Both on-ramp and off-ramp on the boundaries of totally asymmetric simple exclusion

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Abstract. In this paper, both on-ramp and off-ramp on the boundaries of totally asymmetric simple exclusion process (TASEP) were considered. Here, particles can enter and leave the one dimensional lattice on the same site of boundaries by attachment and detachment. The changes of the phase diagrams and density profiles are given by theoretical calculations and the extensive Monte Carlo computer simulations. The result shows that the off-ramp and on-ramp can vary the diagram very obviously.

1. Introduction

As an important tool of out-of-equilibrium statistical mechanics, totally asymmetric simple exclusion process (TASEP) was used to explore vehicular traffic [1-4] and biological transport [5-14].

Recently, ramp as a special geometry was investigated and some important results were presented by TASEP models [15-23] and cellular automata (CA) models [24-28]. For ramp geometry, it is divided into two types, namely on-ramp and off-ramp. In biophysics, the off-ramp (on-ramp) can be replaced by a detachment (attachment), it is employed to express particles randomly attachment to and detachment from lattice [19-23]. Note that comparing with CA models, TASEP models can give the exact solutions by mean field theory. Therefore, our research is studied by TASEP model.

More recently, a single-lane with on-ramp and off-ramp is studied, the result presents that a phase coexistence of the low-density (LD) and high-density (HD) are included in the phase diagram [20]. Through studying the zoned inhomogeneity and on-ramp on ASEP, we find the MC/MC phase can exist in the system and the LD/MC phase vanishes with the fixed on-ramp $q$ ($q < 0.5$) and decreasing hopping rate (from 0.8 to 0.3) [17]. Subsequently, ASEP with off-ramp on the boundaries [16] and on-ramp with constrained resources on ASEP [15] have been explored, respectively. For the former, the off-ramp rate decides the diagram; the system includes three steady phases like normal ASEP when off-ramp rate is less than 0.5. In contrast, there are only two phases in the system. For the latter, the availability of particles becomes a rate-limiting factor for the density of the system. Here, both on-ramp and off-ramp on boundaries of TASEP will be revealed and two typical cases will be discussed. Both numerical and analytical results will be presented. The result will be employed to analyze the effects of density and the diagram under the both on-ramp and off-ramp on boundaries.
2. The model

For both cases, there are $N$ sites in each one-dimensional channel, see Figure 1. In case A, particles can enter and leave the system at the first site (the left boundary) and the last site (the right boundary), respectively. For all sites, the random update rules are as follows:

(i) At the first site, when this site is empty, a particle enters the channel with the entrance rate $\alpha$ ($0 < \alpha < 1$) or the on-ramp rate $q$ ($0 < q < 1$). Meanwhile, when a particle is located on the first site, the particle can leave site 1 in two ways, the particle hops to the site 2 with the hopping rate 1 when the site 2 is empty, or the off-ramp rate $p$ ($0 < p < 1$), see Figure 1 (a);

(ii) At the last site, a particle leaves the channel with the exiting rate $\beta$ ($0 < \beta < 1$), see Figure 1 (a);

(iii) For all other sites, a particle which is at the site $i$ ($1 < i < N$) can move into its neighboring site $i+1$ with moving rate 1, when the state of its neighboring site is not occupied, see Figure 1 (a).

In case B, the random update rules are as follows:

(i) At the first site, a particle enters the channel with the entrance rate $\alpha$ ($0 < \alpha < 1$) if the state of this site is empty, see Figure 1 (b);

(ii) At the last site, particles hop to it from the site $N-1$ with hopping rate 1 or inject it with on-ramp rate $q$, when the state of this site is empty, nevertheless, when a particle is at this site, the particle can leave the channel with the exiting rate $\beta$ ($0 < \beta < 1$) or the off-ramp rate $p$ ($0 < p < 1$), see Figure 1 (b);

(iii) If a particle is at the site $i$ ($1 < i < N$), it will move into the neighboring site $i + 1$ with the moving rate 1, when it’s neighboring site is empty, see Figure 1(b).

![Figure 1](image-url)

Figure 1. Illustration of both on-ramp and off-ramp on the boundaries of TASEP. (a) Case A, both on-ramp and off-ramp on the left boundary; (b) Case B, both on-ramp and off-ramp on the right boundary.

3. Theoretical calculation of case A

First, the results of normal TASEP without on-ramp and off-ramp will be reviewed [1, 29, 30]; they are used in the present theoretical calculation.

