Modelling Correlation Matrices in Multivariate Dyadic Data: Latent Variable Models for Intergenerational Exchanges of Family Support

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Abstract

We define a model for the joint distribution of multiple continuous latent variables which includes a model for how their correlations depend on explanatory variables. This is motivated by and applied to social scientific research questions in the analysis of intergenerational help and support within families, where the correlations describe reciprocity of help between generations and complementarity of different kinds of help. We propose an MCMC procedure for estimating the model which maintains the positive definiteness of the implied correlation matrices, and describe theoretical results which justify this approach and facilitate efficient implementation of it. The model is applied to data from the UK Household Longitudinal Study to analyse exchanges of practical and financial support between adult individuals and their non-coresident parents.

Keywords: Bayesian estimation; Covariance matrix modelling; Item response theory models; Positive definite matrices; Two-step estimation

1 Introduction

In contemporary low-mortality countries, population ageing has led to an increase in the need for help and support for people with age-related functional limitations. At the same time, the need for support may also be increasing among younger people as a result of delayed transitions to adulthood, unstable employment, high cost of living, and rises in divorce and re-partnership rates (Lesthaeghe, 2014; Henretta et al., 2018). With limited public resources available to meet these demands, there is a greater reliance on private transfers of support within families, especially between parents and their adult children. The main ‘currencies’ of such intergenerational exchanges are time (or practical support) and money (e.g. Grundy, 2005). Another form of intergenerational support is coresidence but its overall rate remains low, in spite of a small increase in coresidence between young adults and their parents (e.g. Stone et al., 2011). Transfers of practical and financial support between relatives living in different households are thus a more important component of family exchanges. Understanding the nature of these exchanges is important for anticipating which population sub-groups may be at risk of unmet need

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previous research highlights the importance of reciprocity (or symmetry) in such exchanges, either contemporaneously or over the life course \cite{Hogan1993, Grundy2003, Silverstein2002}, both as a motivating factor for providing support and because of its association with other outcomes. For example, there is evidence that over-benefitting (receiving more than giving) has negative consequences for older parents’ well-being \cite{Davey1998} while balanced exchanges are positively associated with parents’ mental health \cite{Litwin2004}. The extent of reciprocity is likely to depend on individual characteristics. For example, in a cross-national European study, \cite{Mudrazija2016} finds that net transfers from parents to adult children follow a similar age pattern across the majority of countries, with declining positive transfers (parents giving more than they receive) for parents aged 50-79, becoming negative in most countries from age 80. There is also evidence from Europe \cite{Mudrazija2016} and the U.S. \cite{Hogan1993} that reciprocity reflects the geographical proximity of parents and children and gender differences in family roles.

Another question of interest is whether practical and financial support serve as functional substitutes or complements of each other \cite{Mudrazija2016}, and how their interdependence depends on individual characteristics. Among the factors that may play a role are income and geographical distance where better-off adult children or children living at a greater distance from their parents may substitute money for time transfers to parents \cite{Grundy2003}. Alternatively time and money transfers may be positively associated, with a tendency to give or receive both or neither form of support.

Most previous substantive research has either combined practical and financial support, or analysed them separately. The first of these approaches implicitly assumes that the two forms of support are indicators of a common underlying dimension and thus does not allow for differences in their predictors, while the second ignores their interrelationship. Moreover, most earlier work has analysed support given separately from support received, which precludes the analysis of reciprocity of exchanges. Research that has investigated reciprocity has typically modelled a joint categorical outcome for whether exchanges are mutual or one-way \cite{Hogan1993} or modelled the difference between support given and support received \cite{Mudrazija2016}. Both approaches consider reciprocity directly, but then do not permit analysis of the effects of individual characteristics on exchanges in each direction separately. Alternatively, reciprocity can be quantified as the residual correlation in a joint model of support given and support received \cite{Kuha2022, Steele2022}. This approach can be extended to treat financial and practical support as separate but correlated outcomes \cite{Steele2022}.

In this paper we develop a general joint modelling framework that is used to simultaneously investigate predictors of financial and practical support given and received, and predictors of the correlations among these different types of exchange. We analyse cross-sectional multivariate dyadic data from the UK Household Longitudinal Study (UKHLS), which contains 16 questions (‘items’) about exchanges of help on dyads formed of a survey respondent and their non-coresident parent(s). Seven of the items relate to whether or not different kinds of practical help are given to parents (for example, assistance with shopping) and a further seven items indicate the forms of practical help received from parents. The remaining two items indicate whether financial help is given and received. The practical help items are treated as multiple binary indicators of two continuous latent variables which are modelled jointly with latent variables taken to underlie the two indicators of financial exchanges. We also account for zero inflation, which arises from a high proportion of respondents who report giving or receiving none of the types of sup-
port, by including in the model the joint distribution of two binary latent variables for the subpopulations with excess zeros.

The model formulation builds on that of Kuha et al. (2022), who also analysed family exchanges of support using UKHLS data. We extend their model in two ways. First, we separate practical and financial help, which they considered together. Second, and most importantly, we introduce a model for how the correlations of tendencies to give and receive different types of support depend on predictors (covariates). The key advantage of this framework is that it allows us to answer questions not only about the predictors of giving and receiving different forms of support (the mean structure) but also about the predictors of their correlation structure. The latter is of particular interest here because it provides information about the symmetry of exchanges (correlations between giving and receiving help) and complementarity of different forms of help (correlations between giving or receiving financial and practical help) for different population sub-groups. The framework presented here also extends that of Steele et al. (2022) who separated practical and financial support in a joint longitudinal model of bidirectional exchanges using UKHLS data, but with the seven items for practical help given and received collapsed into two binary outcomes, and without predictors for the correlation structure.

Methodologically, this paper contributes to the literature on modelling correlation or covariance matrices given covariates. We review this literature in Section 4. A key challenge here is that the estimated matrices should be positive definite at all relevant values of the covariates (for some approaches the constraint that the diagonal elements of a correlation matrix should be 1 is also challenging, so they model the covariance matrix instead). Broadly, two approaches may be taken to deal with this (Pinheiro & Bates, 1996). ‘Unconstrained’ methods specify a model for some transformation which ensures that the fitted matrix will be positive definite, while ‘constrained’ methods enforce this condition during estimation. A disadvantage of the unconstrained approach is that the parameters of the transformation are often not easily interpretable. Constrained estimation, in contrast, can use interpretable models for the covariances or correlations themselves, but it faces the challenge of how to actually implement the constraint.

We adopt a constrained approach of estimation. We first decompose the covariance matrix into the standard deviations and the correlation matrix, and specify a linear model for each correlation given covariates (in our application we do not use covariates in the model for the standard deviations, but they could easily be included). The estimation is carried out in the Bayesian framework, using a tailored MCMC algorithm. Here the constraint is implemented by checking it at each MCMC sampling step, so that the most recently sampled parameters can only be retained if they imply a positive definite correlation matrix at all relevant values of the covariates. This builds on methods proposed previously without covariates (Barnard et al. 2000; Wong et al. 2003), which we extend to models that include individual-level covariates for the correlations. Theoretical results on properties of correlation matrices which justify this approach and an efficient implementation of it are described in Section 5.

In the rest of the paper, the UKHLS data are introduced in Section 2 and the specification of the joint model is described in Section 3. Section 4 reviews previous literature on modelling covariance and correlation matrices, and Section 5 and Appendix A give the theoretical results that provide the basis of our estimation of the model for the correlations. Estimation of the joint model is described in Section 6 and Appendix B. Results of the analysis of intergenerational exchanges of family support are then described in Section 7 and a concluding discussion is given in Section 8.
2 Data

We use data from the Understanding Society survey, also known as the UK Household Longitudinal Study (UKHLS) [University of Essex, 2013]. This is a long-standing household panel survey. Our analysis does not make use of its longitudinal features, but carries out a cross-sectional analysis of data from one wave of the survey, collected in 2017–19 (wave 9 of UKHLS). This included the ‘family network’ module which collected information on exchanges of help with relatives living outside a respondent’s household.

Respondents who had at least one non-coresident parent were asked whether they ‘nowadays’ ‘regularly or frequently’ gave each of eight types of help to their parent(s): lifts in a car; help with shopping; providing or cooking meals; help with basic personal needs; washing, ironing or cleaning; personal affairs such as paying bills or writing letters; decorating, gardening or house repairs; or financial help. These items are dichotomous, with the response options ‘Yes’ and ‘No’. The same questions were asked about receipt of support from parents, but with ‘personal needs’ replaced by ‘help with childcare’. In the analysis that follows we will distinguish between financial help (measured by a single item for support in each direction) and practical help (measured by the remaining seven items). Where a respondent had both biological and step/adoptive parents alive, the respondents were asked to report on the ones that they had most contact with. Although respondents were asked about giving parents a lift in their car ‘if they have one’, the recorded variable had no missing values for this item. We therefore used other survey information to set this item to missing for respondents who did not have access to a car. Similarly, the childcare item was coded as missing for respondents who did not have coresident dependent children aged 16 or under. For the item on receiving lifts from parents, we do not have information on whether the parents have access to a car, so responses of ‘No’ to this item will include also cases where they do not.

We consider as covariates a range of individual and household demographic and socioeconomic characteristics that aim to capture an adult child’s capacity to give help to their parents and their potential need for support from their parents. Most variables in the survey refer to the respondent (the child in our case), as less information was collected on non-coresident relatives, but we also include a small set of characteristics of the parents that are indicative of their need and capacity for support similarly. The following respondent characteristics were included: age, gender, whether they have a coresident partner, indicators of the presence and age of their youngest biological or adopted coresident children, the number of siblings (as a measure of both alternative sources of support for parents and competition for the receipt of parental support), whether they have a long-term illness that limits their daily activities, employment status (classified as employed or non-employed [unemployed or economically inactive]), education (up to secondary school only, or post-secondary qualifications), household tenure (home-owner or social/private renter), and household income (equivalised, adjusted for inflation using the 2019 Consumer Price Index, and log transformed). The parental characteristics included were the age of the oldest living parent and whether either parent lives alone. We also include the travel time to the nearest parent, dichotomized as 1 hour or less vs. more than 1 hour.

The analysis sample was first restricted to the 15,825 respondents aged 18 or over who had at least one non-coresident parent but no coresident parent. We excluded respondents whose nearest parent lived or worked abroad (1830 of them), because the nature of their exchanges is likely to differ from parents based in the UK, and then also omitted 1792 respondents who had missing data on any covariate or on all the help items. The final sample size for analysis is \( n = 12,203 \). Because of the design of UKHLS, the sample can include some respondents who are siblings to each other. However, preliminary analysis indicated that their number was very small for our analysis sample, so we ignore this
feature and treat all the respondents as independent of each other. Table 1 shows the percentages of positive response on each of the help items for the analysis sample, and Table 2 shows descriptive statistics for the covariates.

Table 1: Percentage of respondents who reported giving help to their parents and receiving help from the parents, by item.

| Item                                         | Help given to parents | Help received from parents |
|----------------------------------------------|-----------------------|----------------------------|
| **Practical help (7 items):**                |                       |                            |
| Lifts in car                                 | 31.1                  | 11.8                       |
| Shopping                                     | 21.5                  | 7.3                        |
| Providing or cooking meals                   | 11.9                  | 12.0                       |
| Basic personal needs (to parents only)       | 3.7                   | –                          |
| Looking after children (from parents only)   | –                     | 39.0                       |
| Washing, ironing or cleaning                 | 7.1                   | 4.9                        |
| Personal affairs                             | 17.1                  | 2.0                        |
| Decorating, gardening or house repairs        | 17.7                  | 7.3                        |
| **Financial help (1 item)**                  | 6.0                   | 12.7                       |
| **At least one of the seven kinds of practical help:** | 43.2 | 33.4 |
| **At least one of any kind of practical or financial help:** | 44.4 | 38.2 |

Data from UKHLS, 2017-19 (Wave 9). The overall sample size is n = 12,203. The percentages for the individual items are based on observed samples for them, excluding cases with missing data. The item on giving lifts to parents is missing for the 17.1% of respondents who have no access to a car, and the item on childcare is missing for the 54.1% respondents who have no co-resident dependent children.
Table 2: Descriptive statistics of the covariates used in the analysis.

| Variable | n   | %   |
|----------|-----|-----|
| **Respondent (child) characteristics:** |     |     |
| Age (years) | Mean=43.7 | SD=11.4 |
| Gender |     |     |
| Female | 7060 | 57.9 |
| Male | 5143 | 42.1 |
| Partnership status |     |     |
| Partnered | 9373 | 76.8 |
| Single | 2830 | 23.2 |
| Age of youngest coresident child |     |     |
| No children | 5002 | 41.0 |
| 0–1 years | 910 | 7.5 |
| 2–4 years | 1231 | 10.1 |
| 5–10 years | 1910 | 15.7 |
| 11–16 years | 1548 | 12.7 |
| 17+ years | 1602 | 13.1 |
| Number of siblings |     |     |
| None | 1235 | 10.1 |
| 1 | 4325 | 35.4 |
| 2 or more | 6643 | 54.4 |
| Longstanding illness |     |     |
| Yes | 1533 | 12.6 |
| No | 10670 | 87.4 |
| Employment status |     |     |
| Employed | 9688 | 79.4 |
| Not employed | 2515 | 20.6 |
| Education (highest qualification) |     |     |
| Secondary or less | 6024 | 49.4 |
| Post-secondary | 6179 | 50.6 |
| Household tenure |     |     |
| Own home outright or with mortgage | 8817 | 72.3 |
| Other (private or social renter) | 3386 | 27.7 |
| Logarithm of household equivalised income | Mean=9.9 | SD=0.79 |
| Parent characteristics: |     |     |
| Age of the oldest living parent (years) | Mean=72.1 | SD=11.2 |
| At least one parent lives alone |     |     |
| Yes | 4641 | 38.0 |
| No | 7562 | 62.0 |
| Child–parent characteristics: |     |     |
| Travel time to the nearest parent |     |     |
| 1 hour or less | 8851 | 72.5 |
| More than 1 hour | 3352 | 27.5 |

Data from UKHLS, 2017-19 (Wave 9). The sample size for all covariates is n = 12,203.
3 Latent variable model for multivariate dyadic data

Here we define the joint model that will be used to analyse the multivariate dyadic data that were described in Section [2]. The model specification is broadly similar to that of Kuha et al. [2022], but with two extensions. First, tendencies to give and receive financial and practical help are represented by separate latent variables, so that the model includes four rather than two such variables for each respondent. Second, the correlations between the latent variables are also modelled as functions of covariates.

Let \((X_i, Y_{Gi}, Y_{Ri})\) be observed data for a sample of dyads \(i = 1, \ldots, n\), where \(X_i\) is a \(Q \times 1\) vector of covariates (including a constant term 1), and \(Y_{Gi} = (Y_{GPi}, Y_{GFi})^T\) and \(Y_{Ri} = (Y_{RPi}, Y_{RFi})^T\) are \((J + 1) \times 1\) vectors of binary indicator variables (items). In our application, the dyads are those between a survey respondent and his or her non-coreident parents. \(Y_{GPi} = (Y_{GPi1}, \ldots, Y_{GPij})^T\) are the respondent’s answers to \(J = 7\) items on different types of practical help given to their parents, \(Y_{Rpi} = (Y_{Rpi1}, \ldots, Y_{RPij})^T\) are the items on practical help received from the parents, and \(Y_{GFi}\) and \(Y_{RFi}\) are the single items on financial help given and financial help received respectively. Each item is coded 1 if that kind of help is given or received, and 0 if not. In other applications, \(Y_{Gi}\) and \(Y_{Ri}\) could be of different lengths and \(Y_{GFi}\) and \(Y_{RFi}\) could also be vectors of multiple indicators, with straightforward modifications of the specifications below.

