SECTIONAL CATEGORY AND THE FIXED POINT PROPERTY

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Abstract. In this work we exhibit an unexpected connection between sectional category theory and the fixed point property. On the one hand, a topological space \( X \) is said to have the fixed point property (FPP) if, for every continuous self-map \( f \) of \( X \), there is a point \( x \) of \( X \) such that \( f(x) = x \). On the other hand, for a continuous surjection \( p : E \to B \), the standard sectional number \( \text{sec}_{\text{op}}(p) \) is the minimal cardinality of open covers \( \{U_i\} \) of \( B \) such that each \( U_i \) admits a continuous local section for \( p \). Let \( F(X,k) \) denote the configuration space of \( k \) ordered distinct points in \( X \) and consider the natural projection \( \pi_{k,1} : F(X,k) \to X \). We demonstrate that a space \( X \) has the FPP if and only if \( \text{sec}_{\text{op}}(\pi_{2,1}) = 2 \). This characterization connects a standard problem in fixed point theory to current research trends in topological robotics.

1. Introduction

A topological space \( X \) has the fixed point property (FPP) if, for every continuous self-map \( f \) of \( X \), there is a point \( x \) of \( X \) such that \( f(x) = x \). We address the following natural question:

Can the FPP be characterized in the category of Hausdorff spaces and continuous maps?

Such characterizations are known in smaller, more restrictive categories. We note that it is known that, for instance, Fadell proved in 1969 (see [4] for references) that, in the category of connected compact metric ANRs:

- If \( X \) is a Wecken space, \( X \) has the FPP if and only if \( N(f) \neq 0 \) for every self-map \( f : X \to X \).
- If \( X \) is a Wecken space satisfying the Jiang condition, \( J(X) = \pi_1(X) \), then \( X \) has the FPP if and only if \( L(f) \neq 0 \) for every self-map \( f : X \to X \).

In this work we characterize the FPP (see Theorem 3.13):

- within the category of Hausdorff spaces.
- in terms of sectional number.

As shown in Section 5 a particularly interesting feature of our characterization comes from its connection to current research trends in topological robotics.

2. Root theory

In this section we give a brief exposition of standard mathematical topics in Root theory: the minimal root number and the Nielsen root number \( NR(f, a) \).

2010 Mathematics Subject Classification. Primary 55M20, 55R80, 55M30; Secondary 68T40.

Key words and phrases. Fixed point property, Configuration spaces, Sectional category, Motion planning problem.

The first author would like to thank grant#2018/23678-6, São Paulo Research Foundation (FAPESP) for financial support.
Our exposition is by no means complete, as we limit our attention to concepts that appear in geometrical and topological questions. More technical details can be found in standard works on root theory, like [1] or [2].

Let \( f : X \to Y \) be a continuous map between topological spaces, and fix \( a \in Y \). A point \( x \in X \) such that \( f(x) = a \) is called a root of \( f \) at \( a \).

In Nielsen root theory, by analogy with Nielsen fixed-point theory, the roots of \( f \) at \( a \) are grouped into Nielsen classes, a notion of essentiality is defined, and the Nielsen root number \( NR(f, a) \) is defined to be the number of essential root classes. The Nielsen root number is a homotopy invariant and measures the size of the root set in the sense that

\[
NR(f, a) \leq MR[f, a] := \min \{ |g^{-1}(a) : g \simeq f| \}.
\]

The number \( MR[f, a] \) is called the minimal root number for \( f \) at \( a \). A classical result of Wecken states that \( NR(f, a) \) is in fact a sharp lower bound in the homotopy class of \( f \) for many spaces, in particular, for compact manifolds of dimension at least 3. Thus, in this case, the vanishing of \( NR(f, a) \) is sufficient to deform a map \( f \) to be root free. Among the central problems in Nielsen root theory (or the theory of root classes) are

- the computation of \( NR(f, a) \),
- the realization of \( NR(f, a) \), i.e., deciding when \( NR(f, a) = MR[f, a] \) holds.

2.1. The Nielsen root number \( NR(f, a) \). We recall from [1] the Nielsen root number \( NR(f, a) \). Let \( f : X \to Y \) be a continuous map between path connected topological spaces, and choose a point \( a \in Y \).

Assume that the set of roots \( f^{-1}(a) \) is non empty. Two such roots \( x_0 \) and \( x_1 \) are equivalent if there is a path \( \alpha : [0, 1] \to X \) from \( x_0 \) to \( x_1 \) such that the loop \( f \circ \alpha \) represents the trivial element in \( \pi_1(Y, a) \). This is indeed an equivalence relation, and an equivalence class is called a root class.

Suppose \( H : X \times [0, 1] \to Y \) is a homotopy. Then a root \( x_0 \in H^{-1}_0(a) \) is said to be \( H \)-related to a root \( x_1 \in H^{-1}_1(a) \) if and only if there is a path \( \alpha : [0, 1] \to X \) from \( x_0 \) to \( x_1 \) such that the loop \( \beta : [0, 1] \to Y \), \( \beta(t) = H(\alpha(t), t) \) represents the trivial element in \( \pi_1(Y, a) \).

Note that a root \( x_0 \) of \( f : X \to Y \) is equivalent to another root \( x_1 \) if and only if \( x_0 \) is related to \( x_1 \) by the constant homotopy at \( f \).

A root \( x_0 \in f^{-1}(a) \) is said to be essential if and only if for any homotopy \( H : X \times [0, 1] \to Y \) beginning at \( f \), there is a root \( x_1 \in H^{-1}_1(a) \) to which \( x_0 \) is \( H \)-related. If one root in a root class is essential, then all other roots in that root class are essential, and we say that the root class itself is essential. The number of essential root classes is called the Nielsen number of \((f, a)\) and is denoted by \( NR(f, a) \). The number \( NR(f, a) \) is a lower bound for the number of solutions of \( f(x) = a \). If \( f' \) is homotopic to \( f \) then \( NR(f, a) = NR(f', a) \). Furthermore, \( NR(f, a) \leq MR[f, a] \).

The order of the cokernel of the fundamental group homomorphism \( f_\# : \pi_1(X) \to \pi_1(Y) \) is denoted by \( R(f) \), that is,

\[
R(f) = \left| \frac{\pi_1(Y)}{f_\#(\pi_1(X))} \right|.
\]

It depends only on the homotopy class of \( f \). There are always at most \( R(f) \) root classes of \( f(x) = a \), in particular, \( R(f) \geq NR(f, a) \).
Lemma 3.6. \cite{13} Let $E$ be a monomorphism. Then $\sec p, a \leq 1$.

