An efficient numerical calculation of gravitational waves from extreme mass ratio inspirals

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Abstract. Gravitational waves from extreme mass ratio inspirals are one of the important sources of LISA. We should calculate these waves so accurately that we can extract physical information of source by data analysis. Recently, we developed an efficient numerical method to compute gravitational waves from binary systems in which a point particle moves in circular orbits on the equatorial plane of the black hole. In this paper, we apply this method to compute gravitational waves from binary systems in which a point particle moves in general bound geodesic orbits of the black hole. We check the accuracy of our code using spherical symmetry of Schwarzschild black hole such that energy flux radiated by a point particle is independent of the inclination angle from the equatorial plane of black hole. We find that the accuracy of our code may be limited only by truncation of $\ell$, $k$ and $n$-modes, where $\ell$ is the degree of the spin-weighted spheroidal harmonics, and $k$ and $n$ are harmonics of the polar and radial motion, respectively. Then we evaluate the rate of change of three constants of motion, energy, angular momentum and the Carter constant, due to the emission of gravitational waves from a particle around Kerr black hole. This is the first time to compute the rate of change of the Carter constant using the adiabatic approximation. We also show that we can calculate gravitational waves accurately even in the case of high eccentric orbits. In this work, we truncate $\ell$ mode up to 20 and estimated that relative accuracy of our numerical results are better than $10^{-5}$ even in the high eccentric case, $e = 0.9$. Our numerical code may be useful to make templates of extreme mass ratio inspirals.

1. Introduction

Gravitational waves radiated from a particle orbiting around a Kerr black hole are one of the main targets of space-based gravitational observatory projects such as the Laser Interferometer Space Antenna (LISA)\cite{1}. Such binary systems are called the extreme mass ratio inspirals (EMRIs). Observing gravitational waves from EMRI, we may be able to obtain information of the central black hole’s spacetime such as mass, spin and the mass distribution of compact objects in the center of galaxy. In order to obtain information from EMRI, we have to achieve the phase accuracy of theoretical gravitational wave forms within one radian over the total cycle of wave, $\sim 10^5$.

Because of its extreme mass ratio, gravitational waves from EMRIs are well approximated by the black hole perturbation formalism. The conventional equation of the black hole perturbation is the Teukolsky equation\cite{2}. There are a lot of previous works that compute gravitational waves from EMRI analytically and numerically\cite{3, 4}. For simple orbits such as circular or equatorial
orbits around black hole, numerical calculations achieved the accuracy around $10^{-5}$ which may be sufficient to detect gravitational waves. For more general orbits, Drasco and Hughes computed gravitational waves from EMRI[5]. However, their computational time and numerical accuracy seem to be insufficient to detect gravitational waves. It might be useful if we can compute gravitational waves more efficiently.

In this paper, we numerically compute the rates of change of constants of motion due to the emissions of gravitational waves from EMRI. The calculations are based on the method which was developed by Fujita and Tagoshi[6, 7] for computation of the homogeneous solutions of the Teukolsky equation. We will show that we can compute gravitational waves from EMRI more accurately and efficiently than past works.

2. Gravitational waves from EMRIs

The Teukolsky equation describes the gravitational perturbation of a Kerr black hole using the Newman-Penrose variables, $\Psi_0$ and $\Psi_4$, which is the gauge-invariant variables corresponding to some tetrad components of the perturbed Weyl curvature[8, 9]. The Weyl scalar $\Psi_4$ is related to the amplitudes of gravitational wave at infinity as

$$\Psi_4 \to \frac{1}{2}(\hat{h}_+ - i \hat{h}_\times), \quad \text{for} \quad r \to \infty. \quad (1)$$

