Nonextensive distribution and factorization of
the joint probability

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Abstract

The problem of factorization of a nonextensive probability distribution is discussed. It is shown that the correlation energy between the correlated subsystems in the canonical composite system can not be neglected even in the thermodynamic limit. In consequence, the factorization approximation should be employed carefully according to different systems. It is also shown that the zeroth law of thermodynamics can be established within the framework of the Incomplete Statistical Mechanics (ISM).

Key words: Statistical mechanics, Nonextensive distribution, factorization approximation
1 Introduction

The nonextensive probability distribution

\[ p_i = \frac{[1 - (1 - q)\beta(E_i - C)]^{\frac{1}{1-q}}}{Z_q} \]  \hspace{1cm} (1)

plays a decisive role for the success of the generalized statistical mechanics [1,2] because it is capable of reproducing unusual distributions of non gaussian type which are met frequently in nature (see reference [1] and the references therein). Very recently, distribution functions of type Eq. (1) were recognized by the so-called eigencoordinates method with high level of authenticity as good description of the amplitude distribution of earthquake noises which are proved to be fractal and strongly correlated [3]. In Eq. (1), \( \beta \) is the generalized inverse temperature, \( E_i \) is the energy of the system in the state \( i \), \( C \) a constant to assure the invariance of the distribution through uniform translation of energy spectrum \( E_i \), and \( Z_q \) is given by

\[ Z_q = \sum_i [1 - (1 - q)\beta(E_i - U_q)]^{\frac{1}{1-q}} \]  \hspace{1cm} (2)

or by

\[ Z_q = \sum_i [1 - (1 - q)\beta(E_i - U_q)]^{\frac{q}{1-q}}, [4] \]  \hspace{1cm} (3)

in Tsallis’ multi-fractal inspired scenario [1] with \( U_q \) the internal energy of the system and \( \beta = \frac{Z_q}{kT} \), and by

\[ Z_q = \left[ \sum_i [1 - (1 - q)\beta E_i]^{\frac{1}{1-q}} \right]^{\frac{1}{q}}. \]  \hspace{1cm} (4)

with \( \beta = \frac{Z_q^{1-q}}{kT} \) in Wang’s incomplete statistics scenario [2] devoted to describe inexact or incomplete probability distribution [5] due to neglected interactions in the system hamiltonian. It is noteworthy that Eq. (1) is a canonical distribution function for isolated system in terms of its total energy \( E_i \).

In this paper, we discuss the factorization problem of Eq. (1) and some of its consequences when the canonical system is composed of correlated subsystems or particles. At the same time, we also comment on some interesting applications of Eq. (1) found in the literature.
2 Factorization approximation

In Boltzmann-Gibbs-Shannon statistics (BGS), thanks to the easy factorization of the exponential distribution functions, i.e.,

$$\sum_i e^{-\beta \sum_j e_{ij}} = \prod_j \sum_i e^{-\beta e_{ij}}. \tag{5}$$

the system distribution function can yield one particle one in terms of the single body energy $e_{ij}$ in the case of independent particles or of mean-field method with $E_i = \sum_{j=1}^{N} e_{ij}$, where $j$ is the index and $N$ the total number of the distinguishable particles.

But this approach is impossible in the case of the nonextensive distribution, because,

$$\sum_i [1 - (1 - q)\beta \sum_j e_{ij}]^{\frac{1}{1-q}} \neq \prod_j \sum_i [1 - (1 - q)\beta e_{ij}]^{\frac{1}{1-q}}. \tag{6}$$

This inequality makes it difficult to apply the nonextensive distribution even to systems of independent particles, because the total partition function cannot be factorized into single particle one. So in the definition of the total system entropy $S_q = f[p(E_i), q]$, $p(E_i)$ must be the probability of a microstate of the system and $E_i$ cannot be replaced by one-body energy $e_{ij}$. However, in the literature, we find applications of Tsallis’ distribution in which $E_i$ is systematically replaced by $e_{ij}$ without explanation [6–11]. The first examples[6,8] are related to the polytropic model of galaxies and the authors have taken the one-body energy of stars and of solar neutrinos ($\epsilon = \Psi + \frac{v^2}{2}$) as $E_i$. Other examples are the peculiar velocity of galaxy clusters ($e \sim v^2$)[11] and the electron plasma turbulence where the electron single site density $n(r)$ was taken as system (electron plasma) distribution function and the total energy was calculated with the one-electron potential $\phi(r)$ [7,9]. The last case is the application of nonextensive blackbody distribution to laser systems where the atomic energy levels were taken as laser system energy in equation (8) of reference [10]. As a matter of fact, in above examples, the calculated entropies and distributions are one-particle ones, as we always do in BGS framework. Consequently, considering Eq. (6), they are only approximate applications of the exact Tsallis’ distribution.

