Class numbers of orders in cubic fields

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Abstract. In this paper it is shown that the sum of class numbers of orders in complex cubic fields obeys an asymptotic law similar to the prime numbers as the bound on the regulators tends to infinity. Here only orders are considered which are maximal at two given primes. This result extends work of P. Sarnak in the real quadratic case. It seems to be the first asymptotic result on class numbers for number fields of degree higher than two.

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**Introduction**

Let $\mathcal{D}$ be the set of all natural numbers $D \equiv 0, 1 \mod (4)$, $D$ not a square. For $D \in \mathcal{D}$ let

$$O_D = \left\{ \frac{x + y\sqrt{D}}{2} \mid x \equiv yD \mod (2) \right\}.$$

Then $O_D$ is an order in the real quadratic field $\mathbb{Q}(\sqrt{D})$. As $D$ varies, $O_D$ runs through the set of all orders of real quadratic fields. Let $\epsilon_D > 1$ be the fundamental unit of $O_D$ and let $h(D)$ be the class number of $O_D$. C.F. Gauß noted in [9] that, as $x > 0$ tends to infinity,

$$\sum_{\substack{D \in \mathcal{D} \\ D \leq x}} h(D) \log(\epsilon_D) = \frac{\pi^2 x^\frac{3}{2}}{18 \zeta(3)} + O(x \log x).$$

This was confirmed later by C.L. Siegel in [18].

Note that $\log \epsilon_D$ equals the regulator $R(O_D)$ of the order $O_D$. For a long time it was believed to be impossible to separate the class number and the regulator. However, in 1981 P. Sarnak showed [17], using the trace formula, that

$$\sum_{\substack{D \in \mathcal{D} \\ D \leq x \\ \epsilon_D \leq x}} h(D) = \text{li}(x^2) + O(x^\frac{3}{2}(\log x)^2),$$

where

$$\text{li}(x) = \int_2^\infty \frac{1}{\log t} dt$$

is the integral logarithm as it appears in the prime number theorem (modulo the Riemann hypothesis). Let $r(O_D) = e^{2R(O_D)}$. Then Sarnak’s results can be rewritten as

$$\sum_{\substack{D \in \mathcal{D} \\ r(O_D) \leq x}} h(D) = \text{li}(x) + O \left( \frac{x^\frac{3}{4}(\log x)^2}{4} \right).$$
Sarnak established this result by identifying the regulators with lengths of closed geodesics of the modular curve $H/SL_2(\mathbb{Z})$ (Theorem 3.1 there) and by using the geodesic prime number theorem for this Riemannian surface.

In this paper we consider the class numbers of orders in cubic fields. To our knowledge, the following is the first asymptotic result on class numbers for number fields of higher degree.

A cubic field is either totally real, or it has one real and two complex embeddings. In the second case, we call it a complex cubic field. Let $p$ and $q$ be two distinct prime numbers and let $C(p, q)$ denote the set of all complex cubic number fields $F$, in which neither $p$ nor $q$ is decomposed; i.e. a complex cubic field $F$ lies in $C(p, q)$ if and only if there is only one prime of $F$ above $p$ and only one above $q$. Next let $O(p, q)$ be the set of all orders $\mathcal{O}$ of fields in $C(p, q)$, which are maximal at $p$ and at $q$. Let $h(\mathcal{O})$ denote the class number of such an order $\mathcal{O}$. Let $\lambda(\mathcal{O})$ be 1, if $p$ and $q$ are ramified in $F$, let it be 3 if only one of them is ramified and 9 if neither is ramified. For $x > 0$ set

$$
\pi_{p,q}(x) = \sum_{\mathcal{O} \in O(p,q) \atop r(\mathcal{O}) \leq x} h(\mathcal{O}) \lambda(\mathcal{O}), 
$$

where $r(\mathcal{O}) = e^{3R(\mathcal{O})}$, and $R(\mathcal{O})$ is the regulator of the order $\mathcal{O}$. The main result of this paper is

$$
\pi_{p,q}(x) \sim \frac{x}{\log x},
$$

as $x$ tends to infinity. More precisely we have

$$
\pi_{p,q}(x) = \text{li}(x) + O \left( \frac{x^{\frac{3}{4}}}{\log x} \right).
$$

The same conclusion holds if the set $\{p, q\}$ is replaced by an arbitrary finite set of primes $S$ with at least two elements. It is this latter version which we give in the text.

This paper grew out of a conversation with Peter Sarnak who urged me to look for applications of the theory developed in [6]. I thank him warmly for drawing my attention into this direction. I also thank Nigel Byott for some useful discussions.
1 The main theorem

Let $\mathcal{O}$ be an order in a number field $F$. Let $I(\mathcal{O})$ be the set of all finitely generated $\mathcal{O}$-submodules of $F$. According to the Jordan-Zassenhaus Theorem, the set of isomorphism classes $[I(\mathcal{O})]$ of elements of $I(\mathcal{O})$ is finite. Let $h(\mathcal{O})$ be the cardinality of the set $[I(\mathcal{O})]$, called the class number of $\mathcal{O}$.

The multiplicative group of invertible elements $F^\times$ of $F$ acts on $I(\mathcal{O})$ by multiplication: $\lambda M = \lambda M = M\lambda$. Since $M$ and $M\lambda$ are isomorphic as $\mathcal{O}$-modules we get a map $I(\mathcal{O})/F^\times \to [I(\mathcal{O})]$ mapping $MF^\times$ to its class $[M]$. We claim that this map is a bijection. It is clearly surjective. So let $M$ and $N$ be elements in $I(\mathcal{O})$ which are isomorphic. Fix an isomorphism $T: M \to N$. Then $T$ extends to an $F$-isomorphism $T_F: F = FM \to FN = F$, so there is $\alpha \in F^\times$ such that $T_F$ is just multiplication by $\alpha$. This implies the claim. It follows that the class number $h(\mathcal{O})$ equals the cardinality of $I(\mathcal{O})/F^\times$.

The class number of the maximal order is also called the class number of the field. In general the class number of $\mathcal{O}$ will be larger than that of $F$.

A number field $F$ of degree 3 over the rationals is also called a cubic field. Consider the set of embeddings of a given cubic field into $\mathbb{C}$. This set has three elements and is permuted by complex conjugation. It follows that either all three embeddings are fixed by complex conjugation, i.e. they are real in which case the field is called totally real or two of them are swapped by complex conjugation and the third is fixed, i.e. one is real and the other two are complex, in which case the field is called complex.