The master equation for $\Psi_4$ can be separated into a radial and an angular part if we expand $\Psi_4$ in Fourier-harmonic modes as

$$(r - ia \cos \theta)^4 \Psi_4 = \sum_{\ell m} \int_{-\infty}^{\infty} d\omega e^{-i\omega t + im\phi} -2S^{\omega \ell m}_4(\theta)R^{\omega \ell m}_r(r), \quad (2)$$

where the angular function $-2S^{\omega \ell m}_4(\theta)$ is the spin-weighted spheroidal harmonic with spin $s = -2$, and $M$ and $aM$ are the mass and the angular momentum of the black hole, respectively. The radial function $R^{\omega \ell m}_r(r)$ satisfies the radial Teukolsky equation,

$$\Delta^2 \frac{d}{dr} \left( \frac{1}{\Delta} \frac{dR^{\omega \ell m}_r}{dr} \right) - V(r)R^{\omega \ell m}_r = T^{\omega \ell m}_r, \quad (3)$$

where $\Delta = r^2 - 2Mr + a^2$ and the source term $T^{\omega \ell m}_r$ is constructed from the tetrad components of the energy momentum tensor[10]. The potential $V(r)$ is given by

$$V(r) = -\frac{K^2 - 2is(r-M)K}{\Delta} - 4is\omega + \lambda, \quad (4)$$

where $K = (r^2 + a^2)\omega - ma$ and $\lambda$ is the eigenvalue of $-2S^{\omega \ell m}_4(\theta)$.

We solve the radial Teukolsky equation by using the Green function method. We introduce two kinds of solution of the homogeneous Teukolsky equation. One of them has purely outgoing property at infinity and the other has purely ingoing property at the horizon. Then the asymptotic property of the solution at the horizon is expressed as

$$R^{\omega \ell m}_r(r \to r_+) \equiv Z^{H}_{\ell m \omega} \Delta^2 e^{-iPr^*}. \quad (5)$$

The solution at infinity is expressed as

$$R^{\omega \ell m}_r(r \to \infty) \equiv Z^{\infty}_{\ell m \omega} r^3 e^{i\omega r^*}. \quad (6)$$

For the bound geodesic orbits, the amplitude of partial wave $Z^{\infty/f}_{\ell m \omega}$ can be expanded by using a Fourier series as[11]

$$Z^{\infty/f}_{\ell m \omega} \equiv \sum_{kn} \hat{Z}^{\infty/f}_{\ell m kn} \delta(\omega - \omega_{mkn}), \quad (7)$$
where $\omega_{mkn} = m\Omega_\phi + k\Omega_\theta + n\Omega_\tau$, and where $\Omega_\tau$ is the frequencies of $r$-motion, $\Omega_\theta$ is the frequencies of $\theta$-motion and $\Omega_\phi$ is the frequencies of $\phi$-motion.

Then the gravitational wave form at infinity is expressed as

$$h_+ - ih_\times = -\frac{2}{r} \sum_{\ell mkn} \frac{\tilde{Z}_{\ell mkn}^\infty}{\omega_{mkn}^2} 2\mathcal{S}^{\omega_{mkn}}_\ell m (\theta) e^{i\omega_{mkn}(r-\xi)+im\phi}.$$  \hspace{1cm} (8)

Moreover, the time-averaged rates of change for the three constants of motion due to the emission of the gravitational waves are expressed as[12]

$$\langle \frac{dE}{dt} \rangle = -\mu^{-1} \sum_{\ell mkn} \frac{1}{4\pi \omega_{mkn}^2} \left( |\tilde{Z}_{\ell mkn}^\infty|^2 + \alpha_{\ell mkn} |\tilde{Z}_{\ell mkn}^H|^2 \right),$$  \hspace{1cm} (9a)

$$\langle \frac{dL_z}{dt} \rangle = -\mu^{-1} \sum_{\ell mkn} \frac{m}{4\pi \omega_{mkn}^2} \left( |\tilde{Z}_{\ell mkn}^\infty|^2 + \alpha_{\ell mkn} |\tilde{Z}_{\ell mkn}^H|^2 \right),$$  \hspace{1cm} (9b)

$$\langle \frac{dC}{dt} \rangle = -2 \langle a^2 \mathcal{E} \cos^2 \theta \rangle \langle \frac{dE}{dt} \rangle + 2 \langle \mathcal{L}_z \cot^2 \theta \rangle \langle \frac{dL_z}{dt} \rangle \hspace{1cm}$$

$$- \sum_{\ell mkn} \frac{k \Upsilon}{2\pi \omega_{mkn}^2} |\tilde{Z}_{\ell mkn}^\infty|^2 + \alpha_{\ell mkn} |\tilde{Z}_{\ell mkn}^H|^2.$$  \hspace{1cm} (9c)

Here $\langle \ldots \rangle$ means the infinite time average.

