Device-independent Randomness Expansion with Entangled Photons

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With the growing availability of experimental loophole-free Bell tests [1–5], it has become possible to implement a new class of device-independent random number generators whose output can be certified [6, 7] to be uniformly random without requiring a detailed model of the quantum devices used [8–10]. However, all of these experiments require many input bits in order to certify a small number of output bits, and it is an outstanding challenge to develop a system that generates more randomness than is consumed. Here, we devise a device-independent spot-checking protocol that consumes only uniform bits without requiring any additional bits with a specific bias. Implemented with a photonic loophole-free Bell test, we can produce 24% more certified output bits (1, 181, 264, 237) than consumed input bits (953, 301, 640). The experiment ran for 91.0 hours, creating randomness at an average rate of 3,606 bits/s with a soundness error bounded by $5.7 \times 10^{-7}$ in the presence of classical side information. Our system will allow for greater trust in public sources of randomness, such as randomness beacons [11], and may one day enable high-quality private sources of randomness as the device footprint shrinks.

In 1964, John Bell showed that measurements on entangled quantum systems may show correlations stronger than those predicted by any local realistic theory [12]. In a loophole-free Bell test, entangled particles are sent to distant stations, referred to as “Alice” and “Bob”, where independent measurements are performed. A violation of local realism occurs if Alice’s and Bob’s measurements produce outcomes incompatible with the predictions of any local realistic theory. In this case, the measurement outcomes must have some randomness even when conditioned on additional or side information available outside the laboratory. Consequently, in addition to testing whether local realistic theories are consistent with nature, a loophole-free Bell test can be used to generate uniformly random bits with respect to any adversary isolated from the laboratory after the protocol starts [6, 7]. Most importantly, the generated random bits can be certified in a device-independent way with a small error.

The first device-independent randomness-generation experiment certified 42 bits of unextracted randomness in data from a Bell test (subject to the locality loophole) with entangled ions [13] acquired over the course of a month. Since then, major improvements in both experimental design [3, 10, 14, 15] and theoretical analysis [16–19] have led to remarkable improvements in the achievable bit rate, the minimum time for generating one random bit, the quality of the certificate, and the type of side information an adversary may have. However, these experiments required far more random bits as input during the experiment than generated. For example, in one run of the recent end-to-end protocol we repeatedly implemented in [10], $k_{\text{out}} = 512$ certified output bits were generated from $k_{\text{in}} = 4.78 \times 10^7$ input bits for measurement settings choices and extractor seed—a ratio of $k_{\text{out}}/k_{\text{in}} = 1.07 \times 10^{-5}$. A long-standing goal has been to achieve randomness expansion, where more certified output bits are generated than input bits consumed.

A key obstacle to achieving randomness expansion in a normal device-independent randomness-generation setup is the requirement that in every “trial” involving a measurement at each station, Alice and Bob must each uniformly at random choose between two measurement settings, thereby consuming a total of two random bits. Instead in our protocol, Alice and Bob only rarely, but ran-
FIG. 1. Schematic of the experiment and trial structure. (a) In our protocol, a source station $S$ sends entangled photons to Alice and Bob to be measured. At the same station, “Spot” randomly signals when it is time to perform a spot-checking trial by counting down from a 17-bit random number $L$. Alice and Bob have no direct or advance knowledge of when a spot-checking trial will take place, and are unable to communicate with one another. If the next trial is to be a spot-checking trial, then Alice’s (Bob’s) settings choice is determined by a random bit from a well-characterized low-latency random number generator (RNG) \[21\]. Otherwise the settings choice is always fixed to a particular setting. Alice and Bob measure photons using the settings determined as above, and record their outcomes. If the protocol concludes successfully, then the outcomes along with a small amount of seed randomness can be sent to a classical extractor to extract the output random bits (not implemented). Otherwise, the protocol fails and no new bits are produced. (b) Each block consists of a random number of trials $L$, and each trial is made up of 8 aggregated pulses (light-blue/red bars). Multiple photons can be detected in any given trial at Alice and Bob (dark-blue/red bars). For the analysis, multiple detection events at Alice (Bob) in a single trial are treated as a single detection event. To allow the Pockels cells, which are part of the measurement setup, to recover after a spot-checking trial, the next 120 trials are ignored before a new block begins. A block always ends with a spot-checking trial.

For our spot-checking strategy, we introduce a trusted third party, “Spot”, who decides when Alice and Bob need to perform the spot-checking trial (see figure 1). Our protocol is based on dividing our experimental trials into blocks of variable length. Each block can have a length that ranges anywhere between one trial and $2^{17}$ trials, with the last trial in the block serving as a spot-checking trial as shown in figure 1. Spot uses 17 bits of public randomness taken from the NIST randomness beacon \[11\] in order to pick the position of the spot-checking trial that ends a block. When the spot-checking trial occurs, Spot sends a signal to Alice and Bob where a settings choice circuit uses one random bit from a well-characterized low-latency random number generator \[21\] to choose each of their measurement settings. Fixed settings are used for the other trials in the block. In contrast to the usual spot-checking strategy proposed in the literature (for example, in Refs. \[13\, 20\]), our block-wise spot-checking strategy does not require converting uniform bits to non-uniform ones with a specific bias to determine when a spot-checking trial is to take place. As neither the adversary nor the untrusted devices used in the experiment can learn in advance when a block ends with a spot-checking trial, all trials in the block contribute to randomness generation. It is therefore possible to produce more randomness than is consumed. In our experiment, each block is estimated to produce on average 32.80 bits of randomness while consuming a total of 19 bits, meaning that expansion is possible, see the Supplementary Information for more details.

In our device-independent setup we do not need to trust the source, and we also do not need to trust most of the equipment at Alice and Bob. However, as with all similar cryptographic protocols, the recording devices (computers and time taggers) must be trusted. If an adversary had access to these devices before or during our experiment, they could compromise the security of our protocol by replacing the experimental records with pre-programmed outputs. Moreover, we assume that Alice’s...
FIG. 2. Source, stations and layout. (a) Our polarization-entangled photons are created by pumping a periodically-poled potassium titanyl phosphate (PPKTP) crystal inside a polarization Mach-Zehnder interferometer made up of beam displacers (BDs) and half-waveplates (HWPs) (see [2]) in the state $|\psi\rangle \approx \cos(14.8^\circ) |HH\rangle + \sin(14.8^\circ) |VV\rangle$. Here $H$ or $V$ represents a horizontally or vertically polarized photon. The pump is a 775 nm pulsed laser operating at 79.6 MHz. A small portion of the pump power is split off and sent to a fast photodiode (FPD) to produce an analog clock signal. This clock signal is used by Spot, who passes it through a divide-by-8 circuit to synchronize with the trials. A countdown circuit takes a 17-bit number from the NIST randomness beacon [11] to determine when a block ends. At the end of the block, a signal is sent to settings-choice generators at Alice and Bob, informing them to perform a spot-checking trial. A divide-by-960 circuit is also used to provide a synchronizing clock to Alice and Bob to calibrate their time taggers. (b) When the settings-choice generators receive a signal from Spot, they perform a spot-checking trial using a bit from a physical random number generator (RNG) with low latency to control a Pockels cell (PC). With the help of two HWPs and one quarter-wave plate (QWP), Alice chooses between measurement angles $a = -4.1^\circ$ and $a' = 25.5^\circ$ and Bob chooses between measurement angles $b = 4.1^\circ$ and $b' = -25.5^\circ$, where the angles are relative to a vertical polarizer. For all non-spot-checking trials the Pockels cells are in settings $a$, $b$. High-efficiency single-photon detectors [23] are used to detect the photons, and their arrival times are recorded on a time tagger and saved to a computer. Green boxes shaded with a dot pattern indicate trusted devices. (c) Locations of Alice (A) and Bob (B), while the source and Spot are co-located at the station S. Alice and Bob are located 194.8 ± 1.0 m apart. The shaded blue circles represent how far local information about Alice’s (Bob’s) settings choice could have propagated at the speed of light when Bob (Alice) have completed the measurement for a trial. Alice’s and Bob’s measurement processes are therefore space-like separated.

and Bob’s settings choices used in the spot-checking trial are independent [2, 22], and that the untrusted equipment at Alice, Bob, and the source do not know in advance when a spot-checking trial will occur or what the settings choices will be. We ensure that the measurement processes of Alice and Bob are space-like separated as shown in figure 2 (c).

We use an experimental setup, shown in figure 2, that is based on those reported in Refs. [2, 8, 10]. Due to our low per-pulse probability of generating a pair of photons from downconversion, it is advantageous to aggregate 8 consecutive pulses into a single trial. This corresponds to a rate of about 10 million total trials per second and an average of 153 spot-checking trials per second. Spot
uses timing information and 17 random bits to determine which trial corresponds to the end of a block as shown in figure 2 (a). For all trials except the last spot-checking trial in a block, Alice’s and Bob’s Pockels cells are turned off. This implements settings choice \(a\) and \(b\) for Alice and Bob respectively as described in figure 2 (b). Over the course of two weeks we collected 110.3 hours worth of data (see Methods for more details).

Our analysis uses the approach of probability estimation [24, 25], according to which probability estimation factors (PEFs) for each trial are multiplied together such that the inverse of the product of the PEFs determines an outcome-probability estimator. Given the set \(\mathcal{C}\) of probability distributions of settings choices and outcomes that are achievable by the devices in a given trial conditional on classical side information, a PEF with “power” \(\beta > 0\) for this trial is a non-negative function \(F\) of settings choices and outcomes satisfying a set of linear inequalities imposed by each distribution in \(\mathcal{C}\) [24, 25]. The coefficients of these inequalities include a \(\beta\)th power of the settings-choice-conditional outcome probabilities. The power \(\beta\) is predetermined and fixed for the whole experiment. For an experiment with \(n\) total trials, let \(F_i\) be the PEF for the \(i\)th trial. Then \((\prod_{i=1}^{n} F_i)^{-1/\beta}\) is an upper bound on the probability of the observed outcomes conditional on settings choices and classical side information at confidence level \(1 - \epsilon\). This can be used for randomness certification as explained in Refs. [24, 25].

A major advantage of probability estimation is that the resulting protocols require significantly less data for randomness generation [24, 25]. Because of how it relates to randomness certificates, the quantity \(W = \sum_{i=1}^{n} \log_2(F_i)/\beta\) is called the running entropy witness up to trial \(n\). The use of PEFs makes it possible to stop the experiment early, as soon as the running entropy witness surpasses the corresponding success threshold [24, 25]. This early-stop feature was exploited in our analysis, see below.

A randomness-generation protocol takes as input the desired number of random bits \(k_{\text{out}}\) and a soundness error \(\epsilon\). When it is run, it either fails or succeeds. If it succeeds, it produces \(k_{\text{out}}\) bits. The behavior is such that within a statistical distance of \(\epsilon\), there exists an “ideal” randomness-generation protocol that has the same probability of success and produces \(k_{\text{out}}\) uniformly random bits conditional on success. The random bits produced by the ideal protocol are independent of settings choices and seed input as well as side information. A consequence is that if the probability of success is at least \(\sqrt{\epsilon}\), then conditioned on success, the random bits produced by the actual protocol are indistinguishable (within a statistical distance of \(\sqrt{\epsilon}\)) from those produced by the ideal protocol.

Before running our analysis, we set a stopping criteria as discussed in the Methods section. This consists of picking a minimum value, \(W_{\min}\), that must be met by the running entropy witness in order for the protocol to succeed. Figure 3 shows some of the tradeoffs involved in choosing \(W_{\min}\). Our choice of \(W_{\min} = 1,616,998,677\) corresponds to an estimated probability of success of \(p_{\text{succ}} \geq 0.9938\). The actual run of the protocol analysis succeeded after 49,977,714 blocks (91.0 hours of experiment time). At this point, we have the ability to generate 1,181,264,237 new random bits at the soundness error \(5.7 \times 10^{-7}\), while consuming 949,576,566 random bits for spot checks and settings choices. In terms of randomness rate, 3,606 new random bits can be generated per second. If we were to extract our bits using Trevisan’s extractor implemented by Ref. [26] with parameters as described in Ref. [8], an additional 3,725,074 seed random bits would be required. We are therefore able to achieve an expansion ratio of \(k_{\text{out}}/k_{\text{in}} = 1.24\). In figure 4 we show the running entropy witness on all the data available for expansion analysis.

Our current analysis is secure against classical side information but not quantum side information. Security against quantum side information can be obtained by the extension of PEFs to quantum estimation factors (QEFs) as done in Refs. [10, 18, 19]. We did not do so here primarily because our current method for constructing QEFs requires too much computation time for our ex-
FIG. 4. Entropies as a function of the number of blocks processed in our protocol run. The running entropy of outputs (the black curve) is the running entropy witness adjusted to account for our soundness error of $5.7 \times 10^{-7}$. After processing 49,977,714 blocks (corresponding to 91.0 hours of experiment time), our success threshold shown as the dashed line (corresponding to the $\sigma_m$ in the Methods section) is reached. The red curve represents the running total of the consumed input bits (including the seed bits needed for the extractor [8, 26]). We compute the running entropies for the remaining blocks after reaching our stopping point to study the performance of our system. Because the running entropy witness must be adjusted to account for our soundness error, we must process 18,196,425 blocks before the running entropy of outputs becomes positive. When we reach our stopping criteria, we are able to achieve an expansion ratio of 1.24, having generated at least 1,181,264,237 new bits after accounting for the required 949,576,566 random bits for spot checks and settings choices as well as 3,725,074 random bits needed as a seed.

As a consumed resource; however, previously used seed bits would have a statistical distance from uniform that needs to be added to the soundness error.

Our current protocol requires a trusted third party, Spot, to determine when Alice and Bob need to perform a spot-checking trial. In principle, our protocol can be modified such that a trusted third party is not needed. However, the modified block-wise spot-checking protocol appears to require a much longer running time for randomness expansion, an issue which deserves further investigation.

Due to the size and complexity of a loophole-free Bell test, the first practical application of device-independent random number generators will likely be as a source of public randomness in randomness beacons. The current NIST randomness beacon operates at a rate of 512 bits/min [11]. In our previous work we were able to generate 512 bits of certified randomness from approximately 5 minutes of data on average [10]. In this work, we are able to certify an average of 3,606 bits/s over the duration of the experiment. It should be noted that we are only able to certify these bits after we reach our stopping criteria, so there is a large latency involved. However, this does show that device-independent randomness generation is now within reach of real-time integration in randomness beacons. We used 3.17 years of bits produced by the NIST randomness beacon, and in 91.0 hours certified enough randomness to in principle power the beacon for the next 3.93 years, provided that they can be kept secret until broadcasting. Furthermore, with two separate device-independent randomness expansion devices it would be possible to greatly increase the overall expansion ratio. The expanded output bits from one device could be used as input for the other device and vice versa. Starting with a relatively small number of uniform input bits, these two systems could “cross-feed” one another to produce vastly more certified output bits [25, 30] for use in public sources of randomness.

