Notes on thermodynamics of super-entropic AdS black holes

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(Dated: January 22, 2020)

The super-entropic black hole, which possesses a noncompact horizon topology and violates the reverse isoperimetric inequality, has been found to satisfy both the thermodynamic first law and the Bekenstein-Smarr mass formula. In this paper, we first derive a new Christodoulou-Ruffini-like squared-mass formula for the four-dimensional Kerr-Newman-AdS super-entropic black hole, and then establish a set of very simple relations between thermodynamic quantities of the super-entropic Kerr-Newman-AdS\textsubscript{4} black hole and its usual counterparts. Using these relations, the thermodynamic quantities of the Kerr-Newman-AdS\textsubscript{4} super-entropic black hole can be obtained from those of the usual pro-type by taking the ultra-spinning limit properly. Then these relations are extended to the singly-rotating Kerr-AdS black holes in arbitrary dimensions and the double-rotating charged black hole in the five-dimensional minimal gauged supergravity. It can be inferred that the thermodynamic quantities of all super-entropic black holes obey similar limiting relations to those of their corresponding conventional rotating AdS black holes, and thus can be obtained by taking the ultra-spinning limit appropriately.

PACS numbers: 04.70.Dy, 04.20.Jb

I. INTRODUCTION

Black hole is the most fundamental and important object predicted by Einstein’s general relativity. Constructing exact black hole solutions to the Einstein equation and studying their properties can deepen our understanding of the nature of gravity and the basic property of spacetime. Recently, a new class of the so-called super-entropic black hole \cite{1,2}, which provides the first example that violates the “reverse isoperimetric inequality” \cite{3,4}, has received considerable interest and enthusiasm. This kind of black hole solution is asymptotical (locally) anti-de Sitter (AdS) and has a finite horizon area, with its horizon topology being noncompact since its topological sphere has two punctures, one at the north and the other at the south.

Remarkably, it has been shown \cite{1} that the super-entropic black hole solution can be alternatively obtained by simply taking the ultra-spinning limit of the usual rotating AdS one. Taking the four-dimensional Kerr-Newman-AdS black hole or an arbitrary dimensional Kerr-AdS black hole as an example, the concrete procedure to obtain the super-entropic black hole is as follows: First rewrite the rotating AdS black hole in a rotating frame at infinity, then boost the rotating angular velocity (only one rotating angular velocity if there exist several rotating axes) to the speed of light, and finally compactify the corresponding azimuthal direction. However, it was claimed \cite{1,5} that the thermodynamic quantities of the super-entropic black hole can not be obtained from those of its corresponding usual rotating AdS black hole by simply taking the ultra-spinning limit. This is simply because some thermodynamic quantities will be divergent or zero when the ultra-spinning limit is taken directly. So far, the thermodynamic quantities of super-entropic black hole have been obtained usually by using the standard calculation method \cite{1,5–9}. As such, a question arises as to whether there are certain relations between the thermodynamic quantities of the usual rotating AdS black hole and their corresponding super-entropic ones, and whether the thermodynamic quantities of the super-entropic black hole can be appropriately derived from those of its usual rotating AdS black hole by taking the ultra-spinning limit. In this paper, we will propose a new, simple way to relate them when the ultra-spinning limit is properly taken. Once this is done, one can greatly simplify the calculations and step toward discussing the super-entropic black hole thermodynamics. Very recently, Appels et al. presented a different method that the super-entropic black hole can also be obtained by running a conical deficit through the usual rotating AdS black hole \cite{10}.

It is well known that thermodynamics of a black hole include three famous mass formulas, i.e., the first law of thermodynamics \cite{11,12}, the Bekenstein-Smarr mass formula \cite{13} and the Christodoulou-Ruffini squared-mass formula \cite{14,15}. Quite recently, these formulas were perfectly extended \cite{16} to the four-dimensional NUT-charged spacetimes. It has been found \cite{1} that the first law of thermodynamics and the Bekenstein-Smarr mass formula can be established for the super-entropic black hole if an extra thermodynamic conjugate pair is introduced. However, a similar Christodoulou-Ruffini squared-mass formula is still absent until now. Actually, the existence of such a squared-mass formula is very important, since it can be used to check whether a new conjugate pair introduced in the first law and the Bekenstein-Smarr mass formula is correct or not. In addition, recent studies have shown that the squared-mass formula satisfied by a black hole can be used to study black hole chemistry \cite{17–19}, thermodynamic Ruppeiner geometry \cite{20–22} and phase transitions \cite{23–28}. Thus, it is desirable to get a similar Christodoulou-Ruffini-like squared-mass formula for the super-entropic black hole too.

In this paper, we first establish a novel Christodoulou-
Ruffini-like squared-mass formula for the four-dimensional Kerr-Newman-AdS super-entropic black hole. Differentiating this formula gives the thermodynamic quantities of this super-entropic black hole, which satisfy both the first law and the Bekenstein-Smarr mass formula. Then, we construct a set of simple relations between thermodynamic quantities of the super-entropic black hole and those of its usual counterpart. Basing upon these relations, the thermodynamic quantities of the super-entropic black hole can be obtained from those of its usual rotating AdS black hole by taking the ultra-spinning limit appropriately. The remaining part of this paper is organized as follows. In Section 2, we first present a brief review of the thermodynamical properties of the Kerr-Newman-AdS super-entropic black hole, and then construct a new Christodoulou-Ruffini-like squared-mass formula for it, from which both the differential and integral mass formulas can be derived via a simple mathematical manipulation without taking into account the chirality condition \( j = M l \). Then, we discuss the impact of the chirality condition on the mass formulas and the super-entropic black hole thermodynamics. After that, we establish a set of simple relations between thermodynamic quantities of the super-entropic Kerr-Newman-AdS black hole and its usual counterparts, by which we can straightforwardly obtain the thermodynamic quantities of the super-entropic black hole when the ultra-spinning limit is taken properly. In Section 3, we turn to discuss the case of the singly-rotating Kerr-AdS black holes in arbitrary dimensions. Then in Section 4, we extend the similar limiting procedure to the double-rotating charged black hole of the five-dimensional minimal gauged supergravity. Finally, we present our conclusions in Section 5. Throughout this paper, the thermodynamic quantities without and with a hat represent to belong to the super-entropic and usual black holes, respectively.

