Long time dynamics of Schrödinger and wave equations on flat tori

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Abstract

We consider a class of linear time dependent Schrödinger equations and quasi-periodically forced nonlinear Hamiltonian wave/Klein Gordon and Schrödinger equations on arbitrary flat tori. For the linear Schrödinger equation, we prove a \( t^\epsilon \) (\( \epsilon > 0 \)) upper bound for the growth of the Sobolev norms as the time goes to infinity. For the nonlinear Hamiltonian PDEs we construct families of time quasi-periodic solutions.

Both results are based on “clusterization properties” of the eigenvalues of the Laplacian on a flat torus and on suitable “separation properties” of the singular sites of Schrödinger and wave operators, which are integers, in space-time Fourier lattice, close to a cone or a paraboloid. Thanks to these properties we are able to apply Delort abstract theorem [Del10] to control the speed of growth of the Sobolev norms, and Berti-Corsi-Procesi abstract Nash-Moser theorem [BCP15] to construct quasi-periodic solutions.

1 Introduction

In the last years many efforts have been made to understand the long time dynamics of Schrödinger and wave equations on compact manifolds. Two important problems regard upper and lower bounds for the possible growth of the Sobolev norms of a solution, and the existence of global in time quasi-periodic solutions, for which the Sobolev norm remains perpetually bounded. Results concerning these problems have been obtained when the underlying manifold is either the standard torus \( \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d \) (see e.g. [Bon99a, Bon99b, Del10] for growth of Sobolev norms for linear systems and [Bon98, Bon05, EK10, GXY11, BB12, BB13, PP15, EGK16, Wan16] for quasi-periodic solutions), a Zoll manifold (see [Del10, MR17, BGM17] for growth and [BBP10, CP16, BK18] for quasi-periodic solutions), a Lie group or a homogeneous space [BPT10, BCP15, CHP15].

The reason is that suitable information about the eigenvalues and eigenfunctions of the Laplace operator are relevant for the long time dynamics.

In this paper we extend some of these results to the case the manifold is any flat torus

\[
\mathbb{T}^d_\mathcal{L} := \mathbb{R}^d/\mathcal{L}
\]

where \( \mathcal{L} \) is a lattice of \( \mathbb{R}^d \), with linearly independent generators \( \mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{R}^d \), i.e.

\[
\mathcal{L} := \left\{ \sum_{i=1}^d m_i \mathbf{v}_i : m_i \in \mathbb{Z} \right\}.
\]

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A particular case of (1.1)-(1.2) is the rectangular torus

\[ T^d_L := (\mathbb{R}/L_1 \mathbb{Z}) \times \cdots \times (\mathbb{R}/L_d \mathbb{Z}) \]  

with generators \( v_a = L_a e_a \) where \( L_a > 0 \) and \( (e_a)_{a=1,\ldots,d} \) denotes the canonical basis of \( \mathbb{Z}^d \).

The equations we consider are the linear time dependent Schrödinger equation

\[ i \partial_t \psi = -\Delta \psi + V(t,x) \psi, \quad x \in T^d_L, \]  

where \( V(t,x) \) is a smooth real valued potential, periodic on the lattice \( \mathcal{L} \), namely \( V(t,x + \sum m_i v_i) = V(t,x), \forall m_i \in \mathbb{Z} \), and the quasi-periodically forced Hamiltonian nonlinear wave (NLW) and nonlinear Schrödinger (NLS) equations

\[ u_{tt} - \Delta u + mu = \epsilon f(\omega t,x,u), \quad i u_t - \Delta u + mu = \epsilon f(\omega t,x,u), \quad x \in T^d_L, \]  

where the nonlinearities \( f, \epsilon \) are sufficiently regular, and the frequency vector \( \omega \in \mathbb{R}^n \) is Diophantine.

For equation (1.4), we will show that, on any flat torus \( T^d_L \), the Sobolev norms of any solution grow at most as \( t^\ell \) (\( \forall \epsilon > 0 \)) when \( t \) goes to infinity, see Theorem 1.1. Concerning the nonlinear wave and Schrödinger equations (1.5) on any flat torus \( T^d_L \), we construct families of quasi-periodic solutions for a large set of frequency vectors \( \omega = \lambda \mathfrak{m} \) with a given Diophantine direction \( \mathfrak{m} \in \mathbb{R}^n \), see Theorem 1.2. In particular, both results hold for rectangular tori.

At the heart of the proofs is the analysis of certain spectral properties of the linear operators \(-\Delta, \partial_t^2 - \Delta + m \) and \( i \partial_t - \Delta + m \) acting on (possibly time quasi-periodic) functions defined on the flat torus \( T^d_L \). The eigenvalues of \(-\Delta \mathcal{L}\) are

\[ \mu_j := \| Wj \|^2, \quad j \in \mathbb{Z}^d, \]  

where \( \| y \|^2 := \sum_{a=1}^d y_a^2 \) is the euclidean norm of \( \mathbb{R}^d \) and \( W \) is the \( d \times d \) invertible matrix

\[ W := V^{-\top}, \quad V := (v_1 \cdots v_d) \]  

(with \( V^{-\top} \) the transpose matrix of \( V^{-1} \) ) associated to the dual lattice. Since \( W \) is invertible we clearly have that \( \mu_j \sim \| j \|_2^2 \). In the particular case of a rectangular torus

\[ W = \text{diag}(\nu_1, \ldots, \nu_d), \quad \nu_a := L_a^{-1}, \quad \mu_j = \sum_{a=1}^d \nu_a^2 j_a^2, \quad j_a \in \mathbb{Z}, \quad j := (j_a)_{a=1,\ldots,d}. \]

We show that \( \mathbb{Z}^d \) may be partitioned into dyadic clusters (i.e. \( \max |j| \leq 2 \min |j| \) for all integers in the same cluster), such that if \( j, j' \in \mathbb{Z}^d \) are in different clusters, then the pairs \( (j, \mu_j), (j', \mu_{j'}) \) are well separated, see Theorem 2.1. This holds true for any flat torus. This result extends Bourgain cluster decomposition [Bou98] to any flat torus (in particular rectangular ones). In order to obtain such clusterization, we prove that a chain of distinct integer vectors of \( \mathbb{Z}^d \) for which the corresponding pairs \( (j, \mu_j) \) have uniformly bounded distance \( \Gamma \) contains at most \( \sim \Gamma^{C_1(d)} \) elements, see Proposition 2.2.

Concerning \( \partial_t^2 - \Delta + m \) and \( i \partial_t - \Delta + m \), restricted to act on subspaces of functions on \( T^d_L \) which are quasi-periodic in time with Diophantine frequency vector \( \omega \in \mathbb{R}^n \), we show that their "singular" sites form time-space clusters which are "well-separated" as well, namely their eigenvalues

\[ -(\omega \cdot \ell)^2 + \mu_j + m, \quad -\omega \cdot \ell + \mu_j + m, \quad \ell \in \mathbb{Z}^n, \quad j \in \mathbb{Z}^d, \]  

(1.8)
which are in modulus less than 1, form chains with a bounded controlled length. This means, roughly speaking, that any sequence \((\ell_q, j_q)_{q=0,\ldots,L} \subset \mathbb{Z}^n \times \mathbb{Z}^d\) of indices such that (i) the correspondingly eigenvalues (1.8) have all modulus smaller than 1 and (ii) the distances between two consecutive indices are uniformly bounded, has necessarily a length \(L\) which is bounded in an appropriate quantitative way, see Propositions 3.1 and 4.1. Notice that in the first case in (1.8) (corresponding to NLW) the singular sites \(\ell, j\) stay near a “cone” (deformed by the lattice), while in the second one (corresponding to NLS) near a “paraboloid”.

The key argument to prove these separation properties consists, following the lines of [Bou05, BB12], to show that certain bilinear forms, naturally associated to the quantities (1.6), (1.8) (see (2.8), (3.5)), are nondegenerate when restricted to the finite dimensional subspace spanned by chains of singular sites. This allows to estimate the projections of the vectors of any chain on the subspace generated by the vectors of the chain itselfs. If such a subspace has maximal dimension, we deduce a bound for all the vectors of the chains. Otherwise, by an inductive argument on the dimension of such subspace, we deduce a bound for the length of the chain. In case of (1.6), the associated bilinear form is positive definite with a quantitative lower bound, for any flat torus, see Lemma 2.7. In the first case (1.8) (corresponding to NLW) the bilinear form has signature, but nevertheless the finite dimensional restrictions (3.11) care about are nondegenerate for any flat torus \(T^d\), for almost all frequency vectors \(\omega = \lambda \omega\) with a prescribed Diophantine direction \(\omega\), see Lemma 3.3 (in [Bou05, BB12, Wan17] further quadratic or higher order Diophantine conditions are required). This is a consequence of the careful expansion (3.12) and the invertibility of the compound matrices formed by the minors of the invertible matrix \(W\) of the lattice.

Once the spectral properties discussed above are proven, we apply Delort abstract theorem [Del10] to control the growth of Sobolev norms for (1.4), and Berti-Corsi-Procesi abstract Nash-Moser theorem [BCP15] to construct quasi-periodic solutions for (1.5).

Finally let us remark that in the last few years there has been an increase of interest in studying the dynamics of Schrödinger equations on rectangular rational or irrational tori. These papers can be roughly divided into two groups: the first one concerns Strichartz estimates and well posedness results [Bou07, CW10, GOW14, BD15, DGG17]. The second group analyzes phenomenon of growth of Sobolev norms [DG17, Den17, SW18], see also [PTV17] for 2 and 3 dimensional arbitrary compact manifolds. It would be interesting to see if the techniques developed in this paper could allow to extend these results for flat tori, as well as those in [MP18, GHHMP18] concerning stability and instability of finite gap solutions of NLS on the standard torus.

We state now more precisely our results.

1.1 Growth of Sobolev norms for linear, time dependent Schrödinger equations

Rescaling the spatial variables, (1.4) is equivalent to the following Schrödinger equation on the standard torus

\[
i \partial_t \psi = -\Delta_x \psi + V(t, x) \psi, \quad x \in \mathbb{T}^d,
\]

with the anisotropic Laplacian

\[
\Delta_x := \sum_{a,b=1}^d \partial_{x_a} [W^T W]_{a,b} \partial_{x_b} = \sum_{a,b,l=1}^d W_l^a W_l^b \partial_{x_a}^2 \partial_{x_b}^2,
\]

where \(W = (W^b_a)_{a,b=1,\ldots,d}\) is the matrix (1.7) associated to the dual lattice (with row index \(a\) and column index \(b\)). In the particular case of the rectangular torus (1.3), the anisotropic Laplacian
reduces to
\[ \Delta \tilde{\varphi} := \sum_{a=1}^{d} \nu_a^2 C_{x_a}, \quad \nu_a = L_a^{-1}, \quad \bar{\nu} := (\nu_a)_{a=1,\ldots,d}. \]

The eigenvalues of \( \Delta_{\mathcal{L}} \) are \( -\mu_j \) with \( \mu_j \) defined in (1.6). In (1.9), for simplicity of notation, we denoted again by \( V \) the potential in the new rescaled variables, so that \( V(t,x) \) is a function in \( C^\infty(\mathbb{R} \times T^d, \mathbb{R}) \). As phase space we consider Sobolev spaces of periodic functions
\[ \mathcal{H}^r = \left\{ \psi = \sum_{j \in \mathbb{Z}^d} \psi_j e^{ij \cdot x} \in L^2(T^d, \mathbb{C}) : \| \psi_j \|^2 := \sum_{j \in \mathbb{Z}^d} \langle j \rangle^{2r} |\psi_j|^2 < \infty \right\} \]
where \( \langle j \rangle := \sqrt{1 + |j|^2} \).

Our first result is an upper bound on the speed of growth of Sobolev norms of (1.9).

**Theorem 1.1 (Growth of Sobolev norms).** Consider the Schrödinger equation (1.9) with potential \( V \) in \( C^\infty(\mathbb{R} \times T^d, \mathbb{R}) \) satisfying
\[ \sup_{(t,x) \in \mathbb{R} \times T^d} \left| e^{it \Delta_{\mathcal{L}}} V(t,x) \right| \leq C_{t,\alpha}, \quad \forall \ell \in \mathbb{N}, \quad \forall \alpha \in \mathbb{N}^d. \] (1.10)
Then, for any \( r > 0 \), for any \( \varepsilon > 0 \), there exists a constant \( C_{r,\varepsilon} > 0 \) such that each solution \( \psi(t) \) of (1.9) with initial datum \( \psi_0 \in \mathcal{H}^r \) fulfills
\[ \| \psi(t) \|_r \leq C_{r,\varepsilon} \langle t \rangle^\varepsilon \| \psi_0 \|_r. \] (1.11)

The proof of Theorem 1.1 is based on a result of “clustering” for the eigenvalues of \( -\Delta_{\mathcal{L}} \) which was proved originally by Bourgain [Bou99b] (see also [Bou98, Bou05]) for the Laplacian on the standard torus \( T^d \), and that we extend here to arbitrary flat tori \( T_{\mathcal{L}}^d \), non just rectangular, see Theorem 2.1. It is interesting that such a result holds for any flat torus, the ultimate reason is that Lemma 2.7 holds for any lattice \( \mathcal{L} \).

Bourgain [Bou99b] used this decomposition result to prove a \( \langle t \rangle^\varepsilon \) upper bound for the growth in time of the Sobolev norms of the solutions of the linear Schrödinger equation (1.3) on \( T^d \), in presence of a smooth potential with arbitrary time dependence. We also mention that, if the potential is analytic and quasi-periodic in time, Bourgain [Bou99a] proved a logarithmic growth for the solutions of (1.3) in \( d = 1 \), and, in \( d = 2 \), under a smallness assumption on the potential. Using KAM techniques, Eliasson-Kuksin [EK99] proved that, for an analytic small potential and a large set of frequencies, the linear Schrödinger equation (1.4) on \( T^d \) can be conjugated to a block-diagonal dynamical system whose solutions have bounded Sobolev norms.

The proof of the result of Bourgain [Bou99b] was greatly simplified by Delort [Del10] who proposed an abstract framework which allows to prove a \( \langle t \rangle^\varepsilon \) bound also on other manifolds, for example Zoll ones; see also [FZ12] for logarithmic bounds in case of potentials on \( T^d \) with Gevrey regularity.

