Automorphism group of the reduced power (di)graph of a finite group

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Abstract

We describe the full automorphism group of the directed reduced power graph and the undirected reduced power graph of a finite group. We compute the full automorphism groups of these graphs of several classes of finite groups. Also, we establish some relation between the automorphism group of the undirected reduced power graph (resp. the directed reduced power graph) and the automorphism group of the undirected power graph (resp. the directed power graph) of a finite group.

Keywords: Group, Power graph, Reduced power graph, Automorphism group.

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1 Introduction

The directed power graph \( \overrightarrow{P}(G) \) of a group \( G \) is the digraph with vertex set \( G \), in which there is an arc from \( u \) to \( v \) if and only if \( u \neq v \) and \( v \) is a power of \( u \); or equivalently \( \langle v \rangle \subseteq \langle u \rangle \). The (undirected) power graph \( P(G) \) of \( G \) is the underlying graph of \( \overrightarrow{P}(G) \). Kelarev and Quinn introduced the directed power graph of a semigroup \([17]\) and a group \([15]\). Chakrabarty et al. defined the power graph of a semigroup \([9]\). The power graph has been studied extensively by many authors; see for instance \([3, 7, 8, 11, 12, 16, 22]\) and a survey \([1]\). The computation of the automorphism group of the power graph of a cyclic group was initiated

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by Alireza et al. [3] and settled by Mehranian et al. [22]. The automorphism group of the power graph of the dihedral group was also computed in [22]. Min Feng et al. [13] described the full automorphism group of $\overrightarrow{P}(G)$ and $P(G)$ for a finite group $G$. By using these, the computation of the automorphism group of $\overrightarrow{P}(G)$ and $P(G)$ when $G$ is cyclic, elementary abelian, dihedral and generalized quaternion groups have been made. Ali Reza Ashrafi et al. [2] computed the automorphism group of the power graph of some more classes of finite groups.

In [23], the authors defined the following: The directed reduced power graph $\overrightarrow{RP}(G)$ of a group $G$ is the digraph with the elements of $G$ as its vertices, and for two vertices $u$ and $v$ in $G$, there is an arc from $u$ to $v$ if and only if $v$ is a power of $u$ and $\langle v \rangle \neq \langle u \rangle$; or equivalently $\langle v \rangle \subset \langle u \rangle$. The (undirected) reduced power graph $RP(G)$ of $G$ is the underlying graph of $\overrightarrow{RP}(G)$. For the study on the reduced power graphs, we refer the reader to [3] [5] [14] [21] [19] [20] [24] [23].

For simplicity, in this paper we call “directed reduced power graph (resp. directed power digraph)” as “reduced power digraph (resp. power digraph)”.

The first aim of this paper is to describe the full automorphism group of the reduced power (di)graph of a finite group. In achieving this, we could see that it requires significantly distinct arguments based on the structure of the reduced power (di)graph in comparison with the arguments used for computing the automorphism group of the power (di)graph of a finite group (cf. [13]). The second aim of this paper is to describe some relations between the automorphism group of the reduced power graph (resp. the reduced power digraph) and the automorphism group of the power graph (resp. the power digraph) of a finite group.

Let $D$ be a digraph. The arc set of $D$ and the automorphism group of $D$ are denoted by $A(D)$ and $Aut(D)$, respectively. The in-degree, the out degree, the set of all in-neighbors and the set of all out-neighbors of a vertex $v$ in $D$ are denoted by $deg_D(v)$, $deg_D^+(v)$, $N_D^-(v)$ and $N_D^+(v)$, respectively. For a vertex $v$ in an undirected graph $\Gamma$, the degree of $v$ and the set of all neighbors of $v$ are denoted by $deg\Gamma(v)$ and $N\Gamma(v)$, respectively.

Let $H \wr K$ denote the wreath product of the groups $H$ over $K$. For an integer $n \geq 1$, $\mathbb{Z}_n$ denotes the additive group of integers modulo $n$. For integers $m, n \geq 1$, $\mathbb{Z}_n^m$ denotes the direct product of $m$ copies of $\mathbb{Z}_n$. The dihedral group of order $2n$ ($n \geq 3$) is given by $D_{2n} = \langle a, b | a^n = e = b^2, \ ab = ba^{-1} \rangle$. The generalized quaternion group of order $4n$ ($n \geq 2$) is given by $Q_{4n} = \langle a, b | a^{2n} = e = b^4, bab^{-1} = a^{-1} \rangle$. The semi-dihedral group of order $8n$ ($n \geq 2$) is given by $SD_{8n} = \langle a, b | a^{4n} = e = b^2, bab^{-1} = a^{2n-1} \rangle$. The modular group of order $p^\alpha$, $p$ a prime and $\alpha \geq 3$ is given by $M_{p^\alpha} = \langle a, b | a^{p^{\alpha-1}} = e = b^p, \ bab^{-1} = a^{p^{\alpha-2}+1} \rangle$. For $n \geq 1$, the group $V_{8n}$ of order $8n$ is defined as $V_{8n} = \langle a, b | a^{2n} = b^4 = e, aba = b^{-1}, ab^{-1}a = b \rangle$ and the group $U_{6n}$ of order $6n$ is defined as $U_{6n} := \langle a, b | a^{2n} = b^3 = e, b^{-1}ab = a^{-1} \rangle$. The identity map on the set $\Omega$ is denoted by $1_\Omega$.

The rest of this paper is arranged as follows. In Section 2, we give some basic constructions for determining the automorphism group of the reduced power (di)graph of a finite group and to introduce a faithful group action on that group. In Section 3, we establish some results on the induced action of the automorphism group of the reduced power (di)graph on the set of all equivalence classes of an equivalence relation defined in Section 2. In Section 4, we give the full structure of $Aut(\overrightarrow{RP}(G))$ and $Aut(RP(G))$ for a finite group $G$. In Section 5, we compute the full automorphism group of of the reduced power (di)graph of several classes.
of finite groups such as cyclic group, dihedral group, generalized quaternion group, semi-dihedral group, the group $V_{6n}$, the group $U_{6n}$, $p$-group with exponent $p$ or non-nilpotent group of order $p^m q$ with all non-trivial elements are of order $p$ or $q$, where $p$, $q$ are distinct primes. In Section 6, we establish the results, which describe some relation between the automorphism group of the reduced power graph (resp. the reduced power digraph) and the automorphism group of the power graph (resp. the power digraph) of a finite group.

2 Two faithful group actions on a group

In the rest of the paper $G$ denotes a finite group. Let $\mathcal{C}(G)$ be the set of all cyclic subgroups of $G$. For $C \in \mathcal{C}(G)$, let $|C|$ denote the set of all generators of $C$. Take $\mathcal{C}(G) = \{C_1, C_2, \ldots, C_k\}$ and $|C_i| = \{[C_{i1}], [C_{i2}], \ldots, [C_{in_i}]\}$. Notice that $G$ is the disjoint union of $[C_1], [C_2], \ldots, [C_k]$.

Let $P(G)$ be the set of all permutations on $\mathcal{C}(G)$ preserving the order, inclusion and non-inclusion with the condition that if the image of $C$ is different from $C$, then it has at least one cyclic subgroup different from the subgroup of $C$. That is, for each $\sigma \in P(G)$, $C_i^\sigma = C_i$ for every $i \in \{1, 2, \ldots, k\}$; $C_i \subseteq C_j$ if and only if $C_i^\sigma \subseteq C_j^\sigma$ and either $C_i^\sigma = C_i$ or the set of all cyclic subgroups of $C_i$ and the set of all cyclic subgroups of $C_i^\sigma$ are different. Let $M(G)$ be the set of all maximal cyclic subgroup of $G$. Take $M(G) = \{C_{i1}, C_{i2}, \ldots, C_{ir}\}$, where $C_{it} = \langle x_i \rangle$ for $t = 1, 2, \ldots, r$. Let $\mathcal{M}(G)$ be the restriction map of $P(G)$ on the set $M(G)$. Then $\mathcal{M}(G)$ is a permutation group on $M(G)$. This group induces a faithful action on the set $G$:

$$\mathcal{M}(G) \times G \to G, \quad (\sigma, x) \mapsto (x_i^\sigma)^j.$$  \hfill (2.1)

Note that for each $x \in G$, $x \in \langle x_i \rangle$ for some $i = 1, 2, \ldots, r$ and so $x = x_i^j$. Suppose $x \in \langle x_i \rangle$ and $\langle x_j \rangle$, where $i \neq j$ and $i, j \in \{1, 2, \ldots, r\}$. Then $x = x_i^j$ and $x = x_j^i$. Since $x^\sigma \in \langle x_i^\sigma \rangle$, $\langle x_j^\sigma \rangle$ and $o(x) = o(x^\sigma)$, so $\langle (x_i^\sigma)^j \rangle = \langle (x_j^\sigma)^i \rangle$. So for each $x \in G$, we fix any one maximal cyclic subgroup $C_{ij} = \langle x_j \rangle$ which contains $x$ and write all the elements of $\langle x \rangle$ as a power of $x_j$.

Now consider the equivalence relation $\simeq$ defined on $G$ as follows [4]: for $x, y \in G$, $x \simeq y$ if and only if $N_{\mathcal{RP}(G)}(x) = N_{\mathcal{RP}(G)}(y)$. It is shown in [4, Proposition 2.2] that $x \simeq y$ if and only if $N^-_{\mathcal{RP}(G)}(x) = N^-_{\mathcal{RP}(G)}(y)$ and $N^+_{\mathcal{RP}(G)}(x) = N^+_{\mathcal{RP}(G)}(y)$. Let $\hat{x}$ denote the $\simeq$-class determined by $x$.

Let $\mathcal{A}(G) = \{\hat{x} \mid x \in G\} = \{\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_t\}$. Notice that $G$ is the disjoint union of $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_t$. For $\Omega \subseteq G$, let $S_\Omega$ denote the symmetric group acting on the set $\Omega$. To prove our main result we need the following faithful group action on $G$ which is analogous to [13, Eqn. (2)].

$$\left( \prod_{i=1}^t S_{\hat{u}_i} \right) \times G \to G, \quad ((\tau_1, \ldots, \tau_t), x) \mapsto x^{\tau_i}, \quad \text{where} \ x \in \hat{u}_i.$$ \hfill (2.2)
3 The induced action of $\text{Aut} (\overrightarrow{RP}(G))$ and $\text{Aut} (\mathcal{RP}(G))$ on $\mathcal{R}(G)$

Consider the equivalence relation $\sim$ defined on $G$ as follows: for $x, y \in G$, $x \sim y$ if and only if $\langle x \rangle = \langle y \rangle$. Let $[x]$ denote the $\sim$-class determined by $x$.