Remark 3.5. $H$ appears in our work, we have that $\alpha \in \alpha_{\ast}$.

Remark 3.4. $\alpha$ is a pullback. Then $\sec p, a \leq 1$.

Remark 3.3. $p$ is a continuous surjection. $\alpha$ is a (homotopy) section of the restriction map $p_{1} : p^{-1}(A) \to A$, i.e., a map $s : A \to E$, such that $p \circ s$ is (homotopic to) the inclusion $A \to B$.

We recall the following definitions.

Definition 3.1. (1) The standard sectional number is the minimal number of elements in an open cover of $B$, such that each element admits a continuous local section to $p$. Let us denote this quantity as $\sec_{\text{op}}(p)$.

(2) The sectional category of $p$, denoted $\text{secat}(p)$, (also called Schwarz genus of $p$) is the minimal number of homotopy continuous local sections of $p$.

Remark 3.1. We have $\secat(p) \leq \sec_{\text{op}}(p)$. Furthermore, if $p$ is a fibration then $\sec_{\text{op}}(p) = \secat(p)$.

Lemma 3.2. \cite{13} Let $p : E \to B$ be a continuous surjection and $R$ be a commutative ring with unit. If there exist cohomology classes $\alpha_{1}, \ldots, \alpha_{k} \in H^{*}(B; R)$ with

$$p^{\ast}(\alpha_{1}) = \cdots = p^{\ast}(\alpha_{k}) = 0 \text{ and } \alpha_{1} \cup \cdots \cup \alpha_{k} \neq 0,$$

then

$$\sec_{\text{op}}(p) \geq k + 1.$$

Remark 3.3. Note that in the case when $B$ is path connected (this case will appear in our work), we have that $\alpha \in H^{*}(B; R)$, $\alpha \neq 0$ with $p^{\ast}(\alpha) = 0$ implies $\alpha \in \bar{H}^{*}(B; R)$.

Remark 3.4. Let $p : E \to B$ be a continuous surjection. If $p_{\ast} : H_{\ast}(E; R) \to H_{\ast}(B; R)$ or $p_{\#} : \pi_{\ast}(E) \to \pi_{\ast}(B)$ are not surjective then

$$\sec_{\text{op}}(p) \geq 2.$$

Remark 3.5. Let $p : E \to B$ be a continuous surjection. If $p$ has a section $s : B \to E$, then $p \circ s = 1_{B}$ and $s^{\ast} \circ p^{\ast} = 1_{H^{*}(B; R)}$. In particular, $p^{\ast} : H^{*}(B; R) \to H^{*}(E; R)$ is a monomorphism.

The following statement is well-known.

Lemma 3.6. \cite{13} Let $p : E \to B$ be a continuous surjection. If the following square

$$\begin{array}{ccc}
E' & \longrightarrow & E \\
p' \downarrow & & \downarrow p \\
B' & \longrightarrow & B
\end{array}$$

is a pullback. Then $\sec_{\text{op}}(p') \leq \sec_{\text{op}}(p)$.
Next, we recall the notion of LS category which, in our setting, is one greater than that given in [3]. For example, the category of a contractible space is one.

**Definition 3.2.** The Lusternik-Schnirelmann category (LS category) or category of a topological space $X$, denoted $\text{cat}(X)$, is the least integer $m$ such that $X$ can be covered by $m$ open sets, all of which are contractible within $X$.

We have $\text{cat}(X) = 1$ iff $X$ is contractible. The LS category is a homotopy invariant, i.e., if $X$ is homotopy equivalent to $Y$ (which we shall denote by $X \simeq Y$), then $\text{cat}(X) = \text{cat}(Y)$.

**Lemma 3.7.** [3] Let $p : E \to B$ be a fibration. Then

1. $\text{sec}_{op}(p) \leq \text{cat}(B)$;
2. If $E$ is contractible, then $\text{sec}_{op}(p) = \text{cat}(B)$.

3.1. **Configuration spaces.** Let $X$ be a topological space and $k \geq 1$. The ordered configuration space of $k$ distinct points on $X$ (see [5]) is the topological space

$$F(X, k) = \{(x_1, \ldots, x_k) \in X^k \mid x_i \neq x_j \text{ whenever } i \neq j\},$$

topologised as a subspace of the Cartesian power $X^k$.

For $k \geq r \geq 1$, there is a natural projection

$$\pi_{k,r}^X : F(X, k) \to F(X, r) \quad \text{where } (x_1, \ldots, x_r, \ldots, x_k) \mapsto (x_1, \ldots, x_r)$$

**Lemma 3.8** (Fadell-Neuwirth fibration [4]). Let $M$ be a connected $m$-dimensional topological manifold (without boundary), where $m \geq 2$. Then, the projection

$$\pi_{k,r}^M : F(M, k) \to F(M, r), \quad k > r \geq 1$$

is a locally trivial bundle with fiber $F(M - Q_r, k - r)$. In particular, $\pi_{k,r}^M$ is a fibration.

**Proposition 3.9.** Let $M$ be a connected $m$-dimensional topological manifold (without boundary), where $m \geq 2$. Then, the projection

$$\pi_{k,r}^M : F(M, k) \to F(M, r), \quad k > r \geq 1$$

has Nielsen root number $NR(\pi_{k,r}^M, a) \leq 1$ for any $a \in F(M, r)$.

**Proof.** The map $\pi_{k,r}^M : F(M, k) \to F(M, r)$ is a fibration with fiber $F(M - Q_r, k - r)$. We note that $F(M - Q_r, k - r)$ is path connected. By the long exact homotopy sequence of the fibration $\pi_{k,r}^M$, we have the induced homomorphism $(\pi_{k,r}^M)_\# : \pi_1 F(M, k) \to \pi_1 F(M, r)$ is an epimorphism. Then, $R(\pi_{k,r}^M) = 1$ and thus the Nielsen root number $NR(\pi_{k,r}^M, a) \leq 1$ for any $a \in F(M, r)$. \qed

**Remark 3.10.** Note that $MR(\pi_{k,1}^X, a) = 0$ (in particular $NR(\pi_{k,1}^X, a) = 0$) for any contractible space $X$.

