Approximation and Schauder bases in M"unzt spaces $M_{\Lambda,C}$ of continuous functions.

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Abstract

M"unzt spaces $M_{\Lambda,C}$ of continuous functions supplied with the absolute maximum norm are considered in this article. Fourier series approximations of functions in M"unzt spaces $M_{\Lambda,C}$ are studied. An existence of Schauder bases in M"unzt spaces $M_{\Lambda,C}$ is investigated.

1 Introduction.

The area of mathematics devoted to topological and geometric properties of topological vector spaces is very important in functional analysis (see, for example, [11, 14, 16, 17, 23]). Particularly, studies of bases in Banach spaces compose a great part of it (see, for example, [11, 14, 15, 18]-[22, 27, 29, 33] and references therein). Many open problems remain for concrete classes of Banach spaces.

Among them M"unzt spaces $M_{\Lambda,C}$ play very important role and there also remain unsolved problems (see [5, 9, 28] and references therein). They are

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defined as completions of the linear span over the real field \( \mathbb{R} \) or the complex field \( \mathbb{C} \) of monomials \( t^\lambda \) with \( \lambda \in \Lambda \) on the segment \([0, 1]\) relative to the absolute maximum norm, where \( \Lambda \subset [0, \infty) \), \( t \in [0, 1] \). It was K. Weierstrass who in 1885 had proved his theorem about polynomial approximations of continuous functions on the segment. But the space of continuous functions possesses the algebra structure. Later on in 1914 C. Müntz had considered generalizations so that his spaces generally had not such algebraic structure.

A problem was whether they have bases [10] [27]. Then a progress was for Müntz spaces satisfying the lacunary condition \( \lim_{n \to \infty} \lambda_{n+1}/\lambda_n > 1 \) with countable \( \Lambda \), but in general this problem was unsolved [9] [28]. It is worth to mention that the monomials \( t^\lambda \) with \( \lambda \in \Lambda \) generally do not form a Schauder basis in the Müntz space \( M_{\Lambda, C} \).

In this article results of investigations of the author on this problem are presented.

In section 2 a Fourier approximation in Müntz spaces \( M_{\Lambda, C} \) of continuous functions on the unit segment \([0, 1]\) supplied with the absolute maximum norm is studied. For this purpose auxiliary Lemmas 3, 4 and Theorem 5 are proved. They are utilized for reducing a consideration to a subclass of Müntz spaces \( M_{\Lambda, C} \) such that a set \( \Lambda \) is contained in the set of natural numbers \( \mathbb{N} \) up to an isomorphism of Banach spaces. It is proved that for Müntz spaces satisfying the Müntz condition and the gap condition their functions belong to Weil-Nagy’s class. Then the theorem about existence of Schauder bases in Müntz spaces \( M_{\Lambda, C} \) under the Müntz condition and the gap condition is proved.

All main results of this paper are obtained for the first time. They can be used for further studies of function approximations and geometry of Banach spaces. This is important not only for progress of mathematical analysis and functional analysis, but also in different applications including measure theory and stochastic processes in Banach spaces.

## 2 Müntz spaces \( M_{\Lambda, C} \).

To avoid misunderstanding we first present our notation and definitions.
1. Notation. Let $C([a, b], F)$ denote the Banach space of all continuous functions $f : [a, b] \rightarrow F$ supplied with the absolute maximum norm
\[ \|f\|_C := \max\{|f(x)| : x \in [a, b]\}, \]
where $-\infty < a < b < \infty$, while $F = \mathbb{R}$ is the real field or $F = \mathbb{C}$ the complex field.

Then $L_p((a, b), F)$ denotes the Banach space of all Lebesgue measurable functions $f : (a, b) \rightarrow F$ possessing a norm as defined by the Lebesgue integral:
\[ \|f\|_{L_p((a, b), F)} := \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} < \infty, \]
where $1 \leq p < \infty$ is a fixed number, and $-\infty \leq a < b \leq \infty$.

Let $Q = (q_{n,k})$ be a lower triangular infinite matrix with matrix elements $q_{n,k}$ having values in the field $F = \mathbb{R}$ or $F = \mathbb{C}$ so that $q_{n,k} = 0$ for each $k > n$, where $k, n$ are nonnegative integers. To each 1-periodic function $f : \mathbb{R} \rightarrow F$ in the space $L_1([a, a + 1], F)$ is counterposed a trigonometric polynomial
\[ (1) \quad U_n(f, x, Q) := \frac{a_0}{2} q_{n,0} + \sum_{k=1}^n q_{n,k} (a_k \cos(2\pi kx) + b_k \sin(2\pi kx)), \]
where $a_k = a_k(f)$ and $b = b_k(f)$ are the Fourier coefficients of a function $f(x)$, whilst on $\mathbb{R}$ the Lebesgue measure is considered.

For measurable 1-periodic functions $h$ and $g$ their convolution is defined whenever it exists:
\[ (2) \quad (h \ast g)(x) := 2 \int_a^{a+1} h(x - t) g(t) \, dt. \]

Putting the kernel of the operator $U_n$ to be:
\[ (3) \quad U_n(x, Q) := \frac{q_{n,0}}{2} + \sum_{k=1}^n q_{n,k} \cos(2\pi kx) \]
one gets
\[ (4) \quad U_n(f, x, Q) = (f \ast U_n(\cdot , Q))(x) = (U_n(\cdot , Q) \ast f)(x). \]

The norms of these operators are:
\[ (5) \quad L_n(Q) := \sup_{\|f\|_C=1} \|U_n(f, x, Q)\|_C = 2 \|U_n(x, Q)\|_{L_1} = 2 \int_a^{a+1} |U_n(t, Q)| \, dt, \]
where $\|\cdot\|_C$ and $\|\cdot\|_{L_1}$ denote norms on Banach spaces $C([a, a + 1], F)$ and $L_1([a, a + 1], F)$ respectively, while $a \in \mathbb{R}$ is a marked real number. These numbers $L_n(Q)$ are called Lebesgue constants of a summation method.
Denote by $C_0([a, a+1], F)$ the subspace of continuous functions $f : [a, a+1] \to F$ satisfying the periodicity condition $f(0) = f(1)$.

