Enhanced Fritz John Stationarity, New Constraint Qualifications and Local Error Bound for Mathematical Programs with Vanishing Constraints

Abeka Khare* and Triloki Nath†

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Abstract

In this paper, we study the difficult class of optimization problem called mathematical programs with vanishing constraints or MPVC. An extensive research has been done for MPVC regarding stationary conditions and constraint qualifications using geometric approaches. We use the Fritz John approach for MPVC to derive the M-stationary conditions under weak constraint qualifications. An enhanced Fritz John type stationarity condition is also derived for MPVC, which provides the notion of enhanced M-stationarity condition under a new and weaker constraint qualification: MPVC-generalized quasinormality. Another new constraint qualification MPVC-CPLD is introduced, which is stronger than former one. A local error bound result is also established under MPVC-generalized quasinormality.

1 Introduction

In this paper, we consider a particular form of optimization problem which attracted the attention of optimization community over the past decade. It has the following form

$$\min \ f(x)$$

s.t. $g_i(x) \leq 0 \ \forall \ i = 1, 2, ..., m$

$h_j(x) = 0 \ \forall \ j = 1, 2, ..., p$

$H_i(x) \geq 0 \ \forall \ i = 1, 2, ..., q$

$G_i(x)H_i(x) \leq 0 \ \forall \ i = 1, 2, ..., q$ (1)

where all functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$, $g_i: \mathbb{R}^n \rightarrow \mathbb{R}$, $h_i: \mathbb{R}^n \rightarrow \mathbb{R}$, $G_i: \mathbb{R}^n \rightarrow \mathbb{R}$, $H_i: \mathbb{R}^n \rightarrow \mathbb{R}$ are continuously differentiable. This problem (1) is called mathematical program with vanishing constraints (or MPVC) due to its implicit sign constraints $G_i(x) \leq 0$, which vanishes immediately whenever $H_i(x) = 0$ (Indeed, $H_i(x) = 0 \Rightarrow G_i(x) \in \mathbb{R}$, i.e. $G_i(x)H_i(x) \leq 0$ is no more a constraint). We denote by $C$, the feasible region for this MPVC throughout the paper.

The above formulation of the problem has been first introduced and studied by Achtziger and Kanzow in [1], where the structure of the problem is discussed in a very lucid and systematic way. Before [1], a few papers in engineering applications have been appeared e.g. [2, 6, 11, 27], which considered particular cases of this general problem, but [1] was the first formal treatment of MPVC, it also featured some applications of MPVC like ground structure and truss structure design. The applications are not limited only to structural and topology optimization, but also applicable in robots motion planning [28, 29]. These applications of MPVC motivate the researchers for further study to evolve the field and to find some new tools to tackle this problem from theoretical as well as algorithmic point of view, e.g. [3, 20]. Subsequent of [1], a lot of collaborative

*Department of Mathematics and Statistics, Dr. Harisingh Gour Vishwavidyalaya, Sagar, Madhya Pradesh-470003, INDIA, Email- abekakhare2012@gmail.com

†Department of Mathematics and Statistics, Dr. Harisingh Gour Vishwavidyalaya, Sagar, Madhya Pradesh-470003, INDIA, Email- tnverma07@gmail.com
work has been done by Hoheisel and Kanzow in [18] [21] [22], which are the detailed study about the constraint qualifications and optimality conditions for MPVC. For more literature on MPVC, we refer to [18] [19] [21] and references therein.

Another well studied class of optimization problem is mathematical programs with equilibrium constraints (MPEC)

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad g_i(x) \leq 0 \quad \forall \ i = 1, 2, \ldots, m \\
& \quad h_j(x) = 0 \quad \forall \ j = 1, 2, \ldots, p \\
& \quad G_i(x) \geq 0 \quad \forall \ i = 1, 2, \ldots, q \\
& \quad H_i(x) \geq 0 \quad \forall \ i = 1, 2, \ldots, q \\
& \quad G_i(x)H_i(x) = 0 \quad \forall \ i = 1, 2, \ldots, q
\end{align*}
\]

where all functions \( f : \mathbb{R}^n \to \mathbb{R}, \ g_i : \mathbb{R}^n \to \mathbb{R}, \ h_i : \mathbb{R}^n \to \mathbb{R}, \ G_i : \mathbb{R}^n \to \mathbb{R}, \ H_i : \mathbb{R}^n \to \mathbb{R} \) are continuously differentiable. The MPEC is known to be a difficult optimization problem due to the violation of the standard constraint qualifications (CQs), e.g. the linear independence constraint qualification (LICQ) and the Mangasarian-Fromovitz constraint qualification (MFCQ) at any feasible point. It possibly happens due to disjunctive and combinatorial nature of characteristic constraints (e.g. \( H_i(x) \geq 0, G_i(x)H_i(x) \leq 0 \) in MPVC). Hence, the classical KKT conditions of standard nonlinear programming are not always necessary optimality conditions for MPEC, even in the case when all constraint functions are affine. In order to find the first order optimality conditions for an MPEC, modified constraint qualifications called MPEC-tailored constraint qualifications are defined. We refer to [30] [39] [43] [48] for a comprehensive overview. One of the optimality conditions called strong stationarity (S-stationarity) is equivalent to the KKT conditions of an MPEC [13] (c.f. [1] for MPVC). Hence, the S-stationarity is not always a necessary optimality condition. A slightly weaker notion called M-stationarity (see [36] [37] [46] [49]) is first-order necessary optimality conditions, which hold under mild assumptions, see [14] [16] [47] . To overcome this difficulty, the violation of standard constraint qualifications, the Fritz John approach is useful. Because, it is classical that Fritz John necessary conditions do not require any constraint qualification. This approach is used in [13] , to provide a simple proof for A-stationarity (weaker than M-stationarity) to be a necessary optimality condition under MPEC-MFCQ. In [47] (see also [13] ), based on the limiting subdifferential and the limiting coderivatives by Mordukhovich, see [42] [54] [59] to grasp such concepts, the Fritz John approach is used to find M-stationarity which is most appropriate necessary optimality condition for MPEC.

To this end, we note that it has been pointed out in [1] that MPVC can always be reformulated as MPEC. But it has some drawbacks, particularly it increases the dimension of the problem, and it involves locally non-unique solutions of corresponding MPEC, also it looses its characteristic of vanishing constraints of MPVC. So, it suggests to investigate MPVC independently taking into account the special structure of vanishing constraints. On the other hand, it also suggests that the whole MPEC machinery or analogous to that can be applied to an MPVC. Thus, Fritz John approach too can be used to exploit the special structure of vanishing constraints of an MPVC. Using the approach for MPVC, we show that M-stationarity conditions hold under MPVC-MFCQ, MPVC-linear-CQ and a weaker constraint qualification MPVC-GMFCQ.

The enhanced Fritz John conditions were first introduced by Bertsekas [7] (a weaker version of this is given by Hestenes [17] ) is stronger than the classical Fritz John conditions. It results in stronger KKT conditions under weaker constraint qualifications. Following [7] , Kanzow and Schwartz [23] established the enhanced KKT type stationarity (called enhanced M-stationarity ) conditions for MPEC under weaker MPEC-constraint qualifications, namely MPEC generalized pseudonormality and quasinormality. These constraint qualifications are MPEC version of that introduced in [9] for standard nonlinear programs. In [45] , the results of [23] has been extended to nonsmooth case. In [23] , it has been shown that pseudonormality is a sufficient condition for existence of local error bound for MPEC. Whereas in [45] , it has been improved to nonsmooth case by showing that the MPEC-generalized quasinormality is sufficient condition for the same under fairly mild assumptions, because pseudonormality implies quasinormality. The MPVC-generalized pseudonormality has been introduced in [23] , we introduce a new constraint qualification: MPVC-generalized quasinormality, its MPEC variant is
known. We prove that an enhanced M-stationarity condition holds under this constraint qualification.

In recent years, it has been shown that many constraint qualifications such as pseudonormality and quasinormality \cite{44, 9}, constant positive linear dependence (CPLD), see \cite{10} and relaxed constant positive linear dependence (RCPLD), see \cite{4}, all to be weaker than MFCQ. Following MPEC-CPLD defined in \cite{15}, we introduce MPVC-CPLD and show that it is stronger than the MPVC-generalized quasinormality. Further, we show that the MPVC-CPLD is also a constraint qualification for the enhanced M-stationarity and provides a sufficient condition for the existence of a local error bound for the MPVC.

We organize the present paper as follows. In Section 2, we recall some well known MPVC tailored constraint qualifications. We also recall some well known stationarity notions and few definitions from nonsmooth analysis. Section 3 is devoted to Fritz John type stationary conditions, that lead to KKT type stationary conditions under suitable constraint qualifications. In section 4, enhanced Fritz John type stationary conditions are investigated in depth. As a result, KKT type enhanced stationary conditions are discussed under the various known and a new constraint qualification: MPVC-generalized quasinormality, another new constraint qualification MPVC-CPLD is introduced. Further it is shown that MPVC-CPLD is stronger than MPVC-generalized quasinormality. The section 4 discusses the error bound results and finally we provide concluding remarks in section 5.

2 Preliminaries

Here, we provide some relevant definitions and background material for the MPVC formulated in \cite{11}, that will be used in subsequent sections of this paper. For an arbitrary feasible point \(x^\ast\) we adopt the following notations for index sets from \cite{21, 23}, which are analogous to MPEC \cite{25, 15}. We define the index sets as follows

\[
I_g(x^\ast) := \{i | g_i(x^\ast) = 0\}, \quad I_+(x^\ast) := \{i | H_i(x^\ast) > 0\}, \quad \text{and} \quad I_0(x^\ast) := \{i | H_i(x^\ast) = 0\}
\]

The index set \(I_+(x^\ast)\) is further divided into two subsets:

\[
I_{+0}(x^\ast) := \{i | H_i(x^\ast) > 0, \ G_i(x^\ast) = 0\}
\]

\[
I_{+\ast}(x^\ast) := \{i | H_i(x^\ast) > 0, \ G_i(x^\ast) < 0\}
\]

and the set \(I_0(x^\ast)\) can be partitioned as follows:

\[
I_{00}(x^\ast) := \{i | H_i(x^\ast) = 0, \ G_i(x^\ast) = 0\}
\]

\[
I_{0+}(x^\ast) := \{i | H_i(x^\ast) = 0, \ G_i(x^\ast) > 0\}
\]

\[
I_{0\ast}(x^\ast) := \{i | H_i(x^\ast) = 0, \ G_i(x^\ast) < 0\}
\]

If the concerned point is understood, we denote the index sets simply by \(I_g, I_+, I_{+0}, I_{00}\) and so on.

Now, we recall some standard constraint qualifications for MPVC based on these notations. The following two constraint qualifications were formally introduced in \cite{21}.

**Definition 2.1.** A vector \(x^\ast \in C\) is said to satisfy MPVC-linearly independent constraint qualification (or MPVC-LICQ) if the gradients

\[
\{\nabla g_i(x^\ast) | i \in I_g(x^\ast)\} \cup \{\nabla h_i(x^\ast) | i = 1, ..., p\} \cup \{\nabla G_i(x^\ast) | i \in I_{+0}(x^\ast) \cup I_{00}(x^\ast)\}
\]

\[
\cup \{\nabla H_i(x^\ast) | i \in I_{0}(x^\ast)\}
\]

are linearly independent.

**Definition 2.2.** A vector \(x^\ast \in C\) for \cite{11} satisfies MPVC-Mangasarian Fromovitz constraint qualification (or MPVC-MFCQ) if the

\[
\nabla h_i(x^\ast) (i = 1, ..., p), \quad \nabla H_i(x^\ast) (i \in I_{+0}(x^\ast) \cup I_{00}(x^\ast))
\]

are linearly independent and there exist a vector \(d \in \mathbb{R}^n\) such that

\[
\nabla h_i(x^\ast) d = 0 \quad (i = 1, ..., p), \quad \nabla H_i(x^\ast)^T d = 0 \quad (i \in I_{+0}(x^\ast) \cup I_{00}(x^\ast))
\]
\[ \nabla g_i(x^*)^T d < 0 \quad (i \in I_g(x^*)), \quad \nabla H_i(x^*)^T d > 0 \quad (i \in I_0-(x^*)), \]
\[ \nabla G_i(x^*)^T d < 0 \quad (i \in I_{+0}(x^*) \cup I_{00}(x^*)) \]

It has been seen that these constraint qualifications are very useful. In [19 Theorem 4.5], MPVC-MFCQ is shown to be a sufficient condition for exactness of MPVC-exact penalty function.

In the spirit of MPEC-GMFCQ [17], the following MPVC-GMFCQ was introduced in [23], and has been shown to be a key in exact penalty results.

**Definition 2.3.** A vector \( x^* \in C \) is said to satisfy MPVC-generalized MFCQ (MPVC-GMFCQ) if there is no multiplier \((\lambda, \mu, \eta^G, \eta^H) \neq 0 \) such that

1. \( \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i}^{p} \mu_i \nabla h_i(x^*) + \sum_{i=1}^{q} \eta^G_i \nabla G_i(x^*) - \sum_{i=1}^{q} \eta^H_i \nabla H_i(x^*) = 0 \)
2. \( \lambda_i \geq 0 \quad \forall \ i \in I_g(x^*), \quad \lambda_i = 0 \quad \forall \ i \notin I_g(x^*) \)
   and \( \eta^G_i = 0 \quad \forall \ i \in I_{+}-(x^*) \cup I_{0}-(x^*) \cup I_{00}(x^*), \quad \eta^H_i \geq 0 \quad \forall \ i \in I_{+0}(x^*) \cup I_{00}(x^*) \)
   \( \eta^H_i \geq 0 \quad \forall \ i \in I_{+0}(x^*) \)

**Remark 2.1.** In [17 Proposition 2.1] it has been established that MPEC-generalized MFCQ is equivalent to NNAMCQ. Analogously, we can show it for MPVC, if NNAMCQ is defined for MPVC analogous to MPEC notion. So we can identify MPVC-GMFCQ and MPVC-NNAMCQ to be same.

We note from [23 Proposition 2.1] that following implications hold:

\[ \text{MPVC-LICQ} \Rightarrow \text{MPVC-MFCQ} \Rightarrow \text{MPVC-GMFCQ}. \]

The following stationarity concepts are widely studied in the literature [1219] and known to be important optimality conditions for the MPVC.

