A SYMPLECTIC PROOF OF VERLINDE FACTORIZATION

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Abstract. We prove a multiplicity formula for Riemann-Roch numbers of reductions of Hamiltonian actions of loop groups. This includes as a special case the factorization formula for the quantum dimension of the moduli space of flat connections over a Riemann surface.

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1. Introduction

The quantization of the moduli space of flat connections over a Riemann surface has been the subject of intensive study from a number of different points of view. Much of the recent work in mathematics has focused on proving formulas for the dimension of the quantization discovered by the physicist E. Verlinde [V] in the context of conformal field theory. Essentially there are two ingredients in Verlinde’s approach: the “factorization theorem” (included by Segal as one of the axioms of conformal field theory [Seg]) describes the quantization of the moduli space associated to a Riemann surface obtained by gluing a second (possibly disconnected) surface along two boundary circles; the “fusion rules” describe the quantization of the moduli space of a three-holed sphere with boundary components marked by irreducible representations of the loop group. Together these give a formula for the dimension of the quantization in terms of a pants decomposition of the surface. Both parts of Verlinde’s approach were carried out rigorously by Tsuchiya-Ueno-Yamada [TUY]. Since then there has been a vast amount of work on improving and understanding the formulas: see in particular the works by Beauville-Laszlo [BL], Bertram-Szenes [BS], Daskalopoulos-Wentworth [DW], Faltings [F], Kumar-Narasimhan-Ramanathan [KNR], Teleman [Te], and Thaddeus [Tha], among others. There is an alternative approach, initiated by Witten [W1] and carried out by Szenes [Sze] and Jeffrey-Kirwan [JK], in which the Verlinde formulas are derived from the cohomology ring of the moduli space. Very recently, yet another approach was outlined by S. Chang [C3], who obtained a character formula for quantizations of Hamiltonian loop group actions and showed that this implies Verlinde’s formula.

In this paper, we derive the factorization theorem in the context of symplectic geometry. To explain the idea (which has appeared in the literature in various forms) let Σ be any compact oriented Riemann surface with boundary, and \( \mathcal{M}(\Sigma) \) the space of flat \( G \)-connections on \( \Sigma \) mod gauge transformations that are trivial on the boundary. The moduli space \( \mathcal{M}(\Sigma) \) carries a Hamiltonian action of \( b \) copies of the loop group \( LG \) where \( b \) is the number of boundary components of \( \Sigma \). If \( \Sigma \) is formed from a second surface \( \hat{\Sigma} \) by gluing together two boundary components, then \( \mathcal{M}(\Sigma) \) is related to \( \mathcal{M}(\hat{\Sigma}) \) by a symplectic reduction by \( LG \). Assuming heuristically that some quantization procedure constructs a projective \( LG^b \)-representation \( Q(\mathcal{M}(\hat{\Sigma})) \) of \( \mathcal{M}(\hat{\Sigma}) \), the “quantization commutes with reduction” principle [GS2] implies that \( Q(\mathcal{M}(\Sigma)) \) should be the \( LG \)-invariant part of \( Q(\mathcal{M}(\hat{\Sigma})) \). A corollary of this principle is a “factorization formula” for the quantizations of finite-dimensional symplectic quotients of \( \mathcal{M}(\Sigma) \) in terms of those of \( \mathcal{M}(\hat{\Sigma}) \).

Our main result Theorem 2.4 is a formula of this type in the general setting of Hamiltonian loop group actions with proper moment maps. The proof is entirely finite-dimensional. Because of the properness assumption, the symplectic quotients are finite-dimensional, and their quantizations can be defined as indices of suitable Dirac operators (Riemann-Roch numbers), using desingularizations if necessary. Using finite-dimensional “symplectic cross-sections” for Hamiltonian loop group actions we reduce the proof to the finite-dimensional version of “quantization commutes with reduction”, which has been proved in general in Meinrenken, Meinrenken-Sjamaar [M2, MS]. An important ingredient is a gluing formula for the behavior of Riemann-Roch numbers under “symplectic surgery” [M2, MW].

Many of the ideas used in our proof are already present in the literature. The construction of the moduli space of a Riemann surface without boundary as a symplectic quotient of the space of all connections by the gauge group action goes back to Atiyah and Bott [AB]. For Riemann surfaces with non-empty boundary the corresponding moduli space was constructed by Donaldson [D] as a special case of moduli spaces of framed connections. The relationship between the Verlinde formula and “quantization commutes with reduction” was outlined by Segal [Seg] and is present both in the algebraic geometry approach and also in the physics literature (e.g. [W1, EMSS]). The symplectic cross-sections used here are generalizations of the “extended moduli space” of Chang [C1], Huebschmann [H] and Jeffrey [J]. Some of our results on Hamiltonian \( LG \)-actions overlap with those of Chang [C2]. The densely-defined torus actions in this paper are related to the work of Goldman [G] and Jeffrey-Weitsman [JW] on the
“twist flows” on the moduli space.

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2. Statement of results

In this section we describe our approach to the factorization property. In order to state the main result as quickly as possible we omit the details of Sobolev topologies and existence of infinite-dimensional quotients which are postponed until Appendix [A].

2.1. Construction of the moduli space. We begin by reviewing Yang-Mills theory for Riemann surfaces with boundary along the lines of Atiyah-Bott [AB] and Donaldson [D]. Let \( \Sigma \) be an oriented compact Riemann surface with \( b \) boundary components. If \( \Sigma \) is connected and has genus \( g \), we will write \( \Sigma = \Sigma^b_g \). We denote by \( \iota : \partial \Sigma \hookrightarrow \Sigma \) the inclusion of the boundary.

Let \( G \) be a connected and simply-connected compact Lie group. We fix an \( \text{Ad} \)-invariant inner product on the Lie algebra \( g \), normalized by the requirement that on each simple summand of \( g \), the norm squared of the coroot corresponding to the highest root is equal to 2. Let \( P \) be a principal \( G \)-bundle over \( \Sigma \), and \( G(\Sigma) \) the group of gauge transformations, i.e. the space of sections \( s \) of the associated bundle \( P \times_{\text{Ad}(G)} G \).

Since \( \pi_0(G) = \pi_1(G) = \{1\} \), the bundle \( P \) is necessarily trivial and we can identify \( G(\Sigma) \) with the space of maps \( \Sigma \to G \) and its Lie algebra with \( \Omega^0(\Sigma,g) \). Let \( A(\Sigma) \sim= \Omega^1(\Sigma,g) \) be the space of principal connections on \( P \). It has a symplectic form given by

\[
\omega_A(a_1, a_2) = \int_{\Sigma} a_1 \wedge a_2
\]

making \( A(\Sigma) \) into an infinite-dimensional symplectic manifold. For all \( A \in A(\Sigma) \), we denote by

\[
d_A = d + [A, \cdot] : \Omega^i(\Sigma,g) \to \Omega^{i+1}(\Sigma,g)
\]

the associated covariant derivative and by \( F_A = dA + \frac{1}{2}[A,A] \in \Omega^2(\Sigma,g) \) its curvature. The natural gauge group action

\[
g \cdot A = \text{Ad}_g(A) - dg g^{-1}
\]

preserves the symplectic structure, and the fundamental vector field corresponding to \( \xi \in \Omega^0(\Sigma,g) \) is given by

\[
\xi_A(\Sigma)(A) = -d_A(\xi).
\]

According to Atiyah and Bott [AB, A], a moment map \( \Phi \) for this action is given by

\[
\langle \Phi(A), \xi \rangle = \int_{\Sigma} F_A \cdot \xi + \int_{\partial \Sigma} \iota^*(A \cdot \xi),
\]

where the second integral is defined with respect to the induced orientation on \( \partial \Sigma \). Let \( G_0(\Sigma) \subset G(\Sigma) \) be the kernel of the restriction map to the boundary so that there is an exact sequence

\[
1 \to G_0(\Sigma) \to G(\Sigma) \to G(\partial \Sigma) \to 1.
\]

The moment map for the action of \( G_0(\Sigma) \) on \( A(\Sigma) \) is \( A \mapsto F_A \) and hence the symplectic quotient of \( A(\Sigma) \) by \( G_0(\Sigma) \) is

\[
M(\Sigma) := A_F(\Sigma)/G_0(\Sigma)
\]

where \( A_F(\Sigma) \subset A(\Sigma) \) is the space of flat connections. If \( \partial \Sigma = \emptyset \) then \( M(\Sigma) \) is a compact, finite dimensional stratified symplectic space (in general singular) [SL]. On the other hand, if \( \partial \Sigma \neq \emptyset \) then according to Donaldson [D] \( M(\Sigma) \) is a smooth infinite-dimensional symplectic manifold. Just as for Riemann surfaces without boundary, \( M(\Sigma) \) admits an alternative description via holonomies (Theorem...
It has a residual Hamiltonian action of the gauge group $\mathcal{G}(\partial \Sigma)$ of the boundary with moment map

$$\Phi : \mathcal{M}(\Sigma) \to \Omega^1(\partial \Sigma, g), \quad [A] \mapsto \iota^* A.$$  

By choosing parametrizations of the boundary components $B_i \cong S^1$ compatible with the induced orientation on $\partial \Sigma$ one obtains an identification $\mathcal{G}(\partial \Sigma) \cong (LG)^b \cong L(G^b)$ where $LG = \text{Map}(S^1, G)$ is the loop group of $G$.

**Remark 2.1.** For any complex structure on $\Sigma$, the Hodge star operator gives rise to an $LG$-invariant complex structure on $\mathcal{M}(\Sigma)$ which makes $\mathcal{M}(\Sigma)$ into an $LG$-Kähler manifold. However, in this paper we will not make use of this complex structure.

**Example 2.2.** The moduli space $\mathcal{M}(\Sigma_1^b)$ for the disk is the space $LG/G = \Omega G$ of based loops in $G$ (fundamental homogeneous space of $L G$). See e.g. [PS, W2].

We now explain how to construct a pre-quantum line bundle $L(\Sigma)$ over $\mathcal{M}(\Sigma)$ which carries an action of a central extension of $\mathcal{G}(\partial \Sigma) \cong LG^b$. Since $\mathcal{A}(\Sigma)$ is an affine space, the trivial line bundle $\mathcal{A}(\Sigma) \times \mathbb{C}$ with connection 1-form

$$\theta_A : T_A \mathcal{A}(\Sigma) \cong \Omega^1(\Sigma, g) \to \mathbb{R}, \quad a \mapsto \frac{1}{2} \int_\Sigma a \wedge A$$

is a pre-quantum line bundle over $\mathcal{A}(\Sigma)$. The central $S^1$-extension $\widetilde{\mathcal{G}(\Sigma)}$ of the gauge group defined by the cocycle

$$(1) \quad \epsilon(g_1, g_2) = \exp \left( -\frac{i}{4\pi} \int_\Sigma g_1^{-1} dg_1 \wedge d g_2 g_2^{-1} \right),$$

acts on $\mathcal{A}(\Sigma) \times \mathbb{C}$ by $\theta$-preserving automorphisms via

$$(g, z) \cdot (A, w) = (g \cdot A, \exp \left( \frac{i}{4\pi} \int_\Sigma g^{-1} dg \wedge A \right) z w).$$

We will show in the appendix, along the lines of the papers [RSW, W2, Mi] that the extension has a canonical trivialization over the subgroup $\mathcal{G}_0(\Sigma) \subset \mathcal{G}(\Sigma)$ so that $\mathcal{G}_0(\Sigma)$ acts on $\mathcal{A}(\Sigma) \times \mathbb{C}$. The quotient

$$L(\Sigma) = (\mathcal{A}(\Sigma) \times \mathbb{C})//\mathcal{G}_0(\Sigma) = (\mathcal{A}_F(\Sigma) \times \mathbb{C})//\mathcal{G}_0(\Sigma)$$

with the induced connection is a pre-quantum line bundle over $\mathcal{M}(\Sigma)$ which carries an action of the central extension $\widetilde{\mathcal{G}(\partial \Sigma)} := \widetilde{\mathcal{G}(\Sigma)}/\mathcal{G}_0(\Sigma)$ of $\mathcal{G}(\partial \Sigma)$. The restriction of $\widetilde{\mathcal{G}(\partial \Sigma)}$ to the boundary is isomorphic to the basic central extension $LG^b$ of the loop group [PS]. We denote by $L^m(\Sigma)$ the $m$-th tensor power of $L(\Sigma)$, and by $M^m(\Sigma)$ the moduli space with $m$ times the equivariant symplectic form, so that $L^m(\Sigma) \to M^m(\Sigma)$ is an $LG^b$-equivariant pre-quantum bundle.

### 2.2. Gluing equals reduction.

Let $\Sigma$ be a compact oriented Riemann surface obtained from a second (possibly disconnected) Riemann surface $\tilde{\Sigma}$ by gluing along two boundary components $B_{\pm} \subset \partial \tilde{\Sigma}$ by the map $B_+ \to B_-; z \mapsto z^{-1}$. (See Figure 4.) In this case $\mathcal{M}(\Sigma)$ can be obtained from $\mathcal{M}(\tilde{\Sigma})$ by a symplectic reduction. Let $LG \to \mathcal{G}(B_+) \times \mathcal{G}(B_-) \subset \mathcal{G}(\partial \tilde{\Sigma})$ denote the anti-diagonal embedding induced by the map $z \mapsto (z, z^{-1})$. The anti-diagonal embedding lifts to the central extension and therefore acts on $L(\tilde{\Sigma}) \to \mathcal{M}(\tilde{\Sigma})$. We have the following theorem which we learned from S. Martin:

**Theorem 2.3.** The moduli space $\mathcal{M}(\Sigma)$ and line bundle $L(\Sigma)$ are given by symplectic reduction by the anti-diagonal action:

$$\mathcal{M}(\Sigma) = \mathcal{M}(\tilde{\Sigma})//LG, \quad L(\Sigma) = L(\tilde{\Sigma})//LG.$$
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Figure 1. The Riemann surfaces $\hat{\Sigma}$ and $\Sigma$

On a set-theoretical level, this follows easily by noting that the moment map for the anti-diagonal action is given by

$$[\hat{A}] \mapsto \iota^+_{\hat{A}} - \iota^-_{\hat{A}}$$

where $\iota_{\pm} : S^1 \to \hat{\Sigma}$ is the composition of the map $z \mapsto z^{\pm 1}$ with the inclusion $B_{\pm} \to \hat{\Sigma}$. Therefore, if an equivalence class $[\hat{A}]$ is in the zero level set then the pullbacks of the representative $\hat{A}$ to $B_{\pm}$ are equal. In this case one can choose a representative $\hat{A}$ so that the two sides patch together to form a connection $A$ on $\Sigma$. We give a rigorous argument in Appendix A.

2.3. Factorization. Let $G$ be a compact Lie group and $M$ a Hamiltonian $G$-space. Suppose that by some quantization procedure one can construct out of these data a virtual representation $Q(M)$ of $G$. The “quantization commutes with reduction” principle (as formulated by Guillemin-Sternberg [GS2]) says that in this case the quantization of the symplectic reduction $M//G$ should equal the invariant part of the quantization:

$$(2) \quad Q(M//G) = Q(M)^G.$$ 

In the context of Kähler quantization [3] was proved in [SS2] for smooth symplectic quotients and in [SR] for singular quotients. Another approach (which does not require the existence of an invariant Kähler structure) is to define $Q(M)$ to be the equivariant index (Riemann-Roch number) of the Spin$^c$-Dirac operator $\partial_L$ associated to a compatible, invariant almost complex structure on $M$ and pre-quantum line bundle $L \to M$ (see Section 4.1)

$$Q(M) = RR(M,L).$$

In this form, the principle has been proved in [Gu, M1, Vg1] in the abelian case and in [M2] for the non-abelian case; the case of singular quotients has been dealt with in [MS].

Heuristically, the factorization property for moduli spaces of flat connections follows from an application of (2) in the setting of Hamiltonian actions of loop groups. Let $\hat{\Sigma}$ be a compact oriented Riemann surface (possibly disconnected) and $\Sigma$ the Riemann surface formed by gluing along two boundary components $B_{\pm} \subset \partial \hat{\Sigma}$. Theorem 2.3 and (2) would imply an isomorphism of $\hat{G}(\partial \Sigma)$-representations

$$Q(\mathcal{M}^m(\Sigma)) = Q(\mathcal{M}^m(\hat{\Sigma}))^{LG} \quad \text{(Factorization)}.$$ 

We want to emphasize that we do not prove the factorization theorem in this form, which would require the construction of the quantization of an infinite dimensional symplectic manifold.