As usually $\mathbb{N} = \{1, 2, ...\}$ denotes the set of all natural numbers, also $c_0(F)$ stands for the Banach space of all converging to zero sequences in $F$ supplied with the absolute supremum norm.

Henceforth the Fourier summation methods given by $\{U_m : m\}$ which converge on $C_0([a, a+1], F)$

$$\lim_{m \to \infty} U_m(f, x, Q) = f(x)$$

uniformly in $x \in [a, a+1]$ will be considered.

2. Definition. Let $\Lambda$ be an increasing sequence in the set $(0, \infty)$.

The completion of the linear space containing all monomials $at^\lambda$ with $a \in F$ and $\lambda \in \Lambda$ and $t \in [\alpha, \beta]$ relative to the absolute maximum norm

$$\|f(t)\|_C := \sup_{t \in [\alpha, \beta]} |f(t)|$$

is denoted by $M_{\Lambda,C}[\alpha, \beta]$, where $0 \leq \alpha < \beta < \infty$. Particularly, for $[\alpha, \beta] = [0, 1]$ it is also shortly written as $M_{\Lambda,C}$. We consider also its subspace

$$M_{\Lambda,C}^0[0, 1] := \{f : f \in M_{\Lambda,C}[0, 1]; f(0) = f(1)\}$$

consisting of 1-periodic functions.

Henceforth it is supposed that the sequence $\Lambda$ satisfies the gap condition

(2) $\inf_k \{\lambda_{k+1} - \lambda_k\} =: \alpha_0 > 0$ and the Müntz condition

(3) $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} =: \alpha_1 < \infty$,

where $\Lambda = \{\lambda_k : k \in \mathbb{N}\}$.

Below Lemmas 3, 4 and Theorem 5 are proved about isomorphisms of Müntz spaces $M_{\Lambda,C}$. Utilizing them we reduce our consideration to a subclass of Müntz spaces $M_{\Lambda,C}$ such that a set $\Lambda$ is contained in the set of natural numbers $\mathbb{N}$.

3. Lemma. For each $0 < \delta < 1$ Müntz spaces $M_{\Lambda,C}([0, 1], F)$ and $M_{\Lambda,C}([\delta, 1], F)$ are isomorphic.
Proof. For every $0 < \delta < 1$ and $0 < \epsilon \leq 1$ and $f \in E$ the norms $\|f\|_{C[0,1]}$ and $\epsilon \|f|_{[0,\delta]}\|_{C[0,\delta]} + \|f|_{[\delta,1]}\|_{C[\delta,1]}$ are equivalent. Due to the Remez-type inequality (see Theorem 6.2.2 in [2] and Theorem 7.4 in [3]) for each $\Lambda$ satisfying the Müntz condition there is a constant $\eta > 0$ so that $\|h|_{[0,\delta]}\|_{C[0,\delta]} \leq \eta \|h|_{[\delta,1]}\|_{C[\delta,1]}$ for each $h \in M_{\Lambda,C}$, where $\eta$ is independent of $h$. This implies that the norms $\|h|_{[\delta,1]}\|_{C[\delta,1]}$ and $\|h\|_{C[0,1]}$ are equivalent on $M_{\Lambda,C}[0,1]$. Each polynomial $a_1 t^{\lambda_1} + ... + a_n t^{\lambda_n}$ on $[\delta,1]$ has the natural extension on $[0,1]$, where $a_1,...,a_n \in F$ are constants and $t$ is a variable. Thus Müntz spaces $M_{\Lambda,C}[0,1]$ and $M_{\Lambda,C}[\delta,1]$ are isomorphic for each $0 < \delta < 1$.

4. Lemma. Müntz spaces $M_{\Lambda,C}$ and $M_{\Xi,\mu_{\Lambda+\beta},C}$ are isomorphic for every $\beta \geq 0$ and $\alpha > 0$ and a finite subset $\Xi$ in $(0,\infty)$.

Proof. The set $\Lambda$ is infinite with $\lim_n \lambda_n = \infty$. By virtue of Theorem 9.1.6 [9] Müntz space $M_{\Lambda,C}$ contains a complemented isomorphic copy of $c_0(F)$. Therefore, $M_{\Lambda,C}$ and $M_{\Xi,\mu_{\Lambda+\beta},C}$ are isomorphic.

The isomorphism of $M_{\alpha_{\Lambda,C}}$ with $M_{\Lambda,C}$ follows from the equality

$$\sup_{t \in [0,1]} |f(t)| = \sup_{t \in [0,1]} |f(t^\alpha)|$$

for each continuous function $f : [0,1] \to F$, since the mapping $t \mapsto t^\alpha$ is the diffeomorphism of the segment $[0,1]$ onto itself. Taking at first $\Lambda_1 = \Lambda \cup \{\delta\}$ and then $\alpha \Lambda_1$ we infer that the Müntz spaces $M_{\Lambda,C}$ and $M_{\alpha_{\Lambda+\beta},C}$ are isomorphic.

5. Theorem. Suppose that increasing sequences $\Lambda$ and $\Upsilon$ of positive numbers satisfy the Müntz condition and the gap condition and $\lambda_n \leq \mu_n$ for each $n$. If

$$\sup_n (\mu_n - \lambda_n) = \delta,$$

where

$$\delta < (8 \sum_{n=1}^{\infty} \lambda_n^{-1})^{-1},$$

then $M_{\Lambda,C}$ and $M_{\Upsilon,C}$ are the isomorphic Banach spaces.

