TWO-LAYERED NUMBERS

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Abstract. In this paper, first, I introduce two-layered numbers. Two-layered numbers are positive integers that their positive divisors except 1 can be partitioned into two disjoint subsets. Similarly, I defined a half-layered number as a positive integer $n$ that its proper positive divisors excluding 1 can be partitioned into two disjoint subsets. I also investigate the properties of two-layered and half-layered numbers and their relation with practical numbers and Zumkeller numbers.

0. Introduction

A perfect number is a positive integer $n$ that equals the sum of its proper positive divisors. Generalizing the concept of perfect numbers, Zumkeller in [1] published a sequence of integers that their divisors can be partitioned into two disjoint subsets with equal sum. Clark et al. in [2] called such integers Zumkeller numbers and investigated some of their properties, and also suggested some conjectures about them. Peng and Bhaskara Rao in [3] introduced half-Zumkeller numbers and provided interesting results about Zumkeller numbers.

In the present paper, I define two-layered numbers based on the concept of perfect numbers and Zumkeller numbers. A two-layered number is a positive integer $n$ that its positive divisors excluding 1 can be partitioned into two disjoint subsets of an equal sum. A partition $\{A, B\}$ of the set of positive divisors of $n$ except 1 is a two-layered partition if each of $A$ and $B$ has the same sum.

In the first section, I investigate the properties of two-layered numbers. For a two-layered number $n$, that sum of its divisors is $\sigma(n)$, the following statements hold (See Proposition 1.4):

Let $\sigma(n)$ be the sum of all positive divisors of $n$. If $n$ is a two-layered number, then

1. $\sigma(n)$ is odd.
2. Powers of all odd prime factors of $n$ should be even.
3. $\sigma(n) \geq 2n + 1$, so $n$ is abundant.

After that, In theorem 1.5 I prove that The integer $n$ is a two-layered number if and only if $\frac{\sigma(n) - 1}{2} - n$ is a sum of distinct proper positive divisors of $n$ excluding 1. I also introduce two methods of generating new two-layered numbers from known two-layered numbers. Suppose that $n$ is a two-layered number and $p$ is a prime number with $(n, p) = 1$, then $np^\alpha$ is a two-layered number for any even positive integer $\alpha$ (See Theorem 1.7). We can also generate two-layered numbers in another way.

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way. Let $n$ be a two-layered number and $p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$ be the prime factorization of $n$. Then for any nonnegative integers $\alpha_1, \ldots, \alpha_m$, the integer

$$p_1^{k_1 + \alpha_1 (k_1 + 1)} p_2^{k_2 + \alpha_2 (k_2 + 1)} \ldots p_m^{k_m + \alpha_m (k_m + 1)}$$

is a two-layered number (See Theorem 1.8).

In the second section of the present paper, I generalize the concept of practical numbers and define semi-practical numbers. A practical number is a positive integer $n$ that every positive integer less than $n$ can be represented as a sum of distinct positive divisors of $n$ [5]. A positive integer $n$ is a semi-practical number if every positive integer $x$ where $1 < x < \sigma(n)$, is a sum of distinct positive divisors of $n$ excluding 1 (See Proposition 2.3). A positive integer $n$ is half-layered if and only if $\sigma(n)$ is odd (See Proposition 2.5).

In section 3, I define a half-layered number. A positive integer $n$ is said to be a half-layered number if the proper positive divisors of $n$ excluding 1 can be partitioned into two disjoint non-empty subsets of an equal sum (See Definition 3.2). A half-layered partition for a half-layered number $n$ is a partition $\{A, B\}$ of the set of proper positive divisors of $n$ excluding 1 so that each of $A$ and $B$ sums to the same value (See Definition 3.2).

After these definitions, I investigate the properties of half-layered numbers. For example, a positive integer $n$ is half-layered if and only if \( \frac{\sigma(n) - n - 1}{2} \) is the sum of some distinct positive proper positive divisors of $n$ (See Proposition 3.3). A positive even integer $n$ is half-layered if and only if \( \frac{\sigma(n) - 2n - 1}{2} \) is the sum (possibly empty sum) of some distinct positive proper divisors of $n$ excluding $n$, \( \frac{n}{2} \), and 1 (See Theorem 3.5). If $n$ is an odd half-layered number, then at least one of the powers of prime factors of $n$ should be even (See Proposition 3.7).

Using the definition of half-Zumkeller numbers, we can derive some of the interesting properties of half-layered numbers. A positive integer $n$ is said to be a half-Zumkeller number if the proper positive divisors of $n$ can be partitioned into two disjoint non-empty subsets of an equal sum. A half-Zumkeller partition for a half-Zumkeller number $n$ is a partition $\{A, B\}$ of the set of proper positive divisors of $n$ so that each of $A$ and $B$ sums to the same value (Definition 3 in [3]). Based on these definition, I prove that if $m$ and $n$ are half-layered numbers with $(m, n) = 1$, then $mn$ is half-layered (See Proposition 3.9).

