Structure Functions of the Nucleon in a Statistical Model

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Abstract

Deep inelastic scattering is considered in a statistical model of the nucleon. This incorporates certain features which are absent in the standard parton model such as quantum statistical correlations which play a role in the propagation of particles when considering Feynman diagrams containing internal lines.

The inclusion of the $O(\alpha_s)$ corrections in our numerical calculations allows a good fit to the data for $x \geq 0.25$. The fit corresponds to values of temperature and chemical potential of approximately $T = 0.067$ GeV and $\mu = 0.133$ GeV. The latter values of parameters, however, give rise, for all $x$, to a large value for $R = \sigma_L/\sigma_T$. 
1. Introduction

In this paper we further develop a statistical model for the structure functions of the nucleon \[1,2,3,4\] based on the MIT-bag model \[5\]. The scattering of high energy leptons from a thermalized gas of quarks and gluons confined to the volume of the nucleon was calculated by considering lowest-order diagrams in perturbative QCD. In refs. \[6,7,8,9\] these considerations were extended to the first order in the strong coupling constant \(\alpha_s\) since the effect of gluons is absent in zeroth order.

The phenomenological success of the parton model indicates that the number of partons in the nucleon is very large due to the \(1/x\) behaviour of the structure functions at small \(x\).

The parton model neglects quantum statistical correlations due to the presence of identical quarks and gluons in the initial and final states. In the statistical model this is taken into account through the use of Fermi-Dirac and Bose-Einstein distributions for quarks and gluons, respectively. Stimulated emission factors for final-state gluons and Pauli-blocking factors for final-state quarks are incorporated. The propagation of particles through a many-body medium is taken into account by using thermal Feynman rules for propagators and vertices. These effects are negligible in lowest order, however, they are important for higher order diagrams especially in the small \(x\) region.

In ref. \[9\], all processes contributing to order \(\alpha_s\) (see Fig. 1) to deep inelastic scattering of leptons off a heat bath of quarks and gluons, were taken into account exactly. All infrared, collinear and ultraviolet divergences cancel in the framework provided by the real-time formalism of finite-temperature quantum field theory. The final expressions from these analytical calculations are given in the appendix.

The presence of the medium leads to a dramatic increase in the length of the calculation. For this reason many authors \[10,11,12,13,14,15\] use special kinematics [e.g. in lepton pair production from a thermalized quark gluon plasma they would use a virtual photon with zero spatial momentum: \(q = (q_0, \vec{q} = 0)\)]. In our case we perform calculations for the case of general kinematics [i.e. \(q = (q_0, \vec{q} \neq 0)\)].

Results from analytical calculations of dilepton production from a quark-gluon plasma to order \(\alpha_s\) (\(q^2 > 0\) for the photon giving rise to the lepton pair), which is closely related to the process of deep inelastic scattering of leptons off a heat bath of quarks and gluons (\(q^2 < 0\) for the photon stemming from the lepton beam), are presented in ref. \[9\]. In Section 4 we present results from numerical calculations of these expressions. There it is shown to what extent recent deep inelastic scattering data can be reproduced by the statistical model.
2. Deep inelastic electron-proton scattering

In our treatment of deep inelastic scattering we consider the scattering of a virtual photon with 4-momentum \( q = (\nu, 0, 0, q_z) \) off the proton (considered as a heat bath of quarks and gluons). To order \( \alpha_s \) the processes depicted in Fig. 1 contribute. According to Fig. 1 there can be more than two particles in the initial state of a reaction so that the rate of reactions (as opposed to a cross section) is the natural quantity to consider when relating the measured structure functions on the hadronic level with the results from our calculations of thermal averages of the fundamental (virtual) photon-quark interaction processes.

The deep inelastic cross section for charged leptons can be written in such a form that it depends linearly on \( F_2 \) and non-linearly and very weakly on \( R \) [16]. Experimentally, \( F_2 \) and \( R \) can be disentangled by measuring deep inelastic cross sections at the same \((x, Q^2)\) point but at different values of the incident beam energy. In practice, as these cross sections depend only weakly on \( R \), all measurements of \( R \) have large errors.

To each of the processes (a) to (f) in Fig. 1 there correspond the two Feynman diagrams in Fig. 2. These two diagrams, as drawn in Fig. 2 with exactly the same four-momentum assignments and directions of arrows, are used for each of the processes (a) to (f) in Fig. 1. When the energy components of the latter four-momenta are all positive the two Feynman diagrams refer to process (a) in Fig. 1 and the energy of a particle in diagrams (b) to (f) in Fig. 1 is considered to be negative when it changes from the left hand to the right hand side or vice versa relative to diagram (a) in Fig. 1.

To each of the diagrams (g) and (h) in Fig. 1 there corresponds the six Feynman diagrams in Fig. 3. These six diagrams, as drawn in Fig. 3 with exactly the same four-momentum assignments and directions of arrows, are used for each of the processes (g) and (h) in Fig. 1. For \( k_0 > 0 \) and \( k'_0 > 0 \) we are considering the three-particle process (g) involving quarks and for \( k_0 < 0 \) and \( k'_0 < 0 \) we are considering the three-particle process (h) involving antiquarks.