Proof. Certainly the Müntz spaces $M_{\Lambda,C}$ and $M_{\Upsilon,C}$ have isometric linear embeddings of into $M_{\Lambda \cup \Upsilon,C}$. Consider a sequence of sets $\Upsilon_k$ satisfying the following restrictions

1. $\Upsilon_k = \{\mu_{k,n} : n = 1,2,...\} \subset \Lambda \cup \Upsilon$ and $\mu_{k,n} \in \{\lambda_n,\mu_n\}$ for each $k = 0,1,2,...$ and $n = 1,2,...$, where $\Upsilon_0 = \Lambda$;

2. $\mu_{k,n} \leq \mu_{k+1,n}$ for each $k = 0,1,2,...$ and $n = 1,2,...$.
(3) \( \{\Delta_{k+1,n} : \Delta_{k+1,n} \neq 0; n = 1, 2, \ldots \} \) is a monotone decreasing subsequence converging to zero obtained from the sequence \( \Delta_{k+1,n} := \mu_{k+1,n} - \mu_{k,n} \) by elimination of zero terms;

(4) \( \{m(k + 1) : k\} \) is a monotone increasing sequence with \( m(k + 1) := \min\{n : \mu_n - \mu_{k+1,n} \neq 0; \forall l < n \mu_l = \mu_{k+1,l}\} \).

For each \( f \in M_{Y_k,C} \) we consider the power series \( f_1(t) = \sum_{l=1}^{\infty} a_n t^{\mu_{k+1,n}} \), where the power series decomposition \( f(t) = \sum_{l=1}^{\infty} a_n t^{\mu_{k,n}} \) converges for each \( 0 \leq t < 1 \), since \( f \) is analytic on \([0,1)\) (see \([5,9]\)). Then we infer that

\[
f(t^2) - f_1(t^2) = \sum_{l=1}^{\infty} a_n t^{\mu_{k,n}} u_n(t) \quad \text{with} \quad u_n(t) := t^{\mu_{k,n}} - t^{\mu_{k,n}+2\Delta_{k+1,n}}
\]

so that \( u_n(t) \) is a monotone decreasing sequence by \( n \) and hence

\[
|f(t^2) - f_1(t^2)| \leq 2|u_{m(k+1)}(t)||f(t)|
\]

according to Dirichlet’s criterium for each \( 0 \leq t < 1 \). Therefore, the function \( f_1(t) \) has the continuous extension onto \([0,1]\) and

\[
\|f - f_1\|_{C([0,1],F)} \leq 4\|f\|_{C([0,1],F)} \Delta_{k+1,m(k+1)}/\lambda_{m(k+1)},
\]

since the mapping \( t \mapsto t^2 \) is the order preserving diffeomorphism of \([0,1]\) onto itself. Thus the series \( \sum_{l=1}^{\infty} a_n t^{\mu_{k+1,n}} \) converges on \([0,1)\). Similarly to each \( g_1 \in M_{Y_{k+1},C} \) a function \( g \in M_{Y_{k+1},C} \) corresponds.

This implies that there exists the linear isomorphism \( T_k \) of \( M_{Y_k,C} \) with \( M_{Y_{k+1},C} \) so that

\[
\|T_k - I\| \leq 4\Delta_{k+1,m(k+1)}/\lambda_{m(k+1)}, \quad T_k : M_{Y_k,C} \to M_{Y_{k+1},C}.
\]

Next we take the sequence of operators

\[
S_n := T_n T_{n-1} \ldots T_0 : M_{A,C} \to M_{Y_{n+1},C} \subset M_{A\cup Y,C}.
\]

The space \( M_{A\cup Y,C} \) is complete and the sequence \( \{S_n : n\} \) converges in the operator norm uniformity to an operator \( S : M_{A,C} \to M_{A\cup Y,C} \) so that

\[
\|S - I\| < 1, \text{ since }
\]

\[
\sum_{k=0}^{\infty} \Delta_{k+1,m(k+1)}/\lambda_{m(k+1)} \leq \delta \sum_{n=1}^{\infty} \lambda_n^{-1} < \infty,
\]

where \( I \) denotes the unit operator. Therefore, the operator \( S \) is invertible.

From Conditions (1 – 4) it follows that \( S(M_{A,C}) = M_{Y,C} \).

6. Remark. In view of Lemmas 3, 4 and Theorem 5 it suffices to consider a set \( \Lambda \) satisfying the gap and M"untz conditions such that \( \Lambda \subset \mathbb{N} \) up to an isomorphism of M"untz spaces.
Next we recall necessary definitions and notations of the Fourier approximation. Then auxiliary Proposition 10 is given which is used for proving Theorem 11 about the property that for Müntz spaces satisfying the Müntz and gap conditions their functions belong to Weil-Nagy’s class.

7. Notation. Suppose that \((\psi(k) : k \in \mathbb{N})\) is a sequence of non-zero numbers tending to zero, \(\beta\) is a marked real number. By \(F\) is denoted the set of all pairs \((\psi, \beta)\), for which

\[
(1) \quad D_{\psi, \beta}(x) := \sum_{k=1}^{\infty} \psi(k) \cos(2\pi kx + \beta\pi/2)
\]

is the Fourier series of some function belonging to \(L_1[0,1]\). Then \(F_1\) denotes the family of all positive sequences \((\psi(k) : k \in \mathbb{N})\) tending to zero with \(\Delta_2 \psi(k) := \psi(k-1) - 2\psi(k) + \psi(k+1) \geq 0\) for each \(k\) so that the series

\[
(2) \quad \sum_{k=1}^{\infty} \frac{\psi(k)}{k} < \infty
\]

converges.