It is the abstract result of Delort [Del10], combined with the novel clustering of the eigenvalues of \( -\Delta_{\mathcal{L}} \) on \( T_{\mathcal{L}}^d \) in Theorem 2.1 that we employ to deduce Theorem 1.1.

Upper bounds of the form (1.11) have a long history. The first results are due to Nenciu [Nen97], who proved, in an abstract framework, a \( \langle t \rangle^\varepsilon \) upper bound for the expected value of the energy in case the system has increasing spectral gaps and bounded perturbation, and by Duclos, Lev and Štovíček [DL08] in case of decreasing spectral gaps.

Such results have been recently improved, in an abstract setup, by Maspero-Robert [MR17] for some unbounded perturbations, and by Bambusi-Grébert-Maspero-Robert [BGRMR17] for a
larger class of unbounded perturbations and without spectral gaps assumptions. Unfortunately these results do not apply to the Schrödinger equation (1.14) on \( \mathbb{T}^d \).

Finally we mention that potentials \( V(t, x) \) which provoke Sobolev norms explosion have been constructed by Bourgain [Bon99a] for a Klein-Gordon and Schrödinger equation on \( \mathbb{T} \), by De- lort [Del14] for the harmonic oscillator on \( \mathbb{R} \), by Bambusi-Grébert-Maspero-Robert [BGMR18] for the Harmonic oscillators on \( \mathbb{R}^d, d \geq 1 \), and by Maspero [Mas18] for systems enjoying an asymptotically constant spectral gap condition.

### 1.2 Quasi-periodic solutions for NLW and NLS equations

As before, we first rescale the spatial variables, recasting (1.15) to the nonlinear wave equation (NLW) and nonlinear Schrödinger equation (NLS) with anisotropic Laplacian on the torus \( \mathbb{T}^d \),

\[
\begin{align*}
    u_{tt} - \Delta_{x} u + mu &= \epsilon f(\omega t, x, u), \\
    iu_t - \Delta_{x} u + mu &= \epsilon f(\omega t, x, u), \quad x \in \mathbb{T}^d.
\end{align*}
\]  

(1.12)

Concerning regularity we assume that the nonlinearity \( f \in \mathcal{C}^6(\mathbb{T}^n \times \mathbb{T}^d \times \mathbb{R}; \mathbb{R}) \), resp. \( f \in \mathcal{C}^6(\mathbb{T}^n \times \mathbb{T}^d \times \mathbb{C}; \mathbb{C}) \) in the real sense (namely as a function of Re(\( u \)), Im(\( u \))), for some \( q \) large enough. We also require that

\[
    f(\omega t, x, u) = \hat{\omega} \hat{f}(\omega t, x, u) \in \mathbb{R}, \quad \forall u \in \mathbb{C},
\]

so that the NLS equation is Hamiltonian.

Concerning the frequency \( \omega \in \mathbb{R}^n \), we constrain it to a fixed direction, namely

\[
    \omega = \lambda \overline{\omega}, \quad \lambda \in \Lambda := [1/2, 3/2], \quad \| \overline{\omega} \|_1 := \sum_{p=1}^{n} |\overline{\omega}_p| \leq 1.
\]

(1.14)

The vector \( \overline{\omega} \in \mathbb{R}^n \) is assumed to be Diophantine, i.e. for some \( \tau_0 \gg n \),

\[
    |\overline{\omega} \cdot \ell| \geq \frac{2\tau_0}{|\ell|^{\tau_0}}, \quad \forall \ell \in \mathbb{Z}^n \setminus \{0\},
\]

(1.15)

where, here and below \( |\ell| := |\ell|_{\infty} = \max(|\ell_1|, \ldots, |\ell_n|) \).

The search for quasi-periodic solutions of (1.12) reduces to finding solutions \( u(\varphi, x) \), periodic in all the variables \( (\varphi, x) \in \mathbb{T}^n \times \mathbb{T}^d \), of the equations

\[
    (\omega \cdot \partial_\varphi)^2 u - \Delta_\varphi u + mu = \epsilon f(\varphi, x, u), \quad i\omega \cdot \partial_\varphi u - \Delta_\varphi u + mu = \epsilon f(\varphi, x, u),
\]

(1.16)

with \( u \) belonging to some Sobolev space \( H^s(\mathbb{T}^n \times \mathbb{T}^d) \) of real valued functions, in the case of NLW, respectively complex valued for NLS.

**Theorem 1.2** (Quasi-periodic solutions of NLW and NLS). Consider the NLW equation (1.12) or the NLS equation (1.12) - (1.13) and assume (1.14) - (1.15). Then there are \( s, q \in \mathbb{R} \) such that, for any \( f, \hat{f} \in \mathcal{C}^q \) and for all \( \epsilon \in [0, \epsilon_0] \) with \( \epsilon_0 > 0 \) small enough, there is a map

\[
    u_\epsilon \in C^1([1/2, 3/2], H^s(\mathbb{T}^n \times \mathbb{T}^d)), \quad \sup_{\lambda \in [1/2, 3/2]} \| u_\epsilon(\lambda) \|_{H^s(\mathbb{T}^n \times \mathbb{T}^d)} \to 0, \quad \text{as } \epsilon \to 0,
\]

and a Cantor-like set \( C_\epsilon \subset [1/2, 3/2] \), satisfying \( \text{meas}(C_\epsilon) \to 1 \) as \( \epsilon \to 0 \), such that, for any \( \lambda \in C_\epsilon \), \( u_\epsilon(\lambda) \) is a solution of (1.16), with \( \omega = \lambda \overline{\omega} \).
In order to prove Theorem 1.2 we verify that the assumptions of the abstract Theorems 2.16–2.18 of [BCP15] are met. For the reader convenience, we report such results in Appendix A.

The key property to prove is that the singular sites of the quasi-periodic wave and Schrödinger operators \( (\omega \cdot \partial_x)^2 - \Delta_x + m \) and \( \omega \cdot \partial_x - \Delta_x + m \), form clusters in space-time Fourier indices which are sufficiently separated. We prove properties of this kind for any flat torus \( T^d \) in sections 3.1 and 4.1.

Existence of quasi-periodic solutions for analytic nonlinear Schrödinger equations with a convolution potential was first proved by Bourgain [Bou98] on \( T^2 \) and then extended in [Bou05] to arbitrary dimension and for wave type pseudo-differential equations. The convolution potential is used as a parameter to fulfill suitable non-resonance conditions. The proof is based on a multiscale approach, originated by Anderson localization theory, to invert approximately the linearized operators arising at each step of a Newton iteration. This approach requires "minimal" non-resonance conditions, and separation properties for the clusters of singular sites. Subsequently existence of quasi-periodic solutions of NLS with convolution potential which are also linearly stable has been proved by Eliasson-Kuksin [EK10], who managed to reduce the linearized equations to a block-diagonal dynamical system, imposing the stronger second order Melnikov non-resonance conditions. For the completely resonant NLS, which has no external parameters, we refer to Wang [Wan16] and Procesi-Procesi [PP15, PP16] who also proved the linear stability of the torus; see also [EGK10] for the beam equation. Unfortunately no reducibility results for NLW on \( T^d \) are available.

Theorem 1.2 extends the results in [BB12, BCP15] valid for a square torus \( T^d \) to the case of an arbitrary flat torus \( T^d \). The technique for proving the separation properties of the singular sites in sections 3.1 and 4.4 could also be applied to extend the existence results of quasi-periodic solutions formulated as in [Bou05]. We find interesting that, with this multiscale approach, the irrationality properties of the lattice generators \( \mathcal{L} \) do not play any role. Such properties could be relevant for proving also reducibility results, i.e. for imposing second order Melnikov non-resonance conditions. We also mention that Theorem 1.2 improves, in the case of NLW, the results in [BB12, BCP15], because it requires \( \omega \) to satisfy only the standard Diophantine condition (1.15), and not additional quadratic Diophantine conditions. This is due to the more careful analysis in section 3.2 to verify separation properties of the singular sites.

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2 Growth of Sobolev norms

As we anticipated in the introduction, the key step for applying Delort theorem [Del10] is to prove the following result about clusterization properties of the eigenvalues \( \mu_j \) of \( -\Delta \) on \( T^d \), i.e. of \( -\Delta_x \) on \( T^d \).

**Theorem 2.1** (Clustering of eigenvalues of \( -\Delta_x \)). There exist constants \( \delta_0(d) > 0, C(\mathcal{L}, d) > 0, c(\mathcal{L}, d) > 0, c(d) > 1 \) such that, for any \( 0 < \delta < \delta_0(d) \), there exists a partition \( (\Omega_\alpha)_{\alpha \in A} \) of \( \mathbb{Z}^d \) satisfying:

\[
1. \forall \alpha \in A, \forall j_1, j_2 \in \Omega_\alpha, \text{ we have } |j_1 - j_2| + |\mu_{j_1} - \mu_{j_2}| \leq C(\mathcal{L}, d) (|j_1| + |j_2|)^{c(d)d}. \quad (2.1)
\]

Moreover each \( \Omega_\alpha \) is finite and, either \( \max_{j \in \Omega_\alpha} |j| \leq C(\mathcal{L}, d) \) or \( \max_{j \in \Omega_\alpha} |j| \leq 2 \min_{j \in \Omega_\alpha} |j| \).
(2) \( \forall \alpha, \beta \in A, \alpha \neq \beta, \forall j_1 \in \Omega_\alpha, \forall j_2 \in \Omega_\beta, \) we have
\[
|j_1 - j_2| + |\mu_{j_1} - \mu_{j_2}| > (|j_1| + |j_2|)^\delta .
\]

We postpone the proof of this result to section 2.1 and we now show that all the assumptions of Delort theorem [Del10] are met. We begin with some preliminary notation. For \( j \in \mathbb{Z}^d \), denote by \( \Pi_j \) the spectral projector
\[
\Pi_j u := \langle u, e_j \rangle_{L^2} e_j , \quad e_j := \frac{e^{ijx}}{(2\pi)^{d/2}} .
\]

**Definition 2.2.** For any \( \sigma \in \mathbb{R} \) we denote by \( A^\sigma \) the space of smooth in time families of continuous operators \( Q(t) \) from \( C^\infty(\mathbb{T}^d) \) to \( \mathcal{D}'(\mathbb{T}^d) \) such that, \( \forall k, N \in \mathbb{N} \), there is \( C_{k,N} > 0 \) such that
\[
\| \Pi_j \partial_t^k Q(t) \Pi_{j'} \|_{L^2(\mathbb{T}^2)} \leq C_{k,N} \frac{(1 + |j| + |j'|)^{\sigma}}{(1 + |j - j'|)^N} , \quad \forall j, j' \in \mathbb{Z}^d , \quad \forall t \in \mathbb{R} .
\]

We denote \( A^{-\infty} := \bigcap_{\sigma \in \mathbb{R}} A^\sigma \).

We remark that each \( Q \) in \( A^\sigma \) extends to a smooth family in \( t \) of bounded operators from \( H^s(\mathbb{T}^d) \) to \( H^{s-\sigma}(\mathbb{T}^d) \). Moreover, if \( Q \in A^\sigma \), then its adjoint \( Q^* \in A^\sigma \) (the adjoint is with respect to the standard \( L^2(\mathbb{T}^d) \) scalar product). Moreover, for any \( \sigma_1, \sigma_2 \in \mathbb{R} \) one has \( A^{\sigma_1} \circ A^{\sigma_2} \subseteq A^{\sigma_1 + \sigma_2} \) and, if \( \sigma_1 \leq \sigma_2 \), then \( A^{\sigma_1} \subseteq A^{\sigma_2} \). Finally, if the potential \( V \in C^\infty(\mathbb{R} \times \mathbb{T}^d, \mathbb{R}) \) fulfills (1.10), then, as a multiplication operator, \( V \in A^0 \).

We now state Delort theorem [Del10], specified to the case of the torus.

**Theorem 2.3** (Delort). Assume that for any \( \sigma \in \mathbb{R} \), there exist subspaces \( A^\sigma_D, A^\sigma_{ND} \) of \( A^\sigma \), invariant under \( Q \mapsto Q^* \), such that the following holds true:

(H1) For any \( Q \in A^\sigma \), there exist \( Q_D \in A^\sigma_D \) and \( Q_{ND} \in A^\sigma_{ND} \) such that \( Q = Q_D + Q_{ND} \).

(H2) There is \( \rho > 0 \) such that for any \( \sigma \in \mathbb{R} \), any \( W \in A^\rho_{ND} \), there exist \( X \in A^\rho_{ND} \) and \( R \in A^{-\infty} \) solving the homological equation
\[
[X, -\Delta_X] = W + R .
\]

Here \([A, B] := AB - BA\) is the commutator of the operators \( A, B \).

(H3) There is an element \( D \in A^2 \), independent of time and commuting with \( \Delta_X \), which defines an equivalent norm on \( H^r(\mathbb{T}^d) \), i.e. for some constants \( c, C > 0 \)
\[
c \| u \|_{H^r(\mathbb{T}^d)} \leq \| D^{r/2} u \|_{L^2(\mathbb{T}^d)} \leq C \| u \|_{H^r(\mathbb{T}^d)} , \quad \forall u \in H^r(\mathbb{T}^d) ,
\]

and, furthermore,
\[
[Q, D] \in A^{-\infty} , \quad \forall Q \in A^\sigma_D .
\]

Then, for all \( r, \epsilon > 0 \), there exists a constant \( C_{r, \epsilon} > 0 \) such that each solution \( \psi(t) \) of the linear Schrödinger equation (119) with initial datum \( \psi_0 \in H^r(\mathbb{T}^d) \) satisfies
\[
\| \psi(t) \|_r \leq C_{r, \epsilon} \| \psi_0 \|_r .
\]
Verification of the assumptions of Delort theorem. Consider the cluster decomposition \((\Omega_\alpha)_{\alpha \in A}\) provided by Theorem 2.1 and define the projector

\[ \Pi_\alpha := \sum_{j \in \Omega_\alpha} \Pi_j, \quad \forall \alpha \in A. \]

Following [Del10], we denote by \(A_\sigma^*\) the subspace of \(A^\sigma\) given by those operators \(Q \in A^\sigma\) such that \(\Pi_\alpha Q \Pi_\beta = 0\) for any \(\alpha, \beta \in A\) with \(\alpha \neq \beta\). Similarly, we denote by \(A_{ND}^\sigma\) the operators satisfying \(\Pi_\alpha Q \Pi_\beta = 0\) for any \(\alpha \in A\). Clearly such subspaces are invariant under \(Q \mapsto Q^\sigma\).

