On the Chow ring of a K3 surface

Arnaud Beauville

Introduction

An important algebraic invariant of a projective manifold $X$ is the Chow ring $\text{CH}(X)$ of algebraic cycles on $X$ modulo linear equivalence. It is graded by the codimension of cycles; the ring structure comes from the intersection product. For a surface we have

$$\text{CH}(X) = \mathbb{Z} \oplus \text{Pic}(X) \oplus \text{CH}_0(X),$$

where the group $\text{CH}_0(X)$ parametrizes 0-cycles on $X$. While the structure of the Picard group $\text{Pic}(X)$ is well understood, this is not the case for $\text{CH}_0(X)$: if $X$ admits a nonzero holomorphic 2-form, it is an infinite-dimensional vector space over $\mathbb{Q}$ ([M], [R]).

The simplest such surfaces are probably the K3 surfaces, which carry a nowhere vanishing holomorphic 2-form. In this case $\text{Pic}(X)$ is a lattice, while $\text{CH}_0(X)$ is very large; the following result is therefore somewhat surprising:

**Theorem.** The image of the intersection product

$$\text{Pic}(X) \otimes \text{Pic}(X) \to \text{CH}_0(X)$$

is infinite cyclic.

In fact we construct a canonical class $\xi_X \in \text{CH}_0(X)$ such that any product of divisors is a multiple of $\xi_X$ (Proposition 1). There is another canonical class in $\text{CH}_0(X)$, namely the second Chern class $c_2(X)$; it seems plausible that it is always a multiple of $\xi_X$, but we are able to check this only in some particular cases (Proposition 2).

Proofs

We work over the complex numbers. A rational curve on a surface is irreducible, but possibly singular.

The Theorem follows from a slightly more precise statement:

**Proposition 1.** Let $X$ be a projective K3 surface. Then all points of $X$ which lie on some rational curve have the same class $\xi_X$ in $\text{CH}_0(X)$; if $D$ and $E$ are divisors on $X$, we have

$$D \cdot E = n \xi_X \quad \text{with } n = \deg(D \cdot E).$$
Proof: Let $R$ be a rational curve on $X$; it is the image of a generically injective map $j : \mathbb{P}^1 \to X$. Put $\xi_R = j_*(p)$, where $p$ is an arbitrary point of $\mathbb{P}^1$. For any divisor $D$ on $X$, we have in $\text{CH}_0(X)$

$$R \cdot D = j_*j^*D = j_*(np) = n \xi_R,$$

with $n = \text{deg}(R \cdot D)$.

Let $S$ be another rational curve. If $\text{deg}(R \cdot S) \neq 0$, the above equality applied to $R \cdot S$ gives $\xi_S = \xi_R$ (recall that $\text{CH}_0(X)$ is torsion free [R]). If $\text{deg}(R \cdot S) = 0$, choose an ample divisor $H$; by a theorem of Bogomolov and Mumford [M-M], $H$ is linearly equivalent to a sum of rational curves. Since $H$ is connected, we can find a chain $R_0, \ldots, R_k$ of distinct rational curves such that $R_0 = R$, $R_k = S$ and $R_i \cap R_{i+1} \neq \emptyset$ for $i = 0, \ldots, k - 1$. We conclude from the preceding case that $\xi_R = \xi_{R_1} = \ldots = \xi_S$.

Thus the class $\xi_R$ does not depend on the choice of $R$; let us denote it by $\xi_X$. We have $R \cdot D = \text{deg}(R \cdot D) \xi_X$ for any divisor $D$ and any rational curve $R$ on $X$. Since the group $\text{Pic}(X)$ is spanned by the classes of rational curves (again by the Bogomolov-Mumford theorem), the Proposition follows. □

Remarks. — 1) The result (and the proof) hold more generally for any surface $X$ such that:

a) The Picard group of $X$ is spanned by the classes of rational curves;

b) There exists an ample divisor on $X$ which is a sum of rational curves.

This is the case when $X$ admits a non-trivial elliptic fibration over $\mathbb{P}^1$ with a section, or for some particular surfaces like Fermat surfaces in $\mathbb{P}^3$ with degree prime to 6 [S].

