Representations of
Quantum Bicrossproduct Algebras

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Abstract

We present a method to construct induced representations of quantum algebras having the structure of bicrossproduct. We apply this procedure to some quantum kinematical algebras in (1 + 1) dimensions with this kind of structure: null-plane quantum Poincaré algebra, non-standard quantum Galilei algebra and quantum kappa Galilei algebra.
1 Introduction

In a recent paper [1] we developed a method to construct induced representations of quantum algebras mainly based on the concepts of module and duality. Since by dualization objects like modules and comodules can be seen as equivalent, then we have not only regular and induced representations but also coregular and coinduced representations. The main result of that work was the possibility of constructing coregular and coinduced representations of a Hopf algebra $U_q(\mathfrak{g})$ when dual bases of it and its dual $\text{Fun}_q(G)$ (or $F_q(G)$) are known, being $\mathfrak{g}$ the Lie algebra of a Lie group $G$.

Now we present a particular study of the induction method for those quantum algebras having a structure of bicrossproduct, which is a generalization of the idea of semidirect product of Lie groups to the quantum case. This kind of structure of semidirect product is well known in physics where many interesting groups, like Euclidean, Galilei and Poincaré have it. The corresponding quantum Lie algebras inherit the ‘semidirect’ structure in the algebra sector and the algebra of functions also has a semidirect product structure in the coalgebra sector. These ideas were generalized by Molnar [2] with the notions of smash product or, more recently by Majid with that of bicrossproduct [3]–[6].

The quantum counterparts of the above mentioned groups and algebras are related with the symmetries of the physical space-time in a noncommutative framework. The study of these quantum symmetries and their representations generalizes the well known and fruitful program started by Wigner in 1939 [7] inside the perspective of the noncommutative geometry [8], which in the last years is finding many applications in physics (see, for instance, [9] and references therein).

In this paper we continue the analysis of the theory of induced representations but now referred to Hopf algebras with bicrossproduct structure, whose first factor is cocommutative and the second one commutative. Our fundamental objective is the description of the representations induced by characters of the commutative sector. We want to avoid the problems derived by the use of pairs of dual bases and to open new ways which allow to do some computations, whose difficulty increases with the number of generators. However, although the first results are also obtained using dual bases, they show the existence of some lying structure connected with some one-parameter flows defined by the cocommutative sector over an object related with the commutative factor. The nature of this structure will be clear after an adequate reinterpretation of the factors of the bicrossproduct. More explicitly, the cocommutative factor will be seen as the enveloping algebra of a certain Lie algebra but the commutative factor will be identified with the algebra of functions of another Lie group. In this way, the action, defining a part of the bicrossproduct structure in the original Hopf algebra, is the result of translating to the algebra of functions the action of a Lie group over another Lie group.

A crucial point in our approach is to describe the Hopf algebra substituting the monomial bases by elements which are product of an element of the group associated to the first factor times a function belonging to the second one. That allows to prove theorems [4] and [5] that are the cornerstone of this paper. The first theorem describes the four regular modules associated to a bicrossproduct Hopf algebra in terms of the regular actions of its components and the action, mentioned above, associated with the bicrossproduct structure. The second theorem allows the construction of the representations induced by characters of the abelian sector and classifies
the equivalence classes of the induced representations in terms of the orbits associated to the action of a certain group. Moreover, a $*$-structure is introduced in such a way that the induced representations are unitary.

The induction procedure, such as it has been formulated by us, has an algebraic character since we use objects like modules, comodules, etc., which are the appropriate tools to work with the algebraic structures exhibited by the quantum groups and algebras [1, 10, 11, 12].

Dobrev has developed [13, 14] a method for constructing representations of quantum groups similar, in some sense, to ours, i.e., both methods emphasize the dual case, closer to the classical one, and the representations are constructed in the algebra sector. We can mention some works that have also extended the induction technique to quantum groups but constructing corepresentations, i.e. representations of the coalgebra sector [15]–[19].

The paper is organized as follows. Section 2 is devoted to reviewing the main ideas and concepts that we will use along the paper, like module, comodule, module algebra, bicrossproduct, etc. The last part of this section presents original results showing how pairs of dual bases and $*$-structures over bicrossproduct Hopf algebras can be obtained starting from those of their factors. The first results about induced representations of quantum algebras with bicrossproduct structure are presented in section 3. We obtain the representations making use of pairs of dual bases. In section 4 we begin to study the induction problem taking into account the deep relation between modules and representations obtaining, in some sense, more deep results from a geometric point of view using the concept of regular co-space. In section 5 we obtain the induced representations of some kinematical quantum algebras making use of the method developed in the previous sections. We end with some comments and conclusions.

2 Mathematical preliminaries

Let $H = (V; m; \eta; \Delta; \epsilon; S)$ be a Hopf algebra with underlying vector space $V$ over the field $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$), multiplication $m : H \otimes H \to H$, coproduct $\Delta : H \to H \otimes H$, unit $\eta : \mathbb{K} \to H$, counit $\epsilon : H \to \mathbb{K}$ and antipode $S : H \to H$.

A Hopf algebra can be considered as a bialgebra with an antilinear map $S$, and a bialgebra can be seen as composed by two ‘substructures’ or ‘sectors’ (the algebra sector $(V, m, \eta)$ and the coalgebra sector $(V, \Delta, \epsilon)$) with some compatibility conditions [20].

On the other hand, the algebras considered in this work are finitely generated although they are infinite dimensional. For this reason the following multi-index notation is very useful [1].

Let us suppose that $A$ is an algebra generated by the elements $(a_1, a_2, \ldots, a_r)$ and the ordered monomials

$$a^n := a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r} \in A, \quad n = (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r, \quad (2.1)$$

form a basis of the linear space underlying to $A$. An arbitrary product of generators of $A$ is written in a normal ordering if it is expressed in terms of the basis $(a^n)_{n \in \mathbb{N}^r}$. In some cases we will use the notation $(a_n := a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r})_{n \in \mathbb{N}^r}$. For $0 = (0, \ldots, 0) \in \mathbb{N}^r$ we have $a^0 \equiv 1_A$. 
Multi-factorials and multi-deltas are defined by
\[ l! = \prod_{i=1}^{n} l_i!, \quad \delta^m_l = \prod_{i=1}^{n} \delta_{l_i}^{m_i}. \tag{2.2} \]

2.1 Duality

It is well known that the dual object of \( V \) is defined as the vector space of its linear forms, i.e., \( V^* = \mathcal{L}(V, K) \). Hence, if \((V, m, \eta)\) is a finite algebra it is natural to define the dual objet as \((V^*, m^*, \eta^*)\) obtaining a coalgebra and viceversa. However, in the infinite dimensional case the spaces \((V \otimes V)^*\) and \(V^* \otimes V^*\) are not isomorphic and some troubles appear with the coproduct as dual of the multiplication map. The concept of pairing solves these difficulties.

A pairing between two Hopf algebras \([20]\), \( H \) and \( H' \), is a bilinear mapping \( \langle \cdot, \cdot \rangle : H \times H' \to K \) that verifies the following properties:
\[ \langle h, m'(h' \otimes k') \rangle = \langle \Delta(h), h' \otimes k' \rangle, \quad \langle h, 1_{H'} \rangle = \epsilon(h), \]
\[ \langle h \otimes k, \Delta'(h') \rangle = \langle m(h \otimes k), h' \rangle, \quad \epsilon'(h') = \langle 1_H, h' \rangle, \]
\[ \langle h, S'(h') \rangle = \langle S(h), h' \rangle. \tag{2.3} \]

Remark that \( \langle h \otimes k, h' \otimes k' \rangle = \delta_{h')} \delta_{k'} \).

The pairing is said to be left (right) nondegenerate if \([\langle h, h' \rangle = 0, \quad \forall h' \in H'] \Rightarrow h = 0 \) \(([\langle h, h' \rangle = 0, \quad \forall h \in H] \Rightarrow h' = 0)\). If the pairing is simultaneously left and right nondegenerate we simply say that it is nondegenerate.

The triplet \((H, H', \langle \cdot, \cdot \rangle)\) composed by two Hopf algebras and a nondegenerate pairing will be called a ‘nondegenerate triplet’.

The bases \((h^m)\) of \( H \) and \((h'_n)\) of \( H' \) are said to be dual with respect to the nondegenerate pairing if
\[ \langle h^m, h'_n \rangle = c_n \delta^m_n, \quad c_n \in K - \{0\}. \tag{2.4} \]

The map \( f^\dagger : H' \to H' \) implicitly defined in terms of the map \( f : H \to H \) by
\[ \langle h, f^\dagger(h') \rangle = \langle f(h), h' \rangle, \tag{2.5} \]
is called the adjoint map of \( f \) with respect to the nondegenerate pairing.

2.2 Modules and comodules

Let us consider the triad \((V, \alpha, A)\), where \( A \) is an associative \( K \)-algebra with unit, \( V \) is a \( K \)-vector space and \( \alpha \) a linear map, \( \alpha : A \otimes_K V \to V \), called action and denoted by \( a \triangleright v = \alpha(a \otimes v) \).

We will say that \((V, \alpha, A)\) (or \((V, \triangleright, A)\)) is a left \( A \)-module if the following two conditions are verified:
\[ a \triangleright (b \triangleright v) = (ab) \triangleright v, \quad 1 \triangleright v = v, \quad \forall a, b \in A, \quad \forall v \in V. \tag{2.6} \]
A morphism of left \(A\)-modules, \((V, \triangleright A)\) and \((V', \triangleright A)\), is a linear map, \(f : V \to V'\), equivarient with respect the action, i.e.,
\[
f(a \triangleright v) = a \triangleright f(v), \quad \forall a \in A, \forall v \in V.
\] (2.7)

Dualizing the concept of \(A\)-module it is obtained the concept of comodule. Thus, if \(C\) is an associative \(K\)-coalgebra with counit, \(V\) a \(K\)-vector space and \(\beta : V \to C \otimes_K V\) a linear map that will be called coaction and denoted by \(v \triangleleft = \beta(v) = v^{(1)} \otimes v^{(2)}\), the triad \((V, \beta, C)\) is said to be a left \(C\)-comodule if the following axioms are verified:
\[
v^{(1)}(1) \otimes v^{(1)}(2) \otimes v^{(2)} = v^{(1)} \otimes v^{(2)}(1) \otimes v^{(2)}(2), \quad \epsilon(v^{(1)})v^{(2)} = v, \quad \forall v \in V,
\] (2.8)
where the coproduct of the elements of \(C\) is symbolically written as \(\Delta(c) = c^{(1)} \otimes c^{(2)}\).

A linear map \(f : V \to V'\) between two \(C\)-comodules, \((V, \triangleright C)\) and \((V', \triangleright C)\) is a morphism if
\[
v^{(1)} \otimes f(v^{(2)}) = f(v^{(1)'} \otimes f(v^{(2)'}, \quad \forall v \in V.
\] (2.9)

Similarly right \(A\)-modules and right \(C\)-comodules are defined.

### 2.3 Module Algebras

When a bialgebra acts or coacts on a vector space equipped with an additional structure of algebra, coalgebra or bialgebra \([2, 21]\) it is usual to demand some compatibility relations for the action. In the following \(B\) and \(B'\) will denote bialgebras, \(A\) an algebra and \(C\) a coalgebra.

The left module \((A, \triangleright B)\) is said to be a \(B\)-module algebra if \(m_A\) and \(\eta_A\) are morphisms of \(B\)-modules, i.e., if
\[
b \triangleright (aa') = (b(1) \triangleright a)(b(2) \triangleright a'), \quad b \triangleright 1 = \epsilon(b)1, \quad \forall b \in B, \forall a, a' \in A.
\] (2.10)

Changing algebra by coalgebra it is obtained the structure of module coalgebra. In this case the left \(B\)-module \((C, \triangleright B)\) is a \(B\)-module coalgebra if \(\Delta_C\) and \(\epsilon_C\) are morphisms of \(B\)-modules, i.e., if
\[
(b \triangleright c)(1) \otimes (b \triangleright c)(2) = (b(1) \triangleright c(1)) \otimes (b(2) \triangleright c(2)), \quad \epsilon_C(b \triangleright c) = \epsilon_B(b)\epsilon_C(c), \quad \forall b, c \in B.
\]

Dualizing these structures two new ones are obtained. The left \(B\)-comodule \((C, \triangleright B)\) is said to be a \(B\)-comodule coalgebra if \(\Delta_C\) and \(\epsilon_C\) are morphisms of \(B\)-comodules, i.e.,
\[
c^{(1)} \otimes c^{(2)}(1) \otimes c^{(2)}(2) = c^{(1)}(1) c^{(2)}(1) \otimes c^{(1)}(2) c^{(2)}(2), \quad \epsilon_C(c^{(1)})\epsilon_C(c^{(2)}) = (\eta_B \circ \epsilon_C)(c).
\]

The left \(B\)-comodule \((A, \triangleright B)\) is a \(B\)-comodule algebra if \(m_A\) and \(\eta_A\) are morphisms of \(B\)-comodules. Explicitly
\[
(aa')(1) \otimes (aa')(2) = a(1)a'(1) \otimes a(2)a'(2), \quad 1_A \triangleleft = 1_B \otimes 1_A.
\] (2.11)
The triad \((B', \triangleright, B)\) is a left \(B\)–module bialgebra if simultaneously is a \(B\)–module algebra and a \(B\)–module coalgebra; \((B', \bowtie, B)\) is a left \(B\)–comodule bialgebra if simultaneously is a \(B\)–comodule algebra and a \(B\)–comodule coalgebra.