After that, I investigate some relations between half-layered and two-layered numbers. For example, let $n$ be even. Then $n$ is half-layered if and only if $n$ admits a two-layered partition such that $n$ and $\frac{n}{2}$ are in distinct subsets. Therefore, if $n$ is an even half-layered number then $n$ is two-layered (See Proposition 3.10). It is also proved that if $n$ is an even two-layered number and if $\sigma(n) < 3n$, then $n$ is half-layered (See Theorem 3.11). Let $n$ be even. Then, $n$ is two-layered if and only if either $n$ is half-layered or $\frac{\sigma(n) - 3n - 1}{2}$ is a sum (possibly an empty sum) of some positive divisors of $n$ excluding $n$, \( \frac{n}{2} \), and 1 (See Proposition 3.12).
Theorem 1.5. The integer $\sigma$ can conclude
Proof. (1) : If $n$ positive divisors of $n$ product $(k_1 \cdot n)$ of $n$ prime factorization of two-layered number, then $\ell n$ of $n$ be an even half-layered number and the prime factorization of $A$ positive integer $Definition 1.1. $A two-layered partition for a two-layered number $Definition 1.2. $A$ is half-layered (See Proposition 3.18). Let $n$ be the sum of all positive divisors of $n$ excluding 1 so that each of $A$ and $B$ sums to the same value.
Example 1.3. The number 36 is a two-layered number and its two-layered partition is $\{A, B\}$, where $A = \{2, 3, 4, 36\}$ and $B = \{6, 9, 12, 18\}$. You can check that each of $A$ and $B$ has the sum of 45. The numbers 72, 144, and 200 are also two-layered. You can find the sequence of two-layered numbers in [4].
Proposition 1.4. Let $\sigma(n)$ be the sum of all positive divisors of $n$. If $n$ is a two-layered number, then
1. $\sigma(n)$ is odd.
2. Powers of all odd prime factors of $n$ should be even.
3. $\sigma(n) \geq 2n + 1$, so $n$ is abundant.
Proof. (1) : If $\sigma(n)$ is even, then $\sigma(n) - 1$ is odd, so it is impossible to partition the positive divisors of $n$ into two disjoint subsets of equal sum.
(2) : using (1), the number of odd positive divisors of $n$ is odd. Suppose that the prime factorization of $n$ is $2^k p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m}$. The number of odd positive divisors of $n$ is $(k_1 + 1)(k_2 + 1) \ldots (k_m + 1)$. All of $k_i$ should be even in order to make the product $(k_1 + 1)(k_2 + 1) \ldots (k_m + 1)$ odd.
(3) : Let $n$ be a two-layered number with two-layered partition $\{A, B\}$. Without loss of generality we may assume that $n \in A$, so the sum in $A$ is at least $n$ and we can conclude $\sigma(n) - 1 \geq 2n$. □
Theorem 1.5. The integer $n$ is a two-layered number if and only if $\frac{\sigma(n) - 1}{2} - n$ is a sum of distinct proper positive divisors of $n$ excluding 1.
Proof. Let $n$ be a two-layered number and its two-layered partition is $\{A, B\}$. Without loss of generality we assume that $n \in A$, so the sum of the remaining elements of $A$ is $\frac{\sigma(n) - 1}{2} - n$. Conversely, if we have a set of proper divisors of $n$ excluding 1 that its sum is $\frac{\sigma(n) - 1}{2} - n$, we can augment this set with $n$ to construct a set of positive divisors of $n$ summing to $\frac{\sigma(n) - 1}{2}$. The complementary set of positive divisors of $n$ sums to the same value, and so these two sets form a two-layered partition for $n$. □
With the help of the next two theorems, we can generate some new two-layered numbers by knowing a two-layered number.
Definition 1.6 (Definition 1 in [3]). A positive integer \( n \) is said to be a Zumkeller number if the positive divisors of \( n \) can be partitioned into two disjoint subsets of equal sum. A Zumkeller partition for a Zumkeller number \( n \) is a partition \( \{A, B\} \) of the set of positive divisors of \( n \) so that each of \( A \) and \( B \) sums to the same value.

Theorem 1.7. Let \( n \) be a two-layered number and \( p \) be a prime number with \((n, p) = 1\), then \( np^a \) is a two-layered number for any even positive integer \( a \).

Proof. Suppose that \( \{A, B\} \) is a Zumkeller partition of \( n \). Then \( \{(A \setminus \{1\}) \cup (pA) \cup (p^2A) \cup \ldots \cup (p^aA), (B \setminus \{1\}) \cup (pB) \cup (p^2B) \cup \ldots \cup (p^aB)\} \) is a two-layered partition of \( np^a \).