An approximation of a function \(f\) by the Fourier series \(S(f, x)\) is estimated by the function

\[
(3) \quad \rho_n(f, x) := f(x) - S_{n-1}(f, x),
\]

where

\[
(4) \quad S_n(f, x) := \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(2\pi kx) + b_k \sin(2\pi kx))
\]

is the partial Fourier sum. That is a trigonometric polynomial approximating a summable (i.e. Lebesgue integrable) 1-periodic function \(f \in L_1[0,1]\).

8. Definitions. Let \(f \in L_1[a,a+1]\) and \(S(f)\) be its Fourier series with coefficients \(a_k\) and \(b_k\), let also \(\psi(k)\) be an arbitrary sequence real or complex. If the function

\[
f_{\psi}^{\beta} := \sum_{k=1}^{\infty} [a_k(f) \cos(2\pi kx + \beta\pi/2) + b_k(f) \sin(2\pi kx + \beta\pi/2)]/\psi(k)
\]

belongs to the space \(L_1[a,a+1]\), then \(f_{\psi}^{\beta} = D_{\psi}^{\beta}f\) is called the Weil \(((\psi, \beta))\) derivative of \(f\).

Let for a Banach space \(\mathcal{N}\) of some functions on \([a,a+1]\):

\[
C_{\psi}^{\beta}\mathcal{N}[a,a+1] := \{f \in \mathcal{N} : \exists f_{\psi}^{\beta} \in C_0[a,a+1]\},
\]
where \((\psi(k) : k)\) is a sequence with non-zero elements for each \(k\) and \(\beta\) is a real parameter.

In particular, let \(C^\psi_\beta M[a, a + 1]\) (or \(C^\psi_\beta[a, a + 1]\) for short) be the space of all continuous 1-periodic functions \(f\) having a continuous Weil derivative \(f^\psi_\beta, \ f^\psi_\beta \in C_0[0, 1]\) and considered relative to the absolute maximum norm and such that

\[
\|f^\psi_\beta\|_C := \max \{|f^\psi_\beta(t)| : t \in [0, 1]\} < \infty.
\]

Particularly, for \(\psi(k) = k^{-r}\) there is the Weil-Nagy class 

\[
W^r_{\beta,C} = W^r_{\beta,C}[a, a + 1] := \{f : f \in C_0[a, a + 1], \exists f^\psi_\beta \in C_0[a, a + 1]\}.
\]

Then let 

\[
E_n(X) := \sup \{\|\rho_n(f; x)\|_{C[a,a+1]} : f \in X\},
\]

\[
E_n(f) := \inf \{\|f - T_n-1\|_{C[a,a+1]} : T_n-1 \in \mathcal{T}_{2n-1}\},
\]

\[
E_n(X) := \sup \{E_n(f) : f \in X\},
\]

where \(X\) is a subset in \(C_0[a, a + 1]\).

10. Proposition. Suppose that an increasing sequence \(\Lambda = \{\lambda_n : n\}\) of natural numbers satisfies the M"{u}ntz condition. Then

\[
(\forall f \in M_{\Lambda,C}[0, 1]) \Rightarrow (df(x)/dx \in L_{1,w}(0, 1)).
\]

Proof. Let \(f \in M_{\Lambda,C}[0, 1]\). In view of Theorem 6.2.3 and Corollary 6.2.4 \[9\] a function \(f\) is analytic on \(\hat{B}_1(0)\) and the series

\[
(1) \ f(z) = \sum_{n=1}^{\infty} a_n z^{\lambda_n}
\]

converges on \(\hat{B}_r(0)\), where \(\hat{B}_r(x) := \{y : y \in \mathbb{C}, |y - x| < r\}\) denotes the open disk in \(\mathbb{C}\) of radius \(r > 0\) with center at \(x \in \mathbb{C}\), where \(a_n \in \mathbb{F}\) is
an expansion coefficient for each \( n \in \mathbb{N} \). That is, the function \( f \) has a holomorphic univalent extension from \([0, 1)\) on \( \hat{B}_1(0) \), since \( \Lambda \subset \mathbb{N} \) (see Theorem 20.5 in [30]).

Using the Riemann integral we have that

\[
\int_x^1 f'(t)dt \quad \text{for each } 0 < x \leq 1
\]

and

\[
\lim_{x \to 0} \int_x^1 f'(t)dt = f(1) - f(0)
\]

due to Newton-Leibnitz’ formula (see, for example, §II.2.6 in [12]), since \( f(x) \) is continuous on \([0, 1)\).

By virtue of the uniqueness theorem for holomorphic functions (see, for example, II.2.22 in [30]), applied to the considered case, if a nonconstant holomorphic function \( g \) on \( U \) has a set \( E(g) = \{ x : g(x) = 0, 0 \leq x < 1 \} \) of zeros in \([0, 1)\), then either \( E(g) \) is finite or infinite with the unique limit point 1. Then we consider a linear function \( s(x) = \alpha + \beta x \) with real coefficients \( \alpha \) and \( \beta \), put \( u(x) = f(x) + s(x) \) and choose \( \alpha \) and \( \beta \) so that \( u(0) = u(1) = 0 \).

On the other hand, \( \min \Lambda = \lambda_1 > 0 \) and hence the function \( f \) is nonconstant. The case \( du(x)/dx = \text{const} \) is trivial. So there remains the variant when \( du(x)/dx \) is nonconstant. Denote by \( x_n \) zeros in \([0, 1)\) of \( du(t)/dt \) of odd order so that \( x_{n+1} > x_n \) for each \( n \in \mathbb{N} \). Therefore,

\[
\int_{x_n}^{x_{n+1}} u'(t)dt \int_{x_{n+1}}^{x_{n+2}} u'(\tau)d\tau < 0
\]

for each \( n \in \mathbb{N} \) according to Theorem II.2.6.10 in [12]. If \( \{ x_n : n \} \) is a finite set, then from Formulas (1, 2) it follows that \( u' \in L_1[0, 1] \) and hence \( u' \in L_{1,w}[0, 1] \).