It is shown in [4, Lemma 2.3] that any $\varnothing$-class $\hat{x}$ can be written as the union of distinct $\sim$-classes $[x_1], [x_2], \ldots, [x_r]$. Also $\hat{x}$ is said to be of

- **Type I**, if $r = 1$, that is $\hat{x} = [x]$;
- **Type II**, if $r \geq 2$ and $\langle x_i \rangle$s are distinct maximal cyclic subgroups of $G$ of same order for all $i = 1, 2, \ldots, r$. We call $(o(x), r)$ as the parameter of $\hat{x}$;
- **Type III**, if $G$ is non-cyclic with $r \geq 2$ and the $\langle x_i \rangle$s are maximal cyclic subgroups of prime orders in $G$ such that $| \{ o(x_i) : i = 1, 2, \ldots, r \} | \geq 2$;
- **Type IV**, if one of the following holds:
  (i) $G$ is non-cyclic with $r = 2$ and $\langle x_1 \rangle, \langle x_2 \rangle$ are non-maximal cyclic subgroups of distinct prime orders in $G$;
  (ii) $G$ is cyclic with $r = 2$ and $\langle x_1 \rangle, \langle x_2 \rangle$ are maximal cyclic subgroups of distinct prime orders in $G$.

In this case we say the pair $(o(x_1), o(x_2))$ as the parameter of $\hat{x}$.

From the above classification, we get the following

**Observation 3.1.** For any $x \in G$, the following hold.

1. If $\hat{x}$ is of Type I, then $| \hat{x} | = \varphi(o(x))$;
2. If $\hat{x}$ is of Type II with parameter $(m, r)$, then $| \hat{x} | = r\varphi(m)$;
3. If $\hat{x}$ is of Type IV with parameter $(p, q)$, then $| \hat{x} | = p + q - 2$.

**Observation 3.2.** (1) Let $x$ and $y$ be two non-trivial elements in $G$ such that $\langle y \rangle \subset \langle x \rangle$.

Then $\hat{x}$ is either of Type I or of Type II and $\hat{y}$ is either of Type I or of Type IV.

(2) Let $x, y \in G$. If $\hat{x}$ and $\hat{y}$ are of Type III, then $\hat{x} = \hat{y}$.

It is easy to see that for each $x \in G$ and $\pi \in \text{Aut} (\overrightarrow{RP}(G))$ (resp. $\pi \in \text{Aut} (\mathcal{RP}(G))$), we have $\hat{x}^\pi = \overrightarrow{x^\pi}$. Hence $\text{Aut} (\overrightarrow{RP}(G))$ (resp. $\text{Aut} (\mathcal{RP}(G))$) induces an action on $\mathcal{R}(G)$ as follows:

$$\text{Aut} (\overrightarrow{RP}(G)) \times \mathcal{R}(G) \to \mathcal{R}(G), \quad (\pi, \hat{x}) \mapsto \hat{x}^\pi$$

and

$$\text{Aut} (\mathcal{RP}(G)) \times \mathcal{R}(G) \to \mathcal{R}(G), \quad (\pi, \hat{x}) \mapsto \overrightarrow{x^\pi}. \quad (3.1)$$

Next we shall show that each orbit of $\text{Aut} (\mathcal{RP}(G))$ on $\mathcal{R}(G)$ consists of some equivalence classes of the same type.
Lemma 3.1. Let \( p, q \) be distinct primes. Let \( x \) and \( y \) be two non-trivial elements in \( G \) such that \( \langle y \rangle \subset \langle x \rangle \). Then we have the following.

1. If \( \hat{x} \) is of Type I, then one of the following holds:
   
   a. \( |\hat{y}| \leq |\hat{x}| \), if \( \hat{y} \) is either of Type I or of Type IV with parameter \( (p, q) \), where \( p, q \neq 2 \). Equality holds if both \( \hat{x} \) and \( \hat{y} \) are of Type I with \( \varphi(o(x)) = 2 \cdot \varphi(o(y)) \) and \( o(y) \) is odd.
   
   b. \( |\hat{y}| > |\hat{x}| \), if \( \hat{y} \) is of Type IV with parameter \( (2, p) \) and \( o(x) = 2p \), where \( p \) is a prime.

2. If \( \hat{x} \) is of Type II, then \( |\hat{y}| < |\hat{x}| \).

Proof. By Observation 3.2(1), \( \hat{x} \) is either of Type I or of Type II, and \( \hat{y} \) is either of Type I or of Type IV.

1. Suppose that \( \hat{x} \) is either of Type I.
   
   If \( \hat{y} \) is of Type I, then \( |\hat{y}| = \varphi(o(x)) \) and also \( |\hat{x}| = r \cdot \varphi(o(x)) \), where \( r \geq 1 \). It is known that for any two positive integers \( m \) and \( n \), if \( m \mid n \), then \( \varphi(m) \leq \varphi(n) \); equality holds if either \( n = m \) or \( m \) is odd and \( n = 2m \). Since \( o(y) \mid o(x) \), it follows that \( \varphi(o(x)) \leq \varphi(o(y)) \).
   
   So \( |\hat{y}| = \varphi(o(y)) \leq \varphi(o(x)) \leq r \cdot \varphi(o(x)) = |\hat{x}| \); equality holds if \( r = 1 \), \( \varphi(o(x)) = 2 \cdot \varphi(o(y)) \) and \( o(y) \) is odd, since \( o(x) \neq o(y) \).
   
   If \( \hat{y} \) is of Type IV with parameter \( (p, q) \), then \( |\hat{y}| = p + q - 2 \). If \( p, q \neq 2 \), then \( p + q - 2 < (p - 1)(q - 1) \). So \( |\hat{y}| < |\hat{x}| \). Suppose \( q = 2 \). Then \( |\hat{y}| = p \). If \( o(x) = 2p \) and \( r = 1 \), then \( |\hat{x}| = (p - 1) < r = |\hat{y}| \). If \( o(x) = 2p \) and \( r > 1 \), then \( |\hat{x}| = r \cdot (p - 1) > r = |\hat{y}| \).

2. If \( \hat{x} \) is of Type II with parameter \( (m, r) \), then \( \hat{y} \) is of Type I. By parts (1) and (2) of Observation 3.1, \( |\hat{x}| = r \cdot \varphi(m) \) and \( |\hat{y}| = \varphi(o(y)) \leq \varphi(m) \). So \( |\hat{y}| < |\hat{x}| \) as desired.

Lemma 3.2. ([4] Lemma 2.1) Let \( e \) be the identity element of \( G \). Let \( S \) be the set of all vertices of \( \mathcal{RP}(G) \) which are adjacent to all other vertices of \( \mathcal{RP}(G) \). Then either \( S = \{e\} \) or \( G = \mathbb{Z}_{2^m}(m \geq 1) \) or \( Q_{2^\alpha}(\alpha \geq 3) \) and \( S = \{e, a\} \), where \( a \) is the unique element of order 2.

Lemma 3.3. (1) Let \( \pi \in \text{Aut}(\mathcal{RP}(G)) \). Then we have the following.

   a. If \( G \cong \mathbb{Z}_{2^m}(m \geq 1) \) and \( Q_{2^\alpha}(\alpha \geq 3) \), then \( \pi \) fixes \( e \).
   
   (ii) If \( G \cong \mathbb{Z}_{2^m}(m \geq 1) \) or \( Q_{2^\alpha}(\alpha \geq 3) \), then \( e^\pi = e \) or \( a \), where \( a \) is the unique element of order 2 in \( G \).

(2) If \( \pi \in \text{Aut}(\widetilde{\mathcal{RP}}(G)) \), then \( \pi \) fixes \( e \).

Proof. (1) Notice that \( \deg_{\mathcal{RP}(G)}(e) = \deg_{\mathcal{RP}(G)}(e^\pi) \). Also, \( \deg_{\mathcal{RP}(G)}(e) = |G| - 1 \). So \( e^\pi \) is adjacent to all the vertices of \( \mathcal{RP}(G) \). Thus the result follows from Lemma 3.2.

(2) Since \( \deg_{\mathcal{RP}(G)}^+(e) = 0 \) and \( \deg_{\mathcal{RP}(G)}^+(x) \geq 1 \) for every \( (e \neq x) \in G \), so \( e^\pi = e \).

Lemma 3.4. ([4] In the proof of Theorem 3.2) Let \( x \) and \( y \) be two non-trivial elements in \( G \) such that they are adjacent in \( \mathcal{RP}(G) \). Then one of the following holds:
Lemma 3.5. (1) Let $\pi \in \text{Aut}(\mathcal{RP}(G))$ and let $x, y \in G$. Then $\langle y \rangle \subset \langle x \rangle$ if and only if $\langle y^\pi \rangle \subset \langle x^\pi \rangle$.

(2) Let $\pi \in \text{Aut}(\mathcal{RP}(G))$ and let $x, y$ be two non-trivial elements in $G$. Then $\langle y \rangle \subset \langle x \rangle$ if and only if $\langle y^\pi \rangle \subset \langle x^\pi \rangle$.

Proof. (1) The result follows from the fact that $(x, y) \in A(\mathcal{RP}(G))$ if and only if $(x^\pi, y^\pi) \in A(\mathcal{RP}(G))$.

(2) Let $x$ and $y$ be two non-trivial elements in $G$ such that $\langle y \rangle \subset \langle x \rangle$. We show that either $\langle x^\pi \rangle \subset \langle y^\pi \rangle$ or $\langle y^\pi \rangle \subset \langle x^\pi \rangle$. Notice that $\hat{x}$ is either of Type I or of Type II and $\hat{y}$ is either of Type I or of Type IV. So we consider the following cases.

Case 1. Let $\hat{x}$ be of Type I. Then by Lemma 3.1, $\hat{y} \succ \hat{x}$, if $\hat{y}$ is of Type IV with parameter $(2, p)$, where $p$ is a prime; $|\hat{x}| \geq |\hat{y}|$, otherwise.

Subcase 1a. If $\hat{y}$ is not of Type IV with parameter $(2, p)$, then $|\hat{x}| \geq |\hat{y}|$. Suppose $\langle \hat{x} \rangle \subset \langle \hat{y} \rangle$. Then $\hat{x}$ is either of Type I or of Type II. By Lemma 3.1, $\hat{x}$ is of Type IV with parameter $(2, q)$ and $\hat{y}$ is of Type I with $o(y) = 2q$, where $q$ is an odd prime. It follows that $|\hat{x}| = (q - 1)$ and $|\hat{y}| = q$ and so $|\hat{x}|$ is odd and $|\hat{x}| \geq |\hat{y}|$. Then by Lemma 3.4, $\langle x \rangle \subset \langle y \rangle$, which is a contradiction. So $\langle y \rangle \subset \langle x \rangle$.

Subcase 1b. If $\hat{y}$ is of Type IV with parameter $(2, p)$, then $\hat{y} = p$. Since $|\hat{x}| \geq |\hat{y}|$, $|\hat{y}| \geq |\hat{x}|$ and $|\hat{y}|$ is odd. By Lemma 3.4, $\langle y \rangle \subset \langle x \rangle$.

Case 2. Suppose $\hat{x}$ is of Type II. By Lemma 3.1, $|\hat{x}| \geq |\hat{y}|$. Then by a similar argument used in Subcase 1a, we get $\langle y \rangle \subset \langle x \rangle$.

By a similar argument, we can show that if $\langle y \rangle \subset \langle x \rangle$, then $\langle y \rangle \subset \langle x \rangle$. $\square$

Proposition 3.1. Let $x \in G$ and $\pi \in \text{Aut}(\mathcal{RP}(G))$. Then we have the following.

