Harmonic analysis on local fields and adelic spaces. II

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Abstract. We develop harmonic analysis in certain categories of filtered Abelian groups and vector spaces. The objects of these categories include local fields and adelic spaces arising from arithmetic surfaces. We prove some structure theorems for quotients of the adéle groups of algebraic and arithmetic surfaces.

Keywords: arithmetic surfaces, higher adèles, harmonic analysis, Fourier transform, Poisson summation formulae.

§ 1. Introduction

This paper is a continuation of [1]. Its goal is twofold. First, we introduce and investigate a category $C^\text{fin}_n$ of filtered Abelian groups which is constructed inductively from the category $C^\text{fin}_0$ of finite Abelian groups as opposed to the category $C_n$ of filtered vector spaces over a field $k$ which was constructed inductively in [2] from the category $C_0$ of finite-dimensional vector spaces over $k$. We also consider some categories of filtered vector spaces over $\mathbb{R}$ or $\mathbb{C}$ and define the mixed categories $C^\text{ar}_0$, $C^\text{ar}_1$, and $C^\text{ar}_2$, which contain some of the previous categories as full subcategories. We then develop harmonic analysis for these categories. This part of our theory is used to study arithmetic surfaces. The resulting formalism is in parallel with the constructions in [1]. A short description of this formalism is given in the introduction of [1], where the origins and history of the whole theory is also explained.

Second, we describe several examples of adelic spaces and show how general categorical notions can be applied to them. Note that [1] contained no concrete examples. Here we carefully study the case of adèle groups for schemes of dimension 1

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(see Example 4 in §3.2 and Example 8 in §4.2) and dimension 2 (see §14 and Example 11 in §5). In particular, we prove a compactness theorem for the adelic quotient groups \( \mathbb{A}_X/(\mathbb{A}_X,0_1 + \mathbb{A}_X,0_2) \) in the case of a projective algebraic surface \( X \) defined over a finite field (see §§14.1, 14.2). This theorem generalizes the classical compactness theorem for the adelic quotient group \( \mathbb{A}_C/\mathbb{A}_C,0 \) in the case of a projective algebraic curve \( C \) defined over a finite field.

When constructing harmonic analysis on groups like \( \mathbb{A}_X^* = \text{GL}(1, \mathbb{A}_X) \) or the multiplicative group of a local field on \( X \), one must take into account the fact that these groups are extensions of discrete groups by groups that are not locally compact. Harmonic analysis on the discrete part is already highly non-trivial (see [6]).

We take this opportunity to correct Corollaries 1, 2 in [1]; see §5.9. We must consider a narrower class of automorphisms there. Namely, we consider only those elements of the automorphism group (or a central extension of it) that interchange the elements of the first filtration of an object of \( C_2 \). This condition holds in many concrete situations, such as in the case of the standard action of the multiplicative group \( K^* \) on a local field \( K \) or the action of \( \mathbb{A}_X^* \) on \( \mathbb{A}_X \).

\[ § 2. \] \textbf{Notation and conventions}

Given any Abelian groups \( A, B \), we write \( \text{Hom}(A, B) \) for the Abelian group of all homomorphisms from \( A \) to \( B \).

By a topological group we always mean a topological group with a Hausdorff topology.

For every locally compact Abelian group \( G \) we denote its Pontryagin dual group by \( \hat{G} \). Thus, \( \hat{G} = \text{Hom}_{\text{cont}}(G, \mathbb{T}) \), where the group \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \) is the one-dimensional torus.

For every topological vector space \( V \) over the field \( \mathbb{C} \) of complex numbers we denote its continuously dual \( \mathbb{C} \)-vector space by \( V' \). Thus, \( V' = \text{Hom}_{\mathbb{C},\text{cont}}(V, \mathbb{C}) \) is the space of all continuous \( \mathbb{C} \)-linear functionals.

\[ § 3. \] \textbf{The categories \( C_{n}^{\text{ar}} \)}

\textbf{3.1. The categories \( C_{n}^{\text{fin}} \).} We recall that the categories \( C_n, n \in \mathbb{N} \), were defined in [2] (see also §§4.1 and 5.1 of [1] for additional properties of the categories \( C_0, C_1 \), and \( C_2 \)).

The categories \( C_n \) are constructed by induction on \( n \), starting from the category \( C_0 \) of finite-dimensional vector spaces over a fixed field \( k \). As mentioned in the introduction of [2], one can similarly construct the categories \( C_{n}^{\text{fin}} \), starting from the category \( C_{n}^{\text{fin}} \) of finite Abelian groups. If \( k \) is a finite field, then \( C_n \) is a full subcategory of \( C_{n}^{\text{fin}} \) for all \( n \in \mathbb{N} \).

We now give a definition of the category \( C_{n}^{\text{fin}} \) by induction on \( n \in \mathbb{N} \).

\textbf{Definition 1.} We say that \((I, F, V)\) is a \textit{filtered Abelian group} if

1) \( V \) is an Abelian group,

2) \( I \) is a partially ordered set such that for all \( i, j \in I \) there are \( k, l \in I \) with \( k \leq i \leq l \) and \( k \leq j \leq l \).

\textsuperscript{2}Note that this correction was already made in the latest electronic version of our paper: http://arxiv.org/abs/0707.1766.
3) $F$ is a function from $I$ to the set of subgroups of $V$ such that $F(i) \subset F(j)$ for all elements $i \leq j$ of $I$,
4) $\bigcap_{i \in I} F(i) = 0$ and $\bigcup_{i \in I} F(i) = V$.

**Definition 2.** We say that a filtered Abelian group $(I, F, V)$ *dominates* another filtered Abelian group $(I_2, F_2, V)$ if there is an order-preserving function $\phi: I_2 \to I_1$ with the following properties:
1) $F_1(\phi(i)) = F_2(i)$ for all $i \in I_2$,
2) for every $j \in I_1$ there are $i_1, i_2 \in I_2$ such that $\phi(i_1) \leq j \leq \phi(i_2)$.

We now use induction to define the objects and morphisms of $C_n^{\text{fin}}$.

**Definition 3.** 1) The category $C_0^{\text{fin}}$ is the category of finite Abelian groups and group homomorphisms.
2) A triple

$$0 \to V_0 \to V_1 \to V_2 \to 0$$

in $C_n^{\text{fin}}$ is *admissible* if it is an exact triple of Abelian groups.

We define the objects of $C_n^{\text{fin}}$ by induction. Suppose that the objects of $C_{n-1}^{\text{fin}}$ have already been defined and the notion of an admissible triple in $C_{n-1}^{\text{fin}}$ has already been introduced.

**Definition 4.** 1) The objects of the category $C_n^{\text{fin}}$, that is, Ob($C_n^{\text{fin}}$), are filtered Abelian groups $(I, F, V)$ with the following additional structures.
   a) For all $i \leq j \in I$ there is a structure $E_{i,j} \in \text{Ob}(C_{n-1}^{\text{fin}})$ on the Abelian group $F(j)/F(i)$.
   b) For all $i \leq j \leq k \in I$, the triple

$$0 \to E_{i,j} \to E_{i,k} \to E_{j,k} \to 0$$

is an admissible triple in $C_{n-1}^{\text{fin}}$.
2) Let $E_1 = (I_1, F_1, V_1)$, $E_2 = (I_2, F_2, V_2)$ and $E_3 = (I_3, F_3, V_3)$ be elements of Ob($C_n^{\text{fin}}$). We say that

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

is an admissible triple in $C_n^{\text{fin}}$ if the following conditions hold.
   a) $0 \to V_1 \to V_2 \to V_3 \to 0$ is an exact triple of Abelian groups.
   b) The filtration $(I_1, F_1, V_1)$ dominates the filtration $(I_2, F'_1, V_1)$, where $F'_1(i) = F_2(i) \cap V_1$ for all $i \in I_2$.
   c) The filtration $(I_3, F_3, V_3)$ dominates the filtration $(I_2, F'_3, V_3)$, where $F'_3(i) = F_2(i)/F_2(i) \cap V_1$.
   d) For all $i \leq j \in I_2$, the triple

$$0 \to F'_1(j) \to F_2(j) \to F'_3(j) \to 0$$

is an admissible triple in $C_{n-1}^{\text{fin}}$. (By the definition of Ob($C_{n}^{\text{fin}}$), every Abelian group in (1) is endowed with the structure of an object in Ob($C_{n-1}^{\text{fin}}$).)

We now define the morphisms in $C_n^{\text{fin}}$ by induction. Suppose that the morphisms in $C_{n-1}^{\text{fin}}$ have already been defined.
Definition 5. Let $E_1 = (I_1, F_1, V_1)$ and $E_2 = (I_2, F_2, V_2)$ be elements of $\text{Ob}(C_n^\text{fin})$. Then the set of morphisms $\text{Mor}_{C_n^\text{fin}}(E_1, E_2)$ consists of all $A \in \text{Hom}(V_1, V_2)$ such that the following conditions hold.

1) For every $i \in I_1$ there is $j \in I_2$ such that $A(F_1(i)) \subset F_2(j)$.
2) For every $j \in I_2$ there is $i \in I_1$ such that $A(F_1(i)) \subset F_2(j)$.
3) For all pairs $i_1 \leq i_2 \in I_1$ and $j_1 \leq j_2 \in I_2$ such that $A(F_1(i_1)) \subset F_2(j_1)$ and $A(F_1(i_2)) \subset F_2(j_2)$, the induced map of Abelian groups

$$A: \frac{F_1(i_2)}{F_1(i_1)} \rightarrow \frac{F_2(j_2)}{F_2(j_1)}$$

belongs to $\text{Mor}_{C_n^\text{fin}-1}(\frac{F_1(i_2)}{F_1(i_1)}, \frac{F_2(j_2)}{F_2(j_1)})$.

The morphisms in $C_n^\text{fin}$ are well defined. This and other properties are corollaries of the following proposition, which is an analogue of Proposition 2.1 in [2].

Proposition 1. Suppose that $E_1 = (I_1, F_1, V_1)$, $E_2 = (I_2, F_2, V_2)$, $E'_1$, $E'_2$ belong to $\text{Ob}(C_n^\text{fin})$ and $A$ is a map belonging to $\text{Hom}(V_1, V_2)$.

1) If the filtered Abelian group $E_1$ dominates the filtered Abelian group $E'_1$ and the filtered Abelian group $E_2$ dominates the filtered Abelian group $E'_2$, then we have $A \in \text{Mor}_{C_n^\text{fin}}(E_1, E_2)$ if and only if $A \in \text{Mor}_{C_n^\text{fin}}(E'_1, E'_2)$.
2) The set $\text{Mor}_{C_n^\text{fin}}(E_1, E_2)$ is an Abelian subgroup of $\text{Hom}(V_1, V_2)$.
3) If $E_3$ is an object in $C_n^\text{fin}$, then

$$\text{Mor}_{C_n^\text{fin}}(E_2, E_3) \circ \text{Mor}_{C_n^\text{fin}}(E_1, E_2) \subset \text{Mor}_{C_n^\text{fin}}(E_1, E_3).$$

The proof is analogous to that of Proposition 2.1 in [2].

Example 1. Let $X$ be an $n$-dimensional scheme of finite type over $\mathbb{Z}$ and let $P(X)$ be the set of points of $X$. Given $\eta, \nu \in P(X)$, we say that $\eta \geq \nu$ if $\nu \in \{\eta\}$. The relation $\geq$ is a partial order on $P(X)$. Let $S(X)$ be the simplicial set induced by $(P(X), \geq)$, that is,

$$S(X)_m = \{(\nu_0, \ldots, \nu_m) \mid \nu_i \in P(X); \nu_i \geq \nu_{i+1}\}$$

is the set of $m$-simplices in $S(X)$ with the $i$th boundary map being deletion of the $i$th point, and the $i$th degeneracy map being duplication of the $i$th point. We write $S(X)^{\text{red}}_m$ for the set of all non-degenerate simplices in $S(X)_m$, that is,

$$S(X)^{\text{red}}_m = \{(\nu_0, \ldots, \nu_m) \mid \nu_i \in P(X); \nu_i < \nu_{i+1}\}.$$

An adelic space $\mathbb{A}(K, \mathcal{F})$ was constructed in [7], [8] for every subset $K \subset S(X)_m$ and every quasi-coherent sheaf $\mathcal{F}$ on $X$. We have

$$\mathbb{A}(K, \mathcal{F}) \subset \prod_{\delta \in K} \mathbb{A}(\delta, \mathcal{F}).$$

Arguing as in the proof of Theorem 1 in [2], we see that the adelic space $\mathbb{A}(K, \mathcal{F})$ is an object of $C_n^\text{fin}$ for every subset $K \subset S(X)^{\text{red}}_m$.

In particular, if $K = S(X)^{\text{red}}_m$ and $\mathcal{F} = \mathcal{O}_X$, then we obtain that the whole adelic space $\mathbb{A}_X = \mathbb{A}(S(X)^{\text{red}}_m, \mathcal{O}_X)$ is an object of $C_n^\text{fin}$.

Suppose that $\delta = (\eta_0 > \cdots > \eta_n) \in S(X)^{\text{red}}_n$ and the point $\eta_n$ is regular on every subscheme $\tilde{\eta}_i$ which is the closure of the point $\eta_i$. Then $K_{\delta} = \mathbb{A}(\delta, \mathcal{O}_V)$ is an
$n$-dimensional local field with finite last residue field; see [9], [10]. We obtain that $K_\delta \in \text{Ob}(C_{n}^{\text{fin}})$.

In particular, we have

$$
F_q((t_n)) \ldots ((t_1)) \in \text{Ob}(C_{n}^{\text{fin}}),
$$

$$
E((t)) \in \text{Ob}(C_2^{\text{fin}}), \quad E\{t\} \in \text{Ob}(C_2^{\text{fin}}),
$$

where $E \supset \mathbb{Q}_p$ is a finite field extension.

3.2. The categories $C_{0}^{\text{ar}}$ and $C_{1}^{\text{ar}}$. We define the category $C_{0}^{\text{ar}}$ in the following way.

**Definition 6.** The category $C_{0}^{\text{ar}}$ is the full subcategory of the category of commutative finite-dimensional real smooth Lie groups such that $G \in \text{Ob}(C_{0}^{\text{ar}})$ if and only if $\pi_0(G)$ is finitely generated. Here the Abelian group $\pi_0(G)$ is the group of connected components of $G$.

We mention the following well-known property.

**Lemma 1.** A group $G$ belongs to $\text{Ob}(C_{0}^{\text{ar}})$ if and only if there is an isomorphism

$$
G \simeq A \times \mathbb{Z}^r \times \mathbb{T}^p \times \mathbb{R}^q,
$$

where $p$, $q$, $r$ are non-negative integers, $A$ is a finite Abelian group and $\mathbb{T} = S^1 = \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus.

**Proof.** If $G$ has the decomposition (2), then we clearly have $G \in \text{Ob}(C_{0}^{\text{ar}})$. Now suppose that $G \in \text{Ob}(C_{0}^{\text{ar}})$ and let $G^0 \subset G$ be the connected component of the identity $e \in G$. Since $G$ is an Abelian group and $G^0$ is connected, the exponential map is a surjective homomorphism from the tangent space at $e \in G$ (which is a Lie group isomorphic to a real vector space) to the group $G^0$. It follows that $G^0 \simeq \mathbb{T}^p \times \mathbb{R}^q$ for some integers $p$, $q$. The discrete group

$$
G/G^0 \simeq \mathbb{Z}^r \times \prod_{1 \leq i \leq l} \mathbb{Z}/n_i \mathbb{Z}
$$

can be written in this way for some integers $n_i (1 \leq i \leq l)$ and $r$. We choose any $i$, $1 \leq i \leq l$, and let $e_i \in G$ be a pre-image of a generator of the group $\mathbb{Z}/n_i \mathbb{Z}$. Then $s_i = e_i^{n_i} \in G^0$. Since $G^0 \simeq \mathbb{T}^p \times \mathbb{R}^q$, there is an element $t_i \in G^0$ such that $t_i^{n_i} = s_i$. Putting $e'_i = e_i t_i^{-1}$, we have $(e'_i)^{n_i} = 1$. Therefore the subgroup of $G$ generated by $e'_i$ is isomorphic to $\mathbb{Z}/n_i \mathbb{Z}$, and we get a lift of the group $\mathbb{Z}/n_i \mathbb{Z}$ to $G$ for every $1 \leq i \leq l$. Lifting the generators of $\mathbb{Z}^r$ to their pre-images in $G$, we isomorphically map the group $\mathbb{Z}^r$ into $G$. The lemma is proved.

The decomposition (2) is non-unique, but we always have the following canonical filtration of $G$ by Lie subgroups:

$$
G \supset G_{\text{tor}} \supset G^0 \supset K,
$$

where $G^0$ is the connected component of the identity $e$ in $G$, $K$ is a maximal compact subgroup of $G^0$, and $G_{\text{tor}}/G^0$ is the maximal torsion subgroup of the discrete group $G/G^0$. The group $G/G^0$ is a finitely generated Abelian group. Then we have the following isomorphisms:

$$
K \simeq \mathbb{T}^p, \quad G^0/K \simeq \mathbb{R}^q, \quad G_{\text{tor}}/G^0 \simeq A, \quad G/G_{\text{tor}} \simeq \mathbb{Z}^r.$$
Hence the non-negative integers $p$, $q$, $r$ in the decomposition (2) are uniquely determined by $G$, and the finite group $A$ is uniquely determined by $G$ up to isomorphism. We also have

$$\pi_0(G) = G/G^0, \quad \text{rank}(\pi_1(G)) = p, \quad \text{dim} G = p + q.$$  

**Definition 7.** We say that a triple in $C^\text{ar}_0$

$$0 \longrightarrow G_1 \longrightarrow G_2 \overset{\phi}{\longrightarrow} G_3 \longrightarrow 0 \quad (4)$$

is an admissible triple in $C^\text{ar}_0$ if $G_1 = \text{Ker}(\phi)$ and $\phi$ is an epimorphism of Lie groups.

**Remark 1.** One can restate Definition 7 as follows. The triple (4) is admissible if and only if it is isomorphic to a triple of the form

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_2/G_1 \longrightarrow 0,$$

where $G_1$ is a closed Lie subgroup of $G_2$. By Cartan's theorem, this is equivalent to saying that $G_1$ is a closed subgroup of $G_2$.

**Example 2.** The category $C^\text{fin}_0$ is a full subcategory of $C^\text{ar}_0$ such that a triple in $C^\text{fin}_0$ is admissible in $C^\text{fin}_0$ if and only if it is admissible in $C^\text{ar}_0$.

For any field $k$ let $C_n(k)$ be the category $C_n$ defined over $k$ (see [2]). We see that $C_n(\mathbb{R})$ is the full subcategory of $C^\text{ar}_0$ such that a triple in $C_n(\mathbb{R})$ is admissible in $C^\text{fin}(\mathbb{R})$ if and only if it is admissible in $C^\text{ar}_0$.

We now define the category $C^\text{ar}_1$.

**Definition 8.** The objects of the category $C^\text{ar}_1$, that is, $\text{Ob}(C^\text{ar}_1)$, are filtered Abelian groups $E = (I,F,V)$ (see Definition 1) with the following additional structure and conditions.

1) For all $i \leq j \in I$ the Abelian group $F(j)/F(i)$ is endowed with the structure of an object $E_{i,j} \in \text{Ob}(C^\text{ar}_0)$.

2) For all $i \leq j \leq k \in I$ the triple

$$0 \longrightarrow E_{i,j} \longrightarrow E_{i,k} \longrightarrow E_{j,k} \longrightarrow 0$$

is admissible in $C^\text{ar}_0$.

3) There are $i_E \leq j_E \in I$ such that $F(i_E)/F(k)$ is a finite Abelian group for all $k \leq i_E$ and $F(l)/F(j_E)$ is a finite Abelian group for all $l \geq j_E$.

**Remark 2.** Condition 3) in Definition 8 is equivalent to the following condition. There is $d_E \in \mathbb{N}$ such that, for every $i \leq j \in I$, the group $F(j)/F(i)$ possesses the following properties as an object of $C^\text{ar}_0$:

$$\text{dim} F(j)/F(i) \leq d_E, \quad \text{rank} \pi_0(F(j)/F(i)) \leq d_E.$$  

**Definition 9.** Suppose that $E_1 = (I_1,F_1,V_1)$ and $E_2 = (I_2,F_2,V_2)$ belong to $\text{Ob}(C^\text{ar}_1)$. Then the set of morphisms $\text{Mor}_{C^\text{ar}_1}(E_1,E_2)$ consists of all $A \in \text{Hom}(V_1,V_2)$ with the following properties.

1) For every $i \in I_1$ there is $j \in I_2$ such that $A(F_1(i)) \subset F_2(j)$.

2) For every $j \in I_2$ there is $i \in I_1$ such that $A(F_1(i)) \subset F_2(j)$.
3) For all pairs \( i_1 \leq i_2 \in I_1 \) and \( j_1 \leq j_2 \in I_2 \) with \( A(F_1(i_1)) \subset F_2(j_1) \) and \( A(F_1(i_2)) \subset F_2(j_2) \), the induced map of Abelian groups
\[
\tilde{A} : \frac{F_1(i_2)}{F_1(i_1)} \longrightarrow \frac{F_2(j_2)}{F_2(j_1)}
\]
belongs to \( \text{Mor}_{C_i^\text{ar}}(\frac{F_1(i_2)}{F_1(i_1)}, \frac{F_2(j_2)}{F_2(j_1)}) \).

We clearly have a complete analogue of Proposition 1 for the category \( C_1^\text{ar} \).

**Proposition 2.** Suppose that \( E_1 = (I_1, F_1, V_1) \), \( E_2 = (I_2, F_2, V_2), \ E'_1, \ E'_2 \) belong to \( \text{Ob}(C_1^\text{ar}) \) and \( A \) is a map belonging to \( \text{Hom}(V_1, V_2) \).

1) If the filtered Abelian group \( E_1 \) dominates \( E'_1 \) and the filtered Abelian group \( E_2 \) dominates \( E'_2 \), then we have \( A \in \text{Mor}_{C_i^\text{ar}}(E_1, E_2) \) if and only if \( A \in \text{Mor}_{C_i^\text{ar}}(E'_1, E'_2) \).

2) The set \( \text{Mor}_{C_i^\text{ar}}(E_1, E_2) \) is an Abelian subgroup of \( \text{Hom}(V_1, V_2) \).

3) If \( E_3 \) is an object of \( C_1^\text{ar} \), then
\[
\text{Mor}_{C_i^\text{ar}}(E_2, E_3) \circ \text{Mor}_{C_i^\text{ar}}(E_1, E_2) \subset \text{Mor}_{C_i^\text{ar}}(E_1, E_3).
\]

The proof is analogous to that of Proposition 2.1 in [2].

**Definition 10.** Suppose that \( E_1 = (I_1, F_1, V_1), \ E_2 = (I_2, F_2, V_2) \) and \( E_3 = (I_3, F_3, V_3) \) belong to \( \text{Ob}(C_1^\text{ar}) \). We say that
\[
0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0
\]
is an admissible triple in \( C_1^\text{ar} \) if the following conditions hold.

a) \( 0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0 \) is an exact triple of Abelian groups.

b) The filtration \( (I_1, F_1, V_1) \) dominates \( (I_2, F'_1, V_1) \), where \( F'_1(i) = F_2(i) \cap V_1 \) for all \( i \in I_2 \).

c) The filtration \( (I_3, F_3, V_3) \) dominates \( (I_2, F'_3, V_3) \), where \( F'_3(i) = F_2(i) / F_2(i) \cap V_1 \).

d) For all \( i \leq j \in I_2 \) the triple
\[
0 \longrightarrow \frac{F'_1(j)}{F'_1(i)} \longrightarrow \frac{F_2(j)}{F_2(i)} \longrightarrow \frac{F'_3(j)}{F'_3(i)} \longrightarrow 0
\]
is admissible in \( C_0^\text{ar} \). (By the definition of \( \text{Ob}(C_1^\text{ar}) \), each Abelian group in (5) possesses the structure of an object in \( \text{Ob}(C_0^\text{ar}) \)).

**Example 3.** The category \( C_1^\text{fin} \) is a full subcategory of \( C_1^\text{ar} \) such that a triple in \( C_1^\text{fin} \) is admissible in \( C_1^\text{fin} \) if and only if it is admissible in \( C_1^\text{ar} \).

The category \( C_0^\text{ar} \) is a full subcategory of \( C_1^\text{ar} \) with respect to the following functor \( I \). For every \( G \in C_0^\text{ar} \) we put \( I(G) = \{ \{0\}, \{1\}, F, G \} \), where \( (0) < (1), F((0)) = e, F((1)) = G, e \) is the trivial subgroup of \( G \) and \( I \) acts as the identity map on the morphisms of \( C_0^\text{ar} \).

If \( G \in \text{Ob}(C_0^\text{ar}) \), then \( G \) is a locally compact Abelian group and, therefore, its Pontryagin dual \( \hat{G} \) is also a locally compact Abelian group. Moreover, we have \( \hat{G} \in \text{Ob}(C_0^\text{ar}) \) by formula (2) since, for all \( n \in \mathbb{N} \),
\[
\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}/n\mathbb{Z}, \quad \hat{T} \simeq \mathbb{Z}, \quad \hat{\mathbb{Z}} \simeq T, \quad \hat{\mathbb{R}} \simeq \mathbb{R}.
\]
Given \( E = (I, F, V) \in \text{Ob}(C^\text{ar}_1) \), we define the dual object \( \hat{E} = (I^0, F^0, \hat{V}) \in \text{Ob}(C^\text{ar}_1) \) as follows. The Abelian group \( \hat{V} \subset \text{Hom}(V, \mathbb{T}) \) is defined as

\[
\hat{V} \overset{\text{def}}{=} \lim_{\overset{\longleftarrow}{j \in \mathbb{N}}} \lim_{i \geq j} \frac{F(i)}{F(j)}.
\] (6)

The set \( I^0 \) is a partially ordered set which equals \( I \) as a set but whose ordering is the reverse of that of \( I \). For \( j \in I^0 \) we define a subgroup

\[
F^0(j) \overset{\text{def}}{=} \lim_{\overset{\longleftarrow}{i \leq j}} \frac{F(i)}{F(j)} \subset \hat{V}.
\] (7)

If \( E_1, E_2 \in \text{Ob}(C^\text{ar}_1) \) and \( \theta \in \text{Mor}_{C^\text{ar}_1}(E_1, E_2) \), then by definition we have a canonical element \( \hat{\theta} \in \text{Mor}_{C^\text{ar}_1}(\hat{E}_2, \hat{E}_1) \). If

\[
0 \longrightarrow E_1 \overset{\alpha}{\longrightarrow} E_2 \overset{\beta}{\longrightarrow} E_3 \longrightarrow 0
\]
is an admissible triple in \( C^\text{ar}_1 \), where \( E_i = (I_i, F_i, V_i) \) for \( 1 \leq i \leq 3 \), then we have the following canonical admissible triple in \( C^\text{ar}_1 \):

\[
0 \longrightarrow \hat{E}_3 \overset{\beta}{\longrightarrow} \hat{E}_2 \overset{\alpha}{\longrightarrow} \hat{E}_1 \longrightarrow 0.
\]

**Definition 11.** We say that an object \( (I, F, V) \in C^\text{ar}_1 \) is a complete object if

\[
V = \lim_{\overset{\longleftarrow}{i \in I}} \lim_{j \leq i} \frac{F(i)}{F(j)}.
\] (8)

We have \( \hat{\hat{G}} = G \) for every locally compact Abelian group \( G \). Hence it follows from the definition that an object \( E \in \text{Ob}(C^\text{ar}_1) \) is complete if and only if \( \hat{\hat{E}} = E \).

We have an obvious functor of completion \( \Psi : C^\text{ar}_1 \longrightarrow C^\text{ar}_1 \). For \( E = (I, F, V) \in \text{Ob}(C^\text{ar}_1) \) we put

\[
\Psi(E) \overset{\text{def}}{=} \left( I, F', \lim_{\overset{\longleftarrow}{i \in I}} \lim_{j \leq i} \frac{F(i)}{F(j)} \right)
\]
and for every \( j \in I \) we put

\[
F'(j) \overset{\text{def}}{=} \lim_{\overset{\longleftarrow}{j \leq i}} \frac{F(i)}{F(j)}.
\]

The map \( \Psi \) extends to the set of morphisms \( \text{Mor}_{C^\text{ar}_1}(E_1, E_2) \), \( E_1, E_2 \in \text{Ob}(C^\text{ar}_1) \), by means of known maps defined on \( \text{Mor}_{C^\text{ar}_0} \).

It is clear from the definition of \( \Psi \) that the object \( \Psi(E) \) is complete for every \( E \in \text{Ob}(C^\text{ar}_1) \), and \( \Psi^2 = \Psi \). Moreover, \( \Psi(E) = \hat{\hat{E}} \).

We denote by \( C^\text{ar}_1, \text{compl} \) the full subcategory of \( C^\text{ar}_1 \) consisting of the complete objects of \( C^\text{ar}_1 \).

Let \( \text{Loc} \) be the category of locally compact Abelian groups. For every group \( G \in \text{Ob}(\text{Loc}) \) there are subgroups \( V \subset U \) of \( G \) with the following properties (see [11], Ch. II, § 2.2).

\((*)\) \( U \) is an open subgroup in \( G \), and \( U \) is generated by a compact neighbourhood of the identity \( e \) (a neighbourhood of \( e \) is a set containing an open neighbourhood of \( e \)).
(**) \( V \) is a compact subgroup in \( G \).

(***) \( U/V \in \text{Ob}(C^{ar}_0) \).

Moreover, \( G \) is the inductive limit of such subgroups \( U \), and every \( U \) is the projective limit of the groups \( U/V \), where \( U \) and \( V \) possess properties (**)–(***). We endow \( G \) with the topology of inductive and projective limits starting from the topological groups \( U/V \) (see [11], Ch. II, §2.2).

Let \( \text{Loc}^{ar} \) be the following full subcategory of \( \text{Loc} \). Given any \( G \in \text{Ob}(\text{Loc}) \), we say that \( G \in \text{Ob}(\text{Loc}^{ar}) \) if there is \( d \in \mathbb{N} \) such that for all pairs \( V \subset U \) of subgroups in \( G \) with properties (**)–(***) we have

\[
\text{dim}(U/V) < d, \quad \text{rank} \pi_0(U/V) < d
\]

(compare this condition with Remark 2).

We now define a functor \( \Phi: C^{ar}_1 \to \text{Loc}^{ar} \). Given \( E = (I, F, V) \in \text{Ob}(C^{ar}_1) \), we put

\[
\Phi(E) \overset{\text{def}}{=} \lim_{i \in I} \lim_{j \leq i} F(i)/F(j)
\]

and endow the group \( \Phi(E) \) with the topology of inductive and projective limits starting from the topological groups \( F(i)/F(j) \in \text{Ob}(C^{ar}_0) \). (Moreover, the condition of Remark 2 becomes the condition (9).) The functor \( \Phi \) is also defined on morphisms because we can glue a compatible system of morphisms from \( C^{ar}_0 \), which is given by Definition 9.

**Proposition 3.** 1) The functor \( \Phi|_{C^{ar}_1,\text{compl}} \) is an equivalence of the categories \( C^{ar}_1,\text{compl} \) and \( \text{Loc}^{ar} \).

2) For every \( E \in \text{Ob}(C^{ar}_1) \) we have \( \Phi(\hat{E}) = \hat{\Phi(E)} \).

3) The functor \( \Phi \) maps the set of admissible triples in \( C^{ar}_1 \) onto the following set of exact triples of locally compact Abelian groups in \( \text{Loc}^{ar} \):

\[
0 \to H \to G \to G/H \to 0,
\]

where \( H \) is a closed subgroup in \( G \).

**Proof.** To prove part 1), we construct a functor

\[
\Lambda: \text{Loc}^{ar} \to C^{ar}_1,\text{compl}.
\]

Given \( G \in \text{Loc}^{ar} \), we put \( \Lambda(G) = (I, F, G) \), where \( I \) is a set of subgroups in \( G \) such that \( K \in I \) if and only if \( K \) coincides either with a subgroup \( U \) or with a subgroup \( V \) possessing properties (**)–(***). The set \( I \) is partially ordered by inclusion of subgroups. The function \( F \) maps every \( K \in I \) to the corresponding subgroup \( K \subset \tilde{G} \). It is clear from the arguments preceding the proposition that \( (I, F, G) \in \text{Ob}(C^{ar}_1,\text{compl}) \).