When the parameters $\alpha, \beta$ satisfy $\alpha < \beta$ and $\alpha < 0.5$, the low-density (LD) phase exist in the model, for which

$$J = \alpha (1 - \alpha), \rho = \alpha, \rho_1 = \alpha, \rho_N = \frac{\alpha (1 - \alpha)}{\beta}$$

(1)

Where the current and the bulk density can be described by the parameters $J$ and $\rho$, respectively. In addition, the particle density at the first (last) site can be expressed by the parameters $\rho_1 (\rho_N)$.

When the parameters $\alpha, \beta$ satisfy $\beta < \alpha$ and $\beta < 0.5$, the high-density (HD) phase exist in the model, for which

$$J = \beta (1 - \beta), \rho = 1 - \beta, \rho_1 = \frac{\beta (1 - \beta)}{\alpha}, \rho_N = 1 - \beta$$

(2)

When the parameters $\alpha, \beta$ satisfy $\alpha > 0.5$ and $\beta > 0.5$, the maximal current (MC) phase exist in the model, for which

$$J = \frac{1}{4}, \rho = \frac{1}{2}, \rho_1 = 1 - \frac{1}{4 \alpha}, \rho_N = \frac{1}{4 \beta}$$

(3)
For our model, there are also three status phases like normal TASEP when the model is in the steady state. The major difference is that both on-ramp and off-ramp are at the first site, as illustrated in Figure 1 (a). Therefore, the entrance rate will be replaced by the effective entrance rate $\alpha_{\text{eff}}$ which is defined as

$$
\alpha_{\text{eff}} = (\alpha + q)(1 - p)
$$

When the system is in the LD phase, the conditions are as following

$$
\alpha_{\text{eff}} < \beta, \alpha_{\text{eff}} < \frac{1}{2}
$$

Substituting Equation (4) in to Equation (5), the parameters ranges of LD steady phase are given by

$$
\alpha < \frac{1}{2(1 - p)} - q, \beta > (\alpha + q)(1 - p)
$$

When the system is in the HD phase, it corresponds to the following conditions

$$
\beta < \alpha_{\text{eff}}, \beta < \frac{1}{2}
$$

Combining Equation (4) and Equation (5) results in the following expressions for HD steady phase

$$
\alpha > \frac{\beta}{1 - p} - q, \beta < \frac{1}{2}
$$

At last, for the MC steady state the parameters satisfy the following conditions

$$
\alpha_{\text{eff}} > \frac{1}{2}, \beta > \frac{1}{2}
$$

According to Equation (4) and Equation (9), the parameters of the MC phase can be obtained as

$$
\alpha > \frac{1}{2(1 - p)} - q, \beta > \frac{1}{2}
$$

The above research can be divided into two conditions (namely, $p < q$ and $p > q$) to analyze. Firstly, the vertical boundary of LD and MC phases will move to the left when $p$ is lower than $q$ and parameter $q$ increases from 0.4 to 0.6. However, the horizontal boundary of HD and MC phases does not change, as shown in Figure 2 (a). The result shows the on-ramp will lead to a wide increase of the system density because it makes the entrance rate of particles increase. Here, the off-ramp rate is fixed (namely, $p = 0.2$), when the on-ramp rate $q$ is more than $1/2(1-p) = 0.625$ (namely, $q = 0.8$), the LD phase vanishes and the diagram includes MC and HD two steady phases only. It implies that the variables $p$ and $q$ satisfy the relation $1/2(1-p) < q$, the LD phase will not exist in the system. Meanwhile, the transition between two steady phases MC and HD is decided by the leaving rate $\beta$, as illustrated in the inset of Figure 2 (a). With this condition, the probability of happening traffic jam is very high.

Similarly, when $p$ is higher than $q$ and parameter $p$ grows from 0.4 to 0.5, the vertical boundary of LD and MC phases will move to the right and the horizontal boundary of HD and MC phases does not change too, as shown in Figure 2 (b). When the variables $p$ and $q$ satisfy the relation $p < 1 - 1/2(1+q)$, the MC phase will not exist in the system. The transition between the two steady phases LD and HD is also determined by the variables $p$ and $q$. Here, when the on-ramp rate is equal to 0.2 ($q = 0.2$) and the off-ramp rate is more than $1 - 1/2(1+q) = 7/12$ (namely, $p = 0.8$), the system includes the two steady phases of LD and HD only, as illustrate in the inset of Figure 2 (b). It is observed that the system is easy in free flow and the probability of jam is very low.
4. Theoretical calculation of case B

