GROTHENDIECK HOMOMORPHISMS IN ALGEBRAICALLY CLOSED VALUED FIELDS

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Abstract. We give a presentation of the construction of motivic integration, that is, a homomorphism between Grothendieck semigroups that are associated with a first-order theory of algebraically closed valued fields, in the fundamental work of Hrushovski and Kazhdan [12]. We limit our attention to a simple major subclass of $\mathcal{V}$-minimal theories of the form $\text{ACVF}_0^S$, that is, the theory of algebraically closed valued fields of pure characteristic 0 expanded by a $(\text{VF}, \Gamma)$-generated substructure $S$ in the language $\mathcal{L}_{\text{RV}}$. The main advantage of this subclass is the presence of syntax. It enables us to simplify the arguments with many new technical details while following the major steps of the Hrushovski-Kazhdan theory.

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1. Introduction

The theory of motivic integration in valued fields has been progressing rapidly since its first introduction by Kontsevich. Early developments by Denef and Loeser et al. have yielded many important results in many directions. The reader is referred to [9] for an excellent introduction to the construction of motivic measure. There have been different approaches to motivic integration. The comprehensive study in Cluckers-Loeser [5] has successfully united the major ones on a general foundation. Their construction may be applied in general to the field of formal Laurent series over a field of characteristic 0 but heavily relies on the Cell Decomposition Theorem of Denef-Pas [6, 16], which is only achieved for valued fields of characteristic 0 that are equipped with an angular component map. However, an angular component map is not guaranteed to exist for just any valued field, for example, algebraically closed valued fields. The Hrushovski-Kazhdan integration theory [12] is a major development that does not require the presence of an angular component map and hence is of great foundational importance. Its basic objects of study are models of $V$-minimal theories. This class of theories encompasses a wide range of first-order expansions of the theory of algebraically closed valued fields of pure characteristic 0 that have been shown to have nice geometrical behaviors. Moreover, by compactness, when integrating a definable object, the theory may be applied to valued fields with large positive residue characteristics.

In this paper, following the major steps of the construction of Grothendieck homomorphisms, that is, homomorphisms between Grothendieck semigroups, but supplying new technical lemmas, we give a presentation of the materials in the first eight chapters of [12]. In doing so, we limit our attention to a simple major subclass of $V$-minimal theories, namely the theory of algebraically closed valued fields of pure characteristic 0 in the language $L_{RV}$ with parameters from the field sort and the (imaginary) value group sort allowed. The main technical differences from the original construction are all results of this restriction. Our principal aim is to reconstruct the Grothendieck homomorphism in [12, Theorem 8.8]. Other similar homomorphisms that involve differential calculus are completely left out. They will be presented in a sequel to this paper that is devoted to the study of Fourier transform in the Hrushovski-Kazhdan integration theory.

1.1. Outline of the construction. The method of the Hrushovski-Kazhdan integration theory is based on a fine analysis of definable subsets up to definable bijections in a first-order language $L_{RV}$ for valued fields. This language has two sorts: the VF-sort and the RV-sort. One of the main features of $L_{RV}$ is that the residue field and the value group are wrapped together in one sort $RV$; see Section 2 for details. Let $(K, \text{val})$ be a valued field and $\mathcal{O}, \mathcal{M}, K$ the corresponding valuation ring, its maximal ideal, and the residue field. Let $RV(K) = K^\times/(1 + \mathcal{M})$ and $rv : K^\times \rightarrow RV(K)$ the quotient map. Note that, for each $a \in K$, val is constant on the subset $a + a \mathcal{M}$ and hence there is a naturally induced map $vrv$ from $RV(K)$ onto the value group $\Gamma$. The situation is illustrated in the following commutative diagram

\[
\begin{array}{cccccc}
\mathcal{O} \setminus \mathcal{M} & \overset{\text{quotient}}{\longleftarrow} & K^\times & \overset{rv}{\longrightarrow} & RV(K) & \overset{vrv}{\longrightarrow} \Gamma \\
\downarrow \quad \quad \\
K^\times & \overset{\text{rv}}{\longleftarrow} & RV(K) & \overset{vrv}{\longrightarrow} \Gamma
\end{array}
\]
where the bottom sequence is exact. Note that the existence of an angular component $\pi : K^\times \to \overline{K}^\times$ is equivalent to the existence of a group homomorphism from $RV(K)$ onto $\overline{K}^\times$ in the diagram. For each $\gamma \in \Gamma$, the fiber $rv^{-1}(\gamma)$ has a natural one-dimensional $\overline{K}$-affine structure, which is denoted as $\overline{K}_\gamma$. The direct sum $\bigoplus_{\gamma \in \Gamma} \overline{K}_\gamma$ may be viewed as a generalized residue field.

Let $ACVF$ be the theory of algebraically closed valued fields in $L_{RV}$. Let $VF_*[\cdot]$ and $RV[\cdot, \cdot]$ be two categories of definable sets with respect to the $VF$-sort and the $RV$-sort, respectively. In order to integrate definable functions with $RV$-sort parameters, the objects in $VF_*[\cdot]$ are exactly the definable subobjects of the products $VF^n \times RV^m$ and the morphisms are just the definable maps. On the other hand, for technical reasons (particularly for keeping track of dimensions), $RV[\cdot, \cdot]$ is formulated in a quite complicated way. All this is explained in Section 6. One of the main goals of the Hrushovski-Kazhdan integration theory is to construct a canonical homomorphism from the Grothendieck semigroup $K^+_VF_*[\cdot]$ to the Grothendieck semigroup $K_+RV[\cdot, \cdot]$ modulo a semigroup congruence relation $I_{sp}$ on the latter. In fact, it will turn out to be an isomorphism. This construction has three main steps.

- **Step 1.** First we define a lifting map $L$ from the objects in $RV[\cdot, \cdot]$ into the objects in $VF_*[\cdot]$; see Definition 6.16. Next we single out a subclass of isomorphisms in $VF_*[\cdot]$, which are called definable special bijections; see Definition 7.5. Then we show that for any object $X$ in $VF_*[\cdot]$ there is a special bijection $T$ on $X$ and an object $Y$ in $RV[\cdot, \cdot]$ such that $T(X)$ is isomorphic to $L(Y)$. This implies that $L$ hits every isomorphism class of $VF_*[\cdot]$. Of course, for this result alone we do not have to limit our means to special bijections. However, in Step 3 below, special bijections become an essential ingredient in computing the congruence relation $I_{sp}$.

- **Step 2.** For any two isomorphic objects $Y_1, Y_2$ in $RV[\cdot, \cdot]$, their lifts $L(Y_1), L(Y_2)$ in $VF_*[\cdot]$ are isomorphic as well. This shows that $L$ induces a semigroup homomorphism from $K_+RV[\cdot, \cdot]$ into $K_+VF_*[\cdot]$, which is also denoted as $L$.

- **Step 3.** In order to invert the homomorphism $L$, we need a precise description of the semigroup congruence relation induced by it. The basic notion used in the description is that of a blowup of an object in $RV[\cdot, \cdot]$; see Definition 11.1. We then show that, for any objects $Y_1, Y_2$ in $RV[\cdot, \cdot]$, there are isomorphic iterated blowups $Y_1^\sharp, Y_2^\sharp$ of $Y_1, Y_2$ if and only if $L(Y_1), L(Y_2)$ are isomorphic. The “if” direction contains a form of Fubini’s Theorem and is the most technically involved part of the construction. Its difficulty will be explained further below when we describe the course of the paper.

The inverse of $L$ thus obtained is a motivic integration; see Theorem 12.2. When the Grothendieck semigroups are formally groupified this integration is recast as an injective ring homomorphism; see Corollary 12.3.

1.2. **Course of the paper.** A remarkable feature of the Hrushovski-Kazhdan integration theory is that model-theoretic study of definable sets plays a fundamental role and yet no advanced results from model theory, say, beyond the first five chapters of [15], are used. In section 2 after introducing the language $L_{RV}$ and the theory $ACVF$, we briefly review some concepts and results in model theory. To
suggest how they may be used later, some of these, especially the various incarnations of the Compactness Theorem, are stated specifically for $L_{RV}$ and ACVF. We also give a syntactical description of what it means to have imaginary elements as parameters in defining sets. In section 3 we establish quantifier elimination for ACVF by one of the standard model-theoretic tests. This is not proved in [12] and the reader is referred to [10]. The theme of the latter is elimination of imaginaries and the relevant results use a much more complicated language than $L_{RV}$, which do not seem to imply quantifier elimination in ACVF in a straightforward fashion. Our proof, except some fundamental facts in the theory of valued fields, is self-contained. In the following two sections we prove some properties that delineate the basic geography of definable sets in ACVF. These properties are used throughout the rest of the paper. As in [12], the key notion here is C-minimality, which was first introduced in [14] and has been further studied in [11]. The main difference between Section 4 and Section 5 is that in the former we work at the level of formulas with real parameters and in the latter we work at the level of types with imaginary parameters allowed.

With the preparatory work done, we are now ready to move on to the actual construction of motivic integration. First of all, we discuss various dimensions, mainly VF-dimension and RV-dimension, and describe the relevant categories of definable sets and the formulation of their Grothendieck semigroups in Section 6. The fundamental lifting map $L$ between VF-categories and RV-categories and the “dummy” functor $E$ between RV-categories are also introduced here. The central topic of Section 7 is RV-products and special bijections on them; see Definition 7.3 and Definition 7.5. The main result is Proposition 7.14 which corresponds to Step 1 above. This section contains the most important technical tool that is not available in [12], namely Proposition 7.13. With its presence, many hard lemmas in [12] have been simplified a great deal (for example, [12] Lemma 7.8, which corresponds to Lemma 10.2 in this paper) or circumvented (among the most notable ones are [12] Lemma 5.5 and the entire [12] Section 3.3).

The notion of a 2-cell is introduced in Section 8 which corresponds to the notion of a bicell in [5]. This notion may look strange and is, perhaps, only of technical interest. It arises when we try to prove some form of Fubini’s Theorem, such as Lemma 11.17. The difficulty is that, although, using C-minimality, the construction of the integration of definable sets of VF-dimension 1 is very functorial (see Lemma 10.3), we are unable to extend this construction to higher VF-dimensions. This is the concern of [12] Question 7.9. It has also occurred in [5] and may be traced back to [7]; see [5] Section 1.7. Anyway, in this situation, the natural strategy of integrating definable sets of higher VF-dimensions is to use the result for VF-dimension 1 and integrate with respect to one VF-sort variable at a time. As in the classical theory of integration, this strategy requires some form of Fubini’s Theorem: for a well-behaved integration, an integral should give the same value when it is evaluated along different orders of VF-sort variables. By induction, this problem is immediately reduced to the case of two VF-sort variables. A 2-cell is a definable subset of $VF^2$ with certain symmetrical geometrical structure that satisfies this Fubini type of requirement. Now the idea is that, if we can find a definable partition for every definable subset such that each piece is a 2-cell indexed by some RV-sort parameters, then, by compactness, every definable subset satisfies the Fubini type of requirement. This kind of partition is achieved in Lemma 8.8.
Section 9 is devoted to showing Step 2 above. The notion of a \( \gamma \)-polynomial is introduced here, which generalizes the relation between a polynomial with coefficients in the valuation ring and its projection into the residue field. This leads to Lemma 9.2, a generalized form of the multivariate version of Hensel’s Lemma. Note that in order to apply Lemma 9.2 to a given definable set we need to find suitable polynomials with a simple common residue root. This is investigated in Lemma 9.4, which does not hold when the substructure in question contains an excessive amount of parameters in the RV-sort. This is the reason why motivic integration is constructed only for theories of the form \( \text{ACVF}^0_S \), where the structure \( S \) is \( (\text{VF}, \Gamma) \)-generated. There is a straightforward remedy for this limitation. For every substructure \( S \) there is a canonical expansion \( S^* \) of \( S \) such that \( S^* \) is \( (\text{VF}, \Gamma) \)-generated and may be embedded into every \( (\text{VF}, \Gamma) \)-generated substructure that contains \( S \); see [12, Proposition 3.51]. Then \( S \) and \( S^* \) are identified for the construction of integration. To keep the conceptual framework simple, we do not include this treatment in the paper.

The key result of Section 10, Lemma 10.3, says that, modulo special bijections, every definable bijection between two definable sets of \( \text{VF} \)-dimension 1 is equal to the lift of an isomorphism in \( \text{RV}^{[*]} \). As has been remarked above, it would be ideal to extend this result to definable sets of all \( \text{VF} \)-dimensions. Being unable to do this, we introduce the notion of a standard contraction, which gives rise to the Fubini type of problem described above; see Definition 10.6. Then in Lemma 10.8 we show that an essential part of Lemma 10.3 holds for 2-cells, which is good enough for the rest of the construction.

The task of identifying the kernel of \( L \), that is, Step 3 above, is carried out in Section 11. We introduce the notion of a blowup and then show that the equivalence relation \( I_{sp}^{[*]} \) it induces on \( \text{RV}^{[*]} \) is indeed a semigroup congruence relation; see Definition 11.1 and Lemma 11.8. We conclude this section with Lemma 11.18, which says that \( I_{sp}^{[*]} \) is the congruence relation induced by the homomorphism \( L \).

In the last section we assemble everything together and deduce the main theorem.

1.3. Technical differences from the original construction. We emphasize again that, in this paper, we do not work at the level of generality as in [12], that is, the whole class of \( V \)-minimal theories. Instead, our construction is specialized for the theories of algebraically closed valued fields of pure characteristic 0 expanded by a substructure \( S \) in the language \( \mathcal{L}_{\text{RV}} \). As has been discussed above, Step 2 of the construction requires \( S \) to be \( (\text{VF}, \Gamma) \)-generated, but other parts of the construction in general do not require this restriction. For this subclass of \( V \)-minimal theories we are able to work with syntax. Very often, in order to grasp the geometrical content of a definable set \( X \), it is a very fruitful exercise to analyze the logical structure of a typical formula that defines \( X \), especially when quantifier elimination is available. Consequently, in the context of this paper, syntactical analysis affords simplifications of many lemmas in [12]. The main technical differences are described here roughly in the order of their first appearances. For this purpose we fix a theory \( \text{ACVF}^0_S \), where \( S \) is \( (\text{VF}, \Gamma) \)-generated.

Let \( \mathcal{L}_v \) be the two-sorted language for valued fields: one sort for the field and the other for the value group. Every model of \( \text{ACVF}^0_S \) may be turned naturally into a structure of \( \mathcal{L}_v \) and consequently any definable subset of any product \( \text{VF}^n \)
in $\text{ACVF}_S^g$ is $S$-definable in $\mathcal{L}_v$. This translation provides the strategy in Section 3 to reduce quantifier elimination in $\mathcal{L}_\text{RV}$ to that in $\mathcal{L}_v$, which has been established by Weispfenning (see Theorem 2.5). Another notable application of it is in Lemma 4.12, whose proof is conceptually much simpler than the corresponding [12, Lemma 3.35].

In almost all sections in the first eight chapters of [12] there are results that we need to more or less reproduce, except [12, Section 3.3], which has been completely dispensed with in this paper. Although, according to [12, Remark (3), p. 34], the lemmas in [12, Section 3.3] are not needed for the construction of integration maps, [12, Lemma 3.26] is used in the very important [12, Lemma 5.5], which is needed for [12, Lemma 5.10], which in turn is directly applied in [12, Lemma 7.24] to settle the Fubini problem described above. Because of Proposition 7.13 and its many consequences, we are still able to reproduce [12, Lemma 5.10], namely Lemma 8.8 without [12, Lemma 5.5]. More details on Proposition 7.13 will be given below.

In Section 5 we follow the syntactical treatment of imaginary elements described in Section 2. In particular, we are able to show that an atomic closed ball or an atomic thin annulus cannot correspond algebraically to an atomic open ball, which implies that one cannot define an atomic closed ball from an atomic open ball; see Lemma 5.9 and Lemma 5.10. These and Lemma 5.11 yield (trivially) a special case of [12, Lemma 3.46].

In order to bypass the notion of measure-preserving isomorphisms in RV-categories (see [12, Definition 5.21]), which requires a discussion of differential calculus, the very simple notion of the weight of an RV-sort tuple (Definition 6.10) is introduced in Section 6. This is used to formulate one of the conditions in the definition of a morphism in RV-categories; see Definition 6.11. The idea is that, a morphism $F : X \rightarrow Y$ in an RV-category should encode the ordering of the volumes of the lifts $L(X)$, $L(Y)$ of $X$, $Y$ so that $F$ itself may be lifted to the corresponding VF-category. To be more concrete, suppose that $X = \{1\}$ and $Y = \{\infty\}$, then $L(X) = \text{rv}^{-1}(1) \times \{1\}$ and $L(Y) = \text{rv}^{-1}(\infty) \times \{\infty\} = \{(0, \infty)\}$ and hence if $F$ is an isomorphism then it is impossible to lift it to an isomorphism. The solution to this is to simply disqualify $F$ as a morphism but allow $F^{-1}$ to be a morphism, which amounts to adopting the alternative definition of RV-categories in [12, Section 3.8.1]. A main advantage of allowing the element $\infty$ in RV-categories is that it makes the discussion of blowups in Section 11 more streamlined.

In [12, Chapter 4], Step 1 of the construction is accomplished through a class of bijections called admissible transformations. Later in [12, Chapter 7] another class of bijections called special bijections are introduced for Step 3. In this paper the two classes are adjusted so that they may be unified into one class and still serve their original purposes; see Definition 7.5. Now we come to Proposition 7.13 which says that, up to isomorphism classes, a polynomial map on an object in a VF-category may be projected down to a morphism between two objects in the corresponding RV-category. To be more precise, let $f(x_1, \ldots, x_n)$ be a polynomial with VF-sort coefficients and $X$ a definable subset of $\text{VF}^n$, then there is a definable special bijection $T$ on $X$ such that there is a function $f_1 : \text{RV}^m \rightarrow \text{RV}$ that makes
the diagram

\[
\begin{align*}
\text{commute. Moreover, this may be carried out simultaneously for any finite number of VF-sort polynomials. Except Section 9, the remainder of the paper heavily relies on Proposition 7.13. Almost all its applications involve the following procedure. Given a morphism } f : X \rightarrow Y \text{ in VF}_r[[\cdot]] \text{ that is defined by a formula } \phi, \text{ we obtain a special bijection } T \text{ on } X \text{ such that for any term in } \phi \text{ of the form } \text{rv}(g(\overline{x})) \text{ there is a commutative diagram as above for } g(\overline{x}) \text{ and hence the morphism } f \circ T^{-1} \text{ in } VF_r[[\cdot]] \text{ may be projected down to a morphism in RV}[[\cdot, \cdot]].
\end{align*}
\]

In Section 8 we give a more detailed treatment of 2-cells than in [12]. The lemmas that lead to Lemma 8.8 should make clear the crucial role of Proposition 7.13.

Let \( t \neq \infty \) be an RV-sort element that is algebraic over some other RV-sort elements. In Lemma 9.4 through analyzing a suitable formula that witnesses this algebraic relation, we find a minimal \( \gamma \)-polynomial for \( t \). This essentially reduces the task of lifting isomorphisms in RV\([\cdot, \cdot]\) to the multivariate version of Hensel’s Lemma. The proof of [12, Proposition 6.1] is thus simplified.

Section 10 and Section 11 more or less correspond to [12, Section 7.2, Section 7.3] and [12, Section 7.4, Section 7.5], respectively. Most of the changes here are made with the hope that the difficult situation may become easier to grasp. For example, unlike in [12, Section 7.5], we do not form additional categories for the computation of the kernel of L. Instead, we work directly with objects in VF_r[[\cdot]] and operations on them called standard contractions, which are a natural conceptual extension of special bijections; see Definition 10.6.

2. Logical preliminaries and the theory ACVF

In this section we review some of the basic concepts and results from model theory that will be used in the construction. In order to make connections with our context as quickly as possible, many of them will be stated in forms that directly involve the language \( \mathcal{L}_{RV} \) and the theory ACVF. The main advantage of being particular here is that it allows us to exemplify the many ways to use compactness in [12]. Since a thorough list of them all is not feasible, hopefully these examples may function as a guide so that every usage of compactness below will be seen as a variation of one of them.

2.1. The setting of \( \mathcal{L}_{RV} \) and ACVF. Let us first introduce the Basarab-Kuhlmann style language \( \mathcal{L}_{RV} \) for algebraically closed valued fields. This style first appeared in [1] and [2] and has been further investigated in [13] and [18]. Its main feature is the use of a countable collection of residue multiplicative structures, which are reduced to just one for valued fields of pure characteristic 0.

**Definition 2.1.** The language \( \mathcal{L}_{RV} \) has the following sorts and symbols:

1. a VF-sort, which uses the language of rings \( \mathcal{L}_R = \{0, 1, +, -, \times\} \);
2. an RV-sort, which uses
   a. the group language \( \{1, \times\} \),
   b. two constant symbols \( 0 \) and \( \infty \),
(c) a unary predicate $\mathbf{K}^x$,
(d) a binary function $+: \mathbf{K}^2 \to \mathbf{K}$ and a unary function $-: \mathbf{K} \to \mathbf{K}$,
where $\mathbf{K} = \mathbf{K}^x \cup \{0\}$,
(e) a binary relation $\leq$;
(3) a function symbol rv from the VF-sort into the RV-sort.

Technically speaking, the constant 0 and the functions $+, -$ in the RV-sort should all be relations. This point of view may be more convenient in some of the statements and arguments below that are of a syntactical nature. For notational convenience, we do not use different symbols for 0 and 1, since which ones are being referred to should always be clear in context.

**Notation 2.2.** The two sorts without the zero elements are denoted as $\text{VF}^x$ and $\text{RV}$, $\text{RV} \setminus \{\infty\}$ is denoted as $\text{RV}^x$, and $\text{RV} \cup \{0\}$ is denoted as $\text{RV}_0$. For any structure $M$ of $\mathcal{L}_{\text{RV}}$ and any formula $\phi$ with parameters in $M$, we write $\phi(M)$ for the subset defined by $\phi$ in $M$. In particular, we write $\text{VF}(M)$, $\text{RV}(M)$, $\text{RV}^x(M)$, $\mathbf{K}(M)$, etc. for the corresponding subsets of $M$. These are simply written as $\text{VF}$, $\text{RV}$, $\text{RV}^x$, $\mathbf{K}$, etc. when the structure in question is clear or when the discussion takes place in an ambient monster model (that is, a universal domain that embeds all “small” models that will occur in the discussion). For any subset $X \subseteq \text{VF}(M)^n \times \text{RV}(M)^m$, we write $\overline{\pi} \in X$ to mean that every element in the tuple $\overline{\pi}$ is in $X$. In particular, we often write $(\overline{\pi}, \overline{t})$ for a tuple of elements in $M$ with the understanding that $\overline{\pi} \in \text{VF}$ and $\overline{t} \in \text{RV}$. For such a tuple $(\overline{\pi}, \overline{t}) = (a_1, \ldots, a_n, t_1, \ldots t_m)$, let

$$rv(\overline{\pi}, \overline{t}) = (rv(a_1), \ldots, rv(a_n), \overline{t})$$

$$rv^{-1}(\overline{\pi}, \overline{t}) = \{\overline{a}\} \times rv^{-1}(t_1) \times \cdots \times rv^{-1}(t_m);$$

similarly for other functions.

Let $M$ be a structure of $\mathcal{L}_{\text{RV}}$. For any subset $A \subseteq M$, the smallest substructure of $M$ containing $A$ is denoted as $\langle A \rangle$. An element $b \in M$ is $A$-definable if there is a tuple $\overline{\pi} \in A$ such that $b$ is $\overline{\pi}$-definable, that is, $b$ is defined by a formula $\phi(\overline{\pi})$. The definable closure of $A$ in $M$, which is the smallest substructure of $M$ containing all the $A$-definable elements, is denoted as $\text{dcl}(A)$. Note that, although in general $\langle A \rangle \neq \text{dcl}(A)$, they may be identified as far as definable sets are concerned. Except in Section 3 this is what we shall do below. An element $b \in M$ is algebraic over $A$, or $A$-algebraic, if it is algebraic over some $\overline{\pi} \in A$, that is, there is a formula $\phi(\overline{\pi})$ that defines a finite subset of $M$ containing $b$. The algebraic closure of $A$ in $M$, which is the smallest substructure of $M$ containing all the $\langle A \rangle$-algebraic elements, is denoted as $\text{acl}(A)$. A basic fact is that, if $M$ models a complete theory in $\mathcal{L}_{\text{RV}}$, then $\text{acl}(A)$ is the same (up to isomorphism, of course) in any other model of the theory that contains $A$.

Let $M$ be a structure of $\mathcal{L}_{\text{RV}}$, $D \subseteq \text{VF}(M)^n \times \text{RV}(M)^m$ a definable subset, and $E$ a definable equivalence relation on $D$. Each equivalence class under $E$ is an imaginary element of $M$ and the collection $D/E$ of the equivalence classes is an imaginary sort of $M$. An imaginary element may occur in a formula as a parameter. Semantically, this means taking union of all the subsets defined by formulas $\phi(\overline{\pi}, \overline{t})$, where the parameters $(\overline{\pi}, \overline{t})$ run through all the “real” elements contained in the equivalence class. Syntactically, it corresponds to an extra existential quantifier and the invariance of the subset that is being defined when a different representative
of the equivalence class is used. Examples will be given below after the imaginary sorts of values and balls have been defined.

Definition 2.3. The theory of algebraically closed valued fields of characteristic 0 in $\mathcal{L}_{RV}$ (hereafter abbreviated as ACVF) states the following:

1. $(\text{VF}, 0, 1, +, -, \times)$ is an algebraically close field of characteristic 0;
2. $(\mathbb{R}^\times, 1, \times)$ is a divisible abelian group, where multiplication $\times$ is augmented by $t \times 0 = 0$ for all $t \in K$ and $t \times \infty = \infty$ for all $t \in RV_0$;
3. $(\mathbb{K}, 0, 1, +, -)$ is an algebraically closed field;
4. the relation $\preceq$ is a preorder on $RV$ with $\infty$ the top element and $K \times \mathbb{K}$ the equivalence class of 1;
5. the quotient $RV / K \times \mathbb{K}$, denoted as $\Gamma \cup \{\infty\}$, is a divisible ordered abelian group with a top element, where the ordering and the group operation are induced by $\preceq$ and $\times$, respectively, and the quotient map $RV \rightarrow \Gamma \cup \{\infty\}$ is denoted as $vrV$;
6. the function $rv : \text{VF}^\times \rightarrow \mathbb{R}^\times$ is a surjective group homomorphism augmented by $rv(0) = \infty$ such that the composite function

$$\text{val} = vrV \circ rv : \text{VF} \rightarrow \Gamma \cup \{\infty\}$$

is a valuation with the valuation ring $O = rv^{-1}(RV^{\geq 1})$ and its maximal ideal $M = rv^{-1}(RV^{> 1})$, where

$$RV^{\geq 1} = \{x \in RV : 1 \leq x\},$$
$$RV^{> 1} = \{x \in RV : 1 < x\};$$

The set $O \setminus M$ of units in the valuation ring is sometimes denoted as $U$. In any model of ACVF, the function $rv|\text{VF}^\times$ may be identified with the quotient map $\text{VF}^\times \rightarrow \text{VF}^\times/(1 + M)$. Hence an RV-sort element $t$ may be understood as a coset of $(1 + M)$. We occasionally treat $t$ as a set and write $a \in t$ to mean that $a \in rv^{-1}(t)$.

Although we do not include the multiplicative inverse function in the VF-sort and the RV-sort, we always assume that, without loss of generality, $\text{VF}(S)$ is a field and $\text{RV}^\times(S)$ is a group for a substructure $S$ of a model of ACVF.

Remark 2.4. Let $\mathcal{L}_v$ be the natural two-sorted language for valued fields: one sort for the field and the other for the value group. With the imaginary $\Gamma$-sort and the valuation map $val$, $\mathcal{L}_{RV}$ may be viewed as an expansion of $\mathcal{L}_v$. Each valued field may be turned naturally into an $\mathcal{L}_{RV}$-structure and hence an $\mathcal{L}_v$-structure. In fact, it is not hard to see that, under the natural interpretations, two valued fields are isomorphic as $\mathcal{L}_{RV}$-structures if and only if they are isomorphic as $\mathcal{L}_v$-structures. Henceforth we shall refer to the two sorts of $\mathcal{L}_v$ as the VF-sort and the $\Gamma$-sort.

In Section 3 we shall establish quantifier elimination for ACVF. The strategy of the proof is to reduce the problem to the following fundamental result of Weispfenning’s [20, Theorem 3.2]:

**Theorem 2.5.** The theory of algebraically closed valued fields of characteristic 0 as formulated in $\mathcal{L}_v$ admits quantifier elimination.

It is equivalent to quantifier elimination that, for any substructure $S$ of a model of ACVF, the theory $\text{ACVF}_S$ — that is, the union of ACVF and the set of all
quantifier-free formulas \(\phi(\overline{a})\) with \(\overline{a} \in S\) that hold in \(S\) — is complete. This implies that, for every integer \(p \geq 0\), the theory \(\text{ACVF}^p = \text{ACVF} \cup \{ \text{char} \overline{K} = p \}\) is complete. It is a basic fact in model theory that monster models (that is, universal domains) are guaranteed to exist for complete theories.

**Convention 2.6.** Henceforth, except in Section 3 we assume that everything happens in an ambient monster model \(\mathcal{C}\) of \(\text{ACVF}^0\), where \(S\) is a fixed “small” substructure of \(\mathcal{C}\). Accordingly, below, in terms such as “definable” (that is, “\(\emptyset\)-definable”), \(\overline{a}\)-definable, “\(\text{acl}(\emptyset)\)”, “\(\mathcal{L}_{\text{RV}}\)”, etc. we shall always mean “\(S\)-definable”, “\(\langle S, \overline{a}\rangle\)"-definable”, “\(\text{acl}(S)\)”, “\(\mathcal{L}_{\text{RV}} \cup S\)” etc. When the additional parameters are not specified, we will just say “parametrically definable”.

The imaginary sort \(\Gamma \cup \{ \infty \}\) is called the \(\Gamma\)-sort. We write \(\overline{t} \in \overline{\gamma}\) to mean that \(\text{rv}(\overline{t}) = \overline{\gamma}\). For any subset \(A\), the assertion that \(\text{rv}^{-1}(\gamma_i) \subseteq A\) for every \(\gamma_i\) in the tuple \(\overline{\gamma}\) is abbreviated as \(\overline{\gamma} \in A\). A subset \(X\) is \(\overline{\gamma}\)-definable if there is a formula \(\phi(\overline{a})\) such that \(X = \bigcup_{\overline{t} \in \overline{\gamma}} X_{\overline{t}}\), where \(X_{\overline{t}}\) is the subset defined by \(\phi(\overline{t})\). Syntactically, \(X\) is defined by any formula of the form

\[\exists \overline{\gamma} \left( \text{rv}(\overline{\gamma}) \leq \overline{t} \wedge \text{rv}(\overline{\gamma}) \geq \overline{t} \wedge \phi(\text{rv}(\overline{\gamma}))\right),\]

where \(\overline{\gamma} \in \overline{\gamma}\) and no element in \(\overline{\gamma}\) occurs in \(\phi(\text{rv}(\overline{\gamma}))\). Accordingly, when a subset \(A \subseteq \text{VF} \cup \text{RV} \cup \Gamma\) is used as a source of parameters, the elements in \(\Gamma(A)\) can only occur in formulas of the above form. Naturally, the definable closure \(\text{dcl}(A)\) of \(A\) also contains those elements that are definable with parameters in \(\Gamma(A)\). Similarly for the algebraic closure \(\text{acl}(A)\) of \(A\).

A substructure \(S\) is VF-generated if \(S = \text{dcl}(A)\) for some \(A \subseteq \text{VF}\). Similarly for RV, \(\Gamma\), and any combination of the three sorts. From now on, unless specified otherwise, a substructure is always \((\text{VF}, \text{RV}, \Gamma)\)-generated.

**Notation 2.7.** Coordinate projection maps are ubiquitous in this paper. To facilitate the discussion, certain notational conventions about them are adopted.

Let \(X \subseteq \text{VF}^n \times \text{RV}^m\). For any \(n \in \mathbb{N}\), let \(I_n = \{1, \ldots, n\}\). First of all, the VF-coordinates and the RV-coordinates of \(X\) are indexed separately. It is cumbersome to actually distinguish them notationally, so we just assume that the set of the indices of the VF-coordinates (VF-indices) is \(I_n\) and the set of the indices of the RV-coordinates (RV-indices) is \(I_m\). This should never cause confusion in context. Let \(I_{n,m} = I_n \uplus I_m\), \(E \subseteq I_{n,m}\), and \(\overline{E} = I_{n,m} \setminus E\). If \(E\) is a singleton \(\{i\}\) then we always write \(E\) as \(i\) and \(\overline{E}\) as \(\overline{i}\). We write \(\text{pr}_E X\) for the projection of \(X\) to the coordinates in \(E\). For any \(\overline{\gamma} \in \text{pr}_E X\), the fiber \(\{\overline{\gamma}: (\overline{\tau}, \overline{\gamma}) \in X\}\) is denoted as \(\text{fib}(X, \overline{\gamma})\). Note that, for notational convenience, we shall often tacitly identify the two subsets \(\text{fib}(X, \overline{\gamma})\) and \(\text{fib}(X, \overline{\gamma}) \times \{\overline{\gamma}\}\). Also, it is often more convenient to use simple descriptions as subscripts. For example, if \(E = \{1, \ldots, k\}\) etc. then we may write \(\text{pr}_{<k}\) etc. If \(E\) contains exactly the VF-indices (respectively RV-indices) then \(\text{pr}_E\) is written as \(\text{prf}\) (respectively \(\text{prv}\)). Suppose that \(E'\) is a subset of the indices of the coordinates of \(\text{pr}_E X\). Then the composition \(\text{pr}_{E'} \circ \text{pr}_E\) is written as \(\text{prf}_{E,E'}\).

Naturally \(\text{prf}_{E'} \circ \text{prf}\) and \(\text{prv}_{E'} \circ \text{prv}\) are written as \(\text{prf}_{E',E'}\) and \(\text{prv}_{E',E'}\).

Suppose that \(X, V, W\) are all definable subsets and \(X \subseteq V \times W\). Sometimes we shall want to investigate the fibers of \(X\) of the form \(\text{fib}(X, \overline{\tau})\) with \(\overline{\tau} \in V\). Note that \(\text{fib}(X, \overline{\tau})\) is in general not definable. Of course it is \(\langle \overline{\tau}\rangle\)-definable. Many properties and notions below depend on the underlying substructure from which the subsets in question are definable. Hence, below, when we study fibers of \(X\),
we shall always assume that the underlying substructure has been expanded in an appropriate way.

