Formula for the widths of the plateaus of the quantum Hall effect

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Abstract

We derive an empirical formula for the width of quantum Hall effect plateaus which is free of adjustable parameters. It describes the integer, fractional and $\nu = 0$ (Wigner insulator) quantum Hall effect in single heterojunctions. The temperature scale of the existence of these three phenomena is the same as the melting temperature of a classical Wigner crystal. We conclude that the basic assumption of the current theory of QHE that the plateau width is determined by the disorder is highly improbable.

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Three phenomena of completely different nature are believed to exist in a 2-dimensional electron system (2DES) in high magnetic fields at low temperatures [1]. The integer quantum Hall effect is thought to be describable as single particle localization (by the disorder) of a noninteracting electron gas [1]. The fractional Hall effect is taught to be due to quasiparticle localization (by the disorder) on top of a many-particle liquid state [1]. A Wigner crystal (many-body effect) exists at low filling factors [2]. The essence of the quantum Hall effect (QHE) is the existence of plateaus of finite width in the Hall resistance [3]. In spite of substantial theoretical efforts devoted to this subject no successful formula has yet been derived for the width of these plateaus. However, since it is believed that the plateau width is determined by localization in the disorder potential, one would expect it to be sample dependent and moreover to decrease and disappear in the low disorder limit.

Here we discuss experimental facts which contradict this assumption. On the basis of the symmetries of the experimental data we derive an empirical formula for the width of the plateaus of the quantum Hall effect (integer and fractional) as well as for the width of the filling factor region of the Wigner solid. We also show that the temperature scales of the integer QHE, fractional QHE and the Wigner solid are the same and close to the classical melting temperature of the Wigner crystal.

The Landau level filling factor of the 2DES of density $n_s$ in a magnetic field $B$ is given by $\nu = n_s hc/eB$. In the absence of the quantum Hall effect (for example at high temperatures) the Hall conductivity is given by the classical value $\sigma_{xy} = enc/B = \nu e^2/h$. The quantum Hall effect plateaus appear at integer and fractional values of $\nu$. The most important factors influencing the width of plateaus are the temperature $T$ and the disorder [4]. We will be interested in the limit of very low temperatures and disorder. We will quantify the meaning of “very low” below.

We start with several observations concerning the shape of the dependence $\sigma_{xy}(\nu)$ in this limit. They are based on visual examination of the available data and should be considered as empirical rules.
1. The transitions between adjacent plateaus are sharp steps.
2. The plateaus extend symmetrically on both sides of the classical line $\sigma_{xy}(\nu) = \nu e^2/h$ (see Fig. 1). We denote the half-width of the plateau at the filling factor $\nu$ by $\Delta \nu$. 
3. The following symmetry relations are fulfilled
   \[ \Delta \nu_{1+\nu} = \Delta \nu_\nu \]  
   \[ \Delta \nu_{1-\nu} = \Delta \nu_\nu \]  
4. The plateaus are grouped in sequences as follows.
A main sequence converging towards $1/2$:
   \[ \nu = p/(2p + 1), \quad p = 0, 1, 2 \ldots \]
and its symmetry partners according to (1). This is the only sequence in 2D hole systems. Two additional secondary sequences converging towards $1/4$ exist in 2D electron systems:
   \[ \nu = p/(4p \pm 1), \quad p = 0, 1, 2 \ldots \]
and their symmetry partners according to (1). The positions of these plateaus is given by the single formula $\nu_{m,p} = p/(2mp \pm 1)$ where $m = 1$ corresponds to the main sequence and $m = 2$ to the secondary sequences.
We would like to stress that the property 4. has been discussed earlier in the literature [5-10]. The symmetry $\nu \rightarrow 1 \pm \nu$ has also been discussed earlier [5-10] although in the present form [5] we give it for the first time. To our knowledge the properties 1. and 2. have not been discussed earlier, although we consider them apparent from the experimental data (see for example [11,6,12]).

Using these facts we find a formula for the widths of the plateaus. Indeed, let $\nu_{m,p}$ and $\nu_{m,p+1}$ be two adjacent plateaus within a particular sequence. As a consequence of 1. and 2., the following recursive equation holds for the plateau widths (see also Fig. 1):

$$\Delta \nu_{m,p} + \Delta \nu_{m,p+1} = |\nu_{m,p+1} - \nu_{m,p}| \tag{2}$$

The solution of this recursive equation is unique for each of the sequences and is given by the formula:

$$\Delta \nu_{p/(2mp \pm 1)} = \frac{1}{8m^2} \left[ 2\psi \left( \frac{p}{2} + \frac{1}{2} \pm \frac{1}{4m} \right) - \psi \left( \frac{p}{2} + 1 \pm \frac{1}{4m} \right) - \psi \left( \frac{p}{2} \pm \frac{1}{4m} \right) \right] \tag{3}$$

where $\psi(z)$ is the digamma function [13]. This is our main result. The staircases obtained from formula (3) are presented in Fig. 2. The main staircase spans the space $0 < \nu < 1/2$ while the secondary staircase spans the space $0 < \nu < 1/3 + \Delta \nu_{1/3} = 0.37187$. The plateau widths of the most prominent plateaus are presented in Tables I-III.