The relation between the measured structure functions on the hadronic level and the results from our calculations of thermal averages of the fundamental (virtual) photon-quark interaction processes is discussed in refs. [2] and [7]. One obtains on the hadronic level, as an example, for the \( \mathcal{O}(\alpha_s) \) process of gluon emission from a quark in Fig. 1(a):

\[
\sigma^{(a)}_\lambda = \frac{V}{2K} \sum_q \int \frac{d^3k}{(2\pi)^32|k_0|} n_F(x_k) \int \frac{d^3k'}{(2\pi)^32|k'_0|} [1 - n_F(x_{k'})] \\
\times \int \frac{d^3p}{(2\pi)^32|p_0|} [1 + n_B(|p_0|)] \langle 2\pi \rangle^4 \delta^4(q + k - k' - p) \\
\times \frac{N^2_c - 1}{2} \sum_{\text{spins}} |M_\lambda|^2 \tag{1}
\]
according to the momentum assignments in Fig. 2 and where $V$ is the volume of the nucleon. In eq. (1) $\lambda$ refers to the helicity of the virtual photon. The matrix element for gluon emission from a quark for a given polarization, $\lambda$, of the space-like ($q^2 < 0$) photon ($\gamma^*$) is given by

$$M_\lambda = M_1 + M_2$$

with

$$M_1 = \bar{u}(k', s')(-ig\gamma_\mu)[\epsilon_\mu^\nu(p)]^*iS^{11}(k + q)(ie\gamma_\nu)\epsilon_\lambda^\nu(q)u(k, s)$$

$$M_2 = \bar{u}(k', s')(ie\gamma_\mu)\epsilon_\mu^\nu iS^{11}(k - p)(-ig\gamma_\nu)[\epsilon_\nu^\nu(p)]^*u(k, s)$$

where we suppress reference to colour factors and fractional charges and where $iS^{11}$ denotes the 11-component of the fermion propagator in the real-time formalism of finite-temperature quantum field theory.

The equations (3) and (4) provide explicit expressions for the matrix element in eq. (1) (with coupling constants, colour factors and fractional charges factored out into explicit normalization constants). We take $N_c = 3$ and include up and down quarks in our model.

Fermion and boson distribution functions can be defined in such a manner that they are valid for both positive and negative energies. They are for a fermion

$$n_F(x_k) = \frac{1}{e^{\beta x_k} + 1}$$

where

$$x_k = |k_0| - \mu \epsilon(k_0)$$

where $\epsilon(k_0)$ gives the sign of $k_0$ and for a boson

$$n_B(|k_0|) = \frac{1}{e^{\beta |k_0|} - 1}.$$ 

### 3. Calculation of the structure functions

The zeroth order in $\alpha_s$ contributions can be calculated in a straightforward manner [2]. The calculation of $O(\alpha_s)$ contributions require a much larger effort. In the calculations of the $O(\alpha_S)$ contributions, the phase space and loop integrations can be analytically reduced [9] to an integration over the two energy variables $K$ and $p_0$ as shown in the final expressions for four-particle processes given by eqs. (B1) and (B2) and for three-particle processes given by eqs. (C1) and (C2). These double integrals cannot be analytically calculated. Their numerical calculation is discussed in Section 4.2.
We distinguish between the zeroth and first order in \( \alpha_s \) contributions by writing:

\[
F_2 = F_2^{(0)} + F_2^{(1)}. \tag{8}
\]

Since we are also interested in the structure function \( R \) which is expressible in terms of \( F_2 \) and \( F_1 \), we also calculate

\[
F_1 = F_1^{(0)} + F_1^{(1)}. \tag{9}
\]

Zeroth order in \( \alpha_s \) expressions for \( F_2 \) and \( F_1 \) in the statistical model are given in ref. [2]:

\[
F_2^{(0)} = \frac{3M^2x^2V}{2\pi^2} \frac{(1 + Mx/\nu)^2}{(1 + Mx/\nu)^3} \sum_q e_q^2 \left\{ \left[ \frac{3}{2} \left( 1 + \frac{2\mu_q}{\nu} \right)^2 - \frac{1}{2} \left( 1 + \frac{Mx}{\nu} \right)^2 \right] f_0(z_q) + \frac{6T}{\nu} \left( 1 + \frac{2\mu_q}{\nu} \right) f_1(z_q) + \frac{6T^2}{\nu^2} f_2(z_q) + \mu_q \rightarrow -\mu_q, z_q \rightarrow \bar{z}_q \right\} \tag{10}
\]

\[
F_1^{(0)} = \frac{3M^2x^2VT}{8\pi^2} \frac{(1 + Mx/\nu)}{(1 + Mx/\nu)^3} \sum_q e_q^2 \left\{ \left[ \left( 1 + \frac{2\mu_q}{\nu} \right)^2 + \left( 1 + \frac{Mx}{\nu} \right)^2 \right] f_0(z_q) + \frac{4T}{\nu} \left( 1 + \frac{2\mu_q}{\nu} \right) f_1(z_q) + \frac{4T^2}{\nu^2} f_2(z_q) + \mu_q \rightarrow -\mu_q, z_q \rightarrow \bar{z}_q \right\} \tag{11}
\]

where the sum over \( q \) is over quark flavours only and where

\[
z_q \equiv \frac{Mx}{2T} - \frac{\mu_q}{T} \tag{12}
\]

and

\[
\bar{z}_q \equiv \frac{Mx}{2T} + \frac{\mu_q}{T} \tag{13}
\]

\[
f_n(z) \equiv \int_z^\infty \frac{y^n dy}{ey + 1}. \tag{14}
\]