We shall frequently need to keep track of the correspondence between the VF-indices and the RV-indices in a subset derived from $X$. It is unduly complicated to describe a precise indexing scheme that is suitable for this task and hence we shall not attempt it here. Instead, we shall give a few typical examples and then rely on the reader’s intuition to figure out the actual indexing in each instance. There is a principle that underlies these examples: coordinates of interest get indices as small as possible. Let

$$c(X) = \{ (\pi, \text{rv}(\pi), \overline{7}) : (\pi, \overline{7}) \in X \} \subseteq VF^n \times RV^{n+m}.$$  

Clearly $X$ is definably bijective to $c(X)$ in a canonical way. This bijection is called the canonical bijection and is denoted as $c$. In $c(X)$, the set of the new RV-indices created by the map rv is $\mathcal{I}_m$. Next, let

$$X^* = \bigcup \{ \text{rv}^{-1}(\overline{7}) \times \{ (\pi, \overline{7}) \} : (\pi, \overline{7}) \in X \} \subseteq VF^{m+n} \times RV^m.$$  

In $X^*$, the set of the new VF-indices created by the “lifting” map $\text{rv}^{-1}$ is $\mathcal{I}_m$. Lastly, let $f : X \rightarrow VF^n \times RV^m$ be a definable function such that, for every $(\pi, \overline{7}) \in X$,

$$(pr_{>1} \circ f)(\pi, \overline{7}) = pr_{>1}(\pi, \overline{7}).$$

Let $Y = (pr_{>1} \circ f)(X) \subseteq VF^{n-1} \times RV^{n+m}$ and $g : Y \rightarrow VF^{n-1} \times RV^{n+m}$ a definable function such that, for every $(\overline{7}, \overline{t}) \in Y$,

$$(pr \circ g)(\overline{7}, \overline{t}) = \overline{t}.$$  

Let $Z = (pr \circ g)(Y) \subseteq VF^{2n+m-1}$. Among the coordinates of $Z$ there are $n$ special ones that correspond to the VF-coordinates of $X$, which have been truncated in the transformation from $X$ to $Z$. These special coordinates are indexed by $1, \ldots, n$.

We now turn to the other important kind of imaginary elements: balls. The open balls form a basis of the valuation topology. Basic properties of balls will be explored in Section 4.

**Definition 2.8.** A subset $b$ of VF is an open ball if there is a $\gamma \in \Gamma$ and an $a \in b$ such that $a \in b$ if and only if $\text{val}(a-b) > \gamma$. It is a closed ball if $a \in b$ if and only if $\text{val}(a-b) \geq \gamma$. It is an rv-ball if $b = \text{rv}^{-1}(t)$ for some $t \in RV$. The value $\gamma$ is the radius of $b$, which is denoted as $\text{rad}(b)$. If $\text{val}$ is constant on $b$ — that is, $b$ is contained in an rv-ball — then $\text{val}(b)$ is the valuative center of $b$; if $\text{val}$ is not constant on $b$, that is, $0 \in b$, then the valuative center of $b$ is $\infty$. The valuative center of $b$ is denoted by $\text{vcr}(b)$.

Note that each point in VF is a closed ball of radius $\infty$. Also, we shall regard VF as a clopen ball of radius $-\infty$.

A ball $b$ may be represented by a triple $(a, b, d) \in VF^3$, where $a \in b$, $\text{val}(b)$ is the radius of $b$, and $d = 1$ if $b$ is open and $d = 0$ if $b$ is closed. A set $\mathfrak{B}$ of balls is a subset of $VF^3$ of triples of this form such that if $(a, b, d) \in \mathfrak{B}$ then for all $a' \in VF$ with $\text{rv}(a-a') \sqsubseteq_d b$, where $\sqsubseteq_d$ is $>'$ if $d = 1$ or $\geq$ if $d = 0$, there is a $b' \in VF$ with $\text{val}(b) = \text{val}(b')$ such that $(a', b', d) \in \mathfrak{B}$. Clearly two triples $(a, b, d)$, $(a', b', d') \in \mathfrak{B}$ represent two different balls, which may or may not be disjoint, if and only if either $(\text{val}(b), d) \neq (\text{val}(b'), d')$ or, in case that they are the same, $\text{rv}(a-a') \sqsubseteq_d b$ does not hold.

We note the following terminological convention. The union of $\mathfrak{B}$, sometimes written as $\bigcup \mathfrak{B}$, is actually the subset $pr_1 \mathfrak{B}$. For any subset $A \subseteq VF$, the assertion that $\bigcup \mathfrak{B} \subseteq A$ may simply be written as $\mathfrak{B} \subseteq A$. We say that $\mathfrak{B}$ is finite if it
contains finitely many distinct balls. A subset of $\mathcal{B}$ is always a set of balls in $\mathcal{B}$. A function $f$ of $\mathcal{B}$ is always a function on the balls in $\mathcal{B}$; that is, $f$ is a relation between $\mathcal{B}$ and a subset $W$ such that for every $b \in \mathcal{B}$ there is a unique $w \in W$ between which and every $(a, b, d) \in b$ the relation holds. Notice that $f$ may or may not be a function on the triples in $\mathcal{B}$.

**Remark 2.9.** In a similar way a ball $b$ may be represented by a triple in $\mathbf{VF} \times \mathbf{RV}^2$. This representation is sometimes more convenient. Below we shall not distinguish these two representations.

We have seen above how to use elements in the imaginary $\Gamma$-sort as parameters in formulas. The idea is the same for balls. Let $b$ be a ball. A subset $X$ is $(b)$-definable if there is a formula $\phi(x, y, z)$ such that $X = \bigcup_{(a, b, d) \in b} X_{(a, b, d)}$, where $(a, b, d) \in \mathbf{VF}^3$ is a representative of $b$ and $X_{(a, b, d)}$ is the subset defined by $\phi(a, b, d)$. Syntactically, $X$ is defined by any formula of the form

$$\exists x, y, z \ (rv(x - a) \square d b \land rv(y) \geq rv(b) \land rv(y) \leq rv(b) \land z = d \land \phi(x, y, z)),$$

where $(a, b, d) \in \mathbf{VF}^3$ is any representative of $b$ and no representative of $b$ occurs in $\phi(x, y, z)$ and $\square d$ is $>1$ if $d = 1$ or $\geq 1$ if $d = 0$. Accordingly, if a subset $A$ contains balls and is used as a source of parameters, then the balls in $A$ can only occur in formulas of the above form. With this understanding, the definable closure $\text{cl}(A)$ and the algebraic closure $\text{acl}(A)$ of $A$ may be defined in the obvious way.

### 2.2. Compactness

The use of the Compactness Theorem in [12] is extensive. Here we prove a few lemmas to illustrate it.

**Definition 2.10.** Let $X, Y$ be definable subsets and $p : X \longrightarrow Y$ a definable function. A definable function $f$ is a $p$-function if there is a $Y' \subseteq Y$ and a partial function $\hat{f}$ on $X$ such that $\text{dom}(\hat{f}) = p^{-1}(Y')$ and $f = p \times \hat{f}$. Let $\Phi(p)$ be a set of $p$-functions. We say that $\Phi(p)$ is $p$-closed if for all $f_1, \ldots, f_n \in \Phi(p)$ there is an $f \in \Phi(p)$ such that $\text{dom}(f) = \bigcup_i \text{dom}(f_i)$ and, for each $\mathbf{y} \in Y$ with $p^{-1}(\mathbf{y}) \subseteq \text{dom}(f)$, there is an $f_i$ such that $f \upharpoonright p^{-1}(\mathbf{y}) = \{\mathbf{y}\} \times (f_i \upharpoonright p^{-1}(\mathbf{y}))$, where $f_i$ is the partial function such that $f_i = p \times f_i$.

Let $X$ be a definable subset and $p$ a definable function such that $X \subseteq \text{dom}(p)$. In this situation a $p$-function with respect to $X$ should always be understood as a $(p \upharpoonright X)$-function.

**Lemma 2.11.** Let $X, Y$ be definable subsets, $p : X \longrightarrow Y$ a definable function, and $\Phi(p)$ a set of $p$-functions that is $p$-closed. Suppose that, for every $\mathbf{y} \in Y$, there is an $f_\mathbf{y} \in \Phi(p)$ such that $f_\mathbf{y}$ is injective on $p^{-1}(\mathbf{y})$. Then there is an $f \in \Phi(p)$ such that $f : X \longrightarrow Y \times Z$ is an injective function for some definable subset $Z$.

**Proof.** Suppose for contradiction that no $f \in \Phi(p)$ is an injective function on $X$ of the required form. Let $\mathcal{L} = \mathcal{L}_{\text{RV}} \cup \{\mathbf{y}\}$, where $\mathbf{y}$ are new constants. Consider the $\mathcal{L}$-theory $T$ that states the following:

1. everything in $\text{ACVF}_S^0$,
2. $\mathbf{y} \in Y$,
3. every $f \in \Phi(p)$ fails to be injective on $p^{-1}(\mathbf{y})$.

If $T$ is not consistent then there is a finite list of functions $f_i \in \Phi(p)$ such that, for all $\mathbf{y} \in Y$, one of the functions $f_i$ is injective on $p^{-1}(\mathbf{y})$. Since $\Phi(p)$ is $p$-closed,
there is a function $f \in \Phi(p)$ on $X$ such that, for each $\overline{y} \in Y$, there is an $f_i$ such that $f(\overline{y}) = (p(\overline{y}), f_i(\overline{y}))$ for every $\overline{y} \in p^{-1}(\overline{y})$. Clearly $f$ is an injective function on $X$ of the required form, contradiction. So $T$ is consistent and there is a model $N \models T$. Since $N$ is also a model of $\text{ACVF}^0$, we have that $\overline{y}^N \in Y$ and, by assumption, there is an $f_{\overline{y}} \in \Phi(p)$ that is injective on $p^{-1}(\overline{y}^N)$, contradiction again. \hfill $\square$

In application, the function $p$ in this lemma is often taken to be the map $r_v$; see, for example, Lemma 4.3. The flexibility of Lemma 2.11 is twofold: on the one hand, injectivity may be replaced by other first-order properties and, on the other hand, restrictions may be imposed on the set $\Phi(p)$ so that we can achieve better control over the form of the function $f$. In the following sections, the phrase “by compactness” often means “by a variation of Lemma 2.11”.

**Lemma 2.12.** Let $\overline{t}, \overline{s} \in \text{RV}$ and $X \subseteq r_v^{-1}(\overline{t})$ a $\overline{t}$-definable subset such that, for every $\overline{p} \in X$, $\overline{s} \in \text{acl}(\overline{p})$. Then $s \in \text{acl}(\overline{t})$.

**Proof.** Let $\mathcal{L} = \mathcal{L}_{\text{RV}} \cup \{\overline{t}, \overline{s}, \overline{r}\}$, where $\overline{r}$ are new constants. Consider the $\mathcal{L}$-theory $T$ that states the following:

1. everything in $\text{ACVF}^0_{\overline{t}, \overline{s}}$,
2. $\overline{t} \in X$,
3. for every $\mathcal{L}$-formula $\phi$ that does not contain $s$ and every integer $k > 0$, either the subset defined by $\phi$ is of size at most $k$ but does not contain $s$ or it is of size greater than $k$.

By the assumption, $T$ must be inconsistent. Therefore, there are integers $k_1, \ldots, k_m$, $\mathcal{L}$-formulas $\phi_1(\overline{x}, y), \ldots, \phi_m(\overline{x}, y)$ that do not contain $s$, and subsets $X_1, \ldots, X_m$ of $X$ defined by $\phi_1^* \ldots, \phi_m^*$, where $\phi_i^*$ is the formula

$$\exists y_1, \ldots, y_k \forall y \left( \phi_i(\overline{x}, y) \rightarrow \bigvee_{1 \leq j \leq k} y = y_j \right),$$

such that $\bigcup_i X_i = X$ and, for every $\overline{x} \in X_i$, the formula $\phi_i(\overline{x}, y)$ defines a finite subset $U_{\overline{x}}$ containing $s$ of size at most $k_i$. Without loss of generality, we may assume that $X_1, \ldots, X_m$ are pairwise disjoint. Then $\bigcap_{\overline{x} \in X} U_{\overline{x}}$ is a $\overline{t}$-definable finite subset that contains $s$. \hfill $\square$

For the proof of the next lemma we need to assume quantifier elimination, which is to be established in Section 3.

**Lemma 2.13.** The exchange principle holds in both sorts:

1. For any $a, b \in VF$, if $a \in \text{acl}(b) \setminus \text{acl}(\emptyset)$ then $b \in \text{acl}(a)$.
2. For any $t, s \in RV$, if $t \in \text{acl}(s) \setminus \text{acl}(\emptyset)$ then $s \in \text{acl}(t)$.

**Proof.** For the first item, let $\phi$ be a quantifier-free formula in disjunctive normal form that witnesses $a \in \text{acl}(b)$. For any term $r_v(g(x))$ in $\phi$, where $g(x) \in VF(\langle b \rangle[x])$, and any $d \in VF$, if $\text{val}(d - a)$ is sufficiently large then $r_v(g(a)) = r_v(g(d))$. On the other hand, clearly $VF$-sort disequalities cannot define nonempty finite subset. Therefore every irredundant disjunct of $\phi$ has a conjunct of the form $f(x, b) = 0$, where $f(x, b) \in VF(\langle b \rangle)[x]$. If $f(a, b) = 0$ then, since $a \notin \text{acl}(\emptyset)$, we must have that $f(x, b) \notin VF(\langle b \rangle)[x]$. So the item follows from the exchange principle in field theory.
Lemma 2.14. Let $f : X \rightarrow Y$ be a definable surjective function, where $X, Y \subseteq \text{VF}$. Then there are definable disjoint subsets $Y_1, Y_2 \subseteq Y$ with $Y_1 \cup Y_2 = Y$ such that $Y_1$ is finite, $f^{-1}(b)$ is infinite for each $b \in Y_1$, and the function $f \upharpoonright f^{-1}(Y_2)$ is finite-to-one.

Proof. For each $b \in Y$, if $f^{-1}(b)$ is infinite then, by compactness, there is an $a \in f^{-1}(b)$ such that $a \notin \text{acl}(b)$. Since $b \in \text{dcl}(a) \subseteq \text{acl}(a)$, by Lemma 2.13 we must have that $b \notin \text{acl}(a) \setminus \text{acl}(\emptyset)$ and hence $b \in \text{acl}(\emptyset)$. Let $\mathcal{L} = \mathcal{L}_{\text{RV}} \cup \{c\}$, where $c$ is a new constant. Consider the $\mathcal{L}$-theory $T$ that states the following:

1. everything in $\text{ACVF}_0^0$,
2. $c \in Y$,
3. $|f^{-1}(c)| > k$ for every integer $k > 0$,
4. for every $\mathcal{L}_{\text{RV}}$-formula $\phi$ and every integer $k > 0$, either the subset defined by $\phi$ is of size at most $k$ but it does not contain $c$ or it is of size greater than $k$.

If $N \models T$ then $c^N \in Y$ and $f^{-1}(c^N)$ is infinite and $c^N \notin \text{acl}(\emptyset)$, contradiction. So $T$ is inconsistent. So there is an $\mathcal{L}_{\text{RV}}$-formula $\phi$ and an integer $k > 0$ such that $\phi(c^N)$ is finite and, for every $b \in Y$, if $|f^{-1}(b)| > k$ then $b \notin \phi(c^N)$. Let $Y_1 = \{b \in Y : f^{-1}(b) \text{ is infinite}\}$ and $Y_2 = Y \setminus Y_1$. Since $Y_1 \subseteq \phi(c^N)$ and $\phi(c^N)$ is finite, clearly $Y_1$ is definable and hence $Y_2$ is definable, as desired. $\square$

Lemma 2.15. Let $f : X \rightarrow Y$ be a definable function, where $X, Y \subseteq \text{VF}$. For every $a \in X$ let $Z_a$ be the intersection of all definable subsets that contain $a$. Suppose that $f \upharpoonright Z_a$ is injective for every $a \in X$. Then there is a finite definable partition $X_1, \ldots, X_n$ of $X$ such that $f \upharpoonright X_i$ is injective for every $i$.

Proof. Let $\mathcal{L} = \mathcal{L}_{\text{RV}} \cup \{c_1, c_2\}$, where $c_1, c_2$ are new constants. Consider the $\mathcal{L}$-theory $T$ that states the following:

1. everything in $\text{ACVF}_0^0$,
2. $c_1, c_2 \in X$ and $c_1 \neq c_2$,
3. $f(c_1) = f(c_2)$,
4. for every $\mathcal{L}_{\text{RV}}$-formula $\phi$, either the subset defined by $\phi$ contains both $c_1$ and $c_2$ or it does not contain either of them.

If $N \models T$ then $c_1^N, c_2^N$ are distinct elements in $X$ and $c_1^N \in Z_{c_2^N}$ and $f(c_1^N) = f(c_2^N)$, contradiction. So $T$ is inconsistent. So there are $\mathcal{L}_{\text{RV}}$-formulas $\phi_1, \ldots, \phi_n$ such that, for every two distinct elements $a_1, a_2 \in X$, if $f(a_1) = f(a_2)$ then $\phi_i(c) \text{ separates } a_1, a_2$ for some $i$. So the partition on $X$ induced by $\phi_1(c), \ldots, \phi_n(c)$ is as desired. $\square$
Naturally injectivity may be replaced by other first-order properties in this lemma.

3. Quantifier elimination in ACVF

We shall show in this section that ACVF admits quantifier elimination. The following model-theoretic test for quantifier elimination will be used; see [19] for a proof.

**Fact 3.1.** For any first-order theory $T$ in a language that has at least one constant symbol, the following are equivalent:

1. $T$ admits quantifier elimination.
2. For any two models $M_1, M_2 \models T$ such that $M_2$ is $\| M_1 \|^+\text{-saturated}$ and any isomorphism $f$ between two substructures $N_1 \subseteq M_1$ and $N_2 \subseteq M_2$, there is a monomorphism $f^* : M_1 \rightarrow M_2$ extending $f$.

Recall that our strategy is to establish the second item in this test for ACVF via reduction to Theorem 2.5; see Remark 2.4.

**Lemma 3.2.** Let $B \subseteq M \models \text{ACVF}$ and $b_0, \ldots, b_n \in \text{VF}(B)$. Let $\overline{F}(X) = \sum_{0 \leq i \leq n} t_i X^i$ be a nonzero polynomial with coefficients in $\text{RV}_0$ such that $t_i = \text{rv}(b_i)$ if $t_i \neq 0$. Let $F(X) = \sum_{0 \leq i \leq n} b_i X^i$. For every $t \in \text{RV}(M)$, if $\overline{F}(t) = 0$ and $\text{rv}(\text{rv}(b_i) t^i) > 0$ for all $t_i = 0$, then there is a $b \in \text{rv}^{-1}(t)$ such that $F(b) = 0$.

**Proof.** Fix a $t \in \text{RV}(M)$ with $\overline{F}(t) = 0$ and $\text{rv}(\text{rv}(b_i) t^i) > 0$ for all $t_i = 0$. Note that, since such a $t$ exists and $\overline{F}(X)$ is not the zero polynomial, we must have that $\overline{F}(X)$ is not a monomial and $t \neq \infty$. Let $m < n$ be the least number such that $t_m \neq 0$. Let $r_1, \ldots, r_n \in \text{VF}(M)$ be the (possibly repeated) roots of $F(X)$. Let $F^*(X) = \sum_{t_i \neq 0} b_i X^i$. For any $b \in t$, if $\text{rv}(b) \neq \text{rv}(r_i)$ for every $i$ then

\[
\begin{cases}
\text{val}(b - r_i) = \text{val}(b), & \text{if } \text{val}(b) < \text{val}(r_i); \\
\text{val}(b - r_i) = \text{val}(r_i), & \text{if } \text{val}(b) \geq \text{val}(r_i).
\end{cases}
\]

So $\prod_i \text{val}(b - r_i) \leq \text{val}(b_m b_m/b_n)$ and hence $\text{val}(F(b)) \leq \text{val}(b_m b_m) = 0$. Since $\text{val}(b_i b^i) > 0$ for all $t_i = 0$, we have that $\text{val}(F^*(b)) = 0$, contradicting the choice of $t$. So $t = \text{rv}(b) = \text{rv}(r_i)$ for some $i$. □

**Notation 3.3.** For a polynomial $F(X) = \sum_i t_i X^i$ with coefficients $t_i \in \text{RV}_0$ it is often convenient to choose a $b_i \in t_i$ for each nonzero $t_i$ and write $F(X)$ as $\sum_i \text{rv}(b_i) X^i$. Below, whenever $F(X)$ is written in this form, it should be understood that $b_i$ is chosen only if $t_i \neq 0$.

For the rest of this section, we fix two models $M_1, M_2 \models \text{ACVF}$ such that $M_2$ is $\| M_1 \|^+$-saturated. Let $S_1 \subseteq M_1$ and $f_1 : S_1 \rightarrow M_2$ a monomorphism.

For any $A \subseteq M \models \text{ACVF}$, we write $\text{VF}(A)^{ac}$, $\overline{\text{K}}(A)^{ac}$, etc. for the corresponding field-theoretic algebraic closures.

**Lemma 3.4.** There is a $P \subseteq M_1$ and a monomorphism $g : P \rightarrow M_2$ extending $f_1$ such that

1. $\text{VF}(P) = \text{VF}(S_1)$,
2. $\overline{\text{K}}(P)$ is the algebraic closure of $\overline{\text{K}}(S_1)$,
3. $\Gamma(P)$ is the divisible hull of $\Gamma(S_1)$.
Proof. First of all, there is a field homomorphism $g_1 : \overline{K}(S_1)^ac \rightarrow \overline{K}(M_2)$ extending $f_1 \upharpoonright \overline{K}(S_1)$. Let $\langle \overline{K}(S_1)^ac, RV(S_1) \rangle = S_2$ and $g_2 : S_2 \rightarrow M_2$ be the monomorphism determined by

$$ts \mapsto g_1(t)f_1(s) \text{ for all } t \in \overline{K}(S_1)^ac \text{ and } s \in RV(S_1).$$

Next, let $n > 1$ be the least integer such that there is a $t_1 \in RV(M_1)$ with $t_1^n \in RV(S_2)$ but $rvv(t_1^n) \notin \Gamma(S_2)$ for every $0 < i < n$. Let $t_2 \in RV(M_2)$ such that $g_2(t_1^n) = t_2^n$. Let $g_3 : \langle S_2, t_1 \rangle \rightarrow M_2$ be the monomorphism determined by

$$t_1s \mapsto t_2g_2(s) \text{ for all } s \in S_2.$$ 

Iterating this procedure the lemma follows. \hfill \square

In the light of this lemma, without loss of generality, we may assume that $\overline{K}(S_1)$ is algebraically closed and $\Gamma(S_1)$ is divisible.

Let $S \subseteq M_1$ be a VF-generated substructure such that

1. $VF(S_1) \subseteq VF(S)$,
2. $RV(S) \subseteq RV(S_1)$,
3. there is a monomorphism $f : S \rightarrow M_2$ with $f \upharpoonright (S \cap S_1) = f_1 \upharpoonright (S \cap S_1)$.

Fix an $e \in VF(M_1)$ such that $rv(e) \in RV(S_1) \setminus RV(S)$. In the next few lemmas, under various assumptions, we shall prove the following claim:

Claim (\ast). RV($\langle S, e \rangle$) $\subseteq RV(S_1)$ and $f$ may be extended to a monomorphism $f^* : \langle S, e \rangle \rightarrow M_2$ such that $f^* \upharpoonright RV(\langle S, e \rangle) = f_1 \upharpoonright RV(\langle S, e \rangle)$.

Lemma 3.5. Let $\overline{F}(x) = x^n + \sum_{0 \leq i < n} rv(a_i)x^i \in K(S)[x]$ be an irreducible polynomial with $rv(a_0) \neq 0$. Suppose that $e \in U(M_1)$ is a root of the polynomial $F(x) = x^n + \sum_{0 \leq i < n} a_ix^i \in O(S)[x]$. If the valued field $\langle VF(S), O(S) \rangle$ is henselian, then Claim (\ast) holds.

Proof. Obviously $rv(e)$ is a root of $\overline{F}(x)$. Also, note that $F(x)$ is irreducible over $VF(S)$. The polynomial

$$f_1(\overline{F}(x)) = x^n + \sum_{0 \leq i < n} f_1(rv(a_i))x^i \in f_1(\overline{K}(S))[x]$$

is irreducible over $f_1(\overline{K}(S))$ and $f_1(rv(e))$ is a root of $f_1(\overline{F}(x))$. By Lemma 3.2 there is a root $d \in VF(M_2)$ of $f(F(x))$ such that $rv(d) = f_1(rv(e))$. By Remark 2.4 Theorem 2.3 and Fact 2.4 there is an $L_v$-monomorphism $f^* : \langle S, e \rangle \rightarrow M_2$ extending $f$. Since $\langle VF(S), O(S) \rangle$ is henselian, without loss of generality, we may assume that $f^*(e) = d$. By Remark 2.4 again, $f^*$ may be treated as an $L_{RV}$-monomorphism extending $f$ with $f^*(rv(e)) = f_1(rv(e))$.

Now, since $[\overline{K}(\langle S, e \rangle) : \overline{K}(S)] = [VF(\langle S, e \rangle) : VF(S)]$, by the fundamental inequality of valuation theory (see [8, Theorem 3.3.4]), we have that

$$\overline{K}(\langle S, e \rangle) = \overline{K}(S)(rv(e)) \subseteq \overline{K}(S_1)$$

$$\Gamma(\langle S, e \rangle) = \Gamma(S).$$

Therefore, $RV(\langle S, e \rangle) = RV(\langle RV(S), rv(e) \rangle) \subseteq RV(S_1)$, which clearly implies that $f^* \upharpoonright RV(\langle S, e \rangle) = f_1 \upharpoonright RV(\langle S, e \rangle)$. \hfill \square

Lemma 3.6. Suppose that $e \notin U(M_1)$, $e^n = a \in VF(S)$ for some integer $n > 1$, and $\text{val}(e^i) \notin \Gamma(S)$ for all $0 < i < n$. If $\langle VF(S), O(S) \rangle$ is henselian, then Claim (\ast) holds.
Proof. By the fundamental inequality of valuation theory and the assumption, we have that
\[ n \leq [\Gamma((S,e)) : \Gamma(S)] \leq [\text{VF}((S,e)) : \text{VF}(S)] \leq n. \]
So \( n = [\Gamma((S,e)) : \Gamma(S)] \) and \( K((S,e)) = K(S) \). Since \( \Gamma(S_1) \) is divisible, we have that \( \text{val}(e) \in \Gamma(S_1) \) and \( \Gamma((S,e)) \subseteq \Gamma(S_1) \).

Any element \( b \in \text{VF}((S,e)) \) may be written as a quotient of two elements of the form \( \sum_{0 \leq i \leq m} b_ie^i \), where \( b_i \in \text{VF}(S) \). Since \( e^n = a \in \text{VF}(S) \), we may assume that \( 0 \leq m < n \).

Claim. For some \( t \in \text{RV}(S) \) and some integer \( 0 \leq k \leq m \), \( \text{rv}(b) = t \cdot \text{rv}(e^k) \).

Proof. We do induction on \( m \). Without loss of generality, we may assume that \( b_m, b_0 \neq 0 \). We claim that \( \text{val}(b_0) \neq \text{val}(e \sum_{j=1}^m b_je^{j-1}) \). Suppose for contradiction that this is not the case. By the inductive hypothesis, \( \text{rv}(\sum_{j=1}^m b_je^{j-1}) = t \cdot \text{rv}(e^k) \) for some \( t \in \text{RV}(S) \) and some integer \( 0 \leq k \leq m - 1 \). So
\[
\text{val} \left( e \sum_{j=1}^m b_je^{j-1} \right) = \text{vrv}(t \cdot \text{rv}(e^{k+1})) = \text{vrv}(\text{rv}(b_0)).
\]

So \( \text{val}(e^{k+1}) \in \Gamma(S) \), which is a contradiction because \( 0 < k + 1 < n \). Now, since \( \text{val}(b_0) \neq \text{val}(e \sum_{j=1}^m b_je^{j-1}) \), either \( \text{rv}(b) = \text{rv}(b_0) \) or \( \text{rv}(b) = \text{rv}(e \sum_{j=1}^m b_je^{j-1}) \) and hence \( \text{rv}(b) \) is of the desired form by the inductive hypothesis. \( \square \)

Therefore, \( \Gamma((S,e)) = \Gamma((\Gamma(S),\text{val}(e))) \) and \( \text{RV}((S,e)) = \text{RV}((\text{RV}(S),\text{rv}(e))) \subseteq \text{RV}(S_1) \).

Note that, since the roots of \( F(x) = x^n - a \) are all of the same value, by the assumption on \( \text{val}(e) \), \( F(x) \) is irreducible over \( \text{VF}(S) \). Let \( a_1, \ldots, a_n \) be the distinct roots of \( F(x) \) in \( M_1 \). We consider the symmetric polynomial
\[
G(y_1, \ldots, y_n) = \prod_{i=1}^n \prod_{j=1}^n \left( y_j - \frac{\text{rv}(a_i)}{\text{rv}(a_j)} \right).
\]

In the expansion of \( G(y_1, \ldots, y_n) \), the coefficient of each monomial is a sum of elements in the residue field and hence may be written as a quotient of two terms:
\[
\frac{\text{rv}(I(a_1, \ldots, a_n))}{\text{rv}(J(a_1, \ldots, a_n))},
\]
where \( I(a_1, \ldots, a_n) \) is a symmetric VF-sort term and hence may be written as \( I(a) \). Moreover, if we substitute \( y/\text{rv}(a_j) \) for \( y_j \) in each monomial then the denominator of its coefficient becomes \( \text{rv}(\prod_i a_i)^n = \text{rv}(a^n) \). So the term \( G(y/\text{rv}(a_1), \ldots, y/\text{rv}(a_n)) \) may be written as a summation \( G(y, a) \) of terms of the form
\[
\frac{\text{rv}(I(a))y^m}{\text{rv}(a^n)},
\]
where \( m \leq n^2 \). Since \( \text{RV}((S,e)) \subseteq \text{RV}(S_1) \), it makes sense to write
\[
S_1 \models G(\text{rv}(e), a) = 0
\]
and hence
\[
f_1(S_1) \models G(f_1(\text{rv}(e)), f(a)) = 0.
\]
So, by Lemma [24], there is a root \( d \in \text{VF}(M_2) \) of the polynomial \( x^n - f(a) \) such that \( \text{rv}(d) = f_1(\text{rv}(e)) \).
As in the previous lemma, there is an $\mathcal{L}_v$-monomorphism $f^*: \langle S, e \rangle \rightarrow M_2$ extending $f$ with $f^*(e) = d$, which may be treated as an $\mathcal{L}_{RV}$-monomorphism extending $f$ with $f^*(rv(e)) = f_1(rv(e))$. Since $RV(\langle S, e \rangle) = RV((RV(S), rv(e)))$, we must have that $f^* | RV(\langle S, e \rangle) = f_1 | RV(\langle S, e \rangle)$. \hfill $\Box$

**Lemma 3.7.** Suppose that $rv(e) \in \overline{K}(S_1)$ is transcendental over $\overline{K}(S)$. If $\Gamma(S)$ is divisible, then Claim (\ast) holds.

**Proof.** Clearly $rv(e)$ does not contain any element that is algebraic over $VF(S)$; in particular, $e$ is transcendental over $VF(S)$. Similarly $f_1(rv(e))$ does not contain any element that is algebraic over $f(VF(S))$. Fix a $d \in VF(M_2)$ with $rv(d) = f_1(rv(e))$.

By the dimension inequality of valuation theory (see [5, Theorem 3.4.3]), the rational rank of $\Gamma(\langle S, e \rangle)/\Gamma(S)$ is 0. Since $\Gamma(S)$ is divisible, we actually have that $\Gamma(\langle S, e \rangle) = \Gamma(S)$. So for every $b \in VF(\langle S, e \rangle)$ there is an $a \in VF(S)$ such that $val(b/a) = 0$. Let $b = \sum_{0 \leq i \leq m} b_i e_i \in VF(\langle S, e \rangle)$, where $b_i \in VF(S)$, and $b^* = \sum_{0 \leq i \leq m} f(b_i)d_i \in VF((f(S), d))$.

**Claim.** If $val(b) = 0$ then

1. $rv(b) \in \overline{K}(S)[rv(e)]$ and $rv(b^*) \in \overline{K}(f(S))[rv(d)]$,
2. $val(b^*) = 0$.