Let $F$ be a complex cubic field. Since the order of the automorphism group of $F$ divides the degree, which is 3, it is either 1 or 3. In the latter case the extension would be galois, so the Galois group would act transitively on the set of embeddings of $F$ into the complex numbers. In particular, all embeddings would be either real or complex. Since this is not the case it follows that the automorphism group of a complex cubic field is trivial.

Let $F$ be a complex cubic field and let $\mathcal{O}$ be an order of $F$. Then the group of units satisfies

$$\mathcal{O}^\times = \pm \epsilon \mathbb{Z}$$

for some base unit $\epsilon$. The image of the base unit under the real embedding
will be of the form \( \pm e^{\pm R(O)} \), where \( R(O) \) is the regulator of \( O \) (see [13]). Set

\[
    r(O) = e^{3R(O)}.
\]

A prime number \( p \) is called non-decomposed in \( F \) if there is only one place in \( F \) lying above \( p \). Fix a finite set \( S \) of prime numbers with at least two elements and let \( C(S) \) be the set of all complex cubic fields \( F \) such that all primes \( p \) in \( S \) are non-decomposed in \( F \). For \( F \in C(S) \) let \( O_F(S) \) be the set of all orders \( \mathcal{O} \) in \( F \) which are maximal at all \( p \in S \), i.e. are such that the completion \( \mathcal{O}_p = \mathcal{O} \otimes \mathbb{Z} \mathbb{Z}_p \) is the maximal order of the field \( F_p = F \otimes \mathbb{Q} \mathbb{Q}_p \) for all \( p \in S \). Let \( O(S) \) be the union of all \( O_F(S) \) where \( F \) ranges over \( C(S) \).

Let \( S_i(F) \) be the set of all \( p \in S \) such that \( p \) is inert in \( F \). Define

\[
    \lambda_S(F) = d^{\left| S_i(F) \right|} = \prod_{p \in S} f_p(F),
\]

where \( f_p(F) \) is the inertia degree of \( p \) in \( F \). For an order \( \mathcal{O} \) in \( F \) let

\[
    \lambda_S(\mathcal{O}) = \lambda_S(F).
\]

The following is our main theorem.

**Theorem 1.1** For \( x > 0 \) let

\[
    \pi_S(x) = \sum_{\substack{\mathcal{O} \in O(S) \\
                        r(O) \leq x}} h(\mathcal{O}) \lambda_S(\mathcal{O}).
\]

Then, as \( x \to \infty \) we have

\[
    \pi_S(x) \sim \frac{x}{\log x}.
\]

More sharply,

\[
    \pi_S(x) = li(x) + O \left( \frac{x^\frac{3}{4}}{\log x} \right),
\]

as \( x \to \infty \) where \( li(x) = \int_2^x \frac{1}{\log t} dt \) is the integral logarithm as in the usual prime number theorem.
As an immediate consequence we get the following corollary.

**Corollary 1.2** Let

\[ \tilde{\pi}_S(x) = \sum_{\mathcal{O} \subseteq \mathcal{O}(S) \atop r(\mathcal{O}) \leq x} h(\mathcal{O}), \]

then

\[ \limsup_{x \to \infty} \frac{\tilde{\pi}_S \log x}{x} \leq 1, \]

and

\[ \liminf_{x \to \infty} \frac{\tilde{\pi}_S \log x}{x} \geq \frac{1}{3^{|S|}}. \]

**Proof:** For any \( \mathcal{O} \) we have 1 \( \leq \lambda_S(\mathcal{O}) \leq 3^{|S|} \), which implies the corollary. Q.E.D.

The theorem will be proved in the following sections.

To clarify the range of the theorem we note:

**Proposition 1.3** For every complex cubic field \( F \) there are infinitely many primes \( p \) which are non-decomposed in \( F \). Every order \( \mathcal{O} \) in \( F \) is maximal at almost all primes \( p \).

**Proof:** The second assertion is well known and so it simply remains to prove the first. Let \( F \) be a complex cubic field. We will show that there are infinitely many primes \( p \) which are non-decomposed in \( F \). Let \( E/\mathbb{Q} \) be the galois hull of \( F \). Then \( \text{Gal}(E/\mathbb{Q}) \) is the permutation group \( S_3 \) in three letters. Let \( \sigma \) be the generator of \( \text{Gal}(E/F) \cong \mathbb{Z}/2\mathbb{Z} \) and let \( p \) be a prime which is unramified in \( E \) and such that there is a prime \( p_1 \) of \( E \) over \( p \) with Frobenius \( \tau \) of order 3. By the Tchebotarev density theorem there are infinitely many such \( p \). Then the stabilizer of \( p_1 \) in \( S_3 \) is \( \langle \tau \rangle \) of order 3, so \( p\mathcal{O}_E = p_1p_2 \), say. As \( \sigma \) does not stabilize \( p_1 \), it interchanges \( p_1 \) and \( p_2 \), so \( p_1 \cap F = p_2 \cap F \) is a prime in \( \mathcal{O}_F \), i.e. \( p \) is non-decomposed in \( F \). Q.E.D.
2 Division algebras of prime degree

Let $d$ be a prime $> 2$. Let $M(\mathbb{Q})$ denote a division algebra of degree $d$ over $\mathbb{Q}$. The dimension of $M(\mathbb{Q})$ is $d^2$. Fix a maximal order $M(\mathbb{Z}) \subset M(\mathbb{Q})$; for any commutative ring with unit $R$ we will write $M(R) = M(\mathbb{Z}) \otimes_{\mathbb{Z}} R$. Further, let $M(R)^\times$ denote the multiplicative group of invertible elements in $M(R)$. We say that $M(\mathbb{Q})$ splits over a prime $p$ if $M(\mathbb{Q}_p)$ is not a division algebra. Since $d$ is a prime we then get an isomorphism $M(\mathbb{Q}_p) \cong \text{Mat}_d(\mathbb{Q}_p)$ (see [14]). Further, since there are no associative division algebras over the reals of degree $d$ it follows that $M(\mathbb{R}) \cong \text{Mat}_d(\mathbb{R})$. For any ring $R$ the reduced norm induces a map $\text{det} : M(R) \to R$. Let $G(R) = \{x \in M(R) | \text{det}(x) = 1\}$. Then $G$ is a simple linear algebraic group over $\mathbb{Z}$. Let $G = G(\mathbb{R})$; then $G$ is isomorphic to $SL_d(\mathbb{R})$. Let $S$ be the set of places of $\mathbb{Q}$ where $M(\mathbb{Q})$ does not split. The set $S$ always is finite, has at least two elements, and contains only finite places. For any prime $p$ we have that $M(\mathbb{Z}_p)$ is a maximal $\mathbb{Z}_p$-order of $M(\mathbb{Q}_p)$.

