The Packing Density of the \( n \)-Dimensional Cross-Polytope

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Abstract The packing density of the regular cross-polytope in Euclidean \( n \)-space is unknown except in dimensions 2 and 4 where it is 1. The only non-trivial upper bound is due to Gravel et al. [Discrete Comput Geom 46(4):799–818, 2011], who proved that for \( n = 3 \) the packing density of the regular octahedron is at most \( 1 - 1.4 \times 10^{-12} \). In this paper, we prove upper bounds for the packing density \( \delta(X^n) \) of the \( n \)-dimensional regular cross-polytope \( X^n \). It turns out that \( \delta(X^n) \) approaches zero exponentially fast with growing dimension. Our bound is non-trivial, that is, less than 1, for \( n \geq 7 \).

Keywords Blichfeldt’s method · Cross-polytope · Density · Packing

1 Introduction

Let \( K \subset \mathbb{R}^n \) be a convex body (a compact convex set with interior points). A family \( K = \{K_1, K_2, \ldots \} \) of congruent copies of \( K \) is a packing in \( \mathbb{R}^n \) if the elements of \( K \)
are non-overlapping (their interiors are pairwise disjoint). The \textit{density} of a packing is essentially the proportion of space covered by elements of the packing. The supremum of the densities of all packings with congruent copies of a convex body $K$ is called the \textit{packing density} of $K$ and it is denoted by $\delta(K)$. For a more detailed introduction into the basic properties of density, see, for example, \cite{21,22}.

One of the central problems of the theory of packing and covering is to determine the packing densities of particular convex bodies. The majority of results concern the $n$-dimensional unit ball $B^n$. However, the exact value of $\delta(B^n)$ is known only in the cases $n = 2$ and 3. In particular, $\delta(B^2) = \pi/\sqrt{12}$, proved by Thue \cite{26,27}, and $\delta(B^3) = \pi/\sqrt{18}$, proved by Hales \cite{17}. Recently, Cohn and Kumar \cite{9} proved that in dimension 24 the density of no sphere packing can exceed the density given by the Leech lattice by more than a multiplicative factor of $1 + 1.65 \times 10^{-30}$. Thus, for any practical purpose, we may consider $\delta(B^{24}) = \pi^{12}/12!$.

The best asymptotic upper bound for $\delta(B^n)$ is due to Kabatjanskii and Levenšteĭn \cite{18}:

$$\delta(B^n) \leq 2^{(-0.599 + o(1)n)}.$$  \hfill (1)

While the bound by Cohn and Elkies \cite{8} is sharper for all $n$ than the bound by Kabatjanskii and Levenšteĭn (see \cite{10}), it yields the same asymptotic upper bound as (1).

Other than the $n$-dimensional ball, the most interesting convex bodies are probably the $n$-dimensional regular polytopes that exist in every dimension: the simplex, the cube, and the cross-polytope. The $n$-cube is a tile, so its packing density is 1. Very little is known about the packing density of the regular simplex and the regular cross-polytope for $n \geq 3$. An exception is the regular cross-polytope in $\mathbb{R}^4$ which tiles the space (cf. \cite{14}, Sect. 22), thus its packing density is 1. The regular simplex does not tile $\mathbb{R}^n$ for $n \geq 3$; for a short proof see \cite[pp. 34–35]{1}. Similarly, the regular cross-polytope does not tile $\mathbb{R}^n$ for $n = 3$ and $n \geq 5$. Very recently, Gravel et al. \cite{15} proved upper bounds for the packing density of the regular tetrahedron ($1 - 2.6 \ldots \times 10^{-25}$) and the regular octahedron ($1 - 1.4 \ldots \times 10^{-12}$). These bounds are certainly not optimal. It seems unclear to the authors of this paper whether the method of Gravel, Elser and Kallus can be extended to higher dimensions.

We note that there has been much work done recently in order to construct efficient packings of regular tetrahedra, octahedra, and other solids in $\mathbb{R}^3$. For a review see Bezdek and Kuperberg \cite{5}, and Torquato and Jiao \cite{29}. The best known lower bound for the packing density of the $n$-dimensional cross-polytope is $2^{-n(1+o(1))}$, which comes from the Minkowski–Hlawka theorem. Rush conjectured in \cite{24} that, in fact, this is best possible. Finally, we remark that, to the best of our knowledge, essentially nothing is known about the covering densities of the regular solids.