We verify now that (H1)–(H3) are met.

Verification of (H1). It is enough to write \(Q = \sum_\alpha \Pi_\alpha Q \Pi_\alpha + \sum_{\alpha \neq \beta} \Pi_\alpha Q \Pi_\beta = Q_D + Q_{ND}\).

Verification of (H2). For any \(Q \in A^\sigma\), denote by \(Q_{j,j'}^{\sigma}\) its matrix elements on the exponential basis \(e_j\) defined in (2.3), i.e. \(Q_{j,j'}^{\sigma} := \langle Q e_j, e_{j'} \rangle_{L^2}\). Thus, the homological equation (2.4) reads

\[ (\mu_{j'} - \mu_j) X_{j,j'}^{\sigma} = W_{j,j'}^{\sigma} + R_{j,j'}^{\sigma}, \quad j \in \Omega_\alpha, j' \in \Omega_\beta, \alpha \neq \beta, \]

and we define its solution

\[ X_{j,j'}^{\sigma} := \begin{cases} \frac{W_{j,j'}}{\mu_{j'} - \mu_j} & \text{if } |\mu_j - \mu_{j'}| \geq \frac{1}{4}(|j| + |j'|)^4, \\ 0 & \text{otherwise} \end{cases} \]

with

\[ R_{j,j'}^{\sigma} := \begin{cases} -\frac{W_{j,j'}}{\mu_{j'} - \mu_j} & \text{if } |\mu_j - \mu_{j'}| < \frac{1}{4}(|j| + |j'|)^4, \\ 0 & \text{otherwise} \end{cases} \]

Since \(W \in A_{ND}^\sigma\) we deduce that \(X \in A_{ND}^\sigma\). To verify that \(R \in A^{-\infty}\), recall that Theorem 2.1 (2) implies that \(|j - j'| + |\mu_j - \mu_{j'}| \geq \frac{1}{4}(|j| + |j'|)^4\), \(\forall j \in \Omega_\alpha, j' \in \Omega_\beta, \alpha \neq \beta\), and therefore \((R_{j,j'}^{\sigma})_{j,j' \in \mathbb{Z}^d}\) is supported on sites satisfying \(|j - j'| \geq \frac{1}{4}(|j| + |j'|)^4\). As a consequence, recalling Definition 2.2, the operator \(R\) is in \(A^m\) for any \(m \in \mathbb{R}\). In conclusion (H2) holds with \(\rho = \delta\).

Verification of (H3). Define the operator

\[ D := \sum_{\alpha \in A} M_\alpha^2 \Pi_\alpha, \quad M_\alpha := \max_j |j|, \]

(the maximum \(M_\alpha\) is attained at some \(j(\alpha)\) since \(\Omega_\alpha\) is finite). Clearly \(D\) commutes with \(\Delta_{\mathcal{L}}\) and also with any operator \(Q \in A_{\mathcal{L}}^\sigma\), so (2.3) is trivially satisfied. By Theorem 2.1 (1) each cluster \(\Omega_\alpha\) is dyadic, i.e.

\[ \forall \alpha \in A, \forall j \in \Omega_\alpha, \quad \text{either } |j| \leq C(\mathcal{L}, d) \quad \text{or} \quad \frac{M_\alpha}{2} \leq |j| \leq M_\alpha, \]

and therefore the operator \(D\) is in \(A^2\) and (2.5) holds.

2.1 Proof of Theorem 2.1

Denote by \(\| \cdot \|_{\mathcal{L}}^2 : \mathbb{R}^d \to \mathbb{R}\) the quadratic positive definite form

\[ \| y \|^2_{\mathcal{L}} := \| Wy \|^2, \quad \forall y \in \mathbb{R}^d, \quad (2.7) \]
Lemma 2.6. For all $j$, each vector $\Gamma$.

Proposition 2.5. The number $L$ is called the length of the chain.

The next proposition provides the key bound for the length of a $\Gamma$-chain. We now introduce the subspace $G$.

Definition 2.4 (Gamma-chain). Given $\Gamma \geq 2$, a sequence of distinct integer vectors $(\bar{\bar{\bar{\bar{j}}}})$ is the second estimate in (2.11).

The first bound in (2.11) directly follows by the definition of $\mu_j$.

Let $g = \dim G$; clearly $1 \leq g \leq d$ (note that $g \geq 1$ because the vectors $\bar{\bar{\bar{\bar{j}}}}$ are all distinct). We consider the orthogonal decomposition of $\mathbb{R}^d$ with respect to the scalar product $(\cdot, \cdot)_\mathbb{R}$.
and we denote by $P_G$ the orthogonal projector on the subspace $G$.

The next step is to derive by (2.11) a bound on the norm of $P_Gj_{q_0}$ for each integer vector $j_{q_0}$, $q_0 = 0, \ldots, L$, of the $\Gamma$-chain. We consider two cases.

Case 1. For any $q_0 \in \{0, \ldots, L\}$ it results that $\text{span}_R \langle j_q - j_{q_0} \rangle$, $|q - q_0| \leq \zeta$ is $G$ where $\zeta := [3(2d + 1)d]^{-1}$. We select a basis of $G$ from $j_q - j_{q_0}$ with $|q - q_0| \leq \zeta$, say

$$f_i := j_q - j_{q_0} \quad \text{with} \quad |q_i - q_0| \leq \zeta, \quad \forall 1 \leq i \leq g.$$  \hfill (2.14)

By the first bound in (2.11) and recalling (2.9) we obtain the estimate

$$|f_i| \leq |q_i - q_0| \Gamma \leq \zeta \Gamma, \quad \forall 1 \leq i \leq g.$$  \hfill (2.15)

To obtain a bound on the norm of $P_Gj_{q_0}$, we decompose $P_Gj_{q_0}$ on the basis $(f_i)_{1 \leq i \leq g}$,

$$P_Gj_{q_0} = \sum_{i=1}^g x_i f_i,$$  \hfill (2.16)

and we look for an upper bound for the coordinates $x := (x_i)_{1 \leq i \leq g}$. The $x$ are determined by solving the linear system $Ax = b$, where

$$A := ((f_i, f_j)_x)_{1 \leq i, j \leq g}$$  \hfill (2.17)

is the $g \times g$-matrix of the scalar products of the basis $(f_i)_{i=1, \ldots, g}$ of $G$ (Gram matrix) and

$$b := (b_i)_{i=1, \ldots, g} \in \mathbb{R}^g, \quad b_i := (P_Gj_{q_0}, f_i)_x = (j_{q_0}, f_i)_x.$$  \hfill (2.18)

By the second estimate in (2.11) and (2.13) the coefficients $b_i$ are estimated by

$$|b_i| \leq c(\mathcal{L}, d)|q_i - q_0|^2 \Gamma^2 \leq c(\mathcal{L}, d)(\zeta \Gamma)^2, \quad \forall l = 1, \ldots, g.$$  \hfill (2.19)

Since the vectors $(f_i)_{i=1, \ldots, g}$ are independent, the Gram matrix $A$ is invertible. We prove in Lemma 2.7 below that that its determinant is bounded away from zero, uniformly with respect to the basis $(f_i)_{i=1, \ldots, g}$ of integer vectors. We introduce some notation.

- For any $g \in \{1, \ldots, d\}$ we denote by $\mathfrak{C}_g(W)$ the $g-th$ compound matrix of the $d \times d$ matrix $W$, defined as the matrix whose entries are the determinants of all possible $g \times g$ minors of $W$, see e.g. [Gan59], Chap. 1 §4. Thus $\mathfrak{C}_g(W)$ is a $\binom{d}{g} \times \binom{d}{g}$ square matrix. The important result is that, for any $g$, the compound matrix of an invertible matrix $W$ is invertible as well (see e.g. [Gan59], Chap. 1 §4) and

$$\mathfrak{C}_g(W)^{-1} = \mathfrak{C}_g(W^{-1}).$$  \hfill (2.20)

- If $A$ is a $m \times n$ matrix, and $1 \leq g \leq m$, we denote by $A_{a_1, \ldots, a_g}$ the $g \times n$ matrix of rows $a_1, \ldots, a_g$ of $A$. If $1 \leq g \leq n$ then $A_{b_1, \ldots, b_g}$ is the $m \times g$ matrix of columns $b_1, \ldots, b_g$ of $A$. Finally if $1 \leq g \leq \min(m, n)$, we denote by $A_{a_1, \ldots, a_g}$ the $g \times g$ matrix of rows $a_1, \ldots, a_g$ and columns $b_1, \ldots, b_g$ of $A$.

**Lemma 2.7.** Let $A$ be the Gram matrix defined in (2.17). Then

(i) There exists $p \in \mathbb{Z}^d \setminus \{0\}$ such that

$$\det A = \|\mathfrak{C}_g(W)p\|^2$$  \hfill (2.21)

where $\mathfrak{C}_g(W)$ is the $g-th$ compound matrix of $W$ and $\|\cdot\|$ is the euclidean norm.
(ii) There exists a constant \( c(\mathcal{L}) > 0 \) such that, for any linearly independent integer vectors \( (f_i)_{i=1,...,g} \),
\[
\det A \geq c(\mathcal{L}) > 0.
\] (2.22)

\textbf{Proof.} Recalling (2.8) we write the Gram matrix \( A \) in (3.11) as
\[
A = (WF)^\top (WF), \quad F := (f_1 | \cdots | f_g).
\] (2.23)

Then applying twice Cauchy-Binet formula we obtain
\[
\det A = \sum_{1 \leq a_1 < \cdots < a_d \leq d} \left( \det([WF]_{a_1 \cdots a_d}) \right)^2
\]
\[
= \sum_{1 \leq a_1 < \cdots < a_d \leq d} \left( \sum_{1 \leq b_1 < \cdots < b_d \leq d} \det(W_{a_1 \cdots a_d}b_{1 \cdots b_d}) \right)^2
\] (2.24)

where \( p_{b_1 \cdots b_d} := \det(F_{b_1 \cdots b_d}) \) are integers because the matrix \( F \) has integer entries. The expression (2.24) is (2.21) with \( p := (p_{b_1 \cdots b_d}) \in \mathbb{Z}^d \). Since the Gram matrix \( A \) of the linearly independent vectors \( (f_i)_{i=1,...,g} \) is invertible, \( \det A \neq 0 \), and, by (2.21), the invertibility of \( \xi_g(W) \) implies that \( p \neq 0 \). Item (ii) follows by item (i) and the invertibility of \( \xi_g(W) \), see (2.20). \( \square \)

As \( (f_i, f_i)_{\mathcal{L}} \leq c(\mathcal{L}, d) ||f_i||_{f_i}^2 \leq c(\mathcal{L}, d)(L^\Gamma)^2 \), using (2.22), (2.15), (2.19), we estimate
\[
x = A^{-1}b \text{ by Cramer rule, obtaining}
\]
\[
|x_i| \leq c(\mathcal{L}, d)(L^\Gamma)^{2d}, \quad \forall 1 \leq i \leq g,
\] (2.25)
and, by (2.16), (2.25), (2.15), we deduce
\[
|P_G j_{q_0}| \leq C(\mathcal{L}, d)(L^\Gamma)^{2d+1}, \quad \forall q_0 \in \{0, \ldots, L\}.
\]

In particular, for all \( q_1, q_2 \in \{0, \ldots, L\} \), we obtain
\[
|j_{q_1} - j_{q_2}| = |P_G j_{q_1} - P_G j_{q_2}| \leq |P_G j_{q_1}| + |P_G j_{q_2}| \leq 2C(\mathcal{L}, d)(L^\Gamma)^{2d+1}.
\]

This estimate, and the fact that \( j_q, q = 0, \ldots, L, \) are distinct integers of \( \mathbb{Z}^d \), imply that their number does not exceed \( C'(\mathcal{L}, d)(L^\Gamma)^{(2d+1)d} \) and therefore
\[
L \leq C(\mathcal{L}, d)(L^\Gamma)^{(2d+1)d}.
\]

Choosing now \( \varsigma \) so small that \( \varsigma(2d+1)d \leq 1/2 \), we obtain (2.16) with \( C_1(d) = 2(2d+1)d \).

\textbf{Case 2.} \( 3q_0 \in \{0, \ldots, L\} \) for which
\[
G_1 := \text{span}_R \langle j_q - j_{q_0}, \quad |q - q_0| \leq L^\varsigma \rangle \subseteq G, \quad g_1 := \dim G_1 \leq g - 1,
\]
so all vectors \( j_q \) with \( |q - q_0| \leq L^\varsigma \) belong to an affine subspace of dimension \( 1 \leq g_1 \leq g - 1 \).

Consider then the subchain \( \{j_q: |q - q_0| \leq L^\varsigma \}, \) which has a length \( L_1 \sim L^\varsigma \). If for any index \( q_1 \) of this subchain it results that \( \text{span}_R \langle j_q - j_{q_1}, \quad |q - q_1| \leq L_1^\varsigma \rangle = G_1 \), then we can repeat the argument of Case 1 and obtain a bound on \( L_1 \) (and hence on \( L \)). Applying this procedure at most \( d \) times, we obtain a bound of the form (2.16), proving Proposition 4.5. \( \square \)

\footnote{\text{If } M \text{ is a } g \times d \text{ matrix, } N \text{ a } d \times g \text{ one, then}
\[
\det(MN) = \sum_{1 \leq i_1 < \cdots < i_g \leq d} \det(A^{i_1 \cdots i_g}) \det(B_{i_1 \cdots i_g}).
\]
2.2 Conclusion

We finally conclude the proof of Theorem 2.1. Fix $\delta_0(d) := \frac{1}{2C_1(d) + 2}$ with $C_1(d)$ given by Proposition 2.5, take an arbitrary $\delta \in (0, \delta_0(d))$ and introduce the equivalence relation on $\mathbb{Z}^d$ defined by

\[ \text{either } j = j', \]
\[ j \sim j' \iff \exists L \in \mathbb{N} \text{ and distinct integer vectors } (j_q)_{q=0,...,L} \subset \mathbb{Z}^d \text{ with } j_0 = j, \]
\[ j_L = j' \text{ and } |\Phi(j_{q+1}) - \Phi(j_q)| \leq (|j_q| + |j_{q+1}|)\delta, \quad \forall q = 0, \ldots, L - 1. \tag{2.26} \]

This equivalence relation provides a partition of $\mathbb{Z}^d$ in classes of equivalence $(\Omega_\alpha)_{\alpha \in A}$, with $\mathbb{Z}^d = \cup_{\alpha \in A} \Omega_\alpha$. We claim that such clusters $\Omega_\alpha$ fulfill (2.1)–(2.2). We split the proof in several steps.

