TOWARDS A GLOBAL SPRINGER THEORY II: 
THE DOUBLE AFFINE ACTION

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ABSTRACT. We construct an action of the graded double affine Hecke algebra (DAHA) on the parabolic Hitchin complex, extending the affine Weyl group action constructed in [YunI]. In particular, we get representations of the degenerate DAHA on the cohomology of parabolic Hitchin fibers. We also generalize our construction to parahoric versions of Hitchin stacks, including the construction of ’tHooft operators as a special case. We then study the interaction of the DAHA action and the cap product action given by the Picard stack acting on the parabolic Hitchin stack.

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1. Introduction

This paper is a continuation of [YunI]. For an overview of the ideas and motivations of this series of papers, see the Introduction of [YunI]. We will use the notations and conventions from [YunI, Sec. 2]. In particular, we fix a connected reductive group $G$ over an algebraically closed field $k$ with a Borel subgroup $B$, a connected smooth projective curve $X$ over $k$ and a divisor $D$ on $X$ of degree at least twice the genus of $X$. Recall from [YunI, Def. 3.1.2] that we defined the parabolic Hitchin moduli stack $\mathcal{M}_{\text{par}} = \mathcal{M}_{\text{par}}^p_{G,X,D}$ as the moduli stack of quadruples $(x, E, \varphi, E^B_x)$ where

- $x \in X$;
- $E$ is a $G$-torsor on $X$ with a $B$-reduction $E^B_x$ at $x$;
- $\varphi \in H^0(X, \text{Ad}(E)(D))$ is a Higgs field compatible with $E^B_x$.

We also defined the parabolic Hitchin fibration (see [YunI, Def. 3.1.6]):

$$f_{\text{par}} : \mathcal{M}_{\text{par}} \to A \times X.$$ 

In [YunI], we have constructed an action of the extended affine Weyl group $\tilde{W}$ on the parabolic Hitchin complex $f_{\text{par}}^* Q$, which justifies to be called the “global Springer action”. As mentioned in [YunI, Sec. 1.2], there are at least three pieces of symmetry acting on the complex $f_{\text{par}}^* Q$; the affine Weyl group action, the cup product action given by certain Chern classes and the cap product action given by the Picard stack $\mathcal{P}$. This paper is devoted to the study of the second and the third action on $f_{\text{par}}^* Q$, as well as the interplay among the three actions.

Recall from [YunI, Rem. 3.5.6] that we have chosen an open subet $A$ of the anisotropic Hitchin base $A_{\text{ani}}$ on which the codimension estimate $\text{codim}_{A_{\text{ani}}}(A_\delta) \geq \delta$ holds for any $\delta \in \mathbb{Z}_{\geq 0}$. Throughout this paper, with the only exception of Sec. 2, we will work over this open subet $A$. All stacks originally over $A_{\text{Hit}}$ or $A_{\text{ani}}$ will be restricted to $A$ without changing notations. Note that when $\text{char}(k) = 0$, we may take $A = A_{\text{ani}}$.

1.1. Main results.

1.1.1. The double affine action. The $\tilde{W}$-action on $f_{\text{par}}^* Q$ constructed in [YunI, Th. 4.4.3] and the Chern class action mentioned above together give a full symmetry of the graded double affine Hecke algebra $\mathbb{H}$ (DAHA) on $f_{\text{par}}^* Q$, which we now define.

For simplicity, let us assume $G$ is almost simple and simply-connected, so that the affine Weyl group $\tilde{W} = X_\text{a}(T) \rtimes W$ is a Coxeter group with simple reflections $\{s_0, s_1, \ldots, s_n\}$. The graded algebra $\mathbb{H}$ is, as a vector space, the tensor product of
the group ring $\mathcal{U}_t(\tilde{W})$ with the polynomial algebra $\text{Sym}_{\mathcal{U}_t}(X^*(\tilde{T})_{\mathbb{Q}_t}) \otimes \mathcal{U}_t[u]$. Here $\tilde{T}$ is the Cartan torus in the Kac-Moody group associated to the loop group $G((t))$ (see Sec. 3.1). The graded algebra structure of $\mathbb{H}$ is uniquely determined by

- $\mathcal{U}_t(\tilde{W})$ is a subalgebra of $\mathbb{H}$ in degree 0;
- $\text{Sym}_{\mathcal{U}_t}(X^*(\tilde{T})_{\mathbb{Q}_t})$ is a subalgebra of $\mathbb{H}$ with $\xi \in X^*(\tilde{T})$ in degree 2;
- $u$ has degree 2, and is central in $\mathbb{H}$;
- For each simple reflection $s_i$ and $\xi \in X^*(\tilde{T})$,

$$s_i \xi - s_i \xi = \langle \xi, \alpha_i^\vee \rangle u$$

Here $\alpha_i^\vee \in X_*(\tilde{T})$ is the coroot corresponding to $s_i$.

We have a decomposition $\tilde{T} = G_{\text{cen}}^m \times T \times G_{\text{rot}}^m$, where $G_{\text{cen}}^m$ is the one dimensional central torus in the Kac-Moody group and $G_{\text{rot}}^m$ is the one dimensional “loop rotation” torus. Let $\delta \in X^*(G_{\text{rot}}^m)$ and $\Lambda_0 \in X^*(G_{\text{cen}}^m)$ be the generators (here we are using Kac’s notation for affine Kac-Moody groups, see [K, 6.5]).

The moduli meaning of $\text{Bun}_{G}^{\text{par}}$ gives a universal $B$-torsor on $\text{Bun}_{G}^{\text{par}}$, and induces a $T$-torsor $\mathcal{L}^i$ on $\text{Bun}_{G}^{\text{par}}$. In particular, for any $\xi \in X^*(T)$, we have a line bundle $L(\xi)$ on $\text{Bun}_{G}^{\text{par}}$ induced from $\mathcal{L}^i$ and the character $\xi$. We can view these line bundles as line bundles on $\mathcal{M}^{\text{par}}$, via the morphism $\mathcal{M}^{\text{par}} \to \text{Bun}_{G}^{\text{par}}$. We also have the determinant line bundle on $\mathcal{M}^{\text{par}}$, which is (up to a power) the pull-back of the canonical bundle $\omega_{\text{Bun}}$ of $\text{Bun}_{G}$.

**Theorem A** (See Th. 3.3.5). There is a graded algebra homomorphism

$$\mathbb{H} \to \bigoplus_{i \in \mathbb{Z}} \text{End}_{\mathcal{A}_X}^G(f_{\text{par}}^*(\mathcal{T}_t))(i)$$

extending the $\tilde{W}$-action in [Yun1, Th. 4.4.3]. The elements $\xi \in X^*(T)$, $\delta, 2h^\vee \Lambda_0$ ($h^\vee$ is the dual Coxeter number of $G$) and $u$ in $\mathbb{H}$ act as cup products with the Chern classes of $\mathcal{L}(\xi), \omega_X, \omega_{\text{Bun}}$ and $\mathcal{O}_X(D)$ respectively, where $D$ is divisor on $X$ that we used to define $\mathcal{M}^{\text{par}}$.

In particular, for any point $(a, x) \in (A \times X)(k)$, we get an action of $\mathbb{H}$ on the cohomology $H^*(\mathcal{M}^{\text{par}}_{a,x})$. It is easy to see that $u$ and $\delta$ acts trivially on $H^*(\mathcal{M}^{\text{par}}_{a,x})$. This gives geometric realizations of representations of the graded DAHA specialized at $u = \delta = 0$.

The above theorem is inspired by the results of Lusztig ([L88]) in the classical situation, where he constructed an action of the graded affine Hecke algebra on the Springer sheaf $\pi_*\mathcal{U}_t$, where $\pi: \tilde{g} \rightarrow g$ is the Grothendieck simultaneous resolution.

1.1.2. Generalizations to the parahoric versions. In classical Springer theory, many constructions for the Grothendieck simultaneous resolution can be generalized to partial Grothendieck resolutions using general parabolic subgroups, see [BM]. In the global situation, the parahoric Hitchin moduli stacks (see Def. 2.5.3) play the role of partial resolutions. The main results of [Yun1] and Th. A above can all be generalized to parahoric Hitchin stacks of arbitrary type $\mathbf{P}$. For example, Th. A generalizes to:

**Theorem B** (see Th. 4.3.2). Fix a standard parahoric subgroup $\mathbf{P} \subset G(F)$ and let $W_\mathbf{P}$ be the Weyl group of the Levi factor of $\mathbf{P}$. Let $\mathbb{H}_\mathbf{P}$ be the subalgebra of
generated by $$\mathbb{H}_P \to \bigoplus_{i \in \mathbb{Z}} \text{End}_{\mathbb{A}_X}^i(f_P, \mathcal{T}_\ell)(i)$$.}

1.1.3. Relation with the cap product action. As we saw in [Yun1, Sec. 3.2], $$\mathcal{M}^{par}$$ has another piece of symmetry, namely the fiberwise action of a Picard stack $$\mathcal{P}$$ over $$\mathcal{A}$$. This action induces an action of the homology complex $$H_1(\mathcal{P}/\mathcal{A})$$ on $$f^{par}_* \mathcal{T}_\ell$$, called the cap product action, see Sec. [1.3]. Since the homology complex $$H_1(\mathcal{P}/\mathcal{A})$$ is the exterior algebra of $$H_1(\mathcal{P}/\mathcal{A})$$ over $$H_0(\mathcal{P}/\mathcal{A})$$, the cap product action is determined by its restriction to $$H_1(\mathcal{P}/\mathcal{A})$$ and $$H_0(\mathcal{P}/\mathcal{A})$$. Also note that $$H_0(\mathcal{P}/\mathcal{A})$$ is isomorphic to $$\mathbb{T}_\ell[\pi_0(\mathcal{P}/\mathcal{A})]$$, the group algebra of the sheaf of fiberwise connected components of $$\mathcal{P} \to \mathcal{A}$$. The next result is about the interplay between the DAHA action constructed in Th. B and the cap product action by the homology of $$\mathcal{P}$$. 

**Theorem C** (see Prop. [5.1.1], Th. [5.2.5], Prop. [5.1.5] and Cor. [5.1.6] respectively).

1. The action of $$H_1(\mathcal{P}/\mathcal{A})$$ on $$f^{par}_* \mathcal{T}_\ell$$ commutes with the action of $$\mathbb{W}$$, $$u$$ and $$\delta$$.

2. The action of $$\mathbb{T}_\ell[\mathbb{X}_*(\mathcal{T})]^\mathcal{W}$$ on $$R^m f^{par}_* \mathcal{T}_\ell$$ given by restricting the $$\mathbb{W}$$-action factors through the action of $$\mathbb{T}_\ell[\pi_0(\mathcal{P}/\mathcal{A})]$$ on $$R^m f^{par}_* \mathcal{T}_\ell$$ via a natural homomorphism $$\mathbb{T}_\ell[\mathbb{X}_*(\mathcal{T})]^\mathcal{W} \to \mathbb{T}_\ell[\pi_0(\mathcal{P}/\mathcal{A})]$$.

3. For a local section $$h$$ of $$H_1(\mathcal{P}/\mathcal{A})$$, and $$\xi \in \mathbb{X}_*(\mathcal{T})$$, their actions on $$f^{par}_* \mathcal{T}_\ell$$ satisfy the commutation relation:

$$[\xi, h] = c_\xi(h_{st})$$

where $$h_{st}$$ is the stable part of $$h$$ (see Def. [1.2.1]), $$c_\xi : H_1(\mathcal{P}/\mathcal{A})_{st} \to H^*(\mathcal{A}/\mathcal{A})(1)$$ is a linear map defined in [1.2.1], and $$c_\xi(h_{st})$$ (a local section of $$H^*(\mathcal{A}/\mathcal{A})(1)$$) acts on $$f^{par}_* \mathcal{T}_\ell$$ by cap product.

4. For any point $$(a, x) \in (\mathcal{A} \times \mathbb{X})(k)$$, the cap product action of $$H_1(\mathcal{P}_a)$$ on $$H^*(\mathcal{M}^{par}_{a,x})$$ commutes with the action of the subalgebra $$\mathbb{T}_\ell[\mathbb{W}] \otimes \text{Sym}(\mathbb{X}_*(\mathcal{T})\mathcal{T}_\ell)$$ of the degenerate graded DAHA $$\mathbb{H}^*/(\delta, u)$$.

1.2. Organization of the paper and remarks on the proofs. In Sec. [2] we define parahoric versions of Hitchin moduli stacks. Many properties of $$\mathcal{M}_{\mathcal{P}}$$ parallel those of $$\mathcal{M}^{par}$$, and we only mention them without giving proofs. We also give examples of parahoric Hitchin moduli stacks in Section [2.6]. This section is used in the proof of Th. A.

In Sec. [3] we construct the graded DAHA action on the parabolic Hitchin complex (i.e., we prove Th. A). For this, we need some knowledge on the line bundles on $$\text{Bun}_2^{par}$$ and the connected components of $$\mathcal{M}^{par}$$, which we review in Sec. [3.2] and Sec. [3.3]. The proof of the relation [1.1] in the DAHA is essentially a calculation of the equivariant cohomology of the Steinberg variety for $$\text{SL}(2)$$, which we carry out in Sec. [3.6].

In Sec. [3] we generalize the main results in [Yun1] and the graded DAHA action to parahoric Hitchin stacks. In particular, we get an action of $$\mathbb{T}_\ell[\mathbb{X}_*(\mathcal{T})]^\mathcal{W}$$ on the usual Hitchin complex $$f^{Hua}_* \mathcal{T}_\ell \otimes \mathcal{T}_\ell$$, which can be viewed as t'Hooft operators in the context of constructible sheaves.
In Sec. 5 we study the relation between the cap product action of $H_*(\mathcal{P}/\mathcal{A})$ on $H_*(\mathcal{Q})$ and the graded DAHA action constructed in the Th. A. The proof of Th. C(2) uses the idea of deforming the product of the affine Grassmannian $Gr_G$ and the usual flag variety $\mathcal{B}$ into the affine flag variety $F\ell_G$, which first appeared in Gaitsgory’s work [G].

In App. A we review the notion of the Pontryagin product and the cap product, which is used in Sec. 5.

In App. B, we prove lemmas concerning the relation between cohomological correspondences and cup/cap products.

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2. Parahoric versions of the Hitchin moduli stack

In this section, we generalize the notion of Hitchin stacks to arbitrary parahoric level structures, not just the Iwahori level structure considered in [YunI, Sec. 3]. Throughout this section, let $F = k((t))$ be the field of formal Laurent series over $k$, and let $O_F = k[[t]]$ be its valuation ring. The reductive group $G$ over $k$ determines a split group scheme $G = G \otimes_{\text{Spec } k} \text{Spec } O_F$ over $O_F$.

Technically speaking, this section is only used in the proof of Th. 3.3.3, but many results about $\mathcal{M}^{par}$ also have their counterparts for parahoric Hitchin stacks, as we will see in Sec. 4.

2.1. Local coordinates. Parahoric subgroups are local notions. In order to make sense of them over a global curve, we have to deal with parahoric subgroups in a “Virasoro-equivariant” way. For this, we need to consider local coordinates on the curve $X$.

2.1.1. The group of coordinate changes. We follow [G] 2.1.2 in the following discussion. Let $\text{Aut}_R$ be the pro-algebraic group of automorphisms of the topological ring $O_F$. More precisely, for any $k$-algebra $R$, $\text{Aut}_R(R)$ is the set of $R$-linear continuous automorphisms of the topological ring $R[[t]] = R \otimes_k O_F$ (with $t$-adic topology). If we define $\text{Aut}_R, n$ to be the algebraic group of automorphisms of $O_F / t^{n+1} O_F$, then $\text{Aut}_R$ is the projective limit of $\text{Aut}_R, n$.

2.1.2. The space of local coordinates. We have a canonical $\text{Aut}_O$-torsor $\text{Coor}(X)$ over $X$, called the space of local coordinates of $X$, defined as follows. For any $k$-algebra $R$, the set $\text{Coor}(X)(R)$ consists of pairs $(x, \alpha)$ where $x \in X(R)$ and $\alpha$ is an $R$-linear continuous isomorphism $\alpha : R[[t]] \rightarrow \hat{O}_x$ (here $\hat{O}_x$ is the completion of $O_{X_R}$ along the graph $\Gamma(x)$, see [YunI] 2.2]) for notations). An element $\sigma \in \text{Aut}_O(R) = \text{Aut}(R[[t]])$ acts on $\text{Coor}(X)(R)$ from the right by

$$(x, \alpha) \cdot \sigma = (x, \alpha \circ \sigma),$$

hence realizing the forgetful morphism $\text{Coor}(X) \rightarrow X$ as a right $\text{Aut}_O$-torsor.

The following fact is well-known.

2.1.3. Lemma. Consider the $\text{Spf } O_F$-bundle $\text{Coor}(X) \times^{\text{Aut}_O} \text{Spf}(O_F)$ associated to the $\text{Aut}_O$-torsor $\text{Coor}(X)$ and the tautological action of $\text{Aut}_O$ on $\text{Spf } O_F$. We have
a natural isomorphism over $X$

$$\text{Coor}(X)^{\times} \times \text{Spf} \mathcal{O}_F \cong \hat{X}^2_\Delta,$$

where the RHS is the formal completion of $X^2$ along the diagonal $\Delta(X) \subset X^2$, viewed as a formal $X$-scheme via the projection to the first factor.

The pro-algebraic group $\text{Aut}_\mathcal{O}$ has the Levi quotient $\mathbb{G}_m$ given by

(2.1) $\text{Aut}_\mathcal{O} \rightarrow \mathbb{G}_m$

$$\sigma \mapsto \sigma(t)/t \text{ mod } t.$$

We call this quotient $\mathbb{G}_m$ the rotation torus, and denote it by $\mathbb{G}_m^{\text{rot}}$. Since we have fixed a uniformizing parameter $t$ of $F$, the quotient map $\text{Aut}_\mathcal{O} \rightarrow \mathbb{G}_m^{\text{rot}}$ admits a section under which $\lambda \in \mathbb{G}_m^{\text{rot}}(k)$ is the automorphism of $\mathcal{O}_F$ given by: $a(t) \mapsto a(\lambda t)$ (for $a(t) \in k[[t]] = \mathcal{O}_F$). We shall also identify $\mathbb{G}_m^{\text{rot}}$ with the subgroup of $\text{Aut}_\mathcal{O}$ given by the image of this section.

From Lem. 2.1.3, we immediately get

2.1.4. Corollary. The $\mathbb{G}_m^{\text{rot}}$-torsor associated to the $\text{Aut}_\mathcal{O}$-torsor $\text{Coor}(X)$ is naturally isomorphic to the $\mathbb{G}_m^{\text{rot}}$-torsor associated to the canonical bundle of $X$, i.e.,

$$\text{Coor}(X)^{\times} \times \mathbb{G}_m^{\text{rot}} \sim \rho_{\omega_X}.$$ (See [Yun] Sec. 2.2 for notations such as $\rho_{\omega_X}$.)

2.2. Parahoric subgroups. The purpose of this subsection is to fix some notations concerning the parahoric subgroups of $G(F)$. Since $G$ has an $\mathcal{O}_F$-model $G$, the group $\text{Aut}(\mathcal{O}_F)$ acts on the (semisimple) Bruhat-Tits building $\mathfrak{B}(G, F)$ of $G(F)$ in a simplicial way, and hence on the set of parahoric subgroups of $G(F)$. We have also fixed a Borel subgroup $B \subset G$, which gives rise to an Iwahori subgroup $I \subset G$. Any parahoric subgroup $P$ containing $I$ is called a standard parahoric subgroup of $G(F)$.

2.2.1. Remark. We make a slight digression on the fixed point set of $\text{Aut}(\mathcal{O}_F)$ on $\mathfrak{B}(G, F)$. The maximal facets that are fixed by $\text{Aut}(\mathcal{O}_F)$ are in bijection with the fixed points of the $\text{Aut}(\mathcal{O}_F)$-action on the $k$-points of the affine flag variety $\mathcal{F}_G = G(F)/I$, which are given by $G(k)\bar{w}I/I$, for $\bar{w} \in \hat{W}$. In other words, the fixed point locus of the $\text{Aut}(\mathcal{O}_F)$ on $\mathfrak{B}(G, F)$ is the $G(k)$-orbit of the standard apartment (given by $T(F) \subset G(F)$, for some maximal torus $T \subset G$ over $k$). In particular, a facet in $\mathfrak{B}(G, F)$ is stable under $\text{Aut}(\mathcal{O}_F)$ if and only if it is pointwise fixed by $\text{Aut}(\mathcal{O}_F)$. Hence, a parahoric subgroup $P \subset G(F)$ is stable under $\text{Aut}(\mathcal{O}_F)$ if and only if its facet $\mathfrak{g}_P \subset \mathfrak{B}(G, F)$ is pointwise fixed by $\text{Aut}(\mathcal{O}_F)$. In particular, all standard parahoric subgroups are stable under $\text{Aut}(\mathcal{O}_F)$.

Let $P \subset G(F)$ be a parahoric subgroup. By Bruhat-Tits theory, $P$ determines a smooth group scheme over $\text{Spec} \mathcal{O}_F$ with generic fiber $G \otimes_k F$ and whose set of $\mathcal{O}_F$-points is equal to $P$ ([11 3.4.1]). We still denote this $\mathcal{O}_F$-group scheme by $P$. Let $\mathfrak{g}_P$ be the Lie algebra of $P$, which is a free $\mathcal{O}_F$-module of rank $\dim_k G$. Let $L_P$ be the Levi quotient of $P$, which is a connected reductive group over $k$. Let $P$ be the stabilizer of $\mathfrak{g}_P$ under $G(F)$ (equivalently, $P$ is the normalizer of $P$ in $G(F)$). Let $\omega_P$ be the finite group $P/P$. 

Let $G((t)) = \text{Res}_{E/k}(G \otimes_k F)$ and $G_P = \text{Res}_{O_F/k} P$ be the (ind-)k-groups obtained by Weil restrictions. We call $G((t))$ the loop group of $G$. For $P = G$, we write $G[[t]]$ instead of $G_G$.

Let $P \subset G(F)$ be a parahoric subgroup stabilized by $\text{Aut}(O_F)$, then $\text{Aut}(O_F)$ naturally acts on the group scheme $P$, lifting its action on $O_F$. Moreover generally, for any $k$-algebra $R$, the group $\text{Aut}_O(R)$ acts on $G_P(R) = P(R[[t]])$ and $G((t))(R) = G(R((t)))$, giving an action of the pro-algebraic group $\text{Aut}_O$ on the group scheme $G_P$ and group ind-scheme $G((t))$. Since $\text{Aut}_O$ acts on $G_P$, it also acts on the Levi quotient $L_P$, and the action necessarily factors through a finite-dimensional quotient of $\text{Aut}_O$. We can form the twisted product

$$L_P := \text{Coor}(X) \overset{\text{Aut}_O}{\times} L_P$$

which is a reductive group scheme over $X$ with geometric fibers isomorphic to $L_P$. Let $\mathfrak{p}$ be the Lie algebra of $L_P$, and let $\mathfrak{l}_P$ be the Lie algebra of $L_P$, which is the vector bundle $\text{Coor}(X) \overset{\text{Aut}_O}{\times} \mathfrak{l}_P$ over $X$.

Let $P$ be a standard parahoric subgroup. Note that the Borel $B$ gives a Borel subgroup $B^P \subset L_P$ whose quotient torus is canonically isomorphic to $T$. Let $W_P$ be the Weyl group of $L_P$ determined by $B^P$ and $T$. Then $W_P$ is naturally a subgroup of $\tilde{W}$. In fact, any maximal torus in $B$ gives an apartment $\mathfrak{a}$ in $\mathfrak{B}(G, F)$, on which $\tilde{W}$ acts by affine transformations. The Weyl group $W_P$ can be identified with the subgroup of $\tilde{W}$ which fixes $\mathfrak{a}_P$ pointwise. The resulting subgroup $W_P \subset \tilde{W}$ is independent of the choice of the maximal torus in $B$.

### 2.3. Bundles with parahoric level structures.

#### 2.3.1. Definition. Let $\text{Bun}_\infty : k - \text{Alg} \to \text{Groupoids}$ be the fpqc sheaf associated to the following presheaf $\text{Bun}_\infty^{\text{pre}} :$ for any $k$-algebra $R$, $\text{Bun}_\infty(R)$ is the groupoid of quadruples $(x, \alpha, \mathcal{E}, \tau_x)$ where

- $x \in X(R)$ with graph $\Gamma(x) \subset X_R$;
- $\alpha : R[[t]] \cong \tilde{O}_x$ a local coordinate;
- $\mathcal{E}$ is a $G$-torsor over $X_R$;
- $\tau_x : G \times \mathcal{D}_x \cong \mathcal{E}|_{\mathcal{D}_x}$ is a trivialization of the restriction of $\mathcal{E}$ to $\mathcal{D}_x = \text{Spec} \tilde{O}_x$.

In other words, $\text{Bun}_\infty$ parametrizes $G$-bundles on $X$ with a full level structure at a point of $X$, and a choice of local coordinate at that point.

#### 2.3.2. Construction. Consider the semi-direct product $G((t)) \times \text{Aut}_O$ formed using the action of $\text{Aut}_O$ on $G((t))$ defined in Sec. 2.2. We claim that this group ind-scheme naturally acts on $\text{Bun}_\infty$ from the right. In fact, for any $k$-algebra $R$, $g \in G(R((t))), \sigma \in \text{Aut}(R[[t]])$ and $(x, \alpha, \mathcal{E}, \tau_x) \in \text{Bun}_\infty(R)$, let

$$R_{g, \sigma}(x, \alpha, \mathcal{E}, \tau_x) = (x, \alpha \circ \sigma, \mathcal{E}^g, \tau_x^g).$$

Let us explain the notations. By a variant of the main result of [BL] (using Tanakian formalism to reduce to the case of vector bundles), to give a $G$-torsor on $X_R$ is the same as to give $G$-torsors on $X_R - \Gamma(x)$ and on $\mathcal{D}_x$ respectively, together with a $G$-isomorphism between their restrictions to $\mathcal{D}_x$. Now let $\mathcal{E}^g$ be the
G-torsor on $X_R$ obtained by gluing $E|_{X_R−Γ(x)}$ with the trivial G-torsor $G × D_x$ via the isomorphism

\[ G × D_x^× \xrightarrow{α_∗^{-1} g α_*} G × D_x^× \xrightarrow{τ_x} E|_{D_x^×} \]

Here $α_* : D_x^× → \text{Spec} R((t))$ is induced by $α$, hence the first arrow is the transport of the left multiplication by $g$ on $G × \text{Spec} R((t))$ to $G × D_x^×$ via the local coordinate $α$. Since left multiplication by $g ∈ G(R((t)))$ is an automorphism of the trivial right $G$-torsor on $\text{Spec} R((t))$, $α_∗^{-1} g α_*$ is an automorphism of the trivial right $G$-torsor on $D_x^×$. The trivialization $τ_0^x$ is tautologically given by the construction of $E^g$.

Let $P$ be a parahoric subgroup of $G(F)$ which is stable under $\text{Aut}(O_F)$.

2.3.3. Definition. The moduli stack $\text{Bun}_P$ of $G$-bundles over $X$ with parahoric level structures of type $P$ is the fpqc sheaf associated to the quotient presheaf $R → \text{Bun}_∞(R)/(G_P × \text{Aut}_F)(R)$.

We will use the notation $(x, E, τ_x \mod P)$ to denote the point in $\text{Bun}_P$ which is the image of $(x, α, E, τ_x) ∈ \text{Bun}_∞(R)$.

