Electromagnetic responses in superconductors provide valuable information on the pairing symmetry as well as physical quantities such as the superfluid density. However, at the superconducting gap energy scale, optical excitations of the Bogoliubov quasiparticles are forbidden in conventional Bardeen-Cooper-Schrieffer superconductors when momentum is conserved. Accordingly, far-infrared optical responses have been understood in the framework of a dirty-limit theory by Mattis and Bardeen for over 60 years. Here we show, by investigating the selection rules imposed by particle-hole symmetry and unitary symmetries, that intrinsic momentum-conserving optical excitations can occur in clean multi-band superconductors when one of the following three conditions is satisfied: (i) inversion symmetry breaking, (ii) symmetry protection of the Bogoliubov Fermi surfaces, or (iii) simply finite spin-orbit coupling with unbroken time reversal and inversion symmetries. This result indicates that clean-limit optical responses are common beyond the straightforward case of broken inversion symmetry. We apply our theory to optical responses in FeSe, a clean multi-band superconductor with inversion symmetry and significant spin-orbit coupling. This result paves the way for studying clean-limit superconductors through optical measurements.
Optical studies have been very important in superconductivity research since the superconducting gap was first observed by far-infrared optical measurements. Not only does the optical absorption gap directly reveal the superconducting gap size, but also the loss of spectral weight of the optical conductivity in the superconducting transition shows the superfluid density. Optical responses in superconductors are well understood by the Mattis–Bardeen theory because Bogoliubov quasiparticles cannot be excited by uniform light when momentum is conserved in the Bardeen–Cooper–Schrieffer (BCS) model. In this paradigm, optical responses are due to impurity scattering and correspond to the Drude responses remaining in the superconducting state. They are thus completely described within a single-band model such as the BCS model and approaches the Drude formula as the photon energy increases above the gap.

On the other hand, there have been cumulative studies revealing the relevance of multiband effects in superconductivity. Strong gap anisotropy and multiple gap signatures due to orbital-dependent pairing have been observed in various superconductors, including elemental metals Nb, Ta, V, and Pb, compound MgB$_2$, strontium titanates, iron pnictides and chalcogenides, and heavy fermion compounds. Multiband effects are also considered to be important in the superconductivity of strontium ruthenates, some half-Heusler compounds with orbitally dependent pairing have been observed in various superconductors. We discuss the optical response of superconducting FeSe, which is closest to this crossover regime.

Results

Setting. Our theory is based on the mean-field theory of superconductors. We assume uniform illumination of light at zero temperature and the conservation of momentum. In momentum

![Image](https://doi.org/10.1038/s41467-021-21905-x)
space, the single-particle mean-field Hamiltonian has the Bogoliubov-de Gennes (BdG) form

$$H(k) = \begin{pmatrix} h(k) & \Delta(k) \\ -\Delta^*(k) & -h^T(-k) \end{pmatrix}$$

(1)

in the basis of the Nambu spinor defined by \( \Psi = (\hat{c}_{\uparrow} e^{i\theta}, \hat{c}_{\downarrow} e^{i\theta}) \), where \( \hat{c}_{\downarrow} \) is the electronic quasiparticle annihilation operator with orbital \( \rho \) and spin \( s = \uparrow, \downarrow \) indices. Here, \( h(k) \) is the normal-state Hamiltonian, and the pairing function \( \Delta(k) \propto (\xi_{\rho} \rho_{\rho} - k) \) satisfies \( \Delta^*(k) = -\Delta^T(-k) \) due to Fermi statistics of electrons. The BdG Hamiltonian always has particle-hole symmetry \( CH(k)C^{-1} = -H(-k) \) under \( C = \tau_y K \), where \( \tau_y \) is a Pauli matrix for the particle-hole indices and \( K \) is the complex conjugation operator.