II. KERR-NEWMAN-ADS\(_4\) SUPER-ENTROPIC BLACK HOLE

A. Thermodynamic quantities

We now present a brief review of thermodynamical properties of a four-dimensional Kerr-Newman-AdS super-entropic black hole, whose metric and Abelian gauge potential are [1, 5]:

\[
\begin{align*}
 ds^2 &= \frac{\Delta(r)}{\Sigma} \left( dt - l \sin^2 \theta d\phi \right)^2 + \left[ \frac{d\nu^2}{\Delta(r)} + \frac{d\theta^2}{\sin^2 \theta} \right] \\
 &= \frac{\sin^2 \theta}{\Sigma} \left[ l dt - (r^2 + l^2) d\phi \right]^2 , \\
 A &= \frac{q r}{\Sigma} (dt - l \sin^2 \theta d\phi) , \\
 \Delta(r) &= (r^2 + l^2)^2 / l^2 - 2mr + \Sigma , \\
 \Sigma &= r^2 + l^2 \cos^2 \theta . 
\end{align*}
\]

Here, \( m \) and \( q \) are the mass and charge parameters, respectively. The azimuthal coordinate \( \phi \) is noncompact and must be compactified by requiring \( \phi \sim \phi + \mu \) with \( \mu \) being a dimensionless parameter related to a new chemical potential \( K \).

The Bekenstein-Hawking entropy is one quarter of the area of the event horizon

\[
S = \alpha / 4 = \mu (r_+^2 + l^2) / 2 ,
\]

where \( r_+ \) is the location of the event horizon. The Hawking temperature is proportional to the surface gravity \( \kappa \) on the event horizon

\[
T = \frac{\kappa}{2\pi} = \frac{\partial_r \Delta(r_+)}{4\pi (r_+^2 + l^2)} = \frac{2 r_+ (r_+^2 + l^2) - ml^2}{2\pi l^2 (r_+^2 + l^2)} .
\]

Note that the super-entropic black hole is rotating with the speed of light at infinity, and on the event horizon its angular velocity is given by

\[
\Omega = - \frac{g\phi}{g\phi} \bigg|_{r=r_+} = \frac{l}{r_+^2 + l^2} .
\]

The electric charge \( Q \) of the black hole can be computed by using the Gauss’ law integral

\[
Q = \frac{1}{4\pi} \int *F = \frac{\mu q}{2\pi} ,
\]

and its corresponding electrostatic potential at the event horizon reads

\[
\Phi = (A_\mu \chi^\mu) \big|_{r=r_+} = \frac{qr_+}{r_+^2 + l^2} ,
\]

where \( \chi = \partial_t + \Omega \partial_\phi \) is the Killing vector normal to the event horizon.

To compute the mass \( M \) and the angular momentum \( J \), the conformal completion method [29] or the Abbott-Deser method [30] is usually used. Here we adopt the former. The idea is to perform a conformal transformation on the metric (1) to remove the divergence in the integrals at the boundary (conformal infinity). After taking the \( r \to \infty \) limit in the line element \( ds^2 / r^2 \), one obtains the boundary metric

\[
d s_{\text{b}}^2 = - \left( \frac{dt}{T} - \sin^2 \theta d\phi \right)^2 + \frac{d\theta^2}{\sin^2 \theta} + \sin^4 \theta d\phi^2 .
\]

Then the conserved charges \( \mathcal{L}[\xi] \) associated with the Killing vector \( \xi \) can be computed by

\[
\mathcal{L}[\xi] = \frac{l^3}{8\pi} \int_{S_{\text{b}}} r N^\alpha N^\beta C_{\alpha \beta}^{\mu} \xi^\nu dS_\mu ,
\]

where \( N^\mu = [0, -r^2 / l^2, 0, 0] \) is the vector normal to the boundary, \( C_{\alpha \beta}^{\mu} \) is the conformal Weyl curvature tensor, and

\[
d S_i = \sin \theta d\theta d\phi / l
\]

is the temporal component of the area vector in the three-dimensional conformal boundary. So one can evaluate the mass \( M \) and the angular momentum \( J \) as

\[
M = \mathcal{L}[\partial_t] = \frac{\mu m}{2\pi} , \quad J = \mathcal{L}[\partial_\phi] = \frac{\mu ml}{2\pi} \equiv M l .
\]
It should be mentioned that the angular momentum \( J \) can also be directly computed by the Komar integral, while the mass can be obtained by the Komar integral after the substraction of a divergence arising from the zero-mass background \([2]\).

In Refs. \([1, 5]\), it has been shown that the above thermodynamic quantities obey the extended differential and integral mass formulas simultaneously
\[
\begin{align*}
\delta M &= TdS + \Delta dJ + VdP + \Phi dQ + Kd\mu, \quad (12) \\
M &= 2(TS + \Delta J - VP) + \Phi Q, \quad (13)
\end{align*}
\]
with the thermodynamic volume and a new chemical potential
\[
\begin{align*}
V &= \frac{2}{3} \mu r_+ (r_+^2 + l^2) \quad (14) \\
K &= \frac{(l^2 - r_+^2)(r_+^2 + l^2)^2 + q^2l^2}{8\pi l^2 r_+ (r_+^2 + l^2)}, \quad (15)
\end{align*}
\]
being conjugate to the pressure \( P = 3/(8\pi l^2) \) and \( \mu \), respectively.

**B. A new squared-mass formula**

The Christodoulou-Ruffini squared-mass formula \([14, 15]\) was initially found for the Kerr-Newman black hole. Later it was generalized to the Kerr-Newman-AdS \(_4\) black hole case \([31]\). Now we try to derive a similar squared-mass formula for the Kerr-Newman-AdS \(_4\) super-entropic black hole.

Note that the event horizon equation \( \Delta (r_+) = 0 \) can be re-expressed as
\[
\frac{S^2}{4\pi l^2} + Q^2 = \frac{\mu M r_+}{\pi}. \quad (16)
\]
From Eq. (3), one can get \( r_+^2 = (2S/\mu) - l^2 \). Substituting it into the squared Eq. (16) and using \( l^2 = 3/(8\pi P) \) and the chirality condition: \( J = Ml \), then after a little algebra we arrive at an identity
\[
M^2 = \frac{1}{2\mu S} \left( \frac{8P}{3} S^2 + \pi Q^2 \right)^2 + \frac{\mu J^2}{2S}, \quad (17)
\]
which is our new Christodoulou-Ruffini-like squared-mass formula for the four-dimensional Kerr-Newman-AdS \(_4\) super-entropic black hole. Obviously, the parameters \( S, J, Q, P \) and \( \mu \) of the black hole form a whole set of energetic extensive parameters for the thermodynamical fundamental functional relation \( M = M(S, J, Q, P, \mu) \). If the chirality condition is not being taken into account, however, see the subsection below for a discussion about its impact on the actual thermodynamics. It is interesting to note that using this squared-mass formula, one can study conveniently the black hole chemistry and thermodynamic phase transition of the Kerr-Newman-AdS \(_4\) super-entropic black hole.