2.3.4. Remark. Instead of taking quotients of $\text{Bun}_∞$, we could also define $\text{Bun}_P$ as the quotient of $\text{Bun}_∞$ by the group $\text{Coor}(X) × G_P$, where $\text{Bun}_∞ = \text{Bun}_∞/\text{Aut}_F$ is the moduli stack of $G$-torsors on $X$ with a full level structure at point of $X$.

We look at several special cases of the above construction. The first case is $P = G = G(O_F)$. In this case we have:

2.3.5. Lemma. There is a canonical isomorphism $\text{Bun}_G \cong \text{Bun}_G × X$, where $\text{Bun}_G$ is the usual moduli stack of $G$-torsors over $X$.

Proof. In other words, we need to show that the forgetful morphism $\text{Bun}_∞ → \text{Bun}_G × X$ is a $G[[t]] × \text{Aut}_G$-torsor. The only not-so-obvious part is the essential surjectivity, i.e., for any $k$-algebra $R$ and any $(x, E) ∈ X(R) × \text{Bun}_G(R)$, we have to find a trivialization of $E|_{D_x}$ locally in the flat or étale topology of $\text{Spec} R$. Since $E|_{Γ(x)}$ is a $G$-torsor, by definition, there is an étale covering $\text{Spec} R' → \text{Spec} R$ which trivializes $E|_{Γ(x)}$, i.e., there is a section $τ_0^x : \text{Spec} R' → E|_{D_x}$ over $\text{Spec} R'$ → $\text{Spec} R = Γ(x) ⊂ D_x$. Since $E|_{D_x}$ is smooth over $D_x$, the section $τ_0^x$ extends to a section $τ' : \text{Spec}(R' ⊗_R O_D) → E|_{D_x}$. In other words, after pulling back to the étale covering $\text{Spec} R' → \text{Spec} R$, $E|_{D_x}$ can be trivialized.

The second case is when $P ⊂ G(O_F)$. Such $P$ are in 1-1 correspondence with parabolic subgroups $P ⊂ G$ over $k$. In this case, using Lem. 2.3.5, it is easy to see that $\text{Bun}_P$ is the moduli stack of $G$-torsors on $X$ with a parabolic reduction of type $P$ at a point of $X$. More precisely, $\text{Bun}_P(R)$ classifies tuples $(x, E, E^P)$, where

- $x ∈ X(R)$ with graph $Γ(x)$;
- $E$ is a $G$-torsor over $X_R$;
- $E^P$ is a $P$-reduction of the $G$-torsor $E|_{Γ(x)}$ over $Γ(x)$.

In particular, if $P = I$, the standard Iwahori subgroup, $\text{Bun}_I$ is what we denoted by $\text{Bun}_G^{\text{par}}$ in [YunI, Sec. 3].
2.4. Properties and examples of $\text{Bun}_P$. We first explore the dependence of $\text{Bun}_P$ on the choice of $P$.

2.4.1. Lemma. There is a natural right action of $\Omega_P = \tilde{P}/P$ on $\text{Bun}_P$. Moreover, suppose two parahoric subgroups $P$ and $Q$ are conjugate under $G(F)$ and are both stable under $\text{Aut}(O_F)$, then there is an isomorphism $\text{Bun}_P \xrightarrow{\sim} \text{Bun}_Q$ which is canonical up to pre-composition with the $\Omega_P$-action on $\text{Bun}_P$ (or up to post-composition with the $\Omega_Q$-action on $\text{Bun}_Q$).

Proof. Let $g \in G(F)$ be such that $Q = g^{-1}Pg$. Let $\text{Bun}_P = \tilde{\text{Bun}}_{\infty}/G_P$, which is an $\text{Aut}_F$-torsor over $\text{Bun}_P$. Since $g^{-1}P = Q$, the natural right action of $g$ on $\text{Bun}_{\infty}$ (see Construction 2.3.2) descends to an isomorphism $R_g : \text{Bun}_P \xrightarrow{\sim} \text{Bun}_Q$. Moreover, for any $\sigma \in \text{Aut}(R[[t]])$, we have a commutative diagram

$$
\begin{array}{c}
\text{Bun}_P(R) \\
\downarrow \sigma
\end{array}
\xrightarrow{R_g}
\begin{array}{c}
\text{Bun}_Q(R) \\
\downarrow \sigma
\end{array}
\tag{2.4}
$$

Since both $P$ and $Q$ are stable under $\sigma$, $\sigma(g)g^{-1}$ normalizes $P$, and hence $\sigma(g)g^{-1} \in \tilde{P}(R)$. The assignment $\sigma \mapsto \sigma(g)g^{-1}$ gives a morphism from the connected pro-algebraic group $\text{Aut}_F$ to the discrete group $\tilde{P}/P$, which must be trivial. Therefore $\sigma(g)g^{-1} \in P(R)$. It is clear that right multiplication by any element in $P$ induces the identity morphism on $\text{Bun}_P$, therefore

$$R_{\sigma(g)} = R_g \circ R_{\sigma(g)g^{-1}} = R_g : \text{Bun}_P(R) \xrightarrow{\sim} \text{Bun}_Q(R).$$

Using diagram (2.4), we conclude that $R_g : \text{Bun}_P(R) \xrightarrow{\sim} \text{Bun}_Q(R)$ is equivariant under $\text{Aut}_F(R)$, hence descends to an isomorphism

$$R_g : \text{Bun}_P \xrightarrow{\sim} \text{Bun}_Q. \tag{2.5}$$

Finally we define the $\Omega_P$-action on $\text{Bun}_P$, and check that the isomorphism (2.4.1) is canonical up to this action. If $g'$ is another element of $G(F)$ such that $Q = g'^{-1}Pg'$, then $g'g^{-1}$ normalizes $P$, hence $g'g^{-1} \in \tilde{P}$. We can write the isomorphism $R_{g'}$ as the composition:

$$\text{Bun}_P \xrightarrow{R_{g'}g^{-1}} \text{Bun}_P \xrightarrow{R_g} \text{Bun}_Q,$$

where the first isomorphism only depends on the image of $g'g^{-1}$ in $\Omega_P = \tilde{P}/P$. Taking $Q = P$, we get the desired $\Omega_P$-action on $\text{Bun}_P$. In general, the isomorphisms between $\text{Bun}_P$ and $\text{Bun}_Q$ given by the various $R_g$ only differ by the action of $\Omega_P$ on $\text{Bun}_P$. \hfill \Box

For parahoric subgroups $P, Q$, let

$$\Omega_{P,Q} := \{ [g] \in P \setminus G(F)/Q | g^{-1}Pg = Q \}.$$

For $g \in G(F)$ such that $g^{-1}Pg = Q$, let $[g] \in \Omega_{P,Q}$ be the corresponding double coset. In particular, $\Omega_P = \Omega_{P,P}$. If $P$ and $Q$ are both stable under $\text{Aut}(O_F)$, the proof of Lem. 2.4.1 gives for each $[g] \in \Omega_{P,Q}$ a canonical isomorphism

$$R_{[g]} : \text{Bun}_P \rightarrow \text{Bun}_Q. \tag{2.6}$$
Suppose we have an inclusion $P \subset Q$ of parahoric subgroups which are both stable under $\text{Aut}(O_F)$, then by construction we have a forgetful morphism

\[
\text{For}^Q_P : \text{Bun}_P \to \text{Bun}_Q
\]

whose fibers are isomorphic to $G_Q/G_P$, which is a partial flag variety of the reductive group $L_Q$. In particular, $\text{For}^Q_P$ is representable, proper, smooth and surjective.

2.4.2. **Corollary.** For any parahoric subgroup $P \subset G(F)$ stable under $\text{Aut}(O_F)$, the stack $\text{Bun}_P$ is an algebraic stack locally of finite type.

**Proof.** By Lem. 2.4.1, we only need to check the statement for standard parahoric subgroups. Since the morphism $\text{For}^I_G : \text{Bun}_I \to \text{Bun}_G \cong \text{Bun}_G \times X$ is representable and of finite type, and $\text{Bun}_G$ is an algebraic stack locally of finite type, $\text{Bun}_I$ is also algebraic and locally of finite type. On the other hand, since $\text{For}^P_I : \text{Bun}_I \to \text{Bun}_P$ is representable, smooth and surjective, $\text{Bun}_P$ is also algebraic and locally of finite type. \(\square\)

We describe examples of $\text{Bun}_P$ for classical groups of type A, B and C.

2.4.3. **Example.** Let $G = \text{SL}(n)$. The standard parahoric subgroups are in 1-1 correspondence with sequences of integers

\[
i = (0 \leq i_0 < \cdots < i_m < n), m \geq 0
\]

For each such sequence $i$, let $P_i$ be the corresponding parahoric subgroup. Then $\text{Bun}_{P_i}$ classifies

\[
(x, \mathcal{E}_{i_0} \supset \mathcal{E}_{i_1} \supset \cdots \supset \mathcal{E}_{i_m} \supset \mathcal{E}_{i_0}(-x), \delta)
\]

where $x \in X$, $\mathcal{E}_{i_j}$ are vector bundles of rank $n$ on $X$ such that $\mathcal{E}_{i_0}/\mathcal{E}_{i_j}$ has length $i_j - i_0$ for $j = 0, 1, \cdots, m$, and $\delta$ is an isomorphism $\text{det}(\mathcal{E}_{i_0}) \cong O_X(-i_0x)$.

2.4.4. **Example.** Let $G = \text{SO}(2n+1)$ (resp. $G = \text{Sp}(2n)$). The standard parahoric subgroups are in 1-1 correspondence with sequences of integers

\[
i = (0 \leq i_0 < \cdots < i_m \leq n), m \geq 0.
\]

For each such sequence $i$, let $P_i$ be the corresponding parahoric subgroup. Then $\text{Bun}_{P_i}$ classifies

\[
(x, \mathcal{E}_{i_0} \supset \cdots \supset \mathcal{E}_{i_m} \supset \mathcal{E}_{i_0}^\perp(-x) \supset \cdots \supset \mathcal{E}_{i_0}^\perp(-x) \supset \mathcal{E}_{i_0}(-x), \sigma)
\]

where $x \in X$, $\mathcal{E}_{i_j}$ are vector bundles of rank $2n+1$ (resp. $2n$) on $X$ such that $\mathcal{E}_{i_0}/\mathcal{E}_{i_j}$ has length $i_j - i_0$ for $j = 0, 1, \cdots, m$, and $\sigma$ is a symmetric (resp. alternating) pairing $\sigma : \mathcal{E}_{i_0} \otimes \mathcal{E}_{i_0} \to O_X$.

For any subsheaf $\mathcal{E} \subset \mathcal{E}_{i_0}$ of finite colength, $\mathcal{E}^\perp$ denotes the subsheaf of the sheaf of rational sections $f$ of $\mathcal{E}$ such that $\sigma(f, \mathcal{E}) \subset O_X$, which is also a vector bundle over $X$.

2.5. **The parahoric Hitchin fibrations.** In this subsection, we define parahoric analogues of $M^{\text{par}}$ and $f^{\text{par}}$ considered in [YunI, Sec. 3.1]. These are analogues of the partial Grothendieck resolutions in the classical Springer theory, cf. [BM].

We first define the notion of Higgs fields in the parahoric situation.
2.5.1. **Construction** (the Higgs fields). For any $k$-algebra $R$ and $(x, \alpha, \mathcal{E}, \tau_x) \in \text{Bun}_\infty(R)$, consider the composition

$$j_* j^* \text{Ad}(\mathcal{E}) \to \text{Ad}(\mathcal{E}) \otimes \tilde{O}_x \overset{\tau_x^{-1}}{\to} g \otimes_k \tilde{O}_x \overset{\alpha^{-1}}{\to} g \otimes_k R((t))$$

where $j : X_R - \Gamma(x) \hookrightarrow X_R$ is the inclusion and the first arrow is the natural embedding. Let $\text{Ad}_P(\mathcal{E})$ be the preimage of $g_P \otimes_k R \subset g \otimes_k R((t))$ under the injection \(2.8\). Sheafifying this procedure, the assignment $(x, \mathcal{E}, \tau_x \mod P) \mapsto \text{Ad}_P(\mathcal{E})$ gives a quasi-coherent sheaf $\text{Ad}_P$ on $\text{Bun}_\infty \times X$.

It is easy to check that

2.5.2. **Lemma.**

1. The quasi-coherent sheaf $\text{Ad}_P$ descends to $\text{Bun}_P \times X$;
2. The quasi-coherent sheaf $\text{Ad}_P$ over $\text{Bun}_P \times X$ is in fact coherent;
3. If both $P$ and $Q$ are stable under $\text{Aut}(O_P)$ and $[g] \in \Omega_P Q$, then there is a canonical isomorphism $R^*_[g] \text{Ad}_Q \cong \text{Ad}_P$ satisfying the obvious transitivity conditions. (Recall the isomorphism $R^*_[g]$ from (2.7)). In particular, $\text{Ad}_P$ has a natural $\Omega_P$-equivariant structure.

**Proof.** (1) Since $g_P$ is stable under $G_P \times \text{Aut}(O_P)$, the subsheaf $\text{Ad}_P(\mathcal{E}) \subset j_* j^* \text{Ad}(\mathcal{E})$ only depends on the image of $(x, \alpha, \mathcal{E}, \tau_x)$ in $\text{Bun}_P(R)$.

(3) Similar to the proof of Lem. 2.4.1

(2) Fix any $k$-algebra $R$ and $(x, \mathcal{E}, \tau_x \mod P) \in \text{Bun}_P(R)$, we want to show that $\text{Ad}_P(\mathcal{E})$ is a coherent sheaf on $X_R$. For $P = G$, clearly $\text{Ad}_G(\mathcal{E}) = \text{Ad}(\mathcal{E})$ is coherent on $X_R$. For $P = I$ and any point $(x, \mathcal{E}, \mathcal{E}_x^P) \in \text{Bun}_I(R)$ (recall $\mathcal{E}_x^P$ is a $B$-reduction of $\mathcal{E}|_{\Gamma(x)}$), we have an exact sequence

$$0 \to \text{Ad}_I(\mathcal{E}) \to \text{Ad}(\mathcal{E}) \to i_* \left(\text{Ad}(\mathcal{E}|_{\Gamma(x)})/\text{Ad}(\mathcal{E}_x^B)\right) \to 0$$

where $i : \Gamma(x) \hookrightarrow X_R$ is the closed inclusion. Since the middle and final terms of the above exact sequence are coherent, $\text{Ad}_I(\mathcal{E})$ is also coherent.

In general, by (3), we can reduce to the case $P \supset I$. In this case we have an embedding $\text{Ad}_I(\mathcal{E}) \hookrightarrow \text{Ad}_P(\mathcal{E})$ whose cokernel is again a finite $R$-module supported on $\Gamma(x)$, hence $\text{Ad}_P(\mathcal{E})$ is a coherent sheaf on $X_R$. \(\square\)

As in the usual definition of the Hitchin moduli stack, we fix a divisor $D$ on $X$ with $\text{deg}(D) \geq 2g_X$.

2.5.3. **Definition.** The Hitchin moduli stack of $G$-bundles over $X$ (with respect to $D$) with parahoric level structures of type $P$ (parahoric Hitchin moduli stack of type $P$ for short) is the fpqc sheaf $\mathcal{M}_P : k - \text{Alg} \to \text{Groupoids}$ which associates to every $k$-algebra $R$ the groupoid of pairs $(\xi, \varphi)$ where

- $\xi = (x, \mathcal{E}, \tau_x \mod P) \in \text{Bun}_P(R)$;
- $\varphi \in H^0(X_R, \text{Ad}_P(\mathcal{E}) \otimes \mathcal{O}_X(D))$.

2.5.4. **Remark.** By Lem. 2.5.2, the group $\Omega_P$ naturally acts on the parahoric Hitchin moduli stack $\mathcal{M}_P$, and $\mathcal{M}_P$ only depends on the the conjugacy class of $P$ up to this action of $\Omega_P$. Therefore we can concentrate on the study of $\mathcal{M}_P$ for standard parahoric subgroups $P \supset I$.

2.5.5. **Lemma.** The Hitchin moduli stack $\mathcal{M}_P$ is an algebraic stack locally of finite type.
Proof. Consider the forgetful morphism \( \mathcal{M}_\mathbf{P} \to \text{Bun}_\mathbf{P} \). The fiber of this morphism over a point \((x, \mathcal{E}, \tau_x \mod \mathbf{P}) \in \text{Bun}_\mathbf{P}(R)\) is the finite \(R\)-module \(H^0(X_R, \text{Ad}_P(\mathcal{E}))(D)\) (the finiteness follows from the coherence of \(\text{Ad}_P(\mathcal{E})\) as proved in Lem. 2.5.2) and the properness of \(X\). Therefore, the forgetful morphism \( \mathcal{M}_\mathbf{P} \to \text{Bun}_\mathbf{P} \) is representable and of finite type. By Cor. 2.4.2, \( \text{Bun}_\mathbf{P} \) is algebraic and locally of finite type, hence so is \( \mathcal{M}_\mathbf{P} \).

2.5.6. **Construction.** We claim that there is a natural morphism

\[
\text{ev}_P : \mathcal{M}_\mathbf{P} \to [\mathfrak{l}_\mathbf{P}/\mathfrak{D}_\mathbf{P}]_D
\]

of “evaluating the Higgs fields at the point of the \( \mathbf{P} \)-level structure”. In fact, to construct \( \text{ev}_P \), it suffices to construct a morphism

\[
\bar{\text{ev}}_P : \text{Bun}_\infty \times_{\text{Bun}_\mathbf{P}} \mathcal{M}_\mathbf{P} \to \mathbb{G}_m \times \rho_D
\]

which is equivariant under \( G_\mathbf{P} \times \text{Aut}_\mathcal{O} \). Here the \( G_\mathbf{P} \times \text{Aut}_\mathcal{O} \)-action on the LHS is on the \( \text{Bun}_\infty \)-factor, and the action on the RHS factors through the \( L_\mathbf{P} \times \text{Aut}_\mathcal{O} \)-action on \( \mathfrak{l}_\mathbf{P} \), with \( L_\mathbf{P} \) acting by conjugation.

For any \( k \)-algebra \( R \) and \((x, \alpha, \mathcal{E}, \tau_x, \varphi) \in (\text{Bun}_\infty \times_{\text{Bun}_\mathbf{P}} \mathcal{M}_\mathbf{P})(R)\), by the definition of \( \text{Ad}_P(\mathcal{E}) \), the maps in (2.8) give

\[
\text{Ad}_P(\mathcal{E}) \to \mathfrak{g}_P \otimes_k R \to \mathfrak{l}_\mathbf{P} \otimes_k R.
\]

Twisting by \( \mathcal{O}_X(D) \), we get

\[
\bar{\text{ev}}_{P,x} : H^0(X_R, \text{Ad}_P(\mathcal{E}))(D) \to \mathfrak{l}_\mathbf{P} \otimes_k x^*\mathcal{O}_X(D).
\]

The assignment \((x, \alpha, \mathcal{E}, \tau_x, \varphi) \mapsto \bar{\text{ev}}_{P,x}(\varphi)\) gives the desired morphism \( \bar{\text{ev}}_P \). It is easy to check that \( \bar{\text{ev}}_P \) is equivariant under \( G_\mathbf{P} \times \text{Aut}_\mathcal{O} \), hence giving the desired morphism \( \text{ev}_P \) in (2.9).

2.5.7. **Morphisms between two parahoric Hitchin stacks.** For two standard parahoric subgroups \( \mathbf{P} \subset \mathbf{Q} \), there is a unique parabolic subgroup \( B^\mathbf{Q}_\mathbf{P} \subset L_\mathbf{Q} \), such that \( \mathbf{P} \) is the inverse image of \( B^\mathbf{Q}_\mathbf{P} \) under the natural quotient \( \mathbf{Q} \to L_\mathbf{Q} \). There is a canonical \( \text{Aut}_\mathcal{O} \)-action on \( B^\mathbf{Q}_\mathbf{P} \) making the embedding \( B^\mathbf{Q}_\mathbf{P} \hookrightarrow L_\mathbf{Q} \) equivariant under \( \text{Aut}_\mathcal{O} \). Let \( \mathfrak{b}^\mathbf{Q}_\mathbf{P} \) be the Lie algebra of \( B^\mathbf{Q}_\mathbf{P} \) and let \( \mathfrak{b}^\mathbf{Q}_\mathbf{P}, \mathfrak{L}^\mathbf{Q}_\mathbf{P} \) be the group scheme and \( L_\mathbf{Q} \)-Lie algebra over \( X \) obtained by applying \( \text{Coo}_{\mathcal{O}}(X)^{\times} \times (-) \).

Since \( B^\mathbf{Q}_\mathbf{P} \) is a quotient of \( \mathbf{P} \), the same construction as in Construction 2.5.6 gives the relative evaluation map

\[
\text{ev}^\mathbf{Q}_P : \mathcal{M}_\mathbf{P} \to [\mathfrak{b}^\mathbf{Q}_\mathbf{P}/\mathfrak{D}^\mathbf{Q}_\mathbf{P}]_D.
\]

Similar to the morphism \( \text{For}^\mathbf{Q}_P : \text{Bun}_\mathbf{P} \to \text{Bun}_\mathbf{Q} \) in (2.7), there is a morphism

\[
\text{For}^\mathbf{Q}_P : \mathcal{M}_\mathbf{P} \to \mathcal{M}_\mathbf{Q}
\]

lifting \( \text{For}^\mathbf{Q}_P \).

2.5.8. **Lemma.** We have a Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{M}_\mathbf{P} & \xrightarrow{\text{ev}^\mathbf{Q}_P} & [\mathfrak{b}^\mathbf{Q}_\mathbf{P}/\mathfrak{D}^\mathbf{Q}_\mathbf{P}]_D \\
\downarrow \text{For}^\mathbf{Q}_P & & \downarrow \text{For}^\mathbf{Q}_P \\
\mathcal{M}_\mathbf{Q} & \xrightarrow{\text{ev}_\mathbf{Q}} & [\mathfrak{l}_\mathbf{Q}/\mathfrak{L}_\mathbf{Q}]_D
\end{array}
\]
In particular, the morphism $\tilde{\text{For}}_Q^\mathcal{P}$ is proper and surjective.

Proof. The first statement follows by tracing down the contructions of the evaluation maps. The morphism $s_\mathcal{P}^Q$ is (locally on $X$) the partial Grothendieck resolution associated to the parabolic subgroup $B_\mathcal{P}^Q$ of $L_Q$, hence proper and surjective. Therefore $\tilde{\text{For}}_Q^\mathcal{P}$ is also proper and surjective. □

Now we define the parahoric Hitchin fibrations. Recall from [YunI Sec. 3.1] that we have the usual Hitchin base space $A^\text{Hit} = H^0(X, \mathfrak{c}_D)$. For any $k$-algebra $R$ and $(x, \mathcal{E}, \tau_x \mod \mathcal{P}) \in \text{Bun}_R(R)$, the natural map of taking invariants $\text{Ad}(\mathcal{E})(D) \to \mathfrak{c}_D$ gives a map

$$\chi_{\mathcal{P}, \mathcal{E}} : H^0(X_R, \text{Ad}_{\mathcal{P}}(\mathcal{E})(D)) \to H^0(X_R - \Gamma(x), \text{Ad}(\mathcal{E})(D))$$

$$H^0(X_R - \Gamma(x), \mathfrak{c}_D).$$

2.5.9. Lemma. The image of the map $\chi_{\mathcal{P}, \mathcal{E}}$ lands in $H^0(X_R, \mathfrak{c}_D)$, hence giving a morphism

$$f_\mathcal{P} : M_\mathcal{P} \to A^\text{Hit} \times X.$$ 

Proof. For $\mathcal{P} \subset G$, $\text{Ad}_{\mathcal{P}}(\mathcal{E}) \subset \text{Ad}(\mathcal{E})$, hence the image of $\chi_{\mathcal{P}, \mathcal{E}}$ obvious lands in $H^0(X_R, \mathfrak{c}_D)$. In particular, the statement holds for $\mathcal{P} = \mathcal{I}$.

In general, we may assume $\mathcal{I} \subset \mathcal{P}$. By Lem. 2.5.8 for any point $(\xi, \varphi) \in M_\mathcal{P}(R)$, after passing to a fpqc base change of $R$, there is always a point $(\tilde{\xi}, \tilde{\varphi}) \in M^{\text{par}}(R)$ mapping to it under $\tilde{\text{For}}_Q^\mathcal{P}$. Since $\chi_{\mathcal{P}, \mathcal{E}}(\varphi) = \chi_{\mathcal{I}, \mathcal{E}}(\tilde{\varphi})$, we conclude that $\chi_{\mathcal{P}, \mathcal{E}}(\varphi) \in H^0(X_R, \mathfrak{c}_D)$. □

2.5.10. Definition. The morphism $f_\mathcal{P} : M_\mathcal{P} \to A^\text{Hit} \times X$ constructed in Lem. 2.5.9 is called the parahoric Hitchin fibration of type $\mathcal{P}$.

Let $\mathfrak{c}_{\mathcal{P}} = l_\mathcal{P} \backslash L_{\mathcal{P}} = t_\mathcal{P} \backslash W_{\mathcal{P}}$ be the GIT quotient of the reductive Lie algebra $l_\mathcal{P}$ over $k$. Recall that the Weyl group $W_{\mathcal{P}}$ of $L_{\mathcal{P}}$ can be identified with a subgroup of $\tilde{W}$. The projection $\tilde{W} \to W$ restricted to $W_{\mathcal{P}}$ induces an injection $W_{\mathcal{P}} \hookrightarrow W_{\mathcal{Q}}$. Moreover, for $\mathcal{P} \subset \mathcal{Q}$, we have an inclusion $W_{\mathcal{P}} \subset W_{\mathcal{Q}}$. Passing to the invariant quotients, we have canonical finite flat morphisms

$$t \overset{\text{id}}{\to} \mathfrak{c}_{\mathcal{P}} \overset{q_{\mathcal{P}}}{\to} \mathfrak{c}_{\mathcal{Q}} \overset{q_{\mathcal{Q}}}{\to} \mathfrak{c}.$$ 

2.5.11. Definition. The enhanced Hitchin base $\tilde{A}_\mathcal{P}$ of type $\mathcal{P}$ is defined by the following Cartesian square

$$\begin{array}{ccc}
\tilde{A}_\mathcal{P} & \to & \mathfrak{c}_{\mathcal{P}, \mathcal{D}} \\
\downarrow q_{\mathcal{P}} & & \downarrow q_{\mathcal{P}} \\
A^\text{Hit} \times X & \to & \mathfrak{c}_D \\
\text{ev} & & \\
\end{array}$$

where “ev” is the evaluation map. Note that $\tilde{A}_\mathcal{I}$ is the universal cameral cover $\tilde{A}$ in [YunI Def. 3.1.7], and $\tilde{A}_G = A^\text{Hit} \times X$.

2.5.12. Lemma. The projection $\text{Coor}(X) \times [l_\mathcal{P}/L_{\mathcal{P}}] \to [l_\mathcal{P}/L_{\mathcal{P}}] \to \mathfrak{c}_{\mathcal{P}}$ factors through the quotient $\text{Coor}(X) \times A^\text{Aut}_\mathcal{O} [l_\mathcal{P}/L_{\mathcal{P}}]$ so that we have a morphism $\chi_{\mathcal{P}} : [l_\mathcal{P}/L_{\mathcal{P}}] \to \mathfrak{c}_{\mathcal{P}}$. 


