LOGNORMAL PROPERTY OF WEAK-LENSING FIELDS

ATSUSHI TARUYA
Research Center for the Early Universe, School of Science, Hongo 7-3-1, University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan; ataruya@utap.phys.s.u-tokyo.ac.jp

MASAHIRO TAKADA\(^1\) AND TAKASHI HAMANA
National Astronomical Observatory, 2-21-1, Osawa, Mitaka-City, Tokyo 181-8588, Japan; mtakada@th.nao.ac.jp, hamana@yukawa.kyoto-u.ac.jp

ISHIA KAYO
Department of Physics, School of Science, Hongo 7-3-1, University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan; kayo@utap.phys.s.u-tokyo.ac.jp

AND

TOSIFUMI FUTAMASE
Astronomical Institute, Tohoku University, Aoba, Sendai 980-8578, Japan; tof@astr.tohoku.ac.jp

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ABSTRACT

The statistical properties of weak-lensing fields are studied quantitatively using ray-tracing simulations. Motivated by an empirical lognormal model that excellently characterizes the probability distribution function of a three-dimensional mass distribution, we critically investigate the validity of the lognormal model in weak-lensing statistics. Assuming that the convergence field \( \kappa \) is approximately described by the lognormal distribution, we present analytic formulae of convergence for the one-point probability distribution function (PDF) and the Minkowski functionals. The validity of the lognormal models is checked in detail by comparing those predictions with ray-tracing simulations in various cold dark matter models. We find that the one-point lognormal PDF can accurately describe the non-Gaussian tails of convergence fields up to \( \nu \sim 10 \), where \( \nu \) is the level threshold given by \( \nu \equiv \kappa / (\kappa_0^2)^{1/2} \), although the systematic deviation from the lognormal prediction becomes manifest at higher source redshift and larger smoothing scales. The lognormal formulae for Minkowski functionals also fit the simulation results when the source redshift is low, \( \z_s \sim 1 \). Accuracy of the lognormal fit remains good even at small angular scales \( 2 \leq \theta \leq 4' \), where the perturbation formulae by the Edgeworth expansion break down. On the other hand, the lognormal model enables us to predict higher order moments, i.e., skewness \( S_3 \), and kurtosis \( S_4 \), and we thus discuss the consistency by comparing the predictions with the simulation results. Since these statistics are very sensitive to the high- and low-convergence tails, the lognormal prediction does not provide a successful quantitative fit. We therefore conclude that the empirical lognormal model of the convergence field is safely applicable as a useful cosmological tool, as long as we are concerned with the non-Gaussianity of \( \nu \lesssim 5 \) for low-\( \z_s \) samples.

Subject headings: cosmology: theory — gravitational lensing — large-scale structure of universe — methods: numerical

1 INTRODUCTION

Cosmic shear, coherent distortions in galaxy images caused by the gravitational field of the intervening large-scale structure, has now been recognized as a powerful cosmological tool (see Mellier 1999; Bartelmann & Schneider 2001 for reviews). Since the signal of cosmic shear reflects the gravitational potential of the total mass distribution, cosmic shear can be a direct probe of dark matter distribution as well as a way to measure cosmological parameters. Recently, measurements of cosmic shear have been independently performed by several groups (Van Waerbeke et al. 2000, 2001b; Wittman et al. 2000; Bacon, Refregier, & Ellis 2000; Kaiser, Wilson, & Luppino 2000; Maoli et al. 2001; Rhodes, Refregier, & Groth 2001). The estimated shear variances from the different observational data sets quantitatively reconcile with each other, which is in good agreement with theoretical predictions based on the cluster-normalized cold dark matter (CDM) model of structure formation. Ongoing and future wide-field surveys using high-resolution CCD cameras promise to dramatically improve the signal-to-noise ratio of weak-lensing signals. Hence, a detailed understanding of weak-lensing statistics will be necessary to extract all the cosmological information present in the data.

Of course, a simple but useful statistical measure of the weak-lensing field is the popular two-point statistics (Blandford et al. 1991; Miralda-Escude 1991; Kaiser 1992); however, a clearer discriminator is needed to break the degeneracy in constraining the cosmological parameters. In this respect, the non-Gaussian features of a weak-lensing field can be very sensitive to the density of the universe, and higher order statistics such as the skewness of the local convergence field are expected to constrain the density parameter \( \Omega_0 \) and even the cosmological constant \( \Lambda \) (Bernaldeau, Van Waerbeke, & Mellier 1997; Jain & Seljak 1997; Van Waerbeke et al. 2001a). On the other hand, topological analysis using the Minkowski functionals of the convergence field has been proposed by Sato et al. (2001) in order to extract non-Gaussian features (see also Matsubara & Jain 2001). Unfortunately, their methodology relies heavily on the validity of the perturbation theory of structure formation and cannot be applied to the weak-lensing field at small angular scales \( \theta \lesssim 4' \), at which the underlying three-
dimensional mass distribution is in the highly nonlinear regime. While the non-Gaussian signature of weak-lensing fields becomes easily measurable on small scales, the non-Gaussian signal primarily reflects the nonlinear growth of density fields, which strongly depends on dark matter clustering properties. It is therefore desirable to investigate the statistical properties of three-dimensional mass density and explore the relation between mass distribution and weak-lensing fields.

Very recently, Kayo, Taruya, & Suto (2001) performed a detailed analysis of the one- and two-point statistics of three-dimensional mass distribution using high-resolution N-body simulations. They found that lognormal models of one- and two-point probability distribution functions (PDFs) can provide an excellent approximation to a nonlinear mass distribution with a Gaussian initial condition, irrespective of the shape of the initial power spectrum. The lognormal distribution has long been known as an empirical prescription characterizing dark matter distribution and/or observed galaxies (e.g., Hamilton 1985; Bouchet et al. 1993; Coles & Jones 1991; Kofman et al. 1994); however, there exists no rigorous explanation for its physical origin. Nevertheless, the lognormal model is now widely utilized in astrophysical contexts such as analytical modeling of dark halo biasing and the Ly$\alpha$ forest (e.g., Mo & White 1996; Taruya & Suto 2000; Bi & Davidsen 1997). In this regard, the result of Kayo et al. (2001) is interesting and can be useful in quantifying dark matter distribution. Furthermore, their results indicate that the lognormal distribution can also describe the weak-lensing field, since the weak-lensing effect primarily reflects the clustering property of dark matter distributions. Then, crucial but fundamental questions arise. How does the lognormal property emerge in the weak-lensing field? Can the lognormal model provide an approximate and reliable prescription for weak-lensing statistics?

In this paper, we quantitatively investigate those issues using ray-tracing simulations. We first consider how the statistical features of three-dimensional mass distribution are related to those of the local convergence, i.e., the two-dimensional projected density field. Assuming that the convergence field is well approximated by the lognormal distribution, we derive analytic formulae for the one-point PDF and the Minkowski functionals. We then perform quantitative comparisons between the lognormal models and the results of ray-tracing simulations. Furthermore, we discuss the consistency of the lognormal model with the higher order statistics on which the previous works have mainly focused.

The plan of the paper is as follows. In § 2 we briefly describe the basic definitions of weak lenses. Then, we discuss the statistical relations between the mass density field and the convergence field in § 3. The analytic expressions for the one-point PDF and the Minkowski functionals are presented. In § 4 we perform a detailed comparison between the lognormal predictions and the ray-tracing simulations. In § 5, we discuss the consistency between lognormal models and previous studies. Finally, § 6 is devoted to conclusions and discussion.