Now we compare with the experiment. The experimental data show that in 2-dimensional hole systems $\Delta \nu_0$ and $\Delta \nu_{1/3}$ belong to the main sequence. Indeed $\Delta \nu_0 = 0.2854$ agrees perfectly well with the experimental value $\sim 0.28$ of the critical filling of the existence of insulating phase $\sigma_{xy} = 0$ [14]. It is also in perfect agreement with the half-width of the $\nu = 1$ plateau $\Delta \nu_1 = \Delta \nu_0 = 0.2854$ [11]. In general the main sequence plateaus (which is the only sequence present in hole systems) are very well described by the formula (3).

In 2-dimensional electron systems also the secondary sequences are observed [11-12]. In fact the experimental data shows that the range $0 < \nu < 1/3 + \Delta \nu_{1/3} = 0.37187$ belongs to the secondary staircase while the range $2/5 - \Delta \nu_{2/5} = 0.38127 < \nu < 1/2$ belongs to the main staircase. For example the width of the Wigner solid region [2] is equal to the half-width of the $\nu = 1$ plateau $\Delta \nu_1 = \Delta \nu_0 = 0.2854$ [11]. In general the main sequence plateaus are presented in Tables I-III.

Further we would like to discuss the conditions under which the real system data is close to the ideal one described above. First we discuss the effect of disorder. In Fig. 3 we present a schematic dependence of the half width of the $\nu = 1$ plateau as a function of the sample mobility $\mu$ at low temperatures and zero magnetic field. It is a summary of our investigation of the available data. The mobility is a measure (although indirect) of the disorder in the sample. At very low mobilities there is no quantum Hall effect. At mobilities in the range of $1-2 \times 10^5$ cm$^2$/V.s the plateau width has a maximum. It is due to the broadening of the plateaus by disorder-induced local electron reservoirs spread in the sample. This broadening is reduced when the sample quality is improved. Above $\mu = 1-2 \times 10^6$ cm$^2$/V.s the plateau width saturates to the value $\Delta \nu_1 \approx 0.19$ (for the best published...
sample $\mu = 1.2 \times 10^7 \text{ cm}^2/\text{V.s}$. If the disorder would be relevant to the width of the plateaus one would expect a gradual decrease to zero of $\Delta \nu_1$ when $\mu$ is increased. A measure of the disorder is the width of the smallest resolvable plateau. For the data published in Ref. [7] it is for example $\Delta \nu_{5/11} \approx 0.004$. In low density systems the effect of disorder is more pronounced. We can estimate it from the difference between the experimental width of the Winger insulator region $\approx 0.19$ and our theoretical value $0.1835$ to be of order of $0.007$.

Next we discuss the effect of temperature. In Fig. 6 we present a combined plot of quantum Hall effect data and Wigner insulator data. On the horizontal axis the filling factor $\nu$ in the case of Wigner insulator and and $1 - \nu$ in the case of QHE is given. The black points are the melting temperature $T$ of the Wigner insulator normalized to the melting temperature of a classical Wigner crystal $T_{cl} = \sqrt{n_se^2/127\varepsilon \pi}$. The data is from the review of Williams et al. [2] and is obtained with different techniques in different groups on samples with densities ranging from $3.4 \times 10^{10}$ cm$^{-2}$ to $10.2 \times 10^{10}$ cm$^{-2}$. No systematic trend of the density dependence of the melting temperature exists. On the same plot we give experimental $\sigma_{xy}(1 - \nu)h/e^2$ taken from Sajoto et al. (solid lines) [16] and the half width of the $\nu = 1$ plateau from Clark et al. (circles). The $\sigma_{xy}(1 - \nu)h/e^2$ dependencies has been offset vertically to the corresponding normalized temperature. The density of the sample is $5.5 \times 10^{10}$ cm$^{-2}$ ($T_{cl} = 380$ mK). The density of the sample of Clark et al. is $1.9 \times 10^{11}$ cm$^{-2}$ corresponding to almost twice higher $T_{cl}$. As it is obvious from the figure the characteristic temperature scale of the existence of both IQHE and Wigner insulator is the melting temperature of the classical Wigner crystal. Moreover, visual inspection of the data of Sajoto et al. and Goldman et al. Ref. [12] shows that also the fractional QHE exists on the same temperature scale. Typical lowest dilution refrigerator temperatures are $20 - 30$ mK $\ll T_{cl}$ which shows that the low temperature limit is reached experimentally. The connection of the melting of a Wigner crystal with the width of the plateaus in the integer QHE was discussed in Ref. [19].