The electromagnetic field couples to the normal-state Hamiltonian through the minimal coupling \( \kappa \rightarrow k + \xi A \), where \( \theta = -e \) (or \( +e \)) for the electron (hole) sector. It follows that the velocity operator is

$$V^a(k) = \frac{1}{\hbar} \frac{\partial H}{\partial A_a} |_{A=0} = \frac{1}{\hbar} \left( \begin{array}{cc} \partial_{k_a} h(k) & 0 \\ 0 & \partial_{k_a} [h^T(-k)] \end{array} \right).$$

Matrix elements of this operator are important in our analysis because they describe the transition amplitudes. In the clean limit, the real part of the optical conductivity tensor is given by

$$\sigma^{\text{ee}}(\omega) = \frac{\pi e^2}{2\hbar} \int \sum_{m,n} f_{nm}(n) V^c_{nm}(k) V^a_{mn}(k) \delta(\omega - \omega_{nm}(k)),$$

(3)

where \( \omega \) is the frequency of light, \( f_{nm} = f_n - f_m \) is the difference between the Fermi distribution of the nth band \( f_n \), \( V^c_{nm} = \langle m|V^c|n\rangle \), and \( \omega_{nm} = \omega_m - \omega_n \) where \( H|n\rangle = \hbar \omega_n |n\rangle \). The delta function is replaced by the Lorentzian distribution when the mean free path is finite.

**Selection rules.** Equation (3) is positive-semidefinite when \( c = a \). Therefore, interband transitions are completely forbidden only when symmetries impose \( V^a_{mn}(k) = 0 \) at every \( k \). The relevant symmetry operators should be \( k \)-local (\( k \rightarrow -k \)). A unitary symmetry imposes selection rules by \( \lambda_m(k) = \lambda(k) \lambda_n(k) \), where \( \lambda_m \) and \( \lambda_n \) are symmetry eigenvalues of \( m, n \) states, and the velocity operator, respectively. We always have \( \lambda_V = 1 \) because \( k \)-local symmetry operations leave \( V^a \) invariant, as one might expect because the velocity operator should transform like \( k \). The selection rules thus simply become

$$V^a_{mn}(k) = 0 \quad \text{when} \quad \lambda_m(k) \neq \lambda_n(k),$$

(4)

meaning that optical excitations are forbidden between two different eigenspaces. Fig. 2a).

Let us consider the transition between two states in the same eigenspace of the unitary symmetry group. The remaining \( k \)-local symmetries come in three types: anti-unitary \( \Sigma \), anti-unitary anti-symmetry \( C \), and unitary anti-symmetry \( S \), where anti-symmetry means that the operator anti-commutes with the Hamiltonian. They form ten EAZ symmetry classes shown in Table 1. \( C \) or \( S \) comes as a combination of \( T \) (or \( C \)) with a \( k \)-reversing unitary operator such as spatial inversion \( P \) in any dimensions or twofold rotation \( C_{2z} \) in two dimensions. \( S \) is the combination \( CE \) up to a phase factor. We find that only \( C \)-type symmetry can additionally exclude transition channels within an eigenspace of the unitary symmetry group. By using that \( C \) is anti-unitary and that the velocity operator is invariant under \( C \) as shown in the “Methods” section 1, we have

$$\langle C \cdot nk|V^a(k)|nk\rangle = 0 \quad \text{when} \quad C^2 = -1.$$  

(5)

This constrains, in particular, the lowest-energy excitations, as illustrated in Fig. 2b, c. If bands are nondegenerate in each eigenspace, Eq. (5) indicates that the excitations across the gap are forbidden when \( C^2 = -1 \) [Fig. 2b]. See class \( C \) and \( CII \) in Table 1.

We find that the absence of optical excitations in single-band metal models, described by a two-band BdG Hamiltonian, can be attributed to the existence of \( C = \tau_y K \) symmetry. A single-band metal has the \( C \)-symmetry in the superconducting state, independent of the pairing symmetry, when it has symmetry \( \xi(k) = \xi(k) \) in the normal state, where \( h(k) = \xi(k) \) is the \( 1 \times 1 \) Hamiltonian. Since the formation of Cooper pairs at the Fermi level requires such a symmetry relating \( k \) and \( -k \), it means that typically no optical excitations can occur in superconductors originating from single-band metals. One can extend this result to show the absence of optical excitations in multiband systems satisfying a generalized single-band pairing condition, the so-called zero superconducting fitness condition (see section 2 in the “Methods”).