Theorem 1.8. Suppose that \( n \) is a two-layered number and \( p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} \) is the prime factorization of \( n \). Then for any nonnegative even integers \( \alpha_1, \ldots, \alpha_m \), the integer
\[
P = p_1^{k_1 + \alpha_1(k_1+1)} p_2^{k_2 + \alpha_2(k_2+1)} \ldots p_m^{k_m + \alpha_m(k_m+1)}
\]
is a two-layered number.

Proof. If we show that \( p_1^{k_1 + \alpha_1(k_1+1)} p_2^{k_2} \ldots p_m^{k_m} \) the proof will be completed. Suppose that \( \{A, B\} \) is a Zumkeller partition of \( n \). If \( D \) is the set of positive divisors of \( n \), then \( (D \setminus \{1\}) \cup (p_1^{k_1+1}D) \cup (p_1^{2(k_1+1)}D) \cup \ldots \cup (p_1^{\alpha_1(k_1+1)}D) \) is the set of positive divisors of \( p_1^{k_1+\alpha_1(k_1+1)} p_2^{k_2} \ldots p_m^{k_m} \) excluding 1. Therefore a two-layered partition for \( p_1^{k_1+\alpha_1(k_1+1)} p_2^{k_2} \ldots p_m^{k_m} \) is \( \{A \setminus \{1\} \cup (p_1^{k_1+1}A) \cup (p_1^{2(k_1+1)}A) \cup \ldots \cup (p_1^{\alpha_1(k_1+1)}A), B \setminus \{1\} \cup (p_1^{k_1+1}B) \cup (p_1^{2(k_1+1)}B) \cup \ldots \cup (p_1^{\alpha_1(k_1+1)}B)\} \) and the proof is complete.

2. SEMI-PRACTICAL NUMBERS AND TWO-LAYERED NUMBERS

Practical numbers have been introduced by Srinivasan in 1948 as what follows:

Definition 2.1. A positive integer \( n \) is a practical number if every positive integer less than \( n \) can be represented as a sum of distinct positive divisors of \( n \).[5]

Because of the structure of two-layered number, if we change the definition of practical numbers and call them semi-practical numbers, we can drive some useful relation between them and two-layered numbers, so I define semi-practical numbers as what follows:

Definition 2.2. A positive integer \( n \) is practical if every positive integer \( x \) where \( 1 < x < n \) can be represented as a sum of distinct positive divisors of \( n \) excluding 1.

Proposition 2.3. Every semi-practical number is divisible by 12.

Proof. Since we can not write 2, 3, and 4 as sums of more than one positive integer greater than 1, they should be divisors of our semi-practical number.

Theorem 2.4. A positive integer \( n \) is is a semi-practical number if and only if every positive integer \( x \) where \( 1 < x < \sigma(n) \), is a sum of distinct positive divisors of \( n \) excluding 1.

Proof. Suppose that \( n \) is a semi-practical number. I introduce an algorithm for writing all positive integer \( x \) between \( n \) and \( \sigma(n) \) as sum of distinct positive divisors of \( n \) excluding 1.
Proposition 2.5. A semi-practical number $n$ is two-layered if and only if $\sigma(n)$ is odd.

Proof. If $n$ is two-layered number, then $\sigma(n)$ is odd by Proposition 2.3. Conversely, if $\sigma(n)$ is odd, then $\frac{\sigma(n) - 1}{2}$ is a positive integer smaller than $\sigma(n)$. Since $n$ is a semi-practical number, using Proposition 2.4. □

Theorem 2.6. Let $n$ be a positive integer and $p$ be a prime with $(n, p) = 1$. Let $D$ be the set of all positive divisors of $n$ including 1. The following conditions are equivalent:

(1) $np$ is two-layered.

(2) There exist two partitions $\{D_1, D_2\}$ and $\{D_3, D_4\}$ of $D \setminus \{1\}$ such that

$$p\left(\sum_{d \in D_1} d - \sum_{d \in D_2} d\right) = \left(\sum_{d \in D_3} d - \sum_{d \in D_4} d\right).$$

(3) There exists a partition $\{D_1, D_2\}$ of $D \setminus \{1\}$ and subsets $A_1 \subseteq D_1$ and $A_2 \subseteq D_2$ such that

$$\frac{p + 1}{2}\left(\sum_{d \in D_1} d - \sum_{d \in D_2} d\right) = \left(\sum_{d \in A_1} d - \sum_{d \in A_2} d\right).$$

Proof. It is clear that $(pD) \cup (D \setminus \{1\})$ is the set of all positive divisors of $np$ excluding 1.

