Conformal Bach flow

Jiaqi Chen¹ · Peng Lu² · Jie Qing³

Received: 28 June 2022 / Accepted: 15 March 2023 / Published online: 30 March 2023
© The Author(s), under exclusive licence to Springer Nature B.V. 2023

Abstract
In this article we introduce conformal Bach flow and establish its well-posedness on closed manifolds. We also obtain its backward uniqueness. To give an attempt to study the long-time behavior of conformal Bach flow, assuming that the curvature and the pressure function are bounded, global and local Shi’s type $L^2$-estimate of derivatives of curvatures is derived. Furthermore, using the $L^2$-estimate and based on an idea from (Streets in Calc Var PDE 46:39–54, 2013) we show Shi’s pointwise estimate of derivatives of curvatures without assuming Sobolev constant bound.

Keywords Conformal Bach flow · Short-time existence · Backward uniqueness · Shi’s type $L^2$ and pointwise estimate of derivatives of curvatures · Finite-time singularity

Mathematics Subject Classification Primary 53C43; Secondary 35K41 · 58J35

1 Introduction
It is well known in the field of conformal geometry that it is very desirable to make use of Bach tensor (cf. (2.2) below) to generate curvature flow. One obvious reason is that Bach tensor is the $L^2$-gradient of Weyl curvature in dimension 4. Note that some versions of Bach flows have been proposed, for example, in [1, (7.7)]. In 2011 Bahuaud and Helliwell [2, Section 5] found a well-posed curvature flow by Bach tensor as follows:

P.L. is partially supported by Simons Foundation through Collaboration Grant 229727. J.Q. is partially supported by NSF DMS-1608782.

✉ Peng Lu
penglul@uoregon.edu
Jiaqi Chen
jche133@ucsc.edu
Jie Qing
qing@ucsc.edu

¹ School of Electrical Engineering and Automation, Xiamen University of Technology, Xiamen 361024, Fujian, China
² Department of Mathematics, University of Oregon, Eugene, OR 97403, USA
³ Department of Mathematics, University of California, Santa Cruz, CA 95064, USA
where $B = B(g)$ is the Bach tensor, $S = S_g$ is the scalar curvature, and $\Delta = \Delta_g$ is the Laplace–Beltrami operator on a manifold of dimensions $n$. The additional term $(\Delta S)g$ with the appropriate coefficient is important and needed for the well-posedness of the flow.

There are many geometric curvature flows studied in the literature. One of them, Ricci flow has made tremendous progress since R. Hamilton’s work in 1982, and several marvelous theorems are proved using the method of Ricci flow starting from G. Perelman’s work in 2003. Another curvature flow we would like to mention here is conformal Ricci flow (CRF in short), which was proposed by Fischer [9] as a modified Ricci flow that keeps the scalar curvature as a unchanged constant by adding a pressure term as follows:

$$\partial_t g = -2 \left( \text{Rc}_g - \frac{s_0}{n} \right) g - 2 p g.$$  

This resembles the Navier–Stokes equations in fluid dynamics and hopefully it produces more efficient approach to find Einstein metrics if possible. Particularly, CRF seems to be more efficient approach to find Einstein metrics if possible. Particularly, CRF seems to be transverse to classes of conformal metrics, which attracts attentions from researchers in conformal geometry (cf. [9, 10, 20, 21, 30]).

Inspired by the idea behind CRF, here we propose the following conformal Bach flow (CBF in short). We always assume dimension $n \geq 4$ below. Suppose that $(M^n, g_0)$ is an $n$-dimensional Riemann manifold with constant scalar curvature $s_0$. CBF is a family of metrics $\{g(t)\}_{t \in [0, T]}$ which satisfies

$$\begin{cases} 
\partial_t g = 2(n - 2) (B(g) + p)g & \text{on } M \times [0, T], \\
S_{g(t)} = s_0 & \text{for } t \in [0, T],
\end{cases}$$

where $p = p(t)$ is a family of functions on $M$. This is a fourth- order evolution equation of $g(t)$. The pressure function $p(t)$ at each time induces the conformal change to keep the scalar curvature as a unchanged constant. Notice that, in contrast to (1.1), the term $(\Delta S)g$ does not make any appearance in (1.2). Similar to CRF, the system (1.2) is equivalent to the following coupled weakly parabolic and elliptic equations:

$$\begin{cases} 
\partial_t g = 2(n - 2) (B(g) + p)g & \text{on } M \times [0, T], \\
((n - 1)\Delta g(t) + s_0) p(t) = -(n - 2) A(g(t)) \cdot B(g(t)) + \nabla^2 g(t) \cdot B(g(t))
\end{cases}$$

for each $t \in [0, T]$. Here $A$ is the Schouten tensor, and we adopt the notations $A \cdot B = A_{ij}B_{ij}$, $\nabla^2 \cdot B = \nabla_i \nabla_j B_{ij}$, and the covariant derivative $\nabla = \nabla_{g(t)}$. In light of Lemma 2.1 below, $\nabla^2 \cdot B$ is in fact of order 4 instead of 6 in term of derivatives of metrics. Nevertheless, CBF (1.3) is a fourth-order geometric flow of $g$. There are indeed growing interests in higher-order geometric flows lately, for instance, among others: Calabi flow [5, 6], Yang-Mills flow [13, 23, 29, 31], Willmore flow [17, 18], $L^2$-Riemann curvature flow [24–28], and Ambient obstruction flow [2, 19].

In this article we initiate the study of CBF. Let us describe the main results of this article. First, we adopt the DeTurck’s trick in Lemma 2.4 below to eliminate the degeneracy from the invariance under diffeomorphisms of the system (1.3). To prove the short-time existence, following the approach used in [20], we first solve $p$ from the second equation in (1.3) and then the existence of a short-time solution to the mixed differential-integral evolution Eq. (2.14) can be established by applying a version of Newton’s method based on the existence theory of systems of linear parabolic differential equations (see Proposition 2.11).
Theorem 1.1 Let $M^n$ be a closed manifold. Assume that $g_0$ is a $C^{4 + \alpha}$ Riemannian metric on $M$ with constant scalar curvature $S_{g_0} = s_0$ and that the elliptic operator $(n - 1)\Delta g_0 + s_0$ is invertible. Then there exist a unique $C^{4 + \alpha, 1 + \frac{\alpha}{2}}$ family of Riemannian metrics $g(t)$ and a $C^{2 + \alpha, 0}$ family of pressure functions $p(t)$ which solve CBF (1.3) for some small $T > 0$ and satisfy the initial condition $g(0) = g_0$.

The metrics $g(t)$ and the functions $p(t)$ are shown to be smooth when the initial metric is smooth (cf. Theorem 2.13). We also show that the backward uniqueness holds for CBF on closed manifolds. Readers are referred to [14] for the history and significance of the backward uniqueness of geometric flows. It is desirable to know whether the assumption that operator $(n - 1)\Delta g(t) + s_0$ is invertible for each $t$ in the following theorem could be removed.

Theorem 1.2 Let $M^n$ be a closed manifold and let $(g(t), p(t))$ and $(\tilde{g}(t), \tilde{p}(t))$, $t \in [0, T]$, be two smooth solutions of (1.3) on $M$. We assume that operator $(n - 1)\Delta g(t) + s_0$ is invertible for each $t$. If $g(T) = \tilde{g}(T)$, then $(g(t), p(t)) = (\tilde{g}(t), \tilde{p}(t))$ for all $t$.

For the study of large time behaviors of solutions to CBF, we follow the approach from [24–28] and establish the integral version of Shi’s type estimate of derivatives of curvature (cf. Theorem 4.1) assuming that the curvature and the pressure function are uniformly bounded in space and time. Moreover, a local version of integral Shi’s type estimate (Theorem 5.5) is also established. Finally, based on an argument in the proof of [28, Theorem 1.3], we obtain the Shi’s pointwise estimate assuming only the boundedness of curvature and pressure function (without uniform Sobolev constant).

Theorem 1.3 Let $M^n$ be a complete manifold and let $(g(t), p(t))$ be a smooth solution to (1.3) in $M \times [0, T]$ with constant scalar curvature $s_0$. We assume that for some constant $K > 0$

$$\sup_{(x, t) \in M \times [0, T]} (|Rm(x, t)| + |p(x, t)|) \leq K. \tag{1.4}$$

Then, for any $m \in \mathbb{N}$ there exists a constant $C = C(n, m)$ such that

$$|\nabla_{g(t)}^m Rm(x, t)|_{g(t)} \leq C \left( K + t^{-\frac{1}{2}} \right)^{1 + \frac{m}{2}} \text{ for } (x, t) \in M \times (0, T].$$

Theorem 1.3 provides a criterion of the finite singular time of CBF and possibly facilitates further studies of long-time behaviors of CBF.

Now we give an outline of the article. In Sect. 2 we introduce CBF and prove Theorem 1.1 using Newton’s method, near the end of the section we give a quick proof of the uniqueness and the improved regularity. In Sect. 3 we prove the backward uniqueness (Theorem 1.2). In Sect. 4 we prove a Shi’s type $L^2$-estimate of derivatives of curvatures for CBF. We also give two quick consequences. One is a characterization of when a singularity develops in CBF, and the other is a compactness theorem for a family of CBF. In Sect. 5 we first prove a local version of Shi’s type $L^2$-estimate of derivatives of curvatures for CBF, then we prove a Shi’s type pointwise estimate of derivatives of curvatures without assuming uniform Sobolev constant bound (Theorem 1.3).

The authors thank the anonymous referee for helpful suggestions which improve the presentation of the paper.
2 Well-posedness of CBF: existence, uniqueness and regularity

In this section we prove Theorem 1.1. This is done in steps. The first step consists of a sequence of conversions. We convert solving (1.3) into solving modified CBF (2.4), then we use the DeTurck’s trick to further convert the solving into solving DeTurck modified CBF system (2.8) (Lemma 2.4), and the last conversion is to use an idea in [20] to reduce to solve the decoupled DeTurck modified CBF (2.14). The second step is to solve the linearized equation of (2.14) by contraction mapping theorem from the solvability of linear parabolic system (see Lemma 2.6) in Sect. 2.3. In the last step we solve (2.14) by Newton’s method in Sect. 2.4 (Proposition 2.11).

In the last two subsections we prove the uniqueness and the improved regularity. We begin this section with some preliminaries of CBF.

2.1 Conformal Bach flow

Let \((M^n, g)\) be a Riemannian manifold. The Schouten tensor \(A = A(g)\) is defined by

\[
A_{ij}(g) \equiv \frac{1}{n-2} \left( R_{ij} - \frac{S}{2(n-1)} g_{ij} \right),
\]

where \(R_{ij}\) is the Ricci tensor and \(S\) is the scalar curvature of metric \(g\). Let \(W\) be the Weyl tensor. The Cotton tensor is defined by

\[
C_{ijk} \equiv \nabla_k A_{ij} - \nabla_j A_{ik} = \frac{n-2}{n-3} \nabla_l W_{lijk}.
\]

The Bach tensor, which is well-known in general relativity, is defined by

\[
B_{ij}(g) \equiv \frac{1}{n-3} \nabla_l W_{klj} + \frac{1}{n-2} R_{kl} W_{klj} = \nabla_k \nabla_l A_{ij} - \nabla_i \nabla_j A_{lk} + A_{lk} W_{klj}.
\]

Metric \(g\) is called Bach flat if \(B(g) = 0\). Note that we adopt the convention that Ricci tensor \(R_{ij} = g_{kl} R_{iklj}\).

Now we list a few basic facts about Bach tensor which will be used later. Bach tensor is trace-free. In dimension 4 Bach tensor is conformally invariant. We have [7, Lemma 5.1]

**Lemma 2.1** For Riemannian manifold \((M^n, g)\), the divergence of Bach tensor

\[
\nabla_j B_{ij} = -\frac{n-4}{(n-2)^2} C_{jki} R_{jk}.
\]

Hence, \(\nabla^2 \cdot B(g) = \nabla_i \nabla_j B_{ij}(g)\) is a differential operator of \(g\) up to fourth derivatives. In particular, in dimension 4 Bach tensor is divergence-free.

Examples of four-dimensional Bach flat metrics are: (i) those locally conformal to Einstein metrics, and (ii) those half-conformally flat [4, Proposition 4.78].

Now we give a proof of the equivalence of systems (1.2) and (1.3).

**Lemma 2.2** Let \(M^n\) be a closed manifold. We define a modified CBF by

\[
\begin{cases}
\partial_t g = 2(n-2) \left( B(g) + \frac{1}{2(n-1)(n-2)} (\Delta g) g + pg \right), \\
((n-1) \Delta g + s_0) p(t) = -(n-2) A(g(t)) \cdot B(g(t)) + \nabla^2_{g(t)} \cdot B(g(t)).
\end{cases}
\]

Then the flows defined by (1.2), (1.3), and (2.4) with initial metric \(g(0) = g_0\) of constant scalar curvature \(s_0\), are all equivalent.
**Proof** We prove the equivalences through three parts. Part 1. Suppose that \( g(t) \) and the associated \( p(t) \) is a solution of (1.2), we prove that the pair \((g(t), p(t))\) is a solution of (1.3) and (2.4) by showing that \( p(t) \) satisfies the second equation in (1.3). As usual, in the computation below \( \Delta, \nabla, R_{ij} \), et al. are defined by metric \( g(t) \).

We apply the first variation formula of scalar curvature (see, for example, [8, p.109] using \( v_{ij}(t) = 2(n - 2)(B_{ij} + pg_{ij}) \) and \( V(t) = g_{ij}(t)v_{ij}(t) \)). We simplify to get

\[
\partial_t S_{g(t)} = 2(n - 2) \left[ -\Delta (g_{ij}(B_{ij} + pg_{ij})) + \nabla_i \nabla_j (B_{ij} + pg_{ij}) - R_{ij}(B_{ij} + pg_{ij}) \right]
= 2(n - 2) \left[ -(n - 1)\Delta p - s_0 p - (n - 2)A_{ij}B_{ij} + \nabla_i \nabla_j B_{ij} \right],
\]

(2.5)

where we have used the facts that Bach tensor is trace-free and \( S_g \) with initial data \( S_{g(0)} = s_0 \). Hence, \( S_{g(t)} = s_0 \).

Part 2. Suppose that pair \((g(t), p(t))\) is a solution of (1.3), we prove that \( g(t) \) associated with \( p(t) \) is a solution of (1.2) and also a solution of (2.4) by showing \( S_{g(t)} = s_0 \). Combining the first equality in (2.5) and the second equation in (1.3), we have

\[
\partial_t S_{g(t)} = 2(n - 2) \left(s_0 - S_{g(t)} \right) p(t)
\]

with initial data \( S_{g(0)} = s_0 \). Hence, \( S_{g(t)} = s_0 \).

