Structural controllability: an undirected graph approach

Madhu N. Belur and Sivaramakrishnan Sivasubramanian*

Abstract

This paper addresses questions regarding controllability for ‘generic parameter’ dynamical systems, i.e. the question whether a dynamical system is ‘structurally controllable’. Unlike conventional methods that deal with structural controllability, our approach uses an undirected graph: the behavioral approach to modelling dynamical systems allows this. Given a system of linear, constant coefficient, ordinary differential equations of any order, we formulate necessary and sufficient conditions for controllability in terms of weights of the edges in a suitable bipartite graph constructed from the differential-algebraic system. A key notion that helps formulate the conditions is that of a ‘redundant edge’. Removal of all redundant edges makes the inferring of structural controllability a straightforward exercise. We use standard graph algorithms as ingredients to check these conditions and hence obtain an algorithm to check for structural controllability. We provide an analysis of the running time of our algorithm. When our results are applied to the familiar state space description of a system, we obtain a novel necessary and sufficient condition to check structural controllability for this description.

Keywords: maximum matching, controllability, behavioral approach

1 Introduction and related work

When dealing with very large dynamical systems, numerical computation is often not feasible. Under the assumption of genericity of parameters, one can answer questions about controllability and ability to achieve arbitrary pole placement using graph theoretic tools. These issues are typically dealt as ‘structural’ issues in control, see [6, 9] and the survey paper [3]. While existing techniques to address structural aspects of control start from a (possibly singular) state space representation of the system, the results in this paper apply to more general models of dynamical systems: linear differential-algebraic equations of possibly higher order. The behavioral theory of systems allows this general approach. While this problem has been studied and analyzed thoroughly since the classical paper by C.-T. Lin [6], this paper handles this problem using an undirected graph. This is possible because in the behavioral approach to systems, variables are not classified as inputs and outputs, and hence the relation between two variables does not have to be a direction of influence of one on another. Further, dealing with higher order differential equations is just as easy as first order. The proposed method is very straightforward and intuitive: we construct a weighted bipartite graph with one vertex set as the equations and another vertex set as the variables. Lack of structural controllability is shown to be equivalent to existence of connected components of a closely related bipartite graph with some edge-weight conditions. The non-existence of such components can be checked using the algorithm we propose (in Section 5), whose running time is quantified using standard graph algorithms.

While structural controllability is the main focus of this paper, the results in this paper are relevant to some other questions about generic properties of polynomial matrices. (See Definition 2 for the precise meaning of genericity of a property.) The first question is under what conditions on a polynomial matrix $M$ can we say that the invariant polynomials of $M$ are generically one. This is nothing but the question as to when are the determinants of all the

*The authors are respectively in the Departments of Electrical Engineering and Mathematics, Indian Institute of Technology Bombay, Powai Mumbai 400076, India. Corresponding author’s email: belur@ee.iitb.ac.in and fax number: +91.22.2572.3707.
maximal minors of a (nonsquare) polynomial matrix generically coprime. Another way to formulate this question is when does the Smith normal form of a univariate polynomial matrix generically have only ones (and zeros) along the diagonal. Finally, this issue is equivalent to the ability to embed a polynomial matrix as a sub-matrix of a unimodular matrix. (A unimodular matrix is defined in Section 2 as a square polynomial matrix whose determinant is a nonzero constant.)

The paper is organized as follows. Some definitions regarding the behavioral approach and some graph theoretic preliminaries are covered in the following section. Section 3 contains some results for square polynomial matrices. A notion called ‘redundant edge’ is introduced here. This notion plays a key role in this paper. Section 4 contains the main results of this paper: two equivalent conditions for checking structural controllability of a dynamical system. Section 5 contains an analysis into the efficiency of the algorithm we propose for checking structural controllability. Section 6 specialises our results to state space systems. In Section 7, we study port-terminal interconnection based models. We also consider the standard interconnection procedures: series, parallel and feedback and show properties of these interconnections. In Section 8, we study the situation when the polynomial matrix does not have full rank. Section 9 has some conclusive remarks.

2 Preliminaries

The following subsection deals with polynomial matrices, the next covers the results about the behavioral approach to modelling and control of dynamical systems, while Subsection 2.3 deals with graph theoretic definitions, in particular matchings in a bipartite graph. Subsection 2.4 relates polynomial matrices and bipartite graphs; this subsection is relevant in the context of genericity of parameters. Subsection 2.5 contains a precise definition of genericity and some simple examples.

2.1 Polynomial matrices

Let \( \mathbb{R}[s] \) be the commutative ring of polynomials in the indeterminate \( s \) with coefficients from the field of real numbers \( \mathbb{R} \). Let \( \mathbb{R}^{p \times v}[s] \) be the ring of polynomial matrices with \( p \) rows and \( v \) columns each of whose entry is a polynomial in \( \mathbb{R}[s] \).

A square polynomial matrix \( M \in \mathbb{R}^{v \times v}[s] \) is said to be nonsingular if \( \det(M) \neq 0 \). The roots of the polynomial \( \det(M) \) are called the zeros of \( M \). Thus the zeros of a square polynomial matrix \( M \) are the complex numbers where \( M \) loses its rank: we use this property to define zeros of a possibly nonsquare polynomial matrix. The zeros of \( M(s) \in \mathbb{R}^{p \times v}[s] \) is defined to be the set of those complex numbers \( \lambda \in \mathbb{C} \) where the rank of the polynomial matrix ‘falls’, more precisely:

\[
\text{zeros}(M) := \{ \lambda \in \mathbb{C} \mid \text{rank} (M(\lambda)) < \text{rank} (M(s)) \}.
\] (1)

The polynomial matrix \( M \in \mathbb{R}^{p \times v}[s] \) is said to be full rank if \( \text{rank} (M) = \min(p, v) \). If \( M \) is a full rank polynomial matrix, \( \text{zeros}(M) \) can be found by computing the roots of the greatest common divisor (gcd) of the determinants of all the maximal minors of \( M \). For a detailed exposition of these notions, we refer to [5].

A polynomial matrix \( U \in \mathbb{R}^{v \times u}[s] \) is called unimodular if \( \det(U) \) is a nonzero constant. These are precisely the square nonsingular polynomial matrices whose zero set is empty, or equivalently, whose inverse is also a polynomial matrix.

2.2 Behavioral approach

A detailed exposition of the behavioural approach can be found in [10]; we briefly cover the results that we need in this paper. A linear time invariant (LTI) dynamical system that is described by a system of ordinary differential
equations can be represented as
\[ M_0w + M_1 \frac{d}{dt} w + \cdots + M_N \frac{d^N}{dt^N} w = 0 \] (2)
for constant matrices \( M_i \in \mathbb{R}^{p \times v} \), with \( M_N \neq 0 \) and for \( w \) a vector-valued, infinitely-often differentiable function \( w : \mathbb{R} \to \mathbb{R}^v \). These \( p \) equations can be written in a shorthand notation by introducing the polynomial matrix \( M(s) := M_0 + M_1 s + \cdots + M_N s^N \). Using the polynomial matrix \( M(s) \in \mathbb{R}^{p \times v}[s] \), the differential equations in (2) can be written as \( M(\frac{d}{dt}) w = 0 \). While the differential equations describing a system are not unique, the set of trajectories that the system allows is intrinsic to the system: we call the set of allowed trajectories as the \textit{behavior} of the system. More precisely, the allowed trajectories are those that satisfy (3)
\[ \mathcal{B} = \{ w \in C^\infty(\mathbb{R}, \mathbb{R}^v) \mid M(\frac{d}{dt}) w = 0 \}. \]
where \( C^\infty(\mathbb{R}, \mathbb{R}^v) \) denotes the space of infinitely often differentiable functions from \( \mathbb{R} \) to \( \mathbb{R}^v \). The set \( \mathcal{B} \) is called the behavior of the system. In the context of (3), \( M(\frac{d}{dt}) w = 0 \) is said to be a \textit{kernel representation} of the behavior. A kernel representation is said to be minimal if the row dimension of \( M \) is the minimum of all kernel representations of \( \mathcal{B} \): in this case \( M \) has full row rank. This corresponds to the case that the equations describing the system are linearly independent over \( \mathbb{R}[s] \).