**Proof.** We do induction on $m$. Without loss of generality, we may assume that $b_m, b_0 \neq 0$. First of all, suppose that $val(b_0) \neq val(e \sum_{j=1}^m b_je^{j-1})$. Then either $val(b) = val(b_0) = 0$ and $val(\sum_{j=1}^m b_je^{j-1}) > 0$ or $val(b) = val(\sum_{j=1}^m b_je^{j-1}) = 0$ and $val(b_0) > 0$. In the former case, let $a \in VF(S)$ be such that $val(a) = val(\sum_{j=1}^m b_je^{j-1})$. By the inductive hypothesis, $val(\sum_{j=1}^m f(b_j)a^{d-1}) = 0$ and hence $val(d \sum_{j=1}^m f(b_j)d^{d-1}) > 0$. So $val(b^*) = val(f(b_0)) = 0$ and $rv(b^*) = rv(f(b_0)) \in \overline{K}(f(S))[rv(d)]$. In the latter case, by the inductive hypothesis, we have that $val(d \sum_{j=1}^m f(b_j)d^{d-1}) = 0$ and $rv(\sum_{j=1}^m f(b_j)d^{d-1}) \in \overline{K}(f(S))[rv(d)]$, which immediately imply that $val(b^*) = 0$ and

$$rv(b^*) = rv \left( d \sum_{j=1}^m f(b_j)d^{d-1} \right) \in \overline{K}(f(S))[rv(d)].$$

Similarly, for $rv(b)$, since either $rv(b) = rv(b_0)$ or $rv(b) = rv(e \sum_{j=1}^m b_je^{j-1})$, clearly $rv(b)$ is of the desired form. Next, if $val(b_0) = val(e \sum_{j=1}^m b_je^{j-1}) < 0$ then, since $val(b/b_0) > 0$, we have that $val(e \sum_{j=1}^m b_je^{j-1}/b_0 + 1) > 0$ and hence

$$rv(e)rv \left( \sum_{j=1}^m \frac{b_je^{j-1}}{b_0} \right) + 1 = 0.$$ 

By the inductive hypothesis, $rv(\sum_{j=1}^m b_je^{j-1}/b_0) \in \overline{K}(S)[rv(e)]$. So the equality implies that $rv(e)$ is algebraic over $\overline{K}(S)$, contradiction. Now the only possibility left is that $val(b_0) = val(e \sum_{j=1}^m b_je^{j-1}) = 0$. In this case,

$$rv(b) = rv(e)rv \left( \sum_{j=1}^m b_je^{j-1} \right) + rv(b_0) \in \overline{K}(S)[rv(e)]$$
by the inductive hypothesis. For the second item, since \( \text{val}(\sum_{j=1}^{m} f(b_j)d^{j-1}) = 0 \) and \( \text{rv}(\sum_{j=1}^{m} f(b_j)d^{j-1}) \in K(f(S))[\text{rv}(d)] \), if \( \text{val}(b^*) > 0 \) then
\[
\text{rv}(d) \text{rv} \left( \sum_{j=1}^{m} f(b_j)d^{j-1} \right) + \text{rv}(f(b_0)) = 0
\]
and hence \( \text{rv}(d) \) is algebraic over \( K(f(S)) \), contradiction. So \( \text{val}(b^*) = 0 \) and hence
\[
\text{rv}(b^*) = \text{rv}(d) \text{rv} \left( \sum_{j=1}^{m} f(b_j)d^{j-1} \right) + \text{rv}(f(b_0)) \in K(f(S))[\text{rv}(d)].
\]

Note that, symmetrically, the claim still holds if \( b \) is replaced by \( b^* \). It follows that the embedding of the field \( VF((S, e)) \) into the field \( VF(M_2) \) determined by \( e \mapsto d \) induces an \( \mathcal{L}_v \)-monomorphism \( f^* : (S, e) \to M_2 \) extending \( f \). As in the previous lemmas, \( f^* \) may be identified as an \( L_{RV} \)-monomorphism. Since \( f^*(\text{rv}(e)) = f_1(\text{rv}(e)) \) and, by the claim, \( \text{RV}((S, e)) = \text{RV}((\text{RV}(S), \text{rv}(e))) \subseteq \text{RV}(S_1) \), we must have that \( f^* \upharpoonright \text{RV}((S, e)) = f_1 \upharpoonright \text{RV}((S, e)) \).

**Lemma 3.8.** Suppose that \( e \) is transcendental over \( VF(S) \) and \( \text{val}(e) \) is of infinite order modulo \( \Gamma(S) \). Then for any \( b = \sum_{0 \leq i \leq m} b_i e^i \in VF((S, e)) \), where \( b_i \in VF(S) \), if \( b \neq 0 \) then \( \text{val}(b) = \min \{ \text{val}(b_i e^i) : 0 \leq i \leq m \} \). Also, \( \Gamma((S, e)) \) is the direct sum of \( \Gamma(S) \) and the cyclic group generated by \( \text{val}(e) \): \( \Gamma((S, e)) = \Gamma(S) \oplus (\mathbb{Z} \cdot \text{val}(e)) \).

**Proof.** This is well-known; see, for example, [17] Lemma 4.8.

**Lemma 3.9.** If \( K(S) = K(S_1) \) and \( \Gamma(S) \) is divisible, then Claim (\( \star \)) holds.

**Proof.** Note that, by the assumption, \( e \notin U(M_1) \), \( K(S) \) is algebraically closed, and \( \text{val}(e) \notin \Gamma(S) \). Since \( \Gamma(S) \) is divisible, clearly \( \text{val}(e) \) is of infinite order modulo \( \Gamma(S) \) and hence \( e \) is transcendental over \( VF(S) \). Choose a \( d \in VF(M_2) \) with \( \text{rv}(d) = f_1(\text{rv}(e)) \). Then \( d \) is transcendental over \( VF(S) \). It is not hard to see that, by Lemma 3.8, the embedding of the field \( VF((S, e)) \) into the field \( VF(M_2) \) determined by \( e \mapsto d \) induces an \( \mathcal{L}_v \)-monomorphism \( f^* : (S, e) \to M_2 \) extending \( f \), which, as above, is identified as an \( L_{RV} \)-monomorphism \( f^* : (S, e) \to M_2 \) extending \( f \). Now, since the rational rank of \( \Gamma((S, e))/\Gamma(S) \) is nonzero and \( K(S) \) is algebraically closed, by the dimension inequality of valuation theory, we have that \( K((S, e)) = K(S) \). By Lemma 3.8 again, \( \Gamma((S, e)) = \Gamma(S) \oplus (\mathbb{Z} \cdot \text{val}(e)) \), that is, \( \Gamma((S, e)) = \Gamma((\Gamma(S), \text{val}(e))) \). So \( \text{RV}((S, e)) = \text{RV}((\text{RV}(S), \text{rv}(e))) \subseteq \text{RV}(S_1) \) and hence \( f^* \upharpoonright \text{RV}((S, e)) = f_1 \upharpoonright \text{RV}((S, e)) \).

**Proposition 3.10.** There is a monomorphism \( f^*_1 : M_1 \to M_2 \) extending \( f_1 \).

**Proof.** First of all, since the henselization \( L \) of \( (VF(S_1), O(S_1)) \) in \( M_1 \) is an immediate extension (in the sense of valuation theory), we have that \( \text{RV}((L, S_1)) = \text{RV}(S_1) \). So we may assume that \( f_1 \) is a monomorphism from \( (L, S_1) \) into \( M_2 \). Now we use Lemma 3.5 to extend \( f_1 \upharpoonright L \) to \( f_2 : S_2 \to M_2 \) by adding all the elements in \( K(S_1) \) that are algebraic over \( K(L) \). Manifestly \( K(S_2) \) is algebraically closed. Then, starting with the least \( n \) such that there is a \( \gamma \in \Gamma(S_2) \) that is not divisible by \( n \), we use Lemma 3.6 to extend \( f_2 \) to \( f_3 : S_3 \to M_2 \) such that \( \Gamma(S_3) \) is divisible. Note that, by the proof of Lemma 3.6, \( K(S_3) = K(S_2) \). Next, we use Lemma 3.7 to extend \( f_3 \) to \( f_4 : S_4 \to M_2 \) by adding an element in \( K(S_1) \) that is transcendental.
over $\overline{K}(S_3)$. Iterating this procedure we may exhaust all elements in $\overline{K}(S_1)$ and hence obtain a monomorphism $f_5 : S_5 \to M_2$ such that $S_5$ satisfies the assumption of Lemma 3.9. Then, a combined application of henzelization, Lemma 3.3, and Lemma 3.9 eventually brings a monomorphism $f^* : S^* \to M_2$ such that $f_1 \subseteq f^*$ and $S^*$ is VF-generated. In this case, the proposition follows from Remark 2.4 and Fact 3.1. 

This proposition and Fact 3.1 immediately yields:

**Theorem 3.11.** The theory ACVF admits quantifier elimination.

4. Basic structural properties

From this section forth the background assumption is resumed: we work in a monster model $\mathcal{C}$ of ACVF$_S^{0}$, where $S$ is a fixed “small” substructure of $\mathcal{C}$.

Although its proof only involves elementary calculations, the following simple lemma is vital to the inductive arguments below. Its failure when $\text{char } \overline{K} > 0$ is one of the major obstacles for generalizing the Hrushovski-Kazhdan integration theory to valued fields of positive residue characteristics.

**Lemma 4.1.** Let $c_1, \ldots, c_k \in \text{VF}$ be distinct elements of the same value $\alpha$ such that their average is 0. Then for some $c_i \neq c_j$ we have that $\text{val}(c_i - c_j) = \alpha$ and hence $\text{rv}$ is not constant on the set $\{c_1, \ldots, c_k\}$.

**Proof.** Suppose for contradiction that $\text{val}(c_i - c_j) > \alpha$ for all $c_i \neq c_j \in A$. Since $c_1 = -(c_2 + \ldots + c_k)$ and $\text{char } \overline{K} = 0$, we have that

$$\alpha = \text{val}(kc_1) = \text{val}((k-1)c_1 - (c_2 + \ldots + c_k)) = \text{val} \left( \sum_{i=2}^{k} (c_1 - c_i) \right) > \alpha,$$

contradiction. 

**Definition 4.2.** Let $A$ be a definable subset of VF$^m$. A definable auxiliary projection of $A$ is a definable function of $A$ of the form

$$(x_1, \ldots, x_m) \mapsto (\text{rv}(g_1), \ldots, \text{rv}(g_k)),$$

where each $g_i : A \to \text{VF}$ is a definable function.

**Lemma 4.3.** Let $A$ be a definable finite subset of VF$^m$. Then there is a definable injective auxiliary projection of $A$.

**Proof.** We do double induction on $n$ and the number $k$ of elements in $A$. For $n = 1$, let $A = \{c_1, \ldots, c_k\} \subseteq \text{VF}$. Let $c = (\sum_{i=1}^{k} c_i)/k$ be the average of $A$. Then there is a definable bijective function from $A$ onto $\{c_1 - c, \ldots, c_k - c\}$. So we may assume that the average of $A$ is 0. Since the set $\text{val}(A)$ is finite, for each $\gamma \in \text{val}(A)$, the set $A \cap \text{val}^{-1}(\gamma)$ is definable. So by the inductive hypothesis we may also assume that $\text{val}$ is constant on $A$; say, $\text{val}(c_1) = \alpha$ for all $c_i \in A$. By Lemma 4.1, $\text{rv}$ is not constant on $A$, that is, $1 < |\text{rv}(A)| \leq k$. So $1 \leq |\text{rv}^{-1}(t) \cap A| < k$ for each $t \in \text{rv}(A)$. By the inductive hypothesis there is a $(t)$-definable injective auxiliary projection $f_t$ of $\text{rv}^{-1}(t) \cap A$ for each $t \in \text{rv}(A)$. It is easy to see that for each $f_t$ there is a definable rv-function $f^*_t$ on a subset of $A$ such that $f^*_t(c_i) = (t, f_t(c_i))$ for each $c_i \in \text{rv}^{-1}(t) \subseteq \text{dom}(f_t)$.

Also, the collection of rv-functions $f$ of $A$ with $\text{ran}(f) \subseteq \text{RV}^m$ for some $m$ is rv-closed. Applying Lemma 2.1, we obtain a definable injective auxiliary projection of $A$. 


Now suppose that \( n > 1 \). By the inductive hypothesis, there is a definable injective auxiliary projection \( g \) of \( \text{pr}_n(A) \) and, for each \( c \in \text{pr}_n(A) \), a \( c \)-definable injective auxiliary projection \( f_c \) of \( \text{fib}(A, c) \). As above, for each \( f_c \),

1. there is a definable \((g \circ \text{pr}_n)\)-function \( f_c^* \) on a subset of \( A \) such that \( f_c^*(a_i) = ((g \circ \text{pr}_n)(a_i), f_c(a_i)) \) for each \( a_i \in \text{fib}(A, c) \),

2. the collection of \((g \circ \text{pr}_n)\)-functions \( f \) with \( \text{ran}(f) \subseteq \text{RV}^m \) for some \( m \) is \((g \circ \text{pr}_n)\)-closed.

Applying Lemma 2.11 we obtain a definable injective auxiliary projection of \( A \). \( \square \)

Note that this proof has nothing to do with algebraic closedness and hence works for the theory of valued fields as naturally formulated in \( \mathcal{L}_{\text{RV}} \).

The role of balls in a motivic measure on a valued field is similar to that of intervals in the Lebesgue measure on the real line. We begin the study of balls with a list of easily seen properties.

**Remark 4.4.** Let \( a \) be an open ball and \( b \) a ball.

1. For any \( c \in \text{VF} \), the subset \( a - c = \{ a - c : a \in a \} \) is an open ball. If \( c \in a \) then \( \text{vcr}(a - c) = \infty \) and \( \text{rad}(a - c) = \text{rad}(a) \) and \( a - c \) is a union of rv-balls. If \( c \notin a \) and \( \text{val}(c) \leq \text{rad}(a) \) then \( \text{vcr}(a - c) \leq \text{rad}(a - c) = \text{rad}(a) \). If \( c \notin a \) and \( \text{val}(c) > \text{rad}(a) \) then \( a - c = a \).

2. \( 0 \notin a \) if and only if \( a \) is contained in an rv-ball if and only if \( \text{vcr}(a) \neq \infty \) if and only if \( \text{rad}(a) \geq \text{vcr}(a) \).

3. The average of finitely many elements in \( a \) is in \( a \), which fails if \( \text{char}(K) > 0 \).

4. For any \( c_1, c_2 \in \text{VF} \), \( (a - c_1) \cap (a - c_2) \neq \emptyset \) if and only if \( a - c_1 = a - c_2 \) if and only if \( \text{val}(c_1 - c_2) > \text{rad}(a) \).

5. If \( a \cap b = \emptyset \) then \( \text{val}(a - b) = \text{val}(a' - b') \) for all \( a, a' \in a \) and \( b, b' \in b \). The subset \( a - b = \{ a - b : a \in a \) and \( b \in b \} \) is a ball that does not contain 0. In fact, for any \( a \in a \) and \( b \in b \), either \( a - b = a - b \) or \( a - b = a - b \).

6. Suppose that \( a \cap b = \emptyset \). Let \( c \) be the smallest closed ball that contains \( a \). Clearly \( \text{vcr}(c) = \text{vcr}(a) \) and \( \text{rad}(c) = \text{rad}(a) \). If \( b \) is a maximal open subball of \( c \), that is, if \( b \) is an open ball contained in \( c \) with \( \text{rad}(b) = \text{rad}(c) \), then \( a - b \) is an rv-ball \( \text{rv}^{-1}(t) \) with \( \text{val}(t) = \text{rad}(a) \). This means that the collection of maximal open subballs of \( c \) admits a \( K \)-affine structure.

7. Let \( f(x) \) be a polynomial with coefficients in \( \text{VF} \) and \( d_1, \ldots, d_n \) the roots of \( f(x) \). Suppose that \( a \) is contained in an rv-ball and does not contain any \( d_i \). Then each \( a - d_i \) is contained in an rv-ball and hence \( f(a) \) is contained in an rv-ball, that is, \( \text{rv}(a) \) is a singleton.

Similar properties are available if \( a \) is a closed ball.

**Definition 4.5.** A subset \( X \) of \( \text{VF} \) is a punctured \((open, closed, rv-) ball if \( X = b \setminus \bigcup_{i=1}^n b_i \), where \( b \) is an \((open, closed, rv-) ball, \( b_1, \ldots, b_n \) are disjoint balls, and \( b_1, \ldots, b_n \subseteq b \). Each \( b_i \) is a hole of \( X \). The radius and the valuative center of \( X \) are those of \( b \). A subset \( Y \) of \( \text{VF} \) is a simplex if it is a finite union of disjoint balls and punctured balls of the same radius and the same valuative center, which are defined to be the radius and the valuative center of \( Y \) and are denoted by \( \text{rad}(Y) \) and \( \text{vcr}(Y) \).

A special kind of simplex is called a thin annulus: it is a punctured closed ball \( b \) with a single hole \( h \) such that \( h \) is a maximal open ball contained in \( b \). For example, an element \( \gamma \in \Gamma \) may be regarded as a thin annulus: it is the punctured closed ball...
with radius $\gamma$ and valuative center $\infty$ and the special maximal open ball containing $0$ removed.

**Remark 4.6.** The theory ACVF$^0$ is $C$-minimal; that is, every parametrically definable subset of VF is a boolean combination of balls. This basically follows from [14] Theorem 4.11 and the easy fact that any subset of VF that is parametrically definable in $\mathcal{L}_v$ is also parametrically definable in the two-sorted language $\mathcal{L}_v$ for valued fields. Hence, for any parametrically definable subset $X$ of VF, there are disjoint balls and punctured balls $a_1, \ldots, a_n$ obtained from a unique set of balls $b_1, \ldots, b_n, b_1, \ldots, b_m$ such that $X = \bigcup_i b_i \setminus \bigcup_j b_j$. If we group $a_1, \ldots, a_n$ by their radii and valuative centers then $X$ may also be regarded as the union of a unique set of disjoint parametrically definable simplexes. Each $b_i$ is a **positive boolean component of $X$** and each $b_j$ is a **negative boolean component of $X$**. It follows that, as imaginary definable subsets, $\Gamma$ is $\sigma$-minimal and the set of maximal open balls contained in a closed ball is strongly minimal.

**Definition 4.7.** Let $b_1, \ldots, b_n$ be the positive boolean components of a subset $X \subseteq VF$. The **positive closure** of $X$ is the set of the minimal closed balls $\{c_1, \ldots, c_m\}$ such that each $c_i$ contains some $b_j$.

Note that, if $X \subseteq VF$ is definable from a set of parameters then its positive closure is definable from the same set of parameters.

Lemma 4.3 is of fundamental importance in the Hrushovski-Kazhdan theory. Other structural properties of functions between or within the two sorts will also be needed below. For example:

**Lemma 4.8.** Let $W$ be a definable subset of $RV^m$ and $f : W \rightarrow VF^n$ a definable function. Then $f(W)$ is finite.

**Proof.** The proof is by induction on $n$. For the base case $n = 1$, suppose for contradiction that $f(W)$ is infinite. By $C$-minimality, $f(W)$ is a union of disjoint balls and punctured balls $b_1, \ldots, b_1$ such that $\text{rad}(b) < \infty$ for some $i$, say $b_i$. Let $\phi$ be a formula that defines $f$. By quantifier elimination, $\phi$ may be assumed to be a disjunction of conjunctions of literals. Since $f(W)$ is infinite, there is at least one disjunct in $\phi$, say $\phi^*$, that does not have an irredundant VF-sort equality as a conjunct. Fix a $b \in b_1$ and a $b \in W$ such that the pair $(\overline{\gamma}, b)$ satisfies $\phi^*$. For any term $\text{rv}(g(x))$ in $\phi^*$, where $g(x) \in VF(\langle \emptyset \rangle)[x]$, and any $d \in VF$, if $\text{val}(d - b)$ is sufficiently large then $\text{rv}(g(b)) = \text{rv}(g(d))$. So there is a $d \in b_1$ such that the pair $(\overline{\gamma}, d)$ also satisfies $\phi^*$, which is a contradiction as $f$ is a function. In general, if $n > 1$, by the inductive hypothesis both $(\text{pr}_1 \circ f)(W)$ and $(\text{pr}_{>1} \circ f)(W)$ are finite, hence $f(W)$ is finite. \qed

**Lemma 4.9.** Let $b \subseteq VF$ be a ball such that $b \cap VF(\text{acl}(\emptyset)) = \emptyset$. For any definable function $f : X \rightarrow RV^n$ with $b \subseteq X$, $f \mid b$ is constant.

**Proof.** Clearly it is enough to show the case $n = 1$. Let $\phi$ be a quantifier-free formula in disjunctive normal form that determines $f \mid b$. We may assume that no disjunct of $\phi$ is redundant and hence $\phi$ does not contain any VF-sort literal. For any term $\text{rv}(g(x))$ in $\phi$, where $g(x) \in VF(\langle \emptyset \rangle)[x]$, and any root $b$ of $g(x)$, since $b \in VF(\text{acl}(\emptyset))$, we have that $b \notin b$ and hence there is a $t \in RV$ such that $b - b \subseteq \text{rv}^{-1}(t)$. So $\text{rv}(g(a_1)) = \text{rv}(g(a_2))$ for all $a_1, a_2 \in b$. It follows that $|f(b)| = 1$. \qed
Lemma 4.10. Let \( f : X \to Y \) be a definable surjective function, where \( X, Y \subseteq VF \). Then there is a definable function \( P : X \to RV^m \) such that, for each \( \bar{F} \in \text{ran} P \), \( f \upharpoonright P^{-1}(\bar{F}) \) is either constant or injective.

Proof. Let \( Y_1, Y_2 \) be a partition of \( Y \) given by Lemma \ref{lem:partition}. By Lemma \ref{lem:injective}, there is an injective function from \( Y_1 \) into \( RV^l \) for some \( l \). The same holds for every \( f^{-1}(b) \) with \( b \in Y_2 \). So the lemma follows from compactness. \( \square \)

Definition 4.11. Let \( \mathcal{B} \) be a finite definable set of (open, closed, rv-) balls \( b_1, \ldots, b_n \). We call \( \mathcal{B} \) an algebraic set of balls, \( \bigcup \mathcal{B} \) an algebraic union of balls, and each \( b_i \) an algebraic (open, closed, rv-) ball. If there is a definable subset \( C \) of \( \bigcup \mathcal{B} \) and a definable surjective function \( f : \mathcal{B} \to C \) such that \( f(b_i) \in b_i \) for every \( b_i \in \mathcal{B} \) then we say that \( \mathcal{B} \) has definable centers and \( C \) is a definable set of centers of \( \mathcal{B} \).

It is not hard to see that, if \( S \) is VF-generated and \( X \) is a \( \gamma \)-definable subset of \( VF^n \), then \( X \) is \( \gamma \)-definable in the two-sorted language \( L_V \).

Lemma 4.12. Suppose that \( S \) is VF-generated and \( \gamma \in \Gamma \). Let \( X \) be a \( \gamma \)-algebraic union of disjoint balls \( b_1, \ldots, b_n \). Then there is a disjunction of VF-sort equalities \( \bigvee_j F_j(x) = 0 \), where \( F_j(x) \in VF(\emptyset)[x] \), such that \( (\bigvee_j F_j(VF) = 0) \cap b_i \neq \emptyset \) for each \( b_i \).

Proof. Without loss of generality we may assume that \( \text{rad} b_i < \infty \) and \( 0 \notin b_i \) for each \( b_i \), that is, each \( b_i \) is an infinite subset and is contained in an rv-ball. Let \( \phi \) be an \( L_V \)-formula such that \( \phi(\gamma, \gamma) \) defines \( X \), where \( \gamma \in VF(\emptyset) \). By Theorem \ref{thm:algebraic}, we may assume that \( \phi \) is quantifier-free and is written in disjunctive normal form. If \( \phi \) does not contain any \( \Gamma \)-sort literal then each disjunct of \( \phi \) must contain a VF-sort equality. In this case the lemma is clear. So let us assume that some disjunct of \( \phi \) contains an irredundant \( \Gamma \)-sort literal and also lacks VF-sort equality. Let \( \Gamma_\gamma \) be the substructure of \( \Gamma \) generated by \( \gamma \). Each \( \Gamma \)-sort literal in \( \phi \) is of the form

\[
\text{val} F(x) \square \text{val} G(x) + \xi,
\]

where \( F(x), G(x) \in VF(\emptyset)[x] \), \( \xi \in \Gamma_\gamma \), and \( \square \) is one of the symbols \( =, \neq, \leq, \) and \( > \). Let \( F_j(x) \) enumerate all polynomials in \( VF(\emptyset)[x] \) that occur in the literals in \( \phi \).

We claim that \( \bigvee_j F_j(x) = 0 \) is as required. Suppose for contradiction that this is not the case, say \( (\bigvee_j F_j(VF) = 0) \cap b_1 = \emptyset \). Let \( R_j \) be the set of the roots of \( F_j(x) \). For each \( r \in \bigcup R_j \), since \( r \notin b_1 \), we have that

\[
\begin{align*}
\text{vcr}(b_1 - r) &< \text{rad} b_1 \leq \infty, & \text{if } b_1 \text{ is a closed ball}; \\
\text{vcr}(b_1 - r) &\leq \text{rad} b_1 < \infty, & \text{if } b_1 \text{ is an open ball}.
\end{align*}
\]

So there is a \( d \in VF \setminus X \) such that

\[
\begin{align*}
(1) \quad \text{val}(d) &= \text{vcr}(b_1), \\
(2) \quad \max \{ \text{vcr}(b_1 - r) : r \in \{0\} \cup \bigcup R_j \} &\leq \text{vcr}(b_1 - d) \leq \text{rad} b_1, \\
(3) \quad \text{vcr}(b_1 - r) &= \text{val}(d - r) \text{ for each } r \in \bigcup R_j, \\
(4) \quad d &\text{ satisfies all VF-sort disequalities in } \phi.
\end{align*}
\]

Since \( b_1 \) is an infinite subset, there is a \( b \in b_1 \) such that \( b \) satisfies a disjunct \( \phi' \) of \( \phi \) and \( \phi' \) lacks VF-sort equality. Then \( d \) also satisfies \( \phi' \), contradiction. \( \square \)
Corollary 4.13. Suppose that $S$ is VF-generated. If $\Gamma(\text{acl}(\emptyset))$ is nontrivial then $\text{acl}(\emptyset)$ is a model of $\text{ACVF}_S$.

Lemma 4.14. Suppose that $S$ is VF-generated. Let $\bar{\gamma} \in \Gamma$ and $\mathfrak{B}$ a $\bar{\gamma}$-algebraic set of balls $b_1, \ldots, b_n$. Then $\mathfrak{B}$ has $\bar{\gamma}$-definable centers.

Proof. The set $\mathfrak{B}$ may be partitioned into subsets $\mathfrak{B}_1, \ldots, \mathfrak{B}_m \subseteq \mathfrak{B}$ such that each $\mathfrak{B}_i$ is an $\bar{\gamma}$-algebraic set of disjoint balls. So without loss of generality we may assume that $\mathfrak{B}$ is a set of disjoint balls. By Lemma 4.12 there is an algebraic subset $C$ of VF such that $C \cap b_i \neq \emptyset$ for every $i$. So the set $\mathfrak{B}$ gives rise to a partition of $C$ and the set of the averages of the parts of this partition is $\bar{\gamma}$-definable. Since $\text{char}(K) = 0$, the corresponding average remains in each $b_i$. □

Lemma 4.15. If $\mathfrak{B}$ is a parametrically definable infinite set of closed balls then there is a parametrically definable map of $\mathfrak{B}$ onto a proper interval of $\Gamma$.

Proof. Since $\Gamma$ is $\sigma$-minimal, any parametrically definable infinite subset of $\Gamma$ contains an interval. Therefore it suffices to show that there is a parametrically definable map of $\mathfrak{B}$ into $\Gamma$ whose image is infinite. If either the subset $\{\text{rad} b : b \in \mathfrak{B}\}$ is infinite or the subset $\{\text{vcr} b : b \in \mathfrak{B}\}$ is infinite then clearly such a map exists. So, without loss of generality, we may assume that both rad and vcr are constant on $\mathfrak{B}$. Since $\mathfrak{B}$ is infinite, obviously $\text{vcr} \mathfrak{B} \neq \emptyset$. Now, by $C$-minimality, the subset $\text{pr}_1 \mathfrak{B}$ is a finite union of disjoint balls $b_1, \ldots, b_n$, some of which may be punctured. Clearly $\text{vcr} b_i = \text{vcr} \mathfrak{B}$ for every $b_i$. Since every $b \in \mathfrak{B}$ is closed and $\mathfrak{B}$ is infinite, we must have that $\text{rad} b > \text{rad} b_i$ for some $b_i$, say $b_1$. Choose a $c \in b_1$ such that the open ball $\{x \in \text{VF} : \text{val}(x - c) > \text{rad} b_1\}$ is contained in $b_1$. Clearly the subset

$$\{\text{vcr}(b - c) : b \in \mathfrak{B} \text{ and } b \subseteq b_1\}$$

is infinite. Hence the parametrically definable map of $\mathfrak{B}$ given by $b \mapsto \text{vcr}(b - c)$ is as desired. □

Lemma 4.16. Suppose that $S$ is $(\text{VF}, \Gamma)$-generated. Let $\bar{t} = (t_1, \ldots, t_n) \in \text{RV}$ and $\mathfrak{B}$ a $\bar{\gamma}$-algebraic set of closed balls. Then $\mathfrak{B}$ has $\bar{\gamma}$-definable centers.

Proof. The proof is by induction on $n$. The base case $n = 0$ is covered by Lemma 4.14. We proceed to the inductive step. First note that for any $\gamma \in \Gamma$ the subset $A_\gamma = \{t \in \text{RV} : \text{vrv}(t) = \gamma\}$ is strongly minimal. Let $\phi$ be a formula that defines $\mathfrak{B}$. Let $\text{vrv}(t_1) = \gamma_1$. For any $\bar{s} = (s_1, t_2, \ldots, t_{n+1})$ with $\text{vrv}(s_1) = \gamma_1$, let $W_\bar{s} \subseteq \text{VF}^\mathfrak{B}$ be the subset defined by $\phi(\bar{s})$. Let $\mathfrak{B}_{\bar{s}} = W_\bar{s}$ if $W_\bar{s}$ is a finite set of closed balls; otherwise $\mathfrak{B}_{\bar{s}} = \emptyset$. Consider the set of closed balls $D = \bigcup \mathfrak{B}_{\bar{s}}$, which contains $\mathfrak{B}$, and the subset

$$D = \bigcup \{\{s\} \times \mathfrak{B}_{\bar{s}} : s \in \gamma_1 \times \{(t_2, \ldots, t_{n+1})\}\},$$

both of which are $(\gamma_1, t_2, \ldots, t_{n+1})$-definable. We claim that $D$ is finite. Suppose for contradiction that $D$ is infinite. Since any two disjoint parametrically definable infinite subsets of $D$ would give rise to two disjoint parametrically definable infinite subsets of $A_{\gamma_1}$, which is a contradiction as $A_{\gamma_1}$ is strongly minimal, we deduce that $D$ is strongly minimal. By Lemma 4.15 there is a parametrically definable map of $D$ onto an interval of $\Gamma$, which must be strongly minimal as well. However, the ordering of $\Gamma$ is linear and dense, and hence no interval of $\Gamma$ is strongly minimal, contradiction. So $D$ is finite. Applying the inductive hypothesis with respect to the
substructure \( \langle \gamma_1 \rangle \) and the tuple \((t_2, \ldots, t_{n+1})\), we conclude that \( \mathfrak{B} \) has \( T \)-definable centers.

**Lemma 4.17.** For any \( t \in RV \), if \( rv^{-1}(t) \) has a definable proper subset then it has definable center.

**Proof.** Let \( X \) be a definable proper subset of \( rv^{-1}(t) \). Let \( b_1, \ldots, b_n \) be the positive boolean components of \( X \) and \( h_1, \ldots, h_m \) the negative boolean components of \( X \). Since \( X \) is a proper subset of \( rv^{-1}(s) \), at least one of these balls is a proper subball of \( rv^{-1}(s) \) and hence its positive closure is also a proper subball of \( rv^{-1}(s) \). Then, by Lemma 5.16 there is a definable finite subset of \( rv^{-1}(s) \) and hence, by taking the average, a definable point in \( rv^{-1}(s) \). \( \square \)

5. **Parametric balls and atomic subsets**

In this section let \( Q \) be a set of parameters that consists of balls of radius \( < \infty \). Without loss of generality, we may assume that no ball in \( Q \) is definable.

**Definition 5.1.** A subset \( X \) generates a complete \( Q \)-type if for all \( Q \)-definable subset \( Y \) either \( X \subseteq Y \) or \( X \cap Y = \emptyset \). An \( Q \)-definable subset \( X \) is atomic over \( \langle Q \rangle \) if it generates a complete \( Q \)-type.

**Lemma 5.2.** Let \( T \) be an \( Q \)-definable set of balls and \( \phi \) a formula such that, for all \( t_1 \neq t_2 \in T \), \( \phi(t_1) \) and \( \phi(t_2) \) define two disjoint balls \( b_{t_1} \) and \( b_{t_2} \). For each \( t \in T \), if \( b_t \) is not \( Q \)-algebraic then it is atomic over \( \langle Q, t \rangle \).

**Proof.** Suppose for contradiction that there is a non-\( Q \)-algebraic \( b_s \) and a formula \( \psi \) such that \( \psi(s) \) defines a proper subset of \( b_s \). For each \( t \in T \), let \( X_t \) be the set defined by \( \psi(t) \) if it is a proper subset of \( b_t \) and \( X_t = \emptyset \) otherwise. Set \( X = \bigcup_{t \in T} X_t \), which is \( Q \)-definable. By \( C \)-minimality, \( X \) is a boolean combination of some balls \( \delta_1, \ldots, \delta_n \). Since the balls \( b_t \) are pairwise disjoint, there are only finitely many balls \( b_t \) that contain some \( \delta_i \). Note that this finite collection of balls is \( Q \)-definable, which does not contain \( b_s \), since \( b_s \) is not \( Q \)-algebraic. On the other hand, since \( b_s \cap X \neq \emptyset \), we must have that \( b_s \subseteq X \). This is a contradiction because the balls \( b_t \) being pairwise disjoint implies that \( b_s \cap X \) is a proper subset of \( b_s \). \( \square \)

**Lemma 5.3.** Let \( X \subseteq VF^n \times RV^m \) be atomic over \( \langle Q \rangle \) and \( \overline{\tau} \in \Gamma \). Then \( X \) is atomic over \( \langle Q, \overline{\tau} \rangle \).

**Proof.** By induction this is immediately reduced to the case that the length of \( \overline{\tau} \) is 1. Suppose for contradiction that there is a formula \( \psi(\gamma) \) that defines a proper subset of \( X \). Then the subset

\[ \Delta = \{ \gamma \in \Gamma : \psi(\gamma) \text{ defines a proper subset of } X \} \]

is nonempty and is \( Q \)-definable. By \( o \)-minimality, some \( \alpha \in \Delta \) is \( Q \)-definable, contradicting the assumption that \( X \) is atomic over \( \langle Q \rangle \). \( \square \)

**Definition 5.4.** Let \( b_1 \) and \( b_2 \) be two (punctured) balls. We say that they are of the same **haecceitistic type** if

1. \( \text{rad}(b_1) = \text{rad}(b_2) \) and \( \text{vcr}(b_1) = \text{vcr}(b_2) \),
2. they are both open balls or both closed balls or both thin annuli.

**Lemma 5.5.** Let \( X \subseteq VF \) be atomic over \( \langle Q \rangle \). Then \( X \) is the union of disjoint balls \( b_1, \ldots, b_n \) of the same haecceitistic type.
Lemma 5.8. Then $\text{on } b$ number of holes. If $\langle b_1, \ldots, b_n \rangle$, because otherwise there would be an $Q$-definable proper subset of $X$ according to $\min \{ \text{vcr}(b_1), \ldots, \text{vcr}(b_n) \}$ or $\min \{ \text{rad}(b_1), \ldots, \text{rad}(b_n) \}$. Similarly either $b_1, \ldots, b_n$ are all closed balls or are all open balls. Also, since the subset of $X$ that contains exactly every unpunctured ball $b_i$ is definable, we have that either $b_1, \ldots, b_n$ are all punctured or are all unpunctured.