Part 3. Suppose that \((g(t), p(t))\) is a solution of (2.4), we prove that \((g(t), p(t))\) is a solution of (1.2) by showing \( S_{g(t)} = s_0 \). Calculating as in Part 1, we have

\[
\partial_t S_{g(t)} = \left(s_0 - S_{g(t)} \right) p - \left(\Delta^2 S_{g(t)} + \frac{1}{n-1} S_{g(t)} \Delta S_{g(t)} \right),
\]

where we have used the equations in (2.4) to get the equality. By the uniqueness of solutions of the fourth-order linear (by treating \( \frac{1}{n-1} S_{g(t)} \) as a known function) parabolic equation on closed manifolds we conclude that \( S_{g(t)} = s_0 \).

**Remark 2.3** In dimension \( n = 4 \), since divergence \( \nabla_i B_{ij} = 0 \), (1.3) simplifies to

\[
\begin{align*}
\partial_t g &= 4 \left( B + pg \right), \\
(3\Delta g(t) + s_0) p(t) &= -2A.g(t) \cdot B(g(t)).
\end{align*}
\]

Note that also in dimension 4 Bach tensor is the negative of the gradient of the \( L^2 \)-norm functional of Weyl tensor (see [4, p.135]), we conclude that \( \int_M |W_g|_g^2 d\mu_g \) is non-increasing under the flow above.

### 2.2 DeTurck modified CBF

First, we give a little motivation for our route of using (2.4) to solve (1.3) described at the beginning of this section. Note that Bach tensor can be written schematically as

\[
B_{ij} = \frac{1}{n-2} \Delta R_{ij} - \frac{1}{2(n-1)(n-2)} (\Delta S)_{g_{ij}} - \frac{1}{2(n-1)} \nabla_i \nabla_j S + \text{lower-order terms,}(2.6)
\]

hence, we may modify the standard choice of the vector field in the DeTurck’s trick for Ricci flow by adding the action of \( \Delta g \) (see the first term in (2.7) below) to eliminate the degeneracy in \( \frac{1}{n-2} \Delta R_{ij} \) caused by the symmetry of diffeomorphisms, the term \(-\frac{1}{2(n-1)(n-2)} (\Delta S)g\) has to be taken care of as in [2] which partly explains why we use (2.4) rather than (1.3), and lastly the term \(-\frac{1}{2(n-1)} \nabla_i \nabla_j S\) is of Hessian-type and can be taken care of by modifying the choice of the vector field (see the second term in (2.7)).
Fixing a metric $\tilde{g}$ with enough smoothness as a background metric and following the choice of the vector field for DeTurck’s trick as in [2, Section 5.2], we define vector field

$$W^i_g = -g^{ij} \Delta_g \left( \Gamma^i_{lj}(g) - \Gamma^i_{lj}(\tilde{g}) \right) + \frac{n-2}{2(n-1)} \nabla g S g^i.$$  \hfill (2.7)

Recall that Lie derivative $(\mathcal{L}_W g)_{ij} = \nabla_i W_j + \nabla_j W_i$. We define the DeTurck modified CBF as

$$\partial_t g = 2(n-2) \left( B(g) + \frac{1}{2(n-1)(n-2)} (\Delta g) g + pg \right) + \mathcal{L}_W g,$$

$$(n-1) \Delta g(t) + s_0 p(t) = -(n-2) A(g(t)) \cdot B(g(t)) + \nabla^2 g(t) \cdot B(g(t)).$$  \hfill (2.8)

We will consider its initial value problem $g(0) = g_0$, assuming scalar curvature $S_{g_0} = s_0$.

If we have a solution $(g(t), p(t))$ of (2.8), using the vector field $W_g$ we may define an one-parameter family of diffeomorphism $\phi_t : M \to M$, $t \in [0, T]$, by solving ordinary differential equations for each $x \in M$

$$\partial_t \phi_t(x) = -W_{g(t)}(\phi_t(x)) \quad \text{with} \quad \phi_0(x) = x. \hfill (2.9)$$

**Lemma 2.4** (i) Let $(g(t), p(t))$ be a complete solution of (2.8) on manifold $M^n$ with initial metric $g_0$ whose scalar curvature $S_{g_0} = s_0$. Let $\phi_t$ be the solution of (2.9). We define $\tilde{g}(t) = \phi_t^* g(t)$ and $\tilde{p}(x, t) = p(\phi_t(x), t)$. Then $(\tilde{g}(t), \tilde{p}(t))$ is a solution of the modified CBF (2.4) with initial condition $g(0) = g_0$.

(ii) Let $(\hat{g}(t), \hat{p}(t))$ be a solution of (2.4) with initial metric $g_0$ whose scalar curvature $S_{g_0} = s_0$. As in [8, p.117] we conclude that Eq. (2.9) is equivalent to fourth-order parabolic equation of maps $\phi_t : M \to M$

$$\partial_t \phi_t = -\Delta_{\tilde{g}(t)} \Delta_{\tilde{g}(t)} \phi_t + \frac{n-2}{2(n-1)} \nabla \tilde{g} S \phi_t,$$  \hfill (2.10)

where $\Delta_{\tilde{g}(t)}$ is the map Laplacian. If we have a solution of (2.10) with $\phi_0 = \text{Id}$, then one can verify that $(\phi_t^{-1})^* \hat{g}(t), \hat{p}(\phi_t^{-1}(x), t))$ is a solution of (2.8) with initial metric $g_0$.

**Proof** (i) A direct computation gives us:

$$\partial_t \tilde{g}(t) = \phi_t^* (\partial_t g(t)) + \partial_s |_{s=0} \left( \phi_t^* g(t) \right)$$

$$= 2(n-2) \phi_t^* \left( B(g) + \frac{1}{2(n-1)(n-2)} (S g) g + pg \right) + \phi_t^* (\mathcal{L}_W g) - \mathcal{L}_{(\phi_{t}^{-1})^* W_g} (\phi_t^* g)$$

$$= 2(n-2) \left( B(\phi_t^* g) + \frac{1}{2(n-1)(n-2)} (\Delta \phi_t^* g S \phi_t^* g) + \hat{p} \phi_t^* g \right).$$

This verifies the first equation in (2.4). The second equation in (2.4) for $\hat{p}(t)$ follows from the second equation in (2.8) for $p(t)$ directly.

(ii) This follows from calculations which are similar to those in (i). \hfill \Box

We need some preparations before we engage into the actual proof of the short-time existence of CBF (1.3). Below the Hölder exponent $\alpha \in (0, 1)$. We have the following inequality (see [20, Lemma 3.5] and [22, p. 175–177], along with the definitions of Hölder spaces and Sobolev spaces for tensors). As a quick reminder, the norm of $h \in C^{k+\alpha,1}(M \times [0, T])$ is the sum of the space $C^{k+\alpha}$-norm of $h$ and the $C^0$ norm of $\partial_t h$.
Lemma 2.5 Assume \( k \geq 2 \). There is a constant \( C \) independent of \( T \) such that for any \( t_1, t_2 \in [0, T] \), we have
\[
\|h(\cdot, t_1) - h(\cdot, t_2)\|_{C^{k+\alpha,1}(M)} \leq C|t_1 - t_2| \cdot \|h\|_{C^{k+\alpha,1}}
\]
for any tensor \( h \in C^{k+\alpha,1}(M \times [0, T]) \).

As the starting point to solve the CBF (1.3) we need the solvability of linear parabolic system (see [11], [20, p.424], or [2, Proposition 3.2 and 3.3].

Lemma 2.6 (i) Let \( L \) be a linear elliptic differential operator of order \( 2m \) with \( C^{\alpha,0} \)-coefficients on a closed manifold \( M^n \) acting on tensors (see [2, p.2195] for the local expression of \( L \)). Then for every \( f \in C^{\alpha,0}(M \times [0, T]) \) there is a unique solution \( u \in C^{2m+\alpha,1}(M \times [0, T]) \) of
\[
\partial_t u = Lu + f, \quad u(0, \cdot) = 0.
\]
And there is a constant \( C \) such that
\[
\|u\|_{C^{2m+\alpha,1}} \leq C\|f\|_{C^{\alpha,0}}
\]
for all \( f \in C^{\alpha,0}(M \times [0, T]) \).

(ii) Moreover if the coefficient functions of operator \( L \) and \( f \) in (i) have better regularity of \( C^{k+\alpha,0} \), then \( u \in C^{2m+k+\alpha,1}(M \times [0, T]) \) and we have Schauder estimate
\[
\|u\|_{C^{2m+k+\alpha,1}} \leq C\|f\|_{C^{k+\alpha,0}}.
\]

Because (2.8) is a coupled system of \((g(t), p(t))\), we further need to reduce it to an equation of \( g(t) \) only and then solve the resulting equation for the short-time existence. The following simple lemma is [20, Lemma 3.4].

Lemma 2.7 (i) Suppose that \( g(t), t \in [0, T] \), is a family of \( C^{1+\alpha,0} \)-metric such that the elliptic operator \((n-1)\Delta_g(t) + s_0\) is invertible for each \( t \). Then there is a constant \( C \) such that for each \( \gamma \in C^{\alpha,0}(M \times [0, T]) \) equation
\[
((n-1)\Delta_g(t) + s_0)\ p(t) = \gamma(t)
\]
has a unique solution \( p \in C^{2+\alpha,0}(M \times [0, T]) \), and
\[
\|p\|_{C^{2+\alpha,0}} \leq C\|\gamma\|_{C^{\alpha,0}}.
\]

(ii) Moreover, if \( g(t) \) is in \( C^{k-1+\alpha,0} \) and \( \gamma \) is in \( C^{k-2+\alpha,0} \) for some \( k \geq 2 \), then we have the solution \( p \in C^{k+\alpha,0}(M \times [0, T]) \) and estimate
\[
\|p\|_{C^{k+\alpha,0}} \leq C\|\gamma\|_{C^{k-2+\alpha,0}}.
\]

Consider metric \( g \in C^{4+\alpha}(M) \) such that the operator \((n-1)\Delta_g + s_0\) is invertible, we may define an operator \( \mathcal{P}(\cdot) \) such that \( \mathcal{P}(g) \in C^{2+\alpha}(M) \) is the solution \( p \) of the equation
\[
((n-1)\Delta_g + s_0)\ p = -(n-2)A(g) \cdot B(g) + \nabla_g^2 B(g) \in C^{\alpha}(M).
\]

Lemma 2.8 Suppose that \( M^n \) is a closed manifold with metrics \( g \in C^{4+\alpha,0}(M \times [0, T]) \). Suppose that the elliptic operator \((n-1)\Delta_g(t) + s_0\) is invertible for each \( t \in [0, T] \). Then there are constants \( C > 0 \) and small \( \delta_0 > 0 \) depending on \( g \) such that
\[
\|\mathcal{P}(g_1) - \mathcal{P}(g_2)\|_{C^{2+\alpha,0}} \leq C\|g_1 - g_2\|_{C^{4+\alpha,0}}
\]
for all $g_i \in C^{4+\alpha,0}(M \times [0, T])$ which satisfies $\|g_i - g\|_{C^{4+\alpha,0}} \leq \delta_0$ for $i = 1, 2$. We also have in Sobolev norm

$$\|\mathcal{P}(g_1) - \mathcal{P}(g_2)\|_{W^{2,2}} \leq C\|g_1 - g_2\|_{W^{4,2}}. \tag{2.13}$$

**Proof** Let operator $T_1(g) \doteq -(n - 2)A(g) \cdot B(g) + \nabla^2 g \cdot B(g)$. From Eq. (2.11) we have

$$(n - 1)\Delta g_1 + s_0)\mathcal{P}(g_1) - \mathcal{P}(g_2)) = (n - 1)\Delta g_2 - \Delta g_1)\mathcal{P}(g_2) + T_1(g_1) - T_1(g_2).$$

Inequality (2.12) follows from Lemma 2.7(i). Using $L^2$-norm we get (2.13).

Let $\mathcal{F}(g) \doteq 2(n - 2) \left( B(g) + \frac{1}{2(n - 1)(n - 2)} (\Delta S_g) g \right) + \mathcal{L}_{W,g} g$. We will consider the following decoupled DeTurck modified CBF of $g(t)$ induced from (2.8)

$$\mathcal{M}(g(t)) \doteq \delta_t g(t) - \mathcal{F}(g(t)) - 2(n - 2)\mathcal{P}(g(t))g(t) = 0 \tag{2.14}$$

with the initial condition $g(0) = g_0$, assuming scalar curvature $S_{g_0} = s_0$. Clearly this flow is equivalent to flow (2.8).

Next we turn to prove the short-time existence of solutions of the linearized equation of (2.14) by contraction mapping theorem.

### 2.3 Linearization of the decoupled flow (2.14)

We now compute the linearization of the decoupled DeTurck modified CBF (2.14) by computing the linearization of operators $\mathcal{F}(g)$ and $\mathcal{P}(g)$ separately. Let $g_s = g + sh$, $s \in (-\epsilon, \epsilon)$, for some $(0,2)$-tensor $h$. Recall that Bach tensor has the schematic expression (2.6), from the linearization formula ([8, (2.47)]) used in the DeTurck trick for Ricci flow, we get (we are sketchy on the details of some tedious but routine calculations here)

$$\delta_h \mathcal{F}(g) \doteq \frac{d}{ds} \bigg|_{s=0} \mathcal{F}(g_s) = -\Delta_g \Delta_{L,g} h + \sum_{a=0}^{3} M_a(g) * \nabla^a h, \tag{2.15}$$

where the Lichnerowicz operator is defined by

$$\Delta_{L,g} h_{ij} = \Delta_{L,g} h_{ij} = \Delta_g h_{ij} + 2R_{iklj}h_{kl} - R_{ik}h_{kj} - R_{jk}h_{ki},$$

$M_a(g)$ depends on $g$ up to the fourth derivatives, and $*$ is some tensor contraction operation by metric $g$ and its inverse.