**Definition 2.4. M-Stationary Condition:** Any feasible point \( x^* \) for \((P)\) is called M-Stationary point for MPVC, if there is a multiplier \((\lambda, \mu, \eta^G, \eta^H) \) such that

\[ 0 = \nabla f(x^*) + \sum_{i \in I_g} \lambda_i \nabla g_i(x^*) + \sum_{i \in I_h} \mu_i \nabla h_i(x^*) + \sum_{i=1}^{q} \eta^G_i \nabla G_i(x^*) - \sum_{i=1}^{q} \eta^H_i \nabla H_i(x^*) \]

and \( \lambda_i \geq 0 \quad \forall \ i \in I_g(x^*), \quad \lambda_i = 0 \quad \forall \ i \notin I_g(x^*) \)
\( \eta^G_i = 0 \quad \forall \ i \in I_{+}-(x^*) \cup I_{0}-(x^*) \cup I_{00}(x^*), \quad \eta^H_i \geq 0 \quad \forall \ i \in I_{+0}(x^*) \cup I_{00}(x^*) \)
\( \eta^H_i \geq 0 \quad \forall \ i \in I_{+0}(x^*) \)

**Definition 2.5. S-Stationary Condition:** Any feasible point \( x^* \) for \((P)\) is called S-Stationary point for MPVC, if there is a multiplier \((\lambda, \mu, \eta^G, \eta^H) \) such that

\[ 0 = \nabla f(x^*) + \sum_{i \in I_g} \lambda_i \nabla g_i(x^*) + \sum_{i \in I_h} \mu_i \nabla h_i(x^*) + \sum_{i=1}^{q} \eta^G_i \nabla G_i(x^*) - \sum_{i=1}^{q} \eta^H_i \nabla H_i(x^*) \]

and \( \lambda_i \geq 0 \quad \forall \ i \in I_g(x^*), \quad \lambda_i = 0 \quad \forall \ i \notin I_g(x^*) \)
\( \eta^G_i = 0 \quad \forall \ i \in I_{+}-(x^*) \cup I_{0}-(x^*) \cup I_{00}(x^*), \quad \eta^G_i \geq 0 \quad \forall \ i \in I_{+0}(x^*) \cup I_{00}(x^*) \)
\( \eta^H_i \geq 0 \quad \forall \ i \in I_{+0}(x^*) \quad \eta^H_i \geq 0 \quad \forall \ i \in I_{00}(x^*) \quad \eta^H_i \geq 0 \quad \forall \ i \in I_{00}(x^*) \)
There are other stationarity concepts also, such as W-stationarity, and T-stationarity, see [12]. The T-stationarity is counter part of C-stationarity for MPEC. It is easy to see that these are weaker than M-stationarity. They all differ in the sign of \( \eta_i^1 \) and \( \eta_i^2 \) for the indices \( i \in I_{00}(x^*) \). In this paper, the W- and T-stationarity will play no role, viewing them typically as too weak for our purposes.

Here, it is worth mentioning that among all stationary conditions, M-stationarity found to be most appropriate stationary condition in the sense that it is the second strongest stationary condition (weaker than S-stationarity). The role of M-stationarity for MPVC is the same as KKT conditions for a standard non linear program.

Now, we recall some basic tools from nonsmooth analysis. We give only concise definitions and results that will be used later to prove our main results. we refer Mordukhovich [34], Rockafellar and Wets [42] and Clarke [10] for more detailed information on the subject.

First we mention that throughout the paper, the following notations will be used. For any set \( S \subseteq \mathbb{R}^n \), \( clS \) and \( S^\diamond \) denote the closure and polar of \( S \) respectively. The symbol \( ||.|| \) and \( ||.||_1 \) denote the Euclidean and max norm on \( \mathbb{R}^n \) respectively. For a function \( g : \mathbb{R}^n \to \mathbb{R} \), \( g^+(x) := \max\{0, g(x)\} \), here \( g^+ \) denotes a vector if max function is defined componentwise.

Here are some definitions of various cones, which are known to be important tools in variational analysis.

**Definition 2.6.** 1. Let \( C \subseteq \mathbb{R}^n \) be a nonempty set. The polar cone of \( C \) is defined as
\[
C^\circ := \{ s \in \mathbb{R}^n | s^T d \leq 0 \ \forall \ d \in C \}
\]

2. Let \( C \subseteq \mathbb{R}^n \) be a nonempty closed set and \( x^* \in C \). The (Bouligand) tangent cone (or contingent cone) of \( C \) at \( x^* \) is defined as
\[
TC(x^*) := \{ d \in \mathbb{R}^n | \exists \{x^k\} \to_C x^*, \{t_k\} \downarrow 0 : \frac{x^k - x^*}{t_k} \to d \}
\]
where \( \{x^k\} \to_C x^* \) denotes a sequence \( \{x^k\} \) converging to \( x^* \) and satisfying \( x^k \in C \forall k \in \mathbb{N} \).

**Definition 2.7.** Let \( C \subseteq \mathbb{R}^n \) be a nonempty closed set and \( x^* \in C \).

1. The Fréchet normal cone of \( C \) at \( x^* \) is defined as
\[
N_C^F(x^*) := TC(x^*)^\circ
\]

2. The convex cone
\[
N_C^\pi(x^*) := \{d \in \mathbb{R}^n | \exists \sigma > 0 \text{ s.t. } \langle d, x - x^* \rangle \leq \sigma ||x - x^*||^2 \ \forall \ x \in C \}
\]
is called the proximal normal cone to \( C \) at \( x^* \).

3. The limiting normal cone of \( C \) at \( x^* \) is defined as
\[
N_C(x^*) := \{ d \in \mathbb{R}^n | \exists \{x^k\} \to_C x^*, d^k \in N_C^F(x^k) : d^k \to d \}
\]\
\[
:= \{ d \in \mathbb{R}^n | \exists \{x^k\} \to_C x^*, d^k \in N_C^\pi(x^k) : d^k \to d \}
\]

3 Fritz John type Stationary Conditions

To obtain first order necessary optimality conditions for a standard non linear program, usually three different approaches are available in the literature: (i) geometrical approach (tangent cone criterion), (ii) exact penalization approach [10] and (iii) Fritz John conditions. The third approach is preferred, some times, over others, since it does not require any constraint qualification. But, it has a disadvantage, it involves Lagrange multiplier associated to objective function, which may be zero. However, this disadvantage, innovatively suggests new constraint qualifications, so that KKT- conditions are satisfied, see [31, 7, 44]. However, they lead
Hence, we obtain
\[ \alpha \nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^q \mu_i \nabla h_i(x^*) + \sum_{i=1}^q \eta_i^G \nabla G_i(x^*) - \sum_{i=1}^q \eta_i^H \nabla H_i(x^*) = 0 \]

Applying Lagrange multiplier rule first obtained by Mordukhovich [35, Theorem 1(b)] (see also [42, Corollary 6.15]), we have
\[ \alpha \]
\[ \lambda_i \geq 0 \quad \forall i \in I_g(x^*) \]
and
\[ \eta_i^G = 0 \quad \forall i \in I_{+}(x^*) \cup I_{0+}(x^*) \]
\[ \eta_i^H = 0 \quad \forall i \in I_{+}(x^*) \]
\[ \eta_i^H \eta_i^G = 0 \quad \forall i \in I_{00}(x^*) \]

**Proof.** We consider MPVC in an equivalent form, called EMPVC defined as follows:

\[
\begin{align*}
\min \quad & f(x) \\
\text{s.t.} \quad & g(x) \leq 0 \\
& h(x) = 0 \\
& H(x) - y = 0 \\
& G(x) - z = 0 \\
& (x, y, z) \in \Omega
\end{align*}
\]

where \( \Omega = \{(x, y, z) \in \mathbb{R}^{n+2q} \mid y_i \geq 0, z_i y_i \leq 0, \quad \forall i = 1, ..., q\} \).

Clearly, \( \Omega \) is nonempty and closed set. It is an optimization problem with equalities, inequalities and an abstract set constraint, suppose we have a local minimum for this problem at \((x^*, y^*, z^*)\), then
\[ y^* = H(x^*) \quad \text{and} \quad z^* = G(x^*) \]

Applying Lagrange multiplier rule first obtained by Mordukhovich [35, Theorem 1(b)] (see also [42, Corollary 6.15]), we have \( \alpha \geq 0, \quad (\alpha, \lambda, \mu, \eta^H, \eta^G) \neq 0 \) such that

\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} = \begin{bmatrix}
\alpha \\
\sum_{i \in I_g(x^*)} \lambda_i \\
\sum_{i=1}^p \mu_i \\
\sum_{i=1}^q \eta_i^H \\
-\sum_{i=1}^q \eta_i^G
\end{bmatrix} \begin{pmatrix}
\nabla f(x^*) \\
\nabla g_i(x^*) \\
\nabla h_i(x^*) \\
\nabla G_i(x^*) \\
\nabla H_i(x^*)
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

and \( \lambda_i \geq 0 \quad \forall i \in I_g(x^*) \)

where \((0, \xi, \zeta)^T \in N_\Omega(x^*, y^*, z^*)\), the limiting normal cone of \( \Omega \) at \((x^*, y^*, z^*)\) and is given by

\[
N_\Omega(x^*, y^*, z^*) = \begin{cases}
0 & \xi_i = 0 = \zeta_i \quad \text{if} \ y_i^* > 0, z_i^* < 0 \\
\xi_i = 0, \zeta_i \geq 0 & \text{if} \ y_i^* > 0, z_i^* = 0 \\
\xi_i \geq 0, \zeta_i = 0 & \text{if} \ y_i^* = 0 = z_i^* \\
\xi_i \leq 0, \zeta_i = 0 & \text{if} \ y_i^* = 0, z_i^* < 0 \\
\xi_i \in \mathbb{R}, \zeta_i = 0 & \text{if} \ y_i^* = 0, z_i^* > 0
\end{cases}
\]

Hence, we obtain
\[
0 = \alpha \nabla f(x^*) + \sum_{i \in I_g} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) - \sum_{i=1}^q \eta_i^H \nabla H_i(x^*) - \sum_{i=1}^q \eta_i^G \nabla G_i(x^*)
\]
Then \( x \) is an M-stationary point.

Proof. Here, we use the standard necessary optimality condition at local minimum \( x^* \), which involves Fréchet normal cone rather than limiting normal cone. Hence, proof follows using above arguments. \( \square \)

Remark 3.1. If we use Fréchet normal cone of \( \Omega \) at local minimum \( (x^*, y^*, z^*) \)

\[
N_{\Omega}^F(x^*, y^*, z^*) = \begin{cases} 
  \xi_i = 0 = \zeta_i ; & \text{if } y_i^* > 0, \ z_i^* < 0 \\
  \xi_i = 0, \zeta_i \geq 0 ; & \text{if } y_i^* > 0, \ z_i^* = 0 \\
  \xi_i \leq 0, \zeta_i = 0 ; & \text{if } y_i^* = 0 = \ z_i^* \\
  \xi_i \leq 0, \zeta_i = 0 ; & \text{if } y_i^* = 0, \ z_i^* < 0 \\
  \xi_i \in \mathbb{R}, \zeta_i = 0 ; & \text{if } y_i^* = 0, \ z_i^* > 0 
\end{cases}
\]

instead of limiting normal cone in the above theorem, then we get a different restriction on multipliers in bi-active set \( I_0(x^*) \). These restrictions yield Fritz John type strong stationary conditions.

Theorem 3.2. A Fritz John type S-stationary condition:

Let \( x^* \) be a local minimum of MPVC \( P \), then there exist \( \alpha \geq 0 \) and \( \lambda, \mu, \eta^H, \eta^G \) such that:

(i) \( \alpha \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{q} \mu_i \nabla h_i(x^*) + \sum_{i=1}^{q} \eta_i^G \nabla G_i(x^*) - \sum_{i=1}^{q} \eta_i^H \nabla H_i(x^*) = 0 \)

(ii) \( \lambda_i \geq 0 \ \forall i \in I_g(x^*) \), \( \lambda_i = 0 \ \forall i \notin I_g(x^*) \)

and \( \eta_i^G = 0 \ \forall i \in I_+(x^*) \cup I_0-(x^*) \cup I_0+(x^*) \), \( \eta_i^G \geq 0 \ \forall i \in I_0+(x^*) \cup I_00(x^*) \)

\( \eta_i^H = 0 \ \forall i \in I_+(x^*) \), \( \eta_i^H \geq 0 \ \forall i \in I_0-(x^*) \) and \( \eta_i^H \) is free \( \forall i \in I_0+(x^*) \)

\( \eta_i^H \geq 0 \), \( \eta_i^G = 0 \ \forall i \in I_00(x^*) \)

Proof. Here, we use the standard necessary optimality condition at local minimum \( x^* \), which involves Fréchet normal cone rather than limiting normal cone. Hence, proof follows using above arguments. \( \square \)

Now, using this result we can establish some well known results of MPEC for MPVC also. As in [17, Corollary 2.1, Proposition 2.1], Ye showed that any local optimizer of MPEC also becomes M-stationarity under (NNAMCQ or MPEC-GMFCQ). Here, first we prove M-stationarity for MPVC under the constraint qualification MPVC-MFCQ. Although this result has been established for MPVC in [19, Corollary 5.3] under the exact penalty condition at the local minimizer, but we are relaxing this exactness condition and using a different approach to prove the result, which is independent and easier than [19]. Indeed, our proof also shows that MPVC-MFCQ need not provide S-stationary condition.

Theorem 3.3. M-stationary condition: Suppose MPVC-MFCQ holds at a local minimizer \( x^* \) of MPVC. Then \( x^* \) will be an M-stationary point.

