LOCALIZED STANDARD VERSUS REDUCED FORMULA AND GENUS ONE LOCAL GROMOV-WITTEN INVARIANTS

Xiaowen Hu

Abstract

For local Calabi-Yau manifolds which are total spaces of vector bundle over balloon manifolds, we propose a formal definition of reduced Genus one Grømov-Witten invariants, by assigning contributions from the refined decorated rooted trees. We show that this definition satisfies a localized version of the standard versus reduced formula, whose global version in the compact cases is due to A. Zinger. As an application we prove the conjecture in a previous article on the genus one Gromov-Witten invariants of local Calabi-Yau manifolds which are total spaces of concave splitting vector bundles over projective spaces. In particular, we prove the mirror formulae for genus one Gromov-Witten invariants of $K_{\mathbb{P}^2}$ and $K_{\mathbb{P}^3}$, conjectured by Klemm, Zaslow and Pandharipande. In the appendix we derive the modularity of genus one Gromov-Witten invariants for the local $\mathbb{P}^2$ as a consequence of the results on Ramanujan’s cubic transformation. Inspired by the localized standard versus reduced formula, we show that the ordinary genus one Gromov-Witten invariants of Calabi-Yau hypersurfaces in projective spaces can be computed by virtual localization after the contribution of a genus one vertex is replaced by a modified one.

1 Introduction

The computations of the Gromov-Witten invariants of Calabi-Yau manifolds play an important role in enumerative geometry and mirror symmetry. In genus zero, for Calabi-Yau complete intersections in projective spaces, we can use the hyperplane property to write the integration of the virtual fundamental class as a twisted Gromov-Witten invariants of the ambient projective spaces, see [12], [14], [23] and the references therein.

In genus one, this approach does not work in a straightforward way, since the hyperplane property does not hold. In [33], [35], A. Zinger defined the reduced genus one Gromov-Witten invariants for symplectic manifolds and found a relation between the reduced and the ordinary Gromov-Witten invariants (standard versus reduced formulae, see theorem 1A and theorem 1B of [35]). Furthermore, J. Li and A. Zinger showed that the reduced genus one Gromov-Witten invariants satisfy the hyperplane property for complete intersections in projective spaces ([24]). So by the standard versus reduced formula, we can reduced the computation of the genus one Gromov-Witten invariants of a complete intersection $X$ in $\mathbb{P}^{n-1}$ to the computation of genus zero Gromov-Witten invariants of $X$, and integrations of some classes on $\overline{\mathcal{M}}_{1,k}(\mathbb{P}^{n-1},d)$, the main component of $\overline{\mathcal{M}}_{1,k}(\mathbb{P}^{n-1},d)$. The latter integrations can be computed by equivariant localizations on a natural desingularization of $\overline{\mathcal{M}}_{1,k}(\mathbb{P}^{n-1},d)$ ([30]). Zinger completed the computations for the genus on Gromov-Witten invariants of Calabi-Yau hypersurfaces in $\mathbb{P}^{n-1}$ by a clever use of properties of symmetric functions and the residue theorem on $S^2$ ([30]). Following this approach, A. Popa computed the genus one Gromov-Witten invariants of Calabi-Yau complete intersections in $\mathbb{P}^{n-1}$ ([28]). To generalize this method to complete intersections in more general spaces (such as toric varieties, flag varieties, ...), there are at least two obstacles:

1. The desingularizations in [30] or [18] can be extended to products of projective spaces in a straightforward way, and may also be extended to toric varieties with some efforts. For Grassmannians and more general flag varieties we need some new ideas.

2. The computations of genus one Gromov-Witten invariants for Calabi-Yau manifolds which are total spaces of vector bundles over balloon manifolds, assign contributions from the refined decorated rooted trees. We show that this definition satisfies a localized version of the standard versus reduced formula, whose global version in the compact cases is due to A. Zinger. As an application we prove the conjecture in a previous article on the genus one Gromov-Witten invariants of local Calabi-Yau manifolds which are total spaces of concave splitting vector bundles over projective spaces. In particular, we prove the mirror formulae for genus one Gromov-Witten invariants of $K_{\mathbb{P}^2}$ and $K_{\mathbb{P}^3}$, conjectured by Klemm, Zaslow and Pandharipande. In the appendix we derive the modularity of genus one Gromov-Witten invariants for the local $\mathbb{P}^2$ as a consequence of the results on Ramanujan’s cubic transformation. Inspired by the localized standard versus reduced formula, we show that the ordinary genus one Gromov-Witten invariants of Calabi-Yau hypersurfaces in projective spaces can be computed by virtual localization after the contribution of a genus one vertex is replaced by a modified one.

1 Introduction

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1. The desingularizations in [30] or [18] can be extended to products of projective spaces in a straightforward way, and may also be extended to toric varieties with some efforts. For Grassmannians and more general flag varieties we need some new ideas.
2. The combinatorial computations in [36] and [28] rely heavily on the $S_n$-symmetry of the toric geometry of $\mathbb{P}^{n-1}$. For more general spaces we have less symmetries so we can not directly make use of the properties of the symmetric rational functions.

The above discussions concern the so called compact cases. For a local Calabi-Yau manifold, which (in a narrow sense) means the total space of an equivariant concave vector bundle $E$ over a toric variety $Y$, one can use virtual localization to compute the genus one Gromov-Witten invariants to any degree; the computations reduce to a purely combinatorial issue. However it is not easy to obtain a closed formula (even via the mirror map). A natural idea is to follow Zinger’s approach in the compact cases. For example, for $E = K_{\mathbb{P}^{n-1}}$, one can try to prove a result similar to [24], to extend the standard versus reduced formula (SvR for short) to the local case, and to make the localization computations over the desingularization of $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^{n-1}, d)$. We may call this approach a geometric approach. This is just the approach proposed in [19]. But when we proceed we find that this geometric approach can be realized in a purely combinatorial way\footnote{Precisely speaking, to say this is a combinatorial realization, one needs to show the formal reduced genus one Gromov-Witten invariants coincide with the geometric one. We don’t pursue this topic in this article.}. More precisely, let $X$ be a local Calabi-Yau manifold, for every decorated one loop graph and every decorated rooted tree $\Gamma$, we associate formally a contribution $\text{Cont}_\Gamma(N_{1,d}^{0,X})$, and define the formal reduced genus one Gromov-Witten invariants of $X$ by

$$N_{1,d}^{0,X} = \sum_{\Gamma \in \mathcal{D}_d} \text{Cont}_\Gamma(N_{1,d}^{0,X}) + \sum_{\Gamma \in \mathcal{D}_{d//}} \text{Cont}_\Gamma(N_{1,d}^{0,X}),$$

where $\mathcal{D}_d$ and $\mathcal{D}_{d//}$ represent the set of decorated one loop graphs and the set of decorated rooted trees respectively, whose precise definitions will be given in the main text. More generally, one can define formal reduced genus one Gromov-Witten invariants with insertions. The first main theorem of this article is

**Theorem 1.** Let $\mu_1, \ldots, \mu_{|J|} \in H_2^+(Y)$. For every decorated rooted tree $\Gamma \in \mathcal{D}_d$, we have

$$\text{Cont}_\Gamma(\langle \mu_1, \ldots, \mu_{|J|} \rangle_{1,d}^X) = \text{Cont}_\Gamma(\langle \mu_1, \ldots, \mu_{|J|} \rangle_{1,d}^{0,X}) + \sum_{m \geq 1, |J| \subset J} \sum_{|J'| = |J| - 2m} \frac{(-1)^m |J'| (m-1)!}{24} \sum_{p=0}^{n+|J'|-2m} \langle \eta_p c_{n+|J'|-2m-p}(TX); \mu_1, \ldots, \mu_{|J'|} \rangle_{(m-|J'|, J', d)}^X.$$  \hspace{1cm} (2)

We refer the reader to section 3 for the precise meaning of this theorem. We call (2) the localized standard versus reduced formula (LSvR for short). As a corollary of (1) and (2), we have

**Corollary 1.**

$$\langle \mu_1, \ldots, \mu_{|J|} \rangle_{1,d}^X = \langle \mu_1, \ldots, \mu_{|J|} \rangle_{1,d}^{0,X} + \sum_{m \geq 1, |J| \subset J} \sum_{|J'| = |J| - 2m} \frac{(-1)^m |J'| (m-1)!}{24} \sum_{p=0}^{n+|J'|-2m} \langle \eta_p c_{n+|J'|-2m-p}(TX); \mu_1, \ldots, \mu_{|J'|} \rangle_{(m-|J'|, J', d)}^X.$$  \hspace{1cm} (3)
we cannot prove the divisor equation directly; but we can deduce the divisor equation for reduced
invariants from corollary 1 and the divisor equation for the ordinary local Gromov-Witten invariants
and for the invariants involve \( \tilde{\eta}_p \)-classes (lemma 3.4 and 3.5).

We prove the LSvR for arbitrary equivariant concave vector bundles over balloon manifolds. This
combinatorial realization of the geometric approach proposed above is more powerful than the latter
itself; for local Calabi-Yau manifolds it bypass the first obstacle discussed above on the desingular-
ization of the moduli spaces of stable maps of genus one.

The corollary 1 enables us to compute genus one Gromov-Witten invariants of

\[
X = \Tot(E = \bigoplus_{k=1}^l \mathcal{O}_{\mathbb{P}^{n-1}}(-a_k) \to \mathbb{P}^{n-1}),
\]

with \( a_k > 0 \) for \( 1 \leq k \leq l \) and \( \sum_{k=1}^l a_k = n \); the crucial point is that while the computation of
the lefthand side of (3) via the virtual localization does not directly lead to a mirror formula, we
can compute the righthand side of (3) to obtain a mirror formula of the lefthand side, via Zinger’s
method in [36]. The genus one mirror formulae for \( K_{\mathbb{P}^2} \) and \( K_{\mathbb{P}^3} \) have been conjectured in [21] (see
also [1]) and [20] respectively. In [19], based on some observations on Zinger’s formulae in compact
cases, the author made a conjecture (generalizing the conjecture for \( K_{\mathbb{P}^2} \) and \( K_{\mathbb{P}^3} \)) on the genus one
mirror formulae for \( X \) of the form (4), which is now the second main theorem of this article. Let

\[
R(w, t) = e^{wt} \sum_{d \geq 0} e^{dt} \prod_{i=1}^l \prod_{s=0}^{d-1} (-a_k w - s) \prod_{s=1}^d (w + s)^n,
\]

and for \( q \geq p \geq 0 \), let

\[
I_{p,q}(t) = \frac{d}{dt} \left( \frac{I_{p-1,q}(t)}{I_{p-1,p-1}(t)} \right).
\]

Denote \( I_p(t) = I_{p,p}(t) \) for \( p \geq 0 \). Let

\[
T = I_{0,1}(t).
\]

**Theorem 2. (=Theorem 3.1)**

\[
\sum_{d=1}^\infty e^{dt} N_{1,d}^X = \frac{n}{48} \left( n - 1 - 2 \sum_{k=1}^l \frac{1}{a_k} \right) (T - t) - \begin{cases} \frac{n+l}{48} \log(1 - \prod_{k=1}^l (-a_k)^{a_k e^t}) + \sum_{p=l}^{n+l-2} \frac{(n+l-2p)^2}{8} \log I_p(e^t), & \text{if } 2 \mid (n+l); \\ \frac{n+l-3}{48} \log(1 - \prod_{k=1}^l (-a_k)^{a_k e^t}) + \sum_{p=l}^{n+l-3} \frac{(n+l-2p)^2-1}{8} \log I_p(e^t), & \text{if } 2 \nmid (n+l). \end{cases}
\]

Since \( I_p(e^t) = 0 \) for \( 0 \leq p \leq l - 1 \), this theorem is equivalent to the conjecture 1 in [19].

For general concave equivariant vector bundles over balloon manifolds, the corollary 1 enables
us to obtain analogs to proposition 5.2 and proposition 5.4. To obtain a mirror formula, however,
one needs additional techniques to overcome the second obstacle discussed above.

For \( X = K_{\mathbb{P}^2} \), the genus one Gromov-Witten potential can be written as a modular form in
suitable modular coordinate on the the modular curve for \( \Gamma(3) \). The derivation of this result in [1]
is a mixture of rigorous mathematics and mirror-symmetry-arguments. To make things clear, we
derive this fact from the results on the Ramanujan’s cubic transformation, which should have been
well-known to experts.
It is interesting to note the interplay between the computations of Gromov-Witten invariants of global (=compact) and of local Calabi-Yau manifolds. In principle, the latter should be easier, and the computation for the local CYs helps us to understand the global CYs. For example, in section 6 of [20], they made use of the localization computation for $K_{P^2}$ to fix the universal behavior at the conifold and thus obtain the genus one Gromov-Witten potential for the CY hypersurface in $P^5$ via the B-model. Conversely, the author used the result on the genus one Gromov-Witten potential for the compact CY complete intersections of [36] and [28] to fix the universal behavior at the conifolds, and further observed that formulation of the group of terms such as $\sum_{l=0}^{n+1} (n+l-2p)^2 \log I_p(e^t)$ in [157] should be ubiquitous, thus made the conjecture in [19]. Now we have known that the standard versus reduced formula in the local case holds in a refined way, what does it feedback to the global theory? We investigate this topic in section 3.6 and make some interesting observations and a conjecture (see conjecture [1] and the remark following it).

This article is organized as follows. In section 2 we first fix the terminology for localization computations. We recall the ordinary virtual localization in genus one, and give the localization data for equivariant integrations over $\mathcal{M}_{(m,d)}(Y,d)$. Then we define the formal reduced genus one local Gromov-Witten invariants. In section 3 we prove the LSvR, from simple cases to the general cases. In section 3.6 we discuss the modified virtual localization for compact Calabi-Yau manifolds. In section 4 we use Givental’s result on genus zero mirror symmetry to compute the difference between the standard and the formal reduced Gromov-Witten invariants of [4]. In section 5 we compute the formal reduced Gromov-Witten invariants of [4], following Zinger’s method and using results from [28] and [29]. In the appendix we see how to deduce the modularity of $\mathcal{F}_1$ for $K_{P^2}$ from Ramanujan’s cubic transformation theory.

**Notations:**

1. For a vector bundle $E$, let $e(E)$ be its Euler class, and for an equivariant vector bundle $E$, let $e(E)$ be its equivariant Euler class. Similarly, we denote by $c_p(E)$ the $p$-th Chern class of $E$, and $c_p(E)$ the $p$-th equivariant Chern class of $E$.

2. Denote the set of positive integers (resp., nonnegative integers ) by $\mathbb{Z}_{>0}$ (resp., $\mathbb{Z}_{\geq 0}$).

3. In section 4 and section 5, we frequently feel convenient to write $e^T = Q$, and $e^t = q$. This $q$ should not be confused with the $q$ appeared in the subscript of $I_{p,q}(e^t)$.

4. When we discuss the refined decorated rooted trees, we try our best to follow the terminology in [30], [36]. However, because we need the Greek letters $\mu$ and $\eta$ at other places, we instead denote the maps $\mu$ and $\eta$ in [30] by their English analogies $m$ and $e$.

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2 **Fixed Loci and localization contributions**

Let $X$ be the total space of a vector bundle $\pi : E \to Y$ where $Y$ is a smooth projective variety over $C$. We assume that the vector bundle $E$ is concave, which means that for every non-constant map from a curve $\gamma : f : C \to Y$ we have $H^0(C, f^*E) = 0$. The total Chern class of $X$ is understood as

$$c(TX) = \pi^*(c(E)c(TY)).$$  \hspace{1cm} (9)$$

When $c_1(E) + c_1(TY) = 0$, we call $X$ a local Calabi-Yau manifold. Since $E$ is concave, every non-constant map from a curve to $X$ actually map the curve into $Y$, and when we compute various

\footnote{For the purpose of this article we can slightly weaken this condition by restricting the genus of $C$ to 0 and 1.}
Gromov-Witten-type invariants of \(X\), we are actually working on the moduli spaces related to \(Y\), so it is convenient to take \(c(TX)\) as a cohomology class (or equivariant cohomology class when we are in an equivariant world) over \(Y\), by an abuse of notation. In the explicit computations in section 4 and section 5, we consider the cases \(X\) of the form \([1\), [2]\)

Now suppose \(Y\) is a balloon manifold of dimension \(n - 1\), which means the following data (see [19, 20]):

- There is a \(\mathbb{T} = (\mathbb{C}^*)^k\)-action on \(Y\) such that there are \(N\) isolated fixed points, named \(P_1, \ldots, P_N\).
- There are finitely many invariant line \((\cong \mathbb{P}^1)\) connecting the fixed points. For each fixed point \(P_i\), where \(1 \leq i \leq N\), there are \(n - 1\) invariant lines \((\cong \mathbb{P}^1)\) connecting \(P_i\) to other fixed points.
- We demand that the weights of these \(n\) invariant lines at \(P_i\) are pairwisely independent. For every pair of distinct fixed points there is at most one invariant line connecting them, if there exists one for \(P_i\) and \(P_j\), we denote it by \(\overline{P_iP_j}\), and say that \(P_i\) and \(P_j\) are neighboring to each other. Let \(\text{Nb}(P_i)\) be the set of fixed points that are neighboring to \(P_i\), and for \(P_j \in \text{Nb}(P_i)\), we denote the weights of \(\overline{P_iP_j}\) at \(P_i\) by \(\alpha_{i,j}\); we have \(\alpha_{i,j} = -\alpha_{j,i}\).

Denote the equivariant cohomology ring of \(Y\) by \(H^*_\mathbb{T}(Y)\). The equivariant cohomology ring of a point is denoted by \(\mathbb{Q}[\alpha_1, \ldots, \alpha_k]\), and its quotient field by \(\mathbb{Q}_\mathbb{a}\). Thus \(\alpha_{i,j}\) are linear combinations of \(\alpha_1, \ldots, \alpha_k\) for \(1 \leq i \leq N, 1 \leq j \leq n\). For every \(P_i\), there is an associated restriction map

\[
|P_i : H^*_\mathbb{T}(Y) \to \mathbb{Q}[\alpha_1, \ldots, \alpha_k].
\]  

Suppose \(E\) is an equivariant concave vector bundle of rank \(l\) over \(Y\), with a \(\mathbb{T}\)-linearization such that the weights of \(E\) at \(P_i\) are \(\varepsilon_{i,1}, \ldots, \varepsilon_{i,l}\).

Following [6], let

\[
H^*_\mathbb{T}(Y) = \{ \beta \in \text{Hom}(\text{Pic}(Y), \mathbb{Z}) : \beta(L) \geq 0 \text{ for } \forall \text{ ample } L \}. \tag{11}
\]

For \(d \in H^*_\mathbb{T}(Y)\), on the moduli stack of stable maps \(\overline{\mathcal{M}_{g,k}}(Y,d)\), let \(\text{ev}_i\) be the \(i\)-th evaluation map, \(\pi : \overline{\mathcal{M}_{g,k+1}}(Y,d) \to \overline{\mathcal{M}_{g,k}}(Y,d)\) be the universal curve and \(f : \overline{\mathcal{M}_{g,k+1}}(Y,d) \to Y\) the universal stable map. Let

\[
\mathcal{U}_g = R^1\pi_*f^*E, \tag{12}
\]

and for \(k \geq 1\) let

\[
\mathcal{U}'_g = \text{ev}_*^1(E) \oplus R^1\pi_*f^*E, \tag{13}
\]

which are both vector bundles over \(\overline{\mathcal{M}_{g,k}}(Y,d)\). The genus \(g\) Gromov-Witten invariants for \(X\) of the form \([1\), [2]\) with (primary) insertions \(\mu_1, \ldots, \mu_k \in H^*(Y)\) are given by

\[
\langle \mu_1, \ldots, \mu_k \rangle^X_{g,k,d} = \left( \bigwedge_{j=1}^k \text{ev}_j^*\mu_j \right) \cap e(\mathcal{U}_g) \cap [\overline{\mathcal{M}_{g,k}}(Y,d)]^{\text{vir}}. \tag{14}
\]

In particular, when \(g = 1\) and \(k = 0\), the genus one Gromov-Witten invariants of \(X\) are given by

\[
N_{1,d}^X = e(\mathcal{U}_1) \cap [\overline{\mathcal{M}_{1,0}}(Y,d)]^{\text{vir}}. \tag{15}
\]

The \(\mathbb{T}\)-action on \(Y\) naturally induces a \(\mathbb{T}\)-action on the moduli stack of stable maps \(\overline{\mathcal{M}_{g,k}}(Y,d)\) and some other related moduli spaces. The linearization of \(E\) naturally induces linearizations of \(\mathcal{U}_g\)

\footnote{This assumption is not necessary, but will make the presentations less complicated.}
The equivariant genus $g$ Gromov-Witten invariants of $X = \text{Tot}(E \to Y)$ with (primary) insertions $\mu_1, \cdots, \mu_k \in H^*_T(Y)$ are given by

$$
\langle \mu_1, \cdots, \mu_k \rangle_{g,k,d}^X = \left( \bigwedge_{j=1}^k \text{ev}_j^* \mu_j \right) \wedge e(U_g) \cap \left[ \overline{M}_{g,k}(Y,d) \right]_{\text{vir}}^T,
$$

where $\left[ \overline{M}_{g,k}(Y,d) \right]_{\text{vir}}^T$ is the equivariant virtual fundamental class.

The $T$-action on $Y$ also induces a $T$-action on some other related moduli spaces as we will see. By [2] and [17], the equivariant integration (against the fundamental cycle when the moduli space is smooth or the virtual moduli cycle when we have a perfect obstruction theory) of the equivariant cohomology classes on these spaces can be computed by (virtual) localization, i.e., every fixed locus contributes to the integration, and summing the contributions we obtain the integration. When the integration we are computing is understood, we call the contribution coming from a fixed locus the localization contribution of this fixed locus (or of the graph which indexes this fixed locus).

In the following part of this section, we describe the fixed loci of three types of moduli spaces $\overline{M}_{1,J}(Y,d)$, $\overline{M}_{g,J}(Y,d)$, the formal fixed loci as an analogy to the truly existing fixed loci of $\tilde{M}_{1,k}(\mathbb{P}^{n-1},d)$, and the corresponding (formal in section 2.3) localization contributions of several Gromov-Witten-type invariants. For convenience, we prefer to use $J$ to represent the set of marked points.

### 2.1 Fixed loci on $\overline{M}_{1,J}(Y,d)$ and localization contributions

The fixed loci on $\overline{M}_{g,J}(Y,d)$ and their localization contributions to the equivariant Gromov-Witten invariants of $X$ are well-known, and the reader may refer to, e.g., [17], [22]. In this subsection we recall the results which are necessary for us and fix the notations.

The fixed loci on $\overline{M}_{g,J}(Y,d)$ are indexed by decorated graphs. We recall the terminology for decorated graphs in [36]. A decorated graph (for $Y$) is a tuple $\Gamma = (\text{Ver}, \text{Edg}; g, m, d, e)$, where

- $(\text{Ver}, \text{Edg})$ is a graph. More precisely, Ver is a finite set, Edg is a finite set of maps, from a finite set Dom(Edg) to the set of two-element subsets of Ver. For a vertex $v \in \text{Ver}$ and an edge $e \in \text{Edg}$, if $v$ lies in the image of $e$, we call $v \in e$ by an abuse of notation. Also, if the image of an edge $e \in \text{Edg}$ is $\{v_1, v_2\}$, we call $e = \{v_1, v_2\}$ by an abuse of notation; we should keep in mind that in general there may be more than one edges with the same image. However in this article we mainly discuss the trees, so no confusions arises. The edges containing the vertex $v$ is denoted by Edg($v$), i.e., Edg($v$) = $\{e \in \text{Edg} : v \in e\}$. In addition, we demand that the graph $(\text{Ver}, \text{Edg})$ is connected in the usual sense. Thus the genus of $(\text{Ver}, \text{Edg})$ is $g(\text{Ver}, \text{Edg}) = 1 - |\text{Ver}| + |\text{Edg}|$. We use Edg$^T(v)$ instead of Edg($v$) when we want to emphasize the underlying graph $\Gamma$.  

- The map $g : \text{Ver} \to \mathbb{Z}^\geq 0$ indicates the genus of the contracted component of the domain curve that a vertex represents.

- The map $m : \text{Ver} \to [N]$ indicates the fixed point which the contracted component maps to. We demand that if $\{v_1, v_2\} \in \text{Edg}$, then
  $$
m(v_1) \neq m(v_2).
$$
• Denote the free semigroup \( \sum(\mathbb{Z}_{\geq 0} \cdot P_i P_j) \) by \( \mathbb{B}(Y) \), where the sum runs over the invariant lines of \( Y \). The map
\[
\mathfrak{d} : \text{Edg} \to \mathbb{B}(Y)
\]
for an edge \( e = \{v_1, v_2\} \in \text{Edg} \) takes values in \( \mathbb{Z}_{>0} \cdot \frac{P_{m(v_1)} P_{m(v_2)}}{P_{m(v_1)} P_{m(v_2)}} \), indicating the degree of the invariant line that an edge represents. For convenience in later use, let \( d(e) \in \mathbb{Z}_{>0} \) such that \( \mathfrak{d}(e) = d(e) \cdot \frac{P_{m(v_1)} P_{m(v_2)}}{P_{m(v_1)} P_{m(v_2)}} \) when \( e = \{v_1, v_2\} \).

• The label map
\[
\mathfrak{e} : J \to \text{Ver}
\]
indicates on which contracted component a marked point lie.

