DIRICHLET PROBLEM OF QUATERNIONIC MONGE-AMPERE EQUATIONS

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Abstract. In this paper, the author studies quaternionic Monge-Ampère equations and obtains the existence and uniqueness of the solutions to the Dirichlet problem for such equations without any restriction on domains. Our paper not only answers to the open problem proposed by Semyon Alesker in [3], but also extends relevant results in [7] to the quaternionic vector space.

1. Introduction

Quaternion and HKT-geometry is an important branch of maths. Mathematicians have discovered some interesting facts from it. It has many applications in mathematical physics. Recently, the question whether there is a quaternionic version of Calabi-Yau Theorem has attracted some experts to do research on it, and they have obtained some results [4][5][6]. Relating to this problem, Dirichlet problem for quaternionic Monge-Ampère (MA) equations on arbitrary strictly pseudoconvex bounded domains is an open problem [3]. In this paper, we solve this issue.

To begin with, we want to describe the background. The classical solvability of the Dirichlet problem for real and complex Monge-Ampère equations were proved under the condition of convexity of domains in [8] and [7], respectively. To general domains in \( \mathbb{C}^n \), Bo Guan managed to obtain the same result in [7] assuming the existence of a subsolution to the corresponding equation [10], and generalized the result to totally real submanifolds [11] and Hermitian manifolds [12][13]. In [7], L. Caffarelli and his co-authors created a subsolution to the Dirichlet problem using the defining function for the strongly pseudo-convex domain. In [14], P. Guan constructed a subsolution to the Dirichlet problem for a degenerate complex Monge-Ampère equation on a special domain with some pieces of the boundary being concave. The Monge-Ampère equation also has many geometric applications, for example, Calabi conjecture [9]. In [16], Yau solved the Calabi conjecture. Yau’s work also shows the existence of Kähler-Einstein metrics on Kähler manifolds with nonpositive first Chern class.

In [3], S. Alesker proved a result on existence and uniqueness of the smooth solution of Dirichlet problem

\[
\begin{aligned}
\det \left( \frac{\partial^2 u}{\partial q_i \partial q_j} \right) &= f(q) \quad \text{in } B, \\
u|_{\partial B} &= \varphi,
\end{aligned}
\]

where \( q \in B \) and \( B \) is the Euclidean ball in \( \mathbb{H}^n \) which denotes the space of \( n \)-tuples of quaternions \((q_1, \ldots, q_n)\). He mainly followed the method in [7], but he made a strong restriction.

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on the domain. He said the reason why he failed to solve \((1.1)\) on general strictly pseudo-convex bounded domains is the fact that the class of diffeomorphisms preserving the class of quaternionic plurisubharmonic(psh) function must be affine transformations. In a word, the priori estimates in both \([3]\) and \([7]\) depend on the positive definiteness of the matrix in the local expression of the boundary, while this positiveness can’t be preserved all the time on quaternionic strictly pseudoconvex domains in general. However, from the statement in the preceding paragraph, we know this positiveness is not necessary. Since the field of quaternions is non-commutative, we need to modify some auxiliary functions in \([11]\) and the proof of \((1.47)\) in \([7]\). At last, we obtain the following:

**Theorem 1.1.** Let \(\Omega\) be any bounded domain with a smooth boundry and \(u \in C^\infty(\Omega)\) be a subsolution such that

\[
\begin{align*}
\det \left( \frac{\partial^2 u}{\partial q_i \partial q_j} \right) &\geq f(q, u), \quad \text{in } \Omega \\
\vert u \vert_{\partial \Omega} &= \varphi \in C^\infty(\partial \Omega),
\end{align*}
\]

where \(f \in C^\infty(\Omega \times \mathbb{R})\) and \(f > 0, f_u = \frac{\partial f}{\partial u} \geq 0\) for any \(q \in \Omega, u \in \mathbb{R}\). Then there exists an unique psh function \(u \in C^\infty(\Omega)\) solving

\[
\begin{align*}
\det(u, u) &= \det \left( \frac{\partial^2 u}{\partial q_i \partial q_j} \right) = f(q, u), \quad \text{in } \Omega \\
\vert u \vert_{\partial \Omega} &= \varphi \in C^\infty(\partial \Omega),
\end{align*}
\]

Proceeding as in \([7]\), we establish the relation between the convexity of domain in \(\mathbb{H}^n\) and the subsolution to the Dirichlet problem of quaternionic MA equations:

**Proposition 1.2.** Suppose that \(\Omega\) is a quaternionic strictly pseudoconvex bounded domain. For any \(q \in \Omega, u \in \mathbb{R}, p \in \mathbb{R}^{4n}\), assume \(f \in C^\infty(\Omega \times \mathbb{R} \times \mathbb{R}^{4n})\) satisfies

\[
(1.4) \quad f > 0, \quad f_u(q, u, \nabla u) \geq 0, \quad |f_p(q, u, \nabla u)| \leq Cf^{1-\frac{n}{2}},
\]

where \(C\) is a constant. Then there exists a subsolution \(u \in C^\infty(\Omega)\) such that

\[
(1.5) \quad \begin{align*}
\det \left( \frac{\partial^2 u}{\partial q_i \partial q_j} \right) &\geq f(q, u, \nabla u), \quad \text{in } \Omega \\
\vert u \vert_{\partial \Omega} &= \varphi \in C^\infty(\partial \Omega),
\end{align*}
\]

where \(p = \nabla u\).

From this property, we have

**Corollary 1.3.** Assume that \(\Omega\) is a quaternionic strictly pseudoconvex bounded domain. If \(f > 0\) and \(f_u \geq 0\), then there exists an unique psh function \(u \in C^\infty(\Omega)\) solving \((1.3)\).

**Remark 1.4.** The restriction “\(f_u \geq 0\)” above is necessary for the uniqueness of the solution, and Corollary \([7]\) settles the “Question 4” in \([3]\).

This article is organized as follows. In section 2, we review some basic definitions and facts from the theory on non-commutative determinants and plurisubharmonic functions of quaternionic variables. In section 3, we prove priori estimates up to second order excepting the boundary estimates for the second order normal derivatives, which are proved more detailedly in section 4. In section 5, we construct a subsolution to clarify Proposition \([1.2]\). In section 6, we consider the dependence of \(f\) on the gradient of \(u\).
2. Quaternionic linear algebra

In the whole of this article, we consider $\mathbb{H}^n$ as right $\mathbb{H}$-vector space, i.e. vectors are multiplied by scalars on the right. The standard theory of vector spaces, basis, and dimension works over $\mathbb{H}^n$, exactly like in the commutative case. However, the theory of non-commutative determinants is quite different and complicated. There are many useful determinants over $\mathbb{H}$, e.g. Dieudonné determinant. Experts are still searching for the best determinant which preserves most of the identities and inequalities known for usual determinant of real and complex matrices. We know the importance of real symmetric and complex Hermitian matrices. Over $\mathbb{H}$, similarly, there is a class of quaternionic matrices called hyperhermitian.

**Definition 2.1.** Let $V$ be a right $\mathbb{H}$-vector space. A hyperhermitian semilinear form on $V$ is a map $a : V \times V \to \mathbb{H}$ satisfying the following properties:

(a) $a$ is additive with respect to each argument;
(b) $a(x, y \cdot q) = a(x, y) \cdot q$ for any $x, y \in V$ and $q \in \mathbb{H}$;
(c) $a(x, y) = \overline{a(y, x)}$,

where $\overline{q}$ denotes the usual quaternionic conjugation of $q \in \mathbb{H}$.

An $n \times n$ quaternionic matrix $A = (a_{ij})$ is called hyperhermitian if $A^* = A$, i.e., $a_{ij} = \overline{a}_{ji}$ for all $i, j$. The proposition below points out the relation between hyperhermitian forms and hyperhermitian matrices.

