Revisiting the Langer–Ambegaokar–McCumber–Halperin theory of resistive transitions in one-dimensional superconductors with exact solutions

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Received 29 June 2011, in final form 19 July 2011
Published 12 August 2011
Online at stacks.iop.org/JPhysCM/23/342203

Abstract

We present an important correction to the Langer–Ambegaokar–McCumber–Halperin theory for the resistive state of a 1D superconductor. We establish that the identification of the saddle on the free energy surface over which Langer and Ambegaokar had claimed the system to move in order to form thermally excited phase slip centres is wrong. With the help of an exact solution we show that the system has to overcome a similar free energy barrier but can actually have vanishing amplitude of the superconducting phase at a point, unlike the Langer–Ambegaokar solution.

In a 1D superconductor, kept below the critical temperature \( T_c \), a current driven transition to a normal (N) phase from the superconducting (SC) phase was seen to occur in experiments as early as 1967 [1, 2]. Such transitions show a finite width in temperature and current strength within which SC and N phases coexist. A theory was proposed by Langer and Ambegaokar (LA) [3], and was subsequently improved by McCumber and Halperin (MH) [4], in order to explain thermal fluctuation induced transitions through metastable states. The Langer–Ambegaokar–McCumber–Halperin (LAMH) theory traces the origin of localized N phases in superconducting 1D (the width is smaller than the coherence length \( \xi \)) samples in the formation of phase slip centres (PSC)—an idea which was originally put forward by Little [5]. Over the past 40 years, the LAMH theory has successfully accounted for the resistivity versus temperature plots observed in this (so-called) resistive regime except for some stretches at the lower temperature end of the resistivity versus temperature plot. At this end (near the \( T_c \)), where the resistivity vanishes and the system moves into the SC phase, some mismatches with the theoretically predicted values have been observed in the early experiments which were believed to be effects of imperfections at the contacts at the ends of the sample [6]. These deviations have also been seen to be system specific, particularly lending support to relating their origin to contact imperfections. Classic experiments done by Webb and Warburton [7] and Newbower et al [8] showed the excellent applicability of LAMH theory to experimental results. Since then, the LAMH theory has remained the basic tool for dealing with the resistive regime of the 1D superconductor near the \( T_c \) regime where quantum effects are negligible.

Such a successful theory, however, has a particular lacuna which is a glaring inconsistency in it. The SC order parameter is a complex number and in 1D it looks like a spiral wound around the wire. The LAMH theory is based on a suggestion by Little, that the amplitude of the SC phase has to locally vanish at a point along the length of the 1D sample in order to have a turn added or removed from it [5]. On the basis of this criterion, one looks for an amplitude modulated solution of the SC phase where the amplitude vanishes at least locally (PSC) allowing the SC order parameter to add or remove turns (change its wavenumber \( q \)). Thus, the total phase along the...
length of the sample can change. An applied voltage across the length of the sample, on the one hand, keeps adding turns to the SC order parameter (increases the wavenumber) to increase the phase. On the other hand, thermal fluctuations make the system access unstable amplitude modulated states like PSCs to give up phase in the middle of the sample. The LA theory basically rests upon a balance between these two. Although this vanishing of the amplitude is crucial for such a phenomenon, the LA solution which corresponds to the passage of the system through a free energy saddle never goes to zero anywhere. It is interesting to note that the LA solution can never actually go to zero because of the infinite free energy cost that it has to incur according to the LA calculations of the free energy and, thus, it is absolutely against the demand of Little’s criterion. LA theory simply claims that the solution that comes close to zero would become zero by some other fluctuations, which is wrong in view of the free energy surface as described by LA theory. Despite this glaring mistake, the LA theory works well because in reality the free energy barriers as calculated from LA theory and the true one (which will be shown) are the same.

In the present paper, we will first clearly identify the error in the identification of the saddle on the free energy surface in LA theory and demonstrate why the already known expression for the variation of the amplitude at the boundary of a superconductor under the no-field condition holds as an exact solution in the case of the LA model as well. In particular, on the basis of this exact solution we will also calculate the corresponding chemical potential profile and the current profile through the sample. We wish to show that the divergence of the current at the point where the amplitude vanishes goes against the LA demand of other fluctuations taking a close to zero amplitude actually to zero, because the LA effective free energy cannot allow that. We would also argue that the addition or subtraction of turns in the middle of the superconducting sample is not a process that takes place just at the point of time at which the amplitude vanishes, but that it should take place rather continuously through the process of formation and relaxation of PSCs.

