On the dynamics of generators of Cauchy horizons

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Abstract
We discuss various features of the dynamical system determined by the flow of null geodesic generators of Cauchy horizons. Several examples with non–trivial (“chaotic”, “strange attractors”, etc.) global behaviour are constructed. Those examples are relevant to the “chronology protection conjecture”, and they show that the occurrence of “fountains” is not a generic feature of Cauchy horizons.

1 Introduction
In considering the question of whether the laws of physics prevent one from constructing a “time machine”, Hawking [1] and Thorne [2] have both stressed the importance of understanding the generic behavior of the null generators of compactly generated Cauchy horizons. In particular it has been suggested (cf. e.g. [2, 3] and references therein) that the onset of quantum instabilities in Cauchy horizons containing “fountains” would prevent the formation of time machines. Here a “fountain” on a future Cauchy horizon is defined as a periodic 1

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1Throughout this paper “periodic” means periodic as a path in spacetime (and not necessarily as a path in the tangent bundle).
generator $\gamma$ of the horizon such that a “nonzero–measure” set of generators of the horizon asymptotically approaches $\gamma$ when followed backwards in time. It is therefore of some interest to enquire whether or not the existence of fountains is a generic property of “compactly generated” Cauchy horizons. In this work we wish to point out that this is unlikely to be true: we construct spacetimes with compactly generated Cauchy horizons for which no fountains occur.

When discussing features of Cauchy horizons, one should focus on features which are stable in an appropriate sense. We show that in the set of all spacetimes with compactly generated Cauchy horizons, there are open sets consisting entirely of spacetimes with nonfountain–like behavior. Unfortunately we are able to make rigorous claims only for compact Cauchy horizons. So the possibility remains open that for spacetimes with compactly generated Cauchy horizons which are not compact, fountains could generically occur. While it is clear to us that this is not true, we note that there is an important technical difference between compact Cauchy horizons and noncompact yet compactly generated Cauchy horizons: As we show in Section 4, if a Cauchy horizon is compactly generated and noncompact, and if further it is contained in an asymptotically flat spacetime (in a technical sense made precise in that Section), then the generators of the Cauchy horizon cannot be continuous. This is one of the difficulties which one has to face when trying to make any rigorous claims about the dynamics of the generators of some non–compact Cauchy horizons.

It is important to note that, following [1, 2], we do not impose any field equations on the spacetimes under consideration. Recall that one expects the existence of a Cauchy horizon to be an unstable feature, when the Einstein field equations (vacuum, or with energy conditions on the source fields) are imposed. It would be interesting to carefully investigate the extent to which the imposition of field equations restricts the allowed dynamics of Cauchy horizon generators; however this problem is not addressed here.

After discussing some preliminary definitions and ideas in Section 2, we focus on verifying the existence of the spacetimes with nonfountain-like dynamics, first for compact Cauchy horizons (Section 3) and then for compactly generated but noncompact Cauchy horizons (Section 4). The discussion of noncompact Cauchy horizons in Section 4 includes the proof that if the spacetime containing it is asymptotically flat, then the generators cannot be continuous.

2 Preliminaries on Cauchy horizons and dynamical systems

We shall consider $C^k$, $(k \geq 3)$ spacetimes $(M^4, g)$ which contain Cauchy horizons (we use the terminology of [4]). Standard results [4] show that a Cauchy horizon is foliated by a congruence of null geodesics. These are called the generators of the horizon. One finds that if one follows a generator of a future Cauchy horizon
into its past then the generator always remains inside the horizon. This is not necessarily true if one follows a generator (on a future Cauchy horizon) into its future. We shall say that a future Cauchy horizon $H^+$ is *compactly generated*, if there exists a compact set $K \subset M$ such that every generator of $H^+$ enters and remains in $K$, when followed into the past.

To discuss the behavior of the generators of a Cauchy horizon, we wish to use some of the language of dynamical systems theory. Recall that a dynamical system $(\Sigma^n, X)$ consists of an $n$–dimensional manifold $\Sigma^n$ and a vector field $X$ specified on $\Sigma^n$. Note that a Cauchy horizon $H$ together with the vector field $T$ of tangents to its generators (normalized in an arbitrary way) constitutes a dynamical system $(H, T)$. We shall always choose the *past directed* orientation of the generators on a future Cauchy horizon. For future Cauchy horizons the past–oriented generators of $H$ are then the orbits of this dynamical system. A distinguished feature of a Cauchy horizon when viewed as a dynamical system is that the vector field $T$ is nowhere vanishing, so none of the orbits of $T$ are fixed points.

A number of issues arise in examining the behavior of the orbits of a given dynamical system $(\Sigma^n, X)$. Of primary interest here is whether or not $(\Sigma^n, X)$ contains any periodic orbits (i.e., orbits which pass repeatedly through the same point). We shall say that a periodic orbit $\lambda$ is an attractor if all the nearby orbits approach it, and a repeller if they all move away. (In general, of course, a periodic orbit is neither a repeller nor an attractor.)

We wish now to briefly describe some specific examples of dynamical systems which we will find useful in our discussion of the dynamics of Cauchy horizons:

**Example 1:** Let $\Sigma^2$ be any two-dimensional manifold, and let $\psi$ be any diffeomorphism from $\Sigma^2$ to itself. Let us recall the *suspension construction* [5] of a three-dimensional dynamical system which has global transverse section $\Sigma^2$ and has Poincaré map $\psi$: For the manifold $\Sigma^3$ of this dynamical system, one chooses the twisted product $\Sigma^2 \times_\psi S^1$, which is defined by quotenting $\Sigma^2 \times \mathbb{R}$ by the map

$$
\Psi : \Sigma^2 \times \mathbb{R} \to \Sigma^2 \times \mathbb{R} ;
$$

$$(p, s) \mapsto (\psi^{-1}(p), s + 1) .$$

So $\Sigma^3 = \Sigma^2 \times_\psi S^1 \equiv (\Sigma^2 \times \mathbb{R}) / \Psi$. Then for the vector field $X$ of the dynamical system, one chooses $X = \rho_*(\partial/\partial s)$, where $\rho$ is the natural projection map $\rho : \Sigma^2 \times \mathbb{R} \to \Sigma^2 \times_\psi S^1$ associated with the definition of the twisted product $\Sigma^2 \times_\psi S^1$, and $\partial/\partial s$ is the vector field tangent to the $\mathbb{R}$ factor of $\Sigma^2 \times \mathbb{R}$. One easily verifies that $(\Sigma^3, X) = (\Sigma^2 \times_\psi S^1, \rho_*(\partial/\partial s))$ has global transverse sections diffeomorphic to $\Sigma^2$, and that $\psi$ is the corresponding Poincaré map.