### 3. Numerical methods and results

Practical numerical calculations of gravitational waves from EMRIs are summarized as follows.

- Calculate constants of motion ($E, L_z, C$) and fundamental frequencies ($\Omega_\phi, \Omega_\theta, \Omega_\tau$) for given orbital parameters ($p, e, \theta_{inc}$), where $p$ is semi-latus rectum, $e$ is eccentricity and $\theta_{inc}$ is inclination angle from the equatorial plane of black hole.
- Derive orbit in frequency domain.
- Calculate two kind of the homogeneous solution of the Teukolsky equation.
- Integrate the Green function over the source term.
- Evaluate gravitational waves using Eq.8 and Eq.9.

Using the methods described in Ref.[11], we evaluate constants of motion, fundamental frequencies and orbit in frequency domain. And we solve the homogeneous Teukolsky equation using Mano-Suzuki-Takasugi(MST) method[13]. In MST method, we expand the homogeneous solution in terms of hypergeometric functions as $R_{\ell m\omega}(r) \sim \sum_n a_n F(\alpha_n, \beta_n, \gamma_n; r)$. Although the application of MST method was limited to the analytical calculation using the low frequency expansion, Fujita and Tagoshi showed that MST method is also useful in numerical calculation[6, 7]. They showed that the homogeneous solutions converges very fast, and evaluated gravitational waves from a compact star moving in circular and equatorial orbits around Kerr black hole very accurately. In this paper, we apply their numerical method to compute the homogeneous solutions of the Teukolsky equation in the case of the generic bound geodesic orbits.

In this section, we show our numerical results such as the time-averaged rates of change of constants of motion, energy $\langle dE/dt \rangle^\infty$, angular momentum $\langle dL_z/dt \rangle^\infty$ and the Carter constant $\langle dC/dt \rangle^\infty$ due to gravitational wave radiated to infinity when a compact star traces generic orbits around Kerr black hole. In numerical calculation, we have to truncate the mode summation in terms of $\ell$, $m$, $k$ and $n$-mode when we estimate gravitational waves using Eq.8 and Eq.9. We truncate the mode summation of $k$ and $n$-mode when the relative accuracy with respect to total flux reaches $10^{-10}$. In Figure 1, we show the behavior of the modal energy flux, $\langle dE_{\ell mkn}/dt \rangle^\infty_{GW}$.
of gravitational waves radiated to infinity in the case of Schwarzschild black hole. From Figure 1, we find that we have to compute larger $n$-mode as eccentricity or $\ell$ becomes large. In Figure 2 and Figure 3, we show the behavior of the modal energy flux, $\langle dE_{\ell m n}/dt \rangle_{GW}$, of gravitational waves radiated to infinity in the case of Schwarzschild black hole. From these figure, we find that the peak location of $k$-mode is around $k = \ell - m$. In Kerr black hole case, these properties of $n$ and $k$-mode are almost the same as that of Schwarzschild black hole, but the spectrum of $k$-mode of Kerr black hole is broader than that of Schwarzschild black hole. As for $m$-mode, we compute all $m$-mode from $\ell$ to $-\ell$. And we set the maximum of $\ell$ to be $\ell_{max} = 20$. The details of the truncation rule are discussed in Ref. [14].

In order to check the accuracy of our numerical code, we compute the rate of change of the energy in the case of Schwarzschild black hole. In the Schwarzschild case, the rate of change of the energy is independent of the inclination angle from the equatorial plane of black hole. In Table 1, we check the accuracy of our code comparing the rate of change of the energy in the case of non-equatorial orbits with that of equatorial orbit. In this table, the values in the square bracket are the estimated truncation errors up to $\ell = 20$-mode. And we find that the relative errors between the equatorial plane case and the non-equatorial plane case are around $10^{-8} - 10^{-11}$. Thus, the relative errors in Table 1 are consistent with the truncation errors of summation over $\ell$, $k$ and $n$-mode. In Table 2, we show the rate of change of three constants of motion in the case of Kerr black hole. This is the first time to compute the rate of change of the Carter constant using the adiabatic approximation. The truncation errors of $\ell$-mode in Table 2 are around $10^{-6} - 10^{-11}$. Thus numerical accuracies in the case of $e = 0.1$ are limited by the truncation of $k$ and $n$-mode, $\sim 10^{-10}$, and in the case of $e = 0.7$ and $e = 0.9$ are limited by the truncation of $\ell$-mode.

