Pion electromagnetic form factor, perturbative QCD, and large-$N_c$ Regge models

Enrique Ruiz Arriola$^1$ and Wojciech Broniowski$^{2,3}$

$^1$Departamento de Física Atómica, Molecular y Nuclear, Universidad de Granada, E-18071 Granada, Spain
$^2$The H. Niewodniczański Institute of Nuclear Physics, PL-31342 Kraków, Poland
$^3$Institute of Physics, Jan Kochanowski University, PL-25406 Kielce, Poland

(Dated: 21 July 2008)

We present a construction of the pion electromagnetic form factor where the transition from large-$N_c$ Regge vector meson dominance models with infinitely many resonances to perturbative QCD is built in explicitly. The construction is based on an appropriate assignment of residues to the Regge poles, which fulfills the constraints of the parton-hadron duality and perturbative QCD. The model contains a slowly falling off non-perturbative contribution which dominates over the perturbative QCD radiative corrections for the experimentally accessible momenta. The leading order and next-to-leading order calculations show a converging pattern which describes the available data within uncertainties, while the onset of asymptotic QCD takes place at extremely high momenta, $Q \sim 10^3 - 10^4\text{GeV}$. The method can be straightforwardly extended to study other form factors where the perturbative QCD result is available.

PACS numbers: 12.38.Lg, 11.30, 12.38.-t

Keywords: Pion electromagnetic form factor, large $N_c$ Regge models, perturbative QCD

I. INTRODUCTION

The composite nature of hadrons can be best seen by studying their electromagnetic form factors at a sufficiently large momentum transfer $|q|^2$. The pion, being the lowest $u$ and $d$ quark-antiquark excitation of the vacuum and identified with the would-be Goldstone massless mode of the spontaneously broken chiral symmetry, provides a simplest candidate to test our present knowledge on hadronic interactions. Due to relativistic and gauge invariance the pion charge form factor (we take $\pi^+$ for definiteness) can be written as

$$\langle \pi^+(p')|J_{\mu}^{\text{em}}(0)|\pi^+(p)\rangle = (p'^\mu + p^\mu) F(q^2)$$

with $q = p' - p$ and $J_{\mu}^{\text{em}}(x) = \sum_{u,d,s,...} e_q \bar{q}(x)\gamma_\mu q(x)$ is the electromagnetic current, with $e_q$ denoting the quark charge in units of the elementary charge. The charge normalization requires $F(0) = 1$.

The pion charge form factor has been the subject of intense experimental efforts. Moreover, it is expected to be measured at TJLAB in the space-like region of $1 \text{GeV}^2 \leq -t \leq 6 \text{GeV}^2$ with unprecedented high precision $\Delta(-tF(t)) \sim 0.02 \text{GeV}^2$. The results might be used as a stringent test of the perturbative QCD (pQCD) radiative corrections. Actually, in the space-like region where $t = -Q^2$, $F(t)$ is real and at large $Q^2$ values the pQCD methods can be applied, yielding asymptotically

$$F(-Q^2) = \frac{16\pi f_\pi^2\alpha(Q^2)}{Q^2} \left[1 + \frac{6.58}{\pi} \frac{\alpha(Q^2)}{Q^2} + \ldots\right].$$

$$Q^2 \gg M^2$$

with $f_\pi = 92.3\text{MeV}$ denoting the pion weak decay constant and $M$ the lowest vector meson mass. Further higher-order power corrections are of the order $O(1/Q^2)$ and correspond to higher twist operators. The form factor depends logarithmically on the scale through the running coupling constant

$$\alpha(Q^2) = \frac{4\pi}{\beta_0 \log(Q^2/\Lambda^2)}$$

$$\beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f.$$
behavior. In the region close to the zero momentum transfer chiral corrections become important \[20, 21\]. For time-like momenta the pion form factor becomes complex and can be related by crossing to the $e^+e^- \rightarrow \pi^+\pi^-$ annihilation amplitude, $\langle \pi^+\pi^- | F^{\mu\nu}(0) \rangle = F(s)(p^\mu + p'^\nu)$, where the final state interactions due to $\pi\pi$ scattering and unitarity play a crucial role \[22\]. While both the time-like and the space-like regions are related by an unsubtracted dispersion relation \[23\],

$$F(t) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{\text{Im} F(t') dt'}{t' - t - i\epsilon}, \quad (7)$$

the well-known time-like region does not determine unambiguously when the onset of the pQCD takes place. Actually, the single VMD model shows that even in the space-like region as low as $Q \sim m_\pi$, the traces of chiral logs and final state interactions are meager.