The corresponding versions at the right are defined in an analogous manner.

By regular module (comodule) we understand an \(A\)–module (\(C\)–comodule) whose vector space is the underlying vector space of the algebra \(A\) (coalgebra \(C\)). The action (coaction) is defined by means of the algebra product (coalgebra coproduct).

For instance, on the regular \(A\)–modules \((A, \triangleright, A)\) and \((A, \triangleleft, A)\) the actions are, respectively,

\[ a \triangleright a' = aa', \quad a' \triangleleft a = a'a, \quad (2.12) \]

If \(B\) is a bialgebra, the regular \(B\)–module \((B, \triangleright, B)\) whose ‘regular’ action is defined by

\[ b \triangleright b' = bb', \quad (2.13) \]

is a module coalgebra. The module \((B^*, \triangleleft, B)\), obtained by dualization, is a module algebra with the ‘regular’ action

\[ \varphi \triangleleft b = \langle \varphi_1, b \rangle \varphi_2, \quad b \in B, \quad \varphi \in B^*. \quad (2.14) \]

It will be also called regular module. The comodule versions can be easily obtained by the reader.

### 2.4 Bicrossproduct structure

The concepts of module algebra and comodule coalgebra allow to describe in a suitable way ‘semidirect’ structures \[4, 21\] as we shall see later.

Let \(H\) be a bialgebra and \((A, \triangleleft, H)\) a right \(A\)–module algebra. The expression

\[ (h \otimes a)(h' \otimes a') = hh'_1(1) \otimes (a \triangleleft h'_2(2))a' \quad (2.15) \]

defines an algebra structure over \(H \otimes A\), denoted by \(H \triangleright A\) and called semidirect product at the right (or simply right semidirect product) of \(A\) and \(H\).

The ‘left’ version is as follows: let \((A, \triangleright, H)\) be a left \(A\)–module algebra. A structure of algebra over \(H \otimes A\), denoted by \(A \triangleright H\) and called left semidirect product of \(A\) and \(H\), is defined by means of

\[ (a \otimes h)(a' \otimes h') = a(h_1(1) \triangleright a') \otimes h_2(2)h'. \quad (2.16) \]

Dual structures of the above ones are constructed in the following way. Let \((C, \bowtie, H)\) be a left \(C\)–comodule coalgebra. A coalgebra structure over \(C \otimes H\), denoted by \(C \triangleright H\) and called left semidirect product, is obtained if

\[ \Delta(c \otimes h) = c_{(1)} \otimes c_{(2)}^{(1)}h_{(1)} \otimes c_{(2)}^{(2)} h_{(2)}(2), \]

\[ \epsilon(c \otimes h) = \epsilon_C(c) \epsilon_H(h). \quad (2.17) \]
When \((C, \triangleright, H)\) is a right \(C\)-comodule coalgebra, the expressions
\[
\Delta(h \otimes c) = h^{(1)} \otimes c^{(1)} \otimes h^{(2)} c^{(2)}, \\
\epsilon(h \otimes c) = \epsilon_C(h) \epsilon_H(c),
\]
characterize a coalgebra structure over \(C \otimes H\) denoted by \(C \triangleright H\) and called right semidirect product of \(C\) and \(H\).

Let \(K\) and \(L\) be two bialgebras, such that \((L, \triangleleft, K)\) is a right \(K\)-module algebra and \((K, \blacktriangledown, L)\) a left \(L\)-comodule coalgebra. The tensor product \(K \otimes L\) is equipped simultaneously with the semidirect structures of algebra \(K \triangleright\triangleleft L\) and coalgebra \(K \blacktriangledown\triangleright L\). If the following compatible conditions are verified
\[
\epsilon(l \triangleleft k) = \epsilon(l) \epsilon(k), \\
\Delta(l \triangleleft k) = (l(1) \triangleleft k(1)) k(2) (1) \otimes k(2) (2), \\
\blacktriangledown (1) = 1 \otimes 1, \\
\blacktriangledown (k k') = (k^{(1)} \triangleleft k'^{(1)}) k'^{(2)} (1) \otimes k(2) k'(2) (2), \\
k(1) (l \triangleleft k(2)) \otimes k(1) (2) = (l \triangleleft k(1)) k(2) (1) \otimes k(2) (2),
\]
then \(K \triangleright\triangleleft L\) and \(K \blacktriangledown\triangleright L\) determine a bialgebra called (right–left) bicrossproduct and denoted by \(K \blacktriangledown\triangleright L\).

If \(K\) and \(L\) are two Hopf algebras then \(K \blacktriangledown\triangleright L\) has also an antipode given by
\[
S(k \otimes l) = (1 \otimes S(k^{(1)} l))(S(k^{(2)}) \otimes 1).
\]

On the other hand, let \(K\) and \(L\) be two bialgebras and \((L, \triangleright, K)\) and \((K, \blacktriangleleft, L)\) a left \(K\)-module algebra and a right \(L\)-comodule coalgebra, respectively, verifying the compatibility conditions
\[
\epsilon(\lambda \blacktriangleleft \kappa) = \epsilon(\lambda) \epsilon(\kappa), \\
\Delta(\lambda \blacktriangleleft \kappa) \equiv (\lambda \blacktriangleleft \kappa)(1) \otimes (\lambda \blacktriangleleft \kappa)(2) = (\lambda^{(1)} \blacktriangleleft \kappa^{(1)}) \otimes \lambda^{(2)} \blacktriangleleft \kappa^{(2)}, \\
\blacktriangleleft (1) = 1 \otimes 1, \\
\blacktriangleleft (\kappa \kappa') = \kappa^{(1)}(1) \kappa'^{(1)}(1) \otimes \kappa^{(1)}(2) \kappa'^{(2)}(2), \\
\lambda^{(2)}(1) \otimes (\lambda^{(1)} \blacktriangleleft \kappa) \lambda^{(2)}(2) = \lambda^{(1)}(1) \otimes \lambda^{(1)}(2) \lambda^{(2)} \blacktriangleleft \kappa.
\]
then \(L \triangleright\blacktriangleleft K\) and \(L \blacktriangleleft\triangleright K\) determine a bialgebra called (left–right) bicrossproduct denoted by \(L \blacktriangleleft\triangleright K\).

If \(K\) and \(L\) are two Hopf algebras then \(L \blacktriangleleft\triangleright K\) has an antipode defined by
\[
S(\lambda \otimes \kappa) = (1 \otimes S(\kappa^{(1)}))(S(\lambda^{(2)}) \otimes 1).
\]

Note that both bicrossproduct structures are related by duality. Effectively, it can be proved that if \(K\) and \(L\) are two finite dimensional bialgebras, and the right \(K\)-module algebra \((L, \triangleleft, K)\) and the left \(L\)-comodule coalgebra \((K, \blacktriangleleft, L)\) verify the conditions \((2.19)\), then \((K \triangleright\blacktriangleleft L)^* = K^* \blacktriangledown\triangleright L^*\).
2.5 Star structures over bicrossproduct Hopf algebras

The following original results show how construct dual bases and *-structures over Hopf algebras with the structure of bicrossproduct when the corresponding objects for the factors of the bicrossproduct are known [11].

**Theorem 2.1.** Let \( H = K \bowtie L \) be a Hopf algebra with structure of bicrossproduct, and \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \) nondegenerate pairings for the pairs \((K, K^*)\) and \((L, L^*)\), respectively. Then the expression

\[
\langle kl, \kappa \lambda \rangle = \langle k, \kappa \rangle_1 \langle l, \lambda \rangle_2. \tag{2.23}
\]

defines a nondegenerate pairing between \( H \) and \( H^* \).

**Proof.** Firstly note that

\[
\langle 1, \kappa \lambda \rangle = \langle 1_K \otimes 1_L, \kappa \lambda \rangle = \langle 1_K, \kappa \rangle \langle 1_L, \lambda \rangle = \epsilon(\kappa)\epsilon(\lambda) = (\epsilon \otimes \epsilon)(\kappa \otimes \lambda) = \epsilon(\kappa \lambda). \tag{2.24}
\]

On the other hand

\[
\langle kl, (\kappa\lambda)(\kappa'\lambda') \rangle = \langle kl, \kappa(\lambda_1 \triangleright \kappa')\lambda_2 \lambda' \rangle = \langle k, \kappa(\lambda_1 \triangleright \kappa') \rangle_1 \langle l, \lambda_2 \lambda' \rangle_2
\]

\[
= \langle k_1, \kappa \rangle_1 \langle k_2, \lambda_1 \triangleright \kappa' \rangle_1 \langle l_1, \lambda_2 \rangle_2 \langle l_2, \lambda' \rangle_2
\]

\[
= \langle k_1, \kappa \rangle_1 \langle k_2, \lambda_1 \rangle \langle l_1, \lambda_2 \rangle_2 \langle l_2, \lambda' \rangle_2
\]

\[
= \langle k_1, \kappa \rangle_1 \langle k_2, \lambda_1 \rangle \langle l_1, \lambda_2 \rangle_2 \langle l_2, \lambda' \rangle_2
\]

\[
= \langle k_1, \kappa \rangle_1 \langle k_2, \lambda_1 \rangle \langle l_1, \lambda_2 \rangle_2 \langle l_2, \lambda' \rangle_2
\]

\[
= \langle k_1, \kappa \rangle_1 \langle k_2, \lambda_1 \rangle \langle l_1, \lambda_2 \rangle_2 \langle l_2, \lambda' \rangle_2
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\[
= \langle k_1, \kappa \rangle_1 \langle k_2, \lambda_1 \rangle \langle l_1, \lambda_2 \rangle_2 \langle l_2, \lambda' \rangle_2
\]

\[
= \langle k_1, \kappa \rangle_1 \langle k_2, \lambda_1 \rangle \langle l_1, \lambda_2 \rangle_2 \langle l_2, \lambda' \rangle_2
\]

Similarly for the identities

\[
\langle kl, 1 \rangle = \epsilon(kl), \quad \langle (kl)(k'l'), \kappa \lambda \rangle = \langle (kl) \otimes (k'l'), \Delta(\kappa \lambda) \rangle. \tag{2.26}
\]

Hence, it is proved that \( \langle \cdot, \cdot \rangle \) is a bialgebra pairing. The pairing is nondegenerate. Effectively, fixed a basis \( \langle l_i \rangle_{i \in I} \) of \( L \), the coproduct can be written as

\[
\Delta(h) = \sum_{i \in I} a_i(h) \otimes l_i, \tag{2.27}
\]

with \( a_i : H \to K \). Let us suppose that

\[
\langle h, \eta \rangle = 0, \quad h \in H, \ \eta \in H^* . \tag{2.28}
\]

Expression (2.28) can be rewritten as

\[
\langle h, \kappa \lambda \rangle = \langle \Delta(h), \kappa \otimes \lambda \rangle = \sum_{i \in I} \langle a_i(h), \kappa \rangle_1 \langle l_i, \lambda \rangle_2 = \sum_{i \in I} \langle a_i(h), \kappa \rangle_1 l_i, \lambda \rangle_2 = 0,
\]
where $\kappa \in K^*$, $\lambda \in L^*$. Since $\langle \cdot, \cdot \rangle_2$ is nondegenerate and $\langle \cdot, \cdot \rangle_1$ is also nondegenerate we get that $a_i(h) = 0$, hence $\Delta(h) = 0$. Finally, using the counit axiom

$$h = (\epsilon \otimes \text{id}) \circ \Delta(h) = 0,$$

we have proved that the pairing is left nondegenerate. In a similar way it is proved that the pairing is nondegenerate at the right. Using the fact that the last equality of (2.29) is a consequence of the two first ones when the pairing is nondegenerate, we conclude that the bilinear form (2.23), which is a bialgebra pairing, is also a pairing of Hopf algebras. \hfill \Box

**Corollary 2.1.** With the pairing and the notation defined in the previous theorem if $(k_m)$ and $(\kappa_m)$ are dual bases for $K$ and $K^*$, and $(l_n)$ and $(\lambda_n)$ are dual bases for $L$ and $L^*$, then $(k_m l_n)$ and $(\kappa_m \lambda_n)$ are dual bases for $H$ and $H^*$. In other words, if $\langle k_m, \kappa_{m'} \rangle = \delta_{m'}^m$ and $\langle l_n, \lambda_{n'} \rangle = \delta_{n'}^n$ then $\langle k_m l_n, \kappa_{m'} \lambda_{n'} \rangle = \delta_{m}^{m'} \delta_{n}^{n'}$.

