On blow up for a class of radial Hartree type equations

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Abstract
We study a class of Hartree type equations and prove a quantitative blow up rate for their blow up solutions. This is an analogue of the result by Merle and Raphaël on 3d cubic NLS.

1 Introduction
We consider the Cauchy problem for the focusing Hartree equation for \( d = 3 \):

\[
\begin{aligned}
    i \partial_t u + \Delta u &= -(V \ast |u|^2)u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\
    u \big|_{t=0} &= u_0 \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^1.
\end{aligned}
\] (1.1)

Here \( V(x) \) is a real valued function, problem (1.1) has three conservation laws:

- **Mass:**
  \[ M(u(t)) = M(u_0) := \int_{\mathbb{R}^3} |u(t)|^2 dx, \] (1.2)

- **Energy:**
  \[ E(u(t)) = E(u_0) := \frac{1}{2} \int |\nabla u(t, x)|^2 dx - \frac{1}{4} \int (V \ast |u|^2)(t, x)|u(t, x)|^2 dx, \] (1.3)

- **Momentum:**
  \[ P(u(t)) = P(u_0) := \text{Im} \int \bar{u} \nabla u(t) dx. \] (1.4)

We will focus on radial \( V \) and radial initial data \( u_0 \) throughout the article.
1.1 Setting of the problem and statement of the main result

For the Eq. (1.1), it describes the dynamics of the mean-field limits of many-body quantum systems, such as coherent states, condensates. In particular, it provides effective model for quantum systems with long-range interactions. Readers can refer to [12, 15, 24] for more information about the physical background of the equation. Besides its physical importance, a lot of mathematical interest for Eq. (1.1) lies in its connection and similarity to cubic NLS,

\[
\begin{aligned}
    i\partial_t u + \Delta u &= -|u|^2 u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\
    u \big|_{t=0} &= u_0.
\end{aligned}
\]  

(1.5)

We should note that the Eq. (1.5) has scaling symmetry, i.e. if \( u(t, x) \) solves (1.5), so does \( u_\lambda(t, x) := \frac{1}{\lambda} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right) \). We also observe that, when \( V(x) = \delta(x) \), then (1.1) formally becomes (1.5). From this perspective, there are two parts we want to research,

Q1: If one obtains some result for (1.5), is it possible to conclude a similar result holds for (1.1).

Q2: Whetner one can generalize a result to a cubic NLS from a Hartree type model.

The above is the main purpose of the current article corresponding to Sects. 2 and 3. However, the Hartree equation differs from cubic NLS mainly in two ways,

- Equation (1.1) does not necessarily enjoy the scaling symmetry, which is one of the most important property for (1.5).
- Due to the convolution structure in (1.1), it is non-local.

The study of blow up problem for focusing NLS (and other nonlinear dispersive PDEs) has been an active research field. Classical virial argument by Glassey in [14] implies the existence of many blow up solutions. Constructive blow up solutions and universality of blow up solutions under certain regime have attracted a lot of researchers, see for examples [27, 29, 31, 32, 39]. But very few can be said about general blow up solutions, i.e. solutions without a size constraint, and it is very hard. In [33], Merle and Raphaël consider the nonlinear Schrödinger equation (1.5), and proved that,

**Theorem 1.1** Let \( u_0 \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^1 \) be radial. Assume that the corresponding solution to (1.5) blows up in finite time \( T \), then there exists a constant \( \gamma > 0 \) such that\(^1\)

\[
||u(t)||_{L^3} \geq C(u_0) |\log(T - t)|^\gamma
\]  

(1.6)

for \( t \) close enough to \( T \).

**Remark 1.2** Merle and Raphaël have proved a more general result for a larger class of Schrödinger equations. Indeed they cover all the mass-supercritical and energy-subcritical cases for \( d \geq 3 \).

**Remark 1.3** \( \dot{H}^{s_c} \) with \( s_c = \frac{1}{2} \) is the critical norm of (1.5), i.e. this norm is invariant under the natural scaling of (1.5). Note that one has, via Sobolev embedding, \( \dot{H}^\frac{1}{2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \).

When the scaling index \( s_c = 0 \), we say it is mass-critical; \( s_c = 1 \) corresponds to the energy-critical case; in this article we focus on the case with \( 0 < s_c < 1 \), i.e. mass supercritical and energy subcritical case.

\(^1\) They obtain (1.6) for \( \gamma = \frac{1}{12} \).
Remark 1.4 The estimate (1.6) may be rephrased as: there is no type II blow up solution to (1.5), for radial initial data in $\dot{H}^{1/2} \cap \dot{H}^{1}$. Thus, all such finite time blow up solutions do not fall into the regime of Soliton Resolution Conjecture, which formally predicts all type II blow up solutions to nonlinear dispersive equation will decouple into solitary wave living at different scales and a regular term.

The main result of the current article is to prove an analogous result for a class of Hartree type equations.

Theorem 1.5 Consider the equation
\[
\begin{aligned}
& i \partial_t u + \Delta u = -(V * |u|^2)u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\
& u \big|_{t=0} = u_0 \in \dot{H}^{1/2} \cap \dot{H}^{1}.
\end{aligned}
\] (1.7)

where $u_0$ is a radial function. Assume $V(x)$ is a radial function satisfying
\[
\sum x_j \partial_{x_j} V(x) \leq -\alpha V(x)
\] (1.8)

for some $\alpha \in (2, +\infty)$, and assume the integrability condition
\[
V(x) \in L^1(\mathbb{R}^3), \quad \sum x_j \partial_{x_j} V(x) \in L^1(\mathbb{R}^3),
\] (1.9)

and the following pointwise bound
\[
|\sum x_j \partial_{x_j} V(x)| \leq \frac{C}{|x|^3}.
\] (1.10)

If $u(t)$ blows up in finite time $T$, then there exists a constant $\gamma > 0$ such that
\[
||u(t)||_{L^3} \geq C(u_0)|\log(T - t)|^\gamma
\] (1.11)

for $t$ close enough to $T$.

Remark 1.6 Assumption (1.9) seems natural. Assumptions (1.8) and (1.10) are due to technical reasons. In order to connect virial identity with energy, we have assumed the condition (1.8) holds. From (1.8), we note that either
\[
V(x) \geq \frac{cV}{|x|^\alpha}
\] (1.12)

at a neighborhood of the origin or $V(x)$ is non-positive. Note that $V(x)$ can not be non-positive because in this case there exist no blow up solutions due to energy conservation law. Under the assumption (1.8), by the following virial identity,
\[
\frac{d^2}{dt^2} \int |x|^2 |u|^2 = 8 \int |\nabla u|^2 + 4 \int x_j |u|^2 \partial_j (V * |u|^2)dx,
\] (1.13)

one could easily derive blow up solutions with negative energy. For the completeness of the article, we have supplemented the proof of this part in Appendix A.

2 A type II blow up solution means its critical norm remains bounded when $t$ tends to the blow up time $T$, i.e., $\limsup_{t \nearrow T} ||u(t)||_{\dot{H}^s} < +\infty$, and a blow up solution is type I if it is not type II.

3 Note that (1.11) implies
\[
||u(t)||_{\dot{H}^{1/2}} \geq C(u_0)|\log(T - t)|^\gamma
\]
in $\dot{H}^{1/2}$ setting and our method could give $\gamma = \frac{2}{31+\varepsilon}, \forall \varepsilon > 0$. 
Remark 1.7 The assumptions on $V(x)$ can be relaxed. Indeed, (1.8) and $V(x) \in L^1(\mathbb{R}^3)$ imply $\sum x_j \partial_{x_j} V(x) \in L^1(\mathbb{R}^3)$.

Remark 1.8 We give an example which satisfies the conditions (1.8), (1.9), and (1.10) i.e. the object of the above analysis is not $\emptyset$,

$$V(Z) = \frac{1}{|Z|^3 \log|Z|^\alpha} \chi(Z), \quad for \ \alpha > 1,$$

with

$$\chi(Z) = \begin{cases} 1 & Z \leq \delta \\ 0 & Z \geq 2\delta \end{cases} \quad for \ some \ \delta \ small \ enough.$$  

We review a series of related work regarding Hartree type equations and NLS,

- mass-critical case

In the 1990s, Merle’s work [27] had given the characteristic in $H^1$ for the blow-up solutions with minimal mass in nonlinear Schrödinger equation,$^4$

$$\begin{cases} i \partial_t u + \Delta u = -|u|^4 u. & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u |_{t=0} = u_0. \end{cases}$$

And Dodson improved this result in $L^2$ setting in [6, 7]. The parallel result in Hartree-equation setting can be found in [24, 35, 36].

When the mass is below the ground state $Q(x)$, Weinstein proved the solution is global and scattering in $H^1$ in [41] and Dodson improved this result to $L^2$ setting in [4].

When the mass is beyond the ground state, Bourgain and Wang in [2] constructed a type of blow up solutions with the blow up rate $||\nabla u(t)||_{L^2} \sim \frac{1}{T-t}$. Besides, another type of blow up solutions, the log-log blow up solutions, are suggested numerically by Landman, Papanocolau, Sulem, Sulem in [21]. This kind of solutions blow up in finite time with the rate $||\nabla u(t)||_{L^2} \sim \left(\log \left|\log(T-t)\right|\right)^{\frac{1}{2}}$. Perelman firstly constructed this kind of solutions in her work [37]. After that, the log-log blow-up solutions have been studied in depth and comprehensively by Merle and Raphaël in a series of work [28–32, 39]. They give a more complete portrayal of the blow up rate in the vicinity of the ground state solution and a classification of the blow up solutions.

We should remark, for the mass-critical focusing NLS, when the mass is much larger than threshold, Merle constructed a $k$-points blow up solution in [26], and readers can also refer to [11, 38] for the study of weakly interacting multi bubbles blow up dynamics for NLS. Besides, Martel and Raphaël in [25] constructed a multi-bubbles blow up solution, with the rate $||\nabla u(t)||_{L^2} \sim \frac{|\log(T-t)|}{T-t}$ due to strong interactions.

- mass-supercritical and energy-subcritical case

$^4$ The ground state associated to (1.15) is the unique positive solution to

$$\Delta Q + Q^{1+\frac{4}{d}} = Q,$$

which supply a stationary solution to (1.15) with $u(t, x) = e^{it}Q(x)$. 

\[ Springer \]
Between this range, for the focusing 3d cubic NLS, Merle and Raphaël in [33] gave a universal blow-up lower bound in the radial case. Besides, towards this model, there is a series of work concerning the following quantity,\(^5\)

\[
M_\Xi := \frac{M[u]E[u]}{M[Q]E[Q]},
\]

readers can refer to for [8, 16, 17] when \(M_\Xi \in (0, 1)\), [9] for \(M_\Xi = 1\) and [1, 10, 36] in the case \(M_\Xi > 1\).

For the focusing Hartree equation, readers may also refer to [13, 42] in this range. Our result is also a step toward understanding some universal blow up behaviour and the connection between NLS and Hartree type equations.

- energy-critical case

For the focusing energy-critical Schrödinger equation, Kenig and Merle [19] used a concentration compactness argument and a rigidity theorem to prove any radial solutions \(u(t)\) in \(d = 3, 4, 5\) which satisfy \(E(u_0) < E(W)\) and \(||u_0||_{\dot{H}^1} < ||W||_{\dot{H}^1}\)\(^6\) must be global and scatter. For the nonradial case, readers can refer to [5] in \(d = 4\) and [19] \(d \geq 5\).

Returning to the Hartree-equation setting, Li et al. [22] treated the focusing case, and proved a parallel result with [19].

### 1.2 A review of Merle and Raphaël’s work in [33] and the connection with our result

Since our work rely on the method developed by Merle and Raphaël in [33], let us review some points and highlight the main quantities in their analysis. There is also some interesting work related to this topic, one can refer to [40] for Navier Stokes equation, [20] for focusing nonlinear Klein–Gordon equation and [3] for inhomogeneous nonlinear Schrödinger equation.

