On A Generalization of “Eight Blocks to Madness”*

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Abstract

We consider a puzzle such that a set of colored cubes is given as an instance. Any cube is the unit length on each edge and its surface is colored so that what we call Surface Color Condition is satisfied. Given a palette of 6 colors, the condition requires that each face should be assigned exactly one color and all faces should be assigned 6 different colors from each other. The puzzle asks to compose a $2 \times 2 \times 2$ cube that satisfies Surface Color Condition from 8 suitable cubes in the instance. Note that cubes and solutions have 30 varieties respectively.

In this paper, we present a collection of our research results on this puzzle. First we present a novel necessary and sufficient condition that a solid of a given variety is composable. This condition immediately leads us to succinct procedures to identify the composability and to decide the assignment of cubes to the corners of the solid. Next we create two extreme instances: a universal instance (i.e., one having all 30 solutions) and an infeasible instance (i.e., one having no solution). We show the minimum size of the universal instance to be 12 and the maximum size of the infeasible instance to be 23. The latter result affirmatively solves one of the open problems that were raised in [E. Berkove et al., “An Analysis of the (Colored Cubes)3 Puzzle,” Discrete Mathematics, 308 (2008) pp. 1033–1045].

1 Introduction

There have been invented various kinds of mathematical puzzles. They often provide not only recreation but also interesting research problems with us. Many types of puzzles such as pencil puzzles [1, 2, 3, 4, 5] and even video games [6, 7, 8] have been shown to be NP-hard, which seems to be a common property that attractive puzzles should have. Hearn and Demaine collected major complexity results in [9].

Puzzle instance creation is one of the possible directions of future research in mathematical puzzles. Ordinary people do not have interest in computational complexity. They just like to play more and more addictive and challenging puzzle instances. Hence it is meaningful to exploit how to create puzzle instances that have desired properties. Nevertheless, there is fewer literature that deals with puzzle instance creation (e.g., [10, 11]). This must be due to its hardness. Creating a puzzle instance is harder than just solving it in general because creation process involves solving. To create an instance, we need not only to check whether a candidate instance is solvable but also to examine whether it has desired properties (e.g., solution uniqueness) or even to search the instance space for better candidates.

With these in mind, we explore how to create an instance for simplest puzzles. In this paper, we take up a certain generalization of a classical puzzle called “Eight Blocks to Madness.” We present a collection of our research results on this puzzle.

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We deal with colored cubes. Suppose that we are given a palette of 6 colors. Any cube is the unit length on each edge and the surface is colored so that the following Surface Color Condition is satisfied.

**Surface Color Condition:** Each face of the cube is assigned exactly one color and all faces are assigned different colors from face to face.

Colored cubes have 30 varieties in all.

In the puzzle that we consider, a puzzle solver is given a set of \( m \) colored cubes as an instance \((m \geq 8)\). The instance may contain multiple cubes for some varieties or may contain no cubes for other varieties. The puzzle asks to compose a \( 2 \times 2 \times 2 \) cube by using 8 suitable cubes in the instance so that the \( 2 \times 2 \times 2 \) cube satisfies Surface Color Condition. We call a \( 2 \times 2 \times 2 \) cube whose surface is colored in that way a solid. Thus there are 30 varieties of solids.

The first highlight is a novel necessary and sufficient condition that a solid of the given variety is composable. The proposed condition is derived from another straightforward condition such that the solid is composable iff there exists an 8-size matching between cubes in the instance and corners of the solid. The condition immediately provides a succinct procedure to identify the composability. Interestingly the procedure decides the composability without finding a maximum matching explicitly. We also present how to determine the assignment of 8 cubes to the 8 corners of the solid.

The other highlight is creation of two extreme instances. Once an instance is given, the varieties that appear as solids are uniquely determined. Note that all the 30 varieties do not necessarily appear. Then it is natural to take interest in an instance such that all the 30 varieties of solids are composable, and an instance such that no solid is composable. We call an instance of the former type a universal instance and an instance of the latter type an infeasible instance. It is easy to construct examples of these types of instances. For example, a 240-size instance that contains 8 cubes for all the 30 varieties is universal. We pursue universal and infeasible instances that are most “distant” from our intuition. It is expected that, the more cubes and the more varieties the instance contains, the more varieties of solids are likely to be composable. It is interesting to explore a smaller universal instance and a larger infeasible instance. We show the minimum size of the universal instance to be 12 and the maximum size of the infeasible instance is 23. In particular the latter gives an affirmative answer to one of the open problems in [12].

The paper is organized as follows. We present previous work related to the puzzle in Section 2. Preparing terminologies and notations in Section 3 we present the novel necessary and sufficient condition that a solid of a given variety is composable in Section 4. We then create minimum universal and maximum infeasible instances in Section 5. We conclude the paper in Section 6, describing open problems on general instance creation.