(1) If $\hat{x}$ is of Type III, then $\hat{x}^\pi = \hat{x}$.

(2) If $\hat{x}$ is of Type II, then $\hat{x}^\pi$ is of Type II with $o(x) = o(x^\pi)$.

(3) If $\hat{x}$ is of Type I, then $\hat{x}^\pi$ is of Type I with $o(x) = o(x^\pi)$.

(4) If $\hat{x}$ is of Type IV with parameter $(p, q)$, then $\hat{x}^\pi$ is of Type IV with parameter $(p, q)$.

Proof. Notice that if $\deg_{\mathcal{RP}(G)}^+(x) > 1$, then $\deg_{\mathcal{RP}(G)}^+(x^\pi) > 1$. So $o(x)$ and $o(x^\pi)$ are composite numbers. By [1, Theorem 3.1], $o(x)$ and $o(x^\pi)$ can be determined from the structure of $\mathcal{RP}(G)$. By Lemma 3.5, if $\langle y \rangle \subset \langle x \rangle$, then $\langle y^\pi \rangle \subset \langle x^\pi \rangle$. It follows that $o(x) = o(x^\pi)$.

(1) If $\hat{x}$ is of Type III, then $N_{\mathcal{RP}(G)}^- (x) = \emptyset$ and $N_{\mathcal{RP}(G)}^+ (x) = \{e\}$. It follows that $N_{\mathcal{RP}(G)}^- (x^\pi) = \emptyset$ and $N_{\mathcal{RP}(G)}^+ (x^\pi) = \{e^\pi\} = \{e\}$, by Lemma 3.3(2). Thus, $\hat{x}^\pi$ is of Type III. Then by Observation 3.2(2), $\hat{x}^\pi = \hat{x}$.
(2) Assume that $\hat{x}$ is of Type II. If \( \deg_{\mathcal{RP}(G)}^+(x) > 1 \), then \( o(x) = o(x^\pi) \) and \( \deg_{\mathcal{RP}(G)}^-(x^\pi) = 0 \). So $\hat{x}^\pi$ is either of Type I or of Type II. By Observation 3.1(2), $|\hat{x}| = |\hat{x}^\pi| = m \cdot o(x)$, where \( m \geq 2 \). It follows that $\hat{x}^\pi$ is of Type II. If \( \deg_{\mathcal{RP}(G)}^+(x^\pi) = 1 \), then \( N_{\mathcal{RP}(G)}^+(x) = N_{\mathcal{RP}(G)}^+(x^\pi) = \{ e \} \). Also, \( \deg_{\mathcal{RP}(G)}^-(x^\pi) = 0 \) and so \( \deg_{\mathcal{RP}(G)}^-(x^\pi) = 0 \). It follows that $\hat{x}^\pi = \hat{x}$. Thus, $\hat{x}^\pi$ is of Type II and \( o(x) = o(x^\pi) \).

(3) Assume that $\hat{x}$ is of Type I. If \( \deg_{\mathcal{RP}(G)}^+(x) > 1 \), then \( o(x) = o(x^\pi) \). Applying (3.1), we get $|\hat{x}^\pi| = \phi(o(x^\pi))$, so $\hat{x}^\pi$ is of Type I. If \( \deg_{\mathcal{RP}(G)}^+(x^\pi) = 1 \), then \( \deg_{\mathcal{RP}(G)}^+(x^\pi) = 1 \). It follows that \( o(x) \) and \( o(x^\pi) \) are primes and so $\hat{x}^\pi$ is one of Type I, Type III or Type IV. Let \( o(x) = p \) and $o(x^\pi) = q$, where \( p \) and \( q \) are primes. If $\hat{x}^\pi$ is of Type II, then by part (2), $\hat{x}^\pi = \hat{x}$. Thus $\hat{x}$ is also of Type II, which is a contradiction. If $\hat{x}^\pi$ is of Type III, then by part (1), $\hat{x}$ is also of Type III, which is a contradiction. If $\hat{x}^\pi$ is of Type IV with parameter \((q, s)\), where \((q \neq s)\) is a prime. Then there exists \( w \in G \) such that \( o(w) = q s \) and \( N_{\mathcal{RP}(G)}^+(w) = \hat{x}^\pi \cup \{ e \} \). It follows that there exists \( y \in G \) such that $\hat{x}y = w$ and \( N_{\mathcal{RP}(G)}^+(y) = \hat{x} \cup \{ e \} \), so $o(y) = p^2$.

Since \( \deg_{\mathcal{RP}(G)}^-(y) > 1 \), by the previous argument, we have \( o(y) = o(w) \), so $p^2 = qr$, which is a contradiction. Thus $\hat{x}^\pi$ is of Type I, so $|\hat{x}^\pi| = q - 1$. Applying (3.1), we have \( o(x) = o(x^\pi) \).

(4) Let $\hat{x}$ be of Type IV with parameter \((p, q)\), where \( p \) and \( q \) are distinct primes. Then $\hat{x} = p + q - 2$ and \( \deg_{\mathcal{RP}(G)}^-(x^\pi) \neq 0 \), \( \deg_{\mathcal{RP}(G)}^-(x^\pi) = 1 \). This implies that \( o(x^\pi) \) is prime and so $\hat{x}^\pi$ is either of Type I or of Type IV. Suppose $\hat{x}^\pi$ is of Type I, then by part (3), $\hat{x}^\pi \cdot 1 = \hat{x}$ is of Type I, which is a contradiction. Thus $\hat{x}^\pi$ is of Type IV. By (3.2), $|\hat{x}^\pi| = p + q - 2$, so $\hat{x}^\pi$ is of Type IV with parameter \((p, q)\).

\section{Full structure of $\text{Aut}(\mathcal{RP}(G))$ and $\text{Aut}(\mathcal{RP}(G))$}

In this section, we determine $\text{Aut}(\mathcal{RP}(G))$ and $\text{Aut}(\mathcal{RP}(G))$.

\textbf{Lemma 4.1.} Let \( \pi \) be a permutation on \( G \).

(1) If \( \pi \in \mathcal{M}(G) \), then \( \langle x^\pi \rangle = \langle x^\pi \rangle \) for each \( x \in G \).

(2) Then \( \pi \in \prod_{i=1}^t S_{\hat{s}_i} \) if and only if \( \hat{x}^\pi = \hat{x} \) for each \( x \in G \).

(3) Suppose for every \( x, y \in G \), \( \pi \) is such that

\begin{enumerate}
    \item \( o(x) = o(x^\pi) \);
    \item if \( \langle x \rangle \) is a maximal subgroup of \( G \) and \( x \neq x^\pi \), then the set of all subgroups of \( \langle x \rangle \) and the set of all subgroups of \( \langle x^\pi \rangle \) are different;
    \item \( \langle y \rangle \subset \langle x \rangle \) if and only if \( \langle y^\pi \rangle \subset \langle x^\pi \rangle \) for every \( x, y \in G \),
\end{enumerate}

then \( \pi \in \mathcal{M}(G) \).

\textbf{Proof.} Parts (1) and (3) follow from (2.1) and part (2) follows from (2.2). \hfill \Box

\textbf{Lemma 4.2.} (1) \( \mathcal{M}(G) \) is a subgroup of $\text{Aut}(\mathcal{RP}(G))$ and $\text{Aut}(\mathcal{RP}(G))$;
(2) \( \prod_{i=1}^{t} S_{\hat{a}_i} \) is a subgroup of \( \text{Aut}(\overrightarrow{RP}(G)) \) and \( \text{Aut}(\mathcal{RP}(G)) \).

**Proof.** (1) Let \( \sigma \in \mathcal{M}(G) \). To show \( \sigma \in \text{Aut}(\overrightarrow{RP}(G)) \), by [2.11], it is enough to show that \((x, y) \in A(\overrightarrow{RP}(G)) \) if and only if \((x^\sigma, y^\sigma) \in A(\overrightarrow{RP}(G)) \). Suppose that \((x, y) \in A(\overrightarrow{RP}(G)) \). Then \( \langle y \rangle \subset \langle x \rangle \). It follows from Lemma 4.4(1) that \( \langle y^\sigma \rangle \subset \langle x^\sigma \rangle \). Therefore, \((x^\sigma, y^\sigma) \in A(\overrightarrow{RP}(G)) \). Hence \( P(G) \) is a subgroup of \( \text{Aut}(\overrightarrow{RP}(G)) \). Since \( \text{Aut}(\overrightarrow{RP}(G)) \subseteq \text{Aut}(\mathcal{RP}(G)) \), it follows that \( \mathcal{M}(G) \) is a subgroup of \( \text{Aut}(\mathcal{RP}(G)) \).

(2) Let \( \tau \in \prod_{i=1}^{t} S_{\hat{a}_i} \) and \((x, y) \in A(\overrightarrow{RP}(G)) \). Then \( \langle y \rangle \subset \langle x \rangle \). By Lemma 4.4(2), \( y^\tau \in \hat{y} \) and \( x^\tau \in \hat{x} \), so \( \langle y^\tau \rangle \subset \langle x^\tau \rangle \). Hence \((x^\tau, y^\tau) \in A(\overrightarrow{RP}(G)) \). Conversely, assume that \((x^\tau, y^\tau) \in A(\overrightarrow{RP}(G)) \). Then \( \langle y^\tau \rangle \subset \langle x^\tau \rangle \). This implies that \( \langle y \rangle \subset \langle x \rangle \), since \( x \in \hat{x}^\tau \) and \( y \in \hat{y}^\tau \). Hence \((x, y) \in A(\overrightarrow{RP}(G)) \). Thus \( \tau \in \text{Aut}(\overrightarrow{RP}(G)) \). It follows that \( \prod_{i=1}^{t} S_{\hat{a}_i} \) is a subgroup of \( \text{Aut}(\overrightarrow{RP}(G)) \). Also, \( \text{Aut}(\overrightarrow{RP}(G)) \subseteq \text{Aut}(\mathcal{RP}(G)) \). This implies that \( \prod_{i=1}^{t} S_{\hat{a}_i} \) is a subgroup of \( \text{Aut}(\mathcal{RP}(G)) \). \( \Box \\

**Lemma 4.3.** \( \mathcal{M}(G) \) is a subgroup of the normalizer of \( \prod_{i=1}^{t} S_{\hat{a}_i} \) in \( \text{Aut}(\overrightarrow{RP}(G)) \) and \( \text{Aut}(\mathcal{RP}(G)) \).

**Proof.** Let \( \pi \in \mathcal{M}(G) \), \( \tau \in \prod_{i=1}^{t} S_{\hat{a}_i} \) and \( x \in G \). Then by Lemma 4.4(2) and \( (3.2) \), \( \hat{x}^{\pi \tau \pi^{-1}} = x^{\pi \tau \pi^{-1}} = \hat{x} \). It follows that \( \pi \tau \pi^{-1} \in \prod_{i=1}^{t} S_{\hat{a}_i} \). Hence the proof. \( \Box \\

**Lemma 4.4.** \( \mathcal{M}(G) \cap \prod_{i=1}^{t} S_{\hat{a}_i} = \{1_G\} \).

**Proof.** Let \( \pi \in \mathcal{M}(G) \cap \prod_{i=1}^{t} S_{\hat{a}_i} \) and \( x \in G \). Since \( x = x_j^i \), where \( \langle x_j \rangle \in M(G) \). Then \( x^\pi = (x_j^i)^\pi \). Since \( \pi \in \prod_{i=1}^{t} S_{\hat{a}_i} \) and \( x_j^i \in \hat{x}_i \), it follows that the set of all subgroups of \( \langle x_j \rangle \) and the set of all subgroups of \( \langle x_j^\pi \rangle \) are same; also \( \pi \in \mathcal{M}(G) \), and so \( \pi \) fixes \( x_j \). Hence \( x^\pi = x_j^i = x \). \( \Box \\

The following result is an immediate consequence of Lemmas 4.2, 4.3 and 4.4.