A behavior \( \mathcal{B} \) is called controllable if for any \( w_1, w_2 \in \mathcal{B} \), there exist \( w_3 \in \mathcal{B} \) and \( T \in \mathbb{R} \) such that
\[ w_3(t) = w_1(t) \quad \text{for} \quad t < 0 \quad \text{and} \quad w_3(t) = w_2(t) \quad \text{for} \quad t > T. \]
A behavior \( \mathcal{B} \) is called autonomous if one can conclude that \( w_1 = w_2 \) whenever \( w_1, w_2 \in \mathcal{B} \) satisfy \( w_1(t) = w_2(t) \) for all \( t \leq 0 \). We state the required results from the behavioral literature in the following proposition for easy reference: see [10] [12].

\begin{proposition}
Consider \( M \in \mathbb{R}^{p \times v}[s] \) and let behavior \( \mathcal{B} \) be described by the kernel representation \( M(\frac{d}{dt}) w = 0 \). Then,
\begin{enumerate}
\item \( \mathcal{B} \) is autonomous if and only if \( M \) has full column rank, i.e. \( \text{rank} (M) = v \).
\item The kernel representation \( M \) is minimal if and only if \( M \) has full row rank, i.e. \( \text{rank} (M) = p \).
\item \( \mathcal{B} \) is controllable if and only if \( M(\lambda) \) has constant row rank for every complex number \( \lambda \in \mathbb{C} \).
\end{enumerate}
\end{proposition}

Thus, using the definition of \textit{zeros}(\( M \)) as in (1) above, a behavior described by \( M(\frac{d}{dt}) w = 0 \) is controllable if and only if the zero set of \( M \) is empty. We use this characterization of controllability and give equivalent graph theoretic conditions under the assumption of genericity of parameters.

### 2.3 Matchings in a bipartite graph

A graph \( G = (V, E) \) in which \( V \) can be partitioned into two non-empty sets \( R \) and \( C \) such that each edge in \( E \) is between a vertex in \( R \) and a vertex in \( C \) is called a bipartite graph. We use \( G = (R, C; E) \) to indicate these two vertex sets and the edge set. For this paper, one vertex set \( R \) denotes the rows and the other \( C \) denotes columns of the polynomial matrix \( M \) describing the differential-algebraic equations of (3). A subgraph in which every vertex has degree at most one is called a matching, i.e. each vertex has at most one edge of this subgraph incident on it. The number of edges in a matching \( M \) is denoted by \( |M| \). For a bipartite graph \( G = (R, C; E) \) with vertex sets \( R \) and \( C \), a matching \( m \) is said to be an \textit{R-saturating matching} if \( |m| = |R| \). We define a \textit{C-saturating matching} analogously. The special case when \( G \) satisfies \( |R| = |C| \), an \( R \)-saturating matching is also \( C \)-saturating matching,
and vice-versa: we call such a matching a perfect matching. A detailed exposition on these notions can be found in [7].

A matching $M$ corresponds to a subset $r$ of $R$, $c$ of $C$ and edges $e \subseteq E$. We use $R(M)$, $C(M)$ and $E(M)$ to denote subset $r$ of $R$, $c \subseteq C$ and the subset of edges that occur in the matching $M$ respectively.

As an example, consider the matrix $M \in \mathbb{R}^{p \times v}[s]$ with its nonzero entries: $e_{ij}$, marked by its row and column indices $i$ and $j$ respectively:

$$M := \begin{bmatrix} 0 & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \end{bmatrix}. \quad (4)$$

Figure 1: Example of a bipartite graph

In the associated bipartite graph (see Figure 1), there are four $R$-saturating matchings, say, $M_1$, $M_2$, $M_3$ and $M_4$ corresponding to the edge pairs: $\{e_{12}, e_{21}\}$, $\{e_{12}, e_{23}\}$, $\{e_{13}, e_{22}\}$ and $\{e_{13}, e_{21}\}$, respectively.

2.4 Bipartite graph associated to polynomial matrices

We frequently need to associate an edge weighted bipartite graph to a given polynomial matrix and vice-versa. For $M \in \mathbb{R}^{p \times v}[s]$, we associate an edge weighted bipartite graph $G = (R,C; E)$ as follows. We set $R$ the rows of $M$ and $C$ as the columns of $M$ and an edge between vertex $v_i$ of $R$ and $v_j$ of $C$ if the $(i,j)$-th entry in $M$ is nonzero.

Further, the degree of the polynomial in the $(i,j)$-the entry is assigned as the weight of this edge. Thus $G$ has as many edges as the number of nonzero entries in $M$ and the weights of the edges are non-negative integers. Note that zero weight does not mean no edge; it means that entry in the matrix $M$ is a nonzero constant, i.e. a polynomial of degree zero. (The zero polynomial is usually said to be of degree $-\infty$; we do not need this in our paper.) We say $G = (R,C; E)$ is the bipartite graph associated to $M$. Conversely, given a bipartite graph $G = (R,C; E)$ with the edges having non-negative integral weights, we can associate polynomial matrices that satisfy the degree specifications in $E$ to the graph $G$.

We need some facts about the relation between the determinants of maximal minors in a polynomial matrix $M$ and $R$-saturating matchings of the bipartite graph $G$ associated to $M$. An elaborate treatment of these issues can be found in [1]. We first assume $M$ is square: suppose $M \in \mathbb{R}^{p \times p}[s]$ is a polynomial matrix. Let $m$ be a perfect matching in $G$. Then $m$ corresponds to a nonzero term in the determinant expansion of $M$. The determinant expansion of $M$ is the sum over all perfect matchings in $G$ (with suitable signs). The matrix $M$ being nonsingular implies that there exists at least one perfect matching in $G$. For the converse, i.e., to conclude that $M$ is nonsingular if there are one or more perfect matchings, one needs the important assumption of ‘genericity’. While this is elaborated in the following subsection, we note here that if there is only one perfect matching in $M$, then $M$ is nonsingular, since no cancellations can occur. An upper triangular, lower triangular, or a diagonal square matrix
has only one contributing perfect matching, and hence the determinant comprises of just one product and is nonzero (assuming the entries along the diagonal are nonzero).

