CHARACTERIZATION OF THE 4-CANONICAL BIRATIONALITY OF ALGEBRAIC THREEFOLDS

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Abstract. In this article we present a 3-dimensional analogue of a well-known theorem of E. Bombieri (in 1973) which characterizes the bi-canonical birationality of surfaces of general type. Let $X$ be a projective minimal 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities and the geometric genus $p_g(X) \geq 5$. We show that the 4-canonical map $\varphi_4$ is not birational onto its image if and only if $X$ is birationally fibred by a family $\mathcal{C}$ of irreducible curves of geometric genus 2 with $K_X \cdot C_0 = 1$ where $C_0$ is a general irreducible member in $\mathcal{C}$.

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers.

In this article we study pluri-canonical systems of projective minimal threefolds $X$ of general type.

Our guidance are results of E. Bombieri [2] on minimal surfaces $S$. For $m \geq 5$, the $m$-canonical map $\Phi_{|mK_S|}$ is always birational. If, for $1 < m < 5$, $\Phi_{|mK_S|}$ is not birational and if the numerical invariants of $S$ are sufficiently large, then $S$ carries a pencil of curves of small genus. For example Bombieri shows the following (see [2], or [1], Theorem 5.4(iv)):

Theorem 1.1. Let $S$ be a smooth minimal projective surface of general type with $K_S^2 \geq 10$ and $p_g(S) \geq 6$. Then the bi-canonical map is not birational onto its image if and only if $S$ has a family of curves of genus 2.

For minimal threefolds $X$ with $\mathbb{Q}$-factorial terminal singularities it is known that there exists some universal constant $r_3$ such that the pluri-canonical map $\varphi_{r_3} := \Phi_{|r_3K_X|}$ is birational (see Tsuji [31], Hacon-McKernan [15] and Takayama [28]). Tsuji [31] has ever proved $r_3 \leq 18(2^9 \cdot 3^7)!$. If one requires in addition that either the invariants of $X$ are big (e.g. for $p_g(X) \geq 4$ see [7]; for $K_X^3 \gg 0$ see [30]) or $X$ is Gorenstein (see [4]), one may take $r_3 = 5$. We remark that $r_3$ can not be 4 because the 4-canonical map of the product of a curve and a...
surface of type \((K^2, p_g) = (1,2)\) is not birational. So it is natural to ask how \(\Phi|_{4KX}\) behaves, provided again that the numerical invariants of \(X\) are large.

1.2. Known works. There are several works that deal with the behaviour of \(\varphi_4\). In 2000, S. Lee [23] proved the base point freeness of \(|4K|\) for minimal Gorenstein threefolds of general type. In 2002, the first author [6] gave a sufficient condition for the birationality of \(\varphi_4\) again for minimal Gorenstein threefolds. Then, in 2005, J. Dong [13] improved upon the method of the first author and gave a sufficient condition for the birationality of \(\varphi_4\) for arbitrary threefolds of general type.

Our main result is the following:

**Theorem 1.3.** Let \(X\) be a minimal projective threefold of general type with \(\mathbb{Q}\)-factorial terminal singularities and the geometric genus \(p_g(X) \geq 5\). Then:

(i) The 4-canonical map \(\varphi_4 := \Phi|_{4KX}\) is generically finite of degree \(\leq 2\).

(ii) \(\varphi_4\) is not birational if and only if \(X\) is birationally fibred by a family \(\mathcal{C}\) of irreducible curves of geometric genus 2 with \(K_X \cdot C_0 = 1\) for a general member \(C_0\) in \(\mathcal{C}\).

(iii) In (ii) the family \(\mathcal{C}\) is birationally, uniquely determined by the given threefold \(X\).

The precise definition of “birationally fibred by a family \(\mathcal{C}\) of irreducible curves” will be given in Definition 2.1. We remark that the condition \(p_g(X) \geq 5\) in Theorem 1.3 is optimal (see Example 6.3).

If a minimal 3-fold \(X\) is birationally fibred by surfaces of type \((K^2, p_g) = (1,2)\), then \(\varphi_4\) is not birational. Here we mean that there exists a birational morphism \(\nu : Z \to X\) and a fibration \(\tilde{f} : Z \to \tilde{B}\) with \(Z\) a smooth threefold and \(\tilde{B}\) a smooth curve, such that a general fiber of \(\tilde{f}\) is a surface of type \((1,2)\). Taking further birational modification of \(Z\) if necessary, we may assume that the relative canonical map \(\tilde{\Psi} : Z \to \mathbb{P}(f_\ast \omega_Z^2)\) is a morphism over \(\tilde{B}\).

Our next result shows how to find the unique family \(\mathcal{C}\) of curves mentioned in Theorem 1.3 from the family of \((1,2)\) surfaces on \(X\).

**Theorem 1.4.** Let \(X\) be a minimal projective threefold of general type with \(\mathbb{Q}\)-factorial terminal singularities and the geometric genus \(p_g(X) \geq 5\). Suppose that \(X\) is birationally fibred by surfaces of type \((K^2, p_g) = (1,2)\). Then the birationally unique family \(\mathcal{C}\) in Theorem 1.3(ii) is on the fibers of \(\tilde{f}\) and induced by \(\tilde{\Psi}\).

Theorem 1.3 has the following application to the effect that minimal threefolds \(X\) of general type with small slope \((K^3/p_g)\) will have non-birational 4-canonical maps \(\varphi_4\).
Theorem 1.5. Let $X$ be a minimal projective threefold of general type with \( \mathbb{Q} \)-factorial terminal singularities. Then $\varphi_4$ is not birational whenever either of the following two conditions is satisfied:

(i) $K_X^3 < \frac{4}{3}(p_g(X) - 2)$ and $p_g(X) \not\in [3, 11]$.

(ii) $X$ is Gorenstein and $K_X^3 < 2p_g(X) - 6$.

There are infinitely many non-trivial examples satisfying the conditions in Theorem 1.5 (see the last section).

2. Notation and the set up

We use the standard notation and terminology in the textbook of Hartshorne. Throughout the paper $D_1 \sim D_2$ (resp. $=\mathbb{Q}$, or $D_1 \equiv D_2$) means that divisors $D_1$ and $D_2$ are linearly equivalent (resp. $rD_1$ and $rD_2$ are linearly equivalent for some positive integer $r$, or $D_1$ and $D_2$ are numerically equivalent). By a minimal variety, we always mean one with a nef canonical divisor $K$ and with terminal singularities. Given a smooth projective threefold $V$ of general type, the 3-dimensional MMP (see [22, 20] for example) says that $V$ has a minimal model $X$ with QFT singularities. Since the birationality of $\Phi_m$ is equivalent to that of $\varphi_m$, we may begin our study from a minimal threefold $X$.

Definition 2.1. Let $Y$ be a $\mathbb{Q}$-Gorenstein (i.e., $rK_Y$ being Cartier for some positive integer $r$) normal projective variety. We say that $Y$ is birationally fibred by a family of curves if there exist both a birational morphism $\pi : Y' \rightarrow Y$ and a fibration $f_{Y'} : Y' \rightarrow W$ whose general fiber is a smooth projective curve, where $Y'$ and $W$ are projective normal varieties. Denote by $C = \{ \pi(Y'_w) | w \in W, Y'_w = f_{Y'}^{-1}(w) \}$.

An element $C_0 \in C$ is called a generic member if $C_0$ is the $\pi$-image of a general fiber $Y'_w = f_{Y'}^{-1}(w)$ of $f_{Y'}$. So the intersection number $(K_Y \cdot C_0) = (\pi^*K_Y \cdot Y'_w)$ is uniquely determined by the curve family $C$ and is independent of the choice of the birational modification $\pi$.

2.2. The canonical map. Suppose $p_g(X) \geq 2$. We may study the canonical map $\varphi_1$ which is only a rational map. First we fix an effective Weil divisor $K_Y \sim K_X$. Take successive blow-ups $\pi : X' \rightarrow X$ (along nonsingular centers), which exists by Hironaka’s big theorem, such that:

(i) $X'$ is smooth;

(ii) the movable part of $|K_{X'}|$ is base point free;

(iii) the support of $\pi^*(K_1)$ is of simple normal crossings.

Denote by $g$ the composition $\varphi_1 \circ \pi$. So $g : X' \rightarrow W' \subseteq \mathbb{P}^{p_g(X)-1}$ is a morphism. Let $X' \xrightarrow{f} B \xrightarrow{\sigma} W'$ be the Stein factorization of $g$. We have the following commutative diagram:
We may write $K_{X'} = \pi^*(K_X) + E_\pi = Q M_1 + Z_1$, where $M_1$ is the movable part of $|K_X|$, $Z_1$ the fixed part and $E_\pi$ an effective $\mathbb{Q}$-divisor which is a $\mathbb{Q}$-sum of distinct exceptional divisors. By $K_{X'} - E_\pi$, we mean $\pi^*(K_X)$. So, whenever we take the round up of $m\pi^*(K_X)$, we always have $\lceil m\pi^*(K_X) \rceil \leq m K_{X'}$ for all positive numbers $m$. We may also write $\pi^*(K_X) = Q M_1 + E_1'$, where $E_1' = Z_1 - E_\pi$ is actually an effective $\mathbb{Q}$-divisor.

If $\dim \varphi_1(X) = 2$, a general fiber of $f$ is a smooth projective curve of genus $\geq 2$. We say that $X$ is canonically fibred by curves.

If $\dim \varphi_1(X) = 1$, a general fiber $S$ of $f$ is a smooth projective surface of general type. We say that $X$ is canonically fibred by surfaces with invariants $(\chi_f(S_0), p_g(S))$, where $S_0$ is the minimal model of $S$. We may write $M_1 \equiv a_1 S$ where $a_1 \geq p_g(X) - 1$.

A generic irreducible element $S$ of $|M_1|$ means either a general member of $|M_1|$ whenever $\dim \varphi_1(X) \geq 2$ or, otherwise, a general fiber of $f$.

**Definition 2.3.** By abuse of terminology, we also define a generic irreducible element $S'$ of an arbitrary linear system $|M'|$ on a general variety $V$ in a similar way. Assume that $|M'|$ is movable. A generic irreducible element $S'$ is defined to be a generic irreducible component in a general member of $|M'|$. (So if $|M'|$ is composed with a pencil (i.e. $\dim \Phi_{|M'|}(V) = 1$), one has $M' \equiv tS'$ for some integer $t \geq 1$. Clearly it may happen that sometimes $tS' \ncong M'$ by our definition.)