Each cluster $\Omega_\alpha$ is bounded. By contradiction suppose that $\Omega_\alpha$ is unbounded, i.e. it contains integer vectors of arbitrary large modulus. For any $j, j' \in \Omega_\alpha$, with $|j'|$ very large (specified later), there exists a sequence $(j_q)_{q=0,...,M}$ satisfying (2.26). Without loss of generality we assume that $|j_q| \leq |j'|$, for all $q$ (otherwise we just replace $j'$ with the $j_q$ having maximum modulus and consider the subchain connecting $j$ and $j_q$). Since

\[ |\Phi(j_{q+1}) - \Phi(j_q)| \leq (|j_q| + |j_{q+1}|)\delta \leq (2|j'|)\delta, \quad \forall 0 \leq q \leq M - 1, \]

it follows that $(j_q)_{q=0,...,M}$ is a $(2|j'|)\delta$-chain according to Definition 2.4, and, therefore, Proposition 2.5 implies that its length $M$ is bounded by $C_2(\mathcal{L}, d)(2|j'|)^{C_1(d)\delta}$. Then

\[ |j' - j| \leq \sum_{q=0}^{M-1} |\Phi(j_{q+1}) - \Phi(j_q)| \leq M (2|j'|)\delta \leq C_2(\mathcal{L}, d)(2|j'|)^{(C_1(d)+1)\delta}, \]

which gives

\[ |j'| \leq |j| + C_2(\mathcal{L}, d)(2|j'|)^{(C_1(d)+1)\delta}. \tag{2.27} \]

As $\delta(C_1(d) + 1) < \delta_0(C_1(d) + 1) \leq 1/2$, inequality (2.27) bounds uniformly $|j'|$ in terms of $|j|$ and $C_2(\mathcal{L}, d)$, giving a contradiction since $j'$ can be chosen to have arbitrary large modulus.

Dyadicity of each cluster $\Omega_\alpha$. Denote by

\[ m_\alpha := \min_{j \in \Omega_\alpha} |j|, \quad M_\alpha := \max_{j \in \Omega_\alpha} |j|. \]

Let $j_\alpha$ be the index which realizes the maximum $M_\alpha = |j_\alpha|$ (it exists since each cluster is bounded). For any $j, j' \in \Omega_\alpha$, let $(j_q)_{q=0,...,L}$ be a sequence of distinct integer vectors in $\Omega_\alpha$ with $j_0 = j, j_L = j'$ satisfying (2.26). Then $(j_q)_{q=0,...,L}$ is a $(2M_\alpha)\delta$-chain according to Definition 2.4 and, by Proposition 2.10

\[ |\Phi(j) - \Phi(j')| \leq L(2M_\alpha)^\delta \leq C_2(\mathcal{L}, d)(2M_\alpha)^{(C_1(d)+1)\delta}. \tag{2.28} \]

Recalling (2.9), we deduce, in particular, that $\operatorname{diam}(\Omega_\alpha) \leq C_2(\mathcal{L}, d)(2M_\alpha)^{(C_1(d)+1)\delta}$. Moreover, since $\delta \leq \delta_0$ with $\delta_0(C_1(d) + 1) = \frac{1}{2}$, we derive that, if $M_\alpha \geq 8(C_2(\mathcal{L}, d))^2$ the we have that

\[ m_\alpha \geq M_\alpha - C_2(\mathcal{L}, d)(2M_\alpha)^{(C_1(d)+1)\delta} \geq \frac{M_\alpha}{2}. \]

Thus, either $\max_{j \in \Omega_\alpha} |j| \leq C(\mathcal{L}, d)$ or $\max_{j \in \Omega_\alpha} |j| \leq 2 \min_{j \in \Omega_\alpha} |j|$. 

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\[ \text{either } j = j', \]
\[ j \sim j' \iff \exists L \in \mathbb{N} \text{ and distinct integer vectors } (j_q)_{q=0,...,L} \subset \mathbb{Z}^d \text{ with } j_0 = j, \]
\[ j_L = j' \text{ and } |\Phi(j_{q+1}) - \Phi(j_q)| \leq (|j_q| + |j_{q+1}|)\delta, \quad \forall q = 0, \ldots, L - 1. \tag{2.26} \]
Separation properties. Let $j_1, j_2 \in \Omega_\alpha$. Consider first the case $M_\alpha \geq C(\mathcal{L}, d)$. Then

$$M_\alpha \leq 2m_\alpha \leq |j_1| + |j_2| \leq 2M_\alpha, \quad \forall j_1, j_2 \in \Omega_\alpha,$$

which together with \[2.28\] imply

$$|j_1 - j_2| + |\mu_{j_1} - \mu_{j_2}| \leq 2|\Phi(j_1) - \Phi(j_2)| \leq C(\mathcal{L}, d) (|j_1| + |j_2|)^{(C_1(d)+1)\delta}.$$

If $M_\alpha \leq C(\mathcal{L}, d)$ and $|j_1| + |j_2| \geq 1$, then

$$|j_1 - j_2| + |\mu_{j_1} - \mu_{j_2}| \leq 2|\Phi(j_1) - \Phi(j_2)| \leq C'(\mathcal{L}, d) \leq C'(\mathcal{L}, d) (|j_1| + |j_2|)^{(C_1(d)+1)\delta}.$$

If $|j_1| + |j_2| < 1$, then the integers $j_1 = j_2 = 0$ and \[2.1\] holds as well. The proof of Theorem \[2.1\] is complete. Concerning item (2), note that if $j_1 \in \Omega_\alpha$, $j_2 \in \Omega_\beta$ and $\alpha \neq \beta$, then $j_1, j_2$ are not in the same equivalence class and therefore

$$|j_1 - j_2| + |\mu_{j_1} - \mu_{j_2}| \geq |\Phi(j_1) - \Phi(j_2)| > (|j_1| + |j_2|)^{\delta}.$$

The proof of Theorem \[2.1\] is complete.

### 3 Quasi-periodic solutions of wave equation

As we already mentioned, we construct quasi-periodic solutions of the nonlinear wave equation in \([1,2]\) by applying Berti-Corsi-Procesi abstract Nash-Moser theorem in \([BCP15]\), which we recall in Appendix \[A\].

First define the nonlinear map

$$F(\epsilon, \lambda, \cdot): H^{s+2}(\mathbb{T}^n \times \mathbb{T}^d, \mathbb{R}) \to H^s(\mathbb{T}^n \times \mathbb{T}^d, \mathbb{R})$$

$$u \mapsto D(\lambda)u - \epsilon f(\varphi, x, u), \quad (3.1)$$

where $D(\lambda): H^{s+2}(\mathbb{T}^n \times \mathbb{T}^d, \mathbb{R}) \to H^s(\mathbb{T}^n \times \mathbb{T}^d, \mathbb{R})$ is the differential operator

$$D(\lambda) := (\hat{\lambda} \varphi \cdot \partial_x)^2 - \Delta_x + m. \quad (3.2)$$

The set $\mathfrak{A}$ in \[A.1\] is then $\mathfrak{A} = \mathbb{Z}^n \times \mathbb{Z}^d \times \{1\}$, that is $\mathfrak{A} = \{1\}$, and the scale of Hilbert spaces \[A.2\] are the Sobolev spaces $H^s(\mathbb{T}^d \times \mathbb{T}^n, \mathbb{C})$ written in Fourier variables.

As in the previous application, the main difficulty is to verify a property of separation of chains of singular sites of the operator $D(\lambda)$. To state it we recall some definitions from Appendix \[A\] (see Definitions \[A.2, A.3\]).

Given $\Gamma \geq 2$, a sequence of distinct integer vectors $(\ell_q, j_q)_{q=0, \ldots, L} \subset \mathbb{Z}^n \times \mathbb{Z}^d$ is a $\Gamma$-chain if

$$\max\{|\ell_{q+1} - \ell_q|, |j_{q+1} - j_q|\} \leq \Gamma, \quad \forall 0 \leq q \leq L - 1;$$

the number $L$ is called the length of the chain (see Definition \[A.2\]).

The operator $D(\lambda)$ in \[3.2\] is represented, in the basis of exponentials $e^{i(\ell \cdot x + j \cdot \varphi)}$, by the infinite dimensional diagonal matrix

$$\text{diag}_{(\ell, j) \in \mathbb{Z}^n \times \mathbb{Z}^d} (D_{\ell, j}(\lambda)), \quad D_{\ell, j}(\lambda) := - (\hat{\lambda} \varphi \cdot \ell)^2 + \mu_j + m,$$

where $\mu_j$ are the eigenvalues of $-\Delta_x$ defined in \[1.9\]. For $\theta \in \mathbb{R}$, we also define

$$D(\lambda, \theta) := \text{diag}_{(\ell, j) \in \mathbb{Z}^n \times \mathbb{Z}^d} (D_{\ell, j}(\lambda, \theta)), \quad D_{\ell, j}(\lambda, \theta) := - (\hat{\lambda} \varphi \cdot \ell + \theta)^2 + \mu_j + m. \quad (3.3)$$

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A site \((t, j) \in \mathbb{Z}^n \times \mathbb{Z}^d\) is singular for \(D(\lambda, \theta)\) if \(|D_{\ell,j}(\lambda, \theta)| < 1\) (see Definition A.3).

For any \(\Sigma \subseteq \mathbb{Z}^n \times \mathbb{Z}^d\) and \(\tilde{j} \in \mathbb{Z}^d\), we denote by \(\Sigma^\tilde{j}\) the section of \(\Sigma\) at fixed \(\tilde{j}\), namely

\[
\Sigma^\tilde{j} := \{(t, \tilde{j}) \in \Sigma\}.
\]

Given \(K > 1\), denote by \(\Sigma_K\) any subset of singular sites of \(D(\lambda, \theta)\) such that the cardinality of the section \(\Sigma^\tilde{j}\) satisfies \(2|\Sigma_K| \leq K\), for any \(\tilde{j} \in \Sigma\) (see Definition A.4).

The key result is a bound on the length of \(\Gamma\)-chains of singular sites of \(D(\lambda, \theta)\).

**Proposition 3.1** (Separation of singular sites for NLW). There exist a constant \(C(\mathcal{L}, d, n, \tau_0) > 0\) and, for any \(N_0 \geq 2\), a set \(\hat{\Lambda} = \hat{\Lambda}(N_0)\) such that for all \(\lambda \in \hat{\Lambda}\), \(\theta \in \mathbb{R}\) and for all \(K > 1\), \(\Gamma \geq 2\) with \(\Gamma K \geq N_0\), any \(\Gamma\)-chain of singular sites of \(D(\lambda, \theta)\) in \(\Sigma_K\) has length \(L \leq (K\Gamma)^{C(\mathcal{L}, d, n, \tau_0)}\).

We postpone the proof to section 5.1 and we first conclude the proof of Theorem 1.2 for NLW.

**Verification of the assumptions of Berti-Corsi-Procesi theorem.** First, (A.4) is trivially satisfied with \(\sigma = 2\). Moreover, provided \(f \in C^3\) for \(q\) large enough and \(\epsilon_0 > (d + n)/2\), the same estimates (11)–(12) hold true.

We verify now Hypothesis 1–3 concerning the operator \(D(\lambda) + \epsilon(\tilde{\partial}_u f)(\varphi, x, u(\varphi, x))\) obtained linearizing (3.1) at a point \(u(\varphi, x) \in H^1'(\mathcal{T}^d \times \mathcal{T}^d, \mathbb{R})\). This operator has the form (A.5) with the multiplication operator \(T(u) := -(\tilde{\partial}_u f)(\varphi, x, u(\varphi, x))\).

Verification of Hypothesis 1. The diagonal matrix \(D(\lambda) = \text{diag}(D_{\ell,j}(\lambda))\) representing (3.2) fulfills the covariance property (A.6) choosing \(D_j(y) := -y^2 + \mu_j + m\).

The multiplication operator \(T(u)\) is represented, in the exponential basis \(e^{i(t\varphi + j \cdot x)}\), by a Töplitz matrix and (A.8)–(A.9) hold with \(\epsilon\).

Verification of Hypothesis 2. Recalling (3.3) one checks that

\[
\{\theta \in \mathbb{R} : |D_{\ell,j}(\lambda, \theta)| \leq N^{-\tau_1}\} \subset I_1 \cup I_2 ,
\]

with intervals \(I_q\) such that \(\text{meas}(I_q) \leq N^{-\tau_1}/\sqrt{m}\).

Therefore Hypothesis 2 holds with \(n = 2([1/\sqrt{m}] + 1)\).

Verification of Hypothesis 3. It is the content of Proposition 3.1.

The measure estimate (A.13) follows exactly as in [BB12, BCP15], and we omit it. Applying Theorem A.3 we prove Theorem 1.2 for the NLW.

**3.1 Proof of Proposition 3.1**

Consider the quadratic form \(Q_{x'} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) defined by

\[
Q_{x'}(x, y) := -x^2 + \|y\|^2 \quad \text{where} \quad \|y\|^2 := \|Wy\|^2
\]

is introduced in (2.4) and the associated symmetric bilinear form \(\phi_{x'} : (\mathbb{R} \times \mathbb{R}^d)^2 \to \mathbb{R}\),

\[
\phi_{x'}((x, y), (x', y')) := -xx' + \langle y, y' \rangle_{x'}, \quad \forall (x, y), (x', y') \in \mathbb{R} \times \mathbb{R}^d,
\]

where \(\langle y, y' \rangle_{x'} := \langle Wy, Wy' \rangle_{\mathbb{R}^d}\) is defined in (2.8).