3.1 Measurement model for the observed items given latent variables

The items in \(Y_{GPi}, Y_{Rpi}, Y_{GFi}\) and \(Y_{RFi}\) are regarded as measures of continuous latent variables \(\eta_{GPi}, \eta_{Rpi}, \eta_{GFi}\) and \(\eta_{RFi}\) respectively. Here we interpret \(\eta_{GPi}\) and \(\eta_{Rpi}\) as an individual’s underlying tendencies to give and to receive practical help respectively, and \(\eta_{GFi}\) and \(\eta_{RFi}\) similarly as tendencies to give and receive financial help.

The data that we analyse have a large number of responses where all the items in \(Y_{Gi}\) or \(Y_{Ri}\) are zero (no help given or received; see Table [1]). The proportions of these all-zero responses may be higher than can be well accounted for by standard latent variable models given the continuous latent variables alone. To allow for this multivariate zero inflation, the model also includes two binary latent class variables \(\xi_{Gi}\) and \(\xi_{Ri}\), for each of which one class represents individuals who are certain not to give (for \(\xi_{Gi}\)) or receive (for \(\xi_{Ri}\)) any kind of help. For giving help, the measurement model for the observed responses \(Y_{Gi}\) given the latent variables \(\eta_{GPi}, \eta_{GFi}, \xi_{Gi}\) is then specified by

\[
p(Y_{Gi} = 0|\xi_{Gi} = 0, \eta_{GPi}, \eta_{GFi}; \phi_G) = p(Y_{Gi} = 0|\xi_{Gi} = 0) = 1 \quad \text{and} \quad (1)
\]

\[
p(Y_{Gi}|\xi_{Gi} = 1, \eta_{GPi}, \eta_{GFi}; \phi_G) = \prod_{j=1}^{J} p(Y_{GPij}|\xi_{Gi} = 1, \eta_{GPi}; \phi_G) \times p(Y_{GFi}|\xi_{Gi} = 1, \eta_{GFi}), \quad (2)
\]

where \(p(\cdot|\cdot)\) denotes a conditional distribution and \(\phi_G\) are measurement parameters. When \(\xi_{Gi} = 0\), respondent \(i\) is thus certain to answer ‘No’ to all items related to giving help. When \(\xi_{Ci} = 1\), the probabilities of responses to \(Y_{GPij}\) are determined by the continuous latent variable \(\eta_{GPi}\) and the response to \(Y_{GFi}\) is determined by \(\eta_{GFi}\). Items \(Y_{GPij}\) \((j = 1, \ldots, J)\) are assumed to be conditionally independent of each other given \(\eta_{GPi}\). If any items in \(Y_{Gi}\) are missing for respondent \(i\), they are omitted from the product in (2). The measurement models for the individual items are specified as

\[
p(Y_{GPij} = 1|\xi_{Gi} = 1, \eta_{GPi}; \phi_G) = \Phi(\tau_{GPj} + \lambda_{GPj} \eta_{GPi}) \quad \text{for} \quad j = 1, \ldots, J, \quad \text{and} \quad (3)
\]

\[
p(Y_{GFi} = 1|\xi_{Gi} = 1, \eta_{GFi}) = \mathbb{1}(\eta_{GFi} > 0), \quad (4)
\]
where Φ(·) is the cumulative distribution function of the standard normal distribution, 1(·) is the indicator function, \( \tau_{GPj} \) and \( \lambda_{GPj} \) are parameters, and we fix \( \tau_{GP1} = 0 \) and \( \lambda_{GP1} = 1 \) for identification of the scale of \( \eta_{GPi} \). Here (3) is a standard latent-variable (item response theory) model for binary items, with probit measurement models, and (4), combined with the normal distribution of \( \eta_{GFi} \) defined below, is a latent-variable formulation of a probit model for the single item \( Y_{GFi} \). Thus \( \phi_G = (\tau_{GP2}, \ldots, \tau_{GPJ}, \lambda_{GP2}, \ldots, \lambda_{GPJ})^T \).

The measurement model for receiving help \( Y_{Ri} \) given \( (\eta_{RFi}, \eta_{RFi}; \xi_{RFi}) \) is defined analogously to (3)–(4), with parameters \( \phi_R \), and \( Y_{Gi} \) and \( Y_{Ri} \) are assumed to be conditionally independent of each other given the latent variables. Let \( \phi = (\phi_G^T, \phi_R^T)^T \).

### 3.2 Structural model for the latent variables given covariates

Let \( \eta_i = (\eta_{GFi}, \eta_{RFi}, \eta_{GFi}, \eta_{RFi})^T \) and \( \xi_i = (\xi_{Gi}, \xi_{RFi})^T \). Their conditional distribution

\[
p(\eta_i, \xi_i | X_i; \psi) = p(\eta_i | X_i; \psi_{\eta}) p(\xi_i | X_i; \psi_{\xi})
\]

is the **structural model** for the latent variables given the covariates. Here \( \eta_i \) and \( \xi_i \) are taken to be conditionally independent, and \( \psi = (\psi_{\eta}^T, \psi_{\xi}^T)^T \) are parameters. The distribution of the latent class variables \( \xi_{i} \) is specified as multinomial, with probabilities

\[
\log \left[ \frac{\pi_{k_1k_2}(X_i)}{\pi_{00}(X_i)} \right] = \gamma_{k_1k_2} X_i,
\]

where \( \pi_{k_1k_2}(X_i) = p(\xi_{Gi} = k_1, \xi_{RFi} = k_2 | X_i; \psi_{\xi}) \) for \( k_1, k_2 = 0, 1 \), so that \( \psi_{\xi} = (\gamma_{01}, \gamma_{10}, \gamma_{11})^T \).

The structural model for the continuous helping tendencies \( \eta_i \) given the covariates \( X_i \) is the main focus of substantive interest in our analysis. Here \( \eta_i \sim N(\mu_i, \Sigma_i) \) is taken to follow a four-variate normal distribution with covariance matrix \( \Sigma_i \) and mean vector

\[
\mu_i = E(\eta_i | X_i; \beta) = \begin{bmatrix} \beta_{GP1}^T X_i \\ \beta_{RP1}^T X_i \\ \beta_{GF1}^T X_i \\ \beta_{RF1}^T X_i \end{bmatrix} = \beta^T X_i
\]

where \( \beta = [\beta_{GP}, \beta_{RP}, \beta_{GF}, \beta_{RF}] \) is a \( Q \times 4 \) matrix of coefficients. In other words, (4) specifies separate linear models for the means of each element of \( \eta_i \). For the covariance matrix, we first decompose it as

\[
\Sigma_i = \text{cov}(\eta_i | X_i; \alpha, \sigma) = S_i R_i S_i^T
\]

where \( \alpha \) are parameters of the correlation matrix \( R_i \) and \( \sigma = (\sigma_{GP}, \sigma_{RP})^T \) are parameters of \( S_i = \text{diag}(\sigma_{GP}, \sigma_{RP}, 1, 1) \), a diagonal matrix of standard deviations where those of \( \eta_{GFi} \) and \( \eta_{RFi} \) are fixed at 1 to identify the measurement model (4) for \( \eta_{GFi} \) and the corresponding model for \( \eta_{RFi} \). Here \( \sigma \) do not depend on the covariates (and thus we could write \( S_i = S \)), but this extension could also be included.

For the correlation matrix, we consider the specification

\[
R_i = R(X_i; \alpha) = \begin{bmatrix} 1 & \rho_{1i} & \rho_{2i} & \rho_{3i} \\ \rho_{1i} & 1 & \rho_{4i} & \rho_{5i} \\ \rho_{2i} & \rho_{4i} & 1 & \rho_{6i} \\ \rho_{3i} & \rho_{5i} & \rho_{6i} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \rho(X_i; \alpha_1) & \rho(X_i; \alpha_2) & \rho(X_i; \alpha_3) \\ \rho(X_i; \alpha_1) & 1 & \rho(X_i; \alpha_4) & \rho(X_i; \alpha_5) \\ \rho(X_i; \alpha_2) & \rho(X_i; \alpha_4) & 1 & \rho(X_i; \alpha_6) \\ \rho(X_i; \alpha_3) & \rho(X_i; \alpha_5) & \rho(X_i; \alpha_6) & 1 \end{bmatrix},
\]

where only the lower triangular part is shown and the \( L = 6 \) distinct correlations are numbered as shown in (5). We specify separate linear models \( \rho_{li} = \rho(X_i; \alpha_i) = \alpha_i^T X_i \) for each \( l = 1, \ldots, L \), i.e.

\[
\rho_i = \alpha_i^T X_i
\]
where \( \mathbf{p}_i = (p_{i1}, \ldots, p_{iL})^T \) and \( \mathbf{a} = [\mathbf{a}_1, \ldots, \mathbf{a}_L] \) is a matrix of coefficients. Note that some variables in \( \mathbf{X}_i \) may be included in only one of the models \( \mathbf{6} \) and \( \mathbf{7} \); if so, some of the corresponding elements of \( \mathbf{\beta} \) or \( \mathbf{a} \) are set to zero. The motivation and interpretation of this choice of model for the correlation matrix is discussed further in Section \( \mathbf{4} \). The full set of parameters of the structural model for \( \eta_i \) is thus \( \psi_i = (\text{vec}(\mathbf{\beta})^T, \sigma^T, \text{vec}(\mathbf{\alpha})^T)^T \), where \( \text{vec}(\cdot) \) denotes the vectorization of a matrix.

Let \( \mathbf{Y} = [\mathbf{Y}_1, \ldots, \mathbf{Y}_n]^T \) denote all the observed data on the items, where \( \mathbf{Y}_i = (\mathbf{Y}_{Gi}^T, \mathbf{Y}_{Ri}^T)^T \), and \( \mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_n]^T \) the data on the covariates. Define \( G_i = 1(\mathbf{Y}_{Gi} \neq \mathbf{0}) \) and \( R_i = 1(\mathbf{Y}_{Ri} \neq \mathbf{0}) \), the indicators for whether responses on giving and on receiving help are not all zero for respondent \( i \). Assuming the observations for different respondents to be independent, the log likelihood function of the model is

\[
\log p(\mathbf{Y}|\mathbf{X}; \phi, \psi) = \sum_{i=1}^N \log \left\{ \pi_{11}(\mathbf{X}_i; \psi) \left[ \int p(\mathbf{Y}_{Gi}|\xi_{Gi} = 1, \eta_{\mathbf{GP}i}, \eta_{\mathbf{GF}i}; \phi_G)p(\mathbf{Y}_{Ri}|\xi_{Ri} = 1, \eta_{\mathbf{RP}i}, \eta_{\mathbf{RF}i}; \phi_R) \right. \\
\times p(\eta|\mathbf{X}_i; \psi) \, d\eta_{\mathbf{GP}i} \, d\eta_{\mathbf{RP}i} \, d\eta_{\mathbf{GF}i} \, d\eta_{\mathbf{RF}i} \right] \\
+ (1 - R_i) \pi_{10}(\mathbf{X}_i; \psi) \left[ \int p(\mathbf{Y}_{Gi}|\xi_{Gi} = 1, \eta_{\mathbf{GP}i}, \eta_{\mathbf{GF}i}; \phi_G) \right. \\
\times p(\eta_{\mathbf{GP}i}, \eta_{\mathbf{GF}i}|\mathbf{X}_i; \psi) \, d\eta_{\mathbf{GP}i} \, d\eta_{\mathbf{GF}i} \right] \\
+ (1 - G_i) \pi_{01}(\mathbf{X}_i; \psi) \left[ \int p(\mathbf{Y}_{Ri}|\xi_{Ri} = 1, \eta_{\mathbf{RP}i}, \eta_{\mathbf{RF}i}; \phi_R) \right. \\
\times p(\eta_{\mathbf{RP}i}, \eta_{\mathbf{RF}i}|\mathbf{X}_i; \psi) \, d\eta_{\mathbf{RP}i} \, d\eta_{\mathbf{RF}i} \right] \\
+ (1 - G_i)(1 - R_i) \pi_{00}(\mathbf{X}_i; \psi) \right\}. 
\]

Estimation of this model is described in Section \( \mathbf{6} \) after some further discussion of questions related to the model for the correlations.

### 4 Modelling correlation and covariance matrices given covariates: Existing approaches

There is a large literature on modelling association structures of multivariate distributions. We review here those parts of it that are most relevant to our work. This means that we consider different approaches to modelling correlation or covariance matrices given covariates, with a particular focus on how the models are specified. This can be combined with different (parametric or other) specifications for the joint distribution as a whole, and different methods of estimating its parameters. Our own model uses a parametric specification of a multivariate normal distribution and Bayesian estimation of its parameters, but the review here is not limited to that case.

We consider only approaches which specify associations in terms of covariances or correlations. This means that we exclude models for conditional associations of some of the variables given the others, such as log-linear models for categorical data or covariance selection models for the inverse covariance matrix of a multivariate normal distribution. We include here models for both correlation and covariance matrices, although our model is for the correlation matrix. We focus on models which are specified directly for these associations or transformations of them. This excludes models where the associations are...
determined indirectly via latent variables, such as random effects models and common factor models. Note, however, that the multivariate response variable whose covariance or correlation matrix is being modelled may itself be a latent variable, as it is in our analysis where we model the correlations of the latent $\eta_i$.

Models for associations may have two broad goals. The first of them is to impose a patterned structure on the covariance or correlation matrix which is more parsimonious than an unstructured matrix that has separate parameters for each pair of variables. This is the case, for example, when an autocorrelation structure is specified for responses that are ordered in time. An extreme version of this occurs in very high-dimensional problems where parsimonious specification is essential for consistent estimation of covariance matrices. We do not consider such regularisation methods here (see Pourahmadi 2011 and Fan et al. 2016 for reviews). The second broad type of model specification considers instead an unstructured model of associations, but specifies how the correlations or covariances in it depend on covariates that are characteristics of the units of analysis, such as the survey respondents in our application. This is the goal of our modelling.

A key question is how to ensure that the estimated matrices will be positive definite. Here Pinheiro & Bates (1996) pointed out a key distinction between two approaches: unconstrained ones where the models are specified for parametrizations (transformations) of the association matrix which are guaranteed to imply a positive definite matrix, and constrained ones where positive definiteness is imposed in the estimation process. Our approach is an instance of constrained estimation, but we summarise first the most important unconstrained methods (see Pourahmadi 2011 and Pan & Pan 2017 for more detailed reviews). They differ in what transformation they use. The most common is the modified Cholesky decomposition of the covariance matrix. It was introduced by Pourahmadi (1999), and general models for it (including covariates) were proposed by Pan & MacKenzie (2006). Other possible transformations include the matrix logarithm of the covariance matrix (Chiu et al. 1996), the 'alternative Cholesky decomposition' of the covariance matrix (Chen & Dunson, 2003), a variant of the modified Cholesky decomposition proposed by Zhang & Leng (2012), parametrizations of the correlation matrix in terms of partial autocorrelations (Wang & Daniels, 2013) or hyperspherical co-ordinates of its standard Cholesky decomposition (Zhang et al. 2015), and the matrix logarithm of the correlation matrix (Archakov & Hansen, 2021; Hu et al. 2021).

