On “time-periodic” black-hole solutions to certain spherically symmetric Einstein-matter systems

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Abstract

This paper explores “black hole” solutions of various Einstein-wave matter systems admitting an isometry of their domain of outer communications taking every point to its future. In the first two parts, it is shown that such solutions, assuming in addition that they are spherically symmetric and the matter has a certain structure, must be Schwarzschild or Reissner-Nordström. Non-trivial examples of matter for which the result applies are a wave map and a massive charged scalar field interacting with an electromagnetic field. The results thus generalize work of Bekenstein [1] and Heusler [12] from the static to the periodic case. In the third part, which is independent of the first two, it is shown that Dirac fields preserved by an isometry of a spherically symmetric domain of outer communications of the type described above must vanish. It can be applied in particular to the Einstein-Dirac-Maxwell equations or the Einstein-Dirac-Yang/Mills equations, generalizing work of Finster, Smoller and Yau [9], [7], [8], and also [6].

For equations of evolution, time-periodic or stationary solutions often correspond to the late time behavior of solutions for a large class of initial data. In the general theory of relativity, time-periodic “black hole” solutions, if they exist, seem to provide reasonable candidates for the final state of gravitational collapse. Such solutions can be defined as those invariant with respect to an isometry of the domain of outer communications which takes every point to its future, or more generally, such that points sufficiently close to infinity are mapped to their future.

In the case of a continuous family of isometries (i.e. stationary and static solutions), this problem has a long history and goes under the name “no hair” conjecture. See [4] for a survey of results and a recent important refinement. Current proofs depend on various extra assumptions and truly satisfactory theorems have only been obtained in the vacuum and electrovacuum static case.

The aim of this paper is to try to generalize some results from the static case to the spherically symmetric “time-periodic” case. The study of periodic solutions to the Einstein equations was initiated in Papapetrou [13], [14]; see also [11]. The analyses indicate that vacuum solutions which are periodic near null infinity should in fact be static there, but they are far from complete, and
depend very much on analyticity assumptions on the nature of null infinity, assumptions which do not appear to be physically valid. This paper appears to be the first to address the issue of the existence of periodic solutions in a non-analytic setting, in particular, in a setting compatible with the evolutionary hypothesis.

After briefly setting some basic assumptions (Section 1) regarding spherical symmetry, we shall show in Section 2 that for a certain class of matter, non-trivial spherically-symmetric black-hole phenomena cannot be described by solutions invariant with respect to a map taking some point to its future. In Section 3, we shall enlarge the class of matter for which the result applies by taking another approach, which in effect reduces the problem to the static case. The method of Section 3 is related to the arguments of [10].

In the spherically symmetric context, the above two sections generalize in particular results of [1] and [12], and Section 2, where it applies, provides a new and easier approach for the static case. Moreover, no assumption of invariance of the matter with respect to the isometry is necessary, nor is any real understanding of the behavior of the isometry on the event horizon. In fact, the results apply equally well when the “periodic” assumption is weakened to an appropriate notion of “almost periodicity”. Key to the results are the monotonicity properties of the area radius or the Hawking mass.

In Section 4, which is independent of Sections 2 and 3, we shall show that Dirac fields preserved by an isometry of the form described above must vanish. The method exploits conservation of the Dirac current. There has been a series of recent work [7], [8], where static spherically symmetric solutions of various coupled Einstein-Dirac-matter systems are considered, and also work [6] where periodic solutions of the Dirac equation on a fixed Reissner-Nordström background are considered. Modulo differences in regularity assumptions, all this previous work follows as a special case of the result of this section, which furthermore excludes non-trivial periodic solutions to a large class of coupled Einstein-Dirac-matter systems.

It should be stressed that the results of this paper suffer from some of the deficiencies of the original “no hair” theorems. A critical discussion of various geometric and regularity assumptions that appear here, and a comparison to [1], [8], [9] and [10], is included in the end (Section 4).

### 1 Some basic assumptions

Let $(M, g)$ be a spacetime on which $SO(3)$ acts by isometry and let $Q$ be the quotient manifold. We assume that the induced metric on $Q$ has bounded curvature. This implies the existence of local null coordinates $u$ and $v$ in a neighborhood of every point of $Q$ such that the induced metric has the form $-\Omega^2 du dv$. Recall (e. g. from [2]) the area radius $r$ and the Hawking mass $m$ defined on $Q$.

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1 We prefer to use the Hawking mass due to its special significance in spherical symmetry. One can equally well work only with area radius.
Our assumption on $Q$ is basically that it strictly contain a complete domain of outer communications $D$, bounded in the future and past by an event horizon, more precisely, that it contain a region whose conformal diagram looks like:

Null geodesics whose endpoint in the above diagram lies on null infinity have in fact infinite affine length; null infinity is thus not contained in $Q$. On the other hand, we assume that the points on the event horizon are contained in $Q$. The event horizon is the union of a future directed null geodesic ray (denoted $H^+$) and a past directed ray (denoted $H^-$). As in the diagram, we require these rays to intersect, i.e. the point $H^+ \cap H^-$ is also in $Q$. Since $Q$ is open, this means that all inextendible null rays emanating from points of $D$ either cross the event horizon and leave $D$ or have infinite affine length. We also assume $r$ is bounded below on $D$ by a positive constant.

