Subordination Results for a Class of Analytic Functions Defined by Convolution

E. A. Adwan

1Common First Year, Saudi Electronic University, Saudi Arabia.

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Abstract

Aims/ Objectives: In this paper, making use the Hadamard product, we introduce drive several interesting subordination results for a new class of analytic function. Furthermore, we mention some known and new results, which follow as special cases of our results.

Keywords: Analytic functions; convolution; subordination; factor sequence.

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1 Introduction

Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( K \) denote the class of functions \( f(z) \in A \) which are convex in \( U \) and let \( S(k, \alpha) \) denote the subclass of \( A \) which satisfies the following
inequality (see [1, 2, 3, 4])

\[ \text{Re} \left( \frac{zf'(z)}{f(z)} + k \frac{z^2f''(z)}{f(z)} \right) > \alpha \quad (k \geq 0; 0 \leq \alpha < 1; z \in \mathbb{U}). \]

The Hadamard product (or convolution) \((f \ast g)(z)\) of the functions \(f(z)\) and \(g(z)\), that is, if \(f(z)\) is given by (1.1) and \(g(z)\) is given by

\[ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (b_n \geq 0), \]

is defined by:

\[ (f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g \ast f)(z). \]

If \(f\) and \(g\) are analytic functions in \(\mathbb{U}\), we say that \(f\) is subordinate to \(g\), written \(f \prec g\) if there exists a Schwartz function \(w\), which (by definition) is analytic in \(\mathbb{U}\) with \(w(0) = 0\) and \(|w(z)| < 1\) for all \(z \in \mathbb{U}\), such that \(f(z) = g(w(z))\), \(z \in \mathbb{U}\). Furthermore, if the function \(g\) is univalent in \(\mathbb{U}\), then we have the following equivalence (cf., e.g., [5, 6]):

\[ f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}). \]

**Definition 1.1.** (Subordinating Factor Sequence) [7]. A sequence \(\{d_n\}_{n=1}^{\infty}\) of complex numbers is said to be a subordinating factor sequence if, whenever \(f\) of the form (1.1) is analytic, univalent and convex in \(\mathbb{U}\), we have the subordination given by

\[ \sum_{n=1}^{\infty} d_n a_n z^n \prec f(z) \quad (z \in \mathbb{U}; a_1 = 1). \]

For \(0 \leq \alpha < 1, k \geq 0\) and for all \(z \in \mathbb{U}\), let \(H(f, g; \alpha, k)\) denote the subclass of \(A\) consisting of functions \(f(z)\) of the form (1.1) and \(g(z)\) of the form (1.2) and satisfying the analytic criterion:

\[ \text{Re} \left\{ \frac{z(f \ast g)'(z)}{(f \ast g)(z)} + k \frac{z^2(f \ast g)''(z)}{(f \ast g)(z)} \right\} > \alpha. \]

The class was introduce and studied by Aouf et al. (see [8]). We note that for suitable choice of \(g\), we obtain the following subclasses studied by various authors.

(1) If we take \(g(z) = \frac{z}{1-z}\), then the class \(H(f, \frac{z}{1-z}; \alpha, k)\) reduces to the class \(S(k, \alpha)\) (see [3]);
(2) If we take \(g(z) = \frac{z}{1-z}\) and \(k = 0\), then the class \(H(f, \frac{z}{1-z}; \alpha, 0)\) reduces to the class \(S'(\alpha)\) (see [9]);
(3) If we take \(g(z) = \frac{z}{(1-z)^2}\) and \(k = 0\), then the class \(H(f, \frac{z}{(1-z)^2}; \alpha, 0)\) reduces to the class \(K(\alpha)\) (see [9]);
(4) If we take

\[ g(z) = z + \sum_{n=2}^{\infty} \sigma_n z^n \]

(or \(b_n = \sigma_n\)), where

\[ \sigma_n = \frac{\Theta \Gamma(a_1 + A_1(n-1)) \ldots \Gamma(a_q + A_q(n-1))}{(n-1)! \Gamma(\beta_1 + B_1(n-1)) \ldots \Gamma(\beta_s + B_s(n-1))} \]

