Pseudo and Strongly Pseudo 2–Factor Isomorphic Regular Graphs

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Abstract

A graph $G$ is pseudo 2–factor isomorphic if the parity of the number of cycles in a 2–factor is the same for all 2–factors of $G$. In [3] we proved that pseudo 2–factor isomorphic $k$–regular bipartite graphs exist only for $k \leq 3$. In this paper we generalize this result for regular graphs which are not necessarily bipartite. We also introduce strongly pseudo 2–factor isomorphic graphs and we prove that pseudo and strongly pseudo 2–factor isomorphic $2k$–regular graphs and $k$–regular digraphs do not exist for $k \geq 4$. Moreover, we present constructions of infinite families of regular graphs in these classes. In particular we show that the family of Flower snarks is strongly pseudo 2–factor isomorphic but not 2–factor isomorphic and we conjecture that, together with the

*This research was financially supported by the Engineering Faculty of Taranto of the Technical University of Bari (Politecnico di Bari), using funds of the Provincia di Taranto for the support of the faculty’s teaching and scientific activities.
Petersen and the Blanuša graphs, they are the only cyclically 4-edge-connected snarks for which each 2-factor contains only cycles of odd length.

1 Introduction

All graphs considered are finite and simple (without loops or multiple edges). We shall use the term multigraph when multiple edges are permitted.

A graph with a 2-factor is said to be 2–factor hamiltonian if all its 2–factors are Hamilton cycles, and, more generally, 2–factor isomorphic if all its 2–factors are isomorphic. Examples of such graphs are \( K_4, K_5, K_{3,3} \), the Heawood graph (which are all 2–factor hamiltonian) and the Petersen graph (which is 2–factor isomorphic).

Several recent papers have addressed the problem of characterizing families of graphs (particularly regular graphs) which have these properties. It is shown in [4, 8] that \( k \)-regular 2–factor isomorphic bipartite graphs exist only when \( k \in \{2,3\} \) and an infinite family of 3–regular 2–factor hamiltonian bipartite graphs, based on \( K_{3,3} \) and the Heawood graph, is constructed in [8]. It is conjectured in [8] that every 3–regular 2–factor hamiltonian bipartite graph belongs to this family. Faudree, Gould and Jacobsen in [7] determine the maximum number of edges in both 2–factor hamiltonian graphs and 2–factor hamiltonian bipartite graphs. In addition, Diwan [6] has shown that \( K_4 \) is the only 3–regular 2–factor hamiltonian planar graph.

In [3] the above mentioned results on regular 2–factor isomorphic bipartite graphs are extended to the more general family of pseudo 2–factor isomorphic graphs i.e. graphs \( G \) with the property that the parity of the number of cycles in a 2–factor is the same for all 2–factors of \( G \). Example of these graphs are \( K_{3,3} \), the Heawood graph \( H_0 \) and the Pappus graph \( P_0 \). In particular, it is proven that pseudo 2–factor isomorphic \( k \)-regular bipartite graphs exist only when \( k \in \{2,3\} \) and that there are no planar pseudo 2–factor isomorphic cubic bipartite graphs. Moreover, it is conjectured in [3] that \( K_{3,3} \), the Heawood graph \( H_0 \) and the Pappus graph \( P_0 \) are are the only 3-edge-connected pseudo 2–factor isomorphic cubic bipartite graphs together with their repeated star products and some partial results towards this conjecture are obtained.

In this paper, we extend the above mentioned results on regular pseudo 2–factor isomorphic bipartite graphs to the not necessarily bipartite case (cf. Section 3). We introduce strongly pseudo 2–factor isomorphic graphs (Definition 2.4(ii)) and we prove that pseudo and strongly pseudo 2–factor isomorphic \( k \)-regular digraphs and \( 2k \)-regular graphs only exist for \( k \leq 3 \) (Theorems 3.1, 3.3 and Corollaries 3.2, 3.4). Moreover, we present four different constructions of infinite classes of regular graphs in these classes (cf.
Section 5). Finally, we deal with snarks and we show that the family of Flower snarks $J(t)$ is strongly pseudo 2–factor isomorphic but not 2–factor isomorphic (Proposition 4.2) and we conjecture that they are, together with the Petersen and the Blanuša2 graphs, the only cyclically 4–edge–connected snarks for which each 2–factor contains only cycles of odd length (Conjecture 4.3).

2 Preliminaries

Let $G$ be a bipartite graph with bipartition $(X,Y)$ such that $|X| = |Y|$, and $A$ be its bipartite adjacency matrix. In general $|det(A)| \leq per(A)$. We say that $G$ is det–extremal if $G$ has a 1–factor and $|det(A)| = per(A)$. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of $G$. For $L$ a 1–factor of $G$ define the sign of $L$, $\text{sgn}(L)$, to be the sign of the permutation of $\{1,2,\ldots,n\}$ corresponding to $L$. Thus $G$ is det–extremal if and only if all 1–factors of $G$ have the same sign.