### 2 Related Work

The original version of “Eight Blocks to Madness” was invented by an Irish named Eric Cross in 1960’s and was issued by Austin Enterprises of Ohio [13]. The puzzle instance has exactly 8 cubes that are of different varieties from each other. Thus a solver has only to arrange the 8 given cubes into a solid. As compared to our generalized case, the solver does not need to struggle to find out the suitable cubes from the instance. Kahan [14] showed that the solid of a unique variety is composable from this original instance. Sobczyk [15] studied a slight generalization of the original puzzle so that an instance can be a set of any 8 cubes, repetition of the same variety cubes being allowed. He showed that at most 6 varieties of solids are composable from such an instance. Our puzzle is a generalization of these previous puzzles in the sense that an instance is a set of any \( m \) cubes \((m \geq 8)\).
Berkove et al. [12] already studied a further generalization. In their puzzle, which they call “(Colored Cubes)\(^3\) Puzzle,” a puzzle solver is given a set of \(m\) cubes as an instance and is asked to compose an \(n \times n \times n\) solid, where \(m \geq n^3\) is assumed. They prove that a solid is always composable when \(n \geq 3\). The scenario of the proof is described as follows; they show that, when \(n \geq 3\), an \(n \times n \times n\) solid is composable if and only if the instance contains such a subset of 8 cubes that can be arranged into a \(2 \times 2 \times 2\) solid. Then they show that any instance of no less than \(3^3 = 27\) cubes has such a subset. Such a subset is significant because, to compose an \(n \times n \times n\) solid, it is most “difficult” to find out 8 cubes from the instance that are assigned to the corners of the solid; these cubes form a \(2 \times 2 \times 2\) solid. We use the term “difficult” to mean that cubes that can be assigned to corners are most restricted since their 3 faces are exposed to the outside. On the other hand, it is comparatively easy to find out cubes to be assigned to other parts of the solid, e.g., we can assign any cube to the inside since no face is exposed to the outside. Thus the \(n = 2\) case plays a significant role in the analysis of the \(n \geq 3\) case. This is why we concentrate on the \(n = 2\) case.

Colored cube is said to be introduced to recreational mathematics by P. A. MacMahon in 1920’s [16, 17]. He invented the first puzzle that deals with colored cubes. In his puzzle, a puzzle solver is given a set of 29 cubes of different varieties except a certain variety, say \(v\). The solver is asked to compose a \(2 \times 2 \times 2\) solid of the variety \(v\) so that Domino Condition is satisfied, i.e., in the inside of the solid, two faces touching each other should have the same color. The research history of this puzzle is summarized in M. Gardner’s famous “Fractal Music, Hypercards and More...” [18]. The most outstanding way to solve the puzzle is one developed by J. H. Conway. He arranged the 30 varieties of cubes in the non-diagonal entries of a \(6 \times 6\) grid, as in Figure 1. When \(v\) is the variety in the row \(i\) and in the column \(j\), we can compose the desired solid from 8 cubes that have varieties in the 4 varieties in the row \(j\) and those in the 4 varieties in the column \(i\), except the symmetric variety with respect to the diagonal, i.e., the one in the row \(j\) and in the column \(i\).

Our puzzle is different from MacMahon’s in three points: in our puzzle, an instance is a multi-set of \(m\) cubes, the variety of a solid is not specified to the solver, and Domino Condition is not imposed on the inside of the solid. Nevertheless, we will use Conway’s Table in the analysis of our puzzle since it has several interesting properties that let us understand the analysis easier.

Among other types of colored cube puzzles, Instant Insanity\(^1\) must be the most well-known puzzle. Recently Demaine et al. analyzed the computational complexity of its variations in [19], where the research history is well-written.

### 3 Preliminaries

A cube has 6 faces, 12 edges and 8 corners. The variety of a cube is specified by how the given 6 colors are assigned to its faces. We index the 30 varieties by means of Conway’s Table in Figure 1. We denote the variety in the row \(i\) and in the column \(j\) by \((i, j)\). For a natural number \(n\), let \([n] = \{1, 2, \ldots, n\}\). We denote the set of all 30 varieties by \(V\_{\text{all}}\), i.e., \(V\_{\text{all}} = \{(i, j) \in [6] \times [6] \mid i \neq j\}\). We call a cube of variety \((i, j)\) an \((i, j)\)-cube.

In order to explain the properties of Conway’s Table, we introduce the notion of corner triple. For each corner of a cube, we define the corner triple as an ordered triple of 3 colors around the corner such that the head element is set to the color of the lexically smallest alphabet among them and the colors are taken in the clockwise order around the corner. Since a cube has 8 corners, any cube (and thus any variety) has 8 corner triples. We denote the set of 8 corner triples of a variety

\(^1\)“Instant Insanity” was originally trademarked by Parker Brothers in 1967. The trademark is now owned by Winning Moves, Inc.
Figure 1: Conway’s Table of the 30 cube varieties. Each variety is represented by how the 6 colors are assigned to the development of a cube. The 6 colors are denoted by \( p, q, r, s, t, u \).

\[(i, j) \text{ by } T_{i,j}. \text{ We show examples as follows.} \]
\[
T_{1,2} = \{(p, q, t), (p, s, u), (p, t, s), (p, u, q), (q, r, t), (q, u, r), (r, s, t), (r, u, s)\},
\]
\[
T_{2,1} = \{(p, t, q), (p, u, s), (p, s, t), (p, q, u), (q, t, r), (q, r, u), (r, t, s), (r, s, u)\}.
\]

Observe that \((6 \times 5 \times 4)/3 = 40\) corner triples are possible in all. This is because, although there are \(6 \times 5 \times 4 = 120\) ordered triples of 3 colors, we deal with triples like \((r, p, q)\) and \((q, r, p)\) as \((p, q, r)\) since \(p\) is lexically smaller than \(q\) and \(r\). Two corner triples are the mirror triples of each other if both contain the same 3 colors but are different ordered triples, e.g., \((p, q, t)\) and \((p, t, q)\) are the mirror triples of each other. Two varieties \((i, j)\) and \((i', j')\) are the mirror varieties of each other if they are the reflexes of each other, e.g., \( (1, 2) \) and \( (2, 1) \) are the mirror varieties of each other. Using the notion of mirror triple, \((i, j)\) and \((i', j')\) are the mirror varieties of each other if, for any triple \( \tau \in T_{i,j} \), there exists a triple \( \tau' \in T_{i',j'} \) that is the mirror triple of \( \tau \).