In this section, the theoretical calculation of case B is presented here. The same method are used to analyze as above. Since the particle can enter and leave the last site $N$ by on-ramp and off-ramp [see Figure 1 (b)], the effective leaving rate $\beta_{\text{eff}}$ which is given as

$$\beta_{\text{eff}} = (1 + q)(\beta + p)$$  \hspace{1cm} (11)

When LD phase exists in the system, the conditions satisfy the following relations

$$\alpha < \beta_{\text{eff}}, \alpha < \frac{1}{2}$$  \hspace{1cm} (12)

Substituting Equation (11) in to Equation (12), the parameters ranges of LD steady phase are given by

$$\alpha < \frac{1}{2}, \beta > -\frac{\alpha}{1 + q} - p$$  \hspace{1cm} (13)

HD phase corresponds to the following relations

$$\beta_{\text{eff}} < \alpha, \beta_{\text{eff}} < \frac{1}{2}$$  \hspace{1cm} (14)

Combining Equation (11) and Equation (14) results in the following expressions for HD steady phase

$$\alpha > (1 + q)(\beta + p), \beta < \frac{1}{2(1 + q)} - p$$  \hspace{1cm} (15)

At last, for the MC steady state, the parameters satisfy the following conditions

$$\alpha > \frac{1}{2}, \beta_{\text{eff}} > \frac{1}{2}$$  \hspace{1cm} (16)

According to Equation (11) and Equation (16), the parameters of the MC phase can be obtained as

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Diagram of steady phases in the model with the variables $p$ and $q$ in case A. (a) $p = 0.2$; (b) $q = 0.2$.}
\end{figure}
It is different from the case B, the variables $p$ and $q$ satisfy either $p < q$ or $p > q$, only the region of HD phase reduces when one of variables fixed and another increases, as shown in Figure 3. For $p < q$, the HD phase will vanish when parameters $p$ and $q$ satisfy $1/2(1+q) < p$. However, when variable $p$ is less than 0.25 (namely $p < 0.25$), the HD phase will always exist in the system with arbitrary value of $q$. Here, the off-ramp rate is fixed (namely $p = 0.4$) and $q = 0.8 > 0.25$, the HD phase vanishes, as illustrated in the inset of Figure 3 (a). The regions of LD and MC phases are equal. Similarly, the HD phase vanishes when the relationship satisfies $1/2(1+q) < p$ with the fixed $q$. Here, when the parameters $p$ and $q$ are equal to 0.8 and 0.2, respectively, the HD phase vanishes; see the inset of Figure 3 (b). The result shows that the system is always in the free flow.

Figure 3. Diagram of steady phases in the model with the variables $p$ and $q$ in case B. (a) $p = 0.2$; (b) $q = 0.2$.

5. Results and analysis

In this section, the theoretical calculations and computer simulations will be compared. For case A and case B, both systems include 1000 sites (namely, $N = 1000$). The density profiles are given by the number of the running time steps in each experiment with $1.1 \times 10^9$. The first $10^8$ time steps will be discarded as transients. For both cases, the density profiles with different on-ramp and off-ramp rates can be simulated by Monte Carlo simulation.

As shown in Figure 4, when variable $p$ is fixed ($p = 0.2$), with the increasing of variable $q$ (from 0.4 to 0.6) the LD steady phase will be replaced by HD and MC phases, see Figure 4 (a) and (b). This result implies that the particles are hard to move with free flow in the system, when the leaving rate $\beta$ is less than 0.5, the system is always in the jam state. Similarly, when variable $q$ is fixed ($q = 0.2$), with the increasing of variable $p$ (from 0.4 to 0.5) the regions of HD and MC phases will be replaced by the region of LD phase, the LD phase is the only phase which shrinks, as illustrated in Figure 4 (c) and (d). The result shows both on-ramp and off-ramp on the right boundary can lead to the system in free flow easily. When the variables $p$ and $q$ are fixed, the average density at a bulk site changes with the increasing of the entrance and leaving rates which is shown in Figure 4 (e) and (f).

As illustrated in Figure 5, when variable $p$ is fixed ($p = 0.2$), with the increasing of variable $q$ (from 0.4 to 0.8), the HD is the only reducing phase, see Figure 5 (a) and (b). Similarly, when variable $q$ is fixed ($q = 0.2$), with the increasing of variable $p$ (from 0.3 to 0.4), the HD is the only reducing phase too, see Figure 5 (c) and (d). This result implies that the particles are can move with free flow or...
The maximal current in the system easily, when the leaving rate $\alpha$ is less than 0.5, the system is always in the free flow. Interestingly, when the HD phase vanishes, the LD and MC phases have the equal region in the system. The boundary between LD and MC phases is at $\alpha = 0.5$, the average density at a bulk site changes with $\alpha$ as shown in Figure 5 (e).