Rather, note the following two corollaries of the principle for compact groups $G$. For any dominant weight $\mu$, let $^*\mu$ be the dominant weight for the dual representation $V^*_\mu$, which by Borel-Weil can be realized as the quantization of the coadjoint orbit $O_{^*\mu} = G \cdot ^*\mu$. The reduction $M_\mu := M \times O_{^*\mu} // G$ is called the reduction of $M$ at $\mu$. As a corollary of the principle [2] one has that $Q(M_\mu) = (Q(M) \otimes V^*_\mu)^G$, so that

$$Q(M) = \oplus_\mu Q(M_\mu) V^*_\mu.$$
Now suppose that $M$ is a compact quantizable Hamiltonian $G \times G$-space, and let $G$ act on $M$ by the diagonal action. Then

$$Q(M//G) = Q(M)^G = \bigoplus_{\mu,\nu} Q(M_{\mu,\nu}) \left( V_{\mu} \otimes V_{\nu} \right)^G = \bigoplus_{\mu} Q(M_{\mu,\mu}).$$

Our main result is a generalization of this formula to the setting of Hamiltonian $LG$-actions with proper moment maps:

**Theorem 2.4.** Let $G$ be a compact connected simply-connected Lie group, and $\mathfrak{A}$ the corresponding fundamental alcove. Let $M$ be a Hamiltonian $L(G \times G)$-Banach manifold with proper moment map at non-zero level $m$ and $LG^2$-equivariant pre-quantum line bundle $L \rightarrow M$. Then

$$\text{RR}(M//L, L//L) = \sum_{\mu \in m \mathfrak{A} \cap \Lambda^*} \text{RR}(M_{\mu,*,\mu}, L_{\mu,*,\mu}).$$

The properness assumption guarantees that all quotients are finite dimensional and compact. Their Riemann-Roch numbers can be defined using desingularization if necessary (see Section 4.3).

As a special case Theorem 2.4 gives

**Theorem 2.5** (Factorization). Let $\hat{\Sigma}$ be a compact oriented Riemann surface (possibly disconnected) with $b \geq 2$ boundary components and $\Sigma$ the Riemann surface formed by gluing along two boundary components $B_{\pm} \subset \partial \hat{\Sigma}$. Given a level $m \in \mathbb{N}$ and dominant weights $\nu = (\nu_1, \ldots, \nu_{b-2})$ at level $m$, one has

$$\text{RR}(\mathcal{M}^m(\Sigma)\nu, L^m(\Sigma)\nu) = \sum_{\mu \in m \mathfrak{A} \cap \Lambda^*} \text{RR}(\mathcal{M}^m(\hat{\Sigma})\nu, L^m(\hat{\Sigma})\nu).$$

A gauge-theoretic interpretation of the reductions $\mathcal{M}^m(\Sigma)\nu$ is given as follows. The fundamental alcove $\mathfrak{A}$ can be viewed as the set of conjugacy classes $\mathfrak{A} \cong G/\text{Ad}(G)$. The Cartan involution $*: \mathfrak{A} \rightarrow \mathfrak{A}$ sends the conjugacy class ($g$) to ($g^{-1}$). For any Riemann surface $\Sigma$ with $b$ boundary components, the reduced space $\mathcal{M}^m(\Sigma)\nu$ for $\nu = (\nu_1, \ldots, \nu_b) \in m \mathfrak{A}^b$ is the moduli space of flat connections on the **marked Riemann surface** $(\Sigma; \nu_1, \ldots, \nu_b)$, for which the holonomies $g_j \in G$ around the boundary components $B_j$ are required to lie in the conjugacy classes $\nu_j/m$.

As examples we review two well-known applications of the factorization theorem.

2.4. **Fusion product.** Let $\Sigma$ be the three-holed sphere and $\mathcal{M}(\Sigma)$ the associated $LG^3$-Hamiltonian manifold. For $\mu, \nu, \alpha \in m \mathfrak{A} \cap \Lambda^*$ define

$$N^m_{\mu,\nu,\alpha} := \text{RR}(\mathcal{M}^m(\Sigma)_{\mu,\nu,*,\alpha}, L^m(\Sigma)_{\mu,\nu,*}).$$

the Riemann-Roch number of the moduli space corresponding to the markings $\mu, \nu, *, \alpha$. The integers $N^m_{\mu,\nu,\alpha}$ are a set of fusion rules. This means that they satisfy the axioms

$$N^m_{\mu,\nu,\alpha} = N^m_{\nu,\mu,*,\alpha} = N^m_{*,*,\nu,*,\alpha},$$

and

$$N^m_{\mu,*,\nu,*,\alpha} = N^m_{*,\mu,*,*,\alpha} = \delta_{\mu,\nu},$$

and

$$\sum_{\alpha} N^m_{\mu,\alpha,\nu} N^m_{\nu,\alpha} = \sum_{\alpha} N^m_{\beta,\alpha,\mu} N^m_{\nu,\rho,\alpha},$$

That (3) satisfies the associativity property (4) follows because by Theorem 2.4, both sides are equal to

$$\text{RR}(\mathcal{M}^m(\hat{\Sigma})s, L^m(\hat{\Sigma})s),$$

where $\hat{\Sigma}$ is the four-holed sphere.
The **fusion product** on the free group $\text{Rep}_m(LG)$ of irreducible $LG$-representations at level $m \in \mathbb{N}$ is the commutative associative product

$$V_\mu \oplus V_\nu = \bigoplus \alpha N_{\mu,\nu}^m V_\alpha.$$  

The holomorphic induction map at level $m$ 

$$\text{Ind}_m : (\text{Rep}(G), \otimes) \to (\text{Rep}_m(LG), \otimes)$$

is a homomorphism of fusion rings. This statement is proved for example in Teleman [Tel] and Faltings [F]. In particular, the fusion coefficients at level $m$ can be expressed in terms of the coefficients for the tensor product on $\text{Rep}(G)$.

2.5. **Verlinde formula.** As noted by Verlinde, in case $b = 0$ and $g \geq 2$ factorization leads to an expression for the quantum dimension of $\mathcal{M}(\Sigma)$ in terms of eigenvalues of a certain symmetric matrix. In this case, $\Sigma$ can be obtained by gluing together $g - 1$ copies of the two-punctured torus $\Sigma^2$. Let $N = \# \mathbb{A} \cap mA^*$ be the number of points in the fundamental alcove and $A$ the $N \times N$ matrix with coefficients

$$A^g_{\beta} = \text{RR}(\mathcal{M}^m(\Sigma^2)_{a,s}, L^m(\Sigma^2)_{a,s}) = \sum_{\mu,\nu} N_{\mu,\nu}^m N_{\mu,\nu}^m.$$  

Theorem 2.4 implies that 

$$\text{RR}(\mathcal{M}^m(\Sigma)_a, L^m(\Sigma)_a) = Tr(A^{g-1}) = \sum \lambda_i^{g-1}$$

where $\lambda_i$ are the eigenvalues of $A$. For information on how to obtain the explicit Verlinde formula from this approach, see Beauville [B].

3. **Hamiltonian loop group actions**

Let $G$ be a connected, simply connected compact Lie group, $T$ a maximal torus and $W = N_G(T)/T$ the Weyl group. We normalize the invariant inner product on $\mathfrak{g}$ as explained in Section 2.1, thereby identifying $\mathfrak{g} \cong \mathfrak{g}^*$ and $\mathfrak{t} \cong \mathfrak{t}^*$. The integral lattice $\exp^{-1}(1) \subset \mathfrak{t}$ will be denoted by $\Lambda$, the weight lattice by $\Lambda^*$, and the affine Weyl group by $W_{\text{aff}} = W \rtimes \Lambda$.

Let the loop group $LG$ be the Banach Lie group consisting of maps $S^1 \to G$ of some fixed Sobolev class $s > \frac{1}{2}$. (By the Sobolev embedding theorem, $LG$ consists of continuous loops, so that multiplication and inversion are defined pointwise.) Recall [FS] that the Lie algebra of the central extension $\widetilde{LG}$ is the product $\widetilde{\mathfrak{g}} = \mathfrak{g} \times \mathbb{R}$, with bracket

$$\left([\xi_1, t_1], [\xi_2, t_2]\right) = \left([\xi_1, \xi_2], \frac{1}{2\pi} \oint \xi_1 \cdot \xi'_2\right)$$

where the prime indicates the derivative with respect to the $S^1$ coordinate. We define $L\mathfrak{g}^*$ to be the space of maps from $S^1$ to $\mathfrak{g}^*$ of Sobolev class $s - 1$. The natural pairing of $L\mathfrak{g}^*$ with $\widetilde{\mathfrak{g}}$ given by integration makes $L\mathfrak{g}^*$ into a subset of the topological dual of $\mathfrak{g}$. The coadjoint action of $LG$ on $\widetilde{\mathfrak{g}}^* := L\mathfrak{g}^* \times \mathbb{R}$ is given by

$$g \cdot (\xi, \lambda) = (\text{Ad}_g(\xi) - \lambda g \cdot g^{-1}, \lambda).$$

It follows that under the identification of $L\mathfrak{g}^*$ with $g$-valued 1-forms of Sobolev class $s - 1$ given by the Hodge $*$-operator and the inner product on $\mathfrak{g}$, the $LG$-actions on elements of $L\mathfrak{g}^*$ considered as connections correspond to the action on the affine hyper-plane $L\mathfrak{g}^* \times \{1\} \subset \widetilde{\mathfrak{g}}^*$. Henceforth we fix a level $\lambda \neq 0$ and identify $L\mathfrak{g}^*$ with the affine hyper-plane $L\mathfrak{g}^* \times \{\lambda\}$. We denote by

$$\text{Hol} : L\mathfrak{g}^* \to G$$
the map that sends $\xi \in Lg^*$ to the holonomy of $\xi/\lambda$, considered as a connection 1-form on $S^1$. Then Hol has the equivariance property $\text{Hol}(g \cdot \xi) = \text{Ad}_{g(0)} \text{Hol}(\xi)$. The restriction of Hol to $g \cong g^* \subset Lg^*$ is given by $\text{Hol}(\xi) = \exp(2\pi \xi/\lambda)$.

We choose a closed positive Weyl chamber $t_+ \subset t$ and let $\mathfrak{A} \subset t_+$ be the corresponding fundamental alcove. There are natural identifications

$$t_+ \cong t/W \cong g/\text{Ad}(G),$$

that is every (co)adjoint orbit meets the positive Weyl chamber in exactly one point. Similarly, for the affine $LG$-action on $Lg^*$ at level $\lambda$ and the action of $W_{\text{aff}} = W \rtimes \Lambda$ on $t$ given by $(w, v) \cdot \xi = w \cdot \xi + \lambda v$ one has

$$\lambda \mathfrak{A} \cong t/W_{\text{aff}} \cong Lg^*/LG,$$

that is, every coadjoint $LG$-orbit at level $\lambda$ meets $\lambda \mathfrak{A} \subset t \subset Lg^*$ in exactly one point.

By a symplectic structure on a Banach manifold $M$ we mean a closed two-form $\omega$ that is weakly non-degenerate, that is, for any $m \in M$ the map $T_m M \to T_m^* M$ induced by $\omega$ is injective (see e.g. [AMR]). A smooth action of $LG$ on $M$ which preserves the symplectic form will be called Hamiltonian at level $\lambda \in \mathbb{R}$ if there exists a moment map $\Phi : M \to Lg^*$ such that the composition of $\Phi$ with the inclusion $Lg^* \to \hat{Lg}^*$ at level $\lambda$ is $LG$-equivariant. Such an action can be considered an action of $\hat{LG}$ with the central circle acting trivially with constant moment map $\lambda$. We emphasize that we require the moment maps for the $LG$-actions to be "sufficiently smooth", i.e. to take values in $Lg^*$, which consists of loops of Sobolev class $s - 1$.

A $\hat{LG}$-equivariant line bundle $L \to M$ with connection is pre-quantum if its curvature is equal to the symplectic form and the action satisfies the pre-quantum condition \([19]\). This requires in particular that $\lambda$ is an integer.

**Example 3.1.** The basic examples of Hamiltonian $LG$-spaces (at level $\lambda \neq 0$) are coadjoint orbits $\mathcal{O}_\xi = LG \cdot (\xi, \lambda)$. Letting $\delta_{\xi/\lambda}$ be the covariant derivative with respect to the connection $\frac{1}{\lambda} \xi$,

$$\delta_{\xi/\lambda} : Lg \to Lg^*, \eta \to \eta' + \frac{1}{\lambda} [\xi, \eta],$$

the fundamental vector field $\eta_{Lg^*}$ for the infinitesimal action of $Lg$ on $Lg^*$ at level $\lambda$ is given by $\eta_{Lg^*} (\xi) = -\lambda \delta_{\xi/\lambda} (\eta)$. The symplectic form $\nu_\xi$ on $\mathcal{O}_\xi$ is given by the usual KKS formula

$$\nu_\xi((\eta_1)_{Lg^*}, (\eta_2)_{Lg^*}) = \frac{\lambda}{2\pi} \int \eta_1 \cdot \delta_{\xi/\lambda} (\eta_2)$$

with moment map as usual the inclusion into $Lg^*$.

For $\xi \in \Lambda^*$, the orbit $\mathcal{O}_\xi, \xi \in \lambda \mathfrak{A}$ admits a pre-quantum line bundle $\Xi(\mathcal{O}_\xi)$ if and only if $\lambda = m \in \mathbb{Z}$ and $\xi \in \Lambda^*$. If $m \in \mathbb{N}$, geometric quantization of $\Xi(\mathcal{O}_\xi) \to \mathcal{O}_\xi$ by the Borel-Weil construction \([PS]\) gives the irreducible positive energy representation of $LG$ at level $m$ with highest weight $\xi$.

**Remark 3.2.**

a. The inversion map $I^* : LG \to LG$ transforms a Hamiltonian action with moment map $(\Phi, \lambda)$ into one with moment map $(\Phi, -\lambda)$. Hence if $M$ is a Hamiltonian $LG \times LG$-manifold with moment map $(\Phi_+, \lambda; \Phi_-, \lambda)$ at level $\lambda \neq 0$ then the anti-diagonal action of $LG$ is at level 0, with moment map $\Phi_+ + \Phi_-.$

b. If $(M, \omega, \Phi)$ is a Hamiltonian $LG$-Banach manifold at non-zero level $\lambda \neq 0$, then $(M, \lambda^{-1} \omega, \lambda^{-1} \Phi)$ is a Hamiltonian $LG$-space at level $+1$. Henceforth, we will always take the level to be $+1$ unless specified otherwise, and identify $Lg^*$ with the affine hyper-plane $Lg^* \times \{1\} \subset \hat{Lg}^*.$

**Example 3.3.** Let $G$ be connected and simply connected. Let $T^* \widehat{LG}$ be the cotangent bundle of the central extension of $LG$. Trivialization by left-invariant one-forms gives a diffeomorphism $T^* \widehat{LG} \cong$
$\hat{L}G \times \text{Hom}(\hat{L}g, \mathbb{R})$ where $\text{Hom}(\hat{L}g, \mathbb{R})$ is the topological dual of $Lg$. The subset $\hat{X} = \hat{L}G \times \hat{L}g^*$ is a Hamiltonian $\hat{L}G \times \hat{L}g$-space, with actions given by

$$L_a(\hat{g}; \xi, \lambda) = (a \hat{g}; \xi, \lambda), \quad R_a(\hat{g}; \xi, \lambda) = (\hat{g} a^{-1}; \text{Ad}_a(\xi, \lambda))$$

and moment maps

$$\tilde{\Phi}^{(L)}(\hat{g}; \xi, \lambda) = \text{Ad}_g(\xi, \lambda), \quad \tilde{\Phi}^{(R)}(\hat{g}; \xi, \lambda) = - (\xi, \lambda).$$

Let $X$ be the reduction of $\hat{X}$ with respect to central circle $S^1$ at moment level 1. Then $X \cong Lg \times Lg^*$, and the induced left and right actions of $Lg$ are Hamiltonian, with moment maps

$$L_a(g, \xi) = (a, g, \xi), \quad R_a(g, \xi) = (g a^{-1}, a \cdot \xi)$$

are Hamiltonian, with moment maps

$$\tilde{\Phi}^{(L)}(g, \xi) = g \cdot \xi, \quad \tilde{\Phi}^{(R)}(g, \xi) = - \xi.$$

(Here we can use the involution $I^*$ to make $\Phi^{(R)}$ into a moment map at level +1.) It has the property that for every Hamiltonian $Lg$-Banach manifold $M$, the reduced space $M \times X//LG$ by the anti-diagonal action is symplectomorphic to $M$ itself.

