We employ non-perturbative flow equations to compute the equation of state for two flavor QCD within an effective quark meson model. Our treatment covers both the chiral perturbation theory domain of validity and the domain of validity of universality associated with critical phenomena. In the vicinity of $T_c$ and zero quark mass we obtain a precision estimate of the universal critical equation of state of the three dimensional $O(4)$ symmetric Heisenberg model. For realistic quark masses the pion correlation length near $T_c$ turns out to be smaller than its zero temperature value.

QCD in a thermal equilibrium situation at sufficiently high temperature differs in important aspects from the corresponding zero temperature or vacuum properties\footnote{Talk given at Eotvos Conference in Science: Strong and Electroweak Matter, Eger, Hungary, 21-25 May 1997.}. A phase transition at some critical temperature $T_c$ or a relatively sharp crossover may separate the high and low temperature physics. Concentrating on the chiral aspects of QCD the transition is related to a qualitative change in the chiral condensate. It was pointed out\footnote{Talk given at Eotvos Conference in Science: Strong and Electroweak Matter, Eger, Hungary, 21-25 May 1997.} that for sufficiently small up and down quark masses, $m_u$ and $m_d$, and for a sufficiently large mass of the strange quark, $m_s$, the chiral transition is expected to belong to the universality class of the three dimensional $O(4)$ Heisenberg model. It was suggested\footnote{Talk given at Eotvos Conference in Science: Strong and Electroweak Matter, Eger, Hungary, 21-25 May 1997.} that a large correlation length may lead to a disoriented chiral condensate\footnote{Talk given at Eotvos Conference in Science: Strong and Electroweak Matter, Eger, Hungary, 21-25 May 1997.} with possible distinctive signatures\footnote{Talk given at Eotvos Conference in Science: Strong and Electroweak Matter, Eger, Hungary, 21-25 May 1997.} in a relativistic heavy–ion collision. The question how small $m_u$ and $m_d$ would have to be in order to see a large correlation length near $T_c$ and if this scenario could be realized for realistic values of the current quark masses remained, however, unanswered. The reason was the missing link between the universal behavior near $T_c$ and zero current quark mass on one hand and the known physical properties at $T = 0$ for realistic quark masses on the other hand. Lattice QCD seems particularly suitable for such a study, however, exploring the universal region is limited by present computer resources\footnote{Talk given at Eotvos Conference in Science: Strong and Electroweak Matter, Eger, Hungary, 21-25 May 1997.}. It is the purpose of this talk to provide the “missing link”. Our approach is based on the use of a non-perturbative flow equation for a scale dependent effective action\footnote{Talk given at Eotvos Conference in Science: Strong and Electroweak Matter, Eger, Hungary, 21-25 May 1997.} $\Gamma_k$, which is the generating functional of the $1PI$ Green
functions in the presence of an infrared cutoff $\sim k$. Varying the infrared cutoff $k$ allows us to consider the relevant physics in dependence on some momentum–like scale. The standard effective action is obtained by removing the infrared cutoff ($k \rightarrow 0$) in the end. The $k$–dependence of the effective average action is given by an exact flow equation\(^7\), which for scalar fields $\Phi_i$ reads

$$k \frac{\partial}{\partial k} \Gamma_k[\Phi] = \frac{1}{2} \text{Tr} \left\{ \left( \Gamma^{(2)}_k[\Phi] + R_k \right)^{-1} k \frac{\partial R_k}{\partial k} \right\}. \quad (1)$$

Here $\Gamma^{(2)}_k$ denotes the matrix of second functional derivatives of $\Gamma_k$ with respect to the field components. We use a momentum dependent infrared cutoff $R_k(q) = Z_{\Phi,k} q^2 e^{-q^2/k^2} / (1 - e^{-q^2/k^2})$ with $Z_{\Phi,k}$ an appropriate wave function renormalization constant. In momentum space the trace reads $\text{Tr} = \int \frac{d^d q}{(2\pi)^d} \sum_i$.

We employ for scales below a “compositeness scale” of $k_\Phi \simeq 600$ MeV a description in terms of quark and scalar mesonic degrees of freedom. This effective quark meson model can be obtained from QCD in principle by “integrating out” the gluon degrees of freedom and by introducing fields for composite operators\(^8\). In this picture the scale $k_\Phi$ is associated to the scale at which the formation of mesonic bound states can be observed\(^8\) in the flow of the momentum dependent four–quark interaction. We imagine that all other degrees of freedom besides the quarks $\psi$ and the scalar and pseudoscalar mesons contained in the complex field $\Phi$ are integrated out. Our truncation corresponds to the ansatz for the effective average action\(^\dagger\)

$$\Gamma_k = \int d^4 x \left\{ Z_{\psi,k} \bar{\psi} i \slashed{\partial} \psi + Z_{\Phi,k} \text{tr} \left[ \partial_\mu \Phi^\dagger \partial^\mu \Phi \right] + U_k(\Phi, \Phi^\dagger) \right\} + \frac{1 + \gamma_5}{2} \Phi_{ab} - \frac{1 - \gamma_5}{2} \Phi_{ab}^\dagger \psi^b - \frac{1}{2} \text{tr} \left( \Phi^\dagger j + j^\dagger \Phi \right) \right\} \quad (2)$$