Our work can be thought of as implementing a simple, but non-trivial, quantum network. Entanglement is distributed and used to perform a task (device-independent randomness expansion) that no classical system is capable of performing. This unique quantum advantage arises from the nonlocal correlations possible in distributed entangled systems, and demonstrates an aspect of the potential power of larger-scale future quantum networks.

**Methods**

**Data Acquisition:** To keep the experiment aligned and well-functioning we collected data in a series of cycles, with each cycle consisting of up to one hour worth of data for expansion analysis. Every cycle began with approximately 2 minutes of calibration data consisting of a standard loophole-free Bell test with the Pockels cells operating at 250 kHz, which was stored in a calibration file. Subsequent data was recorded in expansion files, with each expansion file consisting of $2^{14}$ blocks (approximately 2 minutes). At the end of each expansion file, approximately 5 seconds of additional calibration data...
was saved. After every 5 expansion files, a quick check was made using motorized waveplates to see whether the experiment was still performing well. If the efficiency dropped or the visibility of the entangled state changed, then an automated realignment of the setup was performed, and a new cycle was started. If not, up to 30 expansion files were obtained in the current cycle before proceeding to the next cycle. In this way, enough calibration data was collected to allow our analysis protocol to adapt to experimental drifts in our setup.

Parameter Determination: There are several parameters that must be determined before running our analysis so that the desired number of random bits \( k_{\text{out}} \) at soundness error \( \epsilon \) can be obtained. They are the power \( \beta > 0 \), a maximum number of blocks \( N_b \) to acquire, and a minimum final entropy witness \( W_{\text{min}} \) required for success. The analysis stops with success if the running entropy witness exceeds \( W_{\text{min}} \) and fails if this is not achieved after \( N_b \) blocks. To determine these parameters, we use the first 16 cycles of data, which contains 4,502,276 blocks (about 7.4% of the recorded data), for commissioning and training purposes. After the parameters are determined and fixed, the protocol is run on the non-commissioning data. Blocks are analyzed sequentially using the PEFs with power \( \beta \). The PEFs are constructed and updated using the calibration data that is periodically taken during the experiment, allowing our analysis to adapt to any experimental drifts or changes that occur over time (see the Supplementary Information for details). After processing the blocks in each file, a check is made to see if the running entropy witness surpasses \( W_{\text{min}} \).

The randomness extraction part of the protocol is based on Trevisan’s extractor as described in Refs. \cite{8, 26} and detailed in the Supplementary Information. Its parameters are also determined during commissioning. In addition to the desired number of output bits \( k_{\text{out}} \), they include the length (in bits) of the experimental output string \( m_{\text{in}} \), the extractor error \( \epsilon_{\text{ext}} \) and the number of seed bits \( d_s \) required. We also need to specify the min-entropy (in bits) of the experimental output string required for success \( \sigma_{\text{in}} \), which is determined by the minimum entropy witness \( W_{\text{min}} \), the soundness error \( \epsilon \) and the extractor error \( \epsilon_{\text{ext}} \). The actual experimental output string has a variable length, depending on the actual lengths of the blocks and when the running entropy witness exceeds \( W_{\text{min}} \). This string is zero-filled to length \( m_{\text{in}} \), so \( m_{\text{in}} \) is an upper bound on the maximum length of the output string. Because of the large length of the experimental output string, an implementation of the classical Trevisan’s extractor would take prohibitively long (many months on a large supercomputer) to explicitly extract the desired \( k_{\text{out}} \) random bits. As a result we did not do so for this demonstration.

For demonstration purposes, and because the protocol analysis was performed months after the experimental run and data acquisition were completed, we determined the protocol parameters from both the commissioning data and knowledge of how many blocks were acquired during the experiment. After considering the tradeoffs, we set our soundness error to be \( \epsilon = 5.7 \times 10^{-7} \), corresponding to the 5-sigma criterion, as this enables good expansion with reasonable security. We then constrain the parameters so that a heuristically determined probability of success satisfies \( p_{\text{acc}} \geq 0.9938 \), exceeding the conventional one-sided 2.5-sigma criterion as a compromise between good completeness and expansion. Based on these criteria, we choose \( k_{\text{out}} = 1,181,264,237, \ W_{\text{min}} = 1,616,998,677, \ \beta = 4.7614 \times 10^{-8}, \) and \( N_b = 56,070,910, \) where \( N_b \) is the actual number of non-commissioning blocks acquired in the experiment. The associated extractor parameters are \( m_{\text{in}} = 14,698,652,631,040, \ \epsilon_{\text{ext}} = 1.78 \times 10^{-9}, \) \( d_s = 3,725,074, \) and \( \sigma_{\text{in}} = 1,181,264,480. \) Based on these choices, if all non-commissioning blocks are used for achieving a successful randomness expansion, the expected expansion ratio \( k_{\text{out}}/k_{\text{fin}} \) is 1.105. If success is achieved before processing all blocks, the actual expansion ratio improves as fewer input bits are consumed to certify the same number of output bits. Any random bits used for commissioning or calibration purposes can be obtained from pseudorandom sources, and therefore do not count as consumed randomness in our analysis. We emphasize that the success probability is heuristically determined based on the commissioning data and under the assumption that the entropy witness accumulated over the subsequent non-commissioning blocks for expansion analysis is normally distributed. For more details behind parameter determination, see the Supplementary Information.

Acknowledgements We thank Carl Miller and Scott Glancy for help with reviewing this paper. This work includes contributions of the National Institute of Standards and Technology, which are not subject to U.S. copyright. The use of trade names does not imply endorsement by the U.S. government. The work is supported by the National Science Foundation RAISE-TAQs (Award 1839223); European Research Council (ERC) projects AQUOMET (280169) and ERIDIAN (713682); European Union projects FET Innovation Launchpad UVALITH (809001); the Spanish MINECO projects OCRINA (Grant Ref. PGC2018-097056-B-I00) and Q-CLOCKS (PC12018-092973), the Severo Ochoa programme (SEV-2015-0522); Agència de Gestió d’Ajuts Universitaris i de Recerca (AGAUR) project (2017-SGR-1354); Fundació Privada Cellex and Generalitat de Catalunya (CERCA program); Quantum Technology Flagship project MACQSIMAL (820393); Marie Skłodowska-Curie ITN ZULF-NMR (766402); EMPIR project USOQS (17FUN03).

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SUPPLEMENTARY INFORMATION: DEVICE-INDEPENDENT RANDOMNESS EXPANSION WITH ENTANGLED PHOTONS

This Supplementary Information is structured as follows. In Sect. I we introduce the approach of probability estimation for certifying randomness with respect to classical side information [24][25]. Particularly, we show how to perform probability estimation and certify randomness by means of probability estimation factors (PEFs) with a sequence of blocks. As a consequence, we can design an end-to-end protocol for randomness generation, which is presented in Sect. II. Details on the numerical construction of PEFs are provided in Sect. III. Then, we present our protocol design and data analysis. Specifically, in Sect. IV we explain how to determine the protocol parameters before running our randomness-generation protocol. In Sect. V and Sect. VI we explain how we update PEFs on the basis of calibration data acquired during the experiment and describe the results of running the protocol.

We consider an experiment with a sequence of time-ordered blocks, where each block consists of at most $2^k$ trials executed in time order. Here $k$ is a positive integer. The number of trials in a block (that is, the length of a block) is determined by the value $l$ of a uniform random variable $L$, where $l \in \{1, ..., 2^k\}$. Thus, the average block length is $(1 + 2^k)/2$. In our experiment, each trial in a block uses a pair of quantum devices held by two remote stations, Alice and Bob. Before a trial, the state of the quantum devices can be correlated with the classical side information $E$ possessed by an adversary. In each trial, the binary settings choices $X$ and $Y$ are provided to the devices of Alice and Bob as the trial inputs, and the corresponding binary outcomes $A$ and $B$ are obtained from the devices as the trial outputs. The inputs and outputs of a trial together are called the results of the trial. As is conventional, values of a random variable are denoted by the corresponding lower-case symbol. Thus, $x$ is a value of the random variable $X$, and we have $a, b, x, y \in \{0, 1\}$. The last trial in a block is called the spot-checking trial. For the spot-checking trial, the inputs $X$ and $Y$ are uniformly distributed, while for the other trials in a block, the inputs are fixed to be $X = 0$ and $Y = 0$. In this work, we denote probability distributions by lower-case Greek letters (such as $\mu$ and $\nu$). The expectation and variance of a random variable $X$ according to a probability distribution $\mu$ are denoted by $\mathbb{E}_\mu(X)$ and $\mathbb{V}_\mu(X)$, respectively. The probability of an event $\Phi$ according to $\mu$ is denoted by $P_\mu(\Phi)$.

I. PROBABILITY ESTIMATION

In this section, we first introduce the main concepts of probability estimation [24][25], namely models and probability estimation factors (PEFs). Then we explain how PEFs can certify the smooth conditional min-entropy in the sequence of outputs conditional on the sequence of inputs as well as the classical side information $E$.

A. Models and PEFs

Consider a generic block obtained in the experiment. Let $L$ be the block-length random variable with value $l \in \{1, ..., 2^k\}$. For the $j$'th trial in the block where $j \in \{1, ..., l\}$, the inputs for Alice and Bob are denoted by $X_j$ and $Y_j$, and the corresponding outputs are denoted by $A_j$ and $B_j$. We use $X$ and $Y$ to denote the sequences of inputs of Alice and Bob in the block. Similarly, $A$ and $B$ denote the sequences of the corresponding outputs. For convenience, below we abbreviate the $X_j$ and $Y_j$ together as $Z_j$, and the $A_j$ and $B_j$ together as $C_j$. That is, $Z_j = X_jY_j$ and $C_j = A_jB_j$. We write $Z$ and $C$ for the sequence of $Z_j$'s and $C_j$'s in the block, respectively. The $C_j$ and $Z_j$ together are called the results of the $j$'th trial in the block, and the $C$ and $Z$ together are called the results of the block. As defined so far, the value $l$ of the random variable $L$ is the length of these sequences. Therefore, $C_j$ and $Z_j$ are not yet defined for $j > l$, but for randomness generation we need to pad the sequences with a string of zeros to the maximum length $2^k$ (see Sect. II). For the analysis below, instead we fill the sequences to the maximum length with the special symbol $\ast$ so that $C_j = Z_j = \ast$ for $j > l$. We call the trials that are actually executed real trials and the trials whose results are $\ast$-filled virtual trials. So, a block has a random number $L$ of real trials as well as $(2^k - L)$ virtual trials.

To certify randomness with respect to the classical side information $E$, we need to characterize the set of all possible probability distributions of $CZ$ achievable by a block conditionally on each value $e$ of $E$. That is, we need to characterize the set of all possible probability distributions of $CZ$ which satisfy verifiable physical constraints on device behavior and are achievable by a block. It suffices to characterize a superset $C$ of this set. This superset is called the model for a block. To describe the model for a block, we first consider the case of a single trial in the block. The model for each trial $j$ in the block may depend on the past, particularly on whether the block has not yet ended as expressed by the event $L \geq j$. This trial model is a superset of the set of all possible probability distributions of $C_jZ_j$ which satisfy verifiable physical constraints and are achievable by the $j$'th trial conditionally on the past and classical side information $E$. In our experiment the quantum devices used in each trial are constrained only by quantum mechanics and locality. Given this and the condition $L \geq j$, we can take the model $M_j^{(r)}$ for the $j$'th real
trial to be the set of probability distributions $\mu(C_j Z_j)$ satisfying the following two constraints: 1) The conditional distributions $\mu(C_j | Z_j)$ with $C_j = A_j B_j$ and $Z_j = X_j Y_j$ satisfy the non-signaling conditions [31] and Tsirelson’s bounds [22]. The set of such conditional distributions is denoted as $\mathcal{T}$, which is a convex polytope with 80 extreme points and includes all conditional distributions achievable by quantum mechanics (see Sect. VIII. A of Ref. [23]). 2) The inputs $Z_j = X_j Y_j$ are chosen according to a fixed distribution that depends on the trial position in the block as follows: With probability $q_j = 1/(2^k - j + 1)$ the $j$’th trial is a spot-checking trial and so the distribution of $Z_j$ is uniform, and with probability $(1 - q_j)$ the trial is a non-spot-checking trial and so the inputs $Z_j = X_j Y_j$ are fixed to be $X_j = 0$ and $Y_j = 0$. The experiment is configured so that both the quantum devices and side information $E$ do not know which of the two possibilities will occur at the next trial of the block. The only information available to the devices and $E$ is that the block has not yet ended. Thus, from the point of view of the devices or $E$, conditional on $L \geq j$ the input distribution $\mu(X_j Y_j)$ for each member $\mu$ of $\mathcal{M}^{(i)}$ satisfies that $\mu(X_j = 0, Y_j = 0) = 1 - 3q_j / 4$ and $\mu(X_j = x, Y_j = y) = q_j / 4$ if $x \neq 0$ or $y \neq 0$. So we define the fixed input distribution $\nu_j(X_j Y_j)$ according to

$$\nu_j(X_j = x, Y_j = y) = \begin{cases} 1 - 3q_j / 4 & \text{if } x = y = 0, \\ q_j / 4 & \text{if } x \neq 0 \text{ or } y \neq 0. \end{cases}$$

(S1)

As the input distribution $\nu_j(XY)$ is fixed, the same as $\mathcal{T}$ the model $\mathcal{M}^{(i)}$ is a convex polytope with 80 extreme points. We assume that after the last, spot-checking trial of the block, both the quantum devices and $E$ learn that the block has ended. Given our definitions, the model $\mathcal{M}^{(v)}$ for the virtual trial with $C_j = Z_j = *$ and $j > l$, where $l$ is the actual block length, becomes trivial in the sense that the model has only a fixed and deterministic distribution. Depending on whether the condition $L \geq j$ is satisfied or not, the model for the $j$’th trial in a block is characterized as $\mathcal{M}^{(i)}$ or $\mathcal{M}^{(v)}$ introduced above. We denote the model for the $j$’th trial by $\mathcal{M}_j$ when it is not specified whether the condition $L \geq j$ is satisfied or not.

To perform probability estimation in order to certify randomness in $C_j$ conditional on $Z_j$ and $E$, we introduce probability estimation factors (PEFs) for the trial model $\mathcal{M}_j$. A PEF with a positive power $\beta$ is a non-negative function $F_j : cz \mapsto F_j(cz)$ satisfying the PEF inequality

$$\sum_{cz} \mu(C_j = c, Z_j = z) F_j(cz) \mu(C_j = c| Z_j = z)^\beta \leq 1$$

(S2)

for each probability distribution $\mu(C_j Z_j)$ in the trial model $\mathcal{M}_j$. We note that to satisfy the PEF inequality for all distributions in $\mathcal{M}_j$, it suffices to satisfy this inequality for the extremal distributions of $\mathcal{M}_j$ according to Lemma 14 of Ref. [24]. Furthermore, the constant function $F(cz) = 1$ for all $cz$ is a PEF with power $\beta$ for all models and all $\beta > 0$. We choose this constant function as the PEF for each virtual trial.