**Corollary 4.1.** \( (\prod_{i=1}^{t} S_{\hat{a}_i}) \times \mathcal{M}(G) \) is a subgroup of \( \text{Aut}(\overrightarrow{RP}(G)) \) and \( \text{Aut}(\mathcal{RP}(G)) \).

**Lemma 4.5.** Let \( \pi \in \text{Aut}(\overrightarrow{RP}(G)) \). Then there exists \( \sigma \in \mathcal{M}(G) \) such that \( N_{\overrightarrow{RP}(G)}(x^\pi) = N_{\overrightarrow{RP}(G)}(x^\sigma) \) and \( N_{\mathcal{RP}(G)}(x^\pi) = N_{\mathcal{RP}(G)}(x^\sigma) \) for every \( x \in G \).

**Proof.** Let \( \pi \in \text{Aut}(\overrightarrow{RP}(G)) \). We define a map \( \sigma \) on \( G \) as follows: \( e^\sigma := e \). Let \( x \in G \) with \( x \neq e \). We have the following cases:

**Case 1.** If \( \hat{x} \) is of Type I, then \( w^\sigma := w^\pi \) for every \( w \in \hat{x} \). Then \( o(w) = o(w^\pi) \). If \( \langle x \rangle \) is a maximal subgroup of \( G \), then there is no maximal cyclic subgroup of \( G \) such that the set of all its subgroups is the same as the set of all subgroups of \( \langle w \rangle \), since \( \hat{x} \) is of Type I. Also, \( \hat{x}^\sigma = \hat{x}^\pi = \hat{x} \).

**Case 2.** If \( \hat{x} \) is of Type II, and \( x^\pi \notin \hat{x} \), then by Proposition 3.1(2), \( \hat{x}^\pi \) is also of Type II with \( o(x) = o(x^\pi) \). It follows that if \( \hat{x} = \bigcup_{i=1}^{r} [x_i] \), where \( x_i \in G \), then \( \hat{x}^\pi = \bigcup_{i=1}^{r} [y_i] \), where \( y_i = x_j^i \) for some \( j \in \{1, 2, \ldots, r\} \) and \( o(x_i) = o(y_j) = o(x) \) for \( i, j = 1, 2, \ldots, r \). Then for every \( w \in \hat{x} \), we have \( w = [x_i]_l \) for some \( i \) and \( l \). For such a \( w \), we define \( w^\sigma := [x_i]^\sigma_l = [x_i^\sigma]_l \).
It follows that \( \hat{x}^\sigma = \hat{x}^\pi \) for every \( w \in \hat{x} \). Since \( \hat{x} \neq \hat{x}^\sigma \), the set of all subgroups of \( \langle w \rangle \) and the set of all subgroups of \( \langle w^\pi \rangle \) are not the same for every \( w \in \hat{x} \).

If \( \hat{x} \) is of Type II, and \( x^\pi \in \hat{x} \), then we define \( w^\sigma := w \) for every \( w \in \hat{x} \). It follows that \( \hat{x}^\sigma = \hat{x} \). Then \( \hat{x}^\pi = \hat{x}^\pi \).

Case 3. If \( \hat{x} \) is of Type III, then \( w^\sigma := w \) for every \( w \in \hat{x} \). It follows that \( o(w) = o(w^\pi) \) for every \( w \in \hat{x} \) and \( \hat{x}^\sigma = \hat{x} \). So by Proposition \( \text{3.1(1)} \), \( \hat{x}^\sigma = \hat{x}^\pi \).

Case 4. If \( \hat{x} \) is of Type IV with parameter \( (p, q) \), then by Proposition \( \text{3.1(4)} \), \( \hat{x}^\pi \) is also of Type IV with parameter \( (p, q) \). Then for each \( w \in \hat{x} \), \( w^\sigma := y \), where \( y \in x^\pi \) such that \( o(y) = o(w) \). So, \( \hat{x}^\sigma = \hat{x}^\pi \). Since \( \hat{x} \) is of Type IV, \( \langle w \rangle \) is not a maximal cyclic subgroup.

Summarizing the above argument, we get \( \hat{x}^\sigma = \hat{x}^\pi = \hat{x}^\pi \) for every \( \hat{x} \in \mathcal{B}(G) \).

Since \( \simeq \) is an equivalence relation on \( G \) and \( \pi \) is well defined, it follows that \( \sigma \) is well defined. Let \( A_1, A_2, A_3 \) and \( A_4 \) be the set of all elements in the \( \simeq \)-classes of Type I, II, III, IV, respectively. Then \( G \) is the disjoint union of \( \{e\}, A_1, A_2, A_3, A_4 \). By the definition of \( \sigma \), it is clear that \( o(x) = o(x^\pi) \) and the map \( \sigma |_{A_i} \) is a bijection on \( A_i \) for \( i = 1, 2, 3, 4 \). Thus \( \sigma \) is a bijection on \( G \).

Now let \( x, y \in G \). Then by Lemma \( \text{3.5(2)} \), \( \langle y \rangle \subset \langle x \rangle \) if and only if \( \langle y^\pi \rangle \subset \langle x^\pi \rangle \). It follows that \( \langle y \rangle \subset \langle x \rangle \) if and only if \( \langle y^\sigma \rangle \subset \langle x^\sigma \rangle \), since \( \hat{x}^\sigma = \hat{x}^\pi \). So by Lemma \( \text{4.1(3)} \), \( \sigma \in P(G) \) and \( N_{\mathcal{R}P(G)}(x^\pi) = N_{\mathcal{R}P(G)}(x^\sigma) \), \( N_{\mathcal{R}P(G)}^+(x^\pi) = N_{\mathcal{R}P(G)}^+(x^\sigma) \) for every \( x \in G \).

**Theorem 4.1.** Let \( G \) be a finite group. Then

\[
\text{Aut(}\mathcal{R}P(G)\text{)} = \left( \prod_{i=1}^{t} S_{G_i} \right) \times \mathcal{M}(G),
\]

where \( \mathcal{M}(G) \) and \( \prod_{i=1}^{t} S_{G_i} \) act on \( G \) as in \( \text{(2.1)} \) and \( \text{(2.2)} \), respectively.

**Proof.** Let \( \sigma \in \text{Aut(}\mathcal{R}P(G)\text{)} \). Then by Lemma \( \text{4.5} \), there exists \( \sigma \in \mathcal{M}(G) \) such that \( N_{\mathcal{R}P(G)}(x^\pi) = N_{\mathcal{R}P(G)}(x^\sigma) \) and \( N_{\mathcal{R}P(G)}^+(x^\pi) = N_{\mathcal{R}P(G)}^+(x^\sigma) \) for every \( x \in G \). Thus

\[
N_{\mathcal{R}P(G)}(x^\pi) = N_{\mathcal{R}P(G)}(x^\pi) = N_{\mathcal{R}P(G)}(x^\sigma)
\]

and similarly, \( N_{\mathcal{R}P(G)}^+(x^\pi) = N_{\mathcal{R}P(G)}^+(x^\sigma) \). Therefore, \( \hat{x}^\pi = \hat{x} \). So by Lemma \( \text{4.1(2)} \), \( \pi^\sigma \in \prod_{i=1}^{t} S_{G_i} \). Hence \( \sigma \in (\prod_{i=1}^{t} S_{G_i}, \mathcal{M}(G)) \) and so the result follows from Corollary \( \text{4.1} \). \( \square \)

**Theorem 4.2.** (1) Let \( G \) be a finite group. If \( G \not\cong \mathbb{Z}_{2^m}(m \geq 1) \) and \( Q_{2^\alpha}(\alpha \geq 3) \), then

\[
\text{Aut(}\mathcal{R}P(G)\text{)} = \left( \prod_{i=1}^{t} S_{G_i} \right) \times \mathcal{M}(G),
\]

where \( \mathcal{M}(G) \) and \( \prod_{i=1}^{t} S_{G_i} \) act on \( G \) as in \( \text{(2.1)} \) and \( \text{(2.2)} \), respectively.

(2) \( \text{Aut(}\mathcal{R}P(\mathbb{Z}_{2^m})\text{)} = \mathbb{Z}_2 \times \prod_{i=2}^{m} S_{\mathbb{Z}_{2^i}} \) for \( m \geq 2 \).
(3) \( \text{Aut}(\overrightarrow{\mathcal{R}P}(Q_{2^\alpha})) = \mathbb{Z}_2 \times \left( \prod_{i=2}^{\alpha-1} S_{\varphi(2^i)} \right) \times S_{2^{\alpha-1}} \) for \( \alpha \geq 3 \).

**Proof.** (1) Let \( \pi \in \text{Aut}(\overrightarrow{\mathcal{R}P}(G)) \). Let \( x, y \in G \). Suppose \( x \) and \( y \) are non-trivial elements in \( G \). Then by Lemma 3.3(1), \( \langle y \rangle \subseteq \langle x \rangle \) if and only if \( \langle y^\pi \rangle \subseteq \langle x^\pi \rangle \). Suppose either \( x \) or \( y \) is the trivial element. Without loss of generality, we assume that \( y \in G \) is a non-trivial element. Then by Lemma 3.3(1), \( e^\pi = e \) and so \( \langle y \rangle \subseteq \langle x \rangle \) and \( \langle y^\pi \rangle \subseteq \langle x^\pi \rangle \). It follows that \( \langle x, y \rangle \in \overrightarrow{\mathcal{R}P}(G) \) if and only if \( \langle x^\pi, y^\pi \rangle \in \overrightarrow{\mathcal{R}P}(G) \). Thus \( \pi \in \text{Aut}(\overrightarrow{\mathcal{R}P}(G)) \) and so \( \text{Aut}(\overrightarrow{\mathcal{R}P}(G)) \subseteq \text{Aut}(\overrightarrow{\mathcal{R}P}(G)) \). Thus \( \text{Aut}(\overrightarrow{\mathcal{R}P}(G)) = \text{Aut}(\overrightarrow{\mathcal{R}P}(G)) \), since \( \text{Aut}(\overrightarrow{\mathcal{R}P}(G)) \subseteq \text{Aut}(\overrightarrow{\mathcal{R}P}(G)) \). So the proof follows from Theorem 4.1.