• The genus of a decorated graph \( \Gamma \) is
\[
g(\Gamma) = g(\text{Ver}, \text{Edg}) + \sum_{v \in \text{Ver}} g(v) = 1 - |\text{Ver}| + |\text{Edg}| + \sum_{v \in \text{Ver}} g(v).
\]

• Let
\[
\mathfrak{d}(\Gamma) = \sum_{e \in \text{Edg}} \mathfrak{d}(e) \in \mathbb{B}(Y).
\]
There is a canonical map \( d : \mathbb{B}(Y) \to H^*_T(Y) \). For \( \Gamma \) to represent a fixed locus on \( \mathcal{M}_{1,J}(Y,d) \), we need \( d \circ \mathfrak{d}(\Gamma) = d \); we call \( d \circ \mathfrak{d}(\Gamma) \) the degree of \( \Gamma \).

• The valence of a vertex \( v \in \text{Ver} \) is
\[
\text{val}(v) = |\text{Edg}(v)| + |\text{e}^{-1}(v)|.
\]

• There is a natural projection map \( \pi \) from the set of decorated graphs to the set of graphs, mapping \( \Gamma \) to \( (\text{Ver}, \text{Edg}) \). The automorphism group of \( (\text{Ver}, \text{Edg}) \) acts naturally on the set \( \pi^{-1}(\text{Ver}, \text{Edg}) \), and the stable subgroup associated to \( \Gamma \) is called the automorphism group of \( \Gamma \), denoted by \( \text{Aut}(\Gamma) \).

For \( \mathcal{M}_{1,J}(Y,d) \) there are two types of decorated graphs, the \textit{decorated one-loop graphs} and the \textit{decorated rooted trees}. On a decorated one-loop graph every vertex has genus zero. On a decorated rooted tree every vertex except the root has genus zero, and the root has genus one. So we drop the map \( g \) in the presentations of decorated one-loop graphs and decorated rooted trees. We denote the set of decorated one-loop graphs (resp., decorated rooted trees) of degree \( d \) and with the set of marked points \( J \) by \( \mathcal{DOL}_d^J(Y) \) (resp., \( \mathcal{DRT}_d^J(Y) \)). In the proper context we always have a fixed \( Y \), so we drop the notation for \( Y \) and simply write \( \mathcal{DOL}_d^J \) (resp., \( \mathcal{DRT}_d^J \)).

Let us first consider the contributions from decorated one-loop graphs. For \( \mu_1, \cdots, \mu_{|J|} \in H^*_T(Y) \), the localization contribution of a decorated one-loop graph \( \Gamma \) to
\[
\langle \mu_1, \cdots, \mu_{|J|} \rangle_{1,J,d}^{X}
\]
can be written as a product of contributions from edges and from vertices together with a factor coming from the automorphism group \( \text{Aut}(\Gamma) \), i.e.,
\[
\text{Cont}_\Gamma (\langle \mu_1, \cdots, \mu_{|J|} \rangle_{1,J,d}^{X}) = \frac{1}{|\text{Aut}(\Gamma)|} \cdot \prod_{e \in \text{Ver}} \text{Cont}_{\Gamma,e} (\langle \mu_1, \cdots, \mu_{|J|} \rangle_{1,J,d}^{X}) \prod_{e \in \text{Edg}} \text{Cont}_{\Gamma,e} (\langle \mu_1, \cdots, \mu_{|J|} \rangle_{1,J,d}^{X}).
\]
where the contribution of an vertex \( v \in \text{Ver} \) with \( m(v) = i \) is

\[
\text{Cont}_{v}((\mu_{1}, \cdots, \mu_{J})_{1,J,d}) = \prod_{j \in e^{-1}(v)} \mu_{j} |_{P_{i}}
\]

\[
\cdot \left( \prod_{k=1}^{l} \varepsilon_{i,k} \prod_{P_{j} \in \text{Nb}(P_{i})} \alpha_{i,j} \right)^{|\text{Edg}(v)|-1} \int_{\overline{\mathcal{M}_{0,\text{val}(v)}}} \frac{1}{\prod_{e \in \text{Edg}(v)} \left( \frac{\alpha_{i,e}}{d(e)} - \psi(v,e) \right)}
\]

(20)

where \( \alpha_{(v,e)} = \alpha_{i,j} \) if \( e = \{ v, v' \} \) with \( m(v') = j \), and \( \psi(v,e) \) is the \( \psi \)-class associated to the marked point on \( \overline{\mathcal{M}_{0,\text{val}(v)}} \) corresponding to the edge \( e \). The explicit form of \( \text{Cont}_{v}((\mu_{1}, \cdots, \mu_{J})_{1,J,d}) \) can be computed by the holomorphic Lefschetz formula (3), see e.g., (32); we will not spell out the general formula for this since we don’t need it. For \( X \) of the form (34), the explicit form for \( \text{Cont}_{v}((\mu_{1}, \cdots, \mu_{J})_{1,J,d}) \) is (130).

Note that we always adopt the convention that, the formal integrals over \( \overline{\mathcal{M}_{0,1}} \) and \( \overline{\mathcal{M}_{0,2}} \) are understood as extending the range of \( n \) in the following identity to \( r \geq 1 \):

\[
\int_{\overline{\mathcal{M}_{0,r}}} \frac{1}{\prod_{i=1}^{r} (w_{i} - \psi_{i})} = \frac{1}{\prod_{i=1}^{r} w_{i}} \left( \sum_{i=1}^{r} \frac{1}{w_{i}} \right)^{r-3}.
\]

(21)

Next we consider the contributions from decorated rooted trees. For a decorated rooted tree \( \Gamma \in \mathcal{D}_{j}^{d} \) the root \( v_{0} \) represents a genus one subcurve which is contracted by the stable map. The localization contribution of \( \Gamma \) to (18) can also be written as a product

\[
\text{Cont}_{\Gamma}((\mu_{1}, \cdots, \mu_{J})_{1,J,d}) = \frac{1}{|\text{Aut}(\Gamma)|} \cdot \prod_{e \in \text{Edg}} \text{Cont}_{\text{v}}((\mu_{1}, \cdots, \mu_{J})_{1,J,d}) \prod_{e \in \text{Edg}} \text{Cont}_{\text{v}}((\mu_{1}, \cdots, \mu_{J})_{1,J,d}),
\]

(22)

where for an edge \( e \) and a vertex \( v \in \text{Ver} \setminus \{ v_{0} \} \), the contribution is the same as those in (19) respectively, while the contribution of \( v_{0} \) with \( m(v_{0}) = i \) is

\[
\text{Cont}_{\text{v}_{0}}((\mu_{1}, \cdots, \mu_{J})_{1,J,d}) = \prod_{j \in e^{-1}(v)} \mu_{j} |_{P_{i}}
\]

\[
\cdot \left( \prod_{k=1}^{l} \varepsilon_{i,k} \prod_{P_{j} \in \text{Nb}(P_{i})} \alpha_{i,j} \right)^{|\text{Edg}(v_{0})|-1} \int_{\overline{\mathcal{M}_{1,\text{val}(v_{0})}}} \frac{\prod P_{j} \in \text{Nb}(P_{i}) \Lambda_{j}^{Y} \left( \alpha_{i,j} \right) \prod_{k=1}^{r} \Lambda_{k}^{Y} \left( \varepsilon_{i,k} \right)}{\prod_{e \in \text{Edg}(v_{0})} \left( \frac{\alpha_{i,e}}{d(e)} - \psi(v_{0},e) \right)}
\]

(23)

Summing the contributions, we have

\[
(\mu_{1}, \cdots, \mu_{J})_{1,J,d}^{X} = \sum_{\Gamma \in \mathcal{D}_{j}^{d}} \text{Cont}_{\Gamma}((\mu_{1}, \cdots, \mu_{J})_{1,J,d}) + \sum_{\Gamma \in \mathcal{D}_{j}^{d}} \text{Cont}_{\Gamma}((\mu_{1}, \cdots, \mu_{J})_{1,J,d}).
\]

(24)

### 2.2 Fixed loci on \( \overline{\mathcal{M}(m,J)}(Y,d) \) and localization contributions

In this subsection, we recall the definition of (35) for \( \overline{\mathcal{M}(m,J)}(Y,d) \) and some natural cohomology classes on it. Then we describe the fixed loci on \( \overline{\mathcal{M}(m,J)}(Y,d) \) and their localization contributions.

Let \( Y \) be a smooth projective variety. For \( d = (d_{1}, \cdots, d_{m}) \in (H_{*}^{+}(Y) - \{0\})^{m} \) and finite sets \( J_{1}, \cdots, J_{m} \), define \( \overline{\mathcal{M}(m;J_{1},\cdots,J_{m})}(Y,d) \) by the cartesian diagram

\[
\overline{\mathcal{M}(m;J_{1},\cdots,J_{m})}(Y,d) \quad \prod_{s=1}^{m} \overline{\mathcal{M}_{0,\{0\} \cup J_{s}}(Y,d_{s})}.
\]

(25)

\[
\begin{array}{c}
\text{ev} \downarrow \quad \text{ev}_{0} \downarrow \\
Y \quad \Delta_{Y} \quad Y^{m}
\end{array}
\]
Define
\[ \mathcal{M}_{(m;J_1,\ldots,J_m)}(Y,d)|^{\text{vir}} = \Delta_{Y^\vee} \left( \prod_{s=1}^{m} [\mathcal{M}_{0,s,\cup J_s}(Y,d_s)]^{\text{vir}} \right), \tag{26} \]
where \( \Delta_{Y^\vee} \) is the Gysin map.

Remark 2.1. For flag varieties \( Y = G/P \), since \( \text{ev}_0 : \mathcal{M}_{0,\cup J_s}(Y,d_s) \to Y \) is a smooth morphism and \( \mathcal{M}_{0,1}(Y,d_s) \) is smooth for every \( 1 \leq s \leq m \), \( \mathcal{M}_{(m;J_1,\ldots,J_m)}(Y,d) \) is smooth as well. Thus \( [\mathcal{M}_{(m;J_1,\ldots,J_m)}(Y,d)]^{\text{vir}} = [\mathcal{M}_{(m;J_1,\ldots,J_m)}(Y,d)]^{\text{vir}} \).

For \( d \in H_2^+(Y) \setminus \{0\} \), let
\[ \mathcal{M}_{(m;J_1,\ldots,J_m)}(Y,d) = \prod_{J_1 \sqcup \cdots \sqcup J_m = J} \prod_{d_1,\ldots,d_m > 0} \mathcal{M}_{(m;J_1,\ldots,J_m)}(Y,d), \tag{27} \]
and
\[ [\mathcal{M}_{(m;J_1,\ldots,J_m)}(Y,d)]^{\text{vir}} = \prod_{J_1 \sqcup \cdots \sqcup J_m = J} \prod_{d_1,\ldots,d_m > 0} [\mathcal{M}_{(m;J_1,\ldots,J_m)}(Y,d)]^{\text{vir}}. \tag{28} \]

There are natural projection maps
\[ \pi_s : \mathcal{M}_{(m;J_1,\ldots,J_m)}(Y,d) \to \mathcal{M}_{0,\cup J_s}(Y,d_s) \tag{29} \]
for \( 1 \leq s \leq m \). Let \( \psi_0 \) be the Euler class of the cotangent line bundle on \( \mathcal{M}_{0,\cup J_s}(Y,d_s) \) associated to the marked point \( 0_s \). We define \( \eta_p \in H^{2p}(\mathcal{M}_{0,\cup J_s}(Y,d_s)) \) by the generating function
\[ \sum_{p=0}^{\infty} z^p \eta_p = \prod_{s=1}^{m} \frac{1}{1 - z \pi_s^* \psi_0}. \tag{30} \]

Varying \( d_1 + \cdots + d_m = d \) and \( J_1 \sqcup \cdots \sqcup J_m = J \), we obtain the class \( \eta_p \) over \( \mathcal{M}_{(m;J_1,\ldots,J_m)}(\mathbb{P}^{n-1},d) \).

For every \( j \in J \) there is an evaluation map \( \text{ev}_j : \mathcal{M}_{(m;J)}(Y,d) \to Y \) in an obvious way. There is also an evaluation map \( \text{ev}_0 : \mathcal{M}_{(m;J)}(Y,d) \to Y \) associated to the common \( 0 \)-th marked point. For \( J' \subset J \), let
\[ \langle \eta_p \mu_0; \mu_1,\ldots,\mu_{|J|} \rangle_{(m,J-J',d)}^Y = \frac{1}{m!} \eta_p \prod_{j \in J'} \mu_j \prod_{j \not\in J'} \text{ev}_j^*(\mu_j) \cap [\mathcal{M}_{(m,J-J',d)}(Y,d)]^{\text{vir}}. \tag{31} \]

For \( X = \text{Tot}(E \to Y) \) where \( E \) is a concave vector bundle over \( Y \), to define invariants for \( X \) similar to \( \mathcal{M}_{g,J}(Y,d) \), we need only to replace \( Y \) by \( X \) in the above definitions, and note that \( \mathcal{M}_{g,J}(X,d) = \mathcal{M}_{g,J}(Y,d) \) and \( [\mathcal{M}_{g,J}(X,d)]^{\text{vir}} = U_g \cap [\mathcal{M}_{g,J}(Y,d)]^{\text{vir}} \). More concretely, we have
\[ \langle \eta_p \mu_0; \mu_1,\ldots,\mu_{|J|} \rangle_{(m,J-J',d)}^X = \frac{1}{m!} \prod_{j \not\in J'} \mu_j \prod_{j \in J'} \text{ev}_j^*(\mu_j) \cap [\mathcal{M}_{(m,J-J',d)}(Y,d)]^{\text{vir}}. \tag{32} \]

Now let \( Y \) be a balloon manifold and \( X = \text{Tot}(E \to Y) \) is the total space of a concave equivariant vector bundle \( E \). To describe the fixed loci on \( \mathcal{M}_{(m;J)}(Y,d) \), we need to introduce some notions. For a set \( S \), let \( \mathcal{P}(S) \) be its power set. The set of \( m \)-colored partitions of a finite set \( S \) is defined to be
\[ \mathcal{A}_m(S) = \{ I \in \mathcal{P}(S)^m : I = (I_{(1)},\ldots,I_{(m)}), I_{(1)} \sqcup \cdots \sqcup I_{(m)} = S, |I_{(i)}| > 0, \quad \text{for } 1 \leq i \leq m \}. \]
the set of nonnegative $m$-colored partitions of a finite set $S$ is defined to be

$$\mathcal{A}_m^0(S) = \{ I \in \mathcal{P}(S)^m : I = (I_{(1)}, \ldots, I_{(m)}), I_{(1)} \sqcup \cdots \sqcup I_{(m)} = S, |I_{(i)}| \geq 0, \text{ for } 1 \leq i \leq m\},$$

and the set of $m$-colored partitions of a pair of finite sets $(S, J)$ is defined to be

$$\mathcal{A}_m(S, J) = \mathcal{A}_m(S) \times \mathcal{A}_m^0(J).$$

A $m$-colored decorated rooted tree is a pair $(\Gamma, (I, K))$, where $\Gamma$ is a decorated rooted tree with a root $v_0$, and $(I, K) \in \mathcal{A}_m(\text{Edg}(v_0), \epsilon^{-1}(v_0))$. The notions of the degree of $\Gamma$ and the valence of a vertex is inherited from those of $\Gamma$. The set of $m$-colored decorated rooted trees of degree $d$ and with the set of marked points $J$ is denoted by $m\mathcal{CDRT}_d^d$.

There is a canonical projection map $\pi_\mathcal{CDRT} : m\mathcal{CDRT}_d \to \mathcal{DRT}_d$ with $\pi_\mathcal{CDRT}(\Gamma, (I, K)) = \Gamma$. The automorphism group $\text{Aut}(\Gamma)$ acts on $\pi^{-1}_\mathcal{CDRT}(\Gamma)$ in a natural way, the stable subgroup of $\Gamma$ is called the automorphism group of $\Gamma$. The fixed loci on $\mathcal{M}_{(m, J)}(Y, d)$ are indexed by $\Gamma \in m\mathcal{CDRT}_d^d$ in an obvious way. Two $m$-colored decorated rooted trees $\Gamma_1$ and $\Gamma_2$ index the same fixed locus if and only if $\pi_\mathcal{CDRT}(\Gamma_1) = \pi_\mathcal{CDRT}(\Gamma_2) = \Gamma$ for some $\Gamma$ and $\Gamma_1 = \sigma \Gamma \sigma^{-1}$ for some $\sigma \in \text{Aut}(\Gamma)$.

Let $\mu_0, \mu_1, \ldots, \mu_{|J|} \in H^*_T(Y)$. We denote equivariant version of $\psi$-classes over $\mathcal{M}_{0,J}(Y, d)$ and $\bar{\eta}$-classes over $\mathcal{M}_{(m, J)}(Y, d)$ by the same symbols in the equivariant integration; by the context, no confusion should arise. For $\Gamma = (\Gamma, (I, K), (\bar{\eta}_0, \cdots, \bar{\eta}_d))$, the localization contribution of $\Gamma$ to $\langle \eta_0, c_q(X) ; \mu_1, \cdots, \mu_{|J|} \rangle_{(m, J, d)}^{\mathcal{X}}$ can be written as

$$\text{Contr} (\langle \eta_0, c_q(X) ; \mu_1, \cdots, \mu_{|J|} \rangle_{(m, J, d)}^{\mathcal{X}}) = \frac{1}{m! |\text{Aut}(\Gamma)|} \prod_{e \in \text{Ver}} \text{Contr}_{\text{CDRT}} (\langle \eta_0, c_q(X) ; \mu_1, \cdots, \mu_{|J|} \rangle_{(m, J, d)}^{\mathcal{X}}),$$

$$\text{Contr}_{\text{CDRT}} (\langle \eta_0, c_q(X) ; \mu_1, \cdots, \mu_{|J|} \rangle_{(m, J, d)}^{\mathcal{X}}) = \prod_{j \in \epsilon^{-1}(v_0)} \mu_j |_{P_j},$$

$$\left( \prod_{k=1}^j \sum_{e \in \text{Edg}(v_0)} [x^n] \right) \left( \prod_{k=1}^j \sum_{e \in \text{Edg}(v_0)} [x^n] \right)$$

For an edge $e$ and a vertex $v \in \text{Ver} \setminus \{v_0\}$, the contribution is still the same as (19) respectively. Suppose $m(v_0) = i$, then the contribution of $v_0$ is

$$\text{Contr}_{\text{CDRT}} (\langle \eta_0, c_q(X) ; \mu_1, \cdots, \mu_{|J|} \rangle_{(m, J, d)}^{\mathcal{X}}) = \prod_{j \in \epsilon^{-1}(v_0)} \mu_j |_{P_j},$$

$$\left( \prod_{k=1}^j \sum_{e \in \text{Edg}(v_0)} [x^n] \right) \left( \prod_{k=1}^j \sum_{e \in \text{Edg}(v_0)} [x^n] \right)$$

We briefly explain (34).

- The marked point $0_s$ for $1 \leq s \leq m$ comes from the common node represented by the root $v_0$, which becomes a marked point when we split the domain curve with respect to the $m$ colors.
- The operator $[x^n]$ extracts the equivariant $q$-th Chern class of $X$ restricted to the fixed point $P_i$. The operator $[x^p]$ extracts all the $p$-th monomials of $\psi_{0_s}$ for $1 \leq s \leq m$, and the sum is $\eta_p$ restricted to this fixed curve, by the definition of $\eta_p$.
- By the argument parallel to the proof of the cutting edge axiom in [H], it is easy to show that there is a natural perfect obstruction theory on $\mathcal{M}_{(m, J, d)}(Y, d)$, and the corresponding virtual
fundamental cycle is the same as \( [28] \). The localization contribution is easily read out from this perfect obstruction theory. In particular, when \( Y \) is a flag variety, by remark \( [24] \) it is straightforward to obtain \( [33] \). Note that the deformation of the domain curves should be color-preserved, so the node-smoothing contribution in the usual virtual localization

\[
\int_{\overline{M}_0, \text{val}(v)} \left( \alpha_{v,e} - \psi_{v,e} \right) \cdots
\]

should be replaced by a color-preserved version, which is of the form in \( [34] \).

For later use, we need to write the invariant \( \langle \eta_p c_q (TX); \mu_1, \cdots, \mu_{|J|} \rangle^X_{(m,J,d)} \) as summing over the \( \mathcal{DR}^d \). First note that for every \( \Gamma_1 \in \mathcal{D} \mathcal{R}^d_{J_1} \) and \( J_2 \supset J_1 \), we can attach additional \(|J_2| - |J_1|\) marked points to the root \( v_0 \) of \( \Gamma_1 \), thus obtain \( \Gamma_2 \in \mathcal{D} \mathcal{R}^d_{J_2} \). In this way, we get an injective map

\[
\rho_{J_1,J_2} : \mathcal{D} \mathcal{R}^d_{J_1} \to \mathcal{D} \mathcal{R}^d_{J_2},
\]

such that \( \rho_{J_1,J_2}(\Gamma_1) = \Gamma_2 \).

Then we have

\[
\langle \eta_p c_q (TX); \mu_1, \cdots, \mu_{|J|} \rangle^X_{(m,J,d)} = \sum_{\Gamma \in \mathcal{D} \mathcal{R}^d_{J}} \text{Cont}_\Gamma \left( \langle \eta_p c_q (TX); \mu_1, \cdots, \mu_{|J|} \rangle^X_{(m,J,d)} \right),
\]

where

\[
\text{Cont}_\Gamma \left( \langle \eta_p c_q (TX); \mu_1, \cdots, \mu_{|J|} \rangle^X_{(m,J,d)} \right) = \frac{1}{|\text{Aut}(\Gamma)|} \sum_{\Gamma' \in \mathcal{D} \mathcal{R}^d_{J}} \prod_{e \in \text{Ver}} \text{Cont}_{\Gamma',e} \left( \langle \eta_p c_q (TX); \mu_1, \cdots, \mu_{|J|} \rangle^X_{(m,J,d)} \right) \prod_{e \in \text{Edg}} \text{Cont}_{\Gamma',e} \left( \langle \eta_p c_q (TX); \mu_1, \cdots, \mu_{|J|} \rangle^X_{(m,J,d)} \right).
\]

So if \( \Gamma \) is not in the image of \( \rho_{J',J} \), then \( \text{Cont}_\Gamma \left( \langle \eta_p c_q (TX); \mu_1, \cdots, \mu_{|J|} \rangle^X_{(m,J,d)} \right) = 0 \).

Note that the automorphism factor in \( [36] \) is \( 1/|\text{Aut}(\Gamma)| \) (we have \( \text{Aut}(\Gamma) \cong \text{Aut}(\rho_{J',J}^{\Gamma} \Gamma) \) when \( \Gamma \) is in the image of \( \rho_{J',J} \), not \( 1/|\text{Aut}(\Gamma')| \), because \( g_1, \Gamma \) and \( g_2, \Gamma \) index the same fixed locus when \( g_1 \) and \( g_2 \) are in the same coset of \( \text{Aut}(\Gamma) \) in \( \text{Aut}(\Gamma) \).

### 2.3 Formal fixed loci, formal localization contributions and reduced genus one Gromov-Witten invariants

For \( \mathbb{P}^{n-1} \), the fixed loci on \( \overline{M}_{1,J}(\mathbb{P}^{n-1},d) \) are described in \( [30] \). There are two types of fixed loci. The first type is indexed by the decorated one-loop graphs. The fixed locus indexed by a one-loop graph is exactly the same as that in section 2.1, and for a hypersurface \( X \) in \( \mathbb{P}^{n-1} \), its localization contribution \( \langle \mu_1, \cdots, \mu_{|J|} \rangle^X_{(m,J,d)} \) is the same as the localization contributions to the usual genus one Gromov-Witten invariants. The other type of fixed loci and their localization contributions are described in \( [30] \) (see also \( [36] \) in the cases \(|J| = 0 \) or \( 1 \)).

For a balloon manifold \( Y \), a finite set \( J \) and \( d \in H_2^+(Y) \), as an analogy to \( \mathbb{P}^{n-1} \), we assign two types of fixed loci. The formal fixed loci of the first type for \( (Y,J,d) \) are indexed by \( \mathcal{D} \mathcal{D} \mathcal{G}^d_{J}(Y) \), and their formal localization contributions to the reduced genus one Gromov-Witten invariants...
where \((\text{Ver}, \text{Edg}, v_0; \text{Ver}_+, \text{Ver}_0; m, \varnothing, e)\),

\begin{equation}
\bar{\Gamma} = \text{Ver}, \text{Edg}, v_0; \text{Ver}_+, \text{Ver}_0; m, \varnothing, e),
\end{equation}

where \((\text{Ver}, \text{Edg}, v_0)\) is a rooted tree and \(\bar{\Gamma}\)

(i) \(\text{Ver}_+, \text{Ver}_0 \subseteq \text{Ver} - \{v_0\}\), \(\text{Ver}_+ \neq \emptyset\), \(\text{Ver}_+ \cap \text{Ver}_0 = \emptyset\), \(\{v_0, v\} \in \text{Edg} \text{ for } v \in \text{Ver}_+ \cup \text{Ver}_0\).

(ii) \(\text{Edg} \triangleq \{\{v_0, v\} : v \in \text{Ver}_+\}, \text{Edg}_0 \triangleq \{\{v_0, v\} : v \in \text{Ver}_0\}..\)

(iii) There are three maps

\(m : \text{Ver} \to [N], \quad \varnothing : \text{Edg} - \text{Edg}_0 \to \mathbb{B}(Y), \quad e : J \to \text{Ver}\).

The map

\[\varnothing : \text{Edg} - \text{Edg}_0 \to \mathbb{B}(Y)\]

for an edge \(e = \{v_1, v_2\} \in \text{Edg} - \text{Edg}_0\) takes values in \(\mathbb{Z}_{>0}.P_{m(v_1)}P_{m(v_2)}\). Let \(d(e) \in \mathbb{Z}_{>0}\) such that \(\varnothing(e) = d(e) \cdot P_{m(v_1)}P_{m(v_2)}\) when \(e = \{v_1, v_2\}\). Let

\[\varnothing(\bar{\Gamma}) = \sum_{e \in \text{Edg} - \text{Edg}_0} \varnothing(e) \in \mathbb{B}(Y)\]

For \(\bar{\Gamma}\) to represent a formal fixed locus for \((Y, J, d)\), we need \(d \circ \varnothing(\bar{\Gamma}) = d\); we call \(d \circ \varnothing(\bar{\Gamma})\) the degree of \(\bar{\Gamma}\).