The trivial line bundle $L_X = \hat{X} \times \mathbb{C}$ is a pre-quantum line bundle for $\hat{X}$, with the restriction of the canonical 1-form on $T^* \hat{L}G$ defining a pre-quantum connection; reduction with respect to the $S^1$-action gives an $L(G \times G)$-equivariant pre-quantum bundle $L_X \rightarrow X$ isomorphic to the pull-back of $\hat{L}G \times S^1 \mathbb{C}$. If $L \rightarrow M$ is an $Lg$-equivariant pre-quantum line bundle, the anti-diagonal reduction $L \boxtimes L_X//LG$ is isomorphic to $L$ itself.

We shall see later (Example 3.17) that $X$ is equivariantly symplectomorphic to the moduli space $\mathcal{M}(\Sigma^g_0)$ of the annulus and that $L_X$ is equivariantly isomorphic to $L(\Sigma^g_0)$.

Hamiltonian loop group actions on Banach manifolds with proper moment maps at positive level behave in many respects like Hamiltonian actions of compact groups on finite dimensional symplectic manifolds. This is due to the existence of finite-dimensional “symplectic cross-sections”. These are symplectic analogs of highest-weight modules in representation theory.

### 3.1. Cross-sections for Hamiltonian actions of compact groups

First we review the symplectic cross-section theorem in the setting of compact, connected Lie groups $G$. Choose a maximal torus $T \subset G$ and positive Weyl chamber $\mathfrak{t}_+^* = \{ \tau \subset \mathfrak{t}^* \}$.

For every open face $\sigma$ of $\mathfrak{t}_+^*$, the stabilizer subgroup $G_\xi \subset G$ does not depend on the choice of $\xi \in \sigma$, and is denoted by $G_\sigma$. Since $G_\sigma$ contains the maximal torus $T$, there is a unique $G_\sigma$-invariant splitting of Lie algebras

$$\mathfrak{g} = [\mathfrak{g}_\sigma, \mathfrak{g}_\sigma] \oplus \mathfrak{z}(\mathfrak{g}_\sigma) \oplus \mathfrak{g}_\sigma^+$$

where $\mathfrak{z}(\mathfrak{g}_\sigma)$ is the center of $\mathfrak{g}_\sigma$ and $\mathfrak{g}_\sigma^+$ is a $G_\sigma$-invariant complement. In fact, using an invariant inner product to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, $\mathfrak{g}_\sigma$ is characterized as the kernel of the map $\text{ad}_\xi = [\xi, \cdot]$ and the tangent space at $\xi$ to the coadjoint orbit with its image; moreover $\mathfrak{z}(\mathfrak{g}_\sigma)$ gets identified with the tangent space to $\sigma$.

Let $U_\sigma \subset \mathfrak{g}^*$ be the $G_\sigma$-invariant open subset of $\mathfrak{g}_\sigma^*$ defined by

$$U_\sigma := G_\sigma \cdot \bigcup \tau \subset \sigma$$

Then $G \cdot U_\sigma \cong G \times_{G_\sigma} U_\sigma$, which implies that $U_\sigma$ is a slice for the coadjoint action at any $x \in \sigma$. Now let $N$ be a Hamiltonian $G_\sigma$-space with moment map $\Phi_N$. The symplectic induction construction [GS1] shows that if $\Phi_N(N)$ is contained in $U_\sigma$ then there exists a unique symplectic structure on the associated bundle

$$\text{Ind}^G_{G_\sigma}(N) := G \times_{G_\sigma} N$$
such that the $G$-action is Hamiltonian and the symplectic form $\omega$ and moment map $\Phi$ restrict to the given symplectic form and moment map on $N$. Given a second group $K$ with a Hamiltonian action on $N$ such that this action commutes with $G$, one obtains a Hamiltonian $G \times K$-action on $\text{Ind}^{G}_{K}(N)$. In particular, the action of $G$ always extends to an action of $G \times Z(G_{\sigma})$, where $Z(G_{\sigma})$ is the center of $G_{\sigma}$. The moment map for the $Z(G_{\sigma})$-action is given by the composition of $\Phi$ with the quotient map $q : g^{\ast} \to t_{\sigma}^{\ast}$, followed by projection $t^{\ast} \to \mathfrak{g}(\sigma)^{\ast}$.

The symplectic cross-section theorem [GS1] asserts that conversely, for every Hamiltonian $G$-manifold $M$ with moment map $\Phi : M \to g^{\ast}$, the pre-image $Y_{\sigma} = \Phi^{-1}(U_{\sigma})$ is a $G_{\sigma}$-invariant symplectic submanifold. Consequently, the action of $G$ on $G \times Y_{\sigma}$ extends to a Hamiltonian action of $G \times Z(G_{\sigma})$. We refer to these actions (due to Guillemin-Sternberg [GS3]) as the induded (toric) actions and to the map

$$(8) \quad \Phi = q \circ \Phi : M \to t^{\ast}$$

as the induced (toric) moment map. Notice that if $E \to M$ is a $G$-equivariant vector bundle, the action of $Z(G_{\sigma})$ on the restriction $E|_{\Phi^{-1}(U_{\sigma})}$ extends by $G$-equivariance to an action of $G \times Z(G_{\sigma})$ on the restriction of $E$ to $G \cdot \Phi^{-1}(U_{\sigma})$.

Suppose now that $M$ is a Hamiltonian $G \times G$-manifold (e.g. a product of Hamiltonian $G$-manifolds) with moment map $\Phi = (\Phi_{+}, \Phi_{-})$. Let $M_{0} = M//G$ be the symplectic quotient by the diagonal action, and $\Phi_{0} : M_{0} \to t_{\sigma}^{\ast}$ the residual toric moment map induced by $\Phi_{+}$. The symplectic cross-section $Y_{\sigma,-\sigma} = \Phi^{-1}(U_{\sigma} \times -U_{\sigma})$ is a Hamiltonian $G_{\sigma} \times G_{\sigma}$-space, with moment map the restriction of $\Phi$. There is canonical isomorphism of (possibly singular) symplectic quotients

$$M_{0} \supset \Phi_{0}^{-1}(U_{\sigma}) \cong Y_{\sigma,-\sigma}//G_{\sigma}.$$  

### 3.2. Cross-sections for Hamiltonian actions of loop groups.

We now turn to the discussion of symplectic induction and cross-sections for Hamiltonian $LG$-actions on symplectic Banach manifolds, where $G$ is connected and simply connected. We first need to summarize some properties of the coadjoint action of $LG$ on $Lg^{\ast}$ at level 1. Recall [PS] that the evaluation map $LG \to G$, $g \mapsto g(0)$ maps the isotropy group $(LG)_{\xi}$ of a point $\xi \in Lg^{\ast}$ isomorphically to the stabilizer $G_{\text{Hol}({\xi})}$ of the holonomy of $\xi$, in particular $(LG)_{\xi}$ is compact and connected. For points $\xi \in \mathfrak{g} \subset t \subset Lg^{\ast}$, the inverse map is given by $G_{\text{Hol}({\xi})} \ni (LG)_{\xi}, k \mapsto \text{Ad}(e^{\xi} k)$. It follows that the isotropy group $(LG)_{\xi}$ of a point $\xi \in \mathfrak{g}$ contains $T \subset LG$, and depends only on the open face $\sigma \subset \mathfrak{g}$ containing it. We denote this group by $(LG)_{\sigma}$, the group $G_{\text{Hol}({\xi})}$ by $(LG)_{\sigma}$ and the restriction of the central extension $\widehat{LG}$ to $(LG)_{\sigma}$ by $\widehat{(LG)}_{\sigma}$. Note that

$$\tau \subset \sigma \Rightarrow (LG)_{\tau} \subset (LG)_{\sigma}.$$  

If $\sigma$ contains 0 then $(LG)_{\sigma}$ is contained in the subgroup $G \subset LG$ of constant loops and is equal to the stabilizer group of points in $\sigma$ under the coadjoint action of $G$ on $g^{\ast}$.

Another consequence of the above description is that $(LG)_{\sigma}$ consists only of smooth maps. Alternatively, this follows from the fact that its Lie algebra $(Lg)_{\sigma}$ of $(LG)_{\sigma}$ is equal to the kernel of the elliptic operator $\delta_{\xi} : Lg \to Lg^{\ast}$ defined in (5). The image $\text{im}(\delta_{\xi})$ is equal to the tangent space to the $LG$-orbit through $\xi$, i.e. to the annihilator $(Lg)_{\sigma}^{0}$, so that there are $(LG)_{\sigma}$-invariant direct sum (Hodge) decompositions

$$(9) \quad Lg^{\ast} = (Lg)^{\ast}_{\sigma} \oplus (Lg)^{0}_{\sigma}$$

and

$$(10) \quad Lg = (Lg)_{\sigma} \oplus (Lg)_{\sigma}^{\perp}.$$  

The map $\delta_{\xi}$ induces a $(LG)_{\sigma}$-equivariant Banach space isomorphism $(Lg)_{\sigma}^{\perp} \cong (Lg)^{0}_{\sigma}$.\footnote{The stabilizer groups for the adjoint action are connected by a result of Bott-Samelson described in [BT].}
Remark 3.4. It is important to note that since the \( LG \)-action on \( Lg^* \) is only affine-linear, the action of \((LG)_\sigma \) on \((Lg)^*_\sigma \cong (Lg)^*_0 \times \{1\} \subset \hat{Lg}^* \) in this splitting is not the coadjoint action unless \( \sigma \in \mathfrak{A} \). Indeed, from the description of the isomorphism \((LG)_\sigma \cong G_{\text{Hol}(\xi)} \) and the coadjoint action of \( \hat{Lg} \) given in \( \text{[3]} \), one finds that the action is given by

\[
(LG)_\sigma \times (Lg)^*_\sigma \to (Lg)^*_\sigma, \quad (k, \eta) \mapsto (\text{Ad}_{k^{-1}})^*(\eta - \mu) + \mu,
\]

where \( \mu \) is any element in the affine span of \( \sigma \).

As above, we now define, for every open face \( \sigma \subset \mathfrak{A} \)

\[
U_\sigma = (LG)_\sigma \cdot \bigcup_{\tau \subset \mathfrak{A}} \tau.
\]

Note that \( U_\sigma \) is an open subset of \((Lg)^*_\sigma \), in particular it is finite dimensional and consists only of smooth elements of \( Lg^* \).

Lemma 3.5. The set \( U_\sigma \) is a slice for all \( \xi \in \sigma \) for the action of \( LG \), i.e., the canonical map

\[
LG \times (LG)_\sigma U_\sigma \to LG \cdot U_\sigma
\]

is a diffeomorphism of Banach manifolds.

Proof. The map is bijective because for any \( \eta \in U_\sigma \), the stabilizer \((LG)_\eta \subset (LG)_\sigma \). That the differential is an isomorphism follows from the splitting \( \text{[3]} \).

Remark 3.6. Let \( \sigma = \{ \xi \} \subset \mathfrak{A} \) be a vertex such that \( \exp(-2\pi \xi) \) is contained in the center \( Z(G) \) of \( G \), i.e., \( \overline{(LG)_\sigma} = G \). Then \( f_\xi(\theta) := \exp(-\theta \xi) \) defines an exterior automorphism of \( LG \),

\[
(f_\xi \cdot g)(\theta) = \text{Ad}_{f_\xi(\theta)} g(\theta).
\]

Similarly, there is an automorphism of \( Lg \) by

\[
f_\xi \cdot \eta = \text{Ad}_{f_\xi(\eta)}(\eta) + \xi.
\]

These two automorphisms are compatible, that is

\[
f_\xi \cdot (g \cdot \eta) = (f_\xi \cdot g) \cdot (f_\xi \cdot \eta).
\]

It follows that the slice \( U_\sigma \) is isomorphic to the slice \( U_{\{\eta\}} \). For example, if \( G = SU(n) \) all vertices of \( \mathfrak{A} \) exponentiate to elements of the center.

Lemma 3.7. For all \( \xi \in U_\sigma \), the tangent space \( T_\xi(LG \cdot \xi) \cong \text{im}(\delta_\xi) \) to the coadjoint orbit \( O_\xi \) through \( \xi \) decomposes into an \( \nu_\xi \)-orthogonal direct sum of closed symplectic subspaces,

\[
T_\xi(LG \cdot \xi) \cong T_\xi((LG)_\sigma \cdot \xi) \oplus (Lg)^0_\sigma,
\]

where \((Lg)^0_\sigma \) is the annihilator of \((Lg)_\sigma \) in \( Lg^* \).

Proof. It follows easily from the definition \( \text{[3]} \) of the symplectic form on \( LG \cdot \xi \) that the two subspaces are symplectically orthogonal. The proof is completed by noting that the coadjoint orbit \((LG)_\sigma \cdot \xi \) with the KKS form is a symplectic submanifold.

Theorem 3.8. (Symplectic induction) Let \( \sigma \) be a face of \( \mathfrak{A} \) and \( N \) a symplectic Banach manifold with a Hamiltonian \((LG)_\sigma \)-action and moment map \( \Phi_N : N \to (Lg)^*_\sigma \) such that for some \( \mu \) in the affine span of \( \sigma \), the image \( (\Phi_N + \mu)(N) \subset U_\sigma \). Then there exists a unique \( LG \)-invariant symplectic form \( \omega \) on the Banach manifold

\[
\text{Ind}^{LG}_{(LG)_\sigma}(N) := LG \times (LG)_\sigma N
\]

such the \( LG \)-action is Hamiltonian, with moment map \( \Phi : \text{Ind}^{LG}_{(LG)_\sigma}(N) \to Lg^* \), and such that the pullback of \( \omega \) (resp. \( \Phi \)) to \( N \) is equal to the given symplectic form (resp. moment map \( \Phi_N + \mu \)) on \( N \).
Proof. Let \( M := LG \times_{(LG)_\sigma} N \). By Equation \([11]\) the map \( \Phi_N + \mu : N \to (LG)_\sigma^* \oplus \{1\} \) extends to a unique \( LG \)-equivariant map \( \Phi : M \to LG \times_{(LG)_\sigma} (LG)_\sigma^* \to LG^* \). For \( \Phi \) to be a moment map for \( \omega \), one must have
\[
\omega_x(\eta_M, \zeta_M) = \nu_{\Phi(x)}(\eta_{LG^*}, \zeta_{LG^*}).
\]
This condition, together with \( LG \)-invariance of \( \omega \) and the condition that the pull-back to \( N \) be \( \omega_N \) completely determines \( \omega \), and also implies that \( \omega \) is closed. To show \( M \) is symplectic let \( x \in N \subset M \), and \( \tau \subset \mathfrak{a} \) the open face containing \( \Phi_N(x) \). By \([12]\) together with Lemma \([3.7]\), there is a natural \( \omega \)-orthogonal splitting \( T_xM \cong T_xN \oplus (LG)_\sigma^0 \). Since \( \sigma \subset \tau \), this is in fact a symplectic subspace, which shows that \( T_xM \) is symplectic.

Theorem 3.9. (Symplectic cross-section) Let \((M, \omega)\) be a symplectic Banach manifold, and \( LG \times M \to M \) a Hamiltonian \( LG \)-action with moment map \( \Phi : M \to LG^* \). For every open face \( \sigma \subset \mathfrak{a} \), the symplectic cross-section \( Y_\sigma := \Phi^{-1}(U_\sigma) \) is a symplectic \((LG)_\sigma\)-invariant Banach submanifold, and the action of \((LG)_\sigma\) is Hamiltonian. The restriction \( \Phi|Y_\sigma \) is a moment map for the action of \((LG)_\sigma\), and a moment map for the \((LG)_\sigma\)-action is given by \( \Phi|Y_\sigma - \mu \), for any \( \mu \) in the affine span of \( \sigma \).

Proof. By equivariance and since \( U_\sigma \) is a slice for the \( LG \)-action, \( \Phi \) is transversal to \( U_\sigma \). The implicit function theorem for Banach manifolds thus shows that \( N := \Phi^{-1}(U_\sigma) \) is a smooth Banach submanifold. Since \( T_{\Phi(x)}U_\sigma = (LG)_\sigma^* \subset LG^* \), the tangent space \( T_xN \) at some \( x \in N \) is equal to \( (d_x\Phi)^{-1}(LG)_\sigma^* \). In order to show that \( T_xN \) is symplectic, we show that \( T_xM \cong E \oplus T_xN \) where \( E \) is a closed symplectic complement to \( T_xN \) that is symplectically perpendicular to \( T_xN \).

Let \( E \) be the image of the map \((LG)_\sigma^0 \to T_xM\) sending \( \eta \) to the fundamental vector field \( \eta_M(x) \). A continuous inverse is given by the composition of the map \( d_x\Phi : E \to (LG)_\sigma^0 \) with the isomorphism \((LG)_\sigma^0 \cong (LG)_\sigma^+ \). Hence \( E \) is a closed complement to \( T_xN \).

