Stability Analysis of Linear Symmetric Matrix-Valued Continuous, Discrete, and Impulsive Dynamical Systems - A Unified Approach for the Stability Analysis of Linear Systems

Corentin Briat¹
Independent Researcher
Basel, Switzerland

Abstract
Symmetric matrix-valued dynamical systems are an important class of systems that can describe important processes such as covariance processes or processes on manifolds and Lie Groups. We address here the case of processes that leave the cone of positive semidefinite matrices invariant, thereby including covariance processes. Both the continuous-time and the discrete-time cases are considered. In the LTV case, the obtained stability conditions are expressed as differential and difference Lyapunov conditions which reduce to spectral conditions on the generators in the LTI case. The case of systems with constant delays is also considered for completeness. The results are then extended and unified into an impulsive formulation for which similar results are obtained. The proposed framework is very general and can recover and/or extend almost all the results on the literature of linear systems related to (mean-square) exponential (uniform) stability. Several examples are discussed to illustrate this claim by deriving stability condition for stochastic systems driven by Brownian motion, Markov jump systems, switched systems, impulsive systems, sampled-data systems, and their combinations using the proposed framework.

1. Introduction
Matrix-valued dynamical systems form an important class of systems that can be used to represent several processes such as the evolution of covariances [31], the dynamics of systems on manifolds such as SO(3) [1, 12] or Lie groups [23]. In particular, the class of matrix-valued dynamical systems that leave the cone of positive semidefinite matrices invariant is of special interest because it can represent the dynamics of covariance matrices associated with stochastic dynamical systems. The consideration of the so-called second-order information is not new and has been considered in the past in [31]. More recent works where such [19, 37] also consider second-order information methods for providing new ways to solve existing problems.

The objective of this paper is to provide a clear analysis of linear dynamical systems evolving on the cone of positive semidefinite matrices. Those systems can be seen as a matrix analogue of linear positive systems which leave the nonnegative orthant invariant [17]. Such systems possess very interesting properties which we could hope to be also possessed by more general cone preserving systems [13, 14, 33]. The first step towards a possible full answer to this problem is the development of a stability theory for such matrix-valued dynamical systems. In this paper, we show that the stability analysis of linear symmetric matrix-valued continuous- and discrete-time systems can be analyzed using linear co-positive Lyapunov function that takes nonnegative values for all state values in the cone of positive semidefinite matrices. The theory relies on the use of inner-product of matrices and dual or adjoint operators corresponding to the generator of the systems. This formulation allows for a very simple proof for the main results of the paper. The results are also extended to address the case of matrix-valued impulsive systems. The conditions are very general and take

¹email: corentin@briat.info; url: http://www.briat.info; ORCID Number: 0000-0003-1822-2683
the form of differential matrix-valued equations/inequalities, difference matrix-valued equations/inequalities, and coupled differential/difference matrix-valued equations/inequalities in the continuous-time, the discrete-time, and impulsive case, respectively. In the LTI case, stability is equivalent to a spectral condition for the generator of the systems which can be easily checked using standard numerical tools. The advantage of the Lyapunov approach is that it can be easily extended to cope with uncertainties, nonlinearities, inputs, etc. The analysis of systems with delays is also carried out and it is proven that a given system is exponentially stable if and only if the system with zero-delays is exponentially stable as well. This result parallels that on linear positive systems with delays [6, 22] and can be seen as a generalization of the results in [33]. The approach is then illustrated on a wide variety of stability analysis problems that extend or recover existing results in the literature in a unified manner and with very limited effort.

Outline. Preliminary definitions are given in Section 2. Continuous-time matrix-valued differential equations are considered in Section 3 whereas their discrete-time counterparts in Section 4. The results are unified in a hybrid formulation in Section 5. Application examples are considered in the related sections.

Notations. The set of positive and nonnegative integers are denoted by \( \mathbb{Z}_{>0} \) and \( \mathbb{Z}_{\geq 0} \), respectively. The set of positive and nonnegative real numbers are denoted by \( \mathbb{R}_{>0} \) and \( \mathbb{R}_{\geq 0} \), respectively. Similarly, the cone of positive definite and positive semidefinite matrices of dimension \( n \) are denoted by \( \mathbb{S}_n^{>0} \) and \( \mathbb{S}_n^{\geq 0} \), respectively. The left open right-half plane of the complex plane is denoted by \( \mathbb{C}_{<0} \). The natural basis for the Euclidian space is denoted by \( \{e_1, \ldots \} \). For a square real matrix \( A \), we denote \( \text{Sym}[A] := A + A^T \). For a matrix \( A \) with columns \( \{a_1, \ldots, a_n\} \), the vectorization operator \( \text{vec}(\cdot) \) stacks the columns of a matrix on the top of each other; i.e. \( \text{vec}(A) = [a_1^T \ldots a_n^T]^T \). For a square matrix \( A \), the minimum and maximum eigenvalues of \( A \) are denoted by \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \), respectively. The Hadamard product is denoted by \( \odot \) whereas the Kronecker product and sum are denoted by \( \otimes \) and \( \oplus \), respectively.

2. Preliminaries

We state in this section the essential definitions and results that will be key for deriving the main results of the paper.

Definition 2.1 (Nuclear norm [4]). Let \( A \in \mathbb{R}^{n \times m} \), then its nuclear-norm \( \|A\|_* \) is defined as

\[
\|A\|_* := \min_{i=1}^{\min\{m,n\}} \sigma_i(A)
\]

where \( \sigma_i(A) \) is the \( i \)-th singular values of \( A \). When \( A \in \mathbb{S}_{n \geq 0}^{>0} \), then we have that

\[
\|A\|_* = \text{trace}(A).
\]

The following inner-product, also called Frobenius inner product, extends the concept of scalar product to matrices:

Definition 2.2 (Inner product on \( \mathbb{R}^{n \times n} \)). The inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{R}^{n \times n} \) is defined as

\[
\langle A, B \rangle := \text{trace}(AB^T)
\]

for any \( A, B \in \mathbb{R}^{n \times n} \).

The following result will prove to be instrumental in proving the main stability results of the paper:

Proposition 2.3. Let us consider matrices \( P \in \mathbb{S}_{n > 0}^n \), \( Q \in \mathbb{S}_{n \geq 0}^n \) and \( R \in \mathbb{S}^n \). Then, the following statement holds:

(a) Let \( P \in \mathbb{S}_{n > 0}^n \), then we have that \( \langle P, Q \rangle \geq 0 \) for all \( Q \in \mathbb{S}_{n \geq 0}^n \).
(b) Let $P \in \mathbb{S}^n_{\succ 0}$, then we have that $\langle P, Q \rangle = 0$ for some $Q \in \mathbb{S}^n_{\geq 0}$ implies that $Q = 0$.

(c) Let $R \in \mathbb{S}^n$, then we have that $\langle R, Q \rangle < 0$ (resp. $\langle R, Q \rangle \leq 0$) for all $Q \succeq 0$, $Q \neq 0$, if and only if $R \prec 0$ (resp. $R \preceq 0$).

(d) Let $P \in \mathbb{S}^n_{\succ 0}$, then we have that $\lambda_{\min}(P)\|Q\|_* \leq \langle P, Q \rangle \leq \lambda_{\max}(P)\|Q\|_*$ for all $Q \succeq 0$.

Proof: Throughout this proof, we denote by $\{v_1, \ldots, v_n\}$ the set of orthonormal eigenvectors of the matrix $P$ associated with the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Observe also that the matrix $V = [v_1 \ldots v_n]$ forms an orthonormal basis of $\mathbb{R}^n$.

**Proof of statement (3).** From the symmetric structure of $P$, it can be expressed as

$$P = \sum_{i=1}^{n} \lambda_i v_i v_i^T \quad (4)$$

and, as a result, we have that

$$\langle P, Q \rangle = \sum_{i=1}^{n} \lambda_i \langle v_i v_i^T, Q \rangle = \sum_{i=1}^{n} \lambda_i v_i^T Q v_i \geq 0. \quad (5)$$

This proves statement (3).

**Proof of statement (4).** Assume that $\langle P, Q \rangle = 0$, then

$$\sum_i \lambda_i v_i^T Q v_i = 0 \quad (6)$$

and we necessarily have that $v_i^T Q v_i = 0$ for all $i = 1, \ldots, n$ since $\lambda_i > 0$ for all $i = 1, \ldots, n$. Since $V$ is a basis of $\mathbb{R}^n$, then $V^T Q V$ is similar to $Q$ and the diagonal elements of $V^T Q V$ are all zero. Since the trace of a matrix is the sum of its eigenvalues and since $Q$ is positive semidefinite, then this implies that $Q$ must be the zero matrix. This proves statement (4).

**Proof of statement (5).** This is proven analogously to the statement (3). Assume first that $R \prec 0$. Then, we have that $\langle R, Q \rangle = \sum_{i=1}^{n} \mu_i w_i^T Q w_i$ where $\mu_i$ and $w_i$ are the eigenvalues and the eigenvectors of the matrix $R$. This implies that $\langle R, Q \rangle < 0$ for all $Q \succeq 0$, $Q \neq 0$.

We show now the reverse implication. Assume that $\langle R, Q \rangle < 0$ for all $Q \succeq 0$, $Q \neq 0$. Then, this must hold for all $Q = Q_i = w_i w_i^T \succeq 0$, $i = 1, \ldots, n$. This implies that we must have that $\langle R, Q_i \rangle = \mu_i < 0$. This proves the necessity and the statement (5).

**Proof of statement (6).** Since $Q$ is symmetric, then it can be written as $Q = \sum_{i=1}^{n} \theta_i z_i z_i^T$ where $\theta_i$ and $z_i$ are its eigenvalues and eigenvectors. Then,

$$\langle P, Q \rangle = \sum_{i=1}^{n} \theta_i \langle P, z_i z_i^T \rangle = \sum_{i=1}^{n} \theta_i z_i^T P z_i \leq \lambda_{\max}(P) \sum_{i=1}^{n} \theta_i = \lambda_{\max}(P) \|Q\|_* \quad (7)$$

The lower bound is proven analogously. This proves statement (6).

We will finally need the following definitions:
Definition 2.4 ($\mathcal{S}_{\geq 0}$-copositive definiteness). A continuous function $V : \mathbb{R}_{\geq 0} \times \mathcal{S}_{\geq 0}^n \mapsto \mathbb{R}$ is said to be uniformly $\mathcal{S}_{\geq 0}$-copositive definite if and only if

(a) $V(t, 0) = 0$ for all $t \geq 0$, and

(b) there exists a continuous, strictly increasing function $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$, $\alpha(0) = 0$, such that $V(t, X) \geq \alpha(||X||)$ for all $(t, x) \in \mathbb{R}_{\geq 0} \times \mathcal{S}_{\geq 0}^n$.

Moreover, the function $V$ is said to be radially unbounded if $\alpha(\cdot)$ is such that $\alpha(s) \to \infty$ as $s \to \infty$.

Definition 2.5 (Decrescent function [29]). A continuous function $V : \mathbb{R}_{\geq 0} \times \mathcal{S}_{\geq 0}^n \mapsto \mathbb{R}_{\geq 0}$ is said to be decrescent if there exists a continuous, strictly increasing function $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ with $\alpha(0) = 0$ and $\alpha(s) \to \infty$ as $s \to \infty$ such that $V(t, X) \leq \alpha(||X||)$ for all $(t, X) \in \mathbb{R}_{\geq 0} \times \mathcal{S}_{\geq 0}^n$.

3. Linear Matrix-Valued Symmetric Continuous-Time Systems

This section is devoted to the definition of linear matrix-valued symmetric continuous-time systems that evolves on the cone of positive semidefinite matrices together with their associated generators and state-transition operator. Using those definitions a necessary and sufficient condition for their uniform exponential stability is obtained in terms of a matrix-differential equation or a matrix-differential inequality. Those results are then specialized to the LTI case where additional spectral conditions are also obtained. The case of systems with delays is also considered for completeness and a necessary and sufficient stability condition is also obtained as well. The results are finally illustrated through the analysis of a certain class of linear time-varying stochastic systems subject to Brownian motions and Poissonian jumps, and a class of LTI time-delay systems subject to Brownian motions.

3.1. Preliminaries

Let us consider here the following class of matrix-valued continuous-time dynamical systems

\[
\dot{X}(t) = \text{Sym}[A_0(t)X(t)] + \sum_{i=1}^{N} A_i(t)X(t)A_i(t)^T + \mu(t)X(t), t \geq t_0
\]

\[
X(t_0) = X_0 \in \mathcal{S}_{\geq 0}^n
\]

where $A_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^{n \times n}$, $i = 0, \ldots, N$, and $\mu : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$ are piecewise-continuous and bounded. It is immediate to verify that the right-hand side is uniformly Lipschitz in $X$ which implies that there exists a unique global solution which can be written as

\[
X(t) = S(t, s)X(s), t \geq s \geq 0
\]

where $S(\cdot, \cdot)$ is the state transition operator verifying

\[
\frac{\partial}{\partial t} S(t, s)X = C_t(S(t, s)X), \quad X \in \mathcal{S}_{\geq 0}^n
\]

where $C_t$ is the generator given by

\[
C_t : \mathcal{S}^n \mapsto \mathcal{S}^n
\]

\[
X \mapsto A_0(t)X + XA_0(t)^T + \sum_{i=1}^{N} A_i(t)XA_i(t)^T + \mu(t)X.
\]

The adjoint operator of $C_t$, denoted by $C_t^*$, is given by

\[
C_t^* : \mathcal{S}^n \mapsto \mathcal{S}^n
\]

\[
X \mapsto A_0(t)^TX + XA_0(t) + \sum_{i=1}^{N} A_i(t)^TXA_i(t) + \mu(t)X.
\]

The following result shows that the system (8) leaves the cone $\mathcal{S}_{\geq 0}^n$ invariant:
Proposition 3.1. The matrix-valued differential equation \( S \) leaves the cone of positive semidefinite matrices invariant, that is, for any \( X_0 \succeq 0 \), we have that \( X(t, t_0, X_0) \succeq 0 \) for all \( t \geq 0 \).

Proof: The solution to \( S \) can be shown to satisfy the expression

\[
X(t) = \Phi(t, t_0)X_0\Phi(t, t_0)^T + \sum_{i=1}^{N} \int_{t_0}^{t} \Phi(t, s)A_i(s)X(s)A_i(s)^T \Phi(t, s)^T ds
\]

(13)

where \( \Phi(t, s) \) is the state-transition matrix associated with the system \( \dot{x}(t) = (A_0(t) + \frac{1}{2}\mu(t)I_n)x(t) \). Clearly, if \( X(t_0) = X_0 \succeq 0 \), then \( X(t) \succ 0 \) for all \( t \geq t_0 \). 

Definition 3.2. The zero solution of the system \( S \) is said to be globally uniformly exponentially stable with rate \( \alpha \) if there exist some scalars \( \alpha > 0 \) and \( \beta > 1 \) such that

\[
||X(t)|| \leq \beta e^{-\alpha(t-t_0)}||X_0||
\]

(14)

for all \( t \geq t_0 \), all \( t_0 \geq 0 \), and all \( X_0 \in S^n_{\succeq 0} \).

3.2. Main results

With the previous definitions and results in mind, we are now able to state the following result characterizing the uniform exponential stability of the system \( S \):

Theorem 3.3. The following statements are equivalent:

(a) The system \( S \) is uniformly exponentially stable.