We will assume furthermore that the domain of outer communications admits an isometry $\tau$, which descends to $Q$, i.e., such that it induces an isometry on $D$ such that $\tau^* r = r$, etc. This latter part follows for instance if we assume that the action of $SO(3)$ on $M$ is unique. Moreover, we will assume $\tau$ takes some point $p \in D$ to its future, i.e. $\tau(p) \in I^+(p)$.

The orbit $\tau^n(p)$ must be contained in the timelike line $r = r(p)$. Moreover, since the distance between $\tau(p)$ and $p$ is non-zero and must equal to the distance between $\tau^{(n+1)}(p)$ and $\tau^n(p)$, it follows that $\tau^n(p)$ approaches future timelike infinity, and $\tau^{-n}(p)$ approaches future past timelike infinity. In particular, given any point $q \in D$, there will be a $\tau^i(p)$ in $I^+(q)$, and a $\tau^{-i}(p)$ in $I^-(q)$. Since the causal relation between two points is preserved by an isometry, it follows that $\tau(q)$ and $q$ cannot be connected by an achronal curve. For $\tau^2 q$ could then never be in the future of $\tau^1(p)$. Thus $\tau(q) \in I^+(q)$ for all $q \in D$, and moreover, there are no limit points of the orbits $\tau^n(q)$ in the closure of $D$.

The key behind all our arguments will be to show that the existence of the isometry implies that certain quantities vanish on the event horizon. In the

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Footnotes:

1. The extent to which the assumptions made here can be retrieved from more primitive assumptions will be considered in Section 4.

2. Note that this is the familiar restrictive assumption from the “no-hair” theorems; it excludes in particular the case of the critical $\epsilon = m$ Reissner-Nordström solution. For more, see Section 4.
simplest cases, covered by the following section, this will then uniquely determine the wave matter to be constant on the event horizon, and then constant everywhere, by application of a uniqueness theorem for solutions of the characteristic initial value problem. In the case where the matter satisfies the weak energy condition, the vanishing of this quantity is proven using the monotonicity properties of the Hawking mass.

2 Exploiting the coupling with gravity

We refer the reader to [2] for a derivation of the Einstein equations in spherical symmetry with a general energy-momentum tensor. We will assume here that these equations are satisfied pointwise (i.e. all functions that appear are bounded) in the null coordinate charts of our atlas for an induced energy-momentum tensor $T_{ab}$ on $Q$ which satisfies the energy condition $T_{uu} \geq 0, T_{uv} \geq 0, T_{vv} \geq 0$. Here we always select $v$ such that null geodesic rays from points of $D$ generated by $\partial_v$ are future-directed and have infinite affine length (i.e. “terminate” on null infinity). It follows from

\[
\nabla_a \nabla_b r = \frac{1}{2r}(1 - \partial^a r \partial_c r)g_{ab} - r(T_{ab} - g_{ab}trT)
\]

that $\partial_u r \leq 0$ and $\partial_v r \geq 0$ in $D$ and then, from

\[
\partial_a m = r^2(T_{ab} - g_{ab}trT)\partial_b r
\]

that $\partial_u m \leq 0$ and $\partial_v m \geq 0$.

We then have the following

**Proposition 1** Let $\tau$ be an isometry of the domain of outer communications $D$ as described in the previous section. It follows that $\partial_u r = 0$ and $\partial_v m = 0$ on $H^+$ and $\partial_u r = 0$, $\partial_v m = 0$ on $H^-$. 

**Proof.** The proof is by contradiction. Suppose $p$ and $q$ are two points on $H^+$ such that $r(q) = r(p) + \epsilon$ for $\epsilon > 0$. By continuity of $r$ there exists a point

\[4\] Note that in the spherically symmetric context, one can interchange the notions of spacelike and timelike on the quotient manifold $Q$, making $D = J^+(H^+ \cup H^-)$, and thus, for appropriate equations, data on the event horizon determines solutions throughout $D$, by applying standard theorems [3].

\[5\] The arguments are similar to those of [3]. Assuming one of the inequalities does not hold, one argues by integrating that $r$ will have to become zero after a finite affine length in the direction of a null geodesic that terminates at the event horizon, a contradiction.
\( q' \in \mathbf{D} \) on the ray generated by \( -(\partial_u)_q \) such that \( r(q') > r(p) \).

It follows from the equations \( \partial_u r \leq 0, \partial_v r \geq 0 \) that \( r > r(p) \) in \( \mathbf{D} \cap J^+(q') \). Now consider the point \( p' \) at the intersection of the null ray generated by \( -(\partial_u)_q \) and the null ray generated by \( -(\partial_v)_p \). Again, by the relation \( \partial_u r \leq 0 \) it follows that \( r(p') \leq r(p) \). Now the assumption on \( \tau \) implies that there exists an \( N \) such that \( \tau^n(p') \in \mathbf{D} \cap J^+(q') \) for all \( n > N \). But since \( \tau \) is an isometry \( r(\tau^n(p')) = r(p') \leq r(p) \). This is a contradiction. One can then apply the same argument with \( m \) in place of \( r \), and then for \( H^- \) replacing \( H^+ \), thus completing the proof. □

In virtue of the equation (3) and the boundedness of \( T_{uv} \), it follows from the above proposition that since \( \partial_v m = 0 \) and \( \partial_r r = 0 \) on \( H^+ \) and similarly \( \partial_u m = \partial_u r = 0 \) on \( H^- \), we have \( T_{vv} = 0 \) on \( H^+ \) and \( T_{uu} = 0 \) on \( H^- \).