Proof. Since \( \text{Aut}_\mathcal{O} \) is connected, its action on \( L_\mathcal{P} \) factors through the adjoint action of \( L_\mathcal{P}^{ad} \) on \( L_\mathcal{P} \). Therefore the adjoint GIT quotient \([L_\mathcal{P}/L_\mathcal{P}]_D \xrightarrow{\chi_{\mathcal{P}}} \mathcal{C}_\mathcal{P}, \mathcal{D}\) is automatically \( \text{Aut}_\mathcal{O} \)-invariant and the conclusion follows. \( \square \)

Using the morphism \( \mathcal{M}_\mathcal{P} \xrightarrow{\mathcal{E}_\mathcal{P}} [L_\mathcal{P}/L_\mathcal{P}]_D \xrightarrow{\chi_{\mathcal{P}}} \mathcal{C}_\mathcal{P}, \mathcal{D} \), we get the enhanced parahoric Hitchin fibration of type \( \mathcal{P} \):

\[
\tilde{f}_\mathcal{P} : \mathcal{M}_\mathcal{P} \xrightarrow{(\chi_{\mathcal{P}} \circ \mathcal{E}_\mathcal{P}, f_\mathcal{P})} \mathcal{C}_\mathcal{P}, \mathcal{D} \times \varepsilon_D (A \times X) = \tilde{A}_\mathcal{P}.
\]

2.6. Properties of \( \mathcal{M}_\mathcal{P} \). In this subsection, we list a few properties of \( \mathcal{M}_\mathcal{P} \) and \( f_\mathcal{P} \) parallel to the properties studied in [YunI, Sec. 3.2-3.5] for the parabolic Hitchin scheme (2.11)

We claim that the image of this homomorphism lies in \( \text{Aut}_\mathcal{J} \). For any standard parahoric subgroup \( \mathcal{P} \subset G(F) \), we construct an action of \( \mathcal{P} \) on \( \mathcal{M}_\mathcal{P} \) such that the morphisms \( \text{For}_\mathcal{P} : \mathcal{M}_\mathcal{P} \rightarrow \mathcal{M}_\mathcal{Q} \) are equivariant under \( \mathcal{P} \).

For any \( k \)-algebra \( R \) and an object \((x, \mathcal{E}, \tau_x \mod \mathcal{P}, \varphi) \in \text{Bun}_\mathcal{P}(R) \), we can define the sheaf of automorphisms \( \text{Aut}_\mathcal{P}(\mathcal{E}, \tau_x, \varphi) \) of this object. More precisely, \( \text{Aut}_\mathcal{P}(\mathcal{E}, \tau_x, \varphi) \) is a fpqc sheaf of groups over \( X_R \) such that for any \( u : U \rightarrow X_R \), \( \text{Aut}_\mathcal{P}(\mathcal{E}, \tau_x, \varphi)(U) \) is the set of automorphisms of \( u^* \mathcal{E} \) which preserve the Higgs field \( u^* \varphi \) and the trivialization \( u^* \tau_x \) up to \( G_\mathcal{P} \). For \( \mathcal{P} \subset \mathcal{Q} \) we have an inclusion of sheaves of groups

\[
\text{Aut}_\mathcal{P}(\mathcal{E}, \tau_x, \varphi) \hookrightarrow \text{Aut}_\mathcal{Q}(\mathcal{E}, \tau_x, \varphi).
\]

Moreover, we always have an inclusion

\[
\text{Aut}_\mathcal{P}(\mathcal{E}, \tau_x, \varphi) \hookrightarrow j_* (\text{Aut}(\mathcal{E}, \varphi)|_{X_R - \Gamma(x)}),
\]

where \( j : X_R - \Gamma(x) \hookrightarrow X_R \) is the inclusion.

Let \( a = \mathcal{E}_\mathcal{P}(\mathcal{E}, \varphi) \in \mathcal{A}^{\text{Hitt}}(R) \). Recall from [YunI, Sec. 3.2] we see that the group scheme \( J \) naturally maps to the universal centralizer group scheme over \( g \). Another way to say this is that we have a natural homomorphism \( J_a \in \text{Aut}(\mathcal{E}, \varphi) \). Consider the homomorphism of sheaves of groups

\[
J_a \rightarrow j_* J_a \rightarrow j_* (\text{Aut}(\mathcal{E}, \varphi)|_{X_R - \Gamma(x)}).
\]

We claim that the image of this homomorphism lies in \( \text{Aut}_\mathcal{P}(\mathcal{E}, \tau_x, \varphi) \). In fact, since \( \text{For}_\mathcal{P} \) is surjective by Lem. 2.5.8, we may assume that \((x, \mathcal{E}, \tau_x \mod \mathcal{P}, \varphi) \) comes from a point \((x, \mathcal{E}, \tau_x \mod \mathcal{I}, \varphi) \in \mathcal{M}^{\text{par}}(R) \). From [YunI Lem. 3.2.2], we see that the group scheme \( J \) naturally maps to the centralizer group scheme of \( \mathcal{B} \). Again, this can be rephrased as saying that the image of \( J_a \in \text{Aut}(\mathcal{E}, \varphi) \) lies in \( \text{Aut}_\mathcal{J}(\mathcal{E}, \tau_x, \varphi) \).

By the inclusion (2.12) applied to \( \mathcal{I} \subset \mathcal{P} \), the image of \( J_a \) under (2.13) also lies in \( \text{Aut}_\mathcal{I}(\mathcal{E}, \tau_x, \varphi) \).

With the homomorphism \( J_a \rightarrow \text{Aut}_\mathcal{P}(\mathcal{E}, \tau_x, \varphi) \), we can define the action of \( Q^J \in \mathcal{P}_a(R) \) by

\[
Q^J \cdot (x, \mathcal{E}, \tau_x \mod \mathcal{P}, \varphi) = (x, Q^J_{J_a} (\mathcal{E}, \tau_x \mod \mathcal{P}, \varphi)).
\]

2.6.2. Lemma. The action of \( \mathcal{P} \) on \( \mathcal{M}_\mathcal{P} \) preserves the morphism \( \tilde{f}_\mathcal{P} : \mathcal{M}_\mathcal{P} \rightarrow \tilde{A}_\mathcal{P} \).
Proof. We have a commutative diagram

\[
\begin{array}{c}
P \times M_{\text{par}} \xrightarrow{\text{act}_I} M_{\text{par}} \xrightarrow{\tilde{f}} \tilde{A} \\
\text{id}_P \times \text{For}_I \xrightarrow{\text{act}_P} M_P \xrightarrow{\text{proj}_P} \tilde{A}_P
\end{array}
\]

By [YunI, Lem. 3.2.5], the \( P \)-action on \( M_{\text{par}} \) preserves \( \tilde{f} \), hence

\[
q_I^P \circ \tilde{f} \circ \text{act}_I = q_I^P \circ \tilde{f} \circ \text{proj}_I.
\]

Combining with the diagram 2.14 we get

\[
\tilde{f}_P \circ \text{act}_P \circ (\text{id}_P \times \text{For}_I^P) = \tilde{f}_P \circ \text{proj}_P \circ (\text{id}_P \times \text{For}_I^P).
\]

Since \( \text{For}_I^P : M_{\text{par}} \to M_P \) is surjective, we conclude that

\[
\tilde{f}_P \circ \text{act}_P = \tilde{f}_P \circ \text{proj}_P. \quad \Box
\]

2.6.3. Local counterpart of \( M_P \). For a point \( x \in X(k) \) and an element \( \gamma \in g(F_x) \), we can define the affine Springer fiber of type \( P \) in a similar way as one defines affine Springer fibers in the affine flag varieties. It is a closed sub-ind-scheme \( M_{P,x}(\gamma) \subset F_{\ell P,x} := (\text{Res}_{F_x/k} G) / \text{Res}_{\tilde{O}_x/k} P_x \), here \( P_x \subset G(F_x) \) is the parahoric subgroup corresponding to \( P \) under any choice of local coordinate at \( x \). Again, the group ind-scheme \( P_x(J_a) \) acts on \( M_{P,x}(\gamma) \) whenever \( \chi(\gamma) = a \in \mathcal{O}(\tilde{O}_x) \).

Fix \( (a,x) \in A^\partial(k) \times X(k) \). As in [YunI, Sec. 3.3], from the Kostant section \( \epsilon : A_{\text{Hit}} \to M_{\text{Hit}} \) and the choices of local trivializations we get local data \( \gamma_{a,x} \in g(\tilde{O}_x) \). Analogous to the product formula in [YunI, Prop. 3.3.3], we have:

2.6.4. Proposition (Product formula). Let \( (a,x) \in A^\partial(k) \times X(k) \), and let \( U_a \) be the dense open subset \( a^{-1}c_{\partial}^D \) of \( X \). We have a homeomorphism of stacks:

\[
P_a \times (M_{\text{par}}(\gamma_{a,x}) \times M') \to M_{P,a,x}.
\]

where

\[
P' = \prod_{y \in X - U_a - \{x\}} P_y^{\text{red}}(J_a);
\]

\[
M' = \prod_{y \in X - U_a - \{x\}} M_y^{\text{Hit,red}}(\gamma_{a,y}).
\]

Parallel to [YunI, Prop. 3.4.1], we have

2.6.5. Proposition. Recall \( \deg(D) \geq 2g_X \). Then we have:

(1) The stack \( M_P|_{A^\varphi} \) is smooth;
(2) The stack \( M_P|_{A_{\text{ani}}} \) is Deligne-Mumford;
(3) The morphism \( f_P^{\text{ani}} : M_P|_{A_{\text{ani}}} \to A_{\text{ani}} \times X \) is flat and proper.
2.6.6. Small maps. In classical Springer theory, the morphisms between the various partial Grothendieck resolutions are small, cf. [BM]. According to Lem. 2.5.8 the morphisms \( \widetilde{\text{For}}_P^Q \) between the parahoric Hitchin moduli stacks are base changes of the morphisms between partial Grothendieck resolutions; however, it it not clear that \( eY_Q \) is flat, so that we cannot conclude immediately that \( \widetilde{\text{For}}_P^Q \) is also small. In the following proposition, we prove the smallness of \( \widetilde{\text{For}}_P^Q \) over the locus \( A \subset A^\text{uni} \) where the codimension estimate in [Yun1, Prop. 3.5.5] holds (see [Yun1] Rem. 3.5.6).

2.6.7. Proposition. Let \( P \subset Q \) be standard parahoric subgroups. Then

1. The morphism \( \widetilde{\text{For}}_P^Q : M_P|_A \to M_Q|_A \) is small.
2. The morphism \( \nu_P^Q : A_P \to \widetilde{A}_Q \) is small and birational (i.e., a small resolution of singularities).

Proof. Let us restrict all stacks over \( A^\text{Hi} \) to the open subset \( A \) without changing notations.

(1) For each integer \( d \geq 1 \), let \( Z_{\geq d} \) be the closed subschemes of \( M_Q \) over which the fibers of \( \widetilde{\text{For}}_P^Q \) have dimension \( \geq d \). By Lem. 2.5.8 the fiber of \( \widetilde{\text{For}}_P^Q \) over a point \( (x, \mathcal{E}, \tau_x \mod Q, \varphi) \in M_Q(k) \) is the partial Springer fiber corresponding to the conjugacy class \( \varphi(x) \in [L_Q] \), and is contained in the affine Springer fiber \( M_{A_x}(\gamma) \) (here \( \gamma \in \mathfrak{g}(\mathcal{O}_x) \) is a representative of \( \varphi \) at \( x \) after choosing a trivialization of \( \mathcal{E} \) over \( \mathcal{O}_x \)). Therefore, if \( (x, \mathcal{E}, \tau_x \mod Q, \varphi) \in Z_{\geq d} \), then

\[
\delta(a,x) = \dim M_x^{\text{par}}(\gamma) \geq \dim M_{A_x}(\gamma) \geq \widetilde{\text{For}}_P^{-1}_Q (x, \mathcal{E}, \tau_x \mod Q, \varphi) \geq d
\]

where \( a = f^\text{Hi}(\mathcal{E}, \varphi) \in A(k) \). Therefore the image of \( Z_{\geq d} \to A \times A \) lies in \((A \times A)_{\geq d} \) which has codimension \( \geq d + 1 \) by [Yun1, Cor. 3.5.7].

On the other hand, let \( Y_{\geq d} = \widetilde{\text{For}}_P^{-1}_Q (Z_{\geq d}) \), which maps to \((A \times A)_{\geq d} \) by the above discussion. Since \( f_P : M_P \to A \times A \) is flat by Prop. 2.6.6(4), we have

\[
\text{codim}_{M_P}(Y_{\geq d}) \geq \text{codim}_{A \times A} ((A \times A)_{\geq d}) \geq d + 1.
\]

Therefore

\[
dim(Z_{\geq d}) \leq \dim(Y_{\geq d}) - d \leq \dim(M_P) - (d + 1) - d = \dim(M_Q) - 2d - 1.
\]

This proves the smallness of \( \widetilde{\text{For}}_P^Q \) (we know \( \widetilde{\text{For}}_P^Q \) is surjective by Lem. 2.5.8).

(2) The commutative diagram

\[
\begin{array}{ccc}
\mathbb{L}_P^Q / B_P^Q & \xrightarrow{\chi_P} & \mathbb{L}_P^Q / L_P^Q \\
\pi_P^Q \downarrow & & \downarrow \pi_Q^P \\
\mathbb{L}_Q / L_Q & \xrightarrow{\chi_Q} & \mathbb{L}_Q^Q \\
\end{array}
\]

is Cartesian over \( \mathbb{E}_Q^Q \). Therefore by Lem. 2.5.8 the morphism \( \nu_P^Q \) is also an isomorphism over \((A \times A)^n\), hence birational. Since \( q_P^Q : \widetilde{A}_P \to \widetilde{A}_Q \) is finite and \( \widetilde{\text{For}}_P^Q \) is small by (1), we conclude that \( \nu_P^Q \) is also small. This proves (2). \( \square \)
2.7. Examples in classical groups. In this subsection, we describe the fibers of parahoric Hitchin fibrations for classical groups of type A, B and C.

2.7.1. Example. Let $G = \text{SL}(n)$. According to Example 2.4.3, a standard parahoric subgroup $P \subset \text{SL}(n,F)$ corresponds to a sequence of integers

$$\underline{i} = (0 \leq i_0 < \cdots < i_m < n), m \geq 0.$$

The Hitchin base is

$$A^{\text{Hit}} = \bigoplus_{i=2}^{n} H^0(X, O_X(iD)).$$

For $a = (a_2, \ldots, a_n) \in A^{\text{G}}(k)$ (where $a_i \in H^0(X, O_X(iD)))$, define the spectral curve $Y_a$ as in [Yun] Example 3.1.10. Fix a point $x \in X$. Then the parahoric Hitchin fiber $M_{P,a,x}$ classifies the data

$$(F_{i_0} \supset F_{i_1} \supset \cdots \supset F_{i_m} \supset F_{i_0}(-x), \delta)$$

where $F_{ij} \in \text{Pic}(Y_a)$ such that $F_{i_0}/F_{ij}$ has length $i_0 - i_j$ for $j = 0, 1, \ldots, m$, and $\delta$ is an isomorphism $\det(p_{a,*}F_{ij}) \cong O_X(-i_0x)$.

2.7.2. Example. Let $G = \text{SO}(2n+1)$ (resp. $G = \text{Sp}(2n)$). According to Example 2.4.3, a standard parahoric subgroup $P$ corresponds to a sequence of integers

$$\underline{i} = (0 \leq i_0 < \cdots < i_m \leq n), m \geq 0.$$

The Hitchin base is

$$A^{\text{Hit}} = \bigoplus_{i=1}^{n} H^0(X, O_X(2iD)).$$

For $a = (a_1, \ldots, a_n) \in A^{\text{G}}(k)$ (where $a_i \in H^0(X, O_X(2iD)))$, we have the spectral curve $Y_a$ in the total space of $O_X(D)$ defined by the equation

$$t \sum_{i=0}^{n} a_i t^{2(n-i)} = 0; \quad \text{(resp. } \sum_{i=0}^{n} a_i t^{2(n-i)} = 0\text{)}$$

where $a_0 = 1$. The curve $Y_a$ is equipped with the involution $\tau$ sending $t$ to $-t$. Fix a point $x \in X$. Then the parahoric Hitchin fiber $M_{P,a,x}$ classifies the data

$$(F_{i_0} \supset \cdots \supset F_{i_m} \supset F_{i_0}^\perp(-x) \supset \cdots \supset F_{i_0}^\perp(-x) \supset F_{i_0}(-x), \sigma)$$

where

- $F_{ij} \in \text{Pic}(Y_a)$ such that $F_{i_0}/F_{ij}$ has length $i_0 - i_j$ for $j = 0, 1, \ldots, m$;
- $\sigma : \tau^* F_{i_0} \to F_{i_0}^\perp$ is a map of coherent sheaves on $Y_a$ such that $\tau^* \sigma = \sigma^\vee$ (resp. $\tau^* \sigma = -\sigma^\vee$). Here $(-)^\vee$ means the relative Grothendieck-Serre duality for coherent sheaves on $Y_a$ with respect to the $X$;
- For $j = 0, 1, \ldots, m$, define $F_{ij}^\perp := (\sigma(\tau^* F_{ij}))^\vee$, which naturally contains $F_{i_0}$;
- $\text{coker}(\sigma)$ has length $i_0$.

3. The graded double affine Hecke algebra action

In this section, we enrich the affine Weyl group action constructed in [Yun] Sec. 4] into an action of the graded double affine Hecke algebra (DAHA). We also generalize the graded DAHA action to the case of parahoric Hitchin stacks. To save notations, we assume that $G$ is almost simple throughout this section.
3.1. The Kac-Moody group. In this subsection, we recall the construction of the Kac-Moody group associated to the loop group $G((t))$.

3.1.1. The determinant line bundle. For any $k$-algebra $R$, we have an additive functor

$$
det : D_{perf}(R) \to \text{Pic}(R)$$

Here $D_{perf}(R)$ is the derived category of perfect complexes of $R$-modules and $\text{Pic}(R)$ is the Picard category of invertible $R$-modules. The functor $\det$ sends a projective $R$-module $M$ of finite rank $m$ the invertible $R$-module $\wedge^m M$.

We may define a line bundle $L_{can}$ on $G((t))$, which is pulled back from $\mathcal{G}_G = G((t))/G[[t]]$. For any $R[[t]]$-submodule $\Xi$ of $g \otimes_k R((t))$ which is commensurable with the standard $R[[t]]$-submodule $\Xi_0 := g \otimes_k R[[t]]$ (i.e., $t^N \Xi_0 \subset \Xi \subset t^{-N} \Xi_0$ for some $N \in \mathbb{Z}_{\geq 0}$ and $t^{-N} \Xi_0/\Xi$ and $\Xi/t^N \Xi_0$ are both projective $R$-modules), define the relative determinant line of $\Xi$ with respect to $\Xi_0$ to be:

$$
\det(\Xi : \Xi_0) = (\det(\Xi/\Xi \cap \Xi_0)) \otimes_R (\det(\Xi_0/\Xi \cap \Xi_0))^{\otimes -1}.
$$

For any $g \in G(R((t)))$, consider its action on $g \otimes_k R((t))$ by the adjoint representation. The functor $L_{can}$ then sends $g$ to the invertible $R$-module $\det(\text{Ad}(g) \Xi_0 : \Xi_0)$. Since $\text{Ad}(g) \Xi_0$ only depends on the image of $g$ in $G_{rG}(R)$, the line bundle $L_{can}$ is descended to $G_{rG}$.

Let $G((t)) = \rho_{\mathcal{G}_{can}} \to G((t))$ be the total space of the $\mathbb{G}_m$-torsor associated to the line bundle $L_{can}$. The set $G((t))(R)$ consists of pairs $(g, \gamma)$ where $g \in G(R((t)))$ and $\gamma$ is an $R$-linear isomorphism $R \xrightarrow{\sim} \det(\text{Ad}(g) \Xi_0 : \Xi_0)$. There is a natural group structure on $G((t))$: for $(g_1, \gamma_1)$ and $(g_2, \gamma_2) \in G((t))$, their product $(g_1, \gamma_1) \cdot (g_2, \gamma_2)$ is $(g_1 g_2, \gamma)$, where $\gamma$ is the isomorphism

$$
\gamma_1 \otimes \text{Ad}(g_1)(\gamma_2) : R \otimes_R R \xrightarrow{\sim} \det(\text{Ad}(g_1) \Xi_0 : \Xi_0) \otimes_R \det(\text{Ad}(g_1 g_2) \Xi_0 : \text{Ad}(g_1) \Xi_0) = \det(\text{Ad}(g_1 g_2) \Xi_0 : \Xi_0).
$$

The group $\widehat{G((t))}$ is in fact a central extension

$$
1 \to \mathbb{G}_m^{\text{cen}} \to \widehat{G((t))} \to G((t)) \to 1.
$$

Here we use $\mathbb{G}_m^{\text{cen}}$ to denote the one-dimensional central torus of $\widehat{G((t))}$, which can be identified as the fiber of $\widehat{G((t))}$ over the identity element $1 \in G((t))$. When $g \in G[[t]]$, we have $\text{Ad}(g) \Xi_0 = \Xi_0$, hence a canonical trivialization of $\det(\text{Ad}(g) \Xi_0 : \Xi_0)$. This gives a canonical splitting of the central extension (3.1) over the subgroup $G[[t]] \subset G((t))$.

3.1.2. The completed Kac-Moody group. From the construction it is clear that the action of $\text{Aut}_{\mathcal{O}}$ on $G((t))$ lifts to an action on $G((t))$, hence we can form the semi-direct product

$$
\mathcal{G} := \widehat{G((t))} \rtimes \text{Aut}_{\mathcal{O}}.
$$

We call this object the (complete) Kac-Moody group associated to the loop group $G((t))$.

Let $I^u \subset I$ be the unipotent radical and $G^u_I \subset G_I$ be the corresponding pro-unipotent radical. Let $\text{Aut}_{\mathcal{O}}^\text{pro} \subset \text{Aut}_{\mathcal{O}}$ be the pro-unipotent radical. Consider the
subgroups

\[
\begin{aligned}
G_1 &= \mathbb{G}^\text{cen}_m \times G_1 \times \text{Aut}_O \subset G; \\
G_1^n &= G^n_1 \times \text{Aut}_O^n \subset G_1.
\end{aligned}
\]

We define the universal Cartan torus for the Kac-Moody group \(G\) to be

\[(3.3) \quad \tilde{T} := G_1/G_1^n = \mathbb{G}^\text{cen}_m \times T \times \mathbb{G}^\text{rot}_m.
\]

We will denote the canonical generators of \(X_*(\mathbb{G}^\text{cen}_m), X^*(\mathbb{G}^\text{cen}_m), X_*(\mathbb{G}^\text{rot}_m)\) and \(X^*(\mathbb{G}^\text{rot}_m)\) by \(K_\text{can}, \Lambda_\text{can}, d\) and \(\delta\). Let

\[
\langle \cdot, \cdot \rangle : X^*(\tilde{T}) \times X_*(\tilde{T}) \rightarrow \mathbb{Z}
\]

be the natural pairing.

Let \((\cdot|\cdot)\)\text{can} be the Killing form on \(X_*(T)\):

\[
(3.4) \quad (x|y)\text{can} := \sum_{\alpha \in \Phi} \langle \alpha, x \rangle \langle \alpha, y \rangle.
\]

where \(\Phi \subset X^*(T)\) is the set of roots of \(G\). Let \(\theta \in \Phi\) be highest root and \(\theta^\vee \in \Phi^\vee\) be the corresponding coroot. Let \(\rho\) be half of the sum of the positive roots in \(\Phi\). Let \(h^\vee\) be the dual Coxeter number of \(g\), which is one plus the sum of coefficients of \(\theta^\vee\) written as a linear combination of simple coroots. We have the following fact:

3.1.3. Lemma. \(\frac{1}{2}(\theta^\vee \theta^\vee)\text{can} = 2(\langle \rho, \theta^\vee \rangle + 1) = 2h^\vee\).

Proof. Since \(\theta\) is the highest root, for any positive root \(\alpha \neq \theta\), we have \(\langle \alpha, \theta^\vee \rangle = 0\) or 1 (see [B, Chap VI, 1.8, Prop. 25(iv)]). Hence \(\langle \alpha, \theta^\vee \rangle^2 = \langle \alpha, \theta^\vee \rangle\) for \(\alpha \in \Phi^+ - \{\theta\}\). Therefore

\[
\frac{1}{2}(\theta^\vee \theta^\vee)\text{can} = \sum_{\alpha \in \Phi^+} \langle \alpha, \theta^\vee \rangle^2 = \langle \theta, \theta^\vee \rangle^2 + \sum_{\alpha \in \Phi^+ - \{\theta\}} \langle \alpha, \theta^\vee \rangle
\]

\[
= 4 + 2(\rho - \theta, \theta^\vee) = 2(\langle \rho, \theta^\vee \rangle + 1).
\]

Since \(\langle \rho, \alpha^\vee \rangle = 1\) for every simple coroot \(\alpha^\vee \in \Phi\), we get

\[
\langle \rho, \theta^\vee \rangle + 1 = h^\vee. \quad \Box
\]

3.1.4. The \(\tilde{W}\)-action on \(\tilde{T}\). For any section \(\iota\) of the quotient \(B \rightarrow T\), we can consider the normalizer \(N\) of \(\mathbb{G}^\text{cen}_m \times \iota(T) \times \mathbb{G}^\text{rot}_m\) in \(G(\mathbb{F}_m) \times \mathbb{G}^\text{rot}_m\), and we have a canonical isomorphism \(N/(\mathbb{G}^\text{cen}_m \times \iota(T)[[t]] \times \mathbb{G}^\text{rot}_m) \simeq \tilde{W}\). The conjugation action of \(N\) on \(\mathbb{G}^\text{cen}_m \times \iota(T) \times \mathbb{G}^\text{rot}_m\) induces an action of \(\tilde{W}\) on \(\tilde{T}\), which is independent of the choice of the section \(\iota\). Therefore, we get canonical actions of \(\tilde{W}\) on \(X_*(\tilde{T})\) and \(X^*(\tilde{T})\), denoted by \(\eta \mapsto \tilde{w}\eta\) and \(\xi \mapsto \tilde{w}\xi\). The natural pairing \(\langle \cdot, \cdot \rangle\) is invariant under \(\tilde{W}\): i.e., \(\langle \xi, \eta \rangle = \langle \tilde{w}\xi, \tilde{w}\eta \rangle\).

3.1.5. Lemma. The actions of \(\tilde{W}\) on \(X_*(\tilde{T})\) and \(X^*(\tilde{T})\) are given by:

1. \(w \in W\) fixes \(K_\text{can}, d, \Lambda_\text{can}\) and \(\delta\), and acts in the usual way on \(X_*(T)\) and \(X^*(T)\);

2. \(\lambda \in X_*(T)\) acts on \(\eta \in X_*(\tilde{T})\) and \(\xi \in X^*(\tilde{T})\) by

\[
\lambda \eta = \eta - \langle \delta, \eta \rangle \lambda + \left(\langle \eta|\lambda\rangle\text{can} - \frac{1}{2}\langle \lambda|\lambda\rangle\text{can} \langle \delta, \eta \rangle\right) K_\text{can};
\]

\[
\lambda \xi = \xi - \langle \xi, K_\text{can}\rangle \lambda^* + \left(\langle \xi|\lambda\rangle - \frac{1}{2}\langle \lambda|\lambda\rangle\text{can} \langle \xi, K_\text{can}\rangle\right) \delta.
\]
here $\lambda \mapsto \lambda^*$ is the isomorphism $X_*(T)_{\mathbb{Q}} \cong X^*(T)_{\mathbb{Q}}$ induced by the form $\langle \cdot, \cdot \rangle_{\text{can}}$.