**Proposition 3.11.** [Key lemma] For any $k \geq 2$ and $X$ a Hausdorff space, we have

$$\text{sec}_{op}(\pi_{k,1}^X) \leq k.$$  

**Proof.** Let $(p_1, \ldots, p_k) \in F(X, k)$. For each $i = 1, \ldots, k$, set

$$U_i := X - \{p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k\}$$

...
and \( s_i : U_i \to F(X, k) \) given by

\[
s_i(x) := (x, p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k), \forall x \in U_i.
\]

We note that each \( U_i \) is open (because \( X \) is Hausdorff) and each \( s_i \) is a local section of \( \pi_{k, 1}^X \). Furthermore, \( X = U_1 \cup \cdots \cup U_k \). Thus, \( \text{sec}_{\text{op}}(\pi_{k, 1}^X) \leq k \). \( \square \)

**Definition 3.3.** A topological space \( X \) has the fixed point property (FPP) if for every continuous self-map \( f \) of \( X \) there is a point \( x \) of \( X \) such that \( f(x) = x \).

**Example 3.12.** It is well known that the unit disc \( D^m = \{ x \in \mathbb{R}^m : \| x \| \leq 1 \} \) has the FPP (The Brouwer’s fixed point theorem). The even dimensional projective spaces, \( \mathbb{P}^{2n} \), \( \mathbb{C}P^{2n} \) and \( \mathbb{H}P^{2n} \) have the FPP (see [3]). For the particular case, \( \mathbb{R}P^2 \), see Example 3.18.

Note that the map \( \pi_{2, 1}^X : F(X, 2) \to X \) admits a cross-section if and only if there exists a fixed point free self-map \( f : X \to X \). Thus, we have the following theorem.

**Theorem 3.13.** [Principal theorem] Let \( X \) be a Hausdorff space. The space \( X \) has the FPP if and only if \( \text{sec}_{\text{op}}(\pi_{2, 1}^X) = 2 \).

**Proof.** Suppose \( X \) has the FPP, then \( \text{sec}_{\text{op}}(\pi_{2, 1}^X) \geq 2 \). By the Key Lemma (Proposition 3.11), \( \text{sec}_{\text{op}}(\pi_{2, 1}^X) = 2 \).

Now, suppose \( \text{sec}_{\text{op}}(\pi_{2, 1}^X) = 2 \). In particular, we have \( \text{sec}_{\text{op}}(\pi_{2, 1}^X) \neq 1 \). Hence, \( X \) has the FPP.

**Example 3.14.** No nontrivial topological group \( G \) has the FPP. Indeed, the map \( s : G \to F(G, 2), g \mapsto (g, g_1 g) \) (for some fixed \( g_1 \neq e \in G \)) is a cross-section for \( \pi_{2, 1}^G : F(G, 2) \to G \). The self-map \( G \to G, g \mapsto g_1 g \) is fixed point free.

**Corollary 3.15.** Let \( X \) be a Hausdorff space. If there exist \( \alpha \in H^*(X; R) \) with \( \alpha \neq 0 \) and \( (\pi_{2, 1}^X)^*(\alpha) = 0 \in H^*(F(X, 2); R) \), that is, if the induced homomorphism \( (\pi_{2, 1}^X)^* : H^*(X; R) \to H^*(F(X, 2); R) \) is not injective, then \( \text{sec}_{\text{op}}(\pi_{2, 1}^X) = 2 \). In particular, \( X \) has the FPP.

**Proof.** From Lemma 3.12, \( \text{sec}_{\text{op}}(\pi_{2, 1}^X) \geq 1 + 1 = 2 \). Then, by Proposition 3.11, \( \text{sec}_{\text{op}}(\pi_{2, 1}^X) = 2 \). Thus, the result follows from Theorem 3.13.

**Remark 3.16.** We note that the converse of Corollary 3.15 is not true. For example, we recall that the unit disc \( D^m := \{ x \in \mathbb{R}^m : \| x \| \leq 1 \} \) has the FPP (from the Brouwer’s fixed point theorem) and thus \( \text{sec}_{\text{op}}(\pi_{2, 1}^{D^m}) = 2 \). However, \( \widetilde{H}^*(D^m; R) = 0 \).

**Corollary 3.17.** Let \( X \) be a Hausdorff space. If the induced homomorphisms \( (\pi_{2, 1}^X)^* : H_*(F(X, 2); R) \to H_*(X; R) \) or \( (\pi_{2, 1}^X)^\# : \pi_*(F(X, 2)) \to \pi_*(X) \) are not surjective, then \( \text{sec}_{\text{op}}(\pi_{2, 1}^X) = 2 \). In particular, \( X \) has the FPP.

**Example 3.18.** It is easy to see that \( \pi_2(F(\mathbb{R}P^2, 2)) = 0 \) is trivial and \( \pi_2(\mathbb{R}P^2) = \mathbb{Z} \). Then the induced homomorphism \( (\pi_{2, 1}^{\mathbb{R}P^2})^\# : \pi_3(F(\mathbb{R}P^2, 2)) \to \pi_3(\mathbb{R}P^2) \) is not surjective, and thus \( \text{sec}_{\text{op}}(\pi_{2, 1}^{\mathbb{R}P^2}) = 2 \). In particular, \( \mathbb{R}P^2 \) has the FPP. This part can also be proved by employing Lefschetz’ fixed point theorem.

**Remark 3.19.** For \( k \geq l \geq r \), consider the following diagram
We note that \( \alpha \) exist for any \( x \) variant of \( X \). Similarly, we have (3.1) introduced in [6]. Let \( \pi \) denote the homotopy diagram

\[
\begin{align*}
F(X, k) & \xrightarrow{\pi^X_{k,l}} F(X, l) \\
\pi^X_{r,s} & \xrightarrow{\pi^X_{l,r}} F(X, r)
\end{align*}
\]

It is easy to see that if \( \pi^X_{i} \simeq \text{cte} \), then \( \pi^X_{i} \simeq \text{cte} \) for any \( k \geq l \geq r \). Moreover, we have

\[ MR[\pi^X_{k,r}, a] \geq MR[\pi^X_{k,r}, a] \text{ for any } k \geq l \geq r. \]

**Proposition 3.20.** Let \( X \) be a connected CW complex, and assume one of the following conditions holds:

\begin{enumerate}
  \item \( X \) is non contractible, simply-connected and \( \pi^X_{2,1} \simeq \text{cte} \).
  \item \( MR(\pi^X_{2,1}, x_0) = 0 \) and there exist \( \alpha \in \tilde{H}^*(X; R) \) with \( \alpha \neq 0 \) and \( i^*(\alpha) = 0 \in \tilde{H}^*(X - \{x_0\}; R) \) for some \( x_0 \in X \), that is, \( i^*: \tilde{H}^*(X; R) \to \tilde{H}^*(X - \{x_0\}; R) \) is not injective, where \( i: X - \{x_0\} \hookrightarrow X \) is the inclusion map.
\end{enumerate}

Then \( \text{sec}_{\text{op}}(\pi^X_{2,1}) = 2 \). In particular, \( X \) has the FPP.