Proof. Since MPVC-MFCQ holds at a local minimizer \( x^* \), this implies that there exists a vector \( d \in \mathbb{R}^n \) such that:

\[
\begin{align*}
\nabla h_i(x^*)^T d &= 0 \quad \forall i = 1, ..., p, \\
\nabla H_i(x^*)^T d &= 0 \quad \forall i \in I_{00}(x^*) \cup I_{0+}(x^*) \\
-\nabla g_i(x^*)^T d &> 0 \quad \forall i \in I_g(x^*) \\
-\nabla G_i(x^*)^T d &> 0 \quad \forall i \in I_{+0}(x^*) \cup I_{00}(x^*) \\
\n\nabla H_i(x^*)^T d &> 0 \quad \forall i \in I_{0-}(x^*) 
\end{align*}
\]

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Now, by Motzkin’s theorem
\[
\sum_{i \in I_g} \lambda_i (-\nabla g_i(x^*)) + \sum_{i=1}^{P} \mu_i \nabla h_i(x^*) + \sum_{i \in I_{-0} \cup I_{0+}} \eta_i^G (-\nabla G_i(x^*)) + \sum_{i \in I_{00} \cup I_{0+} \cup I_{-0}} \eta_i^H \nabla H_i(x^*) = 0
\]
has no non zero solution. where
\[
\begin{align*}
\lambda_i & \geq 0 \quad \forall \ i \in I_g(x^*), \lambda_i = 0 \quad \forall \ i \notin I_g(x^*) \\
\eta_i^G & \geq 0 \quad \forall \ i \in I_{+0}(x^*) \cup I_{00}(x^*), \mu_i \in \mathbb{R} \\
\eta_i^H & \geq 0 \quad \forall i \in I_{-0}(x^*), \eta_i^H \in \mathbb{R} \quad \forall i \in I_{00}(x^*) \cup I_{0+}(x^*)
\end{align*}
\]
that is, we can say that
\[
\sum_{i \in I_g} \lambda_i \nabla g_i(x^*) - \sum_{i=1}^{P} \mu_i \nabla h_i(x^*) + \sum_{i \in I_{+0} \cup I_{00}} \eta_i^G \nabla G_i(x^*) - \sum_{i \in I_{00} \cup I_{0+} \cup I_{-0}} \eta_i^H \nabla H_i(x^*) = 0
\]
has no non zero solution satisfying the multiplier conditions (3).

Since \( \mu \) is free, therefore
\[
\sum_{i \in I_g} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{P} \mu_i \nabla h_i(x^*) + \sum_{i \in I_{+0} \cup I_{00}} \eta_i^G \nabla G_i(x^*) - \sum_{i \in I_{00} \cup I_{0+} \cup I_{-0}} \eta_i^H \nabla H_i(x^*) = 0
\]
also has no nonzero solution with restriction on multipliers given in (3).

Now, since \( x^* \) is local minimizer, therefore by the Fritz John type M-stationary condition, there exist a nonzero multiplier \( (\alpha, \lambda, \mu, \eta^H, \eta^G) \) such that
\[
\alpha \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) + \sum_{i=1}^{q} \eta_i^G \nabla G_i(x^*) - \sum_{i=1}^{q} \eta_i^H \nabla H_i(x^*) = 0
\]
where
\[
\lambda_i \geq 0 \quad \forall i \in I_g, \lambda_i = 0 \quad \forall i \notin I_g \quad \text{and} \quad \eta_i^H = 0 \quad \forall i \in I_{+}, \eta_i^G = 0 \quad \forall i \in I_{+0} \cup I_{0-} \cup I_{0+} \quad \text{and} \quad \eta_i^G \geq 0 \quad \forall i \in I_{+0}, \eta_i^H \geq 0 \quad \forall i \in I_{00} \quad \text{and} \quad \eta_i^G \geq 0.
\]

Here if \( \alpha = 0 \), then it contradicts that equation (4), obtained by MPVC-MFCQ, has no nonzero solution with restriction on multipliers given in (3). Hence \( \alpha \neq 0 \) and then by proper scaling in Fritz John type optimality condition we have M-stationary condition at \( x^* \).

The above M-stationarity is also derived in [21, Theorem 3.4] under MPVC-GCQ (MPVC-Guignard constraint qualification), which is much weaker than MPVC-MFCQ. In the above proof, if we use the Fritz John type S-stationarity, then for \( \alpha = 0 \), we may obtain nonzero solutions of (4) with restriction on multipliers given in (3). For illustration, take \( \eta_i^H > 0, \forall i \in I_{00} \) and other multipliers zero, then they satisfy the KKT type S-stationarity conditions, and also (3) has a nonzero solution with restriction given in (3). This contradicts MPVC-MFCQ, hence under MPVC-MFCQ the S-stationarity need not hold.

Our next result shows that the M-stationarity holds under a constraint qualification MPVC-GMFCQ, which is weaker than MPVC-MFCQ

**Theorem 3.4.** If \( x^* \) is a local minimizer of MPVC and MPVC-GMFCQ holds at \( x^* \), then \( x^* \) is an M-stationary point.

**Proof.** If \( x^* \) is local minimizer, then we have by Fritz John type M-stationary condition, there exist \( \alpha \geq 0 \) and \( \lambda, \mu, \eta^H, \eta^G \) such that:
\[ \alpha \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) + \sum_{i=1}^{q} \eta_i^G \nabla G_i(x^*) - \sum_{i=1}^{q} \eta_i^H \nabla H_i(x^*) = 0 \]

where \( \lambda_i \geq 0 \quad \forall \ i \in I_g(x^*), \quad \lambda_i = 0 \quad \forall \ i \notin I_g(x^*) \)

and \( \eta_i^G = 0 \quad \forall \ i \in I_+ (x^*) \cup I_0 (x^*) \cup I_0 (x^*), \quad \eta_i^G \geq 0 \quad \forall \ i \in I_+ (x^*) \cup I_0 (x^*) \)

\[ \eta_i^H = 0 \quad \forall \ i \in I_+ (x^*), \quad \eta_i^H \geq 0 \quad \forall \ i \in I_0 (x^*) \quad \text{and} \quad \eta_i^H \text{ is free} \quad \forall \ i \in I_0 (x^*) \]

If \( \alpha = 0 \), then it violates the MPVC-GMFCQ. Hence \( \alpha \neq 0 \) and then by proper scaling we get M-stationarity at \( x^* \).

By using EMPVC form of MPVC, we can establish M-stationary condition for MPVC. Corresponding result has also been established for OPVIC in [46, Corollary 4.8] and for MPEC in [47]. Here, we are using similar approach of the error bound as in [46, 47] to prove our result. But, firstly analogous to optimization problem with variational inequality constraints (OPVIC) [46, Definition 4.1], we define local error bound for constraint system of EMPVC as follows.

**Definition 3.1. Local error bound property for EMPVC:**

The system of constraints

\[
\begin{align*}
g(z) & \leq 0 \\
h(z) & = 0 \\
H(z) - y & = 0 \\
G(z) - x & = 0 \\
(x, y) & \in \Omega
\end{align*}
\]

where

\[ \Omega = \{(x, y) \in \mathbb{R}^{2q} \mid y_i \geq 0, \ x_i y_i \leq 0, \ \forall \ i = 1, ..., q \} \]

is said to have a local error bound at \( (x^*, y^*, z^*) \), if there exist \( \alpha, \delta, \epsilon > 0 \) such that

\[
d((x, y, z), \Psi(0, 0, 0, 0)) \leq \alpha \|(p, q, r, s)\|
\]

for all \( (p, q, r, s) \in \mathbb{B}_e(0, 0, 0, 0) \) and all \( (x, y, z) \in \Psi(p, q, r, s) \cap \mathbb{B}_e(x^*, y^*, z^*) \), where

\[ \Psi(p, q, r, s) = \{(x, y, z) \in \Omega \times \mathbb{R}^p \mid g(z) + p \leq 0; h(z) + q = 0; H(z) - y + s = 0; G(z) - x + r = 0 \} \]

Usually, in case when all the constraints are affine, KKT necessary optimality conditions hold without any additional constraint qualification, but we can not assure it for MPVC as it is a special and more difficult class of optimization problem. Therefore, to prove our next result, we define one more constraint qualification besides the above local error bound property for EMPVC, which is similar to linear constraint qualification of MPEC case [47, Definition 2.12]

**Definition 3.2. MPVC-linear constraint qualification:** For the MPVC problem, MPVC-linear constraint qualification is said to be satisfied if all the functions \( g, h, G, H \) are affine.

Using these two constraint qualifications, we have the following result.

**Theorem 3.5. (M-stationary condition):** Let \( z^* \) be a local optimal solution for MPVC, where all functions are continuously differentiable at \( z^* \). If either MPVC-GMFCQ or MPVC-linear constraint qualification is satisfied at \( z^* \), then \( z^* \) is M-stationary point.
Proof. Under MPVC-GMFCQ result holds obviously. For the later case, MPVC can be written equivalently as

\[
\begin{align*}
(\text{EMPVC}) \quad & \quad \min f(x) \\
\text{s.t.} \quad & \quad g(x) \leq 0 \\
& \quad h(x) = 0 \\
& \quad H(x) - y = 0 \\
& \quad G(x) - z = 0 \\
& \quad (y, z) \in \Omega
\end{align*}
\]

and \(\Omega = \{(y, z) \in \mathbb{R}^{2q} \mid y_i \geq 0 ; y_i z_i \leq 0\}\).

Now, we consider the set of solutions to the perturbed constraints system for EMPVC.

\[
\Psi(p, q, r, s) = \{(x, y, z) \in \mathbb{R}^n \times \Omega \mid g(x) + p \leq 0; h(x) + q = 0; H(x) - y + s = 0; G(x) - z + r = 0\}
\]

Since all \(g(x), h(x), H(x), G(x)\) are affine, therefore graph of the set valued map \(\Psi(p, q, r, s)\) is a union of polyhedral convex sets and hence \(\Psi(p, q, r, s)\) is a polyhedral multifunction. By [41, Proposition 1] \(\Psi\) is locally upper Lipschitz at each point of \(\mathbb{R}^{2q+m+p}\), in particular at \((0, 0, 0, 0) \in \mathbb{R}^{2q+m+p}\), i.e. there is a neighbourhood \(U\) of \((0, 0, 0, 0)\) and \(\alpha \geq 0\) such that

\[
\Psi(p, q, r, s) \subseteq \psi(0, 0, 0, 0) + \alpha\|(p, q, r, s)\|_{cl\mathbb{B}} \quad \forall \ (p, q, r, s) \in U
\]

where \(cl\mathbb{B}\) denotes the closed unit ball. Equivalently, the constraint system of EMPVC has a local error bound, i.e.

\[
d((x, y, z), \Psi(0, 0, 0, 0)) \leq \alpha\|(p, q, r, s)\|
\]

for all \((p, q, r, s) \in U\) and \((x, y, z) \in \Psi(p, q, r, s)\). Now by Clarke’s principle of exact penalization [10] Proposition 2.4.3 \((x^*, y^*, z^*)\) is also a local optimal solution to the unconstrained problem

\[
\min f(x) + \mu_f d((x, y, z), \Psi(0, 0, 0, 0))
\]

Hence by the local error bound property and the fact that \(x^*\) is local optimal solution to MPVC, we have that \((x^*, 0, 0, 0)\) is a local optimal solution to the following

\[
\min f(x) + \mu_f\alpha\|(p, q, r, s)\|
\]

such that

\[
\begin{align*}
g(x) + p & \leq 0 \\
h(x) + q & = 0 \\
H(x) + s & \geq 0 \\
(G_i(x) + r)(H_i(x) + s) & \leq 0 \quad \forall \ i = 1, \ldots, q
\end{align*}
\]

then at \((x^*, 0, 0, 0, 0)\), MPVC-GMFCQ (or MPVC-NNAMCQ) is satisfied for the above problem. Suppose not, then there is a nonzero multiplier \((\lambda, \mu, \eta^G, \eta^H)\) such that

\[
0 = \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + p + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) + q - \sum_{i=1}^{q} \eta^H_i \nabla (H_i(x^*) + s) + \sum_{i=1}^{q} \eta^G_i \nabla (G_i(x^*) + r)
\]

\[
0 = \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) - \sum_{i=1}^{q} \eta^H_i \nabla H_i(x^*) + \sum_{i=1}^{q} \eta^G_i \nabla G_i(x^*)
\]

\[
+ \sum_{i=1}^{m} \lambda_i \nabla p + \sum_{i=1}^{p} \mu_i \nabla q - \sum_{i=1}^{q} \eta^H_i \nabla s + \sum_{i=1}^{q} \eta^G_i \nabla r
\]

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Theorem 4.1. (Enhanced Fritz John type M-stationary conditions for MPVC.) Necessary optimality conditions for MPVC. We refer Hoheisel and Kanzow [21] for detailed study of these conditions. We follow [25], using limiting and Fréchet normal cone respectively, we obtain M-stationary and S-stationary conditions. Their proof was completely elementary, since they used Fréchet normal cone instead of limiting normal cone [47].

(i) \( \alpha > 0 \)

(ii) \( \lambda_i \geq 0 \) \( \forall i \in \mathcal{I}_0(x^*) \), \( \lambda_i = 0 \) \( \forall i \notin \mathcal{I}_0(x^*) \)

(iii) \( \eta_i^G = 0 \) \( \forall i \in \mathcal{I}_+ - (x^*) \cup \mathcal{I}_0 - (x^*) \cup \mathcal{I}_0 + (x^*) \), \( \eta_i^G \geq 0 \) \( \forall i \in \mathcal{I}_{-+}(x^*) \cup \mathcal{I}_{+-}(x^*) \)

(iv) \( \eta_i^H = 0 \) \( \forall i \in \mathcal{I}_+ (x^*) \), \( \eta_i^H \geq 0 \) \( \forall i \in \mathcal{I}_{-+}(x^*) \) and \( \eta_i^H \) is free \( \forall i \in \mathcal{I}_{+-}(x^*) \)



\[ \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) = 0 \]

and

\[ \eta_i^G = 0 \] \( \forall i \in \mathcal{I}_+ (x^*) \cup \mathcal{I}_0 (x^*) \cup \mathcal{I}_0 + (x^*) \), \( \eta_i^G \geq 0 \) \( \forall i \in \mathcal{I}_{-+}(x^*) \cup \mathcal{I}_{+-}(x^*) \)

\[ \eta_i^H = 0 \] \( \forall i \in \mathcal{I}_+(x^*) \), \( \eta_i^H \geq 0 \) \( \forall i \in \mathcal{I}_{-+}(x^*) \) and \( \eta_i^H \) is free \( \forall i \in \mathcal{I}_{+-}(x^*) \)

\[ \eta_i^G = 0 \] \( \forall i \in \mathcal{I}_0 (x^*) \)

(\( i \) \( \epsilon \) \( k \) \( \eta \) \( \mathbb{N} \) \( \mathbb{R} \) \( \Omega \))

Proof. Firstly, we formulate MPVC equivalently as (EMPVC):

\[
\begin{align*}
\min f(x) \\
\text{s.t.} \quad g(x) &\leq 0 \\
h(x) &= 0 \\
y - H(x) &= 0 \\
z - G(x) &= 0 \\
(x, y, z) &\in \Omega
\end{align*}
\]

Where

\( \Omega = \{(x, y, z) \in \mathbb{R}^{n+q+q} | y_i \geq 0, z_i y_i \leq 0, \forall i = 1, ..., q\} \)

is nonempty and closed set and we have a local minimum at \( (x^*, y^*, z^*) \), then

\[ y^* = H(x^*) \quad \text{and} \quad z^* = G(x^*) \]

Now using the idea of [21] Proposition 2.1, we choose \( \epsilon > 0 \) such that

\[ f(x) \geq f(x^*) \quad \forall (x, y, z) \in \mathcal{S} \]
where 
\[ S = \{ (x, y, z) \mid \| (x, y, z) - (x^*, y^*, z^*) \| \leq \epsilon \} \]

that are feasible for EMPVC.

