A Generalisation of Obata’s theorem

Akhil Ranjan and G. Santhanam

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Abstract

In a complete Riemannian manifold \((M, g)\) if the hessian of a real valued function satisfies some suitable conditions then it restricts the geometry of \((M, g)\). In this paper we characterize all compact rank-1 symmetric spaces, as those Riemannian manifolds \((M, g)\) admitting a real valued function \(u\) such that the hessian of \(u\) has atmost two eigenvalues \(-u\) and \(-u+1/2\), under some mild hypothesis on \((M, g)\). This generalises a well known result of Obata which characterizes all round spheres.

1 Introduction

Lichnerowicz proved in [9] that if \((M, g)\) is a complete Riemannian manifold of dimension \(n \geq 2\) such that the Ricci tensor \(\text{Ric}\) and the metric \(g\) verify the relation \(\text{Ric} \geq lg\) for some \(l > 0\), then the first eigenvalue \(\lambda_1\) of the Laplacian of \((M, g)\) satisfies the inequality \(\lambda_1 \geq \frac{n}{n-1} l\). While characterising the equality case of the above result, Obata proved in [11] that a complete Riemannian manifold \((M, g)\) of dimension \(n \geq 2\) is isometric to the round sphere \((S^n, ds^2)\) if and only if there is a real valued function \(u \in C^2(M)\) such that, the hessian of \(u\), \(\nabla^2 u = -u I_d\).

Recently Robert Molzon and Karen Pinney [16] have proved that a complete Kähler manifold \((M, g, J)\) is isometric to a complex projective space

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if and only if there is a real valued function $u \in C^2(M)$ such that on
$\{p \in M : \nabla u(p) \neq 0\}$, $\nabla^2 u = -u Id + \left(\frac{u-1}{2}\right)(Id - \Pi \nabla h u - \Pi \nabla \overline{h} u)$; here $\nabla h u$ denotes the holomorphic gradient of $u$, $\nabla \overline{h} u$ denotes the antiholomorphic gradient of $u$ and $\Pi_X$ denotes the orthogonal projection on the subbundle $X$. In their paper the authors assume that $\nabla u$ is an eigenvector of $\nabla^2 u$ with eigenvalue $-u$ and also that the multiplicity of the eigenvalue $-u$ is 2. Since the manifold $(M, g, J)$ is assumed to be Kähler, $J \nabla u$ is also an eigenvector of $\nabla^2 u$ with eigenvalue $-u$ and hence the subbundle spanned by $\nabla u$ and $J \nabla u$ becomes a totally geodesic integrable subbundle of $TM$.

In this paper, we drop the Kähler condition and also the condition on the multiplicity of the eigenvalues of $\nabla^2 u$. As a consequence, we characterize all the compact rank-1 symmetric spaces under some mild additional hypothesis. However in Kähler case we have much stronger assertion. (See theorem 3).

Our method is different and uses both geometry and topology.

**Theorem 1** Let $(M, g)$ be a complete Riemannian manifold of dimension $d$. Let $u \in C^2(M)$ be a real valued function such that the hessian of $u$, $\nabla^2 u$, has at most two eigenvalues $-u$ and $-\frac{u+1}{2}$ and $\nabla u$ is an eigenvector of $\nabla^2 u$ with eigenvalue $-u$. Then

1. The multiplicity $k$ of the eigenvalue $-u$ is 1, 2, 4, 8 or $d$.

2. If $k = 1$, then either $(M, g)$ is isometric to $\mathbb{RP}^d$ or $S^d$ with constant sectional curvature $\frac{1}{4}$.

3. If $k = d$, then $(M, g)$ is isometric to $S^d$ with constant sectional curvature 1. (Obata’s theorem)

4. If $k = 2, 4$ or 8, then $(M, g)$ is a pointed Blaschke manifold at $m \in M$, where $m$ is the unique maximum for the function $u$, with totally geodesic cut locus $C(m)$. Moreover $H^*(M, \mathbb{Z}) = H^*(\overline{M}, \mathbb{Z})$ where $\overline{M}$ is a compact rank-1 symmetric space of dimension $kn$ and $k$ is the degree of the generator of $H^*(M, \mathbb{Z})$.

**Theorem 2 (Weak Obata’s Theorem)** Let $(M, g)$, $u$, $m$ and $k \geq 2$ be as in theorem 1. Then $Vol(M) = Vol(\overline{M})$ where $\overline{M}$ is a compact rank-1 symmetric space of dimension $kn$ with sectional curvature $\frac{1}{4} \leq K_{\overline{M}} \leq 1$ and $H^*(M, \mathbb{Z}) = H^*(\overline{M}, \mathbb{Z})$. 

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Theorem 3 Let \((M, g)\) and \(u\) be as in theorem 1. If \((M, g)\) is a Kähler manifold, then \((M, g)\) is isometric to \(\mathbb{CP}^n\) with sectional curvature \(\frac{1}{4} \leq K_{\mathbb{CP}^n} \leq 1\).

Theorem 4 Let \((M, g)\), \(u\), \(m\) and \(k\) be as in theorem 1. If \(k \geq 2\) and \((M, g)\) is a \(P_{2\pi}\) manifold at \(m \in M\), then \((M, g)\) is isometric to \(\overline{M}\) where \(\overline{M}\) is as in theorem 2.

2 Preliminaries

Let \((M, g)\) be a complete Riemannian manifold. For any \(u \in C^2(M)\), let \(X := \frac{\nabla u}{\|\nabla u\|}\) on \(\{p \in M : \nabla u(p) \neq 0\}\). Then we have the following

**Proposition 1** If \(\nabla u\) is an eigenvector of \(\nabla^2 u\) then the integral curves of \(X\) are geodesics and conversely.

**Proof:**

\[
\nabla_X X = \frac{1}{\|\nabla u\|} \nabla_X \nabla u + X\left(\frac{1}{\|\nabla u\|}\right)\nabla u
\]

Then

\[
\nabla_X X = 0
\]

iff

\[
\frac{1}{\|\nabla u\|} \nabla_X \nabla u = -X\left(\frac{1}{\|\nabla u\|}\right)\nabla u
\]

\[
= \frac{X(\|\nabla u\|)}{\|\nabla u\|^2} \nabla u
\]

\[
= \frac{<\nabla \nabla u, \nabla u, \nabla u>}{\|\nabla u\|^3} X
\]

\[
= \frac{1}{\|\nabla u\|} <\nabla_X \nabla u, X> X
\]

iff

\[
\nabla \nabla u = <\nabla_X \nabla u, X> \nabla u
\]

iff \(\nabla u\) is an eigenvector of \(\nabla^2 u\). This completes the proof of the proposition.
Proposition 2 Let $u \in C^2(M)$ be such that the integral curves of $X$ are geodesics. Then $u$ does not have saddle points.

