Properties of Burr distribution and its application to heavy-tailed survival time data

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Abstract. Burr distribution is Burr Type XII distribution which is one among the twelve types of the continuous distributions in Burr system. It has two positive shape parameters, namely $k$ and $c$. It is implied from the probability density function which can be either decreasing or unimodal, and the hazard rate function which can be either decreasing or upside-down bathtub-shaped. The other distributional properties and the moment properties of Burr distribution will be discussed in more detail. By considering these properties, we will study its tail behaviour. To estimate the parameters $k$ and $c$, the maximum likelihood method will be considered. Based on the properties of the data representing the remission time of bladder cancer patients, we infer that Burr distribution is suitable to model the data. The goodness-of-fit using the Kolmogorov–Smirnov test shows that Burr distribution fits well to the data.

Keywords: Burr distribution, distributional properties, heavy tails, moment properties, remission time data

1. Introduction
Normal distribution is the most important and commonly used probability distribution in statistics. It has special properties; that is, its parameters denote its mean and variance directly and its probability density function is symmetrical and bell-shaped [1, 2]. However, it is only suitable for analyzing the symmetrical data. In fact, real data obtained commonly are not always symmetrical. Normal distribution is not recommended for analyzing asymmetrical data because it can result in an inaccurate analysis. Examples of the probability distributions which are suitable for analyzing the asymmetrical data are gamma and skew-normal distributions. Gamma distribution is suitable for analyzing right-skewed data [1], while skew-normal distribution is suitable for analyzing either right- or left-skewed data [3].

The selection of the suitable distribution for analyzing data not only considers its skewness, but also its tail behavior. A distribution which can be an alternative for analyzing the right-skewed and heavy-tailed data is Burr distribution [4]. Burr distribution was first introduced in 1942 by Irving Wingate Burr. It was originally known as Burr Type XII distribution which was one of the twelve types of the continuous distributions in Burr system for modeling lifetime data [5].

Burr distribution has two positive shape parameters, denoted by $k$ and $c$. This results in the probability density function which can be either decreasing or unimodal. In addition, its hazard rate function can be either decreasing or upside-down bathtub-shaped. Hence, it has an important role in survival analysis [6]. It is widely used especially for modeling survival time in medical field. The survival time data
commonly have upside-down bathtub-shaped hazard rate function, for example relapse time of breast cancer patients after surgery [7].

If we want to model data using Burr distribution, information on the parameters is important in order to infer the distributional properties. However, in practice, the parameters are unknown. Therefore, we need to estimate them based on data. In this paper, maximum likelihood method will be considered. This method is powerful for finding the values of the parameters which maximize the likelihood that the distribution produces the data. As an illustration, Burr distribution will be applied to model the data which represent the remission time of bladder cancer patients.

2. Properties of Burr distribution

In 1942, Burr introduced a differential equation [5],

\[
\frac{dF(x)}{dx} = F(x)[1 - F(x)]g(x,F(x))
\]  

(1)

The functions \( F \) and \( g \) are defined on a set \( \{x: 0 \leq F(x) \leq 1\} \) for which the solution of the differential equation 1 exists and the function \( g \) is continuous and positive. For many choices of function \( g \), the differential equation 1 incorporates several properties of the distribution function \( F \) and its formula can be obtained by solving it. Burr obtained the twelve types of the continuous distributions whose distribution functions are given in table 1. These distributions form a system called Burr system.

The most important distribution in Burr system is Burr Type XII distribution which has two positive parameters \( k \) and \( c \). Burr Type XII distribution is discussed in more detail by Burr and has gained special attention [8]. It can be used in the various fields of sciences, including reliability analysis [9, 10], life testing [11], survival analysis [6], actuaries [12], economics [13], forestry [14], hydrology [15], and meteorology [16]. Because of its popularity, Burr Type XII distribution is commonly known as Burr distribution. A random variable \( X \) which has Burr distribution is called Burr random variable.