In the case of left-right bicrossproduct there is a similar result.

**Theorem 2.2.** Let us consider the bicrossproduct Hopf algebra $H = K \rhd L$. Supposing that $K$ and $L$ are equipped with $*$-structures with the following compatibility relation

$$(l \triangleleft k)^* = l^* \triangleleft S(k)^*,$$

(2.30)

Then the expression

$$(kl)^* = l^* k^*, \quad k \in K, \ l \in L,$$

(2.31)

determines a $*$–structure on the algebra sector of $H$.

**Proof.** The definition of a $*$–structure on $H$ has to be consistent with the algebra structure is an antimorphism, i.e.,

$$(lk)^* = k^* l^*, \quad k \in K, \ l \in L.$$

(2.32)

Since the product on $H$ establishes that

$$lk = k_{(1)}(l \triangleleft k_{(2)}),$$

(2.33)

and according to the definition (2.31)

$$(lk)^* = (l \triangleleft k_{(2)})^* (k_{(1)})^*.$$  

(2.34)

Using the product on $H$ one obtains

$$(lk)^* = [(k_{(1)})^*]_{(1)} \{(l \triangleleft k_{(2)})^* \triangleleft [(k_{(1)})^*]_{(2)}\}.$$  

(2.35)

Taking into account (2.30) and that the $*$–structure on $K$ is a coalgebra morphism the equality (2.35) becomes

$$(lk)^* = (k_{(1)})^* \{l^* \triangleleft [S(k_{(3)})^* k_{(2)}^*]\}.$$  

(2.36)

Finally, the property characterizing the antipode reduces this expression to (2.32). \hfill \Box
3 Induced representations for quantum algebras

Since the algebras involved in this work are equipped with a bicrossproduct structure different actions appear. In order to avoid any confusion we will denote them by the following symbols (or their symmetric for the corresponding right actions and coactions):

\(\triangleright (\blacktriangleleft)\): actions (coactions) of the bicrossproduct structure,
\(\vdash \): induced and inducting representations,
\(\succ (\prec)\): regular actions (coactions).

In the following we will show that the problem of the determination of the induced representations is reduced as a last resort to the expression of products in normal ordering. The next result will be very useful for this purpose.

**Proposition 3.1.** Let \(A\) be an associative algebra. The following relations hold:

\[
a^m a' = \sum_{k=0}^{m} \binom{m}{k} \text{ad}^k_a (a') a^{m-k}, \quad \forall a, a' \in A, \quad m \in \mathbb{N},
\]

\[
a'a^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} \text{ad}^k_a (a'),
\]

where

\[
\text{ad}^1_a (a') = aa' - a'a = [a, a'], \quad \text{ad}^0_a (a') = a'a - aa' = [a', a].
\]

**Proof.** The demonstration is by induction. The relations (3.1) are trivial identities for \(m = 0\). Let us suppose that the first expression is true for \(m \in \mathbb{N}\), then for \(m + 1\) we have

\[
a^{m+1} a' = a(a^m a') = a \sum_{k=0}^{m} \binom{m}{k} \text{ad}^k_a (a') a^{m-k}
\]

\[
= \sum_{k=0}^{m} \binom{m}{k} [\text{ad}^k_a (a') a + \text{ad}^{k+1}_a (a')] a^{m-k}
\]

\[
= \sum_{k=0}^{m} \binom{m}{k} \text{ad}^k_a (a') a^{m-k+1} + \sum_{k=0}^{m} \binom{m}{k} \text{ad}^{k+1}_a (a') a^{m-k} \quad (3.3)
\]

\[
= \sum_{k=0}^{m+1} \binom{m+1}{k} \text{ad}^k_a (a') a^{m+1-k+1}
\]

The proof of the second identity (3.1b) is similar. \(\square\)

Note that in an appropriate topological context, where it is allowed the convergence and the reordering of series, expressions (3.1) carry to the usual relation between adjoint action and
exponential mapping:

\[ e^a a' = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} \text{ad}^k_a (a') a^{m-k} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \text{ad}^k_a (a') a^{m-k} \]

\[ = \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \frac{1}{k!(m-k)!} \text{ad}^k_a (a') a^{m-k} = e^{\text{ad}_a (a')} e^a, \tag{3.4} \]

or equivalently

\[ e^a a' e^{-a} = e^{\text{ad}^*_a (a')} . \tag{3.5} \]

For the other adjoint action taking into account that \( \text{ad}_a^* = \text{ad}_a^r \) we get an analogous relation

\[ e^{-a} a' e^a = e^{\text{ad}^*_a (a')} . \tag{3.6} \]

3.1 General case

Let us consider a nondegenerate triplet \((H, \mathcal{H}, \langle \cdot, \cdot \rangle)\). Let \( L \) be a commutative subalgebra of \( H \) and \( \{l_1, \ldots, l_s\} \), a system of generators of \( L \) which can be completed with \( \{k_1, \ldots, k_r\} \), in such a way that \( \langle l_n \rangle_{n \in \mathbb{N}^s} \) is a basis of \( L \) and \( \langle k_m l_n \rangle_{(m,n) \in \mathbb{N}^r \times \mathbb{N}^s} \) a basis of \( H \). Moreover, suppose that there is a system of generators in \( \mathcal{H} \), \( \{\kappa_1, \ldots, \kappa_r, \lambda_1, \ldots, \lambda_s\} \), such that \( \langle \kappa_m \lambda_n \rangle_{(m,n) \in \mathbb{N}^r \times \mathbb{N}^s} \) is a basis of \( \mathcal{H} \) dual of that of \( H \) with the pairing

\[ \langle k_m l_n, \kappa^{m'} \lambda^{n'} \rangle = m! n! \delta_{m}^{m'} \delta_{n}^{n'} . \tag{3.7} \]

We are interested in the description of the representation induced by the character of \( L \) determined by \( a = (a_1, \ldots, a_s) \in \mathbb{K}^s \), i.e.,

\[ 1 \mapsto l_n = a_n = a_1^{n_1} \cdots a_s^{n_s}, \quad n \in \mathbb{N}^s . \tag{3.8} \]

The elements \( f \) of \( \text{Hom}_{\mathbb{K}}(H, \mathbb{K}) \) verifying the invariance condition

\[ f(hl) = f(h) \cdot l, \quad \forall l \in L, \quad \forall h \in H, \tag{3.9} \]

constitute the carrier space \( \mathbb{K}^\uparrow = \text{Hom}_{L}(H, \mathbb{K}) \) of the induced representation. Identifying \( \text{Hom}_{\mathbb{K}}(H, \mathbb{K}) \) with \( \mathcal{H} \) using the pairing, the elements of \( f \in \mathbb{K}^\uparrow \) can be written as

\[ f = \sum_{(m,n) \in \mathbb{N}^r \times \mathbb{N}^s} f_{mn} \kappa^m \lambda^n . \tag{3.10} \]

The equivariance condition \((3.9)\)

\[ \langle hl, f \rangle = \langle h, f \rangle \cdot l, \quad \forall l \in L, \quad \forall h \in H, \tag{3.11} \]

combined with duality gives the following relation between the coefficients \( f_{mn} \)

\[ m! n! f_{mn} = \langle k_m l_n, f \rangle = \langle k_m, f \rangle a_n = m! f_{m0} a_n . \tag{3.12} \]
Hence, the elements of the carrier space of the induced representation are
\[ f = \kappa \psi, \quad \kappa \in \mathcal{K}, \] (3.13)
where \( \psi = e^{a_1 \lambda_1} \cdots e^{a_s \lambda_s} \), and \( \mathcal{K} \) is the subspace of \( \mathcal{H} \) generated by the linear combinations of the ordered monomials \( (\kappa^m)_{m \in \mathbb{N}^r} \). Since \( \psi \) is invertible (it is product of exponentials) there is an isomorphism between the vector spaces \( \mathcal{K} \) and \( \mathbb{K}^\uparrow \) given by \( \kappa \rightarrow \kappa \psi \).

The action of \( h \in H \) over the elements of \( \mathbb{K}^\uparrow \) is determined knowing the action over the basis elements \( (\kappa^p \psi)_{p \in \mathbb{N}^r} \) of this space. So, putting
\[
(k^p \psi) \triangleright h = \sum_{(m,n) \in \mathbb{N}^r \times \mathbb{N}^s} [h]_{mn}^p \kappa^m \lambda^n, \quad p \in \mathbb{N}^r, \] (3.14)
the constants \( [h]_{mn}^p \) can be evaluated by means of duality
\[
m! n! [h]_{mn}^p = \langle (k^p \psi) \triangleright h, k_m l_n \rangle = \langle k^p \psi, hk_m l_n \rangle = \langle k^p \psi, h k_m a_n \rangle. \] (3.15)

The properties of the action allow to compute it only for the generators of \( H \) instead of considering an arbitrary element \( h \) of \( H \). Finally, all that reduces to write the product \( hk_m \) in normal ordering to get the value of the paring in (3.15). However in many cases this task is very cumbersome, for this reason now our objective is to take advantage of the bicrossproduct structure to simplify the job.

### 3.2 Quantum algebras with bicrossproduct structure

In the following we will restrict ourselves to Hopf algebras having a bicrossproduct structure like \( H = \mathcal{K} \ltimes \triangleright \mathcal{L} \), such that the first factor is cocommutative and the second commutative.

We are interested in the construction of the representations induced by ‘real’ characters of the commutative sector \( \mathcal{L} \). We will show that the solution of this problem can be reduced to the study of certain dynamical systems which present, in general, a non linear action.

Let us start adapting the construction presented in the previous subsection 3.1 to the bicrossproduct Hopf algebras \( H = \mathcal{K} \ltimes \triangleright \mathcal{L} \). Let us suppose that the algebras \( \mathcal{K} \) and \( \mathcal{L} \) are finite generated by the sets \( \{k_i\}_{i=1}^r \) and \( \{l_i\}_{i=1}^s \), respectively, such that the generators \( k_i \) are primitive.

We also assume that \( (k_n)_{n \in \mathbb{N}^r} \) and \( (l_m)_{m \in \mathbb{N}^s} \) are bases of the vector spaces underlying to \( \mathcal{K} \) and \( \mathcal{L} \), respectively. Let \( \mathcal{K}^* \) and \( \mathcal{L}^* \) be the dual algebras of \( \mathcal{K} \) and \( \mathcal{L} \), respectively, having dual systems to those of \( \mathcal{K} \) and \( \mathcal{L} \) with analogue properties to them. Hence, duality between \( H \) and \( H^* \) is given by
\[
\langle k_m l_n, k'^{m'} \lambda^{n'} \rangle = m! n! \delta_{m}^{m'} \delta_{n}^{n'}.
\] (3.16)

As we will see later these hypotheses are not, in reality, too restrictive. All these generator systems will be used to described the induced representations.

Let us consider the character of \( \mathcal{L} \) labeled by \( a \in \mathbb{C}^s \)
\[
1 \triangleright l_n = a_n, \quad n \in \mathbb{N}^s, \] (3.17)
the discussion of subsection 3.1 allows us to state the following theorem.
Theorem 3.1. The carrier space, $\mathbb{C}^\uparrow$, of the representation of $H$ induced by the character $a$ of $L$ (see (3.17)) is isomorphic to $K^\ast$ and is constituted by the elements of the form $\kappa \psi$, where $\kappa \in K^\ast$ and

$$\psi = e^{a_1 \lambda_1} e^{a_2 \lambda_2} \cdots e^{a_s \lambda_s}. \tag{3.18}$$

The induced action is given by

$$f \triangleleft h = \sum_{m \in \mathbb{N}^r} \kappa_m \langle h \frac{k_m}{m!}, f \rangle \psi, \quad h \in H, \ f \in \mathbb{C}^\uparrow. \tag{3.19}$$

The action of the generators of $K$ and $L$ in the induced representation will be given in the next theorem, which needs the introduction of some new concepts.