Lemma 2.1 Let $L_1, L_2$ be 1–factors in a bipartite graph $G$ and $t$ be the number of cycles in $L_1 \cup L_2$ of length congruent to zero modulo 4. Then $\text{sgn}(L_1) \text{sgn}(L_2) = (-1)^t$.

Proof. This is a special case of [11, Lemma 8.3.1]. The proof is simple. □

A result of Thomassen [14, Theorem 5.4] implies:

Theorem 2.2 Let $G$ be a 1–extendable det–extremal bipartite graph. Then $G$ has a vertex of degree at most three. □

Another result of Thomassen [13, Theorem 3.2] implies:

Theorem 2.3 Let $G$ be a det–extremal bipartite graph with bipartition $A, B$ and $|A| = |B| = n$. Then $G$ has a vertex of degree at most $\lfloor \log_2 n \rfloor + 1$. □

Definition 2.4 (i) Let $G$ be a graph which contains a 2–factor. Then $G$ is said to be pseudo 2–factor isomorphic if the parity of the number of cycles in a 2–factor is the same for all the 2–factors of $G$. (ii) Let $G$ be a graph which has a 2–factor. For each 2–factor $F$ of $G$, let $t_i^*(F)$ be the number of cycles of $F$ of length $2i$ modulo 4. Set $t_i$ to be the function defined on the set of 2–factors $F$ of $G$ by:

$$t_i(F) = \begin{cases} 0 & \text{if } t_i^*(F) \text{ is even} \\ 1 & \text{if } t_i^*(F) \text{ is odd} \end{cases} (i = 0, 1).$$

Then $G$ is said to be strongly pseudo 2–factor isomorphic if both $t_0$ and $t_1$ are constant functions. Moreover, if in addition $t_0 = t_1$, set $t(G) := t_i(F)$, $i = 0, 1$. 
By definition, if $G$ is strongly pseudo 2–factor isomorphic then $G$ is pseudo 2–factor isomorphic. On the other hand there exist graphs such as the Dodecahedron which are pseudo 2–factor isomorphic but not strongly pseudo 2–factor isomorphic: the 2–factors of the Dodecahedron consist either of a cycle of length 20 or of three cycles: one of length 10 and the other two of length 5.

In [3] we studied pseudo 2–factor isomorphic regular bipartite graphs. In the bipartite case, pseudo 2–factor isomorphic and strongly pseudo 2–factor isomorphic are equivalent.

**Theorem 2.5** Let $G$ be a pseudo 2–factor isomorphic bipartite graph with bipartition $A, B$ and $|A| = |B| = n$. Then $G$ has a vertex of degree at most $\lceil \log_2 n \rceil + 2$.

**Proof.** Since $G$ is pseudo 2–factor isomorphic, it has a 2–factor $X$. Since $G$ is bipartite, $X$ can be partitioned into disjoint 1–factors $L_0, L_1$. Let $t$ be the number of cycles of length congruent to zero modulo four in $Y$. By Lemma 2.1, $\text{sgn}(L)\text{sgn}(L_0) = (-1)^t$. Since $G$ is pseudo 2–factor isomorphic, $t$ is constant for all choices of $L$. Thus all 1–factors of $G$, disjoint from $L_0$, have the same sign. Hence $G - L_0$ is det–extremal. So by Theorem 2.3, $G - L_0$ has minimum degree at most $\lceil \log_2 n \rceil + 1$. Hence $G$ has minimum degree at most $\lceil \log_2 n \rceil + 2$.

In what follows we will denote by $HU, U, SPU$ and $PU$ the sets of 2–factor hamiltonian, 2–factor isomorphic, strongly pseudo 2–factor isomorphic and pseudo 2–factor isomorphic graphs, respectively. Similarly, $HU(k), U(k), SPU(k), PU(k)$ respectively denote the $k$–regular graphs in $HU, U, SPU$ and $PU$.

### 3 Existence theorems

In this section we generalize the results obtained in [3] for bipartite graphs proving results that extend those obtained in [1] and [2].

For $v$ a vertex of a digraph $D$, let $d^+(v)$ and $d^-(v)$ denote the out–degree and in–degree of $v$ respectively. We say that $D$ is $k$–diregular if for all vertices $v$ of $D$, $d^+(v) = k = d^-(v)$.