(b) There exist a differentiable matrix-valued function \( P : \mathbb{R}_{\geq 0} \mapsto S^n_{\succ 0} \) and a continuous matrix-valued function \( Q : \mathbb{R}_{\geq 0} \mapsto S^n_{\succ 0} \) such that

\[
\alpha_1 I \preceq P(t) \preceq \alpha_2 I, \ 0 < \alpha_1 \leq \alpha_2
\]

\[
\beta_1 I \preceq Q(t) \preceq \beta_2 I, \ 0 < \beta_1 \leq \beta_2
\]

(15)

and such that **Lyapunov differential equation**

\[
\dot{P}(t) + C_i^*(P(t)) + Q(t) = 0
\]

(16)

hold for all \( t \geq 0 \).

(c) There exists a differentiable matrix-valued function \( P : \mathbb{R}_{\geq 0} \mapsto S^n_{\succeq 0} \) such that

\[
\alpha_1 I \preceq P(t) \preceq \alpha_2 I, \ 0 < \alpha_1 \leq \alpha_2
\]

(17)

and such that **Lyapunov differential inequality**

\[
\dot{P}(t) + C_i^*(P(t)) \preceq -\alpha_3 I
\]

(18)

hold for all \( t \geq 0 \).

Proof: It is immediate to see that the statements (b) and (c) are equivalent.

**Proof that statement (b) implies statement (c).** To this aim, define the function \( V(t, X) = \langle P(t), X \rangle \) where \( P(\cdot) \) satisfies the conditions of statement (b). Since \( P(t) \) is positive definite, uniformly bounded and uniformly bounded away from 0, then, from Proposition 2.3, we have that \( V \) is \( S^n_{\succeq 0} \)-copositive definite,
radially unbounded and decrescent. Evaluating the derivative of the function $V$ along the trajectories of the system (8) yields
\[
\dot{V}(t, X(t)) = \langle \dot{P}(t), X(t) \rangle + \langle P(t), C_s(X(t)) \rangle
\]
\[
= \langle \dot{P}(t), X(t) \rangle + \langle C_s^*(P(t)), X(t) \rangle
\]
\[
= -\langle Q(t), X(t) \rangle
\]
\[
\leq -\frac{\beta_1}{\alpha_2} \langle P(t), X(t) \rangle = -\frac{\beta_1}{\alpha_2} V(t, X(t)).
\]
Therefore, $V$ is a global, uniform and exponential Lyapunov function for the system (8), which proves that the system (8) is globally uniformly exponentially stable with rate $\beta_1/\alpha_2$.

**Proof that statement (a) implies statement (b).** To this aim, assume that the system (8) is uniformly exponentially stable, then there exist $M, \theta > 0$ such that
\[
\|S(s, t)X\|_s \leq Me^{-\theta(s-t)}\|X\|_s, s \geq t \geq 0
\]
for all $X \in \mathbb{S}_{\geq 0}^n$. We then define $\hat{P} : \mathbb{R}_{\geq 0} \to \mathbb{S}_{\geq 0}^n$ as
\[
\langle \hat{P}(t), X(t) \rangle = \int_t^\infty \langle Q(s), X(s) \rangle ds
\]
where $Q : \mathbb{R}_{\geq 0} \to \mathbb{S}_{\geq 0}^n$ and such that $\beta_1 I \preceq Q(t) \preceq \beta_2 I$ for some $\beta_1, \beta_2 > 0$ and for all $t \geq 0$. We need to prove first that there exist positive constant $\alpha_1, \alpha_2$ such that $\alpha_1 I \preceq Q(t) \preceq \alpha_2 I$ for all $k \geq 0$.

Clearly, we have that
\[
\langle \hat{P}(t), X(t) \rangle = \lim_{\tau \to \infty} \int_t^\tau \langle Q(s), S(s, t)X(t) \rangle ds
\]
\[
\leq \lim_{\tau \to \infty} \beta_2 \|X(t)\|_s \int_t^\tau e^{-\theta(s-t)} ds
\]
\[
\leq \lim_{\tau \to \infty} \frac{\beta_2 M(1 - e^{-\theta \tau})}{\theta} \|X(t)\|_s
\]
\[
= \frac{\beta_2 M}{\theta} \|X(t)\|_s =: \alpha_2 \|X(t)\|_s
\]
which implies that $P(t) \preceq \alpha_2 I$. We need to prove now that there exists an $\alpha_1 > 0$ such that $P(t) \succeq \alpha_1 I$. To this aim, assume that $Q$ is differentiable with uniformly bounded derivative (which is a weak assumption as $Q$ can always be chosen to be a constant). We then obtain
\[
\frac{d}{ds} \langle Q(s), X(s) \rangle = \langle \dot{Q}(s), X(s) \rangle + \langle Q(s), \dot{C}_s(X(s)) \rangle
\]
\[
= \langle \dot{Q}(s), X(s) \rangle + \langle C_s^*(Q(s)), X(s) \rangle
\]
\[
= \langle Q(s) + C_s^*(Q(s)), X(s) \rangle
\]
\[
= \left( \dot{Q}(s) + \text{Sym}[Q(s)A_0(s)] + \sum_{i=1}^N A_i(s)^T Q(s)A_i(s), X(s) \right).
\]
Since both $Q$ and $\dot{Q}$ are uniformly bounded, then there exists a sufficiently large $c > 0$ such that
\[
\frac{d}{ds} \langle Q(s), X(s) \rangle \geq -c \cdot \langle Q(s), X(s) \rangle
\]
for all $X(s) \in \mathbb{S}_{\geq 0}^n$. Integrating both sides from $t$ to $\infty$ yields
\[
\int_t^\infty \frac{d}{ds} \langle Q(s), X(s) \rangle ds \geq -c \cdot \langle \hat{P}(t), X(t) \rangle.
\]
or, equivalently,
\[- \langle Q(t), X(t) \rangle \geq -c \cdot \langle \bar{P}(t), X(t) \rangle \]
where we have used that $X$ is uniformly exponentially stable. Reorganizing the above expression, we get that
\[ \langle Q(t) - c\bar{P}(t), X(t) \rangle \leq 0. \]
(27)
As this inequality holds for all $X(t) \in \mathbb{S}^n_{\geq 0}$, this implies that $\bar{P}(t) \geq Q(t)/c \geq \frac{\bar{P}_1}{c} I =: \alpha_1 I$. So, we have proven that there exist positive constant $\alpha_1, \alpha_2$ such that $\alpha_1 I \leq P(t) \leq \alpha_2 I$ for all $k \geq 0$.

Computing now the time-derivative of (21) yields
\[ \langle \dot{\bar{P}}(t), X(t) \rangle + \langle \bar{P}(t), C_t(X(t)) \rangle = -\langle Q(t), X(t) \rangle \]
which is equivalent to
\[ \langle \dot{\bar{P}}(t) + C^*_t(P(t)) + Q(t), X(t) \rangle = 0. \]
(29)
As this is true for all $X(t) \geq 0$, then this implies that the conditions of statement (b) hold with $P = \bar{P}$ and the same $Q$. This proves the desired result.

The above result demonstrates that the stability of systems of the form (8) can be proven using a specific class of linear $\mathbb{S}^n_{\geq 0}$-copositive Lyapunov functions in a way that is similar to that of the analysis of linear positive systems using linear $\mathbb{R}^n_{\geq 0}$-copositive Lyapunov functions. The proposed approach leads to stability conditions taking the form differential Lyapunov equations of inequalities. In the case of LTI systems, however, we can also connect the stability properties of the system (8) to the spectral properties of a matrix and those of a linear operator as shown below:

**Theorem 3.4 (LTI case).** Assume that the system (8) is time-invariant. Then, the following statements are equivalent:

(a) The system (8) is exponentially stable.

(b) The conditions of Theorem 3.3 and (c) hold with constant matrix-valued functions $P(t) \equiv P \in \mathbb{S}^n_{> 0}$ and $Q(t) \equiv Q \in \mathbb{S}^n_{> 0}$.

(c) The matrix
\[ M_C := F^T \left( I_n \otimes A_0 + A_0 \otimes I_n + \sum_{i=1}^N A_i \otimes A_i + \alpha I_{n^2} \right) F \]
\[ (30) \]
is Hurwitz stable where $F \in \mathbb{R}^{n^2 \times n(n+1)/2}$, $F^TF = I$ is such that
\[ \text{vec}(X) = F\text{vec}(X) \]
\[ (31) \]
where $\text{vec}(X)$ is the vector obtained from $\text{vec}(X)$ where we have removed the redundant entries.

(d) We have that $\text{sp}(C) \subset \mathbb{C}_{< 0}$ where
\[ \text{sp}(C) := \{ \lambda \in \mathbb{C} : C(X) = \lambda X \text{ for some } X \in \mathbb{S}^n, X \neq 0 \} \]
\[ (32) \]
and $C$ is defined as the time-invariant version of the operator (11).

**Proof:** The proof of the equivalence between the two first statements follow from Theorem 3.3 specialized to the LTI case.
Proof that statement (a) is equivalent to statement (d). This follows from the fact that the LTI version of the system \([8]\) is globally exponentially stable if and only if the eigenvalues associated with its dynamics have negative real part. Indeed, the solution to that LTI system is given by

\[
X(t) = \sum_{i=1}^{n(n+1)/2} e^{\lambda_i t} \langle X_0, X_i \rangle \tag{33}
\]

where \((\lambda_i, X_i)\) denotes the \(i\)-th pair of eigenvalues and eigenmatrices.

Proof that statement (c) is equivalent to statement (d). Using Kronecker calculus, Statement (d) is equivalent to saying that

\[
\left( \sum_{i=0}^{N} I_n \otimes A_0 + A_0 \otimes I_n + \sum_{i=1}^{N} A_i \otimes A_i + \alpha I_{n^2} - \lambda I_{n^2} \right) \text{vec}(X) = 0, \quad X \neq 0, X \in \mathbb{S}^n \tag{34}
\]

implies \(\Re(\lambda) < 0\). The eigenvalues depend on the domain of the operator which is not the full space \(\mathbb{R}^{n^2}\) here but the subspace associated with symmetric matrices \(X\). This can be done by restricting the matrix above to that subspace by projection. This can be achieved using the matrix \(F\) as

\[
\left( \mathcal{M}_c - \lambda I_{\overline{h}+1} \right) \text{vec}(X) = 0, \tag{35}
\]

which is equivalent to statement (c).

The above result sheds some light and draws a parallel between standard results for linear systems theory where stability properties are connected to spectral properties of some operators/matrices.

3.3. Systems with delays

Interestingly, it is possible to extend some of the above results to the case of systems with delays, leading to an extension of the results in \([33]\). The LTI time-delay system version of the system \([8]\) is given by

\[
\dot{X}(t) = \text{Sym}[A_0 X(t)] + \sum_{i=1}^{N} A_i X(t) A_i^T + \mu X(t) + \sum_{i=1}^{N} B_i X_i(t - h_i) B_i^T, \quad t \geq 0 \tag{36}
\]

\[
X_0(s) = \Xi(s) \in \mathbb{S}_{\geq 0}^n, \quad s \in [-\overline{h}, 0]
\]

where the delays \(h_i\)'s are positive scalars with \(\overline{h} := \max_i \{h_i\}\), and \(\Xi \in C([-\overline{h}, 0], \mathbb{S}_{\geq 0}^n)\) is the initial condition. As for the system \([8]\), it can be proven that this system leaves the cone of positive semidefinite invariant using exactly the same arguments. Define, moreover, the operators

\[
\mathcal{H}_i : \mathbb{S}^n \mapsto \mathbb{S}^n \quad \quad X \mapsto B_i X B_i^T. \tag{37}
\]

and their adjoints

\[
\mathcal{H}_i^* : \mathbb{S}^n \mapsto \mathbb{S}^n \quad \quad X \mapsto B_i^T X B_i. \tag{38}
\]

The following result demonstrates that the same delay-independent stability property holds for the system \([36]\) as for linear time-invariant positive systems \([6, 9, 22, 26]\):

**Theorem 3.5.** The following statements are equivalent:

(a) The system \([36]\) is exponentially stable for all delay values \(h_i \geq 0, \ i = 1, \ldots, N\).

(b) The system \([36]\) with zero delays (i.e. \(h_i = 0, \ i = 1, \ldots, N\)) is exponentially stable.
(c) There exist matrices \( P, Q_i \in S^n_{>0}, i = 1, \ldots, N \), such that the conditions
\[
C^*(P) + \sum_{i=1}^{N} Q_i < 0 \quad \text{and} \quad \mathcal{H}_i^*(P) - Q_i \preceq 0, \quad i = 1, \ldots, N
\]
hold.

(d) We have that \( \text{sp}(\mathcal{C}) \subset \mathbb{C}_{<0} \) where
\[
\text{sp}(\mathcal{C}) := \left\{ \lambda \in \mathbb{C} : \mathcal{C}(X) + \sum_{i=1}^{N} \mathcal{H}_i(X) = \lambda X \text{ for some } X \in \mathbb{R}^n, X \neq 0 \right\}.
\]

Proof: We know from Theorem\[\text{(33)}\] that the statements \((b)\) and \((c)\) are equivalent. It is also immediate to see that the statement \((d)\) implies the statement \((b)\). So, we need to prove the two remaining implications.

Proof that the statement \((b)\) implies the statement \((c)\). Let us consider the following functional
\[
V(X_t) = \langle P, X_t(0) \rangle + \sum_{i=1}^{N} \int_{-h_i}^{0} \langle Q_i, X_t(s) \rangle \, ds
\]
where \( X_t(s) = X(t+s), s \in [-\bar{h}, 0] \), for which we have
\[
\lambda_{\min}(P) \|X_t(0)\|_* \leq V(X_t) \leq (\lambda_{\max}(P) + \lambda_{\max}(Q))\|X_t\|_{c,*}
\]
where \( \|X\|_{c,*} := \max_{s \in [-\bar{h}, 0]} \|X_t(s)\|_* \). Computing the derivative yields
\[
\dot{V}(X_t) = \langle P, \dot{X}(t) \rangle + \sum_{i=1}^{N} \left[ \langle Q_i, X(t) \rangle - \langle Q_i, X(t-h_i) \rangle \right]
\]
\[
= \langle P, \mathcal{C}(X(t)) \rangle + \sum_{i=1}^{N} \left[ \langle P, \mathcal{H}_i(X(t-h_i)) \rangle + \langle Q_i, X(t) \rangle - \langle Q_i, X(t-h_i) \rangle \right]
\]
\[
= \left( C^*(P) + \sum_{i=1}^{N} Q_i, X(t) \right) + \sum_{i=1}^{N} \langle \mathcal{H}_i^*(P) - Q_i, X(t-h_i) \rangle
\]
Under the conditions of the statement \((b)\), there exists a small enough \( \varepsilon > 0 \) such that \( C^*(P) + \sum_{i=1}^{N} Q_i \preceq -\varepsilon I \) which implies
\[
\dot{V}(X_t) \leq -\varepsilon \|X_t(0)\|_*.
\]
Therefore, the functional \((41)\) is a Lyapunov-Krasovskii functional \([21]\) for the system \((39)\) and the system is exponentially stable regardless the value of the delays as the conditions do not depend on the values of the delays. This proves the implication.