| Type | \( F(x) \) | Interval |
|------|------------|----------|
| I    | \( x \)    | \( (0,1) \) |
| II   | \( (1 + e^{-x})^{-k} \) | \( (-\infty, \infty) \) |
| III  | \( (1 + x^c)^{-k} \) | \( (0, \infty) \) |
| IV   | \( \{1 + [x^{-c}(c - x)]^{\frac{1}{c}}\}^{-k} \) | \( (0, c) \) |
| V    | \( [1 + ce^{-i\arctan(x)}]^{-k} \) | \( (-\pi/2, \pi/2) \) |
| VI   | \( [1 + ce^{-i\arctanh(x)}]^{-k} \) | \( (-\infty, \infty) \) |
| VII  | \( 2^{-k}[1 + \tanh(x)]^k \) | \( (-\infty, \infty) \) |
| VIII | \( [2\pi^{-\frac{1}{2}} \arctan(e^x)]^{-k} \) | \( (-\infty, \infty) \) |
| IX   | \( 1 - 2[2 + c(1 + e^{-x})^k - 1]^{-1} \) | \( (-\infty, \infty) \) |
| X    | \( (1 - e^{-x^k})^k \) | \( (0, \infty) \) |
| XI   | \( [x - (2\pi)^{-\frac{1}{2}} \sin(2\pi x)]^k \) | \( (0, 1) \) |
| XII  | \( 1 - (1 + x^k)^{-k} \) | \( (0, \infty) \) |
As a probability distribution, Burr distribution has the distributional and moment properties. Its distributional properties which will be discussed are the distribution function, the probability density function, the survival function, and the hazard rate function. Its moment properties include the raw and central moments. From these properties, we will study its tail behavior.

2.1. Distributional properties

From table 1, the distribution function $F$ of Burr random variable $X$ is defined by

$$F(x) = P(X \leq x) = 1 - (1 + x^c)^{-k}, \quad x \geq 0. \quad (2)$$

If $X$ denotes the failure time of an item, the value $F(x)$ denotes the probability that item fails before or at time $x$. Otherwise, the probability that item fails after time $x$ is denoted by $S(x) = P(X > x)$. The function $S$ is called the survival function of $X$ defined by

$$S(x) = 1 - F(x) = (1 + x^c)^{-k}, \quad x \geq 0. \quad (3)$$

Graphs of the distribution function $F$ and the survival function $S$ for various values of the parameters $k$ and $c$ are illustrated in figure 1.

From the distribution function defined in equation 1, the probability density function $f$ of Burr random variable $X$ is defined using differentiation; that is,

$$f(x) = \frac{dF(x)}{dx} = kcx^{c-1}(1 + x^c)^{-(k+1)}, \quad x > 0. \quad (4)$$

The value $f(x)$ does not denote a probability. It denotes the failure rate of the item at time $x$, or the instantaneous probability that item fails at time $x$. Graphs of the probability density function $f$ for various values of the parameters $k$ and $c$ are illustrated in figure 2. Figure 2a shows that various values of $k$ result various shapes of the graphs of the probability density function $f$ when the value of $c$ is fixed. Moreover, figure 2b shows that various values of $c$ result various shapes of the graphs of the probability density function $f$ when the value of $k$ is fixed, i.e. decreasing for $0 < c \leq 1$, or unimodal with mode $x_{\text{mod}} = [(c-1)/(kc+1)]^{1/c}$ for $c > 1$. Therefore, $k$ and $c$ are shape parameters. On the other hands, figure 2a and figure 2b also indicate that Burr distribution has right-skewed and heavy-tailed probability density function. This indication will be discussed later in subsection 2.3.

![Figure 1](image1.png)

**Figure 1.** Graphs of the distribution function $F$ and the survival function $S$ of Burr random variable $X$, (a) for a fixed value of $c$ and various values of $k$, and (b) for a fixed value of $k$ and various values of $c$. 
Figure 2. Graphs of the probability density function $f$ of Burr random variable $X$, (a) for a fixed value of $c$ and various values of $k$, and (b) for a fixed value of $k$ and various values of $c$.

Figure 3. Graphs of the hazard rate function of Burr random variable $X$ (a) upside-down bathtub-shaped for $c > 1$ and (b) decreasing for $0 < c \leq 1$.

If we know that the item has not failed until time $x$, the rate that item fails at time $x$ instantaneously is denoted by $h(x)$. The function $h$ is the hazard rate function of Burr random variable $X$ defined by

$$h(x) = \frac{f(x)}{S(x)} = \frac{kcx^{-1}(1+x^c)^{-1(k+1)}}{(1+x^c)^{-k}} = \frac{kcx^{-1}}{1+x^c}, \quad x > 0. \quad (5)$$

Graphs of the hazard rate function $h$ can be upside-down bathtub-shaped for $c > 1$ or decreasing for $0 < c \leq 1$. They are illustrated in figure 3.