The legitimacy of the above mentioned applications depends on the approximation with which we write Eq. (6) as an equality. One of the solutions is the limit of $q \to 1$ where Eq. (6) tends to an equality. But this solution does not hold for the cases of $q$ value very different from unity, as in the above mentioned examples of applications. Another way out is the factorization ap-
proximation proposed in order to obtain the nonextensive Fermi-Dirac and Bose-Einstein distributions [12] which read

\[ \langle n_q \rangle = \frac{g}{[1 + (q - 1)\beta(e - \mu)]^{1-\frac{1}{q}} + 1} \]  

where \( g \) is the degeneracy of the level with energy \( e \). As in the case of the corrected Boltzmann distribution (i.e. \( \frac{n_q}{g} \ll 1 \)), Eq. (7) can be reduced to

\[ p(e) \sim \frac{n_q}{g} = [1 - (1 - q)\beta(e - \mu)]^{1-\frac{1}{q}} \]  

where \( p(e) \) is one-particle probability and \( e \) one-particle energy. So we can say that only in this approximation the above mentioned applications are justified.

It is worth mentioning that, although approximate, the above successful applications were the first proofs of the existence of Tsallis type distributions (Eq. (1)) in nature. In addition, this is a parametrized distribution. So in many cases, what we neglect in the approximation may be compensated (at least partially) by a different value of \( q \) fixed empirically. The interest of the applications mentioned above is to show that this kind of nonextensive (nonadditive) probability can really describe some non-gaussian type peculiar distributions we observe. That is what is important in practice.

Nevertheless, an approximation has sometimes in itself theoretical importance when it concerns the basic foundation of a theory. That is the case of the factorization approximation.

### 3 Factorization of the joint probability

The factorization approximation is a forced marriage between the right-hand side and the left-hand side of Eq. (6), that is we write just like that :

\[ [1 - (1 - q)\beta \sum_j e_j]^{1-\frac{1}{q}} \simeq \prod_j [1 - (1 - q)\beta e_j]^{1-\frac{1}{q}}. \]  

where we keep only the index \( j \) of the particles, as we do from now on in this section. What is neglected in this approximation is the difference \( \Delta \) between the right-hand side and the left-hand side of Eq. (9) which has been investigated in reference [13] with a two-level system for the simple case where \( q > 1 \) (or \( q < 1 \)), \( e_j > 0 \) (or \( e_j < 0 \)) and \( \mu = 0 \). Under these harsh conditions, \( \Delta \) turns out to be very small at normal temperatures for mesoscopic or macroscopic systems (with important particle number \( N \)). But it is not the case for
q < 1 and \( e_j > 0 \) with in addition \( \mu \neq 1 \). So in general, we can not write Eq. (9).

Eqs. (6) and Eq. (9) can be discussed in another way as follows. If we replace \( \sum_j e_j \) by the total energy \( E \) in Eq. (9), we get:

\[
[1 - (1 - q) \beta E]^{1/\mu} = \prod_j [1 - (1 - q) \beta e_j]^{1/\mu}.
\]  \tag{10}

for \( N \) subsystems (or particles with energy \( e_j \) where \( j = 1, 2, \ldots N \)) of a composite system with total energy \( E \) at a given state. This is just the factorization of the joint probability \( p(E) \) as a product of all \( p(e_j) \):

\[
p(E) = \prod_{j=1}^N p(e_j). \tag{11}
\]

With Eq. (10) or (11), strictly speaking, we can not write

\[
E = \sum_j e_j. \tag{12}
\]

But in Tsallis’ scenario, Eq. (12) is necessary for establishing the zeroth law of thermodynamics [14]. Abe studied this problem with ideal gas model and concluded that Eq. (12) can hold for \( e_j > 0, 0 < q < 1 \) and \( N \to \infty \). That is the correlation energy \( E_c = E - \sum_j e_j \) between the maybe strongly correlated subsystems or particles can be neglected. But this is of course not a general conclusion for any \( q \) value or any system. In what follows, we will try to give the general expression of the correlation energy in \( ISM \) because this relation is implicit in Tsallis’ scenario [15]. The following discussion is for \( 0 < q < \infty \), the permitted interval of \( q \) value in \( ISM \) [2].