We define $\mathcal{P}_g'(h) = \delta_h \mathcal{P}(g) \doteq \frac{d}{ds} \bigg|_{s=0} \mathcal{P}(g_s)$, to find $\mathcal{P}_g'(h)$ we compute the linearization of the both sides in Eq. (2.11) with $g = g_s$. Let operator $\mathcal{L} \doteq (n - 1)\Delta_g + s_0$, then

$$\delta_h (\mathcal{L}\mathcal{P}(g)) = -(n - 1)\nabla_i \nabla_j \mathcal{P}(g) h_{ij} - \frac{n - 1}{2} \nabla_j \mathcal{P}(g) \left( 2\nabla_i h_{ij} - \nabla_j H \right) + \mathcal{L}\mathcal{P}'(g)$$

where $H = g_{ij} h_{ij}$ [8, p.547]. From the right-hand side of (2.11), we get

$$\delta_h \left( -(n - 2)A(g) \cdot B(g) + \nabla^2 g \cdot B(g) \right) = \sum_{a=0}^{4} P_a(g) * \nabla^a h,$$

where $P_a(g)$’s are tensors depending on $g$ up to the fourth derivatives. Hence, we have

$$\mathcal{P}_g'(h) = \mathcal{L}^{-1} \left( (n - 1)\nabla_i \nabla_j \mathcal{P}(g) h_{ij} + \frac{n - 1}{2} \nabla_j \mathcal{P}(g) \left( 2\nabla_i h_{ij} - \nabla_j H \right) + \sum_{a=0}^{4} P_a(g) * \nabla^a h \right). \tag{2.16}$$
where we have assumed that operator $L$ is invertible. Hence, we have

$$
\delta h(t) \mathcal{M}(g(t)) = \partial_t h + \Delta_g \Delta_L h \quad \text{for each } \gamma \in C^{\alpha,0}(M \times [0, T]).
$$

The remaining proof for the short-time existence of the decoupled DeTurck modified CBF (2.14) is very close to the proof for that of CRF in [20, Section 3.3.3]. First we need to solve the linearized flow

$$
\begin{cases}
\partial_t h + \Delta_g \Delta_L h - \sum_{a=0}^{3} M_a(g) \ast \nabla^a_{g} h - 2(n-2) P'_g(h) g - 2(n-2) P(g) h = \gamma \\
h(\cdot, 0) = 0
\end{cases}
$$

(2.18)

for each $\gamma \in C^{\alpha,0}(M \times [0, T])$. Using Lemma 2.6(i) we have

**Proposition 2.9** Suppose that $g(t), t \in [0, T]$, is a family of $C^{4+\alpha,0}$-metrics such that the elliptic operator $(n-1)\Delta_g + s_0$ is invertible for all $t$. Then for each $\gamma \in C^{\alpha,0}(M \times [0, T])$ the initial value problem (2.18) has a unique solution $h \in C^{4+\alpha,1}(M \times [0, T])$. Moreover, there is a constant $C$ such that

$$
\|h\|_{C^{4+\alpha,1}} \leq C \|\gamma\|_{C^{\alpha,0}}
$$

(2.19)

for all $\gamma \in C^{\alpha,0}(M \times [0, T])$.

**Proof** To use contraction mapping theorem to prove the existence, we consider the Banach space

$$
E_1([0, T^*]) = \left\{ \tilde{h} \in C^{4+\alpha,0}(M \times [0, T^*]) : \tilde{h}(\cdot, 0) = 0 \right\},
$$

where $T^* \in (0, T]$ is a small constant to be chosen below. By Lemma 2.6(i) for a given $\tilde{h} \in E_1([0, T^*))$ we can solve the system of linear parabolic equations

$$
\begin{cases}
\partial_t h + \Delta_g \Delta_L h - \sum_{a=0}^{3} M_a(g) \ast \nabla^a_{g} h - 2(n-2) P(g) h = \tilde{\gamma} \\
h(\cdot, 0) = 0,
\end{cases}
$$

(2.20)

where $\tilde{\gamma} = \gamma + 2(n-2) P'_g(\tilde{h}) g \in C^{\alpha,0}$; hence, we may define a map

$$
\Psi : E_1([0, T^*)) \to E_1([0, T^*)), \quad \Psi(\tilde{h}) = h \in C^{4+\alpha,1}.
$$

(2.21)

Let $\tilde{h}_i \in E_1([0, T^*)), \ i = 1, 2$. Note that if we set $v \doteq \Psi(\tilde{h}_1) - \Psi(\tilde{h}_2)$, then $v$ satisfies

$$
\begin{cases}
\partial_t v + \Delta_g \Delta_L v - \sum_{a=0}^{3} M_a(g) \ast \nabla^a_{g} v - 2(n-2) P(g) v = 2(n-2)(P'_g(\tilde{h}_1) - P'_g(\tilde{h}_2)) g, \\
v(\cdot, 0) = 0.
\end{cases}
$$

From (2.16) and Lemma 2.7(i), we have for $g \in C^{4+\alpha,0}$

$$
\|P'_g(\tilde{h}_1) - P'_g(\tilde{h}_2)\|_{C^{2+\alpha,0}} \leq C \|\tilde{h}_1 - \tilde{h}_2\|_{C^{4+\alpha,0}},
$$

(2.22)

then it follows from Lemma 2.6(ii) that

$$
\|v\|_{C^{6+\alpha,1}} \leq C \|\tilde{h}_1 - \tilde{h}_2\|_{C^{4+\alpha,0}}.
$$

(2.23)

Hence, by Lemma 2.5 we have

$$
\|v(\cdot, t_1) - v(\cdot, t_2)\|_{C^{4+\alpha}} \leq C \cdot |t_1 - t_2| \cdot \|\tilde{h}_1 - \tilde{h}_2\|_{C^{4+\alpha,0}}.
$$
In particular, using $t_1 = 0$ and $u(\cdot, 0) = 0$ we get
\[
\|\Psi(\tilde{h}_1) - \Psi(\tilde{h}_2)\|_{C^{4+\alpha, 0}} \leq CT^* \|\tilde{h}_1 - \tilde{h}_2\|_{C^{4+\alpha, 0}} \tag{2.24}
\]
for all $\tilde{h}_i \in E_1([0, T^*]), \ i = 1, 2$.

To apply contraction mapping theorem to $\Psi$, we observe that
\[
\|\Psi(\tilde{h})\|_{C^{4+\alpha, 0}} \leq \|\Psi(0)\|_{C^{4+\alpha, 0}} + CT^* \|\tilde{h}\|_{C^{4+\alpha, 0}} \tag{2.25}
\]
by (2.24) and that for some constant $C_0$
\[
\|\Psi(0)\|_{C^{4+\alpha, 1}} \leq C_0 \|\gamma\|_{C^{\alpha, 0}}
\]
by Lemma 2.6(i). Let $R = 2C_0 \|\gamma\|_{C^{\alpha, 0}}$, then when $T^*$ is chosen so that $CT^* \leq \frac{1}{2}$, the map
\[
\Psi: \tilde{B}_R = \{\tilde{h} \in E_1([0, T^*]) : \|\tilde{h}\|_{C^{4+\alpha, 0}} \leq R\} \rightarrow \tilde{B}_R
\]
is a contractive mapping due to (2.24). We get a fixed point of $\Psi$ on $\tilde{B}_R$ which gives the existence of the solution of Eq. (2.18) on time interval $[0, T^*]$.

To see the uniqueness of solutions to (2.18), suppose that $h_1$ and $h_2$ are two solutions, it follows from (2.24) using $h_1 = h_1 = \Psi(h_1)$ and $CT^* \leq \frac{1}{2}$ that $h_1 - h_2 = 0$.

Because (2.18) is linear, there will be no short-time blowup, one may extend its solution from $[0, T^*]$ to $[0, T]$ by steps over time intervals of length $T^*$. Note that when we extend the solution to $[T^*, 2T^*]$, we need to make some simple adjustment to the equations so that the initial condition at $T^*$ for the new equations is 0.

Finally, to see the estimate (2.19), from the estimate (2.25) and $\Psi(h) = h$ for solution $h$ it follows that $\|h\|_{C^{4+\alpha, 0}} \leq 2C_0 \|\gamma\|_{C^{\alpha, 0}}$. Combining this estimate with Eq. (2.18) we get $\|\partial_t h\|_{C^0} \leq C_1 \|\gamma\|_{C^{\alpha, 0}}$. Hence, estimate (2.19) follows.

In summary we have established that the linear operator defined by (2.17)
\[
\delta_\bullet \mathcal{M}(g) : \{h \in C^{4+\alpha, 1}(M \times [0, T]), h(\cdot, 0) = 0\} \rightarrow C^{\alpha, 0}(M \times [0, T]) \tag{2.26}
\]
is an isomorphism, provided that $g = g(t)$ satisfies the assumptions in Proposition 2.9.

### 2.4 Short-time existence of decoupled DeTurck modified CBF (2.14) and proof of the existence part of Theorem 1.1

Now we apply the implicit function theorem [20, Lemma 3.7] to the nonlinear map defined by (2.14)
\[
\mathcal{M} : \{g \in C^{4+\alpha, 1}(M \times [0, T]), g(0) = g_0\} \rightarrow C^{\alpha, 0}(M \times [0, T])
\]
Here $g_0$ is the metric given in Theorem 1.1 and we choose the metric $\tilde{g} = g_0$ which is used in (2.7). We begin with showing that operator $\mathcal{M}$ is continuously differentiable.

**Lemma 2.10** Let $M^n$ be a closed manifold with metrics $g(t) \in C^{4+\alpha, 1}(M \times [0, T])$. Suppose that the elliptic operator $(n-1)\Delta_{g(t)} + s_0$ is invertible for each $t$. Then there is a constant $\delta_0 > 0$ such that we have the following estimate of the norm of the difference of the linear operators
\[
\|\delta_\bullet \mathcal{M}(g_1) - \delta_\bullet \mathcal{M}(g_2)\|_{L(C^{4+\alpha, 1}, C^{\alpha, 0})} \leq C \|g_1 - g_2\|_{C^{4+\alpha, 1}}
\]
for all $g_i \in C^{4+\alpha, 1}(M \times [0, T])$ which satisfies $\|g_i - g\|_{C^{4+\alpha, 1}} \leq \delta_0, \ i = 1, 2$. 

© Springer
Proof For any \( h \in C^{4+\alpha,1}(M \times [0, T]) \) with \( h(\cdot, 0) = 0 \) we have
\[
\delta_h \mathcal{M}(g_1) - \delta_h \mathcal{M}(g_2) = (\Delta_{g_1} \Delta_{g_1} - \Delta_{g_2} \Delta_{g_2}) h - \sum_{a=0}^3 \left( M_{a}(g_1) * g_1 \nabla^a_{g_1} h - M_{a}(g_2) * g_2 \nabla^a_{g_2} h \right) - 2(n-2) \left( \mathcal{P}'_{g_1}(h) g_1 - \mathcal{P}'_{g_2}(h) g_2 \right) - 2(n-2) \left( \mathcal{P}(g_1) - \mathcal{P}(g_2) \right) h.
\]

We will estimate the operator norm of each term in the above expression and omit most of the tedious but routine calculations below.

First, we have
\[
\| (\Delta_{g_1} \Delta_{g_1} - \Delta_{g_2} \Delta_{g_2}) h \|_{C^{\alpha,0}} \leq \| (\Delta_{g_1} - \Delta_{g_2}) \Delta_{g_1} h \|_{C^{\alpha,0}} + \| \Delta_{g_2} (\Delta_{g_1} - \Delta_{g_2}) h \|_{C^{\alpha,0}} \leq C \| g_1 - g_2 \|_{C^{4+\alpha,0}} \cdot \| h \|_{C^{4+\alpha,0}}.
\] (2.27)

For each \( a = 0, 1, 2, 3 \) we have
\[
\| M_{a}(g_1) * g_1 \nabla^a_{g_1} h - M_{a}(g_2) * g_2 \nabla^a_{g_2} h \|_{C^{\alpha,0}} \leq \| (M_{a}(g_1) * g_1 - M_{a}(g_2)) \nabla^a_{g_1} h \|_{C^{\alpha,0}} + \| M_{a}(g_2) * g_2 (\nabla^a_{g_1} h - \nabla^a_{g_2} h) \|_{C^{\alpha,0}} \leq C \| g_1 - g_2 \|_{C^{4+\alpha,0}} \cdot \| h \|_{C^{4+\alpha,0}}.
\] (2.28)

From a proof similar to that on the bottom of [20, p.426], we have
\[
\| \mathcal{P}'_{g_1}(h) - \mathcal{P}'_{g_2}(h) \|_{C^{\alpha,0}} \leq C \| g_1 - g_2 \|_{C^{2+\alpha,0}} \cdot \| h \|_{C^{2+\alpha,0}};
\]

hence, by using (2.22) with \( h_2 = 0 \) we get
\[
\| \mathcal{P}'_{g_1}(h) g_1 - \mathcal{P}'_{g_2}(h) g_2 \|_{C^{\alpha,0}} \leq C \| g_1 - g_2 \|_{C^{2+\alpha,0}} \cdot \| h \|_{C^{2+\alpha,0}} + C \| g_1 - g_2 \|_{C^{\alpha,0}} \cdot \| h \|_{C^{2+\alpha,0}}.
\] (2.29)

Finally, using Lemma 2.8 we have
\[
\| (\mathcal{P}(g_1) - \mathcal{P}(g_2)) h \|_{C^{2+\alpha,0}} \leq C \| g_1 - g_2 \|_{C^{4+\alpha,0}} \cdot \| h \|_{C^{2+\alpha,0}}.
\] (2.30)

The lemma now follows from combining together the inequalities (2.27), (2.28), (2.29), and (2.30).

Next we prove the short-time existence of flow (2.14).

Proposition 2.11 Let \( M^n \) be a closed manifold. Suppose that \( g_0 \) is a \( C^{8+\alpha} \)-Riemannian metric on \( M \) with constant scalar curvature \( s_0 \) and that the elliptic operator \( (n-1)\Delta_{g_0} + s_0 \) is invertible. Then there exists a unique \( C^{4+\alpha,1} \)-solution \( g(t), t \in [0, T], \) of the decoupled DeTurck modified CBF (2.14) for some \( T > 0 \).

Proof Let
\[
\mathcal{M} : \{ g \in C^{4+\alpha,1}(M \times [0, T]), g(0) = g_0 \} \to C^{\alpha,0}(M \times [0, T])
\] (2.31)
be a map defined in (2.14). We define metrics \( \tilde{g} \in C^{4+\alpha,1}(M \times [0, T]) \) by
\[
\tilde{g}(t) = g_0 + t(F(g_0) + 2(n-2)\mathcal{P}(g_0)g_0).
\] (2.32)
We will apply inverse function theorem [20, Lemma 3.7] to map $\mathcal{M}$ around $\bar{g}$ to prove the existence of a solution of equation $\mathcal{M}(g) = 0$. Note that linear operator $(n - 1)\Delta_{\bar{g}(t)} + s_0$ is a small perturbation of $(n - 1)\Delta_{g_0} + s_0$ when $t$ is small. Since operator $(n - 1)\Delta_{g_0} + s_0$ is invertible, we conclude that $(n - 1)\Delta_{\bar{g}(t)} + s_0$ is invertible for each $t \in [0, T]$ when $T$ is small enough. In general, metric $\bar{g}(t)$ does not have constant scalar curvature for $t > 0$.

It follows from a simple calculation that

$$\mathcal{M}(\bar{g}(t)) = \mathcal{F}(g_0) + 2(n - 2)\mathcal{P}(g_0)g_0 - (\mathcal{F}(\bar{g}(t)) + 2(n - 2)\mathcal{P}(\bar{g}(t)) \bar{g}(t)).$$  \hspace{1cm} (2.33)

Using Lemma 2.8 we get

$$\|\mathcal{M}(\bar{g})\|_{C^{\alpha,0}} \leq C T$$  \hspace{1cm} (2.34)

where constant $C$ depends on $\|g_0\|_{C^{8+\alpha}}$. Hence, $\bar{g}(t)$ is an approximate solution of equation $\mathcal{M}(g) = 0$ when $T$ is small.