The \( \mathcal{O}(\alpha_s) \) parts of expressions for \( F_2 \) and \( F_1 \) in the statistical model are:

\[
F_2^{(1)} = \frac{K}{4\pi^2 \alpha} \left( \sigma_T^{(\alpha_s)} + \sigma_L^{(\alpha_s)} \right) \frac{-q^2 \nu}{q^2} \tag{15}
\]

\[
F_1^{(1)} = \frac{K}{4\pi^2 \alpha} M \sigma_T^{(\alpha_s)} \tag{16}
\]

where

\[
\sigma_T^{\alpha_s} = \frac{V}{2} \frac{N_c^2 - 1}{2} \frac{4\alpha_s4\pi\alpha}{2K} \sum_q e_q^2 \left\{ \int M_\Sigma S_4 d\mu + \sum_{W=B,F,F'} \int M_\Sigma^{3W} S_{3W} d\mu_W \right\}
\]
\[
\sum_{W=B,F,F'} \int \bar{M}_0 S_4 d\mu + \sum_{W=B,F,F'} \int \bar{M}_0^3 W S_3 W d\mu \]

with the expressions in curly brackets given in eqs. (C1), (C2), (B1) and (B2). They have to be evaluated for each flavour of quark, since the chemical potential appearing in the statistical factors \( S_4 \) and \( S_3 W \) differ for each flavour as we sum over the flavours of quarks. As explained in Appendices B and C, the \( K, p_0 \) integration regions for four- and three-particle processes are as given in Figs. 4 and 5, respectively, and the integrand for four-particle processes is different in each subregion shown in Fig. 4.

4. Numerical Evaluation

The numerical calculation of the structure functions \( F_2 \) and \( R \) was performed in two parts. The zeroth order is discussed in the next section while the contributions of \( \mathcal{O}(\alpha_s) \) are discussed in Section 4.2. Results from the total calculation, i.e., zeroth order plus \( \mathcal{O}(\alpha_s) \) results are presented in Section 4.3.

4.1. Zeroth Order

We chose to fit to a plot of \( F_2 \) versus \( x \), which is determined by parton distributions which reproduce a wide range of experimental data. Such parton distributions were published in 1990 by Kwiecinski et al. in ref. [17]. They parametrize the distributions at \( Q_0^2 \equiv -q^2 = 4 \text{ GeV}^2 \) and evolve up in \( Q^2 \) in order to test their parton distributions against experimental data. We fit to their so-called \( B_0 \) fit at \( Q^2 = 4 \text{ GeV}^2 \).

The values for temperature and chemical potential which give the best fit for \( F_2 \) versus \( x \) are used to calculate the corresponding plot of \( R \) versus \( x \) which we will compare with experimental data given by Whitlow [18]. In the deep-inelastic scattering limit, \( x, T, \mu_q \) finite with \( \nu \) going to infinity, the expression for \( F_2^{(0)} \) in eq. (10) is dominated by the \( f_0 \) term which can be evaluated analytically. In Ref. [2] the latter expression was used to fit the experimental data by Aubert et al. [19].

In this work we keep the full expression for \( F_2^{(0)} \) as given in eq. (10) and fit to \( F_2 \) as determined by the parton distributions by Kwiecinski et al. in ref. [17] (their \( B_0 \) fit). The values for temperature and chemical potential which produce the best fit to the plot of \( F_2 \) versus \( x \) are used to calculate \( F_1^{(0)} \) in eq. (11) too. Then the structure function \( R \) which is expressible in terms of \( F_2^{(0)} \) and \( F_1^{(0)} \) can
be calculated. The chemical potential for the down quarks was kept fixed in terms of the chemical potential for the up quarks according to

$$\mu_{\text{down}} = \frac{\mu_{\text{up}}}{2^{1/3}}$$ (19)

(as was done in ref. [2]) in order to reproduce the ratio of up to down quarks as being 2 to 1. This is approximately the case for a quark gas at very low temperatures.

As opposed to the $f_0$ function, the functions $f_1$ and $f_2$, which also appear in the expression for $F_2^{(0)}$ in eq. (10), cannot be evaluated analytically. They were evaluated numerically.

For $x \geq 0.4$, the zeroth order in $\alpha_s$ calculation produced a good fit to the $Q^2 = 4\text{GeV}^2$ data of Kwiecinski et al. as shown in Fig. 6.

According to the following rough estimates, the failure of the zeroth order in $\alpha_s$ theory of the statistical model to reproduce the low $x$ behaviour of $F_2$ could be due to the finite volume of the nucleon. From the mass shell condition on the four-momentum $k' = k + q$ of the outgoing quark in the zeroth order in $\alpha_s$ (lowest order Feynman diagram in Fig. 3), one obtains a lower limit on $|\vec{k}|$ of the incoming quark as follows. From the mass shell condition for massless quarks \((k + q)^2 = 0\) and the relation $x = Q^2/(2M\nu)$ one obtains

$$-1 \leq \cos \theta_{\vec{k}q} = \frac{2|\vec{k}|\nu - Q^2}{2|\vec{k}|\nu^2 + 2M\nu x} \leq 1.$$ (20)

The first inequality in the latter line produces

$$|\vec{k}| \geq \frac{Q^2}{2\nu + 2\nu \sqrt{1 + \frac{2Mx}{\nu}}} = \frac{2M\nu x}{2\nu + 2\nu \left[1 + \mathcal{O} \left(\frac{x}{\nu}\right)\right]}$$

$$= \frac{Mx}{2} + \mathcal{O} \left(\frac{x}{\nu}\right).$$ (21)

Another lower limit on $|\vec{k}|$ is obtainable from the Heisenberg uncertainty principle of the general form $\Delta x \Delta k \geq 1$. Taking $\Delta x$ equal to the diameter of the nucleon ($2 \text{ fermi} = 10.14 \text{ GeV}^{-1}$), one obtains $\Delta k \geq (10.14)^{-1}$. From this and the value of the lower limit on $|\vec{k}|$ in eq. (21) one could deduce that our model is only valid for $x$ values satisfying

$$\frac{Mx}{2} \geq (10.14)^{-1},$$ (22)

i.e., $x \geq 0.21$.