We now proceed to outline the more restrictive assumptions on the structure of the matter which will be necessary for our results. The first set of assumptions reflects the structure of the energy-momentum tensor itself. These are:

1. \( T = T(\Psi, F, g) \) where \( F \) is a skew symmetric 2-tensor, and \( \Psi \) takes values in some space endowed with a connection \( \tilde{\nabla} \), such that if \( \tilde{\nabla}_X \Psi = 0 \) identically for all \( X \in T^* M \), then \( T \) corresponds to the energy momentum tensor of a spherically symmetric electric field \( F_{\mu\nu} \) satisfying the source-free Maxwell equations. Here \( \tilde{\nabla}_X \) is the induced connection on \( M \).

2. \( T_{vv} = 0 \) should imply \( \tilde{\nabla}_v \Psi = 0 \), and \( T_{uu} = 0 \) should imply \( \tilde{\nabla}_u \Psi = 0 \).

One example of such a \( T \) is the energy momentum tensor generated by a wave map \( \phi : (M, g) \to (N, h) \) interacting (via the gravitational field only, since it does not carry charge) in an electromagnetic field \( F_{\mu\nu} \):

\[
T_{\mu\nu} = F_{\mu\lambda} F_{\nu\rho} g^{\lambda\rho} - \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} - h_{AB}(\phi^A_{\mu} \phi^B_{\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \phi^A_{\rho} \phi^B_{\sigma}).
\]

A special case of the above is of course when \( N \) is \( \mathbb{R}^n \) with the flat metric and \( \phi \) is then a collection of \( n \) real scalar fields. Another example is the energy momentum tensor generated by a massive complex scalar field \( \phi \) in an electromagnetic
field $F_{\mu
u}$ with electromagnetic potential $A_\mu$

$$T_{\mu
u} = F_{\mu\lambda} F_{\nu\rho} g^{\lambda\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} g^{\rho\lambda} g^{\sigma\kappa}$$

$$- \frac{1}{2} g_{\mu\nu} M^2 \phi \bar{\phi} + \frac{1}{2} (\phi_{\mu} \phi_{\nu} + \bar{\phi}_{\mu} \bar{\phi}_{\nu})$$

$$+ \frac{1}{2} (\phi_{\mu} i e A_{\nu} \phi + \bar{\phi}_{\nu} i e A_{\mu} \phi + \bar{\phi}_{\mu} i e A_{\nu} \phi - \phi_{\nu} i e A_{\mu} \phi)$$

$$- \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} (\phi_{\rho} + i e A_{\rho} \phi) (\bar{\phi}_{\sigma} - i e A_{\sigma} \bar{\phi}) + e^2 A_a A_b \phi \bar{\phi}.$$ 

For $T$ satisfying 1 and 2, it follows that $\nabla_c \Psi = 0$ on $H^+$ and $\nabla_a \Psi = 0$ on $H^-$. Since the restriction of a connection to a one-dimensional set is trivial, we can then choose coordinates for a space representing the degrees of freedom for $\Psi$ such that $\Psi$ is in fact constant on the event horizon. If in local coordinates $x_a \in Q$ the system of equations for $\Psi$, with $F_{\mu\nu}$, $g_{ab}$, and $r$ fixed, is of the form

$$\partial^a \partial_a \Psi = F(\nabla \Psi, \Psi, x_a)$$

with $F$ a sufficiently regular function, then the characteristic initial value problem with initial data on the event horizon is locally well posed, provided $\Psi$ is assumed sufficiently regular. If this equation admits the solution $\Psi = \Psi(H^+ \cup H^-)$, then this must be the only solution in the vicinity of the horizon, and by a continuity argument, this domain of dependence property can be extended to guarantee uniqueness throughout $D$. A sufficient condition for this is clearly $F(0, \Psi, x_a) = 0$. Thus, in view of the fact that spherically symmetric solutions of the Einstein-Maxwell equations are necessarily Reissner-Nordström, we have

**Theorem 1** If $T_{\mu
u}$ satisfies conditions 1 and 2, and $\Psi$ satisfies a system of the form (3), with $F(0, \Psi, x_a)(p) = 0$, and if $\tau$ is as in Proposition 1, it follows that $D$ coincides with the domain of outer communications of a Reissner-Nordström solution.

Note that the above theorem applies to the wave map system, which can be written

$$\partial^a \partial_a \Phi = \Gamma(\Phi)(|\nabla \Phi|^2).$$

where $\Gamma$ is an expression involving the Christoffel symbols of $(N, g)$. (Compare with [12]. The above argument reproves, in particular, the static result, and seems considerably easier, as it does not depend on the geometry of the target.)

Also note that, in the above argument, we have not assumed that $\tau$ preserves $\Psi$, only that it preserves the metric. In fact, it suffice to assume that given any point $p$, then for all $\varepsilon$ there exists an $N(\varepsilon, p)$ such that $|m(\tau^n(p)) - m(p)| < \varepsilon$ for $|n| \geq N$. Such solutions can be called “almost periodic”.

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6Note that for fields which couple directly to the $F_{\mu\nu}$ tensor, there is a regularity assumption on $F$ as well as on $g$ implicit in (3).