By (2.26) we have that for sufficient small $T$

$$\|\delta_\bullet \mathcal{M}(\bar{g})\|_{L(C^{\alpha,0}, C^{4+\alpha,1})} \leq \tilde{C}$$

for some constant $\tilde{C}$. Let

$$\bar{B}(\bar{g}, \delta_0) = \{ g \in C^{4+\alpha,1}(M \times [0, T]) : g(0) = g_0 \text{ and } \| g - \bar{g} \|_{C^{4+\alpha,1}} \leq \delta_0 \}.$$

By the perturbation theory of bounded linear operators and Lemma 2.10, there is a constant $C_0$ and a small number $\delta_0 > 0$ such that the operator norm

$$\|\delta_\bullet \mathcal{M}(g)\|_{L(C^{\alpha,0}, C^{4+\alpha,1})} \leq C_0$$

for all $g \in \bar{B}(\bar{g}, \delta_0)$. By Lemma 2.10 we can choose $\delta_0$ even smaller if necessarily such that for the constant $C_0$ above we have

$$\|\delta_\bullet \mathcal{M}(g_1) - \delta_\bullet \mathcal{M}(g_2)\|_{L(C^{4+\alpha,1}, C^{\alpha,0})} \leq \frac{1}{2C_0}$$

for all $g_1, g_2 \in \bar{B}(\bar{g}, \delta_0)$. From (2.34) we may choose an even smaller $T$ if necessary to get

$$\|\mathcal{M}(\bar{g})\|_{C^{\alpha,0}} \leq \frac{\delta_0}{2C_0}.$$

Now the existence in Proposition 2.11 follows from the inverse function theorem.

We leave the proof of the uniqueness to subsection Sect. 2.5. \hfill \Box

Finally, we can give a quick

*Proof of the existence part of Theorem 1.1.* First, we assume that the initial metric $g_0$ is a $C^{8+\alpha}$, by Proposition 2.11 we have a solution $g \in C^{4+\alpha,1}(M \times [0, T])$ of (2.14) and consequently a function $p \in C^{2+\alpha,0}$. Near the end of Sect. 2.2, we conclude the equivalence between (2.14) and (2.8); hence, we get a solution $(g(t), p(t))$ of (2.8). The better regularity $g \in C^{4+\alpha,1+\frac{1}{4}\alpha}$ follows from the standard parabolic theory. By Lemma 2.4(i) and 2.2 we get the required solution of CBF (1.3).

When the initial metric $g_0$ is in $C^{4+\alpha}$, from the solution of Yamabe problem we may choose a family of $C^{8+\alpha}$-metrics $g_0$ of constant scalar curvature $s_0 \neq 0$ which converges to $g_0$ in $C^{4+\alpha}$-norm. Hence, we have a family of solutions $g_t(t)$ in $C^{4+\alpha,1+\frac{1}{4}\alpha}(M \times [0, T])$ of flow (2.14).

By Lemma 2.10 the operator $\mathcal{M}(g)$ is continuously differentiable in $g \in C^{4+\alpha,1}(M \times [0, T])$. Hence, by the continuous dependence on parameters in the inverse function theorem.

\copyright Springer
and the proof of the existence of $g_t(t)$ in Proposition 2.11 we conclude that the sequence of fixed points $\{g_t(t)\}$ converges in $C^{4+\alpha,1}(M \times [0, T])$-norm to a solution $g_\infty(t) \in C^{4+\alpha,1}(M \times [0, T])$ of (2.14) with initial metric $g_\infty(0) = g_0$. This $g_\infty(t)$ gives rise to the required solution of CBF (1.3) with initial metric $g_0$. □

2.5 Proof of the uniqueness in Theorem 1.1

First, we prove the uniqueness in Proposition 2.11 by the energy method. Since such proof for parabolic equations is well-understood and here the calculation is tedious, we will be sketchy on details below.

Suppose that $g_1(t)$ and $g_2(t)$ are two solutions with initial data $g_1(0) = g_2(0) = g_0$. Let $g_{21}(t) \triangleq g_2(t) - g_1(t)$. Using $\mathcal{F}(g_2) - \mathcal{F}(g_1) = \int_0^1 \frac{d}{ds} \mathcal{F}(g_1 + s(g_2 - g_1)) ds$ and (2.15), we have

$$\partial_t g_{21} = -\Delta^2_{g_1} g_{21} + \sum_{a=0}^3 \bar{A}_a \ast \nabla^a g_{21} - 2(n - 2) P(g_1) g_{21} - 2(n - 2)(P(g_2) - P(g_1)) g_{21}.$$

Let $U(t) \triangleq \int_M \sum_{b=0}^2 |\nabla^b g_{11}(t)|^2_{g_{11}(t)} d\mu_{g_{11}(t)}$ be the energy and let $[A, B] \triangleq AB - BA$ be the commutator of two operators. We compute schematically the derivative of $U(t)$ using integration by parts,

$$\frac{dU(t)}{dt} \leq 2 \int_M \sum_{b=0}^2 \left( \nabla^b g_{11}(t) \partial_t + [\partial_t, \nabla^b g_{11}(t)] \right) g_{21}, \nabla^b g_{11}(t) g_{21}(t)_{g_{11}(t)} d\mu_{g_{11}(t)} + CU(t)

\leq -2 \int_M \sum_{b=0}^2 \left( \nabla^b g_{11}(t) \Delta^2_{g_1} g_{21}, \nabla^b g_{11}(t) g_{21}(t)_{g_{11}(t)} d\mu_{g_{11}(t)}

+ 2 \int_M \sum_{a=0}^5 \sum_{b=0}^2 \left( \bar{A}_{a,b} \ast \nabla^a g_{21}, \nabla^b g_{11}(t) g_{21}(t)_{g_{11}(t)} d\mu_{g_{11}(t)}

- 4(n - 2) \int_M \sum_{b=0}^2 \left( \nabla^b g_{11}(t) (P(g_2) - P(g_1)) g_2, \nabla^b g_{11}(t) g_{21}(t)_{g_{11}(t)} d\mu_{g_{11}(t)} + CU(t)

\leq - \int_M \sum_{b=0}^2 \left| \Delta_{g_1} \nabla^b g_{11}(t) g_{21} \right|^2_{g_{11}(t)} d\mu_{g_{11}(t)} + CU(t).$$

Here we omit all the details to obtain the last inequality, but just point out that (2.13) is used to handle the term containing $P(g_2) - P(g_1)$. Since $U(0) = 0$ by $g_{21}(0) = 0$, we have $U(t) = 0$ from Gronwall inequality. Hence, the uniqueness is proved.

As a consequence we give a

Proof of the uniqueness in Theorem 1.1. The proof is standard as the proof of the uniqueness of Ricci flow on closed manifolds (see, for example, [8, p.117–118]. The basic idea is that given two solutions $(g_i(t), p_i(t))$, $i = 1, 2$, of CBF (1.3), from Lemma 2.4(ii) we have two diffeomorphisms $\varphi_i(t)$ which are solutions of the parabolic Eq. (2.10) corresponding to $g_i(t)$. Then the push-forward metrics $(\varphi_i(t))^* g_i(t)$ are solutions of DeTurck CBF (2.14) satisfying the same initial condition; hence, by the uniqueness in Proposition 2.11 we have $(\varphi_1(t))^* g_1(t) = (\varphi_2(t))^* g_2(t) \equiv g_*(t)$. Note that $\varphi_i(t)$ are the solutions of ODE (2.9) for metric $g_*(t)$ with initial condition $\varphi_1(0) = \varphi_2(0) = \text{Id}_M$. Hence, $\varphi_1(t) = \varphi_2(t)$ and $g_1(t) = \varphi_1(t)^* g_*(t) = \varphi_2(t)^* g_*(t) = g_2(t).$ □

\textcopyright Springer
Remark 2.12 (i) One may give a direct proof of the uniqueness in Theorem 1.1 using B. Kotschwar’s energy technique without relying on DeTurck CBF [15]. More precisely, for two solutions \(g_1(t)\) and \(g_2(t)\) of CBF (1.3) with common initial data \(g_0\) one may use an energy defined by

\[
\mathcal{E}(t) \doteq \int_M |g_1(t) - g_2(t)|^2 d\mu_{g_1(t)} + \int_M |\nabla g_1(t) - \nabla g_2(t)|^2 d\mu_{g_1(t)}
\]

\[
+ \sum_{a=0}^1 \int_M |\nabla^a g_1(t) Rm_{g_1(t)} - \nabla^a g_2(t) Rm_{g_2(t)}|^2 d\mu_{g_1(t)}.
\]

(ii) The idea of uniqueness proof here is similar to the proof of uniqueness for ambient obstruction flow in [3]. The difference is caused from the pressure term \(2(n - 2)p_g\) in the CBF (1.3).

2.6 Regularity

We prove the following theorem.

Theorem 2.13 Suppose that \(g_0\) is smooth. Then the solution \((g(t), p(t))\) of CBF (1.3) is smooth in space and time variables for a short time.

Proof Let metric \(\tilde{g}\) in (2.7) be \(g_0\). Using local coordinates \((x^i)\) we may rewrite the DeTurck modified CBF (2.8) in a schematic way as

\[
\begin{cases}
\partial_t g_{ij} + (g^{kl} \partial_k \hat{a}_l)(g^{pq} \partial_p \hat{a}_q) g_{ij} + \hat{A}_{ij}(g_0, g) - 2(n - 2)p g_{ij} = 0, \\
((n - 1)\Delta_{g(t)} + s_0)p = -(n - 2)A(g(t)) \cdot B(g(t)) + \nabla^2 g_{ij} B(g(t)),
\end{cases}
\]

(2.35)

where \(\hat{A}_{ij}(g_0, g)\) depends on \(\{g_{pq}\}\) up to their third derivatives. We want to prove \(\partial^a g_{ij}\) is in \(C^{4+\alpha,1}(M \times [0,T])\) for each \(a \in \mathbb{N}\) by a bootstrap argument.

Below we only consider the base case \(a = 1\), as an example. By the existence part of Theorem 1.1 solution \(g \in C^{4+\alpha,1}\) and \(p \in C^{2+\alpha,0}\); hence, \(\hat{A}_{ij}(g_0, g) - 2(n - 2)p g_{ij} \in C^{1+\alpha,0}\). It follows from Lemma 2.6(ii) and the smoothness of \(g_0\) that \(g \in C^{5+\alpha,1}\).

After we have improved the spatial regularity to smoothness, we can use the Eq. (2.35) to improve the regularity in time to smoothness. The theorem is proved.

3 The backward uniqueness of CBF

In this section we give a proof of Theorem 1.2 using Agmon-Nirenberg’s energy method. This approach is used by Kotschwar in his second proof of the backward uniqueness of Ricci flow [16], and later by Sun and Zhu in their proof of the backward uniqueness of CRF [30]. Because CBF is a fourth-order system with a pressure term, we need to combine the idea in [16, Section 4] about high-order systems with the idea in [30] about handling the pressure term.

3.1 Estimates of a few basic geometric quantities

Let \((g(t), p(t))\) and \((\tilde{g}(t), \tilde{p}(t))\) be the two solutions in Theorem 1.2. Note that since operator \((n - 1)\Delta_{g(T)} + s_0\) is assumed to be invertible, equality \(g(T) = \tilde{g}(T)\) implies \(p(T) = \tilde{p}(T)\).
Below we will use $g(t)$ as the background metric, in particular, connection $\nabla = \nabla_{g(t)}$, $\Delta = \Delta_{g(t)}$, and measure $d\mu = d\mu_{g(t)}$. Let $\tau \doteq T - t$ be the backward time. We define time-dependent tensors
\[
Y^{(0)} \doteq g(t) - \tilde{g}(t), \quad Y^{(1)} \doteq \nabla - \nabla_{\tilde{g}(t)}, \quad Y^{(k)} \doteq \nabla^{k-1}Y^{(1)} \text{ for } k \geq 2,
\]
\[
X^{(k)} \doteq \nabla^k \text{Rm} - \nabla^k_{\tilde{g}(t)} \text{Rm}_{\tilde{g}(t)} \text{ for } k \geq 0,
\]
\[
q^{(k)} \doteq \nabla^k p(t) - \nabla^k_{\tilde{g}(t)} \tilde{p}(t) \text{ for } k = 0, 1, 2, \text{ and } q^{(k)} \doteq \nabla^{k-2}q^{(2)} \text{ for } k \geq 3.
\]

The following two tensors and the estimates of their associated $L^2$-energies will be used to prove the backward uniqueness.
\[
X \doteq X^{(0)} \oplus X^{(1)} \oplus \cdots \oplus X^{(4)}, \quad Y \doteq Y^{(0)} \oplus Y^{(1)} \oplus \cdots \oplus Y^{(4)}.
\]

**Remark 3.1** In the above $Y^{(k)}$ up to order $k = 4$ is needed because of the following calculation (compare [16, Section 3.1])
\[
\left(\Delta_{\tilde{g}(t)}^2 - \Delta^2\right) \nabla^a \text{Rm}_{\tilde{g}(t)} = \sum_{b=0}^{4} Y^{(b)} \ast T_b,
\]
which is a term shown up in calculating $(\partial_{\tau} + \Delta^2)X^{(a)}$ below. Here $T_b$ depends on $\text{Rm}_{\tilde{g}(t)}$ up to its $(a+4)$th-derivatives. The order of $Y$ then determines the order of $X^{(k)}$ as given above.

We have the following pointwise bounds of derivatives of each components in $X$ and $Y$.

**Lemma 3.2** For $a = 0, 1, 2, 3, 4$, there are constants $C_1$, $C_2$, and $C_3$ depending on $g(t)$ and $\tilde{g}(t)$ such that
\[
|\partial_{\tau} Y^{(a)}| \leq C_1 \sum_{b=0}^{a+2} |X^{(b)}| + C_2 \sum_{b=0}^{a} |Y^{(b)}| + C_3 \sum_{b=0}^{a} |q^{(b)}|,
\]
\[
|\left(\partial_{\tau} + \Delta^2\right)X^{(a)}| \leq C_1 \sum_{b=0}^{a+2} |X^{(b)}| + C_2 \sum_{b=0}^{a} |Y^{(b)}| + C_3 \sum_{b=0}^{a} |q^{(b)}|.
\]

About the proof of this lemma, here we make a general comment which also applies to the proof of Lemma 3.4 below. In geometric analysis it is a standard practice to calculate the evolution equations of geometric quantities associated to the geometric flow, usually such calculations are lengthy but straightforward and their precise schematic forms are very important for the further arguments. In the particular case of Lemma 3.2, as a matter of fact, the evolution of curvatures and their derivatives will be calculated in Sect. 4 for other purpose, while the evolution of the differences of two metrics, the associated two connections, two curvatures, and two derivatives of curvatures can be calculated in a standard fashion.