2. PRELIMINARIES

The inhomogeneous mass distribution of large-scale structure reflects the light-ray trajectory emitted from an angular direction $\theta$ in the source plane by a small angle $d\theta$ in the image plane. The differences between the deflection angles of light rays emitted from a galaxy thus induce a distortion of the galaxy image characterized by the following symmetric matrix,

$$\Phi_{ij} = \frac{\partial \delta \theta_i}{\partial \theta_j} = -2 \int_0^\infty dx \frac{r(x)r(x-\chi)}{r(x)} \partial_i \partial_j \phi(x), \quad i, j = 1, 2, \quad (1)$$

where $\partial_i$ represents a derivative with respect to $\theta_i$, and $\phi$ is the gravitational potential of the three-dimensional mass density field. The variable $\chi$, means the quantity $\chi$ at the source redshift, and the quantities $\chi$ and $r(\chi)$ respectively denote the comoving parts of the radial and the angular-diameter distance,

$$\chi(z) = \int_0^z \frac{c dz'}{H(z')},$$

$$H(z) = H_0 \sqrt{\Omega_0(1+z)^3 + (1 - \Omega_0 - \lambda_0)(1+z)^2 + \lambda_0}, \quad (2)$$

and

$$r(\chi) = \begin{cases} \sin(\sqrt{K}\chi)/\sqrt{K} & K > 0, \\ \chi & K = 0, \\ \sinh(\sqrt{-K}\chi)/\sqrt{-K} & K < 0, \end{cases} \quad (3)$$

with the quantity $K$ being the spatial curvature of the universe, $K = (H_0/c)^2(\Omega_0 + \lambda_0 - 1)$. The matrix $\Phi_{ij}$ is usually decomposed into the trace part, i.e., the convergence field $\kappa_{ij}$, and the tidal shear components $\gamma_{ij}$, defined by

$$\kappa_{ij} = -\frac{1}{2} \left( \Phi_{11} + \Phi_{22} \right),$$

$$\gamma_{ij} = -\frac{1}{2} \left( \Phi_{11} - \Phi_{22} \right) - i \Phi_{12}. \quad (4)$$

The shear field $\gamma_{ij}$ can be directly estimated from the observed ellipticity of the galaxy images. The convergence field $\kappa_{ij}$ can then be reconstructed from the shear map from equations (4) and (5) (e.g., Kaiser & Squires 1993). Since we are interested in the statistical properties of the weak-lensing fields, one can safely employ the Born approximation, where the quantities $\partial_i \partial_j \phi(x)$ are computed along the unperturbed photon trajectory (Blandford et al. 1991; Miralda-Escude 1991; Kaiser 1992). From equations (1) and (4), the convergence field along a line of sight is simply expressed as a weighted projection of the mass density fluctuation field $\delta$ (e.g., Mellier 1999; Bartelmann & Schneider 2001 for reviews),

$$\kappa_{ij}(\theta) = \int_0^\infty dx w(\chi, \chi_5) \delta(r(\chi)) \delta(\theta),$$

$$w(\chi, \chi_5) = \frac{3}{2} \left( \frac{H_0}{c} \right)^2 (1 + z) \Omega_0 \frac{r(\chi)r(\chi_5 - \chi)}{r(\chi_5)}. \quad (6)$$

Note that even in the weak-lensing limit, the density fluctuation $\delta$ is not small but can become much larger than unity.

In the statistical analysis of weak-lensing fields, a smoothing filter is used in order to reduce the noise due to the intrinsic ellipticity of galaxies. Throughout this paper, we adopt a
top-hat smoothing function. The variance of the local convergence can then be expressed as

\[
\langle \kappa^2 \rangle = \int_0^\infty d\chi \left[ w(\chi, \chi_s) \right]^2 \times \int \frac{d^3k}{(2\pi)^3} P_{\text{mass}}(k) W_{\text{th}}^2(k, r(\chi)\theta_{\text{th}}) ,
\]

(8)

where \( P_{\text{mass}}(k) \) is the three-dimensional power spectrum of the mass distribution and the function \( W_{\text{th}}(x) \) represents the Fourier transform of the top-hat smoothing kernel:

\[
W_{\text{th}}(x) = 2 \frac{J_1(x)}{x} .
\]

(9)

Note that equation (8) is derived by employing Limber’s equation (Kaiser 1992). Similarly, variance of the gradient convergence evaluated along an empty beam (see eq. [31]). On the other hand, when the dark matter is composed of smooth microscopic components, as is suggested by the standard CDM scenarios, \( \kappa_{\text{min}} \) becomes larger than the empty-beam value for relevant smoothing scales (see also Jain, Seljak, & White 2000). As is discussed below in § 4.2, \( \kappa_{\text{min}} \) is an important quantity in controlling the lognormal model and is sensitive to the cosmological model.

Even though we have the simple relation in equation (6) between \( \delta \) and \( \kappa \), the relation between the cumulant generating functions of \( \varphi_{\delta} \) and \( \varphi_{\kappa} \) is practically intractable without recourse to some assumptions or approximations. Under the hierarchical Ansatz for the higher order moments of \( \delta \), i.e., \( S_{\kappa} = \text{const}, \) Munshi & Jain (2000) and Valageas (2000) showed that the relation between \( \varphi_{\delta} \) and \( \varphi_{\kappa} \) is greatly reduced and is expressed in a compact form,

\[
\varphi_{\kappa}(x) = \int_0^\infty d\chi \frac{\langle \kappa^2 \rangle}{I_\kappa(\theta)} \frac{w(\chi, \chi_s) L_\delta(\theta)}{\langle \kappa^2 \rangle} x ,
\]

(14)

where the quantity \( I_\kappa \) is defined by

\[
I_\kappa(\theta) = \int \frac{d^3k}{(2\pi)^3} P_{\text{mass}}(k) W_{\text{th}}^2(k, r(\chi)\theta_{\text{th}}) .
\]

(15)

This is valid as long as the small-angle approximation holds. As also proposed by those authors, since the factor \( wL_\delta/\langle \kappa_{\text{min}}(\kappa^2) \rangle \) in equation (14) has typical values of the order of unity, one can use a simple approximation:

\[
\varphi_{\kappa}(x) \simeq \varphi_{\delta}(x) .
\]

(16)

It is easy to check that this approximation holds for the limit of \( z \to 0 \), where the quantities \( wL_\delta/\langle \kappa_{\text{min}}(\kappa^2) \rangle \) and \( \int d\chi \langle \kappa^2 \rangle/I_\kappa(\theta) \) both approach unity.

Once equation (16) is given, the PDF of the convergence field, \( P(\kappa) \), can be directly calculated from the one-point PDF of the three-dimensional mass density field \( P(\delta) \), irrespective of the projection effect (see eq. [6]):

\[
P(\kappa) d\kappa = P \left( \delta \to \frac{\kappa}{\kappa_{\text{min}}} ; \sigma \to \frac{(\kappa^2)^{1/2}}{\kappa_{\text{min}}} \right) \frac{d\kappa}{\kappa_{\text{min}}} .
\]

(17)

Now let us recall the empirical fact that the PDF of local density \( P(\delta) \) is approximately described by the lognormal distribution:

\[
P_{\ln}(\delta) d\delta = \frac{1}{\sqrt{2\pi} \ln(1 + \sigma^2)} \times \exp \left\{ -\frac{\ln(1 + \delta)^2}{2 \ln(1 + \sigma^2)} \right\} \frac{d\delta}{1 + \delta} .
\]

(18)

Substituting equation (18) into equation (17), one finally obtains

\[
P_{\ln}(\kappa) d\kappa = \frac{1}{\sqrt{2\pi} \sigma_{\ln}} \times \exp \left\{ -\frac{\ln(1 + \kappa/\kappa_{\text{min}})^2 + \sigma_{\ln}^2}{2 \sigma_{\ln}^2} \right\} \frac{d\kappa}{\kappa + \kappa_{\text{min}}} .
\]