In the next section, we will prove an upper bound for the packing density of the regular $n$-dimensional cross-polytope using known upper bounds on $\delta(B^n)$ and the ratio of the volumes of the cross-polytope and its insphere. Subsequently, we significantly improve these upper bounds for small dimensions employing a modification of the method of Blichfeldt by Fejes Tóth and Kuperberg \cite{13}. With these methods we establish non-trivial upper bounds on the packing density of the $n$-dimensional cross-polytope for $n \geq 7$, and we also show that the packing density of the cross-polytope approaches 0 exponentially fast as the dimension tends to infinity.
Table 1 Upper bounds on the packing densities of 4-dimensional regular solids obtained from their insphere volume ratios

| n | P       | Upper bound on $\delta(P)$ |
|---|---------|-----------------------------|
| 4 | 120-cell| 0.74972                     |
| 4 | 600-cell| 0.69073                     |

1.1 Upper Bound Using the Insphere

The following proposition is folklore.

*If $K \subset \mathbb{R}^n$ is a convex body, $r(K)$ is the radius of the maximum size ball contained in $K$, and $\text{Vol}(\cdot)$ stands for volume, then*

$$\delta(K) \leq \frac{\text{Vol}(K)}{r(K)^n \text{Vol}(B^n)} \delta(B^n).$$

(2)

Formula (2) may provide a non-trivial upper bound on the packing density of a convex body $K$ whose insphere is sufficiently large in volume compared to $K$. Using this idea, Torquato and Jiao [28,29] derived upper bounds for the packing densities of the regular dodecahedron and the regular icosahedron and for some of the Archimedean solids in $\mathbb{R}^3$, see Tables III and IV in [29]. However, they have not used the insphere volume ratio to investigate the packing densities of convex bodies in higher dimensions.

In $\mathbb{R}^4$, the cube, the cross-polytope, and the 24-cell are tiles, so their packing densities are equal to 1. The insphere volume ratio method gives an upper bound greater than 1 for the packing density of the regular simplex. However, one obtains non-trivial upper bounds for the packing densities of the 120-cell and the 600-cell using the $\delta(B^4) \leq 0.13126 \cdot \pi^2/2$ bound by Cohn and Elkies [8], see the numerical values in Table 1. For precise definitions and basic properties of the regular polytopes in $\mathbb{R}^4$, we refer to the book by Coxeter [12].

In dimensions higher than 4, there exist only three regular solids: the simplex, the cube, and the cross-polytope. The $n$-dimensional cube is a tile in $\mathbb{R}^n$, and thus its packing density is 1. Below, we will use the insphere volume ratio method to obtain an asymptotic upper bound on the density of the $n$-dimensional cross-polytope.

We say that two non-negative sequences $f(n)$ and $g(n)$ are asymptotically equal if

$$\lim_{n \to \infty} f(n)/g(n) = 1.$$  

The asymptotic equality of $f(n)$ and $g(n)$ will be denoted by $f(n) \sim g(n)$. We write $f(n) \ll g(n)$ if there exists a positive real number $\gamma$ such that $f(n) \leq \gamma \cdot g(n)$ for all $n$.

**Theorem 1** There exists a constant $0 < c < 0.87$ such that for the packing density of the $n$-dimensional regular cross-polytope $X^n$

$$\delta(X^n) \ll c^n$$

holds.