Now let \( \phi \in \text{Mor}_{\text{Loc}^{ar}}(G_1, G_2) \). We claim that \( \Lambda(\phi) = \phi \) is well defined as an element of \( \text{Mor}_{C^{ar}_1}(\Lambda(G_1), \Lambda(G_2)) \), that is, \( \phi \) satisfies conditions 1)–3) of Definition 9. Indeed, let \( U_1 \supset V_1 \) be subgroups of \( G_1 \) possessing properties (**)–(***) for the group \( G_1 \). Let \( U_2 \supset V_2 \) be subgroups of \( G_2 \) possessing properties (**)–(***) for the group \( G_2 \). By condition (9) there is no loss of generality in assuming that \( G_2/U_2 \) is a discrete torsion group. The group \( \phi(V_1) \) is compact and, therefore, \( \phi(V_1)+U_2 \) is a finite group. Moreover, \( \phi(U_1)+U_2 \phi(V_1)+U_2 \) is a finite group by formula (2),
whence $\frac{\phi(U_1) + U_2}{U_2}$ is a finite group. It follows that the group $\phi(U_1) + U_2$ belongs to $I$. Thus the map $\phi$ satisfies condition 1) of Definition 9. To show that $\phi$ satisfies condition 2), it suffices to verify condition 1) for the dual map $\phi: \hat{G}_2 \to \hat{G}_1$. Indeed, if $U \supset V$ are subgroups of a group $G \in \text{Ob}(\text{Loc})$ possessing properties $(*)$–(***) for the group $G$, then $V^\perp \supset U^\perp$ are subgroups of $\hat{G}$ possessing properties $(*)$–(***) for the group $\hat{G}$. Here, for every subgroup $H \subset G$ we define the subgroup $H^\perp \subset \hat{G}$ by

$$H^\perp \overset{\text{def}}{=} \{g \in \hat{G} : g|_H \equiv 1\}.$$  

But condition 1) has already been checked for every map. Condition 3) holds for $\phi$ because every continuous homomorphism between Lie groups is smooth.

We now easily see that the functor $\Phi \circ \Lambda$ is isomorphic to the identity functor $\text{Id}_{\text{Loc}^\text{ar}}$, and the functor $\Lambda \circ \Phi$ is isomorphic to the identity functor $\text{Id}_{C_1^\text{ar,compl}}$. This proves part 1).

Part 2) is a corollary of the following observation. If $U \supset V$ are subgroups of a group $G \in \text{Ob}(\text{Loc})$ satisfying properties $(*)$–(***) for the group $G$, then $V^\perp \supset U^\perp$ are subgroups of $\hat{G}$ possessing properties $(*)$–(***) for the group $\hat{G}$. By condition (9) there is no loss of generality in assuming that $V$ is a profinite group. Moreover,

$$\hat{V} = \hat{G}/V^\perp, \quad \hat{U}/V = V^\perp/U^\perp, \quad \hat{G}/U = U^\perp.$$  

Part 3) is a corollary of the following observation. Let

$$0 \to H \to G \xrightarrow{\psi} G/H \to 0$$

be an exact sequence of locally compact Abelian groups, where $H$ is a closed subgroup of $G$. If $U \supset V$ are subgroups of $G$ possessing properties $(*)$–(***) for the group $G$, then $(U \cap H) \supset (V \cap H)$ are subgroups of $H$ possessing properties $(*)$–(***) for the group $H$, and $\psi(U) \supset \psi(V)$ are subgroups of $G/H$ possessing properties $(*)$–(***) for the group $G/H$.

The proposition is proved.

**Definition 12.** Suppose that $E = (I, F, V) \in \text{Ob}(C_1^{\text{ar}})$.

1) We say that $E$ is a compact object if and only if there is an element $i_0 \in I$ such that $F(i_0) = V$ and $F(i_0)/F(i)$ is a compact Lie group for every $i \leq i_0 \in I$.

2) We say that $E$ is a discrete object if and only if there is an element $i_0 \in I$ such that $F(i_0) = \{0\}$ and $F(i)/F(i_0)$ is a discrete Lie group for every $i \geq i_0 \in I$.

**Remark 3.** The following assertions hold by formula (2).

1) $F(i_0)/F(i)$ is a compact Lie group if and only if it is isomorphic to $\mathbb{T}^k \times A$ for some non-negative integer $k$ and some finite Abelian group $A$.

2) $F(i)/F(i_0)$ is a discrete Lie group if and only if it is isomorphic to $\mathbb{Z}^l \times B$ for some non-negative integer $l$ and some finite Abelian group $B$.

**Proposition 4.** Suppose that $E = (I, F, V) \in \text{Ob}(C_1^{\text{ar}})$. Then the following assertions hold.

1) $E$ is a compact object if and only if $\Phi(E)$ is a compact group.

2) $E$ is a discrete object if and only if $\Phi(E)$ is a discrete group.
The group Φ(E) is compact if and only if it is a projective limit of compact groups. The group Φ(E) is discrete if and only if it is an inductive limit of discrete groups. The proposition is proved.

**Example 4.** Given $E_1 = (I_1, F_1, V_1) \in \text{Ob}(C_{1}^{ar})$ and $E_2 = (I_2, F_2, V_2) \in \text{Ob}(C_{1}^{ar})$ we have the following canonical construction:

$$E_1 \times E_2 \overset{\text{def}}{=} (I_1 \times I_2, F_1 \times F_2, V_1 \times V_2) \in \text{Ob}(C_{1}^{ar}),$$

(10)

where $(i_1, i_2) \leq (j_1, j_2) \in I_1 \times I_2$ if and only if $i_1 \leq j_1$ and $i_2 \leq j_2$ and, for every $(k_1, k_2) \in I_1 \times I_2$, we define

$$(F_1 \times F_2)((k_1, k_2)) = F_1(k_1) \times F_2(k_2).$$

We now consider a number field $K$ such that $[K : \mathbb{Q}] = n$. Let $E \subset K$ be the ring of integers in $K$. Then we have an isomorphism of Abelian groups $E \simeq \mathbb{Z}^n$.

Let $p_1, \ldots, p_l$ be the equivalence classes of all Archimedean places of $K$. For every $i$, $1 \leq i \leq l$, let $K_{p_i}$ be the field obtained by completing the field $K$ with respect to the absolute value $p_i$. Then either $K_{p_i} \simeq \mathbb{C}$ or $K_{p_i} \simeq \mathbb{R}$. We have the following isomorphism of Lie groups:

$$\prod_{1 \leq i \leq l} K_{p_i} \simeq \mathbb{R}^n.$$

Following on from Example 3, we consider

$$\left(\{(0), (1)\}, F, \prod_{1 \leq i \leq l} K_{p_i}\right) \in \text{Ob}(C_{1}^{ar}),$$

(11)

where $(0) < (1)$, $F((0)) = e$, $F((1)) = \prod_{1 \leq i \leq l} K_{p_i}$, and $e$ is the trivial subgroup of $\prod_{1 \leq i \leq l} K_{p_i}$.

Using the diagonal embedding of $K$, we get an embedding $E \subset \prod_{1 \leq i \leq l} K_{p_i}$. Hence the triple

$$0 \longrightarrow E \longrightarrow \prod_{1 \leq i \leq l} K_{p_i} \longrightarrow \mathbb{T}^n \longrightarrow 0$$

(12)

is admissible in $C_{0}^{ar}$.

Let $\mathbb{A}_K^{\text{fin}}$ be the ‘finite’ adèle ring of the field $K$, that is, the restricted product of the completions of $K$ over the equivalence classes of all non-Archimedean absolute values with respect to valuation rings. (In the notation of Example 1, $\mathbb{A}_K^{\text{fin}} = \mathbb{A}_{\text{Spec } E}$.) Then we consider

$$(D, H, \mathbb{A}_K^{\text{fin}}) \in \text{Ob}(C_{1}^{ar}),$$

(13)

where $D$ is the set of divisors of Spec $E$. For $d \in D$ we put

$$H(d) = d \prod_{p} \mathcal{O}_{K_p} \subset \mathbb{A}_K^{\text{fin}},$$

where the product is over all equivalence classes of non-Archimedean absolute values and $\mathcal{O}_{K_p}$ is the corresponding valuation ring. (Clearly, $(D, H, \mathbb{A}_K^{\text{fin}}) \in \text{Ob}(C_{1}^{ar})$.)

Let $\mathbb{A}_K = \mathbb{A}_K^{\text{fin}} \times \prod_{1 \leq i \leq l} K_{p_i}$ be the ‘full’ adèle ring of $K$. By formulae (10)–(13) we have

$$Q = (D_0 \cup D_1, H \times F, \mathbb{A}_K) \in \text{Ob}(C_{1}^{ar}),$$

where $D_0 = D \times (0)$ and $D_1 = D \times (1)$. Clearly, $Q$ is a complete object in $C_{1}^{ar}$.
The diagonal embedding of $K$ induces the following exact triple of Abelian groups:

$$0 \rightarrow K \rightarrow \mathbb{A}_K \xrightarrow{\phi} \mathbb{A}_K/K \rightarrow 0.$$  

(14)

For every $i \in D_0$ we have $(H \times F)(i) \cap K = 0$. By the strong approximation theorem (see [12], Ch. II, §15) we have $\phi((H \times F)(j)) = \mathbb{A}/K$ for every $j \in D_1$.

Let $P = (X,Y,K) \in \text{Ob}(C^{ar}_1)$, where $X$ is the set of subgroups of $K$ of the form $(H \times F)(i) \cap K$ for all $i \in D_0 \cup D_1$. The set $X$ is partially ordered by inclusion of subgroups and the function $Y$ sends every element of $X$ to the corresponding subgroup. Then the trivial subgroup $e \subset K$ belongs to $X$. By construction, $Y(j)/e \simeq Y(i) \simeq \mathbb{Z}^n$ for every $j \in X$. If $j \geq i > e \in X$, then $Y(j)/Y(i)$ is a finite Abelian group. Therefore $P$ is a discrete object in $C^{ar}_1$. Clearly, $P$ is a complete object in $C^{ar}_1$.

Let $R = (Z,W,\mathbb{A}_K/K) \in \text{Ob}(C^{ar}_1)$, where $Z$ is the set of subgroups in $\mathbb{A}_K/K$ of the form $\phi((H \times F)(i))$ for all $i \in D_0 \cup D_1$. The set $Z$ is partially ordered by inclusion of subgroups and the function $W$ sends every element of $Z$ to the corresponding subgroup. Then the whole group $g = \mathbb{A}_K/K$ belongs to $Z$. By construction, $g/W(j) \simeq \mathbb{T}^n$ for every $j \in Z$. If $g > j \geq i \in Z$, then $Y(j)/Y(i)$ is a finite Abelian group. Therefore $R$ is a compact object in $C^{ar}_1$. Clearly, $R$ is a complete object in $C^{ar}_1$.

By construction, the triple

$$0 \rightarrow P \rightarrow Q \rightarrow R \rightarrow 0$$

(15)

is admissible in $C^{ar}_1$. It induces the exact triple (14) of Abelian groups.

It is also easy to see from the construction that $\bar{Q} \simeq Q$, $\bar{P} \simeq R$, $\bar{R} \simeq P$.

Remark 4. Formula (11) describes the coarsest filtration of $\prod_{1 \leq i \leq l} K_{p_i} \simeq \mathbb{R}^n$ when $l > 1$. If we introduce any other filtration of $\prod_{1 \leq i \leq l} K_{p_i}$ by $\mathbb{R}$-vector spaces $\prod_{i \in I} K_{p_i}$ (where $I$ is a subset of the finite set $\{1, \ldots, l\}$ and we take the zero subspace for $I = \emptyset$) which dominates the previous filtration and involves $\mathbb{R}$-vector subspaces other than the zero subspace and $\mathbb{R}^n$, then the triple (15) is not admissible in $C^{ar}_1$. Indeed, we have $E \cap \left(\prod_{i \in I} K_{p_i}\right) = 0$ for all $I \neq \{1, \ldots, l\}$, where the intersection is taken inside $\prod_{1 \leq i \leq l} K_{p_i}$. Hence the image of the subspace $\prod_{i \in I} K_{p_i}$ with $I \neq \emptyset$, $I \neq \{1, \ldots, l\}$, in $\mathbb{T}^n$ is non-closed (see the sequence (12)). Thus we get a dense winding of a torus.

We shall use the following technical lemma.

Lemma 2. Let

$$0 \rightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \rightarrow 0$$

be an admissible triple in $C^{ar}_1$.

1) If $D \in \text{Ob}(C^{ar}_1)$ and $\gamma \in \text{Mor}_{C^{ar}_1}(D,E_3)$, then there is an admissible triple

$$0 \rightarrow E_1 \xrightarrow{\gamma \alpha} E_2 \times_{E_3} D \xrightarrow{\gamma \beta} D \rightarrow 0$$

(16)
in $C^1_1$ and a morphism $\beta_\gamma \in \text{Mor}_{C^1_1} (E_2 \times E_3 D, E_2)$ such that the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & E_1 & \overset{\gamma_\alpha}{\longrightarrow} & E_2 \times D & \overset{\gamma_\beta}{\longrightarrow} & D & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_1 & \overset{\alpha}{\longrightarrow} & E_2 & \overset{\beta}{\longrightarrow} & E_3 & \longrightarrow & 0
\end{array}
\]  

(17)

is commutative.

2) If $A \in \text{Ob}(C^1_1)$ and $\theta \in \text{Mor}_{C^1_1} (E_1, A)$, then there is an admissible triple in $C^1_1$

\[
0 \longrightarrow A \overset{\theta_\alpha}{\longrightarrow} A \prod_{E_1} E_2 \overset{\theta_\beta}{\longrightarrow} E_3 \longrightarrow 0
\]

(18)

and a morphism $\alpha_\theta \in \text{Mor}_{C^1_1} (E_2, A \prod_{E_1} E_2)$ such that the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & E_1 & \overset{\alpha}{\longrightarrow} & E_2 & \overset{\beta}{\longrightarrow} & E_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \overset{\theta_\alpha}{\longrightarrow} & A \prod_{E_1} E_2 & \overset{\theta_\beta}{\longrightarrow} & E_3 & \longrightarrow & 0
\end{array}
\]  

(19)

is commutative.

Proof. To prove part 1), we take $E_i = (I_i, F_i, V_i), 1 \leq i \leq 3$, and $D = (K, H, T)$.

We construct $E_2 \times E_3 D = (J, G, W) \in \text{Ob}(C^1_1)$ in the following way. Define an Abelian group

\[
W = V_2 \times T \overset{\text{def}}{=} \{(e, d) \in V_2 \times T: \beta(e) = \gamma(d)\}
\]

and a partially ordered set

\[
J = \{(i, j) \in I_2 \times K: \gamma(H(j)) \subset \beta(F_2(i))\}
\]

where $(i_1, j_1) \preceq (i_2, j_2)$ if and only if $i_1 \leq i_2$ and $j_1 \leq j_2$. Define a function $G$ from $J$ to the set of subgroups of $W$ by putting

\[
G((i, j)) = (F_2(i) \times H(j)) \cap W,
\]

where the intersection is taken inside $V_2 \times T$.

If $(i_1, j_1), (i_2, j_2) \in J$ and $(i_1, j_1) \preceq (i_2, j_2)$, then we have the following commutative diagram of morphisms between Abelian Lie groups:

\[
\begin{array}{cccccc}
0 & \longrightarrow & F_2(i_2) \cap V_1 & \overset{G((i_2, j_2))}{\longrightarrow} & G((i_1, j_1)) & \overset{H(j_2)}{\longrightarrow} & H(j_1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_2(i_2) \cap V_1 & \overset{F_2(i_2)}{\longrightarrow} & F_2(i_1) & \overset{F_2(i_2)/(F_2(i_2) \cap V_1)}{\longrightarrow} & 0
\end{array}
\]

where

\[
\frac{G((i_2, j_2))}{G((i_1, j_1))} = \frac{F_2(i_2)}{F_2(i_1)} \times \frac{F_2(i_2)/(F_2(i_2) \cap V_1)}{F_2(i_1)/(F_2(i_1) \cap V_1)}
\]
and the horizontal triples are admissible in $C_{0}^{ar}$. Hence $E_{2} \times_{E_{1}} D$ is well defined as an object of $C_{1}^{ar}$ and the maps $\beta_{\gamma}$ and $\gamma_{\beta}$ are projections. Moreover, this commutative diagram yields that the triple (16) is admissible in $C_{1}^{ar}$ and the diagram (17) is commutative. Part 1) of the lemma is proved.

We now prove part 2), which is dual to part 1). Write $A = (K', H', T')$.

We construct $A \coprod_{E_{1}} E_{2} = (J', G', W') \in \text{Ob}(C_{1})$ in the following way. Note that $T' \coprod V_{2} = T' \times V_{2}$. Define an Abelian group

$$W' = T' \coprod_{V_{i}} V_{2} \overset{\text{def}}{=} (T' \times V_{2})/E, \quad E = (\theta \times \alpha)V_{1},$$

and a partially ordered set

$$J' = \{(i, j) \in K' \times I_{2} : \theta(F_{2}(j) \cap V_{1}) \subset H'(i)\},$$

where $(i_{1}, j_{1}) \leq (i_{2}, j_{2})$ if and only if $i_{1} \leq i_{2}$ and $j_{1} \leq j_{2}$. Define a function $G'$ from $J'$ to the set of subgroups of $W'$ by putting

$$G'((i, j)) = ((H'(i) \times F_{2}(j)) + E)/E \subset W'.$$

If $(i_{1}, j_{1}), (i_{2}, j_{2}) \in J'$ and $(i_{1}, j_{1}) \leq (i_{2}, j_{2})$, then we have the following commutative diagram of morphisms between Abelian Lie groups:

$$0 \longrightarrow \frac{F_{2}(j_{2}) \cap V_{1}}{F_{2}(j_{1}) \cap V_{1}} \longrightarrow \frac{F_{2}(j_{2})}{F_{2}(j_{1})} \longrightarrow \frac{F_{2}(j_{2})/(F_{2}(j_{2}) \cap V_{1})}{F_{2}(j_{1})/(F_{2}(j_{1}) \cap V_{1})} \longrightarrow 0$$

$$0 \longrightarrow \frac{H'(i_{2})}{H'(i_{1})} \longrightarrow \frac{G'((i_{2}, j_{2}))}{G'((i_{1}, j_{1}))} \longrightarrow \frac{F_{2}(j_{2})/(F_{2}(j_{2}) \cap V_{1})}{F_{2}(j_{1})/(F_{2}(j_{1}) \cap V_{1})} \longrightarrow 0$$

where

$$\frac{G'((i_{2}, j_{2}))}{G'((i_{1}, j_{1}))} = \frac{H'(i_{2})}{H'(i_{1})} \prod_{\frac{F_{2}(j_{2}) \cap V_{1}}{F_{2}(j_{1}) \cap V_{1}}} \frac{F_{2}(j_{2})}{F_{2}(j_{1})}$$

and the horizontal triples are admissible in $C_{0}^{ar}$. Hence $A \coprod_{E_{1}} E_{2}$ is well defined as an object of $C_{1}^{ar}$. Moreover, this commutative diagram yields that the triple (18) is admissible in $C_{1}^{ar}$ and the diagram (19) is commutative. Part 2) is proved. The lemma is proved.

§ 4. Functions and distributions on objects of $C_{0}^{ar}$ and $C_{1}^{ar}$

4.1. Functions and distributions on objects of $C_{0}^{ar}$. Here we introduce the spaces of functions and distributions on objects of $C_{0}^{ar}$.

4.1.1. Definition of basic spaces. For every $G \in \text{Ob}(C_{0}^{ar})$ we define the Schwartz space $S(G)$ of rapidly decreasing functions on $G$ in the following way (see [13], p. 138 and [14], Ch. VII).

Suppose that $G \in \text{Ob}(C_{0}^{ar})$ satisfies

$$G \simeq \mathbb{Z}^{r} \times \mathbb{R}^{q}. \quad (20)$$

We say that a function $p : G \rightarrow \mathbb{C}$ is a polynomial if and only if it is a polynomial in the coordinate functions with respect to the decomposition (20). This definition
is easily seen to be independent of the choice of the isomorphism (20). Now let $G \in \text{Ob}(C^\text{ar}_0)$ be any object. Then we have the canonical filtration (3) on $G$. The subgroup $K$ is an element of this filtration. We denote the subgroup of torsion elements of $G/K$ by $(G/K)_t$. Then $(G/K)/(G/K)_t \simeq \mathbb{Z}^r \times \mathbb{R}^q$ (see the decomposition (2)). Let $u: G \to (G/K)/(G/K)_t$ be the natural map. Then we define the set $P$ of polynomials on $G$ by putting

$$P \overset{\text{def}}{=} \{ p: p = u^*(p') \},$$

where $p'$ runs over the set of all polynomials on $(G/K)/(G/K)_t$ as defined above.

Let $D$ be the set of all invariant differential operators on an object $G \in \text{Ob}(C^\text{ar}_0)$ (note that $G$ is a commutative Lie group). For every $G \in \text{Ob}(C^\text{ar}_0)$ we define a space

$$S(G) = \left\{ \text{all } \mathbb{C}\text{-valued smooth functions } f \text{ on } G: \sup_{x \in G} |p(x)df(x)| < \infty \ \forall p \in P, \ \forall d \in D \right\}.$$

We define a topology on $S(G)$ by the following system of seminorms $\{s_{p,d}\}_{p \in P, d \in D}$:

$$s_{p,d}(f) \overset{\text{def}}{=} \sup_{x \in G} |p(x)df(x)|, \quad p \in P, \quad d \in D, \quad f \in S(G).$$

This means that a basis for this topology is the set of open balls of these seminorms:

$$\hat{B}_{s_{p,d}}(x, r) = \{ y \in G: s_{p,d}(x - y) < r \}, \quad x \in G, \quad r > 0, \quad p \in P, \quad d \in D,$$

and the finite intersections of such balls. Therefore $S(G)$ is a locally convex topological vector space.

Taking only monomials in the coordinate functions (with respect to a fixed decomposition (2)) instead of all polynomials in the definition of the set of seminorms, and taking only monomials formed by the partial derivatives with respect to the coordinates (for the fixed decomposition (2)) instead of all invariant differential operators, we get a new system of seminorms $\{s_i\}_{i \in \mathbb{N}}$, which is countable and equivalent to the previous system $\{s_{p,d}\}_{p \in P, d \in D}$. (This means that this new system generates the same topology on $S(G)$ as the system $\{s_{p,d}\}_{p \in P, d \in D}$.) Hence $S(G)$ is a metrizable vector space whose translation-invariant metric $d(\cdot, \cdot)$ can be defined by the formula

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} \frac{s_i(x - y)}{1 + s_i(x - y)}, \quad x, y \in S(G).$$

(By replacing the system of seminorms by an equivalent one, we can assume without loss of generality that $s_1 \leq s_2 \leq \cdots \leq s_m \leq \cdots$.)

One can also prove that $S(G)$ is a complete topological vector space. Hence $S(G)$ is a Fréchet space.

There are many topologies on the continuous dual $\mathbb{C}$-vector space $V'$ of a locally convex topological $\mathbb{C}$-vector space $V$ (see [15]). We shall always take the weak-$\star$ topology on $V'$. It is generated by the system of seminorms $\{s_v\}_{v \in V}$, $v \in V$, where $s_v(f) = |f(v)|$ for all $f \in V'$. This topology on $V'$ is locally convex. Moreover, the natural map $V \to (V')'$ is always an isomorphism of $\mathbb{C}$-vector spaces (but they are not isomorphic as topological vector spaces).
For every $G \in \text{Ob}(C_{0}^{ar})$ we define the Schwartz space $S'(G)$ of tempered distributions on $G$ by the formula $S'(G) \overset{\text{def}}{=} (S(G))'$. Since the topological vector space $S(G)$ is complete and metrizable, the topological $\mathbb{C}$-vector space $S'(G)$ is sequentially complete.

We now consider examples of Schwartz spaces for various types of objects $G \in \text{Ob}(C_{0}^{ar})$.

**Example 5.** Suppose that $G \simeq \mathbb{R}^{n}$. Then the space $S(\mathbb{R}^{n})$ consists of all smooth $\mathbb{C}$-valued functions $f$ on $\mathbb{R}^{n}$ such that

$$s_{\alpha,\beta}(f) \overset{\text{def}}{=} \sup_{x \in \mathbb{R}^{n}} |x^{\alpha} \partial^{\beta} f(x)| < \infty,$$

where the multi-indices $\alpha = (\alpha_{1}, \ldots, \alpha_{n})$, $\beta = (\beta_{1}, \ldots, \beta_{n})$ consist of non-negative integers, $x^{\alpha} = x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}}$, $\partial^{\beta} = \partial_{x_{1}}^{\beta_{1}} \cdots \partial_{x_{n}}^{\beta_{n}}$. The system of norms $s_{\alpha,\beta}$ (over all multi-indices $\alpha$ and $\beta$) on the space $S(\mathbb{R}^{n})$ is equivalent to the system of seminorms $\{s_{p,d}\}_{p \in \mathbb{P}, d \in \mathbb{D}}$ which was defined above for every $G \in \text{Ob}(C_{0}^{ar})$. We note that the system of norms

$$s'_{\alpha,\beta}(f) \overset{\text{def}}{=} \int_{\mathbb{R}^{n}} |x^{\alpha} \partial^{\beta} f(x)| \, dx$$

(over all multi-indices $\alpha$ and $\beta$) is also equivalent to the previous system of seminorms on $S(\mathbb{R}^{n})$.

**Example 6.** Suppose that $G \simeq \mathbb{T}$. Then the space $S(\mathbb{T})$ consists of all smooth $\mathbb{C}$-valued functions on $\mathbb{T}$. The topology on $S(\mathbb{T})$ is given by the following system of seminorms $\{s_{\alpha}\}_{\alpha \in \mathbb{N}}$:

$$s_{\alpha}(f) \overset{\text{def}}{=} \sup_{t \in \mathbb{T}} |\partial^{\alpha} f(t)| < \infty,$$

where $\alpha \in \mathbb{N}$, $\partial^{\alpha} = \frac{\partial^{\alpha}}{\partial x^{\alpha}}$, $f \in S(\mathbb{T})$. The system of seminorms $\{s_{\alpha}\}_{\alpha \in \mathbb{N}}$ on $S(\mathbb{T})$ is equivalent to the system of seminorms $\{s_{p,d}\}_{p \in \mathbb{P}, d \in \mathbb{D}}$ which was defined above for every $G \in \text{Ob}(C_{0}^{ar})$. We note that the system of seminorms

$$s'_{\alpha}(f) \overset{\text{def}}{=} \int_{\mathbb{T}} |\partial^{\alpha} f(t)| \, dt,$$

where $\alpha \in \mathbb{N}$, is also equivalent to the previous system of seminorms on $S(\mathbb{T})$.

The same argument works for $G \simeq \mathbb{T}^{n}$, $n > 1$.

**Example 7.** Suppose that $G \simeq \mathbb{Z}$. Then $S(\mathbb{Z})$ is the space of two-sided sequences with a certain condition:

$$S(\mathbb{Z}) = \{a_{n} \mid n \in \mathbb{Z}, a_{n} \in \mathbb{C}: |a_{n}| = O(|n^{-k}|) \forall k \in \mathbb{N}\}.$$

The topology on $S(\mathbb{Z})$ is given by the following system of norms $\{s_{k}\}_{k \in \mathbb{N}}$:

$$s_{k}(\{a_{n}\}) = \sup_{n \in \mathbb{Z}} |n^{k}a_{n}|.$$

(This system of norms is equivalent to the system of seminorms $\{s_{p,d}\}_{p \in \mathbb{P}, d \in \mathbb{D}}$ on $S(G)$ which was defined above.) Clearly, $S(\mathbb{Z}) \subset l^{2}(\mathbb{Z})$.

The space $S'(\mathbb{Z})$ is defined to be the following space of sequences:

$$S'(\mathbb{Z}) = \{b_{n} \mid n \in \mathbb{Z}, b_{n} \in \mathbb{C}: \exists k \in \mathbb{N} |b_{n}| = O(|n^{k}|)\}.$$
Clearly, \( S'(\mathbb{Z}) \supset l^2(\mathbb{Z}) \). If \( a = \{a_n\} \in S(\mathbb{Z}) \) and \( b = \{b_n\} \in S'(\mathbb{Z}) \), then the series \( b(a) \overset{\text{def}}{=} \sum_{n \in \mathbb{Z}} b_n a_n \) is absolutely convergent. Moreover, by definition, the linear functional \( b \) is continuous with respect to some norm \( s_k \) on \( S(\mathbb{Z}) \). Then we have \( |b_n| = O(|n^k|) \). Conversely, if \( |b_n| = O(|n^k|) \) for some \( k \in \mathbb{N} \), then \( b = \{b_n\} \) is a continuous linear functional with respect to the norm \( s_{k+2} \) on \( S(\mathbb{Z}) \).

The same argument works for \( G \simeq \mathbb{Z}^n, n > 1 \).

4.1.2. The Fourier transform. Suppose that \( G \in \text{Ob}(C^\text{ar}_0) \). For all \( a \in G \) and \( f \in S(G) \) we define a function \( T_a(f) \in \text{S}(G) \) in the following way:

\[
T_a(f)(b) = f(b + a), \quad b \in G.
\]

For all \( a \in G \) and \( H \in \text{S}'(G) \) we define \( T_a(H) \in \text{S}'(G) \) as follows:

\[
T_a(H)(f) = H(T_{-a}(f)), \quad f \in G.
\]

We define the following \( \mathbb{C} \)-vector space:

\[
\text{S}'(G)^G \overset{\text{def}}{=} \{ H \in \text{S}'(G) : T_a(H) = H \ \forall a \in G \}.
\]

Every \( G \in \text{Ob}(C^\text{ar}_0) \) is a locally compact Abelian group. Hence \( G \) carries a Haar measure \( \nu \), which is uniquely determined up to multiplication by a positive real number. We define an element \( 1_\nu \in \text{S}'(G) \) by putting

\[
1_\nu(f) = \int_G f(x) \, d\nu(x), \quad f \in \text{S}(G).
\]

There is a well-defined map \( 1_\nu : \text{S}(G) \rightarrow \text{S}'(G) \) given by

\[
1_\nu(f)(g) = 1_\nu(fg), \quad f, g \in \text{S}(G).
\]

**Proposition 5.** Suppose that \( G \in \text{Ob}(C^\text{ar}_0) \) and \( \nu \) is a Haar measure on \( G \). Then

1) \( \dim_\mathbb{C} \text{S}'(G)^G = 1 \),

2) \( \text{S}'(G)^G = \mathbb{C} \cdot 1_\nu \) in the interior of \( \text{S}'(G) \).

**Proof.** We can assume that \( G \) is a connected Lie group. We have \( 1_\nu \in \text{S}'(G)^G \) by the definition of a Haar measure.

We now take \( v \in T_{G,e} = \text{Lie} G, v \neq 0 \). The vector \( v \) determines an invariant vector field \( X_v \) on \( G \), that is, a first-order differential operator \( X_v : \text{S}(G) \rightarrow \text{S}(G) \). Since \( X_v \) is continuous, it can be extended to an operator \( X_v : \text{S}'(G) \rightarrow \text{S}'(G) \) by the rule

\[
X_v(H)(f) = -H(X_v(f)).
\]

On the other hand, \( v \) determines a one-parameter semigroup

\[
g_v(t) = \exp(tv) \subset G, \quad t \in \mathbb{R}.
\]

By the mean value theorem and Taylor’s formula, for every \( f \in \text{S}(G) \) we have

\[
\lim_{t \rightarrow 0} T_{g_v(t)}(f) = f, \quad \lim_{t \rightarrow 0} \frac{T_{g_v(t)}(f) - f}{t} = X_v(f).
\]
in the space $S(G)$ for every seminorm $s_{p,d}$, $p \in P$, $d \in D$, on this space. Hence $X(H) = 0$ for any invariant vector field $X$ on $G$ and every $H \in S'(G)^G$. Using induction on $\dim G$ and the standard arguments of the theory of generalized functions, we obtain that $H = c \cdot 1_\nu$ for some $c \in \mathbb{C}$ (see [16], Ch. 1, §2.6 in the case $G = \mathbb{R}$). The proposition is proved.