Next we can construct the model $\mathcal{C}$ for a block as a chain of models $\mathcal{M}_j$ for each trial $j$ in the block. Let $Z_{<j}$ and $C_{<j}$ be the sequences of trial inputs and outputs before the $j$’th trial in the block. Define the random variable $S_j$ as $S_j = 1$ if $L \geq j + 1$ and $S_j = 0$ otherwise. Let $S$ denote the sequence of $S_j$’s for the block. Then the sequence $S$ consists of $(L - 1)$ consecutive 1’s followed by $(2^k - L + 1)$ consecutive 0’s. Equivalently, the block length $L$ is determined by $S$. To specify the distribution of results $\mathcal{C} \mathcal{Z} S$ in a block with length $L$, it is equivalent to know the joint distribution $\mu$ of the variables $\mathcal{C}$, $\mathcal{Z}$ and $S$. Moreover, we have the following two observations: 1) If $Z_j = *$ or $Z_j \neq 0$, $S_j = 0$ because in both cases it is true that $L \leq j$. If $Z_j = 0$, then $L \geq j$ and so the probability that $S_j = 0$ is the probability that the $j$’th trial is the spot-checking trial given that $L \geq j$, which is equal to $1/(2^k - j + 1)$. Therefore, $S_j$ is conditionally independent of $C_{<j}$, $Z_{\leq(j-1)}$ and $S_{\leq(j-1)}$ given $Z_j$. 2) The distribution of $C_j Z_j$ is conditionally independent of $S_{\leq(j-1)}$ given $C_{<j}$. In view of the above two observations and by the chain rule for probability distributions, we have

$$\mu(\mathcal{C} \mathcal{Z} S) = \prod_{j=1}^{2^k} \mu(C_j Z_j S_j | C_{<j} Z_{<j} S_{<j}) = \prod_{j=1}^{2^k} \mu(S_j | C_{<j} Z_{<j} S_{<j}) \mu(C_j Z_j | C_{<j} Z_{<j} S_{<j}) = \prod_{j=1}^{2^k} \mu(S_j | Z_j) \mu(C_j Z_j | C_{<j} Z_{<j} S_{j-1}),$$

(S3)

where to avoid issues, we define conditional probabilities to be 0 if the condition has probability 0. It therefore suffices
to specify \( \mu(C_j Z_j | C_{<j} Z_{<j} S_{j-1}) \), or equivalently, the distributions \( \mu(C_j Z_j | C_{<j} Z_{<j}, L \geq j) \) and \( \mu(C_j Z_j | C_{<j} Z_{<j}, L < j) \). The model \( \mathcal{C} \) is specified by requiring the former to be in \( \mathcal{M}_j^{(c)} \) and the latter to be determined with \( C_j Z_j = ** \).

In general when constructing chained models for the purpose of randomness generation based on PEFs, it is necessary that the next trial’s inputs satisfy a Markov-chain condition to prevent leaking information about the past outputs via the future inputs, see Sect. IV.A of Ref. [25]. Here, the inputs are chosen independently of the past outputs so the necessary conditions are automatically satisfied. Formally, it is necessary to include the condition \( L \geq j \) during a block as a part of the inputs for the trial \( j \) and keep calibration information used during an experiment as private information accessible only to the experimentalists, as explained in Sect. IV.A and Sect. IV.B of Ref. [25].

As for a single trial, for a block we can define the PEF with positive power \( \beta \) as a non-negative function \( G : \mathbf{cz} \mapsto G(\mathbf{cz}) \) satisfying the PEF inequality

\[
\sum_{\mathbf{cz}} \mu(\mathbf{C} = \mathbf{c}, Z = \mathbf{z}) G(\mathbf{cz}) \mu(\mathbf{C} = \mathbf{c} | Z = \mathbf{z})^\beta \leq 1
\]  

(S4)

for each probability distribution \( \mu(\mathbf{Cz}) \) in the model \( \mathcal{C} \) for the block. When the model \( \mathcal{C} \) is obtained by chaining the trial models \( \mathcal{M}_j \) as defined above, the PEFs satisfy the following chaining property: If for each possible trial \( j \) in a block \( F_j \) is a PEF with power \( \beta \) for \( \mathcal{M}_j \), then the product \( G = \prod_{j=1}^{k} F_j \) is a PEF with the same power \( \beta \) for the chained model \( \mathcal{C} \) (see the proof of Thm. 9 in Ref. [24]). Therefore, we need only to construct trial-wise PEFs \( F_j \) in order to obtain a PEF for a block. When we do so, we chose \( F_j = 1 \) when \( j > L \), so that \( G = \prod_{j=1}^{L} F_j \).

We emphasize that in this work the input distribution \( \nu_j(\mathbf{XY}) \) is assumed to be exact as given in Eq. (S1). When the random bits used for determining both the actual block length and the settings choices in the spot-checking trial are not perfectly uniform but have some adversarial biases, the input distribution \( \nu_j(\mathbf{XY}) \) can deviate from Eq. (S1). We may then constrain it in a convex polytope by generalizing the arguments presented in Refs. [10, 25]. In this case, we can still certify randomness by probability estimation but at the cost of a larger number of trials to certify a fixed amount of randomness as compared with the case of no adversarial bias, an issue which deserves further investigation. Allowing for adversarial biases in the input bits for spot checks and settings choices is required but not sufficient for randomness amplification [33, 34]. Randomness amplification with protocols such as ours also requires accounting for adversarial biases in the seed bits used by the extractor. There are extractors that can handle such biases and can be used for randomness amplification [35].

B. Certifying smooth min-entropy

Suppose that the number of blocks actually observed in an experiment is \( n_b \). Denote the sequence of inputs of the \( i \)’th block by \( \mathbf{Z}_i \), and the sequence of the corresponding outputs by \( \mathbf{C}_i \). Furthermore, let \( \bar{\mathbf{Z}} \) be the sequence of inputs of the whole experiment, and \( \bar{\mathbf{C}} \) be the sequence of the corresponding outputs. That is, \( \bar{\mathbf{Z}} = (\mathbf{Z}_1, ..., \mathbf{Z}_{n_b}) \) and \( \bar{\mathbf{C}} = (\mathbf{C}_1, ..., \mathbf{C}_{n_b}) \). For each block indexed by \( i \) we can construct its model \( \mathcal{C}_i \) according to Sect. IA. Each model \( \mathcal{C}_i \) is the same as the model \( \mathcal{C} \) constructed in Sect. IA but with the random variables associated with the \( i \)’th block.

Then, in the same way as we constructed the chained model for a block, we can construct the model \( \mathcal{H} \) for the whole experiment by chaining the models \( \mathcal{C}_i \), \( i \in \{1, ..., n_b\} \). The previously mentioned Markov-chain conditions are again satisfied because the inputs for a block are chosen according to a known distribution independent of the past.

We would like to certify the amount of extractable randomness in the outputs \( \bar{\mathbf{C}} \) conditional on the inputs \( \bar{\mathbf{Z}} \) and the classical side information \( E \). To quantify the amount of extractable randomness, we define the following quantities: 1) Given the joint distribution \( \mu_{\mathbf{Z}} \), outputs \( \bar{\mathbf{C}} \) and classical side information \( E \), the quantity \( \sum_{\mathbf{z}} \mu_{\mathbf{z}}(\mathbf{z}) \max_{\nu}(\mu(\bar{\mathbf{C}} | \mathbf{z}) \nu(\mathbf{z})) \) is called the (average) maximum guessing probability \( P_{\text{guess}}(\bar{\mathbf{C}} | \mathbf{Z}) \nu_{\mu} \) of \( \bar{\mathbf{C}} \) given \( \mathbf{Z} = \mathbf{z} \) and \( E \) according to \( \mu \), and the quantity \( -\log_2(P_{\text{guess}}(\bar{\mathbf{C}} | \mathbf{Z} E; \mu)) \) is called the (classical) \( \mathbf{Z} E \)-conditional min-entropy \( H_{\text{min}}(\bar{\mathbf{C}} | \mathbf{Z} E; \mu) \) of \( \bar{\mathbf{C}} \) according to \( \mu \). For discussions of these quantities, see Refs. [36, 37]. 2) The total-variation distance between two distributions \( \mu \) and \( \nu \) of \( X \) is defined as

\[
\text{TV}(\mu, \nu) = \frac{1}{2} \sum_x |\mu(x) - \nu(x)|.
\]

(S5)

3) The distribution \( \mu_{\bar{\mathbf{C}}} \) of \( \bar{\mathbf{C}} \mathbf{Z} E \) has \( \epsilon_{\nu} \)-smooth maximum guessing probability \( p \) if there exists a distribution \( \nu \) of \( \bar{\mathbf{C}} \mathbf{Z} E \) with \( \text{TV}(\nu, \mu) \leq \epsilon_{\nu} \) and \( P_{\text{guess}}(\bar{\mathbf{C}} | \mathbf{Z} E; \nu) \leq p \). The minimum \( p \) for which \( \mu \) has \( \epsilon_{\nu} \)-smooth maximum guessing probability \( p \) is denoted by \( P_{\text{guess}}(\bar{\mathbf{C}} | \mathbf{Z} E; \mu) \). The quantity \( H_{\text{min}}(\epsilon_{\nu})(\bar{\mathbf{C}} | \mathbf{Z} E; \mu) = -\log_2(P_{\text{guess}}(\bar{\mathbf{C}} | \mathbf{Z} E; \mu)) \) is called the (classical) \( \epsilon_{\nu} \)-
smooth $\mathbf{Z}E$-conditional min-entropy of $\mathbf{C}$ according to $\mu$. This quantity is a specialization of the quantum smooth conditional min-entropy \cite{38} to probability distributions.

We consider an arbitrary joint distribution $\mu$ of $\mathbf{Z}$, $\mathbf{C}$ and $E$ that satisfies the model $\mathcal{H}$ of the experiment. We say that the distribution $\mu(\mathbf{C}\mathbf{Z}E)$ satisfies the model $\mathcal{H}$ if for each value $e$ of $E$, the conditional distribution $\mu(\mathbf{C}\mathbf{Z}|e)$, viewed as a distribution of $\mathbf{C}$ and $\mathbf{Z}$, is in the model $\mathcal{H}$. Given an arbitrary distribution $\mu(\mathbf{C}\mathbf{Z}E)$ satisfying the model $\mathcal{H}$, we quantify the amount of extractable randomness by the smooth conditional min-entropy $H_{\min}^\epsilon(\mathbf{C}E)_\mu$ defined as above. Our goal is to obtain a lower bound on $H_{\min}^\epsilon(\mathbf{C}E)_\mu$ without knowing which particular distribution $\mu(\mathbf{C}\mathbf{Z}E)$ describes the experiment. For this, let the PEF with power $\beta$ for the $i$'th block be $G_i$, which is a function of $C_i$ and $Z_i$, and let $T$ be the product of block-wise PEFs, that is, $T = \prod_{i=1}^{n_b} G_i$. Denote the number of possible outputs for a real trial by $|C|$, and let $|C| = |C|^{n_b \times 2^k}$ be the maximum number of possible outputs after $n_b$ blocks, where each block has at most $2^k$ real trials. A lower bound on the smooth conditional min-entropy is obtained according to Thm. 1 of Ref. \cite{24} with the replacement of the trial-wise PEFs by the block-wise PEFs. We state the theorem for our case of interest as follows:

**Theorem 1.** (Thm. 1 and Thm. 11 of Ref. \cite{24}) Let $1 \geq \epsilon_s > 0$ and $1 \geq p \geq 1/|\mathbf{C}|$. Define $\Phi$ to be the event that $T \geq 1/(p^\beta \epsilon_s)$. Suppose that $\mu$ is an arbitrary joint probability distribution of the inputs $\mathbf{Z}$, the outputs $\mathbf{C}$ and the classical side information $E$ satisfying the model $\mathcal{H}$. Let $\kappa = \mathbb{P}_\mu(\Phi)$ be the probability of the event $\Phi$ according to $\mu$, and denote the distribution of $\mathbf{Z}$, $\mathbf{C}$ and $E$ conditional on $\Phi$ by $\mu_{\mid \Phi}$. Then the smooth conditional min-entropy given $\Phi$ satisfies

$$H_{\min}^\epsilon(\mathbf{C}E)_{\mu_{\mid \Phi}} \geq -\log_2(p) + \frac{1 + \beta}{\beta} \log_2(\kappa). \tag{S6}$$

We remark that the bound in Eq. (S6) implies the following statement: For every $\kappa' > 0$ and for each probability distribution $\mu(\mathbf{C}\mathbf{Z}E)$ satisfying the model $\mathcal{H}$, either the probability of the event $\Phi$ according to $\mu$ is less than $\kappa'$ or the smooth conditional min-entropy given $\Phi$ satisfies

$$H_{\min}^\epsilon(\mathbf{C}E)_{\mu_{\mid \Phi}} \geq -\log_2(p) + \frac{1 + \beta}{\beta} \log_2(\kappa'). \tag{S7}$$

The event $\Phi$ can be interpreted as the event that the experiment succeeds. When the experiment succeeds, we compose the smooth conditional min-entropy bound in Eq. (S6) with a classical-proof strong extractor to obtain near-uniform random bits as detailed in the next section.

## II. RANDOMNESS GENERATION

Our goal is to design a sound randomness-generation protocol, meaning that the protocol has guaranteed performance no matter how low the success probability is. In Sect. \[\text{IIA}\] we introduce the extractor used, which determines the choices of various parameters in the protocol. Then in Sect. \[\text{IIB}\] we formalize the definition of soundness. Finally in Sect. \[\text{IIC}\] we present our randomness-generation protocol and prove its soundness.

### A. Classical-proof strong extractors

An extractor is a function $E : (C, S) \mapsto R$, which extracts near-uniform random bits from the input randomness source $C$ with the help of the seed $S$ and stores the extracted bits in $R$. In this work, we assume that the seed $S$ is uniformly random and independent of all other random variables. Suppose that before applying the extractor the joint distribution of the input $C$, the seed $S$ and the classical side information $E$ is given by the product of distributions $\mu(CE)$ and $\tau(S)$, where $\mu(CE)$ denotes the distribution of $C$ and $E$, and $\tau(S)$ is the uniform distribution of $S$. After applying the extractor, the joint distribution of the output $R$, the seed $S$ and the side information $E$ is denoted by $\pi_\mu(RSE)$, where the subscript $\mu$ indicates that the distribution $\pi$ is obtained by applying the extractor with the distribution $\mu$. The goal is for the distribution $\pi_\mu(RSE)$ to be close to the product of distributions $\tau(RS)$ and $\mu(E)$, where $\tau(RS)$ is the uniform distribution of $R$ and $S$ together, and $\mu(E)$ is the marginal distribution of $E$ according to $\mu(CE)$. Let $|C|$, $|R|$ and $|S|$ denote the numbers of possible values taken by $C$, $R$ and $S$, respectively, and define $m_{\text{in}} = \log_2(|C|)$, $k_{\text{out}} = \log_2(|R|)$ and $d_s = \log_2(|S|)$. Then, a function $E : (C, S) \mapsto R$ is called a classical-proof strong extractor with parameters $(m_{\text{in}}, d_s, k_{\text{out}}, s_{\text{in}}, \epsilon_{\text{ext}})$ if for every distribution $\mu(CE)$ with conditional min-entropy...
$H_{\min}(C|E)_\mu \geq \sigma_{\text{in}}$ bits, the joint distribution of $R$ and $S$ is close to uniform and independent of the classical side information $E$ in the sense that the total-variation distance $\text{TV}(\pi_\mu(RSE), \tau(RS)\mu(E)) \leq \epsilon_{\text{ext}}$.