In this subsection, in order to make the idea of the article more concise and clear, we only review a weaker version of Merle and Raphaël. More precisely, we impose the following non-positive energy assumption for Theorem 1.1,

\[
E(u_0) \leq 0,
\]

and the critical norm under consideration becomes \(\dot{H}^{\frac{1}{2}}\). Towards the nonlinear Schrödinger equation (1.5), for \(t\) close enough to the blow up time \(T\), we renormalize \(u(t)\) with its \(\dot{H}^1\) norm, i.e. let

\[
\lambda_u(t) = \left(\frac{1}{||\nabla u(t)||_{L^2}}\right)^2,
\]

\(^5\) \(M(u) := \int |u|^2\) and \(E(u) := \frac{1}{2} \int |\nabla u|^2 - \frac{1}{4} \int |u|^4\). And \(Q\) is the unique \(H^1\) radial positive solution of

\[
\Delta Q - \frac{1}{2} Q + |Q|^2 Q = 0.
\]

\(^6\) The \(W(x)\) is the unique radial positive solution in \(\dot{H}^1\) to

\[
\Delta W + |W|^{\frac{4}{d-2}} W = 0,
\]

with \(d \geq 3\).
and define the renormalization of $u(t)$:

$$v(\tau, x) = \lambda_u(t) \tilde{u}(t - \lambda_u(t)^2 \tau, \lambda_u(t)x).$$

(1.17)

From local theory of NLS, we note that

$$\lambda_u(t) \lesssim \sqrt{T - t}.$$  

(1.18)

Then a long time behaviour of blow up dynamics has been transformed into a Cauchy problem that satisfies the following special conditions:

$$\begin{cases}
i \partial_v + \Delta v(\tau, x) = |v|^2 v(\tau, x), \\
v(\tau, x)|_{\tau=0} = \lambda_u(t) \tilde{u}(t, \lambda_u(t)x),
\end{cases}$$

(1.19)

with

$$||v(0, x)||_{\dot{H}^{\frac{1}{2}}} = ||u(t)||_{\dot{H}^{\frac{1}{2}}},$$

(1.20)

$$||v(0, x)||_{\dot{H}^1} = 1,$$

(1.21)

and the non-positive energy

$$E(v_0) \leq 0.$$  

(1.22)

Let

$$N(t) = -\log \lambda_u(t),$$

(1.23)

we could conclude, Merle and Raphaël converted Theorem 1.1 into proving the following conclusion, there exists a universal constant $\gamma > 0$ such that

$$||v(0, x)||_{\dot{H}^{\frac{1}{2}}} \geq N^\gamma.$$  

(1.24)

The above analysis reduces the difficulty of the problem to some extent. We know that the blow up behaviour of the solution is a long-time dynamic behaviour, and there are few tools that can be directly applied to characterize the explosion. However, if we are now looking at the local behaviour of the solution, then a rich local theory of the solutions can be applied, making the problem possible.

From the above analysis, it is also a very natural thing to give the connection between the lower bounds of the blow up rate of the Schrödinger equation and Hartree equation, which also gives the possibility to apply the local theory.

To further explain the work by Merle and Raphaël, we now introduce the following notations, the scaling invariant Morrey–Campanato norm

$$\rho(u, R) = \sup_{R' \geq R} \frac{1}{R'} \int_{R' \leq |x| \leq 2R'} |u(x)| dx,$$

(1.25)

a quantity related to the initial data,

$$M_0 = \frac{4||v(0)||_{\dot{H}^{\frac{1}{2}}}}{C_{GN}},$$

where $C_{GN}$ is a universal constant related to Gagliardo-Nirenberg inequality and a similar definition to (1.16),

$$\lambda_v(\tau) = \frac{1}{||v(\tau)||_{\dot{H}^1}^2}.$$  

(1.26)
In order to achieve the goal (1.24), there are three steps need to be implemented. Step 1. Uniform control of the $\rho$ norm and the dispersive estimate.

To achieve the first goal, we give the following proposition,

**Proposition 1.9** Let $v(\tau) \in C([0, e^N], \dot{H}^{1/2} \cap \dot{H}^1)$ be a radially symmetric solution to (1.5) where $N$ is a sufficiently large number and (1.22), (1.21) hold. Then there exist universal constants $C_1, \alpha_1$ and $\alpha_2$ such that the following hold, $\forall \tau_0 \in [0, e^N]$,

(i) the uniform control of the $\rho$ norm:

$$\rho(v(\tau_0), C_1 M_0^{\alpha_1} \sqrt{\tau_0}) \leq C_1 M_0^2.$$  (1.27)

(ii) the dispersive estimate:

$$\int_0^{\tau_0} (\tau_0 - \tau) ||v(\tau)||_{\dot{H}^1}^2 d\tau \leq C_1 M_0^{\alpha_2} \tau_0^{3/2}.$$  (1.28)

Moreover, if we assume $M_0^{\alpha_2} < e^{\sqrt{N}}$, there exist a sequence of $\{\tau_i\}$ with $\tau_i \in [0, e^i]$ and $i \in \{\sqrt{N}, \sqrt{N} + 1, \ldots, N\}$ with the following bounds holds,

$$\sqrt{\tau_i} \leq C_1 M_0^{\alpha_2} \frac{1}{\lambda v(\tau_i)} \in \left[ \frac{1}{10 C_1 M_0^{\alpha_2} e^{i-1}}, \frac{10}{C_1 M_0^{\alpha_2} e^{i}} \right].$$  (1.29)

There are two points we should note,

**Remark 1.10** $\{\alpha_1 = 1, \alpha_2 = 5\}$ is an allowable value of the above proposition, and it is related to the blow up rate in Theorem 1.1, we could see it more clearly to achieve step 2.

**Remark 1.11** The conclusion (1.29) in proposition 1.9 is a direct inference of (1.28) with the help of (1.21). Since (1.28) only gives an estimate of the decay of $||v(\tau)||_{\dot{H}^1}$ in the average sense, we can only get (1.29) at some special time rather than a pointwise estimate.

Step 2. Lower bound on a weighted local $L^2$ norm of $v(0)$.

Now we are ready to give a uniform lower bound on a weighted local $L^2$ norm of $v(0)$.

**Proposition 1.12** Let $v(\tau)$ satisfies the conditions in Proposition 1.9 and $\{\tau_i\}$ are chosen in Proposition 1.9, then the following weighted $L^2$ norm on a sufficiently large ball holds,

$$\frac{1}{\lambda v(\tau_i)} \int_{|x| \leq M_0^{2+2\alpha_2} \lambda v(\tau_i)} |v(0)|^2 \geq c_3,$$  (1.30)

where $c_3$ is a universal constant.

**Remark 1.13** The idea of proof is the following: first we derive the lower bound at time $\tau = \tau_i$ with the aid of non-positive energy constraint, i.e.

$$\frac{1}{\lambda v(\tau_i)} \int_{|x| \leq M_0^{2+2\alpha_2} \lambda v(\tau_i)} |v(\tau_i)|^2 \geq 2c_3.$$  (1.31)

Second, thanks to (1.27) and (1.28), we conclude the difference between $\tau = 0$ and $\tau = \tau_i$ is sufficiently small, i.e. $\exists \varepsilon > 0$ small enough, such that

$$| \frac{1}{\lambda v(\tau_i)} \int_{|x| \leq M_0^{2+2\alpha_2} \lambda v(\tau_i)} |v(\tau_i)|^2 - \frac{1}{\lambda v(\tau_i)} \int_{|x| \leq M_0^{2+2\alpha_2} \lambda v(\tau_i)} |v(0)|^2 | < \varepsilon.$$  (1.32)

Combining (1.31) and (1.32), we derive the final conclusion (1.30).
Remark 1.14 The lower bound \((1.30)\) has implied the choice of \(\gamma\) in Theorem 1.1. A straightforward algebraic computation of our analysis implies

\[
\gamma = \frac{1}{4 + 2\alpha_2 + \epsilon},
\]

for \(\forall \epsilon > 0\), i.e we could derive \(\gamma = \frac{1}{14 + \epsilon}\) in this setting while Merle and Raphaël in [33] could give

\[
\gamma = \frac{1}{12}.
\]  

(1.33)

Step 3. The construction of \(N^\gamma\) disjoint annuli.

Now we are going to construct sufficiently many disjoint annuli. We need to pay attention to the following two points,

- These \(N^\gamma\) annuli are disjoint.
- At each annulus, the similar lower bound still holds in \((1.30)\).

A direct computation by Hölder inequality implies the following choice of the size of annuli satisfies the second point,

\[
\mathcal{C}_i := \left\{ x \in \mathbb{R}^3 \mid \frac{\lambda_v (\tau_i)}{M_0^{2 + \alpha_2}} \leq |x| \leq \lambda_v (\tau_i) M_0^{2 + \alpha_2} \right\}.
\]

In order to satisfy the first point above, we need to choose the number \(p\) such that

\[
\frac{\lambda_v (\tau_i + p)}{M_0^{2 + \alpha_2}} \geq \lambda_v (\tau_i) M_0^{2 + \alpha_2}.
\]

Thanks to \((1.29)\), we can choose \(p\) such that \(e^{\frac{p}{\epsilon^2}} \approx M_0^{4 + 2\alpha_2}\) which gives the derived choice of \(N^\gamma\) disjoint annuli.

Based on the above three steps, we have the following technical remark,

Remark 1.15 The condition \((1.22)\) can be weaken by another three assumptions, and the stronger version will be stated in Sect. 3. Here we are only using condition \((1.22)\) to state this proposition for convenience.

Now we want to talk about the relation in this paper between NLS and Hartree equation,

- Using the above techniques for proving Theorem 1.1, we can obtain result that is parallel in the Hartree equation, which is also the purpose of our article.
- If we get the conclusion in Hartree equation directly, then we can approximate \(\delta\) function by doing a suitable scaling transformation of the potential function \(V(x)\),

\[
V_\epsilon (x) = \frac{1}{\epsilon^3} V \left( \frac{x}{\epsilon} \right), \quad \forall \epsilon > 0.
\]

(1.34)

After using the mature local theory, \(8\) we prove that the Schrödinger equation is also valid in this case.

The above two points give a connection between NLS and Hartree equation, and we supply complete proof in Sect. 3.

\(7\) We should note that the conditions \((1.8), (1.9), (1.10)\) is invariant under the transformation \((1.34)\).

\(8\) The above three steps are local versions for the dynamics of the solutions, so we could apply local theory.
1.3 Strategy and structure of the paper

We use the robust strategy by Merle and Raphaël in [33]. We should note that there are two key points:

(1) The additional error terms.

The condition (1.9) is natural assumption in our problem setting, and the role of the condition (1.8) is to connect the energy (1.3) with the virial identity (1.13). Another assumption (1.10) is technical, because we need to control additional error terms which do not exist in the Schrödinger-equation setting. As in the Schrödinger equation, the “potential function” is \( \delta(x) \). However, in our setting, \( V(x) \) belongs to \( L^1(\mathbb{R}^3) \). So, when the blow-up phenomenon happens, there are some additional error terms we should treat. We give a priori control on two channels which helps us to overcome this difficulty. For more details, one can see step 2 in Proposition 2.2.

(2) The difficulty caused by the no-scaling property of \( V(x) \).

Since we treat the no-scaling case for the potential function \( V(x) \), we should check carefully the constants chosen in Propositions 2.2 and 2.4. After the renormalization of \( u(t), v^{\lambda(t)}(\tau, x) := \lambda_u(t)\bar{u}(t - \lambda_u^2(t)\tau, \lambda_u(t)x) \) at different time \( t \) satisfy different equations

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
& i\partial\tau v^{\lambda(t)} + \Delta v^{\lambda(t)} = -(V_{\lambda(t)} * |v^{\lambda(t)}|^2) v^{\lambda(t)} . \quad (\tau, x) \in [0, \frac{t}{\lambda(t)\tau}) \times \mathbb{R}^3 , \\
& V_{\lambda(t)}(x) = \lambda(t)^3 V(\lambda(t)x) , \\
& v^{\lambda(t)} |_{\tau = 0} = \lambda(t)\bar{u}(t, \lambda(t)x) \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^{1} ,
\end{array}
\right.
\end{align*}
\]

where \( \lambda_u(t) = \left( \frac{1}{||V(u(t))||_L^2} \right)^2 \). For these \( v^{\lambda(t)}(\tau, x) \), we should give a uniform estimate which is independent of the time \( t \). We will deal with this very carefully both in Propositions 2.2 and 2.4, which is also the core of our analysis. Here, we explain from two aspects why the potential function does not have scaling invariant property will not have an essential impact for our analysis.

(i) The renormalization of \( u(t) \)

\( v^{\lambda(t)}(\tau, x) \) defined above satisfies equations (1.35), and it also follows the law of conservation of energy:

\[
E^{\lambda}(v^{\lambda(t)}(\tau)) = E^{\lambda}(v^{\lambda(t)}(0))
\]

\[
= \frac{1}{2} \int |\nabla v^{\lambda(t)}(0, x)|^2 dx - \frac{1}{4} \int (V_{\lambda(t)} * |v^{\lambda(t)}|^2)(0, x)|v^{\lambda(t)}|^2(0, x)dx
\]

\[
= \lambda(t) \left[ \frac{1}{2} \int |\nabla u(t, x)|^2 dx - \frac{1}{4} \int (V * |u|^2)|u|^2(t, x)dx \right]
\]

\[
= \lambda(t) E(u(t)). \tag{1.36}
\]

(1.36) implies we still obtain the scaling invariant property for the energy conservation law although \( V(x) \) does not own it.