\((a_i, A_i > 0, i = 1, \ldots q; \beta_j, B_j > 0, j = 1, \ldots, s; q \leq s + 1; q, s \in \mathbb{N}, N = \{1, 2, 3, \ldots\})\)
and
\[
\Theta = \left\{ \prod_{j=0}^{l-1} \Gamma(\beta_j) \right\}^{1/n} \left\{ \prod_{j=0}^{l-1} \Gamma(\alpha_i) \right\}^{1/m},
\]
(1.8)
then the class \( H(f, z + \sum_{n=2}^{\infty} \sigma_n z^n; \alpha, k) \) reduces to the class \( W^\sigma_\gamma(\alpha, k) \) (see [10])
\[
= \left\{ f \in A : Re \left\{ \frac{z(W^\sigma_\gamma f(z))'}{W^\sigma_\gamma f(z)} + k z^2 (W^\sigma_\gamma f(z))'' \right\} > \alpha, \ 0 \leq \alpha < 1; \ k \geq 0; q, s \in \mathbb{N}; z \in \mathbb{U}, \right\},
\]
(1.9)
where \( W^\sigma_\gamma f(z) \) is the Wright's generalized hypergeometric function (see [11, 12]) which contains well known operators such as the Dziok-Srivastava operator (see [13, 14]), the Carlson-Shaffer linear operator (see [15]), the Bernardi-Libera-Livingston operator (see [16, 17, 18]), Srivastava - Owa fractional derivative operator (see [19]), the Ruscheweyh derivative operator (see [20]) and the Noor integral operator of n-th order (see [21]);

(4) If we take
\[
g(z) = z + \sum_{n=2}^{\infty} \left( \frac{l + 1 + \mu(n - 1)}{l + 1} \right)^m z^n
\]
(1.10)
(or \( b_n = \left( \frac{l+1+\mu(n-1)}{l+1} \right)^m \), \( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( \mu \geq 0, l \geq 0 \), then the class \( H(f, z + \sum_{n=2}^{\infty} \left( \frac{l+1+\mu(n-1)}{l+1} \right)^m z^n; \alpha, k) \) reduces to the class \( m(\mu, l, \alpha, k) \) (see [8]):
\[
= \left\{ f \in A : Re \left\{ \frac{z(I^m(\mu, l) f(z))'}{I^m(\mu, l) f(z)} + k z^2 (I^m(\mu, l) f(z))'' \right\} > \alpha, \right\},
\]
0 \leq \alpha < 1; k \geq 0; m \in \mathbb{N}_0; \mu, l \geq 0, \ z \in \mathbb{U}, \right\},
(1.11)
where \( I^m(\gamma, l) f(z) \) is the extended multiplier transformation (see [22]), for \( l = 0, \gamma \geq 0 \), the operator \( I_m(0, 0) = D_m \) was introduced and studied by Al-Oboudi (see [23]) and for \( l = \gamma = 0 \), the operator \( I_m(0, 0) = D^m \), where \( D^m \) is Salagean differential operator see. [24].

2 Main Results

Unless otherwise mentioned, we shall assume in the reminder of this paper that, \( 0 \leq \alpha < 1, k \geq 0, n \geq 2, z \in \mathbb{U} \) and \( g(z) \) is defined by (1.2). To prove our main results we shall need the following lemmas.

Lemma 2.1. [7]. The sequence \( \{ d_n \}_{n=1}^{\infty} \) is a subordinating factor sequence if and only if
\[
Re \left\{ 1 + 2 \sum_{n=1}^{\infty} d_n z^n \right\} > 0, \ (z \in \mathbb{U}).
\]
(2.1)

Lemma 2.2. [8]. Let the function \( f(z) \) defined by (1.1) satisfy the following condition:
\[
\sum_{n=2}^{\infty} (kn^2 + n - kn - \alpha) b_n |a_n| \leq 1 - \alpha.
\]
(2.2)
Then \( f(z) \in H(f, g; \alpha, k) \).
Let $H^*(f, g; \alpha, k)$ denote the class of functions $f(z) \in A$ whose coefficients satisfy the condition (2.2). We note that $H^*(f, g; \alpha, k) \subseteq H(f, g; \alpha, k)$, $S^*(k, \alpha) \subseteq S(k, \alpha)$, $W^*_2(\alpha, k) \subseteq W^*_2(\alpha, k)$ and $z_\alpha(\mu, l, \alpha, k) \subseteq z_\alpha(\mu, l, \alpha, k)$.