To ensure the proper functioning of an extractor $E$, the parameters $(m_{\text{in}}, d_s, k_{\text{out}}, \sigma_{\text{in}}, \epsilon_{\text{ext}})$ need to satisfy a set of constraints, called extractor constraints. The extractor constraints depend on the specific classical-proof strong extractor used, but these constraints always include that $1 \leq \sigma_{\text{in}} \leq m_{\text{in}}$, $d_s \geq 0$, $k_{\text{out}} \leq \sigma_{\text{in}}$, and $0 < \epsilon_{\text{ext}} \leq 1$. In this work, we use Trevisan’s extractor [39] as implemented by Mauerer, Portmann, and Scholz [26], which is a classical-proof strong extractor requiring a short seed. We refer to this extractor as the TMPS extractor $E_{\text{TPMS}}$. To apply the TMPS extractor, additional extractor constraints [8] are

$$k_{\text{out}} + 4 \log_2(k_{\text{out}}) \leq \sigma_{\text{in}} - 6 + 4 \log_2(\epsilon_{\text{ext}}),$$

$$d_s \leq w^2 \max \left(2, 1 + \left\lceil \frac{\log_2(k_{\text{out}} - e) - \log_2(w - e)}{\log_2(e) - \log_2(e - 1)} \right\rceil \right),$$

where $w$ is the smallest prime larger than $2\log_2(4m_{\text{in}}k_{\text{out}}^2/\epsilon_{\text{ext}}^2)$. We remark that when the extractor error $\epsilon_{\text{ext}}$ is specified in trace distance, Trevisan’s extractor with the above constraints actually works in the presence of quantum side information, as shown in Refs. [26] [27].

**B. Protocol soundness**

Consider a generic randomness-generation protocol $\mathcal{G}$, where there is a binary flag $P$ whose value 0 or 1 indicates failure or success, respectively. Conditional on the success event $P = 1$, the protocol $\mathcal{G}$ produces not only a string of fresh random bits stored in $R$, but also a string of previously used random bits stored in $S'$. The bit string $R$ is of length $k_{\text{out}}$, and the bit string $S'$ is of length $d_s$ consisting of the random seed $S$ used for randomness extraction. The protocol outputs $R$, $S'$ and $P$ are determined not only by the results of the considered experiment, but also by the specific classical-proof strong extractor used and its seed $S$.

Recall that the model for the experiment considered with inputs $\mathcal{Z}$ and outputs $\mathcal{C}$ is $\mathcal{H}$. Consider an arbitrary joint distribution $\mu$ of the inputs $\mathcal{Z}$, the outputs $\mathcal{C}$ and the classical side information $E$ satisfying the model $\mathcal{H}$. Suppose that $\mu$ is the relevant distribution before running the protocol. Let $\pi_\mu$ be the distribution of the protocol outputs $R, S', P$, the experiment inputs $\mathcal{Z}$ and the side information $E$ after running the protocol, where the subscript $\mu$ of $\pi$ indicates that the distribution $\pi$ is induced by $\mu$. The distribution conditional on the success event $P = 1$ is $\pi_\mu(RS'\mathcal{Z}E|P = 1)$. A randomness-generation protocol $\mathcal{G}$ is $\epsilon$-sound for the distribution $\mu(\mathcal{CZ}E)$ if there exists a distribution $\nu(\mathcal{Z}E)$ such that

$$\text{TV}(\pi_\mu(RS'\mathcal{Z}E|P = 1), \tau(RS')\nu(\mathcal{Z}E)) \mathbb{P}_\mu(P = 1) \leq \epsilon,$$

where $\tau(RS')$ is the uniform distribution over all possible values of the variables $R$ and $S'$ together, and $\mathbb{P}_\mu(P = 1)$ is the probability of success according to the distribution $\mu(\mathcal{CZ}E)$. Our goal is to obtain an $\epsilon$-sound protocol for the model $\mathcal{H}$ of our experiment, that is, a protocol which is $\epsilon$-sound for all distributions $\mu(\mathcal{CZ}E)$ satisfying the model $\mathcal{H}$. Note that it may be desirable to have the total-variation distance conditional on success be bounded from above by $\delta$ given that the success probability is larger than some threshold $\kappa$. For this it suffices to choose the soundness error $\epsilon$ as $\epsilon \leq \delta \kappa$. If one wishes to be equally conservative for both $\delta$ and $\kappa$, it makes sense to set $\epsilon = \delta^2$.

We remark that a protocol $\mathcal{G}$ is called $\eta$-complete for a model $\mathcal{H}$ if there exist a distribution $\nu(\mathcal{CZ}E)$ satisfying the model such that the success probability according to $\nu$ satisfies $\mathbb{P}_\nu(P = 1) \geq \eta$. Completeness is an important factor to consider when designing an experiment, while soundness guarantees the performance of the protocol regardless of the actual implementation of the experiment.

**C. PEF-based randomness-generation protocol**

Recall that we consider an experiment with a sequence of time-ordered blocks, where the length of each block is uniformly at random chosen from the set $\{1, \ldots, 2^k\}$ and each block $i$ has inputs $\mathcal{Z}_i$ and outputs $\mathcal{C}_i$. Suppose that in the experiment at most $N_b$ blocks are acquired. Then the maximum length in bits of the outputs $\mathcal{C}$ of the whole experiment is $m_{\text{in}} = N_b \times 2^k \times k_c$, where $k_c$ is the length in bits of the outputs for each real trial in a block. In our experiment, $k_c = 2$ considering that in each real trial there is a binary output for each of Alice and Bob. To extract $k_{\text{out}}$ random bits at soundness error $\epsilon$ from the outputs $\mathcal{C}$ with the help of an extractor $E$, the lower bound $\sigma_{\text{in}}$ on
the conditional min-entropy of \( \bar{C} \) certified using PEFs with power \( \beta \), the seed length \( d_s \), and the extractor error \( \epsilon_{\text{ext}} \) need to be chosen from the set \( X \) defined by

\[
X = \{(\sigma_{\text{in}}, d_s, \epsilon_{\text{ext}}) : \text{the parameters} \ (m_{\text{in}}, d_s, k_{\text{out}}, \sigma_{\text{in}}, \epsilon_{\text{ext}} < \epsilon) \text{ satisfy the extractor constraints for } \mathcal{E},
\quad \text{and} \quad \sigma_{\text{in}} \leq m_{\text{in}} + \frac{1+\beta}{\beta} \log_2(\epsilon) \}.
\] (S10)

The protocol for end-to-end randomness generation is displayed in Protocol 1, where the notation \( 0^{-l} \) denotes a string of \( l \) consecutive zeros and the notation \( c0^{-l} \) denotes a string obtained by padding \( c \) with \( 0^{-l} \). We emphasize that the parameters \( (N_b, m_{\text{in}}, \sigma_{\text{in}}, \beta, d_s, \epsilon_{\text{ext}}) \) specified above are determined before running the protocol. In practice, they are determined using the commissioning data collected before the randomness-generation experiment, see Sect. [IV] for details. We call the difference \( \epsilon_{\text{en}} = \epsilon - \epsilon_{\text{ext}} \) the entropy error. In the rest of this subsection, we use \( Z \) and \( C \) to denote the sequences of inputs and outputs of real trials in a block, and use \( z \) and \( c \) to denote the corresponding sequences of observed inputs and outputs. For each virtual trial in a block, we set its inputs and outputs to be \( 0^{-k_z} \) and \( 0^{-k_c} \), where \( k_z \) and \( k_c \) are the lengths in bits of the inputs and outputs for each real trial.

| Protocol 1: Input-conditional randomness generation. |
|-----------------------------------------------|
| **Input** :                                    |
| • \( k_{\text{out}} \)—the number of fresh random bits to be generated. |
| • \( \epsilon \)—the soundness error satisfying \( \epsilon \in (0, 1] \). |
| **Given** :                                    |
| • \( N_b \)—the maximum number of blocks to be acquired from the experiment. |
| • \( G_i \)—the PEF with power \( \beta \) for each block \( i \). |
| • \( \mathcal{E} \)—the classical-proof strong extractor used. |
| **Output** : \( R, S', P \) as specified in the first paragraph of Sect. [II B] |
| Choose \( (\sigma_{\text{in}}, d_s, \epsilon_{\text{ext}}) \) from the set \( X \) defined in Eq. (S10) ; \ // Ensure the set \( X \) to be non-empty. |
| Get an instance \( s \) of the uniformly random seed \( S \) of length \( d_s \); |
| Set \( \epsilon_{\text{en}} = (\epsilon - \epsilon_{\text{ext}}), q = 2^{-\sigma_{\text{in}}}, \epsilon, \text{ and } t_{\text{min}} = 1/(q^6 \epsilon_{\text{en}}) \); |
| Set \( t = 1 \); |
| **for** \( i \leftarrow 1 \text{ to } N_b \) **do** |
| Run the experiment to acquire a block of real trials with inputs \( z_i \) and outputs \( c_i \); |
| Compute \( g_i = G_i(c_i, z_i) \), and update \( t \) as \( t = t \times g_i \); |
| Set \( c'_i = c_i 0^{-k_c \times (2^k - l_i)} \), where \( l_i \) is the actual length of the \( i \)th block; |
| **if** \( t \geq t_{\text{min}} \) **then** |
| Record the number of blocks actually acquired as \( n_b = i \), and stop acquiring the future blocks; |
| Set \( c'_j = 0^{-k_c \times 2^k} \) for \( j \in \{n_b + 1, \ldots, N_b\} \), and set \( c' = (c'_1, \ldots, c'_{N_b}) \); |
| Return \( P = 1, R = E(c', s), S' = s \) ; \ // Protocol succeeded. |
| **end** |
| **if** \( t < t_{\text{min}} \) **then** |
| Record the number of blocks actually acquired as \( n_b = N_b \); |
| Return \( P = 0, R = 0^{-k_{\text{out}}}, S' = s \) ; \ // Protocol failed. |
| **end** |

Several remarks on the implementation of Protocol 1 are as follows. First, we assume that the set \( X \) defined in Eq. (S10) is non-empty. This assumption needs to be checked before invoking the protocol, and the input parameters can be adjusted to ensure that the assumption holds. Second, to extract uniform random bits from the outputs of an experiment, all extractors studied in literature require the length of the experimental outputs to be fixed beforehand. However, in our experiment the length \( L \) of a block is not prefixed but uniformly at random chosen from the set \( \{1, \ldots, 2^k\} \). So, we pad the outputs \( c \) actually observed in a block with \( 0^{-k_c \times (2^k - l)} \), a string of \( k_c \times (2^k - l) \) consecutive
zeros with \( l \) being the actual block length, to ensure the fixed-length requirement. Third, since the constant function \( F = 1 \) is a valid PEF with any positive power for a trial [24, 25], the constant function \( G = 1 \) is a valid PEF with any positive power for a block of an arbitrary length. Therefore, if the parameter \( t \) in Protocol \( \mathcal{P} \) takes a value larger than \( t_{\text{min}} \) after the \( i \)’th block where \( i < N_b \), we can set the PEFs for all future blocks to be \( G = 1 \) such that the results of the future blocks will not affect the value of \( t \). Equivalently, we can stop acquiring the future blocks and set the outputs of each future block to be \( 0^{-k_c \times 2^k} \) (in order to ensure the fixed-length requirement by the extractor). We call the blocks that are not actually acquired in an experiment virtual blocks. Fourth, for each block \( i \) which is actually acquired, we construct its PEF with power \( \beta \) as \( G_i = \prod_{j=1}^{L_i} F_{ij} \), where \( F_{ij} \) is a PEF with power \( \beta \) for the \( j \)’th real trial in the \( i \)’th block and \( L_i \) is the block length. Before acquiring the \( j \)’th trial in the \( i \)’th block, the PEF \( F_{ij} \) for this trial should be fixed. Otherwise, the soundness of the protocol is not assured.

The soundness of Protocol \( \mathcal{P} \) can be proven by composing Thm. 1 with the classical-proof strong extractor \( \mathcal{E} \) used in the protocol.

**Theorem 2.** Protocol \( \mathcal{P} \) is an \( \epsilon \)-sound randomness-generation protocol for the model \( \mathcal{H} \) of the experiment.

**Proof.** Let the number of blocks actually acquired in the experiment be \( n_b \leq N_b \), and let \( \mu(\mathcal{C}|\mathcal{Z}|E) \) be an arbitrary distribution satisfying the model \( \mathcal{H} \) of the experiment. Here \( \mathcal{C} = (C_1, ..., C_{N_b}) \) and \( \mathcal{Z} = (Z_1, ..., Z_{N_b}) \). Suppose that when running the protocol, the inputs and outputs of the experiment are instantiated to \( \mathcal{C} \) and \( \mathcal{Z} \) according to \( \mu(\mathcal{C}|\mathcal{Z}|E) \).

For each block \( i \) actually acquired, let \( C'_i = C_i^{(l_i/k_c \times 2^k)} \) and \( c'_i = 0^{-k_c \times 2^k} \), where \( k_c \) is the length in bits of the outputs for each real trial and \( l_i \) is the actual length of the \( i \)’th block. Similarly, we have \( Z'_i = Z_i^{(l_i/k_c \times 2^k)} \) and \( z'_i = 0^{-k_c \times 2^k} \), where \( k_c \) is the length in bits of the inputs for each real trial. Moreover, for each virtual block \( j \) with \( j \in \{ (n_b+1), ..., N_b \} \) we set \( C'_j = 0^{-k_c \times 2^k} \) and \( z'_j = 0^{-k_c \times 2^k} \). Define \( \mathcal{C}' = (C'_1, ..., C'_{N_b}) \) and \( \mathcal{Z}' = (Z'_1, ..., Z'_{N_b}) \). The distribution \( \mu(\mathcal{C}'|\mathcal{Z}'|E) \) is fully determined by the distribution \( \mu(\mathcal{C}|\mathcal{Z}|E) \). We emphasize that for each possible distribution \( \mu(\mathcal{C}|\mathcal{Z}|E) \) satisfying the model \( \mathcal{H} \) of the experiment, we can determine the corresponding distribution \( \mu(\mathcal{C}'|\mathcal{Z}'|E) \). The set of all possible such distributions \( \mu(\mathcal{C}'|\mathcal{Z}'|E) \) is defined to be the generalized model \( \mathcal{H}' \) of the experiment.