So it is enough to show that if $b_i$ is punctured then it must be a thin annulus. By atomicity again, if $b_1, \ldots, b_n$ are punctured then each $b_1$ must contain the same number of holes. If $b_i$ has a hole $h_j$ with $\text{rad}(h) < \text{rad}(h_i)$ then $b_i \setminus h^*$ is nonempty, where $h^*$ is the closed ball that has radius $(\text{rad}(h_i) + \text{rad}(h))/2$ and contains $h$. The collection of all such holes $h_1, \ldots, h_m$ is $Q$-definable and hence, if it is not empty, then there would be a proper subset of $X$ that is $Q$-defined by replacing each $h_j$ with $h_j^*$. So each hole in each $b_i$ is a maximal open ball in $b_i$. Suppose for contradiction that $b_i$ contains more than one holes $h_1, \ldots, h_m$. Without loss of generality we may assume that $0 \notin b_1$. Since the subset $b_2 - b_1$ is an $rv$-ball and

$$1 \cdot (h_2 - h_1), \ldots, (m + 1) \cdot (h_m - h_1)$$

are distinct $rv$-balls, for some $1 \leq k \leq m + 1$ we have that $h_1 + k \cdot (h_m - h_1)$ is a maximal open ball in $b_1$ and is disjoint from $\bigcup h_i$. This means that there is a finite $Q$-definable set of maximal open balls in $b_1, \ldots, b_n$ that strictly contains the set of holes in $b_1, \ldots, b_n$. This readily implies that $X$ has a nonempty proper $Q$-definable subset, contradiction.

Note that, in the above lemma, if $Q = \emptyset$ then $X$ cannot be a disjoint union of closed balls of radius $< \infty$, because in that case, by Lemma 4.1, the closed balls would have definable centers. Now, if $X \subseteq VF$ is atomic over $\langle Q \rangle$ then the radius and the valuative center of $X$ are well-defined quantities: they are respectively the radius and the valuative center of the balls $b_1, \ldots, b_n$ in the above lemma. These are also denoted by $\text{rad}(X)$ and $\text{vcr}(X)$. The balls $b_1, \ldots, b_n$ are called the haecceitistic components of $X$.

Corollary 5.6. If $X \subseteq VF$ is atomic over $\langle Q \rangle$ and $b \subseteq X$ is an open (closed) ball then every $a \in X$ is contained in an open (closed) ball $d_a \subseteq X$ with $\text{rad}(d_a) = \text{rad}(b)$.

Lemma 5.7. Let $X \subseteq VF$ be atomic over $\langle Q \rangle$ and $f : X \to VF$ an $Q$-definable injective function. If $X$ has only one haecceitistic component then $f(X)$ also has only one haecceitistic component.

Proof. Let $b_1, \ldots, b_n$ be the haecceitistic components of $f(X)$ given by Lemma 5.5. Suppose that $X$ is an open ball or a closed ball or a thin annulus. Suppose for contradiction that $n > 1$. Then there is exactly one of the components $b_1, \ldots, b_n$, say $b_1$, such that $f^{-1}(b_1)$ contains the punctured ball $X \setminus \bigcup h_j$ for some holes $h_j$. Consequently, since $\text{rad}(f(X))$ is $Q$-definable, the ball $b_1$ and $f^{-1}(b_1)$ are $Q$-definable, contradicting the assumption that $X$ is atomic.

Lemma 5.8. Let $X \subseteq VF$ be atomic over $\langle Q \rangle$ with haecceitistic components $b_1, \ldots, b_n$ and $b$ an open (closed) ball properly contained in some $b_i$. Set $\gamma = \text{rad}(b)$. Then $b$ is atomic over $\langle Q, \gamma, b \rangle$. 

Proof. We assume that $b$ is an open ball, since the proof for closed balls is identical. By Lemma 5.3, $X$ is atomic over $\langle Q, \gamma \rangle$. Since the infinite set of pairwise disjoint balls

$$D = \{ d \subseteq X : d \text{ is an open subball of } X \text{ with } \operatorname{rad}(d) = \gamma \}$$

is $\langle Q, \gamma \rangle$-definable and $\bigcup D = X$, clearly no $d \in D$ is $\langle Q, \gamma \rangle$-algebraic. So, by Lemma 5.2, every $d \in D$ is atomic over $\langle Q, \gamma, d \rangle$. □

Lemma 5.9. Let $\sigma$ be an open ball and $I$ a close ball or a thin annulus such that both $\sigma$ and $I$ are atomic over $\langle Q \rangle$. If $X \subseteq \sigma \times I$ is $Q$-definable then the projection $pr_1 \upharpoonright X$ cannot be finite-to-one.

Proof. We assume that $I$ is a closed ball, since the proof for thin annuli is identical. Suppose for contradiction that there is an $Q$-definable $X \subseteq \sigma \times I$ such that the first coordinate projection on $X$ is finite-to-one. Note that, since $\sigma$ and $I$ are atomic, we must have that $pr_1 X = \sigma$ and $pr_2 X = I$. Let $\mathcal{M}$ be the set of maximal open subballs of $I$, which is $Q$-definable. For any $\tau \in \mathcal{M}$, let $A_{\tau} = pr_1((pr_2 \upharpoonright X)^{-1}(\tau))$. By $C$-minimality each $A_{\tau}$ is a boolean combination of balls. In fact, for any $\tau, \eta \in \mathcal{M}$, $A_{\tau}$ and $A_{\eta}$ must have the same number of boolean components, because otherwise there would be an $Q$-definable proper subset of $I$. Let this number be $k$.

For any $\tau \in \mathcal{M}$, suppose that $\mathcal{B} = \{ b_1, \ldots, b_k \}$ is the set of the boolean components of $A_{\tau}$, we let $\lambda_{\tau} = \min \{ \operatorname{rad}(b_1), \ldots, \operatorname{rad}(b_k) \}$. Moreover, for any $b_i, b_j \in \mathcal{B}$, if $b_i \cap b_j \neq \emptyset$ then let $\rho(b_i, b_j) = \min \{ \operatorname{rad}(b_i), \operatorname{rad}(b_j) \}$, otherwise let $\rho(b_i, b_j) = \operatorname{val}(b_i - b_j)$. Let

$$\rho_{\tau} = \min \{ \rho(b_i, b_j) : (b_i, b_j) \in \mathcal{B}^2 \} :$$

Note that the subsets $\Lambda = \{ \lambda_{\tau} : \tau \in \mathcal{M} \} \subseteq \Gamma$ and $\Delta = \{ \rho_{\tau} : \tau \in \mathcal{M} \} \subseteq \Gamma$ are both $Q$-definable. Since $I$ is atomic, we must have that both $\Lambda$ and $\Delta$ are singletons, say $\Lambda = \{ \lambda \}$ and $\Delta = \{ \rho \}$. Also, we claim that $\lambda < \operatorname{rad}(\sigma)$. To see this, suppose for contradiction $\lambda_{\tau} = \operatorname{rad}(\sigma)$ for every $\tau \in \mathcal{M}$. This means that $A_{\tau}$ has $\sigma$ as a positive boolean component for every $\tau \in \mathcal{M}$. Since $\sigma$ is open, we have that for any $n$ and any $\tau_1, \ldots, \tau_n \in \mathcal{M}$ there is an $a \in \bigcap_{\tau \leq n} A_{\tau}$, and hence there is a $b_i \in \tau_i$ for every $i \leq n$ such that $a, b_i \in X$. Therefore, by compactness, there is an $a \in \mathcal{M}$ such that the fiber $\{ b : (a, b) \in X \}$ is infinite, contradicting the assumption that $pr_1 \upharpoonright X$ is finite-to-one.

Now, fix an $\tau \in \mathcal{M}$. Again, since $\sigma$ is open, there is a proper open subball $\gamma$ of $\sigma$ that properly contains $A_{\tau}$. Let $B_{\gamma} = pr_2((pr_1 \upharpoonright X)^{-1}(\tau))$. Since $B_{\gamma}$ properly contains the maximal open subball $\gamma$ of $I$, by $C$-minimality, either $\gamma$ is a boolean component of $B_{\gamma}$ that is disjoint from any other boolean component of $B_{\gamma}$ or $I$ is a positive boolean component of $B_{\gamma}$. However, the former is impossible, because in that case $B_{\gamma}$ could only have finitely many maximal open subballs of $I$ as its positive boolean components and consequently, since $\Lambda = \{ \lambda \}$ is a singleton, $\gamma$ could not be an open ball, contradiction. So we must have that $I$ is a positive boolean component of $B_{\gamma}$. This means that, by $C$-minimality, $B_{\gamma}$ can only have finitely many maximal open subballs of $I$ as its negative boolean components, say $\tau_1, \ldots, \tau_n$. Again, since $\Lambda = \{ \lambda \}$ and $\lambda < \operatorname{rad}(\sigma)$, $\bigcup_{\tau \leq n} A_{\tau}$ must be a proper subset of $\sigma \setminus 3$ and hence there is a $\eta \in \mathcal{M}$ such that $\eta \subseteq B_{\gamma}$ and $A_{\eta}$ has a boolean component contained in $3$ and another boolean component disjoint from $3$. This implies that $\rho_{\eta} \leq \operatorname{rad}(3)$. On the other hand, since $A_{\tau} \subseteq 3$, we have that $\rho_{\tau} > \operatorname{rad}(3)$. This is a contradiction since $\Delta$ is a singleton. □
Lemma 5.10. Let \( q \) be an open ball such that it is atomic over \( \langle q \rangle \). Let \( X \subseteq VF \) be atomic over \( \langle q \rangle \) such that it only has one haecceitic component. If \( X \) is infinite then it is either an open ball or a thin annulus.

Proof. Suppose for contradiction that \( X \) is a closed ball of radius \( < \infty \). Let \( \psi \) be a quantifier-free formula in disjunctive normal form that defines \( X \). Note that, by Lemma 4.16, \( q \) must occur in \( \psi \). Without loss of generality, \( q \) is represented in \( \psi \) by some \( q \in q \). We claim that any disjunct in \( \psi \) that contains a nontrivial VF-sort equality \( f(x) = 0 \) as a conjunct is redundant: if \( q \) does not occur in \( f(x) \) then, since \( X \) is atomic, \( f(a) \neq 0 \) for any \( a \in X \); if \( q \) does occur in \( f(x) \) then we still have that \( f(a) \neq 0 \) for any \( a \in X \), because otherwise there would be an \( q \)-definable \( Y \subseteq q \times X \) with \( pr_1 | Y \) finite-to-one, contradicting Lemma 5.9. Dually, we may also assume that no disjunct in \( \psi \) contains VF-sort disequality. Similarly, for any term \( rv(g(x)) \) in \( \psi \) with \( g(x) \) nonconstant, we have that \( g(a) \neq 0 \) for any \( a \in X \). It is not hard to see that, since all the roots of all the nonconstant polynomials \( g(x) \) in all the terms of the form \( rv(g(x)) \) in \( \psi \) lie outside \( X \) and \( X \) is a ball, there is a \( b \notin X \) such that

\[
rv(g(b)) = rv(g(a_1)) = rv(g(a_2))
\]

for any \( a_1, a_2 \in X \). So \( b \) also satisfies \( \psi \), contradiction. \( \square \)

Lemma 5.11. Let \( X \subseteq VF \) be an open ball atomic over \( \langle Q \rangle \) and \( f : X \to VF \) an \( Q \)-definable injective function. Then \( f(X) \) is also an atomic open ball.

Proof. By Lemma 5.7, \( f(X) \) is an open ball or a closed ball or a thin annulus. Then, according to Lemma 5.9, \( f(X) \) must be an open ball. \( \square \)

Lemma 5.12. Let \( X \subseteq VF \) generate a complete type. Let \( f : VF \to VF \) be a definable function such that \( f \upharpoonright X \) is injective. Then for every open ball \( b \subseteq X \) the image \( f(b) \) is also an open ball.

Proof. Fix an open ball \( b \subseteq X \) and set \( \gamma = \text{rad}(b) \). We claim that \( b \) is not \( \gamma \)-algebraic. To see this, suppose for contradiction that there is a formula \( \psi(\gamma) \) in disjunctive normal form that defines a finite set \( \mathcal{B} = \{ b_1, \ldots, b_n \} \) of balls such that \( b_1 = b \). Without loss of generality we may assume that every \( b_i \) is an open ball of radius \( \gamma \) and, since \( \mathcal{B} \) is finite, \( \bigcup \mathcal{B} \subseteq X \). So all VF-sort literals in \( \psi(\gamma) \) are redundant. For any term \( rv(g(x)) \) in \( \psi \) with \( g(x) \in VF(\langle \emptyset \rangle)[x] \), clearly if \( g(x) \) is not a constant polynomial then \( g(b) \neq 0 \) for any \( b \in b_i \). Since all the roots of all the nonconstant polynomials \( g(x) \) in all the terms of the form \( rv(g(x)) \) in \( \psi(\gamma) \) lie outside \( \bigcup \mathcal{B} \) and \( \mathcal{B} \) is finite, there is an \( a \notin \bigcup \mathcal{B} \) such that

\[
rv(g(a)) = rv(g(b_1)) = rv(g(b_2))
\]

for any \( b_1, b_2 \in b \). Therefore \( a \) also satisfies \( \psi(\gamma) \), contradiction.

Now, since \( b \) is not \( \gamma \)-algebraic, by Lemma 5.2, \( b \) is atomic over \( \langle \gamma, b \rangle \) and hence, by Lemma 5.11, \( f(b) \) is an open ball. \( \square \)

Proposition 5.13. Let \( X, Y \subseteq VF \) be definable and \( f : X \to Y \) a definable bijection. Then there are definable disjoint subsets \( X_1, \ldots, X_n \subseteq X \) with \( \bigcup X_i = X \) such that, for any open balls \( a \in X_i \) and \( b \in f(X_i) \), both \( f(a) \) and \( f^{-1}(b) \) are open balls.

Proof. For every \( a \in X \) let \( Z_a \subseteq X \) be the intersection of all definable subsets of \( X \) that contains \( a \). So \( Z_a \) generates a complete type. By Lemma 5.12, for every
open ball \( a \subseteq Z_a \), the image \( f(a) \) is an open ball. This open-to-open property may be rephrased as follows: for every \( b \in Z_a \) and \( t \in \text{RV} \) let \( \sigma(b,t) \) be the open ball that contains \( b \) and has radius \( \text{rv}(t) \), if \( \sigma(b,t) \subseteq Z_a \) then \( f(\sigma(b,t)) \) is an open ball. Therefore, by compactness, there is a definable subset \( D_a \subseteq X \) containing \( a \) such that \( f \upharpoonright D_a \) has this open-to-open property. By compactness again, there are definable subsets \( X_1, \ldots, X_m \subseteq X \) with \( \bigcup X_i = X \) such that each \( f \upharpoonright X_i \) has this open-to-open property. Similarly there are definable subsets \( Y_1, \ldots, Y_l \subseteq Y \) with \( \bigcup Y_i = Y \) such that each \( f^{-1} \upharpoonright Y_i \) has this open-to-open property. The partition of \( X \) determined by \( X_1, \ldots, X_m \), \( f^{-1}(Y_1), \ldots, f^{-1}(Y_l) \) is as desired. \( \square \)

Let \( X \subseteq \text{VF}^n \times \text{RV}^m \) and \( i \in \{1, \ldots, n\} \). A subset \( Y \subseteq X \) is an open ball contained in \( X[i] \) if \( Y \) is of the form \( b \times \{\pi\} \), where \( b \) is an open ball and \( \pi \in \text{pr}_i X \). Of course, if \( Y \) is an open ball contained in \( X[i] \) and \( \pi_1 X \) is a singleton then we simply say that \( Y \) is an open ball contained in \( X \). The same goes to closed balls, rv-balls, simplexes, etc.

**Definition 5.14.** Let \( X \subseteq \text{VF}^{n_1} \times \text{RV}^{m_1} \subseteq \text{VF}^{n_2} \times \text{RV}^{m_2} \), and \( f : X \rightarrow Y \) a bijection. Let \( i \in \{1, \ldots, n_1\} \) and \( j \in \{1, \ldots, n_2\} \). For any \( \pi \in \text{pr}_i X \) and any \( \overline{b} \in \text{pr}_j Y \), let

\[
\overline{f} = f \upharpoonright (\text{fib}(X, \pi) \cap f^{-1}(\text{fib}(Y, \overline{b}))).
\]

We say that \( f \) has the \((i,j)\)-open-to-open property if, for every \( \pi \in \text{pr}_i X \) and every \( \overline{b} \in \text{pr}_j Y \), \( \overline{f} \) has the open-to-open property described in Proposition 5.13. If \( f \) has the \((i,j)\)-open-to-open property for every \((i,j) \in \{1, \ldots, n_1\} \times \{1, \ldots, n_2\} \) then \( f \) has the open-to-open property.

With this understanding, Proposition 5.13 may be easily generalized as follows.

**Proposition 5.15.** Let \( X \subseteq \text{VF}^{n_1} \times \text{RV}^{m_1} \subseteq \text{VF}^{n_2} \times \text{RV}^{m_2} \) be definable subsets and \( f : X \rightarrow Y \) a definable bijection. Then there are definable disjoint subsets \( X_1, \ldots, X_n \subseteq X \) with \( \bigcup X_i = X \) such that \( f \upharpoonright X_i \) has the open-to-open property for every \( i \).

**Proof.** First observe that if \( f \) has the \((i,j)\)-open-to-open property then, for every subset \( X^* \subseteq X \), \( f \upharpoonright X^* \) has the \((i,j)\)-open-to-open property. Next, by Proposition 5.13 for any \( \overline{\pi} \in \text{pr}_{i > 1} X \) and \( \overline{b} \in \text{pr}_{j > 1} Y \) there is a \((\overline{\pi}, \overline{b})\)-definable finite partition \( V_1, \ldots, V_n \) of \( \text{dom}(f_{\overline{\pi}\overline{b}}) \) such that each \( f_{\overline{\pi}\overline{b}} \upharpoonright V_i \) has the open-to-open property. Since \( V_1, \ldots, V_n \) may be extended into a definable partition \( V_1^*, \ldots, V_m^* \) of \( X \) such that \( V_i^* \cap \text{dom}(f_{\overline{\pi}\overline{b}}) = V_i \) and for any finite collection of partitions \( P_1, \ldots, P_m \) of \( X \) there is a partition \( P \) of \( X \) such that \( P \) is finer than each \( P_i \), by compactness, we obtain a definable partition \( V_1, \ldots, V_{1,n} \) of \( X \) such that each \( f \upharpoonright V_{1,i} \) has the \((1,1)\)-open-to-open property. Iterating this procedure for each \((i,j) \in \{1, \ldots, n_1\} \times \{1, \ldots, n_2\} \) on each piece of the partition obtained in the previous step, we eventually get a partition of \( X \) that is as desired. \( \square \)

### 6. Categories of definable subsets

Motivic integrals will be constructed as homomorphisms between the Grothendieck semigroups (or semirings) of various categories associated with the theory \( \text{ACVF}_S^0 \).
6.1. **Dimensions.** Before we introduce the categories and their Grothendieck groups, two notions of dimension with respect to the two different sorts are needed.

**Definition 6.1.** Let \( X \subseteq \text{VF}^n \times \text{RV}^m \) be a definable subset. The **VF-dimension** of \( X \), denoted as \( \dim_{\text{VF}} X \), is the smallest number \( k \) such that there is a definable finite-to-one function \( f : X \to \text{VF}^k \times \text{RV}^l \).

**Lemma 6.2.** Let \( X \subseteq \text{VF}^n \times \text{RV}^m \) be a definable subset. Then \( \dim_{\text{VF}} X \leq k \) if and only if there is a definable injection \( f : X \to \text{VF}^k \times \text{RV}^l \) for some \( l \).

**Proof.** Suppose that \( \dim_{\text{VF}} X \leq k \). Let \( g : X \to \text{VF}^k \times \text{RV}^l \) be a definable finite-to-one function. For every \( (\overline{a}, \overline{b}) \in g(X) \), since \( g^{-1}(\overline{a}, \overline{b}) \) is finite, by Lemma 6.3 there is an \( (\overline{a}, \overline{b}) \)-definable injection \( h_{\overline{a}, \overline{b}} : g^{-1}(\overline{a}, \overline{b}) \to \text{RV}^j \) for some \( j \). By compactness, there is a definable function \( h : X \to \text{RV}^j \) for some \( j \) such that \( h \upharpoonright g^{-1}(\overline{a}, \overline{b}) \) is injective for every \( (\overline{a}, \overline{b}) \in g(X) \). Then the function \( f \) on \( X \) given by \( (\overline{a}, \overline{b}) \mapsto (h(\overline{a}, \overline{b}), h(\overline{a}, \overline{b})) \) is as desired. The other direction is trivial. \( \square \)

**Lemma 6.3.** Let \( X \subseteq \text{VF}^n \times \text{RV}^m \) be a definable subset and \( f : X \to \text{RV}^l \) a definable function. Then \( \dim_{\text{VF}} X = \max \{ \dim_{\text{VF}} f^{-1}(\overline{b}) : \overline{b} \in \text{ran}(f) \} \).

**Proof.** Let \( \max \{ \dim_{\text{VF}} f^{-1}(\overline{b}) : \overline{b} \in \text{ran}(f) \} = k \). By Lemma 6.2, for every \( \overline{b} \in \text{ran}(f) \), there is a \( \overline{b} \)-definable injection \( h_{\overline{b}} : f^{-1}(\overline{b}) \to \text{VF}^k \times \text{RV}^j \) for some \( j \). By compactness, there is a definable function \( h : X \to \text{VF}^k \times \text{RV}^j \) for some \( j \) such that \( h \upharpoonright f^{-1}(\overline{b}) \) is injective for every \( \overline{b} \in \text{ran}(f) \). Then the function on \( X \) given by \( (\overline{a}, \overline{b}) \mapsto (h(\overline{a}, \overline{b}), f(\overline{a}, \overline{b})) \) is injective and hence \( \dim_{\text{VF}} X \leq k \). The other direction is trivial. \( \square \)

**Lemma 6.4.** Let \( X \subseteq \text{VF}^n \times \text{RV}^m \) be a definable subset. Suppose that there is an \( (\overline{a}, \overline{b}) \in X \) such that the transcendental degree of \( \text{VF}(\langle \overline{a} \rangle) \) over \( \text{VF}(\langle \overline{b} \rangle) \) is \( k \). Then \( \dim_{\text{VF}} X \geq k \).

**Proof.** Suppose for contradiction that the transcendental degree of \( \text{VF}(\langle \overline{a} \rangle) \) over \( \text{VF}(\langle \overline{b} \rangle) \) is \( k \) for some \( (\overline{a}, \overline{b}) \in X \) but \( \dim_{\text{VF}} X = i < k \). By Lemma 6.2, there is a definable bijection \( f : X \to Y \subseteq \text{VF}^i \times \text{RV}^l \) for some \( l \). Let \( f(\overline{a}, \overline{b}) = (\overline{a}, \overline{b}) \). By quantifier elimination, we have that \( \text{VF}(\langle \overline{a} \rangle)^{ac} \subseteq \text{VF}(\langle \overline{b} \rangle)^{ac} \). So the transcendental degree of \( \text{VF}(\langle \overline{a} \rangle) \) over \( \text{VF}(\langle \overline{b} \rangle) \) is at most \( i \), contradiction. \( \square \)

**Corollary 6.5.** Let \( X \subseteq \text{VF}^n \times \text{RV}^m \) be a definable subset that contains a subset of the form \( \{ (0, \ldots, 0) \} \times \text{RV}^{-1}(\overline{b}) \times \{ \overline{s} \} \) for some \( \overline{b} \in (\text{RV}^*)^k \). Then \( \dim_{\text{VF}} X \geq k \).

**Definition 6.5.** Let \( X \subseteq \text{RV}^m \) be a definable subset. The **RV-dimension** of \( X \), denoted as \( \dim_{\text{RV}} X \), is the smallest number \( k \) such that there is a definable finite-to-one function \( f : X \to \text{RV}^k \) (RV\(^0\) is taken to be the singleton \( \{ \infty \} \)).

**Definition 6.6.** Let \( X \subseteq \text{VF}^m \) be a definable subset. The **RV-fiber dimension** of \( X \), denoted as \( \dim_{\text{RV}} f(X) \), is max \( \{ \dim_{\text{RV}}(\text{fib}(X, \overline{a})) : \overline{a} \in \text{pfv} X \} \).

**Definition 6.7.** Let \( X \subseteq \text{VF}^n \times \text{RV}^m \) be a definable subset. The **RV-fiber dimension** of \( X \), denoted as \( \dim_{\text{RV}} f(X) \), is max \( \{ \dim_{\text{RV}}(\text{fib}(X, \overline{a})) : \overline{a} \in \text{pfv} X \} \).

**Lemma 6.8.** Let \( X \subseteq \text{VF}^n \times \text{RV}^m \) be a definable subset and \( f : X \to \text{VF}^n \times \text{RV}^m \) a definable injection. Then \( \dim_{\text{RV}} f(X) \).

**Proof.** Let \( \dim_{\text{RV}} f(X) = k_1 \) and \( \dim_{\text{RV}} f(X) = k_2 \). Since for every \( \overline{b} \in \text{pfv} f(X) \) there is a \( \overline{b} \)-definable finite-to-one function \( h_{\overline{b}} : \text{fib}(f(X), \overline{b}) \to \text{RV}^{k_2} \), by compactness, there is a definable function \( h : f(X) \to \text{RV}^{k_2} \) such that \( h \upharpoonright \text{fib}(f(X), \overline{b}) \) is finite-to-one for every \( \overline{b} \in \text{pfv} f(X) \). For every \( \overline{a} \in \text{pfv} X \), by Lemma 6.3, the subset
Definition 6.11 (R V-categories) The definable pairs \((\pi,\mathcal{T})\) function \((R V\text{-categories})\) definable and finite-to-one. So \(k_1 \leq k_2\) symmetrically we also have \(k_1 \geq k_2\) and hence \(k_1 = k_2\). □

6.2. Categories of definable subsets. The class of the objects and the class of the morphisms of any category \(\mathcal{C}\) are denoted as \(\text{Ob}\mathcal{C}\) and \(\text{Mor}\mathcal{C}\), respectively.

Definition 6.9 (VF-categories). The objects of the category \(\text{VF}[k,\cdot]\) are the definable subsets of \(\text{VF}\)-dimension \(\leq k\). The morphisms in this category are the definable functions between the objects.

The category \(\text{VF}[k]\) is the full subcategory of \(\text{VF}[k,\cdot]\) of the definable subsets that have RV-fiber dimension 0 (that is, all the RV-fibers are finite). The category \(\text{VF}_*[\cdot]\) is the union of the categories \(\text{VF}[k,\cdot]\). The category \(\text{VF}_*\) is the union of the categories \(\text{VF}[\cdot]\).

Note that, for any definable subset \(X\), by Lemma 6.3 and Lemma 6.4 \(\text{fib}(X,\mathcal{T})\) is finite for any \(\mathcal{T} \in \text{prv}\mathcal{X}\) if and only if \(X \in \text{Ob}\mathcal{VF}[0,\cdot]\).

Definition 6.10. For any tuple \(\mathcal{T} = (t_1,\ldots,t_n) \in \text{RV}\), the weight of \(\mathcal{T}\) is the number \(|\{i \leq n : t_i \neq \infty\}|\), which is denoted as \(\text{wgt}\mathcal{T}\).

Definition 6.11 (RV-categories). The objects of the category \(\text{RV}[k,\cdot]\) are the definable pairs \((U,f)\), where \(U \subseteq \text{RV}^m\) for some \(m\) and \(f : U \rightarrow \text{RV}^k\) is a function (\(\text{RV}^0\) is taken to be the singleton \(\{\infty\}\)). We often denote the projections \(\text{pr}_i\) of \(f_{j_i}\) and write \(f\) as \((f_{j_1},\ldots,f_{j_k})\). The companion \(U_f\) of \((U,f)\) is the subset \(\{(f(\mathcal{T}),\mathcal{T}) : \mathcal{T} \in U\}\).

For any two objects \((U,f),(U',f')\) in this category and any function \(F : U \rightarrow U'\), the companion \(F_{f,f'} : U_f \rightarrow U'_{f'}\) of \(F\) is the function given by

\[
(f(\mathcal{T}),\mathcal{T}) \rightarrow ((f' \circ F)(\mathcal{T}),F(\mathcal{T})).
\]

If, for every \(\mathcal{T} \in U\), \(\text{wgt}(f(\mathcal{T})) \leq \text{wgt}(f' \circ F(\mathcal{T}))\), then we say that \(F\) is volumetric. If \(F\) is definable, volumetric, and, for every \(\mathcal{T} \in \text{ran}(f)\), the subset

\[
(\text{pr}_{\leq k} \circ F_{f,f'})(\mathcal{T}) \times f^{-1}(\mathcal{T})
\]

is finite, then it is a morphism in \(\text{Mor}\text{RV}[k,\cdot]\).

The category \(\text{RV}[k]\) is the full subcategory of \(\text{RV}[k,\cdot]\) of the pairs \((U,f)\) such that \(f : U \rightarrow \text{RV}^k\) is finite-to-one.

Direct sums over these categories are formed naturally:

\[
\text{RV}[*,: ] = \coprod_{0 \leq k} \text{RV}[k,: ], \quad \text{RV}[*] = \coprod_{0 \leq k} \text{RV}[k].
\]

Notation 6.12. We just write \(X\) for the object \((X,\text{id}) \in \text{RV}[k,\cdot]\). By the indexing scheme described in Notation 2.4 for any object \((U,\text{pr}_E) \in \text{RV}[k,\cdot]\) with \(E \subseteq \mathbb{N}\) and \(|E| = k\), we may assume that \((U,\text{pr}_E)\) is \((U,\text{pr}_{\leq k})\) if this is more convenient. This should not cause any confusion in context.

Remark 6.13. One of the main reasons for the peculiar forms of the objects and the morphisms in the RV-categories is that each isomorphism class in these categories may be “lifted” to an isomorphism class in the corresponding VF-category. See Proposition 6.6 and Corollary 6.7 for details.
A subobject of an object \( X \) of a VF-category is just a definable subset. A subobject of an object \((U, f)\) of an RV-category is a definable pair \((X, g)\) with \(X\) a definable subset of \(U\) and \(g = f \upharpoonright X\). Note that the inclusion map is a morphism in both cases.

Notice that the cartesian product of two objects \( X, Y \in \text{VF}[k, \cdot] \) may or may not be in \( \text{VF}[k, \cdot] \). On the other hand, the cartesian product of two objects \((U, f), (U', f') \in \text{RV}[k, \cdot]\) is the object \((U \times U', f \times f') \in \text{RV}[2k, \cdot]\), which is definitely not in \( \text{RV}[k, \cdot] \) if \( k > 0 \). Hence, in \( \text{RV}[*] \) or \( \text{RV}[*] \), multiplying with a singleton in general changes isomorphism class.

**Remark 6.14.** The categories \( \text{VF}_* \) and \( \text{VF}_* \) are formed through union instead of direct sum or other means that induces more complicated structure. The reason for this is that the main goal of the Hrushovski-Kazhdan integration theory is to assign motivic volumes, that is, elements in the Grothendieck groups of the RV-categories, to the definable subsets, or rather, the isomorphism classes of the definable subsets, in the VF-categories, and the simplest categories that contain all the definable subsets that may be “measured” in this motivic way are \( \text{VF}_* \) and \( \text{VF}_* \). In contrast, the unions of the RV-categories are naturally endowed with the structure of direct sum, which gives rise to graded Grothendieck semirings. The ring homomorphisms are obtained by “passing to the limit”; see Corollary [12.8]

**Definition 6.15.** For any \((U, f) \in \text{Ob RV}[k, \cdot]\) and any \( F \in \text{Mor RV}[k, \cdot] \), let \( E_k(f) \) be the function on \( U \) given by \( \pi \mapsto (f(\pi), \infty) \), \( E_k(U, f) = (U, E_k(f)) \), and \( E_k(F) = F \). Obviously \( E_k : \text{RV}[k, \cdot] \to \text{RV}[k + 1, \cdot] \) is a functor that is faithful, full, and injective on objects. For any \( i < j \) let \( E_{i,j} = E_{j-1} \circ \ldots \circ E_i \) and \( E_{i,i} = \text{id} \).

Motivic integrals shall be induced by the following fundamental maps.

**Definition 6.16.** For any \((U, f) \in \text{Ob RV}[k, \cdot]\), let

\[
\mathbb{L}_k(U, f) = \bigcup \left\{ \text{rv}^{-1}(f(\pi)) \times \{\pi\} : \pi \in U \right\}.
\]

The map \( \mathbb{L}_k : \text{Ob RV}[k, \cdot] \to \text{Ob VF}[k, \cdot] \) is called the \( k \)-th canonical RV-lift. The map \( \mathbb{L}_{\leq k} : \text{Ob RV}[\leq k, \cdot] \to \text{Ob VF}[k, \cdot] \) is given by

\[
((U_1, f_1), \ldots, (U_k, f_k)) \mapsto \bigcup_{i \leq k} (\mathbb{L}_k \circ E_{i,k})(U_i, f_i).
\]

The map \( \mathbb{L} : \text{Ob RV}[*, \cdot] \to \text{Ob VF}_*[\cdot] \) is simply the union of the maps \( \mathbb{L}_{\leq k} \).

For notational convenience, when there is no danger of confusion, we shall drop the subscripts and simply write \( \mathbb{E} \) and \( \mathbb{L} \) for these maps.

**Remark 6.17.** Observe that if \((U, f) \in \text{Ob RV}[k]\) then \( \mathbb{L}(U, f) \in \text{Ob VF}[k] \) and hence the restriction \( \mathbb{L} : \text{Ob RV}[k] \to \text{Ob VF}[k] \) is well-defined. Similarly we have the maps

\[
\mathbb{L} : \text{Ob RV}[\leq k] \to \text{Ob VF}[k],
\]

\[
\mathbb{L} : \text{Ob RV}[*] \to \text{Ob VF}_*.
\]

Note that \( \text{rv}(\mathbb{L}(U, f)) = Uf \) for \((U, f) \in \text{Ob RV}[k, \cdot]\).

**Lemma 6.18.** Let \((U, f), (U', f') \in \text{Ob RV}[k, \cdot]\) and \( F : U \to U' \) a definable volumetric function. Suppose that there is a definable function \( F^\dagger : \mathbb{L}(U, f) \to\)
\( L(U', f') \) such that the diagram

\[
\begin{array}{ccc}
L(U, f) & \xrightarrow{rv} & U_f \\
\downarrow F' & & \downarrow F \\
L(U', f') & \xrightarrow{rv} & U'_{f'}
\end{array}
\]

commutes. Then \( F \) is a morphism in \( RV[k, \cdot] \).