**Proof** Consider the regular cross-polytope

$$X^n := \text{conv} (\pm e_1, \ldots, \pm e_n),$$
where $e_i, i = 1, \ldots, n$, are the standard orthonormal basis vectors of $\mathbb{R}^n$. It is clear that

$$\text{Vol}(X^n) = \frac{2^n}{n!},$$

and the inradius of $X^n$ is

$$r_n = r(X^n) = 1/\sqrt{n}.$$

Consider a packing of cross-polytopes in $\mathbb{R}^n$. Then (1) and (2) and the Stirling formula yield that

$$\delta(X^n) \leq \frac{\text{Vol}(X^n)}{r_n^2 \text{Vol}(B^n)} \delta(B^n)$$

$$\leq \frac{\text{Vol}(X^n)}{r_n^2 \text{Vol}(B^n)} 2^{-0.599n(1+o(1))}$$

$$= \frac{2^n \sqrt{n!} \Gamma \left( \frac{n}{2} + 1 \right)}{n! \sqrt{\pi}} 2^{-0.599n(1+o(1))}$$

$$\sim \frac{1}{\sqrt{2}} \left( \frac{e}{\pi^{10.198}} \right)^{\frac{n}{2}}$$

$$\ll 0.86850^n. \tag{3}$$

Note that the upper bound in (3) is asymptotic in nature and it says nothing about the packing density of $X^n$ in specific dimensions. In order to obtain concrete bounds on $\delta(X^n)$, we must use specific upper bounds for $\delta(B^n)$. The Cohn–Elkies bounds on $\delta(B^n)$ for $5 \leq n \leq 36$ \cite[Table 3 on p. 711]{8} and the almost exact value of $\delta(B^{24})$ by Cohn and Kumar \cite{9} yield by simple computations the upper bounds for $\delta(X^n)$ shown in Table 2.

We will improve on these bounds in Sect. 3.
2 Blichfeldt’s Method of Enlargement and Its Extension

Consider a packing of unit balls in \( \mathbb{R}^n \). Replace each ball by a concentric material ball such that the density at distance \( r \) from the center of the ball is

\[
f_0(r) = \begin{cases} 
1 - \frac{r^2}{2} & \text{for } 0 \leq r \leq \sqrt{2}, \\
0 & \text{for } r > \sqrt{2}.
\end{cases}
\]

Blichfeldt [6] observed that then the density of mass at each point of \( \mathbb{R}^n \) is at most 1. It follows that

\[
\delta(B^n) \leq \frac{\text{Vol}(B^n)}{I_n(f_0)},
\]

where \( I_n(f_0) = \int_{\mathbb{R}^n} f_0(|x|) \, dx \) denotes the mass of a material ball. From this one gets by an elementary calculation that \( \delta(B^n) \leq (n + 2)2^{-(n/2) + 2} \). Blichfeldt also noted that using the density function

\[
f^*(r) = \begin{cases} 
f_0(r) & \text{for } r \geq 1, \\
1 - f_0(2 - r) & \text{for } r \leq 1,
\end{cases}
\]

instead of \( f_0(r) \), it still follows that the density of mass is at most 1 at every point of \( \mathbb{R}^n \), which yields a slight improvement of the bound \( \delta(B^n) \leq (n + 2)2^{-(n/2) + 2} \).

Blichfeldt’s idea was widely used to derive density bounds in other situations. A detailed description of the method can be found, for example, in the book [21, pp. 56–59] by Pach and Agarwal. We describe here the method as presented in [13].

Let \( K \subset \mathbb{R}^n \) be a convex body. A non-negative Lebesgue measurable function \( f : [0, \infty) \to \mathbb{R} \) is a Blichfeldt gauge for \( K \) if it satisfies the following conditions:

(i) \( I_n(f) := \int_{\mathbb{R}^n} f(|x|) \, dx < \infty \).

(ii) If \( \{ \varphi_i : i = 1, 2, \ldots \} \) is a set of isometries of \( \mathbb{R}^n \) such that the collection \( \{ \varphi_i K : i = 1, 2, \ldots \} \) is a packing, then for every \( x \in \mathbb{R}^n \) it holds that

\[
\sum_{i=1}^{\infty} f(|\varphi_i^{-1}(x)|) \leq 1.
\]

For technical reasons, we assume that \( f \) satisfies the following condition:

(iii) There exists an \( \hat{r} > 0 \) with \( f(r) = 0 \) for all \( r > \hat{r} \).

