Geodesics for impulsive gravitational waves and
the multiplication of distributions

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Abstract

We consider particle trajectories in the gravitational field of an impulsive pp-wave. Due to the distributional character of the wave profile one inevitably encounters an ambiguous point value $\theta(0)$. We show that this ambiguity may be resolved by imposing covariant constancy of the square of the tangent. Our result is consistent with Colombeau’s multiplication of distributions.

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Introduction

Impulsive pp-waves prominently arise as ultrarelativistic limits of black-hole geometries \[1, 2, 3, 4\]. Since their curvature is completely concentrated on a null hyperplane they are necessarily distributional in nature. Penrose \[5\] gave an intrinsic description of these geometries in terms of his “cut and paste” approach. A different approach to derive the AS-geometry, which represents the ultrarelativistic limit of Schwarzschild, was given by ’t Hooft and Dray \[6\]. They made use of the limit of geodesics in Schwarzschild to derive Penroses junction conditions. We will somehow take the opposite point of view in this work and consider the trajectories of massive and massless particles in geometries generated by an impulsive gravitational wave in their own right. It turns out that the calculation inevitably requires the definition of the point value \(\theta(0)\) at the location of the pulse, thus leaving the usual framework of distribution theory. From a physical point of view this would induce a loss of predictability since the above point value is completely arbitrary. The ambiguity is however resolved if one imposes the physically sensible requirement that a freely falling (geodetic) observer experiences a constant flow of eigentime, i.e. that the norm of the tangent vector remains constant. Our result may be reinterpreted in the sense of Colombeau’s theory of multiplication of distributions if one replaces distributional equality by weak equivalence of Colombeau functions.

In section one we calculate the geodesics for an arbitrary impulsive wave profile. The ambiguity that arises from \(\theta(0)\) is resolved by taking the covariant constancy of the tangent vector into account. Section two gives a brief account of the basic notions of Colombeau’s new generalized functions together with an extension to arbitrary \(C^\infty\) manifolds. Finally, in section three we look at the calculations of section one from the point of view of Colombeau theory thereby justifying the previously obtained results.

1) Geodesics in (impulsive) pp-wave geometries

A pp-wave is usually characterized as being a vacuum geometry which admits a covariantly constant null-vector \(p^a\) \[7\]. However, since we do not want to restrict ourselves to vacuum geometries, we require that the Ricci-tensor is proportional to the tensor square of \(p^a\). From these requirements one shows
that the metric may be written as

\[ g_{ab} = \eta_{ab} + fp_a p_b \]

together with

\[ (p\partial)f = 0, \]

where \( f \) denotes the so-called wave profile and \( \eta_{ab} \) is the flat part of the decomposition. Using the covariant derivative \( \partial_a \) associated with \( \eta_{ab} \) one obtains the difference tensor

\[ C^a_{bc} = \frac{1}{2} \left( p^a \partial_b f p_c + p^a \partial_c f p_b - \partial^a f p_b p_c \right) \]

Choosing a conjugate null vector \( \bar{p}^a \) (i.e. \( \bar{p} \cdot p = -1 \)) allows us to decompose

\[ \partial_a f = -\bar{p}_a (p\partial)f - p_a (\bar{p}\partial)f + \hat{\partial}_a f, \]

where the tilde refers to quantities which are projected in the orthogonal two space of \( p^a \) and \( \bar{p}^a \). This decomposition in turn gives rise to

\[ C^a_{bc} = \frac{1}{2} \left( p^a \tilde{\partial}_b f p_c + p^a \tilde{\partial}_c f p_b - \tilde{\partial}^a f p_b p_c - p^a p_b p_c (\bar{p}\partial)f \right). \]

With respect to an affine parameterization the geodesic equation becomes

\[ \ddot{x}^a + C^a_{bc} \dot{x}^b \dot{x}^c = 0. \]