Now let $L$ be any $2 \times 2$ matrix with integer entries, unit determinant, and eigenvalues with nonunit absolute value — e.g., $L = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$. Based on the lattice quotient definition of the two-torus, any such matrix $L$ defines a diffeomorphism $\psi_L : T^2 \to T^2$ of the two-torus to itself in a standard way. Then,
using the suspension construction described above, we obtain for any such \( L \) a corresponding dynamical system \((\Sigma^3_L, X_L)\) with \( \psi_L \) for its Poincaré map.

One verifies (cf. e.g. [6][pp.156–159]) that for any choice of \( L \), the dynamical system \((\Sigma^3_L, X_L)\) on the compact manifold \( \Sigma^3_L \) (with nowhere vanishing generator \( X_L \)) has the following properties:

1. \( X_L \) has a countable infinity of periodic orbits. The set of all points \( p \in \Sigma^3_L \) which lie on periodic orbits of \( X_L \) is dense in \( \Sigma^3_L \).
2. There are no attracting or repelling periodic orbits.
3. \((\Sigma^3_L,X_L)\) is ergodic (cf. e.g. [7][Ex.5,p.19]).
4. \((\Sigma^3_L,X_L)\) is structurally stable, so that all the properties here are preserved under all sufficiently small \( C^1 \) perturbations of the vector field \( X_L \).

The dynamical systems \((\Sigma^3_L,X_L)\) will be useful for building (stable) families of spacetimes which have compact Cauchy horizons with nonfountain–like generator dynamics (cf. Section 5).

**Example 2:** Consider a so-called DA diffeomorphism \( \psi_{DA} : T^2 \to T^2 \), as defined by Smale [5] (cf. also [6]). We do not wish to describe \( \psi_{DA} \) in detail; however, we wish to note the following. Let \((\Sigma^3_{DA}, X_{DA})\) be obtained by suspension of \( \psi_{DA} \). Then the dynamical system \((\Sigma^3_{DA}, X_{DA})\) (with nowhere vanishing generator \( X_{DA} \)) exhibits the following properties [6][pp.165–169]:

1. There is one repelling orbit \( \Gamma \); there are no attracting orbits.
2. There is a non–periodic attracting set \( \Lambda \) ("strange attractor"), which is locally the product of \( \mathbb{R} \) with a Cantor set. Almost every orbit asymptotically approaches \( \Lambda \), when followed to the future. \( \Lambda \) contains a countable infinity of periodic orbits (none of which are attractors or repellers).
3. The existence and properties of the attracting set \( \Lambda \) above are preserved under all sufficiently small smooth perturbations of the vector field \( X_{DA} \).
4. There exists a neighborhood \( V \) of the repelling orbit \( \Gamma \) such that an arbitrary perturbation of \( X_{DA} \) supported in \( V \) will not affect the existence and the "chaotic" character of the attracting set \( \Lambda \). (Such a perturbation might lead to a different basin of attraction of \( \Lambda \). The new basin of attraction will nevertheless still have nonzero measure.)

We will use Example 2 (and some cutting and pasting) to build spacetimes containing compactly generated, noncompact, “asymptotically flat” Cauchy horizons with nonfountain–like generator dynamics (cf. Section 5).

\(^2\)We are grateful to C. Robinson for pointing out this example to us.
3 Compact Cauchy horizons with nonfountain–like dynamics

In this section we shall show that there exist smooth compact Cauchy horizons with no attracting periodic orbits. [Since a “fountain”, as defined in the Introduction, is precisely an attracting periodic orbit, the existence of spacetimes with Cauchy horizons with nonfountain–like behaviour immediately follows.]

We have the following:

**Proposition 3.1** Let \((\Sigma^3, X)\) be any dynamical system with \(\Sigma^3\) compact and \(X\) nowhere vanishing. There exists a spacetime \((M^4, g)\) (not necessarily satisfying any field equations and/or energy conditions) containing a Cauchy horizon \(H\) which is diffeomorphic to \(\Sigma^3\), and such that the generators of \(H\) are tangent to the orbits of \(X\).

We divide the proof into two main steps, the first of which involves proving the following Lemma:

**Lemma 3.2** Let \((\Sigma^3, X)\) be any dynamical system with \(\Sigma^3\) compact and \(X\) nowhere vanishing. Consider a spacetime \((M^4, g)\) with \(M^4 = \Sigma^3 \times (-\mu, \mu)\) for some \(\mu > 0\), and let \(Z\) be a vector field on \(M^4\) such that \(Z|_{\Sigma^3 \times \{0\}} = X\).

Suppose moreover that the following hold:

1. \(g(Z, Z)|_{\Sigma^3 \times \{0\}} = 0\),
2. \(g^{-1}(dt, dt) < 0\) for all \(t < 0\), where \(t\) parametrizes the interval \((-\mu, \mu)\).

Then:

1. \((\tilde{M}^4, \tilde{g}) \equiv (\Sigma^3 \times (-\mu, 0), g|_{\tilde{M}^4})\) is globally hyperbolic,
2. \(\Sigma^3 \times \{0\}\) is a future Cauchy horizon for \((\tilde{M}^4, \tilde{g})\) in \((M^4, g)\), and
3. \(Z|_{\Sigma^3 \times \{0\}} = X\) is tangent to the null generators of that Cauchy horizon.