Let $F/\mathbb{Q}$ be a finite field extension which embeds into $M(\mathbb{Q})$. By the Skolem-Noether Theorem ([14], Thm 7.21), any two embeddings $\sigma_1, \sigma_2 : F \to M(\mathbb{Q})$ are conjugate by $M(\mathbb{Q})^\times$, i.e. there is a $u \in M(\mathbb{Q})^\times$ such that $\sigma_2(x) = u\sigma_1(x)u^{-1}$ for any $x \in F$.

A prime $p$ is called non-decomposed in the extension $F/\mathbb{Q}$ if there is only one place $v$ of $F$ lying above $p$.

**Lemma 2.1** Let $F/\mathbb{Q}$ be a nontrivial finite field extension. Then $F$ embeds into $M(\mathbb{Q})$ if and only if $[F : \mathbb{Q}] = d$ and every prime $p$ in $S$ is non-decomposed in $F$.

**Proof:** Assume that $F$ embeds into $M(\mathbb{Q})$; then $[F : \mathbb{Q}]$ divides $d$, which is a prime, so that $[F : \mathbb{Q}] = d$. It follows that $F$ is a maximal subfield of $M(\mathbb{Q})$. By Proposition 13.3 in [14] it follows that $F$ is a splitting field of $M(\mathbb{Q})$. Thus, by Theorem 32.15 of [14] it follows that for every $p \in S$ and any place $v$ of $F$ over $p$ we have $[F_v : \mathbb{Q}_p] = d$, which by the degree formula implies that there is only one $v$ over $p$. Conversely, assume that there is only one prime above each $p \in S$. Let $p \in S$ and let $e$ be the ramification index of $p$ in $F$ and $f$ the inertia degree. Then $d = [F : \mathbb{Q}] = ef = [F_v : \mathbb{Q}_p]$.
by the global, respectively local, degree formula. Since this holds for any $p \in S$, Theorem 32.15 implies that $F$ is a splitting field for $M(\mathbb{Q})$. By Proposition 13.3 of [14] we infer that $F$ embeds into $M(\mathbb{Q})$. Q.E.D.

Let $F/\mathbb{Q}$ be a field extension of degree $d$ which embeds into $M(\mathbb{Q})$. Then for any embedding $\sigma : F \to M(\mathbb{Q})$ the set

$$\mathcal{O}_\sigma = \sigma^{-1}(\sigma(F) \cap M(\mathbb{Z}))$$

is an order of $F$. For $p \in S$ let $v_p$ denote the unique place of $F$ over $p$.

**Lemma 2.2** Let $\sigma : F \to M(\mathbb{Q})$ be an embedding of the field $F$. For any $p \in S$ the completion $\mathcal{O}_{\sigma,v_p}$ is a maximal order in $F_{v_p}$. Conversely, let $\mathcal{O} \subset F$ be an order such that for any $p \in S$ the completion $\mathcal{O}_{v_p}$ is maximal. Then there is an embedding $\sigma : F \to M(\mathbb{Q})$, such that $\mathcal{O} = \mathcal{O}_\sigma$.

**Proof:** Let $p \in S$; then $M(\mathbb{Z}_p)$ is a maximal $\mathbb{Z}_p$-order in the division algebra $M(\mathbb{Q}_p)$, and hence by Theorem 12.8 of [14] it coincides with the integral closure of $\mathbb{Z}_p$ in $M(\mathbb{Q}_p)$. Therefore $\mathcal{O}_{\sigma,v_p} = \sigma^{-1}(\sigma(F_{v_p} \cap M(\mathbb{Z}_p)))$ is the integral closure of $\mathbb{Z}_p$ in $F_{v_p}$, which is the maximal order of $F_{v_p}$.

For the converse, let $\mathcal{O}$ be an order of $F$ such that the completion $\mathcal{O}_{v_p}$ is maximal for each $p \in S$. Fix an embedding $F \hookrightarrow M(\mathbb{Q})$ and consider $F$ as a subfield of $M(\mathbb{Q})$. For any $u \in M(\mathbb{Q})^\times$ let $\mathcal{O}_u = F \cap u^{-1}M(\mathbb{Z})u$. We will show that there is a $u \in M(\mathbb{Q})^\times$ such that $\mathcal{O} = \mathcal{O}_u$. This will prove the proposition since one can then take $\sigma$ to be the conjugation by $u$.

Let $\mathcal{O}_1 = F \cap \mathcal{M}(\mathbb{Z})$. Since $\mathcal{O}$ and $\mathcal{O}_1$ are orders, they both are maximal at all but finitely many places. So there is a finite set of primes $T$ with $T \cap S = \emptyset$ and such that with $T_F$ denoting the set of places of $F$ lying over $T$ we have that for any place $v$ of $F$ with $v \notin T_F$ the completion $\mathcal{O}_v$ is maximal and equals $\mathcal{O}_{1,v}$. Let $p \in T$ and fix an isomorphism $M(\mathbb{Q}_p) \to \text{Mat}_d(\mathbb{Q}_p)$. Fix a $\mathbb{Z}_p$-basis of $\mathcal{O} \otimes \mathbb{Z}_p$. This basis then induces an embedding $\sigma_p : F \otimes \mathbb{Q}_p \to \text{Mat}_d(\mathbb{Q}_p) = M(\mathbb{Q}_p)$, such that