For every $G \in \text{Ob}(C^0_{\text{ar}})$ we define a one-dimensional $C$-vector space $\mu(G)$ by putting

$$\mu(G) \overset{\text{def}}{=} S'(G)^G.$$  

The properties of the Haar measure and Proposition 5 imply that if

$$0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 0$$

is an admissible triple in $C^0_{\text{ar}}$, then there is a natural isomorphism

$$\mu(G_1) \otimes_C \mu(G_3) \simeq \mu(G_2). \quad (21)$$

Suppose that $G \in \text{Ob}(C^0_{\text{ar}})$ and let $\hat{G} \in \text{Ob}(C^0_{\text{ar}})$ be the Pontryagin dual group. Take $\mu \in \mu(G)$. Then the Fourier transform $F_\mu : S(G) \rightarrow S(\hat{G})$ is defined by the formula

$$F_\mu(\chi) \overset{\text{def}}{=} \mu(f\chi), \quad \chi \in \hat{G}. \quad (22)$$

The map $F_\mu$ depends linearly on $\mu \in \mu(G)$. Hence there is a well-defined map $F : S(G) \otimes_C \mu(G) \rightarrow S(\hat{G})$ given by

$$F(f \otimes \mu) \overset{\text{def}}{=} F_\mu(f).$$

The Fourier transform on distributions is defined as the map conjugate to the continuous map $F$, that is,

$$F : S'(G) \otimes_C \mu(\hat{G}) \longrightarrow S'(\hat{G})$$

is such that $F_\nu(H) \overset{\text{def}}{=} F(H \otimes \nu), \quad \nu \in \mu(\hat{G}), \quad H \in S'(G).$

It follows from properties of the Fourier transform that for every $\mu \in \mu(G)$ and every $\nu \in \mu(\hat{G})$ we have

$$F_\nu(\mu) = c\delta,$$

where $c \in \mathbb{C}$ is a constant (which depends linearly on $\mu$ and $\nu$) and $\delta \in S'(\hat{G})$ is the Dirac delta function: $\delta(f) \overset{\text{def}}{=} f(0)$ for $f \in S(\hat{G})$. Hence we get a well-defined isomorphism

$$\mu(G) \otimes_C \mu(\hat{G}) \simeq \mathbb{C}. \quad (23)$$

Suppose that $\mu \in \mu(G)$, $\mu \neq 0$. Then we have a well-defined element $\mu^{-1} \in \mu(\hat{G})$ such that $\mu \otimes \mu^{-1} = 1$ with respect to the isomorphism $(23)$.

Given any $f \in S(G)$, we define $\tilde{f} \in S(G)$ by putting

$$\tilde{f}(x) \overset{\text{def}}{=} f(-x), \quad x \in G. \quad (24)$$

Given any $H \in S'(G)$, we define $\tilde{H} \in S'(G)$ by putting

$$\tilde{H}(f) \overset{\text{def}}{=} H(\tilde{f}), \quad f \in S(G). \quad (25)$$

The following proposition contains other well-known properties of the Fourier transform.
Proposition 6. Suppose that $G \in \text{Ob}(C^\text{ar}_0)$, $\mu \in \mu(G)$, $\mu \neq 0$. Then

1) the map $F_\mu$ is an isomorphism between the topological $\mathbb{C}$-vector spaces $S(G)$ and $S(\hat{G})$,

2) the map $F_{\mu^{-1}}$ is an isomorphism between the topological $\mathbb{C}$-vector spaces $S'(G)$ and $S'(\hat{G})$,

3) $F_{\mu^{-1}}F_\mu(f) = \tilde{f}$ for all $f \in S(G)$,

4) $F_\mu F_{\mu^{-1}}(H) = \hat{H}$ for all $H \in S'(G)$.

4.1.3. Direct and inverse images. Consider an admissible triple

$$0 \to G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \to 0 \tag{26}$$

in $C^\text{ar}_0$. We define a map $\beta_* : S(G_2) \otimes_\mathbb{C} \mu(G_1) \to S(G_3)$ by the formula

$$\beta_*(f \otimes \mu)(x) \overset{\text{def}}{=} \mu(f_y),$$

where $f \in S(G_2)$, $\mu \in \mu(G_1)$, $x \in G_3$, $y \in f^{-1}(x)$ and the function $f_y \in S(G_1)$ is given by $f_y(z) \overset{\text{def}}{=} f(z + y)$ for $z \in G_1$. The function $\beta_*(f \otimes \mu)$ is independent of the choice of $y \in f^{-1}(x)$ because $\mu \in S'(G_1)$ is an invariant element.

We easily see that the map $\beta_*$ is continuous. Hence one can define a continuous map $\beta^* : S'(G_3) \otimes_\mathbb{C} \mu(G_1) \to S'(G_2)$ as the map conjugate to $\beta_*$. There is a map $\alpha^* : S(G_2) \to S(G_1)$ defined by

$$\alpha^*(f)(x) \overset{\text{def}}{=} f(\alpha(x)), \quad f \in S(G_2), \quad x \in G_1.$$ 

Since $\alpha^*$ is continuous, one can define a continuous map $\alpha_* : S'(G_1) \to S'(G_2)$ as the map conjugate to $\alpha^*$.

If in the sequence (26) $G_1$ is a compact Lie group and $G_3$ is any object, then there is a map $\beta^* : S'(G_3) \to S'(G_2)$ defined by

$$\beta^*(f)(x) \overset{\text{def}}{=} f(\beta(x)), \quad f \in S(G_3), \quad x \in G_2.$$ 

Since $\beta^*$ is continuous, one can define a continuous map $\beta_* : S'(G_2) \to S'(G_3)$ as the map conjugate to $\beta^*$.

If in the sequence (26) $G_3$ is a discrete Lie group and $G_1$ is any object, then there is a map $\alpha^* : S'(G_2) \to S'(G_1)$ defined by

$$\alpha^*(f)(x) \overset{\text{def}}{=} \begin{cases} 0 & \text{if } x \notin f(G_1), \\ f(y) & \text{if } x = f(y), \end{cases} \quad f \in S(G_1), \quad x \in G_2.$$ 

Since $\alpha^*$ is continuous, one can define a continuous map $\alpha_* : S'(G_2) \to S'(G_1)$ as the map conjugate to $\alpha^*$.

Let

$$0 \to E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \to 0$$

be an admissible triple in $C^\text{ar}_0$ and let

$$0 \to D \xrightarrow{\gamma} E_3 \xrightarrow{\delta} B \to 0$$
be another. There is a commutative diagram

$$
\begin{array}{ccc}
E_2 \times D & \xrightarrow{\gamma \beta} & D \\
\downarrow{\beta} & & \downarrow{\gamma} \\
E_2 & \xrightarrow{\beta} & E_3
\end{array}
$$

where $\gamma \beta$ is an admissible epimorphism (that is, the epimorphic part of some admissible triple in $C^{ar}_0$) and $\beta \gamma$ is an admissible monomorphism (that is, the monomorphic part of some admissible triple in $C^{ar}_0$).

Note that $E_3 \simeq E_2 \coprod_G D$, where $G = E_2 \times E_3$.

**Proposition 7.** 1) For all $f \in S(E_2)$, $H \in S'(D)$, $\mu \in \mu(E_1)$ we have

$$
\gamma^* \beta_*(f \otimes \mu) = (\gamma_\beta)_*(\beta^*_\gamma(f) \otimes \mu),
$$

(27)

$$
\beta^*(\gamma_\gamma(H) \otimes \mu) = (\beta_\gamma)_*\gamma^*_\beta(H \otimes \mu).
$$

(28)

2) If $E_1$ is a compact Lie group, then for all $f \in S(E_3)$, $H \in S'(G)$ we have

$$
\gamma_\beta^* \gamma^*(f) = \beta_\beta^* \beta^*(f),
$$

(29)

$$
\beta_\gamma^*(\gamma_\gamma(H) \otimes \mu) = \gamma_\beta^* \gamma_\beta^*(H \otimes \mu).
$$

(30)

3) If $B$ is a discrete Lie group, then for all $f \in S(G)$, $H \in S'(E_3)$, $\mu \in \mu(E_1)$ we have

$$
\beta_\gamma^*(\beta_\gamma(f) \otimes \mu) = \gamma_\gamma(\gamma_\beta_\gamma(f) \otimes \mu),
$$

(31)

$$
\gamma_\beta^*(\gamma_\gamma(H) \otimes \mu) = \beta_\beta^* \beta_\beta^*(H \otimes \mu).
$$

(32)

4) If $E_1$ is a compact Lie group and $B$ is a discrete Lie group, then for all $f \in S(D)$ and any $H \in S'(E_2)$ we have

$$
\beta_\gamma^*(f) = (\beta_\gamma)_*\gamma_\beta^*(f),
$$

(33)

$$
\gamma_\beta^*(H) = (\gamma_\beta)_*\beta_\beta^*(H).
$$

(34)

**Proof.** For functions this follows from the corresponding definitions, and for distributions it follows from formulae conjugate to those for functions.

Let

$$
0 \rightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \rightarrow 0
$$

be an admissible triple in $C^{ar}_0$, and let

$$
0 \rightarrow L \xrightarrow{\alpha'} H \xrightarrow{\beta'} E_2 \rightarrow 0
$$

be another. Then we have the following admissible triple in $C^{ar}_0$:

$$
0 \rightarrow H \times_{E_2} E_1 \xrightarrow{\beta_\alpha'} H \xrightarrow{\beta_\beta'} E_3 \rightarrow 0.
$$

We put $E = H \times_{E_2} E_1 \in \text{Ob}(C^{ar}_0)$. 

Proposition 8. 1) For all \( f \in S(H), \ G \in S'(E_3), \ \nu \in \mu(L), \ \mu \in \mu(E_1) \) we have
\[
(\beta\beta')(f \otimes (\nu \otimes \mu)) = \beta'_*(\beta'_*(f \otimes \nu) \otimes \mu), \tag{35}
\]
(\beta\beta')^*(G \otimes (\nu \otimes \mu)) = (\beta')^*(\beta^*(G \otimes \mu) \otimes \nu). \tag{36}

2) If \( E_1 \) and \( L \) are compact Lie groups, then so is \( E \) and the following formulae hold for all \( f \in S(E_3), \ G \in S'(H) \):
\[
(\beta\beta')^*(f) = (\beta')^*\beta^*(f), \tag{37}
(\beta\beta')_*(G) = \beta_*(\beta')_*(G). \tag{38}
\]

Proof. The following triple is admissible in \( C^\ar_0 \):
\[
0 \longrightarrow L \xrightarrow{\alpha \alpha'} E \xrightarrow{\alpha \beta'} E_1 \longrightarrow 0. \tag{39}
\]
Hence we canonically have \( \nu \otimes \mu \in \mu(E) \) for all \( \nu \in \mu(L), \ \mu \in \mu(E_1) \) by formula (21). (35) now follows from Fubini’s theorem, and (36) is conjugate to (35).

We see from (39) that if \( E_1 \) and \( L \) are compact Lie groups, then \( E \) is a compact Lie group. (37) now follows from the definitions, and (38) is conjugate to (37). The proposition is proved.

Let
\[
0 \longrightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \longrightarrow 0
\]
be an admissible triple in \( C^\ar_0 \) and let
\[
0 \longrightarrow E_2 \xrightarrow{\alpha'} H' \xrightarrow{\beta'} L' \longrightarrow 0
\]
be another. Then we have the following admissible triple in \( C^\ar_0 \):
\[
0 \longrightarrow E_1 \xrightarrow{\alpha' \alpha} H' \xrightarrow{\alpha' \beta} E_3 \prod_{E_2} H' \longrightarrow 0.
\]

Proposition 9. 1) For all \( f \in S(H'), \ G \in S'(E_1) \) we have
\[
(\alpha' \alpha)^*(f) = \alpha^*(\alpha')^*(f), \tag{40}
(\alpha' \alpha)_*(G) = (\alpha')_*\alpha_*(G). \tag{41}
\]

2) If \( E_3 \) and \( L' \) are discrete Lie groups, then so is \( E_3 \prod_{E_2} H' \) and the following formulae hold for all \( f \in S(E_1), \ G \in S'(H') \):
\[
(\alpha' \alpha)_*(f) = (\alpha')_*\alpha_*(f), \tag{42}
(\alpha' \alpha)^*(G) = \alpha^*(\alpha')^*(G). \tag{43}
\]

Proof. (40) follows from the definitions of the relevant maps, and (41) is conjugate to (40).

The triple
\[
0 \longrightarrow E_3 \xrightarrow{\beta \alpha'} E_3 \prod_{E_2} H' \xrightarrow{\beta \beta'} L' \longrightarrow 0
\]
is admissible in \( C^\ar_0 \). Hence we obtain that if \( E_3 \) and \( L' \) are discrete Lie groups, then so is \( E_3 \prod_{E_2} H' \). (42) now follows at once from the definition, and (43) is conjugate to (42). The proposition is proved.
Let

\[ 0 \to E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \to 0 \]  

(44)

be an admissible triple in \( C^\text{ar}_0 \). Then the following triple of Pontryagin dual groups is admissible in \( C^\text{ar}_0 \):  

\[ 0 \to \widehat{E}_3 \xrightarrow{\beta} \widehat{E}_2 \xrightarrow{\alpha} \widehat{E}_1 \to 0. \]  

(45)

**Proposition 10.** We have commutative diagrams

\[
\begin{array}{cccc}
S(E_2) \otimes_C \mu(E_2) & \xrightarrow{\beta_3 \otimes \text{Id}_{\mu(E_3)}} & S(E_3) \otimes_C \mu(E_3) \\
F & & F \\
S(\widehat{E}_2) & \xrightarrow{\beta^*} & S(\widehat{E}_3) \\
& \downarrow & \\
S(E_2) & \xrightarrow{\alpha^*} & S(E_1)
\end{array}
\]  

(46)

\[
\begin{array}{cccc}
S(\widehat{E}_2) \otimes_C \mu(\widehat{E}_2) & \xrightarrow{\widehat{\alpha}_3 \otimes \text{Id}_{\mu(\widehat{E}_3)}} & S(\widehat{E}_3) \otimes_C \mu(\widehat{E}_3) \\
F \otimes \text{Id}_{\mu(\widehat{E}_2)} & & F \otimes \text{Id}_{\mu(\widehat{E}_1)} \\
S'(E_2) \otimes_C \mu(\widehat{E}_2) & \xrightarrow{\beta^* \otimes \text{Id}_{\mu(\widehat{E}_3)}} & S'(E_3) \otimes_C \mu(\widehat{E}_3) \\
& \downarrow & \\
S'(\widehat{E}_2) & \xrightarrow{\widehat{\beta}_3} & S'(\widehat{E}_3) \\
& \downarrow & \\
S'(E_1) & \xrightarrow{\alpha^*} & S'(E_2)
\end{array}
\]  

(47)

\[
\begin{array}{cccc}
S'(\widehat{E}_1) \otimes_C \mu(\widehat{E}_1) & \xrightarrow{\widehat{\alpha}_3 \otimes \text{Id}_{\mu(\widehat{E}_2)}} & S'(\widehat{E}_2) \otimes_C \mu(\widehat{E}_2) \\
F \otimes \text{Id}_{\mu(\widehat{E}_1)} & & F \otimes \text{Id}_{\mu(\widehat{E}_2)} \\
S'(\widehat{E}_1) & \xrightarrow{\beta^* \otimes \text{Id}_{\mu(\widehat{E}_3)}} & S'(\widehat{E}_3) \otimes_C \mu(\widehat{E}_3) \\
& \downarrow & \\
S'(\widehat{E}_2) & \xrightarrow{\widehat{\alpha}_3} & S'(\widehat{E}_3)
\end{array}
\]  

(48)

\[
\begin{array}{cccc}
S'(\widehat{E}_1) \otimes_C \mu(\widehat{E}_1) & \xrightarrow{\beta^* \otimes \text{Id}_{\mu(\widehat{E}_3)}} & S'(\widehat{E}_3) \otimes_C \mu(\widehat{E}_3) \\
& \downarrow & \\
S'(\widehat{E}_2) & \xrightarrow{\widehat{\alpha}_3} & S'(\widehat{E}_3)
\end{array}
\]  

(49)

**Proof.** We use the natural isomorphisms (21) and (23) for the admissible triples (44), (45) and the pairs of objects \( E_i, \widehat{E}_3 \), \( 1 \leq i \leq 3 \). Then we have

\[ \mu_3(\beta_3(f \otimes \mu_1)\chi) = \mu_3(\beta_3(f\beta^*(\chi) \otimes \mu_1)) = \mu_2(f\beta^*(\chi)), \]  

(50)

where \( \mu_i \in \mu(E_i) \), \( 1 \leq i \leq 3 \), \( \mu_3 = \mu_1 \otimes \mu_2 \), \( \chi \in \widehat{E}_3 \). The first equation in (50) follows from the definition of \( \beta_3 \) and the second from Fubini’s theorem. Formula (50) is equivalent to the diagram (46). The diagram (47) now follows from (46) by part 3) of Proposition 6. The diagram (48) is conjugate to (46) and (49) is conjugate to (47). The proposition is proved.

**Corollary 1** (Poisson formula). Suppose that \( \mu_1 \in \mu(E_1) \), \( \mu_1 \neq 0 \), and \( \mu_3 \in \mu(\widehat{E}_3) \). Put \( \delta_{E_1, \mu_1} \overset{\text{def}}{=} \alpha_3(\mu_1) \in S'(E_2) \) and similarly \( \delta_{\widehat{E}_3, \mu_3} \overset{\text{def}}{=} \beta_3(\mu_3) \in S'(\widehat{E}_2) \). Then

\[ F_{\mu_1}^{-1} \otimes \mu_3(\delta_{E_1, \mu_1}) = \delta_{\widehat{E}_3, \mu_3}. \]
Proof. Using the isomorphisms (21) and (23), we obtain that \( \mu_1^{-1} \otimes \mu_3 \in \mu(\widehat{E}_2) \).
From the definitions, we have \( \delta_{E_1, \mu_1} = \beta^* (\delta_{E_3} \otimes \mu_1) \), where \( \delta_E \in S'(E) \) is defined for every \( E \in \text{Ob}(C^\text{ar}_0) \) by putting \( \delta_E (f) \overset{\text{def}}{=} f(0) \) for all \( f \in S(E) \). We likewise see from the definitions that \( \delta_{\widehat{E}_3, \mu_3} = \widehat{\alpha}^* (\delta_{\widehat{E}_1} \otimes \mu_3) \). Moreover, we have \( F_{\mu_1^{-1}} (\mu) = \delta_{\widehat{E}_1} \).
\( F_{\mu_1^{-1}} (\delta_E) = \mu^{-1} \) for all \( E \in \text{Ob}(C^\text{ar}_0) \) and \( \mu \in \mu(E) \), \( \mu \neq 0 \). The Poisson formula now follows from the diagram (48).

**Proposition 11.** 1) If \( E_1 \) is a compact Lie group, then the following diagrams are commutative:

\[
\begin{array}{ccc}
S(E_3) \otimes_c \mu(E_3) & \overset{\beta^* \otimes \text{Id}_{\mu(E_3)} \otimes 1_{E_1}}{\longrightarrow} & S(E_2) \otimes_c \mu(E_2) \\
F \downarrow & & F \downarrow \\
S(\widehat{E}_3) & \longrightarrow & S(\widehat{E}_2) \\
\end{array}
\]

(51)

where \( 1_{E_1} \in \mu(E_1) \) and, for \( 1 \in S(E_1) \), we define \( 1_{E_1} (1) = 1 \in \mathbb{C} \);

\[
\begin{array}{ccc}
S'(E_2) \otimes_c \mu(\widehat{E}_2) & \overset{\beta_* \otimes \text{Id}_{\mu(\widehat{E}_3)} \otimes 1_{E_1}(\cdot)}{\longrightarrow} & S'(E_3) \otimes_c \mu(\widehat{E}_3) \\
F \downarrow & & F \downarrow \\
S'(\widehat{E}_2) & \longrightarrow & S'(\widehat{E}_3) \\
\end{array}
\]

(52)

where the map \( 1_{E_1}(\cdot) : \mu(\widehat{E}_1) \rightarrow \mathbb{C} \) is determined by the element \( 1_{E_1} \in \mu(E_1) \simeq \mu(\widehat{E}_1)^* \).

2) If \( E_3 \) is a discrete Lie group, then the following diagrams are commutative:

\[
\begin{array}{ccc}
S(E_1) \otimes_c \mu(E_1) & \overset{\alpha^* \otimes \text{Id}_{\mu(E_1)} \otimes \Sigma_{E_3}}{\longrightarrow} & S(E_2) \otimes_c \mu(E_2) \\
F \downarrow & & F \downarrow \\
S(\widehat{E}_1) & \longrightarrow & S(\widehat{E}_2) \\
\end{array}
\]

(53)

where \( \Sigma_{E_3} \in \mu(E_3) \) and, for \( f \in S(E_3) \), we define \( \Sigma_{E_3} (f) = \sum_{x \in E_3} f(x) \);

\[
\begin{array}{ccc}
S'(E_2) \otimes_c \mu(\widehat{E}_2) & \overset{\alpha^* \otimes \text{Id}_{\mu(\widehat{E}_1)} \otimes \Sigma_{E_3}(\cdot)}{\longrightarrow} & S'(E_1) \otimes_c \mu(\widehat{E}_1) \\
F \downarrow & & F \downarrow \\
S'(\widehat{E}_2) & \longrightarrow & S'(\widehat{E}_1) \\
\end{array}
\]

(54)

where the map \( \Sigma_{E_3}(\cdot) : \mu(\widehat{E}_3) \rightarrow \mathbb{C} \) is determined by the element \( \Sigma_{E_3} \in \mu(E_3) \simeq \mu(\widehat{E}_3)^* \).

**Proof.** If \( E \in \text{Ob}(C^\text{ar}_0) \) is a compact Lie group, then \( \widehat{E} \) is a discrete Lie group and \( 1_E \otimes \Sigma_{\widehat{E}} = 1 \) with respect to the isomorphism (23). We also use the isomorphism (21). The diagram (51) follows from Fubini’s theorem and the formula \( F_{1_{E_1}} (c) = c \delta \), where \( c \in \mathbb{C} \), \( \delta \in S(\widehat{E}_1) \) and \( \delta(0) = 1 \), \( \delta(x) = 0 \) if \( x \in \widehat{E}_1 \), \( x \neq 0 \). The diagram (52) is conjugate to (51). The diagram (53) follows from (51) and part 3) of Proposition 6, and (54) is conjugate to (53). The proposition is proved.
4.2. Functions and distributions on objects of $C^\text{ar}_1$. Suppose that $E = (I, F, V) \in \text{Ob}(C^\text{ar}_1)$. By definition, for all $i, j, k \in I$ with $i \leq j \leq k$ we have admissible triples in $C^\text{ar}_0$ of the form

$$0 \longrightarrow F(j)/F(i) \xrightarrow{\alpha_{ijk}} F(k)/F(i) \xrightarrow{\beta_{ijk}} F(k)/F(j) \longrightarrow 0. \quad (55)$$

Hence for all $i, j, k, l \in I$ with $i \leq j \leq k \leq l$ we obtain that

$$F(k)/F(i) \cong (F(l)/F(i)) \times_{F(l)/F(j)} (F(k)/F(j)),$$

$$F(l)/F(j) \cong (F(l)/F(i)) \prod_{F(k)/F(i)} (F(k)/F(j))$$

as objects of $C^\text{ar}_0$. Moreover, by Definition 8 there are $i_E \leq j_E \in I$ such that $F(i_E)/F(k)$ is a finite Abelian group for all $k \leq i_E$ and $F(l)/F(j_E)$ is a finite Abelian group for all $l \geq j_E$. Using the constructions and definitions in §4.1.3, we thus get the following well-defined $\mathbb{C}$-vector spaces:

$$S(E) \overset{\text{def}}{=} \lim_{m \geq j_E} \lim_{n \leq i_E} S(F(m)/F(n)) = \lim_{n \leq i_E} \lim_{m \geq j_E} S(F(m)/F(n)), \quad (56)$$

where the limits are taken with respect to the maps $\beta_{ijk}^*$ and $(\alpha_{ijk})^*$, and

$$S'(E) \overset{\text{def}}{=} \lim_{m \geq j_E} \lim_{n \leq i_E} S'(F(m)/F(n)) = \lim_{n \leq i_E} \lim_{m \geq j_E} S'(F(m)/F(n)), \quad (57)$$

where the limits are taken with respect to the maps $(\beta_{ijk})^*$ and $\alpha_{ijk}^*$.

We see from the last definition that $S(E)$ is a $\mathbb{C}$-subalgebra of the $\mathbb{C}$-algebra $F(V)$ of all $\mathbb{C}$-valued functions on $V$.

**Remark 5.** A similar construction was used by Bruhat to define the corresponding spaces of functions and distributions on locally compact Abelian groups (see [17]).

**Lemma 3.** In the notation of formula (55), for all $i \leq j \leq k \in I$ such that the maps $(\alpha_{ijk})^* : S(F(j)/F(i)) \rightarrow S(F(k)/F(i))$, $\beta_{ijk}^* : S(F(k)/F(j)) \rightarrow S(F(k)/F(i))$ are defined, these maps are injective and the corresponding maps

$$\alpha_{ijk}^* : S'(F(k)/F(i)) \rightarrow S'(F(j)/F(i)),$$

$$(\beta_{ijk})^* : S'(F(k)/F(i)) \rightarrow S'(F(k)/F(j))$$

are surjective.

**Proof.** The injectivity of the maps indicated follows from the definitions. Moreover, we have a map $\alpha_{ijk}^* : S(F(k)/F(i)) \rightarrow S(F(j)/F(i))$ such that $\alpha_{ijk}^* \cdot (\alpha_{ijk})^* = \text{Id}_{S(F(j)/F(i))}$. Taking the conjugate equation, we have $\alpha_{ijk}^* \cdot (\alpha_{ijk})^* = \text{Id}_{S'(F(j)/F(i))}$. Hence the map $\alpha_{ijk}^* : S'(F(k)/F(i)) \rightarrow S'(F(j)/F(i))$ is surjective. We likewise have a map

$$(\beta_{ijk})^* \cdot 1_{F(j)/F(i)} : S(F(k)/F(i)) \rightarrow S(F(k)/F(j)),$$

which is given by

$$(\beta_{ijk})^* \cdot 1_{F(j)/F(i)}(f) \overset{\text{def}}{=} (\beta_{ijk})^* (f \otimes 1_{F(j)/F(i)}),$$
where \( f \in S(F(k)/F(i)) \), \( 1_{F(j)/F(i)} \in \mu(F(j)/F(i)) \) and \( 1_{F(j)/F(i)}(1) = 1 \) because \( F(j)/F(i) \) is a compact Lie group. We have \( (\beta_{ijk})_* \cdot 1_{F(j)/F(i)} \cdot \beta_{ijk} = \text{Id}_{F(k)/F(j)} \).

Considering the conjugate map and taking the conjugate of the last equation, we obtain that the map \( (\beta_{ijk})_* : S'(F(k)/F(i)) \to S'(F(k)/F(j)) \) is surjective. The lemma is proved.

For all \( i \leq j \leq k \in I \) we have a natural non-degenerate \( \mathbb{C} \)-bilinear pairing
\[
\langle \cdot, \cdot \rangle_{i,j} : S(F(j)/F(i)) \times S'(F(j)/F(i)) \to \mathbb{C}.
\]

Using Lemma 3 and the conjugacy properties for direct and inverse images on the Schwartz spaces of objects of \( C_{0}^{\text{ar}} \), we see that there is a non-degenerate \( \mathbb{C} \)-bilinear pairing
\[
\langle \cdot, \cdot \rangle : S(E) \times S'(E) \to \mathbb{C}. \tag{58}
\]

**Example 8.** We again consider the situation in Example 4. There is a number field \( K \) with \([K : \mathbb{Q}] = n\) and we consider the ‘full’ adèlé ring of \( K \),
\[
\mathbb{A}_{K} = \mathbb{A}_{K}^{\text{fin}} \times \prod_{1 \leq i \leq l} K_{p_{i}},
\]
where \( \mathbb{A}_{K}^{\text{fin}} \) is the ‘finite’ adèlé ring of \( K \) and \( p_{1}, \ldots, p_{l} \) are the equivalence classes of all Archimedean places of \( K \). We have \( \prod_{1 \leq i \leq l} K_{p_{i}} \cong \mathbb{R}^{n} \). In Example 4 we constructed an object \( Q = (D_{0} \cup D_{1}, H \times F, \mathbb{A}_{K}) \) of the category \( C_{1}^{\text{ar}} \). Then, by definition,
\[
S(Q) = S(\mathbb{A}_{K}^{\text{fin}}) \otimes_{\mathbb{C}} S\left( \prod_{1 \leq i \leq l} K_{p_{i}} \right),
\]
where \( S(\prod_{1 \leq i \leq l} K_{p_{i}}) = S(\mathbb{R}^{n}) \) (see Example 5) and \( S(\mathbb{A}_{K}^{\text{fin}}) = D(\mathbb{A}_{K}^{\text{fin}}) \) is the space of all locally constant compactly supported functions on \( \mathbb{A}_{K}^{\text{fin}} \) (see [1], §4.2 for the analogous situation in the category \( C_{1}(\mathbb{F}_{q}) \)). Moreover, every \( f \in S(\mathbb{A}_{K}^{\text{fin}}) \) is a finite linear combination of functions \( \bigotimes_{p} f_{p} \), where the product is taken over all equivalence classes of non-Archimedean places of \( K \), \( f_{p} \in S(K_{p}) = D(K_{p}) \) (that is, \( f_{p} \) belongs to the space of all locally constant compactly supported functions on \( K_{p} \)) and \( f_{p} = \delta_{O_{K_{p}}} \) for almost all \( p \), with \( \delta_{O_{K_{p}}}(x) = 1 \) if \( x \in O_{K_{p}} \) and \( \delta_{O_{K_{p}}}(x) = 0 \) if \( x \notin O_{K_{p}} \).

Suppose that \( E = (I, F, V) \in \text{Ob}(C_{1}^{\text{ar}}) \). For all elements \( i \leq j \leq k \in I \) we defined (see §4.1.2) a space \( \mu(F(i)/F(j)) \in S'(F(i)/F(j)) \) such that \( \dim_{\mathbb{C}} \mu(F(i)/F(j)) = 1 \).

For all \( i \leq j \leq k \in I \) the map \( (\beta_{ijk})_* \) determines an isomorphism between the spaces \( \mu(F(k)/F(i)) \) and \( \mu(F(k)/F(j)) \) if \( j \leq i \). Indeed, if \( \mu \in \mu(F(k)/F(j)) \), then formula (21) yields an inclusion \( 1_{F(j)/F(i)} \otimes \mu \in \mu(F(k)/F(i)) \) (see the notation \( 1_{F(j)/F(i)} \) in Proposition 11) and an equality \( (\beta_{ijk})_* (1_{F(j)/F(i)} \otimes \mu) = \mu \).

For all \( i \leq j \leq k \in I \) we similarly see that the map \( \alpha_{ijk} \) determines an isomorphism between the spaces \( \mu(F(k)/F(i)) \) and \( \mu(F(j)/F(i)) \) if \( j \geq i \). Indeed, if \( \mu \in \mu(F(j)/F(i)) \), then formula (21) yields an inclusion \( \mu \otimes \Sigma_{F(k)/F(j)} \in \mu(F(k)/F(i)) \) (see the notation \( \Sigma_{F(k)/F(j)} \) in Proposition 11) and an equality \( \alpha_{ijk} (\mu \otimes \Sigma_{F(k)/F(j)}) = \mu \).

These arguments show that the following space is well defined:
\[
\mu(E) \overset{\text{def}}{=} \lim_{m \geq j \in E} \lim_{n \leq i \in E} \mu(F(m)/F(n)) = \lim_{n \leq i \in E} \lim_{m \geq j \in E} \mu(F(m)/F(n)), \tag{59}
\]
where the limits are taken with respect to the maps \((\beta_{ijk})_*\) and \(\alpha_{ijk}^*\). We have
\[
dim C \mu(E) = 1. \tag{60}
\]

Remark 6. Suppose that \(E = (I, F, V) \in \text{Ob}(C_1^{ar})\). Formulae (56), (57) enable us to define operators \(T_a, a \in V\), on the spaces \(S(E)\) and \(S'(E)\) starting from the corresponding operators on \(S(F(j)/F(i))\) and \(S'(F(j)/F(i))\), where \(i \leq j \in I\) (see the beginning of §4.1.2). Then it is easy to see that \(\mu(E) = S'(E)^V\).

