Supersymmetric Bianchi class A models

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ABSTRACT

The canonical theory of $N = 1$ supergravity is applied to Bianchi class A spatially homogeneous cosmologies. The full set of quantum constraints are then solved with the possible ordering ambiguity taken into account by introducing a free parameter. The wave functions are explicitly given for all the Bianchi class A models in a unified way. Some comments are made on the Bianchi type IX cases.

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1 Introduction

Recently, supersymmetry has been attracting many people in particle physics (see, e.g., ref. [1]). It seems worth studying supersymmetric quantum cosmology by expecting the existence of supersymmetry in the early universe.

Mini-superspace models, i.e. spatially homogeneous relativistic models, have been helpful to understand quantum cosmology. Among them, those of Bianchi class A [2] (its definition explained in section 2) are of particular importance when one investigates them in canonical form, because for Bianchi class ‘B’ models, the rest of the Bianchi types, there is an inconvenience that the equations of motion reduced from the full Einstein equations differ from the ones derived from the reduced action. Since the origin of this disaccord is purely geometrical we expect the same problem will happen also in supergravity. So we restrict our attention to the supersymmetric extension of the class A models. Special cases of such models have been investigated by some authors. Graham studied Bianchi type IX in refs. [3,4], where the full quantum constraints are not explicitly solved. D’Eath, Hawking and Obregón [5] solved the Bianchi type I case, but the Bianchi type I supersymmetric model is so simple that the wave function does not have any interesting structure.

We shall first restrict the canonical theory of supergravity proposed by D’Eath [6] to spatially homogeneous cases, in particular, Bianchi class A models, and then solve the quantum constraints. Then we will obtain the wave function

$$\Psi = const. h^{\frac{3}{2}} \exp \left( \frac{1}{2} \gamma m^{ab} h_{ab} \right) + const. h^{\frac{3}{2}} \exp \left( -\frac{1}{2} \gamma m^{ab} h_{ab} \right) (\psi^A \psi^A)^3,$$

where $h_{ab}$ is a spatial metric with respect to an invariant basis, $h$ its determinant and $m^{ab}$

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3Our result presented below disagrees with [5].

4 After the first submission of this paper it came to our attention that related work [12,13] by some other authors had appeared. They treat the Bianchi type IX cases and the type II cases, respectively. Here, we emphasize it is worth studying all class A models even when considering compactification [14], contrary to the reason explained in ref. [12] for which the author did not study Bianchi models except types I and IX.
the structure constants of a Bianchi class A group. Here, $\gamma$ is a real constant and $\psi^A_a$ a spin-$\frac{3}{2}$ field. The parameter $s$ specifies the ambiguity of the operator ordering (see eqs. (22),(23)). This is a direct generalization of the solutions found in the literature [3,4,5].

2 Spatially homogeneous Bianchi class A models

We write the four-metric as

$$\mathrm{d}s^2 = (N^i N_i - N^2)\mathrm{d}t^2 + 2N_i \mathrm{d}t \mathrm{d}x^i + h_{ij} \mathrm{d}x^i \mathrm{d}x^j,$$

(2)

where $N$ is the lapse function, $N^i$ the shift vector, and $h_{ij}$ the spatial metric. In spatially homogeneous cases, $h_{ij}$ is expanded by an invariant basis of a Bianchi group,

$$h_{ij} = h_{ab}(t) E^a_i E^b_j,$$

(3)

where indices $a, b, \ldots$ as well as the world indices $i, j, \ldots$ run from 1 to 3. The invariant basis $E^a_i$ obeys the following Maurer-Cartan relation:

$$\partial_i E^a_j = C^a_{bc} E^b_i E^c_j,$$

(4)

where $C^a_{bc}$ are the structure constants of the Bianchi group. $C^a_{bc}$ is antisymmetric in $b$ and $c$ and satisfies the Jacobi identity $C^a_{bc} [C^b_{de} C^d_{ce}] = 0$. We can decompose $C^a_{bc}$ as

$$C^a_{bc} = m^{ad} \epsilon_{dce} + \delta^a_b a_c,$$

where $m^{ad}$ is symmetric and $a_c \equiv C^a_{ac}$. The Bianchi class A models are characterized by $a_c = 0$, so that we have

$$C^a_{bc} = m^{ad} \epsilon_{dce}.$$  

(5)

