SEMIPARAMETRIC BEST ARM IDENTIFICATION WITH CONTEXTUAL INFORMATION

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ABSTRACT

We study best-arm identification with a fixed budget and contextual (covariate) information in stochastic multi-armed bandit problems. In each round, after observing contextual information, we choose a treatment arm using past observations and current context. Our goal is to identify the best treatment arm, a treatment arm with the maximal expected reward marginalized over the contextual distribution, with a minimal probability of misidentification. First, we derive semiparametric lower bounds of the misidentification probability for this problem, where we regard the gaps between the expected rewards of the best and suboptimal treatment arms as parameters of interest, and all other parameters, such as the expected rewards conditioned on contexts, as the nuisance parameters. We then develop the “Contextual RS-AIPW strategy,” which consists of the random sampling (RS) rule tracking a target allocation ratio and the recommendation rule using the augmented inverse probability weighting (AIPW) estimator. Our proposed Contextual RS-AIPW strategy is optimal because the upper bound for the probability of misidentification by the strategy matches the semiparametric lower bound, when the budget goes to infinity and the gaps converge to zero.

1 Introduction

The stochastic multi-armed bandit (MAB) problem is a classical abstraction of the sequential decision-making problem (Thompson, 1933; Robbins, 1952; Lai and Robbins, 1985). Best arm identification (BAI) is an instance of the MAB problem, where we consider pure exploration to identify the best treatment arm, a treatment arm that yields the highest expected reward. In this study, we study contextual BAI with a fixed budget, whose goal is to identify the best treatment arm minimizing the probability of misidentification after a fixed number of rounds of adaptive experiments, called a budget (Bubeck et al., 2009, 2011; Audibert et al., 2010). To gain efficiency in this task, we can choose a treatment arm based on a random variable that characterizes the features of treatment arms and can be observed before drawing one of them. This random variable is referred to as contextual information or covariate. Our setting is a generalization of BAI with a fixed budget (Carpentier and Locatelli, 2016). The main focus of this paper is how to employ contextual information for the purpose of efficiently identifying the best-arm maximizing the unconditional mean reward, rather than how to learn optimal context-specific bandit strategies, as studied in the literature of contextual bandit.

In this setting, we develop an asymptotically optimal strategy for BAI with a fixed budget and contextual information under a small-gap regime, where the gaps of expected rewards of the best and suboptimal treatment arms converge to zero. First, we derive lower bounds for the probability of misidentification by extending the distribution-dependent lower bound by Kaufmann et al. (2016) to semiparametric setting under the small-gap regime. Then, we propose our BAI strategy, the Contextual RS-AIPW strategy, which consists of a random sampling (RS) rule using an estimated target allocation ratio and a recommendation rule using an augmented inverse probability weighting (AIPW) estimator.
We prove asymptotic optimality of the proposed strategy by showing that an upper bound of its probability of misidentification matches the efficiency lower bound when the budget goes to infinity under the small-gap regime.

Little has been studied on how to make use of covariate information for BAI with a fixed budget. One of the main research interests in bandit problems is to clarify a tight lower bound on the probability of misidentification, since it enables us to claim that a strategy whose upper bound for the probability of misidentification matches the lower bound is optimal. Glynn and Juneja (2004) proposes a strategy based on optimally selected target allocation ratio. They, however, assume that the optimal target allocation ratio is known in advance, and do not consider the issue of estimating it. Based on the change-of-measure arguments popularized by Lai and Robbins (1985), Kaufmann et al. (2016) derives a distribution-dependent lower bound on the misidentification probability, which is agnostic to the optimal target allocation ratio. Despite the seminal result, a strategy with matching upper and lower bounds given an unknown target allocation ratio has not been proposed (Kaufmann, 2020). We address this issue by proposing the small-gap regime, where we can ignore an estimation error of the optimal target allocation ratio relative to the probability of misidentification.

We note that our study is also a pioneering work for BAI with a fixed budget, as even without contextual information, the existence of an asymptotically optimal BAI strategy is unclear. This study addresses this open question by showing an asymptotically optimal strategy under a small-gap regime because BAI with a fixed-budget and without contextual information is a special case of our setting. Furthermore, we show an analytical solution for the target allocation ratio, which also has been also unknown for a long time.

In BAI, there is another setting called BAI with fixed confidence, where the goal is to stop an adaptive experiment as soon as possible when the best treatment can be recommended with a certain confidence. In both settings of BAI with a fixed budget and confidence, there are few studies using contextual information. However, similar problems are frequently considered in the studies of causal inference, which mainly discuss the efficient estimation of causal parameters the gap between expected outcomes of two treatment arms marginalized over the covariate distribution (van der Laan, 2008; Hahn et al., 2011), rather than BAI. The gap is also called the average treatment effect (ATE) in this literature (Imbens and Rubin, 2015). According to their results, even if the covariates are marginalized, the variance of the estimator can be reduced with the help of covariate information. In BAI with fixed confidence, several recent studies have proposed the use of contextual information to identify a treatment arm with the highest expected reward marginalized over the contextual information (Kato and Ariu, 2021; Russac et al., 2021). These studies are also based on a similar motivation to causal inference literature. In this study, we also follow the motivation and consider BAI with a fixed budget to identify the best treatment arm marginalized over the contextual information. To the best of our knowledge, our study is the first to consider BAI with contextual information in the fixed-budget setting.

From a technical perspective, we develop a small-gap regime, a large-deviation bound, and semiparametric analysis in a derivation of the lower bound. First, we employ the small-gap regime, which implies a situation in which it is difficult to identify the best treatment arm. As explained above, the asymptotic optimality of BAI strategies under a fixed-budget setting is a long-standing open issue. For example, when the gaps are fixed, Carpenter and Locatelli (2016) shows that upper bounds of BAI strategies cannot match lower bounds conjectured by Kaufmann et al. (2016). We consider that a contributing factor is an estimation error of the optimal target allocation ratio, which affects the probability of misidentification. Under the small-gap regime, we can ignore the estimation error relative to the probability of misidentification because identification of the best treatment arm becomes difficult when the gaps are sufficiently small. Thus, this regime makes the asymptotic optimality argument in fixed-budget BAI tractable by allowing the evaluation of optimal allocation probabilities to be ignored. Carpenter and Locatelli (2016) Ariu et al. (2021). Second, to evaluate the probability of misidentification, we develop a new large deviation bound for martingales, as existing bounds, such as the ones in Cramér theorem (Cramér, 1938; Cramér and Touchette, 1994) and Gärtner-Ellis theorem (Gärtner, 1977), cannot be used for stochastic process as the samples in BAI. The derivation is inspired by the results of Fan et al. (2013). Then, we apply the large deviation bounds to obtain the upper bound. Third, we derive the lower bounds by introducing semiparametric analysis. In semiparametric analysis, we can separate parameters into parameters of interest and nuisance parameters, thus enabling us to make a more flexible statistical inference by ignoring unnecessary information. For example, because our target parameter is a treatment arm with the highest marginalized expected reward, the estimation of the distribution of the contextual information is not in our interest.

Organization. This paper is organized as follows. In Section 2, we formulate our problem. In Section 3, we derive the general semiparametric lower bounds for contextual BAI and the target allocation ratio. In Section 4, we propose the contextual RS-AIPW strategy. Then, in Section 5, we show that the proposed strategy is optimal in a sense that the upper bound for the probability of misidentification matches the lower bound. We introduce related work in 6 and discuss several topics in Sections 7. Finally, we present the proof of the semiparametric lower bound in Section 8.
2 Problem Setting

We consider the following setting of BAI with a fixed budget and contextual information. Given a fixed number of rounds \( T \), also called a budget, for each round \( t = 1, 2, \ldots, T \), an agent observes a context (covariate) \( X_t \in \mathcal{X} \) and chooses a treatment arm \( A_t \in [K] = \{1, 2, \ldots, K\} \), where \( \mathcal{X} \subset \mathbb{R}^d \) denotes the context space. Then, the agent immediately receives a reward (or outcome) \( Y_t \) linked to the chosen treatment arm \( A_t \). This setting is called the bandit feedback or Rubin causal model [Neyman, 1923; Rubin, 1974]: that is, a reward in round \( t \) is \( Y_t = \sum_{a \in [K]} 1[A_t = a]Y_{t}^{a} \), where \( Y_{t}^{a} \in \mathbb{R} \) is a potential independent (random) reward, and \( Y_{1}^{1}, Y_{2}^{2}, \ldots, Y_{T}^{K} \) are conditionally independent given \( X_t \). We assume that \( X_t \) and \( Y_{t}^{a} \) are independent and identically distributed (i.i.d.) over \( t \in [T] = \{1, 2, \ldots, T\} \). Our goal is to find a treatment arm with the highest expected reward marginalized over the contextual distribution of \( X_t \) with a minimal probability of misidentification after observing the reward in the round \( T \).

We define our goal formally. Let \( P \) be the joint distribution of \( (Y_{1}^{1}, Y_{2}^{2}, \ldots, Y_{T}^{K}, X_{t}) \). Because \( (Y_{1}^{1}, Y_{2}^{2}, \ldots, Y_{T}^{K}, X_{t}) \) is i.i.d. over \( t \in \{1, 2, \ldots, T\} \), we omit the subscripts and simply denote it as \( (Y_{1}^{1}, Y_{2}^{2}, \ldots, Y_{K}^{K}, X) \) to make it clear that it is time-independent in some case. We call distributions of the potential random variables \( (Y_{1}^{1}, Y_{2}^{2}, \ldots, Y_{K}^{K}, X) \) full-data bandit models [Tsiatis, 2007; Imbens and Rubin, 2015]. For \( P \), let \( \mathbb{P}_{P} \), \( \mathbb{E}_{P} \), and \( \text{Var}_{P} \) be the probability, expectation, and variance in terms of \( P \) respectively and \( \mu^{a}(P) = \mathbb{E}_{P}[Y^{a}] = \mathbb{E}_{P}[\mu^{a}(P)(X)] \) be the expected reward marginalized over the context \( X \), where \( \mu^{a}(P)(x) = \mathbb{E}_{P}[Y^{a}|X=x] \) is the conditional expected reward given \( x \in \mathcal{X} \). Let \( \mathbb{P} \) be a set of all joint distributions \( P \) such that the the best treatment arm \( \pi^{\ast}(P) \) uniquely exists; that is, there exists \( \pi^{\ast}(P) \in [K] \) such that \( \mu^{\ast}(P) = \max_{a\in[K]}\mu^{a}(P) \). An algorithms in BAI is called a strategy, which recommends a treatment arm \( \pi(T) \in [K] \) after sequentially sampling treatment arms in \( t = 1, 2, \ldots, T \). With the sigma-algebras \( \mathcal{F}_{t} = \sigma(X_{1}, A_{1}, Y_{1}, \ldots, X_{t}, A_{t}, Y_{t}) \), we define a BAI strategy as a pair \((\pi(T))_{t \in [T]}, \hat{\pi}(T)\), where

- the sampling rule chooses a treatment arm \( A_{t} \in [K] \) in each round \( t \) based on the past observations \( \mathcal{F}_{t-1} \) and observed context \( X_{t} \).
- the recommendation rule returns an estimator \( \hat{\pi}(T) \) of the best treatment arm \( \pi^{\ast}(P) \) based observations up to round \( T \). Here, \( \hat{\pi}(T) \) is \( \mathcal{F}_{T} \)-measurable.

Let \( \mathbb{P}_{0} \) be the “true” bandit model of the data generating process. Then, our goal is to find a BAI strategy that minimizes the probability of misidentification \( \mathbb{P}_{0}(\hat{\pi}(T) \neq \pi^{\ast}(P)) \).

**Notation.** For all \( a \in [K] \) and \( x \in \mathcal{X} \), let \( \nu^{a}(P)(x) = \mathbb{E}_{P}[|Y^{a}|^2|x] \) and \( \text{Var}_{P}(Y^{a}|x) = (\sigma^{a}(P)(x))^2 \). For the true bandit model \( P_{0} \in \mathbb{P} \), we denote \( \mu^{a}(P_{0}) = \mu^{a}_{0}, \mu^{a}(P_{0})(x) = \mu^{a}_{0}(x), \nu^{a}(P_{0}) = \nu^{a}_{0}, \text{Var}^{a}(P_{0})(x) = \text{Var}^{a}_{0}(P_{0})(x) \). Let \( Y_{t}^{a_{0}} = Y_{t}^{a}, \pi^{a_{0}}(P_{0}) = a_{0}, \mu^{a_{0}}_{0} = \mu_{0}, \text{and} \nu^{a_{0}}_{0} = \nu_{0}. \) For the two Bernoulli distributions with mean parameters \( \mu, \mu' \in [0, 1] \), we denote the KL divergence by \( d(\mu, \mu') = \mu \log(\mu/\mu') + (1-\mu) \log((1-\mu)/(1-\mu')) \) with the convention that \( d(0,0) = d(1,1) = 0 \).

3 Lower Bounds

In this section, we derive lower bounds for the probability of misidentification \( \mathbb{P}_{0}(\hat{\pi}(T) \neq \pi_{a}^{\ast}(P)) \) under a small gap; that is, \( \pi_{a}^{\ast}(P) = \pi_{a}^{\ast}(P_{0}) \rightarrow 0 \) for all \( a \in [K] \). Our lower bounds are semiparametric extensions of distribution-dependent lower bounds shown by [Kaufmann et al., 2016], hence, we call the lower bounds the semiparametric lower bounds.

First, the following conditions for a class of the bandit model \( \mathbb{P} \) are assumed throughout this study.

**Assumption 3.1.** For all \( P, Q \in \mathbb{P} \) and \( a \in [K] \), let \( P^{a} \) and \( Q^{a} \) be the joint distributions of \( (Y^{a}, X) \) of an treatment arm \( a \) under \( P \) and \( Q \) respectively. The distributions \( P^{a} \) and \( Q^{a} \) are mutually absolutely continuous and have density functions with respect to some Lebesgue measure \( m \). The potential outcome \( Y^{a} \) has the first and second moments conditioned on \( x \in \mathcal{X} \). There exist known constants \( C_{\mu}, C_{\nu}, C_{\sigma} > 0 \) such that, for all \( P \in \mathbb{P}, a \in [K], \text{and} x \in \mathcal{X}, |\mu^{a}(P)(x)| \leq C_{\mu}, |\nu^{a}(P)(x)| < C_{\nu}, \text{and max} \{1/(\sigma^{a}(P)(x))^2, (\sigma^{a}(P)(x))^2\} \leq C_{\sigma} \) for all \( x \in \mathcal{X} \).

For a class of bandit models, we consider the location-shift class and equal-variance bandit class defined as follows.

**Definition 3.2** (Location-shift bandit class). A class of bandit models \( \mathbb{P}^{L} \) is a location-shift bandit class if \( \mathbb{P}^{L} = \{P \in \mathbb{P} : (\sigma^{a}(P)(x))^2 = \sigma^{a}(x)^2\} \), where \( \sigma^{a}(x) > 0 \) is a constant.

For simplicity, \( \sigma^{a}(x) \) is denoted by \( \sigma^{a} \).
When the asymptotic variance of the gap estimators is small, the gaps can be estimated more accurately. Theorem 3.6 for any

Then, the analytical solution of this maximization problem and refined lower bound are shown in the following theorem.

In Theorem 3.6, we show the analytical solution of \( \min a \neq a_0 \frac{(\mu_0^a - \mu_0^a)^2}{2\Omega(w)} \) for all \( a \in [K] \), as \( \mu_0^a - \mu_0^a \to 0 \) for all \( a \in [K] \).

The denominator of the first term of the RHS in the lower bound corresponds to a semiparametric analogue of the Cramér-Rao lower bound of the asymptotic variance, called the semiparametric efficiency bound by Diebold et al. [1998]. This result implies that the optimal BAI strategy chooses treatment arms so as to reduce the asymptotic variance of estimators for the gaps (ATEs) between the best and suboptimal treatment arms. Here, when the asymptotic variance of the gap estimators is small, the gaps can be estimated more accurately. Theorem 3.6 will make this implication clearer.

In Theorem 3.6, we show the analytical solution of \( \max_{w \in \mathcal{W}} \min a \neq a_0 \frac{(\mu_0^a - \mu_0^a)^2}{2\Omega(w)} \). This solution exists when \( C_{\sigma^2} \) is sufficiently small. Therefore, the supremum of the RHS in the lower bound can be replaced with the maximum as

\[
\sup_{w \in \mathcal{W}} \min a \neq a_0 \frac{(\mu_0^a - \mu_0^a)^2}{2\Omega(w)} = \max_{w \in \mathcal{W}} \min a \neq a_0 \frac{(\mu_0^a - \mu_0^a)^2}{2\Omega(w)} \quad (1)
\]

Then, the analytical solution of this maximization problem and refined lower bound are shown in the following theorem.

Theorem 3.6. For any \( P_0 \in \mathcal{P}^{H} \), suppose that the same conditions of Theorem 3.3 hold, there exists a constant \( \Delta_0 \) such that \( \mu_0^a - \mu_0^a \leq \Delta_0 \) for all \( a \in [K] \), and there exists a constant \( C > 0 \) such that \( \mu_0^a(x) - \mu_0^a(x) = C (\mu_0^a - \mu_0^a) \) for all \( x \in X \). Then, the maximizer in the RHS of (1) is given as

\[
\bar{w}(a_0|x) = \frac{\sigma(x)}{\sigma^*(x) + \sqrt{\sum b \in [K] \setminus a_0 \frac{(\sigma^b(x))^2}{w(b|x)}}},
\]

\[
\bar{w}(a|x) = \left( 1 - \bar{w}(a_0|x) \right) \frac{(\sigma(a(x))^2)}{\sum b \in [K] \setminus a_0 (\sigma^b(x))^2} \quad \forall a \in [K] \setminus a_0,
\]
Figure 1: An idea in the derivation of the lower bounds. To lower bound the probability of misidentification, or equivalently upper bound $-\frac{1}{T} \log \mathbb{P}_{P_0}(\tilde{a}_T \neq a_0^*)$, it is sufficient to consider a case in the right figure.

And as $\Delta_0 \to 0$, the semiparametric lower bound is given as

$$\limsup_{T \to \infty} -\frac{1}{T} \log \mathbb{P}_{P_0}(\tilde{a}_T \neq a_0^*) \leq \frac{\Delta_0^2}{2 \mathbb{E}_{P_0} \left[ \left( \sigma^*(X_t) + \sqrt{\sum_{a \in [K] \setminus \{a_0^*\}} (\sigma^a(X_t))^2} \right)^2 \right]} + o(\Delta_0^2).$$

The proof is shown in Appendix E.

This result is also consistent with the existing results that many BAI strategies, such as the LUCB strategy (Kalyanakrishnan et al., 2012), choose both an estimated best and second-best treatment arms to discriminate an estimated best treatment arm from the other suboptimal treatment arms with a higher probability. Note that in regret minimization, a strategy is usually designed to sample an estimated best treatment arm more and not to sample other suboptimal treatment arms. Furthermore, because all gaps $\mu^a_0 - \mu^b_0$ are assumed to be upper bounded by $\Delta_0$, we can consider a situation where the expected rewards of all suboptimal treatment arms are in $[\mu^a_0 - \Delta_0, \mu^b_0)$. Moreover, to obtain lower bounds, it is sufficient to consider a case where $\mu^a = \mu^b = \Delta_0$, under which the largest lower bounds are given (Figure 1). In this situation, a strategy whose probability of misidentification matches the lower bound behaves to accurately estimate the gap between the best and a hypothetical second-best treatment arms, which is constructed as a treatment arm with the conditional variance $\sum_{b \in [K] \setminus \{a_0^*\}} (\sigma^a(X_t))^2$ for an allocation ratio $w$. Based on this implication obtained from Theorem 3.6, we construct our strategy in Section 4.

When $K = 2$, the lower bound is given as

$$\limsup_{T \to \infty} -\frac{1}{T} \log \mathbb{P}_{P_0}(\tilde{a}_T \neq a_0^*) \leq \frac{\Delta_0^2}{\mathbb{E}_{P_0} \left[ \left( \sigma^1(X_t) + \sigma^2(X_t) \right)^2 \right]} + o(\Delta_0^2),$$

where the maximizer in the RHS of (1) is given as

$$\tilde{w}(1|x) = \frac{\sigma^1(x)}{\sigma^1(x) + \sigma^2(x)}, \quad \tilde{w}(2|x) = \frac{\sigma^2(x)}{\sigma^1(x) + \sigma^2(x)}.$$ 

This result is also compatible with existing studies on efficient ATE estimation via adaptive experiments, as the maximizer $(\tilde{w}(1|x), \tilde{w}(2|x))$ is used throughout these studies (van der Laan, 2008; Hahn et al., 2011).

Here, note that an allocation ratio $w$ in the supremum corresponds to an expectation of sampling rule $\frac{1}{T} \sum_{t=1}^T 1[A_t = a]$ conditioned on $x$ under a alternative hypothesis $Q \in \mathcal{P} Q \neq P_0$, which is used to derive the lower bound. Therefore, the maximizers $\tilde{w} \in \mathcal{W}$ does not directly imply an allocation ratio used under the true bandit model $P_0 \in \mathcal{P}$. However, the maximizer $\tilde{w}$ still work as conjectures of a target allocation ratio used in our proposed strategy, although there is no logic at this stage to show that a strategy using the allocation ratio has an upper bound for the probability of misidentification matching to the lower bound. In Sections 4 and 5, we show that by allocation samples following the target allocation ratio, the upper bound for the probability of misidentification in our proposed strategy matches the semiparametric lower bound. Thus, we can confirm that these maximizers correspond to the optimal target allocation ratio.

Finally, we show the semiparametric lower bound for bandit models belonging to the equal-variance bandit class.

**Theorem 3.7** (Semiparametric contextual lower bound for the equal-variance bandit class). For any $P_0 \in \mathcal{P}^E$, suppose that the conditions of Theorem 3.5 hold, there exists a constant $\Delta_0$ such that $\mu^a_0 - \mu^a_0 \leq \Delta_0$ for all $a \in [K]$. Then, for
any consistent strategy (Definition 3.2), as \( \Delta_0 \to 0 \),

\[
\limsup_{T \to \infty} -\frac{1}{T} \log \mathbb{P}_{P_0}(\hat{a}_T \neq a_0^*) \leq \frac{\Delta_0^2}{2(a_0^1 + \sigma_0^2)^2} + o(\Delta_0^2),
\]

where the maximizer in the RHS of (1) is given as

\[
\bar{w}(a|x) = 1/K.
\]

This lower bound implies that the uniform-EBA strategy is optimal, where we choose each treatment arm with the same probability (the uniform sampling rule) and recommend a treatment arm with the highest sample average of observed rewards (the empirical best arm (EBA) recommendation rule).

**Remark** (Distribution-dependent lower bounds without contextual information). We review the distribution-dependent lower bound for BAI with a fixed budget when there is no contextual information (Kaufmann et al., 2016). When the potential outcome of each treatment arm \( a \in [K] \) follows the Gaussian distributions, the distribution-dependent lower bound is given as

\[
-\frac{1}{T} \log \mathbb{P}_{P_0}(\hat{a}_T \neq a_0^*) \leq \frac{\Delta_0^2}{2(a_0^1 + \sigma_0^2)^2}.
\]

This lower bound can be derived without localization of an alternative hypothesis and small gap (\( \Delta_0 \to 0 \)). Although the optimal target allocation from this lower bound cannot be derived, by using \( w^*(1) = \frac{\sigma_0^1}{\sigma_0^1 + \sigma_0^2} \) and \( w^*(0) = 1 - w^*(1) \) as a target allocation ratio, we can find an optimal algorithm whose upper bound matches the lower bound when \( \Delta_0 \to 0 \).