**Proposition 2.2.** Fix a basis in a finite dimensional right quaternionic vector space $V$. Then there is a natural bijection between the space of hyperhermitian semilinear form on $V$ and the space $\mathcal{H}_n$ of $n \times n$ hyperhermitian matrices.

We are going to state some basic facts about hyperhermitian matrices as follows.

**Proposition 2.3.** Let $A$ be a matrix of a given hyperhermitian form in a given basis. Assume that $C$ is transition matrix from this basis to another one. Then we have

$$A' = C^* AC,$$

where $(C^*)_ij = \overline{C}_{ji}$ and $A'$ denotes the matrix of the given form in the new basis.

**Remark 2.4.** The matrix $C^* AC$ is hyperhermitian for any hyperhermitian matrix $A$ and any matrix $C$. In particular $C^* C$ is always hyperhermitian.

**Definition 2.5.** A hyperhermitian semilinear form $a$ is called positive definite if $a(x, x) > 0$ for any non-zero vector $x$. Similarly, $a$ is called non-negative definite if $a(x, x) \geq 0$ for any vector $x$.

Let us fix on our quaternionic right space $V$ a positive definite hyperhermitian form $(\cdot, \cdot)$. The space with such a form is called hyperhermitian space. For any quaternionic linear operator $\phi : V \to V$ in hyperhermitian space one can define the adjoint operator $\phi^* : V \to V$ in the usual way, i.e. $(\phi x, y) = (x, \phi y)$ for any $x, y \in V$. Moreover, if one fixes an orthonormal basis in the space $V$, then the operator $\phi$ is selfadjoint if and only if its matrix under this basis is hyperhermitian.

**Proposition 2.6.** For any selfadjoint operator in a hyperhermitian space there exists an orthonormal basis such that its matrix in this basis is diagonal and real.
We are going to introduce the Moore determinant of hyperhermitian matrices. For the definition of the Moore determinant one can refer to [15][1][2]. It is useful to give an explicit formula for the Moore determinant. Let $A = (a_{ij})_{i,j=1}^{n}$ be a hyperhermitian $(n \times n)$-matrix. Suppose that $\sigma$ be a permutation of $1, \ldots, n$. Write $\sigma$ as a product of disjoint cycles such that each cycle starts with the smallest number. Since disjoint cycles commute we can write

$$\sigma = (k_{11} \ldots k_{1j_1})(k_{21} \ldots k_{2j_2}) \cdots (k_{m1} \ldots k_{mj_m}),$$

where for each $i$ we have $k_{i1} < k_{ij}$ for all $j > 1$, and $k_{11} > k_{21} > \ldots > k_{m1}$. This expression is unique. Let $\text{sgn}(\sigma)$ be the parity of $\sigma$. For the next formula one can refer to [1].

**Theorem 2.7.** [1] The Moore determinant of $A$ is

$$\det A = \sum_{\sigma} \text{sgn}(\sigma) a_{k_{11}k_{12}} \cdots a_{k_{1j_1}k_{21}k_{22}} \cdots a_{k_{mjm}k_{m1}},$$

where the sum runs over all permutations.

From now on, we denote the Moore determinant of $A$ by $\det A$. For hyperhermitian matrices, the Moore determinant is the best one, because it has almost all the algebraic and analytic properties of the usual determinant of real symmetric and complex hyperhermitian matrices. Let us state some of them.

**Theorem 2.8.** [3] (1) The Moore determinant of any complex hermitian matrix considered as quaternionic hyperhermitian matrix is equal to its usual determinant. (2) For any hyperhermitian matrix $A$ and any matrix $C$ we have

$$\det(C^*AC) = \det(A) \cdot \det(C^*C).$$

**Definition 2.9.** [2] Let $A = (a_{ij})_{i,j=1}^{n}$ be a quaternionic hyperhermitian matrix. $A$ is called non-negative definite if for every $n$-column of quaternions $\xi = (\xi_i)_{i=1}^{n}$ one has

$$\xi^* A \xi = \sum \bar{\xi}_i a_{ij} \xi_j \geq 0,$$

where $\sum$ denotes a summation over repeated indices. Similarly, $A$ is called positive definite if the above expression is strictly positive unless $\xi = 0$.

From Proposition 2.3 Proposition 2.6 and Theorem 2.8 one can easily check:

**Proposition 2.10.** [2] Let $A$ be a non-negative(resp. positive) definite hyperhermitian matrix. Then $\det A \geq 0$(resp. $\det A > 0$).

The following theorem is a quaternionic generalization of the standard Sylvester criterion.

**Theorem 2.11.** [2] A hyperhermitian $(n \times n)$-matrix $A$ is positive definite if and only if the Moore determinants of all the left upper minors of $A$ are positive.

Let us define now the mixed discriminant of hyperhermitian matrices in analogy with the case of real symmetric matrices studied by A. D. Aleksandrov[1].

**Definition 2.12.** [2] Let $A_1, \ldots, A_n$ be hyperhermitian $(n \times n)$-matrices. Consider the homogeneous polynomial in real variables $\lambda_1, \ldots, \lambda_n$ of degree $n$ equal to

$$\det(\lambda_1 A_1 + \cdots + \lambda_n A_n).$$

The coefficient of the monomial $\lambda_1 \cdots \lambda_n$ divided by $n!$ is called the mixed discriminant of the matrices $A_1, \ldots, A_n$, and it is denoted by $\det(A_1, \ldots, A_n)$. 
Proposition 2.13. [2] The mixed discriminant is symmetric with respect to all variables, and linear with respect to each of them, i.e.

\[
det(\lambda A'_1 + \mu A''_1, A_2, \ldots, A_n) = \lambda \cdot \det(A'_1, A_2, \ldots, A_n) + \mu \cdot \det(A''_1, A_2, \ldots, A_n)
\]

for any real \(\lambda, \mu\). In particular, \(\det(A, \ldots, A) = \det A\).

By Proposition 2.3, Proposition 2.6 and Theorem 2.8, we get the following algebraic identity.

Claim 2.14. [3] For any vector \(a = (a_1, \cdots, a_n)\) we have

\[
\det((a_j u_i) + (a_j u_i)^*, \partial^2 u[n − 1]) = 2(\text{Re} a_n)\det(u_{ij}),
\]

where \((u_{ij})\) is the matrix in Theorem 1.1.

Theorem 2.15. [2] (1) The mixed discriminant of positive(resp. non-negative) definite matrices is positive(resp. non-negative). (2) Fix positive definite hyperhermitian \((n \times n)\)-matrices \(A_1, \ldots, A_{n−2}\). On the real linear space of hyperhermitian \((n \times n)\)-matrices consider the bilinear form

\[
B(X, Y) := \det(X, Y, A_1, \ldots, A_{n−2}).
\]

Then B is non-degenerate quadratic form, and its signature has one plus and the rest are minuses.

Corollary 2.16. [2][Aleksandrov inequality] Let \(A_1, \ldots, A_{n−1}\) be positive definite hyperhermitian \((n \times n)\)-matrices. Then for any hyperhermitian matrix \(X\) we have

\[
\det(A_1, \ldots, A_{n−1}, X)^2 \geq \det(A_1, \ldots, A_{n−1}, A_{n−1}) \cdot \det(A_1, \ldots, A_{n−2}, X, X),
\]

and the equality is satisfied if and only if the matrix \(X\) is proportional to \(A_{n−1}\).