(iv) If \(v_1 \in \text{Ver}_+, v_2 \in \text{Ver} - \text{Ver}_0\) and \(\{v_0, v_2\} \in \text{Edg}\), then

\[\varnothing(\{v_0, v_1\}) = \varnothing(\{v_0, v_2\}) \Leftrightarrow v_2 \in \text{Ver}_+\].

Note that \(\varnothing(\{v_0, v_1\}) = \varnothing(\{v_0, v_2\})\) if and only if \(d(\{v_0, v_1\}) = d(\{v_0, v_2\})\) and \(m(v_1) = m(v_2)\).

(v) If \(\{v_1, v_2\} \in \text{Edg}\), then \(m(v_1) = m(v_2)\) if and only if \(v_1 = v_0\) and \(v_2 \in \text{Ver}_0\), or \(v_2 = v_0\) and \(v_1 \in \text{Ver}_0\).

(vi) If \(v_1 \in \text{Ver}_0\) then \(|\text{Edg}(v_1)| \geq 2\) and \(|\text{val}(v_1)| = |\text{Edg}(v_1)| + |e^{-1}(v_1)| \geq 3\).

Remark 2.2. In the definition of the refined decorated rooted trees in this article we do not include the condition \(\sum_{e \in \text{Edg}_+} d(e) \geq 2\). However, as the proof of lemma \ref{lemma} shows, the localization contribution of a refined decorated rooted tree which does not satisfy this condition is zero. So one can make the choice to include this condition or not; without this condition the summing over graphs becomes slightly easier.

We denote the set of refined decorated rooted trees for \((Y, J, d)\) by \(\mathcal{RDRT}^{\text{rd}}_J(Y)\). As before, when we are discussing a fixed \(Y\) which is clear from the context, we drop the notation \(Y\).

Let \(\bar{\Gamma} \in \mathcal{RDRT}^{\text{rd}}_J(Y)\). For every \(e \in \text{Edg}(v_0)\), there is an associated \textit{strand} \(Z_{\bar{\Gamma}}^e\), which is a decorated tree; we refer the reader to section 1.4 of \cite{30} for the definition. We need also the stacks \(\mathcal{M}_1(I, J)\) for finite sets \(I\) and \(J\), which are blow-ups of \(\mathcal{M}_1(I, J, J)\); we refer the reader to \cite{35} for the

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\(^4\)We highly recommend the reader who is not familiar with the definition of refined decorated rooted trees and some other related notions to refer to section 1.4 of \cite{30} for the case \(Y = \mathbb{P}^{n-1}\).

\(^5\)In \cite{30} the domain of the map \(m\) is \(\text{Ver} - \text{Ver}_0\). In this article we think it is more convenient to extend \(m\) to \(\text{Ver}\).
projection map for this happens we say \( \tilde{\psi} \) have the same underlying decorated rooted tree \( \Gamma \) and acts on the set of refined decorated rooted trees in this sense. Then the automorphism group \( \text{Aut}(\Gamma) \) of \( \tilde{\psi} \) is defined to be

\[
\text{Aut}(\Gamma) = \{ \tilde{\psi} : (\text{Ver}_0, \text{Edg}_0, \{ v_0 \}) \to (\text{Ver}_0, \text{Edg}_0, v_0) \text{ for some } v_0 \in \text{Ver}_0 \}.
\]

For a refined decorated rooted tree \( \tilde{\Gamma} = (\text{Ver}, \text{Edg}, v_0; \text{Ver}_{+}, \text{Ver}_0; m, d, e, I, K) \) we can naturally associate a decorated rooted tree \( \Gamma = (\text{Ver}^r, \text{Edg}^r, v_0; m^r, d^r, e^r) \) such that

- \( \text{Ver}^r = \text{Ver} - \text{Ver}_0 \).
- \( \text{Edg}^r = (\text{Edg} - \text{Edg}_0) \cup \{ (v_0, v) : v \in \text{Edg}(v_1) - \{ v_0 \} \text{ for some } v_1 \in \text{Ver}_0 \} \).
- \( m^r : \text{Ver}^r \to [N] \) is the restriction the map \( m \) to \( \text{Ver} - \text{Ver}_0 \).
- \( d^r : \text{Edg}^r \to \mathbb{B}(Y) \) is a map with \( d^r(e) = d(e) \) for \( e \in \text{Edg} - \text{Edg}_0 \), and \( d(\{ v_0, v \}) = d(\{ v_1, v \}) \text{ where } v_1 \in \text{Ver}_0 \text{ and } v \in \text{Edg}(v_1) - \{ v_0 \} \).
- \( e^r_j : J \to \text{Ver}^r \) is a map with \( e^r_j(j) = e(j) \) for \( j \in \epsilon^{-1}(\text{Ver} - \text{Ver}_0) \) and \( e^r_j(v_0) = v_0 \) for \( j \in \epsilon^{-1}(V_0) \).

By the assumptions in the definition of refined decorated rooted trees, especially \((v)\), we see that \( \Gamma(\text{Ver}^r, \text{Edg}^r, v_0; m^r, d^r, e^r) \) is a decorated rooted tree, which we call the \textit{underlying decorated rooted tree} of \( \tilde{\Gamma} \). There is a canonical projection map \( \pi_{\mathcal{DRT}^d} : \mathcal{DRT}^d \to \mathcal{DRT}^d_j \) which send a refined decorated rooted tree to its underlying decorated rooted tree.

\textbf{An important modification}: for the clearness in counting graphs in the future, we now make a slight modification of the definition of \( \mathcal{DRT}^d_j \). For a refined decorated rooted tree \( \tilde{\Gamma} \) and the corresponding decorated rooted tree \( \Gamma = \pi_{\mathcal{DRT}^d}(\tilde{\Gamma}) \) the edges in \( \text{Edg}^\Gamma(v_0) \) induces a partition of \( \text{Edg}^\Gamma(v_0) \). If \( e \in \text{Edg}^\tilde{\Gamma}(v_0) - \text{Edg}_0 \), then let \( I_e = \{ e \} \). If \( e = \{ v_0, v \} \in \text{Edg}_0 \), then let \( I_e = \text{Edg}(v) - \{ e \} \). Thus \( \{ I_e \}_{e \in \text{Edg}^\Gamma(v_0)} \) canonically corresponds to an \textit{unordered partition} of \( \text{Edg}^\Gamma(v_0) \). Similarly, for every \( e = \{ v_0, v \} \in \text{Edg}^\Gamma(v_0) \), let \( J_e = \epsilon^{-1}(v) \), together with \( \epsilon^{-1}(v_0) \), we obtain an \textit{unordered nonnegative partition} of \( J \). Now we impose the condition that the two partitions are \textit{ordered}. More precisely:

\textbf{Definition 2.1}. A refined decorated rooted tree is a tuple \( \tilde{\Gamma} = (\text{Ver}, \text{Edg}, v_0; \text{Ver}_{+}, \text{Ver}_0; m, d, e, I, K) \), such that \( \tilde{\Gamma} = (\text{Ver}, \text{Edg}, v_0; \text{Ver}_{+}, \text{Ver}_0; m, d, e) \) satisfies the conditions \((i)-(vi)\) above, and \( I \in \mathcal{A}_m(\text{Edg}^\Gamma(v_0)) \) and \( J \in \mathcal{A}_m(J) \) are \textit{colored partitions} which are compatible with the unordered partitions associated to \( \tilde{\Gamma} = (\text{Ver}, \text{Edg}, v_0; \text{Ver}_{+}, \text{Ver}_0; m, d, e) \), where \( m = |\text{Edg}^\Gamma(v_0)| \).

From now on, we adopt this definition of refined decorated rooted trees. We use \( \mathcal{DRT}^d_j \) to represent the set of refined decorated rooted trees in this sense. Then the automorphism group \( \text{Aut}(\Gamma) \) acts on \( \mathcal{DRT}^d_j(\Gamma) \) in a natural way, and the stable subgroup of \( \Gamma \) is called the automorphism group of \( \Gamma \). Two refined decorated rooted trees \( \Gamma_1 \) and \( \Gamma_2 \) index the same fixed locus if and only if they have the same underlying decorated rooted tree \( \Gamma \) and \( \Gamma_1 = g.\Gamma_2 \) for some \( g \in \text{Aut}(\Gamma) \), and when this happens we say \( \Gamma_1 \sim \Gamma_2 \).

Now we define the \textit{formal localization contribution} of \( \tilde{\Gamma} \). Let \( \pi_e : Z_{\tilde{\Gamma}} \to Z_{\Gamma_e} \) be the natural projection map for \( e \in \text{Edg}(v_0) \). On \( Z_{\tilde{\Gamma}} \) there is a universal tangent line bundle associated to the
attachment at \(v_0\), and we denote it by \(L_e\). Let \(L^e = \pi^e_* L_e\) for \(e \in \text{Edg}_+\); by \([35]\) \(\pi^e_* L_e\) as an equivariant line bundle is independent of the choice of \(e \in \text{Edg}_+\). Also, let
\[
F^e_{\tilde{\Gamma}; B} = \bigoplus_{e \in \text{Edg}(v_0) - \text{Edg}_+} \pi^e_* L_e.
\]

Let \(\gamma\) be the tautological line bundle on \(\mathbb{P}(\text{Ver}_+)^{\bullet - 1}\), and \(c_1(\gamma) = -H\). We use the same symbol for the pullbacks of \(\gamma\) and \(L\) to \(\tilde{\Gamma}\) via the natural projection maps.

For each \(e \in \text{Edg}(v_0)\), \(Z^e_{\tilde{\Gamma}}\) is a fixed locus in an appropriate moduli space of genus zero stable maps into \(Y\), thus the virtual normal bundle \(N \tilde{\Gamma}_{\tilde{\Gamma}}\) and the vector bundle \(U^e_0\) corresponding to \(X = \text{Tot}(E \to Y)\) are well-defined.

After these preparation, we define \(e(N \tilde{\Gamma})\) via
\[
\frac{e(N \tilde{\Gamma})}{e(T_{m(v_0)} Y)} = \prod_{e \in \text{Edg}(v_0)} \frac{e(N \tilde{\Gamma}_e)}{e(T_{m(v_0)} Y)} \cdot \frac{e(L^e_\tilde{\Gamma} \otimes F^e_{\tilde{\Gamma}; B} \otimes \gamma^*) e(L \otimes L^\tilde{\Gamma} \otimes \gamma)}{e(L^e_\tilde{\Gamma} \otimes T_{m(v_0)} Y \otimes \gamma^*)}
\]
and define
\[
e(U^e_I) \triangleq e(U_I) \cdot e(E)|_{P_{m(v_0)}} = \prod_{e \in \text{Edg}(v_0)} \pi^e_* e(U^e_0) \cdot e(L^e_\tilde{\Gamma} \otimes E_{\mu(v_0)} \otimes \gamma^*).
\]

We define the formal localization contribution of \(\tilde{\Gamma} = (\text{Ver}, \text{Edg}, v_0; \text{Ver}_+, \text{Ver}_0; m, d, e)\) to 
\[
\langle \mu_1, \cdots, \mu_J \rangle_{1,J,d}^{0,X}
\]
by
\[
\text{Cont}_{\tilde{\Gamma}}(\langle \mu_1, \cdots, \mu_J \rangle_{1,J,d}^{0,X}) = \frac{1}{\text{Aut}(\tilde{\Gamma})} \prod_{j \in J} \mu_j|_{P_{m\in(j)}} \cdot \int_{\tilde{\Gamma}} \frac{e(U_I)}{e(N \tilde{\Gamma})} \cdot \frac{e(U^e_I)}{e(N \tilde{\Gamma})^{\bullet \ne}}
\]
\[
= \frac{1}{\text{Aut}(\tilde{\Gamma})} \prod_{j \in J} \mu_j|_{P_{m\in(j)}} \cdot \int_{\tilde{\Gamma}} \frac{e(U_I)}{e(E)|_{P_{m(v_0)}}} \cdot \frac{e(U^e_I)}{e(N \tilde{\Gamma})^{\bullet \ne}}.
\]

**Definition 2.2.** Let \(Y\) be a balloon manifold and \(E\) a concave equivariant vector bundle over \(Y\). For the local space \(X = \text{Tot}(E \to Y)\) and \(\mu_1, \cdots, \mu_J \in H^*_Y(Y)\) we define
\[
\langle \mu_1, \cdots, \mu_J \rangle_{1,J,d}^{0,X} \triangleq \sum_{\Gamma \in \mathcal{D} \mathcal{D}_d^e} \text{Cont}_{\tilde{\Gamma}}(\langle \mu_1, \cdots, \mu_J \rangle_{1,J,d}^{0,X}) + \sum_{\tilde{\Gamma} \in \mathcal{D} \mathcal{D} \mathcal{D}_d^e/\sim} \text{Cont}_{\tilde{\Gamma}}(\langle \mu_1, \cdots, \mu_J \rangle_{1,J,d}^{0,X}).
\]

In particular, when \(X\) is a local Calabi-Yau space, the reduced genus one degree \(d\) Gromov-Witten invariants of \(X\) is defined by
\[
N^{0,X}_{1,d} = \sum_{\Gamma \in \mathcal{D} \mathcal{D}_d^e} \text{Cont}_{\Gamma}(\langle \mu_1, \cdots, \mu_J \rangle_{1,J,d}^{0,X}) + \sum_{\tilde{\Gamma} \in \mathcal{D} \mathcal{D} \mathcal{D}_d^e/\sim} \text{Cont}_{\tilde{\Gamma}}(\langle \mu_1, \cdots, \mu_J \rangle_{1,J,d}^{0,X}),
\]
where \(\text{Cont}_{\Gamma}(\langle \mu_1, \cdots, \mu_J \rangle_{1,J,d}^{0,X})\) for \(\Gamma \in \mathcal{D} \mathcal{D}_d^e\) and \(\text{Cont}_{\tilde{\Gamma}}(\langle \mu_1, \cdots, \mu_J \rangle_{1,J,d}^{0,X})\) for \(\tilde{\Gamma} \in \mathcal{D} \mathcal{D} \mathcal{D}_d^e/\sim\).
By this definition, when $X$ is a local Calabi-Yau space, $N_{1,d}^{0;X}$ is a priori an element of $\mathbb{Q}_\alpha$. We will see that as a corollary of the LSvR, we have in fact $N_{1,d}^{0;X} \in \mathbb{Q}$.

Now we write \[^{44}\] as a product of contributions of edges and vertices. Since we will finally write the summing

$$\sum_{\bar{\epsilon} \in \mathfrak{D} \mathfrak{D} \mathfrak{T}_1^{\varnothing}} \text{Cont}_\Gamma((\mu_1, \ldots, \mu_1)_{j/1, j,d})$$

into a summing over $\mathfrak{D} \mathfrak{D} \mathfrak{T}_1^{\varnothing}$, we need to gather the contributions from the vertices in $\text{Edg}_0$ and put it into the contribution of the root $v_0$. For this, let us introduce some notations.

Suppose $m(v_0) = i$. For $e = \{v_0, v\} \in \text{Edg}(v_0) \setminus (\text{Edg}_0 \cup \text{Edg}_+)$, let $I_e = \{e\}$, and let $\omega_e = \frac{\alpha_{i,m(v)}}{d(e)}$. For $e = \{v_0, v\} \in \text{Edg}_0$, let $I_e = \text{Edg}(v) \setminus \{e\}$, and for $f = \{v, v'\} \in \text{Edg}(v) \setminus \{e\}$, let $\omega_f = \frac{\alpha_{i,m(v)}}{d(f)}$.

Moreover, let $\omega_+ = \frac{\alpha_{i,m(v)}}{d(e)}$ for any $e = \{v_0, v\} \in \text{Edg}_+$, which is well-defined because $\alpha_{m(v)}$ and $d(e)$ are independent of the choice of $e$ in $\text{Edg}_+$; thus $\omega_+$ is the equivariant Euler class of $L_\Gamma$. Let

$$|\text{Edg}(v_0)| = |\text{Edg}(v_0) \setminus \text{Edg}_0| + \sum_{v \in \text{Edg}_0} (|\text{Edg}(v)| - 1). \quad (47)$$

Equivalently, $|\text{Edg}(v_0)|$ is equal to $|\text{Edg}_1(v_0)|$, where $\Gamma = \pi_{\mathfrak{D} \mathfrak{D} \mathfrak{T}_1}(\bar{\epsilon})$.

Then we define

$$\text{Cont}_{\bar{\epsilon},v_0}((\mu_1, \ldots, \mu_1)_{j/1, j,d}) = \prod_{j \in \epsilon} \mu_{j}^{P_{m(v_0)}}, \prod_{k=1}^{i} \varepsilon_{i,k}, \prod_{P_j \in \text{Nb}(P_i)} \alpha_{i,j}, \frac{1}{|\text{Edg}(v_0)| - 1} \cdot \prod_{e \in \text{Edg}(v_0) \setminus \text{Edg}_+ - \text{Edg}_0} \frac{1}{\omega_e - \omega_+ + H} \prod_{e \in \text{Edg}(v_0) \setminus \text{Edg}_0} \frac{1}{\omega_e - \omega_+ + H} \prod_{f \in \text{Edg}_0} \frac{1}{\omega_f - \omega_f} \cdot \prod_{v \in \text{Ver}_0} \frac{1}{\omega_e - \omega_+ + H} \prod_{f \in \text{Edg}_0} \frac{1}{\omega_f - \omega_f}. \quad (48)$$

For edges in $\text{Edg} \setminus \text{Edg}_0$, and vertices in $\text{Ver} \setminus \{v_0\} \cup \text{Ver}_0$, the contributions are defined as those in \[^{13}\] respectively. Then by \[^{12}\], \[^{13}\], and \[^{14}\], it is not hard to see that, for $\bar{\epsilon} \in \mathfrak{D} \mathfrak{D} \mathfrak{T}_1^{\varnothing}$,

$$\text{Cont}_\bar{\epsilon}(\mu_1, \ldots, \mu_1)_{j/1, j,d} = \frac{1}{\text{Ant}(\bar{\epsilon})} \prod_{v \in \text{Ver}_0} \text{Cont}_{\bar{\epsilon},v}(\mu_1, \ldots, \mu_1)_{j/1, j,d} \prod_{e \in \text{Edg} \setminus \text{Edg}_0} \text{Cont}_{\bar{\epsilon},e}(\mu_1, \ldots, \mu_1)_{j/1, j,d}. \quad (49)$$

There is a canonical bijection between $\text{Ver} \setminus \text{Ver}_0$ and $\text{Ver}(\pi_{\mathfrak{D} \mathfrak{D} \mathfrak{T}_1}(\bar{\epsilon}))$, and between $\text{Edg} \setminus \text{Edg}_0$ and $\text{Edg}(\pi_{\mathfrak{D} \mathfrak{D} \mathfrak{T}_1}(\bar{\epsilon}))$. Using this bijection, from \[^{49}\] we finally obtain

$$\langle \mu_1, \ldots, \mu_1 \rangle_{1, j,d} = \sum_{\bar{\epsilon} \in \mathfrak{D} \mathfrak{D} \mathfrak{T}_1^{\varnothing}} \text{Cont}_\bar{\epsilon}(\mu_1, \ldots, \mu_1)_{j/1, j,d} + \sum_{\bar{\epsilon} \in \mathfrak{D} \mathfrak{D} \mathfrak{T}_1^{\varnothing}} \text{Cont}_\bar{\epsilon}(\mu_1, \ldots, \mu_1)_{j/1, j,d}. \quad (50)$$
where for $\Gamma \in \mathcal{D}_d$,
\[ \text{Cont}_\Gamma((\mu_1, \ldots, \mu_{|J|})^{0;X}_{1,J,d}) = \frac{1}{\text{Aut}(\Gamma)} \cdot \sum_{\tilde{\Gamma} \in \pi_{1,J,d}(\Gamma)} \prod_{v \in \text{Ver}(\Gamma)} \text{Cont}_{\tilde{\Gamma}}((\mu_1, \ldots, \mu_{|J|})^{0;X}_{1,J,d}) \prod_{e \in \text{Edg}(\Gamma)} \text{Cont}_{\tilde{\Gamma} e}((\mu_1, \ldots, \mu_{|J|})^{0;X}_{1,J,d}). \] (51)

**Remark 2.3.** When $Y = \mathbb{P}^{n-1}$, our formal fixed locus associated to a refined decorated rooted tree may be different from that in [30], but the factor $e(\mathcal{N}_G)$ in the localization contribution is the same, as remarked in the footnote 16 in [36].

### 3 Localized standard versus reduced formula

#### 3.1

Let $Y$ be a balloon manifold of dimension $n-1$, $E$ a concave equivariant vector bundle of rank $l$ over $Y$, and $X = \text{Tot}(E \to Y)$. Now we state the first main theorem of this article.

**Theorem 3.1.** Let $\mu_1, \ldots, \mu_{|J|} \in H^*_c(Y)$. For every decorated rooted $\Gamma \in \mathcal{D}_d$, we have the LSvR
\[
\text{Cont}_\Gamma((\mu_1, \ldots, \mu_{|J|})^X_{1,J,d}) = \text{Cont}_\Gamma((\mu_1, \ldots, \mu_{|J|})^{0;X}_{1,J,d}) + \sum_{m \geq 1} \sum_{J' \subset J} \left[ (-1)^{m+|J'|} m! (m-1)! \right] \frac{1}{24} \sum_{p=0}^{n+1-|J'|-2m} \left( \eta_p c_{n+1-|J'|-2m-p}(TX); \mu_1, \ldots, \mu_{|J'|}^X_{m,J-J',d} \right). \] (52)

In particular, when $X$ is a local Calabi-Yau space, for every $\Gamma \in \mathcal{D}_d$ we have
\[
\text{Cont}_\Gamma(N^X_{1;d}) = \text{Cont}_\Gamma(N^{0;X}_{1;d}) + \frac{1}{24} \sum_{m \geq 1} \left[ (-1)^m (m-1)! \right] \sum_{p=0}^{n+1-2m} \left( \eta_p c_{n+1-2m-p}(TX); \mu_1, \ldots, \mu_{|J'|}^X_{m,J-J',d} \right). \] (53)

Note that by definition, for $\Gamma \in \mathcal{D}_d$ we have
\[
\text{Cont}_\Gamma((\mu_1, \ldots, \mu_{|J|})^X_{1,J,d}) = \text{Cont}_\Gamma((\mu_1, \ldots, \mu_{|J|})^{0;X}_{1,J,d}) \] (54)
thus by (29) and (24), (54), (10) we have

**Corollary 3.1.**

\[
(\mu_1, \ldots, \mu_{|J|})^X_{1,J,d} = (\mu_1, \ldots, \mu_{|J|})^{0;X}_{1,J,d} + \sum_{m \geq 1} \sum_{J' \subset J} \left[ (-1)^{m+|J'|} m! (m-1)! \right] \frac{1}{24} \sum_{p=0}^{n+1-|J'|-2m} \left( \eta_p c_{n+1-|J'|-2m-p}(TX); \mu_1, \ldots, \mu_{|J'|}^X_{m,J-J',d} \right). \] (55)

In particular, when $X$ is a local Calabi-Yau space,
\[
N^{0;X}_{1,d} = N^{0;X}_{1,d} + \frac{1}{24} \sum_{m \geq 1} (-1)^m (m-1)! \sum_{p=0}^{n+1-2m} \left( \eta_p c_{n+1-2m-p}(TX); \mu_1, \ldots, \mu_{|J'|}^X_{m,J-J',d} \right). \] (56)
Note that when \( X \) is a local Calabi-Yau space, in our definition of \( N_{d}^{0,X} \), we have fixed a choice of the linearization of \( E \). A consequence of corollary 3.1 is that \( N_{d}^{0,X} \) is independent of the choice of linearization of \( E \).