Decomposition with respect to the conjugate null directions yields

\[ \begin{align*}
px &= 0, \\
\bar{p} \dot{x} + \frac{1}{2} \left( -2(p\dot{x})(\tilde{\partial} f) + (p\dot{x})^2 (\bar{p}\partial)f \right) &= 0, \\
\ddot{x} - \frac{1}{2} (p\dot{x})^2 \tilde{\partial} f &= 0.
\end{align*} \]

The first line of (4) allows us to choose \( px \) as affine parameter unless we are moving in a \( px = const \) surface. This choice leaves us with two equations

\[ \begin{align*}
(px)'' + \left( -x' \tilde{\partial} f + \frac{1}{2} (\bar{p}\partial)f \right) &= 0, \\
\ddot{x} - \frac{1}{2} \tilde{\partial} f &= 0.
\end{align*} \]
where the prime denotes differentiation with respect to \( p x \). Up to now our derivation did not make use of the fact that we are considering impulsive wave profiles, i.e. \( f(p x, \tilde{x}) = \tilde{f}(\tilde{x})\delta(p x) \). By a slight abuse of notation we will drop the tilde on the \( f(\tilde{x}) \) in the following. Physically the above form implies that the geodesics are straight lines above and below the null hyperplane of the pulse

\[
\begin{align*}
\tilde{p}x &= a(p x) + b + \theta(p x) (a^+(p x) + b^+), \\
\tilde{x} &= \tilde{a}(p x) + \tilde{b} + \theta(p x) (\tilde{a}^+(p x) + \tilde{b}^+).
\end{align*}
\]  

(6)

Plugging (6) into the second line of (5) gives

\[
\delta(p x)\tilde{a}^+ + \delta'(p x)\tilde{b}^+ = \frac{1}{2} \delta(p x)\tilde{\partial} f,
\]

which requires

\[
\tilde{b}^+ = 0 \quad \text{and} \quad \tilde{a}^+ = \frac{1}{2} \tilde{\partial} f(\tilde{b}),
\]

whereas the first line

\[
a^+ \delta(p x) + b^+ \delta'(p x) = \frac{1}{2} \tilde{x}'\tilde{\partial} f \delta(p x) + \frac{1}{2} \tilde{\partial}'(p x) f(\tilde{b})
\]

leaves us with

\[
a^+ = \frac{1}{2} (\tilde{a} + \theta(0)\tilde{a}^+) \tilde{\partial} f(\tilde{b}) \quad \text{and} \quad b^+ = \frac{1}{2} f(\tilde{b}).
\]

So the geodesic becomes

\[
\begin{align*}
\tilde{p}x &= a(p x) + b + \theta(p x) \left( \frac{1}{2} (\tilde{a} + \theta(0)\tilde{a}^+) \tilde{\partial} f(\tilde{b})(p x) + \frac{1}{2} f(\tilde{b}) \right), \\
\tilde{x} &= \tilde{a}(p x) + \tilde{b} + \theta(p x) \frac{1}{2} \tilde{\partial} f(\tilde{b})(p x).
\end{align*}
\]  

(7)

At first glance it seems as if we were done, would it not be for the ominous factor \( \theta(0) \) which destroys the predictability of the scattering process. However the physical requirement of the covariantly constant norm of the tangent vector will save the day, i.e.

\[
(\tilde{x} \nabla)(g_{ab}\tilde{x}^a \tilde{x}^b) = 2g_{ab}(\tilde{x} \nabla)\tilde{x}^a \tilde{x}^b = 0 \quad \Rightarrow \quad g_{ab}\tilde{x}^a \tilde{x}^b = \text{const}
\]
For a general pp-wave this gives
\[ \dot{x}^a \dot{x}^b (\eta_{ab} + f p_a p_b) = -2(\dot{p})^2 (\dot{p}) + \dot{x}^2 + f(\dot{p})^2 = \text{const.} \]
\[ \Rightarrow -2(\dot{p}x') + (x')^2 + f = \text{const} \] (8)
Inserting (7) into (8) we obtain immediately
\[ -2(\dot{p}x') + (x')^2 + f = \text{const}, \] (9)
which is only constant if we choose \( \theta(0) = 1/2 \), thereby restoring predictability.