**Theorem 3.1** Let $D$ be a digraph with $n$ vertices and $X$ be a directed 2–factor of $D$. Suppose that either

(a) $d^+(v) \geq \lceil \log_2 n \rceil + 2$ for all $v \in V(D)$, or
(b) $d^+(v) = d^-(v) \geq 4$ for all $v \in V(D)$

Then $D$ has a directed 2-factor $Y$ with a different parity of number of cycles from $X$.

Proof. Suppose that all directed 2-factors $Y$ of $D$ have the same parity of number of cycles. Let $t = 0$ if such a number is even, and $t = 1$ if such a number is odd. Construct the associated bipartite graph $G$ for the digraph $D$ in the following way. For each vertex $u \in V(D)$ make two copies $u'$ and $u''$ in $V(G)$. Each directed $(u, v) \in E(D)$ becomes the undirected edge $(u', v'') \in E(G)$. Additionally we add the edges $(u', u'')$ to $E(G)$ for all $u \in V(D)$. Note that $L_0 = \{(u', u'') : u \in V(D)\}$ is a 1-factor of $G$, and that $\{(u', v'') : (u, v) \in X\}$ is a 1-factor of $G - L_0$.

Let $L$ be a 1-factor of $G$ disjoint from $L_0$. Then $Y' := L \cup L_0$ is a 2-factor in $G$ in which each cycle has alternately edges of $L$ and edges of $L_0$. This 2-factor gives rise to a directed 2-factor $Y$ of $D$ when we contract each edge of $L_0$. Now each cycle of $Y'$ corresponds to exactly one cycle of $Y$ but with twice the length. This implies that for any 1-factor $L$ of $G$ disjoint from $L_0$, the number of cycles in $L \cup L_0$ of length congruent to 0 modulo 4 is equal to the number of even cycles in $Y$, i.e. it is congruent to $t$ modulo 2.

Using Lemma 2.1, we deduce that for any 1-factor $L$ of $G$, disjoint from $L_0$, $\text{sgn}(L)\text{sgn}(L_0) = (-1)^t$. Since $t$ is a constant, we conclude that all 1-factors of $G$, disjoint from $L_0$ have the same sign. Hence $G - L_0$ is det-extremal.

Now (a) and (b) follow directly using Theorem 2.5 and Theorem 2.2 respectively. Notice here that in case (b), because of regularity, $G$ is 1-extendable.

Corollary 3.2

(i) $\text{DSPU}(k) = \text{DPU}(k) = \emptyset$ for $k \geq 4$;

(ii) If $D \in \text{DPU}$ then $D$ has a vertex of out-degree at most $\lceil \log_2 n \rceil + 1$.

Theorem 3.3 Let $G$ be a graph with $n$ vertices and $X$ be a 2-factor of $G$. Suppose that either

(a) $d(v) \geq 2(\lceil \log_2 n \rceil + 2)$ for all $v \in V(G)$, or

(b) $G$ is a $2k$-regular graph for some $k \geq 4$
Then $G$ has a 2-factor $Y$ with a different parity of number of cycles from $X$.

**Proof.** Let $G_1 = G - X$ and $U$ be the set of vertices of odd degree in $G_1$. Let $M$ be a matching between the vertices of $U$. Let $G_2$ be the multigraph obtained by adding the edges of $M$ to $G_1$. Each vertex of $G_2$ has even degree, and hence each component of $G_2$ has an Euler tour. Thus we can construct a digraph $D_2$ by orientating the edges of $G_2$ in such a way that $d^+(D_2(v)) = d^-(D_2(v))$ for all $v \in V(D_2)$. Let $D_1$ be the digraph obtained from $D_2$ by deleting the arcs corresponding to edges in $M$. Thus either

(i) $d^+(D_1(v)) \geq \lfloor \log_2 n \rfloor + 1$, $d^-(D_1(v)) \geq \lfloor \log_2 n \rfloor + 1$ for all $v \in V(D_1)$, or

(ii) $d^+(D_1(v)) = d^-(D_1(v)) \geq 3$ for all $v \in V(D_1)$.

Let $X_1$ be a 1–diregular digraph obtained by directing the edges of $X$ and $D$ be the digraph obtained from $D_1$ by adding the arcs of $X_1$. Then either

(iii) $d^+(D(v)) \geq \lfloor \log_2 n \rfloor + 2$, $d^-(D(v)) \geq \lfloor \log_2 n \rfloor + 2$ for all $v \in V(D)$, or

(iv) $d^+(D(v)) = d^-(D(v)) \geq 4$ for all $v \in V(D)$.