We now deal with the case that $M \in \mathbb{R}^{p \times v}$ is not square. Suppose $p \leq v$ and there exists at least one $R$-saturating matching in the graph $G = (R, C; E)$ associated to $M$. (The case when there does not exist even one $R$-saturating matching is dealt in Section 8 as ‘non-minimal descriptions’. If $p \geq v$ then we will relate full column rank with $C$-saturating matchings of $G$. This case is analogous to the $R$-saturating matching case by taking the transpose of matrix $M$; this is not discussed further for this reason.) Let $m$ be an $R$-saturating matching in $G$, and suppose $C(m)$ is the subset of $C$ corresponding to this matching $m$. Then $m$ contributes one term in the determinant expansion of the maximal minor corresponding to $C(m)$ and $R$. In the situation that $M$ is not square, one needs to deal with possibly different subsets $C(m_j)$ for different $R$-saturating matchings $m_j$ in $G$. This is the crux of this paper and is dealt with in the subsequent sections.

We return to the example of (4) and Figure 1. The maximal minor corresponding to the first two columns has only one nonzero term: product of the entries in the $R$-saturating matching $M_1$ defined after (4). The maximal minor due to the 2nd and 3rd columns has two nonzero terms, namely products of the entries in the matchings $M_2$ and $M_3$. Similarly, product of the entries in the matching $M_4$ corresponds to the maximal minor: columns 1 and 3.

2.5 Genericity of parameters

The notion of structural property makes the key assumption of genericity of parameters. We use the following definition, as in [6] or [9, page 132]. A set $S$ in $\mathbb{R}^n$ is said to be an algebraic variety if $S$ is the solution set of an algebraic equation in $n$ variables. The trivial equation is the zero equation in the variables, in which case the algebraic variety $S = \mathbb{R}^n$: we call this variety trivial. We use the important fact that a nontrivial algebraic variety in $\mathbb{R}^n$ or $\mathbb{C}^n$ is a ‘thin’ set, i.e. a set of measure zero. This is used to define genericity of a property.

**Definition 2** Consider property $P$ in terms of variables $a_1, \ldots, a_n \in \mathbb{R}$. Property $P$ is said to be satisfied generically if the set of values $a_1, \ldots, a_n$ that do not satisfy property $P$ form a nontrivial algebraic variety in $\mathbb{R}^n$.

As a simple example, any two nonzero polynomials are generically coprime. Let $M$ be a square matrix and suppose $G = (R, C; E)$ is its associated bipartite graph. We saw that $M$ is nonsingular if $G$ has only one perfect matching. If $G$ has two or more perfect matchings, then the set of values of the parameters that cause cancellations of all terms, thus causing $M$ to be singular, is a so-called ‘thin’ set, i.e. these values form a non-trivial algebraic variety. More precisely, these values form a set of measure zero in the space of all values that can be attained by these parameters. Due to this reason, we say $M$ is nonsingular generically if there exists at least one perfect matching. A key assumption here is that the nonzero entries in $M$ are chosen ‘independently’ and hence they do not satisfy any nontrivial algebraic relation. In this context, the size of the maximal matching on $G$ denotes the generic rank of $M$ and this rank is also called the term-rank of $M$ (see [9], for example).

3 Polynomial matrices: generic properties

In this section we state and prove some generic properties of polynomial matrices; these are utilized in the next section. The following lemma formulates necessary and sufficient conditions for unimodularity of a polynomial matrix: it is one of the main results of this paper. The conditions for the matrix to be nonsingular is a standard result from the literature, see [9].

**Lemma 3** Consider the edge-weighted bipartite graph $G = (R, C; E)$ constructed from the polynomial matrix $M \in \mathbb{R}^{u \times v}[s]$. 
1. $M$ is generically nonsingular if and only if the corresponding bipartite graph has at least one perfect matching.

2. $M$ is generically unimodular if and only if every perfect matching has only constant entry edges.

**Proof.** (1): Since the determinant is a sum over all perfect matchings, nonsingularity of $M$ implies existence of at least one perfect matching in $G$. Conversely, if one or more perfect matchings exist, then we use genericity to rule out cancellations and conclude that $M$ is nonsingular. This proves Statement 1.

(2): If every perfect matching in $G$ has weight zero, then each product in the determinant expansion is a nonzero constant. Since cancellation of these nonzero constants is ruled out upon addition, the determinant of $M$ is also a nonzero constant, thus proving unimodularity. Conversely, if there exists a perfect matching that has weight one or more, then there is at least one product in the determinant expansion that has degree one or more. Since cancellation is ruled out due to genericity, the determinant of $M$ is of degree at least one, and hence $M$ is not unimodular. This proves Statement 2. □

We introduce the notion of a redundant edge: this plays a key role in our results. Let $M \in \mathbb{R}^{p \times v}[s]$ with $p \leq v$ and consider the weighted bipartite graph $G = (R, C; E)$ constructed from the polynomial matrix $M$. An edge $e$ in $G$ is called redundant if $e$ is not an element of any $R$-saturating matching of $G$. Thus the entry corresponding to $e$ does not play a role in the determinant expansion of any maximal minor of $M$; this means $e$ does not affect the zero set of the polynomial matrix $M$. It turns out to help much in our results to remove the redundant edges in a graph. Consider the graph $G$, and remove all redundant edges: we call the resulting subgraph $G_{nr}$. It is the ‘maximal’ subgraph of $G$ with every edge non-redundant. Clearly, $G$ has an $R$-saturating matching if and only if $G_{nr}$ has one. Moreover, the removal of the non-redundant edges results in a different polynomial matrix, say $M_{nr}$. Due to the genericity assumption on $M$, and since the nonzero entries in $M_{nr}$ are same as those in $M$, we have the genericity property for $M_{nr}$ also.

Using $G_{nr}$, the second statement in Lemma 3 can be restated as follows. The polynomial matrix $M$ is unimodular generically if and only if all edges in $G_{nr}$ have weight zero.

### 4 Structural controllability

We now deal with the case that a polynomial matrix $M \in \mathbb{R}^{p \times v}[s]$ is not square. Consider the bipartite graph $G = (R, C; E)$ associated to $M$. The representation is generically minimal if and only if there exists an $R$-saturating matching. Of course, this requires $|R| \leq |C|$, i.e. $p \leq v$.

After the definition of genericity in Subsection 2.5 above, we saw that two nonzero polynomials $p$ and $q$ are generically coprime. This is just another way of stating that $M = [p, q]$ has its zero set empty generically for nonzero polynomials $p$ and $q$. However, if degree of $p = 0$ (i.e. $p$ is a nonzero constant), then $M$ has its zero set empty even when $q$ is allowed to be zero. These two cases are formulated in more generality in Theorem 4 below.