In supergravity, the basic variables are taken to be the spinor version of the tetrad components $e^{AA'} \mu$ and the spinor valued forms $\psi^A \mu$, $\bar{\psi}^{A'} \mu$ (for spinor conventions, see ref. [6]). $e^{AA'} \mu$ are Grassmann even and $\psi^A \mu$ and $\bar{\psi}^{A'} \mu$ are Grassmann odd. In accordance with (4), we take

$$e^{AA'}_0 = N n^{AA'} + N^i e^{AA'}_i.$$  

(6)
Here, $n_{AA'}$ is the spinor version of the unit vector normal to the surface $t =$const., and it is defined by

$$n_{AA'} e^{AA'}_i = 0, \quad n_{AA'} n^{AA'} = 1. \quad (7)$$

From homogeneity the variables $N$, $N_i$, and $e^{AA'}_i$ are restricted to

$$N = N(t), \quad N_i = N_a(t) E^a_i, \quad e^{AA'}_i = \Omega^{AA'}_a(t) E^a_i. \quad (8)$$

The metric-like matrix $h_{ab}$ defined in (3) is associated with $\Omega^{AA'}_a$ through

$$h_{ab} = -\Omega_{AA'a} \Omega^{AA'}_b. \quad (9)$$

We will use $h_{ab}$ and its inverse $h^{ab}$ to raise and lower the indices $a, b, c, \ldots$. Further, the spin-$\frac{3}{2}$ field $\psi^A_\mu$ will take the following spatially homogeneous form:

$$\psi^A_0 = \psi^A_0(t), \quad \psi^A_i = \psi^A_a(t) E^a_i. \quad (10)$$

Similar restrictions are to be imposed on $\bar{\psi}^{A'}_\mu$.

The full theory of supergravity reduces to a theory with a finite number of degrees of freedom and its Lagrangian is given by the integration of the Lagrangian density $\mathcal{L}(t, x)$ over the spatial coordinates,

$$L(t) = \int d^3x \mathcal{L}(t, x), \quad (11)$$

with $N, N_a, \Omega^{AA'}_a, \psi^A_a, \bar{\psi}^A_0, \bar{\psi}^{A'}_a$, and $\bar{\psi}^{A'}_0$ being the variables of the reduced theory. The reduced action $S \equiv \int dt L(t)$ remains invariant under the three homogeneity preserving local transformations - Lorentz, coordinate, and supersymmetry transformations. Thus, it is possible to get the reduced Hamiltonian $H(t)$ by integrating the Hamiltonian density. This results in a form formally equivalent to the original one:

$$H(t) = NH_\perp + N^a H_a + \psi^A_0 S_A + \bar{S}_A' \bar{\psi}^A_0 - \omega_{AB0} J^{AB} - \bar{\omega}^{A'B'0} \bar{J}^{A'B'}. \quad (12)$$

Here, $N, N^a, \psi^A_0, \bar{\psi}^{A'}_0, \omega_{AB0}$, and $\bar{\omega}^{A'B'0}$ act as Lagrange multipliers which enforce the constraints $H_\perp = 0$, $H_a = 0$, $S_A = 0$, $\bar{S}_A' = 0$, $J^{AB} = 0$, and $\bar{J}^{A'B'} = 0$.

Using the same procedure as shown in ref. [6], we can perform canonical quantization of this system. A quantum state is described by a wave function $\Psi$, which we choose as
an eigenstate of $\psi^A_a$ and $\Omega^{AA'}_a$, i.e., $\Psi = \Psi(\Omega^{AA'}_a, \psi^A_a)$. Corresponding to this choice, $	ilde{\psi}^{A'}_a$ is given by

$$\tilde{\psi}^{A'}_a = -i\hbar D^{AA'}_{ba} \frac{\partial}{\partial \psi^B_a},$$

(13)

and the momentum conjugate to $\Omega^{AA'}_a$, $p_{AA'}^a$, by

$$p_{AA'}^a = -i\hbar \frac{\partial}{\partial \Omega^{AA'}_a} + \frac{1}{2}i\hbar \sigma \epsilon^{abc} \psi^A_b D^{BA'}_{dc} \frac{\partial}{\partial \psi^B_d},$$

(14)

where

$$D^{AA'}_{ab} = \frac{i}{\sigma \hbar} h_{ab} n^{AA'} + \frac{1}{\sigma} \epsilon^{abc} \Omega^{AA'} c$$

(15)

and $\sigma = \int d^3 x E$. We have chosen the operator ordering in $p_{AA'}^a$ so that it becomes hermitian with respect to the formal inner product [6].

