DEFORMATIONS OF LODY-TYPE ALGEBRAS AND THEIR MORPHISMS

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Abstract. We study a formal deformation of multiplications in an operad. This closely resembles Gerstenhaber’s deformation theory for associative algebras. However, this is applicable to various Loday-type algebras and to their twisted analogues. We explicitly describe the cohomology of Loday-type algebras with coefficient in a representation. Finally, the deformation of morphisms between Loday-type algebras are also considered.

1. Introduction

Algebraic deformation theory first appeared in a pioneer work of M. Gerstenhaber in 1964 [15]. In his paper, he study formal one-parameter deformation of associative algebras and show that it is closely related to the Hochschild cohomology of associative algebras. Since then, formal deformation theory has been studied for various other algebras, including Lie algebras, Leibniz algebras and dialgebras [4,24,27]. Later on, Gerstenhaber and Schack also develop a deformation theory of associative algebra morphisms [17] (see also [13]).

Motivated from his deformation theory of associative algebras, Gerstenhaber defined certain operations on the Hochschild complex of associative algebras which induces a rich structure on the cohomology [14]. This structure is now known as Gerstenhaber algebra. In [18] the authors shed a new light on the Gerstenhaber algebra structure on the Hochschild cohomology which led them to relate with Deligne’s conjecture. Given an operad $O$ with a multiplication $\pi \in O(2)$, they showed that the cochain complex induced from the multiplication $\pi$ inherits a homotopy $G$-algebra structure. Hence the cohomology inherits a Gerstenhaber algebra. When one consider the endomorphism operad $\text{End}(A)$ associated to a vector space $A$, a multiplication on $\text{End}(A)$ is precisely an associative algebra structure on $A$ and the corresponding cochain complex is precisely the Hochschild cochain complex. Hence one recovers the result of [14].

Our main objectives in this paper are certain algebras introduced by Loday and his collaborators [2,21–23]. In [22] Loday introduced a notion of (associative) dialgebra which gives rise to a Leibniz algebra in the same way an associative algebra gives rise to a Lie algebra. The Koszul dual operad of dialgebra is given by the operad of dendriform algebra. They can be thought as splitting of associative algebras. These two algebras are surprisingly related to some combinatorial objects, namely with planar binary trees. Later on, Loday and Ronco defined few other algebras (associative trialgebras, dendriform trialgebras) which are related to planar trees (not necessarily binary) in the same way dialgebra and dendriform algebras are related to planar binary trees [23]. Dendriform trialgebras are splitting of associative algebras by three operations. In [2] Aguiar and Loday introduced another class of algebras, called quadri-algebras, which are further splitting of dendriform algebras. In the same spirit, Leroux [21] defined a splitting of dendriform trialgebras and call them ennea-algebras. We call all these algebras as Loday-type algebras.

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Like an associative algebra structure on a vector space $A$ can be described by a multiplication on the endomorphism operad $\text{End}(A)$ associated to $A$, various Loday-type algebras as described above can also be seen as multiplication on suitable operads. The case of dialgebra was given by Majumdar and Mukherjee [25]. The case of associative trialgebra is very similar. In [28] Yau has defined the same for dendriform algebras and dendriform trialgebras, however, there is some inaccuracy in the construction of operads. It has been clarified for dendriform algebras in [9]. We show that other Loday-type algebras (dendriform trialgebras, quadri-algebras and ennea-algebras) can also be described by multiplication on certain operads.

Our aim in this paper is to study deformation of Loday-type algebras. Note that deformation of algebras over quadratic operads has been carried out in [4], however, our approach is more elementary for Loday-type algebras. More precisely, we study deformation of a multiplication in an operad. We also provide a universal deformation formula. Deformation of a Loday-type algebra is defined by a deformation of the corresponding multiplication. In the particular cases of dialgebra or dendriform algebra, our deformations are equivalent to [9,24]. One can also apply this method to certain twisted analogue of Loday-type algebras.

Motivated from the fact that Loday-type algebras can be described by operad with multiplication, we explicitly define the cohomology of Loday-type algebras with coefficients in a representation. As before, our cohomology is more explicit and simple than Balavoine [5] who defined cohomology with coefficients for algebras over operads. In the case of dialgebra, our cohomology coincides with that of Frabetti [12]. We also show that the second cohomology group can be interpreted as equivalence classes of abelian extensions in the category of same Loday-type algebra.

Finally, we also study deformation of Loday-type algebra morphisms. Let $A$ and $B$ be two Loday algebras of same type. A morphism $f : A \rightarrow B$ between them makes $B$ into a representation of $A$ via $f$. We study the deformation of $f$ by deforming the domain and codomain of $f$ as well. It would be clear from the cochain complex (that controls the deformation of $f$) that the operadic approach cannot be directly apply here. In the particular case of dialgebra morphism, we get deformation studied in [29].

All vector spaces, linear maps and tensor products are over a field $K$ of characteristic 0.

2. Operads with multiplication

In this section, we recall some basics on non-symmetric operad equipped with a multiplication. See [18] for more details.

**Definition 2.1.** A non-symmetric operad (non-$\Sigma$ operad in short) in the category of vector spaces is a collection $\mathcal{O} = \{\mathcal{O}(n) | n \geq 1\}$ of vector spaces together with compositions

\[(1) \quad \gamma : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \rightarrow \mathcal{O}(n_1 + \cdots + n_k), \quad f \otimes g_1 \otimes \cdots \otimes g_k \mapsto \gamma(f; g_1, \ldots, g_k)\]

which are associative in the sense that

\[
\gamma(\gamma(f; g_1, \ldots, g_k); h_1, \ldots, h_{n_1+\cdots+n_k}) = \gamma(\gamma(g_1; h_1, \ldots, h_{n_1}); \gamma(g_2; h_{n_1+1}, \ldots, h_{n_1+n_2}), \ldots, \gamma(g_k; h_{n_1+\cdots+n_{k-1}+1}, \ldots, h_{n_1+\cdots+n_k}))
\]

and there is an identity element $\text{id} \in \mathcal{O}(1)$ such that $\gamma(f; \text{id}, \ldots, \text{id}) = f = \gamma(\text{id}; f)$, for $f \in \mathcal{O}(k)$.

Alternatively, a non-$\Sigma$ operad can also be described by partial compositions

\[\circ_i : \mathcal{O}(m) \otimes \mathcal{O}(n) \rightarrow \mathcal{O}(m + n - 1), \quad 1 \leq i \leq m\]
satisfying
\[
\begin{align*}
(f \circ g) \circ i_{i+j-1} h &= f \circ i (g \circ h), & &1 \leq i \leq m, \ 1 \leq j \leq n, \\
(f \circ g) \circ j_{i+j-1} h &= (f \circ h) \circ i g, & &1 \leq i < j \leq m,
\end{align*}
\]
for \(f \in \mathcal{O}(m), \ g \in \mathcal{O}(n), \ h \in \mathcal{O}(p),\) and an identity element \(\text{id} \in \mathcal{O}(1)\) satisfying \(f \circ \text{id} = f = \text{id} \circ f,\) for all \(f \in \mathcal{O}(m)\) and \(1 \leq i \leq m.\) The two definitions of non-symmetric operad are related by
\[
\gamma(f; g_1, \ldots, g_k) = (\cdots ((f \circ_k g_k) \circ_{k-1} g_{k-1}) \cdots) \circ_1 g_1, \quad \text{for } f \in \mathcal{O}(k).
\]
A non-symmetric operad as above may be denoted by \((\mathcal{O}, \gamma, \text{id})\) or \((\mathcal{O}, \circ, \text{id})\). A toy example of a non-symmetric operad is given by the endomorphism operad associated to a vector space. Let \(A\) be a vector space and define \(\text{End}(A^n, A)\) for \(n \geq 1.\) The compositions \((1)\) are substitution of the values of \(k\) operations in a \(k\)-ary operation as inputs. The identity element is given by the identity map on \(A.\)

Let \((\mathcal{O}, \gamma, \text{id})\) be an operad. If \(f \in \mathcal{O}(n),\) we write \(|f| = n - 1.\) In \([19]\) Getzler and Jones has defined the following brace operations
\[
\{f\} \{g_1, \ldots, g_n\} := \sum (-1)^{\varepsilon} \gamma(f; \text{id}, \ldots, \text{id}, g_1, \text{id}, \ldots, \text{id}, g_n, \text{id}, \ldots, \text{id}),
\]
where the summation runs over all possible substitutions of \(g_1, \ldots, g_n\) into \(f\) in the prescribed order and \(\varepsilon = \sum_{p=1}^{m} |g_p|\) with \(|g_p|\) being the total number of inputs in front of \(g_p.\) We denote the circle product \(\circ: \mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(m + n - 1)\) by
\[
f \circ g := \{f\} \{g\} = \sum_{i=1}^{m} (-1)^{|\varepsilon_i - 1||g|} f \circ_i g, \quad \text{for } f \in \mathcal{O}(m).
\]
The braces \((4)\) satisfy certain pre-Jacobi identities, which in particular implies that the circle product \(\circ\) satisfy the pre-Lie identities
\[
(f \circ g) \circ h - f \circ (g \circ h) = (-1)^{|g||h|}((f \circ h) \circ g - f \circ (h \circ g)).
\]
Hence the bracket \([f, g] := f \circ g - (-1)^{|f||g|} g \circ f\) defines a degree \(-1\) graded Lie bracket on \(\otimes_{n \geq 1} \mathcal{O}(n).\)

**Definition 2.2.** A multiplication on an operad \((\mathcal{O}, \gamma, \text{id})\) is an element \(\pi \in \mathcal{O}(2)\) satisfying \(\pi \circ \pi = 0,\) or, equivalently, \(\pi \circ_1 \pi = \pi \circ_2 \pi.\)

Let \((A, \mu)\) be an associative algebra. Then \(\mu\) defines a multiplication on the endomorphism operad associated to \(A.\) In fact, the associativity of \(\mu\) is same as \(\mu \circ_1 \mu = \mu \circ_2 \mu\) in the endomorphism operad. Thus, one might expect that some of the classical results for associative algebras can be extend to any operads equipped with a multiplication.