More generally, we ask when does a polynomial matrix $M \in \mathbb{R}^{p \times v}[s]$ have an empty zero set. We answer this question in Theorem 4 below after we see two examples. Consider polynomial matrices $M_1$ and $M_2 \in \mathbb{R}^{p \times v}[s]$ below, in which the *'s denote nonzero entries, and $p \leq v$.

\[
M_1 = \begin{bmatrix}
p_{11} & * & \cdots & * & \cdots & * \\
0 & * & \cdots & * & \cdots & * \\
& \vdots & \ddots & \vdots & \cdots & \vdots \\
0 & * & \cdots & * & \cdots & *
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
N_{11} & N_{12} \\
0 & N_{22}
\end{bmatrix}
\] (5)
Here, $p_{11}$ is an arbitrary nonzero polynomial and $N_{11}$ is square and nonsingular. Obviously, $\text{zeros}(p_{11}) \subseteq \text{zeros}(M_1)$ and $\text{zeros}(N_{11}) \subseteq \text{zeros}(M_2)$. Thus the existence of a square nonsingular block (after permutation of rows and columns, if necessary) within an upper triangular block matrix means additional conditions on such a block matrix are required to hold for the zero set of the matrix $M_1$ or $M_2$ to be empty. This additional condition is that the zero set of this block also be empty, in other words, that this square and nonsingular block be unimodular. The following theorem states that non-existence of such a block submatrix is also sufficient for the zero set of the polynomial matrix to be empty, under the genericity assumption. A nonsingular submatrix that forms the left-upper triangular block (after permutation of rows/columns, if necessary) within an upper triangular block matrix means additional conditions on such a block submatrix are required to hold for the zero set of the matrix $M_1$ or $M_2$ to be empty. This additional condition is that the zero set of this block also be empty, in other words, that this square and nonsingular block be unimodular.

Theorem 4 Consider the bipartite graph $G = (R, C; E)$ associated to the polynomial matrix $M \in \mathbb{R}^{p \times v}[s]$. Assume $p \leq v$, i.e. $|R| \leq |C|$. Suppose all the redundant edges in $G$ are removed to obtain $G_{nr}$. A necessary and sufficient condition for the zero set of $M$ to be empty generically is as follows.

$P$: If there exist subsets $r \subseteq R$ and $c \subseteq C$ with $|r| = |c|$ such that every $R$-saturating matching $M$ matches $r$ to $c$, then all the edges in $G_{nr}$ incident on $r$ have weight zero.

We note that there are two cases within Property $P$. Either there do not exist subsets $r$ and $c$ such that every $R$-saturating matching matches $r$ to $c$, or there do exist such subsets $r$ and $c$, in which case the edges are required to satisfy a weight condition. The proof makes a distinction between these two cases. Another point to note is that removal of redundant edges from $G$ corresponds to removal of corresponding entries from $M$ to obtain, say, $M_{nr}$. Since the entries that have been removed do not affect any of the maximal minors of $M$, the zero sets of $M$ and $M_{nr}$ are equal. In the context of genericity of parameters in $M_{nr}$, as noted above, since the nonzero entries in $M_{nr}$ are the same as those in $M$, we have the genericity property for $M_{nr}$ also.

Proof. (Necessity:) Suppose there exist subsets $r$ and $c$ such that $|r| = |c|$ and every $R$-saturating matching matches $r$ to $c$, but there are some edges incident on $r$ that have a nonzero weight. After a permutation of elements of $R$ and $C$, the matrix $M$ is now in the form $M_2$ of (5), with the topmost $|r|$ rows corresponding to $r$ and leftmost $|r|$ columns corresponding to $c$. Further, due to genericity, since one or more edges incident on $r$ have weight at least one, the square block $N_{11}$ has determinant a polynomial of degree at least one. This implies that at the roots of the determinant, the matrix $N_{11}$ and hence $M_2$ loses rank, thus showing that the zero set of $M_2$ cannot be empty. This proves necessity of the property $P$.

(Sufficiency:) We now assume that the property $P$ is true, and show that the zero set of $M$ is empty generically. We prove this by induction on $|R|$. Let $|R| = 1$. Since $P$ is true, either there exists a subset $c \subseteq C$ with $|c| = 1$ and every $R$ saturating matching matches $R$ to $c$, or there doesn’t exist such a subset $c$. In the former case, property $P$ forces the entry $M_{ec}$ to be a nonzero constant, and hence the zero set of $M$ is empty. In the latter case, there exist at least two sets (in fact, singleton sets) $c_1$ and $c_2$ with $M_{c_1} \neq 0$ and $M_{c_2} \neq 0$. By genericity, these two polynomials have no common roots, and hence the zero set of $M$ is empty.

We now assume the sufficiency of property $P$ for the zero set of $M$ to be empty when the size of $R \leq k$.

Let $M_{nr} \in \mathbb{R}^{(k+1) \times v}[s]$ have $k + 1$ rows and assume $M$ satisfies the property $P$. Again, we first consider the case when there exist sets $r_1 \subset R$ and $c_1 \subset C$ such that $|r_1| = |c_1|$ and every $R$-saturating matching matches $r_1$ to $c_1$. The case where $r_1 = R$ is not covered by the induction hypothesis, but follows from statement 2 of Lemma 3. After a permutation of the rows and columns of $M_{nr}$, we have $M_{nr}$ as follows

$$M_{nr} = \begin{bmatrix}
M_{r_1 c_1} & 0 \\
0 & M_{r_2 c_2}
\end{bmatrix}$$
where \( r_2 := R - r_1 \) and \( c_2 := C - c_1 \). We now note that \( \text{zeros}(M) = \text{zeros}(M_{r_1 \ c_1}) \cup \text{zeros}(M_{r_2 \ c_2}) \). Further, every \( R \)-saturating matching in \( M_{nr} \) is a union of row-saturating matchings of \( M_{r_1 \ c_1} \) and \( M_{r_2 \ c_2} \). Hence, assumption of property \( P \) for \( M_{nr} \) implies this property for \( M_{r_1 \ c_1} \) and \( M_{r_2 \ c_2} \) also. Since \( M_{r_1 \ c_1} \) and \( M_{r_2 \ c_2} \) have at most \( k \) rows, by the induction hypothesis, they have empty zero sets. Hence \( M_{nr} \) also has an empty zero set. This proves the sufficiency for the case when there exist subsets \( r \) and \( c \) that are matched to each other by every \( R \) saturating matching of \( M_{nr} \).

Consider the other case when there do not exist subsets \( r \) and \( c \) of \( R \) and \( C \) respectively such that every \( R \) saturating matching matches \( r \) and \( c \). Consider the gcd of all \((k + 1) \times (k + 1)\) minors of \( M_{nr} \). Due to the absence of subsets \( r \) and \( c \), the various minors have no common factor arising from the determinant of a fixed square nonsingular submatrix. Due to genericity, there is no other reason that can cause the minors to have a common root, and hence the gcd is equal to 1. Thus the zero set of \( M_{nr} \) is empty. This proves the sufficiency of property \( P \) for the case that \( M_{nr} \) has \( k + 1 \) rows, and by induction this proves the sufficiency part of Theorem 4.

The existence of subsets \( r \) and \( c \) satisfying above property would mean that the determinant of this submatrix is a factor of every maximal minor. The requirement that nonredundant edges incident on \( r \) have weight zero ensures that this submatrix is unimodular, i.e. the determinant is a nonzero constant.

It appears from the above theorem that for the zero set of \( R \) to be generically empty, one requires to check the necessary and sufficient condition for every subset \( r \) of \( R \), thus suggesting exponential running time. However, the absence of redundant edges makes it easy to formulate the existence/non-existence of such a subset in easily verifiable conditions. The following theorem relates the condition to the size of a connected component of \( G \). This theorem, another of the main results of this paper, is one that allows use of standard graph theoretic algorithms to check structural controllability: the algorithm is analyzed in the following section.