Given the fact that the pQCD effects cannot directly be observed at presently available energies, numerous phenomenological QCD-based approaches and model calculations have been suggested in order to understand the transition from the soft to hard scales. They include standard QCD sum rules \[24\], local-duality QCD sum rules \[25, 26\], light-cone QCD sum rules \[27\], nonlocal condensates \[28, 29\], Schwinger-Dyson equations \[30\], instanton-based models \[31, 32\], constituent quark models \[33\], nonlocal quark models \[34, 35\], etc. The scale of the onset of pQCD provoked heated debates in the past. The problem is crucial, as it provides a decisive fingerprint of the underlying quark-gluon substructure of the pion. We note that the upcoming lattice QCD calculations extending the work reported in \[36, 37, 38, 39, 40\] can directly verify this issue without necessarily spanning such a wide energy window as in the experiment. The reason is that a lot of progress has been achieved in extrapolating the lattice data to the chiral limit, which incorporates the enhancement and nonlinearities triggered by the chiral logs.

The class of calculations listed above contains quarks and gluons as explicit dynamical degrees of freedom, and hence requires a detailed knowledge of the pion wave function. On the other hand the parton-hadron duality implies that any hadronic property be describable in the purely hadronic language without an explicit reference to the basic fundamental fields. For instance, the success of the simple VMD fit for the pion charge form factor suggests the inclusion of further radially excited $J^G, J^{PC} = 1^+1^- -$ states, $\rho', \rho'', \rho''' \ldots$ ,

$$F(t) = \sum_{V=\rho, \rho'} c_V \frac{M_V^2}{M_V^2 - t}, \quad (8)$$

This finite sum involves states with a mass below $M_{V,\text{max}}$, the highest allowed vector meson mass which acts as a high energy cut-off. Thus, it could reliably reproduce the data (see below) in a region where $Q^2 < M_{V,\text{max}}^2$, and will only produce inverse integer powers of $Q^2$ asymptotically when $Q^2 \gg M_{V,\text{max}}^2$. This is in formal contradiction with Eq. \[8\], where there is no high energy cut-off and the behavior $1/(Q^2 \log Q^2)$ is obtained. Thus, infinitely many states are clearly needed. This complies to the 't Hooft large-$N_c$ limit \[11\], where any hadronic amplitude can be written in terms of tree diagrams with (infinitely many) mesons and glueballs. In particular, in the large-$N_c$ limit the pion form factor can be written in the form \[8\] with infinitely many resonances.

Based on the success of the Veneziano-Lovelace-Shapiro dual resonance model (see e.g. \[42, 43\] and references therein) Suura \[44\] and Frampton \[45\] proposed analytic models which have recently been resurrected and further elaborated by Dominguez \[46, 47\]. Incidentally, the resulting expressions for the pion charge form factor turn out to be quite similar to the AdS/CFT hard-wall and soft-wall calculations carried out in \[48, 49, 50\]. Despite the successful fit to the data, these calculations do not reproduce the formal asymptotic pQCD behavior, a fact which has been interpreted as an intrinsic limitation of the approach \[49\]. This poses an intriguing puzzle: how do hadronic large-$N_c$ models satisfy the QCD constraints, including the presence of logarithms? Quite generally, pQCD predicts integer powers and logarithms of $Q^2$, whereas the models of Refs. \[44, 45\] are able to generate fractional powers.

In the present paper we analyze the problem for the case of the pion charge form factor and show how the pQCD constraints can judiciously be implemented in a large-$N_c$ Regge model in an exact manner and at the same time preserve the good description of the experimental data. The essence of the approach is a careful assignment of coupling constants to the infinitely many resonances. As a result, the form \[8\] emerges from the infinite sum \[8\]. We term the mechanism the power-log transmutation, which essentially corresponds to a suitable superposition of fractional twist operators in the Regge model of Refs. \[44, 45\]. The present study follows our investigation of the two-point functions \[51, 52\]. In a previous paper \[53\] we have shown how the large-$N_c$ Regge models can be used to deal with the $\gamma^*\pi^0 \rightarrow \gamma$ transition form factor, where the radiative pQCD corrections characterized by the relevant anomalous dimensions are generated with the suitable QCD evolution equations.