If \(\pi\) is a multiplication on an operad \(\mathcal{O},\) then the product
\[
f \cdot g := (-1)^{|f|+1} \{\pi\} \{f, g\}
\]
as differential \(d_\pi: \mathcal{O}(n) \to \mathcal{O}(n + 1),\) \(f \mapsto \pi \circ f - (-1)^{|f|} f \circ \pi,\) makes the graded space \(\oplus_{n \geq 1} \mathcal{O}(n)\) into a differential graded associative algebra. Thus the product passes to the cohomology \(H^\bullet(\mathcal{O}, d_\pi).\) Moreover, it can be shown that the degree \(-1\) graded Lie bracket \([,\,]\) also passes to the cohomology. Finally, the induced product and bracket on the cohomology \(H^\bullet(\mathcal{O}, d_\pi)\) satisfy the graded Leibniz rule to become a Gerstenhaber algebra \([18]\).
The above idea applies to the Hochschild cochain complex of associative algebras, the cochain complex of several other algebras including dialgebras, various other Loday-type algebras, some hom-type algebras and also applicable to singular cochain complex of topological spaces \[7,9,18,25,28\]. Therefore, the cohomology of these algebras inherits a Gerstenhaber structure.

3. Deformation of multiplications

In this section, we define a formal deformation of multiplications in an operad. This is similar to the formal deformation theory of associative algebras developed by Gerstenhaber [15]. The equivalence classes of deformations are described by the moduli spaces of solutions of the Maurer-Cartan equation in a certain dgla.

3.1. Deformation. Let \((\mathcal{O}, \gamma, \text{id})\) be an operad. Consider the space \(\mathcal{O}(n)[[t]]\) of formal power series in a variable \(t\) with coefficient in \(\mathcal{O}(n)\). One can linearly extend the circle products (or brace operations) to \(\bigoplus_{n \geq 1} \mathcal{O}(n)[[t]]\).

Let \(\pi\) be a fixed multiplication on \(\mathcal{O}\). A formal 1-parameter deformation of \(\pi\) is given by a formal sum

\[
\pi_t = \pi_0 + \pi_1 t + \pi_2 t^2 + \cdots \in \mathcal{O}(2)[[t]] \quad \text{with} \quad \pi_0 = \pi,
\]
satisfying \(\pi_t \circ \pi_t = 0\). This is equivalent to a system of equations:

\[
(6) \quad \sum_{i+j=n} \pi_i \circ \pi_j = 0, \quad \text{for} \quad n \geq 0.
\]

For \(n = 0\), we have \(\pi \circ \pi = 0\) which automatically holds from the assumption. For \(n = 1\), we have \(\pi \circ \pi_1 + \pi_1 \circ \pi = 0\), which implies that \(d_2(\pi_1) = 0\). Thus, \(\pi_1\) defines a 2-cocycle in \((\mathcal{O}, d_2)\). The 2-cocycle \(\pi_1\) is called the infinitesimal of the deformation. More generally, if \(\pi_1 = \cdots = \pi_{n-1} = 0\), then \(\pi_n\) is a 2-cocycle. It is called the \(n\)-th infinitesimal of the deformation.

**Definition 3.1.** Two deformations \(\pi_t = \sum_{i \geq 0} \pi_i t^i\) and \(\pi'_t = \sum_{i \geq 0} \pi'_i t^i\) of \(\pi\) are said to be equivalent if there exists a formal sum \(\phi_t = \phi_0 + \phi_1 t + \phi_2 t^2 + \cdots \in \mathcal{O}(1)[[t]]\) (with \(\phi_0 = \text{id} \in \mathcal{O}(1)\)) such that

\[
\phi_t \circ \pi_t = \{\pi'_i\}\{\phi_t, \phi_t\}.
\]

This condition again leads to a system of equations:

\[
(7) \quad \sum_{i+j=n} \phi_i \circ \pi_j = \sum_{i+j+k=n} \{\pi'_i\}\{\phi_j, \phi_k\}, \quad \text{for} \quad n \geq 0.
\]

For \(n = 0\), the relation holds automatically as \(\phi_0 = \text{id}\). However, for \(n = 1\), it gives

\[
\pi_1 + \phi_1 \circ \pi = \{\pi\}\{\text{id}, \phi_1\} + \{\pi\}\{\phi_1, \text{id}\} + \{\pi'_1\}\{\text{id}, \text{id}\},
\]
equivalently, \(\pi_1 - \pi'_1 = \pi \circ \phi_1 - \phi_1 \circ \pi = d_2(\phi_1)\). This shows that the infinitesimals corresponding to equivalent deformations are cohomologous and hence they correspond to same cohomology class in \(H^2(\mathcal{O}, d_2)\).

**Definition 3.2.** A multiplication \(\pi\) is called rigid if any deformation of it is equivalent to the trivial deformation \(\pi_t = \pi\).

**Proposition 3.3.** Let \(\pi_t = \sum_{i \geq 0} \pi_i t^i\) be a deformation of \(\pi\). Then \(\pi_t\) is equivalent to a deformation \(\pi'_t = \pi + \sum_{i \geq p} \pi'_i t^i\), where the first non-vanishing term \(\pi'_p\) is a 2-cocycle but not a coboundary.

**Proof.** Let \(\pi_t = \sum_{i \geq 0} \pi_i t^i\) be a deformation such that \(\pi_1 = \cdots = \pi_{n-1} = 0\) and \(\pi_n\) is the first non-zero term. Then it has been shown that \(\pi_n\) is a 2-cocycle. If \(\pi_n\) is not a coboundary, we are done. If \(\pi_n\) is a coboundary, that is, \(\pi_n = -d_2(\phi_n)\), for some \(\phi_n \in \mathcal{O}(1)\), set \(\phi_t = \text{id} + \phi_n t^n \in \mathcal{O}(1)[[t]]\). We define \(\pi_t = \phi_t^{-1} \circ \pi_t \circ \phi_t\). Then \(\pi_t\) defines a formal deformation of the form

\[
\pi_t = \pi_0 + \pi_1 t + \pi_2 t^2 + \cdots \in \mathcal{O}(2)[[t]] \quad \text{with} \quad \pi_0 = \pi.
\]
Thus, it follows that $\overline{\pi}_{n+1}$ is a 2-cocycle. If this 2-cocycle is not a coboundary, we are done. Otherwise, we apply the same method again. In this way, we can get a required type of equivalent deformation.

As a corollary, we obtain the following.

**Theorem 3.4.** If $H^2(O, d_\pi) = 0$ then the multiplication $\pi$ is rigid.

Let $\pi$ be a multiplication on an operad $(O, \gamma, \text{id})$. A deformation $\pi_t$ is said to be of order $n$ if it is of the form $\pi_t = \sum_{i=0}^{n} \pi_i t^i$. In the following, we assume that $H^2(O, d_\pi) \neq 0$ so that one may obtain non-trivial deformations. Here, we consider the problem of extending a deformation of order $n$ to a deformation of order $n + 1$. Suppose there is an element $\pi_{n+1} \in O(2)$ so that $\overline{\pi}_t = \pi_t + \pi_{n+1} t^{n+1}$ is a deformation of order $n + 1$. Then we say that $\pi_t$ extends to a deformation of order $n + 1$.

Since we assume that $\pi_t = \sum_{i=0}^{n} \pi_i t^i$ is a deformation, it follows from (6) that

$$\pi \circ \pi_t + \pi_t \circ \pi_{t-1} + \cdots + \pi_{t-1} \circ \pi_t + \pi_t \circ \pi = 0,$$

or, equivalently, $-d_\pi(\pi_t) = \sum_{p+q=r+1, p \geq 1} \pi_p \circ \pi_q$, for $i = 1, 2, \ldots , n$. For $\overline{\pi}_t = \pi_t + \pi_{n+1} t^{n+1}$ to be a deformation, one more deformation equation need to satisfy (which is equivalent to)

$$-d_\pi(\pi_{n+1}) = \sum_{i=1}^{n} \pi_i \circ \pi_{n+1-i}. $$

The right hand side of the above equation is called the obstruction to extend the deformation $\pi_t$ to a deformation of order $n + 1$.

**Proposition 3.5.** The obstruction is a 3-cocycle, that is,

$$d_\pi \left( \sum_{i+j=n+1, i,j \geq 1} \pi_i \circ \pi_j \right) = 0.$$

**Proof.** For any $f, g \in O(2)$, it is easy to see that

$$d_\pi(f \circ g) = f \circ d_\pi(g) - d_\pi(f) \circ g + g \cdot f - f \cdot g.$$

(See [15, Theorem 3] for the case of associative algebra.) Therefore,

$$d_\pi \left( \sum_{i+j=n+1, i,j \geq 1} \pi_i \circ \pi_j \right) = \sum_{i+j=n+1, i,j \geq 1} (\pi_i \circ d_\pi(\pi_j) - d_\pi(\pi_i) \circ \pi_j)
= - \sum_{p+q+r=n+1, p,q,r \geq 1} (\pi_p \circ (\pi_q \circ \pi_r) - (\pi_p \circ \pi_q) \circ \pi_r)
= - \sum_{p+q+r=n+1, p,q,r \geq 1} A_{p,q,r} \text{ (say).}$$

The product $\circ$ is not associative, however, they satisfy the pre-Lie identities (5). This in particular implies that $A_{p,q,r} = 0$ whenever $q = r$. Finally, if $q \neq r$ then $A_{p,q,r} + A_{p,r,q} = 0$ by the same identity (5). Hence we have $\sum_{p+q+r=n+1, p,q,r \geq 1} A_{p,q,r} = 0$. \hfill $\square$

It follows from the above proposition that the obstruction defines a cohomology class in $H^3(O, d_\pi)$. If this cohomology class is zero, then the obstruction is given by a coboundary (say $-d_\pi(\pi_{n+1})$). In other words, $\overline{\pi}_t = \pi_t + \pi_{n+1} t^{n+1}$ defines a deformation of order $n + 1$.

As a summary, we get the following.

**Theorem 3.6.** If $H^3(O, d_\pi) = 0$, every finite order deformation of $\pi$ can be extended to a deformation of next order.
3.2. Deformation space. In this subsection, we describe the equivalence classes of formal deformations of $\pi$ as the solutions of the Maurer-Cartan equation in a dgla. See [10] for the case of associative algebra deformations.

