Abstract. We deal with the change point problem in ergodic diffusion processes based on high frequency data. Tonaki et al. (2020, 2021) studied the change point problem for the ergodic diffusion process model. However, the change point problem for the drift parameter when the diffusion parameter changes is still open. Therefore, we consider the change detection and the change point estimation for the drift parameter taking into account that there is a change point in the diffusion parameter. Moreover, we examine the performance of the tests and the estimation with numerical simulations.

1. Introduction

We consider a $d$-dimensional diffusion process $\{X_t\}_{t \geq 0}$ satisfying the stochastic differential equation

$$\begin{cases}
\mathrm{d}X_t = b(X_t, \beta)\mathrm{d}t + a(X_t, \alpha)\mathrm{d}W_t, \\
X_0 = x_0. 
\end{cases} \tag{1.1}$$

where parameter space $\Theta = \Theta_A \times \Theta_B$, which is a compact convex subset of $\mathbb{R}^p \times \mathbb{R}^q$, $\theta = (\alpha, \beta) \in \Theta$ is an unknown parameter and $\{W_t\}_{t \geq 0}$ is an $r$-dimensional standard Wiener process. The diffusion coefficient $a: \mathbb{R}^d \times \Theta_A \rightarrow \mathbb{R}^d \otimes \mathbb{R}^r$ and the drift coefficient $b: \mathbb{R}^d \times \Theta_B \rightarrow \mathbb{R}^d$ are known except for the parameter $\theta$. We assume that the solution of (1.1) exists, and $\mathbb{P}_\theta$ and $\mathbb{E}_\theta$ denote the law of the solution and the expectation with respect to $\mathbb{P}_\theta$, respectively. Let $\{X_t\}_{n=0}^T$ be discrete observations, where $t_i = t_n^i = ih_n$, and $\{h_n\}$ is a positive sequence with $h = h_n \rightarrow 0$, $T = nh \rightarrow \infty$ and $nh^2 \rightarrow 0$ as $n \rightarrow \infty$.

In this paper, we consider the change point problem for ergodic diffusion processes. The change point problem for diffusion processes based on discrete observations has been studied by many researchers, for non-ergodic, see De Gregorio and Iacus (2008), Iacus and Yoshida (2012) and for ergodic, see Song and Lee (2009), Lee (2011), Negri and Nishiyama (2017), Song (2020), Tonaki et al. (2020,2021) and reference therein. De Gregorio and Iacus (2008) considered the test for changes in the diffusion parameter and the change point estimation in non-ergodic case, and Iacus and Yoshida (2012) generalized the estimation method. Song and Lee (2009), Lee (2011), and Song (2020) considered the test for changes in the diffusion parameter in ergodic case. Furthermore, Negri and Nishiyama (2017) and Tonaki et al. (2020) proposed simultaneous test and adaptive tests for changes in the diffusion and drift parameters, respectively, and Tonaki et al. (2021) treated change point estimation for the case where there is a change in the diffusion parameter and for the case where there is no change in the diffusion parameter but there is a change in the drift parameter. The method proposed by Tonaki et al. (2020,2021) is summarized as follows. First, test whether there is any change in the diffusion parameter. If any change is detected, estimate the change point of the diffusion parameter. If not, assume that there is no change in the diffusion parameter, and test whether there is any change in the drift parameter. If any change is detected, estimate the change point of the drift parameter. This method allows us to estimate the change point in the diffusion parameter if a change in the diffusion parameter is detected, and to estimate the change point in the drift parameter if no change in the diffusion parameter is detected, but a change in the drift parameter is detected.

The purposes of the change point problem in the ergodic diffusion processes are to investigate whether there is a change point in the diffusion and drift parameters, and if so, where it is. Given
(a) $\alpha$ changes from 1 to 1.5 at $t = 150$, and $(\beta, \gamma) = (1, 1)$ does not change.

(b) $(\alpha, \beta, \gamma) = (1.2, 1, 1.5)$ does not change.

(c) $\alpha = 1$ does not change, and $(\beta, \gamma)$ changes from $(1, 1)$ to $(0.5, 0.5)$ at $t = 250$.

(d) $\alpha$ changes from 1 to 1.5 at $t = 350$, and $(\beta, \gamma)$ changes from $(1.3, 0.5)$ to $(1.3, 1)$ at $t = 200$.

Figure 1. Sample paths of the Ornstein-Uhlenbeck process $dX_t = -\beta(X_t - \gamma)dt + \alpha dW_t$.

Using the data for the paths as shown in Figure 1, using the method of Tonaki et al. (2020, 2021), we can infer that the diffusion parameter changes at $t = 150$ and $t = 350$ for (A) and (D), respectively, and that there is no change in the diffusion parameter, but the drift parameter changes at $t = 250$ for (C). Furthermore, for (B), it can be inferred that there is no change in either the diffusion or drift parameters. However, Tonaki et al. (2020, 2021) do not provide discussion for the change of the drift parameter when there is a change in the diffusion parameter such as (A) and (D). In other words, we have no way to infer that there is no change point in the drift parameter in (A), and that the drift parameter changes at $t = 200$ in (D). Thus, in this paper, we propose a method for detecting changes in the drift parameter and estimating the change point when the diffusion parameter changes.

This paper is organized as follows. Section 2 reviews the change point inference for the diffusion parameter. In Section 3, we propose test methods to detect changes in the drift parameter when there is a change point in the diffusion parameter, and show the asymptotic properties under the null hypothesis and the consistency of the proposed test statistics. Moreover, we propose change point estimators for the drift parameter in Section 4. In Section 5, we give some examples and simulation studies. Finally, Section 6 is devoted to the proofs of the results of Sections 3 and 4.

2. Change point inference for diffusion parameter

In this section, we review the change point detection and estimation for the diffusion parameter before considering the change point problem for the drift parameter when there is a change point in the diffusion parameter. See Tonaki et al. (2020, 2021) for details.

We set the following notations.

1. For a matrix $M$, $M^T$ denotes the transpose of $M$ and $M^{\otimes 2} = MM^T$.
2. Let $A(x, \alpha) = a(x, \alpha)^{\otimes 2}$ and $\Delta X = X_t - X_{t-1}$.
3. For $k = 1, 2, \ldots$, and $\epsilon \in (0, 1)$, $\{B^0_k(s)\}_{0 \leq s \leq 1}$ denotes a $k$-dimensional Brownian bridge, and $w_k(\epsilon)$ denotes the upper-$\epsilon$ point of $\sup_{0 \leq s \leq 1} \|B^0_k(s)\|$, that is,

$$
P\left( \sup_{0 \leq s \leq 1} \|B^0_k(s)\| > w_k(\epsilon) \right) = \epsilon.$$


4. Let \( \{W(s)\}_{s \geq 0} \) be a two-sided standard Wiener process.
5. Let \( C^1_t(\mathbb{R}^d \times \Theta) \) be the space of all functions \( f \) satisfying the following conditions.
   (i) \( f \) is continuously differentiable with respect to \( x \in \mathbb{R}^d \) up to order \( k \) for all \( \theta \in \Theta \),
   (ii) \( f \) and all its \( x \)-derivatives up to order \( k \) are \( \ell \) times continuously differentiable with respect to \( \theta \in \Theta \),
   (iii) \( f \) and all derivatives are of polynomial growth in \( x \in \mathbb{R}^d \) uniformly in \( \theta \in \Theta \), i.e., \( g \) is of polynomial growth in \( x \in \mathbb{R}^d \) uniformly in \( \theta \in \Theta \) if, for some \( C > 0 \),
   \[
   \sup_{\theta \in \Theta} \|g(x, \theta)\| \leq C(1 + \|x\|)^{\ell}.
   \]

We make the following assumptions.

[C1] There exists a constant \( C > 0 \) such that for any \( x, y \in \mathbb{R}^d \),
   \[
   \sup_{\alpha \in \Theta} \|a(x, \alpha) - a(y, \alpha)\| + \sup_{\beta \in \Theta} \|b(x, \beta) - b(y, \beta)\| \leq C\|x - y\|.
   \]

[C2] \( \sup_x E_0 \left[ \|X_t\|^k \right] < \infty \) for all \( k \geq 0 \) and \( \theta \in \Theta \).

[C3] \( \inf_{x, \alpha} A(x, \alpha) > 0 \).

[C4] \( a \in C_{\alpha}^4(\mathbb{R}^d \times \Theta_A) \) and \( b \in C_{\beta}^4(\mathbb{R}^d \times \Theta_B) \).

[C5] The solution of [1.1] is ergodic with its invariant measure \( \mu_0 \) such that
   \[
   \int_{\mathbb{R}^d} \|x\|^k d\mu_0(x) < \infty
   \]
   for all \( k \geq 0 \) and \( \theta \in \Theta \), and for all measurable function \( f \),
   \[
   \int_{\mathbb{R}^d} f(x) d\mu_0(x) \to \int_{\mathbb{R}^d} f(x) d\mu_0(x)
   \]
as \( \theta_n \to \theta_0 \).

[C6] If there exist \( m \geq 1, \theta_0, \ldots, \theta_{m-1} \in \text{Int } \Theta, \tau_1, \ldots, \tau_{m-1} \in (0, 1) \) such that
   \[
   X_t = \begin{cases} 
   X_t(\theta_j), & t \in [0, \tau_1 T), \\
   X_t(\theta_1), & t \in [\tau_1 T, \tau_2 T), \\
   \vdots \\
   X_t(\theta_{m-1}), & t \in [\tau_{m-1} T, T],
   \end{cases}
   \]
then for any \( f \in C_{\alpha}^{1,1}(\mathbb{R}^d \times \Theta), j = 1, \ldots, m-1, \) and \( \delta \in (1, 2) \) with \( nh^\delta \to \infty \),
   \[
   \max_{[n^{1/\delta}] \leq k \leq [n\tau_j] - [n\tau_{j+1}]} \left\{ \frac{1}{k} \sum_{i=n\tau_j+1}^{[n\tau_j]+k} f(X_{t_i-1}, \theta_j) - \int_{\mathbb{R}^d} f(x, \theta_j) d\mu_0(x) \right\}^p \to 0.
   \]

**Remark 1** If for all \( j = 1, 2, \ldots, m-1, \{X_t^{(j)}\}_{t \geq 0}, X_t^{(j)} = X_t(\theta_j) \) are stationary, then [C6] is satisfied from Lemma 4.3 of Song and Lee (2009).

### 2.1. Test for a change in diffusion parameter.
First, we consider the change detection for the diffusion parameter. For simplicity, we assume that there is a change point under the alternative hypothesis, that is, we consider the following hypothesis testing problem.

\( H_0^\alpha : \alpha \) does not change over \([0, T] \) v.s. \( H_1^\alpha : \) There exists \( \tau_+ \in (0, 1) \) such that
   \[
   \alpha = \begin{cases} 
   \alpha_1^*, & t \in [0, \tau_+ T), \\
   \alpha_2^*, & t \in [\tau_+ T, T],
   \end{cases}
   \]

where \( \alpha_1^*, \alpha_2^* \in \text{Int } \Theta_A \) and \( \alpha_1^* \neq \alpha_2^* \).

We consider the following two cases.

**Case A:** The parameters \( \alpha_1^* \) and \( \alpha_2^* \) depend on \( n \),
**Case B:** The parameters \( \alpha_1^* \) and \( \alpha_2^* \) are fixed and not depend on \( n \).

In Case A, we assume the following convergences hold.
   (i) \( \theta_n = |\alpha_1^* - \alpha_2^*| \to 0 \) and \( n\theta_n^2 \to \infty \) as \( n \to \infty \),
   (ii) \( \alpha_1^* \) and \( \alpha_2^* \) converge to \( \alpha_0 \in \text{Int } \Theta_A \) as \( n \to \infty \).

We make the following assumptions.

[D1] Under \( H_0^\alpha \), there exists an estimator \( \hat{\alpha} \) such that \( \sqrt{n}(\hat{\alpha} - \alpha) = O_p(1) \).
The test statistic for the diffusion parameter is as follows.

\[ T_n^\alpha = \frac{1}{\sqrt{2dn}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \hat{\eta}_i - \frac{k}{n} \sum_{i=1}^{n} \hat{\eta}_i \right|. \]

The following theorem is the result for the asymptotic distribution under the null hypothesis and the consistency of the test \( T_n^\alpha \).

**Theorem 1** (Tonaki et al., 2020,2021) Suppose that [C1]-[C6] hold.

1. If [D1] is satisfied, then \( T_n^\alpha \overset{d}{\rightarrow} \sup_{0 \leq s \leq 1} |B_t^0(s)| \) under \( H_0^\alpha \).
2. If [D2] and [D3] are satisfied, then for \( \epsilon \in (0,1) \), \( P(T_n^\alpha > w_1(\epsilon)) \rightarrow 1 \) under \( H_0^\alpha \) in Case A.
3. If [D4] and [D5] are satisfied, then for \( \epsilon \in (0,1) \), \( P(T_n^\alpha > w_1(\epsilon)) \rightarrow 1 \) under \( H_0^\alpha \) in Case B.

### 2.2. Estimation for a change in diffusion parameter.

If \( H_0^\alpha \) is rejected, we next consider the change point estimation problem for the following stochastic differential equation.

\[ X_t = \begin{cases} X_0 + \int_0^t b(X_s, \beta)ds + \int_0^t a(X_s, \alpha^*_s)dW_s, & t \in [0, \tau^\alpha_0 T), \\ X_{\tau^\alpha_0 T} + \int_{\tau^\alpha_0 T}^t b(X_s, \beta)ds + \int_{\tau^\alpha_0 T}^t a(X_s, \alpha^*_s)dW_s, & t \in [\tau^\alpha_0 T, T], \end{cases} \]

where \( \alpha^*_0, \alpha^*_1 \in \text{Int} \Theta_A, \alpha^*_1 \neq \alpha^*_2, \) and \( \tau^\alpha_0 \in (0,1) \) is unknown.

We make the following assumptions.

[D6] There exist estimators \( \hat{\alpha}_k \) (\( k = 1, 2 \)) such that \( \sqrt{n}(\hat{\alpha}_k - \alpha^*_k) = O_p(1) \).

[D7] \( h/\hat{\sigma}_n^2 \rightarrow \infty \) as \( n \rightarrow \infty \), and \( \hat{\sigma}^{-1} \alpha^*_k - \alpha_0 = O(1) \) in Case A.

Let

\[ \Gamma^\alpha(x, \alpha_1, \alpha_2) = \text{tr}(A^{-1}(x, \alpha_1)A(x, \alpha_2) - I_d) - \log \text{det} A^{-1}(x, \alpha_1)A(x, \alpha_2), \]

and \( Q(x, \theta) \) be the coefficient of \( h^2 \) of \( \mathbb{E}_\theta[(\Delta X)^{\otimes 2}]_{t_0}^{n} \), where \( \gamma_{i-1}^n = \sigma \{ W_s \}_{s \leq t} \).

[D8] \( \inf \Gamma^\alpha(x, \alpha^*_1, \alpha^*_2) > 0 \).

[D9] There exists a constant \( C > 0 \) such that

(a) \( \sup_{x, \alpha_1} |\hat{\sigma}_{\alpha_1}(x, \alpha_1, \alpha_2)\| < C \),

(b) \( \sup_{x, \alpha} \left| \frac{\text{tr} \left( (A^{-1}(x, \alpha_1) - A^{-1}(x, \alpha_2))\partial_{\alpha^*} A(x, \alpha_3) \right)}{\epsilon} \right| < C \),

(c) \( \sup_{x, \theta} |Q(x, \theta)| < C \).
Remark 3 See Section 3 in Tonaki et al. (2021) for how to construct estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$ satisfying [D6]. Tonaki et al. (2021) assumed $\mathbb{T} \partial_{\alpha} \longrightarrow 0$ in addition to condition [D7], but there is no need to assume it under [C6]. The assumption that $h/\partial_{\alpha}^2 \longrightarrow \infty$ in [D7] is not necessary when $\alpha(x, \alpha) = \alpha$.

Let

$$F_i(\alpha) = \text{tr} \left( A^{-1}(X_{t_{i-1}}, \alpha) \frac{(\Delta X_i) \otimes 2}{h} \right) + \log \det A(X_{t_{i-1}}, \alpha),$$

$$\Phi_n(\tau : \alpha_1, \alpha_2) = \sum_{i=1}^{[n\tau]} F_i(\alpha_1) + \sum_{i=[n\tau]+1}^{n} F_i(\alpha_2).$$

The change point estimator for the diffusion parameter is as follows.

$$\hat{\tau}_n^{\alpha} = \arg\min_{\tau \in [0,1]} \Phi_n(\tau : \hat{\alpha}_1, \hat{\alpha}_2).$$

In Case A, let $v \in \mathbb{R}$,

$$e_{\alpha} = \lim_{n \to \infty} \hat{\theta}_{\alpha}^{-1}(\alpha^*_1 - \alpha^*_2),$$

$$\Xi^\alpha(x, \alpha) = \left[ \text{tr} \left( A^{-1} \partial_{\alpha_1} AA^{-1} \partial_{\alpha_2} A(x, \alpha) \right) \right]_{t_1, t_2 = 1},$$

$$J_\alpha = \frac{1}{2} \int_0^1 \int_{\mathbb{R}} \Xi^\alpha(x, \alpha_0) d\mu_{\alpha_0}(x) e_{\alpha},$$

$$F(v) = -2 \frac{J_\alpha^{\alpha}}{v} \mathbb{W}(v) + J_\alpha |v|.$$  

The following results give the asymptotic behavior of the estimator $\hat{\tau}_n^{\alpha}$.

**Theorem 2** (Tonaki et al., 2021) Suppose that [C1]-[C6] and [D6] hold.

1. Under [D7], $n \hat{\tau}_n^{\alpha} (\hat{\tau}_n^{\alpha} - \tau_\alpha^*) \xrightarrow{\mathbb{D}} \arg\min \mathbb{F}(v)$ in Case A.
2. Under [D8] and [D9], $n(\hat{\tau}_n^{\alpha} - \tau_\alpha^*) = O_p(1)$ in Case B.
3. Under [D8], [D9](a) and (b), $n^{\alpha}(\hat{\tau}_n^{\alpha} - \tau_\alpha^*) = O_p(1)$ for $\varepsilon_1 \in [0, \frac{1}{2})$ in Case B.

**Remark 4** Since the 1-dimensional Ornstein-Uhlenbeck process defined by $dX_t = -\beta(X_t - \gamma)dt + \sigma dW_t$ ($\alpha, \beta > 0$, $\gamma \in \mathbb{R}$) does not satisfy [D9](3), but satisfies [D9](1) and (2), we can estimate $\tau_\alpha^*$ by Theorem 2(3) in this model. In contrast, the Hyperbolic diffusion model defined by $dX_t = (\beta - \gamma X_t/\sqrt{1 + X_t^2})dt + \sigma dW_t$ ($\alpha > 0$, $\beta, \gamma \in \mathbb{R}$, $|\beta| < \gamma$) satisfies [D9], and therefore $\tau_\alpha^*$ can be estimated by Theorem 2(2) in this model.

2.3. Test and estimation for the drift parameter when $H_0^\alpha$ is not rejected. If $H_0^\alpha$ is not rejected, i.e., no change in the diffusion parameter is detected, the next step is to investigate the change in the drift parameter. Specifically, the existence of change points in drift parameter is investigated, and if any change is detected, the change point is estimated. In this case, see Subsection 2.2 of Tonaki et al. (2020) for the change detection method and Subsection 2.2 of Tonaki et al. (2021) for the change point estimation method.

3. Change point detection for drift parameter

The purpose of this paper is to infer the change point in the drift parameter when there is a change point in the diffusion parameter. In the following, we consider the situation that $H_0^\alpha$ is rejected in the hypothesis testing problem in Subsection 2.1 and the change point of the diffusion parameter is estimated in Subsection 2.2.