Proof that the statement \((b)\) implies the statement \((c)\). Assume that the system \((36)\) with zero delays (i.e., \( h_i = 0, i = 1, \ldots, N \)) is exponentially stable. This is equivalent to saying that the system
\[
\dot{X}(t) = \mathcal{C}(X(t)) + \sum_{i=1}^{N} \mathcal{H}_i(X(t))
\]
is exponentially stable. This implies that there exist a matrix \( P \in S^n_{>0} \) and an \( \varepsilon > 0 \) such that
\[
C^*(P) + \sum_{i=1}^{N} \mathcal{H}_i^*(P) \preceq -2\varepsilon I.
\]
Define now
\[ Q_i = \mathcal{H}_i^*(P) + \frac{\varepsilon}{N} I_n > 0 \] (47)
and substituting this value in (49) yields
\[
\begin{align*}
\mathcal{C}^*(P) + \sum_{i=1}^{N} Q_i &= \mathcal{C}^*(P) + \sum_{i=1}^{N} \mathcal{H}_i^*(P) \preceq -\varepsilon I < 0, \\
\mathcal{H}_i^*(P) - Q_i &= -\frac{\varepsilon}{N} I_n \preceq 0,
\end{align*}
\]
which implies that the conditions in the statement \((49)\) hold, thereby proving the desired result. The proof is completed. \qed

The above result is the semidefinite analogue of the results obtained for linear positive systems with delays. The stability properties do not depend on the value of the delays and the system is exponentially stable for all possible values of the delays if and only if the system with zero-delays is exponentially stable. It also admits an extension to the periodic systems case using a periodic Lyapunov-Krasovskii functional under the same assumptions as in \([10]\). This extension is omitted for brevity.

3.4. Applications

The objective of this section is to demonstrate that many results from the literature can be recovered and extended using the obtained results. To avoid repetitions, in all the sections below, \(x, x_0 \in \mathbb{R}^n\) denote the state of the system and the initial condition, \(t_0 \geq 0\) is the initial time, \(W(t) = (W_1(t), \ldots, W_N(t))\) is a vector of \(N\) independent Wiener processes which are also independent of the state \(x(t)\) and all the other signals possibly involved in the system. To any of the systems, we can associate a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) with natural filtration satisfying the usual conditions.

3.4.1. Stochastic LTV systems with Poissonian jumps

Let us define the class of stochastic LTV with Poissonian jumps as
\[
\begin{align*}
dx(t) &= A_0(t)x(t)dt + \sum_{i=1}^{N} A_i(t)x(t)dW_i(t) + \sum_{i=1}^{N} (M_i(t) - I)x(t)d\eta_i(t), t \geq t_0 \\
x(t_0) &= x_0
\end{align*}
\] (49)
where \((\eta_1(t), \ldots, \eta_N(t))\) is a vector of \(N\) independent of Poisson processes with rates \((\lambda_1, \ldots, \lambda_N)\) and which are independent of \(x(t)\), \(W(t)\) and of the matrix-valued functions \(A_0, A_1, \ldots, A_N, M_1, \ldots, M_N : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times n}\) which are assumed to be piecewise continuous and bounded.

Several stability notions exist for the system above \([27]\) but we will be mostly interested in the uniform exponential mean-square stability defined below:

**Definition 3.6.** The system \((49)\) is uniformly exponentially mean-square stable if there exist some scalars \(\alpha > 0\) and \(\beta > 1\) such that
\[
\mathbb{E}[\|x(t)\|_2^2] \leq \beta e^{-\alpha(t-t_0)}\mathbb{E}[\|x_0\|_2^2]
\] (50)
holds for all \(t \geq t_0\), all \(t_0 \geq 0\), and all \(x_0 \in \mathbb{R}^n\).

One can observe that \(\mathbb{E}[\|x(t)\|_2^2] = \text{trace} \mathbb{E}[x(t)x(t)^T] \) which means that this notion of stability is about the exponential convergence of the sum of the (nonnegative) eigenvalues of the covariance matrix associated with the system \((49)\) to zero. This remark leads us to the following result:

**Theorem 3.7.** The following statements are equivalent:
(a) The system \((49)\) is uniformly exponentially mean-square stable.
(b) There exist a differentiable matrix-valued function $P : \mathbb{R}_{\geq 0} \rightarrow \mathbb{S}^n_{\geq 0}$ and a continuous matrix-valued function $Q : \mathbb{R}_{\geq 0} \rightarrow \mathbb{S}^n_{\geq 0}$ such that

\[
\begin{align*}
\alpha_1 I & \preceq P(t) \preceq \alpha_2 I, \\
\beta_1 I & \preceq Q(t) \preceq \beta_2 I,
\end{align*}
\]

and such that Lyapunov differential equation

\[
\dot{P}(t) + \text{Sym}[P(t)A_0(t)] + \sum_{i=1}^{N} A_i(t)^T P(t) A_i(t) + \sum_{i=1}^{N} \lambda_i \Delta_i(t) = -Q(t)
\]

where $\Delta_i(t) := M_i(t)^T P(t) M_i(t) - P(t)$, $i = 1, \ldots, N$, hold for some positive constants $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2$ and for all $t \geq t_0$ and all $t_0 \geq 0$.

**Proof**: The system (54) is uniformly exponentially mean-square stable if and only if the dynamics of the covariance matrix $X(t) := \mathbb{E}[x(t)x(t)^T]$ given by

\[
\begin{align*}
\dot{X}(t) &= A_0(t)X(t) + X(t)A_0(t)^T + \sum_{i=1}^{N} A_i(t)X(t)A_i(t)^T + \sum_{i=1}^{N} \lambda_i \Delta_i(t), \quad t \geq t_0 \\
X(t_0) &= X_0 = \mathbb{E}[x_0x_0^T]
\end{align*}
\]

where $\Delta_i(t) := M_i(t)X(t)M_i(t)^T - X(t)$, $i = 1, \ldots, N$, is uniformly exponentially stable. Applying then Theorem 3.3 yields the result. \(\diamondsuit\)

### 3.4.2. A class of diffusion processes with delays

Let us consider the following class of diffusion systems with delays

\[
\begin{align*}
\text{d}x(t) &= A_0x(t)\text{d}t + \sum_{i=1}^{N} A_i x(t-h_i)\text{d}W_i(t) \\
x(s) &= \phi(s), \quad s \in [-\tilde{h}, 0]
\end{align*}
\]

where $\phi \in C([-\tilde{h}, 0], \mathbb{R}^n)$ is the initial condition. The notion of stability we will be interested here is also the mean square exponential stability which we adapt to systems with delays as

**Definition 3.8.** The system (54) is mean-square exponentially stable if there exist some scalars $\alpha > 0$ and $\beta > 1$ such that

\[
\mathbb{E}[\|x(t)\|^2] \leq \beta e^{-\alpha t}\mathbb{E}\left[\max_{s \in [-\tilde{h}, 0]} \|\phi(s)\|^2\right]
\]

holds for all $t \geq 0$ and all initial condition $\phi \in C([-\tilde{h}, 0], \mathbb{R}^n)$.

Once again, this is related to the exponential convergence of the eigenvalues of the covariance matrix, which then leads us to the following result that can be seen as a generalization of Theorem 5 in [33]:

**Theorem 3.9.** The following statements are equivalent:

(a) The system (54) is mean-square exponentially stable for all $h_i \geq 0$, $i = 1, \ldots, N$.

(b) The system (54) with zero-delays (i.e. with $h_i = 0$, $i = 1, \ldots, N$) is mean-square exponentially stable.

**Proof**: Let $X(t) := \mathbb{E}[x(t)x(t)^T]$ to be the covariance matrix of the system (54). The dynamics of this matrix is described by

\[
\dot{X}(t) = A_0X(t) + X(t)A_0^T + \sum_{i=1}^{N} A_i^T X(t-h_i)A_i.
\]

Applying now Theorem 3.3 yields the result. \(\diamondsuit\)
3.4.3. Time-varying Markov jump linear systems

Let us consider the following class of systems described by a linear stochastic differential equation subject to Markovian jumps

\[
\begin{align*}
\dot{x}(t) &= A_{\sigma(t)}(t)x(t)dt + \sum_{i=1}^{N} B_{\sigma(t),i}(t)x(t)dB_i(t), \quad t \geq t_0 \\
x(t_0) &= x_0, \\
\sigma(t_0) &= \sigma_0
\end{align*}
\]

where \(x_0\) is the initial value of the parameter \(\sigma : [t_0, \infty) \rightarrow \{1, \ldots, M\}\) which changes value following a finite Markov chain as

\[
\mathbb{P}(\sigma(t+h) = j|\sigma(t) = i) = \left\{ \begin{array}{ll}
\pi_{ij}h + o(h), & i \neq j, \\
1 + \pi_{ii}h + o(h), & i = j,
\end{array} \right.
\]

where \(\pi_{ij} \geq 0\) for all \(i \neq j\) and \(\pi_{ii} = -\sum_{j \neq i} \pi_{ij}\). We use here the stability definition of Definition 3.6 with the difference that the mean-square convergence must also hold for all initial conditions and paths of the Markov chain. This leads to the following result which can be seen as a generalization of the results in [16, 24]:

**Theorem 3.10.** The system \((57)-(60)\) is uniformly exponentially mean-square stable if and only if there exist differentiable matrix-valued functions \(P_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{S}^n_{\geq 0}, i = 1, \ldots, M,\) and continuous matrix-valued functions \(Q_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{S}^n_{\geq 0}, i = 1, \ldots, M,\) such that

\[
\begin{align*}
\alpha_1 I &\preceq P_i(t) \preceq \alpha_2 I, \\
\beta_1 I &\preceq Q_i(t) \preceq \beta_2 I,
\end{align*}
\]

and the coupled Lyapunov differential equations

\[
\dot{P}_i(t) + A_i(t)^TP_i(t) + P_i(t)A_i(t) + \sum_{j=1}^{N} B_{i,j}(t)^T P_i(t) B_{i,j}(t) + \sum_{j=1}^{M} \pi_{ij} P_j(t) = -Q_i(t)
\]

hold for all \(i = 1, \ldots, M\) and some \(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 > 0\).

**Proof:** Let \(1_i(\sigma(t))\) be the indicator function of the state \(i\) defined as \(1_i(\sigma(t)) = 1\) if \(\sigma(t) = i\), and 0 otherwise. Then, defining \(X_i(t) = \mathbb{E}[x(t)x(t)^T 1_i(\sigma(t))]\), we get that

\[
\dot{X}_i(t) = A_i(t)X_i(t) + X_i(t)A_i(t)^T + \sum_{j=1}^{N} B_{i,j}(t)X_i(t)B_{i,j}(t)^T + \sum_{j=1}^{M} \pi_{ij}X_j(t)
\]

for all \(i = 1, \ldots, M\). While it is readily seen that while \(\mathbb{E}[x(t)x(t)^T] = \sum_{i=1}^{M} X_i(t)\), there is no closed-form expression for the derivative of the covariance matrix associated with the system. Let \(E_i\) be the matrix composed of the columns \(\{e_1, \ldots, e_M\}\) in cyclic ascending order where \(e_1\) is the \(i\)-th column and define \(\bar{\Pi}\) as \(\bar{\Pi} = \sqrt{\pi_{ii}}\). Then, we have that

\[
\begin{align*}
\text{diag}_{i=1}^{M} \left( \sum_{j=1}^{M} \sum_{j \neq i} \pi_{ij}X_j(t) \right) &= \sum_{j=2}^{M} \bar{A}_j X(t)\bar{A}_j^T \\
\end{align*}
\]

where \(\bar{A}_j = (E_i \odot \bar{\Pi}) \odot I_n, j = 2, \ldots, M\). Using this expression, one can rewrite all the matrix-valued differential equations \((61)\) in compact form as

\[
\dot{X}(t) = \bar{A}_0(t)X(t) + X(t)\bar{A}_0(t)^T + \sum_{j=1}^{N} \bar{B}_j(t)X(t)\bar{B}_j(t)^T + \sum_{j=2}^{M} \bar{A}_j X(t)\bar{A}_j^T
\]
where \( X(t) := \text{diag}_{i=1}^{M} \{ X_i(t) \} \),
\[
\bar{A}_0(t) = \text{diag} \left\{ \begin{array}{c}
A_i(t) + \frac{\pi_{ii}}{2} I
\end{array} \right\}, \quad \text{and} \quad \bar{B}_j(t) = \text{diag}(B_{i,j}(t)),
\]
\begin{equation}
(64)
\end{equation}

Using Theorem 3.3, we obtain the stability condition
\[
\dot{P}(t) + \bar{A}_0(t)^T P(t) + P(t) \bar{A}_0(t) + \sum_{j=1}^{N} \bar{B}_j(t)^T P(t) \bar{B}_j(t) + \sum_{j=1}^{M} \bar{A}_j^T P(t) \bar{A}_j = -Q(t).
\]
\begin{equation}
(65)
\end{equation}

As the state of this system has a block-diagonal structure, the state space is actually \( S_{n} \geq 0 \times \ldots \times S_{n} \geq 0 \) (\( M \) times) and not the full space \( S_{nM}^{\geq 0} \). This allows us to also restrict ourselves to block-diagonal matrices \( P(t) = \text{diag}_{i=1}^{M} (P_i(t)) \) and \( Q(t) = \text{diag}_{i=1}^{M} (Q_i(t)) \) which all satisfy the conditions of Theorem 3.3. Exploiting the block-diagonal structure yields
\[
\sum_{j=2}^{M} \bar{A}_j^T P(t) \bar{A}_j = \text{diag} \left\{ \sum_{j=1}^{M} \pi_{ij} P_j(t) \right\}.
\]
\begin{equation}
(66)
\end{equation}
together with the conditions (60) after having expanded the other terms in (65).

\[\Box\]

4. Linear Matrix-Valued Symmetric Discrete-Time Systems

This section is devoted to the definition of linear matrix-valued symmetric discrete-time systems that evolves on the cone of positive semidefinite matrices together with their associated generators and state-transition operator. Using those definitions a necessary and sufficient condition for their uniform exponential stability is obtained in terms of a matrix-difference equation or a matrix-difference inequality. Those results are then specialized to the LTI case where additional spectral conditions are also obtained. The case of systems with delays is also considered for completeness and a necessary and sufficient stability condition is also considered. The results are finally illustrated through the analysis of a certain class of linear time-varying stochastic systems subject to Poissonian jumps, random signals and delays.

4.1. Preliminaries

Let us consider here the following class of matrix-valued discrete-time dynamical systems
\[
X(k+1) = \sum_{i=0}^{N} J_i(k)X(k)J_i(k)^T, \quad k \geq k_0
\]
\[
X(k_0) = X_0 \in S^{\geq 0}_n
\]
\begin{equation}
(67)
\end{equation}

where the matrix-valued functions \( J_i : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^{n \times n}, \ i = 0, \ldots, N \), are bounded. It is clear from the above expression that there is a unique solution to this difference equation, which we denote by \( X(k, k_0, X_0) \). This solution can be written as
\[
X(k) = D_{k,\ell}(X_{\ell}), \quad k \geq \ell \geq k_0
\]
\begin{equation}
(68)
\end{equation}

where
\[
D_{k,\ell}(X) = \begin{cases}
X & \text{if } k = \ell \\
D_{\ell}(X) & \text{if } k = \ell + 1 \\
(D_{k-1} \circ \ldots \circ D_{\ell})(X) & \text{if } k > \ell + 1
\end{cases}
\]
\begin{equation}
(69)
\end{equation}
and

\[
\mathcal{D}_k : S^n \mapsto S^n \\
X \mapsto \sum_{i=0}^{N} J_i(k)XJ_i(k)^T.
\] (70)

The adjoint of the operator \(\mathcal{D}_k\), denoted by \(\mathcal{D}_k^*\), is given by

\[
\mathcal{D}_k^* : S^n \mapsto S^n \\
X \mapsto \sum_{i=0}^{N} J_i(k)^TXJ_i(k).
\] (71)

The following result, which is the discrete-time analogue of Proposition 3.1, shows that the system (67) leaves the cone \(S^n_{\geq 0}\) invariant:

**Proposition 4.1.** The matrix-valued difference equation (67) leaves the cone of positive semidefinite matrices invariant, that is, for any \(X_0 \succeq 0\), we have that \(X(k,k_0,X_0) \succeq 0\) for all \(k \geq k_0\) and all \(k_0 \geq 0\).