7Recall the comment from before that to apply (3), one should first redefine the metric $g_{ab}$ to be its negative, so $D$ becomes $J^+(H^+ \cup H^-)$. 

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3 Constructing a Killing vector and reducing to the static case

Unfortunately, as it stands, the argument of the previous section cannot be applied in the case of a complex scalar field or a massive scalar field, for then the dependence of $F$ on $\Psi$ is not of the type described above. In particular, there do not exist constant non-zero solutions.

It is perhaps instructive to compare here with the static case. The argument of Bekenstein \cite{1}, say for the scalar field $\Box \phi = M \phi$ with $M > 0$, goes roughly as follows: Integrating the equality $\nabla_\alpha (\phi \nabla^\alpha \phi) = \nabla_\alpha \phi \nabla^\alpha \phi + M \phi^2$ using Gauss’s theorem, the boundary contributions along the event horizon vanish, while the contributions along two spacelike curves which are carried to one another by the isometry cancel. Moreover, the divergence is non-negative, since in a static solution $\nabla \phi$ is spacelike, and 0 only if the solution vanishes. Thus, either the solution is identically 0, or there must be a boundary contribution at infinity, i.e. the solution does not decay as $r \to \infty$. (This would imply that the curvature does not decay, and thus the solution would not be asymptotically flat.)

The above Bochner-type method and arguments based on it cannot be applied directly in the periodic case as $\nabla \phi$ may have negative length. With slightly more effort than in Section 2, one can show that for various examples of matter—including the case of a charged massive scalar field, for instance—our initial data determine a static solution, and then apply the above argument to show that this solution must thus not decay at infinity.

The idea is similar in spirit to Theorem 1, except that now we will apply the uniqueness theorem to the solution of the characteristic value problem to a system of second order hyperbolic equations derived from Killing’s equation.

First, we introduce the following new assumptions: Letting $x^A$, $x^B$ denote coordinates on $S^2$, we assume

1. $\partial_v r = 0$ on $H^+$ and $\partial_u r = 0$ on $H^-$
2. $T_{vv} = 0$, $\nabla_u T_{vv} = 0$ on $H^+$ and $T_{uu} = 0$, $\nabla_v T_{uu} = 0$ on $H^-$.\n3. $\nabla_v T_{AB} = 0$ on $H^+$ and $\nabla_u T_{AB} = 0$ on $H^-$.\n
Given an isometry $\tau$ as before, Assumptions 1, 2 and 3 above follow for a large class of matter, including the case of a complex scalar field interacting in an electromagnetic field. (See the Appendix.)

We define $v$ and $u$ on the event horizon so as to yield an affine distance on the event horizon as measured from the point $H^+ \cap H^-$ on $H^+$ and $H^-$ respectively, i.e. we will be assuming that $g_{vv} = -1$ on $H^+ \cup H^-$. We now will define a particular null vector field $K$ on the event horizon, and extend it to $D$ as the unique solution of the initial value problem, with initial data on the event horizon $H^\pm$ for the equation

$$\Box K^\alpha = -K^\beta R_{\beta \gamma} g^{\gamma \alpha}.$$  \hspace{1cm} (4)

Recall the comment in Section 1 about the well-posedness of this problem in spherical symmetry, because of the symmetry between timelike and spacelike directions.
The choice of the definition will be to ensure that $L_K g_{\mu\nu} = 0$ on $H^+ \cup H^-$. For now write $K|_{H^+} = K^v(v)\partial_v$ and $K|_{H^-} = K^u(u)\partial_u$, where we will determine immediately following what $K^v(v)$ and $K^u(u)$ have to be.

Let us concentrate first on $H^+$. In the null coordinates defined above (where in addition $x^A$ and $x^B$ are taken to be normal coordinates), the only non-vanishing Christoffel symbol are $\Gamma^u_{uv}, \Gamma^B_{uA}$ and $\Gamma^v_{AB}$. Outside of $H^+, \Gamma^u_{uu}, \Gamma^v_{vv}, \Gamma^B_{uA}, \Gamma^B_{vA}, \Gamma^v_{AB}$ are the only non-vanishing components. Note also that on $H^+$,

$$R = 2g^{uv}R_{uv} + g^{AB}R_{AB}$$

$$= 2\partial_u \Gamma^u_{vv} + 4g^{AB}\partial_u \Gamma^u_{AB}$$

$$= 2\partial_u \Gamma^u_{vv} + 2(R + 2R_{uv}),$$

and thus, we have that

$$\partial_u \Gamma^u_{vv} = -\frac{1}{2}R - 2R_{uv}.$$

We compute

$$\Box K^u = -2\partial_u \partial_v K^u,$$

$$\Box K^v = -2\partial_u \partial_v K^v + \partial_v K^v(\Gamma^u_{AB}) + K^v(-\partial_u \Gamma^u_{vv}),$$

and thus (4) gives

$$\partial_u \partial_v K^u = 0,$$

(5)

$$\partial_u \partial_v K^v = -\frac{1}{2}\partial_u K^v g^{AB} \Gamma^u_{AB} - \frac{1}{2}(\partial_u \Gamma^u_{vv} + R_{uv})K^v$$

$$= -\frac{1}{2}\partial_v K^v g^{AB} \Gamma^u_{AB} + \frac{1}{4}(R + 2R_{uv})K^v$$

$$= -\frac{1}{4}\partial_v K^v \int_0^v (R + 2R_{uv})dv + \frac{1}{4}K^v(R + 2R_{uv}).$$

