A LOWER BOUND ON THE PROPORTION OF MODULAR ELLIPTIC CURVES
OVER GALOIS CM FIELDS

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Abstract. We calculate an explicit lower bound on the proportion of elliptic curves that are modular over any Galois CM field not containing \( \zeta_5 \). Applied to imaginary quadratic fields, this proportion is at least \( \frac{2}{5} \). Applied to cyclotomic fields \( \mathbb{Q}(\zeta_n) \) with \( 5 \nmid n \), this proportion is at least \( 1 - \epsilon \) with only finitely many exceptions of \( n \), for any choice of \( \epsilon > 0 \).

1. Introduction

Let \( K \) be a number field. For \( X > 0 \), denote \( \mathcal{O}_{K,X} \) to be the set of \( \alpha \in \mathcal{O}_K \) such that for every embedding \( \sigma : K \to \mathbb{C} \) one has \( |\sigma(\alpha)| < X \). We write \( \mathcal{E}_X \) for the set of pairs \( (A, B) \in \mathcal{O}_{K,X} \times \mathcal{O}_{K,X} \) such that \( \Delta(A, B) = -16(4A^3 + 27B^2) \neq 0 \). For \( (A, B) \in \mathcal{E}_X \), we write \( E_{A,B} \) for the elliptic curve over \( K \) given by the equation \( y^2 = x^3 + Ax + B \). We say that an elliptic curve \( E \) over \( K \) is modular if either \( E \) has complex multiplication, or there exists a cuspidal regular algebraic automorphic representation \( \pi \) of \( \text{GL}_2(\mathbb{A}_K) \) such that \( E \) and \( \pi \) have the same \( L \)-function.

The purpose of this article is to prove the following theorem.

Theorem 1. Let \( K \) be a Galois CM field with \( \zeta_5 \notin K \). Then

\[
M_K := \liminf_{X \to \infty} \frac{|\{(A, B) \in \mathcal{E}_X : E_{A,B} \text{ is modular}\}|}{|\mathcal{E}_X|} \geq \left(1 - \frac{1}{5^f}\right)^{2r}
\]

where \( 5\mathcal{O}_K = (p_1 \cdots p_r)^e \) with \( p_i \neq p_j \) and \( e, f \) are the ramification and inertial indices of 5 in \( K \).

This theorem calculates an explicit value for the lower bound, thus offering an improvement, at least in the case of a Galois extension, to the original result by Allen, Khare, and Thorne in Theorem 10.1 of their article [AKT19] which showed that \( M_K > 0 \) for every CM field \( K \) with \( \zeta_5 \notin K \).

2. Preliminaries

Let \( K \) be a number field. Let \( E \) be an elliptic curve over \( K \). Let \( p \) be a prime. Let \( G_K := \text{Gal}(\overline{K}/K) \). Let \( \overline{\rho}_{E,p} : G_K \to \text{Aut}(E[p]) \cong \text{GL}_2(\mathbb{F}_p) \) be the Galois representation of \( G_K \) acting on the \( p \)-torsion points of \( E \) in \( \overline{K} \). Let \( \overline{\rho} : G_K \to \text{GL}_2(\mathbb{F}_p) \) be any continuous representation. We say that a prime \( l \neq p \) is decomposed generic for \( \overline{\rho} \) if it splits completely in \( K \) and for any \( v \mid l \) in \( K \), \( \overline{\rho} \) is unramified at \( v \) and the eigenvalues \( \alpha_v, \beta_v \) of \( \overline{\rho}(\text{Frob}_v) \) satisfy \( \alpha_v \beta_v^{-1} \notin \{1, l, l^{-1}\} \). We say that \( \overline{\rho} \) is decomposed generic if there is a prime \( l \neq p \) that is decomposed generic for \( \overline{\rho} \). The original definition traces back to [CS17].