The proof of theorem 3.1 will occupy sections 3.1-3.4. First note that, in (22), (36) and (51), the factors

\[
\frac{1}{\text{Aut}(\Gamma)} \prod_{v \in \text{Ver} \setminus \{v_{0}\}} \text{Cont}_{\Gamma;v}(\cdot) \prod_{e \in \text{Edg}} \text{Cont}_{\Gamma;e}(\cdot)
\]

are common; in fact by definition, for \( v \in \text{Ver} \setminus \{v_{0}\} \), \( \text{Cont}_{\Gamma;v}(\cdot) \) are equal for the three types of invariants in (53), and so are \( \text{Cont}_{\Gamma;e}(\cdot) \) for \( e \in \text{Edg} \). So it suffices to show

\[
\text{Cont}_{\Gamma;v_{0}}(\langle \mu_{1}, \cdots, \mu_{|J|} \rangle_{1,J,d}^{X}) = \sum_{\Gamma \in \pi_{0}B_{\Delta}^{d}} \sum_{m \geq 1} \sum_{J' \subseteq J} \frac{(-1)^{m+|J'|}m^{|J'|}}{24m} \sum_{p=0}^{n+l-1-|J'|-2m} \sum_{\Gamma^{*} \in \pi_{0}B_{\Delta}^{d}} \text{Cont}_{\Gamma^{*}}(\langle \mu_{1}, \cdots, \mu_{|J'|} \rangle_{J',d}^{X})
\]

This formula concerns only the root \( v_{0} \) together with \( m(v_{0}), \text{Edg}(v_{0}) \) together with their degrees, and the vertices \( v \) neighborhood to \( v_{0} \) together with their labels \( m(v) \). So we only need to show (57) for decorated stars. Here by a decorated star, we mean a decorated rooted tree with \( \text{val}(v) = 1 \) for all \( v \in \text{Ver} \setminus \{v_{0}\} \). The set of decorated stars of degree \( d \) with the set of marked points \( J \) is denoted by \( \mathcal{D}_{J}^{d} \); we have \( J = \varepsilon^{-1}(v_{0}) \). Note further that the three (groups of) terms in (57) has a common factor (recall (47) and the statement following it)

\[
\prod_{j \in \varepsilon^{-1}(v_{0})} \mu_{j} \bigg|_{p_{m(v_{0})}} \cdot \prod_{P_{j} \in \text{Nb}(P_{j})} \alpha_{i,j} \prod_{k=1}^{l} \varepsilon_{i,k} \left[(\text{Edg}(v_{0}) \setminus 1)^{1} \cdot \right].
\]

Therefore, for \( \Gamma \in \mathcal{D}_{J}^{d} \) with \( \mu(v_{0}) = i \), let

\[
\text{Cont}_{\Gamma}^{X} = \int_{M_{1,\text{Edg}(v_{0}) \setminus \varepsilon^{-1}(v_{0})}} \prod_{k=1}^{l} \prod_{P_{j} \in \text{Nb}(P_{j})} \prod_{\Gamma^{*} \in \text{Edg}(v_{0})} \frac{\Lambda_{i}^{\Gamma}(\varepsilon_{i,k})}{\Lambda_{i}^{\Gamma}(\alpha_{i,j})},
\]

\[
\text{Cont}_{\Gamma}^{0,X} = \sum_{\Gamma \in \pi_{0}B_{\Delta}^{d}} \int_{M_{1,\text{Edg}(v_{0}) \setminus \varepsilon^{-1}(v_{0})} \times \mathbb{Z}^{[\text{Edg},|1| - 1)}} \left( \prod_{i=1}^{l}(\varepsilon_{i,k} + |H|) \prod_{P_{j} \in \text{Nb}(P_{j})} (-\omega_{+} + |H|) \right) \prod_{\Gamma \in \text{Edg}(v_{0}) \setminus \text{Edg}_{+,d}} \prod_{v \in \text{Ver}_{0}} \omega_{+} - \omega_{e} - H \prod_{v \in \text{Ver}_{0}} \frac{1}{\omega_{e} - \omega_{+} + H} \prod_{e \in \text{Edg}(v_{0}) \setminus \text{Edg}_{+,d}} \frac{1}{\omega_{e} - \omega_{+} + H} \prod_{f \in I_{e}} (\omega_{f} - |H|),
\]
\[
\mathbb{Cont}^{(m,p,q),X}_{\Gamma, J - J'} = [x^n] \left( \prod_{k=1}^{l} (1 + \varepsilon_{i,k} x) \prod_{P_j \in \text{Nb}(P_i)} (1 + x(\alpha_{i,j})) \right) \\
\cdot \sum_{\Gamma^r \in \pi_1 \mathfrak{M}_d \cap \mathfrak{P}_{J - J', J}^{\Gamma}} \int_{\mathcal{M}(P_0, t(1) \cup K(1) \cup \cdots \cup K(2m)} \frac{1}{\prod_{e \in \text{Edg}(v_0)} (\alpha_{v_0, e} - \psi_{(v_0, e)}) \prod_{s=1}^{m} (1 - x\psi_{v_s})}.
\]

(60)

Then to show (57) it suffices to show for every \( \Gamma \in \mathfrak{D}_d \),

\[
\mathbb{Cont}^X = \mathbb{Cont}^{0,X} + \sum_{m=1}^{2m \leq n+1-1} \sum_{J' \subseteq J} \frac{(-1)^{m+|J'|} m!}{24m} \sum_{p=0}^{n+1-|J'|-2m} \mathbb{Cont}^{(m,p,n+l-1-|J'|-2m),X}_{\Gamma, J - J'}.
\]

(61)

Suppose \( \Gamma \) is a decorated star whose root is labeled by \( i \) with \( |\text{Edg}(v_0)| = r \), the vertices other than \( v_0 \) are labeled by \( j_1, \cdots, j_r \), \( e^{-1}(v_0) = J \), and the degree of the corresponding edges are \( d_1, \cdots, d_r \). Let \( \omega_s = \alpha_{v_0, e} \); since \( Y \) is a balloon manifold, \( \omega_{s_1} = \omega_{s_2} \) if and only if \( j_{s_1} = j_{s_2} \) and \( d_{s_1} = d_{s_2} \). Let \( \psi = (\omega_1, \cdots, \omega_r) \), and we use \( \Gamma_{\psi,j} \) to represent this decorated star. When \( J = \emptyset \), we denote \( \Gamma_{\psi,j} = \Gamma_{\psi,j, J} \) for short. When \( d_s = 1 \) for \( 1 \leq s \leq r \) and \( j_s \) are pairwisely distinct for \( 1 \leq s \leq r \), we call \( \Gamma_{\psi,j} \) a simply decorated star.

In section 3.2, we prove (61) for simply decorated stars with \( J = \emptyset \). In section 3.3, we prove (61) for all decorated stars with \( J = \emptyset \). In section 3.4 we prove for \( J \neq \emptyset \). In section 3.5 we state another form of LSvR and sketch a proof for it.

### 3.2 Simply decorated stars

Let \( \Gamma_{\psi,j} \in \mathfrak{D}_d \) be a simply decorated star with \( J = \emptyset \).

**Lemma 3.1.**

\[
\int_{\mathcal{M}(P_0, t(1) \cup K(1) \cup \cdots \cup K(2m)} \frac{\lambda_1}{\prod_{k=1}^{l} (w_k - \psi_k)} = \frac{1}{24 \prod_{k=1}^{l} w_k} \left( \sum_{k=1}^{r} \frac{1}{w_k} \right)^{r-1}.
\]

(62)

**Proof:** This follows straightforwardly from the proposition 3.1 of [15] and (21), or from the \( \lambda_0 \)-conjecture for \( g = 1 \). \( \square \)

**Lemma 3.2.**

\[
\mathbb{Cont}^{0,X}_{\Gamma_{\psi,j}} = 0.
\]

(63)

**Proof:** Since \( \Gamma_{\psi,j} \) is simply decorated, by the definition of refined decorated rooted trees, for every \( \tilde{\Gamma} \in \pi_1 \mathfrak{M}_d \cap \mathfrak{P}_{J,J, J}^{\Gamma_{\psi,j}} \) we have \( |\text{Edg}(\tilde{\Gamma})| = 1 \) and \( \omega_+ = \alpha_{i,j} \) for some \( P_j \in \text{Nb}(P_i) \). Thus in the right-hand-side (109), the integrand in the second row has a factor \( H \), but \( p^{|\text{Edg}(\tilde{\Gamma})| - 1} \) is a point, so the conclusion follows. \( \square \)
By (62), we have

\[
\text{Cont}_{\Gamma_{i,\omega}}^X = \int_{\mathcal{M}_{1,r}} \prod_{k=1}^l \Lambda_k^y(\varepsilon_{i,k}) \cdot \prod_{P_j \in \text{Nb}(P)} \Lambda_j^y(\alpha_{i,j}) \prod_{k=1}^l (\alpha_{i,j_k} - \psi_k) \\
= \prod_{k=1}^l \varepsilon_{i,k} \prod_{P_j \in \text{Nb}(P)} \alpha_{i,j} \int_{\mathcal{M}_{1,r}} \prod_{k=1}^l (\alpha_{i,j_k} - \psi_k) \\
- [x] \left( \prod_{k=1}^l (x + \varepsilon_{i,k}) \prod_{P_j \in \text{Nb}(P)} (x + \alpha_{i,j}) \right) \cdot \int_{\mathcal{M}_{1,r}} \prod_{k=1}^l (\alpha_{i,j_k} - \psi_k) \\
= \prod_{k=1}^l \varepsilon_{i,k} \prod_{P_j \in \text{Nb}(P)} \alpha_{i,j} \int_{\mathcal{M}_{1,r}} \prod_{k=1}^l (\alpha_{i,j_k} - \psi_k) \\
- [x] \left( \prod_{k=1}^l (x + \varepsilon_{i,k}) \prod_{P_j \in \text{Nb}(P)} (x + \alpha_{i,j}) \right) \cdot \frac{\lambda_1}{24 \prod_{k=1}^l \alpha_{i,j_k} \left( \sum_{k=1}^r \frac{1}{\alpha_{i,j_k}} \right)^{r-1}}.
\]

(64)

On the other hand,

\[
\frac{(-1)^m n + l - 1 - 2m}{24m} \sum_{p=0}^{n+l-1-2m} \text{Cont}_{\Gamma_{i,\omega}}^{(m,p,n+l-1-2m-p),X} = \frac{(-1)^m}{24m} \sum_{I \in \mathcal{A}_m([r])} \int_{\mathcal{M}_{0,(l+1)} \times \cdots \times \mathcal{M}_{0,l(m)} \omega(0_m)} [x^{n+l-1-2m}] \left( \prod_{k=1}^l (1 + \varepsilon_{i,k} x) \prod_{P_j \in \text{Nb}(P)} (1 + \alpha_{i,j} x) \right) \\
= \frac{(-1)^m}{24m} \sum_{I \in \mathcal{A}_m([r])} \int_{\mathcal{M}_{0,(l+1)} \times \cdots \times \mathcal{M}_{0,l(m)} \omega(0_m)} [x^m] \left( \prod_{k=1}^l (x + \varepsilon_{i,k}) \prod_{P_j \in \text{Nb}(P)} (x + \alpha_{i,j}) \right) \\
= \frac{(-1)^m}{24m} \sum_{I \in \mathcal{A}_m([r])} [x^m] \left( \prod_{k=1}^l (x + \varepsilon_{i,k}) \prod_{P_j \in \text{Nb}(P)} (x + \alpha_{i,j}) \right) \\
\cdot \int_{\mathcal{M}_{0,(l+1)} \times \cdots \times \mathcal{M}_{0,l(m)} \omega(0_m)} \prod_{k=1}^l (\alpha_{i,j_k} - \psi_k) \cdot \prod_{s=1}^m (x - \psi_{0_s}) \cdot \prod_{s=1}^m \left( \frac{1}{x} + \sum_{j \in I(s)} \frac{1}{\alpha_{i,j}} \right)^{|I(s)|-2}.
\]

(65)

Some attention should be paid to the equality (65). For a fixed \( I \in \mathcal{A}_m([r]) \), without loss of generality we may assume \( I(s) = \{ j_s \} \) for \( 1 \leq s \leq r_1 \) and \( |I(s)| \geq 2 \) for \( r_1 + 1 \leq s \leq r \). Thus

\[
\int_{\mathcal{M}_{0,(l+1)} \times \cdots \times \mathcal{M}_{0,l(m)} \omega(0_m)} [x^{n+l-1-2m}] \left( \prod_{k=1}^l (1 + \varepsilon_{i,k} x) \prod_{P_j \in \text{Nb}(P)} (1 + \alpha_{i,j} x) \right) \\
= [x^{n+l-1-2m}] \left( \prod_{s=1}^{r_1} \left( 1 + \alpha_{i,j_k} x \right)^{-1} \prod_{k=1}^l (1 + \varepsilon_{i,k} x) \prod_{P_j \in \text{Nb}(P)} (1 + \alpha_{i,j} x) \right) \\
\cdot \int_{\mathcal{M}_{0,(r_1+1)} \times \cdots \times \mathcal{M}_{0,l(m)} \omega(0_m)} \prod_{k=1}^l (\alpha_{i,j_k} - \psi_k) \cdot \prod_{s=r_1+1}^m (1 - x \psi_{0_s}) \cdot \prod_{s=1}^m \left( \frac{1}{x} + \sum_{j \in I(s)} \frac{1}{\alpha_{i,j}} \right)^{|I(s)|-2}.
\]

(66)
By the assumption that $\alpha_{i,j}$ for $1 \leq k \leq r$ are pairwisely distinct, the factor $\prod_{s=1}^{r}(1+\alpha_{i,j},x)$ in the denominator divides the product $\prod_{j \in \text{Nb}(P_i)}(1+\alpha_{i,j},x)$ in the numerator. Thus the righthand-side of (66) as a rational function of $x$ has only poles at $x = 0$ and $x = \infty$, so the equality (*) follows.

For the general decorated stars, the corresponding expression to the righthand-side of (66) has other poles, from which the contributions of refined decorated stars arise.

Now from the following proposition and by lemma 3.2 we see that (61) holds for simply decorated stars with $J = \emptyset$.

**Proposition 3.1.** Let $w_1, \cdots, w_r$ be independent variables. Then we have

$$
\sum_{m=1}^{r} \frac{(-1)^m}{24m} \sum_{I \in A_m([r])} [x^m] \left( \prod_{k=1}^{r} (x + \varepsilon_{i,k}) \prod_{P_j \in \text{Nb}(P_i)}(x + \alpha_{i,j}) \prod_{s=1}^{m} \left( \frac{1}{x} + \sum_{j \in I_{(i)}} \frac{1}{w_j} \right)^{|I_{(i)}|-2} \right) 
= \prod_{k=1}^{r} \varepsilon_{i,k} \prod_{P_j \in \text{Nb}(P_i)} \alpha_{i,j} \int_{A_1,r} \frac{1}{\prod_{k=1}^{r} (w_k \psi_k)}
- [x] \left( \prod_{k=1}^{r} (x + \varepsilon_{i,k}) \prod_{P_j \in \text{Nb}(P_i)}(x + \alpha_{i,j}) \right) \cdot \frac{1}{24r!} \frac{1}{\prod_{k=1}^{r} w_k} \left( \sum_{k=1}^{r} \frac{1}{w_k} \right)^{-r-1}.
$$

The proposition follows immediately from the following lemma.

**Lemma 3.3.** We have

$$
\sum_{m=1}^{r} \frac{(-1)^m}{m} \sum_{I \in A_m([r])} [x^{-1}] \left( \prod_{s=1}^{m} \left( 1 + \sum_{j \in I_{(i)}} x w_j \right)^{|I_{(i)}|-2} \right) = - \left( \sum_{k=1}^{r} w_k \right)^{-r-1},
$$

$$
[x^r] \left( \sum_{m=1}^{r} \frac{(-1)^m}{24m} \sum_{I \in A_m([r])} \prod_{s=1}^{m} \left( 1 + \sum_{j \in I_{(i)}} x w_j \right)^{|I_{(i)}|-2} \right)
= \int_{A_1,r} \frac{1}{\prod_{k=1}^{r} (1 - w_k \psi_k)},
$$

and for $2 \leq p \leq r$,

$$
\sum_{m=1}^{r} \frac{(-1)^m}{m} \sum_{I \in A_m([r])} [x^{r-p}] \left( \prod_{s=1}^{m} \left( 1 + \sum_{j \in I_{(i)}} x w_j \right)^{|I_{(i)}|-2} \right) = 0.
$$

**Proof of the lemma:** Let

$$
H_r(w_1, \cdots, w_r) = \sum_{m=1}^{r} \frac{(-1)^m}{24m} \sum_{I \in A_m([r])} \left( \prod_{s=1}^{m} \left( 1 + \sum_{j \in I_{(i)}} w_j \right)^{|I_{(i)}|-2} \right),
$$

and

$$
F_r(w_1, \cdots, w_r) = \int_{A_1,r} \frac{1}{\prod_{k=1}^{r} (1 - w_k \psi_k)}.
$$

Consider the Taylor expansion of $H_r(w_1, \cdots, w_r)$ at $w_1 = \cdots = w_r = 0$, and for $p \in \mathbb{Z}_{\geq 0}$ let $H_{r,p}(w_1, \cdots, w_r)$ be the degree $p$ part of this expansion, which is a symmetric polynomial in $w_1, \cdots, w_r$ of degree $p$. Then what we need to prove is

$$
H_{r,r-1}(w_1, \cdots, w_r) = - \frac{1}{24} \left( \sum_{k=1}^{r} w_k \right)^{r-1},
$$

20
and for $r \in \mathbb{Z}^{>0}$, and $0 \leq p \leq r-2$

$$H_{r,p}(w_1, \cdots, w_r) = 0.$$  

We prove these identities by induction on $r$. The $r = 1$ case is trivial. For $r > 1$, by the definition of $H_r$, it is not hard to see that for a fixed $I \in A_m([r-1])$ the coefficient of

$$\prod_{s=1}^{m} \left(1 + \sum_{j \in I(s)} w_j\right)^{|I(s)|-2}$$

in $H_r(w_1, \cdots, w_{r-1}, 0)$ is

$$\frac{(-1)^m}{24m} \sum_{s=1}^{m} \left(1 + \sum_{j \in I(s)} w_j\right) + \frac{(-1)^{m+1}}{24(m+1)} \cdot (m+1) = \frac{(-1)^m}{24m} \sum_{k=1}^{r-1} w_k.$$

Therefore

$$H_r(w_1, \cdots, w_{r-1}, 0) = \sum_{k=1}^{r-1} w_k \cdot H_{r-1}(w_1, \cdots, w_{r-1}).$$  

Note that a symmetric polynomial $f(w_1, \cdots, w_r)$ of degree less than $r$ is uniquely determined by $f(w_1, \cdots, w_{r-1}, 0)$, so by induction we obtain (74) and (72). For (73), note that by the string equation, (75) holds for $F_r$, i.e., we have

$$F_r(w_1, \cdots, w_{r-1}, 0) = \sum_{k=1}^{r-1} w_k \cdot F_{r-1}(w_1, \cdots, w_{r-1}).$$

Note also that a symmetric polynomial $f(w_1, \cdots, w_r)$ of degree $r$ is determined by $f(w_1, \cdots, w_{r-1}, 0)$ up to the coefficient of $w_1 \cdots w_r$. So by induction, to show that (74) holds for $r$, it suffices to show that the coefficients of $w_1 \cdots w_r$ in $H_{r,r}(w_1, \cdots, w_r)$ and in $F_r(w_1, \cdots, w_r)$ are equal. But by the definition of $H_r$, it is easy to see that the coefficient of $w_1 \cdots w_r$ in it (this monomial appears only when $m = r$ and $|I(s)| = 1$ for $1 \leq s \leq r$) is

$$\frac{(r-1)!}{24},$$

which is equal to $\int_{M_{1,r}} \psi_1 \cdots \psi_r$, by the dilaton equation.

Remark 3.1. Although the simply decorated stars are the most simple cases, they provide a prototype of the LSvR. As we will see, the contribution from the reduced invariants are correction terms when the assumption of simple decorated stars is not satisfied. The formula (69) can be viewed as a combinatorial solution to the $n$-point function of (25) in genus one. It is interesting to find a higher genera analog of this formula, which may shed light on the computation of Gromov-Witten invariants in higher genera.

3.3 General decorated stars

Let $\Gamma_{i,\vec{\omega}} \in \mathcal{DRT}^{d}$ be a decorated star where $\vec{\omega} = (\omega_1, \cdots, \omega_r)$. Similar to (64), we have

$$\overline{\text{Cont}}_{\Gamma_{i,\vec{\omega}}} = \prod_{k=1}^{l} \varepsilon_k \prod_{p_j \in \text{Nb}(P_i)} \alpha_{i,j} \int_{M_{1,r}} \frac{1}{\prod_{k=1}^{r} (\omega_k - \psi_k)} \cdot [x] \left( \prod_{k=1}^{l} (x + \varepsilon_k) \prod_{p_j \in \text{Nb}(P_i)} (x + \alpha_{i,j}) \right) \cdot \frac{1}{24 \prod_{k=1}^{r} \omega_k} \left( \sum_{k=1}^{r} \frac{1}{\omega_k} \right)^{r-1}.$$  

(76)
Now we consider contributions of the refined decorated rooted trees whose underlying decorated rooted tree is $\Gamma_{i,\omega}$. Define $\tilde{A}_m([r];\vec{\omega})$ to be the set of 3-tuples $(I, U, V) \in \mathcal{P}(|r|)^m \times [m] \times [m]$ satisfying

(i) Writing $I$ as $I = (I(1), \cdots, I(m))$, then $\bigsqcup_{s=1}^m I(s) = [r]$;
(ii) $U \cap V = \emptyset$;
(iii) $|I(s)| = 1$ for $s \in U \cup V$, and $|I(s)| \geq 2$ for $s \not\in U \cup V$;
(iv) $\omega$ are equal to each other for $s \in U$, and $\omega_i \neq \omega_s$ for $s \in U$ and $i \in V$.

For $(I, U, V) \in \tilde{A}_m([r];\vec{\omega})$, denote the common weight $\omega_s$ for $s \in U$ by $\omega_U$ and suppose $I(s) = \{i_s\}$ for $s \in U \cup V$, $I_U = \bigsqcup_{s \in U} I(s)$.

By the definition of refined decorated rooted trees, it is not hard to see that there is a natural 1-1 correspondence between $\tilde{A}_m([r];\vec{\omega})$ and $\pi^{-1}_{\Gamma_{i,\omega}}(\Gamma_{i,\omega})$. Thus by \(59\) and \(39\) we have

$$\text{Cont}^0_{\Gamma,i,\omega} = \sum_{m \geq 1} \sum_{(I, U, V) \in \tilde{A}_m([r];\vec{\omega})} \text{Cont}^0_{\Gamma_{i,\omega}}(I, U, V),$$

where

$$\text{Cont}^0_{\Gamma_{i,\omega}}(I, U, V) = \frac{1}{m!} \int_{\mathcal{M}_1 \times \mathcal{P}^{|U|-1}} \left( \prod_{k=1}^{|U|} \left(-\omega_U - \epsilon_{i,k} + H\right) \prod_{P_j \in \text{Nh}(P)} (-\omega_U + \alpha_{i,j} + H) \right)$$
$$\times \prod_{s \in V} \frac{1}{-\omega_U + H + \omega_s} \prod_{s \in [m] \setminus (U \cup V)} \int_{\mathcal{M}_0 \times [\omega(\omega_s)]} \frac{1}{(-\omega_U - \psi_0 + H) \prod_{j \notin I_s} (\omega_j - \psi_j)}$$
$$= \frac{(-1)^{m+1}(m-1)!}{24m!} \int_{\mathcal{P}^{|U|-1}} \left( \prod_{k=1}^{|U|} \left(H - \omega_U + \epsilon_{i,k}\right) \prod_{P_j \in \text{Nh}(P)} (H - \omega_U + \alpha_{i,j}) \right)$$
$$\times \prod_{s \in [m] \setminus U} \left( \frac{1}{H - \omega_U} + \sum_{j \notin I_s} \frac{1}{\omega_j} \right)^{|I_s| - 2}$$
$$= \frac{(-1)^{m+1}}{24m!} \left[ y^{|U|-1} \right] \left( \prod_{k=1}^{|U|} \left(y - \omega_U + \epsilon_{i,k}\right) \prod_{P_j \in \text{Nh}(P)} (y - \omega_U + \alpha_{i,j}) \right)$$
$$\times \prod_{s \in [m] \setminus U} \left( \frac{1}{y - \omega_U} + \sum_{j \notin I_s} \frac{1}{\omega_j} \right)^{|I_s| - 2}. \tag{78}$$

Note that there is a canonical projection
$$\bar{\pi} : \tilde{A}_m([r];\vec{\omega}) \to A_m([r])$$
defined by $\bar{\pi}(I, U, V) = I$. We are going to write

$$\frac{(-1)^{m} n^{l-1-2m}}{24m!} \sum_{p=0}^m \text{Cont}^0_{\Gamma_{i,\omega}}(m,p,n,l-1-2m-p).X$$

as a sum over contributions from $I \in A_m([r])$, and then compare the contribution of $I$ to \(79\) and the sum of the $\text{Cont}^0_{\Gamma_{i,\omega}}$ for $(I, U, V) \in \bar{\pi}^{-1}(I)$.

Let us temporarily fix $I \in A_m([r])$, and suppose

(i) $|I_s| = 1$ for $s \in \mathcal{M}$ and $|I_s| \geq 2$ for $s \in [m] \setminus \mathcal{M}$, where $\mathcal{M}$ is a subset of $[m]$;
(ii) $\mathcal{M} = \bigsqcup_{k \in \mathcal{M}} U_k$ is a partition according to weight. Precisely speaking, for every $s \in \mathcal{M}$, suppose $I(s) = \{i_s\}$, then for $i_a, i_b \in I(\mathcal{M}) := \bigsqcup_{s \in \mathcal{M}} I(s)$, we have $\omega_{i_a} = \omega_{i_b}$ if and only if $a$ and $b$ belong to the
same $U_k$ for some $k \in \mathfrak{K}$.