**C. The first law of thermodynamics and the Bekenstein-Smarr mass formula**

Differentiating the above squared-mass formula (17) yields the conjugate quantities of \( S, J, Q, P \) and \( \mu \), all of which, as was done in Refs. \([31–33]\), are viewed formally as independent thermodynamical variables at this moment.\(^1\) In doing so, we can arrive at the first law (12) and the Bekenstein-Smarr relation (13), with the conjugate thermodynamic potentials given by the ordinary Maxwell relations as follows. The conjugate quantity of the entropy \( S \) is the Hawking temperature
\[
T = \frac{\partial M}{\partial S} = -\frac{M}{2S} + \frac{8P}{3S} \left( \frac{8P}{3} S^2 + \pi Q^2 \right) = \frac{2r_+(r_+^2 + l^2) - ml^2}{2\pi l^2 (r_+^2 + l^2)}. \quad (18)
\]
The angular velocity and the electrostatic potential, which are conjugate to \( J \) and \( Q \), respectively, are given by
\[
\begin{align*}
\Omega &= \frac{\partial M}{\partial J} = \frac{\mu J}{2SM} = \frac{l}{r_+^2 + l^2}, \quad (19) \\
\Phi &= \frac{\partial M}{\partial Q} = \frac{\mu Q}{SM} \left( \frac{8P}{3} S^2 + \pi Q^2 \right) = \frac{qr_+}{r_+^2 + l^2}. \quad (20)
\end{align*}
\]
These three conjugate quantities are entirely identical to those given in Eqs. (4), (5) and (7). Differentiating the squared-mass formula (17) with respect to the pressure \( P \) and the dimensionless parameter \( \mu \), one can get the thermodynamical volume
\[
\begin{align*}
V &= \frac{\partial M}{\partial P} = \frac{4S}{3S} \left( \frac{8P}{3} S^2 + \pi Q^2 \right) \\
&= \frac{2}{3} \mu r_+ (r_+^2 + l^2), \quad (21)
\end{align*}
\]
and a new conjugate variable
\[
K = \frac{\partial M}{\partial \mu} = \frac{-M^2 S + \mu J^2}{2\mu SM} = \frac{m(l^2 - r_+^2)}{4\pi (r_+^2 + l^2)}, \quad (22)
\]
These two quantities are the same as those obtained in Refs. \([1, 5]\). With the conjugate variables derived from the squared-mass formula (17), the first law of thermodynamics is trivially satisfied whilst the integral Bekenstein-Smarr mass formula is easily checked to be completely obeyed too. Thus, we have verified that the first law of thermodynamics, the Bekenstein-Smarr mass formula and the Christodoulou-Ruffini-like squared-mass formula are all valid for the Kerr-Newman-AdS \(_4\) super-entropic black hole.

**D. Remark on the impact of the chirality condition**

In the last subsection, we have not taken into account the impact of the chirality condition \( J = Ml \) on the thermodynamical relations of the super-entropic black hole. Now let us

\(^1\) However, in fact they are not completely independent of each other by virtue of the existence of the chirality condition \( J = Ml \). We thank the anonymous referee for pointing this out to us. A careful discussion about its impact on the mass formulas is presented in the next subsection.
make a careful discussion about this issue. It should be reminded that the super-entropic black hole is obtained by taking the ultra-spinning limit \((a \to l)\), and it is actually degenerate due to three thermodynamical quantities \((M, J, P)\) satisfying the following constraint

\[
P = \frac{3M^2}{8\pi l^2},
\]

which means that they are not independent. Consequently, the first law \((12)\) and the Bekenstein-Smarr relation \((13)\) depict physically a degenerate thermodynamical system. One can adopt any two of these three variables to describe the genuine thermodynamical properties. However, it is much concise and simpler to present the thermodynamic relations in terms of the enthalpy \(M\) and the pressure \(P\). After eliminating \(J\) from the mass formulas in favor of \(l^2 = \frac{3}{8\pi P}\), the first law \((12)\) and the Bekenstein-Smarr relation \((13)\) degenerate to the following nonstandard forms (their thermodynamic quantities cannot constitute the ordinary conjugate pairs due to the presence of a factor \((1 - \Omega l)\) in front of \(dM\) and \(M\)):

\[
(1 - \Omega l) dM = T dS + V' dP + \Phi dQ + K d\mu, \tag{24}
\]

\[
(1 - \Omega l) M = 2 (TS - V' P) + \Phi Q, \tag{25}
\]

where

\[
V' = V = \frac{J Q}{2P} = V - \frac{4\pi}{3} \Omega M l^3. \tag{26}
\]

Noting that the lowest two eigenvalues of Virasoro algebra: \(L_+ = M\) and \(L_- = 0\) when \(J = Ml\), the above expressions reproduce Eqs. (19-21) in Ref. [1] and Eq. (23) in Ref. [2].

In the same way, by considering that \(J\) is a redundant variable (although it is a real measurable quantity), our squared-mass formula \((17)\) reduces to

\[
2 \mu M^2 \left(S - \frac{3\mu}{16\pi} \right) = \left(\frac{8P}{3} S^2 + \pi Q^2\right)^2. \tag{27}
\]

Taking the positive square root of this formula, one gets a fundamental thermodynamical relation

\[
M \sqrt{2\mu S - \frac{3\mu^2}{8\pi P}} = \frac{8P}{3} S^2 + \pi Q^2, \tag{28}
\]

which coincides with Eq. (24) given originally in Ref. [2].

Eq. (28) suggests that the enthalpy \(M\) be viewed as the functional relation \(M = M(S, Q, P, \mu)\). Similar to the strategy as that in the last subsection, the above nonstandard differential and integral mass formulas can be deduced by exploiting the standard Maxwell rule. Likewise, one perhaps prefers to eliminate \(P\) via Eq. (23) from the beginning, but the resulted expressions would be very complicated, and will not be presented here. It should be pointed out that the discussion made in this subsection can be easily generalized to higher dimensions too.

It is interesting to make a comparison of the above discussions with a recent work [16] on the thermodynamics of four-dimensional Taub-NUT spacetimes, where a new secondary hair \(J_a = mn\) is introduced to perfectly cast their thermodynamics into the standard forms of usual black hole thermodynamics. Without introducing \(J_a = mn\) into the mass formulas, thermodynamical relations would be inconsistent (nonstandard).