(2)-(3) Let \( \pi \in \text{Aut}(\overrightarrow{\mathcal{R}P}(G)) \). By Lemma 3.3(1), \( e^\pi = e \) or \( a \), where \( a \) is the unique element of order 2 in \( G \). Let \( x, y \in G \) be such that \( x, y \neq e, a \). Then \( \text{deg}_{\overrightarrow{\mathcal{R}P}(G)}(x) = \text{deg}_{\overrightarrow{\mathcal{R}P}(G)}(y) \) if and only if \( x = y \). It follows that \( \pi^\pi = \pi \) for every \( x \neq e, a \) and so \( \pi \mid \{G \setminus \{e, a\} \} \subseteq \prod_{i=1}^t S_{\bar{u}_i} \). So we get

\[
\text{Aut}(\overrightarrow{\mathcal{R}P}(G)) = \mathbb{Z}_2 \times \prod_{i=1}^t S_{\bar{u}_i}.
\]

(4.1)

It is not hard to see that \( \mathcal{R}(Z_{2^n}) = \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_t\} \), where \( t = m + 1 \) and \( \bar{u}_i \) is the set of all generators of the unique cyclic subgroup of order \( 2^i - 1 \) in \( Z_{2^n} \) for \( i = 1, 2, \ldots, t \). Let \( a \) be an element of order \( 2^{\alpha-1} \) in \( Q_{2^n} \). Then \( \mathcal{R}(Q_{2^n}) = \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_t\} \), where \( t = \alpha + 1 \), \( \bar{u}_i \) is the set of all generators of the unique cyclic subgroup of order \( 2^i - 1 \) in \( \langle a \rangle \) for \( i = 1, 2, \ldots, t - 1 \) and \( \bar{u}_t \) is the set of all elements in \( Q_{2^n} \setminus \langle a \rangle \). Substituting these in (4.1), we get the results.

5 Examples

In this section, we compute the automorphism groups of the reduced power (di)graphs of several classes of finite groups.

**Example 5.1.** Let \( n \geq 1 \) be an integer and let \( p, q \) be distinct primes. Then

(1) \( \text{Aut}(\overrightarrow{\mathcal{R}P}(Z_n)) \cong \begin{cases} \prod_{i \mid n} S_{\varphi(d)}, & \text{if } n \neq pq; \\ S_{\varphi(pq)} \times S_{p+q-2}, & \text{if } n = pq. \end{cases} \)

(2) \( \text{Aut}(\overrightarrow{\mathcal{R}P}(Z_n)) \cong \begin{cases} \prod_{i \mid n} S_{\varphi(d)}, & \text{if } n \neq pq, 2^\alpha; \\ S_{\varphi(pq)} \times S_{p+q-2}, & \text{if } n = pq; \\ S_2 \times \prod_{i=2}^\alpha S_{\varphi(2^i)}, & \text{if } n = 2^\alpha, \end{cases} \)

where \( \alpha \geq 1 \).

**Proof.** Notice that \( \mathcal{M}(Z_n) \) is a singleton, since \( Z_n \) has exactly one maximal cyclic subgroup. For each \( d \in D(n) \), let \( A_d \) denote the unique cyclic subgroup of order \( d \) in \( Z_n \), where \( D(n) \) denote the set of all positive divisors of \( n \). Then \( S_{[A_d]} \cong S_{\varphi(d)} \). Notice that

\[
\mathcal{R}(Z_n) = \begin{cases} \{[A_d] \mid d \in D(n)\} & \text{if } n \neq pq; \\ \{[A_d] \mid d = 1, n\} \cup \{[A_p] \cup [A_q]\} & \text{if } n = pq; \end{cases}
\]

if \( n \neq pq \); if \( n = pq \); \hspace{1cm} (5.1)

Applying these in Theorems 4.1 and 4.2, we get the result.
Example 5.2. Let \( n \geq 2 \) be an integer and let \( p, q \) be distinct primes. Then

\[
\text{Aut}(\mathcal{R}^p(2n)) = \text{Aut}(\mathcal{R}(2n)) \cong \begin{cases}
\left( \prod_{d|2n} S_{\phi(d)} \right) \times S_n, & \text{if } n \neq pq; \\
S_{\phi(pq)} \times S_{p+q-2} \times S_p, & \text{if } n = pq.
\end{cases}
\]

Proof. Notice that \( D_{2n} = \langle a, b \mid a^n = e = b^2, \ ab = ba^{-1} \rangle = \{ e, a, a^2, \ldots, a^n, b, ab, \ldots, a^{n-1}b \} \). Here \( M(D_{2n}) = \{ \langle a \rangle \} \cup \{ \langle a^i b \rangle \mid 0 \leq i \leq n - 1 \} \). Also, \( o(a^i b) = 2, | \langle a^i b \rangle \cap \langle a \rangle | = 1 \) and \( | \langle a^i b \rangle \cap \langle a^j b \rangle | = 1 \) for \( i \neq j \). Hence \( \mathcal{M}(D_{2n}) = \{ 1_{M(D_{2n})} \} \). Also, \( \mathcal{A}(D_{2n}) = \mathcal{A}(\langle a \rangle) \cup \hat{b} \), where \( \hat{b} = \{ a^i b \mid 1 \leq i \leq n \} \) (cf. [24, Figure 2]). Applying these in Theorems 4.1 and 4.2 we get

\[
\text{Aut}(\mathcal{R}^p(2n)) = \text{Aut}(\mathcal{R}(2n)) = \left( \prod_{\hat{u} \in \mathcal{A}(\langle a \rangle)} S_{\hat{u}} \right) \times S_{\hat{b}}
\]

So the result follows from [5.1].

Example 5.3. Let \( m \geq 2 \) be an integer and let \( p \) be an odd prime. Then

(1) \( \text{Aut}(\mathcal{R}^p(Q_{4m})) \cong \begin{cases}
\left( \prod_{d|2m} S_{\phi(d)} \right) \times S_{2m}, & \text{if } m \neq p; \\
S_{p-1} \times S_p \times S_{2p}, & \text{if } m = p.
\end{cases} \)

(2) \( \text{Aut}(\mathcal{R}(Q_{4m})) \cong \begin{cases}
\left( \prod_{d|2m} S_{\phi(d)} \right) \times S_{2m}, & \text{if } m \neq p, 2^\alpha; \\
S_{p-1} \times S_p \times S_{2p}, & \text{if } m = p; \\
S_2 \times \left( \prod_{i=2}^{\alpha+1} S_{\phi(2^i)} \right) \times S_{2^{\alpha+1}}, & \text{if } m = 2^\alpha,
\end{cases} \)

where \( \alpha \geq 1 \).

Proof. Notice that \( Q_{4m} = \langle a, b \mid a^{2m} = e = b^4, bab^{-1} = a^{-1} \rangle = \{ e, a, a^2, \ldots, a^{2m-1}, b, ab, a^2b, \ldots, a^{2m-1}b \} \). Here \( M(Q_{4m}) = \{ \langle a \rangle \} \cup \{ \langle a^i b \rangle \mid 0 \leq i \leq m - 1 \} \). Clearly, \( o(a) = 2m \). It can be seen that for every \( i = 0, 1, \ldots, m - 1, o(a^i b) = 4 \) and \( \langle a^i b \rangle \) has the subgroup of order 2 as common. Hence \( \mathcal{M}(Q_{4m}) = \{ 1_{M(Q_{4m})} \} \). Also \( \mathcal{A}(Q_{4m}) = \mathcal{A}(\langle a \rangle) \cup \hat{b} \), where \( \hat{b} = \{ a^i b \mid 1 \leq i \leq 2m \} \) (cf. [21, Figure 3]). Hence by Theorem 4.1

\[
\text{Aut}(\mathcal{R}^p(Q_{4m})) = \left( \prod_{\hat{u} \in \mathcal{A}(\langle a \rangle)} S_{\hat{u}} \right) \times S_{\hat{b}}
\]

So the part (1) of this result follows from [5.1].

In view of Theorems 4.1 and 4.2 it follows that \( \text{Aut}(\mathcal{R}^p(Q_{4m})) = \text{Aut}(\mathcal{R}(Q_{4m})) \) for \( m \neq 2^\alpha \), where \( \alpha \geq 2 \). So the part (2) of this result follows from Theorem 4.2(3). \( \square \)
Example 5.4. Let \( n \geq 2 \) be an integer. Then
\[
\text{Aut}(\overrightarrow{\text{RP}}(SD_{8n})) = \text{Aut}(\text{RP}(SD_{8n})) \cong S_{2n} \times S_{2n} \times \prod_{d|4n} S_{\phi(d)}.
\]

Proof. \( SD_{8n} = \langle a, b \mid a^{4n} = e = b^2, bab^{-1} = a^{2n-1} \rangle = \{ e, a, a^2, \ldots, a^{4n-1} \} \cup \{ a^ib \mid 1 \leq i \leq 4n \text{ and } i \text{ is even} \} \cup \{ a^ib \mid 1 \leq i \leq 4n \text{ and } i \text{ is odd} \} \). Here \( M(SD_{8n}) = \{ \langle a \rangle \} \cup \{ \langle a^ib \rangle \mid 1 \leq i \leq 4n \text{ and } i \text{ is even} \} \cup \{ \langle a^ib \rangle \mid 1 \leq i \leq 2n \text{ and } i \text{ is odd} \} \). Notice that \( o(a^ib) = 2 \), where \( 1 \leq i \leq 4n \) and \( i \) is even; \( o(a^ib) = 4 \), where \( 1 \leq i \leq 2n \) and \( i \) is odd; \( \langle a^ib \rangle \cap \langle a \rangle = \langle a^{2n} \rangle \), where \( 1 \leq i, j \leq 2n \), \( i \neq j \) and \( i \) is odd. Hence \( M(SD_{8n}) = \{ 1_{M(SD_{8n})} \} \). Also, \( \mathcal{R}(SD_{8n}) = \mathcal{R}(\langle a \rangle) \cup \hat{b} \cup \hat{ab} \), where \( \hat{b} = \{ a^ib \mid 1 \leq i \leq 4n \text{ and } i \text{ is even} \} \) and \( \hat{ab} = \{ a^ib \mid 1 \leq i \leq 4n \text{ and } i \text{ is odd} \} \) (cf. [24, Figure 4]). Applying these in Theorems [4.1] and [4.2]

\[
\text{Aut}(\overrightarrow{\text{RP}}(SD_{8n})) = \text{Aut}(\text{RP}(SD_{8n})) = \left( \prod_{\hat{a} \in \mathcal{R}(\langle a \rangle)} S_{\hat{a}} \right) \times S_{\hat{b}} \times S_{\hat{ab}} = \left( \prod_{\hat{a} \in \mathcal{R}((Z_{4n}))} S_{\hat{a}} \right) \times S_{\hat{b}} \times S_{\hat{ab}}.
\]

So the result follows from (5.1).

Example 5.5. Let \( G \) be a finite group of order \( n \) which is isomorphic to a \( p \)-group with exponent \( p \) (except \( Z_2 \)) or a non-nilpotent group of order \( n = p^m q \) with all non-trivial elements are of order \( p \) or \( q \), where \( p, q \) are distinct primes. Then \( \text{Aut}(\overrightarrow{\text{RP}}(G)) = \text{Aut}(\text{RP}(G)) \cong S_{n-1} \).