The zeroth order in $\alpha_s$ results for the structure function $R$ are shown in Fig. 7. For $x \geq 0.25$ these results are consistent with the $Q^2 = 4\text{GeV}^2$ experimental data given in Ref. [18].

The zeroth order in $\alpha_s$ calculation is only able to reproduce the data for values of $x \geq 0.4$. In Section 4.2 we investigate to what extent the $\mathcal{O}(\alpha_s)$ corrections are able to extend the fit to values of $x$ smaller than 0.4.
4.2. The zeroth- plus first order in $\alpha_s$ calculation

In this section we take the $O(\alpha_s)$ corrections into account by numerically calculating the quantities given in eqs. (15) and (16) and adding them to zeroth order in $\alpha_s$ results according to eqs. (8) and (9). The zeroth order in $\alpha_s$ quantities given in eqs. (10) and (11) are calculated in the manner discussed in Section 4.3. Due to the modifications introduced by the $O(\alpha_s)$ contributions, we search for new values of temperature and chemical potential which will produce a fit to the plot of $F_2$ versus $x$ of Kwiecinski et al. given in Fig. 6(a).

We keep the chemical potential of the down quarks fixed in terms of the chemical potential of the up quarks according to eq. (19). The values for temperature and chemical potential which produce the best fit to the data are used to calculate $F_1$ in eq. (9). Then the structure function $R$ which is expressible in terms of $F_2$ and $F_1$ can be calculated and compared to the experimental data.

According to the $O(\alpha_s)$ expressions, $F_2^{(1)}$ and $F_1^{(1)}$ in eqs. (15) and (16), one needs to numerically calculate the double integrals in eqs. (B1), (B2),(C1) and (C2). The $K,p_0$ integration regions for four- and three-particle processes are shown in Figs. 4 and 5, respectively. These integrals are devoid of ultraviolet, infrared and collinear singularities as discussed in ref. [9]. However, individual terms in the integrands can still become infinite along certain lines in the $K,p_0$ integration regions. This happens when factors in denominators or arguments of ln functions in some terms become equal to zero along certain lines.

Examples are singularities which arise along the line along which the quantity $b$ is equal to zero. The latter singularity arises in the terms proportional to $\ln|\bar{C}_4|$ and $|G_4|$ in eq. (B1). One of the subregions which contain the line along which $b$ is equal to zero is subregion D in Fig. 4.

Even though singularities from such terms cancel each other, the terms individually give rise to singularities. Therefore one should take care not to approach the line along which $b = 0$ too closely in the numerical integration. For the same reason one should steer clear of other lines along which other individual terms in expressions for three- and four-particle processes are singular. This is accomplished by implementing a trapezium integration method whereby the integrand is evaluated at the discrete points of a two-dimensional lattice constructed in such a way that the two nearest successive lattice points to any of the lines along which singularities arise, are always located an equal distance on both sides of the relevant line. The value of the integral is the stable value attained in successive numerical integrations in which, independently and in steps, the density of lattice points is increased and the range of integration is extended (The presence of thermal distribution factors in terms suppresses contributions at large values of the integration variables.).

4.3. Results

In Section 4.1 it was mentioned that the zeroth order in $\alpha_s$ calculation is able to reproduce the data of Kwiecinski et al. on $F_2$ versus $x$ for values of $x \geq 0.4$ only.
From Fig. 8 it can be seen that, by including $O(\alpha_s)$ corrections, the reproduction of the data of Kwiecinski et al. could be extended to values of $x$ smaller than 0.4. A reasonable fit could be obtained for values of $x \geq 0.25$. This is compatible with our estimate for the limited range of validity of our theory ($x \geq 0.21$ as derived in Section 4.1) due to the finite volume of the nucleon.

For the fixed values of the parameters and the fixed expression for the chemical potential of the down quarks in terms of the chemical potential of the up quarks in eq. (19), a fit to the data of Kwiecinski et al. is possible for values of temperature and chemical potential in the immediate vicinity of $T = 0.067$ GeV and $\mu_{\text{up}} = 0.133$ GeV only.

Figs. 9 and 10 show how the results for $F_2$ versus $x$ from the calculation including zeroth order plus $O(\alpha_s)$ contributions vary as a function of temperature at fixed chemical potential and as a function of chemical potential at fixed temperature, respectively.

From Fig. 8 it can be seen that the $O(\alpha_s)$ corrections to $F_2$ is negative for large $x$. Such negative values could in principle arise from interference terms from three-particle processes since, by neglecting the $O(\alpha_s^2)$ contributions to the amplitude squared of the $O(\alpha_s^0)$ and $O(\alpha_s^1)$ Feynman diagrams for three-particle processes (see Fig. 3), the positive definiteness of the amplitude squared is lost.