**Proof of Lemma:** It follows from hypothesis of this Lemma that the function

\[
T : M^4 = \Sigma^3 \times (-\mu, \mu) \rightarrow \mathbb{R}
\]

is a time function on \(\tilde{M}^4\). Hence we know from Theorem 8.2.2 in Wald \(\text{[8]}\) that the spacetime \((\tilde{M}^4, \tilde{g})\) as defined above is stably causal, and further (cf. the Corollary on p. 199 of \(\text{[8]}\)) that it is strongly causal. Now to show that \((\tilde{M}^4, \tilde{g})\) is globally hyperbolic, it is sufficient (cf. p. 206 of Hawking and Ellis \(\text{[4]}\)) to verify that in addition one has \(J^+(p) \cap J^-(q)\) compact for every \(p, q \in \tilde{M}^4\), where \(J^+(p)\) is the closure of the future of \(p\) in \(\tilde{M}^4\), and \(J^-(q)\) is the closure
of the past of \( q \) in \( \tilde{M}^4 \). But \( J^+(p) \cap J^-(q) \) is certainly closed and it is also the subset of a compact region \( \Sigma^3 \times [T(p), T(q)] \subset \tilde{M}^4 \). Hence \( J^+(p) \cap J^-(q) \) is compact, and it follows that \((M^4, \tilde{g})\) is globally hyperbolic\(^4\).

Now \( Z|_H \) is nowhere vanishing, tangent to \( H \) and null. It follows that the integral curves of \( Z|_H \) are causal curves which never leave \( H \). Hence no subset of \( M^4 \) containing \( \tilde{M}^4 \) and larger than \( \tilde{M}^4 \) can be globally hyperbolic, and consequently \( H \) is a Cauchy horizon. By \(^4\) there is a unique null direction tangent to each point of a smooth Cauchy horizon, with the null generators being tangent to this direction, so it must be that \( Z|_H \) is tangent to the null generators at each point of \( H \).

\[ \square \]

**Proof of Proposition 3.1**: By Lemma 3.2 all we need to do now is show that for any dynamical system \((\Sigma^3, X)\) with \( \Sigma^3 \) compact and \( X \) nonvanishing, we can always find a spacetime \((M^4, g)\) which satisfies the hypotheses of the Lemma. So let \( \mu \) be any positive real number, let \( t \) parametrize the interval \((-\mu, \mu)\) and set \( M^4 = \Sigma^3 \times (-\mu, \mu) \). The vector field \( X \), defined in an obvious way on \( \Sigma^3 \times \{0\} \), may now be Lie–dragged along the flow of \( \partial/\partial t \) to define the vector field \( Z \) on \( M^4 \). By construction we have \( Z|_{\Sigma^3 \times \{0\}} = X \). Note that this construction also guarantees that \( dt(Z) = 0 \).

To construct the appropriate spacetime metric, we first arbitrarily choose the following three fields:

1. Let \( \phi: (-\mu, \mu) \rightarrow \mathbb{R} \) be any monotonically decreasing function such that \( \phi(0) = 0 \). From \( \phi \), we construct \( \chi: M^4 \rightarrow \mathbb{R} \) by setting \( \chi(p, t) = \phi(t) \).
2. Let \( \beta \) be any one-form on \( M^4 \) such that \( \beta(Z) = 1 \) and \( \beta(\partial/\partial t) = 0 \).
3. Let \( \gamma \) be any Riemannian metric on \( \Sigma^3 \); for \( V, W \in T\Sigma \) set
   \[
   \tilde{\gamma}(V, W) \equiv \gamma(V, W) - \frac{1}{2} \{ \gamma(X, V)\beta(W) + \gamma(X, W)\beta(V) \},
   \]
   \[
   \tilde{\gamma}(\partial/\partial t, \partial/\partial t) \equiv \tilde{\gamma}(\partial/\partial t, W) \equiv 0
   \]
   (note that \( \tilde{\gamma}(X, \cdot) = 0 \)); and finally define \( \nu \) as a symmetric \((0, 2)\) tensor field on \( M^4 \) by dragging \( \tilde{\gamma} \) along \( \partial/\partial t \). [Here \( \beta \) has been identified with a form on \( \Sigma \) in the obvious way.] Note that it follows from this definition that \( \nu(Z, Z) = 0 \) and \( \nu(\partial/\partial t, \partial/\partial t) = 0 \).

Using \( \chi \), \( \beta \), and \( \nu \), we define
\[
g = \chi \beta \otimes \beta + dt \otimes \beta + \beta \otimes dt + \nu. \tag{1}
\]
We verify immediately from the properties of \( \chi, \beta \), and \( \nu \) and from the definition of \( Z \) that \( g(Z, Z) = \chi \) everywhere on \( M \), and in particular \( g(Z, Z)|_{\Sigma^3 \times \{0\}} = 0 \), so hypothesis \(^4\) of Lemma 3.2 is satisfied. To verify hypothesis \(^4\) it is useful to

\(^3\)Note that this argument shows, that a spatially compact stably causal spacetime is necessarily globally hyperbolic.
set up local coordinates \((x,y,z,t)\) such that \(\beta = dz\), \(Z = \partial/\partial z\); it follows that \(\nu = \nu_{xx}dx^2 + 2\nu_{xy}dxdy + \nu_{yy}dy^2\). Then the components of the metric \(g\) take the matrix form

\[
g_{\alpha\beta} = \begin{pmatrix}
\nu_{xx} & \nu_{xy} & 0 & 0 \\
\nu_{yx} & \nu_{yy} & 0 & 0 \\
0 & 0 & \chi & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

From this matrix representation, we see that \(g\) is indeed a Lorentz metric (non-degenerate, signature \(+ + + -\)) and we calculate the matrix representation of the inverse metric:

\[
(g^{-1})^{\alpha\beta} = \begin{pmatrix}
\frac{1}{\det \nu_{xx}} & \frac{1}{\det \nu_{xy}} & 0 & 0 \\
\frac{1}{\det \nu_{yx}} & \frac{1}{\det \nu_{yy}} & 0 & 0 \\
0 & 0 & \chi & 1 \\
0 & 0 & 1 & -\chi
\end{pmatrix}.
\]

We see that \(g^{-1}(dt, dt) = -\chi\), which implies that \(g^{-1}(dt, dt) < 0\) for \(t < 0\), as required by hypothesis 2 of Lemma 3.2. So the spacetime \((M^4, g)\) which we have constructed (from the dynamical system \((\Sigma^3, X)\)) satisfies all of the hypotheses of the Lemma, thus completing the proof of the Proposition.