Then Blichfeldt’s method can be summarized as follows:

If \( f \) is a gauge for a convex body \( K \), then

\[
\delta(K) \leq \frac{\text{Vol}(K)}{I_n(f)}.
\]

For \( 0 \leq \varrho \leq r(K) \), the inner parallel domain of \( K \) with radius \( \varrho \) is defined as

\[
K_{-\varrho} := \{ x \in K : \varrho B^n + x \subseteq K \}.
\]
For $x \in \mathbb{R}^n$, let $d(x, K_{-\varrho})$ denote the Euclidean distance of $x$ from $K_{-\varrho}$. It is proved in [13] that if $f$ is a Blichfeldt gauge for $B^n$, then for any $0 < \varrho \leq r(K)$,

$$g_{\varrho}(x) = f\left(\frac{d(x, K_{-\varrho})}{\varrho}\right)$$

is a Blichfeldt gauge for $K$. Thus, writing

$$G(\varrho) = \int_{\mathbb{R}^n} g_{\varrho}(x) \, dx,$$

we have

$$\delta(K) \leq \frac{\text{Vol}(K)}{G(\varrho)}. \quad (6)$$

This method gives an upper bound on $\delta(K)$ for each $0 < \varrho \leq r(K)$. Our objective is to find, or at least estimate, the best such upper bound.

Introduce $\kappa_n = \text{Vol}(B^n)$ and $\omega_n = nk_n$ for the volume and the surface area of the unit ball, respectively. For a convex body $K \subset \mathbb{R}^n$ and a non-negative real number $\lambda$, the radius $\lambda$ parallel domain $K_\lambda = K + \lambda B^n$ of $K$ is the set of points in $\mathbb{R}^n$ whose distance from $K$ is at most $\lambda$. It is a basic fact in the theory of convex bodies that the volume of $K_\lambda$ can be expressed as a polynomial of degree $n$ of $\lambda$ whose coefficients depend only on the convex body $K$. This polynomial is known as Steiner’s formula, and it has different forms depending on how the coefficients are normed. In this paper we will use the following form:

$$\text{Vol}(K + \lambda B^n) = \sum_{j=0}^{n} \lambda^{n-j} \kappa_{n-j} V_j(K),$$

where $V_j(K)$, $j = 0, \ldots, n$, are the intrinsic volumes of $K$ introduced by McMullen [20]. Some of the intrinsic volumes have well-known geometric meaning, for example, $V_n(K) = \text{Vol}(K)$ is the volume, and $2V_{n-1}(K) = S(K)$ is the surface volume of $K$. In general, the intrinsic volumes carry very important geometric information about the convex body $K$. For a simple proof of Steiner’s formula see, for example, [2, pp. 123–125]. For a more extensive treatment of this topic, we refer to the monograph of Schneider [25].

**Lemma 1** Let $I_0(f) := f(0)$ and use the notation introduced above. Then

$$G(\varrho) = \sum_{j=0}^{n} \varrho^j I_j(f) V_{n-j}(K_{-\varrho}). \quad (7)$$

**Outline of the proof** The proof of (7) mimics closely the proof of Steiner’s formula as presented in [2, pp. 123–125]. Since $g_{\varrho}(x)$ is concentrated on $(K_{-\varrho}) + \varrho \hat{r} B^n$, it follows that

$$G(\varrho) = \int_{(K_{-\varrho}) + \varrho \hat{r} B^n} g_{\varrho}(x) \, dx. \quad (8)$$
First, we prove (7) for polytopes. For a polytope $P$, we decompose the parallel domain $(P - \varrho) + \hat{r} B^n$ into disjoint parts using the nearest point map (cf. [2, Formula (6) on p. 124]), and then evaluate the integral (8) separately on each part using the fact that for $j \in \{1, \ldots, n\}$ and $0 < \varrho < r(P)$,

$$\int_{\hat{r} B^j} f \left( \frac{|y|}{\varrho} \right) dy = \varrho^j I_j(f).$$

Finally, we extend the formula for general convex bodies by an approximation argument as in [2, p. 125].

If (7) can be calculated or estimated explicitly, then (6) provides an upper bound for $\delta(K)$. Two such classes of bodies were exhibited in [13]: cylinders and outer parallel domains of segments. We will see in the next section that this technique can provide better bounds for $\delta(X^n)$ than the insphere volume ratio.