Since $L$ is commutative, it can be identified with the algebra of functions $F(\mathbb{R}^s)$ by means of the algebra morphism $L \to F(\mathbb{R}^s)$, $(l \mapsto \tilde{l})$, which maps the generators of $L$ into the canonical projections

$$\tilde{l}_j(x) = x_j, \quad \forall x = (x_1, x_2, \ldots, x_s) \in \mathbb{R}^s, \ j = 1, 2, \ldots, s. \tag{3.20}$$

The $\ast$–structure keeping invariant the generators chosen in $L$ is distinguished in a natural way by the above identification

$$l_j^* = l_j, \quad j = 1, 2, \ldots, s. \tag{3.21}$$

The characters (3.17) compatible with (3.21) are ‘real’, i.e., determined by the elements $a \in \mathbb{R}^n \subset \mathbb{C}^n$. We will restrict to them henceforth. Note that the character (3.17), with $a \in \mathbb{R}^n$, can be written now as

$$1 \triangleleft l = \tilde{l}(a). \tag{3.22}$$

The right action of $K$ on $L$ can be translated to $F(\mathbb{R}^s)$ because the generators of $K$ are primitive and, hence, they act by derivations on the $K$–module algebra of $K \bowtie L$. Thus, the generators $k_i$ induce vector fields, $X_i$, on $\mathbb{R}^s$ determined by

$$X_i \tilde{l} = \tilde{l} \triple<k_i>, \quad i = 1, 2, \ldots, r. \tag{3.23}$$

The corresponding flow, $\Phi_i : \mathbb{R} \times \mathbb{R}^s \to \mathbb{R}^s$, is implicitly defined by

$$(X_i f)(x) = (Df_{x, \Phi_i})(0), \tag{3.24}$$

where $f_{x, \Phi_i}(t) = f \circ \Phi_i^t(x)$ and $D$ is the derivative operator over real variable functions. Notice that, in general, the one-parameter group of transformations associated to the flow $\Phi_i$ is not globally defined.

Proposition 3.2. In the Hopf algebra $H = K \bowtie L$ the following relation holds

$$lk_m = \sum_{p \leq m} \binom{m}{p} k_{m-p} (l \triple<k_p>), \quad \forall l \in L, \forall m \in \mathbb{N}^r. \tag{3.25}$$
where the multi-combinatorial number is defined as product of usual combinatorial numbers or through multi-factorials

\[
\binom{m}{p} = \prod_{i=1}^{r} \binom{m_i}{p_i} = \frac{m!}{p!(m-p)!},
\]  

where the ordered relation over the multi-indices is given by

\[
p \leq m \iff p_1 \leq m_1, \ p_2 \leq m_2, \ldots, \ p_r \leq m_r,
\]

and if \( p \leq m \) the difference between \( m \) and \( p \) is well defined in \( \mathbb{N}^r \) by

\[
m - p = (m_1 - p_1, m_2 - p_2, \ldots, m_r - p_r).
\]

Proof. Let us consider an element \( l \) of \( L \) and a generator \( k_i \) of \( K \) in the associative algebra \( K \triangleleft L \). Taking into account the definition of the product in \( K \triangleleft L \) and that the generators \( k_i \) are primitive we can write

\[
\text{ad}_{k_i}^p(l) = [l, k_i] = l \triangleleft k_i,
\]

Picking out the second formula of (3.1) for \( a' = l \) and \( a = k_i \) we get

\[
l k_i^m = \sum_{p \leq m} \binom{m}{p} k_i^{m-p} (l \triangleleft k_i^p).
\]

This formula is valid for \( m \in \mathbb{N} \). The validity of the expression for a multi-index \( m \in \mathbb{N}^r \) is a direct consequence of the properties of the action \( \triangleleft \) and of the definitions of the multi-objects that has been introduced.

\[\square\]

\textbf{Theorem 3.2.} The explicit action of the generators of \( K \) and \( L \) in the induced representation of Theorem 3.1 realized in the space \( K^* \) is given by the following expressions:

\[
\kappa \triangleleft k_i = \kappa \prec k_i,
\]

\[
\kappa \triangleleft l_j = \kappa \hat{l}_j \circ \Phi_{(\kappa_1, \kappa_2, \ldots, \kappa_r)}(a),
\]

where \( i \in \{1, \ldots, r\} \), \( j \in \{1, \ldots, s\} \), the symbol \( \prec \) denotes the regular action of \( K \) on \( K^* \), and \( \Phi_{(\kappa_1, \kappa_2, \ldots, \kappa_r)} = \Phi_{k_r} \circ \cdots \circ \Phi_{k_2} \circ \Phi_{k_1} \).

Proof. For the first expression we apply (3.19) to the case \( h = k_r \)

\[
(\kappa \psi) \triangleleft k_i = \sum_{m \in \mathbb{N}^r} \kappa^m (k_i, \kappa \psi) \psi = \sum_{m \in \mathbb{N}^r} \kappa^m \langle k_i, k_m \psi \rangle (1_L, \psi) \psi = \sum_{m \in \mathbb{N}^r} \kappa^m \langle k_m \psi \rangle (k_i \psi).
\]

For the third equality we use that \( (1_L, \psi) = 1 \), and the last one is based on the fact that \( \frac{1}{m!} \kappa^m \otimes k_m \) is the \( T \)-matrix [22] of the pair \( (K^*, K) \).
The computation of the action of \( l_j \) is more complicated

\[
(k\psi) \circ l_j = \sum_{m \in \mathbb{N}^r} \kappa^m \langle l_j \frac{m}{m}, \kappa\psi \rangle \psi
\]

\[
= \sum_{m \in \mathbb{N}^r} \kappa^m \left( \sum_{p \leq m} \left( \begin{array}{c} m \\ p \end{array} \right) \frac{1}{m!} k_{m-p}(l_j < k_p), \kappa\psi \rangle \psi
\]

\[
= \sum_{m \in \mathbb{N}^r} \sum_{p \leq m} \frac{1}{p!(m-p)!} \kappa^m \langle k_{m-p}(l_j < k_p), \kappa\psi \rangle \psi
\]

\[
= \sum_{m \in \mathbb{N}^r} \sum_{p \leq m} \frac{1}{p!(m-p)!} \kappa^m \langle k_{m-p}(l_j < k_p), \kappa\psi \rangle \psi
\]

\[
= \sum_{m \in \mathbb{N}^r} \sum_{p \leq m} \frac{1}{p!(m-p)!} \kappa^m \langle k_{m-p}(l_j < k_p), \kappa\psi \rangle \psi
\]

\[
= \sum_{m \in \mathbb{N}^r} \frac{1}{m!} \kappa^m \langle k_m, \kappa\psi \rangle \sum_{p \in \mathbb{N}^r} \frac{1}{p!} \kappa^p (1 - (l_j < k_p)) \psi
\]

\[
= \sum_{m \in \mathbb{N}^r} \frac{1}{m!} \kappa^m \langle k_m, \kappa\psi \rangle \sum_{p \in \mathbb{N}^r} \frac{1}{p!} \kappa^p (1 - (l_j < k_p)) \psi
\]

\[
= \kappa \sum_{p \in \mathbb{N}^r} \frac{1}{p!} \kappa^p (1 - (l_j < k_p)) \psi
\]

\[
= \kappa \left[ \sum_{p \in \mathbb{N}^r} \frac{1}{p!} \kappa^p X_p \bigg| \frac{d^i}{dx^i} \right] \psi.
\]

In the second equality of (3.33) Proposition 3.2 has been used. The next three are simple reorderings of the sums. The sixth equality is a consequence of the equivariance property and of the commutativity in \( \mathcal{K}^* \). The definitions of the duality form in the bicrossproduct structure, of the \( T \)-matrix of the algebra \( \mathcal{K} \), of the identification of \( L \) with the algebra of functions \( F(\mathbb{R}^s) \) are successively applied in the next equalities. Finally, it is defined \( X_p = X_p \mathcal{L} \ldots X_2 \mathcal{L} X_1 \mathcal{L} \) in terms of the vector fields associated to the generators \( k_i \).

On the other hand, from relation (3.24) between the flow \( \Phi_t \) and \( X_1 \) one gets

\[
f \circ \Phi_t(x) = f_{x,\Phi_t}(t) = (e^{tD} f_{x,\Phi_t})(0) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n(D^n f_{x,\Phi_t})(0) = \sum_{n=0}^{\infty} \frac{1}{n!} t^n (X_1^n f)(x), \quad (3.34)
\]

for any regular function \( f \in F(\mathbb{R}^s) \). So, to get the expression of the action established in the theorem it suffices to take \( f = \hat{l}_j \), \( x = a \), and replacing formally the real number \( t \) by \( \kappa^i \), making successively \( i = 1, \ldots, r \), and substitute the relation obtained in (3.33).

Remark that the inverse order in product \( X_p = X_r \mathcal{L} \ldots X_2 \mathcal{L} X_1 \mathcal{L} \) and in the flow composition \( \Phi_{t_1,t_2,...,t_r} = \Phi_{t_r} \ldots \Phi_{t_2} \Phi_{t_1} \) is due to that the action of \( \mathcal{K} \) on \( \mathcal{L} \) is at right.

When \( H \) is the deformed enveloping algebra of a semidirect product with Abelian kernel and the sector \( \mathcal{K} \) is nondeformed then the first expression of (3.31) says that the representation of \( \mathcal{K} \) is the same that in the nondeformed case. On the other hand, the generators of \( \mathcal{L} \) act as multiplication operators affected by the deformation.
4 Modules and representations

The deep relationship between representations and modules (see [1]) allow to reformulate the theory of induced representations for quantum algebras that we have developed in the previous section from the perspective of module theory.

4.1 Regular modules

The objective of this section is to describe the four regular $H$–modules associated to a Hopf algebra $H$: $(H, \prec, H)$, $(H^*, \succ, H)$, $(H, \succ, H)$ and $(H^*, \prec, H)$; $H^*$ is the dual of $H$ in the sense of nondegenerate pairing (see subsection 2.1).

It is well known the existence of theorems proving that, essentially, all the commutative or cocommutative Hopf algebras are of the form $F(G)$ or $\mathbb{K}[G]$ (or $U(g)$) for any group $G$ [21]. So, the kind of bicrossproduct that we will consider can be described as

$$H = \mathbb{C}[K]\boxtimes F(L) \quad \text{or} \quad H = U(\mathfrak{t})\bowtie F(L), \quad (4.1)$$

where $K$ and $L$ are finite groups or Lie groups.

We will focus our attention in the case that both, $K$ and $L$, are Lie groups with associated Lie algebras $\mathfrak{t}$ and $\mathfrak{l}$, respectively. In this way, the dual of $H$ will be

$$H^* = F(K) \triangleright \triangleright U(\mathfrak{l}). \quad (4.2)$$

The clue for an effective description of the regular modules is the use of elements of $H$ and $H^*$ like

$$k\lambda \in H, \quad k \in K, \quad \lambda \in F(L),$$

$$\kappa l \in H^*, \quad \kappa \in F(K), \quad l \in L. \quad (4.3)$$

We will see that these elements describe completely the structures of the regular $H$–modules and are more convenient than the bases of ordered monomials.

**Theorem 4.1.** Let us consider elements $k, k' \in K$, $\lambda, \lambda' \in F(L)$, $\kappa \in F(K)$ and $l \in L$. The action on any of the four regular $H$–modules is:

$$(H, \prec, H) : \quad (k\lambda) \prec k' = kk'(\lambda \triangleright k'), \quad (k\lambda) \prec \lambda' = k\lambda\lambda';$$

$$(H^*, \succ, H) : \quad k' \succ (\kappa l) = (k' \triangleright \kappa)(k' \triangleright l), \quad \lambda' \succ (\kappa l) = \kappa(\lambda' \triangleright l);$$

$$(H, \succ, H) : \quad k' \triangleright (k\lambda) = k'k\lambda, \quad \lambda' \triangleright (k\lambda) = k(\lambda' \triangleright k)\lambda;$$

$$(H^*, \prec, H) : \quad (\kappa l) \prec \lambda' = (\kappa \triangleright k')l, \quad (\kappa l) \prec \lambda' = \kappa((l^{(1)}), \lambda')l^{(2)}l. \quad (4.4)$$

**Proof.** (1) The results relative to the modules $(H, \prec, H)$ and $(H, \succ, H)$ only require the use of the product defined on the semidirect product of algebras $U(\mathfrak{t})\bowtie F(L)$ (remember that for arbitrary elements $k, k' \in U(\mathfrak{t})$ and $\lambda, \lambda' \in F(L)$ such product is given by $(k \otimes \lambda)(k' \otimes \lambda') = kk'(1) \otimes (\lambda \triangleright k'(2))\lambda')$. In order to evaluate the action of $k'$ we take into account that $\Delta(k') = k'k'$. 

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(2) In the module algebra \((H^*, \triangleright, H)\) the action of \(k'\) is obtained by

\[
\langle k\lambda, k' \triangleright (k\lambda) \rangle = \langle (k\lambda) \triangleleft k', k\lambda \rangle = \langle kk'(\lambda \triangleleft k'), k\lambda \rangle \\
= \langle kk', k \rangle(\lambda \triangleleft k', l) = \langle k, k' \triangleright k \rangle(\lambda, k' \triangleright l) \\
= \langle k\lambda, (k' \triangleright k)(k' \triangleright l) \rangle. \tag{4.5}
\]

The action of \(l'\) is obtained in an analogous way

\[
\langle k\lambda, l' \triangleright (k\lambda) \rangle = \langle k\lambda, \triangleleft l', k\lambda \rangle = \langle k\lambda, l' \triangleright k \rangle(\lambda, \triangleright l) \\
= \langle k\lambda, (k' \triangleright k)(k' \triangleright l) \rangle. \tag{4.6}
\]

Notice that in the first expression of (4.5) and of (4.6) the symbol \(\triangleright\) represents the regular action of \((H^*, \triangleright, H)\), but in the last one it denotes the action of \((F(K), \triangleright, U(\mathfrak{g}))\) and of \((F(L), \triangleright, U(\mathfrak{l}))\), respectively.