We say that two different varieties are compatible (resp., incompatible) with each other if they share at least one corner triple (resp., no corner triple). For convenience, we may say that two cubes are compatible (resp., incompatible) when their varieties are compatible (resp., incompatible) with each other. For an arbitrary variety \((i, j)\), let \( V_{i,j} \) (resp., \( V_{i,j}^* \)) denote the set of varieties
compatible (resp., incompatible) with \((i, j)\). Any variety in \(V_{i,j}\) can be obtained by changing the color assignment of \((i, j)\) by either the “swap” operation or the “rotation” operation. By the swap operation, we mean to exchange the 2 colors of adjacent faces that share one edge with each other. By the rotation operation, we mean to rotate the colors of the 3 faces around a corner so that the corresponding corner triple is unchanged. The former yields 12 varieties and the latter yields 8 varieties. The variety set \(V^*_{i,j}\) consists of the mirror variety of \((i, j)\), and the mirrors of the 8 varieties that are generated by performing the rotation operation on \((i, j)\).

**Proposition 1 (Proof of Lemma 2.7 in [12])** For any variety \((i, j) \in V_{\text{all}}\), we have \(|V_{i,j}| = 20\) and \(|V^*_{i,j}| = 9\).

Moreover, whenever two varieties are compatible, they share exactly 2 corner triples.

**Proposition 2 (Lemma 2.7 in [12])** Two different varieties share either 0 or 2 corner triples with each other.

Now we present the properties of Conway’s Table as follows.

**Property 1** The mirror variety of any variety \((i, j)\) is \((j, i)\).

**Property 2** The 5 varieties on the same row or on the same column are incompatible with each other.

From \(|V^*_{i,j}| = 9\), we have

\[
V^*_{i,j} = \{(j, i)\} \cup \{(i, j') \mid j' \in [6] \setminus \{i, j\}\} \cup \{(i', j) \mid i' \in [6] \setminus \{i, j\}\}.
\]

This implies that, to compose an \((i, j)\)-solid, we cannot use cubes of the mirror variety \((j, i)\) or cubes of the varieties that are on the row \(i\) or on the column \(j\), except \((i, j)\) itself. They are all the varieties that we cannot use for the purpose.

An instance is a multi-set of colored cubes. We denote an instance by \(I\). We represent the distribution of varieties in \(I\) by a 6 \(\times\) 6 matrix, which we denote by \(M_I\). Each value \((M_I)_{ij}\) in the matrix represents the number of \((i, j)\)-cubes in the instance. For convenience, we let \((M_I)_{ii} = 0\) for any \(i = 1, \ldots, 6\). We call \(|I|\) the size of \(I\). Clearly we have

\[
|I| = \sum_{(i,j) \in V_{\text{all}}} (M_I)_{ij},
\]

where we assume \(|I| \geq 8\). For a variety \((i, j)\), we define \(\sigma_{I,(i,j)}\) as the number of cubes whose varieties are compatible with \((i, j)\);

\[
\sigma_{I,(i,j)} = \sum_{(i',j') \in V_{i,j}} (M_I)_{i'j'}.
\]

We define \(V(I)\) as the set of varieties that appear in \(I\), that is,

\[
V(I) = \{(i, j) \in V_{\text{all}} \mid (M_I)_{ij} > 0\}.
\]

A solid is a 2 \(\times\) 2 \(\times\) 2 cube that is composed of 8 colored cubes and that satisfies Surface Color Condition. We call a solid a \((k, \ell)\)-solid when its variety is \((k, \ell)\). We say that the \((k, \ell)\)-solid is composable from \(I\) if \(I\) contains a subset of 8 cubes that can be arranged into a \((k, \ell)\)-solid. An instance \(I\) is \(V\)-generatable if the \((k, \ell)\)-solid is composable from \(I\) when and only when \((k, \ell) \in V\). An instance is universal if it is \(V_{\text{all}}\)-generatable. An instance is infeasible if it is \(\emptyset\)-generatable.
4 A Necessary and Sufficient Condition That The \((k, \ell)\)-Solid Is Composable

In this section we present a novel necessary and sufficient condition that the \((k, \ell)\)-solid is composable from a given instance \(I\). The condition immediately leads us to a succinct procedure to decide whether the \((k, \ell)\)-solid is composable. We also present how to determine the assignment of 8 cubes in the instance to the corners of the solid when it is composable.