Now consider the penalized problem approach given by McShane [32] and later elegantly used by Bertsekas [9]

\[
\min_{(x, y, z) \in S \cap \Omega} F_k(x, y, z)
\]

with

\[
F_k(x, y, z) = f(x) + \frac{k}{2} \sum_{i=1}^{m} (g_i^+(x^k))^2 + \frac{k}{2} \sum_{i=1}^{p} h_i(x)^2 + \frac{k}{2} \sum_{i=1}^{q} (y_i - H_i(x))^2 + \frac{k}{2} \sum_{i=1}^{q} (z_i - G_i(x))^2 + \frac{1}{2} \| (x, y, z) - (x^*, y^*, z^*) \| ^2 ; \quad \forall k \in \mathbb{N}
\]

Since \( S \cap \Omega \) is compact and \( F_k \) is continuous, therefore this penalized problem has at least one solution say \((x^k, y^k, z^k)\), \( \forall k \in \mathbb{N} \)

Now we will show that this sequence \{ \((x^k, y^k, z^k)\) \} converges to \((x^*, y^*, z^*)\). Note that

\[
F_k(x^k, y^k, z^k) = f(x^k) + \frac{k}{2} \sum_{i=1}^{m} (g_i^+(x^k))^2 + \frac{k}{2} \sum_{i=1}^{p} h_i(x^k)^2 + \frac{k}{2} \sum_{i=1}^{q} (y_i^k - H_i(x^k))^2 + \frac{k}{2} \sum_{i=1}^{q} (z_i^k - G_i(x^k))^2 + \frac{1}{2} \| (x^k, y^k, z^k) - (x^*, y^*, z^*) \| ^2 \\
\leq F_k(x^*, y^*, z^*) = f(x^*) ; \quad \forall k \in \mathbb{N}
\]

Since \( S \cap \Omega \) is compact, therefore sequence \{ \( f(x^k) \) \} is bounded. This yields

\[
\lim_{k \to \infty} g_i^+(x^k) = 0 ; \quad \forall i = 1, \ldots, m
\]

\[
\lim_{k \to \infty} h_i(x^k) = 0 ; \quad \forall i = 1, \ldots, p
\]

\[
\lim_{k \to \infty} (y_i^k - H_i(x^k)) = 0 ; \quad \forall i = 1, \ldots, q
\]

\[
\lim_{k \to \infty} (z_i^k - G_i(x^k)) = 0 ; \quad \forall i = 1, \ldots, q
\]

Otherwise the left hand side of inequality above would become unbounded and hence every accumulation point of \{ \((x^k, y^k, z^k)\) \} is feasible for the reformulated MPVC \( P \).

The compactness of \( S \cap \Omega \) also ensures the existence of at least one accumulation point. Let \((\bar{x}, \bar{y}, \bar{z})\) be an arbitrary accumulation point of the sequence, then by the continuity:

\[
f(\bar{x}) + \frac{1}{2} \| (\bar{x}, \bar{y}, \bar{z}) - (x^*, y^*, z^*) \| ^2 \leq f(x^*)
\]

and by the feasibility of \((\bar{x}, \bar{y}, \bar{z})\)

\[
f(x^*) \leq f(\bar{x})
\]

Hence

\[
\| (\bar{x}, \bar{y}, \bar{z}) - (x^*, y^*, z^*) \| ^2 = 0
\]

Hence, sequence \{ \((x^k, y^k, z^k)\) \} converges to \{ \((x^*, y^*, z^*)\) \}. Now we may assume without loss of generality that \((x^k, y^k, z^k)\) is an interior point of \( S \) \( \forall k \in \mathbb{N} \). Then by limiting subgradient version [46] Theorem 3.2 of generalized lagrange multiplier rule [10] Theorem 6.1.1, we have

\[
- \nabla F_k(x^k, y^k, z^k) \in N_{\Omega}(x^k, y^k, z^k) ; \quad \forall k \in \mathbb{N}
\]

Where the gradient of \( F_k \) is given by -
\[ -\nabla F_k(x^k, y^k, z^k) = - \left[ \begin{array}{c}
\nabla f(x^k) \\
0 \\
0 
\end{array} \right] + \sum_{i=1}^m k g_i^+(x^k) \left[ \begin{array}{c}
\nabla g_i(x^k) \\
0 \\
0 
\end{array} \right] + \sum_{i=1}^p k h_i(x^k) \left[ \begin{array}{c}
\nabla h_i(x^k) \\
0 \\
0 
\end{array} \right] - \sum_{i=1}^q k(y_i^k - H_i(x^k)) \left[ \begin{array}{c}
\nabla H_i(x^k) \\
e_i \\
0 
\end{array} \right] - \sum_{i=1}^q k(z_i^k - G_i(x^k)) \left[ \begin{array}{c}
0 \\
y^k \\
z^k 
\end{array} \right] + \left[ \begin{array}{c}
x^k \\
y^* \\
z^* 
\end{array} \right] \right] 
\]

and limiting normal cone of \( \Omega \) at \((x^k, y^k, z^k)\) is given by

\[
N_\Omega(x^k, y^k, z^k) = \begin{cases} 
\xi_i = 0 = \zeta_i & : \text{if } y_i^k > 0, \ z_i^k < 0 \\
\xi_i = 0, \zeta_i \geq 0 & : \text{if } y_i^k > 0, \ z_i^k = 0 \\
\xi_i \geq 0, \xi_i, \zeta_i = 0 & : \text{if } y_i^k = 0 = z_i^k \\
\xi_i \leq 0, \zeta_i = 0 & : \text{if } y_i^k = 0, \ z_i^k < 0 \\
\xi_i \in \mathbb{R}, \zeta_i = 0 & : \text{if } y_i^k = 0, \ z_i^k > 0 
\end{cases}
\]

Hence, we obtain

\[
\nabla f(x^k) + \sum_{i=1}^m k g_i^+(x^k) \nabla g_i(x^k) + \sum_{i=1}^p k h_i(x^k) \nabla h_i(x^k) - \sum_{i=1}^q k(y_i^k - H_i(x^k)) \nabla H_i(x^k) \\
- \sum_{i=1}^q k(z_i^k - G_i(x^k)) \nabla G_i(x^k) + (x^k - x^*) = 0
\]

Now we have

If \( y_i^k > 0, \ z_i^k < 0 \) that is if \( i \in I_{+-}(x^*) \) then ;

\[
k(y_i^k - H_i(x^k)) = -(y_i^k - y^*) \\
k(z_i^k - G_i(x^k)) = -(z_i^k - z^*)
\]

If \( y_i^k > 0, \ z_i^k = 0 \) that is \( i \in I_{+0}(x^*) \) then ;

\[
k(y_i^k - H_i(x^k)) = -(y_i^k - y^*)
\]

and \( k(z_i^k - G_i(x^k)) + (z_i^k - z^*) = -\zeta \leq 0 \)

\[
k(z_i^k - G_i(x^k)) \leq -(z_i^k - z^*)
\]

Similarly if \( y_i^k = 0 = z_i^k \) that is \( i \in I_{00}(x^*) \) then ;

\[
k(z_i^k - G_i(x^k)) \leq -(z_i^k - z^*)
\]

If \( y_i^k = 0, \ z_i^k < 0 \) that is \( i \in I_{0-}(x^*) \) then ;

\[
k(y_i^k - H_i(x^k)) \geq -(y_i^k - y^*) \\
k(z_i^k - G_i(x^k)) = -(z_i^k - z^*)
\]

If \( y_i^k = 0, \ z_i^k > 0 \) that is \( i \in I_{0+}(x^*) \) then ;

\[
k(y_i^k - H_i(x^k)) + (y_i^k - y^*) \in \mathbb{R} \\
k(z_i^k - G_i(x^k)) = -(z_i^k - z^*)
\]
Now we define the multipliers, 
\[
\delta_k = \left(1 + \sum_{i=1}^{m}(kg^+_i(x^k))^2 + \sum_{i=1}^{p}(k\bar{h}_i(x^k))^2 + \sum_{i=1}^{q}(ky^k_i - H_i(x^k))^2 + \sum_{i=1}^{q}(kz^k_i - G_i(x^k))^2\right)^{\frac{1}{2}}
\]
and
\[
\alpha_k = \frac{1}{\delta_k}, \quad \lambda^k_i = \frac{k\max\{0, g_i(x^k)\}}{\delta_k}, \quad \forall \; i = 1, \ldots, m \\
\mu^k_i = \frac{kh_i(x^k)}{\delta_k}, \quad \forall \; i = 1, \ldots, p \\
\eta^{H^k}_i = \frac{k(y^k_i - H_i(x^k))}{\delta_k}, \quad \forall \; i = 1, \ldots, q \\
\eta^{G^k}_i = \frac{k(z^k_i - G_i(x^k))}{\delta_k}, \quad \forall \; i = 1, \ldots, q
\]
Since \(\| (\alpha_k, \lambda^k_i, \mu^k_i, \eta^{H^k}_i, \eta^{G^k}_i) \| = 1 \) for all \( k \in \mathbb{N} \). Hence we may assume that this sequence of multipliers converges to some limit \((\alpha, \lambda, \mu, \eta^H, \eta^G) \neq 0\).

Now we will analyze some properties of this limit. Since \( \alpha_k \to \alpha \), therefore sequence \( \{\delta_k\} \) either diverges to \(+\infty\) or converges to some positive value (greater than or equal to one). By continuity of gradients and because of \( x^k \to x^* \), we obtain
\[
\alpha \nabla f(x^*) + \sum_{i=1}^{m}\lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p}\mu_i \nabla \bar{h}_i(x^*) - \sum_{i=1}^{q}\eta^{H}_i \nabla G_i(x^*) - \sum_{i=1}^{q}\eta^{G}_i \nabla H_i(x^*) = 0
\]
Furthermore, \( \alpha \geq 0 \) and \( \lambda_i \geq 0 \) for all \( i \in I_g(x^*) \), additionally we have \( \lambda_i = 0 \) for all \( i \notin I_g(x^*) \). Now remember,
\[
(y^*, z^*) = (H(x^*), G(x^*))
\]
and \((x^k, y^k, z^k) \in \Omega \quad \forall \; k \in \mathbb{N} \)
Now for all \( i \in I_{+0}(x^*) \cup I_{+ -}(x^*) \), where \( y^k_i > 0, z^k_i = 0 \) or \( y^k_i > 0, z^k_i < 0 \), this yields
\[
\eta^{H}_i = \lim_{k \to \infty} \frac{k(y^k_i - H_i(x^k))}{\delta_k} = \lim_{k \to \infty} \frac{-k(y^k_i - y^*)}{\delta_k} = 0
\]
Similarly, for all \( i \in I_{-0}(x^*) \cup I_{- -}(x^*) \cup I_{0+}(x^*) \), we have \( \eta^{G}_i = 0 \)
Now for all \( i \in I_{0 -}(x^*) \)
\[
\eta^{H}_i = \lim_{k \to \infty} \frac{k(y^k_i - H_i(x^k))}{\delta_k} \geq \lim_{k \to \infty} \frac{-(y^k_i - y^*)}{\delta_k} = 0
\]
and for all \( i \in I_{+0}(x^*) \cup I_{00}(x^*) \),
\[
\eta^{G}_i = \lim_{k \to \infty} \frac{k(z^k_i - G_i(x^k))}{\delta_k} \leq \lim_{k \to \infty} \frac{-(z^k_i - z^*)}{\delta_k} = 0
\]
and \( \forall i \in I_{00}(x^*), \eta_i^G \leq 0 \) and \( \eta_i^G, \eta_i^H = 0 \).

Replace \( \eta_i^G \) by \(-\eta_i^G\), then first negative term in eq \( \{5\} \) becomes positive and we get \((i)\), with \( \eta_i^G \geq 0 \forall i \in I_{+0}(x^*) \cup I_{00}(x^*) \) in condition \((ii)\). Finally, assume that \((\lambda, \mu, \eta^G, \eta^H) \neq 0\), then \((\lambda^k, \mu^k, \eta^G_k, \eta^H_k) \neq 0\) for all \( k \in \mathbb{N} \) sufficiently large. Hence by the definition of multipliers, we have \((x^k, y^k, z^k) \neq (x^*, y^*, z^*)\).

Therefore, we have for all \( k \) sufficiently large

\[
f(x^k) < f(x^*) + \frac{1}{2}||[(x^k, y^k, z^k) - (x^*, y^*, z^*)]|| \leq f(x^*)
\]

Hence \( f(x^k) < f(x^*) \). Further, we have the following results for all \( i \) and all \( k \) sufficiently large,

\[
\lambda_i > 0 \Rightarrow \lambda_i^k > 0 \Rightarrow g_i(x^k) > 0 \Rightarrow \lambda_i g_i(x^k) > 0
\]

\[
\mu_i \neq 0 \Rightarrow \mu_i \mu_i^k > 0 \Rightarrow \mu_i h_i(x^k) > 0
\]

Further, if \( \eta_i^H \neq 0 \) for \( i \in \{1, \ldots, q\} \), then

\[
\eta_i^H \eta_i^H > 0
\]

\[
\eta_i^H (y_i^k - H_i(x^k)) > 0
\]

\[
\eta_i^H H_i(x^k) < \eta_i^H y_i^k
\]

Since whenever \( y_i^k > 0 \) for infinitely many \( k \), then \( \eta_i^H = 0 \). Hence, in our case \( y_i^k = 0 \forall k \) sufficiently large, so

\[
\eta_i^H H_i(x^k) < 0
\]

Now if \( \eta_i^G \neq 0 \), then

\[
\eta_i^G \eta_i^G > 0
\]

\[
\eta_i^G (z_i^k - G_i(x^k)) > 0
\]

\[
\eta_i^G G_i(x^k) < \eta_i^G z_i^k
\]

If \( z_i^k \neq 0 \) for infinitely many \( k \), then \( \eta_i^G = 0 \), thus in our case \( z_i^k = 0 \) for all sufficiently large \( k \).