**Proof:** The proof follows from the same lines as the proof of Proposition 3.1 and is thus omitted. 

**Definition 4.2.** The zero solution of the system (67) is globally uniformly exponentially stable with rate \(\rho\) if there exist some constants \(\beta \geq 1\) and \(\rho \in (0,1)\) such that

\[
||X(k,k_0,X_0)|| \leq \beta \rho^{-k-k_0}||X_0||
\] (72)

for all \(k \geq k_0\), all \(k_0 \in \mathbb{Z}_{\geq 0}\), and all \(X_0 \in S^n_{\geq 0}\).

4.2. Main results

With the previous definitions and results in mind, we are now able to state the following result characterizing the uniform exponential stability of the system (67):

**Theorem 4.3 (LTV Case).** The following statements are equivalent:

(a) The system (67) is uniformly exponentially stable.

(b) There exist some matrix-valued functions \(P, Q : \mathbb{Z}_{\geq 0} \mapsto S^n_{\succ 0}\) such that

\[
\alpha_1 I \preceq P(k) \preceq \alpha_2 I, \\
\beta_1 I \preceq Q(k) \preceq \beta_2 I,
\] (73)

and such that Lyapunov difference equation

\[
\mathcal{D}_k^*(P(k+1)) - P(k) + Q(k) = 0
\] (74)

hold for some positive constants \(\alpha_1, \alpha_2, \beta_1, \beta_2\) and for all \(k \geq 0\).

(c) There exists a matrix-valued function \(P : \mathbb{Z}_{\geq 0} \mapsto S^n_{\succ 0}\) such that the

\[
\alpha_1 I \preceq P(k) \preceq \alpha_2 I
\] (75)

and such that Lyapunov difference inequality

\[
\mathcal{D}_k^*(P(k+1)) - P(k) \preceq -\alpha_3 I
\] (76)

hold for some positive constants \(\alpha_1, \alpha_2, \alpha_3\) and for all \(k \geq 0\).
Proof: It is immediate to see that the statements (b) and (c) are equivalent.

Proof that the statement (b) implies the statement (a). Define the function $V(k, X) = \langle P(k), X \rangle$ where $P(\cdot)$ satisfies the conditions of statement (b). Since $P(k)$ is positive definite, uniformly bounded and uniformly bounded away from 0, then, from Proposition 2.3 we have that $V$ is $\mathbb{S}_{\geq 0}$-copositive definite, radially unbounded and decrescent. Evaluating the successive difference of the function $V$ along the trajectories of the system (67) yields

$$V(k + 1, X(k + 1) - V(k, X(k))) = \langle P(k + 1), X(k + 1) \rangle - \langle P(k), X(k) \rangle = \langle P(k + 1), D_k(X(k)) \rangle - \langle P(k), X(k) \rangle = \langle D_k(P(k + 1)) - P(k), X(k) \rangle = -Q(k, X(k)) \leq -\frac{\beta_1}{\alpha_2} \langle P(k), X(k) \rangle = -\frac{\beta_1}{\alpha_2} V(k, X(k)).$$

Therefore, $V$ is a global, uniform and exponential Lyapunov function for the system (67), which proves that the system (67) is globally uniformly exponentially stable with rate $1 - \beta_1/\alpha_2$.

Proof that the statement (b) implies the statement (a). Assume that the system (67) is uniformly exponentially stable, then by definition there exist some $M \geq 1$ and $\rho \in (0, 1)$ such that

$$\|D_{k,k_0}(X)\|_* \leq M \rho^{k-k_0} \|X\|_*$$

holds for all $k \geq k_0$, all $k_0 \geq 0$, and all $X \in \mathbb{S}_{\geq 0}$. Now let $Q : Z_{\geq 0} \rightarrow \mathbb{S}_{\geq 0}$ be such that $\beta_1 I \preceq Q(k) \preceq \beta_2$ for some $\beta_1, \beta_2 > 0$ and define $\bar{P} : Z_{\geq 0} \rightarrow \mathbb{S}_{\geq 0}$ such that it verifies

$$\langle \bar{P}(k), X \rangle = \sum_{\tau=k}^{\infty} \langle Q(\tau), D_{\tau,k}(X) \rangle$$

for all $X \in \mathbb{S}_{\geq 0}$ and all $k \geq 0$, where $D_{k,\tau}$ is defined in (82), (83). We need to prove first that there exist scalars $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 I \preceq \bar{P}(k) \preceq \alpha_2 I$ for all $k \geq 0$.

Clearly, we have that

$$\langle \bar{P}(k), X \rangle = \lim_{T \rightarrow \infty} \sum_{\tau=k}^{T} \langle Q(\tau), D_{\tau,k}(X) \rangle \leq \beta_2 \lim_{T \rightarrow \infty} \sum_{\tau=k}^{T} \|D_{\tau,k}(X)\|_* \leq \beta_2 M \left( \lim_{T \rightarrow \infty} \sum_{\tau=k}^{T} \rho^{\tau-k} \|X\|_* \right) \leq \beta_2 M \left( \lim_{T \rightarrow \infty} \frac{1 - \rho^{T-k+1}}{1 - \rho} \|X\|_* \right) = \frac{\beta_2 M}{1 - \rho} \|X\|_* =: \alpha_2 \|X\|_*.$$
yields
\[ (\mathcal{D}_k^*(P(k+1)) - P(k) + Q(k), X) = 0. \] (83)

Since the above expression holds for all \( X \in \mathbb{S}_n^+ \), then this implies that the Lyapunov equation of statement [6] must hold with \( P = \bar{P} \) and the same \( Q \). This proves the result. \( \diamond \)

The following result is the specialization of the previous one to the LTI case where it is shown that it is possible to connect the stability properties of the system to some spectral properties of a matrix and an operator associated with the LTI version of the system \( [67] \):

**Theorem 4.4 (LTI case).** Assume that the system \( [67] \) is time-invariant. Then, the following statements are equivalent:

(a) The system \( [67] \) is exponentially stable.

(b) The conditions of the Theorem 4.3 in statements [8] and [9] hold with constant matrix-valued functions \( P(t) \equiv P \in \mathbb{S}_n^+ \) and \( Q(t) \equiv Q \in \mathbb{S}_n \).

(c) The matrix
\[
\mathcal{M}_D := F^T \left( \sum_{i=0}^{N} J_i \otimes J_i \right) F
\] (84)
is Schur stable.

(d) We have that \( \text{sp}(\mathcal{D}) \subset \mathbb{D} \) where
\[
\text{sp}(\mathcal{D}) := \{ \lambda \in \mathbb{C} : \mathcal{D}(X) = \lambda X \text{ for some } X \in \mathbb{S}_n, X \neq 0 \}.
\] (85)

and \( \mathcal{D} \) is defined as the time-invariant version of the operator \( [70] \).

**Proof:** The proof of the equivalence between the two first statements follow from Theorem 4.3 specialized to the LTI case.

**Proof that statement (a) is equivalent to statement (d).** This follows from the fact that the LTI version of the system \( [67] \) is exponentially stable if and only if the eigenvalues associated with its dynamics are located inside the unit disc. Indeed, the solution to that LTI system is given by
\[
X(k) = \sum_{i=1}^{n(n+1)/2} \lambda_i^k \langle X_0, X_i \rangle
\] (86)
where \( (\lambda_i, X_i) \) denotes the \( i \)-th pair of eigenvalues and eigenmatrices. This proves the equivalence.

**Proof that statement (c) is equivalent to statement (d).** Using Kronecker calculus, Statement (d) is equivalent to saying that
\[
\left( \sum_{i=0}^{N} J_i \otimes J_i - \lambda I_n^2 \right) \text{vec}(X) = 0, \ X \neq 0, X \in \mathbb{S}_n
\] (87)
implies \( |\lambda| < 1 \). The eigenvalues depend on the domain of the operator which is not the full space \( \mathbb{R}^{n^2} \) here but the subspace associated with symmetric matrices \( X \). This can be done by restricting the matrix above to that subspace by projection. This can be achieved using the matrix \( F \) as
\[
(\mathcal{M}_D - \lambda I_{\text{sym}}) \text{vec}(X) = 0,
\] (88)
which is equivalent to statement (c). \( \diamond \)
4.3. Systems with delays

Interestingly, it is possible to extend some of the above results to the case of systems with delays, leading to an extension of the results in [33] to the discrete-time case with multiple delays. The LTI time-delay system version of the system (67) is given by

\[
X(k + 1) = \sum_{i=0}^{N} J_i X(k) J_i^T + \sum_{i=1}^{N} H_i X(k - \tau_i) H_i^T, \quad k \geq 0
\]

where \( X(k) \) is the state of the system, \( \Xi : \{-\bar{\tau}, \ldots, 0\} \mapsto \mathbb{S}_{n,0}^+ \) is the initial condition and the delays are such that \( \tau_i \geq 0 \), \( i = 1, \ldots, N \), with \( \bar{\tau} = \max_i \{\tau_i\} \). It is also convenient to define the operators \( T_i \) as

\[
T_i : \mathbb{S}^n \mapsto \mathbb{S}^n \quad X \mapsto H_i X H_i^T
\]

and their adjoints \( T_i^* \) as

\[
T_i^* : \mathbb{S}^n \mapsto \mathbb{S}^n \quad X \mapsto H_i^T X H_i.
\]

It can also easily be shown that the solution of the system (89) remains confined in the positive semidefinite cone provided that the initial condition take values inside the positive definite cone.

The following result demonstrates that the same delay-independent stability property holds for the system (89) as for linear time-invariant positive discrete-time systems [2]:

**Theorem 4.5.** The following statements are equivalent:

(a) The system (89) is exponentially stable for any delay values \( \tau_i \geq 0 \), \( i = 1, \ldots, N \).

(b) The system (89) with zero delays (i.e. \( \tau_i = 0 \), \( i = 1, \ldots, N \)) is exponentially stable.

(c) There exist matrices \( P, Q_i \in \mathbb{S}_{n,0}^+ \), \( i = 1, \ldots, N \), such that the conditions

\[
D^*(P) - P + \sum_{i=1}^{N} Q_i < 0 \quad \text{and} \quad T_i^*(P) - Q_i \preceq 0, \quad i = 1, \ldots, N
\]

hold.

(d) We have that \( \text{sp}(\mathcal{D}) \subset \mathbb{D} \) where

\[
\text{sp}(\mathcal{D}) := \left\{ \lambda \in \mathbb{C} : \mathcal{D}(X) + \sum_{i=1}^{N} T_i(X) = \lambda X \text{ for some } X \in \mathbb{S}^n, X \neq 0 \right\}.
\]

**Proof:** We know from Theorem 4.4 that the statements (b) and (d) are equivalent. It is also immediate to see that the statement (c) implies the statement (b). So, we need to prove the two remaining implications.

**Proof that the statement (c) implies the statement (a).** Let us consider the following Lyapunov-Krasovskii functional

\[
V(X_k) = \langle P, X_k(0) \rangle + \sum_{i=1}^{N} \sum_{j=-\tau_i}^{-1} \langle Q_i, X_k(j) \rangle
\]

where \( X_k(s) = X(k + s) \), \( s \in \{-\bar{\tau}, \ldots, 0\} \). Clearly, we have that

\[
V(X_k) \geq \lambda_{\min}(P) \|X_k(0)\|_s
\]
and
\[ V(X_k) \leq (\lambda_{\text{max}}(P) + \lambda_{\text{max}}(Q))||X_k||_s \]  
(96)
where \( ||X_k||_{c,*} := \max_{s \in \{-\bar{\tau}, \ldots, 0\}} ||X(k+s)||_{c,*} \).

The discrete-time derivative of the functional (94) is given by
\[
V(X_{k+1}) - V(X_k) = \langle P, X(k+1) - X(k) \rangle + \sum_{i=1}^{N} \langle Q_i, X(k) \rangle - \langle Q_i, X(k-\tau_i) \rangle  
\]
(97)
\[
= \langle P, D(X(k)) + \sum_{i=1}^{N} T_i(X(k-\tau_i)) - X(k) \rangle + \sum_{i=1}^{N} \langle Q_i, X(k) \rangle - \langle Q_i, X(k-\tau_i) \rangle  
\]
\[
= \langle D^*(P) - P + \sum_{i=1}^{N} Q_i, X(k) \rangle + \sum_{i=1}^{N} \langle T_i^*(P) - Q_i, X(k-\tau) \rangle.  
\]
Under the conditions of the statement (c), there exists a small enough \( \varepsilon > 0 \) such that \( C^*(P) - P + \sum_{i=1}^{N} Q_i \preceq -\varepsilon I \) which implies
\[
V(X_{k+1}) - V(X_k) \leq -\varepsilon ||X_k(0)||_s  
\]
(98)
Therefore, the functional (94) is a discrete-time Lyapunov-Krasovskii functional [18] for the system (89) and the system is exponentially stable regardless the value of the delays as the conditions do not depend on the values of the delays. This proves the implication.

Proof that the statement (b) implies the statement (c). Assume that the system (89) with zero delays is exponentially stable, then the system
\[
X(k+1) = D(X(k)) + \sum_{i=1}^{N} T_i(X(k))  
\]
(99)
is exponentially stable. This implies that there exists a matrix \( P \in S_{>0}^n \) and an \( \varepsilon > 0 \) such that
\[
D^*(P) - P + \sum_{i=1}^{N} T_i^*(P) \preceq -2\varepsilon I.  
\]
(100)
Define now
\[
Q_i = T_i^*(P) + \frac{\varepsilon}{N} I_n \succ 0  
\]
for some \( \varepsilon > 0 \). Substituting that in (102) yields
\[
D^*(P) - P + \sum_{i=1}^{N} Q_i = D^*(P) - P + \sum_{i=1}^{N} T_i^*(P) \preceq -\varepsilon I,  
\]
\[
T_i^*(P) - Q_i = -\frac{\varepsilon}{N} I_n \preceq 0,  
\]
(102)
which implies that the conditions in statement (c) hold, thereby proving the desired result. The proof is completed. \( \Box \)

4.4. Applications

4.4.1. Stochastic LTV systems

We consider here the following stochastic discrete-time varying system
\[
x(k+1) = A_0(k)x(k) + \sum_{i=1}^{N} \nu_i(k)A_i(k)x(k)  
\]
\[
x(k_0) = x_0  
\]
(103)
where \( x(k), x_0 \in \mathbb{R}^n \) are the state of the system and the initial condition, \( k_0 \) is the initial time. The random signals \( \nu_1(k), \ldots, \nu_N(k) \in \mathbb{R} \) are assumed to be independent of each other and from the state \( x(k) \), and they are also assumed to be stationary.