Our main tools will be the following lemmas.
Lemma 2 (Corollary 9.13 [AKT19]). Let $K$ be a CM field and let $E$ be an elliptic curve over $K$ satisfying the following conditions:

1. $\overline{\rho}_{E,5}|_{G_K(\zeta_5)}$ is absolutely irreducible. If $\overline{\rho}_{E,5}(G_K(\zeta_5)) = SL_2(\mathbb{F}_5)$, then $\zeta_5 \notin K$.
2. For each place $p | 5$ of $K$, $E_{K_p}$ is ordinary (i.e. has good ordinary reduction or potentially multiplicative reduction).
3. $\overline{\rho}_{E,5}$ is decomposed generic.

Then $E$ is modular.

Lemma 3 (Lemma 2.3 [AN20]). Let $K/\mathbb{Q}$ be a finite Galois extension and let $\overline{\rho}: G_K \to GL_2(\mathbb{F}_l)$ be a continuous representation with $l > 3$. If $\overline{\rho}(G_K) \supset SL_2(\mathbb{F}_l)$, then $\overline{\rho}$ is decomposed generic.

Fix a norm $\| - \|$ on $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_K^2 = \mathbb{R}^{2[K:\mathbb{Q}]}$. For $X > 0$ and an integer $m$, define the sets
\[
B_K(X) := \{(A, B) \in \mathcal{O}_K^2 : \Delta(A, B) \neq 0, \| (A, B) \| \leq X\}
\]
\[
B_{K,m}(X) := \{(A, B) \in B_K(X) : \overline{\rho}_{E,A,B,m}(G_K) \not\supset SL_2(\mathbb{Z}/m\mathbb{Z})\}
\]

Lemma 4 (Proposition 5.7 [Zyw10]). For a positive integer $m$,
\[
\frac{|B_{K,m}(X)|}{|B_K(X)|} \ll_{K,\| - \|, m} \frac{\log(X)}{X^{[K:\mathbb{Q}]/2}}
\]

Lemma 5 (Theorem 4.1 [Sil09]). Let $q$ be a power of $p$. Let $\mathbb{F}_q$ be a finite field of characteristic $p \geq 3$. Let $E/\mathbb{F}_q$ be an elliptic curve given by a Weierstrass equation
\[
E : y^2 = f(x)
\]
where $f(x) \in \mathbb{F}_q[x]$ is a cubic polynomial with distinct roots in $\mathbb{F}_q$. Then $E$ is supersingular if and only if the coefficient of $x^{p-1}$ in $f(x)^{(p-1)/2}$ is zero.

3. Proof of the main result

Proof of Theorem 7 First, suppose $\overline{\rho}_{E,5}(G_K(\zeta_5)) \supset SL_2(\mathbb{F}_5)$. Then $\overline{\rho}_{E,5}|_{G_K(\zeta_5)}$ is absolutely irreducible. Let $L/K$ be any abelian extension, and suppose $\overline{\rho}_{E,5}(G_K) \supset SL_2(\mathbb{F}_5)$. There is a surjective map
\[
\text{Gal}(L/K) \to \overline{\rho}_{E,5}(G_K)/\overline{\rho}_{E,5}(G_L)
\]
induced by $\overline{\rho}_{E,5}$. Since $\text{Gal}(L/K)$ is abelian, and $SL_2(\mathbb{F}_5)$ is perfect, it follows that $SL_2(\mathbb{F}_5) \subset \overline{\rho}_{E,5}(G_L)$.

By letting $L = K(\zeta_5)$, we can conclude that $\overline{\rho}_{E,5}(G_K) \supset SL_2(\mathbb{F}_5)$ if and only if $\overline{\rho}_{E,5}(G_K(\zeta_5)) \supset SL_2(\mathbb{F}_5)$.