For any \(\varepsilon > 0\), applying Theorem 2.14 to \((\varepsilon X + \frac{1}{\varepsilon} Y)(\varepsilon X + \frac{1}{\varepsilon} Y)^*\), we have

Corollary 2.17. For a fixed \(n \times n\) positive definite hyperhermitian matrix \(A\) and any two \((n \times n)\)-matrices \(X, Y\), we have

\[
|\det(XY^* + XY^*, A[n − 1])| \leq \varepsilon^2\det(XX^*, A[n − 1]) + \frac{1}{\varepsilon^2}\det(YY^*, A[n − 1]).
\]

In the rest of this section, we want to recall some basic definitions and facts from the theory of psh functions of quaternionic variables in [2][3].

Definition 2.18. [3] Let \(\Omega\) be a bounded domain in \(\mathbb{H}^n\). A real valued function \(u : \Omega \to \mathbb{R}\) is called quaternionic plurisubharmonic(psh) if it is upper semi-continuous and its restriction to any right quaternionic line is subharmonic. In particular, we call a \(C^2\)-smooth function \(u : \Omega \to \mathbb{R}\) to be strictly plurisubharmonic(spsh) if its restriction to any right quaternionic line is strictly harmonic (i.e., the Laplacian is strictly positive).

Definition 2.19. [3] An open bounded domain \(\Omega \subset \mathbb{H}^n\) with a smooth boundary \(\partial \Omega\) is called strictly pseudoconvex if for every point \(z_0 \in \partial \Omega\) there exists a neighborhood \(\mathcal{O}\) and a smooth strictly psh function \(h\) on \(\mathcal{O}\) such that \(\Omega \cap \mathcal{O} = h < 0, h(z_0) = 0,\) and \(\nabla h(z_0) \neq 0\).
We usually write a quaternion in the following form
\[ q = t + x \cdot i + y \cdot j + z \cdot k, \]
where \( t, x, y, z \) are real numbers, and \( i, j, k \) satisfy the usual relations
\[ i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \]
The Dirac-Weyl (or Cauchy-Riemann) operator \( \frac{\partial}{\partial q} \) is defined as follows. For any \( \mathbb{H} \)-valued function \( f \)
\[ \frac{\partial}{\partial q} f := \frac{\partial f}{\partial t} + i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}. \]
Let us also define the operator \( \frac{\partial}{\partial q} : \)
\[ \frac{\partial}{\partial q} f := \frac{\partial f}{\partial q} = \frac{\partial f}{\partial t} - \frac{\partial f}{\partial x} i - \frac{\partial f}{\partial y} j - \frac{\partial f}{\partial z} k. \]
In the case of several quaternionic variables, it is easy to check that those two operators above are commutative. For any real valued twice continuously differentiable function \( f \), the matrix \( \left( \frac{\partial^2 f}{\partial q_i \partial q_j} \right)(q) \) is hyperhermitian and we also have another way to define psh function:

**Proposition 2.20.** Let \( f \in C^2(\Omega) \) be real valued. \( f \) is quaternion psh if and only if at everywhere point \( q \in \Omega \) the matrix \( \left( \frac{\partial^2 f}{\partial q_i \partial q_j} \right)(q) \) is non-negative definite.

Last, we want to state the minimum principle to end this section.

**Theorem 2.21.** Let \( \Omega \) be a bounded open set in \( \mathbb{H}^n \). If \( u, v \) are continuous functions on \( \overline{\Omega} \) which are psh in \( \Omega \) and satisfy that
\[ \det \left( \frac{\partial^2 u}{\partial q_i \partial q_j} \right) \leq \det \left( \frac{\partial^2 v}{\partial q_i \partial q_j} \right) \text{ in } \Omega. \]
Then
\[ \min\{u(z) - v(z) | z \in \overline{\Omega}\} = \min\{u(z) - v(z) | z \in \partial \Omega\}. \]

3. \( C^1 \) estimates and partial \( C^2 \) estimates

In order to use the continuity method, it is well known that it suffices to prove priori estimates up to the second-order. We will take three steps to achieve this goal.

**Step 1** Reduce the global 1st-order priori estimates to the boundary ones.

First, let us state the main theorem in step 1.

**Theorem 3.1.** Suppose that a psh function \( u \in C^2(\overline{\Omega}) \) satisfies (1.3). Then
\[ \|u\|_{C^1} \leq C, \]
where the constant \( C \) only depends on \( \|f\|_{C^1} \) and \( \|f\|_{C^0} \).

To prove Theorem 3.1 we need the following important lemma:
Lemma 3.2. Let $D$ be a first-order differential operator of the form $D = \frac{\partial}{\partial x_i}$, where $x_i$ is one of the real coordinate axes in $\mathbb{H}^n$. Then we have
\[
\max_{\Omega} |Du| \leq \max_{\partial \Omega} |Du| + C,
\]
where $C$ is a constant depending only on $\|f\|_{C^0}$, $\|f\|_{C^1}$ and $\Omega$.

Proof. Let $L$ be the linearization of the operator $v \mapsto \log(\det(\partial^2 v)) - \log f(q, v)$ at $u$. Explicitly we can write this operator
\[
L v = n f^{-1} \det(\partial^2 v, \partial^2 u[n-1]) - f^{-1} f_u v \triangleq (L_0 - f^{-1} f_u)v.
\]
Consider the function $\psi = \pm Du + e^{\lambda|q|^2}$, with $\lambda \gg 0$ to be determined. One can easily check
\[
L_0(Du) = n f^{-1} \det(\partial^2 (Du), \partial^2 u[n-1]) = f^{-1} D(\partial^2 u) = D(\log f)
\]
and
\[
L_0(e^{\lambda|q|^2}) = n f^{-1} \det \left( (e^{\lambda|q|^2} \lambda \delta_{ij} + \lambda^2 e^{\lambda|q|^2} q_j q_i), \partial^2 u[n-1] \right)
\]
\[
\geq \lambda e^{\lambda|q|^2} n f^{-1} \det (I, \partial^2 u[n-1]) = \lambda e^{\lambda|q|^2} \sum u^{ii},
\]
where the matrix $(u^{ij})$ and $I$ denote the inverse of the matrix $(u_{ij})$ and the identity matrix, respectively. Then we have
\[
L \psi \geq \pm D(\log f) \mp f^{-1} f_u Du + e^{\lambda|q|^2} (\lambda \sum u^{ii} - f^{-1} f_u) \geq -C + e^{\lambda|q|^2} (n\lambda f^{-\frac{1}{n}} - f^{-1} f_u).
\]
One can choose a large $\lambda$ to make the last expression positive. For such a $\lambda$, by maximum principle, the function $\psi$ achieves its maximum on the boundary $\partial \Omega$. This proves Lemma 3.2.

Proof of Theorem 3.1. By Lemma 3.2, it is sufficient to estimate $\max_{\partial \Omega} |Du|$. Let $h$ be a harmonic function in $\Omega$ which extends $\varphi$. Then $u \leq h$. Besides, we also have $u \geq h$ by minimum principle and the assumption of Theorem 1.1. Hence,
\[
\max_{\partial \Omega} |Du| \leq \max_{\partial \Omega} \{ |\nabla h|, |\nabla u| \}.
\]
Thus Theorem 3.1 is proved.

Step 2 Reduce the global 2nd-order priori estimates to the boundary ones.