Note that we can also take the wave function as an eigenstate of $\tilde{\psi}^{A'}_a$ and $\Omega^{AA'}_a$: $\tilde{\Psi} = \tilde{\Psi}(\Omega^{AA'}_a, \tilde{\psi}^{A'}_a)$, which relates to $\Psi(\Omega^{AA'}_a, \psi^A_a)$ by the fermionic Fourier transform

$$\tilde{\Psi}(\Omega^{AA'}_a, \tilde{\psi}^{A'}_a) = D^(-1)(\Omega) \int \Psi(\Omega^{AA'}_a, \psi^A_a) \exp \left( -\frac{i}{\hbar} C_{AA'}^{ab} \psi^A_a \tilde{\psi}^{A'}_b \right) \prod E, e d\psi^E_v, \quad (16)$$

where

$$C_{AA'}^{ab} = -\sigma \epsilon^{abc} \Omega^{AA'} c$$

(17)

$$D(\Omega) = \det \left( -\frac{i}{\hbar} C_{AA'}^{ab} \right).$$

(18)

With this relation, we can obtain $\tilde{\Psi}$ from $\Psi$, and vice versa.

Physically allowed wave functions are obtained by solving quantum versions of constraint equations:

$$J_{AB} \Psi = 0 , \quad \bar{J}_{A'B'} \Psi = 0 ,$$

(19)

$$S_A \Psi = 0 , \quad \bar{S}_A \Psi = 0 ,$$

(20)

$$H_\perp \Psi = 0 , \quad H_a \Psi = 0 .$$

(21)

Because of the relation $\{ S_A, S_{A'} \} \propto H_\perp n_{AA'} + H_a \Omega^{AA'}_a$ up to terms proportional to $J$ and $\bar{J}$, egs. (21) hold automatically if the conditions (19) and (20) are satisfied. The constraints (13) describe the invariance of $\Psi$ under Lorentz transformations. We therefore
only have to solve $S_A\Psi = 0$ and $\bar{S}_A\Psi = 0$ under the assumption that $\Psi$ is Lorentz invariant. Let us write the explicit forms of $S_A$ and $\bar{S}_A'$:

\begin{align}
S_A &= \sigma m^{ab}\Omega_{AA'a}\psi^A\bigg|_b + \frac{1}{2}i\kappa^2 \left( sp_{AA'}^a\psi^A\big|_a + (1 - s)\psi^A_a p_{AA'}^a \right), \tag{22}
\end{align}

\begin{align}
\bar{S}_A' &= \sigma m^{ab}\Omega_{AA'a}\bar{\psi}^A\big|_b - \frac{1}{2}i\kappa^2 \left( (1 - s)p_{AA'}^a\bar{\psi}^A'\big|_a + s\bar{\psi}^A_a p_{AA'}^a \right), \tag{23}
\end{align}

where $\kappa^2 = 8\pi$ and $s$ parametrizes the ambiguity of the operator ordering, which comes from noncommutativity of $\psi^A_a$, $\bar{\psi}^A_a$ and $p_{AA'}^a$. Note that the orderings in refs. [6] and [5] can be obtained by taking $s$ to be 0 and $\frac{1}{2}$, respectively. Now we write, without specifying the value of $s$, the constraint equations (20) in an explicit form:

\begin{align}
\left( \gamma m^{ab}\Omega_{AA'a}\psi^A\big|_b + \psi^A_a \frac{\partial}{\partial \Omega_{AA'a}} + s\psi^A_a \Omega_{AA'}^a \right) \psi = 0, \tag{24}
\end{align}

\begin{align}
\left( -\gamma m^{ab}\Omega_{AA'a}D^{AB'}_{\,bc} \frac{\partial}{\partial \bar{\psi}^B\big|_c} + \frac{\partial}{\partial \Omega_{AA'a}} \left( D^{AB'}_{\,ba} \frac{\partial}{\partial \psi^B\big|_b} \right) - s\Omega_{AA'}^a D^{AB'}_{\,ba} \frac{\partial}{\partial \psi^B\big|_b} \right) \Psi = 0, \tag{25}
\end{align}

