Magnetic hallmarks of viscous electron flow in graphene

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We propose a protocol to identify spatial hallmarks of viscous electron flow in graphene and other two-dimensional viscous electron fluids. We predict that the profile of the magnetic field generated by hydrodynamic electron currents flowing in confined geometries displays unambiguous features linked to whirlpools and backflow near current injectors. We also show that the same profiles shed light on the nature of the boundary conditions describing friction exerted on the electron fluid by the edges of the sample. Our predictions are within reach of vector magnetometry based on nitrogen-vacancy centers embedded in a diamond slab mounted onto a graphene layer.

Introduction.—Electrical transport1–5, thermal transport6, and scanning gate spectroscopy7 measurements have recently been used to identify signatures of viscous electron flow in high-quality graphene, palladium cobaltate, and GaAs. (For a recent review see e.g. Ref. 8.) In this regime of transport dominated by electron-electron interactions, viscosity determines electron whirlpools in the steady-state current pattern, which have been theoretically studied with great detail and are expected to emerge in confined geometries9–12. So far, a direct experimental observation of electronic whirlpools and associated backflow near current injectors is still lacking.

A promising route to achieve real space imaging of spatial patterns of current flow in two-dimensional (2D) materials is to employ vector magnetometry based on nitrogen-vacancy (NV) centers in diamond13, which combines the benefits of high spatial resolution and competitive magnetic field resolution. NV vector magnetometry optically detects the field-dependent magnetic resonances of an ensemble of NV centers, from which, relying on schemes based on an external magnetic field14,15 or based on optical polarization16, the Cartesian components of the local magnetic field are determined. The capability of this noninvasive imaging technique to access the details of 2D spatial flow patterns has been recently demonstrated in graphene in the diffusive regime15. NV vector magnetometry operates over a wide range of temperatures15, including room temperature15, and its spatial resolution is comparable with the viscosity diffusion length in graphene1–4,9–12. Recently, the electronic spin of a single NV center attached to a scanning tip and operated under ambient conditions was used to image and detect microwave fields in a micron-scale stripline19 and to image charge flow in carbon nanotubes and Pt nanowires20.

In this Rapid Communication we propose to apply NV vector magnetometry to detect viscous spatial flow patterns in graphene. We calculate the magnetic field generated by hydrodynamic currents flowing in a graphene sample of rectangular shape, placed below an array of NV centers and above a metallic back gate. A cartoon of the geometry is shown in Fig. 1. We show that this field carries unambiguous signatures of electron whirlpools.

Transport equations in viscous 2D electron systems.—In the linear response regime and in a steady-state, viscous electron transport in a 2D electron fluid is described9–12 by the continuity

\[ \nabla \cdot \mathbf{J}(x) = 0 \] (1)

and Navier-Stokes

\[ D_0^2 \nabla^2 \mathbf{J}(x) - \sigma_0 \nabla \phi(x) = \mathbf{J}(x) \] (2)

equations. Here, \( x = (x, y) \) describes the position in the
plane where electrons roam, $\mathbf{J}(\mathbf{x})$ the current density, $\phi(\mathbf{x})$ the 2D electrostatic potential, and the characteristic viscosity diffusion length $D_v = \sqrt{\nu \tau}$ has been introduced in Ref. 9, $\nu$ being the kinematic shear viscosity and $\tau$ a phenomenological transport time describing momentum-conserving collisions. In Eq. (2), $\sigma_0 \equiv e^2 \pi \tau / m$ is a Drude-like conductivity, where $\pi$ denotes the equilibrium electron density, which can be controlled by a metallic gate, and $m = h k_F / v_F$ is the electron effective mass in graphene, with $k_F = \sqrt{\pi \tau}$ and $v_F \simeq 10^6$ m/s the Fermi wave number and the Fermi velocity, respectively. Eqs. (1) and (2) can be solved for $\mathbf{J}(\mathbf{x})$ and $\phi(\mathbf{x})$ by introducing suitable boundary conditions $^{6-12}$ These solutions will be used below to study signatures of viscous electron flow, as carried by the magnetic field generated by $\mathbf{J}(\mathbf{x})$.