Note that, to get the second-order priori estimate of $u$, it is sufficient to prove an upper estimate on it. In fact, let $q_i = t + x \cdot i + y \cdot j + z \cdot k$ be one of the quaternionic coordinates. $\triangle u \geq 0$ and the upper estimates on the second derivatives of the form $D^2 u$ imply the lower estimates on them. The estimate on the mixed derivatives also can be obtained easily since
\[
2 u_{tx} = (\frac{\partial}{\partial t} + \frac{\partial}{\partial x})^2 u - u_{tt} - u_{xx}.
\]
Hence we only need to prove an upper estimate of $D^2 u$ on $\partial \Omega$ because of the following lemma:

Lemma 3.3. For a constant $C$ only depending on $\|f\|_{C^0}$, $\|f\|_{C^1}$, $\|f\|_{C^2}$ and $\Omega$, we have
\[
\max_{\Omega} D^2 u \leq \max_{\partial \Omega} D^2 u + C.
\]
Proof. From Lemma 7.5 in [3] or Theorem 1.1.17 in [2], we know
\[ L_0(D^2u) \geq D^2(\log f). \]

This implies
\[
L(D^2u + e^{\lambda|q|^2}) \geq D^2(\log f) - f^{-1}f_u(D^2u) + \lambda e^{\lambda|q|^2} \sum u^{ii} - f^{-1}f_u e^{\lambda|q|^2} \\
= -f^{-2}|Df|^2 + f^{-1}(D(f_qDq) + D(f_u)Du) + e^{\lambda|q|^2}(\lambda \sum u^{ii} - f^{-1}f_u) \\
\geq -C + e^{\lambda|q|^2}(n\lambda f^{-\frac{1}{n}} - f^{-1}f_u) > 0, \text{ for a large } \lambda.
\]

By maximum principle, we complete the proof. \(\square\)

Step 3 Prove the boundary estimates for second derivatives.

In this step, we need to derive priori estimates for three kinds of second derivatives: pure tangential derivatives, mix derivatives and pure normal derivatives. Since the last ones are more complicated, we treat them in the next section. From now on, we will denote the quaternionic units as follows:
\[ e_0 = 1, \quad e_1 = i, \quad e_2 = j, \quad e_3 = k. \]

Fix an arbitrary point \( P \in \partial \Omega \), we can choose such a coordinate system \((q_1, \ldots, q_n)\) near this point that the inner normal to \( \partial \Omega \) at \( P \) coincides with the axis \( x_n^0 \). We can also assume \( P \) to be the origin. For the sake of convenience, we set
\[
t_1 = x_1^0, t_2 = x_2^1, t_3 = x_3^2, \ldots, t_{4(n-1)} = x_{n-1}^3, t_{4n-3} = x_n^1, t_{4n-2} = x_n^2, t_{4n-1} = x_n^3, t_{4n} = x_n^0,
\]
and
\[
t = (t', t_{4n}), \quad t' = (t_1, \ldots, t_\alpha, \ldots, t_\beta, \ldots, t_{4n-1}), \quad \alpha, \beta = 1, \ldots, 4n - 1.
\]

Part I \( |u_{t\alpha t\beta}|(0) < C \sim \|u\|_{C^1}, \|u\|_{C^1}, \|u\|_{C^2}. \)

As in [7][11], we can write \( u - \bar{u} = \tau \sigma \), where \( \tau \) is a smooth function and \( \sigma \) is the defining function of \( \Omega \) with \( |\nabla \sigma| = 1 \). From
\[
(3.1) \quad u_{t_\alpha t_\beta}(0) = \bar{u}_{t_\alpha t_\beta}(0) + \tau(0)\sigma_{t_\alpha t_\beta}(0),
\]
\[
(3.2) \quad (u - \bar{u})_{x_n^0} = \tau_{x_n^0} \sigma + \tau \sigma_{x_n^0}, \quad (u - \bar{u})_{x_n^0}(0) = -\tau(0),
\]
we have \( |u_{t\alpha t\beta}| \sim \|u\|_{C^1}, \|u\|_{C^1}, \|u\|_{C^2}, \Omega. \)

Part II \( |u_{t\alpha x_n^0}|(0) < C \sim \|u\|_{C^1}, \|u\|_{C^1}, \|u\|_{C^2}, \|u\|_{C^3}, \|f\|_{C^0}, \|f\|_{C^1}, \Omega. \)

Firstly, consider
\[
(3.3) \quad h = \pm T(u - \bar{u}) + \sum_{l=1}^{3}(u_{x_n^l} - \bar{u}_{x_n^l})^2, \quad T = \frac{\partial}{\partial t_\alpha} - \frac{\sigma_{t_\alpha}}{\sigma_{x_n^0}} \frac{\partial}{\partial x_n^0}.
\]
By straightforward calculations, we have
\[
|L(T(u - w))| \leq C + Cnf^{-1} \det(I, \partial^2 u[n - 1]) \\
+ nf^{-1} \sum_{l=1}^{3} \det \left( ((u - w)_{x_n}^l (u - w)_{x_n^i,j}) , \partial^2 u[n - 1] \right)
\]
and
\[
L((u_{x_n} - u_{x_n})^2) \geq -C - Cnf^{-1} \det(I, \partial^2 u[n - 1]) \\
+ 2nf^{-1} \det \left( ((u - w)_{x_n}^l (u - w)_{x_n^i,j}) , \partial^2 u[n - 1] \right).
\]
So we obtain
\[
Lh \geq -C - Cnf^{-1} \det(I, \partial^2 u[n - 1]) \\
+ nf^{-1} \sum_{l=1}^{3} \det \left( ((u - w)_{x_n}^l (u - w)_{x_n^i,j}) , \partial^2 u[n - 1] \right).
\]
But the third summand is non-negative. Hence we get
\[
Lh \geq -C - Cnf^{-1} \det(I, \partial^2 u[n - 1]),
\]
where the constant C only depends on \(\|u\|_{C^1}, \|u\|_{C^1}, \|u\|_{C^2}, \|u\|_{C^2}, \|f\|_{C^0}, \|f\|_{C^1}\) and \(\Omega\).

Secondly, set \(\tilde{w} = h - B|q|^2\) with \(B\) to be determined. It is clear that
\[
L\tilde{w} = Lh - BL(|q|^2) \geq -C - (B + C)nf^{-1} \det(I, \partial^2 u[n - 1]) - BC,
\]
where the value of the constant \(C\) might be different from the previous one.

Assuming \(\delta < 1\), we denote \(\Omega \cap B_\delta(P)\) by \(\Omega_\delta\). On \(\partial \Omega \cap B_\delta(P)\), we have \(T(u - \bar{w}) = T(\tau \sigma) \equiv 0\). This implies
\[
|\langle u - \bar{w}, x_k^l \rangle| = | - (u - \bar{w})_{x_n}^l (\sigma_{x_n}^{1})^{-1} \sigma_{x_k}^l | \leq C|q|,
\]
and then we get \(h \leq C|q|^2\). On \(\Omega \cap B_\delta(P)\), it is clear \(|h| \leq C\).

To sum up, we obtain
\[
|h| \leq C|q|^2, \quad \text{on } \partial \Omega \cap B_\delta(P); \\
|h| \leq C, \quad \text{on } \Omega \cap B_\delta(P).
\]

Last, suppose \(d = d(q)\) to be the distance from point \(q\) to the boundary \(\partial \Omega\). Let us take \(w = \tilde{w} + A(u - u - td + \frac{N}{2}d^2)\) as our auxiliary function. Next, we will choose \(t, N, \delta\) to reach \(Lw > 0\). By direct calculation and the properties of the mixed discriminant, we can deal with the items in the bracket above as follows:
\[
L(u - w) \geq \varepsilon_0 nf^{-1} \det(I, \partial^2 u[n - 1]) - n - f^{-1} f_u(u - u) \\
\geq \varepsilon_0 \sum u^{\bar{i}} - C,
\]
where \(\varepsilon_0\) is such a constant that \(u\) satisfies \((u_{\bar{i}}) \geq \varepsilon_0 I\).
\[
tL(d) = tnf^{-1} \det(\partial^2 d, \partial^2 u[n - 1]) - tf^{-1} f Ud \\
\leq tC \sum u^{\bar{i}} + tCd \leq Ct(d + \sum u^{\bar{i}}).
\[
\frac{N}{2} L(d^2) = Nn f^{-1} \det(d(d\gamma) + (d_i d_j, \partial^2 u[n - 1])) - \frac{N}{2} f^{-1} f_u d^2 \\
\geq -NdC \sum u^{\tilde{\sigma}} + N\lambda^{-1}_k - \frac{N}{2} f^{-1} f_u d^2 \\
\geq -NdC \sum u^{\tilde{\sigma}} + \frac{N}{\lambda_n} - 2f^{-1} f_u N d^2,
\]

where \(\lambda_k\) denote the eigenvalues of the matrix \((u_{ij})\) and \(\lambda_1 \leq \cdots \leq \lambda_n\).

