Simple solutions of relativistic hydrodynamics for cylindrically symmetric systems

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Abstract

Simple, self-similar, analytic solutions of 1 + 3 dimensional relativistic hydrodynamics are presented for cylindrically symmetric fireballs corresponding to central collisions of heavy ions at relativistic bombarding energies.

Key words: Relativistic hydrodynamics, cylindrical symmetry, equation of state, Bjorken flow, analytic solutions

1 Introduction

The analytic resolution of 3 dimensional relativistic hydrodynamics is a difficult task because the equations are non-linear and the transverse and the longitudinal motions of the fluid are coupled in a way that is rather complicated to handle.

Historically, the first solutions of relativistic hydrodynamics were found for idealized, one dimensionally expanding sources: Landau’s one-dimensional analytical solution of relativistic hydrodynamics [1] and Bjorken’s boost-invariant solution [2], that correspond to ultrarelativistic bombarding energies. Both of these solutions are frequently utilized for various estimations of observables in ultra-relativistic nucleus-nucleus collisions.

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The repertoire of relativistic hydrodynamics has been reviewed from the point of view of heavy ion physics in ref. [3]. Recently, Biró has found self-similar exact solutions of relativistic hydrodynamics for cylindrically expanding systems [4,5]. Bjorken’s one-dimensional solution can be naturally extended to a three-dimensional solution, assuming spherical symmetry instead of (1+1) dimensional longitudinal dynamics. Biró’s solutions generalize the (1+3) dimensional version of Bjorken’s solution to the case of cylindrical symmetry, interpolating between the one-dimensional and the three-dimensional Bjorken solution. However, Biró’s solutions are valid only when the pressure is independent of space and time, as happens at the softest point of the equation of state.

The numerical solution of the 1+3 dimensional relativistic hydrodynamics is possible with presently available computers and methods, for example the variational method of smoothed particle hydrodynamics has been developed to study event-by-event fluctuations in relativistic heavy-ion collisions [6,7].

Here we describe a new family of exact analytic solutions of relativistic hydrodynamics with a broad family of equations of state, assuming cylindrical symmetry of the particle emitting source. The physical motivation for this study is to consider the time evolution of central collisions in ultra-relativistic heavy-ion physics within the framework of an analytic approach.

Our solution was motivated by the analytic solution of non-relativistic hydrodynamics found by Zimányi, Bondorf and Garpman (ZBG) in 1978 for low-energy heavy-ion collisions with spherical symmetry [8]. This solution has been generalized to the case of ellipsoidal symmetry by Zimányi and collaborators in ref. [9]. In ref. [10,11] a Gaussian parameterization has been introduced to describe the mass dependence of the effective temperature and the radius parameters of the two-particle Bose-Einstein correlation functions in high-energy heavy-ion collisions. Later it has been realized that this phenomenological parameterization of data corresponds to an exact, Gaussian solution of non-relativistic hydrodynamics with spherical symmetry [12]. The spherically symmetric self-similar solutions of non-relativistic hydrodynamics were generalized in ref. [13] in terms of an arbitrary scaling function for the temperature profile, and expressing the density distribution in terms of this scaling function. The spherically symmetric Gaussian solution has been generalized to ellipsoidally symmetric expansions [14], that provided simple analytic insight into the observables of non-central heavy-ion collisions [15]. The family of ellipsoidally symmetric solutions of non-relativistic hydrodynamics has been expressed in terms of a general scaling function for the temperature profile in ref. [16].

Our approach corresponds to a generalization of these recently obtained analytic solutions [12,13,15,16] of non-relativistic fireball hydrodynamics to the
case of relativistic longitudinal and transverse flows. In particular, an analytic approach, the Buda-Lund (BL) model has been developed to parameterize the single particle spectra and the two-particle Bose-Einstein correlations in high-energy heavy-ion physics in terms of hydrodynamically expanding, cylindrically symmetric sources [17]. Here we attempt to find an exact family of solutions of relativistic hydrodynamics that may include the BL model as a particular limiting case.

We attempt here to solve the full (1+3) dimensional relativistic hydrodynamical problem, in trying to overcome two shortcomings of Bjorken’s well known solution. These two shortcomings of Bjorken’s solutions are that i) it describes ultra-relativistic collisions, it is independent of scale parameter in the longitudinal direction and the rapidity distribution is flat; ii) it contains no transverse flow.

We have described recently a new family of (1+1) dimensional solutions [18] that is able to describe an arbitrary rapidity distribution $dn/dy$ by introducing scale-dependent quantities in the longitudinal direction, overcoming shortcoming i), but not addressing shortcoming ii). Here we consider the opposite case, overcoming shortcoming ii), the development of a scale dependent transverse flow in relativistic hydrodynamics, but not addressing shortcoming i). We address both shortcomings simultaneously in a subsequent manuscript [19].

