Constraints on Planet Nine’s Orbit and Sky Position within a Framework of Mean-motion Resonances

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Abstract

A number of authors have proposed that the statistically significant orbital alignment of the most distant Kuiper Belt Objects (KBOs) is evidence of an as-yet undetected planet in the outer solar system, now referred to colloquially as “Planet Nine.” Dynamical simulations by Batygin & Brown have provided constraints on the range of the planet’s possible orbits and sky locations. We extend these investigations by exploring the suggestion of Malhotra et al. that Planet Nine is in small integer ratio mean-motion resonances (MMRs) with several of the most distant KBOs. We show that the observed KBO semimajor axes present a set of commensurabilities with an unseen planet at ~654 au (P ~ 16,725 years) that has a greater than 98% chance of stemming from a sequence of MMRs rather than from a random distribution. We describe and implement a Monte-Carlo optimization scheme that drives billion-year dynamical integrations of the outer solar system to pinpoint the orbital properties of perturbers that are capable of maintaining the KBOs’ apsidal alignment. This optimization exercise suggests that the unseen planet is most consistently represented with mass, m ~ 6–12 M⊕, semimajor axis, a ~ 654 au, eccentricity, e ~ 0.45, inclination, i ~ 30°, argument of periastron, ω ~ 150°, longitude of ascending node, Ω ~ 50°, and mean anomaly, M ~ 180°. A range of sky locations relative to this fiducial ephemeris are possible. We find that the region 30° ≤ R.A. ≤ 50°, −20° ≤ decl. ≤ 20° is promising.

Key words: Kuiper belt: general – planets and satellites: detection – planets and satellites: general

1. Introduction

The astronomical community has once again been captivated by the suggestion that there exists a massive, as-yet undetected trans-Neptunian planet. Interest has been particularly acute in response to the publication of an article by Batygin & Brown (2016a) that explicitly connects the spatial clustering of the perihelia of the most distant Kuiper Belt objects (KBOs) with an object, “Planet Nine,” for which they estimate a mass of m ~ 10 M⊕, a semimajor axis of a ~ 700 au, an eccentricity of e ~ 0.6, and an orbital orientation that is anti-aligned with the roster of distant KBOs.

The prospect of discovering such a world is not new, and indeed, has been held out with varying intermittency and credibility for nearly 170 years. The first quantitative prediction of a planet beyond Neptune was issued in 1848, shortly after the discovery of Neptune itself by Leverrier, Galle & d’Arrest; the physicist Jacques Babinet analyzed discrepancies between the observed orbit of Neptune and the orbit initially derived by Leverrier. Babinet used the outcome of these calculations to posit the existence of a m ~ 13 M⊕ planet at a mean distance of a ~ 47 au, a prediction that was, like most that followed, incorrect. Several decades later, in 1880, George Forbes was the first to propose the existence of trans-Neptunian planets that were somewhat similar to the Planet Nine model that is currently attracting interest. Forbes’ outer planet had a semimajor axis of a ~ 300 au, and drew its location from an analysis of the clustering of the aphelions of distances of periodic comets. An early comparative photographic search for Forbes’ predicted planets was carried out to a then-impressive depth of 15th magnitude, albeit with a null result that was reported in 1892 (Hoyt 1980).

The long-running quest has recently achieved a particularly strong impetus with the realization that unusual properties of the most distant KBOs can be ascribed to the dynamical influence of a sizeable perturbing body. Trujillo & Sheppard (2014) and de la Fuente Marcos & de la Fuente Marcos (2014) observed that the distant scattered objects with a ≥ 150 au and q ≥ 30 au have arguments of perihelion clustered around ~0°, and they considered a super-Earth at several hundred au as a possible explanation. Recently, Batygin & Brown (2016a) observed that the most distant KBOs are clustered not just in their arguments of perihelion but also in physical space. They showed that bodies on highly eccentric, nearly Neptune-crossing orbits can maintain clustering in their perihelia when they are apsidally anti-aligned with a massive, high-eccentricity perturber with a semimajor axis of a ~ 700 au. Following this result, several authors elucidated further details connected to the proposed planet. Planet Nine might have a role in explaining the 6° obliquity of the Sun (Bailey et al. 2016; Lai 2016; Gomes et al. 2017) and the existence of the population of highly inclined, sometimes retrograde trans-Neptunian objects such as Drac and Niku (Gomes et al. 2015; Batygin & Brown 2016a, 2016b; Chen et al. 2016).

Potential formation scenarios for such a distant and eccentric planet include scattering with or capture from nearby stars in the Sun’s birth cluster (Li & Adams 2016; Mustill et al. 2016), in situ accretion (Kenyon & Bromley 2016), or early scattering interactions with the giant planets (Bromley & Kenyon 2016). Fortney et al. (2016), Toth (2016), and Linder & Mordasini (2016) characterized various potential physical properties, such as the estimated size, the temperature, and the apparent brightness, which are useful when considering its detectability.

The search space on the planet’s current sky position has already been significantly constrained. Brown & Batygin (2016) simulated a disk of scattered eccentric planetesimals subject to the influence of Planet Nine over a grid of potential orbital parameters. They searched for consistency with the properties of the observed KBO orbits and combined the
resulting restricted parameter domain with the observational limits of past and present sky surveys. Others have exploited additional sources of data in an effort to constrain the sky position. Holman & Payne (2016a) used the several-decade-long ephemerides of Pluto and other trans-Neptunian objects to locate regions of the sky where the orbital fits are worsened or improved by the tidal acceleration from Planet Nine, thereby ruling out large areas of parameter space. Fienga et al. (2016) and Holman & Payne (2016b) derived similar constraints using the precise measurements of the Earth–Sun distance obtained by the Cassini spacecraft. Holman & Payne (2016b) find a most favored location of R.A. ∼ 40° and decl. ∼ −15°, extending ∼20° in all directions. These results, however, are acknowledged to be very sensitively dependant on the precise solar system model that is employed to make a detailed statement issued by JPL1.

Irrespective of the constraints, the area of search space on the sky is still quite large. To make progress, it is useful to return to the observations (see, e.g., Folkner et al. 2016, and the press release issued by JPL).1

Building on this evidence, Malhotra et al. (2016) noticed that several of the detached KBOs could be in N:1 and N:2 MMRs if the semimajor axis of Planet Nine was ∼665 au, corresponding to a 3:2 MMR with Sedna. Beust (2016) found consistent results using semi-analytical integrations of the resonant secular Hamiltonians, showing that high-eccentricity, apsidally anti-aligned libration islands were possible in a resonant context. He also found, however, that similar libration islands exist in a scenario exhibiting purely secular dynamics, suggesting that MMRs are possible but are not strictly necessary to produce the observed clustering.