(6)

Recalling $(L_K g)_{\alpha\beta} = K_{\alpha;\beta} + K_{\beta;\alpha}$, in view of our knowledge of the Christoffel symbols, and the fact that $K_\nu = 0$ on $H^+$, we obtain

$$(L_K g)_{uv} = \partial_u K_v + \partial_v K_u = -\partial_u K^u - \partial_v K^v,$$  \tag{7}$$

$$(L_K g)_{vv} = 2\partial_v K_v = -2\partial_v K^u = 0,$$

$$(L_K g)_{vA} = 0, (L_K g)_{uA} = 0,$$

$$(L_K g)_{AB} = -\Gamma^u_{AB} K_u - \Gamma^v_{AB} K_v = 0,$$

$$(L_K g)_{uu} = 2(\partial_u K_u - K_u \Gamma^u_{uu}) = -2\partial_u K^v.$$ \tag{8}

Thus, if we are to have $(L_K g)_{\alpha\beta} = 0$ on $H^+$, it follows from (7) that

$$\partial_u K^u = -\partial_v K^v.$$ \tag{9}
Rewriting (5) as $\partial_v \partial_u K^v = 0$, it follows from (9) that

$$\partial_v \partial_u K^v = 0,$$

and thus that $K^v = Cv$. Using (9) again, and the same argument, it follows that $K^u = -Cu$ on $H^-$. Of course, to show that indeed we have $(L_K g)_{\alpha\beta} = 0$ on $H^+$, for the $K$ defined above, it remains to show, in view of (8), that

$$\partial_u K^v = 0.$$ 

Since the above equation is indeed true at $H^+ \cap H^-$, it follows that it is enough to show that $\partial_v \partial_u K^v = 0$, or, by (6), that

$$-\frac{1}{4} \partial_v K^v \int_0^v (R + 2R_uv)dv + \frac{1}{4} K^v (R + 2R_uv) = 0.$$

Assumption 2 together with the conservation of energy-momentum implies that $\partial_v R_{uv} = 0$, and Assumption 3 implies that $\partial_v R = 0$, and thus, $R + 2R_{uv} = c$ where $c = (R + 2R_{uv})|_{H^+ \cap H^-}$. Thus we compute

$$-\frac{1}{4} \partial_v K^v \int_0^v (R + 2R_uv)dv + \frac{1}{4} K^v (R + 2R_uv) = -\frac{1}{4} C(cv) + \frac{1}{4} (Cv)c = 0.$$

We can take then $C = 1$ and we have found a nontrivial vector field vanishing at $H^+ \cap H^-$ satisfying $L_K g = 0$ on $H^+$, and similarly, $L_K g = 0$ on $H^-$ as well.

Denote now the totality of matter by $\Phi$. In terms of this $K$ defined, we assume further

$$4 L_K \Phi = 0$$

on $H^+ \cup H^-$. The quantities $L_K g_{\mu\nu}$ and $L_K \Phi$ satisfy a system of equations which, when everything else is treated as fixed, only admits the zero solution if they vanish on $H^+ \cup H^-$. All our assumptions taken together imply that we have produced a vector field $K$ such that $L_K g = 0$, i.e. a Killing field $K$ on D. Note that a similar argument ensures that $K$ is also a Killing field “downstairs”, i.e. that $K r = 0$. From this, it follows that $K$ must be timelike. Since $K$ does not vanish identically on the event horizon, it follows that there exists a $p \in D$ such that $K(p) \neq 0$. Thus $K$ does not vanish along the line $r = r(p)$, which must be the orbit $\phi_t(p)$ where $\phi_t$ denotes the one parameter group of isometries generated by $K$. Since all future directed constant-$u$ null rays must intersect the line $r = r(p)$, it follows that $K$ can nowhere vanish. For if it did at some point $q$, then choosing a point $s$ on $r = r(p)$ which can be connected to $q$ by a spacelike curve, then $\phi_t(s)$ for large enough $t$ is in the future of $\phi_t(q) = q$, which contradicts the fact that $\phi_t$ is an isometry.

We have thus proven

The expression $L_K \Phi$ can be tricky to define if the equations have a gauge invariance. Typically, this will mean that there is a choice of gauge for which the matter can be expressed by some $\Phi$ for which $L_K \Phi = 0$. See the appendix for the case of a complex scalar field.
Theorem 2 For an Einstein-matter system satisfying Assumptions 1, 2, 3, 4, and 5 above, the domain of outer communications is static.

4 Exploiting a conserved current: the case of the Dirac equation

In the case of Dirac fields, the arguments outlined above do not apply because this matter does not satisfy the positive energy condition. This is related to the fact that the Dirac field probably provides a reasonable model only after second quantization. But in fact, considerations regarding periodic solutions are even easier than in the previous section, and can be studied without applying the coupling with gravity, which played a central role in the previous argument.