### 4 Proposed Strategy: the Contextual RS-AIPW Strategy

This section presents our strategy, which consists of sampling and recommendation rules. For each \( t = 1, 2, \ldots, T \), our sampling rule randomly chooses a treatment arm with a probability identical to an estimated target allocation ratio. In final round \( T \), our recommendation rule recommends a treatment arm with the highest-estimated expected reward.

#### 4.1 Target Allocation Ratio

First, we define a target allocation, which is used to define a sampling rule. We estimate it during an adaptive experiment and employ the estimator as a probability of choosing a treatment arm. We call a target allocation optimal if the upper and lower bounds for the probability of misidentification under our strategy using the allocation ratio match. We conjecture the optimal target allocation ratio using the results of Section 3. In particular, the results of Theorem 3.6 yields the following conjectures for the target allocation ratio \( w^* \in \mathcal{W} \): for each \( x \in \mathcal{X} \),

\[
w^*(a_0^*|x) = \frac{\sigma_0^1(x)}{\sigma_0^1(x) + \sqrt{\sum_{b \in [K] \setminus \{a_0^*\}} (\sigma_0^b(x))^2}},
\]

\[
w^*(a|x) = \left(1 - w^*(a_0^*|x)\right) \frac{(\sigma_0^1(x))^2}{\sum_{b \in [K] \setminus \{a_0^*\}} (\sigma_0^b(x))^2}, \quad \forall a \in [K] \setminus \{a_0^*\}.
\]

Under this conjectured optimal target allocation ratio, we can show that the upper and lower bounds for the probability of misidentification match in Section 3; hence, this target allocation ratio is optimal. This target allocation ratio is unknown when the variances are unknown; therefore, it must be estimated via observations during the bandit process.

#### 4.2 Sampling Rule with Random Sampling (RS) and Estimation

We provide a sampling rule referred to as a random sampling (RS) rule. For \( a \in [K] \) and \( t \in [T] \), let \( \tilde{w}_t(a|x) \) be an estimated target allocation ratio at round \( t \). In each round \( t \), we obtain \( \gamma_t \) from the uniform distribution on \([0, 1]\) and choose a treatment arm \( A_t = 1 \) if \( \gamma_t \leq \tilde{w}_t(1|X_t) \) and \( A_t = a \) for \( a \geq 2 \) if \( \gamma_t \in (\sum_{b=1}^{a-2} \tilde{w}_t(b|X_t), \sum_{b=1}^{a-1} \tilde{w}_t(b|X_t)] \).

As an initialization, we choose a treatment arm \( A_t \) at round \( t \leq K \) and set \( \tilde{w}_t(a|x) = 1/K \) for \( a \in [K] \) and \( x \in \mathcal{X} \). In a round \( t > K \), for all \( a \in [K] \), we estimate the target allocation ratio \( w^* \) using past observations \( \mathcal{F}_{t-1} \), such that for
all \( a \in [K] \) and \( x \in \mathcal{X}, \hat{w}_t(a|x) > 0 \) and \( \sum_{a \in [K]} \hat{w}_t(a|x) = 1 \). Then, in round \( t \), we choose a treatment arm \( a \) with a probability \( \hat{w}_t(a|X_t) \). To construct an estimator \( \hat{w}_t(a|x) \) for all \( x \in \mathcal{X} \) in each round \( t \), we denote a bounded estimator of the conditional expected reward \( \mu^0_t(x) \) by \( \hat{\mu}^0_t(x) \), that of the conditional expected squared reward \( \nu^0_t(x) \) by \( \hat{\nu}^0_t(x) \), and that of the conditional variance \( (\sigma^0_t(x))^2 \) by \( (\hat{\sigma}^0_t(x))^2 \). All estimators are constructed only from samples up to round \( t \). More formally, they are constructed as follows. For \( t = 1, 2, \ldots, K \), we set \( \hat{\mu}^0_t = \hat{\nu}^0_t = (\hat{\sigma}^0_t(x))^2 = 0 \). For \( t > K \), we estimate \( \mu^0(x) \) and \( \nu^0(x) \) using only past samples \( \mathcal{F}_{t-1} \) and converge to the true parameter almost surely.

**Assumption 4.1.** For all \( a \in [K] \) and \( x \in \mathcal{X}, \hat{\mu}^0_t(x) \) and \( \hat{\nu}^0_t(x) \) are \( \mathcal{F}_{t-1} \)-measurable, \( |\hat{\mu}^0_t(x)| \leq C_\mu \) and \( |\hat{\nu}^0_t(x)| \leq C_\nu \), and

\[
\hat{\mu}^0_t(x) \xrightarrow{a.s.} \mu^0(x) \quad \text{and} \quad \hat{\nu}^0_t(x) \xrightarrow{a.s.} \nu^0(x) \quad \text{as} \ t \rightarrow \infty.
\]

For example, we can use nonparametric estimators, such as the nearest neighbor regression estimator and kernel regression estimator, which are proved to converge to the true function almost surely under a bounded sampling probability \( \hat{w}_t \) by Yang and Zhu (2002) and Qian and Yang (2016). As long as these conditions are satisfied, any estimators can be used. Note that we do not assume specific convergence rates for these estimators because we can show the asymptotic optimality without them owing to the unbiasedness of the AIPW estimator (Kato et al., 2021). Let \( \hat{\sigma}^0_t(a|x)^2 = \hat{\nu}^0_t(x) - \hat{\mu}^0_t(x)^2 \) for all \( a \in [K] \) and \( x \in \mathcal{X} \). Then, we estimate the variance \( (\hat{\sigma}^0_t(a|x))^2 \) for all \( a \in [K] \) and \( x \in \mathcal{X} \) in each round \( t \) as \( (\hat{\sigma}^0_t(a|x))^2 = \max\{\min\{((\hat{\sigma}^0_t(x))^2, C_{\sigma^2} \}, 1/C_{\sigma^2} \} \) and define \( \hat{w}_t \) by replacing the variances in \( \nu^0 \) with corresponding estimators; that is, for \( \hat{a}_t \in \arg\max_{a \in [K]} \hat{\mu}^0_t(a) \),

\[
\hat{w}(a|X_t) = \frac{\hat{\sigma}^0_t(a|X_t)}{\hat{\sigma}^0_t(X_t) + \sqrt{\sum_{b \in [K]} \{\hat{\sigma}^0_t(b|X_t)\}^2}},
\]

\[
\hat{w}(a|X_t) = \left(1 - \hat{w}(a|X_t)\right) \sum_{b \in [K]} \{\hat{\sigma}^0_t(X_t)\}^2
\]

If there are multiple elements in \( \arg\max_{a \in [K]} \hat{\mu}^0_t(a) \), we choose one of them as \( \hat{a}_t \) in some way.

We employ this strategy to apply the large deviation expansion for martingales to the estimator of the expected reward, which is the core of our theoretical analysis in Section 5.

### 4.3 Recommendation Rule with the AIPW Estimator

The following section presents our recommendation rule. In the recommendation phase of round \( T \), for each \( a \in [K] \), we estimate \( \mu^a \) for each \( a \in [K] \) and recommend the maximum. To estimate \( \mu^a \), the AIPW estimator is defined as

\[
\hat{\mu}^\text{AIPW}_T,a = 1/T \sum_{t=1}^T \varphi^a(Y_t, A_t, X_t; \hat{\mu}^0_t, \hat{\nu}^0_t, \hat{w}_t), \quad \varphi^a(Y_t, A_t, X_t) = \frac{1[A_t = a](Y_t^a - \hat{\mu}^0_t(X_t))}{\hat{w}_t(a|X_t)} + \hat{\mu}^0_t(X_t). \quad (2)
\]

In the final round \( t = T \), we recommend \( \hat{a}_T = \arg\max_{a \in [K]} \hat{\mu}^\text{AIPW}_T,a \).

\[
(3)
\]

The AIPW estimator has the following properties: (i) its components \( \{\varphi^a(Y_t, A_t, X_t; \hat{\mu}^0_t, \hat{\nu}^0_t, \hat{w}_t)\}_{t=1}^T \) are a martingale difference sequence, thereby allowing us to use the large deviation bounds for martingales; (ii) it has the minimal asymptotic variance among the possible estimators. For instance, we can use other estimators with a martingale property, such as the inverse probability weighting (IPW) estimator (Horvitz and Thompson, 1952), but their asymptotic variance will be larger than that of the AIPW estimator. For the \( t \)-th element of the sum in the AIPW estimator, we use the nuisance parameters estimated from past observations up to the round \( t - 1 \) to make the sequence in the sum a martingale difference sequence. This technique is often used in semiparametric inference for adaptive experiments (van der Laan, 2008; Hadad et al., 2021; Kato et al., 2020, 2021) and also has a similar motivation to double machine learning (Chernozhukov et al., 2018). Note that in double machine learning for a doubly robust (DR) estimator, we usually impose specific convergence rates for the estimators of the nuisance parameter, which are not required in our case owing to the unbiasedness of the AIPW estimator (Assumption 4.1). Also see Kato et al. (2021).

**Remark.** We present the pseudo-code in Algorithm 7. Note that \( C_\mu \) and \( C_{\sigma^2} \) are introduced for technical purposes to bound the estimators. Therefore, any large positive value can be used.
Algorithm 1 Contextual RS-AIPW strategy

Parameter: Positive constants $C_\mu$ and $C_{\sigma^2}$.
Initialization:
for $t = 1$ to $K$ do
    Draw $A_t = t$. For each $a \in [K]$, set $\tilde{w}_t(a|x) = 1/K$.
end for
for $t = K + 1$ to $T$ do
    Observe $X_t$.
    Construct $\tilde{w}_t(1|X_t)$ by using the estimators of the variances.
    Draw $\gamma_t$ from the uniform distribution on $[0, 1]$.
    $A_t = 1$ if $\gamma_t \leq \tilde{w}_t(1|X_t)$ and $A_t = a$ for $a \geq 2$ if $\gamma_t \in \left(\sum_{b=1}^{a-1} \tilde{w}_t(b|X_t), \sum_{b=1}^{a} \tilde{w}_t(b|X_t)\right]$.
end for
Construct $\hat{\mu}_t^{\text{AIPW},a}$ for each $a \in [K]$ following (2).
Recommend $\hat{\alpha}_T$ following (3).

Remark (Remark on the sampling rule). Unlike the sampling rule of Garivier and Kaufmann [2016], our proposed sampling rule does not choose the next treatment arm so that the empirical allocation ratio tracks the optimal target allocation ratio. This is due to the use of martingale properties under the AIPW estimator in the theoretical analysis of the upper bound.

Remark (Sampling for stabilization). In the pseudo-code, only the first $K$ rounds are used for initialization. To stabilize the performance, we can increase the number of samplings in initialization, similarly to the forced-sampling approach employed by Garivier and Kaufmann [2016]. In Section 5 to show the asymptotic optimality, we use almost sure convergence of $\tilde{w}_t$ to $w^*$. As long as $\tilde{w}_t \xrightarrow{a.s.} w^*$, we can adjust $\tilde{w}_t$ appropriately. For instance, we can use $\tilde{w}_t = (1 - r_t)\tilde{w}_t(a|X_t) + r_t/2$ as the sampling probability instead of $\tilde{w}_t$, where $r_t \to 0$ as $t \to \infty$.

Remark (The role of $C_{\sigma^2}$). Assumption 3.7 implies that the sampling probability is bounded by a small constant, $1/(2C_{\sigma^2}) \leq w^*(a|x) \leq C_{\sigma^2}/2$. Thus, it ensures that the variance of the AIPW estimator is finite. Although the role of this constant appears to be similar to the forced sampling (Garivier and Kaufmann [2016]), it is substantially different. We can set $C_{\sigma^2}$ sufficiently large so that it is almost negligible in implementation.

5 Asymptotic Optimality of the Contextual RS-AIPW Strategy

In this section, we derive the following upper bound of the misspecification probability of the RS-AIPW strategy, which implies that the strategy is asymptotically optimal.

5.1 Asymptotic Optimality

We derive the upper bounds for bandit models, where the rewards are sub-exponential random presents our recommendation rule, m variables.

Assumption 5.1. For all $P \in \mathcal{P}$ and $a \in [K]$, $X_t$ is sub-exponential random variable and $Y_t^a$ is conditionally sub-exponential random variable given $X_t = x$; that is, for all $P \in \mathcal{P}$, $a \in [K]$, $t$, $u$, $u' > 0$, and $x \in \mathcal{X}$, there are constants $U, U' > 0$ such that $\mathbb{P}_P(|X_t| > u) \leq 2\exp(-u/U)$ and $\mathbb{P}_P(|Y_t| > u|X_t = x) \leq 2\exp(-u'/U')$

Theorem 5.2 (Upper bound of the RS-AIPW strategy). Suppose that Assumptions 3.7 and 5.7 hold. Then, for any $P_0 \in \mathcal{P}_L$, as $\mu_0 - \mu^*_0 \to 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \log \mathbb{P}_{P_0} (\hat{\alpha}_T \neq a^*_0) \geq \min_{a \neq a^*_0} \frac{(\mu_0 - \mu^*_0)^2}{2\mathbb{E}_{P_0} \left[ \left( \sigma^*_0(X) + \sqrt{\sum_{b \in [K] \setminus \{a_0^*_0\}} (\sigma^b_0(X))^2} \right)^2 + (\mu^*_0(X) - \mu_0^*(X) - (\mu_0^* - \mu_0^*))^2 \right]} - o\left(\frac{(\mu_0 - \mu^*_0)^2}{2}\right).$$

This theorem allows us to evaluate the exponentially small probability of misidentification up to the constant term when $\Delta_0 \to 0$. Moreover, this result also implies that the estimation error of the target allocation ratio $w^*$ is negligible.
Then, for any $\varepsilon > 0$, where $\Delta_0 \to 0$. This is because the upper bound matches the performance of strategies for Gaussian bandit models developed by Glynn and Juneja (2004) given the optimal target allocation ratio. This also means that the estimation error of the target allocation ratio is insensitive to the probability of misidentification in situations where identifying the best treatment arm is difficult due to the small gap.

For $P = P^E$, this upper bound matches the lower bounds in Theorems 3.6 and 5.7 under a small-gap regime.

**Corollary 5.3.** If we also suppose that there exist constants $\Delta_0, C > 0$ such that $|\mu_0^* - \mu_0^a| \geq \Delta_0$ and $|\mu_0^a(x) - \mu_0(x)| \leq C\Delta_0$ for all $a \in [K]$ and $x \in X$, then, as $\Delta_0 \to 0$, the difference variable $Z_t$ can be bounded by any $\varepsilon > 0$, where $\Delta_0$ is obviously optimal.

5.2 Proof of the Upper Bound

Owing to the dependency among samples in BAI, it is also difficult to apply the standard large deviation bound (Dembo and Zeitouni, 2009) to a sample average of some random variable. For example, Gärtner-Ellis theorem (Gärtner, 1977; Ellis, 1984) provides a large deviation bound for dependent samples, but it requires the existence of the cumulant, a logarithmic moment generating function, which is not easily guaranteed for the samples in BAI.

For these problems, we derive a novel Cramér-type large deviation bounds for martingales by extending the results of Grama and Haeusler (2000) and Fan et al. (2013, 2014). Note that their original large deviation bound is only applicable to martingales whose conditional second moment is bounded deterministically; that is, for some martingale difference sequence $\{W_t\}_{t=1}^n$ of some random variable $W$, for any $n > 0$, there exists a real number $0 < \varepsilon < 1/2$ such that $\mathbb{E} \left[ \sum_{a=1}^n \mathbb{E}[W^2|F_{a-1}] - 1 \right] \leq \varepsilon^2$; then, Fan et al. (2013, 2014) derive the upper bound for $P(\sum_{a=1}^n W_a > z)$, where $\varepsilon$ belongs to a range upper bounded by $\varepsilon$. Thus, their large deviation bound holds when $\mathbb{E} \left[ \sum_{a=1}^n \mathbb{E}[W^2|F_{a-1}] - 1 \right]$ can be bounded by any $\varepsilon$ for any $n > 0$. However, in BAI, we cannot usually bound the conditional second moment for any $\varepsilon$ because of the randomness of strategies. This randomness prevents us from applying the original results of Grama and Haeusler (2000) and Fan et al. (2013, 2014). Instead, we consider bounding the conditional second moment for large $T$. We first demonstrate a large deviation bound for martingales by using the mean convergence of the bandit model $P = P^E$, the uniform-EBA strategy is also obviously optimal.

**Step 1: Cramér’s large deviation expansions for the AIPW estimator**

Here, we introduce key elements of our analysis. For each $t \in [T]$, we define the difference variable

$$
\xi_t^a = \frac{\varphi^a\left( Y_t, A_t, X_t; \mu_{t}^a, \hat{w}_t \right) - \varphi^a\left( Y_t, A_t, X_t; \hat{\mu}_{t}^a, \hat{w}_t \right) - (\mu_0^a - \mu_0^a)}{\sqrt{T \eta_t^a}},
$$

where $\eta_t^a = \mathbb{E}_{P_t}[|\sigma_0^a(X) + \sum_{a \in [K] \setminus \{a_0^a\}} (\sigma_0^a(X))^2|^{2}]$. We also define its sum $Z_t^a = \sum_{a=1}^T \xi_t^a$, and a sum of conditional moments $W_t = \sum_{a=1}^T \mathbb{E}_{P_t}[|\xi_t^a|^2|F_{a-1}]$ with initialization $W_0 = 0$. Using the difference variable $\xi_t^a$, we can express the gap estimator as $\sqrt{T}(\hat{\mu}_T - \hat{\mu}_T^{AIPW.a} - (\mu_0^a - \mu_0^a))/\sqrt{\eta_t^a} = \sum_{a=1}^T \xi_t^a = Z_t^a$. Here, $\{(\eta_t^a, F_t)\}_{t=1}^T$ is a martingale difference sequence (Appendix F), using the fact that $\hat{\mu}_t$ and $\hat{w}_t(a | X_t)$ are $F_t$-measurable random variables. Let us also define $V_T = \mathbb{E}_T[|\sum_{t=1}^T \mathbb{E}_{P_t}[|\xi_t^a|^2|F_{a-1}] - 1|]$ and denote the cumulative distribution function of the standard normal distribution by $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x \exp(-t^2/2)dt$.

We obtain the following theorem on the tail probability of $Z_t^a$:

**Theorem 5.4.** Suppose that Assumptions 3.7 and 5.7 and the following condition hold:

Condition A: $\sup_{t \leq T} \mathbb{E}_{P_t}[\exp(C_0 \sqrt{T} \xi_t^a)] |F_{a-1}| \leq C_1$ for some positive constants $C_0, C_1$.

Then, for any $\varepsilon > 0$, there exist $T_0, c_1, c_2 > 0$ such that, for all $T \geq T_0$ and $1 \leq u \leq \sqrt{T} \min\{C_0/4, \sqrt{S_0^2}/(8C_1)\}$,

$$
\frac{\mathbb{P}_{P_t}(Z_t^a \leq -u)}{\Phi(-u)} \leq c_1 u \exp \left( c_2 \left( \frac{u^2}{\sqrt{T}} + \frac{u^2}{T} + u^2(T + \varepsilon) + T_0 \right) \right),
$$

where the constants $c_1, c_2$ depend on $C_0$ and $C_1$ but do not depend on $\{(\eta_t^a, F_t)\}_{t=1}^T$, $u$, and the bandit model $P$. 

Semiparametric Contextual Best Arm Identification with a Fixed Budget
As described by Fan et al. [2014], if $T \mathbb{E}[(\xi^\pi_t)^2 | \mathcal{F}_{t-1}]$ are all bounded from below by a positive constant, Condition A implies the conditional Bernstein condition: for a positive constant $C$, $\mathbb{E}[(\xi^\pi_t)^k | \mathcal{F}_{t-1}] \leq \frac{1}{2} k! (C/\sqrt{T})^{k-2} \mathbb{E}[(\xi^\pi_t)^2 | \mathcal{F}_{t-1}]$ for all $k \geq 2$ and all $t \in [T]$.

From Theorem 5.4, if $(\mu_0^* - \mu_0^0) / \sqrt{\nu^a}$ is some positive function of $u$, $\mathbb{P}(Z_T^r \leq -\sqrt{T}(\mu_0^0 - \mu_0^0) / \sqrt{\nu^a}) = \mathbb{P}_{P_0} \left( \mu_{AIPW}^T, a_0^0 \leq \mu_{AIPW}^T, a \right)$. Then, the probability that we fail to make the correct treatment arm comparison is bounded as

$$
\mathbb{P}_{P_0} \left( \mu_{AIPW}^T, a_0^0 \leq \mu_{AIPW}^T, a \right)
$$

Finally, we consider an approximation of the large deviation bound. Here, we provide the proof sketch of Theorem 5.4. The formal proof is shown in Appendix I.

**Proof sketch of Theorem 5.4** Let us define $r_t(\lambda) = \exp(\lambda u^t) / \mathbb{E}[\exp(\lambda u^t)]$. Then, we apply the change-of-measure technique from Fan et al. (2013, 2014) to transform the bound. In Fan et al. (2013, 2014), the proof is complete up to this procedure. However, in our case, the second moment is also a random variable. Because of the randomness, there remains a term $\mathbb{E}[\exp(\lambda(u) \sum_{t=1}^T \xi^\pi_t) / \prod_{t=1}^T \mathbb{E}[\exp(\lambda(u) \xi^\pi_t)]]$, where $\lambda(u)$ is some positive function of $u$. Therefore, we next consider the bound of the conditional second moment of $\xi^\pi_t$ to apply $L^r$-convergence theorem (Proposition A.3). With some computation, the proof is complete.

**Step 2: Gaussian approximation under a small gap**

Finally, we consider an approximation of the large deviation bound. Here, $\Phi(-u)$ is bounded as

$$
\Phi(-u) \leq \frac{1}{\sqrt{2\pi(1+u)}} \exp(-\frac{u^2}{2}) \leq \Phi(-u) \leq \frac{1}{\sqrt{2\pi(1+u)}} \exp(-\frac{u^2}{2}), \quad u \geq 0 \text{ (see Fan et al. (Section 2.2., 2013)).}
$$

By combining this bound with Theorem 5.4 and Proposition A.5 in Appendix A which shows the rate of convergence in the Central limit theorem (CLT) for $0 \leq u \leq 1$, we have the following corollary.

**Corollary 5.5.** Suppose that Assumptions 3.1 and 5.1 Condition A in Theorem 5.4 and the following conditions hold:

**Condition B:** $(\mu_0^* - \mu_0^0) / \sqrt{\nu^a} \leq \min\{C_0/4, \sqrt{3C_0^2/(8C_1)}\}$;

**Condition C:** $\lim_{T \to \infty} \nu_T = 0$. Then, there exist a constant $c > 0$ such that

$$
\liminf_{T \to \infty} - \frac{1}{T} \log \mathbb{P}_{P_0} \left( \mu_{AIPW}^T, a_0^0 \leq \mu_{AIPW}^T, a \right) \geq \frac{c}{2} \left( \frac{\mu_0^* - \mu_0^0}{\sqrt{\nu^a}} \right)^3 + \left( \frac{\mu_0^* - \mu_0^0}{\sqrt{\nu^a}} \right)^4.
$$

This approximation can be considered a Gaussian approximation because the probability is represented by $\exp(-(-\mu_0^0 - \mu_0^0)^2T/(2\nu^a))$. Condition B is satisfied as $\mu_0^* - \mu_0^0 \to 0$. To use Corollary 5.5, we need to show that Conditions A and C hold. First, the following lemma states that Condition A holds with the constants $C_0$ and $C_1$, which are universal to the problems in $P$.