**Theorem 5** Let \( M \in \mathbb{R}^{p \times v}[s] \) be a polynomial matrix of full row rank. Consider the bipartite graph \( G = (R, C; E) \) constructed from the rows and columns of \( M \) and assume there exists at least one \( R \)-saturating matching. Suppose all redundant edges in the bipartite graph \( G \) are removed to obtain \( G_{nr} \). Let \( g_1, g_2, \ldots, g_c \) be the connected components of \( G_{nr} \). Then \( M(\lambda) \) has full row rank for every complex number \( \lambda \in \mathbb{C} \) generically if and only if

- for every component \( g_i \) satisfying

\[
|R(g_i)| = |C(g_i)|,
\]

all edges in \( g_i \) have weight zero.

**Proof.** The proof becomes simpler if we permute the rows and columns of the polynomial matrix \( M \) such that the matrix assumes a simpler form. We permute the rows and columns of \( M_{nr} \) such that each connected component of \( G_{nr} \) correspond to consecutive rows/columns. Thus \( M_{nr} \) is now in the form:

\[
M_{nr} = \begin{bmatrix}
M_1 & 0 & \cdots & 0 \\
0 & M_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M_c
\end{bmatrix}
\]

with \( M_i \) the submatrices of \( M_{nr} \) corresponding to the connected components \( g_i \). Moreover, \( M_i \) is square if and only if \( |R(g_i)| = |C(g_i)| \). Since there exists an \( R \)-saturating matching in \( M \), and hence in \( M_{nr} \), each \( g_i \) satisfies
[R(g_i)] \leq |C(g_i)|$, and there exists at least one row-saturating matching for each component $g_i$. Further,

$$\text{zeros}(M_{nr}) = \bigcup_{i=1,...,c} \text{zeros}(M_i).$$

With this background, we proceed to the proof.

Only if part: Suppose $M$ has full row rank for every complex number generically, then we show that for every component $g_i$ of $G_{nr}$ satisfying $|R(g_i)| = |C(g_i)|$, we have all edges in $g_i$ have weight zero. Suppose $g_i$ is such that $|R(g_i)| = |C(g_i)|$, and one or more edges in $g_i$ have a nonzero weight. Since each edge is non-redundant, the determinant of $M_i$ is a non-constant. This results in the roots of $\det(M_i)$ causing the zero set of $M$ to be non-empty, thus proving the necessity of the condition by contradiction.

If part: For this part, we need to show that each of the $M_i$ is such that its zero set is empty. There are two cases.

Case 1: $M_i$ is such that $|R(g_i)| = |C(g_i)|$, or

Case 2: $M_i$ is such that $|R(g_i)| < |C(g_i)|$.

In the first case, by assumption, all edges in $g_i$ have weight zero, and hence determinant of $M_i$ is a nonzero constant. This proves that the zero set is empty.

For the second case, since there are multiple row-saturating matchings in $g_i$, there are at least two nonsingular maximal minors in $M_i$. Further, there is no entry that is common to all the terms across all maximal minors: this follows because $g_i$ is connected and every edge is non-redundant. By genericity, the gcd of the two or more maximal minors is one. This proves that the zero set of $M_i$ is empty.

\[\square\]

5 An algorithm and its efficiency

This section contains an analysis of the running time of various algorithms needed for checking structural controllability of a dynamical system by using Theorem 5. We first present the algorithm as pseudocode below. We assume that the input matrix has been mapped into a weighted bipartite graph $G = (R, C; E)$ with $|R| = p, |C| = v$ and weights $w : E \rightarrow \mathbb{Z}_{\geq 0}$.

**Input:** A weighted bipartite graph $G = (R, C; E)$ with $|R| = p, |C| = v$ and weights $w : E \rightarrow \mathbb{Z}_{\geq 0}$.

**Output:** “Structurally controllable” if the system is structurally controllable and “Structurally uncontrollable” otherwise.

1. $G_{nr} = \text{Remove redundant edges}(G)$
2. Let $A_1, A_2, \ldots, A_t$ be the connected components of $G_{nr}$.
   - Comment: Let $A_i$ be a graph with vertex set $V(A_i)$ and edge set $E(A_i)$.
   - Comment: Let $R(A_i) = R \cap V(A_i)$ and $C(A_i) = C \cap V(A_i)$.
3. if all components $A_i$ with $|R(A_i)| = |C(A_i)|$ have $w(e) = 0$ for every $e \in E(A_i)$ then
4. print ‘‘System structurally controllable’’
5. else
6. print ‘‘System structurally uncontrollable’’
7. end if

We analyze the running time of each step of the above algorithm.

**Step 1: Removal of redundant edges:** Recall that an edge $e$ is called redundant if $e$ is not contained in any $R$-saturating matching of $G$. One way to remove redundant edges is by first labelling all edges as redundant

\[\text{We use } \mathbb{Z}_{\geq 0} \text{ to denote the set of non-negative integers.}\]
or non-redundant. In order to label edge \( e = \{x, y\} \) as redundant or non-redundant, consider the subgraph \( G' \) obtained by removing \( e \), the two vertices \( x, y \), and find the size of the maximum cardinality matching in \( G' \). Since \( G' \) is also bipartite with \( R(G') = R - 1 \), it is clear that if the maximum cardinality matching has \( |R| - 1 \) edges, then \( e \) is a non-redundant edge, and if a maximum cardinality saturating matching has strictly less than \( |R| - 1 \) edges, then \( e \) is a redundant edge in \( G \). Do this labelling for all edges \( e \in E \). Finding a maximum cardinality matching in bipartite graphs can be done in \( O(E\sqrt{V}) \) time by finding appropriate augmenting paths. We do not present details of this algorithm as it is standard, instead, we refer the reader to the algorithm of Hopcroft and Karp [4] (see also [2] Page 696)). Since this is done for each edge \( e \in E \), the running time of our algorithm for classification of all edges as redundant or non-redundant takes \( O(E^2\sqrt{V}) \) time.

**Step 2: Decomposition of \( G_{nr} \) into its connected components:** Once all redundant edges in \( G \) have been removed, the algorithm for decomposing \( G_{nr} \) into its connected components is again standard and can be done in \( O(|E_{nr}| \log* (|R| + |C|)) \) time [2] Page 522], where \( |E_{nr}| \) is the number of edges in the subgraph \( G_{nr} \) obtained from \( G \) after removing all its redundant edges.

**Steps 3-7: Connected component checking:** For each connected component \( A_i \) satisfying \( |R(A_i)| = |C(A_i)| \), it takes \( |E(A_i)| \) operations to check the weights of all edges. In other words, in at most these many operations, one can determine whether or not \( A_i \) corresponds to a unimodular submatrix.

**Lemma 6** Given a \( p \times v \) matrix \( M \), let \( E \) be the number of non-zero entries of \( M \). Let \( \mathcal{B} \) be the behaviour of \( M \). Then, there exists an algorithm taking \( O(E^2\sqrt{p + v}) \) time to check if \( \mathcal{B} \) is structurally controllable.