**Magnetic field generated by 2D current profiles.**—A 2D current density, $\mathbf{J}(\mathbf{x})$, confined at $z = 0$, above a metallic gate placed at $z = -d$, generates a magnetic field at $z > 0$, $\mathbf{B}(\mathbf{x}, z) = [B_x(\mathbf{x}, z), B_y(\mathbf{x}, z), B_z(\mathbf{x}, z)]$, with components

$$B_x(\mathbf{x}, z) = \frac{\mu_0}{2} \int d^2 \mathbf{x}' K_{xy}(\mathbf{x} - \mathbf{x}', z) J_y(\mathbf{x}') ,$$  \hfill (3)

$$B_y(\mathbf{x}, z) = -\frac{\mu_0}{2} \int d^2 \mathbf{x}' K_{xy}(\mathbf{x} - \mathbf{x}', z) J_x(\mathbf{x}') .$$  \hfill (4)

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The kernels appearing in the above convolutions read as following:

$$K_{xy}(\mathbf{x} - \mathbf{x}', z) \equiv \frac{z}{2\pi ||\mathbf{x} - \mathbf{x}'||^2 + z^2}$$

$$-\frac{1}{2\pi ||\mathbf{x} - \mathbf{x}'||^2 + (2d + z)^2} ,$$  \hfill (6)

and

$$K_{z}(\mathbf{x} - \mathbf{x}' , z) \equiv \frac{1}{2\pi ||\mathbf{x} - \mathbf{x}'||^2 + z^2}$$

$$-\frac{1}{2\pi ||\mathbf{x} - \mathbf{x}'||^2 + (2d + z)^2} ,$$  \hfill (7)

and $\mu_0$ denotes the free-space permeability. The non-local relations $^{24}$ in Eqs. (3), (4), and (5) are obtained by solving the Poisson equation for the vector potential, accounting for appropriate boundary conditions $^{21}$ Eqs. (3), (4), and (5) describe a one-to-one correspondence between a generic 2D current density and the generated magnetic field.

**Magnetic hallmarks of viscous electron flow in the vicinity resistance geometry.**—In the following we present numerical results for the components of the magnetic field in Eqs. (3) and (4), evaluated at the position of an array of NV centers, assumed to be aligned at $z = d'$. The magnetic field is generated by the 2D current density $\mathbf{J}(\mathbf{x})$ in graphene, in the so-called vicinity resistance geometry—see Refs. 1, 9–11 and below. The graphene sample is modelled as a rectangular stripe of infinite

![FIG. 2. (Color online) Spatial map of the $\hat{z}$ component, $B_z(\mathbf{x}, z = d')$, of the magnetic field (indicated by the color map and in $\mu T$) generated by $J_y(\mathbf{x})$. The current density is represented by the vector field. Panel (a) Ohmic case, $D_v = 0$. Panel (b) viscous case, $D_v = W/4$. The injector is located at position $(0, -W/2)$. The collector (not shown) is set on $B_z(\mathbf{x}, z = 3W/2)$. These horizontal positions have been marked by a solid and a dashed line in Fig. 2, respectively. Solid lines in this plot correspond to the viscous case ($D_v = W/4$), while dashed lines correspond to the Ohmic case ($D_v = 0$).](image-url)
length along the longitudinal direction, $\hat{x}$, while it has a finite width $W = 2 \mu m$ along the transverse direction, $\hat{y}$. We consider that along the lower edge of the sample (set at $y = -W/2$) there is a point-like current source injecting a current $I$ at $x_+ = (0, y = -W/2)$ and a point-like current drain at $x_- = (x_0, y = -W/2)$, while in the remaining points of both edges the normal component of the current density is set at zero, i.e. $J_y(x, \pm W/2) = 0$. An additional boundary condition on the tangential component of the current density is required at the sample edges. Here, we use the free-surface boundary conditions, i.e. we impose $[\partial_y J_x(x, y) + \partial_x J_y(x, y)]_{y=\pm W/2} = 0$. Below we comment on the impact of a different choice. In this case, and following Ref. 10, we can write

$$J(x) = I \left[ \nabla \left[ F(x, y + W/2) - F(x, y - W/2) \right] + \nabla \times \mathbf{\hat{z}} \left[ G(D_v; x, y + W/2) - G(D_v; x, y - W/2) \right] \right],$$

where $F(x) \equiv \ln[\cosh(\pi x/W) - \cos(\pi y/W)]/(2\pi)$, and $G(D_v; x) = 2D_v^2 [\partial_x \partial_y F(x) + S(x)]$, with $S(x) \equiv \sum_{n=1}^{\infty} \sin(n\pi y/W) n\pi \text{sign}(x) e^{-|x|\sqrt{(n\pi/W)^2 + 1}} D_v^2 / (W^2)$. Numerical results have been obtained by setting $d' = 10 \text{ nm, } d = 100 \text{ nm, and } I = 200 \mu A$. Here, we compare the magnetic field generated by viscous flow with a realistic value of the viscosity diffusion length, i.e. $D_v = W/4$, with that generated by Ohmic flow, which is mathematically enforced by setting $D_v = 0$ in Eq. (2).