Denoting \(v = u - u - td + \frac{N}{2} d^2\), those inequalities above imply

\[
Lv \geq - C - C(t + N d^2) d + [\varepsilon_0 - C(t + N d)] \sum u^{\tilde{\sigma}} + \frac{N}{\lambda_n}.
\]

Note that

\[
\frac{\varepsilon_0}{4} \sum u^{\tilde{\sigma}} + \frac{N}{\lambda_n} \geq \frac{n\varepsilon_0}{4} (N\lambda_1^{-1} \cdots \lambda_n^{-1})^{1\over n} = \frac{n\varepsilon_0}{4 f^{-1} N} N^{1\over n} = CN^{1\over n}.
\]

So we can choose \(t, d\) so small that

\[
Lv \geq - C - {\varepsilon_0 \over 2} d + {\varepsilon_0 \over 4} \sum u^{\tilde{\sigma}} + CN^{1\over n}.
\]

Now, we need such a large \(N\) that

\[
(3.7) \quad Lv \geq \frac{\varepsilon_0}{4} (\sum u^{\tilde{\sigma}} + 1).
\]

On one hand, since

\[
\begin{align*}
\begin{cases}
  v = 0, & \text{on } \partial\Omega \cap B_\delta(P); \\
  v \leq -td + \frac{N}{2} d^2 < 0, & \text{on } \Omega \cap \partial B_\delta(P),
\end{cases}
\end{align*}
\]

we can choose \(B\) so large that

\[
(3.8) \quad \begin{cases}
  w \leq C|q|^2 - B|q|^2 < 0, & \text{on } \partial\Omega \cap B_\delta(P); \\
  w \leq C - B|q|^2 < C - B\delta^2 < 0, & \text{on } \Omega \cap \partial B_\delta(P).
\end{cases}
\]

On the other hand, we can also choose such a constant \(A \gg B\) that

\[
(3.9) \quad Lw \geq -C - BC + \frac{A\varepsilon_0}{4} + \left(\frac{A\varepsilon_0}{4} - B - C\right) \sum u^{\tilde{\sigma}} > 0, \text{ in } \Omega_\delta.
\]

By maximum principle, we get

\[
w \leq 0, \text{ on } \Omega_\delta, \quad w(0) = 0.
\]

So

\[
|(T(u - u))_{x^n}|(0) \leq |Av_{x^n}|(0) \leq C.
\]

## 4. The Second Order Normal Derivatives

In this section, our goal is

\[
(4.1) \quad |u_{x^n x^n}|(0) < C \sim \|u\|_{C^1}, \|u\|_{C^1}, \|u\|_{C^2}, \|u\|_{C^3}, \|f\|_{C^0}, \|f\|_{C^1}, \Omega
\]
To reach this goal, in \( \mathbb{C}^n \) case, B. Guan\[11\] used
\[
\widetilde{h} = (u_{y_n} - \varphi_{y_n})^2 - \widetilde{\Phi}
\]
\[
\widetilde{w} = -Av + B|q|^2 - \widetilde{h},
\]
where \( v \) is same as the one in section 3. By choosing an appropriate vector \( \zeta = (\zeta_1, \cdots, \zeta_n) \) and the maximum principle, he get (4.1). However, in \( \mathbb{H}^n \), we need to modify \( \widetilde{h} \) as
\[
\begin{align*}
\widetilde{h}' &= \sum_{k=1}^n \sum_{l=1}^3 (u_{x_k} - u_{x_k}^l)^2 + \frac{1}{2} \sum (u - u)_k \sigma_{k} \xi_i \sigma_{ij} \xi_j - \sum \xi_i u_{ij} \xi_j + u_{ij}(0),
\end{align*}
\]
Remark 4.1. In \( \mathbb{C}^n \) case, our auxiliary function \( h' \) is reduced to
\[
(4.2) \quad h' = \sum_{k=1}^n (u_{y_k} - u_{y_k})^2 + \sum (u - u)_k \sigma_{k} \xi_i \sigma_{ij} \xi_j - \sum \xi_i u_{ij} \xi_j + u_{ij}(0),
\]
where \( z_k = x_k + iy_k \).

Now, let’s start to prove (4.1). First, we only need to prove
\[
0 \leq u_{n\pi}(0) = u_{x_n^2, x_n}(0) + \sum_{1 \leq i \leq 3} u_{x_i x_i}(0) \leq C.
\]
To prove this, we need the lemma below:

Lemma 4.2. If \( \sum u_{ab}^a \xi_a \xi_b \geq C|\xi|^2 \) for any \( \xi \in \mathbb{H}^{n-1} \) and \( 1 \leq a, b \leq n - 1 \). Then
\[
0 \leq u_{n\pi}(0) \leq C.
\]

Proof. By definition, it is clear
\[
\det(u_{ij})(0) = \det(u_{ab})u_{n\pi}(0) + R = f,
\]
where \( R \) denotes the remainder terms and \( u_{ab} \) denotes \( \frac{\partial^2 u}{\partial q_a \partial q_b} \). Then
\[
u_{n\pi}(0) = \frac{f - R}{\det(u_{ab})} \leq C.
\]

By this lemma, it suffices to prove
\[
m_0 \triangleq \min_{P \in \partial \Omega} \min_{\xi \in \mathbb{T}^b \cap \partial \Omega} \xi_i u_{ij} \xi_j \geq C.
\]
Assume \( m_0 \) attains at the origin \( P \in \partial \Omega \) when \( \xi = (1, 0, \cdots, 0) \). Then we only need to prove
\[
m_0 = u_{ij}(0) \geq C > 0.
\]

Since \( u - u = \tau \sigma \), where \( \tau \) is a smooth function and \( \sigma \) is the defining function of \( \Omega \) with \( |\nabla \sigma| = 1 \), we get some basic formulas as follows:
\[
(4.3) \quad (u - u)_i = \tau_i \sigma + \tau \sigma_i, \quad \sum (u - u)_i \sigma_i = \tau_i \sigma \sigma_i + \tau \sigma_i \sigma_i;
\]
\[
(4.4) \quad (u - u)_{ij} = \tau_{ij} \sigma + \sigma_{ij} \tau_i + \tau_i \sigma_i + \tau \sigma_i \sigma_{ij}.
\]
From (4.3) and (4.4), we have

\[(4.5) \quad \tau = \sum_k \frac{(u - u)_{k}\sigma_k + \sigma_k(u - u)_{k}}{2|\nabla \sigma|^2} \quad \text{on} \quad \partial \Omega,\]

and

\[(4.6) \quad \xi_i (u - u)_{ij} = \tau \xi_i \sigma_{ij} \quad \text{for any} \quad q \in \partial \Omega, \xi \in T^q_{\partial} \partial \Omega.\]

So

\[u_{\perp}(0) = u_{\perp}(0) - (u - u)_{x_0} \sigma_{1} \quad \text{on} \quad \partial \Omega, \quad \xi \in T_{\perp} \partial \Omega.\]

We can assume \(u_{\perp}(0) < \frac{1}{2} u_{\perp}(0), \) otherwise, \(m_0 \geq C > 0\) can be obtained immediately.