We denote $I_{U_k} = \bigcup_{s \in U_k} I_s(s)$ for $k \in \mathfrak{K}$, and $I_m = \sum_{k \in \mathfrak{K}} I_{U_k} = \bigcup_{s \in W} I_s(s)$. Then we have

$$\frac{(-1)^m}{24m} \sum_{p=0}^{n+l-1-2m} \sum_{\gamma \in \mathfrak{I}_{m,p,n+l-1-2m,p}} \text{Cont}_{\Gamma_{p,i,j}} \left( \frac{(-1)^m}{24m} \sum_{p=0}^{n+l-1-2m} \text{Cont}_{\Gamma_{p,i,j}} \right),$$

where the contribution of $I \in \mathcal{A}_m([r])$ to

$$\frac{(-1)^m}{24m} \sum_{p=0}^{n+l-1-2m} \sum_{\gamma \in \mathfrak{I}_{m,p,n+l-1-2m,p}} \text{Cont}_{\Gamma_{p,i,j}} \left( \frac{(-1)^m}{24m} \sum_{p=0}^{n+l-1-2m} \text{Cont}_{\Gamma_{p,i,j}} \right),$$

is

$$\frac{(-1)^m}{24m} \sum_{p=0}^{n+l-1-2m} \left[ \sum_{\gamma \in \mathfrak{I}_{m,p,n+l-1-2m,p}} \prod_{s \in \mathfrak{M}} \left( -\omega_s \right)^{P_{i,s}} \right]$$

$$\frac{1}{x^{n+l-1-2m}} \left[ \frac{\prod_{k=1}^{n+l-1-2m} (1 + \varepsilon_{i,t}) \prod_{P_j \in \mathfrak{N}(P)} (1 + \alpha_{i,j})}{\prod_{k \in [r]} \prod_{s \in \mathfrak{M}} (\omega_k - \psi_k) \prod_{s \in \mathfrak{M}} (1 - x \psi_k)} \right],$$

$$\frac{(-1)^m}{24m} \text{Res}_{x=0} \left( \int_{\mathfrak{I}_{m,p,n+l-1-2m,p}} \prod_{k=1}^{n+l-1-2m} (1 + \varepsilon_{i,t}) \prod_{P_j \in \mathfrak{N}(P)} (1 + \alpha_{i,j}) \prod_{k \in [r] \backslash \mathfrak{M}} (\omega_k - \psi_k) \prod_{s \in \mathfrak{M}} (1 - x \psi_k) \right),$$

$$\frac{(-1)^m}{24m} \left( -\text{Res}_{x=-\infty} - \sum_{k \in \mathfrak{K}} \text{Res}_{x=-\omega_k} \right) \left( \int_{\mathfrak{I}_{m,p,n+l-1-2m,p}} \prod_{k=1}^{n+l-1-2m} (1 + \varepsilon_{i,t}) \prod_{P_j \in \mathfrak{N}(P)} (1 + \alpha_{i,j}) \prod_{k \in [r] \backslash \mathfrak{M}} (\omega_k - \psi_k) \prod_{s \in \mathfrak{M}} (1 - x \psi_k) \right),$$

$$\frac{(-1)^m}{24m} \left( \text{Res}_{x=0} + \sum_{k \in \mathfrak{K}} \text{Res}_{x=-\omega_k} \right) \left( \int_{\mathfrak{I}_{m,p,n+l-1-2m,p}} \prod_{k=1}^{n+l-1-2m} (1 + \varepsilon_{i,t}) \prod_{P_j \in \mathfrak{N}(P)} (1 + \alpha_{i,j}) \prod_{k \in [r] \backslash \mathfrak{M}} (\omega_k - \psi_k) \prod_{s \in \mathfrak{M}} (1 - x \psi_k) \right),$$

$$\frac{(-1)^m}{24m} \left( \int_{\mathfrak{I}_{m,p,n+l-1-2m,p}} \prod_{k=1}^{n+l-1-2m} (1 + \varepsilon_{i,t}) \prod_{P_j \in \mathfrak{N}(P)} (1 + \alpha_{i,j}) \prod_{k \in [r] \backslash \mathfrak{M}} (\omega_k - \psi_k) \prod_{s \in \mathfrak{M}} (1 - x \psi_k) \right).$$
\[
\prod_{k \in \mathbb{R}} (x + \omega_{U_k})^{\ell_k} \prod_{k \in \mathbb{R} \setminus \{r\}} (\omega_k - \psi_k) \cdot \prod_{s \in \mathbb{M} \setminus \mathbb{M}_r} (x - \psi_0_s) \\
+ \frac{(-1)^m}{24m} \sum_{k \in \mathbb{R}} [y_{|U_k|-1}] \left( \int_{\mathbb{I} \setminus \mathbb{M}_r} \mathcal{M}_{0, I(s) \setminus \{r\}} \right) \\
\prod_{k \in \mathbb{R}} \left( y - \omega_{U_k} + \varepsilon_{k,t} \right) \prod_{P_j \in \mathbb{N}(P_i)} \left( y - \omega_{U_k} + \alpha_{i,j} \right)
\]

\[
\frac{\left( y - \omega_{U_k} \right)^{m+1}}{24m} \prod_{k \in \mathbb{R} \setminus \{k\}} \left( y - \omega_{U_k} + \omega_{U_k} \right)^{(U)} \prod_{k \in \mathbb{R} \setminus \{r\}} (\omega_k - \psi_k) \prod_{s \in \mathbb{M} \setminus \mathbb{M}_r} (y - \omega_{U_k} - \psi_0_s)
\]

\[
= \frac{(-1)^m}{24m} \left( \prod_{k \in \mathbb{R}} \left( y - \omega_{U_k} + \varepsilon_{k,t} \right) \prod_{P_j \in \mathbb{N}(P_i)} \left( y - \omega_{U_k} + \alpha_{i,j} \right) \\
\right) \prod_{s \in \mathbb{M} \setminus \mathbb{M}_r} \left( \frac{1}{y - \omega_{U_k}} + \sum_{j \in I(s)} \frac{1}{\omega_j} \right) (|I(s)| - 2)
\]

Comparing the last expressions of (78) and (82), we see

\[
\sum_{(I, U, V) \in \mathcal{A}_m([r])} \text{Cont}_{\Gamma_{i,t}, (I, U, V), \omega} + \sum_{I \in \mathcal{A}_m([r])} \text{Cont}_I \left( \frac{(-1)^m}{24m} \sum_{p=0}^{n+l-1-2m} \text{Cont}_{\Gamma_{i,t}, \omega}^{m, p, n+l-1-2m-p, X} \right)
\]

\[
= \frac{(-1)^m}{24m} \left( \prod_{k \in \mathbb{R}} \left( x + \varepsilon_{k,t} \right) \prod_{P_j \in \mathbb{N}(P_i)} \left( x + \alpha_{i,j} \right) \\
\right) \prod_{s \in \mathbb{M} \setminus \mathbb{M}_r} \left( \frac{1}{x} + \sum_{j \in I(s)} \frac{1}{\omega_j} \right) (|I(s)| - 2)
\]

Summing over \( I \in \mathcal{A}_m([r]) \) and \( m \geq 1 \), by proposition 3.1 we obtain (61).

### 3.4 Localized standard versus reduced formula for primary insertions

Let \( \Gamma_{i,t} : J \in \mathcal{D} \mathcal{S}_d \). From the string equation it is easily seen that

\[
\text{Cont}_{\Gamma_{i,t}, \omega} = \left( \sum_{j=1}^{r} \frac{1}{\omega_j} \right)^{|J|} \cdot \prod_{k=1}^{r} \varepsilon_{j,k} \prod_{P_j \in \mathbb{N}(P_i)} \alpha_{i,j} \prod_{s \in \mathbb{M} \setminus \mathbb{M}_r} \frac{1}{(\omega_k - \psi_k)} \\
- [x] \prod_{k=1}^{r} \left( x + \varepsilon_{k,t} \right) \prod_{P_j \in \mathbb{N}(P_i)} \left( x + \alpha_{i,j} \right) \cdot \frac{1}{24m} \prod_{k=1}^{r} \left( \frac{1}{\omega_k} \right)^{|r|-1}
\]

Define \( \tilde{A}_m([r], J; \omega) \) to be the set of 4-tuples \( (I, K, U, V) \in \mathcal{P}([r])^m \times \mathcal{P}(J)^m \times [m] \times [m] \) satisfying

(i) Writing \( I = (I(1), \ldots, I(m)) \), then \( \bigcup_{s=1}^{m} I(s) = [r] \);
(ii) Writing \( J = (K(1), \ldots, K(m)) \), then \( \bigcup_{s=1}^{m} K(s) = J \);
(iii) \( U \cap V = \emptyset \);
(iv) \( |I_s| = 1, K_s = \emptyset \) for \( s \in U \cup V \), and \( |I_s| + |K_s| \geq 2 \) for \( s \notin U \cup V \);
(v) \( \omega_s \) are equal to each other for \( s \in U \), and \( \omega_s \neq \omega_s \) for \( s \in U \) and \( i \in V \).

For \( (I, K, U, V) \in \tilde{A}_m([r], J; \omega) \), denote the common weight \( \omega_s \) for \( s \in U \) by \( \omega_U \) and suppose \( I_s = \{i\} \) for \( s \in U \cup V \), \( I_U = \bigcup_{s \in U} I_s \). There is a natural 1-1 correspondence between \( \prod_{J \subseteq J} \tilde{A}_m([r], J; \omega) \) and \( \tilde{A}_m([r], J; \omega) \); in this correspondence, the subset \( J' \) corresponds to \( \epsilon^{-1}(i_0) \) in a refined decorated rooted tree. By (59) and (39) we have

\[
\text{Cont}_{\Gamma_{i,t}, \omega}^{0, X} = \sum_{J' \subseteq J} \sum_{m \geq 1} \sum_{(I, K, U, V) \in \tilde{A}_m([r], J; \omega)} \text{Cont}_{\Gamma_{i,t}, (I, K, U, V), \omega}^{0, X}
\]
where for $J' \subset J$ and $(I, K; U, V) \in \tilde{A}_m([r], J - J'; \tilde{\omega})$,

\[
\begin{align*}
\text{Cont}^{(\tilde{\omega})}_{(I', U, V)}(I, K, U, V, \tilde{\omega}) &= \frac{1}{m!} \int_{\tilde{M}_{(m, J')} \times \mathbb{P}^{|U|-1}} \left( \prod_{k=1}^{l} (-\omega_U + \varepsilon, k + H) \prod_{p_i \in \text{Nb}(p_j)} (-\omega_U + \alpha, i + j + H) \right) \\
&\cdot \prod_{s \in \mathbb{V}} -\omega_U + H + \omega_s \left( \prod_{I, K, U, V} \int_{\tilde{M}_{(m, J')} \times \mathbb{P}^{|U|-1}} \frac{1}{(\omega_U - \psi - H)} \right) \\
&\cdot \prod_{s \in \mathbb{V}} (-1)^{m+|J'|+1} \frac{|I_s|}{\mathbb{P}^{|U|-1}} \left( \prod_{k=1}^{l} (\omega_U - \psi H) \prod_{p_i \in \text{Nb}(p_j)} (\omega_U - \alpha, i + j + H) \right) \\
&\cdot \prod_{s \in \mathbb{V}} \left( \frac{1}{Y - \omega_U} + \sum_{j \in I_s} 1 \omega_j \right)^{|I_s|+|K_s|+2} \prod_{s \in \mathbb{V}} \left( \frac{1}{Y - \omega_U} + \sum_{j \in I_s} 1 \omega_j \right)^{|I_s|+|K_s|+2}.
\end{align*}
\]

(86)

There is a canonical projection

\[ \tilde{\pi} : \tilde{A}_m([r], J; \tilde{\omega}) \rightarrow A_m([r], J) \]

defined by $\pi(I, K, U, V) = (I, K)$. Let us temporarily fix $J'$ and $(I, K) \in A_m([r], J - J')$, and suppose

(i) $|I_s| = 1$ and $K_s = \emptyset$ for $s \in \mathbb{V}$ and $|I_s| + |K_s| \geq 2$ for $s \in \mathbb{V}$, where $\mathbb{V}$ is a subset of $\mathbb{V}$; 
(ii) $\mathbb{V} = \bigcup_{k \in \mathbb{K}} U_k$ is a partition according to weight. Precisely speaking, for every $s \in \mathbb{V}$, suppose $I_s = \{ i_s \}$, then for $i_a, i_b \in I_{2\mathbb{V}} := \bigcup_{s \in \mathbb{V}} I_s(s)$, we have $\omega_{i_a} = \omega_{i_b}$ if and only if $a$ and $b$ belong to the same $U_k$ for some $k \in \mathbb{K}$.

We denote $I_{U_k} = \bigcup_{s \in U_k} I_s$ for $k \in \mathbb{K}$, and $I_{2\mathbb{V}} = \bigcup_{k \in \mathbb{K}} I_{U_k} = \bigcup_{s \in \mathbb{V}} I_s$. Similar to (81)-(82), we have

\[
\begin{align*}
\frac{(-1)^{m+|J'|+1}}{24m} \sum_{p=0}^{n+|J'|+1} \text{Cont}^{(m, n+|J'|+1)}_{(I, J')}(X) &= \sum_{(I, J') \in A_m([r], J - J')} \text{Cont}^{(m, n+|J'|+1)}_{(I, J')}(X) \\
\end{align*}
\]

(87)

where

\[
\begin{align*}
\text{Cont}^{(m, n+|J'|+1)}_{(I, J')}(X) &= \frac{(-1)^{m+|J'|+1}}{24m} \sum_{p=0}^{n+|J'|+1} \text{Cont}^{(m, n+|J'|+1)}_{(I, J')}(X) \\
&= \frac{(-1)^{m+|J'|+1}}{24m} \sum_{p=0}^{n+|J'|+1} \left( \sum_{\mathbb{V} \ni \mathbb{P}_s \geq 0} \prod_{s \in \mathbb{V}} (-\omega_{i_s})^{p_s} \right) \\
&\cdot \int_{\mathbb{P}^{|U|-1}} \frac{1}{x^{n+|J'|+1} - p} \left( \prod_{k=1}^{l} (\omega_U - \psi H) \prod_{p_i \in \text{Nb}(p_j)} (\omega_U - \alpha, i + j + H) \right). \\
\end{align*}
\]

25
\[
\frac{(-1)^{m+|J'|}m|J'|}{24m}\left[\frac{I_{|\alpha|}^{1m}(x + \varepsilon_{i,t}) \prod_{P_j \in \text{Nb}(P_i)}(x + \alpha_{i,j})}{x^m \prod_{k=1}^{\omega_k}}\right] \\
\cdot \prod_{s=1}^{m} \left(\frac{1}{x} + \sum_{j \in I(\omega)} \frac{1}{\omega_j}\right)^{|I(\omega)|-1}
\]
+ \frac{(-1)^{m+|J'|}m|J'|}{24m} \sum_{k \in \mathbb{R}} \left[\prod_{s \in [m]}^{\omega_k} \left(1 - \frac{1}{y - \omega_k} + \sum_{j \in I(\omega)} \frac{1}{\omega_j}\right)^{|I(\omega)|-1}\right] \cdot \prod_{s \in [m]}^{\omega_k} \left(1 - \frac{x - \varepsilon_{i,t}}{x - \omega_k} \prod_{P_j \in \text{Nb}(P_i)}(y - \omega_k + \alpha_{i,j})\right)
\]  

\[
\text{(88)}
\]

So
\[
\sum_{(I,K,U,V) \in \pi^{-1}(I,K)} \text{Cont}_{(I,K,U,V);J} \left(\frac{(-1)^{m+|J'|}m|J'|}{24m} \sum_{p=0}^{n+1-2m-|J'|} \text{Cont}_{(m,p,n+1-1-2m-|J'|-p),X} \left(\frac{m}{24m} \prod_{k=1}^{\omega_k} \prod_{I \in \mathcal{A}_m([r],J - J')} \left(\frac{1}{x} + \sum_{j \in I(\omega)} \frac{1}{\omega_j}\right)^{|I(\omega)|-2}\right)\right)
\]

Fixing \(I \in \mathcal{A}_m([r])\), summing \ref{89} over \(J'\) and \(K \in \mathcal{A}_m^n(J - J')\) we obtain
\[
\sum_{J' \subset J} \frac{(-1)^m}{24m} \prod_{I \in \mathcal{A}_m([r])} \left(\frac{m}{x} + \sum_{j=1}^{r} \frac{1}{\omega_j}\right) \left(\prod_{s=1}^{m} \frac{1}{x} + \sum_{j \in I(\omega)} \frac{1}{\omega_j}\right)^{|I(\omega)|-2} \prod_{k=1}^{\omega_k} \left(\prod_{j \in I(\omega)} \frac{1}{x} + \sum_{j \in I(\omega)} \frac{1}{\omega_j}\right)^{|I(\omega)|-2} \left(\sum_{J' \subset J} (-1)^{|J'|} \left(\frac{m}{x} + \sum_{j=1}^{r} \frac{1}{\omega_j}\right)^{|J| - |J'|}\right) \prod_{I \in \mathcal{A}_m([r])} \left(\frac{m}{x} + \sum_{j \in I(\omega)} \frac{1}{\omega_j}\right) \prod_{k=1}^{\omega_k} \left(\prod_{j \in I(\omega)} \frac{1}{x} + \sum_{j \in I(\omega)} \frac{1}{\omega_j}\right)^{|I(\omega)|-2}
\]

\[
\text{(90)}
\]

Comparing with \ref{84}, by proposition 3.1 we obtain \ref{61}.

### 3.5 An alternative form of LSvR

In this section, we give another form of LSvR, which corresponds to the theorem 1B of \cite{35}. For this, we need to define the classes \(\tilde{\eta}_p\) over \(\overline{\mathcal{M}}_{(m,J)}(\mathbb{P}^{n-1},d)\). Let us first recall the definition of \(\psi\)-classes \ref{55}. For \(j \in J\), let \(\tilde{\psi}_j \in H^2(\overline{\mathcal{M}}_{g,J}(Y,d))\) be the cohomology class defined by pulling back the \(\psi\)-class on \(\overline{\mathcal{M}}_{g,1}(Y,d)\) via the forgetting map
\[
\overline{\mathcal{M}}_{g,J}(Y,d) \to \overline{\mathcal{M}}_{g,1}(Y,d)
\]
which drops the marked points except the \( j \)-th one and then contracting the unstable components. Let

\[
(m_1 \bar{\psi}_1^{c_1}, \ldots, m_k \bar{\psi}_k^{c_k})_{g,k,d} = \bigwedge_{j=1}^{k} \mu_j \bar{\psi}_j^{c_j} \cap \overline{M}_{g,k}(Y,d)^\text{vir}.
\]  

(91)

The invariants of this type has the advantage that the divisor equation takes a simple form.

**Lemma 3.4.** For \( \gamma \in H^2(Y) \), we have

\[
\langle \gamma, m_1 \bar{\psi}_1^{c_1}, \ldots, m_k \bar{\psi}_k^{c_k} \rangle_{g,k+1,d} = \langle \gamma \cap d \rangle \langle m_1 \bar{\psi}_1^{c_1}, \ldots, m_k \bar{\psi}_k^{c_k} \rangle_{g,k,d}.
\]

(92)

**Proof:** The proof is similar to the usual one, see for example page 264 of [26]; since \( \tilde{\psi}_j \) are defined by pulling back from the moduli spaces with less marked points, there is no additional terms with insertions of the form \( \langle \cdots, (\gamma \wedge m_j) \psi_j c_j \ldots \rangle_{g,k,d} \).

We define \( \tilde{\eta}_p \in H^2(\overline{M}_{(m,J)}(Y,d)) \) by the generating function (restricted to each component \( \overline{M}_{(m,J)}(Y,d) \))

\[
\sum_{p=0}^{\infty} x^p \tilde{\eta}_p = \prod_{s \in [m]} \frac{1}{1 - x \pi_s^* \psi_s}.
\]

(93)

Let

\[
\langle \tilde{\eta}_p \mu_0; \mu_1, \ldots, \mu_{|J|} \rangle_{(m,J,d)}^{Y} = \frac{1}{m!} \tilde{\eta}_p \wedge \text{ev}_0^* (\mu_0) \bigwedge_{j \in J} \text{ev}_j^* (\mu_j) \cap \overline{M}_{(m,J)}(Y,d)^\text{vir}.
\]

(94)

From (92) it is straightforward to deduce

**Lemma 3.5.** For \( \gamma \in H^2(Y) \), we have

\[
\langle \tilde{\eta}_p \mu_0; \gamma, \mu_1, \ldots, \mu_{|J|} \rangle_{(m,J,d)}^{Y} = \langle \gamma \cap d \rangle \langle \tilde{\eta}_p \mu_0; \mu_1, \ldots, \mu_{|J|} \rangle_{(m,J,d)}^{Y}.
\]

(95)

For \( X = \text{Tot}(E \to Y) \) where \( E \) is a concave vector bundle over \( Y \), let

\[
\langle \tilde{\eta}_p \mu_0; \gamma, \mu_1, \ldots, \mu_{|J|} \rangle_{(m,J,d)}^{X} \triangleq \frac{1}{m!} \varepsilon(E)^{m-1} \bigwedge_{i=1}^{m} \pi_i^* (\mathcal{U}_0) \wedge \tilde{\eta}_p \wedge \text{ev}_0^* (\mu_0) \bigwedge_{j \in J} \text{ev}_j^* (\mu_j) \cap \overline{M}_{(m,J)}(Y,d)^\text{vir}.
\]

(96)

The divisor equation (93) still holds.

For finite sets \( I \) and \( J \) with \( I \cap J = \emptyset \) and \(|I| \geq 3\), let

\[
\pi_I : \overline{M}_{0,I,J} \to \overline{M}_{0,I}
\]

(97)

be the map which drops the marked points labelled by elements of \( J \). By the proof the usual string equation, we have

\[
\sum_{j \in J} \frac{1}{w_j} \bigg( \prod_{i \in I} (w_i - \pi_i^* \psi_i) \prod_{j \in J} (w_j - \psi_j) \bigg) = \left( \sum_{j \in J} \frac{1}{w_j} \right)^{|J|} \frac{1}{\prod_{i \in I} w_i \prod_{j \in J} w_j} \left( \sum_{i \in I} \frac{1}{w_i} + \sum_{j \in J} \frac{1}{w_j} \right)^{|I|-3}.
\]

(98)
In the localization contribution, we formally extend this identity to \(|I| \geq 1\).

Now let \(Y\) be a balloon manifold and \(X = \text{Tot}(E \to Y)\) is the total space of a concave equivariant vector bundle \(E\). Let \(\mu_1, \ldots, \mu_{|J|} \in H^*_T(Y)\). For \(\Gamma = (\Gamma, I, K) \in \mathcal{DR}_T^d\), the localization contribution of \(\Gamma\) to \((\tilde{\eta}_p c_q(TX); \mu_1, \ldots, \mu_{|J|}(m, J, d)\) can be written as

\[
\text{Cont}_{\Gamma}((\tilde{\eta}_p c_q(TX); \mu_1, \ldots, \mu_{|J|}(m, J, d)) = \prod_{v \in \text{Ver}} \text{Cont}_{\Gamma, v}((\tilde{\eta}_p c_q(TX); \mu_1, \ldots, \mu_{|J|}(m, J, d))
\]

\[
\prod_{e \in \text{Edg}} \text{Cont}_{\Gamma, e}((\tilde{\eta}_p c_q(TX); \mu_1, \ldots, \mu_{|J|}(m, J, d)). \tag{99}
\]

For an edge \(e\) and a vertex \(v \in \text{Ver} \setminus \{v_0\}\), the contribution is still the same as \(10\) respectively. Suppose \(m(v_0) = i\), then the contribution of \(v_0\) is

\[
\text{Cont}_{\Gamma, v_0}((\tilde{\eta}_p c_q(TX); \mu_1, \ldots, \mu_{|J|}(m, J, d)) = \prod_{j \in e^{-1}(v_0)} \mu_j p_j
\]

\[
\left(\prod_{k=1}^{l} \varepsilon_{i,k} \prod_{p_j \in \text{Ne}(p_i)} \alpha_{i,j} \right) |\text{Edg}(v_0)|^{-1} \cdot [x^q] \left(\prod_{k=1}^{l} (1 + \varepsilon_{i,k} x) \prod_{j \in \{i\} \setminus \{i\}} (1 + \alpha_{i,j} x) \right)
\]

\[
\cdot \frac{1}{\prod_{e \in \text{Edg}(v_0)} (x_0 u e - \psi(v_0, e))} \prod_{s=1}^{m} (1 - x \pi_{i(s), i(s)} \psi_0). \tag{100}
\]

Now we write the invariant \((\tilde{\eta}_p c_q(TX); \mu_1, \ldots, \mu_{|J|}(m, J, d))\) as summing over the \(\mathcal{DR}_T^d\), i.e.,

\[
(\tilde{\eta}_p c_q(TX); \mu_1, \ldots, \mu_{|J|}(m, J, d)) = \sum_{\Gamma \in \mathcal{DR}_T^d} \text{Cont}_{\Gamma}((\tilde{\eta}_p c_q(TX); \mu_1, \ldots, \mu_{|J|}(m, J, d)), \tag{101}
\]

where

\[
\text{Cont}_{\Gamma}((\tilde{\eta}_p c_q(TX); \mu_1, \ldots, \mu_{|J|}(m, J, d))
\]

\[
= \frac{1}{m! |\text{Aut}(\Gamma)|} \sum_{\Gamma \in \mathcal{DR}_T^d} \prod_{e \in \text{Edg}(\Gamma \setminus \{v_0\})} \text{Cont}_{\Gamma, v}((\tilde{\eta}_p c_q(TX); \mu_1, \ldots, \mu_{|J|}(m, J, d))
\]

\[
\prod_{e \in \text{Edg}} \text{Cont}_{\Gamma, e}((\tilde{\eta}_p c_q(TX); \mu_1, \ldots, \mu_{|J|}(m, J, d)). \tag{102}
\]

Now we state another form of the LSvR.