The natural advantage of the unconstrained methods is that they ensure positive definiteness at any values of the covariates. The corresponding disadvantage is that because the models are not specified for the individual association parameters (or even transformations of them), the model parameters are not easily interpretable. All of the interpretations that are available apply only in cases where the response variables have a natural ordering, most obviously in longitudinal data where they are ordered in time. Then the parameters of the modified Cholesky decomposition can be interpreted in terms of an autoregressive model for each variable given its predecessors, and those of the alternative Cholesky decomposition and of Zhang & Leng (2012) similarly in terms of a moving average representation of each variable given error terms of the previous ones (Pourahmadi, 2007; Pan & Pan, 2017), those of Wang & Daniels (2013) as partial autocorrelations of two variables given all the intervening ones, and the hyperspherical co-ordinate parametrization in terms of semi-partial correlations (Ghosh et al. 2021). In our application these properties are not helpful because we consider response variables with no ordering and want to obtain a simple interpretation of coefficients for the correlations themselves.

Turning now to approaches that model individual pairwise association parameters directly, for correlations we could use transformations of them (e.g. Fisher’s $z$) to ensure that the fitted correlations are constrained to $(-1, 1)$. This, however, is not sufficient to
ensure that the correlation matrix as a whole is positive definite, except for a bivariate response (for this case, see e.g. Wilding et al. 2011 and references therein). One pragmatic approach that we could then take is to simply employ such models anyway also more generally, ignoring the possibility of some non-positive definite matrices (see e.g. Yan & Fine 2007). It is plausible that this will work well in some applications, in the best case that the fitted correlation matrices end up being positive definite at all relevant values of the covariates. However, it is not in principle a satisfactory general approach. Luo & Pan (2022) suggest post-hoc adjustments to fitted correlation models to make them positive definite; this, however, is unhelpful when the focus is on interpreting coefficients of the model. A different solution is provided by Hoff & Niu (2012) who propose a model (analogous in form to factor analysis models) where covariances depend on quadratic functions of covariates, and the matrix is automatically positive definite.

Existing literature on constrained estimation focuses on linear models for covariances or correlations. This is not really a limitation even for correlations, because the constraint that the matrix should be positive definite also implies that the individual correlations will be in \((-1, 1)\). The most developed results here are for the linear covariance model for multivariate normal distribution (Anderson, 1973), in which the covariance matrix takes the form \(\Sigma = \sum_k \nu_k \mathbf{G}_k\) where \(\nu_k\) are parameters and \(\mathbf{G}_k\) are known, linearly independent symmetric matrices. Zwiernik et al. (2017) show that although the log-likelihood function for this model typically has multiple local maxima, any hill climbing method initiated at the least squares estimator will converge to its global maximum with high probability. Zou et al. (2017) consider the case where the \(\mathbf{G}_k\) are similarity matrices between the response variables, and propose maximum likelihood estimates and constrained least squares estimators for this model. In these formulations, \(\Sigma\) is the same for all units \(i\).

This is further relaxed by Zou et al. (2022), who allow the values of the similarity matrices in Zou et al. (2017) to depend on unit-specific covariates, thus defining \(\Sigma_i\) as linear combinations of unit-specific similarity matrices.

We will also consider a linear model, as shown in (9), but for the correlations and given unit-specific covariates. We will estimate it in the Bayesian framework and using Markov chain Monte Carlo (MCMC) estimation. As MCMC updates different parameters separately, it is natural to employ the decomposition and model the standard deviations and correlations separately, and this is what has been done in most previous literature that has used a Bayesian approach. MCMC also provides an obvious way to implement constrained estimation, at least in principle. This can be done at each sampling step of the estimation algorithm, by constraining the prior distribution, the proposal distribution from which values of the parameters are drawn, or the acceptance probabilities of the sampled values, in a way which rules out inadmissible values of the sampled parameters. But although the principle is obvious, implementing it is not necessarily easy. Two instances of this approach that we will draw on in particular are those of Barnard et al. (2000) and Wong et al. (2003), and we will return to them below. Other methods of this kind have been proposed by Chib & Greenberg (1998) and Liechty et al. (2004).

What is missing from these existing Bayesian implementations is the inclusion of unit-specific covariates in the models for the correlations, which is our focus. In Section 4 we propose an extended estimation procedure which accommodates such covariates. This in turn requires some further consideration of the constraints on positive definiteness of the correlation matrix, because this now has to hold at different values of the covariates. This question is discussed first, in the next section.
5 Ensuring a positive definite correlation matrix

The key challenge in our constrained estimation is to ensure that the estimated correlation matrices remain positive definite at all relevant values of the covariates. Here we give some further theoretical results which establish when and how this can be achieved.

Let $R = R(\rho)$ denote a symmetric matrix where all the diagonal elements equal 1 and the distinct off-diagonal elements $\rho = (\rho_1, \ldots, \rho_L)^T$ are all in $(-1, 1)$. Let $C_\rho$ denote the set of $\rho$ such that $R(\rho)$ is positive definite, and thus a correlation matrix, for all $\rho \in C_\rho$. It is a convex subset of the hypercube $[-1, 1]^L$ (for the shape of $C_\rho$ in the cases $L = 3$ and $L = 6$, i.e. for $3 \times 3$ and $4 \times 4$ correlation matrices, see Roussieau & Molenberghs [1994]).

We consider model (9) where $\rho = \alpha^T X$. In this section we mostly omit the unit subscript from $X$ and $\rho$, and take $X$ to include only those covariates that are included in the model for $\rho$ (excluding any that are used only for the means $\mu$). It is clear that this $\rho$ cannot be in $C_\rho$ for all values of the parameters $\alpha$ and covariates $X$. We need to limit the scope of the estimated models to the values that do imply $\rho \in C_\rho$. For the covariates, it is useful to introduce here some additional notation. Let $Z$ be the smallest vector of distinct variables, including a constant term 1, which determines $X = X(Z)$. Here $Z$ may be shorter than $X$ if some variables in $X$ are functions of a smaller number of variables in $Z$, e.g. when $X$ includes polynomials or product terms (interactions). Suppose that $Z$ is a $p \times 1$ vector and $X$ a $q \times 1$ vector. Below we denote sets $S_Z \subset \mathbb{R}^p$ and $S_X \subset \mathbb{R}^q$ of $Z$ and $X$ respectively with appropriate subscripts, and also $S_X = X(S_Z) = \{X(Z) \mid Z \in S_Z\}$ when needed to indicate how a set of $Z$ determines that of $X$.

A combination of values $(Z, \alpha)$ is said to be feasible if $\rho = \alpha^T X(Z) \in C_\rho$, and $(X, \alpha)$ to be feasible if $\rho = \alpha^T X \in C_\rho$. We aim to identify known sets of $Z$ and $\alpha$ such that all combinations of values from them are feasible. This will involve the following steps:

1. Specify a set $S_Z$ for $Z$ for which we want the ensure that the estimated correlation matrices are positive definite.

2. Specify a finite test set $S_{XT} = \{X_1, \ldots, X_T\}$ of values for $X$, which will be used for checking positive definiteness during MCMC estimation. The choice of $S_{XT}$ will depend on $S_Z$.

3. Carry out MCMC estimation which includes sampling values of $\alpha$ (together with the other model parameters). At each iteration, carry out checks to ensure that the value of $\alpha$ that is retained from the iteration is feasible with all $X \in S_{XT}$. In the end, this produces an MCMC sample $S_\alpha = \{\alpha_1, \ldots, \alpha_M\}$.

4. Conclude that $(Z, \alpha)$ is feasible for all combinations of $Z \in S_Z$ and values of $\alpha$ in the convex hull of $S_\alpha$.

In step (1), $S_Z$ should include the substantively relevant and interesting values of the covariates for which we want our estimated model to imply valid correlation matrices. For example, this could be a finite set $S_{ZN} = \{Z_1, \ldots, Z_N\}$, normally including at least all the distinct values among the $Z_i$, $i = 1, \ldots, n$, in the observed data. $S_Z$ is always of this form when all the variables in $Z$ are categorical (coded as dummy variables). If $Z$ includes continuous variables, we can also expand $S_{ZN}$ to an infinite set, such as its convex hull $S_{Zn} = \{\sum_{i=1}^N \lambda_i Z_i \mid \sum_{i=1}^N \lambda_i = 1; \lambda_j \geq 0 \text{ for all } j = 1, \ldots, N\}$ or the hyperrectangle $S_{Zr} = \{(Z_1, \ldots, Z_p) \mid Z_j \in [l_s, u_s] \text{ for all } s = 1, \ldots, p\}$, for specified $l_s \leq \min\{Z_{js} \mid j = 1, \ldots, N\}$ and $u_s \geq \max\{Z_{js} \mid j = 1, \ldots, N\}$ for each $s = 1, \ldots, p$. Note that here $S_{ZN} \subset S_{Zn} \subseteq S_{Zr}$.

How the sample $S_\alpha$ in step (3) can be drawn is described at the end of this section and in Section 6. That alone is not yet enough for what we need. This is because in step (3) we can only check feasibility for a finite number of values of $X$ and $\alpha$, whereas the
sets we want to draw conclusions on are infinite for at least \( \alpha \) and possibly also for \( Z \). So some additional theoretical results are needed to justify the conclusion in step (4); this also informs the choice of the test set \( S_{XT} \) in step (2).

Consider first the correlations \( \rho = \alpha^T X \) as they depend on \( X \) rather than \( Z \). Let

\[
C_{\alpha,S_X} = \{ \alpha \in \mathbb{R}^{L \times q} | \rho = \alpha^T X \in C_\rho \text{ for all } X \in S_X \}
\]

be the set of values of \( \alpha \) which are feasible when combined with any \( X \) in \( S_X \). Proposition I gives some basic properties of \( C_{\alpha,S_X} \). Proofs of them are given in Appendix A.

**Proposition 1. Properties of \( C_{\alpha,S_X} \):**

(i) If \( S_{X_2} \subseteq S_{X_1} \), then \( C_{\alpha,S_{X_1}} \subseteq C_{\alpha,S_{X_2}} \).

(ii) \( C_{\alpha,\text{Conv}(S_X)} = C_{\alpha,S_X} \), where \( \text{Conv}(S_X) \) denotes the convex hull of \( S_X \).

(iii) \( 0 \in C_{\alpha,S_X} \).

(iv) Suppose further that there exist \( q \) linearly independent elements in \( S_X \). Then \( C_{\alpha,S_X} \) is bounded.

(v) \( C_{\alpha,S_X} \) is a convex set.

Parts (i) and (ii) of Proposition I explain how \( C_{\alpha,S_X} \) depends on the set \( S_X \) of values considered for \( X \). When step (3) is completed, we know that \( \alpha \in C_{\alpha,S_X} \) for all \( \alpha \in S_\alpha \) then also \( \alpha \in C_{\alpha,\text{Conv}(S_{XT})} \) by (ii). In other words, even though feasibility was checked only for a finite number of values of \( X \), we know that it holds also for the infinite set of their convex hull.

We then need to translate this result for \( X \) back to \( Z \). This is simple if \( X = Z \), so that \( S_{XT} = S_{ZT} \), where \( S_{ZT} \) is a finite test set for \( Z \). Here we need to ensure that \( S_{ZT} \) has been chosen so that \( Z \subseteq \text{Conv}(S_{ZT}) \), i.e. that the convex hull of \( S_{ZT} \) covers \( S_Z \). Then, for any \( \alpha \in S_\alpha \), we have \( \alpha \in C_{\alpha,\text{Conv}(S_{XT})} \) by (ii) as above, and then \( \alpha \in C_{\alpha,S_Z} \) by (i), as required. In terms of the possible target sets \( S_Z \) defined above, the test set \( S_{ZT} \) could be \( S_{ZN} \), which ensures feasibility also for all \( Z \) in \( S_{Zh} \), or \( S_{ZT} \) could consist of the vertices of \( S_{Zr} \), which ensures feasibility in all of \( S_{Zr}, S_{Zh} \) and \( S_{ZN} \).

Some more care is needed when \( X = X(Z) \) includes non-linear functions of \( Z \). If \( S_Z = S_{ZN} \) is finite, a simple pragmatic choice is to set \( S_{XT} = X(S_Z) \) and check all of their values. Otherwise, the forms of these functions need to be considered. This can be seen already in a model for a single correlation \( \rho \), e.g. when it is modelled as a quadratic function \( \rho = \alpha_0 + \alpha_1 Z + \alpha_2 Z^2 \) of a single \( Z \), so that \( X(Z) = (X_1, X_2, X_3)^T = (1, Z, Z^2)^T \). Suppose that \( S_Z = [Z_1, Z_2] \). It is not enough to check just the points \( X_1 = X(Z_1) \) and \( X_2 = X(Z_2) \), because we may have \( \rho \in (-1, 1) \) at \( Z_1 \) and \( Z_2 \) but not at all values between them. It is sufficient to identify one more point \( X_3 = (1, X_{23}, X_{33})^T \) such that the convex hull of \( S_{XT} = \{X_1, X_2, X_3\} \) covers \( X(S_Z) \). For example, this can be obtained by adding the intersection point of the tangents of the curve \( f(Z) = Z^2 \) drawn at \( Z_1 \) and \( Z_2 \), i.e. \( X_{23} = (Z_1 + Z_2)/2 \) and \( X_{33} = Z_1Z_2 \). Note that such a choice depends only on the forms of \( S_Z \) and \( X(Z) \), so it can be used with any value of \( \alpha \) and for any number of correlations \( \rho \). At this point we know that \( \alpha \in C_{\alpha,X(S_{Zr})} \) for all \( \alpha \in S_\alpha \), i.e. that all the values in the MCMC sample \( S_\alpha = \{\alpha_1, \ldots, \alpha_M\} \) are feasible when combined with any value of \( Z \) in the (possibly infinite) target set \( S_Z \). But we still need to extend this conclusion to other values of \( \alpha \) that were not sampled, specifically to the convex hull \( \text{Conv}(S_\alpha) \) of \( S_\alpha \). This is justified by parts (iii)–(v) of Proposition I which concern the values of \( \alpha \) in \( C_{\alpha,S_X} \) given a fixed \( S_X \). Part (iii) shows that this set is non-empty, so some feasible \( \alpha \) always exist, and (iv) states that feasible \( \alpha \) will not drift away, as long as \( S_X \) is not degenerate. Finally,
part (v) shows that \( \alpha \in C_{\alpha,X}(S_Z) \) for all \( \alpha \in \text{Conv}(S_\alpha) \) as required, thus completing step (4) of the process defined above. In particular, the convex hull of the MCMC sample \( S_\alpha \) includes the summary statistics that we will typically use to summarise it for estimation of the parameters in \( \alpha \), such as their (posterior) means and quantile-based interval estimates. These estimates are thus also guaranteed to imply positive definite correlation matrices given any values of the covariates in the pre-specified set of interest \( S_Z \).

We note that part (v) of Proposition 1 would not necessarily hold if the individual correlations \( \rho_l \) were modelled using a nonlinear transformation, for example Fisher’s z transformation which would give the model \( \rho = \tanh(\alpha^T X) \). Inferring feasibility from \( S_\alpha \) to \( \text{Conv}(S_\alpha) \) is thus not necessarily justified for these models. Such transformations are normally used to ensure that individual correlations are constrained to be in the range \((-1,1)\). That, however, is not needed here, because positive definiteness of the matrix as a whole already implies that all of the correlations are in the valid range.