3 The Case of the $n$-Dimensional Cross-Polytope

Let $P \subset \mathbb{R}^n$ be a polytope such that all facets of $P$ are tangent to its insphere. In this case we say that $P$ is circumscribed around its insphere. It is not difficult to see that if $P$ is such a polytope, then for all $0 \leq \varrho < r(P)$, the radius $\varrho$ inner parallel domain $P - \varrho$ of $P$ is a polytope that is similar to $P$ with a similarity ratio $(r(P) - \varrho)/r(P)$.

Since the $j$th intrinsic volume is homogeneous of degree $j$, it holds that

$$V_j(P - \varrho) = \left( \frac{r(P) - \varrho}{r(P)} \right)^j V_j(P), \quad j = 0, \ldots, n.$$

Thus, the right-hand side of (7) becomes a polynomial of degree $n$ of $\varrho$ in the case where $P$ is circumscribed around its insphere, that is,

$$G(\varrho) = \sum_{j=0}^{n} \varrho^j I_j(f) \left( \frac{r(P) - \varrho}{r(P)} \right)^{n-j} V_{n-j}(P) \quad (0 < \varrho < r(P)). \tag{9}$$

In particular, $X^n$ is circumscribed about its insphere. Betke and Henk [4] determined the following formula for the $j$th intrinsic volume of $X^n$:

$$V_j(X^n) = 2^{j+1} \binom{n}{j+1} \cdot \frac{\sqrt{j+1}}{j^\frac{1}{2}} \cdot \gamma(n, j), \tag{10}$$

where

$$\gamma(n, j) = \sqrt{\frac{j+1}{\pi}} \int_0^\infty e^{-(j+1)x^2} \left( 2 \int_0^x e^{-y^2} dy \right)^{n-j-1} dx$$

is the outer angle at a $j$-dimensional face of $X^n$.

Although we only defined $G(\varrho)$ for $0 < \varrho < r_n$, in fact, the polynomial on the right-hand side of (7) is well defined for all real $\varrho$, and thus its derivatives of all orders exist at $\varrho = 0$ and $\varrho = r_n$. 

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Observe that \( \frac{\operatorname{Vol}(X^n)}{G(r_n)} \) is equal to the insphere volume ratio bound that one obtains using the upper bound on \( \delta(B^n) \) from the Blichfeldt gauge \( f \). Indeed, since \( G(r_n) = I_n(f)r_n^n \) we get by (6),

\[
\delta(X^n) \leq \frac{\operatorname{Vol}(X^n)}{G(r_n)} = \frac{\operatorname{Vol}(X^n)}{I_n(f)r_n^n} \leq \frac{\operatorname{Vol}(X^n)}{r_n^n \operatorname{Vol}(B^n)} \cdot \frac{\operatorname{Vol}(B^n)}{I_n(f)}.
\]

In the next lemma we will show that, for sufficiently large \( n \), \( \operatorname{Vol}(X^n)/G(\varrho) \) is strictly monotonically increasing at \( \varrho = r_n \), and thus the minimum of \( \operatorname{Vol}(X^n)/G(\varrho) \) over \([0, r_n]\) is strictly smaller than \( \operatorname{Vol}(X^n)/G(r_n) \); hence the modified Blichfeldt method gives a better upper bound on \( \delta(X^n) \) than the insphere volume ratio combined with the upper bound on \( \delta(B^n) \) coming from the Blichfeldt gauge \( f \).

**Lemma 2** If \( n \) is sufficiently large, then

\[
G'(r_n) < 0.
\]  

*Proof* Elementary calculus yields that

\[
G'(r_n) = -r_n^{n-2}I_{n-1}(f)V_1(X^n) + nr_n^{n-1}I_n(f) = r_n^{n-2}(nr_nI_n(f) - I_{n-1}(f)V_1(X^n)).
\]  

(12)

Böröczky and Henk [7] proved that for any fixed \( j \),

\[
\gamma(n, j) \sim \frac{1}{2} \frac{(j + 1)! (\pi \ln n)^{\frac{j}{2}}}{n^{j+1}} \quad \text{as} \quad n \to \infty.
\]

In particular,

\[
V_1(X^n) \sim \sqrt{\pi} \sqrt{n \ln n} \quad \text{as} \quad n \to \infty.
\]