**Proof.** (1) is clear from the fact that $W \subset G(k) \subset G(O_F)$.

To check (2), we choose a maximal torus in $B$ and call it $T$. We write any element in $\bar{T}$ as $(c, x, \sigma)$ for $c \in G_m^\text{cen}$, $x \in T$ and $\sigma \in G_m^\text{rot}$. We need to check that

$$\text{Ad}(t^\lambda)(c, 1, 1) = (c, 1, 1);$$

(3.5)

$$\text{Ad}(t^\lambda)(1, x, 1) = \left( \prod_{\alpha \in R} x^{\langle \alpha, \lambda \rangle \alpha} x, 1 \right);$$

(3.6)

$$\text{Ad}(t^\lambda)(1, 1, \sigma) = \left( \sigma^{-\frac{1}{2}(\lambda|\lambda)_{\text{can}}, \sigma^{-\lambda}, \sigma} \right)$$

Here, $x^\alpha$ is the image of $x$ under $\alpha : T \to G_m$; similarly $\sigma^{-\lambda}$ is the image of $\sigma$ under $-\lambda : G_m \to T$.

To verify the $G_m^\text{cen}$-coordinates in (3.5) and (3.6), notice that the $G_m^\text{rot}$-coordinate of $\text{Ad}(t^\lambda)(1, x, \sigma)$ is the same as the (scalar) action of $(x, \sigma) \in T \times G_m^\text{rot}$ on the line $\det(\text{Ad}(-\lambda)\mathbb{Z}_0 : \mathbb{Z}_0)$, via the adjoint representation. In terms of the root space decomposition $g = t \oplus (\oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$, we have

$$\det(\text{Ad}(t^\lambda)\mathbb{Z}_0 : \mathbb{Z}_0) = \left( \bigotimes_{\langle \alpha, \lambda \rangle > 0} t^\alpha \mathfrak{g}_{\alpha} \right) \otimes \left( \bigotimes_{\langle \alpha, \lambda \rangle < 0} t^{-\alpha} \mathfrak{g}_{\alpha} \right)$$

Therefore, as a $T \times G_m^\text{rot}$-module, $\det(\text{Ad}(t^\lambda)\mathbb{Z}_0 : \mathbb{Z}_0)$ has weight

$$\left( \sum_{\alpha \in \Phi} \langle \alpha, \lambda \rangle \alpha, - \sum_{\alpha \in \Phi} \frac{1}{2} \langle \alpha, \lambda \rangle^2 - \frac{1}{2} \langle \alpha, \lambda \rangle \right) = \left( \sum_{\alpha \in \Phi} \langle \alpha, \lambda \rangle \alpha, - \frac{1}{2} \langle \lambda|\lambda \rangle_{\text{can}} \right). \quad \square$$

### 3.1.6. Remark.

Let

$$K := 2h^v K_{\text{can}}; \quad \Lambda_0 := \frac{1}{2h^v} \Lambda_{\text{can}}.$$

We see from Lem. 3.1.5 that our definitions of $K, \Lambda_0, d$ and $\delta$ are consistent (up to changing $\lambda$ to $-\lambda$) with the notation for Kac-Moody algebras in [K 6.5]. The simple roots of the complete Kac-Moody group $G$ are $\{\alpha_0 = \delta - \theta, \alpha_1, \ldots, \alpha_n\} \subset X^*(T \times G_m^\text{rot})$; the simple coroots are $\{\alpha_0^* = K - \theta^*, \alpha_1^*, \ldots, \alpha_n^*\} \subset X_*(G_m^\text{cen} \times T)$.

#### 3.2. Line bundles on Bun$_G^\text{par}$.

Let $\omega_{\text{Bun}}$ be the canonical bundle of $\text{Bun}_G$. Since the tangent complex at a point $\mathcal{E} \in \text{Bun}_G(R)$ is $R\Gamma(X_R, \text{Ad}(\mathcal{E}))[1]$, the value of the canonical bundle $\omega_{\text{Bun}}$ at the point $\mathcal{E}$ is the invertible $R$-module $\text{det } R\Gamma(X_R, \text{Ad}(\mathcal{E}))$. Let $\text{Bun}_\infty \to \text{Bun}_\infty$ be the total space of the $G_m$-torsor associated to the pull-back of $\omega_{\text{Bun}}$. More concretely, for any $k$-algebra $R$, $\text{Bun}_\infty(R)$ classifies tuples $(x, \alpha, \mathcal{E}, \tau_x, \epsilon)$ where $(x, \alpha, \mathcal{E}, \tau_x) \in \text{Bun}_\infty(R)$ and $\epsilon$ is an $R$-linear isomorphism $R \cong \text{det } R\Gamma(X_R, \text{Ad}(\mathcal{E}))$.

#### 3.2.1. Construction.

There is a natural action of $G$ on $\text{Bun}_\infty$, lifting the action of $G((t)) \ltimes \text{Aut}_0 \text{Bun}_\infty$ in Construction 2.3.2. In fact, for $(x, \alpha, \mathcal{E}, \tau_x) \in \text{Bun}_\infty(R)$ and $g \in G(R((t)))$, the $G$-torsor $\mathcal{E}^g$ is obtained by gluing the trivial $G$-torsor on $\mathfrak{D}_x \cong \text{Spec } R[[t]]$ (using $\alpha$) with $\mathcal{E}|_{X_R - \Gamma(x)}$ via the identification $\tau_x \circ g$. Hence $\text{Ad}(\mathcal{E}^g)$ is obtained by gluing $g(\mathcal{O}_x) \cong g \otimes_k R[[t]] = \Xi_0$ with $\text{Ad}(\mathcal{E})|_{X_R - \Gamma(x)}$ via the identification $\text{Ad}(\tau_x) \circ \text{Ad}(g)$. In other words, $\text{Ad}(\mathcal{E}^g)$ is obtained by gluing
Ad(g)ξ₀ with Ad(ξ)Xₙ₋₁(γ(x)) via Ad(τₓ). Thus we have a canonical isomorphism of invertible R-modules

$$(\det \mathbf{R}\Gamma(Xₙ, \text{Ad}(\mathcal{E})))(\det \mathbf{R}\Gamma(Xₙ, \text{Ad}(\mathcal{E})))^{-1} \cong \det(\text{Ad}(g)ξ₀ : ξ₀).$$

Therefore, for trivializations $\epsilon : R \to \mathbf{R}\Gamma(Xₙ, \text{Ad}(\mathcal{E}))$ and $\gamma : R \to \mathbf{R}\Gamma(Xₙ, \text{Ad}(\mathcal{E})))^{-1}$, $\epsilon \otimes \gamma$ defines a trivialization of $\text{det}(\mathbf{R}\Gamma(Xₙ, \text{Ad}(\mathcal{E})))$. We then define the action of $\hat{g} = (g, \gamma, \sigma) \in (\mathcal{G}(\hat{t}) \times \text{Aut}_C)(R) = \mathcal{G}(R)$ on $(x, \alpha, \mathcal{E}, \tau_x, \epsilon) \in \text{Bun}_∞(R)$ by

$$R\hat{g}(x, \alpha, \mathcal{E}, \tau_x, \epsilon) = (x, \alpha \circ \sigma, \mathcal{E}, \tau_x^\sigma, \epsilon \otimes \gamma).$$

3.2.2. Construction. We define a natural $\hat{T}$-torsor $\mathcal{L}_{\hat{T}}^\mathcal{F}$ on $\text{Bun}_G^{\text{par}}$, hence line bundles $\mathcal{L}(\xi)$ for $\xi \in X^*(\hat{T})$. Consider the quotient $\mathcal{L}_{\hat{T}}^\mathcal{F} = \text{Bun}_∞/G_t^\mathcal{F}$ (as a fpqc sheaf). The right translation of $\mathcal{G}_t$ on $\text{Bun}_∞$ descends to a right action of $\hat{T} = G_t/G_t^\mathcal{F}$ on $\mathcal{L}_{\hat{T}}^\mathcal{F}$, and realizes the natural projection $\mathcal{L}_{\hat{T}}^\mathcal{F} \to \text{Bun}_G^{\text{par}}$ as a $\hat{T}$-torsor. For each character $\xi \in X^*(\hat{T})$, we define $\mathcal{L}(\xi)$ to be the line bundle on $\text{Bun}_G^{\text{par}}$ associated to $\mathcal{L}_{\hat{T}}^\mathcal{F}$ and the character $\xi$.

We can easily identify the line bundles $\mathcal{L}(\xi)$ for various $\xi \in X^*(\hat{T})$:

3.2.3. Lemma. (1) $\mathcal{L}(\Lambda_{\text{can}})$ is the pull-back of $\omega_{\text{Bun}}$ via the forgetful morphism $\text{Bun}_G^{\text{par}} \to \text{Bun}_G$;

(2) For $\xi \in X^*(T)$, the value of the line bundle $\mathcal{L}(\xi)$ at a point $(x, \mathcal{E}, \mathcal{E}^\mathcal{F}) \in \text{Bun}_G^{\text{par}}(R)$ is the invertible $R$-module associated to the $B$-torsor $\mathcal{E}^\mathcal{F}_x$ over $\Gamma(x) \cong \text{Spec } R$ and the character $B \to T \xleftarrow{\hat{T}} \mathbb{G}_m$;

(3) $\mathcal{L}(\delta)$ is isomorphic to the pull-back of $\omega_X$ via the morphism $\text{Bun}_G^{\text{par}} \to X$ (cf. Lem. 2.1.4).

3.3. The graded double affine Hecke algebra and its action.

3.3.1. Comparison of $\hat{W}$ and $W_{\text{aff}}$. Recall that $\hat{W} = X_{\text{aff}}(T) \rtimes W$ is the extended affine Weyl group associated to $G$ and $W_{\text{aff}} = \mathbb{Z}\Phi^∨ \rtimes W$ is the affine Weyl group, where $\mathbb{Z}Φ^∨$ is the coroot lattice. It is well-known that $W_{\text{aff}}$ is a Coxeter group with simple reflections $X_{\text{aff}} = \{s₀, s₁, \cdots, sₙ\}$, where $s₁, \cdots, sₙ$ are simple reflections of the finite Weyl group $W$ corresponding to our choice of the Borel $B \subset G$. We have an exact sequence

$$1 \to W_{\text{aff}} \to \hat{W} \to \Omega \to 1.$$  

where $\Omega = X_{\text{aff}}(T)/\mathbb{Z}\Phi^∨$. Let $\hat{I}$ be the normalizer of $I$ in $G(F)$, which defines an extension $\hat{G}_I$ of $G_I$ by $\Omega_I = \hat{I}/I$. Let $\hat{G}_I$ be the preimage of $G_I$ in $\hat{G}(\hat{t})$ and $G_I := \hat{G}_I \rtimes \text{Aut}_C$. Then $\hat{G}_I$ normalizes $G_I$ and the conjugation action of $G_I$ on $G_I$, after passing to the quotient $G_I \to \hat{T}$, induces an action of $\Omega_I$ on $\hat{T}$. It is easy to see that this action is a sub-action of the $\hat{W}$-action on $\hat{T}$, hence we can naturally view $\Omega_I$ as a subgroup of $\hat{W}$. It is easy to verify that the composition $\Omega_I \hookrightarrow \hat{W} \to \hat{T}$ is an isomorphism. Therefore we can write $\hat{W}$ as a semi-direct product $\hat{W} = W_{\text{aff}} \rtimes \Omega_I$.

3.3.2. Definition. The graded double affine Hecke algebra (or graded DAHA for short) is an evenly graded $\mathbb{Q}_ℓ$-algebra $\mathcal{H}$ which, as a vector space, is the tensor product

$$\mathcal{H} = \mathcal{G}_I[\hat{W}] \otimes \text{Sym}_{\mathcal{G}_I}(X^*(\hat{T})_{\mathcal{G}_I}) \otimes \mathcal{G}_I[u].$$
Here $\mathcal{O}_\ell[\widetilde{W}]$ is the group ring of $\tilde{W}$, and $\mathcal{O}_\ell[u]$ is a polynomial algebra in the indeterminate $u$. The grading on $\mathbb{H}$ is given by

- $\deg(w) = 0$ for $w \in \tilde{W}$;
- $\deg(u) = \deg(\xi) = 2$ for $\xi \in X^*(\tilde{T})$.

The algebra structure on $\mathbb{H}$ is determined by

1. $\mathcal{O}_\ell[\tilde{W}]$, $\text{Sym}_{\mathbb{Q}}(X^*(\tilde{T})|_{\mathcal{O}_\ell})$ and $\mathcal{O}_\ell[u]$ are subalgebras of $\mathbb{H}$;
2. $u$ is in the center of $\mathbb{H}$;
3. For any simple reflection $s_i \in \Sigma_{\text{aff}}$ (corresponding to a simple root $\alpha_i$) and $\xi \in X^*(\tilde{T})$, we have
   $$s_i\xi - s_i^*s_i = (\xi, \alpha_i^\vee)u;$$
4. For any $\omega \in \Omega_1$ and $\xi \in X^*(\tilde{T})$, we have
   $$\omega\xi = \omega^\vee \xi.$$

3.3.3. **Remark.** When $\tilde{W}$ is replaced by the finite Weyl group $W$, and $\tilde{T}$ is replaced by $T$, the corresponding algebra is the equal-parameter case of the “graded affine Hecke algebras” considered by Lusztig in [L88].

Our main goal in this section is to construct an action of $\mathbb{H}$ on the parabolic Hitchin complex $f^\text{par}_*\mathcal{O}_\ell \in \mathcal{D}_c^b(A \times X)$.

3.3.4. **Construction.** We define the action of the generators of $\mathbb{H}$ on $f^\text{par}_*\mathcal{O}_\ell$.

- The action of $\tilde{W}$ has been constructed in [Yun1, Th. 4.4.3].
- The action of $u$. The Chern class of the line bundle $\mathcal{O}_X(D)$ (after pulling back to $A \times X$) gives a morphism in $\mathcal{D}_c^b(A \times X)$:
  $$c_1(D) : \mathcal{O}_\ell, A \times X \to \mathcal{O}_\ell, A \times X[2](1).$$

In general, for any object $K \in \mathcal{D}_c^b(A \times X)$, the cup product with $c_1(D)$ defines a map $\cup c_1(D) : K \to K[2](1)$. In particular, for $K = f^\text{par}_*\mathcal{O}_\ell$, we get the action of $u$:
  $$u = \cup c_1(D) : f^\text{par}_*\mathcal{O}_\ell \to f^\text{par}_*\mathcal{O}_\ell[2](1).$$

- The action of $X^*(\tilde{T})$. Recall from Construction 3.2.2 that we have a $\tilde{T}$-torus $\mathcal{L}_{\text{aff}}$ over $\text{Bun}^\text{par}_{\mathbb{G}}$, and the associated line bundle $\mathcal{L}(\xi)$ for $\xi \in X^*(\tilde{T})$. We also use $\mathcal{L}(\xi)$ to denote its pull back to $\mathcal{M}^\text{par}$. The Chern class of $\mathcal{L}(\xi)$ gives a map:
  $$c_1(\mathcal{L}(\xi)) : \mathcal{O}_{\ell, \mathcal{M}^\text{par}} \to \mathcal{O}_{\ell, \mathcal{M}^\text{par}}[2](1).$$

We define the action of $\xi$ on $f^\text{par}_*\mathcal{O}_\ell$ to be
  $$\xi = f^\text{par}_*(c_1(\mathcal{L}(\xi))) : f^\text{par}_*\mathcal{O}_\ell \to f^\text{par}_*\mathcal{O}_\ell[2](1).$$

By Lem. 3.2.3 the action of $\Lambda_{\text{can}} \in X^*(\mathbb{G}_{\text{aff}})$ is given by the cup product with (the pull-back of) $c_1(\omega_{\text{aff}})$; the action of $\delta \in X^*(\mathbb{G}_{\text{rot}})$ is given by the cup product with (the pull-back of) $c_1(\omega_X)$.

3.3.5. **Theorem.** The actions of $\tilde{W}$, $u$ and $X^*(\tilde{T})$ on $f^\text{par}_*\mathcal{O}_\ell$ given in Construction 3.3.4 extends to an action of $\mathbb{H}$ on $f^\text{par}_*\mathcal{O}_\ell$. More precisely, we have a graded algebra homomorphism
  $$\mathbb{H} \to \bigoplus_{i \in \mathbb{Z}} \text{End}_{A \times X}^\mathbb{Q}(f^\text{par}_*\mathcal{O}_\ell)(i).$$
such that the image of the elements in \( \widetilde{W} \cup \{ u \} \cup X^*(\widetilde{T}) \subset \mathbb{H} \) are the same as the ones given in Construction 3.3.4.

If we fix a point \((a, x) \in (A \times X)(k)\), we can specialize the above theorem to the action of \( \mathbb{H} \) on the stalk of \( f^*\mathcal{Q}_\ell \) at \((a, x)\), i.e., \( H^*(M^\text{par}_{a,x}) \).

3.3.6. Corollary. For \((a, x) \in (A \times X)(k)\), Construction 3.3.4 gives an action of \( \mathbb{H}/(\delta, u) \) on \( H^*(M^\text{par}_{a,x}) \). In other words, the actions of \( \xi \in X^*(G^\text{cen}_m \times T) \) and \( \tilde{w} \in \tilde{W} \) satisfy the following simple relation:

\[
\tilde{w}\xi = \xi \tilde{w}.
\]

Here \( \xi \mapsto \tilde{w}\xi \) is the action of \( \tilde{w} \) on \( X^*(G^\text{cen}_m \times T) = X^*(\widetilde{T})/X^*(G^\text{rot}_m) \).

Proof. Since the restrictions of \( \mathcal{O}_X(D) \) and \( \omega_X \) to \( M^\text{par}_{a,x} \) are trivial, the actions of \( \delta \) and \( u \) on \( H^*(M^\text{par}_{a,x}) \) are zero. \( \square \)

Sec. 3.5 through Sec. 3.7 are devoted to the proof of Th. 3.3.5. We will check that the four conditions in Def. 3.3.2 hold for the actions defined in Construction 3.3.4.

The condition (1) in Def. 3.3.2 is trivial from construction. The condition (2) is also easy to check. In fact, since the \( \tilde{W} \)-action is constructed from self-correspondences of \( M^\text{par} \) over \( A \times X \), it commutes with the cup product with any class in \( H^*(A \times X) \). In particular, the \( \tilde{W} \)-action commutes with the \( u \)-action and the \( X^*(G^\text{rot}_m) \)-action. On the other hand, the action of \( \xi \in X^*(\widetilde{T}) \) is defined as the cup product with the Chern class \( c_1(\mathcal{L}(\xi)) \in H^2(M^\text{par})(1) \), which certainly commutes with the cup product with the pull-back of \( c_1(D) \in H^2(X)(1) \). Therefore the \( X^*(\widetilde{T}) \)-action also commutes with the \( u \)-action. This verifies the condition (2) in Def. 3.3.2.

We will verify the condition (1) in Sec. 3.5 and the condition (3) in Sec. 3.6 and Sec. 3.7.

3.4. Remarks on Hecke correspondences. In this subsection, we study the relation between the reduced Hecke correspondence \( \mathcal{H}_{\tilde{w}} \) introduced in [Yun1, Def. 4.3.9] and the stratum \( \text{Hecke}_{\tilde{w}}^\text{Bun} \) in the Hecke correspondence \( \text{Hecke}^\text{Bun} \) (see [Yun1, Section 4.2]), for the same subscript \( \tilde{w} \in \tilde{W} \).

3.4.1. Rewriting the Hecke correspondences. Let us first write \( \text{Hecke}^\text{Bun}_{\tilde{w}} \) more precisely using the identification \( \text{Bun}_{G}^\text{par} = \text{Bun}_{\infty}/G_1 \times \text{Aut}_\mathcal{O} \) in Sec. 2.3. We have

\[
\text{Hecke}^\text{Bun}_{\tilde{w}} = \text{Bun}_{\infty} \times_{(G((t))) \rtimes \text{Aut}_\mathcal{O}} G_1 \times \text{Aut}_\mathcal{O}
\]

with the two projections given by

\[
\tilde{b}(\xi, \tilde{g}) = \xi \mod G_1 \times \text{Aut}_\mathcal{O};
\]
\[
\tilde{b}(\xi, \tilde{g}) = R\tilde{g}(\xi) \mod G_1 \times \text{Aut}_\mathcal{O}.
\]
for $\xi \in \widetilde{\text{Bun}}_\infty(S)$ and $\widetilde{g} \in (G((t)) \times \text{Aut}_\mathcal{O})(S)/(G_1 \times \text{Aut}_\mathcal{O})(S)$. The Bruhat decomposition gives

$$G((t)) = \bigsqcup_{\tilde{w} \in \tilde{W}} G_1 \tilde{w} G_1;$$

$$G((t)) \times \text{Aut}_\mathcal{O} = \bigsqcup_{\tilde{w} \in \tilde{W}} (G_1 \times \text{Aut}_\mathcal{O}) \tilde{w} (G_1 \times \text{Aut}_\mathcal{O}).$$

Hence we can write

$$\text{(3.8)} \quad \text{Hecke}^{\text{Bun}}_\tilde{w} = \text{Bun}_\infty \times_{G_1 \times \text{Aut}_\mathcal{O}} ((G_1 \times \text{Aut}_\mathcal{O}) \tilde{w} (G_1 \times \text{Aut}_\mathcal{O}))/G_1 \times \text{Aut}_\mathcal{O}).$$

Recall that we have a morphism (see [Yun1, Diagram (4.3)])

$$\beta : \text{Hecke}^{\text{Par}} \to \text{Hecke}^{\text{Bun}}.$$

3.4.2. Lemma. 

1. The image of $H_{\tilde{w}}^{\text{rs}}$ in $\text{Hecke}^{\text{Bun}}_\tilde{w}$ is contained in $\text{Hecke}^{\text{Bun}}_\tilde{w}$.

2. The image of $H_{\tilde{w}}$ in $\text{Hecke}^{\text{Bun}}_\tilde{w}$ is contained in $\text{Hecke}^{\text{Bun}}_{\tilde{w}}$.

Proof. Since $H_{\tilde{w}}$ is the closure of $H_{\tilde{w}}^{\text{rs}}$ by definition, (2) follows from (1). To check (1), it suffices to check the geometric points (or even the $k$-points). Fix $(a,x) \in (\mathcal{A} \times X)^{\text{rs}}(k)$. Using the local-global product formula [Yun1, Prop. 3.3.3], $\mathcal{M}_{a,x}^{\text{par}}$ is homeomorphic to

$$(3.10) \quad \mathcal{P}_{a}^{\text{red}(J_{\lambda}) \times P'} \times (M_{x}^{\text{par,red}}(\gamma) \times M')$$

where $M'$ and $P'$ are the products of local terms over $y \in X - \{x\}$, and $\gamma \in g(\tilde{O}_x)$ lifting $a(x) \in c(\tilde{O}_x)$. Since $a(x)$ has regular semisimple reduction in $c$, we can conjugate $\gamma$ by $G(\tilde{O}_x)$ so that $\gamma \in t(\tilde{O}_x)$ (here $t \subset b$ is a Cartan subalgebra). The choice of $\gamma$ gives a point $\tilde{x} \in q_a^{-1}(x)$. In particular, we can use $\tilde{x}$ to get an isomorphism $P_x(J_\lambda) = T(F_x)/T(\tilde{O}_x)$.

Fix a uniformizing parameter $t \in \tilde{O}_x$. Now the reduced structure of $M_{x}^{\text{par}}(\gamma) \subset \mathcal{F}_{\ell,\tilde{G},x}$ consists of the $T$-fixed points $\tilde{w}1_{x}/I_x = t^{\lambda_1}w_11_{x}/I_x$ for $\tilde{w} = (\lambda_1,w_1) \in \tilde{W}$. Under the isomorphism $M_{x}^{\text{par}}(\gamma) = M_{x}^{\text{Hit}}(\gamma) \times q_a^{-1}(x)$, $\tilde{w}1_{x}/I_x$ corresponds to the pair $(t^{\lambda_1}G(\tilde{O}_x),G(\tilde{O}_x),w_1^{-1}\tilde{x})$.

We claim that the action of $\tilde{w} = (\lambda,w) \in \tilde{W}$ on $M_{x}^{\text{par}}$, under the product formula $[3.10]$, is trivial on $M'$ and sends $\tilde{w}1_{x}/I_x \in M_{x}^{\text{par}}(\gamma)$ to $\tilde{w}1_{x}/I_x$. In fact, using the definition of the right $\tilde{W}$-action in [Yun1, Cor. 4.3.8], we have

$$\tilde{w}1_{x}/I_x \cdot \tilde{w} = (t^{\lambda_1}G(\tilde{O}_x),G(\tilde{O}_x),w_1^{-1}\tilde{x}) \cdot (\lambda,w)$$

$$= (s_{\lambda}(a,w_1^{-1}\tilde{x})t^{\lambda_1}G(\tilde{O}_x),G(\tilde{O}_x),w_1^{-1}\tilde{x})$$

$$= (s_{w_1\lambda}(a,\tilde{x})t^{\lambda_1}G(\tilde{O}_x),G(\tilde{O}_x),w_1^{-1}\tilde{x})$$

$$= (t^{w_1\lambda_1}G(\tilde{O}_x),G(\tilde{O}_x),w_1^{-1}\tilde{x})$$

$$= \tilde{w}1_{x}/I_x.$$

Clearly, the pair $(\tilde{w}1_{x}/I_x,\tilde{w}1_{x}/I_x) \in \mathcal{F}_{\ell,\tilde{G},x} \times \mathcal{F}_{\ell,\tilde{G},x}$ is in relative position $\tilde{w}$, hence the image of $H_{\tilde{w}}^{\text{rs}}$ in $\text{Hecke}^{\text{Bun}}_\tilde{w}$ is contained in $\text{Hecke}^{\text{Bun}}_{\tilde{w}}$. □

3.5. Connected components of $M^{\text{par}}$. In this subsection, we check the condition [4] in Def. 3.3.2.
3.5.1. Connected components of \( \text{Bun}_{\text{par}} \) and \( \mathcal{M}^{\text{par}} \). It is well-known that the set of connected components of \( \text{Bun}_G \) or \( \text{Bun}_{\text{par}}^G \) is naturally identified with \( \Omega = \mathbb{X}_*(T)/\mathbb{Z}^\vee \), such that the component containing the image of \( \text{Bun}_{G^{sc}} \to \text{Bun}_G \) is indexed by the identity element in \( \Omega \) (here \( G^{sc} \) is the simply-connected form of the derived group \( G^{\text{der}} \) of \( G \)). For any \( \omega \in \Omega \), let \( \text{Bun}_\omega^{\text{par}} \) be the corresponding component. We also write \( \mathcal{M}^{\text{par}}_\omega \) for the preimage of \( \text{Bun}_\omega^{\text{par}} \).

Recall that \( \mathbf{I} \) is the normalizer of \( \mathbf{I} \) in \( G(F) \), and \( \Omega_1 = \mathbf{I}/\mathbf{I} \) can be identified with \( \Omega \) via \( \Omega_1 \to \mathbf{W} \to \Omega \). For any \( \omega \in \Omega_1 \), by Lem. \( 2.4.1 \) we have an automorphism \( R_\omega \) of \( \text{Bun}_{\text{par}}^{\mathbf{I}} \), which sends the connected component \( \text{Bun}_\omega^{\text{par}} \) of \( \text{Bun}_{\omega_1}^{\text{par}} \). Similarly remark applies to the action of \( \Omega_1 \) on \( \mathcal{M}^{\text{par}}_\omega \).