**Lemma 5.6.** Suppose that Assumptions 3.1 and 5.1 hold. For each $C_0 \geq 0$, there exists a positive constant $C_1$ that depends on $C_0, C_0^c, C_0^a$, such that $\sup_{t \in [T]} \mathbb{E}_{P_0} \left[ \exp(\lambda(u) \sum_{t=1}^T \xi^\pi_t) \right] \mathbb{P}_{P_0} \left( \mathcal{F}_{t-1} \right) \leq C_1$.

With regards to Condition C, we introduce the following lemma for the convergence of $\nu_T$, which corresponds to the mean convergence of the variance of the AIPW estimator scaled with $\sqrt{T}$. 

\[10\]
Lemma 5.7. Suppose that Assumptions 3.7 and 5.7 hold. For any \( P \in \mathcal{P} \), \( \lim_{T \to \infty} V_T = 0 \); that is, for any \( \delta > 0 \), there exists \( T_0 \) such that for all \( T > T_0 \), \( \mathbb{E}_{P_0} [\sum_{t=1}^{T} \mathbb{E}_{P_0}[|\xi_t|^2 | F_{t-1}] - 1] \leq \delta \).

The proofs of Lemma 5.6 and Lemma 5.7 are shown in Appendix G and H, respectively.

Finally, the proof of Theorem 5.2 is completed as follows:

\[
\liminf_{T \to \infty} \frac{1}{T} \log \mathbb{P}_n(\hat{a}_T \neq a^*_n) \geq \liminf_{T \to \infty} \frac{1}{T} \log \sum_{a \neq a^*} \mathbb{P}_n(\hat{\mu}_T^{\text{AIPW},\alpha} \geq \mu_T^{\text{AIPW},\alpha}) \\
\geq \liminf_{T \to \infty} \frac{1}{T} \log(K-1) \max_{a \neq a^*} \mathbb{P}_n(\hat{\mu}_T^{\text{AIPW},\alpha} \geq \mu_T^{\text{AIPW},\alpha}) \\
\geq \min_{a \neq a^*} \frac{(\mu_0^a - \mu_0^{a^*})^2}{2V^a} - c \left( \frac{\mu_0^a - \mu_0^{a^*}}{\sqrt{V^a}} \right)^3 + \left( \frac{\mu_0^a - \mu_0^{a^*}}{\sqrt{V^a}} \right)^4.
\]

Remark (CLT). Note that the CLT cannot provide an exponentially small evalulation of the probability of misidentification. It gives an approximation around \( 1/\sqrt{T} \) of the expected reward, but we are interested in an evaluation with constant deviation from the expected reward. However, when the gap converges to zero with \( 1/\sqrt{T} \), our large deviation bound gives the CLT for martingale. In this sense, our result is a generalization of the martingale CLT.

6 Related work

6.1 Additional Literature on BAI

The stochastic MAB problem is a classical abstraction of the sequential decision-making problem (Thompson, 1933; Robbins, 1952; Lai and Robbins, 1985), and BAI is a paradigm of the MAB problem (Even-Dar et al., 2006; Audibert et al., 2010; Bubeck et al., 2011). Though the problem of BAI itself goes back decades, its variants go as far back as the 1950s (Bechhofer et al., 1968).

Kaufmann et al. (2014, 2016) conjectures distribution-dependent lower bounds for BAI. In the BAI literature, there is another setting, known as BAI with fixed confidence (Jennison et al., 1982; Mannor and Tsitsiklis, 2004; Kalyanakrishnan et al., 2012; Wang et al., 2021). For the fixed confidence setting, Garivier and Kaufmann (2016) solves the problem in the sense that they develop a strategy whose upper bound of the sample complexity, an expected stopping time, matches the distribution-dependent lower bound. The result is further developed by Degenne et al. (2019) to solve the two-player game by the no-regret saddle point algorithm. Furthermore, Qin et al. (2017), Shang et al. (2020), and Jourdan et al. (2022) extend the Top Two Thompson Sampling (TTTS), proposed by Russo (2016) and shows the asymptotic optimality of their strategies in the fixed confidence setting. Wang et al. (2021) develops Frank-Wolfe-based Sampling (FWS) to characterize the complexity of fixed-confidence BAI with various types of structures among the arms. See Wang et al. (2021) for techniques in the fixed-confidence setting and a further comprehensive survey.

Russo (2016), Qin et al. (2017), and Shang et al. (2020) propose the Bayesian BAI strategies, which are optimal in the sense of the posterior convergence rate. Although the upper bounds of the sample complexity are shown to match the lower bounds of Kaufmann et al. (2016) in fixed-confidence BAI for some of the methods, the upper bounds for the probability of misidentification do not match that for fixed-budget BAI. Although the rate of the posterior convergence is also optimal in the fixed-budget setting, it does not imply the asymptotic optimality for the probability of misidentification (Kasy and Sautmann, 2021; Ariu et al., 2021). For example, the KL divergence in the lower and upper bounds is flipped between the evaluations of posterior convergence and probability of misidentification. In addition, for the posterior convergence, we consider a convergence of a random variable, while for the probability of misidentification, we consider a convergence of a non-random variable.

Independently, for the fixed-budget BAI without contextual information, Komyama et al. (2022) considers minimax evaluation of the probability of misidentification, and Barrier et al. (2022) considers a nonparametric setting. Their problem setting is related to ours, but the approaches and results differ.

In evaluation, we can use the simple regret. Bubeck et al. (2009) provides a non-asymptotic minimax lower and upper bound of simple regret for bandit models with a bounded support. Following their results, the uniform-EBA strategy is optimal for bandit models with a bounded support. This result is compatible with Theorem 5.7, which implies that the uniform sampling is asymptotically optimal for the equal-variance bandit class. Because Bubeck et al. (2009) does not use other parameters, such as variances, their result does not contradict with Theorem 5.5, which implies that the target allocation ratio using the variances is optimal. Recently, Adusumilli (2021) considers a diffusion process for
the bandit problems \cite{wager2021, fan2021}, including fixed-budget BAI, to evaluate the Bayes and minimax regrets. Furthermore, based on the finding, Adusumilli\cite{adusumilli2022} considers another minimax evaluation of BAI with two-armed bandits. Komiyama et al.\cite{komiyama2021} discusses the optimality of Bayesian simple regret minimization, which is closely related to BAI in a Bayesian setting. They showed that parameters with a small gap make a significant contribution to Bayesian simple regret.

6.2 Literature on Causal Inference

The framework of bandit problems is closely related to the potential outcome framework of \cite{neyman1923, rubin1974}. In causal inference, the gap is often referred to as the average treatment effect, and the estimation is studied in this framework. To estimate the average treatment effect efficiently, van der Laan\cite{van2008}, Hahn et al.\cite{hahn2011}, Tabord-Meehan\cite{tabord2018}, Kato et al.\cite{kato2020}, and Gupta et al.\cite{gupta2021} propose adaptive strategies. The AIPW estimator, which is also referred to as a DR estimator, plays an important role in treatment effect estimation \cite{rohott1994, hahn1998, bang2005, dudik2011, van2014, lubdik2016}. The AIPW estimator also plays an important role in double/debiased machine learning literature because it mitigates the convergence rate conditions of the nuisance parameters \cite{chernozhukov2018, ichimura2022}.

In adaptive experiments for efficient ATE estimation, the AIPW estimator has also been used by van der Laan\cite{van2008} and Hahn et al.\cite{hahn2011}. Karlan and Wood\cite{karlan2014} applied the method of Hahn et al.\cite{hahn2011} to test how donors respond to new information regarding the effectiveness of a charity. These studies have been extended by Tabord-Meehan\cite{tabord2018} and Kato et al.\cite{kato2020}. However, the notion of optimality is based on the analogue of the efficient estimation of the ATE under i.i.d. observations and not complete in adaptive experiments.

When constructing AIPW estimator with samples obtained from adaptive experiments, including BAI strategies, a typical construction is to use sample splitting and martingales \cite{van2008, hadad2020, kato2020, kato2021}. Howard et al.\cite{howard2021}, Kato et al.\cite{kato2020}, and provide non-asymptotic confidence intervals of the AIPW or DR estimator, which do not bound a tail probability in large deviation as ours. The AIPW estimator is also used in the recent bandit literature, mainly in regret minimization \cite{dimakopoulou2021, kim2021, hadad2021, bibaut2021, zhan2021} consider the off-policy evaluation using observations obtained from regret minimization algorithms.

6.3 Difference between Limit Experiments Frameworks

The small-gap regime is inspired by limit experiments framework \cite{lecam1986, van1998, hirano2009}. For a parameter $\theta_0 \in \mathbb{R}$ and $n$ i.i.d. observations for a sample size $n$, the limit experiments framework considers local alternatives $\theta = \theta_0 + h/\sqrt{n}$ for a constant $h \in \mathbb{R}$ \cite{van1991, van1998}. Then, we can approximate the statistical experiment by a Gaussian distribution and discuss the asymptotic optimality of statistical procedures under the approximation. Hirano and Porter\cite{hirano2009} relates the asymptotic optimality of statistical decision rules \cite{manski2000, manski2002, manski2004, deheja2005} to the limit experiment framework. This framework is further applied to policy learning, such as \cite{athey2017}.

Independently, Armstrong\cite{armstrong2022} proposes an application of the local asymptotic framework to a setting similar to BAI by replacing the CLT used in the original framework, such as \cite{van1998}, with that for martingales. In their analysis, the gaps converge to zero with $1/\sqrt{T}$, and a class of BAI strategies is restricted for the second moment of the score to converges to a constant, whereas our gaps converge to zero independently of $T$, and a class of BAI strategies is restricted to be consistent.

Here, note that taking the parameter $\theta = \theta_0 + h/\sqrt{T}$ does not produce the distribution-dependent analysis; that is, the instance is not fixed as $T$ increases. Therefore, a naive application of the distribution-dependent analysis like Proposition B.1 does not provide a lower bounds for BAI in this setting. To match the lower bound of Kaufmann et al.\cite{kaufmann2016}, we need to consider the large deviation bound, rather than CLT. In other words, the limit experiment framework first applies a Gaussian approximation and then evaluates the efficiency under that approximation, where efficiency arguments are complete within the Gaussian distribution. In contrast, we derive the lower bounds of an event under the true distribution in our limit decision-making and approximate it by considering the limit of the gap. Therefore, in limit decision-making, we first consider the optimality for the true distribution and find the optimal strategy in the sense that the upper bound matches the lower bound when the gaps converge to zero.
6.4 Other Related Work

Our small-gap regime is also inspired by lil’UCB (Jamieson et al., 2014), Balsubramani and Ramdas (2016) and Howard et al. (2021) propose sequential testing using the law of iterated logarithms and discuss the optimality of sequential testing based on the arguments of Jamieson et al. (2014).

Ordinal optimization has been studied in the operation research community (Peng and Fu, 2016; Ahn et al., 2021), and a modern formulation was established in the 2000s (Chen et al., 2000; Glynn and Juneja, 2004). Most of these studies consider the estimation of the optimal sampling rule separately from the probability of misidentification.

In addition to Fan et al. (2013, 2014), several studies have employed martingales to obtain tight large deviation bounds (Cappé et al., 2013; Juneja and Krishnasamy, 2019; Howard et al., 2021; Kaufmann and Koolen, 2021). Some of these studies have applied change-of-measure techniques. Tekin and van der Schaar (2015), Guan and Jiang (2018), and Deshmukh et al. (2018) also consider BAI with contextual information, but their analysis and setting are different from those employed in this study.

7 Discussion

7.1 Asymptotic Optimally in BAI with a Fixed Budget

Kaufmann et al. (2016) distributes distribution-dependent lower bounds for BAI with a fixed confidence and budget, based on similar change-of-measure arguments to those found in Lai and Robbins (1985). In BAI with fixed confidence, Garivier and Kaufmann (2016) develops a strategy whose upper bound and lower bounds for the probability of misidentification match. In contrast, in the fixed-budget setting, the existence of a strategy whose upper bound matches the lower bound of Kaufmann et al. (2016) was unclear. We consider that this is because the estimation error of an optimal target allocation ratio is negligible in BAI with a fixed budget, unlike BAI with fixed confidence, where we can draw each treatment arm until the strategy satisfies a condition. Furthermore, there are lower bounds different from Kaufmann et al. (2016), such as Audibert et al. (2010), Bubeck et al. (2011), and Carpentier and Locatelli (2016).

Audibert et al. (2010) proposes the UCB-E and Successive Rejects (SR) strategies. Using the complexity terms $H_1(P) = \sum_{a \in [K] \setminus \{a^*(P)\}} 1/(\Delta^a(P))^2$ and $H_2(P) = \max_{a \in [K] \setminus \{a^*(P)\}} a/(\Delta^a(P))^2$, where $\Delta^a(P) = \mu^a(P) - \mu$, they prove an upper bound for the probabilities of misidentification of the form $\exp(-T/(18H_1(P)))$ and $\exp(-T/(\log(K)H_2(P)))$, for UCB-E with the upper bound on SR, respectively.

Carpentier and Locatelli (2016) discusses the optimality of the method proposed by Audibert et al. (2010) by an effect of constant factors in the exponents of certain bandit models. They proved the lower bound on the probability of misidentification of the form: $\sup_{P \in \mathcal{P}_B} \left\{P_{\hat{a}_T}(\hat{a}_T \neq a^*_T) \exp\left(400T/(\log(K)H_1(P))\right)\right\}$, where for all $P \in \mathcal{P}_B$, there exists a constant $B > 0$ such that $H_1(P) < B$. Our result does not contradict with the result that found by Carpentier and Locatelli (2016), as we consider a small-gap regime, rather than the large-gap regime employed by Carpentier and Locatelli (2016). In the other words, their results are complementary to ours because we consider situations with a small gap.

7.2 Two-stage Sampling Rule

Our Contextual RS-AIPW strategy is also applicable to a setting where we can update the sampling rule in batch, rather than a sequential manner, as well as other BAI strategies in different settings. For example, even in a two-stage setting, where we are allowed to update the sampling rule only once, we can show the asymptotic optimality if the budgets separated into two-stages go to infinity simultaneously. Such a setting has frequently been adopted in the field of economics, such as Hahn et al. (2011) and Kasy and Sautmann (2021).

8 Conclusion

In this study, we considered BAI with a fixed budget and contextual information under a small-gap regime. Subsequently, we derived semiparametric lower bounds for the probability of misidentification by applying semiparametric analysis under the small-gap regime. Then, we proposed the Contextual RS-AIPW strategy. With the help of semiparametric analysis and a new large deviation expansion we developed, we showed that the performance of our proposed Contextual RS-AIPW strategy matches the lower bound under a small gap. We also addressed a long-standing open issue in BAI with a fixed budget; even without contextual information, the existence of an asymptotically optimal BAI strategy...
was unclear. Because BAI with a fixed budget and without contextual information is a special case in our setting, we addressed this question. Furthermore, we demonstrated an analytical solution for the target allocation ratio, which has also been unknown for a long time. Thus, our study serves as a breakthrough in the field of BAI with a fixed budget. Our future direction is to develop BAI strategies for various settings, such as linear (Hoffman et al., 2014; Liang et al., 2019; Katz-Samuels et al., 2020), combinatorial (Chen et al., 2014), and policy learning (Kitagawa and Tetenov, 2018; Athey and Wager, 2017; Dongruo Zhou, 2020).

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I Proof of Theorem A.4: Large Deviation Bound for Martingales

A Preliminaries for the Proof

**Definition A.1.** [Uniform integrability, Hamilton (1994), p. 191] Let $W_t \in \mathbb{R}$ be a random variable with a probability measure $P$. A sequence $\{W_t\}$ is said to be uniformly integrable if for every $\epsilon > 0$ there exists a number $c > 0$ such that

$$E_P[|A_t| \cdot I[|A_t| \geq c]] < \epsilon$$

for all $t$.

The following proposition is from Hamilton (1994), Proposition 7.7, p. 191.

**Proposition A.2** (Sufficient conditions for uniform integrability). Let $W_t, Z_t \in \mathbb{R}$ are random variables. Let $P$ be a probability measure of $Z_t$. (a) Suppose there exist $r > 1$ and $M < \infty$ such that $E_P[Z_t^r] < M$ for all $t$. Then $\{A_t\}$ is uniformly integrable. (b) Suppose there exist $r > 1$ and $M < \infty$ such that $E_P[|Z_t|^r] < M$ for all $t$. If $W_t = \sum_{j=-\infty}^{\infty} h_j Z_{t-j}$ with $\sum_{j=-\infty}^{\infty} |h_j| < \infty$, then $\{W_t\}$ is uniformly integrable.

**Proposition A.3** ($L^r$ convergence theorem, p 165, Loeve (1977)). Let $Z_n$ be a random variable with probability measure $P$ and $z$ be a constant. Let $0 < r < \infty$, suppose that $E_P[|Z_n|^r] < \infty$ for all $n$ and that $Z_n \overset{P}{\to} z$ as $n \to \infty$. The following are equivalent:

(i) $Z_n \to z$ in $L^r$ as $n \to \infty$;

(ii) $E_P[|Z_n|^r] \to E_P[|z|^r] < \infty$ as $n \to \infty$;

(iii) $\{\{Z_n|^r, n \geq 1\}$ is uniformly integrable.

Let $W_t$ be a random variable with probability measure $P$. Let $F_n = \{W_1, W_2, \ldots, W_n\}$.

**Proposition A.4** (Strong law of large numbers for martingales, p 35, Hall et al. (1980)). Let $\{S_n = \sum_{i=1}^{n} W_i, F_n, n \geq 1\}$ be a martingale and $\{U_n, n \geq 1\}$ a nondecreasing sequence of positive r.v. such that $U_n$ is $F_{n-1}$-measurable. Then,

$$\lim_{n \to \infty} U_n^{-1} S_n = 0$$

almost surely on the set $\{\lim_{n \to \infty} U_n = \infty, \sum_{i=1}^{\infty} U_i^{-1} E[|W_i|] < \infty\}$.

**Proposition A.5** (Rate of convergence in the CLT, From the proof of Theorem 3.8, p 89, Hall et al. (1980)). Let $\{S_t = \sum_{s=1}^{t} W_s, F_t, t \geq 1\}$ be a martingale with $F_t$ equal to the $\sigma$-field generated by $W_1, \ldots, W_t$. Let

$$V_t^2 = E\left[\sum_{s=1}^{t} E[W_s^2|F_{s-1}] - 1\right] \quad 1 \leq t \leq T.$$

Suppose that for some $\alpha > 0$ and constants $M, C$ and $D$,

$$\max_{s \leq t} E[\exp(|\sqrt{T}W_s|^\alpha)] < M.$$

Then, for $T \geq 2$,

$$\sup_{-\infty < s < x < \infty} \left| P(S_T \leq x) - \Phi(x) \right| \leq AT^{-1/4}(\log T)^{1+1/\alpha} + V_T^2/\left(DT^{-1/2}(\log T)^{2+2/\alpha}\right),$$

where the constant $A$ depends only on $\alpha, M, C,$ and $D.$
B Proof of Semiparametric Lower Bound (Theorem 3.5)

In this section, we provide proof of Theorem 3.5. Our argument is based on a change-of-measure argument, which has been applied to BAI without contextual information (Kaufmann et al., 2016). In this derivation, we relate the likelihood ratio to the lower bound. Inspired by Murphy and van der Vaart (1997), we expand the semiparametric likelihood ratio, where the gap parameter \( \mu_0 - \mu_0^* \) is regarded as a parameter of interest and the other parameters as nuisance parameters. By using a semiparametric efficient score function, we apply a series expansion to the likelihood ratio of the distribution-dependent lower bound around the gap parameter \( \mu_0 - \mu_0^* \) under a bandit model of an alternative hypothesis. Then, when the gap parameter goes to 0, the lower bound is characterized by the variance of the semiparametric influence function. Our proof is also inspired by van der Vaart (1998) and Hahn (1998). Throughout the proof, for simplicity, \( \mathcal{P}^L \) is denoted by \( \mathcal{P} \).

Precisely, our proof follows these steps. First, the goal is to express the lower bound of the probability of misidentification by using the gap parameter. In Proposition B.1 of Appendix B.1, we introduce a bound for some event based on a change-of-measure argument (Kaufmann et al., 2016). We apply this bound to derive lower bounds for the probability of misidentification in the final step of the proof. Next, we consider distributions of observations. Although we defined distributions of the potential random variables \( (Y^1_t, Y^2_t, \ldots, Y^K_t, X_t) \) (full-data bandit models), we can only observe a reward of a chosen treatment arm, \( Y_{t}^{a_t} \), and context, \( X_t \), and cannot observe other rewards \( (Y_{t}^{a})_{a \in [K] \setminus \{a_t\}} \). Therefore, distributions of observations are different from the full-data bandit models. We induce the former from the latter in Appendix B.2 to discuss optimality. With these preparations, in Appendix B.3, we introduce a parameter into the true nonparametric full-data bandit models to differentiate the log-likelihood around the gap parameter; that is, the gap parameter is introduced so that it corresponds to \( \mu_0 - \mu_0^* \). This parameter is a technical device for the proof, and the parametrized models are called parametric submodels, which are subsets of \( \mathcal{P} \). The derivative is then defined with respect to this parameter, and we consider applying the series expansion to the log likelihood. However, the derivative (score function) is not uniquely defined because it includes nuisance parameters other than the parameter of interest. Therefore, to specify a score function with the tightest lower bound, it is necessary to consider information on the distribution of the observations. To perform these operations, we associate the full-data bandit models with the distribution of the observed data in Appendix B.4. Then, in Appendix B.5, we derive the parametric submodel of distributions of observations from the parametric submodels of the full-data bandit models and define a score function for that the parametric submodel of the distribution of observations. For deriving lower bounds, an alternative hypothesis plays an important role, and we define a class of alternative hypotheses (alternative bandit models) in Appendix B.6. By using the alternative bandit models, we derive a lower bound of the probability of misidentification in Appendix B.7 which depends on the log-likelihood and is related to the gap parameter in the following arguments. For the lower bound, using the score function and alternative bandit models in Appendix B.6, we apply the series expansion to the log-likelihood in Appendix B.8 and characterize the bound in Proposition B.1 of Appendix B.1 with the gap parameter. Then, in Appendix B.9, we derive the information bound of the second moment of the score function; then, in Appendix B.10, we specify a score function whose second moment is equal to the information bound in Appendix B.9. Finally, combining them, we derive the lower bound for the probability of misidentification in Appendix B.11.

B.1 Transportation Lemma

Our lower bound derivation is based on change-of-measure arguments, which have been extensively used in the bandit literature (Lai and Robbins, 1985). Kaufmann et al. (2016) derives the following result based on change-of-measure argument, which is the principal tool in our lower bound. Let us define a density of \((Y^1, Y^2, \ldots, Y^K, X)\) under a bandit model \(P \in \mathcal{P}\) as

\[
p_{P}(y^1, y^2, \ldots, y^K, x) = \prod_{a \in [K]} f_{P}^{a}(y^{a}|x) \zeta_{P}(x)
\]

Let \(f_{P}^{a}\) be denoted by \(f_{P}^{a}\).