Hence

\[
\eta_i^G G_i(x^k) < 0 \quad \text{for all } k \text{ sufficiently large}
\]

that is

\[
-\eta_i^G G_i(x^k) > 0 \quad \text{for all } k \text{ sufficiently large}.
\]

As mentioned earlier that we replace \( \eta_i^G \) by \(-\eta_i^G\). Hence, we obtain

\[
\eta_i^G > 0 \Rightarrow \eta_i^G G_i(x^k) > 0 \quad \text{for all } k \text{ sufficiently large}.
\]

This completes the proof of Theorem. \( \square \)

Again, if we replace limiting normal cone with Fréchet normal cone in above result, then multipliers corresponding to bi-active set will be changed and it yields an S-stationarity type condition.

**Theorem 4.2. (Enhanced Fritz John type S-stationary conditions)**

Let \( x^* \) be a local minimum of \( MPVC \) \((P)\), then there exist multipliers \( \alpha, \lambda, \mu, \eta^H, \eta^G \) such that:

(i) \( \alpha \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) + \sum_{i=1}^{q} \eta_i^G \nabla G_i(x^*) - \sum_{i=1}^{q} \eta_i^H \nabla H_i(x^*) = 0 \)

(ii) \( \alpha \geq 0, \lambda_i \geq 0 \forall i \in I_g(x^*), \lambda_i = 0 \forall i \notin I_g(x^*) \)

and \( \eta_i^G = 0 \forall i \in I_{+-}(x^*) \cup I_{0-}(x^*) \cup I_{0+}(x^*) \), \( \eta_i^G \geq 0 \forall i \in I_{+0}(x^*) \cup I_{00}(x^*) \)

\[
\eta_i^H = 0 \forall i \in I_{+}(x^*), \eta_i^H \geq 0 \forall i \in I_{0-}(x^*) \text{ and } \eta_i^H \text{ is free } \forall i \in I_{0+}(x^*)
\]

\( \eta_i^H \geq 0, \eta_i^G = 0 \forall i \in I_{00}(x^*) \)

\[15\]
(iii) \( \alpha, \lambda, \mu, \eta^G, \eta^H \) are not all equal to zero.

(iv) If \( \lambda, \mu, \eta^G, \eta^H \) are not all equal to zero, then there is a sequence \( \{x^k\} \rightarrow x^* \) such that \( \forall k \in \mathbb{N} \), we have

\[
f(x^k) < f(x^*)
\]

\[
\begin{align*}
\lambda_i &> 0 \Rightarrow \lambda_i g_i(x^k) > 0 \quad \{i = 1, \ldots, m\} \\
\mu_i &\neq 0 \Rightarrow \mu_i h_i(x^k) > 0 \quad \{i = 1, \ldots, p\} \\
\eta^H_i &\neq 0 \Rightarrow \eta^H_i H_i(x^k) < 0 \quad \{i = 1, \ldots, q\} \\
\eta^G_i &> 0 \Rightarrow \eta^G_i G_i(x^k) > 0 \quad \{i = 1, \ldots, q\}
\end{align*}
\]

Proof. To prove this result, we use standard optimality condition in terms of Fréchet normal cone and follow the proof of Theorem 4.1. We replace limiting normal cone with following Fréchet normal cone of \( \Omega \) at optimal point \((x^*, y^*, z^*)\):

\[
N^F_{\Omega}(x^*, y^*, z^*) = \left\{ \begin{array}{ll}
0 = \xi_0 = \zeta & \text{if } y^k > 0, \ z^k < 0 \\
0 = \xi_0, \zeta & \text{if } y^k > 0, \ z^k = 0 \\
\xi, \zeta & \text{if } y^k = 0, \ z^k = 0 \\
\xi, \zeta & \text{if } y^k = 0, \ z^k < 0 \\
\xi, \zeta & \text{if } y^k = 0, \ z^k > 0 \\
\end{array} \right.
\]

Now, by using same procedure as above we get required conditions. \(\square\)

Now, we can define some enhanced stationary conditions associated with enhanced Fritz John type conditions, which we have derived above.

**Definition 4.1.** Enhanced M-stationary condition for MPVC: Let \( x^* \) be a feasible point of MPVC. Then we say that the enhanced M-stationary condition holds at \( x^* \) if and only if there are multipliers \((\lambda, \mu, \eta^G, \eta^H) \neq (0, 0, 0, 0)\) such that

(i) \( 0 = \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) + \sum_{i=1}^{q} \eta^G_i \nabla G_i(x^*) - \sum_{i=1}^{q} \eta^H_i \nabla H_i(x^*) \)

and \( \eta^G_i = 0 \quad \forall \ i \in I_g(x^*), \ \lambda_i = 0 \quad \forall \ i \notin I_g(x^*) \)

(ii) \( \lambda_i \geq 0 \quad \forall \ i \in I_g(x^*), \ \lambda_i = 0 \quad \forall \ i \notin I_g(x^*) \)

and \( \eta^H_i \geq 0 \quad \forall \ i \in I_h(x^*), \ \eta^H_i 
eq 0 \quad \forall \ i \in I_h(x^*) \)

(iii) If \( \lambda, \mu, \eta^G, \eta^H \) not all equal to zero, then there is a sequence \( \{x^k\} \rightarrow x^* \) such that \( \forall k \in \mathbb{N} \), we have

\[
\begin{align*}
\lambda_i &> 0 \Rightarrow \lambda_i g_i(x^k) > 0 \quad \{i = 1, \ldots, m\} \\
\mu_i &\neq 0 \Rightarrow \mu_i h_i(x^k) > 0 \quad \{i = 1, \ldots, p\} \\
\eta^H_i &\neq 0 \Rightarrow \eta^H_i H_i(x^k) < 0 \quad \{i = 1, \ldots, q\} \\
\eta^G_i &> 0 \Rightarrow \eta^G_i G_i(x^k) > 0 \quad \{i = 1, \ldots, q\}
\end{align*}
\]

**Definition 4.2.** Enhanced S-stationary condition for MPVC: Any feasible point \( x^* \) of MPVC is said to satisfy enhanced S-stationary condition thereat if and only if there exist multipliers \((\lambda, \mu, \eta^G, \eta^H) \neq (0, 0, 0, 0)\) such that

(i) \( 0 = \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) + \sum_{i=1}^{q} \eta^G_i \nabla G_i(x^*) - \sum_{i=1}^{q} \eta^H_i \nabla H_i(x^*) \)

(ii) \( \lambda_i \geq 0 \quad \forall \ i \in I_g(x^*), \ \lambda_i = 0 \quad \forall \ i \notin I_g(x^*) \)

and \( \eta^G_i = 0 \quad \forall \ i \in I_g(x^*), \ \lambda_i = 0 \quad \forall \ i \notin I_g(x^*) \)

(iii) If \( \lambda, \mu, \eta^G, \eta^H \) not all equal to zero, then there is a sequence \( \{x^k\} \rightarrow x^* \) such that \( \forall k \in \mathbb{N} \), we have

\[
\begin{align*}
\lambda_i &> 0 \Rightarrow \lambda_i g_i(x^k) > 0 \quad \{i = 1, \ldots, m\} \\
\mu_i &\neq 0 \Rightarrow \mu_i h_i(x^k) > 0 \quad \{i = 1, \ldots, p\} \\
\eta^H_i &\neq 0 \Rightarrow \eta^H_i H_i(x^k) < 0 \quad \{i = 1, \ldots, q\} \\
\eta^G_i &> 0 \Rightarrow \eta^G_i G_i(x^k) > 0 \quad \{i = 1, \ldots, q\}
\end{align*}
\]
These enhanced conditions are stronger than those classic conditions. It is interesting to note that enhanced M-stationarity, being stronger than M-stationarity, is still weaker than S-stationarity (equivalently standard KKT, see [1]), in the sense that $\eta_i^G \eta_i^H = 0 \forall \ i \in I_{00}(x^*)$ and other conditions in enhanced M-stationarity need not imply $\eta_i^H \geq 0, \eta_i^G = 0 \forall \ i \in I_{00}(x^*)$ to be S-stationarity.

**Remark 4.1.** From the above two definitions it is obvious that at any feasible point of MPVC enhanced S-stationarity $\Rightarrow$ enhanced M-stationarity.

Now, we are in a position to consider some more constraint qualifications similar to pseudonormality and quasinormality concepts introduced by Bertsekas and Ozdaglar [9], which essentially explore the behaviour of the problem in a neighbourhood of solution points. These constraint qualifications have been extended for nonsmooth case by Ye and Zhang [44], in terms of limiting subdifferential and also for MPEC as generalized-pseudonormality and generalized-quasinormality in [45], by following the smooth version of MPEC notions, which were first introduced in [23]. These constraint qualifications associated with MPVC are essentially play the same role as in MPEC case. Moreover, one of them will serve as a new constraint qualification and provides a sufficient condition for local error bound for MPVC (see Theorem [17]). The following constraint qualification was introduced in [23], and shown to be a sufficient condition for exactness of the classical $l_1$ penalty function for MPVC under a reasonable assumption.

**Definition 4.3.** A vector $x^* \in C$ is said to satisfy MPVC-generalized pseudonormality, if there is no multiplier $(\lambda, \mu, \eta^H, \eta^G) \neq 0$ such that

(i) $\sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) + \sum_{i=1}^q \eta_i^G \nabla G_i(x^*) - \sum_{i=1}^q \eta_i^H \nabla H_i(x^*) = 0$

(ii) $\lambda_i \geq 0 \ \forall \ i \in I_g(x^*), \ \lambda_i = 0 \ \forall \ i \notin I_g(x^*)$

and $\eta_i^G = 0 \ \forall \ i \in I_{+}(x^*) \cup I_{0-}(x^*) \cup I_{0+}(x^*), \ \eta_i^G \geq 0 \ \forall \ i \in I_{00}(x^*)$

$\eta_i^H = 0 \ \forall \ i \in I_{+}(x^*), \ \eta_i^H \geq 0 \ \forall \ i \in I_{0-}(x^*) \ \text{and} \ \eta_i^H \ \text{is free} \ \forall \ i \in I_{0+}(x^*)$

$\eta_i^H \eta_i^G = 0 \ \forall \ i \in I_{00}(x^*)$

(iii) there is a sequence $\{x^k\} \to x^*$ such that the following is true for all $k \in \mathbb{N}$

$\sum_{i=1}^m \lambda_i g_i(x^k) + \sum_{i=1}^p \mu_i h_i(x^k) + \sum_{i=1}^q \eta_i^G G_i(x^k) - \sum_{i=1}^q \eta_i^H H_i(x^k) > 0.$

Now, we introduce a new constraint qualification, called MPVC-generalized quasinormality analogous to MPEC-generalized quasinormality.

**Definition 4.4.** A vector $x^* \in C$ is said to satisfy MPVC-generalized quasinormality, if there is no multiplier $(\lambda, \mu, \eta^H, \eta^G) \neq 0$ such that

(i) $\sum_{i=1}^m \lambda_i \nabla g_i(x^*) + \sum_{i=1}^p \mu_i \nabla h_i(x^*) + \sum_{i=1}^q \eta_i^G \nabla G_i(x^*) - \sum_{i=1}^q \eta_i^H \nabla H_i(x^*) = 0$

(ii) $\lambda_i \geq 0 \ \forall \ i \in I_g(x^*), \ \lambda_i = 0 \ \forall \ i \notin I_g(x^*)$

and $\eta_i^G = 0 \ \forall \ i \in I_{+}(x^*) \cup I_{0-}(x^*) \cup I_{0+}(x^*), \ \eta_i^G \geq 0 \ \forall \ i \in I_{00}(x^*)$

$\eta_i^H = 0 \ \forall \ i \in I_{+}(x^*), \ \eta_i^H \geq 0 \ \forall \ i \in I_{0-}(x^*) \ \text{and} \ \eta_i^H \ \text{is free} \ \forall \ i \in I_{0+}(x^*)$

$\eta_i^H \eta_i^G = 0 \ \forall \ i \in I_{00}(x^*)$

(iii) There is a sequence $\{x^k\} \to x^*$ such that the following is true $\forall k \in \mathbb{N}$, we have

$\lambda_i > 0 \Rightarrow \lambda_i g_i(x^k) > 0 \ \{i = 1, ..., m\}$

$\mu_i \neq 0 \Rightarrow \mu_i h_i(x^k) > 0 \ \{i = 1, ..., p\}$

$\eta_i^H \neq 0 \Rightarrow \eta_i^H H_i(x^k) < 0 \ \{i = 1, ..., q\}$

$\eta_i^G > 0 \Rightarrow \eta_i^G G_i(x^k) > 0 \ \{i = 1, ..., q\}$
We show, first time, that the MPVC-generalized quasinormality is weakest constraint qualifications which provides a sufficient condition for existence of a local error bound of the MPVC, see Theorem 5.2.

To this end, we note that the MPVC-generalized pseudonormality obviously implies the MPVC-generalized quasinormality, but not conversely. Combining this fact with [23, Proposition 2.1], we have following relationships among these constraint qualifications:

MPVC-LICQ ⇒ MPVC-MFCQ ⇒ MPVC-GMFCQ ⇒ MPVC-generalized pseudonormality
MPVC-generalized quasinormality

**Theorem 4.3.** Let \( x^* \) be a local minimum of (P) satisfying MPVC-generalized-quasinormality. Then \( x^* \) is an enhanced M-stationary point.

**Proof.** Suppose that \( x^* \) is a local minimum of (P), then by M-stationary type enhanced Fritz John necessary optimality condition, we have

\[
\alpha \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) - \sum_{i=1}^{q} \eta^G_i \nabla G_i(x^*) + \sum_{i=1}^{q} \eta^H_i \nabla H_i(x^*) = 0
\]

rest result of optimality condition also holds including \((\alpha, \lambda, \mu, \eta^G, \eta^H) \neq (0, 0, 0, 0, 0)\)

Here if \( \alpha = 0 \), then it shows the existence of nonzero multipliers which violates the MPVC-generalized quasinormality condition. Hence \( \alpha > 0 \), and then by proper scaling we obtain the enhanced M-stationary conditions at \( x^* \).

**Corollary 4.1.** If \( x^* \) is a local minimizer of (P) satisfying MPVC-GMFCQ or MPVC-generalized-pseudonormality, then \( x^* \) is an enhanced M-stationary point.

**Theorem 4.4.** Suppose that \( h_i \) are linear, \( g_j \) are concave, \( G_l \), \( H_l \) are all linear. Then any feasible point of MPVC is MPVC-generalized-pseudonormal.