We refer the reader to [6] for background on this problem in the uncoupled case, and to [7], [8] in the coupled static case. In particular, we recall the Dirac matrices $G^\alpha$, which operate on Dirac fields $\Psi$, which are sections of an appropriate spinor bundle. The precise form of the Dirac equation will depend on the other matter fields present to which the field is coupled. We will simply assume that $\Psi$ satisfies in local coordinates, after fixing the metric and the other matter fields, a linear equation of the form

$$iG^\alpha \partial_\alpha \Psi = F(\Psi),$$

where $F(0) = 0$. Note that by squaring the Dirac operator, it follows that $\Psi$ satisfies a system

$$\Box \Psi = \tilde{F}(\nabla \Psi, \Psi, x_\alpha).$$

We further assume that the vector field $\bar{\Psi} G^\alpha \Psi$ provides a positive current, i.e.

$$\bar{\Psi} G^\alpha \Psi \bar{X}^\beta g_{\alpha\beta} \geq 0$$

when $\bar{X}^\beta$ is future directed and timelike, with equality only in the case where $\Psi$ vanishes, and moreover, this current is conserved:

$$\nabla_\alpha (\bar{\Psi} G^\alpha \Psi) = 0.$$

We do not assume that $\Psi$ is spherically symmetric, but we do assume that it is defined on a spherically symmetric domain of outer communications as before, preserved by an isometry $\tau$, as before, in the sense that

$$\tau_* (\bar{\Psi} G^\alpha \Psi) = \bar{\Psi} G^\alpha \Psi.$$

Moreover, we assume that all other matter fields are spherically symmetric, and thus we can write $\Psi$ as the sum of spherical harmonics each of which satisfy a wave equation of the form (11), but on $Q$, not on $M$.

Consider a spacelike curve $\gamma$ which divides $D$ into two connected components, and intersects the event horizon at $H^+ \cap H^-$. Let $X$ be the future normal vector field to $\gamma$. Fix a point $q$ on the event horizon and $p$ on $\gamma$. We
denote the part of $\gamma$ connecting $p$ with spacelike infinity by $\gamma_p$. Then for an isometry $\tau$ as in Proposition 1, There exists an $n$ such that $\tau^n(p)$ and $q$ can be connected by a spacelike curve $\tilde{\gamma}$:

We will assume $\Psi$ is locally bounded, and that

$$\int_{\gamma} \bar{\Psi} G^\alpha \Psi X^\beta g_{\alpha\beta} < \infty.$$  

The latter assumption is quite reasonable in view of the fact that this integral should equal to the probability of observing the particle on $\gamma$, which should be normalizable to something less than 1. Now integrating the conservation law (12) and applying Gauss' theorem, and since

$$\int_{\gamma} \bar{\Psi} G^\alpha \Psi X^\beta g_{\alpha\beta} > 0$$

it follows that

$$\int_{H^+ \cap J^- (q)} \bar{\Psi} G^\alpha \Psi X^\beta g_{\alpha\beta} \leq \int_{\gamma} \bar{\Psi} G^\alpha \Psi X^\beta g_{\alpha\beta} - \int_{\tau^n(\gamma_p)} \bar{\Psi} G^\alpha \Psi (\tau^n X)^\beta g_{\alpha\beta}$$

$$= \int_{\gamma} \bar{\Psi} G^\alpha \Psi X^\beta g_{\alpha\beta} - \int_{\gamma_p} \bar{\Psi} G^\alpha \Psi X^\beta g_{\alpha\beta}$$

$$= \int_{\gamma \setminus \gamma_p} \bar{\Psi} G^\alpha \Psi X^\beta g_{\alpha\beta},$$

where the second line follows from the fact that $\tau$ is an isometry and (13).

But as $p \to H^+ \cap H^-$, the term on the right hand side approaches 0. Thus, since the left hand side is nonnegative, it follows that

$$\int_{H^+ \cap J^- (q)} \bar{\Psi} G^\alpha \Psi N^\beta g_{\alpha\beta} = 0$$

for all $q$ and consequently, $\bar{\Psi} G^\alpha \Psi N^\beta g_{\alpha\beta} = 0$ identically on $H^+$, and similarly, $\bar{\Psi} G^\alpha \Psi N_-^\beta g_{\alpha\beta} = 0$ on $H^-$, where $N_-$ denotes the null vector tangent to $H^-$. 

11
Since $N_- + N$ at $H^- \cap H^+$ is timelike, it follows by the positivity of the current that $\Psi$ in fact vanishes there.

It turns out that the behavior of $\Psi$ on the event horizon, deduced above, together with the Dirac equation, imply that $\Psi$ vanishes identically on the event horizon:

Choose coordinates $u, v, x^1, \text{and } x^2$ in a neighborhood of $H^+ \cap H^-$, such that, $g = -\Omega^2 dudv + \tilde{g}_{ij} dx^i dx^j$. It follows from the properties deduced above that a spinor representation can be chosen such that $G^\alpha \Psi = 0$, $G^u \partial_v \Psi = 0$, and $G^v \partial_u \Psi = 0$ on $H^+$, while $G^v \Psi = 0$, $G^v \partial_u \Psi = 0$, and $G^u \partial_v \Psi = 0$ on $H^-$. From the anticommutation relations it follows that $G^u G^v = 0$, $G^v G^u = 0$.