Using this condition, we have \((u - u)_{x_0} \sigma_{1} = u_{\perp}(0) - u_{\perp}(0) \geq \frac{1}{2} u_{\perp}(0).\) Then

\[(4.7) \quad \sigma_{1} \geq C > 0.\]

Now, let us consider \(w' = h' - B|q|^2 + Av, \) where

\[(4.8) \quad h' = \sum_{k=1}^{n} \sum_{l=1}^{3} (u_{x_i}^l - u_{x_k})^2 + \Phi,\]

\[(4.9) \quad \Phi = \frac{1}{2} \sum_{i,j} ([u - u]_{k}\sigma_k + \sigma_k(u - u)_{k})_{ij} \xi_i \sigma_{ij} \xi_j - \sum \xi_i u_{ij} \xi_j + u_{\perp}(0).\]

Thanks to Part II, we only need to prove

(J1) \(Lh' \geq -C(1 + \sum_i u_i^2), \) in \( \Omega_{\delta} ; \)

(J2) \(h' \leq C|q|^2, \) on \( \partial \Omega \cap B_{\delta}(P) ; \)

(J3) \(|h| \leq C, \) on \( \Omega \cap B_{\delta}(P). \)

In fact, as in Part II, by these inequalities and maximum principle we can get \(\Phi_{\perp,0}(0) \leq -A u_{\perp,0}(0).\) Equivalently, there exists a constant \(C > 0\) satisfies

\[u_{\perp,0}(0) \sigma_{1}(0) \leq A(u - u)_{x_0} \sigma_{1}(0) + A t \sigma_{1,0}(0) \leq C, \quad u_{\perp,0}(0) \leq C,\]

here we have chosen

\[\xi_i = \begin{cases} \frac{u_i}{\omega}, & \text{if} \quad i = 1; \\ 0, & \text{if} \quad 2 \leq i \leq n - 1; \\ \frac{u_i}{\omega}, & \text{if} \quad i = n, \end{cases}\]

where \(\omega = |\xi|_{H^n}.\)

Since (J3) is obviously satisfied, we will check (J1) and (J2). By (4.6), we have

\[(4.10) \quad \Phi = - \sum_{i,j} \xi_i u_{ij} \xi_j + \phi_{\perp}(0) \leq 0, \quad \text{on} \quad \partial \Omega \cap B_{\delta}(P).\]

Therefore, (J2) follows from (3.5).
In the rest of this section, we derive (J1) to finish Step 3. For simplicity, we set \( \mu = \sum \xi_i \sigma_i \bar{\xi}_i, \mu_k^0 = \sigma_x \mu, \mu_k^l = \sigma_{x^l} \mu, \) we get

\[
L \Phi = \frac{1}{2} \sum L((u - u)_k \sigma_k \mu + \sigma_k (u - u)_k \mu) - L(\sum \xi_i \bar{u}_i \bar{\xi}_i)
\]
\[
= \frac{1}{2} \sum L((u_k \sigma_k + \sigma_k u_k) \mu) - \frac{1}{2} \sum L((u_k \sigma_k + \sigma_k u_k) \mu) - L(\sum \xi_i \bar{u}_i \bar{\xi}_i)
\]
\[
= E + F + G,
\]

where

\[
E = \frac{1}{2} \sum L((u_k \sigma_k + \sigma_k u_k) \mu),
\]
\[
F = -\frac{1}{2} \sum L((u_k \sigma_k + \sigma_k u_k) \mu),
\]
\[
G = L(\sum \xi_i \bar{u}_i \bar{\xi}_i).
\]

It is clear

\[
(4.11) \quad E + G \geq -C(1 + \sum u^5).
\]

As for \( F \), we can write it as

\[
F = -\sum_{k=1}^{n} \sum_{l=1}^{3} L(u_{x^0_k} \mu_k^0 + \mu^l_k u_{x^l_k})
\]
\[
\geq -C - nf^{-1} \sum_{k=1}^{n} \det \left( (u_{x^0_k} \bar{\xi}_k^0 + \mu_k^0 \bar{\xi}_k^0), \partial^2 u[n-1] \right)
\]
\[
- nf^{-1} \sum_{k=1}^{n} \det \left( (u_{x^0_k} \xi_k^0 + \mu_k^0 u_{x^0_k}), \partial^2 u[n-1] \right)
\]
\[
- nf^{-1} \sum_{k=1}^{n} \sum_{l=1}^{3} \det \left( (u_{x^0_k} \sigma_k^l + \mu_k^l u_{x^0_k}), \partial^2 u[n-1] \right)
\]
\[
- nf^{-1} \sum_{k=1}^{n} \sum_{l=1}^{3} \det \left( (u_{x^0_k} \sigma_k^l + \mu_k^l u_{x^0_k}), \partial^2 u[n-1] \right).
\]

We denote the last four terms in the right of the inequality above by \( F_1, F_2, F_3, F_4 \) one by one. Thus it is easy to check \( F_1 + F_3 \geq -C(1 + \sum u^5) \). For \( \forall \varepsilon > 0 \), by Claim 2.14 and Corollary 2.17, we get

\[
F_2 \geq -nf^{-1} \sum_{k=1}^{n} \det \left( (\mu_k^0 u_{x^0_k}) + (\mu_k^0 u_{x^0_k})^*, \partial^2 u[n-1] \right)
\]
\[
+ nf^{-1} \sum_{k=1}^{n} \sum_{l=1}^{3} \det \left( (\mu_k^0 \xi_l u_{x^l_k}) + (\mu_k^0 \xi_l u_{x^l_k})^*, \partial^2 u[n-1] \right)
\]
\[
= -2 \sum_{k=1}^{n} \text{Re}(\mu_k^0) + nf^{-1} \sum_{k=1}^{n} \sum_{l=1}^{3} \det \left( (\mu_k^0 \xi_l u_{x^l_k}) + (\mu_k^0 \xi_l u_{x^l_k})^*, \partial^2 u[n-1] \right),
\]
\[ F_2 + F_1 \geq -C - nf^{-1} \sum_{k=1}^{n} \sum_{l=1}^{3} \det \left( (\mu_{k}^{l} u_{x_{i}^{l}}) + (\mu_{k}^{l} u_{x_{i}^{l}})^*, \partial^2 u[n-1] \right) \]

\[ + nf^{-1} \sum_{k=1}^{n} \sum_{l=1}^{3} \det \left( (\mu_{k}^{l} e_{i} u_{x_{i}^{l}}) + (\mu_{k}^{l} e_{i} u_{x_{i}^{l}})^*, \partial^2 u[n-1] \right) \]

\[ \geq -C(1 + \sum_{i} u_{\Omega}) - (2 - \varepsilon)nf^{-1} \sum_{k=1}^{n} \sum_{l=1}^{3} \det((u_{x_{i}^{l}} u_{x_{j}^{l}}^{\Omega}), \partial^2 u[n-1]), \]

\[ L((u_{x_{i}^{l}} - u_{x_{i}^{l}})^{2}) \geq -C + 2(u_{x_{i}^{l}} - u_{x_{i}^{l}})nf^{-1} \det((u_{x_{i}^{l}} - u_{x_{i}^{l}}), \partial^2 u[n-1]) \]

\[ + 2nf^{-1} \det((u_{x_{i}^{l}} - u_{x_{i}^{l}}), (u_{x_{i}^{l}} - u_{x_{i}^{l}}), \partial^2 u[n-1]) \]

\[ \geq -C(1 + \sum_{i} u_{\Omega}) + 2nf^{-1} \det((u_{x_{i}^{l}} u_{x_{j}^{l}}), \partial^2 u[n-1]) \]

\[ + 2nf^{-1} \det((u_{x_{i}^{l}} u_{x_{j}^{l}}) + u_{x_{i}^{l}} u_{x_{j}^{l}}), \partial^2 u[n-1]) \]

\[ \geq -C(1 + \sum_{i} u_{\Omega}) + (2 - \varepsilon)nf^{-1} \det((u_{x_{i}^{l}} u_{x_{j}^{l}}), \partial^2 u[n-1]). \]

Hence, by the arguments in \[8\] and standard elliptic theory, we complete the proof of Theorem \[1.1\].