(13)

It follows from property (iii) of \( f \) and the Stirling formula that

\[
\frac{I_n(f)}{I_{n-1}(f)} = \frac{\omega_n}{\omega_{n-1}} \frac{\int_0^\infty f(r)r^{n-1}dr}{\int_0^\infty f(r)r^{n-2}dr} \leq \frac{\kappa_n}{\kappa_{n-1}} \frac{n}{n-1} \frac{\int_0^\infty f(r)r^{n-2}\hat{r}dr}{\int_0^\infty f(r)r^{n-2}dr} \sim \hat{r} \sqrt{\frac{2\pi e}{n}} \quad \text{as} \quad n \to \infty,
\]

and thus

\[
I_n(f) \ll \hat{r} \sqrt{\frac{2\pi e}{n}} I_{n-1}(f).
\]

(14)
Combining (12), (13), and (14), we obtain that

\[
G'(r_n) \ll r_n^{n-2} I_{n-1}(f) \left( \hat{r} \sqrt{\frac{2\pi e}{n} nr_n - \sqrt{2\pi \ln n}} \right)
\]

\[
= \sqrt{\pi} r_n^{n-2} I_{n-1}(f) \left( \hat{r} \sqrt{2e - \sqrt{\ln n}} \right)
\]

\[
< 0
\]

for sufficiently large \( n \).

**4 Using the Original Blichfeldt Gauge Function**

In this section, we will use the Blichfeldt gauge \( f^* \) as defined in (5). Then \( I_0(f^*) = 1 \), and for \( n \geq 1 \),

\[
I_n(f^*) = \frac{2\kappa_n}{n+2} (\sqrt{2})^n (1 + b_n),
\]

where

\[
b_n = \frac{1}{(\sqrt{2})^n(n+1)} - (\sqrt{2} - 1)^{n+1} \left( 1 + \frac{\sqrt{2}}{n+1} \right).
\]

**Lemma 3** \( G'(0) = 0 \), and \( G''(0) > 0 \) when \( n \geq 7 \).

**Proof** Routine calculations show that

\[
G'(0) = -\frac{n}{r_n} I_0(f^*) V_n(X^n) + I_1(f^*) V_{n-1}(X^n)
\]

\[
= -n \sqrt{\frac{2^n}{n!}} + 2 \sqrt{n} \frac{2^n}{(n-1)!} \frac{1}{2}
\]

\[
= 0
\]

and

\[
G''(0) = \frac{n(n-1)}{r_n^2} I_0(f^*) V_n(X^n) - 2 \frac{n-1}{r_n} I_1(f^*) V_{n-1}(X^n) + 2 I_2(f^*) V_{n-2}(X^n)
\]

\[
= 2 I_2(f^*) V_{n-2}(X^n) - \frac{n-1}{r_n} I_1(f^*) V_{n-1}(X^n)
\]

\[
= \frac{n^2}{(n-2)!} \left( \frac{\sqrt{n-1}}{2} \arccos \left( 1 - \frac{2}{n} \right) 1.062097 - 1 \right).
\]

Now, it is easy to check that \( G''(0) > 0 \) when \( n \geq 7 \).

Since \( \text{Vol}(X^n)/G(0) = 1 \), it follows by Lemma 3 that \( \text{min} \text{Vol}(X^n)/G(\varrho) < 1 \) for \( n \geq 7 \). Thus we have the following

**Theorem 2** For \( n \geq 7 \) there are constants \( c_n < 1 \) such that \( \delta(X^n) \leq c_n \).
Table 3  Numerically estimated upper bounds on $\delta(X^n)$ using the Blichfeldt gauge $f^*$ for $7 \leq n \leq 36$