If \( N = 1 \), from Eq. (11), we naturally obtain \( E = e_1 \) so \( E_c = 0 \).

If \( N = 2 \), we obtain:

\[
aE = (1 + ae_1)(1 + ae_2) - 1
\]

\[
= ae_1 + ae_2 + a^2 e_1 e_2 = a \sum_{i=1}^2 e_i + a^2 e_1 e_2
\]

where \( a = (q - 1)\beta \). So \( E_c = ae_1 e_2 \).

If \( N = 3 \), we get:
\[ aE = (1 + ae_1)(1 + ae_2)(1 + ae_3) - 1 \] (14)
\[ = a(e_1 + e_2 + e_3) + a^2(e_1e_2 + e_1e_3 + e_2e_3) + a^3(e_1e_2e_3) \]
\[ = a \sum_{i=1}^{3} e_i + a^2 \sum_{i_1 < i_2} e_{i_1}e_{i_2} + a^3 \prod_{i=1}^{3} e_i. \]

and

\[ E_c = a(e_1e_2 + e_1e_3 + e_2e_3). \] (15)

When \( N \to \infty \), this is a infinite product problem. In general, we obtain:

\[ aE = \prod_{i=1}^{N} (1 + ae_i) - 1 \] (16)
\[ = a \sum_{i=1}^{N} e_i + a^2 \sum_{i_1 < i_2} e_{i_1}e_{i_2} \]
\[ + a^3 \sum_{i_1 < i_2 < i_3} e_{i_1}e_{i_2}e_{i_3} + \ldots + a^N \sum_{i_1 < i_2 < \ldots < i_N} \prod_{j=1}^{N} e_{i_j} \]
\[ = \sum_{k=1}^{N} a^k \sum_{i_1 < i_2 < \ldots < i_k} \prod_{j=1}^{k} e_{i_j} \]
\[ = aE_0 + aE_c \]

where \( E_0 \) is the system energy given by Eq. (12) for independent subsystems like the particles of perfect gas. \( E_c \) is given by:

\[ E_c = \prod_{i=1}^{N} (1 + ae_i) - E_0 \] (17)
\[ = \sum_{k=2}^{N} a^{k-1} \sum_{i_1 < i_2 < \ldots < i_k} \prod_{j=1}^{k} e_{i_j} \]

The value of \( E_c \) is in general difficult to estimate. We can discuss it in the following way.

For \( q > 1 \) (or \( 0 < q < 1 \)) and \( e_i > 0 \) (or \( e_i < 0 \)), \( E_c \) may be very big when \( N \to \infty \) because \( \sum_i (ae_i) \) may diverge. So it is in general not negligible.

For \( q > 1 \) (or \( 0 < q < 1 \)) and \( e_i < 0 \) (or \( e_i > 0 \)),
\[ E_c = \sum_{k=2}^{N} (-1)^{k-1} [1 - q]^{k-1} \left( \prod_{i_1 < i_2 < ... < i_k} |e_{i_j}| \right) \]  

The sign of each term varies alternatively, which may turn out to cancel \( E_c \) when \( N \to \infty \). But this is only a possibility. In general, we cannot say that \( E_c \) is negligible.

Considering Eq. (11), the mean value of \( E_c \) is given by the following integral in phase-space \( \Omega(\tau) \):

\[ U_c = \int d\tau p(E) E_c \]

\[ = \sum_{k=2}^{N} (-1)^{k-1} [1 - q]^{k-1} \left( \prod_{i_1 < i_2 < ... < i_k} |U_{i_j}| \right) \]

The conclusion of this section is that \( U_c \) is in general not negligible. So one must be very careful in using nonextensive distribution for correlated subsystems within factorization approximation, especially in the case of the nonextensive quantum statistics derived on the basis of this approximation [12].