Multiplying the Dirac equation (10) by $G^u$, and restricting to $H^+$, one obtains,

$$i(G^u G^v \partial_v \Psi + G^u G^x \partial_x \Psi) = G^u (F(\Psi)).$$

Since $G^u G^v = -G^v G^u$ by the anticommutation relations, it follows from $G^u \Psi = 0$ that $iG^u G^v \partial_v \Psi = G^u F(\Psi)$. Again, from the anticommutation relations, one obtains that $G^u G^v = 2g^{uv} - G^v G^u$, and thus, since $G^v \partial_u \Psi = 0$,

$$\partial_v \Psi = \tilde{f}(\Psi) \quad (14)$$

for a well-behaved function $\tilde{f}$ with $\tilde{f}(0) = 0$.

From the fact shown above that $\Psi = 0$ at $H^+ \cap H^-$, it now follows immediately from (14) that $\Psi$ must vanish identically on $H^+$. One argues in the same way to obtain that $\Psi$ vanishes identically on $H^-$. This condition completely determines initial data for the characteristic initial value problem for each of the spherical harmonics of the Dirac equation, and thus assuming that $\Psi$ is in a space sufficiently regular, all the spherical harmonics must vanish identically in $D$ by uniqueness of the solution of the characteristic initial value problem, and thus $\Psi = 0$ in $D$.

Again, it is clear from the proof that one can replace the assumptions on $\tau$ with the assumption that for all $p, \epsilon$, there exist $N(\epsilon, p)$ such that $|n| \geq N$ implies

$$\left| \int_{\gamma_p} \bar{\Psi} G^\alpha \Psi X^\beta g_{\alpha\beta} - \int_{\gamma_{-\tau(p)}} \bar{\Psi} G^\alpha \Psi X^\beta g_{\alpha\beta} \right| < \epsilon.$$ 

There is thus a sense in which the result holds for “almost periodic solutions” as well.

5 A note on the assumptions

As discussed in the beginning, the motivation for considering “time-periodic” solutions $(Q, g)$ is as “limiting” final states of gravitational collapse. Thus, a priori, it makes sense only to assume that $Q$ be defined to the future of a

\[10\] See the remark in the previous section about the change of sign of the metric $g_{ab}$
spacelike surface $S$.

In view of our assumptions on the existence of an isometry $\tau$, however, given a fundamental domain $F$ such that $S \subset \partial F$, one can construct in an obvious way $\tilde{Q}$ and $\tilde{\tau}$ an isometry of $\tilde{Q}$, such that $\tilde{Q} = Q \cup_i \tau^{-1} F$. In the spherically symmetric case, if the energy momentum tensor satisfies the energy condition, it follows that this spacetime will also have a past boundary. For it is clear by the arguments of Section 2 that since $\partial_v r > 0$ in $D$, it must become less than the infimum of $r$ within finite affine length in the direction $-\partial_v$. Moreover, by arguments similar in spirit to the proof of Proposition 2, this boundary can be shown to have a natural null structure and we can denote it as before by $H^-$. On the original $Q$, it is reasonable to assume only the regularity that might be induced from being a limit of regular “collapsing” spacetimes outside the event horizon $H^+$. By monotonicity, $r$ and $m$ can be extended to $H^+$, but the regularity of matter fields may be quite weak. The regularity implications for $H^-$ may in fact be weaker.

Moreover, there is no guarantee that the spacetime can be extended so that $H^+$ and $H^-$ intersect, much less for fields to have any sort of regularity there. (Compare with the Reissner-Nordström solution with $e = m$.) Our results, on the other hand, depend very much on the existence of the point $H^+ \cap H^-$, and on some assumptions of regularity along these null curves. While both of these aspects of the setup could possibly be weakened, it is clear that some sort of regularity assumption will certainly be an integral part of any argument involving well-posedness of the characteristic value problem.

For another approach to these issues of regularity, it is instructive to compare with the work of Finster, Smoller, and Yau. In [7], [8], and [9], static spherically symmetric solutions to the Einstein-Dirac-Yang/Mills and Einstein-Maxwell-Dirac equations were considered, and in [6], periodic solutions of the Dirac equation on a Reissner-Nordström background.
In the above series of papers, solutions of the Dirac equation are in fact permitted to blow up along the event horizon, in a very specific way, and the normalization condition is relaxed near the event horizon. Thus, from one point of view, their assumptions could be considered weaker. On the other hand, \cite{7}, \cite{8} and \cite{9} introduce assumptions at the level of $C^\infty$ of the metric at the horizon, and various auxiliary coordinated-dependent conditions and power-law assumptions, while \cite{6} depends very much on an assumption on the vanishing of a certain flux over $H^-$. In \cite{6}, the considerations regarding the analysis of $\Phi$ on $H^+$ relate to a particular choice of “extension” of $\Phi$ beyond the event horizon, and they also seem to depend on the conformal geometry of the interior of the Reissner-Nordström black hole, a geometry which is thought to be unstable. It is unclear how any of these conditions are justified if we view $Q$ as “generated” by the process outlined in the beginning of this section, and thus whether anything is gained by allowing \textit{a priori} for a more singular behavior of $\Phi$.

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7 Appendix

We will show that a complex scalar field indeed satisfies the assumptions of Section 3. To reduce the equations to a determined system, we will have to set a gauge. We will require thus that $A_u = 0$ on $H^+$ and $A_u = 0$ on $H^-$, and the components $A_B = 0$ as well, where $x^B$ are coordinates on $S^2$. We will also introduce the notation $D$ for the covariant derivative defined by the connection $A$, i.e. we have $D_\mu \phi = \phi,\mu + ie A_\mu \phi$.