Results of the calculation of the structure function $R$ with zeroth order plus $O(\alpha_s)$ expressions for $F_2$ and $F_1$, as they appear in the expression for $R$ are shown in Fig. 11. In the $x \geq 0.25$ region of interest the results typically lie a factor of 6 above the experimental data for $R$. The absence of data for $x \geq 0.8$ in Fig. 11 is due to the numerical instability encountered as $F_1$, appearing in a denominator in the expression for $R$, goes to zero at large $x$.

5. Summary

In this work, we considered deep inelastic scattering of leptons off a proton in the statistical model proposed in ref. [2]. The interior of the nucleon is viewed as a thermalized assembly of up and down quarks and gluons. This enables one to incorporate features which are absent in the standard parton model. Quantum statistical correlations are incorporated through the use of Fermi-Dirac and Bose-Einstein distributions for initial-state quarks and gluons, respectively. Stimulated emission factors for final-state gluons and Pauli-blocking factors for final-state quarks are incorporated. The propagation of particles through a many-body medium is taken into account by using thermal Feynman rules for propagators and vertices. The statistical model could also be seen as an attempt to describe the interior of the nucleon at a more fundamental level than that attained through the use of arbitrary parton distributions containing many parameters in the parton model.

We chose to fit to a plot of $F_2$ versus $x$, which is determined by parton distributions which reproduce a wide range of experimental data. Such parton distributions were published in 1990 by Kwiecinski et al. in [17]. They decide on
a parametrization of the parton distributions at \( Q_0^2 \equiv -q^2 = 4 \text{ GeV}^2 \) and evolve up in \( Q^2 \) in order to test their parton distributions against experimental data.

Our zeroth order in the strong coupling constant \( \alpha_S \) calculation of the structure function \( F_2 \) versus \( x \) at \( Q^2 = 4 \text{ GeV}^2 \) produced a good fit to the data for \( x \geq 0.4 \). The values for temperature and chemical potential which give the best fit are \( T = 0.04 \text{ GeV} \) and \( \mu_{\text{up}} = 0.21 \text{ GeV} \) (with \( \mu_{\text{down}} = \mu_{\text{up}}/2^{\frac{1}{3}} \)). Other combinations of values for \( T \) and \( \mu_{\text{up}} \) do not produce an improved fit to the data. In consequence, we investigate to what extent \( O(\alpha_S) \) corrections are able to extend the reproduction of the data to values of \( x \) smaller than 0.4. The corresponding zeroth order in \( \alpha_S \) results for the structure function \( R \) are consistent with experimental data Fig. 7.

The inclusion of the \( O(\alpha_S) \) corrections [7] in our numerical calculations allowed the extension of the reproduction of the data of Kwiecinski et al. to values of \( x \) smaller than 0.4. A fit could be obtained for values of \( x \geq 0.25 \). This is compatible with our estimate for the limited range of validity of our theory \((x \geq 0.21)\) due to a finite-size effect arising from the finite volume of the nucleon. The latter fit is possible for values of temperature and chemical potential in the immediate vicinity of \( T = 0.067 \text{ GeV} \) and \( \mu_{\text{up}} = 0.133 \text{ GeV} \) only (with \( \mu_{\text{down}} = \mu_{\text{up}}/2^{\frac{1}{3}} \) and \( \alpha_s = 0.2 \)).

The latter values of parameters, however, do not fit the structure function \( R = \sigma_L/\sigma_T \) to the experimental data presented in Ref. [18]. Even when taking into account that all measurements of \( R \) suffer from large experimental errors due to the weak dependence of the deep inelastic cross section for charged leptons on \( R \), the size of the discrepancy remains unacceptable. This indicates a shortcoming of the statistical model in its present form to reproduce the structure function of the proton.

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Appendix A : Final expressions for three- and four-particle processes

In this appendix we summarize expressions derived for the phase space integrations of the matrix element squared from four- and three-particle processes.

The way in which the final expressions for four-particle processes in eqs. (B1) and (B2) and for three-particle processes in eqs. (C1) and (C2) contribute to the structure functions of the nucleon can be seen by considering eqs. (8),(9) and (15)–(18).

In order to write the expressions in a simplified form, we introduce the abbreviations

\[
\begin{align*}
A &= \frac{(2K + p_0 + \nu)}{2} \\
B &= \frac{(2K - p_0 + \nu)}{2} \\
C &= \frac{(2K + p_0 + q_z \varepsilon(k_0 k'_0 p_0))}{2} \\
D &= \frac{(2K - p_0 + q_z \varepsilon(k_0 k'_0 p_0))}{2} \\
E &= \frac{(\nu + q_z \varepsilon(k_0 k'_0 p_0))}{2} \\
a &= \frac{(2K + p_0 - \nu)}{2} \\
b &= \frac{(2K - p_0 - \nu)}{2} \\
c &= \frac{(2K + p_0 - q_z \varepsilon(k_0 k'_0 p_0))}{2} \\
d &= \frac{(2K - p_0 - q_z \varepsilon(k_0 k'_0 p_0))}{2} \\
e &= \frac{(\nu - q_z \varepsilon(k_0 k'_0 p_0))}{2}
\end{align*}
\]

where \( \varepsilon \) is the sign function.

In the calculation of the \( O(\alpha_s) \) corrections [9], all phase space integrations are analytically reduced to integrations over only the two energy variables \( k_0 \) and \( p_0 \) which are the energy components of the four-momentum assignments \( k \) and \( p \), respectively, in Figs. 2 and 3.