On the other hand, we can view \( \omega \in \Omega_1 \) as an element of \( \mathbf{W} \). Therefore \( \omega \) gives a double coset in \( \Gamma(G(F))/\mathbf{I} \), hence a Hecke correspondence (see the beginning of [YunI Sec. 4.1])

\[
\begin{array}{ccc}
\text{Hecke}_{\omega}^{\text{Bun}} & \overset{\varphi_\omega}{\rightarrow} & \text{Bun}_G^{\text{par}} \\
\downarrow & & \downarrow \quad \quad \downarrow \varphi_\omega \\
\text{Bun}_G^{\text{par}} & \rightarrow & \text{Bun}_G^{\text{par}}
\end{array}
\]

classifying pairs of \( G \)-torsors with Borel reductions at a point of \( X \) which are in relative position \( \omega \).

3.5.2. Lemma. For \( \omega \in \Omega_1 \), the correspondence \( \text{Hecke}_{\omega}^{\text{Bun}} \) is the graph of the automorphism \( R_\omega : \text{Bun}_{\text{par}}^{G} \to \text{Bun}_{\text{par}}^{G} \).

Proof. It is clear that the Schubert cell \( G_1 \omega G_1/G_1 \) consists of one point for any \( \omega \in \Omega_1 \). Therefore the lemma follows from the description \( 3.5.3 \) of \( \text{Hecke}_{\omega}^{\text{Bun}} \). \( \square \)

3.5.3. Corollary. The reduced Hecke correspondence \( \mathcal{H}_\omega \) for the parabolic Hitchin stack \( \mathcal{M}^{\text{par}} \) is the graph of the automorphism \( R_\omega : \mathcal{M}^{\text{par}} \to \mathcal{M}^{\text{par}} \). In particular, the action of \( \omega \in \Omega_1 \subset \mathbf{W} \) on \( f_\omega^{\text{par}} \) defined in [YunI Th. 4.4.3] is the same as \( R_\omega \).

Proof. By the construction of the \( \Omega_1 \)-action on \( \mathcal{M}^{\text{par}} \), for any \( m \in \mathcal{M}^{\text{par}}(R) \) with image \( x \in X(R) \), the Hitchin pairs on \( X_R - \Gamma(x) \) given by restrictions of \( m \) and \( R_\omega m \) are canonically identified. Therefore, there is a natural embedding \( \Gamma(R_\omega) \hookrightarrow \text{Hecke}_{\omega}^{\text{Bun}} \), where \( \Gamma(R_\omega) \) is the graph of \( R_\omega \).

Recall the morphism \( \beta : \text{Hecke}_{\omega}^{\text{Bun}} \to \text{Hecke}_{\omega}^{\text{par}} \) in (3.9). We know from Lem. \( 3.5.2 \) that the \( \beta(\Gamma(R_\omega)) = \text{Hecke}_{\omega}^{\text{Bun}} \). In other words,

\[
\Gamma(R_\omega) \subset \beta^{-1}(\text{Hecke}_{\omega}^{\text{Bun}})^{\text{red}}.
\]

On the other hand, by Lem. \( 3.4.2 \) the reduced structure of \( \beta^{-1}(\text{Hecke}_{\omega}^{\text{Bun}})^{\text{rs}} \) is contained in \( \mathcal{H}^{\text{rs}} \), hence \( \Gamma(R_\omega)^{\text{rs}} \subset \mathcal{H}^{\text{rs}} \). Since both \( \Gamma(R_\omega)^{\text{rs}} \) and \( \mathcal{H}^{\text{rs}} \) are graphs, we must have \( \Gamma(R_\omega)^{\text{rs}} = \mathcal{H}^{\text{rs}} \). Taking closures, we get \( \Gamma(R_\omega) = \mathcal{H}_\omega \). \( \square \)

By Construction \( 3.3.4 \) the action of \( \mathbb{X}^*(\mathbf{T}) \) on \( f_\omega^{\text{par}} \) is defined by the cup product with the Chern classes of the pull-back of the line bundles \( \mathcal{L}(\xi) \) from \( \text{Bun}_G^{\text{par}} \). Now Lem. \( 3.5.3 \) reduces the verification of the condition \( 4 \) in Def. \( 3.3.2 \) to the following fact:
3.5.4. **Lemma.** For each \( \omega \in \Omega_i \) and \( \xi \in \mathcal{X}^*(\mathring{T}) \), there is an isomorphism of line bundles on \( \text{Bun}^\text{par}_G \):

\[
R^*_\omega \mathcal{L}(\xi) \cong \mathcal{L}^*(\omega \xi).
\]

**Proof.** Recall from Construction 3.2.2 that the right action of \( \mathring{T} \) on \( \mathcal{L}^\mathring{\mathcal{S}} \) comes from the right action of \( \mathcal{G}_I \) on \( \text{Bun}_\infty \). On the other hand, the right action of \( \Omega_i \) on \( \text{Bun}^\text{par}_G \) comes from the right action of \( \mathcal{G}_I \) on \( \text{Bun}_\infty \) (see the discussion in the beginning of Sec. 3.3). For any \( g \in \mathcal{G}_I \) and \( \omega \in \Omega_i \), it is clear that:

\[
R_{\text{Ad}(\omega^{-1})g} \circ R_\omega = R_\omega \circ R_g.
\]

Taking the quotient by \( \mathcal{G}_I \), we get an equality of actions on \( \mathcal{L}^\mathring{\mathcal{S}} = \mathcal{Bun}_\infty / \mathcal{G}_I^\mathring{\mathcal{S}} \):

\[
R_{\text{Ad}(\omega^{-1})g} \circ R_\omega = R_\omega \circ R_g, \text{ for } \omega \in \Omega_i, g \in \mathring{T}.
\]

Therefore the \( \mathring{T} \)-torsor \( R^*_\omega \mathcal{L}^\mathring{\mathcal{S}} \) on \( \text{Bun}^\text{par}_G \) is the \( \text{Ad}(\omega) \)-twist of \( \mathcal{L}^\mathring{\mathcal{S}} \). This proves the lemma. \( \square \)

3.6. **Simple reflections—a calculation in \( \mathfrak{sl}_2 \).** In this subsection, we check the condition \( \{11\} \) in Def. 3.3.2 for \( \xi \in \mathcal{X}_s(T \times G^\text{rot}_m) \). The idea is to reduce the problem to a calculation for the Steinberg variety of \( \text{SL}_2 \). For \( i = 0, \cdots, n \), let \( \mathbf{P}_i \) be the standard parahoric subgroup whose Lie algebra \( \mathfrak{g}_i \) is spanned by \( \mathfrak{g} \) and the root space of \( -\alpha_i \). We will abbreviate \( \mathbf{L}_i, \mathbf{P}_i, \mathbf{B}_i^{\mathfrak{P}_i}, \mathbf{b}_i^{\mathfrak{P}_i}, \) etc. by \( L_i, B_i, B_i, b_i, \) etc.

3.6.1. **Lemma.** The reduced Hecke correspondence \( \mathcal{H}_s \) is a closed substack of \( C_i = \mathcal{M}^\text{par}_i \times \mathbf{M}_{\mathbf{P}_i} \mathcal{M}^\text{par}_i \).

**Proof.** For two point \((x, \mathcal{E}_1, \varphi_1, \mathcal{E}_{2,i}) \in \mathcal{M}^\text{par}(R) \) \((i = 1, 2)\) with the same image in \( \mathbf{M}_{\mathbf{P}_i} \), we have a canonical isomorphism \((\mathcal{E}_1, \varphi_1)|_{X_{R_i \sim \Gamma}} \cong (\mathcal{E}_2, \varphi_2)|_{X_{R_i \sim \Gamma}} \).

Therefore, we have a canonical embedding of self-correspondences of \( \mathcal{M}^\text{par}_i \):

\[
\gamma_i : C_i := \mathcal{M}^\text{par}_i \times \mathbf{M}_{\mathbf{P}_i} \mathcal{M}^\text{par}_i \hookrightarrow \mathfrak{Hecke}^\text{par}_i.
\]

Note that \( \mathfrak{Hecke}^\text{Bun}_i = \mathcal{Bun}^\text{par}_G \times \mathbf{B}_{\mathbf{P}_i} \mathcal{Bun}^\text{par}_G \), therefore the image of \( \gamma_i(C_i) \) in \( \mathfrak{Hecke}^\text{Bun}_i \) lies in \( \mathfrak{Hecke}^\text{Bun}_{i,s} \). Then by Lem. 3.3.2, the reduced structure of \( \gamma_i(C_i^{\mathfrak{rs}}) \) must lie in \( \mathfrak{H}^{\mathfrak{rs}}_{\leq s} = \mathfrak{H}^{\mathfrak{rs}}_{\leq s} \bigcup \mathfrak{H}^{\mathfrak{rs}}_s \). By Lem. 2.5.8 applied to \( i \in \mathbf{P}_i \), we see that \( \mathfrak{F}_{\mathbf{P}_i}^{\mathfrak{rs}} : \mathcal{M}^\text{par}_i \rightarrow \mathcal{M}^\text{par}_i \) is an étale double cover. Therefore the two projections \( C_i^{\mathfrak{rs}} \rightarrow \mathcal{M}^\text{par}_i \) are also étale double covers. Since the two projections \( \mathfrak{H}^{\mathfrak{rs}}_{\leq s} = \mathcal{M}^\text{par}_i \) are also étale double covers, \( \gamma_i \) must induce an isomorphism \( C_i^{\mathfrak{rs}} \cong \mathfrak{H}^{\mathfrak{rs}}_{\leq s} \). Taking closures, we conclude that \( \mathcal{H}_s \) lies in \( \gamma_i(C_i) \). This proves the lemma. \( \square \)

By Lem. 2.5.8 and Lem. 3.6.1, we have a Cartesian diagram of correspondences

\[
\begin{array}{cccc}
C_i & \overset{c_i}{\longrightarrow} & [\text{St}_i / \mathcal{L}_i]_D & \overset{[\text{St}_i / \mathcal{L}_i]^1}{\longrightarrow} \\
\mathcal{M}^\text{par}_i & \overset{[\text{B}_i / \mathcal{B}_i]^1}{\longrightarrow} & [\tilde{l}_i / \mathcal{L}_i]_D & \overset{[\tilde{l}_i / \mathcal{L}_i]^1}{\longrightarrow} \\
\mathcal{M}_{\mathbf{P}_i} & \overset{\pi^i}{\longrightarrow} & [\tilde{l}_i / \mathcal{L}_i]_D & \overset{[\tilde{l}_i / \mathcal{L}_i]^1}{\longrightarrow}
\end{array}
\]

(3.11)
We explain the notations. Here $\text{St}_i$ is the Steinberg variety $\tilde{I}_i \times I_i$ of $I_i$ and $\tilde{I}_i$ is the Grothendieck simultaneous resolution of $I_i$. Recall that the action of $\text{Aut}_Q$ on $L_i$ factors through a finite dimensional quotient $Q$. We assume that $Q$ surjects to $\mathbb{G}_m^\text{rot}$. The conjugation action of $L_i$ on $\tilde{I}_i$ and the action of $\text{Aut}_Q$ on $L_i$ gives an action of $L_i \rtimes Q$ on $L_i$, and hence on $I_i$, $I_i$, and $\text{St}_i$. The group $L_i^\sharp$ in the diagram (3.1) is defined as
\[
L_i^\sharp = (L_i \rtimes Q) \times \mathbb{G}_m,
\]
which acts on $I_i$, $I_i$, and $\text{St}_i$ with $\mathbb{G}_m$ acting by dilation.

The natural projection $B^i \to T$ extends to the projection $B^i \rtimes Q \to T \times \mathbb{G}_m^\text{rot}$. Therefore we have a morphism $[I_i/L_i^\sharp] = [B^i/(B^i \rtimes Q \times \mathbb{G}_m^\text{rot}]) \to \mathbb{B}(T \times \mathbb{G}_m^\text{rot})$, which gives a $T \times \mathbb{G}_m^\text{rot}$-torsor on $[I_i/L_i^\sharp]$. The associated line bundles on $[I_i/L_i^\sharp]$ are denoted by $\mathcal{N}(\xi)$, for $\xi \in \mathbb{X}^*(T \times \mathbb{G}_m^\text{rot})$.

Let $\text{St}_i = \text{St}_i^+ \cup \text{St}_i^-$ be the decomposition into two irreducible components, where $\text{St}_i^+$ is the diagonal copy of $I_i$, and $\text{St}_i^-$ is the non-diagonal component. Let $\epsilon$ be the composition
\[
\epsilon : C_i \to [\text{St}_i/L_i]^D \to [\text{St}_i/L_i^\sharp].
\]

### Lemma
For $\xi \in \mathbb{X}^*(T \times \mathbb{G}_m^\text{rot})$, the action of $s_i \xi - s_i \xi s_i - \langle \xi, \alpha_i^\vee \rangle u$ on $f_{2\ell}^\text{par}$ is given by the following cohomological correspondence in $\text{Corr}(C_i; \mathbb{G}_m^\ell(2)(1), \mathbb{G}_m^\ell)$:
\[
\epsilon^* \left( [\text{St}_i^+/L_i^\sharp] \cup \left( \tilde{c}_i^* c_1(\mathcal{N}(\xi)) - \tilde{c}_i^* c_1(\mathcal{N}(s_i \xi)) \right) - [\text{St}_i^-/L_i^\sharp] \cup \langle \xi, \alpha_i^\vee \rangle v \right)
\]
where $v \in H^2([\text{St}_i/L_i^\sharp])((1)$ is the image of the generator of $H^2(\mathbb{B}\mathbb{G}_m)(1)$ (for the $\mathbb{G}_m$ factor in $L_i^\sharp$).

**Proof.** By Construction 3.3.4 and Lem. 3.1.1 about the cup product action on cohomological correspondences, the action of $s_i \xi - s_i \xi s_i - \langle \xi, \alpha_i^\vee \rangle u$ on $f_{2\ell}^\text{par}$ is given by the following cohomological correspondence in $\text{Corr}(C_i; \mathbb{G}_m^\ell(2)(1), \mathbb{G}_m^\ell)$:
\[
[\mathcal{H}_{s_i}] \cup (\tilde{c}_i^* c_1(\mathcal{L}(\xi)) - \tilde{c}_i^* c_1(\mathcal{L}(s_i \xi))) - \langle \xi, \alpha_i^\vee \rangle [\Delta(\mathcal{M}^\text{par})] \cup c_1(D),
\]
where $[\mathcal{H}_{s_i}] \in \text{Corr}(C_i; \mathbb{G}_m^\ell(2), \mathbb{G}_m^\ell)$ is the image of fundamental class of $\mathcal{H}_{s_i}$ via the natural closed embedding $\mathcal{H}_{s_i} \hookrightarrow C_i$ (see Lem. 3.3.1), and $\Delta(\mathcal{M}^\text{par}) \subset C_i$ is the diagonal.

Therefore, to prove the lemma, we have to check
\[
\begin{align*}
(3.12) \quad \epsilon^* \tilde{c}_i^* \mathcal{N}(\xi) &= \tilde{c}_i^* \mathcal{L}(\xi); \\
(3.13) \quad \epsilon^* \tilde{c}_i^* \mathcal{N}(s_i \xi) &= \tilde{c}_i^* \mathcal{L}(s_i \xi); \\
(3.14) \quad \epsilon^* v &= c_1(D) \in H^2(C_i(1)); \\
(3.15) \quad \epsilon^*[\text{St}_i^+/L_i^\sharp] &= [\mathcal{H}_{s_i}] \in \text{Corr}(C_i; \mathbb{G}_m^\ell(2), \mathbb{G}_m^\ell); 
\end{align*}
\]

Let $ev : \mathcal{M}^\text{par} \to [i_i/L_i^\sharp]$ be the evaluation morphism. Then we have $\mathcal{L}(\xi) = ev^* \mathcal{N}(\xi)$. By the first two rows of the diagram (3.11), we have
\[
\epsilon^* \tilde{c}_i^* \mathcal{N}(\xi) = \tilde{c}_i^* \epsilon^* \mathcal{N}(\xi) = \tilde{c}_i^* \mathcal{L}(\xi).
\]

(3.13) is proved in a similar way as (3.12).
By definition, we have a commutative diagram
\[
\begin{array}{ccc}
[\text{St}_i/L_i]_D & \rightarrow & [\text{St}_i/L_i \times Q \times G_m] \\
X & \quad \downarrow \rho_D & \quad B G_m \\
\end{array}
\]

Therefore, the generator \( v \in H^2(BG_m)(1) \) pulls back to \( c_1(D) \in H^2(X)(1) \), which further pulls back to \( c_1(D) \in H^2(C)(1) \).

As a finite type substack of \( \text{Heck} \), \( C \) satisfies (G-2) in [Yun, Def. A.5.1] with respect to \((A \times X)^{\tau s} \subset A \times X\) (see [Yun, Lem. 4.4.4]). By [Yun, Lem. A.5.2], we only need to verify the equality (3.15) over \((A \times X)^{\tau s} \), which is obvious.

By Lem. 3.6.2, the condition (3) for \( \xi \in X^*(T \times G_{m}^{\text{rot}}) \) reduces to the following identity.

3.6.3. Proposition. For each \( \xi \in X^*(T \times G_{m}^{\text{rot}}) \), the following identity hold in \( \text{Corr}([\text{St}_i/L_i], \overline{\Theta}_{\ell}[2](1), \overline{\Theta}_{\ell}) \):

\[
[\text{St}_i/L_i] \cup \left( \overline{\text{st}}_i^* c_1(A(\xi)) - \overline{\text{st}}_i^* c_1(A(\xi)) \right) = [\text{St}_i/L_i]^+ \cup \langle \xi, \alpha_i \rangle u.
\]

Proof. Since the reductive group \( L_i \) has semisimple rank one, we can decompose \( X^*(T \times G_{m}^{\text{rot}}) \) into \( \pm 1 \)-eigenspaces of the reflection \( s_i \):

\[
X^*(T \times G_{m}^{\text{rot}}) = X^*(T \times G_{m}^{\text{rot}})^{s_i} \oplus Q \alpha_i,
\]

where \( \alpha_i \) spans the \(-1\)-eigenspace of \( s_i \).

To prove (3.16), it suffices to prove it for \( \xi \in X^*(T \times G_{m}^{\text{rot}})^{s_i} \) and \( \xi = \alpha_i \) separately.

In the first case, taking Chern class induces an isomorphism

\[
c_1 : X^*(T \times G_{m}^{\text{rot}})^{s_i} \sim H^2(B(L_i \times Q)(1)) \rightarrow H^2(B(B^i \times Q))(1),
\]

Hence \( c_1(A(\xi)) \) lies in the image of the pull-back map

\[
\pi_i^*: H^2(B(L_i^2)(1) \rightarrow H^2(B(L_i^2))(1) \rightarrow H^2(B(B^i \times Q))(1).
\]

Since \( \pi_i \circ \overline{\text{st}}_i = \pi_i \circ \overline{\text{st}}_i \), we conclude that

\[
\overline{\text{st}}_i^* c_1(A(\xi)) = \overline{\text{st}}_i^* c_1(A(\xi)) = \overline{\text{st}}_i^* c_1(A(\xi))
\]

Therefore, the LHS of (3.16) is zero. On the other hand, since \( s_i^* \xi = \xi \), we have \( \langle \xi, \alpha_i \rangle = 0 \), hence the RHS of (3.16) is also zero. This proves the identity (3.16) in the case \( \xi \in X^*(T \times G_{m}^{\text{rot}})^{s_i} \).

Finally we treat the case \( \xi = \alpha_i \). Since \( L_i \times Q \) is connected, the action of \( L_i \times Q \) on \( L_i \) factors through a homomorphism \( L_i \times Q \rightarrow L_i^{\text{ad}} \), where \( L_i^{\text{ad}} \) is the adjoint from of \( L_i \) (isomorphic to \( \text{PGL}(2) \)). Let \( \mathbb{P}^1_i = L_i^{\text{ad}} \) be the flag variety of \( L_i \) or \( L_i^{\text{ad}} \). The pull-back

\[
H^2_{L_i^{\text{ad}}}(\mathbb{P}^1_i) \rightarrow H^2(B(L_i^{\text{ad}} \times Q))(1) = X^*(T \times G_{m}^{\text{rot}})_{\mathbb{P}^1_i}
\]

has image \( \overline{\text{st}}_i \alpha_i \), and the line bundle \( N(\alpha_i) \) on \( L_i \) is the pull-back of the canonical bundle \( \omega_{\mathbb{P}^1_i} \) on \( \mathbb{P}^1_i \). We can therefore only consider the \( L_i^{\text{ad}} \times G_m^{\text{rot}} \)-action on \( L_i \). The
equality (3.16) then reduces to the following identity in the \( L^d_{t_1} \times G_m \)-equivariant Borel-Moore homology group \( H^{BM,L^d_{t_1} \times G_m}_{2d-2}(St_i)(1) \) (\( d = \dim St_i \)):

\[
(3.17) \quad h^{-1}c_1(\omega_{P^1_i \times P^1_i}) \cup [St_i^{-}] = 2v \cup [St_i^{+}],
\]

where \( h^{-} \) is the \( L^d_{t_1} \times G_m \)-equivariant morphism \( h^{-} : St_i^{-} \to P^1_i \times P^1_i \).

We claim that both sides of (3.17) are equal to the fundamental class \( 2[St^{nil}_i] \in H^{BM,L^d_{t_1} \times G_m}_{2d-2}(St_i)(1) \), where \( St^{nil}_i \) is the preimage of the nilpotent cone in \( St_i \) under \( St_i \to i \).

On one hand, \( St^{nil}_i \) is the preimage of the diagonal \( \Delta(P^1_i) \subset P^1_i \times P^1_i \) under \( h^{-} \). Let \( \mathcal{I}_\Delta \) be the ideal sheaf of the diagonal \( \Delta(P^1_i) \), viewed as an \( L^d_{t_1} \)-equivariant line bundle on \( P^1_i \times P^1_i \). We claim that

\[
(3.18) \quad \mathcal{I}_\Delta \otimes 2 \cong \omega_{P^1_i \times P^1_i} \in \text{Pic}_{L^d_{t_1}}(P^1_i \times P^1_i).
\]

In fact, since \( L^d_{t_1} \) does not admit nontrivial characters, we have an isomorphism \( \text{Pic}_{L^d_{t_1}}(P^1_i \times P^1_i) \cong \mathbb{Z} \oplus \mathbb{Z} \) given by taking the degrees along the two rulings of \( P^1_i \times P^1_i \).

Then (3.18) follows by comparing the degrees along the rulings.

Since the Poincaré dual of \( c_1(\mathcal{I}_\Delta) \) is the cycle class \( [\Delta(P^1_i)] \in H^2_{BM,L^d_{t_1}}(P^1_i \times P^1_i)(1) \), we get from (3.18) that

\[
(3.19) \quad c_1(\omega_{P^1_i \times P^1_i}) \cup [P^1_i \times P^1_i] = 2[\Delta(P^1_i)].
\]

Since the morphism \( h^{-} \) is smooth (\( St_i^{-} \) is in fact the total space of a line bundle over \( P^1_i \times P^1_i \)), we can pull-back (3.19) along \( h^{-} \) to get

\[
(3.20) \quad h^{-1}c_1(\omega_{P^1_i \times P^1_i}) \cup [St_i^{-}] = 2[St^{nil}_i] \in H^{BM,L^d_{t_1} \times G_m}_{2d-2}(St_i^{-})(1).
\]

On the other hand, consider the projection \( \tau : i \to t \to t^{ad} \) \((t^{ad} \) is the universal Cartan for \( L^d_{t_1} \), then \( St^{nil}_i = \tau^{-1}(0) \). The class \( [0] \in H^0_{BM,G_m}(t^{ad})(1) \) is the Poincaré dual of \( u \) \((G_m \) acts on the affine line \( t^{ad} \) by dilation). Since \( \tau \) is \( L^d_{t_1} \times G_m \)-equivariant and \( L^d_{t_1} \) acts trivially on \( t^{ad} \), we conclude that

\[
(3.21) \quad [St^{nil}_i] = u \cup [St^{+}_i] \in H^{BM,L^d_{t_1} \times G_m}_{2d-2}(St_i^{+})(1).
\]

If we view both identities (3.20) and (3.21) as identities in \( H^{BM,L^d_{t_1} \times G_m}_{2d-2}(St_i)(1) \), we get the identity (3.17). This completes the proof. \( \square \)

3.7. Completion of the proof of Theorem 3.3.5. To prove Th. 3.3.5 it only remains to verify the relation (3) in Def. 3.3.2 for \( \xi = \Lambda_{can} \).

For each standard parahoric subgroup \( P \subset G(F) \), define a line bundle \( L_{P,can} \) on \( \text{Bun}_P \) which to every point \( (x, \mathcal{E}, x \mod P) \in \text{Bun}_P(R) \) assigns the invertible \( R \)-module \( \det \mathcal{R} \Gamma(X_R, \text{Ad}_P(\mathcal{E})) \). In particular, \( L_{G,can} \) is the pull-back of \( \omega_{\text{Bun}_G} \) from \( \text{Bun}_G \) to \( \text{Bun}_G = \text{Bun}_G \times X \).

3.7.1. Lemma. For each standard parahoric subgroup \( P \subset G(F) \), we have

\[
(3.22) \quad L_{I,can} \otimes L(-2i_P) \cong \text{Pic}^{1,-}L_{P,can} \in \text{Pic}(\text{Bun}_G^{par}).
\]

Here \( 2i_P \) is the sum of positive roots in \( L_P \) (with respect to the Borel \( B^{P}_F \)).

Proof. For \( (x, \alpha, \mathcal{E}, x \mod I) \in \text{Bun}_I(R) \), we have an exact sequence of vector bundles on \( X_R \)

\[
(3.23) \quad 0 \to \text{Ad}_I(\mathcal{E}) \to \text{Ad}_P(\mathcal{E}) \to i_* \mathcal{Q}(\mathcal{E}) \to 0
\]
where $Q(E)$ is a coherent sheaf supported on $\Gamma(x)$. As $E$ varies, we can view $Q$ as a vector bundle over $\text{Bun}_{\text{par}}^G$. Via the local coordinate $\alpha$ and the full level structure $\tau_x$, we can identify $Q(E)$ with the $R$-module $(g_P / g_I) \otimes_k R$. In other words, we have

$$Q \cong \tilde{\text{Bun}}_\infty \times _{G_1 \times \text{Aut}_\mathcal{O}} (g_P / g_I).$$

Taking the determinant, we get

$$\det Q \cong \tilde{\text{Bun}}_\infty \times \text{det}(g_P / g_I).$$

Since the action of $G_1 \times \text{Aut}_\mathcal{O}$ on $\text{det}(g_P / g_I)$ factors through the quotient $G_1 \times \text{Aut}_\mathcal{O} \twoheadrightarrow T \times \mathbb{G}_m$, we conclude that

$$\det Q \cong \mathcal{L}(-2\rho_P).$$

Taking the determinant of the exact sequence (3.23), we get

$$\det \mathcal{R}\Gamma(X_R, \text{Ad}_P(E)) \cong \det \mathcal{R}\Gamma(X_R, \text{Ad}_I(E)) \otimes \det Q(E)$$

Plugging in (3.24), we get the isomorphism (3.22). □

3.7.2. Corollary. For each $i = 0, \cdots, n$, there is an isomorphism of line bundles on $\text{Bun}_{\text{par}}^G$:

$$\text{For}_i^{P, \ast} \mathcal{L}_{P, \text{can}} \cong \text{For}_i^{G, \ast} \omega_{\text{Bun}} \otimes \mathcal{L}(2\rho - \alpha_i)$$

where $2\rho$ is the sum of positive roots in $G$.