Given these clues, we seek to further explore the conjecture that Planet Nine participates in MMRs with one or more detached KBOs. In this picture, Planet Nine’s allowed semimajor axis is strongly constrained, and critically, participation in MMRs potentially permits determination of both Planet Nine’s orbital properties, and its current sky position.

Throughout the majority of this paper, we make the fundamental assumption that Planet Nine exists. The results are thus best interpreted as the constraints on Planet Nine under this hypothesis. Toward the end of the paper, we adopt a contrasting viewpoint and comment on the implications of our results on the evidence for Planet Nine’s existence.

The outline is as follows. In Section 2, we construct a semimajor axis probability distribution under a resonant hypothesis and quantify the likelihood that the observed semimajor axes of the distant KBOs are suggestive of resonant relationships with Planet Nine. In Section 3, we adopt the resonant hypothesis and describe the Monte-Carlo optimization scheme we have used to pinpoint the orbital elements of a planetary perturber that is capable of maintaining clustering in the KBO orbits to the degree that is observed today. In Section 4, we present the results of the calculations. We first examine the orbital element constraints and then demonstrate some key connections between maintenance of orbital clustering and participation in MMRs. In Section 5, we compare our prospective sky locations with previous works and propose a plan to obtain stronger constraints. We also comment on the implications of our results on the evidence for Planet Nine’s existence.

## 2. A Semimajor Axis Probability Distribution

We begin with the assumption that Planet Nine is acting as an exterior perturber in low order MMRs with one or more detached KBOs; we wish to derive a probability distribution for Planet Nine’s semimajor axis conditional on this assumption. Eleven KBOs with semimajor axes of a > 200 au and perihelion distances of q > 30 au are considered. The objects are listed in Table 1 with parameters obtained from the Minor Planet Center. 2004 FE22 has been excluded because its N = 2155 au semimajor axis is over four times larger than the average member of this population. We have also excluded 2013 SM31 because the object’s discovery was not yet announced at the time these calculations were performed.

### Table 1

| Object     | a (au) | q (au) | i(°) | ω(°) | Ω(°) | ω(°) |
|------------|--------|--------|------|------|------|------|
| 2002 GB32  | 217.9  | 35.3   | 14.2 | 37   | 177  | 214  |
| 2000 CR265 | 226.1  | 44.3   | 22.7 | 317.2| 128.3| 85.5 |
| 2001 FP765 | 226.9  | 34.3   | 30.8 | 7    | 179.3| 186.3|
| 2012 VP213 | 260.8  | 80.3   | 24.1 | 292.8| 90.8 | 23.6 |
| 2014 SR340 | 289    | 47.6   | 18   | 341.4| 34.8 | 376.2|
| 2013 FT20  | 310.1  | 43.6   | 17.3 | 40.2 | 217.8| 258  |
| 2004 VN112 | 317.7  | 47.3   | 25.6 | 327.1| 66   | 33.1 |
| 2013 RF98  | 350.0  | 36.1   | 29.6 | 311.8| 67.6 | 379.4|
| 2010 GB174 | 369.7  | 48.8   | 21.5 | 347.8| 130.6| 118.4|
| 2007 TG222 | 483.5  | 35.6   | 18.6 | 285.7| 112.9| 38.6 |
| Sedna      | 499.4  | 76.0   | 11.9 | 311.5| 144.5| 96   |

1 http://www.jpl.nasa.gov/news/news.php?feature=6200
10 $M_\oplus$ and a standard deviation of $\sigma$ (a parameter), and lower and upper bounds of 5 $M_\oplus$ and 20 $M_\oplus$. Varying $\sigma$ allows more or less weight to be placed near the 10 $M_\oplus$ best estimate.

With this mass prior distribution in hand, we create a physically realistic semimajor axis probability distribution as follows. The maximum MMR integer, $N$, and mass distribution standard deviation, $\sigma$, are parameters of this procedure.

1. Draw a mass sample, $m_i$, from the truncated Gaussian prior distribution.
2. Randomly select one of the 11 KBO semimajor axes, $a_{\text{KBO}}$, and one of the allowed MMR integer ratios, $r$. For example, if $N = 4$, one ratio $r$ is selected from the set $\{4/3, 3/2, 2/1, 3/1, 4/1\}$. All resonances are assumed to be equally likely.
3. Calculate the Planet Nine semimajor axis for exact resonance: $a_i = r^{2/3}a_{\text{KBO}}$.
4. Check the Brown & Batygin semimajor axis bound defined by the mass $m_i$. If $a_i$ lies within $[200\text{ au} + 30m_i/M_\oplus, 600\text{ au} + 20m_i/M_\oplus]$, proceed to step 5. If it does not, discard the sample and resume the process from step 1.
5. To account for the MMR libration width surrounding exact resonance, draw a Gaussian perturbation to the value of $a_i$. The perturbation is consistent with the inner test particle’s libration width as defined by Malhotra et al. (2016).2

$$\Delta a_{\text{res}} \approx 0.007 a_{\text{KBO}} \left( \frac{m_i a_{\text{KBO}} A}{3 M_\oplus a_i} \right)^{1/2}.$$ (1)

In general, the coefficient $A$ is a complicated function of the KBO semimajor axis, eccentricity, and inclination. It exhibits, however, only a modest dependence on the particular commensurability being considered. We follow Malhotra et al. and adopt the numerical value $A = 3$.

6. Repeat steps 1–5 until a semimajor axis distribution of the desired size has been constructed.

In Figure 1, we present the semimajor axis distribution for period ratios involving integers up to $N = 5$ and a standard deviation on the mass prior equal to $\sigma = 5 M_\oplus$. The figure is a histogram of two billion semimajor axis samples. We denote all values of perfect resonance with vertical colored lines. There is a sharply defined peak at $a \sim 660$ au, corresponding to a 3:2 resonance with Sedna, among other small integer resonances. The resonance with Sedna occurs at $a \sim 654$ au. Because Sedna has the most well-determined semimajor axis, we will use $a \sim 654$ au to refer to this peak in the remainder of the analysis. The peak is noticeably higher than the others in the distribution, prompting the consideration of its statistical significance. That is, given that the real number line is dense with integer ratios, what are the chances of obtaining a peak as tall or taller at random?