Proof: Suppose the proposition is false.

Let $p \in M$ be a saddle point of $u$. Then $\nabla^2 u(p)$ has both positive and negative eigenvalues. Hence there is a neighbourhood $W$ of $p$ in $M$ such that the flow of $X$ have the form of hyperbolas near $p$ and in this neighbourhood $W$ of $p$ they form a saddle. We may assume that $W := \exp_p(W_1)$ where $W_1$ is a neighbourhood of 0 in $T_pM$. (See [1], [10]).

Let $E_{us} \subseteq T_pM$ denote the eigensubspace of $\nabla^2 u(p)$ on which $\nabla^2 u(p)$ is negative definite and $E^s \subseteq T_pM$ denote the eigensubspace of $\nabla^2 u(p)$ on which $\nabla^2 u(p)$ is positive definite. Let $W_{us} := \exp_p(W_1 \cap E_{us})$ and $W_s := \exp_p(W_1 \cap E^s)$. Then the integral curves of $X$ through any point in $W_{us}$ will start from $p$ and diverge near $p$ in $W$ and the integral curves of $X$ through any point in $W^s$ converge to $p$.

Let $B(p, \epsilon)$ be a strongly geodesically convex neighbourhood of $p$ such that $B(p, \epsilon) \subseteq W$. Let $T_{\frac{\epsilon}{2\sqrt{k}}} W^s$ be the tubular neighbourhood of radius $\frac{\epsilon}{2\sqrt{k}}$ of $W^s$ and $T_{\frac{\epsilon}{2\sqrt{k}}} W_{us}$ be the tubular neighbourhood of radius $\frac{\epsilon}{2\sqrt{k}}$ of $W_{us}$ for $k \geq 1$. Now we choose a point $q_1 \in B(p, \epsilon) \cap (T_{\frac{\epsilon}{2\sqrt{k}}} W^s \setminus W^s)$. Let $\sigma$ be the minimizing geodesic from $q_1$ to $B(p, \epsilon) \cap W^s$ such that $\sigma(0) = q_1$ and $\sigma(1) = q_2 \in B(p, \epsilon) \cap W^s$. Let $\gamma_s$ denote the integral curve of $X$, starting at $\sigma(s)$. If $k$ is large, these geodesics $\{\gamma_s\}$ will pass through the tubular neighbourhood $T_{\frac{\epsilon}{2\sqrt{k}}} W_{us}$ of $W_{us}$ and these geodesics will converge to a geodesic in piecewise $C^1$ limit. The limiting geodesic will pass through $p$ and broken at $p$. Since the geodesics $\gamma_s$ are all minimizing in $B(p, \epsilon)$, the limiting geodesic will also be a minimizing geodesic in $B(p, \epsilon)$. This is a contradiction. Hence $u$ can’t have saddle points. This completes the proof of the proposition.

Since $u$ does not have saddle points, the only possible critical points of $u$ are maxima and minima. To describe these points we first compute the function $u$ along the integral curves of $X$ in the following

Lemma 1 Let $u \in C^2(M)$ be such that $\nabla u$ is an eigenvector of $\nabla^2 u$ with eigenvalue $-u$. Then

1. Along the integral curves $\gamma$ of $X$, the function $u$ is of the form $u(\gamma(t)) = A_\gamma \cos t + B_\gamma \sin t$.

2. The function $u$ has only isolated critical points along $\gamma$. 

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Proof: Let $\gamma$ be an integral curve of $X$. Since $\nabla u$ is an eigenvector with eigenvalue $-u$, we have that

$$u''(\gamma(t)) = <\nabla_{\gamma'(t)} \nabla u, \gamma'(t)>
= -u(\gamma(t)) <\gamma'(t), \gamma'(t)>
$$

whenever $\nabla u(\gamma(t)) \neq 0$. Hence $u(\gamma(t)) = A_\gamma \cos t + B_\gamma \sin t$ whenever $\nabla u(\gamma(t)) \neq 0$. Since $(M, g)$ is a complete Riemannian manifold, any geodesic can be extended for all $t \in \mathbb{R}$ and we can write $u(\gamma(t)) = A_\gamma \cos t + B_\gamma \sin t$ whenever $\nabla u(\gamma(t)) \neq 0$.

Clearly critical points of $u$ are not isolated along $\gamma$ only if $A_\gamma = B_\gamma = 0$. Since $\gamma'(t) = X(\gamma(t))$ for almost all $t$, this will mean that $X = 0$, a contradiction for a unit vector field. This proves the lemma.

Here afterwards we will write $u(t)$ for $u(\gamma(t))$ and $X(t)$ for $X(\gamma(t))$.

**Lemma 2** Let $u$ be as in lemma 1. Then

1. The function $u$ attains its maximum at some point $m \in M$ and $\nabla^2 u$ is non-degenerate at $m$.
2. $u(q) = \cos d(q, m)$ for any point $q \in M$.

Proof: Let $\gamma$ be an integral curve of $X$. We know from lemma 1 that $u(t) = A_\gamma \cos t + B_\gamma \sin t$ for all $t \in \mathbb{R}$. Therefore the function $u$ attains a positive maximum and a negative minimum along the geodesic $\gamma$. We may assume that the function $u$ attains its maximum along $\gamma$ at $t = 0$. Let $\gamma(0) = m$. Clearly $u(m) > 0$. Since $u'(0) = 0$, we have that $B_\gamma = 0$.

Since $\gamma$ is an integral curve of $X$, $\gamma'(t) = X(t) = \frac{\nabla u(t)}{\|\nabla u(t)\|}$ whenever $\nabla u(t) \neq 0$. Therefore

$$u'(t) = <\nabla u(t), \gamma'(t)>
= \|\nabla u(t)\|$$

whenever $\nabla u(t) \neq 0$. Hence $u'(t) = 0$ iff $\|\nabla u(t)\| = 0$. This shows that $t = 0$ is a critical point for the function $u$. Since $u(m) > 0$ and the hessian of $u$ has atmost two eigenvalues $-u$ and $-\frac{u+1}{2}$, we have that $\nabla^2 u(m)$ is negative definite at $m$. Hence $\nabla^2 u$ is non-degenerate at $m$.