(1) $\Rightarrow$ (2). Suppose that $np$ is two-layered. Hence, there is a two-layered partition $\{A, B\}$ of $(pD) \cup (D \setminus \{1\})$. Let $D_1 = \frac{1}{p}(A \cap (pD))$, $D_2 = \frac{1}{p}(B \cap (pD))$, $D_3 = B \cap (D \setminus \{1\})$, $A \cap (D \setminus \{1\})$, then

$$p \sum_{d \in D_1} d + \sum_{d \in D_4} d = p \sum_{d \in D_2} d + \sum_{d \in D_3} d.$$

and the proof is complete.

(2) $\Rightarrow$ (3). Let $A_1 = D_1 \cap D_3$ and $A_2 = D_2 \cap D_4$. We have
\[
\frac{p+1}{2} \left( \sum_{d \in D_1} d - \sum_{d \in D_2} d \right) = \frac{1}{2} \left[ \sum_{d \in D_1} d - \sum_{d \in D_2} d \right] + \frac{1}{2} \left( \sum_{d \in D_1} d - \sum_{d \in D_2} d \right)
\]

\[
= \frac{1}{2} \left[ \sum_{d \in D_3} d - \sum_{d \in D_4} d + \sum_{d \in D_1} d - \sum_{d \in D_2} d \right]
\]

\[
= \frac{1}{2} \left[ \sum_{d \in D_1 \cap D_3} d - \sum_{d \in D_2 \cap D_4} d \right]
\]

\[
= \frac{1}{2} \left[ 2 \left( \sum_{d \in D_1} d \right) - 2 \left( \sum_{d \in D_2} d \right) \right]
\]

\[
= \sum_{d \in A_1} d - \sum_{d \in A_2} d.
\]

(3) ⇒ (1). We can rewrite the equation in (3) as follows:

\[
\frac{p}{2} \sum_{d \in D_1} d + \frac{1}{2} \sum_{d \in A_1} d + \frac{1}{2} \sum_{A_1 \setminus A_1} d = \frac{p}{2} \sum_{d \in D_2} d + \frac{1}{2} \sum_{d \in A_1} d + \frac{1}{2} \sum_{d \in A_2} d.
\]

By multiplying this by 2, we obtain the two-layered partition \{(pD_1) \cup A_2 \cup (D_1 - A_1), (pD_2) \cup A_1 \cup (D_2 - A_2)\} for np, so np is a two-layered number. □

**Proposition 2.7.** Let the positive divisors of \(n\) excluding 1 be written in increasing order as follows: \(a_1 < a_2 < \cdots < a_k = n\). If \(a_{i+1} < 2a_i\) for all \(1 \leq i < k\) and \(\sigma(n)\) is odd, then \(n\) is two-layered.

**Proof.** Let \(b_i = a_i\) or \(a_i\) for each \(i\). I will explain how to choose the sign of \(b_i\) precisely. Then I show that \(\sum_{i=1}^{k} b_i = 0\). Hence, it will imply that \(\sigma(n) - 1\) can be partitioned into two equal-summed subsets.

Let \(b_k = a_k = n\) and let \(b_{k-1} = a_{k-1}\). Note that \(0 < b_k + b_{k-1} < a_{k-1}\) since \(a_{k} < 2a_{k-1}\). Since the current sum \(b_k + b_{k-1}\) is positive, we assign the negative sign to \(b_{k-1}\). Then \(b_{k-1} < b_k + b_{k-1} + b_{k-2} < a_{k-1}a_{k-2}\) since \(a_{k-2} < 2a_{k-2}\). If \(b_k + b_{k-1} + b_{k-2} \geq 0\), we assign the negative sign to \(b_{k-2}\); otherwise, we assign the positive sign to \(b_{k-2}\). Let \(s_i\) be \(\sum_{j=1}^{i} b_j\). In general, the sign assigned to \(b_{i+1}\) is the opposite of the sign of \(s_i\). Let us show inductively that \(|s_i| < a_i\) for \(1 \leq i \leq k\). It is true for \(i = k\). Assume that \(|s_{i+1}| < a_{i+1}\). Since the sign of \(b_i\) is opposite of the sign of \(s_{i+1}\), \(s_i = |s_{i+1}|a_i\). Note that \(a_i < |s_{i+1}|a_i < a_{i+1}a_i < a_i\) since \(a_{i+1} < 2a_i\). Therefore \(|s_i| < a_i\). So \(|s_1| < a_1 = 1\). Since \(\sigma(n) - 1\) is even, \(s_1\), which is obtained by assigning a positive or negative sign to each of the terms in \(\sigma(n) - 1\) is even as well. So \(s_1 = 0\). This implies that \(\sigma(n) - 1\) can be partitioned into two equal-summed subsets. Hence it is two-layered. □