By equivariance of the moment map, one has for all \( \eta, \zeta \in LG_\sigma \)
\[
\omega_x(\eta_M, \zeta_M) = \nu_{\Phi(x)}(\eta_{LG^*}, \zeta_{LG^*}) = \frac{1}{2\pi} \oint \eta \cdot \delta_x \zeta.
\]
By Lemma \([3.7]\) \((LG)_\sigma^0 \) is a symplectic subspace of \( T_{\Phi(x)}(LG_\sigma \Phi(x)) \). It follows that \( d_x\Phi \) restricts to a symplectic isomorphism from \( E \) to \((LG)_\sigma^0 \), and also that \( T_xN \) is symplectically perpendicular to \( E \).

The restriction of \( \Phi \) to \( Y_\sigma \) is a moment map for the \((LG)_\sigma\)-action which is equivariant with respect to the affine action on \((LG)_\sigma^0 \oplus \{1\}\) given in Equation \([11]\). Hence, subtracting \( \mu \) gives a moment map which is equivariant with respect to the usual coadjoint action.

In particular this shows that Hamiltonian actions of loop groups at non-zero level are always proper group actions.

Remark 3.10. If \( M \) is a Hamiltonian \( L(G \times G)\)-manifold with moment map \((\Phi_+, \Phi_-)\), we also define cross-sections
\[
Y_{\sigma, -\sigma} := (\Phi_+, \Phi_-)^{-1}(U_\sigma \times (-U_\sigma)),
\]
which are Hamiltonian \((LG)_\sigma \times (LG)_\sigma\)-spaces. Note that the anti-diagonal action of \((LG)_\sigma \subset LG \) on \( Y_{\sigma, -\sigma} \) is Hamiltonian with moment map the restriction of \( \Phi_+ - \Phi_- \), i.e. no shift is required.

Remark 3.11. In our applications, \( M \) will in fact have an invariant Kähler structure. However, the symplectic cross-sections are usually not Kähler submanifolds of \( M \).

As in the finite dimensional case, every Hamiltonian \( LG \)-manifold has Hamiltonian actions (induced toric flows) of the centers \( Z((LG)_\sigma) \) on \( LG \cdot Y_\sigma \) that commute with the action of \( LG \). The moment maps for these actions are given as the composition of the induced toric moment map
\[\tilde{\Phi} := q \circ \Phi : M \to \mathfrak{a},\]
with the projection \( t^* \to \mathcal{g}((L\mathfrak{g})_\bullet)^* \). Here \( q : L\mathfrak{g}^* \to \mathfrak{a} = L\mathfrak{g}^*/L\mathfrak{g} \) is the quotient map, which can also be written as the composition of the holonomy map with the quotient map \( G \to G/\text{Ad}(G) \). As before, if \( M = M//LG \) is the symplectic reduction of a Hamiltonian \( L(G \times G) \)-manifold \( M \) with moment map \((\Phi_+, \Phi_-)\) by the anti-diagonal action, the maps \( \Phi_\pm \) for \( M \) descend to a map \( \Phi : M \to \mathfrak{a} \), whose \( \mathcal{g}((L\mathfrak{g})_\bullet)^* \)-component is a moment map for the induced action of \( Z((L\mathfrak{g})_\sigma) \) on \( \Phi^{-1}(U_\sigma \cap \mathfrak{a}) \). We refer to these as the residual toric moment map and residual toric flow.

**Proposition 3.12.** Let \( H, G \) be compact connected simply connected groups and \( \mathfrak{a} \) the fundamental alcove for \( G \). Let \( M \) be a Hamiltonian \( L(H \times G \times G) \)-Banach manifold, \( M_0 := M//LG \) the (possibly singular) reduction by the diagonal \( LG \)-action, and \( \Phi_0 : M_0 \to \mathfrak{a} \) the residual toric moment map. For every face \( \sigma \) of \( \mathfrak{a} \) we have a canonical homeomorphism

\[
\Phi_0^{-1}(U_\sigma) \cong Y_{\sigma, -\sigma}//(L\mathfrak{g})_\sigma.
\]

If the anti-diagonal action of \( LG \)-action is free on the zero level set, then \( M_0 \) is a smooth Hamiltonian \( LH \)-Banach manifold and the above identification is a symplectomorphism. If \( H = \{ 1 \} \) and if the moment map is proper then \( M_0 \) is finite dimensional.

**Proof.** Note that \( \Phi_0^{-1}(U_\sigma) \) is equal to the open subset

\[
LG^2 \cdot Y_{\sigma, -\sigma}//LG \subset M_0.
\]

Since \( LG^2 \cdot Y_{\sigma, -\sigma} \) is symplectomorphic to the symplectic induction

\[
\text{Ind}^{LG^2}_{(L\mathfrak{g})_\sigma} (Y_{\sigma, -\sigma}) = LG^2 \times_{(L\mathfrak{g})_\sigma} Y_{\sigma, -\sigma},
\]

the formula (13) follows. Now suppose that the anti-diagonal \( LG \)-action on the zero level set is free. Then the diagonal action of \( (L\mathfrak{g})_\sigma \) on the zero level set in \( Y_{\sigma, -\sigma} \) is free. Since \( (L\mathfrak{g})_\sigma \) is compact, this implies that 0 is a regular value for the moment map for both of these actions, and (13) is a diffeomorphism of Banach manifolds. Also, it is clear that the 2-forms induced by the symplectic form \( \omega \) on both sides of (13) coincide. Since the Meyer-Marsden-Weinstein theorem holds for Hamiltonian actions of compact groups on Banach manifolds, it follows that the two-form on \( M_0 \) is symplectic. \( \square \)

This shows in particular that the Meyer-Marsden-Weinstein theorem holds for anti-diagonal \( LG \)-actions for Hamiltonian \( L(G \times G) \)-actions at positive level. (It does not hold in general for Hamiltonian actions of Banach Lie groups.)

### 3.2.1. Reductions with respect to coadjoint \( LG \)-orbits.

Let \( G \) be a compact connected simply connected Lie group, with fundamental alcove \( \mathfrak{a} \), and let \( M \) be a Hamiltonian \( LG \)-Banach manifold. For each \( \xi \in \mathfrak{a} \) let \( O_\xi = LG \cdot \xi \) be the corresponding loop group orbit. Define the **reduction of** \( M \) **at** \( \xi \) as the symplectic reduction by the anti-diagonal \( LG \)-action

\[
M_\xi = M \times O_\xi//LG.
\]

Letting \( \sigma \) be a face of \( \mathfrak{a} \) such that \( \xi \in U_\sigma \) and \( Y_\sigma \) the corresponding symplectic cross-section of \( M \), one also has

\[
M_\xi = Y_\sigma \times O'_\xi//(L\mathfrak{g})_\sigma,
\]

where \( O'_\xi = O_\xi \cap ((L\mathfrak{g})_\sigma \times \{ 1 \}) = (L\mathfrak{g})_\sigma \cdot \xi \) is the orbit for the compact group \( (L\mathfrak{g})_\sigma \). From (14), one finds in particular that if the moment map is proper then \( M_\xi \) is a finite dimensional symplectic quotient. In the case that the quotient is singular, the local structure of its singularities can be described by normal form theorems as in Sjamaar-Lerman [31].

If \( M \) is a Hamiltonian \( LG \)-manifold at level \( m \in \mathbb{N} \), \( L \to M \) an \( LG \)-equivariant pre-quantum line bundle, and \( \xi \in m\mathfrak{a} \cap \Lambda^* \) then we define

\[
L_\xi := L \boxtimes \Xi(O_\xi)//LG,
\]
where $\Xi(O,\xi)$ is the pre-quantum line bundle of $O,\xi$ as in example 3.1. If the anti-diagonal action of $LG$ is locally free on the zero level set then $L\xi$ is a pre-quantum (orbifold) line bundle over $M\xi$.

3.3. Convexity theorems for Hamiltonian loop group actions. In the following theorem, we use symplectic cross-sections to derive convexity and connectedness properties for the moment map of Hamiltonian $LG$-manifolds. Some of these results have been proved by Chang \cite{C2} using similar methods.

**Theorem 3.13.** Let $G$ be a compact connected simply connected Lie group, and $M$ a connected Hamiltonian $LG$-manifold with proper moment map $\Phi : M \to LG^\ast$.

a. For any face $\sigma$ of the fundamental alcove $A$ such that $\Phi(M) \cap \sigma \neq \emptyset$, the corresponding symplectic cross-section $Y_\sigma$ is finite dimensional and connected.

b. The fibers of $\Phi$ are connected.

c. The intersection $\Phi(M) \cap A$ is a convex polytope.

**Proof.** Finite dimensionality follows from the fact that the restriction of $\Phi$ to the submanifold $Y_\sigma = \Phi^{-1}(U_\sigma)$ is proper as a map to $U_\sigma$. Since $U_\sigma$ is finite dimensional, this is only possible if $\sigma$ is finite dimensional.

We next prove that all non-empty cross-sections $Y_\sigma$ are connected. Since the coadjoint orbits $LG/(LG)_\sigma$ are simply connected, this is the case if and only if all flow-outs $LG \cdot Y_\sigma = LG \times_{(LG)_\sigma} Y_\sigma$ are connected. By properness of $\Phi$, the number of connected components $Y_\sigma^i$ of $Y_\sigma$ is finite. The fact that $LG \cdot Y_\sigma$ is connected follows once we can show the convexity property

$$LG \cdot (Y_\sigma^i \cap Y_\sigma^j) \neq \emptyset, \quad LG \cdot (Y_\sigma^i \cap Y_\sigma^k) \neq \emptyset \implies LG \cdot (Y_\sigma^i \cap Y_\sigma^k) \neq \emptyset,$$

since the collection of all $LG \cdot Y_\sigma^i$ is a finite open covering of $M$, and any two points in $M$ can be joined by a path.

To show (16), let $x \in Y_\sigma^i \cap Y_\sigma^j$ and $y \in Y_\sigma^i \cap Y_\sigma^k$. The fact that the restriction of $\Phi$ to $Y_\sigma^i$ is proper as a map into $U_\sigma$ implies that it has connected fibers and that the image $\Phi(Y_\sigma^i) \subset A$ is convex. This follows from the Condeval-Delzord-Molino technique as explained in \cite{FR}, p. 29 and \cite{HNP}, or by using symplectic cutting \cite{MTW}, Remark 5.2. In particular, the line segment from $\alpha = \Phi(x)$ to $\beta = \Phi(y)$ is contained in $\Phi(Y_\sigma^i)$. It follows that there exists a path $\gamma$ in $LG \cdot Y_\sigma^i$ connecting $x$ and $y$ whose image under $\Phi$ is the line segment $\alpha \beta$. The interior of $\alpha \beta$ intersects $U_\sigma \cap A$ and $U_\kappa \cap A$, and is therefore contained in $U_\sigma \cap U_\kappa \cap A$. It follows that $\gamma^{-1}(\text{int}(\alpha \beta)) \subset LG \cdot (Y_\sigma^i \cap Y_\sigma^k)$.

Since all cross-sections $Y_\sigma$ are connected and all restrictions $\Phi|_{Y_{\sigma}}$ have connected fibers \cite{MTW}, it follows that $\Phi$ has connected fibers. Since $\Phi$ is proper, the number of $(LG)_\sigma$-conjugacy classes of stabilizer groups (orbit types) for the $(LG)_\sigma$-action on $Y_\sigma$ is finite. By \cite{MTW}, Remark 5.2 or the Condeval-Delzord-Molino technique, this implies that $\Phi(Y_\sigma)$ is the intersection of $U_\sigma \cap A$ with a convex polyhedron. Since $M$ is connected, and since a closed connected set is convex if and only if it is locally convex, this shows that the image $\Phi(M)$ is a convex polytope.

**Remark 3.14.** As far as we know the Atiyah-Pressley convexity Theorem \cite{AP}, which is a Kostant-type convexity theorem on projections of coadjoint loop group orbits, does not fit into this framework. Note that the relevant moment map, given by projection of the orbit to $t^\ast$ together with the energy function, is not proper.

**Corollary 3.15.** (Convexity properties of the induced toric moment map.) Let $H, G$ be compact connected simply connected Lie groups, with fundamental alcoves $B, A$, and $M$ a Hamiltonian $L(H \times G \times G)$-manifold with proper moment map. Let $M_0$ be the symplectic reduction with respect to the anti-diagonal $LG$-action, and $\Phi_0 : M_0 \to B \times A$ the product of the induced toric moment map for the induced $LH$-action and the residual toric moment map obtained from the $\{e\} \times \{e\} \times LG$-action on $M$. Then the image of $\Phi_0$ is a convex polytope.
3.4. Application to Yang-Mills theory over a Riemann surface. Let \( \Sigma \) be a compact, connected, oriented Riemann surface of genus \( g \) with \( b \) boundary components, and \( G \) a compact, connected and simply connected Lie group, with fundamental alcove \( A \).

3.4.1. Holonomy description. As in the case \( \partial \Sigma = \emptyset \), the moduli space \( \mathcal{M}(\Sigma) \) of flat \( G \)-connections on \( \Sigma \) admits an alternative description in terms of holonomies.

**Theorem 3.16.** If \( b \geq 1 \), the moduli space \( \mathcal{M}(\Sigma) \) is isomorphic to the set of \( (a, c, \xi) \in G^{2g} \times G^{b-1} \times (L\mathfrak{g}^*)^b \) such that

\[
\prod_{i=1}^{2g} [a_{2i-1}, a_{2i}] = \prod_{i=1}^{b} \text{Ad}_{c_i} \text{Hol}(\xi_i)
\]

where \( c_1 = 1 \). This is a smooth submanifold of \( G^{2g} \times G^{b-1} \times (L\mathfrak{g}^*)^b \) and the identification with \( \mathcal{M}(\Sigma) \) is an \( LG^b \)-equivariant diffeomorphism. Here the action of \( g = (g_1, \ldots, g_b) \in LG^b \) on \( G^{2g} \times G^{b-1} \times (L\mathfrak{g}^*)^b \) is given by

\[
g \cdot a_i = \text{Ad}_{g_{i}(0)} a_i, \quad g \cdot c_j = g_{j}(0) c_j g_{j}(0)^{-1}, \quad g \cdot \xi_j = \text{Ad}_{g_{j}} \xi_j - (g_{j})' g_{j}^{-1}
\]

and the moment map is given by projection to the \( (L\mathfrak{g}^*)^b \)-factor.

**Proof.** The diffeomorphism is given as follows. Let \( B_1, \ldots, B_b \) be the boundary circles of \( \Sigma \) equipped with fixed parametrizations \( B_j \cong S^1 \) compatible with the orientations on \( B_j \). This gives identifications \( G(\partial \Sigma) \cong LG^b \) and \( \Omega^1(\partial \Sigma, \mathfrak{g}) \cong (L\mathfrak{g}^*)^b \) and base points \( x_i \in B_j \). Now let \( \rho_1, \ldots, \rho_{2g} \) be smooth loops based at \( x_1 \), and \( \gamma_2, \ldots, \gamma_b \) smooth paths from \( x_1 \) to \( x_2, \ldots, x_n \), in such a way that \( \pi_1(\Sigma) \) is generated by the \( \rho_i \) together with \( B_1 \) and \( \gamma_j^{-1} B_j \gamma_j \), subject to the relation

\[
\prod_{i=1}^{g} [\rho_{2i-1}, \rho_{2i}] = B_1 (\gamma_2^{-1} B_2 \gamma_2) \cdots (\gamma_b^{-1} B_b \gamma_b).
\]

Consider the map

\[\hat{f} : A_F(\Sigma) \to G^{2g} \times G^{b-1} \times (L\mathfrak{g}^*)^b\]

that takes any flat connection \( A \) to \( (a, c, \xi) \) where \( a_i \) is the holonomy around \( \rho_i \), \( c_j \) the parallel transport along \( \gamma_j \), and \( \xi_j \) the restriction of \( A \) to the boundary loop \( B_j \). Then

\[
\prod_{i=1}^{g} [a_{2i-1}, a_{2i}] = \prod_{j=1}^{b} \text{Ad}_{c_j} \text{Hol}(\xi_j),
\]

where we set \( c_1 := 1 \). The map \( \hat{f} \) is equivariant with respect to the action of the gauge group \( G(\Sigma) \) given on \( G^{2g} \times G^{b-1} \times (L\mathfrak{g}^*)^b \) by

\[
g \cdot a_i = \text{Ad}_{g(x_i)} a_i, \quad g \cdot c_j = g(x_1) c_j g(x_j)^{-1}, \quad g \cdot \xi_j = (g|_{B_j}) \cdot \xi_j.
\]

Moreover, \( \hat{f} \) is surjective onto the set of all \( (a, c, \xi) \) satisfying (17): First, as in the case without boundary, one can construct a smooth connection that has the required holonomies \( a_i \) and \( \text{Ad}_{c_j} \text{Hol}(\xi_j) \). Secondly, one can act by an element \( g \) of \( G(\Sigma) \) with \( g(x_1) = 1 \) (which does not change the holonomies) to obtain the required values of \( \xi_j \). This gives the correct values for the parallel transport \( c_j \) along the curves \( \gamma_j \),...
up to an action of the centralizer of $\text{Hol}(\xi_j)$ from the right. Now evaluation at $x_j$ gives an isomorphism of $Z(\text{Hol}(\xi_j))$ with the stabilizer $(L \mathfrak{g})_{\xi_j}$. Thus finally we can act by an element $g' \in \mathcal{G}(\Sigma)$ with $g'(x_1) = 1$ and $g'(x_j) \in Z(\text{Hol}(\xi_j))$ for $j \geq 2$ to obtain the correct values of $c_j$.

