Asymptotic expansions of weighted prime power counting functions

Jesse Elliott
California State University, Channel Islands
jesse.elliott@csuci.edu

October 6, 2020

Abstract

We prove several asymptotic continued fraction expansions of π(x), Π(x), li(x), Ri(x), and related functions, where π(x) is the prime counting function, Π(x) = \sum_{k=1}^{\infty} \frac{1}{k} \pi(\sqrt[k]{x}) is the Riemann prime counting function, and Ri(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} \text{li}(\sqrt[k]{x}) is Riemann’s approximation to the prime counting function. We also determine asymptotic continued fraction expansions of the function \sum_{p \leq x} p^s for all s ∈ \mathbb{C} with Re(s) > -1, and of the functions \sum_{a^x < p \leq a^{x+1}} \frac{1}{p} and \log \prod_{a^x < p \leq a^{x+1}} (1 - 1/p)^{-1} for all real numbers a > 1. We also determine the first few terms of an asymptotic continued fraction expansion of the function π(ax) − π(bx) for a > b > 0. As a corollary of these results, we determine the best rational approximations of the “linearized” versions of these various functions.

Keywords: prime counting function, asymptotic expansion, continued fraction.

MSC: 11N05, 30B70, 44A15

Contents

1 Introduction 2
  1.1 Summary ................................................. 2
  1.2 Asymptotic continued fraction expansions ................. 3

2 Relative asymptotic expansions 5
  2.1 The Riemann prime counting function ...................... 5
  2.2 Riemann’s approximation to the prime counting function .... 6
  2.3 Prime power counting functions .......................... 11
3 Asymptotic continued fraction expansions
  3.1 Weighted prime power counting functions ............................................ 12
  3.2 Sums of $s$th powers of primes ............................................................. 14
  3.3 Functions related to Mertens’ theorems ..................................................... 17
  3.4 $\pi(ax) - \pi(bx)$ for $a > b > 1$ ............................................................. 23

1 Introduction

1.1 Summary
This paper concerns the asymptotic behavior of the function $\pi : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ that for any $x > 0$ counts the number of primes less than or equal to $x$:

$$\pi(x) = \# \{ p \leq x : p \text{ is prime} \}, \quad x > 0.$$ 

The function $\pi(x)$ is known as the prime counting function. The celebrated prime number theorem, proved independently by de la Vallée Poussin [4] and Hadamard [7] in 1896, states that

$$\pi(x) \sim \frac{x}{\log x} \quad (x \to \infty),$$

where $\log x$ is the natural logarithm. It is well known that this is just the first term of a (divergent) asymptotic expansion of $\pi(x)$, namely,

$$\frac{\pi(x)}{x} \sim \sum_{n=1}^{\infty} \frac{n!}{(\log x)^n} \quad (x \to \infty).$$

As shown in [3, Theorem 1.1], this can be reinterpreted as the (divergent) asymptotic continued fraction expansions

$$\frac{\pi(x)}{x} \sim \frac{1}{\log x} - \frac{1}{1 - \frac{1}{\log x}} - \frac{1}{1 - \frac{2}{\log x}} - \frac{2}{1 - \frac{3}{\log x}} - \frac{3}{1 - \frac{4}{\log x}} - \frac{4}{1 - \frac{5}{\log x}} - \cdots \quad (x \to \infty)$$

and

$$\frac{\pi(x)}{x} \sim \frac{1}{\log x - 1} - \frac{1}{\log x - 3} - \frac{4}{\log x - 5} - \frac{9}{\log x - 7} - \frac{16}{\log x - 9} - \cdots \quad (x \to \infty).$$

In this paper, we prove similar asymptotic continued fraction expansions of various weighted prime counting functions and their smooth approximations.

Specifically, we prove several asymptotic continued fraction expansions of $\pi(x)$, $\Pi(x)$, $\text{li}(x)$, $\text{Ri}(x)$, and related functions, where $\pi(x)$ is the prime counting function, $\Pi(x) = \sum_{k=1}^{\infty} \frac{1}{k} \pi(\sqrt[k]{x})$ is the Riemann prime counting function, and $\text{Ri}(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \text{li}(\sqrt[k]{x})$ is Riemann’s approximation to the prime counting function. We also determine asymptotic continued fraction expansions of the function $\sum_{p \leq x} p^s$ for all $s \in \mathbb{C}$ with $\text{Re}(s) > -1$, and of the functions $\sum_{a^x < p \leq a^{x+1}} \frac{1}{p}$ and $\log \prod_{a^x < p \leq a^{x+1}} (1 - 1/p)^{-1}$ for all real numbers $a > 1$. We also
determine the first few terms of an asymptotic continued fraction expansion of the function 
\( \pi(ax) - \pi(bx) \) for \( a > b > 0 \). As a corollary of these results, we determine the best rational 
approximations of the “linearized” versions of these various functions.