Consider finally step (3), where we want to ensure that the value of parameters \( \alpha \) retained from each MCMC iteration is feasible with all \( X \in S_{XT} \). The key result for it is given here, and it is then implemented as part of our estimation as described in Section 6. Suppose that \( \alpha' \) is a proposed value for \( \alpha \). Since \( S_{XT} \) is finite, it would be possible to simply calculate \( R((\alpha')^T X) \) for all \( X \in S_{XT} \) and retain \( \alpha' \) if these were all positive definite. This, however, could be computationally demanding and inefficient. Instead, we will check and update \( \alpha \) one element at a time. This builds on the result by Barnard et al. (2000) that when starting with a correlation matrix, there exists a continuous interval for each single correlation in it that still yields a positive definite correlation matrix while holding the rest of the correlations fixed at their previous values. Here we extend this idea to apply to the individual coefficients in \( \alpha \) and multiple values of the covariates \( X \). The procedure relies on the following result, the proof of which is given in Appendix A.

**Proposition 2.** Continuous feasible intervals for the coefficients in \( \alpha \): Consider a finite set \( S_{XT} = \{X_j = (X_{j1}, \ldots, X_{jq})^T \mid j = 1, \ldots, T \} \) and any fixed value \( \alpha = [\alpha_1, \ldots, \alpha_L]^T \in C_{\alpha,S_{XT}} \). Denote here (deviating slightly from previous notation) \( \alpha = (\alpha_{lm}, \alpha_{lm}')^T \) where \( \alpha_{lm} \) is the coefficient of \( X_{jm} \) in the model for correlation \( \rho_l \), for any \( m = 1, \ldots, q \) and \( l = 1, \ldots, L \), and \( \alpha_{-lm} \) denotes all other elements of \( \alpha \), \( \rho = (\rho_1, \rho^T_{-l}) \) where \( \rho_{-l} \) denotes all other elements of the distinct correlations \( \rho \) except \( \rho_l \), and \( R(\rho_{-l}, \rho_{-l}) \) the correlation matrix implied by \( \rho \). Let \( \rho^{(j)}_{-l} \) denote the value of \( \rho_{-l} \) for \( \rho_j = \alpha^T X_j \), for \( j = 1, \ldots, T \).

(i) There exists a non-empty interval \((a_{lm}, b_{lm})\) such that \( \alpha' = (\alpha_{lm}', \alpha_{lm})^T \in C_{\alpha,S_{XT}} \) for all \( \alpha_{lm}' \in (a_{lm}, b_{lm}) \).

(ii) Let \( f_{jl}(\rho_l') = |R(\rho_l', \rho^{(j)}_{-l})| \), treated as a function of \( \rho_l' \), where \(| \cdot | \) denotes the determinant of a matrix. If \( X_{jm} \neq 0 \), let

\[
\begin{align*}
    a_{lm}^{(j)} &= \frac{g_{jl} - \sum_{k \neq m} \alpha_{lk} X_{jk} - \text{sgn}(X_{jm}) h_{jl}}{X_{jm}}, \\
    b_{lm}^{(j)} &= \frac{g_{jl} - \sum_{k \neq m} \alpha_{lk} X_{jk} + \text{sgn}(X_{jm}) h_{jl}}{X_{jm}}.
\end{align*}
\]

for each \( j = 1, \ldots, T \), where \( g_{jl} = -d_{jl}/(2c_{jl}) \) and \( h_{jl} = [(d^2_{jl} - 4c_{jl}e_{jl})/(4c_{jl}^2)]^{1/2} \) for \( c_{jl} = [f_{jl}(1) + f_{jl}(-1) - 2f_{jl}(0)]/2, \ d_{jl} = [f_{jl}(1) - f_{jl}(-1)]/2 \) and \( e_{jl} = f_{jl}(0) \). If \( X_{jm} = 0 \), set \( a_{lm}^{(j)} = -\infty \) and \( b_{lm}^{(j)} = +\infty \). Then \((a_{lm}, b_{lm}) = \cap_{j=1}^T (a_{lm}^{(j)}, b_{lm}^{(j)}) \). This interval is non-empty because it contains at least the current value \( a_{lm} \).

Computationally the most demanding part of using this result is the calculation of the necessary determinants. Efficient methods for obtaining them, and other elements of the computations, are discussed in Section 6.2.

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6 Estimation of the model

Following the example and motivation of [Kuha et al. (2022)], we use a two-step procedure to estimate the latent variable model defined in Section 3. The parameters of the measurement model are estimated first, as explained in Section 6.1. They are then fixed at their estimated values in the second step, where the parameters of the structural model are estimated as described in Section 6.2.

6.1 Estimation of the measurement model

In the first step, the measurement parameters $\phi_G$ and $\phi_R$ are estimated separately. For $\phi_G$, the data are the responses $Y_{Gi}$, the measurement model is specified by (1)–(4), and the structural model for $\xi_{Gi}$ and $\eta_{Gi} = (\eta_{G Pi}, \eta_{GF i})^T$ is obtained from (5)–(8) by omitting $\eta_{Ri}$ and the covariates $X_i$. The log likelihood function for $\phi_G$ is then

$$
\log p(Y_G|\phi_G, \pi_G, \mu_{G P}, \mu_{G F}, \sigma_{G P}^2, \rho_{G P}) = \sum_{i=1}^n \log \left[ \pi_G \int p(Y_{Gi}|\xi_{Gi} = 1, \eta_{G Pi}, \eta_{GF i}; \phi_G) p(\eta_{Gi}|\mu_{G Pi}, \mu_{G Fi}, \sigma_{G Pi}^2, \rho_{G Pi}) d\eta_{G Pi} d\eta_{GF i} + (1 - G_i)(1 - \pi_G) \right]
$$

(11)

where $\pi_G = p(\xi_{Gi} = 1)$ and $p(\eta_{Gi}|\mu_{G Pi}, \mu_{G Fi}, \sigma_{G Pi}^2, \rho_{G Pi})$ is a bivariate normal density with $E(\eta_{G Pi}) = \mu_{G Pi}$, $\text{Var}(\eta_{G Pi}) = \sigma_{G Pi}^2$, $E(\eta_{GF i}) = \mu_{G Fi}$, $\text{Var}(\eta_{GF i}) = 1$ and $\text{Corr}(\eta_{G Pi}, \eta_{GF i}) = \rho_{G Pi}$. The estimates $\hat{\phi}_G$ of $\phi_G$ are obtained by maximizing (11), while the estimates of $\mu_{G Pi}, \mu_{G Fi}, \sigma_{G Pi}^2$ and $\rho_{G Pi}$ from this step are discarded. The estimates $\hat{\phi}_R$ of $\phi_R$ are obtained analogously, using the data on $Y_{Ri}$. We have used the Mplus 6.12 software [Muthén & Muthén (2010)] to carry out this first step of estimation.

6.2 Estimation of the structural model

In the second step of estimation, the structural parameters $\psi$ are estimated, treating the measurement parameters fixed at their estimated values $\hat{\phi} = (\hat{\phi}_G, \hat{\phi}_R)^T$. Here we omit $\hat{\phi}$ from the notation for simplicity.

We use a Bayesian approach, implemented using MCMC methods. The estimation algorithm has a data augmentation structure which alternates between imputing the latent variables given the observed variables and values of the parameters, and sampling the parameters from their posterior distributions given the observed and latent variables:

- **Sampling of the latent variables:** Let $\zeta = (\xi, \eta)$, where $\xi$ denotes all the values of the latent $\xi_i$ for the units $i$ in the sample, and $\eta$ all the values of $\eta_i$. Given the observed data $(Y, X)$ and the most recently sampled values of the parameters $\psi$, sample $\zeta$ from

$$
p(\zeta|Y, X, \psi) \propto p(Y|\zeta) p(\zeta|X, \psi).
$$

- **Sampling of the parameters:** Given the observed data on $X$ and the most recently sampled values of the latent variables $\zeta$, sample $\psi$ from the posterior distribution

$$
p(\psi|\zeta, X) \propto p(\zeta|X, \psi) p(\psi),
$$

where

$$p(\zeta|X, \psi) = p(\eta|X; \psi_\eta) p(\xi|X; \psi_\xi)$$
is specified by the structural model and \( p(\psi) \) is the prior distribution of \( \psi = (\psi_\eta^T, \psi_\xi^T)^T \). We take \( \psi_\eta \) and \( \psi_\xi \) to be independent a priori, so that \( p(\psi) = p(\psi_\eta)p(\psi_\xi) \).

The posterior distribution then divides into separate posteriors for \( \psi_\eta \) and \( \psi_\xi \) as

\[
p(\psi|\zeta, X) = p(\psi_\eta|\eta, X)p(\psi_\xi|\zeta, X) \propto [p(\eta|X; \psi_\eta)p(\psi_\eta)] [p(\psi|X; \psi_\xi)p(\psi_\xi)].
\]

These steps further split into separate steps for different components of \( \zeta \) and \( \psi \). For the latent variables \( \zeta \), the regression coefficients \( \psi_\xi \) for \( \xi \) and \( \beta \) for the means of \( \eta \), and the standard deviations \( \sigma \), these steps are similar to the ones in Kuha et al. (2022), with adjustments to allow for the fact that here \( \eta \) has four variables and that their correlations vary by unit \( i \). A description of these steps is given in Appendix B. What is completely new here is the procedure for sampling the coefficients \( \alpha \) of the model (9) for the conditional correlations of \( \eta_i \). It is described in the rest of this section.

When \( \alpha \) is being sampled, all the other quantities are taken as known and fixed at their most recently sampled values. The latent variables \( \eta_i \) are thus also treated as observed response variables in this model for their correlations. The other parameters \( \beta \) and \( \sigma \) of the distribution of \( \eta_i \) are also taken as known, and we will omit them from the notation below.

The posterior distribution that we need is then written as \( p(\alpha|X, \eta) \propto p(\eta|X; \alpha)p(\alpha) \).

As discussed in Section 3, we consider this posterior over a convex and bounded set \( C_{\alpha,SXT} \), where \( S_{XT} \) is a finite test set of values for \( X_j \) which also implies feasibility of correlations over the full set \( X(S_Z) \) that we are interested in. We specify a joint uniform prior distribution \( p(\alpha) \propto 1(\alpha \in C_{\alpha,SXT}) \) for \( \alpha \) over this set.

Let \( \alpha_{lm} \) denote any single element of \( \alpha \), for \( l = 1, \ldots, L \), \( m = 1, \ldots, q \). The sampling algorithm updates one \( \alpha_{lm} \) at a time, taking all the other elements \( \alpha_{-lm} \) fixed at their most recently sampled values. Denote \( R_i(\alpha_{lm}) = R(X_i; \alpha_{lm}, \alpha_{-lm}) \) and define the standardized residuals \( \epsilon = [\epsilon_1, \ldots, \epsilon_n]^T = S^{-1}(\eta - X\beta) \) where \( \eta = [\eta_1, \ldots, \eta_n]^T \) and \( S = \text{diag}(\sigma_{G}, \sigma_{RP}, 1, 1) \). The conditional posterior distribution from which \( \alpha_{lm} \) should be drawn is then

\[
p(\alpha_{lm}|\alpha_{-lm}, \epsilon, X) \propto \prod_{i=1}^n p(\epsilon_i|\alpha, X)p(\alpha_{lm}|\alpha_{-lm})
\]

\[
\propto \prod_{i=1}^n |R_i(\alpha_{lm})|^{-\frac{1}{2}} \exp \left( -\frac{1}{2}^T \epsilon_i R_i(\alpha_{lm})^{-1} \epsilon_i \right) \mathbb{1}(a_{lm} < \alpha_{lm} < b_{lm}),
\]

where \((a_{lm}, b_{lm})\) is the range of \( \alpha_{lm} \) in the subset of \( C_{\alpha,SXT} \) given \( \alpha_{-lm} \). This involves \( n \) matrix determinants and inverse operations, plus further determinants to obtain the interval \((a_{lm}, b_{lm})\) as described in Proposition 2 above. This would be computationally demanding. However, these demands can be reduced because the sampling updates only one parameter \( \alpha_{lm} \) at a time. The calculation of the determinant and inverse of \( R_i(\alpha_{lm}) \) can be avoided by maintaining and updating copies of working determinants and inverses. These features are included in the general elementwise Metropolis-Hastings (MH) procedure that we propose for sampling \( \alpha \). It is given in Algorithm 1 together with Remarks 1–4 below. Assuming the resulting Markov chain satisfies the standard regularity conditions (Tierney, 1994, 1996), in which the detailed balance condition is met by construction, Algorithm 1 has the desired posterior distribution as its unique stationary distribution.

**Remark 1:** Calculating feasible interval for \( \alpha_{lm} \) by Cholesky decomposition.

Proposition 2 and Lemma A.1 in Appendix A describe one way of calculating the interval \((a_{lm}, b_{lm})\). This requires the calculation of \( 3T \) determinants of correlation matrices. An alternative, more efficient procedure for its first steps can be obtained by adapting a method proposed by Wong et al. (2003) Let \( R_j = \Gamma_j^T \Gamma_j \) as defined in the Input statement of Algorithm 1 where \( \Gamma_j = (\gamma_{j_k,k_2}) \). Recall that \( K \) denotes the dimension of \( \Gamma_j \),
Algorithm 1: Elementwise Metropolis-Hastings procedure for sampling $\alpha$

1. **Input:** Current parameters $\alpha = (\alpha_{lm})$ for $l = 1, \ldots, L$, $m = 1, \ldots, q$.

   For units $i = 1, \ldots, n$: Standardized residuals $\varepsilon_i = S^{-1}(\eta_i - \beta^T X_i)$;

   $R^{-1}_i$ and $|R_i|$ for correlation matrices $R_i = R(X_i; \alpha)$.

   For a test set $S_{XT} = \{X_j | j = 1, \ldots, T\}$: Upper triangular matrices $\Gamma_j$ from the Cholesky decompositions $R_j = \Gamma_j^T \Gamma_j$ of $R_j = R(X_j; \alpha)$.

2. **Metropolis-Hastings sampling:**

   for $l = 1, \ldots, L$ do
   
   
   for $m = 1, \ldots, q$ do
   
   **Proposal generation:**
   
   Calculate $a'_{lm}$ based on $\Gamma_1, \ldots, \Gamma_T$. See Remark 1 for more on this.

   Generate $a'_{lm}$ from a proposal distribution $g(a'_{lm} | a_{lm})$. See Remark 2 for more on how the proposal can be created.

   **Rejection:**

   Calculate $R_i(a'_{lm})^{-1}$ by updating $R_i(a_{lm})^{-1}$ and $|R_i(a_{lm})|$ by updating $|R_i(a_{lm})|$, for $i = 1, \ldots, n$; see Remark 3.

   Calculate the acceptance probability

   $$
   \pi(\alpha_{lm} \rightarrow a'_{lm}) = \min \left\{ 1, \frac{p(a'_{lm} | a_{lm}, \epsilon, X) g(a_{lm} | a'_{lm})}{p(a_{lm} | a_{lm}, \epsilon, X) g(a'_{lm} | a_{lm})} \right\}
   $$

   where $p(a_{lm} | a_{lm}, \epsilon, X)$ is given by equation (12).

   Sample $u \sim U(0, 1)$.

   if $u > \pi(\alpha_{lm} \rightarrow a'_{lm})$ then

   Rejected $a'_{lm}$;

   continue

   end

   Accept $a'_{lm}$ and update

   $\alpha_{lm} \rightarrow a'_{lm}$; $R_i(a_{lm})^{-1} \rightarrow R_i(a'_{lm})^{-1}$, $|R_i(a_{lm})| \rightarrow |R_i(a'_{lm})|$.

