Non-unitary HD gravity classically equivalent to Einstein gravity
and its Newtonian limit

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Runaway solutions can be avoided in fourth order gravity by a doubling
of the matter operator algebra with a symmetry constraint with respect to
the exchange of observable and hidden degrees of freedom together with the
change in sign of the ghost and the dilaton fields. The theory is classically
equivalent to Einstein gravity, while its non-unitary Newtonian limit is shown
to lead to a sharp transition, around $10^{11}$ proton masses, from the wavelike
properties of microscopic particles to the classical behavior of macroscopic
bodies, as well as to a trans-Planckian regularization of collapse singularities.
A unified reading of ordinary and black hole entropy emerges as entangle-
ment entropy with hidden degrees of freedom. The emergent picture gives a
substantial agreement with B-H entropy and Hawking temperature.

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I. INTRODUCTION

A possible way out of the so called information loss paradox [1,2] emerging from black hole physics [3] consists in assuming a fundamental non-unitarity [4–8]. In fact it is natural to expect that the decoherence due to black hole formation and evaporation should give rise to a significant modification of the dynamical evolution laws: ”For almost any initial quantum state, one would expect there to be a nonvanishing amplitude for black hole formation and evaporation to occur - at least at a highly microscopic (e.g., Planckian) scale - thereby giving rise to a nonvanishing probability for evolution from pure states to mixed states” [6]. Although such an evolution is incompatible with a cherished principle of quantum theory, which postulates a unitary time evolution of a state vector in a Hilbert space, the crucial issue is to assess if it necessarily gives rise to a loss of quantum coherence or to violations of energy-momentum conservation so large as to be incompatible with ordinary laboratory physics [4–8]. Arguments for such violations were given, starting from the assumption that the effective evolution law governing laboratory physics has a Markovian character [4,5]. On the contrary one would expect that an effective evolution law modeling the process of black hole formation and evaporation, far from being local in time, should retain a long term “memory” [6,8]. In particular the basic idea of the non-Markovian models considered in Ref. [6] is to have the given system interacting with a ”hidden system” with ”no energy of its own and therefore... not... available as either a net source or a sink of energy”.

On the other hand a mechanism for large entropy production in gravitational collapses should most naturally operate in the high curvature region, where one may expect new physics to emerge, while connecting it with the event horizon is somehow puzzling, as the physics on such a manifold has nothing peculiar for a free falling observer. Of course a quantitative model of Bekenstein-Hawking (B-H) entropy [8], along these lines, has to refer to the collapsed matter and, in order to do that, it has to include a mechanism for the elimination of the singularity. This does not mean that one can not identify the entropy carried by Hawking radiation as coming from the horizon within a local viewpoint: the
entropy growth outside the horizon, instead of being directly connected with an entropy produced by a strongly non-unitary dynamics in the region close to the classical singularity, is locally seen as a transformation of entanglement entropy into von Neumann entropy. In fact the relative character of the degrees of freedom involved in a given entanglement entropy is present even in flat space-times, where it can be exhibited explicitly [10]. Of course, in trying to pass from the region close to the horizon, where conventional quantum field theory in curved space-times is expected to work as a good approximation, to the region close to the classical singularity, we have to pay a price. In the absence of a full theory of quantum gravity, we have to rely on partially heuristic arguments and some guessing work, which we intend to show can be carried out by rather natural assumptions.

In looking for a non-unitary theory avoiding the collapse singularity, we are going to start from higher derivative (HD) gravity. From a purely cosmological viewpoint it achieved great popularity since an inflationary solution was obtained without invoking phase transitions in the very early universe, from a field equation containing only geometric terms [11]. More recently a renewed attention towards HD gravity was sparked by the appearance of HD gravitational terms in the low-energy effective action of string theory and in the holographic renormalization group, as well as by a growing interest in the study of brane worlds in HD gravity [12]. However, although HD theories of gravity are natural generalizations of Einstein gravity, already on the classical level they are unstable for the presence of negative energy fields giving rise to runaway solutions. On the quantum level, as to unitarity, a possible optimistic conclusion is that ”the S-matrix will be nearly unitary [13]” [14]. The crucial obstacle in trying to define HD gravity as a sound physical theory, namely the presence of ghosts, seems in fact to be a strong indication, on one side, of non-unitarity and, on the other, of a possible mechanism for avoiding singularities, thanks to short range repulsive terms.

Here a specific non-unitary realization of HD gravity is shown to be compatible with the wavelike properties of microscopic particles, as well as with the assumption of a gravity-induced emergence of classicality [15–23], and seems to give strong indications for the elim-
ination of singularities on a trans-Planckian scale. Parenthetically we are encouraged in our extrapolations by the success of inflationary models, implicitly referring to these scales [24]. The present setting suggests that B-H entropy may be identified with the von Neumann entropy of the collapsed matter, or equivalently with the entanglement entropy between matter and hidden degrees of freedom, both close to the smoothed singularity. This viewpoint is corroborated by the attractive features of the Newtonian limit of the model. In this attempt of evaluating relevant physical quantities by an incomplete theory, and in particular by its Newtonian limit, we are encouraged by well-known precedents, like the amazing quantitative agreement between the analysis of John Mitchell in 1784 and the modern notion of a black hole.

A bonus of the present non-unitary model is the possibility of a unified notion, as von Neumann entropy, both for B-H and ordinary thermodynamic entropy of closed systems. This is not irrelevant, as, "...in order to gain a better understanding of the degrees of freedom responsible for black hole entropy, it will be necessary to achieve a deeper understanding of the notion of entropy itself. Even in flat space-time, there is far from universal agreement as to the meaning of entropy – particularly in quantum theory – and as to the nature of the second law of thermodynamics” [8].

II. STABLE FOURTH ORDER GRAVITY

Long ago deWitt [13] and Stelle [25] analyzed the improved ultraviolet behavior of HD gravity theories stemming from cancellations that are ”analogous to the Pauli-Villars regularization of other field theories” [25]. These cancellations are precisely due to the presence of negative energy fields, which in their turn are the source of instability: energy can flow from negative energy degrees of freedom to positive energy ones and one can have runaway solutions.

In Ref. [14] a remedy for the ghost problem, leading to a non-unitary theory, was suggested by a suitable redefinition of the euclidean path integral. In this paper we mean to
propose an approach directly in real space-time, thus avoiding analytic continuation, which amounts to a very tricky operation outside the realm of a fixed flat geometry. Like in Ref. [14], classical instability is cured at the expense of unitarity. Before treating the physically relevant case, we consider first a simpler fourth order theory for a scalar field $\phi$, which has the same ghostly behavior as HD gravity [14]. Its action is

$$S = \int d^4x \left[ -\frac{1}{2} \phi \Box (\Box - \mu^2) \phi - \lambda \phi^4 + \alpha \psi^\dagger \psi \phi \right] + S_{\text{mat}} [\psi^\dagger, \psi] ,$$

with the inclusion of a matter action $S_{\text{mat}}$ and an interaction with matter, where $\psi^\dagger \psi$ is a shorthand notation for a quadratic scalar expression in matter operators. Defining

$$\phi_1 = \frac{(\Box - \mu^2) \phi}{\mu}, \phi_2 = \frac{\Box \phi}{\mu},$$

the action can be rewritten as

$$S [\phi_1, \phi_2, \psi^\dagger, \psi] = S_{\text{mat}} [\psi^\dagger, \psi]$$

$$+ \int d^4x \left[ \frac{1}{2} \phi_1 \Box \phi_1 - \frac{1}{2} \phi_2 (\Box - \mu^2) \phi_2 - \lambda \left( \frac{\phi_1 - \phi_2}{\mu} \right)^4 + \frac{\alpha}{\mu} \psi^\dagger \psi (\phi_2 - \phi_1) \right].$$

The action of $\phi_2$ has the wrong sign, which classically means that the energy of the $\phi_2$ field is negative. If there were no interaction terms, this negative energy wouldn’t matter because each of the fields would live in its own world and the positive and negative energy worlds would not communicate with each other. However, if there are interaction terms, like $\phi^4$ or $\psi^\dagger \psi \phi$, energy can flow from negative to positive energy degrees of freedom, and one can have runaway solutions, with the positive energy of $\phi_1$ and the negative energy of $\phi_2$ both increasing exponentially in time [14].