For $x \in \mathcal{O}_K$, define $|x|_{\infty} := \max_{\sigma} |\sigma x|$ where $\sigma$ varies over all embeddings $\sigma : K \to \mathbb{C}$ and $| - |$ is the complex absolute value. We want to extend $| - |_{\infty}$ to $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_K$ as a norm. Consider the embedding
\[
\lambda : \mathcal{O}_K \to \mathbb{R}^r \times \mathbb{C}^s \cong \mathbb{R}^n
\]
\[
x \mapsto (\sigma_1 x, \ldots, \sigma_r x, \tau_1 x, \ldots, \tau_s x)
\]
where \( \sigma_1, \ldots, \sigma_r \) are the real embeddings and \( \tau_1, \ldots, \tau_s \) are the non-real embeddings, and \( n = [K : \mathbb{Q}] \).

By choosing the norm \( \| (x_1, \ldots, x_r, z_1, \ldots, z_s) \|_\lambda := \max \{|x_1|, \ldots, |x_r|, |z_1|, \ldots, |z_s|\} \) on \( \mathbb{R}^n \), one has that \( \| x \|_\infty = \| \lambda(x) \|_\lambda \) for all \( x \in \mathcal{O}_K \). Since \( \lambda(\mathcal{O}_K) \) is a lattice in \( \mathbb{R}^n \), the map \( \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_K \to \mathbb{R}^n \) which sends \( t \otimes x \mapsto t \cdot \lambda(x) \) is an isomorphism of \( \mathbb{R} \)-vector spaces. By pulling back the norm \( \| - \|_\lambda \) from \( \mathbb{R}^n \) to \( \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_K \) along this isomorphism, and calling this norm \( \| - \|_\infty \) on \( \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_K \), we obtain that \( \| 1 \otimes x \|_\infty = \| \lambda(x) \|_\lambda = \| x \|_\infty \) and hence we have extended \( \| - \|_\infty \) to \( \| - \|_\infty \) on \( \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_K \). Finally, we define a norm \( \| - \| \) on the product \( \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_K^2 = (\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_K)^2 \) by setting \( \|(A, B)\| := \max\{\|A\|_\infty, \|B\|_\infty\} \). This is the norm we will use with Lemma 4.

Let us define two more sets.

\[
\mathcal{E}'_X = \{(A, B) \in \mathcal{O}_{K,X^4} \times \mathcal{O}_{K,X^6}\} \\
B'_K(X) = \{(A, B) \in \mathcal{O}_K^2 : \|(A, B)\| \leq X\}
\]

Let \( \mathcal{C}_X = \{A \in \mathbb{R}^n : \|A\|_\lambda \leq X\} \) be the closed ball of radius \( X \). Indeed, \( \mathcal{C}_X = \mathcal{X}\mathcal{C}_1 \). Since the boundary of \( \mathcal{C}_1 \) is \((n-1)\)-Lipschitz parameterizable, Lemma 2 from Chapter 6 of [Mar18] tells us that

\[
|\lambda(\mathcal{O}_K) \cap \mathcal{C}_X| = |\lambda(\mathcal{O}_K) \cap \mathcal{X}\mathcal{C}_1| = \frac{\text{vol}(\mathcal{C}_1)}{\text{vol}(\mathbb{R}^n/\lambda(\mathcal{O}_K))} X^n + O(X^{n-1})
\]

Therefore, there are constants \( \kappa_1, \kappa_2 > 0 \) such that

\[
|\mathcal{E}'_X| \sim \kappa_1 X^{4n} X^{6n} \\
|B'_K(X^6)| \sim \kappa_2 X^{6n} X^{6n}
\]

For each \( A \in \mathcal{O}_K \), \( \Delta(A, B) = 0 \) is a quadratic equation in \( B \) satisfied by at most two values of \( \mathcal{O}_K \). Therefore, there are constants \( \kappa_3, \kappa_4 > 0 \) such that

\[
\kappa_1 X^{4n} X^{6n} - \kappa_3 X^{4n} \leq |\mathcal{E}_X| \leq |\mathcal{E}'_X| \sim \kappa_1 X^{4n} X^{6n} \\
\kappa_2 X^{6n} X^{6n} - \kappa_4 X^{4n} \leq |B_K(X^6)| \leq |B'_K(X^6)| \sim \kappa_2 X^{6n} X^{6n}
\]