Suppose that \(E = (I, F, V) \in \text{Ob}(C_1^{ar})\). Then we have a natural isomorphism \(\mu(F(j)/F(i)) \otimes C \mu(F(j)/F(i)) \simeq C\) (see (23)) for all \(i \leq j \in I\). If we also have \(j \leq i_E\), then \(1_{F(j)/F(i)} \otimes \sum_{F(j)/F(i)} \mu\) is 1 under this isomorphism. The formulae (6), (7) and (59), (60) now yield that there is a natural isomorphism
\[
\mu(E) \otimes C \mu(\tilde{E}) \longrightarrow C. \tag{61}
\]

Let
\[
0 \longrightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \longrightarrow 0
\]
be an admissible triple in \(C_1^{ar}\), where \(E_i = (I_i, F_i, V_i), 1 \leq i \leq 3\). By (21) we have the following natural isomorphism for all \(i \leq j \in I_2\):
\[
\mu\left(\frac{F_2(j) \cap V_1}{F_2(i) \cap V_1}\right) \otimes C \mu\left(\frac{\beta(F_2(j))}{\beta(F_2(i))}\right) \longrightarrow \mu\left(\frac{F_2(j)}{F_2(i)}\right). \tag{62}
\]
Moreover, if \(j \leq i_E\), then
\[
1_{\frac{F_2(j) \cap V_1}{F_2(i) \cap V_1}} \otimes 1_{\frac{\beta(F_2(j))}{\beta(F_2(i))}} = 1_{\frac{F_2(j)}{F_2(i)}},
\]
and if \(i \geq j_E\), then
\[
\sum_{\frac{F_2(j) \cap V_1}{F_2(i) \cap V_1}} \otimes \sum_{\frac{\beta(F_2(j))}{\beta(F_2(i))}} = \sum_{\frac{F_2(j)}{F_2(i)}},
\]
under the isomorphism (62).

Therefore formulae (59), (60) yield that there is a natural isomorphism
\[
\mu(E_1) \otimes C \mu(E_3) \longrightarrow \mu(E_2). \tag{63}
\]

Suppose that \(E = (I, F, V) \in \text{Ob}(C_1^{ar})\). For all \(i \leq j \in I\) we have Fourier transforms
\[
F_{i,j}: S(F(j)/F(i)) \otimes C \mu(F(j)/F(i)) \longrightarrow S(F(j)/F(i)),
\]
\[
F_{i,j}: S'(F(j)/F(i)) \otimes C \mu(F(j)/F(i)) \longrightarrow S'(F(j)/F(i)).
\]
Taking the appropriate limits and using formulae (56), (57), (59), (60) and Proposition 11, we get well-defined Fourier transforms (one-dimensional Fourier transforms) constructed from the maps \(F_{i,j}\):
\[
F: S(E) \otimes C \mu(E) \longrightarrow S(\tilde{E}), \quad F: S'(E) \otimes C \mu(\tilde{E}) \longrightarrow S'(\tilde{E}).
\]

Remark 7. It is easy to see that this definition of the Fourier transform \(F: S(E) \otimes C \mu(E) \longrightarrow S(\tilde{E})\) coincides with an analogue of (22):
\[
F(f \otimes \mu) = F_\mu(f), \quad F_\mu(f)(b) = \mu(f\tilde{b}),
\]
where \(f \in S(E), \mu \in \mu(E), b \in \tilde{V} \subset \text{Hom}(V, T)\) and \(S(\tilde{E})\) is regarded as a \(C\)-subspace of the space of all \(C\)-valued functions on \(\tilde{V}\).
Suppose that \( E = (I,F,V) \in \text{Ob}(C_{1}^{\text{ar}}) \), \( f \in \mathcal{S}(E) \), \( H \in \mathcal{S}'(E) \). Using (24), (25) and taking the appropriate limits in (56), (57), we define \( \hat{f} \in \mathcal{S}(E) \), \( \hat{H} \in \mathcal{S}'(E) \) in such a way that \( \hat{f} = f \) and \( \hat{H} = H \). The spaces \( \mathcal{S}(F(j)/F(i)) \) and \( \mathcal{S}'(F(j)/F(i)) \) are topological \( \mathbb{C} \)-vector spaces for all \( i \leq j \in I \). Therefore \( \mathcal{S}(E) \) and \( \mathcal{S}'(E) \) are topological \( \mathbb{C} \)-vector spaces with the topologies of the inductive and projective limit respectively. Using Proposition 6 and taking the corresponding limits in formulae (56), (57), we now obtain the following generalization of Proposition 6.

**Proposition 12.** Suppose that \( E \in \text{Ob}(C_{1}^{\text{ar}}) \), \( \mu \in \mu(E) \), \( \mu \neq 0 \). Then

1) \( \mathbf{F}_\mu \) is an isomorphism between the topological \( \mathbb{C} \)-vector spaces \( \mathcal{S}(E) \) and \( \mathcal{S}(\hat{E}) \),
2) \( \mathbf{F}_{\mu^{-1}} \) is an isomorphism between the topological \( \mathbb{C} \)-vector spaces \( \mathcal{S}'(E) \) and \( \mathcal{S}'(\hat{E}) \),
3) for all \( f \in \mathcal{S}(E) \) and \( G \in \mathcal{S}'(\hat{E}) \) we have \( \langle \mathbf{F}(f), G \rangle = \langle f, \mathbf{F}(G) \rangle \).

If in addition \( E \) is a complete object, then
4) \( \mathbf{F}_{\mu^{-1}} \mathbf{F}_\mu(f) = \hat{f} \) for all \( f \in \mathcal{S}(E) \),
5) \( \mathbf{F}_\mu \mathbf{F}_{\mu^{-1}}(H) = \hat{H} \) for all \( H \in \mathcal{S}'(E) \).

Consider any admissible triple in \( C_{1}^{\text{ar}} \):

\[
0 \rightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \rightarrow 0,
\]

where \( E_i = (I_i,F_i,V_i) \), \( 1 \leq i \leq 3 \). The direct and inverse images of functions and distributions on objects of \( C_{1}^{\text{ar}} \) were constructed in §4.1.3. Using these definitions, formulae (56), (57) and formulae (59), (60) and taking the appropriate inductive or projective limits, we get the following well-defined maps:

1) \( \beta_* : \mathcal{S}(E_2) \otimes_{\mathbb{C}} \mu(E_1) \rightarrow \mathcal{S}(E_3) \),
2) \( \beta^* : \mathcal{S}'(E_3) \otimes_{\mathbb{C}} \mu(E_1) \rightarrow \mathcal{S}'(E_2) \),
3) \( \alpha_* : \mathcal{S}(E_2) \rightarrow \mathcal{S}(E_1) \),
4) \( \alpha^* : \mathcal{S}'(E_2) \rightarrow \mathcal{S}'(E_1) \),
5) \( \beta_* : \mathcal{S}(E_3) \rightarrow \mathcal{S}(E_2) \) if \( E_1 \) is a compact object,
6) \( \beta_* : \mathcal{S}'(E_3) \rightarrow \mathcal{S}'(E_2) \) if \( E_1 \) is a compact object,
7) \( \alpha_* : \mathcal{S}(E_1) \rightarrow \mathcal{S}(E_2) \) if \( E_3 \) is a discrete object,
8) \( \alpha^* : \mathcal{S}'(E_2) \rightarrow \mathcal{S}'(E_1) \) if \( E_3 \) is a discrete object.

We note that, by construction, the maps 1) and 2), 3) and 4), 5) and 6), 7) and 8) are conjugate with respect to the pairing (58).

**Remark 8.** Using the above constructions of direct and inverse images of functions and distributions on objects of \( C_{1}^{\text{ar}} \) and Propositions 7–11, we immediately obtain analogous propositions and an analogue of the Poisson formula for objects of \( C_{1}^{\text{ar}} \). These propositions are stated as above: we merely replace objects of \( C_0^{\text{ar}} \) by objects of \( C_{1}^{\text{ar}} \), compact Lie groups by compact objects of \( C_{1}^{\text{ar}} \) and discrete Lie groups by discrete objects of \( C_{1}^{\text{ar}} \).

Moreover, if \( E = (I,F,V) \) is a compact object of \( C_{1}^{\text{ar}} \), then there is a natural element \( 1_E \in \mu(E) \) which is the projective limit of the elements \( 1_{F(j)/F(i)} \in \mu(F(j)/F(i)), i \leq j \in I \). If \( E = (I,F,V) \) is a discrete object of \( C_{1}^{\text{ar}} \), then there is a natural element \( \Sigma_E \in \mu(E) \) which is the projective limit of the elements \( \Sigma_{F(j)/F(i)} \in \mu(F(j)/F(i)), i \leq j \in I \).
Remark 9. Suppose that \( E_2 = (I_2, F_2, V_2) \in \text{Ob}(C_1^{\text{ar}}) \). Take another object \( E_1 = (I_1, F_1, V_1) \in \text{Ob}(C_1^{\text{ar}}) \) that dominates \( E_2 \) as a filtered Abelian group and has the following property. For all \( i \leq j \in I_2 \) we have the same structure as an object of \( C_0^{\text{ar}} \) on \( F_2(j)/F_2(i) \) and on \( F_1(\phi(j))/F_1(\phi(i)) \) (see Definition 2). Then we say that the object \( E_1 \) dominates \( E_2 \).

Definition 13. Suppose that \( (I, F, V), (J, G, V) \in \text{Ob}(C_1^{\text{ar}}) \). We say that there is an equivalence \( (I, F, V) \sim (J, G, V) \) if one can find objects \( (I_1, F_1, V) \in \text{Ob}(C_1^{\text{ar}}) \), \( 1 \leq l \leq n \), with the following properties.

(i) \( I_1 = I \), \( F_1 = F \), \( I_n = J \), \( F_n = G \).

(ii) For every \( l, 1 \leq l \leq n-1 \), either the object \( (I_l, F_l, V) \) dominates \( (I_{l+1}, F_{l+1}, V) \) or vice versa.

We see from the definitions that if \( E_1 \sim E_2 \), then \( S(E_1) = S(E_2) \), \( S'(E_1) = S'(E_2) \), and \( \mu(E_1) = \mu(E_2) \). These isomorphisms are well defined with respect to the isomorphisms (21) and (23), the constructions of the direct and inverse images and the Fourier transform. (More precisely, one can draw the corresponding commutative diagrams.)

§ 5. The categories \( C_2^{\text{ar}} \)

Definition 14. The objects of the category \( C_2^{\text{ar}} \), that is, \( \text{Ob}(C_2^{\text{ar}}) \), are filtered Abelian groups \( (I, F, V) \) (see Definition 1) with the following additional structures.

(i) For all \( i \leq j \in I \) there is a structure \( E_{i,j} \in \text{Ob}(C_1^{\text{ar}}) \) on the Abelian group \( F(j)/F(i) \).

(ii) For all \( i \leq j \leq k \in I \) the triple

\[
0 \to E_{i,j} \to E_{i,k} \to E_{j,k} \to 0
\]

is admissible in \( C_1^{\text{ar}} \).

Definition 15. Suppose that \( E_1 = (I_1, F_1, V_1) \) and \( E_2 = (I_2, F_2, V_2) \) belong to \( \text{Ob}(C_2^{\text{ar}}) \). Then the set of morphisms \( \text{Mor}_{C_2^{\text{ar}}}(E_1, E_2) \) consists of all elements \( A \in \text{Hom}_k(V_1, V_2) \) such that the following conditions hold.

(i) For every \( i \in I_1 \) there is \( j \in I_2 \) such that \( A(F_1(i)) \subset F_2(j) \).

(ii) For every \( j \in I_2 \) there is \( i \in I_1 \) such that \( A(F_1(i)) \subset F_2(j) \).

(iii) Whenever \( i_1 \leq i_2 \in I_1 \) and \( j_1 \leq j_2 \in I_2 \) with \( A(F_1(i_1)) \subset F_2(j_1) \) and \( A(F_1(i_2)) \subset F_2(j_2) \), the induced \( k \)-linear map

\[
\bar{A}: \frac{F_1(i_2)}{F_1(i_1)} \to \frac{F_2(j_2)}{F_2(j_1)}
\]

is an element of \( \text{Mor}_{C_1^{\text{ar}}}(\frac{F_1(i_2)}{F_1(i_1)}, \frac{F_2(j_2)}{F_2(j_1)}) \).

The following proposition shows that the notion of a morphism in \( C_2^{\text{ar}} \) is well defined.

Proposition 13. Suppose that \( E_1 = (I_1, F_1, V_1), E_2 = (I_2, F_2, V_2), E'_1, E'_2 \in \text{Ob}(C_2^{\text{ar}}) \) and \( A \) is a map belonging to \( \text{Hom}(V_1, V_2) \).

1) If the filtered Abelian group \( E_1 \) dominates the filtered Abelian group \( E'_1 \) and the filtered Abelian group \( E_2 \) dominates the filtered Abelian group \( E'_2 \), then \( A \in \text{Mor}_{C_2^{\text{ar}}}(E_1, E_2) \) if and only if \( A \in \text{Mor}_{C_2^{\text{ar}}}(E'_1, E'_2) \).

2) \( \text{Mor}_{C_2^{\text{ar}}}(E_1, E_2) \) is an Abelian subgroup of \( \text{Hom}(V_1, V_2) \).
3) If $E_3$ is an object of $C_{2}^{\text{ar}}$, then
\[ \text{Mor}_{C_{2}^{\text{ar}}}(E_2, E_3) \circ \text{Mor}_{C_{2}^{\text{ar}}}(E_1, E_2) \subset \text{Mor}_{C_{2}^{\text{ar}}}(E_1, E_3). \]

The proof is analogous to that of Proposition 2.1 in [2] and can be carried out by induction (see Proposition 2).

**Definition 16.** Suppose that $E_1 = (I_1, F_1, V_1)$, $E_2 = (I_2, F_2, V_2)$ and $E_3 = (I_3, F_3, V_3)$ belong to $\text{Ob}(C_{2}^{\text{ar}})$. We say that
\[ 0 \longrightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \longrightarrow 0 \] (64)
is an *admissible triple* in $C_{2}^{\text{ar}}$ if the following conditions hold.

(i) $0 \longrightarrow V_1 \xrightarrow{\alpha} V_2 \xrightarrow{\beta} V_3 \longrightarrow 0$ is an exact triple of Abelian groups.

(ii) The filtration $(I_1, F_1, V_1)$ dominates the filtration $(I_2, F'_1, V_1)$, where $F'_1(i) = F_2(i) \cap V_1$ for all $i \in I_2$.

(iii) The filtration $(I_3, F_3, V_3)$ dominates the filtration $(I_2, F'_3, V_3)$, where $F'_3(i) = F_2(i)/F_2(i) \cap V_1$ for all $i \in I_2$.

(iv) For all $i \leq j \in I_2$, the triple
\[ 0 \longrightarrow F'_1(j) \xrightarrow{\beta} F'_2(j) \xrightarrow{\beta} F'_3(j) \longrightarrow 0 \] (65)
is admissible in $C_{1}^{\text{ar}}$. (By the definition of $\text{Ob}(C_{2}^{\text{ar}})$, each Abelian group in the triple (65) is endowed with the structure of an element of $\text{Ob}(C_{1}^{\text{ar}})$).

We say that the $\alpha$ in an admissible triple (64) is an *admissible monomorphism* and the $\beta$ is an *admissible epimorphism*.

**Example 9.** The category $C_{2}^{\text{fin}}$ is a full subcategory of $C_{2}^{\text{ar}}$ such that a triple in $C_{2}^{\text{fin}}$ is admissible in $C_{2}^{\text{fin}}$ if and only if it is admissible in $C_{2}^{\text{ar}}$.

The category $C_{1}^{\text{ar}}$ is a full subcategory of $C_{2}^{\text{ar}}$ with respect to the following functor $I$. For every $G \in C_{1}^{\text{ar}}$ we put $I(G) = (\{0\}, \{1\}, F, G)$, where $0 < 1$, $F((0)) = e$, $F((1)) = G$, $e$ is the trivial subgroup of $G$ and $I$ acts as the identity map on the morphisms of $C_{1}^{\text{ar}}$.

We say that an object $E_1 = (I_1, F_1, V) \in \text{Ob}(C_{2}^{\text{ar}})$ *dominates* another object $E_2 = (I_2, F_2, V) \in \text{Ob}(C_{2}^{\text{ar}})$ if the following conditions hold.

(i) The filtration $(I_1, F_1, V)$ dominates the filtration $(I_2, F_2, V)$.

(ii) For all $i \leq j \in I_2$ the object $(E_1)_{i,j} = F_2(j)/F_2(i) \in \text{Ob}(C_{1}^{\text{ar}})$ with filtration induced by the $C_{2}^{\text{ar}}$-structure on $E_1$ dominates the object $(E_2)_{i,j} = F_2(j)/F_2(i) \in \text{Ob}(C_{1}^{\text{ar}})$ with filtration induced by the $C_{2}^{\text{ar}}$-structure on $E_2$ (see Remark 9).

We consider the group $\text{Aut}_{C_{2}^{\text{ar}}}(E) \overset{\text{def}}{=} \text{Mor}_{C_{2}^{\text{ar}}}(E, E)^*$ (that is, all invertible elements of the ring $\text{Mor}_{C_{2}^{\text{ar}}}(E, E)$) for every $E \in \text{Ob}(C_{2}^{\text{ar}})$. If an object $E_1 \in \text{Ob}(C_{2}^{\text{ar}})$ dominates an object $E_2 \in \text{Ob}(C_{2}^{\text{ar}})$, then the following equality holds canonically by Proposition 13:

\[ \text{Aut}_{C_{2}^{\text{ar}}}(E_1) = \text{Aut}_{C_{2}^{\text{ar}}}(E_2). \]

Suppose that $E = (I, F, V) \in \text{Ob}(C_{2}^{\text{ar}})$. We define the dual object $\bar{E} = (I^0, F^0, \bar{V}) \in \text{Ob}(C_{2}^{\text{ar}})$ in the following way. The Abelian group $\bar{V} \subset \text{Hom}(V, \mathbb{T})$ is defined as
\[ \bar{V} \overset{\text{def}}{=} \lim_{j \in I} \lim_{i \geq j} E_{j,i}, \] (66)
where the object $\tilde{E}_{j,i} \in \text{Ob}(C_1\text{ar})$ is constructed from the object $E_{j,i} = F(i)/F(j) \in \text{Ob}(C\text{ar})$ (see §3.2). The partially ordered set $I^0$ coincides with $I$ as a set but has the reverse order. For $j \in I^0$ we define a subgroup
\[
F^0(j) \overset{\text{def}}{=} \lim_{i \leq j \in I^0} \tilde{E}_{j,i} \subset \check{V}.
\] (67)

If $E_1, E_2 \in \text{Ob}(C_2\text{ar})$ and $\theta \in \text{Mor}_{C_2\text{ar}}(E_1, E_2)$, then we canonically construct an element $\hat{\theta} \in \text{Mor}_{C_2\text{ar}}(\tilde{E}_2, \tilde{E}_1)$. If
\[
0 \rightarrow E_1 \overset{\alpha}{\rightarrow} E_2 \overset{\beta}{\rightarrow} E_3 \rightarrow 0
\]
is an admissible triple in $C_2\text{ar}$, then we canonically construct the following admissible triple in $C_2\text{ar}$:
\[
0 \rightarrow \tilde{E}_3 \overset{\beta}{\rightarrow} \tilde{E}_2 \overset{\alpha}{\rightarrow} \tilde{E}_1 \rightarrow 0.
\]

**Lemma 4.** Let $0 \rightarrow E_1 \overset{\alpha}{\rightarrow} E_2 \overset{\beta}{\rightarrow} E_3 \rightarrow 0$ be an admissible triple in $C_2\text{ar}$.

1) If $D \in \text{Ob}(C_2\text{ar})$ and $\gamma \in \text{Mor}_{C_2\text{ar}}(D, E_3)$, then there is an admissible triple
\[
0 \rightarrow E_1 \overset{\gamma_\alpha}{\rightarrow} E_2 \times_{E_3} D \overset{\gamma_\beta}{\rightarrow} D \rightarrow 0
\] (68)
in $C_2\text{ar}$ and a $\beta_\gamma \in \text{Mor}_{C_2\text{ar}}(E_2 \times_{E_3} D, E_2)$ such that the diagram
\[
\begin{array}{ccc}
0 & \rightarrow & E_1 \\
\downarrow & & \downarrow \\
0 & \rightarrow & E_1
\end{array}
\]
\[
\begin{array}{ccc}
\gamma_\alpha & \rightarrow & E_2 \\
\beta_\gamma & \rightarrow & D \\
\gamma & \rightarrow & D
\end{array}
\] (69)
is commutative.

2) If $A \in \text{Ob}(C_2\text{ar})$ and $\theta \in \text{Mor}_{C_2\text{ar}}(E_1, A)$, then there is an admissible triple
\[
0 \rightarrow A \overset{\theta_\alpha}{\rightarrow} A \bigsqcup_{E_1} E_2 \overset{\theta_\beta}{\rightarrow} E_3 \rightarrow 0
\] (70)
in $C_2\text{ar}$ and an $\alpha_\theta \in \text{Mor}_{C_2\text{ar}}(E_2, A \bigsqcup_{E_1} E_2)$ such that the diagram
\[
\begin{array}{ccc}
0 & \rightarrow & E_1 \\
\theta & \downarrow & \\
0 & \rightarrow & A
\end{array}
\]
\[
\begin{array}{ccc}
\alpha_\theta & \rightarrow & E_2 \\
\alpha_\theta & \rightarrow & E_3 \\
\alpha_\theta & \rightarrow & E_3
\end{array}
\] (71)
is commutative.

**Proof.** This follows by repeating the proof of Lemma 2 if the fibred product (or amalgamated sum) of the quotients of the filtration in that proof is understood as the fibred product (or amalgamated sum) of objects of $C_1\text{ar}$ as constructed in Lemma 2.

**Definition 17.** We say that $(I_1, F_1, V_1) \in \text{Ob}(C_2\text{ar})$ is a $c$-object if and only if there is an $i \in I_1$ such that $F_1(i) = V_1$. We say that $(I_2, F_2, V_2) \in \text{Ob}(C_2\text{ar})$ is a $d$-object if and only if there is $j \in I_2$ such that $F_2(j) = \{0\}$. 
It follows from Definition 17 that if $E_1 \in \text{Ob}(C^\text{ar}_2)$ is a $c$-object, then $\tilde{E}_1 \in \text{Ob}(C^\text{ar}_2)$ is a $d$-object. If $E_2 \in \text{Ob}(C^\text{ar}_2)$ is a $d$-object, then $\tilde{E}_2 \in \text{Ob}(C^\text{ar}_2)$ is a $c$-object.

**Definition 18.** We say that $(I_1, F_1, V_1) \in \text{Ob}(C^\text{ar}_2)$ is a $cf$-object if $F(i_1)/F(j_1) \in \text{Ob}(C^\text{ar}_1)$ is a compact object for all $i_1 \geq j_1 \in I_1$. We say that $(I_2, F_2, V_2) \in \text{Ob}(C^\text{ar}_2)$ is a $df$-object if $F(i_2)/F(j_2) \in \text{Ob}(C^\text{ar}_1)$ is a discrete object for all $i_2 \geq j_2 \in I_2$.

It follows from Definition 18 that if $E_1 \in \text{Ob}(C^\text{ar}_1)$ is a $cf$-object, then $\tilde{E}_1 \in \text{Ob}(C^\text{ar}_1)$ is a $df$-object. If $E_2 \in \text{Ob}(C^\text{ar}_1)$ is a $df$-object, then $\tilde{E}_2 \in \text{Ob}(C^\text{ar}_1)$ is a $cf$-object.

**Definition 19.** We say that $(I, F, V) \in \text{Ob}(C^\text{ar}_2)$ is a **complete object** if the following conditions hold:

1. The object $E_{j,i} = F(i)/F(j) \in C^\text{ar}_1$ is complete for all $i \geq j \in I$.
2. $V = \lim_{i \in I} \lim_{j \leq i} F(i)/F(j)$.

It follows from the definition of the dual object and Definition 19 that $E \in \text{Ob}(C^\text{ar}_2)$ is a complete object if and only if $\tilde{E} = E$.

We have an obvious functor of completion $\Omega: C^\text{ar}_2 \to C^\text{ar}_2$. For $E = (I, F, V) \in \text{Ob}(C^\text{ar}_2)$ we put

$$\Omega(E) \overset{\text{def}}{=} \left(I, F', \lim_{i \in I} \lim_{j \leq i} \Psi(F(i)/F(j))\right),$$

where for every $j \in I$ we define

$$F'(j) \overset{\text{def}}{=} \lim_{j \leq i} \Psi(F(i)/F(j)),$$

$\Psi$ being the completion functor on $C^\text{ar}_1$ (see §3.2). The functor $\Omega$ is also easily defined on $\text{Mor}_{C^\text{ar}_2}(E_1, E_2)$, $E_1, E_2 \in \text{Ob}(C^\text{ar}_2)$, using known maps defined on $\text{Mor}_{C^\text{ar}_1}$.

It is clear from the definition of $\Omega$ that the object $\Omega(E)$ is complete for every $E \in \text{Ob}(C^\text{ar}_2)$ and $\Omega^2 = \Omega$. Moreover, $\Omega(E) = \tilde{E}$.

**Example 10.** Consider the field $K = \mathbb{R}((t))$ (or the field $K = \mathbb{C}((t))$). Since all finite-dimensional $\mathbb{R}$-vector (or $\mathbb{C}$-vector) spaces are objects of $C^\text{ar}_0$ (and hence objects of $C^\text{ar}_1$ by Example 3), we see that $K$ has a natural structure as an object of $C^\text{ar}_2$ whose filtration is given by fractional ideals in the discrete valuation field $K$.

**Example 11.** Suppose that $X$ is an integral scheme of finite type over $\mathbb{Z}$ with $\text{dim} \, X = 2$ and there is a surjective projective morphism $X \to \text{Spec} \, E$, where $E$ is the ring of integers in a number field $K$, $[K: \mathbb{Q}] = n$. This means that $X$ is an ‘arithmetic surface’.

By Examples 1, 9 we have an inclusion $\mathbb{A}_X \in \text{Ob}(C^\text{fin}_2) \subset \text{Ob}(C^\text{ar}_2)$ (compare this with Example 2.9 in [10]).

Let $p_1, \ldots, p_l$ be the equivalence classes of Archimedean places of $K$. Then the completion field $K_{p_i}$ is isomorphic to either $\mathbb{C}$ or $\mathbb{R}$ for all $i$, $1 \leq i \leq l$. Consider the curve

$$X_K \overset{\text{def}}{=} X \otimes_{\text{Spec} \, E} \text{Spec} \, K,$$

which is a generic fibre of the morphism from $X$ to $\text{Spec} \, E$. Define a ring

$$\mathbb{A}_{X, \infty} \overset{\text{def}}{=} \mathbb{A}_{X_K} \hat{\otimes}_{K} \left( \prod_{1 \leq i \leq l} K_{p_i} \right) = \lim_{D_1} \lim_{D_2 \leq D_1} \left( \frac{\mathbb{A}_{X_K}(D_1)}{\mathbb{A}_{X_K}(D_2)} \otimes_K \left( \prod_{1 \leq i \leq l} K_{p_i} \right) \right), \quad (72)$$
where $\mathbb{A}_{X,K}$ is the adèle ring of the curve $X_K$, $D_2 \leq D_1$ runs over all pairs of the corresponding Cartier divisors on $X_K$ and, for every Cartier divisor $D$ on $X_K$, we have a $K$-vector space

$$\mathbb{A}_{X,K}(D) \overset{\text{def}}{=} \prod_{q \in X_K} \hat{O}_q \otimes_{O_{X_K}} O(D).$$

Here the product is taken over all closed points of $X_K$, the ring $\hat{O}_q$ is the completion of the local ring of a point $q \in X_K$ with respect to the maximal ideal, and $O(D)$ is the invertible sheaf on $X_K$ corresponding to the Cartier divisor $D$ on $X_K$.

Note that

$$\mathbb{A}_{X,K} = \lim_{\rightarrow} \mathbb{A}_{X,K}(D_1) \lim_{\leftarrow} \mathbb{A}_{X,K}(D_2),$$

and $\mathbb{A}_{X,K}(D_1)$ is a finite-dimensional vector space over $K$. Using the isomorphism $\prod_{1 \leq i \leq l} K_{p_i} \simeq \mathbb{R}^n$, we see that $\mathbb{A}_{X,\infty} \in \text{Ob}(\mathcal{C}_{2}^{ar})$ because there is a filtration of $\mathbb{A}_{X,\infty}$ by Cartier divisors on $X_K$ (see formula (72)) and the quotients of this filtration are finite-dimensional vector spaces over $\mathbb{R}$ and belong to $\text{Ob}(\mathcal{C}_{1}^{ar})$ by Examples 3, 9.

We now define the arithmetic adèle ring of $X$:

$$\mathbb{A}_{X}^{ar} \overset{\text{def}}{=} \mathbb{A}_{X} \times \mathbb{A}_{X,\infty}.$$  

We have $\mathbb{A}_{X}^{ar} \in \text{Ob}(\mathcal{C}_{2}^{ar})$ because we take the product filtration on $\mathbb{A}_{X}^{ar}$ induced by the filtrations on $\mathbb{A}_{X}$ and $\mathbb{A}_{X,\infty}$.

Note that

$$\mathbb{A}_{X} \subset \prod_{x \in C} K_{x,C},$$

where the product is taken over all the pairs consisting of integer one-dimensional subschemes $C$ of $X$ and closed points $p$ on $C$. The ring $K_{x,C}$ is a finite product of two-dimensional local fields. If $x$ is a regular point of $C$ and $X$, then $K_{x,C}$ is a two-dimensional local field (see, for example, [10], Theorem 2.10, Proposition 3.4).

Note that closed points of $X_K$ are in a one-to-one correspondence with irreducible ‘horizontal’ curves on $X$, that is, with those integer one-dimensional subschemes of $X$ that are surjectively mapped onto $\text{Spec} E$. (Such an irreducible ‘horizontal’ curve is the closure of a point of a generic fibre.)

From this point of view, we regard the definition of the ring $\mathbb{A}_{X}^{ar}$ as the addition of the local fields $\mathbb{R}((t))$ or $\mathbb{C}((t))$ to the ring $\mathbb{A}_{X}$ in such a way that these fields correspond to pairs consisting of irreducible ‘horizontal’ curves on $X$ and infinite points (or equivalence classes of Archimedean places) on these curves. This is a consequence of the well-known formula

$$L \otimes_K K_p = \prod_{q} L_q,$$

where the field $L$ is a finite extension of $K$, $p$ is an equivalence class of Archimedean places of $K$, and the product is taken over all equivalent classes of Archimedean places $q$ of $L$ that lie over $p$.

**Remark** 10. Proceeding as in the definitions of $\mathcal{C}_{n}^{\text{fin}}$ (see Definitions 4, 5), we could define the categories $\mathcal{C}_{n}^{ar}$ by induction starting from $\mathcal{C}_{2}^{ar}$. In what follows we use only $\mathcal{C}_{0}^{ar}$, $\mathcal{C}_{1}^{ar}$ and $\mathcal{C}_{2}^{ar}$, and so we omit the definition of $\mathcal{C}_{n}^{ar}$ for other $n \in \mathbb{N}$.  