**Proof.** It is enough to show that, for every \( \overline{\alpha} \in U \) and every \( i \leq k \), \( ((f')_i \circ F)(\overline{\alpha}) \in acl(f(\overline{\alpha})) \), which is equivalent to \((pr_i \circ F_j,f')(f(\overline{\alpha}), \overline{\alpha}) \in acl(f(\overline{\alpha})) \). To that end, fix an \( \overline{u} \in U \). Let \( \overline{a} \in rv^{-1}(f(\overline{\alpha})) \) and \( F^1(\overline{a}, \overline{u}) = (b_1, \ldots, b_k, \overline{\alpha}) \). By Lemma 6.18, we have that \( b_i \in acl(\overline{\alpha}) \) and hence \( rv(b_i) \in acl(\overline{\alpha}) \) for each \( i \leq k \). By Lemma 2.12, we conclude that \( rv(b_i) \in acl(f(\overline{\alpha})) \).

**Remark 6.19.** In Lemma 6.18 if both \( F \) and \( F^1 \) are bijections then we may drop the assumption that \( F \) is volumetric, since it is guaranteed by the commutative diagram and Corollary 6.5.

### 6.3. Grothendieck groups

We now introduce the Grothendieck groups associated with the categories defined above. The construction is of course the same for any reasonable category of definable sets of a first-order theory. For concreteness, we shall limit our attention to the present context.

**Convention 6.20.** Let \( f_1, \ldots, f_n \) be definable functions on subsets \( X_1, \ldots, X_n \), respectively. Padding with elements in \( acl(\emptyset) \) if necessary, we may glue \( f_1, \ldots, f_n \) together to form one definable function \( f : \biguplus_i X_i \rightarrow \biguplus_i f_i(X_i) \) in the obvious way. Below, when functions or other kinds of subsets are glued together in this way, we shall always tacitly assume that sufficient padding work has been performed.

Let \( \mathcal{C} \) be a VF-category or an RV-category. For any \( X \in Ob \mathcal{C} \), let \([X]\) denote the isomorphism class of \( X \). The Grothendieck semigroup of \( \mathcal{C} \), denoted as \( K_+ \mathcal{C} \), is the semigroup generated by the isomorphism classes \([X]\) of elements \( X \in Ob \mathcal{C} \), subject to the relation

\[
[X] + [Y] = [X \cup Y] + [X \cap Y].
\]

It is easy to check that \( K_+ \mathcal{C} \) is actually a commutative monoid, the identity element being \([\emptyset]\) or \(([\emptyset], \ldots)\). Since \( C \) always has disjoint unions, the elements of \( K_+ \mathcal{C} \) are precisely the isomorphism classes of \( \mathcal{C} \). If \( \mathcal{C} \) is one of the categories \( VF[\cdot] \), \( VF_+ \), \( RV[\cdot, \cdot] \), and \( RV[\cdot] \) then it is closed under cartesian product. In this case, \( K_+ \mathcal{C} \) has a semiring structure with multiplication given by

\[
[X][Y] = [X \times Y].
\]

Since the symmetry isomorphisms \( X \times Y \rightarrow Y \times X \) and the association isomorphisms \( X \times (Y \times Z) \rightarrow X \times (Y \times Z) \) are always present in these categories, \( K_+ \mathcal{C} \) is always a commutative semiring.

**Remark 6.21.** If \( C \) is either \( VF_+[-] \) or \( VF_+ \) then the isomorphism class of definable singletons is the multiplicative identity element of \( K_+ \mathcal{C} \). If \( C \) is \( RV[\cdot, \cdot] \) then we adjust multiplication when \( RV[0, \cdot] \) is involved as follows. For any \((U, f) \in RV[0, \cdot] \) and \((X, g) \in RV[k, \cdot] \), let

\[
(U, f) \boxtimes (X, g) = (X, g) \boxtimes (U, f) = (U \times X, g^*),
\]
where \( g^* \) is the function on \( U \times X \) given by \((\overline{t}, \overline{s}) \mapsto g(\overline{s}). \) Let
\[
[(U, f)][(X, g)] = [(U, f) \bowtie (X, g)].
\]
It is easily seen that, with this adjustment, \( K_+ \text{ RV}[\ast, \cdot] \) becomes a filtrated semiring and its multiplicative identity element is the isomorphism class of \((\infty, \text{id})\) in \( \text{RV}[0, \cdot]. \) Multiplication in \( K_+ \text{ RV}[\ast] \) is adjusted in the same way.

**Definition 6.22.** A semigroup congruence relation on \( K_+ \mathcal{C} \) is a sub-semigroup \( R \) of the semigroup \( K_+ \mathcal{C} \times K_+ \mathcal{C} \) such that \( R \) is an equivalence relation on \( K_+ \mathcal{C}. \) Similarly, a semiring congruence relation on \( K_+ \mathcal{C} \) is a sub-semiring \( R \) of the semiring \( K_+ \mathcal{C} \times K_+ \mathcal{C} \) such that \( R \) is an equivalence relation on \( K_+ \mathcal{C}. \)

Let \( R \) be a semigroup congruence relation on \( K_+ \mathcal{C} \) and \((x, y), (v, w) \in R. \) Then \((x + v, y + v), (y + v, y + w) \in R \) and hence \((x + v, y + w) \in R. \) Therefore the equivalence classes of \( R \) has a semigroup structure that is induced by that of \( K_+ \mathcal{C}. \) This semigroup is denoted as \( K_+ \mathcal{C}/R \) and is also referred to as a Grothendieck semigroup. Similarly, if \( R \) is a semiring congruence relation on \( K_+ \mathcal{C} \) then \( K_+ \mathcal{C}/R \) is actually a Grothendieck semiring.

**Remark 6.23.** Let \( R \) be an equivalence relation on the semiring \( K_+ \mathcal{C}. \) If for every \((x, y) \in R \) and every \( z \in K_+ \mathcal{C} \) we have that \((x + z, y + z) \in R \) and \((xz, yz) \in R \) then \( R \) is a semiring congruence relation.

Let \((\mathbb{Z}^{K_+ \mathcal{C}}, \oplus)\) be the free abelian group generated by the elements of \( K_+ \mathcal{C} \) and \( \mathcal{C} \) the subgroup of \((\mathbb{Z}^{K_+ \mathcal{C}}, \oplus)\) generated by all elements of \((\mathbb{Z}^{K_+ \mathcal{C}}, \oplus)\) of the types
\[
(1 \cdot x) \oplus ((-1) \cdot x),
\]
\[
(1 \cdot x) \oplus (1 \cdot y) \oplus ((-1) \cdot (x + y)),
\]
where \( x, y \in K_+ \mathcal{C}. \) The Grothendieck group of \( \mathcal{C}, \) denoted as \( K \mathcal{C}, \) is the formal groupification \((\mathbb{Z}^{K_+ \mathcal{C}}, \oplus)/\mathcal{C}\) of \( K_+ \mathcal{C}, \) which is essentially unique by the universal mapping property. Clearly \( K_+ \mathcal{C} \) is canonically isomorphic to a sub-semigroup of \( K \mathcal{C}. \) If \( K_+ \mathcal{C} \) is a semiring then \( K \mathcal{C} \) is a commutative ring.

**Remark 6.24.** It is easily checked that \( E_k \) induces an injective semigroup homomorphisms \( K_+ \text{ RV}[k, \cdot] \longrightarrow K_+ \text{ RV}[k + 1, \cdot] \), which is also denoted as \( E_k. \)

**Notation 6.25.** For any definable subset \( X \subseteq \text{RV}^n, \) we write \([X]_n\) for the isomorphism class \([(X, \text{id})] \in K_+ \text{ RV}[n, \cdot]. \) For any subset \( E \subseteq \mathbb{N} \) with \(|E| = k, \) we write \([X]_E\) for the isomorphism class \([(X, \text{pr}_E')] \in K_+ \text{ RV}[k, \cdot]. \) If \( E = \{1, \ldots, k\} \) etc. then we may write \([X]_{\leq k}\) etc. If \( X \) is a singleton then we just write \([1]_k\) for the isomorphism class \([(X, f)] \in K_+ \text{ RV}[k, \cdot]. \)

7. **RV-products and special bijections**

**Convention 7.1.** Since definably bijective subsets are to be identified, we shall tacitly substitute \( c(X) \) for a subset \( X \) in the discussion if it is necessary or is just more convenient.

**Definition 7.2.** A subset \( p \) is an \((open, closed, rv-) polyball\) if it is of the form \( \prod_{i \leq n} b_i \times \mathbb{I}, \) where each \( b_i \) is an \((open, closed, rv-) ball \) and \( \mathbb{I} \in \text{RV}. \) In this case, the \textit{radius} of \( p, \) denoted as \( \text{rad}(p), \) is \( \min \{\text{rad}(b_i) : i \leq n\}. \)
For any definable subset \( X \), both the subset of \( X \) that contains all the rv-polyballs contained in \( X \) and the superset of \( X \) that contains all the rv-polyballs with nonempty intersection with \( X \) are definable.

**Definition 7.3.** For any subset \( U \subseteq \text{VF}^n \times \text{RV}^m \), the RV-hull of \( U \), denoted by \( \text{RVH}(U) \), is the subset \( \bigcup \{ \text{rv}^{-1}(\bar{t}) \times \{ \bar{v} \} : (\bar{t}, \bar{v}) \in \text{rv}(U) \} \). If \( U = \text{RVH}(U) \), that is, if \( U \) is a union of rv-polyballs, then we say that \( U \) is an RV-product.

**Lemma 7.4.** Let \( X \subseteq (\text{VF}^{n_1} \times \text{RV}^{m_1}) \times (\text{VF}^{n_2} \times \text{RV}^{m_2}) \) be a definable subset such that, for each \( (\bar{\pi}, \bar{t}) \in \text{pr}_{n_1+m_1}X \), \( \text{fib}(X, (\bar{\pi}, \bar{t})) \) is finite. Suppose that \( Y \subseteq \text{VF}^{n_1+n_2} \times \text{RV}^m \) is an RV-product that is definably bijective to \( X \). Then, for any rv-polyball \( \text{rv}^{-1}(t_1, \ldots, t_{n_1+n_2}) \times \{(t_1, \ldots, t_{n_1+n_2}, \bar{s})\} \subseteq Y \), the weight of \( (t_1, \ldots, t_{n_1+n_2}) \) is at most \( n_1 \).

**Proof.** Clearly we have that \( \dim_{\text{VF}} X = \dim_{\text{VF}}(\text{pr}_{n_1+m_1} X) \leq n_1 \). Suppose for contradiction that there is an rv-polyball contained in \( Y \) such that the weight of the tuple in question is greater than \( n_1 \). By Corollary \( 6.5.6 \), \( \dim_{\text{VF}} Y > n_1 \) and hence \( \dim_{\text{VF}} X > n_1 \), contradiction. \( \square \)

**Definition 7.5.** Let \( X \subseteq \text{VF} \times \text{VF} \times \text{RV}^m \). Let \( C \subseteq \text{RVH}(X) \) be an RV-product and \( \lambda : \text{pr}_{>1}(C \cap X) \longrightarrow \text{VF} \) a function such that \( (\lambda(\bar{\pi}, \bar{t}), \bar{\pi}, \bar{t}) \in C \) for every \( (\bar{\pi}, \bar{t}) = (\bar{\pi}_1, \bar{t}_1, \ldots, \bar{t}_m) \in \text{pr}_{>1}(C \cap X) \). Let

\[
C^\lambda = \bigcup_{(\bar{\pi}, \bar{t}) \in \text{pr}_{>1}C} \left( \bigcup \{ \text{rv}^{-1}(t) : \text{rvr}(t) > \text{vrv}(t_1) \} \right) \times (\bar{\pi}, \bar{t}),
\]

\( \text{RVH}(X)^\lambda = C^\lambda \cup (\text{RVH}(X) \setminus C) \).

The centripetal transformation \( \eta : X \longrightarrow \text{RVH}(X)^\lambda \) with respect to \( \lambda \) is defined by

\[ \eta(a_1, \bar{\pi}_1, \bar{t}) = (a_1 - \lambda(\bar{\pi}_1, \bar{t}), \bar{a}_1, \bar{t}) \]

on \( C \cap X \) and \( \eta = \text{id} \) on \( X \setminus C \). Note that \( \eta \) is injective. The inverse of \( \eta \) is naturally called the centrifugal transformation with respect to \( \lambda \). The function \( \lambda \) is called a focus map of \( X \). The RV-product \( C \) is called the locus of \( \lambda \). A special bijection \( T \) is an alternating composition of centripetal transformations and the canonical bijection. The length of a special bijection \( T \), denoted by \( \text{lh} T \), is the number of centripetal transformations in the composition of \( T \). The image \( T(X) \) is sometimes denoted as \( X^\lambda \).

Note that we should have included the index of the targeted VF-coordinate as a part of the data of a focus map. Since it should not cause confusion, below, we shall suppress mentioning it for notational ease.

Clearly if \( X \) is an RV-product and \( T \) is a special bijection on \( X \) then \( T(X) \) is an RV-product. Notice that a special bijection \( T \) on \( X \) is definable if \( X \) and all the focus maps involved are definable. Since we are only interested in definable subsets and definable functions on them, we further require a special bijection to be definable.

**Example 7.6.** Let \( b \subseteq \text{VF} \) be a definable open ball properly contained in \( \text{rv}^{-1}(t) \). By Convention \( 6.1.1 \), \( b \) is identified with the subset \( b \times \{t\} \). By Lemma \( 5.1.7 \), \( \text{rv}^{-1}(t) \) contains a definable element \( a \), which may or may not be in \( b \). Let \( \lambda \) be the focus...
map \( t \mapsto a \). Then the centripetal transformation on \( b \) with respect to \( \lambda \) is given by \((b, t) \mapsto (b - a, t)\).

Let \( F \subseteq \text{rv}^{-1}(t) \times \text{rv}^{-1}(s) \subseteq \text{VF}^2 \) be a definable finite-to-one function, which may be regarded as a focus map of its whose locus is \( \text{rv}^{-1}(t) \times \text{rv}^{-1}(s) \). Let \( \eta_1 \) be the corresponding centripetal transformation. Then \( \eta_1(F) = \text{dom}(F) \times \{0\} \). For each \( b \in \text{ran}(F) \) let \( b^* \) be the average of \( F^{-1}(b) \). Note that, by compactness, the subset \( \{b^* : b \in \text{ran}(F)\} \) is definable. Let \( \lambda_2 : \text{ran}(F) \longrightarrow \text{dom}(F) \) be the focus map given by \((a, b) \mapsto (b^*, b)\). Let \( \eta_2 \) be the corresponding centripetal transformation and \( F^* = (c \circ \eta_2)(F) \). Notice that, by Lemma 4.1, \( \text{rv} \) is not constant on the subset \( F^{-1}(b) - b^* \). Hence \( F^* \) is a function from \( \text{pr}_1 F^* \) onto \( \text{pr}_2 F^* \) such that the maximum size of its fibers on the first \( \text{VF} \)-coordinate is strictly smaller than that of \( F \). This phenomenon will be the basis of many inductive arguments below.

**Definition 7.7.** A subset \( X \) is a deformed \( \text{RV}-\text{product} \) if there is a special bijection \( T \) such that \( T(X) \) is an \( \text{RV}-\text{product} \). In that case, if \( T \) is definable then we say that \( X \) is a definable deformed \( \text{RV}-\text{product} \).

**Lemma 7.8.** Every definable subset \( X \subseteq \text{VF} \times \text{RV}^m \) is a definable deformed \( \text{RV}-\text{product} \).

**Proof.** By compactness, it is enough to show that, for every \((a, \bar{t}) \in X\), there is a special bijection \( T \) on \( X \) such that \( T(a, \bar{t}) \) is contained in an \( \text{rv} \)-polyball \( p \subseteq T(X) \). Fix an \((a, \bar{t}) \in X\). Let \( Z \) be the union of the \( \text{rv} \)-polyballs contained in \( X \), which is a definable \( \text{RV}-\text{product} \). If \((a, \bar{t}) \in Z\) then the canonical bijection is as required. So, without loss of generality, we may assume that \( Z = \emptyset \). By Convention 7.1, the canonical bijection has been applied to \( X \) and hence, for any \( \pi = (s_1, \ldots, s_m) \in \text{prv} X \), the \( \pi \)-definable subset \( \text{fib}(X, \pi) \) is properly contained in the \( \text{rv} \)-ball \( \text{rv}^{-1}(s_1) \).

By \( C \)-minimality, \( \text{fib}(X, \bar{t}) \) is a disjoint union of \( \pi \)-definable simplexes. Let \( s \) be the simplex that contains \((a, \bar{t})\). Let \( b_1, \ldots, b_l, h_1, \ldots, h_n \) be the boolean components of \( s \), where each \( b_i \) is positive and each \( h_i \) is negative. The proof now proceeds by induction on \( n \).

For the base case \( n = 0 \), \( s \) is a disjoint union of balls \( b_1, \ldots, b_l \) of the same radius and valutative center. Without loss of generality, we may assume that \( a \in b_1 \). Let \( \{c_1, \ldots, c_k\} \) be the positive closure of \( s \). Note that this closure is also \( \bar{t} \)-definable. We now start a secondary induction on \( k \). For the base case \( k = 1 \), by Lemma 4.16 there is a \( \bar{t} \)-definable point \( c \in c_1 \). Clearly \( c_1 - c \subseteq \text{rv}^{-1}(\text{rv}(a)) - c \) is a union of \( \text{rv} \)-balls. Let \( C \) be a definable subset of \( \text{RVH}(X) \) and \( \lambda : \text{pr}_{>1}(C \cap X) \longrightarrow \text{VF} \) a definable focus map such that \((a, \bar{t}) \in C \) and \( \lambda(\bar{t}) = c \). Then the centripetal transformation \( \eta \) with respect to \( \lambda \) is as desired. For the inductive step of the secondary induction, by Lemma 4.16 again, there is a \( \bar{t} \)-definable set of centers \( \{c_1, \ldots, c_k\} \) with \( c_1 \in c_i \). Let \( c \) be the average of \( c_1, \ldots, c_k \). Let \( \lambda, \eta \) be as above such that \( \lambda(\bar{t}) = c \). If \( c \in b_1 \) then, as above, the centripetal transformation \( \eta \) with respect to \( \lambda \) is as desired. So suppose that \( c \notin b_1 \). Note that if \( \text{val} \) is not constant on the set \( \{c_1 - c, \ldots, c_k - c\} \) then \( \text{rv} \) is not constant on it and if \( \text{val} \) is constant on it, then, by Lemma 4.1, \( \text{rv} \) is still not constant on it. Consider the special bijection \( \sigma = c \circ \eta \). We have that \( \sigma(a, \bar{t}) = (a - c, r, \bar{t}) \in \sigma(X) \), where \( r = \text{rv}(a - c) \). Observe that the positive closure of the \((r, \bar{t})\)-definable subset \( \text{fib}(\sigma(X), (r, \bar{t})) \) is a proper subset of the set \( \{c_1 - c, \ldots, c_k - c\} \) of closed balls. Hence, by the inductive hypothesis, there is a special bijection \( T \) on \( \sigma(X) \) such that \( T(a - c, r, \bar{t}) \) is contained...
in an rv-polyball $p \subseteq T \circ \sigma(X)$. So $T \circ \sigma$ is as required. This completes the base case $n = 0$.

We proceed to the inductive step. Note that, since $b_1, \ldots, b_l$ are pairwise disjoint, the holes $h_1, \ldots, h_n$ are also pairwise disjoint. Without loss of generality we may assume that all the holes $h_1, \ldots, h_n$ are of the same radius. Let $\{\xi_1, \ldots, \xi_k\}$ be the positive closure of $\bigcup_j h_j$. The secondary induction on $k$ above may be carried out almost verbatim. Only note that, in the inductive step, after applying the special bijection $\sigma$, the number of holes in the fiber that contains $\sigma(a, \overline{T})$ decreases and hence the inductive hypothesis may be applied. □

**Corollary 7.9.** Let $f : X \rightarrow Y$ be a definable surjective function, where $X$, $Y \subseteq VF$. Then there is a definable function $P : X \rightarrow RV^m$ such that, for each $\overline{\tau} \in \text{ran} P$, $P^{-1}(\overline{\tau})$ is an open ball or a point and $f \upharpoonright P^{-1}(\overline{\tau})$ is either constant or injective.

**Proof.** Let $P_1 : X \rightarrow RV^l$ be a function given by Lemma 4.10. Applying Lemma 7.8 to each fiber $P_1^{-1}(\overline{\tau})$, we see that desired function exists by compactness. □

**Remark 7.10.** Corollary 7.9 and Lemma 4.8 imply that the theory $ACVF_b^0$ is $b$-minimal, in the sense of [4].

**Lemma 7.11.** Let $X \subseteq VF \times RV^m$ be a definable subset and $T$ a special bijection on $X$ such that $T(X)$ is an RV-product. Then there is a definable function $\epsilon : (\text{prv} \circ T)(X) \rightarrow VF$ such that, for every $(t, \overline{\tau}) \in (\text{prv} \circ T)(X)$, we have that

$$(\text{prv} \circ T)^{-1}(\text{rv}^{-1}(t) \times \{t, \overline{\tau}\}) = \text{rv}^{-1}(t) + \epsilon(t, \overline{\tau}).$$

**Proof.** We do induction on the length of $T$. For the base case $\text{lh} T = 1$, let $T = \sigma_1 \circ \cdots \circ \sigma_l$, where $\sigma_i$ is a centripetal transformation. Let $\lambda$ and $C \subseteq \text{RVH}(X)$ be the corresponding focus map and its locus. For each $(t, \overline{\tau}) \in (\text{prv} \circ T)(X)$, if $\overline{\tau} \in \text{dom}(\lambda)$ then set $\epsilon(t, \overline{\tau}) = \lambda(\overline{\tau})$, otherwise set $\epsilon(t, \overline{\tau}) = 0$. Clearly $\epsilon$ is as required.

We proceed to the inductive step $\text{lh} T = n > 1$. Let $T = \sigma_l \circ \cdots \circ \sigma_1 \circ \sigma_0$, where $T_1 = \sigma_l \circ \cdots \circ \sigma_1 \circ \sigma_0$. By the inductive hypothesis, for the special bijection $T_1$, there is a function $\epsilon_1 : (\text{prv} \circ T_1)((\sigma_l \circ \cdots \circ \sigma_1)(X)) \rightarrow VF$ as required. Let $\lambda$ and $C \subseteq \text{RVH}(X)$ be the focus map and its locus for the centripetal transformation $\sigma_0$. For each $(t, \overline{\tau}) \in (\text{prv} \circ T)(X)$, if $(\text{prv} \circ T_1)^{-1}(\text{rv}^{-1}(t) \times \{(t, \overline{\tau})\}) = (r, \overline{\tau})$ and $\overline{\tau} \in \text{dom}(\lambda)$ then set $\epsilon(t, \overline{\tau}) = \epsilon_1(t, \overline{\tau}) = (r, \overline{\tau})$, otherwise set $\epsilon(t, \overline{\tau}) = \epsilon_1(t, \overline{\tau})$. Then $\epsilon$ is as required. □

**Remark 7.12.** Note that, in Lemma 7.11 since $\text{dom}(\epsilon) \subseteq RV^l$ for some $l$, by Lemma 4.8 $\text{ran}(\epsilon)$ is actually finite.

The following technical result is very important for the rest of the construction.

**Proposition 7.13.** Let $f_i(\overline{\tau}) = f_i(x_1, \ldots, x_n) \in VF(\emptyset)(\overline{\tau})$ be a finite list of polynomials and $\overline{T} = (t_1, \ldots, t_n) \in RV$ a definable tuple. Then there is a special bijection $T$ on $\text{rv}^{-1}(\overline{T})$ such that, for every rv-polyball $p \subseteq T(\text{rv}^{-1}(\overline{T}))$ and every $f_i(\overline{\tau})$, the subset $f_i(T^{-1}(p))$ is contained in an rv-ball.

**Proof.** We do induction on $n$. For the base case $n = 1$, we write $t, x$ for $\overline{T}, \overline{\tau}$, respectively. By compactness, it is enough to show that for any $a \in \text{rv}^{-1}(t)$ there is a special bijection $T$ on $\text{rv}^{-1}(t)$ such that the image of $\text{RVH}(T(a))$ under every composite map

$$(\text{RVH}(T(a)) \xrightarrow{T^{-1}} \text{rv}^{-1}(t) \xrightarrow{f_i} VF \xrightarrow{\text{rv}} RV).$$
is a singleton. So fix an \( a \in \text{rv}^{-1}(t) \). For any special bijection \( T \) on \( \text{rv}^{-1}(t) \), let \( k(T) \) be the number of elements \((b, \overline{r}) \in \text{RVH}(T(a))\) such that \( f_l(T^{-1}(b, \overline{r})) = 0 \) for some \( l \). It is sufficient to prove the following:

**Claim.** For every special bijection \( T \) on \( \text{rv}^{-1}(t) \) there is a special bijection \( T^* \) on \( T(\text{rv}^{-1}(t)) \) such that the image of \( \text{RVH}((T^* \circ T)(a)) \) under every composite map

\[
\text{RVH}((T^* \circ T)(a)) \xrightarrow{(T^* \circ T)^{-1}} \text{rv}^{-1}(t) \xrightarrow{f_l} \text{VF} \xrightarrow{\text{rv}} \text{RV}
\]

is a singleton.

**Proof.** We do induction on \( k(T) \). For the base case \( k(T) = 1 \), there is a definable subset \( Y \subseteq T(\text{rv}^{-1}(t)) \) such that \( Y \) is the union of those rv-polyballs that contain exactly one \( d \in \text{VF} \) with \( f_1(T^{-1}(d)) = 0 \) for some \( l \). So there is a definable focus map \( \lambda : \text{prv} Y \longrightarrow \text{VF} \) such that for every \( \text{rv}^{-1}(s) \times \{(s, \overline{r})\} \subseteq Y \) we have that \( f_l(T^{-1}(\lambda(s, \overline{r})) = 0 \) for some \( l \). Clearly the special bijection \( T^* \) given by

\[
(b, \overline{r}) \mapsto (b - \lambda(s, \overline{r}), \text{rv}(b - \lambda(s, \overline{r})), s, \overline{r})
\]

for \((b, \overline{r}) \in Y \) is as required. For the inductive step \( k(T) > 1 \), there is a definable subset \( Y \subseteq T(\text{rv}^{-1}(t)) \) such that \( Y \) is the union of those rv-polyballs that contain exactly \( k(T) \) elements \( d \in \text{VF} \) with \( f_1(T^{-1}(d)) = 0 \) for some \( l \). Let \( \text{rv}^{-1}(s) \times \{(s, \overline{r})\} \subseteq Y \) and \( d_1, \ldots, d_{k(T)} \in \text{rv}^{-1}(s) \) enumerate these \( k(T) \) elements. The focus map \( \lambda \) defined above is now modified so that \( \lambda(s, \overline{r}) \) is the average of \( d \) of \( d_1, \ldots, d_{k(T)} \). Let \( T^* \) be defined as above with respect to this \( \lambda \). By Lemma 4.4, \( \text{rv} \) is not constant on the set \( \{d_1 - d, \ldots, d_{k(T)} - d\} \). Hence \( k(T^* \circ T) < k(T) \) and the inductive hypothesis may be applied.

This completes the base case of the induction.

We now proceed to the inductive step. Let \( \overline{x}_1 = (x_2, \ldots, x_n) \) and \( \overline{t}_1 = (t_2, \ldots, t_n) \). For every \( a \in \text{rv}^{-1}(t_1) \), by the inductive hypothesis, there is an \( a \)-definable special bijection \( T_a \) on \( \text{rv}^{-1}(\overline{t}_1) \) such that every function \( \text{rv}(f_l(a, \overline{x}_1)) \) is constant on every subset \( T_a^{-1}(p) \), where \( p \) is an rv-polyball contained in \( T_a(\text{rv}^{-1}(\overline{t}_1)) \). By compactness, there are definable disjoint subsets \( Y_1, \ldots, Y_m \subseteq \text{rv}^{-1}(t_1) \) with \( \bigcup Y_i = \text{rv}^{-1}(t_1) \) and formulas \( \phi_1(x_1), \ldots, \phi_m(x_1) \) such that, for every \( a \in Y_i \), \( \phi_i(a) \) defines a special bijection \( T_a \) on \( \text{rv}^{-1}(\overline{t}_1) \) such that the property described holds with respect to \( T_a \). Applying Lemma 7.3 repeatedly, we obtain a special bijection \( T_1 \) on \( \text{rv}^{-1}(t_1) \) such that each \( T_1(Y_i) \) is an RV-product.

Now, for every \( a \in \text{rv}^{-1}(t_1) \), each locus \( C \) involved in \( T_a \) is determined by an \( a \)-definable subset \( U_C \) of \( \text{RV}^k \) for some \( k \). Let \( \chi(x_1) \) be the formula that defines \( U_C \). Note that if \( T_a \) is defined by \( \phi_i(a) \) then \( \chi(x_1) \) may be taken as a subformula of \( \phi_i(x_1) \). Let \( \chi^*(x_1, \overline{z}) \) be a quantifier-free formula in disjunctive normal form that is equivalent to the formula \( \chi(T_1^{-1}(x_1, \overline{z})) \), where \( \overline{z} \) are RV-sort variables. By compactness and the base case above, there is a special bijection \( \rho_i \) on \( T_1(Y_i) \) such that each term \( \text{rv}(g(x_1)) \) that occurs in \( \chi^*(x_1, \overline{z}) \) is constant on every subset \( \rho_i^{-1}(p) \), where \( p \) is an rv-polyball contained in \( (\rho_i \circ T_1)(Y_i) \). Hence, for each \( a \in (\rho_i \circ T_1)^{-1}(p) \), \( \chi(a) \) defines the same loci for the corresponding centripetal transformations. Consequently, by compactness again, we obtain a special bijection \( \rho \) on \( T_1(\text{rv}^{-1}(t_1)) \) such that \( \rho(T_1)(Y_i) \) is an RV-product for each \( i \) and, for each rv-polyball \( p \subseteq (\rho \circ T_1)(Y_i) \), \( \rho(p) = \overline{z} \), the formula \( \phi_i((\rho \circ T_1)^{-1}(x_1, \overline{z})) \) defines a special bijection on \( p \times \text{rv}^{-1}(\overline{t}_1) \). So \( \phi_i((\rho \circ T_1)^{-1}(x_1, \overline{z})) \) defines a special bijection on \( (\rho \circ T_1)(\text{rv}^{-1}(t_1)) \times \text{rv}^{-1}(\overline{t}_1) \), which is denoted as \( \phi_i \).
It is not hard to see that the special bijections \( \phi_1 \circ \rho \circ T_1, \ldots, \phi_m \circ \rho \circ T_1 \) actually form one special bijection \( T_2 \) on \( \text{rv}^{-1}(\bar{t}) \). Let \( \text{rv}^{-1}(\bar{s}) \times \{(\bar{s}, \bar{t})\} \subseteq T_2(\text{rv}^{-1}(\bar{t})) \), where \( \bar{s} = (s_1, \ldots, s_n) \). By the construction of \( T_2 \), for each \( a_1 \in \text{rv}^{-1}(s_1) \), every function \( \text{rv}(f_1(\bar{s})) \) is constant on the subset 
\[ T_2^{-1}(\{a_1\} \times \text{rv}^{-1}(\bar{s}_1) \times \{(\bar{s}, \bar{t})\}) \].

Let this constant value be \( u^1_{a_1} \). So the function \( h_l : \text{rv}^{-1}(s_1) \rightarrow \text{RV} \) given by \( a_1 \mapsto u^l_{a_1} \) is \( (\bar{s}, \bar{t}) \)-definable. For each \( l \), let \( \psi_l(x_1, z) \) be a quantifier-free formula in disjunctive normal form that defines the function \( h_l \), where \( z \) is an RV-sort variable. We may assume that some conjunct in each disjunct of \( \psi_l(x_1, z) \) is an RV-sort equality. Let \( g_l(x_1) \) enumerate all the polynomials that occur in a term of the form \( \text{rv}(g_l(x_1)) \) in some \( \psi_l(x_1, z) \). By the base case, there is an \( (\bar{s}, \bar{t}) \)-definable special bijection \( T_{s_1} \) on \( \text{rv}^{-1}(s_1) \) such that, for each \( \text{rv}-\text{polyball} \ p \subseteq T_{s_1}(\text{rv}^{-1}(s_1)) \), every term \( \text{rv}(g_l(x_1)) \) is constant on \( T_{s_1}^{-1}(p) \) and hence every function \( h_l \) is constant on \( T_{s_1}^{-1}(p) \). We may identify \( T_{s_1} \) with the function it naturally induces on \( \text{rv}^{-1}(\bar{s}) \times \{(\bar{s}, \bar{t})\} \). Therefore, every function \( \text{rv}(f_l(\bar{s})) \) is constant on the subset 
\[ (T_{s_1} \circ T_2)^{-1}(p \times \text{rv}^{-1}(\bar{s}_1) \times \{(\bar{s}, \bar{t})\}) \].

By compactness, we obtain a special bijection \( T_3 \) on \( T_2(\text{rv}^{-1}(\bar{t})) \) such that the property just described holds for every \( \text{rv}-\text{polyball} \) contained in \( (T_3 \circ T_2)(\text{rv}^{-1}(\bar{t})) \). This completes the inductive step. □

Lemma 7.8 is easily generalized as follows.

**Proposition 7.14.** Every definable subset \( X \subseteq \text{VF}^n \times \text{RV}^m \) is a definable deformed RV-product.

**Proof.** This is by induction on \( n \). The base case \( n = 1 \) is just Lemma 7.8. For the inductive step, by compactness, without loss of generality, we may assume that \( \text{pr}_1 X = \{ (t_1, \ldots, t_n) \in \text{RV} \} \). It is not hard to see that, if we replace the conclusion of Proposition 7.13 with the desired property here, then a simpler version of the argument in the inductive step of the proof of Proposition 7.13 works almost verbatim. It is simpler because the last part of that argument is not needed here. □

**Corollary 7.15.** The map \( \mathbb{L} : \text{Ob} \text{RV}[k,-] \rightarrow \text{Ob} \text{VF}[k,-] \) is surjective on the isomorphism classes of \( \text{VF}[k,-] \).