We consider here the following notion of stability for the system (103):

**Definition 4.6.** The system (103) is uniformly exponentially mean-square stable if there exist some scalars \( \alpha > 0 \) and \( \beta > 1 \) such that

\[
E[|x(k)|^2] \leq \beta e^{\beta \rho(k_0)} E[|x_0|^2]
\]

holds for all \( k \geq t_0, k_0 \geq 0, \) and all \( x_0 \in \mathbb{R}^n \).

As in the continuous-time case, this notion of stability is directly connected to the dynamics of the covariance matrix associated with the system (103). This remark yields the following result:

**Theorem 4.7.** Assume that the random signals \( \nu_1(k), \ldots, \nu_N(k) \) have zero mean and unit variance. Then, the following statements are equivalent:

(a) The system (103) is uniformly exponentially mean-square stable.

(b) There exist some matrix-valued functions \( P, Q : \mathbb{Z}_{\geq 0} \mapsto \mathbb{S}^n_{\geq 0} \) such that

\[
\alpha_1 I \preceq P(k) \preceq \alpha_2 I, \\
\beta_1 I \preceq Q(k) \preceq \beta_2 I,
\]

and such that Lyapunov difference equation

\[
\sum_{i=0}^{N} A_i(k)^T P(k+1) A_i(k) - P(k) + Q(k) = 0
\]

hold for some positive constants \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) and for all \( k \geq 0 \).

**Proof:** Since the \( \nu_i \)'s have zero mean and unit variance, then the covariance matrix associated with the system (103) is described by the matrix difference equation

\[
X(k + 1) = \sum_{i=0}^{N} A_i(k) X(k) A_i(k)^T
\]

(107)

together with \( X(k_0) = E[x(k_0)x(k_0)^T] \). Applying then Theorem 4.3 yields the result. \( \diamond \)

**Theorem 4.8.** Assume that the random signals \( \nu_1(k), \ldots, \nu_N(k) \) follow a Bernoulli distribution with \( P(\nu_i(k) = 1) = p_i \) and \( P(\nu_i(k) = 1) = 1 - p_i, i = 1, \ldots, N \). Then, the following statements are equivalent:

(a) The system (103) is uniformly exponentially mean-square stable.

(b) There exist some matrix-valued functions \( P, Q : \mathbb{Z}_{\geq 0} \mapsto \mathbb{S}^n_{\geq 0} \) such that

\[
\alpha_1 I \preceq P(k) \preceq \alpha_2 I, \\
\beta_1 I \preceq Q(k) \preceq \beta_2 I,
\]

and such that Lyapunov difference equation

\[
\sum_{i=0}^{N} \tilde{A}_i(k)^T P(k+1) \tilde{A}_i(k) - P(k) + Q(k) = 0
\]

(109)

hold for some positive constants \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) and for all \( k \geq 0 \) where

\[
\tilde{A}_0(k) = A_0(k) + \sum_{i=1}^{N} p_i A_i(k), \\
\tilde{A}_i(k) = (p_i(1 - p_j))^{1/2} A_i(k), i = 1, \ldots, N.
\]
Proof: First note that we have $E[\nu_i] = p_i$, $E[\nu_i^2] = p_i$, $E[\nu_i\nu_j] = p_ip_j$, and $\text{Var}(\nu_i) = p_i(1 - p_i)$. The covariance associated with the system (103) is given by

$$
X(k + 1) = A_0(k)X(k)A_0(k)^T + \sum_{i=1}^{N} A_i(k)X(k)A_i(k)^T
$$
(111)

where the matrices are defined in the result. Applying then Theorem 4.3 yields the result. ♦

4.4.2. A class of stochastic linear system with delays

We consider here the following stochastic discrete-time varying system

$$
x(k + 1) = A_0x(k) + \sum_{i=1}^{N} \nu_i(k)A_ix(k - \tau_i)
x(s) = \phi(s), s \in \{-\bar{\tau}, \ldots, 0\}
$$
(112)

where $x(k) \in \mathbb{R}^n$ is the state of the system, $\phi : \{-\bar{\tau}, \ldots, 0\} \rightarrow \mathbb{R}^n$ is the initial condition and the delays $\tau_i$ are such that $\tau_i \geq 0$, $i = 1, \ldots, N$ with $\bar{\tau} := \max_i\{\tau_i\}$. The zero mean and unit variance random signals $\nu_1(k), \ldots, \nu_N(k)$ are assumed to be independent of each other and from the state $x(k)$.

Since the system now involves some time-delays, we need to adapt the notion of mean-square exponential stability to this setup as follows:

**Definition 4.9.** The system (112) is mean-square exponentially stable if there exist some scalars $\rho \in (0, 1)$ and $\beta > 1$ such that

$$
E[||x(k)||^2_2] \leq \beta \rho^k E\left[ \max_{s \in \{-\bar{\tau}, \ldots, 0\}} ||\phi(s)||^2_2 \right]
$$
(113)

holds for all $k \geq 0$ and all initial condition $\phi \in C(\{-\bar{\tau}, \ldots, 0\}, \mathbb{R}^n)$.

We have the following result:

**Theorem 4.10.** The following statements are equivalent:

(a) The system (112) is mean-square exponentially stable for all $\tau_i \geq 0$, $i = 1, \ldots, N$.

(b) The system (112) with zero-delays (i.e. with $\tau_i = 0$ for all $i = 1, \ldots, N$) is mean-square exponentially stable.

Proof: The covariance matrix $X(k)$ associated with the system (112) is governed by

$$
X(k + 1) = A_0(k)X(k)A_0(k)^T + \sum_{i=1}^{N} A_i(k)X(k - \tau_i)A_i(k)^T
$$
(114)

and $X(0) = E[x(0)x(0)^T]$. Applying now Theorem 4.3 yields the result. ♦

4.4.3. Time-varying Markov jump linear systems

We consider here the following class of systems subject to Markovian jumps

$$
x(k + 1) = A_{\sigma(k)}x(k) + \sum_{i=1}^{N} B_{\sigma(k),i}(t)x(k)\nu_i(k), k \geq k_0
$$
x(k_0) = x_0,
\sigma(k_0) = \sigma_0
$$
(115)
where \( x(k), x_0 \in \mathbb{R}^n \) are the state of the system and the initial condition, \( k_0 \) is the initial time. The random signals \( \nu_1(k), \ldots, \nu_N(k) \in \mathbb{R} \) are assumed to be independent of each other and from the state \( x(k) \) and the switching signal \( \sigma(k) \), and are also assumed to be stationary. The switching signal \( \sigma : \mathbb{Z}_{\geq t_0} \mapsto \{1, \ldots, M\} \), with initial value \( \sigma_0 \), evolves according to the finite Markov chain defined by
\[
P(\sigma(k + 1) = j | \sigma(k) = i) = \pi_{ij}
\]
where \( \pi_{ij} \geq 0 \) for all \( i = 1, \ldots, M \), we have that \( \sum_{j=1}^M \pi_{ij} = 1 \). We consider here the stability notion of Definition \( \ref{def:mean_square_stability} \) with the difference that the mean-square convergence must also hold for all paths of the Markov chain. This leads to the following result which can be seen as a generalization of the results in \( \cite{Li} \):

**Theorem 4.11.** The system \( \ref{system} \) is uniformly exponentially mean-square stable if and only if there exist matrix-valued functions \( P_i, Q_i : \mathbb{Z}_{\geq t_0} \mapsto \mathbb{S}^n_{\geq 0} \), \( i = 1, \ldots, M \), such that
\[
\alpha_1 I \preceq P_i(k) \preceq \alpha_2 I,
\]
\[
\beta_1 I \preceq Q_i(k) \preceq \beta_2 I,
\]
and the difference equations
\[
A_i(k)^T \left( \sum_{j=1}^M \pi_{ij} P_j(k + 1) \right) A_i(k) + \sum_{\ell=1}^N B_{i,\ell}(k)^T \left( \sum_{j=1}^M \pi_{ij} P_j(k + 1) \right) B_{i,\ell}(k) - P_i(k) = -Q_i(k)
\]
hold for all \( i = 1, \ldots, M \).

**Proof:** Let \( \mathbb{1}_i(\sigma(k)) \) be the indicator function of the state \( i \) defined as \( \mathbb{1}_i(\sigma(k)) = 1 \) if \( \sigma(k) = i \), and 0 otherwise. Then, defining \( X_i(k) = E[x(k)x(k)^T \mathbb{1}_i(\sigma(k))] \), we get that
\[
X_i(k + 1) = \sum_{j=1}^M \pi_{ji} \left( A_j(k)X_j(k)A_j(k)^T + \sum_{\ell=1}^N B_{j,\ell}(k)X_j(k)B_{j,\ell}(k)^T \right)
\]
for all \( i = 1, \ldots, M \). Let \( E_i \) be the matrix composed of the columns \( \{e_1, \ldots, e_M\} \) in cyclic ascending order where \( e_1 \) is the \( i \)-th column and define the matrices \( A^\pi(k) \) and \( B^\pi(k) \) as
\[
[A^\pi(k)]_{ij} = \sqrt{\pi_{ji}} A_j(k),
\]
\[
[B^\pi(k)]_{ij} = \sqrt{\pi_{ji}} B_{j,\ell}(k).
\]
Then, we have that
\[
\diag_{i=1}^M \left( \sum_{j=1}^M \pi_{ji} A_j(k)X_j(k)A_j(k)^T \right) = \sum_{i=1}^M \tilde{A}_i(k)X(k)\tilde{A}_i(k)^T
\]
and
\[
\diag_{i=1}^M \left( \sum_{j=1}^M \pi_{ji} B_{j,\ell}(k)X_j(k)B_{j,\ell}(k)^T \right) = \sum_{i=1}^M \tilde{B}_{i,\ell}(k)X(k)\tilde{B}_{i,\ell}(k)^T
\]
where \( X(t) := \diag_{i=1}^M \{X_i(t)\} \), \( \tilde{A}_i(k) = A^\pi(k) \odot E_i \) and \( \tilde{B}_{i,\ell} = B^\pi(k) \odot E_i \). Then, one can rewrite the above \( M \) coupled matrix-valued differential equations as
\[
X(k + 1) = \sum_{i=1}^N \tilde{A}_i(k)X(k)\tilde{A}_i(k)^T + \sum_{\ell=1}^N \tilde{B}_{i,\ell}(k)X(k)\tilde{B}_{i,\ell}(k)^T
\]
Using Theorem \( \ref{thm:mean_square_stability} \) we obtain the following stability condition for the above dynamical system
\[
\sum_{i=1}^M \left( \tilde{A}_i(k)^T P(k + 1)\tilde{A}_i(k) + \sum_{\ell=1}^N \tilde{B}_{i,\ell}(k)^T P(k + 1)\tilde{B}_{i,\ell}(k) \right) - P(k) = -Q(k)
\]
for some matrix-valued functions $P, Q$ satisfying the conditions in Theorem 4.3. As the state of this system has a block-diagonal structure, the state space is actually $\mathbb{S}^n_{\geq 0} \times \cdots \times \mathbb{S}^n_{\geq 0}$ ($M$ times), and not the full space $\mathbb{S}^{nM}_{\geq 0}$. Therefore, we can consider, without introducing any conservatism, block-diagonal matrix-valued functions of the form $P(k) = \text{diag}_{i=1}^{M}(P_i(k))$ and $Q(k) = \text{diag}_{i=1}^{M}(Q_i(k))$ where $P_i(k), Q_i(k) \in \mathbb{S}^n_{\geq 0}$, $i = 1, \ldots, M$, all satisfy the boundedness conditions of Theorem 4.3. Using this structure, we obtain

$$\sum_{i=1}^{M} \tilde{A}_i(k)^T P(k+1) \tilde{A}_i(k) = \text{diag}_{i=1}^{M} \left( A_i(k)^T \left( \sum_{j=1}^{M} \pi_{ij} P_j(k+1) \right) A_i(k) \right)$$

(125)

and

$$\sum_{i=1}^{M} \tilde{B}_{i,\ell}(k)^T P(k+1) \tilde{B}_{i,\ell}(k) = \text{diag}_{i=1}^{M} \left( B_{i,\ell}(k)^T \left( \sum_{j=1}^{M} \pi_{ij} P_j(k+1) \right) B_{i,\ell}(k) \right).$$

(126)

Considering, finally, the above expressions and expanding the other terms in (124) yields the conditions [13]. The proof is now completed.

5. Linear Matrix-Valued Symmetric Impulsive Systems

The objective of this section is to merge the results obtained in the case of continuous-time and discrete-time all together to obtain analogous stability conditions in the hybrid setting. A necessary and sufficient condition for the uniform exponential stability of linear matrix-valued symmetric impulsive systems is obtained in terms of coupled matrix-differential and matrix-difference equations or inequalities. The results are finally illustrated through the analysis of a certain class of linear time-varying stochastic systems subject to Poissonian jumps, random signals, switching and sampling.

5.1. Preliminaries

We consider here the following class of matrix-valued impulsive dynamical systems

$$\dot{X}(t) = A_0(t)X(t) + X(t)A_0(t)^T + \sum_{i=1}^{N} A_i(t)X(t)A_i(t)^T + \mu X(t), t \neq t_k, t \geq t^0$$

$$X(t^+)_i = \sum_{i=0}^{\pi} J_i(k)X(t)J_i(k)^T, \quad t = t_k, t \geq t^0$$

$$X(t^0) = X_0 \in \mathbb{S}^n_{\geq 0}$$

(127)

where $X(t), X_0 \in \mathbb{S}^n_{\geq 0}$ is the state of the system and the initial condition, and $t^0 \geq 0$ is the initial time. The matrix-valued functions $A_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n \times n}$, $i = 1, \ldots, N$, are assumed to be piecewise continuous and uniformly bounded whereas the matrix-valued functions $J : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^{n \times n}$ are assumed to be uniformly bounded. The sequence of impulse instants $\{t_k\}_{k \geq 1}$ is assumed to be strictly increasing and to grow unboundedly; i.e. $t_{k+1} > t_k$, $k \geq 1$ and $t_k \rightarrow \infty$ as $k \rightarrow \infty$. Under those assumptions on the data of the system, the impulse sequence and the linearity of the system, we know that there exists a unique complete (i.e. defined for all times $t \geq t^0$) solution, which we denote by $X(t, t^0, X_0)$. This solution can expressed in terms of a state-transition operator corresponding to the generators $\mathcal{G}_t$ and $\mathcal{D}_k$ defined in the previous sections, that is we have that

$$X(t) = S(t, s)X(s), \quad t \geq s \geq t^0$$

(128)

where

$$\frac{\partial}{\partial t} S(t, s)X = \mathcal{G}_t(S(t, s)X), \quad S(s, s) = I, \quad t_{k+1} \geq t \geq s > t_k, \quad k \geq 1$$

(129)

and

$$S(t, t_k)X = S(t, t_k^+) \circ \mathcal{D}_k(X), \quad t \in (t_k, t_{k+1}], \quad k \geq 1.$$

(130)

We will consider in this section the following notion of stability for the system (127):

22
Definition 5.1. The zero solution of the hybrid system (127) is globally uniformly exponentially stable with hybrid rate \((\alpha, \rho)\) if there exist some constants \(\beta \geq 1\), \(\alpha > 0\) and \(\rho \in (0, 1)\) such that
\[
||X(t, t^0, X_0)|| \leq \beta \rho^{\kappa(t, t^0)} e^{-\alpha (t-t^0)} ||X_0||
\]
for all \(t \geq t^0\), all \(t^0 \in \mathbb{R}_{\geq 0}\), and all \(X_0 \in S^n_{\geq 0}\) where \(\kappa(t, t^0)\) denotes the number of jumps in the interval \([t^0, t]\).