3.1. Test for changes in drift parameter. In this subsection, we consider the change detection for the drift parameter when there is a change point in the diffusion parameter, that is, we treat the following stochastic differential equation.

$$X_t = \begin{cases} X_0 + \int_0^t b(X_s, \beta_1) ds + \int_0^t a(X_s, \alpha_1^*) dW_s, & t \in [0, \tau_\alpha^* T), \\ X_{\tau_\alpha^* T} + \int_{\tau_\alpha^* T}^t b(X_s, \beta_2) ds + \int_{\tau_\alpha^* T}^t a(X_s, \alpha_2^*) dW_s, & t \in [\tau_\alpha^* T, T], \end{cases}$$

where $\alpha_1^*, \alpha_2^* \in \text{Int} \Theta_A$, $\alpha_1^* \neq \alpha_2^*$, $\tau_\alpha^* \in (0, 1)$ is unknown, and $\beta_1$ and $\beta_2$ may change, or be equal. We consider the following two hypothesis testing problems.
\[ H^{(1)}_0 : \beta_1 \text{ does not change over } [0, \tau^0_1] \text{ v.s. } H^{(1)}_1 : \text{ not } H^{(1)}_0 \]
and
\[ H^{(2)}_0 : \beta_2 \text{ does not change over } [\tau^1_0, T] \text{ v.s. } H^{(2)}_1 : \text{ not } H^{(2)}_0. \]

We make the following assumptions.

[E1] There exists \( \epsilon_1 \in (0, 1) \) such that \( n^{\epsilon_1} (\hat{\tau}_n^\alpha - \tau^\alpha_0) = o_p(1) \).

[E2] For \( k = 1, 2 \), there exists an estimator \( \hat{\beta}_k \) such that \( \sqrt{T}(\hat{\beta}_k - \beta_k) = O_p(1) \) under \( H^0(k) \).

Remark 5  Let \( \hat{\alpha}_1, \hat{\alpha}_2 \) be estimators of \( \alpha_1^*, \alpha_2^* \). Take into account that if \( n|\alpha_1^* - \alpha_2^*|^2 \rightarrow \infty \), then \( n|\hat{\alpha}_1 - \hat{\alpha}_2|^2 \rightarrow \infty \) in probability, in other words, the probability of \( n|\hat{\alpha}_1 - \hat{\alpha}_2|^2 < 1 \) converges to zero. According to Theorem 3 we can choose \( \epsilon_1 \in (0, 1) \) in [E1], for example, \( n^{\epsilon_1} = (n^{0.5} \wedge n|\hat{\alpha}_1 - \hat{\alpha}_2|^2)^{0.9} \), i.e., \( \epsilon_1 = 0.45 \wedge (0.9 + 1.8 \log n |\hat{\alpha}_1 - \hat{\alpha}_2|) \) for large \( n \). See Kessler (1997), Uchida and Yoshida (2012,2014), Kamatani and Uchida (2015) and Kaino and Uchida (2018) for constructing an estimator \( \hat{\beta}_k \) that satisfies [E2].

Set
\[ \tau_n = \hat{\tau}_n^\alpha - n^{-\epsilon_1}, \quad \tau_n = \hat{\tau}_n^\alpha + n^{-\epsilon_1}. \]

We consider \( r = d \). Let
\[ \xi_{k,i} = (X_{t_{i-1}} - \hat{\alpha}_k)(\Delta X_i - \hat{h}b(X_{t_{i-1}}, \hat{\beta}_k)) \]
for \( k = 1, 2 \). The test statistics for the drift parameter are as follows.

\[ T^{(1)}_{1,n} = \frac{1}{\sqrt{\hat{\sigma}_1^2 T}} \max_{1 \leq s \leq |\tau_n|} \left| \sum_{i=1}^{[n\tau_n]} \xi_{1,i} - \frac{k}{[n\tau_n] \sum_{i=1}^{[n\tau_n]} \xi_{1,i}} \right|, \]

\[ T^{(2)}_{1,n} = \frac{1}{\sqrt{(1 - \tau_n^2)T}} \max_{1 \leq s \leq |\tau_n|} \left| \sum_{i=1}^{[n\tau_n]+k} \xi_{2,i} - \frac{k}{n - [n\tau_n] \sum_{i=[n\tau_n]+1}^{n} \xi_{2,i}} \right|. \]

Theorem 3  Suppose that [C1]–[C6], [D6], [E1] and [E2] hold. Then,
\[ T^{(1)}_{1,n} \xrightarrow{d} \sup_{0 \leq s \leq 1} \mathbf{B}^0_1(s) \text{ under } H^0_1, \]

\[ T^{(2)}_{1,n} \xrightarrow{d} \sup_{0 \leq s \leq 1} \mathbf{B}^0_2(s) \text{ under } H^0_2. \]

\( T^{(1)}_{1,n} \) and \( T^{(2)}_{1,n} \) are simple test statistics, but for the 1-dimensional Ornstein-Uhlenbeck process defined by \( dX_t = -\beta(X_t - \gamma)dt + \alpha dW_t \) (\( \alpha, \beta > 0, \gamma \in \mathbb{R} \)), if \( \beta \) changes and \( \gamma \) does not change, these tests do not satisfy the conditions for the consistency to hold as well as the test statistic proposed by Tonaki et al. (2020). Therefore, we consider other test statistics. Let
\[ \xi_{k,i} = \partial \beta b(X_{t_{i-1}}, \hat{\beta}_k) \mathbf{T} A^{-1}(X_{t_{i-1}}, \hat{\alpha}_k)(\Delta X_i - \hat{h}b(X_{t_{i-1}}, \hat{\beta}_k)), \]

\[ I_{1,n} = \frac{1}{|\tau_n|} \sum_{i=1}^{[n\tau_n]} \partial \beta b(X_{t_{i-1}}, \hat{\beta}_1) \mathbf{T} A^{-1}(X_{t_{i-1}}, \hat{\alpha}_1) \partial \beta b(X_{t_{i-1}}, \hat{\beta}_1), \]

\[ I_{2,n} = \frac{1}{n - [n\tau_n]} \sum_{i=[n\tau_n]+1}^{n} \partial \beta b(X_{t_{i-1}}, \hat{\beta}_2) \mathbf{T} A^{-1}(X_{t_{i-1}}, \hat{\alpha}_2) \partial \beta b(X_{t_{i-1}}, \hat{\beta}_2) \]
for \( k = 1, 2 \). The test statistics for the drift parameter are as follows.

\[ T^{(1)}_{2,n} = \frac{1}{\sqrt{\tau_n^2 T}} \max_{1 \leq s \leq |\tau_n|} \left| T^{(1)}_{1,n} \right| \]

\[ T^{(2)}_{2,n} = \frac{1}{\sqrt{(1 - \tau_n^2)T}} \max_{1 \leq s \leq |\tau_n|} \left| T^{(2)}_{1,n} \right|. \]

We additionally make the following assumptions with respect to the smoothness of the drift coefficient \( b \).

[E3] There exists an integer \( m_1 \geq 3 \) such that \( nh^{m_1/(m_1-1)} \rightarrow \infty \) and \( b \in C^{m_1+1}_1(\mathbb{R}^d \times \Theta_B) \).

[E4] There exists an integer \( m_2 \geq 3 \) such that \( b \in C^{m_2+1}_1(\mathbb{R}^d \times \Theta_B) \) and \( \partial \beta_{m_2+1} \dot{\cdots} \partial \beta_{m_1} b(x, \beta) = 0 \) for \( 1 \leq \ell_1, \ldots, \ell_{m_2+1} \leq q \).
Theorem 4 Suppose that [C1]-[C6], [D6], [E1]-[E3] hold. Then,
\[
T_{2,n}^{(1)} \xrightarrow{d} \sup_{0 \leq s \leq 1} \|B^0_s(\theta)\| \quad \text{under } H^{(1)}_0, \tag{3.3}
\]
\[
T_{2,n}^{(2)} \xrightarrow{d} \sup_{0 \leq s \leq 1} \|B^1_s(\theta)\| \quad \text{under } H^{(2)}_0. \tag{3.4}
\]
Furthermore, (3.3) and (3.4) still hold even if we replace [E3] with [E4].

Remark 6 When the drift parameter changes at the same point as the diffusion parameter, these tests are unable to detect changes in the drift parameter. In other words, even if the null hypotheses $H^{(1)}_0$ and $H^{(2)}_0$ are not rejected, it is possible that the drift parameter changes at the same point as the diffusion parameter. See Subsection 3.3 for the case where neither $H^{(1)}_0$ nor $H^{(2)}_0$ is rejected.

3.2. Consistency of tests. In this subsection, we consider the consistency of the proposed tests. For simplicity, we assume that there is a change point under the alternative hypothesis, that is, for $k = 1, 2$, we make the following assumptions.

[F1] There exist $\beta_k^* \in \Theta_B$ and an estimator $\hat{\beta}_k$ such that $\hat{\beta}_k - \beta_k^* = o_p(1)$ under $H^{(k)}_1$.

[F2] There exist $\beta_k^* \in \Theta_B$ and an estimator $\hat{\beta}_k$ such that $\sqrt{T} (\hat{\beta}_k - \beta_k^*) = O_p(1)$ under $H^{(k)}_1$.

Let $a_k^* \to a_k^*(0) \in \Theta_A$ as $n \to \infty$, where we note that $a_k^*$ may depend on $n$. Let

\[
G_k = \int_{\mathbb{R}^d} 1^d a^{-1}(x, a_k^*(0)) (b(x, \beta_k^* - \beta_k^*) - b(x, \beta_k^*) - b(x, \beta_k^*)) d\mu_{(a_k^*(0), \beta_k^*)}(x),
\]
\[
H_k = \int_{\mathbb{R}^d} \partial_x b(x, \beta_k^*)^T \partial_x^{-1}(x, a_k^*(0)) (b(x, \beta_k^* - \beta_k^*) - b(x, \beta_k^*)) d\mu_{(a_k^*(0), \beta_k^*)}(x).
\]

[F3] $G_{k,1} \neq G_{k,2}$ under $H^{(1)}_1$.

[F4] $H_{k,1} \neq H_{k,2}$ under $H^{(1)}_1$.

[G1] $\eta_{\beta_k} = |\beta_{k,1}^* - \beta_{k,2}^*|$ depends on $n$, and $\eta_{\beta_k} \to 0$, $T \eta_{\beta_k}^2 \to \infty$ as $n \to \infty$ under $H^{(k)}_1$.

[G2] There exists $\beta_k^* \in \Theta_B$ such that $\eta_{\beta_k} (\beta_{k}\ell - \beta_k^*) \to d_k, \ell = 1, 2$ as $n \to \infty$ for $\ell = 1, 2$.

[G3] There exist $\beta_k^*$ such that $\beta_k^* - \beta_k^* = o(1)$ and an estimator $\hat{\beta}_k$ such that $\sqrt{T} (\hat{\beta}_k - \beta_k^*) = O_p(1)$ under $H^{(k)}_1$.

[G4] $\int_{\mathbb{R}^d} 1^d a^{-1}(x, a_k^*) \partial_x b(x, \beta_k^*) d\mu_{(a_k^*, \beta_k^*)}(x) (d_k - d_k, 2) \neq 0$ under $H^{(k)}_1$.

[G5] For each $\epsilon_1 \in (0, 1)$ in [E1], $n^{\epsilon_1^*} \eta_{\beta_k} \to \infty$.

[G6] There exists an integer $m_4 \geq 3$ such that $n^{-m_4} h^{-(m_4 + 1)} = O(1)$, $h^{1/2} \eta_{\beta_k} \to 0$ and $b \in C^{3, m_4 + 1} (\mathbb{R}^d \times \Theta_B)$.

Remark 7 See Uchida and Yoshida (2011) and Tonaki et al. (2020, 2021) for constructing an estimator $\hat{\beta}_k$ that satisfies [F1], [F2] or [G3].

Theorem 5 Suppose that [C1]-[C5], [D6] and [E1] hold. If any one of the following conditions is satisfied

(a) [F1] and [F3],

(b) [G1]-[G4],

then for $\epsilon \in (0, 1)$, $P(T_{1,\epsilon}^{(1)} > w_1(\epsilon)) \to 1$ under $H^{(1)}_1$, and $P(T_{1,\epsilon}^{(2)} > w_1(\epsilon)) \to 1$ under $H^{(2)}_1$. 
Suppose that $[C1]$, $[G6]$ and $[E1]$ hold. If any one of the following conditions is satisfied

(a) $[E4]$, $[F2]$ and $[F4]$,  
(b) $[E5]$, $[F1]$ and $[F4]$,  
(c) $[G1]$, $[G3]$, $[G5]$ and $[G6]$,  

then for $\epsilon \in (0,1)$, $P(T^{(1)}_{2,n} > w_\epsilon(\epsilon)) \rightarrow 1$ under $H^{(1)}_1$, and $P(T^{(2)}_{2,n} > w_\epsilon(\epsilon)) \rightarrow 1$ under $H^{(2)}_1$.

Remark 8 (a) of Theorem $[C5]$, (a) and (b) of Theorem $[G3]$ and (c) of Theorem $[E1]$ are the conditions to satisfy the consistency when the difference of change does not depend on $n$, i.e., Case B described in Section $[G1]$, and (b) of Theorem $[G3]$ and (c) of Theorem $[E1]$ are the conditions to satisfy the consistency when the difference of change depends on $n$ and shrinks, i.e., Case A described in Section $[G1]$.

3.3. Change in diffusion and drift parameters at the same point. Since $T^{(1)}_{1,n}$ and $T^{(2)}_{1,n}$ (or, $T^{(1)}_{2,n}$ and $T^{(2)}_{2,n}$) are tests for the change of the drift parameter in $[0, \tau_{n}T]$ and $[\tau_{n}T, T]$, respectively, neither test can detect the change when the drift parameter changes in $[\tau_{n}T, T]$, i.e., $\tau^*_n = \tau^*_n$.

Therefore, in this subsection, we consider how to investigate whether the drift parameter changes at the same point as the diffusion parameter. In other words, we consider a method for detecting changes in the drift parameter for the following stochastic differential equation.

$$X_t = \begin{cases} 
X_0 + \int_0^t b(X_s, \beta^*_1)ds + \int_0^t a(X_s, \alpha^*_1)dW_s, & t \in [0, \tau^*_n T), \\
X_{\tau^*_n T} + \int_{\tau^*_n T}^t b(X_s, \beta^*_2)ds + \int_{\tau^*_n T}^t a(X_s, \alpha^*_2)dW_s, & t \in [\tau^*_n T, T],
\end{cases}$$

where $\alpha^*_1, \alpha^*_2 \in \int \Theta_A$, $\beta^*_1, \beta^*_2 \in \int \Theta_B$, $\alpha^*_1 \neq \alpha^*_2$, and $\beta^*_1$ and $\beta^*_2$ may be equal or not.

If neither $H^{(1)}_0$ nor $H^{(2)}_0$ is rejected, we construct the estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ for $\beta^*_1$ and $\beta^*_2$ with data from the intervals $[0, \tau_{n}T]$ and $[\tau_{n}T, T]$, respectively. Here, note that the estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ can be constructed to satisfy the following.

$$\sqrt{T}(\hat{\beta}_1 - \beta^*_1) = O_p(1), \quad \sqrt{T}(\hat{\beta}_2 - \beta^*_2) = O_p(1).$$

Then, we have

$$\sqrt{T}|\beta^*_1 - \beta^*_2| \leq \sqrt{T}|\hat{\beta}_1 - \beta^*_1| + \sqrt{T}|\hat{\beta}_2 - \beta^*_2| + \sqrt{T}|\hat{\beta}_1 - \hat{\beta}_2| = O_p(1) + \sqrt{T}|\hat{\beta}_1 - \hat{\beta}_2|$$

and

$$\sqrt{T}|\hat{\beta}_1 - \hat{\beta}_2| \leq \sqrt{T}|\hat{\beta}_1 - \beta^*_1| + \sqrt{T}|\hat{\beta}_2 - \beta^*_2| + \sqrt{T}|\hat{\beta}_1 - \beta^*_2| = O_p(1) + \sqrt{T}|\hat{\beta}_1 - \beta^*_2|.$$ 

Thus, $\sqrt{T}|\hat{\beta}_1 - \hat{\beta}_2| = O_p(1)$ is equivalent to $\sqrt{T}|\beta^*_1 - \beta^*_2| = O(1)$. Note that if $\sqrt{T}|\hat{\beta}_1 - \hat{\beta}_2| \rightarrow \infty$, then $\sqrt{T}|\hat{\beta}_1 - \hat{\beta}_2| \neq O_p(1)$, and if $\sqrt{T}|\hat{\beta}_1 - \hat{\beta}_2|$ is monotone, then $\sqrt{T}|\beta^*_1 - \beta^*_2| \neq O(1)$ is equivalent to $\sqrt{T}|\beta^*_1 - \beta^*_2| \rightarrow \infty$. Hence, we have the following assertions.

(1) If $\sqrt{T}|\hat{\beta}_1 - \hat{\beta}_2| = O_p(1)$, then $\sqrt{T}|\beta^*_1 - \beta^*_2| = O(1)$.

(2) If $\sqrt{T}|\hat{\beta}_1 - \hat{\beta}_2| \rightarrow \infty$, then $\sqrt{T}|\beta^*_1 - \beta^*_2| \rightarrow \infty$.

This implies that if $\sqrt{T}|\hat{\beta}_1 - \hat{\beta}_2|$ is sufficiently large, then we infer that the drift parameter changes at $\tau^*_n T$. Here we note that $\tau^*_n T$ is the same point in time at which the diffusion parameter changes.

Remark 9 The assertion (2) implies that this method can detect any change in the degree that the proposed test statistics $T^{(1)}_{1,n}$ and $T^{(1)}_{2,n}$ can detect it. In other words, the change in the drift parameter that satisfies the assumption $[G1]$ can be detected by the test $T^{(1)}_{1,n}$ or $T^{(1)}_{2,n}$ if the change does not occur at the same time as the diffusion parameter, and can also be detected by this method if the change occurs at the same time. As we saw above, we can theoretically determine whether the drift parameter changes at the same time as the diffusion parameter by investigating $\sqrt{T}|\hat{\beta}_1 - \hat{\beta}_2|$, but it would be difficult to determine whether the drift parameter changes simultaneously with the diffusion parameter in practice. See the numerical simulations in Section $[G1]$.

4. Change point estimation for drift parameter

In this section, we consider the change point estimation for the drift parameter when there is a change point in the diffusion parameter. For simplicity, we assume that there is a change point in
the diffusion and drift parameters, respectively. Namely, there exist $\tau^\alpha_\ast, \tau^\beta_\ast \in (0, 1)$ such that
\[
\alpha^* = \begin{cases} 
\alpha^*_1, & t \in [0, \tau^\alpha_\ast T), \\
\alpha^*_2, & t \in [\tau^\alpha_\ast T, T], 
\end{cases}
\beta^* = \begin{cases} 
\beta^*_1, & t \in [0, \tau^\beta_\ast T), \\
\beta^*_2, & t \in [\tau^\beta_\ast T, T], 
\end{cases}
\]
where $\alpha^*_1, \alpha^*_2 \in \text{Int } \Theta_A, \beta^*_1, \beta^*_2 \in \text{Int } \Theta_B, \alpha^*_1 \neq \alpha^*_2, \beta^*_1 \neq \beta^*_2$ and $\tau^\alpha_\ast \neq \tau^\beta_\ast$.

If $\tau^\beta_\ast < \tau^\alpha_\ast$, then (1.1) can be expressed as follows.
\[
X_i = \begin{cases} 
X_{\tau^\alpha_\ast T} + \int_{\tau^\alpha_\ast T}^t b(x, \beta^*_1)ds + \int_{\tau^\alpha_\ast T}^t a(x, \alpha^*_1)dw_s, & t \in [\tau^\alpha_\ast T, T], \\
X_{\tau^\beta_\ast T} + \int_{\tau^\beta_\ast T}^t b(x, \beta^*_2)ds + \int_{\tau^\beta_\ast T}^t a(x, \alpha^*_2)dw_s, & t \in [\tau^\beta_\ast T, T]. 
\end{cases}
\]

On the other hand, if $\tau^\alpha_\ast < \tau^\beta_\ast$, then (1.1) can be expressed as
\[
X_i = \begin{cases} 
X_{\tau^\alpha_\ast T} + \int_{\tau^\alpha_\ast T}^t b(x, \beta^*_1)ds + \int_{\tau^\alpha_\ast T}^t a(x, \alpha^*_1)dw_s, & t \in [\tau^\alpha_\ast T, T], \\
X_{\tau^\beta_\ast T} + \int_{\tau^\beta_\ast T}^t b(x, \beta^*_2)ds + \int_{\tau^\beta_\ast T}^t a(x, \alpha^*_2)dw_s, & t \in [\tau^\beta_\ast T, T]. 
\end{cases}
\]

We consider the asymptotic properties of the proposed estimators, such as consistency and asymptotic distribution, in the following two cases.