**Corollary 7.16.** Let \( f_l(\bar{s}) \in \text{VF}((\emptyset))[\bar{s}] \) be a finite list of polynomials and \( X \) a definable subset of \( \text{VF}^n \times \text{RV}^m \). Then there is a special bijection \( T \) on \( X \) such that \( T(X) \) is an RV-product and, for every \( \text{rv}-\text{polyball} \ p \subseteq T(X) \), every subset \( f_l(T^{-1}(p)) \) is contained in an \( \text{rv-ball} \).

**Proof.** By Proposition 7.13 there is a special bijection \( T_1 \) on \( X \) such that \( T_1(X) \) is an RV-product. Let \( p = \text{rv}^{-1}(\bar{t}) \times \{(\bar{t}, \bar{s})\} \) be an \( \text{rv-ball} \) contained in \( T_1(X) \). For each \( l \), let \( \psi_l \) be a quantifier-free formula in disjunctive normal form that defines the function \( \text{rv}(f_1(T_1^{-1}(\bar{s}, \bar{t}, \bar{z}))) \) on \( p. \) Clearly we may assume that some conjunct in each disjunct of any \( \psi_l \) is an RV-sort equality. By Proposition 7.13 there is a \( (\bar{t}, \bar{s}) \)-definable special bijection \( T_{\bar{t}, \bar{s}} \) on \( p \) such that each term \( \text{rv}(g_l(\bar{s})) \) that occurs in some \( \psi_l \) is constant on every subset \( T_{\bar{t}, \bar{s}}^{-1}(q) \), where \( q \) is an \( \text{rv-ball} \) contained in \( T_{\bar{t}, \bar{s}}(p) \), and hence \( \text{rv}(f_l(\bar{s})) \) is constant on \( (T_{\bar{t}, \bar{s}} \circ T_1)^{-1}(q). \) By compactness,
there is a special bijection $T_2$ on $T_1(X)$ such that the property just described holds for every rv-polyball contained in $(T_2 \circ T_1)(X)$. □

**Proposition 7.17.** Let $X \subseteq \text{VF}^n \times \text{RV}^m$ be a definable subset. If $\text{pvf} \upharpoonright X$ is finite-to-one then there is a $Y \subseteq \text{RV}^d$ such that $\text{pr}_{\leq n} \upharpoonright Y$ is finite-to-one and $L(Y, \text{pr}_{\leq n})$ is definably bijective to $X$.

**Proof.** By Proposition 7.14 there is a $Y \subseteq \text{RV}^d$ such that there is a definable bijection $T : X \longrightarrow L(Y, \text{pr}_{\leq n})$. Suppose for contradiction that there is a $\bar{t} \in \text{pr}_{\leq n} Y$ such that the subset $\text{fib}(Y, \bar{t})$ is infinite. Fix a tuple $\bar{\sigma} \in \bar{t}$. Consider the $\bar{\sigma}$-definable function $\text{pvf} \circ T^{-1} : \{\bar{\sigma}\} \times \text{fib}(Y, \bar{t}) \longrightarrow \text{pvf} X$. By Lemma 4.8 ran($\text{pvf} \circ T^{-1}$) is finite. Since $\text{pvf} \upharpoonright X$ is finite-to-one, we must have that the subset $T^{-1}(\{\bar{\sigma}\} \times \text{fib}(Y, \bar{t}))$ is finite and hence $\{\bar{\sigma}\} \times \text{fib}(Y, \bar{t})$ is finite, contradiction. □

**Corollary 7.18.** The map $\mathbb{L} : \text{Ob RV}[k] \longrightarrow \text{Ob VF}[k]$ is surjective on the isomorphism classes of $\text{VF}[k]$.

8. 2-Cells

For functions between subsets that have only one VF-coordinate, composing with special bijections on the right and their inverses on the left preserves the open-to-open property.

**Lemma 8.1.** Let $X, Y \subseteq \text{VF}$ be definable and $f : X \longrightarrow Y$ a definable bijection. Then there is a special bijection $T$ on $X$ such that $T(X)$ is an RV-product and, for each rv-polyball $p \subseteq T(X)$, $f \upharpoonright T^{-1}(p)$ has the open-to-open property.

**Proof.** By Proposition 7.13 there is a definable finite partition of $X$ such that the restriction of $f$ to each piece has the open-to-open property. Applying Proposition 7.13 to each piece or its subsequent image yields the desired special bijection. □

**Lemma 8.2.** Let $X, Y \subseteq \text{VF}$ be definable open balls and $f : X \longrightarrow Y$ a definable bijection that has the open-to-open property. Let $\alpha \in \Gamma$ be definable. Then there is a special bijection $T$ on $X$ such that $T(X)$ is an RV-product and, for each rv-polyball $p \subseteq T(X)$, the set

$$\{\text{rad}(b) : b \text{ is an open ball contained in } T^{-1}(p) \text{ with } \text{rad}(f(b)) = \alpha\}$$

is a singleton.

**Proof.** Let $\mathcal{B}$ be the collection of all open balls $b \subseteq X$ with $\text{rad}(f(b)) = \alpha$. Let $\psi(x)$ be a quantifier-free formula in disjunctive normal form that defines the radius function rad on $\mathcal{B}$, where $x$ is the VF-sort variable. By Corollary 7.10 there is a special bijection $T$ on $X$ such that $T(X)$ is an RV-product and each term $\text{rv}(g(x))$ that occurs in $\psi(x)$ is constant on every subset $T^{-1}(p)$, where $p$ is an rv-polyball contained in $T(X)$. So $T$ is as required. □

**Lemma 8.3.** Let $X \subseteq \text{VF}^2$ be a definable subset such that $\text{pr}_1 X$ is an open ball. Let $f : \text{pr}_1 X \longrightarrow \text{pr}_2 X$ be a definable bijection that has the open-to-open property. Suppose that for each $a \in \text{pr}_1 X$ there is a $t_a \in \text{RV}$ such that

$$\text{fib}(X, a) = \text{rv}^{-1}(t_a) + f(a).$$
Then there is a special bijection $T$ on $\text{pr}_1 X$ such that $T(\text{pr}_1 X)$ is an RV-product and, for each rv-polyball $p \subseteq T(\text{pr}_1 X)$, the set
\[
\{ \text{rv}(a - f^{-1}(b)) : a \in T^{-1}(p) \text{ and } b \in \text{fib}(X,a) \}
\]
is a singleton.

**Proof.** For each $a \in \text{pr}_1 X$, let $b_a$ be the minimal closed ball that contains $\text{fib}(X,a)$. Since $\text{fib}(X,a) - f(a) = \text{rv}^{-1}(t_a)$, we have that $f(a) \in b_a$ but $f(a) \notin \text{fib}(X,a)$ if $t_a \neq \infty$. Hence $a \notin f^{-1}(\text{fib}(X,a))$ if $t_a \neq \infty$ and $\{a\} = f^{-1}(\text{fib}(X,a))$ if $t_a = \infty$. Since $f^{-1}(\text{fib}(X,a))$ is a ball, in either case, the function $\text{rv}(a - x)$ is constant on the subset $f^{-1}(\text{fib}(X,a))$. The function $h : \text{pr}_1 X \rightarrow \text{RV}$ given by $a \mapsto \text{rv}(a - f^{-1}(\text{fib}(X,a)))$ is definable. Now we may proceed as in Lemma 8.2 □

**Definition 8.4.** Let $X \subseteq \text{VF}^2$ be such that $\text{pr}_1 X$ is an open ball. Let $f : \text{pr}_1 X \rightarrow \text{pr}_2 X$ be a bijection that has the open-to-open property. We say that $f$ is trapezoidal in $X$ if there are $t_1, t_2 \in \text{RV}$ such that, for each $a \in \text{pr}_1 X$,
\[
\begin{align*}
(1) & \quad \text{fib}(X,a) = \text{rv}^{-1}(t_2) + f(a), \\
(2) & \quad f^{-1}(\text{fib}(X,a)) = a - \text{rv}^{-1}(t_1).
\end{align*}
\]
The elements $t_1, t_2$ are called the paradigms of $X$.

**Remark 8.5.** Let $f$ be trapezoidal in $X$ with respect to $t_1, t_2 \in \text{RV}$. Let $a \in \text{pr}_1 X$, $b$ the minimal closed ball that contains $\text{fib}(X,a)$, and $a$ the minimal closed ball that contains $f^{-1}(\text{fib}(X,a))$. The following properties are easily deduced:
\[
\begin{align*}
(1) & \quad f(a) \notin \text{fib}(X,a) \text{ and hence } a \notin f^{-1}(\text{fib}(X,a)). \\
(2) & \quad \text{rv}(t_1) = \text{rad}(a) = \text{rad}(\text{pr}_1 X) \text{ and } \text{rv}(t_2) = \text{rad}(b) = \text{rad}(\text{pr}_2 X). \\
(3) & \quad f(a) \in b \subseteq \text{pr}_2 X \text{ and } a \in a \subseteq \text{pr}_1 X. \\
(4) & \quad \text{Let } a_f(a) \text{ be the maximal open subballs of } a, b \text{ that contains } a, f(a), \text{ respectively. We have that, for every } a^* \in f^{-1}(a_f(a)), \text{ and hence } a^* - \text{rv}^{-1}(t_1) = a - \text{rv}^{-1}(t_1); \text{ so } a^* \in a_f(a). \text{ Symmetrically, for every } b^* \in f(b_f(a)), \\
& \quad f^{-1}(\text{fib}(X,f^{-1}(b))) = f^{-1}(b^*) - \text{rv}^{-1}(t_1) \\
& \quad = f^{-1}(\text{fib}(X,a)) \\
& \quad \text{and hence } \text{rv}^{-1}(t_2) + b^* = \text{rv}^{-1}(t_2) + f(a); \text{ so } b^* \in a_f(a). \text{ So actually } f(a_f(a)) = a_f(a). \\
(5) & \quad \text{Let } \mathfrak{A}, \mathfrak{B} \text{ be the sets of maximal open subballs of } a, b, \text{ respectively. Then } f \text{ induces a bijection } f_1 : \mathfrak{A} \rightarrow \mathfrak{B}. \\
(6) & \quad \text{For each } a \in \mathfrak{A}, \text{ each } c \in a, \text{ and each } d \in \text{fib}(X,c), \text{ we have that} \\
& \quad \text{fib}(X,c) = \text{rv}^{-1}(t_2) + f_1(a), \\
& \quad \text{fib}(X,d) = f^{-1}(d) + \text{rv}^{-1}(t_1) = a. \\
\end{align*}
\]
So $a - f^{-1}(\text{fib}(X,c)) = \text{rv}^{-1}(t_1)$ and $f_1(a) - \text{fib}(X,c) = -\text{rv}^{-1}(t_2)$. (Hence $f$ is “trapezoidal”.)
(7) The subset $X$ is symmetrical in the following way:

$$\bigcup \{ o \times (rv^{-1}(t_2) + f_1(o)) : o \in \mathbb{A} \}$$

$$= \bigcup \{(f_{j_1}^{-1}(o) + rv^{-1}(t_1)) \times o : o \in \mathbb{B} \}$$

$$= X \cap (a \times VF)$$

$$= X \cap (VF \times b)$$

$$= X \cap (a \times b).$$

**Definition 8.6.** We say that a subset $X$ is a 1-cell if it is either an open ball contained in one rv-ball or a point in $VF$. We say that $X$ is a 2-cell if

1. $X \subseteq VF^2$ is contained in one rv-polyball and $pr_1 X$ is a 1-cell,
2. there is a function $\epsilon : pr_1 X \longrightarrow VF$ and a $t \in RV$ such that, for each $a \in pr_1 X$, $fib(X, a) = rv^{-1}(t) + \epsilon(a)$,
3. one of the following three possibilities occurs:
   a. $\epsilon$ is constant,
   b. $\epsilon$ is injective, has the open-to-open property, and $rad(\epsilon(pr_1 X)) \geq vrv(t)$,
   c. $\epsilon$ is trapezoidal in $X$.

The function $\epsilon$ is called the **positioning function** of $X$ and the element $t$ is called the **paradigm** of $X$.

**Remark 8.7.** A subset $X \subseteq VF \times RV^m$ is a 1-cell if for each $\bar{t} \in prv X$ the fiber $fib(X, \bar{t})$ is a 1-cell in the sense of Definition 8.6. The concept of a 2-cell is generalized in the same way. A cell is definable if all the relevant ingredients are definable.

Suppose that $X$ is a 2-cell. Clearly if its paradigm $t$ is $\infty$ then $X$ and its positioning function $\epsilon$ are identical. It is also easy to see that, if $t \neq \infty$ and $\epsilon$ is not trapezoidal, then $X$ is actually an open polyball.

Notice that Lemma 7.8 implies that for every definable subset $X \subseteq VF \times RV^m$ there is a definable function $P : X \longrightarrow RV^l$ such that, for each $\bar{\pi} \in ran P$, the fiber $P^{-1}(\bar{\pi})$ is a 1-cell. The same holds for 2-cell:

**Lemma 8.8.** For every definable subset $X \subseteq VF^2$ there is a definable function $P : X \longrightarrow RV^m$ such that, for each $\bar{\pi} \in ran P$, the fiber $P^{-1}(\bar{\pi})$ is a 2-cell.

**Proof.** Without loss of generality, we may assume that $X$ is contained in one rv-polyball. For any $a \in pr_1 X$, by Lemma 7.8 there is an $a$-definable special bijection $T_a$ on $fib(X, a)$ such that $T_a(fib(X, a))$ is an RV-product. By Lemma 7.11 there is an $a$-definable function $\epsilon_a : (prv o T_a)(fib(X, a)) \longrightarrow VF$ such that, for every $(t, \bar{\pi}) \in (prv o T_a)(fib(X, a))$, we have that

$$(prv o T_a^{-1})(rv^{-1}(t) \times \{(t, \bar{\pi})\}) = rv^{-1}(t) + \epsilon_a(t, \bar{\pi}).$$

By compactness, we may assume that there is a definable subset $X' \subseteq pr_1 X \times RV^l$ and a definable function $\epsilon : X' \longrightarrow VF$ such that, for each $a \in pr_1 X$, $fib(X', a) = (prv o T_a)(fib(X, a))$ and $\epsilon \upharpoonright fib(X', a) = \epsilon_a$. Since, for each $(t, \bar{\pi}) \in prv X'$, $\epsilon \upharpoonright fib(X', (t, \bar{\pi}))$ may be regarded as a $(t, \bar{\pi})$-definable function from $VF$ into $VF$, by Lemma 4.10, we are reduced to the case that each $\epsilon \upharpoonright fib(X', (t, \bar{\pi}))$ is either constant or injective. If no $\epsilon \upharpoonright fib(X', (t, \bar{\pi}))$ is injective then we can finish by applying Lemma 7.8 to each $fib(X', (t, \bar{\pi}))$ and then compactness.
Suppose that \( \epsilon_{t,\overline{\alpha}} = \epsilon \mid \text{fib}(X', (t, \overline{\alpha})) \) is injective. By Lemma 8.1 we are reduced to the case that \( \text{fib}(X', (t, \overline{\alpha})) \) is an open ball and \( \epsilon_{t,\overline{\alpha}} \) has the open-to-open property. Note that, if \( \text{rad}(\text{ran} \epsilon_{t,\overline{\alpha}}) < \text{rvv}(t) \), then
\[
\text{ran} \epsilon_{t,\overline{\alpha}} = \bigcup_{a \in \text{fib}(X', (t, \overline{\alpha}))} (\text{rv}^{-1}(t) + \epsilon_{t,\overline{\alpha}}(a)).
\]
By Lemma 8.2 we are further reduced to the case that, if \( \text{rad}(\text{ran} \epsilon_{t,\overline{\alpha}}) < \text{rvv}(t) \), then there is a \( \gamma \in \Gamma \) such that \( \text{rad}(\epsilon_{t,\overline{\alpha}}^{-1}(\gamma)) = \gamma \) for every open ball \( b \subseteq \text{ran} \epsilon_{t,\overline{\alpha}} \) with \( \text{rad}(b) = \text{rvv}(t) \). By Lemma 8.3 we are finally reduced to the case that, if \( \text{rad}(\text{ran} \epsilon_{t,\overline{\alpha}}) < \text{rvv}(t) \), then there is an \( r \in \text{RV} \) such that, for every \( a \in \text{fib}(X', (t, \overline{\alpha})) \),
\[
\text{rv}(a - f^{-1}(\text{rv}^{-1}(t) + \epsilon_{t,\overline{\alpha}}(a))) = r
\]
and hence
\[
f^{-1}(\text{rv}^{-1}(t) + \epsilon_{t,\overline{\alpha}}(a)) = a - \text{rv}^{-1}(r).
\]
So, in this case, \( \epsilon_{t,\overline{\alpha}} \) is trapezoidal. Now we are done by compactness. \( \square \)

9. Lifting functions from RV to VF

We shall show that the map \( L \) actually induces homomorphisms between various Grothendieck semigroups when \( S \) is a \((\text{VF}, \Gamma)\)-generated substructure.

Any polynomial in \( O[\overline{\alpha}] \) corresponds to a polynomial in \( K[\overline{\alpha}] \) via the canonical quotient map. The following definition generalizes this phenomenon.

**Definition 9.1.** Let \( \overline{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \Gamma \). A polynomial \( f(\overline{x}) = \sum \gamma_i \overline{x}^i \) with coeffcients \( \alpha_{ij} \in \text{VF} \) is a \( \overline{\gamma} \)-polynomial if there is an \( \alpha \in \Gamma \) such that \( \alpha = \text{val}(\alpha_{ij}) + i_1 \gamma_1 + \ldots + i_n \gamma_n \) for each \( ij = (i_1, \ldots, i_n, j) \). In this case we say that \( \alpha \) is a residue value of \( f(\overline{x}) \) (with respect to \( \overline{\gamma} \)). For a \( \overline{\gamma} \)-polynomial \( f(\overline{x}) \) with residue value \( \alpha \) and a \( \overline{t} \in \text{RV} \) with \( \text{rvv}(\overline{t}) = \overline{\gamma} \), if \( \text{val}(f(\overline{x})) > \alpha \) for all \( \overline{a} \in \text{rv}^{-1}(\overline{t}) \) then \( \overline{t} \) is a residue root of \( f(\overline{x}) \). If \( \overline{t} \in \text{RV} \) is a common residue root of the \( \overline{\gamma} \)-polynomials \( f_1(\overline{x}), \ldots, f_n(\overline{x}) \) but is not a residue root of the \( \overline{\gamma} \)-polynomial \( \det \partial(f_1(\overline{x}), \ldots, f_n(\overline{x})) / \partial(x_1, \ldots, x_n) \), then we say that \( f_1(\overline{x}), \ldots, f_n(\overline{x}) \) are minimal for \( \overline{t} \) and \( \overline{t} \) is a simple common residue root of \( f_1(\overline{x}), \ldots, f_n(\overline{x}) \).

Therefore, according to this definition, every polynomial in \( K[\overline{\alpha}] \) is the projection of some \((0, \ldots, 0)\)-polynomial \( f(\overline{x}) \) with residue value 0.

Hensel’s Lemma is generalized as follows.

**Lemma 9.2 (Generalized Hensel’s Lemma).** Let \( f_1(\overline{x}), \ldots, f_n(\overline{x}) \) be \( \overline{\gamma} \)-polynomials with residue values \( \alpha_1, \ldots, \alpha_n \), where \( \overline{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \Gamma \). For every simple common residue root \( \overline{t} = (t_1, \ldots, t_n) \in \text{RV} \) of \( f_1(\overline{x}), \ldots, f_n(\overline{x}) \) there is a unique \( \overline{\alpha} \in \text{rv}^{-1}(\overline{\gamma}) \) such that \( f_i(\overline{\alpha}) = 0 \) for every \( i \).

**Proof.** Fix a simple common residue root \( \overline{t} = (t_1, \ldots, t_n) \in \text{RV} \) of \( f_1(\overline{x}), \ldots, f_n(\overline{x}) \). Choose a \( c_i \in \text{rv}^{-1}(t_i) \). Changing the coefficients accordingly we may rewrite each \( f_i(\overline{x}) \) as \( f_i(x_1/c_1, \ldots, x_n/c_n) \). Write \( y_i \) for \( x_i/c_i \). Note that, for each \( i \), the coefficients of the \((0, \ldots, 0)\)-polynomial \( f_i(\overline{y}) \) are all of the same value \( \alpha_i \). For each \( i \) choose an \( e_i \in \text{VF} \) with \( \text{val}(e_i) = -\alpha_i \). We have that each \((0, \ldots, 0)\)-polynomial \( f_i^*(\overline{y}) = e_i f_i(\overline{y}) \) has residue value 0 (that is, the coefficients of \( f_i^*(\overline{y}) \) is of value 0). Clearly
\[
(t_1 / \text{rv}(c_1), \ldots, t_n / \text{rv}(c_n)) = (1, \ldots, 1)
\]
is a common residue root of $f_1^*(\gamma), \ldots, f_n^*(\gamma)$; that is, for every $\gamma \in \text{rv}^{-1}(1, \ldots, 1)$ and every $i$ we have that $\text{val} f_i^*(\gamma) > 0$. It is actually a simple root because for every $\gamma \in \text{rv}^{-1}(1, \ldots, 1)$ we have that

$$\det \partial(f_1^*, \ldots, f_n^*)/\partial(y_1, \ldots, y_n)(\gamma) = \left( \prod_i c_i \right) \det \partial(f_1, \ldots, f_n)/\partial(x_1, \ldots, x_n)(\gamma),$$

where $\gamma = (a_1 c_1, \ldots, a_n c_n)$, and hence

$$\text{val}(\det \partial(f_1^*, \ldots, f_n^*)/\partial(y_1, \ldots, y_n)(\gamma)) = \sum_i (-\alpha_i + \gamma_i) + \sum_i \alpha_i - \sum_i \gamma_i = 0.$$

Now the lemma follows from the multivariate version of Hensel’s Lemma (e.g. see [3 Corollary 2, p. 224]).

**Definition 9.3.** Let $X, Y$ be two RV-products, $F$ a subset of $X \times Y$, and $A$ a subset of $\text{rv}(X \times Y)$. We say that $F$ is a $(X, Y)$-lift of $A$ from RV to VF, or just a lift of $A$ for short, if $F \cap (p \times q)$ is a bijective function from $p$ onto $q$ for every $\text{rv}$-polyball $p \subseteq X$ and every $\text{rv}$-polyball $q \subseteq Y$ with $\text{rv}(p \times q) \subseteq A$. A partial lift of $A$ is a lift of any subset of $A$.

It would be ideal to lift all definable subsets of $\text{RV}^n \times \text{RV}^n$ with finite-to-finite correspondence for any substructure $S$. However, the following crucial lemma fails when $S$ is not $(\text{VF}, \Gamma)$-generated.

**Lemma 9.4.** Suppose that $S$ is $(\text{VF}, \Gamma)$-generated. Let $\gamma = (t_1, \ldots, t_n) \in \text{RV}$ be such that $t_n \neq \infty$ and $t_n \in \text{acl}(t_1, \ldots, t_{n-1})$. Let $\text{rv}(\gamma) = (\gamma_1, \ldots, \gamma_n) = \gamma$. Then there is a $\gamma$-polynomial $f(x_1, \ldots, x_n) = f(\gamma)$ with coefficients in $\text{VF}(\{\emptyset\})$ such that the subset $\{r \in \text{RV} : (t_1, \ldots, t_{n-1}, r)$ is a residue root of $f(\gamma)\}$ is finite and $\gamma$ is a residue root of $f(\gamma)$ but is not a residue root of $\partial f(\gamma)/\partial x_n$.

**Proof.** Write $(t_1, \ldots, t_{n-1})$ as $\tilde{t}_n$. Let $\phi(\gamma)$ be a formula such that $\phi(\tilde{t}_n, x_n)$ defines a finite subset that contains $\tilde{t}_n$. By quantifier elimination, there is a conjunction $\psi(\gamma)$ of RV-sort literals such that $\psi(\gamma)$ implies $\phi(\gamma)$ and $\psi(\tilde{t})$ holds. By C-minimality, we may assume that some conjunct $\theta(\gamma)$ in $\psi(\gamma)$ is an RV-sort equality such that $\theta(\tilde{t}_n, x_n)$ defines a finite subset. Since $S$ is $(\text{VF}, \Gamma)$-generated, we may assume that $\theta(\gamma)$ does not contain parameters from $\text{RV}(\{\emptyset\}) \setminus \text{rv}(\text{VF}(\{\emptyset\}))$. Hence it is of the form

$$\gamma^T \cdot \sum_i (\text{rv}(a_{\gamma i}) \cdot \gamma^T) = \text{rv}(a) \cdot \gamma^T \cdot \sum_j (\text{rv}(a_{\gamma j}) \cdot \gamma^T),$$

where $a_{\gamma i}, a_{\gamma j} \in \text{VF}(\{\emptyset\})$. Fix an $s \in \text{RV}$ such that $\text{rv}(s \cdot \gamma^T) = \text{rv}(s \cdot \gamma^T) = 0$. Let $\text{rv}(s) = \delta$. Note that $\delta$ is $\tilde{t}_n$-definable. Let

$$h_1(\gamma, s) = \sum_i (s \cdot \text{rv}(a_{\gamma i}) \cdot \gamma^T)$$

$$h_2(\gamma, s) = \sum_j (-s \cdot \text{rv}(a_{\gamma j}) \cdot \gamma^T).$$

Consider the RV-sort polynomial

$$H(\gamma, s) = h_1(\gamma, s) + h_2(\gamma, s).$$
For any \( r \in RV \), \( H(\overline{t}_n, s, r) = 0 \) if and only if either
\[
\sum_i (rv(a_i) \cdot (\overline{t}_n, r)^i) = \sum_j (rv(a_j) \cdot (\overline{t}_n, r)^j) = 0
\]
or
\[
rv(h_1(\overline{t}_n, s, r)/s) = rv(-h_2(\overline{t}_n, s, r)/s).
\]
Since \( r \neq t_n \) in the former case, by \( C \)-minimality again, the equation \( H(\overline{t}_n, s, x_n) = 0 \) defines a finite subset that contains the subset defined by \( \theta(\overline{t}_n, x_n) \) and is actually \( t_n \)-definable. Let \( m \) be the maximal exponent of \( x_n \) in \( H(\overline{t}, s) \). For each \( i \leq m \) let \( H_i(\overline{t}, s) \) be the sum of all the monomials \( h(\overline{t}, s) \) in \( H(\overline{t}, s) \) such that the exponent of \( x_n \) in \( h(\overline{t}, s) \) is \( i \). Replacing \( s \) with a variable \( y \) and each \( rv(a) \) with \( a \) in \( H_i(\overline{t}, s) \), we obtain a VF-sort polynomial \( H^*_i(\overline{t}, y) \) for each \( i \leq m \). Let
\[
E = \{ i \leq m : rv(H^*_i(\overline{b}, c)) = 0 \text{ for all } (\overline{b}, c) \in rv^{-1}(\overline{t}, s) \}.
\]
Since \( H(\overline{t}, s) = 0 \), clearly \( |E| \neq 1 \). We claim that \( |E| > 1 \). To see this, suppose for contradiction that \( E = \emptyset \). Write \( H^*_i(\overline{t}, y) = yx_i^nG_i(\overline{t}_n) \), where \( \overline{t}_n = (x_1, \ldots, x_{n-1}) \). Let \( \overline{t}_n = (\gamma_1, \ldots, \gamma_{n-1}) \). Clearly each \( G_i(\overline{t}_n) \) is a \( \gamma_n \)-polynomial with residue value \( -\delta - i\gamma_n \). Since \( rv(ch^*_i G_i(\overline{b}_n)) > 0 \) for all \( c \in rv^{-1}(s), b_n \in rv^{-1}(t_n) \), and \( \overline{b}_n \in rv^{-1}(\overline{t}_n) \), we have that \( (\overline{t}_n) \) is a residue root of \( G_i(\overline{t}_n) \). So for all \( r \in RV \) with \( rv(r) = rv(t_n) = \gamma_n \) we have that \( rv(ch^*_i G_i(\overline{b}_n)) > 0 \) for all \( c \in rv^{-1}(s), d \in rv^{-1}(-r), \) and \( \overline{b}_n \in rv^{-1}(\overline{t}_n) \) and hence \( H_i(\overline{t}_n, s, r) = 0 \). So \( H(\overline{t}_n, s, r) = 0 \) for all \( r \in RV \) with \( rv(r) = rv(t_n) = \gamma_n \), which is a contradiction because the equation \( H(\overline{t}_n, s, x_n) = 0 \) defines a finite subset.

Let
\[
H^*(\overline{t}, y) = \sum_{i \in E} H^*_i(\overline{t}, y) = \sum_{i \in E} (yx_i^nG_i(\overline{t}_n)) = yG(\overline{t}).
\]
Since \( (\overline{t}, s) \) is a residue root of \( H^*(\overline{t}, y) \), clearly \( G(\overline{t}) \) is a \( \gamma \)-polynomial with residue value \( -\delta \) such that \( \overline{t} \) is a residue root of it. Also, \( \overline{t}_n \) is not a residue root of any \( G_i(\overline{t}_n) \). It follows that, for some \( k < \max E, \overline{t} \) is a residue root of the \( \gamma \)-polynomial \( \partial G(\overline{t})/\partial^k x_n \) but is not a residue root of the \( \gamma \)-polynomial \( \partial G(\overline{t})/\partial^{k+1} x_n \). \( \square \)

For definable subsets of the residue field, the situation may be further simplified. The following lemma shows that the geometry of definable subsets over the residue field coincides with its algebraic geometry; in other words, each definable subset over the residue field is a constructible subset (in the sense of algebraic geometry) of the Zariski topological space Spec \( K(S)[x_1, \ldots, x_n] \).

**Lemma 9.5.** If \( X \subseteq \overline{K}^m \) is definable then it is a boolean combination of subsets defined by equalities with coefficients in \( \overline{K}(S) \).

**Proof.** Let \( \phi \) be a quantifier-free formula in disjunctive normal form that defines \( X \) and \( \overline{t} = (\gamma_1, \ldots, \gamma_m) \) the \( \Gamma \)-sort parameters in \( \psi \). Without loss of generality \( \gamma_i \notin acl(VF(\{\emptyset\}), RV(\{\emptyset\})) \) for all \( i \). Since \( X \subseteq \overline{K}^m \), each conjunct in each disjunct of \( \phi \) may be assumed to be of the form
\[
\sum_i (rv(a_i) \cdot \gamma_i \cdot \overline{x}^i) \square rv(a) \cdot r \cdot \sum_j (rv(a_j) \cdot \gamma_j \cdot \overline{x}^j),
\]
where \( a_i, a_j \in VF(\emptyset), r, r_i, r_j \in RV(\emptyset) \), and \( \square \) is one of the symbols \( =, \neq, <, \) and \( > \). It is easily seen that the literals involving \( \leq \) or \( > \) are redundant. So each conjunct in \( \phi \) is either an RV-sort equality or an RV-sort disequality. Now the proof
proceeds by induction on \( m \). The base case \( m = 0 \) is clear. For the inductive step, if one of the conjuncts in \( \phi \) is an equality and contains some \( \gamma_i \) as an irredundant parameter then, since \( X \subseteq \overline{\mathbb{K}}^n \), actually \( \gamma_i \) may be defined from other parameters in \( \phi \) and hence, by the inductive hypothesis, the lemma holds. By the same reason, we see that no nontrivial equality with parameters in \( \langle \mathbf{0}, \mathbf{R}^\infty \rangle \) may hold between \( \gamma \) and any \( \overline{\gamma} \in X \). So each disequality in \( \phi \) that contains some \( \gamma_i \) as an irredundant parameter must define either the empty set or a superset of \( X \) and hence is redundant. \( \square \)

**Proposition 9.6.** Suppose that the substructure \( S \) is \( \langle \mathbf{V}, \mathbf{M} \rangle \)-generated. Let \( C \subseteq (\mathbf{R}^\infty \times (\mathbf{R}^\infty)^n \) be a definable subset such that both \( \text{pr}_{\leq n} \upharpoonright C \) and \( \text{pr}_{> n} \upharpoonright C \) are finite-to-one. Then there is a definable subset \( C' \subseteq \mathbf{V}^n \times \mathbf{V}^n \) that lifts \( C \).