4. Summary and future work
We developed an efficient numerical method to compute gravitational waves from bound geodesic orbits. In our method, we compute the homogeneous solution of the Teukolsky equation using Mano-Suzuki-Takahagi method. We checked the accuracy of our code using the fact that the rate of change of a particle’s energy is independent of the inclination angle from the equatorial plane of Schwarzschild black hole. We found that the accuracy of our code may be limited only by the truncation of summation over $\ell$, $k$ and $n$-mode. In this paper, we set the relative errors...
Figure 2. Modal energy flux $\langle dE_{2mk}/dt \rangle_{GW}$ at infinity, radiated by a particle around a Schwarzschild black hole. In this figure, the orbital radius is $10M$, the orbital eccentricity is $0.7$ and the orbital inclination angle is $\theta_{inc} = 20^\circ$.

Figure 3. Modal energy flux $\langle dE_{2mk}/dt \rangle_{GW}$ at infinity, radiated by a particle around a Schwarzschild black hole. In this figure, the orbital radius is $10M$, the orbital eccentricity is $0.7$ and the orbital inclination angle is $\theta_{inc} = 70^\circ$.

Table 1. Time-averaged rates of change of the energy of a particle due to the emission of gravitational waves to infinity from a Schwarzschild black hole. We compare the result for the equatorial plane case with the one for non-equatorial plane. Relative error in square brackets is an order of magnitude estimate for the fractional accuracy for the case of the equatorial plane, which is determined by truncating number of $\ell$-mode. Here we set to $\ell_{\text{max}} = 20$.

| $a/M$ | $p/M$ | $e$  | $\theta_{inc}$ | $\langle dE / dt \rangle_{\infty}$ | Relative error |
|-------|-------|------|-----------------|-----------------------------------|---------------|
| 0     | 10    | 0.1  | $0^\circ$       | -6.31752474714×10$^{-5}$         | [10$^{-11}$]  |
| 0     | 10    | 0.1  | $20^\circ$      | -6.31752474712×10$^{-5}$         | 4.1×10$^{-12}$|
| 0     | 10    | 0.1  | $45^\circ$      | -6.31752474688×10$^{-5}$         | 4.2×10$^{-11}$|
| 0     | 10    | 0.1  | $70^\circ$      | -6.31752474700×10$^{-5}$         | 2.3×10$^{-11}$|
| 0     | 10    | 0.5  | $0^\circ$       | -9.27335012129×10$^{-5}$         | [10$^{-11}$]  |
| 0     | 10    | 0.5  | $20^\circ$      | -9.27335011990×10$^{-5}$         | 1.5×10$^{-10}$|
| 0     | 10    | 0.5  | $45^\circ$      | -9.27335011942×10$^{-5}$         | 2.0×10$^{-10}$|
| 0     | 10    | 0.5  | $70^\circ$      | -9.27335011362×10$^{-5}$         | 8.3×10$^{-10}$|
| 0     | 10    | 0.9  | $0^\circ$       | -4.19426469281×10$^{-5}$         | [10$^{-8}$]   |
| 0     | 10    | 0.9  | $20^\circ$      | -4.19426468367×10$^{-5}$         | 2.2×10$^{-9}$ |
| 0     | 10    | 0.9  | $45^\circ$      | -4.19426468292×10$^{-5}$         | 2.4×10$^{-9}$ |
| 0     | 10    | 0.9  | $70^\circ$      | -4.19426453712×10$^{-5}$         | 3.7×10$^{-8}$ |

of $k$ and $n$-mode are $10^{-10}$ and the maximum value of $\ell$-mode to be $20$. Then we found that the accuracies of our code are better than $10^{-5}$ even in the case of $e = 0.9$. Thus our code will be useful to detect gravitational waves from EMRIs by LISA data analysis.