E. A simple limiting procedure to obtain the thermodynamic quantities of the Kerr-Newman-AdS\(_4\) super-entropic black hole

In the above subsection, we have presented all thermodynamic quantities of a Kerr-Newman-AdS\(_4\) super-entropic black hole. It was claimed in Ref. [1, 5, 10] that these thermodynamic quantities cannot be obtained by taking the \(a \to l\) limit of the usual Kerr-Newman-AdS thermodynamic quantities, due to the singular nature of the ultra-spinning limit. However, we will show that this is not the case and suggest a method that the \(a \to l\) limit can be properly performed. Below, we try to provide a simple way to derive them from those of the usual Kerr-Newman-AdS\(_4\) black hole by taking the ultra-spinning limit appropriately.

It should be reminded that the Kerr-Newman-AdS\(_4\) super-entropic black hole is constructed by taking the \(a \to l\) limit of its corresponding usual Kerr-Newman-AdS\(_4\) black hole in a frame rotating at infinity. However, as is pointed out in [34] that all thermodynamic quantities of the usual Kerr-Newman-AdS\(_4\) black hole that enter the first law and the Bekenstein-Smarr mass formula should be those quantities (especially, the conserved mass, the angular velocity at the horizon relative to the infinity, and the thermodynamic volume) measured in a frame rest at infinity. Transforming all thermodynamic quantities into a frame that rotates at infinity, we have

\[
\hat{M} = \frac{m}{2}, \quad \hat{J} = \frac{ma}{\Sigma}, \quad \hat{Q} = \frac{q}{\Sigma}, \quad \hat{P} = P = \frac{3}{8\pi l^2},
\]

\[
\hat{T} = \frac{\partial_r \hat{A}(r_+)}{4\pi (r_+^2 + a^2)} = \frac{(2r_+^2 + a^2)(2r_+ - ml^2)}{2ml^2(r_+^2 + a^2)}, \tag{29}
\]

\[
\hat{S} = \frac{\pi (r_+^2 + a^2)}{\Sigma}, \quad \hat{\Omega} = \frac{a\Sigma}{r_+^2 + a^2}, \quad \hat{\Phi} = \frac{qr_+}{r_+^2 + a^2},
\]

where

\[
\hat{A}(r) = (1 + r^2/l^2)(r^2 + a^2) = 2mr + q^2, \quad \Sigma = 1 - a^2/l^2.
\]

These thermodynamic quantities still satisfy the Bekenstein-Smarr mass formula

\[
\hat{M} = 2(\hat{T} \hat{S} + \hat{\Omega} \hat{J} - \hat{V} \hat{P}) + \hat{\Phi} \hat{Q}, \tag{30}
\]

however, the first law now boils down to a differential identity

\[
d\hat{M} = T d\hat{S} + \hat{\Omega} d\hat{J} + \hat{V} d\hat{P} + \hat{\Phi} d\hat{Q} + \frac{f}{2a} d\Sigma, \tag{31}
\]

with the thermodynamic volume

\[
\hat{V} = \frac{4}{3} r_+ \hat{S} = \frac{4\pi}{3\Sigma} r_+ (r_+^2 + a^2). \tag{32}
\]

We now want to take the \(a \to l\) limit of the above thermodynamic quantities with an implicit assumption that the mass
and charge parameters remain unchanged when this limit is taken, and so the horizon radius reduces to the one after taking the $a \to l$ limit, which is clear from the horizon equation. Taking straightforwardly the $a \to l$ limit, which makes $\Xi \to 0$, then one can see that the temperature $\hat{T}$ equals to $T$ defined in Eq. (4), the electrostatic potential $\hat{\Phi}$ reduces to $\Phi$ given in Eq. (7), but $(\hat{M}, \hat{J}, \hat{Q}, \hat{V}) \to \infty$, and $\hat{\Omega} \to 0$. This was first observed in [5] that the thermodynamic quantities of the super-entropic black hole can not be obtained directly from those of the usual rotating AdS black hole by taking the $a \to l$ limit.

However, this is only superficial as can be seen by noting that a coordinate transformation on the azimuthal coordinate has to be done in the process of obtaining the super-entropic black hole solution via taking the $a \to l$ limit. That is, before taking the ultra-spinning limit $a \to l$, one needs to redefine a new azimuthal coordinate $\phi = \hat{\phi}/\Xi$ ($\phi$ has a period $2\pi$ to prevent a conical singularity) and to identify it with period $\mu$ to avoid a singular metric in this limit. Only after this coordinate transformation has been done, can the $a \to l$ limit then be taken to obtain a regular super-entropic black hole.

Inspecting into the calculation of the conserved charges by means of the conformal completion method and also of the entropy via the horizon area, one has to perform the integral about the azimuthal coordinates ($\hat{\phi}, \phi$) has a respective period $2\pi, \mu$). It is clear that one should consider the thermodynamic quantities ($\Xi M, \Omega, \Xi J, \Xi Q, \Xi S, \Xi V$) that are all finite in the $a \to l$ limit. Note that the excess factor $\Xi^2$ that appears in the angular momentum and the angular velocity to remove the divergence or the zero is due to the rescaling of the azimuthal coordinate.

Taking into account of the above consideration, we suggest that the following relations

$$
M = \frac{\mu \Xi}{2\pi} \hat{M}, \quad J = \frac{\mu \Xi}{2\pi} \hat{J}, \quad Q = \frac{\mu \Xi}{2\pi} \hat{Q},
$$

$$
S = \frac{\mu \Xi}{2\pi} \hat{S}, \quad \Omega = \frac{1}{\Xi} \hat{\Omega}, \quad V = \frac{\mu \Xi}{2\pi} \hat{V},
$$

$$
T = \hat{T}, \quad \Phi = \hat{\Phi}, \quad P = \hat{P}
$$

should be established between the thermodynamic quantities of the usual Kerr-Newman-AdS$_4$ black hole and their corresponding super-entropic ones when taking the $a \to l$ limit.

Substituting the relations (33) into (30) and taking the $a \to l$ limit yields directly the Bekenstein-Smarr mass formula (13). Considering the differential identity (31), and viewing the dimensional parameter $\mu$ as a thermodynamic variable, we find that the term related to $d\Xi$ just vanishes properly and the identity perfectly reduces to the first law (12) with

$$
K = \frac{M - \Phi Q - 2VP}{2\mu} = \frac{M - TS - \Omega J - \Phi Q}{\mu}.
$$

We now turn to consider the squared-mass formula. To this end, one can deduce the following squared-mass identity:

$$
\frac{\dot{S}}{\pi \Xi} \hat{M}^2 = \hat{P} + \frac{1}{4\pi^2} \left( \hat{S} + \frac{8P}{3} \hat{S}^2 + \pi \hat{Q}^2 \right)^2.
$$

Taking advantage of the relations (33), it becomes

$$
\hat{M}^2 = \frac{1}{2\mu \hat{S}} \left( \frac{\mu \Xi}{2\pi} \hat{S} + \frac{8P}{3} \hat{S}^2 + \pi \hat{Q}^2 \right)^2 + \frac{\mu \hat{J}^2}{2\hat{S}},
$$

which reduces to the squared-mass formula (17) after taking the $a \to l$ limit. Similarly, the chirality condition ($J = M$) can be recovered by the same procedure from the relation: $f = Ma/\Xi$.