(19)

3. LOGNORMAL MODEL PRESCRIPTION

3.1. One-Point PDF

Equation (6) has the simple interpretation that the statistical feature \( \kappa \) is closely related to that of the underlying density field \( \delta \). To show this more explicitly, we consider the one-point PDF, which can be constructed from a full set of the moments. Let us write down the PDF of the mass density field \( \delta \),

\[
P(\delta) d\delta = \int_0^{\infty} d\chi \frac{d\chi}{2\pi \sigma^2} e^{-i \delta (\delta/\sigma + \varphi_\delta(x)/\sigma^2)} d\delta ,
\]

(11)

where \( \sigma \) denotes the rms of \( \delta \), \( \sigma = \langle \delta^2 \rangle^{1/2} \), and \( \varphi_\delta(x) \) is the cumulant generating function:

\[
\varphi_\delta(x) = \sum_{n=2}^\infty S_{n,\delta} \frac{x^n}{n!} , \quad S_{n,\delta} \equiv \frac{\langle \delta^n \rangle}{\langle \delta^2 \rangle^{n-1}} .
\]

(12)

Similarly, the one-point PDF of the local convergence is written as

\[
P(\kappa) d\kappa = \int_0^{\infty} d\chi \frac{d\chi}{2\pi (\kappa^2)} e^{-i \kappa (\kappa/\kappa_{\text{min}}) + \varphi_\kappa(x)/\kappa^2} d\kappa ,
\]

(13)

with the cumulant generating function for \( \kappa \), \( \varphi_\kappa(x) \). Here we have introduced the normalized convergence field \( \kappa \equiv \kappa/\kappa_{\text{min}} \) so as to satisfy the range of the definition being \(-1 < \kappa < +\infty \). The quantity \( \kappa_{\text{min}} \) denotes the minimum value of the convergence field. Note that the actual value of \( \kappa_{\text{min}} \) in the universe should, in principle, depend on the nature of dark matter between the source galaxies and observer (Metcalf & Silk 1999; Seljak & Holtz 1999). If the dark matter is composed of a compact object such as a primordial black hole, \( \kappa_{\text{min}} \) might correspond to the convergence evaluated along an empty beam (see eq. [31]).
where the quantity $\sigma_{ln}$ is defined by

$$\sigma_{ln} = \ln \left( 1 + \frac{\langle \kappa^2 \rangle}{|\kappa_{\text{min}}|^2} \right). \tag{20}$$

Before discussing the application and validity of equation (19), two important points should be mentioned. First, recall that the lognormal distribution violates the assumption of hierarchical Ansatz used in the derivation of equation (14). The resulting expression in equation (19) is thus inconsistent; however, the violation of the hierarchical assumption is fortunately weak on the nonlinear scales of interest here (e.g., Kayo et al. 2001). We therefore expect that as long as we restrict ourselves to the range of applicability, the expression in equation (19) can provide a reasonable approximation and capture an important aspect of the non-Gaussianity in the weak-lensing field. Second, note that the result in equation (19) relies heavily on the validity of the approximation in equation (16). One might suspect that the approximation breaks down for a case with a high source redshift such as $z_s \geq 1$, in which the lensing projection becomes more important, and this is discussed below. Of course, one can directly evaluate the convergence PDF $P(\kappa)$ from equations (13) and (14) assuming the lognormal PDF of $\delta$, although this treatment is not useful in practice. Instead, in this paper we a priori assume the simple analytic prediction in equation (19), whereby we can further derive useful analytic formulae for the Minkowski functionals and obtain a more physical interpretation of the results. We then carefully check our model by comparing the predictions with the numerical simulation results.

### 3.2. Minkowski Functionals

Equation (17) implies that the projection effect is primarily unimportant and is eliminated by rescaling the quantities $\delta \rightarrow \kappa/|\kappa_{\text{min}}|$. That is, the statistical features of $\kappa$ directly reflect those of the three-dimensional density $\delta$. If this is the case, one can further develop a prediction of the weak-lensing field based on the lognormal Ansatz. To investigate this issue, other informative statistics such as higher order correlation and isodensity statistics should be examined.

Among these statistics, it is known that Minkowski functionals can be useful and give a morphological description of the contour map of the weak-lensing field (e.g., Schmalzing & Buchert 1997). In a two-dimensional case, Minkowski functionals are characterized by three statistical quantities: the area fraction $v_0$, circumference per unit length $v_1$, and Euler characteristics per unit area $v_2$. The third functional is equivalent to the famous genus statistics often used in cosmological contexts (Gott, Melott, & Dickinson 1986). These quantities are evaluated for each isocontour as a function of level threshold, $\nu \equiv \kappa/(\kappa^2)$.

The explicit expression of the Minkowski functional $v_0(\nu)$ is given by

$$v_0(\nu) = \Theta \left( \kappa - \nu \langle \kappa^2 \rangle^{1/2} \right), \tag{21}$$

where $\Theta$ denotes the Heaviside step function. The Minkowski functionals $v_1(\nu)$ and $v_2(\nu)$ provide additional information on the convergence field, since the definitions include higher derivative terms such as $\partial_\nu \kappa$ and $\partial_\kappa \partial_\nu \kappa$. According to Matsubara (2000), the Minkowski functionals $v_1$ and $v_2$ in the two-dimensional map can be related to the level-crossing statistics and the genus statistics defined in the two-dimensional surface, $N_1$ and $G_2$ (see also Schmalzing & Buchert 1997):

$$v_1(\nu) = \frac{\pi}{8} N_1(\nu) = \frac{\pi}{8} \left\{ \delta_D \left( \kappa - \langle \kappa^2 \rangle^{1/2} \right) \partial_\nu |\partial_\nu \kappa| \right\}, \tag{22}$$

$$v_2(\nu) = G_2(\nu) = \left\{ \delta_D \left( \kappa - \langle \kappa^2 \rangle^{1/2} \right) \delta_D (\partial_\kappa |\partial_\kappa \kappa|) \right\}. \tag{23}$$

Assuming that the convergence field is approximately described by the lognormal distribution, the analytic expressions for Minkowski functionals can be derived in a straightforward manner. Since the area fraction represents the cumulative probability above the threshold $\nu$, substituting equation (19) into the definition in equation (21) yields

$$v_{0,\ln}(\nu) = \int_{\nu \langle \kappa^2 \rangle^{1/2}}^{\infty} d\kappa P_{\ln}(\kappa) = \frac{1}{2} \text{erfc} \left( \frac{\nu}{\sqrt{2}} \right), \tag{24}$$

with the function $\nu(\nu)$ given by

$$\nu(\nu) = \frac{\sigma_{ln}}{2} + \ln \left( 1 + \frac{\nu \langle \kappa^2 \rangle^{1/2}}{|\kappa_{\text{min}}|} \right). \tag{25}$$

As for the Minkowski functionals $v_1$ and $v_2$, the analytic expressions for the lognormal distribution can be obtained from the local transformation of Gaussian formulae, $N_1(\nu)$ and $G_2(\nu)$. The details of the derivation are described by Taruya & Yamamoto (2001; see also Matsubara & Yokoyama 1996). The final results become

$$v_{1,\ln}(\nu) = \frac{1}{8 \sqrt{2}} \frac{1}{\sqrt{\frac{|\kappa_{\text{min}}|^2 + \langle \kappa^2 \rangle}{\sigma_{ln}}} \frac{\langle \nabla \kappa \rangle^{1/2}}{\nu} e^{-\nu^2/2}, \tag{26}$$

$$v_{2,\ln}(\nu) = \frac{1}{2(2\pi)^{3/2}} \frac{1}{\sqrt{\frac{|\kappa_{\text{min}}|^2 + \langle \kappa^2 \rangle}{\sigma_{ln}}} \frac{\langle \nabla \kappa \rangle^{3/2}}{\nu} y(\nu) e^{-\nu^2/2}. \tag{27}$$