Since the self-correspondence $\mathcal{H}_{\text{can}}$ is over $\mathcal{M}_P$, hence over $\text{Bun}_{P, \text{can}}$. Using Lem. 3.7.2, we conclude that

$$s_i(\Lambda_{\text{can}} + 2\rho - \alpha_i) = (\Lambda_{\text{can}} + 2\rho - \alpha_i)s_i \in \text{Hom}_{\mathcal{A}X}(f_{\text{par}} \mathcal{Q}_\ell, f_{\text{par}} \mathcal{Q}_\ell[2](1)).$$

Observe that for $i = 1, \cdots, n$, we have

$$\langle \Lambda_{\text{can}} + 2\rho - \alpha_i, \alpha_i^\vee \rangle = 2(\rho, \alpha_i^\vee) - 2 = 0.$$

For $i = 0$, we have

$$\langle \Lambda_{\text{can}} + 2\rho - \alpha_0, \alpha_0^\vee \rangle = \langle \Lambda_{\text{can}} + 2\rho - \delta + \theta, K - \theta^\vee \rangle = \langle \Lambda_{\text{can}}, K \rangle - 2(\rho, \theta^\vee) - 2 = 2h^\vee - 2h^\vee = 0.$$

Here we have used the fact that $\langle \Lambda_{\text{can}}, K \rangle = 2h^\vee$ (see Rem. 3.1.6) and $h^\vee = \langle \rho, \theta^\vee \rangle + 1$ (see Lem. 3.1.3). In any case, we have $\langle \Lambda_{\text{can}} + 2\rho - \alpha_i, \alpha_i^\vee \rangle = 0$ for $i = 0, \cdots, r$. This, together with (3.25) means that the relation (3) in Def. 3.3.2 holds for $s_i$ and $\xi = \Lambda_{\text{can}} + 2\rho - \alpha_i$. Since we have already proved the relation (3) in Def. 3.3.2 for $s_i$ and $\xi = 2\rho - \alpha_i \in X^\ast(T \times \mathbb{G}_m)$ in Sec. 3.6, we can subtract this relation from the one for $\xi = \Lambda_{\text{can}} + 2\rho - \alpha_i$, and conclude that the same relation also holds for $s_i$ and $\xi = \Lambda_{\text{can}}$. This completes the proof of relation (3) in Def. 3.3.2 and hence the proof of Th. 3.3.5.

4. Generalizations to parahoric Hitchin moduli stacks

In this section, we generalize the main results in [Yun1 Sec. 4] and Sec. 3 to the case of parahoric Hitchin moduli stacks of arbitrary type $P$. In particular, in the case $P = G$, we get an $\mathcal{Q}_\ell$-analogue of the so-called 'tHooft operators considered by Kapustin-Witten in their gauge-theoretic approach to the geometric Langlands program (see [KW]).
4.1. The action of the convolution algebra. We give two approaches to the parahoric version of [YunI, Th. 4.4.3]. One using Hecke correspondences (Construction 4.1.1) and the other using the smallness of $\tilde{P}_I$ (Construction 4.1.7).

As in the case of $M^\text{par}$, we can define the Hecke correspondence between two parahoric Hitchin moduli stacks $M_P$ and $M_Q$ over $A \times X$:

\[
\begin{array}{ccc}
\overset{\varnothing}{M_P} & \overset{h}{\longrightarrow} & \overset{\varnothing}{M_Q} \\
\overset{f_P}{\downarrow} & & \overset{f_Q}{\downarrow} \\
A \times X & & A \times X
\end{array}
\]

For any scheme $S$, $p\overset{\text{Hecke}}{\longrightarrow}Q(S)$ is the groupoid of tuples

\[(x,\xi_1,\tau_{1,x} \mod P, \varphi_1, \xi_2, \tau_{2,x} \mod Q, \varphi_2, \alpha)\]

where

- $(x,\xi_1,\tau_{1,x} \mod P, \varphi_1) \in M_P(S)$;
- $(x,\xi_2,\tau_{2,x} \mod Q, \varphi_2) \in M_Q(S)$;
- $\alpha$ is an isomorphism of Hitchin pairs $(\xi_1,\varphi_1) \sim_{S \times X - \Gamma(x)} (\xi_2,\varphi_2) \sim_{S \times X - \Gamma(x)}$.

4.1.1. Construction. For every double coset $W_P\tilde{w}W_Q \subset \widetilde{W}$, we will construct a graph-like closed sub-correspondence $\mathcal{H}_{W_P\tilde{w}W_Q}$ of $p\overset{\text{Hecke}}{\longrightarrow}Q$. Let $p\overset{\text{Hecke}}{\longrightarrow}Q_{\tilde{w}}$ be the reduced structure of the restriction of $p\overset{\text{Hecke}}{\longrightarrow}Q$ to $(A \times X)_{\text{rs}}$. By definition, we have an isomorphism

\[
M^{\text{par,rs}} \times M^\text{rs}_P \overset{p\overset{\text{Hecke}}{\longrightarrow}Q}{} \times M^\text{rs}_Q \overset{M^{\text{par,rs}}}{} \cong \mathcal{H}^\text{rs} = \bigsqcup_{\tilde{w} \in \tilde{W}} \mathcal{H}^\text{rs}_{\tilde{w}}.
\]

Recall that each $\mathcal{H}^\text{rs}_{\tilde{w}}$ is the graph of the right $\tilde{w}$-action on $M^{\text{par,rs}}$. Moreover, by Lem. 2.5.8, the projections $\overset{\text{For}}{\longrightarrow}P : M^{\text{par,rs}} \to M^\text{rs}_P$ and $\overset{\text{For}}{\longrightarrow}Q : M^{\text{par,rs}} \to M^\text{rs}_Q$ are the quotients under the right actions of $W_P \subset \tilde{W}$ and $W_Q \subset \tilde{W}$ on $M^{\text{par,rs}}$. If we identify $\mathcal{H}^\text{rs}_{\tilde{w}}$ with $M^{\text{par,rs}}$ via $h_{\tilde{w}}$, we also get a right $\tilde{W}$-action on $\mathcal{H}^\text{rs}_{\tilde{w}}$. By (4.2), the projection $\mathcal{H}^\text{rs}_{\tilde{w}} \to p\overset{\text{Hecke}}{\longrightarrow}Q^\text{rs}$ factors through the quotient

\[
\mathcal{H}^\text{rs}_{\tilde{w}} \to \mathcal{H}^\text{rs}_{\tilde{w}}/(W_P \cap \tilde{w}W_Q \tilde{w}^{-1}) \to p\overset{\text{Hecke}}{\longrightarrow}Q^\text{rs}.
\]

We define $\mathcal{H}_{W_P\tilde{w}W_Q}$ to be the closure of $\mathcal{H}^\text{rs}_{\tilde{w}}/(W_P \cap \tilde{w}W_Q \tilde{w}^{-1})$ in $p\overset{\text{Hecke}}{\longrightarrow}Q$. Clearly, $\mathcal{H}_{W_P\tilde{w}W_Q}$ only depends on the double coset $W_P\tilde{w}W_Q \subset \tilde{W}$. The projections from $\mathcal{H}^\text{rs}_{W_P\tilde{w}W_Q}$ to $M^\text{rs}_P$ and $M^\text{rs}_Q$ are finite étale, hence $\mathcal{H}_{W_P\tilde{w}W_Q}$ is graph-like.

4.1.2. The convolution algebras. To state a generalization of [YunI, Th. 4.4.3] to parahoric Hitchin moduli stacks, we first need to introduce certain convolution algebras. For standard parahoric subgroups $P, Q$, let

\[
\mathcal{E}_\ell[W_P \backslash \tilde{W}/W_Q] \subset \mathcal{E}_\ell[\tilde{W}]
\]

be the subspace of $\mathcal{E}_\ell$-valued functions on $\tilde{W}$ which are nonzero only at finitely many elements of $\tilde{W}$, left invariant under $W_P$ and right invariant under $W_Q$. We
define the convolution product:

(4.3) \( \mathcal{O}_\ell[\widetilde{W}/W_{P}] \otimes \mathcal{O}_\ell[\widetilde{W}/W_{Q}] \to \mathcal{O}_\ell[\widetilde{W}/W_{P}] \)

(4.4) \( f_1 \otimes f_2 \mapsto (f_1 \ast f_2)(\bar{w}) = \sum_{\bar{v} \in \widetilde{W}/W_{Q}} f_1(\bar{v}) f_2(\bar{v}^{-1} \bar{w}). \)

In particular, \( \mathcal{O}_\ell[\widetilde{W}/W_{P}] \) becomes a unital algebra under \( \ast \) with identity element \( 1_{W_{P}} \), the characteristic function of the double coset \( W_{P} \subset \widetilde{W} \). However, the natural embedding \( \mathcal{O}_\ell[\widetilde{W}/W_{P}] \subset \mathcal{O}_\ell[\widetilde{W}] \) is not an algebra homomorphism; it becomes an algebra homomorphism if we divide the inclusion map by \( \# W_{P} \).

4.1.3. Theorem. 

(1) For each pair of standard parahoric subgroups \((P, Q)\), the assignment 

\[ 1_{W_{P}} \ast w_{Q} \mapsto [\mathcal{H}_{W_{P}} \tilde{w} w_{Q}] \# \]

defines a map 

\[ \mathcal{O}_\ell[\widetilde{W}/W_{P}] \to \text{Corr}(\mathcal{H}_{W_{P}}; \mathcal{O}_\ell, \mathcal{O}_\ell) \xrightarrow{(\cdot) \#} \text{Hom}_{A \times X}(f_{Q}, \mathcal{O}_\ell, f_{P}, \mathcal{O}_\ell). \]

Then these maps are compatible with the convolution product in (4.3) and the composition of maps between the complexes \( f_{P}, \mathcal{O}_\ell \) for standard parahoric subgroups \( P \).

(2) In particular, for each standard parahoric subgroup \( P \), there is an algebra homomorphism 

\[ \mathcal{O}_\ell[\widetilde{W}/W_{P}] \to \text{Corr}(\mathcal{H}_{W_{P}}; \mathcal{O}_\ell, \mathcal{O}_\ell) \xrightarrow{(\cdot) \#} \text{End}_{A \times X}(f_{P}, \mathcal{O}_\ell). \]

sending \( 1_{W_{P}} \ast w_{P} \) to \([\mathcal{H}_{W_{P}} w_{P}] \# \). In other words, the convolution algebra 

\[ \mathcal{O}_\ell[\widetilde{W}/W_{P}] \]

acts on the complex \( f_{P}, \mathcal{O}_\ell \).

The proof of this theorem is similar to that of [Yun1, Th. 4.4.3]. The key ingredient is an analogue of [Yun1, Lem. 4.4.4] for \( \mathcal{H}_{W_{P}} \).

We give another way to construct the convolution algebra action in Th. 4.1.3 using the smallness of the forgetful morphisms \( \text{For}_{\mathcal{P}} \).

4.1.4. Construction. We mimic the construction of the classical Springer action reviewed in [Yun1, Construction 4.1.1]. Fix a standard parahoric \( P \). By Prop. 2.6.7 that the morphism \( \text{For}_{\mathcal{P}} : \mathcal{M}_{\text{par}} \to \mathcal{M}_{P} \) is small, therefore the shifted perverse sheaf \( \text{For}_{\mathcal{P}}^{-1} \mathcal{O}_\ell \) is the middle extension of its restriction to \( \mathcal{M}_{\text{par}}^{rs} \). Over \( \mathcal{M}_{\text{par}}^{rs} \), the morphism \( \text{For}_{\mathcal{P}}^{-1} \mathcal{O}_\ell \) is a right \( W_{P} \)-torsor by Lem. 2.5.8 therefore we get a left action of \( W_{P} \) on \( \text{For}_{\mathcal{P}}^{-1} \mathcal{O}_\ell|_{\mathcal{M}_{\text{par}}^{rs}} \), and hence on \( \text{For}_{\mathcal{P}}^{-1} \mathcal{O}_\ell \) by middle extension. Taking direct image along \( f_{P} \), we get a left action of \( W_{P} \) on \( f_{P}, \text{For}_{\mathcal{P}}^{-1} \mathcal{O}_\ell = f_{\text{par}}^{rs} \mathcal{O}_\ell \).

4.1.5. Lemma. The \( W_{P} \)-action on \( f_{\text{par}}^{rs} \mathcal{O}_\ell \) in Construction 4.1.4 coincides with the restriction of the \( \tilde{W} \)-action on \( f_{\text{par}}^{rs} \mathcal{O}_\ell \) in [Yun1, Th. 4.4.3] to \( W_{P} \).

Proof. Over \((A \times X)^{rs} \), it is easy to check that the right \( W_{P} \)-action on \( \mathcal{M}_{\text{par}}^{rs} \) given by Lem. 2.5.8 coincides with the restriction of the right \( \tilde{W} \)-action on \( \mathcal{M}_{\text{par}}^{rs} \)
constructed in [YunI] Cor. 4.3.8. Let $w \in W_P$. Since $\mathcal{H}_w$ is the closure of the $w$-action on $\mathcal{M}_{\mathfrak{rs}}^{\text{par}}$, we have an embedding

$$\bigsqcup_{w \in W_P} \mathcal{H}_w \subset \mathcal{M}_{\mathfrak{rs}}^{\text{par}} \times \mathcal{M}_{\mathfrak{rs}}^{\text{par}} \subset \text{Hecke}_{\mathfrak{rs}}^{\text{par}}.$$ 

Therefore we can view $\mathcal{H}_w$ as a self correspondence of $\mathcal{M}_{\mathfrak{rs}}^{\text{par}}$ over $\mathcal{M}_{\mathfrak{rs}}^{\text{par}}$. The cohomological correspondence $[\mathcal{H}_w]$ then gives an endomorphism $[\mathcal{H}_w]$ of $\widetilde{\mathcal{F}}_{\mathfrak{P}}^\mathfrak{I}, \mathcal{Q}_\ell$, which coincides with the $W_P$-action given in Construction 4.1.4 by the same argument as [YunI] Lem. 4.1.3, using the fact that $\widetilde{\mathcal{F}}_{\mathfrak{P}}^\mathfrak{I}, \mathcal{Q}_\ell$ is a middle extension. On the other hand, taking the direct image of $[\mathcal{H}_w]$ along $f_{\mathfrak{P},*}$, we get the action of $[\mathcal{H}_w]$ on $f_{\mathfrak{P},*}^{\text{par}} \mathcal{Q}_\ell$ considered in [YunI] Th. 4.4.3. This proves the lemma. □

Let $A$ be a finite group acting on an object $\mathcal{F}$ in a Karoubi complete $\mathcal{Q}_\ell$-linear category $\mathcal{C}$, then we have a canonical decomposition

$$(4.5) \quad \mathcal{F} = \bigoplus_{\rho \in \text{Irr}(A)} \mathcal{F}_\rho \otimes V_\rho$$

where $\text{Irr}(A)$ is the set of isomorphism classes of irreducible $\mathcal{Q}_\ell$-representations of $A$. For each $\rho \in \text{Irr}(A)$, $V_\rho$ is the vector space on which $A$ acts as $\rho$. In fact, the decomposition (4.5) is given by the images of the simple idempotents under the map $\mathcal{Q}_\ell[A] \to \text{End}_{\mathcal{C}}(\mathcal{F})$. In particular, we have a canonical direct summand of $\mathcal{F}$ corresponding to the trivial representation of $A$, which we denote by $\mathcal{F}_A$, and call it the $A$-invariants of $\mathcal{F}$.

4.1.6. Lemma. There is a canonical isomorphism in $D^b_c(A \times X)$:

$$(4.6) \quad f_{\mathfrak{P},*} \mathcal{Q}_\ell \cong (f_{\mathfrak{P},*}^{\text{par}} \mathcal{Q}_\ell)^{W_P}$$

Proof. The morphism $\widetilde{\text{For}}_{\mathfrak{P}}^{\mathfrak{I},\mathcal{M}_{\mathfrak{P}}} : \mathcal{M}_{\mathfrak{P}}^{\text{par}} \to \mathcal{M}_{\mathfrak{P}}^{\text{par}}$ gives a map of shifted perverse sheaves

$$(4.7) \quad \mathcal{Q}_\ell,_{\mathcal{M}_{\mathfrak{P}}} \to \widetilde{\text{For}}_{\mathfrak{P}}^{\mathfrak{I},\mathcal{Q}_\ell}.$$ 

Since $\widetilde{\text{For}}_{\mathfrak{P}}^{\mathfrak{I},\mathcal{Q}_\ell}$ is a $W_{\mathfrak{P}}$-torsor, it is clear that the restriction of the map (4.7) to $\mathcal{M}_{\mathfrak{P}}^{\text{par}}$ is the embedding of the $W_{\mathfrak{P}}$-invariants of the RHS. Since both sides of (4.7) are middle extensions from $\mathcal{M}_{\mathfrak{P}}^{\text{par}}$, we conclude that the map (4.7) can be identified with the inclusion of the $W_{\mathfrak{P}}$-invariants on the RHS. Taking $f_{\mathfrak{P},*}$ we get the isomorphism (4.10). □

4.1.7. Construction. Now it is easy to give another proof of Th. 4.1.3. From the $\bar{W}$-action on $f_{\mathfrak{P}}^{\text{par}} \mathcal{Q}_\ell$, we clearly have a map

$$\mathcal{Q}_\ell,_{\mathfrak{P}} \mathcal{W}_{\mathfrak{P}}^{\text{par}} \to \text{End}_{A \times X}(f_{\mathfrak{P}}^{\text{par}} \mathcal{Q}_\ell)^{W_{\mathfrak{P}}}, (f_{\mathfrak{P}}^{\text{par}} \mathcal{Q}_\ell)^{W_{\mathfrak{P}}}.$$ 

By Lem. 4.1.6 this gives a map

$$\mathcal{Q}_\ell,_{\mathfrak{P}} \mathcal{W}_{\mathfrak{P}}^{\text{par}} \to \text{End}_{A \times X}(f_{\mathfrak{P}},_{\mathfrak{P}} \mathcal{Q}_\ell, f_{\mathfrak{P}},_{\mathfrak{P}} \mathcal{Q}_\ell).$$ 

It is easy to check that this map is the same as the one constructed in Th. 4.1.3 and its compatibility with convolutions and compositions is clear from the fact that $\bar{W}$ acts on $f_{\mathfrak{P}}^{\text{par}} \mathcal{Q}_\ell$. 


4.2. The enhanced actions. We can also define the enhanced action on \( \tilde{f}_{P,*}\overline{\mathbb{Q}}_{\ell} \) as we did for \( f_{\mathfrak{f},*}\overline{\mathbb{Q}}_{\ell} \) in [YunII Prop. 4.4.6]. Consider the two projections of \( p\Heckep \) to \( \tilde{\mathbb{A}}_P \):

\[
\begin{array}{c}
p\Heckep \xrightarrow{(\tilde{f}_{P,*}\overline{\mathbb{Q}}_{\ell})} M_P \times_{A \times X} M_P \xrightarrow{(\tilde{f}_{P,*}\overline{\mathbb{Q}}_{\ell})} \tilde{\mathbb{A}}_P \times_{A \times X} \tilde{\mathbb{A}}_P.
\end{array}
\]

Let \( \Heckep_{[e]} \) be the preimage of the diagonal \( \tilde{\mathbb{A}}_P \subset \tilde{\mathbb{A}}_P \times_{A \times X} \tilde{\mathbb{A}}_P \), viewed as a self-correspondence of \( M_P \) over \( \tilde{\mathbb{A}}_P \).

4.2.1. Construction. For \( \lambda \in X_*(T) \), let \( |\lambda|_P \) denote its \( W_P \)-orbit in \( X_*(T) \). For each \( W_P \)-orbit \( |\lambda|_P \), we will construct a graph-like closed substack \( \mathcal{H}_{|\lambda|_P} \subset \Heckep_{|\lambda|_P} \). By the definition of \( \Heckep_{|\lambda|_P} \), we have a morphism

\[
\begin{array}{c}
\Heckep_{|\lambda|_P} \to \Heckep_{|\lambda|_P}
\end{array}
\]

as self-correspondences of \( M_P \) over \( \tilde{\mathbb{A}}_P \). Then we define \( \mathcal{H}_{|\lambda|_P} \) to be the reduced image of \( \mathcal{H}_\lambda \). Clearly, this image only depends on the \( W_P \)-orbit of \( \lambda \in X_*(T) \). The two projections from \( \mathcal{H}_{|\lambda|_P} \) to \( M_P \) are finite étale, hence \( \mathcal{H}_{|\lambda|_P} \) is graph-like.

4.2.2. Proposition. There is a unique algebra homomorphism

\[ (4.8) \quad \overline{\mathbb{Q}}_{\ell}[X_*(T)]^{W_P} \to \text{End}_{\mathbb{A}_P}(\tilde{f}_{P,*}\overline{\mathbb{Q}}_{\ell}), \]

such that \( \text{Av}_{W_P}(\lambda) := \sum_{\lambda' \in |\lambda|_P} \lambda' \) acts by \( [\mathcal{H}_{|\lambda|_P}] \) for any \( \lambda \in X_*(T) \).

Proof. The uniqueness is clear because \( \{\text{Av}_{W_P}(\lambda) \mid \lambda \in X_*(T)\} \) span \( \overline{\mathbb{Q}}_{\ell}[X_*(T)]^{W_P} \). By definition, we have an obvious associative convolution structure \( \Heckep_{|\lambda|_P} \to \Heckep_{|\lambda|_P} \) given by forgetting the middle \( M_P \). By the discussion in [YunII App. A6], this gives algebra structures on \( \text{Corr}(\Heckep_{|\lambda|_P}; \overline{\mathbb{Q}}_{\ell}, \overline{\mathbb{Q}}_{\ell}) \) and \( \text{Corr}(\Heckep_{|\lambda|_P}^{rs}; \overline{\mathbb{Q}}_{\ell}, \overline{\mathbb{Q}}_{\ell}) \). The same argument as [YunII Lem. 4.4.4] shows that any finite type substack of \( \Heckep_{|\lambda|_P} \) satisfies the condition (G-2) with respect to \( \tilde{\mathbb{A}}_P \). Therefore by [YunII Prop. A6.2], it suffices to establish an algebra homomorphism \( \overline{\mathbb{Q}}_{\ell}[X_*(T)]^{W_P} \to \text{Corr}(\mathcal{H}_{|\lambda|_P}^{rs}; \overline{\mathbb{Q}}_{\ell}, \overline{\mathbb{Q}}_{\ell}) \) sending \( \text{Av}_{W_P}(\lambda) \) to \( [\mathcal{H}_{|\lambda|_P}] \).

We have a commutative diagram of correspondences

\[
\begin{array}{ccc}
\mathcal{H}_{|\lambda|_P}^{rs} & \xrightarrow{q_{\mathcal{H}}} & \Heckep_{|\lambda|_P}^{rs} \\
\mathcal{M}_{|\lambda|_P}^{rs} & \xrightarrow{q} & \mathcal{M}_{|\lambda|_P}^{rs} \\
\tilde{\mathbb{A}}_P & \xrightarrow{\tilde{f}} & \mathbb{A}_P
\end{array}
\]

which is a base change diagram over \( \tilde{A}_P^{rs} \). Let \( \mathcal{H}_{|\lambda|_P}^{rs} \) is the reduced structure of \( \Heckep_{|\lambda|_P}^{rs} \). Then \( q_{\mathcal{H}}^{rs} \) gives an embedding of algebras (it is injective because \( q_{\mathcal{H}} \) is surjective)

\[ (4.10) \quad q_{\mathcal{H}}^{rs} : \text{Corr}(\mathcal{H}_{|\lambda|_P}^{rs}; \overline{\mathbb{Q}}_{\ell}, \overline{\mathbb{Q}}_{\ell}) \cong H^0(\mathcal{H}_{|\lambda|_P}^{rs}) \to H^0(\mathcal{H}_{|\lambda|_P}^{rs}) \cong \text{Corr}(\mathcal{H}_{|\lambda|_P}; \overline{\mathbb{Q}}_{\ell}, \overline{\mathbb{Q}}_{\ell}) \cong \overline{\mathbb{Q}}_{\ell}[X_*(T)]. \]

Here we used the fact that \( \tilde{h}_{|\lambda|_P}^{rs} \) and \( \overline{h}_{P,|\lambda|_P}^{rs} \) are étale, so that we can identify their dualizing complexes with the constant sheaf.
By Construction [4.2.1], we have scheme-theoretically that
\[ q^{-1}_H(H^*_\lambda|_P) = \bigcup_{\lambda' \in \lambda|_P} H^*_{\lambda'}. \]
Therefore \( q^*_H(H^*_\lambda|_P) \) is the element \( Av_{W_P}(\lambda) \in \underline{\mathbb{E}}_\ell[X_s(T)] \) under the embedding (4.10). Since the elements \( \{ Av_{W_P}(\lambda)| \lambda \in X_s(T) \} \) span the subalgebra \( \underline{\mathbb{E}}_\ell[X_s(T)]^{W_P} \) of \( \underline{\mathbb{E}}_\ell[X_s(T)]^{W_P} \), we conclude from (4.10) that the elements \( H^*_\lambda|_P \) also span a subalgebra of \( \text{Corr}(H^*_\lambda|_P; \underline{\mathbb{E}}_\ell, \underline{\mathbb{E}}_\ell) \) isomorphic to \( \underline{\mathbb{E}}_\ell[X_s(T)]^{W_P} \). This completes the proof. \( \square \)

4.2.3. **Remark.** The action of \( \underline{\mathbb{E}}_\ell[W_P \backslash \hat{W}/W_P] \) on \( f_{P,\ell} \underline{\mathbb{E}}_\ell \) constructed in Th. 4.1.3 and the action of \( \underline{\mathbb{E}}_\ell[X_s(T)]^{W_P} \) on \( \tilde{f}_{P,\ell} \underline{\mathbb{E}}_\ell \) (hence on \( f_{P,\ell} \underline{\mathbb{E}}_\ell \)) constructed in Prop. 4.2.2 are related by the embedding of algebras
\[ \underline{\mathbb{E}}_\ell[X_s(T)]^{W_P} \rightarrow \underline{\mathbb{E}}_\ell[W_P \backslash \hat{W}/W_P] \]
\[ Av_{W_P}(\lambda) \mapsto 1_{W_P \lambda W_P}. \]

4.2.4. **Remark.** In the special case \( P = G \), Th. 4.1.3 and Prop. 4.2.2 both give the same action of \( \underline{\mathbb{E}}_\ell[X_s(T)]^{W_P} \) on the complex \( f^P_{\ell,\ell} \underline{\mathbb{E}}_\ell \cong \underline{\mathbb{E}}_\ell \) on \( A \times X \). This can be viewed as a realization of \( \acute{t}t Hooft operators \) in the algebraic setting.

4.3. **Parahoric version of the DAHA action.** We also have a version of Th. 3.3.3 for general parahoric Hitchin fibrations.

4.3.1. **Construction.** Fix a standard parahoric subgroup \( P \subset G(F) \). Let \( E_P \) be the subalgebra of \( \mathbb{H} \) generated by \( \underline{\mathbb{E}}_\ell[W_P \backslash \hat{W}/W_P] \subset \underline{\mathbb{E}}_\ell[\hat{W}] \), \( \text{Sym}_{\underline{\mathbb{E}}_\ell}(X^*(\tilde{T})_{\underline{\mathbb{E}}_\ell})^{W_P} \subset \text{Sym}_{\underline{\mathbb{E}}_\ell}(X_s(\tilde{T})_{\underline{\mathbb{E}}_\ell}) \) and \( \underline{\mathbb{E}}_\ell[u] \). We now define the actions of generators of this algebra on \( f_{P,\ell} \underline{\mathbb{E}}_\ell \).