Since $\nabla^2 u(m)$ is negative definite, $m$ is an isolated critical point. Therefore there is a neighbourhood $W$ of $m$ such that the integral curves of $X$
passing through the points in \( W \) will start at \( m \). This proves that any geodesic \( \gamma \) starting at \( m \) is tangent to \( \nabla u \) and hence any \( v \in T_m M \) is an eigenvector of \( \nabla^2 u(m) \) with eigenvalue \(-u(m)\). Therefore all the eigenvalue of \( \nabla^2 u(m) \) must be equal and hence \( A_{\gamma} = 1 \) for all geodesics \( \gamma \) starting at \( m \) and \( u(t) = \cos t \) along any geodesic \( \gamma \) starting at \( m \).

Since \((M, g)\) is a complete Riemannian manifold, for any point \( q \in M \) there is a distance minimizing geodesic \( \gamma \) from \( m \) to \( q \). Hence \( u(q) = \cos d(m, q) \). This completes the proof of the lemma.

Now we describe the set \( C := \{ q \in M : u(q) = \min_{p \in M} u(p) \} \) in the following

**Lemma 3** The function \( u \) has atmost two maximum \( m \) and \( m' \).

1. If \( m \neq m' \), then the cut locus of \( m = \{ m' \} \) and the cut locus of \( m' = \{ m \} \). Moreover \( M \) is homeomorphic to \( S^{\dim M} \) and the multiplicity of the eigenvalue \(-u\) is 1.

2. (a) If \( m' = m \), then \((M, g)\) is a pointed Blaschke manifold at \( m \in M \), \( C \) is the cut locus of \( m \) and \( C := \{ q \in M : d(q, m) = \pi \} \). Further \( C \) is totally geodesic.
   
   (b) The multiplicity \( k \) of the eigenvalue \(-u\) is 1, 2, 4, 8 or \( \dim M \).
   
   (c) If \( k = 1 \) then \( \pi_1(M) = \mathbb{Z}_2 \) and \( M \) has the homotopy type of \( \mathbb{R} \mathbb{P}^d \). If \( k = d \), then \( M \) has the homotopy type of \( S^d \). If \( k = 2, 4 \) or 8 then \( M \) is simply connected and \( H^*(M, \mathbb{Z}) = H^*(\overline{M}, \mathbb{Z}) \) where \( \overline{M} \) is as in theorem 1.

**Proof:** Let \( C_M := \exp_m(S(0, 2\pi)) \), the image of the sphere of radius \( 2\pi \) in \( T_m M \). Then any point \( q \in C_M \) is a point of maximum for the function \( u \). i.e., \( u(q) = 1 \) for all \( q \in C_M \). Since \( C_M \) is connected and \( \nabla^2 u(q) \) is non-degenerate for \( q \in C_M \), we have that \( C_M \) is a singleton. Let \( C_M = \{ m' \} \). This proves that the function \( u \) has atmost two points of maximum and also that \(-1\) is the minimum for the function \( u \). Hence \( C := \{ q \in M : d(q, m) = \pi \} \) and the eigenvalues of \( \nabla^2 u(q) \) are atmost \(-1\) and \(-\frac{u(q)+1}{2} = 0 \) for all \( q \in C \).

We have seen above that \( C = \exp_m(S(0, \pi)) \). Since \( C \) is connected and \( \nabla^2 u \) has atmost two eigenvalues \(-1\) and 0 on \( C \), the rank of \( \nabla^2 u \) is constant on \( C \). Let us denote this constant by \( k \). Then \( C \) is a \((d-k)\)- dimensional submanifold of \( M \) and the normal bundle of \( C \) is spanned by limiting vectors \( \frac{\nabla u}{\| \nabla u \|} \) as we move towards \( C \).
Since $\nabla u \neq 0$ on the open geodesic ball $B(m, \pi) \setminus \{m\}$ and the flow of $\nabla u$ are geodesics, for any geodesic $\gamma$ starting at $m$, the cut point of $m$ along the geodesic $\gamma$ does not occur on $B(m, \pi) \setminus \{m\}$. Hence $\exp_m \mid_{B(0, \pi)} : B(0, \pi) \to M$ is a diffeomorphism of the open ball $B(0, \pi)$ of radius $\pi$ to $T_m M$ on to the geodesic ball $B(m, \pi)$.

If $m' \neq m$, then, since $\nabla u \neq 0$ on $B(m', \pi) \setminus \{m'\}$ and the flow of $\nabla u$ are geodesics, the set $B(m', \pi) \setminus \{m'\}$ will be free of cut points of $m$. Therefore the cut locus of $m$ is contained in $C \cup \{m\}$. Now, since $M = \overline{B(m, \pi)} \cup B(m', \pi)$ and $M$ is a smooth manifold $C$ is an $(d-1)$- dimensional submanifold of $M$. Therefore $\exp_m : S(0, \pi) \to C$ is either one-one or a two sheeted covering. If $\exp_m : S(0, \pi) \to C$ is a two sheeted covering, then all the geodesics starting at $m$ will stop minimizing beyond $C$. This is a contradiction. Hence $\exp_m : S(0, \pi) \to C$ is one-one. This shows that $\text{diam} M > \pi$. Now the flow of $\nabla u$ will move towards $m$ for points $q$ at distance $< \pi$ from $m$ and the flow of $\nabla u$ will move towards another maximum $m'$ for points $q$ at distance $> \pi$ from $m$. This proves that $M$ is homeomorphic to $S^d$. The proof also shows that the cut-locus of $m$ is $\{m'\}$ and the cut-locus of $m'$ is $\{m\}$.

Since $C$ is a submanifold of dimension $(d - 1)$, the multiplicity $k$ of the eigenvalue $-u$ is 1.

If $m' = m$, then $C$ is the cut locus of $m$ i.e., $C = C(m)$. Since the tangential cut locus of $m$ is spherical, it follows from \cite{12, 13} that $(M, g)$ is a Blaschke manifold at $m \in M$. (See also \cite{2}). Now we prove that $C$ is totally geodesic.