(ii) The scaling for the potential function \( V(x) \).

A direct computation implies

\[
\sum x_j \partial x_j V(\lambda x) = \lambda^3 \left| \sum \lambda x_j \lambda x_j V(\lambda x) \right| \leq \lambda^3 \cdot \frac{C}{|\lambda x|^3} = \frac{C}{|x|^3} , \tag{1.37}
\]

under the assumption (1.10) which is a universal upper bound independent of \( \lambda \). Besides, we only use \( L^1 \) norm for the potential function \( V(x) \) which is also a scaling invariant norm under the renormalization, i.e. \( ||V_\lambda(x)||_{L^1} = ||V(x)||_{L^1} \).
The paper is organized as follows. In Sect. 2, we will prove Theorem 1.5, which gives an answer to Q1. More specifically, we will give a uniform control of the \( \rho \) norm, which is a suitable scaling invariant Morrey–Campanato norm, and a lower bound on a weight local \( L^2 \) norm, then we use a blackbox by applying Propositions 2.2 and 2.4 to finish the proof for Theorem 1.5. In Sect. 3, we will prove Theorem 1.1 by our result, which answers Q2. Properly speaking, we generalize the result for Schrödinger equation from the Hartree type equation in Sect. 2. In Appendix A, for readers’ convenience, we give some examples of blow up solutions and some direct observations for the blow up rate for \( V(x) \) with higher integrability conditions. However, we can not give any special examples \( V(x) \) which lead to this kind of blow up solutions. Appendix B is devoted to give some standard result of stability theory needed in Sect. 3.

2 Blow-up rate for the blow-up solution

In this section, we focus on the proof of Theorem 1.5, i.e. we consider the equation

\[
\begin{cases}
i \partial_t u + \Delta u = -(V \ast |u|^2)u, & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\
u |_{t=0} = u_0 \in \dot{H}^{1/2} \cap \dot{H}^1.
\end{cases}
\]

with \( V(x) \) satisfies the conditions in Theorem 1.5. We will give a lower bound for the blow-up rate.

First, we prove the main propositions at the heart of the proof of Theorem 1.5. Let \( u_0 \in \dot{H}^{1/2} \cap \dot{H}^1 \) with radial symmetry and assume that the corresponding solution \( u(t) \) to (2.1) blows up in finite time \( 0 < T < +\infty \). We can pick \( t \) close enough to \( T \). Let

\[
\lambda_u(t) = \left( \frac{1}{||\nabla u(t)||_{L^2}} \right)^2, \tag{2.2}
\]

then from the local theory,

\[
\lambda_u(t) \lesssim \sqrt{T - t}. \tag{2.3}
\]

We define the renormalization of \( u(t) \) by

\[
v^{(t)}(\tau, x) = \lambda_u(t) \tilde{u}(t - \lambda_u^2(t) \tau, \lambda_u(t) x). \tag{2.4}
\]

In this subsection, to clarify the notations, we omit the dependence of \( \lambda \) on \( u(t) \) and define \( v^\lambda := v^{(t)}(\tau, x) \).

Then it is not hard to check that \( v^\lambda \) satisfies the following equation

\[
\begin{cases}
i \partial_\tau v^\lambda + \Delta v^\lambda = -(V_\lambda \ast |v^\lambda|^2) v^\lambda, & (\tau, x) \in [0, \frac{T}{\lambda^2}] \times \mathbb{R}^3, \\
v^\lambda |_{\tau=0} = \lambda \tilde{u}(t, \lambda x) \in \dot{H}^{1/2} \cap \dot{H}^1.
\end{cases}
\]

Because of the no-scaling property of the potential function \( V(x) \), the renormalization \( v^\lambda \) satisfies different equation at different time \( \tau \). So we need some universal properties for these \( v^\lambda \) which are independent of \( \lambda \).

We first give needed definitions in our following propositions and recall some elementary inequalities.

Define a smooth radially symmetric cut-off function,

\[
\psi(x) = \begin{cases} 
\frac{|x|^2}{2}, & |x| \leq 2, \\
0, & |x| \geq 3,
\end{cases} \tag{2.6}
\]

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with $\psi_R(x) := R^2 \psi \left( \frac{x}{R} \right)$ and we state the following lemma in [33]:

**Lemma 2.1 (Radial Gagliardo-Nirenberg inequality)**

(i) There exists a universal constant $C > 0$ such that for all $u \in L^3$,

$$\forall R > 0, \quad \frac{1}{R} \int_{|y| \leq R} |u|^2 dy \leq C ||u||^2_{L^3}. \quad (2.7)$$

(ii) For all $\eta > 0$, there exists a constant $C_\eta > 0$ such that for all $u \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^{1}$ with radial symmetry, for all $R > 0$,

$$\int_{|x| \geq R} |u|^4 \leq \eta ||\nabla u||^2_{L^2(|x| \geq R)} + \frac{C_0}{R} [\rho(u, R)^2 + \rho(u, R)^3]. \quad (2.8)$$

And from the definition of energy, we know

$$E^\lambda(v^\lambda) := \frac{1}{2} \int |\nabla v^\lambda|^2 dx - \frac{1}{4} \int (V_\lambda * |v^\lambda|^2)|v^\lambda|^2 dx$$

$$\geq \frac{1}{2} \left( 1 - \left( \frac{||v^\lambda||_{L^3}^3}{c_V} \right)^2 \right) \int |\nabla v^\lambda|^2 dx, \quad (2.9)$$

for some $c_V > 0$ which is independent of $\lambda$.

With these prepared knowledge, we prove the following,

**Proposition 2.2** Let $\tau_* > 0$ and $v^\lambda(\tau) \in C([0, \tau_*], \dot{H}^{\frac{1}{2}} \cap \dot{H}^{1})$ be a radially symmetric solution to (2.5) and assume

$$\tau_*^{\frac{1}{2}} \max(E^\lambda(v^\lambda_0), 0) < 1, \quad (2.10)$$

and

$$M_0^{\lambda} := \frac{4||v^\lambda_0||_{L^3}}{c_V} \geq 2, \quad (2.11)$$

then there exist $C_1, \alpha_1, \alpha_2 > 0$ which are independent of $\lambda$ such that

$$\rho(v^\lambda(\tau_*), (M_0^{\lambda})^{\alpha_1} \sqrt{\tau_*}) \leq C_1 (M_0^{\lambda})^{\alpha_2}, \quad (2.12)$$

and

$$\int_0^{\tau_*} (\tau_* - \tau)||\nabla v^\lambda(\tau)||^2_{L^2} d\tau \leq (M_0^{\lambda})^{\alpha_2} \tau_*^{\frac{3}{2}}. \quad (2.13)$$

**Proof** Step 1. Local radial virial estimate.

We define

$$V_a(\tau) := \int_{R^3} a(x)|v^\lambda(\tau, x)|^2 dx, \quad (2.14)$$

and

$$P_a(\tau) := \frac{d}{d\tau} V_a(\tau) = 2Im \int_{R^3} \nabla a(x) \nabla v^\lambda(\tau, x)v^\lambda(\tau, x)dx. \quad (2.15)$$

The a direct computation gives
\[
\frac{1}{4} \frac{d}{dt} P_a(\tau) \leq \alpha E^\lambda(v^\lambda(\tau)) - \left(\frac{\alpha}{2} - 1\right) \int |\nabla v^\lambda|^2 dx + \frac{C}{R^2} \int_{2R \leq |x| \leq 3R} |v^\lambda|^2 dx
\]

\[
+ \frac{1}{4} \int \int [(\partial_{x_j} \psi_R(x) - \partial_{y_j} \psi_R(y)) - (x_j - y_j)] \partial_{x_j} x \cdot y
\]

\[
\times V_\lambda(x - y)|v^\lambda(\tau, y)|^2 |v^\lambda(\tau, x)|^2 dxy
\]

\[
= \alpha E^\lambda(v^\lambda(\tau)) - \left(\frac{\alpha}{2} - 1\right) \int |\nabla v^\lambda|^2 dx + \frac{C}{R^2} \int_{2R \leq |x| \leq 3R} |v^\lambda|^2 dx + \Phi(\tau),
\]

where we substitute \(a(x)\) with \(\psi_R(x)\) and use the condition (1.8) with

\[
|\Phi(\tau)| = \frac{1}{4} \int \int [[(\partial_{x_j} \psi_R(x) - \partial_{y_j} \psi_R(y)) - (x_j - y_j)] \partial_{x_j} x \cdot y
\]

\[
\times V_\lambda(x - y)|v^\lambda(\tau, y)|^2 |v^\lambda(\tau, x)|^2 dxy |
\]

\[
\leq \int \int_{|x| \geq R, |y| \geq R} [[(\partial_{x_j} \psi_R(x) - \partial_{y_j} \psi_R(y)) - (x_j - y_j)] \partial_{x_j} x \cdot y
\]

\[
\times V_\lambda(x - y)|v^\lambda(\tau, y)|^2 |v^\lambda(\tau, x)|^2 dxdy
\]

\[
+ \int \int_{|x| \leq R, |y| \geq 2R} [[(\partial_{x_j} \psi_R(x) - \partial_{y_j} \psi_R(y)) - (x_j - y_j)] \partial_{x_j} x \cdot y
\]

\[
\times V_\lambda(x - y)|v^\lambda(\tau, y)|^2 |v^\lambda(\tau, x)|^2 dxdy
\]

\[
+ \int \int_{|y| \leq R, |x| \geq 2R} [[(\partial_{x_j} \psi_R(x) - \partial_{y_j} \psi_R(y)) - (x_j - y_j)] \partial_{x_j} x \cdot y
\]

\[
\times V_\lambda(x - y)|v^\lambda(\tau, y)|^2 |v^\lambda(\tau, x)|^2 dxdy
\]

\[
:= (I) + (II) + (III).
\]

We should point out, the term \(\Phi(\tau)\) is different from the error term in Schrödinger-equation setting and this is the reason to assume the additional condition (1.10). We summarize the estimate of \(\Phi(\tau)\) as follows,

**Lemma 2.3** Assume (1.9), (1.10) in Theorem 1.5 hold, then \(\forall \eta > 0, 0 < R_1 < R\), there exists \(C(\eta) > 0\) such that

\[
|\Phi(\tau)| \leq \eta \|
\nabla v^\lambda(\tau)\|_{L^2}^2 + C(\eta)R^{-1}[\rho(v^\lambda(\tau), R) + \rho(v^\lambda(\tau), R)^3
\]

\[
+ \rho(v^\lambda(\tau), R_1)\rho(v^\lambda(\tau), R)] + \|
\nabla v^\lambda\|_{L^2}^2 \cdot \frac{R^2}{R^2} \cdot \rho(v^\lambda(\tau), R). \tag{2.16}
\]

If we assume Lemma 2.3 holds, then we conclude

\[
\frac{1}{4} \frac{d}{dt} P_a(\tau) \leq \alpha E^\lambda(v^\lambda(\tau)) - \frac{1}{2} \left(\frac{\alpha}{2} - 1\right) \int |\nabla v^\lambda|^2 dx
\]

\[
+ C(\eta)R^{-1}[\rho(v^\lambda(\tau), R) + \rho(v^\lambda(\tau), R)^3 + \rho(v^\lambda(\tau), R_1)
\]

\[
\rho(v^\lambda(\tau), R)] + \|
\nabla v^\lambda\|_{L^2}^2 \cdot \frac{R^2}{R^2} \cdot \rho(v^\lambda(\tau), R). \tag{2.17}
\]

Now we return to the proof of Lemma 2.3,
Proof For (I),
\[
(I) \leq \|x \cdot \nabla V(\lambda x)\|_{L^1} \|v^\lambda(\tau, x) \cdot \chi_{\{|x| \geq R\}}\|_{L^4}^2 \|v^\lambda(\tau, y) \cdot \chi_{\{|y| \geq R\}}\|_{L^4}^2 \\
\leq \|v^\lambda(\tau, x) \cdot \chi_{\{|x| \geq R\}}\|_{L^4}^2 \|v^\lambda(\tau, y) \cdot \chi_{\{|y| \geq R\}}\|_{L^4}^2 \\
\lesssim \eta \|\nabla v^\lambda\|_{L^2}^2 + \frac{C_H}{R} [\rho(v^\lambda, R)^2 + \rho(v^\lambda, R)^3] \tag{2.18}
\]
by Lemma 2.1.