5. Construction of a subsolution

In this section, we will construct a subsolution to \[1.5\] in a strictly pseudoconvex domain by the method in \[7\]. To prove Proposition \[1.2\] we first show that under the assumption of Proposition \[1.2\] if \( u \leq m \), then there exists a constant \( C=C(m) \) satisfies

\[ f(q, u, p) \leq C(1 + |p|^n) \quad \text{for} \quad z \in \Omega. \]

Indeed, if \( \omega \leq m \), then

\[ f(q, \omega, \eta) = f(q, 0, 0) + \int_{0}^{1} \frac{d}{dt} f(q, t\omega, t\eta) dt \]

\[ \leq C + m \int_{0}^{1} f_{u}(q, t\omega, t\eta) dt + \sum_{i=1}^{4n} |p_{i}| \int_{0}^{1} |f_{p_{i}}(q, t\omega, t\eta)| dt \]

\[ \leq C(1 + \max_{0 \leq i \leq 1} f^{1 - \frac{1}{n}}(q, t\omega, t\eta)(C + |p|)). \]

Setting \( \Lambda = \max_{\omega \leq m} f(q, \omega, \eta) \), we get

\[ f(q, u, p) \leq \Lambda \leq C(1 + |p|^n). \]

Now, we define

\[ \bar{u} = \varphi + s(e^{kr} - 1), \quad k, s > 0, \]

where \( r \) is a strictly psh defining function for \( \Omega \). Extending \( \varphi \) as a psh \( C^\infty \) function in \( \overline{\Omega} \), we get

\[ \det(\bar{u}_{\bar{r}}) \geq (skr^{k})^{n} \det(r_{\bar{r}} + kr_{\bar{r}_{i}}). \]

Let \( \alpha > 0 \) be such that \( (r_{\bar{r}}) \geq \alpha I \). Then

\[ \det(r_{\bar{r}} + kr_{\bar{r}_{i}}) \geq \alpha^{n-1}(\alpha + k|\nabla r|^2). \]
To check this at a point $z^0 \in \Omega$ choose coordinate such that $r_i(z^0) = 0$ for $i < n$, then $|\nabla r(z^0)| = |r_n(z^0)|$ and the above inequality follows. Now we have
\[
\det(u_{ij}) \geq (sk\alpha e^{kr})^n(1 + \frac{k}{\alpha} |\nabla r|^2),
\]
and
\[
|\nabla u| \leq \max |\nabla \varphi| + sk\alpha e^{kr} |\nabla r|,
|\nabla u|^n \leq C_1 + C_2 (sk\alpha e^{kr})^n |\nabla r|^n.
\]
Choose $k$ so large that
\[
C(m) C_2 |\nabla r|^n \leq k\alpha^{n-1} |\nabla r|^2,
\]
then we can choose $s$ so that
\[
f(q, u, \nabla u) \leq C(m)(1 + |\nabla u|^n) \leq C(m)(1 + C_1 + C_2 (sk\alpha e^{kr})^n |\nabla r|^n) \leq C(m)(1 + C_1) + (sk\alpha e^{kr})^n \frac{k}{\alpha} |\nabla r|^2 \leq (sk\alpha e^{kr})^n (1 + \frac{k}{\alpha} |\nabla r|^2) \leq \det(u_{ij}),
\]
where $\Lambda = \max \varphi$. Hence we prove Proposition 1.2.

6. Dependence of $f$ on the gradient of $u$

In this section, we consider
\[
det(u_{ij}) = f(q, u, \nabla u) \in C^\infty(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^4), \quad \text{in} \ \Omega
\]
\[
u|_{\partial \Omega} = \varphi \in C^\infty(\partial \Omega).
\]
We are going to establish the following result.

Theorem 6.1. Let $\Omega$ be any bounded domain with a smooth boundary and $u \in C^\infty(\Omega)$ be a subsolution satisfying (1.5). About $f$, elementally, we assume $f > 0$, $f_u \geq 0$. Further, suppose that there exists a constant $C$ such that
\[
|\partial_q f|, |f_u(q, u, \nabla u)|, |f_{p_i}(q, u, \nabla u)| \leq C f^{1 - \frac{1}{m}},
\]
where $q = (q_1, \ldots, q_n) = (x_1^0, x_1^1, x_1^2, \ldots, x_n^0, x_n^1, x_n^2)$, $p_i := u_{t_i}$ and $\partial_q$ denotes differentiation with respect to $t_i$. Then there exists an unique psh function $u \in C^\infty(\overline{\Omega})$ solving (6.1).

The proof of this theorem can also be divided into three steps as in section 3. The assumptions (6.2) are only used in Step 1, while Step 3 can be proceeding as in section 3 and section 4. Precisely, the differences are merely the proofs of Lemma 3.2 and Lemma 3.3. So we will only prove these two lemmas under our new restrictions in this section.

Lemma 6.2. Suppose that a psh function $u \in C^\infty(\overline{\Omega})$ satisfies (6.1). Let $D$ be a first-order differential operator of the form $D = \frac{\partial}{\partial x_i}$, where $x_i$ is one of the real coordinate axes in $\mathbb{H}^n$. Then we have
\[
\max_{\Omega} |Du| \leq \max_{\partial \Omega} |Du| + C,
\]
where $C$ is a constant depending only on $\|f\|_{C^0}$, $\|f\|_{C^1}$ and $\Omega$. 

Proof. Here $L$ is the linearization of the operator $v \mapsto \log(\det(\partial^2 v)) - \log f(q, v, \nabla v)$ at $u$. Explicitly we can write $L$ as

$$Lv = n f^{-1} \det(\partial^2 v, \partial^2 u[n - 1]) - f^{-1}(f_u v + f_p v_i)$$

(6.4)

$$\triangleq \left(L_0 - f^{-1}(f_u + f_p \partial_{i})\right) v,$$

where $\nabla u = (u_1, \cdots, u_{4n}) = (p_1, \cdots, p_{4n})$ and $t_i$ are defined as in Theorem 6.1. Consider the function $\psi = \pm Du + e^{\kappa |q|^2}$, with $\kappa \gg 0$ to be determined. One can easily check

$$L\psi \geq -C f^{-\frac{1}{2}} + e^{\kappa |q|^2} \kappa^2 n f^{-1} \det((q_j \bar{q}_i), \partial^2 u[n - 1])$$

$$+ e^{\kappa |q|^2} \kappa \left(\sum u^{\bar{i}} - 2 f^{-1} f_p t_i\right) - e^{\kappa |q|^2} f^{-1} f_u$$

(6.5)

$$\geq -C f^{-\frac{1}{2}} - e^{\kappa |q|^2} f^{-\frac{1}{2}} + e^{\kappa |q|^2} \kappa^2 n f^{-1} \det((q_j \bar{q}_i), \partial^2 u[n - 1])$$

$$+ e^{\kappa |q|^2} \kappa \left(\sum u^{\bar{i}} - \tilde{C} f^{-\frac{1}{2}}\right),$$

where $C, \tilde{C}$ are constants. Suppose that $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of $(u^{\bar{i}})$. Now we continue to calculate $L\psi$ in two cases.

Case (1). $\lambda_1 \leq (2\tilde{C})^{1-n} f^{-\frac{1}{2}}$.