We answer this question with a Monte-Carlo simulation. In each trial of the simulation, we generate a semimajor axis distribution using the steps outlined above. However, instead of using the true KBO semimajor axes, we use synthetic KBO semimajor axes randomized in the range defined by the semimajor axes of the true KBOs. Each trial uses just one set of randomized semimajor axes. Whereas Figure 1 was generated with two billion samples, the trial semimajor axis distributions use only 200,000 samples for computational feasibility. The reduction in sample size has little impact on the calculation we wish to perform. The 200,000 samples of a given trial distribution are organized into 250 bins. An example synthetic trial distribution is shown in the right panel of Figure 2. The left panel shows the true distribution using the actual KBO semimajor axes.

We wish to quantify the percentage of trial distributions that contain a peak as significant as the $\sim 654$ au peak in the true distribution. For each trial distribution, we thus measure the

\[\Delta a_{\text{res}} \approx 0.007 a_{\text{KBO}} \left( \frac{m_i a_{\text{KBO}} A}{3 M_\oplus a_i} \right)^{1/2}.\]
significance of the highest peak as the number of standard deviations of the peak bin height above the median height. A visual depiction of this quantity is shown in Figure 2.

Likewise, we perform the same peak significance measurements on the true semimajor axis distribution (that which corresponds to the observed KBO semimajor axes) and find the mean of 200 measurements. Finally, we calculate the percentage of the synthetic trials that had peak significances greater than the mean peak significance of the true distribution.

Figure 3 displays histograms of the peak significances for 5000 Monte-Carlo trials for the cases of $N = 5$ and $N = 6$ and $\sigma = 5 M_\odot$ in each case. For $N = 5$, only 1.6% of synthetic samples have peak significances greater than the mean true peak significance, and for $N = 6$, it is 1.7%. To investigate the strength of this result, we repeated our analysis for a collection of other reasonable parameters. Table 2 lists the percentages for $\sigma = 5 M_\odot$ and for a case with less weight near $m = 10 M_\odot$, given by $\sigma = 7 M_\odot$. The percentages are quite similar, though in general, the tighter the mass prior distribution, the stronger the result. Finally, to ensure that the result is not an artifact of

![Figure 2](image-url)  
*Figure 2. Semimajor axis distribution histograms to illustrate the Monte-Carlo peak significance simulation. On the left is the true distribution corresponding to the observed KBO semimajor axes. (In effect, this histogram is simply a coarser binning of Figure 1.) On the right is an example Monte-Carlo trial distribution corresponding to a set of semimajor axes randomly drawn within the range of the true KBO semimajor axes. Both histograms correspond to $N = 5$ and $\sigma = 5$, and they each have 200,000 samples and 250 bins. The horizontal dashed lines are at the medians of the distributions. The “x” markers on the vertical dashed line correspond to standard deviations above the median. The labeled significance measurements are thus the number of standard deviations of the highest peak above the median.*

![Figure 3](image-url)  
*Figure 3. Histograms of the significances of the highest peak in 5000 synthetically generated semimajor axis distributions. The cases of $N = 5$ and $N = 6$ are shown with $\sigma = 5$ in each case. The green line indicates the mean significance of the $a \sim 660$ au peak in the true distribution with the light band giving $1\sigma$ uncertainties. For $N = 5$, 1.6% of samples lie above the peak significance in the true distribution. For $N = 6$, it is 1.7%.*

| $N$  | $\sigma = 5 M_\odot$ | $\sigma = 7 M_\odot$ |
|------|-------------------|-------------------|
| 5    | 1.6%              | 1.9%              |
| 6    | 1.7%              | 1.7%              |
| 7    | 1.6%              | 3.1%              |

*Table 2: Percentage of Monte-Carlo Synthetic Trial Distributions with Peak Significances Greater Than the Significance of the True Distribution*
the assumed resonant width, we decreased $\Delta a_{\text{res}}$ to half its size. On average, this results in percentages less than 0.5% larger.

In summary, when taking into account prior estimates of Planet Nine’s mass and mass/semimajor axis relationship, it appears that the KBO semimajor axes exhibit commensurabilities with a Planet Nine semimajor axis of $a \sim 654$ au. We have quantified the low probability that the relation arises by chance. Furthermore, placing a narrower prior on the $10 M_\oplus$ estimate most favored by Brown & Batygin only strengthens the case. We therefore devote the rest of our analysis to exploring the plausibility of Planet Nine residing in an $a \sim 654$ au orbit. We stress that our results rest on this fundamental assumption and that they lose their import if the peak in the semimajor axis distribution has arisen by chance.

### 3. Numerical Methods

If Planet Nine has $a \sim 654$ au and participates in MMRs with a subset of the distant KBOs, the MMRs necessarily impose constraints on both its orbital parameters and its current location. For example, given that Sedna is currently near its perihelion approach, a 3:2 resonant configuration would imply that Planet Nine’s mean anomaly is close to one of three locations: $M = 60^\circ$, $M = 180^\circ$, or $M = 300^\circ$. Moreover, direct numerical calculations can be used to explore the consequences of the requirement that resonances exist, thereby increasing the constraints on all of Planet Nine’s orbital parameters.

Most orbital configurations that are consistent with Planet Nine’s nominal parameters lead to rapid orbital evolution of the most distant KBOs. For example, de la Fuente Marcos et al. (2016) show that when the baseline Planet Nine model of Batygin & Brown (2016a) is adopted, the orbits of 2004 VN112, 2007 TG422, and 2013 RF10 are destabilized within a few tens of megayears. Furthermore, the apsidal alignment is not maintained for arbitrary instances that are consistent with the nominal model. However, de la Fuente Marcos et al. (2016) show that slight modifications that enforce $\Delta \varpi = 180^\circ$ in the Batygin & Brown (2016a) model lead to improved KBO stability. If Planet Nine has $a \sim 654$ au, orbital parameter domains must exist where the KBOs are relatively stable and where apsidal alignment is maintained for long periods.