This toy model shares with HD gravity theories some of the mentioned improvements on the ultraviolet behavior. In fact there is a complete cancellation of all infinities coming from the $\psi^\dagger \psi \phi$ interaction and corresponding to self-energy and vertex graphs [25], owing to the difference in sign between $\phi_1$ and $\phi_2$ propagators. A key feature of the non-interacting theory ($\lambda = \alpha = 0$), making it classically viable, can be considered to be its symmetry under the transformation $\phi_2 \rightarrow -\phi_2$, by which symmetrical initial conditions, i.e. with $\phi_2 = \dot{\phi}_2 = 0$,
produce symmetrical solutions, thus in particular avoiding the runaway ones. We are going now to extend this symmetry to the interacting theory. If one symmetrizes the action (3) as it is, this eliminates the direct interaction between the ghost field and the matter altogether and then the mentioned cancellations. A possible procedure to get a symmetric action while keeping Pauli-Villars-like cancellations is suggested by previous attempts \cite{14} and by the information loss paradox \cite{6}, both pointing to a non-unitary theory, where tracing out hidden degrees of freedom results in general in mixed states. In particular the most natural way to make the hidden degrees of freedom ”not... available as either a net source or a sink of energy” \cite{6} is to constraint them to be an exact copy of the observable ones. Accordingly we introduce a (meta-)matter algebra that is the product of two equivalent copies of the observable matter algebra, respectively generated by the $\psi^\dagger, \psi$ and $\tilde{\psi}^\dagger, \tilde{\psi}$ operators, and a symmetrized action

$$S_{Sym} = \frac{1}{2} \left\{ S \left[ \phi_1, \phi_2, \psi^\dagger, \psi \right] + S \left[ \phi_1, -\phi_2, \tilde{\psi}^\dagger, \tilde{\psi} \right] \right\},$$

(4)

which is invariant under the symmetry transformation

$$\phi_2 \longrightarrow -\phi_2, \ \psi \longrightarrow \tilde{\psi}, \ \tilde{\psi} \longrightarrow \psi.$$  

(5)

This duplication is formally analogous to what is done in thermo-field dynamics \cite{27}, where in particular it can be used to describe the irreversible evolution of open systems \cite{28}. If the symmetry constraint is imposed on quantum states, i.e. the state space is restricted to those states $|\Psi\rangle$ that are generated from the vacuum by symmetrical operators, then

$$\langle \Psi | F \left[ \phi_2, \psi^\dagger, \psi \right] |\Psi\rangle = \langle \Psi | F \left[ -\phi_2, \tilde{\psi}^\dagger, \tilde{\psi} \right] |\Psi\rangle \ \forall F.$$  

(6)

This implies that, as usual with constrained theories, allowed states do not give a faithful representation of the original algebra, which is then larger than the observable algebra. In particular the constrained state space cannot distinguish between $F \left[ \psi^\dagger, \psi \right]$ and $F \left[ \tilde{\psi}^\dagger, \tilde{\psi} \right]$, by which the $\tilde{\psi}$ operators are referred to hidden degrees of freedom, according to a standard terminology for non-unitary models \cite{6}, while only the $\psi$ operators represent matter degrees
of freedom. On a classical level the constraint implies that $\psi$ and $\tilde{\psi}$ are to be identified while the $\phi_2$ field vanishes and, as a consequence, the classical constrained action is that of an ordinary second order scalar theory interacting with matter:

$$S_{Cl} = \int d^4x \left[ \frac{1}{2} \phi_1 \Box \phi_1 - \lambda \left( \frac{\phi_2}{\mu} \right)^4 - \frac{\alpha}{\mu} \phi_1 \tilde{\psi} \psi \right] + S_{mat}[\psi, \tilde{\psi}]. \quad (7)$$

Consider then the classical action for a fourth order theory of gravity including matter

$$S = S_G[g_{\mu\nu}] + S_{mat}[g_{\mu\nu}, \psi, \tilde{\psi}]$$

$$= -\int d^4x \sqrt{-g} \left[ \alpha R_{\mu\nu} R^{\mu\nu} - \beta R^2 + \frac{1}{16\pi G} R \right] + \int d^4x \sqrt{-g} L_{mat}, \quad (8)$$

where $L_{mat}$ denotes the matter Lagrangian density in a generally invariant form. In terms of the contravariant metric density $\sqrt{32\pi G} h_{\mu\nu} = \sqrt{-g} g_{\mu\nu} - \eta_{\mu\nu}$, the Newtonian limit of the static field gives

$$h_{00} \sim \frac{1}{r} + \frac{4}{3} \frac{e^{-\mu_0 r}}{r} - \frac{4}{3} \frac{e^{-\mu_2 r}}{r}, \quad (9)$$

where $\mu_0 = [32\pi G (3\beta - \alpha)]^{-1/2}$, $\mu_2 = [16\pi G \alpha]^{-1/2}$, while a complete analysis for the whole metric can be found in Ref. [29]. From Stelle’s linearized analysis, the first term in Eq. (8) corresponds to the usual massless graviton, the second one to a massive scalar and the third one to a negative energy spin-two field. In fact, in analogy with Eq. (2), one can introduce an explicit transformation from the initial field $g_{\mu\nu}$ appearing in the fourth order form of the action to a new metric tensor $\tilde{g}_{\mu\nu}$, a massless scalar field $\chi$ dilatonically coupled to the metric and a spin-two massive field $\phi_{\mu\nu}$, this transformation leading to the second order form [30]. To be specific, following Ref. [30] (see Eq. (6.9) apart from the matter term), the action (8) can be rewritten as the sum of the Einstein-Hilbert action $S_{EH}$ for $\tilde{g}_{\mu\nu}$, an action $S_{gh}$ for the traceless symmetric ghost field $\phi_{\mu\nu}$ and the scalar field $\chi$ coupled to the metric $\tilde{g}_{\mu\nu}$, and a matter action $S_{mat}$, where $g_{\mu\nu}$ is expressed in terms of $\tilde{g}_{\mu\nu}$, $\phi_{\mu\nu}$ and $\chi$ (replacing $g_{\mu\nu}$ by $e^\chi g_{\mu\nu}$ in Eq. (4.12) in Ref. [30]):

$$S \left[ \tilde{g}_{\mu\nu}, \phi_{\mu\nu}, \chi, \psi, \tilde{\psi} \right]$$

$$= S_{EH} [\tilde{g}_{\mu\nu}] + S_{gh} [\tilde{g}_{\mu\nu}, \phi_{\mu\nu}, \chi] + S_{mat} [g_{\mu\nu} (\tilde{g}_{\sigma\tau}, \phi_{\sigma\tau}, \chi), \psi, \tilde{\psi}]. \quad (10)$$
In $S_{gh}$ above the quadratic part in $\phi_{\mu\nu}$ has the wrong sign \[30\], just as for $\phi_2$ in Eq. \(3\). As before, a simple way to get rid of classical instability would be to symmetrize the action with respect to the transformation $\phi_{\mu\nu} \rightarrow -\phi_{\mu\nu}$ and to introduce the symmetry constraint with respect to this transformation. This however would eliminate the corresponding repulsive term in Eq. \(3\), which is a possible candidate in avoiding the singularity in gravitational collapse. Once one accepts non-unitarity, it is rather natural to assume that one can cure the instability, while keeping the short-range repulsive term, by introducing hidden degrees of freedom as above, i.e. from a quantum viewpoint to accept that the operator algebra involved in defining the dynamics is larger than the observable algebra. To be specific, once again, we double the matter algebra by taking a meta-matter algebra which is the product of two copies of the observable matter algebra, respectively generated by the operators $\psi^\dagger, \psi$ and $\tilde{\psi}^\dagger, \tilde{\psi}$. We then define the symmetrized action \[31\]

$$S_{Sym} = \frac{1}{2} \left\{ S \left[ g_{\mu\nu}, \phi_{\mu\nu}, \chi, \psi^\dagger, \psi \right] + S \left[ \bar{g}_{\mu\nu}, -\phi_{\mu\nu}, -\chi, \tilde{\psi}^\dagger, \tilde{\psi} \right] \right\} ,$$

(11)

which is symmetric under the transformation

$$\phi_{\sigma\tau} \rightarrow -\phi_{\sigma\tau}, \chi \rightarrow -\chi, \psi \rightarrow \tilde{\psi}, \tilde{\psi} \rightarrow \psi, \bar{g}_{\mu\nu} \rightarrow \bar{g}_{\mu\nu}.$$  

(12)