Using the example dynamical systems from Section 2, together with this Proposition (and some of the constructions outlined in its proof), we can easily construct a large numbers of spacetimes containing compact Cauchy horizons with nonfountain–like dynamics. Here the only essential restriction is, that the vector field \(X\) generating the dynamical system be nowhere vanishing — this excludes examples like e.g. (a compactified version of) the Lorenz attractor [9] or of the geometric model thereof [10], but clearly allows for interesting dynamics. Models with e.g. “horseshoes” can be constructed on \(S^2 \times S^1\) using the periodically perturbed nonlinear pendulum equation or the periodically perturbed Duffing equation.

We wish to stress that these examples can be constructed in such a way that the nontrivial properties of the dynamics are stable under small smooth variations of the metric. For example, let \((\Sigma^3, X)\) be the Anosov flow discussed in Example 1 of the previous Section. The metric \(g\) constructed in the proof of Proposition 3.1 can be chosen to satisfy the stability criterion of [11], so that small smooth variations of the metric will lead to small \(C^k\) variations of the Cauchy horizon. This in turn will lead to a small \(C^{k-1}\) variation of the field of null tangents to the generators, and the stability of the resulting dynamical systems follows from stability of Anosov flows.

Note that all examples discussed so far have \(\Sigma^3\) defined as a twisted product of a two–dimensional manifold with the circle. Do all compact Cauchy horizons with nonfountain–like behavior have this sort of topology? Certainly not. In

\[4k\] here may be made arbitrarily large (but probably not \(k = \infty\)) by appropriately choosing \(g\).
the next Section we shall see how to construct Cauchy horizons with interesting dynamical behaviour of the generators by using the connected sum operation. It would be of some interest to find out whether or not there are spacetimes with nonfountain–like behavior in a Cauchy horizon of arbitrary (compact, three-dimensional) topology.

4 Noncompact compactly generated horizons

It is relatively easy to construct a spacetime \((M^4, g)\) which has a compactly-generated but noncompact Cauchy horizon with nonfountain–like dynamics. First, one chooses a compact dynamical system \((\Sigma^3, X)\) which has nonfountain–like dynamics and also has a repelling periodic orbit: the DA system as discussed in example 2 will do. Then, one uses Proposition 3.1 to construct a spacetime containing a Cauchy horizon diffeomorphic to \(\Sigma^3\) with generators matching the orbits of \(X\). Finally one removes the repelling orbit from the horizon in the spacetime. The Cauchy horizon \(\mathcal{H}\) of the resulting spacetime is clearly not compact. On the other hand, one verifies that \(\mathcal{H}\) is compactly-generated by noting that if one defines the set \(\mathcal{K} = \mathcal{H} \setminus \tilde{S}\), where \(\tilde{S}\) is a small open thickening of the removed orbit, then since the removed orbit was repelling, all past-directed null generators of \(\mathcal{H}\) enter and remain in \(\mathcal{K}\), which is compact. [Note that this example shows that the inequality \(f \geq 0\), which according to \([1]\) holds for any periodic generator of a Cauchy horizon, is not correct.]

The above example is rather artificial, and it is natural to enquire about the existence of smooth compactly generated horizons in asymptotically flat spacetimes. By way of example, consider \(M = \mathbb{R}^4\) with a metric \(g\) which is the standard Minkowski metric outside of a compact set \(C\). Let us moreover assume that there exist periodic time-like curves in \(C\). [An explicit example of such a spacetime can be found in \([1]\).] \(M\) will have a Cauchy horizon \(\mathcal{H}^+\), which is the boundary of the domain of dependence of any standard \(t = \text{const}\) plane lying in \(M \setminus J^+(C)\). Now \(\mathcal{H}^+\) can be “sandwiched” between \(\partial J^+(p)\) and \(\partial J^+(q)\), where \(p, q\) are any two points such that \(C \subset J^+(p)\), and \(C \cap J^+(q) = \emptyset\); by “sandwiched” here we mean that \(\mathcal{H}^+ \subset \{I^- (\partial J^+(q)) \cap I^+ (\partial J^+(p))\}\). It is then easily seen that for all \(R \in \mathbb{R}\) large enough the world tube \(T = \{(t, \vec{x}) : t \in \mathbb{R}, |\vec{x}| = R\}\) intersects each of the generators of \(\mathcal{H}^+\) transversally.

Based on this example, we shall say that a compactly generated Cauchy horizon \(\mathcal{H}\) in an orientable and time–orientable spacetime \((M, g)\) is of asymptotically flat type if the boundary set \(\partial (\mathcal{K} \cap \mathcal{H})\) consists of a finite number \(I\) of spheres \(S_i\), with each generator of \(\mathcal{H} \setminus \mathcal{K}\) intersecting one of the \(S_i\)’s transversally. Here \(\mathcal{K}\) is one of the compact sets in \(M\) which characterizes \(\mathcal{H}\) as compactly generated. It is easy to convince oneself that the behaviour described in this definition should occur, e.g., for compactly generated Cauchy horizons in spacetimes which admit a sufficiently regular compactification in lightlike directions (the number \(I\) above corresponds then to the number of connected components of \(\text{Scri}\)).
We would like to find spacetimes with compactly-generated, asymptotically flat type Cauchy horizons with nonfountain-like dynamics. The following result is an obstacle to the construction of such spacetimes:

**Proposition 4.1** Let $\mathcal{H}^+$ be a compactly generated future Cauchy horizon of asymptotically flat type. Then the field $X$ of directions tangent to the generators of $\mathcal{H}^+$ cannot be continuous.