**Proof.** By compactness, the lemma is reduced to showing that for every \( \langle \overline{t}, \overline{x} \rangle \in C \) there is a definable lift of some subset of \( C \) that contains \( \langle \overline{t}, \overline{x} \rangle \). Fix a \( \langle \overline{t}, \overline{x} \rangle \in C \) and set \( \langle \overline{y}, \overline{\delta} \rangle = \text{rv}(\overline{t}, \overline{x}) \). Let \( \phi(\overline{x}, \overline{y}) \) be a formula that defines \( C \). Consider the formulas \( \exists \overline{y}_i \phi(\overline{x}, \overline{y}) \) and \( \exists \overline{\tau}_i \phi(\overline{x}, \overline{y}) \), where \( \overline{y}_i = \overline{y} \setminus y_i \) and \( \overline{\tau}_i = \overline{x} \setminus x_i \). By Proposition 9.4 for each \( y_i \) there is a \( (\overline{\gamma}, \overline{\delta}) \)-polynomial \( \mu_i(\overline{x}, y_i) \) with coefficients in \( \mathbf{V} \setminus \mathbf{F} \) such that \( (\overline{t}, \overline{s}_i) \) is a residue root of \( \mu_i(\overline{x}, y_i) \) but is not a residue root of \( \mu(\overline{x}, y_i) \). Similarly we obtain such a \( (\gamma_i, \overline{\delta}) \)-polynomial \( \nu_i(x_i, \overline{\gamma}) \) for each \( x_i \). For each \( i \), let \( a_i(\overline{xy})^{\overline{x}_i} \) and \( b_i(\overline{xy})^{\overline{x}_i} \) be two monomials with \( a_i, b_i \in \mathbf{V} \) such that

\[
\mu_i^*(\overline{x}, \overline{y}) + \nu_i^*(\overline{x}, \overline{y}) = a_i(\overline{xy})^{\overline{x}_i} \mu_i(\overline{x}, y_i) + b_i(\overline{xy})^{\overline{x}_i} \nu_i(x_i, \overline{\gamma})
\]

is a \( (\overline{\gamma}, \overline{\delta}) \)-polynomial. Let \( a_i \) be the residue value of \( \mu_i^*(\overline{x}, \overline{y}) + \nu_i^*(\overline{x}, \overline{y}) \). Note that for any \( \langle \overline{y}, \overline{b} \rangle \in \mathbf{V}^n \setminus \mathbf{V} \) we have

\[
\text{val}(\partial \mu_i^*/\partial y)(\overline{y}, \overline{b}) = \text{val}(a_i \overline{b})^k + \text{val}(\partial \mu_i/\partial y)(\overline{y}, \overline{b}) = \alpha_i - \delta_i
\]

and for \( j \neq i \) we have

\[
\text{val}(\partial \mu_j^*/\partial y)(\overline{y}, \overline{b}) = \text{val}(a_j) + \text{val}(\partial (\overline{xy})^{\overline{x}_i}/\partial y)(\overline{y}, \overline{b}) + \text{val}(\mu_i(\overline{y}, \overline{b})) > \alpha_i - \delta_j.
\]

Therefore,

\[
\text{val det}(\partial(\mu_1^*, \ldots, \mu_n^*)/\partial(y_1, \ldots, y_n))(\overline{y}, \overline{b}) = \text{val} \prod_i (\partial \mu_i^*/\partial y_i)(\overline{y}, \overline{b}) = \sum_i \alpha_i - \sum_i \delta_i.
\]

This shows that \( \overline{y} \) is a simple common residue root of \( \mu_1^*(\overline{x}, \overline{y}), \ldots, \mu_n^*(\overline{x}, \overline{y}) \) for any \( \overline{y} \in \mathbf{V} \setminus \mathbf{M} \). Similarly \( \overline{t} \) is a simple common residue root of \( \nu_1^*(\overline{x}, \overline{y}), \ldots, \nu_n^*(\overline{x}, \overline{y}) \) for any \( \overline{y} \in \mathbf{V} \setminus \mathbf{M} \). Now, it is not hard to see that we may choose integers \( d_i, e_i \) and form a \( (\overline{\gamma}, \overline{\delta}) \)-polynomial

\[
\tau_i(\overline{x}, \overline{y}) = d_i \mu_i^*(\overline{x}, \overline{y}) + e_i \nu_i^*(\overline{x}, \overline{y})
\]

such that \( \overline{y} \) is a simple common residue root of \( \tau_1(\overline{x}, \overline{y}), \ldots, \tau_n(\overline{x}, \overline{y}) \) for any \( \overline{y} \in \mathbf{V} \setminus \mathbf{M} \) and \( \overline{t} \) is a simple common residue root of \( \tau_1(\overline{x}, \overline{y}), \ldots, \tau_n(\overline{x}, \overline{y}) \) for any \( \overline{y} \in \mathbf{V} \setminus \mathbf{M} \). By the generalized Hensel’s Lemma 9.2 for each \( \overline{y} \in \mathbf{V} \setminus \mathbf{M} \) there is a unique \( \overline{y} \in \mathbf{V} \setminus \mathbf{M} \) such that \( \bigwedge_i \tau_i(\overline{y}, \overline{b}) = 0 \), and vice versa. \( \square \)
Corollary 9.7. Suppose that the substructure $S$ is $(VF, \Gamma)$-generated. The map $L$ induces homomorphisms between various Grothendieck semigroups: $K_+ RV[k, :] \to K_+ VF[k, :], K_+ RV[k] \to K_+ VF[k]$, etc.

Proof. For any $RV[k, :]$-isomorphism $F : (U, f) \to (V, g)$ and any $\pi \in U$, by definition, $\text{wgt}(\pi) = \text{wgt}(g \circ F)(\pi)$. Therefore, $L(U, f)$ and $L(V, g)$ are $VF[k, :]$-isomorphic by Proposition 9.6.

10. Contracting to RV

Definition 10.1. Let $X \subseteq VF^n \times RV^m$ be an $RV$-product and $f : X \to Y$ a function, where $Y$ is also an $RV$-product. We say that $f$ is contractible if for every $rv$-polyball $p \subseteq X$ the subset $f(p)$ is contained in an $rv$-polyball.

Clearly, for two (definable) $RV$-products $X$ and $Y$, if $f : X \to Y$ is an (definable) contractible function, then there is a unique (definable) function $f_1 : rv(X) \to rv(Y)$ such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow {rv} & & \downarrow {rv} \\
rv(X) & \xrightarrow{f_1} & rv(Y)
\end{array}
\]

commutes. Note that, in this case, if both $f$ and $f_1$ are bijective then $f$ is a lift of $f_1$. Equivalently, if $f$ is bijective and both $f$ and $f^{-1}$ are contractible then $f$ is a lift of $f_1$.

Lemma 10.2. Let $X \subseteq VF^{n_1} \times RV^{m_1}$ and $Y \subseteq VF^{n_2} \times RV^{m_2}$ be definable subsets and $f : X \to Y$ a definable function. Then there exist special bijections $T_X, T_Y$ on $X, Y$ such that the function $T_Y \circ f \circ T_X^{-1}$ is contractible.

Proof. Recall that by Convention 7.1 the canonical bijection is automatically applied to all subsets. By Proposition 7.14 there is a special bijection $T_Y$ on $Y$ such that $T_Y(Y)$ is an $RV$-product. So we may assume that $Y$ is an $RV$-product. Fix a sequence of quantifier-free formulas $\psi_1, \ldots, \psi_{m_2}$ that define the functions $f_i = \text{pr}_i \circ \text{prv} \circ T_Y \circ f$ for $1 \leq i \leq m_2$. Let $g_i(\pi)$ enumerate all the $VF$-sort polynomials that occur in $\psi_1, \ldots, \psi_{m_2}$ in the form $\text{rv}(g_i(\pi))$. By Proposition 7.14 and Corollary 7.10 there is a special bijection $T_X$ on $X$ such that $T_X(X)$ is an $RV$-product and the function $\text{rv}(g_i(T_X^{-1}(\pi)))$ is constant on every $rv$-polyball $p \subseteq T_X(X)$ for every $i$. So on such an $rv$-polyball every $f_i \circ T_X^{-1}$ is constant.

Lemma 10.3. Let $X \subseteq VF \times RV^{m_1}$ and $Y \subseteq VF \times RV^{m_2}$ be definable subsets and $f : X \to Y$ a definable bijection. Then there exist special bijections $T_X : X \to X^*$ and $T_Y : Y \to Y^*$ such that, in the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{T_X} & X^* \\
\downarrow {f} & & \downarrow {f^*} \\
Y & \xrightarrow{T_Y} & Y^*
\end{array}
\]

$T^*_X$ is bijective and hence $f_1^*$ is a lift of it.
Proof. By Proposition 5.15, there is a definable partition $X_1, \ldots, X_n$ of $X$ such that each $f \mid X_i$ has the open-to-open property. Therefore, applying Lemma 10.2 to each $f \mid X_i$ or its subsequent image, we may assume that $X, Y$ are RV-products and $f$ is contractible and has the open-to-open property. In particular, for each rv-polyball $p \subseteq X$, $f(p)$ is an open ball contained in an rv-polyball $p^* \subseteq Y$. By Lemma 10.2 again, there is a special bijection $T_Y : Y \rightarrow Y^2$ such that $(T_Y \circ f)^{-1}$ is contractible. Let

$$T_Y = c \circ \eta_n \circ \cdots \circ c \circ \eta_1,$$

where each $\eta_i$ is a centripetal transformation and $c$ is the canonical bijection.

Now, by induction on $n$, we construct a special bijection

$$T_X = c \circ \eta_n^* \circ \cdots \circ c \circ \eta_1^*$$
on $X$ such that, for each $i$, both $L_i \circ f \circ (L_i^*)^{-1}$ and $(T_Y \circ f \circ L_i^*)^{-1}$ are contractible, where

$$L_i = c \circ \eta_i \circ \cdots \circ c \circ \eta_1,$$

$$L_i^* = c \circ \eta_i^* \circ \cdots \circ c \circ \eta_1^*.$$

Then $T_X, T_Y$ will be as desired. To that end, suppose that $\eta_i^*$ has been constructed for each $i \leq k < n$. Let $Z_k = L^*_k(X)$ and $Z_k^2 = L_k(Y)$. Let $C \subseteq Z^2_k$ be the locus of $\eta_{k+1}$ and $\lambda$ the corresponding focus map. Since $L_k \circ f \circ (L_k^*)^{-1}$ is contractible and has the open-to-open property, each rv-polyball $p \subseteq Z_k^2$ is the union of disjoint subsets of the form $(L_k \circ f \circ (L_k^*)^{-1})(q)$, where $q \subseteq Z_k$ is an rv-polyball. For each $T = (t_1, t_2) \in \text{dom}(\lambda)$, let

$$O_T = \{ q \subseteq Z_k : q \text{ is an rv-polyball and } (L_k \circ f \circ (L_k^*)^{-1})(q) \subseteq \text{rv}^{-1}(t_1) \times \{ t_1 \} \}.$$

Then, for each $T = (t_1, t_2) \in \text{dom}(\lambda)$, there is an open subball $a_T \subseteq \text{rv}^{-1}(t_1) \times \{ t_1 \} \subseteq C$ and a $q_T \in O_T$ such that $(\lambda(T), T) \in a_T$ and $i(T) = (L_k \circ f \circ (L_k^*)^{-1})(q_T)$. Let $C^* = \bigcup \{ q_T : T \in \text{dom}(\lambda) \} \subseteq Z_k$ and, for each $T \in \text{dom}(\lambda)$,

$$a_T = (L_k \circ f \circ (L_k^*)^{-1})(\lambda(T), T) \in q_T.$$

Let $\lambda^* : \text{pr}_{\geq 1} C^* \rightarrow \text{VF}$ be the corresponding focus map given by $\lambda^*(\text{pr}_{\geq 1} q_T) = a_T$. Note that both $C^*$ and $\lambda^*$ are definable. Let $\eta_{k+1}^*$ be the centripetal transformation determined by $C^*$ and $\lambda^*$. For each $T \in \text{dom}(\lambda)$, the restriction of $L_{k+1} \circ f \circ (L_{k+1}^*)^{-1}$ to $c(q_T - a_T)$ is a bijection between the RV-products $c(q_T - a_T)$ and $c(q_T - \lambda(T))$ that is contractible in both ways. So, by the construction of $L_{k+1}$, $(T_Y \circ f \circ L_{k+1}^*)^{-1}$ is contractible. Also, for each $T \in \text{dom}(\lambda)$ and any $q \in O_T$ with $q \neq q_T$,

$$(L_{k+1} \circ f \circ (L_{k+1}^*)^{-1})(c(q))$$
is an open polyball contained in an rv-polyball. So $L_{k+1} \circ f \circ (L_{k+1}^*)^{-1}$ is contractible.

Corollary 10.4. Let $(X_1, g_1), (X_2, g_2) \in \text{Ob RV}[1, :]$ be such that $L(X_1, g_1)$ is definably bijective to $L(X_2, g_2)$. Then there are special bijections $T_1, T_2$ on $L(X_1, g_1)$, $L(X_2, g_2)$ such that $(X_1^*, \text{pr}_1)$ and $(X_2^*, \text{pr}_1)$ are isomorphic, where

$$(X_1^*, \text{pr}_1) = ((\text{pr}_1 \circ T_1)(L(X_1, g_1)), \text{pr}_1),$$

$$(X_2^*, \text{pr}_1) = ((\text{pr}_1 \circ T_2)(L(X_2, g_2)), \text{pr}_1).$$
Proof. By Lemma 10.3 there are special bijections \( T_1, T_2 \) on \( L(X_1, g_1), L(X_2, g_2) \) such that there are definable bijections
\[
F : (rv \circ T_1)(L(X_1, g_1)) \to (rv \circ T_2)(L(X_2, g_2))
\]
and \( F^\dagger \) is a lift of \( F \). Since \( F \) is a bijection between the companions of \( (X_1^*, pr_1) \) and \( (X_2^*, pr_1) \), by Remark 6.14 the natural projection of \( F \) is an isomorphism between the two.

**Definition 10.5.** Let \( X \subseteq VF^n \times RV^{m_1} \) and \( Y \subseteq VF^n \times RV^{m_2} \) and \( f : X \to Y \) a bijection. Let \( E \subseteq \mathbb{N} \) be the set of the indices of the VF-coordinates. We say that \( f \) is relatively unary if there is an \( i \in E \) such that \( (pr_{E_i} \circ f)(\overline{x}) = pr_{E_i}(\overline{x}) \), where \( E_i = E \setminus \{i\} \). In this case we say that \( f \) is unary relative to the coordinate \( i \). If, in addition, \( f \upharpoonright \text{fib}(X, \overline{x}) \) is a special bijection on \( \text{fib}(X, \overline{x}) \) for every \( \overline{x} \in pr_{E_i}X \) then we say that \( f \) is special relative to the coordinate \( i \).

Obviously the inverse of a relatively unary bijection is a relatively unary bijection.

Let \( X \subseteq VF^n \times RV^{m} \), \( C \subseteq RVH(X) \) an RV-product, \( \lambda \) a focus map with respect to \( C \), and \( \eta \) the centripetal transformation with respect to \( \lambda \). Clearly \( \eta \) is unary relative to the coordinate 1. It follows that every special bijection \( T \) on \( X \) of length 1 is a relatively special bijection and hence every special bijection \( T \) on \( X \) is a composition of relatively special bijections.

Suppose that \( X \) is definable. Let \( i \leq n \) and \( E_i = \{1, \ldots, n\} \setminus \{i\} \). By Proposition 7.14 for every \( \overline{a} \in pr_{E_i}X \), there is an \( \overline{a} \)-definable special bijection \( I_\overline{a} \) such that \( I_\overline{a}(\text{fib}(X, \overline{x})) \) is an RV-product and hence, by compactness, there is a special bijection \( I_i \) relative to the coordinate \( i \) such that \( I_i(\text{fib}(X, \overline{a})) \) is an RV-product for every \( \overline{a} \in pr_{E_i}X \).

Let
\[
X_i = \{(\overline{a}, (pr \circ I_i)(a_i, \overline{a}, \overline{t}), \overline{t}) : (a_i, \overline{a}, \overline{t}) \in X \} \subseteq VF^{n-1} \times RV^{m+1}.
\]
We write \( \hat{I}_i : X \to X_i \) for the function induced by \( I_i \). Let \( j \leq n \) with \( j \neq i \). Repeating the above procedure for \( X_i \) with respect to \( j \), we obtain a subset \( X_j \subseteq VF^{n-2} \times RV^{m+2} \) and a function \( \hat{I}_j : X_i \to X_j \), which depend on the relatively special bijection \( I_j \). Continuing this procedure, we see that, for any permutation \( \sigma \) of \( \{1, \ldots, n\} \), there is a sequence of relatively special bijections \( I_{\sigma(1)}, \ldots, I_{\sigma(n)} \) and a corresponding function \( \hat{I}_\sigma : X \to RV^{m+n} \) such that there are an \( E \subseteq \mathbb{N} \) with \( |E| = n \) and a composition of relatively special bijections
\[
I_\sigma = I_{\sigma(n)} \circ \cdots \circ I_{\sigma(1)} : X \to L(\hat{I}_\sigma(X), pr_E).
\]
According to our indexing scheme, we may always assume that \( E = \{1, \ldots, n\} \).

**Definition 10.6.** The function \( \hat{I}_\sigma \) is called a standard contraction of \( X \).

Let \( X \subseteq VF^n \times RV^{m} \) and \( \hat{I}_{id} \) a standard contraction of \( X \) such that \( I_{id}(X) \) is of the form \( rv^{-1}(t_i) \times \{(\overline{0}, t_i, \overline{\infty}, \overline{\infty})\} \), where \( \overline{0} \) is a tuple of 0 of length \( n-1 \) and \( \overline{\infty} \) is a tuple of \( \infty \) of length \( n-1 \). Let \( I_{id} = I_0 \circ \cdots \circ I_1 \) and \( I_{\leq i} = I_1 \circ \cdots \circ I_{i-1} \). Clearly \( I_{\leq i}(X) \) is of the form \( rv^{-1}(t_i) \times \{(\overline{0}, \overline{0}, t_i, \overline{\infty}, \overline{\infty})\} \), where \( \overline{0} \) is a tuple of 0 of length \( n-1 \), \( \overline{0} \in VF \) is a tuple of length \( n-i \), and \( \overline{\infty} \) is a tuple of \( \infty \) of length \( i-1 \). So for any distinct \( (a, \overline{0}, t_i, \overline{\infty}, \overline{\infty}), (b, \overline{0}, t_i, \overline{\infty}, \overline{\infty}) \in I_{id}(X) \) we have that
\[
(pr_{E_i} \circ I_{id}^{-1})(a, \overline{0}, t_i, \overline{\infty}, \overline{\infty}) \neq (pr_{E_i} \circ I_{id}^{-1})(b, \overline{0}, t_i, \overline{\infty}, \overline{\infty}).
\]
This simple observation is used to prove the following:
Lemma 10.7. Let $X \subseteq \text{VF}^n \times \text{RV}^{m_1}$, $Y \subseteq \text{VF}^n \times \text{RV}^{m_2}$ be definable subsets and $f : X \to Y$ a definable bijection. Then there is a definable partition $X_1, \ldots, X_k$ of $X$ such that each $f \upharpoonright X_i$ is a composition of definable relatively unary bijections.

Proof. We do induction on $n$. Since the base case $n = 1$ holds vacuously, we proceed to the inductive step directly. By Lemma 7.14 for each $\overline{a} = (a_1, \ldots, a_{n-1}) \in \text{pvf}_{<n} X$, there is a $\overline{a}$-definable standard contraction $\overline{I}_{\overline{a}, \overline{a}}$ on $\text{fib}(X, \overline{a})$ such that $(\overline{I}_{\overline{a}, \overline{a}} \circ f)(\overline{I}_{\overline{a}, \overline{a}})$ is an RV-product. By Lemma 7.13 in each tuple $(\overline{t}, \overline{a}) = (t_1, \ldots, t_n, \overline{a}) \in \text{prv} Z_{\overline{a}}$, there is at most one $i \leq n$ such that $t_i \neq \infty$, that is, each rv-polynomial contained in $Z_{\overline{a}}$ is of the form $\overline{t} = (\overline{t}, t_i, \overline{a}, \overline{a}) = (\overline{t}, t_i, \overline{a}, \overline{a}) = (\overline{t}, t_i, \overline{a}, \overline{a}) = (\overline{t}, t_i, \overline{a}, \overline{a}) = (\overline{t}, t_i, \overline{a}, \overline{a}) = (\overline{t}, t_i, \overline{a}, \overline{a}) = (\overline{t}, t_i, \overline{a}, \overline{a}) = (\overline{t}, t_i, \overline{a}, \overline{a}) = (\overline{t}, t_i, \overline{a}, \overline{a})$ for some $i \leq n$. So there is an $\overline{a}$-definable partition $A_1, \ldots, A_n$ of $\text{fib}(X, \overline{a})$ such that if $(\overline{t}, \overline{a}_n, \overline{a}) \in A_i$ then $(\overline{I}_{\overline{a}, \overline{a}} \circ f)(\overline{t}, \overline{a}_n, \overline{a})$ is of the form $(b_1, \overline{0}, t_i, \overline{a}, \overline{a})$. By the observation above, if $(b_1, \overline{0}, t_i, \overline{a}, \overline{a})$, $(b_1', \overline{0}, t_i, \overline{a}, \overline{a})$ are distinct elements in $Z_{\overline{a}}$, then 

$$
(\text{pvf}_i \circ I_{\overline{a}, \overline{a}})(b_1, \overline{0}, t_i, \overline{a}, \overline{a}) \neq (\text{pvf}_i \circ I_{\overline{a}, \overline{a}})(b_1', \overline{0}, t_i, \overline{a}, \overline{a}).
$$

Let $g_{\overline{a}, i}$ be the function on $A_i$ given by 

$$(\overline{t}, a_n, \overline{a}) \mapsto (\overline{t}, a_i, \overline{a}, \overline{a}),$$

where $(\text{pvf}_i \circ I_{\overline{a}, \overline{a}})(\overline{t}, a_n, \overline{a}) = (t_1, \overline{0}, t_i, \overline{a}, \overline{a})$ and $(\text{pvf}_i \circ f)(\overline{t}, a_n, \overline{a}) = d_i$. Therefore, after reindexing the VF-coordinates in each $A_i$ separately, each $g_{\overline{a}, i}$ is an $\overline{a}$-definable unary bijection on $A_i$ relative to the coordinate $i$ such that $\text{pvf}_i \circ f = \text{pvf}_i \circ g_{\overline{a}, i}$. By compactness, there are a definable partition $B_1, \ldots, B_n$ of $X$ and definable unary bijections $g_i$ on $B_i$ relative to the coordinate $i$ such that $\text{pvf}_i \circ f = \text{pvf}_i \circ g_i$. 

For each $i \leq n$ let $h_i$ be the function on $g_i(B_i)$ such that $f \upharpoonright B_i = h_i \circ g_i$. For each $a \in (\text{pvf}_i \circ g_i)(B_i)$, since $h_i(\text{fib}(g_i(B_i), a)) = \text{fib}(f(B_i), a)$, by the inductive hypothesis, there is a $a$-definable partition $D_1, \ldots, D_i$ of $g_i(B_i), a)$ such that each $h_i \upharpoonright D_j$ is a composition of $a$-definable relatively unary bijections. So the inductive step holds by compactness. \qed

Lemma 10.8. Let $X \subseteq \text{VF}^2$ be a definable 2-cell. Let 12, 21 denote the permutations of 1, 2. Then there are standard contractions $\overline{I}_{12}$ and $\overline{I}_{21}$ of $X$ such that $(\overline{I}_{12}(X), \text{pvf}_2) \cong (\overline{I}_{21}(X), \text{pvf}_2)$ are isomorphic.

Proof. Let $\epsilon$ be the positioning function of $X$ and $t \in \text{RV}$ the paradigm of $X$. If $t = \infty$ then $X$ is the function $\epsilon : \text{pr}_1 X \to \text{pr}_2 X$, which is either a constant function or a bijection. In the former case, since $X$ is essentially just an open ball, the lemma simply follows from Lemma 7.8. In the latter case, there are special bijections $I_2, J_1$ on $X$ relative to the coordinates 2, 1 such that 

$$
I_2(X) = \text{pr}_1(X) \times \{(0, \infty)\},
$$

$$
J_1(X) = \{0\} \times \text{pr}_2(X) \times \{\infty\}.
$$

So the lemma simply follows from Lemma 10.3. For the rest of the proof we assume that $t \neq \infty$.

If $\epsilon$ is not trapezoidal in $X$ then $X = \text{pr}_1(X) \times \text{pr}_2(X)$ is an open polyball, where $\text{pr}_1(X)$, $\text{pr}_2(X)$ are definable open balls. By Lemma 7.8 there are special bijections $T_1$, $T_2$ on $\text{pr}_1 X$, $\text{pr}_2 X$ such that $T_1(\text{pr}_1 X)$, $T_2(\text{pr}_2 X)$ are RV-products. In this case the standard contractions determined by $(T_1, T_2)$ and $(T_2, T_1)$ are essentially the same.

Suppose that $\epsilon$ is trapezoidal in $X$. Let $r$ be the other paradigm of $X$. Recall that $\epsilon : \text{pr}_1 X \to \text{pr}_2 X$ is again a bijection. Let $I_2$ be the special bijection on
$X$ relative to the coordinate 2 given by $(a, b) \mapsto (a, b - \epsilon(a))$ and $J_1$ the special bijection on $X$ relative to the coordinate 1 given by $(a, b) \mapsto (a - \epsilon^{-1}(b), b)$, where $(a, b) \in X$. Clearly

$$I_2(X) = \text{pr}_1(X) \times \text{rv}^{-1}(t) \times \{t\},$$

$$J_1(X) = \text{rv}^{-1}(r) \times \text{pr}_2(X) \times \{r\}.$$  

So, again, the lemma follows from Lemma 10.8 \hfill $\square$

**Corollary 10.9.** Let $X \subseteq \text{VF}^2 \times \text{RV}^m$ be a definable subset. Let 12, 21 denote the permutations of $\{1, 2\}$. Then there is a definable bijection $f : X \rightarrow \text{VF}^2 \times \text{RV}^l$ such that $f$ is unary relative to both coordinates and there are standard contractions $\hat{I}_{12}$ and $\hat{J}_{21}$ of $f(X)$ such that $(\hat{I}_{12}(f(X)), \text{pr}_{\leq 2})$ and $(\hat{J}_{21}(f(X)), \text{pr}_{\leq 2})$ are isomorphic.

**Proof.** By Lemma 8.8 there is a definable function $f : X \rightarrow \text{VF}^2 \times \text{RV}^l$ such that $f(X)$ is a 2-cell and, for each $(\bar{t}, \bar{r}) \in X$, $f(\bar{t}, \bar{r}) = (\bar{t}, \bar{r}, \bar{s})$ for some $\bar{s} \in \text{RV}^m$. By Lemma 10.8 and compactness, there are standard contractions $\hat{I}_{12}$ and $\hat{J}_{21}$ of $f(X)$ such that there is a commutative diagram:

\[
\begin{array}{ccc}
\text{RV}^l & \overset{F}{\longrightarrow} & \hat{f}(X) \\
\downarrow & \quad & \downarrow \text{pr}_{\leq 2} \\
\hat{I}_{12}(f(X)) & \overset{f}{\longrightarrow} & \hat{J}_{21}(f(X)) \\
\end{array}
\]

where $F$ is a definable bijections. Applying Lemma 2.12 and Lemma 4.8 as in the proof of Lemma 6.18, it is not hard to see from the proof of Lemma 10.8 that $F$ satisfies the condition of finite-to-finite correspondence of isomorphisms in $\text{RV}$-categories with respect to the projection map $\text{pr}_{\leq 2}$ and hence is an isomorphism between $(\hat{I}_{12}(f(X)), \text{pr}_{\leq 2})$ and $(\hat{J}_{21}(f(X)), \text{pr}_{\leq 2})$. \hfill $\square$

11. The kernel of $\mathbb{L}$

We identify all the semigroup homomorphisms induced by $\mathbb{L}$ with $\mathbb{L}$. We shall show that the kernel of $\mathbb{L}$, that is, the semigroup (semiring) congruence relation induced by $\mathbb{L}$ on the domain of $\mathbb{L}$, is in effect definable and hence the inverse of $\mathbb{L}$ modulo the congruence relation is definable.

11.1. Blowups in $\text{RV}$ and the congruence relation $\mathbb{L}_{sp}$.

**Definition 11.1.** Let $(Y, f) \in \text{Ob} \text{RV}[k, \cdot]$ be such that, for all $\bar{t} \in Y$, $f_{|k}(\bar{t}) \in \text{acl}(f_{|1}(\bar{t}), \ldots, f_{|k-1}(\bar{t}))$ and $f_{|k}(\bar{t}) \neq \infty$. Let $(Y, f)^{\sharp} = (Y^{\sharp}, f^{\sharp}) \in \text{Ob} \text{RV}[k, \cdot]$ be such that $Y^{\sharp} = Y \times \text{RV}^{>1}$ and, for any $(\bar{t}, s) \in Y^{\sharp}$,

$$f^{\sharp}_{|i}(\bar{t}, s) = f_{|i}(\bar{t}) \text{ if } 1 \leq i < k,$$

$$f^{\sharp}_{|k}(\bar{t}, s) = sf_{|k}(\bar{t}).$$

The object $(Y, f)^{\sharp}$ is an elementary blowup of $(Y, f)$. An elementary blowup of any subobject of $(Y, f)$ is an elementary sub-blowup of $(Y, f)$.

Let $(X, g) \in \text{Ob} \text{RV}[k, \cdot]$ and $(C, g \mid C) \in \text{Ob} \text{RV}[k, \cdot]$ a subobject of $(X, g)$. Let $F : (Y, f) \rightarrow (C, g \mid C)$ be an isomorphism. Then

$$(Y, f)^{\sharp} \uplus (X \setminus C, g \mid (X \setminus C)) = (Y^{\sharp} \uplus (X \setminus C), f^{\sharp} \uplus (g \mid (X \setminus C)))$$
is the blowup of \((X, g)\) via \(F\), written as \((X, g)^F\), where the subscript \(F\) may be dropped when it is not needed. The subset \(C\) is called the blowup locus of \((X, g)^F\). Let \((Z, h) \in \text{Ob } R V[k, \cdot]\) be isomorphic to a subobject of \((X, g)\). Then the blowup of \((Z, h)\) induced by \(F\), that is, the disjoint union of an elementary sub-blowup of \((Y, f)\) and a subobject of \((Z, h)\), is a sub-blowup of \((X, g)\) via \(F\).

An iterated blowup is a composition of finitely many blowups. The length of an iterated blowup is the length of the composition, that is, the number of the blowups involved.

Note that, for any \((Y, f) \in \text{Ob } R V[k, \cdot]\) and a coordinate of \(f(Y)\), if there is an elementary blowup of \((Y, f)\) with respect to that coordinate then it is unique. We should have included the index of the “blown up” coordinate as a part of the data involved.

**Remark 11.2.** Let \((Y^k, f^k)\) be an elementary blowup of \((Y, f) \in \text{Ob } R V[k, \cdot]\). Since \(r_k \in \text{acl}(\tau_k)\) for each \((\tau_k, r_k) \in f(Y)\), by compactness, the subset \(\text{fib}(f(Y), \tau_k)\) is finite for each \(\tau_k \in (pr_{<k} \circ f)(Y)\). By compactness again, we see that actually the set

\[\{\text{fib}(f(Y), \tau_k) : \tau_k \in (pr_{<k} \circ f)(Y)\}\]

is bounded. By definition, for any \((\tau_k, u) \in f^k(Y^k)\) and \((\tau, s) \in (f^k)^{-1}(\tau_k, u)\), we have that

\[\tau_k = (f_{t1}(\tau), \ldots, f_{tk-1}(\tau))\]

\[u = sf_{tk}(\tau),\]

where \(f_{tk}(\tau) \in \text{fib}(f(Y), \tau_k)\). So the projection map \(pr_{\leq k} : Y^k \rightarrow Y\) is an \(RV[k, \cdot]\)-morphism. Also, since

\[(f^k)^{-1}(\tau_k, u) = \bigcup_{r_k \in \text{fib}(f(Y), \tau_k)} \{f^{-1}(\tau_k, r_k) \times \{s\} : u = sr_k\},\]

clearly if \((Y, f) \in RV[k]\) then \((Y^k, f^k) \in RV[k]\). So any iterated blowup of an object in \(\text{Ob } RV[k]\) is an object in \(\text{Ob } RV[k]\).

**Definition 11.1** is stated relative to the underlying substructure \(S\). If an object \((X, f)\) is \(\bar{\alpha}\)-definable for some extra parameters \(\bar{\alpha}\), then the iterated blowups of \((X, f)\) should be \(\bar{\alpha}\)-definable.

The results below will be stated only for the more general categories \(RV[k, \cdot]\), \(RV[*, \cdot]\), etc. But, by Remark 11.2 they are easily seen to hold when restricted to \(RV[k]\), \(RV[*, \cdot]\), etc. as well.

**Lemma 11.3.** Let \((Y_1, f_1), (Y_2, f_2) \in \text{Ob } RV[k, \cdot]\) and \((Y_1, f_1)^k, (Y_2, f_2)^k\) two elementary blowups. If \((Y_1, f_1), (Y_2, f_2)\) are isomorphic then \((Y_1, f_1)^k, (Y_2, f_2)^k\) are isomorphic.

**Proof.** Let \(F : (Y_1, f_1) \rightarrow (Y_2, f_2)\) be an isomorphism. Let \(F^k : (Y_1^k, f_1^k) \rightarrow (Y_2^k, f_2^k)\) be the bijection given by \((\bar{t}, s) \mapsto (F(\bar{t}), s)\). We claim that \(F^k\) is an isomorphism.

We first check the condition of finite-to-finite correspondence. By compactness, for any \(\tau_k = (r_1, \ldots, r_{k-1}) \in (pr_{<k} \circ f_1)(Y_1)\), the subset \(\text{fib}(f_1(Y_1), \tau_k)\) is finite and does not contain \(\infty\). For any \((\tau_k, u) \in f^k(Y^k)\) and any \((\bar{t}, s) \in Y_1^k = Y_1 \times RV^{>1}\)
with \( f_1^\sharp(t, s) = (\tau_k, u) \), there is an \( r_k \in \text{fib}(f_1(Y_1), \tau_k) \) such that \( f_{1|k}(\tau) = r_k \) and \( u = sr_k \). Let \( A = \{ r_k \in \text{fib}(f_1(Y_1), \tau_k) : \text{there is an } s \in \text{RV}^{>1} \text{ such that } sr_k = u \} \).

We have that
\[
(f_2^\sharp \circ F^\sharp \circ (f_1^\sharp)^{-1})(\tau_k, u)
= (f_2^\sharp \circ F^\sharp) \left( \bigcup_{r_k \in A} \left( f_1^{-1}(\tau_k, r_k) \times \left\{ \frac{u}{r_k} \right\} \right) \right)
\]
\[
= f_2^\sharp \left( \bigcup_{r_k \in A} \left( (F \circ f_1^{-1})(\tau_k, r_k) \times \left\{ \frac{u}{r_k} \right\} \right) \right)
\]
\[
= \bigcup_{r_k \in A} \left\{ f_2^\sharp \left( \frac{u}{r_k} \right) : \tilde{t} \in (F \circ f_1^{-1})(\tau_k, r_k) \right\}
\]
\[
= \bigcup_{r_k \in A} \left\{ \left( \text{pr}_{<k} \circ f_2(\tilde{t}), \frac{u_{f_2(k)\tilde{t}}}{r_k} \right) : \tilde{t} \in (F \circ f_1^{-1})(\tau_k, r_k) \right\}.
\]
Since the subset \((f_2 \circ F \circ f_1^{-1})(\tau_k, r_k)\) is finite for each \( r_k \in A \), it follows that the subset \((f_2^\sharp \circ F^\sharp \circ (f_1^\sharp)^{-1})(\tau_k, u)\) is also finite. Similarly for the other direction \( f_1^\sharp \circ (F^\sharp)^{-1} \circ (f_2^\sharp)^{-1} \).