In this case, $\lambda_n \geq 2\tilde{C} f^{-\frac{1}{2}}$ and so $\sum u^{\bar{i}} - \tilde{C} f^{-\frac{1}{2}} \geq \tilde{C} f^{-\frac{1}{2}}$. Consequently,

$$L\psi \geq -C f^{-\frac{1}{2}} - e^{\kappa |q|^2} f^{-\frac{1}{2}} + \kappa e^{\kappa |q|^2} \tilde{C} f^{-\frac{1}{2}} \geq 0 \quad \text{for a large } \kappa.$$

Case (2). $\lambda_1 \geq (2\tilde{C})^{1-n} f^{-\frac{1}{2}}$.

In this case, we get

$$L\psi \geq -C f^{-\frac{1}{2}} - e^{\kappa |q|^2} f^{-\frac{1}{2}} + \kappa e^{\kappa |q|^2} f^{-\frac{1}{2}} (\kappa (2\tilde{C})^{1-n} |q|^2 - \tilde{C}) \geq 0 \quad \text{for a large } \kappa.$$

Here we have choose coordinates such that $|q| > c_0 > 0$, for translating coordinates preserves $|Du|$. Hence, in either case, for $\kappa$ large we have $L\psi \geq 0$ and the maximum principle yields (6.5). \hfill \Box

In the rest of this paper, we will prove the last lemma below.

**Lemma 6.3.** Let $u \in C^{\infty}(\overline{\Omega})$ be a a psh function solving (6.1). For a constant $C$ only depending on $\|f\|_{C^0}$, $\|f\|_{C^1}$, $\|f\|_{C^2}$ and $\Omega$, we have

$$\max_{\partial \Omega} D^2 u \leq \max_{\partial \Omega} D^2 u + C.$$  

(6.6)

To prove the lemma above, we need the following lemma.

**Lemma 6.4.** Let $u \in C^{\infty}(\overline{\Omega})$ be a a psh function solving (6.1). Then

$$\max_{\overline{\Omega}} \sum_{|\xi| = 1} \xi_i u_{\bar{\eta} \xi_j} \leq C.$$  

(6.7)

**Remark 6.5.** Generally speaking, in $\mathbb{H}^n$, we don’t have

$$u_{\bar{\eta} \xi_k} = u_{k \bar{\eta} \xi_j}$$

and we can’t represent $q = (x^0, x^1, x^2, x^3)$ by $(q, \bar{q})$. Therefore the proof in [7] need to be modified.
Proof. Let $M$ be defined by

$$M = \max_{\Omega} \sum_{ij} \xi_i u_{ij} \xi_j \exp(A \sum_k |u_k|^2),$$

where $A$ is a large constant to be determined. Let $M$ attains its maximum at $(q^0, \xi^0)$. Assume $q^0 \in \Omega$ (Otherwise, (6.7) follows from the boundary estimates easily). We can choose coordinates such that $\xi^0 = (1, 0, \cdots, 0)$ and $u_{ij}(q^0) = 0$ for $i \neq j$. Then

$$\psi = \log u_{11} + A \sum_k |u_k|^2 = \log (u_{1t1} + u_{t2t2} + u_{t3t3} + u_{t4t4}) + A \sum_{j=1}^{4n} u_{tj}^2$$

achieves its maximum at $q^0$. Hence at this point, we have

$$\frac{u_{1T}}{u_{11}} = -2A \sum_k u_{tk} u_{tk},$$

and

$$L_0 \psi = n f^{-1} \det((\Delta_1 u)_{ij}, \partial^2 u[n-1]) + A \sum_{k=1}^{4n} n f^{-1} \det((u_{tk}^2)_{ij}, \partial^2 u[n-1]),$$

where

$$D_1 u := u_{t1} + u_{t2} + u_{t3} + u_{t4}, \quad F := \log f,$$

$I := n f^{-1} \det((\Delta_1 u)_{ij}, \partial^2 u[n-1])$

$$= n f^{-1} \det \left(\frac{(\Delta_1 u)^2}{\Delta_1 u}, \partial^2 u[n-1] \right) - n f^{-1} \det \left(\frac{(\Delta_1 u)^2}{\Delta_1 u}, \partial^2 u[n-1] \right)$$

$$= \frac{\Delta_1 F}{u_{11}} + \frac{1}{u_{11}} (n f^{-1} \det(\partial^2 (D_1 u), \partial^2 u[n-1]))^2$$

$$- \frac{1}{u_{11}} (n(n-1) f^{-1} \det(\partial^2 (D_1 u)[2], \partial^2 u[n-2])) - \sum_{i=1}^n \frac{|(\Delta_1 u)|^2}{(u_{Tj})^2 u_{Tj}}$$

$$\geq \frac{\Delta_1 F}{u_{11}}.$$

$$\Delta_1 F = \frac{\Delta_1 F}{u_{11}}.$$
and

\[
II = A \sum_{k=1}^{4n} n f^{-1} \det((u_{ik}^2)_{ij}, \partial^2 u[n - 1]) = A \sum_{k=1}^{4n} u_{ik}^2 u_{ik} + 2A \sum_{k=1}^{4n} \sum_{i=1}^{n} \frac{u_{ik} u_{ik}}{u_{ij}^2} \\
= 2A \sum_{k=1}^{4n} u_{ik} F_{ik} + 2A \sum_{i=1}^{n} u_{ij} + 2A \sum_{j=1}^{n} u_{ij}^2 - 2A \sum_{j=1}^{n} u_{ij}^2.
\]

(6.15)

From (6.11) we get

\[
I \geq \frac{1}{u_{1T}} \left( \sum_{j=1}^{4n} F_{p_j p_i} u_{i j} u_{ij} + \sum_{p_j} F_{p_j} u_{1T j} + F_{1j} |u_{1j}|^2 + F_{u|u_1|} - C - C \sum_{j=1}^{4n} |u_{ij}| \right) \\
\geq -\frac{1}{u_{1T}} C - C - C u_{1T} - C \sum_{j=1}^{4n} |u_{ij}| - C \sum_{j=1}^{4n} |u_{ij}|^2 - 2A \sum_{j=1}^{4n} u_{ij} F_{ik}.
\]

(6.16)

Thus for a large \( A \), we have

\[
0 \geq L_0 \psi = I + II \geq 2A \sum_{i=1}^{n} u_{ij} - \frac{1}{u_{1T}} C - AC - Cu_{1T}
\]

(6.17)

and it follows that \( u_{1T}(q^0) \) bounded, hence \( M \) is bounded and therefore the \( (u_{ij}) \) are bounded on \( \Omega \).

Proof of Lemma [6.3] Consider \( \Psi = D^2 u + A(Du)^2 + e^{\kappa_0 |q|^2} \) with \( \kappa \gg 0 \) to be determined. By Lemma 6.4, we have

\[
\sum_{i,j} \zeta_i u_{ij} \geq c_0 |\zeta|^2 \quad \text{for} \quad c_0 > 0.
\]

(6.18)
Then, by direct calculations, one can get
\[
L \Psi \geq -C f^{-\frac{1}{n}} - C \sum |Du_j|^2 + 2A \sum |Du_i|^2 u^i - AC
+ e^{\kappa_0 |q|^2} \left( \kappa_0 \sum u^i + \kappa_0^2 \sum u^i |q_i|^2 - C f^{-\frac{1}{n}} \kappa_0 \right)
\geq -C - C \sum |Du_j|^2 + 2A_0 \sum |Du_i|^2 - AC
+ e^{\kappa_0 |q|^2} \left( n f^{-\frac{1}{2}} \kappa_0 + |q| (c_0 \kappa_0^2 |q| - 2 \kappa_0 f^{-\frac{1}{n}}) \right)
\geq 0 \quad \text{for a large } \lambda.
\] (6.19)

Last, by the maximum principle, (6.6) follows. \qed

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