**II. MESON DOMINANCE**

In this preparatory section we introduce the basic definitions and notation for the pion form factor in VMD models. The electromagnetic current is written as $J_{\mu,\text{em}}^\pi(x) = B^\mu(x)/2 + J_{\mu,\text{v}}^\pi(x)$ with $B^\mu(x) = \sum \bar{q}(x)\gamma^\mu q(x)/N_c$ being the baryon current and $J_{\mu,\text{v}}^\pi(x) = \sum \bar{q}(x)\gamma^\mu q(x)/2$ the isovector current. Using the isospin invariance, assumed throughout, we
have
\[
\langle \pi^n(p')|J_{V}^{\mu}(0)|\pi^n(p)\rangle = \epsilon^{abc}(p'^{\mu} + p'^{\nu}) F(q^{2}),
\]  
(9)
with \(|\pi^n(p)\rangle\) denoting a pion state, and \(a, b, c\) the Cartesian isospin indices. In the large-\(N_c\) limit the meson dominance of the pion charge form factor is the statement that one can parameterize the (iso-vector) current as a superposition of vector meson fields, \(\rho_{n,\mu}(x)\),
\[
J_{V}^{\mu,\alpha}(x) = \sum_{n} F_{V}(n)M_{V}(n)\rho_{n,\mu}^{\alpha}(x),
\]  
(10)
where \(n = 0\) corresponds to the ground state \(\rho(770)\) meson, and higher values of \(n\) to excited states. Correspondingly, the matrix element between the vacuum and the one-vector meson state is
\[
\langle 0|J_{V}^{\mu}(0)|\rho_{n}^{\alpha}(\epsilon)\rangle = \delta^{\alpha\beta} M_{V}(n)F_{V}(n)\epsilon^{\mu},
\]  
(11)
with \(\epsilon_{\mu}\) denoting the vector-meson polarization. The coupling constants may be determined from the electromagnetic decay \(\rho_{n} \rightarrow e^+e^-\) using the formula
\[
\Gamma(\rho_{n} \rightarrow e^+e^-) = \frac{4\pi\alpha^2 F_{V}(n)^2}{3 M_{V}(n)},
\]  
(12)
for the partial decay rate. For \(M_{V} = m_{\rho} = 770\text{MeV}\) and \(\Gamma(\rho \rightarrow e^+e^-) = 6.5\text{keV}\) one gets \(F_{V} = F_{\rho} = 150\text{MeV}\).

The two-point vector-current-vector current correlator is defined as
\[
\Pi_{V}^{\mu,\nu}(q) = i \int d^4xe^{-iq\cdot x} \langle 0|T \{ J_{V}^{\mu}(x)J_{V}^{\nu}(0)\} |0\rangle = \Pi_{V}(q^{2}) (q^{\mu}q^{\nu} - g^{\mu\nu}q^{2}) g^{\mu
u},
\]  
(13)
where
\[
\Pi_{V}(t) = \sum_{n} \frac{F_{V}(n)^2}{M_{V}(n)^2 - t},
\]  
(14)
The quark-hadron duality for large values of \(t\) in \(11\) requires the Regge model – parton model matching condition \(51,52\) for asymptotically large values of the radial quantum number \(n\),
\[
\frac{dF_{V}(n)^2}{dn^{2}} \sim \frac{N_{c}}{24\pi^{2}}.
\]  
(15)
For the radial Regge model (see next Section) \(dF_{V}^{2}(n)/dn = a = \text{const.}\), hence at large \(n\) we must have \(F_{V}(n) = \text{const.}\). \(51,52\)

The vector meson-pion-pion amplitude is
\[
\langle \pi^n(p')|\rho_{n,\mu}^{\beta}(0)|\pi^n(p)\rangle = (p'^{\mu} + p^{\mu}) \epsilon^{abc}\frac{g_{V\pi\pi}(n)}{M_{V}(n)^2 - t},
\]  
(16)
with \(g_{V\pi\pi}(n)\) the coupling constant. This yields the \(\rho_{n} \rightarrow \pi\pi\) partial decay rate
\[
\Gamma(\rho_{n} \rightarrow \pi\pi) = \frac{g_{V\pi\pi}^2 M_{V}}{48\pi}\left(1 - \frac{4m_{\pi}^2}{M_{V}^2}\right)^{1/2}.
\]  
(17)
For the \(\rho(770)\) meson one gets \(g_{\rho\pi\pi} \simeq 6\) for \(\Gamma(\rho \rightarrow 2\pi) = 150\text{MeV}\).

For the pion electromagnetic form factor we have
\[
F(t) = \sum_{n} \frac{c_{n}M_{V}(n)^2}{M_{V}(n)^2 - t},
\]  
\[
c_{n} = \frac{F_{V}(n)g_{V\pi\pi}(n)}{M_{V}(n)},
\]  
(18)
With the adopted conventions we note that \(F_{V}(n)\) has the dimension of energy, while \(g_{V\pi\pi}(n)\) and \(c_{n}\) are dimensionless. Note that the signs of the residues appearing in the form factor \(18\) may be priori be positive or negative, while all contributions to the two-point correlator \(13\) are positive. The possibility of different signs in Eq. \(18\) provides a mechanism for cancellation. The form factor satisfies the dispersion relation \(17\) with the spectral density
\[
\frac{1}{\pi} \text{Im} F(t) = \sum_{n} c_{n} \frac{M_{V}(n)^2}{M_{V}(n)^2 - t} \delta(t - M_{V}(n)^2).
\]  
(19)
Note that with the previously listed parameters for the lowest \(\rho(770)\) resonance one has \(c_{\rho} = g_{\rho\pi\pi}F_{\rho}/m_{\rho} = 1.17\). Because of charge conservation this requires higher states with negative \(c_{n}\) coefficients. In fact, taking Eq. \(3\) with the physical vector meson masses, \(m_{\rho} = 770\text{MeV}, m_{\rho'} = 1459(10)\text{MeV}, m_{\rho''} = 1720(20)\text{MeV}\) and \(m_{\rho'''} = 2000(30)\text{MeV}\) and using the coupling constants \(c_{n}\) as fit parameters to the electromagnetic form factor data in the intermediate \(Q^2\) range, \(0.6\text{GeV}^2 < Q^2 < 2.4\text{GeV}^2\), yields \(c_{\rho} = 1.25, c_{\rho'} = -0.17\) for two resonances, \(c_{\rho} = 1.39, c_{\rho'} = -0.53, c_{\rho''} = 0.26\) for three resonances, and \(c_{\rho} = 1.39, c_{\rho'} = -0.53\) , \(c_{\rho''} = 0.26, c_{\rho'''} = -0.004\) for four resonances. Such an approach, although phenomenologically appealing and numerically stable for the lowest energy states, can only yield an integer power fall-off and, as already mentioned, does not match to pQCD, Eq. \(8\) at high energies.