Note that the only non-vanishing components of the electromagnetic tensor $F_{\mu\nu}$ are $F_{uv}$ and the collection $F_{AB}$, where $A$ and $B$ range over coordinates on $S^2$.

Thus,

$$T_{vv} = D_v \phi \overline{D_v \phi}, \quad T_{uu} = D_u \phi \overline{D_u \phi},$$

$$T_{uv} = -\frac{1}{4} g_{uv} F_{AB} F_{CD} g^{AC} g^{BD} - \frac{1}{2} g_{uv} (D_u \phi \overline{D_v \phi} + D_v \phi \overline{D_u \phi}),$$

$$T_{AB} = F_{AC} F_{BD} g^{CD} - \frac{1}{4} g_{AB} F_{MN} F_{CD} g^{MC} g^{ND}.$$  

In particular, the existence of $\tau$ implies that $T_{vv} = 0$ on $H^+$, $T_{uu} = 0$ on $H^-$, and thus $\partial_v \phi = D_v \phi = 0$ on $H^+$ and $\partial_u \phi = D_u \phi = 0$ on $H^-$. Moreover, since

\[11\] The motivation for this seems to be more the global behavior of the special $(r, t)$ coordinate system and its associated gauge, than observations which could be made by local observers, employed here.
\( \nabla_u T_{vu} = \partial_u T_{vu} \), applying \( \partial_u \) to (15) yields \( \nabla_u T_{vu} = 0 \) on \( H^+ \) and similarly \( \nabla_v T_{uv} = 0 \) on \( H^- \). Assumptions 1 and 2 of Section 3 thus hold.

Now, Maxwell’s equations

\[
F_{\mu\nu;\rho} g^{\rho\sigma} - ie\phi\bar{D}_\mu \phi + ie\bar{\phi} D_\mu \phi = 0,
\]

restricted to \( H^+ \), yields the equation

\[
\partial_v F_{uv} = 0,
\]

and on \( H^- \), the equation

\[
\partial_u F_{vu} = 0.
\]

Similarly, the equation \( F_{[AB,v]} = 0 \) yields

\[
\partial_v F_{AB} = 0,
\]

and

\[
\partial_u F_{AB} = 0,
\]

throughout \( D \). Thus we have \( \partial_v T_{AB} = 0 \) on \( H^+ \) and \( \partial_u T_{AB} = 0 \) on \( H^- \), and this implies Assumption 3.

To write a determined system of equations, we impose the equation \( \nabla^\alpha A_\alpha = 0 \). Note that this equation, together with the condition that \( A_u = 0 \) on \( H^+ \) and \( A_v = 0 \) on \( H^- \) implies that \( L_K A_\mu = 0 \) on \( H^+ \cup H^- \). For, on \( H^+ \) we obtain that \( A_{u,v} = \frac{1}{2} F_{uv} \), and thus

\[
(L_K A)_u = K^\nu A_{\mu,u} + K^\mu_{,u} A_\mu = K^\nu A_{u,v} + K^u_{,u} A_u = K^\nu A_{u,v} - K^v_{,u} A_u = \frac{1}{2} K^\nu F_{uv} - \frac{1}{2} \partial_v K^\nu F_{uv,v} = \frac{1}{2} C_v F_{uv} - \frac{1}{2} C_v F_{uv} = 0,
\]

\[
(L_K A)_v = K^\nu A_{v,u} + K^u_{,u} A_u + K^v_{,u} A_v = 0 + 0 + 0 = 0.
\]

Our equations for the matter \( \Phi = (A_\mu, F_{\mu\nu}, \phi) \) thus now become

\[
\nabla^\alpha A_\alpha = 0,
\]

\[
A_{(\mu,\nu)} = F_{\mu\nu},
\]

\[
\nabla^\alpha F_{\alpha\mu} = ie(\phi \bar{D}_\mu \phi - \bar{\phi} D_\mu \phi),
\]

\[
g^{\mu\nu} D_\mu D_\nu \phi = 0.
\]

Applying \( L_K \) to these equations, and using the equation (4) yields

\[
\nabla^\alpha (L_K A)_\alpha = L(\nabla L_K g) + L(L_K g),
\]
\[(L_K A)_{(\mu, \nu)} = L(L_K \phi) + L(L_K g),\]
\[\Box (L_K \phi) = L(\nabla L_K (g)) + L(L_K A) + L(L_K g) + L(L_K F).\]
Here, the notation \(L(x)\) means terms linear in \(x\). Noting that
\[L_K T = L(g) + L(\phi) + L(A) + L(F),\]
and that \(L_K F_{\mu \nu} = (L_K A)_{(\mu, \nu)}\), we have that given \(g, A, F,\) and \(K\), the above system coupled with the equation
\[\Box L_K g = L(L_K T) + L(L_K g)\]
can be written as a closed linear hyperbolic system in 1+1 dimensions for \(L_K A\), \((L_K \phi)\), and \((L_K g)\), with vanishing initial data on \(H^+ \cup H^-\), and for which 0 is a solution. Since 0 is a solution of this system it must be the only solution, by uniqueness of this initial value problem, i.e. the final assumption of Section 3 is also verified.

The argument clearly also applies to the massive case, and to more general so-called Higgs fields.

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