For each block \( i \leq n_b \) which is actually acquired in the experiment, as the constant function \( F = 1 \) is a valid PEF with any positive power for a trial [24, 25], the function \( G_i(C, Z) \) is a PEF with power \( \beta \) for the block with results \( C'_i Z'_i \). For the same reason, the constant function \( G_j = 1 \) is a PEF with power \( \beta \) for each virtual block \( j \) with \( j > n_b \). Therefore, if we set \( T(\mathcal{C}'|\mathcal{Z}') = \prod_{i=1}^{n_b} G_i(C_i, Z_i) \) and write the event \( \Phi = \{ Z|\mathcal{C}' : T(\mathcal{C}'|\mathcal{Z}') \geq t_{\text{min}} = 1/(q^\beta \epsilon_{\text{en}}) \} \), then when \( \mathcal{C}'|\mathcal{Z}' \in \Phi \), the protocol succeeds, that is, \( P = 1 \). Let the probability of success according to \( \mu \) be \( \kappa = \mathbb{P}(\Phi) \), and denote the joint distribution of \( \mathcal{C}'|\mathcal{Z}' \) and \( E \) conditional on success by \( \mu(\mathcal{C}'|\mathcal{Z}'|E|\Phi) \), abbreviated as \( \mu|\Phi \).

We first consider the case \( \kappa \in [\epsilon, 1] \). With the substitutions \( \mathcal{H} \rightarrow \mathcal{H}', \epsilon_s \rightarrow \epsilon_{\text{en}}/\kappa \) and \( p \rightarrow q \epsilon_{\text{en}}/\kappa \), the event \( \Phi \) in the statement of Thm. 1 becomes the same as the event \( \Phi \) defined above, and so the parameter \( \kappa \) in the statement of Thm. 1 becomes the same as the parameter \( \kappa \) introduced above. To apply Thm. 1 we need to check that the parameter \( p \) in the statement of Thm. 1 satisfies

\[
p \rightarrow q \epsilon_{\text{en}}^{1/\beta} = 2^{-\sigma_{\text{in}} \epsilon \kappa^{1/\beta}} \geq 2^{-\sigma_{\text{in}} \epsilon^{1+1/\beta}} \geq 1/2^{m_{\text{en}}} = 1/|\mathcal{C}'|,
\]

where the equation in the first line is according to the specification \( q = 2^{-\sigma_{\text{en}} \epsilon} \) in Protocol \( \mathcal{P} \). The inequality in the second line follows from \( \kappa \geq \epsilon \), and the inequality in the last line follows from \( \sigma_{\text{in}} \leq m_{\text{en}} + 1/\beta \log_2(\epsilon) \) as required in the definition of the set \( X \) (see Eq. S10). Therefore, we can apply Thm. 1 with the above substitutions to obtain

\[
H_{\text{min}}^{\epsilon_{\text{en}}/\kappa}(\mathcal{C}'|\mathcal{Z}'|E|\Phi) \geq -\log_2(q/\kappa).
\]

Considering that \( \kappa \geq \epsilon \) and that Protocol \( \mathcal{P} \) sets \( q = 2^{-\sigma_{\text{en}} \epsilon} \), Eq. S11 implies that

\[
H_{\text{min}}^{\epsilon_{\text{en}}/\kappa}(\mathcal{C}'|\mathcal{Z}'|E|\Phi) \geq \sigma_{\text{in}}.
\]

According to the definition of smooth conditional min-entropy (see the second paragraph of Sect. 1B), there exists a distribution \( \nu(\mathcal{C}'|\mathcal{Z}'|E) \) such that

\[
TV(\mu(\mathcal{C}'|\mathcal{Z}'|E|\Phi), \nu(\mathcal{C}'|\mathcal{Z}'|E)) \leq \epsilon_{\text{en}}/\kappa,
\]
and

\[ H_{\text{min}}(\mathcal{C}(\vec{Z}^\text{en})_{\nu} \geq \sigma_{\text{in}}. \]

(S14)

Because the parameters \( (m_{\text{in}}, d_s, k_{\text{out}}, \sigma_{\text{in}}, \epsilon_{\text{ext}}) \) satisfy the extractor constraints for \( \mathcal{E} \), we can apply the classical-proof strong extractor \( \mathcal{E} \) with the distribution \( \nu(\mathcal{C}^\text{en}\vec{Z}^\text{en}) \) and the resulting joint distribution \( \pi_{\nu} \) of the extractor output \( R \), the seed \( S \) and the classical side information \( \vec{Z}^\text{en} \) satisfies

\[ \text{TV}(\pi_{\nu}(RS\vec{Z}^\text{en}), \tau(RS)\nu(\vec{Z}^\text{en})) \leq \epsilon_{\text{ext}}, \]

(S15)

where \( \tau(RS) \) is the uniform distribution of \( R \) and \( S \), and \( \nu(\vec{Z}^\text{en}) \) is the marginal distribution of \( \vec{Z}^\text{en} \) and \( E \) according to \( \nu(\mathcal{C}^\text{en}\vec{Z}^\text{en}) \). Moreover, since the total-variation distance satisfies the data-processing inequality, Eq. (S13) implies that

\[ \text{TV}(\pi_{\mu}(RS\vec{Z}^\text{en}|\Phi), \pi_{\nu}(RS\vec{Z}^\text{en}|\Phi)) \leq \text{TV}(\mu(\mathcal{C}^\text{en}\vec{Z}^\text{en}|\Phi), \nu(\mathcal{C}^\text{en}\vec{Z}^\text{en}|\Phi)) \leq \epsilon_{\text{en}}/\kappa. \]

(S16)

The triangle inequality for the total-variation distance together with Eqs. (S15) and (S16) yield

\[ \text{TV}(\pi_{\mu}(RS\vec{Z}^\text{en}|\Phi), \tau(RS)\nu(\vec{Z}^\text{en})) \leq \text{TV}(\pi_{\mu}(RS\vec{Z}^\text{en}|\Phi), \pi_{\nu}(RS\vec{Z}^\text{en}))+\text{TV}(\pi_{\nu}(RS\vec{Z}^\text{en}), \tau(RS)\nu(\vec{Z}^\text{en})) \leq \epsilon_{\text{ext}}+\epsilon_{\text{en}}/\kappa. \]

We multiply both sides by \( \kappa \) to obtain the soundness statement

\[ \text{TV}(\pi_{\mu}(RS\vec{Z}^\text{en}|\Phi), \tau(RS)\nu(\vec{Z}^\text{en})) \kappa \leq \epsilon_{\text{ext}} \kappa + \epsilon_{\text{en}} \leq \epsilon_{\text{ext}} + \epsilon_{\text{en}} = \epsilon. \]

For the case \( \kappa < \epsilon \), since the total-variation distance cannot be larger than one,

\[ \text{TV}(\pi_{\mu}(RS\vec{Z}^\text{en}|\Phi), \tau(RS)\nu(\vec{Z}^\text{en})) \kappa \leq \kappa < \epsilon, \]

where \( \tau(RS) \) is the uniform distribution as in Eq. (S15) and \( \nu(\vec{Z}^\text{en}) \) is an arbitrary distribution of \( \vec{Z}^\text{en} \) and \( E \).

Therefore, the condition for \( \epsilon \)-soundness is satisfied for \( \mu(\mathcal{C}^\text{en}\vec{Z}^\text{en}) \), independently of the value of the success probability \( \kappa \). Equivalently, the condition for \( \epsilon \)-soundness is satisfied for \( \mu(\mathcal{C}Z^\text{en}) \), as the distribution \( \mu(\mathcal{C}^\text{en}\vec{Z}^\text{en}) \) is fully determined by \( \mu(\mathcal{C}Z^\text{en}) \). Because \( \mu(\mathcal{C}^\text{en}\vec{Z}^\text{en}) \) is an arbitrary distribution satisfying the model \( \mathcal{H} \), Protocol 1 is \( \epsilon \)-sound for the model \( \mathcal{H} \).

We remark that the above soundness proof, motivated by the soundness proof in the presence of quantum side information in Refs. [18, 19], is simpler than that presented in our previous work’s protocol Q, Thm. 21 of Ref. [25]. There we designed the protocol and proved its soundness using a linear strong extractor rather than a classical-proof strong extractor. By taking advantage of the linearity of the extractor, the soundness proof in Ref. [25] works with the parameter \( q = 2^{-\sigma_{\text{in}}} \), that is without the additional factor \( \epsilon \) as specified in Protocol 1. This has the effect of reducing the success threshold \( t_{\text{min}} \) for the product of block-wise PEFs by a factor of \( \epsilon^\beta \). Here we choose not to take advantage of this improvement to simplify the soundness proof and presentation of the protocol.

III. CONSTRUCTION OF BLOCK-WISE PEFs

In this section, we formulate the PEF optimization problem for randomness generation with block-wise PEFs. For randomness generation, the central task is to maximize the number of random bits generated after a fixed number of blocks, while for randomness expansion, the goal is different as we care about the difference between the number of random bits generated and the number of random bits consumed. The optimization problem for randomness expansion is based on the optimization problem formulated in this section and will be detailed in Sect. IV.

A. Formulation of block-wise PEF optimization

Let \( G_i(C_iZ_i) \) be a PEF with power \( \beta \) for the \( i \)'th block in an experiment. The experiment successfully implements Protocol 1 if the block-wise PEFs satisfy the condition \( \prod_{i=1}^{N_i} G_i(C_iZ_i) \geq t_{\text{min}} = 1/(q^\beta \epsilon_{\text{en}}) \) with \( q = 2^{-\sigma_{\text{in}}} \epsilon \), or
equivalently,
\[ \sum_{i=1}^{N_b} \log_2(G_i(C, Z_i))/\beta + \log_2(\epsilon_{en})/\beta + \log_2(\epsilon) \geq \sigma_{in}. \]  
(S17)

In this work, we call the quantity \( \sum_{i=1}^{N_b} \log_2(G_i(C, Z_i))/\beta \) the output entropy witness after \( N_b \) blocks. We call the quantity on the left-hand side of the above equation the adjusted output entropy witness, where the adjustment is for both the entropy error \( \epsilon_{en} \) and the soundness error \( \epsilon \). This adjusted output entropy witness is the one plotted in Fig. 4 of the main text. Hence, for randomness generation we aim to obtain a large expected value of the adjusted output entropy witness with as few blocks as possible, supposing that the quantum devices to be used perform as designed. The input-conditional output distribution \( \nu_h(C|Z) \) of these “honest” devices is thus known, independent and identical for each trial given the inputs, and anticipated to be the same as the actual device behavior in the absence of faults or interference. In our experiment, the (presumed current) honest-device input-conditional output distribution is determined by maximum likelihood estimation with data collected in Sect. VII B of Ref. [25]. Before the experiment, we can choose values for \( \sigma_{in} \) and \( \epsilon_{en} \) (see Protocol 1) and optimize over the block-wise PEFs and the power \( \beta \) such that the number \( N_{\text{exp}, b} \) of blocks required for success with honest devices, as defined below, is minimized. Then we fix the number of blocks \( N_b \) available in the experiment to be a number larger than the minimum number of blocks, so that the actual experiment succeeds with high probability if the quantum devices used are honest. Because for honest devices, the results of each block are independent and identically distributed (i.i.d.), in the pre-experiment optimization we set all \( G_i \) to be the same. In reality, we update the block-wise PEFs to be used for future blocks according to incoming calibration data during the experiment. The optimization of block-wise PEFs is the same in all cases, except that during the experiment parameters such as \( \beta \) and \( \epsilon_{en} \) are fixed.

Consider a generic next block with results \( CZ \) and model \( C \). With honest devices where the input-conditional output distribution for each trial is fixed to be \( \nu_h(C|Z) \), in principle we can determine the distribution \( \nu(CZ) \) for the next block’s results. However, the explicit expression for \( \nu(CZ) \) is somewhat involved. Here we just use the basics. For determining \( \nu(CZ) \), in view of Eq. \( \text{(S17)} \), it suffices to know the conditional distributions \( \nu(C_j Z_j|C_{<j} Z_{<j}, L \geq j) \) and \( \nu(C_j Z_j|C_{<j} Z_{<j}, L < j) \), where \( L \) is the block-length variable. We implicitly assume that the results \( C_{<j} Z_{<j} \) are consistent with whether the condition \( L \geq j \) holds or not. Given that \( L \geq j \), the distribution of \( C_j Z_j \) is independent of \( C_{<j} Z_{<j} \) for honest devices. So,

\[
\nu(C_j Z_j|C_{<j} Z_{<j}, L \geq j) = \nu(C_j |Z_j, L \geq j) \nu(Z_j |L \geq j) = \nu_h(C_j |Z_j) \nu(Z_j),
\]

where for the equality in the last line we used the facts that the probability \( \nu(C_j = c|Z_j = z, L \geq j) \) is fixed to be \( \nu_h(C = c|Z = z) \), independent of \( j \), and that the input distribution \( \nu(Z_j |L \geq j) \) is given by \( \nu(Z_j) \) of Eq. \( \text{(S1)} \) with \( Z_j = X_j Y_j \). On the other hand, when \( j > L \) the results of the \( j \)'th trial are fixed to be \( Z_j = C_j = * \), independent of the past results \( C_{<j} Z_{<j} \), according to model construction. So,

\[
\nu(C_j Z_j|C_{<j} Z_{<j}, L < j) = \delta_{C_j,*} \delta_{Z_j,*},
\]

where \( \delta \) is the Kronecker delta function.

For honest devices with distribution \( \nu(CZ) \) for each block, after \( N_b \) blocks the adjusted output entropy witness (that is, the left-hand side of Eq. \( \text{(S17)} \)) has expectation \( \mathbb{E}_\nu(N_b \log_2(G(CZ))/\beta + \log_2(\epsilon_{en})/\beta + \log_2(\epsilon)) \). Here, \( G(CZ) \) is a block-wise PEF with power \( \beta \) for the model \( C \) for each block. Thus, one way to optimize block-wise PEFs is as follows:

\[
\begin{align*}
\max_{\mathcal{G}} : & \quad \mathbb{E}_\nu\left(N_b \log_2(G(CZ))/\beta + \log_2(\epsilon_{en})/\beta + \log_2(\epsilon)\right) \\
\text{subject to: } & \quad 1) \ G(cz) \geq 0, \text{ for all } cz, \\
& \quad 2) \ \sum_{cz} \mu(C = c, Z = z) G(cz) \mu(C = c|Z = z) \beta \leq 1, \text{ for all } \mu(CZ) \in \mathcal{C}.
\end{align*}
\]

(S20)

Here, the maximum is over all possible block-wise PEFs \( G(CZ) \) with power \( \beta \) for \( C \), not only over the block-wise PEFs constructed by chaining trial-wise PEFs with power \( \beta \). When the block-wise PEFs are constrained to those constructed by chaining trial-wise PEFs and when the PEFs for all possible trial positions in a block are constrained to those constructed by a well-performing liner-interpolation method as detailed in the next subsection, the block-wise PEF optimization is effectively solved. We emphasize that in the absence of additional constraints on \( G \), the block-wise PEF returned by the optimization problem of Eq. \( \text{(S20)} \) is optimal at \( \nu \) given the model \( C \), but every feasible block-wise PEF, for example, the solution returned by the method described in the next subsection, is valid...
by definition regardless of the actual distributions of the blocks’ results as long as the possible distributions of these results given the past are in $\mathcal{C}$.