Spatial maps of the components $B_x$ and $B_y$ of the magnetic field generated by viscous and Ohmic flows, computed from the current density in Eq. (8) by using Eqs. (3) and (4), are reported in Fig. 2 and Fig. 4, respectively. By considering the drain at $x_0 \to -\infty$, we are able to focus on the electron whirlpool on the right of the current injector, as seen in Figs. 2(b) and 4(b).

In Fig. 2 we clearly see that in the viscous case the negative lobes of $B_x(x, z = d')$ near to the current injector are much more extended spatially for finite $D_v$ than for $D_v = 0$. A contraction of the size of these lobes by increasing temperature and therefore reducing $D_v$, crossing over from the hydrodynamic to the Ohmic regime, signals the occurrence of such smooth transition. We now note that the effect of a finite viscosity is much more pronounced than what is seen in the color map. Indeed, it is enough to look at Fig. 3, where we present one-dimensional cuts of the 2D spatial map taken along the vertical lines $x = W/2$ and $x = 3W/2$. We clearly see that, in the Ohmic case, both the profiles of $B_x$ and $J_y$ are concave functions of $y/W$. On the contrary, in the viscous case and in the presence of a whirlpool, the profiles of $B_x$ and $J_y$ have an opposite convexity in an extended range of values of $y/W$.

A spatial map of $B_y(x, z = d')$ is reported in Fig. 4, for a vanishing—panel (a)—and finite—panel (b)—viscosity diffusion length. A clearer signal of viscosity is seen in this figure, in comparison with the map of $B_x(x, z = d')$ reported in Fig. 2. In the viscous case, the negative regions of $B_y(x, z = d')$ near the current injector are more collimated than in the Ohmic $D_v = 0$ case. Also, positive regions of $B_y(x, z = d')$ are present to the right of the current injector, where $B_y(x, z = d')$ is instead negative definite in the Ohmic case. One-dimensional cuts of $B_y(x, z = d')$ are shown in Fig. 5, together with $J_y(x)$. Within the shown regions, we clearly see that the profile of $B_y(x, z = d')$ generated by Ohmic flow is monotonic. On the contrary, the presence of a current whirlpool in the viscous case generates a magnetic field with a $B_y$ profile featuring clear non-monotonicity. We conclude that current whirlpools stemming from viscous electron flow in confined geometries determine clear-cut trends in the spatial maps of the generated magnetic field.

**Longitudinal flow.**—We now consider the situation in which no current is injected or extracted laterally at the edges of the graphene sample. In this case, we take current flowing only along the longitudinal direction $\hat{x}$. Following Ref. 9, the current density $J(y) = [J_x(y), 0]$, stemming from the solution of Eqs. (1) and (2), is uniform along $\hat{x}$, and reads as follows:

$$J_x(y) = \frac{I}{W} \left[ 1 - D_v \cosh(y/D_v) / \xi \right],$$

where $\xi \equiv \ell_b \sinh(W/2D_v) + D_v \cosh(W/2D_v)$. This current density profile has been obtained by describing friction exerted by the edges of the device via the generic boundary conditions $[\partial_y J_x(x, y) + \partial_x J_y(x, y)]_{y=\pm W/2} = \mp J_x(x, y = \pm W/2)/\ell_b$. Here, the boundary scattering length $\ell_b$ allows us to interpolate between the no-slip ($\ell_b \to 0$) and free-surface ($\ell_b \to +\infty$) boundary conditions. Plots of Eq. (9) for different values of $\ell_b$ and $D_v$ are shown in Fig. 6(c). The transition from trans-
verse uniform flow—occurring for free-surface boundary conditions—to Poiseuille flow\(^26\)—occurring for no-slip boundary conditions is clearly visible.