This paper is a sequel to the paper \([3]\), and the definitions and results therein will be 
assumed here. Thus, for example, we require the notion of an asymptotic expansion, and 
that of an asymptotic continued fraction expansion, over some unbounded subset \( \mathcal{X} \) of \( \mathbb{C} \). We 
also require the notions of a Jacobi continued fraction, a Stieltjes continued fraction, and a 
best rational function approximation of a function.

The paper \([3]\) focuses on divergent asymptotic continued fraction expansions. This paper 
deals also with convergent asymptotic continued fraction expansions. In Section 1.2, we make 
a few general observations about such expansions. In Section 2, we prove various asymptotic 
expansions of weighted prime power counting functions relative to each other. Some of these 
asymptotic expansions are easily verified (e.g., \( \Pi(x) \sim \sum_{n=1}^{\infty} \frac{1}{n} \pi(\sqrt[n]{x}) \) \( x \to \infty \)), but others, 
especially Propositions 2.5 and 2.7 are undoubtedly worth making explicit. Finally, in 
Section 3, we apply the results of Section 2 and of the paper \([3]\) to prove several asymptotic 
continued fraction expansions of various weighted prime power counting functions and their 
smooth approximations.

1.2 Asymptotic continued fraction expansions

The following result is an immediate corollary of \([3\) Theorems 2.4 and 2.9].

**Proposition 1.1.** Let \( f(z) \) be a complex-valued function defined on some unbounded subset 
\( \mathcal{X} \) of \( \mathbb{C} \), and let \( \mu \) be a measure on \( \mathbb{R} \) with infinite support and finite moments \( \mu_k = m_k(\mu) = \int_{-\infty}^{\infty} t^k d\mu \in \mathbb{R} \). Then the following conditions are equivalent.

1. One has the asymptotic expansion 
\[ f(z) \sim \sum_{k=0}^{\infty} \frac{\mu_k}{z^{k+1}} \quad (z \to \infty) \]

of \( f(z) \) over \( \mathcal{X} \).

2. \( f(z) \) has an asymptotic Jacobi continued fraction expansion 
\[ f(z) \sim \frac{a_1}{z + b_1} - \frac{a_2}{z + b_2} - \frac{a_3}{z + b_3} - \cdots \quad (z \to \infty) \]

such that the \( n \)th approximant \( w_n(z) \) of the continued fraction for all \( n \geq 1 \) has the 
asymptotic expansion 
\[ w_n(z) \sim \sum_{k=0}^{2n-1} \frac{\mu_k}{z^{k+1}} \quad (z \to \infty) \]

of order \( 2n \) at \( z = \infty \), where \( a_n, b_n \in \mathbb{R} \) and \( a_n > 0 \) for all \( n \).

If the conditions above hold, then \( f(z) \) and the sequences \( \{a_n\} \) and \( \{b_n\} \) satisfy the equivalent 
conditions (2)(a)-(e) of \([3\) Theorem 2.4], and so, for example, the \( w_n(z) \) are precisely the 
best rational approximations of \( f(z) \) over \( \mathcal{X} \).
The hypotheses on \( f(z) \) of the proposition can be achieved, at least over any subset \( \mathfrak{x} \) of \( C_{\delta,\varepsilon} = \{ z \in \mathbb{C} : \delta \leq |\text{Arg}(z)| \leq \pi - \varepsilon \} \), for any fixed \( \delta, \varepsilon > 0 \), if one has

\[
f(z) = S_\mu(z) + O\left(\frac{1}{z^k}\right) \quad (z \to \infty)_{\mathfrak{x}}
\]

for all integers \( k \), where \( S_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{z-t} \) denotes the **Stieltjes transform** of \( \mu \). However, in cases relevant to various prime counting functions one seeks an asymptotic expansion of some function \( f(x) \) over \( \mathbb{R}_{>0} \), not over \( C_{\delta,\varepsilon} \). In these cases, one would have to verify the asymptotic expansion \( f(x) \sim \sum_{k=0}^{\infty} \frac{\mu_k}{x^{k+1}} \quad (x \to \infty) \) over \( \mathbb{R}_{>0} \) through some other means. As discussed in [3], this exact situation occurs, for example, with the function \( f(x) = \pi(e^x) \), since the asymptotic expansion \( \pi(e^x) \sim \sum_{k=0}^{\infty} \mu_k(\gamma_0) x^{k+1} \quad (x \to \infty) \) follows from the prime number theorem with error term, where \( \gamma_0 \) is the exponential distribution with weight parameter 1 supported on \([0, \infty)\).