Before the experiment, we aim to minimize the number of blocks $N_{\text{exp,b}}$ required for successful randomness generation with honest devices. For this, we define the quantity

$$g_b(\beta) = \max_G \mathbb{E}_\nu \left( \log_2 (G(CZ)) / \beta \right),$$

where the maximum is over all possible block-wise PEFs $G(CZ)$ with power $\beta$ for $\mathcal{C}$. Thus, for honest devices with distribution $\nu(CZ)$ for each block, the expectation of $\sum_i N_i \log_2 (G_i(C_iZ_i)) / \beta$ can be as high as $N_h g_b(\beta)$. Success requires that $\sum_{i=1}^{N} \log_2 (G_i(C_iZ_i)) \geq \log_2 (t_{\text{min}})$, where $t_{\text{min}} = 1/(\sqrt{q^3} \epsilon_{\text{en}})$ with $q = 2^{-\sigma_{\text{en}} \epsilon}$. For adequate probability of success we therefore need $\log_2 (t_{\text{min}}) / \beta$ smaller than $N_h g_b(\beta)$ by an amount of order $\sqrt{N_h}$ that is asymptotically small compared to $N_h g_b(\beta)$. For the present analysis, we simply define the minimum number of blocks $N_{\text{exp,b}}(\beta)$ required using block-wise PEFs with power $\beta$ by the identity

$$N_{\text{exp,b}}(\beta) = \log_2 (t_{\text{min}}) / \beta = \sigma_{\text{en}} - \log_2 (\epsilon_{\text{en}}) / \beta - \log_2 (\epsilon).$$

Minimizing $N_{\text{exp,b}}(\beta)$ over $\beta > 0$ gives the minimum number of blocks $N_{\text{exp,b}}(\beta)$ required according to this simplification:

$$N_{\text{exp,b}} = \inf_{\beta > 0} N_{\text{exp,b}}(\beta).$$

The lower bound $N_{\text{exp,b}}$ may be considered tight to lowest order in the sense that one needs only to increase the number of blocks used in practice by an amount that is asymptotically small compared to $N_{\text{exp,b}}$, in order to guarantee sufficient probability of success. The probability of success can be estimated according to the distribution of a sum of the i.i.d. random variables $\log_2 (G_i)$, where $G_i$ is the PEF used for the $i$th block. We compute the mean and variance of $\log_2 (G_i)$ in the next subsection and estimate the probability of success accordingly in Sect. IV.B. In the next subsection, we will provide an effective method for determining a lower bound of $g_b(\beta)$. Accordingly, we can obtain an upper bound of $N_{\text{exp,b}}$.

We remark that for each fixed $\beta$, the quantity $g_b(\beta)$ defined in Eq. (S21) can be identified as the maximum asymptotic randomness rate per block witnessed by block-wise PEFs with power $\beta$ when each block has the same distribution $\nu(CZ)$ and is described by the same model $\mathcal{C}$. Therefore, the maximum asymptotic randomness rate per block witnessed by all possible block-wise PEFs for $\mathcal{C}$ is $g_{b,\text{max}} = \sup_{\beta > 0} g_b(\beta)$. The justification of the above identification is the same as that for the case of trial-wise PEFs as detailed in Refs. [24, 25].

### B. Simplified and effective block-wise PEF optimization

Consider a generic block with the $j$th trial model $\mathcal{M}_j$ for $j \leq L$. Given PEFs $F_j$ with power $\beta$ for trial models $\mathcal{M}_j$, by PEF chaining the product $G = \prod_{j=1}^L F_j$ is a PEF with power $\beta$ for the model $\mathcal{C}$ for the block. In view of the optimization problems formulated in the previous subsection, we wish to choose $F_j$ so as to maximize the expectation of $\log_2 (G(CZ)) / \beta$ for honest devices with distribution $\nu(CZ)$. Instead of optimizing $F_j$ for each $j$, here we develop a well-performing linear-interpolation method to construct the PEFs for all possible trial positions in a block given the optimized PEFs for only three trial positions. We find that the trial-wise PEFs thus obtained perform almost as well as the optimized PEFs for each trial position.

We begin by computing the quantity $g_b(G, \beta) = \mathbb{E}_\nu \left( \log_2 (G(CZ)) / \beta \right)$. This quantity can be interpreted as the asymptotic randomness rate witnessed by the block-wise PEF $G(CZ)$ with power $\beta$ when each block has the distribution $\nu(CZ)$. Since we choose $F_j(C_jZ_j) = 1$ for $j > L$, or equivalently, for $C_j = Z_j = 0$, $\log_2 (F_j) = 0$ for $j > L$ and so

$$\mathbb{E}_\nu \left( \log_2 (G(CZ)) \right) = \mathbb{E}_\nu \left( \sum_{j=1}^L \log_2 (F_j(C_jZ_j)) \right)$$

$$= \mathbb{E}_\nu \left( \sum_{j=1}^{2^L} \log_2 (F_j(C_jZ_j)) \right)$$

$$= \sum_{j=1}^{2^L} \mathbb{E}_\nu \left( \log_2 (F_j(C_jZ_j)) \right).$$

(S24)
In view of the law of total expectation and considering that the distribution of the block-length variable \( L \) is determined by \( \nu(CZ) \), we can continue where we left off to get

\[
\mathbb{E}_\nu \left( \log_2(G(CZ)) \right) = \sum_{j=1}^{2^k} \left( \mathbb{P}_\nu(j \leq L) \mathbb{E}_\nu \left( \log_2(F_j(C_jZ_j)) | j \leq L \right) + \mathbb{P}_\nu(j > L) \mathbb{E}_\nu \left( \log_2(F_j(C_jZ_j)) | j > L \right) \right)
\]

\[
= \sum_{j=1}^{2^k} \mathbb{P}_\nu(j \leq L) \mathbb{E}_\nu \left( \log_2(F_j(C_jZ_j)) \right)
\]

\[
= \sum_{j=1}^{2^k} \omega_j \mathbb{E}_\nu \left( \log_2(F_j(C_jZ_j)) \right),
\]

where \( \omega_j = (2^k - j + 1)/2^k \) is the probability that the block-length variable \( L \) is larger than or equal to \( j \), and the distribution \( \nu \) in the subscript is the abbreviation of the distribution \( \nu(C_jZ_j|C < j Z < j, L \geq j) \), which is given by Eq. (S18) for honest devices. Therefore, we have

\[
g_b(G, \beta) = \sum_{j=1}^{2^k} \omega_j g_{b,j}(F_j, \beta),
\]

(S26)

where \( g_{b,j}(F_j, \beta) = \mathbb{E}_{\nu_j} \left( \log_2(F_j(C_jZ_j))/\nu \right) \). Hence, to maximize \( g_b(G, \beta) \) over trial-wise PEFs, it suffices to maximize each \( g_{b,j}(F_j, \beta) \) independently over the PEFs \( F_j \) for the trial model \( M_j \) constructed under the condition \( L \geq j \). We denote the maximum of \( g_b(G, \beta) \) over \( G = \prod_{j=1}^{L} F_j \) by \( g_b^*(\beta) \) and the maximum of \( g_{b,j}(F_j, \beta) \) over \( F_j \) by \( g_{b,j}(\beta) \). Then, according to Eq. (S26), we have

\[
g_b^*(\beta) = \sum_{j=1}^{2^k} \omega_j g_{b,j}(\beta).
\]

Moreover, \( g_b^*(\beta) \) is a lower bound of the quantity \( g_b(\beta) \) defined in Eq. (S21). That is, we have

\[
g_b(\beta) \geq g_b^*(\beta) = \sum_{j=1}^{2^k} \omega_j g_{b,j}(\beta).
\]

(S27)

For the purpose of estimating the probability of success in our protocol implementation, we need the variance of \( \log_2(G(CZ)) \) with respect to \( \nu(CZ) \). Although the variance is not needed for optimizing trial-wise PEFs, we obtain an expression for it here which will be used in Sect. IVB. In the same way as deriving Eq. (S24), we have

\[
\mathbb{E}_\nu \left( \log^2_2(G(CZ)) \right) = \mathbb{E}_\nu \left( \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} \log_2(F_i(C_iZ_i)) \log_2(F_j(C_jZ_j)) \right)
\]

\[
= \sum_{i=1}^{2^k} \sum_{j=1}^{2^k} \mathbb{E}_\nu \left( \log_2(F_i(C_iZ_i)) \log_2(F_j(C_jZ_j)) \right).
\]

(S28)

Consider the case \( i \leq j \). The case \( i > j \) can be computed by exchanging \( i \) and \( j \). By splitting the expression conditional on \( L < i, i \leq L < j \) and \( j \leq L \) and taking into account that the product \( \log_2(F_i(C_iZ_i)) \log_2(F_j(C_jZ_j)) = 0 \) for the first two cases, we get

\[
\mathbb{E}_\nu \left( \log_2(F_i(C_iZ_i)) \log_2(F_j(C_jZ_j)) \right) = \mathbb{P}_\nu(j \leq L) \mathbb{E}_\nu \left( \log_2(F_i(C_iZ_i)) \log_2(F_j(C_jZ_j)) \right) | j \leq L).
\]

(S29)

If \( i = j \leq L \),

\[
\mathbb{E}_\nu \left( \log_2(F_i(C_iZ_i)) \log_2(F_j(C_jZ_j)) \right) | j \leq L) = \mathbb{E}_{\nu_{ij}} \left( \log^2_2(F_j(C_jZ_j)) \right).
\]

(S30)

If \( i < j \leq L \), considering that in this case the inputs \( Z_i \) must be equal to 00 and that with honest devices the output distributions for the trials \( i, j \) are independent conditionally on their respective inputs, we have

\[
\mathbb{E}_\nu \left( \log_2(F_i(C_iZ_i)) \log_2(F_j(C_jZ_j)) \right) | j \leq L) = \mathbb{E}_{\nu_{ij}|Z_i=00} \left( \log_2(F_i(C_iZ_i)) \right) \mathbb{E}_{\nu_j} \left( \log_2(F_j(C_jZ_j)) \right).
\]

(S31)

In view of the above four equations as well as Eq. (S25), we can compute the variance of \( \log_2(G(CZ)) \).
Next, we observe that both the optimization problem of Eq. (S20) for updating block-wise PEFs during an experiment and the optimization problem of Eq. (S23) for minimizing the number of blocks required for success before the experiment are based on determining the quantity $g_0(\beta)$. Here we focus on an effective method for lower-bounding $g_0(\beta)$. For this, we first use Eq. (S27) to reduce the problem of finding $g_0'(\beta)$ to the problem of finding $g_{b_j}(\beta)$ for each possible trial position $j$ in the block. For each $j$, the optimization problem can be formulated as a sequential quadratic program and so can be effectively solved (see Refs. 24, 25 for details). However, a block usually consists of a large number of trials. So, the individual optimizations for all trial positions in a block still take much time. To save time, it is better to solve the optimization problems for only a few trial positions, and then construct valid but maybe suboptimal PEFs for the other trial positions by an efficient method. In this way, we obtain a lower bound of $g_0(\beta)$. For this, we take advantage of the similarity among the trial models $\mathcal{M}_j$ constructed under the condition $L \geq j$, particularly the following proposition:

**Proposition 1.** Let $F_j(C_jZ_j)$ be a PEF with power $\beta$ for the trial model $\mathcal{M}_j$ constructed under the condition $L \geq j$, where the distribution of inputs $Z_j = X_jY_j$ according to $\mathcal{M}_j$ is fixed to be $\nu_j(X_jY_j)$ of Eq. (S1). Then, $4\nu_j(Z_j)F_j(C_jZ_j)$ is a PEF with power $\beta$ for the trial model $\mathcal{M}_{j=2^k}$.

**Proof.** In view of the model construction detailed in the paragraph including Eq. (S1), the input-conditional output distributions according to the trial model $\mathcal{M}_j$ constructed conditionally on $L \geq j$ form the same set $\mathcal{T}$, independent of $j$. Since the input distribution $\nu_j(Z_j)$ according to $\mathcal{M}_j$ is fixed, each distribution of $\mathcal{M}_j$ can be expressed as $\mu_j(C_jZ_j) = \nu_j(Z_j)\mu(C_jZ_j)$ where $\mu(C_jZ_j) \in \mathcal{T}$. In view of the definition in Eq. (S2), a PEF with power $\beta$ for $\mathcal{M}_j$ is a non-negative function $F_j : cz \mapsto F_j(cz)$ satisfying the linear inequality

$$\sum_{c z} \nu_j(Z_j = z)F_j(cz)\mu(C_j = c|Z_j = z)^{1+\beta} \leq 1,$$

for each distribution $\mu(C_j|Z_j) \in \mathcal{T}$. Considering that when $j = 2^k$ the input probability is $\nu_j(Z_j = z) = 1/4$ for each $z$, the statement in the proposition follows. □

We remark that Prop. 1 applies to an arbitrary trial model $\mathcal{M}$ as long as according to $\mathcal{M}$ the input distribution $\nu(Z)$ is fixed with $\nu(Z = z) > 0$ for all $z$ and the input-conditional output distributions form the set $\mathcal{T}$.