The dynamics of a 1D superconductor in the resistive regime is given by the time dependent Ginzburg–Landau (TDGL) equation [4] as

\[ i\hbar \frac{\partial \psi}{\partial t} + \mu \psi = \psi_{xx} + \left( \alpha - \frac{\beta}{2} |\psi|^2 \right) \psi \]

(1)

\[ j = \text{Im}(\psi^* \nabla \psi) - \mu \psi. \]

(2)

In this model, \( \psi \) is the superconducting order parameter which is complex valued. The system being 1D, \( x \) is the distance along the wire from some arbitrary origin. Let us consider the length of the wire as \( L \), with cross-sectional area \( \sigma \). \( \mu \) is the electrochemical potential which can be considered as the order parameter of the N phase within the scope of GL phenomenology. The constants \( \alpha \) and \( \beta \) measure the free energy density differences of the SC and N phases, as \( g_N - g_S = \alpha^2 / 2 \beta = a(\Delta T^2) \), where \( a \) is another constant that depends upon the density of states at the Fermi surface \( N(0) \), and the Boltzmann constant, as \( a = 4.7N(0)k_B^2 / j \) is the current (density) through the sample and the subscripts \( t \) and \( x \) on \( \psi \) and \( \mu \) indicate partial derivatives.

Equation (1) has two stationary solutions: (1) \( \psi = 0 \) and (2) \( \psi = Ae^{i\phi(x)} \) where \( x = (\alpha - q^2) / \beta, j = A^2 q \), which is the superconducting state when \( \mu = 0 \) [3]. The SC order parameter can be visualized as a spiral wound around the x-axis (along the wire). If there are \( N \) turns along the length \( L \), there is a total phase difference of \( \phi = 2\pi N / L \). Thus, the wavenumber \( q \) of the SC phase is a measure of the number of turns that the system has on a unit length, since \( q = 2\pi N / L \). The expression for the corresponding GL free energy, when the steady state solution is purely superconducting, is given by

\[ F = L\sigma \left[ (q^2 - \alpha)A^2 + \frac{\beta}{2} A^4 \right] \]

(3)

where \( \sigma \) is the cross-section of the wire. The form of \( F \) clearly indicates that the SC states with smaller \( q \) values are energetically favoured. In other words, the spiralling SC order parameter would tend to lose its turns to go to a lower free energy state. Adopting the ansatz \( \psi = A(x, t)e^{i\phi(x)} \) where \( \partial \phi / \partial x = q(x) \) in equation (1) and separating the real and imaginary parts (where we consider \( A(x, t) \) and \( \phi(x, t) \) real functions) we get

\[ \frac{\partial A}{\partial t} - \frac{\partial^2 A}{\partial x^2} - (\alpha - q^2)A + \beta A^3 = 0 \]

(4)

\[ A \frac{\partial q}{\partial x} + 2q \frac{\partial A}{\partial x} - \mu A = 0 \]

(5)

and along with the two equations mentioned above, one has to take into account equation (2) to get a complete picture of the situation.

Before we go into the exact solution and calculation of the barriers, let us first have a look at what went wrong in the LA calculations of the barrier. The stationary equation (4) can be rewritten in the form

\[ \frac{\partial^2 A}{\partial x^2} = -\frac{\delta f(\alpha - q^2, A^2)}{\delta A} = -\frac{\delta U}{\delta A}. \]

(6)

The above expression clearly shows that the effective potential \( U \) goes to zero as \( A = 0 \), whereas, in the LA theory, because of replacing the \( q^2 A^2 / 2 \) with \( -\beta / 2A^2 \) in the expression for \( U \), it diverges at \( A = 0 \) for all nonzero \( j \). This replacement of \( q^2 A^2 / 2 \) with \( \beta / 2A^2 \) is clearly wrong because the amplitude in the above expression is not the supercurrent amplitude; rather it is a modulated form of it and the current density in such a case is not the supercurrent density corresponding to the constant amplitude SC phase.

Figures 1(a) and (b) show a schematic comparison of \( U \) as obtained by LA theory and us. In our case, the \( A = 0 \) point would be visited by the trajectory from the point \( A^2 = (\alpha - q^2) / \beta \) where the state is metastable at the cost of an increase of the velocity equivalent \( \frac{\partial \phi}{\partial t} \). It is interesting to note that the point \( A^2 = (\alpha - q^2) / \beta \) is the peak of \( U \), similarly to the one considered in LA theory. But the \( U \) of
LA theory does not allow the system to reach $A = 0$ because that would require an infinite storage of the kinetic energy equivalent of $(\frac{d\mu}{dx})^2/2$. The actual GL free energy containing a gradient square term would then have to achieve infinite free energy storage somewhere, in the form of a diverging amplitude gradient, in order to allow the system to ever reach the $A = 0$ point. But, the constant amplitude SC phase does not allow for such a storage. To circumvent this inconsistency, LA theory proposes this barrier to be a saddle such that the system can escape through other dimensions of the space in which the free energy is defined. A replacement of $q$ by $j$ and $A$ still keeps the space two-dimensional and there we see that the $j$ has to fall faster than $A$ and necessarily vanish to keep the free energy divergence free as the $A$ vanishes. This goes against the very concept of a current driven origin of the resistive state. Since the LA calculations mean that $A = 0$ is never accessible for a nonzero $j$, the subsequent claim of LA theory that other fluctuations would make an LA solution which goes closer to zero actually reach zero is untenable. As a consequence of a wrong replacement of the wavenumber, the saddle identified is not the lowest barrier and that will become clear in the following where we will show a smaller barrier.