Since (II) and (III) are symmetric, we only estimate (II). In fact, this term does not appear in the Schrödinger equation. We should divide the area \(\{|x| \leq R, |y| \geq 2R\}\) into \(\{|x| \leq R_1, |y| \geq 2R\}\) and \(\{|x| \leq 2 R_1, |y| \geq 2R\}\) and choose \(R_1\) carefully in next step. The term in the area \(\{|x| \leq R_1, |y| \geq 2R\}\) can be controlled by \(\frac{R_1^2}{R^2} \|\nabla v^\lambda\|_{L^2}^2 \rho(v^\lambda(\tau), R)\) and the other is controlled by \(\frac{1}{R} \rho(v^\lambda(\tau), R_1) \rho(v^\lambda(\tau), R)\). The details are the followings.

In the area \(\{|x| \leq 2R, |y| \geq 2R\}\), using the property (1.10), we conclude
\[
\|[(\partial_{x_j} \psi_R(x) - \partial_{y_j} \psi_R(y)) - (x_j - y_j)] \partial_{x_k} V_\lambda(x - y)\| \lesssim \lambda^3 \cdot \frac{1}{(\lambda |x - y|)^3} \tag{2.19}
\]
which leads to
\[
(II) \lesssim \int \int_{|x| \leq R, |y| \geq 2R} \frac{1}{|y|^3} |v^\lambda(\tau, y)|^2 |v^\lambda(\tau, x)|^2 dxdy \\
\lesssim \int_{|x| \leq R} |v^\lambda(\tau, x)|^2 dx \left[ \sum_{j=0}^{+\infty} \frac{1}{(2jR)^3} \int_{2jR \leq |y| \leq 2j+1R} |v^\lambda(\tau, y)|^2 dy \right] \\
\lesssim \left[ \int_{|x| \leq R} |v^\lambda(\tau, x)|^2 dx + \int_{R_1 \leq |x| \leq R} |v^\lambda(\tau, x)|^2 dx \right] \cdot \frac{1}{R^2} \rho(v^\lambda(\tau), R) \\
\lesssim \left[ R_1^2 \|\nabla v^\lambda\|_{L^2}^2 + R \rho(v^\lambda(\tau), R_1) \right] \cdot \frac{1}{R^2} \rho(v^\lambda(\tau), R). \tag{2.20}
\]

From (2.18) and (2.20), we finish the proof of Lemma 2.3. \( \square \)

Step 2. A priori control of the \(\rho\) norm on parabolic space time interval.

In this step, we need to choose \(R_1\) appropriately. On the one hand, we should choose \(R_1\) small enough so that \(\frac{R_1^2}{R^2} \cdot \rho(v^\lambda(\tau), R) \ll \frac{1}{2}\). On the other hand, we can not choose \(R_1\) too small otherwise we can not control \(R^{-1} \rho(v^\lambda(\tau), R_1) \rho(v^\lambda(\tau), R)\). However, the above analysis is not self-contradictory. A natural idea is, we choose \(R_1\) with \(\frac{R_1^2}{R^2} \lesssim \epsilon^{c_1}\) such that \(\frac{R_1^2}{R^2} \cdot \rho(v^\lambda(\tau), R) \ll \frac{1}{2}\). At the same time, let \(R \geq R_1 \gg (M_0^\lambda)^{\frac{1}{2}}\) so that \(R^{-1} \rho(v^\lambda(\tau), R_1) \rho(v^\lambda(\tau), R)\) still can be treated as an error term. Once this difficulty is overcome, the rest of the analysis is similar to the one in [33].

Let \(\epsilon, \delta > 0\) be a small enough constant to be chosen later. Let
\[
G_\epsilon = (M_0^\lambda)^{\frac{1}{2}}, \quad A_{\epsilon 1} = \left( \frac{\epsilon G_\epsilon}{(M_0^\lambda)^2} \right)^{\frac{1}{2}}, \quad A_{\epsilon 2} = \left( \frac{\epsilon G_\epsilon}{(M_0^\lambda)^2} \right)^{\frac{1}{2}} \cdot \left( \frac{\delta}{M_0^\lambda} \right). \tag{2.21}
\]
Recall from Lemma 2.1, there exists a universal constant \(C > 0\) such that
\[
\forall R > 0, \forall u \in L^3, \rho(u, R) \leq C \|u\|_{L^3}^2.
\]
From the regularity of the flow \( v^\lambda \in C([0, \tau^*], \dot{H}^{1/2} \cap \dot{H}^1) \) and the definition of \( M_0^\lambda \), we may consider the largest time \( \tau_1 \in [0, \tau^*] \) such that

\[
\forall \tau_0 \in [0, \tau_1], \quad [M_1^\lambda(A_{\xi_1}, \tau_1)]^2 = \max_{\tau \in [0, \tau_1]} \rho(v^\lambda(\tau), A_{\xi_1} \sqrt{\tau}) \leq \frac{(M_0^\lambda)^2}{\varepsilon}, \tag{2.22}
\]

and

\[
[M_2^\lambda(A_{\xi_2}, \tau_1)]^2 = \max_{\tau \in [0, \tau_1]} \rho(v^\lambda(\tau), A_{\xi_2} \sqrt{\tau}) \leq \frac{(M_0^\lambda)^5}{\varepsilon^{10}}, \tag{2.23}
\]

and

\[
\int_0^{\tau_0} (\tau_0 - \tau) ||\nabla v^\lambda(\tau)||^2_{L^2} d\tau \leq G_\varepsilon \varepsilon^{-1/2} \tau_0^{3/4}. \tag{2.24}
\]

We claim that:

\[
\forall \tau_0 \in [0, \tau_1], \quad [M_1^\lambda(A_{\xi_1}, \tau_1)]^2 = \max_{\tau \in [0, \tau_1]} \rho(v^\lambda(\tau), A_{\xi_1} \sqrt{\tau}) \leq \frac{(M_0^\lambda)^2}{\varepsilon}, \tag{2.25}
\]

and

\[
[M_2^\lambda(A_{\xi_2}, \tau_1)]^2 = \max_{\tau \in [0, \tau_1]} \rho(v^\lambda(\tau), A_{\xi_2} \sqrt{\tau}) \leq \frac{(M_0^\lambda)^5}{\varepsilon^{10}}, \tag{2.26}
\]

and

\[
\int_0^{\tau_0} (\tau_0 - \tau) ||\nabla v^\lambda(\tau)||^2_{L^2} d\tau \leq \frac{G_\varepsilon}{2} \tau_0^{3/4} \tag{2.27}
\]

provided \( \varepsilon, \delta > 0 \) has been chosen small enough, and (2.12), (2.13) follow.

Proof of (2.25) and (2.26). \( \forall \tau_0 \in [0, \tau_1] \), we integrate twice in time between 0 and \( \tau_0 \) from (2.17) and get:

\[
\int \psi_R |v^\lambda(\tau_0)|^2 + C_1 \int_0^{\tau_0} (\tau_0 - \tau) ||\nabla v^\lambda(\tau)||^2_{L^2} d\tau
\]

\[
\leq \int \psi_R |v^\lambda(0)|^2 + \tau_0 \left[ \text{Im} \left( \int \nabla \psi_R(x) \nabla v^\lambda(0) \bar{v}^\lambda(0) dx \right) + E(v^\lambda) \right]
\]

\[
+ \frac{1}{R} \int_0^{\tau_0} (\tau_0 - \tau) [\rho(v^\lambda(\tau), R) + \rho(v^\lambda(\tau), R)]^3 + \rho(v^\lambda(\tau), R_1) \rho(v^\lambda(\tau), R)] d\tau
\]

\[
+ \frac{R_1^2}{R^2} \int_0^{\tau_0} (\tau_0 - \tau) ||\nabla v^\lambda||^2_{L^2} \rho(v^\lambda(\tau), R) d\tau. \tag{2.28}
\]

We let \( R \geq A_{\xi_1} \sqrt{\tau_0} \) for \( \forall \tau_0 \in [0, \tau_1] \) and choose \( R_1 \) as \( R_1 := \frac{\delta}{M_0^\lambda} R \), then a direct computation yields

\[
\frac{R_1^2}{R^2} \int_0^{\tau_0} (\tau_0 - \tau) ||\nabla v^\lambda||^2_{L^2} \rho(v^\lambda(\tau), R) d\tau
\]

\[
\leq 2 \left( \frac{\delta}{M_0^\lambda} \right)^2 \frac{(M_0^\lambda)^2}{\varepsilon} \int_0^{\tau_0} (\tau_0 - \tau) ||\nabla v^\lambda||^2_{L^2} d\tau
\]

\[
\leq 2 \frac{\delta^2}{\varepsilon} \int_0^{\tau_0} (\tau_0 - \tau) ||\nabla v^\lambda||^2_{L^2} d\tau. \tag{2.29}
\]

Now we can choose \( \delta \) small enough such that \( 2 \frac{\delta^2}{\varepsilon} \ll C_1^0 \). For the sake of simplicity, if we define \( \delta(\varepsilon) = \varepsilon \) and \( \varepsilon \) small enough, substitute (2.29) into (2.28) and we get
\[
\int \psi_R |v^\lambda(\tau_0)|^2 + \frac{C'_1}{2} \int_0^{\tau_0} (\tau_0 - \tau) ||\nabla v^\lambda(\tau)||_{L^2}^2 d\tau
\leq \int \psi_R |v^\lambda(0)|^2 + \tau_0 |Im(\int \nabla \psi_R(x) \nabla v^\lambda(0)v^\lambda(0)dx) + E^\lambda(v_0^\lambda)\tau_0|
+ \frac{1}{R} \int_0^{\tau_0} (\tau_0 - \tau)[(\rho(v^\lambda(\tau), R) + \rho(v^\lambda(\tau), R)^3 + 3\rho(v^\lambda(\tau), R_1)\rho(v^\lambda(\tau), R)]d\tau.
\]

(2.30)

We directly estimate the right hand side of (2.30) except the second term:

\[
\int \psi_R |v^\lambda(\tau_0)|^2 \leq \left( \int (\psi_R)^3 dx \right)^{\frac{1}{3}} \left( \int |v_0^\lambda|^3 dx \right)^{\frac{2}{3}} \leq R^3 (M_0^\lambda)^2,
\]

(2.31)

\[
\frac{1}{R} \int_0^{\tau_0} (\tau_0 - \tau)[(\rho(v^\lambda(\tau), R) + \rho(v^\lambda(\tau), R)^3 + \rho(v^\lambda(\tau), R_1)\rho(v^\lambda(\tau), R)]d\tau
\leq \frac{\tau_0^2}{R} \left( \frac{(M_0^\lambda)^6}{\varepsilon^3} + \frac{(M_0^\lambda)^7}{\varepsilon^11} \right)
\leq 2 \frac{\tau_0^2}{R} \frac{(M_0^\lambda)^7}{\varepsilon^11}.
\]

(2.32)

For the second term, we: \( \forall \tau_0 \in [0, \tau_1], \ \forall A \geq A_{\varepsilon_1}, \) let \( R = A\sqrt{\tau_0}, \) then

\[
|Im(\int \nabla \psi_R \cdot \nabla v^\lambda(0)v^\lambda(0) + E^\lambda(v_0^\lambda)\tau_0| \leq C \frac{(M_0^\lambda)^2 A^3}{\varepsilon^{\frac{4}{3}}} \tau_0^{\frac{1}{3}}.
\]

(2.33)

In step 2 and step 3, we assume (2.33) holds and derive the desired upper bounds (2.25), (2.26) and (2.27). In step 4, we give the proof of (2.33) and thus finish the whole proof.

Since \( R \geq A_{\varepsilon_1} \sqrt{\tau_0} \) and the definition, \( R_1 = \frac{\varepsilon}{M_0^\lambda} R, \) we deduce \( R_1 \geq A_{\varepsilon_2} \sqrt{\tau_0}. \)

We divide (2.30) by \( R^3, \) and get

\[
\frac{1}{R} \int_{R \leq |x| \leq 2R} |v^\lambda(\tau_0)|^2 dx \leq C \left[ (M_0^\lambda)^2 + \frac{(M_0^\lambda)^2}{\varepsilon^{\frac{3}{2}}} + \frac{(M_0^\lambda)^7}{\varepsilon^{11}} \frac{1}{A_{\varepsilon_1}^4} \right] \leq \frac{(M_0^\lambda)^2}{\varepsilon},
\]

(2.34)

by the definition of \( G_\varepsilon \) and \( A_{\varepsilon_1}. \)

Similarly, we divide (2.30) by \( R_1^3, \) and get

\[
\frac{1}{R_1} \int_{R_1 \leq |x| \leq 2R_1} |v^\lambda(\tau_0)|^2 dx \leq C \left[ \left( \frac{R}{R_1} \right)^3 (M_0^\lambda)^2 + \frac{(M_0^\lambda)^5}{\varepsilon^{3+\frac{2}{3}}} + \frac{(M_0^\lambda)^7}{\varepsilon^{11}} \frac{1}{A_{\varepsilon_1} A_{\varepsilon_2}^3} \right]
\leq C \left[ \frac{(M_0^\lambda)^5}{\varepsilon^{3+\frac{2}{3}}} + \frac{(M_0^\lambda)^5}{\varepsilon^{3+\frac{2}{3}}} + \frac{(M_0^\lambda)^7}{\varepsilon^{11}} \frac{1}{A_{\varepsilon_1} A_{\varepsilon_2}^3} \right]
\leq \frac{(M_0^\lambda)^5}{\varepsilon^{10}}.
\]

(2.35)

By standard continuity argument, we conclude (2.12) holds. We should note that the choice of \( C_1 \) and \( \alpha_1 \) is independent of \( \lambda \) from (2.22) and the definition of \( A_{\varepsilon_1}. \)

Step 3. Self similar decay of the gradient.