**Case A:** The parameters $\beta^*_1$ and $\beta^*_2$ depend on $n$.
**Case B:** The parameters $\beta^*_1$ and $\beta^*_2$ are fixed and not depend on $n$.

We make the following assumptions.

**[H1]** The test $T_{1,n}^{(1)}$ or $T_{2,n}^{(1)}$ detects a change in the drift parameter.

**[H2]** The test $T_{1,n}^{(2)}$ or $T_{2,n}^{(2)}$ detects a change in the drift parameter.

**[H3]** There exist estimators $\hat{\beta}_k (k = 1, 2)$ such that $\sqrt{T}(\hat{\beta}_k - \beta^*_k) = O_p(1)$.

**[A1]** $\hat{\vartheta}_\beta = |\beta^*_1 - \beta^*_2|$ depends on $n$, and
\[
\vartheta_\beta \rightarrow 0, \quad T\sqrt{\vartheta}_\beta^2 \rightarrow \infty
\]
as $n \rightarrow \infty$. Furthermore, $\sqrt{T}(\hat{\beta}_k - \beta_0) = O(1)$ holds for some $\beta_0 \in \text{Int } \Theta_B$ and $k = 1, 2$.

**[A2]** There exists an integer $m_4 \geq 3$ such that $n h^{m_4/(m_4-1)} \rightarrow \infty$, $h^{-1/2} \vartheta_\beta^{m_4-1} \rightarrow 0$ and $b \in \mathbb{C}^{4,m_4+1}_4(\mathbb{R}^d \times \Theta_B)$.

Let $\alpha^*_k \rightarrow \alpha^{(0)}_k \in \text{Int } \Theta_A$ as $n \rightarrow \infty$, and
\[
\Gamma^\beta(x, \alpha, \beta_1, \beta_2) = \text{tr}[A^{-1}(x, \alpha)(b(x, \beta_1) - b(x, \beta_2))^\otimes 2].
\]

**[B1]** $\inf_x \Gamma^\beta(x, \alpha^{(0)}_1, \beta^*_1, \beta^*_2) > 0$.

**[B2]** $\inf_x \Gamma^\beta(x, \alpha^{(0)}_2, \beta^*_1, \beta^*_2) > 0$.

**[B3]** There exists a constant $C > 0$ such that
\[
(a) \quad \sup_{x, \alpha, \beta_k} |\partial_{(\alpha, \beta_1, \beta_2)} \Gamma^\beta(x, \alpha, \beta_1, \beta_2)| < C,
(b) \quad \sup_{x, \alpha, \beta_k} \left| \text{tr}\left(A^{-1}(x, \alpha)\partial_{\beta} b(x, \beta_1)(b(x, \beta_2) - b(x, \beta_3))\right)\right|_{i=1}^q < C.
\]

**Remark 10** See Section 3 in Tonaki et al. (2021) for the construction of estimators that satisfy [H3]. Tonaki et al. (2021) assumed $T\vartheta_\beta^4 \rightarrow 0$ in addition to condition [A1], but there is no need to assume it under [C6].

We consider $\tau^\beta_\ast < \tau^\alpha_\ast$. Suppose that [H1] holds. Then, we set
\[
G_i(\beta | \alpha) = \text{tr}\left(A^{-1}(X_{t_{i-1}}, \alpha)(\Delta X_i - h b(X_{t_{i-1}}, \beta))^\otimes 2\right),
\]
\[ \Psi_{1,n}(\tau : \beta_1, \beta_2 | \alpha) = \sum_{i=1}^{[n\tau]} G_i(\beta_1 | \alpha) + \sum_{i=[n\tau]+1} G_i(\beta_2 | \alpha) \]

and propose

\[ \hat{\tau}_{1,n}^{\beta} = \arg\min_{\tau \in [0, \infty]} \Psi_{1,n}(\tau : \hat{\beta}_1, \hat{\beta}_2 | \hat{\alpha}_1) \]

as an estimator of \( \tau_1^{\beta} \).

We consider \( \tau_{1}^{\alpha} < \tau_{1}^{\beta} \). Suppose that [H2] holds. Then, we set

\[ \Psi_{2,n}(\tau : \beta_1, \beta_2 | \alpha) = \sum_{i=1}^{[n\tau]} G_i(\beta_1 | \alpha) + \sum_{i=[n\tau]+1} G_i(\beta_2 | \alpha) \]

and propose

\[ \hat{\tau}_{2,n}^{\beta} = \arg\min_{\tau \in [0, \infty]} \Psi_{2,n}(\tau : \hat{\beta}_1, \hat{\beta}_2 | \hat{\alpha}_2) \]

as an estimator of \( \tau_2^{\beta} \).

In Case A, we set for \( k = 1, 2 \) and \( v \in \mathbb{R} \),

\[ e_{\beta} = \lim_{n \to \infty} \theta^{-1}(\beta_1^{*} - \beta_2^{*}), \]

\[ \Xi^{\beta}(x, \alpha, \beta) = \left[ \partial_{\beta_1} b(x, \beta)^{T} A^{-1}(x, \alpha) \partial_{\beta_2} b(x, \beta) \right]_{\ell_1, \ell_2 = 1}^{g} \]

\[ \mathcal{J}_{k,\beta} = e_{\beta}^{T} \int_{\mathbb{R}^{g}} \Xi^{\beta}(x, \alpha_{k}, \beta_{0}) d\mu_{(\alpha_{k}, \beta_{0})}(x) e_{\beta}, \]

\[ \mathcal{G}_{k}(v) = -2 \mathcal{J}_{k,\beta}^{1/2} \mathcal{W}(v) + \mathcal{J}_{k,\beta} |v| \]

**Theorem 7** Let \( \tau_{1}^{\alpha} \leq \tau_{1}^{\beta} \). Suppose that [C1]-[C6], [D6], [E1], [H1] and [H3] hold.

(1) Under [A1] and [A2], in Case A,

\[ T \theta_{\beta}^{2}(\hat{\tau}_{1,n}^{\beta} - \tau_{1}^{\beta}) \xrightarrow{d} \arg\min_{v \in \mathbb{R}} \mathcal{G}_{2}(v). \]

(2) Under [B1] and [B3], in Case B,

\[ T(\hat{\tau}_{1,n}^{\beta} - \tau_{1}^{\beta}) = O_{p}(1). \]

**Theorem 8** Let \( \tau_{2}^{\alpha} < \tau_{2}^{\beta} \). Suppose that [C1]-[C6], [D6], [E1], [H2] and [H3] hold.

(1) Under [A1] and [A2], in Case A,

\[ T \theta_{\beta}^{2}(\hat{\tau}_{2,n}^{\beta} - \tau_{2}^{\beta}) \xrightarrow{d} \arg\min_{v \in \mathbb{R}} \mathcal{G}_{2}(v). \]

(2) Under [B2] and [B3], in Case B,

\[ T(\hat{\tau}_{2,n}^{\beta} - \tau_{2}^{\beta}) = O_{p}(1). \]

**Remark 11** In the case of the 1-dimensional Ornstein-Uhlenbeck process defined by \( dX_t = -\beta(X_t - \gamma)dt + \alpha dW_t \) \((\alpha, \beta > 0, \gamma \in \mathbb{R})\), [B1]-[B3] are satisfied if \( \beta \) does not change and \( \gamma \) changes. Therefore, Theorems 7 and 8 (2) enable us to estimate the change point of the drift parameter in Case B. Moreover, in the case of the hyperbolic diffusion model defined by \( dX_t = (\beta - \gamma X_t / \sqrt{1 + X_t^2})dt + \alpha dW_t \) \((\alpha > 0, \beta \in \mathbb{R}, |\beta| < \gamma)\), since [B1]-[B3] hold, we can estimate the change point in time of the drift parameter in Case B. On the other hand, in Case A, the change point of the drift parameter can be estimated in both cases of the 1-dimensional Ornstein-Uhlenbeck process and the hyperbolic diffusion model.

5. **Examples and simulation results**

We consider the following stochastic differential equation with a change point in the diffusion parameter.

\[ X_t = \begin{cases} 
X_0 + \int_0^t b(X_s, \beta)ds + \int_0^t a(X_s, \alpha_1^{*})dW_s, & t \in [0, \tau_{1}^{\alpha}], \\
X_{\tau_{1}^{\alpha} T} + \int_{\tau_{1}^{\alpha} T}^t b(X_s, \beta)ds + \int_{\tau_{1}^{\alpha} T}^t a(X_s, \alpha_2^{*})dW_s, & t \in [\tau_{1}^{\alpha} T, T], 
\end{cases} \]
where $\alpha^*_1 \neq \alpha^*_2$, and $\beta$ may change in $[0, T]$.

In this section, we consider the following three situations and confirm the results of Sections 3 and 4 by numerical simulations.

(i) The drift parameter $\beta$ does not change over $[0, T]$.
(ii) The drift parameter $\beta$ changes from $\beta_1^*$ to $\beta_2^*$ at $\tau^*_1$, where $\tau^*_1 \in (0, 1)$, $\tau^*_2 \neq \tau^*_1$.
(iii) The drift parameter $\beta$ changes from $\beta_1^*$ to $\beta_2^*$ at the same point as the diffusion parameter.

We perform numerical simulations with the following procedures.

**All situations:** Perform the following six steps in all situations (i)-(iii).

1. Test for a change in the diffusion parameter in $[0, T]$ to check whether there is a change point or not. See Subsection 2.1 for the test statistic.
2. If a change is detected in (1), choose $\tau_1, \tau_2 \in (0, 1)$ such that the test detects the change in $[\tau_1 T, \tau_2 T]$ to estimate $\alpha^*_1$ and $\alpha^*_2$. See Section 3 of Tonaki et al. (2021).
3. Construct estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$ for $\alpha^*_1$ and $\alpha^*_2$ from $[0, \tau_1 T]$ and $[\tau_2 T, T]$, respectively.
4. Estimate $\tau^*_n$ with the estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$ of (3). Let $\hat{\tau}^*_n$ be the estimator of $\tau^*_n$. See Subsection 2.2 for the estimator of $\tau^*_n$.
5. Choose $\epsilon_1 > 0$ such that $n^{1/2} \sup_{0 \leq s \leq 1} |B^0(s)| > n^{0.3} |\hat{\alpha}_2 - \hat{\alpha}_1|^2$, and set $\tau_n = \tau^*_n - n^{-\epsilon_1}$ and $\tau_n = \tau^*_n + n^{-\epsilon_1}$.
6. Test for a change in the drift parameter in $[0, \tau_n T]$ and $[\tau_n T, T]$ to check whether there is a change point or not. See Subsection 3.1 for the test statistics.

**Situations (i) and (iii):** After (1)-(6), perform the following step in situations (i) and (iii).

7. If neither of the tests in (6) detects a change, construct estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ from $[0, \tau_n T]$ and $[\tau_n T, T]$, respectively, and investigate $\sqrt{T} |\hat{\beta}_1 - \hat{\beta}_2|$ to check whether the drift parameter changes at the same point as the diffusion parameter. See Subsection 3.2.

**Situation (ii):** After (1)-(6), perform the following step in situation (ii).

7. If a change is detected in (6), estimate $\tau^*_n$. Note that the estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ for $\beta_1^*$ and $\beta_2^*$ are constructed in the same way as (2)-(3). See Section 4 for the estimator of $\tau^*_n$.

All simulations are conducted at significance level 0.05 and the corresponding critical values are obtained from the following: the Brownian bridge is generated by taking $10^4$ points on the interval $[0, 1]$, and the maximum value of its norm is recorded. This is repeated $10^4$ times. As a result, we have

$$P \left( \sup_{0 \leq s \leq 1} |B^0_1(s)| > 1.3617 \right) = 0.05, \quad P \left( \sup_{0 \leq s \leq 1} \|B^0_2(s)\| > 1.5736 \right) = 0.05,$$

i.e., the corresponding critical values are 1.3617 and 1.5736 for the 1-dimensional and 2-dimensional Brownian bridges, respectively.

5.1. **Model 1.** We consider the 1-dimensional Ornstein-Uhlenbeck process defined by

$$dX_t = -\beta(X_t - \gamma)dt + \alpha \, dW_t, \quad X_0 = x_0,$$

where $\alpha, \beta > 0$ and $\gamma \in \mathbb{R}$. For simulations of the test statistics and the estimator, we study the following stochastic differential equation

$$X_t = \begin{cases} X_0 - \int_0^t \beta(X_s - \gamma)ds + \alpha^*_1 W_t, & t \in [0, \tau^*_1 T), \\ X_{\tau^*_1 T} - \int_{\tau^*_1 T}^t \beta(X_s - \gamma)ds + \alpha^*_2(W_t - W_{\tau^*_1 T}), & t \in [\tau^*_1 T, T], \end{cases}$$

where $x_0 = 2$, $\tau^*_1 = 0.8$, $\alpha^*_1 = 1$, $\alpha^*_2 = 1.2$, $\beta = (\beta, \gamma)$.

We consider the following three situations.

(i) The drift parameter $\beta = (1, 2)$ does not change over $[0, T]$.
(ii) The drift parameter $\beta$ changes from $\beta^*_1 = (1, 2 - \vartheta_\beta)$ to $\beta^*_2 = (1, 2)$ at $\tau^*_1 = 0.4$ (Case A).
(iii) The drift parameter $\beta$ changes from $\beta^*_1 = (1, 2 - \vartheta_\beta)$ to $\beta^*_2 = (1, 2)$ at $\tau^*_1 = \tau^*_2 = 0.8$.

The number of iteration is 1000. We set that the sample size of the data $\{X_i\}_{i=0}^n$ is $n = 10^6$ or $10^7$, $h = n^{-0.52}$, $T = nh = n^{0.48}$, $nh^2 = n^{-0.04}$, $\vartheta_\beta = n^{-0.1}$.

We test for changes in the diffusion parameter in the interval $[0, T]$. The results show that in all situations (i)-(iii), the change is detected in all 1000 iterations. In order to estimate the parameters before and after the change, we test for the change in the diffusion parameter in the
interval \([0.125T, 0.875T]\). The results indicate that the change is detected in all 1000 iterations for all situations (i)-(iii). Therefore, we estimate \(\alpha_1^*\) from \([0, 0.125T]\) and \(\alpha_2^*\) from \([0.875T, T]\), and estimate \(\tau_n^*\) using the estimators \(\hat{\alpha}_1\) and \(\hat{\alpha}_2\). The estimates of \(\alpha_1^*, \alpha_2^*\) and \(\tau_n^*\) are reported in Table 1. In this case, we chose \(\epsilon_1 = 0.45\) for all iterations.

**Table 1.** Mean and standard deviation of the estimators. True values: \(\alpha_1^* = 1, \alpha_2^* = 1.2, \tau_n^* = 0.8\).

| \(n\) | \(T\)     | \(h\)             | \(\hat{\alpha}_1\) | \(\hat{\alpha}_2\) | \(\tilde{\tau}_n^\alpha\) |
|-------|-----------|-------------------|---------------------|---------------------|-----------------------------|
| \(10^6\) | 758.58    | \(7.59 \times 10^{-4}\) | (i) 1.00018         | 1.20018             | 0.79878                     |
|        |           |                   | (0.00207)           | (0.00235)           | (0.0016)                    |
|        |           |                   | (ii) 1.00018        | 1.20018             | 0.79879                     |
|        |           |                   | (0.00207)           | (0.00235)           | (0.0016)                    |
|        |           |                   | (iii) 1.00018       | 1.20018             | 0.79879                     |
|        |           |                   | (0.00207)           | (0.00235)           | (0.0016)                    |
| \(10^7\) | 2290.87   | \(2.29 \times 10^{-4}\) | (i) 1.00005         | 1.20010             | 0.79961                     |
|        |           |                   | (0.00207)           | (0.00235)           | (0.0016)                    |
|        |           |                   | (ii) 1.00005        | 1.20010             | 0.79961                     |
|        |           |                   | (0.00207)           | (0.00235)           | (0.0016)                    |
|        |           |                   | (iii) 1.00005       | 1.20010             | 0.79961                     |
|        |           |                   | (0.00207)           | (0.00235)           | (0.0016)                    |

Next, we test for changes in the drift parameter in the intervals \([0, \tau_nT]\) and \([\tau_nT, T]\). The results of the tests for changes in the drift parameter are summarized in Table 2 and Figures 2, 4 and 6. From Table 2, Figures 2 and 6, we can see that in (i) and (iii), the proportions of the test statistics that exceed the critical values are close to the significance level 0.05, and the distribution of the test statistics almost correspond with the null distribution. This implies that the test statistics have good performance. Therefore, in (i) and (iii), we construct estimators \(\hat{\beta}_1 = (\hat{\beta}_1, \hat{\gamma}_1)\) and \(\hat{\beta}_2 = (\hat{\beta}_2, \hat{\gamma}_2)\) from the intervals \([0, \tau_nT]\) and \([\tau_nT, T]\), respectively when the test statistics \(\hat{T}_{1,n}^{(1)}\) and \(\hat{T}_{1,n}^{(2)}\) do not exceed the critical value, and investigate \(\sqrt{T} \|\hat{\beta}_1 - \hat{\beta}_2\|\). The results of the estimates of \(\hat{\beta}_1^*\) and \(\hat{\beta}_2^*\) in (i) and (iii) are summarized in Table 3 and Figure 3 and Table 5 and Figure 7, respectively. From Figure 3, \(\sqrt{T} \|\hat{\beta}_1 - \hat{\beta}_2\|\) does not diverge when increasing from \(n = 10^6\) to \(n = 10^7\) in (i). In this case, it appears that \(\sqrt{T} \|\hat{\beta}_1 - \hat{\beta}_2\|\) is bounded in probability. Meanwhile, from Figure 7, \(\sqrt{T} \|\hat{\beta}_1 - \hat{\beta}_2\|\) diverges when increasing from \(n = 10^6\) to \(n = 10^7\) in (iii). According to Subsection 3.3, we can infer that the drift parameter changes at the same time as the diffusion parameter in (iii). However, as we can see by comparing Figures 3(B) and 7(B), it would be difficult to determine whether the drift parameter changes at the same point as the diffusion parameter when \(n = 10^7\). In this case, it would be possible to determine whether there is a change when \(n = 10^9\), but this is not realistic.

**Table 2.** Proportions over the corresponding critical value.

| \(n\) | \(T\)     | \(h\)             | \(\hat{T}_{1,n}^{(1)}\) | \(\hat{T}_{1,n}^{(2)}\) | \(\hat{T}_{2,n}^{(1)}\) | \(\hat{T}_{2,n}^{(2)}\) |
|-------|-----------|-------------------|---------------------|---------------------|-----------------------------|-----------------------------|
| \(10^6\) | 758.58    | \(7.59 \times 10^{-4}\) | (i) 0.040           | 0.034               | 0.045                      | 0.048                      |
|        |           |                   | (ii) 0.784          | 0.704               | 0.045                      | 0.046                      |
|        |           |                   | (iii) 0.040         | 0.034               | 0.046                      | 0.048                      |
| \(10^7\) | 2290.87   | \(2.29 \times 10^{-4}\) | (i) 0.048           | 0.053               | 0.040                      | 0.046                      |
|        |           |                   | (ii) 0.981          | 0.944               | 0.040                      | 0.045                      |
|        |           |                   | (iii) 0.048         | 0.053               | 0.040                      | 0.046                      |

On the other hand, according to Table 2 and (A)-(D) of Figure 4, it can be seen that the proportions of the test statistics \(\hat{T}_{1,n}^{(1)}\) and \(\hat{T}_{2,n}^{(2)}\) that exceed the critical value approach 1.000 as \(n\) increases, and the distribution of the test statistics diverges in (ii). We can also see from (E)-(F) of
Figure 2. Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in (i).

Table 3. Mean and standard deviation of the estimators in (i). True values: $\beta^* = 1$, $\gamma^* = 2$.

| $n$  | $T$    | $h$     | $\hat{\beta}_1$ | $\hat{\gamma}_1$ | $\hat{\beta}_2$ | $\hat{\gamma}_2$ |
|------|--------|---------|------------------|-------------------|------------------|------------------|
| $10^6$ | 758.58 | $7.59 \times 10^{-4}$ | 1.00700         | 1.99850          | 1.02695         | 1.99989          |
|      |        |         | (0.05871)       | (0.04084)        | (0.12165)       | (0.09591)        |
| $10^7$ | 2290.87 | $2.29 \times 10^{-4}$ | 1.00211         | 2.00024          | 1.00837         | 1.99797          |
|      |        |         | (0.03321)       | (0.02422)        | (0.06541)       | (0.05436)        |

Figure 3. Histogram of $\sqrt{T}||\hat{\beta}_1 - \hat{\beta}_2||$ in (i).