   Update $\Gamma_j(a_{lm}) \rightarrow \Gamma_j(a'_{lm})$ for $j = 1, \ldots, T$; see Remark 4.

   end

end

3. **Output:** Updated $\alpha$, $R^{-1}_i$, $|R_i|$ and $\Gamma_j$.

and assume that $p_l$ in Lemma [A.1] corresponds to the $(K, K-1)$th element of $R_j$. We then have $g_{jl} = \sum_{k=1}^{K-2} \gamma_{j,k-1} \gamma_{k,k}$ and $h_{jl} = \gamma_{j,K-1} - (1 - \sum_{k=1}^{K-2} (\gamma_{j,k})^2)^{1/2}$, and $(a_{lm}, b_{lm})$ can be obtained by plugging in $g_{jl}$ and $h_{jl}$ into (10) in Proposition 2 as before. If $p_l$ is not the $(K, K-1)$th element of $R_j$, we can permute the indices with a permutation matrix $P$ so that it is the $(K, K-1)$th element of the matrix $P^T R_j P = (\Gamma_j P)^T (\Gamma_j P)$, followed by a Givens rotation by an orthogonal matrix $Q$ such that $Q \Gamma_j P = \tilde{\Gamma}_j$, where $\tilde{\Gamma}_j$ is upper-triangular and $P^T R_j P = \tilde{\Gamma}_j^T \tilde{\Gamma}_j$. Then apply the calculation above to $\tilde{\Gamma}_j$.

**Remark 2:** Generating proposal values for $\alpha_{lm}$.

We have used a simple random walk Metropolis sampler. It generates the proposal through an independent Gaussian increment to the previous value, as $a'_{lm} = a_{lm} + \gamma_m \delta$, where $\delta$ is drawn from the standard normal distribution. Thus $a_{lm} | a'_{lm} \sim N(a'_{lm}, \gamma_m^2)$. The step size $\gamma_m$ should be chosen to achieve a good balance between rejection rate and mixing efficiency. We have used $\gamma_m = C(\sqrt{n/||X_{1m}, \ldots, X_{nm}||_x}^2)^{-1}$, where $|| \cdot ||_x$ is the infinity norm and $C$ is a constant chosen, the same for all $\gamma_m$, which is used to control
rejection rates in the range 0.7–0.8. The sampler was efficient enough in our real data analysis when the step sizes were chosen appropriately.

An alternative would be to use the ARMS algorithm [Gils et al., 1993] to adaptively construct the proposal function of \( \alpha_{lm} \) in \((a_{lm}, b_{lm})\). This can improve the acceptance rate but the algorithm may require the likelihood function \( p(\epsilon|\alpha, X) \) to be evaluated multiple times based on the rejection condition, whereas in the random walk Metropolis method it needs to be calculated at most once in each iteration. Other methods for improving the acceptance rate exist (Chib & Greenberg, 1998), but their implementation is more complex and relies heavily on tuning.

**Remark 3: Updating the determinant and inverse of correlation matrix.**

Here we want to update the determinant and inverse of \( R_i(\alpha_{lm}) \) to those of \( R_i'(\alpha'_{lm}) \). Suppose that the correlation parameter \( p_i \) corresponds to the \((k_1, k_2)\)th element of \( R_i(\alpha_{lm}) \). Let \( \varepsilon_{ilm} = (\alpha'_{lm} - \alpha_{lm})X_{ilm} \), and denote by \( w_1 \) and \( w_2 \) the \( K \times 1 \) vectors which are zero except that the \( k_1 \)th element of \( w_1 \) and the \( k_2 \)th element of \( w_2 \) are \( \sqrt{\varepsilon_{ilm}} \). Then

\[
R_i'(\alpha'_{lm}) = \left[R_i(\alpha_{lm}) + (\text{sgn}(\varepsilon_{ilm})w_1^Tw_1^T) + w_2(\text{sgn}(\varepsilon_{ilm})w_1)^T\right].
\]

Since this is of the form \((A + uv^T) + vu^T\), \( R_i'(\alpha'_{lm})^{-1} \) can be computed efficiently with two rank-1 updates by applying twice the Sherman-Morrison formula

\[
(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}
\]

and \(|R_i(\alpha_{lm})|\) can be calculated by updating \(|R_i'(\alpha'_{lm})|\) through two applications of \(|A + uv^T| = (1 + v^TA^{-1}u)|A|\), the second of which employs the first update of the inverse. These steps reduce the computation complexity of \( p(\epsilon|\alpha, X) \) from \( O(K^3) \) to \( O(K^2) \).

**Remark 4: Updating the Cholesky decomposition of a correlation matrix.**

Let \( \varepsilon_{jlm} = (\alpha'_{lm} - \alpha_{lm})X_{jlm} \). Let \( w_1 \) and \( w_2 \) be defined as in Remark 3, and define \( w \) as the \( K \times 1 \) vector where the \( k_1 \)th and \( k_2 \)th elements are \( \sqrt{\varepsilon_{jlm}} \) and the other elements are zero. Then we can write

\[
R_j'(\alpha'_{lm}) = \left[R_j(\alpha_{lm}) + \text{sgn}(\varepsilon_{jlm})ww^T - \text{sgn}(\varepsilon_{jlm})w_1w_1^T - \text{sgn}(\varepsilon_{jlm})w_2w_2^T\right].
\]

The Cholesky decomposition of \( R_j'(\alpha'_{lm}) \) can be computed efficiently from this, with three rank-1 updates for the Cholesky decomposition of the form \( A + uu^T \) or \( A - uu^T \) (Seeger, 2008); built-in functions for this are available in Matlab and linear algebra libraries like Eigen (Guennebaud et al., 2010). This updating rule reduces the computation complexity of the Cholesky decomposition \( R_j = \Gamma_j^T\Gamma_j \) from \( O(K^3) \) to \( O(K^2) \).

Alternatives to Algorithm 1 could also be considered. In cases when \( X = X(Z) \) is a complex function such as a cubic spline, for better efficiency the element-wise MH algorithm could be replaced by a blockwise algorithm where subvectors of \( \alpha \) can be proposed and rejected together. Apart from the MH algorithm we use in this paper, we note that the “gridded Gibbs” sampler discussed in Barnard et al. (2001) also works here in principle, where the feasible intervals for each \( \alpha_{lm} \) can be discretized into grids. However, the computational efficiency for evaluating posterior function over these grids may suffer.
7 Analysis of child-parent exchanges of support

7.1 Introduction and research questions

The model defined in Section 3 was fitted to the UKHLS data on exchanges of support between respondents and their non-coresident parents that were introduced in Section 2, using the method of estimation that was described in Section 6. Here receiving and giving help are modelled jointly, treating practical and financial support as distinct but correlated outcomes. We investigate the following research questions:

(a) What individual characteristics are associated with higher or lower levels of giving practical and financial help to the parents, and receiving such help from the parents?
(b) To what extent are exchanges reciprocated and how does reciprocity vary according to individual characteristics?
(c) Are practical and financial support substitutes for one another or are they complementary, and how does this depend on individual characteristics?

Questions (b) and (c) refer to within-person correlations between the helping tendencies. For (b), higher levels of reciprocity would correspond to positive correlations between giving and receiving help. For (c), positive correlations between the tendencies to give (or to receive) practical and financial help would suggest that the two types of support are complementary (i.e. given together), and negative correlations that they are substitutes.

Estimates \( \hat{\phi} \) of the parameters of the measurement model were obtained first, as explained in Section 6.3. They are shown in Table S1 of the supplementary materials. The loading parameters are positive, so the latent variables \( \eta_{GP} \) and \( \eta_{RP} \) are defined so that larger values of them imply higher tendencies to give and receive practical help (and the same is true by construction for the financial help variables \( \eta_{GF} \) and \( \eta_{RF} \)). The measurement parameters were then fixed at \( \hat{\phi} \) in the estimation of the rest of the model below.

The structural model for the joint distribution of the latent variables was estimated using the MCMC algorithm described in Section 6.2 and Appendix B. Estimated parameters and some predicted values for these models are shown in Tables 3-6 and in Tables S2–S3 of the supplementary materials. They are based on a sample of 380,000 draws of the parameters \( \psi \), obtained by pooling two MCMC chains of 200,000 iterations each, with a burn-in sample of 10,000 omitted from each chain. Convergence was assessed by visual inspection of trace plots of the two chains which suggested adequate mixing. In the role of the set of interest \( S_Z \) for the covariates (as defined in Section 5), we used the simple choice of the set of all the \( n \) observed values of \( Z_i \) in the data, and as the test set \( S_{XT} \) all the distinct values of \( X_i = X(Z_i) \) implied by them.

Estimated parameters of the multinomial logistic model (5) for the joint distribution of the binary latent class variables \( \xi_G, \xi_R \) are shown in Table S2, and fitted class probabilities \( p(\xi_G = 1) \) and \( p(\xi_R = 1) \) from it given different values of the covariates in Table S3. This model component is included primarily to allow for zero-inflation in the observed item responses, so it is not our main focus. We could, however, also interpret the classes defined by \( \xi_G = 1 \) and \( \xi_R = 1 \) as latent sub-populations of ‘givers’ and ‘receivers’ of help respectively. The estimated overall proportions of these classes, averaged over the sample distribution of the covariates, are 0.67 for ‘givers’ and 0.62 for ‘receivers’.

The focus of interest is the model for the joint distribution of \( \eta = (\eta_{GP}, \eta_{GF}, \eta_{RP}, \eta_{RF})^\top \), which we interpret as continuous latent tendencies for the adult respondents to give practical and financial help to and receive help from their non-coresident parents, after accounting for the zero-inflation. We consider first results for the linear model (8) for the means of \( \eta \), which is used to answer research question (a), and then discuss estimates of the model (9–10) for their correlations, corresponding to questions (b) and (c).
7.2 Predictors of levels of giving and receiving help

Table 3 shows the estimated coefficients of the predictors of the means of practical ($\eta_{GP}$) and financial ($\eta_{GF}$) help given by respondents to parents. There is little evidence that the respondent’s partnership status or the presence or age of their children are associated with the tendency to give help. Women tend to give more practical help than men, but there is no gender difference in giving financial help. Indicators of lower socioeconomic status or a more difficult economic situation of the respondent (lower education, not being a homeowner, lower household income, and not being employed) are associated with a higher tendency to give practical help, while having more education and higher household income predict a higher tendency to give financial help. These results are consistent with a pattern where children give help to the best of their ability, with less well-off children giving, on average, relatively more practical support and less financial support (this does not, however, tell us about possible substitution of types of help by the same person; for that, we will turn to the within-person correlations in the next section). However, the results for household tenure and employment status (where home owners and the employed also tend to give less financial help) deviate from this pattern, after controlling for education and income. There is also some evidence that respondents with one sibling give less help than those with none, which could suggest some sharing of support between the siblings (although there is no similar reduction for those with more siblings).

Having a parent who lives alone and older parental age are positively associated with giving both forms of help, with the positive association between age and financial help emerging when the oldest parent reaches their early 70s. Both of these findings are consistent with children giving help according to parental need. After controlling for parental age, the (correlated) respondent’s age has an inverse U-shaped relationship with giving both practical and financial help, with highest levels of giving at around ages 43 and 49 respectively. Finally, respondents who live more than an hour away from the nearest parent have a lower tendency to give practical help, but a higher tendency to give financial help.

As for the effects of socioeconomic status, the different directions of these associations suggest differences in the mix of different types of help related to the giver’s circumstances, in this case according to how feasible it is to provide practical help.

Covariate effects on levels of practical and financial help that the respondents receive from their parents (variables $\eta_{RP}$ and $\eta_{RF}$) are shown in Table 4. Women tend to receive more of both types of support than men. Expected levels of support from parents are also higher for respondents who are not employed, have less education, or have no coresident partner, all of which can be taken to indicate higher levels of need for support. Respondents with two or more siblings tend to receive less of either form of help than those from one or two-child families, which may reflect greater competition for parental resources in larger families. For financial help, the tendency to receive such help is higher for respondents who have lower household income or who rent rather than own their homes, as well as for those with very young or secondary school age children. These associations are also consistent with parents providing financial assistance to children who are most in need.

Levels of both practical and financial help received decline with the respondent’s age, which is consistent with reduced need by respondents. As a function of the oldest parent’s age, receipt of practical help also declines from age 67 onwards, but the tendency to receive financial help increases with parental age. This may be interpreted as another instance of the balance of different types of help depending on the giver’s capacities, in this case with older parents being more able to give financial than practical support. Finally, longer travel time between the respondent and their nearest parent is associated with less practical and more financial help, as it was also for help from respondents to parents.
7.3 Models for the correlations: Predictors of symmetry in exchanges and complementarity of practical and financial help

Estimated coefficients ($\hat{\alpha}$) of the model (8)–(9) for the residual correlations of $\eta = (\eta_{GP}, \eta_{GF}, \eta_{RP}, \eta_{RF})$ are shown in Table 5 and some fitted correlations from these models in Table 6. Here we included as covariates the respondent’s age, age squared, gender, household income, and travel time to the nearest parent. Whereas the models in Section 7.2 concern the mean of each helping tendency separately, these correlations focus on their joint distribution for a given child-parent dyad, over and above the levels predicted by the mean models. They can be used to investigate research questions (b) and (c) above.

The four correlations between the tendencies to give and receive help (of the same or different type) can be viewed as measures of reciprocity or symmetry in exchanges between children and their parents (question b). Results for them are given in the first four columns of each table, where for ease of interpretation we focus on the fitted values in Table 6. Consider first the correlations averaged over the sample distribution of the covariates, shown on the first row. There is a moderate positive correlation of 0.38 between giving and receiving practical help ($GP \leftrightarrow RP$). In other words, when a child has a high tendency to give practical help to their parent(s), relative to what would be predicted by their own and the parents’ characteristics, they also tend to receive a higher-than-average level of support from the parents. This suggests a fair amount of reciprocity in practical help. The other three correlations are weaker, indicating little dyad-level reciprocity in anything other than practical help. What is not observed here are any substantial negative correlations. They would indicate that when the tendency to help is high in one direction it is low in the other, as would happen for example if help was given only in the direction of greater need. This is not seen here even for giving and receiving financial help, even though we might have expected financial exchanges to be largely unidirectional. One possible explanation of this is that the single financial support item covers also small sums of money, which may be exchanged more frequently and symmetrically than large ones.

The ($GP \leftrightarrow RP$) correlation is also the one for which we see the most noticeable covariate effects, as illustrated by the other rows of Table 6. It declines sharply with age, and it is significantly higher for men than for women and among parents and children who live farther apart. Reciprocity in practical support is highest at younger ages of the adult children. We note that this captures a different aspect of the effects of age than the mean models in Section 7.2. There respondent’s age was negatively associated with tendency to give practical help and positively associated (up to age around 43) with tendency to receive it. Thus younger individuals tend to give less practical help and receive more of it, and the expected balance of support is more toward help from parents to children, than is the case at other ages (comparable conclusions were reached in a different way by Mudrazija 2016, who considered net financial values of the differences between these two directions). The residual correlations show that, around these expected levels, for younger respondents the level of practical help that they do (or do not) give is particularly strongly predictive of how much support they receive. Similarly, the gender difference in the correlation suggests that men are more likely than women to engage in two-way exchanges or not exchange practical help at all.