We substitute (2.25), (2.26) and (2.33) at \( A = A_{\varepsilon_1} \) into (2.30), then we derive
\[
\int_0^{\tau_0} (\tau_0 - \tau) ||\nabla v^\lambda(\tau)||_{L^2}^2 d\tau
\leq R^3 (M_0^\lambda)^2 + C \frac{(M_0^\lambda)^2 A_{\lambda_1}^3}{\varepsilon^3} \tau_0^3 + \frac{\tau_0^2 (M_0^\lambda)^7}{\varepsilon^{11}}
\leq \frac{\tau_0^3 (M_0^\lambda)^2 A_{\lambda_1}^3}{\varepsilon^3} \leq \frac{G_\varepsilon}{2} \tau_0^3,
\] (2.36)

which finishes the proof of (2.13). From (2.21), we know the choice of \(\alpha_2\) in (2.13) is also independent of \(\lambda\).

Step 4. Proof of the momentum estimate (2.33).

We can not directly use the interpolation estimate,
\[
|\text{Im} \left( \int \nabla \psi_R(x) \nabla v^\lambda(0) \overline{v^\lambda(0)} dx \right) | \lesssim ||v_0^\lambda||_{L^3}^2 \left| \int \psi_R |v_0^\lambda|^2 d\sigma \right|^{1/2} \lesssim R^3 ||v_0^\lambda||_{L^3}^2 M_0^\lambda.
\] (2.37)

since we only know the information about \(||v_0^\lambda||_{L^3}\) rather than \(||v_0^\lambda||_{\dot{H}^{1/2}}\). On the other hand, if we use Hölder inequality to estimate
\[
|\text{Im} \left( \int \nabla \psi_R(x) \nabla v^\lambda(0) \overline{v^\lambda(0)} dx \right) | \leq R^3 ||\nabla v^\lambda(0)||_{L^3} \left( \int \psi_R v_0^\lambda \right)^{1/2} \leq R^3 ||\nabla v_0^\lambda||_{L^2} M_0^\lambda.
\]

Although we have used the information of \(||v_0^\lambda||_{L^3}\), we can not take full advantage of the information from (2.13) at the initial time \(\tau = 0\). It is because that we hope \(||\nabla v^\lambda(\tilde{\tau})||_{L^2}\) has the asymptotic behaviour with
\[
||\nabla v^\lambda(\tilde{\tau})||_{L^2} \leq \frac{C}{\tilde{\tau}^{1/4}},
\] (2.38)

for some special large enough time \(\tilde{\tau}\) from (2.13). With the above analysis, a natural idea is to choose a time \(\tilde{\tau}\) carefully to obtain the decay estimate (2.38) at \(\tilde{\tau}\). Then we use (2.15) to conclude the difference between \(\text{Im}(\int \nabla \psi_R(x) \nabla v^\lambda(0) \overline{v^\lambda(0)} dx)\) and \(\text{Im}(\int \nabla \psi_R(x) \nabla v^\lambda(\tilde{\tau}) \overline{v^\lambda(\tilde{\tau})} dx)\) is suitably small, which helps us to finish the proof (2.33). In order to explain the above content more clearly, we have divided the proof into three parts.

Part I: Choose a suitable time to obtain the decay estimate (2.38).

Let \(\tau_0 \in [0, \tau_1]\) \(A \geq A_{\varepsilon_1}\) and \(R = A \sqrt{\tau_0}\). First, we should choose the proper \(\tilde{\tau}\) as above.

To be more specific, we claim the following fact: there exists a universal constant \(K > 0\) which is independent of \(\lambda\) and \(\tau_0\) such that
\[
\tilde{\tau}_0 \in \left[ \frac{\varepsilon^{3/4}}{4} \tau_0, \frac{\varepsilon^{3/2}}{2} \tau_0 \right] \text{ with } ||\nabla v^\lambda(\tilde{\tau}_0)||_{L^2}^2 \leq \frac{KG \varepsilon}{\tau_0^{1/2}}.
\] (2.39)

Proof of (2.39). By contradiction, let \(\tilde{\tau} = \varepsilon^{3/2} \tau_0\), then
\[
\int_{\frac{\varepsilon^{3/4}}{4} \tau_0}^{\varepsilon^{3/2} \tau_0} ||\nabla v^\lambda(\sigma)||_{L^2}^2 d\sigma \geq KG \varepsilon \int_{\frac{\varepsilon^{3/4}}{4} \tau_0}^{\varepsilon^{3/2} \tau_0} \frac{d\sigma}{\sigma^{1/2}} \geq CKG \varepsilon \tilde{\tau}^{1/2}.
\] (2.40)

Moreover, \(\tilde{\tau} = \varepsilon^{3/2} \tau_0 \leq \tau_0 \leq \tau_1\) and (2.24) implies:
\[ G_\varepsilon \tilde{\tau}^3 \geq \int_0^{\tilde{\tau}} (\tilde{\tau} - \sigma) \| \nabla \psi^\lambda (\sigma) \|_{L^2}^2 d\sigma \geq \frac{\tilde{\tau}^2}{2} \int_0^{\tilde{\tau}} \| \nabla \psi^\lambda (\sigma) \|_{L^2}^2 d\sigma \geq CK_\varepsilon \tilde{\tau}^3. \quad (2.41) \]

From (2.41), we conclude a contradiction for \( K > 0 \) large enough.

Part II: Derive the desired upper bound for the quantity \( \text{Im} (\int \nabla \psi_R(x) \nabla \psi^\lambda (\tilde{\tau}_0) \bar{v}^\lambda (\tilde{\tau}_0) dx) \).

Define \( R = A_1 \sqrt{\tilde{\tau}_0} = A_1 \sqrt{\bar{\tau}_0} \) and thus

\[ \frac{\varepsilon^{\frac{3}{2}}}{16} \leq \frac{A}{A_1} \leq \varepsilon^{\frac{3}{4}}. \quad (2.42) \]

We claim:

\[ \mid \text{Im} \int \nabla \psi_R \cdot \nabla \psi^\lambda (\tilde{\tau}_0) \bar{v}^\lambda (\tilde{\tau}_0) \mid \leq C \frac{(M_0^\lambda)^2 A^3}{\varepsilon^{\frac{3}{2}}} \tau_0^\frac{1}{2}. \quad (2.43) \]

Proof of (2.43). Before we derive (2.43), we show the following inequality

\[ \frac{1}{R^3} \int \psi_R |v^\lambda (\tilde{\tau}_0)|^2 \leq C \left[ (M_0^\lambda)^2 + \frac{G_\varepsilon}{A_1^3} \right]. \quad (2.44) \]

In fact, from (2.15) and Cauchy-Schwarz inequality, we obtain

\[ \left| \frac{d}{d\tau} \int \psi_R |v^\lambda|^2 \right| = 2 \mid \text{Im} \left( \int \nabla \psi_R \cdot \nabla \psi^\lambda v^\lambda \right) \mid \leq C \| \nabla \psi^\lambda \|_{L^2} \left( \int |\psi_R |v^\lambda|^2 \right)^{\frac{1}{2}}. \]

Therefore, we integrate this differential inequality from 0 to \( \tilde{\tau}_0 \) and get:

\[ \int \psi_R |v^\lambda (\tilde{\tau}_0)|^2 \leq C \left[ \int \psi_R |v^\lambda (0)|^2 + \left( \int_0^{\tilde{\tau}_0} \| \nabla \psi^\lambda (\sigma) \|_{L^2}^2 d\sigma \right)^2 \right] \]

\[ \leq C \left[ R^3 (M_0^\lambda)^2 + \bar{\tau}_0 \int_0^{\tilde{\tau}_0} \| \nabla \psi^\lambda (\sigma) \|_{L^2}^2 d\sigma \right]. \quad (2.45) \]

Since \( 2\tilde{\tau}_0 \leq \varepsilon^{\frac{3}{2}} \tau_0 \leq \tau_1 \) and using (2.24), we obtain

\[ \int_0^{\tilde{\tau}_0} \| \nabla \psi^\lambda (\sigma) \|_{L^2}^2 d\sigma \leq \frac{1}{\tilde{\tau}_0} \int_0^{2\tilde{\tau}_0} (2\tilde{\tau}_0 - \sigma) \| \nabla \psi^\lambda (\sigma) \|_{L^2}^2 d\sigma \leq C G_\varepsilon \tilde{\tau}_0^3. \quad (2.46) \]

Thus, recalling that \( R = A_1 \sqrt{\tilde{\tau}_0} \), we combine (2.45) and (2.46) to get

\[ \int \psi_R |v^\lambda (\tilde{\tau}_0)|^2 \leq C [R^3 (M_0^\lambda)^2 + G_\varepsilon \tilde{\tau}_0^3] = CR^3 \left[ (M_0^\lambda)^2 + \frac{G_\varepsilon}{A_1^3} \right] \]

and concludes the proof of (2.44).

Now we control the virial quantity (2.43) at time \( \tilde{\tau}_0 \):

\[ | \text{Im} \int \nabla \psi_R \cdot \nabla \psi^\lambda (\tilde{\tau}_0) \bar{v}^\lambda (\tilde{\tau}_0) | \leq R^{\frac{3}{2}} \| \nabla \psi^\lambda (\tilde{\tau}_0) \|_{L^2} \left( \frac{1}{R^3} \int |v^\lambda (\tilde{\tau}_0)|^2 \right)^{\frac{1}{2}} \]

\[ \leq CR^{\frac{3}{2}} \frac{G_\varepsilon^{\frac{1}{2}}}{\tilde{\tau}_0^{\frac{1}{4}}} \left[ (M_0^\lambda)^2 + \frac{G_\varepsilon}{A_1^3} \right]^{\frac{1}{2}} \]

\[ \leq CAA_1^{\frac{1}{2}} \tau_0^{\frac{1}{2}} G_\varepsilon^{\frac{1}{2}} \left[ M_0^\lambda + \frac{G_\varepsilon}{A_1^2} \right] \]
from $A \geq A_{\varepsilon 1}$ and the choice of $\bar{\tau}_0$. From the definition of $G_{\varepsilon}$, $A_{\varepsilon 1}$ and $A_1$ in (2.21) and (2.42), we derive
\[
|Im \int \nabla \psi_R \cdot \nabla \nu^\lambda (\bar{\tau}_0) \overline{\nu^\lambda (\bar{\tau}_0)}| \\
\leq C(M_0^\lambda)^2 A^3 \tau_0^{\frac{1}{2}} \frac{1}{\varepsilon^{\frac{1}{2}}} \frac{1}{\varepsilon^{\frac{3}{2}}} = C(M_0^\lambda)^2 A^3 \tau_0^{\frac{1}{2}} \varepsilon^{\frac{1}{2}},
\]
which finishes the proof of (2.43).

Part III: Prove the initial control on the virial quantity (2.33).