We have three ways of generating nontrivial optical excitations in an eigenspace. When bands are nondegenerate within an eigenspace, one can (i) break \( C \)-symmetry (EAZ class A, AI, AII, and AIII) or (ii) realize \( C^2 = +1 \) (class D and BDI). (iii) Or, when bands are Kramer degenerate due to \( C \)-symmetry satisfying \( \Sigma^2 = -1 \), lowest-energy excitations are generally allowed irrespective of the sign of \( C^2 \) (class DIII and CII). The first condition (i) just means breaking inversion symmetry when other unitary symmetries do not exist, which was demonstrated in ref. 30. The second (ii) implies that the superconductor may host stable BFSs. Secondly, \( C^2 = +1 \) protects \( 0D \) \( Z \) topological charges, \( Z \)-stable nodal surfaces/lines/points in 3D/2D/1D can appear after superconducting pairing on the Fermi surfaces, respectively, which we call as BFSs without distinguishing their dimension. Let us note that these are twofold degenerate BFSs. On the other hand, a stable nondegenerate BFS can appear in the EAZ classes A, AI, and AII. For instance, a superconductor with broken inversion and time-reversal symmetries can host stable BFSs. Their stability is guaranteed by the change of the number of occupied BdG bands across the BCS, which is a \( Z \)-topological charge. Since these classes correspond to the case (i), the symmetry protection of the stable BFSs, whether it is twofold degenerate or not, indicates that the lowest-energy optical excitations are possible. The last possibility (iii) is realized in T-

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**Table 1 Tenfold way classification of the lowest optical excitations in superconductors.**

| EAZ class | \( Z^2 \) | \( C^2 \) | \( S^2 \) | Lowest excitation | BFS stability |
|-----------|----------|----------|----------|-----------------|--------------|
| A         | 0 0 0 0  | Yes*     | Z        |                 |              |
| AI        | 1 0 0 0  | Yes*     | Z        |                 |              |
| All       | 1 -1 0 0 | Yes*     | Z        |                 |              |
| AllII      | 0 0 1 1  | Yes      | Z        |                 |              |
| D         | 0 1 0 1  | Yes      | \( Z_2^* \) |                 |              |
| BDI       | 1 1 1    | Yes      | \( Z_2 \) |                 |              |
| C         | 0 -1 -1 0| No        | 0        |                 |              |
| CI        | 1 -1 1 0 | No        | 0        |                 |              |
| DIII      | -1 1 0 1 | Yes      | Z        |                 |              |
| CII       | -1 -1 1 1| Yes      | Z        |                 |              |

Anti-unitary \( \Sigma \), anti-unitary anti-symmetry \( K \), and unitary anti-symmetry \( S \) operators that do not change the momentum define ten effective Altland-Zirnbauer (EAZ) symmetry classes at a given generic momentum. \( O \) in the second set of columns indicates that no corresponding symmetry exists within the eigenspace of interest. When both \( Z \) and \( \Sigma \) symmetries exist, \( \Sigma^2 = Z \). In classes A, AI, and All, the lowest possible excitation energy within an eigenspace may correspond to the direct superconducting gap, because the states with the lowest possible and the highest negative energies may have different symmetry eigenvalues. The asterisk (*) in the third column means that the excitation cannot occur when there is only one band in an eigenspace. The last column shows the stability of Bogoliubov Fermi surfaces (BFSs), meaning nodal surface/line/point in 3D/2D/1D superconductors.
**Fig. 2** Selection rules in clean superconductors. (a) Selection rule by unitary symmetry. \( \lambda_{1,2} \) are eigenvalues of a \( \mathbf{k} \)-local unitary symmetry operator. No optical excitation occurs between two states with different eigenvalues. (b) Selection rule by \( \mathcal{C} \) symmetry. The case with \( \mathcal{C} = \text{PT} \) is shown. No optical transition occurs between PC-related states when \( (\text{PT})^2 = -1 \). (c) Optical excitation channels in \( \mathcal{C} \)-symmetric superconductors in the clean limit. At low photon energies comparable to the superconducting gap \( 2\Delta \), the relevant excitations are spectrum-inversion-symmetric (SIS) ones, i.e., from energy \( -E \) to \( E \). For nondegenerate bands, they are transitions between \( \mathcal{C} \)-related pairs. (d, e) Optical excitations in spin-degenerate systems with and without spin–orbit coupling, respectively. (Here, the order of \( d \) and \( e \) has been changed in order to match the label in the figure.) In \( d \), \( \mathcal{C} \) is PT symmetry with \( (\text{PT})^2 = -1 \) imposes Kramers degeneracy. As a state \( |n\rangle \) can be excited to one of two SIS states, \( |PC|n\rangle \) and \( |PT|n\rangle \), the excitation from \( |n\rangle \) is possible even when one transition channel, from \( |n\rangle \) to \( |PC|n\rangle \), is blocked by \( (\text{PC})^2 = -1 \). The same applies to the excitation from \( |PT|n\rangle \). In \( e \), the boxes labeled up and down indicate the spin up and down eigenspace \( (\lambda_{\uparrow} = h/2 \) and \( -h/2 \), respectively). Since \( C \) reverses the spin (the anti-particle of a spin-up electron carries the down spin) while \( P \) does not change the spin, \( PC \) reverses the spin. Its combination with the spin rotation around the \( y \)-axis, which is \( \sigma_y \), for spin-singlet pairing, acts within a sector. For spin-triplet pairing, the spin rotation around the \( y \)-axis acts on the particle and hole sector with an opposite sign due to the spin carried by the Cooper pair, so the additional \( \sigma_y \) is introduced (see section 3 in the “Methods”). Optical excitations are forbidden when \( \mathcal{C} \) defined within a spin sector, which is \( -i\sigma_y \), for singlet pairing \(-i\sigma_x \sigma_y \) for triplet pairing), satisfies \( \mathcal{C}^2 = -1 \). \( E_F \) is the Fermi level in all figures.