Strictly speaking one should consider in Eq. \(18\) the leading large-\(N_c\) contributions to the vector meson parameters. According to Ref. \(41\) one has \(M_{V}(n) \sim N_{c}^{0}\), \(F_{V}(n) \sim \sqrt{N_{c}}\), and \(g_{V\pi\pi}(n) \sim 1/\sqrt{N_{c}}\) such that \(F(t) \sim N_{c}^{0}\). Corrections to this behavior are generally \(\sim 1/N_{c}\) suppressed relative to the leading order and hence we expect at worst a 30% detuning of the physical values. The large-\(N_{c}\) dependence of meson parameters has been studied in unitarized chiral perturbation theory approaches yielding a larger value for the vector meson mass when \(m_{\rho}^{N_{c}} \rightarrow m_{\rho}^{\infty} \sim 1.2m_{\rho}^{2}\) \(54,55\) Chiral quark models at the one loop level are large \(N_{c}\) motivated. The Spectral Quark Model \(56\) reproduces by construction the simple VMD result, Eq. \(3\), for the pion form factor providing, in addition, the value \(m_{\rho}^{2} = 24\pi^{2}f_{\rho}^{2}/N_{c}\) which for \(f_{\pi} = 92.3\text{MeV}\) yields \(m_{\rho} \sim 820\text{MeV}\), a larger value than the physical mass. The trend to an increased value of the \(\rho\)-meson mass can also be traced when in a fit of the two resonance version of the generalized VMD, Eq. \(8\), the lowest mass state is allowed.
III. REGGE MODELS

We now proceed to review the Regge models in the scope necessary for our analysis, in particular regarding the pion electromagnetic form factor. The radial Regge trajectories are

$$M_n^2 = M^2 + an.$$  \hspace{1cm} (20)

The slope of the radial Regge trajectory, $a$, may be identified with the string tension, $\sigma = a/(2\pi)$, which for heavy quarks corresponds to the confining potential $V(r) = \sigma r$. Acceptable values are in the range $\sigma = 420 - 500\text{MeV}$ [57]. In this work we use for definiteness

$$\sigma = 450\text{ MeV}, \ M = 820\text{ MeV}. \hspace{1cm} (21)$$

As mentioned above, these parameters need not exactly reproduce the physical values, as the accuracy of the present large-$N_c$ Regge approach is not expected to be better than the large-$N_c$ expansion itself. Fortunately, the pion electromagnetic form factor turns out not to be very sensitive to the details of the radial Regge trajectory. Following Refs. [44, 45, 46, 47], we consider the function

$$f_b(t) = \frac{B(b - 1, \frac{M^2 - t}{a})}{B(b - 1, \frac{M^2}{a})}, \hspace{1cm} (22)$$

with $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ denoting the Euler Beta function. The function (22) fulfills the normalization condition

$$f_b(0) = 1. \hspace{1cm} (23)$$

For $x \gg y$ one has $B(x, y) \sim \Gamma(y)x^{-y}$, hence in the asymptotic region of $M^2 - t \gg (b - 1)a$ we find

$$f_b(t) \sim \frac{\Gamma(\frac{M^2}{a} + b - 1)}{\Gamma(\frac{M^2}{a})} \left(\frac{a}{M^2 - t}\right)^{b-1}. \hspace{1cm} (24)$$

Moreover, this function is positive on the real axis, $t < 0$, and has single poles at $t = M_n^2 = M^2 + an$, with residuals read off from the expansion

$$f_b(t) = \frac{a}{B(b - 1, \frac{M^2}{a})} \times \sum_{n=0}^{\infty} \frac{\Gamma(2 - b + n)}{\Gamma(n + 1)\Gamma(2 - b) an + M^2 - t}. \hspace{1cm} (25)$$