**Proof.** i): The assumption that \( X \) is a non contractible simply-connected CW complex implies that there exist \( \alpha \in \tilde{H}^*(X; R) \) with \( \alpha \neq 0 \). From \( \pi^X_{2,1} \simeq \text{cte} \), we have \( (\pi^X_{2,1})^* = 0 \) is trivial. Then, by Corollary 3.13, \( \text{sec}_{\text{op}}(\pi^X_{2,1}) = 2 \).

ii): From \( MR(\pi^X_{2,1}, x_0) = 0 \), there exist a continuous map \( \varphi : F(X, 2) \to X \) such that \( \varphi^{-1}(x_0) = \emptyset \) and \( \varphi \simeq \pi^X_{2,1} \). We have the following commutative (up to homotopy) diagram

\[
\begin{array}{ccc}
F(X, 2) & \xrightarrow{\pi^X_{2,1}} & X \\
\varphi \downarrow & & \uparrow i \\
X - \{x_0\} & & \\
\end{array}
\]

The fact \( \pi^X_{2,1} \simeq i \circ \varphi \) implies \( \varphi^* \circ i^* = (\pi^X_{2,1})^* \). In particular, \( (\pi^X_{2,1})^*(\alpha) = \varphi^* \circ i^*(\alpha) = 0 \). Therefore, there exist \( \alpha \in \tilde{H}^*(X; R) \) with \( \alpha \neq 0 \) and \( (\pi^X_{2,1})^*(\alpha) = 0 \in \tilde{H}^*(F(X, 2); R) \), then \( \text{sec}_{\text{op}}(\pi^X_{2,1}) = 2 \). \( \square \)

**Example 3.21.** For \( \pi^{S^2 \vee S^1}_{2,1}: F(S^2 \vee S^1, 2) \to S^2 \vee S^1 \), we have \( MR[\pi^{S^2 \vee S^1}_{2,1}, x_0] \geq 1 \) for any \( x_0 \in S^2 \vee S^1 \). Indeed, we consider \( S^2 \vee S^1 = S^2 \times b_0 \cup a_0 \times S^1 \). Set \( s: S^2 \vee S^1 \to F(S^2 \vee S^1, 2) \) given by the formulae

\[
\begin{align*}
s(a_0, b) &= ((a_0, b), (-a_0, b)) \text{ for any } b \in S^1 \text{ and} \\
s(a, b_0) &= ((a, b_0), (-a, b_0)) \text{ for any } a \in S^2.
\end{align*}
\]

We note that \( s \) is a cross-section of \( \pi^{S^2 \vee S^1}_{2,1} \). Thus, \( \text{sec}_{\text{op}}(\pi^{S^2 \vee S^1}_{2,1}) = 1 \). Also, there exist \( \alpha \in \tilde{H}^1(S^2 \vee S^1; R) \) with \( \alpha \neq 0 \) and \( i^*(\alpha) = 0 \in \tilde{H}^1(S^2; R) \). From Proposition 3.20 ii), \( MR[\pi^{S^2 \vee S^1}_{2,1}, x_0] \neq 0 \).

We next relate our results to Farber’s topological complexity, a homotopy invariant of \( X \) introduced in [6]. Let \( PX \) denote the space of all continuous paths \( \gamma: [0, 1] \to X \) in \( X \) and \( c_{0,1}: PX \to X \times X \) denote the map associating to any path
\( \gamma \in PX \) the pair of its initial and end points, i.e., \( \epsilon_{0,1}(\gamma) = (\gamma(0), \gamma(1)) \). Equip the path space \( PX \) with the compact-open topology.

**Definition 3.4.** The topological complexity of a path-connected space \( X \), denoted by \( TC(X) \), is the least integer \( m \) such that the cartesian product \( X \times X \) can be covered with \( m \) open subsets \( U_i \),

\[
X \times X = U_1 \cup U_2 \cup \cdots \cup U_m,
\]

such that for any \( i = 1, 2, \ldots, m \) there exists a continuous local section \( s_i : U_i \to PX \) of \( \epsilon_{0,1} \), that is, \( \epsilon_{0,1} \circ s_i = id \) over \( U_i \). If no such \( m \) exists we will set \( TC(X) = \infty \).

We have \( TC(X) = 1 \) if and only if \( X \) is contractible. The TC is a homotopy invariant, i.e., if \( X \simeq Y \) then \( TC(X) = TC(Y) \). Moreover,

\[
\operatorname{cat}(X) \leq TC(X) \leq 2\operatorname{cat}(X) - 1
\]

for any path connected CW complex \( X \).

**Proposition 3.22.** Let \( X \) be a non contractible path connected CW complex. If \( \pi^{X}_{2,1} \simeq x_0 \) for some \( x_0 \in X \) then \( X - \{x_0\} \) is contractible in \( X \). In particular, \( \operatorname{cat}(X) = 2 \) and \( 2 \leq TC(X) \leq 3 \).

**Proof.** Let \( H : F(X,2) \times [0,1] \to X \) be a homotopy between \( \pi^{X}_{2,1} \) and \( x_0 \). Set \( G : (X - \{x_0\}) \times [0,1] \to X \) given by the formula \( G(x,t) = H((x,x_0),t) \). We have \( G(x,0) = x \) and \( G(x,1) = x_0 \) for any \( x \in X - \{x_0\} \). Thus \( X - \{x_0\} \) is contractible in \( X \).

The converse of Proposition 3.22 is not true; take for instance \( X = S^n \).

**Remark 3.23.** It is well known that \( \operatorname{cat}(X) = 2 \) corresponds to the case in which \( X \) is a co-H-space. This is a large class of spaces which includes all suspensions. In addition there are well-known examples of co-H-spaces that are not suspensions.

**Example 3.24.** We recall that the odd-dimensional projective spaces \( \mathbb{R}P^{2n+1} \) has not the FPP, because there is a continuous self-map \( h : \mathbb{R}P^{2n+1} \to \mathbb{R}P^{2n+1}, \) given by the formula

\[
h([x_1 : y_1 : \cdots : x_{n+1} : y_{n+1}]) = [-y_1 : x_1 : \cdots : -y_{n+1} : x_{n+1}],
\]

without fixed point. Thus, \( \operatorname{sec}_{op}(\pi^{\mathbb{R}P^{2n+1}}_{2,1}) = 1 \).