If, however, \( \mu \) a finite measure on \( \mathbb{R} \) with infinite and compact support, then \( \mu \) has finite moments, and the asymptotic continued fraction expansion of \( S_\mu(z) \) in holds over \( \mathbb{C} \), not just over \( C_{\delta,\varepsilon} \), that is, one has

\[
S_\mu(z) \sim \frac{a_1}{z + b_1} - \frac{a_2}{z + b_2} - \frac{a_3}{z + b_3} - \cdots \quad (z \to \infty).
\]

In this case, a function \( f(z) \) has the asymptotic Jacobi continued fraction expansion

\[
f(z) \sim \frac{a_1}{z + b_1} - \frac{a_2}{z + b_2} - \frac{a_3}{z + b_3} - \cdots \quad (z \to \infty)_{\mathfrak{x}}
\]

over some unbounded subset \( \mathfrak{x} \) of \( \mathbb{C} \) if and only if

\[
f(z) = S_\mu(z) + O\left(\frac{1}{z^k}\right) \quad (z \to \infty)_{\mathfrak{x}}
\]

for all integers \( k \). Also in this case, \( S_\mu(z) \) is analytic at \( \infty \) with Laurent expansion

\[
S_\mu(z) = \sum_{k=0}^{\infty} \frac{\mu_k}{z^{k+1}}, \quad |z| \gg 0
\]

and Stieltjes continued fraction expansion

\[
S_\mu(z) = \frac{a_1}{z + b_1} - \frac{a_2}{z + b_2} - \frac{a_3}{z + b_3} - \cdots, \quad z \in \mathbb{C} \setminus \mathbb{R} \text{ or } |z| \gg 0.
\]

A proposition analogous to Proposition 1.1, along with similar comments, hold for Stieltjes continued fractions, as a consequence of [3, Theorem 2.6 and 2.8].

**Example 1.2.** For a simple example that will be relevant in Section 3.2, consider the uniform distribution \( \mu \) on \([-1, 0]\). This measure has Stieltjes transform

\[
\log \left(1 + \frac{1}{z}\right) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \sum_{k=1}^{\infty} \frac{1}{2k}, \quad z \in \mathbb{C} \setminus [-1, 0], \quad (1.1)
\]
and moments
\[ m_n(\mu) = \int_{-1}^{0} t^n dt = \frac{(-1)^n}{n+1}, \]
which yields the asymptotic expansion
\[
\log \left( 1 + \frac{1}{z} \right) \sim \sum_{n=1}^{\infty} \frac{(-1)^n}{nz^n} (z \to \infty),
\]
which of course is also valid as an exact Laurent expansion for \(|z| > 1\) (where \(\log\) is the principal branch of the logarithm). Consequently, one also has the asymptotic expansion
\[
\log \left( 1 + \frac{1}{z} \right) \sim \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{4}{4z^4} + \frac{4}{5z^5} + \frac{9}{6z^6} + \frac{9}{7z^7} + \frac{16}{8z^8} + \cdots (z \to \infty). \tag{1.2}
\]
The expansion (1.1) is well-known and is re-expressed in the form
\[
\log(1 + z) = \frac{z}{1+2+3+4+5+6+7+8+9+\cdots}, \quad z \in \mathbb{C} \setminus (-\infty, -1]
\]
via the transformation \(z \mapsto 1/z\).

Further examples, as they relate to the prime counting function, are provided in Section 3.