Here we skip the calculations used to prove Lemma 3.2, instead we mention two key points involved in the calculations.

(I) For any tensor $W$ we hope to express $\nabla^a_{\tilde{g}(t)} W$ in terms of $\nabla^a W$ and $Y^{(b)}$ for $a \leq 4$ and $b \leq 4$, in particular, for $a = 2$ we have
\[
\nabla^2_{\tilde{g}(t)} W = \nabla^2 W + Y^{(2)} \ast W + Y^{(1)} \ast \nabla W + Y^{(1)} \ast W \ast T_1,
\]
where $T_1$ is some tensor depending on $g(t)$ and $\tilde{g}(t)$.
(II) A careful calculation is needed to see that the upper bound of $| (\partial_s + \Delta^2) X^{(4)} |$ does not need $|Y^{(5)}|$. More precisely, in taking the difference of Eq. (4.4) below for $g(t)$ and $\tilde{g}(t)$ with $a = 4$, we need to bound the difference
\[
\left| \nabla^a (T (\nabla^2 p)) - \nabla^a (T (\nabla^2_{\tilde{g}(t)} \tilde{p})) \right|
\]
without using $|Y^{(5)}|$. For $a \leq 4$ we actually have
\[
\nabla^{a+2} p - \nabla^{a+2}_{\tilde{g}(t)} \tilde{p} = \left( \nabla^a - \nabla^a_{\tilde{g}(t)} \right) \nabla^2_{\tilde{g}(t)} p + \nabla^a (\nabla^2 p - \nabla^2_{\tilde{g}(t)} \tilde{p}) = \sum_{b=0}^{a} Y^{(b)} \neq \tilde{T} b + \sum_{b=0}^{a+2} q^{(b)},
\]
where $\tilde{T} b$ depends on $g(t)$, $\tilde{g}(t)$, $p(t)$, and $\tilde{p}(t)$. This calculation is different from the calculation in the proof of [30, Lemma 2]. It is this calculation which forces us to use $q^{(b)}$ as defined above rather than $\nabla b q^{(0)}$ used in [30].

Lemma 3.2 can be summarized to

**Lemma 3.3** We have
\[
| (\partial_s X + \Delta^2) X | + | \partial_s Y | \leq C \left( |Y| + |X| + |X^{(5)}| + |X^{(6)}| \right) + C \sum_{b=0}^{6} |q^{(b)}|. \tag{3.1}
\]

To apply Agmon-Nirenberg’s energy method we need to control $\sum_{b=0}^{6} |q^{(b)}|$ by $|Y| + |X| + |X^{(5)}| + |X^{(6)}|$, actually controlling the $L^2$-norm of $\sum_{b=0}^{6} |q^{(b)}|$ is enough.

**Lemma 3.4** We have
\[
\int_M \sum_{b=0}^{2} |q^{(b)}|^2 d\mu \leq C \int_M \left( \sum_{b=0}^{2} |X^{(b)}|^2 + \sum_{c=0}^{1} |Y^{(c)}|^2 \right) d\mu,
\]
\[
\int_M |\nabla q^{(a+2)}|^2 d\mu \leq C \int_M \left( \sum_{b=0}^{a+2} |X^{(b)}|^2 + \sum_{c=0}^{\max\{a,2\}} |Y^{(c)}|^2 \right) d\mu \text{ for } a = 1, 2, 3, 4.
\]

**Proof** (sketch) (compare to the proof of Lemma 4.5 below) The first inequality follows from $W^{2,2}$-estimate for the elliptic equation of $((n-1)\Delta + s_0) (p - \tilde{p})$ derived from taking the difference of second equation in (1.3) for $p$ and $\tilde{p}$ (see the proof of Lemma 2.8). Here we use the assumption that $(n-1)\Delta g(t) + s_0$ is invertible. The second inequality follows from $W^{a,2}$-estimate for the elliptic equation of $((n-1)\Delta + s_0) (\nabla^2 p - \nabla^2_{\tilde{g}(t)} \tilde{p})$ which equals to an expression of $Y^{(0)}$, $Y^{(1)}$, $Y^{(2)}$, and $X$. \square

### 3.2 Proof of Theorem 1.2

Define
\[
E(t) \doteq \int_M (|X|^2 + |Y|^2) d\mu, \quad F(t) \doteq \int_M |\Delta X|^2 d\mu,
\]
and their quotient $N(t) \doteq \frac{F(t)}{E(t)}$. Using Lemma 3.3 and 3.4 and by a calculation as done in [16, Sections 2 and 4] we have
\[
\frac{dE}{d\tau} \leq CE + 2F + C \int_M \left( |X^{(5)}|^2 + |X^{(6)}|^2 \right) d\mu.
\]
In the calculation we use the backward and forward parabolic operators \( \mathcal{L}_B = \partial_t - \Delta^2 \) and \( \mathcal{L}_F = \partial_t + \Delta^2 \). As in [16, Section 4] (see also Sect. 4.2 below) we may use interpolation inequalities to bound \( \int_M (|X^{(5)}|^2 + |X^{(6)}|^2) d\mu \) by \( \int_M |\Delta X|^2 d\mu \) and \( \int_M |X|^2 d\mu \); hence, we get
\[
\frac{dE}{d\tau} \leq C(E + F) \leq C(N + 1)E. \tag{3.2}
\]
Using Lemma 3.4 and following the proof in [16, Section 4] we can show
\[
\frac{dF}{d\tau} \geq -C(E + F) + \frac{1}{2} \left( \int_M |\mathcal{L}_F X|^2 d\mu - \int_M |\mathcal{L}_B X|^2 d\mu \right),
\]
\[
\frac{dN}{d\tau} \geq -C(N + 1) - \frac{1}{2} E \left( \int_M |\mathcal{L}_B X|^2 d\mu + \int_M |\partial_\tau Y|^2 d\mu \right).
\]
The remaining argument for the backward uniqueness is the same as that in [16, Section 4].

## 4 Integral version of Shi’s type estimate for CBF

We will prove the following Shi’s type \( L^2 \)-estimate of derivatives of curvature for CBF (1.3). The estimate will be used in Sect. 4.6 to characterize the time when CBF develops a singularity and in Sect. 4.7 to show the compactness of a sequence of solutions of CBF.

**Theorem 4.1** Let \((g(t), p(t))\), \(t \in [0, T)\), be a smooth solution of CBF on closed manifold \(M^n\) with constant scalar curvature \(s_0\). Let constant \(\alpha > 0\). We assume that there is a constant \(K > 0\) such that the curvature of \(g(t)\) and potential function \(p(t)\) satisfy
\[
\sup_{(x,t) \in M \times [0, \min\{\frac{\alpha}{K}, T\}]} (|\text{Rm}(x, t)| + |p(x, t)|) \leq K.
\]
We also assume that operator \((n - 1)\Delta g(t) + s_0\) is invertible for each \(t\) with \(\|((n - 1)\Delta g(t) + s_0)^{-1}\|_{L(C^{\alpha}, C^{2+\alpha})} \leq K\). Then for any \(m \in \mathbb{N}\) there exists a constant \(C = C(n, s_0, \alpha, K, m)\) such that for all \(t \in (0, \min\{\frac{\alpha}{K}, T\})\) we have
\[
\int_M |\nabla_{g(t)}^m \text{Rm}(\cdot, t)|^2_{g(t)} d\mu_{g(t)} \leq \frac{C \cdot \int_M |\text{Rm}(\cdot, 0)|^2_{g(0)} d\mu_{g(0)}}{t^{m/2}}. \tag{4.1}
\]
The technique to derive the above estimate is standard (see, for example, [24, Theorem 5.4]. We will prove the estimate through several subsections.

### 4.1 Evolution equation of \(\nabla_{g(t)}^d \text{Rm}_{g(t)}\)

First, we derive the evolution equation of \((0, 4)\)-curvature tensor \(\text{Rm} = \text{Rm}_{g(t)}\), i.e., a formula for \((\partial_t + \Delta^2_{g(t)}) \text{Rm}\). We introduce a convenient notation
\[
B^k_i(Rm) = \sum_{i_1 + \cdots + i_s = k} C_{i_1, \ldots, i_s} \nabla^{i_1} \text{Rm} \ast \cdots \ast \nabla^{i_s} \text{Rm} \tag{4.2}
\]
for some constants \(C_{i_1, \ldots, i_s}\).
Lemma 4.2 Let \((g(t), p(t))\) be a smooth solution of CBF (1.3) on manifold \(M^n\) with constant scalar curvature \(s_0\). Then

\[
\partial_t \text{Rm} + \Delta^2 \text{Rm} = B^2_{23}(\text{Rm}) + B^0_3(\text{Rm}) + 2(n-2)p \text{Rm} + (n-2)T(\nabla^2 p), \tag{4.3}
\]

where \((0,4)\)-tensor \(T(\nabla^2 p)\) is defined by

\[
T(\nabla^2 p)_{ijkl} = g_{jl} \nabla^i \nabla_k p - g_{jk} \nabla_i \nabla_l p - g_{il} \nabla_j \nabla_k p + g_{ik} \nabla_j \nabla_l p.
\]

**Proof** From [8, (2.67)] we have

\[
\partial_t R_{ijkl} = \frac{1}{2} \left( \nabla_i \nabla_k h_{jl} - \nabla_i \nabla_l h_{jk} - \nabla_j \nabla_k h_{il} + \nabla_j \nabla_l h_{ik} \right) + \frac{1}{2} \left( R_{ijkl} h_{pl} + R_{ijpl} h_{pk} \right),
\]

where \(h_{ij} = 2(n-2)(B_{ij}(g) + pg_{ij})\). From (2.6) and scalar curvature \(S_{g(t)} = s_0\) we have

\[
\partial_t R_{ijkl} = \nabla_i \nabla_k \Delta R_{jl} - \nabla_i \nabla_l \Delta R_{jk} - \nabla_j \nabla_k \Delta R_{il} + \nabla_j \nabla_l \Delta R_{ik} + B^2_{23}(\text{Rm}) + B^0_3(\text{Rm}) + 2(n-2)p R_{ijkl} + (n-2)\nabla^a(\nabla^2 p)\right).
\]

Here we treat \(s_0\) as a \(\text{Rm}\) factor. The equality (4.3) then follows from Ricci identity ([8, (1.30)]) and a well-known formula ([8, (2.64)]).

From (4.3) we may derive the following by a direct calculation which we omit.

**Lemma 4.3** Let \((g(t), p(t))\) be a smooth solution of CBF (1.3) on manifold \(M^n\) with constant scalar curvature \(s_0\). Then

\[
\partial_t (\nabla^a \text{Rm}) + \Delta^2 (\nabla^a \text{Rm}) = B^{2+a}_{23}(\text{Rm}) + B^a_3(\text{Rm}) + 2(n-2)p \nabla^a \text{Rm} + \sum_{b=1}^{a} \nabla^b \text{Rm} \cdot \nabla^{a-b} \text{Rm} + (n-2)\nabla^a(T(\nabla^2 p)). \tag{4.4}
\]

### 4.2 An interpolation inequality

We will need the following inequality (compare [18, Corollary 5.5] and [24, Lemma 10.3].

**Lemma 4.4** Let \((M^n, g)\) be a closed Riemannian manifold. Let integers \(0 \leq i_1, \ldots, i_r \leq a \in \mathbb{N}\) and \(i_1 + \cdots + i_r = 2a\). Then for any \(\epsilon > 0\) and nonnegative integers \(r_1\) and \(r_2\) there is a constant \(C = C(n, a, r, r_1, r_2, \epsilon)\) such that for any tensor \(W\) of type \((r_1, r_2)\) on \(M\) we have

\[
\left| \int_M \nabla^{i_1} W \ast \cdots \ast \nabla^{i_r} W \, d\mu_g \right| \leq \epsilon \int_M |\nabla^{a+1} W|^2 d\mu_g + C \|W\|^{(a+1)(r-2)} \cdot \int_M |W|^2 d\mu_g, \tag{4.5}
\]

where \(\|W\|_{\infty}\) is the \(C^0\)-norm.

**Proof** Assume \(i_1, \ldots, i_{l} \geq 1\) and \(i_{l+1}, \ldots, i_r = 0\). By the Hölder inequality we have

\[
\left| \int_M \nabla^{i_1} W \ast \nabla^{i_2} W \ast \cdots \ast \nabla^{i_r} W \, d\mu_g \right| \leq \|W\|_{\infty}^{-l} \cdot \prod_{j=1}^{l} \left( \int_M |\nabla^{j} W|^2 d\mu_g \right)^{\frac{i_j}{2a}}. \tag{4.6}
\]

By using

\[
\left( \int_M |\nabla^{j} W|^2 d\mu_g \right)^{\frac{i_j}{2a}} \leq C \|W\|_{\infty}^{-l} \left( \int_M |\nabla^{a} W|^2 d\mu_g \right)^{\frac{i_j}{2a}} \tag{6.6}
\]
from [12, Corollary 12.6] we have
\[
\left| \int_M \nabla^i W \ast \cdots \ast \nabla^i W \, d\mu_g \right| \leq C\|W\|_{\infty}^{r-2} \cdot \int_M \nabla^a W^2 \, d\mu_g. \tag{4.7}
\]

It follows from [12, Corollary 12.7] and Young’s inequality that for any \( A > 0 \) and \( 1 \leq i \leq a \) we have
\[
A \int_M |\nabla^i W|^2 \, d\mu_g \leq \epsilon \int_M |\nabla^{a+1} W|^2 \, d\mu_g + CA^\frac{a+1}{a+2} \int_M |W|^2 \, d\mu_g, \tag{4.8}
\]
where constant C depends on \( \epsilon, i, a \) but not on A. The lemma follows from combining (4.7) and (4.8).

\[\square\]

4.3 \( L^2 \)-estimate of the derivatives of pressure function \( p \)

We have

**Lemma 4.5** Let \((g(t), p(t))\) be a smooth solution of CBF (1.3) on closed manifold \( M^n \) with constant scalar curvature \( s_0 \). Then for any \( \epsilon \in (0, 1), a \in \mathbb{N}, \) and each time \( t \) the pressure function \( p \) satisfies the following energy estimate
\[
\int_M |\nabla^a p(t)|^2 \, d\mu \leq \epsilon \int_M |\nabla^{a+2} \text{Rm}|^2 \, d\mu + C_1 \| \text{Rm}(\cdot, t) \|_2^2 + C_2 \cdot \| p(t) \|_2^2. \tag{4.9}
\]
Here constant \( C_1 \) depends on \( a, \| \text{Rm}(\cdot, t) \|_\infty \), \( \sup_{x \in M} |\text{Rm}(\cdot, t)|_{g(t)} \), and constant \( C_2 \) depends on \( a, s_0, \| \text{Rm}(\cdot, t) \|_\infty \), and \( \| p(t) \|_\infty \).