Proof. Let \( G \) be any one of the groups mentioned in this example. According to [10], every non-trivial element of \( G \) is of order either \( p \) or \( q \). Hence all the proper subgroups of \( G \) are maximal cyclic subgroups of prime order and the set of all proper subgroups of each of these maximal subgroups are the same, which is clearly \( \{ e \} \). Hence \( M(G) = \{ 1_{M(G)} \} \). Also, \( \mathcal{R}(G) = \{ \{ e \}, G \setminus \{ e \} \} \). Applying these in Theorems [4.1] and [4.2] we get

\[
\text{Aut}(\overrightarrow{\text{RP}}(G)) = \text{Aut}(\text{RP}(G)) = S_{G\setminus\{e\}} = S_{n-1}.
\]

So the result follows.

Example 5.6. Let \( n \geq 1 \) be an integer and \( p \) be a prime. Then
\[
\text{Aut}(\overrightarrow{\text{RP}}(\mathbb{Z}p^m)) = \text{Aut}(\text{RP}(\mathbb{Z}p^m)) \cong (S_{p-1} \times S_{p^{m(p-1)}}) \wr S_m, \text{ where } m = \frac{p^m-1}{p-1}.
\]

Proof. Notice that \( \mathbb{Z}p^m \) has \( m = \frac{p^m-1}{p-1} \) subgroups of order \( p \). Let them be \( \langle x_i \rangle \) for \( i = 1, 2, \ldots, m \). Also, each \( \langle x_i \rangle \) is contained in exactly \( p^{m-1} \) cyclic subgroups of order \( p^2 \). So \( M(G) \) is the set of all cyclic subgroups of order \( p^2 \). It follows that \( \mathcal{M}(\mathbb{Z}p^m) \cong S_m \). Also, \( \mathcal{R}(\mathbb{Z}p^m) = \{ e, \hat{x_1}, \hat{x_2}, \ldots, \hat{x_m}, \hat{b_1}, \hat{b_2}, \ldots, \hat{b_m} \} \), where \( \langle b_i \rangle \) is the cyclic subgroup of order \( p^2 \)
This completes the proof. 

\textbf{Example 5.7.} Let \( \alpha \geq 3 \) be an integer and \( p \) be a prime. Then we have the following.

1. \( \text{Aut}(\hat{\mathcal{RP}}(\mathbb{M}_8)) = \text{Aut}(\mathcal{RP}(\mathbb{M}_8)) = S_2 \times S_4 \).

2. \( \text{Aut}(\hat{\mathcal{RP}}(\mathbb{M}_{p^\alpha})) = \text{Aut}(\mathcal{RP}(\mathbb{M}_{p^\alpha})) = S_{p^\alpha - 1(p-1)} \times S_{p(p-1)} \times G_1 \times G_2, \)

where \( p^\alpha \neq 8 \) and \( G_j = \prod_{i=1}^{\alpha - j - 1} S_{p^{\alpha - i - 2}(p-1)i} \) for \( j = 1, 2 \).

\textit{Proof.} (1) From the subgroup lattice of \( \mathbb{M}_8 \) (cf. [6, Figure 3]), it is clear that \( \mathbb{M}_8 \) contains four maximal cyclic subgroups of order 2 and a unique maximal cyclic subgroup of order 4. It follows that \( \mathcal{M}(\mathbb{M}_8) = I_{M(M_8)} \). Also, \( \mathcal{R}(\mathbb{M}_8) = \mathcal{R}(\mathbb{Z}_4) \cup \{b\} \), where \( \hat{b} = \{b, a^2b, ab, a^3b\} \). Applying these in Theorems 4.1 and 4.2, we get the result (1).

(2) From the subgroup lattice of \( \mathbb{M}_{p^\alpha} \), where \( p^\alpha \neq 8 \) (cf. [6, Figure 4]), it is clear that \( \mathbb{M}_{p^\alpha} \) has \( p \) number of cyclic subgroups of order \( p^i \) for \( i = 2, 3, \ldots, p^n \) and \( p + 1 \) cyclic subgroups of order \( p \). Hence \( M(G) \) is the set of all cyclic subgroups of order \( p^\alpha \). Moreover, all the maximal cyclic subgroups have the same set of proper cyclic subgroups. So \( \mathcal{M}(G) = \{1_{M(G)}\} \). Also,

\[ \mathcal{R}(\mathbb{M}_{p^\alpha}) = \{\hat{a}, \hat{b}, e\} \bigcup_{i=1}^{\alpha - 2} \{a^p\} \bigcup_{i=1}^{\alpha - 3} \{a^p b\}, \]

where \( o(b) = p \), \( o(a^p) = p^{\alpha - (i + 1)} = o(a^{p^i} b), \hat{a} = \bigcup_{j=1}^{p - 1} [a^{p^j}] \cup [a], \hat{b} = \bigcup_{j=1}^{p - 1} [a^{\alpha - 2 p^j}] \cup [b], \hat{a^p} = [a^p], \)

and \( \hat{a^p b} = \bigcup_{j=1}^{p - 1} [a^{p^i} b^j] \) for \( i = 1, 2, \ldots, \alpha - 2 \).

Applying these in Theorems 4.1 and 4.2, we get

\[ \text{Aut}(\hat{\mathcal{RP}}(\mathbb{M}_{p^\alpha})) = \text{Aut}(\mathcal{RP}(\mathbb{M}_{p^\alpha})) = S_{\hat{a}} \times S_{\hat{b}} \times \left( \prod_{i=1}^{\alpha - 2} S_{a^p} \right) \times \left( \prod_{i=1}^{\alpha - 3} S_{a^p b} \right) \]

\[ = S_{p^\alpha - 1(p-1)} \times S_{p(p-1)} \times G_1 \times G_2, \]

where \( G_j = \prod_{i=1}^{\alpha - j - 1} S_{p^{\alpha - i - 2}(p-1)i} \) for \( j = 1, 2 \). \qed
Note that the subgroup lattice of \( \mathbb{Z}_{p^{\alpha} - 1} \times \mathbb{Z}_p \) is isomorphic to the subgroup lattice of \( \mathcal{M}_{p^{\alpha}}(p^\alpha \neq 8) \). As a consequence, we get the following result by using Example 5.8 (2).

**Example 5.8.** Let \( \alpha \geq 2 \) be an integer and \( p \) be a prime. Then
\[
\text{Aut}(\overrightarrow{\mathcal{RP}}(\mathbb{Z}_{p^{\alpha} - 1} \times \mathbb{Z}_p)) = \text{Aut}(\mathcal{RP}(\mathbb{Z}_{p^{\alpha} - 1} \times \mathbb{Z}_p)) = S_{p^{\alpha} - 1(p-1)} \times S_{p(p-1)} \times G_1 \times G_2,
\]
where \( p^\alpha \neq 8 \) and \( G_j = \prod_{i=1}^{\alpha-j-1} S_{p^{\alpha-1} - 2(p-1)} \) for \( j = 1, 2 \).

**Example 5.9.** Let \( n \) be an odd positive integer. Then
\[
\text{Aut}(\overrightarrow{\mathcal{RP}}(V_{8n})) = \text{Aut}(\mathcal{RP}(V_{8n})) = S_{2n} \times S_{2n} \times \left( \prod_{d|2n} S_{\varphi(d)} \right) \times \left( \prod_{d|2, d|n} S_{\varphi(d)} \right) \triangleleft S_2.
\]

**Proof.** Notice that
\[
M(V_{8n}) = \{ \langle ab^2 \rangle, \langle a \rangle, \langle a^2 b^2 \rangle \} \bigcup \{ \langle a^j b \rangle \mid j \text{ is even} \} \bigcup \{ \langle a^j b^k \rangle \mid j \text{ is odd, } k = 1 \text{ or } 3 \},
\]
where \( o(ab^2) = o(a) = o(a^2 b^2) = 2n \) and \( o(a^j b) = 4 \), if \( j \) is even; and \( o(a^j b^k) = 2 \), if \( j \) is odd and \( k = 1 \) or \( 3 \). Also, \( \langle a^j b \rangle \cap \langle a b^2 \rangle = \langle b^2 \rangle \). It follows that each \( \sigma \in \mathcal{M}(V_{8n}) \) fixes all the elements in \( \{ \langle a^j b \rangle \mid j \text{ is even} \} \) and \( \{ \langle a^j b^k \rangle \mid j \text{ is odd, } k = 1 \text{ or } 3 \} \). Notice that \( \langle a^2 b^2 \rangle \) contains a subgroup \( \langle b^2 \rangle \) of order 2 which is contained in \( n \) number of subgroups of order 4; but \( \langle a \rangle \) and \( \langle a b^2 \rangle \) does not contain such a subgroup. It turns out that \( \mathcal{M}(V_{8n}) \cong S_2 \).

Suppose \( Z_n \) denotes the reduced power graph of a cyclic group of order \( n \). Then the structure of \( \mathcal{RP}(V_{8n}) \) is as shown in [2, Figure 3]. From this structure, we have
\[
\mathcal{R}(V_{8n}) = \hat{a}^2 b \cup \hat{a}b \cup \mathcal{R}(\langle a^2 b^2 \rangle) \bigcup [\mathcal{R}(\langle ab^2 \rangle) \setminus \mathcal{R}(\langle a^2 \rangle)] \bigcup [\mathcal{R}(\langle a \rangle) \setminus \mathcal{R}(\langle a^2 \rangle)],
\]
where \( \hat{a}^2 b = \{ a^j b^k \mid j \text{ is even, } k = 1 \text{ or } 3 \} \) and \( \hat{a}b = \{ a^j b^k \mid j \text{ is odd, } k = 1 \text{ or } 3 \} \).

Applying these in Theorems 1.1 and 1.2 we get
\[
\text{Aut}(\overrightarrow{\mathcal{RP}}(V_{8n})) = \text{Aut}(\mathcal{RP}(V_{8n}))
\]
\[
= S_{2n} \times S_{2n} \times \left( \prod_{\hat{u} \in \mathcal{R}(\langle a^2 b^2 \rangle)} S_{\hat{u}} \right) \times \left( \prod_{\hat{u} \in \mathcal{R}(\langle ab^2 \rangle), \hat{u} \notin \mathcal{R}(\langle a^2 \rangle)} S_{\hat{u}} \right) \times \left( \prod_{\hat{u} \in \mathcal{R}(\langle a \rangle), \hat{u} \notin \mathcal{R}(\langle a^2 \rangle)} S_{\hat{u}} \right) \times S_2
\]
\[
= S_{2n} \times S_{2n} \times \left( \prod_{\hat{u} \in \mathcal{R}(Z_{2n})} S_{\hat{u}} \right) \times \left( \prod_{\hat{u} \in \mathcal{R}(\langle a \rangle), \hat{u} \notin \mathcal{R}(\langle a^2 \rangle)} S_{\hat{u}} \right) \times S_2
\]
\[
= S_{2n} \times S_{2n} \times \left( \prod_{d|2n} S_{\varphi(d)} \right) \times \left( \prod_{d|2, d|n} S_{\varphi(d)} \right) \times S_2
\]
\[
= S_{2n} \times S_{2n} \times \left( \prod_{d|2n} S_{\varphi(d)} \right) \times \left( \prod_{d|2, d|n} S_{\varphi(d)} \right) \triangleleft S_2.
\]

This completes the proof. \( \square \)
Example 5.10. (1) Let \( n = 2^k t \), where \( t \) is an odd positive integer and \( k \) is a positive integer with \( n \neq 2 \). Then \( \text{Aut}(\overline{\mathcal{RP}}(V_{8n})) = \text{Aut}(\mathcal{RP}(V_{8n})) = S_{2n} \times S_{2n} \times S_{2n} \times \left( \prod_{d | 2^k t} S_{\varphi(d)} \right) \times \left( \prod_{d | 2^k t} S_{\varphi(d)} \right) \).