Consider now the case when the set \( \{ x_n : n \} \) is infinite. We take the closed disc \( V = \{ u \in \mathbb{C} : |u - 1/2| \leq 1/2 \} \), hence \([0, 1] \subset V \). According to Cauchy’s formula 21(5) in [30]

\[
f'(z) = \frac{1}{2\pi i} \int_{\omega} \frac{f(\xi)}{\xi - z}d\xi
\]

for each \( z \in \text{Int}(V) \), where \( \omega \) is a rectifiable path encompassing once a point \( z \) in the positive direction so that \( \omega \subset V \), where \( \text{Int}(V) \) denotes the interior of \( V \). For \( 3/4 < x < 1 \) a circle can be taken with center at \( x \) and of radius
0 < r < 1 - x with r ↑ (1 - x) whilst x ↑ 1. Using the homotopy theorem and the continuity of f on the compact disc V one can take simply the circle ω = ∂V = \{u ∈ C : |u - 1/2| = 1/2\}. Since \(\max_{z ∈ V} |f(z)| =: G < ∞\), then from the estimate of Cauchy’s integral (4) it follows that

\[
|f'(x)| ≤ G/(2πx)
\]

for each 3/4 < x < 1, hence \(f'(x) ∈ L_{1,w}(3/4, 1)\) and consequently \(u'(t) ∈ L_{1,w}(3/4, 1)\). Therefore, from the inequality (5) we infer that

\[
\sup_{t > 0} y\mu\{t : t ∈ (0, 1], |f'(t)| ≥ y\} ≤ \sup_{t > 0} y\mu\{t : t ∈ (0, 3/4], |f'(t)| ≥ y\} + \sup_{t > 0} y\mu\{t : t ∈ [3/4, 1], |f'(t)| ≥ y\} < ∞,
\]

where \(\mu\) denotes the Lebesgue measure on \([0, 1]\). The latter means that \(df(x)/dx ∈ L_{1,w}[0, 1]\).

11. Theorem. Let an increasing sequence \(Λ = \{λ_n : n\} ⊂ \mathbb{N}\) of natural numbers satisfy the M"untz condition and let \(f ∈ M_{Λ,C}[0, 1]\). Then for each 0 < γ < 1 there exists β = β(γ) ∈ \(\mathbb{R}\) so that \(v ∈ W_{γ, β}^{L∞}(l, l + 1)\) if and only if there exists a function \(φ = φ_{h, γ, β}\) which is 1-periodic on \(\mathbb{R}\) and Lebesgue integrable on \([0, 1]\) such that

\[
(1) \quad h(x) = \frac{a_0(h)}{2} + (φ * D_{ψ, β})(x),
\]

where \(a_0(h) = 2 \int_{0}^{1} h(t)dt\) (see §§7 and 8).

We take a sequence \(U_n(t, Q)\) given by Formula 1(3) so that

\[
\lim_m g_{m,k} = 1 \text{ for each k and } \sup_m L_m(Q) < ∞ \text{ and } \sup_{m,k} |g_{m,k}| < ∞ \quad \text{and write for short } U_n(t) \text{ instead of } U_n(t, Q). \quad \text{Under these conditions the limit exists}
\]

\[
(2) \lim_n (v * U_n)(x) = v(x)
\]

in \(L_{∞}(0, 1)\) norm for each \(v ∈ L_{∞}((0, 1), \mathbf{F})\) according to Chapters 2 and 3 in [32] (see also [11, 36]).
On the other hand, Formula I(10.1) \cite{32} provides
\[(3)\ S[(y_{\psi_1}^{\psi_1})_{\beta_1-\beta_2}] = S[(y_{\psi_2}^{\psi_2})_{\beta_2}],\]
where \(S[y]\) is the Fourier series corresponding to a function \(y \in L_{\beta_2}^{\psi_2}\), when \((\psi_1, \beta_1) \leq (\psi_2, \beta_2)\).

Put \(\theta(k) = k^{\gamma-1}\) for all \(k \in \mathbb{N}\). Then \(D_{\theta, -\beta} \in L_1(0, 1)\) for each \(\beta \in \mathbb{R}\) due to Theorems II.13.7, V.1.5 and V.2.24 \cite{36} (or see \cite{11}). This is also seen from chapters I and V in \cite{32} and Formulas (1) and (3) above. In view of Dirichlet’s theorem (see §430 in \cite{3}) the function \(D_{\theta, -\beta}(x)\) is continuous on the segment \([\delta, 1 - \delta]\) for each \(0 < \delta < 1/4\).

According to formula 2.5.3.(10) in \cite{25}
\[
\int_0^\infty x^{a-1} \left( \frac{\sin(bx)}{\cos(bx)} \right) dx = b^{-a} \Gamma(\alpha) \left( \frac{\sin(\pi \alpha/2)}{\cos(\pi \alpha/2)} \right)
\]
for each \(b > 0\) and \(0 < Re(\alpha) < 1\). On the other hand, the integration by parts gives:
\[
\int_a^\infty x^{a-1} \left( \frac{\sin(bx)}{\cos(bx)} \right) dx = b^{-a-1} \left( \frac{\cos(ab)}{-\sin(ab)} \right) b^{-1}(\alpha-1) \int_a^\infty x^{a-2} \left( \frac{-\cos(bx)}{\sin(bx)} \right) dx
\]
for every \(a > 0\), \(b > 0\) and \(0 < Re(\alpha) < 1\). From formulas V(2.1), theorems V.2.22 and V.2.24 in \cite{36} (see also \cite{4, 24}) we infer the asymptotic expansions
\[
\sum_{n=1}^\infty n^{-\alpha} \sin(2\pi nx) \approx (2\pi x)^{\alpha-1} \Gamma(1 - \alpha) \cos(\pi \alpha/2) + \mu x^\alpha,
\]
\[
\sum_{n=1}^\infty n^{-\alpha} \cos(2\pi nx) \approx (2\pi x)^{\alpha-1} \Gamma(1 - \alpha) \sin(\pi \alpha/2) + \nu x^\alpha
\]
in a small neighborhood \(0 < x < \delta\) of zero, where \(0 < \delta < 1/4\), \(0 < \alpha < 1\), \(\mu\) and \(\nu\) are real constants. Taking \(\beta = \alpha = 1 - \gamma\) we get that \(D_{\theta, -\beta}(x) \in L_\infty(0, 1)\).