Like above, if the state space is restricted to those states $|\Psi\rangle$ that are generated from the vacuum by symmetrical operators, then

$$\langle \Psi | F \left[ g_{\mu\nu}, \phi_{\mu\nu}, \chi, \psi^\dagger, \psi \right] |\Psi\rangle = \langle \Psi | F \left[ \bar{g}_{\mu\nu}, -\phi_{\mu\nu}, -\chi, \tilde{\psi}^\dagger, \tilde{\psi} \right] |\Psi\rangle \quad \forall F.$$  

(13)

Like for the previous toy model, the constrained state space does not distinguish between $F \left[ \psi^\dagger, \psi \right]$ and $F \left[ \tilde{\psi}^\dagger, \tilde{\psi} \right]$, by which the $\tilde{\psi}$ operators are referred to hidden degrees of freedom, while only the $\psi$ operators represent observable matter. On a classical level the constraint implies that $\psi$ and $\tilde{\psi}$ are to be identified, while the $\phi_{\mu\nu}$ and $\chi$ fields vanish and, as a consequence, the classical constrained action is that of ordinary matter coupled to ordinary gravity:

$$S_{Cl} \left[ g_{\mu\nu}, \psi^\dagger, \psi \right] = S_{EH} \left[ \bar{g}_{\mu\nu} \right] + S_{mat} \left[ \bar{g}_{\mu\nu}, \psi^\dagger, \psi \right],$$

(14)
as $S_{gh} [\bar{g}_{\mu\nu}, 0, 0] = 0$ (Eq. (6.9) in Ref. [30]) and $g_{\mu\nu}(\bar{g}_{\sigma\tau}, 0, 0) = \bar{g}_{\mu\nu}$ (Eq. (4.12) in Ref. [30] with $e^\chi g_{\mu\nu}$ replacing $g_{\mu\nu}$). While this modification of fourth order gravity is expected to affect its ultraviolet behavior, still it does not worsen it for one-loop meta-matter to meta-matter amplitudes, at variance with the trivial symmetrization of the original theory. It should also be remarked that one could limit symmetrization to the ghost field only, without involving the scalar field, especially if one were concerned with the cosmological implications of keeping the dilatonic scalar field in the classical action.

A final remark is in order as to the possible rereading of the present model as a bimetric HD theory where two worlds interact only by means of a coupling between the metrics (of the fourth order formalism) [32]. In fact the model can be defined by replacing $\bar{g}_{\mu\nu}$, $-\phi_{\mu\nu}$, and $-\chi$ in the second term of the right hand side of Eq. (11) by three independent fields and adding an interaction, including Lagrange multipliers, leading to the necessary identifications. However we do not commit ourselves to the prevailing view pointing to underlying higher dimensional theories, even though a natural setting could appear to be an extension of the Randall-Sundrum model [33], with two positive tension flat branes separated by one intermediate negative tension flat brane [34], where $\psi$ and $\tilde{\psi}$ meta-matters reside on distinct positive tension branes.

III. NEWTONIAN LIMIT AND GRAVITATIONAL LOCALIZATION

Of course the elimination of classical runaway solutions is only a first step in assessing the consistency of the ensuing non-unitary theory. A further natural step consists in studying its main implications for ordinary laboratory physics. In order to do that, consider the Newtonian limit of such a theory with non-relativistic meta-matter and instantaneous action at a distance interactions. Looking at the signs in Eq. (11), we see the following. The interactions due to the massless graviton field $\bar{g}_{\mu\nu}$ are always attractive, whereas those due to the scalar field $\chi$ are attractive but for the ones between observable and hidden meta-matter; finally those due to the massive field $\phi_{\mu\nu}$ are repulsive within observable and within hidden
meta-matter, due to its ghostly character (see the sign of the third term in Eq. (11)), and are otherwise attractive, since the ghostly character is offset by the difference in sign in its coupling with observable and hidden meta-matter. The corresponding (meta-)Hamiltonian operator is then

\[ H_G = H_0[\psi^\dagger, \psi] + H_0[\tilde{\psi}^\dagger, \tilde{\psi}] \]

acting on the product \( F_\psi \otimes F_{\tilde{\psi}} \) of the Fock spaces of the (non-relativistic counterparts of the) \( \psi \) and \( \tilde{\psi} \) operators. Here two couples of non-relativistic meta-matter operators \( \psi_j^\dagger, \psi_j \) and \( \tilde{\psi}_j^\dagger, \tilde{\psi}_j \) appear for every particle species and spin component, while \( m_j \) is the mass of the \( j \)-th particle species and \( H_0 \) is the matter Hamiltonian in the absence of gravity.

The \( \tilde{\psi}_j \) operator obeys the same statistics as the corresponding operators \( \psi_j \), while \( [\psi, \tilde{\psi}]_- = [\psi, \tilde{\psi}^\dagger]_- = 0 \). Though never appearing in our formulae, the electromagnetic potential in the Coulomb gauge should be included in the original degrees of freedom, even though, in the non-relativistic setting, it is not involved in the gravitational interaction.

With reference to Eq. (11), observe that the action at a distance counterpart of the field-theoretic cancellations mentioned above is the possibility of avoiding normal ordering in the last two terms. It would correspond, in fact, to the subtraction of the finite operator \( G(\mu_0 - 4\mu_2) \sum_j m_j^2 \int dx \psi_j^\dagger(x) \psi_j(x)/12 \) and its hidden correspondent, which in a fixed particle-number space correspond to irrelevant finite constants.

To be specific, the meta-particle state space \( S \) is the subspace of \( F_\psi \otimes F_{\tilde{\psi}} \) including the meta-states obtained from the vacuum \( |0\rangle \rangle = |0\rangle_\psi \otimes |0\rangle_{\tilde{\psi}} \) by applying operators built in terms of the products \( \psi_j^\dagger(x) \tilde{\psi}_j^\dagger(y) \) and symmetrical with respect to the interchange \( \psi^\dagger \leftrightarrow \tilde{\psi}^\dagger \), which, then, have the same number of \( \psi \) and \( \tilde{\psi} \) meta-particles of each species. As the observable
algebra is identified with the $\psi$ operator algebra, expectation values can be evaluated by preliminarily tracing out the $\tilde{\psi}$ operators. In particular, for instance, the most general meta-state corresponding to one-particle states is represented by

$$
|f\rangle\rangle = \int dx\int dy f(x,y)\psi_1^{\dagger}(x)\tilde{\psi}_1^{\dagger}(y)|0\rangle \rangle , \quad f(x,y) = f(y,x).
$$

(16)

This is a consistent definition since $H_G$ generates a group of (unitary) endomorphisms of $S$. A pure $n$-particle state, represented in the traditional setting by

$$
|g\rangle = \int d^n x g(x_1, x_2, ..., x_n)\psi_{j_1}^{\dagger}(x_1)\psi_{j_2}^{\dagger}(x_2)...\psi_{j_n}^{\dagger}(x_n)|0\rangle
$$

(17)

is represented in $S$ by the only meta-state that, by tracing out $\tilde{\psi}$ operators, gives the state $|g\rangle\langle g|$, with $|g\rangle$ as in Eq. (17), namely by

$$
||g\otimes g\rangle\rangle \propto \int d^n x d^n y g(x_1, ..., x_n)g(y_1, ..., y_n)\psi_{j_1}^{\dagger}(x_1)...\psi_{j_n}^{\dagger}(x_n)\tilde{\psi}_{j_1}^{\dagger}(y_1)...\tilde{\psi}_{j_n}^{\dagger}(y_n)|0\rangle \rangle .
$$

(18)

It should be remarked that, when our initial knowledge of the system state is characterized by a density matrix, there is no unique prescription to associate it with a pure meta-state. In such a case one has to consider the possibility of using mixed meta-states to encode our incomplete knowledge.