2 Relative asymptotic expansions

2.1 The Riemann prime counting function

The Riemann prime counting function is given by
\[
\Pi(x) = \sum_{n=1}^{\infty} \frac{1}{n} \pi(\sqrt[n]{x}) = \sum_{n=1}^{\infty} \frac{1}{n} \pi(\sqrt[n]{x}), \quad x > 0.
\]
It is a weighted prime power counting function, where each power \(p^n > 1\) of a prime \(p\) is weighted by \(\frac{1}{n}\). Since \(\pi(x) = \Pi(x) = 0\) if \(x < 2\), and \(\sqrt[n]{x} < 2\) if \(n > \log_2 x\), one has
\[
\Pi(x) = \sum_{n \leq \log_2 x} \frac{1}{n} \pi(\sqrt[n]{x}), \quad x > 0. \tag{2.1}
\]

Proposition 2.1. One has the asymptotic expansion
\[
\Pi(x) \sim \sum_{n=1}^{\infty} \frac{1}{n} \pi(\sqrt[n]{x}) (x \to \infty).
\]
Proof. By (2.1), for any positive integer $N$, one has
$$\frac{1}{N}\pi(x^{1/N}) \leq \Pi(x) - \sum_{k=1}^{N-1} \frac{1}{n}\pi(x^{1/n}) \leq \frac{1}{N}\pi(x^{1/N}) + \frac{1}{N+1}(\log_2 x)\pi(x^{1/(N+1)})$$
for all $x > 2^N$, and therefore
$$1 \leq \frac{\Pi(x) - \sum_{n=1}^{N-1} \frac{1}{n}\pi(x^{1/n})}{\frac{1}{N}\pi(x^{1/N})} \leq 1 + \frac{1}{N+1}(\log_2 x)\pi(x^{1/(N+1)})$$
as $x \to \infty$. It follows that
$$\lim_{x \to \infty} \frac{\Pi(x) - \sum_{n=1}^{N-1} \frac{1}{n}\pi(x^{1/n})}{\frac{1}{N}\pi(x^{1/N})} = 1.$$ 

The proposition follows. \hfill \Box

Corollary 2.2. One has
$$\Pi(x) - \pi(x) = \sum_{n=2}^{\infty} \sum_{p^n \leq x} \frac{1}{n} \sim \frac{1}{2} \pi(\sqrt{x}) \sim \sqrt{x} \log x \ (x \to \infty)$$
and
$$\Pi(x) - \pi(x) - \frac{1}{2} \pi(\sqrt{x}) \sim \frac{1}{3} \pi(\sqrt{x}) \sim \frac{\sqrt{x}}{\log x} \ (x \to \infty).$$

As is well known, by M"obius inversion one has
$$\pi(x) = \sum_{n \leq \log_2 x} \frac{\mu(n)}{n} \Pi(\sqrt{x}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \Pi(\sqrt{x}), \quad x > 0.$$ 

A proof similar to that of Proposition 2.1 yields the following.

Proposition 2.3. One has the asymptotic expansion
$$\pi(x) \sim \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \Pi(\sqrt{x}) \ (x \to \infty).$$

2.2 Riemann’s approximation to the prime counting function

The logarithmic integral function $\text{li}(x) = \int_0^x \frac{dt}{\log t}$ can be extended to a complex function by setting
$$\text{li}(z) = \text{Ei}(\log z),$$
where
$$\text{Ei}(z) = \gamma + \log z - \text{Ein}(-z) = \gamma + \log z + \sum_{k=1}^{\infty} \frac{z^k}{k \cdot k!}$$
and where
\[ \Ein(z) = \int_0^z (1 - e^{-t}) \frac{dt}{t} = \sum_{k=1}^{\infty} \frac{(-1)^k z^k}{k \cdot k!} \]
is entire. Let
\[ \pi_0(x) = \lim_{\epsilon \to 0} \frac{\pi(x + \epsilon) + \pi(x - \epsilon)}{2} \]
and
\[ \Pi_0(x) = \lim_{\epsilon \to 0} \frac{\Pi(x + \epsilon) + \Pi(x - \epsilon)}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \pi_0(x^{1/n}). \]
By Möbius inversion one has
\[ \pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \Pi(x^{1/n}), \]
and likewise for \( \pi_0(x) \). Riemann’s explicit formula for \( \Pi_0 \) states that
\[ \Pi_0(x) = \text{li}(x) - \sum \text{li}(x^\rho) - \log 2, \quad x > 1, \]
where the sum runs over all of the zeros \( \rho \) of the Riemann zeta function \( \zeta(s) \) (the non-trivial zeros taken in conjugate pairs in order of increasing imaginary part and repeated to multiplicity). Riemann’s explicit formula for \( \pi_0 \) states that
\[ \pi_0(x) = \text{Ri}(x) - \sum \text{Ri}(x^\rho), \quad x > 1, \]
where \( \text{Ri}(x) \) is Riemann’s function
\[ \text{Ri}(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \text{li}(x^{1/n}), \quad x > 0. \]
It follows that the function \( \text{li}(x) \) is properly considered an approximation for \( \Pi(x) \), while Riemann’s function \( \text{Ri}(x) \) is the analogous approximation for \( \pi(x) \).