Let $v \in T_q C$. We extend $v$ to a vectorfield $V$ in a neighbourhood of $q \in M$ such that $V$ is tangential to the level sets of $u$. Now we write $V = V_1 + V_2$ where $V_1$ is an eigenvectorfield of $\nabla^2 u$ with eigenvalue $-u$ and $V_2$ is an eigenvectorfield of $\nabla^2 u$ with eigenvalue $-\frac{u+1}{2}$. Then

$$\nabla_V X = \nabla_{V_1} X + \nabla_{V_2} X$$

$$= \frac{u}{\|\nabla u\|} V_1 - \frac{u+1}{2\|\nabla u\|} V_2$$

Therefore, since $V_2(q) = v$ and $V_1(q) = 0$, we have that

$$<\nabla_V X, V>(q) = -\frac{u(q)}{\|\nabla u\|} \|V_1(q)\|^2 - \frac{u(q)+1}{2\|\nabla u\|} \|V_2(q)\|^2$$

$$= 0$$
This proves that $C$ is a totally geodesic submanifold of $(M, g)$.

Since $(M, g)$ is Blaschke manifold at $m$, it follows from Bott-Samelson’s theorem [3], that all geodesics starting at $m$ have same index $\lambda = 0, 1, 3, 7$ or $d - 1$. If $\lambda > 0$, then we have the following possibilities.

1. $\lambda = 1$, $d = 2n$ and $M$ has the homotopy type of $\mathbb{C}P^n$.

2. $\lambda = 3$, $d = 4n$ and $M$ has the integral cohomology ring of $\mathbb{H}P^n$.

3. $\lambda = 7$, $d = 16$ and $M$ has the integral cohomology ring of $\mathbb{C}a\mathbb{P}^2$.

4. $\lambda = d - 1$ and $M$ has the homotopy type of $S^d$.

If $\lambda = 0$, then $\pi_1(M) = \mathbb{Z} \mathbb{Z}_2$ and $M$ has the homotopy type of $\mathbb{R}P^d$.

When $\lambda > 0$, the cut locus of $m$ coincides with the conjugate locus of $m$. Since the cut-locus coincides with the conjugate locus, we have that $\lambda = k - 1$, where $k$ is the rank of the hessian of $u$, $\nabla^2 u$ on $C$. This proves that the multiplicity of the eigenvalue $-u$ is 1, 2, 4, 8 or $\dim M$. This completes the proof.

**Remark:** If $k = d$, then $-\frac{u + 1}{2}$ is not an eigenvalue of $\nabla^2 u$. Hence $\nabla^2 u = -uId$ and $C$ is singleton. This is Obata’s theorem.

### 3 Proof of theorem 1

**Proof of 1(1) and 1(4):** Proof of theorem 1(1) and theorem 1(4) follows from lemma 3.

**Proof of 1(2):** Now we prove that if $k = 1$, then either $(M, g)$ is isometric to $S^d$ or $\mathbb{R}P^d$ with constant sectional curvature $\frac{1}{4}$.

Since the multiplicity of the eigenvalue $-u$ is 1, any vector $E \perp \nabla u$ is an eigenvector of $\nabla^2 u$ with eigenvalue $-\frac{u + 1}{2}$. Hence the eigensubbundle $E_{-\frac{u + 1}{2}} := \{ E \in TM : \nabla^2 u(E) = -\frac{u + 1}{2}E \}$ is parallel along $X$.

Let $\gamma$ be a geodesic starting at $m$ and let $J$ be the Jacobi field describing the variation of the geodesic $\gamma$ such that $J(0) = 0$ and $J'(0) = E \in E_{-\frac{u + 1}{2}}$ is of unit norm. Since $[J, X] = 0$ along $\gamma$, we have that $\nabla_X J = \nabla_{J, X}$. Since $u = \cos t$ along the geodesics $\gamma$ starting at $m$, $\nabla u = -\sin t \frac{\partial}{\partial t}$ and
\[ X = \frac{\nabla u}{\| \nabla u \|} = -\frac{\partial}{\partial t}. \] Therefore

\[-J' = \frac{1}{\| \nabla u \|} \nabla_J \nabla u = \frac{-u + 1}{2} \frac{1}{\| \nabla u \|} J = \frac{-1}{2} \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} J \]

and

\[ \frac{\langle J', J \rangle}{\| J \|^2} = \frac{1}{2} \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} \]

Therefore

\[ \frac{d}{dt} \log \frac{\| J \|}{\sin \frac{t}{2}} = 0 \]

and hence \[ \frac{\| J \|}{\sin \frac{t}{2}} \bigg|_{t=0} = 2. \] This shows that \[ \| J \| = 2 \sin \frac{t}{2}. \] Since \[ E_{-u+1} \]

is parallel along \( X \), we can write \( J(t) = 2 \sin \frac{t}{2} E(t) \) where \( E \) is a parallel vector field along \( X \). Therefore

\[ R(J, X)X = -J'' = \frac{1}{4} J \]

This proves that \( E_{-u+1} \) is an eigensubbundle of \( R(\cdot, X)X \) with eigenvalue \[ \frac{1}{4} \]

and \[ K(E, X) = \frac{1}{4} \] for \( E \in E_{-u+1} \) of unit norm.

First we prove that if \( M \) is homeomorphic to \( S^d \), then \( (M, g) \) is isometric to \( S^d \) with constant sectional curvature \[ \frac{1}{4} \].

We choose a point \( \overline{m} \in S^d \) and fix a linear isometry \( i : T_m M \to T_{\overline{m}} S^d \). Now we define a map \( \Phi : M \to S^d \) by \( \Phi(q) := \exp_{\overline{m}} \circ i \circ \exp_m^{-1}(q) \). Then \( \Phi \) maps the geodesics \( \gamma \) starting at \( m \) in \( M \) on to the geodesics \( \overline{\gamma} \) starting at \( \overline{m} \) in \( S^d \) and it also maps the geodesic spheres around \( m \) in \( M \) on to the geodesic spheres around \( \overline{m} \) in \( S^d \). To complete the proof, we only have to show that \( d\Phi \) is norm preserving. But this follows easily from the fact that
the Jacobi field along the geodesics $\gamma$ starting at $m$ in $M$ are same as that of the Jacobi fields along the geodesics $\gamma$ starting $m$ in $S^d$. This completes the proof.

When $M$ has the homotopy type of $\mathbb{R}P^d$, a similar argument as above shows that $(M, g)$ is isometric to $\mathbb{R}P^d$ with constant sectional curvature $\frac{1}{4}$.

**Proof of 1(3):** Since $k = d$, we have that $\nabla^2 u = -uI$. Now an argument similar to the proof of 1(3) shows that $M$ is isometric to $S^d$ with constant sectional curvature 1.