We have conducted a comprehensive parameter search by numerically integrating a set of KBOs under the influence of Planet Nine and seeking out domains of orbital parameter space where KBO large-scale stability and apsidal alignment are maintained. In our integrations, we included the 11 KBOs with semimajor axes $a > 200$ au and perihelion distances $q > 30$ au listed in Table 1. However, we restricted the constraint on apsidal alignment to only seven of these with $a > 250$ au. The only member with $a > 250$ au that we exclude is 2013 FT28, which is anti-aligned from the rest of the KBOs (Sheppard & Trujillo 2016). Provided the results of the KBO orbital evolution under the influence of the distant planet, we wish to observe these seven KBOs cluster in their arguments and longitudes of perihelion. We first define simple merit functions that allow us to parameterize the degree to which clustering is maintained in a time-averaged sense.

#### 3.1. Quantifying Apsidal Confinement

A useful quantification of clustering stems from the degree to which the KBO longitudes of perihelion, $\varpi_i$, are spread about anti-alignment with Planet Nine. If

$$|\varpi_i - \varpi_{\text{ref}} - 180^\circ| = |\Delta \varpi_i - 180^\circ|$$

is the absolute deviation of the $i$th KBO’s longitude of perihelion from perfect anti-alignment with Planet Nine (in a range from $0^\circ$ to $180^\circ$), then

$$m = \text{median} \left\{ |\Delta \varpi_i - 180^\circ| \right\}_{1 \leq i \leq 7}$$

is a measure of the typical deviation of the seven KBOs at one slice in time. The median over $T$ time slices,

$$s = \text{median} \{ m \}_{1 \leq t \leq T}$$

is thus an average measure of the scatter across the duration of the integration. We wish to favor configurations that produce small values of $s$. A suitable merit function is

$$f(s) = e^{-s^2/1000},$$

where the constant 1000 was chosen through experimentation.

A conservative merit function only demands KBO clustering rather than both clustering and orbital anti-alignment with Planet Nine. This may be parameterized by alignment of the KBO argument of perihelion vectors,

$$e_i = \{ \cos \omega_i \hat{i} + \sin \omega_i \hat{j} \}.$$  

Provided a set of orbital evolutions for seven KBOs, we quantify the clustering as follows. We first calculate the seven bodies’ unit argument of perihelion vectors $e_i(t)$ and their mean unit vector $\bar{e}_m(t)$ at all times $t$. We compute the sum of the time-varying dot products,

$$S_\omega(t) = \sum_{i=1}^{7} e_i(t) \cdot \bar{e}_m(t),$$

which quantifies the degree to which the vectors are clustered about their mean. If a KBO gets ejected from the system before the integration is over, we set $e_i(t)$ to zero, such that this KBO stops contributing to $S_\omega(t)$. We also evaluate the time-varying anti-alignment with respect to Planet Nine as

$$A_\omega(t) = \bar{e}_m(t) \cdot e_{\text{ref}}(t).$$

If the seven bodies are perfectly anti-aligned with respect to Planet Nine for all times, then $S_\omega(t) = 7$ and $A_\omega(t) = -1$. The present-day value of $S_\omega$ for the seven KBOs we are considering is $S_\omega(0) = 6.52$. The time average $\langle S_\omega(t) \rangle$ is an effective measure of confinement. Assuming that the present-day observed configuration of the KBOs is not a unique moment in time, $\langle S_\omega(t) \rangle$ should be close to $S_\omega(0)$. Furthermore, an integration that maintains KBO anti-alignment with P9 will yield $\langle A_\omega(t) \rangle \approx -1$.

Analogous metrics may be constructed with the longitude of perihelion vectors,

$$e_\varpi = \{ \cos \varpi \hat{i} + \sin \varpi \hat{j} \},$$

for which the present-day value is $S_\varpi(0) = 5.63$. A successful configuration must produce clustering in both $\omega$ and $\varpi$ in order to match the observations. We note that it is insufficient to enforce clustering in $\varpi$ or the Runge–Lenz eccentricity vectors alone because these metrics do not guarantee confinement in $\omega$. 

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**References**

Millholland & Laughlin (2017 March)
In the analysis that follows, the merit function $f(s)$ will be used to guide a Markov Chain Monte-Carlo procedure; $\langle S_a(t), \langle A_\omega(t), \langle S_\varpi(t) \rangle, \langle A_\varpi(t) \rangle \rangle$ will be used to further analyze the resulting samples.

### 3.2. Differential Evolution Markov Chain Monte-Carlo (DE-MCMC) Parameter Space Exploration

Having defined merit functions by which we can judge the KBO orbital evolution associated with a given set of Planet Nine parameters, we next discuss how these parameters are sampled. We begin with initial parameter distributions that are motivated by the strong suggestion of commensurability in Section 2 and by the constraints provided by the simulations described by Brown & Batygin (2016). The distributions are listed in Table 3, where we define $N(\mu, \sigma, a, b)$ as a truncated Gaussian probability distribution with mean $\mu$, standard deviation $\sigma$, and lower and upper bounds, $a$ and $b$. $N(\mu, \sigma)$ and $U[a,b]$ are Gaussian and uniform distributions, respectively.

The dependence of the $m$ distribution on $a$ corresponds to the constraints derived by Brown & Batygin (2016). This mass-semimajor axis relationship was also employed in Section 2 when we used mass constraints within the calculation of the MMR semimajor axis distribution. Similarly, the relationship of the eccentricity distribution to the mass has been adapted from Brown & Batygin’s constraints. The main feature is the negative correlation between $e$ and $m$. Finally, we note that the longitude of perihelion has been constrained to $\omega \sim N(230^\circ, 25^\circ)$ in accord with anti-alignment with the KBOs. $\omega$ and $\Omega$ are allowed to vary through a full $360^\circ$, subject to the constraint that $\omega = \omega + \Omega$.

Using these initial parameter distributions to seed a collection of Markov Chains, we carried out a parameter exploration using a DE-MCMC algorithm (Ter Braak 2006). The DE-MCMC technique is most often used to sample from a posterior parameter distribution, typically for Bayesian inference; it iterates many Markov Chains in parallel and uses the inter-chain differences to inform the parameter jumps from one iteration to the next. In our analysis, however, the merit function takes the place of a likelihood function. As a result, this technique will not produce true posterior parameter distributions because the merit function is not a true probability density function. Rather, the DE-MCMC is simply a tool that we use to efficiently sample the seven-dimensional parameter space, assuming large parameter covariances and smooth parameter space structure, both of which DE-MCMC is helpful in accounting for.