**Proof:** Suppose that the field $X$ is continuous. Now consider the compact manifold $\hat{\mathcal{H}}$ constructed by adding a point $p^\infty$ to each of the “asymptotic ends” $S_i \times \mathbb{R} \subset \mathcal{H}^+$. We can deform the field of generators on each of the ends $S_i \times \mathbb{R}$ to obtain a continuous vector field $\hat{X}$ on $\hat{\mathcal{H}}$ which is nowhere vanishing except at the points $p^\infty_i$. At each of those points the index of $X$ will be equal to $+1$; consequently the index of $\hat{X}$ will be equal to $I \neq 0$. Note that $\mathcal{H}$ is orientable because $(M, g)$ has been assumed to be time–orientable and orientable. This, however, contradicts the fact that the index of a continuous vector field on a compact, three–dimensional, orientable manifold vanishes. \(\square\)

This result makes it difficult to systematically study the dynamics of the null generators in spacetimes containing compactly generated Cauchy horizons of asymptotically flat type. In particular, the construction carried out in Proposition 3.1 encounters various obstacles. However, as it has been suggested \[2, 12, 1\] that there exist compactly generated Cauchy horizons $\mathcal{H}$ of asymptotically flat type which are smooth on an open dense set $\mathcal{U}$ (the complement of which has zero measure), we believe the following result should be of interest:

**Proposition 4.2** Let $\mathcal{H}^+ = \partial^+ \mathcal{D}(\Sigma)$ be a future Cauchy horizon in a space-time $(M, g)$. Suppose that there exists an open subset $\mathcal{U}$ of $\mathcal{H}^+$ such that $\mathcal{U}$ is a smooth submanifold of $M$. Suppose moreover that there exists a smooth time function $\tau$ on $\mathcal{D}(\Sigma)$ such that $\lim_{p \to \mathcal{U}} \nabla \tau(p)$ exists, and is a smooth, nowhere vanishing vector field on $\mathcal{U}$. Then there exists a space-time $(M', g')$ with a future Cauchy horizon $\mathcal{H}'$ diffeomorphic to $\mathcal{H}^+ \# \Sigma^3_{DA}$ (where $\Sigma^3_{DA}$ is the manifold discussed in Example 2, Section 4), and with non–trivial long–time dynamics of the generators of $\mathcal{H}'$. [Here $\#$ denotes the connected sum.] Moreover, $\mathcal{H}'$ will share certain overall properties of $\mathcal{H}^+$; in particular if $\mathcal{H}^+$ is compact, or compactly generated, or of asymptotically flat–type, then the same will be true of $\mathcal{H}'$.

**Remarks:** We believe that the inclusion in Proposition 4.2 of the hypothesis that the function $\tau$ exists should be unnecessary, for the following reasons:

1. We consider it likely that the remaining conditions of Proposition 4.2 are sufficient to guarantee that such a function can be constructed.

\[5\] Here we define $\partial^+ \mathcal{D}(\Sigma) = \overline{\mathcal{D}(\Sigma)} \setminus (\mathcal{D}^+(\Sigma) \cup \Sigma)$, similarly for $\partial^- \mathcal{D}(\Sigma)$. We use the convention in which the domains of dependence are open sets; in particular they do not include the Cauchy horizons.
2. We have written the proof below in such a way that the existence of the function \( \tau \) is essentially used in one place only. We believe that it should be possible to replace that step of the argument by one which does not require the existence of the function \( \tau \).

**Proof:** Let \( \Sigma \) be a partial Cauchy surface in \((M, g)\) such that \( \mathcal{H}^+ = \partial^+ \mathcal{D}(\Sigma) \).

Replacing \( M \) by a subset thereof if necessary we may assume that \( \mathcal{H}^- \equiv \partial^- \mathcal{D}(\Sigma) = \emptyset \). Let \( \mathcal{U} \) be a smooth subset of \( \mathcal{H}^+ \). Passing to a subset of \( \mathcal{U} \) if necessary we may without loss of generality assume that: 1) the closure \( \mathring{\mathcal{U}} \) of \( \mathcal{U} \) is compact, and 2) that the generators of \( \mathcal{H}^+ \) have a cross-section \( S \) in \( \mathcal{U} \), and 3) that \( \mathcal{U} \approx S \times (-1, 1) \), with \( S \) being a smooth two-dimensional embedded submanifold. We claim that we can find a defining function \( \varphi \) for \( \mathcal{U} \), defined on a conditionally compact neighborhood \( \mathcal{O} \) of \( \mathcal{U} \), such that \( \varphi|_{\mathcal{O} \cap \mathcal{D}^+ (\Sigma)} \) is a time function. [Recall that \( \varphi : \mathcal{O} \to \mathbb{R} \) is a defining function for \( \mathcal{U} \) if \( d\varphi \) is nowhere vanishing on \( \mathcal{U} \) and if we have \( p \in \mathcal{U} \cap \mathcal{O} \Leftrightarrow \varphi(p) = 0 \).] If we have a time function \( \tau \), as assumed in the hypotheses of Proposition 4.2, we set \( \varphi = \tau|_{\mathcal{O} \cap \mathcal{D}^+ (\Sigma)} \), and we are done.