Remind that for Lebesgue measurable functions \(f : \mathbb{R} \rightarrow \mathbb{R}\) and \(g : \mathbb{R} \rightarrow \mathbb{R}\) there is the equality \(f_{-\infty}^\infty f(x - t)\chi_{[0, \infty)}(x - t)g(t)\chi_{[0, \infty)}(t)dt = \int_0^\infty f(x - t)g(t)dt\) for each \(x > 0\) whenever this integral exists, where \(\chi_A\) denotes the characteristic function of a subset \(A\) in \(\mathbb{R}\) such that \(\chi_A(y) = 1\) for each \(y \in A\), also \(\chi_A(y) = 0\) for each \(y\) outside \(A\), \(y \in \mathbb{R} \setminus A\). In particular, if \(0 < x \leq T\), where \(0 < T < \infty\) is a constant, then \(f_0^T f(x - t)g(t)dt = f_0^T f(x - t)\chi_{[0, T]}(x - t)g(t)\chi_{[0, T]}(t)dt\) (see also \cite{8, 13}). This can be applied to formula 1(2) putting \(\alpha = 0\) there and with the help of the equality
\[ f_0^1 f(x-t)g(t)dt = f_0^x f(x-t)g(t)dt + f_0^{1-x} f_1((1-x) - v)g_1(v)dv \]

for each \( 0 \leq x \leq 1 \) and 1-periodic functions \( f \) and \( g \) and using also that

\[ \|f|_{[a,b]}\| \leq \|f|_{[0,1]}\| \]

for the considered here types of norms for each \( [a, b] \subset [0, 1] \), where \( f_1(t) = f(-t) \) and \( g_1(t) = g(-t) \) for each \( t \in \mathbb{R} \), since

\[ f_0^1 f(x-t)g(t)dt = f_0^{1-x} f(v - 1 + x)g(1 - v)dv. \]

Mention that according to the weak Young inequality

\[ (4) \| \xi * \eta \|_r \leq K_{p,q} \| \xi \|_p \| \eta \|_{q,w} \]

for each \( \xi \in L_p \) and \( \eta \in L_{q,w} \), where \( 1 \leq p, q, r \leq \infty \) and \( p^{-1} + q^{-1} = 1 + r^{-1} \), \( K_{p,q} > 0 \) is a constant independent of \( \xi \) and \( \eta \) (see theorem 9.5.1 in [6], §IX.4 in [26]).

With the help of Proposition 10 and (2) we define the function \( s(x) \) such that

\[ s(x) = \lim_{\eta \downarrow 0} \lim_n \eta^{-1} \int_0^\eta (D_{\theta,-\beta} * U_n) * v'(x - t)dt. \]

In virtue of the weak Young inequality (4) and Proposition 10 this function \( s \) is in \( L_\infty(0, 1) \). Therefore \( \phi_{\nu, \gamma, \beta} = s \) and \( D_\beta^\nu v = s \) according to (1) and (3). Thus \( v \in W_\beta^\gamma L_\infty(0, 1) \). On the other hand, \( v \) is analytic on \( (0, 1) \), 1-periodic and continuous on \( [0, 1] \). Therefore, \( \|v\|_{L_\infty(0, 1)} = \|v\|_{C(0,1)} \) and hence \( v \in W_\beta^\gamma \alpha_{C}[0, 1] \).

**12. Lemma.** If an increasing sequence \( \Lambda \) of natural numbers satisfies the Müntz condition, also \( 0 < \gamma < 1 \),

\[ X = \{ v : v \text{ is } 1 \text{-periodic and } \forall t \in [0, 1) \ v(t) = f(t) + (f(0) - f(1))t, \]

\[ f \in M_\Lambda C; \|v\|_{C[0,1]} \leq 1 \}, \]

then a positive constant \( \omega = \omega(\gamma) \) exists so that

\[ (1) \ E_n(X) \leq \mathcal{E}_n(X) \leq \omega n^{-\gamma} \ln n \]

for each natural number \( n \in \mathbb{N} \).

**Proof.** Due to Theorem 11 the inclusion is valid \( v \in W_\beta^\gamma \alpha_{C}[0, 1] \) for the 1-periodic extension \( v \) of \( v(t) = f(t) + (f(0) - f(1))t \) on \( [0, 1] \) for each \( f \in M_\Lambda C \).

Then estimate (1) follows from Theorems 3.12.3 and 3.12.3' in [32].