We start with the following notations. Let $\mathfrak{g}$ be a dgla. Consider the new dgla $L = \mathfrak{g} \otimes (t)$, where $(t) \subset \mathcal{K}[[t]]$ is the ideal generated by $t$. Therefore, degree $n$ elements of $L$ are of the form $\gamma = f_1 t + f_2 t^2 + \cdots$, where each $f_i \in \mathfrak{g}^n$. The dgla structure on $L$ is induced from the dgla structure on $\mathfrak{g}$. An element $\gamma = f_1 t + f_2 t^2 + \cdots \in L^1 = \mathfrak{g}^1 \otimes (t)$ is Maurer-Cartan if it satisfies

$$d\gamma + \frac{1}{2} [\gamma, \gamma] = 0 \Leftrightarrow df_k + \frac{1}{2} \sum_{i+j=k} [f_i, f_j] = 0, \text{ for all } k \geq 1.$$ 

Denote by $\text{MC}(\mathfrak{g})$ the set of Maurer-Cartan elements in $L$. Moreover, the gauge group of $\mathfrak{g}$, defined as

$$G(\mathfrak{g}) = \exp(L^0),$$

where $L^0 = \mathfrak{g}^0 \otimes (t)$ is the Lie algebra and $\exp(L^0)$ denotes the group whose underlying space is $L^0$ and the multiplication given by the Baker-Campbell-Hausdorff formula. The gauge group $G(\mathfrak{g})$ acts on $L^1 = \mathfrak{g}^1 \otimes (t)$ by

$$x \cdot \gamma = \exp(x) \cdot \gamma \cdot \exp(-x), \text{ for } x \in G(\mathfrak{g}), \gamma \in L^1.$$ 

The gauge group preserves the space $\text{MC}(\mathfrak{g})$ of Maurer-Cartan elements (see [10] for details). The quotient space

$$\mathcal{D}ef(\mathfrak{g}) = \text{MC}(\mathfrak{g})/G(\mathfrak{g})$$

is called the moduli space of solutions of the Maurer-Cartan equation in $L = \mathfrak{g} \otimes (t)$.

Let $\pi$ be a multiplication in an operad $\mathcal{O}$ and consider the dgla $\mathfrak{g} = (\mathcal{O}(1) + 1), [\ , \ ], d_\pi)$. Then

$$\text{MC}(\mathfrak{g}) = \{ \pi_t = \pi_1 t + \pi_2 t^2 + \cdots | \pi_i \in \mathcal{O}(2) \text{ and } d_\pi(\pi_k) + \frac{1}{2} \sum_{i+j=k} [\pi_i, \pi_j] = 0, \forall k \geq 1 \}$$

$$= \{ \pi_t = \pi_1 t + \pi_2 t^2 + \cdots | \pi_i \in \mathcal{O}(2) \text{ and } -d_\pi(\pi_k) = \sum_{i+j=k, i,j \geq 1} \pi_i \circ \pi_j, \forall k \geq 1 \}.$$ 

Therefore, $\text{MC}(\mathfrak{g})$ is the set of all formal deformations of $\pi$. Observe that, in this example, the group $G(\mathfrak{g})$ is isomorphic to the group

$$H = \{ \phi_t = \text{id} + \phi_1 t + \phi_2 t^2 + \cdots | \phi_i \in \mathcal{O}(1) \}$$

via $\exp : G(\mathfrak{g}) \to H$ and the inverse is given by $\log : H \to G(\mathfrak{g})$. Note that two formal deformations $\pi_t = \pi_1 t + \pi_2 t^2 + \cdots$ and $\pi'_t = \pi'_1 t + \pi'_2 t^2 + \cdots$ and equivalent (i.e. define same element in $\mathcal{D}ef(\mathfrak{g})$) if and only if

$$\exp(x) \cdot (\pi_1 t + \pi_2 t^2 + \cdots) = (\pi_1 t + \pi_2 t^2 + \cdots)(\exp(x) \otimes \exp(x)).$$ 

Hence we conclude that two formal deformations are equivalent in the sense of Definition 3.1 if and only if they lie in the same orbit of the gauge group action. In other words, $\mathcal{D}ef(\mathfrak{g})$ is the equivalence classes of formal deformations of $\pi$.

3.3. A universal deformation. In this subsection, we gave a universal deformation of multiplications. When one consider the deformation of an associative algebra (i.e. deformation of a multiplication in the endomorphism operad), one recovers the universal deformation constructed by Gerstenhaber [16]. We start with the following lemma, the proof of which is similar to [6].
Lemma 3.7. Let $D \in \mathcal{O}(1)$ be such that $d_\pi(D) = 0$. Then for any $\overline{D} \in \mathcal{O}(1)$,
\[
 D^p \circ (D^n \cdot \overline{D}^j) = \sum_{j=0}^{p} \left( \begin{array}{c} p \\ j \end{array} \right) D^{n+j} \cdot (D^{p-j} \circ \overline{D}^j) 
\]
\[
 D^p \circ (\overline{D}^j \cdot D^j) = \sum_{j=0}^{p} \left( \begin{array}{c} p \\ j \end{array} \right) (D^j \circ \overline{D}^j) \cdot D^{q+p-j}. 
\]

Theorem 3.8. Let $D, \overline{D} \in \mathcal{O}(1)$ be such that $d_\pi(D) = d_\pi(\overline{D}) = 0$. Further, if $D \circ \overline{D} = \overline{D} \circ D$ then
\[
 (9) 
\]
\[
 \pi_t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \{D^n, \overline{D}^n\} = -\sum_{n=0}^{\infty} \frac{t^n}{n!} D^n \cdot \overline{D}^n 
\]
defines a deformation of $\pi$.

Proof. Here $\pi_t = -\frac{1}{n} D^n \cdot \overline{D}^n$. To prove that $\pi_t$ defines a deformation of $\pi$, one has to verify relations (6). First observe that
\[
 \sum_{i+j=n} \pi_i \circ_1 \pi_j = \sum_{i=0}^{n} \pi_i \circ_0 \pi_{n-i} = \sum_{i=0}^{n} \frac{1}{i!(n-i)!} (D^i \circ (D^{n-i} \cdot \overline{D}^{n-i})) \cdot \overline{D}^j 
\]
\[
 = \sum_{i=0}^{n} \frac{1}{i!(n-i)!} \sum_{j=0}^{i} \left( \begin{array}{c} i \\ j \end{array} \right) D^{n-i+j} \cdot (D^{i-j} \circ \overline{D}^{n-i}) \cdot \overline{D}^j
\]
\[
 = \sum_{i,j=0,1,\ldots,n,i \geq j} \frac{1}{(n-i)!j!(i-j)!} D^{n-j} \cdot (D^i \circ \overline{D}^{n-i}) \cdot \overline{D}^j. 
\]

Similarly,
\[
 \sum_{i+j=n} \pi_i \circ_2 \pi_j = \sum_{i=0}^{n} \pi_i \circ_2 \pi_{n-i} = \sum_{i=0}^{n} \frac{1}{i!(n-i)!} (D^i \circ (\overline{D}^j \circ (D^{n-i} \cdot \overline{D}^{n-i}))) 
\]
\[
 = \sum_{i=0}^{n} \frac{1}{i!(n-i)!} \sum_{j=0}^{i} \left( \begin{array}{c} i \\ j \end{array} \right) D^j \cdot (\overline{D}^j \circ (D^{n-i} \cdot \overline{D}^{n-i}))) 
\]
\[
 = \sum_{i,j=0,1,\ldots,n,i \geq j} \frac{1}{(n-i)!j!(i-j)!} D^j \cdot (\overline{D}^j \circ (D^{n-i} \cdot \overline{D}^{n-i})). 
\]

By replacing the dummy variables $i \leftrightarrow n-j$ and $j \leftrightarrow n-i$ and using the fact that $D, \overline{D}$ commute, we get the same expression as in (10). Thus we obtain $\sum_{i+j=n} \pi_i \circ_1 \pi_j = \sum_{i+j=n} \pi_i \circ_2 \pi_j$ which is equivalent to $\sum_{i+j=n} \pi_i \circ \pi_j = 0$. Hence the proof.

4. LODAY-TYPE ALGEBRAS: DEFORMATIONS

It is shown in [9,25] that dialgebra and dendriform algebra structure on a vector space can be seen as a multiplication in suitable operads. It is very easy to verify that deformation of these algebras as developed in [9,24] are equivalent to deformation of corresponding multiplications. We show that other Loday-type algebras (e.g. dendriform trialgebras, quadri-algebras, ennea-algebras) can also be described by multiplication in certain operads, and by definition, their deformation are given by deformation of respective multiplications.
More precisely, for various Loday-type algebras, there is a sequence of non-empty sets $U = \{U_n | n \geq 1\}$ and collection of ‘nice’ functions

$$R_0(m; 1, \ldots, 1, n, 1, \ldots, 1) : U_{m+n-1} \to U_m$$

$$R_i(m; 1, \ldots, 1, n, 1, \ldots, 1) : U_{m+n-1} \to \mathbb{K}[U_n].$$

It turns out that, for any vector space $A$, the collection of spaces $\mathcal{O}(n) = \text{Hom}(\mathbb{K}[U_n] \otimes A^\otimes n, A)$, for $n \geq 1$, with the partial compositions

$$(f \circ g)(r; a_1, \ldots, a_{m+n-1}) = f(R_0(m; 1, \ldots, 1, n, 1, \ldots, 1)r; a_1, \ldots, a_{i-1}, g(R_i(m; 1, \ldots, 1, n, 1, \ldots, 1)r; a_i, \ldots, a_{i+n-1}), \ldots, a_{m+n-1}),$$

for $f \in \mathcal{O}(m)$, $g \in \mathcal{O}(n)$, $r \in U_{m+n-1}$ and $a_1, \ldots, a_{m+n-1} \in A$, forms an operad. The identity element $id \in \mathcal{O}(1)$ is given by $id(r; a) = a$, for all $r \in U_1$ and $a \in A$. The functions $\{R_0, R_i\}$ are called the structure functions for the operad. Finally, a Loday-type algebra structure on $A$ is equivalent to a multiplication on the operad $\mathcal{O}$. For example, in the case of dialgebras, $U_n = Y_n$ is the set of planar binary trees with $(n+1)$ leaves and for dendriform algebras, $U_n = C_n$ is the set of first $n$ natural numbers.