Note that in the limit $\langle \kappa^2 \rangle^{1/2}/|\kappa_{\text{min}}| \ll 1$, the above predictions recover the Gaussian formulae:

$$v_{0,G}(\nu) = \frac{1}{2} \text{erfc} \left( \frac{\nu}{\sqrt{2}} \right), \tag{28}$$

$$v_{1,G}(\nu) = \frac{1}{8 \sqrt{2}} \left( \frac{\langle \nabla \kappa \rangle^{1/2}}{\langle \kappa^2 \rangle} \right)^{1/2} e^{-\nu^2/2}, \tag{29}$$

$$v_{2,G}(\nu) = \frac{1}{2(2\pi)^{3/2}} \left( \frac{\langle \nabla \kappa \rangle^{3/2}}{\langle \kappa^2 \rangle} \right) \nu e^{-\nu^2/2}. \tag{30}$$

Apart from the PDF $P(\kappa)$ and the area fraction $v_0$, there exists no clear reason that the extension of the lognormal model to the Minkowski functionals $v_1$ and $v_2$ still provides a good approximation in the real universe. At present, the predictions in equations (26) and (27) are just regarded as an extrapolation from the one-point PDF $P(\kappa)$ and should be checked by numerical simulations.

### 4. COMPARISON WITH RAY-TRACING SIMULATIONS

In this section, we quantitatively examine the validity of the lognormal model by comparing its predictions with simulation results. A brief description of ray-tracing simulation is presented in §4.1. In §4.2 the model parameters in the lognormal formulae are checked in detail using the simulation
data. Some important numerical effects are also discussed. Then, § 4.3 describes our main results, i.e., the comparisons of one-point PDFs and Minkowski functionals between the lognormal predictions and the simulation data.

4.1. Ray-tracing Simulations

In order to investigate the lognormal properties of convergence fields, we use a series of ray-tracing simulations in three cold dark matter models (SCDM, LCDM, and OCDM for the standard, lambda, and open CDM models, respectively). The cosmological parameters used here are summarized in Table 1.

To perform a ray-tracing simulation, a light-cone data set is first generated by a particle-mesh (PM) N-body code. The PM code uses $256^2 \times 512$ particles and is performed in a periodic rectangular box of size $(L, L, 2L)$ with the force mesh $256^2 \times 512$. The initial conditions are generated according to the transfer functions of Bond & Efstathiou (1984). Then, the light cone of the particles is extracted from each simulation during the run (Hamana, Colombi, & Suto 2001). The final set of light-cone data is created so as to cover a field of view of $5 \times 5 \text{deg}^2$, and the box sizes of each output are chosen so as to match to the convergence of the light-ray bundle. As a result, the angular resolution of the ray-tracing simulation, which is basically limited by the spatial resolution of the PM N-body code, becomes almost constant: $\theta_{\text{res}} \approx 15'$ from the observer at $z = 0$ (Hamana & Mellier 2001).

Once the light-cone data are obtained, ray-tracing simulations are next performed using a multiple lens plane algorithm (e.g., Schneider, Ehlers, & Falco 1992; Jain et al. 2000; Hamana, Martel, & Futamase 2000). In our calculation, the intervals between lens planes are fixed to a comoving length, $80 \ h^{-1} \text{Mpc}$. The $512^2$ light rays are then traced backward from the observer’s point to the source plane. The initial ray directions are set on $512^2$ grids, and the two-dimensional deflection potentials are calculated by solving ray-bundle equations keeping the same grids in each lens plane. We obtained 40 realizations by randomly shifting the simulation boxes.

After constructing the weak-lensing map in the image plane at $z = 0$, the smoothed convergence fields are finally computed on $512^2$ grids by convolving the top-hat smoothing kernel. All the statistical quantities such as one-point PDFs and Minkowski functionals are evaluated from the data. In a subsequent analysis, we use the convergence data set fixing the source redshifts to $z_s = 1$ and 2.

4.2. On the Lognormal Model Parameters

Since the lognormal predictions presented in the previous section rely heavily on the three parameters $\kappa_{\text{min}}, (\kappa^2)$, and $((\nabla \kappa)^2)$, we first check them in some detail using the ray-tracing simulations.

| TABLE 1 |
| --- |
| **Cosmological Parameters Used in N-Body Simulations** |
| **Model** | $\Omega_0$ | $\lambda_0$ | $h$ | $\sigma_8$ |
| SCDM | 1.0 | 0.0 | 0.5 | 0.6 |
| LCDM | 0.3 | 0.7 | 0.7 | 0.9 |
| OCDM | 0.3 | 0.0 | 0.7 | 0.85 |

Figure 1 shows the minimum value of the convergence $\kappa_{\text{min}}$ plotted against the smoothing angle. The error bars indicate $1 \sigma$ errors around each mean, where the mean value $\kappa_{\text{min}}$ is obtained from 40 realizations of the data. As stated in § 3.1, the convergence $\kappa$ evaluated along an empty beam is theoretically considered as a possibility of $\kappa_{\text{min}}$ and is expressed from equation (6) as

$$
\kappa_{\text{empty}} = -\int_0^{\chi_s} d\chi w(\chi, \chi_s),
$$

which implies that the light ray propagates through empty space with $\delta = -1$ everywhere along the line of sight. In Figure 1, thick lines represent the theoretical prediction from equation (31) for each model. Clearly, these predictions give much smaller values than the simulation results in both the $z_s = 1.0$ and 2.0 cases. In addition to the systematic cosmological model dependences, the scale dependence of $\kappa_{\text{min}}$ also appears.

These results mean that none of the light rays become completely empty beam (see also Jain et al. 2000). Nevertheless, one cannot exclude the possibility that some or even the majority of lines of sight become empty at scales smaller than the mean separation angle between particles of N-body simulations. In this sense, the minimum value $\kappa_{\text{min}}$ could be affected by the finite sampling from a limiting survey size of the convergence map. Based on this consideration, we propose the following intuitive way to explain the behaviors of $\kappa_{\text{min}}$ shown in Figure 1. Assuming that the original PDF of infinitesimal light rays obeys the lognormal model with the minimum value $\kappa_{\text{empty}}$, one can roughly estimate $\kappa_{\text{min}}$ from the condition that the expectation number of an independent sampling area for $\kappa_{\text{empty}} < \kappa < \kappa_{\text{min}}$ becomes unity in the field of view, $5^\circ \times 5^\circ$. The details of estimation are...
described in Appendix A. For a given angular scale $\theta_{\text{res}}$ and cosmological model, the quantity $\kappa_{\text{min}}$ is evaluated by solving equation (A2) with the prior PDF (eq. [A1]). The thin lines in Figure 1 depict the estimation based on this prescription, which reasonably agrees with simulations. This result is interesting in the sense that we can analytically predict $\kappa_{\text{min}}$ and the resulting minimum value is sensitive to cosmological parameters, especially $\Omega_0$. The practical possibility is discussed again in the final section.