Next we check that \( F^\sharp \) is volumetric, that is, the condition on weight. For any \((\tilde{t}, s) \in Y_1^\sharp\), if \( s \neq \infty \) then
\[
\text{wgt} \ f_1^\sharp(\tilde{t}, s) = \text{wgt} \ f_1(\tilde{t})
\]
\[
\text{wgt} (f_2^\sharp \circ F)(\tilde{t}, s) = \text{wgt} (f_2 \circ F)(\tilde{t}).
\]
Since \( \text{wgt} \ f_1(\tilde{t}) = (f_2 \circ F)(\tilde{t}) \), we deduce that \( \text{wgt} \ f_1^\sharp(\tilde{t}, s) = \text{wgt} (f_2^\sharp \circ F^\sharp)(\tilde{t}, s) \). If \( s = \infty \) then
\[
\text{wgt} \ f_1^\sharp(\tilde{t}, s) = \text{wgt} \ f_1(\tilde{t}) - 1
\]
\[
\text{wgt} (f_2^\sharp \circ F)(\tilde{t}, s) = \text{wgt} (f_2 \circ F)(\tilde{t}) - 1,
\]
and hence \( \text{wgt} \ f_1^\sharp(\tilde{t}, s) = \text{wgt} (f_2^\sharp \circ F^\sharp)(\tilde{t}, s) \). \( \square \)

**Corollary 11.4.** Let \((X_1, g_1), (X_2, g_2) \in \text{Ob RV}[k, -]\) be isomorphic. Let \((X_1, g_1)^\sharp, (X_2, g_2)^\sharp\) be two blowups of \((X_1, g_1), (X_2, g_2)\) with isomorphic blowup loci. Then \((X_1, g_1)^\sharp, (X_2, g_2)^\sharp\) are isomorphic.

**Lemma 11.5.** Let \((X_1, g_1), (X_2, g_2) \in \text{Ob RV}[k, -]\) be isomorphic. Let \((Z_1, h_1), (Z_2, h_2)\) be two iterated blowups of \((X_1, g_1), (X_2, g_2)\) of length \( l_1, l_2 \), respectively. Then there are isomorphic iterated blowups \((Z_1^\sharp, h_1^\sharp), (Z_2^\sharp, h_2^\sharp)\) of \((Z_1, h_1), (Z_2, h_2)\) of length \( l_2, l_1 \), respectively.

**Proof.** Fix an isomorphism \( I : (X_1, g_1) \rightarrow (X_2, g_2) \). We do induction on the sum \( l = l_1 + l_2 \). For the base case \( l = 1 \), without loss of generality, we assume that \( l_2 = 0 \). Let \( C \) be the blowup locus of \((Z_1, h_1)\). Clearly \((X_2, g_2)\) may be blown up by using the same elementary blowup as \((Z_1, h_1)\), where the blowup locus is changed to \( I(C) \). So the base case holds.

We proceed to the inductive step. Let \((X_1, g_1)^\sharp, (X_2, g_2)^\sharp\) be the first blowups in \((Z_1, h_1), (Z_2, h_2)\) and \( C_1, C_2 \) their blowup loci, respectively. Let \((Y_1, f_1)^\sharp, (Y_2, f_2)^\sharp\)
be the corresponding elementary blowups. If, say, \( l_2 = 0 \), then by the argument in the base case \((X_2, g_2)\) may be blown up to an object that is isomorphic to \((X_1, g_1)\) and hence the inductive hypothesis may be applied. So let us assume that \( l_1, l_2 > 0 \).

Let \( A_1 = C_1 \cap I^{-1}(C_2) \) and \( A_2 = I(C_1) \cap C_2 \). Since \((A_1, g_1 \mid A_1)\) and \((A_2, g_2 \mid A_2)\) are isomorphic, by Lemma 11.3 the elementary sub-blowups of \((Y_1, f_1)^\sharp\), \((Y_2, f_2)^\sharp\) that correspond to \((A_1, g_1 \mid A_1)\) and \((A_2, g_2 \mid A_2)\) are isomorphic. Then, it is not hard to see that the blowup \((X_1, g_1)^{\sharp\sharp}\) of \((X_1, g_1)^\sharp\) using the locus \( I^{-1}(C_2) \) \( \setminus \) \( C_1 \) and its corresponding elementary sub-blowup of \((Y_2, f_2)^\sharp\) and the blowup \((X_2, g_2)^{\sharp\sharp}\) of \((X_2, g_2)^\sharp\) using the locus \( I(C_1) \) \( \setminus \) \( C_2 \) and its corresponding elementary sub-blowup of \((Y_1, f_1)^\sharp\) are isomorphic.

Applying the inductive hypothesis to the iterated blowups \((X_1, g_1)^{\sharp\sharp}, (Z_1, h_1)\) of \((X_1, g_1)^\sharp\), we obtain an iterated blowup \((X_1^*, g_1^*)\) of \((X_1, g_1)^{\sharp\sharp}\) of length \( l_1 - 1 \) and a blowup \((Z_1, h_1)^\sharp\) of \((Z_1, h_1)\) such that \((X_1^*, g_1^*)\) and \((Z_1, h_1)^\sharp\) are isomorphic. Similarly, we obtain an iterated blowup \((X_2^*, g_2*)\) of \((X_2, g_2)^\sharp\) of length \( l_2 - 1 \) and a blowup \((Z_2, h_2)^\sharp\) of \((Z_2, h_2)\) such that \((X_2^*, g_2*)\) and \((Z_2, h_2)^\sharp\) are isomorphic. Now, applying the inductive hypothesis to the iterated blowups \((X_1^*, g_1^*), (X_2^*, g_2*)\) of \((X_1^*, g_1^*)\) of length \( l_2 - 1 \) and an iterated blowup \((X_2^*, g_2*)\) of \((X_2^*, g_2*)\) of length \( l_1 - 1 \) such that \((X_1^*, g_1^*)\) and \((X_2^*, g_2*)\) are isomorphic. Finally, applying the inductive hypothesis to the iterated blowups \((X_1^*, g_1^*), (Z_1, h_1)^\sharp\) of \((X_1^*, g_1^*)\), \((Z_1, h_1)^\sharp\) and the iterated blowups \((X_2^*, g_2*), (Z_2, h_2)^\sharp\) of \((X_2^*, g_2*)\), \((Z_2, h_2)^\sharp\), we obtain an iterated blowup \((Z_1^*, h_1)^\sharp\) of \((Z_1, h_1)^\sharp\) of length \( l_2 - 1 \) and an iterated blowup \((Z_2^*, h_2)^\sharp\) of \((Z_2, h_2)^\sharp\) of length \( l_1 - 1 \) such that \((X_1^*, g_1^*), (Z_1^*, h_1)^\sharp\) are isomorphic and \((X_2^*, g_2*), (Z_2^*, h_2)^\sharp\) are isomorphic. This process is illustrated as follows:

\[
\begin{array}{cccccc}
(X_1, g_1) & \cdots & (X_1, g_1) & \cdots & (X_1, g_1) & \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
(X_1, g_1)^{\sharp\sharp} & \cdots & (X_1^*, g_1^*) & \cdots & (X_2^*, g_2*) & \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
(X_1, g_1)^\sharp & \cdots & (Z_1, h_1)^\sharp & \cdots & (Z_1, h_1)^\sharp & \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
(X_2, g_2) & \cdots & (Z_2, h_2) & \cdots & (Z_2, h_2) & \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
(X_2, g_2)^{\sharp\sharp} & \cdots & (X_2^*, g_2*) & \cdots & (Z_2^*, h_2)^\sharp & \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
(X_2, g_2)^\sharp & \cdots & (Z_2, h_2)^\sharp & \cdots & (Z_2, h_2)^\sharp & \cdots \\
\end{array}
\]

So \((Z_1^*, h_1)^\sharp\) and \((Z_2^*, h_2)^\sharp\) are as desired. \(\square\)

**Definition 11.6.** Let \( I_{sp}[k, \cdot] \) be the subclass of \( \text{Ob RV}[k, \cdot] \times \text{Ob RV}[k, \cdot] \) of those pairs \(((X_1, g_1), (X_2, g_2))\) such that there exist isomorphic iterated blowups \((X_1, g_1)^\sharp\), \((X_2, g_2)^\sharp\). Let

\[
I_{sp}^{[\ast, \cdot]} = \prod_{0 \leq i} I_{sp}^{[i, \cdot]},
\]

\[
I_{sp}[k] = I_{sp}[k, \cdot] \cap (\text{Ob RV}[k] \times \text{Ob RV}[k]),
\]

\[
I_{sp}[\ast] = I_{sp}[\ast, \cdot] \cap \prod_{0 \leq i} (\text{Ob RV}[i] \times \text{Ob RV}[i]).
\]

We will just write \( I_{sp} \) for all these classes if there is no danger of confusion. When the underlying substructure \( S \) is expanded with some extra parameters \( \varpi \) we shall write \( I_{sp}(\varpi) \) for the accordingly expanded classes.
Remark 11.7. By Lemma 11.5 $I_{sp}$ may be thought of as a binary relation on isomorphism classes.

Lemma 11.8. $I_{sp}[k, \cdot]$ is a semigroup congruence relation and $I_{sp}[*, \cdot]$ is a semiring congruence relation.

Proof. Clearly $I_{sp}[k, \cdot]$ is reflexive and symmetric. Suppose that $([(X_1, g_1)], [(X_2, g_2)])$, $([(X_2, g_2)], [(X_3, g_3)])$ $\in I_{sp}[k, \cdot]$. Then, by Lemma 11.5 there are iterated blowups $(X_1, g_1)^2$ of $(X_1, g_1)$, $(X_2, g_2)^2_1$ and $(X_2, g_2)^2_2$ of $(X_2, g_2)$, and $(X_3, g_3)^2$ of $(X_3, g_3)$ such that they are all isomorphic. So $I_{sp}[k, \cdot]$ is transitive and hence is an equivalence relation. For any $[(Z, h)] \in K_+ RV[k, \cdot]$, it is easily checked that

$([(X_1, g_1) \cup (Z, h)], [(X_2, g_2) \cup (Z, h)]) \in I_{sp}[k, \cdot]$

$([(X_1, g_1) \times (Z, h)], [(X_2, g_2) \times (Z, h)]) \in I_{sp}[*, \cdot]$.

So the lemma follows from Remark 6.23.

\[\square\]

11.2. Blowups and special bijections.

Lemma 11.9. Let $(Y, f) \in Ob RV[k, \cdot]$ and $\eta$ a centripetal transformation on $L(Y, f)$ with respect to a focus map $\lambda$ such that the locus of $\lambda$ is $L(Y, f)$. Let $Z = (prv \circ c \circ \eta)(L(Y, f))$. Then $Z, pr_{\leq k} \in Ob RV[k, \cdot]$ is isomorphic to an elementary blowup of $(Y, f)$.

Proof. Suppose that $dom(\lambda) = pr_{> 1} L(Y, f)$. Without loss of generality, we may assume that $0 \notin ran(\lambda)$, that is, $\infty \notin pr_1 f(Y)$. Since $\lambda$ is a function, for every $(r_1, \tau_1) \in f(Y)$ and every $\tau_1 \in rv^{-1}(\tau_1)$ we have that $r_1 \in acl(\tau_1)$ and hence, by Lemma 2.12 $r_1 \in acl(\tau_1)$. So the elementary blowup $(Y^2, f^2)$ of $(Y, f)$ with respect to the first coordinate of $f(Y)$ does exist. Note that, by Convention 7.1 $pr_{\geq k} Z$ is the companion $Y_f$ of $(Y, f)$. Clearly the function $F : Z \longrightarrow Y^2$ given by

$(r_1, \tau_1, f(t_1, \bar{r}_1, t_1, \bar{\tau}_1) \longmapsto (t_1, \bar{r}_1, r_1/f_1(t_1, \bar{\tau}_1))$

is an isomorphism between $(Z, pr_{\leq k})$ and $(Y^2, f^2)$, where $(t_1, \bar{r}_1) \in Y$ and $f(t_1, \bar{\tau}_1) = (f_1(t_1, \bar{\tau}_1), \tau_1)$.

\[\square\]

Corollary 11.10. Let $(X, g) \in Ob RV[k, \cdot]$ and $T$ a special bijection on $L(X, g)$. Let $G = (prv \circ T)(L(X, g))$. Then $Z, pr_{\leq k} \in Ob RV[k, \cdot]$ is isomorphic to an iterated blowup of $(Y, g)$.

Proof. By induction on the length $lh T$ of $T$ and Lemma 11.5 this is immediately reduced to the case $lh T = 1$, which follows from Lemma 11.9.

\[\square\]

Corollary 11.11. Let $(X_1, g_1), (X_2, g_2) \in Ob RV[1, \cdot]$ be such that $L(X_1, g_1)$ is definably bijective to $L(X_2, g_2)$. Then $([(X_1, g_1)], [(X_2, g_2)]) \in I_{sp}$.

Proof. Let $(prv \circ T_1)(L(X_1, g_1)) = Z_1$ and $(prv \circ T_2)(L(X_2, g_2)) = Z_2$. By Corollary 10.4 and Remark 6.19 there are special bijections $T_1, T_2$ on $L(X_1, g_1), L(X_2, g_2)$ such that $(Z_1, pr_1)$ and $(Z_2, pr_1)$ are isomorphic. So the corollary follows from Corollary 11.10.

\[\square\]

Lemma 11.12. Suppose that the substructure $S$ is $(VF, \Gamma)$-generated. Let $Y^2, f^2$ be an elementary blowup of $(Y, f) \in Ob RV[k, \cdot]$. Then there is a special bijection
$T$ of length 1 on $L(Y,f)$ such that there is a commutative diagram

$$
\begin{array}{ccc}
L(Y,f) & \xrightarrow{T} & T(L(Y,f)) \\
\downarrow F & & \downarrow F_1 \\
L(Y^2, f^2) & \xrightarrow{\text{prv}} & Y^2
\end{array}
$$

where $Z = (\text{prv} \circ T)(L(Y,f))$ and both $F$ and $F_1$ are definably bijective.

**Proof.** For any $\mathbf{t} = (\mathbf{t}_k, t_k) = (t_1, \ldots, t_{k-1}, t_k) \in f(Y)$ and any centripetal transformation $\eta$ on $\text{rv}^{-1}(\mathbf{t})$ with respect to a focus map $\lambda$ on $\text{rv}^{-1}(\mathbf{t}_k)$, the function

$$F_\mathbf{t} : (c \circ \eta)(\text{rv}^{-1}(\mathbf{t}) \times f^{-1}(\mathbf{t})) \longrightarrow L(f^{-1}(\mathbf{t}) \times \text{RV}^{>1}, f^t \mid (f^{-1}(\mathbf{t}) \times \text{RV}^{>1}))$$

given by

$$(c \circ \eta)(\mathbf{a}_k, a_k, \mathbf{a}) \mapsto (\mathbf{a}_k, a_k - \lambda(\mathbf{a}_k), \mathbf{a}, \text{rv}(a_k - \lambda(\mathbf{a}_k))/t_k)$$

is a bijection as required. So, by compactness, it is enough to show that there is a $\mathbf{t}$-definable focus map $\lambda$ such that $\text{rv}^{-1}(\mathbf{t}_k) \times f^{-1}(\mathbf{t}) \subseteq \text{dom}(\lambda)$. Let $\text{rvr}(\mathbf{t}) = (\gamma_1, \ldots, \gamma_k) = \mathbf{a}$. Since $t_k \in \text{acl}(t_1, \ldots, t_{k-1})$ and $t_k \neq \infty$, by Lemma 9.4 there is a $\mathbf{a}$-polynomial $p(x_1, \ldots, x_k) = p(\mathbf{a})$ with coefficients in $\mathbf{VF}(\mathbf{a})$ such that $\mathbf{t}$ is a residue root of $p(\mathbf{a})$ but is not a residue root of $\partial p(\mathbf{a})/\partial x_k$. This means that, for every $\mathbf{a}_k \in \text{rv}^{-1}(\mathbf{t}_k)$, $t_k$ is a simple residue root of the $\gamma_k$-polynomial $p(\mathbf{a}_k, x_k)$ and hence, by the generalized Hensel’s Lemma 9.2, there is a unique $a_k \in \text{rv}^{-1}(t_k)$ such that $p(\mathbf{a}_k, a_k) = 0$. So there exists a focus map as desired. \qed

**Remark 11.13.** By the conclusion of Lemma 11.2, $F$ is a lift of $F_1$ and hence, by Remark 6.19, $F_1 \in \text{Mor} \mathbf{RV}[k, \cdot]$. This gives an alternative proof of Lemma 11.3 for the case that the substructure $S$ is $(\mathbf{VF}, \Gamma)$-generated.

**Corollary 11.14.** Suppose that the substructure $S$ is $(\mathbf{VF}, \Gamma)$-generated. Let $(Y, f)$ be an object in $\mathbf{RV}[k, \cdot]$ and $(Y^t, f^t)$ an iterated blowup of $(Y, f)$ of length 1. Then $L(Y^t, f^t) \in \mathbf{VF}[k, \cdot]$ is $(\mathbf{VF}, \Gamma)$-generated.

**Proof.** By induction this is immediately reduced to the case $l = 1$, which follows from Lemma 11.2 and Corollary 9.4. \qed

**Lemma 11.15.** Let $X_1 \subseteq \mathbf{VF}^n \times \mathbf{RV}^{m_1}$, $X_2 \subseteq \mathbf{VF}^n \times \mathbf{RV}^{m_2}$ be two definable subsets such that $\text{prv} X_1 = \text{prv} X_2 = A$. Suppose that there is an $E \subseteq \mathbf{N}$ such that

$$(|\text{fib}(X_1, \mathbf{a})|_E, |\text{fib}(X_2, \mathbf{a})|_E) \in I_{\text{sp}}(\mathbf{a})$$

for every $\mathbf{a} \in A$. Let $\mathcal{I}_\sigma$, $\mathcal{I}_\tau$ be two standard contractions of $X_1$, $X_2$ and $E' = E \cup \{1, \ldots, n\}$. Then

$$(|\mathcal{I}_\sigma(X_1)|_{E'}, |\mathcal{I}_\sigma(X_2)|_{E'}) \in I_{\text{sp}}.$$ 

**Proof.** By induction on $n$ this is immediately reduced to the case $n = 1$. By an argument similar to the one in the proof of Lemma 9.2, there is a special bijection $T_A$ on $A$ such that the following hold.

1. $T_A(A) = A^2$ is an RV-product.
2. For every $\text{rv}$-polyball $p \subseteq A^2$ and every $a_1, a_2 \in A$ with $T_A(a_1), T_A(a_2) \in p$, $\text{fib}(X_1, a_1) = \text{fib}(X_1, a_2)$ and $\text{fib}(X_2, a_1) = \text{fib}(X_2, a_2)$.
(3) Let $h_1 = T_A \circ (\text{prf} \upharpoonright X_1)$ and $h_2 = T_A \circ (\text{prf} \upharpoonright X_2)$. For any rv-polyball $p = \text{rv}^{-1}(t) \times \{(t, \overline{\alpha})\} \subseteq A^2$, 
\[
h_1^{-1}(p) = A_p \times U_{p,1} \quad \text{and} \quad h_2^{-1}(p) = A_p \times U_{p,2},
\]
where $A_p \subseteq A$ and, for any $a \in A_p$, $U_{p,1} = \text{fib}(X_1, a)$ and $U_{p,2} = \text{fib}(X_2, a)$. Moreover, there is a formula $\phi$ such that, for any $a \in A_p$, $\phi(a)$ defines the same iterated blowups that witness $([U_{p,1}]_E, [U_{p,2}]_E) \in \text{Isp}(a)$. Note that, for each $b \in \text{rv}^{-1}(t)$, by Lemma 4.18 the subsets $h_1^{-1}(\text{fib}(A^2, b))$, $h_2^{-1}(\text{fib}(A^2, b))$ are finite and hence, by Lemma 2.12, the aforementioned iterated blowups defined by $\phi(a)$ with $a \in A_p$ are also $t$-definable. Therefore, $([U_{p,1}]_E, [U_{p,2}]_E) \in \text{Isp}(t)$.

Now let
\[
X^1_2 = \bigcup \{\{T_A(a)\} \times \text{fib}(X_1, a) : a \in A\},
\]
\[
X^2_2 = \bigcup \{\{T_A(a)\} \times \text{fib}(X_2, a) : a \in A\}.
\]
Clearly, for any $\overline{t} \in \text{pr}_E X_1$, $\text{fib}(X^1_2, \overline{t})$ is an RV-product that is $\overline{t}$-definably bijective to $\text{fib}(X_1, \overline{t})$ and hence to the $\overline{t}$-definable RV-product $\text{fib}(I_\sigma(X_1), \overline{t})$. By Corollary 11.11 we have that
\[
([\text{prf}\text{fib}(X^1_2, \overline{t})]_1, [\text{fib}(I_\sigma(X_1), \overline{t})]_1) \in \text{Isp}(\overline{t})
\]
and hence, by compactness,
\[
([\text{prf}X^1_2]_{E'}, [\overline{I}_\sigma(X_1)]_{E'}) \in \text{Isp}.
\]
Symmetrically we have that
\[
([\text{prf}X^2_2]_{E'}, [\overline{I}_\sigma(X_2)]_{E'}) \in \text{Isp}.
\]
On the other hand, in the notation of the third item above, we have that $p \times U_{p,1} \subseteq X^1_2$, $p \times U_{p,2} \subseteq X^2_2$, and, by the third item above,
\[
([\text{prf}(p \times U_{p,1})]_E, [\text{prf}(p \times U_{p,2})]_E) \in \text{Isp}(t).
\]
So, by compactness, we deduce that
\[
([\text{prf}X^1_2]_{E'}, [\text{prf}X^2_2]_{E'}) \in \text{Isp}.
\]
Since $\text{Isp}$ is a congruence relation, the lemma follows. 

**Corollary 11.16.** Let $X_1 \subseteq VF^n \times RV^m$, $X_2 \subseteq VF^n \times RV^{m_2}$ be two definable subsets and $f : X_1 \rightarrow X_2$ a unary bijection relative to the coordinate $i$. Then for any permutation $\sigma$ of $\{1, \ldots, n\}$ with $\sigma(1) = i$ and any standard contractions $\overline{I}_\sigma$, $\overline{J}_\sigma$ of $X_1$, $X_2$,
\[
([\overline{I}_\sigma(X_1)]_{\leq n}, [\overline{J}_\sigma(X_2)]_{\leq n}) \in \text{Isp}.
\]

**Proof.** Let $E = \{1, \ldots, n\} \setminus \{i\}$. For any $\overline{\alpha} \in \text{pr}_E X_1 = \text{pr}_E X_2$ and any $\overline{\alpha}$-definable standard contractions $\overline{I}$, $\overline{J}$ of $\text{fib}(X_1, \overline{\alpha})$, $\text{fib}(X_2, \overline{\alpha})$, by Corollary 11.11 we have that
\[
([\overline{I}(\text{fib}(X_1, \overline{\alpha}))]_1, [\overline{J}(\text{fib}(X_2, \overline{\alpha}))]_1) \in \text{Isp}(\overline{\alpha}).
\]
Then the corollary follows from Lemma 11.15. 

\qed
Lemma 11.17. Let $X \subseteq VF^n \times RV^m$ be a definable subset. Let $i, j \in \{1, \ldots, n\}$ be distinct and $\sigma_1, \sigma_2$ two permutations of $\{1, \ldots, n\}$ such that $\sigma_1(1) = \sigma_2(2) = i$, $\sigma_1(2) = \sigma_2(1) = j$, and $\sigma_1 \upharpoonright \{3, \ldots, n\} = \sigma_2 \upharpoonright \{3, \ldots, n\}$. Then, for any standard contractions $\hat{I}_{\sigma_1}, \hat{I}_{\sigma_2}$ of $X$,

$$(\lceil \hat{I}_{\sigma_1}(X) \rceil_{\leq n}, \lceil \hat{I}_{\sigma_2}(X) \rceil_{\leq n}) \in I_{sp}.$$

Proof. Let $ij, ji$ denote the permutations of $\{i, j\}$ and $E = \{1, \ldots, n\} \setminus \{i, j\}$. By compactness and Lemma 11.15 it is enough to show that, for any standard contractions $\hat{I}_{ij}, \hat{I}_{ji}$ of $\text{fib}(X, \bar{\pi})$,

$$(\lceil \hat{I}_{ij}(\text{fib}(X, \bar{\pi})) \rceil_{\leq 2}, \lceil \hat{I}_{ji}(\text{fib}(X, \bar{\pi})) \rceil_{\leq 2}) \in I_{sp}.$$  

To that end, fix an $\bar{\pi} \in \text{pr}_E X$ and let $Y = \text{fib}(X, \bar{\pi})$. By Corollary 10.9, there are a definable bijection $f : Y \to VF^2 \times RV^l$ that is unary relative to both coordinates and two standard contractions $\hat{J}_{ij}, \hat{J}_{ji}$ of $f(Y)$ such that

$$[\hat{J}_{ij}(f(Y))]_{\leq 2} = [\hat{J}_{ji}(f(Y))]_{\leq 2}$$

in the corresponding RV-category with respect to $(\bar{\pi})$. So the desired property follows from Corollary 11.16. \hfill $\Box$

If the substructure $S$ is $(VF, \Gamma)$-generated then the congruence relation $I_{sp}$ is the congruence relation induced by $L$:

Proposition 11.18. Suppose that the substructure $S$ is $(VF, \Gamma)$-generated. Let $(X, g), (Y, f) \in \text{Ob} RV[k, \cdot]$. Then

$$[L(X, g)] = [L(Y, f)] \text{ if and only if } [(X, g)] = [(Y, f)] \in I_{sp}.$$ 

Proof. For the “only if” direction, suppose that $F : L(X, g) \to L(Y, f)$ is a definable bijection. By Lemma 10.7, there is a definable partition $X_1, \ldots, X_n$ of $L(X, g)$ such that each $F_i = F \upharpoonright X_i$ is a composition of relatively unary bijections. By Lemma 11.14 there are special bijections $T_1, T_2$ on $L(X, g), L(Y, f)$ such that $T_1(X_i), (T_2 \circ F)(X_i)$ are RV-products for each $i$. Let

$$G_i = (T_2 \upharpoonright F(X_i)) \circ F_i \circ (T_1^{-1} \upharpoonright T_1(X_i)).$$

Note that each $G_i$ is a composition of relatively unary bijections. By Corollary 11.10 it is enough to show that, for each $i$,

$$(\lceil \text{pr}_V \circ T_1 \rceil(X_i))_{\leq k}, (\lceil \text{pr}_V \circ T_2 \circ F \rceil(X_i))_{\leq k}) \in I_{sp}.$$ 

This follows from Corollary 11.16 and Lemma 11.14. The “if” direction follows from Corollary 11.14 and Corollary 9.7. \hfill $\Box$

12. Motivic Integration

In this section we assume that the substructure $S$ is $(VF, \Gamma)$-generated.

As before, the results will be stated for the more general categories $RV[k, \cdot], RV[\ast, \cdot]$, etc. By Remark 11.2 it is not hard to see that analogous results may be derived for the restricted categories $RV[k], RV[\ast], etc. if the arguments are accordingly restricted.
Proposition 12.1. For each $k \geq 0$ there is a canonical isomorphism of Grothendieck semigroups

$$
\phi^+_k : K_+ VF[k, \cdot] \longrightarrow K_+ RV[k, \cdot]/I_{sp}
$$

such that

$$
\phi^+_k [X] = [(U, f)]/I_{sp} \text{ if and only if } [X] = [\mathbb{L}(U, f)].
$$

Proof. By Corollary 9.7, $L$ induces a canonical semigroup homomorphism

$$
\mathbb{L} : K_+ RV[k, \cdot] \longrightarrow K_+ VF[k, \cdot].
$$

By Corollary 7.15, $L$ is surjective. By Proposition 11.18, the semigroup congruence relation induced by $L$ is precisely $I_{sp}$ and hence $K_+ RV[k, \cdot]/I_{sp}$ is canonically isomorphic to $K_+ VF[k, \cdot]$. \hfill \square

For each $k > 0$ let $K_+ RV^x[k, \cdot]$ be the sub-semigroup of $K_+ RV[k, \cdot]$ that contains $[0]_k$ and those elements $[(U, f)]$ with $f(U) \subseteq (RV^x)_k$. For $k = 0$ let $K_+ RV^x[0, \cdot] = K_+ RV[0, \cdot]$. We have the direct sums:

$$
K_+ RV^x[k, \cdot] = \bigoplus_{i \leq k} K_+ RV^x[i, \cdot] \subseteq \bigoplus_{0 \leq k} K_+ RV^x[k, \cdot] = K_+ RV^x[*, \cdot].
$$

For each $k \geq 0$, let $E_k$ be the obviously surjective semigroup homomorphism

$$
\bigoplus_{i \leq k} E_{i,k} : K_+ RV^x[\leq k, \cdot] \longrightarrow K_+ RV[k, \cdot].
$$

It is easily seen from the condition on weight in Definition 6.11 that $E_k$ is injective as well. For every $k \geq 0$ we have a commutative diagram:

$$
\begin{array}{ccc}
K_+ RV^x[\leq k, \cdot] & \longrightarrow & K_+ RV^x[\leq k + 1, \cdot] \\
\downarrow E_k & & \downarrow E_{k+1} \\
K_+ RV[k, \cdot] & \longrightarrow & K_+ RV[k + 1, \cdot]
\end{array}
$$

Let $I_{sp}^x[\leq k, \cdot]$ be the semigroup congruence relation on $K_+ RV^x[\leq k, \cdot]$ induced by $E_k$ and $I_{sp}$. It is easy to see that $I_{sp}^x[\leq k, \cdot]$ is the restriction of $I_{sp}^x[\leq k + 1, \cdot]$ to $K_+ RV^x[\leq k, \cdot]$. So

$$
I_{sp}^x[*, \cdot] = \bigcup_{0 \leq k} I_{sp}^x[\leq k, \cdot]
$$

is a semiring congruence relation on $K_+ RV^x[*, \cdot]$. As above, all these congruence relations shall be simply denoted as $I^x_{sp}$. For every $k \geq 0$, Proposition 12.1 induces a commutative diagram:

$$
\begin{array}{ccc}
K_+ VF[k, \cdot] & \longrightarrow & K_+ VF[k + 1, \cdot] \\
\downarrow f_+ & & \downarrow f_+ \\
K_+ RV^x[\leq k, \cdot]/I_{sp}^x & \longrightarrow & K_+ RV^x[\leq k + 1, \cdot]/I_{sp}^x
\end{array}
$$

Putting these together we obtain:
Theorem 12.2. There is a canonical isomorphism of Grothendieck semirings
\[ \int_+: \mathbf{K} \mathbf{V F}_+[*] \longrightarrow \mathbf{K} \mathbf{R V}^\times_+[*]/I_{sp}^\times \]
such that
\[ \int_+[X] = [(U, f)]/I_{sp}^\times \text{ if and only if } [X] = \mathbb{L}(U, f). \]

In the groupification \( \mathbf{K} \mathbf{R V}^\times_+[*] \) of \( \mathbf{K} \mathbf{R V}^\times_+[*] \), \( I_{sp}^\times \) determines uniquely an ideal \( \mathbb{I} \). We shall give a simple algebraic description of this ideal as follows. First observe that
\[ ([1], [1] \oplus [(\mathbf{R V}^\times)^{>1}]_1) \in I_{sp}^\times, \]
where \( (\mathbf{R V}^\times)^{>1} = \mathbf{R V}^{>1} \setminus \{\infty\} \). Let \( [(Y, f)] \in \mathbf{K} \mathbf{R V}^\times_+[*] \) and \( (Y^\sharp, f^\sharp) \) an elementary blowup of \( (Y, f) \). Let
\[ Y^\sharp_1 = Y \times \{\infty\}, \quad Y^\sharp_2 = Y^\sharp \setminus Y^\sharp_1, \]
\[ f^\sharp_1 = \mathbb{E}_{k-1}(f^\sharp \upharpoonright Y^\sharp_1), \quad f^\sharp_2 = f^\sharp \upharpoonright Y^\sharp_2. \]
It is easily seen from Remark 11.2 that
\[ ([Y^\sharp_2, f^\sharp_1]) \times [1]_1 = [(Y, f)] \]
\[ ((Y^\sharp_2, f^\sharp_1)) \times (\mathbf{R V}^\times)^{>1} \mathbf{K} = [(Y^\sharp_2, f^\sharp_2)]. \]
Hence
\[ ([Y^\sharp_2, f^\sharp_1]) \oplus ([Y^\sharp_2, f^\sharp_2]) = ([Y^\sharp_2, f^\sharp_1]) \oplus ([(Y^\sharp_2, f^\sharp_1)] \times (\mathbf{R V}^\times)^{>1} \mathbf{K}) \]
\[ = ([Y^\sharp_2, f^\sharp_1]) \times ([1]_0 \oplus (\mathbf{R V}^\times)^{>1} \mathbf{K}) \]
\[ = I_{sp}^\times ([Y^\sharp_2, f^\sharp_1]) \times [1]_1 \]
\[ = [(Y, f)]. \]
This shows that, as a semiring congruence relation on \( \mathbf{K} \mathbf{R V}^\times_+[*] \), \( I_{sp}^\times \) is generated by \( ([1], [1]_0 \oplus (\mathbf{R V}^\times)^{>1} \mathbf{K}) \) and hence its corresponding ideal \( \mathbb{I} \) in the graded ring \( \mathbf{K} \mathbf{R V}^\times_+[*] \) is generated by the element \([1]_0 \oplus (\mathbf{R V}^\times)^{>1} \mathbf{K} - [1]_1 \). Let
\[ \mathbb{J} = [1]_1 - (\mathbf{R V}^\times)^{>1} \mathbf{K}. \]
We now compute in \( \mathbf{K} \mathbf{R V}^\times_+[*] \):
\[ [(Y, f)] = [(Y, f)] \times [1]_0 = \mathbb{J} [(Y, f)] \times [1]_1 = [(Y, f)] \times (\mathbf{R V}^\times)^{>1} \mathbf{K} = [(Y, f)] \times \mathbb{J}. \]
Iterating this computation we see that
\[ \mathbf{K} \mathbf{R V}^\times_+[*]/\mathbb{I} \cong \lim_k \mathbf{K} \mathbf{R V}^\times_+ k, \]
where the maps of the direct limit system are given by \( [(Y, f)] \longrightarrow [(Y, f)] \times \mathbb{J}. \) Consequently, the groupification of the isomorphism \( \int_+ \) may be understood as
\[ \int^\mathbf{K} : \mathbf{K} \mathbf{V F}_+[*] \longrightarrow \lim_k \mathbf{K} \mathbf{R V}^\times_+ k. \]
Since this direct limit may be embedded into the zeroth graded piece of the graded ring \( \mathbf{K} \mathbf{R V}^\times_+ [k, \cdot] \mathbb{J}^{-1} \) via the map determined by
\[ [(X, g)] \longrightarrow [(X, g)] \times \mathbb{J}^{-k} \]
for \( [(X, g)] \in \mathbf{K} \mathbf{R V}^\times_+ k, \cdot \), we have the following:
Corollary 12.3. The Grothendieck semiring isomorphism \( \int_\tau \) induces an injective ring homomorphism

\[
\int^K : KVF_* \rightarrow KRV^\times[*_\cdot][J^{-1}].
\]

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