The magnetic field generated by the current distribution in Fig. 6(c) and evaluated at a distance \(d' = 10\) nm from the electron fluid— where we assume that the NV centers are placed—is shown in Figs. 6(a) and (b). Fig. 6(a) shows that the profile of \(B_y\) generated by viscous flow and no-slip boundary conditions displays a local minimum at the center of the sample, distinguishing it from the profile generated by a transversally-uniform current density, \(J/W\), obtained enforcing free-surface boundary conditions. The positive kinks observed in all the profiles of \(B_y\), located near \(\pm \gamma = \pm \sqrt{W^2 + (2D)^2}/2\), close to the edges of the sample, arise from the contribution of the image current density at the back gate, placed at \(z = -2d\). Similarly, Fig. 6(b) shows that the sharpness and the amplitude of the profile of \(B_y\) decreases with increasing \(D\) by enforcing no-slip boundary conditions, while the profile generated by a transversally-uniform current density is sharply peaked at \(\pm \gamma\). The detection of the magnetic field generated by a longitudinal flow may therefore enable to determine the suitable boundary conditions for the tangential component of the current density. Before concluding, we enlighten that in the vicinity of the Ohmic case (\(D_o = W/4\)), while dashed lines correspond to the Ohmic case (\(D_o = 0\)).

In summary, we have calculated the magnetic field generated by viscous flow in two-dimensional conductors, showing that it displays unambiguous features linked to current whirlpools and backflow near current injectors.

We have also shown that the same quantity sheds light on the nature of the boundary conditions describing friction exerted on the electron fluid by the edges of the sample. We believe that our predictions can be tested by carrying out nitrogen-vacancy vector magnetometry on two-dimensional hydrodynamic electron fluids, a technique with spatial resolution that can greatly enrich our understanding of hydrodynamic transport in solid-state physics.
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21 The Poisson equation for the vector potential in the Coulomb gauge reads \( \nabla^2 \mathbf{A}(q, z) + \partial^2 \mathbf{A}(q, z)/\partial z^2 = -\mu_0 \mathbf{J}(q) \delta(z) \), and has general solution \( \mathbf{A}(q, z) = [\mu_0 J(q) e^{-|q|\sqrt{2}}/(2q)] + \mathbf{a}_+(q) e^{qz} + \mathbf{a}_-(q) e^{-qz} \), with \( \mathbf{J}(q) \) and \( \mathbf{A}(q, z) \) denoting the Fourier transforms of the current density and vector potential with respect to the in-plane coordinate, respectively. Appropriate boundary conditions need to be imposed on the general solution: i) \( \mathbf{a}_+ = 0 \), which imposes that the vector potential vanishes at \( z = \infty \), ii) \( \mathbf{a}_+(q) = \mathbf{t} q \cdot \mathbf{a}_+(q)/q \), which imposes that the vector potential satisfies the Coulomb gauge, and iii) \( B_z(x, z = -d) = 0 \), i.e., the perpendicular component of the magnetic field vanishes at the position of the back gate, which is treated as a perfect conductor. From i) and iii) it follows that \( \mathbf{a}_-(q) = -\mu_0 \mathbf{J}(q) e^{-2qd}/(2q) \). Then the solution of the Poisson equation, accounting for all boundary conditions reads as follows: \( \mathbf{A}(q, z) = \mu_0 \mathbf{J}(q) e^{-|q|\sqrt{2d}} - e^{-2qd} e^{-qz}/(2q) \). Eqs. (3) and (4) in the main text are obtained by Fourier transforming into real space, \( B_z(x, z) = -\partial \mathbf{A}_x(q, z)/\partial z + B_y(q, z) = \partial \mathbf{A}_y(q, z)/\partial z \), respectively, where we have used the 2D Fourier transform \( \mathbf{A}(q) = \mathbf{f} \mathbf{d} x \mathbf{e}^{-i q \mathbf{x}}/2\pi |\mathbf{x}|^2 + z^2 \). Eq. (5) in the main text follows from \( B_y(q, z) = i \mathbf{q} \times \mathbf{A}(q, z) \) and \( e^{-qz}/q = \mathbf{f} \mathbf{d} x \mathbf{e}^{-i q \mathbf{x}}/2\pi |\mathbf{x}|^2 + z^2 \).
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