According to the mechanism by means of which collinear and infrared divergences cancel [9], it is appropriate to transform from the variables \( (k_0, p_0) \) to the variables \( (K, p_0) \) where

\[
K = k_0 + \frac{\nu - p_0}{2}
\]

Appendix B : Final expressions for four-particle processes

The convention concerning the sign of the energy of a particle discussed in Section 2, enables one to view all contributing four-particle processes simultaneously in the two-dimensional energy plane shown in Fig. 4. The Latin letters in Fig. 4 serve to label the subregions over which one integrates in the \( K, p_0 \) plane. The numbers in Fig. 4 label the classes of subregions. The final analytical expressions are different for each class. The classes arise as follows: For some regions of the \( k_0, k'_0 \) plane \( (k'_0 = k_0 + q_0 - p_0) \), the restrictions imposed by the energy–momentum conserving Dirac delta function in the phase space integration, causes one to replace the usual upper limit of 1 and/or the usual lower limit of \(-1\) for \( \cos \theta \) and/or \( \cos \theta' \) (where \( \theta \) is the angle between \( \vec{q} \) and \( \vec{K} \) and \( \theta' \) the angle between \( \vec{q} \) and \( \vec{K}' \)) by expressions in terms of \( k_0 \) and \( k'_0 \). The classes of subregions in the \( k_0, k'_0 \) plane are distinguished according to the combination of upper and lower limits for \( z \) and \( z' \) which are obtained for each class. In consequence, the integrations over \( z \) and \( z' \) will lead to different analytical results for each class.
The final results for four-particle processes after the cancellation of collinear
singularities are

\[
\int \tilde{M}_\Sigma S_4 d\mu = \frac{1}{4\pi^3 q_z} \int dK dp_0 S_4 \left\{ \frac{q^2}{2p_0} \ln |\bar{A}_4| + |q^2| \left[ \frac{|p_0| q^2}{4 A^2} - \frac{\epsilon(p_0)}{2 A} \right] \ln |\bar{B}_4| \\
+ |q^2| \left[ \frac{|p_0| q^2}{4 b^2} + \frac{\epsilon(p_0)}{2 b} \right] \ln |\bar{C}_4| \\
+ \epsilon(k_0 k'_0) \left[ E_4 + \frac{(B - p_0)(c - E)(C - e)}{2 A} |F_4| \\
+ \frac{(a + p_0)(d + e)(D + E)}{2 b} |G_4| \right] \right\}
\]

(B1)

and

\[
\int \tilde{M}_0 S_4 d\mu = \frac{1}{4\pi^3 q_z} \int dK dp_0 S_4 \left\{ \frac{q^2}{2p_0} \left[ \frac{|p_0| q^2}{4 q_z^2 A} - \epsilon(p_0) \frac{q^2 - 2 B \nu}{2 q_z^2} \right] \ln |\bar{B}_4| \\
+ \frac{q^2}{2p_0} \left[ \frac{C c + D d}{q_z^2} \ln |\bar{A}_4| + \frac{|q^2|}{2b} \left[ \frac{|p_0| q^2}{4 b q_z^2} + \epsilon(p_0) \frac{q^2 + 2 a \nu}{2 q_z^2} \right] \ln |\bar{C}_4| \\
+ \epsilon(k_0 k'_0) \frac{q^2}{2 q_z^2} \left[ E_4 + \frac{(B - p_0)(c - E)(C - e)}{2 A} |F_4| \\
+ \frac{(a + p_0)(d + e)(D + E)}{2 b} |G_4| \right] \right\},
\]

(B2)

with quantities as defined in Table 1 and

\[
S_4 = \left[ \frac{1}{2} - \frac{\epsilon(k_0)}{2} (1 - 2 n_F(x_k)) \right] \left[ \frac{1}{2} + \frac{\epsilon(k'_0)}{2} (1 - 2 n_F(x'_k)) \right] \\
\left[ \frac{1 + \epsilon(p_0)}{2} + n_B(|p_0|) \right].
\]

(B3)

These expressions are valid for all the four-particle processes with the $K, p_0$ integration
region given in Fig. 4.

The way in which these expressions contribute to the structure functions of
the nucleon can be seen by considering eqs. (A2), (B1) and (15)–(18).

Appendix C : Final expressions for three-particle processes

Similar to our discussions for the four-particle processes, both the quark
and antiquark three-particle processes can be viewed simultaneously in the two-
dimensional energy plane shown in Fig. 5. From the figure it can be seen that
the $K, p_0$ integration region for three-particle processes is not subdivided into sub-
regions as in the case of four-particle processes (see Fig. 4). For three-particle
processes, subdivisions of the energy plane need only to be considered when performing angular integrations in intermediate steps and these subdivisions disappear after the angular integrations.