Here $\Gamma_k$ is invariant under the chiral flavor symmetry $SU_L(2) \times SU_R(2)$ in absence of the explicit symmetry breaking through the source term $j \sim \bar{m} = (m_u + m_d)/2$. We will consider the flow of the most general form of the potential term $U_k$ consistent with the symmetries. For $k \rightarrow 0$ the potential $U_k$ encodes the equation of state. At non–zero temperature the ansatz\(^\dagger\) therefore allows to study the complete non–analytic behavior of the effective potential, or equivalently the free energy, near the critical temperature of the second order phase transition. On the other hand, our approximations for the kinetic terms are rather crude and parameterized by only two running wave function renormalization constants, $Z_{\Phi,k}$ and $Z_{\psi,k}$. The same holds for the effective
Yukawa coupling $h_k$. We further neglect the scalar triplet $a_0$ and the pseudoscalar singlet (associated with the $\eta'$) for $k < k_\Phi$. This can be achieved in a chirally invariant way and leads to the $O(4)$ symmetric linear sigma model for the pions and the sigma resonance, however, coupled to quarks now. The non-perturbative flow equations for the quark–meson model are obtained from eq. (1) generalized to include fermions and using the ansatz (2) for $\Gamma_k$. We solve them numerically.

A reliable quantitative derivation of the effective quark meson model from QCD is still missing. We emphasize, however, that the quantitative aspects of this derivation will be of minor relevance for our practical calculations in the mesonic sector: If the effective Yukawa coupling between the quarks and the mesons turns out to be strong at the compositeness scale we observe a fast approach of the scale dependent effective couplings to approximate partial infrared fixed points. As a consequence, the detailed form of the meson potential at $k_\Phi$ becomes unimportant, except for the value of one relevant scalar mass parameter $m_{k_\Phi}$. Here we fix $m_{k_\Phi}$ from phenomenological input such that $f_\pi = 92.4$ MeV (for $m_\pi = 135$ MeV) which sets our unit of mass for two flavor QCD. The only other input parameter we use is the constituent quark mass $M_q$ to determine the scale $k_\Phi$. We consider a range $300 \text{ MeV} < M_q \lesssim 350 \text{ MeV}$ and find a rather weak dependence of our results on the precise value of $M_q$. We point out that though a strong Yukawa coupling at $k_\Phi$ is phenomenologically suggested by the comparably large value of the constituent quark mass $M_q$ it enters our description as a (consistent) assumption.

The equation of state expresses $\langle \bar{\psi} \psi \rangle$ as a function of $T$ and $\hat{m}$ where the chiral condensate is related to the expectation value $\langle \Phi \rangle$ by

$$\langle \bar{\psi} \psi \rangle = -2m^2 k_\Phi [\langle \Phi \rangle - \hat{m}] .$$

Curve (a) of figure 1 gives the temperature dependence of $\langle \bar{\psi} \psi \rangle$ in the chiral limit $\hat{m} = 0$. Here the lower curve is the full result for arbitrary $T$ whereas the upper curve corresponds to the universal scaling form of the equation of state for the $O(4)$ Heisenberg model. We see perfect agreement of both curves for $T$ sufficiently close to $T_c = 100.7$ MeV. This demonstrates the capability of our method to cover the critical behavior and, in particular, to reproduce the critical exponents of the $O(4)$–model. The curves (b), (c) and (d) are for non–vanishing values of the average current quark mass $\hat{m}$. Curve (c) corresponds to $\hat{m}_{\text{phys}}$ or, equivalently, $m_\pi(T = 0) = 135$ MeV. One observes a crossover in the range $T = (1.2 - 1.5)T_c$. In order to facilitate comparison

\footnote{The present investigation for the two flavor case does not take into account a speculative “effective restoration” of the axial $U_A(1)$ symmetry at high temperature.}
Figure 1: The plot shows the chiral condensate $\langle \bar{\psi} \psi \rangle$ as a function of temperature $T$. Lines (a), (b), (c), (d) correspond at zero temperature to $m_\pi = 0, 45\text{ MeV}, 135\text{ MeV}, 230\text{ MeV}$, respectively. For each pair of curves the lower one represents the full $T$-dependence of $\langle \bar{\psi} \psi \rangle$ whereas the upper one shows for comparison the universal scaling form of the equation of state for the $O(4)$ Heisenberg model. The critical temperature for zero quark mass is $T_c = 100.7\text{ MeV}$. The chiral condensate is normalized at a scale $k_\Phi \simeq 620\text{ MeV}$.