## 3. The key technical results

In this section we will collect all the technical results needed to prove our theorems. First we recall two lemmas about the so-called Tankeev’s principle which will be tacitly used throughout our paper.

**Lemma 3.1.** ([29], Lemma 2) Let $V$ be a nonsingular projective variety. Let $D$ and $M$ are two divisors on $V$. Assume that $|M|$ is base point free, $\dim \Phi_{|M'|}(V) \geq 2$ and $|D| \neq \emptyset$. Denote by $T$ a general member of $|M'|$. If $\Phi_{|D+M|}(V) = 1$, then $\Phi_{|D+M|}(T)$ is not birational either.

**Lemma 3.2.** ([5], 2.1) Let $V$ be a nonsingular projective variety. Let $D$ and $M$ are two divisors on $V$. Assume that $|M|$ is base point free,
3.3. Fact. Let $|M|$ be a base point free linear system on any variety $V$ with $\dim \Phi_{|M|}(V) = 1$. Denote by $S$ a generic irreducible element of $|M|$. Then clearly $O_S(M|_S) \cong O_S$ and $O_S(S|_S) \cong O_S$.

From now on we present a technical, but key theorem which is a generalized form of Theorem 2.6 in [7] and will be frequently applied in our proof.

3.4. Assumption. We keep the same notation as in 2.2. Assume that $X_0$ is a smooth model of $X$ and that $J \leq K_{X_0}$ is an effective divisor with $n_J := h^0(X_0, O_{X_0}(J)) \geq 2$. Recall that we have already a birational morphism $\pi : X' \to X$ in 2.2.

We may take further blow-ups to get a birational map $\pi_J : X'' \to X$ such that the following three conditions are satisfied:

(i) $X''$ is smooth and $X''$ dominates both $X'$ and $X_0$, i.e. there are two birational morphisms $\pi' : X'' \to X'$ and $\pi_0 : X'' \to X_0$, $\pi_J := \pi \circ \pi'$.

(ii) The movable part $|M_J|$ of $|\pi_0^*(\pi_J)|$ is base point free.

(iii) The support of $\pi_J^*(K_1)$ is of simple normal crossings.

Denote by $g_J$ the composition $\phi_{J} \circ \pi_J$. So $g_J : X'' \to W'_J \subseteq \mathbb{P}^{n_J-1}$ is a morphism. Let $g_J : X'' \xrightarrow{f_J} B_J \xrightarrow{h_J} W'_J$ be the Stein factorization of $g_J$. Since $M_J \leq \pi^*(M_1) \leq \pi_J^*(K_X)$, we can write $\pi_J^*(K_X) = Q M_J + E'_J$ where $E'_J$ is an effective $Q$-divisor. Set $d_J := \dim(B_J)$. Denote by $S_J$ a generic irreducible element of $|M_J|$. Then $S_J$ is a smooth projective surface of general type. When $d_J = 1$, one has $M_J \equiv a_J S_J$ where $a_J \geq n_J - 1$.

(**) Whenever $p_g(X) = n_J$, we simply take $X'' := X'$ and adopt our setting in 2.2. So $M_J$, $\pi_J$, $g_J$, $f_J$, $s_J$, $S_J$, $B_J$, $W'_J$ and $E'_J$ are respectively $M_1$, $\pi$, $g$, $f$, $s$, $S$, $B$, $W'$ and $E'_1$ just as in 2.2.

3.5. Key notations. We place ourselves in the above situation, and consider $S_J$, a generic irreducible element of $|M_J|$. We define $p$ to be 1 if $d_J \geq 2$ and $a_J$ otherwise. We assume further that $S_J$ is equipped with a movable linear system $|G|$ and that a generic irreducible element $C$ of $|G|$ is smooth. We define $\xi := \pi_J^*(K_X) \cdot C$. We fix an integer
We consider the sub-system
\[ |K_{S_j} + \gamma (m - 1)\pi^*_j(K_X) - S_j - \frac{1}{p}E_j^\gamma|_{S_j}. \]
Then the following theorem holds:

**Theorem 3.6.** (1) If \( L_m \) separates different generic irreducible elements of \( |G| \) (namely \( \Phi_{L_m}(C_1) \neq \Phi_{L_m}(C_2) \) where \( C_1, C_2 \) are different generic irreducible elements of \( |G| \)) and \( \beta \) is a rational number such that \( \pi^*_j(K_X) - \beta C \) is numerically equivalent to an effective \( \mathbb{Q} \)-divisor, then \( \varphi_m \) is birational if one of the following conditions is satisfied where we set \( \alpha := (m - 1 - \frac{1}{p} - \frac{1}{\beta})\xi \) and \( \alpha_0 := \gamma \alpha^\gamma:

i. \( \alpha > 2; \)
ii. \( \alpha_0 \geq 2 \) and \( C \) is non-hyperelliptic;
iii. \( \alpha > 0 \), \( C \) is non-hyperelliptic and \( C \) is an even divisor on \( S_J \).

(2) One has the inequality \( \xi \geq \frac{2g(C) - 2 + \alpha_0}{m} \) if one of the following conditions is satisfied:

iv. \( \alpha > 1; \)
v. \( \alpha > 0 \) and \( C \) is an even divisor on \( S_J \).

**Proof.** We consider the sub-system
\[ |K_{X''} + \gamma (m - 1)\pi^*_j(K_X) - \frac{1}{p}E_j^\gamma| \subset |mK_{X''}|. \]

Let \( S'_j \) and \( S''_j \) be two different generic irreducible elements of \( |M_j| \). Clearly one has
\[ K_{X''} + \gamma (m - 1)\pi^*_j(K_X) - \frac{1}{p}E_j^\gamma \geq K_{X''} + \gamma (m - 2)\pi^*_j(K_X) \geq M_j. \]

So \( |K_{X''} + \gamma (m - 1)\pi^*_j(K_X) - \frac{1}{p}E_j^\gamma| \) can separate \( S'_j \) and \( S''_j \) if either \( \dim(B_J) \geq 2 \) or \( \dim(B_J) = 1 \) and \( g(B_J) = 0 \). For the case \( \dim(B_J) = 1 \) and \( g(B_J) > 0 \), one has \( a_J \geq n_J \geq 2 \). Thus \( p \geq 2 \). Since
\[ (m - 1)\pi^*_j(K_X) - \frac{2}{p}E_j - S'_j - S''_j \equiv (m - 1 - \frac{2}{p})\pi^*_j(K_X) \]
is nef and big, the Kawamata-Viehweg vanishing theorem gives a surjective map:
\[
H^0(X'', K_{X''} + \gamma (m - 1)\pi^*_j(K_X) - \frac{2}{p}E_j^\gamma) 
\rightarrow H^0(S'_j, K_{S'_j} + \gamma (m - 1)\pi^*_j(K_X) - \frac{2}{p}E_j^\gamma|_{S'_j}) \oplus 
H^0(S''_j, K_{S''_j} + \gamma (m - 1)\pi^*_j(K_X) - \frac{2}{p}E_j^\gamma|_{S''_j}).
\]
The last two groups are non-zero. Therefore,
\[ |K_{X''} + \gamma (m - 1)\pi^*_j(K_X) - \frac{1}{p}E_j^\gamma| \]
separates $S'_j$ and $S''_j$. By Lemma 3.1 and Lemma 3.2 it suffices to prove that $|mK_{X''}|_S$ gives a birational map.

Noting that $(m-1)\pi_j^*(K_X) - \frac{1}{p}E_j - S_j$ is nef and big, the vanishing theorem gives a surjective map

$$H^0(X'', K_{X''} + \gamma (m-1)\pi_j^*(K_X) - \frac{1}{p}E_j)$$

$$\rightarrow H^0(S_j, K_{S_j} + \gamma (m-1)\pi_j^*(K_X) - S_j - \frac{1}{p}E_j|_{S_j}).$$

We are reduced to prove that $|K_{S_j} + \gamma (m-1)\pi_j^*(K_X) - S_j - \frac{1}{p}E_j|_{S_j}$ gives a birational map.

Now consider a generic irreducible element $C \in |G|$. By our assumption there is an effective $\mathbb{Q}$-divisor $H$ on $S_j$ such that

$$\frac{1}{\beta}\pi_j^*(K_X)|_{S_j} \equiv C + H.$$

By the vanishing theorem, we have the surjective map

$$H^0(S_j, K_{S_j} + \gamma (m-1)\pi_j^*(K_X) - S_j - \frac{1}{p}E_j)|_{S_j} - H|_{S_j} \rightarrow H^0(C, K_C + D),$$

where $D := \gamma (m-1)\pi_j^*(K_X) - S_j - \frac{1}{p}E_j|_{S_j} - C - H|_{S_j}$ is a divisor on $C$. Noting that

$$(m-1)\pi_j^*(K_X) - S_j - \frac{1}{p}E_j|_{S_j} - C - H \equiv (m-1 - \frac{1}{p} - \frac{1}{\beta})\pi_j^*(K_X)|_{S_j}$$

and that $C$ is nef on $S$, we have $\deg(D) \geq \alpha$ and thus $\deg(D) \geq \alpha_0$. Whenever $C$ is non-hyperelliptic, $m - 1 - \frac{1}{p} - \frac{1}{\beta} > 0$ and $C$ is an even divisor on $S$, $\deg(D) \geq 2$ automatically follows and thus $|K_C + D|$ gives a birational map.

Whenever $\deg(D) \geq 3$, then $|K_C + D|$ gives a birational map and, by assumption the linear system $L_m$ separates different irreducible elements of $|G|$, the Tankeev principle again says that

$$|K_{S_j} + \gamma ((m-1)\pi_j^*(K_X) - S_j - \frac{1}{p}E_j)|_{S_j} - H||_C$$

gives a birational map. Since

$$|K_{S_j} + \gamma ((m-1)\pi_j^*(K_X) - S_j - \frac{1}{p}E_j)|_{S_j} - H|$$

$$\subset |K_{S_j} + \gamma (m-1)\pi_j^*(K_X) - S_j - \frac{1}{p}E_j|_{S_j}|,$$

the latter linear system gives a birational map. So $\varphi_m$ of $X$ is birational.

Whenever $\deg(D) \geq 2$, $|K_C + D|$ is base point free by the curve theory. Denote by $|M_m|$ the movable part of $|mK_{X''}|$ and by $|N_m|$ the
satisfied with the blow down onto the smooth minimal model. We are done.