It is interesting to note that this stability notion may be relaxed to the cases \(\alpha > 0\), \(\rho = 1\), and \(\alpha = 0\), \(\rho \in (0, 1)\) which consists to the case of persistent flowing and persistent jumping, respectively (see e.g. [20]) under some additional conditions on the impulse time sequences. Indeed, persistent flowing requires that the time spent by the system flowing is infinite while persistent jumping requires that all the impulses arrive in finite time; i.e. \(t_{k+1} - t_k\) is finite for all \(k \geq 0\), \(t_0 = t^0\). This will be further discussed in the next section.

5.2. Main stability result

We have the following result:

Theorem 5.2. The following statements are equivalent:

(a) The matrix-valued linear dynamical system (127) is globally uniformly exponentially stable with hybrid rate \((\alpha, \rho)\) for some \(\alpha > 0\) and \(\rho \in (0, 1)\).

(b) There exist a piecewise differentiable matrix-valued function \(P : \mathbb{R}_{\geq 0} \mapsto S^n_{\geq 0}\), a piecewise continuous matrix-valued function \(Q : \mathbb{R}_{\geq 0} \mapsto S^n_{\geq 0}\), and a matrix-valued function \(R : \mathbb{Z}_{\geq 0} \mapsto S^n_{\geq 0}\) such that
\[
\begin{align*}
\alpha_1 I & \leq P(t) \leq \alpha_2 I, \\
\beta_1 I & \leq Q(t) \leq \beta_2 I, \\
\gamma_1 I & \leq R(k) \leq \gamma_2 I,
\end{align*}
\]

such that
\[
\begin{align*}
\dot{P}(t) + C(t)P(t) & = -Q(t), \quad t \neq t_k, \ t \geq t^0, \ k \geq 1 \\
D_k(P(t^+_k)) - P(t_k) & = -R(k), \quad t_k > t^0, \ k \geq 1
\end{align*}
\]
hold for some scalars \(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 > 0\).

(c) There exists a piecewise differentiable matrix-valued function \(P : \mathbb{R}_{\geq 0} \mapsto S^n_{\geq 0}\) such that
\[
\alpha_1 I \leq P(t) \leq \alpha_2 I,
\]
and
\[
\begin{align*}
\dot{P}(t) + C(t)P(t) & \leq -\alpha_3 I, \quad t \neq t_k, \ t \geq t^0, \ k \geq 1, \\
D_k(P(t^+_k)) - P(t_k) & \leq -\alpha_4 I, \quad t_k > t^0, \ k \geq 1
\end{align*}
\]
hold for some scalars \(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \gamma_1, \gamma_2 > 0\).

Proof : The statements (b) and (c) are readily seen to be equivalent. Therefore, we need to focus on the connection with the statement (a).

Proof that the statement (b) implies the statement (a). Assume that the condition of the statement (b) hold and define the function \(V(t, X) = \langle P(t), X \rangle\). From the conditions in statement (b), this function is \(S^n_{\geq 0}\)-copositive definite, radially unbounded and decrescent. Computing now its derivative along the flow of the system (127) yields
\[
\begin{align*}
\dot{V}(t, X(t)) & = \langle \dot{P}(t), X(t) \rangle + \langle P(t), \dot{C}(t)X(t) \rangle \\
& = \langle \dot{P}(t) + C(t)P(t), X(t) \rangle \\
& = -\langle Q(t), X(t) \rangle \\
& \leq -\frac{\beta_1}{\alpha_2} V(t, X(t)).
\end{align*}
\]
Similarly, evaluating the discrete-time derivatives at jump times yields
\[
V(t_k^+, X(t_k^+)) - V(t_k, X(t_k)) = \langle P(t_k^+), \mathcal{D}_k(X(t_k)) \rangle - \langle P(t_k), X(t_k) \rangle = \langle D_k^*(P(t_k^+)) - P(t_k), X(t_k) \rangle \\
= -\langle R(k), X(t_k) \rangle \\
\leq -\frac{\alpha}{\alpha_2} V(t_k, X(t_k)).
\] (137)

Letting \( \alpha := \frac{\beta_1}{\alpha_2} > 0 \) and \( \rho = 1 - \frac{\gamma_1}{\alpha_2} \in (0, 1) \), we get that
\[
\dot{V}(t, X(t)) \leq -\alpha V(t, X(t)), \quad t \neq t_k, \quad t \geq t^0, \quad k \geq 1
\]
\[
V(t_k^+, X(t_k^+)) \leq \rho V(t_k, X(t_k)), \quad t_k > t^0, \quad k \geq 1.
\] (138)

Integrating this expression yields
\[
V(t, X(t)) \leq V(0, X_0) e^{-\alpha(t-t_0)} \rho^k(t,t_0)
\] (139)
which implies
\[
\|X(t, t_0, X_0)\| \leq \frac{\alpha_2}{\alpha_1} e^{-\alpha(t-t_0)} \rho^k(t,t_0) \|X_0\|
\] (140)
which proves the uniform hybrid exponential convergence. This proves the desired implication.

**Proof that the statement (a) implies the statement (c).** Assume that the conditions of statement (a) hold and define \( \tilde{P}(t) \) as
\[
\langle \tilde{P}(t), X(t) \rangle = \int_t^\infty \langle Q(s), X(s) \rangle ds + \sum_{t \leq t_i} \langle R(i), X(t_i) \rangle
\] (141)
We need to show first that \( \|P(t)\| \) is uniformly bounded and bounded away from zero. From the definition of uniform hybrid exponential stability, i.e. Definition 5.1, we have that
\[
\langle \tilde{P}(t), X(t) \rangle = \lim_{\tau \to \infty} \left[ \int_t^\tau \langle Q(s), S(s, t)X(t) \rangle ds + \sum_{\tau > t_i \geq t} \langle R(i), S(t_i, t)X(t) \rangle \right]
\leq M\|X(t)\|_\tau \lim_{\tau \to \infty} \left( \beta_{X,2} \int_t^\tau e^{-\alpha(s-t)} \rho^{k(s,t)} ds + \beta_{Y,2} \sum_{\tau > t_i \geq t} \rho^{k(t_i, t)} e^{-\alpha(t_i-t)} \right)
\leq M\|X(t)\|_\tau \lim_{\tau \to \infty} \left( \beta_{X,2} \int_t^\tau e^{-\alpha(s-t)} ds + \beta_{Y,2} \sum_{\tau > t_i \geq t} \rho^{k(t_i, t)} \right)
= M\|X(t)\|_\tau \left( \frac{\beta_{X,2}}{\alpha} + \frac{\beta_{Y,2}}{1-\rho} \right).
\] (142)
Hence, there exists an \( \alpha_2 > 0 \) such that \( \tilde{P}(t) \leq \alpha_2 I \).

Let us now prove that there exists an \( \alpha_1 > 0 \) such that \( \tilde{P}(t) \geq \alpha_1 I \). To this aim, we consider
\[
\frac{d}{ds} \langle Q(s), X(s) \rangle = \langle \dot{Q}(s), X(s) \rangle + \langle Q(s), C_s(X(s)) \rangle
= \langle \dot{Q}(s), X(s) \rangle + \langle C_s^*(Q(s)), X(s) \rangle
= \langle Q(s) + C_s^*(Q(s)), X(s) \rangle
\] (143)
where we have assumed that \( Q(s) \) is piecewise differentiable with bounded derivative. This is not a restrictive assumption as it can be chosen as a constant without loss of generality. Now, using the fact that \( Q(s) \) is bounded with bounded derivative, then there exists a large enough \( c > 0 \) such that
\[
\langle \dot{Q}(s) + C_s^*(Q(s)), X(s) \rangle \geq -c\langle Q(s), X(s) \rangle.
\] (144)
This implies that for some sufficiently large \( c > 0 \), we have that

\[
\frac{d}{ds} \langle Q(s), X(s) \rangle \geq -c \langle Q(s), X(s) \rangle
\]

(145)

for all \( X(s) \in S^n \) and for all \( t_k < s \leq t \leq t_{k+1}, k \in \mathbb{Z}_{\geq 0} \).

Now assume that \( t_{k-1} < t \leq t_k \) and observe that

\[
\begin{align*}
\int_t^{t_k} \frac{d}{ds} \langle Q(s), X(s) \rangle ds &= \langle Q(t_k), X(t_k) \rangle - \langle Q(t), X(t) \rangle \\
\int_t^{t_{k+1}} \frac{d}{ds} \langle Q(s), X(s) \rangle ds &= \langle Q(t_{k+1}), X(t_{k+1}) \rangle - \langle Q(t^+_i), X(t^+_i) \rangle \\
&\geq -c \int_t^{t_k} \langle Q(s), X(s) \rangle ds - c \int_t^{t_{k+1}} \langle Q(s), X(s) \rangle ds.
\end{align*}
\]

(146)

Summing all the above terms gives

\[
- \langle Q(t), X(t) \rangle + \sum_{i=k}^{\infty} \left( \langle Q(t_i), X(t_i) \rangle - \langle Q(t^+_i), X(t^+_i) \rangle \right) \geq -c \int_t^{\infty} \langle Q(s), X(s) \rangle ds.
\]

(147)

Adding \(-c \sum_{i=k}^{\infty} \langle R(i), X(t_i) \rangle = -c \sum_{i=k}^{\infty} \langle R(i), X(t_i) \rangle \) on both sides yields

\[
- \langle Q(t), X(t) \rangle + \sum_{i=k}^{\infty} \left( \langle Q(t_i), X(t_i) \rangle - \langle Q(t^+_i), X(t^+_i) \rangle \right) - c \sum_{i=k}^{\infty} \langle R(i), X(t_i) \rangle \geq -c \langle \bar{P}(t), X(t) \rangle.
\]

(148)

where we have used the fact that \( t_{k-1} < t \leq t_k \). Noting that \( X(t^+_i) = D_i(X(t_i)) \), then we get that

\[
\sum_{i=k}^{\infty} \left( \langle Q(t_i), X(t_i) \rangle - \langle Q(t^+_i), X(t^+_i) \rangle \right) = \sum_{i=k}^{\infty} \left( \langle Q(t_i) - cR(i), X(t_i) \rangle - \langle Q(t^+_i), D_i(X(t_i)) \rangle \right) = \sum_{i=k}^{\infty} \langle Q(t_i) - cR(i) - D_i^*(Q(t^+_i)), X(t_i) \rangle
\]

(149)

Therefore, and since the matrix-valued functions \( J_j \) are uniformly bounded, then there exists a large enough \( c > 0 \) (which stays compatible with our previous choice for \( c \)) such that \( Q(t_i) - cR(i) - D_i^*(Q(t^+_i)) < 0 \). Therefore, with such a \( c > 0 \), we get that

\[
- \langle Q(t), X(t) \rangle \geq -c \langle \bar{P}(t), X(t) \rangle
\]

(150)

and, as a result, \( \bar{P}(t) \geq Q(t)/c \geq \beta_1 I/c =: \alpha_1 I \). This proves the second inequality for \( \bar{P}(t) \).

We now prove that \( \bar{P}(t) \) in (141) verifies the equalities in (133). Computing the derivative of \( \bar{P}(t) \) with respect to time yields

\[
\frac{d}{dt} \langle \bar{P}(t), X(t) \rangle = \langle \dot{\bar{P}}(t), X(t) \rangle + \langle \bar{P}(t), \dot{X}(t) \rangle = \langle \dot{\bar{P}}(t), X(t) \rangle + \langle \bar{P}(t), C_t(X(t)) \rangle = \langle \dot{\bar{P}}(t), X(t) \rangle + \langle C_t^*(\bar{P}(t)), X(t) \rangle = \langle Q(t), X(t) \rangle.
\]

(151)

Therefore, \( \langle \dot{\bar{P}}(t) + C_t^*(\bar{P}(t)) + Q(t), X(t) \rangle = 0 \) and (141) verifies the first inequality in (133) with \( P = \bar{P} \).
Similarly, evaluating $\bar{P}(t)$ at $t_k$ and $t_k^+$ yields

$$
\langle \bar{P}(t_k^+), X(t_k^+) \rangle = \sum_{i=k+1}^{\infty} \langle R(i), X(t_i) \rangle + \int_{t_k}^{\infty} \langle Q(s), X(s) \rangle ds
$$

$$
\langle \bar{P}(t_k), X(t_k) \rangle = \langle \bar{P}(t_k), X(t_k) \rangle = \sum_{i=k}^{\infty} \langle R(i), X(t_i) \rangle + \int_{t_k}^{\infty} \langle Q(s), X(s) \rangle ds
$$

$$
= \langle R(k), X(t_k) \rangle + \sum_{i=k+1}^{\infty} \langle R(i), X(t_i) \rangle + \int_{t_k}^{\infty} \langle Q(s), X(s) \rangle ds
$$

$$
= \langle R(k), X(t_k) \rangle + V(t_k^+, X(t_k^+)).
$$

This implies that $\langle \bar{P}(t_k^+), X(t_k^+) \rangle - \langle \bar{P}(t_k), X(t_k) \rangle + \langle R(k), X(t_k) \rangle = 0$. However, we also have that

$$
\langle \bar{P}(t_k^+), X(t_k^+) \rangle - \langle \bar{P}(t_k), X(t_k) \rangle + \langle R(k), X(t_k) \rangle = \langle \bar{P}(t_k^+), D_k(X(t_k)) \rangle - \langle \bar{P}(t_k), X(t_k) \rangle + \langle R(k), X(t_k) \rangle
$$

$$
= \langle D_k^2(\bar{P}(t_k^+)), X(t_k) \rangle + \langle -\bar{P}(t_k) + R(k), X(t_k) \rangle
$$

$$
= \langle D_k^2(\bar{P}(t_k^+)) - \bar{P}(t_k) + R(k), X(t_k) \rangle
$$

$$
= 0,
$$

which shows that $\bar{P}$ in (141) also verifies the second inequality in (133) with $P = \bar{P}$. This proves the desired result.

\[\text{\hfill \Box}\]

5.3. Relaxed stability results

The above results can be relaxed in two possible ways. The first one, called persistent flowing relaxation, relaxes the strict decrease of the Lyapunov function at jumps:

**Theorem 5.3 (Persistent flowing).** Assume that $D_k(X) \preceq X$ for all $k \geq 0$ and all $X \in \mathbb{S}_{\geq 0}^n$. Then, the following statements are equivalent:

- **(a)** The matrix-valued linear dynamical system (157) is globally uniformly exponentially stable hybrid rate $(\alpha, 1)$ for some $\alpha > 0$.

- **(b)** There exist a piecewise differentiable matrix-valued function $P : \mathbb{R}_{\geq 0} \to \mathbb{S}_{\geq 0}^n$ and a continuous matrix-valued function $Q : \mathbb{R}_{\geq 0} \to \mathbb{S}_{\geq 0}^n$ such that

\[
\alpha_1 I \preceq P(t) \preceq \alpha_2 I,
\]

\[
\beta_1 I \preceq Q(t) \preceq \beta_2 I,
\]

such that

\[
\dot{P}(t) + C_t(P(t)) = -Q(t), \quad t \neq t_k, \quad t \geq t_0, \quad k \geq 1
\]

\[
D_k(P(t_k^+)) - P(t_k) = 0, \quad t_k > t_0, \quad k \geq 1
\]

hold for some scalars $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$.