Considering, for notational simplicity, particles of one and the same species, the time derivative of the matter canonical momentum in a space region $\Omega$ in the Heisenberg picture reads

$$
\frac{d\mathbf{\hat{p}}_\Omega}{dt} = -i\hbar \frac{d}{dt} \int d^3x \psi^{\dagger}(x)\nabla \psi(x) \equiv \left. \frac{d\mathbf{\hat{p}}_\Omega}{dt} \right|_{G=0} + \mathbf{\hat{F}}_G = -\frac{i}{\hbar} \left[ \mathbf{\hat{p}}_\Omega, H_0[\psi^{\dagger}, \psi] \right]
$$

$$
+ \frac{G}{2} m^2 \int \Omega d^3x \psi^{\dagger}(x)\psi(x) \nabla_x \int R^3 dy \frac{\tilde{\psi}(y)\tilde{\psi}(y)}{|x-y|} \left( 1 - \frac{1}{3} e^{-\mu_0|x-y|} + \frac{4}{3} e^{-\mu_2|x-y|} \right)
$$

$$
+ \frac{G}{2} m^2 \int \Omega d^3x \psi^{\dagger}(x)\psi(x) \nabla_x \int R^3 dy \frac{\psi^{\dagger}(y)\psi(y)}{|x-y|} \left( 1 + \frac{1}{3} e^{-\mu_0|x-y|} - \frac{4}{3} e^{-\mu_2|x-y|} \right). \quad (19)
$$

The expectation of the gravitational force can be written as

$$
\left\langle \mathbf{\hat{F}}_G \right\rangle =
$$
\[ \frac{G}{2} m^2 \left\langle \int_{\Omega} dx \psi^\dagger(x) \psi(x) \nabla_x \int_{\Omega} dy \frac{\tilde{\psi}^\dagger(y) \tilde{\psi}(y)}{|x-y|} \left( 1 - \frac{1}{3} e^{-\mu_0|x-y|} + \frac{4}{3} e^{-\mu_2|x-y|} \right) \right\rangle \]
\[ + \frac{G}{2} m^2 \left\langle \int_{\Omega} dx \psi^\dagger(x) \psi(x) \nabla_x \int_{R^3 \setminus \Omega} dy \frac{\tilde{\psi}^\dagger(y) \tilde{\psi}(y)}{|x-y|} \left( 1 - \frac{1}{3} e^{-\mu_0|x-y|} + \frac{4}{3} e^{-\mu_2|x-y|} \right) \right\rangle \]
\[ + \frac{G}{2} m^2 \left\langle \int_{\Omega} dx \psi^\dagger(x) \psi(x) \nabla_x \int_{R^3 \setminus \Omega} dy \frac{\psi^\dagger(y) \psi(y)}{|x-y|} \left( 1 + \frac{1}{3} e^{-\mu_0|x-y|} - \frac{4}{3} e^{-\mu_2|x-y|} \right) \right\rangle, \] (20)

where, on allowed states, the first term vanishes for the antisymmetry of the kernel \( \nabla_x \left[ (1 - e^{-\mu_0|x-y|}/3 + 4e^{-\mu_2|x-y|}/3) / |x-y| \right] \) and the symmetry constraint on the state, while the third one vanishes, as it should be for self-gravitating matter, just as a consequence of the antisymmetry of the corresponding kernel. As is usual with the evaluation of forces between macroscopic bodies, we can then approximate \( \left\langle \psi^\dagger(x) \psi(x) \tilde{\psi}^\dagger(y) \tilde{\psi}(y) \right\rangle \) and \( \left\langle \psi^\dagger(x) \psi(x) \psi^\dagger(y) \psi(y) \right\rangle \) respectively by \( \left\langle \psi^\dagger(x) \psi(x) \right\rangle \left\langle \tilde{\psi}^\dagger(y) \tilde{\psi}(y) \right\rangle \) and \( \left\langle \psi^\dagger(x) \psi(x) \right\rangle \left\langle \psi^\dagger(y) \psi(y) \right\rangle \), as \( x \in \Omega \) and \( y \in R^3 \setminus \Omega \). Finally, as \( \left\langle \psi^\dagger(y) \tilde{\psi}(y) \right\rangle = \left\langle \psi^\dagger(y) \psi(y) \right\rangle \), we get
\[ \left\langle \tilde{F}_G \right\rangle \simeq Gm^2 \int_{\Omega} dx \left\langle \psi^\dagger(x) \psi(x) \right\rangle \nabla_x \int_{R^3 \setminus \Omega} dy \frac{\psi^\dagger(y) \psi(y)}{|x-y|}, \] (21)

denominating the classical aspects of the interaction are the same as for the traditional Newton interaction, consistently with the classical equivalence of the original theory to Einstein gravity [31].

Although we are using the general Newtonian limit (33), it is worthwhile to remark that we are mainly interested to two opposite specialized limits.

The ordinary Newtonian limit, for ordinary laboratory physics, corresponds to taking \( \mu_0, \mu_2 \to \infty \), if \( \mu_0^{-1} \) and \( \mu_2^{-1} \) are assumed, as usual, of the order of the Planck length, in which case the meta-Hamiltonian \( H_G \) can be rewritten in the form
\[ H_G = H[\psi^\dagger, \psi] + H[\tilde{\psi}^\dagger, \tilde{\psi}] - \frac{G}{2} \sum_{j,k} m_j m_k \int dx dy \frac{\psi_j^\dagger(x) \psi_j(x) \tilde{\psi}_j^\dagger(y) \tilde{\psi}_j(y)}{|x-y|}, \] (22)

where \( H[\psi^\dagger, \psi] \) and \( H[\tilde{\psi}^\dagger, \tilde{\psi}] \) respectively include the halved (normal ordered) Newton interaction within observable and hidden meta-matter. In this form we have a well defined
non-unitary model of Newtonian gravity without any free parameter. Tracing out the \( \tilde{\psi} \) operators from the meta-state evolving according to the unitary meta-dynamics generated by \( H_G \) results in a non-Markov non-unitary physical dynamics for the ordinary matter algebra [35].

The trans-Planckian Newtonian limit concerns the use we are going to make of the model with reference to gravitational collapse, where the model replaces the classical singularity with a trans-Planckian structure. To this end we consider the opposite limit \( \mu_0, \mu_2 \to 0 \), leading to a Hamiltonian \( H_G \) as in (22) with \( H \) and \( G/2 \) respectively replaced by \( H_0 \) and \( G \). The rationale for the use of this bold extension of the Newtonian limit outside its typical applicability range, though within a merely heuristic approach, resides in part in the soundness of its physical consequences, as shown in the following.

A general new feature of the model with respect to the usual inclusion of Newtonian gravity in QM is the localization due to the presence of an effective self-interaction. Consider in fact in the traditional setting a physical body in a given quantum state whose wave function \( \Psi_{CM}(X)\Psi_{INT}(x_i - x_j) \) is the product of the wave function of the center of mass and of an internal wave function. In particular \( \Psi_{CM} \) can be chosen, for simplicity, in such a way that the corresponding meta-wave function \( \Psi_{TOT} = \Psi_{CM}(X)\Psi_{INT}(x_i - x_j)\Psi_{CM}(Y)\Psi_{INT}(y_i - y_j) \) can be rewritten as:

\[
\Psi_{TOT} = \tilde{\Psi}_{CM}(\frac{X + Y}{2})\tilde{\Psi}_{INT}(X - Y)\Psi_{INT}(x_i - x_j)\Psi_{INT}(y_i - y_j),
\]

where \( y_i, Y \) denote the hidden correspondents of \( x_i, X \). As to \( \tilde{\Psi}_{INT}(X - Y) \), we choose it as the ground state of the relative motion of the two interpenetrating meta-bodies, which is formally equivalent to the plasma oscillations of two opposite charge distributions. The corresponding potential energy, if the body is spherically symmetric and not too far from being a homogeneous distribution of radius \( \Xi \) and mass \( M \), has the form \( \xi GM^2 f (|X - Y|) \), where

\[
f (r) = \begin{cases} 
-1/r & \text{for } r \geq 2\Xi \\
\frac{1}{2}a r^2/\Xi^3 & \text{for } r \ll \Xi
\end{cases},
\]

\begin{equation}
(24)
\end{equation}
with $\xi = 1/2, 1$ respectively for the ordinary and the trans-Planckian limit, and $\alpha \sim 10^6$ a dimensionless constant. We are interested here to the case of small relative displacements. The relative ground state is represented by

$$\tilde{\Psi}_{\text{INT}}(X - Y) = (\Lambda^2 \pi)^{-3/4} e^{-\frac{|X - Y|^2}{2\Lambda^2}}; \quad \Lambda = (2\hbar^2 \Xi^3 / \alpha \xi GM^3)^{1/4}. \tag{25}$$

Then, if we choose $\Psi_{CM}(X) \propto \exp\left[-X^2/\Lambda^2\right]$, we get

$$\tilde{\Psi}_0(X, Y) = \Psi_{CM}(X)\Psi_{CM}(Y) = \tilde{\Psi}_{\text{INT}}(X - Y)\tilde{\Psi}_{\text{INT}}(X + Y). \tag{26}$$

In particular for body densities $\sim 10^{24} m_p/cm^3$, where $m_p$ denotes the proton mass, $\Lambda \sim (m_p/M)^{1/2} cm$, which shows that the small displacement approximation is acceptable already for $M \sim 10^{12} m_p$, when $\Lambda \sim 10^{-6} cm$, whereas the body dimensions are $\sim 10^{-4} cm$ [35].