The result now follows from (iii),(iv) and Theorem 3.1. □

**Corollary 3.4**

(i) If $G \in PU$ then $G$ contains a vertex of degree at most $2 \lfloor \log_2 n \rfloor + 3$;

(ii) $PU(2k) = SPU(2k) = \emptyset$ for $k \geq 4$. □

We know that $PU(3)$, $SPU(3)$, $PU(4)$ and $SPU(4)$ are not empty (cf. table in Section 5) and we conjectured in [1] that $HU(4) = \{K_5\}$. There are many gaps in our knowledge even when we restrict attention to regular graphs. Some questions arise naturally.

**Problem 3.5** Is $PU(2k + 1) = \emptyset$ for $k \geq 2$?

In particular we wonder if $PU(7)$ and $PU(5)$ are empty.

**Problem 3.6** Is $PU(6)$ empty?
Problem 3.7 Is $K_5$ the only 4–edge–connected graph in $PU(4)$?

In Section 5 we present examples of 2–edge–connected graphs in $PU(4)$.

Of course a major problem is to find some sort of classification of the elements of $PU(3)$. A general resolution of this problem is unlikely since we have no classification of the bipartite elements of $PU(3)$. A first step might be to attempt to classify the near bipartite elements of $PU(3)$ (a non-bipartite graph is near bipartite if it can be made bipartite by the deletion of exactly two edges). The cubic near bipartite graph obtained from the Petersen graph by adding an edge joining two new vertices in two edges at maximum distance apart is not in $PU(3)$. On the other hand, if a vertex of $K_{3,3}$ is inflated to a triangle the resulting graph is near bipartite and belongs to $PU(3)$.

Problem 3.8 Do there exist near bipartite graphs of girth at least four in $PU(3)$?

In section 4 we have taken a different direction in examining elements of $PU(3)$ which contain only ‘odd 2–factors’.

We close this section with some remarks on the operation of star products of cubic graphs.

Let $G, G_1, G_2$ be graphs such that $G_1 \cap G_2 = \emptyset$. Let $y \in V(G_1)$ and $x \in V(G_2)$ such that $d_{G_1}(y) = 3 = d_{G_2}(x)$. Let $x_1, x_2, x_3$ be the neighbours of $y$ in $G_1$ and $y_1, y_2, y_3$ be the neighbours of $x$ in $G_2$. If $G = (G_1 - y) \cup (G_2 - x) \cup \{ x_1 y_1, x_2 y_2, x_3 y_3 \}$, then we say that $G$ is a star product of $G_1$ and $G_2$ and write $G = (G_1, y) \ast (G_2, x)$, or $G = G_1 \ast G_2$ for short, when we are not concerned which vertices are used in the star product. The set $\{ x_1 y_1, x_2 y_2, x_3 y_3 \}$ is a 3–edge cut of $G$ and we shall also say that $G_1$ and $G_2$ are 3–cut reductions of $G$.

Star products preserve the property of being 2–factor hamiltonian, 2–factor isomorphic, pseudo 2–factor isomorphic and, obviously, strongly pseudo 2–factor isomorphic in the family of cubic bipartite graphs (cf. [8],[4],[3]). Note that the converse is not true for 2–connected pseudo 2–factor isomorphic bipartite graphs [3].

In general for graphs not necessarily bipartite, star products do not preserve the property of being 2–factor hamiltonian graphs, since it is easy to check that $K_4 \ast K_4$ is not 2–factor hamiltonian. Hence, 2–factor isomorphic, pseudo 2–factor isomorphic and strongly pseudo 2–factor isomorphic non–bipartite graphs are also not preserved under star products.

Still, it is easily proved that the cubic graph $G := (G_1, x) \ast (G_1, y)$ is 2–factor hamiltonian if and only if $G_1$ and $G_2$ are 2–factor hamiltonian and
the 3–edge cut $E_1(x, y) = \{x_1y_1, x_2y_2, x_3y_3\}$ is tight (i.e. every 1–factor of $G$ contains exactly one edge of $E_1(x, y)$, c.f. e.g [11, p. 295])

However, if $G_1, G_2$ and $G := (G_1, x) \ast (G_2, y)$ are pseudo 2–factor isomorphic graphs for some $x \in V(G_1)$ and $y \in V(G_2)$, then $E_1(x, y)$ is not necessarily tight. For example, if $G_1 = K_4$ and $G_2$ is the Petersen graph, they are both pseudo 2–factor isomorphic, and so is their star product which contains 2–factors of type $(3, 9)$ and $(5, 7)$, but the 3–edge cut is not tight, since the 2–factor of type $(3, 9)$ contains no edges of the 3–edge cut.

4 Snarks

A snark (cf. e.g. [9]) is a bridgeless cubic graph with edge chromatic number four. (By Vizing’s theorem the edge chromatic number of every cubic graph is either three or four so a snark corresponds to the special case of four). In order to avoid trivial cases, snarks are usually assumed to have girth at least five and not to contain a non–trivial 3–edge cut. The Petersen graph $P$ is the smallest snark and Tutte conjectured that all snarks have Petersen graph minors. This conjecture was confirmed by Robertson, Seymour and Thomas (unpublished, see [12]). Necessarily, snarks are non–hamiltonian.