Let $A$ be a fixed Loday-type algebra with associated multiplication given by $\pi \in \mathcal{O}(2) = \text{Hom}(\mathbb{K}[U_2] \otimes A^\otimes 2, A)$. Thus, a deformation of $A$ is given by a formal sum $\pi_t = \pi + \pi_1 t + \pi_2 t^2 + \cdots \in \mathcal{O}(2)[[t]]$ satisfying $\pi_t \circ \pi_t = 0$. Cohomological interpretations of deformation will be remarked in the next section when we introduce the cohomology of Loday-type algebras (Remark 5.4).

### 4.1. Dialgebras

The operad for dialgebras was constructed in [25] and the deformation theory was studied in [24].

**Definition 4.1.** [22] A dialgebra is a vector space $A$ together with two bilinear maps $\lhd, \rhd : A \otimes A \to A$ satisfying the following relations

- $a \lhd (b \rhd c) = (a \lhd b) \rhd c = a \lhd (b \rhd c)$,
- $(a \lhd b) \lhd c = a \lhd (b \rhd c)$,
- $(a \lhd b) \rhd c = a \lhd (b \rhd c) = (a \rhd b) \rhd c$, for all $a, b, c \in A$.

Let $Y_n$ be the set of planar binary trees with $(n+1)$ leaves, one root and each vertex is trivalent. For each $y \in Y_n$, we label the $(n+1)$ leaves by $\{0, 1, \ldots, n\}$ from left to right. We define maps $d_i : Y_n \to Y_{n-1}$ ($0 \leq i \leq n$) which is obtained by deleting the $i$-th leaf. Finally, the structure maps are given by

$$R_0(m; 1, \ldots, 1, n, 1, \ldots, 1) := \hat{d}_0 \circ \hat{d}_1 \circ \cdots \circ \hat{d}_{i-1} \circ \hat{d}_i \circ \cdots \circ \hat{d}_{i+n-2} \circ \hat{d}_{i+n-1} \cdots \circ \hat{d}_{m+n-1} : Y_{m+n-1} \to Y_m,$$

$$R_i(m; 1, \ldots, 1, n, 1, \ldots, 1) := \hat{d}_0 \circ \hat{d}_1 \circ \cdots \circ \hat{d}_{i-2} \circ \hat{d}_{i-1} \cdots \circ \hat{d}_{i+n-1} \circ \hat{d}_{i+n} \circ \cdots \circ \hat{d}_{m+n-1} : Y_{m+n-1} \to \mathbb{K}[Y_n].$$

In such a case, the image of $R_i(m; 1, \ldots, 1, n, 1, \ldots, 1)$ entirely lies in $Y_n$. It is shown in [25] that for any vector space $A$, the spaces $\mathcal{O}(n) := \text{Hom}(\mathbb{K}[Y_n] \otimes A^\otimes n, A)$, for $n \geq 1$, defines an operad whose partial compositions are given by (11). Moreover, if $(A, \lhd, \rhd)$ is a dialgebra, then the element $\pi \in \mathcal{O}(2)$ given by $\pi(\hat{Y}; a, b) = a \lhd b$ and $\pi(\hat{Y}; a, b) = a \rhd b$ defines a multiplication on $\mathcal{O}$.

A deformation of a dialgebra $(A, \lhd, \rhd)$ in the sense of [24] is given by two formal power series

$$\lhd_t = \lhd_0 + \lhd_1 t + \lhd_2 t^2 + \cdots \quad \text{and} \quad \rhd_t = \rhd_0 + \rhd_1 t + \rhd_2 t^2 + \cdots$$
(with $\cdot_0 = -$ and $\cdot_t = \cdot$) of binary operations on $A$ such that $(A[[t]], \cdot_t, \cdot_t)$ forms a dialgebra. Then it follows that the formal sum $\pi_t = \pi_0 + \pi_1 t + \pi_2 t^2 + \cdots \in O(2)[[t]]$ defines a deformation of $\pi$, where $\pi_t(\bar{Y}; a, b) = a \cdot b$ and $\pi_t(\bar{Y}; a, b) = a \cdot_i b$, for all $i \geq 0$. This follows as the deformation equations of [24] are equivalent to the deformation equations (6).

4.2. Associative trialgebras. This type of algebra is formed by three binary operations $\cdot_t, \cdot, \cdot_0$ which satisfy 11 associative-style identities [23]. They are related to planar trees (not necessarily binary) exactly in the same way dialgebras are related to planar binary trees.

Let $T_n$ be the set of planar trees with $n + 1$ leaves and one root in which each vertex has valence at least 2. Then $T_2$ has 3 elements and $T_3$ has 11 elements [23]. Exactly, in the same way as above, the sets $O(n) := Hom(K[T_n] \otimes A^{\otimes n}, A)$, for $n \geq 1$, inherits a structure of an operad. Further, if $(A, \cdot_t, \cdot, \cdot_0)$ is an associative trialgebra, then the element $\pi \in O(2)$ given by $\pi(\bar{Y}; a, b) = a \cdot b$, $\pi(\bar{Y}; a, b) = a \cdot b$ defines a multiplication on the operad $O$ [28]. Note that the condition $(\pi \circ \pi)(y; a, b, c) = 0$, for all $y \in T_3$ correspond to 11 defining identities.

Thus, a deformation of $(A, \cdot_t, \cdot, \cdot_0)$ is given by a formal sum $\pi_t = \pi + \pi_1 t + \pi_2 t^2 + \cdots \in O(2)[[t]]$ satisfying $\pi_t \circ \pi_t = 0$. Explicitly, it is given by three formal series $\pi_t = \cdot_0 + \cdot_1 t + \cdot_2 t^2 + \cdots$, $\pi_t = \cdot_0 + \cdot_1 t + \cdot_2 t^2 + \cdots$, $\pi_t = \cdot_0 + \cdot_1 t + \cdot_2 t^2 + \cdots$ (with $\cdot_0 = \cdot$, $\cdot_0 = \cdot$ and $\cdot_0 = \cdot$) of binary operations on $A$ such that $(A[[t]], \cdot_t, \cdot, \cdot_0)$ is an associative trialgebra. These two interpretations are related by $\pi_t(\bar{Y}; a, b) = a \cdot b$, $\pi_t(\bar{Y}; a, b) = a \cdot b$ and $\pi_t(\bar{Y}; a, b) = a \cdot b$, for all $i \geq 0$.

4.3. Dendriform algebras. Dendriform algebras are Koszul dual to dialgebras and they can be thought as a certain splitting of associative algebras [22]. The corresponding operad for dendriform algebras and deformation theory was explicitly studied in [9].

**Definition 4.2.** A dendriform algebra is a vector space $A$ together with two bilinear maps $\prec, \succ: A \otimes A \rightarrow A$ satisfying

$$(a \prec b) \prec c = a \prec (b \prec c + b \succ c),$$

$$(a \succ b) \prec c = a \succ (b \prec c),$$

$$(a \prec b + a \succ b) \succ c = a \succ (b \succ c), \text{ for all } a, b, c \in A.$$

It follows from the definition that the sum operation $a \star b = a \prec b + a \succ b$ is associative. To define the operad, let $C_n$ be the set of first $n$ natural numbers. Since we will treat them as certain symbols, we denote them by $C_n = \{[1], \ldots, [n]\}$. For any $m, n \geq 1$ and $1 \leq i \leq m$, we define maps $R_0(m; 1, \ldots, n, \ldots, 1) : C_{m+n-1} \rightarrow C_m$ and $R_i(m; 1, \ldots, n, \ldots, 1) : C_{m+n-1} \rightarrow K[C_n]$ by

$$R_0(m; 1, \ldots, n, \ldots, 1)([r]) = \begin{cases} [r] & \text{if } r \leq i - 1 \\ [i] & \text{if } i \leq r \leq i + n - 1 \\ [r - n + 1] & \text{if } i + n \leq r \leq m + n - 1. \end{cases}$$

$$R_i(m; 1, \ldots, n, \ldots, 1)([r]) = \begin{cases} [1] + [2] + \cdots + [n] & \text{if } r \leq i - 1 \\ [r - i + 1] & \text{if } i \leq r \leq i + n - 1 \\ [1] + [2] + \cdots + [n] & \text{if } i + n \leq r \leq m + n - 1. \end{cases}$$

We may view these functions by the following combinatorial way. Put the first $(m+n-1)$ natural numbers into $m$ boxes in the following way

$$1 \ 2 \ \cdots \ i-1 \ i \ i+1 \ \cdots \ i+n-1 \ i+n \ \cdots \ m+n-1.$$
With these notations, the map \( R_0(m; 1, \ldots, n, \ldots, 1)([r]) \) gives the number of the box where \( r \) appear and \( R_i(m; 1, \ldots, n, \ldots, 1)([r]) \) gives the position of \( r \) in the \( i \)-th box (if \( r \) lies in the \( i \)-th box) and \([1] + \cdots + [n]\) otherwise.

Let \( A \) be a vector space. For any \( n \geq 1 \), we define \( O(n) := \text{Hom}(\mathbb{K}[C_n] \otimes A^\otimes n, A) \). Then it is shown in [9] that \( O \) inherits a structure of an operad with structure functions are given by (11).

Note that if \((A, \prec, \succ) \) is a dendriform algebra, then the element \( \pi \in O(2) = \text{Hom}(\mathbb{K}[C_2] \otimes A^\otimes 2, A) \) given by \( \pi([1]; a, b) = a \prec b \) and \( \pi([2]; a, b) = a \succ b \), defines a multiplication in the operad. It has been implicitly shown in [9] that a formal deformation of \((A, \prec, \succ) \) is equivalent to a deformation of \( \pi \).

4.4. Dendriform trialgebra. These algebras are Koszul dual to associative trialgebra and can be thought as a splitting of an associative algebra by three operations [23].