$$\sigma_p^{-1}(\sigma_p(F \otimes \mathbb{Q}_p) \cap \text{Mat}_d(\mathbb{Z}_p)) = \mathcal{O} \otimes \mathbb{Z}_p.$$
By the Noether-Skolem Theorem [10] 7.21 there is a \(\tilde{u}_p \in M(\mathbb{Q}_p^\times)\) such that 
\[ \mathcal{O} \otimes \mathbb{Z}_p = F_p \cap \tilde{u}_p^{-1}M(\mathbb{Z}_p)\tilde{u}_p. \]
For \(p \notin T\) set \(\tilde{u}_p = 1\) and let \(\tilde{u} = (\tilde{u}_p) \in M(\mathbb{A}_{fin})\), where \(\mathbb{A}_{fin}\) is the ring of the finite adeles over \(\mathbb{Q}\), i.e. the restricted product over the local fields \(\mathbb{Q}_p\), where \(p\) ranges over the primes. By strong approximation \(M(\mathbb{Q})^\times\) is dense in \(M(\mathbb{A}_{fin})\) and so there is \(u \in M(\mathbb{Q})^\times\) such that 
\[ M(\hat{\mathbb{Z}})u = M(\hat{\mathbb{Z}})\tilde{u}, \]
where \(\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p\). It follows that \(\mathcal{O} = \mathcal{O}_u\) for this \(u\).

Q.E.D.

Let \(F\) be a field extension of \(\mathbb{Q}\) of degree \(d\) which embeds into \(M(\mathbb{Q})\). Let \(\mathcal{O}\) be an order of \(F\), which is maximal at each place in \(S\). By Lemma 2.2 we know that there is an embedding \(\sigma\) of \(F\) into \(M(\mathbb{Q})\) such that \(\mathcal{O} = \mathcal{O}_\sigma\). Let \(u \in M(\mathbb{Z})^\times\) and let \(u^\sigma = u\sigma u^{-1}\). Then \(\mathcal{O}_u = \mathcal{O}_\sigma\), so the group \(M(\mathbb{Z})^\times\) acts on the set \(\Sigma(\mathcal{O})\) of all \(\sigma\) with \(\mathcal{O} = \mathcal{O}_\sigma\).

**Lemma 2.3** The quotient \(\Sigma(\mathcal{O})/M(\mathbb{Z})^\times\) is finite and has cardinality equal to the product \(h(\mathcal{O})\lambda_S(\mathcal{O})\).

**Proof:** Fix an embedding \(F \rightarrow M(\mathbb{Q})\) and consider \(F\) as a subfield of \(M(\mathbb{Q})\) such that \(\mathcal{O} = F \cap M(\mathbb{Z})\). For \(u \in M(\mathbb{Q})^\times\) let 
\[ \mathcal{O}_u = F \cap u^{-1}M(\mathbb{Z})u. \]
Let \(U\) be the set of all \(u \in M(\mathbb{Q})^\times\) such that 
\[ \mathcal{O} = F \cap M(\mathbb{Z}) = F \cap u^{-1}M(\mathbb{Z})u. \]
Then \(F^\times\) acts on \(U\) by multiplication from the right and \(M(\mathbb{Z})^\times\) acts by multiplication from the left. It is clear that 
\[ |M(\mathbb{Z})^\times \setminus U/F^\times| = |M(\mathbb{Z})^\times \setminus \Sigma(\mathcal{O})|. \]
So we only have to show that the left hand side equals \(h(\mathcal{O})\lambda_S(\mathcal{O})\). For \(u \in U\) let 
\[ I_u = F \cap M(\mathbb{Z})u. \]
Then \(I_u\) is a finitely generated \(\mathcal{O}\)-module in \(F\). We claim that the map \(\Psi:\)
\[ M(\mathbb{Z})^\times \setminus U/F^\times \rightarrow I(\mathcal{O})/F^\times \]
\[ u \mapsto I_u, \]

is surjective and $\lambda_S(O)$ to one. We will show this through localization and strong approximation. So, for a prime $p$ let $U_p$ be the set of all $u_p \in M(Q_p) \times$ such that $O_p = F_p \cap M(Z_p) = F_p \cap u_p^{-1}M(Z_p)u_p$. We have to show the following:

1. For $p \notin S$ the localized map $\Psi_p : M(Z_p) \times \backslash U_p/F_p^\times \to I(O_p)/F_p^\times$ is injective.

2. For $p \in S$ the map $\Psi_p$ is $f_p(F)$ to one.

3. The map $\Psi$ is surjective.

For ‘1.’ let $p \notin S$, let $u_p, v_p \in U_p$, and assume

$$F_p \cap M(Z_p)u_p = F_p \cap M(Z_p)v_p.$$ 

Let $z_p = v_pu_p^{-1}$. Elementary divisor theory implies that there are $x, y \in M(Z_p)^\times = Mat_d(Z_p)^\times$ such that

$$z_p = x\text{diag}(p^{k_1}, \ldots, p^{k_d})y,$$

where diag denotes the diagonal matrix and $k_1 \leq k_2 \leq \cdots \leq k_d$ are integers. Replacing $u_p$ by $yu_p$ and $v_p$ by $x^{-1}v_p$ we may assume that $z$ equals the diagonal matrix. The assumptions then easily imply $k_1 = \cdots = k_d = 0$, which gives the first claim.

For ‘2.’ let $p \in S$ and recall that $F_p$ is a local field and so $h(O_p) = 1$. So the claim is equivalent to

$$|M(Z_p)^\times \backslash M(Q_p)^\times /F_p^\times| = f_p(F).$$

By Proposition 17.7 of [11], it follows that

$$|M(Z_p)^\times \backslash M(Q_p)^\times| = d = e(M(Q_p)/Q_p),$$

where $e$ denotes the ramification index. If $F_p/Q_p$ is ramified, then $f_p(F) = 1$ and $|M(Z_p)^\times \backslash M(Q_p)^\times /F_p^\times| = 1$. If $F_p/Q_p$ is unramified, then $f_p(F) = d$ and $|M(Z_p)^\times \backslash M(Q_p)^\times /F_p^\times| = d$ as claimed.
For the surjectivity of $\Psi$ let $I \subset O$ be an ideal. We shall show that there is a $u \in M(\mathbb{Q})^\times$ such that

$$F \cap u^{-1}M(\mathbb{Z}) u = F \cap M(\mathbb{Z})$$

and

$$I = I_u = F \cap M(\mathbb{Z}) u.$$

We shall do this locally. First note that, since $I$ is finitely generated, there is a finite set of primes $T$ with $T \cap S = \emptyset$ such that for any $p \notin T \cup S$ the completion $I_p$ equals $O_p$ which is the maximal order of $F_p$. For these $p$ set $\tilde{u}_p = 1$.