The function depends on three parameters: the lowest-lying meson mass, $M$, the string tension, $\sigma = a/(2\pi)$, and the asymptotic fall-off parameter, $b$. An interesting feature is the fact that for non-integer values of $b$ a large-$t$ expansion in powers of $1/t$ has zero coefficients. For integer $b = N + 1$ the formula corresponds to exactly $N$ mesons,

$$f_{N+1}(t) = \prod_{n=0}^{N} \left(\frac{M^2 + an}{M^2 + an - t}\right). \hspace{1cm} (26)$$

Particular expressions corresponding to $b = 2, 3, 4$ are

$$f_2(t) = \frac{M^2}{M^2 - t}, \hspace{1cm} (27)$$
$$f_3(t) = \frac{M^2(M^2 + a)}{(M^2 - t)(M^2 + a - t)}, \hspace{1cm} (28)$$
$$f_4(t) = \frac{M^2(M^2 + a)(M^2 + 2a)}{(M^2 - t)(M^2 + a - t)(M^2 + 2a - t)}. \hspace{1cm} (29)$$

At asymptotic values of $Q^2$ Eq. (25) yields

$$f_b(t = -Q^2) = \frac{\Gamma(\frac{M^2}{a} + b - 1)}{\Gamma(\frac{M^2}{a})} \left(\frac{a}{Q^2}\right)^{b-1}. \hspace{1cm} (30)$$

Thus, the value of $b$ controls the asymptotic fall-off in the $Q^2$ variable.

IV. FROM POWERS TO LOGARITHMS

In this Section we carry out the construction of the pion electromagnetic form factor which complies to the asymptotic pQCD constraints. For the pion charge form factor one has the leading power behavior,

$$F(Q^2) = \frac{16\pi^2\alpha(Q^2)}{Q^2} \sum_{n=0}^{\infty} c_n \alpha(Q^2)^n, \hspace{1cm} (31)$$

with the coefficients $c_n$ calculable in pQCD (albeit this perturbative series may diverge). We have at LO $c_0 = 1$

1 One might think that taking $N \to \infty$ the general result would be recovered, but according to the product formula for the Gamma function,

$$\Gamma(z) = \lim_{N \to \infty} N^{\frac{z}{N}} \prod_{k=1}^{N} \frac{k}{(k + z)}, \hspace{1cm} (32)$$

we see that this is not the case, since

$$\lim_{N \to \infty} \prod_{n=1}^{N} \frac{M^2 + an}{M^2 + an - t} = \lim_{N \to \infty} N^{t/a} \frac{\Gamma((M^2 - t)/a)}{\Gamma(M^2/a)}$$

and the result is ambiguous. This function has the poles located at the same place as in (22). The ambiguity is manifest in the choice of the parameter $b$. Generally speaking, the result for non-integer $N < b < N + 1$ has infinitely many resonances but it is closer to the case of finite $N$ rather than to $N \to \infty$. This suggests that a method based on truncating the tower of mesons is not expected to be convergent for increasing $t$ values.
which is stable in the large-$N_c$ limit, as $\alpha \sim 1/N_c$ and $f_\pi \sim \sqrt{N_c}$. For large momenta $F(Q^2)$ is bounded as follows:

$$\frac{C}{Q^4} < F(Q^2) < \frac{C'}{Q^2}.$$  

Thus, according to [23], the admissible possible power dependence effectively corresponds to $2 < b < 3$. A fit to the data yields with $b = 2.3(1)$ [17] in agreement with the above expectations. This is a remarkable result, for it indicates on the one hand that even at energies where pQCD does not clearly set in, there seems to be some indirect information on the best possible fractional power behavior. On the other hand, note that in order to have a fractional power from the leading twist pQCD result (29) we need a non-analytic dependence of the form $e^{-c/a(Q^2)} \sim (Q^2/\Lambda^2)^{-4\pi c/\beta_0}$, which is clearly out of reach for standard perturbation theory. The previous considerations suggest that pQCD and large $N_c$ Regge models are mutually incompatible. As we discuss shortly this is not necessarily so.

Now we come to the core of our construction. In order to generate the asymptotic dependence (29) we superpose the Regge model formula (22) over the values of $b$,

$$F(t) = \int_2^\infty db \, \rho(b) f_b(t),$$  

where the density is given by

$$\rho(b) = \rho_{\text{high}}(b) + \rho_{\text{low}}(b),$$  

and

$$\rho_{\text{high}}(b) = \frac{4\pi}{\beta_0} \frac{16\pi T^2}{\Lambda^2} \left( \frac{a}{\Lambda^2} \right)^1 \left( \frac{M^2}{a} \right)^{b-1} \times \sum_{n=0}^\infty \frac{c_n}{n!} \left( \frac{4\pi}{\beta_0} \right)^n (b-2)^n.$$  