**Crossover to clean-limit optical responses.** In real materials, the disorder is present even in very clean samples. We thus need to compare the magnitude of the disorder-mediated and intrinsic responses to characterize the detectability of the latter. We estimate the magnitude of the intrinsic response by counting the dimension of the conductivity tensor in Eq. (3), which gives \( \sigma_{\text{int}}(\omega) \approx \frac{c_0}{\hbar} \frac{(2\Delta)^2}{E_F} \alpha^2 \) above the superconducting gap, where \( k_F \) and \( E_F \) are the Fermi wave number and Fermi energy. Here, \( 0 \leq \alpha \leq 1 \) is the ratio between the dominant pairing \( \Delta \) and the pairing that are responsible for the optical conductivity. Comparing this with the disorder-mediated response, we obtain

\[
\frac{\sigma_{\text{int}}(\omega)}{\sigma_{\text{dis}}(\omega)} \sim \frac{\omega}{2\Delta} \frac{(2\Delta)^2}{E_F} \alpha^2
\]

above the superconducting gap, where we use that \( \sigma_{\text{dis}}(\omega) = \sigma_0(\omega)^{5,7} \) as shown in Fig. 1d, where \( \sigma_0 \) is the Drude conductivity in the normal state (see section 5 in the Methods). We thus find that \( \sigma_{\text{int}} \geq \sigma_{\text{dis}} \) at \( \omega \sim 2\Delta \) when

\[
x \equiv \frac{1}{\lambda_{z}} > x_c = (k_F \xi_0)^2 \alpha^{-2}.
\]

Since \( x_c \gg 1 \) in general because \( k_F \xi_0 \sim E_F / \Delta \gg 1 \) (\( E_F / \Delta \) is about \( 10^4 \) for pure metals and \( 10^2 \) for most unconventional superconductors), the clean limit for optical responses is realized in samples much cleaner than that are usually thought to be clean, just satisfying \( x > 1 \). This explains how the Mattis–Bardeen-type theories have successfully calculated optical conductivity even in clean superconductors.