It is well known that \( \text{li}(x) \) has the series representation
\[ \text{li}(x) = \gamma + \log \log x + \sum_{k=1}^{\infty} \frac{(\log x)^k}{k \cdot k!}, \quad x > 1, \tag{2.2} \]
Similarly, \( \text{Ri}(x) \) has the series representation
\[ \text{Ri}(x) = 1 + \sum_{k=1}^{\infty} \frac{(\log x)^k}{k \cdot k! \zeta(k+1)}, \quad x > 1, \]
which is the well-known Gram series representation of \( \text{Ri}(x) \).
Let
\[ R(x) = \sum_{n \leq \log x} \frac{\mu(n)}{n} \text{li}(x^{1/n}), \quad x > 1, \]
so that
\[ R(e^x) = \sum_{n \leq x} \frac{\mu(n)}{n} \text{li}(e^{x/n}), \quad x > 0. \] (2.3)

**Lemma 2.4** (III). One has
\[ \text{Ri}(x) = R(x) + O((\log \log x)^2) \quad (x \to \infty), \quad x > e. \]

**Proof.** The series representation (2.2) for li(x) implies that
\[ \text{li}(t) = \gamma + \log \log t + O(\log t), \quad 1 < t < e, \]
hence also
\[ \text{li}(x^{1/n}) = \gamma + \log \log x - \log n + O\left(\frac{\log x}{n}\right) \quad (x \to \infty), \quad n > \log x, \]
where the implicit constant does not depend on n. Therefore, using also the facts that \( \sum_{n=N} \frac{1}{n^2} \sim \frac{1}{N} \quad (N \to \infty), \sum_{n=1}^{N} \frac{1}{n} \sim \log N \quad (N \to \infty), \sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0, \) and \( \sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n} = -1, \) for \( x > e \) we have
\[ \sum_{n > \log x} \frac{\mu(n)}{n} \text{li}(x^{1/n}) = \sum_{n > \log x} \frac{\mu(n)}{n} (\gamma + \log \log x - \log n) + O\left(\sum_{n > \log x} \frac{\log x}{n^2}\right) \]
\[ = \sum_{n > \log x} \frac{\mu(n)}{n} (\gamma + \log \log x - \log n) + O(1) \]
\[ = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} (\gamma + \log \log x - \log n) + O((\log \log x)^2) \]
\[ = 0 + 0 - (1) + O((\log \log x)^2) \]
\[ = O((\log \log x)^2) \quad (x \to \infty). \]

The lemma follows.

**Proposition 2.5.** One has the asymptotic expansion
\[ \text{Ri}(x) \sim \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \text{li}(x^{1/n}) \quad (x \to \infty). \]
Proof. Let $N > 1$ be a fixed positive integer. By the lemma, for $x > e^N$ one has

$$
\text{Ri}(x) = \sum_{n \leq \log x} \frac{\mu(n)}{n} \text{li}(x^{1/n}) + O((\log \log x)^2)
$$

$$
= \sum_{n=1}^{N} \frac{\mu(n)}{n} \text{li}(x^{1/n}) + \sum_{N < n \leq \log x} \frac{\mu(n)}{n} \text{li}(x^{1/n}) + O((\log \log x)^2)
$$

$$
= \sum_{n=1}^{N} \frac{\mu(n)}{n} \text{li}(x^{1/n}) + O\left(\text{li}(x^{1/(N+1)}) \log x\right) + O((\log \log x)^2)
$$

$$
= \sum_{n=1}^{N} \frac{\mu(n)}{n} \text{li}(x^{1/n}) + O\left(x^{1/(N+1)}\right)
$$

$$
= \sum_{n=1}^{N-1} \frac{\mu(n)}{n} \text{li}(x^{1/n}) + O\left(\frac{\mu(N)}{N} \text{li}(x^{1/N})\right) \quad (x \to \infty).
$$

The proposition follows. \qed

Corollary 2.6. One has

$$
\text{li}(x) - \text{Ri}(x) \sim \frac{\sqrt{x}}{\log x} \quad (x \to \infty)
$$

and

$$
\text{li}(x) - \text{Ri}(x) - \frac{1}{2} \text{li}(\sqrt{x}) \sim \frac{1}{3} \text{li}(\sqrt[3]{x}) \sim \frac{\sqrt[3]{x}}{\log x} \quad (x \to \infty).
$$