Figure 4 that the distribution of the test statistic $T_{1,n}^{(2)}$ almost corresponds with the null distribution. Therefore, we estimate the drift parameters before and after the change by the same procedure as Steps (2)-(3), and also estimate the change point of the drift parameter when the test statistic $T_{1,n}^{(1)}$ exceeds the critical value. Here, we construct the estimators by looking for the intervals with no change point. Specifically, we first test for changes in the drift parameter in $[0.25\tau_n T, 0.75\tau_n T]$. If the change is detected, we construct $\hat{\beta}_1$ from $[0, 0.25\tau_n T]$ and $\hat{\beta}_2$ from $[0.75\tau_n T, \tau_n T]$. If no change is detected, we next expand the test interval to $[0.125\tau_n T, 0.875\tau_n T]$, $[0.0625\tau_n T, 0.9375\tau_n T]$, and...
Figure 4. Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in (ii).

$[0.01 \tau, 0.99 \tau]$ and if the change is detected in the expanded interval, we estimate $\beta_1^*$ and $\beta_2^*$ using the data in the intervals that are not used in the test. The results of these estimates are shown in Table 4 and Figure 5. We can see that the distribution of the estimator almost corresponds with the asymptotic distribution and the estimator has good performance from Figure 5. In this simulation, we considered the situation that the difference between $\gamma_1^*$ and $\gamma_2^*$ shrinks.

As we mentioned in Remark 11, the change point can also be estimated when the difference is fixed.

Table 4. Mean and standard deviation of the estimators in (ii). True values: $\beta^* = 1$, $\gamma_2^* = 2$, $\tau_1^B = 0.4$, $\gamma_1^* \approx 1.7488$ and $1.8005$ for $n = 10^6$ and $10^7$, respectively.

| $n$     | $T$     | $h$     | $\hat{\beta}_1$ | $\hat{\gamma}_1$ | $\hat{\beta}_2$ | $\hat{\gamma}_2$ | $\hat{\tau}_1^B$ |
|---------|---------|---------|------------------|-------------------|------------------|-------------------|-----------------|
| $10^6$  | 758.58  | $7.59 \times 10^{-4}$ | 1.09318           | 1.73533           | 1.11949          | 2.00802           | 0.40789         |
|         |         |         | (0.331866)       | (0.14934)         | (0.46379)        | (0.17179)         | (0.12373)       |
| $10^7$  | 2290.87 | $2.29 \times 10^{-4}$ | 1.01099           | 1.79835           | 1.02131          | 2.00017           | 0.40229         |
|         |         |         | (0.09569)        | (0.05623)         | (0.09199)        | (0.06371)         | (0.07034)       |
Figure 5. Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) with $n = 10^7$ in (ii).

Table 5. Mean and standard deviation of the estimators in (iii). True values: $\beta^* = 1$, $\gamma_2^* = 2$, $\gamma_1^* \approx 1.74881$ and $1.80047$ for $n = 10^6$ and $10^7$, respectively.

| $n$  | $T$        | $h$         | $\hat{\beta}_1$ | $\hat{\gamma}_1$ | $\hat{\beta}_2$ | $\hat{\gamma}_2$ | $\hat{\beta}_1$ | $\hat{\gamma}_1$ | $\hat{\beta}_2$ | $\hat{\gamma}_2$ |
|------|------------|-------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| $10^6$ | 758.58     | $7.59 \times 10^{-4}$ | 1.00714          | 1.74722          | 1.02731          | 1.99986          | (0.05874)       | (0.04078)       | (0.12144)       | (0.09612)       |
| $10^7$ | 2290.87    | $2.29 \times 10^{-4}$ | 1.00215          | 1.80070          | 1.00840          | 1.99795          | (0.03331)       | (0.02423)       | (0.06540)       | (0.05432)       |

Figure 6. Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in (iii).
5.2. Model 2. We consider the hyperbolic diffusion model defined by
\[ dX_t = \left( \beta - \frac{\gamma X_t}{\sqrt{1 + X_t^2}} \right) dt + \alpha dW_t, \quad X_0 = x_0, \]
where \( \alpha > 0, \beta \in \mathbb{R} \) and \( |\beta| < \gamma \). In order to investigate the asymptotic performance of the test statistics and the estimator, we treat the following stochastic differential equation
\[ X_t = \begin{cases} 
X_0 + \int_0^t \left( \beta - \frac{\gamma X_s}{\sqrt{1 + X_s^2}} \right) ds + \alpha_1^* W_t, & t \in [0, \tau_*^\alpha T), \\
X_{\tau_*^\alpha T} + \int_{\tau_*^\alpha T}^t \left( \beta - \frac{\gamma X_s}{\sqrt{1 + X_s^2}} \right) ds + \alpha_2^* (W_t - W_{\tau_*^\alpha T}), & t \in [\tau_*^\alpha T, T], 
\end{cases} \]
where \( x_0 = 1, \tau_*^\alpha = 0.4, \alpha_1^* = 1 + n^{-0.36}, \alpha_2^* = 1, \beta = (\beta, \gamma) \).

We consider the following three situations.
(i) The drift parameter \( \beta = (1, 2) \) does not change over \([0, T]\).
(ii) The drift parameter \( \beta \) changes from \( \beta_1^* = (1, 2) \) to \( \beta_2^* = (0.5, 2) \) at \( \tau_*^\beta = 0.7 \) (Case B).
(iii) The drift parameter \( \beta \) changes from \( \beta_1^* = (1, 2) \) to \( \beta_2^* = (0.5, 2) \) at \( \tau_*^\beta = \tau_*^\alpha = 0.4 \).

The number of iteration is 1000. We set that the sample size of the data \( \{X_t\}_{t=0}^n \) is \( n = 10^6 \) or \( 10^7 \), \( h = n^{-0.625}, \quad T = nh = n^{0.375}, \quad nh^2 = n^{-0.25} \).

| \( n \) | \( T \) | \( h \) | \( \hat{\alpha}_1 \) | \( \hat{\alpha}_2 \) | \( \hat{\tau}_n^\alpha \) |
|------|------|------|------|------|------|
| \( 10^6 \) | 177.83 | 1.78 \times 10^{-4} | (i) | 1.00702 | 1.00002 | 0.39858 | (0.00170) | (0.00172) | (0.06628) |
| | | | (ii) | 1.00712 | 0.99992 | 0.39286 | (0.00176) | (0.00152) | (0.05996) |
| | | | (iii) | 1.00702 | 1.00003 | 0.39922 | (0.00172) | (0.00173) | (0.06767) |
| \( 10^7 \) | 421.70 | 4.22 \times 10^{-4} | (i) | 1.00304 | 1.00001 | 0.39855 | (0.00045) | (0.00050) | (0.02576) |
| | | | (ii) | 1.00304 | 1.00003 | 0.39871 | (0.00045) | (0.00047) | (0.02728) |
| | | | (iii) | 1.00304 | 1.00002 | 0.39855 | (0.00045) | (0.00051) | (0.02635) |

We test for changes in the diffusion parameter in the interval \([0, T]\) in 1000 iterations. In all situations (i)-(iii), the change was detected 990 times when \( n = 10^6 \) and 1000 times when \( n = 10^7 \). If the change in the diffusion parameter is detected, the next step is to estimate the parameters \( \alpha_1^* \) and \( \alpha_2^* \) in the same way to estimate \( \beta_1^* \) and \( \beta_2^* \) in situation (ii) of model 1, and estimate \( \tau_*^\alpha \) using...
the estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$. The estimates of $\alpha^*_1$, $\alpha^*_2$ and $\tau^\alpha$ are shown in Table 6. In this case, we chose $\epsilon_1 = 0.9 + 1.8 \log_n |\hat{\alpha}_1 - \hat{\alpha}_2|$ for all iterations.

Next, we test for changes in the drift parameter in the intervals $[0, \tau_n T]$ and $[\tau_n T, T]$. Table 7 and Figures 8, 10 and 12 show the simulation results of the tests for changes in the drift parameter. In (i) and (iii), it can be seen that the test statistics have good performance from Table 7, Figures 8 and 12. Hence, in (i) and (iii), we construct $\hat{\beta}_1$ and $\hat{\beta}_2$ from the intervals $[0, \tau_n T]$ and $[\tau_n T, T]$, respectively when the test statistics $T^{(1)}_{1,n}$ and $T^{(2)}_{1,n}$ do not exceed the critical value, and investigate $\sqrt{T} \|\hat{\beta}_1 - \hat{\beta}_2\|$. The results of the estimates of $\beta^*_1$ and $\beta^*_2$ in (i) and (iii) are summarized in Table 8 and Figure 9, and Table 10 and Figure 13, respectively. It can be seen that $\sqrt{T} \|\hat{\beta}_1 - \hat{\beta}_2\|= O_p(1)$ in (i), and $\sqrt{T} \|\hat{\beta}_1 - \hat{\beta}_2\|$ diverges in (iii) from Figures 9 and 13, respectively.

| $n$  | $T$     | $h$     | $T_{1,n}^{(1)}$ | $T_{2,n}^{(1)}$ | $T_{1,n}^{(2)}$ | $T_{2,n}^{(2)}$ |
|------|---------|---------|----------------|----------------|----------------|----------------|
| $10^6$ | 177.83  | $1.78 \times 10^{-4}$ | (i) 0.035 0.043 0.060 | (35/990) (43/990) (59/990) | (50/990) |
|      |         |         | (ii) 0.034 0.060 0.510 | (34/990) (59/990) (505/990) | (410/990) |
|      |         |         | (iii) 0.043 0.071 0.052 | (43/990) (70/990) (51/990) | (64/990) |
| $10^7$ | 421.70  | $4.22 \times 10^{-4}$ | (i) 0.038 0.040 0.038 | (35/990) (43/990) (59/990) | (50/990) |
|      |         |         | (ii) 0.040 0.040 0.941 | (34/990) (59/990) (505/990) | (410/990) |
|      |         |         | (iii) 0.043 0.049 0.044 | (43/990) (70/990) (51/990) | (64/990) |

Table 7. Proportions of the corresponding critical values exceeded.

Figure 8. Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in (i).

From Table 7 and (A)-(B) of Figure 10, we see that in (ii), the distribution of the test statistic $T_{2,n}^{(1)}$ almost corresponds with the null distribution. Moreover, it can be seen from Table 7 and (C)-(F) of Figure 10 that the proportions of the test statistics $T_{1,n}^{(2)}$ and $T_{2,n}^{(2)}$ that exceed the
critical value approach $1.000$ as $n$ increases, and the distribution of the test statistics $T_{2,n}^{(2)}$ diverges in (ii). Therefore, in (ii), we estimate the drift parameters before and after the change, and also estimate the change point when the test statistic $T_{1,n}^{(1)}$ exceeds the critical value. Here, we construct the estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ in the same way as in situation (ii) of model 1. The results of these estimates are shown in Table 9 and Figure 11. From Figure 11, we can see that the distribution of the estimator does not diverge when increasing from $n = 10^6$ to $n = 10^7$, which implies that the estimator has good performance.

**Table 9.** Mean and standard deviation of the estimators in (ii). True values: $\beta_1^* = 1$, $\beta_2^* = 0.5$, $\gamma_1^* = 2$, $\tau_\beta^* = 0.7$.

| $n$ | $T$ | $h$ | $\hat{\beta}_1$ | $\hat{\gamma}_1$ | $\hat{\beta}_2$ | $\hat{\gamma}_2$ | $\hat{\tau}_\beta^*$ |
|-----|-----|-----|------------------|------------------|------------------|------------------|------------------|
| $10^6$ | 177.83 | $1.78 \times 10^{-4}$ | 1.72974 | 3.09513 | 0.52996 | 2.91408 | 0.70629 |
| | | | (1.81075) | (2.15126) | (0.84344) | (1.83324) | (0.13175) |
| $10^7$ | 421.70 | $4.22 \times 10^{-4}$ | 1.14360 | 2.25463 | 0.51434 | 2.11352 | 0.69866 |
| | | | (0.59646) | (0.82162) | (0.21015) | (0.45589) | (0.06982) |

**Table 10.** Mean and standard deviation of the estimators in (iii). True values: $\beta_1^* = 1$, $\beta_2^* = 0.5$, $\gamma_1^* = 2$.

| $n$ | $T$ | $h$ | $\hat{\beta}_1$ | $\hat{\gamma}_1$ | $\hat{\beta}_2$ | $\hat{\gamma}_2$ |
|-----|-----|-----|------------------|------------------|------------------|------------------|
| $10^6$ | 177.83 | $1.78 \times 10^{-4}$ | 1.07202 | 2.15235 | 0.52224 | 2.07371 |
| | | | (0.42817) | (0.61731) | (0.27310) | (0.47234) |
| $10^7$ | 421.70 | $4.22 \times 10^{-4}$ | 1.01424 | 2.03951 | 0.50641 | 2.02274 |
| | | | (0.13744) | (0.21292) | (0.07602) | (0.15020) |
Figure 10. Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in (ii).

Figure 11. Histogram of $T(\hat{\tau}_n^\beta - \tau_\beta)$ in (ii).
6. Proofs

We set the following notations.
1. $\mathcal{F}^n_{t-1} = \sigma \{ [W_s] s \leq t^n \}.$
2. $C, C_1, C_2, \ldots > 0$ denote universal constants.
3. For a measurable set $A$ and an integrable random variable $X$, we define
   \[
   \mathbb{E}[X : A] = \int_A X(\omega) dP(\omega).
   \]
4. For a function $f$ on $\mathbb{R}^d \times \Theta$, we define $f_{t-1}(\theta) = f(X_{t-1}, \theta)$.
5. We define
   \[
   A \otimes x^\otimes k = \sum_{\ell_1, \ldots, \ell_k = 1}^{d_1} A^{\ell_1, \ldots, \ell_k} x^{\ell_1} \ldots x^{\ell_k}, \quad \text{for } A \in \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_1}, \ x \in \mathbb{R}^{d_1}.
   \]
Lemma 1 Let $0 \leq \tau_1 < \tau_2 \leq 1$, where $\tau_1, \tau_2$ may depend on $n$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence with $r_n^2([n\tau_2] - [n\tau_1])h \longrightarrow 0$, and $\{M_n\}_{n=1}^{\infty}$ be a martingale with $E[\|M_n\|^2] \leq Ch$. If $[n\tau_1] < n_k \leq [n\tau_2]$ and $[n\tau_1] \leq k_n < [n\tau_2]$ on $\Omega_n$ with $P(\Omega_n) \longrightarrow 1$, then
\[
\lim_{n \to \infty} \left( \sum_{i=[n\tau_1]+1}^{k_n} M_i \right) = a_p(1), \quad \lim_{n \to \infty} \left( \sum_{i=\ell_n+1}^{[n\tau_2]} M_i \right) = a_p(1). \tag{6.1}
\]

Proof. Let
\[
S_n = r_n \left| \sum_{i=[n\tau_1]+1}^{k_n} M_i \right|.
\]
For all $\epsilon > 0$,
\[
P(S_n > \epsilon) \leq P(S_n > \epsilon, \Omega_n) + P(\Omega_n^c) \leq \frac{1}{\epsilon^2} E[|S_n^2| : \Omega_n] + P(\Omega_n^c). \tag{6.2}
\]

From Burkholder inequality, we have
\[
E[|S_n^2| : \Omega_n] \leq r_n^2 E\left[ \max_{[n\tau_1] < k \leq [n\tau_2]} \left\| \sum_{i=[n\tau_1]+1}^{k} M_i \right\|^2 : \Omega_n \right] \\
\leq r_n^2 \frac{\epsilon^2}{\epsilon} E\left[ \max_{[n\tau_1] < k \leq [n\tau_2]} \left\| \sum_{i=[n\tau_1]+1}^{k} M_i \right\|^2 \right] \\
\leq C r_n^2 \sum_{i=[n\tau_1]+1}^{[n\tau_2]} E[\|M_i\|^2] \\
= O(r_n^2([n\tau_2] - [n\tau_1])h) = o(1). \tag{6.3}
\]

Therefore, from (6.2), (6.3) and $P(\Omega_n^c) \longrightarrow 0$, we have the first part of (6.1). According to
\[
\left\| \sum_{i=\ell_n+1}^{[n\tau_2]} M_i \right\|^2 \leq 2 \left( \left\| \sum_{i=[n\tau_1]+1}^{[n\tau_2]} M_i \right\|^2 + \left\| \sum_{i=\ell_n+1}^{[n\tau_2]} M_i \right\|^2 \right),
\]
the second part of (6.1) is obtained in the same way. \hfill \square

Let $\theta_k = (\alpha_k^*, \beta_k)$, $\tau_n^L = \tau_n^* - 2n^{-\epsilon_1}$, $\tau_n^U = \tau_n^* + 2n^{-\epsilon_1}$, $m_n = [n\tau_n^L]$ and $M_n = [n\tau_n^U]$. 

Proof of Theorem 3 We first prove (5.1). Let
\[
\mathcal{L}_{1,n}^{(1)} = \frac{1}{\sqrt{d_{n\tau_n}}} \max_{1 \leq k \leq m_n} \sum_{i=1}^{k} \xi_{1,i} - \frac{\epsilon}{n\tau_n} \sum_{i=1}^{[n\tau_n]} \xi_{1,i},
\]
\[
\mathcal{U}_{1,n}^{(1)} = \frac{1}{\sqrt{d_{n\tau_n}}} \max_{1 \leq k \leq [n\tau_n]} \sum_{i=1}^{k} \xi_{1,i} - \frac{\epsilon}{n\tau_n} \sum_{i=1}^{[n\tau_n]} \xi_{1,i}
\]
and $D_n = \{n^{\epsilon_1} | \tau_n^* - \tau_n^o| \leq 1\}$. Note that the probability of $D_n$ converges to one from [E1], and $m_n \leq [n\tau_n^L] \leq [n\tau_n^*] \leq [n\tau_n] \leq M_n$ on $D_n$. Since $\mathcal{L}_{1,n}^{(1)} \leq \mathcal{T}_{1,n}^{(1)} \leq \mathcal{U}_{1,n}^{(1)}$ on $D_n$, if
\[
\mathcal{L}_{1,n}^{(1)} \longrightarrow \sup_{0 \leq s \leq 1} |B_1^0(s)|, \tag{6.4}
\]
\[
\mathcal{U}_{1,n}^{(1)} \longrightarrow \sup_{0 \leq s \leq 1} |B_1^0(s)|, \tag{6.5}
\]
then we have
\[
\lim_{n \to \infty} P(\mathcal{T}_{1,n}^{(1)} \leq x, D_n) \leq \lim_{n \to \infty} P(\mathcal{L}_{1,n}^{(1)} \leq x, D_n) \leq \lim_{n \to \infty} P(\mathcal{L}_{1,n}^{(1)} \leq x) \leq P\left( \sup_{0 \leq s \leq 1} |B_1^0(s)| \leq x \right),
\]
\[
\lim_{n \to \infty} P(\mathcal{T}_{1,n}^{(1)} > x, D_n) \leq \lim_{n \to \infty} P(\mathcal{U}_{1,n}^{(1)} > x, D_n) \leq \lim_{n \to \infty} P(\mathcal{U}_{1,n}^{(1)} > x) \leq P\left( \sup_{0 \leq s \leq 1} |B_1^0(s)| > x \right).
\]
Since
\[
\lim_{n \to \infty} P(\mathcal{T}_{1,n}^{(1)} > x) \leq \lim_{n \to \infty} P(\mathcal{T}_{1,n}^{(1)} > x, D_n),
\]
we see
\[
P \left( \sup_{0 \leq s \leq 1} |B_1^0(s)| \leq x \right) \leq \lim_{n \to \infty} P(T_{1,n}^{(1)} \leq x, D_n)
\]
\[
\leq \lim_{n \to \infty} P(T_{1,n}^{(1)} \leq x, D_n) \leq P \left( \sup_{0 \leq s \leq 1} |B_1^0(s)| \leq x \right),
\]
i.e.,
\[
\lim_{n \to \infty} P(T_{1,n}^{(1)} \leq x, D_n) = P \left( \sup_{0 \leq s \leq 1} |B_1^0(s)| \leq x \right). \tag{6.6}
\]
Hence, from (6.6), \( P(D_n) \to 1 \) and \( P(T_{1,n}^{(1)} \leq x) = P(T_{1,n}^{(1)} \leq x, D_n) + P(T_{1,n}^{(1)} \leq x, D_n^c) \), we obtain
\[
T_{1,n}^{(1)} \xrightarrow{d} \sup_{0 \leq s \leq 1} |B_1^0(s)|.
\]
From the above, it suffices to show (6.4) and (6.5).