2 The equations of relativistic hydrodynamics

We solve the relativistic continuity and energy-momentum conservation equation:

$$\partial_\mu (nu^\mu) = 0,$$
$$\partial_\nu T^{\mu\nu} = 0.$$  

Here the $n \equiv n(t, r)$ is the number density, the four-velocity is denoted by $u^\mu \equiv u^\mu(t, r) = \gamma(1, v)$, normalized to $u^\mu u_\mu = \gamma^2(1 - v^2) = 1$, and the energy-momentum tensor is denoted by $T^{\mu\nu}$. We assume perfect fluid,

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu - pg^{\mu\nu},$$

where $\epsilon$ stands for the relativistic energy density and $p$ denotes the pressure.

We close this set of relativistic hydrodynamical equations with the equations of state. We assume a gas containing massive conserved quanta,

$$\epsilon = mn + \kappa p,$$
$$p = nT.$$
The equations of state have two free parameters, $m$ and $\kappa$. Non-relativistic hydrodynamics of ideal gases corresponds to the limiting case of $m \gg T$, $v^2 \ll 1$ and $\kappa = \frac{3}{2}$.

The energy-momentum conservation equations can be projected into a component parallel to $u^{\mu}$ and components orthogonal to $u^{\mu}$, which are respectively the relativistic energy and Euler equations:

\begin{align*}
   u^{\mu} \partial_{\mu} \epsilon + (\epsilon + p) \partial_{\mu} u^{\mu} &= 0, \\
   u_{\nu} u^{\mu} \partial_{\mu} p + (\epsilon + p) u^{\mu} \partial_{\mu} u_{\nu} - \partial_{\nu} p &= 0.
\end{align*}

(6) \hspace{1cm} (7)

With the help of the equations of state and the continuity equation, the energy equation can be rewritten as an equation for the temperature,

$$u^{\mu} \partial_{\mu} T + \frac{1}{\kappa} T \partial_{\mu} u^{\mu} = 0.$$ 

(8)

We adopt the following notational conventions: the coordinates are $x^{\mu} = (t, r) = (t, r_x, r_y, r_z)$, $x_{\mu} = (t, -r_x, -r_y, -r_z)$ and the metric tensor is $g^{\mu\nu} = g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

We solve 5 independent equations, the continuity, the (3 spatial components of) relativistic Euler, and the temperature equation, eqs. (1,7,8). The equations of state, eq. (4,5) close this system of equations in terms of 5 variables, $n$, $T$ and $v = (v_x, v_y, v_z)$.

3 Self-similar, cylindrically symmetric solutions

As we are primarily interested in the effects of finite transverse size and the development of transverse flow, we assume that the longitudinal flow component is that of Bjorken’s type,

$$v_z(t, r_z) = \frac{r_z}{t}.$$ 

(9)

A similar assumption has been made when a new family of (1+1) dimensional solutions of relativistic hydrodynamics was obtained in ref. [18]. However, in contrast to what has been done in [18], here we assume scale-invariance in the longitudinal direction.

We search for self-similar solutions, that are scale dependent in the transverse directions, and depend only on the transverse radius variable $r_t = \sqrt{r_x^2 + r_y^2}$ and the longitudinal proper time $\tau_z = \sqrt{t^2 - r_z^2}$. Let us introduce the scaling
variable $x$ as
\[ x = \frac{r_x^2 + r_y^2}{R^2}, \tag{10} \]
and assume that, in the frame where $v_z = 0$ (longitudinal proper frame),
the transverse motion corresponds to a Hubble type of self-similar transverse expansion,
\[ v^x_z(\tau_z, r_z) = \frac{\dot{R}(\tau_z)}{R(\tau_z)} r_x, \tag{11} \]
\[ v^y_z(\tau_z, r_z) = \frac{\dot{R}(\tau_z)}{R(\tau_z)} r_y, \tag{12} \]
where $\dot{R} = dR(\tau_z)/d\tau_z$ and hereafter we will designate by starred symbols
the variables in the longitudinal proper frame. We assume that the scale $R$
depends on time only trough the longitudinal proper time, $\tau_z$.