[Had we not made the assumption of the existence of \( \tau \), the existence of \( \varphi \) could be established as follows: Let \( \psi \) be any defining function for \( \mathcal{U} \) defined on some neighborhood \( \mathcal{O} \), and let \( X \) be any future-directed timelike vector field on \( \mathcal{O} \). If

\[
[X^\mu \nabla_\mu (\nabla^\nu \psi \nabla_\nu \psi)]_{|\mathcal{U}} > 0,
\]

then passing to a subset of \( \mathcal{O} \) if necessary we shall have \( \nabla^\nu \psi \nabla_\nu \psi|_{\mathcal{D}^+ (\Sigma) \cap \mathcal{O}} < 0 \), and then setting \( \varphi = \psi \) we are done. If \((2)\) does not hold, consider any smooth function \( \alpha \) on \( \mathcal{O} \); we have

\[
X^\mu \nabla_\mu (\nabla^\nu (\alpha \psi) \nabla_\nu (\alpha \psi)) \big|_{\mathcal{U}} = \alpha^2 [X^\mu \nabla_\mu (\nabla^\nu \psi \nabla_\nu \psi) + X^\nu \nabla_\nu \psi \nabla^\mu \psi \nabla_\mu (\log^2 \alpha)]
\]

\((3)\)

Note that \( X^\nu \nabla_\nu \psi \) is nowhere vanishing on \( \mathcal{U} \), as \( X^\nu \) is time-like and \( \nabla^\nu \psi \) is null. Let \( \hat{\alpha} : \mathcal{U} \to \mathbb{R} \) be any strictly positive solution of the equation

\[
[\nabla^\mu \psi \nabla_\mu (\log^2 \hat{\alpha})] = (X^\mu \nabla_\mu \psi)^{-1} [1 - X^\mu \nabla_\mu (\nabla^\nu \psi \nabla_\nu \psi)] \big|_{|\mathcal{U}}
\]

and let \( \alpha \) be any strictly positive extension of \( \hat{\alpha} \) to \( \mathcal{O} \). Setting \( \varphi = \alpha \psi \) the desired defining function then follows. Passing to a subset of \( \mathcal{O} \) we may moreover assume that \( d\varphi \) is nowhere vanishing on \( \mathcal{O} \). Changing \( \varphi \) to \( -\varphi \) if necessary we may suppose that \( \nabla^\nu \varphi \) is past-directed on \( \mathcal{D}^+ (\Sigma) \cap \mathcal{O} \).

Let \( \mathcal{B}_4p \subset \mathcal{U} \) be a closed coordinate ball of radius \( 4p \) covered by coordinates \( x^i \), with the \( x^i \)’s chosen so that \( g^{i\nu} \varphi_{\nu}|_{\mathcal{U}} = \frac{\partial \varphi}{\partial x^i} \), and with \( \mathcal{B}_4p \) compact in \( \mathcal{U} \). If we choose a timelike future directed vector field \( T \) on \( \mathcal{O} \), then by dragging the coordinates \( x^i \) along the integral curves of \( T \) we obtain a coordinate system \((x^0, x^i) = (\varphi, x^i) \) on a compact set \([-\delta, \delta] \times \mathcal{B}_4p \), for some \( \delta > 0 \). Since \( \varphi \) is a
time function on $[-\delta, 0) \times B_{4\rho}$, the sets $\{s\} \times B_{4\rho}$ are spacelike for $s \in [-\delta, 0)$. In this coordinate system the metric takes the form

$$g_{\mu\nu}dx^\mu dx^\nu = 2g_{30}dx^0 dx^3 + g_{00}(dx^0)^2 + 2g_{0A}dx^0 dx^A + g_{AB}dx^A dx^B + O(\varphi),$$  \hspace{1cm} (4)

where the labels $A, B$ run over 1, 2, and where $O(\varphi)$ indicates terms which vanish at least as fast as $|\varphi|$ for small values of $|\varphi|$. Consequently,

$$\det g = -(g_{30})^2 \det(g_{AB}) + O(\varphi),$$  \hspace{1cm} (5)

from which it follows that if $\delta$ is sufficiently small, then $g_{30}$ does not change sign. From the above construction, it follows that in fact $g_{30}$ is positive.

Let $t$ be any strictly negative time function on $D(\Sigma)$ such that 1) the level sets of $t$ are Cauchy surfaces for $D(\Sigma)$, and 2) $t(p) \to 0$ as $p \to \partial^+ D(\Sigma)$. Then set

$$\epsilon = \inf_{\partial D(\Sigma)} t(p) | p \in \{-\delta\} \times B_{2\rho}.$$  \hspace{1cm} (6)

Clearly, $\Sigma = \{t = -\epsilon/2\}$ is a partial Cauchy surface in $M$ such that $\partial^+ D(\Sigma; M_1) = \mathcal{H}^+$ is a future Cauchy horizon for $\Sigma$. Here we use the notation $D(\Sigma; M_1)$ for the domain of dependence of $\Sigma$ in $(M_1, g^1)$; we shall use a similar convention for $J^\pm$, etc.

Let $\Sigma^3_{DA}$ be the manifold discussed in Example 2, Section 2. Let $\Gamma$ be the repelling orbit and $\mathcal{V}$ be the designated neighborhood of $\Gamma$ as discussed in that Example, and let $B_{4\rho} \subset \mathcal{V}$ be a closed coordinate ball covered by coordinates $y_i$. Finally let $\psi$ be the inversion map:

$$U \supset \hat{B}_{2\rho} \setminus B_{\rho/2} \ni x^i \psi \rightarrow y^i = -\frac{x^i}{r(x)^2} \in \hat{B}_{2\rho} \setminus B_{\rho/2} \subset \Sigma^3_{DA},$$

where $r(x) = \sqrt{\sum (x^i)^2}$. It is easily shown that one can find a nowhere vanishing vector field $X$ on $\Sigma^3_{DA} \setminus B^3_{\rho}$ such that

$$(\psi^{-1})_* X \big|_{B_{2\rho}\setminus B_{\rho}} = \nabla \varphi,$$

and

$$X \big|_{\Sigma^3_{DA} \setminus B^3_{\rho/2}} = X_{DA},$$

where $X_{DA}$ is the generator of the DA flow discussed in Section 2, and where we have used “hats” to denote the interior of a set: $\hat{B}_\rho = \text{int} B_\rho$, etc. Now let $g^{DA}$ be any Lorentzian metric constructed on $M_{DA} \equiv (-\delta, \delta) \times \Sigma^3_{DA} \setminus B^3_{\rho/2}$ as
described in the Proof of Proposition 3.1. On \((-\delta, 0) \times \{\Sigma_{\Delta A} \setminus B_{\rho/2}\} \subset M_{\Delta A}\), we can define a time function \(y^0\) by
\[
y^0(s, p) = s.
\]