**Figure 4.** Density profiles of case A in (a)-(f). The square describes the simulations. Solid line indicates the theoretical calculations.

**Figure 5.** Density profiles of case A in (a)-(e). The square describes the simulations. Solid line indicates the theoretical calculations.
6. Summaries
In this paper, we used theoretical calculation and computer simulation to analyze both on-ramp and off-ramp on the boundaries of TASEP. The difference with on-ramp (off-ramp) on TASEP models is that on-ramp and off-ramp are at the site on the boundaries. For both cases, the on-ramp and off-ramp rates $q$ and $p$ decided the diagram of the system. For case A, when the variables $p$ and $q$ satisfy $p < q$ and $1/2(1-p) < q$, the LD phase vanishes. The boundary between HD and MC phases is decided by the ejection rate $\beta$. In contrast, if the parameters $p$ and $q$ satisfy $p > q$ and $p < 1-1/2(1+q)$, the system includes only two phase (namely, LD and HD phase). Meanwhile, the boundary between HD and LD phases is also determined by the parameters $p$ and $q$.

However, the HD phase is the only vanishing phase when the variables $p$ and $q$ satisfy either $p < q$ (or $p > q$) and $1/2(1+q) < p$ for case B. Interestingly, when $p < q$ and the variable $p$ is less than 0.25 (namely $p < 0.25$), the HD phase will always exist in the system with arbitrary value of $q$. Comparing with case A, case B is easy to make particles move in free flow.

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References
[1] Chowdhury D, Santen L and Schadschneider A 2000 Phys. Rep. 329 199
[2] Popkov V, Santen L, Schadschneider A and Schutz G 2001 J. Phys. A: Math. Gen. 34 L45
[3] Xiao S and Bai J 2013 Mod. Phys. Lett. B. 27 1
[4] Xiao S, Liu M and Shang J 2012 Mod. Phys. Lett. B. 26 600
[5] Macdonald C, Gibbs J and Pipkin A 1968 Biopolymers. 6 1
[6] MacDonald C and Gibbs J 1969 Biopolymers 7 707
[7] Chou T 2003 Biophys. 85 755
[8] Shaw L, Zia R and Lee K 2003 Phys. Rev. E. 68 125
[9] Chou T and Lakatos, G 2004 Phys. Rev. Lett. 93 198101
[10] Dong J, Schmittmann B and Zia R 2006 J. Stat. Phys. 128 21
[11] Dong J, Schmittmann B and Zia R 2007 Phys. Rev. E. 76 168
[12] Howard J 2001 Mechanics of Motor Proteins and the Cytoskeleton Sunderland of Sinauer Associates Inc
[13] Kolomeisky A and Fisher M 2007 Ann. Re. Phys. Chem. 58 675
[14] Basu A and Chowdhury D 2007 Phys. Rev. E. 75 135
[15] Liu Y, Xiao W and Dong P 2016 Renew. Sust. Energ. Rev. 62 815
[16] Xiao S, Wu S and Tang L 2012 Mod. Phys. B 26 1275
[17] Xiao S and Bai J 2013 Mod. Phys. Lett. B 27 1
[18] Qiu K, Yang X and Zhang W 2007 Physica A 373 1
[19] Xiao S, Liu M and Shang J 2012 Mod. Phys. Lett. B 26 1150036
[20] Parmeggiani A., Franosch T and Frey E 2003 Phys. Rev. Lett. 90 137
[21] Xiao S, Liu M and Shang J 2012 Chin. Phys. B 21 218
[22] Lipowsky R, Klumpp S and Nieuwenhuizen, T 2001 Phys. Rev. Lett. 87 108101
[23] Yang X, Qiu K and Zhang W 2007 Physica A 379 595
[24] Ezzahraoui H, Benrihane Z and Benyoussef A 2004 Int. J. Modern Phys. B 18 2347
[25] Jia B, Jiang R and Wu Q 2004 Phys. Rev. E 69 056105
[26] Diedrich G, Santen L, Schadschneider A and Zittartz J 2000 Int. J. Modern Phys. C 11 335
[27] Jiang R., Wu Q S and Wang B H 2002 Phys. Rev. E 66 036104
[28] Chen J, Lin L and Jiang R 2017 Physica A 465 347
[29] Derrida B, Evans M R, Hakim V and Pasquier V 1993 J. Phys. A: Math. Gen. 26 1493
[30] Schütz G M and Domany E 1993 J. Stat. Phys. 72 227