On the other hand, we know that an even-dimensional projective spaces \( \mathbb{R}P^{2n} \) has the FPP. Thus, \( \operatorname{sec}_{op}(\pi^{\mathbb{R}P^{2n}}_{2,1}) = 2 \). Analogous facts hold for complex and quaternionic projective spaces.

**Example 3.25.** The spheres \( S^n \) does not have the FPP, because the antipodal map \( A : S^n \to S^n, x \mapsto -x \) has not fixed points. Thus, \( \operatorname{sec}_{op}(\pi^{S^n}_{2,1}) = 1 \).

**Example 3.26.** We know that any closed surface \( \Sigma \), except for the projective plane \( \Sigma \neq \mathbb{R}P^2 \), has not the FPP. Thus, \( \operatorname{sec}_{op}(\pi^{\Sigma}_{2,1}) = 1 \).

**Remark 3.27.** Let \( X \) be a topological space. The map

\[
\pi^X_{k-1} : F(X,k) \to X
\]

has a continuous section, i.e., \( \operatorname{sec}_{op}(\pi^X_{k-1}) = 1 \) if and only if there exist \( k - 1 \) fixed point free continuous self-maps \( f_2, \ldots, f_k : X \to X \) which are non-coincident, that is, \( f_i(x) \neq f_j(x) \) for any \( i \neq j \) and \( x \in X \).
Example 3.28. Let $G$ be a topological group with order $|G| \geq k$. Then $\text{sec}_{\text{op}}(\pi^{G}_{k,1}) = 1$, because the map $s : G \to F(G, 2)$, $g \mapsto (g, g_1 g, \ldots, g_{k-1} g)$ is a cross section for $\pi^{G}_{k,1}$ (for some fixed $(g_1, \ldots, g_{k-1}) \in F(G - \{e\}, k - 1)$).

Example 3.29. [5] Let $M$ be a topological manifold (without boundary) and $Q_m \subset M$ be a finite subset with $m$ elements. Then $\text{sec}_{\text{op}}(\pi^{M - Q_m}_{k,1}) = 1$ for any $m \geq 1$.

Proposition 3.30. For any $k > r \geq 1$ and $X$ a Hausdorff space, we have

$$\text{sec}_{\text{op}}(\pi^{X}_{k,r}) \leq \binom{k}{r},$$

where $\binom{k}{r} = \frac{k!}{r!(k-r)!}$ is the standard binomial coefficient.

Proof: Let $(p_1, \ldots, p_k) \in F(X, k)$ be a fixed $k$–tuple. Set $Q_k := \{p_1, \ldots, p_k\}$ and for each $I_r \subseteq Q_k$ with $|I_r| = r$, we define $Q_{I_r} := Q_k - I_r = \{p_{j_1}, \ldots, p_{j_{k-r}}\}$ with $j_1 < \cdots < j_{k-r}$.

Set

$$U_{I_r} := F(M - Q_{I_r}, r)$$

and $s_{I_r} : U_{I_r} \to F(X, k)$ given by

$$s_{I_r}(x_1, \ldots, x_r) := (x_1, \ldots, x_r, p_{j_1}, \ldots, p_{j_{k-r}}), \forall x \in U_{I_r}.$$ 

We note $U_{I_r}$ is open in $F(M, r)$ and each $s_l$ is a local section of $\pi^{X}_{k,r}$. Furthermore, $F(M, r) = \bigcup_{l \subseteq Q_k, |l| = r} U_{I_l}$. Then, $\text{sec}_{\text{op}}(\pi^{X}_{k,r}) \leq \binom{k}{r}$. \hfill \Box

Corollary 3.31. Let $M$ be a connected topological manifold (without boundary) of dimension at least two. Then

$$\text{sec}_{\text{op}}(\pi^{M}_{k,r}) \leq \min\{\binom{k}{r}, \text{cat}(F(M, r))\}.$$ 

Example 3.32. Let $M$ be a contractible topological manifold (without boundary) of dimension at least two. Then $\text{sec}_{\text{op}}(\pi^{M}_{k,1}) \leq \min\{k, \text{cat}(M) = 1\} = 1$. In particular, $M$ does not have the FPP.

Remark 3.33. Consider the following diagram

(3.2) \hspace{1cm} \begin{tikzcd}
F(X, k) & F(X, k - 1) \\
X \\
\end{tikzcd}

It is easy to see that if $\pi^{X}_{k,1}$ admits a section, then $\pi^{X}_{k-1,1}$ also admits a section. Thus $\text{sec}_{\text{op}}(\pi^{X}_{k,1}) = 1$ implies $\text{sec}_{\text{op}}(\pi^{X}_{k-1,1}) = 1$ for any $r \leq k$. Furthermore, when $X = S^d$, we also consider the following diagram

(3.3) \hspace{1cm} \begin{tikzcd}
F(S^d, k) & F(S^d, 2) \\
S^d \\
\end{tikzcd}
and recall that \( \pi^{2d}_{k,1} \) always admits a section. Hence, if \( \pi^{2d}_{k,2} \) admits a section, so does \( \pi^{2d}_{k,1} \). The converse is also true, i.e., if \( \pi^{2d}_{k,1} \) admits a section, so does \( \pi^{2d}_{k,2} \).

**Proposition 3.34.** For any \( k > 2 \) and \( d \) even. We have

\[
\sec_{op}(\pi^{2d}_{k,r}) = \text{cat}(F(S^d, r)) = 2, \text{ for } r = 1 \text{ or } 2.
\]

**Proof.** First, we show that \( \sec_{op}(\pi^{2d}_{k,r}) \geq 2 \) for any \( d \) even, \( k \geq 3 \) and \( r = 1 \) or 2. (This part can also be proved by employing Lefschetz’s theory of coincidences.) Indeed, by the above diagrams, it suffices now to show that \( \sec_{op}(\pi^{2d}_{3,1}) \geq 2 \) for \( d \) even, that is, \( \pi^{2d}_{3,1} \) does not admit a cross section. If a cross section existed, it would generate a map \( f : S^d \to S^d \) such that \( f(x) \neq x \) and \( f(x) \neq -x \) for any \( x \in S^d \). Since \( f(x) \neq -x \) for every \( x \in S^d \), it is easy to see that \( f \) is a homeomorphism of \( S^d \) which is a contradiction. (We recall that \( f : S^d \to S^d \) has no fixed points.) Thus, \( \pi^{2d}_{3,1} \) does not admit a cross section.