**Proposition B.1** (Lemma 1 in Kaufmann et al. (2016)). Suppose that Assumption 3.7 holds. Then, for any two bandit model \(P, Q \in \mathcal{P}\) with \(K\) treatment arms such that for all \(a \in [K]\), \(f_{P}^{a}(y^{a}|x) \zeta_{P}(x)\) and \(f_{Q}^{a}(y^{a}|x) \zeta_{Q}(x)\) are mutually absolutely continuous,

\[
\mathbb{E}_{Q} \left[ \sum_{t=1}^{T} \mathbb{I}[A_{t} = a] \log \left( \frac{f_{Q}^{a}(Y_{t}^{a}|X_{t}) \zeta_{Q}(X_{t})}{f_{P}^{a}(Y_{t}^{a}|X_{t}) \zeta_{P}(X_{t})} \right) \right] \geq \sup_{\mathbb{P} \in \mathcal{F}} d(\mathbb{P}_{Q}(\mathcal{E}), \mathbb{P}_{P}(\mathcal{E})).
\]

Recall that \(d(p, q)\) indicates the KL divergence between two Bernoulli distributions with parameters \(p, q \in (0, 1)\).
This “transportation” lemma provides the distribution-dependent characterization of events under a given bandit model \(P\) and corresponding perturbed bandit model \(P'\).

Between the true bandit model \(P_0 \in \mathcal{P}\) and a bandit model \(Q \in \mathcal{P}\), we define the log-likelihood as

\[
L_T = \sum_{t=1}^{T} \sum_{a \in [K]} \mathbb{I}[A_t = a] \log \left( \frac{f_Q^a(Y_t^a | X_t) \zeta_Q(X_t)}{f_{P_0}^a(Y_t^a | X_t) \zeta_{P_0}(X_t)} \right).
\]

For this log-likelihood ratio, from Lemma B.1 between the true model \(P_0\), we have

\[
\mathbb{E}_Q[L_T] \geq \sup_{\mathcal{E} \in \mathcal{F}} d(\mathbb{P}_Q(\mathcal{E}), \mathbb{P}_{P_0}(\mathcal{E})).
\]

We consider an approximation of \(\mathbb{E}_Q[L_T]\) under an appropriate alternative hypothesis \(Q \in \mathcal{P}\) when the gaps between the expected rewards of the best treatment arm and suboptimal treatment arms are small.

### B.2 Observed-Data Bandit Models

Next, we define a semiparametric model for observed data \((Y_t, A_t, X_t)\), as we can only observe the triple \((Y_t, A_t, X_t)\) and cannot observe the full-data \((Y_1^1, Y_2^2, \ldots, Y_T^K, X_t)\).

For each \(x \in \mathcal{X}\), let us define the average allocation ratio under a bandit model \(P \in \mathcal{P}\) and a BAI strategy as

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left[ \mathbb{I}[A_t = a | X_t = x] = \kappa_{T,P}(a|x) \right]
\]

This quantity represents the average sample allocation to each treatment arm \(a\) under a strategy. Then, we first show the following lemma. We show the proof in Appendix C.

**Lemma B.2.** Suppose that Assumption 3.1 holds. For \(P_0, Q, P \in \mathcal{P}\),

\[
\frac{1}{T} \mathbb{E}_P[L_T] = \sum_{a \in [K]} \mathbb{E}_P \left[ \mathbb{E}_P \left[ \log \frac{f_Q^a(Y_t^a | X_t) \zeta_Q(X)}{f_{P_0}^a(Y_t^a | X_t) \zeta_{P_0}(X)} | X_t \right] \kappa_{T,P}(a | X_t) \right].
\]

Based on Lemma B.2 for some \(\kappa \in \mathcal{W}\), we consider the following samples \(\{(Y_t, A_t, X_t)\}_{t=1}^{T}\), instead of \(\{(Y_t, A_t, X_t)\}_{t=1}^{T}\), generated as

\[
\{(Y_t, A_t, X_t)\}_{t=1}^{T} \overset{i.i.d}{\sim} r(y, d, x) = \prod_{a \in [K]} \{f_P^a(y^a | x) \kappa(a | x) \}^{1[|d=a]} \zeta_P(x),
\]

where \(\kappa(a | x) \kappa(a | x)\) corresponds to the conditional expectation of \(\mathbb{I}[A_t = a]\) given \(X_t\). The expectation of \(L_T\) for \(\{(Y_t, A_t, X_t)\}_{t=1}^{T}\) on \(P\) is identical to that for \(\{(Y_t, A_t, X_t)\}_{t=1}^{T}\) from the result of Lemma B.2 when \(\kappa = \kappa_{T,P}\). Therefore, to derive the lower bound for \(\{(Y_t, A_t, X_t)\}_{t=1}^{T}\), we consider that for \(\{(Y_t, A_t, X_t)\}_{t=1}^{T}\). Note that this data generating process is induced by a full-data bandit model \(P \in \mathcal{P}\); therefore, we call it an observed-data bandit model.

Formally, for a bandit model \(P \in \mathcal{P}\) and some \(\kappa \in \mathcal{W}\), by using a density function of \(P\), let \(\overline{\mathcal{R}}_P^\kappa\) be a distribution of an observed-data bandit model \(\{(Y_t, A_t, X_t)\}_{t=1}^{T}\) with the density given as

\[
\overline{\tau}_P^\kappa(y, d, x) = \prod_{a \in [K]} \{f_P^a(y^a | x) \kappa(a | x) \}^{1[|d=a]} \zeta_P(x).
\]

We call it an observed-data distribution. To avoid the complexity of the notation, we will denote \(\{(Y_t, A_t, X_t)\}_{t=1}^{T}\) in the following arguments. Let \(\mathcal{R} = \{\overline{\mathcal{R}}_P : P \in \mathcal{P}\}\) be a set of all observed-data bandit models \(\overline{\mathcal{R}}_P\). For \(P_0 \in \mathcal{P}\), let \(\overline{\mathcal{R}}_{P_0} = \overline{\mathcal{R}}_0\), and \(\tau_{P_0}^\kappa = \tau_0^\kappa\).

### B.3 Parametric Submodels for the Full-Data Bandit Models

The purpose of this section is to introduce parametric submodels for the true full-data bandit model \(P_0 \in \mathcal{P}\), which is indexed by a real-valued parameter and a set of distributions contained in the larger set \(\mathcal{P}\), and define the derivative of the parametric submodels.
In Section B.5, we define parametric submodels for observed-data bandit models under the true full-data bandit model, which is a set of distributions contained in the larger set $\mathcal{R}_0$, by using the parametric submodels for full-data bandit models. These definitions of parametric submodels are preparations for the series expansion of the log-likelihood; that is, we consider approximation of the log-likelihood $L_T = \sum_{t=1}^{T} \sum_{a \in [K]} I[A_t = a] \log \left( \frac{f_{P_0}(Y_t^a|X_t)_{Q_0}(X_t)}{f_{P_0}(Y_t^a|X_t)_{Q}(X_t)} \right)$ using $\mu_0^* - \mu_0^a$, where $Q \in \mathcal{P}$ is an alternative bandit model.

This section consists of the following two parts. In the first part, we define parametric submodels as (4) with condition (5).

Then, in the following part, we confirm the differentiability (12) and define score functions.

**Definition of parametric submodels for the observed-data distribution** First, we define parametric submodels for the true full-data bandit model $P_0$ with the density function $p_{P_0}(y^1, \ldots, y^K, x)$ by introducing a parameter $\varepsilon = (\varepsilon^a)_{a \in [K] \setminus \{a_0^*\}} \in \Theta$ with some compact space $\Theta$. We construct our parametric submodels so that the parameter can be interpreted as the gap parameter of a parametric submodel. For $P \in \mathcal{P}$, we define a set of parametric submodels $\{P_\varepsilon : \varepsilon \in \Theta^{K-1}\} \subset \mathcal{P}$ as follows: for a set of some functions $(g^a)_{a \in [K] \setminus \{a_0^*\}}$ such that $g^a : \mathbb{R} \times \mathcal{X} \to \mathbb{R}$, a parametric submodel $P_\varepsilon$ has a density such that for each $a \in [K] \setminus \{a_0^*\}$,

$$p_\varepsilon(y^a, x) = (1 + \varepsilon^a g^a(\phi_\varepsilon^a(y, x), \phi_\varepsilon^a(y, x), x)) p_{P_0}(y^a, x),$$

(4)

where for a constant $\tau > 0$ and each $d \in [K]$, $\phi_\varepsilon^d : \mathbb{R} \times \mathcal{X} \to (-\tau, \tau)$ is a truncation function such that for $\varepsilon^a < c(\tau)$,

$$\phi_\varepsilon^d(y, x) = y \mathbb{I}[|y| < \tau] - \mathbb{E}_{P_0}[Y^d \mathbb{I}[|Y^d| < \tau]|X_t = x] + \mu_0^d(x),$$

$$|\varepsilon^a g^a(\phi_\varepsilon^a(y), \phi_\varepsilon^a(y), x)| < 1,$

and $c(\tau)$ is some decreasing scalar function with regard to $\tau$ such that for the inverse $c^{-1}(e) = \tau$, $\tau \to \infty$ as $e \to 0$. Let $\phi_\varepsilon^a$ be denoted by $\phi_\varepsilon^a$. This is a standard construction of parametric submodels with unbounded random variables (Hansen [2022]). For $a \in [K] \setminus \{a_0^*\}$, this parametric submodel must satisfy $\mathbb{E}_{P_0}[g^a(\phi_\varepsilon^*(Y_t, X_t), \phi_\varepsilon^*(Y_t, X_t), X_t)]=0$, $\mathbb{E}_{P_0}[|g^a(\phi_\varepsilon^*(Y_t, X_t), \phi_\varepsilon^*(Y_t, X_t), X_t)|^2] < \infty$, and

$$\int (y^* - y^a)^2 p_\varepsilon(y^a, x) dy^a dy^a dx = \mu_0^a + e^a.$$ 

(5)

In Section B.9, we specify functions $(g^a)_{a \in [K] \setminus \{a_0^*\}}$ and confirm that the specified $g^a$ satisfies (5). Note that the parametric submodels are usually not unique. For each $a \in [K] \setminus \{a_0^*\}$, the parametric submodel $p_\varepsilon(y^a, y^g, x)$ is equivalent to $p_{P_0}(y^a, y^g, x)$ when $\varepsilon^a = 0$ for any $(\varepsilon^a)_{a \in [K] \setminus \{a_0^*\}}$.

For each $a \in [K] \setminus \{a_0^*\}$ and a parametric submodel $P_\varepsilon$, let $f^a_\varepsilon(y|x), f^a_0(y|x) = f^a_0(y|x)$ and $\zeta_\varepsilon(x)$ be the conditional densities of $Y_t^a$ and $Y_0^a$ given $X_t = x$ and the density of $Y_t$, which satisfies (4) and (5) as

$$p_\varepsilon(y^a, y^g, x) = f^a_\varepsilon(y|x) f^a_0(y|x) \zeta_\varepsilon(x),$$

$$\int (y^* - y^a)^2 f^a_\varepsilon(y|x) f^a_0(y|x) \zeta_\varepsilon(x) dy^a dy^a dx = \mu_0^a + \mu_0^a + e^a.$$ 

According to the definition of the parametric submodels, $f^a_0(y|x) = f^a_{P_0}(y|x)$, $f^a_0(y|x) = f^a_0(y|x) = f^a_{P_0}(y|x)$ and $\zeta_\varepsilon(x) = \zeta_{P_0}(x)$.

**Differentiability and score functions of the parametric submodels for the observed-data distribution.** Next, we confirm the differentiability of $p_\varepsilon(y^a, y^g, x)$.

Because $\sqrt{p_\varepsilon(y^a, y^g, x)}$ is continuously differentiable for every $(y^a, y^g, x)$, and $\int \left( \frac{p_\varepsilon(y^a, y^g, x)}{p_\varepsilon(y^a, y^g, x)} \right)^2 p_\varepsilon(y^a, y^g, x) dm$ are all defined and continuous in $\varepsilon$, where $m$ is some reference measure on $(y^a, y^g, x)$, from Lemma 7.6 of [van der Vaart 1998], we see that the parametric submodel has the score function $g^a$ in the $L_2$ sense; that is, the density $p_\varepsilon(y^a, y^g, x)$ is differentiable in quadratic mean (DQM): for $a \in [K] \setminus \{a_0^*\}$, and any $(e^b)_{b \in [K] \setminus \{a_0^*\}}$,

$$\int \left[ p^b_\varepsilon(y^a, y^g, x) - p^b_{P_0}(y^a, y^g, x) - \frac{1}{2} \varepsilon^a g^a(\phi_\varepsilon^a(y, x), \phi_\varepsilon^a(y, x), x) p_{P_0}^{1/2}(y^a, y^g, x) \right]^2 dm = o(\varepsilon^a).$$

(6)

This relationship is derived from

$$\frac{\partial}{\partial \varepsilon^a} \Bigg|_{\varepsilon^a=0} \log p_\varepsilon(y^a, y^g, x) = \frac{g^a(\phi_\varepsilon^a(y, x), \phi_\varepsilon^a(y, x), x)}{1 + \varepsilon^a g^a(\phi_\varepsilon^a(y, x), \phi_\varepsilon^a(y, x), x)} \Bigg|_{\varepsilon^a=0} = g^a(\phi_\varepsilon^a(y, x), \phi_\varepsilon^a(y, x), x),$$

for any $(e^b)_{b \in [K] \setminus \{a_0^*\}}$. 

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To clarify the relationship between \( g^a \) and a score function, for each \( a \in [K] \setminus \{a^*_0\} \), and any \( (e^b)_{b \in [K] \setminus \{a^*_0\}} \), we express the score function as

\[
g^a(\phi^*_a(y, x), \phi^a(y, x), x) = \left. \frac{\partial}{\partial e^a} \log p_e(y^*, y^a, x) \right|_{e^a = 0} = S^a_0(y|x) + S^a_0(y|x) + S^a_0(x),
\]

where

\[
S^a_0(y|x) = \left. \frac{\partial}{\partial e^a} \log f^a_e(y|x) \right|_{e^a = 0}, \quad S^a_0(y|x) = \left. \frac{\partial}{\partial e^a} \log f^a_e(y|x) \right|_{e^a = 0}, \quad S^a_0(x) = \left. \frac{\partial}{\partial e^a} \log \zeta_e(x) \right|_{e^a = 0}.
\]

**B.4 Mapping from Observed-Data to Full-Data Bandit Models**

According to Section 7.2 of Tsiatis (2007), we define a mapping from full-data to observed-data as \((y, x) = \mathcal{T}^d(y^*, y^a, x)\), where \( \mathcal{T}^d: \mathbb{R}^2 \times \mathcal{X} \to \mathbb{R} \times \mathcal{X} \) is a known many-to-one function, which maps the full-data \((y^*, y^a, x)\) to observed-data bandit models \((y^d, x)\). We only consider a case where \((Y^*_t, Y^a_t, X_t)\) is continuous and define a function \(V^d: \mathbb{R} \to \mathbb{R}\) as a counterpart value of the observation; that is, \(V^d(Y^*_t, Y^a_t) = ((Y^*_t)^{b \in \{a^*_0, a\} \setminus \{d\}})\). Then, the mapping

\[
(Y^*_t, Y^a_t, X_t) \mapsto \{(\mathcal{T}^d(Y^*_t, Y^a_t, X_t), V^d(Y^*_t, Y^a_t))\}
\]
is one-to-one for all \( a \in [K] \setminus \{a^*_0\} \) and \( d \in \{a^*_0, a\} \). For \( a \in [K] \setminus \{a^*_0\} \), \( d \in \{a^*_0, a\} \), \( \tau^d = (y^d, x) \), and \( \nu^d = ((y^b)_{b \in \{a^*_0, a\} \setminus \{d\}}) \), which correspond to \( \mathcal{T}^d \) and \( V^d \) respectively, we define the inverse transformation as

\[
(y^*, y^a, x) = H^d(\tau^d, \nu^d), \tag{7}
\]

Then, by the standard formula for change of variables, let us define the density of \((\tau^d, \nu^d)\) under \( \mathcal{T}^d \) and \( V^d \) as

\[
p_{\mathcal{T}^d, V^d}(\tau^d, \nu^d) = p_P(H^d(\tau^d, \nu^d))J(\tau^d, \nu^d), \tag{8}
\]

where \( J \) is the Jacobian of \( H^d \) with respect to \((\tau^d, \nu^d)\). To find the density of the observed data \( \tau_P^d(y, d, x) \), we can use

\[
\tau_P^d(y, d, x) = \int \tau_P^{d, V^d}(\tau^d, d, \nu^d) d\nu^d, \tag{9}
\]

where

\[
\tau_P^{d, V^d}(\tau^d, d, \nu^d) = \kappa(d|x)p_{\mathcal{T}^d, V^d}(\tau^d, \nu^d). \tag{10}
\]

Consequently, using (8) and (10), we can rewrite (9) as

\[
\tau_P^d(y, d, x) = \int \kappa(d|x)p_P(H^d(\tau^d, \nu^d))J(\tau^d, \nu^d)d\nu^d. \tag{11}
\]

**B.5 Parametric Submodels for the Observed-Data Bandit Models and Tangent Space**

This section consists of the following three parts. In the first part, we define parametric submodels as \( \mathcal{P} \) with condition \( \mathcal{P} \). Then, in the following part, we confirm the differentiability (12) and define score functions. Finally, we define a set of score functions, called a tangent set in the final paragraph.

By using the parametric submodels and tangent set, in Section B.8 we demonstrate the series expansion of the log-likelihood (Lemma B.6). In this section and Section B.8 we abstractly provide definitions and conditions for the parametric submodels and do not specify them. However, in Sections B.9 and B.10 we show a concrete form of the parametric submodel by finding score functions satisfying the conditions imposed in this section.

By using the parametric submodels for the true full-data bandit model \( P_0 \in \mathcal{P} \) in Section B.3 we define parametric submodels for observed-data bandit models under the true full-data bandit model \( P_0 \in \mathcal{P} \). Because we define the density functions of the parametric submodel of the true full-data bandit model, the parametric submodels for the observed-data bandit models are given as follows:

\[
r^a_e(y, a, x) = f^a_e(y|x)\kappa(a|x)\zeta_e(x) \quad \forall a \in [K] \setminus \{a^*_0\},
\]

\[
r^e_0(y, a_0, x) = f^a_e(y|x)\kappa(a^*_0|x)\zeta_e(x).
\]
Appendix D.

This also implies that for all \( y, d, \epsilon \) given \( d \in [K] \), and \( \int \left( \frac{p_\epsilon^{\ast}(y,d,d)}{p_\epsilon^{\ast}(y,d,d)} \right)^2 \tau_\epsilon^\ast(y,d,d) \ dm \) are well defined and continuous in \( \epsilon \), where \( m \) is some reference measure on \( (y,d,d) \), from Lemma 7.6 of van der Vaart (1998), we see that the parametric submodel has the score function \( g^\ast \) in the L2 sense; that is, the density \( \tau_\epsilon^\ast(y,d,d) \) is differentiable in quadratic mean (DQM): for \( a \in [K] \backslash \{ a_0^\ast \}, d \in \{ a_0^\ast, a \}, \) and any \( (\epsilon^b)_{b \in [K] \backslash \{ a_0^\ast, a \}} \).

Then we show the differentiability in quadratic mean at \( \epsilon^a \) = 0 of \( \tau_\epsilon^{1/2} \) in the following lemma. We show the proof in Appendix D.

**Lemma B.3.** Under Assumption B.1 for \( a \in [K] \{ a_0^\ast \} \) and \( d \in \{ a_0^\ast, a \} \),

\[
\int \left[ \tau_\epsilon^{1/2}(y,d,d) - \frac{1}{2} \epsilon^a S^\ast(y,d,d) \tau_\epsilon^{1/2}(y,d,d) \right]^2 \ dm = o(\epsilon^a).
\]

where

\[
S^\ast(y,d,d) = \mathbb{E}_{P_0} \left[ g^\ast(\phi^\ast_0(Y_t^\ast, X_t), \phi^\ast(Y_t^\ast, X_t), X_t) | T^d(Y_t^\ast, Y_t^\ast, X_t) = (y,d) \right].
\]

In the following section, we specify a measurable function \( S^\ast \) with \( g^\ast \), satisfying the conditions (4) and (5), which corresponds to a score function of \( \tau_\epsilon^{1/2} \) on \( (a_0^\ast, a) \) and \( \tau_\epsilon^{1/2} \) on \( (a_0^\ast, a) \). To clarify the relationship between \( g^\ast \) and a score function, for each \( a \in [K] \{ a_0^\ast \}, \) and any \( (\epsilon^b)_{b \in [K] \{ a_0^\ast, a \}} \), we denote the score function as

\[
S^\ast(y,d,d) = \frac{\partial}{\partial \epsilon^a} \log \tau_\epsilon^{1/2}(y,d,d) = 1 [d = a_0^\ast] S_f^{a_0^\ast, a_0^\ast}(y|x) + 1 [d = a] S_f^{a, a}(y|x) + S_\epsilon^\ast(x) \quad \forall d \in \{ a_0^\ast, a \}.
\]

Note that \( \frac{\partial}{\partial \epsilon^a} \log \kappa(a|x) = 0 \).

**Definition of the tangent set.** Recall that parametric submodels and corresponding score functions are not unique. Here, we consider a set of score functions. For a set of the parametric submodels \( \{ \mathcal{R}_x : \epsilon \in \Theta^K \} \), we obtain a corresponding set of score functions \( g^\ast \) in the Hilbert space \( L_2(\mathcal{R}_Q) \), which we call a tangent set of \( \mathcal{R} \) at \( \mathcal{R}_Q \) and denote it by \( \mathcal{R}^\ast \). Because \( \mathbb{E}_{P_0} \left[ (g^\ast(\phi^\ast_0(Y_t, X_t), A_t, X_t))^2 \right] \) is automatically finite, the tangent set can be identified with a subset of the Hilbert space \( L_2(\mathcal{R}_Q^\ast) \), up to equivalence classes. For our parametric submodels, the tangent set at \( \mathcal{R}_Q^\ast \) in \( L_2(\mathcal{R}_Q^\ast) \) is given as

\[
\mathcal{R}^\ast = \left\{ 1[d = a_0^\ast] S_f^{a_0^\ast, a_0^\ast}(y|x) + 1[d = a] S_f^{a, a}(y|x) + S_\epsilon^\ast(x) \right\}.
\]

**B.6 Alternative Bandit Model**

Then, we define a class of alternative hypotheses. To derive a tight lower bound by applying the change-of-measure arguments, we use an appropriately defined alternative hypothesis. Our alternative hypothesis is defined using the parametric submodel of \( P_0 \) as follows:

**Definition B.4.** Let \( \text{Alt}(P_0) \subset P \) be alternative bandit models such that for all \( Q \in \text{Alt}(P_0), a^\ast(Q) \neq a_0^\ast \), and \( \mathcal{R}_x = \mathcal{R}_x^{Q_0^\ast, Q} \), where \( \epsilon = (a^\ast)_{a \in [K] \backslash \{ a_0^\ast \}}, \epsilon^a = \left( \mu^a_0(Q) - Q^\ast(Q) \right) = (\mu^a_0 - \mu^a_0) \).