The final results for three-particle processes after the cancellation of collinear singularities are [Declaration of symbols appears after eq. (C2)]:

\[
\sum_{W=B,F,F'} \int \tilde{M}_W^3 S_3W d\mu_W
\]

\[
= \frac{1}{4\pi^3 q_z} \int dKdp_0 \theta(1-y)\theta(1+y) \left\{ S_{3B}(−|p_0|) \left[ 1 + \frac{a}{A} + \frac{A}{a} + \frac{q^2}{2} \left( \frac{1}{a^2} + \frac{1}{A^2} \right) \right] \\
+ S_{3F} \left[ -\frac{\epsilon(b)q^2}{2p_0} \ln |A_F| - \epsilon(b)|q^2| \left[ \frac{p_0}{4A^2} - \frac{1}{2a} \right] \ln |\tilde{C}_F| \\
+ \epsilon(b)|q^2| q_z \left[ \frac{3 + a + A}{4q_z^2} (a + A - 2p_0) \right] \ln |K_F| + |b| \left[ \frac{1}{2} + \frac{A^2 - a^2}{q_z^2} + \frac{q^2}{2a^2} + \frac{A}{a} \right] \right] \\
+ S_{3F'} \left[ -\frac{\epsilon(B)q^2}{2p_0} \ln |A_{F'}| - \epsilon(B)|q^2| \left[ \frac{p_0}{4A^2} - \frac{1}{2A} \right] \ln |B_{F'}| \\
- \frac{\epsilon(B)q^2}{q_z} \left[ \frac{3 + a + A}{4q_z^2} (a + A - 2p_0) \right] \ln |K_{F'}| \\
+ |B| \left[ \frac{1}{2} + \frac{A^2 - a^2}{q_z^2} + \frac{q^2}{2a^2} + \frac{a}{A} \right] \right] + S_2 \frac{p_0|p_0|(K + p_0/2)}{p_0^2 + T^2} \right\}. \tag{C1}
\]

The corresponding final result for the longitudinal polarization of the virtual photon is:

\[
\sum_{W=B,F,F'} \int \tilde{M}_0^3 W S_3W d\mu_W
\]

\[
= \frac{1}{4\pi^3 q_z} \int dKdp_0 \theta(1-y)\theta(1+y) \left\{ -S_{3B} \frac{|p_0|}{q_z^2} \left[ -\nu^2 + \frac{q^2}{2} \left[ \frac{a}{A} + \frac{A}{a} + \frac{q^2}{2} \left( \frac{1}{a^2} + \frac{1}{A^2} \right) \right] \right] \\
+ S_{3F} \left[ -\epsilon(b) \frac{|q^2|}{4p_0q_z^2} \ln |A_F| |q^2 + 4K^2 + p_0^2 - \nu^2| - \epsilon(b) \frac{|q^2|}{2aq_z^2} \left[ p_0 \left( \frac{q^2}{4a} + \nu \right) \\
- \frac{q^2 + 2\nu a}{2} \right] \ln |\tilde{C}_F| + \epsilon(b) \frac{|q^2|}{q_z^2} \left[ \frac{1}{4} - \frac{a + A}{4q_z^2} (a + A - 2p_0) \right] \ln |K_F| \\
+ \epsilon(b) \frac{b}{q_z^2} \left[ -\frac{\nu^2}{2} - A^2 + a^2 + \frac{q^2}{2} \left( \frac{A}{a} + \frac{q^2}{2a^2} \right) \right] \right] \\
+ S_{3F'} \left[ -\epsilon(B) \frac{|q^2|}{4p_0q_z^2} \ln |A_{F'}| |q^2 + 4K^2 + p_0^2 - \nu^2| - \epsilon(B) \frac{|q^2|}{2Aq_z^2} \left[ p_0 \left( \frac{q^2}{4A} - \nu \right) \\
- \frac{q^2 - 2\nu A}{2} \right] \ln |B_{F'}| - \epsilon(B) \frac{|q^2|}{q_z^2} \left[ \frac{1}{4} - \frac{a + A}{4q_z^2} (a + A - 2p_0) \right] \ln |K_{F'}| \right\}.
\]

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\[ + \epsilon(B) \frac{B}{q_z^2} \left[ -\frac{\nu^2}{2} + A^2 - a^2 + \frac{q^2}{2} \left( \frac{a}{A} + \frac{q^2}{2A^2} \right) \right] + S_2 \frac{p_0|p_0|(K + p_0/2)}{p_0^2 + T^2} \right\}. \]  (C2)

where

\[ S_{3B} = S_2 \left[ 1 + 2n_B(|p_0|) \right] \frac{1}{2} \]  (C3)

\[ S_{3F} = S_2 \left[ 1 - 2n_F(x_k-p) \right] \frac{1}{2} \]  (C4)

\[ S_{3F'} = S_2 \left[ 1 - 2n_F(x_{k'-p}) \right] \frac{1}{2} \]  (C5)

\[ S_2 = \left\{ \frac{1}{2} - \frac{\epsilon(k_0)}{2} (1 - 2n_F(x_k)) \right\} \left\{ \frac{1}{2} + \frac{\epsilon(k_0+\nu)}{2} (1 - 2n_F(x_{k+q})) \right\} \]  (C6)

\[ x_k = |k_0| - \mu(\epsilon(k_0)). \]  (C7)

\[ \bar{A}_F = -\frac{b^2}{Dd} \]  (C8)

\[ K_F = \frac{(q_z+\nu)(2K-p_0-q_z)}{(q_z-\nu)(2K-p_0+q_z)} \]  (C9)

\[ \bar{C}_F = \frac{b^2}{p_0^2} \]  (C10)

\[ \bar{A}_{F'} = -\frac{B^2}{Dd} \]  (C11)

\[ K_{F'} = \frac{(q_z+\nu)(2K-p_0+q_z)}{(q_z-\nu)(2K-p_0-q_z)} \]  (C12)

\[ \bar{B}_{F'} = \frac{B^2}{p_0^2} \]  (C13)

and where, in the step functions \( \theta \),

\[ y = \frac{q^2 + 2k_0\nu}{2k_0q_z}. \]  (C14)