**Theorem 2.3.** Let $f \in H^*(f, g; \alpha, k)$, $b_n \geq b_2 > 0$ $(n \geq 2)$. Then for every convex function $\phi \in K$, we have

$$\frac{(2k - \alpha + 2) b_2}{2[(2k - \alpha + 2) b_2 + (1 - \alpha)]} (f \ast \phi)(z) < \phi(z),$$

and

$$\text{Re}\{f(z)\} > -\frac{(2k - \alpha + 2) b_2 + (1 - \alpha)}{(2k - \alpha + 2) b_2}.$$  

The constant

$$\frac{(2k - \alpha + 2) b_2}{2[(2k - \alpha + 2) b_2 + (1 - \alpha)]}$$

is the best estimate.

**Proof.**

Let $f(z) \in H^*(f, g; \alpha, k)$ and let $\phi(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$. Then we have

$$\frac{(2k - \alpha + 2) b_2}{2[(2k - \alpha + 2) b_2 + (1 - \alpha)]} (f \ast \phi)(z) = \frac{(2k - \alpha + 2) b_2}{2[(2k - \alpha + 2) b_2 + (1 - \alpha)]} \left(z + \sum_{n=2}^{\infty} a_n c_n z^n\right).$$  

Thus, by Definition 1, the subordination result (2.3) will hold true if the sequence

$$\left\{\frac{(2k - \alpha + 2) b_2}{2[(2k - \alpha + 2) b_2 + (1 - \alpha)]} a_n \right\}_{n=1}^{\infty},$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalent to the following inequality:

$$\text{Re}\left\{1 + \sum_{n=1}^{\infty} \frac{(2k - \alpha + 2) b_2}{[(2k - \alpha + 2) b_2 + (1 - \alpha)]} a_n z^n\right\} > 0.$$  

(2.7)

Now, since

$$(kn^2 + n - kn - \alpha) b_n,$$

is an increasing function of $n$ $(n \geq 2)$, we have

$$\text{Re}\left\{1 + \sum_{n=1}^{\infty} \frac{(2k - \alpha + 2) b_2}{(2k - \alpha + 2) b_2 + (1 - \alpha)} a_n z^n\right\} = \text{Re}\left\{1 + \frac{(2k - \alpha + 2) b_2}{(2k - \alpha + 2) b_2 + (1 - \alpha)} z\right\} + \frac{1}{(2k - \alpha + 2) b_2 + (1 - \alpha)} \sum_{n=2}^{\infty} (2k - \alpha + 2) b_2 a_n z^n\right\} \geq 1 - \frac{(2k - \alpha + 2) b_2}{(2k - \alpha + 2) b_2 + (1 - \alpha) r} - \frac{1}{(2k - \alpha + 2) b_2 + (1 - \alpha)} \sum_{n=2}^{\infty} (kn^2 + n - kn - \alpha) b_n |a_n| r^n.$$
where we have also made use of assertion (2.2) of Lemma 2. Thus (2.7) holds true in \( U \). This proves the inequality (2.3). The inequality (2.4) follows from (2.3) by taking the convex function \( \phi(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in K \). To prove the sharpness of the constant \( \frac{(2k - \alpha + 2)b_2}{2[(2k - \alpha + 2)b_2 + (1 - \alpha)]} \), we consider the function \( f_0(z) = z - \frac{(1 - \alpha)}{(2k - \alpha + 2)b_2} z^2 \). (2.8)