We next show that the fiber of $\hat{f}$ over $(a, c, \xi)$ is equal to $G_{\partial}(\Sigma)$. To see this, one may assume after acting by $\mathcal{G}(\Sigma)$ that $\xi$ is smooth. Let $A_1, A_2 \in \hat{f}^{-1}(a, c, \xi)$. For $x \in \Sigma$, choose a smooth path $\lambda$ from $x_1$ to $x$, and let $g(x) \in G$ defined by parallel transport along $\lambda$ by $A_1$, followed by parallel transport along $\lambda^{-1}$ by $A_2$. This is independent of the choice of $\lambda$, and therefore gives a smooth function $g : \Sigma \to G$ with $g \in G_{\partial}(\Sigma)$ and $g \cdot A_1 = A_2$. By a similar argument, one shows that the kernel of the tangent map to $\hat{f}$ is equal to the tangent space to the $G_{\partial}(\Sigma)$-orbit. Hence $\hat{f}$ descends to a smooth embedding

$$f : \mathcal{M}(\Sigma) \to G^{2g} \times G^{b-1} \times (L \mathfrak{g}^*)^b$$

with image equal to the set \([17]\).

It is immediate from this description that the moment map is proper. Equivalently, the cross-sections are finite-dimensional. This fact can also be proved in the gauge theory description by writing the tangent space to the cross-section as the solution space to an elliptic boundary value problem. Since $G$ is a product of simple groups, thus $(G, G) = G$, the holonomy description also shows that for $g \geq 1$ the moment map is surjective onto $(L \mathfrak{g}^*)^b$ so that the moment polytope is simply $\mathfrak{X}^g$ and the convexity Theorem \[13\] is vacuous. On the other hand, in the case $g = 0$ of a $b$-punctured sphere, \([17]\) shows that the moment polytope is equal to the image of the subset

$$\{d \in G^b | \prod_{j=1}^b d_j = 1\}$$

under the quotient map $G^b \to \mathfrak{X}^b$.

**Example 3.17.** By \([17]\), the moduli space $\mathcal{M}(\Sigma_0^b)$ of the two-punctured sphere (annulus) is equivariantly diffeomorphic to the subset $G \times L \mathfrak{g}^* \times L \mathfrak{g}^* \ni (c, \xi_1, \xi_2)$ defined by $\text{Hol}(\xi_1) \text{ Ad}_c \text{ Hol}(\xi_2) = 1$. Consider the Hamiltonian $L(G \times G)$-manifold $X \cong L \mathfrak{g} \times L \mathfrak{g}^*$ introduced in Example \[33\]. The map

$$X \to \mathcal{M}(\Sigma_0^b), (g, \eta) \mapsto (g(0), g \cdot \eta, -\eta)$$

is an equivariant diffeomorphism, preserving the moment maps. Since $\mathcal{M}(\Sigma_0^b)$ and $X$ are multiplicity free, i.e., since all reduced spaces $X_{\xi_1, \xi_2}$ are points, this map is necessarily a symplectomorphism.

Along similar lines, one can prove the holonomy description for the moduli spaces $\mathcal{M}(\Sigma)_{\xi_1, \ldots, \xi_b}$ (see e.g. \[AM\]):

**Theorem 3.18.** Let $\xi_1, \ldots, \xi_b \in \mathfrak{A}$, and $C_1, \ldots, C_b \subset G$ the corresponding conjugacy classes. The moduli space $\mathcal{M}(\Sigma)_{\xi_1, \ldots, \xi_b}$ is homeomorphic to the the quotient of the subset $(a, d) \in G^{2g} \times C_1 \times \cdots \times C_b$ given by

$$\prod_{i=1}^g \prod_{j=1}^b [a_{2i-1}, a_{2i}] = \prod_{j=1}^b d_j,$$

by the conjugation action of $G$.

3.4.2. Symplectic cross-sections. Consider once again the Hamiltonian $L \mathfrak{g}^b$-manifold $\mathcal{M}(\Sigma)$ for $b > 0$. In the holonomy description, the cross-sections $\mathcal{Y}_{\sigma_1 \cdots \sigma_b}$ for faces $\sigma = (\sigma_1, \ldots, \sigma_b)$ of $\mathfrak{A}^b$ are the smooth submanifolds of $G^{2g} \times G^{b-1} \times (L \mathfrak{g}^*)^b$ given by the condition \([17]\) together with the requirement $\xi_j \in U_{\sigma_j}$. For the dimension of $\mathcal{Y}_{\sigma_1 \cdots \sigma_b}$, we find

$$\dim(\mathcal{Y}_{\sigma_1 \cdots \sigma_b}) = (2g + b - 1) \dim G + \sum \dim U_{\sigma_j} - \dim G$$

$$= (2g - 2) \dim G + \sum (\dim G + \dim (L \mathfrak{g})_{\sigma_j}).$$
The extended moduli spaces of Chang [C1], Huebschmann [H] and Jeffrey [J] are defined, for a Riemann surface with a single boundary component, as the subset of $G^g \times \mathfrak{g}^*$ given by the condition $\prod[a_{2i-1}, a_{2i}] = \exp(2\pi \xi)$. On a neighborhood of $\xi = 0$, this is a smooth submanifold and can be given a symplectic structure; however for larger $\xi$ one encounters singularities and degeneracies of the symplectic form. One can view the extended moduli space as the symplectic cross-section corresponding to $\sigma = \{0\}$. The degeneracies find a natural explanation in the full space $\mathfrak{g}^*$ not being a slice for $LG$, whence the cross-section may not be chosen too big. In [J], more general extended moduli spaces are defined for all central elements $c \in Z(G)$ by the modified condition $\prod[a_{2i-1}, a_{2i}] = c \exp(2\pi \xi)$; these may be identified, using Remark 3.6, with the cross-sections corresponding to $\sigma = \{\eta\}$, where $\eta \in \mathfrak{a}$ is the unique vertex such that $\exp(-2\pi \eta) = c$.

3.4.3. Pants decomposition and Goldman twist flows. For every compact connected Riemann surface $\Sigma$ of genus $g$ with $b$ boundary components, except for the cases $(g,b) = (0,0), (0,1), (0,2), (1,0)$, there exist embedded circles $C_1, \ldots, C_r \subset \Sigma$ that decompose $\Sigma$ into a union of 3-holed spheres (pairs of pants) $\Sigma_1, \ldots, \Sigma_r$. For the number $l$ of pants and $r$ of separating curves in this decomposition, one has $3l = b + 3r$, where:

\[
\begin{align*}
    r &= 3g + b - 6 & \text{if } & b \geq 1, g \geq 2, \\
    r &= 3g - 3 & \text{if } & b = 0, g \geq 2, \\
    r &= b & \text{if } & b \geq 1, g = 1, \\
    r &= b - 1 & \text{if } & b \geq 3, g = 0.
\end{align*}
\]

By repeated application of Theorem 2.3, the moduli space $\mathcal{M}(\Sigma)$ is obtained as a symplectic reduction

\[
\mathcal{M}(\Sigma) = \mathcal{M}(\Sigma_1) \times \ldots \times \mathcal{M}(\Sigma_r)/LG^r.
\]

As in the finite dimensional case, this induces a residual toric flow on $\mathcal{M}(\Sigma)$, i.e. a densely defined, effective Hamiltonian $T^{r+b}$-action which commutes with the action of $G(\partial\Sigma)$, known as the Goldman twist flow. The corresponding residual toric moment map, called here the Goldman map, is

\[
\mathfrak{G} : \mathcal{M}(\Sigma) \to \mathfrak{a}^{r+b}, [A] \to (G \cdot \text{Hol}_{B_1}(A), \ldots, G \cdot \text{Hol}_{B_r}(A), G \cdot \text{Hol}_{C_1}(A), \ldots, G \cdot \text{Hol}_{C_r}(A)).
\]

As a special case of Corollary 2.12, we have:

**Theorem 3.19.** The image $\mathfrak{G}(\mathcal{M}(\Sigma)) \subset \mathfrak{a}^{b+r}$ of the Goldman map $\mathfrak{G}$ is a $(b + r) \cdot \dim T$-dimensional convex polytope.

This polytope is given explicitly as follows. Let $P \subset \mathfrak{a}^3$ denote the polytope which is the moment polytope for the three-holed sphere. For each $\Sigma_j$ let $B_1^j, B_2^j, B_3^j \subset \partial \Sigma_j$ denote its boundary components, and let $\mathfrak{a}_j^\nu$ be a copy of the fundamental alcove associated to $B_j^\nu$. Define an involution $\kappa : \mathfrak{a}^{3l} \to \mathfrak{a}^{3l}$ by

\[
(\kappa(x))^\nu = \left\{ \begin{array}{ll}
    x_l^\mu & \text{if } B_j^\nu = B_l^\mu \\
    x_j^\nu & \text{otherwise.}
\end{array} \right.
\]

Then $\mathfrak{G}(\mathcal{M}(\Sigma))$ is the image of the intersection $P^l \cap (\mathfrak{a}^{3l})^\kappa$ where $(\mathfrak{a}^{3l})^\kappa$ denotes the fixed point set of the involution $\kappa$, under the projection $\mathfrak{a}^{3l} \to \mathfrak{a}^{b+r}$.

**Remark 3.20.** For the case $G = SU(2)$ and $\partial \Sigma = \emptyset$ the dimension $(2g - 2) \dim G$ of the moduli space $\mathcal{M}(\Sigma)$ is precisely twice the dimension of this polytope, which means that the Goldman flow gives a completely integrable torus action on a dense subset of $\mathcal{M}(\Sigma)$. This has been studied extensively in Jeffrey-Weitsman [JW].

By Proposition 2.13 if $\sigma$ is a face of the fundamental alcove $\mathfrak{a}$ then we can write $\mathfrak{G}^{-1}(U_\sigma \cap \mathfrak{a}) \subset \mathcal{M}(\Sigma)$ as a symplectic reduction of finite-dimensional symplectic manifolds:
Lemma 3.21. Let \( \hat{\Sigma} \) be a compact oriented Riemann surface (possibly disconnected) and \( \Sigma \) the Riemann surface formed by gluing along two boundary components \( B_\pm \subset \partial \hat{\Sigma} \), and let \( \Phi_\pm : M(\Sigma) \to LG^* \) denote the moment maps for the LG-actions corresponding to \( B_\pm \). Then there is a symplectic diffeomorphism

\[
\Phi^{-1}(U_\sigma \cap \mathcal{A}) = \Phi^{-1}_+(U_\sigma) \cap \Phi^{-1}_-(U_\sigma) / (LG)_\sigma.
\]

4. Quantization

In this section, we define the quantization of moduli spaces of flat connections using the equivariant index (Riemann-Roch number) of Spin-c-Dirac operators, and introduce techniques for dealing with singular quotients.

4.1. Riemann-Roch numbers for almost complex orbifolds. Let \( M \) be a compact almost complex \( G \)-manifold. Then \( M \) has a canonical \( G \)-invariant Spin-c-structure. Given a Hermitian orbifold vector-bundle \( E \to M \), one can consider the \( G \)-equivariant Spin-c-Dirac operator \( \hat{\phi}_E \) corresponding to this Spin-c-structure, twisted by \( E \) (see e.g. [LM]). The equivariant index of \( \hat{\phi}_E \) is a virtual representation of \( G \) which we call the \textbf{equivariant Riemann-Roch number},

\[
\chi = \text{RR}(M, E) := \text{Ind}_G(\hat{\phi}_E) \in \text{Rep}(G),
\]

where \( \text{Rep}(G) \) denotes the representation ring for \( G \). An explicit expression for \( \text{RR}(M, E) \) is given by the Equivariant Index Theorem of Atiyah-Segal-Singer.

If \( M \) is an almost complex \( G \)-orbifold, and \( E \to M \) a \( G \)-orbifold vector bundle\(^2\) the same definition applies. A formula for the character is given by the orbifold index theorem of Kawasaki [Ka] (the equivariant version of which was proved by Vergne [Vergne]). Riemann-Roch numbers satisfy the following functorial properties:

a. (Products) If \( E_i \to M_i \) are \( G_i \)-equivariant vector bundles over compact almost complex \( G_i \)-orbifolds \((i = 1, 2)\), then

\[
\text{RR}(M_1 \times M_2, E_1 \boxtimes E_2) = \text{RR}(M_1, E_1) \boxtimes \text{RR}(M_2, E_2).
\]

b. (Conjugation) Let \( E \to M \) be a \( G \)-equivariant orbifold bundle over a compact almost complex \( G \)-orbifold. Let \( M^* \) denote \( M \) with the opposite almost complex structure. Then

\[
\text{RR}(M^*, E^*) = \text{RR}(M, E)^*,
\]

where, for all \( \chi \in \text{Rep}(G) \), \( \chi^* \in \text{Rep}(G) \) denotes the dual representation.

c. (Induction) Let \( \sigma \subset t^*_+ \) be an open face of the positive Weyl chamber, with corresponding stabilizer group \( G_\sigma \). Let \( \Lambda^*_\sigma, + \supset \Lambda^*_+ \) be the dominant weights for \( G_\sigma \), and

\[
\text{Ind}_{G_\sigma}^G : \text{Rep}(G_\sigma) \to \text{Rep}(G)
\]

denote holomorphic induction. Let \( G/G_\sigma \) be equipped with the standard complex structure coming from its identification with a coadjoint orbit \( G \cdot \mu \), for any \( \mu \in \sigma \). For every almost complex \( G_\sigma \)-orbifold \( Y_\sigma \), the associated bundle \( G \times_{G_\sigma} Y_\sigma \) has a naturally induced almost complex structure. Given a \( G_\sigma \)-invariant orbifold vector bundle \( E_\sigma \to Y_\sigma \), we have

\[
\text{RR}(G \times_{G_\sigma} Y_\sigma, G \times_{G_\sigma} E_\sigma) = \text{Ind}_{G_\sigma}^G \text{RR}(Y_\sigma, E_\sigma).
\]

For a proof, see e.g. [LM].

If \( M \) is a Hamiltonian \( G \)-orbifold with moment map \( \Phi : M \to g^* \), and \( E \to M \) a \( G \)-equivariant vector bundle, one can always choose a \( G \)-invariant compatible almost complex structure \( J \) on \( M \) to define \( \text{RR}(M, E) \). Since any two \( J \)'s are equivariantly homotopic, this definition does not depend on the choice of \( J \). We will be mostly interested in the case that \( E = L \) is a pre-quantum line bundle, i.e., \( L \) is a \( G \)-equivariant Hermitian line bundle equipped with invariant connection whose curvature is \(-2\pi i \)}
times the symplectic form, and the fundamental vector fields on \( L \) and \( M \) are related by the Kostant formula

\[
(19) \quad \xi_L = \text{Lift}(\xi_M) + 2\pi\langle \Phi, \xi \rangle \frac{\partial}{\partial \phi}
\]

where \( \frac{\partial}{\partial \phi} \) is the generating vector field for the scalar circle action on the fiber.

4.2. Desingularization for quotients by compact groups. In this section we define Riemann-Roch numbers for singular symplectic quotients by compact groups and state the “quantization commutes with reduction” Theorem, referring to [MS] for more details.

Suppose that \( M \) is a Hamiltonian \( H \times G \)-orbifold \((H, G \text{ compact})\), such that the \( G \)-action has proper moment map \( \Phi : M \to \mathfrak{g}^* \), and let \( L \to M \) be a \( H \times G \)-equivariant pre-quantum bundle. Let \( M_0 = M//G \) be the symplectic quotient of \( G \) and \( L//G = L_0 := L|\Phi^{-1}(0)/G \) the quotient bundle.

It can be shown ([MS]) that if \( \Phi^{-1}(0) \) is contained in a fixed infinitesimal \( G \)-orbit type stratum (that is, if the dimension of the isotropy group \( G_x \) does not jump as \( x \) varies in \( \Phi^{-1}(0) \)) the level set \( \Phi^{-1}(0) \) is an orbifold and consequently \( M_0 \) is a symplectic orbifold and \( L_0 \) with its induced connection an orbifold pre-quantum line bundle.