2) Weak equality and Colombeau’s product of distributions

Although the result of the previous section is physically satisfactory it suffers from mathematical deficiencies. This is due to the fact that we implicitly multiplied discontinuous functions with (singular) distributions resulting in the ambiguous point value \( \theta(0) \). Therefore this section is devoted to a brief summary of Colombeau’s new generalized functions \[ \text{[10, 11, 12]} \], together with an extension to arbitrary manifolds, since they provide a natural framework for the multiplication of distributions on a mathematically rigorous basis. From the physical point of view Colombeau objects may be regarded as (arbitrary) regularisations of singular distributions. More precisely\[ \text{[1]} \] one considers one-parameter families \( (f_\epsilon)_{0<\epsilon<1} \) of \( C^\infty \) functions with moderate growth in the parameter \( \epsilon \), namely
\[ \mathcal{E}_M = \{(f_\epsilon) | f_\epsilon \in C^\infty(\mathbb{R}^n) \ \forall K \subset \mathbb{R}^n \text{compact}, \forall \alpha \in \mathbb{N}^n \} \]
\[ \exists N \in \mathbb{N}, \exists \eta > 0, \exists c > 0 \ \text{s.t.} \sup_{x \in K} |D^\alpha f_\epsilon(x)| \leq \frac{c}{\epsilon^N} \ \forall 0 < \epsilon < \eta \}. \] (10)
Addition, multiplication and differentiation are defined as pointwise operations, turning \( \mathcal{E}_M \) into an algebra. \( C^\infty \) functions \( f \) are naturally embedded into \( \mathcal{E}_M \) as constant families, i.e. \( f_\epsilon = f \), whereas continuous functions or elements of \( \mathcal{D}' \) require the use of a so called “mollifier” \( \varphi \) for their embedding
\[ f_\epsilon(x) = \int d^n y \frac{1}{\epsilon^n} \varphi \left( \frac{y-x}{\epsilon} \right) f(y) \quad \int d^n y \varphi(y) = 1, \] (11)
\[ \text{[1]} \] For a motivation of the growth condition see [10]
together with additional conditions on the momenta of $\varphi$ \cite{10, 11}. With regard to distributions the above convolution integral is formal. In order to identify the different embeddings of $C^\infty$ functions one takes the quotient of the algebra $E_M$ with respect to the ideal

$$\mathcal{N} = \{ (f_\epsilon) \mid f_\epsilon \in C^\infty(\mathbb{R}^n), \forall K \subset \mathbb{R}^n \text{ compact}, \forall \alpha \in \mathbb{N}^n, \forall N \in \mathbb{N} \} \ni \exists \eta > 0, \exists c > 0, \sup_{x \in K} |D^\alpha f_\epsilon(x)| \leq c\epsilon^N \forall 0 < \epsilon < \eta \}. \quad (12)$$

thereby obtaining the Colombeau algebra $\mathcal{G}$. Elements of $\mathcal{G}$ are therefore equivalence classes of one-parameter families of $C^\infty$ functions. In order to make contact with distribution theory one considers a coarser equivalence relation which is usually called weak equality or association. It is intuitively motivated by the fact that the limit $\epsilon \to 0$ (if it exists) should reproduce the corresponding distributional object. More precisely, two Colombeau-objects $(f_\epsilon)$ and $(g_\epsilon)$ are associated if

$$\lim_{\epsilon \to 0} \int d^n x (f_\epsilon(x) - g_\epsilon(x)) \varphi(x) = 0 \forall \varphi \in C^\infty_0. \quad (13)$$