(3) When the regular module \((H^*, \triangleleft, H)\) is taken in consideration, the following chains of equalities determine the action of \(k'\) and \(l'\), respectively:

\[
\langle (k\lambda) \triangleleft k', k\lambda \rangle = \langle k\lambda, k' \triangleright (k\lambda) \rangle = \langle k\lambda, k'k\lambda \rangle \\
= \langle k, k'k \rangle(l, \lambda) = \langle k \triangleleft k', k \rangle(l, \lambda) = \langle (k \triangleleft k')l, k\lambda \rangle; \tag{4.7}
\]

\[
\langle (k\lambda) \triangleleft l', k\lambda \rangle = \langle k\lambda, l' \triangleright (k\lambda) \rangle = \langle k\lambda, l' \triangleleft k \rangle(\lambda \triangleleft k) \\
= \langle k, k \rangle(l, (\lambda' \triangleleft k)\lambda) = \langle k, k \rangle(l, (\lambda' \triangleleft k))l, \lambda) \\
= \langle k, k \rangle(l^{(1)}, \lambda')(l^{(2)}, k)l, \lambda) = \langle k\lambda, (k\lambda) \rangle(l^{(1)}, \lambda')(l, \lambda) \\
= \langle k\lambda, l, k\lambda \rangle(l^{(1)}, \lambda') = \langle k\lambda(l^{(1)}), \lambda' l^{(2)} l, k\lambda \rangle. \tag{4.8}
\]

\(\square\)

Note that: (i) the action (4.8) is described in terms of the structure of \(U(\mathfrak{l})\) as right \(F(K)\)–comodule; and (ii) except the term \((l^{(1)}, \lambda')(l^{(2)})\) including a coaction, the action on the regular modules appears described by means of other actions, most of them regular.

From a computational point of view the following proposition and its corollary are very useful, since they allow to reduce the description of the regular modules to the study of the action of \(K\) on \(L\) associated to the structure of \(U(\mathfrak{g})\)–module of \(F(L)\).

Let us start fixing the notation to be used. Let \(r\) and \(s\) be the dimensions of the groups \(K\) and \(L\), respectively. Let us consider the basis \((k_i)_{i=1}^r\) of \(\mathfrak{k}\) and \((l_j)_{j=1}^s\) of \(\mathfrak{l}\), and the local coordinate systems of second kind associated to the above bases \((\kappa_i)_{i=1}^r\) and \((\lambda_j)_{j=1}^s\). Remember that using multi-index notation one has

\[
\langle k_n, \kappa_{n'} \rangle = n! \delta_n^{n'}, \quad \langle l_m, \lambda_{m'} \rangle = m! \delta_m^{m'}, \quad n, n' \in \mathbb{N}^r, \quad m, m' \in \mathbb{N}^s. \tag{4.9}
\]

Finally, let us denote by \(k\) the inverse map of the coordinate system \((\kappa_i)\), i.e.,

\[
k : \mathbb{R}^r \rightarrow K \\
t \mapsto e^{t_1 k_1} e^{t_2 k_2} \cdots e^{t_r k_r}. \tag{4.10}
\]
Proposition 4.1. For every $\lambda \in F(L)$ and $l \in L$ the following relation holds
\[
\langle l^{(1)}, \lambda \rangle l^{(2)} = \sum_{n \in \mathbb{N}} \frac{1}{n!} (k_n \triangleright l, \lambda) \kappa^n = \lambda(k(\kappa) \triangleright l).
\] (4.11)

Proof. Let us rewrite the coaction at the right of $F(K)$ on $l \in L$ as
\[
\triangleright l \equiv l^{(1)} \otimes l^{(2)} = \sum_{(m,n) \in \mathbb{N}^s \times \mathbb{N}^r} [l]^m_n l_m \otimes \kappa^n.
\] (4.12)
The pairing defined in the bicrossproduct in accordance with Theorem 2.1 allows to obtain the coordinates of $\triangleright l$ in terms of the action (dual of the coaction) of $U(k)$ on $U(l)$
\[
[l]^m_n = \frac{1}{m!n!} \langle \lambda^m \otimes k_n, \triangleright l \rangle = \frac{1}{m!n!} \langle \lambda^m, k_n \triangleright l \rangle.
\] (4.13)
Inserting the last expression in (4.12) one easily gets
\[
\langle l^{(1)}, \lambda \rangle l^{(2)} = \sum_{(m,n) \in \mathbb{N}^s \times \mathbb{N}^r} \frac{1}{m!n!} \langle \lambda^m, k_n \triangleright l \rangle \langle l_m, \lambda \rangle \kappa^n.
\] (4.14)
The sum on $m$ gives account of the action of the $T$–matrix associated to the pair $(U(l), F(L))$ and, hence, the expression (4.14) is simplified getting
\[
\langle l^{(1)}, \lambda \rangle l^{(2)} = \sum_{n \in \mathbb{N}^r} \frac{1}{n!} (k_n \triangleright l, \lambda) \kappa^n.
\] (4.15)
On the other hand, since
\[
\lambda(k(t) \triangleright l) = \langle k(t) \triangleright l, \lambda \rangle = \sum_{n \in \mathbb{N}^r} \frac{1}{n!} (k_n \triangleright l, \lambda) t^n,
\] (4.16)
in order to get $\langle l^{(1)}, \lambda \rangle l^{(2)}$ it is enough to perform the formal substitution $t_i \to \kappa_i$ in the expression $\lambda(k(t) \triangleright l)$. \hfill \Box

The above proposition allows to give an expression for $\langle l^{(1)}, \lambda \rangle l^{(2)}$ completely independent of the bases chosen in the algebras.

Corollary 4.1. Let $\hat{l}$ be the map
\[
\hat{l} : K \longrightarrow L \quad k \mapsto k \triangleright l
\] (4.17)
projecting the group $K$ on the orbit passing through $l \in L$. Then, for any $\lambda \in F(L)$ and any $l \in L$ one has
\[
\langle l^{(1)}, \lambda \rangle l^{(2)} = \lambda \circ \hat{l}.
\] (4.18)
Taking into account that in \((U(l),\succ,F(L))\) it is verified that
\[\lambda \succ l = \lambda(l)l, \quad \forall \lambda \in F(L), \forall l \in L\]  
(4.19)

Theorem 4.1 can be rewritten in a more explicit way.

**Theorem 4.2.** The action on each of the four regular \(H\)-modules is:

\[
\begin{align*}
(H, \prec, H) & : \quad (k\lambda) \prec k' = kk'(\lambda \prec k'), & (k\lambda) \prec \lambda' = k\lambda\lambda'; \\
(H^*, \succ, H) & : \quad k' \succ (k\lambda) = (k' \succ \kappa)(k' \succ l), & \lambda' \succ (k\lambda) = \lambda'(l)k\lambda; \\
(H, \succ, H) & : \quad k' \succ (k\lambda) = k'k\lambda, & \lambda' \succ (k\lambda) = k(\lambda' \prec k)\lambda; \\
(H^*, \prec, H) & : \quad (k\lambda) \prec k' = (\kappa \prec k')l, & (k\lambda) \prec \lambda' = \kappa(\lambda' \circ \hat{l})l;
\end{align*}
\]

where \(k, k' \in K\), \(\lambda, \lambda' \in F(L)\), \(\kappa \in F(K)\) and \(l \in L\).

The result of this theorem does not make reference to the nature of Lie groups \(K\) and \(L\), since it is formulated in terms of the regular actions and associated ones to the bicrossproduct structure. Thus, the theorem may be applied to other kinds of groups.

Note that, in general, the action of \(K\) on \(L\) is not globally defined. Hence, \(\hat{l}\) (4.17) only projects, in reality, a neighbourhood of the identity into the orbit of \(l\). Henceforth, \(\lambda \circ \hat{l}\) does not define, in general, a map over the whole \(K\) and the expression \((k\lambda) \prec \lambda' = \kappa(\lambda' \circ \hat{l})l\) only has sense enlarging the space \(F(K)\), for instance, including it inside spaces of formal series.

As a conclusion, we can say that in the description of the regular actions the computation of the left action of the group \(K\) on the group \(L\) is really the most important fact. From this point of view, the deformations used in this work may be interpreted as one-parameter families of nonlinear actions homotopically equivalent to the linear actions of the nondeformed cases.

### 4.2 Co-spaces and induction

In the context of noncommutative geometry the manifold \(X\) is replaced by the algebra \(F(X)\) of \(\mathcal{C}^\infty\) \(\mathbb{C}\)-valued functions on \(X\) as well as the Lie group \(G\) by the enveloping algebra \(U(\mathfrak{g})\) of its Lie algebra \(\mathfrak{g}\). Since \((F(X),\mathfrak{v},U(\mathfrak{g}))\) is a module algebra over the Hopf algebra \(U(\mathfrak{g})\), we can generalize the concept of \(G\)-space in algebraic terms [12].

Let \(H\) be a Hopf algebra. A left (right) \(H\)-co-space is a module algebra \((A,\triangleright, H)\) \((A,\triangleleft, H)\).

The morphisms among \(H\)-co-spaces are the morphisms of \(H\)-modules and the concepts of subco-space or quotient co-space are equivalent to module subalgebra or quotient module algebra, respectively. We have adopted the term of co-space instead of space to stress the dual character of \(A\) as way of describing the initial geometric object.

Given a pair of algebras with a non-degenerate pairing \((H, H', \langle , \rangle)\), we obtain, via dualization of the regular actions, the regular \(H\)-co-spaces \((H', \succ, H)\) and \((H', \prec, H)\).

The explicit description of the four regular modules studied in the previous subsection allows a complete analysis of the representations of the algebra \(H = U(\mathfrak{t})\triangleright F(L)\) induced by the one-dimensional modules of the commutative sector. As we will see, the left co-space \((H^*, \succ, H)\)
characterizes the carrier space of the induced representation and the right co-space \((H^*, \prec, H)\) determines the induced action of \(H\) on the carrier space.

Firstly, remember that the set of characters of the algebra \(F(L)\) is its spectrum. An important theorem by Gelfand and Naimark \(^{23}\) establishes the following isomorphism

\[
\text{Spectrum } F(L) \simeq L. \tag{4.21}
\]

Fixed \(l \in L\), the character (or the corresponding right \(F(L)\)-module over \(\mathbb{C}\)) is given by

\[
1 \mapsto \lambda = \lambda(l), \quad \lambda \in F(L). \tag{4.22}
\]

In order to construct the representation of \(H = U(\mathfrak{k}) \bowtie F(L)\) induced by (4.22) let us start determining the carrier space \(\mathbb{C}^\uparrow \subset H^*\). The element \(f \in H^*\) satisfies the equivariance condition if it verifies

\[
\lambda > f = \lambda(l)f, \quad \forall \lambda \in F(L). \tag{4.23}
\]

Expanding \(f\) in terms of the bases of \(\mathfrak{k}\) and \(I\)

\[
f = \sum_{(m,n) \in \mathbb{N}^r \times \mathbb{N}^s} f_m^n l_n, \tag{4.24}
\]

the equivariance condition gives the following relation among the coefficients \(f_m^n\)

\[
f_m^n = \frac{1}{m!n!} f_0^m \lambda^n(l), \quad m \in \mathbb{N}^r, n \in \mathbb{N}^s. \tag{4.25}
\]

Hence, the general solution is:

\[
f = \left( \sum_{m \in \mathbb{N}^r} \frac{1}{m!} f_0^m \lambda^m(l) \right) \left( \sum_{n \in \mathbb{N}^s} \frac{1}{n!} \lambda^n(l) l_n \right), \quad f_0^m \in \mathbb{C}. \tag{4.26}
\]

Taking into account the definition of the second kind coordinates \(\lambda_j\) over the group \(L\), the expression (4.26) can be rewritten in a more compact form

\[
f = \kappa l, \quad \kappa \in F(K). \tag{4.27}
\]

In other words, the carrier space of the induced representation admits a natural description in terms of products \textit{function/element}, introduced in (1.3), instead of terms of monomial bases.