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§ 6. Virtual measures

Suppose that $E = (I, F, V) \in \text{Ob}(C^a_{2r})$. For all $i, j \in I$ we define a one-dimensional $\mathbb{C}$-vector space of virtual measures,

$$
\mu(F(i) | F(j)) \overset{\text{def}}{=} \lim_{l \leq i, l \leq j} \text{Hom}_\mathbb{C}(\mu(F(i)/F(l)), \mu(F(j)/F(l))). \quad (73)
$$

To take the inductive limit here, we use the following identities derived from (63): for $l' \leq l \in I$,

$$
\mu(F(i)/F(l')) = \mu(F(i)/F(l)) \otimes_\mathbb{C} \mu(F(l)/F(l')),
$$

$$
\mu(F(j)/F(l')) = \mu(F(j)/F(l)) \otimes_\mathbb{C} \mu(F(l)/F(l')).
$$

The maps in the limit above are given by

$$
f \in \text{Hom}_\mathbb{C}(\mu(F(i)/F(l)), \mu(F(j)/F(l))) \mapsto f' \in \text{Hom}_\mathbb{C}(\mu(F(i)/F(l')), \mu(F(j)/F(l'))),
$$

where $f'(a \otimes c) \overset{\text{def}}{=} f(a) \otimes c$, $a$ is any element of $\mu(F(i)/F(l))$, and $c$ is any element of $\mu(F(l)/F(l'))$. These maps are isomorphisms.

**Proposition 14.** For all $i, j, l \in I$ there is a canonical isomorphism

$$
\gamma: \mu(F(i) | F(j)) \otimes_\mathbb{C} \mu(F(j) | F(l)) \rightarrow \mu(F(i) | F(l))
$$

such that the associativity diagram

\[
\begin{array}{ccc}
\mu(F(i) | F(j)) \otimes_\mathbb{C} \mu(F(j) | F(l)) \otimes_\mathbb{C} \mu(F(l) | F(n)) & \longrightarrow & \mu(F(i) | F(l)) \otimes_\mathbb{C} \mu(F(l) | F(n)) \\
\downarrow & & \downarrow \\
\mu(F(i) | F(j)) \otimes_\mathbb{C} \mu(F(j) | F(n)) & \longrightarrow & \mu(F(i) | F(n))
\end{array}
\]

which is constructed by means of the maps $\gamma$, is commutative for all $i, j, l, n \in I$.

**Proof.** We have a canonical map

$$
\text{Hom}_\mathbb{C}(\mu(F(i)/F(l')), \mu(F(j)/F(l'))) \otimes_\mathbb{C} \text{Hom}_\mathbb{C}(\mu(F(j)/F(l')), \mu(F(l)/F(l'))) \rightarrow \text{Hom}_\mathbb{C}(\mu(F(i)/F(l')), \mu(F(l)/F(l'))),
$$

(74)

which satisfies the associativity diagram. This map commutes with the inductive limit in (73). The map $\gamma$ is now obtained by taking the inductive limit in (74). The proposition is proved.

**Remark 11.** We have the following canonical isomorphisms. For all $i, j \in I$,

$$
\mu(F(i) | F(i)) = \mathbb{C}, \quad \mu(F(i) | F(j)) = \mu(F(j) | F(i))^*.
$$

If $i \leq j \in I$, then

$$
\mu(F(i) | F(j)) = \mu(F(j)/F(i)), \quad \mu(F(j) | F(i)) = \mu(F(j)/F(i))^*.
$$
§ 7. Basic spaces

Suppose that $E = (I, F, V) \in \text{Ob}(C^\text{ar}_2)$ and $i \geq j \geq l \geq n \in I$. Then $F(i) \supset F(j) \supset F(l) \supset F(n)$ and the definition of objects of $C^\text{ar}_2$ yields the following equivalences between objects of $C^\text{ar}_1$:

$$F(j)/F(n) \sim (F(i)/F(n)) \otimes_{F(i)/F(l)} (F(j)/F(l)), $$

$$F(i)/F(l) \sim (F(i)/F(n)) \prod_{F(j)/F(n)} (F(j)/F(l)).$$

If $i \geq j \geq l \in I$, then we have the following admissible triples in $C^\text{ar}_1$:

$$0 \rightarrow F(j)/F(l) \xrightarrow{\alpha_{lji}} F(i)/F(l) \xrightarrow{\beta_{lji}} F(i)/F(j) \rightarrow 0.$$ 

Fix an element $o \in I$. By §4.1.3, the following spaces over $\mathbb{C}$ are well defined.

1) The space

$$\mathcal{S}_{F(o)}(E) \overset{\text{def}}{=} \lim_{q \leq p \in I} \lim_{q \in I} \mathcal{S}(F(p)/F(q)) \otimes_\mathbb{C} \mu(F(q) | F(o))$$

$$= \lim_{q \leq p \in I} \lim_{q \in I} \mathcal{S}(F(p)/F(q)) \otimes_\mathbb{C} \mu(F(q) | F(o))$$

with respect to the maps

$$\left(\beta_{lji}\right)_* \otimes \text{Id}_{\mu(F(j) | F(o))}: $$

$$\mathcal{S}(F(i)/F(l)) \otimes_\mathbb{C} \mu(F(l) | F(o)) \rightarrow \mathcal{S}(F(i)/F(j)) \otimes_\mathbb{C} \mu(F(j) | F(o))$$

and the maps

$$\alpha_{lji}^* \otimes \text{Id}_{\mu(F(l) | F(o))}: $$

$$\mathcal{S}(F(i)/F(l)) \otimes_\mathbb{C} \mu(F(l) | F(o)) \rightarrow \mathcal{S}(F(j)/F(l)) \otimes_\mathbb{C} \mu(F(l) | F(o)).$$

2) The space

$$\mathcal{S}'_{F(o)}(E) \overset{\text{def}}{=} \lim_{q \leq p \in I} \lim_{q \in I} \mathcal{S}'(F(p)/F(q)) \otimes_\mathbb{C} \mu(F(o) | F(q))$$

$$= \lim_{q \leq p \in I} \lim_{q \in I} \mathcal{S}'(F(p)/F(q)) \otimes_\mathbb{C} \mu(F(o) | F(q))$$

with respect to the maps

$$\beta_{lji}^* \otimes \text{Id}_{\mu(F(o) | F(l))}: $$

$$\mathcal{S}'(F(i)/F(j)) \otimes_\mathbb{C} \mu(F(o) | F(j)) \rightarrow \mathcal{S}'(F(i)/F(l)) \otimes_\mathbb{C} \mu(F(o) | F(l))$$

and the maps

$$\left(\alpha_{lji}\right)_* \otimes \text{Id}_{\mu(F(o) | F(l))}: $$

$$\mathcal{S}'(F(j)/F(l)) \otimes_\mathbb{C} \mu(F(o) | F(l)) \rightarrow \mathcal{S}'(F(i)/F(l)) \otimes_\mathbb{C} \mu(F(o) | F(l)).$$

Using the definitions of these spaces, we see that for every $o_1 \in I$ there is a canonical isomorphism

$$\mathcal{S}_{F(o)}(E) \otimes_\mathbb{C} \mu(F(o) | F(o_1)) \rightarrow \mathcal{S}_{F(o_1)}(E)$$
such that the diagram
\[
\begin{array}{c}
\mathcal{S}_{F(o)}(E) \otimes \mu(F(o) \mid F(o_1)) \otimes \mu(F(o_1) \mid F(o_2)) \\
\downarrow
\end{array}
\longrightarrow
\begin{array}{c}
\mathcal{S}_{F(o_1)}(E) \otimes \mu(F(o_1) \mid F(o_2)) \\
\downarrow
\end{array}
\]
\[
\mathcal{S}_{F(o)}(E) \otimes \mu(F(o) \mid F(o_2))
\longrightarrow
\mathcal{S}_{F(o_2)}(E)
\]
is commutative for every \( o_2 \in I \).

Dually, for every \( o_1 \in I \) there is a canonical isomorphism
\[
\mu(F(o_1) \mid F(o)) \otimes \mathcal{S}'_{F(o)}(E) \rightarrow \mathcal{S}'_{F(o_1)}(E)
\]
such that the diagram
\[
\begin{array}{c}
\mu(F(o_2) \mid F(o_1)) \otimes \mu(F(o_1) \mid F(o)) \otimes \mathcal{S}'_{F(o)}(E) \\
\downarrow
\end{array}
\longrightarrow
\begin{array}{c}
\mu(F(o_2) \mid F(o_1)) \otimes \mathcal{S}'_{F(o_1)}(E) \\
\downarrow
\end{array}
\]
\[
\mu(F(o_2) \mid F(o)) \otimes \mathcal{S}'_{F(o)}(E)
\longrightarrow
\mathcal{S}'_{F(o_2)}(E)
\]
is commutative for every \( o_2 \in I \).

Remark 12. Let \( \{B_l\} \), \( l \in J \), be a projective system of vector spaces over a field \( k \) with surjective transition maps \( \phi_{l_1,l_2} : B_{l_1} \rightarrow B_{l_2} \) for all elements \( l_1 \geq l_2 \in J \). Let \( J^0 \) be a partially ordered set which equals \( J \) as a set but has the reverse order. Let \( \{A_l\} \), \( l \in J^0 \), be an inductive system of \( k \)-vector spaces with injective transition maps \( \psi_{l_2,l_1} : A_{l_2} \rightarrow A_{l_1} \). Suppose that for every \( l \in J \) we have a non-degenerate \( k \)-linear pairing
\[
\langle \cdot , \cdot \rangle_l : B_l \times A_l \rightarrow k
\]
that satisfies the following condition for all \( l_1 \geq l_2 \in I \), \( x \in B_{l_1} \), \( y \in A_{l_2} \):
\[
\langle \phi_{l_1,l_2}(x), y \rangle_{l_2} = \langle x, \psi_{l_2,l_1}(y) \rangle_{l_1}.
\]
Then we canonically obtain a non-degenerate \( k \)-linear pairing between the \( k \)-vector spaces \( \lim_{\leftarrow l \in J} B_l \) and \( \lim_{\rightarrow l \in J^0} A_l \). It is induced by the pairings \( \langle \cdot , \cdot \rangle_l \), \( l \in J \).

If \( E = (I,F,V) \in \text{Ob}(\mathbb{C}_2^\mathbb{R}) \), then we have non-degenerate \( \mathbb{C} \)-linear pairings between the spaces \( \mathcal{S}(F(i)/F(j)) \) and \( \mathcal{S}'(F(i)/F(j)) \) for all \( i \geq j \in I \) (see (58)). Moreover, we easily see from the definitions and the corresponding facts about direct and inverse images in \( C_0^\mathbb{R} \) that the transition maps in the projective system (75) are surjective and the transition maps in the inductive system (76) are injective.

Therefore, applying Remark 12 twice for the formulae (75), (76), we see that there is a non-degenerate pairing
\[
\langle \cdot , \cdot \rangle_{\mathcal{S}_{F(o)}(E)} : \mathcal{S}_{F(o)}(E) \times \mathcal{S}'_{F(o)}(E) \rightarrow \mathbb{C}.
\]

Remark 13. Let \( E = (I,F,V) \in \text{Ob}(\mathbb{C}_2^\mathbb{R}) \) be a cf-object. By Remark 8, for every pair of elements \( i \geq j \in I \) there is a canonical element
\[
1_{ij} \overset{\text{def}}{=} 1_{F(i)/F(j)} \in \mu(F(i)/F(j))
\]
(it can also be defined in terms of the non-degenerate pairing (58): \( \langle 1, 1_{ij} \rangle = 1 \)). Hence for any elements \( k,l \in I \) there is a canonical element \( 1_{kl} \in \mu(F(k)/F(l)) \) such that for all \( k,l,n \in I \) we obtain from Proposition 14 that
\[
\gamma(1_{kl} \otimes 1_{ln}) = 1_{kn}.
\]
The existence of elements $1_{lo}$ and $1_{ol}$ for $l, o \in I$ enables us to omit the 1-dimensional $\mathbb{C}$-spaces $\mu(F(l) \mid F(o))$ and $\mu(F(o) \mid F(l))$ from the formulae (75) and (76) which determine the $\mathbb{C}$-spaces $S_{F(o)}(E)$ and $S'_{F(o)}(E)$. These $\mathbb{C}$-spaces are independent of the choice of $o \in I$.

We similarly suppose that $E = (I, F, V) \in \text{Ob}(C^\text{ar})$ is a $df$-object. By Remark 8, for every pair of elements $i \geq j \in I$ there is a canonical element

$$\delta_{ij} \overset{\text{def}}{=} \sum_{F(j)/F(i)} \mu(F(i)/F(j))$$

(it can also be defined in terms of the non-degenerate pairing (58): $\langle f, \delta_{ij} \rangle = \sum_{x \in V_{ij}} f(x)$, where $f \in S(E)$, $F(i)/F(j) = (I_{ij}, F_{ij}, V_{ij}) \in \text{Ob}(C^\text{ar})$). Hence for any elements $k, l \in I$ there is a canonical element $\delta_{kl} \in \mu(F(k) \mid F(l))$ such that for all $k, l, n \in I$ we obtain from Proposition 14 that

$$\gamma(\delta_{kl} \otimes \delta_{ln}) = \delta_{kn}.$$ 

The existence of the elements $\delta_{lo}$ and $\delta_{ol}$ for $l, o \in I$ enables us to omit the 1-dimensional $\mathbb{C}$-spaces $\mu(F(l) \mid F(o))$ and $\mu(F(o) \mid F(l))$ from the formulae (75) and (76) which determine the $\mathbb{C}$-spaces $S_{F(o)}(E)$ and $S'_{F(o)}(E)$. These $\mathbb{C}$-spaces are independent of the choice of $o \in I$.

§ 8. The Fourier transform

Let $E = (I, F, V) \in \text{Ob}(C^\text{ar})$ be a complete object. Then we have $\check{E} = E$. We fix an element $o \in I$.

8.1. The maps $f \mapsto \check{f}, G \mapsto \check{G}$. Take any elements $i \geq j \in I$. By § 4.2 there are $\mathbb{C}$-linear maps

$$f \otimes \mu \in S(F(i)/F(j)) \otimes_{\mathbb{C}} \mu(F(j) \mid F(o))$$

$$\mapsto \check{f} \otimes \mu \in S(F(i)/F(j)) \otimes_{\mathbb{C}} \mu(F(j) \mid F(o)),$$

$$G \otimes \mu' \in S'(F(i)/F(j)) \otimes_{\mathbb{C}} \mu(F(o) \mid F(j))$$

$$\mapsto \check{G} \otimes \mu' \in S'(F(i)/F(j)) \otimes_{\mathbb{C}} \mu(F(o) \mid F(j)).$$

These maps commute with direct and inverse images if we replace $i, j \in I$ by $i', j' \in I$, $i' \geq i$, $j' \leq j$. Hence formulae (75), (76) show that the $\mathbb{C}$-linear maps

$$f \in S_{F(o)}(E) \mapsto \check{f} \in S_{F(o)}(E), \quad G \in S'_{F(o)}(E) \mapsto \check{G} \in S'_{F(o)}(E)$$

are well defined. The square of each is the identity map. Using the corresponding (obvious) formulae for $C^\text{ar}$, we now get

$$\langle \check{f}, G \rangle_{S_{F(o)}(E)} = \langle f, \check{G} \rangle_{S'_{F(o)}(E)} \quad \forall f \in S_{F(o)}(E), \quad \forall G \in S'_{F(o)}(E).$$

8.2. Two-dimensional Fourier transforms. As described in § 5, there are dual objects $\bar{E} = (I^0, F^0, \check{V}) \in \text{Ob}(C^\text{ar})$. For all elements $i \geq j \in I$ the object $E_{j,i} = F(i)/F(j) \in \text{Ob}(C^\text{ar})$ determines an object $\check{E}_{j,i} = F^0(j)/F^0(i) \in \text{Ob}(C^\text{ar})$.

For all $l, n \in I$ we have

$$\mu(F(l) \mid F(n)) = \mu(F^0(l) \mid F^0(n)). \quad (77)$$
Indeed, let \( k \in I \) be such that \( k \leq l \) and \( k \leq n \). Then Proposition 14, Remark 11 and formula (61) yield that
\[
\mu(F(l) | F(n)) = \mu(F(l) | F(k)) \otimes \mathbb{C} \mu(F(k) | F(n))
\]
\[
= \mu(F(l)F(k))^* \otimes \mathbb{C} \mu(F(n)F(k)) = \mu(F^0(k)/F^0(l)) \otimes \mathbb{C} \mu(F^0(k)/F^0(n))^*
\]
\[
= \mu(F^0(l) | F^0(k)) \otimes \mathbb{C} \mu(F^0(k) | F^0(n)) = \mu(F^0(l) | F^0(n)).
\]

Therefore we have the following maps (Fourier transforms) for all \( i \geq j \in I \):
\[
\mathbf{F} \otimes \mu(F(i) | F(o)):
\]
\[
S(F(i)/F(j)) \otimes \mathbb{C} \mu(F(j) | F(o)) \rightarrow S(F^0(j)/F^0(i)) \otimes \mathbb{C} \mu(F^0(i) | F^0(o)), \; (78)
\]
\[
\mathbf{F} \otimes \mu(F(o) | F(i)):
\]
\[
S'(F(i)/F(j)) \otimes \mathbb{C} \mu(F(o) | F(j)) \rightarrow S'(F^0(j)/F^0(i)) \otimes \mathbb{C} \mu(F^0(o) | F^0(i)). \; (79)
\]

We now use Remark 8 or, more precisely, an analogue of Proposition 10 for the category \( C^\text{ar}_1 \). This remark connects the Fourier transform with direct and inverse images. We use it with the elements \( i, j \in I \) replaced by \( i', j' \in I \), \( i' \geq i, \; j' \leq j \), and taking limits according to formulae (75), (76). Thus we obtain that the following \( \mathbb{C} \)-linear maps (two-dimensional Fourier transforms) are well defined:
\[
\mathbf{F}: \mathcal{S}_{\mathcal{F}(o)}(E) \rightarrow \mathcal{S}'_{\mathcal{F}(0)}(\hat{E}), \; (80)
\]
\[
\mathbf{F}: \mathcal{S}'_{\mathcal{F}(o)}(E) \rightarrow \mathcal{S}'_{\mathcal{F}(0)}(\hat{E}). \; (81)
\]

Remark 14. For all \( i \geq j \in I \), the spaces \( \mathcal{S}(F(i)/F(j)) \) and \( \mathcal{S}'(F(i)/F(j)) \) are topological \( \mathbb{C} \)-vector spaces and \( \dim_{\mathbb{C}} \mu(F(j) | F(o)) = 1 \). Therefore \( \mathcal{S}(E) \) and \( \mathcal{S}'(E) \) are also topological \( \mathbb{C} \)-vector spaces endowed with the topology of projective and inductive limits by formulae (75), (76) respectively (see [15]).

**Proposition 15.**
1) The maps \( \mathbf{F} \) defined in (80), (81) are isomorphisms of topological \( \mathbb{C} \)-vector spaces.

2) For all elements \( f \in \mathcal{S}_{\mathcal{F}(o)}(E) \) and \( G \in \mathcal{S}'_{\mathcal{F}(0)}(\hat{E}) \) we have
\[
\mathbf{F} \circ \mathbf{F}(f) = \tilde{f}, \quad \mathbf{F} \circ \mathbf{F}(G) = \hat{G}.
\]

3) For all elements \( f \in \mathcal{S}_{\mathcal{F}(o)}(E) \) and \( G \in \mathcal{S}'_{\mathcal{F}(0)}(\hat{E}) \) we have
\[
\langle \mathbf{F}(f), G \rangle_{\mathcal{S}'_{\mathcal{F}(0)}(\hat{E})} = \langle f, \mathbf{F}(G) \rangle_{\mathcal{S}_{\mathcal{F}(o)}(E)}.
\]

**Proof.** This follows from the construction of the two-dimensional Fourier transform (described in this section) and the properties of the Fourier transform in \( C^{\text{ar}}_1 \) (see §4.2 and Proposition 12) applied to the functions and distributions on \( F(i)/F(j) \in \text{Ob}(C^{\text{ar}}_1) \) for all \( i \geq j \in I \).

**§ 9. A central extension and its representations**

In this section we construct a central extension of a certain subgroup of the automorphism group of an object in \( C^{\text{ar}}_2 \) and consider representations of this group on the spaces of functions and distributions on this object.
9.1. Canonical isomorphisms. Suppose that an object \( E_1 = (I_1, F_1, V) \in \text{Ob}(C_2^{ar}) \) dominates \( E_2 = (I_2, F_2, V) \in \text{Ob}(C_2^{ar}) \) (see §5) and fix an element \( o \in I_2 \). Then we get canonical isomorphisms

\[
S_{F(o)}(E_1) = S_{F(o)}(E_2), \quad S'_{F(o)}(E_1) = S'_{F(o)}(E_2).
\]

Indeed, it follows from (75), (76) that the inductive and projective limits occurring in the definitions of these spaces are the same on sets of indices depending on \( I_1 \) or \( I_2 \). Moreover, for the same reason, the two-dimensional Fourier transform coincides for spaces depending on \( E_1 \) or \( E_2 \) (compare this with Remark 9 and the argument after Definition 13).

9.2. A subgroup of the automorphism group. Given \( E = (I, F, V) \in \text{Ob}(C_2^{ar}) \), we define the following subgroup of the group \( \text{Aut}_{C_2^{ar}}(E) \).

**Definition 20.** The set \( \text{Aut}_{C_2^{ar}}(E)' \) consists of all elements \( g \in \text{Aut}_{C_2^{ar}}(E) \) with the following properties.

1) For every \( i \in I \) there is \( j \in I \) such that \( gF(i) = F(j) \).
2) For all \( i \geq j \in I \), if \( g(F(i)) = F(p) \) and \( g(F(j)) = F(q) \), then \( p \geq q \in I \).
3) The element \( g^{-1} \in \text{Aut}_{C_2^{ar}}(E) \) satisfies conditions analogous to 1), 2).

Clearly, the set \( \text{Aut}_{C_2^{ar}}(E)' \) is a subgroup of \( \text{Aut}_{C_2^{ar}}(E) \).

We now construct an object \( \widetilde{E} \in \text{Ob}(C_2^{ar}) \), which dominates the object \( E \in \text{Ob}(C_2^{ar}) \).

First, we fix any elements \( i \geq j \in I \) and consider the natural object \( E_{j,i} = (I_{j,i}, F_{j,i}, F(i)/F(j)) \in \text{Ob}(C_1^{ar}) \) arising from the \( C_2^{ar} \)-structure on \( E \). We define the following object \( \widetilde{E}_{j,i} \in \text{Ob}(C_1^{ar}) \) that dominates \( E_{j,i} \). Consider the filtered Abelian group \( \widetilde{E}_{j,i} = (G, \text{Id}, F(i)/F(j)) \), where \( G \) is the set of subgroups \( gF_{I,k}(n) \) of the group \( F(i)/F(j) \) such that \( g \in \text{Aut}_{C_2^{ar}}(E)' \), \( k \geq l \in I \), \( n \in I_{l,k} \), \( g(F(k)) = F(i) \), \( g(F(l)) = F(j) \), \( F_{I,k}(n) \subseteq F(k)/F(l) \). The set \( G \) is partially ordered by inclusion of subgroups. The function \( \text{Id} \) sends each element of \( G \) to the corresponding subgroup of \( F(i)/F(j) \). By Definition 15, for every such subgroup \( gF_{I,k}(n) \) one can find elements \( p \geq q \in I_{j,i} \) such that

\[
F_{j,i}(p) \supset gF_{I,k}(n) \supset F_{j,i}(q),
\]

and \( gF_{I,k}(n) \) is a closed subgroup of the Lie group \( F_{j,i}(p)/F_{j,i}(q) \). Hence we obtain in accordance with Remark 1 that the filtered Abelian group \( \widetilde{E}_{j,i} \) determines a well-defined object of \( C_1^{ar} \) and the object \( \widetilde{E}_{j,i} \) dominates \( E_{j,i} \).

We see immediately from these constructions that the new object \( \widetilde{E} = (I, F, V) \), whose groups \( F(i)/F(j), i \geq j \in I \), are endowed with the \( C_2^{ar} \)-structure of the object \( E_{j,i} \), is well defined as an object of \( C_2^{ar} \). Moreover, the object \( \widetilde{E} \) dominates \( E \) in the category \( C_2^{ar} \). Therefore \( \text{Aut}_{C_2^{ar}}(\widetilde{E}) = \text{Aut}_{C_2^{ar}}(E) \). We obtain from the definition of \( \widetilde{E} \) that

\[
\text{Aut}_{C_2^{ar}}(\widetilde{E})' = \text{Aut}_{C_2^{ar}}(E)' \quad (82)
\]
as subgroups \( \text{Aut}_{C_2^{ar}}(E) \).

9.3. The central extension. Suppose that \( E = (I, F, V) \in \text{Ob}(C_2^{ar}) \). Then the space \( \mu(F(i_1) \mid F(i_2)) \) of virtual measures is defined for all \( i_1, i_2 \in I \). For every
element \( g \in \text{Aut}_{C^r_2}(E)' \) and every \( i \in I \) we see from the definition of the group \( \text{Aut}_{C^r_2}(E)' \) that \( gF(i) = F(p) \) for some element \( p \in I \). Hence a one-dimensional \( \mathbb{C} \)-vector space \( \mu(gF(i_1) \mid hF(i_2)) \) is well defined for all \( g, h \in \text{Aut}_{C^r_2}(E)' \) and \( i_1, i_2 \in I \).

Using the isomorphism \((82)\) and \(\S\ 9.1\), we obtain the following isomorphism for every element \( g \in \text{Aut}_{C^r_2}(E)' \) and all \( p \geq q \in I \):

\[
\begin{align*}
\varphi_g : S(F(p) / F(q)) &\rightarrow S(gF(p) / gF(q)), & (83) \\
\end{align*}
\]

where \( \varphi_g(f)(v) \equiv f(g^{-1}v) \) for all \( v \in gF(p) / gF(q) \) and \( f \in S(F(p) / F(q)) \). (Here we use the fact that it suffices to consider \( \tilde{E} \) instead of \( E \) and \( S(F(i) / F(j)) \) is a \( \mathbb{C} \)-subalgebra of the space \( F(F(i) / F(j)) \) of all \( \mathbb{C} \)-valued functions on \( V \), where \( i \geq j \in I \) are arbitrary elements.)

The following isomorphism is obtained dually to formula \((83)\) and holds for all \( g \in \text{Aut}_{C^r_2}(E)' \) and all \( p \geq q \in I \):

\[
\begin{align*}
\varphi'_g : S'(F(p) / F(q)) &\rightarrow S'(gF(p) / gF(q)). & (84) \\
\end{align*}
\]

(In other words, the map \( \varphi'_g \) is conjugate to the map \( \varphi_{g^{-1}} \) with respect to the pairing \((58)\).)

Using Remark 6, we obtain that the map \( \varphi'_g \) induces an isomorphism

\[
\begin{align*}
m_g : \text{Hom}_\mathbb{C}(\mu(F(p) / F(s)), \mu(F(q) / F(s))) &\rightarrow \text{Hom}_\mathbb{C}(\mu(gF(p) / gF(s)), \mu(gF(q) / gF(s))), \\
\end{align*}
\]

where, for every \( f \in \text{Hom}_\mathbb{C}(\mu(F(p) / F(s)), \mu(F(q) / F(s))) \),

\[
m_g(f) \equiv n_g \circ f \circ n_{g^{-1}}.
\]

Applying the isomorphism \( m_g \) to the inductive limit in \((73)\), we get \( \mathbb{C} \)-linear isomorphisms

\[
\begin{align*}
l_g : \mu(F(p) \mid F(q)) &\rightarrow \mu(gF(p) \mid gF(q)) \\
\end{align*}
\]

for all elements \( p, q \in I \).

We have \( l_{g_1g_2} = l_{g_1}l_{g_2} \) for all \( g_1, g_2 \in \text{Aut}_{C^r_2}(E)' \). Moreover, the following equation holds for all \( a \in \mu(F(p) \mid F(q)), b \in \mu(F(q) \mid F(s)) \) and any \( p, q, s \in I \):

\[
\gamma(l_g(a) \otimes l_g(b)) = l_g(\gamma(a \otimes b))
\]

(see Proposition 14, where the isomorphism \( \gamma \) was introduced).

Given any \( p, q \in I \) and any element \( \mu \in \mu(F(p) \mid F(q)), \mu \neq 0 \), we canonically define an element \( \mu^{-1} \in \mu(F(q) \mid F(p)) \) such that \( \mu \otimes \mu^{-1} = 1 \) with respect to the canonical isomorphism

\[
\mu(F(p) \mid F(q)) \otimes \mathbb{C} \mu(F(q) \mid F(p)) = \mathbb{C}.
\]

Suppose that \( E = (I, F, V) \in \text{Ob}(C^r_2) \) and fix some element \( o \in I \). Then we have the central extension of groups

\[
1 \rightarrow \mathbb{C}^* \rightarrow \overline{\text{Aut}_{C^r_2}(E)}_{F(o)} \rightarrow \text{Aut}_{C^r_2}(E)' \rightarrow 1, \tag{85}
\]

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where
\[
\text{Aut}_{C_2^r}(E)'_{F(o)} \overset{\text{def}}{=} \{ (g, \mu) : g \in \text{Aut}_{C_2^r}(E)', \mu \in \mu(F(o) | gF(o)), \mu \neq 0 \}.
\]

Here \( \Lambda((g, \mu)) = g \). The operations \((g_1, \mu_1) \cdot (g_2, \mu_2) = (g_1g_2, \gamma(\mu_1 \otimes l_{g_1}(\mu_2))) \) and \((g, \mu)^{-1} = (g^{-1}, l_{g^{-1}}(\mu^{-1})) \) determine the structure of a group on the set \( \text{Aut}_{C_2^r}(E)'_{F(o)} \). (The identity of this group is \((e, 1)\), where \(e\) is the identity of \( \text{Aut}_{C_2^r}(E)' \).)

**Remark 15.** For every \( o_1 \in I \) there is a canonical isomorphism
\[
\alpha_{o,o_1} : \text{Aut}_{C_2^r}(E)'_{F(o)} \to \text{Aut}_{C_2^r}(E)'_{F(o_1)}.
\]

Indeed, fix any element \( \nu \in \mu(F(o_1) | F(o)), \nu \neq 0 \). Then
\[
\alpha_{o,o_1}((g, \mu)) \overset{\text{def}}{=} (g, \gamma(\nu \otimes \mu) \otimes l_g(\nu^{-1})).
\]
The map \( \alpha_{o,o_1} \) is independent of the choice of \( \nu \in \mu(F(o_1) | F(o)), \nu \neq 0 \).

**9.4. A representation of the central extension.** Suppose that \( E = (I, F, V) \in \text{Ob}(C_{2r}^\text{ar}) \). In § 9.2 we constructed an object \( \bar{E} \in \text{Ob}(C_{2r}^\text{ar}) \) which dominates \( E \).

We have an isomorphism \((83)\) for every \( g \in \text{Aut}_{C_2^r}(E)' \) and all \( p \geq q \in I \). Fix an element \( o \in I \). Using formula \((83)\), we define a map
\[
R_{\bar{g}} : S(F(p)/F(q)) \otimes_C \mu(F(q) | F(o)) \to S(gF(p)/gF(q)) \otimes_C \mu(gF(q) | F(o)) \quad (86)
\]
for every \( \bar{g} = (g, \mu) \in \text{Aut}_{C_2^r}(E)'_{F(o)} \) and all elements \( p \geq q \in I \). By definition, \( R_{\bar{g}} \) is the composite of \( r_g \otimes l_q \) and multiplication by \( \mu^{-1} \in \mu(gF(o) | F(o)) \).

For every \( \bar{g} \in \text{Aut}_{C_2^r}(E)'_{F(o)} \) we can apply formula \((86)\) to \((75)\) to get a map
\[
R_{\bar{g}} : S_{F(o)}(\bar{E}) \to S_{F(o)}(\bar{E})
\]
such that \( R_{\bar{g}}R_{\bar{h}} = R_{\bar{gh}} \). Therefore, arguing as in § 9.1, we obtain a representation of the group \( \text{Aut}_{C_2^r}(E)'_{F(o)} \) by the maps \( R_{\bar{g}} \) on the \( C \)-space \( S_{F(o)}(E) \).

Let \( o \in I \). Dually to formula \((86)\), we can use the isomorphisms \( r'_g \) given by \((84)\) to define a map
\[
R'_{\bar{g}} : S'(F(p)/F(q)) \otimes_C \mu(F(o) | F(q)) \to S'(gF(p)/gF(q)) \otimes_C \mu(F(q) | gF(q)) \quad (87)
\]
for every \( \bar{g} = (g, \mu) \in \text{Aut}_{C_2^r}(E)'_{F(o)} \) and all elements \( p \geq q \in I \). By definition, \( R'_{\bar{g}} \) is the composite of \( r'_g \otimes l_g \) and multiplication by \( \mu \in \mu(F(o) | gF(o)) \).