We ran in parallel a large collection of DE-MCMC instances of five chains. We used jumping parameters $\gamma = 0.1$ and $\gamma_a = 0.001$ (Ter Braak 2006), such that one chain’s proposal jump during a given iteration was approximately one-tenth of the vector between two other randomly selected chains. As stated above, the merit function $f(s)$ took the place of the likelihood. The sampling of $e$ and $\omega$ used the transformations $e \cos(\omega)$ and $e \sin(\omega)$; the other orbital elements were unaltered.

For all of our $N$-body dynamical integrations, we used the hybrid symplectic-Bulirsch–Stoer algorithm from the Mercury $N$-body integration package (Chambers 1999). We modeled the gravitational potential of the inner three giant planets (Jupiter, Saturn, and Uranus) as an orbit-averaged quadrupole by giving the Sun a physical radius equal to Uranus’ orbital radius and a $J_2$ moment equal to (Burns 1976)

$$J_2 = \frac{m_Ia_I^2 + m_Sa_S^2 + m_Ua_U^2}{2M_\odot a_U^2}.$$  

Neptune was followed in direct $N$-body fashion, and the integration timestep was set to one-tenth of Neptune’s orbital period. The integrations were carried out for one billion years. While this is too short to ensure apsidal confinement lasting over the age of the solar system, it is certainly long enough to observe KBO orbital instability and loss of apsidal confinement due to differential precession (that is, within a configuration in which Planet Nine’s orbital parameters are unsuitable).

### 3.3. KBO Cloning and Multiplexing

While monitoring the orbital clustering of the seven KBOs is straightforward, the KBOs’ own orbital uncertainties must also be accounted for. Inspection of the bodies listed in Table 1 using the the JPL Small-Body Database Browser indicates that the semimajor axis uncertainties for the distant KBOs are of similar magnitude to the resonant libration widths. If MMRs are indeed the primary mechanism for the orbital clustering, the uncertainties must be considered when evaluating whether the bodies can coexist in resonance.

To accommodate the uncertainties, all of our integrations were performed not only on the KBOs themselves, but also on a set of “clones.” For each integration, we randomly generated five clones per KBO with identical values for $e_i, i, \Omega_i, \omega_i, M_i$ but with a Gaussian perturbation to the semimajor axis, $a_i$, consistent with the currently estimated semimajor axis uncertainties. Ideally, one would draw perturbations on all the KBO orbital elements, but the semimajor axis is clearly the principal parameter in determining the proximity to MMR. We retained one copy of the original KBO with no perturbation, for a total of six clones per KBO, or 42 bodies total to be followed in each integration.

A benefit of representing seven KBOs as 42 independent bodies is a dramatic increase in sample size enabled through a multiplexing process. In keeping with the hypothesis of Malhotra et al. (2016), we expect that the clone with semimajor axis closest to exact resonance will generally perform the best at maintaining apsidal alignment in time. For a given integration, we seek the set of seven best-performing clones.  

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3 After generating the data set presented in this paper, we ran separate calculations in which the clones’ full set of orbital elements were sampled from the KBO uncertainty covariance matrices. We generated a sample size that is one-third as large as the data set presented here. We found no significant differences between the results that only varied the clone semimajor axes and those that sampled all orbital elements.
There are \( 6^7 = 279,936 \) such sets of possible combinations. In effect, each dynamical integration of 42 bodies that is performed can be recast as a set of 279,936 independent integrations of seven bodies, each of them with the same set of parameters for Planet Nine but with different parameters for the seven KBOs of interest.

To summarize, we a posteriori generated the 279,936 multiplexed samples from each billion-year integration that was performed using the procedure discussed in Section 3.2. In each multiplexed sample’s combination of seven KBOs, we calculated the measures of argument and longitude of perihelion vector clustering and anti-alignment that were defined in Section 3.1.

4. RESULTS

4.1. Orbital Element Constraints

We now present the results of \( \sim \)250,000 individual billion-year orbital integrations. Rather than considering all \( 6^7 \) multiplexed samples for each integration, which would produce a prohibitively large set of 70 billion one-Gyr integrations to analyze, we first consider a subset of the best-performing samples. This subset consists of each integration’s single multiplexed sample that exhibits the largest value of \( \langle S_\omega(t) \rangle \), defined in Section 3.1 as the time average of the sum of the dot products of the seven KBO argument of perihelion vectors with their mean vector. In this procedure, we opted to maximize \( \langle S_\omega(t) \rangle \) as opposed to \( \langle S_\omega(t) \rangle \) because, as we will show, obtaining clustering close to the present-day observations is much more challenging for \( \omega \) than for \( \varpi \).

Figures 4–7 show the performance of the KBOs in maintaining clustering in their arguments of perihelion as a function of Planet Nine’s orbital elements. In each scatter plot, the x-axis is the argument of perihelion clustering measure \( \langle S_\omega(t) \rangle \), with the dashed line marking the present-day value, \( S_\omega(0) = 6.52 \). The colorbar variable \( \langle A_\omega(t) \rangle \), as defined in Section 3.1, quantifies the degree to which the KBOs maintained anti-alignment with Planet Nine, with \( \langle A_\omega(t) \rangle = -1 \) being perfect anti-alignment. Figures 4 and 7 include an additional scatter plot using \( \langle S_\omega(t) \rangle \), the longitude of perihelion clustering measurement, as the colorbar.

We first make some general observations. The figures indicate that an optimization scheme of this nature is a viable technique for constraining Planet Nine’s orbital elements. Obtaining large values of \( \langle S_\omega(t) \rangle \) is challenging, and the vast majority of orbital configurations do not work out, yet there are constrained regions of the parameter space where significant clustering can be maintained.

\( \langle S_\omega(t) \rangle \) is generally anti-correlated with \( \langle A_\omega(t) \rangle \), though the samples with the largest values of \( \langle S_\omega(t) \rangle \) are not exclusively anti-aligned with Planet Nine. We note that anti-alignment with Planet Nine is not required by the KBO observations, which simply exhibit an observed clustering in \( \omega \) and \( \varpi \). It is difficult to conceive, however, of a configuration that is not anti-aligned and that can survive for billions of years. Either secular or resonant confinement mechanisms require \( \Delta \varpi \approx 180^\circ \), with \( \Delta \omega \approx 180^\circ \) implying a nodal asymmetry. When longer time integrations are performed, we expect that there will be a decrease in the number of samples that perform well without an anti-aligned configuration.