We say that a graph $G$ is odd 2–factored if for each 2–factor $F$ of $G$ each cycle of $F$ is odd. By definition, an odd 2–factored graph $G$ is strongly pseudo 2–factor isomorphic.

Lemma 4.1 Let $G$ be a cubic 3–connected odd 2–factored graph then $G$ is a snark.

Proof. Since $G$ is odd 2–factored, the chromatic index of $G$ is at least four. Hence, by Vizing’s Theorem, $G$ has chromatic index 4. ∎

Question: Which snarks are odd 2–factored?

Let $t \geq 5$ be an odd integer. The Flower snark (cf. [10]) $J(t)$ is defined in much the same way as the graph $A(t)$ described in [1]. The graph $J(t)$ has vertex set

$$V(t) = \{h_i, u_i, v_i, w_i : i = 1, 2, \ldots, t\}$$

and edge set

$$E(t) = \{h_iu_i, h_iv_i, h_iw_i, u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1}, : i = 1, 2, \ldots, t - 1\}$$

$$\cup \{u_{i+1}v_1, u_1u_1, w_1w_1\}$$
For \( i = 1, 2, \ldots, t \) we call the subgraph \( IC_i \) of \( J(t) \) induced by the vertices \( \{h_i, u_i, v_i, w_i\} \) the \( i^{th} \) interchange of \( J(t) \). The vertices \( h_i \) and the edges \( \{h_iu_i, h_iv_i, h_iw_i\} \) are called respectively the hub and the spokes of \( IC_i \). The set of edges \( \{u_iu_{i+1}, v_{i+1}, w_iw_{i+1}\} \) linking \( IC_i \) to \( IC_{i+1} \) are said to be the \( i^{th} \) link \( L_i \) of \( J(t) \). The edge \( u_iu_{i+1} \in L_i \) is called the \( u \)-channel of the link. The subgraph of \( J(t) \) induced by the vertices \( \{u_i, v_i : i = 1, 2, \ldots, t\} \) and \( \{w_i : i = 1, 2, \ldots, t\} \) are respectively cycles of length \( 2t \) and \( t \) and are said to be the base cycles of \( J(t) \).

Recall that in a cubic graph \( G \), a 2–factor, \( F \), determines a corresponding 1–factor, namely \( E(G) - F \). In studying 2–factors in \( J(t) \) it is more convenient to consider the structure of 1–factors.

**Proposition 4.2** Let \( t \geq 5 \) be an odd integer. Then \( J(t) \) is odd 2–factored. Moreover, \( J(t) \) is strongly pseudo 2–factor isomorphic but not 2–factor isomorphic.

**Proof.** If \( L \) is a 1–factor of \( J(t) \) each of the \( t \) links of \( J(t) \) contain precisely one edge from \( L \). This follows from the argument in \([1]\) Lemma 4.7. Then, a 1–factor \( L \) may be completely specified by the ordered \( t \)–tuple \((a_1, a_2, \ldots, a_t)\) where \( a_i \in \{u_i, v_i, w_i\} \) for each \( i = 1, 2, \ldots, t \) and indicates which edge in \( L_i \) belongs to \( L \). Together these edges leave a unique spoke in each \( IC_i \) to cover its hub. Note that \( a_i \neq a_{i+1}, i = 1, 2, \ldots, t \). To read off the corresponding 2–factor \( F \) simply start at a vertex in a base cycle at the first interchange. If the corresponding channel to the next interchange is not banned by \( L \), proceed along the channel to the next interchange. If the channel is banned, proceed via a spoke to the hub (this spoke cannot be in \( L \)) and then along the remaining unbanned spoke and continue along the now unbanned channel ahead. Continue until reaching a vertex already encountered, so completing a cycle \( C_1 \). At each interchange \( C_1 \) contains either 1 or 3 vertices. Furthermore as \( C_1 \) is constructed iteratively, the cycle \( C_1 \) is only completed when the first interchange is revisited. Since \( C_1 \) uses either 1 or 3 vertices from \( IC_1 \) it can revisit either once or twice. If \( C_1 \) revisits twice then \( C_1 \) is a hamiltonian cycle which is not the case. Hence it follows that \( F \) consists of two cycles \( C_1 \) and \( C_2 \). Let \( k_1 \) and \( k_3 \) be respectively the number of interchanges which contain 1 and 3 vertices of \( C_1 \). Then the length of \( C_1 \) is \( k_1 + 3k_3 \). Since \( C_1 \) visits iteratively each of the \( t \) interchanges, \( k_1 + k_3 \) is odd. Thus, the length of \( C_1 \) is odd and so is the length of \( C_2 \). Hence \( J(t) \) is odd 2–factored and \( J(t) \in SPU(3) \).