The factor \( S_2 \) describes the statistical factors for the external either quarks or antiquarks in three-particle processes. The last term in each of eqs. (C1) and (C2) has been introduced [8] to make the integrations over positive and negative \( p_0 \) separately convergent for the purpose of numerical calculation. From the definition of \( K \) it follows that \( K + p_0/2 \) is independent of \( p_0 \) and therefore that the last term in each of eqs. (C1) and (C2) contributes zero when added at a positive and negative value of \( p_0 \).
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Table 1: Definitions of the quantities appearing in eqs. (B1) and (B2) [9]

| Class | $\tilde{A}_4$ | $\tilde{B}_4$ | $\tilde{C}_4$ | $E_4$ | $F_4$ | $G_4$ |
|-------|--------------|--------------|--------------|-------|-------|-------|
| 1     | 1            | 1            | 1            | $p_0$ | $\frac{p_0}{A(B - p_0)}$ | $\frac{p_0}{b(a + p_0)}$ |
| 2     | $\frac{B^2}{Dd}$ | $\frac{B^2}{p_0^2}$ | $\frac{q^2}{4dD}$ | $B$ | $\frac{B}{A(B - p_0)}$ | $\frac{B}{(d + e)(D + E)}$ |
| 3     | $\frac{aB}{cD}$ | $\frac{EB}{p_0c}$ | $\frac{Ea}{p_0D}$ | $E$ | $\frac{E}{A(c - E)}$ | $\frac{E}{b(D + E)}$ |
| 4     | $\frac{cB}{ad}$ | $\frac{BC}{p_0E}$ | $\frac{p_0e}{ad}$ | $c$ | $\frac{c}{A(c - E)}$ | $\frac{c}{(d + e)(a + p_0)}$ |
| 5     | $\frac{aD}{BC}$ | $\frac{p_0 - e}{c}$ | $\frac{aD}{p_0E}$ | $D$ | $\frac{D}{(B - p_0)(C - e)}$ | $\frac{D}{b(D + E)}$ |
| 6     | $\frac{aB}{Cd}$ | $\frac{eB}{p_0C}$ | $\frac{ea}{p_0d}$ | $p_0 - e$ | $\frac{p_0 - e}{(C - e)(B - p_0)}$ | $\frac{p_0 - e}{(a + p_0)(d + e)}$ |
| 7     | $\frac{a^2}{Cc}$ | $\frac{q^2}{4Cc}$ | $\frac{a^2}{p_0^3}$ | $a$ | $\frac{a}{(C - e)(c - E)}$ | $\frac{a}{b(a + p_0)}$ |
Figure Captions

Fig. 1 Processes contributing to deep inelastic scattering to first order in $\alpha_s$.

Fig. 2 Diagrams for four-particle processes.

Fig. 3 Diagrams for three-particle processes.

Fig. 4 Regions of support for all four-particle processes in the $K, p_0$ plane [9].

Fig. 5 Regions of support for all three-particle processes in the $K, p_0$ plane.

Fig. 6 Plots of $F_2$ versus $x$ at $Q^2 = 4$ GeV$^2$; solid line reproduces the parton distributions of Kwiecinski et al. [17] ($B_-$ fit) and dashed line represents the zeroth order in $\alpha_s$ expression given in eq. (10).

Fig. 7 Plot of $R$ versus $x$ at $Q^2 = 4$ GeV$^2$ for $\alpha_s = 0.2$, $T = 0.04$ GeV, $\mu_{up} = 0.21$ GeV and $\mu_{down} = \mu_{up}/2^\frac{1}{3}$.

Fig. 8 Plots of $F_2$ versus $x$ at $Q^2 = 4$ GeV$^2$; short dashed line represents the zeroth order in $\alpha_s$ expression given in eq. (10) and the long dashed line represents the zeroth order plus $O(\alpha_s)$ expression given in eqs. (8) and (15) with $\alpha_s = 0.2$. The solid line is the $B_-$ fit of Ref. [17].

Fig. 9 Plots of $F_2$ versus $x$ at $Q^2 = 4$ GeV$^2$ as a function of temperature at fixed chemical potential $\mu_{up} = 0.133$ GeV according to the zeroth order plus $O(\alpha_s)$ expression given in eqs. (8) and (15) with $\alpha_s = 0.2$. The temperature is 0.043 GeV for the lowest dashed curve 0.055 GeV for the next upper one, 0.067 GeV for the third dashed line and 0.079 GeV for the highest dashed line. The solid line is the $B_-$ fit of Ref. [17].

Fig. 10 Plots of $F_2$ versus $x$ at $Q^2 = 4$ GeV$^2$ as a function of temperature at fixed temperature $T = 0.067$ GeV according to the zeroth order plus $O(\alpha_s)$ expression given in eqs. (8) and (15) with $\alpha_s = 0.2$. The chemical potential $\mu_{up}$ is 0.109 GeV for the lowest dashed curve 0.133 GeV for the next upper one, 0.157 GeV for the third dashed line and 0.181 GeV for the highest dashed line. The solid line is the $B_-$ fit of Ref. [17].

Fig. 11 Plot of $R$ versus $x$ at $Q^2 = 4$ GeV$^2$ with $\alpha_s = 0.2$, $T = 0.067$ GeV, $\mu_{up} = 0.133$ GeV and $\mu_{down} = \mu_{up}/2^\frac{1}{3}$.