When applying Theorem 3.6 another technical problem is about how to find a suitable \( \beta \) once we have chosen a linear system \(|G|\). The following lemma will be useful in this context.

\[\text{Lemma 3.7. Keep the same notation as in 3.4 and with } p \text{ as in 3.6.} \]

Assume that \( B_f = \mathbb{P}^1 \). Let \( f_j : X'' \to \mathbb{P}^1 \) be the induced fibration. Then one can find a sequence of rational numbers \( \{\beta_n\} \) with \( \lim_{n \to +\infty} \beta_n = \frac{p}{p+1} \) such that \( \pi_j(K_X)|_{F_0} - \beta_n \sigma^*(K_{F_0}) \sim \mathbb{Q} N_{a_0} \) with an effective \( \mathbb{Q} \)-divisor \( N_{a_0} \), where \( \sigma : F \to F_0 \) is the blow down onto the smooth minimal model.

\[\text{Proof. This is a generalized version of an established statement of the first author (see Lemma 3.4 in [11]).} \]

One has \( \mathcal{O}_{B_f}(p) \to f_j_* \omega_{X''} \) and therefore \( f_j_* \omega_{X''/B_f} \to f_j_* \omega_{X''/p+8} \).

For any positive integer \( k \), denote by \( M_k \) the movable part of \( |kK_{X''}| \).

Note that \( f_j_* \omega_{X''/B_f} \) is generated by global sections since it is semi-positive. So any local section can be extended to a global one. On the other hand, \( |4p\sigma^*(K_{F_0})| \) is base point free and is exactly the movable part of \( |4pK_F| \) by Bombieri [2] or Reider [20]. Applying Lemma 2.7 of [5], one has the following, where \( a_0 := 4p + 8 \) and \( b_0 := 4p \):

\[a_0 \pi_j(K_X)|_F \geq M_{4p+8}|_F \geq b_0 \sigma^*(K_{F_0}).\]

This means that there is an effective \( \mathbb{Q} \)-divisor \( E'_0 \) such that

\[a_0 \pi_j(K_X)|_F = \mathbb{Q} b_0 \sigma^*(K_{F_0}) + E'_0.\]

Thus \( \pi_j(K_X)|_F = \mathbb{Q} \frac{p}{p+2} \sigma^*(K_{F_0}) + E_0 \) with \( E_0 = \frac{1}{a_0} E'_0 \).

We first consider the case \( p \geq 2 \).

Assume that we have defined \( a_n \) and \( b_n \) such that the following is satisfied with \( l = n \):

\[a_l \pi_j(K_X)|_F \geq b_l \sigma^*(K_{F_0}).\]

We shall define \( a_{n+1} \) and \( b_{n+1} \) inductively such that the above display is satisfied with \( l = n + 1 \). One may assume from the beginning that \( a_n \pi_j(K_X) \) supports on a divisor with normal crossings. Then the Kawamata-Viehweg vanishing theorem implies the surjective map

\[H^0(K_{X''} + \gamma a_n \pi_j(K_X) \gamma + F) \to H^0(F, K_F + \gamma a_n \pi_j(K_X) \gamma|_F). \]

That means

\[|K_{X''} + \gamma a_n \pi_j(K_X) \gamma + F||_F = |K_F + \gamma a_n \pi_j(K_X) \gamma|_F| \]

\[\subset |K_F + b_n \sigma^*(K_{F_0})| \]

\[\subset (b_n + 1) \sigma^*(K_{F_0}).\]
Denote by $M'_{an+1}$ the movable part of $|(a_n + 1)K_{X^*} + F|$. Applying Lemma 2.7 of \cite{5} again, one has $M'_{an+1}|_F \geq (b_n + 1)\sigma^*(K_{F_0})$.

**Claim.** For any integer $t > 0$, $|K_{X^*} + M'_{an+t} + F||_F \supset |(b_n + t + 1)\sigma^*(K_{F_0})|$. 

**Proof.** Re-modifying our original $\pi_J$ such that $|M'_{an+1}|$ is base point free. In particular, $M'_{an+1}$ is nef. According to \cite{7}, $|mK_X|$ gives a birational map whenever $m \geq 8$. Thus $M'_{an+1}$ is big. The Kawamata-Viehweg vanishing theorem gives

$$|K_{X^*} + M'_{an+1} + F||_F = |K_F + M'_{an+1}|_F$$

$$\supset |K_F + (b_n + 1)\sigma^*(K_{F_0})|$$

$$\supset |(b_n + 2)\sigma^*(K_{F_0})|.$$ 

Therefore $t = 1$ case is true.

Assume we have proved that $|K_{X^*} + M'_{an+t-1} + F||_F \supset |(b_n + t)\sigma^*(K_{F_0})|$. Denote by $M'_{an+t}$ the movable part of $|K_{X^*} + M'_{an+t-1} + F|$. Then $M'_{an+t}|_F \supset |(b_n + t)\sigma^*(K_{F_0})|$. Similarly we may assume that $|M'_{an+t}|$ is base point free. The vanishing theorem gives the surjective map:

$$|K_{X^*} + M'_{an+t} + F||_F = |K_F + M'_{an+t}|_F$$

$$\supset |K_F + (b_n + t)\sigma^*(K_{F_0})|$$

$$\supset |(b_n + t + 1)\sigma^*(K_{F_0})|.$$ 

\[\square\]

Take $t = p - 1$. Noting that

$$|K_{X^*} + M'_{an+p-1} + F| \subset |(a_n + p + 1)K_{X^*}|$$

and applying Lemma 2.7 of \cite{5} again, one has

$$a_{n+1} \pi'_J(K_X)|_F \geq M_{an+p+1}|_F \geq M'_{an+p}|_F \geq b_{n+1} \sigma^*(K_{F_0}).$$

Here we set $a_{n+1} := a_n + p + 1$ and $b_{n+1} = b_n + p$. Note that $a_n = n(p + 1) + a_0$ and $b_n = np + b_0$. Set $\beta_n = \frac{b_n}{a_n}$. Then $\lim_{n \to +\infty} \beta_n = \frac{p}{p+1}$.

The case $p = 1$ can be proved similarly, but with a simpler induction. We omit the details and leave it as an exercise. \[\square\]

4. Proof of the main theorem, Part I

We now begin the study of the birationality of $\varphi_d$. Set $d = d(X) = \dim \varphi_1(X)$. In the rest of the section, we shall prove:

**Theorem 4.1.** Let $X$ be a minimal threefold of general type with $p_g(X) \geq 5$. Then $\varphi_d$ is not birational if and only if one of the following cases occurs, where $f : X' \to B$ is as in Section 2.

(i) $d(X) = 2$, $g(C) = 2$ and $\pi^*(K_X) \cdot C = 1$ for a general fiber $C$ of $f$.

(ii) $d(X) = 1$ and $(K_{S_0}^2, p_g(S_0)) = (1, 2)$; here $S_0$ is a smooth minimal model of a general fiber $S$ of $f$. 
In both cases, \( \varphi_4 \) is generically finite of degree 2 and there is a family \( \mathcal{C} \) (on \( X \)) of irreducible curves of geometric genus 2 with \( K_X \cdot C_0 = 1 \), where \( C_0 \) is a general member of \( \mathcal{C} \).

In the case (ii), a general \( C_0 \) is the image on \( X \) of a curve in the movable part of \( |K_S| \) with \( S \) a general fiber of \( f \).

We prove the theorem according to the value of \( d = d(X) \).

4.2. The case \( d = d(X) = 3 \). Assume \( p_g(X) \geq 5 \). We shall show that \( \varphi_4 \) is birational.

We set \( J := K_{X'} \) to run Theorem 3.6. Then \( \pi = \pi_J \). According to the setting in Theorem 3.6, we have \( p = 1 \). For a generic irreducible element \( S \) of \( |M_1| \), the linear system \( |M_1|_S \) is not composed with a pencil of curves. Take \( G := M_1|_S \). For all \( m \geq 4 \), it is clear that \( K_S + \tau(m-2)\pi^*(K_X)|_S \geq G \). So Tankeev’s principle (Lemma 3.1 and Lemma 3.2) implies that \( K_S + \tau(m-2)\pi^*(K_X)|_S \) separates different generic irreducible elements of \( |G| \). On the other hand, since \( \pi^*(K_X)|_S \cong M_1|_S \sim C \) where \( C \) is an irreducible curve on \( S \). Take \( \beta = 1 \) and the conditions of Theorem 3.6(1) is satisfied. One has \( \pi^*(K_X) \cdot S^2 = \pi^*(K_X)|_S \cdot S|_S \geq (M_1|_S \cdot M_1|_S) = S^3 \). Denote by \( \Lambda \) the subsystem \( |S|_S \subset |S|_S \). If \( \Phi_\Lambda \) is birational for a general \( S \), so is \( \varphi_4 \). Otherwise since \( \Lambda \) gives a generically finite map \( S \to \mathbb{P}^{p_g(X)-2} \) onto a non-degenerate surface of degree \( \geq 2 \), one has \( (S|_S)^2 \geq 2(p_g(X) - 3) \geq 4 \). So \( \xi = \pi^*(K_X) \cdot S^2 \geq 4 \).

If we take \( m = 4 \), then

\[
\alpha = (m-1 - \frac{1}{p} - \frac{1}{\beta}) \xi \geq \xi > 2.
\]

Theorem 3.6(1) tells us that \( \varphi_4 \) is birational onto its image. (This method, however, gives no information on the birationality of \( \varphi_3 \).)

4.3. The case \( d = d(X) = 2 \). Assume \( p_g(X) \geq 5 \). We shall show by the arguments up to 4.3 that \( \varphi_4 \) is birational unless the case of Theorem 3.1 (i) occurs. Set \( J := K_{X'} \). Let \( S \) be a generic irreducible element of \( |M_1| \).