If \( \Phi^{-1}(0) \) and \( M_0 \) have more serious singularities, we define \( RR(M_0, L_0) \) as the \( H \)-equivariant Riemann-Roch number of an orbifold line bundle \( \tilde{L}_0 \to \tilde{M}_0 \) obtained from \( L_0 \to M_0 \) by means of partial desingularization. One procedure to obtain a desingularization is due to Kirwan [K]. Roughly speaking, this is an inductive procedure involving a sequence of symplectic blow-ups on \( M \), resulting in a new Hamiltonian \( H \times G \)-orbifold \( \tilde{M} \), followed by a symplectic quotient \( \tilde{M}_0 = \tilde{M}//G \). Since blow-ups in the symplectic category depend on both the choice of Darboux charts around the blow-up locus and on the “size” of the blow-up corresponding to the cohomology class of the symplectic form on the exceptional divisor (see [McS] for details), the partial desingularization \( \tilde{M}_0 \) defined this way is unique only up to symplectic homotopy, and the bundle \( \tilde{L}_0 \to \tilde{M}_0 \) is unique up to isomorphism. This, however, is enough to obtain a well-defined Riemann-Roch number \( RR(\tilde{M}_0, \tilde{L}_0) \).

A simpler, but less canonical way of desingularizing a symplectic quotient is to shift the value of the moment slightly. For all \( \alpha \in \mathfrak{g}^* \), let \( M_\alpha = \Phi^{-1}(\alpha)/G_\alpha \) (with the induced \( H \)-action) be the symplectic quotient. For \( \alpha \in \Phi(M) \) close to zero and general\(^3\), one can take \( (L|\Phi^{-1}(\alpha))/G_\alpha \to M_\alpha \) as a desingularization of \( L_0 \to M_0 \).

One should keep in mind that the diffeotype of the desingularization obtained in this way will in general depend on the choice of the shift. In fact, if \( G \) non-abelian, the two spaces may even have different dimension (so it is perhaps misleading to call \( M_\alpha \) a desingularization of \( M_0 \) in this case.) It turns out that the desingularization procedures described above give the same Riemann-Roch numbers:

**Proposition 4.1.** ([MS]) Suppose \((M, \omega)\) is a Hamiltonian \( H \times G \)-orbifold, such that the \( G \)-action has proper moment map \( \Phi : M \to \mathfrak{g}^* \), and let \( L \to M \) be an \( H \times G \)-equivariant pre-quantum line bundle. Then there is a neighborhood \( U \) of \( 0 \in \mathfrak{g}^* \) such that for all generic \( \alpha \in U \cap \Phi(M) \), one has \( RR(M_\alpha, L_0) = RR(M_\alpha, (L|\Phi^{-1}(\alpha))/G_\alpha) \) as representations of \( H \), where the first RR-number is defined by Kirwan’s partial desingularization.

**Theorem 4.2** (Quantization commutes with reduction). ([M2] [MS]) Let \((M, \omega)\) be a compact Hamiltonian \( G \times H \)-orbifold and \( L \to M \) a \( G \times H \)-equivariant pre-quantum line bundle. Then the \( G \)-invariant part of \( RR(M, L) \) equals the Riemann-Roch number of the symplectic quotient:

\[
RR(M, L)^G = RR(M//G, L//G)
\]

\(^3\)We call a value \( \alpha \) of the moment map \( \Phi \) generic if the restriction of \( \Phi \) to \( \Phi^{-1}(\alpha) \) has maximal rank. This implies that \( \Phi^{-1}(\alpha) \) is a smooth suborbifold and that the dimension of \( G_x \) for \( x \in \Phi^{-1}(\alpha) \) is constant.
where the right-hand side is defined via partial desingularization if 0 is not a regular value of the $G$-moment map.

If $\mu \in \Lambda^*_+ \subset \mathfrak{g}^*$ is a dominant weight, the coadjoint orbit $O_\mu = G / \mu$ with its KKS form has a pre-quantum line bundle $\Xi(O_\mu) \to O_\mu$. Given a $G$-equivariant pre-quantum line bundle $L \to M$ as above, we define the reduced pre-quantum line bundle by

$$L_\mu := (L \otimes \Xi(O_\mu))//G \to M_\mu = (M \times O_\mu)//G.$$  

As a direct corollary to Theorem 4.2, the multiplicity $N(\mu)$ for the weight $\mu$ to occur in $\text{RR}(M, L)$ is given by $\text{RR}(M_\mu, L_\mu)$, defined by canonical desingularization if necessary.

### 4.3. Desingularizations for loop group quotients

We now define desingularizations for reductions by loop group actions. Let $G$ be a compact connected simply-connected Lie group and $M$ a Hamiltonian $L(G \times G)$-manifold with proper moment map at level $m$ and $L(G \times G)$-equivariant pre-quantum line bundle $L$. We will show that the reduction $L//L G \to M//L G$ by the anti-diagonal action can be written in a canonical way as a reduction in finite dimensions, so the desingularization of $L//L G$ can be carried out as in the previous section.

Let $\mathcal{M}^m(\Sigma_0^3)$ denote the moduli space for the three-holed sphere at level $m$, $L^m(\Sigma_0^3)$ its pre-quantum line bundle and

$$M^{(1)} = M \times \mathcal{M}^m(\Sigma_0^3)/L G \times L G, \quad L^{(1)} = L \times L^m(\Sigma_0^3)/L G \times L G$$

the quotient by the product of the two anti-diagonal $L G$-actions, defined by pairing each $L G$-factor for $M$ with an $L G$-factor for $\mathcal{M}^m(\Sigma_0^3)$. Since these actions are free, $M^{(1)}$ is a Hamiltonian $L G$-manifold and $L^{(1)}$ an $L G$-equivariant pre-quantum line bundle. Because the reduction of $\mathcal{M}^m(\Sigma_0^3)$ at 0 is the moduli space $\mathcal{M}^m(\Sigma_0^3)$, the reduction of $M^{(1)}$ at 0 is given by

$$M^{(1)}_0 = (M \times \mathcal{M}^m(\Sigma_0^3)//L G)//L G = M//L G$$

by Example 3.3 and 3.17. Let $Y_{\{0\}}^{(1)}$ be the symplectic cross-section for $M^{(1)}$ at $\{0\} \subset \mathfrak{g}$. Then

$$M//L G \cong M^{(1)}_0 \cong Y_{\{0\}}^{(1)}//G, \quad L//L G \cong L^{(1)}_0 \cong L^{(1)}_0 | Y_{\{0\}}^{(1)}//G$$

which proves the claim. In particular, the Riemann-Roch numbers of the line bundles in (15) can be defined by desingularization. Note that in the case of moduli spaces, the above procedure corresponds to “adding a puncture” to the Riemann surface.

### 5. Symplectic Surgery

Recall from Subsection 4.4.3 that if $\Sigma$ is a Riemann surface with $b$ boundary components and $\xi_1, \ldots, \xi_b \in \mathfrak{g}$, the Goldman functions corresponding to a pants decomposition of $\Sigma$ generate a densely defined torus action on the moduli space $\mathcal{M}(\Sigma)_{\xi_1, \ldots, \xi_b}$. In this section, we describe how to use this torus action in conjunction with Lerman’s symplectic cutting procedure to decompose the moduli space into pieces which can be expressed as symplectic reductions in finite dimensions.

#### 5.1. Symplectic cutting

We briefly recall Lerman’s cutting construction [L]. Let $M$ be a Hamiltonian $S^1$-orbifold, with moment map $\psi : M \to \mathbb{R}$. Consider the action of $S^1$ on the product $M \times \mathbb{C}^-$ given by $e^{i\phi} \cdot (m, z) = (e^{i\phi} \cdot m, e^{-i\phi} z)$, with moment map

$$\tilde{\psi}(m, z) = \psi(m) - |z|^2.$$  

The zero level set of $\tilde{\psi}$ is a union of $\psi^{-1}(0) \times \{0\}$ and the set of all $(m, z)$ with $\psi(m) = |z|^2 > 0$. Suppose that 0 is a regular value of $\psi$. Then 0 is also a regular value of $\tilde{\psi}$, and the reduced space $M_+ := M \times \mathbb{C}^-//S^1$ is a symplectic orbifold. As a topological space, $M_+$ is obtained from the manifold with
boundary $\psi^{-1}(\mathbb{R}_{>0})$ by collapsing the boundary by the nullfoliation of the pullback of the symplectic form.

It was shown by Lerman that this is also true symplectically:

**Proposition 5.1** (Lerman). Let

$$M_+ := (M \times \mathbb{C})/S^1$$

be the cut space. The canonical maps

$$i_0 : M_0 \to M_+, \quad i_{>0} : \psi^{-1}(\mathbb{R}_{>0}) \to M_+$$

are smooth symplectic embeddings, and the normal bundle of $M_0$ in $M_+$ is canonically isomorphic to the associated bundle $\psi^{-1}(0) \times_{S^1} \mathbb{C}$, where $S^1$ acts on $\mathbb{C}$ with weight $-1$. Given a Hamiltonian action of a Lie group $G$ on $M$, with moment map $\Phi : M \to \mathfrak{g}^*$, such that this action commutes with the action of $S^1$, there is a naturally induced Hamiltonian $G$-action on $M_+$, which agrees with the given actions and moment maps on $M_0$ and $\psi^{-1}(\mathbb{R}_{>0})$.

By reversing the $S^1$-action, one can also define a cut-space $M_-$, which is the union of $M_0$ and $\psi^{-1}(\mathbb{R}_{<0})$.

The orbifold structure on $M_{\pm}$ depends only on the circle action on $\psi^{-1}(0)$; replacing the given circle by a covering introduces additional orbifold singularities. This shows that cutting is a local operation, that is, the construction only depends on the existence of the circle action on $\psi^{-1}(0)$. For example, one can cut the two-torus $S^1 \times S^1$, with its area form as a symplectic form, along the circle $S^1 \times \{1\}$ by using a locally defined Hamiltonian for the $S^1 \times \{1\}$-action. Note that in this case the “cut space” $M_{\text{cut}}$ (which is the disjoint union $M_+ \cup M_-$ for global Hamiltonian $S^1$-actions) is connected, and in fact just the two-sphere.

Returning to the case of a globally defined Hamiltonian $S^1$-action, suppose that we are given an $S^1 \times G$-equivariant complex vector bundle $E \to M$. By pulling $E$ back to $M \times \mathbb{C}^-$, restricting to $\psi^{-1}(0)$ and taking the quotient

$$E_+ := (\text{pr}_1^* E|_{\psi^{-1}(0)})/S^1$$

we obtain a $G$-vector bundle over $M_+$. There are canonical equivariant isomorphisms

$$i_0^* E_+ \cong E_0, \quad i_{>0}^* E_+ \cong E|_{\psi^{-1}(\mathbb{R}_{>0})}.$$ 

In particular, one obtains $G$-equivariant “cut bundles” $E_+ \to M_+$ and $E_- \to M_-$ even if the $S^1$-action is only defined near $\psi^{-1}(0)$. (Of course, the cut bundle will depend on the choice of a lift of the $S^1$-action to $E$.) To include the case that the cut space is connected, we denote the cut bundle by $E_{\text{cut}} \to M_{\text{cut}}$.

If $E = L$ is a pre-quantum line bundle, i.e. has a Hermitian metric and connection with curvature the symplectic form, then $L_{\pm} = (L \boxtimes \mathbb{C})/S^1$ with induced metric and connection is a pre-quantum line bundle. The following result was proved (for the case of global $S^1$-actions) by Siye Wu.

**Proposition 5.2.** Let $E \to M$ be a $G$-equivariant vector bundle, such that the $G$-action on $M$ preserves the symplectic form. Let $E_0 \to M_0$ be the reduced bundle and $E_{\text{cut}} \to M_{\text{cut}}$ the cut bundle, with respect to a local Hamiltonian $S^1$-action. The Riemann-Roch numbers of the cut bundles satisfy the gluing rule

$$RR(M, E) = RR(M_{\text{cut}}, E_{\text{cut}}) − RR(M_0, E_0).$$

By iterating the cutting operation, it is possible to obtain more general cut-spaces. The idea is to take a “suborbifold with corners” $N \subset M$ with the property that the nullfoliation on the boundary faces of $N$ are orbifold-torus bundles; the cut space $M_N = N/\sim$ is obtained by collapsing the boundary faces.

We make this precise as follows. Let $N \subset M$ be a closed subset together with a finite collection of open subsets $U_j, \ j \in J$ of $M$ equipped with Hamiltonian $S^1$-actions with moment maps $\psi_j : U_j \to \mathbb{R},$
and an open subset \( V \subset M \) equipped with a Hamiltonian action of a torus \( H \), with moment map \( \Psi : V \to h^* \).

**Definition 5.3.** We will call a closed subset \( N \subset M \) together with the collection \( (V, \Psi), \{(U_j, \psi_j)\} \) a

**sub-orbifold with reducible corners** if the following holds:

a. Locally, near any point \( n, N \) is given as the intersection 
\[
\bigcap_{U_j \ni n} \psi_j^{-1}(\mathbb{R}_{\geq 0}) \cap \Psi^{-1}(0).
\]

b. On each non-empty intersection \( U_{j_1} \cap \ldots \cap U_{j_k} \cap V \), the \( S^1 \)-actions for the moment maps \( \psi_{j_\nu} \) commute with each other and with \( H \), and the corresponding action of \((S^1)^k \times H\) is locally free.

The **boundary faces** of \( N' \subset N \) are obtained by setting some of the “boundary defining functions” \( \psi_j \) equal to zero. By including these into \( \Psi \), one sees that every boundary face is also a sub-orbifold with reducible corners.

By reducing with respect to the \( H \)-action and collapsing the boundary facets \( N \cap \psi_j^{-1}(0) \) as above, one obtains a cut space \( M_N \) of dimension \( \dim M - 2 \dim H \). Given a complex vector bundle \( E \to V \) together with lifts of the local \( H \)-resp. \( S^1 \)-actions so that the lifts commute on the intersections \( U_{j_1} \cap \ldots \cap U_{j_k} \cap V \), one obtains cut bundles \( E_N \to M_N \).

**Remark 5.4.**

a. Even though the nullfoliation of the boundary faces is intrinsically defined, the cut space depends, as an orbifold, on the choice of the local \( S^1 \)-actions.

b. In case \( M \) is a Hamiltonian \( G \)-orbifold and \( N \) a \( G \)-invariant sub-orbifold with reducible corners, the cut space \( M_N \) is a Hamiltonian \( G \)-orbifold if the local \( H \)-and \( S^1 \)-actions commute with \( G \).

c. If \( E = L \) is a pre-quantum line bundle and if the local lifts of the local circle actions satisfy the pre-quantum condition then \( L_N \) is a pre-quantum line bundle.

The following Theorem shows how the Riemann-Roch number of the original bundle decomposes into Riemann-Roch numbers for cut bundles.

**Theorem 5.5.** (Gluing Formula) Let \( M \) be a compact Hamiltonian \( G \)-orbifold, \( E \to M \) a \( G \)-equivariant complex orbifold vector bundle, and \( \mathcal{N} = \{N\} \) a collection of sub-orbifolds with reducible corners of \( M \), together with \( G \)-equivariant lifts to \( E \) of the local \( S^1 \)- and \( H \)-actions. We assume that

a. The collection \( \mathcal{N} \) covers \( M \).

b. For each \( N \in \mathcal{N} \), all boundary faces of \( N \) are also in \( \mathcal{N} \).

c. The intersection of any two \( N \in \mathcal{N} \) is either empty or is a boundary face of each.

Then the \( G \)-equivariant Riemann-Roch numbers of the cut spaces satisfy the gluing rule

\[
RR(M, E) = \sum_{N \in \mathcal{N}} (-1)^{\text{codim}(N)} RR(M_N, E_N).
\]  

(21)

For sub-orbifolds with reducible corners constructed from induced toric maps for Hamiltonian actions of compact groups \( G \) the above gluing formula was proved in [MW]; the general result will be discussed in a separate paper [MW].

### 5.2. Induced toric actions for compact groups.

Let \( G \) be a compact connected Lie group, and \( M \) a Hamiltonian \( G \)-orbifold, with moment map \( \Phi : M \to g^* \), and let \( \tilde{\Phi} = q_0 \circ \Phi \) be the corresponding induced toric map. Recall from Section 5.1 that for every open face \( \sigma \subset t_+^* \), there is a \( G \)-equivariant Hamiltonian action of the center \( Z(\mathcal{G}_\sigma) \subset G_{\sigma} \) on \( G \cdot Y_\sigma = \Phi^{-1}(U_\sigma) \), with moment map the \( j(\mathfrak{g}_\sigma)^* \)-component of \( \tilde{\Phi} \).