**Theorem 3.2.** Let \(\mu_1, \ldots, \mu_{|J|} \in H^*_T(Y)\). For every decorated rooted tree \(\Gamma \in \mathcal{DR}_T^d\), we have the LSvR

\[
\text{Cont}_{\Gamma}(\mu_1, \ldots, \mu_{|J|}(m, J, d)) = \text{Cont}_{\Gamma}(\mu_1, \ldots, \mu_{|J|}(m, J, d))
\]

\[
+ \frac{1}{24} \sum_{m \geq 1} (-1)^m (m - 1)! \sum_{p=0}^{m+l-1-2m} \text{Cont}_{\Gamma}(\tilde{\eta}_p c_{n+l-1-2m-p}(TX); \mu_1, \ldots, \mu_{|J|}(m, J, d)). \tag{103}
\]
In particular, when $X$ is a local Calabi-Yau space, for every $\Gamma \in \mathcal{D}^{d}_{\emptyset}$ we have

$$\text{Cont}_{\Gamma}(N^{X}_{l,d}) = \text{Cont}_{\Gamma}(N^{0,X}_{l,d}) + \frac{1}{24} \sum_{m \geq 1} \left[ (1)^{m}(m-1)! \right]_{n+l-1-2m} \sum_{p=0}^{\text{Cont}_{\Gamma}\left((\check{\eta}_{\mu}c_{n+l-1-2m-p}(TX);X_{(m,\emptyset,d)}) \right)}^{n+l-1-2m} \text{Cont}_{\Gamma}\left((\check{\eta}_{\mu}c_{n+l-1-2m-p}(TX);X_{(m,\emptyset,d)}) \right).$$

(104)

Sketch of the proof: By theorem 3.1 it suffices to show

$$\sum_{p=0}^{n+l-1-2m} \text{Cont}_{\Gamma}\left((\check{\eta}_{\mu}c_{n+l-1-2m-p}(TX);X_{(m,\emptyset,d)}) \right)$$

$$= \sum_{J \subset J} (-1)^{|J'|-|J'|} \sum_{p=0}^{n+l-1-|J'|-2m} \text{Cont}_{\Gamma}\left((\check{\eta}_{\mu}c_{n+l-1-|J'|-2m-p}(TX);\mu_{1},\cdots,\mu_{|J'|}X_{2m-2m-p},J_{d}) \right).$$

(105)

As before, we need only to show (105) for decorated stars. Let $\Gamma = \Gamma_{t\in\mathcal{D}_{\emptyset}}$. Define

$$\text{Cont}_{\Gamma}^{(m,p,q),X} = [x^{q}](\prod_{k=1}^{l}(1+\varepsilon_{i,k}x) \prod_{j \in [n]\setminus\{i\}} (1+\alpha_{i,j}x))$$

$$\cdot \prod_{p \in \mathcal{D}^{d}_{\emptyset}(\Gamma)} \int_{M(0,0,l_{1}) \cup K_{1}(0,0)} \cdots \int_{M(0,0,l_{m}) \cup K_{m}(0,0)} \frac{1}{\prod_{e \in \text{Edg}(v_{0})}(\psi(e) - \psi(v_{0},e)) \prod_{s=1}^{m}(1-x\pi^{(1)}_{(e),(0,i)(0,j)})}.$$

(106)

By eliminating the common factors as in section 3.1, it suffices to show

$$\sum_{p=0}^{n+l-1-2m} \text{Cont}_{\Gamma}^{(m,p,n+l-1-2m-p),X}$$

$$= \sum_{J \subset J} (-1)^{|J'|-|J'|} \sum_{p=0}^{n+l-1-|J'|-2m} \text{Cont}_{\Gamma,J_{d}}^{(m,p,n+l-1-|J'|-2m-p),X}. \quad (107)$$

But by (105) and

$$\sum_{J \subset J} \sum_{K \in \mathcal{A}_{\emptyset}^{d}(J-J')} \text{Cont}_{\Gamma,K}(I,K) \left((-1)^{m+|J'|} \sum_{p=0}^{n+l-1-|J'|} \text{Cont}_{\Gamma,J_{d}}^{(m,p,n+l-1-|J'|-p),X} \right)$$

$$= \sum_{J \subset J} \sum_{K \in \mathcal{A}_{\emptyset}^{d}(J-J')} (-1)^{|J'|}m^{J'} \left(\prod_{k=1}^{l}(1+\varepsilon_{i,k}x) \prod_{p \in \mathcal{N}_{X}(P_{i})}(1+\alpha_{i,j}x) \prod_{s=1}^{m}(1+x) \prod_{j \in I_{(s)}} \frac{1}{\omega_{j}} \right)$$

$$\cdot \prod_{k=1}^{l} \omega_{k} \left(\prod_{s=1}^{m} \frac{1}{\omega_{j}} \right) \left(\prod_{i=1}^{m} \frac{1}{\omega_{j}} \right)$$

$$= \left[ x^{n+l-1-2m} \right] \left(\prod_{k=1}^{l}(1+\varepsilon_{i,k}x) \prod_{p \in \mathcal{N}_{X}(P_{i})}(1+\alpha_{i,j}x) \prod_{s=1}^{m}(1+x) \prod_{j \in I_{(s)}} \frac{1}{\omega_{j}} \right)$$

$$\cdot \sum_{J \subset J} \sum_{K \in \mathcal{A}_{\emptyset}^{d}(J-J')} (-x)^{m}m^{J'} \left(\prod_{s=1}^{m} \frac{1}{\omega_{j}} \right) \left(\prod_{i=1}^{m} \frac{1}{\omega_{j}} \right) \left(\prod_{s=1}^{m} \frac{1}{\omega_{j}} \right). \quad (108)$$

29
In particular, when \( P \) is a complete intersection. The contribution of the root of a vertex other than the root is as the usual genus zero Gromov-Witten of Corollary 3.2.

Let \( \Gamma \in \mathcal{CDRT}_d \) be a generic smooth section of \( \mathcal{CDRT}_d \) by (43) by

\[
\sum_{j \in I(c)} \left( \prod_{l=1}^{n-l} \left( 1 + \varepsilon_{i,l}x \right) \prod_{p \in \mathbb{N}} \left( x + \sum_{j \in I(c)} \frac{1}{\omega_j} \right) \right),
\]

(108)

it is easy to deduce (107).

\[ \blacksquare \]

Corollary 3.2.

\[
\langle \mu_1, \ldots, \mu_j \rangle^{X}_{1,d} = \langle \mu_1, \ldots, \mu_j \rangle^{0,X}_{1,d} + \frac{1}{24} \sum_{m \geq 1} (-1)^m (m - 1)! \sum_{p=0}^{n+l-1-2m} \langle \tilde{\eta}_p c_{n+l-1-2m-p}(TX); \mu_1, \ldots, \mu_j \rangle^{X}_{(m,0,d)},
\]

(109)

In particular, when \( X \) is a local Calabi-Yau space,

\[
N^{0;X}_{1,d} = X^{0;X}_{1,d} + \frac{1}{24} \sum_{m \geq 1} (-1)^m (m - 1)! \sum_{p=0}^{n+l-1-2m} \langle \tilde{\eta}_p c_{n+l-1-2m-p}(TX); \rangle^{X}_{(m,0,d)},
\]

(110)

\[ \blacksquare \]

3.6 Modifying the virtual localization in genus one for Calabi-Yau complete intersections

Let \( E \) be an equivariant convex splitting vector bundle over a balloon manifold \( Y \). The weights of \( E \) at the fixed point \( P_i \) are still denoted by \( \varepsilon_{i,1}, \ldots, \varepsilon_{i,l} \). We assume that for every \( 1 \leq k \leq l \), and every \( P_j \in \text{Nb}(P_i) \), \( \varepsilon_{i,k} \) is linear independent to \( \alpha_{i,j} \). For \( X = \text{Tot}(E \to Y) \), it is natural to define the invariants

\[
\langle \tilde{\eta}_p \mu_0; \mu_1, \ldots, \mu_j \rangle^{X}_{(m,j'-j,0,d)} = \langle \tilde{\eta}_p \mu_0; \mu_1, \ldots, \mu_j \rangle^{W}_{(m,j'-j,0,d)},
\]

where \( W \) is a generic smooth section of \( E \). For \( (\Gamma, I, K) \in m \mathcal{CDRT}_d(Y) \), the localization contribution of an edge or of a vertex other than the root is as the usual genus zero Gromov-Witten of complete intersections. The contribution of the root \( v_0 \) is replacing (43) by

\[
\text{Contr}_{v_0} \left( \langle \tilde{\eta}_p \mu_0(TX); \mu_1, \ldots, \mu_j \rangle^{X}_{(m,j',j,0,d)} \right) = \prod_{j \in \varepsilon^{-1}(v_0)} \mu_j \bigg|_{P_i} \left( \prod_{P_j \in \text{Nb}(P_i)} \frac{1}{\varepsilon_{i,j}} \left| \varepsilon_{i,k} \right| \left( \prod_{k=1}^{p} \frac{1 + \alpha_{i,j}x}{1 + \varepsilon_{i,k}x} \right) \right) \left( \prod_{p \in \text{Edg}(v_0)} \left( \frac{1}{\varepsilon_{i,k}} - \varepsilon_{i,k}x \right) \right),
\]

(112)

We define the localization contribution of \( \Gamma \in \mathcal{CDRT}_d \) to \( \langle \tilde{\eta}_p \mu_0(TX); \mu_1, \ldots, \mu_j \rangle^{X}_{(m,j',j,0,d)} \) as in section 2.2.

To define the formal reduced genus one Gromov-Witten invariants of \( X \), it is natural to replace (103) by

\[
e(U'_1) \triangleq \left( \frac{1}{e(E)} \bigg|_{P_m(v_0)} \prod_{c \in \text{Edg}(v_0)} \pi_c^* e(U'_0) \cdot \frac{1}{e(L^*_{\Gamma}) \otimes E_{\mu(v_0)} \otimes \gamma^*},
\]

(113)
where
\[ e(U_0) = e(\pi_* f^* E) / e(E). \] (114)

Thus when \( E = \bigoplus_{k=1}^{l} \mathcal{O}(a_k) \to \mathbb{P}^{n-1} \), where \( a_k > 0 \) and \( \sum_{k=1}^{l} a_k = n \), the formal reduced genus one Gromov-Witten invariants of \( X \) are no other than the reduced genus one Gromov-Witten invariants of \( W \), where \( W \) is a complete intersection of \( \mathbb{P}^{n-1} \) with multiple degree \( (a_1, \ldots, a_l) \), by the localization contributions given in [36] and [28].

On the other hand, we cannot compute the genus one Gromov-Witten invariants of \( W \) by the virtual localization, so it does not make sense to say whether the LSvR holds for \( W \) (or for \( X \)). But since the global SvR holds ([35]), it is reasonable to formally write the genus one Gromov-Witten invariants of \( W \) as summing over \( \mathcal{D} \mathcal{O} \mathcal{L} \) and \( \mathcal{D} \mathcal{O} \mathcal{R} \), such that the LSvR holds. For simplicity, we consider the case \( E = \mathcal{O}_{\mathbb{P}^{n-1}(n)} \). Then \( \alpha_{i,j} = \alpha_i - \alpha_j \), and we can linearize \( E \) such that the weight of \( E \) at \( P_i \) is \( \alpha_i \). The localization contribution of \( \Gamma \in \mathcal{D} \mathcal{O} \mathcal{L} \) to \( \mathcal{N}_W^{\Gamma} \) is as in the usual virtual localization. For \( \Gamma \in \mathcal{D} \mathcal{O} \mathcal{R} \), the problem arise only at the root \( v_0 \). As in the concave cases, we can put aside the common factors, and focus on \( \text{Cont}^W \), where \( \Gamma \) is a decorated star. Let \( \Gamma_{i,:,v} \) be a simply decorated star, where \( \omega_s = \alpha_i - \alpha_j \) for \( 1 \leq s \leq r \). When \( \Gamma \) is a simply decorated star, we have \( \text{Cont}^W_{\Gamma_{i,:,v}} = 0 \), so we define

\[
\text{Cont}^W_{\Gamma_{i,:,v}} = \sum_{m \geq 1} \frac{(-1)^m}{24m} \sum_{\mathcal{I} \in \mathcal{A}_m(\mathcal{I})} \left\{ \sum_{i = 1}^{\mu} \frac{1}{\alpha_i} \right\}^{\mathcal{I}(\mathcal{I}) - 2}.
\]

In contrast to the concave cases, the rational function of \( x \) in the big brackets has a pole at \( -\frac{1}{\alpha_i} \).

It is not hard to see

\[
\text{Cont}^W_{\Gamma_{i,:,v}} = \sum_{m \geq 1} \frac{(-1)^m}{24m} \sum_{\mathcal{I} \in \mathcal{A}_m(\mathcal{I})} [x^m] \left( \frac{\prod_{j \in [n] \setminus \{i\}} (x + \alpha_i - \alpha_j)}{(x + \omega_1 \cdot \alpha) \prod_{k=1}^{r} (\alpha_i - \alpha_k - \psi_k)} \cdot \prod_{s=1}^{m} \left( 1 + \sum_{j \in \mathcal{I}(s)} \frac{1}{\alpha_i - \alpha_j} \right) \right)^{|\mathcal{I}(\mathcal{I})| - 2}.
\]

By the lemma 3.3 we have

\[
\sum_{m \geq 1} \frac{(-1)^m}{24m} \sum_{\mathcal{I} \in \mathcal{A}_m(\mathcal{I})} [x^m] \left( \frac{\prod_{j \in [n] \setminus \{i\}} (x + \alpha_i - \alpha_j)}{(x + \alpha_i \cdot \alpha) \prod_{k=1}^{r} (\alpha_i - \alpha_k - \psi_k)} \cdot \prod_{s=1}^{m} \left( 1 + \sum_{j \in \mathcal{I}(s)} \frac{1}{\alpha_i - \alpha_j} \right) \right)^{|\mathcal{I}(\mathcal{I})| - 2}
\]

\[
= \prod_{j \in [n] \setminus \{i\}} (\alpha_i - \alpha_j) \int_{\mathcal{M}_{1,1}} \frac{1}{\prod_{k=1}^{r} (\omega_k - \psi_k)} \cdot [x] \left( \frac{\prod_{j \in [n] \setminus \{i\}} (x + \alpha_i - \alpha_j)}{\alpha_i \cdot \alpha - x} \right) \cdot \frac{1}{24 \prod_{k=1}^{r} \omega_k} \left( \frac{1}{\sum_{k=1}^{r} \omega_k} \right)^{r-1}.
\]

Thus we have

\[
\text{Cont}^W_{\Gamma_{i,:,v}} = \int_{\mathcal{M}_{1,1}} \frac{\prod_{j \in [n] \setminus \{i\}} \Lambda^Y_j (\alpha_i - \alpha_j)}{\Lambda^Y_1 (\alpha_i \cdot \alpha) \prod_{k=1}^{r} (\omega_k - \psi_k)} \cdot H_r \left( -\frac{\alpha_i}{\omega_1}, \ldots, -\frac{\alpha_i}{\omega_r} \right).
\] (118)
where the function $H_r$ is defined by (71). For general decorated stars $\Gamma; v$, taking into account the contribution $\text{Cont}^{SW}_{\Gamma; v}$ as in the preceding sections, we naturally define $\text{Cont}^{SW}_{\Gamma; v}$ still by (118). For a finite set of variables $S = \{w_1, \cdots, w_r\}$, let
\[
H(S) = H_r(w_1, \cdots, w_r).
\]
(119)
For $\Gamma \in \mathcal{D}^d_{\mathcal{D}^d_{\Phi}}$, we define
\[
\text{Cont}_r(N_{1,d}^W) = \frac{1}{|\text{Aut}(\Gamma)|} \prod_{v \in \text{Ver}} \text{Cont}_{r,v}(N_{1,d}^W) \prod_{e \in \text{Edg}} \text{Cont}_{r,e}(N_{1,d}^W),
\]
(120)
where
\[
\text{Cont}_{r,v_0}(N_{1,d}^W) = \left( \prod_{j \in [n] \setminus \{i\}} \frac{(\alpha_i - \alpha_j)}{n\alpha_i} \right)^{|\text{Edg}(v_0)|-1} \left( \int_{M_{1,\text{val}(v_0)}} \prod_{j \in [n] \setminus \{i\}} \Lambda_j^\gamma(\alpha_i - \alpha_j) \right.
\]
\[
\left. \prod_{e \in \text{Edg}(v_0)} \left( \frac{(\alpha_{v_0,e} - d(e))}{\psi(v_0,e)} \right) \right) \prod_{j \in [n] \setminus \{i\}} \left( -n\alpha_i + \alpha_i - \alpha_j \right) \prod_{k=1}^{r+1} \frac{\omega_k}{\alpha_{v_0,e}} \cdot H \left( \prod_{e \in \text{Edg}(v_0)} \right),
\]
(121)
and $\text{Cont}_{r,v}(N_{1,d}^W)$, $\text{Cont}_{r,e}(N_{1,d}^W)$, as well as the localization contribution of $\Gamma \in \mathcal{D}^d_{\Phi}$, are defined as in the genus zero virtual localization. By the SvR for $W$ (35), we have
\[
N_{1,d}^W = \sum_{\Gamma \in \mathcal{D}^d_{\mathcal{D}^d_{\mathcal{D}^d_{\Phi}}}} \text{Cont}_r(N_{1,d}^W) + \sum_{\Gamma \in \mathcal{D}^d_{\mathcal{D}^d_{\mathcal{D}^d_{\Phi}}}} \text{Cont}_r(N_{1,d}^W).
\]
(122)
We call (121) the modified contribution of the genus one vertex $v_0$. With the third row of (121) dropped, we call the resultant contribution the naive contribution. Then we can summarize the above as

**Theorem 3.3.** Assigning the contribution of the genus one vertex to be the modified contribution, we can compute the genus one Gromov-Witten invariants of the Calabi-Yau hypersurface $W$ in $\mathbb{P}^{n-1}$ by (the modified) virtual localization.

It is routine to deduce parallel results for the complete intersections in $\mathbb{P}^{n-1}$. It is reasonable to make the following conjecture.

**Conjecture 1.** Theorem 3.3 holds for complete intersections in balloon manifolds.

We leave the precise formulation of this conjecture to the reader.

**Remark 3.2.** It would be fascinating if one could find a direct geometric interpretation for (116).

Note that by the properties of $H(S)$ shown in the proof of lemma 3.3, it is straightforward to see the sum of the terms in the square bracket of (116) has no factors $n\alpha_i$ in the denominator. In higher genera, we conjecture that there are also some natural correction terms which cancel the negative powers of the weight $E_r\ell$, such that the corresponding modified virtual localization gives the correct Gromov-Witten invariants.

4 The difference between standard and formal reduced genus one Gromov-Witten invariants of $X = \text{Tot}(\bigoplus_{k=1}^l \mathcal{O}(-a_k) \to \mathbb{P}^{n-1})$

4.1 We follow the description on the equivariant cohomology of $\mathbb{P}^{n-1}$ in the section 1.1 of 36. We recollect the facts and terminology of 36 that we need in the following.
• The equivariant cohomology ring of $\mathbb{P}^{n-1}$ is
\begin{equation}
H^*_T(\mathbb{P}^{n-1}) = \mathbb{Q}[x, \alpha_1, \cdots, \alpha_n]/ \prod_{i=1}^{n}(x - \alpha_i).
\end{equation}
(123)

• The restriction map on the equivariant cohomology induced by $P_i \mapsto \mathbb{P}^{n-1}$ is given by
\begin{equation}
\mathbb{Q}[x, \alpha_1, \cdots, \alpha_n]/ \prod_{i=1}^{n}(x - \alpha_i) \rightarrow \mathbb{Q}[\alpha_1, \cdots, \alpha_n], \ x \mapsto \alpha_i,
\end{equation}
and the localized equivariant cohomology ring is
\begin{equation}
\mathcal{H}^*_T(\mathbb{P}^{n-1}) = \mathbb{Q}_n[x]/ \prod_{i=1}^{n}(x - \alpha_i),
\end{equation}
(125)
where $\mathbb{Q}_n = \mathbb{Q}(\alpha_1, \cdots, \alpha_n)$ is the field of fractions of $\mathbb{Q}[\alpha_1, \cdots, \alpha_n]$.

• The tautological line bundle $\gamma_{n-1}$ ($\cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$) is linearized such that the equivariant Euler class restricted to the fixed points are
\begin{equation}
e(\gamma_{n-1})|_{P_i} = -\alpha_i
\end{equation}
for $1 \leq i \leq n$.

• The equivariant Poincaré dual of $P_i$ is
\begin{equation}
\phi_i = \prod_{j \in [n] \setminus \{i\}} (x - \alpha_j).
\end{equation}
(127)

The tangent bundle $T \mathbb{P}^{n-1}$ is linearized such that the equivariant Euler class restricted to the fixed points are
\begin{equation}
\text{e}(T \mathbb{P}^{n-1})|_{P_i} = \prod_{j \in [n] \setminus \{i\}} (\alpha_i - \alpha_j) = \phi_i|_{P_i}.
\end{equation}
(128)
for $1 \leq i \leq n$.

This $(\mathbb{C}^*)^n$-action can be lifted to the vector bundle $E = \prod_{k=1}^{l} \mathcal{O}(-a_k)$, and there are many choices of liftings. In this article we fix the lifting such that as an equivariant vector bundle we have $E \cong \bigotimes_{k=1}^{l} \gamma_{\alpha_k}$. Thus the equivariant Euler class of $E$ restricted to the fixed points are
\begin{equation}
\text{e}(E)|_{P_i} = \prod_{k=1}^{l} (-a_k \alpha_i).
\end{equation}
(129)

The localization contributions have been described as in section 2, except that the contribution of an edge $e = \{v_1, v_2\}$ with $m(v_1) = i$ and $m(v_2) = j$ is
\begin{equation}
\text{Contribution}(\mu_1, \cdots, \mu_j)_{|_{i,j,d}}^X = \prod_{k=1}^{l} \alpha_k \prod_{a=1}^{d(e)-1} (-a_k \alpha_j + a \alpha_{d(e)}) / \alpha_j \cdot (d(e))d(e)(\alpha_j - \alpha_j)d(e) \prod_{k \neq i,j} \alpha_k \prod_{a=0}^{d(e)} (\alpha_i - \alpha_k + \alpha_{d(e)}) / \alpha_i \cdot (d(e))d(e).
\end{equation}
(130)

\footnote{As shown in \textbf{[17], [20], [21] and [19]}, for certain extremal cases of $X$, other choices of the liftings will make the localization computations extremely simple.}

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4.2

As in (36), let

$$\mathcal{F}(\alpha, x, Q) = \sum_{d=1}^{\infty} Q^d (ev_* e(U_1)) \in (H^{n-2}_T(\mathbb{P}^{n-1}))[[Q]],$$

(131)

we have

$$\mathcal{F}(\alpha, x, Q) = \mathcal{F}_0(Q)x^{n-2} + \mathcal{F}_1(\alpha, Q)x^{n-3} + \cdots + \mathcal{F}_{n-2}(\alpha, Q),$$

(132)

where $\mathcal{F}_p(\alpha, Q) \in \mathbb{Q}[[Q]][\alpha_1, \cdots, \alpha_n]$ is of degree $p$ and symmetric in $\alpha_1, \cdots, \alpha_n$. By the divisor equation and a degree counting we have

$$\frac{d}{dT} \sum_{d=1}^{\infty} e^{dT} N_i^{X,d} = \mathcal{F}_0(e^T).$$

(133)

To determine $\mathcal{F}(\alpha, x, Q)$ (and thus $\mathcal{F}_0(Q)$), it suffices to compute $\mathcal{F}(\alpha, \alpha_i, Q)$. By the Atiyah-Bott localization theorem (2) on $\mathbb{P}^{n-1}$,

$$\mathcal{F}(\alpha, \alpha_i, Q) = \sum_{d=1}^{\infty} Q^d \langle \phi_i \rangle_{1,1,d}.$$  

(134)

By corollary 3.2,

$$\langle \phi_i \rangle_{1,1,d} = \langle \phi_i \rangle_{1,1,d}^0 X + \frac{1}{24} \sum_{m \geq 1} (-1)^m (m-1)! \sum_{p=0}^{n+l-1-2m} (\eta_p e_{n+l-1-2m-p}(TX) ; \phi_i)_{(m,1,d)} X.$$  

(135)

Let

$$C_i(Q) = \sum_{d=1}^{\infty} Q^d \frac{1}{24} \sum_{m \geq 1} (-1)^m (m-1)! \sum_{p=0}^{n+l-1-2m} (\eta_p e_{n+l-1-2m-p}(TX) ; \phi_i)_{(m,1,d)} X.$$  

(136)

The same reasoning shows that there exist $C(\alpha, x, Q) \in (H^{n-2}_T(\mathbb{P}^{n-1}))[[Q]]$ such that

$$C(\alpha, x, Q) = C_0(Q)x^{n-2} + C_1(\alpha, Q)x^{n-3} + \cdots + C_{n-2}(\alpha, Q),$$

(137)

where $C_p(\alpha, Q)$ is of degree $p$ and symmetric in $\alpha_1, \cdots, \alpha_n$, and

$$C_i(Q) = \mathcal{C}(\alpha, \alpha_i, Q).$$

(138)

By the divisor equation (35), we have

$$\mathcal{C}_0(e^{dT}) = \frac{d}{dT} \left( \sum_{d=1}^{\infty} Q^d \frac{1}{24} \sum_{m \geq 1} (-1)^m (m-1)! \sum_{p=0}^{n+l-1-2m} (\eta_p e_{n+l-1-2m-p}(TX) )_{(m,0,d)} \right).$$

(139)

By (132), (134), (135), (137) and (138), there exist $\mathcal{F}_0^0(\alpha, x, Q) \in (H^{n-2}_T(\mathbb{P}^{n-1}))[[Q]]$ such that

$$\mathcal{F}_0^0(\alpha, x, Q) = \mathcal{F}_0^0(Q)x^{n-2} + \mathcal{F}_1^0(\alpha, Q)x^{n-3} + \cdots + \mathcal{F}_{n-2}^0(\alpha, Q),$$

(140)

where $\mathcal{F}_p^0(\alpha, Q)$ is of degree $p$ and symmetric in $\alpha_1, \cdots, \alpha_n$, and

$$\sum_{d=1}^{\infty} Q^d \langle \phi_i \rangle_{1,1,d}^0 = \mathcal{F}_0^0(\alpha, \alpha_i, Q).$$

(141)
By (133), (139) and the LSvR for \( N^X_{1,d} \), we deduce the divisor equation for the reduced genus one Gromov-Witten invariants of \( X \):

\[
\mathfrak{g}^0_0(e^T) = \mathfrak{g}^0_0(e^T) - \mathfrak{c}^0_0(e^T) = \frac{d}{dt} \sum_{d=1}^{\infty} e^{dT} N^0_{1,d}. \tag{142}
\]

In subsection 4.1, we use the known result on the generating function of one-point genus zero Gromov-Witten invariants for \( X \) to obtain a formula for

\[
\sum_{d=1}^{\infty} Q^d \frac{1}{24} \sum_{m \geq 1} (-1)^m (m - 1)! \sum_{p=0}^{n+l-1-2m} \langle \eta_{p,c_{n+l-1-2m-p}(TX)} \rangle^X_{(m,\varnothing,d)}. \tag{143}
\]

In section 5, we compute the righthand-side of (142). By theorem 4.3 and theorem 5.1, we have

**Theorem 4.1.**

\[
\sum_{d=1}^{\infty} Q^d N^X_{1,d} = \frac{n}{48} \left(n - 1 - 2 \sum_{k=1}^{l} \frac{1}{a_k}\right) (T - t) - \left\{ \begin{array}{ll}
\frac{n+l}{48} \log(1 - \prod_{k=1}^{l} (-a_k)^{a_k} q) + \frac{n+l}{48} \sum_{p=1}^{l} \frac{(n+l-2p)^2}{8} \log I_p(q), & \text{if } 2 | (n+l); \\
\frac{n+l-3}{48} \log(1 - \prod_{k=1}^{l} (-a_k)^{a_k} q) + \frac{n+l-3}{48} \sum_{p=1}^{l} \frac{(n+l-2p+1)^2}{8} \log I_p(q), & \text{if } 2 \nmid (n+l).
\end{array} \right. \tag{144}
\]

**4.3**

Let

\[
Z_r(Q) = \prod_{k=1}^{l} (-a_k)^{\infty} \sum_{d=1}^{n+l-3} Q^d \langle \psi^{e_1^* H^{n+l-3-r}} \rangle^X_{\{M_{0,1}(\mathbb{P}^{n-1},d)\}}. \tag{145}
\]

for \( 0 \leq r \leq n + l - 3 \). We would like to formally write

\[
Z_r(Q) = \sum_{d=1}^{\infty} Q^d \langle \psi^{e_1^* H^{n+l-3-r}} \rangle^X_{\{M_{0,1}(\mathbb{P}^{n-1},d)\}}. \tag{146}
\]

Givental gave a mirror formula for the generating function of \( Z_r(Q) \).