**Proposition 2.8** (Proposition 1 in [3]). Let the prime factorization of \(n\) be \(\prod_{i=1}^{m} p_i^{k_i}\). Then

\[
\sigma(n) = \prod_{i=1}^{m} \frac{p_i^{k_i+1} - 1}{p_i - 1}
\]

and

\[
\frac{\sigma(n)}{n} = \prod_{i=1}^{m} \frac{p_i^{k_i+1} - 1}{p_i^{k_i}(p_i - 1)} < \prod_{i=1}^{m} \frac{p_i}{p_i - 1}
\]
**Proposition 2.9.** Let the prime factorization of an odd number \( n \) be \( p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m} \), where \( 3 \leq p_1 < p_2 < \cdots < p_m \). If \( n \) is two-layered, then
\[
\prod_{i=1}^{m} \frac{p_i}{p_i - 1} > 2,
\]
and \( m \) is at least 3. In particular:

1. If \( m \leq 6 \), then \( p_1 = 3 \), \( p_2 = 5 \), \( 7 \) or \( 11 \).
2. If \( m \leq 4 \), then \( p_1 = 3 \), \( p_2 = 5 \) or \( 7 \).
3. If \( m = 3 \), then \( p_1 = 3 \), \( p_2 = 5 \), and \( p_3 = 7 \) or \( 11 \) or \( 13 \).

**Proof.** If \( n \) is two-layered, then by Propositions 1.4 and 2.8,
\[
2^{p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}} = 2n < \sigma(n) = m \prod_{i=1}^{m} (p_i^{j_i} - 1).
\]
Dividing both sides by \( p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m} \), we get
\[
2 < \prod_{i=1}^{m} (\sum_{j=0}^{k_i} p_i^{-j_i}) < \prod_{i=1}^{m} \frac{p_i}{p_i - 1}.
\]
If \( m \leq 2 \), then
\[
\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{3}{2} < 2.
\]
Therefore \( m \geq 3 \). The parts of 1–3 follows by verifying the condition \( \prod_{i=1}^{m} \frac{p_i}{p_i - 1} > 2 \) directly as given below.

1. Let \( m \leq 6 \). If \( p_1 \neq 3 \), then \( p_1 \geq 5 \) and
\[
\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{5}{4} \times \frac{7}{6} \times \frac{11}{10} \times \frac{13}{12} \times \frac{17}{16} \times \frac{19}{18} < 2.
\]
Therefore, \( p_1 = 3 \). If \( p_2 > 11 \), then \( p_2 \geq 13 \) and
\[
\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{3}{2} \times \frac{13}{12} \times \frac{17}{16} \times \frac{19}{18} \times \frac{23}{22} \times \frac{29}{28} < 2.
\]
Hence, \( p_2 \leq 11 \). This implies that \( p_2 = 5 \), \( 7 \) or \( 11 \).

2. Let \( m \leq 4 \). By 1, \( p_1 = 3 \). If \( p_2 > 7 \), then \( p_2 \geq 11 \), so
\[
\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{3}{2} \times \frac{11}{10} \times \frac{13}{12} \times \frac{17}{16} < 2.
\]
Therefore, \( p_2 \leq 7 \). This implies that \( p_2 = 5 \) or \( 7 \).

3. Let \( m = 3 \). By 1, \( p_1 = 3 \). If \( p_2 \neq 5 \), then \( p_2 \geq 7 \) and \( p_3 \geq 11 \). So
\[
\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{3}{2} \times \frac{7}{6} \times \frac{11}{10} < 2.
\]
Hence \( p_2 = 5 \).

If \( p_3 \geq 17 \), then
\[
\prod_{i=1}^{m} \frac{p_i}{p_i - 1} \leq \frac{3}{2} \times \frac{5}{4} \times \frac{17}{16} < 2.
\]
Hence, \( p_3 < 17 \) and consequently \( p_3 = 7 \), \( 11 \) or \( 13 \).
3. HALF-LAYERED NUMBERS

**Definition 3.1.** A positive integer \( n \) is said to be a half-layered number if the proper positive divisors of \( n \) excluding 1 can be partitioned into two disjoint non-empty subsets of equal sum.

**Definition 3.2.** A half-layered partition for a half-layered number \( n \) is a partition \( \{ A, B \} \) of the set of proper positive divisors of \( n \) excluding 1 so that each of \( A \) and \( B \) sums to the same value.

**Proposition 3.3.** A positive integer \( n \) is half-layered if and only if \( \frac{\sigma(n) - n - 1}{2} \) is the sum of some distinct positive proper positive divisors of \( n \).

**Example 3.4.** In Example 1.3, we saw that 36 was a two-layered number. It is also a half-layered number and its half-layered partition is \( \{ A, B \} \), where \( A = \{ 2, 3, 4, 18 \} \) and \( B = \{ 6, 9, 12 \} \). You can check that each of \( A \) and \( B \) has the sum of 27. The numbers 72, 105, and 144 are also half-layered. You can find the sequence of half-layered numbers in [6].