Because Burr distribution has the hazard rate function which can be either decreasing or upside-down bathtub-shaped, Burr distribution has important role widely in survival analysis. Burr distribution can be used to model the survival time whose hazard rate function decreases, for example the survival time of bullet-hit soldiers undergoing surgery [17]. In addition, it can also be used to model the survival time whose hazard rate function is upside-down bathtub-shaped, for example the relapse time of breast cancer patients after surgery. In that case, the recurrence rate increases drastically over a time period due to the risk of infection after surgery, then decreases slowly during therapy [7].

2.2. Moment properties

For a fixed positive integer $m$, the $m$th raw moment of Burr random variable $X$ is defined by

$$E(X^m) = \int_{-\infty}^{\infty} x^mf(x)dx = k\int_{0}^{\infty} x^m[(1+x^c)^{-1}]^{k+1} \cdot cx^{-2} \cdot dx. \quad (6)$$
Let \( u = (1 + x^c)^{-1} \) or \( x = [(1 - u)/u]^{1/c} \), then \( du = -cx^{-1}(1 + x^c)^{-2} dx = -cx^{-1}u^{-2} dx \) or \( cx^{-1}dx = -u^{-2}du \). It is clear that \( u \to 1 \) as \( x \to 0 \), and \( u \to 0 \) as \( x \to \infty \). Therefore, equation 6 becomes

\[
E(X^n) = k \int_0^1 \left( \frac{1-u}{u} \right)^m u^{k-1}(-u^{-2})du = k \int_0^1 u^{(k-m/c)+1}(1-u)^{(1+m/c)-1}du.
\]  

(7)

When \( k-m/c > 0 \) and \( 1+m/c > 0 \) (or when \( m < kc \)), the integral part in equation 7 exists and is equal to \( B(k-m/c, 1+m/c) \), where \( B(\cdot, \cdot) \) is beta function. Because \( B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \) for any \( a, b > 0 \), where \( \Gamma(\cdot) \) is gamma function, the \( m \)th raw moment of \( X \) can be written as

\[
E(X^m) = kB(k-m/c, 1+m/c) = \frac{\Gamma(k-m/c)\Gamma(1+m/c)}{\Gamma(k)}, \quad m < kc.
\]  

(8)

From equation 8, for \( m = 1 \), we obtain the first raw moment which is called mean and denoted by \( \mu \). Therefore, the mean of \( X \) is defined by

\[
\mu = E(X) = \frac{\Gamma(k-1/c)\Gamma(1+1/c)}{\Gamma(k)}, \quad kc > 1.
\]  

(9)

For \( m = 2 \), we obtain the second raw moment \( E(X^2) \). On the other hands, the second central moment or variance can be defined as \( \sigma^2 = E[(X - \mu)^2] = E(X^2) - \mu^2 \). Hence, we obtain

\[
\sigma^2 = \frac{\Gamma(k-2/c)\Gamma(1+2/c)\Gamma(k) - \Gamma^2(k-1/c)\Gamma^2(1+1/c)}{\Gamma^2(k)}, \quad kc > 2.
\]  

(10)

2.3. Tail behavior

According to [18], the tail of a probability distribution (or more properly the right tail) is the portion of the distribution corresponding to large value of the random variable. The large possible values have great impact, so it is important to understand the tail properties of the distribution. A random variable which tends to assign higher probability to larger values is said to be heavier tailed.

There are several ways to classify the tail heaviness of a distribution. One of the ways is based on the existence of all of its raw moments. A distribution is indicated to be light-tailed if its \( m \)th raw moment exists for any positive integer \( m \). Otherwise, it is indicated to be heavy-tailed if its \( m \)th raw moment exists for a certain positive integer \( m \) only. From equation 8, the \( m \)th moment of Burr distribution exists for \( m < kc \) only. Therefore, it can be indicated that Burr distribution is heavy-tailed. The existence of the raw moment of Burr distribution for \( m < kc \) only is also implied by the behavior of its tail function (or survival function). Its tail function \( S(x) = (1 + x^c)^{-k} \) is proportional to the tail function of Pareto distribution [4]. Here, \( S(x) = (1 + x^c)^{-k} - x^{-kc} \) which indicates that the tail of Burr distribution decreases algebraically and, hence, implies that the \( m \)th raw moment of Burr distribution is infinite (or does not exist) for \( m \geq kc \). We can say that the value \( kc \) is the tail index of Burr distribution.