Next let $p \in S$. Let $v_p$ be the unique place of $F$ over $p$. Then $O_p = \mathcal{O}_{v_p}$ is maximal, so it is the valuation ring to $v_p$ and $I_p = \pi^k_p O_p$ for some $k \geq 0$, where $\pi_p$ is a uniformizing element in $O_p$. It follows that at this $p$, the element $\tilde{u}_p = \pi^k_p Id$ will do the job.

Next let $p \in T$. Then $M(\mathbb{Z}_p) = \text{Mat}_d(\mathbb{Z}_p)$. Let $\overline{O}_p = O_p/pO_p$ and $\overline{T}_p = I_p/pI_p$. Then $\overline{O}_p$ is a commutative algebra over the field $\mathbb{F}_p$ with $p$ elements, which implies that $\overline{O}_p \cong \bigoplus_{i=1}^s F_i$, where each $F_i$ is a finite field extension of $\mathbb{F}_p$. Let $n_i$ be the degree of $F_i/\mathbb{F}_p$. Then there is an embedding $\overline{O}_p \hookrightarrow \text{Mat}_d(\mathbb{F}_p)$ whose image lies in $\text{Mat}_{n_1}(\mathbb{F}_p) \times \cdots \times \text{Mat}_{n_s}(\mathbb{F}_p) \subset \text{Mat}_d(\mathbb{F}_p)$. According to the Noether-Skolem Theorem there is a $\tilde{S} \in GL_d(\mathbb{F}_p)$ such that $\tilde{S}\overline{O}_p\tilde{S}^{-1} \subset \text{Mat}_{n_1}(\mathbb{F}_p) \times \cdots \times \text{Mat}_{n_s}(\mathbb{F}_p) \subset \text{Mat}_d(\mathbb{F}_p)$. The $\overline{O}_p$-ideal $\overline{T}_p$ must be of the form

$$\overline{T}_p = \bigoplus_{i=1}^s \epsilon_i F_i,$$

where $\epsilon_i \in \{0, 1\}$. Let $S$ be a matrix in $GL_d(\mathbb{Z}_p)$ which reduces to $\tilde{S}$ modulo $p$ and let $\tilde{u}_p = S^{-1}(p^{\epsilon_1}Id_{n_1} \times \cdots \times p^{\epsilon_s}Id_{n_s})S$ in $\text{Mat}_d(\mathbb{Z}_p)$. By abuse of notation we also write $\tilde{u}_p$ for its reduction modulo $p$. Then we have

$$\overline{T}_p = \overline{O}_p \cap \text{Mat}_d(\mathbb{F}_p) \tilde{u}_p.$$

Let

$$I_{\tilde{u}_p} = F \cap M(\mathbb{Z}_p) \tilde{u}_p.$$

Then it follows that

$$\overline{T}_p \cong \overline{T}_{\tilde{u}_p} = I_{\tilde{u}_p}/pI_{\tilde{u}_p}$$
and by Theorem 18.6 of [16] it follows that $I_p \cong I_{\tilde{u}_p}$, which implies that there is some $\lambda \in F_p$ such that $I_p = I_{\tilde{u}_p} \lambda$. Replacing $\tilde{u}_p$ by $\tilde{u}_p \lambda$ and setting $\tilde{u} = (\tilde{u}_p)_p \in M(A_{\text{fin}})$ we get

$$I = F \cap M(\mathbb{Z})\tilde{u}.$$ 

By strong approximation there is a $u \in M(\mathbb{Q})^\times$ such that $M(\hat{\mathbb{Z}})u = M(\hat{\mathbb{Z}})\tilde{u}$ and therefore $I = I_u$.

Q.E.D.

We will summarize the results of this section in the following proposition.

**Proposition 2.4** Let $d$ be a prime $> 2$ and let $F/\mathbb{Q}$ be an extension of degree $d$. Then $F$ embeds into the division algebra $M(\mathbb{Q})$ of degree $d$ if and only if every prime $p$ at which $M(\mathbb{Q})$ does not split is non-decomposed in $F$. Every embedding $\sigma : F \to M(\mathbb{Q})$ gives by intersection with $M(\mathbb{Z})$ an order $O_\sigma$ in $F$. Every order $O$ of $F$, which is maximal at each $p$ where $M(\mathbb{Q})$ is non-split, occurs in this way. The number of $M(\mathbb{Z})^\times$-conjugacy classes of embeddings giving rise to the same order $O$ is equal to $h(O)\lambda_S(O)$.

### 3 Analysis of the Ruelle zeta function

From now on we restrict to the case $d = 3$. Let $\Gamma = \mathcal{G}(\mathbb{Z})$; then $\Gamma$ forms a discrete subgroup of $G = \mathcal{G}(\mathbb{R}) \cong SL_3(\mathbb{R})$. Since $M(\mathbb{Q})$ is a division algebra it follows that $\mathcal{G}$ is anisotropic over $\mathbb{Q}$ and so $\Gamma$ is cocompact in $G$. Let $g_0$ be the real Lie algebra of $G$ and let $g = g_0 \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. A subgroup $\Sigma$ of $G$ is called weakly neat if it is torsion free and for each $\sigma \in \Sigma$ the adjoint $Ad(\sigma) \in GL(g)$ does not have a root of unity except 1 as an eigenvalue. In other words, a torsion free group $\Sigma$ is weakly neat if for every $\sigma \in \Sigma$ and every $n \in \mathbb{N}$ the connected component $G^0_\sigma$ of the centralizer of $\sigma$ coincides with the connected component $G^0_{\sigma^n}$ of the centralizer of $\sigma^n$. Further an element $x$ of $G$ is called regular if its centralizer is a torus.