The right regular action describes the action on the induced module, which can be translated to \(F(K)\) using the isomorphism \(F(K) \rightarrow \mathbb{C}^\uparrow (\kappa \mapsto \kappa l)\):

\[
\kappa \cdot l = \kappa \cdot l = \kappa(\lambda \circ l). \tag{4.28}
\]

Comparing these expressions with those of Theorem \(^3\\text{3.2}\) we observe that the action of the subalgebra \(U(\mathfrak{k})\) is given by the regular action. The action of the subalgebra \(F(L)\) is of multiplicative kind and the evaluation of the corresponding factor, from a computational point of view, is essentially reduced to obtain the one-parameter flows associated to the action of \(K\) on \(L\) derived of the bicrossproduct structure of the algebra \(H\).
4.3 Equivalence and unitarity of the induced representations

Let $\triangleright_l$ be the representation of $H = U(\mathfrak{t})\triangleright F(L)$ induced by $l \in L$ and $f_k$ the automorphism of $F(K)$ given by the regular action of an element $k \in K$, i.e., $f_k(\kappa) = k \triangleright \kappa$. Since
\[
[k \triangleright (\lambda \circ \hat{l})(k')](\lambda \circ \hat{l}(k') = \lambda((k'k) \triangleright l) = [\lambda \circ \hat{k} \triangleright l](k'),
\]
then one has that
\[
f_k(\kappa \triangleright_l \lambda) = k \triangleright [\kappa(\lambda \circ \hat{l})] = (k \triangleright \kappa)[k \triangleright (\lambda \circ \hat{l})] = f_k(\kappa)[\lambda \circ \hat{k} \triangleright l] = f_k(\kappa) \triangleright_l \lambda.
\]
Taking into account, besides, that the action of the subalgebra $U(\mathfrak{t})$ on the induced module is not affected by the choice of the element $l$ in $L$, we conclude that the $H$–modules $(\mathbb{C}^\uparrow, \triangleright_l, H)$ and $(\mathbb{C}^\uparrow, \triangleright_{k\circ \hat{l}}, H)$ are isomorphic via $f_k$.

The problem of the unitarity of the induced representation passes, firstly, for choosing a $*$–structure in $H$. The usual determination is to consider ‘hermitian operators’ a family of generators of $H$, but troubles, related with the real or complex nature of the deformation parameter, may appear [19, 23]. The point of view adopted here allows a simple solution of the problem: $U(\mathfrak{t})$ and $F(L)$ carry associated $*$–structures in a natural way. Explicitly,
\[
k^* = k^{-1}, \quad \forall k \in K,
\]
\[
\lambda^*(l) = \overline{\lambda(l)}, \quad \forall \lambda \in F(L), \quad \forall l \in L.
\]
Choosing in $H$ the $*$–structure associated to those given by (4.29), according to Theorem 2.2, the problem of the unitarization is easily solved. Firstly, the action of the elements $k \in K$ shows that the space $F(K)$ has to be restricted to the square-integrable functions with respect to the right invariant Haar measure $\mu$ over $K$ (i.e., $\mu(k \triangleright A) = \mu(A)$ with $A$ a $\mu$–measurable set in $K$).

In fact, it is necessary to restrict the space $\mathcal{H} = L^2(K, \mu)$ and to consider only the space $\mathcal{H}_\infty$ of $C^\infty$ functions, since the Lie algebra $U(\mathfrak{t})$ acts by means of differential operators over these functions. On the other hand, the elements of $F(L)$ act by a multiplicative factor and impose a new restriction in $\mathcal{H}_\infty$ because only the functions $\kappa$ such that $\kappa(\lambda \circ \hat{l})$ is also square-integrable (supposing that the action is global in the orbit of $l$) will be admissible. If $K$ is compact all that is automatically verified and in the opposite case there is a condition over the vanishing order of $\kappa$ at the infinity points. The results of this discussion are summarized in the following theorem.

**Theorem 4.3.** Let us consider an element $l \in L$ supporting a global action of the group $K$. The carrier space, $\mathbb{C}^\uparrow$, of the representation of $H$ induced by the character determined by $l$ is the set of elements of $H^*$ of the form
\[
k \lambda l, \quad \kappa \in F(K).
\]

There is an isomorphism between $\mathbb{C}^\uparrow$ and $F(K)$ given by the map $\kappa \mapsto k \lambda l$. The action induced by the elements of the form $k \in K$ and $\lambda \in F(L)$ in the space $F(K)$ is
\[
k \lambda l = \kappa(\lambda \circ \hat{l}).
\]
The modules induced by $l$ and $k ⋾ l$ are isomorphic. So, the induction algorithm establishes a correspondence between the space of orbits $L/K$ and the set of equivalence classes of representations.

If the group $K$ is compact the induced representation is unitary in the space $L^2(K)$, of square-integrable functions with respect to the right invariant Haar measure, when the $*$–structure given by Theorem 2.2, applied to the natural structures of the factors of the bicrossproduct $H = U(\mathfrak{k}) ⋿ ▶_F L$, is considered.

4.4 Local representations

The called local representations [25] in the deformed version appear when one induces from representations of the subalgebra $U(\mathfrak{k})$. Let us consider the following character of $U(\mathfrak{k})$

$$\kappa \in \text{Spectrum } U(\mathfrak{k}) \subset F(K), \quad k \vdash 1 = \kappa(k). \quad (4.32)$$

Since the Hopf algebra $U(\mathfrak{k})$ is, in general, non commutative, the set of characters may be very reduced, even it may be generated only by the counit. For this reason an interesting problem to be researched in the future is the study of the representations induced by representations of $U(\mathfrak{k})$ of dimension greater that one.

The carrier space of the representation induced by $\kappa$ is determined by the following equivariance condition

$$f \prec k = \kappa(k)f, \quad \forall k \in U(\mathfrak{k}). \quad (4.33)$$

The algebra $H^*$ can be consider as a left free $F(K)$–module and, hence, it is possible to fix a basis $(l_j)_{j \in J}$ of $U(l)$ such that $f \in H^*$ can be expressed in a unique form as

$$f = \sum_{j \in J} \kappa_j l_j, \quad \kappa_j \in F(K). \quad (4.34)$$

The equivariance condition can be written now as

$$\sum_{j \in J} (\kappa_j \prec k) l_j = \sum_{j \in J} \kappa(k)\kappa_j l_j, \quad \forall k \in U(\mathfrak{t}). \quad (4.35)$$

Taking into account that the elements $l_j$ constitute a basis of the $F(K)$–module $H^*$ the corresponding coefficients can be equating, obtaining

$$\kappa_j \prec k = \kappa(k)\kappa_j, \quad \forall k \in U(\mathfrak{t}). \quad (4.36)$$

The previous equality (4.36) implies that

$$\kappa_j(kk') = \kappa(k)\kappa_j(k'), \quad \forall k, k' \in K. \quad (4.37)$$

Choosing $k'$ equal to the identity element $e \in K$, one gets

$$\kappa_j = \kappa_j(e)\kappa, \quad \forall j \in J. \quad (4.38)$$
in this way the elements of $\mathbb{C}^\uparrow$ are of the form
\[ f = \kappa l, \quad l \in U(1). \] (4.39)
The map $U(1) \to \mathbb{C}^\uparrow$, defined by $l \mapsto \kappa l$, is an isomorphism of vector spaces. The representation can be realized in this way over $U(1)$ and the final result is
\[ k \mapsto l = \kappa(k) \kappa(l), \]
\[ \lambda \mapsto l = \lambda(l)l. \] (4.40)

5 Examples

5.1 Null-plane quantum Poincaré algebra

The null-plane quantum deformation of the $(1+1)$ Poincaré algebra, $U_z(p(1,1))$, is a $q$–deformed Hopf algebra that in a null-plane basis, $\{P_+, P_-, K\}$, has the form [26, 27]
\[ [K, P_+] = \frac{1}{z}(e^{-2zP_+} - 1), \quad [K, P_-] = -2P_-, \quad [P_+, P_-] = 0; \]
\[ \Delta P_+ = P_+ \otimes 1 + 1 \otimes P_+, \quad \Delta X = X \otimes 1 + e^{-2zP_+} \otimes X, \quad X \in \{P_-, K\}; \]
\[ \epsilon(X) = 0, \quad X \in \{P_\pm, K\}; \]
\[ S(P_+) = -P_+, \quad S(X) = -e^{2zP_+}X, \quad X \in \{P_-, K\}. \] (5.1)

It has also the structure of bicrossproduct [28]
\[ U_z(p(1,1)) = K \triangleright \bowtie \ L, \]
where $K$ is a commutative and cocommutative Hopf algebra generated by $K$, and $L$ is the commutative Hopf subalgebra of $U_z(p(1,1))$ generated by $P_+$ and $P_-$. The right action of $K$ on $L$ comes determined by
\[ P_+ \triangleright K = \frac{1}{z}(e^{-2zP_+} - 1), \quad P_- \triangleright K = 2P_- . \] (5.2)
The left coaction of $L$ over the generator of $K$ is
\[ K \triangleright = e^{-2zP_+} \otimes K. \] (5.3)

In the dual Hopf algebra $F_z(P(1,1)) = K^* \triangleright \varphi^* \bowtie \varphi^*$ let us denote by $\varphi$ the generator of $K^*$ and by $a_\pm$ those of $L^*$. The left action of $L^*$ on $K^*$ is given by
\[ a_+ \triangleright \varphi = 2z(e^{-\varphi} - 1), \quad a_- \triangleright \varphi = 0, \] (5.4)
and the right coaction of $K^*$ over the generators of $L^*$ by
\[ \triangleright a_\pm = a_\pm \otimes e^{\mp 2\varphi}. \] (5.5)
With these actions we obtain the Hopf algebra structure of $F_z(P(1,1))$

\[
[a_+, a_-] = -2za_-, \quad [a_+, \varphi] = 2z(e^{-\varphi} - 1), \quad [a_-, \varphi] = 0;
\]

\[
\Delta a_\pm = a_\pm \otimes e^{\mp 2\varphi} + 1 \otimes a_\pm, \quad \Delta \varphi = \varphi \otimes 1 + 1 \otimes \varphi;
\]

\[
\epsilon(f) = 0, \quad f \in \{a_\pm, \varphi\};
\]

\[
S(a_\pm) = -a_\pm e^{\mp \varphi}, \quad S(\varphi) = -\varphi.
\]  

(5.6)

Theorem 2.1 allows to obtain easily a pair of dual bases in such a way that the duality between $U_z(p(1,1))$ and $F_z(P(1,1))$ is explicitly given by the pairing

\[
\langle K^m P^n P^p_+, \varphi^q a_r^s a^\delta_\ep \rangle = m!n!p! \delta^m_q \delta^n_r \delta^p_\delta.
\]  

(5.7)

Now let us consider the bicrossproduct structure of $U_z(p(1,1))$ as follows

\[
U_z(p(1,1)) = U(k) \bowtie F(T_z, 2),
\]  

(5.8)

where $k$ is the one-dimensional Lie algebra generated by $K$ and the group $T_z, 2$ is a deformation of the additive group $\mathbb{R}^2$ defined by the law

\[
(a' - e^{-2za'_+}, a'_+ + a_+) = (a_-, a_+, a_+).
\]  

(5.9)

The functions

\[
P_-(a_-, a_+) = a_-, \quad P_+(a_-, a_+) = a_+
\]  

(5.10)

define a global chart on $T_z, 2$. The $U(k)$–module algebra structure of $F(T_z, 2)$ taking part in the bicrossproduct is given by

\[
P_- \triangleleft K = 2P_-, \quad P_+ \triangleleft K = \frac{1}{z}(e^{-2za_+} - 1).
\]  

(5.11)

Hence, the vector field associated to $K$ is

\[
\hat{K} = 2P_- \frac{\partial}{\partial P_-} + \frac{1}{z}(e^{-2za_+} - 1) \frac{\partial}{\partial P_+}.
\]  

(5.12)

5.1.1 One-parameter flow

The vector field $\hat{K}$ has a unique equilibrium point at $(0,0)$, which has hyperbolic nature. The function

\[
h = P_-(e^{-2za_+} - 1)
\]  

(5.13)

is a first integral of $\hat{K}$. The computation of the integral curves require to solve the differential system

\[
\dot{a}_- = 2a_-, \quad \dot{a}_+ = \frac{1}{z}(e^{-2za_+} - 1).
\]  

(5.14)
If $z > 0$ the integral curves placed in the region $\alpha_+ < 0$ are given by

$$\alpha_-(s) = c_1 e^{2s}, \quad \alpha_+(s) = \frac{1}{2z} \ln(1 - e^{-2(s-c_2)}). \quad (5.15)$$

The second order system associated to them is

$$\ddot{\alpha}_-(s) = 4\alpha_-(s), \quad \ddot{\alpha}_+(s) = -\frac{2}{z} e^{-2\alpha_+}(e^{-2\alpha_+} - 1). \quad (5.16)$$

These equations may be interpreted as particles moving over a straight line under the action of repulsive potentials. From the expression of the integral curves we get the following flow

$$\Phi^s(\alpha_-, \alpha_+) = \left( \alpha_-, e^{2s} \frac{1}{2z} \ln(1 - e^{-2s}(1 - e^{2\alpha_+})) \right). \quad (5.17)$$

If, for example, we suppose that $z > 0$ then the curve that starts at the point $(\alpha_-, \alpha_+)$ is defined in the interval

$$s \in \begin{cases} 
\left( \frac{1}{2} \ln(1 - e^{2\alpha_+}), +\infty \right) & \alpha_+ < 0, \\
(-\infty, +\infty) & \alpha_+ \geq 0.
\end{cases} \quad (5.18)$$

Hence, the expression

$$e^{sK} (\alpha_-, \alpha_+) = (\alpha_- e^{2s}, \frac{1}{2z} \ln(1 - e^{-2s}(1 - e^{2\alpha_+}))) \quad (5.19)$$

defines a local action (except in the nondeformed limit $z \to 0$, where the action is global) of $R$ (the Lie group associated to the Lie algebra $\mathfrak{k}$) on $T_{z,2}$. The action decomposes $T_{z,2}$ in three strata:

i) the point at the origin, whose isotropy group is $R$,

ii) the four orbits constituted by the semi-axes,

iii) the rest of the set $T_{z,2}$. This last stratum has a foliation by one-dimensional orbits: deformed hyperbolic branches.