Note that at a constant $q$ the equation (5) gets a form which had been arrived at by one of the authors [9, 10] previously through a separation of length scales. Equation (5) gives the corresponding $\mu$ profile for an amplitude modulation obtained from equation (4). In what follows we will stick to this constant $q$ scenario because (a) equation (4) that actually admits the amplitude modulations is independent of any variation of $q$, (b) even with constant $q$ we will be able to show an exact form of the PSC which has a free energy barrier smaller than that proposed by LA theory and that serves our present purpose and (c) this is a good approximation because normally a single turn gets added or removed at the formation of the PSC and that changes the $q$ by an order $1/L$ which is quite small. Considering the time independence of the amplitude, equation (4) admits a solution $A = A_0 \tanh x/\sqrt{2}\xi$ with a couple of conditions: (1) $(\alpha - q^2) - 1/\xi^2 = 0$ and (2) $\beta A_0^2 - 1/\xi^2 = 0$ where $\xi$ is the bulk GL coherence length. These conditions immediately identify that $A_0^2 = (\alpha - q^2)/\beta$, which is the standard amplitude of the steady superconducting state. From the above conditions we recover the standard bulk relation between the $\alpha$ and $\xi$ also. The above mentioned solution is a well known one, which one observes at the boundary of a bulk superconductor in no-field conditions. The LA free energy denies it simply because of the divergence of the effective free energy at zero amplitude.

The corresponding $\mu$ profile of the superposed normal phase comes out as $\mu = (2q/\sqrt{2}\xi)(1/\tanh x/\sqrt{2}\xi - \tanh x/\sqrt{2}\xi)$. This is the exact solution which would correspond to an energy barrier that is the same as that in LA theory. The barrier height would become even smaller with a nonzero $q$, as is expected from GL free energy. Corresponding to our exact solutions for $A$ and $\mu$, the current can be evaluated from equation (2) for a constant $q$. It’s important to note that this current has a value $A_0^2 q^2/2$ far from the origin and it diverges at the origin as $1/\tanh^2 x/\sqrt{2}\xi$. So, it has a part which grows as $1/A^2$ where $A \rightarrow 0$ and there is no question of the current falling faster than $A$ for the divergence of the effective free energy $U$ of LA theory to remain analytic. So, according to the free energy expression of the LA theory, this exact solution of the system does not qualify for overcoming the free energy barrier, but experimental studies done so far basically confirm having such solutions that vanish locally.

At the limit of the supercurrent density $j \rightarrow 0$ the very complicated looking LA solution obtained through the LA theory mentioned saddle of the free energy surface actually overcomes a free energy barrier somewhat bigger than $\Delta F = (8\sqrt{2}/3K_BT)(g_n - g_s)\sigma_2 [3, 6, 11]$. The free energy barrier corresponding to our exact PSC solution $A = A_0 \tanh x/\sqrt{2}\xi$ can be calculated by putting $\psi = A_0(\tanh x/\sqrt{2}\xi)e^{i\psi_0}$ and $A_0e^{i\psi_0}$ respectively in the expression for the free energy (corresponding to equation (1)) shown below and subtracting the latter free energy from the former:

$$F = \sigma \int_{-\infty}^{\infty} dx \left[ |\nabla \psi|^2 - \alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 \right].$$

If we make use of the conditions (1) and (2) as mentioned above in connection with the derivation of the PSC, we can immediately show that the free energy barrier can be exactly calculated as

$$\Delta F = (8\sqrt{2}/3K_BT)(g_n - g_s)\sigma_2 \xi - \frac{26\sqrt{2}\alpha}{3\beta\xi^2} q^2 + \frac{28\sigma_2}{\sqrt{2}\beta} q^4.$$