Information about the tail behavior of Burr distribution can also be revealed by its hazard rate function. According to [18], a distribution with decreasing hazard rate function is indicated to be heavy-tailed. From figure 3, we have seen that the hazard rate function of Burr distribution can decrease on \((0, \infty)\), or can have a shape of upside-down bathtub which decreases at large values after reaching a peak. This indicates that Burr distribution is suitable to model the heavy-tailed survival time data whose hazard rate function is either decreasing or upside-down bathtub-shaped.
3. Parameter estimation

When we model the given data, it is important to know the values of the parameters of the model. However, in practice, they are unknown. Therefore, we need to estimate the parameters based on the data. In this section, we will perform the parameter estimation for Burr distribution which can help us in modeling the data in section 4. Let \( X_1, X_2, \ldots, X_n \) denote a random sample of size \( n \) from Burr random variable \( X \) whose parameters \( k \) and \( c \) are positive. Let \( x_1, x_2, \ldots, x_n \) be their corresponding realizations or observation values. The probability that \( X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n \) is the joint probability density function \( L \) of \( X_1, X_2, \ldots, X_n \) defined by

\[
L(k, c; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; k, c) = \prod_{i=1}^{n} k c x_i^{c-1} (1 + x_i^c)^{-(k+1)} = k^n c^n \left( \prod_{i=1}^{n} x_i^{c-1} \right) \left( \prod_{i=1}^{n} (1 + x_i^c)^{-(k+1)} \right).
\]

The function \( L \) is also called the likelihood function of the random sample \( X_1, X_2, \ldots, X_n \). The good estimates for \( k \) and \( c \) can be obtained by finding the values of \( k \) and \( c \) which maximize the probability \( L(k, c; x_1, x_2, \ldots, x_n) \). This probability can be maximized by setting \( \partial L(k, c; x_1, x_2, \ldots, x_n)/\partial k = 0 \) and \( \partial L(k, c; x_1, x_2, \ldots, x_n)/\partial c = 0 \), then solving the resulting equations. We know that the likelihood function \( L \) and its logarithm, i.e. the log-likelihood function \( \ln(L) \), are maximized at the same values of \( k \) and \( c \) [19]. Therefore, it will be easier to solve the equations \( \partial \ln[L(k, c; x_1, x_2, \ldots, x_n)]/\partial k = 0 \) and \( \partial \ln[L(k, c; x_1, x_2, \ldots, x_n)]/\partial c = 0 \), rather than solving the equations \( \partial L(k, c; x_1, x_2, \ldots, x_n)/\partial k = 0 \) and \( \partial L(k, c; x_1, x_2, \ldots, x_n)/\partial c = 0 \). Here, the log-likelihood function \( \ln(L) \) is defined by

\[
\ln[L(k, c; x_1, x_2, \ldots, x_n)] = \ln(k^n) + \ln(c^n) + \ln \left( \prod_{i=1}^{n} x_i^{c-1} \right) + \ln \left( \prod_{i=1}^{n} (1 + x_i^c)^{-(k+1)} \right) = n \ln(k) + n \ln(c) + (c - 1) \sum_{i=1}^{n} \ln x_i - (k + 1) \sum_{i=1}^{n} \ln(1 + x_i^c). \tag{12}
\]

Let the likelihood function, and the log-likelihood function, have the maximum value at \( \hat{k} \) and \( \hat{c} \). When we evaluate \( \partial \ln[L(k, c; x_1, x_2, \ldots, x_n)]/\partial k = 0 \) and \( \partial \ln[L(k, c; x_1, x_2, \ldots, x_n)]/\partial c = 0 \) at \( \hat{k} \) and \( \hat{c} \), we have

\[
0 = \frac{\partial \ln[L(k, c; x_1, x_2, \ldots, x_n)]}{\partial k} \bigg|_{k = \hat{k}, c = \hat{c}} = \frac{n}{\hat{k}} - \sum_{i=1}^{n} \ln(1 + x_i^\hat{c}) \tag{13}
\]

and

\[
0 = \frac{\partial \ln[L(k, c; x_1, x_2, \ldots, x_n)]}{\partial c} \bigg|_{k = \hat{k}, c = \hat{c}} = \frac{n}{\hat{c}} + \sum_{i=1}^{n} \ln(x_i) - (\hat{k} + 1) \sum_{i=1}^{n} \frac{x_i^\hat{c} \ln(x_i)}{1 + x_i^\hat{c}} \tag{14}
\]

and we obtain

\[
\hat{k} = \frac{n}{\sum_{i=1}^{n} \ln(1 + x_i^\hat{c})} \tag{15}
\]
The solutions of \( \hat{k} \) and \( \hat{c} \) from equation 15 and equation 16 are the maximum likelihood estimates for the parameters \( k \) and \( c \), respectively. However, these equations cannot be solved analytically. It is more convenient to use numerical method, such as Newton–Raphson method, to solve these equations numerically.