By (2.3) and Möbius inversion, one has

$$
\text{li}(e^x) = \sum_{n \leq x} \frac{1}{n} \text{R}(e^{x/n}), \quad x > 0
$$

and therefore

$$
\text{li}(x) = \sum_{n \leq \log x} \frac{1}{n} \text{R}(x^{1/n}), \quad x > 1.
$$

However, for all $x > 0$, one has $\lim_{x \to \infty} \text{Ri}(x^{1/n}) = \text{Ri}(1) = 1$, so that the sum $\sum_{n=1}^{\infty} \frac{1}{n} \text{Ri}(x^{1/n})$ diverges for all $x > 0$. Nevertheless, one has the following.

Proposition 2.7. One has the (divergent) asymptotic expansion

$$
\text{li}(x) \sim \sum_{n=1}^{\infty} \frac{1}{n} \text{Ri}(x^{1/n}) \quad (x \to \infty).
$$
Proof. Let $N$ be a fixed positive integer. For $x > e^N$ one has

$$\text{li}(x) - \sum_{n=1}^{N} \frac{1}{n} \text{Ri}(x^{1/n}) = \sum_{N < n \leq \log x} \frac{1}{n} \text{Ri}(x^{1/n}) + \sum_{n \leq \log x} \frac{1}{n} (R(x^{1/n}) - \text{Ri}(x^{1/n}))$$

$$= \sum_{N < n \leq \log x} \frac{1}{n} \text{Ri}(x^{1/n}) + O \left( \sum_{n \leq \log x} \frac{\log \log x - \log n}{n} \right)$$

$$= \sum_{N < n \leq \log x} \frac{1}{n} \text{Ri}(x^{1/n}) + O((\log \log x)^3)$$

$$\sim \frac{1}{N+1} \text{Ri}(x^{1/(N+1)}) \ (x \to \infty).$$

The proposition follows.

Remark 2.8. Using Riemann’s approximation $\text{Ri}(x)$ to $\pi(x)$, we can provide a plausible explanation for Legendre’s approximation $L \approx 1.08366$ of the Legendre constant $L = \lim_{x \to \infty} A(x) = 1$, where

$$A(x) = \log x - \frac{x}{\pi(x)}, \ x > 0$$

is the unique function such that $\pi(x) = \frac{x}{\log x - A(x)}$ for all $x > 0$. Figure 1 compares Riemann’s approximation $\text{Ri}(x)$ with Gauss’s approximation $\text{li}(x)$, on a lin-log scale. Notice that the graph of $x - 1 - \frac{e^x}{\text{Ri}(e^x)}$ consistently traces the “center” of the wiggly graph of $A(e^x) - 1 = x - 1 - \frac{e^x}{\pi(e^x)}$ and is a better approximation, at least for small $x$, than is $x - 1 - \frac{e^x}{\text{li}(e^x)}$. Figure 2 compares the functions $x - \frac{e^x}{\text{Ri}(e^x)}$ and $A(e^x) = x - \frac{e^x}{\pi(e^x)}$ on a smaller interval. It is interesting to observe that the function $\log x - \frac{x}{\Ri(x)}$, which is Riemann’s approximation to $A(x)$, appears to attain a global maximum of approximately 1.08356 at $x \approx 216811 \approx e^{12.2871}$, with a very small derivative nearby that appears to attain a local (and perhaps even global) minimum of only about $-3.68 \times 10^{-9}$ somewhat near the point $(475000, 1.0828)$. These features offer a plausible explanation of how Legendre was led to his approximation $L \approx 1.08366$. See Figure 3 for a graph of the derivative of $\log x - \frac{x}{\Ri(x)}$ near its apparent local minimum.
2.3 Prime power counting functions

For all $x > 0$, let

$$\pi^*(x) = \sum_{k=1}^{\infty} \sum_{p^k \leq x} 1$$

denote the number of prime powers (excluding 1) less than or equal to $x$, so that

$$\pi^*(x) = \sum_{n=1}^{\infty} \pi(\sqrt[n]{x}) = \sum_{n \leq \log_2 x} \pi(\sqrt[n]{x})$$

for all $x > 0$, and also let

$$\tilde{\pi}(x) = \pi^*(x) - \pi(x) = \sum_{n=2}^{\infty} \sum_{p^n \leq x} 1 = \sum_{n=2}^{\infty} \pi(\sqrt[n]{x}) = \sum_{1 \leq n \leq \log_2 x} \pi(\sqrt[n]{x})$$

denote the number of composite prime powers less than or equal to $x$. By Möbius inversion, one has

$$\pi(x) = \sum_{n \leq \log_2 x} \mu(n) \pi^*(\sqrt[n]{x}) = \sum_{n=1}^{\infty} \mu(n) \pi^*(\sqrt[n]{x}).$$