**Application to FeSe.** In FeSe, however, intrinsic optical responses can make up a significant portion of the observed signal in far-infrared optical measurements. FeSe is a clean quasi-two-dimensional material that has a remarkably large ratio \( \Delta / E_F \gtrsim 0.1 \) with significant spin–orbit coupling comparable to the Fermi energy \( E_F \) and strongly orbital-dependent pairing \( 14 \), such that \( \alpha \sim 1 \) is expected. It, therefore, satisfies all the requirements for significant intrinsic optical responses. Here, we use the low-energy model of FeSe in ref. \( 39 \) to demonstrate our theory, focusing on the Fermi surface near \( \Gamma = (0, 0) \) for simplicity (see section 6 in the “Methods”). We consider six constant pairing functions \( \Delta_1, \Delta_2, \Delta_3, \Delta_{4a}, \Delta_{4b} \), and \( \Delta_5 \) that preserve time-reversal symmetry whose matrix forms and symmetries are given in the “Methods” and Table 2. All of
them have even parity, but spin-triplet pairing can occur due to their multi-orbital nature \((\Delta_3\) and \(\Delta_{4a,4b})\). As we show in the "Methods" section 7, for even parity pairing, optical transitions are not forbidden within each \(M_i\) eigenspace in spin–orbit coupled systems. For \(\Delta = 1,2,3,5\) pairing, \(\mathcal{C}\) symmetry does not exist within a mirror sector. On the other hand, for \(\Delta = 4a,4b\) \(\mathcal{C}\) symmetry exists but satisfies \(C^2 = 1\) in each mirror sector. In accordance with our analysis, all multi-band pairing \(\Delta = 2,3,4a,4b,5\) allow for non-zero optical responses [Fig. 3a–d]. In the case of \(\Delta = 4a,4b\) and \(\Delta_5\) pairing, optical conductivity tensors are non-zero down to zero frequency because of their gapless spectrum due to the BFS and Dirac points, respectively [Fig. 3d, e].

In experiments, a highly anisotropic pairing gap was observed\(^{4,15}\) having a sinusoidal shape with \(2\sim 3\) meV peak at \(k_y = 0\) and almost zero deep at \(k_y = 0\). Supposing that the gap function belongs to the trivial representation of the symmetry group, we can obtain a similar anisotropic gap with various combinations of \(\Delta_3\), \(\Delta_4\), and \(\Delta_5\). For example, we obtain (i) \(\Delta_1 = 4.09 \text{ meV}, \Delta_2 = 4.82 \text{ meV}, \text{ and } \Delta_3 = 1.93 \text{ meV}\) and (ii) \(\Delta_1 = 8.98 \text{ meV}, \Delta_2 = 9.39 \text{ meV}, \text{ and } \Delta_3 = 0 \text{ meV}\), respectively, by least-square-fitting with and without spin–orbit coupled pairing \(\Delta_5\) to the function \(\Delta(\theta) = 2.06 + 1.42 \cos(2\theta) – 0.44 \cos(4\theta)\) (shown as a black curve) that was obtained in ref.\(^{15}\) from experimental data.

Conductivity with pairing functions used in (e), \(\sigma_{ie}\) is the internal optical conductivity in the superconducting state (solid lines), and \(\sigma_{int}\) is the Drude conductivity in the normal state (dashed lines). The disorder-mediated conductivity in the superconducting state is expected to be comparable to \(\sigma_{int}\).

**Discussion**

Our theory establishes the existence of the true clean-limit optical responses beyond the Mattis–Bardeen theory in multiband superconductors. While we focus on linear responses in the current work, our classification of optical transitions applies to nonlinear optical responses also\(^{30}\). Since nonlinear optical conductivity tensors have more components than the linear counterpart, they give richer information on the symmetry of the system. For instance, it is hard to detect inversion symmetry breaking from linear optical responses. On the other hand, since second-order optical responses are allowed only when inversion symmetry is broken, they directly reveal the presence of inversion symmetry\(^{30}\).

As such, various optical measurements can be used in the study of clean multiband superconductors. We anticipate an immediate impact of our work on the optical study of the exotic superconductivity in FeSe. Furthermore, our theory may be relevant to the recently discovered 2D superconductivities reaching \(\Delta/\varepsilon_F > 0.1\) in twisted trilayer graphene\(^{46,41}\) and ZrNCl\(^{42}\). As the synthesis of extremely clean superconductors advances further, our results will become relevant to more materials.