Using (2.17), we derive the crude estimate which connects $\tau = 0$ with $\tau = \bar{\tau}_0$:
\[
\left| \frac{d}{d\tau} \right| \left| \int \nabla \psi_R \cdot \nabla \nu^\lambda (\bar{\tau}_0) \overline{\nu^\lambda (\bar{\tau}_0)} \right| \\
\leq C \left| [E^\lambda (v_0^\lambda)] + \int |\nabla \nu^\lambda|^2 dx + R^{-1}[\rho(v^\lambda (\tau), R) \right.
\left. + \rho(v^\lambda (\tau), R_1)\rho(v^\lambda (\tau), R)\] \right). \tag{2.47}
\]

Since
\[
R^{-1}[\rho(v^\lambda (\tau), R) + \rho(v^\lambda (\tau), R_1) + \rho(v^\lambda (\tau), R_1)\rho(v^\lambda (\tau), R)]
\leq \frac{C}{R \varepsilon^{\frac{11}{2}}} \leq \frac{1}{\bar{\tau}_0^{\frac{3}{2}}}
\]
for $\varepsilon$ small enough, we conclude from (2.47)
\[
\left| \frac{d}{d\tau} \right| \left| \int \nabla \psi_R \cdot \nabla \nu^\lambda (\bar{\tau}_0) \overline{\nu^\lambda (\bar{\tau}_0)} \right| \leq C \left\{ |E^\lambda (v_0^\lambda)] + \int |\nabla \nu^\lambda|^2 dx + \frac{1}{\bar{\tau}_0^{\frac{3}{2}}} \right\}. \tag{2.48}
\]

We integrate (2.48) in time from 0 to $\bar{\tau}_0$ and get
\[
|Im \int \nabla \psi_R \cdot \nabla \nu^\lambda (0) \overline{\nu^\lambda (0)} | \leq |Im \int \nabla \psi_R \cdot \nabla \nu^\lambda (\bar{\tau}_0) \overline{\nu^\lambda (\bar{\tau}_0)} |
\]
\[
+ C \left[ \int_0^{\bar{\tau}_0} ||\nabla \nu^\lambda (\sigma)||^2 L_2 d\sigma + |E^\lambda (v_0^\lambda)] |\bar{\tau}_0 + \bar{\tau}_0^{\frac{3}{2}} \right]. \tag{2.49}
\]

Thanks to (2.43) and (2.46), we conclude
\[
Im \int \nabla \psi_R \cdot \nabla \nu^\lambda (0) \overline{\nu^\lambda (0)} + E^\lambda (v_0^\lambda) \tau_0 \leq \frac{C(M_0^\lambda)^2 A^3}{\varepsilon^{\frac{3}{2}}} \tau_0^{\frac{1}{2}} + CG_{\varepsilon} \tau_0^{\frac{1}{2}}
\]
\[
+ E^\lambda (v_0^\lambda) \tau_0 + C \varepsilon^2 |E^\lambda (v_0^\lambda)] |\bar{\tau}_0. \tag{2.50}
\]

Our main task is to estimate the right-hand side in (2.50) term by term. Since the first term has been controlled, we estimate the second term
\[
G_{\varepsilon} \tau_0^{\frac{1}{2}} \leq CA_{\varepsilon 1}^3 (M_0^\lambda)^2 \frac{\varepsilon^{\frac{3}{2}}}{\varepsilon} \tau_0^{\frac{1}{2}} \leq \frac{C(M_0^\lambda)^2 A^3}{\varepsilon^{\frac{3}{2}}} \tau_0^{\frac{1}{2}}, \tag{2.51}
\]
where we use the definition of $G_{\varepsilon}$, $A_{\varepsilon 1}$ in (2.21) and the choice of $\bar{\tau}_0$ in (2.39). Lastly, to control the remaining two terms in (2.50), we observe that
\[
E^\lambda (v_0^\lambda) \tau_0 + C |E^\lambda (v_0^\lambda)] |\bar{\tau}_0 \leq [E^\lambda (v_0^\lambda) + C \varepsilon^2 |E^\lambda (v_0^\lambda)] |\tau_0
\]
\[
\leq C \max [E^\lambda (v_0^\lambda), 0] \tau_0 \leq C \tau_0^{\frac{1}{2}}
\]
for $\varepsilon > 0$ small enough and we have used the condition (2.10). Summing up the above estimates, we finish the desired proof (2.33).

The following proposition implies a nontrivial repartition of the $L^2$ mass of the initial data. This repartition helps us to get uniform lower bound on sufficient disjoint annuli.

**Proposition 2.4** (Lower bound on a weighted local $L^2$ norm of $v^\lambda(0)$)

Let $\tau_* > 0$ and $v^\lambda(\tau) \in C([0, \tau_*], \dot{H}^1 \cap \dot{H}^1)$ be a radially symmetric solution to (2.5) and assume (2.10) and (2.11) in Proposition 2.2 hold. Then there exist universal constants $\alpha_3, c_3 > 0$ which are independent of $\lambda$ such that the following holds true. Let

$$\tilde{\lambda}_v(\tau) = \left( \frac{1}{||\nabla v^\lambda(\tau)||_{L^2}} \right)^2,$$

and let

$$\tau_0 \in [0, \frac{\tau_*}{2}],$$

$$\tilde{\lambda}_v(\tau_0) E^\lambda(v^\lambda_0) \leq \frac{1}{4}.$$

Let

$$F_* = \frac{\sqrt{\tau_0}}{\tilde{\lambda}_v(\tau_0)},$$

and

$$D_* = (M^\lambda_0)^{\alpha_3} \max\{1, F_*^3\},$$

then

$$\frac{1}{\tilde{\lambda}_v(\tau_0)} \int_{|x| \leq D_* \tilde{\lambda}_v(\tau_0)} |v^\lambda(0)|^2 \geq c_3.$$

**Proof** Step 1. Energy constraint and lower bound on $v^\lambda(\tau_0)$.

We claim that there exist universal constants $C_3, c_3 > 0$ which are independent of $\lambda$ such that:

$$\frac{1}{\tilde{\lambda}_v(\tau_0)} \int_{|x| \leq A_* \tilde{\lambda}_v(\tau_0)} |v^\lambda(\tau_0)|^2 \geq c_3$$

with

$$A_* = C_3 \max\{(M^\lambda_0)^{\alpha_1} F_*, (M^\lambda_0)^6\}. $$

Now we prove the claim (2.58). Consider a renormalization of $v^\lambda(\tau_0)$:

$$w(x) = \tilde{\lambda}_v(\tau_0) v^\lambda(\tau_0, \tilde{\lambda}_v(\tau_0)x),$$

then from (2.52), (2.54) and the conservation of the energy:

$$||\nabla w||_{L^2} = 1$$

and
\[ E^{\lambda, \tilde{\lambda}_v(\tau_0)}(w) = \frac{1}{2} \int |\nabla w|^2 dx - \frac{1}{4} \int (V_{\lambda, \tilde{\lambda}_v(\tau_0)} w^2)(x) |w(x)|^2 dx \]
\[ = \tilde{\lambda}_v(\tau_0) \left( \frac{1}{2} \int |\nabla w|^2 dx - \frac{1}{4} \int (V_{\lambda, \tilde{\lambda}_v(\tau_0)} w^2)(x) |w(x)|^2 dx \right) \]
\[ = \tilde{\lambda}_v(\tau_0) E^{\lambda}(v^\lambda(\tau_0)) = \tilde{\lambda}_v(\tau_0) E^{\lambda}(v^\lambda_0) \leq \frac{1}{4}, \quad (2.62) \]
we conclude
\[ \iint_{\mathbb{R}^3 \times \mathbb{R}^3} V_{\lambda, \tilde{\lambda}_v(\tau_0)}(x-y)|w(x)|^2|w(y)|^2 dxdy = 4 \left( \frac{1}{2} ||\nabla w||^2_{L^2} - E^{\lambda, \tilde{\lambda}_v(\tau_0)}(w) \right) \geq 1. \]

Pick now \( \varepsilon > 0 \) small enough and let
\[ A_\varepsilon = C_\varepsilon \max[(M_0^\lambda)^{\alpha_1} F_\ast, (M_0^\lambda)^{\alpha}] \]
for \( C_\varepsilon \) large enough to be chosen. First we observe that
\[ \rho(w, A_\varepsilon) \leq \rho(w, (M_0^\lambda)^{\alpha_1} F_\ast) = \rho(v^\lambda(\tau_0), (M_0^\lambda)^{\alpha_1} \tilde{\lambda}_v(\tau_0) F_\ast) \]
\[ = \rho(v^\lambda(\tau_0), (M_0^\lambda)^{\alpha_1} \sqrt{\tau_0}) \leq C_1 (M_0^\lambda)^2. \]

Thus, from (1.9), (2.8)
\[ \iint_{|x| \geq A_\varepsilon, |y| \geq A_\varepsilon} V_{\lambda, \tilde{\lambda}_v(\tau_0)}(x-y)|w(x)|^2|w(y)|^2 dxdy \]
\[ \leq ||w(x) \cdot \chi_{|x| \geq A_\varepsilon}||^2_{L^4} ||w(y) \cdot \chi_{|y| \geq A_\varepsilon}||^2_{L^4} \]
\[ \leq ||w(x) \cdot \chi_{|x| \geq A_\varepsilon}||^4_{L^4} \]
\[ \leq \varepsilon ||\nabla w||^2_{L^2} + \frac{C(\varepsilon)}{A_\varepsilon} [\rho(w, A_\varepsilon) + \rho(w, A_\varepsilon)^3] \]
\[ \leq 2\varepsilon \quad (2.64) \]
for \( C_\varepsilon \) large enough and
\[ \iint_{|x| \leq A_\varepsilon, |y| \leq A_\varepsilon} \iint_{|x| \geq A_\varepsilon, |y| \leq A_\varepsilon} \iint_{|x| \leq A_\varepsilon, |y| \geq A_\varepsilon} V_{\lambda, \tilde{\lambda}_v(\tau_0)}(x-y)|w(x)|^2|w(y)|^2 dxdy \geq \frac{1}{2}. \quad (2.65) \]

By the Pigeon house principle, there must exist at least one term that is bigger than \( \frac{1}{6} \).
Without loss of generality, we assume
\[ \iint_{|x| \leq A_\varepsilon, |y| \geq A_\varepsilon} V_{\lambda, \tilde{\lambda}_v(\tau_0)}(x-y)|w(x)|^2|w(y)|^2 dxdy \geq \frac{1}{6}. \]

Then by the Hölder inequality, Young inequality and Sobolev embedding theory,
\[ \frac{1}{6} \leq C ||\chi_{|x| \leq A_\varepsilon} w||^3_{L^2} ||\nabla w||_{L^2}^3, \]
which implies that
\[ \int_{|x| \leq A_\varepsilon} |w(x)|^2 dx \geq c_3 > 0 \]
for some constant $c_3 > 0$ which is independent of $\lambda$. By (2.60), this is
\[
\frac{1}{\lambda_v(\tau_0)} \int_{|x| \leq A\lambda_v(\tau_0)} |v^\lambda(\tau_0)|^2 \geq c_3
\]
which finishes the claim (2.58).

Step 2. Backwards integration of the $L^2$ fluxes.
We claim: $\forall \varepsilon > 0$, there exists $\tilde{C}_\varepsilon > 0$ such that $\forall D \geq D_\varepsilon$ with
\[
D_\varepsilon = \tilde{C}_\varepsilon \max[F_*, F^2_\varepsilon] \cdot \max[(M^0_\lambda)^{a_1}, (M^0_\lambda)^{2+a_2}],
\]
let
\[
\tilde{R} = \tilde{R}(D, \tau_0) = D\tilde{\lambda}_v(\tau_0),
\]
and $\chi_{\tilde{R}}(r) = \chi \left( \frac{r}{\tilde{R}} \right)$ for some smooth radially symmetric cut-off function $\chi(r) = 1$ for $r \leq 1$, $\chi(r) = 0$ for $r \geq 2$, then:
\[
\frac{1}{\lambda_v(\tau_0)} \int \chi_{\tilde{R}}|v^\lambda(\tau_0)|^2 - \frac{1}{\lambda_v(\tau_0)} \int \chi_{\tilde{R}}|v^\lambda(0)|^2 < \varepsilon.
\]
(2.58) and (2.68) now imply (2.57).