Figure 4 presents the results for the eccentricity. Samples with the largest values of \( \langle S_\omega(t) \rangle \) have \( e \sim 0.4–0.5 \), with the left panel showing that anti-aligned configurations have \( e \sim 0.4–0.5 \). This is slightly lower but not in disagreement with the eccentricities favored by Brown & Batygin (2016) for \( a \sim 600–700 \) au. The right panel of Figure 4 depicts similar results, except the color bar now shows clustering in the KBO longitudes of perihelion. Given that \( S_\omega(0) = 5.63 \), clustering is evidently much easier for \( \varpi \) than for \( \omega \). Interestingly, a large degree of clustering in \( \varpi \) is most easily attainable for highly eccentric orbits.

Figure 5 presents the distributions of \( \omega \), \( \Omega \), and \( \varpi \). As discussed in Section 3.2, the initial distribution restricted \( \varpi \) to initial anti-alignment with the KBOs but let \( \omega \) and \( \Omega \) vary through \( 0^\circ–360^\circ \). It is not surprising to see strongest anti-alignment occurring for \( \omega \sim 150^\circ \) because this corresponds to present-day anti-alignment with the KBOs. It is still significant, however, as in the absence of the perturbing planet, the KBOs randomize their alignment in tens of millions of years due to
differential precession. Moreover, the region where $\omega \sim 150^\circ$ also produces the strongest clustering in the KBO arguments of perihelion. The longitude of the ascending node displays strongest clustering and anti-alignment for $\varpi \sim 205^\circ$. By extension, the node is best at $\Omega \sim 50^\circ$.

The inclination distribution in Figure 6 illustrates that low inclinations $i \lesssim 20^\circ$ are consistently disfavored. Strongest anti-alignment occurs for $30^\circ \lesssim i \lesssim 40^\circ$, in agreement with the Brown & Batygin (2016) estimate, whereas clustering is strongest for very high inclinations. To understand this apparent preference for high inclination configurations, we examined the KBO trajectories of representative samples of this population. The KBOs in these configurations typically undergo large eccentricity oscillations and a rapid increase to highly inclined or retrograde orbits. In general, the bodies are only able to maintain inclinations close to their present-day values for $i \lesssim 40^\circ$, though some configurations with $i < 40^\circ$ can still cause the KBOs to evolve to high inclinations. This evolution to high inclination and retrograde orbits was discussed in Shankman et al. (2016) and in Batygin & Brown (2016b), who showed that Planet Nine is capable of producing the observed population of highly inclined, $a < 100$ au trans-Neptunian objects through a process involving Kozai-Lidov cycles and close encounters with Neptune. We also found a moderate correlation between KBO evolution to highly inclined orbits and nodal clustering, possibly providing a connection to the observed nodal clustering in the population of $a < 100$ au TNOs reported in Chen et al. (2016). An extended exploration of these inclination dynamics is saved for a future discussion.

In contrast to some of the other orbital elements, the mass distribution in Figure 7 exhibits less constraint. The KBO clustering and anti-alignment is possible for masses throughout the range $6 \, M_\oplus \lesssim m \lesssim 12 \, M_\oplus$. In agreement with previous studies, masses less than $\sim 6 \, M_\oplus$ are strongly disfavored. The semimajor axis distribution does not display much structure and is not shown here. The KBO semimajor axes are constantly perturbed, however, through the cloning procedure, meaning that the semimajor axis ratio $a_P / a_{KBO}$ or period ratio $P_P / P_{KBO}$ are more physically informative than $a_P$. We pursue an exploration of the period ratio in Section 4.2.

4.2. Resonant KBOs

If MMRs are responsible for the observed apsidal clustering of the KBOs, then the configurations that exhibit better maintenance of clustering than others should preferentially place the bodies in period ratio commensurabilities. We investigated this hypothesis in the following manner. For each of the $\sim 250,000$ integrations that was performed, we extracted the single multiplexed sample combination of seven KBOs (with the largest value of $\mathcal{S}_T$). We also extracted one random combination of KBOs from the multiplexed samples. In the left panels of Figures 8 and 9, we plot the clustering merit $\mathcal{S}_T$ versus the period ratio $P_P / P_{KBO}$ at $t = 0$ for the best-performing multiplexed samples. The right panels show the analogous plots for the random multiplexed samples. The diagrams confirm that MMRs are in fact an influential mechanism in the observed KBO clustering. The vertical overdensities in the left panels appear at distinct period ratios, signaling that the configurations that perform the best are preferentially near these commensurabilities. We have labeled the commensurabilities that manifest most strongly with their integer ratios. To further guide the eye, differenced histograms...
of the period ratios are plotted as black lines in the left panels. The differenced histograms are made by generating histograms of the period ratios of the best-performing samples (those in the left panels) with \( \langle S_\lambda(t) \rangle > 4 \) and subtracting from them Gaussian fits to the period ratio histograms of the random samples (right panels). The peaks in the resulting differenced histograms indicate the locations of the overdensities.

2012 VP\(_{113}\) exhibits the strongest feature at 4:1, with significant features also in 2010 GB\(_{174}\) at 9:4, 7:3, and 5:2, and in Sedna at 3:2. Interestingly, Sedna displays several other overdense striations at nearby, higher order resonances.

Although these period ratio diagrams establish the importance of proximity to orbital period commensurabilities, we have not yet demonstrated true resonant libration. If the KBOs are in resonance with Planet Nine, their motions will display librations of a critical resonant angle of the form

\[
\phi_r = (p + q) \lambda_{p9} - p \lambda_{KBO} - (q - r) \varpi_{p9} - r \varpi_{KBO}
\]

where \( \lambda \) is the mean longitude and \( \varpi \) is the longitude of perihelion. \( p \) and \( q \) are integers that are fixed for a given \( (p + q) \) resonance, and \( r \) is an integer ranging from 0 to \( q \).

We monitored the resonant angles of the KBOs that appear most strongly in Figures 8 and 9: Sedna in 3:2, 2012 VP\(_{113}\) in 4:1, and 2010 GB\(_{174}\) in 9:4, 7:3, and 5:2. We also monitored 2004 VN\(_{112}\) in 3:1, 2014 SR\(_{149}\) in 7:2, 2007 TG\(_{122}\) in 8:5, 2001 FP\(_{185}\) in 5:1, and 2000 CR\(_{105}\) in 5:1. (The last two are members of the 200 au < \( a_{KBO} < 250 \) au set of KBOs that we integrated but did not include in the \( \langle S_\lambda(t) \rangle \) clustering measurements.) Crucially, within KBO semimajor axis uncertainties, all of these resonances can coexist with \( a_{p9} \sim 654 \) au.