**13. Lemma.** If \( \psi \in F_1 \) and \( (\psi, \beta) \in F \) (see §7), then \( C_\beta^{\psi,0}[0, 1] := \{ f \in C_\beta^{\psi}[0, 1] : \int_0^1 f(t)dt = 0 \} \) is the Banach space relative to the norm given by
the formula:

\[(1) \quad \|f\|_{C^\psi} := \|f^\psi\|_C.\]

**Proof.** We have that \(C^\psi_{\beta}[0,1]\) is the \(F\)-linear space and hence \(C^\psi_{\beta,0}[0,1]\) is such also as the kernel of the linear functional \(\phi(f) := \int_0^1 f(t)dt\), since each \(f \in C^\psi_{\beta}[0,1]\) is integrable. Therefore, the assertion of this lemma follows from Propositions I.8.1 and I.8.3 [32], since each \(f \in C^\psi_{\beta}[0,1]\) has the convolution representation:

\[(2) \quad f(x) = a_0(f) + 2 \int_0^1 f^\psi(x-t)D^\psi_{\psi,\beta}(t)dt\]

for each \(x \in [0,1]\), but \(a_0(f) = 0\) for each \(f \in C^\psi_{\beta,0}[0,1]\), while the convolution \(h \ast u\) is continuous for each \(h \in C[0,1]\) and \(u \in L_1[0,1]\) so that \(\|h \ast u\|_C \leq 2\|h\|_C\|u\|_{L_1}\), where \(D^\psi_{\psi,\beta}\) is defined by Formula 7(1).

To prove Theorem 15 about existence of a Schauder basis the following proposition is useful.

**14. Proposition.** Let \(X\) be a Banach space over \(\mathbb{R}\) and let \(Y\) be its Banach subspace so that they fulfill conditions (1 – 4) below:

1. there is a sequence \((e_i : i \in \mathbb{N})\) in \(X\) such that \(e_1, \ldots, e_n\) are linearly independent vectors and \(\|e_n\|_X = 1\) for each \(n\) and

2. there exists a Schauder basis \((z_n : n \in \mathbb{N})\) in \(X\) such that

\[z_n = \sum_{k=1}^n b_{k,n}e_k \text{ for each } n \in \mathbb{N}, \text{ where } b_{k,n} \text{ are real coefficients;}\]

3. for every \(x \in Y\) and \(n \in \mathbb{N}\) there exist \(x_1, \ldots, x_n \in \mathbb{R}\) so that

\[\|x - \sum_{i=1}^n x_ie_i\|_X \leq s(n)\|x\|,\]

where \(s(n)\) is a strictly monotone decreasing positive function with

\[\lim_{n \to \infty} s(n) = 0\]

and

4. \(u_n = \sum_{l=m(n)}^{k(n)} u_{n,l}e_l,\)

where \(u_{n,l} \in \mathbb{R}\) for each natural numbers \(k\) and \(l\), where a sequence \((u_n : n \in \mathbb{N})\) of normalized vectors in \(Y\) is such that its real linear span is everywhere dense in \(Y\) and \(1 \leq m(n) \leq k(n) < \infty\) and \(m(n) < m(n+1)\) for each \(n \in \mathbb{N}\).

Then \(Y\) has a Schauder basis.

**Proof.** Without loss of generality one can select and enumerate

5. vectors \(u_1, \ldots, u_n\) so that they are linearly independent in \(Y\) for each natural number \(n\). By virtue of Theorem (8.4.8) in [23] their real linear
span \( \text{span}_R(u_1, \ldots, u_n) \) is complemented in \( Y \) for each \( n \in \mathbb{N} \). Put \( L_{n,\infty} := \text{cl}_X \text{span}_R(u_k : k \geq n) \) and \( L_{n,m} := \text{cl}_X \text{span}_R(u_k : n \leq k \leq m) \), where \( \text{cl}_X A \) denotes the closure of a subset \( A \) in \( X \), where \( \text{span}_R A \) denotes the real linear span of \( A \). Since \( Y \) is a Banach space and \( u_k \in Y \) for each \( k \), then \( L_{n,\infty} \subset Y \) and \( L_{n,m} \subset Y \) for each natural numbers \( n \) and \( m \). Then we infer that

\[
L_{n,j} \subset \text{span}_R(e_l : m(n) \leq l \leq k_{n,j}),
\]

where \( k_{n,j} := \max(k(l) : n \leq l \leq j) \).

Take arbitrary vectors \( f \in L_{1,j} \) and \( g \in L_{j+1,q} \), where \( 1 \leq j < q \). Therefore, there are real coefficients \( f_i \) and \( g_i \) such that

\[
f = \sum_{i=1}^{k_{j,1}} f_i e_i \quad \text{and} \quad g = \sum_{i=m(j)+1}^{k_{j,q}} g_i e_i.
\]

Hence due to condition (2):

\[
\|f - \sum_{i=m(j)}^{m(j)} f_i e_i\|_X \leq s(m(j))\|f\| \quad \text{and} \quad \|g - \sum_{i=k_{j,1}}^{k_{j,q}} g_i e_i\|_X \leq s(k_{j,1} + 1)\|g\|_X.
\]

On the other hand,

\[
f = \sum_{i=1}^{m(j)} f_i e_i + \sum_{i=m(j)+1}^{k_{j,1}} f_i e_i, \quad \text{consequently,} \quad \|f^{[j+1]}\| \leq s(m(j + 1))\|f\|, \quad \text{where}
\]

\[
f^{[j+1]} := \sum_{i=m(j)+1}^{k_{j,1}} f_i e_i \quad \text{and} \quad \sum_{i=1}^{b} f_i e_i := 0, \quad \text{when} \ a > b.
\]

When \( 0 < \delta < 1/4 \) and \( s(m(j) + 1) < \delta \) we infer using the triangle inequality that \( \|f^{[j+1]} - h\|_X \leq \delta\|f^{[j+1]}\|_X/(1-\delta) \leq \delta s(m(j+1)-1)\|f\|_X/(1-\delta) \) for the best approximation \( h \) of \( f^{[j+1]} \) in \( L_{j+1,\infty} \), since \( m(j) < m(j+1) \) for each \( j \). Therefore, the inequality \( \|f - g\|_X \geq \|f - f^{[j+1]}\|_X - \|f^{[j+1]} - g\|_X \) and \( s(n) \downarrow 0 \) imply that there exists \( n_0 \) such that the inclination of \( L_{1,j} \) to \( L_{j+1,\infty} \) is not less than \( 1/2 \) for each \( j \geq n_0 \). Condition (4) implies that \( L_{1,n_0} \) is complemented in \( Y \). By virtue of Theorem 1.2.3 [9] a Schauder basis exists in \( Y \).