**Lemma 3.1** The group $\Gamma$ is torsion-free and every $\gamma \in \Gamma$ with $\gamma \neq 1$ is regular. In particular, it follows that $\Gamma$ is weakly neat.
Proof: Let $\gamma \in \Gamma$, $\gamma \neq 1$; then the centralizer $M(\mathbb{Q})_\gamma$ of $\gamma$ in $M(\mathbb{Q})$ is a division subalgebra of $M(\mathbb{Q})$ and so it is either $\mathbb{Q}$, a cubic number field or $M(\mathbb{Q})$. It cannot be $\mathbb{Q}$ since it contains $\gamma$ and if $\gamma$ is in $\mathbb{Q}$, it is central so its centralizer is $M(\mathbb{Q})$. It can neither be $M(\mathbb{Q})$, since then $\gamma$ would be in $\mathbb{Q}1$, say $\gamma = r1$ and then $1 = \det(\gamma) = r^3$, thus $\gamma = 1$ which was excluded. So it follows that $M(\mathbb{Q})_\gamma$ is a cubic field $F$. Note that this implies that $\gamma$ cannot be a root of unity, since a cubic field does not contain roots of unity other than $\pm 1$. This shows that $\Gamma$ is torsion free. Moreover, we get $M(\mathbb{R})_\gamma = M(\mathbb{Q})_\gamma \otimes \mathbb{Q} \mathbb{R}$ so $G_\gamma = (F \otimes \mathbb{R})^1$, the norm one elements, this is a torus, so $\gamma$ is regular.

Q.E.D.

We have $\mathfrak{g}_0 = sl_3(\mathbb{R})$ and $\mathfrak{g} = sl_3(\mathbb{C})$. Let $b$ be the Killing form on $\mathfrak{g}$. Then

$$b(X,Y) = \text{tr} \text{ad}(X)\text{ad}(Y) = 6 \text{tr} (XY).$$

Let $K \subset G$ be the maximal compact subgroup $SO(3)$. Let $\mathfrak{k}_0 \subset \mathfrak{g}_0$ be its Lie algebra and let $\mathfrak{p}_0 \subset \mathfrak{g}_0$ be the orthogonal space of $\mathfrak{k}_0$ with respect to the form $b$. Then $b$ is positive definite on $\mathfrak{p}_0$ and thus defines a $G$-invariant metric on $X = G/K$, the symmetric space attached to $G$.

Since $\Gamma$ is torsion-free it acts without fixed points on the contractible space $X$, so $X_\Gamma = \Gamma \backslash X$ is the classifying space of $\Gamma$, in particular, it follows that $\Gamma$ is the fundamental group of $X_\Gamma$. We thus obtain a natural bijection

$$\{\text{free homotopy classes of maps } S^1 \to X_\Gamma\} \leftrightarrow \{\text{conjugacy classes } [\gamma] \text{ in } \Gamma\}.$$  

For each conjugacy class $[\gamma]$ let $X_\gamma$ denote the union of all geodesics lying in the homotopy class $[\gamma]$ of $\gamma$. It is known [7] that all geodesics in $[\gamma]$ have the same length $l_\gamma$ and $X_\gamma$ is a submanifold of $X_\Gamma$ diffeomorphic to $\Gamma_\gamma \backslash G_\gamma/K_\gamma$, where $G_\gamma$ and $\Gamma_\gamma$ are the centralizers of $\gamma$ and $K_\gamma$ is a maximal compact subgroup of $G_\gamma$.

An element $\gamma \in \Gamma$ is called primitive if for $\sigma \in \Gamma$ and $n \in \mathbb{N}$ the equation $\sigma^n = \gamma$ implies that $n = 1$. Since every closed geodesic is a positive power of a unique primitive one it is easy to see that every $\gamma \in \Gamma$ with $\gamma \neq 1$ is a positive power of a unique primitive element $\gamma_0$. We write $\gamma = \gamma_0^{\mu(\gamma)}$ and call $\mu(\gamma)$ the multiplicity of $\gamma$. Clearly primitivity is a property of the conjugacy class.
Up to conjugacy the group $G$ has two Cartan subgroups, namely the group of diagonal matrices and the group $H = AB$, where

$$A = \left\{ \begin{pmatrix} a & \cdot & \cdot \\ \cdot & a & \cdot \\ \cdot & \cdot & a^{-2} \end{pmatrix} \middle| a > 0 \right\}$$

and

$$B = \begin{pmatrix} SO(2) \\ 1 \end{pmatrix}.$$ 

Let $P$ denote the parabolic $\begin{pmatrix} * & * \\ 0 & 0 & * \end{pmatrix}$. It has a Langlands decomposition $P = MAN$ and $B$ is a compact Cartan subgroup of

$$M \cong SL_2^\pm(\mathbb{R}) = \{ x \in Mat_2(\mathbb{R}) | \det(x) = \pm 1 \}.$$ 

Let

$$H_1 = \frac{1}{6} \begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} \in \mathfrak{a}_0 = \text{Lie } A.$$ 

Then it follows that $b(H_1) = b(H_1, H_1) = 1$. Let $A^- = \{ \exp(tH_1) | t > 0 \}$ and let $\mathcal{E}_P(\Gamma)$ be the set of all conjugacy classes $[\gamma]$ in $\Gamma$ such that $\gamma$ is conjugate in $G$ to an element $a_\gamma b_\gamma$ of $A^- B$, and let $\mathcal{E}_P^0(\Gamma)$ the set of all primitive elements therein. For $s \in \mathbb{C}$ with $\text{Re}(s) > 0$ let

$$R_\Gamma(s) = \prod_{[\gamma] \in E_P^0(\Gamma)} (1 - e^{-sl_\gamma})$$

be the *Ruelle-zeta function* attached to $P$. In [3] it is shown that $R_\Gamma(s)$ converges for $\text{Re}(s) > 0$ and that it extends to a meromorphic function of finite order on the plane. We will show:

**Theorem 3.2** The function $R_\Gamma(s)$ has a simple zero at $s = 1$. Apart from that, all poles and zeros of $R_\Gamma(s)$ are contained in the union of the interval $[-1, \frac{3}{2}]$ with the three vertical lines given by $-\frac{1}{2} + i\mathbb{R}$, $i\mathbb{R}$ and $\frac{1}{2} + i\mathbb{R}$.
Proof: For any finite dimensional representation \( \sigma \) of \( M \) let

\[
Z_{P,\sigma}(s) = \prod_{[\gamma] \in \mathcal{E}_P(G)} \prod_{n \geq 0} \det(1 - e^{-st}, \sigma(b_{\gamma})S^n(a_{\gamma}b_{\gamma}|n))
\]

where \( n = \text{Lie}_C(N) \) and \( S^n(a_{\gamma}b_{\gamma}|n) \) is the \( n \)-th symmetric power of the adjoint action of \( a_{\gamma}b_{\gamma} \) on \( n \). In [3], Theorem 2.1 it is shown that \( Z_{P,\sigma} \) extends to a meromorphic function and that all its poles and zeros lie in \( \mathbb{R} \cup \left( \frac{1}{2} + i\mathbb{R} \right) \).