Finally, \( J(t) \notin U(3) \) since it has 2–factors of types \((t, 3t)\) and \((t+4, 3t-4)\). Indeed, if \((a_1, a_2, \ldots, a_t)\) is such that \( a_i \in \{u_i, v_i\} \), we obtain a 2–factor of type \((t, 3t)\) in \( J(t) \). On the other hand, if \((a_1, a_2, \ldots, a_t)\) is such that \( a_j = w_j \), for some \( j \in \{1, \ldots, t\} \), and \( a_i \in \{u_i, v_i\} \), for all \( i \neq j \), we obtain a 2–factor of type \((t+4, 3t-4)\) in \( J(t) \). \( \square \)
A set $S$ of edges of a graph $G$ is a *cyclic edge cut* if $G - S$ has two components each of which contains a cycle. We say that a graph $G$ is *cyclically $m$–edge–connected* if each cyclic edge cut of $G$ has size at least $m$. We consider graphs without cyclic edge cuts to be cyclically $m$–edge–connected for all $m \geq 1$. Thus, for instance $K_4$ and $K_{3,3}$ are cyclically $m$–edge–connected for all $m \geq 1$.

We have the following information about some well–known snarks:

| Snark                      | Odd 2–factored | 2–Factor Types          |
|----------------------------|----------------|-------------------------|
| Blanuša snark 1            | No             | (5, 5, 8) et al.        |
| Blanuša snark 2            | Yes            | (5, 13) and (9, 9)      |
| Loupekine snark 1          | No             | (5, 8, 9) et al.        |
| Loupekine snark 2          | No             | (5, 8, 9) et al.        |
| Celmins-Swart snark 1      | No             | (5, 5, 8, 8) et al.     |
| Double Star snark          | No             | (7, 7, 16) et al.       |
| Szekeres snark             | No             | (5, 5, 40) et al.       |

We have also checked all known snarks up to 22 vertices and all the named snarks up to 50 vertices and they are all not odd 2–factored, except for the Petersen graph, Blanuša 2, and the Flower snark $J(t)$. We tentatively and possibly wildly suggest the following:

**Conjecture 4.3** A cyclically 4–edge–connected snark is odd 2–factored if and only if $G$ is the Petersen graph, Blanuša 2, or a Flower snark $J(t)$, with $t \geq 5$ and odd.

## 5 Appendix: 2–edge–connected constructions

In this section we present some sporadic examples and some constructions for graphs in $HU(k), U(k), SPU(k)$ and $PU(k)$, for $k = 3, 4$. The sporadic examples will be presented in a table, and since some platonic solids belong to some of these classes we have included them all (even those that do not belong to any of these sets). Lists of numbers (if present), in the last column of the table, represent the types of 2–factors of the corresponding graph.
Some of these sporadic examples will be used as seeds for the following 2–edge–connected constructions. Firstly we describe a family of pseudo 2–factor isomorphic cubic graphs based on a construction used in [3] for 2–factor isomorphic bipartite graphs. Here we show that this construction preserves pseudo 2–factor isomorphic not necessarily bipartite graphs but not strongly pseudo 2–factor isomorphic ones. Then we present a specific construction of strongly pseudo 2–factor isomorphic cubic graphs which are not 2–factor isomorphic. Finally we present two infinite families of 2–edge–connected 4–regular graphs which are strongly pseudo 2–factor isomorphic.

(1) We construct an infinite family of graphs in $PU(3)$.
Let $G_i$ be a cubic graph and $e_i = (x_i, y_i) \in E(G_i)$, $i = 1, 2, 3$. Let $G^* = (G_1, e_1) \circ (G_2, e_2) \circ (G_3, e_3)$ be the 3–regular graph called 3–joins (cf. [3] p. 440) defined as follows:

$$V(G^*) = \left( \bigcup_{i=1}^{3} V(G_i) \right) \cup \{u, v\}$$

$$E(G^*) = \left( \bigcup_{i=1}^{3} (E(G_i) - \{e_i\}) \right) \cup \left( \bigcup_{i=1}^{3} \{(x_i, u), (y_i, v)\} \right),$$

$G^*$ is 2–edge–connected but not 3–edge connected. In [3] Proposition 3.18 we proved that if $G_i$ are 2–factor hamiltonian cubic bipartite graphs, then $G^*$ is 2–factor isomorphic.
Proposition 5.1 Let $G_i$ $(i = 1, 2, 3)$ be pseudo 2–factor isomorphic cubic graphs. Then $G^*$ is a cubic pseudo 2–factor isomorphic graph.