From Proposition 3.31, \( \sec_{op}(\pi^{2d}_{k,r}) \leq \min\left(\frac{d}{2}, \text{cat}(F(S^d, r)) = \text{cat}(S^d) = 2\right) \). Then \( \sec_{op}(\pi^{2d}_{k,r}) = 2 \) for any \( k \geq 3, r \in \{1, 2\} \) (\( d \) even).

**Proposition 3.35.**

(1) Let \( X \) be a topological space. If there exist a retract \( L \) of \( X \), then

\[
\sec_{op}(\pi^L_{k,1}) \geq \sec_{op}(\pi^X_{k,1}).
\]

(2) \([5]\) If \( M \) is differentiable and admits a non-vanishing vector field, then

\[
\sec_{op}(\pi^M_{k,1}) = 1 \text{ for every } k.
\]

**Proof.** (1) Let \( r : X \to L \) be a retraction, i.e., \( r \circ i = 1_L \), where \( i : L \to X \) is the inclusion map. Note that \( r^{-1}(A) \subset A \) for any \( A \subset L \). We have the following commutative diagram

\[
\begin{array}{ccc}
F(L, k) & \overset{i^k}{\longrightarrow} & F(X, k) \\
\pi^L_{k,1} \downarrow & & \downarrow \pi^X_{k,1} \\
L & \overset{i}{\longrightarrow} & X
\end{array}
\]

Suppose \( U \subset L \) is an open set of \( L \) with local section \( s : U \to F(L, k) \) of \( \pi^L_{k,1} \). Set \( V = r^{-1}(U) \subset X \) and consider \( \sigma : V \to F(X, k) \) given by

\[
\sigma = i^k \circ s \circ r.
\]

Since, \( r^{-1}(U) \subset U \), we have that \( \sigma \) is a local section of \( \pi^X_{k,1} \). Therefore, \( \sec_{op}(\pi^L_{k,1}) \geq \sec_{op}(\pi^X_{k,1}) \).

**Corollary 3.36.** \([5]\) If \( M \) is compact and the first Betti number of \( M \) does not vanish, then \( \sec_{op}(\pi^M_{k,1}) = 1 \) for every \( k \).

**Corollary 3.37.** \([5]\) If \( M \) is an odd dimensional differentiable manifold, \( \sec_{op}(\pi^M_{k,1}) = 1 \) for every \( k \).

**Corollary 3.38.** For any \( k > 2 \) and \( d \) odd. We have

\[
\sec_{op}(\pi_{k,r} : F(S^d, k) \to F(S^d, r)) = 1 \text{ for } r = 1 \text{ or } 2.
\]
4. Topological complexity of a map

Recall that $PE$ denotes the space of all continuous paths $\gamma : [0, 1] \to E$ in $E$ and $e_{0,1} : PE \to E \times E$ denotes the map associating to any path $\gamma \in PE$ the pair of its initial and end points $\pi(\gamma) = (\gamma(0), \gamma(1))$. Equip the path space $PE$ with the compact-open topology.

Let $p : E \to B$ be a continuous surjection between path-connected spaces, and let

$$e_p : PE \to E \times B, \ e_p = (1 \times p) \circ e_{0,1}. \quad (4.1)$$

**Definition 4.1.** The topological complexity of the map $p$, denoted by $TC(p)$, is the sectional number $sec_{op}(e_p)$ of the map $e_p$, that is, the least integer $m$ such that the cartesian product $E \times B$ can be covered with $m$ open subsets $U_i$,

$$E \times B = U_1 \cup U_2 \cup \cdots \cup U_m,$$

such that for any $i = 1, 2, \ldots, m$ there exists a continuous local section $s_i : U_i \to PE$ of $e_p$, that is, $e_p \circ s_i = id$ over $U_i$. If no such $m$ exists we will set $TC(p) = \infty$.

We use a definition of topological complexity which generally is not the same that given in [12]. However, under certain conditions, these two definitions coincides (see [12]).

The proof of the following statement proceeds by analogy with [12].

**Proposition 4.1.** For any map $p : E \to B$, we have

$$TC(p) \geq \max\{cat(B), sec_{op}(p)\}. \quad (4.1)$$

**Proof.** Let $U \subset E \times B$ be an open subset and $s : U \to PE$ be a partial section of $e_p$. Fix $x_0 \in E$ and consider the inclusion $i_0 : B \to E \times B$, given as $i_0(b) = (x_0, b)$. Set $V = i_0^{-1}(U) \subset B$, it is an open subset of $B$.

Consider the map $H : V \times [0, 1] \to B$ given by $H(b, t) = p(s(x_0, b(t))$. It is easy to check that $H$ is a null-homotopy. We conclude that $TC(p) \geq cat(B)$.

On the other hand, consider the map $\sigma : V \to E$ defined by $\sigma(b) = s(x_0, b(1))$. One can easily see that $\sigma$ is a partial section over $V$ to $p$. Therefore, $TC(p) \geq sec_{op}(p)$. \hfill $\square$

The proof of the following statement proceeds by analogy with [12].

**Proposition 4.2.** Consider the diagram of maps $E' \xrightarrow{p'} E \xrightarrow{p} B \xrightarrow{p''} B'$. If $p$ admits a section, then

\begin{align*}
\text{a)} \quad & TC(p'') \leq TC(p''p), \\
\text{b)} \quad & TC(p'p) \leq TC(p').
\end{align*}

In particular, $TC(B) \leq TC(p) \leq TC(E)$.

**Proof.** Let $s : B \to E$ be a section to $p$.

\begin{enumerate}
\item Suppose $\alpha_{p''p} : U \to PE$ is a partial section of $e_{p''p}$ over $U \subset E \times B'$.
\end{enumerate}

Set $V := (s \times 1_{B'})^{-1}(U) \subset B \times B'$. Then we can define the continuous map $\alpha_{p'} : V \to PB$ by

$$\alpha_{p'}(b, b')(t) := \begin{cases} b, & \text{for } 0 \leq t \leq \frac{1}{2}; \\
p(\alpha_{p''p}(s(b), b')(2t - 1)), & \text{for } \frac{1}{2} \leq t \leq 1.
\end{cases}$$

Since $\alpha_{p'}$ is a partial section of $e_{p'}$ over $V$, we conclude that $TC(p'') \leq TC(p''p)$. 

b) Let \( \alpha_{p'} : U \rightarrow PE' \) be a partial section to \( e_{p'} : PE' \rightarrow E' \times E \) over \( U \subset E' \times E \). Set \( V := (1_{E'} \times s)^{-1}(U) \subset E' \times B \) and define the continuous map \( \alpha_{pp'} : V \rightarrow PE' \) given by
\[
\alpha_{pp'}(e', b) := \alpha_{p'}(e', s(b)).
\]
It follows that \( \alpha_{pp'} \) is a partial section of \( e_{pp'} \) over \( V \). This implies \( TC(pp') \leq TC(p') \). □

**Theorem 4.3.** Let \( X \) be a Hausdorff space.

i) If \( X \) has the FPP, then \( TC(\pi_{k,1}^X) \geq \max\{\text{cat}(X), 2\} \) for any \( k \geq 2 \).

ii) If \( TC(\pi_{2,1}^X) < TC(X) \) or \( TC(\pi_{2,1}^X) > TC(F(X, 2)) \), then \( \text{sec}_{op}(\pi_{2,1}^X) = 2 \). In particular, \( X \) has the FPP.

iii) If \( X \) is a non-contractible space which does not have the FPP, then the configuration space \( F(X, 2) \) is not contractible.