By our definition in 3.5 one has \( p = 1 \). Note that \( \pi^*(K_X)|_S \geq M_1|_S \equiv a_2 C \), where \( C \) is a general fiber of \( f : X' \to B \) and \( a_2 \geq h^0(S, M_1|_S) - 1 \geq p_g(X) - 2 \). Take \( G := M_1|_S \). Then \( C \) as a generic irreducible element of \( |G| \), is smooth. So we can take \( \beta := a_2 \geq 3 \). Pick up two different general fibers \( C_1 \) and \( C_2 \) of \( f \). Since we have

\[
\pi^*(K_X)|_S \equiv a_2 C + E_1|_S,
\]

for all \( m \geq 4 \),

\[
(m-2)\pi^*(K_X)|_S \equiv -2 E_1|_S \equiv 2 E_1|_S \equiv (m-2 - \frac{2}{a_2}) \pi^*(K_X)|_S
\]

for all \( m \geq 4 \).
is nef and big. So the Kawamata-Viehweg vanishing theorem ([19, 32]) gives a surjective map
\[ H^0(S, K_S + \gamma(m - 2)\pi^*(K_X)|S - \frac{2}{a_2}E_1'|S') \]
\[ \rightarrow \ H^0(C_1, K_{C_i} + D_i) \oplus H^0(C_2, K_{C_2} + D_2), \]
where \( D_i := \gamma(m - 2)\pi^*(K_X)|S - \frac{2}{a_2}E_1'|S'| \) for \( i = 1, 2, \) and \( H^0(C_1, K_{C_i} + D_i) \neq 0 \) because deg \( (D_i) \geq (m - 2 - \frac{2}{a_2})(\pi^*(K_X)|S \cdot C_i) > 0 \) for \( i = 1, 2. \) This means that \( |K_S + \gamma(m - 2)\pi^*(K_X)|S - \frac{2}{a_2}E_1'|S'| \) separates \( C_1 \) and \( C_2. \) So does \( |K_S + \gamma(m - 2)\pi^*(K_X)|S'|. \) Thus conditions in Theorem 3.6(1) are always satisfied. Now we may run Theorem 3.6.

Take a sufficiently large integer \( m \) such that \( \alpha > 1, \) one has then the inequality
\[ m\xi \geq 2g(C) - 2 + (m - 1 - \frac{1}{p} - \frac{1}{\beta})\xi, \]
which gives
\[ \xi \geq \frac{3(2g(C) - 2)}{7}. \] (4.1)

Take \( m = 4. \) Then
\[ \alpha = (2 - \frac{1}{\beta})\xi \geq \begin{cases} 10 \beta > 1, & g(C) = 2; \\ 2 \beta > 2, & g(C) \geq 3. \end{cases} \]

So Theorem 3.6 says that \( \xi \geq 1 \) whenever \( g(C) = 2; \) that \( \xi \geq \frac{7}{2}, \)
\( \varphi_4 \) is birational whenever \( g(C) \geq 3. \) We will study the case \( g(C) = 2 \)
which is slightly complicated.

Claim 4.4. (1) If \( g(C) = 2, \) \( p_g(X) \geq 6 \) and \( \xi > 1, \) then \( \xi \geq \frac{5}{2} \) and \( \varphi_4 \)
is birational onto its image.
(2) If \( g(C) = 2, \) \( p_g(X) = 5 \) and \( \xi > 1, \) then \( \xi \geq \frac{9}{2}. \) Whenever \( \xi > \frac{9}{2}, \)
\( \varphi_4 \) is birational onto its image (which also implies \( \xi \geq \frac{5}{2}. \))

Proof. (1) We have \( p = 1 \) and \( \beta \geq 4 \) by assumption. We can always find a positive integer \( l_0 > 4 \) such that \( \xi \geq \frac{l_0 + 1}{l_0}. \) Take \( m_0 = l_0 - 1 \geq 4. \) We hope to run Theorem 3.6. We have:
\[ \alpha - (l_0 - 3) = (l_0 - 2 - \frac{1}{p} - \frac{1}{\beta})\xi - (l_0 - 3) \geq (l_0 - 3 - \frac{1}{4})\frac{l_0 + 1}{l_0} - (l_0 - 3) = \frac{1}{4}(3 - \frac{13}{l_0}) > 0. \]
Thus one has \( \alpha_0 \geq l_0 - 2. \) Theorem 3.6 gives \( \xi \geq \frac{l_0}{l_0 - 1}. \) One may proceed by induction as long as \( l_0 - 1 \geq 4. \) So Theorem 3.6 simply gives \( \xi \geq \frac{4}{2}. \)

Take \( m_1 = 4. \) We get \( \alpha = (4 - 2 - \frac{1}{\beta})\xi \geq \frac{7}{4} \cdot \frac{5}{4} > 2. \) Therefore \( \varphi_4 \) is birational onto its image.
(2) The same argument as in (1) shows that if $\beta \geq 3$ then $\alpha - (l_0 - 3) \geq \frac{2a_0 - 10}{3a_0}$ and hence $a_0 \geq l_0 - 2$ if $l_0 > 5$. If we take $l_0 = 6$ and $m_0 = l_0 - 1 = 5$. Then $a_0 \geq 4$. Theorem 3.6(2) gives $\xi \geq \frac{5}{6}$.

Take $m_2 = 4$. We get $\alpha = (4 - 2 - \frac{1}{3})\xi \geq \frac{5}{3} \cdot \frac{5}{6} = 2$. Therefore, $\varphi_4$ is birational onto its image by Theorem 3.6 whenever $\xi > \frac{5}{6}$. Furthermore Theorem 3.6(2) gives $\xi \geq \frac{5}{6}$ whenever $\xi > \frac{5}{6}$.

\textbf{Claim 4.5.} The situation with $d = 2$, $g(C) = 2$, $p_g(X) = 5$ and $\xi = \frac{6}{5}$ does not occur.

\textit{Proof.} Assume $d = 2$, $g(C) = 2$, $p_g(X) = 5$ and $\xi = \frac{6}{5}$. We want to deduce a contradiction.

Consider a general member $S \in |M_1|$. We have seen that $M_1|_S \equiv a_2C$ where $a_2 \geq p_g(X) - 2 \geq 3$. If $a_2 > 3$, then we may take $\beta = 4$ as in the proof of [14]. Then Theorem 3.6 will give $\xi > \frac{5}{6}$, a contradiction. So we may assume $a_2 = 3$. Consider the induced fibration $f : X' \rightarrow B$ where $B$ is a normal surface given by the Stein factorization; see the notation in Section 2. We can write $S = f^*(H_B)$ for some ample divisor $H_B$ on $B$ and $H_B = s^*(H')$ for a hyperplane $H'$ on $W' \subset \mathbb{P}^4$.

Then $3 = a_2 = H_B^2 = \deg(s) \deg(W')$. Since $W'$ is non-degenerate, $\deg(W') \geq 3$. This means that $\deg(s) = 1$ and $s$ is a finite morphism of degree 1. Hence $B$ is the normalization of $W'$. According to Del Pezzo, or Nagata (see Theorem 7 in [24]), or Reid (see Exercise 19 of Chapter 2, page 30 in [25]), $W'$ is either a cone over a smooth rational base curve of degree 3 in $\mathbb{P}^3$, or the ruled surface $\mathbb{F}_e$. In particular, $W'$ is a normal rational surface and $W'$ has at worst a single singularity. We know that a minimal resolution $\bar{W}$ of $W'$ is the ruled surface $\mathbb{F}_e$ with $e > 0$. Denote by $\rho : W \rightarrow W'$ the minimal resolution. Set $\bar{H} = \rho^*(H')$. Then $\bar{H}^2 = (H')^2 = \deg(W') = 3$. So it is clear that $W$ can not be $\mathbb{F}_0$. Thus we have $W = \mathbb{F}_e$ with $e > 0$.

If necessary we can modify our $X'$ (by further blowing ups) so that we get a fibration $f : X' \rightarrow B$ with $B$ smooth so that $g = s \circ f$ in notation of Section 2; further, $s = \rho \circ \tau$, where $\tau : B \rightarrow \mathbb{F}_e$ and $\rho : \mathbb{F}_e \rightarrow W'$. We have $H_B = \tau^*\bar{H}$. Now we can perform the computation on $\mathbb{F}_e$ with $e > 0$. Noting that $\bar{H}$ is nef and big on $\mathbb{F}_e$, we can write

$$\bar{H} \sim \mu G_0 + nT$$

where $G_0$ is the unique section with $G_0^2 = -e$, $\mu$ and $n$ are integers and $T$ is the general fiber of the ruling on $\mathbb{F}_e$. The property of $\bar{H}$ being nef and big implies $\mu > 0$ and $n \geq \mu e$. If $e = 1$, the equality $(\mu G_0 + nT)^2 = 3$ implies $\mu = 1$ and $n = 2$. If $e > 1$, clearly $n \geq 2\mu \geq 2$. Now let $\alpha_e : \mathbb{F}_e \rightarrow \mathbb{P}^1$ be the ruling, whose fibers are all smooth rational curves. Set $f_0 := \alpha_e \circ \tau \circ f : X' \rightarrow \mathbb{P}^1$, which is a fibration with connected fibers. Denote by $F$ a general fiber of $f_0$. We have

$$M_1 \sim f^*(H_B) = f^*\tau^*(\bar{H}) \geq nF$$
with $n \geq 2$.

Note that a general fiber $F$ of $f_0$ is fibred by curves $C$ (also as fibers of $f$) with $g(C) = 2$. In our situation, we may take $p = 2$ and consider $J := 2F$ on $X'$. This fits the setting for Lemma 3.7. Note that we actually have $f_J = f_0$ and $\pi_J = \pi$ under this situation. So Lemma 3.7 says that $\pi^*(K_X)|_F - \beta_0\sigma^*(K_{F_0})$ is pseudo-effective for $\beta_0 \mapsto \frac{2}{3}$. One has

$$\frac{6}{5} = \pi^*(K_X) \cdot C = (\pi^*(K_X)|_F \cdot C)_F \geq \frac{2}{3}\sigma^*(K_{F_0}) \cdot C.$$

This implies that $\sigma^*(K_{F_0}) \cdot C = 1$. Note that the uniqueness of the Zariski decomposition says that the nef divisor $\pi^*(K_X)|_F$ can be added some effective $\mathbb{Q}$-divisors to become the maximal nef part $\sigma^*(K_{F_0})$ in $K_F$. Thus $\sigma^*(K_{F_0}) = \pi^*(K_X)|_F$ is pseudo-effective. One has $\frac{6}{5} = (\pi^*(K_X)|_F \cdot C)_F \leq \sigma^*(K_{F_0}) \cdot C = 1$, which is absurd. We are done. \hfill \Box

**Proposition 4.6.** If $g(C) = 2$, $p_g(X) \geq 5$ and $\xi = 1$, then $\varphi_4$ is generically finite of degree 2.

**Proof.** Recall that we have $K_{X'} = \pi^*(K_X) + E_\pi$. On $X$ we set $Z := \pi_*(Z_1)$ and $N := \pi_*(M_1)$. Clearly $K_X \sim N + Z$. Then there is an effective $\mathbb{Q}$-divisor $E_1$, which is supported by some exceptional divisors, such that $\pi^*(N) = M_1 + E_1$. Therefore $E'_1 = \pi^*(Z) + E_1$. For a general member $S$ of $|M_1|$, we have $K_X|_S = \pi^*(K_X)|_S + E_\pi|_S = (M_1|_S + E'_1|_S) + E_\pi|_S$. So $E'_1|_S = \pi^*(Z)|_S + E_1|_S$. One knows that $E_\pi$ is composed of all those exceptional divisors of $\pi$. Thus $E_1 \leq E_\pi$ and $E_1|_S \leq E_\pi|_S$.