(2) \( \text{Aut}(\overline{\mathcal{RP}}(V_{16})) = \text{Aut}(\mathcal{RP}(V_{16})) = S_5 \times (S_4 \cup S_2) \).

Proof. (1) Assume that \( n \neq 2 \). It can be seen that

\[
M(V_{8n}) = \left\{ \langle a \rangle, \langle a^{2k+1}b^2 \rangle \right\} \cup \left\{ \langle a^{2i}b^2 \rangle \mid 0 \leq i \leq k \right\} \cup \left\{ \langle a^ib \rangle \mid j \text{ is even} \right\} \cup \left\{ \langle a^ib^k \rangle \mid j \text{ is odd, } k = 1 \text{ or } 3 \right\},
\]

where \( o(a) = 2^{k+1}t \), \( o(a^{2k+1}b^2) = 2^kt \), \( o(a^{2i}b^2) = 2^{k+2-i}t \) for \( 0 \leq i \leq k \); \( o(a^ib) = 4 \), if \( j \) is even and \( k = 1 \) or \( 3 \); \( o(a^ib^k) = 2 \), if \( j \) is odd and \( k = 1 \) or \( 3 \). Also, \( \langle a^ib \rangle \cap \langle a^ib^k \rangle = \langle b^2 \rangle \). It follows that each \( \sigma \in \mathcal{M}(V_{8n}) \) fixes all the elements in \( \{ \langle a^ib \rangle \mid j \text{ is even} \} \) and \( \{ \langle a^ib^k \rangle \mid j \text{ is odd, } k = 1 \text{ or } 3 \} \). Notice that \( o(a) = o(ab^2) \). However, the proper cyclic subgroups of \( \langle a \rangle \) and \( \langle ab^2 \rangle \) are the same, so \( \langle a \rangle \) and \( \langle ab^2 \rangle \) are fixed by each \( \sigma \in \mathcal{M}(V_{8n}) \).

Also, for \( 1 \leq i \leq k \), \( \langle a^{2i}b^2 \rangle \) is the only maximal cyclic subgroup of order \( 2^{k+2-i}t \). It follows that \( \langle a^{2i}b^2 \rangle \) is fixed by each \( \sigma \in \mathcal{M}(V_{8n}) \). Hence \( \mathcal{M}(V_{8n}) = \{ 1_{M(V_{8n})} \} \).

Suppose \( Z_n \) denotes the reduced power graph of a cyclic group of order \( n \). Then the structure of \( \mathcal{RP}(V_{8n}) \) is the same as the graph shown in [4, Figure 5]. From this structure, we have

\[
\mathcal{R}(V_{8n}) = \overline{a^2b} \cup \overline{ab} \cup \overline{a} \cup \overline{a^2b} \left[ \mathcal{R}(\langle a^2 \rangle) \right] \cup \mathcal{R}(\langle a^{2k-1} \rangle) \bigcup_{i=1}^{k} \mathcal{R}(\langle a^{2i-2} \rangle) \bigcup_{i=1}^{k} \mathcal{R}(\langle a^{2i+2} \rangle) \bigcup_{i=1}^{k} \mathcal{R}(\langle a^{2i+2} \rangle) \bigcup_{i=1}^{k} \mathcal{R}(\langle a^{2i+2} \rangle),
\]

where \( a^2b = \{ a^ib^k \mid j \text{ is even, } k = 1 \text{ or } 3 \} \); \( ab = \{ a^ib \mid j \text{ is odd, } k = 1 \text{ or } 3 \} \) and \( \widehat{a} = [a] \cup [ab^2] \). Applying these in Theorems 4.1 and 4.2 we get

\[
\text{Aut}(\overline{\mathcal{RP}}(V_{8n})) = \text{Aut}(\mathcal{RP}(V_{8n}))
\]

\[
= S_{2n} \times S_{2n} \times S_{2n} \times \left( \prod_{\widehat{u} \in \mathcal{R}(\langle a^2 \rangle)} S_{\widehat{u}} \right) \times \left( \prod_{\widehat{u} \in \mathcal{R}(\langle a^{2k-1} \rangle)} S_{\widehat{u}} \right)
\]

\[
\times \left( \prod_{\widehat{u} \in \mathcal{R}(\langle a^{2i+2} \rangle)} S_{\widehat{u}} \right)
\]

\[
= S_{2n} \times S_{2n} \times S_{2n} \times \left( \prod_{\widehat{u} \in \mathcal{R}(\langle a^{2k} \rangle)} S_{\widehat{u}} \right) \times \left( \prod_{\widehat{u} \in \mathcal{R}(\langle Z_t \rangle)} S_{\widehat{u}} \right)
\]
Applying these in Theorems 4.1 and 4.2, we get
\[ S_{G} \times S_{G} = S_{2n} \times S_{2n}, \]

(2) Assume that \( n = 2 \). Notice that
\[ M(V_{8n}) = \{ \langle a \rangle, \langle b \rangle, \langle ab \rangle, \langle a^2b \rangle, \langle ab^2 \rangle, \langle ab^3 \rangle, \langle a^2b^2 \rangle, \langle a^3b^3 \rangle \}, \]
where \( o(a) = o(b) = o(ab^2) = o(ab) = 2 \) and \( o(ab) = o(a^3b) = o(ab^3) = o(a^2b^2) = 2 \). Also \( \langle a \rangle \cap \langle ab^2 \rangle = \langle a^2 \rangle \) and \( \langle b \rangle \cap \langle ab^2 \rangle = \langle b^2 \rangle \). It follows that \( \mathbb{M}(V_{8n}) \cong S_2 \). Since \( \mathcal{A}(V_{16}) = \{ \hat{a}, \hat{b}, \hat{ab} \} \), where \( \hat{a} = [a] \cup [ab^2], \hat{b} = [b] \cup [a^2b] \) and \( \hat{ab} = [ab] \cup [a^3b] \). Applying these in Theorems 4.1 and 4.2 we get
\[ \text{Aut}(\overrightarrow{\mathcal{P}}(V_{8n})) = \text{Aut}(\mathcal{RP}(V_{8n})) = S_{ab} \times S_{\hat{a}} \times S_{\hat{b}} \times S_2 \]
\[ = S_5 \times (S_4 \times S_4) \times S_2 \]
\[ = S_5 \times (S_4 \times S_2). \]
This completes the proof. \( \square \)

6 Relation between \( \text{Aut}(\mathcal{RP}(G)) \) (resp. \( \text{Aut}(\overrightarrow{\mathcal{P}}(G)) \)) and \( \text{Aut}(\mathcal{P}(G)) \) (resp. \( \text{Aut}(\overrightarrow{\mathcal{P}}(G)) \))

Let \( G \) be a group. Consider the equivalence relation \( \approx \) on \( G \) defined as follows: for every \( x, y \in G \), \( x \approx y \) if and only if \( N_{\mathcal{P}(G)}[x] = N_{\mathcal{P}(G)}[y] \). Let \( \pi \) denote the \( \approx \)-class determined by \( x \) (cf. [7]).

Theorem 6.1. Let \( G \) be a finite group. Then we have the following:

(1) \( \text{Aut}(\mathcal{RP}(G)) \subseteq \text{Aut}(\mathcal{P}(G)) \) if and only if for every \( x \in G \), \( \hat{x} \) is either of Type I or of Type II with parameter \( (2, r) \), where \( r \geq 2 \);

(2) \( \text{Aut}(\mathcal{P}(G)) \subseteq \text{Aut}(\mathcal{RP}(G)) \) if and only if \( \pi = [x] \) for every \( x \in G \).

Proof. (1) Suppose that for every \( x \in G \), \( \hat{x} \) is of Type II with parameter \((m, r)\), where \( m \neq 2 \). Then \( \hat{x} = \bigcup_{i=1}^{r} [x_i] \), where \( o(x_i) = m \) for \( i = 1, 2, \ldots, r \). Consider the function \( \pi \) on \( G \) defined by \( x_1 = x_2 = x_2 \) and \( y = y \). Then \( \pi \) is a bijection and \( N_{\mathcal{RP}(G)}(x_1) = N_{\mathcal{RP}(G)}(x_2) \), so \( \pi \in \text{Aut}(\mathcal{RP}(G)) \). Since \( m \neq 2 \), there exists \( y \in G \) such that \( \langle y \rangle = \langle x_1 \rangle \) and \( \langle y \rangle \neq \langle x_2 \rangle \). So \( (y, x_1) \in A(\mathcal{P}(G)) \). However, \( \langle y \rangle, x_1 \rangle \notin A(\mathcal{P}(G)) \). Consequently, \( \text{Aut}(\mathcal{RP}(G)) \nsubseteq \text{Aut}(\mathcal{P}(G)) \). Similarly, we can show that, if \( \hat{x} \) is of Type III or Type IV, then \( \text{Aut}(\mathcal{RP}(G)) \nsubseteq \text{Aut}(\mathcal{P}(G)) \).

Next we assume that for every \( x \in G \), \( \hat{x} \) is either of Type I or of Type II with parameter \((2, r)\), where \( r \geq 2 \). Let \( \pi \in \text{Aut}(\mathcal{RP}(G)) \) and let \( x, y \in G \). Suppose that \( x \) and \( y \) are
adjacent in $\mathcal{P}(G)$. Then $\langle x \rangle \subseteq \langle y \rangle$ or $\langle y \rangle \subseteq \langle x \rangle$. If $\langle x \rangle \neq \langle y \rangle$, then $x$ and $y$ are also adjacent in $\mathcal{R}\mathcal{P}(G)$ and so $x^\pi$ and $y^\pi$ are adjacent in $\mathcal{R}\mathcal{P}(G)$. It follows that $x^\pi$ and $y^\pi$ are adjacent in $\mathcal{P}(G)$.

Suppose $x$ and $y$ are not adjacent in $\mathcal{P}(G)$. Then $\langle x \rangle \not\subseteq \langle y \rangle$ and $\langle y \rangle \not\subseteq \langle x \rangle$. So $x$ and $y$ are also not adjacent in $\mathcal{R}\mathcal{P}(G)$. It follows that $x^\pi$ and $y^\pi$ are not adjacent in $\mathcal{R}\mathcal{P}(G)$.

If $\langle x^\pi \rangle \not\subseteq \langle y^\pi \rangle$, then by a similar argument used above, we get $\langle x^\pi \rangle = \langle y^\pi \rangle$. So $x$ and $y$ are adjacent in $\mathcal{P}(G)$, which is a contradiction. Thus $\langle x^\pi \rangle \not\subseteq \langle y^\pi \rangle$ and $\langle y^\pi \rangle \not\subseteq \langle x^\pi \rangle$ and so $x^\pi$ and $y^\pi$ are not adjacent in $\mathcal{P}(G)$.