Note that \( M \) is isomorphic with \( \text{SL}_2^\pm(\mathbb{R}) \), the group of real \( 2 \times 2 \) matrices of determinant \( \pm 1 \). Let \( \sigma_0 : M \to GL_2 \) denote the standard representation. In [6], Theorem 4.1 it is shown that

\[
\mathcal{R}_\Gamma(s) = \frac{Z_{P,1}(s)Z_{P,1}(s+1)}{Z_{P,\sigma_0}(s+\frac{1}{2})}.
\]

So to complete the proof, it suffices to show the following proposition.

**Proposition 3.3** For \( \sigma = 1 \) the poles and zeros of \( Z_{P,\sigma}(s) \) lie in \( [\frac{1}{4}, \frac{3}{4}] \cup \left( \frac{1}{2} + i\mathbb{R} \right) \cup \{0, 1\} \) and the function \( Z_{P,\sigma}(s) \) has a simple zero at \( s = 1 \).

For \( \sigma = \sigma_0 \) the poles and zeros of \( Z_{P,\sigma} \) all lie in \( [0, 1] \cup \left( \frac{1}{2} + i\mathbb{R} \right) \).

Proof: Let \( \hat{G} \) denote the set of all isomorphism classes of irreducible unitary representations of \( G \). The group \( G \) acts on the Hilbert space \( L^2(\Gamma\backslash G) \) by translations from the right. Since \( \Gamma\backslash G \) is compact this representation decomposes discretely:

\[
L^2(\Gamma\backslash G) = \bigoplus_{\pi \in \hat{G}} N_\Gamma(\pi)\pi,
\]

with finite multiplicities \( N_\Gamma(\pi) \). For \( \pi \in \hat{G} \) let \( \pi_K \) denote the \((g, K)\)-module of \( K \)-finite vectors in \( \pi \). Then the Lie algebra \( n \) acts on \( \pi_K \) and we denote by \( H^q(n, \pi_K) \) the corresponding Lie algebra cohomology [1]. Let \( m \) denote the complexified Lie algebra of \( M \) and let \( m = \mathfrak{k}_M \oplus \mathfrak{p}_M \) be its polar decomposition, where \( \mathfrak{k}_M \) is the complexified Lie algebra of \( K_M = K \cap M \).

In [3] Theorem 2.1 it is shown that all poles and zeros of \( Z_{P,\sigma} \) lie in \( \mathbb{R} \cup (\frac{1}{2} + i\mathbb{R}) \) and that for \( \lambda \in \mathfrak{a}^* \) the (vanishing-) order of \( Z_{P,\sigma} \) at \( s = \lambda(H_1) \) is

\[
\sum_{\pi \in \hat{G}} N_\Gamma(\pi) \sum_{p, q \geq 0} (-1)^{p+q} \dim(H^q(n, \pi_K) \otimes \wedge^p \mathfrak{p}_M \otimes \mathfrak{v}_\lambda|_{\mathfrak{k}M})^\mathbb{K}_M,
\]
where \( (\cdot)_\lambda \) denotes the generalized \( \lambda \)-eigenspace. Note that the torus \( A \) acts trivially on all tensor factors except \( H^q(n, \pi_K) \). Let \( \pi \in \hat{G} \) and let \( \wedge_\pi \in \mathfrak{h}^* \) be a representative of its infinitesimal character. Corollary 3.23 of [10] says that every character of the \( a \)-action on \( H^q(n, \pi_K) \) is of the form \( \mu = w \wedge_\pi + \rho_P \) for some \( w \in W(g, \mathfrak{h}) \).

To show the proposition we concentrate on \( \sigma \) being trivial since the other case is similar. Since \( \rho(H_1) = - \frac{1}{2} \) we have to show that for every \( \pi \) which has a nonzero contribution, all eigenvalues \( \lambda \) of \( a \) on \( H_p(n, \pi_K) \) satisfy \( -\frac{3}{2} \rho_P \leq \text{Re}(\lambda) \leq -\frac{1}{2} \). By the isomorphism of \( AM \)-modules [10], p. 57:

\[
H_p(n, \pi_K) \cong H^{2-p}(n, \pi_K) \otimes \wedge^2 n
\]

this becomes equivalent to \( \frac{1}{2} \leq \text{Re}(\lambda) \leq \frac{3}{2} \rho_P \) whenever \( \lambda \) is an eigenvalue of \( a \) on \( H_p(n, \pi_K) \). Now fix \( \pi \in \hat{G} \) and let \( \wedge_\pi \in \mathfrak{h}^* \) be a representative of its infinitesimal character. Then, according to Corollary 3.32 of [10] we have to show that \( -\frac{1}{2} \rho_P \leq \text{Re}(\wedge_\pi|_a) \leq \frac{1}{2} \rho_P \).

In the case when \( \pi \) is induced from the minimal parabolic it follows that its distributional character \( \Theta_\pi \) is zero on \( AB \). By the construction of the test function in [3], p. 903, this implies that the contribution of \( \pi \) is zero. Therefore, by the classification of the unitary dual in [19], it remains to consider the case of the trivial representation and the case when \( \pi \) is unitarily induced from \( P = MAN \). So let \( \pi = \pi_{\xi,\nu} \) be induced from \( P \), where \( \nu \) is imaginary. Then we may assume that \( \xi \) is not induced, since otherwise the double induction formula would lead us back to the previous case. Let \( w \in W \) and \( \wedge_\xi \) be the infinitesimal character of \( \xi \). We lift \( \wedge_\xi \) to \( \mathfrak{d} \) by defining it to be zero on \( a \). Then the Weyl group \( W \) will act on \( \wedge_\xi \). We have to show that

\[
-\frac{1}{2} \rho_P \leq \text{Re}(w \wedge_\xi|_a) \leq \frac{1}{2} \rho_P.
\]

Let us start with \( \xi \) being the trivial representation. Then \( \wedge_\xi \left( \begin{array}{cc} t & -t \\ a & b \end{array} \right) \).