The condition is based on a graph \(G_{I,(k,\ell)} = (T_{k,\ell}, E_{I,(k,\ell)})\), which is defined as follows. We take \(T_{k,\ell}\) as the node set. Each corner triple in \(T_{k,\ell}\) is a node in the graph. We have 8 nodes in all. Each edge in \(E_{I,(k,\ell)}\) is associated with an \((i, j)\)-cube in \(I\) such that \((i, j)\) is compatible with \((k, \ell)\), i.e., \((i, j) \in V_{k,\ell}\). The edge connects two nodes \(\tau\) and \(\tau'\) such that \(T_{i,j} \cap T_{k,\ell} = \{\tau, \tau'\}\). Recall Proposition 2 which states that any \((i, j)\)-cube shares exactly 2 corner triples with \((k, \ell)\). Then the number \(|E_{I,(k,\ell)}|\) of edges is equal to \(\sigma_{I,(k,\ell)}\), that is, the number of cubes that are compatible with \((k, \ell)\). The graph may have multi-edges. Using the graph \(G_{I,(k,\ell)}\), we have the following condition.

**Theorem 1** The \((k, \ell)\)-solid is composable from an instance \(I\) iff \((M_I)_{k,\ell}\) is no less than the number of trees in the graph \(G_{I,(k,\ell)}\).

Before going on to the proof, let us show illustrative examples and how the theorem works. In Figure 2 we show graphs \(G_{I,(1,2)}\) and \(G_{I,(2,3)}\) for the instance \(I\) that has the following matrix representation:

\[
M_I = \begin{pmatrix}
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}.
\]

In the figure, a node (resp., an edge) is labeled with its corresponding corner triple (resp., cube variety). We see that the \((1,2)\)-solid is composable since the number of trees in \(G_{I,(1,2)}\) is 1 and \((M_I)_{12} = 2\). On the other hand, the \((2,3)\)-solid is not composable since the number of trees in \(G_{I,(2,3)}\) is 1 but \((M_I)_{23} = 0\).
We prove Theorem 1 by means of proving the following Lemma 1. Theorem 1 is immediate from the lemma.

**Lemma 1** The \((k, ℓ)\)-solid is not composable from an instance \(I\) iff \((M_I)_{k,ℓ}\) is smaller than the number of trees in the graph \(G_{I,(k,ℓ)}\).

We prove the lemma by utilizing another necessary and sufficient condition, which is rather straightforward. We define a bipartite graph \(B_{I,(k,ℓ)} = (I ∪ T_{k,ℓ}, F_{I,(k,ℓ)})\) as follows; \(I\) and \(T_{k,ℓ}\) are the two node sets and \(F_{I,(k,ℓ)} \subseteq I × T_{k,ℓ}\) is the edge set. Recall that \(I\) is the instance given, a multi-set of cubes. We regard each cube in \(I\) as a node. We denote an \((i, j)\)-cube by \(c_{i,j}\). We also regard each corner triple in \(T_{k,ℓ}\) as a node. Then the edge set \(F_{I,(k,ℓ)}\) is defined as follows.

\[
F_{I,(k,ℓ)} = \{(c_{i,j}, τ) ∈ I × T_{k,ℓ} \mid τ ∈ T_{i,j}\}.
\]

Hence \((c_{i,j}, τ) \in F_{I,(k,ℓ)}\) indicates that cube \(c_{i,j}\) can be allocated to the corner having \(τ\). From Proposition 2, the degree of \(c_{i,j}\) is 0 if \((i,j) \in V^*_{k,ℓ}\), it is 2 if \((i, j) \in V_{k,ℓ}\), and it is 8 if \((i, j) = (k, ℓ)\).

A matching in a graph is a subset of edges such that no two edges in the subset have an endpoint in common. Then any matching in \(B_{I,(k,ℓ)}\) represents admissible allocation of cubes to corners, where any cube in \(I\) is allocated to at most one corner, and any corner is assigned at most one cube. Since \(|T_{k,ℓ}| = 8\), the matching number (i.e., the size of maximum matching) is at most 8. The following theorem is obvious.

**Theorem 2** The \((k, ℓ)\)-solid is composable from an instance \(I\) iff the matching number of \(B_{I,(k,ℓ)}\) is 8.

We introduce more notations on the bipartite graph \(B_{I,(k,ℓ)}\). Let \(T ⊆ T_{k,ℓ}\) be any subset of corner triples. We denote by \(I(T) ⊆ I\) the set of nodes that are connected to nodes in \(T\), i.e., \(I(T)\) is the set of cubes that have at least one corner triple in \(T\). We define a set function \(f : 2^{T_{k,ℓ}} → \mathbb{Z}\) as \(f(T) = |I(T)| - |T|\). One easily sees that \(f\) is submodular, that is, for any \(T, T' ⊆ T_{k,ℓ}\), \(f(T) + f(T') ≥ f(T ∪ T') + f(T \cap T')\).

We describe the strategy to prove the necessity of Lemma 1. Suppose that the \((k, ℓ)\)-solid is not composable. From Theorem 2, the matching number of \(B_{I,(k,ℓ)}\) is less than 8. Then from Hall’s Marriage Theorem [21], there is a subset \(T ⊆ T_{k,ℓ}\) such that \(f(T) < 0\). Let \(\mathcal{T}\) be the family of minimizers of \(f\), that is,

\[
\mathcal{T} = \{T^* ∈ 2^{T_{k,ℓ}} \mid f(T^*) = \min_{T ⊆ T_{k,ℓ}} f(T)\}.
\]