The above method is a very simple, effective approach to obtain easily all the thermodynamic quantities of the super-entropic black hole from their counterparts of the usual Kerr-Newman-AdS$_4$ black hole.

### III. SINGLY-ROTATING KERR-ADS SUPER-ENTROPIC BLACK HOLES IN ARBITRARY DIMENSIONS

To demonstrate that the relations (33) are also applicable to the case of black holes in arbitrary dimensions, let us now consider the singly-rotating $d$-dimensional Kerr-AdS spacetimes in the frame rotating at infinity [35]

$$
ds^2 = -\frac{\hat{\Delta}}{\Xi} \left( dt - \frac{a}{\Xi} \sin^2 \theta d\phi \right)^2 + \frac{\Xi}{\hat{\Delta}} dr^2 + \frac{\Xi}{\hat{\Delta}} \frac{d\theta^2 + \sin^2 \theta (d\phi - \frac{a}{\Xi} dt)^2}{r^2 - a^2/d^2},
$$

where

$$
\hat{\Delta} = \left( r^2 + a^2 \right) \left( 1 + r l^2 / 2 ight) - 2mr^{d-2}, \quad \Xi = 1 - a l^2 - \Delta_0 = 1 - a l^2 - \cos^2 \theta, \quad \hat{\Sigma} = r^2 + a^2 \cos \theta.
$$

The thermodynamic quantities of these black holes have the following expressions in the extended phase space

$$
\hat{\mathcal{M}} = \frac{\omega_{d-2}}{8\pi \Xi} (d-2) m, \quad \hat{f} = \frac{\omega_{d-2}}{4\pi \Xi} ma, \quad \hat{\Omega} = \frac{a \Xi}{r^2 + a^2},
$$

$$
\hat{\Sigma} = \frac{\omega_{d-2}}{4\Xi} \left( r^2 + a^2 \right)^{d-4}, \quad \hat{v} = \frac{\omega_{d-2}}{\left( d-1 \right) \Xi} \left( r^2 + a^2 \right)^{d-3},
$$

$$
\hat{T} = \frac{(d-1)r^2 + (d-3)(a^2 + l^2)r^2}{4\pi r^2 + (d^2 + a^2)l^2},
$$

where $\omega_{d-2}$ is the volume of the unit $(d-2)$-sphere.

The above thermodynamic quantities satisfy an extended Bekenstein-Smarr mass formula

$$
(d-3)\hat{M} = (d-2) \left( \hat{T} \hat{\Sigma} + \hat{\Omega} \hat{f} + \hat{v} \hat{P} \right),
$$

however the first law boils down to a differential identity as before

$$
d\hat{M} = \hat{T} d\hat{S} + \hat{\Omega} d\hat{J} + \hat{\Sigma} d\hat{P} + \frac{\hat{f}}{2\hat{a}} d\Xi,
$$

where the pressure conjugate to the thermodynamic volume is

$$
P = -\frac{\hat{A}}{8\pi} \frac{(d-1)(d-2)}{16\pi l^2}.
$$
Three steps to construct the super-entropic versions of the higher-dimensional singly-rotating Kerr-AdS black holes are [1, 5]: (1) redefine the angle coordinate $\phi$ by multiplying it with a factor $\Xi$; (2) take the $a \to l$ limit; (3) compactify the $\phi$-direction with a period of the dimensional parameter $\mu$. Using the relations (33) and taking the ultra-spinning limit $a \to l$ (with the assumption that the mass parameter $m$ remains unchanged again), we can easily obtain the thermodynamical quantities of the singly-rotating Kerr-AdS super-entropic black holes in arbitrary dimensions [1, 5]

\[
M = \frac{\mu \Omega_{d-2}}{16\pi^2} (d-2)m, \quad J = \frac{\mu \Omega_{d-2}}{8\pi^2} ml = \frac{2Ml}{d-2},
\]
\[
S = \frac{\mu \Omega_{d-2}}{8\pi} \left( r_+^2 + l^2 \right)^{d-4}, \quad T = \frac{(d-1)r_+^2 + (d-5)l^2}{4\pi r_+^2 l^2},
\]
\[
\Omega = \frac{l}{r_+^2 + l^2}, \quad V = \frac{\mu \Omega_{d-2}}{2\pi(d-1)} \left( r_+^2 + l^2 \right)^{d-3}. \tag{42}
\]

In the four-dimensional case ($d = 4$) where $\mu = 2\pi$, the above expressions (42) reduce to those given in the last section. It is not difficult to check that these thermodynamic quantities satisfy both the first law of thermodynamics and the Bekenstein-Smarr mass formula

\[
dM = TdS + \Omega dJ + VdP + Kd\mu, \tag{43}
\]
\[
(d-3)M = (d-2)(TS + \Omega J) - 2VP, \tag{44}
\]

where

\[
K = \frac{M - TS - \Omega J}{\mu} = \frac{m\Omega_{d-2}(d-2)(l^2 - r_+^2)}{8\pi [(d-2)r_+^2 + (3d-10)l^2]}, \tag{45}
\]

is the conjugate quantity of the variable $\mu$. Similar to what has been done in the last section, one can also derive the above expression for $K$ via taking the $a \to l$ limit in the differential identity (40).

We now turn to a generalization of the Christodoulou-Ruffini-like squared-mass formula (17) to higher dimensions.

Following the procedure used in the above section, we can rewrite the horizon equation: $(r_+^2 + l^2)^2 - 2ml^2S_+^2 - d = 0$ as

\[
r_+^{d-3} = \frac{16P}{(d-1)\mu \Omega_{d-2} M^2}, \tag{46}
\]

and then by expressing $r_+$ as a function of $S$, we can finally deduce a new mass formula

\[
\left[ \frac{16P}{(d-1)\mu \Omega_{d-2} M^2} \right]^{\frac{1}{d-4}} = \frac{4S}{\mu \Omega_{d-2}} \left[ \frac{16P}{(d-1)\mu \Omega_{d-2} M^2} \right]^{\frac{1}{d-4}} - \frac{l^2(D-2)^2}{4M^2}. \tag{47}
\]

for the singly-rotating Kerr-AdS super-entropic black holes in arbitrary dimensions.