The coefficients $c_n$ are precisely the same as in Eq. (29). Note that Eq. (33) corresponds to a Borel transformation of the original perturbative series, a feature which is welcome in view that the pQCD series is generally believed to be divergent but Borel-summable (see, e.g., Ref. 58 and references therein). The formula

$$\int_0^\infty d\epsilon \, \epsilon^n \, x^{-\epsilon} = \frac{n!}{\log^{n+1} x}$$  

is the key ingredient in the power-to-log transmutation, where $\epsilon = b - 2$. Note that by taking the spectral density (33) we get the right pQCD asymptotics when the large-$Q^2$ behavior of the Regge model is used. The lower limit of integration in Eq. (33) controls the power of $Q^2$ in front of the RHS of Eq. (29). In fact, it is the behavior of $\rho(b)$ in the vicinity of $b = 2$ that determines the asymptotic behavior of $F(Q^2)$, thus $\rho(b)$ is not determined uniquely away from $b = 2$. One could attempt to use the form (33) for all values of $b$. However, according to the charge conservation we have to fulfill the normalization condition

$$F(0) = \int_0^\infty db \, \rho(b) = 1.$$  

Fixing the scale $\Lambda_{\text{QCD}} = 250$ MeV, we get both at LO and NLO

$$Z_{\text{high}} = \int_2^\infty db \, \rho_{\text{high}}(b) < 1.$$  

To account for the missing strength we add an extra non-perturbative contribution, $\rho_{\text{low}}(b)$, which has support away from $b = 2$. For simplicity it is taken in the form of a delta function,

$$\rho_{\text{low}}(b) = (1 - Z_{\text{high}}) \delta(b - b_0),$$  

with $b_0 = 2.3$, as in the fit of Ref. [17], although other less singular distributions could also be used. Certainly, the presence of $\rho_{\text{low}}(b)$ is not affecting the asymptotics of $F(Q^2)$, which is governed by the behavior near $b = 2$, but it modifies $F(Q^2)$ at lower momenta.

Explicitly, at LO and NLO we use

$$\rho_{\text{high}}^{\text{LO}}(b) = \frac{4\pi}{\beta_0} \frac{16\pi T^2}{\Lambda^2} \left( \frac{a}{\Lambda^2} \right)^{b-1} \left( \frac{M^2}{a} \right)^{b-1} \times \sum_{n=0}^\infty \frac{c_n}{n!} \left( \frac{4\pi}{\beta_0} \right)^n (b-2)^n.$$  

which with parameters (21) and $N_f = 3$ yield

$$Z_{\text{high}}^{\text{LO}} = 0.27, \quad Z_{\text{high}}^{\text{NLO}} = 0.39.$$  

The spectral densities (38) are plotted in Fig. (1) with the dashed and solid lines representing the LO and NLO formulas, respectively. The $\rho_{\text{low}}(b)$ contribution is represented by the vertical line at $b = b_0 = 2.3$. 

![FIG. 1: The density $\rho_{\text{high}}^{\text{high}}(b)$ at LO (dashed line) and NLO (solid line). The $\rho_{\text{low}}^{\text{low}}(b)$ contribution is represented by the vertical line at $b = b_0 = 2.3$.](image-url)
We note that the strength of the spectral density is practically contained in the interval between 2 and 3, and at large \( b \) we have a very fast fall-off, \( \rho_{\text{high}}(b) \sim b^{3/2-M^2/a}(a/\Lambda^2)^{-b}e^{-b\log(b/e)} \). More generally, we might also include a finite upper limit of integration using the formula

\[
\int_{b_1}^{b_2} db \left( \frac{M_V^2 - t}{a} \right)^{1-b} \frac{1}{\log(M_V^2 - t)/a} \times \left[ \left( \frac{M_V^2 - t}{a} \right)^{1-b_1} - \left( \frac{M_V^2 - t}{a} \right)^{1-b_2} \right].
\]

The resulting value for \( Z_{\text{high}} \) changes by a small amount for \( b_1 > 3 \), depending on its precise value. Actually, by extending the integration to infinity we are maximizing the impact of perturbative corrections, and as we see, they are not large. Thus not much change is expected from cutting off the integral above \( b = 3 \).

V. POLE-RESIDUE ASSIGNMENT

Our procedure is equivalent to imposing pQCD constraints for the pole-residue assignment in the spectral representation of the pion charge form factor. Looking formally at the problem, we need to form the spectral density \([13]\) in such a manner that the asymptotic pQCD constraints are satisfied (apart for other constraints, such as normalization). In the preceding section we have demonstrated explicitly that it is possible to accomplish this goal. More generally, in the large-\( N_c \), Regge model we have to choose the location of poles and fix their residues. Admittedly, there is a redundancy between shifting the poles or the residues. We decide to keep the poles fixed at their original location \([20]\) because they are phenomenologically well described by the Regge trajectories. For the residues the prescription of the previous section is equivalent to taking

\[
c_n = \int_2^\infty db \frac{\alpha}{B(b-1, \frac{M^2}{a})} \frac{\Gamma(2 - b + n)}{\Gamma(n + 1)\Gamma(2 - b)}. \quad (41)
\]