**Proof** Using Lemma 2.1 we may rewrite schematically the second equation in (1.3) as
\[
(-(n-1)\Delta - s_0) p = B_2^2(\text{Rm}) + B_3^0(\text{Rm}). \tag{4.10}
\]
and hence,
\[
(-(n-1)\Delta - s_0) \nabla^{a-1} p = B_2^{a+1}(\text{Rm}) + B_3^{a-1}(\text{Rm}) + \sum_{b=1}^{a-1} B_2^{a-b-1}(\text{Rm}) \ast \nabla^b p. \tag{4.11}
\]

Multiplying the above equation by \( \nabla^{a-1} p \) and using integration by parts we get
\[
(n-1) \int_M |\nabla^a p|^2 \, d\mu = s_0 \int_M |\nabla^{a-1} p|^2 \, d\mu + \int_M B_2^{a+1}(\text{Rm}) \ast \nabla^{a-1} p \, d\mu
\]
\[
+ \int_M B_3^{a-1}(\text{Rm}) \ast \nabla^{a-1} p \, d\mu + \sum_{b=1}^{a-1} \int_M \nabla^{a-b-1} \text{Rm} \ast \nabla^b p \ast \nabla^{a-1} p \, d\mu. \tag{4.12}
\]

Below we estimate each terms in the right-hand side of the above equality.

Note that by (4.8) we have
\[
\left| s_0 \int_M |\nabla^{a-1} p|^2 \, d\mu \right| \leq \epsilon \int_M |\nabla^a p|^2 \, d\mu + C|s_0|^a \int_M \phi^2 \, d\mu.
\]
The next two terms can be estimated by using Hölder inequality, (4.6), and (4.8),
\[
\left| \int_M B_2^{a+1}(\text{Rm}) \ast \nabla^{a-1} p \, d\mu \right| + \left| \int_M B_3^{a-1}(\text{Rm}) \ast \nabla^{a-1} p \, d\mu \right|
\]
Remark 4.7

Note that this is the only place where we use the assumption that \( n = 1 \). We have the following

\[
\left| \int_M \nabla^{a-b-1} Rm \ast \nabla^{b} p \ast \nabla^{a-1} p \, d\mu \right|
\]

\[
\leq \left( \int_M |\nabla^{a-b-1} Rm|^{2} \, d\mu \right)^{\frac{2b}{n}} \cdot \left( \int_M |\nabla^{b} p|^{2} \, d\mu \right)^{\frac{b}{n}} \cdot \left( \int_M |\nabla^{a-1} p|^{2} \, d\mu \right)^{\frac{1}{2}}
\]

\[
\leq C \|Rm\|_{\infty}^{2b} \int_M |\nabla^{a-1} Rm|^{2} \, d\mu + C \|p\|_{\infty}^{2b} \int_M |\nabla^{a-1} p|^{2} \, d\mu
\]

\[
\leq \epsilon \int_M |\nabla^{a+2} Rm|^{2} \, d\mu + C \|Rm\|_{\infty}^{2b} \int_M |\nabla^{a-1} Rm|^{2} \, d\mu + C \|p\|_{\infty}^{2b} \int_M |\nabla^{a-1} p|^{2} \, d\mu
\]

where we have used (4.8) to get the last inequality. Hence, the lemma is proved.

Next we bound \( \int_M p^2 \, d\mu \).

**Lemma 4.6** Let \((g(t), p(t))\) be a smooth solution of CBF (1.3) on closed manifold \( M^n \) with constant scalar curvature \( s_0 \). Assume that for each \( t \) operator \((n - 1)\Delta g(t) + s_0 \) is invertible, then for any \( \epsilon \in (0, 1) \) there exists a positive constant \( C \) depending on \( n \) and \( \epsilon \) such that for each time \( t \) the pressure function satisfies

\[
\|p(t)\|_2^2 \leq \epsilon \int_M |\nabla^3 Rm|^{2} \, d\mu + C \left( \|Rm(\cdot, t)\|_{\infty}^{2} + \|Rm(\cdot, t)\|_{\infty}^{6} \right) \|Rm\|_2^2.
\]

**Proof** Since the operator \((n - 1)\Delta + s_0 \) is invertible, by Eq. (4.10) and the standard elliptic theory we have the following

\[
\|p(t)\|_2^2 \leq C \left( \|B_2^2(Rm)\|_2^2 + \|B_3^0(Rm)\|_2^2 \right)
\]

\[
\leq C \left( \|Rm(\cdot, t)\|_{\infty}^{2} \int_M |\nabla^2 Rm|^{2} \, d\mu + \|Rm(\cdot, t)\|_{\infty}^{6} \int_M |\nabla^2 Rm|^{2} \, d\mu \right).
\]

The lemma now follows from applying (4.8) to \( \|Rm(\cdot, t)\|_{\infty}^{2} \int_M |\nabla^2 Rm|^{2} \, d\mu \).

**Remark 4.7** Note that this is the only place where we use the assumption that \((n - 1)\Delta + s_0 \) is invertible in the proof of Theorem 4.1. We may replace the assumption by a upper bound of \( \|p(t)\|_2 \).

### 4.4 Differential inequality about \( \frac{d}{dt} \int_M |\nabla^a Rm|^{2} \, d\mu \)

We have

\[ \frac{d}{dt} \int_M |\nabla^a Rm|^{2} \, d\mu. \]
Proposition 4.8 Let \((g(t), p(t))\) be a smooth solution of CBF \((1.3)\) on closed manifold \(M^n\) with constant scalar curvature \(s_0\). Then for any \(\epsilon \in (0, 1)\) and any \(a \in \mathbb{N}\) there exist constants \(C_1\) and \(C_2\) depending on \(n, \epsilon, a, s_0, \|\text{Rm}(\cdot, t)\|_\infty\), and \(\|p(t)\|_\infty\) such that

\[
\frac{d}{dt} \int_M |\nabla^a \text{Rm}|^2 d\mu + (2 - \epsilon) \int_M |\nabla^{a+2} \text{Rm}|^2 d\mu \leq C_1 \|\text{Rm}\|_2^2 + C_2 \|p\|_2^2. \tag{4.13}
\]

When \(a = 0\) we have

\[
\frac{d}{dt} \int_M |\text{Rm}|^2 d\mu + 2 \int_M |\nabla^2 \text{Rm}|^2 d\mu \leq C \left(1 + \|\text{Rm}(\cdot, t)\|_2^2 + \|p(t)\|_\infty\right) \cdot \|\text{Rm}\|_2^2, \tag{4.14}
\]

where \(C\) is a constant depending only on \(n\).

**Proof** From (4.4) we have

\[
\frac{d}{dt} \int_M |\nabla^a \text{Rm}|^2 d\mu = -2 \int_M |\nabla^{a+2} \text{Rm}|^2 d\mu + \int_M B_3^{2+2a} (\text{Rm}) d\mu + \int_M B_4^{2a} (\text{Rm}) d\mu + \int_M \nabla^a \text{Rm} \ast \nabla^a (T(\nabla^2 p)) d\mu + \sum_{b=0}^a \int_M B_2^{2a-b} (\text{Rm}) \ast \nabla^b p d\mu, \tag{4.15}
\]

where we have used the following integration by parts to get the first term on the right-hand side above

\[
\int_M \langle \nabla^a \text{Rm}, \Delta^2 (\nabla^a \text{Rm}) \rangle d\mu = \int_M |\nabla^{a+2} \text{Rm}|^2 d\mu + \int_M B_3^{2+2a} (\text{Rm}) d\mu.
\]

For the terms behind the summation sign in (4.15) and for \(1 \leq b \leq a - 1\), we use Hölder inequality and (4.6) to get

\[
\left| \int_M \nabla^a \text{Rm} \ast \nabla^b p \ast \nabla^{a-b} \text{Rm} d\mu \right| \\
\leq \left( \int_M |\nabla^a \text{Rm}|^2 d\mu \right)^{b \over 2} \cdot \left( \int_M |\nabla^{a-b} \text{Rm}|^{2a \over 2a-b} d\mu \right)^{b \over 2a} \cdot \left( \int_M |\nabla^b p|^{{2a \over a+b}} d\mu \right)^{{a \over a+b}} \\
\leq C \|\text{Rm}\|_\infty^{b \over 2a} \cdot \|p\|_\infty^{a \over 2} \left( \int_M |\nabla^a \text{Rm}|^2 d\mu \right)^{b \over 2a} \\
\leq C \|\text{Rm}\|_\infty^{b \over 2a} \cdot \|p\|_\infty^{a \over 2} \int_M |\nabla^a \text{Rm}|^2 d\mu + C \int |\nabla^a p|^2 d\mu \\
\leq 2\epsilon \int_M |\nabla^{a+2} \text{Rm}|^2 d\mu + C(\epsilon) \|\text{Rm}\|_\infty^{b \over 2a} \cdot \|p\|_\infty^{a \over 2} \int_M |\text{Rm}|^2 d\mu \\
+ C_1(\epsilon, a, \|\text{Rm}\|_\infty) \int_M |\text{Rm}|^2 d\mu + C_2(\epsilon, a, s_0, \|\text{Rm}\|_\infty \cdot \|p\|_\infty) \int_M p^2 d\mu,
\]

where we have used Young’s inequality to get the second to last inequality, and (4.8) and (4.9) to get the last inequality. When \(b = a\) or \(b = 0\), the above estimate still holds and the proof is actually simpler.

By using interpolation inequality (4.5) and integration by parts, we have

\[
\left| \int_M B_3^{2+2a} (\text{Rm}) d\mu \right| + \left| \int_M B_4^{2a} (\text{Rm}) d\mu \right|
\]



\(\circ\) Springer
≤ \epsilon \int_M |\nabla^{a+2} Rm|^2 d\mu + C(\epsilon)\|Rm\|^{\alpha+2}_\infty \int_M |Rm|^2 d\mu.

By using integration by parts and (4.9), we have
\[ \left| \int_M \nabla^a Rm * \nabla^{a+2} p d\mu \right| \leq \epsilon \int_M |\nabla^{a+2} Rm|^2 d\mu + (4\epsilon)^{-1} \int |\nabla^a p|^2 d\mu \]
\[ \leq \epsilon \int_M |\nabla^{a+2} Rm|^2 d\mu + C_1(\epsilon, a, \|Rm\|_\infty) \int_M |Rm|^2 d\mu \]
\[ + C_2(\epsilon, a, s_0, \|Rm\|_\infty, \|p\|_\infty) \int_M p^2 d\mu. \]

Putting together all calculations above, we obtain estimate (4.13). A careful check of the above calculation will produce the inequality for \( a = 0 \).

4.5 Proof of Theorem 4.1

We use the idea in the proof of [24, Theorem 5.4]. From Proposition 4.8 with \( \epsilon = \frac{1}{4} \) we have the following. For \( a \geq 2 \)
\[ \frac{d}{dt} \int_M |\nabla^a Rm|^2 d\mu + \frac{7}{4} \int_M |\nabla^{a+2} Rm|^2 d\mu \leq C_1 \|Rm\|^2_2 + C_2 \|p\|^2_2. \]

Let \( \bar{T} \doteq \min\{\frac{a}{K}, T\} \). Below \( t \in (0, \bar{T}) \). Note that by applying Lemma 4.6 to bound the last term in the above inequality and using (4.8) to bound \( \int_M |\nabla^3 Rm|^2 d\mu \) by \( \frac{1}{4} \int_M |\nabla^4 Rm|^2 d\mu \), plus something else, we get
\[ \frac{d}{dt} \int_M |\nabla^a Rm|^2 d\mu + \frac{3}{2} \int_M |\nabla^{a+2} Rm|^2 d\mu \leq \frac{1}{2a+1} \bar{T}^{-a-1} \|\nabla^4 Rm\|^2_2 + C_3(a, \bar{T}) \|Rm\|^2_2. \]
(4.16)

where constant \( C_3(a, \bar{T}) \) depends on \( C_1 \) and \( C_2 \). As a special case but using different factor in front of \( \|\nabla^4 Rm\|^2_2 \), we have
\[ \frac{d}{dt} \int_M |\nabla^2 Rm|^2 d\mu + \int_M |\nabla^4 Rm|^2 d\mu \leq -\frac{1}{2} \|\nabla^4 Rm\|^2_2 + C_4(\bar{T}) \|Rm\|^2_2. \]
(4.17)

For \( a \geq 1 \) we define function
\[ f_a(t) \doteq \sum_{j=0}^a \frac{t^j}{j!} \int_M |\nabla^{2j} Rm|^2 d\mu. \]

Using (4.16), (4.17), and (4.14), we compute
\[ \frac{d}{dt} f_a(t) = \sum_{j=1}^a \frac{t^j}{j!} \left( \frac{d}{dt} \int_M |\nabla^{2j} Rm|^2 d\mu + \int_M |\nabla^{2j+2} Rm|^2 d\mu \right) \]
\[ + \left( \frac{d}{dt} \int_M |Rm|^2 d\mu + \int_M |\nabla^2 Rm|^2 d\mu \right) - \frac{t^a}{a!} \int_M |\nabla^{2a+2} Rm|^2 d\mu \]
\[ \leq \|Rm\|^2_2 \sum_{j=2}^a \left( \frac{t^j}{j!} C_3(2j, \bar{T}) \right) + \|\nabla^4 Rm\|^2_2 \left( \sum_{j=2}^a \left( \frac{t^j}{j!} \frac{1}{2^{2j+1}\bar{T}^{2j-1}} \right) - \frac{t^a}{2} \right) \]
\[+ C_4(\bar{T}) \cdot t \| Rm \|_2^2 + C(1 + \| Rm \|_\infty^2 + \| p \|_\infty) \cdot \| Rm \|_2^2 \leq C(n, s_0, \bar{T}, \| Rm \|_\infty^2, \| p \|_\infty, \alpha) \cdot \| Rm \|_2^2. \] (4.18)

This implies (4.1) when \( m \) is even.

For (4.1) with odd \( m \), we use the following interpolation to get it,

\[\int_M |\nabla^{2a+1} Rm|^2 d\mu \leq \left( \int_M |\nabla^{2a} Rm|^2 d\mu \right)^{1/2} \left( \int_M |\nabla^{2a+2} Rm|^2 d\mu \right)^{1/2} \]

This finishes the proof of Theorem 4.1.

We end this section with two direct applications of Theorem 4.1.