- In Th. 4.1.3(2), we have constructed an action of \( \underline{\mathbb{E}}_\ell[W_P \backslash \hat{W}/W_P] \) on \( f_{P,\ell} \underline{\mathbb{E}}_\ell \).
- \( u \) still acts by cup product with the pull-back of \( c_1(O_X(D)) \).
- Let \( P^u \subset P \) be the pro-unipotent radical and let \( G^u_P = G_{P^u} \rtimes \text{Aut}_{\mathcal{O}}^u \subset G_P \rtimes \text{Aut}_{\mathcal{O}} \subset G \). Let \( \tilde{G}_P \) be the preimage of \( G_P \) in \( \hat{G}(t) \). Consider the map \( \text{Bun}_{\infty}/G_P^u \rightarrow \text{Bun}_{\infty}/\tilde{G}_P \rtimes \text{Aut}_{\mathcal{O}} = \text{Bun}_{P^u} \), which is a right torsor under the reductive group \( L_P := \tilde{G}_P \rtimes \text{Aut}_{\mathcal{O}} / G_P^u \). The group \( L_P \) is isomorphic to \( G_{m}^\text{cen} \times L_P \rtimes G_{m}^\text{rot} \). The characteristic classes of this \( L_P \)-torsor are given by
\[ H^*(\mathbb{B}L_P) \cong H^*(\mathbb{B}\tilde{T})^{W_P} \cong \text{Sym}(X^*(\tilde{T})_{\underline{\mathbb{E}}_\ell}[-2](-1))^{W_P}. \]
These characteristic classes give a graded action of \( \text{Sym}(X^*(\tilde{T})_{\underline{\mathbb{E}}_\ell})^{W_P} \) on \( f_{P,\ell} \underline{\mathbb{E}}_\ell \) by cup product.

4.3.2. **Theorem.** There is a unique graded algebra homomorphism:
\[ H_P \rightarrow \bigoplus_{i \in \mathbb{Z}} \text{End}_{A \times X}(f_{P,\ell} \underline{\mathbb{E}}_\ell)(i) \]
such that \( f_{P,\ell} \underline{\mathbb{E}}_\ell[W_P \backslash \hat{W}/W_P], u \) and \( \text{Sym}(X^*(\tilde{T})_{\underline{\mathbb{E}}_\ell})^{W_P} \) acts as in Construction 4.3.1.

This can be proved by a similar argument as Th. 3.3.3. We omit the proof here.
4.3.3. Remark. Let \( 1_{WP} \in \overline{Q}_l[\tilde{W}] \) be the characteristic function of the subset \( WP \subset \tilde{W} \), then \( \frac{1}{#WP} 1_{WP} \) is an idempotent in \( \mathbb{H} \). It is not hard to check that
\[
\mathbb{H}_P = 1_{WP} \mathbb{H} 1_{WP}.
\]
Therefore, \( \mathbb{H}_P \) naturally acts on \( f^*_P, \overline{Q}_l = (f^*_P, \overline{Q}_l)^{WP} \) (see Lem. 4.1.6). This gives another proof of Th. 4.3.2.

5. Relation with the Picard stack action

In this section, we study the interaction between the graded DAHA action on \( f^*_P, \overline{Q}_l \) and the cap product action on it by the homology complex of the Picard stack \( P \). In Sec. 5.1, we study the commutation relation between the cap product action and the graded DAHA action. In Sec. 5.2, we relate the central part of the \( \overline{Q}_l[\tilde{W}] \)-action on the parabolic Hitchin complex to the action by the component groups of the Picard stack. For background on the cap product, we refer the readers to App. A.

5.1. The cap product and the double affine action. We apply the general discussions in Sec. A.3 to the situation of the \( P \)-action on \( M^\text{par} \) over \( \tilde{A} \) or over \( A \times X \).

Define the morphisms \( p \) and \( \tilde{p} \) as:
\[
\tilde{A} \longrightarrow A \times X \longrightarrow p \quad \tilde{p}
\]

Then we have the cap product actions
\[
\cap : \quad p^*H_*(P/A) \otimes f^*_P, \overline{Q}_l \rightarrow f^*_P, \overline{Q}_l,
\]
\[
\cap : \quad \tilde{p}^*H_*(P/A) \otimes \tilde{f}_*, \overline{Q}_l \rightarrow \tilde{f}_*, \overline{Q}_l.
\]

In this section, we study the relationship between these cap product actions and the graded double affine Hecke algebra action constructed in Th. 3.3.5.

5.1.1. Proposition. The cap product action of \( p^*H_*(P/A) \) on \( f^*_P, \overline{Q}_l \) commutes with the \( \tilde{W} \)-action defined in [Yun], Th. 4.4.3; the action of \( \tilde{p}^*H_*(P/A) \) on \( \tilde{f}_*, \overline{Q}_l \) commutes with the \( \tilde{W} \)-equivariant structure defined in [Yun], Prop. 4.4.6.

Proof. We give the proof of the first statement; the proof of the second one is similar. In Lem. B.2.3 of the Appendix, we give a sufficient condition for a cohomological correspondence to commute with the cap product. We want to apply this lemma to our situation.

The Picard stack \( P \) acts on \( Hecke^\text{par} \) by twisting the two parabolic Hitchin data by the same \( J_a \)-torsor. This makes \( h \) and \( \tilde{h} \) both \( P \)-equivariant. We want to apply Lem. B.2.3 to the correspondences \( H_{\tilde{w}} \) which are of finite type. To this end we have to check two things

1. The correspondence \( H_{\tilde{w}} \) is stable under the action of \( P \);
2. The fundamental class \( [H_{\tilde{w}}] \), viewed as an element of Corr(\( H_{\tilde{w}} ; \overline{Q}_l, \overline{Q}_l \)), is \( P \)-invariant (cf. Def. B.2.1).

We show (1). We first show that \( H_{\tilde{w}}^{rs} \) is stable under the \( P \)-action. Since \( H_{\tilde{w}}^{rs} \) is the graph of the right \( \tilde{w} \)-action on \( M^\text{par,rs} \), it suffices to show that the right \( \tilde{W} \)-action is \( P \)-equivariant. But this follows immediately from the explicit formula of the right \( \tilde{W} \)-action defined in [Yun], Cor. 4.3.8.,
We then show that \( H_\tilde{w} \) is stable under \( \mathcal{P} \). Since \( \mathcal{P} \) is smooth over \( \mathcal{A} \times X \), \( \mathcal{P} \times_{\mathcal{A} \times X} H_\tilde{w} \) is smooth, and in particular flat over \( H_\tilde{w} \). Since \( H_\tilde{w}^n \) is dense in \( H_\tilde{w} \), we conclude that \( \mathcal{P} \times_{\mathcal{A} \times X} H_\tilde{w}^n \) is dense in \( \mathcal{P} \times_{\mathcal{A} \times X} H_\tilde{w} \). Consider the action map \( \text{act} : \mathcal{P} \times_{\mathcal{A} \times X} H_\tilde{w} \to \text{Hecke}_{\text{par}} \). We already know from above that \( \text{act}(\mathcal{P} \times_{\mathcal{A} \times X} H_\tilde{w}^n) \) scheme-theoretically lands in \( H_\tilde{w}^n \), therefore by the density property we just observed, \( \text{act}(\mathcal{P} \times_{\mathcal{A} \times X} H_\tilde{w}) \) also lands scheme-theoretically in \( H_\tilde{w} \). This proves (1).

(2) follows from (1): both \( \text{act}^! \) and \( \text{proj}^! \) are the fundamental class \( [\mathcal{P} \times_{\mathcal{A} \times X} H_\tilde{w}] \).

Next we study the relation between the cap product by \( H_*(\mathcal{P} / \mathcal{A}) \) and the cup product by the Chern classes of \( \mathcal{L}(\xi) \).

5.1.2. Rewriting the Chern classes. Suppose \( \xi \in X^*(T) \). Recall from [Yun1] Construction 3.2.8 that we have a tautological \( T \)-torsor \( Q^T \) over \( \mathcal{P} \). Let \( Q(\xi) \) be the line bundle associated to \( Q^T \) and the character \( \xi : T \to G_m \). The Chern class of \( Q(\xi) \) gives a map

\[ c_1(Q(\xi)) : \overline{\mathcal{P}}_{\mathcal{A}} \to H^*(\overline{\mathcal{P}}/\overline{\mathcal{A}})[2](1) \]

Since \( \overline{\mathcal{A}} \) and \( \mathcal{A} \) are both smooth, we have

\[ \overline{\mathcal{P}}_{\mathcal{A}} \cong \overline{\mathcal{P}}[2](-1) \]

Let \( \beta : \overline{\mathcal{P}} \to \mathcal{P} \) be the projection. Since both \( \overline{\mathcal{P}} \) and \( \mathcal{P} \) are smooth, we have \( \beta^*\overline{\mathcal{P}}_{\mathcal{A}} \cong \overline{\mathcal{P}}[2](1) \). By proper base change, we have \( g_*\overline{\mathcal{P}} \cong g_*\beta^*\overline{\mathcal{P}}[2](-1) \cong \overline{\mathcal{P}}[2](-1) \), where \( g : \overline{\mathcal{P}} \to \overline{\mathcal{A}} \) and \( g : \mathcal{P} \to \mathcal{A} \) are the structure morphisms. Therefore

\[ H^*(\overline{\mathcal{P}}/\overline{\mathcal{A}}) \cong \overline{\mathcal{P}}^*H^*(\mathcal{P}/\mathcal{A})[2](-1) \]

Using (5.5) and (5.6), we can rewrite (5.4) as

\[ c_1(Q(\xi)) : \overline{\mathcal{P}}_{\mathcal{A}} \to \overline{\mathcal{P}}^*H^*(\mathcal{P}/\mathcal{A})[2](1) \]

By adjunction, this gives a map

\[ c_1(Q(\xi)) : H_*(\overline{\mathcal{A}}/\mathcal{A}) \to H^*(\mathcal{P}/\mathcal{A})[2](1) \]

5.1.3. Lemma. Under the natural decomposition \( [\mathcal{A}, \mathcal{A}] \), the map \( c_1(Q(\xi))^2 \) factors through

\[ c_1(Q(\xi))^2 : H_*(\overline{\mathcal{A}}/\mathcal{A}) \to H^*(\mathcal{P}/\mathcal{A})[1](1) \subset H^*(\mathcal{P}/\mathcal{A})[2](1) \]

Proof. By construction, the line bundle \( Q(\xi) \) on \( \overline{\mathcal{A}} \times \mathcal{A} \mathcal{P} \) is the pull-back of the Poincaré line bundle on \( \overline{\mathcal{A}} \times \mathcal{A} \mathcal{P} \mathcal{I}(\overline{\mathcal{A}}/\mathcal{A}) \) using the morphism

\[ \mathcal{P} \xrightarrow{p} \mathcal{P}_{\mathcal{I}}(\overline{\mathcal{A}}/\mathcal{A}) \xrightarrow{I_\xi} \mathcal{P}_{\mathcal{I}}(\overline{\mathcal{A}}/\mathcal{A}) \]

where \( I_\xi \) sends a \( T \)-torsor to the induced line bundle associated to the character \( \xi \). By Lem. 6.1.4 below, the component group of \( \mathcal{I}(\mathcal{A}_a) \) is torsion-free for any \( a \in \mathcal{A} \). Since \( e_{\mathcal{Q}_a}^{\mathcal{P}_a} \) is finite for \( a \in \mathcal{A} \), the morphism \( (5.8) \) necessarily factors through the neutral component \( \mathcal{P}_{\mathcal{I}}(\overline{\mathcal{A}}/\mathcal{A})^0 \) of \( \mathcal{P}_{\mathcal{I}}(\overline{\mathcal{A}}/\mathcal{A}) \). Therefore \( c_1(Q(\xi))^2 \) factors through

\[ H_*(\overline{\mathcal{A}}/\mathcal{A}) \to H^*(\mathcal{P}_{\mathcal{I}}(\overline{\mathcal{A}}/\mathcal{A})^0/\mathcal{A})[2](1) \to H^*(\mathcal{P}/\mathcal{A})[2](1) \]

Since \( \mathcal{P}_{\mathcal{I}}(\overline{\mathcal{A}}/\mathcal{A})^0 \to \mathcal{A} \) has connected fibers, the above map has to land in the stable part of \( H^*(\mathcal{P}/\mathcal{A})[2](1) \).
By the definition of the tautological line bundle $\mathcal{Q}(\xi)$, for each integer $N \in \mathbb{Z}$, we have

$$(\text{id}_\mathbb{A} \times [N])^* \mathcal{Q}(\xi) \cong \mathcal{Q}(N\xi) \cong \mathcal{Q}(\xi)^{\otimes N}.$$ 

Therefore we have a commutative diagram

$$
\begin{array}{ccc}
\mathbb{H}_* (\mathbb{A}/\mathbb{A}) & \xrightarrow{c_1(\mathcal{Q}(\xi))} & \mathbb{H}^*(\mathcal{P}/\mathcal{A})[2](1) \\
\downarrow N_{c_1(\mathcal{Q}(\xi))} & & \downarrow [N]^*
\end{array}
$$

This implies that $c_1(\mathcal{Q}(\xi))$ factors through the eigen-subcomplex of $\mathbb{H}^*(\mathcal{P}/\mathcal{A})_{\text{st}}[2](1)$ with eigenvalue $N$ under the endomorphism $[N]^*$, i.e., $\mathbb{H}^1(\mathcal{P}/\mathcal{A})_{\text{st}}[1](1)$ (cf. Rem. A.1.2).

5.1.4. **Lemma.** For any reduced projective curve $C$ over $k$, the component group $\pi_0(\text{Pic}(C))$ is a free abelian group of finite rank.

**Proof.** Let $\pi : \tilde{C} \to C$ be the normalization, then we have an exact sequence of groups over $C$:

$$1 \to \mathbb{G}_m \to \pi_* \mathbb{G}_m \to \bigoplus_{c \in C^{\text{sing}}} \mathcal{R}_c \to 1$$

where each $\mathcal{R}_c$ is a connected commutative algebraic group, viewed as an étale sheaf supported at the singular point $c \in C^{\text{sing}}$. This gives an exact sequence of Picard stacks

$$\prod_{c \in C^{\text{sing}}} \mathcal{R}_c \to \text{Pic}(C) \xrightarrow{\pi_*} \text{Pic}(\tilde{C}) \to 1$$

Since $\prod_{c \in C^{\text{sing}}} \mathcal{R}_c$ is connected, we have $\pi_0(\text{Pic}(C)) \cong \pi_0(\text{Pic}(\tilde{C}))$. Since $\tilde{C}$ is a disjoint union of smooth connected projective curves, $\pi_0(\text{Pic}(\tilde{C})) \cong \mathbb{Z}^{\text{tr}(C)}$ is a free abelian group of finite rank.

Dually, we can also write the map $c_1(\mathcal{Q}(\xi))^2$ in (5.7) as

$$c_\xi := D(c_1(\mathcal{Q}(\xi))^2) : H_1(\mathcal{P}/\mathcal{A})_{\text{st}} \to H^*(\mathbb{A}/\mathbb{A})[1](1).$$

5.1.5. **Proposition.**

(1) The cap product action of $p^*H_*(\mathcal{P}/\mathcal{A})$ on $f^*_{\text{par}}\mathcal{L}_\xi$ commutes with the actions of $u$ and $\delta$.

(2) Suppose $h$ is a section of $H_1(\mathcal{P}/\mathcal{A})$ over an étale chart $U \to \mathcal{A}$, which acts on $f^*_{\text{par}}\mathcal{L}_\xi|_U$ via cap product. Then we have the following commutation relation between the $h$-action and the $\xi$-action on $f^*_{\text{par}}\mathcal{L}_\xi|_U$:

$$[\xi, h] := \xi(h\cap) - (h\cap)\xi = c_\xi(h_{\text{st}})$$

where $h_{\text{st}}$ is the stable part of $h$ and $c_\xi(h_{\text{st}}) \in H^1(\mathbb{A}_U/U)(1)$ acts on $f^*_{\text{par}}\mathcal{L}_\xi$ by cap product.

**Proof.** (1) For $\xi = \delta$ or $u$, $\mathcal{L}(\xi)$ is the pull-back of a line bundle on $X$. Since the action of $\mathcal{P}$ preserves the base $\mathcal{A} \times X$, the cap product by $p^*H_*(\mathcal{P}/\mathcal{A})$ commutes with $\cup c_1(\mathcal{L}(\xi))$ on $f^*_{\text{par}}\mathcal{L}_\xi$.

(2) By the commutative diagram in [Yun, Lem. 3.2.9], we have an isomorphism of line bundles on $\tilde{\mathcal{P}} \times _\mathcal{A} \mathcal{M}^{\text{par}}$:

$$\text{act}^*\mathcal{L}(\xi) \cong \mathcal{Q}(\xi) \otimes _\mathcal{A} \mathcal{L}(\xi).$$
Unfolding the definition of the cap product, we get a commutative diagram
\[
\begin{array}{ccc}
(g \times f_{\text{par}})(\bar{\mathbb{Q}}_\ell \boxtimes \bar{\mathbb{Q}}_\ell) & \xrightarrow{c_1(\mathbb{Q}(\xi)) \otimes \text{id} + \text{id} \otimes c_1(\mathbb{L}(\xi))} & (g \times f_{\text{par}})(\bar{\mathbb{Q}}_\ell \boxtimes \bar{\mathbb{Q}}_\ell)[2](1) \\
\downarrow & & \downarrow \\
f_{\text{par}}^* \text{act} \text{proj} \bar{\mathbb{Q}}_\ell & & f_{\text{par}}^* \text{act} \text{proj} \bar{\mathbb{Q}}_\ell[2](1) \\
\phi \downarrow & & \phi \downarrow \\
f_{\text{par}}^* \text{act} \text{proj} \bar{\mathbb{Q}}_\ell & & f_{\text{par}}^* \text{act} \text{proj} \bar{\mathbb{Q}}_\ell[2](1) \\
\downarrow \text{ad.} & & \downarrow \text{ad.} \\
f_{\text{par}}^* \bar{\mathbb{Q}}_\ell & & f_{\text{par}}^* \bar{\mathbb{Q}}_\ell[2](1)
\end{array}
\]

In other words, for a local section \( h \) of \( H_1(\mathcal{P}/\mathcal{A}) \) and a local section \( \gamma \) of \( f_{\text{par}}^* \bar{\mathbb{Q}}_\ell \), we have
\[
(h \cup c_1(\mathbb{Q}(\xi))) \cap \gamma + h \cap (\gamma \cup c_1(\mathbb{L}(\xi))) = (h \cap \gamma) \cup c_1(\mathbb{L}(\xi)).
\]
This, together with (5.9) implies (5.10). \( \square \)

5.1.6. Corollary. For \((a, x) \in \mathcal{A}(k) \times X(k)\), the cup-product action of \( H_*(\mathcal{P}_a) \) on \( H^*(\mathcal{M}_{a, \text{par}}^*) \) commutes with the action of the subalgebra \( \bar{\mathbb{Q}}_\ell[W] \otimes \text{Sym}(X^*(T)_\ell) \subset H^*(\delta, u) \).

Proof. By Prop. 5.1.1 and Prop. 5.1.5(1), it only remains to show that the cup product commutes with \( \xi \in X^*(T) \). Let \( h \in H_1(\mathcal{P}_a) \). Restricting to a point \((a, x)\), the cohomology class \( c_\xi(h_a) \in H^1(X_a)(1) \in H^1(q_a^{-1}(x))(1) = 0 \) must be 0. Therefore \( h \) also commutes with \( \xi \in X^*(T) \) by (5.10). \( \square \)

5.1.7. Question. The commutation relation between the cap product by \( H_*(\mathcal{P}/\mathcal{A}) \) and the cap product by \( c_1(\mathcal{L}_{\text{can}}) \) on \( f_{\text{par}}^* \bar{\mathbb{Q}}_\ell \) remains unclear to the author.

5.2. Comparison of the \( \bar{\mathbb{Q}}_\ell[X_*(T)]^W \)-action and the \( \pi_0(\mathcal{P}/\mathcal{A}) \)-action. The cap product \( f_{\text{par}}^* \bar{\mathbb{Q}}_\ell \) in particular gives an action of \( p^* H_*(\mathcal{P}/\mathcal{A}) = p^* \bar{\mathbb{Q}}_\ell[\pi_0(\mathcal{P}/\mathcal{A})] \) on the complex \( f_{\text{par}}^* \bar{\mathbb{Q}}_\ell \). On the other hand, the center \( \bar{\mathbb{Q}}_\ell[X_*(T)]^W \) also acts on the complex \( f_{\text{par}}^* \bar{\mathbb{Q}}_\ell \) by \[ \text{Th. 4.4.3} \]. The idea of relating the \( \pi_0(\mathcal{P}/\mathcal{A}) \)-action to the \( \bar{\mathbb{Q}}_\ell[X_*(T)]^W \)-action is suggested to the author by B-C.Ngô.

5.2.1. Comparison for the Hitchin complex. Recall from Prop. 4.2.2 that we have a \( \bar{\mathbb{Q}}_\ell[X_*(T)]^W \)-action on \( f_{\text{Hit}}^* \bar{\mathbb{Q}}_\ell \boxtimes \bar{\mathbb{Q}}_\ell \in D^b_{\text{par}}(\mathcal{A} \times X) \), which can be written as (since \( p \) is smooth)
\[
\bar{\mathbb{Q}}_\ell[X_*(T)]^W \otimes p^! f_{\text{Hit}}^* \bar{\mathbb{Q}}_\ell \rightarrow p^! f_{\text{Hit}}^* \bar{\mathbb{Q}}_\ell.
\]
Applying the adjunction \((p_!, p^!)\) and the projection formula, we get
\[
\alpha : \bar{\mathbb{Q}}_\ell[X_*(T)]^W \otimes H_*(X) \otimes f_{\text{Hit}}^* \bar{\mathbb{Q}}_\ell \rightarrow f_{\text{Hit}}^* \bar{\mathbb{Q}}_\ell.
\]
Decomposing \( \alpha \) according to \( H_*(X) = \oplus_{i=0}^\infty H_i(X)[i] \), we get
\[
\alpha_i : \bar{\mathbb{Q}}_\ell[X_*(T)]^W \otimes H_i(X) \otimes f_{\text{Hit}}^* \bar{\mathbb{Q}}_\ell \rightarrow f_{\text{Hit}}^* \bar{\mathbb{Q}}_\ell[-i].
\]
The goal of this subsection is to describe the effects of \( \alpha_i \) in terms of the cap product of \( H_*(\mathcal{P}/\mathcal{A}) \).
5.2.2. Proposition. There is a natural map
\[ \sigma : \mathcal{Q}_\ell[X_*(T)]^W \otimes H_*((A \times X)^{rs}/A) \to H_*(_\mathcal{P}/A) \]
such that the following diagram is commutative
\[
\begin{array}{ccc}
\mathcal{Q}_\ell[X_*(T)]^W \otimes H_*((A \times X)^{rs}/A) & \xrightarrow{\sigma \otimes \text{id}} & H_*(_\mathcal{P}/A) \otimes f_*^\text{Hit}\mathcal{Q}_\ell \\
\text{id} \otimes \rho \otimes \text{id} & & \cap \\
\mathcal{Q}_\ell[X_*(T)]^W \otimes H_*(X) \otimes f_*^\text{Hit}\mathcal{Q}_\ell & \xrightarrow{\alpha} & f_*^\text{Hit}\mathcal{Q}_\ell
\end{array}
\]
where \( j : (A \times X)^{rs} \hookrightarrow A \times X \) is the open inclusion.

Proof. Recall from [YunI, Rem. 4.3.7] that we have a morphism:
\[ s : X_*(T) \times \mathcal{P} \to \mathcal{G}_{\mathcal{P}} \to \mathcal{P}. \]
This gives a push-forward map on homology
\[ s_! : \mathcal{Q}_\ell[X_*(T)] \otimes H_*(_\mathcal{A}^{rs}/A) \to H_*(_\mathcal{P}/A) \]
which is \( W \)-invariant (\( W \) acts trivially on the two factors on the LHS and acts trivially on the RHS). Therefore, it factors through the \( W \)-coinvariants of \( \mathcal{Q}_\ell[X_*(T)] \otimes H_*(_\mathcal{A}^{rs}/A) \). In particular, if we restrict to \( \mathcal{Q}_\ell[X_*(T)]^W \), the map \( s_! \) factors through a map
\[
\mathcal{Q}_\ell[X_*(T)]^W \otimes (H_*(_\mathcal{A}^{rs}/A))_W \to H_*(_\mathcal{P}/A)
\]
Since \( g^{rs} : _\mathcal{A}^{rs} \to (A \times X)^{rs} \) is an étale \( W \)-cover, we have \( (H_*(_\mathcal{A}^{rs}/A))_W = H_*((A \times X)^{rs}/A) \). Therefore the map (5.12) gives the desired map \( \sigma \). The diagram (5.11) is commutative because the \( X_*(T) \)-action on \( _\mathcal{P}/A \mathcal{A} \) comes from the following morphism
\[ X_*(T) \times _\mathcal{A}^{rs} \to \mathcal{M}^{\text{Hit}} \xrightarrow{\delta} _\mathcal{P} \times \mathcal{A} \mathcal{M}^{\text{Hit}} \xrightarrow{\text{act}} \mathcal{M}^{\text{Hit}}. \]

Passing to the level of (co)homology sheaves, we get

5.2.3. Corollary. The map \( \sigma \) induces maps \( \sigma_i \) (\( i = 0, 1, 2 \)) on homology sheaves:
\[ \sigma_i : \mathcal{Q}_\ell[X_*(T)]^W \otimes H_i((A \times X)^{rs}/A) \to H_i(_\mathcal{P}/A) \]
such that the following diagram is commutative for each \( m \in \mathbb{Z}, i = 0, 1, 2 \):
\[
\begin{array}{ccc}
\mathcal{Q}_\ell[X_*(T)]^W \otimes H_((A \times X)^{rs}/A) \otimes R^mf_*^\text{Hit}\mathcal{Q}_\ell & \xrightarrow{\sigma_i \otimes \text{id}} & H_i(_\mathcal{P}/A) \otimes R^mf_*^\text{Hit}\mathcal{Q}_\ell \\
\text{id} \otimes \rho \otimes \text{id} & & \cap \\
\mathcal{Q}_\ell[X_*(T)]^W \otimes H_*(X) \otimes R^mf_*^\text{Hit}\mathcal{Q}_\ell & \xrightarrow{\alpha_i} & R^mf_*^\text{Hit}\mathcal{Q}_\ell
\end{array}
\]
Since \( H_0((A \times X)^{rs}/A) = \mathcal{Q}_\ell[A] \), \( \sigma_0 \) gives a homomorphism of sheaves of algebras
\[ \sigma_0 : \mathcal{Q}_\ell[X_*(T)]^W \to \mathcal{Q}_\ell[\pi_0(_\mathcal{P}/A)]. \]

5.2.4. Corollary. The action of \( \mathcal{Q}_\ell[X_*(T)]^W \) on \( R^mf_*^\text{Hit}\mathcal{Q}_\ell \otimes \mathcal{Q}_\ell[X_*(T)]^W \) constructed in Prop. 4.2.2 factors through the \( \mathcal{Q}_\ell[\pi_0(_\mathcal{P}/A)] \)-action on \( R^mf_*^\text{Hit}\mathcal{Q}_\ell \) via the map \( \sigma_0 \) in (5.14).