One easily verifies the following analogue of Propositions 2.1 and 2.3.
Proposition 2.9. One has the asymptotic expansions
\[ \pi^*(x) \sim \sum_{n=1}^{\infty} \pi(\sqrt[n]{x}) \ (x \to \infty) \]
and
\[ \pi(x) \sim \sum_{n=1}^{\infty} \mu(n) \pi^*(\sqrt[n]{x}) \ (x \to \infty) \]

In general, for any \( O \) bound, one may seek explicit \( O \) constants. For example, by [11, Lemma 3], one has
\[ \pi(x) \leq \pi^*(x) \leq \pi(x) + \pi(\sqrt{x}) + 3\sqrt{x} \]
for all \( x \geq 9621 \). Thus, explicit \( O \) constants can be sought for any of the terms of any of the asymptotic expansions proved in [3] and in this paper. We do not pursue this extensive line of research here, since we are interested in pursuing asymptotic expansions rather than explicit inequalities.

3  Asymptotic continued fraction expansions

3.1  Weighted prime power counting functions

It follows from [3, Lemma 2.1], the prime number theorem with error term, and our results in Section 3 that, with respect to the asymptotic sequence \( \{1/(\log x)^n\} \), the functions \( \pi(x), \Pi(x), \text{li}(x), \text{and Ri}(x) \) all have the same asymptotic continued fraction expansions, as described by [3 Theorems 1.1 and 1.2]. Similarly, one has the following.

**Theorem 3.1.** Let \( n \) be a positive integer, and let \( f(x) \) be any of the following functions.

1. \( \Pi(x) \approx \sum_{k=1}^{n-1} \frac{1}{k} \pi(\sqrt[k]{x}) = \sum_{k=n}^{\infty} \frac{1}{k} \pi(\sqrt[k]{x}). \)
2. \( \text{li}(x) \approx \sum_{k=1}^{n-1} \frac{1}{k} \text{Ri}(\sqrt[k]{x}). \)
3. \( \mu(n) \left( \pi(x) - \sum_{k=0}^{n-1} \frac{\mu(k)}{k} \Pi(\sqrt[k]{x}) \right) = \mu(n) \left( \sum_{k=n}^{\infty} \frac{\mu(k)}{k} \Pi(\sqrt[k]{x}) \right). \)
4. \( \mu(n) \left( \text{Ri}(x) - \sum_{k=1}^{n-1} \frac{\mu(k)}{k} \text{li}(\sqrt[k]{x}) \right) = \mu(n) \left( \sum_{k=n}^{\infty} \frac{\mu(k)}{k} \text{li}(\sqrt[k]{x}) \right). \)
5. \( \frac{1}{n} \pi(\sqrt[n]{x}). \)
6. \( \frac{1}{n} \Pi(\sqrt[n]{x}). \)
7. $\frac{1}{n} \text{Ri}(\sqrt{x})$.

8. $\frac{1}{n} \text{li}(\sqrt{x})$.

9. $\frac{1}{n} \pi_n^*(x)$, where $\pi_n^*(x) = \sum_{k=n}^{\infty} \pi(\sqrt[k]{x}) = \sum_{k=n}^{\infty} \sum_{p^k \leq x} 1$.

10. $\sum_{k=n}^{\infty} \frac{1}{k} \pi(\sqrt[k]{x}) = \sum_{k=n}^{\infty} \sum_{p^k \leq x} \frac{1}{k}$.

One has the asymptotic continued fraction expansions

$$f(x) \sim \frac{\sqrt{x}}{\log x} \frac{n}{1} \frac{n}{1} \frac{2n}{1} \frac{2n}{1} \frac{3n}{1} \frac{3n}{1} \cdots \ (x \to \infty)$$

and

$$f(x) \sim \frac{\sqrt{x}}{\log x - n} \frac{n^2}{\log x - 3n} \frac{(2n)^2}{\log x - 5n} \frac{(3n)^2}{\log x - 7n} \frac{(4n)^2}{\log x - 9n} \cdots \ (x \to \infty).$$

Consequently, the best rational approximations of the function $e^{-x/n} f(e^x)$ are precisely the approximants $w_k(x)$ of the continued fraction

$$\frac{1}{x - n} \frac{n^2}{x - 3n} \frac{(2n)^2}{x - 5n} \frac{(3n)^2}{x - 7n} \frac{(4n)^2}{x - 9n} \cdots .$$

Moreover, one has

$$e^{-x/n} f(e^x) - w_k(x) \sim \frac{(n^k k!)^2}{x^{2k+1}}$$

for all $n \geq 0$.