We will now need further assumptions on the map $\pi$. We may take the $\pi$ to be the composition $X' \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_0} X$ where $\pi_0$ is the resolution of the indeterminacy of $\varphi_1$, $\pi_1$ is the resolution of those isolated singularities on $X_1$ which are away from all exceptional locus of $\pi_0$ and $\pi_2$ is the minimal further modification such that $\pi^*(K_1)$ has the support of simple normal crossings (recall here that $K_1 \sim K_X$ is a fixed Weil divisor as in [22]). Set $\pi_3 := \pi_0 \circ \pi_1$. By abuse of notations we will have a set of divisors for $\pi_3$ similar to that for $\pi$. For example we may write $K_{X_2} = \pi^*_3(K_X) + E_{\pi_3}$ where $E_{\pi_3}$ is an effective $\mathbb{Q}$-divisor. The movable part $|K_{X_2}|$ of $|K_{X_2}|$ is already base point free. Write $\pi^*_3(N) = M_{\pi_3} + E_{1,\pi_3}$ and $\pi^*_3(K_X) = M_{\pi_3} + E'_{1,\pi_3}$ where $E_{1,\pi_3}$ and $E'_{1,\pi_3}$ are both effective $\mathbb{Q}$-divisors. Clearly $E'_{1,\pi_3} = \pi^*_3(Z) + E_{1,\pi_3}$. By the definition of $\pi_3$, $E_{\pi_3}$ is the union of two parts $E_{\pi_3}^\prime + E_{\pi_3}''$ where $E_{\pi_3}''$ consists all those components over the indeterminacy of $\varphi_1$ and $E_{\pi_3}''$ is totally disjoint from $E_{\pi_3}^\prime$. Denote by $S_{\pi_3}$ a general member of $|M_{\pi_3}|$. Then $|M_{\pi_3}|_{S_{\pi_3}}$ is a free pencil of genus 2 with a general member $C_{\pi_3}$. As we have seen $\text{Supp}(E_{\pi_3}''|_{S_{\pi_3}}) = 0$ and so $\text{Supp}(E_{1,\pi_3}|_{S_{\pi_3}}) = \text{Supp}(E_{1,\pi_3}|_{S_{\pi_3}})$. Now we see $1 = \pi^*(K_X) \cdot C = \pi^*_3(K_X) \cdot \pi_{2,\pi_3}(C) = \pi^*_3(K_X) \cdot C_{\pi_3}$. Since $2 = \text{deg}(K_{C_{\pi_3}}) = (\pi^*_3(K_X) + E_{\pi_3}) \cdot C_{\pi_3}$ and $\pi^*_3(K_X)|_{S_{\pi_3}} \cdot C_{\pi_3} = 1$, we get $E_{\pi_3}|_{S_{\pi_3}} \cdot C_{\pi_3} = E_{\pi_3} \cdot C_{\pi_3} = 1$. Therefore $E_{1,\pi_3} \cdot C_{\pi_3} > 0$. Noting that
\( \pi^*_{\ell}(E_{1, \pi_\ell}) \leq E_1 \) one sees
\[
E_{1|S} \cdot C \geq \pi^*_{\ell}(E_{1, \pi_\ell})|_{S} \cdot C = E_{1, \pi_\ell}|_{S_{\pi_\ell}} \cdot C_{\pi_\ell} > 0
\]  
(4.2)
which is what we want to show in this paragraph.

Since \( d(X) > 1 \), the linear system \( |4K_X| \) (and also \( |K_X| \)) separates different irreducible elements of \( |M_1| \). Therefore, \( \Phi_4 \) is birational if and only if \( \Phi_4|_S \) is birational for a general \( S \). Now on a general surface \( S \), we have a pencil \( |M_1|_S \) and \( \Phi_4|_S \) separates different generic irreducible elements of \( |M_1|_S \) (see [13] and the paragraph below). So similarly \( \Phi_4|_S \) is birational if and only if \( (\Phi_4|_S)|_C \) is birational. We will show that \( (\Phi_4|_S)|_C = \Phi_{[2K_C]} \). The latter is, however, not birational.

Denote by \( M_4 \) the movable part of \( |4K_X| \). Then, for a general \( S \) in \( |M_1| \), \( M_4|_S \leq 4\pi^*(K_X)|_S \) and \( M_4|_S \cdot C \leq 4\pi^*(K_X)|_S \cdot C = 4 \). By Kawamata-Viehweg vanishing, we have a surjective map:
\[
H^0(X', K_{X'} + 2\pi^*(K_X)|_S) \rightarrow H^0(S, K_S + 2\pi^*(K_X)|_S).
\]
Since
\[
2\pi^*(K_X)|_S - C - \frac{1}{a_2}E_1'|_S \equiv (2 - \frac{1}{a_2})\pi^*(K_X)|_S
\]
is nef and big, the vanishing theorem gives a surjective map:
\[
H^0(S, K_S + 2\pi^*(K_X)|_S - \frac{1}{a_2}E_1'|_S) \rightarrow H^0(C, K_C + D),
\]
where
\[
D := 2\pi^*(K_X)|_S - \frac{1}{a_2}E_1'|_S|_C = (2 - \frac{1}{a_2})E_1'|_S|_C
\]
and \( \deg(D) \geq (2 - \frac{1}{a_2}) \geq 2 - \frac{1}{3} > 1 \), noting that \( E_1'|_C = \pi^*K_X|_C = \xi = 1 \). So \( |K_C + D| \) is base point free. Denote by \( M_4' \), \( N_4' \) the movable parts of \( |K_X| + 2\pi^*(K_X)|_S \), \( |K_S + 2\pi^*(K_X)|_S - \frac{1}{a_2}E_1'|_S|_C \) respectively. Then, by Lemma 2.7 of [3], one has
\[
M_4|_C := (M_4|_S)|_C \geq (M_4'|_S)|_C \geq N_4'|_C \geq K_C + D.
\]
So \( 4 = 4\pi^*(K_X)|_S \cdot C = M_4 \cdot C = \deg(K_C + D) \geq 4 \). This means \( M_4|_C \sim K_C + D \) and \( \deg(D) = 2 \). On the other hand, we have shown \( |M_4||_C \supset |K_C + D| \). Thus \( \Phi_4|_C = \Phi_{[K_C + D]} \). Since \( \deg(\Phi_{[K_C]}|_C) = 2 \), we have \( \deg(\Phi_4|_C) \leq 2 \). So \( \Phi_4 \) is either birational or of degree two.

We have:
\[
K_C \sim (K_X|_S + S|_S)|_C = (\pi^*(Z)|_S)|_C + (E_1|_S)|_C + (E_\pi|_S)|_C. \quad (4.2)
\]
Since \( 2 = \deg(K_C) = (\pi^*K_X + E_\pi)|_C \) and \( E_1'|_S \cdot C = \pi^*(K_X)|_S \cdot C = 1 \), we get \( E_\pi|_S \cdot C = E_\pi \cdot C = 1 \). So \( \text{Supp}(E_\pi|_S) \) is either one of the following situations:

**Case 1.** a single point \( P \);

**Case 2.** two points \( P + Q \), where \( P, Q \) are different points on \( C \).

We consider **Case 1** and **Case 2** separately and note that \( E_1'|_S = \pi^*(Z)|_S + E_1|_S \) and \( \text{Supp}(E_1|_S) \subset \text{Supp}(E_\pi|_S) \).
Suppose we are in Case 1. Then \((E_1|_S)|_C = P\). If \(\text{Supp}((\pi^*(Z)|_S)|_C + E_1|_S)\) contains a point other than \(P\) (say a point \(R\)), then \(\pi^*(Z)|_S|_C + (E_1|_S)|_C + \pi^*(E_1|_S)|_C = P + R\) and \(R\) is not contained in \(\text{Supp}(E_1|_S)\) otherwise \(R\) is in \(\text{Supp}(E_1|_S)\), a contradiction. Thus \(R \leq \pi^*(Z)|_S|_C\) as an integral part because \(\pi^*(Z)|_S|_C + (E_1|_S)|_C + \pi^*(E_1|_S)|_C\) is an integral divisor. This says that \(\text{deg}(\pi^*(Z)|_S|_C) + (E_1|_S)|_C\) is not contained in \(\text{Supp}(E_1|_S)\). Therefore, \(R \leq \pi^*(Z)|_S|_C\) is in \(\text{Supp}(E_1|_S)\), a contradiction. Thus \(R \leq \pi^*(Z)|_S|_C\) is an integral divisor. This says that \(\text{deg}((E_1|_S)|_C) > 0\) by the relation (4.2), which is absurd. Thus \(\pi^*(Z)|_S|_C + (E_1|_S)|_C = P\) and \(K_C \sim 2P\). In this case, we have \(D = 2P\) and \(\varphi_4|_C = \Phi_{[2K_C]}\) is not birational. So \(\varphi_4\) is not birational onto its image.

Suppose we are in Case 2. The right hand side of (4.2) must be \(P + Q\) and \(K_C \sim P + Q\). We also know that \(\text{Supp}((\pi^*(Z)|_S)|_C + (E_1|_S)|_C) = P + Q\). So \(D = P + Q\). And thus \(\varphi_4|_C = \Phi_{[2K_C]}\) is not birational. □

From now on, we study the case \(d = d(X) = 1\). We shall show that \(\varphi_4\) is birational unless the case of Theorem 4.1 (ii) occurs. Let \(b\) be the genus of \(B = f(X')\). Let \(S\) be a general fibre of \(f : X' \to B\) and \(\sigma : S \to S_0\) the smooth blow down to a minimal model. From now on within this section, we always set \(J := K_X\) to run Theorem 3.7.