Thus from (2.3), we have

\[
\frac{(2k - \alpha + 2)b_2}{2[(2k - \alpha + 2)b_2 + (1 - \alpha)]} f_0(z) < \frac{z}{1-z}.
\]

(2.9)

Moreover, it can easily be verified for the function \( f_0(z) \) given by (2.8) that

\[
\min_{|z| \leq r} \left\{ \text{Re} \left( \frac{(2k - \alpha + 2)b_2}{2[(2k - \alpha + 2)b_2 + (1 - \alpha)]} f_0(z) \right) \right\} = -\frac{1}{2}.
\]

(2.10)

This shows that the constant \( \frac{(2k - \alpha + 2)b_2}{2[(2k - \alpha + 2)b_2 + (1 - \alpha)]} \) is the best possible. This completes the proof of Theorem 1.

Putting \( g(z) = z + \sum_{n=2}^{\infty} \sigma_n z^n \), where \( \sigma_n \) is defined by (1.7), in Lemma 2 and Theorem 1, we obtain the following corollary:

**Corollary 2.4.** Let \( f \) defined by (1.1) be in the class \( W_{r,q}^s(\alpha, k) \) and satisfy the condition

\[
\sum_{n=2}^{\infty} (kn^2 + n - k - \alpha) \sigma_n |a_n| \leq 1 - \alpha.
\]

Then for every function \( \phi \in K \), we have

\[
\frac{(2k - \alpha + 2)\sigma_2}{2[(2k - \alpha + 2)\sigma_2 + (1 - \alpha)]} (f * \phi)(z) < \phi(z),
\]

and

\[
\text{Re} \{ f(z) \} > -\frac{(2k - \alpha + 2)\sigma_2 + (1 - \alpha)}{(2k - \alpha + 2)\sigma_2}.
\]

The constant \( \frac{(2k - \alpha + 2)\sigma_2}{2[(2k - \alpha + 2)\sigma_2 + (1 - \alpha)]} \) is the best estimate.

Putting \( g(z) = z + \sum_{n=2}^{\infty} \left( \frac{1 + \mu(n-1)}{l+1} \right)^m z^n \), in Lemma 2 and Theorem 1, we obtain the following corollary:

**Corollary 2.5.** Let \( f \) defined by (1.1) be in the class \( m_{\mu}(\mu, l, \alpha, k) \) and satisfy the condition

\[
\sum_{n=2}^{\infty} (kn^2 + n - k - \alpha) \left( \frac{1 + \mu(n-1)}{l+1} \right)^m |a_n| \leq 1 - \alpha.
\]
Then for every function $\phi \in K$, we have
\[
\frac{(2k - \alpha + 2) \left( \frac{1+i+\mu}{1+i} \right)^m}{2[(2k - \alpha + 2) \left( \frac{1+i+\mu}{1+i} \right) + (1 - \alpha)]} (f * \phi)(z) < \phi(z),
\]
and
\[
\text{Re}\{f(z)\} > \frac{(2k - \alpha + 2) \left( \frac{1+i+\mu}{1+i} \right)^m + (1 - \alpha)}{(2k - \alpha + 2) \left( \frac{1+i+\mu}{1+i} \right)^m}.
\]

The constant
\[
\frac{(2k - \alpha + 2) \left( \frac{1+i+\mu}{1+i} \right)^m}{2[(2k - \alpha + 2) \left( \frac{1+i+\mu}{1+i} \right) + (1 - \alpha)]}
\]
is the best estimate.

Remark 2.1. (i) Putting $g(z) = \frac{z}{(1-z)^2}$ (or $b_0 = 1$) and $k = 0$ in Theorem 1, we obtain the result obtained by Frasin [25], Corollary 2.3;
(ii) Putting $g(z) = \frac{z}{(1-z)^2}$ and $k = 0$ in Theorem 1, we obtain the result obtained by Frasin [25], Corollary 2.6.

3 Conclusions

In this work we presented some new drive several interesting subordination results for a new class of analytic function defined by convolution. This theorem leave open several possibilities that are worth investigating.

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Competing Interests

The author declare that there are no conflicts of interest regarding the publication of this paper.

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