4 Proof of theorem 2

Let $S(m, r) := \{q \in M : u(q) = \cos r\}$. Then $L := \frac{\nabla^2 u}{\|\nabla u\|}$ is the second fundamental form the level sets $S(m, r)$ of the function $u$, with respect to the inward unit normal. (See [8]). Hence the mean curvature of $S(m, r)$ at any point $p \in S(m, r)$ is

$$Tr(L(p)) = \sum_{i=1}^{kn-1} <L(p)(e_i), e_i>$$

where $e_1, e_2, \ldots, e_{kn-1}$ is an orthonormal basis of $T_p S(m, r)$. Since $-u$ and $-\frac{u+1}{2}$ are the only eigenvalues of $\nabla^2 u$ and the multiplicity of the eigenvalue $-u$ is $k$, we have that

$$Tr(L(p)) = \sum_{i=1}^{kn-1} <L(p)(e_i), e_i>$$

$$= -\{(k-1) \cot t + \frac{(kn-k)(1 + \cos t)}{2 \sin t}\}$$

$$= -\{(k-1) \cot t + \frac{kn-k}{2} \cot \frac{t}{2}\}$$

On the other hand, we know that $Tr(L(p)) = \frac{\theta'_m(t)}{\theta_m(t)}$ where $\theta_m(t)$ is the Riemannian volume density function of $(M, g)$ in geodesic polar coordinates centred at the point $m$ in $M$ (See [8]). This proves that

$$\theta_m(t) = 2^{kn-k} \sin^{kn-k} t \frac{t}{2} \sin^{k-1} t$$

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Therefore

\[
Vol(M) = \int_{U_m M} \int_0^{\pi} \theta_m(t) dt \, d\theta
\]

\[
= \int_{U_m M} \int_0^{\pi} 2^{kn-k} \sin^{kn-k} t \, \frac{t}{2} \sin^{-1} t
\]

\[
= Vol(M)
\]

This completes the proof of theorem 2.

5 Proof of Theorem 3

First we note that, since \((M, g)\) is a Kähler manifold, the even betti numbers are positive.(See [3]). Hence \(H^2(M, \mathbb{Z}) \neq 0\). This proves that the multiplicity of the eigenvalue \(-u\) is 2 and \(H^2(M, \mathbb{Z}) = H^2(\mathbb{C}P^n, \mathbb{Z})\).

For each point \(q \in C\), the cut-locus of \(m\), we denote by \(\sum_q\), the union of all geodesics from \(m\) to \(q\). Then \(\sum_q\) is a smooth surface except possibly at \(m\) and \(\sum_q\) is totally geodesic at \(q\). (See [13]). We write the induced metric on \(\sum_q\) by

\[
ds^2 = dr^2 + f_q(r, \theta) d\theta^2
\]

where \(f_q(r, \theta)\) is a continuous function which is smooth except possibly at \(m\) and \(f_q(0, \theta) = 0 = f_1(\pi, \theta)\) and \(f_q'(\pi, \theta) = -1\). We, now, prove the following

Lemma 4

1. \(\sum_q\) is a smooth totally geodesic surface in \(M\).

2. \(\sum_q\) is isometric to \(S^2\) with constant curvature 1.

Proof: Let \(\gamma\) be a geodesic segment joining \(m\) and \(q\) and \(J\) be the Jacobi field describing the variation of \(\gamma\) such that \(J(0) = 0\) and \(J(\pi) = 0\). We normalise \(J\) such that \(\|J'(\pi)\| = 1\). Since \(J(t) \subseteq T\sum_q\) and \(\sum_q\) is totally geodesic at \(q\), \(J'(\pi) \in T\sum_q\).

Since \(J\) is a Jacobi field along \(\gamma\), we have that \([J, X] = 0\) along \(\gamma\). Hence

\[
\nabla_X J = \nabla_J X
\]

and

\[
- < J', J > = \frac{1}{\|\nabla u\|} < \nabla_J \nabla u, J >
\]
Since the eigenvalues $-\frac{u+1}{2}$ and $-u$ of $\nabla^2 u$ satisfy the inequality $-\frac{u+1}{2} \leq -u$, it follows that

$$-< J', J > \leq -\frac{u}{\| \nabla u \|} \| J \|^2$$

and

$$\frac{< J', J >}{\| J \|^2} \geq \frac{\cos t}{\sin t}$$

Therefore

$$\frac{d}{dt} \log \frac{\| J \|}{\sin t} \geq 0$$

This shows that

$$\frac{\| J \|}{\sin t} \leq \frac{\| J \|}{\sin t} \big|_{t=\pi} = 1$$

i.e., $\frac{\| J \|}{\sin t} \leq \sin t$. Hence

$$\text{Area}(\sum_q) = \int_{S_m} \sum_q \int_0^\pi \| J(t) \| dt \, d\theta$$

$$\leq \int_{S_m} \sum_q \int_0^\pi \sin t \, dt \, d\theta$$

$$= 4\pi$$

This proves that $\text{Area}(\sum_q) \leq 4\pi$. Therefore

$$\min_{q \in C} \text{Area}(\sum_q) \leq 4\pi$$

and equality holds iff

1. The Jacobi field $J$ is an eigenvector field of $\nabla^2 u$ with eigenvalue $-u$, $\| J \| = \sin t$ and $\sum_q$ is a smooth totally geodesic surface in $M$.

2. $\sum_q$ is isometric to $S^2$ with sectional curvature $1$. 
3. \(\exp_m : S(0, \pi) \to C\) is a great circle fibration; here \(S(0, \pi)\) denotes the sphere of radius \(\pi\) in \(T_m M\).

We will now show, using Kähler condition, that \(\text{Area}(\Sigma_q) = 4\pi\) for each \(q \in C\).

Let \(\omega\) be the Kähler form of \((M, g)\). Then we know that \(\frac{\omega^n}{n!}\) is the volume form of \((M, g)\). (See [3]). On the other hand we know that 

\[
\text{Vol}(M) = \frac{4\pi^n}{n!}. 
\]

Therefore

\[
\int_m \frac{\omega^n}{n!} = \frac{4\pi^n}{n!}
\]

i.e., \(f_M(\frac{\omega}{4\pi})^n = 1\).