In a relativistic notation, the above form may be parametrized as
\[ u^\mu = (\cosh \zeta \cosh \xi, \sinh \xi \frac{r_x}{r_t}, \sinh \xi \frac{r_y}{r_t}, \sinh \zeta \cosh \xi), \tag{13} \]
\[ \tanh \xi = \frac{\dot{R}(\tau_z)}{R(\tau_z)} r_t = v_t^* = \gamma_l v_t, \tag{14} \]
\[ \cosh \xi = \frac{1}{\sqrt{1 - \dot{R}^2 \tau_z}} \equiv \gamma_t^*, \tag{15} \]
\[ \cosh \zeta = \frac{t}{\tau_z} \equiv \gamma_l. \tag{16} \]
The space-time rapidity $\eta$ is defined as
\[ \eta = 0.5 \log \left( \frac{t + r_z}{t - r_z} \right). \tag{17} \]
For a scaling longitudinal Bjorken flow we obtain
\[ \zeta = \eta. \tag{18} \]

Using the above ansatz for the flow velocity distribution, we find that the
continuity equation is solved by the form
\[ n(t, r_x, r_y, r_z) = n_0 \left( \frac{\tau_{z0} R_0^2}{\tau_z R^2} \right) \frac{1}{\cosh \xi} G(x), \tag{19} \]
where $G(x)$ is an arbitrary non-negative function of the scaling variable $x$ and
$n_0$, $\tau_{z0}$ and $R_0$ are normalization constants. We use the convention $n_0 = n(t_0, 0, 0, 0)$, $\tau_{z0} = \tau_z(t_0, r_{z0})$ and $R_0 = R(\tau_{z0})$, where $r_{z0}$ is such that, together
with \( t_0 \), satisfies eq. (9). This implies that \( \mathcal{G}(x = 0) = 1 \). The temperature equation, eq. (8), is solved by

\[
T(t, r_x, r_y, r_z) = T_0 \left( \frac{\tau_z R_0^2}{\tau_x R^2} \right)^{1/\kappa} \mathcal{F}(x).
\]

(20)

The constants of normalization are \( T_0 = T(t_0, 0, 0, 0) \) and \( \mathcal{F}(0) = 1 \). We find that the solution is independent of the form of the function \( \mathcal{F}(x) \), provided that \( \mathcal{F}(x) > 0 \).

Using the ansatz for the flow profile and the solution for the density and the temperature profile, the relativistic Euler equation reduces to a complicated non-linear equation that contains \( \dot{R} \), \( \ddot{R} \) and \( \dot{R} \) and the variable \( x \). Taking this equation at \( x = 0 \) we express \( \dot{R} \) as a function of \( R \) and \( \dot{R} \). Substituting this back to the Euler equation we obtain a transcendental equation for \( \dot{R}^2 \), and \( x \). This equation has a particular solution if

\[
\dot{R} = \dot{R}_0.
\]

(21)

In this case, the acceleration of the radius parameter vanishes, \( \ddot{R} = 0 \), and the solution is \( R = R_0 + \dot{R}_0(\tau_z - \tau_{z0}) \). Then the relativistic Euler equation reduces to

\[
\left( 1 + \frac{1}{\kappa} \right) \left( \frac{\dot{R}}{\tau_z} + 3\dot{R}^2 \right) = 2(1 - x\dot{R}^2) \left[ \log \mathcal{G}(x)\mathcal{F}(x) \right]' ,
\]

(22)

where the lhs depends only on \( \tau_z \) while the rhs is only a function of the variable \( x \), hence both sides are constant. This implies that

\[
\frac{R}{\tau_z} = \dot{R}_0 ,
\]

(23)

thus \( R_0 = \dot{R}_0 \tau_{z0} \). Thus the origin of the time axis (fixed by the assumption of the scaling longitudinal Bjorken flow profile) coincides with the vanishing value of the transverse radius parameters.

The solutions can be casted in a relatively simple form by introducing the proper time \( \tau \),

\[
\tau = \sqrt{\tau_z^2 - \tau_t^2} = \sqrt{t^2 - r_x^2 - r_y^2 - r_z^2}.
\]

(24)

(25)

Using this natural variable we find that

\[
v = \frac{r}{t},
\]

(26)

\[
u^\mu = \frac{x^\mu}{\tau}.
\]

(27)
Thus the velocity field of our solution corresponds to the flow field of the spherically symmetric scaling solution. However, in the scaling solution the temperature and the pressure distributions are dependent only on the proper time \( \tau \), while in our case both the density and the temperature distributions are generally dependent on the scale variable \( x \) in the transverse direction.

As the solution is relativistic, and it is defined in the positive light-cone, given by \( \tau \geq 0 \), we obtain a constraint for the transverse coordinate, \( r_t \leq \tau_z \). This together with eq. (23), the solution for the scale \( R \), implies that the scaling variable has to satisfy the constraint \( x\dot{R}_0^2 \leq 1 \), which corresponds to the limitation that the velocity of the fluid can not exceed the speed of light.

By replacing eqs. (21,23) into the Euler equation, eq.(22), one obtains

\[
\frac{d}{dx} \log \left[ (1 - x\dot{R}_0^2)^{2(1+1/\kappa)} G(x)F(x) \right] = 0, \tag{28}
\]

which gives, together with the condition \( G(0)F(0) = 1 \),

\[
G(x)F(x) = (1 - \dot{R}_0^2 x)^{-2(1+1/\kappa)}. \tag{29}
\]

In this family of solutions, the scaling functions for the temperature and the density distribution are thus not independent. However, a constraint is given for their product, hence one of them can be chosen as an arbitrary positive function.