This also implies that for all \( Q \in \text{Alt}(P_0) \), for all \( a \in [K] \backslash \{ a_0^\ast \}, \mu_0^a - \mu_0^a > 0 \) and there exists \( a \in [K] \backslash \{ a_0^\ast \} \) such that \( \mu_0^a(Q) - Q^\ast(Q) < 0 \). Let \( \mu_0^a(Q) \) be denoted by \( Q^\ast(Q) \).

**B.7 Derivation of a Lower Bound of the Probability of Misidentification**

Here, we derive a lower bound for the probability of misidentification as follows, which is refined later:

**Lemma B.5.** Under Assumption B.1 for any \( P_0 \in P \) and \( Q \in \text{Alt}(P_0) \), any consistent strategy satisfies

\[
\lim_{T \to \infty} \sup_{\epsilon} \sup_{w \in \Theta} \min_{a_0^\ast, a} \inf_{\epsilon \leq \epsilon^a < (\mu_0^a - \mu_0^a)} \sum \mathbb{E}_{\mathcal{R}_x} \left[ \mathbb{E}_{\mathcal{R}_x} \left[ \log \frac{f_{P_0}^{\ast}(y,t) | X_t) \zeta_{\epsilon}(X)}{f_{P_0}^{\ast}(y,t) | X_t) \zeta_{\epsilon}(X)} | X_t \right] w(a | X_t) \right].
\]
We consider series expansion of the log-likelihood $L_T$. To prove this lemma, for $\text{Lemma B.6.}$ Suppose that Assumption 3.1 holds. For $\text{Proof of Lemma B.5.}$ By using $\epsilon$, $\text{Proof of Lemma B.6.}$

under a small-gap regime (small $\epsilon \in [0, 1)$, there exists $t_0(\epsilon)$ such that for all $T \geq t_0(\epsilon)$, $P_0(\epsilon) \leq \epsilon \leq P_0(\epsilon)$. Then, for all $T \geq t_0(\epsilon)$, $E_0[L_T] \geq d(\epsilon, 1 - P_0(\epsilon)) = \epsilon \log \frac{1}{1 - P_0(\epsilon)} + (1 - \epsilon) \log \frac{1}{P_0(\epsilon)}$. Then, taking the limsup and letting $\epsilon \to 0$,

$$\limsup_{T \to \infty} - \frac{1}{T} \log P_0(\epsilon) \neq a_0^* \leq \inf_{Q \in \text{Alt}(P_0)} \limsup_{T \to \infty} \frac{1}{T} E_0[L_T] \leq \inf_{Q \in \text{Alt}(P_0)} \limsup_{T \to \infty} \sum_{a \in [K]} E_0 \left[ E_0 \left[ \log \frac{f_Q^0(Y_t^a \mid X_t) \zeta_Q(X_t)}{f_Q^0(Y_t^a \mid X_t) \zeta_P(X_t)} | X_t \right] \right] \Leq \sup_{w \in W} \inf_{Q \in \text{Alt}(P_0)} \sum_{a \in [K]} E_0 \left[ E_0 \left[ \log \frac{f_Q^0(Y_t^a \mid X_t) \zeta_Q(X_t)}{f_Q^0(Y_t^a \mid X_t) \zeta_P(X_t)} | X_t \right] w(a | X_t) \right] = \sup_{w \in W} \min_{\mu^* \in [K] \setminus \{a_0^*\}} \inf_{\mu^* \in [K] \setminus \{a_0^*\}} \sum_{a \in [K]} E_0 \left[ E_0 \left[ \log \frac{f_Q^0(Y_t^a \mid X_t) \zeta_Q(X_t)}{f_Q^0(Y_t^a \mid X_t) \zeta_P(X_t)} | X_t \right] w(a | X_t) \right].$$

By using $\epsilon^a = (\mu^* - \mu^a(Q)) - (\mu^0 - \mu^0) < (\mu^0 - \mu^0) \in W$ for the parametric submodel,

$$\sup_{w \in W} \min_{\mu^* \in [K] \setminus \{a_0^*\}} \inf_{\mu^* \in [K] \setminus \{a_0^*\}} \sum_{a \in [K]} E_0 \left[ E_0 \left[ \log \frac{f_Q^0(Y_t^a \mid X_t) \zeta_Q(X_t)}{f_Q^0(Y_t^a \mid X_t) \zeta_P(X_t)} | X_t \right] w(a | X_t) \right] = \sup_{w \in W} \min_{\mu^* \in [K] \setminus \{a_0^*\}} \inf_{\mu^* \in [K] \setminus \{a_0^*\}} \sum_{a \in [K]} E_{\pi_w} \left[ E_{\pi_w} \left[ \log \frac{f_{\pi_w}^0(Y_t^a \mid X_t) \zeta_{\pi_w}(X_t)}{f_{\pi_w}^0(Y_t^a \mid X_t) \zeta_{\pi_w}(X_t)} | X_t \right] w(a | X_t) \right].$$

The proof is complete.

**B.8 Semiparametric Likelihood Ratio**

For $a \in [K] \setminus \{a_0^*\}$, let $\epsilon \in (0, \ldots, 0, \epsilon^a, 0, \ldots, 0)$. Let us also define

$$L_{a}^0 = \sum_{t=1}^{T} \left\{ \mathbb{I}[A_t = a_0^* \log \left( \frac{f_{\sigma}^0(Y_t^a \mid X_t)}{f_{\sigma}^0(Y_t^a \mid X_t)} \right) + \mathbb{I}[A_t = a \log \left( \frac{f_{\sigma}^0(Y_t^a \mid X_t)}{f_{\sigma}^0(Y_t^a \mid X_t)} \right) + \log \left( \frac{\zeta_{\sigma}(X_t)}{\zeta_{\sigma}(X_t)} \right) \right\}.$$ 

We consider series expansion of the log-likelihood $L_{a}^0$ defined between $P_0 \in \mathcal{P}$ and $Q \in \text{Alt}(P_0)$, where $E_0[L_T]$ works as a lower bound for the probability of misidentification as shown in Section B.7. We consider an approximation of $L_{a}^0$ under a small-gap regime (small $\mu^0 - \mu^0$), which is upper-bounded by the variance of the score function. Our argument is inspired by that in [Murphy and van der Vaart (1997)].

Then, we prove the following lemma:

**Lemma B.6.** Suppose that Assumption 3.1 holds. For $P_0 \in \mathcal{P}$, $Q \in \text{Alt}(P_0)$, and each $a \in [K] \setminus \{a_0^*\}$,

$$\frac{1}{T} E_0[L_{a}^0] = \frac{(\epsilon^a)^2}{2} E_{P_0} \left[ (S^a(Y_t, A_t, X_t))^2 \right] + o \left( (\epsilon^a)^2 \right).$$

To prove this lemma, for $a \in [K] \setminus \{a_0^*\}$ and $d \in [K]$, we define

$$\ell_{\epsilon^a}^0(y, d, x) = \mathbb{I}[d = a_0^* \log f_{\epsilon^a}^0(y \mid x) + 1[d = a \log f_{\epsilon^a}^0(y \mid x) + \log \zeta_{\epsilon^a}(x).$$

Note that if $\epsilon^a = 0$, then

$$\ell_{\epsilon^a}^0(y, d, x) = \mathbb{I}[d = a_0^* \log f_{0}^0(y \mid x) + 1[d = a \log f_{a}^0(y \mid x) + \log \zeta_{\epsilon^a}(x).$$

**Proof of Lemma B.6** By using the parametric submodel defined in the previous section, from the series expansion,

$$L_{a}^0 = \sum_{t=1}^{T} \left\{ \mathbb{I}[A_t = a_0^* \log \left( \frac{f_{\sigma}^0(Y_t^a \mid X_t)}{f_{\sigma}^0(Y_t^a \mid X_t)} \right) + \mathbb{I}[A_t = a \log \left( \frac{f_{\sigma}^0(Y_t^a \mid X_t)}{f_{\sigma}^0(Y_t^a \mid X_t)} \right) + \log \left( \frac{\zeta_{\sigma}(X_t)}{\zeta_{\sigma}(X_t)} \right) \right\}.$$
\[ = \sum_{t=1}^{T} \left\{ \frac{\partial}{\partial a} \epsilon_a^o(Y_t, A_t, X_t) \epsilon_a + \frac{\partial^2}{\partial (\epsilon_a)^2} \epsilon_a^o(Y_t, A_t, X_t) \frac{(\epsilon_a)^2}{2} + o\left((\epsilon_a)^2\right) \right\}. \]

Here, we fix \((\epsilon^b)_{b \in [K] \setminus \{a^*_\}}\), where \(\epsilon^b = 0\). Note that
\[
\frac{\partial}{\partial a} \epsilon_a^o(y, d, x) = S^a(y, d, x)
\]
\[
\frac{\partial}{\partial (\epsilon_a)^2} \epsilon_a^o(y, d, x) = -\left(S^a(y, d, x)\right)^2.
\]

Let \(\bar{R}_{Q}^{T,Q} = R_{Q},\), \(r_{Q}^{T,Q}(y, d, x) = \tau_{Q}(y, d, x)\), and \(\tau_{Q}^{o,T}(y, d, x) = \tau_0(y, d, x)\). Because the density \(\tau_{Q}(y, d, x)\) is DQM \([12]\).

\[
\mathbb{E}_Q[S^a(Y_t, A_t, X_t)] = \mathbb{E}_{\tau_{Q}}[S^a(Y_t, A_t, X_t)]
\]
\[
= \mathbb{E}_{\tau_{Q}}[S^a(Y_t, A_t, X_t)] - \sum_{d \in [K]} \int S^a(y, d, x) \left(1 + \frac{1}{2} \epsilon_a S^a(y, d, x)\right)^2 \tau_0(y, d, x) \mathrm{d}y \mathrm{d}x
\]
\[
+ \sum_{d \in [K]} \int S^a(y, d, x) \left(1 + \frac{1}{2} \epsilon_a S^a(y, d, x)\right)^2 \tau_0(y, d, x) \mathrm{d}y \mathrm{d}x
\]
\[
= \sum_{d \in [K]} \int S^a(y, d, x) \left\{ \tau_{Q}(y, d, x) - \left(1 + \frac{1}{2} \epsilon_a S^a(y, d, x)\right)^2 \tau_0(y, d, x) \right\} \mathrm{d}y \mathrm{d}x
\]
\[
+ \sum_{d \in [K]} \int S^a(y, d, x) \left(1 + \frac{1}{2} \epsilon_a S^a(y, d, x)\right)^2 \tau_0(y, d, x) \mathrm{d}y \mathrm{d}x
\]
\[
= o(\epsilon_a) + \mathbb{E}_{P_0} [S^a(Y_t, A_t, X_t)] + \epsilon_a \mathbb{E}_{P_0} [\left(S^a(Y_t, A_t, X_t)\right)^2],
\]
where we used
\[
\sum_{d \in [K]} \int S^a(y, d, x) \tau_0(y, d, x) \mathrm{d}y \mathrm{d}x
\]
\[
= \sum_{d \in [K]} \int \left\{ \mathbb{I}[d = a_0^*] \phi_f^{a_0^*, T}(y|x) + \mathbb{I}[d = a] \phi_f^{a_0}(y|x) + S^\alpha(x) \right\} \tau_0(y, d, x) \mathrm{d}y \mathrm{d}x.
\]

Similarly,
\[
-\mathbb{E}_Q \left[(S^a(Y_t, A_t, X_t))^2\right] = o(\epsilon_a) - \mathbb{E}_{P_0} \left[(S^a(Y_t, A_t, X_t))^2\right] + \epsilon_a \mathbb{E}_{P_0} \left[(S^a(Y_t, A_t, X_t))^3\right].
\]

By using these expansions, we approximate \(\mathbb{E}_Q[L_T]\). Here, by definition, \(\mathbb{E}_{P_0} [S^a(Y_t, A_t, X_t)] = 0\). Then, we approximate the likelihood ratio as follows:
\[
\mathbb{E}_Q[L_T^2] = \frac{(\epsilon_a)^2}{2} T \mathbb{E}_{P_0} \left[(S^a(Y_t, A_t, X_t))^2\right] + o\left(T(\epsilon_a)^2\right).
\]

\[\square\]

**B.9 Observed-Data Semiparametric Efficient Influence Function**

Our remaining task is to specify the score function \(S^a\). Because there can be several score functions for our parametric submodel due to directions of the derivative, we find a parametric submodel that has a score function with the largest variance, called a least-favorable parametric submodel \([\text{van der Vaart} 1998]\).

In this section, instead of the original observed-data bandit model \(\bar{R}_{Q}^{T,Q}\), we consider an alternative observed-data bandit model \(\bar{R}_{Q}^{T,Q†}\), which is a distribution of \(\{(\phi_f^a(Y_t, X_t), A_t, X_t)\}_{t=1}^{T}\). Let \(\bar{R}_{Q}^{T,Q†}\) be parametric submodel defined as well as Section B.5 \(\bar{R}_{Q}^{T,Q}\), \(\bar{R}_{Q}^{T,Q†}\) be a set of all \(\bar{R}_{Q}^{T,Q}\), and \(r_{Q}^{T,Q†}(y, d, x) = f_{Q}^{a,T}(y|x)\nu_{T,Q}(d|x)\zeta_{Q}(x)\). For each \(a \in [K]\setminus\{a^*_\}\), let \(S^a(y, d, x)\) and \(\bar{R}_{a}^{T,Q}\) be a corresponding score function and tangent space, respectively.
As a preparation, we define a parameter \( \mu^*(Q) - \mu^*(Q) \) as a function \( \psi^a : \mathcal{R}^\kappa_{\mathbf{y}, \mathbf{z}} \rightarrow \mathbb{R} \) such that \( \psi^a(\mathcal{R}^\kappa_{\mathbf{y}, \mathbf{z}}) = \mu^a - \mu^a_0 + \varepsilon^a \). The information bound for \( \psi^a(\mathcal{R}^\kappa_{\mathbf{y}, \mathbf{z}}) \) of interest is called semiparametric efficiency bound. Let \( \bar{\mathcal{R}}_{\mathbf{y}, \mathbf{z}} \) be the closure of the tangent space. Then, \( \psi^a(\mathcal{R}^\kappa_{\mathbf{y}, \mathbf{z}}) = \mu^a - \mu^a_0 + \varepsilon^a \) is pathwise differentiable relative to the tangent space \( \bar{\mathcal{R}}_{\mathbf{y}, \mathbf{z}} \) if and only if there exists a function \( \tilde{\psi}^a \in \bar{\mathcal{R}}_{\mathbf{y}, \mathbf{z}} \) such that

\[
\frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon = 0} \psi^a(\mathcal{R}^\kappa_{\mathbf{y}, \mathbf{z}}) \left( \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon = 0} \mu^a_0 - \mu^a_0 + \varepsilon^a = 1 \right) = \mathbb{E}_{\mathcal{R}^\kappa_{\mathbf{y}, \mathbf{z}}} \left[ \tilde{\psi}^a(Y_t, A_t, X_t)S^a(\mathbf{y}, \mathbf{A}) \right].
\]

This function \( \tilde{\psi}^a \) is called the semiparametric influence function.

Then, we prove the following lemma on the lower bound for \( \mathbb{E}_{\mathbf{P}_0} \left[ (S^a(\mathbf{y}, \mathbf{A})^2 \right] \), which is called the semiparametric efficiency bound:

**Lemma B.7.** Any score function \( S^a \in \bar{\mathcal{R}}_{\mathbf{y}, \mathbf{z}} \) satisfies

\[
\mathbb{E}_{\mathbf{P}_0} \left[ (S^a(\mathbf{y}, \mathbf{A})^2 \right] \geq \frac{1}{\mathbb{E}_{\mathbf{P}_0} \left[ \tilde{\psi}^a(\mathbf{y}, \mathbf{A})^2 \right]}.\]

**Proof.** From the Cauchy-Schwarz inequality, we have

\[
1 = \mathbb{E}_{\mathbf{P}_0} \left[ \tilde{\psi}^a(\mathbf{y}, \mathbf{A})S^a(\mathbf{y}, \mathbf{A}) \right] \leq \sqrt{\mathbb{E}_{\mathbf{P}_0} \left[ \tilde{\psi}^a(\mathbf{y}, \mathbf{A})^2 \right]} \sqrt{\mathbb{E}_{\mathbf{P}_0} \left[ S^a(\mathbf{y}, \mathbf{A})^2 \right]}.\]

Therefore,

\[
\sup_{S^a \in \bar{\mathcal{R}}_{\mathbf{y}, \mathbf{z}}} \frac{1}{\mathbb{E}_{\mathbf{P}_0} \left[ (S^a(\mathbf{y}, \mathbf{A})^2 \right] \leq \mathbb{E}_{\mathbf{P}_0} \left[ \tilde{\psi}^a(\mathbf{y}, \mathbf{A})^2 \right].\]

For \( a \in [K] \setminus \{a_0\} \) and \( d \in [K] \setminus \{a_0, a\} \), let us define a semiparametric efficient score function \( S^a_{\text{eff}}(y, d, x) \in \bar{\mathcal{R}}_{\mathbf{y}, \mathbf{z}} \) as

\[
S^a_{\text{eff}}(y, d, x) = \frac{\tilde{\psi}^a(y, d, x)}{\mathbb{E}_{\mathbf{P}_0} \left[ \tilde{\psi}^a(\mathbf{y}, \mathbf{A})^2 \right]}.\]

Next, we consider finding \( \tilde{\psi}^a \in \bar{\mathcal{R}}_{\mathbf{y}, \mathbf{z}} \). We can use the result of [Hahn, 1998]. Let us guess that for each \( a \in [K] \setminus \{a_0\} \) and \( d \in \{a_0, a\} \), \( \tilde{\psi}^a(y, d, x) \) is given as follows:

\[
\tilde{\psi}^a(y, d, x) = \begin{cases} 1[d = a] \phi^a(y, x) - \mu^a_0(x), & a \in [K] \setminus \{a_0\} \setminus \{a_0, a\} \\ 1[d = a] \phi^a(y, x) - \mu^a_0(x) - \mu^a_0(x), & \{a_0, a\}. \end{cases}
\]

Then, as shown by [Hahn, 1998], the condition \( 1 = \mathbb{E}_{\mathcal{R}^\kappa_{\mathbf{y}, \mathbf{z}}} \left[ \tilde{\psi}^a(Y_t, A_t, X_t)S^a(\mathbf{y}, \mathbf{A}) \right] \) holds under (14) when for each \( a \in [K] \setminus \{a_0\} \) and \( d \in \{a_0, a\} \), the semiparametric efficient score functions are given as

\[
S^a_{\text{eff}}(y, d, x) = \begin{cases} (\phi^a_0(y, x) - \mu^a_0(x)) \right/ \psi_0^a(\kappa, \mathbf{y}, \mathbf{z} \tau), & a \in [K] \setminus \{a_0\} \setminus \{a_0, a\} \\ (\phi^a_0(y, x) - \mu^a_0(x)) \right/ \psi_0^a(\kappa, \mathbf{y}, \mathbf{z} \tau) + (\mu^a_0(x) - \mu^a_0(x) - \mu^a_0(x)). \end{cases}
\]
We also note that we specify our score function
We also define the limit of the semiparametric efficient influence function when
According to Lemma B.7, we can conjecture that if we use the semiparametric efficient score function for our score
in Lemma B.7. However, we cannot use the semiparametric efficient score function because it is derived for
misidentification. Note that the variance of the semiparametric efficient score function is equivalent to the lower bound
for
For
Here, note that for each $d \in [K]$, 
$$
E_{P_0} \left[ \left( \phi_d^\ast(Y_t, X_t) - \mu_d^0(X_t) \right)^2 \right] = E_{P_0} \left[ \left( Y_t^d 1[|Y_t^d| < \tau] - E_{P_0}[Y_t^d 1[|Y_t^d| < \tau]|X_t]|X_t) \right)^2 \right] = E_{P_0} \left[ \left( Y_t^d 1[|Y_t^d| < \tau] - (E_{P_0}[Y_t^d 1[|Y_t^d| < \tau]|X_t]|X_t) \right)^2 \right].
$$
We also note that
$$
E_{R_{\tau}^{\ast}} \left[ \left( S_0^\ast(Y_t, A_t, X_t) \right)^2 \right] = \bar{V}_0^\ast(\kappa_T, \tau) = \left( E_{R_{\tau}^{\ast}} \left[ \left( \bar{\psi}^\ast(Y_t, A_t, X_t) \right)^2 \right] \right)^{-1}.
$$
Summarizing the above arguments, we obtain the following lemma.

**Lemma B.8.** For $a \in [K] \setminus \{a_0^\ast\}$ and $d \in [K] \setminus \{a_0^\ast, a\}$, the semiparametric efficient influence function is
$$
\bar{\psi}^\ast(y, d, x) = \frac{\mathbb{I}[d = a_0^\ast](\phi_d^\ast(y, x) - \mu_d^0(x)) - \mathbb{I}[d = a](\phi_d^\ast(y, x) - \mu_d^0(x))}{\kappa_T, Q(a_0^\ast|X)} + \mu_d^0(x) - \mu_d^0(x) - (\mu_a^0 - \mu_0^0).
$$
We also define the limit of the semiparametric efficient influence function when $\tau \to \infty$ and the variance as
$$
\bar{\psi}_0^\ast(y, d, x) = \frac{\mathbb{I}[d = a_0^\ast](Y_t^d - \mu_d^0(x)) - \mathbb{I}[d = a](Y_t^d - \mu_d^0(x))}{\kappa_T, Q(a|X)} + \mu_d^0(x) - \mu_d^0(x) - (\mu_a^0 - \mu_0^0),
$$
$$
\bar{V}_0^\ast(\kappa_T, \tau) = E_{P_0} \left[ \left( \bar{\psi}_0^\ast(Y_t, A_t, X_t) \right)^2 \right] = E_{P_0} \left[ \left( \frac{\sigma_0^\ast(X_t)}{\kappa_T, Q(a_0^\ast|X)} \right)^2 + \left( \frac{\sigma_0^\ast(X_t)}{\kappa_T, Q(a|X)} \right)^2 + \left( (\mu_0^a(X_t) - \mu_d^0(X_t)) - (\mu_a^0 - \mu_0^0) \right)^2 \right] = \Omega_0^\ast(\kappa_T, \tau) + C(\mu_a^0 - \mu_0^0),
$$
where $C > 0$ is a constant.

**B.10 Specification of the Observed-Data Score Function**

According to **Lemma B.7**, we can conjecture that if we use the semiparametric efficient score function for our score function, we can obtain a tight upper bound for $E_{P_0}[L_T^\ast]$, which is related to a lower bound for the probability of misidentification. Note that the variance of the semiparametric efficient score function is equivalent to the lower bound in **Lemma B.7**. However, we cannot use the semiparametric efficient score function because it is derived for $R_{\tau}^{\ast}$, rather than $R_{\tau}^{\ast}$. Furthermore, if we use the semiparametric efficient score function for our score function, the constant $[\] is not satisfied. Therefore, based on our obtained result, we specify our score function, which differs from the semiparametric efficient score function, but they match when $\tau \to \infty$.