**Lemma 4.7.** If \(d = 1\) and \(b > 0\), then \(\pi^*(K_X)|_S \sim \sigma^*(K_{S_0})\).

**Proof.** We use the idea of Lemma 14 in Kawamata’s paper [18]. By Shokurov’s theorem in [27], each fiber of \(\pi : X' \to X\) is rationally chain connected. Therefore, \(f(\pi^{-1}(x))\) is a point for all \(x \in X\). Considering the image \(G \subset (X \times B)\) of \(X'\) via the morphism \((\pi \times f) \circ \triangle_{X'}\), where \(\triangle_{X'}\) is the diagonal map \(X' \to X' \times X'\), one knows that \(G\) is a projective variety. Let \(g_1 : G \to X\) and \(g_2 : G \to B\) be two projections. Since \(g_1\) is a projective morphism and even a bijective map, \(g_1\) must be both a finite morphism of degree 1 and a birational morphism. Since \(X\) is normal, \(g_1\) must be an isomorphism. So \(f\) factors as \(f_1 \circ \pi\) where \(f_1 := g_2 \circ g_1^{-1} : X \to B\) is a well defined morphisms. In particular, a general fiber \(S_0\) of \(f_1\) must be smooth minimal. So it is clear that \(\pi^*(K_X)|_S \sim \sigma^*(K_{S_0})\) where \(\sigma\) is nothing but \(\pi|_S\). □

**4.8. The case** \(d = 1, b > 0\). Assume \(p_g(X) \geq 2\) and \(S\) is not of type \((c_1^2, p_g) = (1, 2)\). We shall show that \(\varphi_4\) is birational.

One has \(M_1 \equiv a_1S\) where \(a_1 \geq p_g(X) \geq 2\). Clearly, \(|4K_{X'}|\) separates different generic irreducible elements of \(|M_1|\). In fact, if \(S_1\) and \(S_2\) are two different fibers of \(f\), Kawamata-Viehweg vanishing gives a

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1C. D. Hacon and J. McKernan (see [16]) have recently extended V. V. Shokurov’s result to any dimension and without assuming the MMP.
surjective map

\[ H^0(X', K_{X'} + \gamma 3^*\pi^*(K_X) - \frac{2}{a_1}E'_1) \rightarrow H^0(S_1, K_{S_1} + \gamma 3^*\pi^*(K_X) - \frac{2}{a_1}E'_1|_{S_1}) \oplus H^0(S_2, K_{S_2} + \gamma 3^*\pi^*(K_X) - \frac{2}{a_1}E'_1|_{S_2}). \]

One has

\[ K_{S_i} + \gamma 3^*\pi^*(K_X) - \frac{2}{a_1}E'_1|_{S_i} \geq K_{S_i} + 2\sigma^*_{S_i} > 0 \]

which means that \(|K_{X'} + \gamma 3^*\pi^*(K_X) - \frac{2}{a_1}E'_1|\) separates \(S_1\) and \(S_2\). So does \(|4K_{X'}|\).

We study another subsystem

\[ |K_{X'} + \gamma 3^*\pi^*(K_X) - \frac{1}{a_1}E'_1| \subset |4K_{X'}|. \]

Pick up a general fiber \(S\) of \(f : X' \rightarrow B\). Since \(3^*\pi^*(K_X) - \frac{1}{a_1}E'_1 - S \equiv (3 - \frac{1}{a_1})\pi^*(K_X)\) is nef and big, the Kawamata-Viehweg vanishing theorem gives a surjective map

\[ H^0(X', K_{X'} + \gamma 3^*\pi^*(K_X) - \frac{1}{a_1}E'_1) \rightarrow H^0(S, K_S + \gamma 3^*\pi^*(K_X) - \frac{1}{a_1}E'_1|_S) \]

\[ \subset H^0(S, K_S + \gamma (3^*\pi^*(K_X) - \frac{1}{a_1}E'_1)|_S) \]

\[ = H^0(S, K_S + 2\sigma^*(K_{S_0}) + \gamma (1 - \frac{1}{a_1})E'_1|_S). \]

Thus, by Tankeev’s principle, it suffices to show the birationality of \(\Phi|_{K_{S_0} + 2\sigma^*(K_{S_0}) + \gamma (1 - \frac{1}{a_1})E'_1|_S}\).

If \((K_{S_0}^2, p_g(S_0)) \neq (2, 3)\), the statement is clear by Bombieri ([2]). Otherwise it is a corollary of Lemma 1.3 in [11]. (In fact, this is an easy exercise by Kawamata-Viehweg vanishing, just noting the fact (see page 227 in [1]) that \(|K_{S_0}|\) has no base points and that \(|K_{S_0}|\) gives a generically finite morphism of degree 2.)

**4.9. Claim.** Consider the case \(d = 1, b = 0\). Assume that \(p_g(X) \geq 5\) and that a general fiber \(S\) of the induced fibration \(f : X' \rightarrow B\) is not of type \((1,2)\). Then \(\varphi_4\) is birational.

Since \(p_g(X) > 0\), we have \(p_g(S) > 0\) for a general fiber \(S\). Thus by an established theorem (see Bombieri [2], Reider [26], Catanese-Ciliberto
[3], and P. Francia [14] or directly refer to Theorem 3.1 in the survey article by Ciliberto [12]), |2K_{S_0}| is always base point free. We classify S into two types separately in order to organize our proof ((1,2) type surfaces excluded):

(A) (K^2_{S_0}, p_g(S)) \neq (2, 3);
(B) (K^2_{S_0}, p_g(S)) = (2, 3).

To prove the claim, we consider these two case separately.

**Case (A).** We want to verify the conditions in Theorem 3.6. We take G := 2\sigma^*(K_{S_0}). Since |G| is base point free, a generic irreducible element C of |G| is smooth and |G| is not composed with a pencil.

By the assumption, we have an inclusion \( \mathcal{O}(4) \hookrightarrow f^*\omega_X' \). So there is another inclusion \( f^*\omega^2_{X'/P^1} \hookrightarrow f^*\omega^3_X \). (4.3)

The sheaf on the left is semi-positive and is clearly generated by global sections. On the other hand, \( 3\pi^*(K_X) \geq M_3 \) where \( M_3 \) is the movable part of |3K_X|.

Noting that any local sections along a general fiber S can be extended to a global section, we clearly have \( 3\pi^*(K_X)|_S \geq 2\sigma^*(K_{S_0}) \sim G \). So we have

\[ K_S + \frac{3}{a_1}E_1'|_S \geq \gamma E_\pi|_S + (1 - \frac{1}{a_1})E_1'|_S + 3\pi^*(K_X)|_S \geq G. \]

So one of the conditions in Theorem 3.6(1) is satisfied.

By Lemma 3.7, we may take a \( \beta \) (arbitrarily close to \( \frac{2}{5} \)) such that \( \pi^*(K_X)|_S - \beta C \) is pseudo-effective. Noting that G is an even divisor and that C is non-hyperelliptic (by the birationality of \( \Phi|_{3K_{S_0}} \)), we only have to verify \( \alpha > 0 \) in order to apply Theorem 3.6. We have \( p = 4 \). Then take \( m = 4 \). Then

\[ \alpha = (4 - 1 - \frac{1}{4} - \frac{1}{\beta})\xi > 0. \]

So \( \varphi_4 \) is birational by Theorem 3.6.

**Case (B).** We take \( G = \sigma^*(K_{S_0}) \). Since |G| is base point free, a generic irreducible element C of |G| is a smooth curve of genus 3. Since |2\sigma^*(K_{S_0})| is base point free, the similar argument in Case (A) shows that conditions in Theorem 3.6(1) are satisfied.

We have \( p = 4 \). Again by Lemma 3.7, we may take a \( \beta \) (arbitrarily near \( \frac{2}{5} \)) such that \( \pi^*(K_X)|_S - \beta C \) is pseudo effective. We have \( \xi \geq \beta C^2 \Rightarrow \frac{8}{5} \). In fact, taking the limit, one has \( \xi \geq \frac{8}{5} \). Take \( m = 4 \). Then

\[ \alpha = (4 - 1 - \frac{1}{p} - \frac{1}{\beta})\xi \geq \frac{12}{5} > 2. \]

Theorem 3.6 tells us that \( \varphi_4 \) is birational onto its image. This proves the claim.

4.10. **The case** \( d = 1 \), \( (K^2_{S_0}, p_g(S_0)) = (1, 2) \). Assume \( p_g(X) \geq 3 \). We shall show that the case of Theorem 4.1 (ii) occurs.

Since the 4-canonical map of S is not birational by Bombieri (see [2] or [1]), it is clear that \( \varphi_4 \) of X is not birational either. It is however
easy to see by inclusion (4.3) that $\varphi_4$ is at worst generically finite of degree 2 since $\Phi_{|2K_S|}$ is generically finite of degree 2.

We now show the existence of the family $\mathcal{C}$ of curves of genus 2 on $X$ with the property $K_X \cdot C_0 = 1$.

We may consider the relative canonical map $\Psi : X' \rightarrow \mathbb{P}(f_*\omega_{X'/B}^\vee)$ over $B$. By taking further birational modifications we may assume that $\Psi$ is a morphism over $B$. So we have the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\Psi} & \mathbb{P}(f_*\omega_{X'/B}^\vee) \\
\pi \downarrow & & \downarrow p \\
X & \xrightarrow{f} & B \\
\varphi_1 \downarrow & & \downarrow \\
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Since $\Psi|_S = \Phi_{|K_S|}$, we see that a general fiber of $\Psi$ is a genus 2 curve which is nothing but the movable part of $|K_S|$. By [1], one knows that $|K_{S_0}|$ has one base point and its generic irreducible element is a curve of genus 2. So on $S$ we take $G$ to be the movable part of $|\sigma^*(K_{S_0})|$. Pick up a generic irreducible element $C$ of $|G|$, which is also a smooth fiber of $\Psi$. We will apply Theorem 3.6 to estimate $\xi = \pi^*(K_X)|_S \cdot C$. Note that $\xi \leq \sigma^*(K_{S_0}) \cdot C = K_{S_0}^2 = 1$ because $\pi^*(K_X)|_S$ is nef and $\sigma^*(K_{S_0})$ is the nef part of the Zariski decomposition of $K_S$.

Applying Lemma 3.7 once more, we may take a $\beta$ (arbitrarily near $\frac{4}{5}$) such that $\pi^*(K_X)|_S - \beta \sigma^*(K_{S_0})$ is pseudo effective. Then $\xi \geq \frac{4}{5}$. We have $p \geq 4$. Take $m = 4$. Then $\alpha = (4 - 1 - \frac{1}{p} - \frac{1}{\beta}) \xi \geq \frac{6}{5} > 1$. Theorem 3.6 gives $\xi \geq 1$.

So the only possibility is $\pi^*(K_X)|_S \cdot C = \xi = 1$. We take the family $\mathcal{C}$ on $X$ to be the set of images (via $\pi$) of those fibers of $\Psi$. For a general member $C_0 \in \mathcal{C}$, one has $K_X \cdot C_0 = \pi^*(K_X) \cdot C = 1$. So the case in Theorem 4.1 (ii) follows. This completes the proof of Theorem 4.1.