Our final goal in this subsection is to prove:
5.2.5. **Theorem.** The action of $\mathcal{H}_i[X_*(T)]^W$ on $\mathbf{R}^m f_\ast^\text{par} \mathcal{Q}_\ell$ constructed in [Yun] Th. 4.4.3 factors through the $p^\ast \mathcal{Q}_\ell[p_0(\mathcal{P}/\mathcal{A})]$ action on $\mathbf{R}^m f_\ast^\text{par} \mathcal{Q}_\ell$ via the map $p^\ast p_0 : \mathcal{H}_i[X_*(T)]^W \to p^\ast \mathcal{Q}_\ell[p_0(\mathcal{P}/\mathcal{A})]$.

5.2.6. **Remark.** The other hand, we will see in [YunIII] Sec. 5.2 that the action of the whole lattice $X_*(T)$ on $\mathbf{R}^m f_\ast^\text{par} \mathcal{Q}_\ell$ does not factor through a finite quotient: the action can be unipotent.

5.2.7. **Hecke modifications at two points.** To prove the Th. 5.2.5 we consider a more general Hecke correspondence which combines the two situations we considered in [Yun] Sec. 4.1 and Sec. 4.2.

\[(5.15)\]

\[
\begin{array}{ccc}
\mathcal{M}_{\text{par}} \times X & \xrightarrow{f_{\text{par}} \times \text{id}_X} & \mathcal{M}_{\text{par}} \times X \\
\downarrow & & \downarrow \\
\mathcal{A} \times X^2 & \xrightarrow{f_{\text{par}} \times \text{id}_X} & \mathcal{A} \times X^2
\end{array}
\]

For any scheme $S$, $\mathcal{H}_{\text{Hecke}}'(S)$ is the groupoid of tuples $(x, y, E_1, \varphi_1, E_2, \varphi_2, E_{2,x}, \alpha)$ where

- $(x, E_i, \varphi_i, E_{i,x}) \in \mathcal{M}_{\text{par}}(S)$;
- $y \in X(S)$ with graph $\Gamma(y)$;
- $\alpha$ is an isomorphism of Hitchin pairs $(E_1, \varphi_1)|S \times X - \Gamma(y) \sim (E_2, \varphi_2)|S \times X - \Gamma(y)$

For a point $(a, x, y) \in (\mathcal{A} \times X^2)(k)$ such that $x \neq y$, the fibers of $\overline{h}$ and $\overline{g}$ over $(a, x, y)$ are isomorphic to the product of $M_{\text{Hit}}^\gamma(\gamma_{a,y})$ and a Springer fiber in $G/B$ corresponding to $\gamma_{a,x}$ (see the discussion in [Yun] Sec. 3.3); while if we restrict to the diagonal $\Delta_X : \mathcal{A} \times X \subset \mathcal{A} \times X^2$, $\mathcal{H}_{\text{Hecke}}'|_{\Delta_X}$ is the same as $\mathcal{H}_{\text{Hecke}}\,_{\text{par}}$. The reader may notice the analogy between our situation and the situation considered by Gaitsgory in [G], where he uses Hecke modifications at two points to deform the product $\mathcal{G}_G \times G/B$ to $\mathcal{F}_G$.

As in the case of $\mathcal{H}_{\text{Hecke}}\,_{\text{par}}$, we have a morphism $\mathcal{H}_{\text{Hecke}}' \to \mathcal{M}_{\text{par}} \times_{\mathcal{A} \times X} \mathcal{M}_{\text{par}} \to \overline{\mathcal{A}} \times_{\mathcal{A} \times X} \overline{\mathcal{A}}$.

Let $\mathcal{H}_{\text{Hecke}}'_{[\alpha]}$ be the preimage of the diagonal $\overline{\mathcal{A}} \subset \overline{\mathcal{A}} \times_{\mathcal{A} \times X} \overline{\mathcal{A}}$. We have a commutative diagram of correspondences

\[(5.16)\]

\[
\begin{array}{ccc}
\mathcal{H}_{\text{Hecke}}'_{[\alpha]} & \xrightarrow{q'} & \mathcal{G}_{\text{Hecke}}^G \\
\overline{h}_{[\alpha]} & \downarrow & \overline{h}_G \\
\mathcal{M}_{\text{par}} \times X & = & \mathcal{M}_{\text{G}} = \mathcal{M}_{\text{Hit}} \times X \\
\overline{\mathcal{A}} \times X & \xrightarrow{q \times \text{id}_X} & \mathcal{A} \times X
\end{array}
\]

By [Yun] Lem. 3.5.4, this is a base change diagram if we restrict the base spaces to $\overline{\mathcal{A}}^0 \times X \to \mathcal{A} \times X$. Recall from Construction 4.2.1 that for each $W$-orbit $|\lambda|$ in $X_*(T)$, we have a graph-like closed substack $\mathcal{H}_{|\lambda}| \subset \mathcal{G}_{\text{Hecke}}^G$. Let $\mathcal{H}_{|\lambda}' \subset \mathcal{H}_{\text{Hecke}}'_{[\alpha]}$.
be closure of the preimage of $\mathcal{H}^r_{[\lambda]}$ under $q'$. By the same argument as Prop. 4.2.2 we can prove:

5.2.8. **Lemma.** There is a unique action $\alpha'$ of $\overline{\Omega}_\mathfrak{g}[\mathfrak{X}_s(T)]^W$ on the complex $(f^\text{par} \times \text{id}_X)_\mathfrak{T} = f^\text{par} \overline{\Omega}_\mathfrak{T} \otimes \overline{\Omega}_{\mathfrak{X},X}$ on $\mathfrak{A} \times X^2$ such that $\text{Av}_W(\lambda)$ acts as $[\mathcal{H}^r_{[\lambda]}]_#$. 

On the other hand, $p^*\sigma_0(\pi_0(P/A))$ acts on $f^\text{par} \overline{\Omega}_\mathfrak{T} \otimes \overline{\Omega}_{\mathfrak{X},X}$ via its action on the first factor, here $p' : \mathfrak{A} \times X^2 \to \mathfrak{A}$ is the projection. Similar to the proof of Prop. 5.2.2 and Cor. 5.2.2 we have

5.2.9. **Lemma.** The action $\alpha'$ of $\overline{\Omega}_\mathfrak{g}[\mathfrak{X}_s(T)]^W$ on $H^m(f^\text{par} \overline{\Omega}_\mathfrak{T} \otimes \overline{\Omega}_{\mathfrak{X},X}) = R^m f^\text{par} \overline{\Omega}_\mathfrak{T} \otimes \overline{\Omega}_{\mathfrak{X},X}$ constructed in Lem. 5.2.8 factors through the $p^*\sigma_0(\pi_0(P/A))$ action on $f^\text{par} \overline{\Omega}_\mathfrak{T} \otimes \overline{\Omega}_{\mathfrak{X},X}$ via the homomorphism $p^*\sigma_0 : \overline{\Omega}_\mathfrak{g}[\mathfrak{X}_s(T)]^W \to p^*\overline{\Omega}_\mathfrak{T}(\pi_0(P/A))$.

Now we are ready to prove the theorem.

**Proof of Th. 5.2.4.** We denote by $\alpha$ the action of $\overline{\Omega}_\mathfrak{g}[\mathfrak{X}_s(T)]^W$ on $f^\text{par} \overline{\Omega}_\mathfrak{T}$ given by restricting the $\overline{\Omega}_\mathfrak{T}[W]$-action. We will define another action of $\overline{\Omega}_\mathfrak{g}[\mathfrak{X}_s(T)]^W$ on $f^\text{par} \overline{\Omega}_\mathfrak{T}$.

Restricting the correspondence diagram (5.15) to the diagonal $\Delta_X : \mathfrak{A} \times X \to \mathfrak{A} \times X^2$, we recover the correspondence $\text{Hecke}^\text{par}$. Restricting the commutative diagram (5.16) to the diagonal, we recover the commutative diagram (4.9). The $\Delta_X$-restriction of the $f^\text{par} \overline{\Omega}_\mathfrak{T} \otimes \overline{\Omega}_{\mathfrak{X},X}$ action of $\alpha'$ on $f^\text{par} \overline{\Omega}_\mathfrak{T} \otimes \overline{\Omega}_{\mathfrak{X},X}$ constructed in Lem. 5.2.8 gives an action of $\overline{\Omega}_\mathfrak{g}[\mathfrak{X}_s(T)]^W$ on $f^\text{par} \overline{\Omega}_\mathfrak{T} = \Delta_X^*(f^\text{par} \overline{\Omega}_\mathfrak{T} \otimes \overline{\Omega}_{\mathfrak{X},X})$. We denote this action by $\alpha'_\Delta$.

We claim that the actions $\alpha$ and $\alpha'_\Delta$ are the same. In fact, by [YunI, Lem. A.2.1], the action of $\alpha'_\Delta(\text{Av}_W(\lambda))$ is given by the class $\Delta_\mathfrak{T}[\mathcal{H}^r_{[\lambda]}] \in \text{Corr}(\text{Hecke}^\text{par} ; \overline{\Omega}_\mathfrak{T}, \overline{\Omega}_\mathfrak{T})$. On the other hand, the action of $\alpha(\text{Av}_W(\lambda))$ is given by the class $\sum_{X \in [\lambda]}[\mathcal{H}_X] \in \text{Corr}(\text{Hecke}^\text{par} ; \overline{\Omega}_\mathfrak{T}, \overline{\Omega}_\mathfrak{T})$. When restricted to $(A \times X)^r$ both classes coincide with the fundamental class of $Q'[\mathcal{H}^r_{[\lambda]}]$ (cf. diagram (4.9)). Since both classes are supported on a graph-like substack of $\text{Hecke}^\text{par}$ (see [YunI, Lem. 4.4.4]), their coincidence over $(A \times X)^r$ ensures that their actions on $f^\text{par} \overline{\Omega}_\mathfrak{T}$ are the same, by [YunI, Lem. A.5.2].

By Lem. 5.2.9 the action $\alpha'_\Delta$ of $\overline{\Omega}_\mathfrak{g}[\mathfrak{X}_s(T)]^W$ on $R^m f^\text{par} \overline{\Omega}_\mathfrak{T}$ factors through $p^*\sigma_0 : \overline{\Omega}_\mathfrak{g}[\mathfrak{X}_s(T)]^W \to \Delta_X^* p^*\overline{\Omega}_\mathfrak{T}(\pi_0(P/A)) = p^*\overline{\Omega}_\mathfrak{T}(\pi_0(P/A))$. Since this action is the same as $\alpha$, the theorem is proved. 

**Appendix A. Generalities on the cap product**

In this appendix, we recall the formalism of cap product by the homology sheaf of a commutative smooth group scheme, partially following [NOS 7.4].

**A.1. The Pontryagin product on homology.** Let $P$ be a commutative smooth group scheme (or Deligne-Mumford Picard stack such as $\mathcal{P}$) of finite type over a scheme $S$. Let $g : P \to S$ be the structure map and let $H_*(P/S)$ be the homology complex of $P$ on $S$.

**A.1.1. Lemma.** There is a canonical decomposition in $D^b(S)$:

$$H_*(P/S) \cong \bigoplus_{i \geq 0} H_i(P/S)[i].$$  

Proof. Take any \( N \in \mathbb{Z} \) which is coprime to the cardinalities of \( \pi_0(P_s) \) for all \( s \in S \) (such an integer exists because there are only finitely many isomorphism types of \( \pi_0(P_s) \)). The \( N \)-th power map \([N] : P \to P\) induces an endomorphism \([N]_s\) on \( H_1(P/S)\). Let \( H_i\) be the direct summand of \( H_1(P/S)\) on which the eigenvalues of \([N]_s\) have archimedean norm \( N^i\) for any embedding \( \overline{Q}_\ell \to \mathbb{C}\). It is easy to see that \( H_i\) is independent of the choice of \( N\). Then \( H_s(P/S)\) is the direct sum of \( H_i\) and each \( H_i\) is isomorphic to \( H_i(P/S)[i]\). \( \square \)

The multiplication map \( \text{mult} : P \times_S P \to P\) induces a Pontryagin product

\[ H_s(P/S) \otimes H_s(P/S) \to H_s(P/S), \]

which, in turn, induces a Pontryagin product on the homology sheaves \( H_s(P/S)\). Since the multiplication map is compatible with the \( N \)-th power map in the obvious sense, the decomposition \([A.1]\) respects the Pontryagin product on the homology complex and the Pontryagin product on the homology sheaves.

We have the following facts about the homology sheaves of \( P/S\):

- \( H_0(P/S) \cong \overline{Q}_\ell[\pi_0(P/S)]\). Recall from \([N08\,6.2]\) that there is a sheaf of abelian groups \( \pi_0(P/S)\) on \( S\) for the étale topology whose fiber at \( s \in S\) is the finite group of connected components of \( P_s\). Therefore the group algebra \( \overline{Q}_\ell[\pi_0(P/S)]\) is a \( \overline{Q}_\ell\)-sheaf of algebras on \( S\) whose fiber at \( s \in S\) is the \( 0^{th}\) homology of \( P_s\). This algebra structure is the same as the one induced from the Pontryagin product.
- If \( P_s\) is connected for some \( s \in S\), the stalk of \( H_1(P/S)\) at \( s\) is the \( \overline{Q}_\ell\)-Tate module \( V_\ell(P_s) = T_\ell(P_s) \otimes_{\mathbb{Z}_\ell} \overline{Q}_\ell\) of \( P_s\). Moreover, the Pontryagin product induces an isomorphism

\[ \bigwedge V_\ell(P_s) = \bigwedge H_1(P_s) \cong H_i(P_s). \]  

(A.2)

A.1.2. Remark. If we work with cohomology rather than homology, the \( N \)-th power map also gives a natural decomposition

\[ H^*(P/S) \cong \bigoplus_i H^i(P/S)[-i]. \]  

(A.3)

This decomposition respects the cup product on the cohomology complex and the cup product on the cohomology sheaves.

A.2. The stable parts. We have seen from the decomposition \([A.1]\) and the fact \( H_0(P/S) = \overline{Q}_\ell[\pi_0(P/S)]\) that \( \pi_0(P/S)\) acts on the homology complex \( H_*(P/S)\) and the cohomology complex \( H^*(P/S)\).

A.2.1. Definition. The stable part of \( H_i(P/S)\) (resp. \( H^i(P/S)\)) is the direct summand on which the action of \( \pi_0(P/S)\) is trivial. We denote the stable parts by \( H_i(P/S)_{st}\) and \( H^i(P/S)_{st}\). Let \( H_s(P/S)_{st} = \oplus_i H_i(P/S)_{st}[i]\) and \( H^s(P/S)_{st} = \bigoplus_i H^i(P/S)_{st}[-i]\) be the corresponding decompositions of \( H_s(P/S)_{st}\) and \( H^s(P/S)_{st}\).

A.2.2. Remark. To make sense of the invariants of a sheaf under the action of another sheaf of finite abelian groups, we refer to \([N06\,Prop.\,8.3]\).

It is clear that the stable part \( H_i(P/S)_{st}\) (resp. \( H^i(P/S)_{st}\)) inherits a Pontryagin product (resp. a cup product) from that of \( H_s(P/S)\) (resp. \( H^s(P/S)\)).
Let $P^0 \subset P$ be the Deligne-Mumford substack over $S$ of fiberwise neutral components of $P/S$ (which exists as an open substack of $P$, cf. [N06 Prop. 6.1]). Let $V_\ell(P^0/S)$ be the sheaf of $\mathbb{Q}_\ell$-Tate modules of $P^0$ over $S$.

A.2.3. Lemma.

(1) The embedding $P^0 \subset P$ and the Pontryagin product gives a natural isomorphism of $\mathbb{Q}_\ell[\pi_0(P/S)]$-algebra objects in $D_b^c(S)$:

$$\mathbb{Q}_\ell[\pi_0(P/S)] \otimes H_*(P^0/S) \cong H_*(P/S).$$

(2) The natural embedding $P^0 \subset P$ followed by the projection onto the stable part gives a natural isomorphism of algebra objects in $D_b^c(S)$:

$$\bigwedge(V_\ell(P^0/S)[1]) \cong H_*(P^0/S) \to H_*(P/S) \to H_*(P/S)_{\text{st}}.$$  

Proof. Both maps (A.4) and (A.5) are direct sums of maps between (shifted) sheaves. To check they are isomorphisms, it suffices to check on the stalks. Fix a geometric point $s \in S$. Since all connected components of $P_s$ are isomorphic to $P^0_s$, we have a $\pi_0(P_s)$-equivariant isomorphism

$$H_*(P_s) \cong H_*(P^0_s) \otimes \mathbb{Q}_\ell[\pi_0(P_s)]$$

on which $\pi_0(P_s)$ acts via the regular representation on $\mathbb{Q}_\ell[\pi_0(P_s)]$. This proves (A.4). Using (A.6), the natural embedding $P^0_s \subset P_s$ followed by the projection onto the stable part

$$H_*(P^0_s) \hookrightarrow H_*(P_s) \to H_*(P_s)_{\text{st}}$$

becomes the tensor product of the identity map on $H_*(P^0_s)$ with the map

$$\mathbb{Q}_\ell \cdot e \hookrightarrow \mathbb{Q}_\ell[\pi_0(P_s)] \to \mathbb{Q}_\ell[\pi_0(P_s)]^\pi_0(P_s)$$

where $e \in \pi_0(P_s)$ is the identity element. Now the composition of the maps in (A.8) is obviously an isomorphism, hence the composition of the maps in (A.7) is also an isomorphism. To obtain the first isomorphism in (A.5), we only need to apply the isomorphism (A.2) to the connected Picard stack $P^0/S$. □

A.2.4. Remark. If $P/S$ is smooth and proper, then the above lemma easily dualize to a similar statement about the cohomology complex $H^*(P/S)$. In particular, we have an isomorphism of algebra objects in $D_b^c(S)$ (with the cup product on the LHS and the wedge product on the RHS):

$$H^*(P/S)_{\text{st}} \cong H^*(P^0/S) \cong \bigwedge(V_\ell(P^0/S)^*[−1]).$$

A.3. The cap product. Suppose $P$ acts on a Deligne-Mumford stack $M$ over $S$, with the action and projection morphisms

$$P \times_S M \xrightarrow{\text{act}} M.$$  

Suppose $\mathcal{F}$ is a $P$-equivariant complex on $M$, then in particular we are given an isomorphism

$$\phi : \text{act}^! \mathcal{F} \cong \text{proj}^! \mathcal{F}.$$  

Therefore we have a map

$$\text{act} : \text{proj}^! \mathcal{F} \xrightarrow{\phi^{-1}} \text{act}^! \mathcal{F} \xrightarrow{\text{adj}} \mathcal{F}.$$
Let $f : M \to S$ be the structure map. Using Künneth formula ($P$ is smooth over $S$), we get

\[(A.11) \quad H^* (P/S) \otimes f_* \mathcal{F} = g_* \mathcal{D}_{P/S} \otimes f_* \mathcal{F} \cong (g \times f)_! \text{proj}_1^! \mathcal{F} = f_* \text{act}_! \text{proj}_1^! \mathcal{F}.
\]

Applying $f_!$ to the map \((A.10)\) and combining with the isomorphism \((A.11)\), we get the cap product:

\[(A.12) \quad \cap : H^* (P/S) \otimes f_* \mathcal{F} \to f_* \mathcal{F}
\]
such that $f_* \mathcal{F}$ becomes a module over the algebra $H^* (P/S)$ under the Pontryagin product. Using the decomposition \((A.1)\) we get the actions

\[
\cap_i : H^i (P/S) \otimes f_* \mathcal{F} \to f_* \mathcal{F}[-i]; \\
\cap^m : H^* (P/S) \otimes 
\]

When $i = 0$, the cap product $\cap_0$ gives an action of $\mathbb{Z} [\pi_0 (P/S)]$ on $f_* \mathcal{F}$. By the isomorphism \((A.4)\), to understand the cap product, we only need to understand $\cap_0$ and $\cap_1$.

**Appendix B. Complement on cohomological correspondences**

This appendix is a complement to [YunI, App. A]. We continue to use the notations from loc. cit. In particular, we fix a correspondence diagram

\[
\begin{array}{ccc}
C & \xrightarrow{c} & X \\
\downarrow & & \downarrow f \\
Y & \xleftarrow{\bar{c}} & S
\end{array}
\]

**B.1. Cup product and correspondences.** In this subsection, we study the interaction between the cup product and cohomological correspondences. For each $i \in \mathbb{Z}$, let

\[
\text{Corr}^i (C; \mathcal{F}, \mathcal{G}) = \text{Corr} (C; \mathcal{F}[i], \mathcal{G})
\]

\[
\text{Corr}^* (C; \mathcal{F}, \mathcal{G}) = \oplus_i \text{Corr}^i (C; \mathcal{F}, \mathcal{G}).
\]

We have a left action of $H^* (X)$ and a right action of $H^* (Y)$ on $\text{Corr}^* (C; \mathcal{F}, \mathcal{G})$. More precisely, for $\alpha \in H^j (X), \beta \in H^i (Y)$ and $\zeta \in \text{Corr}^j (C; \mathcal{F}, \mathcal{G})$, we define $\alpha \cdot \zeta, \zeta \cdot \beta \in \text{Corr}^{i+j} (C; \mathcal{F}, \mathcal{G})$ as

\[
\alpha \cdot \zeta : \bar{c}^* \mathcal{G} \xrightarrow{\zeta} \bar{c}^i \mathcal{F} [i] \xrightarrow{\tau^! (\cup \alpha)} \bar{c}^i \mathcal{F} [i+j]; \\
\zeta \cdot \beta : \bar{c}^* \mathcal{G} \xrightarrow{\tau^* (\cup \beta)} \bar{c}^* \mathcal{G} [j] \xrightarrow{\zeta} \bar{c}^i \mathcal{F} [i+j].
\]

The following lemma is obvious.

**B.1.1. Lemma.** For $\alpha \in H^j (X), \beta \in H^i (Y)$ and $\zeta \in \text{Corr}^j (C; \mathcal{F}, \mathcal{G})$, we have

\[
(\alpha \cdot \zeta)_{\#} = f_* (\cup \alpha) \circ \zeta_{\#}; \quad (\zeta \cdot \beta)_{\#} = \zeta_{\#} \circ g_! (\cup \beta).
\]

On the other hand, $H^* (C)$ acts on $\text{Corr}^* (C; \mathcal{F}, \mathcal{G}) = \text{Ext}^* (\bar{c}^* \mathcal{G}, \bar{c}^i \mathcal{F})$ by cup product, which we denote simply by $\cup$. 

B.1.2. Lemma. Let $\alpha \in H^*(X), \beta \in H^*(Y)$ and $\zeta \in \text{Corr}^*(C; F, G)$, then we have

$$\alpha \cdot \zeta = \zeta \cup (\overline{\alpha}^* \beta); \quad \zeta \cdot \beta = \zeta \cup (\overline{\beta}^* \gamma).$$

Proof. The second identity is obvious from definition. We prove the first one. By the projection formula and adjunction, we have a map

$$\overline{c_1} (\overline{\alpha}^! F \otimes \overline{\beta}^* K) \cong (\overline{c_1} \overline{\alpha}^! F) \otimes K \to F \otimes K$$

bifunctorial in $F, K \in D_c(X, \mathbb{Q}_l)$. Applying the adjunction $(\overline{\zeta}, \overline{\eta})$ again, we get a bifunctorial map

$$\overline{c_1} F \otimes \overline{\beta}^* K \to \overline{c_1} (F \otimes K).$$

Now taking $K = \mathbb{Q}_l$, and view $\alpha \in H^i(X)$ as a map $\alpha : K \to K[i]$. The functoriality of the map [B.3] in $K$ implies a commutative diagram

$$
\begin{array}{ccc}
\overline{c_1} F \otimes \overline{\beta}^* K & \longrightarrow & \overline{c_1} (F \otimes K) \\
\downarrow \text{id} \otimes \overline{\alpha} & & \downarrow \overline{c_1} (\text{id} \otimes \overline{\alpha}) \\
\overline{c_1} F \otimes \overline{\beta}^* K[i] & \longrightarrow & \overline{c_1} (F \otimes K[i])
\end{array}
$$

which is equivalent to the first identity in (B.2). \qed

If we have a base change diagram of correspondences induced from $S' \to S$ as in [Yun1] App. A.2], then the pull-back map

$$\gamma^* : \text{Corr}^*(C; F, G) \to \text{Corr}^*(C'; \phi^* F, \psi^* G)$$

commutes with the actions of $H^*(X), H^*(Y)$ and $H^*(C)$ in the obvious sense.

B.2. Cap product and correspondences. In this subsection, we study the interaction between the cap product (see Sec. A.3) and cohomological correspondences. Suppose a group scheme $P$ (or a Picard stack which is Deligne-Mumford) over $S$ acts on the correspondence diagram (B.1), i.e., $\overline{c}$ and $\overline{\beta}$ are $P$-equivariant. We use “act” to denote the action maps by $P$ and “proj” to denote the projections along $P$, and add subscripts to indicate the space on which $P$ is acting, e.g., $\text{act}_C : P \times_S C \to C$. Let $F, G$ be $P$-equivariant complexes on $X$ and $Y$.

B.2.1. Definition. For $\zeta \in \text{Corr}(C; F, G)$, we say $\zeta$ is $P$-invariant if the pull-backs $\text{act}_C^* \zeta$ and $\text{proj}_C^* \zeta$ correspond to each other under the isomorphism

$$\text{Corr}(P \times_S C; \text{act}_X^* F, \text{act}_Y^* G) \cong \text{Corr}(P \times_S C; \text{proj}_X^* F, \text{proj}_Y^* G)$$

given by the equivariant structures of $F$ and $G$.

B.2.2. Remark. Here we use the !-pull-back rather than the *-pull-back of cohomological correspondences defined in [Yun1] App. A.2]. Since the action morphisms and the projections are smooth, the !- and *-pull-backs only differ by a shift and a twist, so that the results in [Yun1] App. A.2] are still applicable in this situation.

B.2.3. Lemma. Suppose $X$ is proper over $S$ so that $f_* F = f_* F$. Let $\zeta \in \text{Corr}(C; F, G)$ be $P$-invariant, then the cap product action of $H_*(P/S)$ commutes with $\zeta_!$, i.e.,
we have a commutative diagram
\[
\begin{array}{ccc}
\mathbb{H}_\ast(P/S) \otimes g_!G & \xrightarrow{id \otimes \zeta} & \mathbb{H}_\ast(P/S) \otimes f_!F \\
\downarrow{\phi} & & \downarrow{\phi} \\
g_!G & \xleftarrow{\zeta} & f_!F
\end{array}
\]

Proof. Consider the correspondence of \(P \times_S C\) between \(P \times_S X\) and \(P \times_S Y\) over \(P\) as the base change from the correspondence diagram (B.1) by the action maps \(\text{act}\). Let \(h: P \to S\) be the structure morphism. By [YunL, Lem. A.2.1], we get a commutative diagram (note that the action maps are smooth)

\[
\begin{array}{ccc}
g_!\text{act}_Y^!G & \xrightarrow{h_!h^!g_!G} & h_!h^!f_!F \\
\downarrow{\text{ad.}} & & \downarrow{\text{ad.}} \\
g_!G & \xleftarrow{\zeta} & f_!F
\end{array}
\]

By assumption, we have \(\text{act}_C^!\zeta = \prod_{C}^!\zeta\). Therefore we can identify the top row of the diagram (B.5) with
\[
\begin{array}{ccc}
\mathbb{H}_\ast(P/S) \otimes g_!G & \xrightarrow{h_!h^!g_!G} & \mathbb{H}_\ast(P/S) \otimes f_!F \\
\downarrow{\phi} & & \downarrow{\phi} \\
g_!G & \xleftarrow{\zeta} & f_!F
\end{array}
\]

This identifies the outer quadrangle of the diagram (B.5) with the diagram (B.4).

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