We specify our score function $S^a(y, d, x) = \mathbb{I}[d = a_0^\ast]S_{f}^{a, a_0^\ast}(y|x) + \mathbb{I}[d = a]S_{f}^{a, a}(y|x) + S_{c}^{a}(x)$ as follows:
$$
S_{f}^{a, a_0^\ast}(y|x) = \frac{(\phi_d^\ast(y, x) - \mu_d^0(x))}{\kappa_T, Q(a_0^\ast|X)} / V_0^a(\kappa_T, \tau) = S_{f, eff}^a(y|x) V_0^a(\kappa_T, \tau) / V_0^a(\kappa_T, \tau),
$$
$$
S_{f}^{a, a}(y|x) = \frac{(\phi_d^\ast(y, x) - \mu_d^0(x))}{\kappa_T, Q(a|X)} / V_0^a(\kappa_T, \tau) = S_{f, eff}^a(y|x) V_0^a(\kappa_T, \tau) / V_0^a(\kappa_T, \tau),
$$
$$
S_{c}^{a}(x) = (\mu_0^a(x) - \mu_d^0(x)) / V_0^a(\kappa_T, \tau) = S_{c, eff}^a(x) V_0^a(\kappa_T, \tau) / V_0^a(\kappa_T, \tau),
$$
where
$$
V_0^a(\kappa_T, \tau) = \bar{V}_0^a(\kappa_T, \tau) + \sum_{d \in \{a_0^\ast, a\}} \mathbb{E}_{P_0} \left[ \frac{\mu_d^0(X_t)E_{P_0}[Y_t^d 1[|Y_t^d| < \tau]|X_t] - (E_{P_0}[Y_t^d 1[|Y_t^d| < \tau]|X_t)]^2}{\kappa_T, Q(d|X_t)} \right].
$$
We note that $V_0^a(\kappa; \tau) \to \tilde{V}_0^a(\kappa; \tau)$ as $\varepsilon^a \to 0$ and $\tau \to \infty$.

From the definition of the parametric submodel, we have

$$g^a(\phi^*_r(y, x), \phi^*_r(y, x), x) = S^a(y, x) + S^a(y, x) + S^a(y, x)$$

Then, we can also confirm that condition (5) holds for our specified $g^a$:

$$\int \left( y^* - y^a \right) (1 + \varepsilon^a g^a(\phi^*_r(y, x), \phi^*_r(y, x), x)) p_0(a_0^a, a, x) dy^* dy^a dx$$

$$= \mu^*_0 - \mu_0^a + \varepsilon^a \left( \int (y^* - y^a) g^a(\phi^*_r(y, x), \phi^*_r(y, x), x) p_0(y, a_0^a, x) dy^* dy^a dx \right)$$

$$= \mu^*_0 - \mu_0^a + \varepsilon^a,$$

where we used the definition of the variance (15).

In summary, from Lemmas B.6 under our specified score function, we obtain the following lemma:

Lemma B.9. Suppose that Assumption 3.7 holds. For $P_0 \in P$ and $Q \in Alt(P_0)$,

$$\frac{1}{T} \mathbb{E}_{\tilde{P}_0} I^a_T = \frac{(\varepsilon^a)^2}{2V_0^a(\kappa; \tau)} + o\left( (\varepsilon^a)^2 \right).$$

B.11 Proof of Theorem 3.5

Combining above arguments and refining the lower bound in Lemma B.5, we prove Theorem 3.5

Proof of Theorem 3.5. From the inequality in Lemma B.5, taking the limsup and letting $\epsilon \to 0$,

$$\limsup_{T \to \infty} \frac{1}{T} \log \mathbb{P}_0(\tilde{a}_T \neq a^*_T)$$

$$\leq \sup_{w \in W} \min_{a \in [K] \setminus \{a^*_T\}} \inf_{a_{w \in [K] \setminus \{a^*_T\}}} \varepsilon^a \left( \mu_0^a - \mu_0^a \right) \sum_{a \in [K]} \mathbb{E}_{\tilde{P}_0} \left[ \mathbb{E}_{\tilde{P}_T} \left[ \log f^a_{\tilde{P}_0}(Y^a_t | X_t) \zeta(X_t) I(1) | X_t \right] w(a) | X_t \right]$$

$$\leq \sup_{w \in W} \min_{a \in [K] \setminus \{a^*_T\}} \inf_{a_{w \in [K] \setminus \{a^*_T\}}} \varepsilon^a \left( \mu_0^a - \mu_0^a \right) \frac{(\varepsilon^a)^2}{2} \mathbb{E}_{\tilde{P}_0} \left[ (S^a(Y_t, A_t, X_t))^2 \right] + o\left( (\varepsilon^a)^2 \right)$$

$$\leq \sup_{w \in W} \min_{a \in [K] \setminus \{a^*_T\}} \left( \mu_0^a - \mu_0^a \right) \frac{(\varepsilon^a)^2}{2} \mathbb{E}_{\tilde{P}_0} \left[ (S^a(Y_t, A_t, X_t))^2 \right] + o\left( (\varepsilon^a)^2 \right).$$

Here, for $\inf_{\varepsilon^a \leq \left( \mu_0^a - \mu_0^a \right)}\frac{\varepsilon^a}{2} \mathbb{E}_{\tilde{P}_0} \left[ (S^a(Y_t, A_t, X_t))^2 \right]$, we set $\varepsilon^a = - (\mu_0^a - \mu_0^a)$, which indicates a situation where $\mu^*(Q) - \mu^*(Q)$ is sufficiently close to 0. Then, after $\mu_0^a - \mu_0^a \to 0$, by letting $\tau \to \infty$, we obtain $V_0^a(w; \tau) \to \Omega_0^a(w)$, which is the semiparametric efficiency bound in Lemmas B.7 and B.8. The proof is complete. □
C  Proof of Lemma B.2

Proof.

\[
\mathbb{E}_Q[L_T] = \sum_{t=1}^{T} \mathbb{E}_Q \left[ \sum_{a \in [K]} \mathbb{I} \{ A_t = a \} \log \frac{f^a_Q(Y^a_t | X_t)}{f^a_{P_0}(Y^a_t | X_t)} \right]
\]

\[
= \sum_{t=1}^{T} \mathbb{E}_Q^{X_t,F_{t-1}} \left[ \sum_{a \in [K]} \mathbb{E}_Q^{Y^a_t | A_t} \left[ \mathbb{I} \{ A_t = a \} \log \frac{f^a_Q(Y^a_t | X_t)}{f^a_{P_0}(Y^a_t | X_t)} \right] \right]
\]

\[
= \sum_{t=1}^{T} \mathbb{E}_Q^{X_t,F_{t-1}} \left[ \sum_{a \in [K]} \mathbb{E}_Q \left[ \mathbb{I} \{ A_t = a \} \mathbb{I} \{ X_t = x \}, F_{t-1} \right] \mathbb{E}_Q^{Y^a_t | X_t} \left[ \log \frac{f^a_Q(Y^a_t | X_t)}{f^a_{P_0}(Y^a_t | X_t)} \right] \right]
\]

\[
= \sum_{t=1}^{T} \int \left( \sum_{a \in [K]} \mathbb{E}_Q^{Y^a_t | X_t} \left[ \log \frac{f^a_Q(Y^a_t | X_t)}{f^a_{P_0}(Y^a_t | X_t)} \right] \mathbb{I} \{ X_t = x \}, F_{t-1} \right) \mathbb{E}_Q \left[ \mathbb{I} \{ A_t = a \} \mathbb{I} \{ X_t = x \}, F_{t-1} \right] \zeta(x) dx
\]

where \( \mathbb{E}_Q^X \) denotes an expectation of random variable \( Z \) over the distribution \( Q \). We used that the observations \( (Y^1_t, \ldots, Y^K_t, X_t) \) are i.i.d. across \( t \in \{ 1, 2, \ldots, T \} \). \( \square \)

D  Proof of Lemma B.3

Proof. For the parametric submodel of the observed-data bandit models, the log-likelihood for the observed data is

\[
\log \mathcal{L}_e(y, d, x) = \log \int \kappa(d|x)p_e(H^d(\tau^d, v^d))J(\tau^d, v^d)dv^d, \tag{16}
\]

where note that \( p_e(H^d(\tau^d, v^d)) = p_e(y^a, y^a, x) \). Then, for \( d \in \{ a_0, a \} \),

\[
S^a(y, d, x) = \frac{\partial}{\partial x} \left[ \log \int \kappa(d|x)p_e(H^d(\tau^d, v^d))J(\tau^d, v^d)dv^d \right]_{x=0} \tag{17}
\]

\[
= \int \frac{\partial}{\partial x} \kappa(d|x)p_e(H^d(\tau^d, v^d))J(\tau^d, v^d)dv^d \tag{18}
\]

Dividing and multiplying by \( p_e(H^d(\tau^d, v^d))J(\tau^d, v^d)dv^d \) in the integral of the numerator of (17) yields

\[
\int \frac{\partial}{\partial x} \kappa(d|x)p_e(x=0(H^d(\tau^d, v^d))J(\tau^d, v^d)dv^d \tag{19}
\]

\[
= \int g^a(\phi^a_e(y, x), \phi^a_e(y, x), x)p_e(x=0(H^d(\tau^d, v^d))J(\tau^d, v^d)dv^d \tag{20}
\]

\[
= \int g^a(\phi^a_e(y, x), \phi^a_e(y, x), x)p_H(H^d(\tau^d, v^d))J(\tau^d, v^d)dv^d \tag{21}
\]

Hence,

\[
S^a(y, d, x) = \mathbb{E}_{P_0} \left[ g^a(\phi^a_e(Y^a_t, X_t), \phi^a_e(Y^a_t, X_t), X_t) | T^d(Y^a_t, Y^a_t, X_t) = (y, x) \right] \tag{22}
\]

This concludes the proof. \( \square \)
E  Proof of Theorem 3.6

Proof. From Theorem 3.5 and (1), the lower bounds are characterized by

\[
\max_{w \in \mathcal{W}} \min_{a \neq a_0} \frac{(\mu_0 - \mu_0^*)^2}{2 \mathbb{E}_{\mu_0} \left( \frac{(\sigma_1(X))^2}{w(a_0 | x)} + \frac{(\sigma_2(X))^2}{w(a | x)} \right)}.
\]

From the condition, there exists \( \Delta_0 \) such that \((\mu_0^* - \mu_0^*)^2 \leq \Delta_0^2\). Therefore, to solve the optimization problem, for each \( x \in \mathcal{X} \), it is enough to consider the following non-linear programming:

\[
\min_{R \in \mathbb{R}, w \in \mathcal{W}} R
\]

\[
\text{s.t.} \quad R \geq \frac{(\sigma_0^a(x))^2}{w(a_0^* | x)} + \frac{(\sigma_0^a(x))^2}{w(a | x)} \quad \forall a \in [K] \setminus \{a_0^*\}
\]

\[
\sum_{c \in [K]} w(c | x) = 1, \quad \forall a \in [K] \setminus \{a_0^*\}
\]

For this problem, we derive the first-order condition, which is sufficient for the global optimality of such a convex programming problem. For Lagrangian multipliers \( \lambda^a \geq 0 \) and \( \gamma \in \mathbb{R} \), we consider the following Lagrangian function:

\[
L(\lambda) = R + \sum_{a \in [K] \setminus \{a_0^*\}} \lambda^a \left\{ \frac{(\sigma_0^a(x))^2}{w(a_0^* | x)} + \frac{(\sigma_0^a(x))^2}{w(a | x)} - R \right\} - \gamma \left\{ \sum_{c \in [K]} w(c | x) - 1 \right\}
\]

Then, the optimal target allocation is given as the optimal solution \( w^* \in \mathcal{W} \), which satisfies that for \( (\lambda^a)_{a \in [K] \setminus \{a_0^*\}} \) and \( \gamma \),

\[
1 - \sum_{a \in [K] \setminus \{a_0^*\}} \lambda^a = 0
\]

\[
-2 \sum_{a \in [K] \setminus \{a_0^*\}} \lambda^a \frac{(\sigma_0^a(x))^2}{w(a | x)} = \bar{\gamma}
\]

\[
\lambda^a \left\{ \frac{(\sigma_0^a(x))^2}{w(a_0^* | x)} + \frac{(\sigma_0^a(x))^2}{w(a | x)} - R^* \right\} = 0
\]

\[
\bar{\gamma} \left\{ \sum_{c \in [K]} \bar{w}(c | x) - 1 \right\} = 0.
\]

Here, (23) implies \( \lambda^a > 0 \) for some \( a \in [K] \setminus \{a_0^*\} \). With \( \lambda^a > 0 \), since \(-\frac{(\sigma_0^a(x))^2}{w(a_0^* | x)} < 0\) for all \( a \in [K] \), it follows that \( \gamma < 0 \). This also implies that \( \lambda^a > 0 \) for each \( a \in [K] \setminus \{a_0^*\} \). Then, (24) implies that

\[
\frac{(\sigma_0^a(x))^2}{w(a_0^* | x)} + \frac{(\sigma_0^a(x))^2}{w(a | x)} = \bar{R} \quad \forall a \in [K] \setminus \{a_0^*\}.
\]

This implies that for \( a, b \in [K] \setminus \{a_0^*\} \)

\[
\frac{(\sigma_0^a(x))^2}{w(a | x)} = \frac{(\sigma_0^b(x))^2}{w(b | x)}.
\]

Therefore, for \( a \in [K] \setminus \{a_0^*\} \)

\[
\bar{w}(a | x) = (1 - \bar{w}(a_0^* | x)) \frac{(\sigma_0^a(x))^2}{\sum_{b \in [K] \setminus \{a_0^*\}} (\sigma_0^b(x))^2}.
\]
Finally, we solve
\[
\min_{w \in \mathcal{W}} \frac{(\sigma_0^*(x))^2}{w(a_0^*|x)} + \frac{\sum_{b \in [K]\setminus\{a_0^*\}} (\sigma_b^*(x))^2}{1 - w(a_0^*|x)}.
\]
By solving this, we have
\[
\tilde{w}(a_0^*|x) = \frac{\sigma_0^*(x)}{\sigma_0^*(x) + \sqrt{\sum_{b \in [K]\setminus\{a_0^*\}} (\sigma_b^*(x))^2}},
\]
\[
\tilde{w}(a|x) = (1 - \tilde{w}(a_0^*|x)) \frac{(\sigma_0^*(x))^2}{\sum_{b \in [K]\setminus\{a_0^*\}} (\sigma_b^*(x))^2} \quad \forall a \in [K]\setminus\{a_0^*\}.
\]

**F** \((\xi_t^a, \mathcal{F}_t)\) is martingale difference sequences

**Proof.** Clearly, \(E_{P_0}[|\xi_t^a|] < \infty\). For each \(t \in [T]\),
\[
E_{P_0} [\xi_t^a | \mathcal{F}_{t-1}] = \frac{1}{\sqrt{TV}} \mathbb{E}_{P_0} \left[ \varphi_0^a \left( Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{\alpha}_t \right) - \varphi^0 \left( Y_t, A_t, X_t; \tilde{\mu}_t^0, \tilde{\alpha}_t \right) - (\mu_0^a - \mu_0^0) \right]_{\mathcal{F}_{t-1}} \]
\[
= \frac{1}{\sqrt{TV}} \mathbb{E}_{P_0} \left[ \mathbb{E}_{P_0} [Y_t^* - \tilde{\mu}_t^a(X_t)] | X_t, \mathcal{F}_{t-1} \right]_{\mathcal{F}_{t-1}} - \mathbb{E}_{P_0} [Y_t^* - \tilde{\mu}_t^0(X_t)] | X_t, \mathcal{F}_{t-1} - \tilde{\mu}_t^0(X_t) - (\mu_0^a - \mu_0^0) \right]_{\mathcal{F}_{t-1}} \]
\[
= 0.
\]

**G** Proof of Lemma [5.6]

**Proof.** For the simplicity, let us denote \(E_{P_0} \) by \(E\). Recall that \(\varphi^a \left( Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{\alpha}_t \right)\) is constructed as
\[
\varphi^a \left( Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{\alpha}_t \right) = \frac{1}{\tilde{w}_t(a|X_t)} \left( Y_t^* - \tilde{\mu}_t(X_t) \right) + \tilde{\mu}_t^a(X_t).
\]
For each \(t = 1, \ldots, T\), we have
\[
E \left[ \exp \left( C_0 \sqrt{T} |\xi_t^a| \right) | \mathcal{F}_{t-1} \right] \]
\[
= E \left[ \exp \left( \frac{C_0}{\sqrt{Va}} \left( \varphi_0^a \left( Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{\alpha}_t \right) - \varphi^a \left( Y_t, A_t, X_t; \tilde{\mu}_t^0, \tilde{\alpha}_t \right) - (\mu_0^a - \mu_0^0) \right) \right) | \mathcal{F}_{t-1} \right] \]
\[
\leq E \left[ \exp \left( \frac{C_0}{\sqrt{Va}} \left( \varphi_0^a \left( Y_t, A_t, X_t; \tilde{\mu}_t^0, \tilde{\alpha}_t \right) - \varphi^0 \left( Y_t, A_t, X_t; \tilde{\mu}_t^0, \tilde{\alpha}_t \right) + \frac{C_0(\mu_0^a - \mu_0^0)}{\sqrt{Va}} \right) \right) | \mathcal{F}_{t-1} \right] \]
\[
\leq E \left[ \exp \left( \frac{C_0}{\sqrt{Va}} \varphi_0^a \left( Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{\alpha}_t \right) - \varphi^a \left( Y_t, A_t, X_t; \tilde{\mu}_t^0, \tilde{\alpha}_t \right) + \frac{2C_0C_\mu}{\sqrt{Va}} \right) | \mathcal{F}_{t-1} \right] \]
\[
\overset{(a)}{=} C_1 E \left[ \exp \left( \frac{C_0}{\sqrt{Va}} \left( \varphi_0^a \left( Y_t, A_t, X_t; \tilde{\mu}_t^0, \tilde{\alpha}_t \right) - \varphi^a \left( Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{\alpha}_t \right) \right) \right) | \mathcal{F}_{t-1}, \mathbb{P}(A_t = a_0^0 | \mathcal{F}_{t-1}) \right] \]
\[
+ C_1 E \left[ \exp \left( \frac{C_0}{\sqrt{Va}} \left( \varphi_0^a \left( Y_t, A_t, X_t; \tilde{\mu}_t^0, \tilde{\alpha}_t \right) - \varphi^a \left( Y_t, A_t, X_t; \tilde{\mu}_t^0, \tilde{\alpha}_t \right) \right) \right) | \mathcal{F}_{t-1}, \mathbb{P}(A_t = a | \mathcal{F}_{t-1}) \right].
\]
where for $(a)$, we denote $\tilde{C}_1 = \exp \left( 2 C_0 C_{\mu} / \tilde{V}^a \right)$. Since $X_{a,i}$ is a sub-exponential random variable (Assumption 5.1), there exists some universal constant $C > 0$ such that for all $P_0 \in \mathcal{P}$, for all $\lambda \geq 0$ such that $0 \leq \lambda \leq 1 / C$, $\mathbb{E}[\exp(\lambda (X_{a,t} - \mu_0^a))] \leq \exp(C^2 \lambda^2)$ (Vershynin [2013], Proposition 2.7.1). Note that from the assumptions that $|\mu_k^a| \leq C_{\mu}$, $\max(\{\sigma_0^a\}^2, 1 / \{\sigma_0^a\}^2) \leq C_{\sigma^2}$, and $|\tilde{w}_t(a|X_t)| \geq C_w$ for all $t \in \{1, \ldots, T\}$, where $C_w > 0$ is a constant that depends on $C_{\sigma^2}$. Therefore, there exists a positive constant $C_1(C_0, C_{\mu}, C_{\sigma^2})$ such that

$$\mathbb{E} \left[ \exp(C_0 \sqrt{T} |\xi|^a) \bigg| \mathcal{F}_{t-1} \right] \leq C_1(C_0, C_{\mu}, C_{\sigma^2}).$$

This concludes the proof.

\[ \square \]

### H Proof of Lemma 5.7

Assumptions 4.1 and the continuity of $w^*$ with respect to $\text{Var}_{P_0}(Y^a|x)$ directly implies the following corollary, which states the almost sure convergence of $\tilde{w}_t(a|X_t)$.

**Lemma H.1.** Under the RS-AIPW strategy, for each $a \in [K]$ and $x \in \mathcal{X}$,

$$\tilde{w}_t(a|x) \xrightarrow{a.s.} w^*(a|x),$$

Then, we present the following results on the convergence of the second moment. Recall we defined

$$\sqrt{V^a} = \sqrt{\mathbb{E}_{P_0} \left[ \frac{(\sigma_0^a(X))^2}{w^*(a_0^a|X)} + \frac{(\sigma_0^a(X))^2}{w^*(a|X)} + (\mu_0^a(X) - \mu_0^a(X) - (\mu_0^a - \mu_0^a))^2 \right]}$$

**Lemma H.2.** Under the RS-AIPW strategy, for each $a \in [K] \setminus \{a_0^a\}$, with probability 1,

$$\lim_{t \to \infty} \mathbb{P}_{P_0} \left[ \mathbb{E}_{P_0} \left[ \left( \varphi^a(Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{w}_t) - \varphi^a(Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{w}_t) - (\mu_0^a - \mu_0^a) \right)^2 \bigg| \mathcal{F}_{t-1} \right] \right] = 0.$$

**Proof.**

\[
\mathbb{E}_{P_0} \left[ \left( \varphi^a(Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{w}_t) - \varphi^a(Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{w}_t) - (\mu_0^a - \mu_0^a) \right)^2 \bigg| \mathcal{F}_{t-1} \right] = \mathbb{E}_{P_0} \left[ \left( \frac{1}{\tilde{w}_t(a_0^a|X_t)} - \frac{1}{\tilde{w}_t(a|X_t)} \right)^2 \bigg| \mathcal{F}_{t-1} \right]
\]

\[
= \mathbb{E}_{P_0} \left[ \left( \frac{1}{\tilde{w}_t(a_0^a|X_t)} - \frac{1}{\tilde{w}_t(a|X_t)} \right)^2 \bigg| \mathcal{F}_{t-1} \right] + 2 \left( \frac{1}{\tilde{w}_t(a_0^a|X_t)} - \frac{1}{\tilde{w}_t(a|X_t)} \right) \left( \tilde{\mu}_t^a(X_t) - \tilde{\mu}_t^a(X_t) - (\mu_0^a - \mu_0^a) \right)
\]

\[
= \mathbb{E}_{P_0} \left[ \left( \frac{1}{\tilde{w}_t(a_0^a|X_t)} - \frac{1}{\tilde{w}_t(a|X_t)} \right)^2 \bigg| \mathcal{F}_{t-1} \right] + \mathbb{E}_{P_0} \left[ \left| \tilde{\mu}_t^a(X_t) - \tilde{\mu}_t^a(X_t) - (\mu_0^a - \mu_0^a) \right| \right]
\]

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We also have
\[
\left( \hat{\mu}_t(X_t) - \hat{\mu}_t^0(X_t) \right)^2 | F_{t-1}
\]
\[
= \mathbb{E}_{P_0} \left[ \frac{(Y_t - \hat{\mu}_t(X_t))^2}{\hat{w}_t(a|X_t)} | F_{t-1} \right] + \mathbb{E}_{P_0} \left[ \frac{(Y_t - \hat{\mu}_t^0(X_t))^2}{\hat{w}_t(a|X_t)} | F_{t-1} \right]
\]
\[
- \mathbb{E}_{P_0} \left[ \left( \hat{\mu}_t^0(X_t) + \hat{\mu}_t(X_t) - (\mu_0 - \mu_0^0) \right)^2 | F_{t-1} \right].
\]