The only other clearly significant covariate effects on the correlations that relate to reciprocity are those between within-dyad distance and the ($GP \leftrightarrow RF$), ($GF \leftrightarrow RF$) and ($GP \leftrightarrow RP$) correlations. Recall that the models for the means showed that the balance of the expected levels of different types of help moves towards more financial and less practical support when the child and the parent(s) live further apart. Of the residual correlations here, ($GP \leftrightarrow RP$) is quite strongly positive when the distance is longer vs. less positive when it is shorter, while ($GP \leftrightarrow RF$) is near zero vs. moderately positive
and (GF ↔ RF) moderately negative vs. near zero similarly (and GF ↔ RP is always small). One possible interpretation of these different patterns is that among children and parents who live further apart providing practical support requires a greater effort and the tendency to give such support may be higher when reciprocated. For such dyads, financial help may also more often involve one-way (and perhaps larger) transfers which are less often (and less easily) reciprocated by practical help.

The two remaining correlations, between the tendencies to give financial and practical help and between the tendencies to receive financial and practical help, are used to examine whether one form of help that a person may give serves as a substitute for the other or whether they are complementary, and whether this varies according to individual characteristics (research question c). Here the mean models in Section 7.2 also give information about one version of this question, when they show that the expected balance of the two types of help is, on average, different for dyads with different characteristics. This is most obvious when the coefficient of a covariate has different signs for practical and financial help, as it does for example for the distance between respondent and their parents (a similar result for expected levels of financial vs. time assistance given distance was found by Bonsang 2007 in a cross-national European study). However, this is again not the same as the question of substitution for a person, i.e. whether the level of one kind of help that he or she gives predicts higher or lower levels of the other kind of help.

Results for the correlations that address this question are given in the final two columns of Tables 5 and 6. The fitted correlations are positive overall and in all sub-groups defined by the covariates. This indicates clearly that within a person the types of help are not supplementary but complementary: a person (child or parent) who has a high tendency to give one kind of help (relative to what would be expected given the characteristics of their dyad) also has a high tendency to give the other kind of help. The most noticeable covariate effect that holds for both children and the parents is that the degree of complementarity in practical and financial help is greater when the child-parent distance is small. For help received from the parents, complementarity also declines with the respondent’s (and thus in effect also the parents’) age. This suggests that at older ages the parents more often tend to limit the support that they give to one of these types (mostly likely financial help, in light of the results in Table 4) rather than both of them.

In conclusion, we return to the research questions that were stated in Section 7.1. The first question was addressed by the models for the mean levels of helping tendencies in Section 7.2. Their results may be summarised in terms of two broad types of characteristics: the capacities of a giver of support and the level of need of the recipient. The model results indicate clearly that recipients with higher level of need (such as children with less privileged socioeconomic status or parents who are older or live alone) tend to receive more support. For capacities of giving, the results are more subtle. There is no strong evidence that lower capacity is associated with less help given in some overall sense. Instead, different types of individuals tend to give the types of help that they are best able to give, e.g. with less wealthy children giving relatively more practical than financial help to their parents, and older parents providing relatively more financial help to their children.

The other two research questions correspond to the models for residual correlations in this section. The results show evidence of reciprocity between children and parents in practical help, and of within-person complementarity in giving different types of help. A prominent covariate effect on these correlations was found for the distance between children and their parents, with different patterns of correlations between helping tendencies of different types and directions for children who lived far from rather than close to their parents.
Table 3: Estimated parameters of the linear model for the expected value of the tendency to give practical help ($\eta_{GP}$) and to give financial help ($\eta_{GF}$) to individuals’ non-coresident parents. The estimates are posterior means from MCMC samples (with posterior standard deviations in parentheses).

| Estimated coefficients: | Giving practical help | Giving financial help |
|-------------------------|-----------------------|-----------------------|
|                         | $\hat{\beta}_{GP}$    | $\hat{\beta}_{GF}$   |
| Intercept               | $-0.70^{***}$ (0.18)  | $-2.35^{***}$ (0.31) |
| **Respondent (child) characteristics** |                       |                       |
| Age $^\dagger$ (×10)   | 0.03 (0.03)           | 0.12$^{***}$ (0.04)  |
| Age squared$^\dagger$ (×10³) | $-0.60^{***}$ (0.12) | $-0.70^{***}$ (0.19) |
| Gender                  |                       |                       |
| Female (vs. Male)       | 0.41$^{***}$ (0.03)   | 0.03 (0.04)           |
| Partnership status      |                       |                       |
| Partnered (vs. Single)  | $-0.04$ (0.03)        | 0.01 (0.05)           |
| Age of youngest coresident child (vs. No children): |                       |                       |
| 0–1 years               | $-0.08$ (0.06)        | $-0.05$ (0.09)        |
| 2–4 years               | 0.01 (0.05)           | 0.03 (0.08)           |
| 5–10 years              | 0.02 (0.04)           | 0.09 (0.07)           |
| 11–16 years             | $-0.04$ (0.05)        | $-0.10$ (0.07)        |
| 17+ years               | 0.03 (0.04)           | $-0.03$ (0.06)        |
| Number of siblings (vs. None) | $-0.08^*$ (0.04) | $-0.12^*$ (0.07) |
| 2 or more               | 0.00 (0.04)           | 0.06 (0.07)           |
| Longstanding illness (vs. No) | 0.07$^*$ (0.04) | 0.07 (0.06)           |
| Employment status (vs. Employed) |                       |                       |
| Not employed            | 0.21$^{***}$ (0.03)   | 0.11$^{**}$ (0.05)   |
| Post-secondary          | $-0.05^{**}$ (0.03)   | 0.12$^{***}$ (0.04)  |
| Education (vs. Secondary or less) |                       |                       |
| Own home outright or with mortgage | $-0.17^{***}$ (0.03) | $-0.19^{***}$ (0.05) |
| Logarithm of household equilivalised income | $-0.04^{**}$ (0.02) | 0.09$^{***}$ (0.03) |
| **Parent characteristics** |                       |                       |
| Age of the oldest living parent$^\dagger$ (×10) | 0.28$^{***}$ (0.02) | $-0.02$ (0.04) |
| Age of the oldest parent squared$^\dagger$ (×10³) | 0.52$^{***}$ (0.11) | 0.63$^{***}$ (0.17) |
| At least one parent lives alone (vs. No) | 0.33$^{***}$ (0.03) | 0.24$^{***}$ (0.04) |
| **Child-parent characteristics** |                       |                       |
| Travel time to the nearest parent |                       |                       |
| More than 1 hour (vs. 1 hour or less) | $-0.43^{***}$ (0.04) | 0.14$^{**}$ (0.05) |
| **Residual s.d.:**      | $\hat{\sigma}_{GP}$  | 1                     |
|                         | 0.73 (0.01)           |                       |

The posterior credible interval excludes zero at level 90% (*), 95% (**) or 99% (***)

$^\dagger$ Age of respondent is centered at 40, and age of oldest living parent at 70.
Table 4: Estimated parameters of the linear model for the expected value of the tendency to receive practical help ($\eta_{RP}$) and to receive financial help ($\eta_{RF}$) from individuals’ non-coresident parents. The estimates are posterior means from MCMC samples (with posterior standard deviations in parentheses).

| Estimated coefficients:       | Receiving practical help | Receiving financial help |
|-------------------------------|---------------------------|--------------------------|
|                               | $\hat{\beta}_{RP}$       | $\hat{\beta}_{RF}$      |
| Intercept                     | $-2.17^{***}$ (0.23)      | $1.03^{***}$ (0.34)      |
| **Respondent (child)**        |                           |                          |
| characteristics               |                           |                          |
| Age $^\dagger$ ($\times 10$) | $-0.26^{***}$ (0.03)      | $-0.28^{***}$ (0.04)     |
| Age squared$^\dagger$ ($\times 10^3$) | $-0.16$ (0.18)       | $-0.46^*$ (0.23)         |
| Gender                        |                           |                          |
| Female (vs. Male)             | $0.27^{***}$ (0.03)       | $0.15^{***}$ (0.04)      |
| Partnership status            |                           |                          |
| Partnered (vs. Single)        | $-0.35^{***}$ (0.04)      | $-0.30^{***}$ (0.05)     |
| Age of youngest coresident child (vs. No children): |                           |                          |
| 0–1 years                     | $0.02$ (0.05)              | $0.14^*$ (0.07)          |
| 2–4 years                     | $-0.03$ (0.05)             | $0.07$ (0.06)            |
| 5–10 years                    | $-0.09^{**}$ (0.04)       | $-0.02$ (0.06)           |
| 11–16 years                   | $-0.11^*$ (0.06)          | $0.18^{**}$ (0.07)      |
| 17– years                     | $-0.12$ (0.07)             | $0.02$ (0.09)            |
| Number of siblings (vs. None) |                           |                          |
| 1                             | $0.00$ (0.05)              | $-0.07$ (0.07)           |
| 2 or more                     | $-0.14^{***}$ (0.05)      | $-0.25^{***}$ (0.06)    |
| Longstanding illness (vs. No) |                           |                          |
| Not employed                  | $0.03$ (0.05)              | $0.06$ (0.06)            |
| Employment status (vs. Employed) |                        |                          |
| Not employed                  | $0.21^{***}$ (0.04)       | $0.14^{**}$ (0.05)      |
| Education (vs. Secondary or less) |                      |                          |
| Post-secondary                | $-0.06^{**}$ (0.03)       | $-0.06$ (0.04)           |
| Household tenure (vs. Renter) |                           |                          |
| Own home outright or with mortgage | $0.08^{**}$ (0.03) | $-0.34^{***}$ (0.05)   |
| Logarithm of household equivalised income | $0.01$ (0.02) | $-0.14^{***}$ (0.03) |
| **Parent characteristics**    |                           |                          |
| Age of the oldest living parent$^\dagger$ ($\times 10$) | $-0.03$ (0.03)  | $0.22^{***}$ (0.04)    |
| Age of the oldest parent squared$^\dagger$ ($\times 10^3$) | $-0.44^{***}$ (0.15) | $0.28$ (0.19)         |
| At least one parent lives alone (vs. No) | $-0.05$ (0.03) | $0.04$ (0.04)          |
| **Child-parent characteristics** |                        |                          |
| Travel time to the nearest parent |                      |                          |
| More than 1 hour (vs. 1 hour or less) | $-0.42^{***}$ (0.05) | $0.27^{***}$ (0.06)    |
| **Residual s.d.:**            |                           |                          |
| $\hat{\sigma}_{RP}$          | $0.68$ (0.02)              | $1$                      |

The posterior credible interval excludes zero at level 90% (*), 95% (**) or 99% (***)

$^\dagger$ Age of respondent is centered at 40, and age of oldest living parent at 70.
Table 5: Estimated coefficients ($\hat{\alpha}$) of the model for the residual correlations of the tendencies to give and receive practical help (GP and RP) and to give and receive financial help (GF and RF). The estimates are posterior means from MCMC samples (with posterior standard deviations in parentheses).

| Covariate Setting                          | GP→RP | GP→RF | GF→RP | GF→RF | GP→GF | RP→RF |
|--------------------------------------------|-------|-------|-------|-------|-------|-------|
| Intercept                                  | 0.087 | 0.166 | -0.133| -0.126| 0.475***| 0.148 |
| Age of respondent†                         | -0.014***| 0.004* | 0.003 | -0.001 | -0.002 | -0.009***|
| Age squared† (×10³)                        | -0.277** | -0.137 | 0.001 | 0.159 | -0.112 | -0.251*|
| Female                                     | -0.151***| -0.025 | -0.119*| -0.103*| -0.080* | 0.044 |
| Travel time to nearest parent > 1 hr       | 0.141***| -0.206***| -0.119 | -0.226***| -0.273***| -0.252***|
| Log(household income)                      | 0.044***| 0.007 | 0.025 | 0.017 | 0.003 | 0.017 |

The posterior credible interval excludes zero at level 90% (*), 95% (**) or 99% (***)..
† Age of respondent is centered at 40.

Table 6: Fitted residual correlations calculated using the parameter estimates in Table 5 averaged over parameter values in the MCMC samples and over covariate values in the analysis sample. The ‘Overall’ values are averaged over sample values of all the covariates, and the other fitted values over the sample values of all the covariates except for the one fixed at the specified value.

| Covariate Setting                          | GP→RP | GP→RF | GF→RP | GF→RF | GP→GF | RP→RF |
|--------------------------------------------|-------|-------|-------|-------|-------|-------|
| Overall                                    | 0.38  | 0.16  | 0.02  | -0.06 | 0.36  | 0.20  |
| Age of respondent                          |       |       |       |       |       |       |
| 35 years                                   | 0.53  | 0.14  | 0.00  | -0.07 | 0.39  | 0.31  |
| 45 years                                   | 0.39  | 0.18  | 0.03  | -0.08 | 0.37  | 0.22  |
| 55 years                                   | 0.20  | 0.19  | 0.06  | -0.06 | 0.32  | 0.08  |
| Gender                                     |       |       |       |       |       |       |
| Female                                     | 0.31  | 0.14  | -0.03 | -0.10 | 0.32  | 0.22  |
| Male                                       | 0.47  | 0.17  | 0.09  | 0.00  | 0.40  | 0.18  |
| Travel time to the nearest parent          |       |       |       |       |       |       |
| > 1 hr                                     | 0.48  | 0.01  | -0.06 | -0.22 | 0.16  | 0.02  |
| ≤ 1 hr                                     | 0.34  | 0.21  | 0.05  | 0.00  | 0.43  | 0.27  |
| Logarithm of household equivalised income  |       |       |       |       |       |       |
| 25th percentile                            | 0.37  | 0.15  | 0.02  | -0.06 | 0.36  | 0.20  |
| 50th percentile                            | 0.38  | 0.16  | 0.02  | -0.06 | 0.36  | 0.20  |
| 75th percentile                            | 0.39  | 0.16  | 0.03  | -0.05 | 0.36  | 0.21  |
8 Conclusions

We have proposed methods for analysing the levels and correlations of intergenerational help and support. This involved defining a model for the joint distribution of latent variables which represent individuals’ tendencies of giving and receiving different types of support. A particular focus of the paper was on developing models for how the correlations of these variables depend on covariates. A linear model was specified for each correlation, and the estimation procedure was designed so that it ensures that the estimated model implies positive definite correlation matrices over the relevant range of the covariates. This builds on literature on such ‘constrained’ methods of estimation for models for correlations, which are here extended to include unit-level covariates. The estimation is carried out using a tailored MCMC algorithm which includes an efficient Metropolis-Hastings sub-procedure for estimating the correlation model.

The model was used to study exchanges of practical and financial support between adult individuals and their non-coresident parents in the UK, using survey data from the UK Household Longitudinal Study. The modelling framework allows us to model both the conditional means and correlations of different helping tendencies. The mean levels are broadly positively associated with many characteristics of the recipients that indicate higher need, and with characteristics of givers that indicate their higher capacity to give help. These results are, arguably, fairly encouraging about patterns of intergenerational support in this population. Less positively, however, a very substantial proportion of both adult individuals and their parents do not typically give any of the kinds of help considered here. The estimated correlations indicate reciprocity, where those who tend to give high levels of practical help also tend to receive much of it, and complementarity, where those who tend to give high levels of one kind of help (practical or financial) also tend to give much of the other kind. This suggests a picture of a general culture of helpfulness within some families, and general lack of it in others, rather than a sort of zero-sum game where help would flow only in one direction at a time and one kind of help would reduce the amount of other kinds.

This work could be extended in a number of ways in future research. Methodologically, the proposed modelling approach for the correlation matrix could be embedded into other covariance modelling tasks, such as the copula model (Hoff, 2007; Murray et al., 2013). The computational efficiency and mixing rates of the simple element-wise Metropolis-Hastings MCMC sampler that was used here could perhaps be improved by using other approaches, for example adaptive MCMC (Haario et al., 2001; Andrieu & Thoms, 2008) which proposes multiple parameters from an adaptive proposal in each iteration.