Next, we turn to focus on the variances of the convergence and its gradient fields. In Figure 2, the measured amplitudes $\langle \kappa^2 \rangle^{1/2}$ and $\langle (\nabla \kappa)^2 \rangle^{1/2}$ are plotted in linear and log scale and are compared with linear (short-dashed lines) and nonlinear (solid lines) predictions. For the rms of the convergence $\langle \kappa^2 \rangle^{1/2}$, the nonlinear prediction based on the Peacock & Dodds (1996) formula faithfully reproduces the simulation result over the smoothing angles $\theta > \theta_{\text{res}} = 1.5$. On the other hand, in the case of $\langle (\nabla \kappa)^2 \rangle$, nonlinear predictions systematically deviate from simulations. The discrepancy remains even at larger smoothing angles, $\theta \sim 10^\circ$, and causes $30\%-40\%$ error.

While the theoretical predictions based on equations (8) and (10) were computed assuming that the mass power spectrum $P_{\text{mass}}(k)$ is continuous and has infinite resolution, the simulation data are practically affected by the finite resolution. In our case, the limitation of PM force mesh could be attributed to the finite resolution or the cutoff of Fourier modes, which becomes influential even on larger smoothing angles $\theta > \theta_{\text{res}}$. In fact, compared to $\langle \kappa^2 \rangle$, the dominant contribution to the quantity $\langle (\nabla \kappa)^2 \rangle$ comes from the short-wavelength modes of three-dimensional density fluctuations, sensitively depending on the choice of the smoothing filter. To show the significance of this effect, the nonlinear prediction is modified according to the finite resolution of the PM code (see Appendix B). Then, the Fourier integrals in equations (8) and (10) are discretized so as to mimic the PM $N$-body resolution as follows (eqs. [B1] and [B2]).

$$\int_0^\infty dk_\perp \Delta^2(k_\perp)W^2_{\text{th}}(k_\perp, \theta)$$
$$= \sum_{n=1}^{N_{\text{max}}} \Delta k_\perp \Delta^2(k_{\perp, n})W^2_{\text{th}}(k_{\perp, n}, \theta),$$

$$\Delta^2(k_\perp) \equiv \begin{cases} (k_\perp/2\pi)P_{\text{mass}}(k_\perp) & \langle \kappa^2 \rangle, \\ (k_\perp/2\pi)^2P_{\text{mass}}(k_\perp) & \langle (\nabla \kappa)^2 \rangle, \end{cases}$$

where the $n$th Fourier mode $k_{\perp, n}$ is given by $k_{\perp, n} = \Delta k_{\perp}$ and the interval $\Delta k_{\perp}$ is set to $\Delta k_{\perp} = 2\pi/L_{\text{box}}(z)$.

The long-dashed lines in Figure 2 show the results including the finite-resolution effect. In each panel, the number of Fourier modes $N_{\text{max}}$ is chosen as 90 (see the discussion in Appendix B). It is apparent that the prediction of $\langle (\nabla \kappa)^2 \rangle^{1/2}$ has systematically reduced power at all scales, and it almost coincides with the simulation result, although the result of $\langle \kappa^2 \rangle^{1/2}$ remains unchanged. We have also examined various cases by changing the maximum number to $64 \leq N_{\text{max}} \leq 128$, but obtained qualitatively similar behavior. Note that the incorrect prediction of $\langle (\nabla \kappa)^2 \rangle$ leads to a systematic error in predicting the amplitude of the Minkowski functionals $v_1$ and $v_2$ from their definitions in equations (26) and (27), while it does not alter the non-Gaussian shapes of those functionals with respect to the level threshold.

Keeping the above remarks in mind in what follows, to make a comparison with the lognormal models transparent, we use the parameters $\langle \kappa^2 \rangle$, $\langle (\nabla \kappa)^2 \rangle$, and $\kappa_{\text{min}}$, estimated directly from the simulations when plotting the lognormal predictions in equations (19), (24), (26), and (27).

### 4.3. Results

#### 4.3.1. One-Point PDF

As a quick view of the validity of the lognormal model, we first deal with the one-point PDF $P(\kappa)$. Figures 3 and 4 show the one-point PDFs of the local convergence in various CDM models with the smoothing angles $\theta = 2^\circ$, $4^\circ$, and $8^\circ$. Here the PDF data is constructed by binning the data with $\Delta \kappa = 0.01$. The source redshifts are fixed to $z_s = 1.0$ in Figure 3 and 2.0 in Figure 4. Clearly, the functional form of the one-point PDF becomes broader with increasing source...
redshift $z_s$. On small angular scales, the PDF significantly deviates from the Gaussian PDF denoted by dashed lines. Although local convergences on larger smoothing scales tend to approach the Gaussian form, they still exhibit a non-Gaussian tail.

In Figures 3 and 4, solid lines represent the lognormal predictions in equation (19), whose parameters $\kappa_{\text{min}}$ and $\langle \kappa^2 \rangle$ are directly estimated from simulations. The agreement between the lognormal model and the simulation results is generally good. In particular, at lower source redshift $z_s = 1.0$, the lognormal PDF accurately describes the non-Gaussian tails in the high-convergence region up to $\kappa \lesssim 10\langle \kappa^2 \rangle^{1/2}$. In the case of higher source redshift $z_s = 2.0$, the discrepancy becomes evident at a larger smoothing scale. The lognormal PDF overpredicts in the high-convergence region and underpredicts in the low-convergence region. This discrepancy might be ascribed to the projection effect of gravitational lensing (see eq. [6]), since the lognormal PDF is obtained based on the approximation in equation (17), which cannot be validated with increasing source redshift $z_s$. Nevertheless, even in that case, a better agreement between the lognormal model and the simulations was found at small angular scales. These features are also seen in Figure 5, in which the differences between the simulated PDF and lognormal PDF normalized by the simulated PDF, $|P_{\text{sim}}(\kappa) - P_{\text{ln}}(\kappa)|/P_{\text{sim}}(\kappa)$, are plotted as a function of level threshold $\nu = \kappa/(\kappa^2)^{1/2}$, in the case of the LCDM model.

The accurate lognormal fit in the low-$z_s$ case is amazing and is regarded as a considerable success. On a closer look at the non-Gaussian tails, however, the lognormal PDF slightly underpredicts the simulations in the low-density region, $\kappa < 0$ (see Fig. 3 and the top panel of Fig. 5). Furthermore, at $\theta = 2^\circ$, the simulation results generally tend to deviate from the lognormal PDF in the highly non-Gaussian tails $\kappa \gtrsim 10\langle \kappa^2 \rangle^{1/2}$, although the prediction still remains consistent within a 1 $\sigma$ error. In particular, in the case of the OCDM model, the discrepancy becomes apparent even at $\kappa \gtrsim 8\langle \kappa^2 \rangle^{1/2}$. The systematic deviation seen in non-Gaussian tails might be ascribed to the presence of virialized objects. Note that the angular scale $\theta = 2^\circ$ corresponds to the effective smoothing scale $R \sim 0.5$ h$^{-1}$ Mpc at mean redshift.

![Figure 3](image-url)
This indicates that a high-\(\kappa\) value is attained by a light ray propagating through a high-density region in very massive halos with \(M \gtrsim 10^{14} \, h^{-1} \, M_\odot\). Indeed, the highly non-Gaussian tails of convergence PDFs sensitively depend on the detailed structure of nonlinear objects, as pointed out by Kruse & Schneider (2000). They construct an analytical model of a one-point PDF based on the universal profile of dark matter halos and the Press-Schechter theory for halo abundance. Since the virialized halos induce the highly non-Gaussian tails of the local convergence, their treatment would be helpful in describing the non-Gaussian tails of PDFs. In contrast, due to the lack of physical bases, no reliable prediction is expected from the empirical lognormal model.