Here, we used
\[
\mathbb{E}_{P_0} \left[ \frac{1[A_t = a]}{\hat{w}_t(a|X_t)} (Y_t - \hat{\mu}_t(X_t))^2 | F_{t-1} \right] = \mathbb{E}_{P_0} \left[ \frac{\hat{w}_t(a|X_t)^2}{\hat{w}_t(a|X_t)} | F_{t-1} \right]
\]
and
\[
\mathbb{E}_{P_0} \left[ \frac{1[A_t = a]}{\hat{w}_t(a|X_t)} (Y_t - \hat{\mu}_t(X_t))^2 | F_{t-1} \right] = \mathbb{E}_{P_0} \left[ \frac{\hat{w}_t(a|X_t)^2}{\hat{w}_t(a|X_t)} | F_{t-1} \right]
\]
\[
= \mathbb{E}_{P_0} \left[ \frac{(Y_t - \hat{\mu}_t^0(X_t))^2}{\hat{w}_t(a|X_t)} | F_{t-1} \right] + \mathbb{E}_{P_0} \left[ \frac{(Y_t - \hat{\mu}_t^0(X_t))^2}{\hat{w}_t(a|X_t)} | F_{t-1} \right]
\]
\[
- \mathbb{E}_{P_0} \left[ \left( \hat{\mu}_t^0(X_t) + \hat{\mu}_t(X_t) - (\mu_0 - \mu_0^0) \right)^2 | F_{t-1} \right].
\]

We also have
\[
\mathbb{E}_{P_0} \left[ \frac{(Y_t - \hat{\mu}_t(X_t))^2}{\hat{w}_t(a|X_t)} | F_{t-1} \right]
\]
\[
= \mathbb{E}_{P_0} \left[ \frac{(Y_t - \hat{\mu}_t^0(X_t))^2}{\hat{w}_t(a|X_t)} | F_{t-1} \right] + \mathbb{E}_{P_0} \left[ \frac{(Y_t - \hat{\mu}_t^0(X_t))^2}{\hat{w}_t(a|X_t)} | F_{t-1} \right]
\]
\[
- \mathbb{E}_{P_0} \left[ \left( \hat{\mu}_t^0(X_t) + \hat{\mu}_t(X_t) - (\mu_0 - \mu_0^0) \right)^2 | F_{t-1} \right].
\]

Then,
\[
\mathbb{E}_{P_0} \left[ \frac{(Y_t - \hat{\mu}_t^{a_0}(X_t))^2}{\hat{w}_t(a_0^0|X_t)} | F_{t-1} \right] + \mathbb{E}_{P_0} \left[ \frac{(Y_t - \hat{\mu}_t^{a_0}(X_t))^2}{\hat{w}_t(a_0|X_t)} | F_{t-1} \right]
\]
\[
- \mathbb{E}_{P_0} \left[ \left( \hat{\mu}_t^{a_0}(X_t) + \hat{\mu}_t(X_t) - (\mu_0 - \mu_0^0) \right)^2 | F_{t-1} \right]
\]
\[
= \mathbb{E}_{P_0} \left[ \mathbb{E}_{P_0} \left[ (Y_t^a)^2 | X_t \right] - (\mu_0^a(X_t))^2 + (\mu_0^a(X_t) - \hat{\mu}_t^0(X_t))^2 \right]
\]
\[
+ \mathbb{E}_{P_0} \left[ \mathbb{E}_{P_0} \left[ (Y_t^a)^2 | X_t \right] - (\mu_0^a(X_t))^2 + (\mu_0^a(X_t) - \hat{\mu}_t^0(X_t))^2 \right]
\]
\[
- \mathbb{E}_{P_0} \left[ \left( \hat{\mu}_t^{a_0}(X_t) + \hat{\mu}_t(X_t) - (\mu_0 - \mu_0^0) \right)^2 \right].
\]

Because \( \hat{\mu}_t^0(x) \xrightarrow{a.s.} \mu_0^0(x) \) and \( \hat{w}_t(a|x) \xrightarrow{a.s.} w_s^*(a|x) \), for each \( x \in \mathcal{X} \), with probability 1,
\[
\lim_{t \to \infty} \left( \mathbb{E}_{P_0} \left[ (Y_t^a)^2 | X_t \right] - (\mu_0^a(x))^2 + (\mu_0^a(x) - \hat{\mu}_t^0(x))^2 \right)
\]
\[
+ \left( \mathbb{E}_{P_0} \left[ (Y_t^a)^2 | X_t \right] - (\mu_0^a(x))^2 + (\mu_0^a(x) - \hat{\mu}_t^0(x))^2 \right)
\]
\[
- \left( \hat{\mu}_t^{a_0}(x) + \hat{\mu}_t(x) - (\mu_0 - \mu_0^0) \right)^2
\]
\[
= 34
\]
This concludes the proof.

Then, we obtain the following decomposition:

\[
\begin{align*}
&-\left(\frac{(\sigma_0^2(x))^2}{w^*(a_0^*X)} + \frac{(\sigma_0^2(X))^2}{w^*(aX)} + (\mu_0^*(x) - \mu_0^*(X)) \right) \\
&\leq \lim_{t \to \infty} \frac{\mathbb{E}_{P_0}(Y_t^a)^2|x} - \mu_0^*(x) - \frac{(\sigma_0^2(x))^2}{w^*(a_0^*X)} + \lim_{t \to \infty} \left| \mathbb{E}_{P_0}(Y_t^a)^2|x \right| - \mu_0^*(X) - \frac{(\sigma_0^2(X))^2}{w^*(aX)} \\
&+ \lim_{t \to \infty} \frac{(\mu_0^*(x) - \mu_0^*(x))^2}{w^*(a_0^*X)} + \lim_{t \to \infty} \left| \mathbb{E}_{P_0}(Y_t^a)^2|x \right| - \mu_0^*(x) - \frac{(\sigma_0^2(x))^2}{w^*(a_0^*X)} \\
&= 0.
\end{align*}
\]

Note that \(\mathbb{E}_{P_0}(Y_t^a)^2|x - (\mu_0^*(x))^2 = (\sigma_0^2(x))^2\). This directly implies that

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{P_0}\left[ \left(\varphi_0^a\left(Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{w}_t\right) - \varphi^a\left(Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{w}_t\right) - (\mu_0^* - \mu_0^*) \right)^2 \right] \to 0,
\]

Then, from Proposition A.3 for any \(\delta\), there exists \(T_0\) such that for all \(T > T_0\)

\[
\mathbb{E}_{P_0}\left[ \frac{1}{T} \sum_{t=1}^{T} \left(\varphi_0^a\left(Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{w}_t\right) - \varphi^a\left(Y_t, A_t, X_t; \tilde{\mu}_t^a, \tilde{w}_t\right) - (\mu_0^* - \mu_0^*) \right)^2 \right] \leq \delta.
\]

This concludes the proof.

\[\square\]

1 Proof of Theorem 5.4: Large Deviation Bound for Martingales

For brevity, let us denote \(P_{P_0}\) and \(\mathbb{E}_{P_0}\) by \(\mathbb{P}\) and \(\mathbb{E}\), respectively. For all \(t = 1, \ldots, T\), let us define

\[
r_t(\lambda) = \frac{\exp(\lambda \xi_t)}{\mathbb{E}[\exp(\lambda \xi_t)]},
\]

and

\[
\eta_t(\lambda) = \xi_t - b_t(\lambda),
\]

where

\[
b_t(\lambda) = \mathbb{E}[r_t(\lambda) \xi_t^2].
\]

Then, we obtain the following decomposition:

\[
Z_T^a = U_T(\lambda) + B_T(\lambda),
\]

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We find that
\[ U_T(\lambda) = \sum_{t=1}^{T} \eta_t(\lambda) \]
and
\[ B_T(\lambda) = \sum_{t=1}^{T} b_t(\lambda). \]

Let \( \Psi_T(\lambda) = \sum_{t=1}^{T} \log E[\exp(\lambda \xi_t^a)]. \)

Before showing the proof of Theorem 5.4, we show the following lemmas. In particular, Lemma 1.3 in Appendix I is our novel result to bound \( E[\exp(\lambda(\sum_{t=1}^{T} \xi_t^a))/ (\prod_{t=1}^{T} E[\exp(\lambda(\xi_t^a))])]. \) Lemmas 1.1–1.3 are modifications of the existing results of [Fan et al. (2013), 2014].

**Lemma 1.1.** Under Condition A,
\[ E[|\xi_t^a|^k | F_{t-1}] \leq k! \left( C_0 T^{1/2}\right)^{-k} C_1, \quad \text{for all} \quad k \geq 2. \]

**Proof.** Applying the elementary inequality \( x^k/k! \leq \exp(x), \forall x \geq 0, \) to \( x = C_0 |\sqrt{T} \xi_t^a|, \) for \( k \geq 2, \)
\[ |\xi_t^a|^k \leq k!(C_0 T^{1/2})^{-k} \exp(C_0 |\sqrt{T} \xi_t^a|). \]

Taking expectations on both sides, with Condition A, we obtain the desired inequality. Recall that Condition A is
\[ \sup_{1 \leq t \leq T} E_{P_0} \left[ \exp \left( C_0 \sqrt{T} |\xi_t^a| \right) \right] | F_{t-1} \right] \leq C_1 \]
for some positive constants \( C_0 \) and \( C_1. \)

**Lemma 1.2.** Under Condition A, there exists some constant \( C > 0 \) such that for all \( 0 \leq \lambda \leq \frac{1}{3} C_0 \sqrt{T}, \)
\[ |B_T(\lambda) - \lambda| \leq C \left( \lambda^3 + \lambda^2 / \sqrt{T} \right). \]

**Proof.** By definition, for \( t = 1, \ldots, T, \)
\[ b_t(\lambda) = \frac{E[\xi_t^a \exp(\lambda \xi_t^a)]}{E[\exp(\lambda \xi_t^a)]}. \]
Jensen’s inequality and \( E[\xi_t^a] = E[E[\xi_t^a | F_{t-1}]] = 0 \) implies that \( E[\exp(\lambda \xi_t^a)] \geq 1 \) and
\[ E[\xi_t^a \exp(\lambda \xi_t^a)] = E[\xi_t^a (\exp(\lambda \xi_t^a) - 1)] \geq 0, \quad \text{for} \ \lambda \geq 0. \]
We find that
\[ B_T(\lambda) \leq \sum_{t=1}^{T} E[\xi_t^a \exp(\lambda \xi_t^a)] \]
\[ = \lambda E[W_T] + \sum_{t=1}^{T} \sum_{k=2}^{\infty} \sum_{k=2}^{\infty} E \left[ \frac{\xi_t^a (\lambda \xi_t^a)^k}{k!} \right], \]
by the series expansion for \( \exp(x). \) Recall that \( W_T = \sum_{t=1}^{T} E_{P_0} [\xi_t^a | F_{t-1}] \) is the sum of the conditional second moment. Here, using Lemma 1.1 and \( E[\xi_t^{k+1}] = E[E[\xi_t^{k+1} | F_{t-1}]], \) for some constant \( C_2, \)
\[ \sum_{t=1}^{T} \sum_{k=2}^{\infty} \left| E \left[ \frac{\xi_t^a (\lambda \xi_t^a)^k}{k!} \right] \right| \leq \sum_{t=1}^{T} \sum_{k=2}^{\infty} \left| E \left[ \xi_t^{k+1} \right] \right| \frac{\lambda^k}{k!} \]
\[ \leq \sum_{t=1}^{T} \sum_{k=2}^{\infty} (k + 1)! \left( C_0 T^{1/2} \right)^{-(k+1)} C_1 \frac{\lambda^k}{k!}. \]
Therefore,

\[ B_T(\lambda) = \lambda + \lambda V_T + C_2 \lambda^2 / \sqrt{T}. \]

Next, we show the lower bound of \( B_T(\lambda) \). First, by using Lemma I.11 using some constant \( C_3 > 0 \), for all \( 0 \leq \lambda \leq \frac{1}{4} C_0 \sqrt{T} \),

\[ \mathbb{E} \left[ \exp(\lambda \xi^a_T) \right] \leq 1 + \sum_{k=1}^{\infty} \left[ \frac{C_4 \lambda^2 T^{-1}}{k!} \right] = 1 + C_1 \sum_{k=1}^{\infty} \lambda^k (C_0 \sqrt{T})^{-k} \leq 1 + C_3 \lambda^2 T^{-1}. \]

This inequality together with (25) implies the lower bound of \( B_T(\lambda) \): for some positive constant \( C_4 \),

\[
B_T(\lambda) = \sum_{t=1}^{T} \frac{\mathbb{E}[\xi^a_t \exp(\lambda \xi^a_t)]}{\mathbb{E}[\exp(\lambda \xi^a_t)]} \\
\geq \left( \sum_{t=1}^{T} \frac{\mathbb{E}[\xi^a_t \exp(\lambda \xi^a_t)]}{\sum_{i=0}^{\infty} \mathbb{E}[\xi^a_t \exp(\lambda \xi^a_t)^i]} \right) (1 + C_3 \lambda^2 T^{-1})^{-1} \\
= \left( \lambda W_T + \sum_{t=1}^{T} \sum_{i=0}^{\infty} \mathbb{E}[\xi^a_t \exp(\lambda \xi^a_t)^i]/k! \right) (1 + C_3 \lambda^2 T^{-1})^{-1} \\
\geq \left( \lambda W_T - \sum_{t=1}^{T} \sum_{i=0}^{\infty} \mathbb{E}[\xi^a_t \exp(\lambda \xi^a_t)^i]/k! \right) (1 + C_3 \lambda^2 T^{-1})^{-1} \\
\geq (\lambda - \lambda V_T - C_2 \lambda^2 / \sqrt{T}) (1 + C_3 \lambda^2 T^{-1})^{-1} \\
\geq \lambda - \lambda V_T - C_4 \lambda^2 / \sqrt{T}. \\
\]

This concludes the proof.

**Lemma 1.3.** Assume Condition A. There exists some constant \( C > 0 \) such that for all \( 0 \leq \lambda \leq \frac{1}{4} C_0 \sqrt{T} \),

\[ \left| \Psi_T(\lambda) - \frac{\lambda^2}{2} \right| \leq C \left( \lambda^3 / \sqrt{T} + \lambda^2 V_T \right). \]

**Proof.** First, we have \( \mathbb{E}[\exp(\lambda \xi^a_t)] \geq 1 \) from Jensen’s inequality. Using the series expansion of \( \log(1 + \varphi) \), \( \varphi \geq 0 \), there exists \( 0 \leq \varphi^t \leq \mathbb{E}[\exp(\lambda \xi^a_t)] - 1 \) (for \( t = 1, \ldots, T \)) such that

\[
\Psi_T(\lambda) = \log \prod_{t=1}^{T} \mathbb{E}[\exp(\lambda \xi^a_t)] \\
= \sum_{t=1}^{T} \left( \mathbb{E}[\exp(\lambda \xi^a_t)] - 1 \right) - \frac{1}{2} \left( \mathbb{E}[\exp(\lambda \xi^a_t)] - 1 \right)^2. \\
\]

Because \( \langle \xi^a_t \rangle \) is a martingale difference sequence, \( \mathbb{E}[\xi^a_t] = \mathbb{E}[\mathbb{E}[\xi^a_t | F_{t-1}]] = 0 \). Therefore,

\[
\Psi_T(\lambda) - \frac{\lambda^2}{2} \mathbb{E}[W_T] \\
= \sum_{t=1}^{T} \left( \mathbb{E}[\exp(\lambda \xi^a_t)] - 1 \right) - \frac{1}{2} \left( \mathbb{E}[\exp(\lambda \xi^a_t)] - 1 \right)^2 - \sum_{t=1}^{T} \left( \lambda \mathbb{E}[\xi^a_t] + \lambda^2 \mathbb{E}[(\xi^a_t)^2] \right) \\
= \sum_{t=1}^{T} \left( \mathbb{E}[\exp(\lambda \xi^a_t)] - 1 \right) - \frac{1}{2} \left( \mathbb{E}[\exp(\lambda \xi^a_t)] - 1 \right)^2 - \sum_{t=1}^{T} \left( \lambda \mathbb{E}[\xi^a_t] + \lambda^2 \mathbb{E}[(\xi^a_t)^2] \right). \\
\]

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Then, by using \( E[\exp(\lambda \xi^a_t)] \geq 1 \), we have

\[
\left| \Psi_T(\lambda) - \frac{\lambda^2}{2} E[W_T] \right| \leq \sum_{t=1}^T \left| \mathbb{E}[\exp(\lambda \xi^a_t)] - 1 - \lambda \mathbb{E}[\xi^a_t] - \frac{\lambda^2}{2} \mathbb{E}[(\xi^a_t)^2] \right| + \frac{1}{2} \sum_{t=1}^T (\mathbb{E}[\exp(\lambda \xi^a_t)] - 1)^2
\]

\[
\leq \sum_{t=1}^T \sum_{k=3}^{+\infty} \frac{\lambda^k}{k!} \mathbb{E}[(\xi^a_t)^k] + \frac{1}{2} \sum_{t=1}^T \left( \sum_{k=1}^{+\infty} \frac{\lambda^k}{k!} \mathbb{E}[(\xi^a_t)^k] \right)^2.
\]

From Lemma I.1 for a constant \( C_3 \),

\[
\left| \Psi_T(\lambda) - \frac{\lambda^2}{2} E[W_T] \right| \leq C_3 \lambda^3 / \sqrt{T}
\]

In conclusion, we have

\[
\left| \Psi_T(\lambda) - \frac{\lambda^2}{2} \right| \leq C_3 \lambda^3 / \sqrt{T} + \frac{\lambda^2}{2} (E[W_T - 1]) \leq C_3 \lambda^3 / \sqrt{T} + \frac{\lambda^2}{2} E[|W_T - 1|].
\]

Recall that \( V_T = E[|W_T - 1|] \). Then,

\[
\left| \Psi_T(\lambda) - \frac{\lambda^2}{2} \right| \leq C \left( \lambda^3 / \sqrt{T} + \lambda^2 V_T \right).
\]

\[\square\]

**Lemma I.4.** Assume Condition A. For any \( \epsilon > 0 \) there exists \( T_0 > 0 \) and some constants \( \tilde{C}_2, \tilde{C}_3, \tilde{C}_4 > 0 \) such that for all \( T \geq T_0 \) and \( 0 \leq \lambda \leq \frac{C_0}{\tilde{C}_4} \sqrt{T} \),

\[
\mathbb{E}\left[ \exp\left( \frac{X}{T} \sum_{t=1}^T \xi^a_t \right) \right] \leq \exp\left( \frac{\tilde{C}_2 \lambda^4}{T} + \frac{\tilde{C}_3 \lambda^3}{\sqrt{T}} + \frac{\tilde{C}_4 T_0 + \epsilon \lambda^2}{T} \right).
\]

**Proof.** Here, we have

\[
\mathbb{E}\left[ \exp\left( \frac{X}{T} \sum_{t=1}^T \xi^a_t \right) \right] = \mathbb{E}\left[ \prod_{t=1}^T \mathbb{E}\left[ \exp(\lambda \xi^a_t) | F_{t-1} \right] \right].
\]

Then, by using Lemma I.1 for each \( t = 1, \ldots, T \),

\[
\mathbb{E}\left[ \exp(\lambda \xi^a_t) | F_{t-1} \right] \leq 1 + \frac{\lambda^2}{2} \mathbb{E}\left[ (\xi^a_t)^2 | F_{t-1} \right] + \sum_{k=3}^{+\infty} \frac{\lambda^k}{k!} \mathbb{E}\left[ (\xi^a_t)^k | F_{t-1} \right]
\]

\[
\leq 1 + \frac{\lambda^2}{2} \mathbb{E}\left[ (\xi^a_t)^2 | F_{t-1} \right] + \sum_{k=3}^{+\infty} \frac{\lambda^k}{k!} C_1(C_0 \sqrt{T})^{-k}
\]

\[
\leq 1 + \frac{\lambda^2}{2} \mathbb{E}\left[ (\xi^a_t)^2 | F_{t-1} \right] + O \left( \lambda^3 / T^{3/2} \right).
\]

Therefore,

\[
\mathbb{E}\left[ \exp\left( \frac{X}{T} \sum_{t=1}^T \xi^a_t \right) \right] \leq \mathbb{E}\left[ \prod_{t=1}^T \left( 1 + \frac{\lambda^2}{2} \mathbb{E}\left[ (\xi^a_t)^2 | F_{t-1} \right] + O \left( \lambda^3 / T^{3/2} \right) \right) \right]
\]

\[
\leq \mathbb{E}\left[ \prod_{t=1}^T \exp\left( \frac{\lambda^2}{2} \mathbb{E}\left[ (\xi^a_t)^2 | F_{t-1} \right] + O \left( \lambda^3 / T^{3/2} \right) \right) \right].
\]

Similarly, by using Lemma I.1 and constants \( c, \tilde{C} > 0 \), we have

\[
\mathbb{E}\left[ \exp \left( \frac{X \xi^a_t}{T} \right) \right] = \exp \left( \log \mathbb{E}\left[ \exp \left( \frac{X \xi^a_t}{T} \right) \right] \right)
\]

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\[
\begin{align*}
&= \exp \left( \log \left( 1 + \sum_{k=2}^{\infty} \mathbb{E} \left[ \frac{(\lambda \xi^a_k)^k}{k!} \right] \right) \right) \\
&= \exp \left( \frac{\lambda^2}{2} \mathbb{E} \left[ (\xi^a)^2 \right] + \sum_{k=3}^{\infty} \mathbb{E} \left[ \frac{(\lambda \xi^a_k)^k}{k!} \right] - \frac{1}{2} \left( \sum_{k=2}^{\infty} \mathbb{E} \left[ \frac{(\lambda \xi^a_k)^k}{k!} \right] \right)^2 + \frac{1}{3} \left( \sum_{k=2}^{\infty} \mathbb{E} \left[ \frac{(\lambda \xi^a_k)^k}{k!} \right] \right)^3 + \cdots \right) \\
&\geq \exp \left( \frac{\lambda^2}{2} \mathbb{E} \left[ (\xi^a)^2 \right] - \frac{1}{2} \left( \sum_{k=2}^{\infty} \mathbb{E} \left[ \frac{(\lambda \xi^a_k)^k}{k!} \right] \right)^2 - \frac{1}{3} \left( \sum_{k=2}^{\infty} \mathbb{E} \left[ \frac{(\lambda \xi^a_k)^k}{k!} \right] \right)^3 + \cdots \right) \\
&\geq \exp \left( \frac{\lambda^2}{2} \mathbb{E} \left[ (\xi^a)^2 \right] - c \lambda^3 / T^{3/2} - \frac{1}{2} \left( \frac{4C_1 \lambda^2}{3C_0^2 T} \right)^2 - \frac{1}{3} \left( \frac{4C_1 \lambda^2}{3C_0^2 T} \right)^3 - \frac{1}{4} \left( \frac{4C_1 \lambda^2}{3C_0^2 T} \right)^4 - \cdots \right) \\
&\geq \exp \left( \frac{\lambda^2}{2} \mathbb{E} \left[ (\xi^a)^2 \right] - c \lambda^3 / T^{3/2} - \frac{1}{2} \left( \frac{4C_1 \lambda^2}{3C_0^2 T} \right)^2 \right) \\
&\geq \exp \left( \frac{\lambda^2}{2} \mathbb{E} \left[ (\xi^a)^2 \right] - c (\lambda / \sqrt{T})^3 - c \lambda^4 / T^2 \right).
\end{align*}
\]