Applying formula \((87)\) to \((76)\) and arguing as in § 9.1, we obtain a representation of the group \( \text{Aut}_{C_2^r}(E)'_{F(o)} \) on the \( C \)-space \( S'_{F(o)}(E) \) by maps \( R'_{\bar{g}} \).

By the construction of \( R_{\bar{g}} \) and \( R'_{\bar{g}} \), the following formula holds for all \( H \in S'_{F(o)}(E), f \in S_{F(o)}(E) \) and \( \bar{g} \in \text{Aut}_{C_2^r}(E)'_{F(o)} \):
\[
\langle R'_{\bar{g}}(H), R_{\bar{g}}(f) \rangle_{S_{F(o)}(E)} = \langle H, f \rangle_{S_{F(o)}(E)}.
\]
9.5. The Fourier transform and the representation of the central extension. Let $E = (I, F, V) \in \text{Ob}(C^\text{ar}_2)$ be a complete object. Then we have a dual object $\tilde{E} = (I^0, F^0, \tilde{V})$ and $\tilde{E} = E$. For every $g \in \text{Aut}_{C^\text{ar}_2}(E)$ there is a canonically defined element $\tilde{g} \in \text{Aut}_{C^\text{ar}_2}(\tilde{E})$. Thus we get an isomorphism of groups

$$g \in \text{Aut}_{C^\text{ar}_2}(E) \longleftrightarrow \tilde{g}^{-1} \in \text{Aut}_{C^\text{ar}_2}(\tilde{E}),$$

which induces an isomorphism of groups

$$\text{Aut}_{C^\text{ar}_2}(E)' \rightarrow \text{Aut}_{C^\text{ar}_2}(\tilde{E})'.$$

Hence we have the following isomorphism of groups for every $o \in I$:

$$(g, \mu) \in \text{Aut}_{C^\text{ar}_2}(E)'_{F(o)} \longrightarrow (\tilde{g}^{-1}, \mu) \in \text{Aut}_{C^\text{ar}_2}(\tilde{E})'_{F^0(o)},$$

where we have used the isomorphism (77) to conclude that for every $g \in \text{Aut}_{C^\text{ar}_2}(E)$ we can canonically have

$$\mu(F(o) | gF(o)) = \mu(F^0(o) | \tilde{g}^{-1}F^0(o)).$$

Using the isomorphism (89), we obtain representations of the group $\text{Aut}_{C^\text{ar}_2}(\tilde{E})'_{F(o)}$ on the $\mathbb{C}$-spaces $S_{F^0(o)}(\tilde{E})$ and $S'_{F^0(o)}(\tilde{E})$.

**Proposition 16.** Let $E = (I, F, V) \in \text{Ob}(C^\text{ar}_2)$ be a complete object. Then the Fourier transform $F$ determines an isomorphism between the representations of the group $\text{Aut}_{C^\text{ar}_2}(\tilde{E})'_{F(o)}$ on the $\mathbb{C}$-spaces $S_{F(o)}(E)$, $S_{F^0(o)}(\tilde{E})$ and $S'_{F(o)}(E)$, $S'_{F^0(o)}(\tilde{E})$.

**Proof.** Using the definition of the two-dimensional Fourier transform $F$ in § 8.2 and formulae (75), (76), we reduce the assertions of this proposition to the corresponding assertions about isomorphisms of objects of $C^\text{ar}_1$ and the one-dimensional Fourier transform between them. The latter assertions follow from the definition and properties of the one-dimensional Fourier transform or, alternatively, can be further reduced to the case of objects of $C^\text{ar}_0$ by formulae (56), (57) using the definition of the one-dimensional Fourier transform in § 4.2. The proposition is proved.

**Remark 16.** In this section we have used the subgroup $\text{Aut}_{C^\text{ar}_2}(E)'$ instead of the whole group $\text{Aut}_{C^\text{ar}_2}(E)$ for $E \in \text{Ob}(C^\text{ar}_2)$. There are many important examples where the groups under consideration are subgroups of $\text{Aut}_{C^\text{ar}_2}(E)$ if $E$ has a sufficiently fine filtration. For example, $E$ may be constructed from a two-dimensional local field $K$ whose last residue field is finite, from $K \cong \mathbb{R}((t))$ or from $E = K \cong \mathbb{C}((t))$. In these cases, $K^* \subset \text{Aut}_{C^\text{ar}_2}(E)'$ (see Examples 1, 10). In another example, $E$ may be the arithmetic adèle ring $\mathbb{A}^\text{ar}_X$, where $X$ is an ‘arithmetic surface’, and the group of invertible elements (the idèle group) $(\mathbb{A}^\text{ar}_X)^*$ is contained in $\text{Aut}_{C^\text{ar}_2}(E)'$ (see Example 11).

**§ 10. Direct and inverse images**

In this section we construct direct and inverse images of spaces of functions and distributions on objects of $C^\text{ar}_2$.

The following data are fixed throughout the section. Let

$$0 \longrightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \longrightarrow 0$$

(90)
be an admissible triple in $C_1^\ar$, where $E_i = (I_i, F_i, V_i)$, $1 \leq i \leq 3$. By definition, we have order-preserving functions

(i) $\gamma : I_2 \to I_3$ such that $\beta(F_2(i)) = F_3(\gamma(i))$ for all $i \in I_2$,
(ii) $\varepsilon : I_2 \to I_1$ such that $F_2(i) \cap V_1 = F_1(\varepsilon(i))$ for all $i \in I_2$.

10.1. The case when $E_1$ is a c-object.

**Proposition 17.** Suppose that $E_1$ is a c-object in (90). Let $o \in I_2$. Then there is a direct image morphism

$$\beta_* : S_{F_2(o)}(E_2) \otimes_C \mu(F_1(\varepsilon(o)) \mid V_1) \to S_{F_3(\gamma(o))}(E_3).$$

**Proof.** For all $i, j \in I_2$ we canonically have

$$\mu(F_2(i) \mid F_2(j)) = \mu(F_1(\varepsilon(i)) \mid F_1(\varepsilon(j))) \otimes_C \mu(F_3(\gamma(i)) \mid F_3(\gamma(j))). \quad (91)$$

Let $k \in I_2$ be such that $F_1(\varepsilon(k)) = V_1$. Then for any elements $i \geq j \in I_2$ with $i \geq k \geq j$ we have an admissible triple in $C_1$ of the form

$$0 \to V_1 / F_1(\varepsilon(j)) \to F_2(i) / F_2(j) \xrightarrow{\beta_{ij}} F_3(\gamma(i)) / F_3(\gamma(j)) \to 0.$$

There is a well-defined map

$$(\beta_{ij})_* : S(F_2(i) / F_2(j)) \otimes_C \mu(F_1(\varepsilon(j)) \mid V_1) \to S(F_3(\gamma(i)) / F_3(\gamma(j))),$$

where we have used the formula $\mu(F_1(\varepsilon(j)) \mid V_1) = \mu(V_1 / F_1(\varepsilon(j)))$. Hence we get a well-defined map

$$(\beta_{ij})_* \otimes \text{Id}_{\mu(F_3(\gamma(j)) \mid F_3(\gamma(o)))} : S(F_2(i) / F_2(j)) \otimes_C \mu(F_1(\varepsilon(j)) \mid F_3(\gamma(j))) \otimes_C \mu(F_3(\gamma(j)) \mid F_3(\gamma(o)))$$

$$\to S(F_3(\gamma(i)) / F_3(\gamma(j))) \otimes C \mu(F_3(\gamma(j)) \mid F_3(\gamma(o))).$$

We see from (91) that

$$\mu(F_3(\gamma(j)) \mid F_3(\gamma(o))) = \mu(F_2(j) \mid F_2(o)) \otimes_C \mu(F_1(\varepsilon(o)) \mid F_1(\varepsilon(j))).$$

Therefore,

$$\mu(F_1(\varepsilon(j)) \mid V_1) \otimes_C \mu(F_3(\gamma(j)) \mid F_3(\gamma(o))) = \mu(F_2(j) \mid F_2(o)) \otimes_C \mu(F_1(\varepsilon(o)) \mid V_1).$$

Thus we obtain a well-defined map

$$(\beta_{ij})_* \otimes \text{Id}_{\mu(F_3(\gamma(j)) \mid F_3(\gamma(o)))} :$$

$$S(F_2(i) / F_2(j)) \otimes_C \mu(F_2(j) \mid F_2(o)) \otimes C \mu(F_1(\varepsilon(o)) \mid V_1)$$

$$\to S(F_3(\gamma(i)) / F_3(\gamma(j))) \otimes C \mu(F_3(\gamma(j)) \mid F_3(\gamma(o))).$$

We now take the projective limits used to construct the spaces $S_{F_2(o)}(E_2)$ and $S_{F_3(\gamma(o))}(E_3)$. This yields a well-defined map $\beta_*$ constructed from the maps $(\beta_{ij})_* \otimes \text{Id}_{\mu(F_3(\gamma(j)) \mid F_3(\gamma(o)))}$. The proposition is proved.

**Remark 17.** Dually to the maps above, we obtain a well-defined map

$$\beta^* : S'_{F_3(\gamma(o))}(E_3) \otimes_C \mu(F_1(\varepsilon(o)) \mid V_1) \to S'_{F_2(o)}(E_2)$$

such that $\beta_*$ and $\beta^*$ are conjugate with respect to the pairings $\langle \cdot, \cdot \rangle_{S_{F_2(o)}(E_2)}$ and $\langle \cdot, \cdot \rangle_{S_{F_3(\gamma(o))}(E_3)}$. 
10.2. The case when $E_3$ is a $d$-object.

**Proposition 18.** Suppose that $E_3$ is a $d$-object in (90). Let $o \in I_2$. Then there is an inverse image morphism

$$
\alpha^*: \mathcal{S}_{F_2(o)}(E_2) \otimes \mathbb{C} \mu(\mathcal{F}_3(\gamma(o)) \mid \{0\}) \rightarrow \mathcal{S}_{F_1(\varepsilon(o))}(E_1),
$$

where $\{0\}$ is the zero subgroup of $V_3$.

**Proof.** For all $i, j \in I_2$ we canonically have

$$
\mu(F_2(i) \mid F_2(j)) = \mu(F_1(\varepsilon(i)) \mid F_1(\varepsilon(j))) \otimes \mathbb{C} \mu(F_3(\gamma(i)) \mid F_3(\gamma(j))). \quad (92)
$$

Since $\beta$ is an admissible epimorphism and $E_3$ is a $d$-object, there is an element $k \in I_2$ such that $F_3(\gamma(k)) = \{0\}$. For any elements $i \geq j \in I_2$ with $i \geq k \geq j$ we have an admissible triple in $C^*_k$ of the form

$$
0 \rightarrow F_1(\varepsilon(i)) / F_1(\varepsilon(j)) \overset{\alpha_{ij}}{\longrightarrow} F_2(i) / F_2(j) \rightarrow F_3(\gamma(i)) \rightarrow 0. \quad (93)
$$

There is a well-defined map

$$
\alpha_{ij}^*: \mathcal{S}(F_2(i) / F_2(j)) \rightarrow \mathcal{S}(F_1(\varepsilon(i)) / F_1(\varepsilon(j))).
$$

Therefore we get a well-defined map

$$
\alpha_{ij}^* \otimes \text{Id}_{\mu(F_1(\varepsilon(j)) \mid F_1(\varepsilon(o)))}:
$$

$$
\mathcal{S}(F_2(i) / F_2(j)) \otimes \mathbb{C} \mu(F_1(\varepsilon(j)) \mid F_1(\varepsilon(o)))
\rightarrow \mathcal{S}(F_1(\varepsilon(i)) \mid F_1(\varepsilon(j))) \otimes \mathbb{C} \mu(F_1(\varepsilon(j)) \mid F_1(\varepsilon(o))).
$$

We see from (92) and (93) that

$$
\mu(F_2(o) \mid F_2(j)) = \mu(F_1(\varepsilon(o)) \mid F_1(\varepsilon(j))) \otimes \mathbb{C} \mu(F_3(\gamma(o)) \mid \{0\}).
$$

Hence

$$
\mu(F_1(\varepsilon(j)) \mid F_1(\varepsilon(o))) = \mu(F_2(j) \mid F_2(o)) \otimes \mathbb{C} \mu(F_3(\gamma(o)) \mid \{0\}).
$$

Thus we get a well-defined map

$$
\alpha_{ij}^* \otimes \text{Id}_{\mu(F_1(\varepsilon(j)) \mid F_1(\varepsilon(o)))}:
$$

$$
\mathcal{S}(F_2(i) / F_2(j)) \otimes \mathbb{C} \mu(F_2(j) / F_2(o)) \otimes \mathbb{C} \mu(F_3(\gamma(o)) \mid \{0\})
\rightarrow \mathcal{S}(F_1(\varepsilon(i)) \mid F_1(\varepsilon(j))) \otimes \mathbb{C} \mu(F_1(\varepsilon(j)) \mid F_1(\varepsilon(o))).
$$

We now take the projective limits (with respect to $i \geq k \geq j$) used to construct the spaces $\mathcal{S}_{F_2(o)}(E_2)$ and $\mathcal{S}_{F_1(\varepsilon(o))}(E_1)$. Then we obtain a well-defined map $\alpha^*$ constructed from the maps $\alpha_{ij}^* \otimes \text{Id}_{\mu(F_1(\varepsilon(j)) \mid F_1(\varepsilon(o)))}$. The proposition is proved.

**Remark 18.** Dually to the maps above, we obtain a well-defined map

$$
\alpha_*: \mathcal{S}'_{F_1(\varepsilon(o))}(E_1) \otimes \mathbb{C} \mu(F_3(\gamma(o)) \mid \{0\}) \rightarrow \mathcal{S}'_{F_2(o)}(E_2)
$$

such that $\alpha^*$ and $\alpha_*$ are conjugate with respect to the pairings $\langle \cdot, \cdot \rangle_{\mathcal{S}_{F_2(o)}(E_2)}$ and $\langle \cdot, \cdot \rangle_{\mathcal{S}_{F_1(\varepsilon(o))}(E_1)}$. 
10.3. The case when \( E_1 \) is a \( cf \)-object.

**Proposition 19.** Suppose that \( E_1 \) is a \( cf \)-object in (90). Let \( o \in I_2 \). Then there is an inverse image morphism

\[
\beta^*: S_{F_3(\gamma(o))}(E_3) \to S_{F_2(o)}(E_2).
\]

**Proof.** For all elements \( i, j \in I_2 \) we canonically have

\[
\mu(F_2(i) | F_2(j)) = \mu(F_1(\varepsilon(i)) | F_1(\varepsilon(j))) \otimes_{C} \mu(F_3(\gamma(i)) | F_3(\gamma(j))).
\]

(94)

Since \( E_1 \) is a \( cf \)-object, for any elements \( i \geq j \in I_2 \) we have the canonical element \( 1_{ij} \in \mu(F_1(\varepsilon(i))/F_1(\varepsilon(j))) \) constructed in Remark 13. Therefore for any elements \( i, j \in I_2 \) we obtain a canonical element \( 1_{ij} \in \mu(F_1(\varepsilon(i)) | F_1(\varepsilon(j))) \) such that \( 1_{ij} \otimes 1_{jk} = 1_{ik} \) for all \( i, j, k \in I_2 \). Thus we have a canonical isomorphism

\[
\mu(F_1(\varepsilon(i)) | F_1(\varepsilon(j))) = C \quad \text{for any} \quad i, j \in I_2 \quad \text{(see also Remark 13)}.
\]

We obtain from (94) that the following equality holds for all \( i, j \in I_2 \):

\[
\mu(F_2(i) | F_2(j)) = \mu(F_3(\gamma(i)) | F_3(\gamma(j))).
\]

(95)

Given any elements \( i \geq j \in I_2 \), we have an admissible triple in \( C^\text{ar}_1 \) of the form

\[
0 \to \frac{F_1(\varepsilon(i))}{F_1(\varepsilon(j))} \to \frac{F_2(i)}{F_2(j)} \beta_{ij} \to \frac{F_3(\gamma(i))}{F_3(\gamma(j))} \to 0.
\]

We obtain a well-defined map

\[
\beta_{ij}^*: S(F_3(\gamma(i))/F_3(\gamma(j))) \to S(F_2(i)/F_2(j))
\]

because \( \frac{F_1(\varepsilon(i))}{F_1(\varepsilon(j))} \) is a compact object of \( C^\text{ar}_1 \).

Formula (95) yields that

\[
\mu(F_2(j) | F_2(o)) = \mu(F_3(\gamma(j)) | F_3(\gamma(o))).
\]

Therefore we get a map

\[
\beta_{ij}^* \otimes \text{Id}_{\mu(F_2(j) | F_2(o))} : S(F_3(\gamma(i))/F_3(\gamma(j))) \otimes_{C} \mu(F_3(\gamma(j)) | F_3(\gamma(o)))
\]

\[
\to S(F_2(i)/F_2(j)) \otimes_{C} \mu(F_2(j) | F_2(o)).
\]

These maps (over all elements \( i \geq j \in I_2 \)) are compatible if we take projective limits according to the formulæ that define \( S_{F_3(\gamma(o))}(E_3) \) and \( S_{F_2(o)}(E_2) \). Therefore we obtain a map \( \beta^* \) constructed from the maps \( \beta_{ij}^* \otimes \text{Id}_{\mu(F_2(j) | F_2(o))} \). The proposition is proved.

**Remark 19.** Dually to the maps above, we obtain a well-defined map

\[
\beta_*: S'_{F_2(o)}(E_2) \to S'_{F_3(\gamma(o))}(E_3)
\]

such that \( \beta^* \) and \( \beta_* \) are conjugate with respect to the pairings \( \langle \cdot, \cdot \rangle_{S_{F_2(o)}(E_2)} \) and \( \langle \cdot, \cdot \rangle_{S_{F_3(\gamma(o))}(E_3)} \).
10.4. The case when $E_3$ is a df-object.

Proposition 20. Suppose that $E_3$ is a df-object in (90). Let $o \in I_2$. Then there is a direct image morphism

$$\alpha_* : S_{F_1(\varepsilon(o))}(E_1) \to S_{F_2(o)}(E_2).$$

Proof. For any elements $i, j \in I_2$ we canonically have

$$\mu(F_2(i) \mid F_2(j)) = \mu(F_1(\varepsilon(i)) \mid F_1(\varepsilon(j))) \otimes_C \mu(F_3(\gamma(i)) \mid F_3(\gamma(j))). \tag{96}$$

Since $E_3$ is a df-object, for any elements $i \geq j \in I_2$ we have the canonical element $\delta_{ij} \in \mu(F_3(\gamma(i)) \mid F_3(\gamma(j)))$ constructed in Remark 13. Therefore for any elements $i, j \in I_2$ we obtain a canonical element $\delta_{ij} \in \mu(F_3(\gamma(i)) \mid F_3(\gamma(j)))$ such that $\delta_{ij} \otimes \delta_{jk} = \delta_{ik}$ for all $i, j, k \in I_2$. Hence we canonically have $\mu(F_3(\gamma(i)) \mid F_3(\gamma(j))) = C$ for all $i, j \in I_2$ (see also Remark 13).

Thus we obtain from (96) that the following equality holds for all $i, j \in I_2$:

$$\mu(F_2(i) \mid F_2(j)) = \mu(F_1(\varepsilon(i)) \mid F_1(\varepsilon(j))). \tag{97}$$

For any elements $i \geq j \in I_2$ there is an admissible triple in $C_1^{ar}$ of the form

$$0 \to \frac{F_1(\varepsilon(i))}{F_1(\varepsilon(j))} \xrightarrow{\alpha_{ij}} \frac{F_2(i)}{F_2(j)} \to \frac{F_3(\gamma(i))}{F_3(\gamma(j))} \to 0.$$

We obtain a well-defined map

$$\left(\alpha_{ij}\right)_* : S\left(F_1(\varepsilon(i))/F_1(\varepsilon(j))\right) \to S(F_2(i)/F_2(j))$$

because $\frac{F_3(\gamma(i))}{F_3(\gamma(j))}$ is a discrete object in $C_1^{ar}$. We see from (97) that

$$\mu(F_2(j) \mid F_2(o)) = \mu(F_1(\varepsilon(j)) \mid F_1(\varepsilon(o))).$$

Hence we have a map

$$\left(\alpha_{ij}\right)_* \otimes \text{Id}_{\mu(F_2(j) \mid F_2(o))} : S\left(F_1(\varepsilon(i))/F_1(\varepsilon(j))\right) \otimes_C \mu(F_1(\varepsilon(j)) \mid F_1(\varepsilon(o))) \to S(F_2(i)/F_2(j)) \otimes_C \mu(F_2(j) \mid F_2(o)).$$

These maps (over all elements $i \geq j \in I_2$) are compatible if we take projective limits according to the formulae that define the spaces $S_{F_1(\varepsilon(o))}(E_1)$ and $S_{F_2(o)}(E_2)$. Thus we get a map $\alpha_*$ constructed from the maps $\left(\alpha_{ij}\right)_* \otimes \text{Id}_{\mu(F_2(j) \mid F_2(o))}$. The proposition is proved.

Remark 20. Dually to the maps above, we obtain a well-defined map

$$\alpha_*' : S'_{F_2(o)}(E_2) \to S'_{F_1(\varepsilon(o))}(E_1)$$

such that $\alpha_*$ and $\alpha_*'$ are conjugate with respect to the pairings $\langle \cdot, \cdot \rangle_{S_{F_2(o)}(E_2)}$ and $\langle \cdot, \cdot \rangle_{S'_{F_1(\varepsilon(o))}(E_1)}$. 
§ 11. Composition of maps and base-change rules

11.1. Base-change rules. Let

\[ 0 \longrightarrow E_1 \overset{\alpha}{\longrightarrow} E_2 \overset{\beta}{\longrightarrow} E_3 \longrightarrow 0 \]

be an admissible triple in \( C^r_2 \), where \( E_i = (I_i, F_i, V_i) \), \( 1 \leq i \leq 3 \). Let

\[ 0 \longrightarrow D \overset{\gamma}{\longrightarrow} E_3 \overset{\delta}{\longrightarrow} B \longrightarrow 0 \]

be an admissible triple in \( C^r_2 \), where \( D = (R, S, Y) \), \( B = (T, U, W) \). Then we have the following commutative diagram of morphisms in \( C^r_2 \):

\[
\begin{array}{cccccc}
0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to E_1 \overset{\gamma_\alpha}{\to} E_2 \times D \overset{\gamma_\beta}{\to} D \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to E_1 \overset{\alpha}{\to} E_2 \overset{\beta}{\to} E_3 \to 0
\end{array}
\]

(98)

Put \( X' = E_2 \times E_3 \) \( D = (N, Q, X) \) as an object of \( C^r_2 \). The two horizontal triples and two vertical triples in the diagram (98) are admissible in \( C^r_2 \).

Take any elements \( i \leq j \in I_2 \). Then the diagram (98) induces the following commutative diagram of morphisms in \( C^r_1 \) (see the construction of a fibred product in Lemma 4):

\[
\begin{array}{cccccc}
0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to F_2(j) \cap V_1 \overset{\gamma_{\alpha,j_i}}{\to} F_2(j) \cap X \overset{\gamma_{\beta,j_i}}{\to} \beta(F_2(j)) \cap Y \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to F_2(j) \cap V_1 \overset{\alpha_{j_i}}{\to} F_2(j) \overset{\beta_{j_i}}{\to} \beta(F_2(j)) \to 0
\end{array}
\]

(99)
Here the two horizontal triples and two vertical triples are admissible in $C_{1}^{\text{ar}}$. We also have an equality

$$\frac{F_{2}(j) \cap X}{F_{2}(i) \cap X} = \frac{F_{2}(j)}{F_{2}(i)} \times \frac{\beta(F_{2}(j)) \cap Y}{\beta(F_{2}(i)) \cap Y}$$

of objects of $C_{1}^{\text{ar}}$.

**Proposition 21.** In the notation of the diagram (98) let $E_{1}$ be a $c$-object and let $B$ be a $d$-object. Take $o \in I_{2}$, $\mu \in \mu(F_{2}(o) \cap V_{1} \mid V_{1})$, $\nu \mu(\delta \beta(F_{2}(o)) \mid \{0\})$. Then

1) for every $f \in S_{F_{2}(o)}(E_{2})$,

$$\gamma^{*}(\beta_{*}(f \otimes \mu) \otimes \nu) = (\gamma_{*})_{*}(\beta_{*}(f \otimes \nu) \otimes \mu), \quad (100)$$

2) for every $G \in S_{\beta(F_{2}(o)) \cap Y}(D)$,

$$\beta^{*}(\gamma_{*}(G \otimes \nu) \otimes \mu) = (\beta_{*})_{*}(\gamma_{*}(G \otimes \mu) \otimes \nu). \quad (101)$$

**Remark 21.** The statement of Proposition 21 is well defined (with respect to the definition of the direct and inverse image morphisms) since $\alpha = \beta_{*} \circ \gamma_{*}$, $\beta_{*} = \delta \circ \beta$.  

**Proof of Proposition 21.** We first prove (100). Let $j_{0} \in I_{2}$ be such that $F_{2}(j_{0}) \cap V_{1} = V_{1}$ (such an element exists since $E_{1}$ is a $c$-object). Let $i_{0} \in I_{2}$ be such that $\beta\delta(F_{2}(i_{0})) = \{0\}$ (such an element exists since $B$ is a $d$-object). We consider any elements $j \geq i \in I_{2}$ with $j \geq j_{0}$, $i \leq i_{0}$. Then the base-change formula on objects of $C_{1}^{\text{ar}}$ holds for the maps $\beta_{ji}$, $\gamma_{ji}$, $(\beta_{*})_{ji}$, $(\gamma_{*})_{ji}$ from the Cartesian square of the diagram (99). (We must use an analogue of formula (27) for objects of $C_{1}^{\text{ar}}$; see Remark 8.) We multiply the resulting formula by the corresponding spaces of measures and take the projective limit with respect to all $j \geq i \in I_{2}$, $j \geq j_{0}$, $i \leq i_{0}$. Using the constructions of the direct and inverse image maps $\beta_{*}$ and $\gamma_{*}$ in Propositions 17, 18, we obtain the base-change formula (100).

Formula (101) is dual to (100) and can be proved in the same way if we take the inductive limit with respect to $j, i \in I_{2}$ and use an analogue of (28) instead of the analogue of (27) (see Remark 8). The proposition is proved.

**Proposition 22.** In the notation of the diagram (98) let $E_{1}$ be a $cf$-object and let $B$ be a $df$-object. Take $o \in I_{2}$. Then

1) for all $f \in S_{\beta(F_{2}(o)) \cap Y}(D)$,

$$\beta^{*}(\gamma_{*}(f) = (\beta_{*})_{*}(\gamma_{*}(f), \quad (102)$$

2) for all $G \in S_{F_{2}(o)}(E_{2})$,

$$\gamma^{*}(\beta_{*}(G) = (\beta_{*})_{*}(\beta_{*}^{*}(G). \quad (103)$$

**Proof.** We first prove (102). Consider arbitrary elements $j \geq i \in I_{2}$. Then the base-change formula on objects of $C_{1}^{\text{ar}}$ holds for the maps $\beta_{ji}$, $\gamma_{ji}$, $(\beta_{*})_{ji}$, $(\gamma_{*})_{ji}$ from the Cartesian square of the diagram (99). (We must use an analogue of formula (33) for objects of $C_{1}^{\text{ar}}$; see Remark 8.) Taking the projective limit with respect to such elements $j \geq i \in I_{2}$ and using the construction of the inverse and direct image maps $\beta^{*}$ and $\gamma_{*}$ in Propositions 19, 20, we obtain the base-change formula (102).

Formula (103) can be obtained in a similar way if we take the inductive limit with respect to $i, j \in I_{2}$ and use an analogue of (34) instead of the analogue of (33) (see Remark 8). The proposition is proved.
Proposition 23. In the notation of the diagram (98) let $E_1$ be a $c$-object and let $B$ be a $df$-object. Take $o \in I_2$ and $\mu \in \mu(F_2(o) \cap V_1 | V_1)$. Then
1) for all $f \in S_{F_2(o)}(X')$ and $\mu \in \mu(E_1)$,
$$\beta_\ast((\beta_\gamma)_\ast(f \otimes \mu)) = \gamma_\ast((\beta_\gamma)_\ast(f \otimes \mu)), \quad (104)$$

2) for all $G \in S'_{F_2(o)}(E_3)$ and $\mu \in \mu(E_1)$,
$$\gamma_\beta_\ast(\gamma_\ast(G) \otimes \mu) = \beta_\gamma_\ast(\beta_\ast(G) \otimes \mu). \quad (105)$$

The proof is analogous to the proofs of Propositions 21, 22. It can be reduced to appropriate analogues of formula (31) for objects of $C_1^{ar}$ (to prove formula (104)) and formula (32) for objects of $C_1^{ar}$ (to prove formula (105)); see Remark 8. The proposition is proved.

Proposition 24. In the notation of the diagram (98) let $E_1$ be a $cf$-object and let $B$ be a $d$-object. Take $o \in I_2$ and $\nu \in \mu(\delta_\beta(F_2(o)) | \{0\})$. Then
1) for all $f \in S_{\beta(F_2(o))}(E_3)$,
$$\gamma_\beta_\ast(\gamma_\ast(f \otimes \nu)) = \beta_\gamma_\ast(\beta_\ast(f \otimes \nu)), \quad (106)$$

2) for all $G \in S'_{F_2(o)}(X')$,
$$\beta_\ast((\beta_\gamma)_\ast(G \otimes \nu)) = \gamma_\ast((\gamma_\beta)_\ast(G) \otimes \nu). \quad (107)$$

The proof is analogous to the proofs of Propositions 21, 22. It can be reduced to appropriate analogues of formula (29) for objects of $C_1^{ar}$ (to prove formula (106)) and formula (30) for objects of $C_1^{ar}$ (to prove formula (107)); see Remark 8. The proposition is proved.

11.2. Composition of maps. We again consider the diagram (98). If $E_1$ and $D$ are $c$-objects, then so is $E_2 \times_{E_3} D$. This is a consequence of the following admissible triple in $C_2^{ar}$:

$$0 \to E_1 \xrightarrow{\gamma_\alpha} E_2 \times_{E_3} D \xrightarrow{\gamma_\beta} D \to 0. \quad (108)$$

Take $o \in I_2$. Then (108) yields a canonical isomorphism
$$\mu(F_2(o) \cap X | X) = \mu(F_2(o) \cap V_1 | V_1) \otimes_\mathbb{C} \mu(\beta(F_2(o)) \cap Y | Y). \quad (109)$$

(The subspaces appearing in (109) are elements of the filtration of the corresponding objects of $C_2^{ar}$. Hence the spaces of virtual measures are well defined; see § 6.)

The diagram (98) yields the following admissible triple in $C_2^{ar}$:

$$0 \to E_2 \times_{E_3} D \xrightarrow{\beta_\gamma} E_2 \xrightarrow{\delta_\beta} B \to 0. \quad (110)$$

Proposition 25. In the notation of the diagram (98) let $E_1$ and $D$ be $c$-objects. Take $o \in I_2$, $\mu \in \mu(F_2(o) \cap V_1 | V_1)$, $\nu \in \mu(\beta(F_2(o)) \cap Y | Y)$. Then
1) for all $f \in S_{F_2(o)}(E_2)$,
$$((\delta_\beta)_\ast(f \otimes (\mu \otimes \nu))) = \delta_\ast(\beta_\ast(f \otimes \mu) \otimes \nu), \quad (110)$$

2) for all $G \in S'_{\beta(F_2(o))}(B)$,
$$((\delta_\beta)_\ast(G \otimes (\mu \otimes \nu))) = \beta_\ast((\delta_\ast(G) \otimes \nu) \otimes \mu). \quad (111)$$

Remark 22. We have $\mu \otimes \nu \in \mu(F_2(o) \cap X | X)$ by (109).
Proof of Proposition 25. Let $j_0 \in I_2$ be such that $F_2(j_0) \cap V_1 = V_1$ and $\beta(F_2(j_0)) \cap Y = Y$. We take arbitrary elements $j \geq i \in I_2$ such that $j \geq j_0$. Then an analogue of formula (35) for objects of $C_1^{ar}$ (see Remark 8) holds for the maps $\beta_{ji}$, $\delta_{ji}$ and $(\beta_{ji})_{ji} = \delta_{ji} \beta_{ji}$ (see the diagram (99)). We multiply the resulting formula by the corresponding spaces of measures and take the limit with respect to the elements $j \geq i \in I_2$, $j \geq j_0$. Using the explicit construction of the direct image morphism in Proposition 17, we get formula (110).