Viewing Eq. (47) as the fundamental relation for the thermodynamical function $M = M(S, J, P, \mu)$ and differentiating it with respect to its variables as was done in the last section, their conjugate thermodynamic quantities can be correctly gotten as those given above. This confirms that using the relations (33), the thermodynamic quantities of the singly-rotating Kerr-AdS super-entropic black holes in arbitrary dimensions can also be obtained from those of the singly-rotating Kerr-AdS black holes by properly taking the ultra-spinning limit.

The above discussions can be readily extended to the general case with multiple rotation parameters in higher dimensions. It has been mentioned [5] that the ultra-spinning limit can be taken in one and only one rotating axis. Here we additionally point out that all of the remaining rotation axes should be put in the frame rest at infinity in order to ensure that all thermodynamic quantities measured in this frame obey both the first law and the Bekenstein-Smarr mass formula simultaneously. Otherwise, the “first law” will boil down to a differential identity only. In the next section, we will illustrate this issue by considering an exact double-rotating black hole solution to the five-dimensional Einstein-Maxwell-Chern-Simons gauged supergravity theory.

IV. SUPER-ENTROPIC BLACK HOLE OF FIVE-DIMENSIONAL MINIMAL GAUGED SUPERGRAVITY

In this section, we will generalize the relations (33) to the case of a double-rotating charged black hole in the five-dimensional minimal gauged supergravity and then check the validity of these relations. The solution to the five-dimensional Einstein-Maxwell-Chern-Simons gauged supergravity is given by [36]

\[
ds^2 = - \frac{\tilde{\Delta}_r}{r^2(r^2 + y^2)} X^2 + (r^2 + y^2)^2 \left( \frac{r^2 dr^2}{\tilde{\Delta}_r} + \frac{dy^2}{H} \right) + \frac{H}{r^2 + y^2} X^2 + \frac{1}{r^2 y^2} \left( abZ + \frac{qy^2}{r^2 + y^2} X^2 \right), \tag{48}
\]
\[
A = \frac{\sqrt{3} q}{2(r^2 + y^2)} X, \tag{49}
\]

where

\[
X = 1 - y^2 l^{-2} \frac{\Xi_a \Xi_b}{\Xi_a \Xi_b} dt - \frac{a(a^2 - y^2)}{(a^2 - b^2) \Xi_a} d\phi - \frac{b(b^2 - y^2)}{(b^2 - a^2) \Xi_b} d\psi, \tag{50}
\]
\[
Y = 1 + r^2 l^{-2} \frac{\Xi_a \Xi_b}{\Xi_a \Xi_b} dt - \frac{a(r^2 + a^2)}{(a^2 - b^2) \Xi_a} d\phi - \frac{b(r^2 + b^2)}{(b^2 - a^2) \Xi_b} d\psi, \tag{51}
\]
\[
Z = (1 + r^2 l^{-2})(1 - y^2 l^{-2}) \frac{\Xi_a \Xi_b}{\Xi_a \Xi_b} dt - \frac{(r^2 + a^2)(a^2 - y^2)}{(a^2 - b^2) a \Xi_a} d\phi - \frac{(r^2 + b^2)(b^2 - a^2) b \Xi_b}{(b^2 - a^2) \Xi_a} d\psi, \tag{52}
\]
\[
\tilde{\Delta}_r = (1 + r^2 l^{-2})(r^2 + a^2)(r^2 + b^2) - 2mr^2 + q^2 + 2qab, \tag{53}
\]
\[
H = -(y^2 - l^{-2})(a^2 - y^2)(b^2 - y^2), \tag{54}
\]
\[
\Xi_a = 1 - a^2 l^{-2}, \quad \Xi_b = 1 - b^2 l^{-2}. \tag{55}
\]

The above solution is presented in the frame where both the $\phi$-axis and the $\psi$-axis are rest at infinity [36]. Thermodynamical properties of this solution are consistent only in this rest frame [36]. When considering the extended phase space of a variable cosmological constant, their consistent thermodynamic quantities were given in Ref. [37] and subsequently in Ref. [3].
We would like to boost the rotating angular velocity of the \( \phi \)-axis to the speed of light, then we need in advance to make a coordinate transformation: \( \phi \to \phi + at/l^2 \) to a special frame where the \( \phi \)-axis is rotating whilst the \( \psi \)-axis is non-rotating at infinity. Transforming the thermodynamic quantities given in Refs. \([3, 36, 37]\) into this special frame, the expressions for the mass, electric charge, two angular momenta, Bekenstein-Hawking entropy, Hawking temperature, electrostatic potential, and two angular velocities are given as follows:

\[
\dot{M} = \frac{\pi(2 + \Xi_a)(m + qab)^{-2}}{4\Xi_a\Xi_b}, \quad \dot{Q} = \frac{\sqrt{3}\pi q}{4\Xi_a\Xi_b},
\]
\[
\dot{J}_o = \frac{\pi(2ma + qbl(2 - \Xi_a))}{4\Xi_a\Xi_b},
\]
\[
\dot{J}_\psi = \frac{\pi(2mb + qa(2 - \Xi_b))}{4\Xi_a\Xi_b},
\]
\[
\dot{S} = \frac{\pi^2[(r_a^2 + a^2)(r_b^2 + b^2) + qab]}{2\Xi_a\Xi_br_+},
\]
\[
\dot{T} = \frac{(2r_a^2 + a^2 + b^2 + l^2)l^4 - (q^2 + 2qab + a^2b^2)l^2}{2\pi l^2 r_+(r_a^2 + a^2)},
\]
\[
\Phi = \frac{\sqrt{3}qr_+}{(r_a^2 + a^2)(r_b^2 + b^2) + qab},
\]
\[
\dot{\Omega}_o = \frac{\Xi_a[a(r_a^2 + b^2)^2 + qbl]}{(r_a^2 + a^2)(r_b^2 + b^2) + qab},
\]
\[
\dot{\Omega}_\psi = \frac{b(1 + r_a^2 l^{-2})(r_a^2 + a^2) + qa}{(r_a^2 + a^2)(r_b^2 + b^2) + qab},
\]

where \( \dot{\Omega}_o \) represents the horizon angular velocity around the \( \phi \)-axis, whilst \( \dot{\Omega}_\psi \) is the one around the \( \psi \)-axis, which is evaluated at horizon relative to the infinity.