**Theorem 3.5.** A positive even integer \( n \) is half-layered if and only if \( \frac{\sigma(n) - 2n - 1}{2} \) is the sum (possibly empty sum) of some distinct positive divisors of \( n \) excluding \( n, \frac{n}{2}, \) and 1.

**Proof.** An even number \( n \) is half-layered if and only if there exists a which is the sum (possibly empty sum) of some positive divisors of \( n \) excluding \( n, \frac{n}{2}, \) and 1 such that
\[
\frac{n}{2} + a = \frac{\sigma(n) - n - 1}{2}.
\]
Therefore, \( a = \frac{\sigma(n) - 2n - 1}{2} \).□

**Example 3.6.** The number \( 3^4 \times 2^4 \) is a half-layered number, since
\[
\frac{\sigma(3^4 \times 2^4) - 2(3^4 \times 2^4) - 1}{2} = 579 = 432 + 108 + 36 + 3
\]
is a sum of positive divisors of \( 3^4 \times 2^4 \) excluding \( 3^4 \times 2^4, 3^4 \times 2^3 \), and 1. Hence, by Theorem 3.5 it is a half-layered number.

**Proposition 3.7.** If \( n \) is an odd half-layered number, then at least one of the powers of prime factors of \( n \) should be even.

**Proof.** If \( n \) is odd and half-layered, then \( \sigma(n)n - 1 \) must be even and \( \sigma(n) \) must be even. Let the prime factorization of \( n \) be \( \prod_{i=1}^{m} P_i^{k_i} \). Then \( \sigma(n) = \prod_{i=1}^{m} (\sum_{j=0}^{k_i} P_i^j) \). If \( \sigma(n) \) is odd, then there exists one \( k - i \) which is odd.□

**Definition 3.8** (Definition 3 in [3]). A positive integer \( n \) is said to be a half-Zumkeller number if the proper positive divisors of \( n \) can be partitioned into two disjoint non-empty subsets of an equal sum. A half-Zumkeller partition for a half-Zumkeller number \( n \) is a partition \( \{ A, B \} \) of the set of proper positive divisors of \( n \) so that each of \( A \) and \( B \) sums to the same value.

**Proposition 3.9.** If \( m \) and \( n \) are half-layered numbers with \((m, n) = 1\), then \( mn \) is half-layered.
Proof. Let \( M \) be the set of proper positive divisors of \( m \) and let \( \{M_1, M_2\} \) be a half-Zumkeller partition for \( m \). Let \( N \) be the set of proper positive divisors of \( n \) and let \( \{N_1, N_2\} \) be a half-Zumkeller partition for \( n \). Since \((m, n) = 1\), then the set of proper positive divisors of \( mn \) is \((MN) \cup (nM) \cup (mN)\). Observe that \( \{(M_1, N \setminus \{1\}) \cup (mN_1) \cup (nM_1), (M_2N \setminus \{1\}) \cup (mN_2) \cup (nM_2)\} \) is a half-layered partition for \( mn \). Therefore \( mn \) is half-layered.

**Proposition 3.10.** Let \( n \) be even. Then \( n \) is half-layered if and only if \( n \) admits a two-layered partition such that \( n \) and \( \frac{n}{2} \) are in distinct subsets. Therefore, if \( n \) is an even half-layered number then \( n \) is two-layered.

Proof. Let \( n \) be even. Let \( D \) be the set of all positive divisors of \( n \) excluding 1. The number \( n \) is half-layered if and only if there exists \( A \subseteq D \setminus \{n, \frac{n}{2}\} \) such that \[
\frac{n}{2} + \sum_{a \in A} a = \sum_{b \in D, b \notin \{n, \frac{n}{2}\}} b.
\]
That is,
\[
n + \sum_{a \in A} a = \frac{n}{2} + \sum_{b \in D, b \notin \{n, \frac{n}{2}\}} b.
\]
This is equivalent to saying that \( n \) admits a two-layered partition such that \( n \) and \( \frac{n}{2} \) are in distinct subsets.

**Theorem 3.11.** Let \( n \) be an even two-layered number. If \( \sigma(n) < 3n \), then \( n \) is half-layered.

Proof. Since \( n \) and \( \frac{n}{2} \) together sum to more than \( \frac{\sigma(n)}{2} \), they must be in different subsets in any two-layered partition for \( n \). Therefore, by Proposition 3.10, \( n \) is half-layered.

**Proposition 3.12.** Let \( n \) be even. Then, \( n \) is two-layered if and only if either \( n \) is half-layered or \( \frac{\sigma(n) - 3n - 1}{2} \) is a sum (possibly an empty sum) of some positive divisors of \( n \) excluding \( n, \frac{n}{2}, \) and 1.