We can take advantage of Prop. 1 for constructing PEFs for all trial positions in a block by interpolation. For this purpose, let $\tilde{\nu}_q(Z)$ be a distribution of $Z$ parameterized by a positive number $q \in (0, 4/3)$ such that $\tilde{\nu}_q(Z = 00) = 1 - 3q/4$ and $\tilde{\nu}_q(Z = z) = q/4$ if $z \neq 00$. Then $\nu_j(Z) = \tilde{\nu}_q(Z)$, where $q_j = 1/(2^k - j + 1)$. Define $\tilde{F}(q; CZ) = 4\tilde{\nu}_q(Z)F_{2^k+1-1/q}(CZ)$ so that $\tilde{F}(q; CZ) = 4\nu_j(Z)F_j(CZ)$. Here, we implicitly allow non-integer suffixes for $F$, but we will not explicitly refer to non-integer suffixed $F$. Below, $q$ is the parameter to be interpolated on. According to Prop. 1 for each $j$, $\tilde{F}(q_j; CZ)$ is constrained to be a PEF with power $\beta$ for the model $\mathcal{M}_{j=2^k}$. This fact motivates us to construct a function $\tilde{F}(q; CZ)$ with $q \in (0, 4/3)$ such that this function is always a PEF with power $\beta$ for $\mathcal{M}_{j=2^k}$. Suppose that both $\tilde{F}(q'; CZ)$ and $\tilde{F}(q''; CZ)$ with $q' < q''$ are PEFs with power $\beta$ for $\mathcal{M}_{j=2^k}$. According to the PEF definition, any convex combination of $\tilde{F}(q'; CZ)$ and $\tilde{F}(q''; CZ)$ is also a PEF with power $\beta$ for $\mathcal{M}_{j=2^k}$. Therefore, to ensure that the function $\tilde{F}(q; CZ)$ with $q \in [q', q'']$ is a PEF with power $\beta$ for $\mathcal{M}_{j=2^k}$, we can construct this function as the linear interpolant

$$\tilde{F}(q; CZ) = L_{q', q''}(q; CZ) = \frac{q'' - q}{q'' - q'} \tilde{F}(q'; CZ) + \frac{q - q'}{q'' - q'} \tilde{F}(q''; CZ),$$

(S32)

a convex combination of $\tilde{F}(q'; CZ)$ and $\tilde{F}(q''; CZ)$. Considering that the parameter $q_j$ depends on $j$ monotonically, the above linear interpolation provides a way of constructing PEFs $F_j(CZ)$ with power $\beta$ for the trial model $\mathcal{M}_j$ as follows: Given two PEFs $F_{j_1}(CZ)$ and $F_{j_2}(CZ)$ with power $\beta$ for $\mathcal{M}_{j_1}$ and $\mathcal{M}_{j_2}$ respectively, we first compute the corresponding $\tilde{F}(q_{j_1}; CZ)$ and $\tilde{F}(q_{j_2}; CZ)$, and then we can determine a PEF $F_j(CZ)$ with power $\beta$ for $\mathcal{M}_j$, for any $j$ between $j_1$ and $j_2$, from the $\tilde{F}(q; CZ)$ constructed as the linear interpolant $L_{q_{j_1}, q_{j_2}}(q; CZ)$ according to $F_j(CZ) = \tilde{F}(q_j; CZ)/(4\nu_j(Z))$. To construct the PEFs with power $\beta$ for all trial positions in a block, we first find the optimal PEFs witnessing $g_{b_j}(\beta)$ for three trial positions $j = 1, j_{\text{mid}}, 2^k$ and compute the corresponding $\tilde{F}(q_{j=1}; CZ)$, $\tilde{F}(q_{j=\text{mid}}; CZ)$ and $\tilde{F}(q_{j=2^k}; CZ)$, where the choice of $j_{\text{mid}}$ can be optimized (see the next paragraph). Then we construct the function $\tilde{F}(q; CZ)$ by linear interpolation as

$$\tilde{F}(q; CZ) = \begin{cases} L_{q_{j=1}, q_{j=\text{mid}}}(q; CZ) & \text{if } q \in [q_{j=1}, q_{j=\text{mid}}], \\ L_{q_{j=\text{mid}}, q_{j=2^k}}(q; CZ) & \text{if } q \in [q_{j=\text{mid}}, q_{j=2^k}]. \end{cases}$$

(S33)
We refer to the above method as the linear-interpolation method, where the PEFs for the trial positions \( j = 1, j_{\text{mid}}, 2^k \) are optimized.

Using the trial-wise PEFs obtained by the above linear-interpolation method and in view of Eq. (S27), we can compute a lower bound on the quantity \( g'_b(\beta) \) with honest devices. We observe that there exists an optimal middle trial position \( j_{\text{mid,opt}} \) such that the computed lower bound on \( g'_b(\beta) \) is as high as possible. The optimal position \( j_{\text{mid,opt}} \) depends not only on the honest-device distribution \( \nu_0(C|Z) \) but also on the power \( \beta \) of trial-wise PEFs, and it can be found by a line search. Furthermore, we observe that the trial-wise PEFs obtained by linear interpolation with the optimal position \( j_{\text{mid,opt}} \) witness a lower bound of \( g'_b(\beta) \) which is at least 99.99\% of \( g'_b(\beta) \). Therefore, the trial-wise PEFs obtained by linear interpolation can perform almost as well as the optimal trial-wise PEFs that witness the value of \( g'_b(\beta) \). In our numerical analysis for randomness expansion detailed in the next three sections, we used the above linear-interpolation method with \( j_{\text{mid,opt}} = 53,478 \) where the maximum block length is \( 2^{17} \). In this way, we need only to find the optimal PEFs for three trial positions in order to construct a well-performing PEF for a block with length \( 2^{17} \).

IV. PROTOCOL DESIGN AND COMMISSIONING

Protocol design and commissioning consists of choosing the protocol parameters based on anticipated experiment performance. For our demonstration, the first task was to pick the maximum block length that was used in our experiment. This had to be done before acquiring the data because it affects the experiment itself. The remaining parameters were determined after the data was acquired but before the data was unblinded. However, for production-quality implementations, all protocol parameters should be chosen before acquiring data.

A. Block-length determination

Our randomness-expansion experiment was performed in August of 2018. Before that, in July of 2018, we acquired about 8 minutes of experimental Bell-trial data at the rate of approximately 100,000 trials per second, where the raw counts are summarized in Table I. This data was used to determine the optimal choice for the maximum block length of our randomness-expansion experiment under the assumption that the devices perform as inferred from this data. We estimated the input-conditional output distribution for a trial by maximum likelihood (as detailed in Sect. VIII B of Ref. [25]). The estimate is denoted by \( \nu_0(C|Z) \) and shown in Table I. For each possible choice \( 2^k \) for the maximum block length, we formulated the following optimization problem: Assuming that the quantum devices used are honest with the input-conditional output distribution \( \nu_0(C|Z) \) for each trial, minimize the number of blocks required for randomness expansion with soundness error \( \epsilon \). Denote this minimum number of blocks by \( N_{b,\min}(k) \), where the dependence on \( k \) is explicit and the dependence on \( \epsilon \) is implicit. The minimum number of blocks \( N_{b,\min}(k) \) can be found by a binary search provided that for each \( N_b \) and \( k \) we can determine whether \( N_b \) blocks suffice for randomness expansion with maximum block length \( 2^k \). See the steps of Algorithm 2 for details. Once \( N_{b,\min}(k) \) is known, we can choose \( k \) to minimize the expected number of trials \( N_{t,\min}(k) = N_{b,\min}(k) \times (1 + 2^k)/2 \) in \( N_{b,\min}(k) \) blocks. Define \( k_{\text{opt}} \) to be the minimizing value of \( k \). The minimum number of trials \( N_{t,\min}(k) \) required for randomness expansion when \( k = k_{\text{opt}} \) is abbreviated as \( N_{t,\min} \).

### TABLE I. Counts of measurement settings \( xy \) and outcomes \( ab \) used for finding the optimal choice for the maximum block length.

| \( xy \) | \( ab \) | 00 | 10 | 01 | 11 |
|---|---|---|---|---|---|
| 00 | 11183694 | 11345 | 12229 | 28730 |
| 10 | 11092860 | 98100 | 10996 | 29439 |
| 01 | 11094694 | 11817 | 98213 | 27771 |
| 11 | 10982482 | 125705 | 123749 | 2306 |

With the procedure outlined in the previous paragraph and the input-conditional output distribution \( \nu_0(C|Z) \) given in Table II, we determined the minimum number of trials \( N_{t,\min} \) required and the associated optimal choice \( 2^{k_{\text{opt}}} \) for the maximum block length, in order to achieve randomness expansion with a soundness error \( \epsilon \) varying from \( 10^{-3} \) to \( 10^{-12} \). Here we consider the case where the inputs \( X = 0 \) and \( Y = 0 \) are used in each non-spot-checking
Algorithm 2: Determine whether $N_b$ blocks suffice for randomness expansion with maximum block length $2^k$ and soundness error $\epsilon$, given that the input-conditional output distribution for each trial is $\nu(C|Z)$.

1. Maximize the expected net number of random bits $\sigma_{\text{net, opt}}(\beta)$ over $\beta > 0$ by a local search, where $\sigma_{\text{net, opt}}(\beta)$ is computed as follows:

   (a) Determine the trial-wise PEFs with power $\beta$ according to the linear-interpolation method of Sect. III.B, where the choice for the middle trial position $m_{\text{mid}}$ required is optimized.

   (b) Compute the randomness rate per block $g_b(\beta, 2^k)$ at $\nu(C|Z)$ witnessed by these PEFs.

   (c) Maximize $\sigma_{\text{net}}(\beta, \epsilon_{\text{en}})$ over $\epsilon_{\text{en}}$ by a line search, where $\sigma_{\text{net}}(\beta, \epsilon_{\text{en}})$ is the expected net number of random bits at entropy error $\epsilon_{\text{en}}$ computed as follows:

      i. Compute the adjusted output entropy witness $\sigma_{\text{in}}$ expected after $N_b$ blocks, $\sigma_{\text{in}} = N_b g_b(\beta, 2^k) + \log_2(\epsilon_{\text{en}})/\beta + \log_2(\epsilon)$ (Eq. (S22)).

      ii. Set the extractor error as $\epsilon_{\text{ext}} = \epsilon - \epsilon_{\text{en}}$.

      iii. Determine the number $k_{\text{out}}$ of extractable random bits and the number $d_s$ of seed bits required according to the extractor constraints of Eq. (S8) with $m_{\text{in}} = N_b \times 2^k \times 2$. Note: The $X$ of Eq. (S10) is guaranteed to be non-empty.

      iv. Set $\sigma_{\text{net}}(\beta, \epsilon_{\text{en}}) = k_{\text{out}} - d_s - N_b(k + 2)$. Note: Each block consumes $k$ random bits for determining its length and 2 random bits for the settings choices of the spot-checking trial.

   (d) Set the maximum of $\sigma_{\text{net}}(\beta, \epsilon_{\text{en}})$ over $\epsilon_{\text{en}}$ as $\sigma_{\text{net, opt}}(\beta)$.

2. Set $\beta_{\text{opt}}$ to be the $\beta$ that maximizes $\sigma_{\text{net, opt}}(\beta)$.

3. If $\sigma_{\text{net, opt}}(\beta_{\text{opt}}) \geq 0$, then randomness expansion with parameters $(N_b, 2^k, \epsilon)$ at $\nu(C|Z)$ is possible.

The results are summarized in Table III. Several interesting points illustrated by the results in Table III are as follows: 1) The optimal choice for the maximum block length is $2^{17}$, independent of the soundness error. 2) Both the optimal power $\beta_{\text{opt}}$ and the optimal error-splitting ratio (that is, the ratio of the optimal entropy error to the optimal extractor error) are independent of the soundness error. 3) The minimum number of trials $N_{t, \text{min}}$ required for randomness expansion scales linearly with the logarithm of the soundness error. In addition, we observed that the optimal choice for the maximum block is independent of the particular inputs used in every non-spot-checking trial. However, the optimal choice for the maximum block length, as well as the optimal PEF power and the optimal error-splitting ratio, depends on the input-conditional output distribution $\nu(C|Z)$. Particularly, we observed that these optimal parameters are well correlated with the statistical strength for rejecting local realism, which is the minimum Kullback-Leibler divergence of the true distribution $\nu(C|Z)/4$ of Bell-trial results from the local realistic distributions [10, 11] supposing that the inputs of Bell trials are uniformly randomly distributed. For the distribution $\nu(C|Z)$ given in Table II, the statistical strength for rejecting local realism is $7.19 \times 10^{-6}$.
The fixed inputs $X = 0$ and $Y = 0$ are used in each non-spot-checking trial. Here $N_{t, \text{min}}$ is the minimum number of trials required, $2^{k_{\text{opt}}}$ is the optimal choice for the maximum block length, $\beta_{\text{opt}}$ is the optimal power of the trial-wise PEFs used, and $\epsilon_{\text{en, opt}}$ is the associated optimal entropy error.

| $\epsilon$       | $N_{t, \text{min}}$ | $2^{k_{\text{opt}}}$ | $\beta_{\text{opt}}$ | $\epsilon_{\text{en, opt}}$ |
|------------------|---------------------|-----------------------|-----------------------|----------------------------|
| $1 \times 10^{-3}$ | $1.90 \times 10^{11}$ | $2^{17}$              | $1.32 \times 10^{-7}$ | $9.78 \times 10^{-4}$     |
| $1 \times 10^{-6}$ | $3.80 \times 10^{11}$ | $2^{17}$              | $1.32 \times 10^{-7}$ | $9.78 \times 10^{-7}$     |
| $1 \times 10^{-9}$ | $5.70 \times 10^{11}$ | $2^{17}$              | $1.32 \times 10^{-7}$ | $9.78 \times 10^{-10}$    |
| $1 \times 10^{-12}$| $7.60 \times 10^{11}$ | $2^{17}$              | $1.32 \times 10^{-7}$ | $9.78 \times 10^{-13}$    |

**B. Parameter determination**

After choosing $2^{17}$ as the maximum block length, we ran the randomness-expansion experiment and collected 110.3 hours worth of data over the course of two weeks. The data was acquired in a series of cycles. Each cycle began with about 2 minutes of calibration trials, generated at the rate of approximately 250,000 trials per second, which were stored in a calibration file, and then proceeded with a varying number of expansion files. Each expansion file recorded about 2 minutes of block data, generated at the rate of approximately 153 blocks per second, which were followed by about 5 seconds of calibration trials, generated at the rate of approximately 250,000 trials per second and recorded at the end of the file. For calibration trials, the input settings were chosen uniformly. Note that pseudorandom settings choices would suffice for calibration purposes. For non-spot-checking trials in a block the inputs were fixed to be $X = 0$ and $Y = 0$. Non-spot-checking trials with no detections, namely those satisfy $a = 0$ and $b = 0$, were not explicitly recorded. For non-spot-checking trials with detections and for the spot-checking trial in a block, their positions and outcomes, as well as the settings choices used in the spot-checking trial, were recorded. Because the probability of detections at inputs $X = 0$, $Y = 0$ is less than 0.0046, this recording method saves space.

For commissioning and training purposes, we unblinded the first 16 cycles, which contains 4,502,276 blocks (about 7.4% of the recorded block data). We refer to this data as the training set and the remaining blinded data as the analysis set. We use the training set to choose the protocol input parameters ($k_{\text{out}}$, $\epsilon$) and the required pre-analysis-run parameters ($\beta$, $t_{\text{min}}$, $\epsilon_{\text{en}}$), as well as the associated extractor parameters. To make our choices, we took advantage of prior knowledge of the number of blocks $N_b$ available in the analysis set. We found that $N_b = 56,070,910$. In addition we were aware of specific properties of the analysis set such as which cycles had reduced-length calibration files. We remark that in production-quality implementations, such specific knowledge is not available.

First, we fixed the soundness error to be $\epsilon = 5.7 \times 10^{-7}$, corresponding to the 5-sigma criterion. We estimated the input-conditional output distribution $\nu(C|Z)$ by maximum likelihood (Sect. VIII B of Ref. [25]) from the calibration trials recorded in the first 6 cycles of the training set. We used only the first 6 cycles for this purpose because there were indications that they were representative of a stable setup. The counts used to infer $\nu(C|Z)$ are shown in Table [V] and the values of $\nu(C|Z)$ inferred are in Table [V]. We then determined the PEF power $\beta$ and entropy error $\epsilon_{\text{en}}$ by setting them to the optimizing quantities in Algorithm 2 with input parameters $k_{\text{out}}$, $\epsilon$ found above different from what were obtained in the original run of Algorithm 2 which resulted in our choosing $2^{17}$ as the maximum block length. In the original run of Algorithm 2 if we had used the distribution in Table [V] determined from the calibration trials recorded in the first 6 cycles, we would have chosen $2^{18}$ as the maximum block length.