4 Re-establishment of the zeroth law of thermodynamics

In this section, we will discuss an application of the so-called Incomplete Statistical Mechanics (ISM) [2], a new version of the nonextensive statistical mechanics (NSM) based on the normalization condition \( \sum_i p_i^q = 1 \) where \( q \) is a positive parameter. \( q \neq 1 \) corresponds to the fact that the probability distributions \( \{p_i\} \) with incomplete random variables [5] do not sum to one. If \( q = 1 \), the random variables of the problem become complete and we recover the conventional normalization (CN) \( \sum_i p_i = 1 \). ISM was proposed as a consequence of the study of the fundamental theoretical problems [1,4] of Tsallis version of NSM with escort probability [1]. The fundamental philosophy of ISM is that CN is difficult to be applied to the systems with important complicated interactions or correlations because, in this case, our “incomplete” knowledge about the states of the systems does not permit us to sum all the (exact) probabilities. With simpler systems, this human ignorance could be neglected and CN holds.

In our opinion, ISM is not just another form of Tsallis nonextensive statistics,
contrary to what some scientists thought. Some of the significant differences is
discussed in Ref.[4,15]. In what follows, we will re-establish, in a precise way,
the zeroth law which was established within Tsallis theory only in the case of
factorization approximation in neglecting the correlation energy.

In incomplete statistical mechanics, we have following equations for a com-
posite system containing two correlated subsystems $A$ and $B$:

$$S_q(A + B) = S_q(A) + S_q(B) + \frac{q-1}{k} S_q(A) S_q(B),$$  \hspace{1cm} (20)

$$E_{ij}(A + B) = E_i(A) + E_j(B) + (q-1)\beta E_i(A) E_j(B),$$  \hspace{1cm} (21)

$$U_q(A + B) = U_q(A) + U_q(B) + (q-1)\beta U_q(A) U_q(B),$$  \hspace{1cm} (22)

$$S_q = k\frac{Z_q^{q-1} - 1}{q-1} + k\beta Z_q^{q-1} U_q,$$  \hspace{1cm} (23)

and

$$\beta = \frac{Z_q^{1-q} k}{k T}. \hspace{1cm} (24)$$

In Eq. (20), there is a plus sign “+” before the correlated term. But it was
minus sign “-” in Tsallis formalism. Eqs. (21) and (22) show the same $q$-
dependence of the correlation terms as Eq. (20). They do not exist in Tsallis
formalism with escort probability [15]. The $q$-dependence of the above equa-
tions is of crucial importance for the establishment of the zeroth law.

From Eq. (20), a small variation of the total entropy can be written as :

$$\delta S_q(A + B) = [1 + \frac{q-1}{k} S_q(B)] \delta S_q(A) + [1 + \frac{q-1}{k} S_q(A)] \delta S_q(B)$$  \hspace{1cm} (25)

$$= [1 + \frac{q-1}{k} S_q(B)] \frac{\partial S_q(A)}{\partial U_q(A)} \delta U_q(A)$$

$$+ [1 + \frac{q-1}{k} S_q(A)] \frac{\partial S_q(B)}{\partial U_q(B)} \delta U_q(B).$$

And from Eq. (22), the variation of the total internal energy is given by :

$$\delta U_q(A + B) = [1 + \frac{q-1}{k} U_q(B)] \delta U_q(A) + [1 + \frac{q-1}{k} U_q(A)] \delta U_q(B).$$  \hspace{1cm} (26)

It is supposed that the total system $(A + B)$ is completely isolated. So $\delta U_q(A + B) = 0$ which leads to :
\[
\frac{\delta U_q(A)}{1 + \frac{q-1}{k} U_q(A)} = -\frac{\delta U_q(B)}{1 + \frac{q-1}{k} U_q(B)} \tag{27}
\]