**Proof.** We prove this result by contradiction. Suppose there is a feasible point \( x^* \) that is not MPVC-generalized-pseudonormal. Then there is a nonzero multiplier \((\lambda, \mu, \eta^G, \eta^H)\) such that

\[
\sum_{j=1}^{m} \lambda_j \nabla g_j(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) + \sum_{l=1}^{q} \eta^G_l \nabla G_l(x^*) - \sum_{l=1}^{q} \eta^H_l \nabla H_l(x^*) = 0 \tag{8}
\]

and \( \lambda_j \geq 0 \quad \forall j \in I_0(x^*), \quad \lambda_j = 0 \quad \forall j \notin I_0(x^*) \)

\( \eta^G_l = 0 \quad \forall l \in I_{0-}(x^*) \cup I_{0+}(x^*) \cup I_{00}(x^*) \quad \eta^G_l \geq 0 \quad \forall l \in I_{00}(x^*) \cup I_{0+}(x^*) \)

\( \eta^H_l = 0 \quad \forall l \in I_0(x^*) \quad \eta^H_l \geq 0 \quad \forall l \in I_{0-}(x^*) \quad \eta^H_l \) is free \( \forall l \in I_{0+}(x^*) \)

and there is a sequence \( \{x^k\} \to x^* \) such that the following is true for all \( k \in \mathbb{N} \)

\[
\sum_{j=1}^{m} \lambda_j g_j(x^k) + \sum_{i=1}^{p} \mu_i h_i(x^k) + \sum_{l=1}^{q} \eta^G_l G_l(x^k) - \sum_{l=1}^{q} \eta^H_l H_l(x^k) > 0 \tag{9}
\]

Now by the linearity of \( h_i, G_l, H_l \) and by the concavity of \( g_j \), we have for all \( x \in \mathbb{R}^n \)

\[
\begin{align*}
h_i(x) &= h_i(x^*) + \nabla h_i(x^*)^T (x - x^*) \quad ; \quad i = 1, \ldots, p \\
G_l(x) &= G_l(x^*) + \nabla G_l(x^*)^T (x - x^*) \quad ; \quad l = 1, \ldots, q \\
H_l(x) &= H_l(x^*) + \nabla H_l(x^*)^T (x - x^*) \quad ; \quad l = 1, \ldots, q \\
g_j(x) &\leq g_j(x^*) + \nabla g_j(x^*)^T (x - x^*) \quad ; \quad j = 1, \ldots, m
\end{align*}
\]
By multiplying these relations with \( \mu_i, \eta_l^G, \eta_l^H \) and \( \lambda_j \) and adding over \( i, l, l \) and \( j \) respectively, we obtained \( \forall x \in \mathbb{R}^n \)

\[
\sum_{j=1}^{m} \lambda_j \nabla g_j(x) + \sum_{i=1}^{p} \mu_i \nabla h_i(x) + \sum_{l=1}^{q} \eta_l^G \nabla G_l(x) - \sum_{l=1}^{q} \eta_l^H \nabla H_l(x) \leq m \sum_{j=1}^{m} \lambda_j \nabla g_j(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) + \sum_{l=1}^{q} \eta_l^G \nabla G_l(x^*) - \sum_{l=1}^{q} \eta_l^H \nabla H_l(x^*)
\]

\[
+ \left[ \sum_{j=1}^{m} \lambda_j \nabla g_j(x^*)^T + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*)^T + \sum_{l=1}^{q} \eta_l^G \nabla G_l(x^*)^T - \sum_{l=1}^{q} \eta_l^H \nabla H_l(x^*)^T \right] (x-x^*)
\]

\[
= \left[ \sum_{j=1}^{m} \lambda_j \nabla g_j(x^*) + \sum_{i=1}^{p} \mu_i \nabla h_i(x^*) + \sum_{l=1}^{q} \eta_l^G \nabla G_l(x^*) - \sum_{l=1}^{q} \eta_l^H \nabla H_l(x^*) \right]^T (x-x^*)
\]

the last inequality holds, because we have

\[
\mu_i h_i(x^*) = 0 \quad \forall i \quad \text{and} \quad \sum_{j=1}^{m} \lambda_j g_j(x^*) = 0
\]

\[
\sum_{l=1}^{q} \eta_l^G G_l(x^*), \quad \sum_{l=1}^{q} \eta_l^H H_l(x^*) = 0
\]

Now by the condition (8), we have

\[
\sum_{j=1}^{m} \lambda_j \nabla g_j(x) + \sum_{i=1}^{p} \mu_i \nabla h_i(x) + \sum_{l=1}^{q} \eta_l^G \nabla G_l(x) - \sum_{l=1}^{q} \eta_l^H \nabla H_l(x) \leq 0 \quad ; \quad \forall x \in \mathbb{R}^n
\]

But, it contradicts condition (9), hence \( x^* \) is MPVC-generalized pseudonormal.

Now, we introduce the notion of Constant Positive Linear Dependence condition for MPVC as MPVC-CPLD, which is weaker than MPVC-MFCQ. The CPLD notion was introduced by Qi and Wei in [40] for standard nonlinear programs. This notion has been generalized by Hoheisel et al. in [20] for MPEC and later also, it has been employed to analyze these problems [20] [45].

**Definition 4.5.** MPVC-CPLD : A feasible point \( x^* \) is said to satisfy MPVC-CPLD if and only if for any indices set \( I_h \subset B = \{1, ..., p\}, J_0 \subset I_g(x^*), L_0^H \subset I_{0+}(x^*) \cup I_{00}(x^*) \cup I_{0-}(x^*), L_0^G \subset I_{0+}(x^*) \cup I_{00}(x^*) \), whenever there exist \( \lambda_j, \mu_i \geq 0, \forall j \in J_h, i \in I_h, \eta_l^H, \eta_l^G \) not all zero such that

\[
\sum_{j \in J_h} \lambda_j \nabla g_j(x^*) + \sum_{i \in I_h} \mu_i \nabla h_i(x^*) + \sum_{l \in L_0^G} \eta_l^G \nabla G_l(x^*) - \sum_{l \in L_0^H} \eta_l^H \nabla H_l(x^*) = 0
\]

and \( \eta_l^G \eta_l^H = 0 \quad ; \quad \forall l \in I_{00}(x^*) \), then there is a neighbourhood \( U(x^*) \) of \( x^* \) such that for any \( x \in U(x^*) \), the vectors

\[
\{ \nabla h_i(x) | i \in I_h \}, \{ \nabla g_j(x) | j \in J_0 \}, \{ \nabla G_l(x) | l \in L_0^G \}, \{ \nabla H_l(x) | l \in L_0^H \}
\]

are linearly dependent.

The following Lemma from [41] Lemma 1] is crucial to prove our next result.

**Lemma 4.1.** If \( x = \sum_{i=1}^{m+p} \alpha_i v_i \) with \( v_i \in \mathbb{R}^n \) for every \( i, \{v_i\}_{i=1}^{m} \) is linearly independent and \( \alpha_i \neq 0 \) for every \( i = m + 1, ..., m + p \), then there exist \( J \subset \{m + 1, ..., m + p\} \) and scalars \( \tilde{\alpha}_i \) for every \( i \in \{1, ..., m\} \cup J \) such that

(i) \( x = \sum_{i \in \{1, ..., m\} \cup J} \tilde{\alpha}_i v_i \),

(ii) \( \alpha_i \tilde{\alpha}_i > 0 \) for every \( i \in J \),

(iii) \( \{v_i\}_{i \in \{1, ..., m\} \cup J} \) is linearly independent.
Now, based on this definition of MPVC-CPLD and above Lemma, we have the following important result.

**Theorem 4.5.** Let \( x \) be a feasible solution of MPVC such that MPVC-CPLD holds. Then \( x \) is an MPVC-generalized quasinormal point.

**Proof.** Here, we deal only with vanishing constraints. Assume that \( x \) is a feasible point and the MPVC-CPLD condition holds at \( x \).

If \( x \) satisfies MPVC-GMFCQ, then MPVC-generalized quasinormality obviously holds.

Suppose, MPVC-GMFCQ does not hold, then there is a nonzero vector \( (\eta^G, \eta^H) \in \mathbb{R}^q \times \mathbb{R}^q \) such that

\[
- \sum_{l=1}^{q} \eta^H_l \nabla H_l(x) + \sum_{l=1}^{q} \eta^G_l \nabla G_l(x) = 0
\]

and \( \eta^H_l = 0, \ \forall \ l \in I_+(x), \ \eta^G_l = 0, \ \forall \ l \in I_-(x) \cup I_0(-x), \ \eta^H_l = 0, \ \forall \ l \in I_0(x), \) and \( \eta^H_l \) is free for all \( l \in I_+(x) \cup I_0(x), \ \eta^G_l \geq 0, \ \forall \ l \in I_+(x) \cup I_0(x) \) and \( \forall \ l \in I_0(x), \ \eta^G_l \eta^H_l = 0 \)

Now we define the index sets

\[
A^H_+(x) = \{ l \in I_+(x) \cup I_0(x) | \eta^H_l > 0 \} \\
A^H_+(x) = \{ l \in I_+(x) \cup I_0(x) | \eta^H_l < 0 \} \\
A^G_+(x) = \{ l \in I_+(x) \cup I_0(x) | \eta^G_l > 0 \} \\
I^0_+(x) = \{ l \in I_0(x) | \eta^H_l > 0, \eta^G_l = 0 \} \\
I^0_0(x) = \{ l \in I_0(x) | \eta^H_l < 0, \eta^G_l = 0 \} \\
I^0_- = \{ l \in I_0(x) | \eta^H_l = 0, \eta^G_l > 0 \} \\
I^0_{00}(x) = \{ l \in I_0(x) | \eta^H_l = 0, \eta^G_l < 0 \}
\]

Since \( \{\eta^H, \eta^G\} \) is nonzero vector, therefore the union of the above sets must be nonempty and we may write

\[
0 = - \left[ \sum_{l \in A^H_+(x)} \eta^H_l \nabla H_l(x) + \sum_{l \in A^H_-(x)} \eta^H_l \nabla H_l(x) \right] + \sum_{l \in A^G_+(x)} \eta^G_l \nabla G_l(x)
- \left[ \sum_{l \in I^0_+(x)} \eta^H_l \nabla H_l(x) + \sum_{l \in I^0_0(x)} \eta^H_l \nabla H_l(x) \right] + \left[ \sum_{l \in I^0_-} \eta^H_l \nabla G_l(x) + \sum_{l \in I^0_{00}} \eta^G_l \nabla G_l(x) \right]
\]

first we assume that \( A^H_+(x) \) is nonempty. Let \( l_1 \in A^H_+(x) \), then

\[
-\eta^H_{l_1} \nabla H_{l_1}(x) = \left[ \sum_{l \in A^H_+(x) \setminus \{l_1\}} \eta^H_l \nabla H_l(x) + \sum_{l \in A^G_+(x)} \eta^H_l \nabla H_l(x) \right] - \sum_{l \in A^G_+(x)} \eta^G_l \nabla G_l(x)
+ \left[ \sum_{l \in I^0_+(x)} \eta^H_l \nabla H_l(x) + \sum_{l \in I^0_0(x)} \eta^H_l \nabla H_l(x) \right] - \left[ \sum_{l \in I^0_-} \eta^H_l \nabla G_l(x) + \sum_{l \in I^0_{00}} \eta^G_l \nabla G_l(x) \right]
\]

If \( \nabla G_{l_1}(x) = 0 \), then \( \{\nabla G_{l_1}(x)\} \) is linearly dependent. Then, by MPVC-CPLD, set \( \{\nabla G_{l_1}(y)\} \) must be linearly dependent for all \( y \) in some neighbourhood of \( x \). Therefore \( \nabla G_{l_1}(y) = 0 \) for all \( y \) in an open neighbourhood of \( x \). Since \( G_{l_1}(x) = 0 \Rightarrow G_{l_1}(y) = 0 \) for all \( y \) in the neighbourhood of \( x \). Hence, for any sequence \( x^k \to x \), \( G_{l_1}(x^k) = 0 \) for all sufficiently large \( k \) always holds, i.e for sequence \( x^k \to x \), \( -\eta^H_l H_l(x^k) > 0 \) never holds. Therefore, MPVC-generalized quasinormality holds at \( x \).

Now if \( \nabla G_{l_1} \neq 0 \), then \( \nabla G_{l_1}(x) \) is linearly independent and then by lemma there exists index sets

\[
\hat{A}^H_+ \subset A^H_+(x)/\{l_1\}, \ \hat{A}^H_-(x) \subset A^H_-(x), \ \hat{A}^G_+(x) \subset A^G_+(x), \ \hat{I}^0_0(x) \subset I^0_0(x), \ \hat{I}^0_- \subset I^0_-(x), \ \hat{I}^0_0(x) \subset I^0_0(x), \ \hat{I}^0_+(x) \subset I^0_+(x), \ \hat{I}^0_- \subset I^0_-(x), \ \hat{I}^0_{00}(x) \subset I^0_{00}(x),
\]
such that the vectors
\[
\begin{align*}
\{\nabla H_l(x)\}_{l \in \tilde{A}_+^H(x)}, & \quad \{\nabla H_l(x)\}_{l \in \tilde{A}_-^H(x)}, & \quad \{\nabla G_l(x)\}_{l \in \tilde{A}_+^G(x)}, & \quad \{\nabla G_l(x)\}_{l \in \tilde{I}_0^+ G(x)}, & \quad \{\nabla H_l(x)\}_{l \in \tilde{I}_0^- G(x)},
\end{align*}
\]
are linearly dependent and

\[
-\eta_l^H \nabla H_l(x) = \left[ \sum_{l \in \tilde{A}_+^H(x)} \tilde{\eta}_l^H \nabla H_l(x) + \sum_{l \in \tilde{A}_-^H(x)} \tilde{\eta}_l^H \nabla H_l(x) \right] - \left[ \sum_{l \in \tilde{A}_+^G(x)} \tilde{\eta}_l^G \nabla H_l(x) + \sum_{l \in \tilde{I}_0^+ G(x)} \tilde{\eta}_l^G \nabla H_l(x) \right]
\]

\[
+ \left[ \sum_{l \in \tilde{I}_0^- G(x)} \tilde{\eta}_l^H \nabla H_l(x) + \sum_{l \in \tilde{I}_0^- G(x)} \tilde{\eta}_l^G \nabla H_l(x) \right] - \left[ \sum_{l \in \tilde{I}_0^+ G(x)} \tilde{\eta}_l^G \nabla G_l(x) + \sum_{l \in \tilde{I}_0^- G(x)} \tilde{\eta}_l^G \nabla G_l(x) \right]
\]

with \( \tilde{\eta}_l^H > 0 \) \( \forall l \in \tilde{A}_+^H(x) \), \( \tilde{\eta}_l^H < 0 \) \( \forall l \in \tilde{A}_-^H(x) \), \( \tilde{\eta}_l^G > 0 \) \( \forall l \in \tilde{A}_+^G(x) \), \( \tilde{\eta}_l^H > 0 \), \( \tilde{\eta}_l^G = 0 \) \( \forall l \in \tilde{I}_0^+ G(x) \), \( \tilde{\eta}_l^H < 0 \), \( \tilde{\eta}_l^G < 0 \) \( \forall l \in \tilde{I}_0^- G(x) \).