Another simple case corresponds to masses lower than $10^{10} m_p$, where the two metabodies can be approximated as point particles and their ground state wave function, in the ordinary Newtonian limit, is

$$\Psi(X - Y) \propto e^{-|X - Y|/a}; \quad a = 4\hbar^2 \xi^{-1}G^{-1}M^{-3} \sim 10^{25} (M/m_p)^{-3} cm, \tag{27}$$

by which gravitational localization, consistently with recent experiments, can be ignored for all practical purposes even for particles much larger than fullerene [36][37]. The ensuing situation corresponds then to a rather sharp localization mass threshold $M_t \sim \hbar^{3/5}G^{-3/10} \rho^{1/10}$, which is very robust with respect to mass density variation.

It is easily seen that the present framework actually is compatible with the way terrestrial gravity appears in QM. A crucial experiment, dating back to 1975, exhibits in fact in a striking manner how terrestrial gravity enters the Schrödinger equation in the usual way, i.e. just as a Coulomb external field [38]. To this end the calculation of the average gravitational force acting over a lump performed above does not suffice since it can explain only e.g. the free fall of a microscopic particle by means of classical equations ( Ehrenfest theorem) where $\hbar$ does not appear.
Consider the problem of a large, for simplicity spherically symmetric, massive body (the Earth) in some irrelevant internal state in interaction with an external microscopic particle. Define the meta-Hamiltonian of the Earth-particle system as

\[
    H = -\frac{\hbar^2}{2M} \sum_{i=1,2} \nabla^2_{R_i} - GM^2 f(|R_1 - R_2|) - \frac{\hbar^2}{2m} \sum_{i=1,2} \nabla^2_{x_i} - GmM \sum_{i,j=1,2} \frac{1}{|x_i - R_j|} \tag{28}
\]

where \(M\) and \(m\) are respectively the mass of the Earth and of the particle, \(R_1, R_2\) and \(x_1, x_2\) respectively the center of meta-mass coordinates of the two Earth and particle copies.

Let’s start with a meta-state of the meta-Earth system corresponding to the fundamental (or a not too highly excited) one with respect to the relative motion of the two copies and choosing the initial CM meta-state of the bound system of the two copies just as above. The localization length is in this case of the order \(\Lambda_{\text{Earth}} \sim 10^{-26}\) cm. Having in mind that the particle is described by a wave packet whose size \(a\) is in any case much larger than \(\Lambda_{\text{Earth}}\), we can approximate the squared modulus of Earth’s meta-wave-function by a product of delta functions \(\delta^3(r) \delta^3(R)\), where \(r, R\) respectively denote the internal and the CM coordinates of the meta-Earth bound system. As a consequence \(x_i - R_j\) in the Newton potential can be replaced by \(x_i\). Of course, since the spreading time of the Earth’s CM wave function over a region of the size \(a \gtrsim 10^{-10}\) cm is given by \(a\Lambda_{\text{Earth}} M/\hbar \gtrsim 10^{19}\) s, the approximation is justified in any physically relevant situation, and actually even much better than what appears from this analysis, as we are ignoring the spreading of the particle wave function. As a result the gravitational interaction enters in the particle dynamics simply by the presence of the usual external Newton potential.

IV. EVOLUTION FROM PURE TO MIXED STATES

It should be stressed that, while in the ensuing dynamics the constraint on the hidden degrees of freedom to have the same average energy as the observable matter avoids them to be ”available as either a net source or a sink of energy”, only the meta-Hamiltonian is strictly conserved. If we include in the physical energy the usual Newtonian interaction between observable degrees of freedom, the physical energy operator
\[ H_{Ph}[\psi^+, \psi] = H_0[\psi^+, \psi] - \frac{G}{2} \sum_{j,k} m_j m_k \int dx dy \frac{\psi_j^+(x) \psi_j(x) \psi_k^+(y) \psi_k(y)}{|x - y|} \quad (29) \]

is not the generator of time evolution. To be specific, in the ordinary Newtonian limit, the generator of the meta-dynamics can be written

\[ H_G = H_{Ph}[\psi^+, \psi] + H_{Ph}[\tilde{\psi}^+, \tilde{\psi}] \quad (30) \]

\[-\frac{G}{4} \sum_{j,k} m_j m_k \int dx dy \left[ \frac{2\psi_j^+(x) \psi_j(x) \psi_k^+(y) \psi_k(y)}{|x - y|} \right] \]

\[ + \frac{G}{4} \sum_{j,k} m_j m_k \int dx dy \left[ : \psi_j^+(x) \psi_j(x) \psi_k^+(y) \psi_k(y) : + \tilde{\psi}_j^+(x) \tilde{\psi}_j(x) \tilde{\psi}_k^+(y) \tilde{\psi}_k(y) : \right] / |x - y|, \]

from which we see that \( H_G \) and \( H_{Ph}[\psi^+, \psi] + H_{Ph}[\tilde{\psi}^+, \tilde{\psi}] \) in general are different only due to correlations. The two sums above have approximately equal expectations and fluctuate around the classical gravitational energy. On one side these energy fluctuations have to be present in any model leading to dynamical wave function localization, which in itself requires a certain injection of energy \[^{[39]}\]. On the other hand these fluctuations, though irrelevant on a macroscopic scale, are precisely what can lead to thermodynamical equilibrium in a closed system if thermodynamic entropy is identified with von Neumann entropy \[^{[7]}\]. In fact, due to the interaction with the hidden degrees of freedom, a pure eigenstate of the ordinary energy \( H_{Ph} \) is expected to evolve into a microcanonical ensemble.

As a simple example showing how a pure state can evolve into a mixed one, consider a free spherically symmetric body of ordinary matter above localization threshold, initially described by a gaussian wave packet, whose size is chosen as above in such a way that the particle-copy system is in its ground state, thus recovering the meta-wave-function \[^{[23]}\] \[^{[40]}\]. The factor depending on the center of meta-mass of the \( \psi \) and \( \tilde{\psi} \) meta-bodies in \( \tilde{\Psi}_0(X, Y) \) \[^{[26]}\], for \( M \gtrsim 10^{12} m_p \), spreads in time as usual for a body of mass \( 2M \), so that after a time \( t \), the meta-wavefunction becomes

\[ \tilde{\Psi}_t(X, Y) \propto \exp \left[ -\frac{|X - Y|^2}{2\Lambda^2} \right] \exp \left[ \frac{-|X + Y|^2/4}{\Lambda^2/2 + i\hbar t/M} \right] \equiv e^{-\alpha|X-Y|^2} e^{-\alpha_1|X+Y|^2}. \quad (31) \]

In order that this be compatible with the assumption that gravity continuously forces localization \[^{[13]} \quad ^{[23]}\], the spreading of the physical state must be the outcome of the entropy
growth. This initially vanishes, as the initial meta-wavefunction (26) is unentangled and then the physical state, obtained by tracing out $Y$, is pure. If one evaluates the physical state $\rho_t(X, X') = \int dY \tilde{\Psi}_t(Y) \tilde{\Psi}_t^*(X', Y)$, one finds that the space probability density reads

$$\rho_t(X, X) = \left[ \frac{8\alpha_0(\alpha_t + \tilde{\alpha}_t)}{\pi(\alpha_t + \tilde{\alpha}_t + 2\alpha_0)} \right]^{3/2} \exp \left[ -\frac{8\alpha_0(\alpha_t + \tilde{\alpha}_t)}{(\alpha_t + \tilde{\alpha}_t + 2\alpha_0)} X^2 \right] \propto \exp \frac{-2\Lambda^2X^2}{\Lambda^4 + 2\hbar^2t^2/M^2}. \quad (32)$$