Proof. Let \( n \) be even. If \( n \) is two-layered but not half-layered, then by Proposition 3.10 any two-layered partition of the positive divisors of \( n \) must have \( n \) and \( \frac{n}{2} \) in the same subsets. In other words, there exists \( a \) which is a sum (possibly an empty sum) of some positive divisors of \( n \) excluding \( n, \frac{n}{2}, \) and 1 such that
\[
2(n + \frac{n}{2} + a) = \sigma(n) - 1
\]
So, \( a = \frac{\sigma(n) - 3n - 1}{2} \). Therefore, the number \( \frac{\sigma(n) - 3n - 1}{2} \) is a sum (possibly an empty sum) of some positive divisors of \( n \) excluding \( n, \frac{n}{2}, \) and 1.

If \( n \) is half-layered, then \( n \) is two-layered by Proposition 3.10. If \( \frac{\sigma(n) - 3n - 1}{2} \) is a sum (possibly an empty sum) of some positive divisors of \( n \) excluding \( n, \frac{n}{2}, \) and 1, then
\[
\frac{\sigma(n) - 2n - 1}{2} = \frac{\sigma(n) - 3n - 1}{2} + \frac{n}{2}
\]
is a sum of some positive divisors of \( n \) excluding \( n \), and 1. By Theorem 1.5, the number \( n \) is two-layered.
Proposition 3.13. If 6 divides \( n \) and \( \sigma(n) < \frac{10n}{3} \), then \( n \) is half-layered.

Proof. If \( n \) is not half-layered, by Proposition 3.12, \( \frac{\sigma(n) - 3n - 1}{2} \) is a sum (might be an empty sum) of some positive divisors of \( n \) excluding \( n, \frac{n}{2}, \) and 1. Then,

\[
\frac{\sigma(n) - 2n - 1}{2} = \frac{\sigma(n) - 3n - 1}{2} + \frac{n + n}{6}.
\]

Since \( \sigma(n)/n < \frac{10}{3} \) we have that \( \frac{\sigma(n) - 3n - 1}{2} < \frac{n}{3} \). Hence \( \frac{\sigma(n) - 2n - 1}{2} \) is a sum of some positive divisors of \( n \) excluding \( n, \frac{n}{2}, \) and 1. By Proposition 3.3, \( n \) is half layered. This is a contradiction.

Proposition 3.14. If \( n \) is two-layered, then \( 2n \) is half-layered.

Proof. Let \( n = 2^kL \) with \( k \) a nonnegative integer and \( L \) an odd number, be a two-layered number. Then all positive divisors of \( n \) excluding 1 can be partitioned into two disjoint equal-summed subsets \( D_1 \) and \( D_2 \). Observe that every positive divisor of \( 2n \) which is not a positive divisor of \( n \) can be written as \( 2^{k+1}\ell \) where \( \ell \) is a positive divisor of \( L \). Therefore all positive divisors of \( 2n \) which are not positive divisors of \( n \) except 2n itself. This procedure will yield an equal-summed partition of all positive divisors of \( 2n \) except \( 2n \) itself. This shows that \( 2n \) is half-Zumkeller.

Corollary 3.15. Let \( n \) be even and the prime factorization of \( n \) be \( 2^k p_1^{k_1} \cdots p_m^{k_m} \). If \( n \) is two-layered but not half-layered, then \( 2^k p_1^{k_1} \cdots p_m^{k_m} \) is not two-layered for any \( i \leq k - 1 \), and \( 2^k p_1^{k_1} \cdots p_m^{k_m} \) is half-layered for any \( i \geq k + 1 \).

Proposition 3.16. Let \( n \) be an even half-layered number and \( p \) be a prime with \( (n, p) = 1 \). Then \( np^\ell \) is half-layered for any positive integer \( \ell \).

Proof. Since \( n \) is an even half-layered number, the set of all positive divisors of \( n \), excluding 1, denoted by \( D_0 \) can be partitioned into two disjoint subsets \( A_0 \) and \( B_0 \) so that the sums of the two subsets are equal and \( \frac{n}{2} \) are in distinct subsets (by Proposition 3.10).

Group the positive divisors of \( np^\ell \) except 1 into \( \ell + 1 \) groups \( D_0, D_1, \ldots, D_\ell \) according to how many positive divisors of \( p \) they admit, i.e., \( D_i \) consists of all positive divisors of \( np^\ell \) admitting \( i \) positive divisors of \( p \). Then each \( D_i \) can be partitioned into two disjoint subsets so that the sums of the two subsets are equal and \( \frac{np^\ell}{2} \) are in distinct subsets according to the two-layered partition of the set \( D_0 \). Therefore all positive divisors of \( np^\ell \) excluding 1 can be partitioned into two disjoint subsets so that the sum of these two subsets equal and \( np^\ell \) and \( \frac{np^\ell}{2} \) are in distinct subsets. By Proposition 3.10, \( np^\ell \) is half-layered.