where $\gamma \equiv 2\sigma/\hbar\kappa^2$. Moreover, using the Fourier transform (16), eqs. (24) and (25) can be recast into the conditions for $\tilde{\Psi}(\Omega_{AA'}^a, \bar{\psi}^A_a)$:

\begin{align}
\left( -\gamma m^{ab}\Omega_{AA'a}D^{AB'}_{\,bc} \frac{\partial}{\partial \bar{\psi}^B\big|_c} - \frac{\partial}{\partial \Omega_{AA'a}} \left( D^{AB'}_{\,ba} \frac{\partial}{\partial \psi^B\big|_b} \right) + (2 - s)\Omega_{AA'}^a D^{AB'}_{\,ba} \frac{\partial}{\partial \psi^B\big|_b} \right) \tilde{\Psi} = 0, \tag{26}
\end{align}

\begin{align}
\left( \gamma m^{ab}\Omega_{AA'a}\bar{\psi}^A'\big|_b - \bar{\psi}^A_a \frac{\partial}{\partial \Omega_{AA'}^a} + s\bar{\psi}^A_a \Omega_{AA'}^a \right) \tilde{\Psi} = 0. \tag{27}
\end{align}

### 3 Solving quantum constraints

Because of the Lorentz invariant property, $\Psi$ can only contain an even number of $\psi^A_a$. Thus, we can decompose $\Psi$ into $\psi^0$, $\psi^2$, $\psi^A$, $\psi^6$ parts, which we write as $\Psi_0$, $\Psi_2$, $\Psi_4$, $\Psi_6$, respectively, so that we have $\Psi = \Psi_0 + \Psi_2 + \Psi_4 + \Psi_6$. These terms are related to the $\bar{\psi}^6$, $\bar{\psi}^4$, $\bar{\psi}^2$, $\bar{\psi}^0$ parts of $\tilde{\Psi}$ by the Fourier transform, respectively. We write $\tilde{\Psi} = \tilde{\Psi}_6 + \tilde{\Psi}_4 + \tilde{\Psi}_2 + \tilde{\Psi}_0$, where the subscripts represent the number of $\bar{\psi}^A_a$ contained in each term.

We shall give these terms separately by solving the constraint equations along lines similar to ref. [5].
For $\Psi_0$, eq. (24) is trivial because $\Psi_0$ has no fermionic variables. Bearing in mind that Lorentz invariance imposes a condition $\Psi_0 = \Psi_0(h_{ab})$, eq. (24) reduces to

$$\left( \gamma m^{ab} - 2 \frac{\partial}{\partial h_{ab}} + s h^{ab} \right) \Psi_0 = 0,$$

so that we obtain

$$\Psi_0 = \text{const.} h^{\frac{s}{2}} \exp \left( \frac{1}{2} \gamma m^{ab} h_{ab} \right),$$

where $h \equiv \det(h_{ab})$.

Next, let us look at $\Psi_2$. Solving eqs. (24) and (25) under the assumption that $\Psi_2$ is Lorentz invariant, we get

$$\Psi_2 = 0$$

as solutions except for $s = 0, -4/3$ and $m^{ab} = 0$. In these exceptional cases, there are the formal solutions

$$\Psi_2 = \text{const.} \psi'^{\prime 2} \quad (s = 0),$$

$$\Psi_2 = \text{const.} (\psi''^2 + 2 \psi'^2) \quad (s = -4/3),$$

where $\psi'^2 \equiv \psi^A_a \psi^A_a$ and $\psi''^2 \equiv i h^{-1/2} \epsilon^{abc} \Omega_{A A'} c_{n B} A' \psi^A_a \psi^B_b$. However, since they cease to be solutions when considering a small perturbation, we exclude them from our consideration.

For $\Psi_4$, or equivalently, $\Psi_2$, using a similar method to the case of $\Psi_2$, eqs. (26) and (27) can be solved to yield $\Psi_2 = 0$, i.e.,

$$\Psi_4 = 0.$$

Although there are also exceptional formal solutions when $s = 0, 10/3$ and $m^{ab} = 0$, we discard them for the same reason as above.