Definition 4.3. A dendriform trialgebra is a vector space \( A \) endowed with three binary operations \( \prec \) (left), \( \succ \) (right), and \( \cdot \) (middle) satisfying the following set of 7 identities

\[
\begin{align*}
(a \prec b) \prec c &= a \prec (b \prec c + b \succ c + b \cdot c), \\
(a \succ b) \prec c &= a \succ (b \prec c), \\
(a \prec b + a \succ b + a \cdot b) \succ c &= a \succ (b \succ c), \\
(a \succ b) \cdot c &= a \succ (b \cdot c), \\
(a \prec b) \cdot c &= a \cdot (b \succ c), \\
(a \cdot b) \prec c &= a \cdot (b \prec c), \\
(a \cdot b) \cdot c &= a \cdot (b \cdot c), \quad \text{for all } a, b, c \in A.
\end{align*}
\]

It turns out that \((A, \prec + \cdot, \succ) \) is a dendriform algebra and hence \((A, \prec + \succ + \cdot) \) is an associative algebra. To define the corresponding operad, let \( P_n \) be the set of all non-empty subsets of \{1, 2, \ldots, n\}. Thus \( P_2 = \{\{1\}, \{2\}, \{1, 2\}\} \) and \( P_3 = \{\{1\}, \{2\}, \{3\}, \{2, 3\}, \{1, 3\}, \{1, 2\}, \{1, 2, 3\}\} \).

For any \( m, n \geq 1 \) and \( 1 \leq i \leq m \), we define the structure functions as

\[
\begin{align*}
R_0(m; 1, \ldots, n, \ldots, 1)(X) &= \{r \mid \exists x \in X \text{ which is in } r\text{-th box} \}, \\
R_i(m; 1, \ldots, n, \ldots, 1)(X) &= \begin{cases} 
\sum_{S \in P_n} S & \text{if } X \cap (i\text{-th box}) = \phi \\
\{k_1 - (i - 1), \ldots, k_l - (i - 1)\} & \text{if } X \cap (i\text{-th box}) = \{k_1, \ldots, k_l\},
\end{cases}
\end{align*}
\]

for any \( X \in P_{m+n-1} \). For any vector space \( A \), we define \( O(n) = \text{Hom}(\mathbb{K}[P_n] \otimes A^\otimes n, A) \), for \( n \geq 1 \).

It turns out that \( O \) is an operad with the structure functions as defined above and the partial compositions are given by (11).

If \((A, \prec, \succ, \cdot) \) is a dendriform trialgebra, we define an element \( \pi \in O(2) = \text{Hom}(\mathbb{K}[P_2] \otimes A^\otimes 2, A) \) by \( \pi(\{1\}; a, b) = a \prec b \), \( \pi(\{2\}; a, b) = a \succ b \), and \( \pi(\{1, 2\}; a, b) = a \cdot b \). The 7 defining identities of a dendriform trialgebra is equivalent to the fact that \( \pi \) defines a multiplication on \( O \). Note that the element \( \pi \) can be understood by the following Hasse diagram of the set of all non-empty subsets of \{1, 2\} ordered by inclusion:

\[
\begin{align*}
\{1, 2\} &= \cdot \\
\{1\} &= \prec \\
\{2\} &= \succ
\end{align*}
\]
In view of previous discussions, a deformation of a dendriform trialgebra \((A, \prec, \succ, \cdot)\) is given by three formal series

\[
\prec_t = \prec_0 + \prec_1 t + \prec_2 t^2 + \cdots, \quad \succ_t = \succ_0 + \succ_1 t + \succ_2 t^2 + \cdots \quad \text{and} \quad \cdot_t = \cdot_0 + \cdot_1 t + \cdot_2 t^2 + \cdots
\]

(with \(\prec_0 = \prec, \succ_0 = \succ\) and \(\cdot_0 = \cdot\)) of binary operations on \(A\) such that \((A[[t]], \prec_t, \succ_t, \cdot_t)\) is a dendriform trialgebra.

**Remark 4.4.** Let \((A, \prec, \succ, \cdot)\) be a dendriform trialgebra and \((\prec_t, \succ_t, \cdot_t)\) be a deformation of it. Then \((A[[t]], \prec_t + \cdot_t, \succ_t)\) is a dendriform algebra which provides a deformation of the corresponding dendriform algebra \((A, \prec + \cdot, \succ)\). Moreover, the pair \((A[[t]], \prec_t + \succ_t + \cdot_t)\) is a deformation of the corresponding associative algebra \((A, \prec + \succ + \cdot)\).

A Rota-Baxter algebra is an associative algebra \((A, \mu)\) together with a \(\mathbb{K}\)-linear map \(R : A \to A\) which satisfies

\[
\mu(R(x), y) = R(\mu(x, R(y))) + \mu(R(x), y) + \lambda \mu(x, y), \quad \text{for all} \ x, y \in A.
\]

Here \(\lambda \in \mathbb{K}\) is fixed and is called the weight of the Rota-Baxter algebra. It follows from [11] that a Rota-Baxter algebra \((A, \mu, R)\) of weight \(\lambda\) induces a dendriform trialgebra \((A, \prec, \succ, \cdot)\) where

\[
x \prec y = \mu(x, R(y)), \quad x \succ y = \mu(R(x), y) \quad \text{and} \quad x \cdot y = \lambda \mu(x, y).
\]

Therefore, one gets a dendriform dialgebra \((A, \prec', \succ')\) with

\[
x \prec' y = x \prec y + x \cdot y \quad \text{and} \quad x \succ' y = x \succ y.
\]

A deformation of a Rota-Baxter algebra \((A, \mu, R)\) of weight \(\lambda\) is given by a deformation \(\mu_t = \sum_{i \geq 0} \mu_i t^i\) of the associative algebra \((A, \mu)\) and a formal sum \(R_t = \sum_{i \geq 0} R_i t^i\) where each \(R_i \in \text{End}(A, A)\) with \(R_0 = R\) such that \((A[[t]], \mu_t, R_t)\) is a Rota-Baxter algebra of the same weight \(\lambda\). Thus, a deformation of a Rota-Baxter algebra induces a dendriform trialgebra \((A[[t]], \prec_{A[[t]]}, \succ_{A[[t]]}, \cdot_{A[[t]]})\) and a dendriform dialgebra \((A[[t]], \prec'_{A[[t]]}, \succ'_{A[[t]]})\). In other words, the triplet \((\prec_t, \succ_t, \cdot_t)\) is a deformation of the corresponding dendriform trialgebra \((A, \prec + \cdot, \succ)\) where

\[
x \prec_t y = \sum_{i+j=n} \mu_i(x, R_j(y)) t^n,
\]
\[
x \succ_t y = \sum_{i+j=n} \mu_i(R_j(x), y) t^n,
\]
\[
x \cdot_t y = \sum_n \lambda \mu_n(x, y) t^n.
\]

Thus, the pair \((\prec'_t, \succ'_t)\) is a deformation of the corresponding dendriform dialgebra \((A, \prec', \succ')\) where

\[
x \prec'_t y := x \prec_t y + x \cdot_t y = \left( \sum_{i+j=n} \mu_i(x, R_j(y)) + \lambda \mu_n(x, y) \right) t^n,
\]
\[
x \succ'_t y := x \succ_t y = \sum_{i+j=n} \mu_i(R_j(x), y) t^n.
\]

**4.5. Quadri-algebras.** Quadri-algebras are further splitting of dendriform algebras that arise naturally on the space of linear endomorphisms of an infinitesimal bialgebra and from two commuting Rota-Baxter operators [2]. These algebras are given by 4 binary operations \(\prec\) (north-west), \(\succ\) (north-east), \(\preceq\) (south-west), \(\succeq\) (south-east) and satisfying 9 identities. See [2] for definition. We show that a quadri-algebra structure on a vector space can be seen as a multiplication in a certain operad in which the structure functions are cartesian product of the structure functions of the operad defined for dendriform algebras.
Let $Q_n = \{1, \ldots, n\} \times \{1, \ldots, n\}$ be the cartesian product of first $n$ natural numbers with itself. We write the elements of $Q_n$ as $\{(1,1), \ldots, (1,n), \ldots, (n,1), \ldots, (n,n)\}$. Thus the cardinality of $Q_n$ is $n^2$. It is useful to think that the $n^2$ elements of $Q_n$ have been allotted in a $n \times n$ square matrix.

For any $m,n \geq 1$; $1 \leq i \leq m$ and $(r,s) \in Q_{m+n-1}$, we define the structure maps as

$$R_0(m;1,\ldots,n,\ldots,1)(r,s) = \begin{cases} (r,s) & \text{if } 1 \leq r \leq i - 1 \text{ and } 1 \leq s \leq i - 1 \\ (i,s) & \text{if } i \leq r \leq i + n - 1 \text{ and } 1 \leq s \leq i - 1 \\ (r-n+1,s) & \text{if } i + n \leq r \leq m + n - 1 \text{ and } 1 \leq s \leq i - 1 \\ (r,i) & \text{if } 1 \leq r \leq i - 1 \text{ and } i \leq s \leq i + n - 1 \\ (r-n+1,i) & \text{if } i + n \leq r \leq m + n - 1 \text{ and } i \leq s \leq i + n - 1 \\ (r,s-n+1) & \text{if } 1 \leq r \leq i - 1 \text{ and } i + n \leq s \leq m + n - 1 \\ (i,s-n+1) & \text{if } i \leq r \leq i + n - 1 \text{ and } i + n \leq s \leq m + n - 1 \\ (r-n+1,s-n+1) & \text{if } i + n \leq r \leq m + n - 1 \text{ and } i + n \leq s \leq m + n - 1, \end{cases}$$

and

$$R_i(m;1,\ldots,n,\ldots,1)(r,s) = \begin{cases} \sum_{(u,v) \in P_i} (u,v) & \text{if } 1 \leq r \leq i - 1 \text{ and } 1 \leq s \leq i - 1 \\ (r-i+1,1) + \cdots + (r-i+1,n) & \text{if } i \leq r \leq i + n - 1 \text{ and } 1 \leq s \leq i - 1 \\ \sum_{(u,v) \in P_i} (u,v) & \text{if } i + n \leq r \leq m + n - 1 \text{ and } 1 \leq s \leq i - 1 \\ (1, s-i+1) + \cdots + (n, s-i+1) & \text{if } 1 \leq r \leq i - 1 \text{ and } i \leq s \leq i + n - 1 \\ (r-i+1, s-i+1) & \text{if } i \leq r \leq i + n - 1 \text{ and } i \leq s \leq i + n - 1 \\ (1, s-i+1) + \cdots + (n, s-i+1) & \text{if } i + n \leq r \leq m + n - 1 \text{ and } i \leq s \leq i + n - 1 \\ \sum_{(u,v) \in P_i} (u,v) & \text{if } 1 \leq r \leq i - 1 \text{ and } i + n \leq s \leq m + n - 1 \\ (r-i+1,1) + \cdots + (r-i+1,n) & \text{if } i \leq r \leq i + n - 1 \text{ and } i + n \leq s \leq m + n - 1 \end{cases}$$

One observes that these functions are cartesian product of the respective functions defined for dendriform algebras. It follows that, for any vector space $A$, the spaces $\mathcal{O}(n) = \text{Hom}(\mathbb{K}[Q_n] \otimes A^{\otimes n}, A)$, for $n \geq 1$, inherits a structure of an operad whose structure functions are defined above and partial compositions are given by (11).