| $n$ | $\delta(X^n) \leq$ | $n$ | $\delta(X^n) \leq$ |
|-----|---------------------|-----|---------------------|
| 7   | 0.99805             | 22  | 0.37264             |
| 8   | 0.98606             | 23  | 0.33850             |
| 9   | 0.96188             | 24  | 0.30697             |
| 10  | 0.92730             | 25  | 0.27794             |
| 11  | 0.88500             | 26  | 0.25129             |
| 12  | 0.83754             | 27  | 0.22690             |
| 13  | 0.78705             | 28  | 0.20462             |
| 14  | 0.73524             | 29  | 0.18448             |
| 15  | 0.68339             | 30  | 0.16586             |
| 16  | 0.63247             | 31  | 0.14908             |
| 17  | 0.58317             | 32  | 0.13398             |
| 18  | 0.53596             | 33  | 0.12017             |
| 19  | 0.49116             | 34  | 0.10770             |
| 20  | 0.44896             | 35  | 0.09647             |
| 21  | 0.40944             | 36  | 0.08635             |

We note that since $G'(r_n) < 0$, the minimum of $\text{Vol}(X^n)/G(\varrho)$ is attained at an interior point of the interval $[0, r_n]$ and the method yields an upper bound on $\delta(X^n)$ that is better than the one obtained from the insphere volume ratio combined with the Blichfeldt upper bound on $\delta(X^n)$.

Although the quantities in (10) cannot be calculated explicitly, they can be approximated by numerical methods. By such numerical calculations, one obtains for $G(\varrho)$ a degree $n$ polynomial in $\varrho$ whose maximum may be approximated (again by numerical methods). The results of our calculations are described in Tables 3 and 4, and in Fig. 1 they are compared to the upper bounds in Table 2.

Table 4  Numerically estimated upper bounds on $\delta(X^n)$ using the Blichfeldt gauge $f^*$

| $n$ | $\delta(X^n) \leq$ | $n$ | $\delta(X^n) \leq$ |
|-----|---------------------|-----|---------------------|
| 40  | $5.52108 \times 10^{-2}$ | 140 | $1.98099 \times 10^{-7}$ |
| 50  | $1.72421 \times 10^{-2}$ | 150 | $5.36214 \times 10^{-8}$ |
| 60  | $5.19017 \times 10^{-3}$ | 160 | $1.44520 \times 10^{-8}$ |
| 70  | $1.52250 \times 10^{-3}$ | 170 | $3.88033 \times 10^{-9}$ |
| 80  | $4.38143 \times 10^{-4}$ | 180 | $1.03837 \times 10^{-9}$ |
| 90  | $1.24242 \times 10^{-4}$ | 190 | $2.77031 \times 10^{-10}$ |
| 100 | $3.48195 \times 10^{-5}$ | 200 | $7.37113 \times 10^{-11}$ |
| 110 | $9.66572 \times 10^{-6}$ | 250 | $9.51567 \times 10^{-14}$ |
| 120 | $2.66200 \times 10^{-6}$ | 500 | $2.25312 \times 10^{-28}$ |
| 130 | $7.28254 \times 10^{-7}$ | 750 | $4.01494 \times 10^{-43}$ |
| 1000 | $6.36493 \times 10^{-58}$ |
Fig. 1 Comparison of upper bounds on $\delta(X^n)$ obtained from insphere volume ratio (diamonds) and the Blichfeldt method with $f^*$ (dots) for $7 \leq n \leq 36$.

We note that the numerical calculations suggest that the value of $\rho$ at which the minimum of $\text{Vol}(X^n)/G(\rho)$ is attained approaches $\frac{2}{3\sqrt{n}}$ as $n \to \infty$. Furthermore, it also appears from numerical calculations that for sufficiently large $n$, the terms in which the exponent of $\rho$ is around $2n/3$ dominate $G\left(\frac{2}{3\sqrt{n}}\right)$.

Finally, we remark that we fitted an exponential function on the numerical results obtained from $f^*$ and got the following approximate asymptotics:

$$\delta(X^n) \ll 0.87434^n.$$

5 Concluding Remarks

The upper bounds obtained from the Blichfeldt method depend on the gauge function used. In Sect. 4, we used $f^*$ because $I_n(f^*)$ can be evaluated explicitly for all $n$. However, $f^*$ is not the best possible such function, although in small dimensions it provides better upper bounds on $\delta(X^n)$ than other known Blichfeldt gauges for $B^n$. For example, it is remarked in [13] that Levensteın [19] introduced the following Blichfeldt gauge derived from spherical codes. Let $M(n,\phi)$ denote the maximum number of points on $S^{n-1}$ with the property that their pairwise angular distances are not smaller than $\phi$. Then the following function

