Collisionless Dynamics in Two-Dimensional Bosonic Gases

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We study the dynamics of dilute and ultracold bosonic gases in a quasi two-dimensional (2D) configuration and in the collisionless regime. We adopt the 2D Landau-Vlasov equation to describe a three-dimensional gas under very strong harmonic confinement along one direction. We use this effective equation to investigate the speed of sound in quasi 2D bosonic gases, i.e., the sound propagation around a Bose-Einstein distribution. We derive coupled algebraic equations for the real and imaginary parts of the sound velocity, which are then solved taking into account the equation of state of the 2D bosonic system. Above the Berezinskii-Kosterlitz-Thouless critical temperature we find that there is rapid growth of the imaginary component of the sound velocity which implies a strong Landau damping. Quite remarkably, our theoretical results are in good agreement with very recent experimental data obtained with a uniform 2D Bose gas of $^{87}$Rb atoms.

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Introduction. The Boltzmann-Vlasov equation is the most relevant tool to investigate the kinetics of three-dimensional (3D) quantum gases made of out-of-condensate atoms. In the collisionless regime this equation reduces to the Landau-Vlasov equation, where the collisional integral is neglected but the mean-field interaction potential is still present and supports collective modes. In the case of fermionic gases the speed of sound in this collisionless regime is the well-known zero-sound velocities of fermions around the Fermi-Dirac distribution. In the collisionless regime this effective equation is used to investigate the collisionless regime by using an effective 2D Landau-Vlasov equation. We assume that the bosonic system is under external confinement given by the trapping potential $U_{ext}(r, z) = U(r) + \frac{1}{2}m\omega_z^2z^2$, (1) that is the sum of a generic potential $U(r)$ in the plane $x - y$ with $r = (x, y)$ the 2D position and a harmonic confinement along the $z$ axis.

An effective two-dimensional (2D) configuration can be realized when the harmonic confinement along the $z$ axis is tight enough. In order to effectively constrain atoms on a plane, the energy $\hbar\omega_z$ of longitudinal confinement must be much larger than the planar average kinetic energy $(p_x^2 + p_y^2)/(2m)$ with $p = (p_x, p_y)$ the planar linear momentum, a condition actual experiments can provide quite easily. The 3D system is then forced to occupy the longitudinal ground state along the confining axis and one finds that the planar distribution $f(r, p)$ of atoms in the 4D single-particle phase space $(r, p) = (x, y, p_x, p_y)$ satisfies the effective 2D Landau-
Vlasov equation \[ \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} - \nabla_{\mathbf{r}} \left( U + U_{\text{mf}} \right) \cdot \nabla_{\mathbf{p}} \right] f(\mathbf{r}, \mathbf{p}, t) = 0 , \quad (2) \]

where

\[
U_{\text{mf}}(\mathbf{r}, t) = g_{2D} \int \frac{d^2 \mathbf{p}}{(2\pi \hbar)^2} f(\mathbf{r}, \mathbf{p}, t) \]

(3)
is the self-consistent Hartree-Fock dynamical mean-field term \[6,13,14\], and the memory of the original 3D character of the system is encoded in the renormalized 2D term \[6,18,19\], and the memory of the original 3D character of the system is encoded in the renormalized 2D coupling constant

\[
g_{2D} = \frac{\sqrt{8\pi} \hbar^2}{m} \left( \frac{a_x}{a_z} \right) \quad (4)
\]

with \( a_z = \sqrt{\hbar/(m a_z)} \) the characteristic length of the axial harmonic confinement.

Performing the Fourier transform of this equation according to

\[
\tilde{f}(\mathbf{k}, \mathbf{p}, \omega) = \int dt \int d^2 \mathbf{r} \delta f(\mathbf{r}, \mathbf{p}, t) \exp \left( i(\mathbf{k} \cdot \mathbf{r} - \omega t) \right) \text{ with } \mathbf{k} \text{ a 2D wavevector and } \omega \text{ the angular frequency}, \text{ one finds an implicit formula for the dispersion relation \[9\], given by}
\]

\[
1 - g_{2D} \int \frac{d^2 \mathbf{p}}{(2\pi \hbar)^2} \frac{\mathbf{k} \cdot \nabla_{\mathbf{r}} f_0(\mathbf{p})}{\mathbf{p} \cdot \mathbf{k}/m - \omega} = 0 . \quad (7)
\]

Note that this equation is nothing else than the condition to find the pole of the dynamic response function of the system within the random-phase approximation (RPA) \[10\]. Equation (6) is also called linearized Boltzmann transport equation without collisional term. In Ref. \[20\] it has been solved numerically by preparing the system at equilibrium in the presence of a weak stationary potential generating a sinusoidal density modulation of a given wavelength. Then the potential has been suddenly removed to generate a damped time-dependent oscillation and hence the speed of sound.