Proof of (2.68). Pick $\varepsilon > 0$. We compute the $L^2$ fluxes from (2.15) with $\chi_{\tilde{R}}$:
\[
\left| \frac{d}{d\tau} \int \chi_{\tilde{R}}|v^\lambda|^2 \right| = 2 \left| \text{Im} \left( \int \nabla \chi_{\tilde{R}} \cdot \nabla v^\lambda \bar{v}^\lambda \right) \right|
\leq \frac{C}{R} \|\nabla v^\lambda(\tau)\|_{L^2} \left( \int_{|x| \leq 2\tilde{R}} |v^\lambda(\tau)|^2 \right)^{\frac{1}{2}}
\leq \frac{C}{R^2} \|\nabla v^\lambda(\tau)\|_{L^2} \left( \rho(v^\lambda(\tau), \tilde{R}) \right)^{\frac{1}{2}}.
\]
Now observe from (2.55), (2.66) and (2.67) that:
\[
\forall \tau \in [0, \tau_0], \quad \tilde{R} = D\tilde{\lambda}_v(\tau_0) \geq D_\varepsilon \frac{\tilde{\lambda}_v(\tau_0)}{\sqrt{\tau_0}} \geq \frac{D_\varepsilon}{F_*} \sqrt{\tau} \geq (M^0_\lambda)^{a_1} \sqrt{\tau},
\]
and thus (2.12) and the monotonicity of $\rho$ ensure:
\[
\forall \tau \in [0, \tau_0], \quad \rho(v^\lambda(\tau), \tilde{R}) \leq \rho(v^\lambda(\tau), (M^0_\lambda)^{a_1} \sqrt{\tau}) < C_1 (M^0_\lambda)^2.
\]
(2.69)
Now we derive that
\[
\forall \tau \in [0, \tau_0], \quad \left| \frac{d}{d\tau} \int \chi_{\tilde{R}}|v^\lambda|^2 \right| \leq \frac{C M^0_\lambda \cdot \tilde{R} \cdot \tilde{\lambda}_v(\tau_0)}{R^2} \|\nabla v^\lambda(\tau)\|_{L^2}.
\]
We integrate this between 0 and $\tau_0$, divide by $\tilde{R}$ and use (2.13) to get:
\[
\frac{1}{\lambda_v(\tau_0)} \int \chi_{\tilde{R}}|v^\lambda(\tau_0)|^2 - \frac{1}{\lambda_v(\tau_0)} \int \chi_{\tilde{R}}|v^\lambda(0)|^2 \leq \frac{C \cdot M^0_\lambda \cdot \tilde{R}}{\tilde{\lambda}_v(\tau_0) \sqrt{\tau_0}} \left( \tau_0 \int_0^{\tau_0} \|\nabla v^\lambda(\tau)\|_{L^2}^2 d\tau \right)^{\frac{1}{2}}
\]
\[ \leq \frac{CM_0^\lambda}{D^2 \cdot \tilde{\lambda}_v(\tau_0)^{\frac{3}{2}}} \left( \int_0^{2\tau_0} (2\tau_0 - \tau) ||\nabla v(\tau)||_2^2 \, d\tau \right)^{\frac{1}{2}} \leq \frac{CM_0^\lambda}{D^2 \cdot \tilde{\lambda}_v(\tau_0)^{\frac{3}{2}}} \left( M_0 \right)^{\frac{3}{2}} \cdot \tau_0^{\frac{3}{2}} \]

\[ \leq \frac{C(M_0^\lambda)^{1 + \frac{3}{2}} \|F_*\|^2}{D^2 \varepsilon} \leq \varepsilon \]

for \( \tilde{\varepsilon} \) large enough which ends the proof. \( \square \)

In the end of this section, we state a proposition as a blackbox to finish the proof of Theorem 1.5,

**Proposition 2.5** Let \( v_\varepsilon(\tau, x) \in C([0, e^N], \dot{H}^{\frac{1}{2}} \cap \dot{H}^{1}) \)\(^9\) be a radial solution to (2.1) with the potential function

\[ V_\varepsilon(x) := \frac{1}{\varepsilon^3} V \left( \frac{x}{\varepsilon} \right), \ \forall \varepsilon > 0, \]

where \( V(x) \) is a fixed potential function satisfying the conditions in Theorem 1.5 and initial data satisfies

\[ ||v_\varepsilon(0, x)||_{\dot{H}^1} = 1, \quad (2.70) \]

Besides, we assume the following conditions\(^{10}\)

\[ e^\frac{N}{2} \cdot \max(E^V(v_\varepsilon(0), 0)) < 1, \quad (2.71) \]

\[ M_0 := \frac{4||v_\varepsilon(0)||_{L^3}}{c_V} \geq 2, \quad (2.72) \]

and

\[ \forall \tau_0 \in [0, e^N], \quad \frac{E^V(v_\varepsilon(0))}{||v_\varepsilon(\tau_0)||_{\dot{H}^1}^2} \leq \frac{1}{4}, \quad (2.73) \]

then there exists a universal constant \( \gamma_1 > 0 \) which is independent of \( \varepsilon \) such that

\[ ||v_\varepsilon(0)||_{L^3} \geq N^{\gamma_1}. \]

Using this proposition to Eq. (2.5) with \( \tau = 0 \), it is direct to see

\[ ||u(t, x)||_{L^3} = ||v^\lambda(0, x)||_{L^3} \geq |\log(T - t)|^{\gamma'}, \]

and we finish the proof of Theorem 1.5.

### 3 The connection between Schrödinger equation and Hartree equation

In Sect. 2, we have proved Theorem 1.5. In this section, we aim to show the our result also implies Theorem 1.1 as said in Sect. 1.2. Before our analysis, we state a stronger version proved by Merle and Raphaël in [33], which is also claimed in Remark 1.15,

\(^9\) \( N \) is a large enough fixed number.

\(^{10}\) \( E^V(v_\varepsilon(0)) := \frac{1}{2} \int |\nabla v_\varepsilon(0)|^2 - \frac{1}{4} \int (V_\varepsilon * |v_\varepsilon(0)|^2)|v_\varepsilon(0)|^2. \)
**Proposition 3.1** Let \( v(\tau, x) \in C([0, e^N], \dot{H}^{1/2} \cap \dot{H}^1) \) be a radial solution to Eq. (1.5) with the initial data satisfying

\[
\|v(0, x)\|_{\dot{H}^1} = 1,
\]

and satisfies the following conditions\(^{11}\)

\[
e^{\frac{N}{2}} \cdot \max(E(v_0), 0) < 1,
\]

\[
M_0 := \frac{4\|v_0\|_{L^3}^3}{C_{GN}} \geq 2,
\]

and

\[
\forall \tau_0 \in [0, e^N], \quad \frac{E(v_0)}{\|v(\tau_0)\|_{\dot{H}^1}^2} \leq \frac{1}{4},
\]

then there exists a universal constant \( \gamma > 0 \) such that

\[
\|v_0\|_{L^3} \geq N^\gamma.
\]

Now we want to derive proposition 3.1 by applying our conclusion, proposition 2.5. First we state a result of standard stability theory, and supply the proof in Appendix B,

**Lemma 3.2** We consider the nonlinear Schrödinger equation (1.5) and Hartree equation (2.1) with the same radial initial data \( v_0 \in \dot{H}^{1/2} \cap \dot{H}^1 \), and assume the normalization condition

\[
\|V(x)\|_{L^1} = \|V_\varepsilon(x)\|_{L^1} = 1.
\]

We define \( T_{\text{max}} \) as the lifetime of \( v(\tau) \) and claim the following holds true, \( \forall \delta > 0, \forall T \in [0, T_{\text{max}}) \), \( \exists \varepsilon^* = \varepsilon^*(\delta, T) > 0 \), such that \( \forall 0 < \varepsilon < \varepsilon^* \),

\[
\|v_\varepsilon(\tau) - v(\tau)\|_{L^\infty([0, T], \dot{H}^1)} \leq \delta.
\]

Combining Lemma 3.2 and Proposition 2.5, we will prove Proposition 3.1. The main task is to verify the above conditions in Proposition 2.5 hold true item by item.

**Proof** From the standard analysis knowledge and the definition of \( E(v_0), E^{V_\varepsilon}(v_0) \), \( \forall m_1 > 0, \exists m_2 > 0 \), such that \( \forall 0 < \varepsilon < m_2 \),

\[
|E(v_0) - E^{V_\varepsilon}(v_0)| < m_1.
\]

Since \( \tau_* \) is a fixed number, condition (2.71) is naturally established. The assumption (2.72) is verified by condition (3.3). Lastly, combining Lemma 3.2 and (3.8), we conclude (2.73) holds. The conclusion of the Proposition 2.5 then leads directly to the conclusion (3.5) as we want.

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\(^{11}\) \( E(v_0) := \frac{1}{2} \int |\nabla v_0|^2 - \frac{1}{4} \int |v_0|^4. \)
Appendix A: A virial argument and The applications of local theory

In this section, we give complete analysis claimed in Remark 1.6 and some applications of local theory. First, through a classical virial argument, we prove the existence of the blow up solutions, and state it as follows,

**Proposition A.1** Let $v(t, x) \in C([0, T), \dot{H}^{1/2} \cap \dot{H}^1)$ be a solution to Eq. (1.1) where $V(x)$ satisfies the condition (1.8) and $V(x) \in L^1(\mathbb{R}^3)$, if the initial data $v_0$ satisfies

$$E^V(v_0) = \frac{1}{2} \int |\nabla v_0|^2 - \frac{1}{4} \int (V * |v_0|^2)|v_0|^2 < 0,$$

then $v(t, x)$ blows up in finite time, i.e. $T < +\infty$.

**Proof** Since $V(x)$ is a radial function, then

$$2 \int x_j |v(t, x)|^2 \partial_{x_j} \int V(x - y)|v(t, y)|^2 dy dx$$

$$= \int \int (x_j - y_j) \partial_{x_j} V(x - y)|v(t, x)|^2 |v(t, y)|^2 dy dx.$$

So we can rewrite the virial identity (1.13) as the follows

$$\frac{d^2}{dt^2} \int |x|^2 |v|^2 = 8 \int |\nabla v|^2 + 2 \int \int (x_j - y_j) \partial_{x_j} V(x - y)|v(t, x)|^2 |v(t, y)|^2 dy dx$$

$$= 16 \left( \frac{1}{2} \int |\nabla v|^2 + \frac{1}{8} \int \int (x_j - y_j) \partial_{x_j} V(x - y)|v(t, x)|^2 |v(t, y)|^2 dy dx \right)$$

$$:= 16 K^V(v(t)).$$

Using the condition (1.8), we conclude

$$K^V(v(t)) \leq E^V(v(t)) = E^V(v_0) < 0,$$

and we finish the proof.

**Proposition A.2** Assume $u(t)$ is a solution in $C([0, T), \dot{H}^{1/2} \cap \dot{H}^1)$ to Eq. (1.1) with

$$\begin{cases}
u_0 \in \dot{H}^{1/2} \cap \dot{H}^1, \\
||u_0||_{\dot{H}^1} \leq M,
\end{cases}$$

and $V(x) \in L^1(\mathbb{R}^3) \cap L^\frac{3}{2}(\mathbb{R}^3)$, then, if $u(t)$ blows up in finite time $T < \infty$, there holds

$$\lim_{t \to T} ||u(t)||_{\dot{H}^{1/2}} = +\infty.$$

**Proof** step 1. The energy is well-defined.

From the energy conservation law,

$$E^V(u(t)) = E^V(u_0) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx - \frac{1}{4} \int (V * |u|^2)(t, x)|u(t, x)|^2 dx.$$

Also, with the aid of Sobolev embedding theorem in $\mathbb{R}^3$, $\dot{H}^{1/2} \hookrightarrow L^3$ and $\dot{H}^1 \hookrightarrow L^6$. 

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\[
\int (V \ast |u|^2)(t, x)|u(t, x)|^2 \, dx \lesssim \|(V \ast |u|^2)\|_{L_3^2} \|u\|_{L_3^1}^2 \lesssim \|V\|_{L_3^2} \|u\|_{L_3^1}^4 \lesssim \|u\|_{H_2^1}^4.
\]  

(A.3)

So \(E^V(u(t))\) is well defined.

step 2. Proof by contradiction.

From step 1,

\[
E^V(u(t)) \geq \frac{1}{2} \int |\nabla u(t, x)|^2 \, dx - C \|u\|_{H_2^1}^4.
\]  

(A.4)

If \(u(t)\) blows up at \(T\), then

\[
\lim_{t \nearrow T} \|u(t)\|_{H_2^1} = +\infty.
\]

(Otherwise we can continues the solution to \([t, t + C(M)]\) with \(t + C(M) > T\), which is a contradiction.)

The fact \(E^V(u(t)) = E^V(u_0) < +\infty\) shows that

\[
\lim_{t \nearrow T} \|u(t)\|_{H_2^1} = +\infty.
\]

\(\square\)

**Proposition A.3** Assume the assumptions in Proposition A.2 hold, and \(u_0 \in H^1(\mathbb{R}^3), V(x) \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\). Then \(u(t)\) can not blow up in finite time \(T < \infty\).

**Proof** Recall

\[
E^V(u(t)) = E^V(u_0) = \frac{1}{2} \int |\nabla u(t, x)|^2 \, dx - \frac{1}{4} \int (V \ast |u|^2)(t, x)|u(t, x)|^2 \, dx < +\infty
\]

(A.5)

and

\[
\int (V \ast |u|^2)(t, x)|u(t, x)|^2 \, dx \lesssim \|(V \ast |u|^2)\|_{L_\infty^2} \|u\|_{L_2^3}^4 \lesssim \|V\|_{L_\infty^2} \|u\|_{L_2^\infty}^4 \lesssim \|u_0\|_{L_2^3}^4 \lesssim C(u_0),
\]

where we also use mass conservation.

The above gives

\[
\int |\nabla u(t, x)|^2 \, dx \leq \frac{1}{2} \int (V \ast |u|^2)(t, x)|u(t, x)|^2 \, dx + 2E^V(u_0) \leq C(u_0),
\]

(A.6)

which is a contradiction to \(\lim_{t \nearrow T} \|u(t)\|_{H_2^1} = +\infty\). \(\square\)

In fact, we also have a stronger characterization about the blow-up rate of critical norm, and state it as follows:

**Proposition A.4** Under the assumptions in Proposition A.2 and assume \(u(t)\) blows up in finite time \(T < +\infty\), then \(\exists C = C(u_0)\) s.t.

\[
\|u(t)\|_{H_2^1} \geq \frac{C}{(T - t)^{\frac{1}{4}}},
\]

(A.7)
and

\[ \|u(t)\|_{H^\frac{1}{2}} \geq \frac{C}{(T-t)^{\frac{1}{8}}} \quad \text{(A.8)} \]

for \( t \) close enough to \( T \).