The spreading is extremely slow, as its typical time, for bodies of density $\sim 10^{24}m_p/cm^3$, is $\sim 10^3$ sec independently from the mass, as can be checked by means of Eqs. (32) and (25). If it is due to entropy growth only, rather than to the spreading of the wave function, the entropy $S_t$ is expected to depend approximately on the ratio between the final and the initial space volumes roughly occupied by the two Gaussian densities, according to

$$S_t \sim K_B \frac{3}{2} \ln \left[ \frac{\alpha_t + \tilde{\alpha}_t + 2\alpha_0}{2(\alpha_t + \tilde{\alpha}_t)} \right], \quad (33)$$

at least for large enough times. (Linear momentum probability density does not depend on time.) Of course this corresponds to the approximation of the mixed state by means of an ensemble of $N$ equiprobable localized states, which is legitimate if $N$ turns out to be large enough. In order to evaluate the entropy of the state represented by $\rho_t(X, X')$ and to check Eq. (33), we use the possibility, in this approximation, of linking the entropy

$$S_t = -K_B \text{Tr} [\rho_t \ln \rho_t] = K_B \ln N \quad (34)$$

with the purity

$$\text{Tr} [\rho_t^2] = 1/N; \quad \rho_t^2(X, X') = \int dX'' \rho_t(X, X'') \rho_t(X'', X'). \quad (35)$$

By an explicit computation we get

$$\text{Tr} [\rho_t^2] = \int dX \rho_t^2(X, X) = \frac{[4\alpha_0(\alpha_t + \tilde{\alpha}_t)]^3}{\left( (2\alpha_t\tilde{\alpha}_t + 6\alpha_t\alpha_0 + 6\tilde{\alpha}_t\alpha_0 + 2\alpha_0^2)^2 - 4(\tilde{\alpha}_t - \alpha_0)^2(\alpha_t - \alpha_0)^2 \right)^{3/2}} \quad (36)$$

and, for large times, one can keep just the leading term in $\alpha_t/\alpha_0$, that is
\[ Tr \left[ \rho_t^2 \right] \sim \left( \frac{\alpha t + \bar{\alpha} t}{2\alpha_0} \right)^{3/2} \]  

which, by using Eqs. (34,35), gives

\[ S_t \sim -K_B \frac{3}{2} \ln \left( \frac{\alpha t + \bar{\alpha} t}{2\alpha_0} \right) = K_B \frac{3}{2} \ln \left( \frac{\Lambda^4 + 4\hbar^2 t^2/M^2}{\Lambda^4} \right) \]  

which differs from the leading term in Eq.(33) by an irrelevant quantity \((3/2)K_B \ln 2\). This validates our view of the free motion of a macroscopic body, at variance with the rather unphysical stationary localized states of the Schrödinger-Newton (S-N) model, whose initial linear momentum uncertainty does not give rise to a spreading of the probability density \([41–45]\). More generally, while that non-linear generalization of QM was considered to be a reasonable mean field approximation of an unspecified theory, by its unitarity it cannot model any fundamental gravitational decoherence. It is remarkable that the S-N model can be actually obtained as the \(N \to \infty\) limit of the \(N\) color generalization of the present Newtonian limit \([46]\).

It should be stressed that the notion of coarse graining entropy, often taken as the starting point in dealing with the quantum foundations of the second law of thermodynamics \([47]\), can be easily connected with the present approach. Consider, for simplicity, a non-degenerate physical state

\[ \rho_{Ph} = \sum_j p_j |j\rangle \langle j|, \quad p_j \in R, \quad p_j = p_k \Rightarrow j = k. \]  

The most general pure meta-state vector giving rise to \(\rho_{ph}\) is

\[ ||\Psi\rangle\rangle_\varphi = \sum_j e^{i\varphi_j} \sqrt{p_j} |j\rangle \langle j|, \]  

where \(|j\rangle \langle j|\) denotes the tensor product of two corresponding vectors in the two Fock spaces and the \(\varphi_j \in [0,2\pi]\) are arbitrary real parameters. The indistinguishability of the corresponding meta-states, due to the restriction of the physical algebra, induces in the meta-state space an unambiguous coarse graining, at variance with the rather vague one in the traditional approaches. To be specific, it is natural to introduce the macro-meta-state
\[ \rho_{CG} \equiv \int \prod_{j} \frac{d\varphi_j}{2\pi} \langle\langle \Psi \rangle\rangle_{\varphi} \langle\Psi\rangle = \sum_{j} p_j |j\rangle \langle j|, \]

(41)
corresponding to the equiprobability of the micro-meta-states \( |\langle\rangle_{\varphi} \rangle\rangle \). The corresponding coarse graining entropy is

\[ S_{CG} = -K_B Tr [\rho_{CG} \ln \rho_{CG}] = -K_B \sum_{j} p_j \ln p_j, \]

(42)
which coincides with the von Neumann entropy of the physical state \( \rho_{Ph} \).

Vice versa, if we assume that a specific pure meta-state \( |\langle\rangle_{\varphi} \rangle\rangle \) is given, the Schmidt decomposition theorem allows us to write it in terms of orthonormal vectors as

\[ |\langle\rangle_{\varphi} \rangle\rangle = \sum_{j} \sqrt{p_j} |j\rangle |j^\prime\rangle, \]

(43)
with the \( p_j \) positive, for simplicity distinct, real numbers. By the symmetry constraint on the meta-state space one can choose the relative phases in such a way that \( |j\rangle \) and \( |j^\prime\rangle \) can be taken as corresponding vectors in the two Fock spaces, thus reproducing \( |\langle\rangle_{\varphi} \rangle\rangle \) in eq. (40) for \( \varphi = 0 \). Although this amounts to the knowledge of a definite microstate, the entropy of the corresponding physical state \( \rho_{Ph} \) is non-vanishing and coincides with the coarse graining entropy of the corresponding macrostate \( \rho_{CG} \). This shows the objective and non-conventional character of the notion of entropy in the present approach, since it does not depend on a subjective characterization based on the notion of a macroscopic observer [47].

V. WAVE FUNCTION REDUCTION

In an interaction representation of the ordinary Newtonian limit, where the free meta-Hamiltonian is \( H[\psi^\dagger, \psi] + H[\tilde{\psi}^\dagger, \tilde{\psi}] \), the time evolution of an initially untangled meta-state \( |\langle\rangle_{0} \rangle\rangle \) is represented by

\[ |\langle\rangle_{t} \rangle\rangle = T \exp \left[ \frac{i}{\hbar} Gm^2 \int dt \int dx dy \frac{\psi^\dagger(x,t) \psi(x,t) \tilde{\psi}^\dagger(y,t) \tilde{\psi}(y,t)}{|x-y|} \right] |\langle\rangle_{0} \rangle\rangle \]

\[ \equiv U(t) |\langle\rangle_{0} \rangle\rangle \equiv U(t) |\Phi(0)\rangle_{\psi} \otimes |\Phi(0)\rangle_{\chi}. \]

(44)
Then, by a Stratonovich-Hubbard transformation \cite{11}, we can rewrite $U(t)$ as

\[
U(t) = \int D[\varphi_1, \varphi_2] \exp \frac{ic^2}{2\hbar} \int dt dx \left[ \varphi_1 \nabla^2 \varphi_1 - \varphi_2 \nabla^2 \varphi_2 \right] \\
T \exp \left[ -i \frac{mc}{\hbar} \sqrt{2\pi G} \int dt dx \left[ \varphi_1(x, t) + \varphi_2(x, t) \right] \psi^\dagger(x, t)\psi(x, t) \right] \\
T \exp \left[ -i \frac{mc}{\hbar} \sqrt{2\pi G} \int dt dx \left[ \varphi_1(x, t) - \varphi_2(x, t) \right] \tilde{\psi}^\dagger(x, t)\tilde{\psi}(x, t) \right] \tag{45}
\]

namely as a functional integral over two auxiliary real scalar fields $\varphi_1$ and $\varphi_2$.