Association behaves like equality on the level of distributions. It is compatible with addition and differentiation and it allows multiplication with $C^\infty$ functions. However, it does not respect multiplication of Colombeau objects, as might have been expected. The classical example in this context is provided by the powers of the $\theta$ function, which upon naive multiplication would lead to contradictions about the point value of $\theta(0)$. Specifically

$$(\theta(x))^n = \theta(x) \Rightarrow n\theta(0)\theta'(x) = \theta'(x)$$

which cannot hold for arbitrary $n$ since $\theta(0)$ is $n$-independent. Looking at the above equality from the Colombeau point of view we have

$$(\theta(x))^n \approx \theta(x) \Rightarrow n(\theta(x))^{n-1}\theta'(x) \approx \theta'(x)$$

which holds separately for each $n$ and thereby allows us to replace $(\theta)^{n-1}\theta'$ by $(1/n)\theta'$. The contradiction is avoided because multiplication by theta would break the association.

The above concepts made explicit use of the additive (group) structure of $\mathbb{R}^n$. Specifically the convolution integral used for the embedding of continuous functions and distributions in general lacks generalization to arbitrary
manifolds. In the following we will give a coordinate–independent formulation of the Colombeau algebra. To begin with let us briefly recall the definition of distributions on arbitrary (paracompact) $C^\infty$ manifolds. The natural generalization of distribution space as the dual of a suitable test-function space $[13]$ is achieved by replacing test functions by test forms. That is we define a distribution as (continuous) a linear functional on the space of $C^\infty$ $n$-forms with compact support together with the usual locally convex vector space topology $[14]$. Locally integrable functions $f$ give rise to regular functionals via

$$\tilde{\varphi} \mapsto (f, \tilde{\varphi}) := \int_M f \tilde{\varphi} \quad \forall \tilde{\varphi} \in C_0^\infty(M)$$

The natural generalization of the derivative of a distribution is given by the notion of the Lie-derivative with respect to an arbitrary $C^\infty$ vector-field $X$, i.e.

$$(L_X f, \tilde{\varphi}) := (-1)(f, L_X \tilde{\varphi}).$$

The above definition shows that distribution space does not require additional structure but is purely a concept which depends on the differentiable structure of the manifold. It reduces to the usual notion of distribution space on $\mathbb{R}^n$ if one makes use of the natural volume form $d^n x$ and decomposes every test-form $\tilde{\varphi} = \varphi(x) d^n x$. Let us now try to do the same for the Colombeau algebra. The $C^\infty$ functions of moderate growth in $\epsilon$ are easily generalized

$$E_M(M) = \{(f_\epsilon) \in C^\infty(M) | \forall K \subset M \text{ compact}, \forall \{X_1, \ldots, X_p\}$$

$$X_i \in \Gamma(TM), [X_i, X_j] = 0, \exists N \in \mathbb{N}, \exists \eta > 0, \exists c > 0$$

$$\text{s.t. } \sup_{x \in K} |L_{X_1} \ldots L_{X_p} f_\epsilon(x)| \leq \frac{c}{\epsilon^N} \quad 0 < \epsilon < \eta \}$$

In the same manner we may extend the ideal $\mathcal{N}$

$$\mathcal{N}(M) = \{(f_\epsilon) \in C^\infty(M) | \forall K \subset M \text{ compact}, \forall \{X_1, \ldots, X_p\}$$

$$X_i \in \Gamma(TM), [X_i, X_j] = 0, \forall q \in \mathbb{N}, \exists \eta > 0, \exists c > 0$$

$$\text{s.t. } \sup_{x \in K} |L_{X_1} \ldots L_{X_p} f_\epsilon(x)| \leq c\epsilon^q \quad 0 < \epsilon < \eta \}$$

As usual the Colombeau-algebra $\mathcal{G}(M)$ is defined to be the quotient of $E_M(M)$ with respect to $\mathcal{N}(M)$. In order to generalize the embedding of continuous