**Methods**

**Selection rule by \(\mathcal{C}\) symmetry.** Equation (5) can be simply derived as follows.

\[
\langle \mathcal{C} \cdot nk | V^\text{in}(k) | nk \rangle = \langle \mathcal{C} V^\text{in}(k) \cdot nk | \mathcal{C} \cdot nk \rangle = \mathcal{C} \cdot nk | V^\text{in}(k) \cdot \mathcal{C}^{-1} | nk \rangle
\]

\[
= \epsilon_{kV} \mathcal{C} \cdot nk | V^\text{in}(k) | \mathcal{C} \cdot nk \rangle
\]

We use that \(\mathcal{C}\) is a anti-unitary operator in the first line, use that \(\mathcal{C}^2 = \pm 1\) is a number, and \(V^\text{in}\) is Hermitian in the second line, and define \(\epsilon_{kV} = \pm 1\) by \(\mathcal{C} V^\text{in}(k) | \mathcal{C}^{-1}\rangle = \epsilon_{kV} V^\text{in}(k)\) in the third line. Let us recall that \(V^\text{in}(k) = i \tau_z h(k)\) for a BdG Hamiltonian \(H(k)\). \(\mathcal{C}\) anti-commutes with \(\tau_z\), because \(C\) anti-commutes with \(\tau_z\) while a physical unitary operator \(U_U\) that combine to define \(C = U_U C\) commutes with \(\tau_z\). Since the \(\mathcal{C}\) symmetry condition imposes \(\mathcal{C} H(k) | \mathcal{C}^{-1}\rangle = -H(k)\), we obtain \(\epsilon_{kV} = 1\), i.e. \(\mathcal{C}\) commutes with \(V(k)\). Equation (5) then follows.

We note that normal-state systems with an emergent \(\mathcal{C}\) symmetry follow a different selection rule because they satisfy \(\tau_v \mathcal{C} = -\mathcal{C} \tau_v\). The difference comes from the fact that, in the normal state, all quasi-particles are electronic quasi-particles that couple to the gauge field with the equal charge \(-e\) (i.e., the emergent \(\mathcal{C}\) does not reverse the gauging charge). In this case, the velocity operator is \(V(k) = \partial_k h(k)\),...
where $h$ is the $C$-symmetric normal-state Hamiltonian, so that $\mathcal{G}(k)|\xi_0\rangle = \delta_{\xi_0} \xi_0(k) |\xi_0\rangle = -v_F(k)$. The selection rule is then $\xi_0(k) v_F(k) |\xi_0\rangle = 0$ when $\xi_0 = 1$, which is opposite to the superconducting case.

**Optical excitations with a single-band condition.** Let us suppose that the normal state is described by a single band, i.e., $h(k) = \xi_0(k)$ in a 1 x 1 matrix. We consider the normal state having time-reversal symmetry or inversion symmetry (or other symmetries whose action is equivalent to them) because only then pairing between two electrons $e$ and $c_\alpha$ effectively occurs at the Fermi level. Then, $\xi(k) = \xi_0(k)$ such that $\mathcal{G}(k) = h_d(k) \xi_0(k)$ has vanishing inter-band matrix components for all $k$, where $\xi_0$ is the 2 x 2 identity matrix with the particle–hole indices. It follows that superconductivity in a single-band metal cannot exhibit nontrivial optical conductivity in the clean limit. The same is true when the band has spin degeneracy at every $k$, because $h(k) = \xi_0(k)$ is again proportional to the identity matrix such that the velocity operator is diagonal.

These constraints can be understood from the $C$ symmetry. Let us note that no identity term appears in the BdG Hamiltonian because we consider $\xi(k) = \xi_0(k)$. Thus, general two-band BdG Hamiltonian takes the form $H = g_d k_d + g_s k_s + \text{tr}$. It always satisfies $\mathcal{G} H(k)|\xi_0\rangle = -H|\xi_0\rangle$ for $\xi_0(k)$ such that $\xi_0 = 1$. This symmetry blocks optical transitions by Eq. (5). Let us consider the case where the normal states, such that $\xi_0(k) = \xi(k) = \xi_0(k)$ and $\xi^\pm = \xi_0(k)$, is the 2 x 2 identity matrix with the particle–hole indices. It effectively occurs at the Fermi level. Then, after we take $\xi_0 = \xi_0(k)$ and $\xi^\pm = \xi_0(k)$, it is proportional to the identity matrix. After we take $\xi_0 = \xi_0(k)$ and $\xi^\pm = \xi_0(k)$, it is proportional to the identity matrix.