This shows that \((\frac{\omega}{4\pi})^n\) is a generator of \(H^{2n}(M, \mathbb{Z})\). Now we write \(\frac{\omega}{4\pi} = c\theta\) where \(c > 0\) and \(\theta\) is a generator of \(H^2(M, \mathbb{Z})\). Since \(f_M(\frac{\omega}{4\pi})^n = 1\) and \(\theta^n\) is a generator of \(H^{2n}(M, \mathbb{Z})\), we get that \(c = 1\). Thus we have proved that \(\frac{\omega}{4\pi}\) is a generator of \(H^2(M, \mathbb{Z})\). Hence

\[
\int_{\Sigma_q} \frac{\omega}{4\pi} = 1
\]

But from Wirtinger’s inequality (See [7]), it follows that

\[
\frac{\text{Area}(\Sigma_q)}{4\pi} \geq \int_{\Sigma_q} \left(\frac{\omega}{4\pi}\right) = 1
\]

This proves that \(\text{Area}(\Sigma_q) = 4\pi\). Hence the proof of the lemma.

Let \(E_{-u} := \{ E \in TM : \nabla^2 u(E) = -uE\}\). Then from lemma 4, we see that \(E_{-u}\) is parallel along \(\gamma\) for any geodesic \(\gamma\) from \(m\) to \(q\) and \(E_{-u} \mid \Sigma_q\) is the tangent bundle of the surface \(\Sigma_q\) for any \(q \in C\).

Let \(J\) be the Jacobi field along a geodesic \(\gamma\) from \(m\) to \(q\) such that \(J(0) = 0\) and \(J(\pi) = 0\). Then, since \(\Sigma_q\) is isometric to \(S^2\) with constant curvature 1, the Jacobi field \(J\) is of the form \(J(t) = \sin tE(t)\) where \(E(t)\) is parallel along \(\gamma\) and \(E(t) \in E_{-u}\). This shows that

\[
R(J, X)X = -J'' = J
\]

This proves that \(E_{-u}\) is an eigensubbundle of \(R(\cdot, X)X\) with eigenvalue 1.
Since $E_u$ is parallel along $X$, the eigensubbundle $E_{u+\frac{1}{2}} := \{ E \in TM : \nabla^2 u(E) = -\frac{u+1}{2} \}$ is also parallel along $X$. An easy computation shows that $E_{u+\frac{1}{2}}$ is also eigensubbundle of $R(.,X)X$ with eigenvalue $\frac{1}{4}$. This shows that any Jacobi field $J$ describing the variation of a geodesic $\gamma$ such that $J(0) = 0$ and $J'(0) \in E_{u+\frac{1}{2}}$ is given by $J(t) = 2 \sin \frac{t}{2} E(t)$ where $E(t)$ is a parallel field along $X$. (See also proof of theorem 1 in section 3).

Now we prove the following

**Lemma 5** $\exp_m : S(0,\pi) \to C$ is congruent to Hopf fibration.

**Proof:** We know from lemma 4 that $\exp_m : S(0,\pi) \to C$ is a great circle fibration. Here we will show that this fibration is Riemannian. Then it will follow from the classification of Riemannian submersions of round spheres with connected totally geodesic fibres that this fibration is congruent to Hopf fibration. (See [4], [14], [5]).

Let $W \in T_m C$ be a unit vector. Let $\gamma$ be a geodesic from $m$ to $q$. Then, since $E_{u+\frac{1}{2}}$ is parallel along $\gamma$, there exists a parallel vectorfield $E(t)$ along $\gamma$ such that $E(0) = E$ and $E(\pi) = W$. Now the unit vector $E$ is tangential to $S(0,\pi)$ and orthogonal to the fibre through $\pi \gamma'(0) = \pi v$ where $\gamma'(0) = v$.

Let $J(t) := d(\exp_m)_{tv}(tE)$. Then $J$ is a Jacobi field along $\gamma$ such that $J(0) = 0$ and $J'(0) = E$. But we have seen above that any such Jacobi field is given by $J(t) = 2 \sin \frac{t}{2} E(t)$. Therefore $d(\exp_m)_{\pi v}(\pi E) = J(\pi) = W$. This proves that, up to a constant factor $\pi$, $\exp_m : S(0,\pi) \to C$ is a Riemannian submersion with totally geodesic fibres and hence $\exp_m : S(0,\pi) \to C$ is congruent to Hopf fibration.

Now we come to the proof of theorem 3.

**Proof of theorem 3:** Let us fix a point $\overline{m} \in C \mathbb{P}^n$. Then we know that $\exp_{\overline{m}} : S(0,\pi) \to C(\overline{m})$ is the standard Hopf fibration; here $C(\overline{m})$ denotes the cut locus of $\overline{m}$. Now since, $\exp_m : S(0,\pi) \to C$ is congruent to Hopf fibration there is a linear isometry $i : T_m M \to T_{\overline{m}} C \mathbb{P}^n$ such that $i$ carries the fibres of the fibration $\exp_m : S(0,\pi) \to C$ to the fibres of $\exp_{\overline{m}} : S(0,\pi) \to C(\overline{m})$.

Now we define $\Phi : M \to C \mathbb{P}^n$ by $\Phi(q) := \exp_{\overline{m}} \circ i \circ \exp_{\overline{m}}^{-1}(q)$. Then for any geodesic $\gamma$ through $m \in M$, $\overline{\gamma} := \Phi(\gamma)$ is a geodesic through $\overline{m} \in C \mathbb{P}^n$. To complete the proof, we have to show that $d\Phi$ preserves the lengths of Jacobi fields along $\gamma$. 

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Let $\gamma$ be a geodesic starting at $m \in M$. Let $\gamma'(0) = v$ and let $E(t)$ be the parallel vectorfield along $\gamma$ such that $E(t) \in E_{-u}$. Then $E(t)$ is given by $E(t) = \frac{1}{\sin t}d(\exp_m)_{tv}(E(0))$ and $d\Phi_{\gamma(t)}$ maps $d(\exp_m)_{tv}(E(0))$ to $d(\exp_m)_{ti(\gamma)}(i(E(0)))$. Since the isometry preserves the fibration, the vector $d(\exp_m)_{ti(\gamma)}(i(E(0))) \in E_1 := \{w \in T\mathcal{IP}^n : R(w,\gamma)^\gamma = w\}$; here $R$ denotes the Riemannian curvature tensor of $\mathcal{IP}^n$. We know all the Jacobi fields on $\mathcal{IP}^n$ along the geodesics $\gamma$ and they are of the form $J(t) = \sin t E(t)$ for $E \in E_1$ and $J(t) = 2 \sin \frac{t}{2} E(t)$ for $E \in E_{\pm 1}$ where $E_{\pm 1} := \{w \in T\mathcal{IP}^n : R(w,\gamma)^\gamma = \pm \frac{1}{4}\}$. Hence we see that $d(\exp_m)_{ti(\gamma)}(i(E(0))) = \sin t i(E(0))$. This shows that $d\Phi$ is norm preserving on $E_{-u}$.