If $(A, \langle, \rangle, \langle, \rangle, \langle, \rangle)$ is a quadri-algebra, then it can be shown that the element $\pi \in \mathcal{O}(2) = \text{Hom}(\mathbb{K}[Q_2] \otimes A^{\otimes 2}, A)$ defined by $\pi((1,1); a,b) = a \langle b, \pi((1,2); a,b) = a \langle b, \pi((2,1); a,b) = a \langle b and $\pi((2,2); a,b) = a \langle b$ is a multiplication on the operad $\mathcal{O}$. The correspondence between $\pi$ and $4$ operations of the quadri-algebra $A$ can be understood by the following diagram

```
  (1,1)  N
    / \
   /    \
(1,2)  W
    \
   /    \
(2,1)  E
    \
    S  (2,2)
```
A deformation of a quadri-algebra \((A, \kappa, \gamma, \nu, \omega)\) is a deformation of \(\pi\) in the operad \(O\). One can also explicitly write the deformation of \(A\) by \(4\) formal power series \((\kappa_t, \gamma_t, \nu_t, \omega_t)\) of binary operations on \(A\) such that \((A[[t]], \kappa_t, \gamma_t, \nu_t, \omega_t)\) is a quadri-algebra.

4.6. Ennea-algebras. Like quadri-algebras are splitting of dendriform algebras, ennea-algebras are splitting of dendriform trialgebras \([21]\). These algebras are given by \(9\) binary operations and satisfying \(49\) relations. See the above reference for definition.

We have seen that the structure functions for the operad of quadri-algebras are cartesian product of the structure functions for the operad of dendriform algebras. Similarly, the structure functions for the operad of ennea-algebras are cartesian product of the structure functions for the operad of dendriform trialgebras. More precisely, define \(E_n = P_n \times P_n\), where \(P_n\) is the set of all non-empty subsets of \(\{1, \ldots, n\}\). It follows that the cardinality of \(E_2\) is \(9\) and that of \(E_3\) is \(49\). We define structure functions \(R_0(m; 1, \ldots, 1, n, 1, \ldots, 1): E_{m+n-1} \to E_m\) and \(R_i(m; 1, \ldots, 1, n, 1, \ldots, 1) = \): \(E_{m+n-1} \to K[E_n]\) to be the cartesian product of the structure functions defined for dendriform trialgebras. The \(9\) elements of \(E_2\) correspond to \(9\) binary operations and \(49\) elements of \(E_3\) correspond to \(49\) defining relations of an ennea-algebra. More explicitly, an ennea-algebra structure on a vector space \(A\) is equivalent to a multiplication on the operad \(O(n) = \text{Hom}(K[E_n] \otimes A^\otimes n, A)\), for \(n \geq 1\), whose structure functions are defined above. A deformation of an ennea-algebra can be defined in an analogous way.

4.7. Hom analogue of Loday-type algebras. Recently hom-type algebras have been studied by many authors. In these algebras, the identities defining the structures are twisted by one homomorphism (or two commuting homomorphisms). See \([3, 20, 26]\) for more details. In this subsection, we describe various Loday-type algebras twisted by homomorphisms as multiplications in certain operads.

Let \((O, \gamma, \text{id})\) be an operad with partial compositions \(\circ_1\). Let \(\alpha, \beta \in O(1)\) be such that \(\alpha \circ \beta = \beta \circ \alpha\). Consider

\[O_{\alpha, \beta}(n) = \{f \in O(n) | \gamma(f; \alpha, \ldots, \alpha) = \alpha \circ f \text{ and } \gamma(f; \beta, \ldots, \beta) = \beta \circ f\}, \quad n \geq 1.\]

Define twisted partial compositions \(\circ'_1: O_{\alpha, \beta}(m) \otimes O_{\alpha, \beta}(n) \to O_{\alpha, \beta}(m + n - 1)\) by

\[(f \circ'_1 g) = \gamma(f; \alpha^{n-1}, \ldots, \alpha^{n-1}, g, \beta^{n-1}, \ldots, \beta^{n-1}), \quad \text{in } i\text{-th place},\]

for \(f \in O_{\alpha, \beta}(m)\) and \(g \in O_{\alpha, \beta}(n)\).

**Proposition 4.5.** With the above notations \((O_{\alpha, \beta}, \circ'_1, \text{id})\) forms an operad.

The proof of the above proposition is simple and can be found in \([7, 8]\) when \(O\) is the endomorphism operad associated to a vector space. The operad \((O_{\alpha, \beta}, \circ'_1, \text{id})\) is called the twisted variation of \(O\) twisted by \(\alpha\) and \(\beta\). One observes that when \(\alpha = \beta = \text{id} \in O(1)\), the twisted variation \(O_{\text{id}, \text{id}}\) is same as the operad \(O\).

Thus it follows from the previous proposition that one can construct twisted version of various operads as defined in previous subsections. Multiplications of these twisted operads are called twisted Loday-type algebras. For example, twisted associative algebra structure on \(A\) is given by a multiplication on the twisted variation \(\text{End}_{\alpha, \beta}(A)\). One can construct twisted Loday-type algebra structures as follows. If \(\pi \in O(2) = \text{Hom}(K[U_2] \otimes A^\otimes 2, A)\) defines a fixed Loday-type algebra structure on \(A\), and \(\alpha, \beta\) be two commuting algebra morphisms, then \(\{\pi\} \{\alpha, \beta\} \in O_{\alpha, \beta}(2)\) given by

\[\{\pi\} \{\alpha, \beta\}(r; a, b) := \pi(r; \alpha(a), \beta(b)),\]

for \(r \in U_2, \ a, b \in A\), defines a twisted Loday-type algebra structure on \(A\).
In the same analogy, a deformation of a twisted Loday-type algebra $A$ is by definition a deformation of the corresponding multiplication in the twisted variation operad. For hom-associative algebras, it recovers the deformation described in [3].

5. Representations of Loday-type algebras

In this section, we study representations and cohomology of Loday-type algebras.

5.1. Representations. Let $A$ be a fixed Loday-type algebra. Assume that the Loday-type algebra structure on $A$ can be given by a multiplication $\pi$ on the operad $\mathcal{O}$ in which the structure functions are given by $\{R_0, R_1\}$ on the sets $\{U_n | n \geq 1\}$.

**Definition 5.1.** A representation of $A$ is given by a vector space $M$ together with maps

$$\theta_1 : \mathbb{K}[U_2] \otimes (A \otimes M) \to M, \quad \theta_2 : \mathbb{K}[U_2] \otimes (M \otimes A) \to M$$

satisfying

$$\theta_1(R_0(2; 1, 2)y; a, \theta_1(R_2(2; 1, 2)y; b, m)) = \theta_1(R_0(2; 2, 1)y; \pi(R_1(2; 2, 1)y; a, b), m),$$

$$\theta_2(R_0(2; 1, 2)y; m, \pi(R_2(2; 1, 2)y; a, b)) = \theta_2(R_0(2; 2, 1)y; \theta_2(R_1(2; 2, 1)y; m, a), b),$$

$$\theta_1(R_0(2; 1, 2)y; a, \theta_2(R_2(2; 1, 2)y; m, b)) = \theta_2(R_0(2; 2, 1)y; \theta_1(R_1(2; 2, 1)y; a, m), b),$$

for all $y \in U_3$ and $a, b \in A, m \in M$.

There are $3 \sharp(U_3)$ relations to define a representation. Moreover, it follows that $A$ is a representation of itself with $\theta_1 = \theta_2 = \pi$.

Let $A$ be a fixed Loday-type algebra. An ideal of $A$ is a subspace $I \subset A$ satisfying $\pi(r; A, I) \subset A$ and $\pi(r; I, A) \subset I$, for all $r \in U_2$. Any ideal of $A$ is a representation with $\theta_1 = \theta_2 = \pi$. Any vector space $M$ is a representation of $A$ with $\theta_1 = \theta_2 = 0$.

**Proposition 5.2.** (Semi-direct product) Let $A$ be a fixed Loday-type algebra and $M$ be a representation of $A$. Then the direct sum $A \oplus M$ inherits a Loday algebra structure of same type. The multiplication is given by

$$\pi_{A \oplus M}(r; (a, m), (b, n)) = (\pi(r; a, b), \theta_1(r; a, n) + \theta_2(r; m, b)),$$

for $r \in U_2$ and $(a, m), (b, n) \in A \oplus M$.

**Proof.** Straightforward.

5.2. Cohomology with coefficients. Let $A$ be a Loday-type algebra and $M$ be a representation of it. Define

$$C^n(A, M) := \text{Hom}(\mathbb{K}[U_n] \otimes A^\otimes n, M),$$

for $n \geq 1$.