Thus, assuming nondegenerate states, we immediately see that all transitions from $\xi_0$ to $\xi_0$ are forbidden.

**Optical excitations with spin rotation symmetries.** Let us consider the EAZ classes within spin rotation sectors for example. We assume that time reversal, spatial inversion, a spin U(1) rotation and a spin $\pi$ rotation (around an axis perpendicular to the U(1) axis) symmetries are all present. Let us take energy eigenstates such that they carry a definite spin along the $z$ direction. In the case of triplet pairing, this means that we take the $z$-direction as the triplet spin direction because continuous spin rotation symmetries around other directions are broken. Within each spin sector, bands are nondegenerate at generic momenta. As we show in Fig. 2d, CP flips the spin because of particles-hole conjugation. However, the combination of spin $\pi$ rotation, which is $-i\sigma_y$ for singlet pair ($-i\sigma_x$ for triplet pair), and PC acts within a spin sector, so symmetry under this $C$-type operation constraints optical excitations through Eq. (5). Optimal excitations between related pairs are allowed when $\xi^\pm = \xi_0(k)$, which is $\xi^\pm = \xi_0(k)$, respectively, for singlet and triplet pairing, where BfIs are stable. Since (PC)$^2 = +1$ for even- and odd-parity pairing (see section 4 in the “Methods” below), this requires odd-parity singlet pairing or even-parity triplet pairing, which is possible only when multiple-orbital pairing, e.g., an orbital triplet, is realized. Alternately, if inversion symmetry is broken or spin-orbit coupling is not negligible, optical excitations are allowed with fully gapped superconductivity. Let us note that $s$-spin-preserving spin-orbit coupling is enough to allow optical excitations, because it breaks spin rotation symmetries around other axes such that $\xi_0$ is stable. In each spin sector, general, spin–orbit coupling breaks all spin rotation symmetries, so two excitation channels are allowed [Fig. 2e].

**Symmetry operator and pairing symmetry.** Let $\mathcal{U}_g$ be a unitary operator that acts on space as $\xi_0 \rightarrow \xi_0$. Suppose that it is a symmetry operator of the normal state, i.e., $\mathcal{U}_g h(k) |\xi_0\rangle = h(k)|\xi_0\rangle$, and the pairing function has eigenvalues $\epsilon^{\pm}$ under $\mathcal{U}_g$, i.e., $\mathcal{U}_g |\Phi^{\pm}\rangle = \epsilon^{\pm}|\Phi^{\pm}\rangle$. Due to the non-trivial transformation of the pairing function, the BdG Hamiltonian is symmetric under

$$U_g = \begin{pmatrix} u_2 & 0 \\ 0 & e^{i\theta_1} u_1 \end{pmatrix},$$

which rotates the hole sector by $e^{i\theta_1}$ more

$$U_g H(k) U_g^{-1} = H(k),$$

and $U_g$ satisfies the commutation relation with the particle–hole conjugation operator $C$

$$U_g C = e^{i\theta_1} C U_g.$$

Let us take two examples.

1. $U_g = P$ is spatial inversion: $e^{i\theta_1} = 1$ and $-1$ indicates even-parity and odd-parity pairing. Thus, $PC = +CP$ (PC = -CP) for even-parity (odd-parity) pairing.

2. $U_g$ is a spin rotation around the $y$-axis by $\pi$: $e^{i\theta_1} = +1$ always for a spin-singlet pairing, and $e^{i\theta_1} = -1$ (−1) when the pairing function is a spin-triplet with its spin parallel (perpendicular) to the $y$-axis.