First we notice that (as in the proof of [BB12, Lemma 4.2]) it is enough to bound the length of a \(\Gamma\)-chain \((\ell_q, j_q)_{q=0,\ldots,L}\) of singular sites of \(D(\lambda, 0)\). Indeed, consider a \(\Gamma\)-chain of singular sites \((\ell_q, j_q)_{q=0,\ldots,L}\) for \(D(\lambda, \theta)\), i.e.,

\[
|\omega \cdot \ell_q + \theta|^2 - \mu_{j_q} - m | < 1 , \quad \forall q = 0, \ldots, L , \quad \omega = \lambda \gamma.
\]

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Suppose first that \( \theta = \omega \cdot \overline{t} \) for some \( \overline{t} \in \mathbb{Z}^n \). Then the translated \( \Gamma \)-chain \((\ell_q + \overline{t}, j_q)_{q=0,\ldots,L}\) is formed by singular sites for \( D(\lambda, 0) \), namely
\[
|\omega \cdot (\ell_q + \overline{t})|^2 - \mu_{j_q} - m| < 1, \quad \forall \ell = 0, \ldots, L.
\]

For any \( \theta \in \mathbb{R} \), we consider an approximating sequence \( \omega \cdot \overline{t}_i \to \theta, \overline{t}_i \in \mathbb{Z}^n \). A \( \Gamma \)-chain of singular sites for \( D(\lambda, \theta) \) (see (3.6)), is, for \( i \) large enough, also a \( \Gamma \)-chain of singular sites for \( D(\lambda, \omega \cdot \overline{t}_i) \).

Then we bound its length arguing as in the above case.

Let \((\ell_q, j_q)_{q=0,\ldots,L}\) be a \( \Gamma \)-chain of singular sites of \( D(\lambda, 0) \). Setting
\[
x_q := \omega \cdot \ell_q = \lambda \overline{w} \cdot \ell_q, \quad \forall q = 0, \ldots, L,
\]
by the definition of singular sites and (3.4) we have
\[
|Q_{\mathcal{G}}(x_q, j_q) + m| < 1, \quad \forall q = 0, \ldots, L.
\]

**Lemma 3.2.** For all \( q_0, q \in \{0, \ldots, L\} \) we have
\[
\left| \phi_{\mathcal{G}} \left( (x_{q_0}, j_{q_0}), (x_q - x_{q_0}, j_q - j_{q_0}) \right) \right| \leq C(L, d, n)|q - q_0|^2 \Gamma^2.
\]

**Proof.** By bilinearity
\[
Q_{\mathcal{G}}(x_q, j_q) = Q_{\mathcal{G}}(x_{q_0}, j_{q_0}) + 2\phi_{\mathcal{G}} \left( (x_{q_0}, j_{q_0}), (x_q - x_{q_0}, j_q - j_{q_0}) \right) + Q_{\mathcal{G}}(x_q - x_{q_0}, j_q - j_{q_0})
\]

and then (3.8) follows combining (3.7) and the estimate
\[
|Q_{\mathcal{G}}(x_q - x_{q_0}, j_q - j_{q_0})| \leq |\omega \cdot (\ell_q - \ell_{q_0})|^2 + \|j_q - j_{q_0}\|^2 \leq c(L, d, n)\Gamma^2|q - q_0|^2
\]
which follows by the definition of \( \Gamma \)-chain. \( \square \)

Let us introduce the following subspace of \( \mathbb{R}^{d+1} \):
\[
G := \text{span}_\mathbb{R} \langle (x_q - x_{q'}, j_q - j_{q'}) , \ 0 \leq q, q' \leq L \rangle
\]
\[
= \text{span}_\mathbb{R} \langle (x_q - x_{q_0}, j_q - j_{q_0}) , \ q = 0, \ldots, L \rangle, \quad \forall q_0 = 0, \ldots, L.
\]

Let \( g := \text{dim} G \). We have \( 1 \leq g \leq d + 1 \). We consider two cases.

**Case 1.** For any \( q_0 \in \{0, \ldots, L\} \) it results that \( \text{span}_\mathbb{R} \langle (x_q - x_{q_0}, j_q - j_{q_0}) , \ |q - q_0| \leq L \rangle = G \) for some \( \varsigma := \varsigma(L, d, n, \tau_0) > 0 \) small enough, fixed later. We select a basis of \( G \) from \((x_q - x_{q_0}, j_q - j_{q_0})\) with \(|q - q_0| \leq L\), say
\[
f_i := (x_q - x_{q_0}, j_q - j_{q_0}) = (\omega \cdot (\ell_q - \ell_{q_0}), j_q - j_{q_0}), \quad \forall 1 \leq i \leq g.
\]

Then, since \((\ell_q, j_q)_{q=0,\ldots,L}\) is a \( \Gamma \)-chain,
\[
|f_i| \leq \varsigma |\ell_q - \ell_{q_0}| + |j_q - j_{q_0}| \leq \varsigma |q - q_0| \Gamma \leq C(d, n)L^\varsigma \Gamma, \quad \forall 1 \leq i \leq g.
\]

In order to derive from (3.8) a bound for \((x_q, j_q)\) we need a non degeneracy property for the restriction \( (\phi_{\mathcal{G}})|G \) of \( \phi_{\mathcal{G}} \) to the subspace \( G \). Then we consider the \( g \times g \) symmetric matrix
\[
A_{\mathcal{G}, \Lambda} := A_{\mathcal{G}, \Lambda} := (A^g_{i,j})_{i,j=1}^g, \quad A^g_{i,j} := \phi_{\mathcal{G}}(f_i, f_j),
\]
which represents the restriction of the bilinear form \( \phi_{\mathcal{G}} \) defined in (3.5) to the subspace \( G \). The next key lemma, which is the technical part of our argument, shows that, provided \( \lambda \) is chosen in a set \( \Lambda \) of large measure, the matrix \( A_{\mathcal{G}, \Lambda} \) is invertible and the modulus of its determinant is bounded away from zero.
Lemma 3.3. Let $A_{\mathcal{L}}$ be the matrix defined in $(3.11)$. Then

(i) \[ \det A_{\mathcal{L}} = \|c_g(W)p\|^2 - \lambda^2 \|c_{g-1}(W)(\mathcal{R} \otimes m)\|^2 \] (3.12)

where $c_g(W)$ is the $g$-th compound matrix of $W$, $\|\|$ the euclidean norm, $p \in \mathbb{Z}^d$ and $\mathcal{R} \otimes m := (\mathcal{R} \cdot m_\alpha)_\alpha \in \mathbb{R}^{(s_n+1)n}$, $m_\alpha \in \mathbb{Z}^n$, $\alpha = (a_1, \ldots, a_{g-1})$ with $1 \leq a_1 < \ldots < a_{g-1} \leq d$. Each vector $m_\alpha$ satisfies

\[ |m_\alpha| \leq C(d)(\Gamma \lambda)^g. \] (3.13)

(ii) \[ \det A_{\mathcal{L}} \text{ is not identically zero as function of } \lambda^2. \]

(iii) For any $N_0$ sufficiently large, there exists a set $\overline{\Lambda} := \Lambda(N_0) \subset \Lambda$ with $\text{meas}(\Lambda \setminus \overline{\Lambda}) \leq O(N_0^{-1})$

such that for any $\lambda \in \overline{\Lambda}$ one has, for some $c(d) > 0$ and $\tau_2 := \tau_2(d, n, \tau_0)$,

\[ |\det A_{\mathcal{L}},\lambda| \geq \frac{c(d)}{N_0(\Gamma \lambda)^{\tau_2}}. \] (3.14)

We postpone the proof of Lemma 3.3 to section 3.2, first concluding the proof of Proposition 3.1.

Take $\lambda \in \overline{\Lambda}$, so that, by the previous lemma, the matrix $A_{\mathcal{L}}$ is invertible. Therefore the symmetric bilinear form $\phi_\mathcal{L}|G$ is nondegenerate and it induces the splitting

\[ \mathbb{R}^{d+1} = G \oplus G^{1,\mathcal{L}} \quad \text{where} \quad G^{1,\mathcal{L}} := \{ z \in \mathbb{R}^{d+1} : \phi_\mathcal{L}(z, f) = 0 \ \forall f \in G \} . \]

We denote by $P_G : \mathbb{R}^{d+1} \to G$ the corresponding projector on $G$.

In order to obtain bounds for the projection of each $P_G(x_{q_0}, j_{q_0})$, we decompose it on the basis $(f_i)_{1 \leq i \leq g}$,

\[ P_G(x_{q_0}, j_{q_0}) = \sum_{i=1}^{q} z_i f_i, \] (3.15)

and we look for bounds of the coordinates $z = (z_i)_{1 \leq i \leq g}$. The $z$ are determined by solving the linear system $A_{\mathcal{L}} z = b$, where $A_{\mathcal{L}}$ is the matrix in $(3.11)$ and $b := (b_i)_{1 \leq i \leq g} \in \mathbb{R}^g$, $b_i := \phi_\mathcal{L}(P_G(x_{q_0}, j_{q_0}), f_i) = \phi_\mathcal{L}((x_{q_0}, j_{q_0}), f_i)$.

Then the Cramer rule, the estimates $(3.3)$, $(3.4)$ and $|A_i| \leq c(\mathcal{L}, d, n)|f_i|^2 \leq c(\mathcal{L}, d, n)(\Gamma \lambda)^2$, by $(3.10)$, give

\[ |z_i| \leq N_0 C(\mathcal{L}, d, n)(\Gamma \lambda)^{2g+w_2}, \quad \forall 1 \leq i \leq g. \] (3.16)

From $(3.15)$, $(3.16)$, $(3.10)$, we deduce

\[ |P_G(x_{q_0}, j_{q_0})| \leq N_0 C(\mathcal{L}, d, n)(\Gamma \lambda)^{2g+w_2+1}, \quad \forall q_0 \in \{0, \ldots, L\}. \]

Therefore we get that, for all $q_1, q_2 \in \{0, \ldots, L\}$,

\[ |(x_{q_1}, j_{q_1}) - (x_{q_2}, j_{q_2})| = |P_G(x_{q_1}, j_{q_1}) - P_G(x_{q_2}, j_{q_2})| \leq N_0 C(\mathcal{L}, d, n)(\Gamma \lambda)^{2g+w_2+1}. \]

In particular, for all $q_1, q_2 \in \{0, \ldots, L\}$ we have also $|j_{q_1} - j_{q_2}| \leq N_0 C(\mathcal{L}, d, n)(L \Gamma)^{2d+w_2+1}$, and so the sequence $\{j_q\}_{q=0}^L$ is contained in a ball of diameter $N_0 C(\mathcal{L}, d, n)(L \Gamma)^{2d+w_2+1}$. Since they are integer vectors, their number (counted without multiplicity), does not exceed $N_0^2 C(\mathcal{L}, d, n)(L \Gamma)^{(2d+w_2+1)d}$. By the assumptions of Proposition 3.1 the $\Gamma$-chain $(\ell_q, j_q)_{q=0}^L$
is in $\Sigma_K$, thus, for any $q_0 \in \{0, \ldots, L\}$, the cardinality of the set of singular sites of the chain with fixed $j_{q_0}$ is bounded by

$$Z\{(\ell_q, j_q)_{q=0,\ldots,L} : j_q = j_{q_0}\} \leq K,$$

and we deduce that

$$L \leq N_0^n C(\mathcal{L}, d, n)(\Gamma^\ell)^{(2d + \tau_2 + 1)d} K \leq (\Gamma K)^d C(\mathcal{L}, d, n)(\Gamma^\ell)^{(2d + \tau_2 + 1)d}.$$

By choosing $\epsilon$ such that $\epsilon(2d + \tau_2 + 1)d \leq 1/2$, we deduce that $L \leq C(\mathcal{L}, d, n)(\Gamma K)^d C(\mathcal{L}, d, n)(\Gamma^\ell)^{(2d + \tau_2 + 1)d}$.

Case 2. $\exists q_0 \in \{0, \ldots, L\}$ for which

$$G_1 := \text{span}_k \langle (x_q - x_{q_0}, j_q - j_{q_0}) \rangle, \quad |q - q_0| \leq L^\epsilon \subseteq G, \quad g_1 := \dim G_1 \leq g - 1,$$

so all vectors $(x_q - x_{q_0}, j_q - j_{q_0})$ belong to an affine subspace of dimension $1 \leq g_1 \leq g - 1$. Then we repeat the argument of Case 1 on the subchain $\{(\ell_q, j_q) : |q - q_0| \leq L^\epsilon\}$, obtaining a bound on $L^\epsilon$, thus for $L$. Applying this procedure at most $d + 1$ times, we obtain a bound on $L$ of the claimed form $L \leq (\Gamma K)^d C(\mathcal{L}, d, n, \tau_0)$.

3.2 Proof of Lemma 3.3

Write the bilinear form $\phi_\mathcal{L}$ in (3.5) as the sum of two symmetric bilinear forms:

$$\phi_\mathcal{L} = -\phi_1 + \phi_2,$$

$$\phi_1((x, y), (x', y')) := xx', \quad \phi_2((x, y), (x', y')) := (y, y')_\mathcal{L},$$

where $(y, y')_\mathcal{L} = \langle Wy, Wy' \rangle_{\mathcal{L}^d}$ and $W = V^{-1}$ is defined in (1.7).

Correspondingly we decompose the symmetric matrix $A_{\mathcal{L}, \lambda}$ in (3.11), as $A_{\mathcal{L}, \lambda} = -S + R$ where the symmetric matrices $S = (S^i_\ell)_{1 \leq i, \ell \leq g}$ and $R = (R^i_\ell)_{1 \leq i, \ell \leq g}$ are given by

$$S^i_\ell := \phi_1(f_{\ell'}, f_i) = (\omega \cdot l_\ell)(\omega \cdot l_i), \quad l_i := \ell_{q_i} - \ell_{q_0} \in \mathbb{Z}^n,$$

$$R^i_\ell := \phi_2(f_{\ell'}, f_i) = (k_{\ell'}, k_i)_\mathcal{L}, \quad k_i := j_{q_i} - j_{q_0} \in \mathbb{Z}^d.$$ (3.19)

Since $(\ell_q, j_q)$, with indices $|q - q_0| \leq L^\epsilon$, is a $\Gamma$-chain, we have

$$\max\{|l_i|, |k_i|\} \leq |q_i - q_0| \Gamma \leq L^\epsilon \Gamma, \quad \forall 1 \leq i \leq g.$$ (3.20)

We write $S = (S_1|\cdots|S_g)$ and $R = (R_1|\cdots|R_g)$ where $S_i, R_i \in \mathbb{R}^g$, $i = 1, \ldots, g$ denote the columns of $S$ and $R$. The matrix $S$ has rank 1 since all its columns $S_i \in \mathbb{R}^g$ are colinear:

$$S_i = (\omega \cdot l_i)1, \quad \forall i = 1, \ldots, g, \quad \text{where} \quad 1 := \begin{pmatrix} \omega \cdot l_1 \\ \vdots \\ \omega \cdot l_g \end{pmatrix}.$$ (3.21)

By the multilinearity of the determinant along each column, we develop $\det A_{\mathcal{L}, \lambda}$ as

$$\det A_{\mathcal{L}, \lambda} = \det(-S + R) = \det(R - \det(S_1|R_2|\cdots|R_g) - \cdots - \det(R_1|\cdots|R_{g-1}|S_g))$$ (3.22)

where we used that the determinant of a matrix with two colinear columns $S_i, S_i'$ is null.