For clarity, let us introduce new forms of the scaling functions as

\[
T(x) = F(x)(1 - \dot{R}_0^2 x)^{2/\kappa}, \tag{30}
\]

\[
V(x) = G(x)(1 - \dot{R}_0^2 x)^2. \tag{31}
\]

Then the constraint can be casted to the simple form of

\[
V(x)T(x) = 1. \tag{32}
\]

This construction for the scaling functions of the transverse density and temperature profiles coincides with the method, that we developed for the solution of the relativisitic hydrodynamical equations in the \((1+1)\) dimensional problem, but here the transverse flow has a two-dimensional distribution, so the exponents and the scaling variables had to be re-defined accordingly.

Let us summarize our new family of solutions of the 1+3 dimensional relativistic hydrodynamics for cylindrically symmetric systems by substituting the results to the density, temperature and pressure profiles.
We obtain
\[ v = \frac{r}{t}, \quad \text{for} \quad |r| \leq t, \quad (33) \]
\[ x = \frac{r^2}{R_0^2 \tau^2}, \quad \text{for} \quad r_t \leq \tau, \quad (34) \]
\[ n(t, r) = n_0 \left( \frac{\tau_0}{\tau} \right)^3 \mathcal{V}(x), \quad (35) \]
\[ p(t, r) = p_0 \left( \frac{\tau_0}{\tau} \right)^{3+3/\kappa}, \quad (36) \]
\[ T(t, r) = T_0 \left( \frac{\tau_0}{\tau} \right)^{3/\kappa} \frac{1}{\mathcal{V}(x)}, \quad (37) \]

where \( p_0 = n_0 T_0 \). Note that the scaling variable \( x \) is invariant for boosts in the longitudinal direction, and it is rotation-invariant in the transverse direction, but \( x \) is not boost-invariant in the transverse directions. Hence we have generated cylindrically symmetric, longitudinally boost invariant solutions of relativistic hydrodynamics. In the longitudinal direction, these solutions are homogeneous, boost-invariant and also scale-invariant. Due to this reason, the observable rapidity distribution is
\[ \frac{dn}{dy} = \text{const}, \quad (38) \]
a flat distribution, corresponding to the ultra-relativistic nature of the solution in the longitudinal direction (where \( y = 0.5 \log[(E + k_z)/(E - k_z)] \) is the rapidity of a particle with four-momentum \((E, k)\) and \(dn/dy\) is the rapidity distribution of particle density).

A new hydrodynamical solution is assigned to each non-negative function \( \mathcal{V}(x) \), similarly to the cases of the non-relativistic solutions of ref. [13] and the 1+1 dimensional relativistic solution of ref. [18]. Note that the solutions are valid also for massive particles, the form of the solution is independent of the value of the mass \( m \). The form of solutions depends parameterically on \( \kappa \), that characterizes the equation of state.

We have obtained new solutions of the (1+3) dimensional relativistic hydrodynamical equations which describe a self-similar, streaming flow. In the case of \( \dot{R} = 1 \) and \( \mathcal{V}(x) = 1 \) we recover the spherically symmetric scale-invariant solution. This means that, in this limiting case, the pressure, the density and the temperature profiles depend only on the proper time \( \tau \).

In the general case, our solution depends on the scale \( R \) (or we can say it contains some characteristic scale \( R \)) and also on an arbitrary scaling function \( \mathcal{V}(x) \).
4 Summary

We have found a new family of solutions of 1+3 dimensional relativistic hydrodynamics with cylindrical symmetry. This family solves the continuity equation and the conservation of the energy-momentum tensor of a perfect fluid, assuming a simple equation of state, similarly to the case of the recently found 1+1 dimensional solutions. The mass of the particles \( m \) and the constant of proportionality between the kinetic energy density and the pressure, \( \kappa \), are free parameters of the solution.

As compared to the well-known case of the scale-invariant solution, we have solved one more equation, the continuity equation. We have considered equations of state that have two free parameters, the mass \( m \) and \( \kappa = \partial \epsilon / \partial p \), while the scale-invariant solution is obtained in the \( m = 0 \) approximation. Interestingly, our generalizations resulted in additional freedom in the solution. The new solutions, similarly to the scale-invariant case, prescribe scaling 3-dimensional flow and pressure distribution. However, in our case, the pressure is a product of the local number density and the local temperature, hence one of these can be chosen in an arbitrary manner.

The essential result of our paper is that we found a rich family of exact analytic solutions of relativistic hydrodynamics that contain both a longitudinal Bjorken flow (that is frequently utilized in estimations of observables in high energy heavy ion collisions) and a relativistic transverse flow (whose existence is evident from the analysis of the single particle spectra at RHIC and SPS energies).

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