Corollary 3.17. If \( n \) is an even half-layered number and \( m \) is a positive integer with \( (n, m) = 1 \), then \( nm \) is half-layered.

Theorem 3.18. Let \( n \) be an even half-layered number and the prime factorization of \( n \) be \( p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m} \). Then for nonnegative integers \( \ell_1, \ldots, \ell_m \), the integer

\[
p_1^{k_1 + \ell_1(k_1 + 1)} p_2^{k_2 + \ell_2(k_2 + 1)} \cdots p_m^{k_m + \ell_m(k_m + 1)}
\]

is half-layered.
Proof. It is sufficient to show that \( p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} \) is half-layered if \( p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} \) is an even half-layered number. Assume that \( p_1^{k_1} p_2^{k_2} \ldots p_m^{k_m} \) is even and half-layered, then the set of all positive divisors of \( n \) excluding 1, denoted by \( D_0 \), can be partitioned into two disjoint subsets \( A_0 \) and \( B_0 \) so that the sums of the two subsets are equal and \( \frac{n}{2} \) are in distinct subsets (by Proposition 3.10). Note that the positive divisors of \( p_1^{k_1+\ell_1(k_1+1)} p_2^{k_2} \ldots p_m^{k_m} \) excluding 1 can be partitioned into \( \ell_1 + 1 \) disjoint groups \( D_i, 0 \leq i \leq \ell_1 \), where elements in \( D_i \) are obtained by multiplying \( p_i^{\ell_1(k_i+1)} \) with elements in \( D_0 \). Using the partition \( A_0, B_0 \) of \( D_0 \) we can partition every \( D_i \) into two disjoint subsets \( A_i \) and \( B_i \) so that the sums of the corresponding subsets are equal and \( np_1^{\ell_1(k_1+1)} \) and \( \frac{np_1^{\ell_1(k_1+1)}}{2} \) are in distinct subsets. Therefore, the set of all positive divisors of \( p_1^{k_1+\ell_1(k_1+1)} p_2^{k_2} \ldots p_m^{k_m} \) excluding 1 can be partitioned into two disjoint equal-summed subsets and \( p_1^{k_1+\ell_1(k_1+1)} p_2^{k_2} \ldots p_m^{k_m} \) and \( \frac{np_1^{\ell_1(k_1+1)}}{2} p_2^{k_2} \ldots p_m^{k_m} \) are in distinct subsets. By Proposition 3.10, \( p_1^{k_1+\ell_1(k_1+1)} p_2^{k_2} \ldots p_m^{k_m} \) is half-layered.

**Theorem 3.19.** Let \( n \) be an even integer and \( p \) be a prime with \( (n, p) = 1 \). Let \( D \) be the set of all positive divisors of \( n \) excluding 1. Then the following conditions are equivalent:

1. \( np \) is half-layered.
2. There exist two partitions \( \{D_1, D_2\} \) and \( \{D_3, D_4\} \) of \( D \) such that \( n \) is in \( D_1 \), \( \frac{n}{2} \) is in \( D_2 \), and

\[
p(\sum_{d \in D_1} d - \sum_{d \in D_2} d) = \sum_{d \in D_3} d - \sum_{d \in D_4} d.
\]

3. There exists a partition \( \{D_1, D_2\} \) of \( D \) and subsets \( A_1 \subseteq D_1 \) and \( A_2 \subseteq D_2 \) such that \( n \) is in \( D_1 \), \( \frac{n}{2} \) is in \( D_2 \), and

\[
p + 1 \left( \sum_{d \in D_1} d - \sum_{d \in D_2} d \right) = \sum_{d \in A_1} d - \sum_{d \in A_2} d.
\]

**Proof.** By Proposition 3.10, \( np \) is half-layered if and only if there is a two-layered partition \( \{A, B\} \) of \( (pD) \cup D \) such that \( n \in A \) and \( \frac{n}{2} \in B \). The rest of the proof follows along the lines of the proof of Theorem 2.7. □

**Proposition 3.20.** If \( a_1 < a_2 < \cdots < a_k = n \) are all positive divisors of an even number \( n \) excluding 1 with \( a_{i+1} < 2a_i \) for all \( i \) and \( \sigma(n) \) is odd, then \( n \) is half-layered.

**Proof.** Note that in the proof of Proposition 2.7, \( b_k = n \) and \( b_{k1} = -\frac{n}{2} \) have different signs. So we get a two-layered partition of \( n \) such that \( n \) and \( \frac{n}{2} \) are in distinct subsets. By Proposition 3.10, \( n \) is half-layered. □

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