- **(c)** There exists a piecewise differentiable matrix-valued function $P : \mathbb{R}_{\geq 0} \to \mathbb{S}_{\geq 0}^n$ such that

\[
\alpha_1 I \preceq P(t) \preceq \alpha_2 I
\]

and

\[
\dot{P}(t) + C_t(P(t)) \preceq -\alpha_3 I, \quad t \neq t_k, \quad t \geq t_0, \quad k \geq 1
\]

\[
D_k(P(t_k^+)) - P(t_k) \preceq 0, \quad t_k > t_0, \quad k \geq 1
\]

hold for some scalars $\alpha_1, \alpha_2, \alpha_3 > 0$. 

26
Proof: The proof follows from the same lines as the proof of Theorem 5.2. The only difference is in the proof of the implication that the statement (a) implies the statement (c). In the current proof, we consider a continuous matrix-valued function $Q$ and define the matrix-valued function $\bar{P}(t)$ as

$$\langle \bar{P}(t), X(t) \rangle := \int_{t}^{\infty} \langle Q(s), X(s) \rangle ds.$$  \hspace{1cm} (158)

As a result, the expression (147) becomes

$$- \langle Q(t), X(t) \rangle + \sum_{i=k}^{\infty} \langle Q(t_i), X(t_i) - X(t_i^+) \rangle \geq -c \langle \bar{P}(t), X(t) \rangle.$$  \hspace{1cm} (159)

Under the assumption that $D \leq C$, we then get that

$$- \langle Q(t), X(t) \rangle \geq -c \langle \bar{P}(t), X(t) \rangle$$  \hspace{1cm} (160)

which shows, in turn, that $\bar{P}(t) \geq \beta_1/cI$. The rest of the proof follows from the same lines.

The second one, called persistent jumping relaxation, relaxes the strict decrease of the Lyapunov function along the flow of the system:

**Theorem 5.4 (Persistent jumping).** Assume that $C_t(X) \leq 0$ for all $t \geq 0$ and all $X \in S^n_{\geq 0}$. Then, the following statements are equivalent:

(a) The matrix-valued linear dynamical system (147) is globally uniformly exponentially stable with hybrid rate $(0, \rho)$ for some $\rho \in (0, 1)$.

(b) There exist a piecewise differentiable matrix-valued function $P : \mathbb{R}_{\geq 0} \mapsto S^n_{>0}$ and a matrix-valued function $R : \mathbb{Z}_{\geq 0} \mapsto S^n_{\geq 0}$ such that

$$\alpha_1 I \preceq P(t) \preceq \alpha_2 I,$$

and

$$\gamma_1 I \preceq R(k) \preceq \gamma_2 I,$$

and

$$\bar{P}(t) + C_t(P(t)) = 0, \quad t \neq t_k, \quad t \geq t_0, \quad k \geq 1$$

$$D_k(P(t_k^+)) - P(t_k) = -R(k), \quad t_k > t_0, \quad k \geq 1$$

hold for some $\alpha_1, \alpha_2, \gamma_1, \gamma_2 > 0$.

(c) There exists a piecewise differentiable matrix-valued function $P : \mathbb{R}_{\geq 0} \mapsto S^n_{\leq 0}$ such that

$$\alpha_1 I \preceq P(t) \preceq \alpha_2 I,$$

and

$$\bar{P}(t) + C_t(P(t)) \preceq 0, \quad t \neq t_k, \quad t \geq t_0, \quad k \geq 1$$

$$D_k(P(t_k^+)) - P(t_k) \preceq -\alpha_3 I, \quad t = t_k, \quad t \geq t_0$$

hold for some $\alpha_1, \alpha_2, \alpha_3 > 0$.

Proof: The proof follows from the same lines as the proof of Theorem 5.2. The only difference is in the proof of the implication that the statement (a) implies the statement (c). In the current proof, we define the matrix-valued function $P(t)$ as

$$\langle P(t), X(t) \rangle := \sum_{t \leq t_i} \langle R(i), X(t_i) \rangle.$$  \hspace{1cm} (165)

Clearly, we have that $\langle P(t), X(t) \rangle \geq \langle R(s_i), X(s_i) \rangle$ where $s_i := \min\{i : t \leq t_i\}$. Using now the assumption that $C_t(X) \leq 0$ for all $t \geq 0$, we get that $X(s_i) \leq X(t)$, which implies that

$$\langle P(t), X(t) \rangle \geq \langle R(s_i), X(t) \rangle$$  \hspace{1cm} (166)

and that $\bar{P}(t) \geq \gamma_1 I$. This proves the lower bound for $\bar{P}(t)$. The rest of the proof follows from the same lines as the proof of Theorem 5.2. \hfill \Box
5.4. Applications

We illustrate in this section that several results from the literature can be recovered and/or extended using the proposed approach. In particular, we cover the cases of stochastic impulsive systems with deterministic impulse times \[8, 24, 28\], stochastic switched system with deterministic impulse times \[7, 8, 20, 28, 30\], stochastic sampled-data systems \[8, 20, 28\], and impulsive Markov jump linear systems \[32\]. This covers a broad range of systems of the literature.

5.4.1. A class of stochastic LTV systems with deterministic impulses

We consider here the following class of stochastic LTV systems with deterministic jumps as

\[
\begin{align*}
\dot{x}(t) &= A_0(t)x(t)dt + \sum_{i=1}^{N} A_i(t)x(t)dW_i(t), \quad t \neq t_k, \ k \geq 1 \\
x(t^+_k) &= J_0(k)x(t_k) + \sum_{i=1}^{N} J_i(k)x(t_k)\nu_i(k), \ k \geq 1, \ t_k \geq t^0 \\
x(t^0) &= x_0
\end{align*}
\]

where \(x, x_0 \in \mathbb{R}^n\) is the state of the system and the initial condition, \(t^0\) is the initial time. The vector \(W(t) \coloneqq (W_1(t), \ldots, W_N(t))\) is a vector of \(N\) independent Wiener processes which are also independent of the state \(x(t)\) whereas the vector \(\nu(k) \coloneqq (\nu_1(k), \ldots, \nu_N(k))\) contains \(N\) independent random processes which are independent of the state \(x(t_k)\) and which have zero mean and unit variance. The sequence of impulse times \(t_{k \geq 1}\) is assumed to be increasing and to grow unboundedly. To this system, we can associate a filtered probability space \((\Omega, \mathcal{F}, F_{t,k}, \mathbb{P})\) with hybrid filtration \[34\] which satisfies the usual conditions.

The following notion of stability for the system \((167)\) is considered here:

**Definition 5.5.** The system \((167)\) is said to be uniformly exponentially mean-square stable with hybrid rate \((\alpha, \rho)\) if there exist an \(M > 0\), a \(\rho \in (0, 1)\) and an \(\alpha > 0\) such that we have

\[
\mathbb{E}[||x(t)||^2] \leq M\rho^{\alpha(t-t^0)}e^{-\alpha(t-t^0)}\mathbb{E}[||x_0||^2]
\]

for all \(t \geq t^0\) and all \(t^0 \geq 0\) where \(\kappa(t, s)\) denotes the number of jumps in the interval \([s, t]\).

We then have the following result which is an extension of the results in \[8\]:

**Theorem 5.6.** The following statements are equivalent:

(a) The system \((167)\) is uniformly exponentially mean-square stable with hybrid rate \((\alpha, \rho)\) for some \(\alpha > 0\) and \(\rho \in (0, 1)\).

(b) There exist a piecewise differentiable matrix-valued function \(P : \mathbb{R}_{\geq 0} \rightarrow \mathbb{S}^n_{\geq 0}\), a piecewise continuous matrix-valued function \(Q : \mathbb{R}_{\geq 0} \rightarrow \mathbb{S}^n_{\geq 0}\), and a matrix-valued function \(R : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{S}^n_{\geq 0}\) such that

\[
\begin{align*}
\alpha_1 I &\leq P(t) \leq \alpha_2 I, \\
\beta_1 I &\leq Q(t) \leq \beta_2 I, \\
\gamma_1 I &\leq R(k) \leq \gamma_2 I,
\end{align*}
\]

such that

\[
\dot{P}(t) + \text{Sym}(P(t)A_0(t)) + \sum_{i=1}^{N} A_i(t)^TP(t)NA_i(t) = -Q(t), \quad t \neq t_k, \ t \geq t^0, \ k \geq 1
\]

\[
\sum_{i=1}^{N} J_i(k)^TP(t^+_k)J_i(k) - P(t_k) = -R(k), \quad t_k > t^0, \ k \geq 1
\]

hold for some scalars \(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 > 0\).
Proof: Observing that the covariance $X(t) = \mathbb{E}[x(t)x(t)^T]$ of the state of the system is governed by the system
\begin{align*}
\dot{X}(t) &= A_0(t)X(t) + X(t)A_0(t)^T + \sum_{i=1}^{N} A_i(t)X(t)A_i^T, \ t \neq t_k, \ k \geq 1, \ t \geq t^0, \\
X(t_k^+) &= \sum_{i=0}^{N} J_i(k)X(t_k)J_i(k)^T, \ k \geq 1, \ t \geq t^0, \\
X(t_0^+) &= X(t_0) = X(0),
\end{align*}

then an immediate application of Theorem 5.2 yields the result. \hfill \Box

5.4.2. A class of stochastic LTV switched systems with deterministic switchings

We illustrate in this section that the results obtained in this paper allow to retrieve and extend those developed in [7]. To this aim, let us consider here the following class of LTV stochastic switched systems

\begin{align}
\begin{aligned}
dx(t) &= A_{\sigma(t),0}(t)x(t)dt + \sum_{i=1}^{N} A_{\sigma(t),i}(t)x(t)dW_i(t) \\
x(t_0) &= x_0 \\
\sigma(t_0) &= \sigma_0
\end{aligned}
\end{align}

(172)

where $x, x_0 \in \mathbb{R}^n$ are the state of the system and the initial condition, and $t_0 = 0$ is the initial time. The Wiener processes $W_1(t), \ldots, W_N(t)$ are assumed to be independent of each other and of the state $x(t)$ and the switching signal $\sigma(t)$ of the system. The switching signal $\sigma : \mathbb{R}_{\geq 0} \rightarrow \{1, \ldots, M\}$, with initial value $\sigma_0$, is assumed to be deterministic and piecewise constant. The switching signal is assumed to change values at time $t_k \geq 0$, $k \geq 1$, and this sequence is assumed to be increasing and to grow unboundedly. The dwell-time $T_k$ is defined as $T_k := t_{k+1} - t_k$, $k \geq 0$. The notion of mean-square stability in Definition 5.5 can be straightforwardly adapted to the system above. This leads us to the following result:

**Theorem 5.7 (Fixed dwell-time).** Assume that there exist differentiable matrix-valued functions $P_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{S}^n_{\succ 0}$, $\alpha_1 I \leq P_i(\cdot) \leq \alpha_2 I$, $i = 1, \ldots, M$, such that
\begin{align}
\dot{P}_i(\tau) + A_{0,i}(\tau)^TP_i(\tau) + P_i(\tau)A_{0,i}(\tau) + \sum_{j=1}^{N} A_{i,j}^TP_i(\tau)A_{i,j} &\leq -\alpha_3 I, \quad i = 1, \ldots, M, \quad \tau \in [0, T_k], \\
P_i(0) - P_j(T_k) &\leq 0, \quad i, j = 1, \ldots, M, \quad i \neq j
\end{align}

(173)

hold for some $\alpha_1, \alpha_2, \alpha_3 > 0$.

Then, the LTI version of the system (172) is mean-square exponentially stable under the fixed dwell-time sequence $\{T_k\}_{k \geq 0}$.

Proof: The dynamics of the covariance matrix $X(t) = \mathbb{E}[x(t)x(t)^T]$ is given by
\begin{align}
\begin{aligned}
\dot{X}(t) &= A_{\sigma(t),0}(t)X(t) + A_{\sigma(t),0}(t)^TX(t) + \sum_{j=1}^{N} A_{\sigma(t),j}^TX(t)A_{\sigma(t),j} \\
X(t_k^+) &= X(t_k)
\end{aligned}
\end{align}

(174)

Since, $D_k(X) - X \preceq 0$, then we can apply Theorem 5.3 with the matrix $P(t_k + \tau) = P_\tau(\tau)$ whenever $\sigma(t_k^+) = i$ where $P_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{S}^n_{\succ 0}$. Substituting those expressions in the stability conditions yields the conditions of the result. \hfill \Box

When the sequence of dwell-times satisfies a certain condition, we can specialize the above result to obtain more tractable stability conditions. The next result applies to the case when the constant dwell-time case, that is, when $T_k = \bar{T}$, $k \geq 0$, for some constant dwell-time value $\bar{T} > 0$. 

29
Corollary 5.8 (Constant dwell-time). Assume that there exist differentiable matrix-valued functions $P_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{S}_{\geq 0}^n$, $i = 1, \ldots, M$, such that

$$\dot{P}_i(\tau) + A_{i,0}^T P_i(\tau) + P_i(\tau) A_{i,0} + \sum_{j=1}^{N} A_{i,j}^T P_i(\tau) A_{i,j} \preceq -\alpha I, \quad \tau \in [0, \bar{T}], \ i = 1, \ldots, M, \quad (175)$$

hold for some $\alpha > 0$.

Then, the LTI version of the system (172) is mean-square exponentially stable under eventual constant dwell-time $\bar{T}$.

Proof: The proof simply follows from substituting $T_k$ by $\bar{T}$ in Corollary 5.7.

The following result addresses the case where the dwell-time sequence satisfies $T_k \geq \bar{T}$, $k \geq 1$, for some minimum dwell-time value $\bar{T}$.

Corollary 5.9 (Minimum dwell-time). Assume that there exist differentiable matrix-valued functions $P_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{S}_{\geq 0}^n$, $i = 1, \ldots, M$, such that

$$A_{i,0}^T P_i(\bar{T}) + P_i(\bar{T}) A_{i,0} + \sum_{j=1}^{N} A_{i,j}^T P_i(\bar{T}) A_{i,j} \preceq -\alpha I, \quad i = 1, \ldots, M, \quad (176)$$

$$\dot{P}_i(\tau) + A_{i,0}^T P_i(\tau) + P_i(\tau) A_{i,0} + \sum_{j=1}^{N} A_{i,j}^T P_i(\tau) A_{i,j} \preceq -\alpha I, \quad \tau \in [0, \bar{T}], \ i = 1, \ldots, M,$$

$$P_i(0) - P_j(\bar{T}) \preceq 0, \quad i, j = 1, \ldots, M,$$

hold for some $\alpha > 0$.

Then, the LTI version of the system (172) is mean-square exponentially stable under minimum dwell-time $\bar{T}$.

Proof: The proof follows from considering all the matrix valued functions $P_i(\tau)$ such that $P_i(\tau) = P_i(\bar{T})$ for all $\tau \geq \bar{T}$. Even though those functions are not differentiable at $\tau = \bar{T}$, then it is possible to shown that if the conditions in the result hold, then there exist differentiable functions that satisfy the conditions of the result; see e.g. [23].