$$f_n(x) = \begin{cases} \frac{1}{M(n,\phi)} & \text{for } 0 \leq |x| < \sqrt{\frac{2}{1-\cos\phi}}, \\ 0 & \text{for } \sqrt{\frac{2}{1-\cos\phi}} \leq |x| \end{cases}$$

is a Blichfeldt gauge for $B^n$ if $\pi/3 \leq \phi \leq \pi$. For a brief explanation why $f_n$ is a Blichfeldt gauge, see [13, p. 726]. Kabatjanskı and Levensteın (cf. Formula (52) in [18]) proved that
\[ M(n, \varphi) \leq \frac{4(k+n-2)}{1-t_{1,k+1}^{\alpha,\alpha}} \quad \text{if} \quad \cos \varphi \leq t_{1,k}^{\alpha,\alpha}, \tag{15} \]

where \( t_{1,k}^{\alpha,\alpha} \) denotes the largest root of the Jacobi polynomial of degree \( k \) with parameter \( \alpha = (n-3)/2 \). For a definition of Jacobi polynomials see, for example, Formula (23) in [18]. Using (15), one can obtain Blichfeldt gauge functions for \( X^n \) in any dimension which yield concrete upper bounds on \( \delta(X^n) \). However, these Blichfeldt gauges do not give better bounds on \( \delta(X^n) \) than \( f^* \) up to (at least) dimension 300. On the other hand, in dimension 500, one obtains a better bound using (15) than with \( f^* \). We note that the calculations with (15) become computationally demanding for higher dimensions.

Kabatjanskii and Levenšteǐn [18] derived from (15)

\[ M(n, \varphi) \leq \left( \sin(\varphi/2) \right)^{-n} 2^{-\sqrt{0.599 + o(1)n}}, \tag{16} \]

which holds for \( \varphi \leq 63^0 \) and is the best asymptotic upper bound on \( M(n, \varphi) \). We note that if one uses the Blichfeldt gauge \( f_n \), then (16) and Blichfeldt’s theorem yield the Kabatjanskii–Levenšteǐn upper bound (1) on \( \delta(B^n) \). Together with (11), this indicates that the Blichfeldt method may provide a better asymptotic upper bound on \( \delta(X^n) \) than the insphere volume ratio combined with (1).

We calculated upper bounds on \( \delta(X^n) \) for \( n \leq 1000 \) using the Blichfeldt gauge \( f_n \) derived from (16) (omitting the unknown \( o(n) \) term). The calculations suggest that the minimum of \( \text{Vol}(X^n)/G(\varphi) \) is attained at a value \( \varphi \) which tends to be roughly \( 0.767 \ldots \times r_n \) as \( n \to \infty \). We fitted an exponential function on the results, based on which we conjecture that

\[ \delta(X^n) \ll 0.82886^n. \]

In order to prove this we would need two things: an estimate on the \( o(n) \) term in (16) and asymptotic formulae for the intrinsic volumes \( V_j(X^n) \) for all \( j = 0, \ldots, n \). To the best of our knowledge, no such asymptotic formulae are known at present.

Finally, we believe that the modified Blichfeldt method will not provide a non-trivial upper bound on the packing density of the regular \( n \)-simplex \( T^n \). This belief is supported by the following evidence. Using the lower bound \( 2(n-1)2^{-n}\zeta(n) \) of Ball [3] on \( \delta(B^n) \) (\( \zeta(n) \) denotes the Riemann zeta function), a similar calculation as in the proof of Theorem 1 shows that the insphere volume ratio method gives an upper bound for \( \delta(T^n) \) that is larger than 1 in any dimension and it diverges to infinity exponentially fast as \( n \to \infty \). Similar formulas to (10) are known (cf. Ruben [23] and Hadwiger [16]) for the intrinsic volumes of \( T^n \). We made some numerical experiments in specific dimensions using \( f^* \) and \( f_n \) and have found that the upper bound on \( \delta(T^n) \) given by the modified Blichfeldt method was larger than 1 in each case.

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