5. Proof of the main theorem, Part II

5.1. Assumption. Assume that a minimal threefold $X$ of general type is birationally fibred by a family $\mathcal{C}$ of curves of geometric genus 2 with $K_X \cdot C_0 = 1$ for a general member $C_0 \in \mathcal{C}$. Suppose $p_g(X) \geq 5$. We will show that $\varphi_4$ is not birational and that the above family of curves is uniquely determined. This should conclude Theorem 1.3 by virtue of Theorem 4.1.

After successive blow-ups, we may assume that $X'$ is the birational model of $X$ which is fibred by genus two curves over a base $W$. Denote by $f_\mathcal{C} : X' \rightarrow W$ the fibration. By assumption, a general fiber $C'$ of $f_\mathcal{C}$ is a smooth curve of genus 2. Take a very ample divisor $A$ on $W$. Set $H := f_\mathcal{C}^*(A)$. Then $|H|$ is a base point free linear system on $X'$. 
We now argue according to the value of \( d = d(X) = \dim \varphi_1(X) \).

5.2. Claim. \( d < 3 \).

Suppose the contrary that \( d = 3 \). Then \( \dim \varphi_1(H) = 2 \) for a general \( H \). Note that \( |M_1|_H \supseteq |M_1||H \). We already know that \( \Phi_{|M_1||H|} \) is generically finite. So is \( \Phi_{|M_1||H|} \). Thus, for a generic fiber \( C' \) of \( f \) with \( C' \subset H \), \( \dim \Phi_{|M_1||H|}(C') = 1 \). The Riemann-Roch and Clifford’s theorem on \( C' \) imply then that \( (M_1 \cdot C')_X = (M_1||H| \cdot C')_H \geq 2 \). So \( K_X \cdot C_0 = \pi^*(K_X) \cdot C' \geq M_1 \cdot C' \geq 2 \), a contradiction.

5.3. Claim. \( \varphi_4 \) is not birational whenever \( d = 2 \).

We consider a generic fiber \( C' \subset H \) for a general \( H \). The map \( \varphi_1|_H \) is exactly defined by the linear system \( |M_1||H| \subset |M_1||H| \). If \( \dim \varphi_1|_H(C') = 1 \), then the Riemann-Roch and Clifford’s theorem on \( C' \) implies

\[
K_X \cdot C_0 = \pi^*(K_X) \cdot C' \geq M_1 \cdot C' = (M_1||H| \cdot C')_H \geq 2,
\]

a contradiction. Thus, \( \varphi_1 \) maps a general \( C' \) to a point. So Lemma 14 of [13] implies that \( f : X' \to B \) birationally factors through \( f_\xi \). Take further birational modifications to \( X' \) such that \( f \) factors through \( f_\xi : X' \to W \) and a surjective morphism \( W \to B \) between normal projective surfaces. Then the uniqueness of the Stein factorization says that \( W \to B \) is birational and we may assume \( f = f_\xi \) (after birationally modifying the base of \( f_\xi \)). This simply means that the family \( \xi \) is exactly the canonically induced family. Since \( \pi^*(K_X)|_S \cdot C = K_X \cdot C_0 = 1 \), the argument in [4.6] implies that \( \varphi_4 \) is generically finite of degree 2.

5.4. Claim. \( \varphi_4 \) is not birational whenever \( d = 1 \).

As we have shown above that \( \dim \varphi_1|_H(C) = 0 \), \( \varphi_1 \) contracts a general fiber \( C \) of \( f_\xi \) to a point. Similarly we may suppose that \( f : X' \to B \) factors through \( f_\xi : X' \to W \) and \( \theta : W \to B \).

![Diagram](attachment:image.png)

For a general point \( b \in B \), the fiber \( S = f^{-1}(b) \) has a natural fibration \( f_b : S \to W_b = \theta^{-1}(b) \). A general fiber of \( f_b \) lies in the same numerical class as that of \( C \). Pick up a smooth fiber \( C_b \) of \( f_b \). By the assumption, \( \pi^*(K_X)|_S \cdot C_b = \pi^*(K_X)|_S \cdot C = K_X \cdot C_0 = 1 \). Since \( p_g(X) \geq 5 \), Lemma 5.7 tells us that one may take a \( \beta \) (arbitrarily near \( \frac{1}{5} \)) such that \( \pi^*(K_X)|_S - \beta \sigma^*(K_{S_0}) \) is pseudo effective, where \( \sigma : S \to S_0 \) is the smooth blow down to a minimal model. So \( \sigma^*(K_{S_0}) \cdot C_b \leq \frac{5}{4} \pi^*(K_X)|_S \cdot
$C = \frac{5}{4}$. This means that $K_{S_0} \cdot L = \sigma^*(K_{S_0}) \cdot C_b = 1$ where $L := \sigma_*(C_b)$. This implies that $L^2 > 0$ and in fact $K_{S_0}^2 = L^2 = 1$ and $L \equiv K_{S_0}$ by the Hodge index theorem. Now the surface theory tells us that $q(S_0) = 0$ and $1 \leq p_a(S_0) \leq 2$ (by the Noether inequality). Note that $g(W_b) \leq q(S) = 0$, so $W_b$ is a smooth rational curve. Hence $h^0(S, C_b) = 2$. The computation on $S_0$ shows that $p_a(L) = 2 = g(C_b)$. Thus $L \cong C_b$ is a smooth curve of genus 2. If $S_0$ has the invariants $(K_{S_0}^2, p_g(S_0)) = (1, 1)$ (resp. $(1, 2)$), then the Neron Severi group of $S_0$ has no torsion by Bombieri [2] (resp. $S_0$ is simply connected by [1]). Therefore, $K_{S_0} \sim L$, where the latter is movable. Hence $S_0$ must be of type $(1,2)$. This already shows that $\varphi_4$ is not birational. Since $h^0(S_0, L) = p_g(S) = 2 = h^0(S, C_b)$, our $|C_b|$ is clearly the movable part of $|\sigma^*(K_{S_0})|$. Since $C_b \equiv C$ and since $S$ is simply connected, we have $C_b \sim C$. Hence the family of fibers $C$ of $f_\sigma$, when restricted on $S$, is exactly the canonically induced family on $S$.

5.5. Conclusion to the main theorem. Theorem 1.3 is uniquely determined by $X$. We are done.

5.6. Proof of Theorem 1.4

Proof. We now prove Theorem 1.4. Set $\tilde{b} := g(B)$.

The existence of $\tilde{f}$ implies that $\varphi_4$ is not birational. Therefore $d = \dim \varphi_4(X) < 3$ by [12]. The proof of Theorem 1.3 also implies the existence of the family $\mathscr{C}$ of curves of genus 2. By taking further birational modification, if necessary, we may suppose $X' = Z$. We also have the birational modification $\pi = \nu : X' = Z \longrightarrow X$. Denote by $\tilde{S}$ a general fiber of $\tilde{f}$. Let $\hat{\sigma} : \tilde{S} \longrightarrow \tilde{S}_0$ be the blow-down onto the smooth minimal model.

Case 1. $\tilde{b} > 0$.

By the proof of Lemma 1.7 we see that $\tilde{f}$ factors through $X$ and thus $\nu^*(K_X)|_\tilde{S} \sim \hat{\sigma}^*(K_{\tilde{S}_0})$.

We can actually get the family $\mathscr{C}$ by considering the morphism $\tilde{\Psi} : Z \longrightarrow \mathbb{P}(\tilde{f}_*\omega_{Z/B})$. Let $\tilde{C} \subset \tilde{S}$ be a general fiber of the induced fibration by taking the Stein factorization of $\tilde{\Psi}$, where $\tilde{S}$ is a smooth fiber of $\tilde{f}$. We have

$$\pi^*(K_X) \cdot \tilde{C} = \nu^*(K_X) \cdot \tilde{C} = \hat{\sigma}^*(K_{\tilde{S}_0}) \cdot \tilde{C} = 1.$$ 

So the uniqueness of the family $\mathscr{C}$ says that $\mathscr{C}$ can be obtained by taking those images (via $\nu = \pi$) of all fibers of $\tilde{\Psi}$.

Case 2. $\tilde{b} = 0$.

Let $\tilde{C} \subset \tilde{S}$ be a general fiber of the induced fibration by taking the Stein factorization of $\tilde{\Psi}$, where $\tilde{S}$ is a smooth fiber of $\tilde{f}$. Then $\tilde{C}$ is a smooth curve of genus 2. We want to show $\pi^*(K_X) \cdot \tilde{C} = 1.$
Considering the natural map
\[ H^0(X', M_1) \rightarrow H^0(\tilde{S}, M_1|\tilde{s}) \subset H^0(\tilde{S}, K_{\tilde{S}}), \]
we have \( \pi^*(K_X) \geq M_1 \geq 2\tilde{S} \) by the fact that the vector space on the right has the dimension at most 2 and the assumption \( p_g(X) \geq 5 \). We take \( J := 2\tilde{S} \) and will apply Theorem 3.6. Note that we have actually \( \pi_J = \pi \) and \( f_J = \tilde{f} \). Clearly \( p = 2 \). So Lemma 3.7 implies that one can take \( \beta \mapsto \zeta \) such that \( \pi^*(K_X)|_{\tilde{s}} - \beta \tilde{\sigma}^*(K_{\tilde{S}_0}) \) is pseudo-effective. Thus one has
\[ \pi^*(K_X) \cdot \tilde{C} = \pi^*(K_X)|_{\tilde{s}} \cdot \tilde{C} \geq \beta \tilde{\sigma}^*(K_{\tilde{S}_0}) \cdot \tilde{C} = \beta. \]
So \( \zeta = \pi^*(K_X) \cdot \tilde{C} \geq \frac{2}{3} \). Now we can apply Theorem 3.6 to estimate \( \zeta \).
Take \( m_1 = 5 \). Then \( \alpha_1 = (m_1 - 1 - \frac{1}{p} - \frac{1}{\beta}) \zeta \geq \frac{4}{3} \). So Theorem 3.6 gives \( \tilde{\zeta} \geq \frac{4}{3} \). Take \( m_0 = 6 \). Theorem 3.6 implies \( \tilde{\zeta} \geq \frac{8}{6} \). In fact, an induction shows \( \tilde{\zeta} \geq \frac{n+1}{n} \) for all \( n \geq 8 \). Thus \( \tilde{\zeta} \geq 1 \). Since \( \tilde{\zeta} \leq \tilde{\sigma}^*(K_{\tilde{S}_0}) \cdot \tilde{C} = 1 \), we have \( \pi^*(K_X) \cdot \tilde{C} = 1 \).