**Proof.** Using the steps listed above and the running time involved for individual steps, it is clear that the running time of the algorithm is at most \( O(E^2\sqrt{p + v}) + O(E \log^* (p + v)) + O(E) \) which is \( O(E^2\sqrt{p + v}) \), thus completing the proof. \( \square \)

It is evident that one requires significantly lesser operations than mentioned in Step 1 for classification of edges into redundant and non-redundant: in the process of marking an edge \( e_1 \), if \( e_1 \) is non-redundant because it could be completed to an \( R \)-saturating matching in \( G \), then one marks all other edges in that \( R \)-saturating matching also in \( G \) as non-redundant. Hence the number of edges remaining to be marked is fewer in the next sweep, if \( e_1 \) is non-redundant. On the other hand, if an edge is redundant, then its immediate removal from \( G \) will quicken the procedure for marking other edges. It appears that for dense matrices, there will be significant improvement by such multiple-marking/intermediate-removal procedure. Thus the complexity could be significantly better than \( O(E^2\sqrt{V}) \). A precise count of the run-time complexity is an interesting problem worth exploring.

### 6 State space systems

The results in this paper simplify determination of structural controllability for the situation \( \frac{dx}{dt} = Ax + Bu \), with \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \). In this section we use the main results of this paper to obtain a novel method for checking structural controllability of the regular state space system.

Construct the polynomial matrix \( M(s) := \begin{bmatrix} sI - A & B \end{bmatrix} \). For the dimensions assumed on \( A \) and \( B \), the polynomial matrix \( M(s) \in \mathbb{R}^{n \times (n + m)}[s] \). Construct the bipartite graph \( G \) corresponding to \( M \) and remove all redundant edges to obtain \( G_{nr} \). The following theorem states that structural controllability of \((A, B)\) is equivalent to each state in \( G_{nr} \) being connected to some input vertex.

**Theorem 7** Consider the bipartite graph \( G \) constructed for \( M(s) = \begin{bmatrix} sI - A & B \end{bmatrix} \in \mathbb{R}^{n \times (n + m)}[s] \). Obtain \( G_{nr} \) by removal of all redundant edges. The system \((A, B)\) is generically controllable if and only if \( G_{nr} \) has the property that each state is connected to some input vertex.
Before we proceed to the proof, it is noteworthy that the above theorem is just application of generic rank check to the Popov-Belevitch-Hautus (PBH) test for controllability. While methods in structural controllability literature have used the rank condition on $[B\ AB\ \cdots \ A^{n-1}B]$, the PBH test has not been explored with as much depth. When applying the generic rank condition to the PBH test, the removal of redundant edges turns out to result in checking connectivity of state vertices in the undirected graph $G_{nr}$. The classical methods treat first order systems using a directed graph approach, and this is the result of the view that each variable in the systems is either an input or output; consequently, the input is to be utilized to control the output in a desired fashion. The behavioral approach allows studying control without having to classify variables into inputs/outputs. This results in an undirected graph. Further, higher order dynamical systems are as easily dealt as first order systems in this approach.

The proof of the above result becomes easier after some notation that is relevant to the state space case. We index the $R$-vertex set of the graph by $x_i$ for $i = 1, \ldots, n$, while the $C$-vertex set has vertices corresponding to states and inputs: $C$ is indexed by $x_i$ for $i = 1, \ldots, n$ and $u_j$ for $j = 1, \ldots, m$. Further, since we are dealing with a system of first order differential equations, all the edges have weight either zero or one. We use a thick edge for an edge of weight one, and dotted edge for edge of weight zero. Further, the thick edges are precisely the ‘parallel edges’: namely the ones that connect $x_i$ to $x_i$, while the dotted edges are the non-parallel edges: those that connect $x_i$ to $x_j$ for $i \neq j$ and also those that connect $x_i$ to $u_k$. In other words, the parallel edges correspond to the diagonal entries in $M(s) := [sI - A \ B]$, i.e. the degree one entries in $M(s)$. There are exactly $n$ parallel edges, and these form one $R$-saturating matching in $G$, and hence all the parallel edges are also in $G_{nr}$. It is sufficiently many non-parallel edges connecting $u_k$ to the states that help controllability of $(A,B)$: this is the intuitive idea of the proof.

**Proof of Theorem 7**. Due to all the parallel edges being non-redundant, and due to each $x_i$ being connected to $x_i$, we first infer that vertices $x_i$ and $x_i$ lie in the same connected component $g$. Hence, for each connected component $g$, the condition $|R(g)| = |C(g)|$ is equivalent to the absence of any input vertex $u$ in $C(g)$.

(Only if part:) We assume that there exists a state $x_i$ such that $x_i$ is not connected to any input vertex in $G_{nr}$, and show that $(A,B)$ is not controllable. Consider $g$, the connected component of $G_{nr}$ which contains $x_i$. Due to the fact that each $x_j$ of $R(G_{nr})$ is connected to $x_j$ (of $C(G_{nr})$), the assumption on $x_i$ implies that there is no input vertex in $C(g)$. This implies that for $g$, we have $|R(g)| = |C(g)|$. This means every $R$-saturating matching in $G_{nr}$ matches $R(g)$ and $C(g)$. Since the parallel edges in $g$ have weight one each, the condition in Theorem 5 is not satisfied. Thus $(A,B)$ is not controllable.

(If part:) We now show that if every state $x_i$ in $G_{nr}$ is connected to some input vertex $u_j$, then every connected component $g_k$ of $G_{nr}$ satisfies the condition $|R(g_k)| < |C(g_k)|$; from Theorem 5 above, it then follows that the system $(A,B)$ is generically controllable.

Consider the connected components of $G_{nr}$. Due to the $n$ parallel edges in $G$ that connect each state $x_j$ and $x_j$ forming an $R$-saturating matching, we noted above that each parallel edge is non-redundant, and hence in $G_{nr}$. Further, this also causes $|R(g)| \leq |C(g)|$ for each component $g$ of $G_{nr}$. Note that $|C(g)| - |R(g)|$ is precisely the number of input vertices in $g$. Thus a state $x_i$ is connected to some input vertex if and only if $|R(g)| < |C(g)|$ for the component $g$ that contains $x_i$. Hence assuming that $G_{nr}$ is such that every state is connected to some input vertex implies that $|R(g)| < |C(g)|$ for every connected component of $G_{nr}$. By Theorem 5 above, this implies that $(A,B)$ is structurally controllable. □
We use our method to determine structural controllability of some state space models below.

**Example 8** Consider \( A = \begin{bmatrix} 0 & p & 0 \\ 0 & q & 0 \\ 0 & r & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} \), for real numbers \( p, q, r \) and \( s \). This example has been shown to be structurally uncontrollable in [11] using the method developed there. By constructing the graph \( G = (R, C; E) \) for \( [sI - A \ B] \), one can check that the edges corresponding to real numbers \( p \) and \( r \) do not occur in any \( R \)-saturating matching, and hence \( G_{nr} \) does not contain these edges. Consequently, \( x_1 \) and \( x_3 \) are not connected to any input vertex.

**Example 9** Consider \( A = \begin{bmatrix} 0 & p & 0 \\ 0 & q & 0 \\ r & 0 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \), for real numbers \( p, q, r \) and \( s \). This example has been obtained from the previous one just by a change of location of entry \( r \). The corresponding graph \( G_{nr} \) is now such that each state is connected to the input vertex.