with lattice simulations which are typically performed for larger values of $m_\pi$ we also present results for $m_\pi(T = 0) = 230\text{ MeV}$ in curve (d). One may define a “pseudocritical temperature” $T_{pc}$ associated to the smooth crossover as the inflection point of $\langle \bar{\psi} \psi \rangle(T)$. The value for the pseudocritical temperature for $m_\pi = 230\text{ MeV}$ is $T_{pc} \simeq 150\text{ MeV}$. For realistic quark mass, or $m_\pi = 135\text{ MeV}$, we obtain $T_{pc} \simeq 130\text{ MeV}$.

A second important result of our investigations is the temperature dependence of the space–like pion correlation length $m_\pi^{-1}(T)$. The plot for $m_\pi(T)$ in figure 2 again shows the second order phase transition in the chiral limit $\hat{m} = 0$. In this limit the behavior for small positive $T - T_c$ is characterized by the critical exponent $\nu$, i.e. $m_\pi(T) = (\xi^+)^{-1} T_c \left( (T - T_c)/T_c \right)^\nu$ and we obtain $\nu = 0.787, \xi^+ = 0.270$. For $\hat{m} > 0$ we find that $m_\pi(T)$ remains almost constant for $T \lesssim T_c$. For $T > T_c$ the correlation length decreases rapidly and for $T \gg T_c$ the precise value of $\hat{m}$ becomes irrelevant. The overall size of the pion correlation length near the critical temperature is given by $m_\pi(T_{pc}) \simeq 1.7 m_\pi(0)$ for the realistic value $\hat{m}_{phys}$.

We point out two important answers one obtains from this study: First of all, for a thermal equilibrium situation the chiral transition gives no indication for strong fluctuations of pions with long wavelength. It should be emphasized, however, that a tricritical behavior with a massless excitation remains possible for three flavors.\end{flushleft}
Figure 2: The plot shows $m_\pi$ as a function of temperature $T$ for three different values of the average light current quark mass $\hat{m}$. The solid line corresponds to the realistic value $\hat{m} = \hat{m}_{\text{phys}}$ whereas the dotted line represents the situation without explicit chiral symmetry breaking, i.e., $\hat{m} = 0$. The intermediate, dashed line assumes $\hat{m} = \frac{\hat{m}_{\text{phys}}}{10}$.

model independent, arguments for a description of the chiral phase transition. Despite the observed comparably short correlation lengths at non-zero temperature, we find the approximate validity of the $O(4)$ scaling behavior over a large temperature interval near and above $T_c$ even for quark masses somewhat larger than the realistic ones. The present approach can be extended to the case with three light quark flavors.

Acknowledgments

I want to thank D.-U. Jungnickel and C. Wetterich for collaboration on the subject presented in this talk. I am grateful to the organizers of this conference for providing a most stimulating environment.

References

1. J.C. Collins and M.J. Perry, Phys. Rev. Lett. 34 (1975) 1353.
2. R.D. Pisarski and F. Wilczek, Phys. Rev. D29 (1984) 338.
3. K. Rajagopal and F. Wilczek, Nucl. Phys. B399 (1993) 395; B404 (1993) 577; K. Rajagopal, in Quark-Gluon Plasma 2, edited by R. Hwa (World Scientific, 1995) [hep-ph/9504310].
4. See e.g. J. D. Bjorken, K. L. Kowalski and C. C. Taylor, preprint SLAC-PUB-6109 [hep-ph/9309235] and references therein.
5. For a review see A. Ukawa, these proceedings.
6. J. Berges, D.-U. Jungnickel and C. Wetterich, preprint HD–THEP–97–20 [hep-ph/9705474].
7. C. Wetterich, Phys. Lett. B301 (1993) 90 and references therein.
8. U. Ellwanger and C. Wetterich, Nucl. Phys. B423 (1994) 137.
9. C. Wetterich, Z. Phys. C48 (1990) 693.
10. D.-U. Jungnickel and C. Wetterich, Phys. Rev. D53 (1996) 5142.
11. J. Berges, N. Tetradis and C. Wetterich, Phys. Rev. Lett. 77 (1996) 873.
12. S. Gavin, A. Gocksch and R.D. Pisarski, Phys. Rev. D49 (1994) 3079.
13. J. Berges, D.-U. Jungnickel and C. Wetterich, in preparation.