Now remark that $\det R$ is the same that we computed in Lemma 2.7 (substituting $k_i \sim f_i$ in (2.17)), so we obtain

$$\det R = \|\mathcal{E}_g(W)p\|^2$$ (3.23)
Lemma 3.4. One has
\[ \det(S_1 | R_2 | \cdots | R_g) + \cdots + \det(R_1 | \cdots | R_{g-1} | S_g) \]

for some vector \( p \in \mathbb{Z}_g \) (notice that \( p \) could be zero since the vectors \( k_1, \ldots, k_g \) may not be independent).

In the next lemma we compute \( \det(S_1 | R_2 | \cdots | R_g) + \cdots + \det(R_1 | \cdots | R_{g-1} | S_g) \).

**Lemma 3.4.** One has
\[ \det(S_1 | R_2 | \cdots | R_g) + \cdots + \det(R_1 | \cdots | R_{g-1} | S_g) \]

\[ = \lambda^2 \sum_{1 \leq c_1 < \cdots < c_{g-1} \leq d} \left( \sum_{1 \leq a_1 < \cdots < a_{g-1} \leq d} \det(W_{c_1, \ldots, c_{g-1}}^{a_1, \ldots, a_{g-1}}) \right)^2 \]  \( (3.24) \)

with integer coefficients \( m_{a_1, \ldots, a_{g-1}} \in \mathbb{Z}^n \) satisfying
\[ |m_{a_1, \ldots, a_{g-1}}| \leq C(d)(L^2 \Gamma)^g \]  \( (3.25) \)

**Proof.** Developing the matrix \( R \) in (3.19) (recall (2.8) and (1.7)) we get
\[ R = \sum_{a, b, c=1}^d W_c^a W_c^b R_s^{(ab)}, \quad R^{(ab)} := \begin{pmatrix} k_1^{(a)} \\ \vdots \\ k_g^{(a)} \end{pmatrix}, \quad a, b = 1, \ldots, d, \quad s = 1, \ldots, g \] 

where \( k_i^{(a)} \) is the \( a \)-th component of the vector \( k_i \). Note that the \( g \times g \) symmetric matrix \( R^{(ab)} \) has colinear columns, with the \( s \) column given by
\[ R_s^{(ab)} = k_s^{(b)} k^{(a)}, \quad k^{(a)} := \begin{pmatrix} k_1^{(a)} \\ \vdots \\ k_g^{(a)} \end{pmatrix}, \quad a, b = 1, \ldots, d, \quad s = 1, \ldots, g \]  \( (3.26) \)

Writing the columns of \( R \) as \( R_s = \sum_{a, b, c=1}^d W_c^a W_c^b R_s^{(ab)} \) and substituting in the first line of (3.24), we get, by the multilinearity of the determinant,
\[ \det(S_1 | R_2 | \cdots | R_g) + \cdots + \det(R_1 | \cdots | R_{g-1} | S_g) \]

\[ = \sum_{m_1, \ldots, m_g=1}^d \prod_{2 \leq h \leq g} W_{oh}^{m_h} W_{oh}^{m_h} \cdot \det(S_1 | R_2^{(m_2 m_3)} | \cdots | R_g^{(m_3 m_g)} ) \]

\[ + \cdots + \sum_{m_1, \ldots, m_{g-1}=1}^d \prod_{1 \leq h \leq g-1} W_{oh}^{m_h} W_{oh}^{m_h} \cdot \det(R_1^{(m_1 n_1)} | \cdots | R_g^{(n_1 n_2 n_3 \cdots n_{g-1})} | S_g) \]

\[ = \sum_{m_1, \ldots, m_{g-1}=1}^d \prod_{1 \leq h \leq g-1} W_{oh}^{m_h} W_{oh}^{m_h} \cdot (\omega \cdot l_1) k_2^{(n_1)} \cdots k_g^{(n_{g-1})} \det(l | k^{(m_1)} | \cdots | k^{(m_{g-1})}) \]

\[ + \cdots + \sum_{m_1, \ldots, m_{g-1}=1}^d \prod_{1 \leq h \leq g-1} W_{oh}^{m_h} W_{oh}^{m_h} \cdot (\omega \cdot l_g) k_1^{(n_1)} \cdots k_g^{(n_{g-1})} \det(l | k^{(m_1)} | \cdots | k^{(m_{g-1})}). \]
Note that, in the last equality, we have renamed the indices in the first sum and reordered the columns of the matrix inside the determinant to have the vector \( \mathbf{1} \) always in the first position. Now we notice that the only non zero determinants are those where the indices \( m_i \) are all different, otherwise at least two columns are colinear. Thus the sums above are on distinct indices, namely in each sum \( m_i \neq m_{i'} \) for \( i \neq i' \). Therefore the \( m_i \)'s in each monomial are just a permutation of some \( 1 \leq a_1 < \cdots < a_{g-1} \leq d \). Denote by \( \sigma_\mathcal{S} \) the permutation of these \( g-1 \) indices defined by \( \sigma_\mathcal{S}(a_i) = m_i \quad \forall i \), and by \( \text{sign} \sigma_\mathcal{S} \) its signature. By the alternating property of the determinant, we have

\[
\det \left( |k^{(a_1)}| \cdots |k^{(a_{g-1})}| \right) = \det \left( |l^{(a_1)}| \cdots |l^{(a_{g-1})}| \right) \text{sign} \sigma_\mathcal{S}.
\]

Therefore \( \det(S_1|R_2| \cdots |R_g) + \cdots + \det(R_1| \cdots |R_{g-1}|S_g) \) equals

\[
\sum_{1 \leq a_1 < \cdots < a_{g-1} \leq d} \det \left( |k^{(a_1)}| \cdots |k^{(a_{g-1})}| \right) \prod_{1 \leq h \leq g-1} W_{a_h}^{\sigma_\mathcal{S}(a_h)} \left( \omega \cdot l_i k^{(n_1)}_2 \cdots k^{(n_{g-1})}_g + \cdots + (-1)^{g-1} (\omega \cdot l_i) k^{(n_1)}_1 \cdots k^{(n_{g-1})}_g \right)
\]

where \( S_{g-1} \) denotes the group of all possible permutations of the indices \( a_1 < \cdots < a_{g-1} \). Now one has

\[
\sum_{\sigma \in S_{g-1}} \text{sign} \sigma_\mathcal{S} W_{a_1}^{\sigma_\mathcal{S}(a_1)} \cdots W_{a_{g-1}}^{\sigma_\mathcal{S}(a_{g-1})} = \det(W_{a_1}^{a_1} \cdots W_{a_{g-1}}^{a_{g-1}})
\]

and this determinant is not zero only if the \( o_i \) are permutation of some indices \( 1 \leq c_1 < \cdots < c_{g-1} \leq d \). As above, we denote by \( \sigma_\mathcal{C} \) the permutation such that \( \sigma_\mathcal{C}(e_i) = a_i \), for all \( i \). We obtain

\[
3.27 = \sum_{1 \leq e_1 < \cdots < e_{g-1} \leq d} \det \left( |k^{(e_1)}| \cdots |k^{(e_{g-1})}| \right) \det(W_{e_1}^{e_1} \cdots W_{e_{g-1}}^{e_{g-1}})
\]

\[
\times \sum_{n_1, \ldots, n_{g-1} = 1}^d \left[ (\omega \cdot l_i) k^{(n_1)}_2 \cdots k^{(n_{g-1})}_g + \cdots + (-1)^{g-1} (\omega \cdot l_i) k^{(n_1)}_1 \cdots k^{(n_{g-1})}_g \right]
\]

\[
\times \sum_{\sigma \in S_{g-1}} \text{sign} \sigma_\mathcal{C} W_{c_1}^{n_1} \cdots W_{c_{g-1}}^{n_{g-1}}.
\]

Again, the last line equals

\[
\sum_{\sigma \in S_{g-1}} \text{sign} \sigma_\mathcal{C} W_{c_1}^{n_1} \cdots W_{c_{g-1}}^{n_{g-1}} = \det(W_{c_1}^{n_1} \cdots W_{c_{g-1}}^{n_{g-1}})
\]

and the determinant is not zero only if the \( n_i \) are permutation \( \sigma_\mathcal{C} \) of some indices \( 1 \leq b_1 < \cdots < b_{g-1} \leq d \). We obtain thus

\[
(3.28) = \sum_{1 \leq b_1 < \cdots < b_{g-1} \leq d} \det \left( |k^{(a_1)}| \cdots |k^{(a_{g-1})}| \right) \det(W_{b_1}^{b_1} \cdots W_{b_{g-1}}^{b_{g-1}}) \det(W_{c_1}^{b_1} \cdots W_{c_{g-1}}^{b_{g-1}})
\]

\[
\times \sum_{\sigma \in S_{g-1}} \left[ (\omega \cdot l_i) k^{(b_1)}_2 \cdots k^{(b_{g-1})}_g + \cdots + (-1)^{g-1} (\omega \cdot l_i) k^{(b_1)}_1 \cdots k^{(b_{g-1})}_g \right].
\]
Now, the expression in the last line of (3.29) is nothing but the determinant of the matrix

\[
\begin{pmatrix}
\omega \cdot l_1 & k_1^{(b_1)} & \ldots & k_1^{(b_{\gamma-1})} \\
\omega \cdot l_2 & k_2^{(b_1)} & \ldots & k_2^{(b_{\gamma-1})} \\
\vdots & \vdots & \ddots & \vdots \\
\omega \cdot l_\gamma & k_\gamma^{(b_1)} & \ldots & k_\gamma^{(b_{\gamma-1})}
\end{pmatrix}
\]

(3.30)

as one sees easily by a Laplace expansion of the determinant along the first column and recalling the Leibnitz formula for the determinant. In conclusion we have proved that

\[
\text{Formula (3.23), Lemma 3.4 and (3.22) imply (3.12).}
\]

\[
\det (1|k^{(a_1)}| \ldots |k^{(a_{\gamma-1})}) = \omega \cdot m_{a_1 \ldots a_{\gamma-1}} = \lambda \omega \cdot m_{a_1 \ldots a_{\gamma-1}}
\]

where \( m_{a_1 \ldots a_{\gamma-1}} \) is a vector in \( \mathbb{Z}^n \) satisfying

\[
|m_{a_1 \ldots a_{\gamma-1}}| \leq C(d) \max_{1 \leq i \leq g} |l_i| \max_{1 \leq i \leq g} |k_i|^{\gamma-1} \leq C(d)(L^\gamma)^g
\]

by (3.20).

\[
\text{Formula (3.23), Lemma 3.4 and (3.22) imply (3.12).}
\]

Lemma 3.5. Consider

\[
P_{p,m}(\lambda^2) := \|\mathcal{E}_g(W)p\|^2 - \lambda^2 \|\mathcal{E}_{g-1}(W)(\overline{\omega} \otimes m)\|^2
\]

where \( p \in \mathbb{Z}^d \) and \( m := (m_2) \in \mathbb{Z}^{(g-1)n} \), \( \overline{a} = (a_1, \ldots , a_{\gamma-1}) \) with \( 1 \leq a_1 \leq \ldots \leq a_d \leq g \), and \( \overline{\omega} \otimes m := (\overline{\omega} \cdot m_2)_{\overline{a}} \in \mathbb{R}^{(g-1)n} \). Let \( |m| := \max |m_i| \). Assume that \( \overline{\omega} \in \mathbb{R}^n \) is Diophantine, according to (1.15). Then there exists \( \Lambda(\mathcal{L}) > 0 \) such that for \( \tau \geq \tau_1(d,n,\gamma_0) \) large enough, for any \( \gamma \in [0,\frac{1}{2}\epsilon(\mathcal{L})] \), the set

\[
\tilde{\Lambda} := \left\{ \lambda \in \Lambda : |P_{p,m}(\lambda^2)| \geq \frac{\gamma}{1 + |m|}, \forall (p,m) \neq (0,0) \right\}
\]

has large measure, more precisely

\[
\text{meas}(\Lambda \setminus \tilde{\Lambda}) \leq C(\mathcal{L}, \overline{\omega}, d, \gamma_0)\gamma.
\]

Proof. For \( p \in \mathbb{Z}^d \) and \( m \in \mathbb{Z}^{(g-1)n} \), let

\[
\eta_p := \|\mathcal{E}_g(W)p\|^2, \quad \zeta_m(\overline{\omega}) := \|\mathcal{E}_{g-1}(W)(\overline{\omega} \otimes m)\|^2, \quad P_{p,m}(\xi) = \eta_p - \xi \zeta_m(\overline{\omega}), \quad \xi := \lambda^2.
\]
We have that \( \text{meas}(\Lambda/\tilde{\Lambda}) \leq C \sum_{(p,m) \neq (0,0)} \text{meas}(R_{p,m}) \), where
\[
R_{p,m} := \left\{ \xi = \lambda^2 \in \left[ \frac{1}{4}, \frac{9}{4} \right] : |P_{p,m}(\xi)| < \frac{\gamma}{1 + |m|^\tau} \right\}.
\]
We distinguish two cases.