The following result addresses the case where the dwell-time sequence satisfies $T_k \in [T_{\min}^{\sigma(1)}, T_{\max}^{\sigma(1)}], k \geq 1$, for some range dwell-time value $[T_{\min}^{1}, T_{\max}^{1}] \times \ldots \times [T_{\min}^{M}, T_{\max}^{M}]$.

Corollary 5.10 (Mode-dependent range dwell-time). Assume that there exist differentiable matrix-valued functions $P_i : \mathbb{R}_{\geq 0} \mapsto \mathbb{S}_{\geq 0}^n$, $i = 1, \ldots, M$, such that

$$\dot{P}_i(\tau) + A_{0,i}(\tau)^T P_i(\tau) + P_i(\tau) A_{0,i}(\tau) + \sum_{j=1}^{N} A_{i,j}^T P_i(\tau) A_{i,j} \preceq -\alpha I, \quad i = 1, \ldots, M, \ \tau \in [0, T_i^{\min}],$$

$$P_i(0) - P_j(\theta) \preceq 0, \quad i, j = 1, \ldots, M, \ i \neq j, \ \theta \in [T_i^{\min}, T_i^{\max}], \quad (177)$$

hold for some $\alpha > 0$.

Then, the LTI version of the system (172) is mean-square exponentially stable under the mode-dependent range dwell-time $[T_{\min}^{1}, T_{\max}^{1}] \times \ldots \times [T_{\min}^{M}, T_{\max}^{M}]$.

Proof: The proof simply consists of considering $T_k^i$ to lie within the interval $[T_{\min}^{i}, T_{\max}^{i}], i = 1, \ldots, M$, in the conditions.
5.4.3. Sampled-data stochastic LTV systems

We illustrate in this section that the results obtained in this paper allow to retrieve and extend those developed in [3]. To this aim, let us consider here the following class of LTV stochastic sampled-data systems

Let us consider the following LTV stochastic process

\[ \begin{align*}
\dot{x}(t) & = (A_0(t)x(t) + B_0(t)u(t))dt + \sum_{i=1}^{N} (A_i(t)x(t) + B_i(t)u(t))dW_i(t) \\
x(t^{0}) & = x_0
\end{align*} \] (178)

and we assume that the control input \( u \) is given by

\[ u(t) = K_1(k)x(t_k) + K_2(k)u(t_k), \quad t \in (t_k, t_{k+1}]. \] (179)

As for the other systems, \( x, x_0 \in \mathbb{R}^n \) are the state of the system and the initial condition, and \( t^0 \geq 0 \) is the initial time. The Wiener processes \( W_i(t), \ldots, W_N(t) \) are assumed to be independent of each other and of the state \( x(t) \). The matrix-valued functions describing the system are also assumed to be uniformly bounded.

We then have the following result which consists of an extension of the results in [3]:

**Theorem 5.11.** The following statements are equivalent:

(a) The LTV stochastic sampled-data system (178)-(179) is uniformly mean-square exponentially stable.

(b) There exists a differentiable matrix-valued function \( P : \mathbb{R}_{\geq 0} \rightarrow \mathbb{S}^n_{\geq 0}, \alpha_1 I \preceq P(\cdot) \preceq \alpha_2 I \), such that

\[ \dot{P}(t) + \bar{A}_0(t)^T P(t) + P(t) \bar{A}_0(t) + \sum_{i=1}^{N} \bar{A}_i(t)^T P(t) \bar{A}_i(t) \leq -\alpha_3 I, \quad t \neq t_k, \quad t \geq t^0 \] (180)

and

\[ \dot{J}(k)^T P(t_k^+) \bar{J}(k) - P(t_k) \leq -\alpha_4 I, \quad k \geq 1, \quad t_k > t^0 \] (181)

hold for some \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0 \) where

\[ \bar{A}_i(t) := \begin{bmatrix} A_i(t) & B_i(t) \\ 0 & 0 \end{bmatrix}, \quad \bar{J}(k) := \begin{bmatrix} I & 0 \\ K_1(k) & K_2(k) \end{bmatrix} \] (182)

**Proof:** The system (178)-(179) can be reformulated as the following impulsive system

\[ \begin{align*}
\dot{z}(t) & = \bar{A}_0(t)z(t)dt + \sum_{i=1}^{N} \bar{A}_i(t)z(t)dW_i(t) \\
z(t^{+}) & = \bar{J}z(t_k)
\end{align*} \] (183)

and

\[ z(t_k^+) = \bar{J}z(t_k) \] (184)

where \( A_i, i = 1, \ldots, M, \) and \( \bar{J}(k) \) are defined in the result. The covariance system associated with the above impulsive system is given by

\[ \begin{align*}
\dot{X}(t) & = \bar{A}_0(t)X(t) + X(t)\bar{A}_0(t)^T + \sum_{i=1}^{N} \bar{A}_i(t)X(t)\bar{A}_i(t)^T \\
X(t_k^+) & = \bar{J}X(t_k)\bar{J}^T.
\end{align*} \] (185) (186)

Applying then Theorem 5.2 yields the result.

The following results can be seen as generalizations of those in [3, 8]:

31
Corollary 5.12 (Periodic sampling). The following statements are equivalent:

(a) The LTI version of the sampled-data system (178) - (179) is mean-square exponentially stable under periodic sampling \(T\).

(b) There exist a differentiable matrix-valued function \(P : [0, T] \mapsto S^n_{\geq 0}\) such that

\[
\dot{P}(\tau) + \bar{A}_0^T P(\tau) + P(\tau)\bar{A}_0 + \sum_{i=1}^{N} \bar{A}_i^T P(\tau)\bar{A}_i \preceq -\alpha_1 I, \quad \tau \in [0, T] \tag{187}
\]

and

\[
\dot{J}^T P(0)\bar{J} - P(\bar{T}) \preceq -\alpha_2 I \tag{188}
\]

hold for some \(\alpha_1, \alpha_2 > 0\).

Proof: Due to periodicity of the impulse times period, we can choose \(P(t_k + \tau) = P(\tau), \quad \tau \in (0, T]\), in the conditions of Theorem 5.11 which proves the result. \(\diamondsuit\)

Corollary 5.13 (Aperiodic sampling). The following statements are equivalent:

(a) The LTI version of the sampled-data system (178) - (179) is mean-square exponentially stable under aperiodic sampling \([T_{\min}, T_{\max}]\).

(b) There exist a differentiable matrix-valued function \(P : [0, T] \mapsto S^n_{\geq 0}\) such that

\[
\dot{P}(\tau) + \bar{A}_0^T P(\tau) + P(\tau)\bar{A}_0 + \sum_{i=1}^{N} \bar{A}_i^T P(\tau)\bar{A}_i \preceq -\alpha_1 I, \quad \tau \in [0, \max]\]

and

\[
\dot{J}^T P(0)\bar{J} - P(\theta) \preceq -\alpha_2 I, \quad \theta \in [T_{\min}, T_{\max}] \tag{190}
\]

hold for some \(\alpha_1, \alpha_2 > 0\).

Proof: Due to periodicity of the impulse times period, we can choose \(P(t_k + \tau) = P(\tau), \quad \tau \in (0, T_{\max}]\), in the conditions of Theorem 5.11 which proves the result. \(\diamondsuit\)

5.4.4. A class of stochastic LTV impulsive systems with stochastic impulses and switching

We consider here the following class of systems that extends the systems analyzed in [32] to the stochastic and time-varying case:

\[
dx(t) = A_{0,\sigma(t)}x(t)dt + \sum_{i=1}^{N} A_{i,\sigma(t)}(t)x(t)dW_i(t), \quad \sigma(t) \in \{1, \ldots, M_c\},
\]

\[
x(t_k^+) = J_{0,\sigma(t_k)}(k)x(t_k) + \sum_{i=1}^{N} J_{i,\sigma(t_k)}(k)\nu_i(k)x(t_k), \quad \sigma(t_k) \in \{M_c + 1, \ldots, M_c + M_d\} \tag{191}
\]

\[
x(t^0) = x_0 \quad \sigma(t^0) = \sigma_0
\]

where \(M_c\) is the number of continuous modes and \(M_d\) is the number of discrete modes. As for the other systems \(x, x_0 \in \mathbb{R}^n\) are the state of the system and the initial condition, and \(t^0 \geq 0\) is the initial time. The Wiener processes \(W_1(t), \ldots, W_N(t)\) are assumed to be independent of each other, of the state \(x(t)\), and the switching signal \(\sigma(t)\) of the system. Similarly, the random sequences \(\nu_1(k), \ldots, \nu_N(k)\) are independent of
each other, of \( x(t_k) \), and the switching signal \( \sigma(t_k) \). The matrix-valued functions describing the system are also assumed to be uniformly bounded. The switching signal \( \sigma : \mathbb{R}_{\geq 0} \rightarrow \{1, \ldots, M\} \), with initial value \( \sigma_0 \), is assumed to be stochastically varying and piecewise constant. We also have that for Theorem 5.14.

Proof: The complete proof is omitted but can be seen as a combination of the proofs of Theorem 5.6, Theorem 3.10, and Theorem 4.11.

\( \square \)

6. Concluding statements and future works

A unified formulation for the analysis of a broad class of linear systems in terms of the analysis of matrix-valued differential equations has been introduced. The approach is shown to generalize and unify most of the results on the literature on linear systems including continuous-time, discrete-time, impulsive, switched and sampled-data systems, and certain systems with delays. The natural next step is the consideration of systems with inputs and the derivation of a dissipativity theory for such systems [11, 32, 36]. This would lead, for instance, to the derivation of necessary and sufficient conditions for the passivity, \( L_2 \)-gain, etc. for such systems. The generalization of the approach to nonlinear matrix-valued monotone systems [3] would also be of great interest.
References

[1] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications*. Springer Science+Business Media New York, 1988.

[2] A. Y. Aleksandrov and O. Mason. Diagonal Lyapunov–Krasovskii functionals for discrete-time positive systems with delay. *Systems & Control Letters*, 63:63–67, 2014.

[3] D. Angeli and E. D. Sontag. Monotone control systems. *IEEE Transactions on Automatic Control*, 48(10):1684–1698, 2003.

[4] R. Bhatia. *Matrix Analysis*. Springer, 1997.

[5] C. Briat. Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive and sampled-data systems under dwell-time constraints. *Automatica*, 49(11):3449–3457, 2013.

[6] C. Briat. Robust stability and stabilization of uncertain linear positive systems via integral linear constraints - $L_1$- and $L_\infty$-gains characterizations. *International Journal of Robust and Nonlinear Control*, 23(17):1932–1954, 2013.

[7] C. Briat. Convex conditions for robust stabilization of uncertain switched systems with guaranteed minimum and mode-dependent dwell-time. *Systems & Control Letters*, 78:63–72, 2015.

[8] C. Briat. Stability analysis and stabilization of stochastic linear impulsive, switched and sampled-data systems under dwell-time constraints. *Automatica*, 74:279–287, 2016.

[9] C. Briat. Stability and performance analysis of linear positive systems with delays using input-output methods. *International Journal of Control*, 91(7):1669–1692, 2018.

[10] C. Briat. Stability and $L_1 \times \ell_1$-to-$L_1 \times \ell_1$ performance analysis of uncertain impulsive linear positive systems with applications to the interval observation of impulsive and switched systems with constant delays. *International Journal of Control*, 93(11):2634–265, 2020.

[11] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland. *Dissipative Systems Analysis and Control*. Springer-Verlag, 2007.

[12] F. Bullo and A. D. Lewis. *Geometric Control of Mechanical Systems*. Springer Science+Business Media New York, 2005.

[13] S. Bundfuss and M. Dür. Copositive Lyapunov functions for switched systems over cones. *Systems & Control Letters*, 58(5):342–345, 2009.

[14] Y. Chen, P. Bolzern, P. Colaneri, Y. Bo, and B. Du. Stability and stabilization for markov jump linear systems in polyhedral cones. In *57th IEEE Conference on Decision and Control*, pages 4779–4784, Miami Beach, USA, 2018.

[15] O. L. V. Costa, M. D. Fragoso, and R. P. Marques. *Discrete-Time Markov Jump Linear Systems*. Springer-Verlag, London, UK, 2005.

[16] O. L. V. Costa, M. D. Fragoso, and M. G. Todorov. *Continuous-Time Markov Jump Linear Systems*. Springer-Verlag, Berlin, Heidelberg, 2013.

[17] L. Farina and S. Rinaldi. *Positive Linear Systems: Theory and Applications*. John Wiley & Sons, 2000.

[18] E. Fridman. *Introduction to Time-Delay Systems*. Birkhäuser, Springer International Publishing Basel Switzerland, 2014.

[19] A. Gattami and B. Bamieh. Simple covariance approach to $\langle \infty$ analysis. *IEEE Transactions on Automatic Control*, 61(3):789–794, 2016.
[20] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems. Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[21] K. Gu, V. L. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Birkhäuser, Boston, 2003.

[22] W. M. Haddad and V. Chellaboina. Stability theory for nonnegative and compartmental dynamical systems with time delay. *Systems & Control Letters*, 51(5):355–361, 2004.

[23] T. Holicki and C. W. Scherer. Stability analysis and output-feedback synthesis of hybrid systems affected by piecewise constant parameters via dynamic resetting scalings. *Nonlinear Analysis: Hybrid Systems*, 34:179–208, 2019.

[24] Y. Ji and H. J. Chizeck. Controllability, stabilizability, and continuous-time markovian jump linear quadratic control. *IEEE Transactions on Automatic Control*, 777-788, 1990.

[25] V. Jurdjevic and H. J. Sussmann. Control systems on lie groups. *Systems & Control Letters*, 12(2):313–329, 1992.

[26] T. Kaczorek. Stability of positive continuous-time linear systems with delays. *Bulletin of the Polish Academy of Sciences - Technical sciences*, 57(4):395–398, 2009.

[27] R. Khasminskii. *Stability of Stochastic Differential Equations*. Springer, 2012.

[28] A. N. Michel, L. Hou, and D. Liu. *Stability of dynamical systems - Continuous, discontinuous and discrete systems*. Birkhäuser, Boston, 2008.

[29] R. M. Murray, Z. Li, and S. S. Sathry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.

[30] U. Shaked and E. Gershon. Robust $H_\infty$ control of stochastic linear switched systems with dwell-time. *International Journal of Robust and Nonlinear Control*, 24:1664–1676, 2014.

[31] R. E. Skelton, T. Iwasaki, and K. M. Grigoriadis. *A Unified Algebraic Approach to Linear Control Design*. Taylor & Francis, 1997.

[32] M. Souza, A. R. Fioravanti, and V. S. P. Araujo. Impulsive markov jump linear systems: Stability analysis and $\mathcal{Z}_2$ control. *Nonlinear Analysis: Hybrid Systems*, 42:101089, 2021.

[33] T. Tanaka, C. Langbort, and V. Ugrinovskii. DC-dominant property of cone-preserving transfer functions. *Systems & Control Letters*, 62(8):699–707, 2013.

[34] A. R. Teel, A. Subbaraman, and A. Sferlazza. Stability analysis for stochastic hybrid systems: A survey. *Automatica*, 50(10):2435–2456, 2014.

[35] A. van der Schaft. *$L_2$-Gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag, London, 2000.

[36] J. C. Willems. *The analysis of feedback systems*. MIT Press, Cambridge, MA, USA, 1971.

[37] S. You, A. Gattami, and J. C. Doyle. Primal robustness and semidefinite cones. In *54th IEEE Conference on Decision and Control*, pages 6227–6232, Osaka, Japan, 2015.