**Proof of (6.4)** We have
\[
\sum_{i=1}^{k} \xi_{1,i} - \frac{k}{|n \Sigma_n|} \sum_{i=1}^{|n \Sigma_n|} \xi_{1,i}
\]
\[
= \sum_{i=1}^{k} \xi_{1,i} - \frac{k}{m_n} \sum_{i=1}^{m_n} \xi_{1,i} + \frac{k}{m_n} \left( 1 - \frac{m_n}{|n \Sigma_n|} \right) \sum_{i=1}^{m_n} \xi_{1,i} - \frac{k}{|n \Sigma_n|} \sum_{i=m_n+1}^{n \Sigma_n} \xi_{1,i}. \tag{6.7}
\]
Let \( \xi_{k,i} = 1_{\hat{d}^{-1}}(\alpha_k)(\Delta X_i - h b_{i-1}(\hat{\beta}_k)) \), \( \mathcal{M}_{k,i} = \xi_{k,i} - \mathbb{E}_{\theta_k}[\xi_{k,i} | \mathcal{G}^{n-1}_n] \). Here, noting that
\[
\xi_{k,i} = 1_{\hat{d}^{-1}}(\alpha_k)(\Delta X_i - h b_{i-1}(\hat{\beta}_k))
\]
\[
= 1_{\hat{d}^{-1}}(\alpha_k)(\Delta X_i - h b_{i-1}(\hat{\beta}_k))
\]
\[
+ \frac{1}{\sqrt{\hat{\theta}}} \int_0^1 \partial_\alpha \left( 1_{\hat{d}^{-1}}(\alpha)(\Delta X_i - h b_{i-1}(\hat{\beta}_k)) \right) \bigg|_{\alpha = \hat{\alpha}_k + u(\hat{\alpha}_k - \alpha)} du \sqrt{\hat{\theta}}(\hat{\alpha}_k - \alpha)
\]
\[
= 1_{\hat{d}^{-1}}(\alpha_k)(\Delta X_i - h b_{i-1}(\hat{\beta}_k)) - h 1_{\hat{d}^{-1}}(\alpha_k)(b_{i-1}(\hat{\beta}_k) - b_{i-1}(\hat{\beta}_k))
\]
\[
+ \frac{1}{\sqrt{\hat{\theta}}} \int_0^1 \partial_\beta \left( 1_{\hat{d}^{-1}}(\alpha_k)(b_{i-1}(\beta) - b_{i-1}(\beta)) \right) \bigg|_{\beta = \hat{\beta}_k + u(\hat{\beta}_k - \beta)} du \sqrt{\hat{\theta}}(\hat{\beta}_k - \beta)
\]
\[
= \xi_{k,i} - \mathcal{M}_{k,i} + \mathbb{E}_{\theta_k}[\xi_{k,i} | \mathcal{G}^{n-1}_n] + O_p \left( \frac{1}{\sqrt{n}} \right)
\]
\[
= \mathcal{M}_{k,i} + \mathbb{E}_{\theta_k}[\xi_{k,i} | \mathcal{G}^{n-1}_n] + O_p \left( \frac{1}{\sqrt{n}} \right).
\]
we have
\[
\frac{1}{\sqrt{T}} \max_{1 \leq k \leq m_n} \left| \frac{k}{m_n} \left( 1 - \frac{m_n}{|n \Sigma_n|} \right) \sum_{i=1}^{m_n} \xi_{1,i} \right|
\]
\[
\leq \frac{1}{\sqrt{T}} \left| \frac{|n \Sigma_n| - m_n}{|n \Sigma_n|} \sum_{i=1}^{m_n} \xi_{1,i} \right|
\]
\[
\leq \frac{1}{\sqrt{T}} \left| \frac{|n \Sigma_n| - m_n}{|n \Sigma_n|} \left( \sum_{i=1}^{m_n} \mathcal{M}_{1,i} \right) + m_n O_p \left( \frac{1}{\sqrt{n}} \right) \right|
\]
\[
= \frac{m_n}{|n \Sigma_n|} n^{1-\epsilon_1} \left| \frac{|n \Sigma_n| - m_n}{n^{1-\epsilon_1}} \sum_{i=1}^{m_n} \mathcal{M}_{1,i} \right| + \frac{n^{1-\epsilon_1}}{\sqrt{T}} O_p \left( \frac{1}{\sqrt{n}} \right) \tag{6.8}
\]
Since $\frac{m_n}{n^{1-\varepsilon_1}} [n^{\tau_n}] - m_n = O_p(1)$, $\frac{n^{1-\varepsilon_1}}{\sqrt{T}} \sqrt{\frac{K}{n}} = n^{-\varepsilon_1} \rightarrow 0$, $\mathbb{E}_0 |\mathcal{M}_{1,i}^2| \leq Ch$ and

$$\frac{n^{2-2\varepsilon_1} m_n h}{Tm_n^2} = O(n^{-2\varepsilon_1}) = o(1),$$

we see

$$\frac{1}{\sqrt{T}} \max_{1 \leq k \leq m_n} \left| \frac{k}{m_n} \left( 1 - \frac{m_n}{[n^{\tau_n}]} \right) \sum_{i=1}^{[n^{\tau_n}] - 1} \hat{\xi}_{1,i} \right| = o_p(1) \quad (6.9)$$

from Lemma 1 and (6.8). In the same way as (6.8), we have

$$\frac{1}{\sqrt{T}} \max_{1 \leq k \leq m_n} \left| \frac{k}{[n^{\tau_n}] \sum_{i=m_n+1}^{[n^{\tau_n}] - 1} \hat{\xi}_{1,i} \right| \leq \frac{m_n}{[n^{\tau_n}]} \frac{1}{\sqrt{T}} \left| \sum_{i=m_n+1}^{[n^{\tau_n}] - 1} \mathcal{M}_{1,i} \right| + o_p(1).$$

Since $[n^{\tau_n}] \leq [n^{\tau_n}]$ on $D_n$ and

$$\frac{([n^{\tau_n}] - m_n) h}{T} = O(n^{-\varepsilon_1}) = o(1),$$

we obtain

$$\frac{1}{\sqrt{T}} \max_{1 \leq k \leq m_n} \left| \frac{k}{[n^{\tau_n}]} \sum_{i=m_n+1}^{[n^{\tau_n}] - 1} \hat{\xi}_{1,i} \right| = o_p(1) \quad (6.10)$$

from Lemma 1. According to (6.7), (6.9) and (6.10), we can express that

$$\mathcal{L}_{1,n}^{(1)} = \frac{1}{\sqrt{d_{\tau_n} T}} \max_{1 \leq k \leq m_n} \left| \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} - \frac{[n^{\tau_n}]}{k} \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} \right|$$

and

$$= \frac{1}{\sqrt{d_{\tau_n} T}} \max_{1 \leq k \leq m_n} \left| \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} - \frac{[n^{\tau_n}]}{k} \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} \right| + o_p(1).$$

In the same proof as Theorem 2 of Tonaki et al. (2020), we obtain

$$= \sqrt{\frac{1}{2} \frac{1}{d_{\tau_n}} \frac{1}{\sqrt{d_{\tau_n} T}}} \max_{1 \leq k \leq m_n} \left| \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} - \frac{[n^{\tau_n}]}{k} \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} \right| \overset{d}{\rightarrow} \sup_{0 \leq s \leq 1} \left| B^0_{\mathcal{L}}(s) \right|,$$

which concludes the proof of (6.4).

Proof of (6.5). We have

$$\sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} = \frac{[n^{\tau_n}]}{[n^{\tau_n}]} \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i}$$

and

$$\sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} = \frac{[n^{\tau_n}]}{[n^{\tau_n}]} \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} + \left( 1 - \frac{[n^{\tau_n}]}{[n^{\tau_n}]} \right) \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} + \frac{[n^{\tau_n}]}{[n^{\tau_n}]} \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i},$$

Therefore, by the same argument, we obtain

$$\mathcal{U}_{1,n}^{(1)} = \frac{1}{\sqrt{d_{\tau_n} T}} \max_{1 \leq k \leq [n^{\tau_n}]} \left| \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} - \frac{[n^{\tau_n}]}{k} \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} \right|$$

and

$$= \frac{1}{\sqrt{d_{\tau_n} T}} \max_{1 \leq k \leq [n^{\tau_n}]} \left| \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} - \frac{[n^{\tau_n}]}{k} \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} \right| + o_p(1)$$

and

$$= \sqrt{\frac{1}{2} \frac{1}{d_{\tau_n}} \frac{1}{\sqrt{d_{\tau_n} T}}} \max_{1 \leq k \leq [n^{\tau_n}]} \left| \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} - \frac{[n^{\tau_n}]}{k} \sum_{i=1}^{[n^{\tau_n}]} \hat{\xi}_{1,i} \right| + o_p(1).$$
Next, we prove (3.2). Let

\[ L_{1,n}^{(2)} = \frac{1}{d(1 - \tau_n)} \max_{1 \leq k \leq n - M_n} \left| \sum_{i = [n\tau_n] + 1}^{[n\tau_n] + k} \xi_{2,i} \right|, \]

\[ U_{1,n}^{(2)} = \frac{1}{d(1 - \tau_n)} \max_{1 \leq k \leq n - [n\tau_n]} \left| \sum_{i = [n\tau_n] + 1}^{[n\tau_n] + k} \bar{\xi}_{2,i} \right|. \]

Since \( L_{1,n}^{(2)} \leq U_{1,n}^{(2)} \leq U_{1,n}^{(2)} \) on \( D_n \), it is enough to show

\[ L_{1,n}^{(2)} \xrightarrow{d} \sup_{0 \leq s \leq 1} |B_i^{(s)}(s)|, \quad (6.11) \]

\[ U_{1,n}^{(2)} \xrightarrow{d} \sup_{0 \leq s \leq 1} |B_i^{(s)}(s)|, \quad (6.12) \]

similar to the proof of (3.1).

**Proof of (6.11).** We can express that

\[
\begin{align*}
\sum_{i = [n\tau_n] + 1}^{[n\tau_n] + k} \xi_{2,i} &= \sum_{i = [n\tau_n] + 1}^{[n\tau_n] + k} \xi_{2,i} \\
&= \sum_{i = M_n + 1}^{M_n + k} \xi_{2,i} + \sum_{i = [n\tau_n] + 1}^{M_n + k} \xi_{2,i} - \sum_{i = [n\tau_n] + 1}^{M_n + k} \xi_{2,i} \\
&= \sum_{i = M_n + 1}^{M_n + 1} \xi_{2,i} + \sum_{i = [n\tau_n] + 1}^{M_n + 1} \xi_{2,i} - \sum_{i = [n\tau_n] + 1}^{M_n + 1} \xi_{2,i} \\
&+ \sum_{i = M_n + 1}^{M_n + 1} \xi_{2,i} + \sum_{i = [n\tau_n] + 1}^{M_n + 1} \xi_{2,i} \\
&= \sum_{i = M_n + 1}^{M_n + 1} \xi_{2,i} + \sum_{i = [n\tau_n] + 1}^{M_n + 1} \xi_{2,i} \\
&+ \sum_{i = M_n + 1}^{M_n + 1} \xi_{2,i} + \sum_{i = [n\tau_n] + 1}^{M_n + 1} \xi_{2,i}.
\end{align*}
\]

Here, we have

\[
\frac{1}{\sqrt{T}} \max_{1 \leq k \leq n - M_n} \left| \frac{k}{n - M_n} \left( 1 - \frac{n - M_n}{n - [n\tau_n]} \right) \sum_{i = M_n + 1}^{n} \xi_{2,i} \right|
\]

\[
\leq \frac{1}{\sqrt{T}} \left| n - [n\tau_n] - M_n \right| \sum_{i = M_n + 1}^{n} \xi_{2,i}
\]

\[
\leq \frac{1}{\sqrt{T}} n \left( n - [n\tau_n] \right) \left( \sum_{i = M_n + 1}^{n} \mathcal{M}_{2,i} \right) + (n - M_n) O_p \left( \sqrt{\frac{h}{n}} \right)
\]

\[
= \frac{n - M_n}{n - [n\tau_n]} n^{\epsilon_1 - 1} [n\tau_n] - M_n \left( \frac{n^{1 - \epsilon_1}}{\sqrt{T(n - M_n)}} \sum_{i = M_n + 1}^{n} \mathcal{M}_{2,i} \right) + \frac{n^{1 - \epsilon_1}}{\sqrt{T}} O_p \left( \sqrt{\frac{h}{n}} \right). \quad (6.14)
\]

Since \( \frac{n - M_n}{n - [n\tau_n]} n^{\epsilon_1 - 1} [n\tau_n] - M_n = O_p(1) \), \( \mathbb{E}_2[\mathcal{M}_{2,i}] \leq C h \) and

\[
\frac{n^{2 - 2\epsilon_1} h (n - M_n)}{T(n - M_n)^2} = O(n^{-2\epsilon_1}) = o(1),
\]

we have

\[
\frac{1}{\sqrt{T}} \max_{1 \leq k \leq n - M_n} \left| \frac{k}{n - M_n} \left( 1 - \frac{n - M_n}{n - [n\tau_n]} \right) \sum_{i = M_n + 1}^{n} \xi_{2,i} \right| = o_p(1) \quad (6.15)
\]
from Lemma \[1\] and (6.14). Similarly, we see

\[
\frac{1}{\sqrt{T}} \max_{1 \leq k \leq n-M_n} \left| \frac{1}{n - [n\tau_n]} \sum_{i=[n\tau_n]+1}^{M_n} \xi_{2,i} \right| \leq \frac{1}{\sqrt{T}} \max_{1 \leq k \leq n-M_n} \left| \sum_{i=[n\tau_n]+1}^{M_n} M_{2,i} \right| + o_p(1).
\]

Since \([n\tau_n^2] \leq [n\tau_n]\) on \(D_n\) and

\[
h(M_n - [n\tau_n^2]) = O(n^{-\epsilon_1}) = o(1),
\]

we obtain

\[
\frac{1}{\sqrt{T}} \max_{1 \leq k \leq n-M_n} \left| \frac{1}{n - [n\tau_n]} \sum_{i=[n\tau_n]+1}^{M_n} \xi_{2,i} \right| = o_p(1) \tag{6.16}
\]

from Lemma \[1\]. In the same way as (6.14), we have

\[
\frac{1}{\sqrt{T}} \max_{1 \leq k \leq n-M_n} \left| \sum_{i=[n\tau_n]+k+1}^{M_n+k} \xi_{2,i} \right| \leq \frac{1}{\sqrt{T}} \max_{1 \leq k \leq n-M_n} \left| \sum_{i=[n\tau_n]+k+1}^{M_n+k} M_{2,i} \right| + o_p(1)
\]

=: \(Q_n + o_p(1)\). \tag{6.17}

For all \(\epsilon > 0\),

\[
P(Q_n > 2\epsilon) \leq P(Q_n > 2\epsilon, D_n) + P(D_n^c).
\]

Here, the first term on the right hand side can be transformed as follows.

\[
P(Q_n > 2\epsilon, D_n)
\]

\[
\leq P \left( \frac{1}{\sqrt{T}} \max_{[n\tau_n^2] \leq \ell \leq M_n} \max_{1 \leq k \leq n-M_n} \left| \sum_{i=\ell+k+1}^{M_n+k} M_{2,i} \right| > 2\epsilon, D_n \right)
\]

\[
\leq P \left( \frac{1}{\sqrt{T}} \max_{[n\tau_n^2] \leq \ell \leq M_n} \max_{1 \leq k \leq n-M_n} \left| \sum_{i=\ell+k+1}^{M_n+k} M_{2,i} \right| > 2\epsilon \right)
\]

\[
= P \left( \frac{1}{\sqrt{T}} \max_{[n\tau_n^2] \leq \ell \leq M_n} \max_{1 \leq k \leq n-M_n} \left| \sum_{i=[n\tau_n]^2+k+1}^{M_n+k} M_{2,i} - \sum_{i=[n\tau_n]^2+k+1}^{\ell+k} M_{2,i} \right| > 2\epsilon \right)
\]

\[
\leq P \left( \frac{1}{\sqrt{T}} \max_{1 \leq k \leq n-M_n} \left| \sum_{i=[n\tau_n]^2+k+1}^{M_n+k} M_{2,i} \right| + \frac{1}{\sqrt{T}} \max_{1 \leq k \leq n-M_n} \left| \sum_{i=[n\tau_n]^2+k+1}^{\ell+k} M_{2,i} \right| > 2\epsilon \right)
\]

\[
\leq P \left( \frac{1}{\sqrt{T}} \max_{1 \leq k \leq n-M_n} \left| \sum_{i=[n\tau_n]^2+k+1}^{M_n+k} M_{2,i} \right| > \epsilon \right) + P \left( \frac{1}{\sqrt{T}} \max_{1 \leq k \leq n-M_n} \left| \sum_{i=[n\tau_n]^2+k+1}^{\ell+k} M_{2,i} \right| > \epsilon \right) \tag{6.19}
\]

We choose \(r > \frac{2-\epsilon_1}{\epsilon_3}\). Noting that \(\epsilon_1 r > 2 - \epsilon_3 > 1\), we see, from Theorem 2.11 of Hall and Heyde (1980), convex inequality and \(E_{\theta_d} [M_{2,i}^r] \leq C h^r\),

\[
P \left( \frac{1}{\sqrt{T}} \max_{1 \leq k \leq n-M_n} \left| \sum_{i=[n\tau_n]^2+k+1}^{M_n+k} M_{2,i} \right| > \epsilon \right)
\]

\[
= P \left( \bigcup_{k=1}^{n-M_n} \left\{ \frac{1}{\sqrt{T}} \left| \sum_{i=[n\tau_n]^2+k+1}^{M_n+k} M_{2,i} \right| > \epsilon \right\} \right)
\]

\[
\leq \sum_{k=1}^{n-M_n} P \left( \frac{1}{\sqrt{T}} \left| \sum_{i=[n\tau_n]^2+k+1}^{M_n+k} M_{2,i} \right| > \epsilon \right)
\]

\[
\leq \sum_{k=1}^{n-M_n} \frac{1}{\sqrt{T} \epsilon r} \epsilon r \left[ \sum_{i=[n\tau_n]^2+k+1}^{M_n+k} M_{2,i} \right]^{2r}.
\]
\[
\begin{align*}
&\leq \sum_{k=1}^{n-M_n} \frac{C_1}{T_\tau \epsilon^{2r}} \mathbb{E}_{\theta_2} \left[ \left( \sum_{i=[n\tau^n]+k+1}^{M_n+k} \mathcal{M}_{2,i} \right)^{\tau} \right] \\
&\leq \sum_{k=1}^{n-M_n} \frac{C_2}{T_\tau \epsilon^{2r}} \left[ n^{1-\epsilon_1} r \right] \frac{1}{\lceil n^{1-\epsilon_1} \rceil} \sum_{i=[n\tau^n]+k+1}^{M_n+k} \mathbb{E}_{\theta_2}[\mathcal{M}_{2,i}^{2r}] \\
&= O\left( \frac{n^{1+r-\epsilon_1} r^r}{T_\tau} \right) = O(n^{1-\epsilon_1 r}) = o(1) \\
\end{align*}
\]