### 5.1.2 Regular co-spaces

The elements of $F_z(P(1,1))$ can be written as

$$\phi(\alpha_-, \alpha_+), \quad \phi \in F_z(R), \quad (\alpha_-, \alpha_+) \in T_{z,2} \quad (5.20)$$

instead of the monomials $\varphi^a a_+^r a_+^s$. The expression $\phi(\alpha_-, \alpha_+)$ does not denote a function, $\phi$, at the point $(\alpha_-, \alpha_+)$ but the product of these two elements in the algebra $F_z(P(1,1))$.

The structure of the regular co-space $(F_z(P(1,1)), \cdot, U_z(p(1,1)))$ is immediately obtained using Theorem 4.2. So,

$$(\phi(\alpha_-, \alpha_+)) \cdot e^{sK} = \phi(e^{sK} \cdot)(\alpha_-, \alpha_+),$$

$$(\phi(\alpha_-, \alpha_+)) \cdot P_- = \phi\alpha_- e^{2\varphi}(\alpha_-, \alpha_+),$$

$$(\phi(\alpha_-, \alpha_+)) \cdot P_+ = \phi \frac{1}{2z} \ln(1 - e^{-2\varphi}(1 - e^{2\alpha_+}))(\alpha_-, \alpha_+), \quad (5.21)$$
The action on \((F_z(P(1,1)), \succ, U_z(p(1,1)))\) is given by
\[
e^{sK} \succ (\phi(\alpha_-, \alpha_+)) = \phi(\cdot e^{sK})(\alpha_- e^{2s}, \frac{1}{2z} \ln(1 - e^{-2s}(1 - e^{2s\alpha_+}))),
\]
\[P_- \succ (\phi(\alpha_-, \alpha_+)) = \alpha_- \phi(\alpha_-, \alpha_+),
\]
\[P_+ \succ (\phi(\alpha_-, \alpha_+)) = \alpha_+ \phi(\alpha_-, \alpha_+).\] 

In the above expressions the dot stands for the argument of the function \(\phi = \phi(\cdot)\), and \(\varphi\) denotes the natural coordinate function over the group \(\mathfrak{g}\).

Note that the elements \((\alpha_-, \alpha_+) \in T_{z,2}\) describe the subalgebra of \(F_z(P(1,1))\) generated by \(a_-\) and \(a_+\). The pair \((\alpha_-, \alpha_+)\) is an eigenvector of the endomorphisms associated to the action (5.22) of the generators \(P_-\) and \(P_+\). This fact, together with the action of \(\mathfrak{g}\) on \(T_{z,2}\), guarantees that the subalgebra generated by \(a_-\) and \(a_+\) is stable under the action (5.22).

### 5.1.3 Induced representations

The representation of \(U_z(p(1,1))\) induced by the character \((\alpha_-, \alpha_+) \in T_{z,2}\) is given according to Theorem 4.3 by the following expressions
\[
\phi \triangleleft K = \phi',
\]
\[
\phi \triangleleft P_- = \phi \alpha_- e^{2\varphi},
\]
\[
(5.23)
\]
\[
\phi \triangleleft P_+ = \phi \frac{1}{2z} \ln(1 - e^{-2\varphi}(1 - e^{2s\alpha_+})).
\]

Choosing a representative in each orbit one gets a representative of every equivalence classes of induced representations. For instance, the representation induced by the equilibrium point \((0,0) \in T_{z,2}\) is
\[
\phi \triangleleft K = \phi',
\]
\[
\phi \triangleleft P_\mp = 0.
\]

The local representations induced by the character
\[
K^m \triangleright 1 = e^m,
\]

of the subalgebra \(U(\mathfrak{k})\) are given, according to (1.40), by
\[
e^{sK} \triangleright (\alpha_-, \alpha_+) = e^{sc}(\alpha_- e^{2s}, \frac{1}{z} \ln(1 - e^{-2s}(1 - e^{2s\alpha_+}))),
\]
\[P_- \triangleright (\alpha_-, \alpha_+) = \alpha_- (\alpha_-, \alpha_+),
\]
\[P_+ \triangleright (\alpha_-, \alpha_+) = \alpha_+ (\alpha_-, \alpha_+).
\]
5.2 Non-standard quantum Galilei algebra

The non-standard quantum Galilei algebra $U_z(\mathfrak{g}(1,1))$ is isomorphic to the quantum Heisenberg algebra $H_q(1)$ [29, 30] and to the deformed Heisenberg–Weyl algebra $U_\rho(HW)$ [31]. It can be obtained by contraction [31] of a non-standard deformation of the Poincaré algebra [27] (the null-plane quantum Poincaré).

The deformed Hopf algebra $U_z(\mathfrak{g}(1,1))$ has the following structure

$$[H, K] = -\frac{1 - e^{-4zP}}{4z}, \quad [P, K] = 0, \quad [H, P] = 0;$$
$$\Delta P = P \otimes 1 + 1 \otimes P, \quad \Delta X = X \otimes 1 + e^{-2zP} \otimes X, \quad X \in \{H, K\};$$
$$\epsilon(X) = 0, \quad X \in \{H, P, K\};$$
$$S(P) = -P, \quad S(X) = -e^{2zP}X, \quad X \in \{H, K\}. \tag{5.27}$$

In [28] it was proved that $U_z(\mathfrak{g}(1,1))$ has structure of bicrossproduct

$$U_z(\mathfrak{g}(1,1)) = K \triangleleft \bowtie L,$$

where $L$ is the commutative and non-cocommutative Hopf subalgebra $U_z(t_2)$ generated by $P$ and $H$, and $K$ is the commutative and cocommutative Hopf algebra (it is not a Hopf subalgebra of $U_z(\mathfrak{g}(1,1))$) generated by $K$.

The right action of $K$ on $L$ is given by

$$P \triangleleft K = [P, K] = 0, \quad H \triangleleft K = [H, K] = -\frac{1 - e^{-4zP}}{4z}. \tag{5.28}$$

The left coaction of $L$ over the generator of $K$ is

$$K \bowtie = e^{-2zP} \otimes K. \tag{5.29}$$

The corresponding function algebra $F_z(G(1,1))$ has a bicrossproduct structure dual of the above one

$$F_z(G(1,1)) = K^* \triangleright \bowtie L^*.$$ 

Let $v, x$ and $t$ be the generators dual of $K, P$ and $H$. The action of $L^*$ on $K^*$ is

$$x \triangleright v = -2zv, \quad t \triangleright v = 0, \tag{5.30}$$

and the coaction of $K^*$ over the generators of $L^*$ is

$$x \triangleleft = 1 \otimes x, \quad t \triangleleft = 1 \otimes t. \tag{5.31}$$

Action and coaction allow to obtain the Hopf algebra structure of $F_z(G(1,1))$

$$[t, v] = 0, \quad [x, v] = -2zv, \quad [t, x] = 2zt;$$
$$\Delta t = t \otimes 1 + 1 \otimes t, \quad \Delta x = x \otimes 1 + 1 \otimes x - t \otimes v, \quad \Delta v = v \otimes 1 + 1 \otimes v;$$
$$\epsilon(f) = 0, \quad f \in \{t, x, v\};$$
$$S(v) = -v, \quad S(x) = -x - tv, \quad S(t) = -t. \tag{5.32}$$
The nondegenerate pairing between $U_z(g(1,1))$ and $F_z(G(1,1))$ is given by

$$\langle K^m H^n P^p, v^q r^s \rangle = m!n!p! \delta_q^m \delta_r^n \delta_s^p. \quad (5.33)$$

In [1] we constructed the induced representations of $U_z(g(1,1))$, however now we will recover the same results but making use of its bicrossproduct structure

$$U_z(g(1,1)) = U(v) \rtimes F(T_z, 2), \quad (5.34)$$

where $v$ is the Lie algebra of the one-dimensional galilean boosts group and $T_z, 2$ is a deformation of the additive group $\mathbb{R}^2$ defined by

$$(b', a')(b, a) = (b' + e^{-2za'}b, a' + a). \quad (5.35)$$

In this definition we have assume that the deformation parameter is real. Note that the composition law (5.35) is obtained from the expression of the coproduct (5.27). The elements of $T_z, 2$ can be factorized as $(b, a) = (b, 0)(0, a)$. The coordinates on $T_z, 2$ will be denoted by $H$ and $P$, so

$$H(b, a) = b, \quad P(b, a) = a. \quad (5.36)$$

The $U(v)$–module algebra $F(T_z, 2)$ is described by the action

$$H \triangleright K = -\frac{1}{4z}(1 - e^{-4za}), \quad P \triangleleft K = 0. \quad (5.37)$$

The vector field associated to this action on $T_z, 2$ is

$$\hat{K} = -\frac{1}{4z}(1 - e^{-4za}) \frac{\partial}{\partial H}. \quad (5.38)$$

5.2.1 One-parameter flow

Let us observe that the vector field $\hat{K}$ has infinite fixed points $((b, 0), \ b \in \mathbb{R})$, and $P$ is an invariant. The integral curves

$$\dot{b} = -\frac{1}{4z}(1 - e^{-4za}), \quad \dot{a} = 0 \quad (5.39)$$

determine the autonomous system

$$b(s) = -\frac{1}{4z}(1 - e^{-4za})s + c_2, \quad a(s) = c_1. \quad (5.40)$$

The flow associated to the vector field $\hat{K}$, deduced from its integral curves, is

$$\Phi^s(b, a) = (b - \frac{1}{4z}(1 - e^{-4za})s, a). \quad (5.41)$$

It is defined for any value of $s$, giving a global action of $\mathcal{V}$ (the Lie group associated to $v$) on $T_z, 2$

$$e^{sK} \triangleright (b, a) = (b - \frac{1}{4z}(1 - e^{-4za})s, a). \quad (5.42)$$

The group $T_z, 2$ is decomposed in two strata under this action:

i) The set of points $(b, 0)$. Each of them is an orbit with stabilizer the group $\mathcal{V}$.

ii) The other stratum, constituted by the remaining elements of $T_z, 2$, is a foliation with sheets $\mathcal{O}_a = \{(b, a) | a \in \mathbb{R}^*, \ b \in \mathbb{R}\}$. The isotopy group of the point $(0, a) \in \mathcal{O}_a$ is $\{e\}$.
5.2.2 Regular co-spaces

Theorem 4.2 allows to construct the regular co-spaces in a direct and immediately way. Remember that $F_z(G(1,1))$ can be described considering elements of the form

$$\phi(b,a), \quad \phi \in F(\mathcal{M}), \quad (b,a) \in T_{z,2}, \quad (5.43)$$

instead of the monomial elements $t^q x^r$.

For $(F_z(G(1,1)), \prec, U_z(\mathfrak{g}(1,1)))$ one has

$$\phi(b,a) \prec e^K = \phi(e^K \cdot (b,a)), \quad (5.44)$$

and for $(F_z(G(1,1)), \succ, U_z(\mathfrak{g}(1,1)))$

$$e^K \succ (\phi(b,a)) = \phi(\cdot e^K)(b - \frac{1}{4z}(1 - e^{-4za})s,a), \quad H \succ (\phi(b,a)) = b\phi(b,a), \quad P \succ (\phi(b,a)) = a\phi(b,a). \quad (5.45)$$

The elements $(b,a) \in T_{z,2}$ describe the subalgebra of $F_z(G(1,1))$ generated by $t$ and $x$ which, as in the previous case, is stable under the action (5.43).