Note that the empty set \(∅\) does not belong to \(\mathcal{T}\) since \(f(∅) = 0\). Then there are minimal non-empty subsets in \(\mathcal{T}\) with respect to set-inclusion. In fact, a minimal subset is unique; Since \(f\) is submodular, \(\mathcal{T}\) is a distributive lattice [21]. Then minimal subsets should be pairwise disjoint. Suppose that there are two minimal subsets in \(\mathcal{T}\), denoted by \(T\) and \(T^* (T \neq T^*)\). Since \(T ∩ T^* = ∅\), we have \(|T ∪ T^*| = |T| + |T^*|\). Then;

\[
f(T ∪ T^* ) = |I(T ∪ T^*)| - |T ∪ T^* |
= |I(T) ∪ I(T^*)| - (|T| + |T^*|)
= |I(T)| + |I(T^*)| - (|I(T) ∩ I(T^*)| - |T| - |T^*|)
= f(T) + f(T^*) - |I(T) ∩ I(T^*)| < f(T^*).
\]

The last inequality comes from \(f(T) = f(T^*) < 0\). It contradicts the fact that \(T^*\) is a minimizer of \(f\).

Let \(T^*\) denote the unique minimal subset in \(\mathcal{T}\). The cube set \(I(T^*)\) can contain not only \((k, ℓ)\)-cubes but also \((i, j)\)-cubes such that \((i, j)\) and \((k, ℓ)\) are compatible. By Proposition 2 \(|T_{i,j} ∩ T_{k,ℓ}| = 2\). From the definition of \(I(T^*)\), \(|T_{i,j} ∩ T^*| > 0\). Since \(T^* ⊆ T_{k,ℓ}\), \(T_{i,j} ∩ T^* ⊆ T_{i,j} ∩ T_{k,ℓ}\), but we can show that both intersections are equivalent.
Proposition 3  Let $c_{i,j}$ be any cube node in $I(T^*)$ such that $(i, j) \neq (k, \ell)$. Then we have $T_{i,j} \cap T^* = T_{i,j} \cap T_{k,\ell}$.

Proof: Suppose that $T_{i,j} \cap T^* \subseteq T_{i,j} \cap T_{k,\ell}$ holds. Then $|T_{i,j} \cap T^*| = 1$. The node $c_{i,j}$ has 2 neighbors. We denote them by $\tau, \tau' \in T_{k,\ell}$. Without loss of generality, we let $\tau \in T^*$ and $\tau' \in T_{k,\ell} \setminus T^*$. Let us denote $T = T^* \setminus \{\tau\}$. Since $I(T)$ does not contain $c_{i,j}$, we have $|I(T)| \leq |I(T^*)| - 1$. Then $f(T) = |I(T)| - |T| \leq (|I(T^*)|-1)-(|T^*|-1) = f(T^*)$, which contradicts the minimality of the subset $T^*$. □

The proposition states that, in $B_{I,(k,\ell)}$, both of the 2 neighbors of $c_{i,j}$ belong to $T^*$. Now we are ready to prove Lemma 1.

Proof of Lemma 1 We show the necessity. We turn our attention back to the graph $G_{I,(k,\ell)}$. In the graph $T^* \subseteq T_{k,\ell}$ is a subset of nodes. By Proposition 3 any node in $T^*$ and any node out of $T^*$ are not connected. We partition $T^*$ into $T^* = T_1^* \cup T_2^* \cup \cdots \cup T_d^*$ so that each $T_p^*$ $(p = 1, 2, \ldots, d)$ induces a connected component in $G_{I,(k,\ell)}$. We claim that each connected component induced by $T_p^*$ should be a tree. Suppose that the connected component induced by a certain $T_p^*$ is not a tree. Denoting by $q_p$ the number of edges in the component, $|T_p^*| \leq q_p$ holds. Then we have;

\[ f(T^*) = |I(T^*)| - |T^*| = (M_{I})_{k\ell} + \sum_{p=1}^{d} q_p - \sum_{p=1}^{d} |T_p^*| = (M_{I})_{k\ell} - d < 0, \]

which indicates that $(M_{I})_{k\ell}$ should be smaller than the number of trees in $G_{I,(k,\ell)}$. For the sufficiency, let $T^* \subseteq T_{k,\ell}$ be the set of nodes that constitute the trees in $G_{I,(k,\ell)}$. It is easy to see $|I(T^*)| < |T^*|$ from the discussion so far. □

We note that Theorem 1 should provide a succinct way to decide whether the $(k, \ell)$-solid is composable from a given $I$. Theorem 2 tells that, to make the decision, we have only to compare the number $M_{k\ell}$ of $(k, \ell)$-cubes with the number of trees in $G_{I,(k,\ell)}$ that has only 8 nodes and $\sigma_{I,(k,\ell)}$ edges, whereas Theorem 2 requires us to compute a maximum matching in $B_{I,(k,\ell)}$ that has $8 + |I|$ nodes and $8(M_{I})_{k\ell} + 2\sigma_{I,(k,\ell)}$ edges. Even though $|I|$ is not so huge in practice (e.g., $|I| \ll 10^2$), we dare to point out their time complexities that are derived from asymptotic analyses; the former takes only $\Theta(|I|)$ time since it can be done by a typical graph search, while the latter takes $O(|I|^{3/2})$ time even by the best algorithm so far [22]. This may indicate the efficiency of our decision procedure.

When the $(k, \ell)$-solid is composable from $I$, we can determine an assignment of 8 cubes to the corners by the following procedure.

1. For every tree in $G_{I,(k,\ell)}$, put a self-loop on any node. The edge of the self-loop corresponds to a $(k, \ell)$-cube.