On the contrary, here we directly solve Eq. (6) by a fully analytical approach.

In Eq. (7) there is a singularity on the integration path for \( \omega = \mathbf{p} \cdot \mathbf{k}/m \). In order to attach a meaning to the integral, we must interpret \( \omega \) as a complex quantity, i.e. \( \omega = \omega_R + i\omega_I \), where \( \omega_I > 0 \) in order to avoid an exponential growth of the perturbation \[8\].

Eq. (11) can be further simplified by assuming, without loss of generality, that \( \mathbf{k} \parallel \mathbf{\hat{e}}_z \), i.e. \( \mathbf{k} = (k, 0) \). In this way one finds

\[
1 - g_{2D} \int \frac{d p_x}{(2\pi \hbar)^2} \frac{\partial f_0(p_x)}{\partial p_x} \left| \frac{1}{m - c} \right| = 0 . \quad (8)
\]

where \( c = \omega/k \) and \( f_0(p_x) = \int f_0(p_x, p_y) dp_y/(2\pi \hbar) \). Clearly, from Eq. (8) one can extract the speed \( c \) of sound in our collisionless regime. This velocity is, in general, a complex number such that \( c = \omega/k = c_R + ic_I \) with \( c_R = \omega_R/k \) and \( c_I = \omega_I/k \).

In the limit of weakly damped wave, i.e. \( c_I \ll c_R \), an elegant formulation is provided for the real and imaginary part of \( c \[21\]. In particular, one finds two coupled equations for the real part \( c_R \) and the imaginary part \( c_I \) of the speed of sound. The equation derived from the real part of Eq. (8) reads

\[
1 - g_{2D} \mathcal{P} \int \frac{d p_x}{(2\pi \hbar)^2} \left| \frac{\partial f_0(p_x)}{\partial p_x} \right| \frac{\partial \phi(c)}{\partial c} \left|_{c_R} \right| = 0 , \quad (9)
\]

where we denote \( \phi(c) = \frac{m g \delta f_0}{(2\pi \hbar)^2} \int \frac{d p_x}{(2\pi \hbar)^2} \frac{\partial f_0(p_x)}{\partial p_x} \bigg|_{p_x = mc} \) and \( \mathcal{P} \) means principal value.

The equation derived from the imaginary part of Eq. (8) is instead given by

\[
c_I = \frac{\pi \frac{\partial f_0(p_x)}{\partial p_x} \bigg|_{p_x = mc_R}}{\mathcal{P} \int \frac{d p_x}{(2\pi \hbar)^2} \left| \frac{\partial f_0(p_x)}{\partial p_x} \right| \frac{p_x}{p_x/m - c_R}} . \quad (10)
\]
By inserting Eq. (15) in Eq. (14) we get an equation for $c_R$. This equation can be easily solved numerically and, taking into account Eq. (13), one finds the real part of the zero-sound velocity as a function of temperature $T$ and adimensional interaction strength $\tilde{g}_{2D}$. In Fig. 1 we compare the solution of Eq. (14) with the experimental data reported in Ref. [16]. The agreement between our results and the experimental points is excellent in the low-temperature regime and still good close to the superfluid threshold given by the Berezinskii-Kosterlitz-Thouless critical temperature $T_c$. The velocity $c_R$ does not display any discontinuity at the critical temperature $T_c$. This feature marks a crucial difference with respect to first-sound and second-sound velocities calculated within the superfluid Landau-Khalatnikov model, which intrinsically relies upon a collisional dynamics of the normal component [22, 23]. Despite the similar behaviour exhibited far below $T_c$ by the second-sound velocity $c_2$ [17] and our collisionless velocity $c_R$, the former is related to the superfluid density and consequently it jumps to zero at $T_c$ [17].