\(^2\text{A more detailed version will be given in [15].}\)
functions and more generally of distributions, we have to find an analogue of the smoothing kernel $\varphi$. The immediate problem we are facing is that the expression used in $\mathbb{R}^n$

$$f_\epsilon(x) = \int f(y)\varphi\left(\frac{y - x}{\epsilon}\right)\frac{1}{\epsilon^n}d^n y$$

(14)

makes explicit use of the linear structure of $\mathbb{R}^n$ as may be seen from the argument of $\varphi$ in (14). Moreover, in order to allow the identification of the two types of embeddings of $C^\infty$ functions, as discussed at the beginning of this section, one has to require

$$\int y^\alpha \varphi(y)d^n y = 0 \quad 1 \leq |\alpha| \leq q,$$

for some finite but arbitrary $q \in \mathbb{N}$. Once again this expression depends explicitly on the chosen coordinates. At first glance it seems that the above conditions have to be modified to allow a generalization to an arbitrary manifold. However, if one makes use of the tangent bundle $TM$ of the manifold $M$ the above conditions may be taken as they are. In order to show how this can be done let us consider local coordinates on $TM$ that are induced by coordinates on $M$, i.e. a bundle chart. Let $(U, x)$ denote a local coordinate system of $M$ then $(\pi^{-1}(U), (x, \lambda))$ will denote the corresponding bundle chart, where $TM \xrightarrow{\pi} M$ refers to the canonical projection and $\lambda$ to the coordinates along the fibres. Diffeomorphisms $M \xrightarrow{\mu} M$ induce fibre-preserving diffeomorphisms on $TM \xrightarrow{(\mu, (\partial \mu/\partial x))} TM$, namely so called bundle morphisms. The smoothing kernels will now be taken to be $C^\infty$ $n$-forms on $TM$, which allow a local ADM-like representation

$$\tilde{\varphi} = \varphi(x, \lambda)(d\lambda^i + N_i^j dx^j)^n.$$

Making use of the embedding map

$$i_x : T_x M \rightarrow TM \quad \lambda \mapsto (x, \lambda)$$

we require $\tilde{\varphi}$ to obey

$$\int_{T_x M} i_x^* \tilde{\varphi} = \int_{T_x M} \varphi(x, \lambda)d^n \lambda = 1, \quad i_x^* \tilde{\varphi} \in \Omega_0(T_x M),$$

$$\int_{T_x M} \lambda^\alpha i_x^* \tilde{\varphi} = \int_{T_x M} \lambda^\alpha \varphi(x, \lambda)d^n \lambda = 0 \quad \forall \ 1 \leq |\alpha| \leq q.$$
All of the conditions (15) are invariant under $M$-diffeomorphisms, since the latter act linearly on the fibres. Moreover, rescaling the smoothing kernel is a well defined concept, since the transformation

$$\phi_\epsilon : (x, \lambda) \mapsto (x, \frac{1}{\epsilon} \lambda) \quad \bar{\phi}_\epsilon := \phi_\epsilon^* \bar{\varphi}$$

is a specific case of the left action of the structure group $GL(n, \mathbb{R})$ of $TM$. Having solved the first part of the problem we now have to decide how to generalize the convolution integral necessary for the embedding of the continuous functions in the Colombeau algebra. Let us therefore choose a local coordinate system on $M$, which at the same time induces coordinates on every tangent space in its domain. Let us, moreover, denote the representative of a $C^\infty(M)$ function with respect to the above coordinates by $f(x)$. Identification of the local coordinates on $M$ with those of the fibre attached at $x$ allows us to “lift” $f$ to $T_xM$ by defining the value of the lift at $\lambda$ to be $f(x + \lambda)$. The corresponding smoothened function $f_\epsilon$ on $M$ will then be defined to be

$$f_\epsilon(x) := \int_{T_xM} f(x + \lambda) \varphi(x, \lambda) \frac{1}{\epsilon^n} d^n\lambda = \int_{T_xM} f(x + \epsilon \lambda) \varphi(x, \lambda) d^n\lambda$$