**Example 10** Consider \( A = \begin{bmatrix} 0 & r & 0 \\ 0 & 0 & p \\ 0 & 0 & q \end{bmatrix} \) and \( B = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \), for real numbers \( p, q, r \) and \( s \). The significance of this example is that it has been obtained from the previous one by just a permutation of the states: this form is closer to the controller canonical form and makes controllability of \((A, B)\) evident.

### 7 Special models

In this section we study some special cases that are encountered often in studying dynamical systems. We first deal with dynamical systems whose models can be constructed using the fact that they are made up of subsystems that have a particular property. We refer to these as bond-graph-type systems and study them in Subsection 7.1. The following subsection deals with analyzing the three most basic methods of constructing a complex system: interconnection using series, parallel and feedback connection. We prove a result about structural controllability for these basic constructions. In Subsection 7.3 we consider the state space system in controller-canonical form and the Gilbert’s canonical form and apply our main results to that case.
7.1 Port-terminal interconnection based models

In this subsection, we consider an important class of mathematical models for which we obtain finer bounds on the number of edges in the concerned bipartite graph. These are systems comprised of subsystems that are interconnected in a particular fashion. Many physical devices can be modeled as having a certain ‘port-behavior’ and such that when they are interconnected, ‘energy’ is exchanged across their ports. The interconnection of such subsystems gives rise to larger systems that retain this property of port-based energy exchange. These systems can also be analyzed using bond-graph-based tools ([8]). See also [13] for port-based methods of studying this large class of physical systems.

The significance of such models for our paper is that one can estimate the number of edges in the polynomial matrix $M$ using the following heuristic arguments. The matrix $M$ is composed of various differential/algebraic equations corresponding to system laws. The system laws are typically of three types, first: ‘device laws’. Each device law has typically two or three variables. Secondly, interconnection laws of the current/flow type (Kirchoff’s current law, for example). These are laws indicating that the net flow at each junction is zero. Since each junction typically involves three or four variables, the number of edges corresponding to these constraints also are very few. The third important type of constraints are the ‘voltage drop’ kind of equations, this is akin to the Kirchoff’s voltage laws. These equations arise from constraints that the net change of ‘across variables’ around each loop is zero; such constraints can involve a fairly large number of variables, and these contribute to a large number of edges. However, the number of such equations correspond to the number of independent ‘loops’, and hence these constraints are themselves typically small in number. Such arguments can be made in not just electrical networks, but in any system that allows a bond-graph framework for modelling.

We assume $|E| \approx 3|R|$ to obtain heuristic estimates on the running time of the algorithm. In this case, we get from Lemma 6, that the running time is $O(R^2\sqrt{C})$.

7.2 Signal flow graphs

In this subsection we consider the models for dynamical systems constructed using signal-flow graphs. We show that the three basic building blocks of complex interconnections: the series, parallel and feedback connection do not introduce any redundant edge in the resulting larger mathematical model’s bipartite graph $G$. The absence of redundant edges results in significant improvement in the runtime complexity of the algorithm proposed in Section 5; this is elaborated before the beginning of Section 5. The larger question whether arbitrary combination of the three building blocks still leads to: 1) structural controllability, and 2) no redundant edges, remains an important open question.

The following theorem states that series, parallel and feedback interconnection of two systems retains structural controllability, and moreover, there are no redundant edges in the resulting bipartite graph.

**Theorem 11** Let $S_1$ and $S_2$ be two Single Input Single Output (SISO) systems. Consider the system $S_3$ obtained by any one of the following interconnection procedures:

- series interconnection,
- parallel interconnection,
- feedback interconnection.

Then $S_3$ is structurally controllable and the bipartite graph constructed for equations describing $S_3$ has no redundant edges.

**Proof.** Systems $S_1$ and $S_2$ are assumed to have transfer functions $\frac{q_1(s)}{p_1(s)}$ and $\frac{q_2(s)}{p_2(s)}$ respectively. We prove the result for the feedback interconnection, and give only the main features for the other two interconnections.

Let $p_1(\frac{d}{dt})y = q_1(\frac{d}{dt})e$ and $p_2(\frac{d}{dt})v = q_2(\frac{d}{dt})y$ be the differential equations describing $S_1$ and $S_2$. Feedback interconnection results in the additional equation: $e = r - v$. Collating these three equations into a matrix, we get
\[ M(s)w = 0 \]

with

\[
M_{\text{fd}} = \begin{bmatrix}
p_1 & -q_1 \\
q_2 & -p_2
\end{bmatrix}
\quad \text{and} \quad
w_{\text{fd}} = \begin{bmatrix}
y \\
e \\
v \\
r
\end{bmatrix}.
\]

The blank entries in the polynomial matrix \( M_{\text{fd}} \) are all zero. It is straightforward to see that each nonzero entry in \( M_{\text{fd}}(s) \) occurs in some term of a suitable \( 3 \times 3 \) minor of \( M_{\text{fd}} \). This means that the bipartite graph constructed from \( M_{\text{fd}} \) has no redundant edges, thus proving the theorem for this interconnection configuration.

For \( S_1 \) and \( S_2 \) connected in series and in parallel, we can write the two systems of equations in matrix form as follows:

\[
M_{\text{ser}} = \begin{bmatrix}
q_1 & -p_1 \\
q_2 & -p_2
\end{bmatrix}, \quad w_{\text{ser}} = \begin{bmatrix}
r \\
v \\
y
\end{bmatrix}
\quad \text{and} \quad
M_{\text{par}} = \begin{bmatrix}
p_1 & -q_1 \\
q_2 & -p_2 \\
1 & 1 & -1
\end{bmatrix},
\]

\[
w_{\text{par}} = \begin{bmatrix}
u \\
v \\
r \\
y
\end{bmatrix}.
\]

Like the feedback interconnection case, non-redundancy of every edge is verified by checking that each nonzero entry constitutes a term in one or more maximal minors. This proves non-redundancy of every edge. The structural controllability is verified by using Theorem 5.

### 7.3 State space canonical forms

In this subsection we show that the familiar controller canonical state space form also has this property: there are no redundant edges in the bipartite graph constructed from \( M(s) := [sI - A \ B] \), when the \((A, B)\) pair is in the controller canonical form. We also show that the Gilbert’s canonical form also has no redundant edges and displays structural controllability explicitly.

The next important situation when there are no redundant edges is the familiar state space case when the pair \((A, B)\) is in the controller canonical form.

**Theorem 12** Consider \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^n \) as in the equation below:

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}.
\]

Then the polynomial matrix \( M(s) := [sI - A \ B] \) is structurally controllable and has no redundant edges.

**Proof.** The structural controllability is straightforward: from the matrix \( M(s) \), we see that there is one \( n \times n \) minor whose determinant is equal to one; namely the columns corresponding to the last \( n \) columns. This proves the full rank condition for every complex number \( \lambda \) of \( M(\lambda) \).

We now prove the absence of any redundant edge. Consider \( M(s) = [sI - A \ B] \). The diagonal entries in \( M \) form a matching and hence are all non-redundant. The non-redundance of the ones along the superdiagonal follows from the previous paragraph. It remains to show that each of the \( a_i \)'s correspond to non-redundant edges. This can be seen by expansion of the determinant of \( sI - A \) as follows. Since each of the \( a_i \)'s appear in the determinantal
expansion (in fact, they are the coefficients of the characteristic polynomial), for each edge \( a_i \), there is a maximal matching of size \( n \) that contains this edge. This proves that these edges are also non-redundant. This proves the theorem.