Proof. All the 2–factors $F$ in $G^*$ are composed from 2–factors $F_1, F_2, F_3$ of $G_1, G_2, G_3$ such that, for some $\{i, j, k\} = \{1, 2, 3\}$, we have $e_i \notin F_i$, $e_j \in F_j$, and $e_k \in F_k$. Let $C_j$ and $C_k$ be the cycles of $F_j, F_k$, containing the edges $e_j, e_k$ respectively. Then the cycles of $F$ are all the cycles from $F_1, F_2$ and $F_3$, except for $C_j$ and $C_k$, and the cycle $C = (C_j \cup C_k) - \{e_j, e_k\} \cup \{x_ju, y_jv, x_kv, y_kv\}$. Therefore, the parity of the number of cycles in a 2–factor $F$ of $G^*$ is $t(F) = t(F_1) + t(F_2) + t(F_3) - 1 \pmod{2}$. Since $t(F_i)$ is constant for each $i = 1, 2, 3$, then $t(F)$ is also constant and $G^*$ is pseudo 2–factor isomorphic. \hfill $\square$

A brief analysis of the values of $t_0$ and $t_1$ over all 2–factors of $G^*$, with respect to the values of $t_0$ and $t_1$ in $G_i$, for $i = 1, 2, 3$, gives rise to the following proposition.

Proposition 5.2 Let $G_i$ be strongly pseudo 2–factor isomorphic graphs such that in any 2–factor of $G_i$ all cycles have even length, $i = 1, 2, 3$. Then $G^*$ is strongly pseudo 2–factor isomorphic. \hfill $\square$

However, in general, strongly pseudo 2–factor isomorphism is not preserved under this construction. A counterexample can be built from the Flower snark $J(5)$ (cf. Section 4). In fact, the graph $J(5)^*$, obtained as a 3–join of $G_i := J(5)$ and $e_i := v_7u_1$, $i = 1, 2, 3$, is not strongly pseudo 2–factor isomorphic since it contains 2–factors of types $(5, 5, 5, 15, 32)$ and $(5, 5, 11, 15, 26)$.

(2) We construct an infinite family of graphs $H(n)$ in $SPU(3)$.

Let $H(n)$, be the family of cubic graphs on $n \geq 14$ vertices, $n$ even, defined as follows. Let $K^{*}_{3,3}$ and $K^{*}_{4}$ be the graphs obtained by deleting exactly one edge from $K_{3,3}$ and $K_{4}$ respectively. Set $n \equiv 2j \pmod{8}$, $j = 0, 1, 2, 3$. Set $\theta := j + 2 \pmod{4}$ where $0 \leq \theta \leq 3$. Then $H(n)$ is an infinite family of cubic graphs on $n \geq 14$ vertices, $n$ even, obtained from a cycle of length $(n - 2\theta)/4$ by “inflating” $\theta$ of the vertices of the cycle into copies of $K^{*}_{3,3}$ and $(n - 6\theta)/4$ of the vertices of the cycle into copies of $K^{*}_{4}$ (cf. e.g. picture below for $H(14)$).

Proposition 5.3 The family of cubic graphs $H(n)$ is strongly pseudo 2–factor isomorphic but not 2–factor isomorphic.

Proof. By construction $H(n)$ has 2–factors $F_1 := F_1(n)$, where $F_1$ consists of $\theta$ cycles of length 6 and $(n - 6\theta)/4$ cycles of length 4, and $F_2 := F_2(n)$, where $F_2$ consists of a cycle of length $n$ (i.e. it is hamiltonian). Hence $H(n)$ is not 2–factor isomorphic.

First suppose $n \equiv 0 \pmod{4}$. Then $j = 0$ or 2 and $\theta = 2$ or 0, respectively. Therefore, $\theta$ is even and $(n - 6\theta)/4$ is odd. Thus, the number of cycles
in a 2–factor of $H(n)$ is odd, and all such cycles have even length. Thus $H(n) \in \mathcal{P}(3)$. Moreover, it is easy to check that $t_0$ and $t_1$ are constant. Hence $H(n) \in \mathcal{S}(3)$.

Now suppose $n \equiv 2 \pmod{4}$. Then $j = 1$ or 3 and $\theta = 3$ or 1, respectively. Therefore, $\theta$ is odd and $(n - 6\theta)/4$ is even. Thus, the number of cycles in a 2–factor of $H(n)$ is odd, and all such cycles have even length. Thus $H(n) \in \mathcal{P}(3)$. Again it is easily checked that $t_0$ and $t_1$ are constant. Hence $H(n) \in \mathcal{S}(3)$. \hfill \Box

(3) We construct an infinite family of graphs $H^*(5(2k+1))$ in $\mathcal{S}(4)$.