2. Color all the 8 nodes white.

3. Repeat the followings while the graph contains a cycle. Choose a cycle $\tau_1 \to \tau_2 \to \cdots \to \tau_n \to \tau_1$. Let $c_k$ be the cube that is associated with an edge between $\tau_k$ and the next node, which is $\tau_{k+1}$ when $1 \leq k < n$ and is $\tau_1$ when $k = n$. For each $k = 1, 2, \ldots, n$, if $\tau_k$ is a white node, assign the cube $c_k$ to $\tau_k$. Color the white nodes on the cycle black. Remove the $n$ edges in the cycle from the graph.
4. Repeat the followings while the graph contains a white node. Choose a white node \( \tau_1 \) and focus on the connected component to which \( \tau_1 \) belongs. Choose a black node \( \tau_n \) in the component. Suppose a path from \( \tau_1 \) to \( \tau_n \), say \( \tau_1 \rightarrow \tau_2 \rightarrow \cdots \rightarrow \tau_n \). For each \( k = 1, 2, \ldots, n-1 \), if \( \tau_k \) is a white node, assign the cube \( c_k \) to \( \tau_k \), where \( c_k \) is associated with an edge between \( \tau_k \) and \( \tau_{k+1} \). Color the nodes on the path black and remove the edges in the path from the graph.

The procedure certainly halts, finding an assignment of cubes to corners whenever the \((k, \ell)\)-solid is composable; Just after we finish 3, every connected component is a tree and has at least one black node. Every black node (corner) has been assigned a cube and the edge corresponding to the cube has been removed. These properties remain to hold after we finish any iteration of 4, where at least one white node is turned into black.

Figure 3 illustrates how the procedure assigns cubes in the instance of \((1, 2)\)-solid. The three figures show how the graph \( G_{f,(1,2)} \) becomes after 2, 3 and 4, respectively. The orientation of a removed edge represents the direction of the chosen cycle or path. We see that the \((1, 2)\)-cube is assigned to \((p, s, u)\), the two \((5, 6)\)-cubes are assigned to \((p, q, t)\) and \((p, u, q)\), the \((6, 5)\)-cube is assigned to \((r, u, s)\), and so on.
5 Minimum Universal and Maximum Infeasible Instances

In this section, we create universal and infeasible instances that are most “distant” from our intuition in terms of the instance size.

5.1 Minimum Universal Instance

**Lemma 2** The size of the universal instance is at least 12.

**Proof:** Let $I$ be a universal instance. For any variety $(k, \ell)$, since the $(k, \ell)$-solid is composable, the number $|E_I| + (M_I)_{k\ell} = \sigma_{I,(k,\ell)} + (M_I)_{k\ell}$ should be at least 8. Then the sum of this number over the 30 varieties is at least $30 \times 8 = 240$. On the other hand, each $(i,j)$-cube in $I$ contributes to the sum in exactly 21 varieties; once for $G_{I, (i,j)}$ and 20 times for $G_{I, (k,\ell)}$-s such that $(i,j) \in V_{k,\ell}$. Hence we have $|I| \geq 240/21 > 11$. □

**Lemma 3** There is a universal instance that consists of 12 cubes.

**Proof:** The instance $I$ that has the following matrix representation is universal. The size of $I$ is exactly 12;

$$M_I = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}.$$  \(2\)

One can check the universality by using the procedure that we explained in the last section. □

**Theorem 3** The size of the minimum universal instance is 12.

We found a minimum universal instance in \(2\) by means of the integer programming. We describe how the integer programming can be applied to instance creation in the concluding section.

5.2 Maximum Infeasible Instance

**Lemma 4** There is an infeasible instance of size 23.

**Proof:** We show an instance $I$ that has the following matrix representation to be infeasible;

$$M_I = \begin{pmatrix}
0 & 7 & 7 & 7 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  \(3\)

The size of $I$ is exactly 23. We claim that the graph $G_{I, (k,\ell)}$ for every $(k,\ell) \in V_{all}$ does not satisfy the composable condition of Theorem\(^1\). Recall Properties\(^1\) and\(^2\) of Conway’s Table. For every $(1,\ell)$ ($\ell = 2, \ldots, 6$), the graph $G_{I, (1,\ell)}$ consists of eight isolated points, and thus has eight trees, while $(M_I)_{1\ell}$ is smaller than 8.
Suppose $k, \ell = (2, 1)$. We claim that the graph $G_{I, (2, 1)}$ should contain a tree. We have $(1, 2) \notin V_{2, 1}$ and $(1, 3), \ldots, (1, 6) \in V_{2, 1}$. The edges appearing in $G_{I, (2, 1)}$ come from the cubes of the varieties $(1, 3), \ldots, (1, 6)$. The edges coming from the same variety cubes form multi-edges, and those coming from different variety cubes do not touch with each other since they are pairwise incompatible. There are four connected components in $G_{I, (2, 1)}$, each of which consists of two nodes. Among these, two connected components contain only one edge respectively, which are trees. Since $(M_I)_{21} = 0$, the $(2, 1)$-solid is not composable. All the remaining varieties $(k, \ell)'$-s are analogous. □

Berkove et al. [12] showed that the size of the maximum infeasible instance is at least 22, presenting an infeasible instance of size 22 as the certificate. The $I'$ is a subset of the above mentioned $I$ such that $(M_{I'})_{1\ell} = (M_I)_{1\ell}$ for $\ell = 2, \ldots, 5$ and $(M_{I'})_{16} = 0$. Also they conjectured the maximum size to be 23, without presenting the evidence explicitly. They left the problem of deciding the maximum size open.