Based on the map \(\Psi\) we define
\[
(-\delta, \delta) \times \{\hat{B}_{2\rho} \setminus B_{\rho/2}\} \ni (t, p) \rightarrow \Psi(t, p) = (t, \psi(p)) \in (-\delta, \delta) \times \{\hat{B}_{1\rho}^1 \setminus B_{\rho/2}^1\}.
\]

Then, in local coordinates on \((-\delta, \delta) \times \{\hat{B}_{2\rho} \setminus B_{\rho/2}\}\) the metric \(\Psi^*g^{\Delta A}\) takes the form
\[
g^{\Delta A}_{\mu\nu} dx^\mu dx^\nu = 2\beta_\mu dx^0 dx^\mu + g^{\Delta A}_{ij} x^i x^j,
\]
where
\[
\beta_0 = 0, \quad \beta_i dx^i = \psi^* \beta.
\]

On \(\hat{B}_{2\rho} \setminus B_{\rho}\), we have
\[
\beta_3 = \beta_\mu \nabla^\mu \rho = \langle \psi^* \beta, \nabla \rho \rangle = \langle \beta, \Psi_* \nabla \varphi \rangle = \langle \beta, \beta \rangle = 1.
\]

Since the metric \(g^{\Delta A}\) is \(y^0\)-independent, \((\hat{B}_{2\rho} \setminus B_{\rho/2})\) actually holds on \((-\delta, \delta) \times \{\hat{B}_{2\rho} \setminus B_{\rho}\}\).

A similar calculation shows that
\[
g^{\Delta A}_{33} = 0
\]
on \((-\delta, \delta) \times \{\hat{B}_{2\rho} \setminus B_{\rho}\}\). Now let \(\phi \in C^\infty(\mathbb{R}^4)\) be any non–negative function such that \(\phi = 0\) in \(\mathbb{R} \times B_{\rho}\), and \(\phi = 1\) in \(\mathbb{R}^4 \setminus (\mathbb{R} \times B_{2\rho})\). On \(M_1\) we may define the smooth metric \(g^2\) to coincide with \(g^1\) on \(M \setminus \{(-\delta, \delta) \times \hat{B}_{2\rho}\}\), and to be given by
\[
g^2_{\mu\nu} dx^\mu dx^\nu = \phi g_{\mu\nu} dx^\mu dx^\nu + (1 - \phi) g^{\Delta A}_{\mu\nu} dx^\mu dx^\nu
\]
on \((-\delta, \delta) \times \{\hat{B}_{2\rho} \setminus B_{\rho/2}\}\) (it is easily seen from eqs. (4), (7)–(10) and from eq. (8) with \(g^2\) substituted for \(g\), that \((\hat{B}_{2\rho} \setminus B_{\rho/2})\) indeed defines a Lorentzian metric). Note that \(\varphi = x^0\) is still a time function for this new metric in \((-\delta, 0) \times \{\hat{B}_{2\rho} \setminus B_{\rho/2}\}\).

The desired spacetime \(M'\) will now be obtained by gluing together \((M_1, g^2)\) and \((M_{\Delta A}, g^{\Delta A})\): Specifically, we choose
\[
M' = \left[\{M_1 \setminus \{(-\delta, \delta) \times \hat{B}_{\rho/2}\}\} \sqcup M_{\Delta A}\right] / \Psi
\]

Since the metrics \(g^2\) and \(\Psi^*g^{\Delta A}\) coincide on \(B_{\rho} \setminus B_{\rho/2}\), a metric \(g'\) can be defined on \(M'\) in the obvious way. There is a natural identification between points in \(M_1 \setminus \{(-\delta, \delta) \times \hat{B}_{\rho/2}\}\) and an appropriate subset of \(M'\), and similarly for points in \(M_{\Delta A}\), with another subset of \(M'\). We shall use this identification without mentioning it explicitly in what follows. Let us note that the function \(\varphi'\), defined as
\[
\varphi'(p) = \begin{cases} \varphi(p) = x^0(p), & p \in (-\delta, 0) \times \{B_{\rho} \setminus \hat{B}_{\rho/2}\}, \\ y^0(p), & p \in (-\delta, 0) \times \{\Sigma_{\Delta A} \setminus B_{\rho/2}\}. \end{cases}
\]
is a smooth time function on the interior of the set on which it has been defined.

We now claim that the submanifold $\hat{M}$ of $M'$ defined by

$$\hat{M} = \left[ \{D(\Sigma; M_1) \setminus [(-\delta, 0) \times \hat{B}_{\rho/2}] \} \cup \{(-\delta, 0) \times [\Sigma_{DA} \setminus \hat{B}_{\rho/2}^{1}] \} \right] / \Psi,$$

with the metric obtained from $g'$ by restriction, is globally hyperbolic. First, we wish to show that for all $p, q \in \hat{M}$, the set $J^+(p; \hat{M}) \cap J^-(q; \hat{M})$ is compact. To do this it is convenient to consider various cases, according to whether $p \in D(\Sigma; M_1) \setminus [(-\delta, 0) \times \hat{B}_\rho]$, $p \in (-\delta, 0) \times [\hat{B}_\rho \setminus B_{\rho/2}]$, or $p \in (-\delta, 0) \times [\Sigma_{DA} \setminus \hat{B}_{\rho/2}^{1}]$, similarly for $q$. Suppose, e.g., that $p, q \in M_1 \setminus [(-\delta, 0) \times \hat{B}_\rho]$. We define