When the composite system \((A + B)\) is in equilibrium, \(\delta S_q(A + B) = 0\). In this case, Eqs. (25) and (27) lead us to:

\[
\frac{1 + \frac{q-1}{k} U_q(A)}{1 + \frac{q-1}{k} S_q(A)} \frac{\partial S_q(A)}{\partial U_q(A)} = \frac{1 + \frac{q-1}{k} U_q(B)}{1 + \frac{q-1}{k} S_q(B)} \frac{\partial S_q(B)}{\partial U_q(B)} \tag{28}
\]

With the help of Eqs. (23) and (24), it is straightforward to show that, in general:

\[
\frac{1 + \frac{q-1}{k} U_q}{1 + \frac{q-1}{k} S_q} = Z_q^{1-q} \tag{29}
\]

which recasts Eq. (28) as follows:

\[
Z_q^{1-q}(A) \frac{\partial S_q(A)}{\partial U_q(A)} = Z_q^{1-q}(B) \frac{\partial S_q(B)}{\partial U_q(B)} \tag{30}
\]

or

\[
\beta(A) = \beta(B) \tag{31}
\]

where \(\beta\) is the generalized inverse temperature defined in Eq. (24). Eq. (30) or (31) is the generalized zeroth law of thermodynamics which describes the thermodynamic relations between different nonextensive systems in thermal equilibrium.

One of the important meanings of Eq. (31) is that the thermal equilibrium of a system is now characterized by \(\beta\) but not \(T\). This allows us to have an explicit distribution with Eq. (1), which becomes implicit in Tsallis formalism with Eq. (2) and (3).

## 5 Conclusion

Our conclusion is that all applications of Tsallis’ distribution relative to correlated subsystems or particles should be considered to be valid only under the factorization approximation and the condition \(n_q \ll 1\). This factorization approximation must be employed carefully because the correlation energy between the correlated subsystems is in general not negligible even in the
thermodynamic limit ($N \to \infty$). The zeroth law of thermodynamics can be established in the framework of *Incomplete Statistical Mechanics* without any approximation.

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[4] Eqs. (2) and (3) lead to a peculiar equality

$$\sum_i [1 - (1 - q)\beta(E_i - U_q)]^{1\over1-q} = \sum_i [1 - (1 - q)\beta(E_i - U_q)]^q\over\sum_i [1 - (1 - q)\beta(E_i - U_q)]^{1\over1-q}.$$  

This equality is a basic relation of the theory and must hold for arbitrary value of $\beta$ and $E_i$. A simple calculation we performed with a two level system showed that the only constant solution for $q$ of the above peculiar equality with arbitrary $\beta$, $E_i$ and $p_i$ is $q = 1$. This result is indeed a little surprising because it means that the theory loses its generality while a generalized distribution is given by Eqs.(1)-(3). This is evidently a consequence of the double-normalization, i.e. $\sum_i p_i = 1$ and $\sum_i P_i = 1$ with $P_i = p_i^q\over\sum_i p_i^q$, referred to as *escort probability*.

The escort probability was proposed to avoid originally unnormalized expection value (i.e. $<C>\neq C$ while $C$ is a constant) and to recover energy translational invariance of the distribution [1]. We would like to indicate here that the above peculiar equality can be avoided in ISM because the theory is single-normalized with $<C> = C$. The energy translational invariance of the
generalized distributions, in our opinion, is a more complicated topic and perhaps requires long discussions. Our first idea is that it could be recovered by defining relative energy value (indeed, absolute energy does not exist and all energy value is relative) or adding an energy translation invariance constraint in the maximization of entropy, provided that this invariance is worthy of being kept in addition to the scale invariance for especially the well-known power-laws distributions observed around the critical points of phase transitions or in financial market.

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[15] For a composite system containing two correlated subsystems $A$ and $B$, from the factorization of joint probability $p_{ij}(A + B) = p_i(A)p_j(B)$ in Tsallis scenario with escort probability, we have:

$$E_{ij}(A + B) - U_q(A + B) = \frac{E_i(A) - U_q(A)}{\sum_i p_i(A)} + \frac{E_j(B) - U_q(B)}{\sum_j p_j(B)} + \frac{(q-1)\beta[E_i(A) - U_q(A)][E_j(B) - U_q(B)]}{\sum_i p_i(A) \sum_j p_j(B)}.$$  

This equation makes it impossible to find the correlation energy $E_c(A, B)$ and its meanvalue.