For $\Psi_0$, the solution is easily found to be

$$\tilde{\Psi}_0 = \text{const.} h^{-\frac{s}{2}} \exp \left( -\frac{1}{2} \gamma m^{ab} h_{ab} \right)$$

as in the case of $\Psi_0$. In other words, we have

$$\Psi_6 = \text{const.} h^{\frac{s}{2}} \exp \left( -\frac{1}{2} \gamma m^{ab} h_{ab} \right) \left( \psi^A_a \psi_{Aa} \right)^3,$$

where $h^{\frac{s}{2}}$.
where $\psi^A_d\psi^a_A = \psi^A_d\psi^b_A\delta^{ab}$.

It is rather remarkable that all the solutions (1) for Bianchi class A models are found in a unified way. We emphasize that these solutions are actually exact.

Recently, D’Eath claimed that the full $N = 1$ supergravity has no quantum corrections [7]. The reduced $N = 1$ supergravity presented above also has no quantum corrections and looks like a free theory. If one wants to see the effect of interactions, one probably has to explore $N = 1$ supergravity coupled to supermatter.

### 4 An example: Bianchi type IX case

One specific case would be of particular interest, i.e. Bianchi type IX representing homogeneous $S^3$ universe.

Giving the invariant basis for it as

$$E^1 = \sin \beta \cos \gamma d\alpha - \sin \gamma d\beta, \quad E^2 = \sin \beta \sin \gamma d\alpha + \cos \gamma d\beta, \quad E^3 = \cos \beta d\alpha + d\gamma, \quad (36)$$

where $E^a \equiv E^a_i dx^i$, $x^1 = \alpha$, $x^2 = \beta$ and $x^3 = \gamma$, the structure constants are given by $m^{ab} = \text{diag}(1,1,1)$. And taking the range of variables as

$$0 \leq \alpha \leq 4\pi, \quad 0 \leq \beta \leq \pi, \quad 0 \leq \gamma \leq 2\pi, \quad (37)$$

we can easily find $\sigma = 16\pi^2$, so that we have

$$I \equiv \frac{1}{2}\gamma m^{ab} h_{ab} = \frac{2\pi}{\hbar} \text{Tr}(h_{ab}). \quad (38)$$

For the semi-classical forms of the quantum states, it seems natural to ask if the states $\Psi_0$ and $\Psi_6$ are the Hartle-Hawking state [8] and the wormhole state [9], respectively. To answer this question, we note that vacuum Bianchi type IX metrics are diagonalizable [2]. In such cases it is fortunately known [10,11] that the asymptotically Euclidean four-metrics which correspond to the classical flow of $I$ are found to be

$$ds^2 = F^{-1/2}d\rho^2 + F^{1/2}\frac{\rho^2}{4} \left[ \left(1 - \frac{a_1^4}{\rho^4}\right)^{-1} (E^1)^2 + \left(1 - \frac{a_2^4}{\rho^4}\right)^{-1} (E^2)^2 + \left(1 - \frac{a_3^4}{\rho^4}\right)^{-1} (E^3)^2 \right], \quad (39)$$
where
\[
F = \left(1 - \frac{a_1^4}{\rho^4}\right)\left(1 - \frac{a_2^4}{\rho^4}\right)\left(1 - \frac{a_3^4}{\rho^4}\right)
\]
and \(a_1, a_2, a_3\) are constants. At large distances, the metrics (39) are well-behaved and \(\Psi_6\) dumps rapidly. Hence \(\Psi_6\) is certainly the ground quantum wormhole state. On the other hand, though \(\Psi_0\) itself is regular at small three-geometries, the classical flow of \(I\) is singular when \(\rho \to 0\). This means the quantum state \(\Psi_0\) is not the Hartle-Hawking state [12].

The ‘filled’ state \(\Psi_6\) corresponds to the wormhole state in our description above and this seems to be in contradiction with the literature [3,4,12], where the bosonic state \(\Psi_0\) corresponds to the wormhole state. However, another possibility exists because of the indefiniteness of the sign of \(\sigma\), which is defined as \(\int d^3x E\), not \(\int d^3x |E|\). In fact, by the following antipodal transformation
\[
\beta \to \beta + \pi,
\]
we have \(\sigma \to -\sigma,\ m^{ab} \to m^{ab}\), so that \(I \to -I\). Correspondingly this time the bosonic state \(\Psi_0\) becomes the wormhole state. We conclude that there are two distinct solutions of the constraints in the case of Bianchi type IX, which are mutually related by the antipodal transformation.

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