$$K = J^+(p; M_1) \cap J^-(q; M_1) \cap [(-\delta, 0) \times \partial B_\rho].$$

$K$ is easily seen to be compact by global hyperbolicity of $(D(\Sigma; M_1), g^1)$. If $K = \emptyset$ we have $J^+(p; \hat{M}) \cap J^-(q; \hat{M}) = J^+(p; M_1) \cap J^-(q; M_1)$ and we are done; otherwise we have

$$-\delta < s_- = \inf \varphi(p) < s_+ = \sup \varphi(p) < 0$$

where the sup and the inf are taken over $p \in K$. Since we have a time function $\varphi'$ on $\hat{M} \setminus \{M_1 \setminus [(-\delta, 0) \times \hat{B}_\rho]\}$ which agrees with $\varphi$ on $K$, it follows that $J^+(p; \hat{M}') \cap J^-(q; \hat{M}')$ can be covered by the compact sets $J^+(p; M_1) \cap J^-(q; M_1), [-s_-, s_+] \times [\hat{B}_\rho \setminus \hat{B}_{\rho/2}^{1}]$ and $[-s_-, s_+] \times [\hat{B}_{\rho/2}^{1} \setminus \hat{B}_\rho]$. Using similar arguments one shows compactness of $J^+(p; \hat{M}) \cap J^-(q; \hat{M})$ for the remaining cases.

To prove strong causality of $\hat{M}$, we use the existence of the time function $\tau$ on $D(\Sigma, M)$: Indeed, the function $\tau'$ defined by:

$$\tau'(p) = \begin{cases} \tau(p), & p \in M_1 \setminus [(-\delta, 0) \times \hat{B}_{\rho/2}], \\ g^0(p), & p \in M_{DA}, \end{cases}$$

is a smooth time function on $\hat{M}$. This ensures strong causality of $\hat{M}$, and global hyperbolicity of $\hat{M}$ follows. [Let us emphasize, that this is the only point at which the hypothesis of existence of $\tau$ is needed in our argument.]

We wish to show now that the set $\mathcal{H}'$, defined as the boundary of $\hat{M}$ in $M'$, is a Cauchy horizon. Note first that the generators of $\mathcal{H}^+$ in $B_\rho \setminus \hat{B}_{\rho/2}$ are the integral curves of $\nabla \varphi = \nabla' \varphi$, where $\varphi$ is the defining function for $\mathcal{H}^+$ on $U$ defined at the beginning of this proof, and $\nabla'$ is the gradient with respect to the metric $g'$. These curves are smoothly continued by the null (with respect to the metric $g'$) integral curves of $\nabla' \varphi$. Consider now any subset $\hat{M}$ of $M'$ which contains $\hat{M}$ as a proper subset. It follows that $\hat{M} \cap \mathcal{H}' \neq \emptyset$. Let $p \in \hat{M} \cap \mathcal{H}'$. We claim that there exists a past directed causal curve $\lambda$
through \( p \) which never enters \( \tilde{M} \): If \( p \in M_1 \), then consider a generator \( \Gamma \) of \( \mathcal{H}^+ \) through \( p \). If \( \Gamma \) never enters \( B_{\rho} \) when followed backwards with respect to the time orientation, then the connected component of \( \Gamma \cap \tilde{M} \) which contains \( p \) provides the desired curve \( \lambda \). If \( \Gamma \) enters \( B_{\rho} \), let \( \tilde{\Gamma} \) be the segment of \( \Gamma \) up to the point \( \tilde{p} \) where it first enters \( B_{\rho} \). \( \tilde{\Gamma} \) can be smoothly continued at \( \tilde{p} \) by the integral curve \( \Gamma' \) of \( \nabla' \phi' \). If this curve never exits \( B_{\rho} \) through the sphere \( \partial B_{\rho} \), we can set \( \lambda \) to be the connected component of \( (\Gamma \cup \Gamma') \cap \tilde{M} \) which contains \( p \).

If \( \Gamma \) exits \( B_{\rho} \) through the sphere \( \partial B_{\rho} \), then it can be smoothly continued by a segment \( \Gamma'' \) of a generator of \( \mathcal{H}^+ \). If \( \Gamma'' \) never reenters \( B_{\rho} \), we define \( \lambda \) to be the connected component of \( (\Gamma \cup \Gamma' \cup \Gamma'') \cap \tilde{M} \) that contains \( p \).

This shows the existence of an inextendible, past-directed causal curve \( \lambda \subset \mathcal{H}' \cap \tilde{M} \) through all \( p \in \mathcal{H}' \cap \tilde{M} \). Hence it follows that

\[
\mathcal{H}' = \partial^+ D(\Sigma'; M')
\]

for some \( \Sigma' \subset M' \).

Clearly we have

\[
\mathcal{H}' \approx \mathcal{H}^+ \# \Sigma_{DA}^3,
\]

where \( \# \) denotes the connected sum. Moreover the generators of \( \mathcal{H}' \) coincide with

1. those of \( \mathcal{H}^+ \) on \( \mathcal{H}^+ \setminus B_{\rho} \), and with

2. the integral curves of the suspension of the DA–diffeomorphism on \( \Sigma_{DA}^3 \setminus B_{3\rho/2}^1 \).

From what has been said in Example 2, Section 2, it follows that a “non-zero measure” set of generators of \( \mathcal{H}' \) will be attracted to a “strange attractor”, when followed backwards in time.

We expect that some of the examples constructed as in Proposition 4.2 are stable in the dynamical sense. However, we have no proof of this assertion. To establish stability one would need to prove that small smooth variations of the metric lead to small \( C^2 \) variations of the horizon on (perhaps an open subset of) \( \mathcal{U} \). Now it is not difficult to show that, for an appropriately chosen \((M, g)\), the metric \( g' \) on \( M' \) can be constructed so that small variations of \( g' \) will indeed lead to small \( C^0 \) variations of the horizon \( \mathcal{H}' \) (cf. e.g. [13] for various results of this kind). The transition from \( C^0 \) to \( C^2 \) seems, however, to be a non-trivial matter. In particular, we have not been able to generalize the methods of [11] from compact Cauchy horizons with global cross–sections to noncompact Cauchy horizons.
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