Second, we need to choose the success threshold $t_{\text{min}}$ for running Protocol II or equivalently, the success threshold $W_{\text{min}}$ for the output entropy witness stated in the main text. In view of the success condition $\prod_{i=1}^{N_b} G_i(C_i, Z_i) \geq t_{\text{min}}$ and the definition of the output entropy witness below Eq. (S17), $W_{\text{min}}$ is related with $t_{\text{min}}$ by $W_{\text{min}} = \log_2(t_{\text{min}}) / \beta$. To determine the value for $W_{\text{min}}$ used in our expansion analysis, we studied the dependence of the success probability in an honest implementation of the protocol on the threshold $W_{\text{min}}$. Since the number of blocks available for expansion analysis is $N_b = 56,070,910$ and the randomness rate per block estimated in the previous paragraph is
$g_b = 36.06$ bits, the output entropy witness after $N_b$ blocks with honest devices, where the input-conditional output distribution is fixed to be the $\nu(C|Z)$ in Table V is expected to be $N_bg_b = 2,021,917,014$ bits. To estimate the success probability, we also need to know the variance $\sigma^2$ of the output entropy witness after $N_b$ blocks. The variance is given as $\sigma^2 = N_b \nu(\log_2(G(CZ))|/\beta)$, where $\nu_v(\log_2(G(CZ))|/\beta$ is the variance of the variable $\log_2(G(CZ))$ according to the distribution $\nu(CZ)$ determined by the honest-device input-conditional output distribution $\nu(C|Z)$. By the results presented in the third paragraph of Sect. [III.B] we found that $\nu_v(\log_2(G(CZ))|/\beta) = 4.6729 \times 10^8$ and so $\sigma^2 = 2.6201 \times 10^{16}$. In view of the central limit theorem, we assume that the output entropy witness after $N_b$ blocks with honest devices is normally distributed with mean $N_bg_b$ and variance $\sigma^2$. Thus, given that the success threshold for the output entropy witness is $W_{\min}$, the success probability is estimated to be $p_{\text{succ}} = Q(- (N_bg_b - W_{\min})/\sigma)$. Here the function $Q$ is the tail distribution function of the standard normal distribution. For our expansion analysis, we chose the success probability $p_{\text{succ}} = 0.9938$ such that $(N_bg_b - W_{\min})/\sigma = 2.5$, matching the conventional one-sided 2.5-sigma criterion. Consequently, we determined that $W_{\min} = 1,616,998,677$ and so $t_{\min} = 2^{77}$. The completeness calculation just performed is heuristic in that we assume that the output entropy witness at the end is sufficiently close to normally distributed for the tail calculation to be accurate. On the other hand, it is somewhat pessimistic because it does not account for the possibility that the threshold is reached early but is not exceeded at the end in the absence of an early stop.

Third, we determined the number of seed bits required for applying the TMPS extractor. Considering that $t_{\min} = 1/(q^a \epsilon_{\text{en}})$ with $q = 2^{-\sigma_{\text{en}}}$ in Protocol 1, we set $\sigma_{\text{in}} = \log_2(t_{\min})/\beta + \log_2(\epsilon_{\text{en}})/\beta + \log_2(\epsilon) = 1,181,264,480$. Moreover, we set the length in bits of the extractor input to be $m_{\text{in}} = N_b \times 2^{17} \times 2 = 14,698,652,631,040$ and the extractor error to be $\epsilon_{\text{ext}} = \epsilon - \epsilon_{\text{en}} = 1.7800 \times 10^{-9}$. The condition $\sigma_{\text{in}} \leq m_{\text{in}} + \frac{14}{7} \beta \log_2(\epsilon)$ required for defining the set $X$ of Eq. (S10) is satisfied. Therefore, according to the TMPS extractor constraints in Eq. (S8), $d_s = 3,725,074$ seed bits are needed, and conditional on success $k_{\text{out}} = 1,181,264,237$ new random bits can be extracted. As the protocol is designed to consume $k_{\text{in}} = 1,069,072,364$ random bits, including $1,065,347,290$ random bits for spot checks and settings choices as well as $3,725,074$ seed bits, the expected expansion ratio conditional on success according to this calculation is $k_{\text{out}}/k_{\text{in}} = 1.105$. However, if the threshold for success is reached early, the expansion ratio is higher, as witnessed by the final results of our protocol run.

In the same way as above, we can vary the value of the desired success probability $p_{\text{succ}}$ and study the dependence of the expected expansion ratio at the soundness error $\epsilon = 5.7 \times 10^{-7}$ on $p_{\text{succ}}$ (see Fig. 3 of the main text). Moreover, we can also vary the value of the desired soundness error $\epsilon$ and study the dependence of the expected expansion ratio on $\epsilon$ given the fixed success probability $p_{\text{succ}} = 0.9938$. The results are illustrated in Fig. 5.

![Fig. 5](image)

**Fig. 5.** The expected expansion ratio as a function of the soundness error, when fixing the desired success probability to be 0.9938. Given the input-conditional output distribution $\nu(C|Z)$ of Table V expected for each trial, we first optimized over the PEF power $\beta$ and the entropy error $\epsilon_{\text{en}}$, in order to maximize the expected net number of random bits extractable from $N_b = 56,070,910$ blocks of analysis data at a chosen soundness error (according to Algorithm 2). Then, with the optimal solutions found above we computed the expected expansion ratio (see the text for details), supposing that the success probability is 0.9938. Our choice for the soundness error and the corresponding expected expansion ratio is labelled by the cross.
TABLE IV. Counts of measurement settings $xy$ and outcomes $ab$ in the calibration trials collected over the first 6 unblinded cycles.

| $xy$ | $ab$ | 00 | 10 | 01 | 11 |
|------|------|----|----|----|----|
| 00   |      | 62824397 | 64859 | 71896 | 153039 |
| 10   |      | 62360267 | 524745 | 60696 | 165193 |
| 01   |      | 62385836 | 64579 | 506557 | 153310 |
| 11   |      | 61772852 | 672142 | 642597 | 16105 |

TABLE V. The input-conditional output distribution $\nu(C|Z)$ with $C=AB$ and $Z=XY$ by maximum likelihood using the raw data in Table IV. They are used for determining the PEF power $\beta$ and the entropy error $\epsilon_{en}$ for expansion analysis, not to make a statement about the actual distribution when running the experiment.

| $xy$ | $ab$ | 00 | 10 | 01 | 11 |
|------|------|----|----|----|----|
| 00   |      | 0.995404388386381 | 0.001026519904493 | 0.001141638426253 | 0.002427453282873 |
| 10   |      | 0.988123719866393 | 0.008307188424481 | 0.000959649740469 | 0.002609441968657 |
| 01   |      | 0.988527138911423 | 0.001024397388010 | 0.008018887901211 | 0.002429575799356 |
| 11   |      | 0.978890595338359 | 0.010660940961074 | 0.01019277426503 | 0.000255689432064 |

V. PEF UPDATING DURING THE ANALYSIS

Based on the training set, the need for periodic realignment during the experiment and the reports from the experimenters, we anticipated that the input-conditional output distribution drifted significantly during the experiment. We therefore decided to update block-wise PEFs used for each next cycle based on calibration data preceding the block data of the cycle. We decided to always use at least $n_{calib, min} = 22,200,000$ calibration trials for this purpose. For 14 of the 140 cycles of the analysis set, the cycle’s calibration file did not contain sufficiently many trials. For these cycles, we used also calibration trials from the last expansion files of the previous cycle to obtain at least $n_{calib, min}$ calibration trials in total. From the calibration data, we determined the (presumed current) honest-device input-conditional output distribution $\nu(C|Z)$ by maximum likelihood (Sect VIII B of Ref. [25]) and obtained the trial-wise PEFs with power $\beta = 4.7614 \times 10^{-8}$ for all possible trial positions in a block by the linear-interpolation method (Sect. III B). The block-product of these PEFs is a PEF with the same power $\beta$ for the block. To verify that PEFs obtained in this way achieve close to optimal performance for the true distribution, we performed simulations. The simulation and its results are described in Fig. 6 and show that the PEFs obtained perform close to optimal with high probability.

Before running the protocol on the analysis set, we tested the performance of the protocol with cycle-updated PEFs on the training set. The training set contains 4,502,276 blocks. The empirical randomness rate per block from this test run is 35.6851 bits, consistent with the prediction of 36.0558 bits based on the input-conditional output distribution determined from the first 6 cycles in the training set (Table V). The empirical randomness rate per block was obtained by dividing the final value of the running output entropy witness by the number of blocks processed. The empirical variance of the per-block output entropy witnesses is $4.4786 \times 10^6$, similar to the predicted variance of $4.6729 \times 10^8$.

VI. ANALYSIS RESULTS

Our implementation of Protocol [1] for randomness analysis is shown in Protocol [3]. There were two independent runs of expansion analysis. The primary one was performed using MATLAB which took 45.4 hours on a personal computer, while the secondary one used Python which took also about 45 hours on a personal computer. The main purpose of the secondary run was as a consistency check during training and analysis. The two independent runs
FIG. 6. Performance of PEFs constructed using random samples. We assumed that the true distribution of calibration trials is $\nu(C|Z)/4$, where $\nu(C|Z)$ is the input-conditional output distribution determined from the calibration trials collected over the first 6 unblinded cycles (Table V). From this true distribution, we determined that the randomness rate per block is 36.0558 bits (given the PEF power $\beta = 4.7614 \times 10^{-8}$). Then we drew 20,000 random samples according to the true distribution, where each sample has 30,000,000 trials corresponding to 2 minutes of calibration trials at the rate of 250,000 trials per second. For each sample, we determined a block-wise PEF with power $\beta = 4.7614 \times 10^{-8}$ following the same procedure as described in the first paragraph of Sect. V. Then we computed the randomness rate per block witnessed by this block-wise PEF. For this computation, we assumed that each trial in a block has the same input-conditional output distribution $\nu(C|Z)$ according to the true distribution of sampled results. The results demonstrate that the randomness rate per block witnessed is almost independent of the random sample used. This observation justifies that it is sufficient to update the block-wise PEF using only 2 minutes of calibration trials.
Protocol 3: Protocol as implemented.

**Input**
- Analysis Data—calibration and expansion data acquired successively in a series of cycles. Each cycle begins with a calibration file and then proceeds with a varying number of expansion files. Each expansion file records a varying number of blocks, which are followed by a few seconds of calibration trials.

**Given**
- $W_{\text{min}}$—the success threshold for the running output entropy witness (chosen to be $1,616,998,677$, see Sect. IV B).
- $N_b$—the number of blocks available in the analysis set (set to be $56,070,910$, see Sect. IV B).
- $\beta$—the PEF power (chosen to be $4.7614 \times 10^{-8}$, see Sect. IV B).
- $j_{\text{mid}}$—the middle trial position used by the linear-interpolation method (fixed to be $53,478$, see Sect. III B).
- $n_{\text{calib,min}}$—the minimum number of calibration trials to be used for PEF updating (set to be $22,200,000$, see Sect. V).

**Output**
$P$— a binary flag indicating success ($P = 1$) or failure ($P = 0$).

Initialize the binary flag as $P = 0$;
Initialize the running output entropy witness as $W_{\text{run}} = 0$;
Initialize the running number of blocks processed $N_{\text{run}} = 0$;

```plaintext
for cycle in Analysis Data do
    n_{\text{calib,act}} \leftarrow \text{actual number of trials in the calibration file;}
    \{\text{Data}_{\text{calib}}(cz)\}_{cz} \leftarrow \text{counts of measurement settings } z = xy \text{ and outcomes } c = ab \text{ in the calibration file;}
    \text{if } n_{\text{calib,act}} < n_{\text{calib,min}} \text{ then}
        \text{Load the last expansion file in the previous cycle;}
        \text{while } n_{\text{calib,act}} < n_{\text{calib,min}} \text{ do}
            \text{Increment } n_{\text{calib,act}} \text{ by the number of calibration trials in the loaded expansion file;}
            \text{Increment } \{\text{Data}_{\text{calib}}(cz)\}_{cz} \text{ with the counts of settings and outcomes of calibration trials in the loaded expansion file;}
            \text{Load the previous expansion file;}
        \text{end}
    \text{end}

    Use \{Data_{\text{calib}}(cz)\}_{cz} to determine an honest-device input-conditional output distribution } \nu(C|Z) \text{ by maximum likelihood (see Sect. VIII B of Ref. [25]);}

    Determine the trial-wise PEFs \{F_j(C_j Z_j)\}_{j=1}^{2^7} \text{ with power } \beta \text{ for all possible trial positions } j \text{ in a block according to the linear-interpolation method of Sect. III B;}

    for expansion file in cycle do
        N_{\text{run}} \leftarrow N_{\text{run}} + 1;
        for trial j in block do
            Update $W_{\text{run}} \leftarrow W_{\text{run}} + \log_2(F_j(c_j z_j))/\beta$, where $c_j$ and $z_j$ are the settings and outcomes observed at the $j$‘th trial;
        end
        if $W_{\text{run}} \geq W_{\text{min}}$ then
            Record the number of blocks required for successful expansion as $n_b = N_{\text{run}}$;
            Return $P = 1$; // Protocol succeeded.
        end
    end
end
```
We processed the 56,070,910 blocks of the 140 cycles in the analysis set successively. After processing 49,977,714 blocks corresponding to running the experiment for 91.0 hours, the running output entropy witness exceeded the success threshold $W_{\text{min}}$. At this point, we consumed 949,576,566 random bits for spot checks and settings choices. If we applied the TMPS extractor with the chosen extractor parameters, we would consume an additional 3,725,074 random bits for seed and output 1,181,264,237 bits with soundness error $5.7 \times 10^{-7}$ for an actual expansion ratio of 1.24. We continued processing the remaining blocks and accumulating the running output entropy witness. After processing all blocks, we observed that the empirical randomness rate per block witnessed by cycle-updated block-wise PEFs with power $\beta = 4.7614 \times 10^{-8}$ is 32.8028 bits, lower than the randomness rate $36.0558$ bits per block predicted by the input-conditional output distribution $\nu(C|Z)$ of Table V. The empirical variance of the per-block output entropy witnesses is $4.3264 \times 10^8$, also lower than the predicted variance of $4.6729 \times 10^8$. The complete dynamics of the adjusted output entropy witness is illustrated in Fig. 4 of the main text. During the expansion analysis, we also tracked the drifts of the randomness rate per block as well as the statistical strength for rejecting local realism [40, 41], see Fig. [7] These results suggest that the randomness rate per block and statistical strength are positively correlated.

![FIG. 7. The statistical strength and randomness rate per block predicted by the calibration data used in each cycle. Based on at least 22,000,000 of the most recent calibration trials preceding the blocks in each cycle, we estimated the input-conditional output distribution for a trial. Assuming that the estimated distribution is the true one for all trials of the next cycle, we computed the statistical strength for rejecting local realism [40] [41] and randomness rate per block witnessed by the cycle-updated block-wise PEF with power $\beta = 4.7614 \times 10^{-8}$.](image)