In this special frame, the above thermodynamic quantities (50) satisfy the integral Bekenstein-Smarr mass formula:

\[
2\dot{M} = 3(\dot{T} \dot{S} + \dot{\Omega}_o \dot{J}_o + \dot{\Omega}_\psi \dot{J}_\psi) + 2(\dot{\Phi} \dot{Q} - \dot{V} \dot{P}),
\]

whilst the differential mass formula merely represents a differential identity only

\[
d\dot{M} = \dot{P} d\dot{S} + \dot{\Omega}_o \dot{J}_o + \dot{\Omega}_\psi \dot{J}_\psi + \dot{\Phi} d\dot{Q} + \dot{V} d\dot{P} + \frac{\dot{J}_o}{2a} d\Xi_a,
\]

where the thermodynamic volume \( \dot{V} \) conjugate to the pressure \( \dot{P} = P = 3/(4\pi l^2) \) has the form

\[
\dot{V} = \frac{\pi^2}{6\Xi_a\Xi_b} \left\{ 3(r_a^2 + a^2)(r_b^2 + b^2) + 2qab \right. \\
+ \left. b(2mb + qa(2 - \Xi_b)) \right\}.
\]

Now assuming that the ultra-spinning direction is along the \( \phi \)-axis of the above special frame, and then taking the \( a \to l \) limit (after defining a new angle coordinate \( \phi \) by multiplying the old one with a factor \( \Xi_a \)) yields the expected super-entropic black hole solution presented in Ref. \([5]\). Then we take the same ultra-spinning limit \( a \to l \) on the above thermodynamic quantities of the double-rotating charged black hole in the five-dimensional minimal gauged supergravity theory. Now the relations (33) in the singly-rotating case should be generalized as follows:

\[
M = \frac{\mu \Xi_a}{2\pi} \dot{M}, \quad Q = \frac{\mu \Xi_a}{2\pi} \dot{Q}, \quad J_o = \frac{\mu \Xi_a^2}{2\pi} \dot{J}_o,
\]
\[
J_\psi = \frac{\mu \Xi_a}{2\pi} \dot{J}_\psi, \quad \Omega_o = \frac{1}{\Xi_a} \dot{\Omega}_o, \quad S = \frac{\mu \Xi_a}{2\pi} \dot{S},
\]
\[
V = \frac{\mu \Xi_a}{2\pi} \dot{V}, \quad T = \dot{T}, \quad \Omega_\psi = \dot{\Omega}_\psi, \quad \Phi = \dot{\Phi}
\]

together with \( P = \dot{P} \). Similarly, we have assumed that the mass parameter \( m \), the electric charge parameter \( q \), and one rotation parameter \( b \) remain unchanged.

Then taking the \( a \to l \) limit, one can get straightforwardly the thermodynamic quantities of the corresponding super-entropic black hole

\[
M = \frac{\mu(2 + \Xi_a)(m + qbl/l)}{8\Xi_a^2}, \quad Q = \frac{\sqrt{3}\mu q}{8\Xi_a^2},
\]
\[
J_o = \frac{\mu(ml + qb)}{4\Xi_b}, \quad J_\psi = \frac{\mu[mb + ql(2 - \Xi_b)]}{8\Xi_b},
\]
\[
\Omega_o = \frac{[l(r_a^2 + b^2) + qb]}{(r_a^2 + l^2)(r_a^2 + b^2) + qbl},
\]
\[
\Omega_\psi = \frac{bl^{-2}(r_a^2 + l^2)^2 + qbl}{(r_a^2 + l^2)(r_a^2 + b^2) + qbl},
\]
\[
S = \frac{\pi\mu[(r_a^2 + l^2)^2 + qbl]}{4\Xi_br_+},
\]
\[
T = \frac{l^4[2 + (2r_a^2 + b^2)(l^{-2}) - (q + bl)^2]}{2\pi l^2[(r_a^2 + l^2)(r_a^2 + b^2) + qbl]},
\]
\[
\Phi = \frac{\sqrt{3}qr_+}{(r_a^2 + l^2)(r_a^2 + b^2) + qbl},
\]
\[
V = \frac{\pi\mu}{12\Xi_b} \left\{ 3(r_a^2 + l^2)(r_a^2 + b^2) + 2qbl \right. \\
+ \left. b[2mb + ql(2 - \Xi_b)] \right\}.
\]

These expressions are the same ones as those initially obtained in Ref. \([5]\). They satisfy both the differential and integral mass formulas simultaneously

\[
dM = TdS + \Omega_o dJ_o + \Omega_\psi dJ_\psi + \Phi dQ + VdP,
\]
\[
2M = 3(TS + \Omega_o dJ_o + \Omega_\psi dJ_\psi) + 2(\Phi Q - VP),
\]

if \( \mu \) is really a constant. Dealing with the integral mass formula (51) and the differential identity (52) for a constant \( \mu \) also leads to the above first law and the Bekenstein-Smarr mass formula. Therefore, our limiting procedure is also applicable to coping with the case of black holes carrying multiple rotation parameters as well.

\[\textbf{V. CONCLUSIONS}\]

The super-entropic black hole has spurred an increasing deal of recent interest, due to the discovery that it has a non-
compact horizon topology and violates the “reverse isoperimetric inequality”, but its thermodynamic quantities satisfy both the first law and the Bekenstein-Smarr mass formula. In this paper, we obtain firstly a new Christodoulou-Ruffini-like squared-mass formula for the four-dimensional Kerr-Newman-AdS super-entropic black hole and its extension to higher dimensions with just one rotation parameter. Differentiating the squared-mass formulas yields the conjugate partners of their corresponding thermodynamic variables, which satisfy both the first law and the Bekenstein-Smarr mass formula. Then the impact of the chirality condition on the actual thermodynamics is discussed. After that, we construct a set of very simple relations between thermodynamic quantities of the usual Kerr-Newman-AdS\(_4\) black hole and those of its super-entropic counterpart. Using these relations, we find that the thermodynamic quantities of the super-entropic Kerr-Newman-AdS\(_4\) black hole can be derived from those of its corresponding usual black hole by taking the ultra-spinning limit appropriately. Our method is then generalized to the singly-rotating Kerr-AdS black hole in arbitrary dimensions and the double-rotating charged black hole of the five-dimensional minimal gauged supergravity. From our discussions completed in this article, it is natural to infer that our method can be used to obtain the thermodynamic quantities of all super-entropic black holes from those of their usual counterparts by taking the ultra-spinning limit properly and is in accordance with the spirit of obtaining these solutions by taking the same limit.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (NSFC) under Grant No. 11775077, No. 11690034, No. 11435006, No. 11675130 and No. 11275157, and by the Science and Technology Innovation Plan of Hunan province under Grant No. 2017XK2019.