The uniqueness of the family \( \mathcal{C} \) means that \( \mathcal{C} \) can be obtained by taking those images (via \( \nu = \pi \)) of all fibers of \( \tilde{\Psi} \). We are done. \( \square \)

6. Application, new examples and open problems

6.1. Proof of Theorem 1.5

Proof. (i) For a minimal threefold \( X \) of general type with \( K_X^3 < \frac{4}{3}(p_g(X) - 2) \), one has \( p_g(X) > 2 \) since \( K_X^3 > 0 \). By assumption, we have \( p_g(X) \geq 12 \). We still consider the canonical map and keep the same notation as above.

If \( d = 3 \), then Kobayashi [21] proved \( K_X^3 \geq 2(p_g(X) - 6) \geq \frac{4}{3}(p_g(X) - 2) \), a contradiction.

If \( d = 2 \) and \( g(C) \geq 3 \), we have shown in [4, 5] that \( \xi \geq \frac{7}{4} \). So one has \( K_X^3 \geq \frac{7}{4}(p_g(X) - 2) > \frac{4}{3}(p_g(X) - 2) \), also a contradiction.

If \( d = 2 \), \( g(C) = 2 \) and \( \xi > 1 \), we have shown in [4, 5] that \( \xi \geq \frac{5}{4} \) since \( p_g(X) > 5 \). We may use Theorem 3.6 to go on estimating \( \xi \). Recall that we have \( p = 1 \). Since \( p_g(X) \geq 12 \), we may take \( \beta \geq 10 \). Take \( m = 3 \). Then \( \alpha = (3 - 1 - \frac{1}{p} - \frac{1}{\beta}) \xi \geq \frac{8}{8} > 1 \). Theorem 3.6 gives \( \xi \geq \frac{4}{3} \).
So \( K_X^3 \geq \frac{4}{3}(p_g(X) - 2) \), a contradiction.

If \( d = 1 \) and \( (K_{S_0}^2, p_g(S_0)) \neq (1, 2) \), then the results in [9] (Theorem 3.2(2), 3.4, 3.5) show that \( K_X^3 \geq \frac{2}{3}p_g(X) - \frac{5}{2} > \frac{4}{3}(p_g(X) - 2) \), a contradiction.

Therefore, it is true that either \( d = 2 \), \( g(C) = 2 \) and \( \xi = 1 \) (since \( p_g(X) > 5 \)) or \( d = 1 \) and \( (K_{S_0}^2, p_g(S_0)) = (1, 2) \). Theorem 3.6 implies that \( \varphi_4 \) is not birational.

(ii) By Theorem 4.1 in [8], it is true that either \( d = 2 \) and \( g(C) = 2 \), or \( d = 1 \) and \( (K_{S_0}^2, p_g(S_0)) = (1, 2) \) whenever \( K_X^3 < 2(p_g(X) - 6) \). We have to exclude the case with \( d = 2 \) and \( \xi > 1 \). Noting that \( \xi \) must
be an integer, we have $\xi \geq 2$. So, in that case, we still have $K_X^2 \geq 2(p_g(X) - 2)$, a contradiction. Therefore, Theorem 3.6 implies that $\varphi_4$ is not birational. □

**Example 6.2.** M. Kobayashi (see Proposition 3.2 in [21]) has constructed a family of canonically polarized smooth threefolds $Y$ satisfying the equality

$$K_Y^3 = \frac{4}{3}p_g(Y) - \frac{10}{3}$$

where $p_g(Y) = 7, 10, 13, \ldots$.

Theorem 1.5 says that all examples above have non-birational 4-canonical maps.

The example below shows that the assumption $p_g(X) \geq 5$ in Theorem 1.3 is optimal.

**Example 6.3.** On $\mathbb{P}^3_C$, take a smooth hypersurface $S$ of degree 10. $S \sim 10H$ where $H$ is a hyperplane. Let $\tau : X = \text{Spec} \oplus_{i=0}^1 \mathcal{O}(-5iH) \to \mathbb{P}^3_C$ be the double cover branched along $S$. Then $X$ is a nonsingular canonical threefold with $K_X = \tau^*H$, $K_X^3 = 2$ and $p_g(X) = 4$ and $\Phi_1$ is a finite morphism onto $\mathbb{P}^3$ of degree 2. One may easily check that $\Phi_4$ is also a finite morphism of degree 2. Indeed, let $C$ be the inverse on $X$ of a general line. Then $C$ is a hyperelliptic curve of genus 4 with $(4K_X)|_C = K_C + P + Q$, where $\tau(P) = \tau(Q)$. Hence $\phi_4|_C$ is a degree 2 map (see Section 6.5 of Iitaka’s book [17]).

It is not difficult to see that, for a generic irreducible curve $C_0$ in any family of curves on $X$, $K_X \cdot C_0 \geq 2$.

There are still several natural and unsolved problems:

**6.4. Open problems.** Let $X$ be a minimal projective threefold of general type with $\mathbb{Q}$FT singularities and with $p_g(X) \geq 5$.

1. Is it true that $\varphi_4$ is not birational if and only if $X$ is birationally fibred by a family of surfaces of type $(c_1^2, p_g) = (1, 2)$?
2. Is it possible to characterize the birationality of $\varphi_3$?
3. Is it true that $\dim \varphi_2(X) \geq 2$ when $p_g(X)$ is bigger?

We have only a partial answer to Problems 6.4(1). The above problems might be very difficult, but very interesting. There has not been any counter example to Open problems 6.4.

If $X$ is birationally fibred by surfaces of type $(1, 2)$, then one surely has the non-birationality of $\varphi_4$, which is hence a natural condition in the result below.

**Theorem 6.5.** Let $X$ be a minimal projective threefold of general type with $\mathbb{Q}$FT singularities and with $p_g(X) \geq 5$. Suppose that $\varphi_4$ is not birational. Then $X$ is birationally fibred by surfaces of type $(1, 2)$ in the sense of Theorem 1.3 if and only if either
Proof. By Theorem 4.1, we may assume that \( \dim \nu C \pi \) is a pencil \( \tilde{h} \) surface to a curve) with connected fiber. Let \( \pi X C F \) and \( \ell \) of a hyperplane \( g \) is a general fiber of \( M p \) pencil \( F \) where the surface blow down to a minimal model.

\[ \ast (\ast) \]

We assert that \( H \Phi (H F) = 0 \) by \([1]\) Set \( F_1, F_2 \) of \( f \): \( 0 \to \mathcal{O}(K_{X'} - F_1) \to \mathcal{O}(K_{X'}) \to \mathcal{O}(K_{F_1}) \to 0 \)

Thus \( F_1 + F_2 \leq M_1 \) (the movable part of \( K_{X'} \), or \( \pi^* K_X \)). By Theorem 4.1 the general fibers \( C \) of \( f : X' \to B \) are curves of genus 2 with \( \pi^* K_X \cdot C = 1 \). Since \( C \cdot 2F \leq C \cdot M_1 \leq C \cdot \pi^* K_X = 1 \), one has \( C \cdot F = 0 \), so the general \( C \) are contained in general fibers \( F \). By Lemma 14 in Kawamata [18] (after further blowing up \( X' \) and \( B \) if necessary), \( f \) factors as \( f = \tau \circ f \) where \( \tau : B \to \tilde{B} \) is a surjective morphism (from a surface to a curve) with connected fiber. Let \( M_1 = f^* H_B, F_1 = f^{-1}(b) \) and \( \ell = \tau^{-1}(b) \). Then \( F_1 = f^* \ell_1 \) and \( H_B \geq \ell_1 + \ell_2 \). Also \( \ell \) is a smooth rational curve because \( g(\ell_1) \leq q(F_1) \) and \( F_1 \) is of type \((1,2)\) (so \( q(F_1) = 0 \) by \([1]\)). Set \( H_B = s^* H_{W'} \) with \( H_{W'} = H' |_{W'} \), the restriction of a hyperplane \( H' \) on \( \mathbb{P}^{p_1(X)-1} \).

We assert that \( H_B \cdot \ell_1 = 1 \), or equivalently \( M_1 \cdot F_1 = C_1 \) where \( F_1 \) is a general fiber of \( f \). Indeed, \( K_{F_1} = K_{X'} \mid F_1 = (M_1 + E_1 + E_2) \mid F_1 \geq M_1 \mid F_1 = f^* (H_B \cdot \ell_1) = \sum_{i=1}^{l} C_i \) Here \( C_i \) are fibers of the rational free pencil \( F_1 \to \ell_1 \) and, \( e \geq 1 \) because \( H_B \) is nef and big. Since \( p_1(F_1) = 2 \), we have \( e = 1 \). The assertion is proved.

By the projection formula, \( 1 = s^* H_{W'} \cdot \ell_j = H_{W'} \cdot \ell'_j = H' \cdot \ell'_j \), where \( \ell'_j := s_*(\ell_j) \). So \( \ell'_j \) is irreducible and smooth; indeed it is a line in \( \mathbb{P}^{p_1(X)-1} \); also \( \ell_j \to \ell'_j \) is birational and finite and hence an isomorphism.

Conversely, suppose that \( H_B = s^* H_{W'} \geq \ell_1 + \ell_2 \) where \( \ell_j \) are generic irreducible members in a free pencil \( \Lambda \), the movable part of \( s^* \Lambda \) (on the surface \( B \); here \( B \) and \( X' \) are further blown up if necessary) parametrized by a smooth curve \( \tilde{B} \). Then \( M_1 = f^* H_B \geq F_1 + F_2 \), where the surface \( F_j := f^* \ell_j \) is again parametrized by \( \tilde{B} \). We have only to show that \( F_j \) is of type \((1,2)\). Let \( \sigma : F = F_j \to F_0 \) be the smooth blow down to a minimal model.
We assert that $C \cdot \sigma^* F_0 = 1$ with $C$ a general fiber of $f|_{F_j} : F_j \to \ell_j = f(F_j)$ (and also a general fiber of $f$). By Theorem 4.1, $C \cdot \pi^* K_X = 1$. So the assertion is clear. If $g(\tilde{B}) = 0$, then by Lemma 3.7, we have $1 \leq \sigma^* K_{F_0} \cdot C \leq \frac{3}{2} \pi^* K_X \cdot C = 3/2$. Hence the assertion is true. Now the assertion and the argument in Claim 5.4 imply that $F$ is of type $(1, 2)$. This proves the theorem.

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