In Fig. 2 we show the values of \( c_n \) for the three considered models: the model with fixed \( b \) of Dominguez \([17]\), with \( \rho(b) = \delta(b - 2.3) \) (circles), and our model for the LO (squares) and NLO (diamonds) cases. We note a strong similarity between all cases. In particular, the first residue, \( c_0 \), is positive and the remaining residues are negative, which leads to the desired cancellation. At very large values of \( n \) (not displayed) the LO and NLO residues have a larger magnitude than for the model with fixed \( b \). Despite this similarity, we note that our LO and NLO models do satisfy the asymptotic pQCD constraints, while the fixed-\( b \) model does not. This reflects the subtlety of the cancellation in the power-to-log transmutation mechanism. We stress that within our scheme we may achieve the goal of reproducing pQCD without modifying the spectrum; our spectral method features an effective way of implementing QCD radiative corrections by appealing to a modification of the meson wave functions.

At this point it is also interesting to display the values of the resulting \( g_{V\pi\pi} \) couplings. This requires some knowledge on the vector meson-photon coupling \( F_V(n) \). As mentioned above, quark-hadron duality for large \( t \) at the level of the two point vector correlator requires the Regge model – parton model matching condition, Eq. \([15]\), which for the mass formula, Eq. \([20]\), becomes

\[
2\pi\sigma = 24\pi^2 F_V^2 / N_c. \quad (42)
\]

This formula works reasonably well already for the lowest \( \rho(770) \) state, where \( F_V = 150 \text{MeV} \) yields \( \sqrt{\sigma} = 530\text{MeV} \), while we expect \( \sigma = 420 - 500\text{MeV} \) \([57]\). Following previous works \([51, 52]\) the formula \([42]\) will be assumed to be valid for all \( n \) disregarding possible non-linearities which are not very relevant within the present context \(^2\). With \( F_V = 150 \text{MeV} \) and \( c_n \) from Fig. 2 with Eq. \([18]\) we get

\[
g_{\rho\pi\pi} = 4.3(4.4), \quad g_{\rho'\pi\pi} = -2.3(-2.6), \quad g_{\rho''\pi\pi} = -0.6(-0.6), \quad g_{\rho'''\pi\pi} = -0.4(-0.4). \quad (43)
\]

where the first values are for the LO model, and the values in parenthesis for the NLO model.

\(^2\) The sensitivity to details of the Regge trajectory depends on the computed observable. While for the pion form factor analyzed here there is some freedom, condensates with proper signs are crucially dependent on these details, as shown in Ref. \([51, 52]\).
VI. PION CHARGE FORM FACTOR RESULTS

In Fig. 3 we display the results of our model for the pion charge form factor and compare them to the TJLAB [6, 7, 8] and Cornell [3] data. The three lines close to one another and to the data are the NLO model (solid line), the LO model (dashed line), and the model with fixed $b = 2.3$. The two lower curves correspond to the NLO (solid) and (LO) asymptotic pQCD results. We note that in the range of momenta accessible to experiments all the considered models yield very close predictions and describe the data well. These predictions depart from one another at very high values of $Q^2$, as can be seen from Fig. 4, where we plot the LO result (dashed line) and the asymptotic LO pQCD expression (solid line). We note that the curves meet at $Q^2 = 10^8$ GeV$^2$, which is a very high scale. For comparison we also plot the result of the fixed $b$ model (dotted line), which with the chosen value $b_0 = 2.3$ decays as $(1/Q^2)^{1.3}$. Figure 5 shows the same study for the NLO calculation, with the model denoted by the dashed, and the NLO pQCD calculation by the solid lines, respectively. The two curves meet at somewhat lower scales, $Q^2 = 10^7$ GeV$^2$, than for the LO case of Fig. 4.

VII. CONCLUSIONS

There have been countless attempts to understand the delayed onset of pQCD in the pion charge form factor. The standard VMD model is known to fit the available data remarkably well, but shows no obvious link to pQCD. In the present paper we have approached the problem from the viewpoint of the large-$N_c$ Regge models. Our approach exploits explicitly the quark-hadron duality at a non-perturbative level and has the genuine advantage that much of the discussion can be carried out without an explicit reference to the light-cone wave functions and/or parton distribution amplitudes; many uncertainties in current calculations seem related to our lack of the detailed knowledge of these non-perturbative objects used in the description of exclusive processes. Our generalized VMD model includes infinitely many resonances, describes the data and simultaneously complies to the known short-distance pQCD constraints. The present framework requires a suitable modeling both of the spectrum and the vector meson coupling to the electromagnetic current. While it describes the so far experimentally explored space-like momentum region, it is rather hard to provide estimates of the systematic error of the calculation. The important feature which has been clearly identified several times in the past in the analysis of the data is that at large $Q^2$ the pion form factor seems to have a non-integer power fall-off, which actually turns out to be in the expected range for the best possible pQCD power-log behavior, but still is qualitatively different from the theoretical expectations based on pQCD. We have shown that there is no contradiction between both behaviors. Actually, we have spelled out a simple mechanism where a suitable superposition of non-integer power fall-offs may transmute into the de-
sired asymptotic pQCD behavior, including the presence of logarithms. We have shown that such a procedure does not spoil the good agreement in the so far experimentally accessible region down the low energy region where chiral corrections cause sizable distortions from any large-$N_c$ calculation. Moreover, we are able to reproduce simultaneously the high-energy pQCD behavior, providing some confidence on the range where pQCD sets in. We find that about 1/4 for the LO and about 1/3 for the NLO case of the pion charge is due to the high-energy pQCD tail in our approach. Finally, the present calculations suggest that non-perturbative contributions dominate in the region corresponding to the present and planned experimental data, and would saturate the full result only at extremely high values, $Q^2 \sim 10^7 - 10^8$ GeV$^2$.