In order to investigate both temporary and long-lasting resonant libration, we divided the billion-year integrations into 10 sections of 100 Myr and checked for libration in each slice of time. Remarkably, all KBOs we monitored were capable of experiencing resonant libration given relatively similar Planet Nine parameters. While the existence of resonant libration should not be surprising, we note that this is the first direct numerical confirmation that these observed high-eccentricity KBOs are capable of librating stably in an MMR with a distant perturber.

Figure 10 depicts the coexistence of resonant libration. We first restricted the sample space to the subset of Planet Nine configurations for which at least one of 2012 VP\(_{113}\)’s critical resonant angles, \( \phi_r \), was librating with \( z < 1.95 \) in the first 100 Myr. Here, \( z \equiv \min \{ \Delta \cos \phi_r, \Delta \sin \phi_r \} \), where \( \Delta \cos \phi_r \equiv (\cos \phi_r)_{\max} - (\cos \phi_r)_{\min} \) and likewise for \( \Delta \sin \phi_r \). For each KBO and within each 100 Myr section of time, we then calculated the percentage of samples for which at least one clone was librating during that 100 Myr slice of time. The results are shown in Figure 10. The scales of the percentages for each KBO are relatively unimportant because they depend on the prior distributions of \( a_{p9} \) and \( 2a_{KBO} \), but the shapes of the curves are more informative. The curves for Sedna demonstrate that it is capable of libration from the beginning of the integration, starting at 0–100 Myr, in a configuration in which 2012 VP\(_{113}\) is also librating from \( t = 0 \). The sharp increases in the percentages for most of the other resonances indicate that these bodies are difficult to maintain in resonance starting from \( t = 0 \), but they can be captured into libration as time advances.

We also examined long-lived libration, that is, resonance starting from \( t = 0 \) and lasting for an arbitrary length of time, rather than libration within a 100 Myr section of time. Figure 11 shows the decay in the maintenance of resonant libration of Sedna and 2012 VP\(_{113}\) as a function of time. The percentage of samples with the bodies remaining in resonance decays predictably as time advances, though there is a substantial fraction of the initial population that maintains libration for the full billion-year integration. The libration zones of Sedna and 2013 VP\(_{113}\) are illustrated on an \( a_{p9}/e_{p9} \) diagram in Figure 12. Here, the spread in \( a_{p9} \) for the resonant configurations reflects both the intrinsic libration width and the uncertainties propagated through the KBO semimajor axes. The libration zones shrink in time because there is a decay in the number of bodies in resonance from \( t = 0 \). However, the overlap in the libration zones remains at values of \( a_{p9}/e_{p9} \) that are consistent between the two resonances.

5. Discussion

5.1. The Resonant Perturbation Mechanism and Sky Location Constraints

Batygin & Brown (2016a) were the first to propose that the mean-motion resonant perturbation may be responsible for the perihelion clustering of the distant KBOs. Subsequently, Malhotra et al. (2016) observed that many KBOs would exhibit...
small integer commensurabilities if the semimajor axis of Planet Nine was at \( \sim 654 \text{ au} \). Our work has provided evidence that supports both of these claims.

The \( a \sim 654 \text{ au} \) resonant hypothesis will be rapidly strengthened or undermined by future KBO observations. Participation in resonant libration generates predicted bounds on the KBO semimajor axes. These bounds are substantially smaller than current measurement uncertainties such as those for 2012 VP\(_{113}\), which we have shown can exhibit long-lasting libration in a 4:1 resonance. Repeated observations of these KBOs to refine their period ratios can therefore permit rapid testing of the predictions, to either refute or substantially strengthen the validity of our assumed scenario. Moreover, the semimajor axes of as-yet-undiscovered, distant KBOs and their proximity to commensurabilities with \( a \sim 654 \text{ au} \) will continue to test the claim.

The MMRs impose restrictions not only on the orbit of the perturbing planet, but also on its position within the orbit. We can therefore evaluate the constraints our results yield on the sky position of Planet Nine, and compare the results with those of other studies.

In Figure 13, we display with large black points the sky positions of configurations where the KBOs maintained strong anti-alignment with Planet Nine, \( \langle \vec{A}_i(t) \rangle < -0.97 \), and strong clustering in the arguments of perihelion, \( \langle \vec{\omega}_i(t) \rangle > 5.2 \). The large gray points correspond to resonant configurations with the following constraints: Sedna and 2012 VP\(_{113}\) librating from \( t = 0 \) for at least 100 Myr, 2010 GB\(_{174}\) librating in one of its three dominant resonances, and \( 0.4 < e_P < 0.5 \). The faint gray points are the sky locations of all samples, which is not a uniform distribution in R.A. due to the orientation of the orbit in space from the restrictions on \( \vec{\omega} \). Finally, the orbit of our fiducial model is shown with coloration according to distance.\(^5\)

Though the plot is sparse, it is clear that the region with the highest density of favored points occurs at R.A. \( \sim 30^\circ \) and decl. \( \sim -10^\circ \), or near the aphelion of our fiducial orbit.

\(^5\) A variety of Planet Nine orbits relative to the fiducial model are possible, and no orbit with representative round-numbered orbital elements is expected to be the perfect solution. When the fiducial model is integrated for four billion years with cloned copies of the KBOs that are consistent with observed uncertainties, the majority of the bodies maintain orbital stability and low inclinations relative to the ecliptic. Many of the bodies also maintain apsidal clustering, with the best multiplexed sample’s 1 Gyr averaged values of \( \langle \vec{A}_i(t) \rangle \) and \( \langle \vec{\omega}_i(t) \rangle \) being 5.36 and 5.24, respectively. In general, the results are better than the outcomes of similar Planet Nine orbits that we tested for 4 Gyr.
Furthermore, analytic considerations of the KBO resonances also point toward aphelion as a likely location. In our integrations, we found that resonant configurations were the most stable and long-lasting when the critical resonant angle \( \phi_q = (p + q) \lambda_{\text{KBO}} - p \lambda_{\text{KBO}} + q \lambda_{\text{KBO}} \) exhibited libration about centers at \( \phi_q = 0^\circ \) or \( \phi_q = 180^\circ \). We calculated histograms of \( \phi_q \) libration amplitudes exhibited by the KBOs when they were in resonance. Then, under the assumption of libration about \( \phi_q = 0^\circ \) or \( \phi_q = 180^\circ \), we transformed these observed distributions of KBO libration amplitudes into a joint probability distribution of the present-day mean anomaly for Planet Nine. The resulting most likely locations are perihelion or aphelion, which may be understood as a symmetry resulting from the fact that the KBOs are all currently near their perihelion passages. Moreover, observational constraints significantly disfavor perihelion for Planet Nine, making aphelion the most probable location.