Figure 1. Schematic diagrams for the comparison of effective potentials on arbitrary scales: (a) $U$ as considered by LA theory and (b) $U$ that we consider here.
We can see that at $q = 0$, which corresponds well with the $j = A'q = 0$ condition, the free energy barrier is clearly the same as the LA value when the relevant length scale of the amplitude modulation is equal to the bulk GL coherence length $\xi$. Taking roughly $q \rightarrow 1/\xi$, one recovers the $\Delta T^{2/3}$ power law in all the terms of the above mentioned free energy expression. Note that consideration of a nonzero $q$ even reduces the free energy barrier. It is important to note that $q = 0$ actually sets $\mu = 0$, which is a violation of the relation

$$\frac{4\pi e}{h} \Delta V = \frac{\partial \triangle \arg \psi}{\partial t}$$

where $\Delta V$ is the applied voltage drop across the system. In the literature, the limit at which $\Delta T$ becomes $\Delta T_C$ is $j/j_C \rightarrow 0$ where $j_C = 2\alpha^{3/2}/3\sqrt{3}$. This limit can be seen as $q \rightarrow 0$ limit also, but $q$ cannot actually vanish for the theory to be consistent and so is $j$. Taking into account that a nonzero $q$ is essential, one can easily infer that a nonzero $A_0$ and $j$, however small, should accompany a PSC and that the LA solution for the PSC would actually encounter an enormous barrier.

Let us discuss the implication of the present results. The LAMH theory is the most useful and accepted framework in understanding the classical PSC induced origin of the resistive regime of the 1D superconductor. It matches fairly well the experimentally obtained resistivity versus temperature plots. Our present analysis helps solve the riddle of how such a successful theory can actually not predict a proper PSC solution, and establishes LAMH theory on an even stronger foundation more than 40 years after its introduction. The LA theory, however, uses the correct numerical value despite identifying the wrong barrier, because it had stuck to the conservation of energy in getting the saddle. Here, we actually show that no such saddle exists, because the current density cannot go to zero at a point as the amplitude does. We have shown that the good old solution for the amplitude modulations at the boundary of a bulk superconductor holds perfectly well in the LA case as well.

Our present analysis, besides resolving this PSC riddle, also clarifies and sheds light on a few other points. First of all, we have considered here a constant $q$ during the PSC formation. This not only simplifies the calculations but also turns out to be correct because, in general, only a turn addition or removal happens by the formation of a PSC. Moreover, as we have already shown, there is no reason to think of a coupling between the amplitude and the wavenumber in the modulated amplitude phase, just as in the SC phase, and that has severe consequences. It has to be explored what other forms of PSC-like or other solutions exist with a spatially varying $q$. Nevertheless, in the present context, with a constant $q$ we are able to point out the discrepancy in the calculations of the saddle in LA theory. Our solution, which satisfies Little’s criterion of amplitude vanishing, clearly has the same barrier to overcome as LA theory’s saddle and in that sense is a better representative for PSC. Note that our exact solution is unstable for $\tanh^2(x/\sqrt{2}\xi) < 1/3$ to any infinitesimal uniform perturbation. So, fluctuations would start growing at the core of the PSC and would make it relax back to the constant amplitude SC phase.

Another important point to note is that the claim that the LA solution approximates to a form proportional to $\tanh|x|$ [11] is somewhat in conflict with equation (4) because this form would produce a delta function contribution at the origin in equation (4) resulting in the invalidity of such a form as a solution. This is why our exact solution is of the form $\tanh(x/\sqrt{2}\xi)$, which induces a global phase shift of $\pi$ on half of it, unlike the $\tanh|x|$ form. It is interesting to note that equation (1) is invariant under a constant global phase shift. As a result, such a phase shift should not be energetically unfavourable and can happen with a locally vanishing amplitude, particularly because the single-valued nature of the SC order parameter is not compromised. But this marks a small departure from Little’s proposition that a phase change of $2\pi$ or its multiple would actually happen right at the vanishing of the amplitude. Rather, we see here that a phase change of half of that required to add/lose a turn is happening at the time of formation of the PSC and the other half has to happen during relaxation of the PSC to keep the order parameter single valued. One way of looking at this scenario would be to consider that, as the PSC forms, it produces strain locally for the turns and that strain relaxes by rotating half of the spiral. By the time the PSC has occurred, half of the spiral is rotated by an angle $\pi$. Now a relaxation of the PSC might find it energetically favourable or retain some memory causing it to continue effectively rotating the same half in the same direction, with the outcome that when the relaxation of the PSC is complete, the system loses/gains a turn. A detailed investigation using the dynamics could be revealing.

We conclude by saying that we have exactly solved the TDGL model for the resistive state of a 1D superconductor in order to explicitly show an energy barrier similar to the LA saddle one. On the basis of that, we identify the error in the LA calculation of the energy barrier. We put the LAMH theory on an even firmer basis.

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