This definition is not coordinate invariant because the action of an $M$-diffeomorphism $\mu$ yields

$$\bar{f}_\epsilon(x) = \int_{T_xM} \bar{f}(\bar{x} + \bar{\lambda}) \bar{\varphi}(\bar{x}, \frac{\bar{\lambda}}{\epsilon}) \frac{1}{\epsilon^n} d^n\bar{\lambda} = \int_{T_xM} \bar{f}(\bar{x} + \epsilon \bar{\lambda}) \bar{\varphi}(\bar{x}, \bar{\lambda}) d^n\bar{\lambda}$$

$$= \int_{T_xM} f(\mu^{-1}(\mu(x) + \epsilon \partial_\mu \partial x \lambda)) \varphi(x, \lambda) d^n\lambda, \quad (16)$$

which obviously differs from the definition in the unbarred system

$$f_\epsilon(x) = \int_{T_xM} f(x + \epsilon \lambda) \varphi(x, \lambda) d^n\lambda \quad (17)$$

The difference of (16) and (17) is of order $\epsilon^{q+1}$ if all the momenta of $\varphi$ up to order $q$ vanish, which is a necessary condition to belong to the ideal $\mathcal{N}$. Thus the corresponding Colombeau object is well-defined since the coordinate
change only generates a motion within its equivalence class. Finally, the concept of association is also without problems, i.e.

\[(f_\varepsilon) \approx (g_\varepsilon) \text{ iff } \lim_{\varepsilon \to 0} \int_M (f_\varepsilon - g_\varepsilon) \tilde{\varphi} = 0 \forall \tilde{\varphi} \in \Omega_0(M).\]

We will say that the Colombeau object \((f_\varepsilon)\) gives rise to a distribution \(T\) if

\[(T, \tilde{\varphi}) := \lim_{\varepsilon \to 0} \int_M f_\varepsilon \tilde{\varphi}\]

defines a (continuous) linear functional over the space of test-forms. The correspondence will in general be many to one. It is easy to see that two different distributions with associated Colombeau objects require the latter to be also different. As in the case of \(\mathbb{R}^n\), association of Colombeau-objects becomes equality on the level of distributions and all vector space operations together with Lie-derivatives do not break the association.

**3) Geodesic equation and Colombeau theory**

Let us now look at the geodesic equation from the point of view of Colombeau theory. That is, replace the delta-function appearing in (1) by an arbitrary regularisation \(\delta\) and do the same for the \(\theta\)'s that enter in (6). Taking into account the three Killing-vectors of the general impulsive wave-profile \([9, 17]\)

\[
\xi^a_1 = p^a \quad \xi^a_\mathbf{\tilde{t}} = (\mathbf{\tilde{t}} x)p^a - (px)(\mathbf{\tilde{t}})_a, \tag{18}
\]

where \(\mathbf{\tilde{t}}\) denotes an arbitrary vector in the subspace transversal to \(p^a, \bar{p}^a\), yields

\[
p\dot{x} \approx \text{const}
\]

\[
(\mathbf{\tilde{t}} x)p\dot{x} - (px)(\mathbf{\tilde{t}} \dot{x}) \approx \text{const}. \tag{19}
\]

The first line of (13) just tells us that \(px\) is an affine parameter, while the second line fixes the transversal shift \(\tilde{b}^+\) in (\(\mathfrak{B}\)) to be zero. The requirement that the length of tangent vector is covariantly constant along the geodesic becomes

\[-2a + \bar{a}^2 + (-2a^+ + (\bar{a}^+)^2)\theta + (-2b^+ + f(\bar{b}))\theta' \approx \text{const}, \tag{20}\]
which implies that the coefficients of $\theta$ and $\theta'$ have to be zero

$$
a^+ = \tilde{a}a^+ + \frac{1}{2}(\tilde{a}^+)^2, \\
b^+ = \frac{1}{2}f(\tilde{b}).
$$