15. Theorem. If an increasing sequence \( \Lambda \) of positive numbers satisfies the Müntz condition and the gap condition, then the Müntz space \( M_{\Lambda,C}[0,1] \) has a Schauder basis.

Proof. By virtue of Lemma 4 and Theorem 5 it is sufficient to prove an existence of a Schauder basis in the Müntz space \( M_{\Lambda,C} \) for \( \Lambda \subset \mathbb{N} \). According to §4 the Banach spaces \( M_{\Lambda,C}^0 \) and \( M_{\Lambda,C} \) are isomorphic.

The functional

\[
(1) \quad \phi(h) := \int_0^1 h(\tau) d\tau
\]
is continuous on $C^\psi_\beta[0,1]$, where $\psi$ and $\beta$ satisfy conditions of Lemma 13. Then $coker(\phi) = F$. Therefore, $C^\psi_\beta[0,1] = F \oplus C^\psi_\beta,0[0,1]$.

In view of Theorem 6.2.3 and Corollary 6.2.4 [9] each function $g \in M_{A,C}[0,1]$ has an analytic extension on $\hat{B}_1(0)$ and hence

$$g(z) = \sum_{n=1}^{\infty} c_n z^{\lambda_n} = \sum_{k=1}^{\infty} p_k u_k(z)$$

are the convergent series on the unit open disk $\hat{B}_1(0)$ in $\mathbb{C}$ with center at zero (see §12), where $\Lambda \subset \mathbb{N}$ and $c_n = c_n(g) \in \mathbb{N}$, $p_n = p_n(g) = c_1 + \ldots + c_n$, $u_1(z) := z^{\lambda_1}$, $u_{n+1}(z) := z^{\lambda_{n+1}} - z^{\lambda_n}$ for each $n = 1, 2, \ldots$.

Take the finite dimensional subspace $X_n := \text{span}_{\mathbb{R}}(u_1, \ldots, u_n)$ in $X := M^0_{A,C}$, where $n \in \mathbb{N}$. Due to Lemma 4 the Banach space $X \oplus X_n$ exists and is isomorphic with $M_{A,C}$.

Consider the trigonometric polynomials $U_m(f, x, Q)$ for $f \in X$, where $m = 1, 2, \ldots$ (see §§1 and 8 also). Put $Y_{K,n}$ to be the $C[0,1]$ completion of the linear span $\text{span}_{\mathbb{R}}(U_m(f, x, Q) : (m, f) \in K)$, where $K \subset \mathbb{N} \times (X \oplus X_n)$, $m \in \mathbb{N}$, $f \in (X \oplus X_n)$.

There exists a countable subset $\{f_n : n \in \mathbb{N}\}$ in $X$ such that $f_n = D_{\psi,\beta}*g_n$ with $g_n \in L(0,1)$ for each $n \in \mathbb{N}$ and so that $\text{span}_{\mathbb{R}}(f_n : n \in \mathbb{N})$ is dense in $X$, since $X$ is separable. From Formulas (1,2) and Theorem 11 and Lemma 12 we infer that a countable set $K$ and a sufficiently large natural number $n_0$ exist so that the Banach space $Y_{K,n_0}$ is isomorphic with $(X \oplus X_{n_0})$ and $Y_{K,n_0}(0,1) \subset W^\beta_{\gamma,C}(0,1)$, where $0 < \gamma < 1$ and $\beta = 1 - \gamma$. Thus the Banach space $Y_{K,n_0}$ is the $C[0,1]$ completion of the real linear span of a countable family $(s_l : l \in \mathbb{N})$ of trigonometric polynomials $s_l$.

Without loss of generality this family can be refined by induction such that $s_l$ is linearly independent of $s_1, \ldots, s_{l-1}$ over $F$ for each $l \in \mathbb{N}$. With the help of transpositions in the sequence $\{s_l : l \in \mathbb{N}\}$, the normalization and the Gaussian exclusion algorithm we construct a sequence $\{r_l : l \in \mathbb{N}\}$ of trigonometric polynomials which are finite real linear combinations of the initial trigonometric polynomials $\{s_l : l \in \mathbb{N}\}$ and satisfying the conditions

1. $\|r_l\|_{C(0,1)} = 1$ for each $l$;
2. the infinite matrix having $l$-th row of the form ..., $a_{l,k}, b_{l,k}, a_{l,k+1}, b_{l,k+1}, \ldots$
for each $l \in \mathbb{N}$ is upper trapezoidal (step), where

$$r_l(x) = \frac{a_{l,0}}{2} + \sum_{k=m(l)}^{n(l)} [a_{l,k} \cos(2\pi kx) + b_{l,k} \sin(2\pi kx)]$$

with $a_{l,m(l)}^2 + b_{l,m(l)}^2 > 0$ and $a_{l,n(l)}^2 + b_{l,n(l)}^2 > 0$, where $1 \leq m(l) \leq n(l)$, $\text{deg}(r_l) = n(l)$, or $r_1(x) = \frac{a_{1,0}}{2}$ when $\text{deg}(r_1) = 0$; $a_{l,k}, b_{l,k} \in \mathbb{R}$ for each $l \in \mathbb{N}$ and $0 \leq k \in \mathbb{Z}$.

Then as $X$ and $Y$ in Proposition 14 we take $X = C[0,1]$ and $Y = Y_{K,n_0}$. In view of Proposition 14 and Lemma 4 the Schauder basis exists in $Y_{K,n_0}$ and hence also in $M_{\Lambda,C}[0,1]$.

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