A sub-orbifold with reducible corners \( N \subset M \) can be constructed as follows. Let \( Q \subset t^* \) be a simple rational polytope, with the property that for any two open faces \( \sigma \subset t_+^* \) and \( F \) of \( Q \) with non-empty intersection, the tangent space to \( F \) contains the space perpendicular to \( \sigma \), i.e. the space \( \mathfrak{g}_{\sigma}^0 \cap t^* \).
Examples of polytopes $Q$ with these properties are given in Figure 2 below. We claim that the pre-image $N := \tilde{\Phi}^{-1}(Q)$ is a sub-orbifold with reducible corners for generic $Q$.

For each open face $F \subset Q$, let $T_F$ denote the the torus with Lie algebra the annihilator of the tangent space to $F$. In particular, let $H := T_{\text{int}(Q)}$, and define

$$\Psi := \text{pr}_h \cdot (\tilde{\Phi} - \mu)$$

where $\mu \in Q \cap t_+^*$ and $\text{pr}_{h^*}$ is the projection to $h^*$.

By assumption, $H \subset Z(G_\sigma)$ for every face $\sigma$ that meets $Q$. It follows that $\Psi$ is smooth on a $G$-invariant neighborhood $V$ of $N$ and generates a $G$-equivariant Hamiltonian $H$-action.

Similarly, given any codimension 1-face $F \subset Q$ with $F \cap t_+^* \neq \emptyset$, let $v_F \in \Lambda$ be a lattice vector such that $h \oplus \mathbb{R} \cdot v_F = \mathbb{R} \cdot (F - \mu_F)^0$ where $\mu_F \in F$. On a neighborhood $U_F$ of $\tilde{\Phi}^{-1}(F)$, the map

$$\psi_F = \langle \tilde{\Phi} - \mu_F, v_F \rangle : U_F \to \mathbb{R}$$

is smooth and generates an $G \times H$-equivariant $S^1$-action. Moreover, the $S^1$-actions on all overlaps $U_{F_1} \cap \ldots \cap U_{F_k}$ commute. The condition for the above data to define a sub-orbifold with reducible corners is that for each open face $F \subset Q$, the $T_F \subset T \subset G$-action on $\Psi^{-1}(F \cap t_+^*)$ is locally free. In this case, we call the polytope $Q$ admissible and denote the cut space by $M_Q = M_N$. Note that if $Q = \{\mu\}$ with $\mu \in \text{int}(t_+^*)$ then $Q$ is admissible if and only if $\mu$ is a regular value of $\Phi$, and in this case $M_Q$ is just the reduced space $M_\mu$.

![Figure 2. Examples of $Q$ for $G = SU(3)$](image)

If $E \to M$ is a $G$-equivariant vector bundle then as explained in Section 3.1 there is a canonical lift of the $Z(G_\sigma) \times G$-action on $\tilde{\Phi}^{-1}(U_\sigma) = G \cdot Y_\sigma$. Therefore one also has a canonical lift of the local $H$- resp. $S^1$-actions, and a corresponding cut bundle $E_N$. However, if $E = L$ is a $G$-equivariant pre-quantum bundle then this lift does not in general satisfy the pre-quantum condition, and therefore the cut bundle $L_N$ is not pre-quantum.

However, if $Q$ is a lattice polytope then $\text{pr}_{h^*}(\mu)$ lies in the lattice for $H \subset T$, and also $\langle \mu_F, v_F \rangle \in \mathbb{Z}$. One therefore obtains local pre-quantum lifts by multiplying the given $H$- resp. $S^1$-actions by the character $\exp(2\pi i \langle \text{pr}_h(\mu), \xi \rangle)$ resp. $\exp(2\pi i \langle \mu_F, v_F \rangle \phi)$. The cut bundle $L_Q$ for the modified actions is a $G$-equivariant pre-quantum line bundle. In case $Q$ has rational vertices, choose any covering $\tilde{T} \to T$ such that $Q$ is a lattice polytope in the refined lattice $\Lambda^* \supset \Lambda^*$. The local actions of the corresponding covers $\tilde{H} \to H$ and $S^1 \to S^1$ admit pre-quantum lifts as before. Notice again that passing to a cover introduces additional orbifold singularities.

A family $R$ satisfying the hypotheses of Theorem 5.3 can be constructed as follows. Let $M$ be a compact Hamiltonian $G$-orbifold, and $\mu \in \text{int}(t_+^*)$. Recall that the dual cone $C_\sigma$ to $t_+^*$ at an open face
σ ⊂ t∗ is defined to be the set of all vectors v ∈ t∗ such that the minimum of the inner product (v, x) for x ∈ t∗ is achieved by x ∈ σ. For each face τ ⊂ σ let Qσ,τ denote the polytope µ + τ − Cσ, and Q the collection of these polytopes. For generic choices of µ, all of these polytopes are admissible.

![Figure 3. The collection Q for G = SU(3)](image)

Next consider the case that M is a Hamiltonian G × G-orbifold, with moment map (Φ+, Φ−) and assume that 0 is a regular value for the diagonal G-action. Let M0 = M//G be the reduced space and ˜Φ0 : M0 → t∗ the residual toric moment map induced from the map ˜Φ+ = q ◦ Φ+ on the first factor. Given any polytope Q as above the subset N := ˜Φ−1(0) ∩ ∗ ˜Φ−(M0) is a sub-orbifold with reducible corners.

If E → M is a G × G-equivariant vector bundle, the quotient bundle E0 has a canonical lift of the local Z(Gσ)-actions on M0, so that there is a canonical cut bundle. The same argument as above shows that if E = L is a pre-quantum line bundle and Q is a rational polytope, one can modify the given lift of the local actions in such a way that the cut bundle is a pre-quantum line bundle.

As an application, we prove:

**Proposition 5.6.** Let G be a compact connected Lie group and M a Hamiltonian G × G-orbifold with proper moment map Φ = (Φ+, Φ−). Let L → M be a G × G-equivariant pre-quantum line bundle. If the symplectic quotient M0 = M//G by the diagonal action is compact, the Riemann-Roch number of the reduced bundle L0 = L//G is given by

\[ RR(M0, L0) = \sum_{\mu \in A^*_+} RR(M_{\mu, *\mu}, L_{\mu, *\mu}). \]  

(22)

(Here all RR-numbers are defined by partial desingularization if necessary.)

**Proof.** If M is compact, both sides are equal, by the “quantization commutes with reduction” Theorem 4.2. If M is not compact, let ˜Φ0 : M0 → t∗ denote the residual toric moment map induced by ˜Φ+. Its image is given by

\[ ˜Φ0(M0) = ˜Φ+(M) \cap * ˜Φ−(M). \]

Choose a compact admissible rational polytope Q ⊂ t∗ × t∗, such that ˜Φ0(M0) × * ˜Φ0(M0) is contained in the relative interior of Q ∩ (t∗ × t∗). We can assume that Q has rational vertices, so that the cut bundle LQ → M0 is a G-equivariant pre-quantum bundle. Since M0 is compact and L//G = LQ//G, this reduces the proof to the case where M is compact. □
5.3. **Induced toric actions for loop groups.** Now let \( G \) be a compact, connected, simply-connected Lie group, and \( \mathfrak{A} \) the corresponding fundamental alcove. Let \( M \) be a Hamiltonian \( L(G \times G) \)-manifold with proper moment map \( \Phi = (\Phi_+, \Phi_-) \) (at level \( \lambda = +1 \)). We assume that the anti-diagonal \( LG \)-action is locally free on its zero level set, so that the symplectic quotient \( M_0 = M/LG \) is a finite dimensional, compact symplectic orbifold. Let \( \tilde{\Phi}_0 : M_0 \to \mathfrak{A} \) denote the residual toric moment map induced from \( q \circ \Phi_+ \).

For each face \( \sigma \subset \mathfrak{A} \), the open set \( \tilde{\Phi}_0^{-1}(U_\sigma) \subset M_0 \) carries a Hamiltonian action of \( Z((LG)_\sigma) \), with moment map given by \( \tilde{\Phi}_0 \) followed by projection to \( \mathfrak{z}((Lg)_\sigma)^* \).

Let \( \mathfrak{A}_\epsilon := (1 - \epsilon)\mathfrak{A} + \epsilon \mu \) for some small \( \epsilon \in \mathbb{R}_{>0} \) and \( \mu \in \text{int}(\mathfrak{A}) \). This is the shaded region in Figure 4.

![Figure 4](image-url)  

**Figure 4.** Cutting the fundamental alcove for \( G = SU(3) \)

For each face \( \sigma \) of \( \mathfrak{A} \), let \( \sigma_\epsilon = (1 - \epsilon)\sigma + \epsilon \mu \) denote the corresponding face of \( \mathfrak{A}_\epsilon \). For any face \( \tau \subset \mathfrak{T} \) let \( Q_{\tau, \sigma} = \tau - C_\sigma \). Let \( Q \) be the collection of polytopes \( Q_{\tau, \sigma} \) (see Figure 4). For generic choices of \( \epsilon, \mu \), each \( Q \in Q \) gives rise to a sub-orbifold with reducible corners \( N = \tilde{\Phi}_0^{-1}(Q) \) and cut space \( (M_0)_Q := (M_0)_N \).

Also, for every \( \hat{LG}^2 \)-equivariant vector bundle \( E \to M \), any lift of the local \( Z((LG)_\sigma) \)-actions to \( E_0 = E/K \) defines a cut-bundle \( (E_0)_Q \). Theorem 5.5 gives an expression for \( RR(M_0, E_0) \) in terms of the Riemann-Roch numbers \( RR((M_0)_Q, (E_0)_Q) \).

If \( L \to M \) is a \( \hat{LG}^2 \)-equivariant pre-quantum line bundle, and \( \epsilon \in \mathbb{Q}, \mu \in \Lambda^* \otimes \mathbb{Z} \) are chosen so that each \( Q \) is a polytope with rational vertices, one can pass to a cover \( \tilde{T} \to T \) and arrange the lifts in such a way that one obtains a pre-quantum cut-bundle \( (L_0)_Q \).

6. **Proof of Theorem 2.4**

As in the statement of the theorem, let \( G \) be compact, connected and simply connected, and \( \mathfrak{A} \) its fundamental alcove. Let \( M \) be a Hamiltonian \( L(G \times G) \)-manifold with proper moment map at level \( m \in \mathbb{N} \) and pre-quantum line bundle \( L \to M \), and \( L/LG \to M/LG \) the symplectic reduction with respect to the anti-diagonal \( LG \)-action.

The basic idea of the proof of 2.4 is to cut the symplectic quotient \( M/LG \) into pieces, all of which are obtained by diagonal reduction of a finite-dimensional symplectic manifold by a Hamiltonian action of a compact group. The result then follows by applying the “quantization commutes with reduction” Theorem for compact groups, together with the gluing formula. If the quotient \( M/LG \) is singular, we have to combine this idea with partial desingularization.

Let us first assume that \( 0 \) is a regular value for the anti-diagonal action, so that \( M/LG \) is a compact symplectic orbifold, and \( L/LG \to M/LG \) a pre-quantum line bundle. Let \( Q \) be a collection of admissible
rational polytopes in $\mathfrak{A}$ defined in the previous subsection. By the Gluing Formula, Equation \(24\),
\[
\text{RR}(M//LG, L//LG) = \sum_{Q \in \mathcal{Q}} (-1)^{\text{codim}(Q)} \text{RR}((M//LG)_{Q}, (L//LG)_{Q}).
\]
Each cut space $(M//LG)_{Q}$ can be written as a reduction in finite dimensions: Let $\sigma \subset \mathfrak{A}$ be an open face such that $Q \cap \mathfrak{A}$ is contained in $\mathcal{U}_\sigma$, and let $Y_{\sigma,-\sigma}$ denote the corresponding cross-section of $M$. The restriction to $Y_{\sigma,-\sigma}$ of $\Phi_+$ defines an induced toric action on $Y_{\sigma,-\sigma}$; let $(Y_{\sigma,-\sigma})_Q$ be the corresponding cut space. Then
\[
(M//LG)_Q = (Y_{\sigma,-\sigma})_Q K_\sigma, \quad (L//LG)_Q = (L|Y_{\sigma,-\sigma})_Q K_\sigma.
\]
By Proposition 5.6 applied to $(Y_{\sigma,-\sigma})_Q$ we obtain that
\[
\text{RR}((M//LG)_Q, (L//LG)_Q) = \sum_{\mu \in \mathfrak{Q}(L)} \text{RR}(M_{\mu,*,\mu}, L_{\mu,*,\mu})
\]
where $\Lambda^*_m = \Lambda^* \cap m\mathfrak{A}$ is the set of dominant weights for $LG$ at level $m$. Finally, applying the Euler identity $\sum_{Q \in \mathfrak{Q}} (-1)^{\text{codim}(Q)} = 1$,
\[
\text{RR}(M//LG, L//LG) = \sum_{Q \in \mathfrak{Q}} (-1)^{\text{codim}(Q)} \sum_{\mu \in \mathfrak{Q}(L) \cap \Lambda^*_m} \text{RR}(M_{\mu,*,\mu}, L_{\mu,*,\mu})
\]
which proves the result in this case.

In the case where $0$ is not a regular value, we use a desingularization by “adding a puncture”, as explained in section 4.3. Let
\[
M^{(2)} := M \times \mathcal{M}(\Sigma^3_0) // LG
\]
and
\[
M^{(1)} = M^{(2)} // LG = M \times \mathcal{M}(\Sigma^3_0) // LG \times LG,
\]
equipped with their pre-quantum bundles. Correspondingly, we will use cross-sections $Y^{(1)}_{(0),\sigma,-\sigma}$ for $M^{(1)}$, $Y^{(2)}_{(0),\sigma,-\sigma}$ for $M^{(2)}$, and $Y_{\sigma,-\sigma}$ for $M$. To compactify $Y^{(1)}_{(0),\sigma,-\sigma}$, we choose an admissible polytope $Q_1$ with rational vertices with $0 \in \text{int}(Q_1)$, and use the cut space $(Y^{(1)}_{(0),\sigma,-\sigma})_1$. We also choose an admissible family of polytopes $Q_2$ to cut $Y_{\sigma,-\sigma}$ and $(Y^{(2)}_{(0),\sigma,-\sigma})_1$ into pieces. Then
\[
(Y^{(2)}_{(0),\sigma,-\sigma})_1 \times Q_2 // G \cong (Y_{\sigma,-\sigma})_2, \quad (Y^{(2)}_{(0),\sigma,-\sigma})_1 \times Q_2 // (LG)_\sigma = (Y^{(1)}_{(0),\sigma,-\sigma})_1 \times Q_2.
\]
Using the gluing formula, together with repeated application of “quantization commutes with reduction” we compute (omitting the line bundles from the notation):
\[
\text{RR}(M//LG) = \text{RR}((Y^{(1)}_{(0),\sigma,-\sigma})_1 \times Q_2 // G) = \text{RR}((Y^{(1)}_{(0),\sigma,-\sigma})_1 // G)
\]
\[
= \sum_{Q_2 \in Q_2} (-1)^{\text{codim}(Q_2)} \text{RR}((Y^{(1)}_{(0),\sigma,-\sigma})_1 \times Q_2 // G)
\]
\[
= \sum_{Q_2 \in Q_2} (-1)^{\text{codim}(Q_2)} \text{RR}((Y^{(2)}_{(0),\sigma,-\sigma})_1 \times Q_2 // (LG)_\sigma)
\]
\[
= \sum_{Q_2 \in Q_2} (-1)^{\text{codim}(Q_2)} \text{RR}((Y_{\sigma,-\sigma})_2 // (LG)_\sigma)
\]
\[
= \sum_{Q_2 \in Q_2} (-1)^{\text{codim}(Q_2)} \sum_{\mu \in \mathfrak{Q}(L) \cap \Lambda^*_m} \text{RR}(M_{\mu,*,\mu})
\]
\[
= \sum_{\mu \in \Lambda^*_m} \text{RR}(M_{\mu,*,\mu})
\]
(Euler Identity).
APPENDIX A. THE GAUGE-THEORETIC CONSTRUCTION OF $\mathcal{M}(\Sigma)$

Let $\Sigma$ be a compact Riemann surface of genus $g$ with $b$ boundary components, and $G$ a connected, simply connected compact Lie group. Let $\mathcal{A}(\Sigma) \cong \Omega^1(\Sigma, g)$ be the space of $G$-connections of Sobolev class $s > \frac{1}{2}$. Recall that for any manifold $X$ with boundary, the restriction map $C^\infty(X) \to C^\infty(\partial X)$ to the boundary extends for $t > \frac{1}{2}$ to a continuous surjection

$$H_t(\Sigma) \to H_{t-\frac{1}{2}}(\partial \Sigma).$$

(see e.g. [3W], chapter 11). Hence there is a surjective map from $\mathcal{A}(\Sigma)$ to the space $\mathcal{A}(\partial \Sigma) \cong \Omega^1(\partial \Sigma, g)$ of connections over $\partial \Sigma$ of Sobolev class $s - \frac{1}{2}$. The space $\mathcal{G}(\Sigma)$ of maps from $\Sigma$ to $G$ of Sobolev class $s + 1$ consists of continuous maps, and is a Banach Lie group acting smoothly on the space $\mathcal{A}(\Sigma)$. Restriction to the boundary gives a surjective map from $\mathcal{G}(\Sigma)$ to the space $\mathcal{G}(\partial \Sigma)$ of maps $\partial \Sigma \to G$ of Sobolev class $s + \frac{1}{2}$, and as before we define $\mathcal{G}_0(\Sigma)$ to be the kernel.