### 4.6 Characterizing finite time singularities

Recall that for a closed Riemannian manifold \((M^n, g)\) the Sobolev constant \(C_S(g)\) is the smallest constant \(C\) such that for any function \(u \in C^1(M)\) we have

\[\|u\|_{L^p} \leq C \left( \|\nabla u\|_2 + \text{Vol}_g^{-\frac{1}{n}} \|u\|_2 \right), \] (4.19)

where \(\|u\|_p\) is the \(L^p\)-norm and \(\text{Vol}_g\) is the volume of \(M\). The first application of Theorem 4.1 is

**Theorem 4.9** Let \((g(t), p(t)), t \in [0, T)\), be a smooth solution of CBF (1.1) on closed manifold \(M^n\) with constant scalar curvature \(s_0\). We assume that there is a constant \(K > 0\) such that the curvature of \(g(t)\) and potential function \(p(t)\) satisfy the following conditions

(i) \(\sup_{(x,t) \in M \times [0,T)} (|\text{Rm}(x,t)|_{g(t)} + |p(x,t)|) \leq K\),

(ii) the operator norm \(\|(n-1)\Delta_{g(t)} + s_0\|_{L^1(C^0, C^{2+m})} \leq K\) for \(t \in [0, T)\), and

(iii) the Sobolev constant \(C_S(g(t)) \leq K\) for \(t \in [0, T)\).

Then \((g(t), p(t))\) can be extended to a smooth solution of CBF on \([0, T+\delta]\) for some \(\delta > 0\).

**Proof** From assumption (i), (ii), and Theorem 4.1 we get uniform bounds on \(L^2\)-norm of any \(m\)-th derivatives of the curvature, \(\|\nabla^m \text{Rm}(\cdot, t)\|_{L^2} \leq C(m, T, K)\) for \(t \in [T/2, T)\). Then using assumption (iii) about the Sobolev constant and arguing as in the proof of [24, Theorem 6.2] we get uniform \(C^k\)-norm bounds of curvatures of \(g(t)\) for \(t \in [T/2, T)\).

Note that the uniform \(C^4\)-norm bounds imply the uniform equivalence of metric \(g(t), t \in [T/2, T]\), with \(g(T/2);\) hence, we have a uniform lower bound of injectivity radius \(\text{inj}_{g(t)} \geq \delta > 0\). By the Cheeger-Gromov compactness theorem of Riemannian manifolds, we conclude that \(g(t)\) converges smoothly to a metric called \(g(T)\) as \(t \to T^-\) (no diffeomorphisms needed). Then we can use assumption (ii) and Theorem 1.1 to extend metric \(g(T)\) to a solution \(g(t)\) of CBF for \(t \in [T, T+\delta]\). \(\Box\)

**Remark 4.10** It follows from Remark 4.7 that we may replace condition (ii) used to prove the uniform \(C^k\)-norm bounds of curvatures of \(g(t)\) in Theorem 4.9 by assuming \(\|p(t)\|_2 \leq K\) for all \(t \in [0, T)\), this follows from (i) and \(\frac{d}{dt} \text{Vol}_{g(t)}(M) = n(n-2) \int_M p \, d\mu_{g(t)}\). However, we still need to use (ii) to extend the flow beyond \(t = T\).

### 4.7 Compactness theorem for CBF

The proof of the following theorem is standard and is also similar to that of [24, Theorem 7.1], we omit it.
Theorem 4.11 Let \( \{(g_i(t), p_i(t))\} \), \( t \in (\alpha, \omega) \), be a family of smooth solutions of CBF (1.3) on closed manifolds \( M^n_i \) with constant scalar curvature \( s_0(i) \), where \(-\infty < \alpha < 0 < \omega < \infty\).

Let \( \{q_i \in M_i\} \) be a sequence of points. We assume that there is a constant \( K > 0 \) such that for each \( i \) the curvature of \( g_i(t) \) and potential function \( p_i(t) \) satisfy the following conditions

(i) \( \sup_{(x,t) \in (\alpha, \omega)} (|Rm_{g_i}(x,t)|_{g_i(t)} + |p_i(x,t)|) \leq K \),
(ii) the operator norm \( \|((n-1)\Delta_{g_i(t)} + s_0(i))^{-1}\|_{L^1(C^0, C^{2+\varepsilon})} \leq K \) for \( t \in (\alpha, \omega) \),
(iii) the Sobolev constant \( C_S(g_i(t)) \leq K \) for \( t \in (\alpha, \omega) \), and
(iv) \( \lim_{t \to \alpha} \int_{M_i} |Rm_{g_i}(\cdot,t)|_{g_i(t)}^2 \, d\mu_{g_i(t)} \leq K \).

Then sequence \( \{(M_i, g_i(t), p_i(t), q_i)\} \) sub-converges in pointed Cheeger-Gromov \( C^\infty \)-topology to a complete solution \( (M_\infty^n, g_\infty(t), p_\infty(t), q_\infty) \), \( t \in (\alpha, \omega) \), of CBF (1.3).

Remark 4.12 It follows from Remark 4.7 that we may replace condition (ii) in Theorem 4.11 by assuming \( \|p_i(t)\|_2 \leq K \) for all \( i \) and \( t \in (\alpha, \omega) \).

5 Local integral and pointwise Shi’s type estimate

In this section we prove a local version of the integral Shi’s type estimate (Theorem 5.5). At the end we give a proof of Theorem 1.3. The basic ideas of the two proofs are from [28, Theorem 4.4] and [28, Section 5], respectively.

5.1 A property about cutoff functions and localized interpolation inequalities

We will need the following when we use cutoff functions to localize later.

Lemma 5.1 Let \( (M^n, g(t)) \), \( t \in [0, T] \), be a smooth family of Riemannian manifolds. Then there are constants

\[
C_1 = C_1 \left( \sup_{t \in [0,T]} |\partial_t g|_{g(t)}, T \right) \quad \text{and} \quad C_2 = C_2 \left( \sup_{t \in [0,T]} (|\partial_t g|_{g(t)} + |\nabla \partial_t g|_{g(t)}), T \right)
\]

such that for any function \( \eta \in C^\infty(M) \) we have pointwise estimates

\[
|\nabla \eta|_{g(t)} \leq C_1 |\nabla \eta|_{g(0)} \quad \text{and} \quad |\nabla^2 \eta|_{g(t)} \leq C_2 \left( |\nabla \eta|_{g(0)} t + |\nabla^2 \eta|_{g(0)} \right) \quad \text{for} \quad t \in [0, T].
\]

Proof The first estimate follows from

\[
\partial_t |\nabla \eta|^2 = \partial_t g_{ij}(t) \partial_i \eta \partial_j \eta \leq |\partial_t g| \cdot |\nabla \eta|^2.
\]

For the second estimate we express the Hessian in local coordinates as

\[
(\nabla^2 \eta)_{ij}(t) = \partial_i \partial_j \eta - \Gamma^k_{ij}(t) \partial_k \eta,
\]

and we compute the derivative

\[
\partial_t |\nabla^2 \eta|^2 = \nabla \partial_t g \ast \nabla \eta \ast \nabla^2 \eta + \partial_t g \ast \nabla^2 \eta \ast \nabla^2 \eta
\]

\[
\leq C(n) \sup_{t \in [0,T]} |\nabla \partial_t g| \cdot \sup_{t \in [0,T]} |\nabla \eta| \cdot |\nabla^2 \eta| + \sup_{t \in [0,T]} |\partial_t g| \cdot |\nabla^2 \eta|^2.
\]

Using the first estimate and applying the Gronwall-type inequality we get the second estimate.

The following local interpolation inequalities are simple consequence from [18, Section 5] (see also [24, Section 10]) which are analog of (4.8) and (4.5).
Lemma 5.2 Let \( (M^n, g) \) be a Riemannian manifold and let \( \eta \) be a \( C^1 \) function which satisfies \( 0 \leq \eta \leq 1 \) and \( |\nabla \eta| \leq \Lambda \). We assume that set \( \{ x \in M, \eta(x) > 0 \} \) is precompact in \( M \). Let \( a \in \mathbb{N} \) and let \( W \) be any \( C^{a+1} \)-tensor of type \( (r_1, r_2) \). Then

(i) For \( i = 1, 2, \cdots, a \), and constants \( A > 0, \epsilon > 0, s \geq 2a \), we have

\[
A \int_M |\nabla_i W|^2 \eta^s d\mu \leq \epsilon \int_M |\nabla^{a+1} W|^2 \eta^{s+2a+2-2i} d\mu + CA^{a+1} \| W \|_{2, \eta>0}^2,
\]  

(5.1)

where constant \( C = C(n, a, r_1, r_2, s, \Lambda, \epsilon) \).

(ii) For \( 0 \leq i_1, \cdots, i_r \leq a \in \mathbb{N} \) with \( i_1 + \cdots + i_r = 2a \), and constants \( \epsilon > 0, s \geq 2a \), we have

\[
\left| \int_M \nabla^{i_1} W \cdots \nabla^{i_r} W \eta^s d\mu \right| \leq \epsilon \int_M |\nabla^{a+1} W|^2 \eta^{s+2} d\mu + C \| W \|_{(a+1)(r-2)} \| W \|_{2, \eta>0}^2,
\]  

(5.2)

where constant \( C = C(n, a, r, r_1, r_2, s, \Lambda, \epsilon) \).

5.2 Localized integral version of Shi’s type estimate

Let \( (g(t), p(t))_{t \in [0, T]} \) be a local smooth solution of CBF \((1.3)\) on manifold \( M^n \) with constant scalar curvature \( s_0 \), and let \( \eta \) be a \( C^1 \) function which satisfies \( 0 \leq \eta \leq 1 \) and \( |\nabla \eta|_{g(t)} \leq \Lambda \).

In this subsection we further assume that \( \{ x \in M, \eta(x) > 0 \} \) is precompact in \( M \). The logic steps of the proof of Theorem 5.5 is similar to that in Sect. 4.

First we prove a localized version of (4.9).

Lemma 5.3 For any \( \epsilon \in (0, 1) \), \( a \in \mathbb{N} \), and \( s \geq 2a \) we have for each time \( t \)

\[
\int_M |\nabla^a p(t)|^2 \eta^s d\mu \leq \epsilon \int_M |\nabla^{a+2} \operatorname{Rm} |^2 \eta^{s+4} d\mu + C_1 \| \operatorname{Rm} \|_{2, \eta>0}^2 + C_2 \| p \|_{2, \eta>0}^2,
\]  

(5.3)

where constant \( C_1 \) depends on \( n, a, s_0, \| \operatorname{Rm}(\cdot, t) \|_{\infty, \eta>0}, s, \Lambda \) and constant \( C_2 \) further depends on \( \| p(t) \|_{\infty, \eta>0} \).

Proof We start with multiplying Eq. (4.11) by \( \nabla^{a-1} p \cdot \eta^s \) and using integration by parts. Note that the \( \nabla \eta \) term is bounded \( \Lambda \). To finish the estimate we need to use interpolation inequalities in Lemma 5.2, and to be careful with the power of \( \eta \) when applying Hölder inequality. \( \Box \)

Next we establish a differential inequality about \( \frac{d}{dt} \int_M |\nabla^a \operatorname{Rm} |^2 \eta^s d\mu \).

Proposition 5.4 We assume that \( |\nabla^2 \eta(x)|_{g(t)} \leq \Lambda \). For \( a \in \mathbb{N} \) and \( s \geq 2a \) we have

\[
\frac{d}{dt} \int_M |\nabla^a \operatorname{Rm} |^2 \eta^s d\mu + (2 - \epsilon) \int_M |\nabla^{a+2} \operatorname{Rm} |^2 \eta^s d\mu \leq C_1 \| \operatorname{Rm} \|_{2, \eta>0}^2 + C_2 \| p \|_{2, \eta>0}^2,
\]  

(5.4)

where constants \( C_1 \) and \( C_2 \) are as in Lemma 5.3.

Proof The proof is similar to Proposition 4.8, here we only give a rough sketch. From Eq. (4.4) and that \( \eta \) is independent of \( t \), we have

\[
\frac{d}{dt} \int_M |\nabla^a \operatorname{Rm} |^2 \eta^s d\mu + 2 \int_M |\nabla^{a+2} \operatorname{Rm} |^2 \eta^s d\mu
\]
for some constant $K$ defined by

\[
\text{Proof by } (\Lambda_1/\Lambda_1)
\]

By the localized interpolation inequalities in Lemma 5.2, we have

\[
\text{equality and the localized interpolation inequalities, we have }
\]

\[
C \text{ is a constant which depends on } n
\]

where we have applied integration by parts to $\int_M (\nabla^a Rm, \Delta^2 (\nabla^a Rm)) \eta^s d\mu$.

Let

\[
L_{\epsilon} \doteq \epsilon \int_M |\nabla^{a+2} Rm|^2 \eta^s d\mu + C_1 \| \nabla^a Rm \|_{2, \eta > 0}^2 + C_2 \| p \|_{2, \eta > 0}^2.
\]

By the localized interpolation inequalities in Lemma 5.2, $p(t) \leq \| p(t) \|_{\infty, \eta > 0}$, and $|\nabla \eta| \leq \Lambda$, we conclude that all of the terms on the right-hand side of the above display are bounded by $(\Lambda + 1) L_{\epsilon}$ except those terms which contain function $p$. For them, by Cauchy–Schwarz inequality and the localized interpolation inequalities, we have

\[
\int_M \left( |\nabla^a Rm * \nabla^{a+2} p \eta^s| + |\nabla^a Rm * \nabla^b p * \nabla^{a-b} Rm \eta^s| \right) d\mu
\]

\[
\leq L_{\epsilon} + C(\epsilon, \Lambda) \int_M |\nabla^a p|^2 \eta^s d\mu.
\]

Note that by Lemma 5.3 we have $\int_M |\nabla^a p|^2 \eta^s d\mu \leq L_{\epsilon}$; hence, inequality (5.4) follows from combining the inequalities above. \hfill \Box

Finally, we give a proof of

**Theorem 5.5** Let $(M^n, g(t), p(t))_{t \in [0, T]}$ be a local smooth solution of CBF (1.3). Fix $r > 0$ and $q \in M$ we assume ball $B_{g(0)}(q, 2r)$ is precompact in $M$ and

\[
\sup_{B_{g(0)}(q, 2r) \times [0, T]} \left( |\nabla^m Rm(\cdot, t)| + |p(\cdot)| + |\partial_t g| + |\nabla \partial_t g| \right) \leq K \tag{5.5}
\]

for some constant $K > 0$. Then for $m \in \mathbb{N}$ we have

\[
\int_{B_{g(0)}(q, r)} |\nabla^m Rm(\cdot, t)|^2 d\mu \leq \frac{C}{r^{m/2}},
\]

where $C$ is a constant which depends on $n, m, r, g(0), T, K$, and

\[
\sup_{t \in [0, T]} \left( \| \nabla^m Rm(\cdot, t) \|_{L^2(B_{g(0)}(p, 2r))}^2 + \| p(t) \|_{L^2(B_{g(0)}(p, 2r))}^2 \right).
\]

**Proof** First we show the estimate for even $m = 2a$. Let $\eta$ denote a smooth cutoff function defined by

\[
\eta(x) = \begin{cases} 
0 & \text{if } x \notin B_{g(0)}(q, 2r), \\
\in [0, 1] & \text{if } x \in B_{g(0)}(q, 2r) \setminus B_{g(0)}(q, r), \\
1 & \text{if } x \in B_{g(0)}(q, r).
\end{cases}
\]

Obviously $\eta$ satisfies $|\nabla \eta|_{g(0)} + |\nabla^2 \eta|_{g(0)} \leq C(r, g(0))$. By Lemma 5.1 and (5.5) we have

\[
\sup_{t \in [0, T]} |\nabla \eta|_{g(t)} + |\nabla^2 \eta|_{g(t)} \leq \Lambda(n, T, K, r, g(0)).
\]
Let \( \beta_a = 1 \) and let \( \beta_j, \ j = 0, 1, \ldots, a - 1 \), be nonnegative constants to be determined below. We define

\[
f_a(t) \doteq \sum_{j=0}^{a} \beta_j t^j \int_M |\nabla^{2j} \text{Rm}(\cdot, t)|^2 \eta^{j+2} d\mu.
\]

It follows from Proposition 5.4 with \( \epsilon = \frac{1}{2} \) that

\[
\frac{d}{dt} f_a(t) \leq \sum_{j=1}^{a} \int_M |\nabla^{2j} \text{Rm}(\cdot, t)|^2 \eta^{j+2} d\mu \cdot \left( -\frac{3}{2} \beta_{j-1} t^{j-1} + C_3(j) \beta_j t^j + \beta_j t^{j-1} \right) \\
+ C \left( 1 + \|\text{Rm}(\cdot, t)\|_{2, \eta > 0}^2 + \|p(t)\|_{2, \eta > 0} \right) \|p\|_{2, \eta > 0}^2 \eta > 0
\]

where

\[
C_3(j) = C_1(j) \|\text{Rm}(\cdot, t)\|_{2, \eta > 0}^2 + C_2(j) \|p(t)\|_{2, \eta > 0}^2
\]

and \( C_1(j) \) and \( C_2(j) \) are given in Proposition 5.4. It is clear that by an appropriate inductive choice of the constants \( \beta_i \) starting from \( \beta_a = 1 \) we obtain

\[
\frac{d}{dt} f_a(t) \leq C \left( 1 + \|\text{Rm}(\cdot, t)\|_{2, \eta > 0}^2 + \|p(t)\|_{2, \eta > 0} \right) \|p\|_{2, \eta > 0}^2.
\]

Integrating this ODE, we get the required estimate for even \( m = 2a \).