Another reason for discrepancy might be the choice of the parameter \(\kappa_{\text{min}}\). Strictly speaking, the minimum value of \(\kappa\) seen in the simulated PDF represents a rare event for all data sets, which are not rigorously equivalent to the averaged value of 40 realizations of the data shown in Figure 1. We have also examined the lognormal fit adopting the minimum value of the PDF data for \(\kappa_{\text{min}}\). The thin lines in Figure 5 show the results in the LCDM case adopting the actual minimum value of \(\kappa\) for each PDF data set. The results seem somehow improved at small angle \(\theta = 2' - 4'\) in high-\(z_s\) cases, but we rather recognize the fact that the lognormal fit to the high-density region is sensitive to the choice of \(\kappa_{\text{min}}\).

Except for these details, the lognormal model of a one-point PDF remains a fairly accurate model of the convergence field, at least up to \(\kappa \sim 5(\kappa^*)^{1/2}\), and is indeed applicable, even at small angular scales such as \(2' \lesssim \theta \lesssim 4'\), irrespective of the assumptions in equations (16) or (17).

4.3.2. Minkowski Functionals

Having recognized the successful lognormal fit to the one-point PDF, we next investigate the lognormal model of Minkowski functionals. For this purpose, we restrict our analysis to the source redshift \(z_s = 1.0\). Here the Minkowski functionals for a simulated convergence map are calculated by the method developed by Winitzki & Kosowsky (1997).

Figure 6 shows the results in various CDM models fixing the smoothing angle \(\theta = 2'\). The Minkowski functionals are plotted against the level threshold \(\nu = \kappa / (\kappa^*)^{1/2}\) with the interval \(\Delta \nu = 0.5\), so that the data in each bin are approximately regarded as statistically independent. The solid lines depict the lognormal predictions in equations (24), (26), and (27).
Fig. 5.—Differences of one-point PDFs normalized by the simulated PDF, $[P_{\text{sim}}(\kappa) - P_{\text{th}}(\kappa)]/P_{\text{th}}(\kappa)$ as a function of level threshold $\nu = \kappa / \langle \kappa^2 \rangle$ in the LCDM model. Top: Results fixing the source redshift to $z_s = 1.0$. Bottom: Results fixing the source redshift to $z_s = 2.0$. The solid, short-dashed, and long-dashed lines represent the cases with smoothing angle $\theta = 2^\circ$, $4^\circ$, and $8^\circ$, respectively. In plotting the ratios, the lognormal prediction adopting the averaged minimum value $\nu_{\text{min}}$ (see Fig. 1) is used for the thick lines, while the thin lines represent the results adopting the minimum value of each PDF data set.

Similar to the one-point PDF, a significant non-Gaussian signature is detected from the asymmetric shape of the Minkowski functionals, especially from the Euler characteristic $v_2$. In marked contrast to the Gaussian predictions, the lognormal predictions reproduce the simulation results remarkably well: not only the shape dependences, but also the amplitudes. The agreement between the lognormal prediction and the simulations still remains accurate over the rather wider range $-4 < \nu < 4$, in which the discrepancy seen in the one-point PDF of the OCDM model is not observed. Since the prediction has no adjustable parameter and only uses the information from output data, this agreement is successful.

Figure 7 depicts the results with various smoothing angles, fixing the cosmology to an LCDM model. For illustrative purposes, the amplitudes of $v_1$ and $v_2$ at the smoothing angles $\theta = 4^\circ$ and $8^\circ$ are artificially changed by multiplying by the factors indicated in each panel, in order to make the non-Gaussianity manifest. The Minkowski functionals tend to approach the Gaussian prediction with increasing smoothing angle, consistent with the behaviors of the one-point PDF. The results in other cosmological models are also similar, and the agreement between the lognormal models and simulations is satisfactory.

To manifest the accuracy of the lognormal fit in contrast to the other existing analytical models, let us now consider the perturbation predictions. Employing the Edgeworth expansion, the perturbative expressions for Minkowski functionals are derived analytically (Matsubara 2000; Sato et al. 2001):

$$v_0(\nu) \simeq v_{0,G}(\nu) + (\langle \kappa^2 \rangle)^{1/2} \frac{1}{6\sqrt{2\pi}} e^{-\nu^2/2} S_3^{(0)} H_2(\nu),$$ (32)

$$v_1(\nu) \simeq v_{1,G}(\nu) \left\{ 1 + (\langle \kappa^2 \rangle)^{1/2} \left[ \frac{S_3^{(0)}}{6} H_3(\nu) + \frac{S_3^{(1)}}{3} H_1(\nu) \right] \right\},$$ (33)

$$v_2(\nu) \simeq v_{2,G}(\nu) \left\{ 1 + (\langle \kappa^2 \rangle)^{1/2} \left[ \frac{S_3^{(0)}}{6} H_4(\nu) + \frac{2S_3^{(1)}}{3} H_2(\nu) + \frac{S_3^{(2)}}{3} \right] \right\}. \quad (34)$$

Note that in the case of the three-dimensional density field, the above expansion is valid up to the rms of $\delta$, $\sigma < 0.3$ (Matsubara & Yokoyama 1996; Matsubara & Suto 1996), which can be translated into the condition $\langle \kappa^2 \rangle^{1/2} < 0.3\nu_{\text{min}}$. Here the function $H_n(\nu)$ denotes the $n$th order Hermite polynomial $H_n(\kappa) \equiv (-1)^n e^{\kappa^2/2} (d/d\kappa)e^{-\kappa^2/2}$, and the quantities $S_3^{(n)}$ represent the skewness parameters, defined as

$$S_3^{(0)} = \frac{\langle \kappa^5 \rangle}{\langle \kappa^2 \rangle^2},$$ (35)

$$S_3^{(1)} = \frac{3}{4} \frac{\langle \kappa^2 \nabla^2 \kappa \rangle}{\langle \kappa^2 \rangle^2},$$ (36)

$$S_3^{(2)} = - \frac{3}{4} \frac{\langle \nabla\kappa\nabla^2\kappa \rangle}{\langle \nabla\kappa \rangle^2}. \quad (37)$$

Equations (32)–(34) imply that in the weakly nonlinear regime, the non-Gaussian features on the Minkowski functionals can be completely described by the above parameters, which can be evaluated by the second-order perturbation theory of structure formation (e.g., Bernardeau et al. 1997). It is worth noting that in the case of the lognormal model, all the skewness parameters $S_3^{(n)}$ are equal to 3 (Hikage, Taruya, & Suto 2001).

Figure 8 plots the perturbation results in the LCDM model. The source redshift is fixed to $z_s = 1.0$. The solid lines represent the results in which the skewness parameters are evaluated using second-order perturbation theory (perturb 1), while the dashed lines depict the results using those estimated from simulations (perturb 2). In both cases, the variances $\langle \kappa^2 \rangle$ and $\langle (\nabla\kappa)^2 \rangle$ in equations (32)–(34) are estimated from simulations. Also, in Figure 9 the comparison between various model predictions is summarized, introducing the fractional error, defined by

$$\text{Err}(v_i(\nu)) \equiv \frac{v_i^{(\text{sim})}(\nu) - v_i^{(\text{model})}(\nu)}{v_i^{(\text{max})}},$$ (38)

with the quantity $v_i^{(\text{max})}$ being the maximum value of $v_i(\nu)$ among the mean values of the simulation for each Minkowski functional: $v_{0,\text{max}}^{(\text{sim})} = 1.0$, $v_{1,\text{max}}^{(\text{sim})} = 0.040$, and $v_{2,\text{max}}^{(\text{sim})} = 0.0042$ at angular scale $\theta = 2^\circ$, for instance. As a reference, the 1 $\sigma$ error of the simulation results is plotted as error bars around the zero mean in each panel.
At the large smoothing angle $\theta = 8^\circ$, both of the perturbation results tend to reconcile with each other and fit the simulation results well within a $1\sigma$ error. As the smoothing angle decreases, however, the perturbation results cease to fit the simulations, because the rms of the local convergence reaches $r_{\text{rms}}^2 \gtrsim 2.0$, and the Edgeworth expansion breaks down. Furthermore, the second-order perturbations underpredict the skewness parameters, compared to those estimated from simulations, which leads to different predictions (compare perturb 1 with perturb 2 in the left and middle panels in Figs. 8 and 9). In particular, the circumference $v_1$ shows a peculiar behavior, $v_1 < 0$, which is not allowed by definition. Note that even in these cases, the fractional error $\text{Err}(v_1(\nu))$ in the lognormal prediction still remains smaller, although the systematic deviations in every model are not so large.