Let us fix a Riemannian metric on $\Sigma$, and parametrizations $B_i \cong S^1$ of the boundary components compatible with the metric and orientation. As before we identify $G \times B_i$ to the space $\mathcal{G}(\Sigma)$ of maps from $\Sigma$ to $G$ of Sobolev class $s - 1$, the moment map for the $\mathcal{G}(\Sigma)$-action becomes a continuous map

$$\mathcal{A}(\Sigma) \to \Omega^2(\Sigma, g) \oplus \Omega^1(\partial \Sigma, g), \ A \mapsto (F_A, \iota^* A).$$

An atlas for the moduli space $\mathcal{M}(\Sigma) = \mathcal{A}_F(\Sigma)/\mathcal{G}_0(\Sigma)$ for $b > 0$ is constructed, as in the case without boundary, from local slices for the gauge group action. Let $A \in \mathcal{A}(\Sigma)$ be any connection. By the implicit function theorem, any connection $A + a$, with a small, can be gauge transformed by a unique element $g \in \mathcal{G}_0(\Sigma)$ into Coulomb gauge with respect to $A$, that is, so that

$$d_A^*(g \cdot (A + a) - A) = 0.$$ (23)

In other words, a neighborhood of $A$ in $A + \ker(d_A^*)$ is a slice for the $\mathcal{G}_0(\Sigma)$-action on $\mathcal{A}(\Sigma)$. For any $A \in \mathcal{A}_F(\Sigma)$ one defines a local moduli space to be a neighborhood of $A$ in $\mathcal{A}_F(\Sigma)$ of the form

$$V_A \subset \{ A + a \in \Omega^1(\Sigma, g) | F_{A+a} = 0, d_A^* a = 0 \}.$$ (24)

Using the implicit function theorem again, one shows that if $V_A$ is taken sufficiently small, $V_A \subset \mathcal{A}_F(\Sigma)$ is a smooth Banach submanifold locally homeomorphic to its tangent space at $0$

$$T_0(V_A) = \{ a \in \Omega^1(\Sigma, g) | d_A a = 0, d_A^* a = 0 \}.$$ (25)

The sets $V_A$ together with the coordinate mappings $V_A \to T_0(V_A)$ give an atlas for $\mathcal{M}(\Sigma)$. Since the tangent space $T_0(V_A)$ is invariant under the Hodge $*\cdot$-operator, it is a complex (and therefore symplectic) subspace of $T_A(\mathcal{A}(\Sigma))$ and this shows that $\mathcal{M}(\Sigma)$ is a Banach Kähler manifold.

Similarly, the products $V_A \times \mathbb{C} \cong T_0(V_A) \times \mathbb{C}$ are slices for the $\mathcal{G}_0(\Sigma)$-action on $\mathcal{A}_F(\Sigma) \times \mathbb{C}$, and give local bundle charts for the line bundle $L(\Sigma)$.

Remark A.1. For a Riemann surface $\Sigma$ without boundary, the moduli space $\mathcal{M}(\Sigma)$ is singular in general. Its smooth part $\mathcal{M}(\Sigma)^{\text{smooth}}$ is given by gauge equivalence classes of irreducible connections, i.e. those with trivial stabilizer. The above proof applies to the smooth part and shows that it is a symplectic manifold.

We are now in position to prove Theorem 2.3.

Theorem A.2. Let $\Sigma$ be a compact oriented Riemann surface with $b > 0$ boundary components. Suppose $\Sigma$ is obtained from a compact oriented Riemann surface $\bar{\Sigma}$ by gluing along two boundary components $B_\pm \subset \partial \Sigma$. Let $\pi : \bar{\Sigma} \to \Sigma$ be the gluing map. Then the map

$$\mathcal{M}(\Sigma) \to \mathcal{M}(\bar{\Sigma}) \langle LG \rangle, [A] \mapsto LG \cdot [\pi^* A]$$
is an \( \mathcal{L}G^b \)-equivariant symplectic diffeomorphism. Similarly, there is an \( \mathcal{L}G^b \)-equivariant isomorphism of line bundles

\[
L(\Sigma) = L(\hat{\Sigma})/\mathcal{L}G.
\]

**Proof.** Choose a parametrization \( B_\pm \cong S^1 \) as in Section 2 and consider the corresponding anti-diagonal \( \mathcal{L}G \)-action on \( \mathcal{M}(\Sigma) \). Let \( A \in \mathcal{A}_F(\Sigma) \) be any fixed connection, and \( V_A \subset \mathcal{A}_F(\Sigma) \) the corresponding local moduli space. Define a map

\[
\psi : V_A \times \mathcal{L}G \to \mathcal{M}(\hat{\Sigma}), \quad (A + a, h) \mapsto h \cdot \pi^*(A + a).
\]

Notice that this map is \( \mathcal{L}G \)-equivariant and that its image is contained in the zero level set of \( \mathcal{M}(\hat{\Sigma}) \). To show that \( \psi \) is a diffeomorphism from a small neighborhood of \( (0, e) \) onto its image, we can assume after acting by a suitable element of \( \mathcal{G}(\Sigma) \) that \( A \) is smooth. Since the map in (24) is a homeomorphism it suffices to prove that \( \psi \) is an immersion, that is, the tangent map to \( \psi \) at \( (0, e) \) is injective and has closed image. In terms of the local moduli space \( V_{\pi^*A} \), the map \( \psi \) is given as

\[
\psi : V_A \times \mathcal{L}G \to V_{\pi^*A}, \quad (A + a, h) \mapsto h \cdot \pi^*(A + a)
\]

where \( h \in \mathcal{G}(\Sigma) \) is the unique gauge transformation such that \( h|B_\pm = h \) and \( h|\partial \Sigma - (B_+ \cup B_-) = 1 \), and such the right-hand side of (25) lies in \( V_{\pi^*A} \). The tangent map is given by

\[
d_{(0,e)}\psi(b,\eta) = \pi^*b - d_{\pi^*A}\tilde{\eta}
\]

where \( \tilde{\eta} \in \Omega^0(\hat{\Sigma}, \mathfrak{g}) \) is the unique solution of the Dirichlet problem

\[
d^*_{\pi^*A}d_{\pi^*A}\tilde{\eta} = 0, \quad \iota^*_\pm \tilde{\eta} = \eta|C, \quad \tilde{\eta}|(\partial \hat{\Sigma} - B_+ - B_-) = 0.
\]

Suppose that \( (a, \eta) \in \ker d_{(0,e)}\psi \). Since \( \tilde{\eta} \) is continuous, and since the restrictions of \( \tilde{\eta} \) to \( B_\pm \) are equal, \( \tilde{\eta} = \pi^*\zeta \) is the pullback of a continuous function \( \zeta \in C^0(\Sigma, \mathfrak{g}) \). Then \( d_A\zeta = a \) in the sense of distributions, which by ellipticity of \( d_A \) on 0-forms implies that the \( \zeta \) is smooth. Since \( d_A d_A\zeta = 0 \) and \( \zeta|\partial \Sigma = 0 \) it follows that \( \zeta = 0 \) and hence \( a = 0 \), as required. This shows the first claim. The image of \( d\psi \) is closed by ellipticity of \( d_{\pi^*A} \).

**Remark A.3.** In the case that \( \Sigma \) has empty boundary so that \( \mathcal{M}(\Sigma) \) is possibly singular, the above proof gives a symplectomorphism between the smooth parts of \( \mathcal{M}(\hat{\Sigma})/\mathcal{L}G \) and \( \mathcal{M}(\Sigma) \). In order to extend this to an isomorphism of stratified symplectic spaces in the sense of Sjamaar-Lerman \cite{SL}, it would be necessary to define a Poisson algebra of “smooth” functions on \( \mathcal{M}(\Sigma) \). In this paper, we do this by representing \( \mathcal{M}(\Sigma) \) as a symplectic quotient \( \mathcal{M}(\Sigma) = \mathcal{M}(\Sigma')/\mathcal{L}G \), where \( \Sigma' \) denotes \( \Sigma \) with one puncture added, and use symplectic cross-sections to replace \( \mathcal{M}(\Sigma')/\mathcal{L}G \) by a finite dimensional symplectic quotient. Similarly, the singular symplectic quotient \( \mathcal{M}(\Sigma)/\mathcal{L}G \) is defined by adding a puncture.

**Appendix B. The Action of \( \hat{\mathcal{G}}(\Sigma) \) on \( \mathcal{A}(\Sigma) \times \mathbb{C} \)**

Let \( \Sigma \) be a compact, connected oriented Riemann surface with boundary, \( G \) a connected, simply connected compact Lie group and \( \mathcal{A}(\Sigma) \) the space of flat \( G \)-connections (of Sobolev class \( s > \frac{1}{2} \)). In this appendix, we will explain, following Mickelsson [M], Ramadas-Singer-Weitsman \cite{RSW}, and Witten \cite{W}, how to obtain an explicit construction of the canonical central extension \( \hat{\mathcal{G}}(\Sigma) \) of the gauge group and of its action on the pre-quantum line bundle \( \mathcal{A}(\Sigma) \times \mathbb{C} \).

Let us briefly recall some facts about pre-quantization. Let \( (\mathcal{M}, \omega) \) be a connected symplectic manifold, and \( L \to M \) a pre-quantum line bundle with pre-quantum connection \( \nabla \). By Kostant’s theorem
the group $\text{Aut}_\theta(L)$ of connection preserving Hermitian bundle automorphisms is a central extension by $S^1$ of the group $\text{Diff}_c(M)$ of symplectomorphisms of $M$. Hence, given a Lie group $K$ and a symplectic action $K \to \text{Diff}_c(M)$ one obtains a central extension
\begin{equation}
1 \to S^1 \to \hat{K} \to K \to 1,
\end{equation}

\( \text{to} \) together with an action $\hat{K} \to \text{Aut}_\theta(L)$. If the action of $K$ is in fact Hamiltonian, with equivariant moment map $\Phi : M \to \mathfrak{k}^*$, the formula $\hat{K}$ defines a Lie algebra splitting $\mathfrak{k} \cong \mathbb{R} \oplus \mathfrak{k}$ or equivalently an invariant flat connection on the bundle (26). If $\pi_0(K) = \{1\}$ and if the holonomy of this connection is trivial, one obtains in this way a lift $K \to \text{Aut}_\theta(L)$.

\begin{itemize}
\item Let $\Sigma$ be any compact oriented 3-manifold with boundary $\partial \Sigma$. Let $\hat{\Sigma}$ be the Riemann surface $\Sigma$ without boundary obtained from $\Sigma$ by “capping off” the boundary components, and let $\hat{\Sigma} = \Sigma^\#$. Then $\pi_0(\hat{\Sigma}) = \{1\}$ implies that any loop in $\hat{\Sigma}$ can be deformed to a loop in $\Sigma$.
\item Suppose first that $\partial \Sigma = \emptyset$, so that $\mathcal{G}_\theta(\Sigma) = \mathcal{G}(\Sigma)$. Let $\mathcal{L} \in \Omega^3(\hat{\Sigma})$ be the closed bi-invariant 3-form on $\hat{\Sigma}$ given by
\begin{align*}
\lambda &= 1 \overline{g^{-1} dg \wedge [g^{-1} dg, g^{-1} dg]}.
\end{align*}
\end{itemize}

Then $\frac{1}{2\pi} \lambda$ (using the normalized inner product defined in Section 2) defines an integral cohomology class. Let $B$ be any compact oriented 3-manifold with boundary $\partial B = \Sigma$ and $\hat{g} \in \text{Map}(B, \hat{\Sigma})$ an extension of $g$ of Sobolev class $C^2$ and $B$ of Sobolev class $C^2$. The integral over $B$ of $\frac{1}{2\pi} \hat{g}^*(\lambda)$ is independent of $B$ mod $\mathbb{Z}$, and therefore
\begin{align*}
\Gamma : \mathcal{G}(\Sigma) \to S^1, \ g \mapsto \exp \left( \frac{i}{4\pi} \int_B \hat{g}^*(\lambda) \right)
\end{align*}
a well-defined smooth map. One verifies the coboundary property
\begin{align*}
\Gamma(g_1, g_2) = \Gamma(g_1) \Gamma(g_2) c(g_1, g_2).
\end{align*}

The group homomorphism $\mathcal{G}(\Sigma) \to \hat{\mathcal{G}}(\Sigma)$, $g \mapsto (g, \Gamma(g))$ defines the trivialization.

In the case $\partial \Sigma \neq \emptyset$, consider the Riemann surface $\hat{\Sigma}$ without boundary obtained from $\Sigma$ by “capping off” the boundary components, and let $\hat{\Gamma} : \mathcal{G}(\Sigma) \to S^1$ be defined as before. Let $\mathcal{G}_c(\Sigma) \subset \mathcal{G}(\Sigma)$ be the subgroup of gauge transformations in the kernel of the restriction map $\mathcal{G}(\Sigma) \to \mathcal{G}((\Sigma - \Sigma))$. Then the restriction $\hat{\Gamma}$ of $\Gamma$ to $\mathcal{G}_c(\Sigma)$ satisfies $\hat{\Gamma}_c(\Sigma)$ and therefore shows that the extension is trivial over $\mathcal{G}_c(\Sigma)$. Now consider $\mathcal{G}_\theta(\Sigma) \supset \mathcal{G}_c(\Sigma)$. One sees easily that any loop in $\mathcal{G}_\theta(\Sigma)$ can be deformed to a loop in $\mathcal{G}_c(\Sigma)$.

On the Lie algebra level, $\text{Lie}(\hat{\mathcal{G}}(\Sigma))$ is the central extension by $\mathbb{R}$ of $\text{Lie}(\mathcal{G}(\Sigma))$ defined by the cocycle
\begin{align*}
(\xi_1, \xi_2) &\mapsto \frac{1}{2\pi} \int_{\Sigma} d\xi_2 \wedge d\xi_2 - \frac{1}{2\pi} \int_{\partial \Sigma} \xi_1 d\xi_2,
\end{align*}

that is, $\text{Lie}(\hat{\mathcal{G}}(\Sigma)) = \text{Lie}(\mathcal{G}(\Sigma)) \oplus \mathbb{R}$ with bracket
\begin{align*}
[(\xi_1, t_1), (\xi_2, t_2)] = \left( \xi_1, \xi_2, \frac{1}{2\pi} \int_{\partial \Sigma} \xi_1 d\xi_2 \right).
\end{align*}

One checks that the pre-quantum lift (19) corresponds to the inclusion of $\text{Lie}(\mathcal{G}(\Sigma))$ in $\text{Lie}(\hat{\mathcal{G}}(\Sigma))$ as the first summand. Notice however that since the moment map for the $\mathcal{G}(\Sigma)$-action is equivariant only over the subgroup $\mathcal{G}_\theta(\Sigma)$, the connection on $\hat{\mathcal{G}}(\Sigma)$ obtained in this way is flat only over $\mathcal{G}_\theta(\Sigma)$. We will now check that this flat connection has trivial holonomy, or equivalently that the restriction to $\mathcal{G}_\theta(\Sigma)$ of the above cocycle is a coboundary.
i.e. that the natural map $\pi_1(G_c(\Sigma)) \to \pi_1(G_\partial(\Sigma))$ is surjective. It follows that the central extension is trivial over $G_\partial(\Sigma)$ as well.

The quotient $\hat{G}(\partial \Sigma) = \hat{G}(\Sigma)/G_\partial(\Sigma)$ is the (unique) central extension of $G(\partial \Sigma)$ defined by the Lie algebra cocycle $(\xi_1, \xi_2) \mapsto \int_{\partial \Sigma} \xi_1 \cdot d\xi_2$. It follows that $\hat{G}(\partial \Sigma)$ is just the basic central extension $\hat{L}G_b \to L\hat{G}$. (In fact, the above construction with $\Sigma$ the two-disk is precisely Mickelsson’s construction $M$.)

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