Case 1: \( p \neq 0 \). Therefore
\[
\eta_p = \|e_p(W)p\|^2 \geq \frac{1}{\|e_p(W)^{-1}\|^2} \|p\|^2 \geq c(L)p^2 \geq c(L).
\]
If \( R_{p,m} \neq \emptyset \) then, since \( |\xi| \leq 9/4 \) and \( \gamma \in \left[ 0, \frac{1}{4}c(L) \right] \), we deduce that
\[
|\zeta_m(\omega)| \geq \frac{c(L)}{3}, \quad |p|^2 \leq 3|\zeta_m(\omega)| \leq c(L,\omega,d)|m|^2, \quad \text{meas}(R_{p,m}) \leq \frac{2\gamma}{1 + |m|^\tau} |\zeta_m(\omega)|.
\]
Hence, for \( \tau := \tau(d,n) \) sufficiently large,
\[
\sum_{p \neq 0, m} \text{meas}(R_{p,m}) = \sum_{0 < |p| < C|m|, m} \text{meas}(R_{p,m}) \leq C(L,\omega,d,n) \gamma.
\]

Case 2: \( p = 0 \). In this case \( m \neq 0 \) and, by the invertibility of the compound matrix \( e_{g-1}(W) \) and the diophantine condition \( \|A\| \leq 1.15 \), we get
\[
|\zeta_m(\omega)| \geq \epsilon_{g-1}(L) \|\omega \otimes m\|^2 \geq \epsilon_{g-1}(L) |\omega \otimes m|^2 \geq \epsilon_{g-1}(L) \frac{\gamma^2}{|m|^{2\tau_0}}.
\]
We deduce that, for \( \tau \geq 2\tau_0 \),
\[
R_{0,m} \subseteq \left( 0, \frac{\gamma}{1 + |m|^\tau}, \frac{|m|^{2\tau_0}}{\gamma_0 c(L)} \right) \subseteq \left( 0, \frac{\gamma}{\gamma_0 c(L)} \right).
\]
This inclusion and (3.32) prove (3.31).

Proof of Lemma 3.3 (i)–(iii). Item (i) follows combining (3.22) with (3.23) and Lemma 3.4
To prove item (ii) it is sufficient to notice that \( \text{det} A_{\omega,\lambda} \) is not zero when evaluated at \( \lambda \).
Indeed the matrix \( A_{\omega,\lambda} \) is the Gram matrix of the vectors \( f_i \) with respect to the scalar product \( \phi_1 + \phi_2 \). Being \( f_1, \ldots, f_g \) linearly independent, \( \text{det} A_{\omega,\lambda} > 0 \). In particular the integer vectors \( (p,m) \neq (0,0) \). Finally item (iii) follows by Lemma 3.5 and the bound (3.13) for \( |m_2| \), choosing \( N_0 = \gamma^{-1} \) and \( \tau_2 := g\tau_1(d,n,n_0) \).

\[\Box\]

4 Quasi-periodic solutions of Schrödinger equation

It is convenient to consider the nonlinear Schrödinger equation in (4.12) coupled with its complex conjugate equation, so we look for zeros of the nonlinear operator
\[
F(\epsilon, \lambda, \cdot) : H^{s+2}(\mathbb{T}^m, \mathbb{C}) \times H^{s+2}(\mathbb{T}^m, \mathbb{C}) \to H^s(\mathbb{T}^m, \mathbb{C}) \times H^s(\mathbb{T}^m, \mathbb{C}),
\]
\[
\left( \begin{array}{c} u^+ \\ u^- \end{array} \right) \mapsto D(\lambda) \left( \begin{array}{c} u^+ \\ u^- \end{array} \right) - \epsilon f(u^+, u^-),
\]
where \( u^\pm \) are functions of the periodic variables \( (\varphi, x) \in \mathbb{T}^n \times \mathbb{T}^d \), \( D(\lambda) \) is the differential operator
\[
D(\lambda) := \left( \begin{array}{cc} 1\lambda \nabla \cdot \hat{\varphi} - \Delta \varphi + m & 0 \\ 0 & -i\lambda \nabla \cdot \hat{\varphi} - \Delta \varphi + m \end{array} \right).
\]
and $f(u^+, u^-)$ is the nonlinearity

$$f(u^+, u^-) := \left( \begin{array}{c} \mathfrak{f}^+(\varphi, x, u^+, u^-) \\ \mathfrak{f}^-(\varphi, x, u^+, u^-) \end{array} \right);$$

here $\mathfrak{f}^\pm(u, v)$ are two extensions (in the real sense) of $\mathfrak{f}(\varphi, x, u)$ so that $\mathfrak{f}^+(u, \overline{u}) = \overline{\mathfrak{f}^-(u, \overline{u})} = \mathfrak{f}(u)$ and $\partial_u \mathfrak{f}^+(u, \overline{u}) = \partial_u \mathfrak{f}^-(u, \overline{u}) = 0$ and $\partial_v \mathfrak{f}^+(u, \overline{u}) = \partial_v \mathfrak{f}^-(u, \overline{u}) = 0$ and $\partial_v \mathfrak{f}^+(u, \overline{u}) = \partial_v \mathfrak{f}^-(u, \overline{u}) = 0$ and $\partial_v \mathfrak{f}^+(u, \overline{u}) = \partial_v \mathfrak{f}^-(u, \overline{u}) = 0$.

We look for zeros of $F$ in the subspace

$$\mathcal{U} := \left\{ u = (u^+, u^-) \in H^s(\mathbb{T}^{n+d}, \mathbb{C}) \times H^s(\mathbb{T}^{n+d}, \mathbb{C}) : u^+ = u^- \right\}.$$

The set $\mathcal{R}$ of (4.1) is then $\mathcal{R} = \mathbb{Z}^n \times \mathbb{Z}^d \times \{1, -1\}$ and the scale of Hilbert spaces (4.2) is $H^s(\mathbb{T}^{n+d}, \mathbb{C}) \times H^s(\mathbb{T}^{n+d}, \mathbb{C})$ written in Fourier variables.

As in the application to NLS, the main difficulty is to verify a bound for the length of chains of singular sites of $D(\lambda)$. We begin by specializing Definitions (4.2) to the case of NLS.

First, given $\Gamma \geq 2$, a sequence of distinct integer vectors $(\ell_q, \beta_q)_{q=0,...,L} \subset \mathbb{Z}^n \times \mathbb{Z}^d \times \{1, -1\}$ is a $\Gamma$-chain if $\max\{|\ell_q - \ell_q|, |\beta_{q+1} - \beta_q|, |\alpha_{q+1} - \alpha_q|\} \leq \Gamma$, $\forall 0 \leq q \leq L-1$. The number $L$ is called the length of the chain (see Definition (4.2)).

The operator $D(\lambda)$ in the basis of exponentials

$$e^{it\varphi} \mathbf{e}_{\ell, a}(x), \quad \mathbf{e}_{\ell, a}(x) := \begin{cases} (e^{ijx}, 0) & \text{if } a = +1 \\ (0, e^{ijx}) & \text{if } a = -1, \end{cases}$$

is represented by the infinite dimensional matrix

$$D(\lambda) := \text{diag}_{\ell, j, a}(D_{\ell, j, a}(\lambda)), \quad D_{\ell, j, a}(\lambda) := -a \lambda \varphi \cdot \ell + \mu_j + m, \quad (\ell, j, a) \in \mathbb{Z}^n \times \mathbb{Z}^d \times \{1, -1\},$$

where $\mu_j$ are defined in (1.6). For $\theta \in \mathbb{R}$ define also

$$D(\lambda, \theta) := \text{diag}(D_{\ell, j, a}(\lambda, \theta)), \quad D_{\ell, j, a}(\lambda, \theta) := -a \lambda \varphi \cdot \ell + \theta + \mu_j + m.$$

Now recall that a site $(\ell, j, a) \in \mathbb{Z}^n \times \mathbb{Z}^d \times \{1, -1\}$ is called singular if $|D_{\ell, j, a}(\lambda, \theta)| < 1$ (see Definition (4.3)).

For any $\Sigma \subseteq \mathbb{Z}^n \times \mathbb{Z}^d \times \{1, -1\}$ and $\gamma \in \mathbb{Z}^d$, we denote by $\Sigma^\gamma$ the section of $\Sigma$ at fixed $\gamma$, namely $\Sigma^\gamma := \{(\ell, j, a) \in \Sigma\}$. Given $K > 1$, denote by $\Sigma_K$ any subset of singular sites of $D(\lambda, \theta)$ such that the cardinality of the section $\Sigma^\gamma$ satisfies $|\Sigma^\gamma| \leq K$, for any $\gamma \in \Sigma$ (see Definition (4.4)).

The key result is the following bound on the length of $\Gamma$-chains of singular sites.

**Proposition 4.1** (Separation of singular sites for NLS). There exists a constant $C(\mathcal{Z}, d, n) > 0$ such that for all $\lambda \in \Lambda \subset \mathbb{R}$ and for all $K > 1$, $\Gamma \geq 2$, any $\Gamma$-chain of singular sites of $D(\lambda, \theta)$ in $\Sigma_K$ has length $L \leq (K \Gamma)^{C(\mathcal{Z}, d, n)}$.

We postpone the proof to the next section [4.4] and we first prove Theorem (4.2) for NLS.

**Verification of the assumptions of Berti-Corsi-Procesi theorem.** First, (4.4) is trivially satisfied with $\sigma = 2$. Moreover, provided $f \in C^3$ for $q$ large enough and $s_0 > (d + n)/2$, the tame estimates (11)–(12) hold true.

We verify now Hypothesis 1–3 concerning the operator $D(\lambda) + c T(u)$,

$$T(u) = \left( \begin{array}{cc} p(\varphi, x) q(\varphi, x) \\ q(\varphi, x) p(\varphi, x) \end{array} \right), \quad p(\varphi, x) = -\partial_u f^+(\varphi, x, u^+(\varphi, x), u^-(\varphi, x)), \quad q(\varphi, x) = -\partial_u f^-(\varphi, x, u^+(\varphi, x), u^-(\varphi, x)).$$

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obtained linearizing (4.1) at a point \((u^+, u^-) \in \mathcal{U}\). By the Hamiltonian structure (4.13), the conditions on \(\ell \pm\) and the request \((u^+, u^-) \in \mathcal{U}\), the function \(p(\varphi, x)\) is real and \(q(\varphi, x)\) is complex valued.

**Verification of Hypothesis 1.** The covariance property \([A.6]\) holds with \(\mathcal{D}_{\ell,j}(y) = -ay + \mu_j + m\). The multiplication operator \(T(u)\), is represented, in the exponential basis (4.2), by the Töplitz matrix
\[
T^{(\ell', j', \sigma)}_{(\ell, j, a)} = p(j - j', \ell - \ell') \quad \text{if} \quad \sigma = \pm 1, \quad T^{(\ell', j', -1)}_{(\ell, j, 1)} = q(j - j', \ell - \ell'), \quad T^{(\ell', j', -1)}_{(\ell, j, -1)} = \overline{q(j - j', \ell - \ell')},
\]
where \(p(j, \ell), q(j, \ell)\) are the Fourier coefficients of the functions \(p(\varphi, x), q(\varphi, x)\). Then \([A.7]\) holds and \([A.8]-(A.9)\) hold with \(\sigma_0 = 0\).

**Verification of Hypothesis 2.** By a direct computation Hypothesis 2 is met with \(n = 2\).

**Verification of Hypothesis 3.** It is the content of Proposition 4.1.

The verification of the measure estimates \([A.13]\) follows exactly as in [BB13], and we omit. Applying Theorem \([A.5]\) we prove Theorem 1.2 for NLS.

### 4.1 Proof of Proposition 4.1

Let \((\ell_q, j_q, a_q)_{q=0,\ldots, L}\) be a \(\Gamma\)-chain of singular sites for \(D(\lambda, \theta)\); in particular
\[
\max_{q=0,\ldots, L-1} \{|\ell_{q+1} - \ell_q|, |j_{q+1} - j_q|\} \leq \Gamma. \tag{4.3}
\]
As in Section 2.1 we introduce the quadratic form and the associated scalar product
\[
\|y\|_2^2 := \|Wy\|^2, \quad \langle y, y' \rangle_{\mathcal{X}} = \langle Wy, Wy' \rangle_{\mathbb{R}^d}, \quad y, y' \in \mathbb{R}^d.
\]
We have the following lemma.

**Lemma 4.2.** For all \(q_0, q \in \{0, \ldots, L\}\) we have
\[
|\langle j_{q_0}, j_q - j_{q_0} \rangle_{\mathcal{X}}| \leq C(\mathcal{X}, d, n)\|q - q_0\|_2^2 \Gamma^2. \tag{4.4}
\]

**Proof.** By the definition of singular sites, for all \(q \in \{0, \ldots, L\},\)
\[
\begin{cases}
|\mu_{j_q} - \omega \cdot \ell_q + m - \theta| < 1 & \text{if } a_q = +1, \\
|\mu_{j_q} + \omega \cdot \ell_q + m + \theta| < 1 & \text{if } a_q = -1,
\end{cases}
\]
which leads to one of the following \(\theta\)-independent inequalities
\[
|\pm \omega \cdot (\ell_{q+1} \pm \ell_q) + \mu_{j_{q+1}} \pm \mu_{j_q}| \leq 2(|m| + 1).
\]
By (4.13) we deduce
\[
|\mu_{j_{q+1}} \pm \mu_{j_q}| \leq 2(|m| + 1) + n|\omega| \leq C_1 \Gamma
\]
for some \(C_1 := C_1(m, n).\) Since \(|\mu_{j_{q+1}} - \mu_{j_q}| \leq |\mu_{j_{q+1}} + \mu_{j_q}|,\) in any case we obtain the bound
\[
|\mu_{j_{q+1}} - \mu_{j_q}| \leq C_1 \Gamma, \quad \forall q \in \{0, \ldots, L\}.
\]
Therefore, for any \(q, q_0 \in \{0, \ldots, L\}\) we get
\[
\|j_{q_0}\|_2^2 - \|j_q\|_2^2 = |\mu_{j_{q_0}} - \mu_{j_q}| \leq C_1 |q - q_0| \Gamma.
\]
Since \(\|j_q\|_2^2 = \|j_{q_0}\|_2^2 + 2\langle j_{q_0}, j_q - j_{q_0} \rangle_{\mathcal{X}} + \|j_q - j_{q_0}\|_2^2,\) we deduce
\[
|\langle j_{q_0}, j_q - j_{q_0} \rangle_{\mathcal{X}}| \leq \|j_{q_0}\|_2^2 - \|j_q\|_2^2 + \|j_q - j_{q_0}\|_2^2 \leq C(\mathcal{X}, d, n)\|q - q_0\|_2^2 \Gamma \|q - q_0\|_2^2
\]
which is (4.4).
At this point the proof closely follows that in Section 2.3 and we shall be short. Consider the subspace
\[ G := \text{span}_\mathbb{R} \langle j_q - j_{q'} \rangle_{0 \leq q, q' \leq L} = \text{span}_\mathbb{R} \langle j_q - j_{q_0} \rangle, \quad q = 0, \ldots, L, \quad \forall q_0 = 0, \ldots, L, \]
with dimension \( g := \dim G, \ 1 \leq g \leq d \). Decompose \( \mathbb{R}^d = G \oplus G^\perp \) as in \([231]\), and let \( P_G \) denote the orthogonal projector on \( G \). As in the proof of Proposition 2.3 in order to bound the norm of each \( P_G j_{q_0} \), we consider two cases.