For (a), we used Jensen’s inequality for \( m = 2, 3, \ldots \) as
\[-(-1)^m \frac{1}{m} \left( \sum_{k=2}^{\infty} \mathbb{E} \left[ \frac{(|\lambda \xi^a_k|^k)}{k!} \right] \right)^m \geq - \frac{1}{m} \left( \sum_{k=2}^{\infty} \mathbb{E} \left[ \frac{(|\lambda \xi^a_k|^k)}{k!} \right] \right)^m \geq - \frac{1}{m} \left( \sum_{k=2}^{\infty} \mathbb{E} \left[ \frac{(|\lambda \xi^a_k|^k)}{k!} \right] \right)^m.
\]

For (b), we used the fact there exist a constant \( c > 0 \) such that
\[
\mathbb{E} \left[ \sum_{k=2}^{\infty} \frac{|\lambda \xi^a_k|^k}{k!} \right] \leq \sum_{k=2}^{\infty} \frac{\lambda^k}{k!} \cdot k! \cdot C_1 \frac{1}{(C_0 \sqrt{T})^k} = C_1 \sum_{k=2}^{\infty} \left( \frac{\lambda}{C_0 \sqrt{T}} \right)^k = \frac{C_1 \lambda^2}{C_0^2 T} \frac{1}{1 - \frac{\lambda}{C_0 \sqrt{T}}}
\]
and
\[
\mathbb{E} \left[ \sum_{k=3}^{\infty} \frac{|\lambda \xi^a_k|^k}{k!} \right] \leq \sum_{k=3}^{\infty} \frac{\lambda^k}{k!} \cdot k! \cdot C_1 \frac{1}{(C_0 \sqrt{T})^k} \leq c \left( \frac{\lambda}{\sqrt{T}} \right)^3.
\]

Here, for (c), we used Lemma 1 and for (d), we used (27). Then, by combining the above upper and lower bounds, with some constant \( C_0, C_1 > 0 \),
\[
\frac{\mathbb{E} \left[ \exp \left( \sum_{t=1}^{T} \xi^a_t \right) \right]}{\prod_{t=1}^{T} \mathbb{E} \left[ \exp (\lambda \xi^a_t) \right]} \leq \frac{\mathbb{E} \left[ \exp \left( \frac{\lambda^2}{2} \mathbb{E} \left[ (\xi^a)^2 | \mathcal{F}_{t-1} \right] + O \left( \left( \frac{\lambda}{\sqrt{T}} \right)^3 \right) \right) \right]}{\prod_{t=1}^{T} \exp \left( \frac{\lambda^2}{2} \mathbb{E} \left[ (\xi^a)^2 \right] \right)} \\
\leq \exp \left( \frac{\lambda^2}{2} \mathbb{E} \left[ (\xi^a)^2 \right] - c \left( \frac{\lambda}{\sqrt{T}} \right)^3 - c \lambda^4 / T^2 \right) \mathbb{E} \left[ \left( \sum_{t=1}^{T} \xi^a_t \right) \right] / 2 \right)
\]

Using Hölder’s inequality,
\[
\frac{\mathbb{E} \left[ \exp \left( \sum_{t=1}^{T} \xi^a_t \right) \right]}{\prod_{t=1}^{T} \mathbb{E} \left[ \exp (\lambda \xi^a_t) \right]} \leq \exp \left( \frac{\lambda^2}{2} \mathbb{E} \left[ (\xi^a)^2 \right] \right) \mathbb{E} \left[ \left( \sum_{t=1}^{T} \xi^a_t \right) \right] / 2 \right).
\]

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\[
\leq \exp \left( C_0 \frac{X^4}{T} + C_1 \frac{X^3}{\sqrt{T}} \right) \mathbb{E} \left[ \prod_{t=1}^T \exp \left( \frac{X^2}{2} \left( \mathbb{E}[(\xi_t^a)^2 | \mathcal{F}_{t-1}] - \mathbb{E}[(\xi_t^a)^2] \right) / 2 \right) \right] \\
\leq \exp \left( C_0 \frac{X^4}{T} + C_1 \frac{X^3}{\sqrt{T}} \right) \prod_{t=1}^T \left( \mathbb{E} \left[ \exp \left( \frac{TX^2}{2} \left( \mathbb{E}[(\xi_t^a)^2 | \mathcal{F}_{t-1}] - \mathbb{E}[(\xi_t^a)^2] \right) / 2 \right) \right] \right)^{\frac{1}{2}} 
\]  

(26)

Note that the term
\[
\frac{X^2}{2} \left( \mathbb{E}[(\xi_t^a)^2 | \mathcal{F}_{t-1}] - \mathbb{E}[(\xi_t^a)^2] \right)
\]

is bounded by some constant because \( \bar{w}_t (a|X_t) \) and \( \hat{\mu}_t^a \) are bounded and \( X \leq \sqrt{T} \min \left\{ C_0, \sqrt{\frac{3C_2^2}{8C_1}} \right\} \). Then, from Lemma [1,2] with probability one,
\[
\mathbb{E} \left[ \left( \phi^a (Y_t, A_t, X_t; \hat{\mu}_t^a, \bar{w}_t) - \phi^a (Y_t, A_t, X_t; \hat{\mu}_t^a, \bar{w}_t) - (\mu_0^a - \mu_0^a)^2 \right) \mathcal{F}_{t-1} \right] \\
- \mathbb{E} \left[ \left( \phi^a (Y_t, A_t, X_t; \hat{\mu}_t^a, \bar{w}_t) - \phi^a (Y_t, A_t, X_t; \hat{\mu}_t^a, \bar{w}_t) - (\mu_0^a - \mu_0^a)^2 \right) \mathcal{F}_{t-1} \right] \\
\leq \mathbb{E} \left[ \left( \phi^a (Y_t, A_t, X_t; \hat{\mu}_t^a, \bar{w}_t) - \phi^a (Y_t, A_t, X_t; \hat{\mu}_t^a, \bar{w}_t) - (\mu_0^a - \mu_0^a)^2 \right) \mathcal{F}_{t-1} \right] - \sqrt{V_a} \\
+ \mathbb{E} \left[ \left( \phi^a (Y_t, A_t, X_t; \hat{\mu}_t^a, \bar{w}_t) - \phi^a (Y_t, A_t, X_t; \hat{\mu}_t^a, \bar{w}_t) - (\mu_0^a - \mu_0^a)^2 \right) \mathcal{F}_{t-1} \right] - \sqrt{V_a} \\
= \mathbb{E} \left[ \left( \phi^a (Y_t, A_t, X_t; \hat{\mu}_t^a, \bar{w}_t) - \phi^a (Y_t, A_t, X_t; \hat{\mu}_t^a, \bar{w}_t) - (\mu_0^a - \mu_0^a)^2 \right) \mathcal{F}_{t-1} \right] - \sqrt{V_a} \\
+ \mathbb{E} \left[ \left( \phi^a (Y_t, A_t, X_t; \hat{\mu}_t^a, \bar{w}_t) - \phi^a (Y_t, A_t, X_t; \hat{\mu}_t^a, \bar{w}_t) - (\mu_0^a - \mu_0^a)^2 \right) \mathcal{F}_{t-1} \right] - \sqrt{V_a} \\
\rightarrow 0,
\]

Here, let us define an event \( \mathcal{E} \) such that
\[
\mathcal{E} = \{ T \left( \mathbb{E}[(\xi_t^a)^2 | \mathcal{F}_{t-1}] - \mathbb{E}[(\xi_t^a)^2] \right) \rightarrow 0 \text{ as } t \rightarrow \infty \},
\]

which occurs with probability one. On the event \( \mathcal{E} \), for all \( \varepsilon > 0 \), there exists \( T_0 \geq 0 \) such that for all \( T > T_0 \),
\[
\exp \left( \frac{TX^2}{2} \left( \mathbb{E}[(\xi_t^a)^2 | \mathcal{F}_{t-1}] - \mathbb{E}[(\xi_t^a)^2] \right) / 2 \right) \leq \exp \left( \frac{X^2}{2} \varepsilon \right).
\]

Because this event occurs with probability one and \( \mathbb{E}[(\xi_t^a)^2 | \mathcal{F}_{t-1}] - \mathbb{E}[(\xi_t^a)^2] \) is bounded, for all \( \varepsilon > 0 \), there exists \( T_0 \) such that for all \( T > T_0 \),
\[
\mathbb{E} \left[ \exp \left( \frac{TX^2}{2} \left( \mathbb{E}[(\xi_t^a)^2 | \mathcal{F}_{t-1}] - \mathbb{E}[(\xi_t^a)^2] \right) / 2 \right) \right] \leq \exp \left( \frac{X^2}{2} \varepsilon \right).
\]

From this result, for all \( T > T_0 \),
\[
\mathbb{E} \left[ \exp \left( \frac{TX^2}{2} \left( \mathbb{E}[(\xi_t^a)^2 | \mathcal{F}_{t-1}] - \mathbb{E}[(\xi_t^a)^2] \right) / 2 \right) \right]^{1/T} \leq \exp \left( \frac{X^2}{2} \varepsilon / T \right).
\]

Therefore, in (26), from the boundedness of the random variables, for a constant \( C > 0 \)
\[
\prod_{t=1}^T \left( \mathbb{E} \left[ \exp \left( \frac{TX^2}{2} \left( \mathbb{E}[(\xi_t^a)^2 | \mathcal{F}_{t-1}] - \mathbb{E}[(\xi_t^a)^2] \right) / 2 \right) \right] \right)^{\frac{1}{2}}
\]

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The definition of $\lambda$.

Proof of Theorem 5.4. There exists some constant $C > 0$ such that for all $1 \leq u \leq \sqrt{T} \min \left\{ \frac{1}{4}C_0, \sqrt{\frac{3C_2}{8C_1}} \right\}$,

$$
\mathbb{P} (Z_T^u > u) = \int \left( \prod_{t=1}^{T} \frac{\exp (\lambda \xi_t^u)}{\mathbb{E} \left[ \exp (\lambda \xi_t^u) \right]} \right) \left( \prod_{t=1}^{T} \frac{\exp (\lambda \xi_t^u)}{\mathbb{E} \left[ \exp (\lambda \xi_t^u) \right]} \right)^{-1} \mathbb{1} [Z_T^u > u] \, d\mathbb{P}
$$

$$
= \int \left( \prod_{t=1}^{T} \frac{\exp (\lambda \xi_t^u)}{\mathbb{E} \left[ \exp (\lambda \xi_t^u) \right]} \right) \exp (-\lambda \sum_{t=1}^{T} \xi_t^u \log \left( \prod_{t=1}^{T} \mathbb{E} \left[ \exp (\lambda \xi_t^u) \right] \right) \mathbb{1} [Z_T^u > u] \, d\mathbb{P}
$$

$$
= \int \left( \prod_{t=1}^{T} \frac{\exp (\lambda \xi_t^u)}{\mathbb{E} \left[ \exp (\lambda \xi_t^u) \right]} \right) \exp (-\lambda Z_T^u + \Psi_T(\lambda)) \mathbb{1} [Z_T^u > u] \, d\mathbb{P}
$$

$$
= \int \left( \prod_{t=1}^{T} \frac{\exp (\lambda \xi_t^u)}{\mathbb{E} \left[ \exp (\lambda \xi_t^u) \right]} \right) \exp (-\lambda U_T(\lambda) - \lambda B_T(\lambda) + \Psi_T(\lambda)) \mathbb{1} [U_T(\lambda) + B_T(\lambda) > u] \, d\mathbb{P},
$$

$$
\leq \int \left( \prod_{t=1}^{T} \frac{\exp (\lambda \xi_t^u)}{\mathbb{E} \left[ \exp (\lambda \xi_t^u) \right]} \right) \exp \left( -\lambda U_T(\lambda) - \frac{\lambda^2}{2} + C(\lambda^3 + \lambda^2/\sqrt{T}) \right)
$$

$$
\cdot \mathbb{1} \left[ U_T(\lambda) + \lambda + C(\lambda V_T + \lambda^2/\sqrt{T}) > u \right] \, d\mathbb{P},
$$

where for the last inequality, we used Lemma 1.2 and Lemma 1.3. Let $\overline{\lambda} = \overline{\lambda}(u)$ be the largest solution of the equation

$$
\lambda + C(\lambda V_T + \lambda^2/\sqrt{T}) = u.
$$

The definition of $\overline{\lambda}$ implies that there exist $C' > 0$ such that, for all $1 \leq u \leq \sqrt{T} \min \left\{ \frac{1}{4}C_0, \sqrt{\frac{3C_2}{8C_1}} \right\}$,

$$
C'u \leq \overline{\lambda}(u) = \frac{2u}{\sqrt{(1 + CV_T)^2 + 4Cu/\sqrt{T} + CV_T + 1}} \leq u
$$

and there exists $\theta \in (0, 1]$ such that

$$
\overline{\lambda}(u) = u - C(\overline{\lambda} V_T + \overline{\lambda}^2/\sqrt{T})
$$

$$
= u - C\theta(u V_T + u^2/\sqrt{T}) \in \left[ C', \sqrt{T} \min \left\{ \frac{1}{4}C_0, \sqrt{\frac{3C_2}{8C_1}} \right\} \right].
$$
Then, we obtain for all $1 \leq u \leq \sqrt{T}\min\left\{ \frac{1}{4}C_0, \sqrt{\frac{3\epsilon_3^2}{8\epsilon_1}} \right\}$,

$$P(Z_T^u > u) \leq \exp\left(C \lambda^3 T^{-1/2} + \lambda^2 T \right) \int \prod_{t=1}^T \frac{\exp \left( \frac{1}{\lambda^2} \right)}{\mathbb{E} \left[ \exp \left( \frac{1}{\lambda^2} \right) \right]} \exp \left( -\lambda U_T(\lambda) \right) \mathbb{I}_{U_T(\lambda) > 0} d\mathbb{P}.$$ 

Here, we have

$$\int \prod_{t=1}^T \frac{\exp \left( \frac{1}{\lambda^2} \right)}{\mathbb{E} \left[ \exp \left( \frac{1}{\lambda^2} \right) \right]} \exp \left( -\lambda U_T(\lambda) \right) \mathbb{I}_{U_T(\lambda) > 0} d\mathbb{P} = \mathbb{E} \left[ \prod_{t=1}^T \frac{\exp \left( \frac{1}{\lambda^2} \right)}{\mathbb{E} \left[ \exp \left( \frac{1}{\lambda^2} \right) \right]} \exp \left( -\lambda U_T(\lambda) \right) \mathbb{I}_{U_T(\lambda) > 0} \right].$$

We also define another measure $\tilde{P}_\lambda$ as

$$d\tilde{P}_\lambda = \frac{\prod_{t=1}^T \exp \left( \frac{1}{\lambda^2} \right)}{\mathbb{E} \left[ \exp \left( \frac{1}{\lambda^2} \right) \right]} d\mathbb{P} = \frac{\exp \left( \lambda \sum_{t=1}^T \xi_t^2 \right)}{\mathbb{E} \left[ \exp \left( \lambda \sum_{t=1}^T \xi_t^2 \right) \right]} d\mathbb{P}.$$

Note that $\tilde{P}_\lambda$ is a probability measure, as the following holds

$$\int d\tilde{P}_\lambda = \int \frac{\exp \left( \lambda \sum_{t=1}^T \xi_t^2 \right)}{\mathbb{E} \left[ \exp \left( \lambda \sum_{t=1}^T \xi_t^2 \right) \right]} d\mathbb{P} = \frac{1}{\mathbb{E} \left[ \exp \left( \lambda \sum_{t=1}^T \xi_t^2 \right) \right]} \mathbb{E} \left[ \exp \left( \lambda \sum_{t=1}^T \xi_t^2 \right) \right] = 1.$$

We further denote $\tilde{E}_\lambda$ as the expectation under the measure $\tilde{P}_\lambda$. In the same way as (37) and (38) in Fan et al. (2013), it is easy to see that

$$\mathbb{E} \left[ \prod_{t=1}^T \frac{\exp \left( \frac{1}{\lambda^2} \right)}{\mathbb{E} \left[ \exp \left( \frac{1}{\lambda^2} \right) \right]} \exp \left( -\lambda U_T(\lambda) \right) \mathbb{I}_{U_T(\lambda) > 0} \right] = \mathbb{E} \left[ \prod_{t=1}^T \frac{\exp \left( \frac{1}{\lambda^2} \right)}{\mathbb{E} \left[ \exp \left( \frac{1}{\lambda^2} \right) \right]} \exp \left( -\lambda U_T(\lambda) \right) \mathbb{I}_{U_T(\lambda) > 0} \right]$$ 

$$= \mathbb{E} \left[ \exp \left( \lambda \sum_{t=1}^T \xi_t^2 \right) \right] \mathbb{E} \left[ \prod_{t=1}^T \frac{\exp \left( \frac{1}{\lambda^2} \right)}{\mathbb{E} \left[ \exp \left( \frac{1}{\lambda^2} \right) \right]} \exp \left( -\lambda U_T(\lambda) \right) \mathbb{I}_{U_T(\lambda) > 0} \right]$$ 

$$= \mathbb{E} \left[ \exp \left( \lambda \sum_{t=1}^T \xi_t^2 \right) \right] \int_0^\infty \lambda \exp \left( -\lambda y \right) \tilde{P}_\lambda(0 < U_T(\lambda) < y) dy.$$ 

Besides, for a standard Gaussian random variable $\mathcal{N}$,

$$\mathbb{E} \left[ \exp \left( -\lambda \mathcal{N} \right) \mathbb{I}_{\mathcal{N} > 0} \right] = \int_0^\infty \lambda \exp \left( -\lambda y \right) \mathbb{P}(0 < \mathcal{N} < y) dy.$$ 

(30)

Then, from (29) and (30),

$$\left| \mathbb{E}_\lambda \left[ \exp \left( -\lambda U_T(\lambda) \right) \mathbb{I}_{U_T(\lambda) > 0} \right] - \mathbb{E} \left[ \exp \left( -\lambda \mathcal{N} \right) \mathbb{I}_{\mathcal{N} > 0} \right] \right| \leq 2 \sup_g \left| \tilde{P}_\lambda(U_T(\lambda) \leq g) - \Phi(g) \right|.$$ 

(31)
Therefore,
\[
\mathbb{P}(Z_T^u > u) 
\leq \frac{\mathbb{E}\left[\exp\left(\lambda \sum_{t=1}^T \xi_t^u\right)\right]}{\prod_{t=1}^T \mathbb{E}\left[\exp(\lambda \xi_t^u)\right]} \cdot \left(1 - \Phi(\bar{\lambda})\right) + \bar{\lambda}\left(1 - \Phi(\bar{\lambda})\right)\exp\left(C \frac{\bar{\lambda}^2}{\sqrt{T} + \bar{\lambda}^2 V_T}\right)
\]
\[
\leq \frac{\mathbb{E}\left[\exp\left(\lambda \sum_{t=1}^T \xi_t^u\right)\right]}{\prod_{t=1}^T \mathbb{E}\left[\exp(\lambda \xi_t^u)\right]} \cdot \left(1 - \Phi(\bar{\lambda})\right) \exp\left(C \frac{\bar{\lambda}^2}{\sqrt{T} + \bar{\lambda}^2 V_T}\right),
\]
where \(c = \sqrt{2\pi(1 + C')/C'}\), and \(\bar{\lambda}\) is chosen to be \(\bar{\lambda} \geq (1 + \lambda c)\) (Note that \(\bar{\lambda} \geq C'\) from (27)).

From Lemma I.4 for any \(\varepsilon > 0\), there exists \(T_0 > 0\) such that for all \(T \geq T_0\),
\[
\mathbb{E}\left[\exp\left(\lambda \sum_{t=1}^T \xi_t^u\right)\right] \leq \exp\left(\tilde{C}_2\bar{\lambda}^4 \big/ T + \tilde{C}_3\bar{\lambda}^3 \big/ \sqrt{T} + \tilde{C}_4T_0 + \varepsilon \bar{\lambda}^2\right).
\]

In summary, by (31) and (32), for all \(1 \leq u \leq \sqrt{T} \min \left\{\frac{1}{C_0}, \sqrt{\frac{\bar{\lambda}^2}{\pi T}}\right\}\),
\[
\mathbb{P}(Z_T^u > u) \leq \tilde{C}\bar{\lambda}\exp\left(\tilde{C}_2\bar{\lambda}^4 \big/ T + \tilde{C}_3\bar{\lambda}^3 \big/ \sqrt{T} + \tilde{C}_4T_0 + \varepsilon \bar{\lambda}^2\right).
\]

Next, we compare \(1 - \Phi(\bar{\lambda})\) with \(1 - \Phi(u)\). Recall the following upper bound and lower bound on \(1 - \Phi(x) = \Phi(-x)\):
\[
\frac{1}{\sqrt{2\pi(1 + x)}} \exp\left(-\frac{x^2}{2}\right) \leq \Phi(-x) \leq \frac{1}{\sqrt{\pi(1 + x)}} \exp\left(-\frac{x^2}{2}\right), x \geq 0.
\]

For all \(1 \leq u \leq \sqrt{T} \min \left\{\frac{1}{C_0}, \sqrt{\frac{\bar{\lambda}^2}{\pi T}}\right\}\),
\[
1 \leq \frac{\int_{-\infty}^{\bar{\lambda}} \exp(-t^2/2)dt}{\int_0^{\infty} \exp(-t^2/2)dt}
\]
\[
\frac{1}{\sqrt{2\pi(1+\lambda)}} \exp(-\lambda^2/2) \\
\frac{1}{\sqrt{2\pi(1+u)}} \exp(-u^2/2) \\
= \sqrt{2} \frac{1 + u}{1 + \lambda} \exp((u^2 - \lambda^2)/2).
\]

From (28), we have
\[
u_2 - \lambda^2 = (u + \lambda)(u - \lambda) \\
\leq 2u(C\theta(uV_T + u^2/\sqrt{T})) \\
= 2C\theta(u^2 V_T + u^3/\sqrt{T}).
\]

Therefore, with some constant \(\tilde{C}_4 > 0\)
\[
\int_{-\infty}^{\infty} \exp(-t^2/2)dt \leq \exp \left( \tilde{C}_4 \left( u^2 V_T + u^3/\sqrt{T} \right) \right).
\]

We find that
\[
1 - \Phi(\bar{\lambda}) \leq (1 - \Phi(u)) \exp \left( \tilde{C}_4 \left( u^2 V_T + u^3/\sqrt{T} \right) \right).
\]

By combining (33), (34), and (27), for any \(\varepsilon > 0\) all \(1 \leq u \leq \sqrt{T} \min \left\{ \frac{1}{\tilde{C}_5}, \sqrt{\frac{3C^2}{8\varepsilon^2T}} \right\}, \) there exist \(T_0 > 0\) and \(\tilde{C}_5 > 0\) such that for all \(T \geq T_0,\)
\[
\Pr (Z_T > u) \leq \tilde{C}\lambda \exp \left( C \left( \lambda^3/\sqrt{T} + \lambda^2 V_T \right) + \tilde{C}_2 \lambda^4/T + \tilde{C}_5 \lambda^3/\sqrt{T} + \tilde{C}_4 \left( u^2 V_T + u^3/\sqrt{T} + T_0 \right) + \varepsilon u^2 \right)
\leq \tilde{C} u \exp \left( \tilde{C}_5 \left( u^2 V_T + u^3/\sqrt{T} + u^4/T + T_0 \right) + \varepsilon u^2 \right)
= \tilde{C} u \exp \left( \tilde{C}_5 \left( u^2 (V_T + \varepsilon) + u^3/\sqrt{T} + u^4/T + T_0 \right) \right).
\]

Applying the same argument to the martingale \(-Z_T^a\), we conclude the proof.