Substantively, the choices of this analysis were constrained by the available data. Although we were able to consider practical and financial support separately, the single indicator of financial support leaves us unable to examine varieties of it in more detail. Because the data were collected from the adult children only, we have limited information about their parents. Both of these limitations could be relaxed by richer data, but collecting it would be correspondingly more demanding. Another promising direction would be to extend these models to longitudinal data. This would allow us, for example, to examine questions of reciprocity and complementarity of help over time as well as contemporaneously as was done here. These areas of further research remain to be pursued.

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Appendix A  Proofs of the propositions in Section 5

Proof of Proposition 1

(i) Let \( \alpha \in C_{\alpha, S_X^1} \), so that \( \alpha^T X \in C_\rho \) for all \( X \in S_{X_1} \). Since \( S_{X_2} \subseteq S_{X_1} \), in particular, for all \( X \in S_{X_2} \subseteq S_{X_1} \), \( \alpha^T X \in C_\rho \), and thus \( \alpha \in C_{\alpha, S_{X_2}} \).

(ii) Since \( S_X \subseteq \text{Conv}(S_X) \), we have \( C_{\alpha, \text{Conv}(S_X)} \subseteq C_{\alpha, S_X} \) by (i). So we just need to prove the other direction. Suppose that \( \alpha \in C_{\alpha, S_X} \), for any \( X' \in \text{Conv}(S_X) \), there exist a finite number of points \( X_1, \ldots, X_r \in S_X \) and \( \lambda_1, \ldots, \lambda_r \geq 0 \), \( \sum \lambda_j = 1 \), such that \( X' = \sum \lambda_j X_j \). Then we have \( \alpha^T X' = \alpha^T (\sum \lambda_j X_j) = \sum \lambda_j (\alpha^T X_j) \in C_\rho \), i.e., \( \alpha \in C_{\alpha, \text{Conv}(S_X)} \), which holds because \( \alpha^T X_j \in C_\rho \) for all \( j = 1, \ldots, r \), and \( C_\rho \) is a convex set.

(iii) \( \alpha = 0 \) gives \( \rho = 0 \). This implies the identity correlation matrix, which is in \( C_\rho \).

(iv) Under the further assumption stated in (iv), we can find a set \( S_{X*} = \{X_1, \ldots, X_q\} \subseteq S_X \) such that the matrix \( X_* = [X_1, \ldots, X_q] \) is non-singular. Suppose that \( \alpha \in C_{\alpha, S_{X*}} \), and let \( \alpha^T X_* = [\rho_1, \ldots, \rho_q] \). Then \( \alpha^T = [\rho_1, \ldots, \rho_q] X_*^{-1} \). This is bounded, because all elements of \( \rho_1, \ldots, \rho_q \) are bounded (moreover, \( \rho \in [-1, 1]^q \)). Finally, since \( S_{X*} \subseteq S_X \), we have \( C_{\alpha, S_X} \subseteq C_{\alpha, S_{X*}} \) by (ii), and thus \( C_{\alpha, S_X} \) is also bounded.

(v) Suppose that \( \alpha_1, \alpha_2 \in C_{\alpha, S_X} \) and that \( 0 \leq \lambda \leq 1 \). Then \( (\lambda \alpha_1 + (1 - \lambda) \alpha_2)^T X = \lambda \alpha_1^T X + (1 - \lambda) \alpha_2^T X = \lambda \rho_1 + (1 - \lambda) \rho_2 \in C_\rho \), where the last equation holds since \( C_\rho \) is a convex set. Thus \( \lambda \alpha_1 + (1 - \lambda) \alpha_2 \in C_{\alpha, S_X} \).

The proof of Proposition 2 builds on the key ideas of Barnard et al. (2000), extended to the case of models with covariates that we consider.

Lemma A.1. Let \( R(\rho) = R(\rho_1, \rho_{-1}) \) be the positive definite correlation matrix defined by distinct correlations \( \rho = (\rho_1, \rho_{-1})^T \). Consider \( f_l(\rho_l^1) = |R(\rho_l^1, \rho_{-1})| \) as a univariate function of \( \rho_l^1 \in [-1, 1] \). Then \( f_l(\rho_l^1) \) is a quadratic function of \( \rho_l^1 \) with negative second order coefficient. The matrix \( R_l = R(\rho_l^1, \rho_{-1}) \) is positive definite if and only if \( f_l(\rho_l^1) > 0 \).

Proof of Lemma A.1. \( R_l \) is a symmetric matrix where \( \rho_l^1 \) appears once in both its upper and lower triangles, so \( f_l(\rho_l^1) \) is a quadratic function. Suppose that \( R_l \) is a \( K \times K \) matrix. Without loss of generality, assume that \( \rho_l^1 \) is in its \( K \)th row, first column (and first row, \( K \)th column), as we can always swap both row and column without changing the positive definiteness and determinant value. Thus, the coefficient of \( (\rho_l^1)^2 \) in \( f_l(\rho_l^1) \) is \( c_l = (-1)^{2K+1} |R_{l,l}| \), where \( R_{l,l} \) is the submatrix of \( R_l \) obtained by deleting the first and last rows and columns. Here \( R_{l,l} \) is a correlation matrix, obtained by deleting from \( R(\rho) \) all those correlations which involve either of the two variables whose correlation is \( \rho_l^1 \). Thus \( R_{l,l} \) is positive definite, \( |R_{l,l}| > 0 \), and \( c_l < 0 \).

\( R_l \) is positive definite if and only if \( |R_{l,k}| > 0 \) for all \( k = 1, \ldots, K \), where \( R_{l,k} \) is the \( k \)th primary submatrix of \( R_l \) (Sylvester’s criterion). Here \( \rho_l^j \) only affects \( |R_{l,K}| = |R_{l,l}| \). Because \( R_{l,1}, \ldots, R_{l,K-1} \) are equal to the corresponding submatrices of the positive definite correlation matrix \( R(\rho) \), we have \( |R_{l,k}| > 0 \), for \( k = 1, \ldots, K - 1 \). So \( R_l = R(\rho_l^1, \rho_{-1}) \) is positive definite if and only if \( f_l(\rho_l^1) = |R_l| > 0 \).

Proof of Proposition 2

From Lemma A.1 we know that \( R_{j,l} = R(\rho_l^j, \rho_{-j}) \) is positive definite if and only if \( f_{j,l}(\rho_l^j) = |R_{j,l}| > 0 \). We can write \( f_{j,l}(\rho_l^j) = c_{j,l}(\rho_l^j)^2 + d_{j,l}\rho_l^j + e_{j,l} \), where \( c_{j,l} = f_{j,l}(1) + \cdots + f_{j,l}(r) \).
f_{jl}(-1) - 2 f_{jl}(0)}/2, d_{jl} = [f_{jl}(1) - f_{jl}(-1)]/2, e_{jl} = f_{jl}(0). The set of values for \( \rho' \) for which \( f_{jl}(\rho') > 0 \) is a finite interval because \( c_{jl} < 0, f_{jl}(0) = |R(0, \rho_{-1})| > 0, \) and \( f_{jl}(\rho') \) is a continuous function. Let us denote the roots of \( f_{jl}(\rho') = 0 \) by \( x_{jl1} > x_{jl2} \), and define

\[
\begin{align*}
g_{jl} &= \frac{x_{jl1} + x_{jl2}}{2} = \frac{d_{jl}}{2c_{jl}}, \\
h_{jl} &= \frac{x_{jl1} - x_{jl2}}{2} = \sqrt{\frac{d_{jl}^2 - 4c_{jl}e_{jl}}{4c_{jl}^2}}.
\end{align*}
\] (A1)

\( R_{jl} \) is positive definite when \( \rho' \in (g_{jl} - h_{jl}, g_{jl} + h_{jl}) \).

Consider now \( \rho_j = \alpha^T X_j \) as specified by model (9), as functions of coefficients \( \alpha \) and covariates \( X_j \). Consider \( \rho'_l = \alpha'_{lm} X_{jm} + \sum_{k \neq m} \alpha_{lk} X_{jk} \) as implied by this model, treating \( \alpha'_{lm} \) for a single \( m = 1, \ldots, q \) as the argument of the function and fixing all the other elements of \( \alpha \) and \( X_j \) at the values which defined \( \rho_j \). Solving the end points of the feasible interval of \( \rho'_l \) for \( \alpha'_{lm} \), and taking into account the sign of \( X_{jm} \) gives the feasible interval for \( \alpha'_{lm} \) with end points \( a_{lm}^{(j)} \) and \( b_{lm}^{(j)} \) as shown in (10) in Proposition 2 when \( X_{jm} \neq 0 \). When \( X_{jm} = 0, f_{jl}(\rho') \) does not depend on \( \alpha'_{lm} \) and the interval can be taken to be infinite. The interval for \( \alpha_{lm} \) which is feasible for all of the \( X_j \in S_{X_T} \) is then 

\( (a_{lm}, b_{lm}) = \cap_j (a_{lm}^{(j)}, b_{lm}^{(j)}) \).

\( \square \)

**Appendix B  Details of the MCMC algorithm**

Here we describe the MCMC sampling algorithm for estimating the structural-model parameters of the model which was introduced in Section 3. The general idea of this estimation was outlined in Section 6.2, where we also described the sampling steps for the parameters \( \alpha \) of the model for the correlations of \( \eta_i \). As discussed there, the steps for the other elements of the model are the same or very similar to the ones proposed in Kuha et al. (2022). Their details are also given here in order to keep this description self-contained.

The algorithm has been packed into an R [R Core Team, 2020] package [which will be included in the supplementary materials and made available open source on an author’s GitHub page]. The algorithm was programmed in R with core functions implemented in C++, where two techniques are used to speed up the procedure. First, for sampling steps with non-standard distributions, adaptive rejection sampling (Gilks et al., 1995) is used, exploiting log-concavity of the posterior density functions. Second, parallel sampling is used within each MCMC iteration where possible. The parallelization is implemented through the OpenMP C++ API (Dagum & Menon, 1998).

Different elements of \( \zeta \) and \( \psi \) are sampled one at a time, as scalars or vectors as appropriate. In the notation below, those quantities that are not being sampled in a given step are taken to be observed and fixed at their most recently sampled values.

**Sampling the latent variables**: Generate values for the latent variables \( \zeta_i = (\xi_i^T, \eta_i^T)^T \), given the observed data and current values of the parameters \( \psi \). This can be parallelised, because \( \zeta_i \) for different units \( i \) are conditionally independent.

(1) Sampling \( \xi \) from \( p(\xi|\eta, Y, X, \psi) \): Draw \( \xi_i = (\xi_{Gi}, \xi_Ri)^T \) independently for \( i = 1, \ldots, n \), from multinomial distributions with probabilities

\[
p(\xi_G = j, \xi_R = k|\eta, Y_i, X_i, \psi) \propto p(Y_{Gi}|\xi_G = j, \eta_{Gi}) p(Y_{Ri}|\xi_R = k, \eta_{Ri}) p(\xi_G = j, \xi_R = k|X_i; \psi_{\xi})
\] (B1)
for \(j, k = 0, 1\), where the measurement model is specified by (11)-43 for \(Y_{Gi}\) and similarly for \(Y_{Ri}\), and the structural model for \(\xi_i\) is specified by (5).

(2) Sampling \(\eta\) from \(p(\eta|\xi, Y, X, \psi)\): Draw \(\eta_i = (\eta_{GP_i}, \eta_{RP_i}, \eta_{GF_i}, \eta_{RF_i})^T\) independently for \(i = 1, \ldots, n\), from

\[
\begin{align*}
  p(\eta_{GP}|\eta_{-GP_i}, \xi_i, Y_i, X_i, \psi) & \propto p(Y_{GP}|\xi_i, \eta_{GP})p(\eta_{GP}|\eta_{-GP_i}, X_i; \psi) \quad (B2) \\
p(\eta_{GF}|\eta_{-GF_i}, \xi_i, Y_i, X_i, \psi) & \propto p(Y_{GF}|\xi_i, \eta_{GF})p(\eta_{GF}|\eta_{-GF_i}, X_i; \psi) \quad (B3) \\
p(\eta_{RP}|\eta_{-RP_i}, \xi_i, Y_i, X_i, \psi) & \propto p(Y_{RP}|\xi_i, \eta_{RP})p(\eta_{RP}|\eta_{-RP_i}, X_i; \psi) \quad (B4) \\
p(\eta_{RF}|\eta_{-RF_i}, \xi_i, Y_i, X_i, \psi) & \propto p(Y_{RF}|\xi_i, \eta_{RF})p(\eta_{RF}|\eta_{-RF_i}, X_i; \psi). \quad (B5)
\end{align*}
\]

Here \(\eta_{-GP_i}\) denotes \((\eta_{GP_1}, \eta_{RP_1}, \eta_{RF_1})\) and \(\eta_{-GF_i}\), \(\eta_{-RP_i}\), and \(\eta_{-RF_i}\) are defined similarly. The conditional distributions for the \(\eta\)-variables on the right hand sides of (B2)-(B5) are the univariate conditional normal distributions implied by the joint normal distribution given by (36)-(38). The sampling distributions depend on the values of the \(\xi\)-variables. When \(\xi_{Gi} = 0\), in which case always \(Y_{Gi} = 0\), we have \(p(Y_{GP_i}|\xi_i, \eta_{GP}) = p(Y_{GF_i}|\xi_i, \eta_{GF}) = 1\) and \(\eta_{GP_i}\) and \(\eta_{GF_i}\) are drawn directly from the conditional normal distributions. When \(\xi_{Gi} = 1\), adaptive rejection sampling is used for \(\eta_{GP_i}\) and truncated normal sampling for \(\eta_{GF_i}\). The sampling of \(\eta_{RP_i}\) and \(\eta_{RF_i}\) is analogous.

**Sampling the parameters of the structural model**: Generate values for the parameters \(\psi\) from their distributions given the observed variables and current imputed values of the latent variables \(\zeta\). These have the form of posterior distributions of these structural parameters when both \(\zeta\) and \(X\) are taken to be observed data (this step does not depend on \(Y\)). The prior distributions are taken to be of the form \(p(\psi) = p(\psi_\xi)p(\sigma)p(\alpha)\), i.e. independent for different blocks of parameters; their specific forms are given below. The sampling steps for \(\psi_k\) and \(\psi_\eta\) do not depend on each other, so they can be carried out in either order or in parallel.

(3) Sampling \(\psi_k = (\gamma_{011}^T, \gamma_{101}^T, \gamma_{111}^T)^T\) from \(p(\psi_k|X, \xi) \propto p(\xi|X; \psi_k) p(\psi_k)\). This is the posterior distribution of the coefficients of the multinomial logistic model (5) for \(\eta_i\) given \(X_i\). Define \(\gamma = (\gamma_{011}^T, \gamma_{101}^T, \gamma_{111}^T)^T\), where \(\gamma_0 = \mathbf{0}\). Let \(\gamma_{jkr}\) denote the coefficient of \(X_{jkr}\) in the model for \(\xi = \mathbf{0}\), and \(\gamma_{-jkr}\) denote the vector obtained by omitting \(\gamma_{jkr}\) from \(\gamma\). We take the prior distributions of each non-zero \(\gamma_{jkr}\) to be independent of each other, with \(p(\gamma_{jkr}) \sim N(0, \sigma^2)\) with \(\sigma^2 = 100\). The sampling is done using conditional Gibbs sampling, one parameter at a time. We cycle over all \(r = 1, \ldots, Q\) and over \((j, k) = (0, 1), (1, 0), (1, 1)\) to draw \(\gamma_{jkr}\) from

\[
p(\gamma_{jkr}|\gamma_{-jkr}, X, \xi) \propto \left[ \prod_{r=0}^n \frac{\exp(\gamma^T_{r1} X_i)}{\sum_{r,s=0}^{1} \exp(\gamma^T_{rs} X_i)} \right] p(\gamma_{jkr}) \quad (B6)
\]

where \(\delta_{ijk} = 1(\xi_{Gi} = j, \xi_{Ri} = k)\). These are sampled using adaptive rejection sampling.