**Theorem 4.2.** (A. Givental)

\[
e^{Tw}(1 + \sum_{r=0}^{n+l-3} Z_r(Q)w^{r+2}) = R(w, t) \mod (w^{n+l}). \tag{147}
\]

**Proof:** Let \(^7\)

\[
J(h, \alpha, Q) = 1 + \frac{1}{h} \sum_{d=1}^{\infty} Q^d \psi^* \left( \frac{e_T(H)}{h - \psi} \right). \tag{148}
\]

where \( \psi^* \) is the push-forward map in the equivariant cohomology and

\[
I(h, \alpha, q) = \sum_{d=0}^{\infty} Q^d \prod_{k=1}^{l} \prod_{i=1}^{d-1} (-a_k x - \alpha_i + sh). \tag{149}
\]
The genus zero mirror theorem (Theorem 4.2 in [15]) says

$$J(h, \alpha, Q) = e^{-\frac{\pi i \ell(q)}{2 \hbar}} I(h, \alpha, q).$$

(150)

Since $x$ is not a zero divisor in $H^*_{\text{tr}}(\mathbb{P}^{n-1})$, from (150) we have

$$\prod_{k=1}^{\infty} \sum_{d=1}^{\infty} Q^d \text{ev}_s \left( \frac{e_T(U_0)}{\hbar - \psi} \right) = \left( e^{-z f(q)} \prod_{d=0}^{\infty} \sum_{k=1}^{\infty} \prod_{a_k=0}^{a_k-1} (-a_k x - s \hbar) - 1 \right) / x^l.$$  

(151)

The lefthand-side of (151) lies in $H^*_G(\mathbb{P}^{n-1})$; when $l = 1$ this is obvious, and when $l = 2$, $f(q) = 0$. Taking the nonequivariant limit, it is easy to deduce (147) from (151).

In the following we use (147) to compute (143).

**Theorem 4.3.**

$$\sum_{m \geq 1} (-1)^m (m - 1)! \sum_{p=0}^{n+1-2m} \langle \eta_p c_{n+1-2m-p}(TX) \rangle^X_{(m, \varnothing, d)}.$$  

$$= - \frac{1}{\prod_{k=1}^{\infty} (-a_k)} \text{Res}_{w=0} \left\{ \prod_{k=1}^{l} (1 - a_k w) \cdot ((1 + w)^n - w^n) \left( -Tw + \ln R(w, t) \right) \right\}.$$  

(152)

**Proof:** The proof is parallel to the proof of lemma 2.2 in [35].

$$\sum_{m \geq 1} (-1)^m (m - 1)! \sum_{p=0}^{n+1-2m} \langle \eta_p c_{n+1-2m-p}(TX) \rangle^X_{(m, \varnothing, d)}$$  

$$= \sum_{p=2}^{n+1-2m} \sum_{m=1}^{2m-p} (-1)^m (m - 1)! \langle \eta_{2m-p} c_{n+1-2m-p}(TX) \rangle^X_{(m, \varnothing, d)}$$  

$$= \sum_{p=2}^{n+1-1} \sum_{m=1}^{2m-p} \langle (1 + w)^n \prod_{k=1}^{l} (1 - a_k w) \rangle \sum_{p=0}^{2m-p} (-1)^m (m - 1)! \langle \eta_{p-2m} H^{n+l-1-p} \rangle^X_{(m, \varnothing, d)}.$$  

(153)

By the normalization sequence,

$$\sum_{m=1}^{2m-p} (-1)^m (m - 1)! \langle \eta_{p-2m} H^{n+l-1-p} \rangle^X_{(m, \varnothing, d)}$$  

$$= (-1)^m \sum_{m=1}^{2m-p} \frac{1}{m} \langle \text{ev}_0^* H^{n+l-1-p} \left( \prod_{k=1}^{l} (-a_k H) \right)^{m-1} \prod_{i=1}^{m} \frac{e(U_0)}{1 - \psi_0}, [M_{(m, \varnothing)}(\mathbb{P}^{n-1}, d)] \rangle$$  

$$= (-1)^m \sum_{m=1}^{2m-p} \left( \prod_{k=1}^{l} (-a_k) \right)^{m-1} \langle \text{ev}_0^* H^{n+l-1-p} \prod_{i=1}^{m} \frac{e(U_0)}{1 - \psi_0}, [M_{(m, \varnothing)}(\mathbb{P}^{n-1}, d)] \rangle.$$  

(154)
By the decomposition of the diagonal in \((\mathbb{P}^{n-1})^m\),

\[
\sum_{d=1}^{\infty} Q^d \sum_{m=1}^{2m \leq p} \left( -\prod_{k=1}^{l} (-a_k) \right)^m (\text{ev}^* H^{n-lm-1-p} \prod_{i=1}^{m} \frac{e(U_0)}{1 - \psi_0} \pi_{l-i} \mathcal{M}(\mathbb{P}^{n-1}, d_i))
\]

\[
= \sum_{m=1}^{2m \leq p} \left( -\prod_{k=1}^{l} (-a_k) \right)^m \sum_{d=1}^{\infty} Q^d \sum_{m \geq 0} \prod_{i=1}^{m} (\text{ev}^* H^{n-1-p_i} e(U_0)) \mathcal{M}(\mathbb{P}^{n-1}, d_i))
\]

\[
= \sum_{m=1}^{2m \leq p} (-1)^m \sum_{d=1}^{\infty} Q^d \sum_{m \geq 0} \prod_{i=1}^{m} (\text{ev}^* H^{n-1-p_i} e(U_0)) \mathcal{M}_{0,1}(\mathbb{P}^{n-1}, d_i))
\]

\[
= \sum_{m=1}^{2m \leq p} (-1)^m \sum_{m \geq 0} Z_{p_i-2}(Q)
\]

\[
= -[w^p] \left( \ln \left( 1 + \sum_{r=0}^{n+l-3} Z_r(Q)w^{r+2} \right) \right).
\]

By (153), (154), (155) and (147), noting that \(-T w + \ln R(w, t) \equiv 0 \mod (w^l)\), we obtain (152).

5 Formal reduced genus one Gromov-Witten invariants of

\(X = \text{Tot} \left( \bigoplus_{k=1}^{l} \mathcal{O}(-a_k) \rightarrow \mathbb{P}^{n-1} \right)\)

Let \(A_i, \tilde{A}_{ij}, B_i, \tilde{B}_{ij}\) be the four types of decorated one loop graph and refined decorated rooted trees defined in [28]. We write \(\langle \phi_{i} \rangle_{1,1,d}^{0,X}\) as the sum of localization contributions of these graphs.

\[
\langle \phi_{i} \rangle_{1,1,d}^{0,X} = A_{i}(Q) + \sum_{j=1}^{n} \tilde{A}_{ij}(Q) + B_{i}(Q) + \sum_{j=1}^{n} \tilde{B}_{ij}(Q).
\]

Let \(\mathbb{Q}[\alpha]^{S_n}, \mathcal{I} \subset \mathbb{Q}[\alpha]^{S_n}, \tilde{\mathbb{Q}[\alpha]^{S_n}}, \mathbb{Q}[\alpha]^{S_{n-1}}\) and \(\mathcal{K}_i\) be defined as in [28]. We will frequently make use of the lemma 5.1 and lemma 5.2 of [28]. We adopt the notation \(\equiv_i\) of [28]. \(F \equiv_i G\) means \(F - G \in \mathcal{K}_i\). We call \(A_{i}(Q) + \sum_{j=1}^{n} \tilde{A}_{ij}(Q)\) the type A contributions, and \(B_{i}(Q) + \sum_{j=1}^{n} \tilde{B}_{ij}(Q)\) the type B contributions. In the following of this section, after some preparation on some properties of hypergeometric series, we will compute the two types of contributions modulo \(\mathcal{K}_i\) separately.

In the computation of the type A contributions we can make use of Popa’s results in [28] in a dual way to simplify our proof. Although Popa’s computation is for \(\bigoplus_{k=1}^{l} \mathcal{O}(a_k)\) where \(\sum_{k=1}^{l} a_k = n\) and \(a_k \geq 2\) for \(1 \leq k \leq l\), the last assumption \(a_k \geq 2\) occurs only because of the factors of the form \(a_k \alpha + \hat{h}\) in the denominators of some rational functions. In the local cases, we don’t have factors of this form in the denominators, so we are able to apply, e.g., lemma 5.4 of [28] to our space \(X\).

By proposition 5.3 and 5.5 we obtain
Theorem 5.1.

\[
\sum_{d=1}^{\infty} e^{dT}N_{d}^{0,X} = \frac{n}{48} \left(n - 1 - 2 \sum_{k=1}^{l} \frac{1}{a_{k}}\right)(T - t) - \left\{ \begin{array}{ll}
\frac{n+l}{48} \log(1 - \prod_{k=1}^{l}(-a_{k})^{n}e^{t}) + \sum_{\rho=1}^{n+l-2} \frac{(n+l-2p)^{2}}{8} \log I_{p}(e^{t}) & \text{if } 2 \mid (n+l) \\
\frac{n+l-3}{48} \log(1 - \prod_{k=1}^{l}(-a_{k})^{n}e^{t}) + \sum_{\rho=1}^{n+l-2} \frac{(n+l-2p)^{2}-1}{8} \log I_{p}(e^{t}) & \text{if } 2 \nmid (n+l)
\end{array} \right.
\]

\[
+ \frac{1}{24} \prod_{k=1}^{l}(-a_{k}) \text{Res}_{w=0} \left\{ \prod_{k=1}^{l}(1 - a_{k}w) \cdot \left((1 + w)^{n} - w^{n}\right) \right\}_{w^n+l} (-Tw + \ln R(w,t)) \right\}. \quad (157)
\]

5.1 Some properties of certain hypergeometric series

Following [34], we denote by

\[ \mathcal{P} \subset 1 + q\mathbb{Q}(w)[[q]] \]

the subgroup of power series in \( q \) with constant term 1 whose coefficients are rational functions in \( w \) which are regular at \( w = 0 \), and define a map \( \mathcal{M} : \mathcal{P} \to \mathcal{P} \) by

\[ \mathcal{M}F(w, q) = \left(1 + \frac{q}{w} \frac{\partial}{\partial q} \right) \frac{F(w, q)}{F(0, q)} \]

Note that for \( F(w, q) \in \mathcal{P} \), we have a well-defined series \( \log F(w, q) \in \mathbb{Q}(w)[[q]] \). Let \( \mathcal{P}_{1} \) be the subset of \( \mathcal{P} \) such that \( F(w, q) \in \mathcal{P}_{1} \) if and only if every coefficient of the power series \( \log F(w, q) \) is \( O(w) \) as \( w \to \infty \). We recall the following lemma from [34]

Lemma 5.1. If \( F \in \mathcal{P} \) and \( \mathcal{M}^{k}F = F \) for some \( k > 0 \), then \( F \in \mathcal{P}_{1} \). \( \Box \)

The proof of this lemma in [34] shows also

Lemma 5.2. If \( F \in \mathcal{P} \), then \( \mathcal{M}F \in \mathcal{P}_{1} \) if and only if \( IF \in \mathcal{P}_{1} \). \( \Box \)

Now following [28], we define

\[
\mathcal{F}_{-l}(w, q) = \sum_{d=0}^{\infty} q^{d} \prod_{k=1}^{l} \prod_{r=0}^{d-1} \frac{(a_{k}w + r)}{((w + r)^{n} - w^{n})}, \quad (158)
\]

and for \( p > -1 \),

\[
\mathcal{F}_{p} = \mathcal{M}^{l+p} \mathcal{F}_{-l}. \quad (159)
\]

We have

\[
\mathcal{F}_{0}(w, q) = \sum_{d=0}^{\infty} q^{d} \prod_{k=1}^{l} \prod_{r=0}^{d} \frac{(a_{k}w + r)}{((w + r)^{n} - w^{n})},
\]

and by lemma 4.1 of [28], we have

\[ \mathcal{M}^{n} \mathcal{F}_{0} = \mathcal{F}_{0}. \]

Thus by lemma 5.1 and lemma 5.2 we have \( \mathcal{F}_{p} \in \mathcal{P}_{1} \) for \( p \geq -l \). By the definition of \( \mathcal{P}_{1} \), as \( w \to \infty \), \( \mathcal{F}_{p}(w, q) \) has an asymptotic expansion of the form

\[
\mathcal{F}_{p}(w, q) \sim e^{\mu_{p}(q)w} \sum_{s=0}^{\infty} \Phi_{p,s}(q)w^{-s}. \quad (160)
\]
Let
\[ L(q) = (1 - \prod_{i=1}^{l} a_i q)^{-\frac{1}{l}}. \]  
(161)

**Proposition 5.1.** The power series \( \mu_{-l}(q) \) and \( \Phi_{-l,0}(q) \) defined in (160) are given by
\[ \mu_{-l}(q) = \int_0^q \frac{L(u) - 1}{u} du, \quad \Phi_{-l,0}(q) = L^{\frac{1}{1+l}}. \]  
(162)

**Proof:** Since \( F_p(0, q) = 0 \) for \( -l \leq p \leq -1 \), we have
\[ F_p(w, q) = \left( 1 + \frac{q}{w} \frac{\partial}{\partial q} \right)^{1+p} F_{-l} \]
for \( -l \leq p \leq 0 \). Thus let \( 1 + \frac{q}{w} \frac{\partial}{\partial q} \) operates on both sides of (160), we obtain
\[ \mu_{p+1}(q) = \mu_p(q), \quad \Phi_{p+1,0}(q) = (1 + q \mu_p'(q)) \Phi_p,0(q) \]  
(163)
for \( -l \leq p \leq -1 \). By the proposition 4.3 of [28] we have
\[ \mu_0(q) = \int_0^q \frac{L(u) - 1}{u} du, \quad \Phi_{0,0}(q) = L^{\frac{1}{1+l}}. \]
from which (162) follows. \( \square \)

From now on, we denote \( \mu(q) = \mu_{-l}(p) = \cdots = \mu_0(q) \). For convenience to consult results from [28], we define
\[ \tilde{I}_p(q) = \mathcal{M}^p F_0(w, q)|_{w=0}. \]  
(164)
Thus
\[ I_{p+1}(q) = \tilde{I}_p((-1)^n q), \]  
(165)
for \( 0 \leq p \leq n - 1 \).

### 5.2 Some properties of genus zero generating functions

Let
\[ Z_{ij}^*(h, Q) = \sum_{d=1}^{\infty} Q^d \int_{\mathcal{M}_{0,2}((3n-1), d)} \frac{e(U'_0)}{h - \psi_1} e_{1}^{\phi_1} e_{2}^{\phi_2}, \]  
(166)
\[ Z_{ij}^*(h, Q) = h^{-1} \sum_{d=1}^{\infty} Q^d \int_{\mathcal{M}_{0,2}((3n-1), d)} \frac{e(U'_0)}{h - \psi_1} e_{1}^{\phi_1} e_{2}^{\phi_2}, \]  
(167)
\[ Z_{ij}^*(h, Q) = \frac{1}{2h_1 h_2} \sum_{d=1}^{\infty} Q^d \int_{\mathcal{M}_{0,2}((3n-1), d)} \frac{e(U'_0)}{(h - \psi_1)(h - \psi_2)} e_{1}^{\phi_1} e_{2}^{\phi_2}, \]  
(168)
\[ Z_{ij}^*(h, Q) = \sum_{d=1}^{\infty} Q^d \int_{\mathcal{M}_{0,1}((3n-1), d)} \frac{e(U'_0)}{h - \psi_1} e_{1}^{\phi_1}. \]  
(169)
By the string equation we have
\[ Z_i^*(h, Q) = h^{-1} Z_i^*(h, Q). \] (170)

Define
\[ \eta_i(Q) = \text{Res}_{h=0} \left\{ \log \left( 1 + Z_{ij}^*(h, Q) \right) \right\}, \] (171)
and
\[ \Phi_0(\alpha_i, Q) = \text{Res}_{h=0} \left\{ h^{-1} e^{-\eta_i(Q)/h} \left( 1 + Z_i^*(h, Q) \right) \right\}. \] (172)

**Lemma 5.3.** \( e^{-\eta_i(Q)/h} \left( 1 + Z_i^*(h, Q) \right) \) is holomorphic at \( h = 0 \), and thus
\[ \Phi_0(\alpha_i, Q) = e^{-\eta_i(Q)/h} \left( 1 + Z_i^*(h, Q) \right) \bigg|_{h=0}. \] (173)

**Proof:** The proof is the same as the proof of lemma 2.3 in [28]. The only change is to replace \( \text{e}(V_0^0) \) there by \( \text{e}(U_0^0) \), and note that as in (2.10) of that proof, by the normalization sequence we still have
\[ \text{e}(U_0^0) = \prod_{e \in \text{Edg}(\nu_0)} \pi_i^* \text{e}(U_0^0). \] (174)

\[ \square \]

Let
\[ f(q) = \begin{cases} \sum_{d=1}^{\infty} q^{d-1} \frac{(ad)!}{d}, & \text{if } l = 1; \\ 0, & \text{if } l \geq 2. \end{cases} \] (175)

In any case the mirror map is
\[ T = t + f(e^t). \] (176)

**Lemma 5.4.**
\[ 1 + Z_i^*(h, Q) - e^{-f(q) \frac{n_i}{h}} \mathcal{F}_{-l} \left( \frac{\alpha_i}{h} \right) (-1)^n q \in \mathcal{I} \cdot \hat{\mathbb{Q}}[\alpha]^S_{n-1}[[q]], \] (177)
\[ \alpha_i^{n-2} \alpha_j + h Z_{ij}^*(h, Q) - \alpha_i^{n-2} \alpha_j e^{-f(q) \frac{\alpha_i}{h}} \frac{\mathcal{M} \mathcal{F}_{-l} \left( \frac{\alpha_i}{h} \right) (-1)^n q}{I_{1,1}(q)} \in \mathcal{K}_i[[q]]. \] (178)

**Proof:** This follows from the theorem 4.7 of [28]; the argument is almost the same as in the proof of the first part of the lemma 5.3 of [29], so we omit it. \( \square \)

**Lemma 5.5.**
\[ n \alpha_i^{n-1} + 2(h_1 + h_2) h_1 h_2 \bar{Z}_i^*(h_1, h_2, Q) \]
\[ -\alpha_i^{n-1} e^{-f(q) \frac{\alpha_i}{h_1}} \mathcal{F} \left( \frac{\alpha_i}{h_1}, \frac{\alpha_i}{h_2}, (-1)^n q \right) \in \mathcal{I} \cdot \hat{\mathbb{Q}}[\alpha]^S_{n-1}[[q]], \] (179)

where
\[ \mathcal{F}(w_1, w_2, q) = \sum_{p=0}^{n-1-l} \frac{M^{n-1-l-p} \mathcal{F}_0(w_1, q) M^{n-1-l-p} \mathcal{F}_0(w_2, q) \mathcal{I}_{n-1-l-p,n-1-l-p}(q)}{\mathcal{I}_{p,l}(q) \mathcal{I}_{n-1-l-p,n-1-l-p}(q)} \]
\[ + \sum_{p=0}^{l} \frac{M^{n-1-l-p} \mathcal{F}_0(w_1, q) M^{n-1-l-p} \mathcal{F}_0(w_2, q) \mathcal{I}_{n-1-l-p,n-1-l-p}(q)}{\mathcal{I}_{n-1-l-p,n-1-l-p}(q) \mathcal{I}_{n-1-l-p,n-1-l-p}(q)}. \] (180)
Proof: By the corollary 4.6, theorem 4.7 and remark 4.4 of [29], it is not hard to find that

\[
\begin{aligned}
& n\alpha_i^{n-1} + 2(h_1 + h_2)h_1h_2\tilde{Z}_I(h_1, h_2, Q) - a_i^{n-1}e^{-f(q)}\alpha_i \left(\frac{1}{h} + \frac{1}{H} \right) \\
& = \sum_{p=0}^{n-1-l} \frac{M^p F_0(w_1, (-1)^n q)}{I_p, p((-1)^n q)} M^{n-1-l-p} F_0(w_2, (-1)^n q) \\
& + \sum_{p=1}^{n-1} \frac{M^{n-1-l+p} F_0(w_1, (-1)^n q)}{I_{n-1-l+p, n-1-l-p}((-1)^n q)} F_{-p}(w_2, (-1)^n q) \in \mathcal{I} \cdot \tilde{Q} [\alpha] S_n^{-1}[q].
\end{aligned}
\]

(181)

Moreover, in the proof of lemma 4.5 of [28] we see that

\[
\frac{\mathcal{F}_{n-l}(w, q)}{I_{n-l,n-l}(q)} = \mathcal{F}_{-l}(w, q).
\]

(182)

Furthermore by (4.7) of [28] we see \( \tilde{I}_{p,p}(q) = 1 \) for \( n - l + 1 \leq p \leq n - 1 \), therefore by (182) we have

\[
\mathcal{F}_{-p}(w, q) = \mathcal{F}_{n-p}(w, q) = \frac{\mathcal{F}_{n-p}(w, q)}{I_{n-p,n-p}(q)}
\]

(183)

for \( 1 \leq p \leq l - 1 \). Substituting (182) and (183) into (181) we obtain (179). \( \square \)

By (160) and (177) we have

Corollary 5.1.

\[
\eta_i(Q) \equiv (\mu((-1)^n q) - f(q))\alpha_i \in \mathcal{I} \cdot \tilde{Q}[\alpha] S_n^{-1}[q],
\]

(184)

\[
\Phi_0(\alpha_i, Q) - \Phi_{-l,0}((-1)^n q) \in \mathcal{I} \cdot \tilde{Q}[\alpha] S_n^{-1}[q].
\]

(185)

\( \square \)

Lemma 5.6.

\[
\text{Res}_{h=0} \left\{ h^{-1} e^{-\mu(q)} \frac{M^{n-1} F_{-1}(\alpha_i, h)}{h} \right\} = L(q) \Phi_{-l,0}(q),
\]

(186)

Proof: The \( p = -l \) case of (163) gives (180). \( \square \)

For later use, we record the lemma 5.4 of [28] as follows.