If the planet has the size and the reflectivity of Neptune, our fiducial model suggests that it will have \( V \sim 23 \) at aphelion, with a parallactic motion of \( 0.5 \) hr\(^{-1} \). To visualize the favored...
Figure 14. 3D plot of the orbits of the distant KBOs and our fiducial Planet Nine model. The transparent gray plane is the ecliptic. The orbits in white correspond to all KBOs with $a > 250$ au, including the oppositely aligned 2013 FT$_{38}$ and the long-period 2004 FE$_{22}$ and uO3L91, which were not included in the simulations. The gray dots on the KBO orbits are the objects’ current positions. The colored orbit is the fiducial model with $a = 654$ au, $e = 0.45$, $i = 30^\circ$, $\omega = 150^\circ$, and $\Omega = 50^\circ$. Planet Nine has been placed at aphelion, its favored location. The points along the orbit are colored according to distance and are evenly spaced in time. (Note that the coloration is not a one-to-one correspondence with Figure 13.) We have labeled the distances and R.A./decl. coordinates of five selected locations along the orbit. A manipulable version of the 3D plot is available at the following URL: https://smillholland.github.io/P9_Orbit/.

Our current estimate of Planet Nine’s sky location is consistent with previously derived constraints. Brown & Batygin (2016) used the observational limits of previous and present sky surveys to delineate regions of phase-space where Planet Nine should have been or will be detected. Figure 10 of their paper shows that the zone of R.A. $\sim 30^\circ$–$90^\circ$ corresponds to a largely unsurveyed region on the sky. Fienga et al. (2016) considered the nominal Batygin & Brown (2016a) Planet Nine with a range of orbital positions. They performed fits of the decade-long Cassini measurements of the Earth–Saturn distance to constrain the current location of Planet Nine. They rule out Planet Nine positions near perihelion, given by R.A. $< 2^\circ$ and R.A. $> 255^\circ$, that worsen the fit to the Cassini residuals. The orbital position that leads to a maximum improvement of the residuals is located at R.A. $\sim 30^\circ$ and decl. $\sim -20^\circ$, which is near our favored region. Holman & Payne (2016b) used a dynamical model of the tidal influence of Planet Nine to fit the Cassini ranging data. They significantly rule out large areas of the sky and find a most favored location of R.A. $\sim 40^\circ$ and decl. $\sim -15^\circ$, extending $\sim 20^\circ$ in all directions. Although these results depend sensitively on the assumed solar system model (Folkner et al. 2016), it is perhaps worth noting that the region correlates well with the highest density of favored points in Figure 13.

The sparseness of the resonant configurations in Figure 13 is a reflection of the computational infeasibility of using N-body integrations alone to pinpoint the perturber parameters that allow multiple KBOs to coexist in resonance. If the resonant hypothesis is correct, an optimization scheme that more heavily exploits the MMRs will effectively constrain Planet Nine’s orbital elements and sky position. Specifically, we are interested in converging upon the parameters that allow all KBOs to exhibit long-lasting resonant libration simultaneously. This will require not just orbital integrations but also a better dynamical understanding of how these high-eccentricity resonances are arranged in phase-space.

5.2. An Alternate Interpretation of the Results

The orbital element constraints in Section 4.1 were presented as constraints on Planet Nine conditional upon its existence. We now release this assumption and comment on the implications of the results for the evidence of the planet’s existence. In this paradigm, there is opportunity for a much less optimistic interpretation.

As shown in Figure 4, for example, the time-averaged measure of clustering in the KBO arguments of perihelion, \langle S_w(t) \rangle, is almost always smaller than the present-day value, \langle S_w(0) \rangle = 6.52. In other words, sustained clustering that matches the present-day value is exceptionally difficult to obtain. The fact that it is so challenging could be viewed as evidence against the planet’s existence.

One possible explanation of the discrepancy is that \langle S_w(t) \rangle might not be a suitable comparison measurement to \langle S_w(0) \rangle. This time-averaged quantity favors clustering throughout the entire integration, even though the bodies experience perturbations in their perihelion distances that would render them unobservable.
to present-day surveys. Moreover, it might be the case that these high-\(a\), high-\(e\) KBOs are more clustered than as-yet unobserved counterparts with similarly large \(a\) but smaller \(e\). If that is the case, a variant of \(\langle S_\omega(t) \rangle\) that is restricted to bodies with perihelion distances below some limit of observability would be a more suitable comparison metric to \(S_\omega(0)\). However, we checked the integrations for a connection between observability (i.e., small perihelion distances) and clustering in \(\omega\) and \(\varpi\) and found little evidence for such a connection. There is thus no reason to expect that \(\langle S_\omega(t) \rangle\) is a biased merit function, so the tension between \(\langle S_\omega(t) \rangle\) and \(S_\omega(0)\) remains.

One way to reconcile the difficulty the KBOs have in maintaining clustering is to take it as evidence against the existence of Planet Nine. An alternate, more optimistic interpretation involves two ideas: (1) we might be observing the KBOs at a time period in which they are more clustered than average, such that \(\langle S_\omega(t) \rangle\) need not match \(S_\omega(0)\); and (2) it is possible that only a small but nonzero domain of Planet Nine parameters are capable of producing long-lasting clustering in the observed KBOs.

### 5.3. In Conclusion

The prospect that our solar system harbors an as-yet undetected super-Earth mass planet has generated an extraordinary wave of interest and attention. Two facets of the hypothesis seem particularly compelling. If the planet does exist, and especially if its mass is less than \(10 M_\oplus\), it can provide a detailed ground-truth verification that theoretical models of the atmospheres and structures of planets in unexplored mass and temperature regimes are on the right track. Furthermore, there is a satisfyingly dramatic prospect of resolution. If a trans-Neptunian planet with the proposed properties is actually out there, it will be detected very soon.

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