(4) Sampling \(\psi_\eta = (\text{vec}(\beta)^T, \sigma^T, \text{vec}((\alpha)^T))^T\) from \(p(\psi_\eta|X, \eta) \propto p(\eta|X; \psi_\eta) p(\psi_\eta)\). Here the sampling of \(\alpha\) has been described in Section (82). For \(\beta\), the sampling is from the posterior distribution \(p(\text{vec}(\beta)|X, \eta) \propto p(\eta|X; \psi_\eta) p(\beta|\text{vec}(\beta))\) where \(\sigma\) and \(\alpha\) are regarded as known. This means that the conditional covariance matrices \(\Sigma_i = \text{cov}(\eta_i|X_i; \sigma, \alpha)\) are also known here. We specify \(p(\text{vec}(\beta)) \sim N(0, \sigma^2 \text{I}_Q)\) with \(\sigma^2 = 100\). The sampling is done separately for each of the four subvectors of \(\beta\). Let \(\beta_1\) denote one of them, say \(\beta_1 = \beta_{GP}\), and \(\beta_2\) the rest of them, say \(\beta_2 = [\beta_{RP}, \beta_{GF}, \beta_{RF}]\), and let \(\psi_{\eta(\beta_1)}\) denote all the elements of \(\psi_{\eta}\) other than \(\beta_1\). Let \(\eta_i\) be partitioned correspondingly into \(\eta_{1i}\) and \(\eta_{2i}\), and \(\Sigma_i\) into the blocks \(\Sigma_{11i}, \Sigma_{12i}\) and \(\Sigma_{22i}\). The conditional distribution \(p(\eta_{1i}|\eta_{2i}, X_i; \psi_{\eta})\) is then univariate normal with mean \(\beta_1^T X_i + d_{2i}\), where \(d_{2i} = \Sigma_{12i}^{-1}(\eta_{2i} - \beta_2^T X_i)\), and variance \(\sigma_{2i}^2 = \Sigma_{11i} - \Sigma_{12i} \Sigma_{22i}^{-1} \Sigma_{12i}\). Let \(V_1 = \text{diag}((\sigma_{11i}^2, \ldots, \sigma_{1n}^2))\) and \(e_1 = (\eta_{11} - d_{21}, \ldots, \eta_{1n} - d_{2n})^T\).
The value of $\beta_1$ is then sampled from $p(\beta_1|X, \eta, \psi_{\eta(\beta_1)}) \sim N(V_{\beta_1}(X^TV_1^{-1}e_1), V_{\beta_1})$ where $V_{\beta_1} = (X^TV_1^{-1}X + I_Q/\sigma^2_{\beta_1})^{-1}$. This is repeated with each of the four subvectors of $\beta$ in turn in the role of $\beta_1$.

For sampling of the standard deviation parameters $\sigma$, denote here $\sigma_1 = \sigma_{GP}$ and $\sigma_2 = \sigma_{RP}$. For both of them we use the prior distribution Inv-Gamma$(\alpha_0, \beta_0)$ with $\alpha_0 = \beta_0 = 10^{-5}$, independently for $\sigma^2_1$ and $\sigma^2_2$. This implies the priors $p(\sigma_k) \propto \sigma_k^{-2\alpha_0 - 1} \exp(-\beta_0/\sigma^2_k)$ for $k = 1, 2$. Denote by $\psi_{\eta(\sigma)}$ all other parameters in $\psi_\eta$ apart from $\sigma_k$. Recall that this means that in $\Sigma_i = S R_i S$, where $S = \text{diag}(\sigma_1, \sigma_2, 1, 1)$, the correlation matrix $R_i = R(X_i; \alpha)$ is also treated as known here. Let $e_i = (e_{i1}, e_{i2}, e_{i3}, e_{i4})^T = \eta_i - \beta^T X_i$. The parameter $\sigma_k$ is then drawn from

$$p(\sigma_k|X, \eta, \psi_{\eta(\sigma)}) \propto \prod_{i=1}^n p(\eta_i|X_i; \psi_\eta)p(\sigma_k) \propto \prod_{i=1}^n \sigma_k^{-1} \exp\left(-\frac{1}{2} e_i^T \Sigma_i^{-1} e_i\right) p(\sigma_k)$$

$$\propto \sigma_k^{-\alpha - 1} \exp\left(-\beta_1/\sigma^2_k - 2\beta_2/\sigma_k\right),$$

where $\alpha = n + 2\alpha_0$, $\beta_1 = \beta_0 + (\sum_{i=1}^n e_{ik}^2 w_{kii})/2$, $\beta_2 = \sum_{i=1}^n e_{ik}(\sum_{j \neq k} w_{kj} e_{ij}/\sigma_j)/2$, and $w_{kji}$ is the $(k, j)$th element of $R_i^{-1}$. Then random-walk Metropolis sampler or the adaptive rejection Metropolis sampler (ARMS, Gilks et al., 1995) can be used to sample $\sigma_1$ and $\sigma_2$.

**Appendix**

Table 7: Estimated parameters (measurement loadings and intercepts) of the measurement models for survey items on help given by respondents to their parents and on help received from the parents.

| Item                                      | Giving practical help | Receiving practical help |
|-------------------------------------------|------------------------|--------------------------|
|                                           | loading | intercept | loading | intercept |
| Lifts in car                              | 1.12    | 0.83      | 1.14    | 1.54      |
| Shopping                                  | 2.38    | 1.02      | 1.70    | 2.08      |
| Providing or cooking meals                | 1.24    | -0.28     | 1.15    | 1.57      |
| Basic personal needs (to parent only)     | 1.32    | -1.32     | -       | -         |
| Looking after children (from parents only)| -       | -         | 0.89    | 2.25      |
| Washing, ironing or cleaning              | 1.32    | -0.77     | 1.15    | 0.82      |
| Personal affairs                          | 1.00    | 0.00      | 1.00    | 0.00      |
| Decorating, gardening or house repairs    | 0.57    | -0.22     | 0.74    | 0.37      |
Table 8: Estimated coefficients of the multinomial logistic model for the zero-inflation latent classes ($\xi_G; \xi_R$). The coefficients $\gamma_{00}$ are fixed at 0 for identification. The estimates are posterior means from MCMC samples (with posterior standard deviations in parentheses).

| Covariate | $\gamma_{01}$ | $\gamma_{10}$ | $\gamma_{11}$ |
|-----------|---------------|---------------|---------------|
| Intercept | -3.98*** (0.76) | -1.73* (1.08) | 1.56*** (0.52) |
| **Respondent (child) characteristics** | | | |
| Age (centered at 40) ($\times 10$) | -0.13 (0.21) | -0.44* (0.24) | -0.44*** (0.09) |
| Age squared ($\times 10^3$) | -2.63 (1.72) | 1.34 (0.93) | 1.54*** (0.46) |
| Gender | | | |
| Female (vs. Male) | 1.47*** (0.30) | 0.21 (0.17) | 0.06 (0.09) |
| Partnership status | | | |
| Partnered (vs. Single) | 0.13 (0.22) | 0.71*** (0.20) | -0.05 (0.10) |
| Age of youngest co-resident child (vs. No children): | | | |
| 0–1 years | 0.51 (0.35) | -0.60 (0.53) | -0.01 (0.17) |
| 2–4 years | 0.50 (0.32) | 0.10 (0.40) | 0.45*** (0.16) |
| 5–10 years | 0.43* (0.26) | -1.69*** (0.66) | 0.16 (0.14) |
| 11–16 years | -0.59* (0.30) | -0.03 (0.22) | -0.32** (0.14) |
| 17+ years | -0.24 (0.39) | -0.20 (0.24) | -0.11 (0.17) |
| Number of siblings (vs. None) | | | |
| 1 | -0.11 (0.28) | -0.49** (0.23) | 0.02 (0.15) |
| 2 | -0.70** (0.27) | -0.41* (0.22) | -0.23 (0.15) |
| Long standing illness (vs. No) | 0.00 (0.23) | -0.37* (0.21) | -0.27** (0.11) |
| Employment status (vs. Employed) | | | |
| Not employed | -0.25 (0.21) | 0.36* (0.19) | -0.34*** (0.10) |
| Education (vs. Secondary or less) | | | |
| Post-secondary | 0.63*** (0.18) | -0.05 (0.16) | 0.26** (0.09) |
| Household tenure (vs. Renter) | | | |
| Own home outright or by mortgage | -0.23 (0.21) | 0.36* (0.22) | 0.06 (0.10) |
| Logarithm of household equivalised income | 0.37*** (0.09) | 0.01 (0.11) | -0.01 (0.05) |
| **Parent characteristics** | | | |
| Age of the oldest living parent (centered at 70) ($\times 10$) | 0.37* (0.20) | 1.58*** (0.42) | 0.17** (0.07) |
| Squared Age of the oldest parent ($\times 10^3$) | -12.72*** (2.41) | -2.91** (1.50) | -0.37 (0.37) |
| At least one parent lives alone (vs. No) | -0.84*** (0.21) | 1.13*** (0.17) | 0.25*** (0.09) |
| **Child-parent characteristics** | | | |
| Travel time to the nearest parent | | | |
| More than 1 hour (vs. 1 hour or less) | -1.65*** (0.24) | -1.73*** (0.20) | -1.99*** (0.10) |

The posterior credible interval excludes zero at level 90% (*), 95% (**) or 99% (**).
Table 9: Fitted membership probabilities of the zero-inflation latent classes \((\xi_G; \xi_R)\), from the estimated model in Table S2. The fitted probabilities are averaged over parameter values in MCMC samples and over covariate values in the observed sample (for all covariates for the “Overall” figures, and for all but the fixed covariate for the rest. The odds ratios (OR) calculated from these averages are also shown.

| Covariate setting | \(p(\xi_G = j; \xi_R = k)\) | Marginal probabilities [with difference (and its SD)] | \(p(\xi_G = 1)\) | \(p(\xi_R = 1)\) |
|-------------------|-----------------|------------------|-----------------|-----------------|
| Overall           | .24 .09 .14 .53 | 10.6 .67          | 0.22/*       |                  |
| Respondent (child) characteristics | | | | |
| Age               | | | | |
| 35 years          | .22 .09 .16 .54 | 9.1 .70           | 0.28/*       |                  |
| 45 years          | .28 .10 .14 .48 | 9.5 .62           | -0.07*** (0.02) | 0.58 -0.05** (0.02) |
| 55 years          | .31 .07 .14 .48 | 18.2 .62          | -0.07*** (0.03) | 0.55 -0.08* (0.04) |
| Gender            | | | | |
| Female            | .23 .13 .14 .50 | 6.5 .65           | 0.26/*       |                  |
| Male              | .26 .04 .13 .56 | 30.0 .70          | +0.05*** (0.02) | 0.60 -0.03 (0.02) |
| Partnership status | | | | |
| Single            | .25 .08 .09 .57 | 19.7 .66          | 0.24/*       |                  |
| Partnered         | .24 .09 .15 .52 | 9.1 .67           | +0.01 (0.02) | 0.61 -0.05** (0.02) |
| Age of youngest coresident child | | | | |
| No children       | .24 .08 .15 .52 | 10.5 .68           | +0.05 (0.03) | 0.60 -0.05 (0.04) |
| 0-1 years         | .25 .12 .10 .53 | 13.1 .63          | 0.25/*       |                  |
| 2-4 years         | .19 .10 .13 .59 | 9.7 .72           | +0.09*** (0.03) | 0.68 +0.03 (0.04) |
| 5-10 years        | .24 .11 .04 .60 | 51.0 .64           | +0.02 (0.03) | 0.71 +0.06* (0.04) |
| 11-16 years       | .29 .06 .18 .48 | 13.8 .65           | +0.02 (0.03) | 0.54 -11*** (0.04) |
| 17+ years         | .26 .07 .14 .52 | 14.6 .66           | +0.03 (0.04) | 0.59 -0.06 (0.05) |
| Number of siblings| | | | |
| No sibling        | .21 .11 .17 .51 | 6.3 .68           | 0.21/*       |                  |
| 1 sibling         | .23 .10 .12 .55 | 10.3 .67           | -0.01 (0.02) | 0.65 +0.03 (0.03) |
| 2 or more         | .26 .07 .14 .52 | 13.0 .66           | -0.01 (0.02) | 0.59 -0.03 (0.03) |
| Longstanding illness | | | | |
| Yes               | .28 .10 .13 .49 | 11.4 .62          | 0.24/*       |                  |
| No                | .24 .09 .14 .53 | 10.4 .67           | +0.05*** (0.02) | 0.62 +0.02 (0.02) |
| Employment status | | | | |
| Not employed      | .27 .09 .18 .46 | 8.3 .65           | 0.27/*       |                  |
| Employed          | .24 .09 .13 .55 | 11.8 .67          | +0.03 (0.02) | 0.64 +0.09*** (0.02) |
| Education         | | | | |
| Secondary or less | .26 .07 .15 .52 | 13.4 .67          | 0.26/*       |                  |
| Post-secondary    | .23 .10 .13 .54 | 9.6 .67           | -0.00 (0.01) | 0.64 +0.05*** (0.02) |
| Household tenure  | | | | |
| Own home          | .24 .08 .15 .53 | 10.7 .68          | 0.25/*       |                  |
| Renter            | .25 .10 .12 .53 | 11.5 .64          | -0.03* (0.02) | 0.63 +0.02 (0.02) |
| Logarithm of household equivalised income | | | | |
| 25 percentile     | .25 .08 .14 .53 | 12.0 .67          | 0.25/*       |                  |
| 50 percentile     | .24 .09 .14 .53 | 10.7 .67           | -0.01** (0.00) | 0.62 +0.00 (0.00) |
| 75 percentile     | .24 .10 .14 .52 | 9.6 .66           | -0.01** (0.01) | 0.62 +0.00 (0.01) |
| Parent characteristics | | | | |
| Age of the oldest living parent | | | | |
| 65 years          | .29 .11 .04 .56 | 46.3 .60          | 0.29/*       |                  |
| 70 years          | .25 .15 .08 .52 | 12.6 .60           | +0.00 (0.01) | 0.67 +0.00 (0.01) |
| 80 years          | .22 .06 .20 .52 | 9.7 .72           | +12*** (0.02) | 0.58 -10*** (0.03) |
| At least one parent lives alone | | | | |
| Yes               | .22 .05 .19 .55 | 14.5 .74          | 0.22/*       |                  |
| No                | .27 .11 .09 .53 | 14.0 .62           | -11*** (0.01) | 0.64 +0.05** (0.02) |
| Child-parent characteristics | | | | |
| Travel time to the nearest parent | | | | |
| &gt; 1 hour       | .51 .07 .11 .39 | 22.7 .42          | 0.51/*       |                  |
| &lt;= 1 hour       | .15 .10 .15 .61 | 6.2 .76           | +34*** (0.02) | 0.70 +32*** (0.02) |

The posterior credible interval excludes zero at level 90% (*), 95% (**) or 99% (**).
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