2) Let $A$ be an abelian surface. According to [Bl], the image of the product map $\text{Pic}(A) \otimes \text{Pic}(A) \to \text{CH}_0(A)$ has finite index, so the situation looks rather different from the K3 case. There is however an analogue to the Proposition. Let us work for simplicity in the $\mathbb{Q}$-vector space $\text{CH}_Q(A) := \text{CH}(A) \otimes \mathbb{Q}$. Let $\text{Pic}^+(A)$ be the subspace of $\text{Pic}_Q(A)$ fixed by the action of the involution $a \mapsto -a$. We have a direct sum decomposition

$$\text{Pic}_Q(A) = \text{Pic}^+(A) \oplus \text{Pic}_Q^0(A),$$

so that $\text{Pic}^+(A)$ is canonically isomorphic to $\text{NS}_Q(A)$. Now we claim that the image of the map $\mu : \text{Pic}^+(A) \otimes \text{Pic}^+(A) \to \text{CH}_Q(A)$ is $\mathbb{Q}[0]$. This is a direct consequence of the decomposition of $\text{CH}_Q(A)$ described in [B]: let $k$ be an integer $\geq 2$, and let $k$ be the multiplication by $k$ in $A$. We have $k^*D = k^2D$ for any element $D$ of $\text{Pic}^+(A)$, thus $k^*\xi = k^4\xi$ for any element $\xi$ in the image of $\mu$; but the latter property characterizes the multiples of $[0]$. □
Proposition 1 provides a distinguished class $\xi_X$ of degree 1 in the Chow group $\text{CH}_0(X)$: by definition, any 0-cycle of degree $d$ whose support is contained in a finite union of rational curves has class $d\xi_X$ in $\text{CH}_0(X)$. On the other hand, there is another canonical element in that group, namely the Chern class $c_2(X)$; we are led to ask whether this class is proportional to $\xi_X$. In all cases we were able to check the answer is positive:

**Proposition 2.** The relation $c_2(X) = 24\xi_X$ in $\text{CH}_0(X)$ holds in the following cases:

a) $X$ is a complete intersection in a product of projective spaces;
b) $X$ admits an elliptic fibration;
c) $X$ is a Kummer surface.

**Proof:**

a) We observe more generally that if a projective manifold $V$ is such that $c_2(V)$ is a rational combination of products of divisors, the same property holds for a smooth hypersurface $Y$ in $V$: this follows at once from the exact sequence

$$0 \to \mathcal{O}_Y(-Y) \to \Omega^1_{V|Y} \to \Omega^1_Y \to 0.$$ 

b) Let $f: X \to \mathbb{P}^1$ be an elliptic fibration. We have an exact sequence

$$0 \to f^*\omega_{\mathbb{P}^1}(V) \to \Omega^1_X \to \mathcal{I}_Z \otimes f^*\omega_{\mathbb{P}^1}^{-1}(-V) \to 0,$$

where $V$ is a sum of smooth rational curves contained in the fibres of $f$, and $\mathcal{I}_Z$ is the ideal sheaf of a finite subscheme $Z \subset X$ contained in the locus where $f$ is not smooth. Let $[Z]$ be the class of the corresponding 0-cycle in $\text{CH}_0(X)$, and $F \in \text{Pic}(X)$ the class of a fibre; from the exact sequence we obtain $c_2(X) = [Z] - (2F - V)^2$. But $Z$ is supported by (smooth) rational curves, hence is proportional to $\xi_X$.

c) Let $X$ be a Kummer surface. There exist an abelian surface $A$ and a diagram

$$\begin{array}{ccc}
\hat{A} & \xrightarrow{\varepsilon} & A \\
\pi \downarrow & & \downarrow \\
X & & \\
\end{array}$$

where $\varepsilon: \hat{A} \to A$ is the blowing up of the points of order $\leq 2$ of $A$ and $\pi: \hat{A} \to X$ is the quotient map by the involution of $\hat{A}$ deduced from $a \mapsto -a$. Let $E$ be the exceptional divisor in $\hat{A}$. Using the exact sequences

$$0 \to \pi^*\Omega^1_X \to \Omega^1_{\hat{A}} \to \mathcal{O}_E(-E) \to 0$$

$$0 \to \varepsilon^*\Omega^1_A \to \Omega^1_{\hat{A}} \to \Omega^1_E \to 0$$

2
a straightforward computation gives $\pi^*c_2(X) = -3E^2$. Let $C$ be the divisor $\pi_*E$ (sum of the 16 $(-2)$-curves of the Kummer surface). We have $\pi^*C = 2E$ and therefore $\pi^*C^2 = 4E^2$, from which we get $c_2(X) = -\frac{3}{4}C^2 = 24\xi_X$.

It is tempting to conjecture that the relation $c_2(X) = 24\xi_X$ always holds, but the evidence is not overwhelming; the crucial case would be when Pic$(X)$ has rank 1 (in which case the theorem is trivial).

There seems to be no reason to expect that the theorem still holds for other surfaces, say for regular surfaces of general type. But finding a counter-example is probably hard: we know of no way to prove that an explicitly given 0-cycle of degree 0 on such a surface has nonzero class in the Chow group.

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Arnaud BEAUVILLE
Laboratoire J.-A. Dieudonné
UMR 6621 du CNRS
UNIVERSITÉ DE NICE
Parc Valrose
F-06108 NICE Cedex 02