The physical state corresponding to the meta-state \cite{14} is given by

\[
\rho_{Ph}(t) \equiv Tr_\tilde{\psi} \left| \Phi(t) \right\rangle \left\langle \Phi(t) \right| = \sum_k \tilde{\psi} \left| k \right\rangle \left\langle k \right| \Phi(t) \left\langle \Phi(t) \right| k \rangle \tag{46}
\]

and, by using Eq. \cite{15}, we can write

\[
\tilde{\psi} \left| k \right\rangle \left\langle \Phi(t) \right| = \int D[\varphi_1, \varphi_2] \exp \frac{ic^2}{2\hbar} \int dt dx \left[ \varphi_1 \nabla^2 \varphi_1 - \varphi_2 \nabla^2 \varphi_2 \right] \\
T \exp \left[ -i \frac{mc}{\hbar} \sqrt{2\pi G} \int dt dx \left[ \varphi_1(x, t) - \varphi_2(x, t) \right] \tilde{\psi}^\dagger(x, t)\tilde{\psi}(x, t) \right] \left| \Phi(0) \right \rangle_{\tilde{\psi}} \\
T \exp \left[ -i \frac{mc}{\hbar} \sqrt{2\pi G} \int dt dx \left[ \varphi_1(x, t) + \varphi_2(x, t) \right] \psi^\dagger(x, t)\psi(x, t) \right] \left| \Phi(0) \right \rangle_{\tilde{\psi}}. \tag{47}
\]

Then the final expression for the physical state at time $t$ is given by

\[
\rho_{Ph}(t) = \int D[\varphi_1, \varphi_2, \varphi'_1, \varphi'_2] \exp \frac{ic^2}{2\hbar} \int dt dx \left[ \varphi_1 \nabla^2 \varphi_1 - \varphi_2 \nabla^2 \varphi_2 - \varphi'_1 \nabla^2 \varphi'_1 + \varphi'_2 \nabla^2 \varphi'_2 \right] \\
\psi \left| \Phi(0) \right\rangle T^{-1} \exp \left[ \frac{mc}{\hbar} \sqrt{2\pi G} \int dt dx \left[ \varphi'_1 - \varphi'_2 \right] \psi^\dagger \psi \right] \\
T \exp \left[ -i \frac{mc}{\hbar} \sqrt{2\pi G} \int dt dx \left[ \varphi_1 - \varphi_2 \right] \psi^\dagger \psi \right] \left| \Phi(0) \right \rangle_{\psi} \\
T \exp \left[ -i \frac{mc}{\hbar} \sqrt{2\pi G} \int dt dx \left[ \varphi_1 + \varphi_2 \right] \psi^\dagger \psi \right] \left| \Phi(0) \right \rangle_{\psi} \\
\psi \left| \Phi(0) \right\rangle T^{-1} \exp \left[ \frac{mc}{\hbar} \sqrt{2\pi G} \int dt dx \left[ \varphi'_1 + \varphi'_2 \right] \psi^\dagger \psi \right] \tag{48}
\]

where, due to the constraint on the meta-state space, $\tilde{\psi}$ operators were replaced by $\psi$ operators, and the meta-state vector $\left| \Phi(0) \right \rangle_{\tilde{\psi}}$ by $\left| \Phi(0) \right \rangle_{\psi}$. This expression can even be taken
as an independent equivalent definition of the non-unitary dynamics, free from any reference to the extended algebra including unobservable degrees of freedom.

Consider an initial linear, for simplicity orthogonal, superposition of \( N \) localized states of a macroscopic body, existing, as shown above, as pure states corresponding to unentangled bound meta-states for bodies of ordinary density and a mass \( M \) higher than \( \sim 10^{11} m_p \) [13]:

\[
|\Phi(0)\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |z_j\rangle
\]  

(49)

where \( |z\rangle \) represents a localized state centered in \( z \). We consider the localized states as approximate eigenstates of the particle density operator, i.e. \( \psi^\dagger(x, t)\psi(x, t)|z\rangle \simeq n(x-z)|z\rangle \), where time dependence is irrelevant, consistently with these states being stationary both in the gravity-free and in the interacting Schrödinger pictures apart from a slow spreading, which, as shown below, is much slower than the computed time for wave function reduction.

According to Eq. (48), the density matrix elements are then given by

\[
\langle z_h | \rho_{ph}(t) | z_k \rangle
= \int D[\varphi_1, \varphi_2, \varphi'_1, \varphi'_2] \exp \frac{i c^2}{2 \hbar} \int dx \int dt \left[ \varphi_1 \nabla^2 \varphi_1 - \varphi_2 \nabla^2 \varphi_2 - \varphi'_1 \nabla^2 \varphi'_1 + \varphi'_2 \nabla^2 \varphi'_2 \right]
\]

\[
\frac{1}{N^2} \sum_{j=1}^{N} \exp \left[ -i \frac{mc}{\hbar} \sqrt{2\pi G} \int dx dt \left[ [\varphi_1 - \varphi_2] n(x-z_j) - [\varphi'_1 - \varphi'_2] n(x-z_j) \right] \right]

\exp \left[ -i \frac{mc}{\hbar} \sqrt{2\pi G} \int dx dt \left[ [\varphi_1 + \varphi_2] n(x-z_h) - [\varphi'_1 + \varphi'_2] n(x-z_k) \right] \right] 
\]

(50)

and, after integrating out the scalar fields,

\[
\langle z_h | \rho_{ph}(t) | z_k \rangle = \frac{1}{N^2} \sum_{j=1}^{N} \exp \frac{i}{\hbar} G m^2 t \int dx dy \left[ \frac{n(x-z_j)n(y-z_h)}{|x-y|} - \frac{n(x-z_j)n(y-z_k)}{|x-y|} \right]
\]

(51)

which shows that, while diagonal elements are given by \( \langle z_h | \rho_{ph}(t) | z_h \rangle = 1/N \), the coherences, under reasonable assumptions on the linear superposition in Eq. (48), of a large number of localized states, approximately vanish, due to the random phases in Eq. (51). This makes the state \( \rho_{ph}(t) \), for times \( t \gtrsim T_G \sim 10^{20}(M/m_p)^{-5/3} \) sec, which are consistently
short with respect to the time of the entropic spreading $\sim 10^3$ sec, equivalent to an ensemble of localized states:

$$\rho_{ph}(t) \simeq \frac{1}{N} \sum_{j=1}^{N} |z_j \rangle \langle z_j|.$$  \hspace{1cm} (52)

It is worthwhile to remark that the extremely short localization time of a macroscopic body may make its unlocalized states unobservable for all practical purposes. The above analysis is also supported by numerical evidence independently from the particular assumptions made here on the initial unlocalized state $|49\rangle$ $50$. In such a way one gets a gravity-induced dynamical reduction of the wave function, which up to now was assumed to follow, possibly, from a future theory of quantum gravity $13$. It is worthwhile to remark that the order of magnitude of decoherence times in Eq. (51) agrees with the one obtained by previous numerological arguments for gravity-induced localization $19$: ” Although a detailed estimate of $T_G$ would require a full theory of quantum gravity... it is reasonable to expect that for non-relativistic systems ...” $43$. What is new here in this regard is a fully defined dynamical model without any free parameter, which in principle allows for the explicit evaluation of any physically relevant quantity and for addressing crucial questions like the search for (gravitational-)decoherence free states of the physical operator algebra $51$.

To be more specific, we have derived the first unified model for Newtonian gravity and gravity-induced decoherence. If the states $|z_j \rangle$ in Eq. (49) are the pointer states of a measurement apparatus and $|e_j \rangle$ are the measurement eigenstates of a microscopic system, the product state

$$|z_0 \rangle \otimes \sum_j c_j |e_j \rangle$$ \hspace{1cm} (53)

according to the traditional von Neumann model for the interaction between the two systems, is transformed into an entangled state $47$

$$\sum_j c_j |z_j \rangle \otimes |e_j \rangle.$$ \hspace{1cm} (54)
Obviously our previous analysis of the effect of the gravitational (self-)interaction on the quantum motion of the macroscopic body is not affected by the presence of the microscopic system, by which the reduction of the wave function occurs:

$$\sum_{j,k} c_j \bar{c}_k |z_j\rangle \otimes |e_j\rangle \langle e_k| \otimes \langle z_k| \longrightarrow \sum_j |c_j|^2 |z_j\rangle \otimes |e_j\rangle \otimes \langle z_j|.$$  \hspace{1cm} (55)

Of course one can look in principle for a collapse model [18,52] in terms of a stochastic dynamics for pure states, which, when averaged, leads to Eq. (55). Apart, in principle, from the non uniqueness of the stochastic realization [52], stochastic models can certainly be useful as computational tools [53]. However the view advocated here considers density matrices arising from gravitational decoherence as the fundamental characterization of the system state and not just as a bookkeeping tool for statistical uncertainties. The fact that the apparent uniqueness of the measurement result seems to imply a real collapse is perhaps more an ontological than a physical problem, and presumably, if one likes it, that can be addressed by a variant of the many-world interpretation [54,55].