and
\[
\begin{align*}
&\frac{1}{\sqrt{T}} \max_{|n\tau^n|+1 \leq k \leq n-M_n} \max_{i=[n\tau^n]+k+1} \left| \sum_{k=1}^{M_n} \sum_{k=1}^{n-M_n} \frac{1}{\sqrt{T}} \sum_{i=[n\tau^n]+k+1}^{\ell+k} \mathcal{M}_{2,i} \right| > \epsilon \\
&= P\left( \frac{1}{\sqrt{T}} \max_{|n\tau^n|+1 \leq k \leq n-M_n} \left\{ \sum_{k=1}^{M_n} \sum_{k=1}^{n-M_n} \frac{1}{\sqrt{T}} \sum_{i=[n\tau^n]+k+1}^{\ell+k} \mathcal{M}_{2,i} \right\} > \epsilon \right) \\
&\leq \sum_{\ell=|n\tau^n|+1}^{M_n} \sum_{k=1}^{n-M_n} P\left( \frac{1}{\sqrt{T}} \sum_{i=[n\tau^n]+k+1}^{\ell+k} \mathcal{M}_{2,i} > \epsilon \right) \\
&\leq \sum_{\ell=|n\tau^n|+1}^{M_n} \sum_{k=1}^{n-M_n} \frac{C_1}{T_\tau \epsilon^{2r}} \mathbb{E}_{\theta_2} \left[ \left( \sum_{i=[n\tau^n]+k+1}^{M_n+k} \mathcal{M}_{2,i} \right)^{2r} \right] \\
&\leq \sum_{\ell=|n\tau^n|+1}^{M_n} \sum_{k=1}^{n-M_n} \frac{C_2}{T_\tau \epsilon^{2r}} \left[ n^{1-\epsilon_1} \right] \frac{1}{\lceil n^{1-\epsilon_1} \rceil} \sum_{i=[n\tau^n]+k+1}^{M_n+k} \mathbb{E}_{\theta_2}[\mathcal{M}_{2,i}^{2r}] \\
&\leq \sum_{\ell=|n\tau^n|+1}^{M_n} \sum_{k=1}^{n-M_n} \sum_{k=1}^{n-M_n} \frac{C_3}{T_\tau \epsilon^{2r}} \left[ n^{1-\epsilon_1} \right] h^r \\
&= O(n^{1-\epsilon_1+r-\epsilon_1 r}) = O(n^{2-\epsilon_1-\epsilon_1 r}) = o(1). 
\end{align*}
\]
Here, taking into account that
\[
\sum_{i=1}^{m} \hat{\zeta}_{1,i} = \frac{k}{n} \sum_{i=1}^{[n\tau_{\theta}]} \hat{\zeta}_{1,i} = \frac{k}{n} \sum_{i=1}^{[n\tau_{\theta}]} \zeta_{1,i} + \frac{[n\tau_{\theta}]}{n} \sum_{i=1}^{m} \hat{\zeta}_{1,i} = \frac{k}{n} \sum_{i=1}^{[n\tau_{\theta}]} \zeta_{1,i} + \frac{[n\tau_{\theta}]}{n} \sum_{i=1}^{m} \hat{\zeta}_{1,i},
\]
we obtain, by the same argument,
\[
U_{1,n}^{(2)} = \frac{1}{\sqrt{d(1 - \tau_{\theta})T}} \max_{1 \leq k \leq n - [n\tau_{\theta}]} \left| \sum_{i=1}^{[n\tau_{\theta}]} \hat{\zeta}_{1,i} - \frac{k}{n} \sum_{i=1}^{[n\tau_{\theta}]} \zeta_{1,i} \right| + o_p(1)
\]
\[
= \sqrt{\frac{1 - \tau_{\theta}^{\alpha}}{1 - \tau_{\theta}^{\alpha}} \frac{1}{\sqrt{d(1 - \tau_{\theta})T}} \max_{1 \leq k \leq n - [n\tau_{\theta}]} \left| \sum_{i=1}^{[n\tau_{\theta}]} \hat{\zeta}_{1,i} - \frac{k}{n} \sum_{i=1}^{[n\tau_{\theta}]} \zeta_{1,i} \right| + o_p(1)
\]
\[
d \xrightarrow[0 \leq s \leq 1]{} \sup |B_{\theta}^{(s)}(s)|.
\]

**Proof of Theorem 2** Let
\[
L_{2,n}^{(1)} = \frac{1}{\sqrt{d(1 - \tau_{\theta})T}} \max_{1 \leq k \leq n - [n\tau_{\theta}]} \left| I_{1,n}^{-1/2} \left( \sum_{i=1}^{k} \hat{\zeta}_{1,i} - \frac{k}{n} \sum_{i=1}^{[n\tau_{\theta}]} \zeta_{1,i} \right) \right|, \quad \mathcal{L}_{2,n}^{(1)} \xrightarrow[0 \leq s \leq 1]{} \|B_{\theta}^{(s)}(s)\|,
\]
\[
U_{2,n}^{(1)} = \frac{1}{\sqrt{d(1 - \tau_{\theta})T}} \max_{1 \leq k \leq n - [n\tau_{\theta}]} \left| I_{1,n}^{-1/2} \left( \sum_{i=1}^{k} \hat{\zeta}_{1,i} - \frac{k}{n} \sum_{i=1}^{[n\tau_{\theta}]} \zeta_{1,i} \right) \right|, \quad U_{2,n}^{(1)} \xrightarrow[0 \leq s \leq 1]{} \|B_{\theta}^{(s)}(s)\|.
\]

On $D_n$, we have $L_{2,n}^{(1)} \leq \mathcal{L}_{2,n}^{(1)} \leq U_{2,n}^{(1)}$. Similar to the proof of (3.1), it is enough to show
\[
L_{2,n}^{(1)} \xrightarrow[0 \leq s \leq 1]{} \|B_{\theta}^{(s)}(s)\|, \quad (6.23)
\]
\[
U_{2,n}^{(1)} \xrightarrow[0 \leq s \leq 1]{} \|B_{\theta}^{(s)}(s)\|, \quad (6.24)
\]

**Proof of (6.23).** We have
\[
\sum_{i=1}^{k} \hat{\zeta}_{1,i} - \frac{k}{n} \sum_{i=1}^{[n\tau_{\theta}]} \zeta_{1,i}
= \sum_{i=1}^{k} \hat{\zeta}_{1,i} - \frac{m_{n}}{m} \sum_{i=1}^{m} \zeta_{1,i} + \frac{k}{m} \sum_{i=m+1}^{[n\tau_{\theta}]} \hat{\zeta}_{1,i} = \frac{k}{m} \sum_{i=m+1}^{[n\tau_{\theta}]} \hat{\zeta}_{1,i}.
\]
Let
\[
Z_{k,i}^{[\theta]} = \frac{1}{\sqrt{d(1 - \tau_{\theta})T}} \max_{1 \leq k \leq n - [n\tau_{\theta}]} \left| \sum_{i=1}^{k} \hat{\zeta}_{1,i} - \frac{k}{n} \sum_{i=1}^{[n\tau_{\theta}]} \zeta_{1,i} \right|, \quad N_{k,i}^{[\theta]} = \mathbb{E}_{\theta}[Z_{k,i}^{[\theta]}|\theta_{n}^{(k)}].
\]
Here, taking into account that
\[
\hat{\zeta}_{1,i} = \partial_{\theta} b_{n-1}(\hat{\beta}_{k})^{T} A_{n-1}^{-1}(\alpha_{k})(\Delta X_{i} - h b_{n-1}(\beta_{k}))
= \partial_{\theta} b_{n-1}(\hat{\beta}_{k})^{T} A_{n-1}^{-1}(\alpha_{k})(\Delta X_{i} - h b_{n-1}(\beta_{k}))
+ \frac{1}{\sqrt{n}} \int_{0}^{1} \partial_{\alpha} \left( \partial_{\beta} b_{n-1}(\hat{\beta}_{k})^{T} A_{n-1}^{-1}(\alpha)(\Delta X_{i} - h b_{n-1}(\hat{\beta}_{k})) \right)_{\alpha=\alpha_{k}+u(\hat{\alpha}_{k}-\alpha_{k})} du \sqrt{n}(\hat{\alpha}_{k} - \alpha_{k})
= \partial_{\theta} b_{n-1}(\hat{\beta}_{k})^{T} A_{n-1}^{-1}(\alpha_{k})(\Delta X_{i} - h b_{n-1}(\beta_{k})) - h \partial_{\theta} b_{n-1}(\hat{\beta}_{k})^{T} A_{n-1}^{-1}(\alpha_{k})(b_{n-1}(\hat{\beta}_{k}) - b_{n-1}(\beta_{k}))
\]
\[
\frac{1}{\sqrt{n}} \int_0^1 \partial_\alpha \left( \partial_\beta b_{i-1}(\hat{\beta}) \right) \left( \Delta X_i - h b_{i-1}(\beta) \right) \bigg|_{\alpha = \alpha_k + u(\hat{\beta}_k - \alpha_k^*)} \, du \sqrt{n}(\hat{\beta}_k - \alpha_k^*) \\
= \sum_{j=0}^{m-1} \frac{1}{T^{j/2}} z_{k,i}^{(j)} \otimes (\sqrt{T}(\hat{\beta}_k - \beta))^{\otimes j} \\
+ \frac{1}{T^{m/2}} \int_0^1 \left( 1 - u \right)^{m-1} (m - 1)! \, \partial_\beta \left( \partial_\beta b_{i-1}(\hat{\beta}) \right) \left( \Delta X_i - h b_{i-1}(\beta) \right) \bigg|_{\beta = \hat{\beta}_k + u(\hat{\beta}_k - \beta_k)} \, du \sqrt{T}(\hat{\beta}_k - \beta_k) \\
+ \frac{1}{n} \int_0^1 \partial_\alpha \left( \partial_\beta b_{i-1}(\hat{\beta}) \right) \left( \Delta X_i - h b_{i-1}(\beta) \right) \bigg|_{\alpha = \alpha_k + u(\hat{\beta}_k - \alpha_k^*)} \, du \sqrt{n}(\hat{\beta}_k - \alpha_k^*) \\
= \sum_{j=0}^{m-1} \frac{1}{T^{j/2}} z_{k,i}^{(j)} \otimes (\sqrt{T}(\hat{\beta}_k - \beta))^{\otimes j} + O_p \left( \frac{\sqrt{n}}{T^{m/2}} \sqrt{\frac{h}{n}} \right) \\
= \sum_{j=0}^{m-1} \frac{1}{T^{j/2}} z_{k,i}^{(j)} \otimes (\sqrt{T}(\hat{\beta}_k - \beta))^{\otimes j} + \sum_{j=0}^{m-1} \frac{1}{T^{j/2}} \mathbb{E}_k (z_{k,i}^{(j)} \otimes \varepsilon_{i-1}^n) \otimes (\sqrt{T}(\hat{\beta}_k - \beta))^{\otimes j} \\
+ O_p \left( \frac{\sqrt{n}}{T^{m/2}} \sqrt{\frac{h}{n}} \right) \\
= \sum_{j=0}^{m-1} \frac{1}{T^{j/2}} z_{k,i}^{(j)} \otimes (\sqrt{T}(\hat{\beta}_k - \beta))^{\otimes j} + O_p \left( \sqrt{\frac{n}{h}} \right) \\
= \sum_{j=0}^{m-1} \frac{1}{T^{j/2}} z_{k,i}^{(j)} \otimes (\sqrt{T}(\hat{\beta}_k - \beta))^{\otimes j} + O_p \left( \sqrt{\frac{n}{h}} \right),
\]

we have

\[
\frac{1}{\sqrt{T}} \max_{1 \leq k \leq m} \left\| \frac{k}{m} \left( 1 - \frac{m_n}{n \Sigma_n} \right) \sum_{i=1}^{n} \tilde{\xi}_{i,k} \right\| \\
\leq \frac{1}{\sqrt{T}} \left\| \frac{n \Sigma_n - m_n}{n \Sigma_n} \right\| \sum_{j=1}^{m} \tilde{\xi}_{j,k} \\
\leq \frac{1}{\sqrt{T}} \left\| \frac{n \Sigma_n - m_n}{n \Sigma_n} \right\| \left( \sum_{j=0}^{m-1} \frac{1}{T^{j/2}} \sum_{i=1}^{m} z_{k,i}^{(j)} \right) \left( \sqrt{T}(\hat{\beta}_1 - \beta_1) \right)^j + m_n O_p \left( \sqrt{\frac{h}{n}} \right) \\
\leq \frac{m_n}{n \Sigma_n} n^{1-1} \left| \frac{n \Sigma_n - m_n}{n \Sigma_n} \right| - m_n \left( \sum_{j=0}^{m-1} \frac{n^{1-1}}{T^{(j+1)/2} m_n} \right) \left\| \sum_{j=1}^{m} z_{k,i}^{(j)} \right\| \left( \sqrt{T}(\hat{\beta}_1 - \beta_1) \right)^j + O_p(n^{-\epsilon_1}). \quad (6.26)
\]

Since \( \frac{m_n}{n \Sigma_n} n^{1-1} \left| \frac{n \Sigma_n - m_n}{n \Sigma_n} \right| = O_p(1), \mathbb{E}_k \left( \| z_{k,i}^{(j)} \|^2 \right) \leq C h \) and

\[
\frac{n^{2-2\epsilon_1} m_n h}{T^{(j+1)/2} m_n^2} = O(n^{-2\epsilon_1}) = o(1)
\]

for \( 0 \leq j \leq m - 1 \), we have

\[
\frac{1}{\sqrt{T}} \max_{1 \leq k \leq m} \left\| \frac{k}{m} \left( 1 - \frac{m_n}{n \Sigma_n} \right) \sum_{i=1}^{n} \tilde{\xi}_{i,k} \right\| = o_p(1) \quad (6.27)
\]

from Lemma 1 and (6.26). By the same argument, we also have

\[
\frac{1}{\sqrt{T}} \max_{1 \leq k \leq m} \left\| \frac{k}{n \Sigma_n} \sum_{i=1}^{n} \tilde{\xi}_{i,k} \right\| = o_p(1). \quad (6.28)
\]
Therefore, according to (6.25), (6.27), (6.28) and Theorem 3 of Tonaki et al. (2020), we obtain

\[ L_{2,n}^{(1)} = \frac{1}{\sqrt{T_n}} \max_{1 \leq k \leq m_n} \left\| T_{1,n}^{-1/2} \left( \sum_{i=1}^{k} \hat{\zeta}_{i,i} - \frac{k}{[n \tau_n]} \sum_{i=1}^{[n \tau_n]} \hat{\zeta}_{i,i} \right) \right\| \]

\[ = \frac{1}{\sqrt{T_n}} \max_{1 \leq k \leq m_n} \left\| T_{1,n}^{-1/2} \left( \sum_{i=1}^{k} \hat{\zeta}_{i,i} - \frac{k}{m_n} \sum_{i=1}^{m_n} \hat{\zeta}_{i,i} \right) \right\| + o_p(1) \]

\[ = \frac{1}{\sqrt{T_n}} \frac{1}{\sqrt{T_n}} \max_{1 \leq k \leq m_n} \left\| T_{1,n}^{-1/2} \left( \sum_{i=1}^{k} \hat{\zeta}_{i,i} - \frac{k}{m_n} \sum_{i=1}^{m_n} \hat{\zeta}_{i,i} \right) \right\| + o_p(1) \]

\[ \xrightarrow{d} \sup_{0 \leq s \leq 1} \| B_q^0(s) \|. \]

Proof of (6.24). We have

\[ \sum_{i=1}^{k} \hat{\zeta}_{i,i} - \frac{k}{[n \tau_n]} \sum_{i=1}^{[n \tau_n]} \hat{\zeta}_{i,i} \]

\[ = \sum_{i=1}^{k} \hat{\zeta}_{i,i} - \frac{k}{[n \tau_n]} \sum_{i=1}^{[n \tau_n]} \hat{\zeta}_{i,i} + \frac{k}{[n \tau_n]} \left( 1 - \frac{[n \tau_n]}{[n \tau_n] + 1} \right) \sum_{i=1}^{[n \tau_n]} \hat{\zeta}_{i,i} + \frac{k}{[n \tau_n]} \sum_{i=[n \tau_n]+1}^{[n \tau_n]+1} \hat{\zeta}_{i,i} \]

By the same argument, we obtain

\[ U_{2,n}^{(1)} = \frac{1}{\sqrt{T_n}} \max_{1 \leq k \leq [n \tau_n]} \left\| T_{1,n}^{-1/2} \left( \sum_{i=1}^{k} \hat{\zeta}_{i,i} - \frac{k}{[n \tau_n]} \sum_{i=1}^{[n \tau_n]} \hat{\zeta}_{i,i} \right) \right\| \]

\[ = \frac{1}{\sqrt{T_n}} \max_{1 \leq k \leq [n \tau_n]} \left\| T_{1,n}^{-1/2} \left( \sum_{i=1}^{k} \hat{\zeta}_{i,i} - \frac{k}{[n \tau_n]} \sum_{i=1}^{[n \tau_n]} \hat{\zeta}_{i,i} \right) \right\| + o_p(1) \]

\[ = \frac{1}{\sqrt{T_n}} \frac{1}{\sqrt{T_n}} \max_{1 \leq k \leq [n \tau_n]} \left\| T_{1,n}^{-1/2} \left( \sum_{i=1}^{k} \hat{\zeta}_{i,i} - \frac{k}{[n \tau_n]} \sum_{i=1}^{[n \tau_n]} \hat{\zeta}_{i,i} \right) \right\| + o_p(1) \]

\[ \xrightarrow{d} \sup_{0 \leq s \leq 1} \| B_q^0(s) \|. \]

Next, we show (6.4). Let

\[ L_{2,n}^{(2)} = \frac{1}{\sqrt{(1 - \tau_n)T}} \max_{1 \leq k \leq n - M_n} \left\| T_{2,n}^{-1/2} \left( \sum_{i=[n \tau_n]+1}^{[n \tau_n]+k} \hat{\zeta}_{i,i} - \frac{k}{[n \tau_n]} \sum_{i=[n \tau_n]+1}^{n} \hat{\zeta}_{i,i} \right) \right\|, \]

\[ U_{2,n}^{(2)} = \frac{1}{\sqrt{(1 - \tau_n)T}} \max_{1 \leq k \leq n - [n \tau_n]} \left\| T_{2,n}^{-1/2} \left( \sum_{i=[n \tau_n]+1}^{[n \tau_n]+k} \hat{\zeta}_{i,i} - \frac{k}{[n \tau_n]} \sum_{i=[n \tau_n]+1}^{n} \hat{\zeta}_{i,i} \right) \right\|. \]

Since \( L_{2,n}^{(2)} \leq T_{2,n}^{(2)} \leq U_{2,n}^{(2)} \) on \( D_n \), it is sufficient to prove

\[ L_{2,n}^{(2)} \xrightarrow{d} \sup_{0 \leq s \leq 1} \| B_q^0(s) \|, \quad U_{2,n}^{(2)} \xrightarrow{d} \sup_{0 \leq s \leq 1} \| B_q^0(s) \|. \]

These can be proved in the same way as (6.11) and (6.12).

Proof of Theorem 5 Step 1. We first prove that \( P(T_{1,n}^{(1)} > w_1(\varepsilon)) \) converges to one as \( n \to \infty \) under \( H_1^{(1)} \).

(a) If we prove

\[ \frac{1}{\tau_n^{p_1}} \sum_{i=1}^{[n \tau_n]} \hat{\xi}_{1,i} \xrightarrow{p} \mathcal{G}_{1,1}, \quad \frac{1}{(\tau_n - \tau_n^{p_1})} \sum_{i=[n \tau_n]+1}^{[n \tau_n]+k} \hat{\xi}_{1,i} \xrightarrow{p} \mathcal{G}_{1,2}, \]

(6.29)

(6.30)
then
\[
\frac{1}{\mathcal{L}_n T} \sum_{i=1}^{[\frac{n}{\tau_n^*}]} \tilde{\xi}_{i,i} = \frac{\tau_{\xi}^\beta}{\mathcal{L}_n \tau_n^\beta T} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i} + \frac{\tau_n^\beta - \tau_{\xi}^\beta}{(\tau_n^\beta - \tau_{\xi}^\beta) T} \sum_{i=1}^{[\frac{n}{\tau_n^\beta}]} \tilde{\xi}_{i,i} \rightarrow \left(1 - \frac{\tau_{\xi}^\beta}{\tau_n^\beta}\right) G_{1,1}, \\
\frac{1}{\mathcal{L}_n T} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i} - \frac{[\frac{n\tau_n^\beta}{\tau_n^*}]}{[\frac{n}{\tau_n^*}]} \left(\frac{\tau_{\xi}^\beta}{\mathcal{L}_n \tau_n^\beta T} \sum_{i=1}^{[\frac{n}{\tau_n^*}]} \tilde{\xi}_{i,i} - \frac{[\frac{n}{\tau_n^*}]}{[\frac{n\tau_n^\beta}{\tau_n^*}]} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i}\right) \rightarrow \frac{\tau_{\xi}^\beta}{\tau_n^\beta} \left(G_{1,1} - \left(1 - \frac{\tau_{\xi}^\beta}{\tau_n^\beta}\right) G_{1,2}\right) \quad (6.30)
\]
Therefore, we have
\[
\mathcal{T}_{1,n}^{(1)} = \frac{1}{\sqrt{\mathcal{L}_n T}} \max_{1 \leq k \leq [\frac{n}{\tau_n^*}]} \left| \frac{1}{\mathcal{L}_n T} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i} - \frac{[\frac{n\tau_n^\beta}{\tau_n^*}]}{[\frac{n}{\tau_n^*}]} \left(\frac{\tau_{\xi}^\beta}{\mathcal{L}_n \tau_n^\beta T} \sum_{i=1}^{[\frac{n}{\tau_n^*}]} \tilde{\xi}_{i,i} - \frac{[\frac{n}{\tau_n^*}]}{[\frac{n\tau_n^\beta}{\tau_n^*}]} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i}\right) \right| \\
\geq \frac{1}{\sqrt{\mathcal{L}_n T}} \left| \frac{1}{\mathcal{L}_n T} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i} - \frac{[\frac{n\tau_n^\beta}{\tau_n^*}]}{[\frac{n}{\tau_n^*}]} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i}\right| \\
= \sqrt{\frac{\mathcal{L}_n T}{d}} \left| \frac{1}{\mathcal{L}_n T} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i} - \frac{[\frac{n\tau_n^\beta}{\tau_n^*}]}{[\frac{n}{\tau_n^*}]} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i}\right| \\
\rightarrow \infty,
\]
which implies \( P(\mathcal{T}_{1,n}^{(1)} > u_1(\epsilon)) \rightarrow 1. \) (6.29) can be shown similarly to (4.54) of Tonaki et al. (2020).