From these discussions, we conclude that the empirical lognormal models can provide a good approximation to the Minkowski functionals, compared to the current existing models.

5. CONSISTENCY WITH HIGHER ORDER MOMENTS

Since the skewness of the local convergence has been proposed as a simple statistical estimator of the non-Gaussian signature used to determine cosmological parameters, various authors have investigated the usefulness of this quantity using ray-tracing simulations. These analyses have revealed that the skewness at small angle $\theta \lesssim 10^\circ$ exhibits a significant influence of nonlinear clustering, and a more reliable theoretical model beyond perturbation theory is needed. According to these facts, nonperturbative predictions based on “hyperextended perturbation theory” (Scoccimarro & Frieman 1999) or the nonlinear fitting formula of the bispectrum (Scoccimarro & Couchman 2001) are examined (Hui 1999; Van Waerbeke et al. 2001a).

In general, the one-point PDF as well as the Minkowski functionals characterizes a family of higher order statistics. Hence, as a consistency check of the lognormal prediction, it seems natural to analyze the higher order moments of local convergence. From an empirical lognormal PDF (eq.

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**Fig. 6.**—Minkowski functionals as a function of level threshold $\nu = \kappa / |\kappa|^2$ at the smoothing angle $\theta = 2^\circ$. The source redshift is fixed to $z_s = 1.0$. Solid lines show the lognormal predictions based on eqs. (24), (26), and (27), in which all the parameters are estimated from simulations. The dashed lines show the Gaussian predictions obtained from eqs. (28), (29), and (30). Left: SCDM model. Middle: LCDM model. Right: OCDM model.
The skewness and the kurtosis of the local convergence, defined by 
\[ S_3 = \frac{1}{\kappa_{\text{min}}} \left( 3 + \frac{\langle \kappa^2 \rangle}{\kappa_{\text{min}}} \right), \]  
and 
\[ S_4 = \frac{1}{\kappa_{\text{min}}} \left[ 16 + 15 \left( \frac{\langle \kappa^2 \rangle}{\kappa_{\text{min}}} \right)^2 + 6 \left( \frac{\langle \kappa^2 \rangle}{\kappa_{\text{min}}} \right)^3 \right], \]
are respectively given by

Figure 10 shows the direct measurement of \( S_{3,\kappa} \) and \( S_{4,\kappa} \), fixing the source redshift at \( z_s = 1.0 \). The error bars indicate a 1 \( \sigma \) error estimated from the 40 realizations of the data. The lognormal predictions in equations (39) and (40) are depicted as solid lines.

In practice, the convergence field \( \kappa \) in a numerical simulation does not extend for the entire range between \( \kappa_{\text{min}} \) and \( +\infty \) but is limited to \( \kappa_{\text{min}} < \kappa < \kappa_{\text{max}} \), because of the finite sampling effect from a limited amount of simulation data (see Kayo et al. 2001 in the case of a three-dimensional density field \( \delta \)). Thus, the \( n \)th order moments of \( \kappa \) given by

\[ \langle \kappa^n \rangle = \int_{\kappa_{\text{min}}}^{\kappa_{\text{max}}} d\kappa P_{\text{ln}}(\kappa)\kappa^n \]

may provide a better description in evaluating the skewness \( S_{3,\kappa} \) and the kurtosis \( S_{4,\kappa} \).

The dashed lines in Figure 10 show the lognormal prediction based on equation (41), where the cutoff parameter \( \kappa_{\text{max}} \) is adopted as the averaged maximum value from each realization of a data set. In contrast to the accurate fit seen in the one-point PDF, the lognormal fit of skewness and kurtosis seems very poor. The prediction without the cutoff \( \kappa_{\text{max}} \) tends to overpredict with increasing smoothing angle, which shows the opposite behavior compared to the simulations. As for the lognormal model based on equation (41), while the predictions reduce their amplitudes and broadly agree with the simulations in the case of \( S_{4,\kappa} \), they still exhibit some systematic discrepancies in \( S_{3,\kappa} \). We have also examined the lognormal fit adopting the minimum and
maximum values of the PDF data itself, but the result is not drastically improved.

At first glance, these discrepancies seem to contradict the results in the one-point PDF; however, a closer look at the one-point PDF reveals that the lognormal PDFs slightly underpredict the simulation results in the low-density region. This tiny discrepancy may be ascribed to the overestimation of skewness. In other words, the skewness as well as the kurtosis is very sensitive to rare events, i.e., the high- and low-convergence parts of the non-Gaussian tails. This sensitivity is clearly shown in the OCDM model. The simulated PDF at smoothing angle $\theta = 2'$ exhibits a highly non-Gaussian tail, and it overshoots the lognormal prediction (see top right panel in Fig. 3). The resulting skewness and kurtosis yield values larger than those of lognormal prediction. On the other hand, in the SCDM and LCDM models, lognormal PDFs accurately fit the highly non-Gaussian tails, and thereby the predictions of skewness and kurtosis become relatively consistent with the simulations, at least on small scales, $2' \leq \theta \leq 4'$.

Therefore, the lognormal model of convergence does not provide an accurate prediction for statistics sensitive to rare events. This disagreement simply reflects the fact that the empirical lognormal model does not correctly describe the projected structure of dark matter halos. On the other hand, a sophisticated nonperturbative model based on hyperextended perturbation theory or the fitting formula of the bispectrum is constructed so as to reproduce the $N$-body results of higher order moments, which can provide an accurate prediction for the convergence skewness (Hui 1999; Van Waerbeke et al. 2001a). Hence, for the prediction of higher order moments, the nonperturbative model is more useful and reliable than the lognormal model. In contrast, the lognormal model fairly describes non-Gaussianity around the peak of the PDFs over the broad range, $\kappa \leq 5\langle \kappa^2 \rangle^{1/2}$, which cannot be described by such a nonper-

Fig. 8.—Edgeworth expansion of Minkowski functionals compared with the simulations. In each panel, the results of the $z_s = 1$ case in the LCDM model are shown. Solid lines (perturb 1) indicate the perturbation results, where the skewness parameters are calculated via perturbation theory, while the skewness parameters in the predictions shown by dashed lines (perturb 2) are estimated from simulation data directly. In plotting both cases, the variances $\langle \kappa^2 \rangle$ and $\langle (\nabla \kappa)^2 \rangle$ are fitted to the numerical simulations. Note that the amplitudes of $v_1$ and $v_3$ at the smoothing angles $\theta = 4'$ and $8'$ are enhanced in order to clarify the differences as indicated in each panel.
6. CONCLUSION AND DISCUSSION

In the present paper, we have quantitatively investigated the extent to which the lognormal model fairly describes the statistics of weak-lensing fields on linear and nonlinear scales using ray-tracing simulations. The validity of the lognormal model has been checked in detail by comparing the lognormal predictions of the one-point PDF and the Minkowski functionals with their simulation results.