VI. BLACK HOLE HEURISTIC

Our first aim is to evaluate within our model the finite linear dimension of a collapsed matter lump, replacing the classical singularity. In order to do that we boldly use Eq. (15) for lengths smaller than \( \mu_0 \) and \( \mu_2 \), namely in the limit \( \mu_0, \mu_2 \to 0 \). This corresponds to the replacement of our meta-Hamiltonian with the model meta-Hamiltonian in Ref. [35], where there is no gravitational interaction within observable and within hidden matter, while there is a Newton interaction between observable and hidden matter. This interaction is effective in lowering the gravitational energy of a matter lump as far as the localization length \( \Lambda \sim (\hbar^2 \Xi^3/GM^3)^{1/4} \) is fairly smaller than the lump radius \( \Xi \). The highest possible density then corresponds roughly to \( (\hbar^2 \Xi^3/(GM^3))^{1/4} = \Xi \), namely to

$$\Xi = \frac{\hbar^2}{GM^3}.$$  \hspace{1cm} (56)
As to the space-time geometry, the Schwarzschild metric in ingoing Eddington-Finkelstein coordinates \((v, r, \theta, \phi)\) covers the two regions of the Kruskal maximal extension that are relevant to gravitational collapses \([56]\):

\[
ds^2 = - \left[1 - 2MG/(rc^2)\right] dv^2 + 2drdv + r^2 \left[d\theta^2 + \sin^2 \theta d\phi^2\right]. \tag{57}
\]

If in the region beyond the horizon we put \(x = v - \int dr \left[1 - 2MG/(rc^2)\right]^{-1}\), then

\[
ds^2 = \left[1 - 2MG/(rc^2)\right]^{-1} dr^2 - \left[1 - 2MG/(rc^2)\right] dx^2 + r^2 \left[d\theta^2 + \sin^2 \theta d\phi^2\right] \tag{58}
\]

If we trust \([56]\) as the minimal length involved in the collapse, a future full theory of quantum gravity should include a mechanism avoiding the singularity at \(r = 0\) by the introduction of \(\Xi\) as a regularization length. In particular, to characterize the region occupied by the collapsed lump, consider that for time-like geodesics at constant \(\theta\) and \(\phi\) one can show that \(|dx/dr| \sim r^{3/2}\) as \(r \to 0\). This implies that the \(x\) coordinate difference \(\Delta x\) of two material points has a well defined limit as \(r \to 0\), by which it is natural to assume that the \(x\) width of the collapsed matter lump is \(\Delta x \sim \Xi\). As to the apparent inconsistency of matter occupying just a finite \(\Delta x\) interval with \(\partial/\partial x\) being a Killing vector, one should expect on trans-Planckian scales substantial quantum corrections to the Einstein equations that the model gives on a classical level, with the dilaton and the ghost fields, though vanishing in the average, playing a crucial role. On the other hand we are proceeding according to the usual assumption, or fiction, of QM on the existence of a global time variable, at least in the region swept by the lump. In fact the most natural way to regularize \([58]\) is to consider it as an approximation for \(r > \Xi\) of a regular metric, whose coefficients for \(r \to 0\) correspond to the ones in \([58]\) with \(r = \Xi\), in which case there is no obstruction in extending the metric to \(r < 0\), where taking constant coefficients makes \(\partial/\partial r\) a time-like Killing vector. As a consequence, the relevant space metric in the region swept by the collapsed lump is

\[
ds^2_{SPACEx} \sim 2MG/(\Xi c^2) \ dx^2 + \Xi^2 \left[d\theta^2 + \sin^2 \theta d\phi^2\right].
\]

The volume of the collapsed matter lump is then:
\[ V \sim \Xi^2 \Delta x \sqrt{MG/\Xi c^2} = \left[ \frac{\hbar^2}{GM^3} \right]^{5/2} \sqrt{MG/c^2} = \frac{\hbar^5 M^{-7}}{(G^2 c)} . \quad (59) \]

According to the above view, thermodynamical equilibrium is reached, due to the gravitational interaction generating entanglement between the observable and hidden meta-matter, by which the matter state is a microcanonical ensemble corresponding to the energy

\[ E = M c^2 + G M^2 / \Xi = M c^2 + G M^2 \left[ G M^3 / \hbar^2 \right] \sim G^2 M^{5/2} / \hbar^2 , \quad \text{if} \quad M \gg M_P , \quad (60) \]

where \( M_P = \sqrt{\hbar c / G} \) is the Planck mass, and to the energy density

\[ \varepsilon = E / V \sim G^4 c M^{12} / \hbar^7 . \quad (61) \]

For simplicity we treat the collapsed lump as a three-dimensional bulk, since treating it more properly, for the presence of the huge dilation factor in the \( x \) direction, as a string-like structure gives unchanged results. As this energy density corresponds to a very high temperature, not to be mistaken for the Hawking temperature, the matter can be represented by massless fields, whose equilibrium entropy is given by

\[ S \sim \left( \frac{K_B}{\hbar^{3/4} c^{3/4}} \right) \varepsilon^{3/4} V = G^2 K_B / (\hbar c) . \quad (62) \]

Of course this result can be trusted at most for its order of magnitude, the uncertainty in the number of species being just one part of an unknown numerical factor. With this proviso, common to other approaches [8], Eq. (62) agrees with B-H entropy.

Our heuristic assumption of taking as gravitational energy of the collapsed lump just the expression given above in Eq. (59) is consistent with the connection existing between the temperature of the collapsed lump and Hawking temperature on purely thermodynamical grounds. In fact, if we take for granted that a future theory of quantum gravity will account for black hole evaporation, we can connect the temperature

\[ T \sim \sqrt[4]{\varepsilon \hbar^3 c^5} / K_B \sim c G M^3 / K_B \hbar \quad (63) \]

of our collapsed matter lump with the (spectral) temperature of the radiation at infinity. If we model radiation by massless fields, emitted for simplicity at a constant temperature as
we are interested just in orders of magnitude, this temperature is defined in terms of the ratio \( E_\infty/S_\infty \) of its energy \( E_\infty \) and its entropy \( S_\infty \). It is natural to assume that, "once" thermodynamical equilibrium is reached due to the highly non-unitary dynamics close to the classical singularity, no entropy production occurs during evaporation, by which \( S_\infty = S \). Then, if \( E_\infty = M c^2 \) is the energy of the total Hawking radiation spread over a very large space volume, its temperature agrees with Hawking temperature, i.e.

\[
T_\infty = (E_\infty/E) T \sim (c^3 \hbar/MGK_B).
\] (64)

VII. CONCLUDING REMARKS

Of course the reversibility of the unitary meta-dynamics makes entropy decrease conceivable too \[57\], so that a derivation of the entropy-growth for a closed system (in principle the whole universe), in the present context, must have recourse to the choice of suitable initial conditions, like unentanglement between the observable and the hidden algebras. While the assumption of special initial conditions dates back to Boltzmann, only a non-unitary dynamics makes it a viable starting point, within a quantum context, for the microscopic derivation of the second law of thermodynamics, in terms of von Neumann entropy, for a genuinely closed system. This is meant without introducing generalized microcanocity conditions, and then renouncing isolation \[58\].

It should be remarked that, for a realistic physical setting, most of the in principle observable degrees of freedom are yet out of our control and non-unitarity is the result of interactions with both fundamentally hidden degrees of freedom and with the environment. Environment-induced decoherence \[59\], in most cases, may overshadow fundamental decoherence, even though the recent amazing experimental achievements in preserving and measuring quantum coherences make the detection of gravity-induced decoherence a less despairing task \[36,37,60,62\]. In this respect the most natural experimental setting to look for gravitationally-induced decoherence seems to be that of Bose-Einstein (B-E) condensation, due to the unprecedented scale of controlled quantum coherence achieved there \[63\].
In particular the localization mass-threshold is not too far from the present experimental limits, while its robustness with respect to the mass density variations may be a typical signature of our gravitational self-interaction.

In conclusion, if we define a suitable non-unitary modification of fourth order gravity, by doubling the matter algebra and introducing a suitable constraint in order not to enlarge the observable algebra, we get the following outcomes:

1) classical runaway solutions are absent and the ensuing classically stable theory may be made equivalent to Einstein gravity;

2) the Newtonian limit is classically equivalent to ordinary Newton gravity;

3) from a quantum viewpoint this non-unitary limit implies gravity induced localization and decoherence, which are compatible both with the wavelike behavior of microscopic particles and the classicality of the center of mass motion of macroscopic bodies;

4) the model strongly supports the interpretation of the thermodynamic entropy of a closed system as von Neumann entropy and paves the way for the quantum foundations of the second law of thermodynamics;

5) a bold use of the action at a distance limit of the model together with some geometric insight coming from Einstein gravity allows us to ascribe to the smoothed singularity of a black hole a finite entropy, which apart from an undetermined numerical factor coincides with the B-H entropy, and a very high temperature that is compatible with the much lower Hawking evaporation temperature.

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