Therefore, \( |\mathcal{E}_X| \sim \kappa_1 X^{4n} X^{6n} \) and \( |B_K(X^6)| \sim \kappa_2 X^{6n} X^{6n} \). By combining our calculations with the estimate given in Lemma 4, we can obtain the following asymptotic bound.

\[
\left\{ (A, B) \in \mathcal{E}_X : \mathbf{p}_{E,A,B,5}(G_K) \not\equiv \text{SL}_2(\mathbb{F}_5) \right\} \ll \frac{|B_{K,5}(X^6)|}{X^{6n/2}} \frac{1}{X^{2n}} \sim \log(X^6) X^{2n/2} \to 0
\]

Therefore, condition (1) of Lemma 2 is automatically satisfied for 100% of elliptic curves over any number field \( K \). When \( K \) is Galois, Lemma 3 immediately yields that the same curves satisfying condition (1) also satisfies condition (3). Therefore, by Lemma 2, for a Galois CM field \( K \) with \( \zeta_5 \not\in K \), the limit that is calculated in Theorem 1 depends only on the proportion of elliptic curves \( E \) over \( K \) such that \( E_{K_p} \) is ordinary for each place \( p | 5 \).
Let $E : y^2 = x^3 + Ax + B$ with $(A, B) \in \mathcal{O}_K^2$. Let $5\mathcal{O}_K = (p_1 \cdots p_r)^e$ be the unique factorization. For $E$ to have good reduction at $p$ for each $p \mid 5$ it is sufficient that $\Delta(A, B) = -16(4A^3 + 27B^2) \neq 0 \pmod{p_i}$ for all $1 \leq i \leq r$. Note that this is not a necessary condition because the equation for $E$ need not be a minimal integral model at $p_i$ for each $1 \leq i \leq r$. Let $R = p_1 \cdots p_r$ denote the radical of $5\mathcal{O}_K$.

\[
\left(\frac{\mathcal{O}_K}{R}\right)^2 = \left(\frac{\mathcal{O}_K}{p_1}\right)^2 \times \cdots \times \left(\frac{\mathcal{O}_K}{p_r}\right)^2 = \left(\frac{\mathcal{O}_K}{p_1}\right)^2 \times \cdots \times \left(\frac{\mathcal{O}_K}{p_r}\right)^2 = \mathbb{F}_{5^r} \times \cdots \times \mathbb{F}_{5^r}^2.
\]

Working over the finite field $\mathbb{F}_{5^r}$, the equation $\Delta(A, B) = 0$ defines a singular elliptic curve over $\mathbb{F}_{5^r}$ with a single cusp at the origin. By Exercise 3.5 of [Sil09], $\Delta(A, B) \mid \mathbb{O}^2_{5^r}$ includes the point at infinity, and does not include the singular point $(0, 0)$. It follows that the number of pairs in $(A, B) \in \mathbb{F}_{5^r}^2$ such that $\Delta(A, B) = 0$ over $\mathbb{F}_{5^r}$ is $5^r - 1 + 1 = 5^r$ and hence $(5^r)^2 - 5^r$ pairs in $\mathbb{F}_{5^r}^2$ describe non-singular curves over $\mathbb{F}_{5^r}$. Therefore, the number of classes in $(\mathcal{O}_K/R)^2$ describing elliptic curves over $K$ with good reduction at every $p \mid 5$ is at least

\[
((5^r)^2 - 5^r)^r = (5^r(5^r - 1))^r.
\]

Next, we use Lemma 5 to find out which of these elliptic curves with good reduction are also ordinary.