Thus $\pi \in \text{Aut}(\mathcal{P}(G))$ and so $\text{Aut}(\mathcal{R}\mathcal{P}(G)) \subseteq \text{Aut}(\mathcal{P}(G)).$

(2) Suppose there exists $x \in G$ such that $\overline{x} \neq [x]$. Then there exists $y \in \overline{x}$ such that $\langle x \rangle \neq \langle y \rangle$, and so $o(x) \neq o(y)$. Consider the function $\pi$ on $G$ defined by $x^\pi = y$, $y^\pi = x$, and $w^\pi = w$ for every $w \in G$ with $w \neq x, y$. Then for every $u, v \in G$, $(u, v) \in A(\mathcal{P}(G))$ if and only if $(u^\pi, v^\pi) \in A(\mathcal{P}(G))$. So $\pi \in \text{Aut}(\mathcal{P}(G))$. Since $\overline{x} \neq [x]$, it follows that one of the following holds: (i) either $o(x)$ or $o(y)$ is a composite number or (ii) $x = e$ and $o(y)$ is a prime.

Suppose that either $o(x)$ or $o(y)$ is a composite number. Without loss of generality, we assume that $o(x)$ is a composite number. Then by parts (2) and (3) of Proposition 5.11, $\pi \notin \text{Aut}(\mathcal{R}\mathcal{P}(G))$, since $o(x) \neq o(x^\pi)$. Thus $\text{Aut}(\mathcal{P}(G)) \not\subseteq \text{Aut}(\mathcal{R}\mathcal{P}(G))$. Suppose that $x = e$ and $o(y)$ is a prime. Then by Lemma 5.3, $G \cong \mathbb{Z}_{2m} (m \geq 1)$ or $Q_{2\alpha} (\alpha \geq 3)$. By Example 5.1(2) and Example 5.1(iii), it follows that $\text{Aut}(\mathcal{P}(\mathbb{Z}_{2m})) \not\subseteq \text{Aut}(\mathcal{R}\mathcal{P}(\mathbb{Z}_{2m})).$

Now we show that $\text{Aut}(\mathcal{P}(Q_{2\alpha})) \not\subseteq \text{Aut}(\mathcal{R}\mathcal{P}(Q_{2\alpha})).$ Let $a^2, a^4 \in Q_{2\alpha}$. Then $o(a^2) = 2^{\alpha+1}$ and $o(a^4) = 2^\alpha$. Consider the function $\pi$ on $Q_{2\alpha}$ defined by $(a^2)^\pi = a^4$, $(a^4)^\pi = a^2$, and $x^\pi = x$ for all $x \neq a^2, a^4$. Since $N_{\mathcal{P}(G)}(a^2) = N_{\mathcal{P}(G)}(a^4)$, it follows that $\pi \in \text{Aut}(\mathcal{P}(Q_{2\alpha})).$ But $\text{deg}_{\mathcal{R}\mathcal{P}(Q_{2\alpha})}(a^2) \neq \text{deg}_{\mathcal{R}\mathcal{P}(Q_{2\alpha})}(a^4)$ and hence $\pi \notin \text{Aut}(\mathcal{R}\mathcal{P}(Q_{2\alpha})).$ Thus $\text{Aut}(\mathcal{P}(Q_{2\alpha})) \not\subseteq \text{Aut}(\mathcal{R}\mathcal{P}(Q_{2\alpha})).$

Assume that for every $x \in G$, $\overline{x} = [x]$. Let $\pi \in \text{Aut}(\mathcal{P}(G))$ and let $x, y \in G$. Suppose either $\overline{x}$ or $\overline{y}$ is $\overline{y}$. Then clearly, $\langle x \rangle \subseteq \langle y \rangle$ if and only if $\langle x^\pi \rangle \subseteq \langle y^\pi \rangle$. This implies that if $(x, y) \in A(\mathcal{R}\mathcal{P}(G))$, then $(x^\pi, y^\pi) \in A(\mathcal{R}\mathcal{P}(G))$. Now we assume that $\overline{x} \neq \overline{y}$. If $(x, y) \in A(\mathcal{R}\mathcal{P}(G))$, then $(x, y) \in A(\mathcal{P}(G))$ and so $(x^\pi, y^\pi) \in A(\mathcal{P}(G))$. It follows that any one of the following holds: $\langle x^\pi \rangle \subseteq \langle y^\pi \rangle$, $\langle y^\pi \rangle \subseteq \langle x^\pi \rangle$ or $\langle x^\pi \rangle = \langle y^\pi \rangle$. Suppose $\langle y^\pi \rangle = \langle x^\pi \rangle$. Then $x^\pi = y^\pi$ and so $\overline{x} = \overline{y}$. Hence $[x] = [y]$ and so $\langle x \rangle = \langle y \rangle$, which is a contradiction to the fact that $(x, y) \in A(\mathcal{R}\mathcal{P}(G))$. If $\langle x^\pi \rangle \subseteq \langle y^\pi \rangle$ or $\langle y^\pi \rangle \subseteq \langle x^\pi \rangle$, then $(x^\pi, y^\pi) \in A(\mathcal{P}(G))$.

Suppose $(x, y) \notin A(\mathcal{R}\mathcal{P}(G))$. Then either $\langle x \rangle \not\subseteq \langle y \rangle$ and $\langle y \rangle \not\subseteq \langle x \rangle$, or $\langle x \rangle = \langle y \rangle$. If $\langle x \rangle \not\subseteq \langle y \rangle$ and $\langle y \rangle \not\subseteq \langle x \rangle$, then $(x, y) \notin A(\mathcal{P}(G))$. This implies that $(x^\pi, y^\pi) \notin A(\mathcal{R}\mathcal{P}(G))$, since $\mathcal{R}\mathcal{P}(G)$ is a subgraph of $\mathcal{P}(G)$. If $\langle x \rangle = \langle y \rangle$, then $\overline{x} = \overline{y}$, and so $x^\pi = y^\pi$. It follows that $[x^\pi] = [y^\pi]$. Then $\langle x^\pi \rangle = \langle y^\pi \rangle$ and $(x^\pi, y^\pi) \notin A(\mathcal{R}\mathcal{P}(G))$. Thus $\pi \in \text{Aut}(\mathcal{R}\mathcal{P}(G))$ and so $\text{Aut}(\mathcal{P}(G)) \subseteq \text{Aut}(\mathcal{R}\mathcal{P}(G)).$

The following result is an immediate consequence of Theorem 6.1.

**Corollary 6.1.** Let $G$ be a finite group. Then $\text{Aut}(\mathcal{R}\mathcal{P}(G)) = \text{Aut}(\mathcal{P}(G))$ if and only if for every $x \in G$, $\overline{x} = [x]$ and $\widehat{x}$ is either of Type I or of Type II with parameter $(2, r)$, where $r \geq 2$. 

\[\square\]
Theorem 6.2. Let $G$ be a finite group. Then we have the following:

(1) $\text{Aut}(\overrightarrow{P}(G)) \subseteq \text{Aut}(\overrightarrow{RP}(G))$;

(2) $\text{Aut}(\overrightarrow{RP}(G)) \subseteq \text{Aut}(\overrightarrow{P}(G))$ if and only if for every $x \in G$, $\hat{x}$ is either of Type I or of Type II with parameter $(2, r)$, where $r \geq 2$.

Proof. (1) Let $\pi \in \text{Aut}(\overrightarrow{P}(G))$ and let $x, y \in G$. Then $\langle x \rangle = \langle y \rangle$ if and only if $\langle x^\pi \rangle = \langle y^\pi \rangle$; and $\langle y \rangle \subseteq \langle x \rangle$ if and only if $\langle y^\pi \rangle \subseteq \langle x^\pi \rangle$. It follows that $(x, y) \in A(\overrightarrow{RP}(G))$ if and only if $(x^\pi, y^\pi) \in A(\overrightarrow{RP}(G))$. Thus $\pi \in \text{Aut}(\overrightarrow{RP}(G))$, and so $\text{Aut}(\overrightarrow{P}(G)) \subseteq \text{Aut}(\overrightarrow{RP}(G))$.

(2) Consider the function $\pi$ defined in the proof of Theorem 6.1(1). It is shown that $\pi \in \text{Aut}(\overrightarrow{RP}(G))$. Hence, if there exists $x \in G$ such that $\hat{x}$ is of Type II with parameter $(2, r)$, where $r \neq 2$, or Type III or Type IV, then by a similar argument used in the proof of Theorem 6.1(1), we get $\text{Aut}(\overrightarrow{RP}(G)) \neq \text{Aut}(\overrightarrow{P}(G))$.

Suppose that for every $x \in G$, $\hat{x}$ is either of of Type I or of Type II with parameter $(2, r)$, where $r \geq 2$. Let $\pi \in \text{Aut}(\overrightarrow{RP}(G))$ and let $x, y \in G$. Then by Lemma 3.5(1), $\langle y \rangle \subseteq \langle x \rangle$ if and only if $\langle y^\pi \rangle \subseteq \langle x^\pi \rangle$. If $\langle x \rangle = \langle y \rangle$, then $\hat{x} = \hat{y}$. It follows by assumption that $\hat{x}^\pi = \hat{y}^\pi$. Hence $\hat{x}^\pi = \hat{y}^\pi$. By Proposition 3.1(3), $[x^\pi] = [y^\pi]$ and so $\langle x^\pi \rangle = \langle y^\pi \rangle$. Similarly, if $\langle x^\pi \rangle = \langle y^\pi \rangle$, then $\langle x \rangle = \langle y \rangle$. It follows that $\pi \in \text{Aut}(\overrightarrow{P}(G))$. So $\text{Aut}(\overrightarrow{RP}(G)) \subseteq \text{Aut}(\overrightarrow{P}(G))$. 

The following result is an immediate consequence of Theorem 6.2.

Corollary 6.2. Let $G$ be a finite group. Then $\text{Aut}(\overrightarrow{RP}(G)) = \text{Aut}(\overrightarrow{P}(G))$ if and only if for every $x \in G$, $\hat{x}$ is either of Type I or of Type II with parameter $(2, m)$, where $m \geq 1$.

Example 6.1. $\text{Aut}(\overrightarrow{RP}(U_{6n})) = \text{Aut}(\overrightarrow{RP}(U_{6n})) = \text{Aut}(\overrightarrow{P}(U_{6n})) = \text{Aut}(\overrightarrow{P}(U_{6n}))$.

Proof. Notice that $\text{Aut}(\overrightarrow{P}(U_{6n}))$ is given in [2, Theorem 2.6]. Suppose $Z_n$ denotes the reduced power graph of a cyclic group of order $n$. Then the structure of $\overrightarrow{RP}(U_{6n})$ is the same as the graph shown in [2, Figure 2]. From the structure of $\overrightarrow{P}(U_{6n})$ and $\overrightarrow{RP}(U_{6n})$, we have $\overrightarrow{x_1} = \overrightarrow{x_2} = [x]$. Hence by Corollaries 6.1 6.2 and [13, Proposition 5.2], we get the result.

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