By similar arguments we can show that $d\Phi$ is norm preserving on $E_{\pm 1}$. Hence $\Phi : M \setminus C \to \mathcal{IP}^n \setminus C(\mathcal{P})$ is an isometry. Now it follows from the uniform continuity that $\Phi : M \to \mathcal{IP}^n$ is an isometry. This completes the proof of theorem 3.

6 Proof of theorem 4

In this section we assume that $(M, g)$ is a $P_{2\pi}$ manifold at $m$ and $k \geq 2$.

We define the eigensubbundle $E_{-u} := \{E \in TM : \nabla^2 u(E) = -u E\}$. First we prove that $E_{-u}$ and $E_{\pm 1}$ are parallel along $X$ and from this we deduce that $E_{-u}$ and $E_{\pm 1}$ are also eigensubbundles of $R(\cdot, X)X$ with eigenvalues 1 and $\frac{1}{4}$ respectively in the following

**Lemma 6**  
1. $\exp_m : S(0, \pi) \to C$ is a great sphere fibration.
2. $E_{-u}$ and $E_{\pm 1}$ are parallel along $X$.
3. $R(E, X)X = E$ if $E \in E_{-u}$ and $R(E, X)X = \frac{E}{4}$ if $E \in E_{\pm 1}$.

**Proof:** Since $k \geq 2$, we know from lemma 3 that the cut locus $C$ of $m$ coincides with the conjugate locus of $m$ and the multiplicity of each conjugate point $q \in C$ is $k - 1$.

Let $\gamma$ be a geodesic starting at $m$ and let $J$ be a Jacobi field describing the variation of the geodesic $\gamma$ such that $J(0) = 0 = J(\pi)$ and $\|J'(\pi)\| = 1$. Let $\gamma(\pi) = q \in C$ and $\text{seg}(q, m)$ denote the set of all minimizing geodesics from $q$ to $m$. Since $(M, g)$ is a Blaschke manifold at $m$ and $C$ is the cut locus of $m$, we know from Omori [12], Nakagawa-Shiohama [13] that $\Lambda(m, q) := \{\gamma'(0) : \}$
\(\gamma \in \text{seg}(q, m)\), the link of \(q\) and \(m\), is a great subsphere of \(T_q M\) orthogonal to \(T_q C\). We denote by \(\Sigma_q\), the union of all geodesics from \(q\) to \(m\). Then \(\Sigma_q\) is a smooth \(k\)-dimensional submanifold of \(M\) except possibly at \(m \in M\) and \(\Sigma_q\) is totally geodesic at \(q\). Since \(J(t) \subseteq T \Sigma_q\) and \(\Sigma_q\) is totally geodesic at \(q\), we have that \(J'(\pi) \in T_q \Sigma_q\).

Let \(\sigma(\theta) := \cos \theta \gamma'(\pi) + \sin \theta J'(\pi)\), the great circle in the plane spanned by \(\gamma'(\pi)\) and \(J'(\pi)\). Let \(\gamma_{\theta}\) denote the geodesic from \(q\) to \(m\) such that \(\gamma'_{\theta}(0) = \sigma(\theta)\). Since \((M, g)\) is a \(P_{2\pi}\) manifold at \(m \in M\), these geodesics \(\gamma_{\theta}\)'s are all smoothly closed at \(m\). Let \(\sigma(\theta)\) be a curve in \(U_m M\) defined by \(\sigma(\theta) := \sigma(\theta)\).

Now we have a variation \(H(t, \theta) := \exp_m(t \sigma(\theta))\) of the geodesics \(\gamma_{\theta}\) for a fixed \(\theta\). Then the Jacobi field \(J_{\theta}\) along \(\gamma_{\theta}\) is given by

\[
\nabla_{X} J_{\theta} = \nabla_{J_{\theta}} X
\]

and

\[
- \langle J'_{\theta}, J_{\theta} \rangle = \frac{1}{\|\nabla u\|} < \nabla J_{\theta} \nabla u, J_{\theta} >
\]

Since the eigenvalues \(-u\) and \(-\frac{u+1}{2}\) of \(\nabla^2 u\) are such that \(-\frac{u+1}{2} \leq -u\), it follows, as in lemma 4, that

\[
\| J_{\theta} \|_{\sin t} \bigg|_{t=0} \leq \| J_{\theta} \|_{\sin t} \bigg|_{t=\pi} = 1
\]

i.e., \(\| J'_{\theta}(0) \| \leq 1\). Now

\[
J'_{\theta}(0) = \frac{\partial}{\partial \theta} \big|_{t=0} \exp_m(t \sigma(\theta)) = \frac{\partial}{\partial \theta} \big|_{t=0} \exp_m(t \sigma(\theta)) = \frac{\partial}{\partial \theta} d(\exp_m)_{0}(\sigma(\theta)) = \frac{\partial}{\partial \theta} \sigma(\theta)
\]

Therefore the length of the curve \(\sigma(\theta)\) is

\[
l(\sigma(\theta)) = \int_{0}^{2\pi} \| \frac{\partial}{\partial \theta} \sigma(\theta) \| = \int_{0}^{2\pi} \| J'_{\theta}(0) \| \leq \frac{16}{2\pi}
\]
On the other hand, since \((M, g)\) is a \(P_{2\pi}\) manifold at \(m\), it follows that \(\tilde{\sigma}(\theta)\) and \(\tilde{\sigma}(-\theta)\) are antipodal points in \(U_m M\) for each \(\theta\). Hence \(l(\tilde{\sigma}(\theta)) \geq 2\pi\). This shows that \(l(\tilde{\sigma}(\theta)) = 2\pi\) and \(\tilde{\sigma}(\theta)\) is a great circle. Thus we have proved that

1. \(\| J'_\theta(0) \| = 1 \) and \(\| J_\theta \| = \sin t\). Hence \(J_\theta\) is an eigenvectorfield of \(\nabla^2 u\) with eigenvalue \(-u\) for each \(\theta\).

2. \(\exp_m : S(0, \pi) \to C\) is a great sphere fibration.

Now we prove that \(E_{-u}\) is parallel along \(X\) and also an eigensubbundle of \(R(\cdot, X)X\) with eigenvalue 1.