**Estimates of disorder-mediated and intrinsic responses in the clean regime.** The disorder-mediated response in the superconducting state is comparable to the Drude response in the normal state. When the light frequency $\omega$ is much larger than the inverse relaxation time $\Gamma$, which is the case in the clean regime $\omega \ll \Delta \ll \hbar \omega$, the Drude conductivity is $\sigma_{\alpha\alpha}(\omega) = \frac{e^2}{\hbar \omega} \approx \sigma_{\alpha\alpha}$. Here, $\sigma_{\alpha\alpha} \approx \frac{e^2}{\hbar \omega} \approx \frac{e^2}{\hbar \omega} (-\hbar^2 \omega) \Gamma$, and $\Gamma \approx \hbar^2 \omega$. Since $\sigma_{\alpha\alpha}$ and $\sigma_{\alpha\alpha}$, we have

$$\sigma_{\alpha\alpha}(\omega) \sim \frac{e^2}{\hbar \omega} \approx \frac{e^2}{\hbar \omega} \approx \frac{e^2}{\hbar \omega} \Gamma^2,$$

where we use $E_F = h v_F k_F$, $\Delta = h v_F \xi_0$, and $\Gamma = h v_F \xi_0^\alpha$. To estimate the intrinsic response, let us note that the inter-band velocity operator is linear in the leading order of $\Delta / E_F$, where $\Delta$ is the largest multiband pairing that is allowed to generate interband transitions by the selection rules in Eqs. (4) and (5). The conductivity tensor in Eq. (3) is thus proportional to $\Delta^2$. 

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Using that the delta function has the dimension $\omega^{-1}$, and using the Fermi energy and wave number as other scales, we obtain

$$\sigma_{m}(\omega) \sim \frac{e^2}{h} \frac{1}{\hbar} \frac{1}{E_F} \frac{1}{\omega^2 a^2},$$

where $a = |\Delta|/\Delta$ is equal to or smaller than one since $\Delta$ is the dominant pairing strength by definition. It follows that

$$\sigma_{m}(\omega) \sim \frac{e^2}{h} \frac{1}{\hbar} \frac{1}{E_F} \frac{1}{\omega^2 a^2},$$

above the superconducting gap frequency.

### FeSe model at the $\Gamma$ point

If we regard FeSe as a 2D system, it has three Fermi surfaces around $\Gamma = (0,0)$, $X = (\pi,0)$, and $Y = (0,\pi)$, respectively, in the 1Fe Brillouin zone. Here, we consider the Fermi surface near $\Gamma$. The Fermi surface of FeSe at $\Gamma$ consists mainly of two orbital degrees of freedom $\delta_{\alpha n}$ and $\delta_{\alpha n}$, so we take $\nu_{\alpha}(k)$ as the basis state. At zero temperature, the normal-state Hamiltonian has the form

$$H_1 = H_0 + h_{\text{soc}} + h_{\text{nem}}.$$

### Optical excitations in $M_r, P$, and $T$-symmetric 2D superconductors

In two-dimensional systems perpendicular to the $z$ axis, $M_r$ symmetry divides eigenstates into two distinct eigenstates with $M_r$ eigenvalues $\lambda = \pm 1$. It imposes a selection rule.

Let $M_r C = \eta M_{C}, \quad \eta = \pm 1$, and $M_r n = \lambda_n |n\rangle$. Then, $M_r \frac{PC}{n} \frac{PC}{n} = \eta \frac{PC}{n} \frac{PC}{n} = \eta \frac{PC}{n} \frac{PC}{n}$. Thus, $|n\rangle$ and $PC|n\rangle$ have the different eigenvalues when $\eta = 1$ and has the same eigenvalues when $\eta = -1$. In the former case, the optical transition from $|n\rangle$ to $PC|n\rangle$ is forbidden by $M_r$ symmetry because they are indifferent eigenstates (and the velocity operator does not change the eigenstate), but the transition from $|n\rangle$ to $S|n\rangle$ is allowed. On the other hand, in the latter case, the transition from $|n\rangle$ to $PC|n\rangle$ within the same mirror sector is forbidden when the pairing is odd-parity such that $PC|n\rangle = -|n\rangle$. In summary, optical transitions between particle-hole- and chiral-related states are forbidden in $P$ and $T$-symmetric systems when $M_r C = -CM_r$ and $PC = -CP$. Similar constraints can appear in one-dimensional systems also due to mirror symmetries.

### Data availability

The data support that the findings of this study are available from the corresponding author upon reasonable request.

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Author contributions
J.A. conceived the original idea and performed the theoretical analysis. N.N. supervised the project and noticed the relevance of the theory to FeSe. Both authors discussed the data and wrote the paper.

Competing interests
The authors declare no competing interests.

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