Lemma 4 provides a concrete example of a 23-size infeasible instance. Furthermore, we find that there is no infeasible instance of size 24 by means of computer search, which gives an affirmative answer to Berkove et al.’s open problem.

**Lemma 5** An instance is not infeasible if the size is no less than 24.

**Proof:** We investigated all the 24-size instances to see whether an infeasible instance of that size exists. For this purpose, we wrote a program code in C.

Below we outline the algorithm of this program. The algorithm consists of two phases of enumeration. The first phase enumerates all the partitions of the integer 24. A partition of an integer $m$ is defined as a sequence $(m_1, \ldots, m_k)$ of integers such that $m_1 + \cdots + m_k = m$ and $1 \leq m_1 \leq \cdots \leq m_k$. Then for each partition $(m_1, \ldots, m_k)$ of 24, the second phase enumerates all the 24-size instances such that the 24 cubes have $k$ different varieties and that the distribution of the numbers of the cubes of the $k$ varieties is $(m_1, \ldots, m_k)$, when it is sorted in the non-decreasing order.

If the algorithm were naïvely implemented, it would take too much time for us to see the result. In particular, in the first phase, the number of partitions of 24 is $S(24, 1) + S(24, 2) + \cdots + S(24, 24) \approx 4.5 \times 10^{17}$, where $S(m, k)$ denotes the Stirling number of the second kind, the number of partitions of $m$ elements into $k$ non-empty subsets. To improve the efficiency, we introduce necessary conditions that a partition should satisfy in the first phase so that an infeasible instance could be generated in the second phase. If a partition does not satisfy at least one of the conditions, then we discard it since there is no possibility to generate an infeasible instance. We describe the list of the conditions below.

- The largest number in the partition should be at most 7, i.e., $m_k \leq 7$. If not so, no infeasible instance would be generated in the second phase; each $m_i$ becomes the number of cubes of a certain variety in the second phase. If $m_k \geq 8$, a solid can be composed from 8 cubes of the same variety. As a consequence, the size of a partition should be at least 4, i.e., $k \geq 4$.

- The size of a partition should be at most 9, i.e., $k \leq 9$. In our preliminary experiments, we found that any 10-size instance $I$ such that all the cubes have different varieties is not infeasible. The number of such instances is $(10) \approx 3.0 \times 10^7$. It is not so large, and we investigated whether each $I$ is infeasible or not, by using Theorem [2]. Then we find that no $I$ is infeasible. Obviously any superset of $I$ is not infeasible either. Then, any $I'$ with $|V(I')| \geq 10$ cannot be infeasible.

2The author is ready to provide the program code with those who are interested.
Lemma 4.1 in [12] provides necessary conditions that the partition should satisfy. For example, if \( k \geq 7 \), then \( m_1 = \cdots = m_{k-6} = 1 \). See [12] for detail.

It is only 78 partitions that satisfy all of the above conditions.

For each of the 78 partitions \( (m_1, \ldots, m_k)'s \), we investigate all the instances such that the matrix representations are obtained by assigning each \( m_i \) to non-diagonal entries of a 6 \( \times \) 6 zero matrix. A rough estimate of the number of such instances could be \((30)_k = 30 \times 29 \times \cdots \times (30-k+1)\), which amounts to \( 5.2 \times 10^{12} \) when \( k = 9 \). The number is so huge, but we do not need to investigate all of them; we can restrict ourselves to an assignment such that:

\[
(M_I)_{12} \geq (M_I)_{13} \geq \cdots \geq (M_I)_{16} \quad \text{and} \quad \sum_{(1,\ell) \in V_{all}} (M_I)_{1\ell} \geq \sum_{(k,\ell) \in V_{all}} (M_I)_{k\ell} \quad (k = 2, 3, \ldots, 6).
\]

This is because, from any instance, permuting the 6 colors appropriately, we can construct an instance whose matrix representation satisfies (3). Permuting the 6 colors invokes rearrangement of the 30 numbers in the non-diagonal entries of the 6 \( \times \) 6 matrix representation. Since there are 6! = 720 permutations of colors, there are 720 rearrangements of the 30 numbers. The numbers are arranged as follows: Suppose partitioning the 30 numbers into 6 subsets so that each subset contains the 5 numbers from the same row. Permuting the 6 colors, the 5 numbers in a certain subset are moved to the first row. Furthermore, there are 5! = 120 permutations of these numbers. Now we see 6 \( \times \) 120 = 720 permutations.

Over the 78 partitions, it is only 2.8 \( \times \) 10^7 instances that the program searched. The search takes less than 6 minutes for our computer that carries a 3.40GHz CPU and an 8GB main memory. As a result, we did not find an infeasible instance. □

Theorem 4 The size of the maximum infeasible instance is 23.

6 Concluding Remarks

We have considered a simplest generalization of “Eight Blocks to Madness” puzzle. The main contributions of the paper are a novel necessary and sufficient condition that a solid of a given variety is composable from a given instance (Theorem 1), and creation of minimum universal and maximum infeasible instances (Theorems 3 and 4, respectively).