5.2.3 Induced representations

A representative of each equivalence class of induced representations, obtained according to the Theorem 4.3, is:

i) Considering the character given by $(b,0)$:

$$\phi \vdash e^K = \phi(e^K \cdot), \quad \phi \vdash H = \phi b, \quad \phi \vdash P = 0. \quad (5.46)$$

ii) Taking the character associated to $(0,a)$ the induced representation is

$$\phi \vdash e^K = \phi(e^K \cdot), \quad \phi \vdash H = \phi \frac{1}{4z}(1 - e^{-4za})v, \quad \phi \vdash P = \phi a. \quad (5.47)$$

The local representations induced by the character of $U(\mathfrak{so}(2))$ given by

$$K^m \vdash 1 = e^n, \quad (5.48)$$

are obtained applying the result (1.40):

$$e^K \vdash (b,a) = e^c(b - \frac{1}{4z}(1 - e^{-4za})s,a), \quad (5.49)$$

$$H \vdash (b,a) = b(b,a), \quad P \vdash (b,a) = a(b,a).$$

The actions of the generators in the way that they were presented in [1] can be easily deduced from these expressions.
5.3 Quantum kappa–Galilei algebra

A contraction of the quantum algebra $U_q(su(2))$ gives the deformation $U_\kappa(g(1,1))$ of the enveloping Galilei algebra in $(1 + 1)$ dimensions \[32\]. This quantum algebra is characterized by the following commutation relations and structure mappings:

\[
[H,K] = -P, \quad [P,K] = \frac{P^2}{2\kappa}, \quad [H,P] = 0;
\]
\[
\Delta H = H \otimes 1 + 1 \otimes H, \quad \Delta X = X \otimes 1 + e^{-H/\kappa} \otimes X, \quad X \in \{P,K\};
\]
\[
\epsilon(X) = 0, \quad X \in \{H,P,K\};
\]
\[
S(H) = -H, \quad S(X) = -e^{H/\kappa}X, \quad X \in \{P,K\}. \tag{5.50}
\]

The bicrossproduct structure of $U_\kappa(g(1,1))$ is

\[
U_\kappa(g(1,1)) = K\triangleright\bowtie L,
\]
with $L$ the commutative and non-cocommutative Hopf subalgebra $U_\kappa(t_2)$ spanned by $P$ and $H$, and $K$ the commutative and cocommutative Hopf subalgebra generated by $K$ (it is not a Hopf subalgebra of $U_\kappa(g(1,1))$). The right action of $K$ on $L$ is given by

\[
P \triangleright K = [P,K] = \frac{P^2}{2\kappa}, \quad H \triangleright K = [H,K] = -P, \tag{5.51}
\]
and the left coaction of $L$ over the generator of $K$ is

\[
K \bowtie = e^{-H/\kappa} \otimes K. \tag{5.52}
\]

The dual algebra has also a bicrossproduct structure

\[
F_\kappa(G(1,1)) = K^* \triangleright\bowtie L^*,
\]
where $K^*$ is generated by $v$ and $L^*$ by $x$ and $t$. The left action of $L^*$ on $K^*$ is defined by

\[
x \triangleright v = \frac{v^2}{2\kappa}, \quad t \triangleright v = -v/\kappa, \tag{5.53}
\]
and the right coaction of $K^*$ on $L^*$ is:

\[
\triangleright t = t \otimes 1, \quad \triangleright x = x \otimes 1 - t \otimes v. \tag{5.54}
\]

The above action and coaction allow to recover the Hopf algebra structure of $F_\kappa(G(1,1))$:

\[
[t,x] = -x/\kappa, \quad [x,v] = \frac{v^2}{2\kappa}, \quad [t,v] = -v/\kappa;
\]
\[
\Delta t = t \otimes 1 + 1 \otimes t, \quad \Delta x = x \otimes 1 + 1 \otimes x - t \otimes v, \quad \Delta v = v \otimes 1 + 1 \otimes v;
\]
\[
\epsilon(f) = 0, \quad f \in \{v,t,x\};
\]
\[
S(v) = -v, \quad S(x) = -x - tv, \quad S(t) = -t. \tag{5.55}
\]

The pairing between $U_\kappa(g(1,1))$ and $F_\kappa(G(1,1))$ is now given by

\[
\langle K^m P^n H^p, v^q x^r t^s \rangle = m!n!p! \delta^m_q \delta^p_s, \tag{5.56}
\]

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Let us interpret the algebraic structure of bicrossproduct of the quantum \( \kappa \)-Galilei algebra as

\[
U_\kappa(g(1,1)) = U(v) \bowtie F(T_{\kappa,2}),
\]

(5.57)

where \( T_{\kappa,2} \) is the group, deformation of the additive \( \mathbb{R}^2 \) group, defined by

\[
(a', b')(a, b) = (a' + e^{-b'/\kappa}a, b' + b).
\]

(5.58)

The elements \((a, b)\) can be factorized in the form \((a, b) = (a, 0)(0, b)\). The functions \( P \) and \( H \) defined by

\[
P(a, b) = a,
\]

\[
H(a, b) = b,
\]

(5.59)

determine a global chart on \( T_{\kappa,2} \).

The action of the generator \( K \) on the \( U(v) \)-module algebra \( F(T_{\kappa,2}) \) is given by

\[
H \bowtie K = -P,
\]

\[
P \bowtie K = \frac{P^2}{2\kappa}.
\]

(5.60)

Hence, the induced vector field is

\[
\dot{K} = \frac{P^2}{2\kappa} \frac{\partial}{\partial P} - P \frac{\partial}{\partial H}.
\]

(5.61)

### 5.3.1 One-parameter flow

The invariant points of the vector field \( \dot{K} \) are \((0, b)\). To get an invariant function under the action of \( \dot{K} \) is sufficient to determine firstly the one–forms, \( \eta \), verifying \( \dot{K} \rfloor \eta = 0 \). The general solution is

\[
\eta_\alpha = \alpha(dP + \frac{1}{2\kappa}PdH), \quad \alpha \in F(T_{\kappa,2}).
\]

(5.62)

Choosing \( \alpha_0 = 1/P \), the one–form \( \eta_\alpha \) is exact and invariant under \( \dot{K} \). So, the invariant function is \( h = Pe^{H/2\kappa} \). The autonomous system

\[
\dot{a} = \frac{a^2}{2\kappa}; \quad \dot{b} = -a,
\]

(5.63)

which determines the integral curves, is easily integrated. For the curves placed in the region \( a < 0 \) we find the following expressions:

\[
a(s) = - \frac{1}{c_1 + \frac{s}{2\kappa}},
\]

\[
b(s) = 2\kappa \ln(c_1 + \frac{s}{2\kappa}) + c_2.
\]

(5.64)

The associated second order equations

\[
\ddot{a} - \frac{a^3}{2\kappa^2} = 0, \quad \ddot{b} + \frac{b^2}{2\kappa} = 0,
\]

(5.65)
can be interpreted as particles moving in a straight line under forces depending on the position or the velocity, respectively. From the expression of the integral curves the flow associated to $\hat{K}$ is obtained:

$$\Phi^s(a,b) = \left( \frac{a}{1 - \frac{sa}{2\kappa}}, b + 2\kappa \ln(1 - \frac{sa}{2\kappa}) \right).$$

(5.66)

Note that the action of $\mathfrak{U}$ on $T_{\kappa,2}$,

$$e^{sK} \triangleright (a,b) = \left( \frac{a}{1 - \frac{sa}{2\kappa}}, b + 2\kappa \ln(1 - \frac{sa}{2\kappa}) \right),$$

(5.67)

is not global. The space $T_{\kappa,2}$ is decomposed in two strata under this action:

i) The set points of the form $(0, b)$. Each of them constitutes a 0–dimensional orbit with stabilizer $\mathfrak{U}$.

ii) The other stratum, constituted by the rest of the space, presents a foliation by one-dimensional sheets.

### 5.3.2 Regular co-spaces

The action on the regular co-space $(F_{\kappa}(G(1,1)), \prec, U_{\kappa}(\mathfrak{g}(1,1)))$ is obtained applying Theorem 4.2

$$\phi(a,b) \prec e^{sK} = \phi(e^{sK} \cdot)(a,b),$$

$$\phi(a,b) \prec P = \phi \left( \frac{a}{1 - \frac{sa}{2\kappa}} \right)(a,b),$$

$$\phi(a,b) \prec H = \phi \left( b + 2\kappa \ln(1 - \frac{sa}{2\kappa}) \right)(a,b),$$

(5.68)

with $\phi \in F(\mathfrak{U})$ and $(a,b) \in T_{\kappa,2}$.

The co-space $(F_{\kappa}(G(1,1)), \succ, U_{\kappa}(\mathfrak{g}(1,1)))$ is analogously described by

$$e^{sK} \succ (\phi(a,b)) = \phi(\cdot e^{sK})(a,b),$$

$$P \succ (\phi(a,b)) = a\phi(a,b),$$

$$H \succ (\phi(a,b)) = b\phi(a,b).$$

(5.69)

Similar comments to those of subsections 5.1.2 and 5.2.2 may be done here.

### 5.3.3 Induced representations

According to Theorem 4.3 each element $(a,b) \in T_{\kappa,2}$ induces a representation given by

$$\phi \triangleright e^{sK} = \phi(e^{sK} \cdot),$$

$$\phi \triangleright P = \phi \left( \frac{a}{1 - \frac{sa}{2\kappa}} \right),$$

$$\phi \triangleright H = \phi \left( b + 2\kappa \ln(1 - \frac{sa}{2\kappa}) \right),$$

(5.70)
which effectively coincides with that was obtained in \[1\].

The local representations induced by the character of $U(\mathfrak{u})$ given by

$$K^m \vdash 1 = c^m,$$

(5.71)

are obtained applying the result (4.40)

$$e^{sK} \vdash (a, b) = e^{sc}(\frac{a}{1 - \frac{as}{2\kappa}}, b + 2\kappa \ln(1 - \frac{as}{2\kappa})),

(5.72)$$

$$P \vdash (a, b) = a(b, a),

$$H \vdash (a, b) = b(b, a).$$

From these expressions the actions of the generators are easily obtained.

6 Concluding remarks

Remember that in \[1\] we introduced an algebraic method for constructing (co)induced representations of Hopf algebras based on the existence of a triplet composed by two Hopf algebras and a nondegenerate pairing between them such that there exists a paring of dual bases. However, the difficulty of the computation of the normal ordering of a product of elements increases with the number of algebra generators. In this work we avoid these troubles when the quantum algebra has a bicrossproduct structure.

We are able to define structures over a bicrossproduct Hopf algebra $H = K \triangleright \triangleleft L$ in terms of those of its components $K$ and $L$. So, theorem 2.1 gives a procedure to obtain dual bases of the pair $(H, H^*)$ starting from the dual bases of the components. Analogously, theorem 2.2 characterizes a $*$-structure for the algebra sector of $H$ from the $*$-structures defined on $K$ and $L$.

Our induction procedure is not a generalization of the induction method for Lie groups. We introduce the concept of co-space, which generalizes in an algebraical way the concept of $G$-space (being $G$ a transformation group), and we establish the connection between induced representations and regular co-spaces. There are different procedures for compute regular co-spaces but the introduction of the endomorphisms associated to the regular actions and the use of adjoint operators respect to the duality form simplifies extraordinarily the computations \[1\]. Note that vector fields have been used to compute commutators and the advisability of using exponential elements instead of monomial bases. For bicrossproduct Hopf algebras, like $H = K \triangleright \triangleleft L$, with $K$ cocommutative and $L$ commutative, theorems 3.1 and 3.2 establish a connection between the representations of $H$ induced by characters of $L$ and certain one-parameter flows. Although the proof is based on the use of pairs of dual bases the results so obtained are, essentially, independent of the bases used. Moreover, we can associate, in some sense, quantum bicrossproduct groups and dynamical systems via these flows. These relation we will be analyzed more detailed in a forthcoming paper.

The bicrossproduct Hopf algebras like $H = K \triangleright \triangleleft \mathcal{L}$, ($\mathcal{K}$ and $\mathcal{L}$ commutative and cocommutative, respectively, infinite dimensional algebras), has been studied interpreting $\mathcal{K}$ as the enveloping algebra $U(\mathfrak{h})$ of a Lie group $K$ and $\mathcal{L}$ as the algebra of functions over a Lie group $L$.  

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From this point of view, certain families of Hopf algebras that are deformations of semidirect products can be seen as homotopical deformations of the original actions.

The description of the regular co-spaces associated to $H$ may be done without monomial bases. Theorem 4.1 proves that the action on such co-spaces may be obtained using the action, deduced from the bicrossproduct structure, of the group $K$ over the group $L$. In this way the problems derived from the use of dual bases and the high dimension of the algebra $H$ are avoided.

The description of the (induced) representations appears as a corollary of the above mentioned theorems. The bicrossproduct algebra $H = (t)\triangleright F(L)$ gives, in a natural way, a $*$-structure for which the representations are, essentially, unitary. Theorem 4.3 discusses the equivalence of the induced representations establishing a correspondence among classes of induced representations and orbits of $L$ under the action of $K$. This result is in some sense analogous to the Kirillov orbits method [33]. The problem of the irreducibility of the representations is still open. Partial results for particular cases have been obtained; for instance, see ref. [10] for the standard quantum $(1+1)$ Galilei algebra and [14] for the quantum extended $(1+1)$ Galilei algebra.

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