Lifting \( \wedge_\xi \) to \( \mathfrak{d} \) we get

\[
\wedge_\xi \left( \begin{array}{cc} a & b \\ b & c \end{array} \right) = \frac{1}{2}(a - b).
\]

This vanishes on \( A \), so it’s real part is zero. This deals with the case when \( w = 1 \). For \( w \in W \) being the transposition interchanging \( b \) and \( c \) we get

\[
\wedge_\xi \left( \begin{array}{cc} a & b \\ b & c \end{array} \right) = \frac{1}{2}(a - c),
\]
which, restricted to $\mathfrak{a}$, coincides with $\frac{1}{2}\rho$. This implies the claim for this $w$. All other Weyl group elements are treated similarly.

It remains to consider the case when $\xi$ is a (limit of) discrete series representation, so $\xi = \mathcal{D}_0^+ \oplus \mathcal{D}_0^-$, where the notation is as in [12]. Let $\tau \in \hat{K}_M$ and let $P_\tau: V_\xi \to V_\xi(\tau)$ be the projection onto the $\tau$-isotype. For any function $f$ on $G$ which is sufficiently smooth and of sufficient decay the operator $\pi(f)$ is of trace class. Its trace is

$$\sum_{\tau \in \hat{K}_M} \int_K \int_{MAN} a^{\nu+\rho} f(k^{-1} m \kappa) \text{tr} P_\tau \xi(m) P_\tau d m \, da.$$ 

Plugging in the test function $f = \Phi$ constructed in [3], p.903, this gives

$$\int_{A^+} a^{\nu+\rho} l_{\alpha}^{j+1} e^{-s l_{\alpha}} \text{tr} \xi(f_1) da,$$

where $f_1$ is the Euler-Poincaré function on $M$ attached to the trivial representation. Then

$$\text{tr} \xi(f_1) = \sum_{p=0}^{\dim \mathfrak{p}_M} (-1)^p \dim(V_\xi \otimes \bigwedge^p \mathfrak{p}_M)^{K_M}.$$ 

Now $K_M \cong O(2)$ and the $K_M$-types can be computed explicitly. Thus one sees that $\text{tr} \xi(f_1)$ can only be nonzero if $n = 1$ or $n = 2$. This means that either $\wedge \xi \left( t \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ equals 0 or $t$ respectively, which, in a similar way to the above, implies the claim. In the case $\sigma = \sigma_0$ the function $f_1$ is replaced by $f_{\sigma_0}$ and one proceeds in the same fashion. This takes care of all induced representations. By the classification of the unitary dual of $GL_3(\mathbb{R})$ in [19] it follows that it remains to worry about the trivial representation only, so let $\pi = \text{triv}$ be the trivial representation of $G$; then the space $H_0(\mathfrak{n}, \pi_K) = \pi_K/\mathfrak{n}\pi_K$ is one dimensional with trivial $\mathfrak{a}$-action. This gives a simple zero at $s = 1$.

Q.E.D.
4 Asymptotics of closed geodesics

For $\gamma \in \Gamma$ let $N(\gamma) = e^{l_\gamma}$ and define for $x > 0$:

$$\pi(x) = \# \{ [\gamma] \in \mathcal{E}_P(\Gamma) | N(\gamma) \leq x \}.$$ 

The geodesic prime number theorem in our context is

**Theorem 4.1** For $x \to \infty$ we have the asymptotic

$$\pi(x) \sim \frac{x}{\log x}.$$ 

More sharply we have that

$$\pi(x) = li(x) + O \left( \frac{x^{\frac{3}{4}}}{\log x} \right)$$

as $x \to \infty$ where $li(x) = \int_2^x \frac{1}{\log t} \, dt$ is the integral logarithm.

**Proof:** To simplify the notation in what follows we write $\gamma$ for an element of $\mathcal{E}_P(\Gamma)$ and $\gamma_0$ for a primitive element. If $\gamma$ and $\gamma_0$ occur in the same formula it is understood that $\gamma_0$ will be the primitive element underlying $\gamma$. Unless otherwise specified, all sums will run either over $\gamma$ or $\gamma_0$. For $x > 0$ let

$$\psi(x) = \sum_{N(\gamma) \leq x} l_{\gamma_0}.$$ 

For $\text{Re}(s) > 1$ we have

$$\frac{R'_\Gamma(s)}{R_\Gamma(s)} = \sum_{\gamma_0} \frac{l_{\gamma_0} e^{-sl_{\gamma_0}}}{1 - e^{-sl_{\gamma_0}}}$$

$$= \sum_{\gamma_0} l_{\gamma_0} \sum_{n=1}^{\infty} e^{-snl_{\gamma_0}}$$

$$= \sum_{\gamma} l_{\gamma_0} e^{-sl_{\gamma}}.$$
From this point on the argumentation is, up to minor changes the same as in \cite{15}, which leads to
\[\psi(x) = x + O(x^{3/4}).\]

From this, the theorem is deduced by standard techniques \cite{15} of analytic number theory.

Q.E.D.

We now finish the proof of the main theorem (1.1). For this we have to find a division algebra \(M(\mathbb{Q})\) such that for a given set of primes \(S\) with at least two elements we have \(\pi_S(x) = \pi(x)\). Firstly, there is a division algebra \(M(\mathbb{Q})\) of degree 3 such that the set of places at which \(M(\mathbb{Q})\) is non-split, coincides with the set \(S\). This algebra is obtained by taking the local Brauer-invariants at \(p \in S\) to be equal to \(\frac{1}{3}\) or \(\frac{2}{3}\) and zero everywhere else in such a way that they sum to zero in \(\mathbb{Q}/\mathbb{Z}\) (\cite{14}, Theorem 18.5). Next, we have a bijection
\[\mathcal{E}_P(\Gamma) \rightarrow O(S),\]
given by
\[ [\gamma] \mapsto F_{\gamma} \cap M(\mathbb{Z}), \]
where \(F_{\gamma}\) is the centralizer of \(\gamma\) in \(M(\mathbb{Q})\). Under this bijection the length \(l_\gamma\) is transferred to \(\log r(O)\). The theorem follows.

Q.E.D.

**Corollary 4.2** We finally note that for \(M(\mathbb{Q})\) chosen as above the Ruelle zeta function of Theorem 3.2 takes the form:
\[R_\Gamma(s/3) = \prod_{\mathcal{O} \in O(S)} \left(1 - e^{-sR(O)}\right)^{h(O)\lambda_\mathcal{O}(\mathcal{O})}\]
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