**Proof.** i) We have \( TC(\pi_{k,1}^X) \geq \text{sec}_{op}(\pi_{k,1}) \geq 2 \). We recall that, \( \text{sec}_{op}(\pi_{2,1}) = 2 \) implies \( \text{sec}_{op}(\pi_{k,1}) \geq 2 \), for any \( k \geq 2 \).

ii) This follows from Proposition 4.2.

iii) By Proposition 4.2 we have \( 1 < TC(X) \leq TC(\pi_{2,1}) \leq TC(F(X, 2)) \) and thus \( F(X, 2) \) is not contractible. □

**Remark 4.4.** Theorem 4.3 iii) gives a partial generalization of the work [14].

**Example 4.5.** We know that the unit disc \( D^m \) has the FPP. Then \( TC(\pi_{2,1}^{D^m}) \geq 2 \), for any \( k \geq 2 \).

The following Lemma generalizes the statement given in ([12], pg. 19).

**Lemma 4.6.** If \( p : E \rightarrow B \) is a fibration and \( p' : B' \rightarrow B' \) is a continuous map, then the following diagram is a pullback
\[
\begin{array}{ccc}
PE & \xrightarrow{p#} & PB \\
\downarrow \scriptstyle e_{p'} & & \downarrow \scriptstyle e_{p'} \\
E \times B' & \xrightarrow{p \times 1_{B'}} & B \times B'
\end{array}
\]

**Proof.** For any \( \beta : X \rightarrow PB \) and any \( \alpha : X \rightarrow E \times B' \) satisfying \( e_{p'} \circ \beta = (p \times 1_{B'}) \circ \alpha \), we will check that there exists \( H : X \rightarrow PE \) such that \( e_{p' \circ p} \circ H = \alpha \) and \( p# \circ H = \beta \).

Indeed, note that we have the following commutative diagram:
\[
\begin{array}{c}
X \xrightarrow{p_1 \circ \alpha} E \\
\downarrow \scriptstyle i_0 & & \downarrow \scriptstyle p \\
X \times I & \xrightarrow{\beta} & B
\end{array}
\]
where \( p_1 \) is the projection onto the first coordinate. Because \( p \) is a fibration, there exists \( H : X \times I \to E \) satisfying \( H \circ i_0 = p_1 \circ \alpha \) and \( p \circ H = \beta \), thus we do. \( \square \)

The following statement is well-known. It was proved in [12], however we give an elemental proof in our context.

**Proposition 4.7.** If \( p : E \to B \) is a fibration, then \( \text{TC}(p'p) \leq \text{TC}(p') \) for any \( p' : B \to B' \). In particular, \( \text{TC}(p) \leq \text{TC}(B) \).

**Proof.** Since \( p : E \to B \) is a fibration, the following diagram is a pullback (see Lemma 4.6).

\[
\begin{array}{ccc}
PE & \xrightarrow{p\#} & PB \\
\downarrow{\epsilon_{p'}} & & \downarrow{\epsilon'} \\
E \times B' & \xrightarrow{p \times 1_{B'}} & B \times B'
\end{array}
\]

This implies \( \text{TC}(p'p) = \text{sec}_{op}(\epsilon_{p'p}) \leq \text{sec}_{op}(\epsilon_{p'}) = \text{TC}(p') \). \( \square \)

**Corollary 4.8.** If \( p : E \to B \) is a fibration that admits a section, then \( \text{TC}(p) = \text{TC}(B) \). In particular, \( \text{TC}(p) = 1 \) if and only if \( B \) is contractible.

5. **The \((k, r)\) Robot Motion Planning Problem**

In this section we use the results above within a particular problem in robotics.

Recall that, in general terms, the **configuration space** or **state space** of a system \( S \) is defined as the space of all possible states of \( S \) (see [9] or [10]). Investigation of the problem of simultaneous collision-free motion planning for a multi-robot system consisting of \( k \) distinguishable robots, each with state space \( X \), leads us to study the ordered configuration space \( F(X, k) \) of \( k \) distinct points on \( X \). Explicitly,

\[
F(X, k) = \{ (x_1, \ldots, x_k) \in X^k \mid x_i \neq x_j \text{ for } i \neq j \},
\]

topologised as a subspace of the cartesian power \( X^k \). Note that the \( i \)-th coordinate of a point \( (x_1, \ldots, x_n) \in F(X, k) \) represents the configuration of the \( i \)-th moving object, so that the condition \( x_i \neq x_j \) reflects the collision-free requirement.

The **(k, r) robot motion planning problem** consists in controlling simultaneously these \( k \) robots without collisions, where one is interested in the initial positions of the \( k \) robots and **only interested in the final position of the first \( r \) robots** (\( k \geq r \)) (see Figure 1).

An **algorithm** for the \((k, r)\) robot motion planning problem is a function which assigns to any pair of configurations \((A, B) \in F(X, k) \times F(X, r)\) consisting of an initial state \( A = (a_1, \ldots, a_k) \in F(X, k) \) and a desired state \( B = (b_1, \ldots, b_r) \in F(X, r) \), a continuous motion of the system starting at the initial state \( A \) and ending at the desired state \( B \) (see Figure 2).

The central problem of modern robotics, the **motion planning problem**, consists of finding a motion planning algorithm.

We note that an algorithm to the \((k, r)\) robot motion planning problem is a (not necessarily continuous) section \( s : F(X, k) \times F(X, r) \to PF(X, k) \) of the map

\[
e_{s_{X, k, r}} : PF(X, k) \to F(X, k) \times F(X, r), \quad e_{s_{X, k, r}}(\alpha) = (\alpha(0), \pi_{k, r}^X(\alpha(1))),
\]

where \( \pi_{k, r}^X : F(X, k) \to F(X, r) \) is the projection of the first \( r \) coordinates.
Figure 1. The $(2, 1)$ robot motion planning problem: we need to move Robots 1 and 2, simultaneously and avoiding collisions, from the initial positions $(a_1, a_2)$ to a final position $b_1$ of Robot 1. We are only interested in the final position of the first robot.