The convergence field seen in the one-point PDF and the Minkowski functionals displays non-Gaussian features and significantly deviates from Gaussian predictions. We have shown that the analytic formulae for lognormal models are useful and accurately describe the simulation results on both small and large smoothing angular scales in the case of low-$z_s$ data, while the perturbative prediction by Edgeworth expansion fails to reproduce the simulation results on small scales because of the nonlinearity of the underlying three-dimensional density field. A detailed comparison revealed that the lognormal model does not provide an accurate prediction for the statistics sensitive to rare events such as the skewness and kurtosis of the convergence. We therefore conclude that as long as we are concerned with the appropriate range of the convergence, $\kappa \lesssim 5(\kappa^2)^{1/2}$, the lognormal model empirically but quantitatively gives a useful approximation characterizing the non-Gaussian features of the convergence field.

The results obtained here will lead to an important improvement in the estimation of cosmological parameters using Minkowski functionals (Sato et al. 2001). Although the original methodology has been proposed with the use of the Edgeworth formulae (eqs. [32]–[34]), a more reliable estimation of cosmological parameters will be possible using
influence of finite sampling on the estimation of $\kappa_{\text{min}}$

The minimum value of the convergence $\kappa_{\text{min}}$ is theoretically expected to be equal to $\kappa_{\text{empty}}$ (eq. [31]). In practice, however, the minimum values estimated from the simulation data can be systematically larger than those obtained from the empty beam, due to the finite sampling from the limiting survey area.

To see the influence of the finite-sampling effect, let us roughly estimate the minimum value $\kappa_{\text{min}}$. Assuming that the prior one-point PDF $P^{(\text{prior})}(\kappa)$ is approximately described by the lognormal PDF, in which the minimum value is characterized by $\kappa_{\text{empty}}$ instead of the actual value $\kappa_{\text{min}}$,

$$P^{(\text{prior})}(\kappa) = \frac{1}{\sqrt{2\pi}\sigma_{\ln}} \exp \left\{ -\frac{[\ln(1 + \kappa/|\kappa_{\text{empty}}|) + \frac{\sigma_{\ln}^2}{2}]^2}{2\sigma_{\ln}^2} \right\} \frac{d\kappa}{\kappa + |\kappa_{\text{empty}}|}, \quad (A1)$$

with the quantity $\sigma_{\ln}^2 = \ln(1 + \langle \kappa^2 \rangle/\langle |\kappa| \rangle^2)$. Then, the probability that the observed minimum value $\kappa_{\text{min}}$ systematically deviates from $\kappa_{\text{empty}}$ is given by

$$\int_{-|\kappa_{\text{empty}}|}^{-|\kappa_{\text{min}}|} d\kappa \ P^{(\text{prior})}(\kappa) = \frac{\pi \theta_{\text{in}}^2}{\theta_{\text{field}}} \ . \quad (A2)$$

The statistics of cosmic shear directly reflect the statistical feature of mass distribution, and using this fact, one might even discriminate the nature of dark matter. Furthermore, weak-lensing statistics have the potential to reveal the non-linear and stochastic properties of galaxy biasing. In any case, we expect that the lognormal property of the weak-lensing field will be helpful and will play an important role in extracting various cosmological information.

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where $\theta_{\text{th}}$ is the top-hat smoothing angle and $\theta_{\text{field}}$ is the field-of-view angle. The right-hand side of equation (A2) represents the lower limit of probability determined from the expectation number of the independent sampling area. The minimum value $\kappa_{\text{min}}$ is obtained by solving equation (A2).

Based on equation (A2), the resulting values of $\kappa_{\text{min}}$ are summarized in Figure 1 (thin lines). Here, the field-of-view angle $\theta_{\text{field}}$ is fixed to 5°, and the parameters in the prior lognormal PDF, $\langle \kappa^2 \rangle$ and $\kappa_{\text{empty}}$, are computed according to the theoretical predictions in equations (8) and (31), respectively.

**APPENDIX B**

**EFFECT OF FINITE RESOLUTION AND VARIANCES ($\kappa^2$) AND $\langle (\nabla \kappa)^2 \rangle$**

Statistical analysis based on N-body simulations with the PM algorithm should be carefully treated if we deal with statistics on small scales. In our ray-tracing simulation, the box size of each simulation $L_{\text{box}}(z)$ is determined so as to satisfy the resolution angle $\theta_{\text{res}} \approx 1/5$, i.e., $L_{\text{box}}(z) \approx r(\theta_{\text{res}})$. The mesh of the PM algorithm is fixed to $252 \times 512$ in each simulation box. Among these parameters, the finite force mesh number severely restricts the Fourier modes of mass fluctuations, which could affect the evaluation of $\langle \kappa^2 \rangle$ and $\langle (\nabla \kappa)^2 \rangle$, depending on the choice of smoothing filter.

In order to investigate the significance of finite mesh size, the binomial probabilities in equations (8) and (10) are modified according to the PM N-body code. Since the influence of finite mesh size mainly affects the high-frequency part of the fluctuations, the Fourier integrals in equations (8) and (10) are modified to

$$\int_0^\infty dk_\perp \Delta^2(k_\perp) W_{\text{ih}}(k_\perp r \theta) \approx \sum_{n=1}^{N_{\text{max}}} \Delta k_\perp \Delta^2(k_\perp n) W_{\text{ih}}^2(k_\perp nr \theta),$$

where

$$\Delta^2(k_\perp) \equiv \left\{ \begin{array}{ll} (k_\perp/2\pi) P_{\text{mass}}(k_\perp) & \langle \kappa^2 \rangle, \\ (k_\perp^2/2\pi) P_{\text{mass}}(k_\perp) & \langle (\nabla \kappa)^2 \rangle. \end{array} \right.$$  

(B2)

The $n$th Fourier mode $k_{\perp, n}$ is given by $k_{\perp, n} = \Delta k_{\perp, n}$, and the interval $\Delta k_{\perp}$ is set to $\Delta k_{\perp} = 2\pi/L_{\text{box}}(z)$.

In the above modification, the number of Fourier modes $N_{\text{max}}$ might be crucial in evaluating the quantity sensitive to the high-frequency mode, which is related to the force mesh number $N_{\text{mesh}} = 256$. Recall that the Nyquist frequency restricts the high-frequency mode to $k_{\text{Nyq}} = (\Delta k_{\perp}/2)N_{\text{mesh}}$, which implies $N_{\text{max}} = N_{\text{mesh}}/2$. Furthermore, the number of independent Fourier modes is reduced by the factor 4 in evaluating the power spectrum. Thus, the high-frequency cutoff is roughly given by $k_{\text{cut}} \approx 2\Delta k_{\perp}(N_{\text{mesh}}/4)$, which yields

$$N_{\text{max}} = \frac{N_{\text{mesh}}}{2\sqrt{2}} \approx 90.$$  

(B3)

The long-dashed lines in Figure 2 represent the results taking into account the finite mesh size. Here, the nonlinear mass power spectrum $P_{\text{mass}}(k)$ by Peacock & Dodds (1996) is used in evaluating the discretized Fourier integral in equation (B1). We also examined the various cases by changing the maximum number, $64 < N_{\text{max}} < 128$, but obtained qualitatively similar behavior: the prediction $\langle (\nabla \kappa)^2 \rangle$ has systematically reduced power over all scales, while the amplitude of $\langle \kappa^2 \rangle$ remains almost unchanged. Furthermore, similar modification to the $z$ integral has been made so as to match the number of lens planes in the multiple-lens plane algorithm; however, this does not affect the final results.

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