We now consider \( A \) and \( B \) in the Gilbert’s canonical form (with \( n = 3 \) for simplicity):

\[
A = \begin{bmatrix} \lambda_1 & 1 \\ \lambda_1 & 0 \\ 0 & \lambda_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ b_2 \\ 0 \end{bmatrix}.
\]

It is a routine matter to verify that each edge in the corresponding bipartite graph is non-redundant, and that the state-space test for structural controllability (Theorem 7) allows conclusion of structural controllability for this \((A, B)\) pair.

It is noteworthy that the controller canonical form’s realization (in terms of the classical series interconnection of \( n \) integrators; see [5, page 39], for example) is just a combination of series and feedback configuration of several SISO subsystems. An interesting open problem would be to prove that there would be no redundant edges and one would have structural controllability for arbitrary interconnection of SISO systems using one or more of series, parallel and feedback interconnection configurations. In other words, it appears (and remains to be proved) that the conventional signal flow graph satisfies the key properties of no redundant edges and structural controllability.

The significance of the absence of redundant edges is that the runtime complexity of the algorithm proposed in Section 5 can be significantly improved. Referring to that algorithm, Step 1 would be inessential. Hence the running time is bounded by \( O(E \log^*(p + v) + E) \), which is \( O(E \log^*(p + v)) \), where recall that \( p \) and \( v \) are the number of equations and variables, and \( E \) is the number of edges in \( G \). Note that \( G \) is same as \( G_{nr} \).

8 Non-minimal descriptions

One important case that we have not addressed so far was when one or more equations describing the system are repeated or, more generally, a linear combination of the other equations. This description of the system is called non-minimal in behavioral literature. While manipulation of equations to obtain an equivalent minimal set of equations is always possible when exact equations are specified, this is not possible in the context of structural controllability checks. This is because in checking structural controllability, we only assume the structure of the system of equations is given, and further, we make the key assumption that the parameters in the equations are algebraically independent. This key assumption will fail to hold if manipulation of equations is allowed to obtain a equivalent and minimal description of the system. This section deals with such non-minimal description of systems.

The first important point to note is that the rank of a polynomial matrix is the size of a nonsingular minor of largest size. If \( M(\frac{d}{dt})w = 0 \) is a kernel representation of a system, then we are dealing with the case when rank \((M) < \) row dimension of \( M \). Hence there does not exist an \( R \)-saturating matching in the bipartite graph \( G = (R, C; E) \) constructed from \( M \). Suppose the size of the matching with largest size is equal to \( r_1 \). Then controllability of the system described by \( M(\frac{d}{dt})w = 0 \) is equivalent to coprimeness of all the \( r_1 \times r_1 \) minors of \( M \). Since checking generic coprimeness of all maximal minors is the main subject of this paper, the results of this paper can easily be modified to handle the case of non-minimal descriptions of linear dynamical systems. The key modification is that we now call an edge in \( G \) redundant if it does not exist in any matching of maximal size.

**Corollary 13** Let \( M(\frac{d}{dt})w = 0 \) be a description, possibly non-minimal, of a dynamical system. Construct the bipartite graph \( G = (R, C; E) \) from \( M \). Let \( r_1 \) be the size of the maximal matching in \( G \). Construct \( G_{nr} \) from \( G \)
by removal of every redundant edge, i.e. an edge that doesn’t occur in any \( r_1 \) sized matching. Resolve \( G_{nr} \) into its connected components \( g_i \). Then, the following are equivalent.

- The system is structurally controllable,
- For every component \( g_i \) in \( G_{nr} \) satisfying \( |R(g_i)| = |C(g_i)| \), all edges in \( g_i \) have weight zero.

The situation of non-minimal description of systems does not happen in the state space case, and hence has not been addressed in the literature. However, as noted in Section 9, this is relevant in the context of Smith normal form of polynomial matrices: non-minimal description means that the polynomial matrix does not have full rank.

9 Conclusive remarks

We developed a method to check structural controllability of a system of differential equations. As mentioned at the beginning of this paper, this work is formulating conditions on a polynomial matrix \( M \) under which its invariant polynomials are generically one, i.e. checking when the Smith normal form of \( M \) has no nonconstant polynomials along its diagonal. This related problem has also been addressed for the case when the polynomial matrix does not have full rank: the so-called non-minimal description of Section 9. In addition to providing necessary and sufficient conditions to check these properties, we also provided run-time complexity of an algorithm to check this. For the more familiar state-space description of a dynamical system, this gives a novel method to check controllability. The central notion that we used at various edges was that of a redundant edge: removal of redundant edges reveals structural properties easily. It may be noted that redundancy here is akin to the fact that the off-diagonal entries do not affect the determinant of an upper-triangular (or lower-triangular) matrix.

In this paper, we addressed only the controllability aspect of dynamical systems. The close relation between the methods to check controllability and observability leave no reason to address or pay any special attention to graph theoretic methods to check structural observability. This is true for both the state space and the behavioral description of dynamical systems.

Acknowledgments: We thank S.R Khare, S. Krishnan and D. Chakraborty for useful discussions.

References

[1] L. Babai and P. Frankl, *Linear Algebraic Methods in Combinatorics*, Department of Computer Science, University of Chicago, 1992.
[2] T.H. Cormen, C.E. Leiserson, R.L. Rivest and C. Stein *Introduction to Algorithms*, M.I.T. Press, 2001.
[3] J.-M. Dion, C. Commault and J. van der Woude, Generic properties and control of linear structured systems: a survey, Automatica, vol 39, pp. 1125-1144, 2003.
[4] J.E. Hopcroft and R.M. Karp, An \( n^{5/2} \) algorithm for maximum matchings in bipartite graphs, *SIAM Journal on Computing*, vol. 2, pp. 225–231, 1973.
[5] T. Kailath, *Linear Systems*, Englewood Cliffs: Prentice-Hall, 1980.
[6] C.-T. Lin, Structural controllability, *IEEE Transactions on Automatic Control*, vol. 19, pp. 201–208, 1974.
[7] L. Lovasz and M.D. Plummer, *Matching Theory*, Amsterdam: North Holland, 1986.
[8] B.M. Maschke, A.J. van der Schaft and P.C. Breedveld, The dynamics of electrical LC-circuits in the light of bond graphs, *Proceedings IMACS 13th World Congress on Computational and Applied Mathematics*, Dublin, Ireland, 1991.

[9] K. Murota, *Systems Analysis by Graphs and Matroids: Structural Solvability and Controllability*. Springer-Verlag, Berlin, 1987.

[10] J.W. Polderman and J.C. Willems, *Introduction to Mathematical Systems Theory: a Behavioral Approach*, New York: Springer, 1997.

[11] K.J. Reinschke, *Multivariable Control: a Graph-theoretic approach*, Berlin: Springer-Verlag, 1988.

[12] J.C. Willems, Paradigms and puzzles in the theory of dynamical systems, *IEEE Transactions on Automatic Control*, vol. 36, pp. 259–294, 1991.

[13] J.C. Willems, Terminals and ports, *Proceedings of the IEEE International Symposium on Circuits and Systems (ISCAS)*, Paris, France, pp. 81–84, 2010.