We also computed the rate of change of the Carter constant using adiabatic approximation. This is the first time to evaluate the rate of change of the Carter constant accurately.

Using our code, we will estimate gravitational waves from more general orbits such that eccentricity is higher than 0.9 and inclination angle is $90^\circ$ in the future work. And we will compute gravitational waves including adiabatic evolution of constants of motion and construct
Table 2. Time-averaged rates of change of three constants of motion, energy \( \langle d\mathcal{E} \rangle / dt \), angular momentum \( \langle d\mathcal{J}_z \rangle / dt \) and the Carter constant \( \langle d\mathcal{C} \rangle / dt \) due to gravitational wave radiated to infinity per unit mass in the case of some generic orbits around Kerr black hole. The orbital parameters used here are the same as those used by Drasco and Hughes in Ref. [5] except for the case of \( e = 0.9 \), which they did not show. Our results are consistent with their ones, except for the rate of change of the Carter constant, \( \langle d\mathcal{C} \rangle / dt \). Each number in square brackets is an order of magnitude estimate for the fractional accuracy of the preceding number, which is determined by truncating number of \( \ell \)-mode. Here we set to \( \ell_{\text{max}} = 20 \). We note that the case of \( q = 0.9M, p = 6M, e = 0.9 \) and \( \theta_{\text{inc}} = 80^\circ \) is out of angle of stable bound orbits.

| \( e \) | \( \theta_{\text{inc}} \) | \( \langle d\mathcal{E} \rangle / dt \rangle \) | \( \langle d\mathcal{J}_z \rangle / dt \rangle \) | \( \langle d\mathcal{C} \rangle / dt \rangle \) |
|------|--------|-------------------|-------------------|-------------------|
| 0.1  | 20°    | \(-5.87363797768 \times 10^{-4}\) \(10^{-11}\) | \(-8.53727877914 \times 10^{-3}\) | \(-5.24019849268 \times 10^{-3}\) |
| 0.1  | 40°    | \(-6.1832941189 \times 10^{-4}\) \(10^{-11}\) | \(-7.63099400995 \times 10^{-3}\) | \(-2.0271137751 \times 10^{-2}\) |
| 0.1  | 60°    | \(-6.83348194356 \times 10^{-4}\) \(10^{-11}\) | \(-6.07829110810 \times 10^{-3}\) | \(-4.32194649967 \times 10^{-2}\) |
| 0.1  | 80°    | \(-8.05858116215 \times 10^{-4}\) \(10^{-10}\) | \(-3.62538057535 \times 10^{-3}\) | \(-7.18520475478 \times 10^{-2}\) |
| 0.7  | 20°    | \(-7.73120284923 \times 10^{-4}\) \(10^{-11}\) | \(-6.69307191178 \times 10^{-3}\) | \(-3.88677844044 \times 10^{-3}\) |
| 0.7  | 40°    | \(-8.74643775649 \times 10^{-4}\) \(10^{-11}\) | \(-6.52867843731 \times 10^{-3}\) | \(-1.6232556695 \times 10^{-2}\) |
| 0.7  | 60°    | \(-1.14681348743 \times 10^{-3}\) \(10^{-11}\) | \(-6.37991880771 \times 10^{-3}\) | \(-4.14597930483 \times 10^{-2}\) |
| 0.7  | 80°    | \(-2.71871174658 \times 10^{-3}\) \(10^{-10}\) | \(-8.40111270562 \times 10^{-3}\) | \(-1.34124912753 \times 10^{-1}\) |
| 0.9  | 20°    | \(-3.22207770684 \times 10^{-4}\) \(10^{-10}\) | \(-2.36899718598 \times 10^{-3}\) | \(-1.35639225718 \times 10^{-3}\) |
| 0.9  | 40°    | \(-3.81207517166 \times 10^{-4}\) \(10^{-10}\) | \(-2.43210898466 \times 10^{-3}\) | \(-5.96021788095 \times 10^{-3}\) |
| 0.9  | 60°    | \(-5.54821575240 \times 10^{-4}\) \(10^{-10}\) | \(-2.67958515939 \times 10^{-3}\) | \(-1.7080085390 \times 10^{-2}\) |

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efficient templates for LISA data analysis.