Formula (111) can be obtained in a similar way if we use an analogue of formula (36) for objects of $C_1^{ar}$ (see Remark 8). The proposition is proved.

In the notation of the diagram (98) suppose that $E_1$ and $D$ are $cf$-objects. Then the admissible triple (108) in $C_2^{ar}$ yields that $E_2 \times_{E_3} D$ is a $cf$-object.

**Proposition 26.** In the notation of the diagram (98) let $E_1$ and $D$ be $cf$-objects. Take $o \in I_2$. Then

1) for all $f \in S_{\delta \beta(F_2(o))}(B)$,

$$ (\delta \beta)^*(f) = \beta^* \delta^*(f), \quad (112) $$

2) for all $G \in S'_{F_2(o)}(E_2)$,

$$ (\delta \beta)_*(G) = \delta_* \beta_*(G). \quad (113) $$

**Proof.** Consider arbitrary elements $j \geq i \in I_2$. Then the analogue of formula (37) for objects of $C_1^{ar}$ (see Remark 8) holds for the maps $\beta_{ji}$, $\delta_{ji}$ and $(\beta_{ji})_{ji} = \delta_{ji} \beta_{ji}$ (see the diagram (99)). We multiply this formula by the appropriate spaces of measures and take the limit with respect to $j \geq i \in I_2$. Using the explicit construction of the inverse image morphism in Proposition 19, we obtain (112).

Formula (113) is proved in a similar way using the analogue of formula (38) for objects of $C_1^{ar}$ (see Remark 8). The proposition is proved.

Modifying (and adapting the notation in) the diagram (98), we get the following commutative diagram of morphisms between objects of $C_2^{ar}$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E_1 & \xrightarrow{\alpha} & E_2 & \xrightarrow{\beta} & E_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \beta_{o'} & & \\
0 & \longrightarrow & E_1 & \xrightarrow{\alpha' \alpha} & H' & \xrightarrow{\alpha' \beta} & E_3 \bigoplus_{E_2} H' & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \theta & & \\
L' & \longrightarrow & L' & \longrightarrow & 0 & & & & \\
\end{array}
$$

(114)

The two horizontal and two vertical triples in (114) are admissible in $C_2^{ar}$. 

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Let $E_i = (I_i, F_i, V_i)$, $1 \leq i \leq 3$, $H' = (J', T', W')$ be objects of $C^a'_2$. Using the triple

$$0 \longrightarrow E_3 \xrightarrow{\beta_{\alpha'}} E_3 \coprod_{E_2} H' \xrightarrow{\theta'} L' \longrightarrow 0,$$  \hspace{1cm} (115)

which is admissible in $C^a'_2$, we see that if $E_3$ and $L'$ are $d$-objects, then so is $E_3 \coprod_{E_2} H'$.

Let $o \in J'$ be any element. Then (115) canonically yields that

$$\mu(\alpha'_\beta(T'(o) | \{0\})) = \mu(\beta(T'(o) \cap V_2 | \{0\}) \otimes \mu(\theta(T'(o)) | \{0\}).$$  \hspace{1cm} (116)

(The subspaces appearing in (116) are elements of the filtration of the corresponding objects of $C^a'_2$. Therefore the spaces of virtual measures are well defined; see § 6.)

The diagram (114) yields an admissible triple in $C^a'_2$ of the form

$$0 \longrightarrow E_1 \xrightarrow{\alpha'_\alpha} H' \xrightarrow{\alpha'_\beta} E_3 \coprod_{E_2} H' \longrightarrow 0.$$

**Proposition 27.** In the notation of the diagram (114) let $E_3$ and $L'$ be $d$-objects. Suppose that $o \in J'$, $\mu \in \mu(\beta(T'(o) \cap V_2 | \{0\})$, $\nu \in \mu(\theta(T'(o)) | \{0\})$. Then

1) for every $f \in S_{T'(o)}(H')$,

$$(\alpha'\alpha)^*(f \otimes (\mu \otimes \nu)) = \alpha^*(\alpha'^* (f \otimes \nu) \otimes \mu),$$  \hspace{1cm} (117)

2) for every $G \in S'_{T'(o) \cap V_1}(E_1)$,

$$(\alpha'\alpha)_*(G \otimes (\mu \otimes \nu)) = (\alpha')_* (\alpha_*(G \otimes \mu) \otimes \nu).$$  \hspace{1cm} (118)

**Remark 23.** We have $\mu \otimes \nu \in \mu(\alpha'_\beta(T'(o) | \{0\})$ because of (116).

**Proof of Proposition 27.** Formula (117) can be reduced to the analogue of formula (40) for objects of $C^a'_1$ (see Remark 8) if we take the projective limit with respect to elements $j \geq i \in J'$ with $i \leq i_0$, where $i_0$ is chosen in such a way that $\theta(T'(i_0)) = \{0\}$ and $\beta(T'(i_0) \cap V_2) = \{0\}$. Here we use the explicit construction of the inverse image morphism in Proposition 18 (compare with the proof of Proposition 26).

Formula (118) is proved in a similar way using the analogue of formula (41) for objects of $C^a'_1$ (see Remark 8) and taking the inductive limit. The proposition is proved.

In the notation of the diagram (114) suppose that $E_3$ and $L'$ are $df$-objects. Then the triple (115) (which is admissible in $C^a'_2$) yields that $E_3 \coprod_{E_2} H'$ is a $df$-object.

**Proposition 28.** In the notation of the diagram (114) let $E_3$ and $L'$ be $df$-objects. Suppose that $o \in J'$. Then

1) for every $f \in S_{T'(o) \cap V_1}(E_1)$,

$$(\alpha'\alpha)_*(f) = (\alpha')_* \alpha_*(f),$$  \hspace{1cm} (119)

2) for every $G \in S'_{T'(o)}(H')$,

$$(\alpha'\alpha)^*(G) = \alpha^*(\alpha')^*(G).$$  \hspace{1cm} (120)
Proof. Formula (119) can be reduced to the analogue of formula (42) for objects of $C_{\lambda r}^1$ (see Remark 8) if we choose elements $j \geq i \in J'$ and take the projective limit with respect to elements $j \geq i \in J'$. Here we use the explicit construction of the direct image morphism in Proposition 20.

Formula (120) is proved in a similar way taking the inductive limit with respect to $j \geq i \in J'$ and using the analogue of (43) instead of the analogue of (42) (see Remark 8). The proposition is proved.

§ 12. The Fourier transform and the direct and inverse images

Let

$$0 \longrightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \longrightarrow 0$$

be an admissible triple in $C_{\lambda r}^2$ with $E_i = (I_i, F_i, V_i), 1 \leq i \leq 3$. Suppose that $E_i, 1 \leq i \leq 3$, are complete objects of $C_{\lambda r}^2$. Then $\tilde{E}_i = E_i, 1 \leq i \leq 3$.

By definition there are order-preserving functions

(i) $\gamma: I_2 \rightarrow I_3$ such that $\beta(F_2(i)) = F_3(\gamma(i))$ for all $i \in I_2$,

(ii) $\varepsilon: I_2 \rightarrow I_1$ such that $F_2(i) \cap V_1 = F_1(\varepsilon(i))$ for all $i \in I_2$.

We consider the following admissible triple in $C_{\lambda r}^2$:

$$0 \longrightarrow \tilde{E}_3 \xrightarrow{\beta} \tilde{E}_2 \xrightarrow{\alpha} \tilde{E}_1 \longrightarrow 0.$$ We have $\tilde{E}_i = (I_i^0, F_i^0, \tilde{V}_i), 1 \leq i \leq 3$. Recall that the partially ordered set $I_i^0, 1 \leq i \leq 3$, is equal to $I_i$ as a set, but its order is the reverse of that of $I_i, 1 \leq i \leq 3$. We obtain that, for every element $l \in I_2$,

$$\tilde{\alpha}(F_2^0(l)) = F_1^0(\varepsilon(l)), \quad F_2^0(l) \cap \tilde{V}_3 = F_3^0(\gamma(l)).$$

(121)

For all $i, j \in I_2$ we see from (77) that

$$\mu(F_1(\varepsilon(i)) | F_1(\varepsilon(j))) = \mu(F_1^0(\varepsilon(i)) | F_1^0(\varepsilon(j))),$$

$$\mu(F_3(\gamma(i)) | F_3(\gamma(j))) = \mu(F_3^0(\gamma(i)) | F_3^0(\gamma(j))).$$

(122)

We have the following assertions.

Proposition 29. Suppose that $a \in I_2$.

1) If $E_1$ is a $c$-object, then the diagram

$$S_{F_2(a)}(E_2) \otimes \mathbb{C} \mu(F_1(\varepsilon(o)) | V_1) \xrightarrow{\beta^*} S_{F_3(\gamma(o))}(E_3)$$

$$\downarrow \quad F \otimes \text{Id}_{\mu(F_1(\varepsilon(o)) | V_1)}$$

$$S_{F_2^0(a)}(\tilde{E}_2) \otimes \mathbb{C} \mu(F_1^0(\varepsilon(o)) | \{0\}) \xrightarrow{(\tilde{\beta})^*} \mathcal{D}_{F_3^0(\gamma(o))}(\tilde{E}_3)$$

is commutative.

2) If $E_3$ is a $d$-object, then the diagram

$$S_{F_2(a)}(E_2) \otimes \mathbb{C} \mu(F_3(\gamma(o)) | \{0\}) \xrightarrow{\alpha^*} S_{F_1(\varepsilon(o))}(E_1)$$

$$\downarrow \quad F \otimes \text{Id}_{\mu(F_3(\gamma(o)) | \{0\})}$$

$$S_{F_2^0(a)}(\tilde{E}_2) \otimes \mathbb{C} \mu(F_3^0(\gamma(o)) | \tilde{V}_3) \xrightarrow{(\tilde{\alpha})^*} S_{F_1^0(\varepsilon(o))}(\tilde{E}_1)$$

is commutative.
3) If $E_1$ is a $c$-object, then the diagram
\[
S'_{F_3(\gamma(o))}(E_3) \otimes \mu(F_1(\varepsilon(o)) \mid V_1) \xrightarrow{\beta^*} S'_{F_2(o)}(E_2) \\
F \otimes \text{Id}_{\mu(F_1(\varepsilon(o)) \mid V_1)} \downarrow F \\
S'_{F_3^0(\gamma(o))}(\tilde{E}_3) \otimes \mu(F_1^0(\varepsilon(o)) \mid \{0\}) \xrightarrow{(\beta)^*} S'_{F_2^0(o)}(\tilde{E}_2)
\]
is commutative.

4) If $E_3$ is a $d$-object, then the diagram
\[
S'_{F_1(\varepsilon(o))}(E_1) \otimes \mu(F_3(\gamma(o)) \mid \{0\}) \xrightarrow{\alpha^*} S'_{F_2(o)}(E_2) \\
F \otimes \text{Id}_{\mu(F_3(\gamma(o)) \mid \{0\})} \downarrow F \\
S'_{F_1^0(\varepsilon(o))}(\tilde{E}_1) \otimes \mu(F_3^0(\gamma(o)) \mid \tilde{V}_3) \xrightarrow{(\alpha)^*} S'_{F_2^0(o)}(\tilde{E}_2)
\]
is commutative.

5) If $E_1$ is a $cf$-object, then the diagram
\[
S_{F_3(\gamma(o))}(E_3) \xrightarrow{\beta^*} S_{F_2(o)}(E_2) \\
F \downarrow F \\
S_{F_3^0(\gamma(o))}(\tilde{E}_3) \xrightarrow{(\beta)^*} S_{F_2^0(o)}(\tilde{E}_2)
\]
is commutative.

6) If $E_3$ is a $df$-object, then the diagram
\[
S_{F_1(\varepsilon(o))}(E_1) \xrightarrow{\alpha^*} S_{F_2(o)}(E_2) \\
F \downarrow F \\
S_{F_1^0(\varepsilon(o))}(\tilde{E}_1) \xrightarrow{(\alpha)^*} S_{F_2^0(o)}(\tilde{E}_2)
\]
is commutative.

7) If $E_1$ is a $cf$-object, then the diagram
\[
S'_{F_2(o)}(E_2) \xrightarrow{\beta^*} S'_{F_3(\gamma(o))}(E_3) \\
F \downarrow F \\
S'_{F_2^0(o)}(\tilde{E}_2) \xrightarrow{(\beta)^*} S'_{F_3^0(\gamma(o))}(\tilde{E}_3)
\]
is commutative.

8) If $E_3$ is a $df$-object, then the diagram
\[
S'_{F_2(o)}(E_2) \xrightarrow{\alpha^*} S'_{F_1(\varepsilon(o))}(E_1) \\
F \downarrow F \\
S'_{F_2^0(o)}(\tilde{E}_2) \xrightarrow{(\alpha)^*} S'_{F_1^0(\varepsilon(o))}(\tilde{E}_1)
\]
is commutative.
Proof. This follows from the formulae (121), (122), the constructions of two-dimensional direct and inverse images in §10, the construction of the two-dimensional Fourier transform in §8 and the corresponding analogues of Propositions 10, 11 for objects of $C_{1}^{ar}$ (see Remark 8), which connect the direct and inverse image morphisms with the Fourier transform on objects of $C_{1}^{ar}$.

§ 13. Two-dimensional Poisson formulae

13.1. Poisson formula I. Let $0 \rightarrow E_{1} \xrightarrow{\alpha} E_{2} \xrightarrow{\beta} E_{3} \rightarrow 0$ be an admissible triple in $C_{2}^{ar}$ with $E_{i} = (I_{i}, F_{i}, V_{i})$, $1 \leq i \leq 3$. Suppose that $E_{i}$, $1 \leq i \leq 3$, are complete objects of $C_{2}^{ar}$. By definition there are order-preserving functions

(i) $\gamma: I_{2} \rightarrow I_{3}$ such that $\beta(F_{2}(i)) = F_{3}(\gamma(i))$ for all $i \in I_{2},$

(ii) $\varepsilon: I_{2} \rightarrow I_{1}$ such that $F_{2}(i) \cap V_{1} = F_{1}(\varepsilon(i))$ for all $i \in I_{2}.$

Let $o \in I_{2}$. If we suppose that $E_{1}$ is a $c$-object, then, given any $\mu \in \mu(F_{1}(\varepsilon(o)) | V_{1})$, we canonically construct $1_{\mu} \in S'_{F_{1}(\varepsilon(o))}(E_{1})$ in the following way. By definition, we have

$$S'_{F_{1}(\varepsilon(o))}(E_{1}) = \lim_{k \in I_{1}} S'(V_{1}/F_{1}(k)) \otimes C \mu(F_{1}(\varepsilon(o)) | k)$$  

(123)

and $\mu(F_{1}(\varepsilon(o)) | V_{1}) \subset S'(V_{1}/F_{1}(\varepsilon(o))).$ Therefore $\mu \in S'(V_{1}/F_{1}(\varepsilon(o))).$ We put $k = \varepsilon(o)$ in (123). Then $\mu(F_{1}(\varepsilon(o)) | k) = C$ and we take $1 \in \mu(F_{1}(\varepsilon(o)) | \varepsilon(o)).$ Therefore $\mu$ determines $1_{\mu} \in S'_{F_{1}(\varepsilon(o))}(E_{1})$ by formula (123).

Let $o \in I_{2}$. If we suppose that $E_{3}$ is a $d$-object, then, given any $\nu \in \mu(F_{3}(\gamma(o)) | \{0\})$, we canonically construct $\delta_{\nu} \in S'_{F_{3}(\gamma(o))}(E_{3})$ in the following way. By definition, we have

$$S'_{F_{3}(\gamma(o))}(E_{3}) = \lim_{k \in I_{2}} S'(F_{3}(k)) \otimes C \mu(F_{3}(\gamma(o)) | \{0\}).$$  

(124)

We put $k = \gamma(o)$ in (124). Take $\delta \in S'(F_{3}(\gamma(o)))$ in accordance with (124), where $\langle \delta, f \rangle \overset{\text{def}}{=} f(0), f \in S(F_{3}(\gamma(o))),$ with respect to the pairing (58). (One can also define $\delta$ as the projective limit of Dirac delta functions in accordance with (57).) We also put $\nu \in \mu(F_{3}(\gamma(o)) | \{0\})$ in (124). Thus we have constructed $\delta_{\nu} \in S'_{F_{3}(\gamma(o))}(E_{3}).$

Note that we have

$$F(1_{\mu}) = \delta_{\mu}, \quad F(\delta_{\nu}) = 1_{\nu}$$  

(125)

under the maps

$$F: S'_{F_{1}(\varepsilon(o))}(E_{1}) \rightarrow S'_{F_{1}(\varepsilon(o))}(E_{1}), \quad F: S'_{F_{3}(\gamma(o))}(E_{3}) \rightarrow S'_{F_{3}(\gamma(o))}(E_{3}).$$

Here we have used the equalities

$$\mu(F_{1}(\varepsilon(o)) | V_{1}) = \mu(F_{1}^{0}(\varepsilon(o)) | \{0\}), \quad \mu(F_{3}(\gamma(o)) | \{0\}) = \mu(F_{3}^{0}(\gamma(o)) | V_{3}).$$

We now suppose that $E_{1}$ is a $c$-object and $E_{3}$ is a $d$-object. Let $o \in I_{2}.$ Then, given any elements $\mu \in \mu(F_{1}(\varepsilon(o)) | V_{1})$ and $\nu \in \mu(F_{3}(\gamma(o)) | \{0\})$, we have a well-defined characteristic function

$$\delta_{E_{1}, \mu \otimes \nu} \overset{\text{def}}{=} \alpha_{*}(1_{\mu} \otimes \nu) \in S'_{F_{2}(\varepsilon)}(E_{2}).$$  

(126)
Lemma 5. We have $\delta_{E_1,\mu\otimes\nu} = \beta^*(\delta_\nu \otimes \mu)$.

Proof. This follows from the constructions and the corresponding assertions about the admissible triple

$\begin{array}{c}
0 \longrightarrow V_1 \\
\quad F_1(\varepsilon(k_2)) \\
\quad F_2(k_2) \\
\quad F_3(\gamma(k_1)) \\
\end{array}$

in $C^a_1$, where $k_1 \geq o \geq k_2 \in I_2$ and $F_1(\varepsilon(k_1)) = V_1$, $F_3(\gamma(k_2)) = \{0\}$.

We now have our first two-dimensional Poisson formula.

Theorem 1 (Poisson formula I). Suppose that $E_1$ is a $c$-object and $E_3$ is a $d$-object. Take $o \in I_2$. Then for all $\mu \in \mu(F_1(\varepsilon(o)) \mid V_1)$ and $\nu \in \mu(F_3(\gamma(o)) \mid \{0\})$ we have

$F(\delta_{E_1,\mu\otimes\nu}) = \delta_{E_3,\nu\otimes\mu}$.

Proof. This follows from Lemma 5, formulæ (125) and Proposition 29, which connects the direct and inverse image morphisms with the two-dimensional Fourier transform.

Definition 21. Suppose that $E = (I, F, V) \in \text{Ob}(C^a_2)$. We say that an element $g \in \text{Aut}_{C^a_2}(E)$ satisfies condition (*) if and only if the following conditions hold.

1) $g \in \text{Aut}_{C^a_2}(E)^\prime$ (see Definition 20).

2) Suppose that $i \geq j \in I$ are any elements with $g(F(i)) = F(p)$ and $g(F(j)) = F(q)$ for some elements $p \geq q \in I$. Let $E_{j,i} = (I_{j,i}, F_{j,i}, \mu_1/F(j))$ and $E_{q,p} = (I_{q,p}, F_{q,p}, \mu_2/F(q))$ be the corresponding objects of $C^a_1$ constructed from $E$. Then for every $l \in I_{i,j}$ there is $r_l \in I_{q,p}$ such that $gF_{j,i}(l) = F_{q,p}(r_l)$ and we have $r_{l_1} \leq r_{l_2} \in I_{q,p}$ whenever $l_1 \leq l_2 \in I_{i,j}$.

Remark 24. It is easy to see that the subgroups of $\text{Aut}_{C^a_2}(E)$ considered in Remark 16 for $E \in \text{Ob}(C^a_2)$ consist of elements satisfying condition (*).

Take any admissible triple of complete objects in $C^a_2$:

$\begin{array}{c}
0 \longrightarrow E_1 \overset{\alpha}{\longrightarrow} E_2 \overset{\beta}{\longrightarrow} E_3 \longrightarrow 0,
\end{array}$

where $E_i = (I_i, F_i, V_i)$, $1 \leq i \leq 3$, $E_1$ is a $c$-object and $E_3$ is a $d$-object. Suppose that $g \in \text{Aut}_{C^a_2}(E_2)$ satisfies condition (*) (see Definition 21). Then we have the following admissible triple in $C^a_2$:

$\begin{array}{c}
0 \longrightarrow gE_1 \overset{g(\alpha)}{\longrightarrow} gE_2 \overset{g(\beta)}{\longrightarrow} gE_3 \longrightarrow 0,
\end{array}$

(127)

where $gE_2 = E_2$, $gE_3 = E_2/gE_1$.

We canonically have from (127) that

$\mu(F_2(0) \mid gF_2(o)) = \mu(F_2(o) \cap gV_1 \mid gF_1(\varepsilon(o))) \otimes \mu(g(\beta)(F_2(o)) \mid gF_3(\gamma(o)))$.

For every $a \in \mu(F_2(0) \mid gF_2(0))$ we consider $a = b \otimes c$, where

$b \in \mu(F_2(o) \cap gV_1 \mid gF_1(\varepsilon(o)))$, \hspace{1cm} c \in \mu(g(\beta)(F_2(o)) \mid gF_3(\gamma(o)))$.

Using the hypotheses and notation of Theorem 1 we have

$g\mu \in \mu(gF_1(\varepsilon(o)) \mid gV_1)$, \hspace{1cm} g\nu \in \mu(gF_3(\gamma(o)) \mid \{0\})$.

Since $\tilde{g} = (g, a) \in \text{Aut}_{C^a_2}(E_2)^\prime_{F_2(o)}$, we obtain that

$R^2_g(\delta_{E_1,\mu\otimes\nu}) = \delta_{gE_1,g(\mu\otimes b)\otimes g(\nu\otimes c)} = \delta_{gE_1,g\mu\otimes g\nu\otimes a}$.
Corollary 2. Suppose that $g \in \Aut_{C_2^{ar}}(E_2)$ satisfies condition $(\ast)$ of Definition 21 and $(g,a) \in \Aut_{C_2^{ar}}(E_2)(F_2(o))$. Then we have

$$F(\delta g E_1.g \mu \odot g \nu \odot a) = \delta \tilde{g}^{-1}(E_3).\tilde{g}^{-1}(\nu) \odot \tilde{g}^{-1}(\mu) \odot a.$$ 

Proof. This follows from Theorem 1 and §9.5.

13.2. Poisson formula II. Let $0 \to E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \to 0$ be an admissible triple in $C_2^{ar}$ with $E_i = (I_i,F_i,V_i)$, $1 \leq i \leq 3$. Suppose that $E_i$, $1 \leq i \leq 3$, are complete objects of $C_2^{ar}$. By definition, there are order-preserving functions

(i) $\gamma: I_2 \to I_3$ such that $\beta(F_2(i)) = F_3(\gamma(i))$ for all $i \in I_2$,

(ii) $\varepsilon: I_2 \to I_1$ such that $F_2(i) \cap V_1 = F_1(\varepsilon(i))$ for all $i \in I_2$.

If $E_1$ is a $cf$-object, then, using Remark 13, we have

$$S(E_1) = \lim_{i \geq j \in I_1} S(F_1(i)/F_1(j)).$$

Since $F_1(i)/F_1(j)$ is a compact object of $C_1^{ar}$ (for all elements $i \geq j \in I_1$), we have $1 \in S(F_1(i)/F_1(j))$. Taking the projective limit, we see that the element

$$1 \in S(E_1)$$

is well defined in this case.

If $E_3$ is a $df$-object, then, using Remark 13, we have

$$S(E_3) = \lim_{i \geq j \in I_3} S(F_3(i)/F_3(j)).$$

Since $F_3(i)/F_3(j)$ is a discrete object of $C_1^{ar}$ (for all elements $i \geq j \in I_3$), we have $\delta_0 \in S(F_3(i)/F_3(j))$, where $\delta_0(0) = 1$ and $\delta_0(x) = 0$ for $x \neq 0$ (here $x$ belongs to the underlying Abelian group of the object $F_3(i)/F_3(j)$ of $C_1^{ar}$). In this case, taking the projective limit, we get a well-defined element

$$\delta_0 \in S(E_3).$$

Note that we have

$$F(1) = \delta_0, \quad F(\delta_0) = 1$$

under the maps

$$F: S(E_1) \to S(\tilde{E}_1), \quad F: S(E_3) \to S(\tilde{E}_3).$$

We now suppose that $E_1$ is a $cf$-object and $E_3$ is a $df$-object. Let $o \in I_2$. Then the element

$$\delta_{E_1}^{\text{def}} = \alpha_*(1) \in S_{F_2(o)}(E_2)$$

is well defined.

Lemma 6. We have $\delta_{E_1} = \beta^*(\delta_0)$.

Proof. This follows from the constructions and the corresponding assertions for the admissible triples

$$0 \to \frac{F_1(\varepsilon(i))}{F_1(\varepsilon(j))} \to \frac{F_2(i)}{F_2(j)} \to \frac{F_3(\gamma(i))}{F_3(\gamma(j))} \to 0$$

in $C_1^{ar}$ (for any elements $i \geq j \in I_2$), where $\frac{F_1(\varepsilon(i))}{F_1(\varepsilon(j))}$ is a compact object of $C_1^{ar}$ and $\frac{F_3(\gamma(i))}{F_3(\gamma(j))}$ is a discrete object of $C_1^{ar}$.
**Theorem 2** (Poisson formula II). Suppose that $E_1$ is a $cf$-object and $E_3$ is a $df$-object. Let $o \in I_2$. Then
\[ F(\delta_{E_1}) = \delta_{E_3}. \]

**Proof.** This follows from Lemma 6, formulae (130) and Proposition 29 on morphisms of the direct and inverse images and the two-dimensional Fourier transform.

Consider any admissible triple of complete objects of $C^\text{ar}_2$:
\[ 0 \rightarrow E_1 \xrightarrow{\alpha} E_2 \xrightarrow{\beta} E_3 \rightarrow 0, \]
where $E_i = (I_i, F_i, V_i), 1 \leq i \leq 3$, $E_1$ is a $cf$-object and $E_3$ is a $df$-object. Suppose that $g \in \text{Aut}_{C^\text{ar}_2}(E_2)$ satisfies condition $(\ast)$ of Definition 21. Then we have the following admissible triple in $C^\text{ar}_2$:
\[ 0 \rightarrow gE_1 \xrightarrow{g(\alpha)} gE_2 \xrightarrow{g(\beta)} gE_3 \rightarrow 0, \quad (132) \]
where $gE_2 = E_2$, $gE_3 = E_2/gE_1$.

**Corollary 3.** Suppose that $g \in \text{Aut}_{C^\text{ar}_2}(E_2)$ satisfies condition $(\ast)$ of Definition 21. Then
\[ F(\delta_{gE_1}) = \delta_{g^{-1}(E_3)}. \]

**Proof.** This follows from Theorem 2 and § 9.5.

**§ 14. An example**

Here we compute some quotient groups of the adèlle group $\mathbb{A}_X$ of a two-dimensional scheme $X$.

14.1. Some quotient groups of the adèle group of an algebraic surface. We recall that the group $\mathbb{A}_K/K$ for a number field $K$ was computed in Example 4.

Let $D$ be an irreducible projective curve over a field $k$. Let $p \in D$ be a smooth $k$-rational point. Arguing as in Example 4, we get an exact sequence
\[ 0 \rightarrow \prod_{q \in D, q \neq p} \hat{O}_q \rightarrow \mathbb{A}_D/k(D) \rightarrow K_p/A_p \rightarrow 0, \quad (133) \]
where $k(D)$ is the field of rational functions on $D$, $\hat{O}_q$ is the completion of the local ring $\mathcal{O}_q$ of a point $q \in D$ by the maximal ideal $m_q$, $K_p = k(D)_p$ is the field of fractions of the ring $\hat{O}_p$ if $p$ is a smooth point (in the general case, $K_p$ is the localization of the ring $\hat{O}_p$ with respect to the multiplicative system $\mathcal{O}_p \setminus 0$) and $A_p$ is the ring of regular functions on the affine curve $D \setminus p$. Note that $K_p \simeq k((t))$.

If the field $k$ is finite, then all the terms of (133) possess the structure of compact objects of $C^\text{ar}_1$, whence (133) is an admissible triple in $C^\text{ar}_1$.

Note that the adèlle ring $\mathbb{A}_D$ and the field of rational functions $k(D)$ are groups occurring in the adèlle complex of $D$. Indeed, for every quasi-coherent sheaf $\mathcal{F}$ on $D$ there is an adèlle complex (see [10], § 3.1, [7]–[9])
\[ \mathbb{A}_{D,0}(\mathcal{F}) \oplus \mathbb{A}_{D,1}(\mathcal{F}) \rightarrow \mathbb{A}_{D,01}(\mathcal{F}) \]
whose cohomology groups coincide with the cohomology groups \( H^*(D, \mathcal{F}) \). For the sheaf \( \mathcal{F} = \mathcal{O}_D \) we have

\[
\mathbb{A}_{D,0}(\mathcal{O}_D) = k(D), \quad \mathbb{A}_{D,1}(\mathcal{O}_D) = \prod_{q \in D} \hat{\mathcal{O}}_q, \quad \mathbb{A}_{D,01}(\mathcal{O}_D) = A_D = \prod_{q \in D} K_q,
\]

where \( \prod' \) stands for the restricted (adéle) product. Thus we obtain that

\[
\mathbb{A}_D/k(D) = \mathbb{A}_{D,01}(\mathcal{O}_D)/\mathbb{A}_{D,0}(\mathcal{O}_D).
\]

Now let \( X \) be a two-dimensional scheme. For every quasi-coherent sheaf \( \mathcal{F} \) on \( X \) there is an adéle complex (see [9], [10], §3.3)

\[
\mathbb{A}_X,0(\mathcal{F}) \oplus \mathbb{A}_X,1(\mathcal{F}) \oplus \mathbb{A}_X,2(\mathcal{F}) \to \mathbb{A}_X,0(\mathcal{F}) \oplus \mathbb{A}_X,02(\mathcal{F}) \oplus \mathbb{A}_X,12(\mathcal{F}) \to \mathbb{A}_X,012(\mathcal{F})
\]

whose cohomology groups coincide with the cohomology groups \( H^*(X, \mathcal{F}) \).

Let \( X \) be a smooth irreducible surface over a field \( k \) and \( C \) a divisor on \( X \). For the sheaf \( \mathcal{F} = \mathcal{O}_X(C) \) we define the components of the adéle complex (135) as subgroups of \( \prod_{x \in D} K_{x,D} \), where the product is taken over all pairs consisting of an irreducible curve \( D \) on \( X \) and a point \( x \) of \( D \). For every such pair \( x, D, x \in D \), the ring \( K_{x,D} \) is a finite product of two-dimensional local fields. If \( x \) is a smooth point of \( D \), then \( K_{x,D} = k(x)((u))((t)) \) is a two-dimensional local field (see [10], §§2.2, 3.3). We now define

\[
\mathbb{A}_{X,012}(\mathcal{O}_X(C)) = A_X = \prod_{x \in D} K_{x,D},
\]

where the adéle product \( \prod' f_{x,D} \) is given inside the ordinary product \( \prod f_{x,D} \) by the following conditions.

1) If \( D \subset X \) is a fixed irreducible curve and \( t_D = 0 \) is a local equation of \( D \) on some open affine subset \( U \subset X \), then \( \{f_{x,D}\} \in \mathbb{A}_D((t_D)) \).

2) We have \( \{f_{x,D}\} \in \mathbb{A}_D[[t_D]] \) for all but finitely many irreducible curves \( D \subset X \).