(21)

From the projection of the geodesic onto the orthogonal complement of $p^\alpha$ and $\bar{p}^\alpha$

$$
\ddot{x}'' - \frac{1}{2}\partial f \delta \approx 0
$$

(22)

together with the above results we obtain

$$
\tilde{a}^+ = \frac{1}{2}\tilde{\partial} f(\tilde{b})
$$

Since all the plus-parameters are determined in terms of the initial data, the remaining equation has to be satisfied consistently.

$$
(\bar{p}x)'' - (x'\tilde{\partial})f \delta - \frac{1}{2}f \delta' \approx 0 \\
(a^+(px) + b^+)\theta'' + 2a^+\theta' - \frac{1}{2}(\tilde{a} + ((px)\theta)'(\tilde{a}^+ \tilde{\partial}) f) \delta - \frac{1}{2}f(\tilde{b}) \delta' \approx 0 \\
b^+\theta'' + a^+\theta' - \frac{1}{2}(\tilde{a}\tilde{\partial})f(\tilde{b})\theta' - \frac{1}{2}(\tilde{a}^+ \tilde{\partial}) f ((px)\theta)' \delta - \frac{1}{2}f(\tilde{b}) \theta'' \approx 0 \\
\frac{1}{8}(\tilde{\partial} f(\tilde{b})^2) \theta' - \frac{1}{4}(\tilde{\partial} f(\tilde{b}) \tilde{\partial}) f ((px)\theta)' \delta \approx 0.
$$

(23)

Taking into account that $((px)\theta)'$ is also a Theta-function, which will be denoted by $\tilde{\theta}$ one has $\tilde{\theta} \delta \approx C\theta'$ as may be seen from

$$
(\tilde{\theta} \delta, \varphi) = \int_{-\infty}^{\infty} du \int_{-\infty}^{u} dv \varphi(u) \varphi(v) \epsilon(u) \epsilon(v) \phi, \psi \in C_0\infty
$$

$$
= \int_{-\infty}^{\infty} du \phi(u) \int_{-\infty}^{u} dv \varphi(\epsilon u) = C \varphi(0) + O(\epsilon).
$$

(24)

In (24) $\phi$ and $\psi$ respectively denote the regularisations of $\delta$ and $\tilde{\theta}'$. Therefore (23) becomes

$$
\theta' \frac{1}{8}(\tilde{\partial} f)^2(\tilde{b})(1 - 2C) \approx 0,
$$
which requires $C$ to be $1/2$. However, this is precisely the type of condition we encountered in section one. It restricts the regularisations one is allowed to choose for $\delta$ and $\theta$ in order to obtain a consistent (regularisation-independent) result.

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Conclusion

In this work we solved the geodesic equation for arbitrary impulsive pp-wave geometries. Due to the singular character of the wave profile, which contains a delta-function, one inevitably leaves the framework of (classical) distribution theory. This fact manifests itself in the appearance of the ambiguous pointvalue $\theta(0)$. However, since we made use of an affine parametrization, the length of the tangent vector remains covariantly constant along the geodesic. This requirement fixed $\theta(0)$ to be $1/2$. In order to justify our result mathematically we made use of the recently developed framework of Colombeau’s new generalized functions, which is designed to allow a (consistent) multiplication of distributions. Although the original definition of the Colombeau algebra $\mathcal{G}$ made use of $\mathbb{R}^n$-specific concepts we showed that a generalisation to an arbitrary manifold $M$ is possible by employing the corresponding tangent bundle $TM$. Finally we showed that our condition on $\theta(0)$ may be justified from the point of view of Colombeau-theory as condition on the “regularizations” used for the pulse and the geodetic trajectory, respectively.

As a next step we will investigate semiclassical scattering, namely various wave equations in a general impulsive pp-background with specific emphasis on those geometries arising as ultrarelativistic limits of black-hole geometries.
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