**Proof** If not, \( \exists \{t_n\} \) with \( \lim_{n \to +\infty} t_n = T \) such that \( \|u(t_n)\|_{\dot{H}^1} \leq \frac{1}{n(T-t)^{\frac{1}{8}}} := M_n \).

At the time \( t_n \),

\[
\begin{align*}
&i \partial_t u + \Delta u = -(V \ast |u|^2) u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\
&u |_{t=t_n} = u(t_n, x) \in \dot{H}^1 \cap \dot{H}^\frac{1}{2}.
\end{align*}
\quad \text{(A.9)}
\]

From local theory, \( u(t) \in C([t_n, t_n + T_n], \dot{H}^\frac{1}{2} \cap \dot{H}^1) \) with \( T_n \geq \frac{C}{M_n} \geq n^4(T - t_n) \).

However, \( t_n + n^4(T - t_n) > T \) for \( n \) large enough, which is a contradiction.

From the energy conservation law,

\[
E^V(u(t)) = E^V(u_0) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx - \frac{1}{4} \int (V \ast |u|^2)(t, x)|u(t, x)|^2 dx
\]

\[
\geq \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 - C\|u(t)\|_{H^\frac{1}{2}}^4.
\]

From the (A.9), \( \|u(t)\|_{\dot{H}^1} \geq \frac{C}{(T-t)^{\frac{1}{8}}} \), so \( \|u(t)\|_{H^\frac{1}{2}} \geq \frac{C}{(T-t)^{\frac{1}{8}}} \). \quad \square

**Remark A.5** In Proposition A.4, we have used the fact \( V(x) \in L^\frac{3}{2} (\mathbb{R}^3) \).

**Remark A.6** Although we can give some characterizations about the blow-up rate, we cannot give a concrete example of the blow-up solution. This is a weakness for Propositions A.2–A.4.

**Appendix B: The proof of Lemma 3.2**

**Proof** We define \( u_\varepsilon := v_\varepsilon - v \), i.e. \( v_\varepsilon(\tau, x) = u_\varepsilon(\tau, x) + v(\tau, x) \), then \( u_\varepsilon(\tau, x) \) satisfies

\[
\begin{align*}
&i \partial_\tau u_\varepsilon(\tau, x) + \Delta u_\varepsilon(\tau, x) = -(V_\varepsilon \ast |v_\varepsilon(\tau, x)|^2)v_\varepsilon(\tau, x) + |v|^2 v, \quad (\tau, x) \in \mathbb{R} \times \mathbb{R}^3, \\
&u_\varepsilon(\tau, x) |_{\tau=0} = 0.
\end{align*}
\quad \text{(B.1)}
\]

From the integral equation,

\[
u_\varepsilon(\tau, x) = i \int_0^\tau e^{i(\tau-s)\Delta}[(V_\varepsilon \ast |v_\varepsilon|^2)v_\varepsilon - |v|^2 v](s)ds,
\quad \text{(B.2)}
\]

which implies

\[
\|u_\varepsilon(\tau, x)\|_{L^\infty_T([0,T], \dot{H}^1)} = || \int_0^\tau e^{i(\tau-s)\Delta}[(V_\varepsilon \ast |v_\varepsilon|^2)v_\varepsilon - |v|^2 v](s)ds\|_{L^\infty_T([0,T], \dot{H}^1)}
\]

\[
\leq || \int_0^\tau e^{i(\tau-s)\Delta} \nabla[(V_\varepsilon \ast |v_\varepsilon|^2)v_\varepsilon - (V_\varepsilon \ast |v_\varepsilon|^2)v](s)ds\|_{L^\infty_T([0,T], L^2)}
\]

\[
+ || \int_0^\tau e^{i(\tau-s)\Delta} \nabla( [(V_\varepsilon \ast |v_\varepsilon|^2) - |v|^2 v](s)ds\|_{L^\infty_T([0,T], L^2)}
\]
Next we will estimate these two terms.

\[(I) \leq \left\| \nabla ((V_\epsilon * |v_\epsilon|^2) v_\epsilon - (V_\epsilon * |v_\epsilon|^2) v) \right\|_{L^6_T(L^5_x)} \]
\[\leq \left\| V_\epsilon * (\nabla |v_\epsilon|^2) (v_\epsilon (s) - v(s)) \right\|_{L^6_T(L^5_x)} + \left\| V_\epsilon * ((|v_\epsilon|^2) \nabla (v_\epsilon (s) - v(s))) \right\|_{L^6_T(L^5_x)} \]
\[\leq \left\| V_\epsilon * (\nabla |v_\epsilon|^2) \right\|_{L^\infty_T(L^5_x)} \left\| (v_\epsilon (s) - v(s)) \right\|_{L^6_T(L^5_x)} \cdot T^{1/2} \]
\[+ \left\| V_\epsilon * ((|v_\epsilon|^2) \nabla (v_\epsilon (s) - v(s))) \right\|_{L^6_T(L^5_x)} \cdot T^{1/2} \]
\[\leq \left\| V_\epsilon * (\nabla |v_\epsilon|^2) \right\|_{L^\infty_T(L^5_x)} \left\| u_\epsilon (\tau, x) \right\|_{L^\infty_T(H^1)^{1/2}} \]
\[\leq \left\| u_\epsilon (\tau, x) \right\|_{L^\infty_T(H^1)^{1/2}} \left\| v_\epsilon (\tau, x) \right\|_{L^\infty_T(H^1)^{1/2}} \cdot T^{1/2}. \]

Before we estimate part \((II)\), let us recall some elementary knowledge,

\[\forall f \in L^p(\mathbb{R}^3), \; p \in (1, +\infty), \; \lim_{\varepsilon \searrow 0^+} \left\| V_\epsilon * f - f \right\|_{L^p(\mathbb{R}^3)} = 0. \quad (B.3)\]

From the local theory, \(v(\tau, x) \in C([0, T], \dot{H}^1)\), so \(\mathcal{V} \delta > 0\). \(\exists \delta_1 > 0, \text{ s.t. } \forall \tau_1, \tau_2 \in [0, T] \text{ with } |\tau_2 - \tau_1| < \delta_1, \left\| v(\tau_1, x) - v(\tau_2, x) \right\|_{\dot{H}^1} < \delta\). Similarly, we have \(\left\| v(\tau_1, x) - v(\tau_2, x) \right\|_{L^6} < \delta\) and \(\left\| v(\tau_1, x) - v(\tau_2, x) \right\|_{L^2} < \delta\).

Now we estimate the part \((II)\):

\[(II) = \left\| \int_0^T e^{i(\tau-s)\Delta} \nabla ((V_\epsilon * |v_\epsilon|^2) - |v|^2) v(s) ds \right\|_{L^6_T(L^2_x)} \]
\[\leq \left\| \nabla ((V_\epsilon * |v|^2) - |v|^2) v(s) \right\|_{L^6_T(L^5_x)} \]
\[\leq \left\| \nabla ((V_\epsilon * u_\epsilon + \bar{v} u_\epsilon)) v(s) \right\|_{L^6_T(L^5_x)} \]
\[+ \left\| \nabla ((V_\epsilon * |u_\epsilon|^2) v(s)) \right\|_{L^6_T(L^5_x)} \]
\[:= (i) + (ii) + (iii). \]

For the term \((ii)\),

\[\text{(ii)} \leq \left\| u_\epsilon \right\|_{L^\infty_T(H^1)} \left\| v \right\|_{L^\infty_T(H^1)} T^{1/2}. \quad (B.4)\]

and the computation is also direct for \((iii)\),

\[\text{(iii)} \leq \left\| u_\epsilon \right\|_{L^\infty_T(H^1)} \left\| v \right\|_{L^\infty_T(H^1)} T^{1/2}. \quad (B.5)\]

Finally, we estimate term \((i)\):

\[(i) \leq \left\| ((V_\epsilon * |v|^2) - |v|^2) \nabla v(s) \right\|_{L^6_T(L^5_x)} \]
\[\leq \left\| ((V_\epsilon * |v|^2) - |v|^2) \nabla v(s) \right\|_{L^6_T(L^5_x)} \]
\[+ \left\| ((V_\epsilon * |v|^2) - |v|^2) \right\|_{L^\infty_T(L^2_x)} \left\| \nabla v(s) \right\|_{L^\infty_T(L^2_x)} T^{1/2} \]
\[+ \left\| ((V_\epsilon * |v|^2) - |v|^2) \right\|_{L^\infty_T(L^2_x)} \left\| v(s) \right\|_{L^\infty_T(L^6_x)} T^{1/2}. \]
\[ \leq \delta_2 \| \nabla v(s) \|_{L^\infty_T (0, T), L^2} T^{\frac{1}{2}}, \]

where we used the prepared knowledge in the last inequality and \( \delta_2 \) is a small enough constant to be defined later. Summing up the above three estimations, we conclude, \( \forall \) fixed \( T \in [0, T_{max}] \), \( \forall \delta_2 > 0, \exists \varepsilon_1 > 0 \), such that \( \forall \varepsilon \in (0, \varepsilon_1) \).

\[
\| u_\varepsilon \|_{L^\infty_T (0, T), \dot{H}^1} \leq C \| u_\varepsilon \|_{L^\infty_T (0, T), \dot{H}^1}^3 T^{\frac{1}{2}} + \| u_\varepsilon \|_{L^\infty_T (0, T), \dot{H}^1}^2 \| v \|_{L^\infty_T (0, T), \dot{H}^1} T^{\frac{1}{2}} + \| u_\varepsilon \|_{L^\infty_T (0, T), \dot{H}^1} \| v \|_{L^\infty_T (0, T), \dot{H}^1} T^{\frac{1}{2}} + \delta_2 \| v \|_{L^\infty_T (0, T), \dot{H}^1} T^{\frac{1}{2}}. 
\]  

(B.6)

Dual to the fact \( v(t) \in C([0, T], \dot{H}^1) \), there exists \( M = M(T) \) such that \( \| \nabla v(\tau) \|_{L^\infty_T (0, T), L^2} < M \). We can also choose \( \delta_2 \) small enough such that \( \delta_3 \) small and \( \delta_3 := \delta_2 M^2 \). Then (B.6) can be rewritten as

\[
\| u_\varepsilon \|_{L^\infty_T (0, T), \dot{H}^1} \leq C \| u_\varepsilon \|_{L^\infty_T (0, T), \dot{H}^1}^3 T^{\frac{1}{2}} + \| u_\varepsilon \|_{L^\infty_T (0, T), \dot{H}^1}^2 \| \dot{H}^1 \|_{L^\infty_T (0, T), \dot{H}^1} M T^{\frac{1}{2}} + \| u_\varepsilon \|_{L^\infty_T (0, T), \dot{H}^1} \| v \|_{L^\infty_T (0, T), \dot{H}^1} T^{\frac{1}{2}} + \delta_3 T^{\frac{1}{2}}. 
\]  

(B.7)

We can divide \( T \) into \( N(T) \) intervals and the length of each interval is \( T_1 \) such that \( CT_1^{\frac{2}{3}} \ll 1 \), \( CMT_1^{\frac{1}{2}} \ll \frac{1}{100} \) and \( CM^2 T_1^{\frac{2}{3}} \ll \frac{1}{100} \).

From each interval, (B.7) leads to

\[
\| u_\varepsilon \|_{L^\infty_T ([\tau_i, \tau_{i+1}], \dot{H}^1)} \leq \| u_\varepsilon (\tau_i) \|_{\dot{H}^1} + \frac{1}{100} \| u_\varepsilon \|_{L^\infty_T ([\tau_i, \tau_{i+1}], \dot{H}^1)}^3 + \frac{1}{100} \| u_\varepsilon \|_{L^\infty_T ([\tau_i, \tau_{i+1}], \dot{H}^1)}^2 + \delta_3. 
\]  

(B.8)

On the one hand, we choose \( \delta_3 \) such that \( 2^{\frac{N(T)}{2}} \delta_3 \ll 1 \). On the other hand, with the aid of standard continuity argument, we adjust \( \delta_3 \) such that

\[
\| u_\varepsilon \|_{L^\infty_T (0, T), \dot{H}^1} \leq 2^{\frac{N(T)}{2}} \delta_3 \leq \delta, 
\]  

(B.9)

which ends the proof.

\[ \square \]

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