The coboundary operator $\delta : C^n(A, M) \to C^{n+1}(A, M)$ is given by

$$\langle \delta f \rangle(y; a_1, \ldots, a_{n+1})$$

$$= \theta_1(R_0(2; 1, n)y; a_1, f(R_2(2; 2, 1)y; a_2, \ldots, a_{n+1}))$$

$$+ \sum_{i=1}^n (-1)^i f(R_0(n; 1, \ldots, 2, \ldots, 1)y; a_1, \ldots, a_{i-1}, \pi(R_1(1, \ldots, 2, \ldots, 1)y; a_i, a_{i+1}), a_{i+2}, \ldots, a_{n+1})$$

$$+ (-1)^{n+1} \theta_2(R_0(2; n, 1)y; f(R_1(2; n, 1)y; a_1, \ldots, a_n), a_{n+1}),$$

for $y \in U_{n+1}$ and $a_1, \ldots, a_{n+1} \in A$.

When we consider the case of a dialgebra, our cohomology coincides with that of Frabetti [12] and in the case of a dendriform algebra, it coincides with [9].
Remark 5.3. When $M = A$ with the representation given by $\theta_1 = \theta_2 = \pi$, the above coboundary map coincides with the one induced from the multiplication $\pi$. Therefore, it follows from the discussion of section 2 that the cohomology of $A$ (with coefficients in itself) inherits a Gerstenhaber algebra structure. The first observation also ensures that $\delta^2 = 0$ for the coboundary map defined above with coefficients in any arbitrary representation.

Remark 5.4. In the previous section, we define deformation of a Loday-type algebra $A$ as deformation of the corresponding multiplication. It follows from the previous remark that the results of section 3 can be stated for Loday-type algebras as follows. The vanishing of the second cohomology of the Loday algebra $A$ implies that $A$ is rigid and vanishing of the third cohomology allows one to extend a finite order deformation of $A$ to next order.

5.3. Abelian extensions. In this subsection, we show that second cohomology group of a Loday-type algebra can be described by equivalence classes of abelian extensions. We start with the following definition.

Definition 5.5. Let $A$ and $B$ be two Loday algebras of same type. A morphism between them is given by a linear map $f : A \to B$ satisfying

$$f(\pi_A(r; a, a')) = \pi_B(r; f(a), f(a')),$$

for all $r \in U_2$, $a, a' \in A$, where $\pi_A$ and $\pi_B$ denote the multiplications corresponding to the algebra structures on $A$ and $B$, respectively.

Let $A$ be a Loday-type algebra and $M$ be a vector space. Note that $M$ can be considered as a Loday algebra of same type with the trivial multiplication $\pi_M = 0$.

Definition 5.6. An abelian extension of $A$ by $M$ is given by an extension

$$0 \to M \xrightarrow{i} E \xrightarrow{j} A \to 0$$

of Loday algebras (of same type) such that the sequence is split over $K$.

An abelian extension induces an $A$-representation on $M$ via the actions

$$\theta_1(r; a, m) = \pi_E(r; s(a), i(m)) \quad \text{and} \quad \theta_2(r; m, a) = \pi_E(r; i(m), s(a)),$$

for $r \in U_2$, $a \in A$, $m \in M$ and $s : A \to E$ is any section corresponding to the $K$-splitting. One can easily verify that this action is independent of the choice of $s$.

Two such abelian extensions are said to be equivalent if there is a morphism $\phi : E \to E'$ between Loday algebras which makes the following diagram commute

$$\begin{array}{ccc}
0 & \to & M \\
& \searrow^i & \downarrow \phi & \nearrow^j \\
& & E & \\
& \searrow^{i'} & \downarrow \phi & \nearrow^{j'} \\
& & E' & \\
& \downarrow & & \downarrow \\
& A & \to & 0.
\end{array}$$

Next, fix an $A$-representation $M$. We denote by $\mathcal{E}xt(A, M)$ the equivalence classes of abelian extensions of $A$ by $M$ for which the induced representation on $M$ is the prescribed one.

Theorem 5.7. There is a bijection: $H^2(A, M) \cong \mathcal{E}xt(A, M)$. 
Proof. Given a 2-cocycle \( f \in C^2(A, M) \), we consider the \( \mathbb{K} \)-module \( E = A \oplus M \) with the following Loday algebra structure

\[
\pi_E(r; (a, m), (b, n)) = (\pi(r; a, b), \theta_1(r; a, n) + \theta_2(r; m, b) + f(r; a, b)).
\]

(Observe that when \( f = 0 \) this is the semi-direct product.) Using the fact that \( f \) is a 2-cocycle, it is easy to verify that \( \pi_E \) defines a Loday algebra structure (of same type) on \( E \). Moreover, \( 0 \to M \to E \to A \to 0 \) defines an abelian extension with the obvious splitting. Let \( \pi_E' \) be the multiplication on \( E = A \oplus M \) associated to the cohomologous 2-cocycle \( f - \delta(g) \), for some \( g \in C^1(A, M) \). The equivalence between abelian extensions \( (E, \pi_E) \) and \( (E, \pi_E') \) is given by \( (a, m) \mapsto (a, m + g(a)) \).

Therefore, the map \( H^2(A, M) \to \mathcal{E}xt(A, M) \) is well defined.

Conversely, given an extension \( 0 \to M \xrightarrow{j} E \xrightarrow{\pi} A \to 0 \) with splitting \( s \), we may consider \( E = A \oplus M \) and \( s \) is the map \( s(a) = (a, 0) \). With respect to the above splitting, the maps \( i \) and \( j \) are the obvious ones. Since \( j \circ \pi_E(r; (a, 0), (b, 0)) = \pi(r; a, b) \) as \( j \) is an algebra map, we have \( \pi_E(r; (a, 0), (b, 0)) = (f(r; a, b), \pi(r; a, b)) \), for some \( f \in C^2(A, M) \). Since \( \pi_E \) defines a Loday algebra structure on \( E \), it follows that \( f \) is a 2-cocycle. Similarly, one can observe that any two equivalent extensions are related by a map \( E = A \oplus M \xrightarrow{\phi} A \oplus M = E', (a, m) \mapsto (a, m + g(a)) \) for some \( g \in C^1(A, M) \). Since \( \phi \) is an algebra morphism, we have

\[
\phi \circ \pi_E(r; (a, 0), (b, 0)) = \pi_E'(r; \phi(a, 0), \phi(b, 0))
\]

which implies that \( f'(r; a, b) = f(r; a, b) - \delta(g)(r; a, b) \). Here \( f' \) is the 2-cocycle induced from the extension \( E' \). This shows that the map \( \mathcal{E}xt(A, M) \to H^2(A, M) \) is well defined. Moreover, these two maps are inverses to each other. □

6. Deformation of Loday-type algebra morphisms

In this section, we study deformation of morphisms between Loday algebras of same type. The results of this section are parallel to the classical cases (see, for example [17,29]).

Let \( A \) and \( B \) be two Loday algebras of same type and \( f : A \to B \) be a morphism. Then \( B \) can be considered as a representation of \( A \) via \( f \) in the following way:

\[
\theta_1(r; a, b) = \pi_B(r; f(a), b) \quad \theta_2(r; b, a) = \pi_B(r; b, f(a)),
\]

for all \( r \in U_2 \), \( a \in A \) and \( b \in B \).

In such a case, we define a new cochain complex whose \( n \)-th cochain group is given by

\[
C^n(f, f) = C^n(A, A) \times C^n(B, B) \times C^{n-1}(A, B)
\]

\[
= \text{Hom}(\mathbb{K}[U_n] \otimes A^{\otimes n}, A) \times \text{Hom}(\mathbb{K}[U_n] \otimes B^{\otimes n}, B) \times \text{Hom}(\mathbb{K}[U_{n-1}] \otimes A^{\otimes n-1}, B).
\]

The differential \( \delta_f : C^n(f, f) \to C^{n+1}(f, f) \) is defined by

\[
\delta_f(\phi, \psi, \zeta) = (d_A \phi, d_B \psi, f \circ \phi - \psi \circ f^{\otimes n} - \delta \zeta)
\]

where \( d_A, d_B \) denote the coboundary map defining the cohomology of \( A \) and \( B \), respectively.

Proposition 6.1. \( (C^n(f, f), \delta) \) is a cochain complex.

Proof. We have

\[
(\delta_f)^2(\phi, \psi, \zeta) = \delta_f(d_A \phi, d_B \psi, f \circ \phi - \psi \circ f^{\otimes n} - \delta \zeta)
\]

\[
= (d_A^2 \phi, d_B^2 \psi, f \circ d_A \phi - (d_B \psi) \circ f^{\otimes (n+1)} - \delta(f \circ \phi - \psi \circ f^{\otimes n} - \delta \zeta)).
\]

It follows from a direct verification that \( f \circ d_A \phi = \delta(f \circ \phi) \) and \( (d_B \psi) \circ f^{\otimes (n+1)} = \delta(\psi \circ f^{\otimes n}) \). Hence \( (\delta_f)^2 = 0 \) as \( d_A, d_B \) and \( \delta \) are differentials. □
Proposition 6.2. If $H^n(A, A)$, $H^n(B, B)$ and $H^{n-1}(A, B)$ are all trivial, then $H^n(f, f)$ is so.

Proof. Let $(\phi, \psi, \zeta) \in C^n(f, f)$ be an $n$-cocycle. Then it follows that $\phi \in C^n(A, A)$, $\psi \in C^n(B, B)$ are $n$-cocycles and $f \circ \phi - \psi \circ f^{\otimes n} - \delta \zeta = 0$. Hence by the hypothesis, there exist $(n-1)$-cochains $\phi' \in C^{n-1}(A, A)$ and $\psi' \in C^{n-1}(B, B)$ such that $\phi = d_A^f \phi'$ and $\psi = d_B^f \psi'$. Moreover,

$$
\delta(f \circ \phi' - \psi' \circ f^{\otimes (n-1)} - \zeta) = f \circ d_A^f \phi' - (d_B^f \psi') \circ f^{\otimes n} - \delta \zeta
$$

Hence, $f \circ \phi' - \psi' \circ f^{\otimes (n-1)} - \zeta \in C^{n-1}(A, B)$ is a $(n-1)$-cocycle. By the hypothesis, there exist $\zeta' \in C^{n-2}(A, B)$ such that $f \circ \phi' - \psi' \circ f^{\otimes (n-1)} - \zeta = \delta \zeta'$. Thus, it follows that $(\phi, \psi, \zeta) = \delta f(\phi', \psi', \zeta')$ is a coboundary. \qed

Unlike the case of deformation of Loday algebras, the deformation of morphisms cannot describe by using operad with multiplication. The reason is the appearance of the third factor in the deformation complex of $f$.

