \in [0,T]} |H(t)| \), we obtain the conclusion, that is,
\[
H(t) \cdot \frac{H(t_j)}{|H(t_j)|} \geq |H(t)| - 2L \frac{T}{m} \geq S_*|H(t)|, \quad \forall t \in [t_j, t_{j+1}], \forall j = 0 \ldots, m - 1.
\]

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