Case 1. For any \( q_0 \in \{0, \ldots, L\} \), it results that \( \text{span}_\mathbb{R} \langle j_q - j_{q_0} \rangle \leq L \rangle = G \) with \( \zeta := [3(d + 2)d]^{-1} \). Then we select a basis \( f_1, \ldots, f_q \) of \( G \) extracted from \( j_q - j_{q_0}, \ |q - q_0| \leq L \). Proceeding as in the proof of Proposition 2.3 and using estimate \([4.4]\) in place of \([2.11]\), one proves the bound
\[ 2 \{ j_q : 0 \leq q \leq L \} \leq C(\mathcal{L}, d, n)(L^\Gamma)^{(d+2)d}. \]
By the assumptions of Proposition 2.3 the \( \Gamma \)-chain belongs to \( \Sigma_K \), thus the cardinality of the set of singular sites in the \( \Gamma \)-chain with fixed \( j_{q_0} \), is bounded by \( K \), namely
\[ 2 \{ (\ell_q, j_q, a_q)_{q=0, \ldots, L} : \ j_q = j_{q_0} \} \leq K, \]
and therefore we get that
\[ L \leq C(\mathcal{L}, d, n)(L^\Gamma)^{(d+2)d} K. \]
Since \( \zeta(d + 2)d \leq 1/2 \) we finally deduce that \( L \leq (K\Gamma)^{C(\mathcal{L}, d, n)} \).

Case 2. \( \exists q_0 \in \{0, \ldots, L\} \) for which \( q_1 := \dim \text{span}_\mathbb{R} \langle j_q - j_{q_0} \rangle \leq L \rangle = g - 1 \). Then one argues as in Case 2 of Proposition 2.3, obtaining a bound of the form \( L \leq (K\Gamma)^{C(\mathcal{L}, d, n)} \) as claimed, completing the proof of Proposition 4.3.

A Berti-Corsi-Procesi abstract Nash-Moser theorem

We state here Berti-Corsi-Procesi abstract theorem \([130]\) specified to the case when the index of the space-Fourier component runs all over \( \mathbb{Z}^d \) (which is the situation in case the spatial variable \( x \) belongs to \( \mathbb{T}^d \)). Consider a scale of Hilbert sequence spaces defined in the following way. Define first the index set
\[ \mathfrak{R} := \mathbb{Z}_n \times \mathbb{Z}^d \times \mathfrak{A} \ni (\ell, j, a) = k \quad \text{(A.1)} \]
where the set \( \mathfrak{A} = \{1\} \) (for NLW) or \( \mathfrak{A} = \{1, -1\} \) (for NLS). If \( \mathfrak{A} = \{1\} \) we simply write \( k = (\ell, j) \).

For \( k = (\ell, j, a), k' = (\ell', j', a') \in \mathfrak{R} \) we set
\[ \text{dist}(k, k') := \begin{cases} 1 & \quad \text{if } \ell = \ell', j = j', a = a' \vspace{0.2cm} \\ \max\{||\ell| - |\ell'||, |j - j'||\} & \quad \text{otherwise} \end{cases} \]

here \( |\ell| := \max\{||\ell_1||, \cdots, ||\ell_n||\} \), \( |j| := \max\{||j_1||, \cdots, ||j_d||\} \). For any \( s \geq 0 \), define the Sobolev space
\[ H^s := H^s(\mathfrak{R}) := \{ u = (u_k)_{k \in \mathfrak{R}}, \ u_k \in C : ||u||_s^2 := \sum_{k \in \mathfrak{R}} \langle w_k \rangle^{2s} |u_k|^2 < \infty \} \quad \text{(A.2)} \]
where the weights \( \langle w_k \rangle := \max\{1, |\ell|, |j|\} \).

A bounded linear operator \( M : H^s \rightarrow H^s \) is represented by an infinite dimensional matrix \( (M^k_{k'})_{k, k' \in \mathfrak{R}} \). We define now a norm on such operators.
Definition A.1 (s-decay norm). We say that a linear operator $M$ has finite $s$-decay norm if
\[ |M|^2 := \sum_{(\ell,j) \in \mathbb{Z}^d} \max(1,|\ell|,|j|)^{2s} \sup_{\substack{t_1-t_2=\ell \atop j_1-j_2=\ell}} \|M^{(t_2,j_2)}\|_0 < \infty \]
where $M^{(t_2,j_2)} := \left\{ M^{(t_2,j_2,a_2)} \right\}_{a_2 \in \mathbb{A}}$ and $\|\cdot\|_0$ is the operator norm.

Consider a nonlinear operator $F(\epsilon, \lambda, \cdot) : H^{s+\sigma} \to H^s$ of the form
\[ F(\epsilon, \lambda, u) = D(\lambda)u + \epsilon f(u) \tag{A.3} \]
where $\epsilon > 0$ is a small parameter, $\lambda \in \Lambda \subset [1/2, 3/2]$ and $D(\lambda)$ is a linear diagonal operator $D(\lambda) : H^{s+\sigma} \to H^s$ fulfilling
\[ \|D(\lambda)h\|_s, \|D(\lambda)h\|_s \leq C(s) \|h\|_{s+\sigma}. \tag{A.4} \]
The nonlinearity $f$ is assumed to be at least of class $C^2(B_{s_0}^1, H_{s_0})$ for some $s_0 > (d+n)/2$, where $B_{s_0}^1$ is the ball of center zero and radius 1 in $H_{s_0}$. We assume that the following tame estimates hold: given $S' > s_0$, for all $s \in [s_0, S')$ there exists a constant $C(s) > 0$ such that for any $\|u\|_{s_0} \leq 2$,
1. $\|d(f(u)[h])\|_s \leq C(s)(\|u\|_s \|h\|_{s_0} + \|h\|_s)$,
2. $\|d^2f(u)[h,v]\|_s \leq C(s)(\|u\|_s \|h\|_{s_0} \|v\|_{s_0} + \|h\|_s \|v\|_{s_0} + \|h\|_{s_0} \|v\|_s).

The main assumptions are on the operator which is obtained by linearizing (A.3) at a point $u \in H^s$: we denote it by
\[ L = L(\lambda, \epsilon, u) := D(\lambda) + \epsilon T(u) \tag{A.5} \]
where $T(\epsilon) := df(u)$.

Hypothesis 1. Let $\mathfrak{A} \in \mathbb{R}^d$ satisfy (1.15). There exist a function $\mathfrak{D} : \mathbb{Z}^d \times \mathfrak{A} \times \mathbb{R} \to \mathbb{C}$ and $\sigma_0 > 0$ such that for all $\|u\|_{s_0}, \|u'\|_{s_0} \leq 2$ and $s_0 + \sigma_0 < s < S'$, one has
1. (Covariance) $D_{(\ell,j,a)}(\lambda) = \mathfrak{D}_{(j,a)}(\mathfrak{A} \cdot \ell)$, \quad $\forall \lambda \in \Lambda$, \tag{A.6}
2. (Töplitz in time) $T_{(\ell,j,a)}^{(\ell',j',a')} = T_{(j,a)}^{(j',a')}(\ell - \ell')$, \tag{A.7}
3. (Off diagonal decay) $|T(\ell)|_{s-\sigma_0} \leq C(s)(1 + \|\epsilon\|_s)$, \tag{A.8}
4. (Lipschitz) $|T(\ell) - T(\ell')|_{s-\sigma_0} \leq C(s)(\|u - u'\|_s + (\|u\|_s + \|u'\|_s)\|u - u'\|_{s_0})$. \tag{A.9}

In order to state the next hypothesis, we define for any $\theta \in \mathbb{R}$ the infinite matrices
\[ D(\lambda, \theta) := \text{Diag}(D_{(\ell,j,a)}(\lambda, \theta)), \quad D_{(\ell,j,a)}(\lambda, \theta) := \mathfrak{D}_{(j,a)}(\mathfrak{A} \cdot \ell + \theta), \]
\[ L(\lambda, \theta, u) := D(\lambda, \theta) + \epsilon T(u). \]

Hypothesis 2. There is $n \in \mathbb{N}$ such that for all $\tau_1 > 1, N > 1, \lambda \in \Lambda, (\ell,j,a) \in \mathfrak{A}$ the set
\[ \{ \theta \in \mathbb{R} : |D_{(\ell,j,a)}(\lambda, \theta)| \leq N^{-\tau_1} \} \subseteq \bigcup_{q=1}^n I_q \quad \text{intervals with meas}(I_q) \leq N^{-\tau_1}. \tag{A.10} \]

The last hypothesis that is needed concerns separation properties of clusters of singular sites. To state it, we need some preliminary definitions.
Definition A.2 (Γ-chain). Given $\Gamma \geq 2$, a sequence of distinct integer vectors $(k_q)_{q=0}^{\ldots,L} \subset \mathbb{R}$ is a Γ-chain if
\[
\text{dist}(k_{q+1}, k_q) \leq \Gamma, \quad \forall 0 \leq q \leq L - 1.
\]
The number $L$ is called the length of the chain.

Definition A.3 (Singular sites). We say that $k \in \mathbb{R}$ is a singular site for the matrix $D := \text{Diag}(D_k)$ if $|D_k| < 1$.

For any $\Sigma \subseteq \mathbb{R}$ and $\bar{j} \in \mathbb{Z}^d$, we denote by $\Sigma^{\bar{j}}$ the section of $\Sigma$ at fixed $\bar{j}$, namely
\[
\Sigma^{\bar{j}} := \{k = (\ell, \bar{j}, a) \in \Sigma\}.
\]

Definition A.4. Let $\theta, \lambda$ be fixed and $K > 1$. We denote by $\Sigma_K$ any subset of singular sites of $D(\lambda, \theta)$ such that the cardinality of the sections $\Sigma^{\bar{j}}$ satisfies $|\Sigma^{\bar{j}}| \leq K$, for any $\bar{j} \in \Sigma$.

Hypothesis 3. There exist a constant $C(\mathcal{L}, d, n) > 0$ and for any $N_0 \geq 2$, a set $\tilde{\Lambda} = \tilde{\Lambda}(N_0)$ such that for all $\lambda \in \tilde{\Lambda}$, $\theta \in \mathbb{R}$ and for all $K > 1$, $\Gamma \geq 2$ with $\Gamma K \geq N_0$, any Γ-chain of singular sites of $D(\lambda, \theta)$ in $\Sigma_K$ (as in Definition A.3) has length $L \leq (K\Gamma)^{C(\mathcal{L}, d, n)}$.

Given a family of matrices $L(\theta)$ parametrized by a parameter $\theta \in \mathbb{R}$ and $N > 1$, for any $k = (\ell, j, a) \in \mathbb{R}$ we denote by $L_{N, \ell, j}(\theta)$ the sub-matrix of $L(\theta)$ centered at $(\ell, j)$, i.e.,
\[
L_{N, \ell, j}(\theta) := \left\{L_k^{\ell'}(\theta) : \text{dist}(k, k') \leq N\right\}.
\]

For $\tau > 0$, $N_0 \geq 1$, set
\[
\overline{\lambda} := \overline{\lambda}(N_0, \tau) := \{\lambda \in \Lambda : |D_k(\lambda)| \geq N_0^{-\tau} \Rightarrow \forall k = (\ell, j, a) \in \mathbb{R} : \max\{|\ell|, |j|\} \leq N_0\}.
\]

Theorem A.5 (Berti-Corsi-Procesi). Let $\epsilon > d + n + 1$. Assume that $F$ in (A.3) satisfies (A.4), (f1)–(f2) and Hypotheses 1–3 with $S'$ large enough, depending on $\epsilon$. Then, there are $\tau_1 > 1$, $N_0 \in \mathbb{N}$, $s_1, S \in (s_0 + \sigma_0, S' - \sigma_0)$ with $s_1 < S$ (all depending on $\epsilon$) and $c(S) > 0$ such that for all $N_0 \geq N_0$, if the smallness condition
\[
\epsilon N_0^{\delta} < c(S)
\]
holds, then the following holds:

1. (Existence) There exist a function $u_0 \in C^1(\Lambda, H^s + \sigma)$ with $u_0(\lambda) = 0$, and a set $\mathcal{C}_\epsilon \subset \Lambda$ (defined only in terms of $u_0$), such that, for all $\lambda \in \mathcal{C}_\epsilon$, we have $F(\epsilon, \lambda, u_0(\lambda)) = 0$.

2. (Measure estimate) Let $N_0 = \lfloor \epsilon^{-1/(S+1)} \rfloor$ with $\epsilon$ small enough so that (A.12) holds. Assume that for all $N \geq N_0$,
\[
\text{meas}(\Lambda \setminus \overline{\Sigma}_N), \text{meas}(\Lambda \setminus \overline{\Sigma}_N) = O(N^{-1}), \quad \text{meas}(\Lambda \setminus (\overline{\lambda} \setminus \overline{\Lambda})) = O(N_0^{-1}),
\]
where $\overline{\lambda} = \tilde{\Lambda}(N_0)$ is defined in Hypothesis 3, $\overline{\lambda}$ in (A.11), and, for all $N \in \mathbb{N}$,
\[
\overline{\Sigma}_N := \left\{\lambda \in \Lambda : \|L_{N,0,0}\|_{0} \leq N^{-\tau_1/2}\right\},
\]
\[
\overline{\mathcal{B}}_N := \left\{\lambda \in \Lambda : \forall j_0 \in \mathbb{Z}^d \text{ there is a covering}\right\}
\]
\[
\overline{B}_{N}(j_0, \epsilon, \lambda) \subseteq \bigcup_{q=1}^{N^s} I_q, \text{ with } I_q = I_{q}(j_0) \text{ intervals with } \text{meas}(I_q) \leq N^{-\tau_1}
\]
where
\[
\overline{B}_{N}(j_0, \epsilon, \lambda) := \left\{\theta \in \mathbb{R} : \|L_{N,0,j_0}\|_{0} > N^{-\tau_1/2}\right\}.
\]
Then $\mathcal{C}_\epsilon$ satisfies, for some $K > 0$, the measure estimate $\text{meas}(\Lambda \setminus \mathcal{C}_\epsilon) \leq K \epsilon^{1/(S+1)}$. 26
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