Lemma 5.7.

\[
\text{Res}_{h_1=0} \text{Res}_{h_2=0} \left\{ e^{-\mu(q)} \alpha_i \left(\frac{1}{h_1} + \frac{1}{h_2} \right) \right\} = \frac{2}{\alpha_i L(q)} \cdot q \frac{d\tilde{A}(q)}{dq},
\]

(187)

where

\[
\tilde{A}(q) = \frac{n}{48} \left( n - 1 - 2 \sum_{k=1}^{l} \frac{1}{a_k} \right) \mu(q)
\]

\[
- \left\{ \begin{array}{ll}
\frac{n+1}{48} \log(1 - \prod_{k=1}^{l} a_k^{a_k q}) + \sum_{p=0}^{n-l-2} \frac{(n-2)p}{8} \log \tilde{I}_{p,p}(q), & \text{if } 2 \mid (n-l); \\
\frac{n-2}{48} \log(1 - \prod_{k=1}^{l} a_k^{a_k q}) + \sum_{p=0}^{n-l-2} \frac{(n-2)p}{8} \log \tilde{I}_{p,p}(q), & \text{if } 2 \nmid (n-l).
\end{array} \right.
\]

(188)
5.3 Summing the type A contributions

The same argument as in the proof of proposition 1.1 in [36] shows

Proposition 5.2.

\[ A_i(Q) = \frac{1}{\Phi_0(\alpha_i, Q)} \text{Res}_{h=0} \left\{ e^{-\eta_1(Q)/h_1} e^{-\eta_2(Q)/h_2} \bar{Z}_{h_1, h_2}^\alpha(Q) \right\}, \]  

(189)

\[ \tilde{A}_{ij}(Q) = \frac{A_j(Q)}{\prod_{k \in [n]\setminus\{j\}} (\alpha_j - \alpha_k)} \text{Res}_{h=0} \left\{ e^{-\eta_1(Q)/h} (1 + \bar{Z}_{h}^\alpha(Q)) \right\}. \]  

(190)

Since

\[ \text{Res}_{h=0} \left\{ \text{Res}_{h=0} \left\{ e^{-\eta_1(Q)/h_1} e^{-\eta_2(Q)/h_2} \right\} \right\} = 0, \]

by (189), (179), (187) (184) and (185) we have

\[ A_i(Q) - \alpha_i^n - \frac{1}{L((-1)^n q) \Phi_{-L,0}((-1)^n q)} \cdot q \frac{d\bar{A}((-1)^n q)}{dq} \in \mathcal{I} \cdot \bar{Q}_i[\alpha]^{S_{n-1}[[q]]. \]

(191)

By (190), (178), (184) and (185) we have

\[ \sum_{j=1}^n \tilde{A}_{ij}(Q) - \sum_{j=1}^n \left( \frac{\alpha_j^{n-2}}{\prod_{k \in [n]\setminus\{j\}} (\alpha_j - \alpha_k)} \frac{L((-1)^n q) \Phi_{-L,0}((-1)^n q)}{I_{1,1}(q)} \cdot q \frac{d\bar{A}((-1)^n q)}{dq} \right) \in \mathcal{K}_i[[q)\]

(192)

Combining (191) and (192) we obtain

\[ A_i(Q) + \sum_{j=1}^n \tilde{A}_{ij}(Q) - \alpha_i^{n-2} \frac{1}{I_{1,1}(q)} \cdot q \frac{d\bar{A}((-1)^n q)}{dq} \in \mathcal{K}_i[[q]]\]

(193)

Since \( I_{1,1}(q) = q\frac{dq}{dq} \), we obtain

Proposition 5.3.

\[ A_i(Q) + \sum_{j=1}^n \tilde{A}_{ij}(Q) \equiv_i \alpha_i^{n-2} Q \frac{d}{dq} \left[ \frac{n}{48} \left( n - 1 - 2 \sum_{k=1}^l \frac{1}{a_k} \right) \mu((-1)^n q) \right. \]

\[- \left\{ \frac{n+1}{48} \log(1 - \prod_{k=1}^l (-a_k)^a q) + \sum_{p=1}^{n-2} \frac{(n+l-2p)^2}{8} \log I_{p, p}(q), \quad \text{if} \quad 2 \mid (n-l) \right\} \]

\[- \left\{ \frac{n-2}{48} \log(1 - \prod_{k=1}^l (-a_k)^a q) + \sum_{p=1}^{n-3} \frac{(n+l-2p)^2}{8} \log I_{p, p}(q), \quad \text{if} \quad 2 \nmid (n-l) \right\}. \]

(194)
5.4 Summing the type B contributions

Proposition 5.4.

\[ B_\ell(Q) = \frac{1}{24\alpha_i \prod_{k=1}^l \alpha_k} \text{Res}_{h=0,\infty} \left\{ \prod_{k=1}^l (h - a_k \alpha_k) \prod_{j=1}^n (\alpha_i - \alpha_j + h) \frac{Z^n_j(h, Q)}{1 + Z^n_j(h, Q)} \right\}. \]  

(195)

\[ \hat{B}_{ij}(Q) = -\frac{1}{24 \prod_{k=1}^l (a_k \cdot \alpha_j \cdot \prod_{k \in [n] \setminus \{j\}} (\alpha_j - \alpha_k))} \cdot \text{Res}_{h=0,\infty} \left\{ \prod_{k=1}^l \frac{(h - a_k \alpha_j) \prod_{j=1}^n (\alpha_i - \alpha_j + h)}{h^2} \frac{Z^n_j(h, Q)}{1 + Z^n_j(h, Q)} \right\}. \]  

(196)

Proof: Let

\[ \Psi_i(h, \psi) = \frac{\prod_{k=1}^l (-a_k \alpha_i + h) \prod_{j \in [n] \setminus \{i\}} (\alpha_i - \alpha_j + h)}{h + \psi}. \]

Then

\[ \int_{Z_\Gamma} \frac{e^{(U_\alpha)\psi}\phi_\alpha}{e(N Z_\Gamma)} = \int_{\tilde{M}_{1, [\text{val}(v_0)]} \times \mathbb{P}^{m-1}} \left\{ \Psi_i(h, \psi) \prod_{e \in \text{Edg}_+} \left( \int_{Z_\Gamma \alpha} \frac{e^{(U_\alpha)\psi}\phi_\alpha}{e(N Z_\Gamma)} \right) \right\} \text{Res}_{h=0,\infty} \left\{ \prod_{k=1}^l (h - a_k \alpha_i) \prod_{j=1}^n (\alpha_i - \alpha_j + h) \frac{Z^n_j(h, Q)}{1 + Z^n_j(h, Q)} \right\}. \]

The sum of above terms with only \( m \equiv |\text{Edg}(v_0)| \) and \((\mu(\Gamma), \partial(\Gamma))\) fixed is

\[ \int_{\tilde{M}_{1, [\text{val}(v_0)]}} \text{Res}_{z=\psi T} \left( \Psi_i(z, \psi) \prod_{e \in \text{Edg}(v_0)} Z^n_i(z, u) \right). \]

Since \( |\text{val}(v_0)| = m + 1 \), we have

\[ \int_{\tilde{M}_{1, [\text{val}(v_0)]}} \text{Res}_{z=\psi T} \left( \Psi_i(z, \psi) \prod_{e \in \text{Edg}(v_0)} Z^n_i(z, u) \right) = \frac{(-1)^m m!}{24} \text{Res}_{z=v_0} \left\{ (z - a_k \alpha_i) \prod_{j \neq i} (\alpha_i - \alpha_j + z) \cdot (Z^n_i(z, u))^m \right\}. \]

Therefore summing over \( m \geq 1 \) and all possible \((\mu(\Gamma), \partial(\Gamma))\) and taking into the symmetric group \( S_m \) we have

\[ B_\ell(u) = \frac{1}{24 \prod_{k=1}^l (-a_k \cdot \alpha_i) \prod_{r \in [n] \setminus \{i\}} d=1, m=1} \left\{ \prod_{k=1}^l (z - a_k \alpha_i) \prod_{j \in [n] \setminus \{i\}} (\alpha_i - \alpha_j + z) \frac{Z^n_j(z, u)}{z^2} \right\} \]

\[ = -\frac{1}{24 \prod_{k=1}^l (-a_k \cdot \alpha_i) \prod_{r \in [n] \setminus \{i\}} d=1} \left\{ \prod_{k=1}^l (z - a_k \alpha_i) \prod_{j=1}^n (\alpha_i - \alpha_j + z) \frac{Z^n_j(z, u)}{1 + Z^n_j(z, u)} \right\}. \]

43
Then by the residue theorem we obtain (195).

Besides the groups of symmetries, the contribution of type $\tilde{B}_{ij}$ with only $|\text{Edg}(v_0)| = m + 1$ and $(\mu(\Gamma), \partial(\Gamma))$ fixed is

$$\sum_{m=1}^{\infty} \frac{1}{24} \prod_{k=1}^{l} (-a_k) \cdot \alpha_j^m \prod_{k \in [n] \setminus \{j\}} (\alpha_j - \alpha_k) \int_{\hat{\mathcal{M}}_{1,|\text{val}(v_0)|}} \text{Res}_{z=\frac{\alpha_j - \alpha_k}{-m}} \left( \Psi_j(z, \tilde{\psi})(Z_j^n(z, Q))^m \cdot z Z_j^*(z, Q) \right).$$

We have

$$\int_{\hat{\mathcal{M}}_{1,|\text{val}(v_0)|}} \text{Res}_{z=\psi \Gamma} \left( \Psi_j(z, \tilde{\psi})(Z_j^n(z, Q))^m \cdot z Z_j^*(z, Q) \right) = \frac{(-1)^m m!}{24} \text{Res}_{z=\psi \Gamma} \left\{ z^{-(m+2)} \prod_{k=1}^{l} (z - a_k \alpha_i) \prod_{\alpha \in [n] \setminus \{ j \}} (\alpha_j - \alpha_s + z) \cdot (Z_j^n(z, Q))^m \cdot Z_j^*(z, Q) \right\}.$$

Summing over $m \geq 1$ and all possible $(\mu(\Gamma), \partial(\Gamma))$ and taking into the account of the group of symmetries $S_m$, we have

$$\tilde{B}_{ij}(Q) = \frac{1}{24} \prod_{k=1}^{l} (-a_k) \cdot \alpha_j^m \prod_{k \in [n] \setminus \{j\}} (\alpha_j - \alpha_k) \sum_{r \in [n] \setminus \{j\}} \sum_{d=1}^{\infty} \text{Res}_{z=\frac{\alpha_j - \alpha_k}{-m}} \left\{ \prod_{k=1}^{l} (z - a_k \alpha_i) \prod_{\alpha j=1}^{n} (\alpha_j - \alpha_s + z) \cdot \frac{1}{1 + Z_j^*(z, Q)} \cdot Z_j^*(z, Q) \right\}.$$

Again by the residue theorem we obtain (199). \qed

**Proposition 5.5.**

$$B_i(Q) + \sum_{j=1}^{n} \tilde{B}_{ij}(Q) = \alpha_i^{n-2} Q \frac{d}{dQ} \left[ -\frac{n}{48} (n - 1 - 2 \sum_{k=1}^{l} \frac{1}{a_k}) \left( -f(q) + \mu((-1)^n q) \right) + \frac{1-l}{48} \ln \left( 1 - \prod_{k=1}^{l} (-a_k)^{a_k} q \right) + \frac{1}{24} \prod_{k=1}^{l} (-a_k) \cdot \text{Res}_{w=0} \left\{ \prod_{k=1}^{l} (1 - a_k w) \cdot \left( (1 + w)^n - w^n \right) \left( -T w + \ln R(w, t) \right) \right\} \right].$$

(197)

**Proof:** By (195) and (174)

$$B_i(Q) = \frac{1}{24 \alpha_i^l} \prod_{k=1}^{l} (-a_k) \cdot \mathfrak{H}_{h=0, \infty} \left\{ \prod_{k=1}^{l} (h - a_k \alpha_i) \prod_{n=1}^{n} (\alpha_i - \alpha_k + h) \frac{Z_j^*(h, Q)}{1 + Z_j^*(h, Q)} \right\}$$

$$\equiv \frac{1}{24 \alpha_i^l} \prod_{k=1}^{l} (-a_k) \cdot \text{Res}_{h=0, \infty} \left\{ \prod_{k=1}^{l} (h - a_k \alpha_i) \cdot \left( (\alpha_i + h)^n - \alpha_i^n \right) \right\} \frac{\alpha_i^{n-2+l} \prod_{k=1}^{l} \left( \frac{\alpha_i}{\alpha_k} - a_k \right) \left( (1 + \frac{\alpha_i}{\alpha_k})^n - 1 \right)}{\left( \frac{\alpha_i}{\alpha_k} \right)^3}$$

\(\equiv \frac{1}{24 \alpha_i^l} \prod_{k=1}^{l} (-a_k) \cdot \text{Res}_{h=0, \infty} \left\{ \alpha_i^{n-3+l} \prod_{k=1}^{l} \left( \frac{\alpha_i}{\alpha_k} - a_k \right) \left( (1 + \frac{\alpha_i}{\alpha_k})^n - 1 \right) \right\} \)
\[
\frac{e^{-f(q)}\frac{\alpha_i}{\mathcal{F}^{-1}(\frac{Q}{\hbar}, (-1)^n q)}}{e^{-f(q)}\mathcal{F}^{-1}(\frac{Q}{\hbar}, (-1)^n q)}
\]

\[
\equiv_i \frac{\alpha_i^{n-2}}{24 \prod_{k=1}^n(-a_k)} \text{Res}_{h=0,\infty} \left\{ \prod_{k=1}^n(h - a_k)(1 + h)^n - 1 \right\} \cdot \frac{e^{-f(q)}\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q) - 1}{e^{-f(q)}\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q)}/h^3 \right\}
\]

By (196) and (177), (178),

\[
\hat{B}_{ij}(Q) = -\frac{1}{24 \prod_{k=1}^n(-a_k) \cdot \prod_{k \in [n] \setminus \{j\}}(\alpha_j - \alpha_k)} \cdot \text{Res}_{h=0,\infty} \left\{ \prod_{k=1}^n(h - a_k \alpha_j) \prod_{k=1}^n(h - a_k + h) \right\} / (1 + Z^*_i(h, Q))
\]

\[
\equiv_i -\frac{\alpha_i^{n-2} \alpha_j^{n-1}}{24 \prod_{k=1}^n(-a_k) \cdot \prod_{k \in [n] \setminus \{j\}}(\alpha_j - \alpha_k)} \cdot \text{Res}_{h=0,\infty} \left\{ \prod_{k=1}^n(h - a_k)((1 + h)^n - 1) (1 + hQ \frac{d}{dQ}) \left[ e^{-\frac{f(q)}{\pi}}\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q) - 1 \right] \right\}/h^3 \right\}
\]

where we have used

\[
(1 + hQ \frac{d}{dQ}) \left[ e^{-\frac{f(q)}{\pi}}\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q) \right] e^{-\frac{f(q)}{\pi}}\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q)
\]

\[
= e^{-\frac{f(q)}{\pi}}\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q) + (hQ \frac{d}{dQ}) e^{-\frac{f(q)}{\pi}}\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q) + e^{-\frac{f(q)}{\pi}}\left( hQ \frac{d}{dQ} \mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q) \right)
\]

\[
= e^{-\frac{f(q)}{\pi}}\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q) + \left( -\frac{I_{1,1}(q) - 1}{I_{1,1}(q)} \right) e^{-\frac{f(q)}{\pi}}\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q)
\]

\[
+ e^{-\frac{f(q)}{\pi}} \frac{1}{I_{1,1}(q)} \frac{d}{dq} \mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q)
\]

\[
= e^{-\frac{f(q)}{\pi}} M_{\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q)}
\]

Summing over \(j\) we have

\[
\sum_{j=1}^n \hat{B}_{ij}(Q) \equiv_i -\frac{\alpha_i^{n-2}}{24 \prod_{k=1}^n(-a_k)} \text{Res}_{h=0,\infty} \left\{ \prod_{k=1}^n(h - a_k)((1 + h)^n - 1) \right\} / h^3 \right\}
\]

\[
\cdot \left[ (1 + hQ \frac{d}{dQ}) \left[ e^{-\frac{f(q)}{\pi}}\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q) - 1 \right] \right]
\]

\[
\equiv_i \frac{\alpha_i^{n-2}}{24 \prod_{k=1}^n(-a_k)} \text{Res}_{h=0,\infty} \left\{ \prod_{k=1}^n(h - a_k)((1 + h)^n - 1) \right\} / h^3 \right\}
\]

Therefore combining (198) and (202) we obtain

\[
B_i(Q) + \sum_{j=1}^n \hat{B}_{ij}(Q) \equiv_i -\frac{\alpha_i^{n-2}}{24 \prod_{k=1}^n(-a_k)} \text{Res}_{h=0,\infty} \left\{ \prod_{k=1}^n(h - a_k)((1 + h)^n - 1) \right\} / h^2 \right\}
\]

\[
\cdot \left[ Q \frac{d}{dQ} \left[ e^{-\frac{f(q)}{\pi}}\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q) \right] \right] / e^{-\frac{f(q)}{\pi}}\mathcal{F}^{-1}(\frac{1}{\hbar}, (-1)^n q)
\]

\[
= \alpha_i^{n-2} Q \frac{d}{dQ} \left( \hat{B}_0(q) + \hat{B}_\infty(q) \right)
\]

\[
(203)
\]
For \( X \), we have
\[
\tilde{B}_w(q) = -\frac{1}{24 \prod_{k=1}^l (-a_k)} \text{Res}_{h=0, \infty} \left\{ \prod_{k=1}^l (h - a_k)(1 + h)^n - 1 \right\} 
\cdot \log \left( e^{-\frac{q}{h}} \mathcal{F}_{-l}(\frac{1}{h}, (-1)^n q) \right),
\]
for \( w = 0 \) or \( \infty \). Now we use results in section 5.1 to compute \( \tilde{B}_0(q) \) and \( \tilde{B}_\infty(q) \). By lemma 5.2 and 10.2, we have
\[
\tilde{B}_0(q) = -\frac{1}{24 \prod_{k=1}^l (-a_k)} \left[ \prod_{k=1}^l (-a_k) \left( \frac{n(n-1)}{2} - n \sum_{k=1}^l \frac{1}{a_k} \right) (-f(q) + \mu((-1)^n q)) \right] 
+ n \prod_{k=1}^l (-a_k) \ln \Phi_{-l,0}((-1)^n q)
\]
\[
= -\frac{n}{48} (n - 1 - 2 \sum_{k=1}^l \frac{1}{a_k}) (-f(q) + \mu((-1)^n q)) - \frac{n}{24} \cdot \frac{1 - l}{2n} \ln (1 - \prod_{k=1}^l (-a_k)^n q)
\]
\[
= -\frac{n}{48} (n - 1 - 2 \sum_{k=1}^l \frac{1}{a_k}) (-f(q) + \mu((-1)^n q)) + \frac{1 - l}{48} \ln (1 - \prod_{k=1}^l (-a_k)^n q),
\]
where we have used
\[
\prod_{k=1}^l (z - a_k) \cdot \frac{(1 + z)^n - 1}{z^2} = \frac{n \prod_{k=1}^l (-a_k)}{z} + \prod_{k=1}^l (-a_k) \left( \frac{n(n-1)}{2} - n \sum_{k=1}^l \frac{1}{a_k} \right) + O(z).
\]
On the other hand,
\[
\tilde{B}_\infty(q) = \frac{1}{24 \prod_{k=1}^l (-a_k)} \text{Res}_{w=0} \left\{ \prod_{k=1}^l (1 - a_k w) \cdot ((1 + w)^n - w^n) \right\} 
\cdot \ln \left( e^{-T_w e^{tw} \mathcal{F}_{-l}(w, (-1)^n q)} \right)
\]
\[
= \frac{1}{24 \prod_{k=1}^l (-a_k)} \text{Res}_{w=0} \left\{ \prod_{k=1}^l (1 - a_k w) \cdot ((1 + w)^n - w^n) \right\} \left( -T_w + \ln R(w, t) \right),
\]
where we have used the fact \( e^{tw} \mathcal{F}_{-l}(w, (-1)^n q) \equiv R(w, t) \mod w^{n+l} \).

A The modularity of the genus one Gromov-Witten potential for the local \( \mathbb{P}^2 \)

For \( X = K_{\mathbb{P}^2} \), the mirror map is \( Q = q e^{f(q)} \), where
\[
f(q) = \sum_{d=1}^{\infty} q^d \frac{(-1)^d 3! (3d-1)!}{(d!)^3}.
\]

Let \( \psi = -\frac{1}{3} q^3 + \psi \), the genus one free energy given in 11 is
\[
\mathcal{F}_1 = -\frac{1}{2} \log \left( \frac{dT}{d\psi} \right) - \frac{1}{12} \log (1 - \psi^3).
\]
Up to a constant, we have

\[ \mathcal{F}_1 = - \frac{1}{12} \log q - \frac{1}{2} \log (I_{1,1}) - \frac{1}{12} \log (1 + 27q), \tag{209} \]

where

\[ I_{1,1}(q) = 1 + qf'(q) = \sum_{d=0}^{\infty} q^d \frac{(-1)^d (3d)!}{(d!)^3} = 2F_1(1/3, 2/3; 1; -27q). \tag{210} \]

Note that the meaning of the physicists’ genus one free energy is, to get the generating series \( \sum_{d=1}^{\infty} Q^d N^d_{1,d} \) of the Gromov-Witten invariants, one needs to add \( \frac{1}{2} \log Q \) to cancel the log-term in \( \mathcal{F}_1 \). After doing this, we see that \( \mathcal{F}_1 \) coincides with \( \mathcal{I} \).

Now we recall the definition of the modular coordinate in [1] and then show that \( \mathcal{F}_1 \) is a modular form in this coordinate. Let \( q = \exp(2\pi it) \) be the coordinate on the modular curve of \( \Gamma(3) \). From the mathematical viewpoint, [1] gives two ways to relate \( q \) to \( q \). One way is through

\[ j(\tau) = \frac{27q^3(8 + q^3)^3}{(1 - q^3)^3} = \frac{(216q - 1)^3}{q(27q + 1)^3}, \tag{211} \]

Let us first take a look at another way. First we have

\[ \sum_{d=1}^{\infty} dN_{0,d} e^{dT} = - \frac{1}{3} \frac{x^2}{h^2} \left( e^{- \frac{f(q)}{h}} Y(x, h, q) \right) \]

\[ = - \frac{1}{3} \left( e^{-xf(q)} \sum_{d=0}^{\infty} q^d \frac{\prod_{s=0}^{3d-1} (x + s)^3}{\prod_{s=1}^{d} (x + s)^3} \right) \]

\[ = \frac{f(q)^2}{6} - \frac{1}{3} \sum_{d=1}^{\infty} q^d \frac{(-1)^d (3d - 1)!}{(d!)^3} \left( \sum_{s=d+1}^{3d-1} \frac{1}{s} \right), \tag{212} \]

and thus

\[ \frac{d^2}{dT^2} \sum_{d=1}^{\infty} dN_{0,d} e^{dT} = \frac{f(q)}{3} - \frac{f(q) + \sum_{d=1}^{\infty} q^d \frac{(-1)^d (3d)!}{(d!)^3} \left( \sum_{s=d+1}^{3d-1} \frac{1}{s} \right)}{3I_{1,1}(q)} \]

\[ = \frac{f(q)}{3} - \frac{\sum_{d=1}^{\infty} q^d \frac{(-1)^d (3d)!(3d)!}{(d!)^3} \left( \sum_{s=d+1}^{3d} \frac{1}{s} \right)}{I_{1,1}(q)}. \tag{213} \]

The second way to relate \( q \) to \( q \) is via

\[ \tau = -3 \cdot 2\pi i \frac{d}{dT} \left( \frac{1}{6} \left( \frac{T}{2\pi i} \right)^2 + \frac{1}{6} \left( \frac{T}{2\pi i} \right) \right) \]

\[ = -3 \cdot \frac{1}{2\pi i} \cdot \left( \frac{f(q)}{3} - \frac{\sum_{d=1}^{\infty} q^d \frac{(-1)^d (3d)!}{(d!)^3} \left( \sum_{s=d+1}^{3d} \frac{1}{s} \right)}{I_{1,1}(q)} \right) \]

\[ = -\frac{1}{2} + \log q, \tag{214} \]

So

\[ q = -q \exp \left( \frac{\sum_{d=1}^{\infty} q^d \frac{(-1)^d (3d)!}{(d!)^3} \sum_{s=d+1}^{3d} \frac{1}{s} \left( \Psi(3d) - \Psi(3d - 2\Psi(d + 1)) \right)}{\sum_{d=0}^{\infty} q^d \frac{(-1)^d (3d)!(3d)!}{(d!)^3}} \right), \tag{215} \]
where
\[ \Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{k=1}^{\infty} \left( \frac{1}{k} - \frac{1}{k + z - 1} \right). \] (216)

By Chap. 11, entry 26 and example 2 of entry 27 in [7], we obtain
\[ q = \exp \left( - \frac{2\pi}{\sqrt{3}} \frac{\partial T}{\partial \psi} \right). \] (217)

The equivalence of the two ways is insured by the following lemma (by abuse of notation we write \( j \) as a function of \( q \)).

**Lemma A.1.**
\[ j \left( \exp \left( - \frac{2\pi}{\sqrt{3}} \frac{\partial T}{\partial \psi} \right) \right) = \frac{(216q - 1)^3}{q(1 + 27q)^2}. \]

**Proof:** This is (2.8) of [8]. □

The modularity of \( F_1 \) in the coordinate \( q \) is a consequence of Ramanujan’s cubic transformation. First we write \( F_1 \) as
\[ F_1 = -\frac{1}{24} \log \left( \left( \frac{\partial T}{\partial \psi} \right)^{12} (1 - \psi^3)^3 \right) + \frac{1}{24} \log(1 - \psi^3) \]
\[ = -\frac{1}{24} \log (q(1 + 27q)^3) + \frac{1}{24} \log \left( \frac{1 + 27q}{27q} \right) + \text{Const}. \] (218)

By (210), (217), and (2.1), (2.5) of [8], it is not hard to show

**Lemma A.2.**
\[ \Delta(q) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \text{ is the Ramanujan's tau function}. \] (219)

Following [1], let
\[ d(q) = \left( \theta_2 \left( \frac{\pi}{6}, q^{1/2} \right) \right)^3, \]
Then by the infinite product formula for Jacobi theta functions (see for example [31])
\[ \theta_2 \left( \frac{\pi}{6}, q^{1/2} \right) = \sqrt{3q^{1/2}} \prod_{n=1}^{\infty} (1 - q^{3n}), \]
the infinite product formula for the Dedekind’s eta function
\[ \eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \]
together with (2.7) of [11], we obtain
\[ \frac{1 + 27q}{27q} = -\frac{27\eta^{12}(q)}{d^4(q)}. \] (220)

So by (215), (219) and (220) we have

**Theorem A.1.** Up to a constant,
\[ F_1 = -\frac{1}{6} \log \left( d(q) \eta^{3}(q) \right). \] (221) □
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Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China

E-mail address: huxw08@mails.tsinghua.edu.cn