6.1. Infinitesimal deformations. In this subsection, we describe formal deformation of morphisms between Loday algebras of same type and show that the above defined cohomology controls the deformation.

A deformation of $f$ is given by a triple $\theta_t = (\pi_{A,t}, \pi_{B,t}, f_t)$ in which

- $\pi_{A,t} = \sum_{i \geq 0} \pi_{A,t}^i$ is a deformation of $A$;
- $\pi_{B,t} = \sum_{i \geq 0} \pi_{B,t}^i$ is a deformation of $B$;
- $f_t = \sum_{i \geq 0} f_t^i : A[[t]] \to B[[t]]$ is a morphism of Loday algebras, where each $f_t : A \to B$ is a $\mathbb{K}$-linear map and $f_0 = f$.

Two deformations $\theta_t = (\pi_{A,t}, \pi_{B,t}, f_t)$ and $\theta'_t = (\pi'_{A,t}, \pi'_{B,t}, f'_t)$ of $f$ are said to be equivalent if there is a map $\Phi_t = (\phi_{A,t}, \phi_{B,t})$ where

- $\phi_{A,t} : A[[t]] \to A[[t]]$ is an equivalence between $\pi_{A,t}$ and $\pi'_{A,t}$;
- $\phi_{B,t} : B[[t]] \to B[[t]]$ is an equivalence between $\pi_{B,t}$ and $\pi'_{B,t}$;
- $\phi_{B,t} \circ f_t = f'_t \circ \phi_{A,t}$.

Proposition 6.3. The linear part $(\pi_{A,1}, \pi_{B,1}, f_1)$ of a deformation $\theta_t$ is a 2-cocycle in $(C^2(f, f), \delta_f)$ whose cocycle class is determined by the equivalence class of $\theta_t$.

Proof. Let $\theta_t = (\pi_{A,t}, \pi_{B,t}, f_t)$ be a deformation of $f$. Since $\pi_{A,t} = \sum_{i \geq 0} \pi_{A,t}^i$ and $\pi_{B,t} = \sum_{i \geq 0} \pi_{B,t}^i$ are deformations of $A$ and $B$, respectively, we have $\pi_{A,1} \in \text{Hom}(\mathbb{K}[U_2] \otimes A^{\otimes 2}, A)$ and $\pi_{B,1} \in \text{Hom}(\mathbb{K}[U_2] \otimes B^{\otimes 2}, B)$ are 2-cocycles of $A$ and $B$, respectively. Moreover, $f_1 : A[[t]] \to B[[t]]$ is a Loday algebra morphism implies that

$$
f_t(\pi_{A,t}(r; a, b)) = \pi_{B,t}(r; f_t(a), f_t(b)),
$$

for all $r \in U_2$ and $a, b \in A$. By equating coefficients of $t$, we get

$$
\sum_{i+j=n} f_t(\pi_{A,j}(r; a, b)) = \sum_{i+j+k=n} \pi_{B,i}(r; f_{j}(a), f_{k}(b)), \quad \text{for all } n \geq 0.
$$

(For $n = 0$, the identity is equivalent to the fact that $f : A \to B$ is a Loday algebra morphism.) For $n = 1$, we get

$$
f(\pi_{A,1}(r; a, b)) + f_1(\pi_A(r; a, b)) = \pi_B(r; f(a), f_1(b)) + \pi_B(r; f_1(a), f(b)) + \pi_{B,1}(r; f(a), f(b)).
$$
This is equivalent to
\[ f \circ \pi_{A,1} - \pi_{B,1} \circ f^{\otimes 2} - \delta(f_1) = 0. \]

Therefore, we conclude that \( \delta f(\pi_{A,1}, \pi_{B,1}, f_1) = 0. \)

Finally, let \( \theta_t = (\pi_{A,t}, \pi_{B,t}, f_t) \) and \( \theta_t' = (\pi_{A,t}', \pi_{B,t}', f_t') \) be two equivalent deformations of \( f \) and the equivalence is given by \( \Phi_t = (\phi_{A,t}, \phi_{B,t}). \) Since \( \phi_{A,t} : A[[t]] \to A[[t]] \) is an equivalence between the deformations \( \pi_{A,t} \) and \( \pi_{A,t}' \), we have \( \pi_{A,1} - \pi_{A,1}' = d_A(\phi_{A,1}), \) for some \( \phi_{A,1} \in C^1(A, A). \) Similarly, \( \pi_{B,1} - \pi_{B,1}' = d_B(\phi_{B,1}), \) for some \( \phi_{B,1} \in C^1(B, B). \) Finally, the condition \( \delta f_t \circ f_t = f_t' \circ \phi_{A,t} \) implies that
\[ f_1 - f_1' = f \circ \phi_{A,1} - \phi_{B,1} \circ f. \]

Thus it follows that \( (\pi_{A,1}, \pi_{B,1}, f_1) - (\pi_{A,1}', \pi_{B,1}', f_1') = \delta f(\phi_{A,1}, \phi_{B,1}, 0). \) Hence the infinitesimals are cohomologous.

**Definition 6.4.** A morphism \( f : A \to B \) between Loday algebras of same type is called rigid if every deformation of \( f \) is equivalent to the trivial deformation \( \theta_t = (\pi_A, \pi_B, f). \)

**Proposition 6.5.** A deformation \( \theta_t \) of \( f \) is equivalent to a deformation \( \theta_t' \) in which the first non-zero term \( (\pi_{A,p}, \pi_{B,p}, f_t') \) is a 2-cocycle but not a coboundary.

**Proof.** Let \( \theta_t = (\pi_{A,t}, \pi_{B,t}, f_t) \) be a deformation of \( f \) in which
\[ (\pi_{A,1}, \pi_{B,1}, f_1) = \cdots = (\pi_{A,n-1}, \pi_{B,n-1}, f_{n-1}) = (0, 0, 0) \]
and \( (\pi_{A,n}, \pi_{B,n}, f_n) \) is the first non-zero term. Then \( (\pi_{A,n}, \pi_{B,n}, f_n) \) is a 2-cocycle in \( C^2(f, f). \) Assume that it is a coboundary, say \( (\pi_{A,n}, \pi_{B,n}, f_n) = \delta(\phi, \psi, 0). \) Hence \( \pi_{A,n} = d_A\phi, \pi_{B,n} = d_B\psi \) and \( f_n = f \circ \phi - \psi \circ f. \) Setting
\[ \phi_{A,t} = \text{id}_A + \phi t^n \quad \text{and} \quad \phi_{B,t} = \text{id}_B + \psi t^n. \]
Define \( \theta_t' = (\pi_{A,t}', \pi_{B,t}', f_t') \) where \( \pi_{A,t}' = \phi_{A,t} \circ \pi_{A,t} \circ \phi_{A,t}^{-1}, \pi_{B,t}' = \phi_{B,t} \circ \pi_{B,t} \circ \phi_{B,t}^{-1} \) and \( f_t' = \phi_{B,t} \circ f_t \circ \phi_{A,t}^{-1}. \) Then \( \theta_t' \) is a deformation of \( f \) in which \( (\pi_{A,1}', \pi_{B,1}', f_1') = \cdots = (\pi_{A,n}', \pi_{B,n}', f_n') = 0. \)

If the first non-zero term is not a coboundary, we are done. If not, we can apply the same method to obtain a required type of equivalent deformation.

**Theorem 6.6.** If \( H^2(f, f) = 0 \) then \( f \) is rigid.

In the same spirit of section 3, we would like to extend a deformation of finite order to a deformation of next order. A deformation \( \theta_t = (\pi_{A,t}, \pi_{B,t}, f_t) \) is said to be of order \( n \) if it is of the form
\[ \theta_t = (\pi_{A,t}) = \sum_{i=0}^{n} \pi_{A,i} t^i, \quad (\pi_{B,t}) = \sum_{i=0}^{n} \pi_{B,i} t^i, \quad f_t = \sum_{i=0}^{n} f_i t^i. \]

Suppose there is a 2-cochain \( \theta_{n+1} = (\pi_{A,n+1}, \pi_{B,n+1}, f_{n+1}) \in C^2(f, f) \) such that
\[ \theta_{n+1} = \theta_t + \theta_{n+1} t^{n+1} = (\sum_{i=0}^{n+1} \pi_{A,i} t^i, \sum_{i=0}^{n+1} \pi_{B,i} t^i, \sum_{i=0}^{n+1} f_i t^i) \]
is a deformation of order \( n+1. \) It turns out that the first two components of \( \theta_{n+1} = (\pi_{A,n+1}, \pi_{B,n+1}, f_{n+1}) \) must satisfy
- \( -d_A(\pi_{A,n+1}) = \sum_{i+j=n+1, i,j \geq 1} \pi_{A,i} \circ \pi_{A,j}, \)
- \( -d_B(\pi_{B,n+1}) = \sum_{i+j=n+1, i,j \geq 1} \pi_{B,i} \circ \pi_{B,j}. \)
One may also define a map \( \theta(f) : \mathbb{K}[U_2] \otimes A^{2} \rightarrow B \) by
\[
\theta(f)(r; a, b) := \sum_{i, j, k = 0}^{n+1, 0 \leq i, j, k < n+1} \pi_{B, i}(r; f_{j}(a), f_{k}(b))\]
for \( r \in U_2 \) and \( a, b \in A \). The triplet
\[
\text{Ob}(\theta) = (\text{Ob}_A, \text{Ob}_B, \theta(f)) \in C^3(f, f)
\]
is called the obstruction to extend the deformation.

The proof of the following proposition is a tedious calculation. See [29] for the case of dialgebra.

**Proposition 6.7.** The obstruction \( \text{Ob}(\theta) \) is a 3-cocycle.

Hence we obtain the following.

**Theorem 6.8.** If \( H^3(f, f) = 0 \), every finite order deformation of \( f \) can be extended to a deformation of next order. In such a case, every 2-cocycle in \( C^2(f, f) \) is the infinitesimal of some deformation.

When we consider the case of dialgebra morphisms, our theory compactify the descriptions of [28]. In a similar manner, we can apply the results of this section to morphisms between other Loday algebras.

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