The dashed line of Fig. 1 is obtained by using Eq. (14) with $c_I = 0$. Comparing the dashed line with the solid line, which is instead derived solving the coupled equations of state, relating the shifted chemical potential $\tilde{\mu}$ to the number density $n = N/L^2$, is simply derived from the normalization condition

$$N = \int \frac{d^2 r d^2 p}{(2\pi \hbar)^2} f_0(p),$$

resulting in

$$\tilde{\mu} = k_B T \ln \left(1 - e^{-T_B/T}\right)$$

where $k_B T_B = 2\pi \hbar^2 n/m$ is the temperature of Bose degeneracy and clearly $\tilde{\mu} < 0$.

In order to describe the behaviour of the quasi-2D uniform Bose gas below or just above the critical temperature, we choose the Bose-Einstein distribution function

$$f_0(p) = \frac{1}{L^2} \frac{1}{e^{\beta\left(p^2/m + g_{2D} n - \mu\right)} - 1}$$

from which $L^2 f_0(p) = k_B T / (\sqrt{2/p^2} - \tilde{\mu})$ from which $L^2 f_0(p) = k_B T / (\sqrt{2/p^2} - \tilde{\mu})$. Consequently, the coupled Eqs. (9) and (10) for the real and imaginary part of the zero-sound velocity respectively read

$$1 + \frac{\tilde{g}_{2D} k_B T}{2\pi} \left[ \frac{2}{mc_R^2 - 2\tilde{\mu}} + \frac{\sqrt{mc_R^4 - 2\tilde{\mu} - \sqrt{mc_R^4}}}{\sqrt{mc_R^4 - 2\tilde{\mu} + \sqrt{mc_R^4}}} \right] \left[ \frac{\sqrt{mc_R^4 - 2\tilde{\mu} - \sqrt{mc_R^4}}}{\sqrt{mc_R^4 - 2\tilde{\mu} + \sqrt{mc_R^4}}} \right] + \frac{\tilde{g}_{2D} k_B T c_I}{\sqrt{m(mc_R^2 - 2\tilde{\mu})^{3/2}}} = 0, \quad (14)$$

$$c_I = -\frac{\frac{6c_B}{(mc_R^2 - 2\tilde{\mu})^{3/2}} + \frac{2(mc_R^2 + \tilde{\mu})}{\sqrt{m(mc_R^2 - 2\tilde{\mu})^{3/2}}} \ln \left(\frac{\sqrt{mc_R^4 - 2\tilde{\mu} - \sqrt{mc_R^4}}}{\sqrt{mc_R^4 - 2\tilde{\mu} + \sqrt{mc_R^4}}}\right)}{\sqrt{m(mc_R^2 - 2\tilde{\mu})^{3/2}}}.$$
Eqs. (14) and (15), one clearly sees the increasingly relevant role played by the imaginary part $c_I$ (the so-called Landau damping) above $T_c$.

In Fig. 2 we report the absolute value of $c_I$ as a function of $T/T_c$ for $g_{2D} \simeq 0.16$. The solid black line is obtained from Eq. (15) where $c_R$ has derived by solving Eq. (14). Inset: Ratio between the imaginary and the real part of $c$ as a function of the temperature.

Conclusions. We have analyzed the sound propagation in collisionless bosonic gases assuming a 2D configuration. By solving the linearized 2D Landau-Vlasov equation in the degenerate regime, where bosonic statistical effects play a relevant role, we have derived an integral equation for the speed of sound as a function of temperature and interaction strength. From this integral equation we have obtained two coupled algebraic equations for the real and imaginary part of the sound velocity. We have then compared our theoretical results with experimental data of a recent experiment [16], where the $^{87}$Rb atoms of the bosonic cloud are expected to be in the collisionless regime. This expectation is fully confirmed: the agreement between our theory and the experiment is very encouraging. Our theoretical analysis strongly suggests that the density perturbation used in the experiment of Ref. [16] has excited the “bosonic zero sound”, i.e. the sound of a collisionless bosonic fluid. For a superfluid system, a density perturbation can be used to excite the second sound only if the system is weakly interacting and collisional [17]. By increasing the interaction strength $g_{2D}$ the 2D bosonic system enters in the collisional regime where the Landau-Vlasov equation (2) loses its validity. The collisional regime is in fact correctly described by the two-fluid model of Landau-Khalatnikov which reduces to the usual hydrodynamics above the critical temperature $T_c$.

During the final stage of this work, a theoretical preprint on the same topic appeared [20]. The conclusions of Ref. [20], based on stochastic Gross-Pitaevskii equation and dynamic response function, are similar to ours.

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