For odd \( m = 2a - 1 \) the estimate follows from the interpolation inequality and the even case.

\[\square\]

**Remark 5.6** From the CBF Eq. (1.3) we know that \( |\partial_t g| + |\nabla \partial_t g| \) in (5.5) can be bounded by assuming

\[
\max_{a=0,1, \ldots, 3} \sup_{B_{\epsilon(0)}(q,2r) \times [0,T]} |\nabla^a \text{Rm}(\cdot, \cdot)| \leq K.
\]

### 5.3 Shi’s type pointwise estimate

Recall that in the proof of Theorem 4.9 we use the assumption of Sobolev constant’s bound to get \( C^0 \)-bound of the derivatives of curvatures. In Theorem 1.3 we remove the assumption. Now we give a proof of the theorem which is similar to that of [28, Theorem 4.4].

We define function

\[
f_m(x, t, g) \doteq \sum_{j=1}^{m} \left| \nabla^j \text{Rm}(x, t) \right|^{\frac{2}{j+2}} \text{ on } M \times (0, T].
\]

We claim that for all sufficiently large \( m \geq 3 \) there is a constant \( C = C(n, m) \) such that

\[
f_m(x, t, g) \leq C \left( K + t^{\frac{1}{2}} \right).
\]

The claim implies the theorem and we will prove it by a contradiction argument.

Suppose the claim is false, then there is a sequence \( \{(M^n_i, g_t(t), p_t(t))\}_{i \in [0, T]} \) of complete solutions to CBF (1.3) with constant scalar curvature \( s_0 \) which satisfy the assumption of the theorem, together with points \( (x_i, t_i) \) such that

\[
\lim_{i \to \infty} \frac{f_m(x_i, t_i, g_t)}{K + t_i^{\frac{1}{2}}} = \infty.
\]
Without loss of generality, we may choose the points \((x_i, t_i)\) such that
\[
\frac{f_m(x_i, t_i, g_i)}{K + \frac{1}{2} t_i^{-\frac{1}{2}}} \geq \frac{1}{2} \sup_{M_t \times (0, T_i]} \frac{f_m(x, t, g_i)}{K + \frac{1}{2} t^{-\frac{1}{2}}}. \tag{5.6}
\]

Let constants
\[
\lambda_i \equiv f_m(x_i, t_i, g_i) \quad \text{and} \quad \tilde{s}_{0i} = \lambda_i^{-1} s_{0i}.
\]

We define the scaled metrics and potential functions by
\[
\tilde{g}_i(\tilde{t}) \equiv \lambda_i g \left( t_i + \frac{\tilde{t}}{\lambda_i^2} \right), \quad \tilde{p}_i(\tilde{t}) = \lambda_i^{-2} p_i \left( t_i + \frac{\tilde{t}}{\lambda_i^2} \right).
\]

By the scaling property of CBF \((\tilde{g}_i(\tilde{t}), \tilde{p}_i(\tilde{t}))\) satisfies CBF \((1.3)\) with constant scalar curvature \(\tilde{s}_{0i}\), and exists on \(\tilde{t} \in [-t_i \lambda_i^2, 0]\). These solutions have the following simple properties.

(P1) By the definition of \(f_m\), we have
\[
f_m(x, \tilde{t}, \tilde{g}_i) = \lambda_i^{-1} f_m \left( x, t_i + \frac{\tilde{t}}{\lambda_i^2}, g_i \right);
\]

hence, \(f_m(x_i, 0, \tilde{g}_i) = 1\).

(P2) Note that
\[
t_i^{-\frac{1}{2}} \lambda_i = \frac{f_m(x_i, t_i, g_i)}{t_i^{-\frac{1}{2}}} \geq \frac{f_m(x_i, t_i, g_i)}{K + t_i^{-\frac{1}{2}}} \to \infty,
\]

we conclude that the solutions \((\tilde{g}_i(\tilde{t}), \tilde{p}_i(\tilde{t}))\) exist on \([-1, 0]\) for \(i\) sufficiently large.

(P3) By assumption \((1.4)\) we have that for any \((x, \tilde{t}) \in M_t \times [-1, 0] \]
\[
|\text{Rm}_{\tilde{g}_i}(x, \tilde{t})| \leq \frac{K}{\lambda_i} \to 0 \quad \text{as} \quad i \to \infty.
\]

(P4) From (P1) we have that for \((\tilde{x}, \tilde{t}) \in M_t \times [-1, 0] \]
\[
f_m(\tilde{x}, \tilde{t}, \tilde{g}_i) = \frac{f_m \left( \tilde{x}, \tilde{t}_i + \frac{\tilde{t}}{\lambda_i^2}, g_i \right)}{\lambda_i} \leq \frac{f_m \left( \tilde{x}, \tilde{t}_i + \frac{\tilde{t}}{\lambda_i^2}, g_i \right)}{f_m(x_i, t_i, g_i)} \leq 2 \cdot \frac{K + \left( t_i + \frac{\tilde{t}}{\lambda_i^2} \right)^{-\frac{1}{2}}} {K + t_i^{-\frac{1}{2}}} \leq 3,
\]

where we have used \((5.6)\) and (P2) to get the first inequality above.

Let metric \(\tilde{h}_i(\tilde{t})\), defined on some ball \(B(0, 2r) \subset \mathbb{R}^n\) with \(\tilde{t} \in [-1, 0]\), be a local lifting of \(\tilde{g}_i(\tilde{t})\) near \(x_i\) by exponential map \(\exp_{\tilde{g}_i(0)}\). By (P4) and \((1.4)\) we have a uniform bound of curvatures
\[
\sum_{j=0}^{m} |\nabla^j \text{Rm}_{\tilde{h}_i(\tilde{t})}(\tilde{x})|_{\tilde{h}_i(\tilde{t})} \leq C \quad \text{for} \quad \tilde{x} \in B(0, 2r).
\]

This implies that metrics \(\tilde{h}_i(\tilde{t})\) are uniformly equivalent to Euclidean metric on \(B(0, 2r)\). Hence, by Cheeger–Gromov compactness theorem of Riemannian manifolds the sequence \(\{\tilde{h}_i(\tilde{t})\}\) sub-converges to \(\tilde{h}_\infty(\tilde{t})\) in \(C^{m-2-\alpha}\)-topology with \(\tilde{t} \in [-1, 0]\). We can improve the
convergence of \(\tilde{h}_i(t)\) to \(C^\infty\)-convergence for \(\tilde{t} \in [-\frac{1}{2}, 0]\) by using Theorem 5.5. Note that by (1.4) and the scaling property we have uniform upper bounds for
\[
\sup_{(\tilde{x}, \tilde{t}) \in B(0,2r) \times [-1,0]} |\tilde{p}_i(\tilde{x}, \tilde{t})| \quad \text{and} \quad \|\tilde{p}_i(\tilde{t})\|_{L^2(B(0,2r))}.
\]
By Remark 5.6 and (P4) we may apply Theorem 5.5 to \(\tilde{h}_i(t)\) on \(B(0,2r) \times [-1,0]\) to conclude that for each fixed \(a \in \mathbb{N}\)
\[
\int_{B(0,r)} |\nabla_{\tilde{h}_i(t)}^a Rm_{\tilde{h}_i(t)}(\cdot)|^2 d\mu_{\tilde{h}_i(t)}
\]
has a uniform upper bound.

Since metrics \(\tilde{h}_i(0)\) are uniformly equivalent to the Euclidean metric on \(B(0, r)\), it follows that the Sobolev constants for metric \(\tilde{h}_i(0)\) on \(B(0, 2r)\) are uniformly bounded. As in the proof of Theorem 4.9 it follows that the \(C^a\)-norms of the curvature of \(\tilde{h}_i(0)\) on \(B(0, \frac{1}{2}r)\) are uniformly bounded for all \(a\). Thus, by taking a further sub-sequence we conclude that \(\tilde{h}_i(0)\) is sub-convergent to \(\tilde{h}_\infty(0)\) in \(C^\infty\); hence, \(f_m(0,0, \tilde{h}_\infty) = 1\) from (P1). However, \(Rm_{\tilde{h}_\infty} = 0\) on \(B(0, \frac{1}{2}r)\), we get a contradiction and the theorem is proved.

References
1. Bakas, I., Bourliot, F., Lüst, D., Petropoulos, M.: Geometric flows in Hořava–Lifshitz gravity. J. High Energy Phys. 2010, 58 (2010)
2. Bahuaud, E., Helliwell, D.: Short-time existence for some higher-order geometric flows. Comm. PDE 36, 2189–2207 (2011)
3. Bahuaud, E., Helliwell, D.: Uniqueness for some higher-order geometric flows. Bull. Lond. Math. Soc. 47, 980–995 (2015)
4. Besse, A.: Einstein manifolds. Springer-Verlag, Berlin (1987)
5. Calabi, E.: Extremal Kähler metrics II. In: Differential geometry and complex analysis, pp. 95–114. Springer (1985)
6. Chen, X.X.: Calabi flow in Riemann surfaces revisited: a new point of view. Int. Math. Res. Not. 2001, 275–297 (2001)
7. Cao, H.D., Chen, Q.: On Bach-flat gradient shrinking Ricci solitons. Duke Math. J. 162, 1149–1169 (2013)
8. Chow, B., Lu, P., Ni, L.: Hamilton’s Ricci flow. Grad. Studies Math, vol. 77. American Mathematical Society, Providence, RI (2006)
9. Fischer, A.: An introduction to conformal Ricci flow. Class. Quantum Grav. 21, S171–S218 (2004)
10. Fischer, A., Moncrief, V.: Conformal volume collapse of 3-manifolds and the reduced Einstein flow, pp. 463–522. Geometry, mechanics and dynamics, Springer, New York (2002)
11. Hamilton, R.: Harmonic maps of manifolds with boundary. Lectures Notes in Math, vol. 471. Springer, Berlin (1975)
12. Hamilton, R.: Three-manifolds with positive Ricci curvature. J. Diff. Geom. 17, 255–306 (1982)
13. Hong, M., Tian, G.: Global existence of the m-equivariant Yang-Mills flow in four dimensional spaces. Comm. Anal. Geom. 12, 183–211 (2004)
14. Kotschwar, B.L.: Backwards Uniqueness for the Ricci Flow. Int. Math. Res. Not. 2010, 4064–4097 (2010)
15. Kotschwar, B.: An energy approach to the problem of uniqueness for the Ricci flow. Comm. Anal. Geom. 22, 149–176 (2014)
16. Kotschwar, B.: A short proof of backward uniqueness for some geometric evolution equations. Int. J. Math. 27, 1650102 (2016)
17. Kuwert, E., Schätzle, R.: The Willmore flow with small initial energy. J. Diff. Geom. 57, 409–441 (2001)
18. Kuwert, E., Schätzle, R.: Gradient flow for the Willmore functional. Comm. Anal. Geom. 10, 307–339 (2002)
19. Lopez, C.: Ambient obstruction flow. Trans. Amer. Math. Soc. 370, 4111–4145 (2018)
20. Lu, P., Qing, J., Zheng, Y.: A note on the conformal Ricci flow. Pacific J. Math. 268, 413–434 (2014)
21. Lu, P., Qing, J., Zheng, Y.: Conformal Ricci flow on asymptotically hyperbolic manifolds. Sci. China Math. 62, 157–170 (2019)
22. Lunardi, A.: Analytic semigroups and optimal regularity in parabolic problems. Birkhäuser, Boston (1995)
23. Schlatter, A., Struwe, M., Tahvildar-Zadeh, A.: Global existence of the equivariant Yang-Mills flow in four space dimensions. Am. J. Math. 120, 117–128 (1998)
24. Streets, J.: The gradient flow of \( \int |Rm|^2 \). J. Geom. Anal. 18, 249–271 (2008)
25. Streets, J.: The gradient flow of the \( L^2 \) curvature energy on surfaces. Int. Math. Res. Not. 2011, 5398–5411 (2011)
26. Streets, J.: The gradient flow of the \( L^2 \) curvature energy near the round sphere. Adv. Math. 231, 328–356 (2012)
27. Streets, J.: The gradient flow of the \( L^2 \) curvature functional with small initial energy. J. Geom. Anal. 22, 691–725 (2012)
28. Streets, J.: The long time behavior of fourth order curvature flows. Calc. Var. PDE 46, 39–54 (2013)
29. Struwe, M.: The Yang-Mills flow in four dimensions. Calc. Var. PDE 2, 123–150 (1994)
30. Sun, X.M., Zhu, A.Q.: Backward uniqueness for the conformal Ricci flow. Diff. Geom. Appl. 56, 110–119 (2018)
31. Waldron, A.: Long-time existence for Yang-Mills flow. Invent. Math. 217, 1069–1147 (2019)

**Publisher’s Note**  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.