Figure 2. An algorithm for the $(2, 1)$ robot motion planning problem

A motion planning algorithm $s$ is called \textit{continuous} if and only if $s$ is continuous. Absence of continuity will result in instability of the behavior of the motion planning. In general, there is not a global continuous motion planning algorithm, and only local continuous motion plans may be found. This fact gives, in a natural way, the use of the numerical invariant $TC(\pi_{k,r}^X)$. Recall that $TC(\pi_{k,r}^X)$ is the minimal number of \textit{continuous} local motion plans to $e_{\pi_{k,r}^X}$ (i.e., continuous local sections for $e_{\pi_{k,r}^X}$), which are needed to construct an algorithm for autonomous motion planning of the $(k,r)$ robot motion planning problem. Any motion planning algorithm $s := \{s_i : U_i \to PE\}_{i=1}^n$ is called \textit{optimal} if $n = TC(\pi_{k,r}^X)$. 
Theorem 5.1. Let $M$ be a connected topological manifold without boundary of dimension at least 2, and let $\pi^X_{k,r} : F(M,k) \to F(M,r)$ be the Fadell-Neuwirth fibration.

1. If $M$ does not have the FPP, then $\text{TC}(\pi^M_{2,1}) = \text{TC}(M)$. Hence, the complexity for the $(2,1)$ robot motion planning problem is the same complexity to the manifold $M$. More general, if $\text{sec}_\text{op}(\pi^M_{k,r}) = 1$, then $\text{TC}(\pi^M_{k,r}) = \text{TC}(F(M,r))$.

2. If $M$ has the FPP, then $\max\{2, \text{cat}(M)\} \leq \text{TC}(\pi^M_{k,1}) \leq \text{TC}(M)$, for any $k \geq 2$. In particular, $M$ is not contractible.

Example 5.2. We recall that the $n$–dimensional sphere $S^n$ does not have the FPP. Then,

$$\text{TC}(\pi^{S^n}_{2,1}) = \text{TC}(S^n) = \begin{cases} 2, & \text{for } n \text{ odd;} \\ 3, & \text{for } n \text{ even.} \end{cases}$$

Furthermore, we have that any contractible topological manifold $M$ without boundary does not have the FPP. Hence, $\text{TC}(\pi^M_{2,1}) = \text{TC}(M) = 1$.

Example 5.3. 

- The odd dimensional projective spaces $\mathbb{RP}^m$ does not have the FPP, then $\text{TC}(\pi^m_{2,1}) = \text{TC}(\mathbb{RP}^m)$. By [12], the topological complexity $\text{TC}(\mathbb{RP}^m)$ for any $m \neq 1, 3, 7$, coincides with the smallest integer $k$ such that the projective space $\mathbb{RP}^m$ admits an immersion into $\mathbb{R}^{k-1}$.
- It is known that any closed surface $\Sigma$ except the projective plane $\Sigma \neq \mathbb{RP}^2$, does not have the FPP. Thus, $\text{TC}(\pi^m_{2,1}) = \text{TC}(\Sigma)$.
- We have that the projective plane $\mathbb{RP}^2$ has the FPP. Furthermore, it is well known $\text{cat}(\mathbb{RP}^2) = 3$ and $\text{TC}(\mathbb{RP}^2) = 4$ [7]. Then, $3 = \text{cat}(\mathbb{RP}^2) \leq \text{TC}(\pi^m_{k,1}) \leq \text{TC}(\mathbb{RP}^2) = 4$, for $k \geq 2$.
- For any connected compact Lie group, the Fadell-Neuwirth fibration

$$\pi^{G \times \mathbb{R}^m}_{k,k-1} : F(G \times \mathbb{R}^m,k) \to F(G \times \mathbb{R}^m,k-1)$$

admits a continuous section (for $m \geq 2$). Then $\text{TC}(\pi^{G \times \mathbb{R}^m}_{k,k-1}) = \text{TC}(F(G \times \mathbb{R}^m,k-1))$. By [13], the topological complexity $\text{TC}(F(G \times \mathbb{R}^m,2)) = 2\text{TC}(G)$. Hence, $\text{TC}(\pi^{G \times \mathbb{R}^m}_{3,2}) = 2\text{TC}(G) = 2\text{cat}(G)$.
- Any connected Lie group has not the FPP and $\text{cat}(G) = \text{TC}(G)$. Then, $\text{TC}(\pi^G_{k,1}) = \text{TC}(G) = \text{cat}(G)$. In general, $\text{TC}(\pi^G_{k,1}) = \text{TC}(G) = \text{cat}(G)$ for any $k \geq 2$.

Example 5.4. 

- We have that the sectional number $\text{sec}_\text{op}(\pi^{d}_{k,r}) = \text{cat}(F(S^d,r)) = 2$, for any $k \geq 3$ and $d$ even, and $r = 1, 2$. Then $2 = \text{sec}_\text{op}(\pi^{d}_{k,r}) \leq \text{TC}(\pi^{d}_{k,r}) \leq \text{TC}(F(S^d,r)) = \text{TC}(S^d) = 3$.
- For any $k \geq 2$ and $d$ odd, and $r = 1, 2$. We have $\text{sec}_\text{op}(\pi^{d}_{k,r}) = 1$. Hence, $\text{TC}(\pi^{d}_{k,r}) = \text{TC}(F(S^d,r)) = \text{TC}(S^d) = 2$.

Proposition 5.5. [12] Let $p : E \to B$ be a fibration between ANR spaces. Then

$$\text{cat}(B) \leq \text{TC}(p) \leq \min\{\text{cat}(E) + \text{cat}(E)\text{sec}_\text{op}(p) - 1, \text{TC}(B), \text{cat}(E \times B)\}.$$ 

In particular, $\text{TC}(p) = 1$ if and only if $B$ is contractible.
Theorem 5.6. Let $M$ be a connected topological manifold without boundary of dimension at least 2. If $M$ has the FPP, then
\[ \max\{2, \text{cat}(M)\} \leq \text{TC}(\pi_2(M)) \leq \min\{3\text{cat}(F(M, 2)) - 1, \text{TC}(M), \text{cat}(F(M, 2) \times M)\}. \]

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