Let \( \hat{\mathcal{O}}_D \) be the completion of the local ring of an irreducible curve \( D \subset X \), \( K_D \) its field of fractions, \( \hat{\mathcal{O}}_x \) the completion of the local ring of a point \( x \in X \), \( K_x = \hat{\mathcal{O}}_x \cdot k(X) \) inside the field of fractions of \( \hat{\mathcal{O}}_x \), and \( \hat{\mathcal{O}}_{x,D} \) the product of the discrete valuation rings of the two-dimensional local fields belonging to the ring \( K_{x,D} \).

The following groups are diagonally embedded in the group \( A_X \):

\[
\mathbb{A}_{X,0}(\mathcal{O}_X(C)) = k(X), \quad \mathbb{A}_{X,1}(\mathcal{O}_X(C)) = \prod_{D \subset X} \hat{\mathcal{O}}_D \otimes_{\mathcal{O}_X} \mathcal{O}_X(C),
\]

\[
\mathbb{A}_{X,2}(\mathcal{O}_X(C)) = \prod_{x \in X} \hat{\mathcal{O}}_x \otimes_{\mathcal{O}_X} \mathcal{O}_X(C),
\]

\[
\mathbb{A}_{X,01}(\mathcal{O}_X(C)) = \prod_{D \subset X} K_D := \left( \prod_{D \subset X} K_D \right) \cap A_X,
\]

\[
\mathbb{A}_{X,12}(\mathcal{O}_X(C)) = \prod_{x \in D} \hat{\mathcal{O}}_{x,D} \otimes_{\mathcal{O}_X} \mathcal{O}_X(C) := \left( \prod_{x \in D} \hat{\mathcal{O}}_{x,D} \otimes_{\mathcal{O}_X} \mathcal{O}_X(C) \right) \cap A_X.
\]

In what follows we do not indicate the sheaf in adélic notation if it is the structure sheaf \( \mathcal{O}_X \). Then we use simpler notation: \( A_X = A_{X,012}, A_{X,0}, A_{X,1}, A_{X,2}, A_{X,01}, A_{X,02}, A_{X,12} \).
The following expression is an analogue of formula (134) for the case of a projective algebraic surface $X$:

$$\frac{\mathbb{A}_X}{\mathbb{A}_X,01 + \mathbb{A}_X,02} = \frac{\mathbb{A}_X,012(O_X)}{\mathbb{A}_X,01(O_X) + \mathbb{A}_X,02(O_X)}.$$  \hspace{1cm} (136)

Starting from (136), we shall obtain an exact sequence which is analogous to (133) and, in some cases, to the sequences in Example 4 for arithmetic surfaces.

**Theorem 3.** Let $X$ be a projective smooth irreducible algebraic surface over a field $k$. Let $C_i$, $1 \leq i \leq k$, be irreducible curves on $X$. Suppose that $C = \bigcup_{1 \leq i \leq k} C_i$ is an ample divisor on $X$. Then there is an exact sequence

$$0 \to \prod_{x \in D, D \not\subset C} \hat{\mathcal{O}}_{x,D} \to \prod_{x \in X, x \not\in C} \hat{\mathcal{O}}_x \to \frac{\mathbb{A}_X}{\mathbb{A}_X,01 + \mathbb{A}_X,02} \to 0,$$ \hspace{1cm} (137)

where $B_x$ is the following subring of $K_x$:

$$B_x \overset{\text{def}}{=} \bigcap_{D \subset X, D \not\subset C} (K_x \cap \hat{\mathcal{O}}_{x,D}).$$

(The intersection $K_x \cap \hat{\mathcal{O}}_{x,D}$ is taken in the ring $K_{x,D}$.)

**Remark 25.** Theorem 3 is an elaboration of constructions introduced in [4], [18].

**Remark 26.** The first term in (137) can be rewritten as

$$\frac{\prod_{x \in D, D \not\subset C} \hat{\mathcal{O}}_{x,D}}{\prod_{D \subset X, D \not\subset C} \hat{\mathcal{O}}_D + \prod_{x \in X, x \not\in C} \hat{\mathcal{O}}_x} = \frac{\prod_{D \subset X, D \not\subset C} (\prod_{x \in D} \hat{\mathcal{O}}_{x,D})}{\prod_{x \in X, x \not\in C} \hat{\mathcal{O}}_x}. \hspace{1cm} (138)$$

For a fixed irreducible curve $D \subset X$ with $D \not\subset C$ we have

$$\prod_{x \in D} \hat{\mathcal{O}}_{x,D} = \mathbb{A}_D[[t_D]], \quad \hat{\mathcal{O}}_D = k(D)[[t_D]].$$

Hence,

$$\frac{\prod_{x \in D} \hat{\mathcal{O}}_{x,D}}{\hat{\mathcal{O}}_D} = (\mathbb{A}_D/k(D))[[t]]. \hspace{1cm} (139)$$

Suppose that $k$ is a finite field. Then it follows from the exact sequence (133) that the group $\mathbb{A}_D/k(D)$ has the structure of a compact object of $C_1^{\text{fr}}$. More precisely, it is an object of $C_1^{\text{fin}}$. Therefore the group $(\mathbb{A}_D/k(D))[[t]]$ also has the structure of a compact object of $C_1^{\text{fin}}$. (We must use the structure of an object of $C_1^{\text{fin}}$ similar to the Tikhonov topology on an infinite product of compact topological spaces.) For every point $x \in X$, the group $\hat{\mathcal{O}}_x$ is profinite and, therefore, has the structure of a compact object of $C_1^{\text{fin}}$. Hence we obtain that the group given by (138) has the structure of a compact object of $C_1^{\text{fin}}$. 

Proof of Theorem 3. We define an open set \( U = X \setminus C \). By hypothesis, \( C \) is an ample divisor on \( X \). Therefore the quasi-coherent sheaf \( \mathcal{F} \overset{\text{def}}{=} \lim_{n \in \mathbb{N}} \mathcal{O}_X(nC) \) satisfies

\[
H^1(X, \mathcal{F}) = 0, \quad H^2(X, \mathcal{F}) = 0
\]

since \( H^i(X, \mathcal{F}) = \lim_{n \in \mathbb{N}} H^i(X, \mathcal{O}_X(nC)) \) and

\[
H^i(X, \mathcal{O}_X(nC)) = 0
\]

if \( i > 0 \) and \( n \in \mathbb{N} \) is sufficiently large. By definition (see \([7]-[10]\)) the adèle groups of quasi-coherent sheaves commute with inductive limits. Hence we obtain that the following properties hold in the interior of the group \( \mathbb{A}_{X,012}(\mathcal{F}) = \mathbb{A}_{X,012} = \mathbb{A}_X \):

\[
\begin{align*}
\mathbb{A}_{X,1}(\mathcal{F}) &= \mathbb{A}_{X,12}(\mathcal{F}) \cap \mathbb{A}_{X,01}(\mathcal{F}), \\
\mathbb{A}_{X,2}(\mathcal{F}) &= \mathbb{A}_{X,12}(\mathcal{F}) \cap \mathbb{A}_{X,02}(\mathcal{F})
\end{align*}
\]

because, by construction, these properties hold for the adèle groups of the sheaves \( \mathcal{O}_X(nC), n \in \mathbb{N} \). Since \( H^1(X, \mathcal{F}) = 0 \), we obtain from the adèle complex (135) that, in the interior of the group \( \mathbb{A}_X \),

\[
\begin{align*}
\mathbb{A}_{X,12}(\mathcal{F}) \cap (\mathbb{A}_{X,01} + \mathbb{A}_{X,02}) &= (\mathbb{A}_{X,12}(\mathcal{F}) \cap \mathbb{A}_{X,01}) + (\mathbb{A}_{X,12}(\mathcal{F}) \cap \mathbb{A}_{X,02}) \quad (140)
\end{align*}
\]

because \( \mathbb{A}_{X,01}(\mathcal{F}) = \mathbb{A}_{X,01} \) and \( \mathbb{A}_{X,02}(\mathcal{F}) = \mathbb{A}_{X,02} \). By definition,

\[
\begin{align*}
\mathbb{A}_{X,12}(\mathcal{F}) &= \prod_{x \in D, D \subset C} \tilde{\mathcal{O}}_{x,D} + \prod_{1 \leq i \leq k} \prod_{x \in C_i} \tilde{\mathcal{O}}_{x,C_i}, \\
\mathbb{A}_{X,01} &= \prod_{D \subset C, D \subset C} \mathcal{O}_D + \prod_{1 \leq i \leq k} \mathcal{O}_{C_i}, \\
\mathbb{A}_{X,02} &= \prod_{x \in U} \mathcal{O}_x + \prod_{x \in C} \mathcal{O}_x
\end{align*}
\]

Using (140), we deduce from these formulae that

\[
\begin{align*}
\mathbb{A}_{X,12}(\mathcal{F}) \cap (\mathbb{A}_{X,01} + \mathbb{A}_{X,02}) &= \left( \prod_{D \subset X, D \subset C} \tilde{\mathcal{O}}_D + \prod_{1 \leq i \leq k} \mathcal{O}_{C_i} \right) + \left( \prod_{x \in U} \tilde{\mathcal{O}}_x + \prod_{x \in C} \mathcal{O}_x \right) \\
&= \prod_{x \in D, D \subset C} \tilde{\mathcal{O}}_{x,D} \quad (144)
\end{align*}
\]

We now consider the natural map

\[
\phi: \prod_{x \in D, D \subset C} \tilde{\mathcal{O}}_{x,D} \rightarrow \frac{\mathbb{A}_X}{\mathbb{A}_{X,01} + \mathbb{A}_{X,02}}. \\
\]

By definition,

\[
\text{Ker } \phi = \left( \prod_{x \in D, D \subset C} \tilde{\mathcal{O}}_{x,D} \right) \cap (\mathbb{A}_{X,01} + \mathbb{A}_{X,02}) \subset \prod_{D \subset X, D \subset C} \tilde{\mathcal{O}}_D + \prod_{x \in U} \tilde{\mathcal{O}}_x.
\]

Formula (141) shows that \( \mathbb{A}_{X,12}(\mathcal{F}) \supset \prod_{x \in D, D \subset C} \tilde{\mathcal{O}}_{x,D} \). Therefore,

\[
\mathbb{A}_{X,12}(\mathcal{F}) \cap (\mathbb{A}_{X,01} + \mathbb{A}_{X,02}) \supset \text{Ker } \phi.
\]

Hence we obtain from (144) that

\[
\text{Ker } \phi = \prod_{D \subset X, D \subset C} \tilde{\mathcal{O}}_D + \prod_{x \in U} \tilde{\mathcal{O}}_x. \\
\]

(146)
because $\prod_{x \in D, D \not\subseteq C} \hat{\mathcal{O}}_{x,D} \supset \Ker \phi$ and the elements of $\prod_{x \in D, D \not\subseteq C} \hat{\mathcal{O}}_{x,D}$ have zero projections on all the subgroups $K_{x,C_i}$, $x \in C_i$, $1 \leq i \leq k$, of the adèle group $\mathbb{A}_X$ while every non-zero element of $\prod_{x \in C} K_{C_i}$ or $\prod_{x \in B} B_x$ has a non-zero projection on some subgroup $K_{x,C_i}$, $x \in C_i$, $1 \leq i \leq k$, of the adèle group $\mathbb{A}_X$.

It now follows from (145) and (146) that the beginning of the sequence (137) is exact.

Since $H^2(X, \mathcal{F}) = 0$, we obtain from the adèle complex (135) that

$$\mathbb{A}_X = \mathbb{A}_{X,12}(\mathcal{F}) + \mathbb{A}_{X,01} + \mathbb{A}_{X,02}. \quad (147)$$

Consider the subgroup $\mathbb{A}_{X,12}^U(\mathcal{F}) \overset{\text{def}}{=} \prod_{x \in D, D \not\subseteq C} \hat{\mathcal{O}}_{x,D}$ of the group $\mathbb{A}_X$. Then formula (141) may be rewritten as

$$\mathbb{A}_{X,12}(\mathcal{F}) = \mathbb{A}_{X,12}^U(\mathcal{F}) \oplus \prod_{1 \leq i \leq k} \prod_{x \in C_i} K_{x,C_i}. \quad (148)$$

Formula (147) yields an obvious exact sequence

$$\begin{array}{c}
\frac{\mathbb{A}_{X,12}(\mathcal{F}) \cap (\mathbb{A}_{X,12}^U(\mathcal{F}) + \mathbb{A}_{X,01} + \mathbb{A}_{X,02})}{\mathbb{A}_{X,12}^U(\mathcal{F})} \xrightarrow{\mathbb{A}_{X,12}(\mathcal{F})} \mathbb{A}_{X,12}(\mathcal{F}) \xrightarrow{\mathbb{A}_{X,12}^U(\mathcal{F})} \mathbb{A}_X \\
\xrightarrow{\mathbb{A}_{X,12}(\mathcal{F})} \frac{\mathbb{A}_{X,12}^U(\mathcal{F}) + \mathbb{A}_{X,01} + \mathbb{A}_{X,02}}{\mathbb{A}_{X,12}(\mathcal{F})}. \quad (149)
\end{array}$$

Consider the group

$$G \overset{\text{def}}{=} \frac{\mathbb{A}_{X,12}(\mathcal{F}) \cap (\mathbb{A}_{X,01} + \mathbb{A}_{X,02})}{\mathbb{A}_{X,12}^U(\mathcal{F}) \cap (\mathbb{A}_{X,01} + \mathbb{A}_{X,02})}$$

and the map

$$\psi: G \to \frac{\mathbb{A}_{X,12}(\mathcal{F}) \cap (\mathbb{A}_{X,12}^U(\mathcal{F}) + \mathbb{A}_{X,01} + \mathbb{A}_{X,02})}{\mathbb{A}_{X,12}^U(\mathcal{F})}. \quad (150)$$

Clearly, $\psi$ is an isomorphism.

We claim that

$$G \simeq \prod_{1 \leq i \leq k} K_{C_i} + \prod_{x \in C} B_x. \quad (151)$$

Indeed, by definition, we have

$$\mathbb{A}_{X,12}^U(\mathcal{F}) \cap (\mathbb{A}_{X,01} + \mathbb{A}_{X,02}) \supset \prod_{D \subseteq X, D \not\subseteq C} \hat{\mathcal{O}}_D + \prod_{x \in U} \hat{\mathcal{O}}_x. \quad (152)$$

Formula (144) yields a natural map

$$\xi: \prod_{1 \leq i \leq k} K_{C_i} + \prod_{x \in C_i} B_x \to G.$$ 

It follows from (144) and (152) that $\xi$ is surjective. On the other hand, $\xi$ is injective since its image does not contain the subgroup $\mathbb{A}_{X,12}^U(\mathcal{F})$. Thus $\xi$ is an isomorphism and formula (151) is proved.
We see from (148) that
\[
\frac{\mathbb{A}_{X,12}(\mathcal{F})}{\mathbb{A}_{X,12}^0(\mathcal{F})} \simeq \prod_{1 \leq i \leq k} \prod'_{x \in C_i} K_{x,C_i}.
\]
Combining this with (149)–(151), we obtain that the sequence (137) is exact at the middle and the end. The theorem is proved.

14.2. A more precise calculation. In this subsection we calculate the last non-zero term of (137) more explicitly.

**Theorem 4.** Suppose that \(X\) is a projective smooth irreducible algebraic surface over a field \(k\) and \(C_i, 1 \leq i \leq k,\) are irreducible curves on \(X.\) Put \(C = \bigcup_{1 \leq i \leq k} C_i\) and \(p = \bigcap_{1 \leq i \leq k} C_i.\) Then we have an isomorphism
\[
\frac{\prod_{1 \leq i \leq k} \prod'_{x \in C_i} K_{x,C_i}}{\prod_{1 \leq i \leq k} \prod'_{x \in C} B_x} \simeq \frac{\prod_{1 \leq i \leq k} K_{p,C_i}}{\prod_{1 \leq i \leq k} B_{C_i} + B_p},
\]
where the subring \(B_{C_i}, 1 \leq i \leq k,\) of the field \(K_{C_i}\) is given by
\[
B_{C_i} \overset{\text{def}}{=} \bigcap_{x \in C_i \setminus p} (K_{C_i} \cap B_x).
\]
(The intersection \(K_{C_i} \cap B_x\) is taken in the ring \(K_{x,C_i}.)\)

Before proving Theorem 4, we establish the following lemma.

**Lemma 7.** Under the hypotheses of Theorem 4 we have the following isomorphisms for all \(i, 1 \leq i \leq k:\)
\[
K_{C_i}/B_{C_i} \simeq \prod'_{x \in C_i \setminus p} K_{x,C_i}/B_x.
\]

**Proof.** Fix an \(i, 1 \leq i \leq k.\) We have a diagonal embedding \(K_{C_i} \to \prod'_{x \in C_i \setminus p} K_{x,C_i}\) which induces a map
\[
\eta: K_{C_i}/B_{C_i} \to \prod'_{x \in C_i \setminus p} K_{x,C_i}/B_x.
\]
It follows from the definition of the ring \(B_C\) that \(\eta\) is injective.

We now prove that \(\eta\) is surjective. Let \(J_{C_i}\) be the ideal sheaf of the curve \(C_i\) on the surface \(X.\) For every \(n \in \mathbb{N}\) we consider the scheme
\[
Y_n = (C_i \setminus p, \mathcal{O}_X/J_{C_i}^{n} | C_i \setminus p).
\]
It is an infinitesimal neighbourhood of the affine curve \(C_i \setminus p\) in \(X.\) For all \(l, m \in \mathbb{Z}\) with \(l < m\) we take a coherent sheaf \(J_{C_i}^{l}/J_{C_i}^{m} | C_i \setminus p\) on the scheme \(Y_{m-l}.\) We have \(\dim Y_{m-l} = 1\) and \(H^1(Y_{m-l}, J_{C_i}^{l}/J_{C_i}^{m} | C_i \setminus p) = 0\) for \(l < m \in \mathbb{Z}\) since \(Y_{m-l}\) is an affine scheme. Therefore the adèle complex (see [7], [10]) gives us the following exact sequence \(\mathcal{K}_{l,m}\) for all \(l < m \in \mathbb{Z}:\)
\[
0 \to H^0(Y_{m-l}, J_{C_i}^{l}/J_{C_i}^{m} | C_i \setminus p) \to \mathbb{A}_{Y_{m-l},0} J_{C_i}^{l}/J_{C_i}^{m} | C_i \setminus p \oplus \mathbb{A}_{Y_{m-l},1} J_{C_i}^{l}/J_{C_i}^{m} | C_i \setminus p \to \mathbb{A}_{Y_{m-l},01} J_{C_i}^{l}/J_{C_i}^{m} | C_i \setminus p \to 0.
\]
For every fixed $l \in \mathbb{Z}$, the projective system of Abelian groups
\[
(H^0(Y_{m-l}, J^l_{C_i} / J^m_{C_i} | C_i \setminus p), m > l)
\]
satisfies the condition ML (the Mittag–Leffler condition) because the map
\[
H^0(Y_{m-l+1}, J^l_{C_i} / J^{m+1}_{C_i} | C_i \setminus p) \to H^0(Y_{m-l}, J^l_{C_i} / J^m_{C_i} | C_i \setminus p)
\]
is surjective for every $m > l$ in view of the exact sequence
\[
0 \to J^m_{C_i} / J^{m+1}_{C_i} | C_i \setminus p \to J^l_{C_i} / J^{m+1}_{C_i} | C_i \setminus p \to J^l_{C_i} / J^m_{C_i} | C_i \setminus p \to 0
\]
of sheaves on $Y_{m-l+1}$, and $H^1(Y_{m-l+1}, J^m_{C_i} / J^{m+1}_{C_i} | C_i \setminus p) = 0$ because $J^m_{C_i} / J^{m+1}_{C_i}$ is a coherent sheaf on the affine scheme $Y_{m-l+1}$.

We now consider the sequence $\mathcal{K} := \lim_{l \in \mathbb{Z}} \lim_{m \geq l} \mathcal{K}_{l,m}$. It is exact because the inductive limit preserves exactness and so does the projective limit because of the condition ML. The sequence $\mathcal{K}$ is of the form
\[
0 \to M \to A_0 \oplus A_1 \to A_{01} \to 0,
\]
where the subgroups of $A_X$ satisfy
\[
A_0 = K_{C_i}, \quad A_1 = \prod_{x \in C_i \setminus p} B_x, \quad A_{10} = \prod_{x \in C_i \setminus p} K_{x,C_i}.
\]

The exactness of (153) now yields that $\eta$ is surjective. The lemma is proved.

Proof of Theorem 4. Suppose that $k = 1$, that is, $C = C_1$. We have
\[
\prod_{x \in C} K_{x,C} / K_C + \prod_{x \in C} B_x = \prod_{x \in C \setminus p} K_{x,C} \oplus K_{p,C} / K_C + \prod_{x \in C \setminus p} B_x \oplus B_p = \prod_{x \in C \setminus p} K_{x,C} / K_C \oplus \prod_{x \in C \setminus p} K_{p,C} / B_p = B_C + B_p,
\]
where the third equation follows from Lemma 7.

If $k > 1$, then we must perform an analogous calculation applying Lemma 7 $k$ times to the curves $C_i$, $1 \leq i \leq k$. For example, when $k = 2$ we have
\[
\frac{\prod_{x \in C_1} K_{x,C_1} \oplus \prod_{x \in C_2} K_{x,C_2}}{K_{C_1} \oplus K_{C_2} + \prod_{x \in C} B_x}
= \frac{(\prod_{x \in C_1 \setminus p} K_{x,C_1}) \oplus K_{p,C_1} \oplus (\prod_{x \in C_2 \setminus p} K_{x,C_2}) \oplus K_{p,C_2}}{K_{C_1} \oplus K_{C_2} + (\prod_{x \in C_1 \setminus p} B_x) \oplus B_p \oplus (\prod_{x \in C_2 \setminus p} B_x)} = \frac{K_{p,C_1} \oplus K_{p,C_2}}{B_{C_1} \oplus B_{C_2} + B_p},
\]
where the last equality follows from Lemma 7. The theorem is proved.
14.3. The case of an arithmetic surface. Suppose that $X = C_1 \times_k C_2$ and $C = C_1 \cup C_2$, $p = C_1 \cap C_2$. The curves $C_1$ and $C_2$ are transversal on $X$. Assume that $C_1$ and $C_2$ are projective lines over $k$ with coordinates $t$ and $u$ such that the point $p$ is given by the local equations $t = 0$, $u = 0$.

By definition we have the following rings and fields:

$$
k(C_1) = k(t), \quad k(C_2) = k(u),
$$

$$
K_{p,C_1} = k((t))((u)), \quad K_{p,C_2} = k((u))((t)),
$$

$$
B_{C_1} = k[t^{-1}]((u)), \quad B_{C_2} = k[u^{-1}]((t)),
$$

$$
B_p = \lim_{m \in \mathbb{N}, n \in \mathbb{N}} u^{-m} t^{-n} k[[u, t]].
$$

Consider the quotient group (see Theorem 4)

$$
F = \frac{K_{p,C_1} \oplus K_{p,C_2}}{B_{C_1} \oplus B_{C_2} + B_p},
$$

where the group $B_p$ is diagonally embedded in $K_{p,C_1} \oplus K_{p,C_2}$. We recall that $F$ can be computed by reduction to a single two-dimensional local field, say, to $K_{p,C_2}$. Namely, we have $0 \oplus K_{p,C_2} + B_{C_1} \oplus B_{C_2} + B_p = K_{p,C_1} \oplus K_{p,C_2}$. Therefore,

$$
0 \longrightarrow K_{p,C_2} \cap (B_{C_1} \oplus B_{C_2} + B_p) \longrightarrow K_{p,C_2} \rightarrow F \longrightarrow 0,
$$

(154)

where the intersection is taken inside the group $K_{p,C_1} \oplus K_{p,C_2}$ and $K_{p,C_2} = 0 \oplus K_{p,C_2}$ is considered inside the same group. The first non-zero group in sequence (154) is equal to $B_{C_2} + K_{p,C_2} \cap (B_{C_1} + B_p)$ since $B_{C_2}$ belongs only to $K_{p,C_2}$ as a subgroup. The group $K_{p,C_2} \cap (B_{C_1} + B_p)$ is equal to the subgroup $B_{C_1} \cap B_p$ of the group $K_{p,C_1} \oplus 0$. This subgroup is embedded in $0 \oplus K_{p,C_2}$ via the inclusion $B_p \subset K_{p,C_2}$.

We finally get

$$
F = \frac{K_{p,C_2}}{B_{C_2} + (B_{C_1} \cap B_p)} = \frac{k((u))((t))}{k[u^{-1}]((t)) + k((u))[t^{-1}]} \simeq k[[u, t]]ut.
$$

For an arithmetic surface we have the following analogue of this calculation. Regard the surface $X$ as a fibration with projection onto the curve $C_2$. We want to compare the surface $X$ fibred over $C_2$ with the simplest arithmetic surface $P^1$ over Spec $\mathbb{Z}$. The point $p \in C_2$ will correspond to the place $\infty$ added to Spec $\mathbb{Z}$. The curve $C_1$ corresponds to the non-existent closed fibre over $\infty$. We have the analogies

$$
k(C_2)_p = k((u)) \sim \mathbb{R}, \quad K_{p,C_2} \sim \mathbb{R}((t)),
$$

$$
B_{C_2} = \{\text{regular functions on } C_2 \setminus p\}((t)) \sim \mathbb{Z}((t)),
$$

$$
B_{C_1} \cap B_p = k((u))[t^{-1}] \sim \mathbb{R}[t^{-1}]
$$

based on the classical analogy between algebraic surfaces and arithmetic surfaces (see, for example, [19]).

Using this dictionary, we obtain the following analogue of the group $F$:

$$
F \sim \frac{\mathbb{R}((t))}{\mathbb{Z}((t)) + \mathbb{R}[t^{-1}]} = (\mathbb{R}/\mathbb{Z})[[t]]t \simeq \mathbb{T}[[t]]t.
$$

Clearly, the group $\mathbb{T}[[t]]t$ has the structure of an object of $C_2^{ar}$ which is both a $c$-object and a $cf$-object.
Remark 27. It is also interesting to consider an analogue of the first non-zero term of (137) in the case of an arithmetic surface. We recall the situation of Example 11. Consider a regular two-dimensional scheme $X$ with a projective surjective morphism $X \to \text{Spec} E$, where $E$ is the ring of integers in a number field $K$, $[K: \mathbb{Q}] = n$. This means that $X$ is an arithmetic surface. Let $X_K$ be the generic fibre of this morphism and let $C \subset X$ be a ‘horizontal’ irreducible arithmetic curve, that is, an integral one-dimensional subscheme $C$ which is mapped surjectively onto $\text{Spec} E$.

The subscheme $C$ corresponds to a closed point $p_C \in X_K$. We recall that $p_1, \ldots, p_l$ are the Archimedean places of $K$, with the corresponding completion fields $K_{p_i}$, $1 \leq i \leq l$. Then, according to the definition of the adèle ring $\mathbb{A}_X^\text{ar}$ (see Example 11), the following group $\Psi$ is an analogue of the first non-zero term in (137):

$$
\Psi \overset{\text{def}}{=} \frac{\left( \prod_{x \in D, D \not\subset C} \hat{O}_{x,D} \right) \oplus \left( \prod_{q \in X_K, q \neq p_C} \hat{O}_q \otimes_K \left( \prod_{1 \leq i \leq l} K_{p_i} \right) \right)}{\prod_{D \subset X, D \not\subset C} \hat{O}_D + \prod_{x \in X, x \not\in C} \hat{O}_x},
$$

where $x \in D$ runs over all pairs consisting of an integral one-dimensional subscheme $D$ in $X$ and a closed point $x$ on $D$. The point $q$ is a closed point of $X_K$ and the ring $\hat{O}_q$ is the completion of the local ring of $q$ on $X_K$ with respect to the maximal ideal of $q$. By definition we have

$$
\hat{O}_q \otimes_K \left( \prod_{1 \leq i \leq l} K_{p_i} \right) = \lim_{\leftarrow n \in \mathbb{N}} \left( \hat{O}_q \otimes_K \left( \prod_{1 \leq i \leq l} K_{p_i} \right) \right),
$$

where $\hat{m}_q$ is the maximal ideal of the local ring $\hat{O}_q$.

We now explain how to embed the subgroups (155) inside one another. We note that for a fixed ‘horizontal’ irreducible arithmetic curve $D \subset X$ there is a diagonal embedding

$$
\hat{O}_D \hookrightarrow \left( \prod_{x \in D} \hat{O}_{x,D} \right) \oplus \left( \hat{O}_{p_D} \otimes_K \left( \prod_{1 \leq i \leq l} K_{p_i} \right) \right),
$$

where the point $p_D \in X_K$ corresponds to the irreducible arithmetic curve $D \subset X$ and, by construction, $\hat{O}_D = \hat{O}_{p_D}$.

For a fixed ‘vertical’ irreducible curve $D \subset X$ (that is, $D$ is defined over a finite field) we have only the diagonal embedding

$$
\hat{O}_D \hookrightarrow \prod_{x \in D} \hat{O}_{x,D}.
$$

For a fixed closed point $x \in X$ we also have only the diagonal embedding

$$
\hat{O}_x \hookrightarrow \prod_{D \ni x} \hat{O}_{x,D}.
$$

As was done in (138), we can now rewrite the group $\Psi$ as

$$
\Psi = \frac{\Psi_1 \oplus \Psi_2}{\prod_{x \in X, x \not\in C} \hat{O}_x},
$$

(156)
where
\[ \Psi_1 = \prod_{D \subset X, D \not\subset C} \left( \prod_{x \in D} \hat{\mathcal{O}}_{x,D} \right) \oplus \left( \hat{\mathcal{O}}_{p_D} \hat{\otimes}_K \left( \prod_{1 \leq i \leq l} K_{p_i} \right) \right) / \hat{\mathcal{O}}_D, \]
\[ \Psi_2 = \prod_{D \subset X, D \not\subset C} \left( \prod_{x \in D} \hat{\mathcal{O}}_{x,D} \right) / \hat{\mathcal{O}}_D. \]

For a fixed irreducible ‘horizontal’ arithmetic curve \( D \subset X \) let \( k(D) \) be the field of rational functions on \( D \) and let \( t_D = 0 \) be the local equation of \( D \) on some open subset of \( X \). We have
\[ \left( \prod_{x \in D} \hat{\mathcal{O}}_{x,D} \right) \oplus \left( \hat{\mathcal{O}}_{p_D} \hat{\otimes}_K \left( \prod_{1 \leq i \leq l} K_{p_i} \right) \right) = \left( \mathbb{A}_{k(D)}/k(D) \right)[[t_D]]. \]

We obtain from Example 4 that \( \mathbb{A}_{k(D)}/k(D) \) is a compact object of \( C_1^\text{ar} \). Hence \( \left( \mathbb{A}_{k(D)}/k(D) \right)[[t_D]] \) (and, therefore, \( \Psi_1 \)) has the structures of a \( c \)-object and a \( cf \)-object of \( C_2^\text{ar} \) (compare with Remark 26).

For a fixed irreducible ‘vertical’ curve \( D \subset X \) we have
\[ \left( \prod_{x \in D} \hat{\mathcal{O}}_{x,D} \right) = \lim_{\rightarrow \infty} \mathbb{A}_{D_n} / \mathbb{A}_{D_n,0}, \]
where the closed subscheme \( D_n \subset X \) is given by \( D_n = (D, \mathcal{O}_X/J_{p_D}^n) \) and \( J_D \) is the ideal sheaf of the curve \( D \) on \( X \). Note that \( D_1 = D \). Using induction on \( n \) and arguing as at the beginning of §14.1, we obtain that \( \mathbb{A}_{D_n}/\mathbb{A}_{D_n,0} \) has the structure of a compact object of \( C_1^\text{ar} \). Hence \( \Psi_2 \) has the structures of a \( c \)-object and a \( cf \)-object of \( C_2^\text{ar} \).

As in Remark 26, the group \( \prod_{x \in X, x \not\in C} \hat{\mathcal{O}}_x \) has the structures of a \( c \)-object and a \( cf \)-object of \( C_2^\text{ar} \). Using this and (156), we obtain that the group \( \Psi \) has the structures of a \( c \)-object and a \( cf \)-object of \( C_2^\text{ar} \) (compare with Remark 26).

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