\[
(x^3 + Ax + B)^2 = x^6 + 2Ax^4 + 2Bx^3 + A^2x^2 + 2ABx + B^2
\]

Again, let us first work over the finite field $\mathbb{F}_{5^r}$. Suppose $\Delta(A, B) \neq 0$. Looking at the $x^4$ coefficient in the above expansion, we see that $E$ is ordinary if and only if $A \neq 0$. Consequently, $E$ is supersingular if and only if $A = 0$ whence necessarily $B \neq 0$. Therefore, out of the $(5^r)^2 - 5^r$ pairs in $\mathbb{F}_{5^r}^2$ describing non-singular curves over $\mathbb{F}_{5^r}$, at most $5^r - 1$ of them correspond to supersingular curves. It follows that at least \((5^r)^2 - (5^r - 1) = (5^r - 1)^2\) pairs in $\mathbb{F}_{5^r}^2$ describe non-singular curves over $\mathbb{F}_{5^r}$ which are also ordinary. Therefore, the number of classes in $(\mathcal{O}_K/R)^2$ describing elliptic curves over $K$ with good and ordinary reduction at every $p \mid 5$ is at least

\[
(5^r - 1)^2r
\]

Finally, just divide by the number of elements in $(\mathcal{O}_K/R)^2$.

\[
\frac{(5^r - 1)^2r}{(5^r)^{2r}} = \left(1 - \frac{1}{5^r}\right)^{2r}
\]

4. Applications

**Corollary 6.** Let $K$ be a Galois CM field with $\zeta_5 \notin K$. Then

$$M_K \geq (4/5)^{2[K:Q]}$$

**Corollary 7.** Let $K = \mathbb{Q}(\sqrt{-n})$ with $n > 0$ be an imaginary quadratic field. Then

1. $M_K \geq (4/5)^4 \geq 0.409$ if 5 splits in $K$.
2. $M_K \geq (24/25)^2 \geq 0.921$ if 5 is inert in $K$. 

(3) $M_K \geq (4/5)^2 = 0.64$ if 5 ramifies in $K$.

Corollary 8. Let $K = \mathbb{Q}(\zeta_n)$ with $n \geq 2$ be a cyclotomic field and $5 \nmid n$. Then for every $\varepsilon > 0$ there exists $N$ such that for all $n \geq N$, $M_K \geq 1 - \varepsilon$.

Proof. Indeed, $f$ is the multiplicative order of 5 (mod $n$). Trivially, $5^f \geq n + 1$ and $f \geq \log_5(n + 1)$. Therefore, $r = \varphi(n)/f \leq n/\log_5(n + 1)$.

$$
\left(1 - \frac{1}{5^f}\right)^{2r} \geq \left(1 - \frac{1}{n + 1}\right)^{2n/\log_5(n+1)}
$$

This function is increasing, and tends to 1 as $n \to \infty$. \qed

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References

[AKT19] Patrick B. Allen, Chandrashekhar Khare, and Jack A. Thorne. Modularity of GL$_2(F_p)$-representations over CM fields, 2019.

[AN20] Patrick B. Allen and James Newton. Monodromy for some rank two Galois representations over CM fields. Doc. Math., 25:2487–2506, 2020.

[CS17] Ana Caraiani and Peter Scholze. On the generic part of the cohomology of compact unitary Shimura varieties. Ann. of Math. (2), 186(3):649–766, 2017.

[Mar18] Daniel A. Marcus. Number fields. Universitext. Springer, Cham, 2018. Second edition of [ MR0457396], With a foreword by Barry Mazur.

[Sil09] Joseph H. Silverman. The arithmetic of elliptic curves, volume 106 of Graduate Texts in Mathematics. Springer, Dordrecht, second edition, 2009.

[Zyw10] David Zywina. Elliptic curves with maximal Galois action on their torsion points. Bull. Lond. Math. Soc., 42(5):811–826, 2010.