[1] R.A. Hennigar, R.B. Mann, and D. Kubizňák, Entropy inequality violations from ultraspinning black holes, Phys. Rev. Lett. 115, 031101 (2015).
[2] D. Klemm, Four-dimensional black holes with unusual horizons, Phys. Rev. D 89, 084007 (2014).
[3] M. Cvetič, G.W. Gibbons, D. Kubizňák, and C.N. Pope, Black hole entrophy and an entropy inequality for the thermodynamic volume, Phys. Rev. D 84, 024037 (2011).
[4] B.P. Dolan, D. Kastor, D. Kubizňák, R.B. Mann, and J. Traschen, Thermodynamic volumes and isoperimetric inequalities for de Sitter black holes, Phys. Rev. D 87, 104017 (2013).
[5] R.A. Hennigar, D. Kubizňák, R.B. Mann, and N. Musoke, Ultraspinning limits and super-entropic black holes, J. High Energy Phys. 1506, 096 (2015).
[6] R.A. Hennigar, D. Kubizňák, R.B. Mann, and N. Musoke, Ultraspinning limits and rotating hyperboloid membranes, Nucl. Phys. B 903, 400 (2016).
[7] S.M. Noorbaksh and M. Ghominejad, Ultra-spinning gauged supergravity black holes and their Kerr/CFT correspondence, Phys. Rev. D 95, 046002 (2017).
[8] S.M. Noorbaksh and M.H. Vahidinia, Extremal vanishing horizon Kerr-AdS black holes at ultraspinning limit, J. High Energy Phys. 1801, 042 (2018).
[9] S.M. Noorbaksh and M. Ghominejad, Higher dimensional charged AdS black holes at ultra-spinning limit and their 2d CFT duals, arXiv:1702.03448.
[10] M. Appels, L. Cuspinera, R. Gregory, P. Krtouš, and D. Kubizňák, Are Superentropic black holes superentropic?, arXiv:1911.12817.
[11] J.D. Bekenstein, Black holes and entropy, Phys. Rev. D 7, 2333 (1973).
[12] S.W. Hawking, Black holes and thermodynamics, Phys. Rev. D 13, 191 (1976).
[13] L. Smarr, Mass formula for Kerr black holes, Phys. Rev. Lett. 30, 71 (1973); 30, 521(E) (1973).
[14] D. Christodoulou, Reversible and irreversible transformations in black hole physics, Phys. Rev. Lett. 25, 1596 (1970).
[15] D. Christodoulou and R. Ruffini, Reversible transformations of a charged black hole, Phys. Rev. D 4, 3552 (1971).
[16] S.Q. Wu and D. Wu, Thermodynamical hairs of the four-dimensional Taub-Newman-Uni-Tamburino spacetimes, Phys. Rev. D 100, 101501 (2019).
[17] D. Kastor, S. Ray, and J. Traschen, Entropy and the mechanics of AdS black holes, Classical Quantum Gravity 26, 195011 (2009).
[18] A.M. Frassino, R.B. Mann, and J.R. Mureika, Lower-dimensional black hole chemistry, Phys. Rev. D 92, 124069 (2015).
[19] D. Kubizňák, R.B. Mann, and M. Teo, Black hole chemistry: thermodynamics with Lambda, Classical Quantum Gravity 34, 063001 (2017).
[20] G. Ruppeiner, Riemannian geometry in thermodynamic fluctuation theory, Rev. Mod. Phys. 67, 605 (1995); 68, 313(E) (1996).
[21] G. Ruppeiner, Stability and fluctuations in black hole thermodynamics, Phys. Rev. D 75, 024037 (2007).
[22] G. Ruppeiner, Thermodynamic curvature and phase transitions in Kerr-Newman black holes, Phys. Rev. D 78, 024016 (2008).
[23] S.W. Wei, Y.X. Liu, and R.B. Mann, Ruppeiner geometry, phase transitions, and the microstructure of charged AdS black holes, Phys. Rev. D 100, 124033 (2019).
[24] S.W. Wei, Y.X. Liu, and R.B. Mann, Repulsive interactions and universal properties of charged anti-de Sitter black hole microstructures, Phys. Rev. Lett. 123, 071103 (2019).
[25] S.W. Wei and Y.X. Liu, Null geodesics, quasinormal modes, and thermodynamic phase transition for charged black holes in asymptotically flat and dS spacetimes, arXiv:1909.11911.
[26] P. Wang, H.W. Wu, and H.T. Yang, Thermodynamic geometry of AdS\(_d\) black holes and black holes in a cavity, arXiv:1910.07874.
[27] Z.M. Xu, B. Wu, and W.L. Yang, The fine micro-thermal structures for the Reissner-Nordström black hole, arXiv:1910.03378.
[28] S.W. Wei and Y.X. Liu, Intriguing microstructures of five-dimensional neutral Gauss-Bonnet AdS\(_d\) black hole, arXiv:1910.04528.
[29] W. Chen, H. Lü, and C. N. Pope, Mass of rotating black holes in gauged supergravities, Phys. Rev. D 73, 104036 (2006).
[30] L. F. Abbott and S. Deser, Stability of gravity with a cosmological constant, Nucl. Phys. B 195, 76 (1982).
[31] M.M. Caldarelli, G. Cognola, and D. Klemm, Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories, Classical Quantum Gravity 17, 399 (2000).
[32] S.Q. Wu, New formulation of the first law of black hole thermodynamics: A stringy analogy, Phys. Lett. B 608, 251 (2005).
[33] D.C. Wright, Black holes and the Gibbs-Duhem relation, Phys. Rev. D 21, 884 (1980).
[34] G.W. Gibbons, M.J. Perry, and C.N. Pope, The first law of thermodynamics for Kerr-anti-de Sitter black holes, Classical Quantum Gravity 22, 1503 (2005).
[35] S.W. Hawking, C.J. Hunter, and M.M. Taylor-Robinson, Rotation and the AdS/CFT correspondence, Phys. Rev. D 59, 064005 (1999).
[36] Z.W. Chong, M. Cvetič, H. Lü, and C.N. Pope, General non-extremal rotating black holes in minimal five-dimensional gauged supergravity, Phys. Rev. Lett. 95, 161301 (2005).
[37] S.Q. Wu, Separability of massive field equations for spin-0 and spin-1/2 charged particles in the general non-extremal rotating charged black holes in minimal five-dimensional gauged supergravity, Phys. Rev. D 80, 084009 (2009).