[1] M. Gourdin, Phys. Rept. 11, 29 (1974).
[2] C. J. Bebek et al., Phys. Rev. D13, 25 (1976).
[3] C. J. Bebek et al., Phys. Rev. D17, 1693 (1978).
[4] S. R. Amendolia et al., Phys. Lett. B138, 454 (1984).
[5] S. R. Amendolia et al. (NA7), Nucl. Phys. B277, 168 (1986).
[6] J. Volmer et al. (The Jefferson Lab F(pi)), Phys. Rev. Lett. 86, 1713 (2001), nucl-ex/0010009.
[7] T. Horn et al. (Jefferson Lab F(pi)-2), Phys. Rev. Lett. 97, 192001 (2006), nucl-ex/0607005.
[8] V. Tadevosyan et al. (Jefferson Lab F(pi)), Phys. Rev. C75, 055205 (2007), nucl-ex/0607007.
[9] H. P. Blok, G. M. Huber, and D. J. Mack (2002), nucl-ex/0208011.
[10] S. J. Brodsky and G. R. Farrar, Phys. Rev. Lett. 31, 1153 (1973).
[11] S. J. Brodsky and G. R. Farrar, Phys. Rev. D11, 1309 (1975).
[12] G. R. Farrar and D. R. Jackson, Phys. Rev. Lett. 43, 246 (1979).
[13] A. V. Radyushkin (1977), hep-ph/0410276.
[14] A. V. Efremov and A. V. Radyushkin, Theor. Math. Phys. 42, 97 (1980).
[15] A. V. Efremov and A. V. Radyushkin, Phys. Lett. B94, 245 (1980).
[16] V. M. Braun, A. Khodjamirian, and M. Maul, Phys. Rev. D61, 073004 (2000), hep-ph/9907495.
[17] S. S. Ageev, Phys. Rev. D72, 074020 (2005), hep-ph/0509345.
[18] B. Melic, B. Nizic, and K. Passek, Phys. Rev. D60, 074004 (1999), hep-ph/9802204.
[19] J. Sakurai, Currents and Mesons (University of Chicago, Chicago, 1969).
[20] J. Gasser and H. Leutwyler, Nucl. Phys. B250, 517 (1985).
[21] H. Leutwyler (2002), hep-ph/0212324.
[22] J. Nieves and E. Ruiz Arriola, Nucl. Phys. A679, 57 (2000), hep-ph/9907469.
[23] J. F. Donoghue and E. S. Na, Phys. Rev. D56, 7073 (1997), hep-ph/9611418.
[24] B. L. Ioffe and A. V. Smilga, Phys. Lett. B114, 353 (1982).
[25] V. A. Nesterenko and A. V. Radyushkin, Phys. Lett. B115, 410 (1982).
[26] V. Braguta, W. Lucha, and D. Melikhov, Phys. Lett. B661, 334 (2008), 0710.5461.
[27] V. M. Braun and I. E. Halperin, Phys. Lett. B328, 457 (1994), hep-ph/9402270.
[28] A. P. Bakulev and A. V. Radyushkin, Phys. Lett. B271, 223 (1991).
[29] A. P. Bakulev, K. Passek-Kumericki, W. Schroers, and N. G. Stefanis, Phys. Rev. D70, 033014 (2004).
[30] P. Maris and C. D. Roberts, Phys. Rev. C58, 3659 (1998), nucl-th/9804062.
[31] P. Faccioli, A. Schwenk, and E. V. Shuryak, Phys. Rev. D67, 113009 (2003), hep-ph/0202027.
[32] S.-i. Nam and H.-C. Kim, Phys. Rev. D77, 094014 (2008), 0709.1745.
[33] F. Cardarelli et al., Phys. Rev. D53, 6682 (1996).
[34] H. Pagels and S. Stokar, Phys. Rev. D20, 2947 (1979).
[35] H. Ito, W. W. Buck, and F. Gross, Phys. Rev. C45, 1918 (1992).
[36] F. D. R. Bonnet, R. G. Edwards, G. T. Fleming, R. Lewis, and D. G. Richards (Lattice Hadron Physics), Phys. Rev. D72, 054506 (2005), hep-lat/0410129.