4. Application to real data

In this section, a real data set is considered as the illustration of the application of Burr distribution. The data set consists of 137 observations which represent the remission time (in years) of bladder cancer patients [17]. The descriptive statistics of the data are given in table 2 and the histogram is given in figure 4a. The comparison between the mean, median, and modes shows that the distribution of the data is asymmetrical. It can be inferred from the positive value of the skewness that the data are right-skewed. The large value of the kurtosis shows that it is also heavy-tailed. We can see these properties of the data from the histogram in figure 4a. Therefore, Burr distribution can be used to model the data because it can capture the distributional properties and the tail behavior of the data.

| Statistic | Value   | Statistic | Value   |
|-----------|---------|-----------|---------|
| Mean      | 0.779   | Minimum   | 0.007   |
| Median    | 0.521   | Maximum   | 6.588   |
| Modes     | 0.168, 0.224, 0.252, 0.280, 0.361, 0.443, 0.451 | 0.25-quantile | 0.280 |
| Variance  | 0.743   | 0.50-quantile | 0.521 |
| Skewness  | 3.248   | 0.75-quantile | 0.983 |
| Kurtosis  | 18.464  |           |         |

![Figure 4](image-url)
To estimate the parameters $k$ and $c$ of Burr distribution, we use the Newton–Raphson method to solve equation 15 and equation 16, simultaneously, using Wolfram Mathematica version 11.1. With computational time for 0.106 seconds, we obtain $\hat{k} = 2.065$ and $\hat{c} = 1.434$ as the estimates for the parameters $k$ and $c$ of Burr distribution, respectively, based on the remission time data. The fit between the data and Burr distribution can be observed graphically in figure 4b and figure 4c. Figure 5b shows that the probability density function of Burr distribution fits well to the remission time data which are right-skewed and heavy-tailed. Moreover, Figure 5c shows that the distribution function of Burr distribution fits well to the empirical distribution function based on the remission time data.

Furthermore, the Kolmogorov–Smirnov test can show that Burr distribution fits well to the remission time data. Let $F$ be the actual distribution function of the data. Suppose that $\hat{F}$ is the estimated distribution function of Burr distribution with $\hat{k} = 2.065$ and $\hat{c} = 1.434$, and $F_n$ is the empirical distribution function based on the data. To test the hypothesis $H_0: \hat{F} = F$ against the alternative hypothesis $H_1: \hat{F} \neq F$ using Kolmogorov–Smirnov test, we use the test statistic

$$
KS = \sup \{|\hat{F}(x) - F_n(x)|: -\infty < x < \infty\}.
$$

At the significance level $\alpha = 0.05$, $H_0$ will be rejected if $KS > KS_{0.95}$, where $KS_{0.95} = 0.116$ is the 0.95th quantile of the test statistic $KS$, or if $p$-value $< \alpha$. It can be obtained that $KS = 0.034 < 0.116 = KS_{0.95}$, and $p$-value $0.996 > 0.05 = \alpha$. Therefore, $H_0$ fails to be rejected. Hence, it can be concluded that the remission time data have Burr distribution with the estimated parameters $\hat{k} = 2.065$ and $\hat{c} = 1.434$.

To perform how good Burr distribution as the model of the remission time data, we can generate a random sample from Burr distribution which have same size with the real (remission time) data. By generating a random sample $u_1, u_2, \ldots, u_n$ of size $n = 137$ from uniform distribution over the interval $(0,1)$, hence, we obtain the generated data $x_1, x_2, \ldots, x_n$ from Burr distribution using the quantile transformation $x_i = [(1-u_i)^{1/k} - 1]^{1/c}$, $i = 1, 2, \ldots, n$. Histogram and empirical distribution function of the generated data are compared in figure 5. We can see that the generated data do not differ from the real data.

5. Conclusion
Burr distribution was originally known as Burr Type XII distribution which was one of the twelve types of the continuous distributions in Burr system. It has two shape parameters $k$ and $c$, which implies that
its probability density function and hazard rate function can be either decreasing or non-monotone. This non-monotone hazard rate function has an important role in survival analysis. On the other hands, Burr distribution has a certain moment only because of its tail behavior. It has a tail index like Pareto distribution. This fact implies that Burr distribution is heavy-tailed. Therefore, it can be an alternative model for a heavy-tailed survival time data. By using the maximum likelihood method, the parameter estimates of Burr distribution cannot be computed analytically, so numerical methods are required. The Kolmogorov–Smirnov test shows that Burr distribution fits well to the data which represent the remission time of bladder cancer patients.

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