Proof of (6.30). We can prove
\[
\frac{1}{(\tau_n^\beta - \tau_{\xi}^\beta) T} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i} \rightarrow G_{1,2} \quad (6.31)
\]
with the same argument as (4.55) of Tonaki et al. (2020). We have
\[
\Delta_n = \left| \frac{1}{(\tau_n^\beta - \tau_{\xi}^\beta) T} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i} - \frac{1}{(\tau_n^\beta - \tau_{\xi}^\beta) T} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i}\right| \\
= 1 \left| \left(\frac{1}{\mathcal{L}_n - \tau_n^\beta} - \frac{1}{\tau_n^\beta - \tau_{\xi}^\beta}\right) \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i} + \frac{1}{\tau_n^\beta - \tau_{\xi}^\beta} \left(\sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i} - \sum_{i=1}^{[\frac{n}{\tau_n^*}]} \tilde{\xi}_{i,i}\right) \right| \\
\leq \left| \frac{\tau_n^\alpha - \tau_n^\beta}{(\tau_n^\beta - \tau_{\xi}^\beta)(\tau_n^\beta - \tau_{\xi}^\beta)} \right| \left| \frac{1}{(\tau_n^\beta - \tau_{\xi}^\beta) T} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i} + \frac{1}{(\tau_n^\beta - \tau_{\xi}^\beta) T} \sum_{i=1}^{[\frac{n}{\tau_n^*}]} \tilde{\xi}_{i,i} - \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i}\right| \\
= \left| \frac{\tau_{\xi}^\beta - \tau_n^\beta}{(\tau_n^\beta - \tau_{\xi}^\beta)(\tau_n^\beta - \tau_{\xi}^\beta)} \right| \left| \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i} + \frac{1}{(\tau_n^\beta - \tau_{\xi}^\beta) T} \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i} - \sum_{i=1}^{[\frac{n\tau_n^\beta}{\tau_n^*}]} \tilde{\xi}_{i,i}\right| \\
= \left| \frac{\tau_{\xi}^\beta - \tau_n^\beta}{(\tau_n^\beta - \tau_{\xi}^\beta)(\tau_n^\beta - \tau_{\xi}^\beta)} \right| \mathcal{S}_n + \frac{1}{\tau_n^\alpha - \tau_{\xi}^\beta} \mathcal{Q}_n.
\]
If we prove \( \mathcal{S}_n \rightarrow 0 \) and \( \mathcal{Q}_n \rightarrow 0 \), then, from \( n^{-1}(\tau_n^\alpha - \tau_n^\beta) = O_p(1) \) and (6.31), we have \( \Delta_n \rightarrow 0 \) and (6.30). In the following, we prove them.
Set $\mathcal{Y}_{k,i} = \mathcal{T}_n^{-1}(\alpha_k^*) \Delta X_i$, $\mathcal{M}_{k,i} = \mathcal{Y}_{k,i} - \mathbb{E}_{\theta_k} [\mathcal{Y}_{k,i} | g_{n-1}^n]$. Notice that
\[
\xi_{k,i} = \mathcal{T}_n^{-1}(\alpha_k^*) (\Delta X_i - h_b - 1(\hat{\beta}_k)) = \mathcal{T}_n^{-1}(\alpha_k^*) (\Delta X_i - h_b - 1(\hat{\beta}_k)) + \frac{1}{\sqrt{n}} \int_0^1 \partial_n \left( \mathcal{T}_n^{-1}(\alpha) (\Delta X_i - h_b - 1(\hat{\beta}_k)) \right) \bigg|_{\alpha = \alpha_k^* + u(\hat{\alpha}_k - \alpha_k^*)} du \sqrt{n}(\hat{\alpha}_k - \alpha_k^*)
\]
and
\[
\frac{1}{\sqrt{n}} \int_0^1 \partial_n \left( \mathcal{T}_n^{-1}(\alpha) (\Delta X_i - h_b - 1(\hat{\beta}_k)) \right) \bigg|_{\alpha = \alpha_k^* + u(\hat{\alpha}_k - \alpha_k^*)} du \sqrt{n}(\hat{\alpha}_k - \alpha_k^*) = \mathcal{M}_{k,i} + \mathbb{E}_{\theta_k} [\mathcal{Y}_{k,i} | g_{n-1}^n] + O_p(h) = \mathcal{M}_{k,i} + O_p(h).
\]
We see
\[
\mathcal{S}_n = \frac{n^{1-c_1}}{T} \sum_{i=1}^{[n \tau_n^a] + 1} \mathcal{M}_{1,i} + O_p \left( \frac{n^{1-c_1}}{T} \right) = \mathcal{S}_n + o_p(1)
\]
and
\[
\mathcal{Q}_n = \frac{1}{T} \sum_{i=[n \tau_n^a] + 1}^{[n \tau_n^b]} \mathcal{N}_{1,i} = \frac{1}{T} \sum_{i=[n \tau_n^a] + 1}^{[n \tau_n^b]} \mathcal{M}_{1,i} + O_p(n^{-c_1}) =: \mathcal{Q}_n + o_p(1)
\]
on $D_n$. Since $[n \tau_n^a] \leq [n \tau_n^a]$, $m_n \leq [n \tau_n^a]$ on $D_n$,
\[
\frac{n^{-2c_1}([n \tau_n^a] - [n \tau_n^b])h}{T^2} = O \left( \frac{n^{-2c_1}}{T} \right) = o(1), \quad \frac{([n \tau_n^a] - m_n)h}{T^2} = O \left( \frac{n^{-c_1}}{T} \right) = o(1),
\]
we have $\mathcal{S}_n \overset{p}{\rightarrow} 0$ and $\mathcal{Q}_n \overset{p}{\rightarrow} 0$ from Lemma [1]. Hence, we obtain the desired results.

(b) According to
\[
\mathcal{T}^{(1)}_i = \frac{1}{\sqrt{\tau_n^a T}} \max_{1 \leq k \leq [n \tau_n]} \left| \sum_{i=1}^{k} \hat{\xi}_{1,i} - k \frac{[n \tau_n]}{[n \tau_n]} \sum_{i=1}^{[n \tau_n]} \hat{\xi}_{1,i} \right|
\]
\[
\geq \frac{1}{\sqrt{\tau_n^a T}} \left| \sum_{i=1}^{[n \tau_n]} \hat{\xi}_{1,i} - \frac{[n \tau_n]}{[n \tau_n]} \sum_{i=1}^{[n \tau_n]} \hat{\xi}_{1,i} \right| = \sqrt{\frac{T \theta_{\beta_1}}{\tau_n^a}} \frac{1}{T \theta_{\beta_1}} \sum_{i=1}^{[n \tau_n]} \hat{\xi}_{1,i} - \frac{[n \tau_n]}{[n \tau_n]} \frac{1}{T \theta_{\beta_1}} \sum_{i=1}^{[n \tau_n]} \hat{\xi}_{1,i}
\]
\[
\mathcal{K}^{(1)}_n = \frac{1}{T \theta_{\beta_1}} \sum_{i=1}^{[n \tau_n]} \hat{\xi}_{1,i} - \frac{[n \tau_n]}{[n \tau_n]} \frac{1}{T \theta_{\beta_1}} \sum_{i=1}^{[n \tau_n]} \hat{\xi}_{1,i} \overset{p}{\rightarrow} c.
\]
Note that there exists $c \neq 0$ such that
\[
\mathcal{K}^{(2)}_n = \frac{1}{T \theta_{\beta_1}} \sum_{i=1}^{[n \tau_n]} \hat{\xi}_{1,i} - \frac{[n \tau_n]}{[n \tau_n]} \frac{1}{T \theta_{\beta_1}} \sum_{i=1}^{[n \tau_n]} \hat{\xi}_{1,i} \overset{p}{\rightarrow} c'.
\]
as in the proof of Proposition 2 of Tonaki et al. (2021). Meanwhile, we see
\[
\Delta_n = |\mathcal{K}^{(1)}_n - \mathcal{K}^{(2)}_n|
\]
\[
= \frac{1}{T \theta_{\beta_1}} \left| \sum_{i=1}^{[n \tau_n]} \hat{\xi}_{1,i} - \frac{[n \tau_n]}{[n \tau_n]} \sum_{i=1}^{[n \tau_n]} \hat{\xi}_{1,i} \right|
\]
Therefore, from (1) we have
\[ -\tau^2 \leq \frac{\tau^2}{\tau_n} T_{\beta_1} \leq \frac{1}{\theta_{\beta_1}} \sum_{i=1}^{n} \hat{\xi}_{2,i} \leq \frac{1}{\theta_{\beta_1}} \sum_{i=1}^{n} \hat{M}_{1,i} \] + \text{O}_p \left( \frac{1}{n^{\epsilon_1}} \right) =: S_n + o_p(1),

and
\[ \hat{\mathcal{S}}_n = \frac{n^{\epsilon_1}}{\theta_{\beta_1}} \sum_{i=1}^{n} \hat{M}_{1,i} \] + \text{O}_p \left( \frac{1}{n^{\epsilon_1}} \right) =: S_n + o_p(1),

on \( D_n \). Since \( m_n \leq \lfloor \tau_n \rfloor \) on \( D_n \),
\[ \frac{n^{-2\epsilon_1} \tau^2}{\theta_{\beta_1}^2} = O \left( \frac{n^{-2\epsilon_1}}{\theta_{\beta_1}^2} \right) = o(1), \quad \frac{[\tau_n^2] - m_n \tau^2}{\theta_{\beta_1}^2} = O \left( \frac{n^{-\epsilon_1}}{\theta_{\beta_1}^2} \right) = o(1), \]
we have \( S_n \xrightarrow{p} 0 \) and \( \hat{Q}_n \xrightarrow{p} 0 \) from Lemma [ ]. That is, we obtain \( \hat{S}_n \xrightarrow{p} 0 \) and \( \hat{Q}_n \xrightarrow{p} 0 \). Therefore, from
\[ \hat{S}_n = \frac{n^{\epsilon_1}}{\theta_{\beta_1}} \sum_{i=1}^{n} \hat{M}_{1,i} \] + \text{O}_p \left( \frac{1}{n^{\epsilon_1}} \right) =: S_n + o_p(1),

we have \( \Delta_n \xrightarrow{p} 0 \). This and (6.33) lead to (6.32).

Step 2. Next, we prove \( P(T_{1_n}^2 > w_1(\varepsilon)) \) converges to one as \( n \to \infty \) under \( H_1^{(2)} \).

(a) If we prove
\[ \frac{1}{(1 - \tau^2)} \sum_{i=[n\tau_n]+1}^{n} \hat{\xi}_{2,i} \xrightarrow{p} G_{2,1}, \quad \frac{1}{1 - \tau^2} \sum_{i=[n\tau_n]+1}^{n} \hat{\xi}_{2,i} \xrightarrow{p} G_{2,2}, \quad (6.34) \]

then
\[ \frac{1}{(1 - \tau^2)} \sum_{i=[n\tau_n]+1}^{n} \hat{\xi}_{2,i} = \frac{\tau^2 - \tau_n}{1 - \tau_n} \frac{1}{(1 - \tau^2)T} \sum_{i=[n\tau_n]+1}^{n} \hat{\xi}_{2,i} + \frac{1 - \tau^2}{1 - \tau_n} \sum_{i=[n\tau_n]+1}^{n} \hat{\xi}_{2,i} \]
\[ \xrightarrow{p} \frac{\tau^2 - \tau_n}{1 - \tau^2} \left( G_{2,1} - \frac{\tau^2 - \tau_n}{1 - \tau^2} \left( G_{2,1} + \left( 1 - \frac{\tau^2 - \tau_n}{1 - \tau_n} \right) G_{2,2} \right) \right) \]
\[ = \frac{\tau^2 - \tau_n}{1 - \tau_n} \left( 1 - \frac{\tau^2 - \tau_n}{1 - \tau_n} \left( G_{2,1} - G_{2,2} \right) \right) \neq 0, \]

and
\[ T_{1_n}^{(2)} = \frac{1}{\sqrt{T(1 - \tau_n)T}} \max_{1 \leq k \leq n - [n\tau_n]} \frac{\sum_{i=[n\tau_n]+1}^{n} \hat{\xi}_{2,i} - \frac{k}{n - [n\tau_n]} \sum_{i=[n\tau_n]+1}^{n} \hat{\xi}_{2,i}}{\sum_{i=[n\tau_n]+1}^{n} \hat{\xi}_{2,i}}. \]
can be proved in the same way as (6.30) and (4.55) of Tonaki et al. (2020).

(b) According to

\[
\mathcal{P}_{1,n} = \mathcal{P}_{1,n}^{(2)} = \frac{1}{\sqrt{d(1 - \tau_n)T}} \max_{1 \leq k \leq n - [n\tau_n]} \left| \sum_{i=[n\tau_n]+1}^{[n\tau_n]+k} \xi_{2,i} - \frac{[n\tau_n]^2 - [n\tau_n]}{n - [n\tau_n]} \sum_{i=[n\tau_n]+1}^{n} \xi_{2,i} \right| \]

(6.34)

it suffices to prove that there exists \( c \neq 0 \) such that

\[
\frac{1}{T \vartheta_{\beta_2}} \sum_{i=[n\tau_n]+1}^{[n\tau_n]} \xi_{2,i} - \frac{[n\tau_n]^2 - [n\tau_n]}{n - [n\tau_n]} \sum_{i=[n\tau_n]+1}^{n} \xi_{2,i}, \quad p \to c.
\]

This can be derived in the same way as (6.32).

\[\square\]

\textbf{Proof of Theorem 6} \ Step 1. First, we prove \( P(\tau_{1,n}^{(1)} > \omega_1(\epsilon)) \to 1 \) under \( H_1^{(1)} \).

(a) If we prove

\[
\frac{1}{\tau_n^{\beta}} \sum_{i=1}^{[n\tau_n]} \xi_{1,i} \xrightarrow{p} \mathcal{H}_{1,1},
\]

(6.35)

\[
\frac{1}{(\tau_n - \tau_n^{\beta})T} \sum_{i=[n\tau_n]+1}^{[n\tau_n]} \xi_{1,i} \xrightarrow{p} \mathcal{H}_{1,2},
\]

(6.36)

then

\[
\frac{1}{\tau_n T} \sum_{i=1}^{[n\tau_n]} \xi_{1,i} = \frac{\tau_n^{\beta}}{\tau_n^{\beta} T} \sum_{i=1}^{[n\tau_n]} \xi_{1,i} + \frac{\tau_n - \tau_n^{\beta}}{(\tau_n - \tau_n^{\beta})T} \sum_{i=[n\tau_n]+1}^{n} \xi_{1,i} \xrightarrow{p} \frac{\tau_n^{\beta}}{\tau_n^{\alpha}} \mathcal{H}_{1,1} + \left( 1 - \frac{\tau_n^{\beta}}{\tau_n^{\alpha}} \right) \mathcal{H}_{1,2},
\]

and

\[
\tau_{1,n}^{(1)} = \frac{1}{\sqrt{\tau_n T}} \max_{1 \leq k \leq [n\tau_n]} \left| T_{1,n}^{-1/2} \left( \sum_{i=1}^{k} \xi_{1,i} - \frac{k}{[n\tau_n]} \sum_{i=1}^{[n\tau_n]} \xi_{1,i} \right) \right|
\]
Note that with the same argument as (4.60) of Tonaki et al. (2020). We have

\[
(6.35) \text{ can be shown similarly to (4.59) of Tonaki et al. (2020).}
\]

**Proof of (6.36).** We can prove

\[
\frac{1}{(\tau^\alpha - \tau^\beta)^T} \sum_{i=[n\tau^\alpha]+1}^{[n\tau^\beta]} \tilde{\xi}_{1,i} \xrightarrow{p} \mathcal{H}_{1,2}
\]

with the same argument as (4.60) of Tonaki et al. (2020). We have

\[
\Delta_n = \left| \frac{1}{(\tau^\alpha - \tau^\beta)^T} \sum_{i=[n\tau^\alpha]+1}^{[n\tau^\beta]} \tilde{\xi}_{1,i} - \frac{1}{(\tau^\alpha - \tau^\beta)^T} \sum_{i=[n\tau^\beta]+1}^{[n\tau^\beta]} \tilde{\xi}_{1,i} \right|
\]

\[
= \frac{1}{T} \left( \frac{n\tau^\alpha - \tau^\beta}{\tau^\alpha - \tau^\beta} \right) \left| \sum_{i=[n\tau^\alpha]+1}^{[n\tau^\beta]} \tilde{\xi}_{1,i} + \frac{1}{(\tau^\alpha - \tau^\beta)^T} \sum_{i=[n\tau^\beta]+1}^{[n\tau^\beta]} \tilde{\xi}_{1,i} \right|
\]

\[
= \frac{n\tau^\alpha (\tau^\alpha - \tau^\beta)}{(\tau^\alpha - \tau^\beta)^2} \left| \sum_{i=[n\tau^\alpha]+1}^{[n\tau^\beta]} \tilde{\xi}_{1,i} + \frac{1}{(\tau^\alpha - \tau^\beta)^T} \sum_{i=[n\tau^\beta]+1}^{[n\tau^\beta]} \tilde{\xi}_{1,i} \right|
\]

\[
= \frac{n\tau^\alpha (\tau^\alpha - \tau^\beta)}{(\tau^\alpha - \tau^\beta)^2} \left| \tilde{\xi}_{1,i} \right|
\]

\[
= \frac{1}{\tau^\alpha - \tau^\beta} \Delta_n.
\]

If we prove \( \tilde{R}_n \xrightarrow{p} 0 \) and \( \tilde{V}_n \xrightarrow{p} 0 \), then, from \( n\tau^\alpha (\tau^\alpha - \tau^\beta) = O_p(1) \) and (6.37), we have \( \Delta_n \xrightarrow{p} 0 \) and (6.36). In the following, we prove them.