Let $K^*_5 = K^*_5 - e$. Take an odd cycle $C_{2k+1}$. Let $H^*(5(2k+1))$, $k \geq 1$ be the graph of degree 4 obtained by inflating each vertex of $C_{2k+1}$ to a graph isomorphic to $K^*_5$. The 2–factors of $H^*(5(2k+1))$ are $F_1 = (5(2k+1))$ and $F_2 = (5, 5, \ldots, 5)$ with $2k+1$ cycles of size 5. Therefore, $t^*(H^*(5(2k+1))) = 0$ and $H^*(5(2k+1))$ is a 4–regular 2–edge–connected strongly pseudo 2–factor isomorphic but not 2–factor isomorphic (cf. e.g. picture below for $H^*(15)$). Notice that adding any edge to $H^*(5(2k+1))$ results in a graph which is not pseudo 2–factor isomorphic.

(4) We construct a second infinite family of graphs in $\mathcal{S}(4)$.

In [1, p. 400] we defined an edge $e$ belonging to a 2–factor of a graph $G$ to be loyal if for each 2–factor $F$ containing $e$, the cycle to which $e$ belongs had constant length, independently of the choice of $F$. We used graphs containing
a loyal edge to define an infinite family of 2-connected 4-regular 2-factor isomorphic graphs [1, Construction (1), p. 400]. We extend this construction to the strongly pseudo 2-factor isomorphic case.

Let $G$ be a graph and let $e$ be one of its edges such that there are 2-factors $F, F'$ of $G$ containing and avoiding $e$ respectively. We now define $e$ to be pseudo loyal if for each 2-factor $F$ containing $e$, the cycle to which $e$ belongs has constant length modulo 4, independently of the choice of $F$.

Let $G \in SPU(4)$ and let $e$ be a pseudo loyal edge in $G$, and let $c$ be the length (modulo 4) of the cycle containing $e$ in a 2-factor of $G$ containing $e$. Let $G_1, G_2, G_3, G_4$ be four isomorphic copies of $G$ and $e_i = x_iy_i$ be the loyal edge in $G_i$ corresponding to $e$. We construct a 4-regular graph $G'$ called a 4-seed graft of $G$ by taking

$$V(G') = \left(\bigcup_{i=1}^{4} V(G_i)\right) \cup \{u, v\}$$

and

$$E(G') = \left(\bigcup_{i=1}^{4} (E(G_i) - \{e_i\})\right) \cup \left(\bigcup_{i=1}^{4} \{(x_i, u), (y_i, v)\}\right)$$

We call the new vertices $u, v$ clips and we refer to $G$ as a seed for $G'$.

**Proposition 5.4** Let $G \in SPU(4)$ and let $e$ be a pseudo loyal edge in $G$. Then the 4-regular seed graft $G'$ of $G$ is strongly pseudo 2-factor isomorphic, has connectivity 2 and each edge of $G'$ which is adjacent to a clip is pseudo loyal.

**Proof.** By construction $G'$ is not 3-edge connected thus $G'$ has connectivity 2. Let $F$ be a 2-factor of $G'$. Relabeling if necessary, we may suppose that $\{ux_1, ux_2, vy_1, vy_2\} \subseteq F$. Then $(F \cap G_i) + e_i$ are 2-factors of $G_i$ containing $e_i$ for $i = 1, 2$, and $F \cap G_j$ is a 2-factor of $G_j$ avoiding $e_j$ for $j = 3, 4$. The cycle of $F$ containing the clips is $C = (C_1 - e_1) \cup (C_2 - e_2) \cup \{x_1u, y_1v, x_2u, x_2v\}$ and it has constant length $2c + 2(\text{mod} 4)$, independently of the choice of $F$, where $c$ is the length (modulo 4) of the cycle containing $e$ in a 2-factor of $G$ containing $e$. Then, each edge of $G'$ adjacent to a clip is pseudo loyal. This also implies that the values $t_0$ and $t_1$ are constant over all 2-factors of $G'$, independently of the choice of $F$. Hence, $G' \in SPU(4)$. \qed

**Note:** In [1, p. 400] the only seed we had for the family of graphs with loyal edges was $K_5 \in U(4)$, in which each edge is loyal. In the family $H^*(5(2k+1))$ the edges of the cycle $C_{2k+1}$ are pseudo loyal, and if $k$ is even, then all edges of the graph are pseudo loyal. Therefore, Proposition 5.4 gives rise to an infinite family of 2-connected graphs in $SPU(4)$ starting from $H^*(5(2k+1))$ for each value of $k$. 

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