Now by the linear independence of the vectors and by the continuity arguments, the vectors
\[
\begin{align*}
\{\nabla H_l(y)\}_{l \in \tilde{A}_+^H(x)}, & \quad \{\nabla H_l(y)\}_{l \in \tilde{A}_-^H(x)}, & \quad \{\nabla G_l(y)\}_{l \in \tilde{A}_+^G(x)}, & \quad \{\nabla G_l(y)\}_{l \in \tilde{I}_0^+ G(x)}, & \quad \{\nabla H_l(y)\}_{l \in \tilde{I}_0^- G(x)}, & \quad \{\nabla G_l(y)\}_{l \in \tilde{I}_0^- G(x)},
\end{align*}
\]
are linearly independent for all \( y \) in neighbourhood of \( x \) and by the MPVC-CPLD assumption, the vectors
\[
\begin{align*}
\eta_l^H \nabla H_l(y), & \quad \{\nabla H_l(y)\}_{l \in \tilde{A}_+^H(x)}, & \quad \{\nabla H_l(y)\}_{l \in \tilde{A}_-^H(x)}, & \quad \{\nabla G_l(y)\}_{l \in \tilde{A}_+^G(x)}, & \quad \{\nabla G_l(y)\}_{l \in \tilde{I}_0^+ G(x)}, & \quad \{\nabla H_l(y)\}_{l \in \tilde{I}_0^- G(x)}, & \quad \{\nabla G_l(y)\}_{l \in \tilde{I}_0^- G(x)},
\end{align*}
\]
are linearly dependent for all \( y \) in neighbourhood of \( x \). Hence, \( \eta_l^H \nabla H_l(y) \) must be a linear combination of all the remaining vectors for all \( y \) in neighbourhood of \( x \).

Now by [5 Lemma 3.2], there exist a smooth function \( \psi \) defined in a neighbourhood of \((0, ..., 0)\) such that, for all \( y \) in a neighbourhood of \( x \),

\[
-\eta_l^H \nabla H_l(y) = \psi(\{\nabla H_l(y)\}_{l \in \tilde{A}_+^H(x)}, \{\nabla H_l(y)\}_{l \in \tilde{A}_-^H(x)}, \{\nabla G_l(y)\}_{l \in \tilde{A}_+^G(x)}, \{\nabla G_l(y)\}_{l \in \tilde{I}_0^+ G(x)}, \{\nabla H_l(y)\}_{l \in \tilde{I}_0^- G(x)}, \{\nabla G_l(y)\}_{l \in \tilde{I}_0^- G(x)}), \]

and

\[
\nabla \psi(0, ..., 0) = (\{\tilde{\eta}_l^H\}_{l \in \tilde{A}_+^H(x)}, \{\tilde{\eta}_l^H\}_{l \in \tilde{A}_-^H(x)}, \{\tilde{\eta}_l^G\}_{l \in \tilde{A}_+^G(x)}, \{\tilde{\eta}_l^G\}_{l \in \tilde{I}_0^+ G(x)}, \{\tilde{\eta}_l^H\}_{l \in \tilde{I}_0^- G(x)}, \{\tilde{\eta}_l^G\}_{l \in \tilde{I}_0^- G(x)}).
\]

Now suppose \( \{x^k\} \) is an infeasible sequence that converges to \( x \) and such that

\[
\begin{align*}
H_l(x^k) < 0 & \quad \forall \ l \in \tilde{A}_+^H(x) \\
H_l(x^k) > 0 & \quad \forall \ l \in \tilde{A}_-^H(x) \\
G_l(x^k) < 0 & \quad \forall \ l \in \tilde{A}_+^G(x) \\
H_l(x^k) < 0 & \quad \forall \ l \in \tilde{I}_0^+ G(x) \\
H_l(x^k) > 0 & \quad \forall \ l \in \tilde{I}_0^- G(x) \\
G_l(x^k) < 0 & \quad \forall \ l \in \tilde{I}_0^- G(x) \\
G_l(x^k) > 0 & \quad \forall \ l \in \tilde{I}_0^- G(x)
\end{align*}
\]
Now by the Taylor’s expansion of $\psi$ at $(0, ..., 0)$ we have for $-\eta_i^H \nabla H_i(x^k)$ by the above sequence
\[-\eta_i^H \nabla H_i(x^k) = \psi(0, ..., 0) + \left(\{\nabla H_1(x^k)\}_{l \in \tilde{\Lambda}_{H_i}^o(x)} \ldots \{\nabla G_l(x^k)\}_{l \in \tilde{\Lambda}_{G_i}^o(x)}\right)
\]
\[(\{\tilde{\eta}_i^H\}_{l \in \tilde{\Lambda}_{H_i}^o(x)} \ldots \{\tilde{\eta}_i^G\}_{l \in \tilde{\Lambda}_{G_i}^o(x)})\]
\[= \psi(0, ..., 0) + \sum_{l \in \tilde{\Lambda}_{H_i}^o(x)} H_i(x^k) \tilde{\eta}_i^H + \sum_{l \in \tilde{\Lambda}_{G_i}^o(x)} G_l(x^k) \tilde{\eta}_i^G + \sum_{l \in F^0(x)} H_i(x^k) \hat{\eta}_i^H + \sum_{l \in G^0(x)} G_l(x^k) \hat{\eta}_i^G
\]
then for all $k$ large enough, we must have $-\eta_i^H \nabla H_i(x^k) \leq 0$, therefore for the sequence $x^k \to x$, $-\eta_i^H \nabla H_i(x^k) > 0$ does not hold. The proofs for the remaining cases are entirely same as for the above case. Hence, MPVC-generalized quasinormality holds.  

Since, we already have seen that enhanced M-stationarity is a consequence of MPVC-generalized quasinormality. So, we have the following.

**Corollary 4.2.** Let $x^*$ be a local minimizer of MPVC. If $x^*$ satisfies MPVC-CPLD, then $x^*$ is an enhanced M-stationary point.

## 5 Local Error Bound

Since local error bound property is also a constraint qualification, hence much attention has been paid in this context to standard nonlinear programs [33], MPEC [25], OPVIC [46] etc. Probably, for MPVC, [19] Proposition 3.4] is the first result on existence of local error bound where it has been proved, but in context of a more general problem, that calmness of some perturbed map is equivalent to the existence of local error bounds. In [12], the local error bound result is derived under the MPVC-constant rank in the subspace of components (MPVC-CRSC). To find the relationship between MPVC-generalized-quasinormality and MPVC-CRSC in [12], it needs a separate discussion, we do not discuss it in this paper. We prove that the MPVC-generalized quasinormality is sufficient condition for the existence of local error bound in Theorem 5.2.

In order to prove Theorem 5.2 we collect some more results, which are necessary for the proof. The first result ensures the quasinormality in a whole neighbourhood.

**Lemma 5.1.** If a feasible point $x^*$ is MPVC-generalized-quasinormal, then all feasible points in a neighbourhood of $x^*$ are MPVC-generalized-quasinormal.

**Proof.** Suppose contrary that there is a sequence $\{x^k\}$ such that $x^k \neq x$ for all $k$ and $x^k \to x$ and $x^k$ is not MPVC-generalized-quasinormal for all $k$. Therefore there exist scalars $(\lambda^k, \mu^k, \eta^G, \eta^H) \neq (0, 0, 0, 0)$ and a sequence $\{x^{k,t}\}$ such that
\[
\begin{align*}
(i) \quad & \sum_{i=1}^m \lambda_i^k g_i(x^k) + \sum_{i=1}^p \mu^k_i \nabla h_i(x^k) + \sum_{i=1}^q \eta_i^G \nabla G_i(x^k) - \sum_{i=1}^q \eta_i^H \nabla H_i(x^k) = 0 \\
(ii) \quad & \lambda_i^k \geq 0 \quad \forall \ i \in I_g(x^k), \quad \lambda_i^k = 0 \quad \forall \ i \not\in I_g(x^k) \\
\eta_i^G \geq 0 \quad & \forall \ i \in I_{+o}^1(x^k), I_{-o}^1(x^k) \cup I_{o+}^1(x^k) \quad \eta_i^G \geq 0 \quad \forall \ i \in I_{0o}^1(x^k) \cup I_{-o}^1(x^k) \\
\eta_i^H \geq 0 \quad & \forall \ i \in I_{+o}^1(x^k), \quad \eta_i^H \geq 0 \quad \forall \ i \in I_{-o}^1(x^k) \quad \eta_i^H \text{ is free} \quad \forall \ i \in I_{o+}^1(x^k) \\
\eta_i^{Hk} \eta_i^{Hk} \geq 0 \quad & \forall \ i \in I_{0o}^1(x^k) \\
(iii) \quad & \lambda_i^k g_i(x^{k,t}) > 0 \quad ; \quad \forall \lambda_i^k > 0 \\
\mu_i^k h_i(x^{k,t}) > 0 \quad & ; \quad \forall \mu_i^k > 0 \\
-\eta_i^H H_i(x^{k,t}) > 0 \quad & ; \quad \forall \eta_i^H \neq 0 \\
\eta_i^{Gk} G_i(x^{k,t}) > 0 \quad & ; \quad \forall \eta_i^{Gk} > 0 \\
\end{align*}
\]

Let for each $k$,

$$
\hat{\lambda}^k = \frac{\lambda^k}{\| (\lambda^k, \mu^k, \eta^{G_k}, \eta^{H_k}) \|}
$$

$$
\hat{\mu}^k = \frac{\mu^k}{\| (\lambda^k, \mu^k, \eta^{G_k}, \eta^{H_k}) \|}
$$

$$
\hat{\eta}^{G_k} = \frac{\eta^{G_k}}{\| (\lambda^k, \mu^k, \eta^{G_k}, \eta^{H_k}) \|}
$$

$$
\hat{\eta}^{H_k} = \frac{\eta^{H_k}}{\| (\lambda^k, \mu^k, \eta^{G_k}, \eta^{H_k}) \|}
$$

then without loss of generality we assume that

$$
(\hat{\lambda}^k, \hat{\mu}^k, \hat{\eta}^{G_k}, \hat{\eta}^{H_k}) \to (\lambda^*, \mu^*, \eta^{G*}, \eta^{H*})
$$

dividing both sides of (i), (ii) and (iii) by $\| (\lambda^k, \mu^k, \eta^{G_k}, \eta^{H_k}) \|$ and taking the limit, we have

(i) \[ \sum_{i=1}^{m} \lambda^*_i \nabla g_i(x^*) + \sum_{i=1}^{p} \mu^*_i \nabla h_i(x^*) + \sum_{i=1}^{q} \eta^{G*}_i \nabla G_i(x^*) - \sum_{i=1}^{q} \eta^{H*}_i \nabla H_i(x^*) = 0 \]

(ii) \[ \lambda^*_i > 0 \quad \forall \ i \in I_g(x^*), \quad \mu^*_i = 0 \quad \forall \ i \notin I_g(x^*) \]

and \[ \eta^{G*}_i = 0 \quad \forall \ i \in I_{+}(x^*) \cup I_{0+}(x^*) \cup I_{0-}(x^*) \]

\[ \eta^{H*}_i = 0 \quad \forall \ i \in I_{+}(x^*) \]

\[ \eta^{G*}_i H_i(x^*) > 0 \quad \forall \ i \in I_{0-}(x^*) \]

\[ \eta^{H*}_i > 0 \quad \forall \ i \in I_{0+}(x^*) \]

(iii) A sequence \{ $\xi^k$ \} \to $x^*$ such that the following holds for each $i$ as $k \to \infty$,

\[ \lambda^*_i g_i(\xi^k) > 0 \quad ; \quad \forall \ \lambda^*_i > 0 \]

\[ \mu^*_i h_i(\xi^k) > 0 \quad ; \quad \forall \ \mu^*_i \neq 0 \]

\[ -\eta^{H*}_i H_i(\xi^k) > 0 \quad ; \quad \forall \ \eta^{H*}_i \neq 0 \]

\[ \eta^{G*}_i G_i(\xi^k) > 0 \quad ; \quad \forall \ \eta^{G*}_i > 0 \]

since \( (\lambda^k, \mu^k, \eta^{H_k}, \eta^{G_k}) \) is non zero, therefore \( (\lambda^*, \mu^*, \eta^{H*}, \eta^{G*}) \neq (0, 0, 0, 0) \). Here is a contradiction to the fact that $x^*$ is MPVC-generalized quasinormal. Hence, all feasible points in the neighbourhood of $x^*$ are MPVC-generalized quasinormal.

The following result gives a representation for limiting normals at a point to the constraint region in terms of quasinormal multipliers

**Theorem 5.1.** If \( \bar{x} \) is MPVC-generalized quasinormal for \( C \), then

\[ N_C(\bar{x}) \subset \left\{ \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) + \sum_{i=1}^{p} \mu_i \nabla h_i(\bar{x}) + \sum_{i=1}^{q} [\eta^{G*}_i \nabla G_i(\bar{x}) - \eta^{H*}_i \nabla H_i(\bar{x})] : (\lambda, \mu, \eta^{G}, \eta^{H}) \in M(\bar{x}) \right\} \]

where $M(\bar{x})$ denotes the set of MPVC-generalized quasinormal multipliers corresponding to the point $\bar{x}$.

**Proof.** Here, for sake of the simplicity, we omit the equality and inequality constraints and consider only vanishing constraints, which needs to be handled. Let $v$ be an element of set $N_C(\bar{x})$. By definition of limiting normal cone, there are sequences $x^l \to \bar{x}$ and $v^l \to v$ with $v^l \in N_C^{F}(x^l)$ and $x^l \in C$.