Set

\[
Z_{k,i}^{[j]} = \frac{1}{j!} \partial_{\beta}^j \left( \partial_{\beta} b_{i-1}(\beta)^T A_{i-1}^{-1}(\alpha_k^\beta) \Delta X_i \right) \big|_{\beta = \beta_k^\alpha},
\]

\[
\mathcal{N}_{k,i}^{[j]} = Z_{k,i}^{[j]} - \mathbb{E}_{\theta_k} [Z_{k,i}^{[j]} | \theta_{i-1}].
\]

Note that

\[
\tilde{\xi}_{k,i} = \partial_{\beta} b_{i-1}(\beta)^T A_{i-1}^{-1}(\alpha_k^\beta)(\Delta X_i - h b_{i-1}(\beta_k))
\]

\[
= \partial_{\beta} b_{i-1}(\beta)^T A_{i-1}^{-1}(\alpha_k^\beta)(\Delta X_i - h b_{i-1}(\beta_k))
\]

\[
+ \frac{1}{\sqrt{n}} \int_0^1 \partial_{\alpha} \left( \partial_{\beta} b_{i-1}(\beta)^T A_{i-1}^{-1}(\alpha)(\Delta X_i - h b_{i-1}(\beta_k)) \right) \big|_{\alpha = \alpha_k^\beta + u(\alpha_k^\beta - \alpha_k^\beta)} du \sqrt{n} (\alpha_k^\beta - \alpha_k^\beta)
\]

\[
= \partial_{\beta} b_{i-1}(\beta)^T A_{i-1}^{-1}(\alpha_k^\beta)(\Delta X_i - h b_{i-1}(\beta_k))
\]

\[
+ \frac{1}{\sqrt{n}} \int_0^1 \partial_{\alpha} \left( \partial_{\beta} b_{i-1}(\beta)^T A_{i-1}^{-1}(\alpha)(\Delta X_i - h b_{i-1}(\beta_k)) \right) \big|_{\alpha = \alpha_k^\beta + u(\alpha_k^\beta - \alpha_k^\beta)} du \sqrt{n} (\alpha_k^\beta - \alpha_k^\beta)
\]

\[
= \partial_{\beta} b_{i-1}(\beta)^T A_{i-1}^{-1}(\alpha_k^\beta)(\Delta X_i) + O_p(h)
\]

\[
= \sum_{j=0}^{m_{k,i} - 1} Z_{k,i}^{[j]} \otimes (\tilde{\beta}_k - \beta_k^\alpha)^{\otimes j} + O_p \left( h \vee \frac{\sqrt{n}}{T^{m_{k,i}^2/2}} \right)
\]

\[
= \sum_{j=0}^{m_{k,i} - 1} (\mathcal{N}_{k,i}^{[j]} + \mathbb{E}_{\theta_k} [Z_{k,i}^{[j]} | \theta_{i-1}]) \otimes (\tilde{\beta}_k - \beta_k^\alpha)^{\otimes j} + O_p \left( h \vee \frac{\sqrt{n}}{T^{m_{k,i}^2/2}} \right)
\]

\[
= \sum_{j=0}^{m_{k,i} - 1} \mathcal{N}_{k,i}^{[j]} \otimes (\tilde{\beta}_k - \beta_k^\alpha)^{\otimes j} + O_p(h).
\]
Let
\[ \mathcal{R}_n^{[j]} = \frac{n^{-c_1}}{T} \left\| \sum_{i=[n\tau^n_j]+1}^{[n\tau^n_j]} \mathcal{N}_{1,i}^{[j]} \right\|. \]

We have
\[ \mathcal{R}_n \leq \sum_{j=0}^{m_1-1} \mathcal{R}_n^{[j]}|\hat{\beta}_1 - \beta'_1|^j + O_p\left(\frac{n^{1-c_1}h}{T}\right) = \sum_{j=0}^{m_1-1} \mathcal{R}_n^{[j]}|\hat{\beta}_1 - \beta'_1|^j + o_p(1), \]
and
\[ \hat{\nu}_n = \frac{1}{T} \left\| \sum_{i=[n\tau^n]}^{[n\tau^n]+1} \zeta_{i,1} \right\| \leq \sum_{j=0}^{m_1-1} \nu_n^{[j]}|\hat{\beta}_1 - \beta'_1|^j + O_p(n^{-c_1}) = \sum_{j=0}^{m_1-1} \nu_n^{[j]}|\hat{\beta}_1 - \beta'_1|^j + o_p(1) \]
on $D_n$. Let $E_n = \{|\hat{\beta}_1 - \beta'_1| \leq 1\}$. Noting that $P(E_n^c) \to 0$ from [F1], for all $\epsilon > 0$,

\[
P(\mathcal{R}_n > (m_1 + 1)\epsilon) \leq P \left( \sum_{j=0}^{m_1-1} \mathcal{R}_n^{[j]}|\hat{\beta}_1 - \beta'_1|^j > m_1\epsilon \right) + o(1)
\leq P \left( \sum_{j=0}^{m_1-1} \mathcal{R}_n^{[j]}|\hat{\beta}_1 - \beta'_1|^j > (m_1 + 1)\epsilon \right) + o(1)
= P(D_n^c) + o(1) + o(1)
= P(D_n^c) + o(1),
\]

\[
P(\hat{\nu}_n > (m_1 + 1)\epsilon) \leq P(\hat{\nu}_n > (m_1 + 1)\epsilon, D_n) + P(D_n^c)
\leq P \left( \sum_{j=0}^{m_1-1} \nu_n^{[j]}|\hat{\beta}_1 - \beta'_1|^j > (m_1 + 1)\epsilon \right) + o(1)
\leq P \left( \sum_{j=0}^{m_1-1} \nu_n^{[j]}|\hat{\beta}_1 - \beta'_1|^j > m\epsilon \right) + P(D_n^c) + o(1)
\leq P(D_n^c) + o(1),
\]

\[
\mathbb{E}_{\theta_1}[(\mathcal{R}_n^{[j]})^2] \leq C h,
\]

\[
\frac{n^{-2c_1}([n\tau^n_j] - [n\tau^n_j]^2)h}{T^2} = O \left( \frac{n^{-2c_1}}{T^2} \right) = o(1), \quad \frac{([n\tau^n_j] - m_n)h}{T^2} = O \left( \frac{n^{-c_1}}{T^2} \right) = o(1),
\]
we have $\mathbb{E}_{\theta_1}[(\mathcal{R}_n^{[j]})^2] = o(1)$ and $\mathbb{E}_{\theta_1}[(\nu_n^{[j]})^2] = o(1)$ for $0 \leq j \leq m_1 - 1$ as in Lemma 1.

Hence, we get the desired results.

(b) Notice that
\[
\tilde{\zeta}_{k,i} = \sum_{j=0}^{m_2-1} \mathcal{N}_{k,i}^{[j]} \otimes (\hat{\beta}_k - \beta'_k)^{\otimes j} + O_p(h)
\]
from [E4]. Then, the desired result can be obtained in the same way as in (a).
(c) According to
\[
T_{2,n}^{(1)} = \frac{1}{\sqrt{\Sigma I}} \max_{1 \leq k \leq [n\tau_n]} \left| I_{1,n}^{-1/2} \left( \sum_{i=1}^{k} \zeta_{1,i} - \frac{k}{[n\tau_n]} \sum_{i=1}^{[n\tau_n]} \zeta_{1,i} \right) \right|
\]
\[
\geq \frac{1}{\sqrt{\Sigma I}} \left| I_{1,n}^{-1/2} \left( \sum_{i=1}^{[n\tau_n]} \zeta_{1,i} - \frac{[n\tau_n]}{[n\tau_n]} \sum_{i=1}^{[n\tau_n]} \zeta_{1,i} \right) \right|
\]
\[
= \sqrt{\frac{T \vartheta_{\beta_1}}{\Sigma_n}} \left| I_{1,n}^{-1/2} \left( \frac{1}{T \vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_n]} \zeta_{1,i} - \frac{[n\tau_n]}{[n\tau_n]} \frac{1}{T \vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_n]} \zeta_{1,i} \right) \right|
\]
it is sufficient to prove that there exists \( c \neq 0 \) such that
\[
K_{n}^{(1)} = \frac{1}{T \vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_n]} \zeta_{1,i} - \frac{[n\tau_n]}{[n\tau_n]} \frac{1}{T \vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_n]} \zeta_{1,i} \xrightarrow{p} c. \tag{6.38}
\]
Notice that there exists \( c' \neq 0 \) such that
\[
K_{n}^{(2)} = \frac{1}{T \vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_n]} \zeta_{1,i} - \frac{[n\tau_n]}{[n\tau_n]} \frac{1}{T \vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_n]} \zeta_{1,i} \xrightarrow{p} c'. \tag{6.39}
\]
as in the proof of Proposition 3 of Tonaki et al. (2021). Meanwhile, we see
\[
\Delta_n = \| K_{n}^{(1)} - K_{n}^{(2)} \|
\]
\[
\leq \left( \frac{[n\tau_n]}{T \vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_n]} \zeta_{1,i} - \frac{[n\tau_n]}{[n\tau_n]} \sum_{i=1}^{[n\tau_n]} \zeta_{1,i} \right) + n^{c_1} \frac{[n\tau_n]}{[n\tau_n]} \sum_{i=1}^{[n\tau_n]} \zeta_{1,i}
\]
\[
= : \frac{[n\tau_n]}{[n\tau_n]} \tilde{Y}_n + n^{c_1} \frac{[n\tau_n]}{[n\tau_n]} \frac{[n\tau_n]}{[n\tau_n]} \tilde{R}_n
\]
and
\[
\zeta_{1,i} = \sum_{j=0}^{m_3-1} N_{i,j}^{(\beta_1 - \beta_1')^{\otimes j}} + O_p(h)
\]
from
\[
G_3. \text{ Here, we set}
\]
\[
\gamma_n^{(j)} = \frac{1}{T \vartheta_{\beta_1}} \left\| \frac{[n\tau_n]}{[n\tau_n]} \sum_{i=1}^{[n\tau_n]} N_{1,i}^{(j)} \right\|, \quad R_n^{(j)} = \frac{n^{c_1}}{T \vartheta_{\beta_1}} \left\| \sum_{i=1}^{[n\tau_n]} N_{1,i}^{(j)} \right\|
\]
and have, from
\[
G_5,
\]
\[
\tilde{Y}_n = \frac{1}{T \vartheta_{\beta_1}} \left\| \sum_{i=1}^{[n\tau_n]} \zeta_{1,i} \right\| \leq \sum_{j=0}^{m_3-1} \gamma_n^{(j)} |\hat{\beta}_1 - \beta_1'|^j + O_p \left( \frac{n^{1-c_1} h}{T \vartheta_{\beta_1}} \right) = \sum_{j=0}^{m_3-1} \gamma_n^{(j)} |\hat{\beta}_1 - \beta_1'|^j + o_p(1)
\]
and
\[
\tilde{R}_n \leq \sum_{j=0}^{m_3-1} R_n^{(j)} |\hat{\beta}_1 - \beta_1'|^j + O_p \left( \frac{n^{1-c_1} h}{T \vartheta_{\beta_1}} \right) = \sum_{j=0}^{m_3-1} R_n^{(j)} |\hat{\beta}_1 - \beta_1'|^j + o_p(1)
\]
on \( D_n \). Thus, for all \( \epsilon > 0 \),
\[
P(\tilde{Y}_n > (m_3 + 1)\epsilon) \leq P(\tilde{Y}_n > (m_3 + 1)\epsilon, D_n) + P(D_n^c)
\]
\[
\leq P \left( \sum_{j=0}^{m_3-1} \gamma_n^{(j)} |\hat{\beta}_1 - \beta_1'|^j > m_3 \epsilon, D_n \right) + o(1)
\]
\[ \leq P \left( \sum_{j=0}^{m_3-1} \mathcal{V}_n^{[j]} | \hat{\beta}_1 - \beta_1^{[j]} > m_3 \epsilon, D_n \cap E_n \right) + P(E_n^c) + o(1) \]

\[ \leq P \left( \sum_{j=0}^{m_3-1} \mathcal{V}_n^{[j]} > m_3 \epsilon, D_n \cap E_n \right) + o(1) \]

\[ \leq \sum_{j=0}^{m_3-1} P(\mathcal{V}_n^{[j]} > \epsilon, D_n) + o(1) \]

\[ \leq \frac{1}{\epsilon^2} \sum_{j=0}^{m_3-1} E_{\theta_1}(\mathcal{V}_n^{[j]} | D_n)^2 + o(1), \]

\[ P(\mathcal{R}_n > (m_3 + 1) \epsilon) \leq P \left( \sum_{j=0}^{m_3-1} \mathcal{R}_n^{[j]} | \hat{\beta}_1 - \beta_1^{[j]} > m_3 \epsilon \right) + o(1) \]

\[ \leq P \left( \sum_{j=0}^{m_3-1} \mathcal{R}_n^{[j]} > m_3 \epsilon, E_n \right) + O(1) \]

\[ \leq \sum_{j=0}^{m_3-1} P(\mathcal{R}_n^{[j]} > \epsilon) + o(1) \]

\[ \leq \frac{1}{\epsilon^2} \sum_{j=0}^{m_3-1} E_{\theta_1}(\mathcal{R}_n^{[j]} | D_n)^2 + o(1). \]

Since

\[ \frac{([n \tau_1^\alpha] - m_n)h}{T^2 \beta_1^2} = O \left( \frac{n^{-c_1}}{T^2 \beta_1^2} \right) = o(1), \quad \frac{n^{-2c_1} [n \tau_1^\alpha] h}{T^2 \beta_1^2} = O \left( \frac{n^{-2c_1}}{T^2 \beta_1^2} \right) = o(1), \]

we have

\[ E_{\theta_1}(\mathcal{V}_n^{[j]} | D_n) = o(1) \] and

\[ E_{\theta_1}(\mathcal{R}_n^{[j]} | D_n) = o(1) \]

for \( 0 \leq j \leq m_3 - 1 \) as in Lemma 1 that is, we obtain \( \mathcal{V}_n \overset{p}{\to} 0 \) and \( \mathcal{R}_n \overset{p}{\to} 0 \), which implies \( \Delta_n \overset{p}{\to} 0 \). This and (6.39) yield (6.38).

Step 2. Next, we show \( P(T_{2,n}^{(2)} > w_1(\epsilon)) \to 1 \) under \( H_1^{(2)} \).

(a) If we prove

\[ \frac{1}{(T_{2,n}^{(2)} - \tau_n)T} \sum_{i=[n \tau_2^\alpha]+1}^{[n \tau_2^\alpha]} \tilde{\zeta}_{2,i} \overset{p}{\to} \mathcal{H}_{2,1}, \quad \frac{1}{(1 - \tau_n)T} \sum_{i=[n \tau_2^\alpha]+1}^{[n \tau_2^\alpha]} \tilde{\zeta}_{2,i} \overset{p}{\to} \mathcal{H}_{2,2}, \]

then

\[ \frac{1}{(1 - \tau_n)T} \sum_{i=[n \tau_2^\alpha]+1}^{[n \tau_2^\alpha]} \tilde{\zeta}_{2,i} = \left( \frac{\tau_n^\beta - \tau_n^\alpha}{1 - \tau_n} \frac{1}{\tau_n^\alpha} \right) \mathcal{H}_{2,1} + \left( \frac{1 - \tau_n^\beta - \tau_n^\alpha}{1 - \tau_n^\alpha} \right) \mathcal{H}_{2,2}, \]

\[ = \left( \frac{\tau_n^\beta - \tau_n^\alpha}{1 - \tau_n^\alpha} \right) \left( \mathcal{H}_{2,1} + \left( \frac{1 - \tau_n^\beta - \tau_n^\alpha}{1 - \tau_n^\alpha} \right) \mathcal{H}_{2,2} \right), \]

\[ = \left( \frac{\tau_n^\beta - \tau_n^\alpha}{1 - \tau_n^\alpha} \right) (\mathcal{H}_{2,1} - \mathcal{H}_{2,2}), \]

we have

\[ \mathcal{H}_{2,1} - \mathcal{H}_{2,2} \neq 0, \]
and

\[ \tau_{2,n}^{(2)} = \frac{1}{\sqrt{(1 - \tau_n)T}} \max_{1 \leq k \leq n - [n \tau_n]} \left\| T_{2,n}^{-1/2} \left[ \sum_{i=[n \tau_n]+1}^{[n \tau_n]+k} \tilde{\zeta}_{2,i} \right] - \frac{k}{n - [n \tau_n]} \sum_{i=[n \tau_n]+1}^{n} \tilde{\zeta}_{2,i} \right\| \]

\[ \geq \frac{1}{\sqrt{(1 - \tau_n)T}} \left\| T_{2,n}^{-1/2} \left( \sum_{i=[n \tau_n]+1}^{[n \tau_n]+k} \tilde{\zeta}_{2,i} - \frac{[n \tau_n] - [n \tau_n]}{n - [n \tau_n]} \sum_{i=[n \tau_n]+1}^{n} \tilde{\zeta}_{2,i} \right) \right\| \]

\[ = \sqrt{(1 - \tau_n)T} \left\| T_{2,n}^{-1/2} \left( \frac{1}{(1 - \tau_n)T} \sum_{i=[n \tau_n]+1}^{[n \tau_n]+k} \tilde{\zeta}_{2,i} - \frac{[n \tau_n] - [n \tau_n]}{n - [n \tau_n]} \frac{1}{(1 - \tau_n)T} \sum_{i=[n \tau_n]+1}^{n} \tilde{\zeta}_{2,i} \right) \right\| \]

\[ \xrightarrow{p \to \infty}. \]

(6.40) can be proved in the same way as (6.36) and (4.60) of Tonaki et al. (2020).

(c) According to

\[ \tau_{2,n}^{(2)} = \frac{1}{\sqrt{(1 - \tau_n)T}} \max_{1 \leq k \leq n - [n \tau_n]} \left\| T_{2,n}^{-1/2} \left[ \sum_{i=[n \tau_n]+1}^{[n \tau_n]+k} \tilde{\zeta}_{2,i} \right] - \frac{k}{n - [n \tau_n]} \sum_{i=[n \tau_n]+1}^{n} \tilde{\zeta}_{2,i} \right\| \]

\[ \geq \frac{1}{\sqrt{(1 - \tau_n)T}} \left\| T_{2,n}^{-1/2} \left( \sum_{i=[n \tau_n]+1}^{[n \tau_n]+k} \tilde{\zeta}_{2,i} - \frac{[n \tau_n] - [n \tau_n]}{n - [n \tau_n]} \sum_{i=[n \tau_n]+1}^{n} \tilde{\zeta}_{2,i} \right) \right\| \]

\[ = \sqrt{(1 - \tau_n)T} \left\| T_{2,n}^{-1/2} \left( \frac{1}{T \vartheta_{2}} \sum_{i=[n \tau_n]+1}^{[n \tau_n]+k} \tilde{\zeta}_{2,i} - \frac{[n \tau_n] - [n \tau_n]}{n - [n \tau_n]} \frac{1}{T \vartheta_{2}} \sum_{i=[n \tau_n]+1}^{n} \tilde{\zeta}_{2,i} \right) \right\| , \]

it is sufficient to show that there exists \( c \neq 0 \) such that

\[ \frac{1}{T \vartheta_{2}} \sum_{i=[n \tau_n]+1}^{[n \tau_n]+k} \tilde{\zeta}_{2,i} - \frac{[n \tau_n] - [n \tau_n]}{n - [n \tau_n]} \frac{1}{T \vartheta_{2}} \sum_{i=[n \tau_n]+1}^{n} \tilde{\zeta}_{2,i} \xrightarrow{p \to c}, \]

which can be derived in the same way as (6.38).

\[ \square \]

**Proof of Theorem 7** We have

\[ \Psi_{1,n}(\tau : \beta_1, \beta_2 | \alpha) - \Psi_{1,n}(\tau^{\beta} : \beta_1, \beta_2 | \alpha) = \sum_{i=1}^{[n \tau]} G_i(\beta_1 | \alpha) + \sum_{i=[n \tau]+1}^{[n \tau_\beta]} G_i(\beta_2 | \alpha) - \sum_{i=1}^{[n \tau_\beta]} G_i(\beta_1 | \alpha) - \sum_{i=[n \tau_\beta]+1}^{[n \tau]} G_i(\beta_2 | \alpha) \]

\[ = \sum_{i=[n \tau_\beta]+1}^{[n \tau]} \left( G_i(\beta_1 | \alpha) - G_i(\beta_2 | \alpha) \right) \]

for \( \tau^{\beta} < \tau < \tau^{\alpha} \), and

\[ \Psi_{1,n}(\tau : \beta_1, \beta_2 | \alpha) - \Psi_{1,n}(\tau^{\beta} : \beta_1, \beta_2 | \alpha) = \sum_{i=[n \tau]+1}^{[n \tau_\beta]} \left( G_i(\beta_2 | \alpha) - G_i(\beta_1 | \alpha) \right) \]

for \( \tau < \tau^{\beta} \). Therefore, by the same proof of Theorems 3 and 4 of Tonaki et al. (2021), we obtain

\[ T \vartheta_{2}^{(2)}(\hat{\tau}_{1,n}^{\beta} - \tau^{\beta}) \xrightarrow{d} \arg \min_{v \in \mathbb{R}} G_1(v) \]

in Case A, and \( T(\hat{\tau}_{1,n}^{\beta} - \tau^{\beta}) = O_p(1) \) in Case B.

\[ \square \]

Theorem 8 can be proved in the same way as the proof of Theorem 7.

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