Since \(J_\theta\) is an eigenvectorfield of \(\nabla^2 u\) with eigenvalue \(-u\) and \(\| J_\theta \| = \sin t\), we can write \(J_\theta(t) = \sin t E(t)\) where \(E\) is a unit eigenvectorfield in \(E_{-u}\).

Since \(E\) is a unit field \(E' \perp E\). But

\[
-J'_\theta = \nabla_X J_\theta \\
\quad = \nabla J_\theta X \\
\quad = \frac{1}{\| \nabla u \|} \nabla J_\theta \nabla u \\
\quad = -\frac{u}{\| \nabla u \|} J_\theta
\]

i.e., \(J'_\theta\) is also along \(E\).

On the other hand \(J'_\theta = \cos t E + \sin t E'\). This shows that \(\sin t E' = J'_\theta - \cos t E\) is along \(E\). Hence \(E' = 0\) along \(X\). i.e., \(E\) is a parallel vectorfield along \(X\). This proves that \(E_{-u}\) is parallel along \(X\). Hence as in lemma 4 we have that \(E_{-u}\) is also an eigensubbundle of \(R(\cdot, X)X\) with eigenvalue 1 and hence \(K(E, X) = 1\) for \(E \in E_{-u}\) of unit norm.

Since \(E_{-u}\) is parallel along \(X\), \(E_{\frac{-u+1}{2}}\) is also parallel along \(X\). Now by an easy computation we can show that \(\tilde{R}(E, X)X = \frac{E}{4}\) for \(E \in E_{\frac{-u+1}{2}}\). Hence \(K(E, X) = \frac{1}{4}\) for \(E \in E_{\frac{-u+1}{2}}\) of unit norm. This shows that any Jacobi field \(J\) describing the variation of a geodesic \(\gamma\) such that \(J(0) = 0\) and \(J'(0) \in E_{\frac{-u+1}{2}}\) is given by \(J(t) = 2 \sin \frac{t}{2} E(t)\) where \(E(t)\) is a parallel field along \(X\). (See also proof of theorem 1 in section 3).

Now we prove the following

**Lemma 7** \(\exp_m : S(0, \pi) \to C\) is congruent to Hopf fibration.
Proof: The proof is same as that of lemma 5.

Proof of theorem 2: Proof follows along the same lines of the proof of theorem 3.

Concluding Remarks:

1. In theorem 2 it is enough to assume that for each $q \in C$ there exists at least one periodic geodesic $\gamma_q$ of period $2\pi$ from $m$ to $q$.

2. It appears that an alternative to the hypothesis $P_{2\pi}$ at $m$ could be that $\text{diam}(M) = \pi$. Since this forces $m$ to be critical point for each $d_q$ for $q \in C$.

3. The assumption $P_{2\pi}$ at $m$ is sufficient for our subsequent work. (See the following remark.)

4. Antonio Ros has proved in [15] that if $(M, g)$ is an $n$-dimensional $P_{2\pi}$ manifold such that the Ricci tensor $\text{Ric}$ and the metric $g$ verify the relation $\text{Ric} \geq lg$, where $l$ is a real constant, then the first eigenvalue $\lambda_1$ of the Laplacian of $(M, g)$ satisfies the inequality $\lambda_1 \geq \frac{1}{3}(2l + n + 2)$. Further the equality holds iff for any $\lambda_1$ eigenfunction $f$ on $M$ and for any $u$ in $UM$ we have $f(\gamma_u(t)) = A_u \cos t + B_u \sin t + C_u$. We will discuss in detail the relation between the results of [15] and our results elsewhere.

Question: Can one drop the assumption that $(M, g)$ is $P_{2\pi}$ at $m$? In this context, it may be remarked that it is possible to construct fibrations of spheres with almost all fibres of diameter $< 2\pi$.

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References

[1] V. I. Arnold, *Ordinary Differential Equations*, Springer-Textbook, Springer-Verlag, 1992.

[2] Arthur L. Besse, *Manifolds all of whose geodesics are closed*, A Series of Modern Surveys in Mathematics 93, Springer-Verlag, 1978.
[3] M. Berger, P. Gauduchon and E. Mazet, *Le Spectre d’une variété Riemannienne*, LNM Vol 194, Springer-verlag, Berlin-Heidelberg-NewYork, 1971.

[4] R. H. Escobales, *Riemannian submersions with totally geodesic fibres*, Journal of Differential Geometry, 10 (1975), 253-276.

[5] H. Gluck, F. Warner and W. Ziller, *Fibrations of spheres by parallel great spheres and Berger’s Rigidity theorem*, Annals of Global Analysis and Geometry, 5 (1987), 53-82.

[6] P. A. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John-Wiley& Sons, 1978.

[7] F. R. Harvey, *Spinors and Calibrations*, Perspectives in Mathematics, 1992.

[8] H. Karcher, *Riemannian Comparison Constructions*, MAA Studies in Math. 27 Global Differential Geometry, ed.by S. S. Chern. The Mathematical Association of America, 1989.

[9] A. Lichnerowicz, *Geometrie des groupes de transformation*, Paris, Dunod, 1958.

[10] J. Milnor, *Morse Theory*, Annals of Mathematics studies, Princeton university press, Princeton, 1963.

[11] M. Obata, *Certain conditons for a Riemannian manifold to be isometric to a sphere*, Journal of Mathematical Society of Japan, 14 (1962), 333-340.

[12] H. Omori, *A Class of Riemannian metrics on a manifold*, Journal of Differential Geometry, 2 (1968), 253-262.

[13] H. Nakagawa and K. Shiohama, *On Riemannian manifolds with certain cut loci I, II*, Tohoku Mathematica Journal, 22 (1970), 14-23, 357-361.

[14] A. Ranjan, *Riemannian submersions of spheres with totally geodesic fibres*, Osaka Journal of Mathematics, 22 (1985), 243-260.
[15] Antonio Ros, *Eigenvalue Inequalities for Minimal Submanifolds and P-manifolds* Mathematische Zeitschrift, 187 (1984), 393-404.

[16] Robert Molzon and Karen Pinney, *A Characterization of Complex projective space up to biholomorphic isometry*, Preprint 1992.

Akhil Ranjan, Department of Mathematics, Indian Institute of Technology, Bombay- 400 076, India. e-mail: aranj@cc.iitb.ernet.in

G. Santhanam, School of Mathematics, Tata Institute of Fundamental Research, Bombay- 400 005, India. e-mail: santhana@tifrvax.tifr.res.in, santhana@math.tifr.res.in