Step I. By the lemma 5.1 $x^l$ is generalized quasi-normal for sufficient large $l$. Now by Theorem 6.11, for each $l$, there exist a smooth function $\psi^l$ that has a strict global minimizer $x^l$ over $C$ with $-\nabla \psi^l(x^l) = v^l$. Since $x^l$ is MPVC-generalized quasi normal point of $C$, therefore by Theorem 4.3 enhanced M-stationary condition holds for problem

\[ \min \ \psi^l(x) \]
That is, there exists a vector $(\eta^G, \eta^H)$ such that

\[ v^l \in \sum_{i=1}^{q} \eta^G_i \nabla G_i(x^l) - \sum_{i=1}^{q} \eta^H_i \nabla H_i(x^l) \]  

(10)

with

\[ \eta^G_i = 0 \quad \forall \ i \in I_+ - (x^l) \cup I_0 - (x^l) \cup I_0 + (x^l), \quad \eta^G_i \geq 0 \quad \forall \ i \in I_{00}(x^l) \cup I_{+0}(x^l) \]

\[ \eta^H_i = 0 \quad \forall \ i \in I_+ (x^l), \quad \eta^H_i \geq 0 \quad \forall \ i \in I_0 - (x^l), \quad \text{and} \quad \eta^H_i \text{ is free} \quad \forall \ i \in I_{0+}(x^l) \]

Moreover, let $G^l = \{ i | \eta^G_i > 0 \}, \ H^l = \{ i | \eta^H_i \neq 0 \}$ then there is a sequence \( \{ x^{k,l} \} \to x^k \) as \( k \to \infty \) such that \( \forall \ k \in \mathbb{N} \)

\[ \eta^G_i G_i(x^{k,l}) > 0 ; \quad \forall \ i \in G^l \]

\[ -\eta^H_i H_i(x^{k,l}) > 0 ; \quad \forall \ i \in H^l \]

Step II. Now we will show that the sequence \( \{ \eta^G, \eta^H \} \) is bounded. To prove this suppose contrary that \( \{ \eta^G, \eta^H \} \) is unbounded. Now for all \( l \), denote

\[ \tilde{\eta}^G = \frac{\eta^G}{|| (\eta^G, \eta^H) ||} \]

\[ \tilde{\eta}^H = \frac{\eta^H}{|| (\eta^G, \eta^H) ||} \]

then without loss of generality we can assume that

\[ (\eta^G, \eta^H) \to (\eta^G^*, \eta^H^*) \]

Dividing both sides of eq (10) by \( || (\eta^G, \eta^H) || \) and then taking the limit, we obtain similarly to the proof of lemma (5.1) :

\[ 0 \in \sum_{i=1}^{q} \eta^G_i \nabla G_i(\bar{x}) - \sum_{i=1}^{q} \eta^H_i \nabla H_i(\bar{x}) \]  

(11)

and

\[ \eta^G_i = 0 \quad \forall \ i \in I_+ - (\bar{x}) \cup I_0 - (\bar{x}) \cup I_0 + (\bar{x}), \quad \eta^G_i \geq 0 \quad \forall \ i \in I_{00}(\bar{x}) \cup I_{+0}(\bar{x}) \]

\[ \eta^H_i = 0 \quad \forall \ i \in I_+ (\bar{x}), \quad \eta^H_i \geq 0 \quad \forall \ i \in I_0 - (\bar{x}), \quad \text{and} \quad \eta^H_i \text{ is free} \quad \forall \ i \in I_{0+}(\bar{x}) \]

and a sequence \( \{ \zeta^l \} \to \bar{x} \) as \( l \to \infty \), and for each \( l \),

\[ \eta^G_i G_i(\zeta^l) > 0 ; \quad \forall \ i \in G^l \]

\[ -\eta^H_i H_i(\zeta^l) > 0 ; \quad \forall \ i \in H^l \]

Now here is a violation of the fact that \( \bar{x} \) is MPVC-quasi normal, hence sequence \( \{ \eta^G, \eta^H \} \) must be bounded.

Step III. Without loss of generality we can assume now \( \{ \eta^G, \eta^H \} \to \{ \eta^G^*, \eta^H^* \} \) as \( l \to \infty \)

\[ v \in \sum_{i=1}^{q} \eta^G_i \nabla G_i(\bar{x}) - \sum_{i=1}^{q} \eta^H_i \nabla H_i(\bar{x}) \]  

(12)
and
\[
\begin{align*}
\eta_i^G &= 0 \quad \forall i \in I_{+} - (\bar{x}) \cup I_{0} - (\bar{x}) \cup I_{0+} (\bar{x}), \quad \eta_i^G \geq 0 \quad \forall i \in I_{00} (\bar{x}) \cup I_{+0} (\bar{x})
\eta_i^H &= 0 \quad \forall i \in I_{+} (\bar{x}), \quad \eta_i^H \geq 0 \quad \forall i \in I_{0} - (\bar{x}), \text{ and } \eta_i^H \text{ is free } \forall i \in I_{0+} (\bar{x})
\eta_i^G \cdot \eta_i^H &= 0 \quad \forall i \in I_{00} (\bar{x})
\end{align*}
\]
and we can find a subsequence \( \{ \zeta^l \} \) converges to \( \bar{x} \) as \( l \to \infty \) and for each \( l \)
\[
\begin{align*}
\eta_i^G G_i (\zeta^l) &> 0; \quad \forall i \in G^l
\eta_i^H H_i (\zeta^l) &> 0; \quad \forall i \in H^l
\end{align*}
\]
Hence, the proof is complete. \( \Box \)

In [23 Theorem 4.5], it has been established that MPEC-generalized pseudonormality is a sufficient condition for the existence of a local error bound for smooth MPEC. Ye and Zhang [45 Theorem 3.1] has improved it for nonsmooth MPEC under the assumptions that \( g \) are to be only subdifferentially regular around the concerned point. In the context of MPVC, we confine ourselves only to smooth case, see [33]. However, one can show that the result also holds under the assumptions of [45 Theorem 3.1].

**Theorem 5.2.** Let \( x^* \in \mathcal{C} \) the feasible region of MPVC. If \( x^* \) is MPVC-generalized quasinormal, then there are \( \delta, \epsilon > 0 \) such that
\[
\text{dist}_\mathcal{C} (x) \leq c \left( ||h(x)||_1 + ||g^+(x)||_1 + \sum_{i=1}^q \text{dist}_\Delta (G_i(x), H_i(x)) \right)
\]
holds \( \forall \ x \in \mathbb{B}(x^*, \delta/2) \),

where \( \Delta := \{(a, b) \in \mathbb{R}^2 | b \geq 0, ab \leq 0 \} \), and \( \text{dist}_\Delta (x) \) is the distance in \( l_1 \)-norm from \( x \) to set \( \Delta \).

**Proof.** For the sake of simplicity, we omit the equality constraints. Here, we check only the case when \( x^* \) is on the boundary because assertion may fail there for \( x \notin \mathcal{C} \). For \( x \in \text{int} \mathcal{C} \) assertion is always true.

we choose some sequences \( \{ y^k \} \) and \( \{ x^k \} \) such that \( y^k \to x^* \), \( y^k \notin \mathcal{C} \), and \( x^k = \prod_\mathcal{C} (y^k) \), the projection of \( y^k \) onto the set \( \mathcal{C} \). Since \( ||x^k - y^k|| \leq ||y^k - x^*|| \), therefore \( x^k \to x^* \). We may assume here that both the sequences \( \{ y^k \} \) and \( \{ x^k \} \) belong to \( \mathbb{B}(x^*, \delta_0) \).

Since \( y^k - x^k \in N^+_\mathcal{C} (x^k) \subset N^+_\mathcal{C} (x^*) \), we have \( y^k = \frac{x^k - x^k}{||y^k - x^k||} \in N^+_\mathcal{C} (x^k) \). Since \( x^* \) is MPVC-generalized quasi-normal, therefore \( x^k \) is also MPVC-generalized quasi-normal (by Lemma 5.1) for all sufficiently large \( k \) and then without loss of generality, we may assume that all \( x^k \) are MPVC-generalized quasi-normal. Then by Theorem 5.1 there exist a sequence of scalars \( (\lambda^k, \eta^G, \eta^H) \) such that
\[
\begin{align*}
\eta^k &\in \sum_{i=1}^m \lambda^k_i \nabla g_i(x^k) + \sum_{i=1}^q \eta^G_i \nabla G_i(x^k) - \sum_{i=1}^q \eta^H_i \nabla H_i(x^k) 
\end{align*}
\]
and
\[
\begin{align*}
\lambda^k_i &> 0 \quad \forall i \in I_{g}(x^k), \quad \lambda^k_i = 0 \quad \forall i \notin I_{g}(x^k)
\eta^G_i &> 0 \quad \forall i \in I_{+} - (x^k) \cup I_{0} - (x^k) \cup I_{0+} (x^k), \quad \eta^G_i \geq 0 \quad \forall i \in I_{00} (x^k) \cup I_{+0} (x^k)
\eta^H_i &> 0 \quad \forall i \in I_{+} (x^k), \quad \eta^H_i \geq 0 \quad \forall i \in I_{0} - (x^k), \text{ and } \eta^H_i \text{ is free } \forall i \in I_{0+} (x^k)
\eta^G_k \cdot \eta^H_k &= 0 \quad \forall i \in I_{00} (x^k)
\end{align*}
\]
and there exist a sequence \( \{ x^{k,s} \} \in \mathbb{R}^n \) such that \( x^{k,s} \to x^k \) as \( s \to \infty \) and for which we have \( \forall s \)
\[
\begin{align*}
\lambda^k_i g_i(x^{k,s}) &> 0
\eta^G_i G_i(x^{k,s}) &> 0
\eta^H_i H_i(x^{k,s}) &> 0
\end{align*}
\]
Now, similar to the proof of Theorem 5.1, we can show that the MPVC-generalized quasi normality of $x^*$ shows the boundedness of sequence $\{\lambda^k, \eta^G, \eta^H\}$, and hence we may assume that $\{\lambda^k, \eta^G, \eta^H\}$ converges to some vector $\{\lambda^*, \eta^G^*, \eta^H^*\}$. Then there exist a number $M_0 > 0$ such that for all $k$, $\|\{\lambda^*, \eta^G^*, \eta^H^*\}\| \leq M_0$.

Without loss of generality, we may assume that $y^k \in B(x^*, \frac{M_0}{2})$ and $x^k \in B(x^*, \delta_0)$ for all $k$. Now we set $(\bar{\lambda}^k, \bar{\eta}^G, \bar{\eta}^H) = 2(\lambda^k, \eta^G, \eta^H)$, then from 14, for each $k$,

$$
\eta^k = \frac{y^k - x^k}{||y^k - x^k||} = \sum_{i=1}^{m} \lambda_i \nabla g_i(x^k) + \sum_{i=1}^{q} \eta^G_i \nabla G_i(x^k) - \sum_{i=1}^{q} \eta^H_i \nabla H_i(x^k)
$$

or

$$
\frac{y^k - x^k}{||y^k - x^k||} = \frac{x^k - y^k}{||y^k - x^k||} + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^k) + \sum_{i=1}^{q} \eta^G_i \nabla G_i(x^k) - \sum_{i=1}^{q} \eta^H_i \nabla H_i(x^k)
$$

Now we have from the above discussion

$$
||y^k - x^k|| = \left< \frac{y^k - x^k}{||y^k - x^k||}, y^k - x^k \right>
$$

$$
= \left< \frac{x^k - y^k}{||y^k - x^k||}, y^k - x^k \right> + \sum_{i=1}^{m} \lambda_i \rho \nabla g_i(x^k) - \sum_{i=1}^{q} \eta^G_i \nabla G_i(x^k) - \sum_{i=1}^{q} \eta^H_i \nabla H_i(x^k)
$$

$$
\leq \sum_{i=1}^{m} \lambda_i \nabla g_i(x^k) + \sum_{i=1}^{q} \eta^G_i \nabla G_i(x^k) + \sum_{i=1}^{q} \eta^H_i \nabla H_i(x^k)
$$

$$
\leq 2 \left[ \sum_{i=1}^{m} \lambda_i g_i(y^k) + \sum_{i=1}^{q} \eta^H_i H_i(y^k) - \sum_{i=1}^{q} \eta^G_i G_i(y^k) \right] + \frac{1}{2} ||y^k - x^k||
$$

Now, without loss of generality, we may assume that, for sufficiently large $k$

$$
o(||y^k - x^k||) \leq \frac{1}{4(M_0 + 1)} ||y^k - x^k||
$$

then we have

$$
||y^k - x^k|| \leq 2 \left[ \sum_{i=1}^{m} \lambda_i g_i(y^k) + \sum_{i=1}^{q} \eta^H_i H_i(y^k) - \sum_{i=1}^{q} \eta^G_i G_i(y^k) \right] + \frac{1}{2} ||y^k - x^k||
$$

Now, all the above discussion implies that

$$
\text{dist}_C(y^k) = ||y^k - x^k|| \leq 4M_0 \left( \sum_{i=1}^{m} g_i^+(y^k) + \phi(G(y^k), H(y^k)) \right)
$$
where
\[
\phi(G(y^k), H(y^k)) = \sum_{i=1}^{q} \max \{0, -H_i(y^k), \min \{G_i(y^k), H_i(y^k)\}\}
\]

Hence, for any sequence \( \{y^k\} \in \mathbb{R}^n \) converging to \( x^* \) there is a number \( c > 0 \) such that
\[
\text{dist}_C(y^k) \leq c \left( ||g^+(y^k)||_1 + \sum_{i=1}^{q} \text{dist}_\Delta(G_i(y^k), H_i(y^k)) \right), \quad \forall \ k = 1, 2, ...
\]

This implies the error bound property at \( x^* \). Indeed, suppose the contrary. Then there exists a sequence \( \tilde{y}^k \to x^* \) such that \( \tilde{y}^k \notin C \) and
\[
\text{dist}_C(\tilde{y}^k) > c \left( ||g^+(\tilde{y}^k)||_1 + \sum_{i=1}^{q} \text{dist}_\Delta(G_i(\tilde{y}^k), H_i(\tilde{y}^k)) \right), \quad \forall \ k = 1, 2, ...
\]

which is a contradiction. □

6 Concluding Remarks

We have used the Fritz John approach for MPVC, first time, to derive the M-stationary conditions under weak constraint qualifications. The derivations for stationarity conditions given in section 2 are simpler than others, available in the literature. Further, the enhanced M-stationarity has been shown to be a new stationarity condition for MPVC. The enhanced stationarity motivated to introduce two new constraint qualifications: MPVC-generalized quasinormality and MPVC-CPLD, later is stronger than former. An error bound result has been found using these constraint qualifications. However, it remains to discuss the relationship of these new constraint qualifications with other known MPVC- constraint qualifications, such as MPVC-GCQ and MPVC-ACQ. We hope that these relationships will open up some new paths for MPVC field.

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