We would like to mention that here we have neglected the effect of dissipation ($\sigma_{xx} = 0$) and related to it reentrant phases [12,18].

At the end we would like to discuss the consequences of the existence of formula (3).

1. Integer and fractional Hall effect seem to have the same origin as well as the low-filling factor insulating phase (Wigner crystal) which can be viewed as $\nu = 0$ quantum Hall effect.
2. Disorder is highly improbable to be the reason for the finite width of the plateaus.

These two conclusions cast serious doubts on the validity of the standard picture [1] of the behavior of 2DEG in magnetic field.

The symmetry relations $\nu \rightarrow 1 \pm \nu$ as well as recursive relations of the type of (3) are natural for the properties of electrons moving in periodic potential [20]. It has been shown, however, [21] that if one uses a Kubo formula for the Hall conductivity of independent spinless electrons in external periodic potential, one can not obtain the fractional QHE. The problem we see with the Kubo formula is that it is a rewritten fluctuation-dissipation theorem and it is not clear if it could be applied for description of a dissipationless phenomenon like QHE. For example one can not obtain superconductivity by applying the Kubo formula to a noninteracting electron gas.

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FIGURES

FIG. 1. Schematic drawing of two adjacent QHE plateaus at filling factors $\nu_a$ and $\nu_b$. The plateaus are symmetrically extended on both sides of the classical line.

FIG. 2. The $\sigma_{xy}(\nu)$ plot of the main sequence staircase $m = 1$ (the dotted line) and the secondary staircase $m = 2$ (the solid line) obtained using equation (3) in the region $0 < \nu < 1/2$.

FIG. 3. The $\sigma_{xy}(\nu)$ plot of the staircase obtained using equation (3) and symmetry relations (1) in the region $0 < \nu < 3$.

FIG. 4. The $\rho_{xy}(1/\nu)$ plot of the staircase obtained using equation (3) and symmetry relations (1).

FIG. 5. Schematic dependence of the plateau half-width $\Delta\nu_1$ on the mobility.

FIG. 6. Combined plot of quantum Hall effect data and Wigner insulator data. On the horizontal axis is given the filling factor $\nu$ in the case of Wigner insulator and and $1 - \nu$ in the case of QHE. The black points are the melting temperature $T$ of the Wigner insulator normalized to the melting temperature of a classical Wigner crystal $T_{cl} = \sqrt{n_s e^2/127\pi}\sqrt{\pi}$. Solid lines are the experimental $\sigma_{xy}(1 - \nu)h/e^2$ taken from Sajoto et al.. Circles are the half width of the $\nu = 1$ plateau from Clark et al.. The $\sigma_{xy}(1 - \nu)h/e^2$ dependencies has been offset vertically to the corresponding normalized temperature.
TABLES

TABLE I. Plateau half-widths of the main sequence $\nu = p/(2p + 1)$.
\begin{tabular}{|l|cccccc|}
\hline
$\nu$ & 0 & 1/3 & 2/5 & 3/7 & 4/9 & 5/11 \\
\hline
$\Delta \nu_\nu$ & 0.2854 & 0.0479 & 0.0187 & 0.0098 & 0.0060 & 0.0041 \\
\hline
\end{tabular}

TABLE II. Plateau half-widths of the secondary sequence $\nu = p/(4p + 1)$.
\begin{tabular}{|l|cccc|}
\hline
$\nu$ & 0 & 1/5 & 2/9 & 3/13 \\
\hline
$\Delta \nu_\nu$ & 0.1835 & 0.0165 & 0.0057 & 0.0028 \\
\hline
\end{tabular}

TABLE III. Plateau half-widths of the secondary sequence $\nu = p/(4p - 1)$.
\begin{tabular}{|l|ccccc|}
\hline
$\nu$ & 1/3 & 2/7 & 3/11 & 4/15 \\
\hline
$\Delta \nu_\nu$ & 0.0385 & 0.0091 & 0.0039 & 0.002 \\
\hline
\end{tabular}