Our future work includes the problem of instance creation in a general setting. The general problem is summarized as follows.

| Instance Creation Problem |
|---------------------------|
| **Input:** A subset \( V \subseteq V_{all} \) of varieties. |
| **Question:** Is there a \( V \)-generatable instance? If the answer is yes, output a \( V \)-generatable instance of the minimum size (or of the maximum size). |

In this paper we gave answers to the extreme cases of \( V = V_{all} \) and \( V = \emptyset \), yielding the most “nontrivial” instances in terms of the instance size. Our main concerns are the following problems. We leave them open.

**Problem 1.** Does there a \( V \)-generatable instance exist for any \( V \subseteq V_{all} \)?

**Problem 2.** When a \( V \)-generatable instance exists, what are the minimum and maximum sizes?

In particular, what is the condition that the minimum size is more than 8 and what is the condition that the maximum size is finite?
We conjecture that the answer to Problem 1 should be no. We suspect that $V = \{(1,2),\ldots,(1,6)\}$ (i.e., the set of 5 varieties in the same row of Conway’s Table) should be a certificate to our conjecture.

Let us describe why we conjecture so. In our preliminary studies, to gain insight into Instance Creation Problem, we formulated the problem by the integer programming so that we are asked to find a $V$-generatable instance of the minimum size for a given $V \subseteq V_{all}$. In our formulation, we have two types of variables. One is integral variables, $x_{i,j}$-s ($((i,j) \in V_{all})$, that represent the numbers of $(i,j)$-cubes in the instance $I$. Thus we have 30 integral variables. The other is binary variables that are used to constrain the non-composability of $(k',\ell')$-solid ($(k',\ell') \notin V_{all}$), which will be explained soon. Recall Theorem 2 that states a necessary and sufficient condition that a variables that are used to constrain the non-composability of $(k',\ell')$-solid is composable in terms of the matching number of the bipartite graph $B_{I,(k,\ell)}$. Denoting the matching number by $\mu(B_{I,(k,\ell)})$, the problem is described as follows.

\[
\text{minimize } z \\
\text{subject to } z = \sum_{(i,j) \in V_{all}} x_{i,j} \geq 8 \\
\mu(B_{I,(k,\ell)}) = 8 \quad \text{for } \forall (k,\ell) \in V \\
\mu(B_{I,(k',\ell')}) \leq 7 \quad \text{for } \forall (k',\ell') \in V_{all} \setminus V \\
x_{i,j} \in \mathbb{Z}^+ \cup \{0\} \quad \text{for } \forall (i,j) \in V_{all}
\]

We explain how to express the conditions (4) and (5) in terms of linear inequalities. For every $(k,\ell) \in V$, the condition (4) can be stated as follows; for any subset $T \subseteq T_{k,\ell}$ of corner triples,

\[
|I(T)| = \sum_{(i,j) \in V_{all}: T_{i,j} \cap T \neq \emptyset} x_{i,j} \geq |T|.
\]

For every $(k',\ell') \in V_{all} \setminus V$, the condition (5) is rewritten as follows; there is a subset of $T \subseteq T_{k',\ell'}$ of triples such that $|I(T)| \leq |T| - 1$ holds. Let us introduce a binary variable $y_T$ for each subset $T \subseteq T_{k',\ell'}$. Using a sufficiently large constant $C$, the condition (5) is expressed by a set of linear inequalities as follows; for any subset $T \subseteq T_{k',\ell'}$,

\[
Cy_T - \left( \sum_{(i,j) \in V_{all}: T_{i,j} \cap T \neq \emptyset} x_{i,j} - (|T| - 1) \right) \geq 0,
\]

and

\[
\sum_{T \subseteq T_{k',\ell'}} y_T \leq 2^{|T_{k',\ell'}| - 1} - 2^8 - 1 = 255.
\]

One can easily show that there is a subset $T \subseteq T_{k',\ell'}$ that satisfies $|I(T)| \leq |T| - 1$ iff there is an assignment of 0 or 1 to the binary variables $y_T$-s over all subsets $T'$-s of $T_{k',\ell'}$ such that (4) and (5) are satisfied together with $x_{i,j}$-s.

We solved the integer programming problems for several $V'$-s, using IBM ILOG CPLEX [24], a state-of-the-art optimization software. Setting $V = V_{all}$, we found the minimum universal instance in (2). We also found that, for any $V$ such that $|V| \leq 4$, there is a $V$-generatable instance. In most cases, the optimal solution is found within a few seconds. On the other hand, when $V = \{(1,2),\ldots,(1,6)\}$ with $|V| = 5$, the computation terminates due to memory shortage, without finding any $V$-generatable instance, after a couple of hours. The problem symmetricity must be one of the unsuccessful reasons. Based on these, we conjecture that there is $V \subseteq V_{all}$ such that no $V$-generatable instance exists.

Problem 2 must be also an interesting issue. For example, when $V = V_{all}$, the minimum size is 12 (Theorem 3), and the maximum size is infinite; any superset is also universal. On the other
hand, when \( V = \emptyset \), the minimum size is 8 (e.g., an instance \( I \) such that \( (\mathcal{M}_I)_{12} = (\mathcal{M}_I)_{21} = 4 \)) and the maximum size is 23, which is finite (Theorem 4). The general case is open.

Although being quite simple and classical, colored cube puzzles still provide us with many mathematical problems. We hope that the paper forms a basis of future research in recreational mathematics.

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