**Corollary 3.2.** Let $f(x)$ be any of the following functions.

1. $\Pi(x) - \pi(x) = \sum_{k=2}^{\infty} \frac{1}{k} \pi(\sqrt[k]{x}) = - \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \Pi(\sqrt[k]{x})$.

2. $\text{li}(x) - \text{Ri}(x) = - \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \text{li}(\sqrt[k]{x})$.

3. $\frac{1}{2} \tilde{\pi}(x)$, where $\tilde{\pi}(x) = \sum_{k=2}^{\infty} \pi(\sqrt[k]{x}) = \sum_{k=2}^{\infty} \sum_{p^k \leq x} 1$.

4. $\frac{1}{2} \pi(\sqrt{x})$.

5. $\frac{1}{2} \Pi(\sqrt{x})$. 
6. \( \frac{1}{2} \text{Ri}(\sqrt{x}) \).

7. \( \frac{1}{2} \text{li}(\sqrt{x}) \).

One has the asymptotic continued fraction expansions

\[
f(x) \sim \frac{\sqrt{x}}{\log x - 1} \frac{2}{\log x} \frac{2}{\log x} \frac{4}{\log x} \frac{4}{\log x} \frac{6}{\log x} \frac{6}{\log x} \cdots \quad (x \to \infty)
\]

and

\[
f(x) \sim \frac{\sqrt{x}}{\log x - 2} \frac{4}{\log x - 6} \frac{4 \cdot 4}{\log x - 10} \frac{4 \cdot 9}{\log x - 14} \frac{4 \cdot 16}{\log x - 18} \cdots \quad (x \to \infty).
\]

Consequently, the best rational approximations of the function \( e^{-x^2/2} f(e^x) \) are precisely the approximants of the continued fraction

\[
\frac{1}{x - 2} \frac{4}{x - 6} \frac{4 \cdot 4}{x - 10} \frac{4 \cdot 9}{x - 14} \frac{4 \cdot 16}{x - 18} \cdots.
\]

### 3.2 Sums of \( s \)-th powers of primes

Consider the function

\[
\pi_s(x) = \sum_{p \leq x} p^s, \quad x > 0
\]

for complex values of \( s \) (so of course \( \pi(x) = \pi_0(x) \)). The following \( O \) bound is proved using the prime number theorem with error term and Abel’s summation formula.

**Proposition 3.3.** For all \( s \in \mathbb{C} \) with \( \text{Re}(s) > -1 \) and all \( t > 0 \), one has

\[
\pi_s(x) = -E_1(-(s + 1) \log x) + O \left( x^{\text{Re}(s)+1} (\log x)^{-t} \right) \quad (x \to \infty).
\]

Consequently, [3, Theorem 1.1, Lemma 2.1, and Corollary 3.1] yield the following.

**Theorem 3.4.** Let \( s \in \mathbb{C} \) with \( \text{Re}(s) > -1 \). One has the asymptotic expansion

\[
\pi_s(x) \sim \sum_{k=0}^{\infty} \frac{k! x^{s+1}}{(s+1) \log x)^{k+1}} \quad (x \to \infty)
\]

and the asymptotic continued fraction expansions

\[
\pi_s(x) \sim \frac{x^{s+1}}{(s+1) \log x} \frac{1}{1 -} \frac{1}{(s+1) \log x} \frac{2}{1 -} \frac{2}{(s+1) \log x} \frac{3}{1 -} \frac{3}{(s+1) \log x} \cdots \quad (x \to \infty).
\]

and

\[
\pi_s(x) \sim \frac{x^{s+1}}{(s+1) \log x - 1} \frac{1}{(s+1) \log x - 3} \frac{4}{(s+1) \log x - 5} \frac{9}{(s+1) \log x - 7} \cdots \quad (x \to \infty).
\]
Let $w_n(x)$ for any nonnegative integer $n$ denote the $n$th approximant of the continued fraction

$$\frac{1}{x-1} - \frac{1}{x-3} - \frac{4}{x-5} - \frac{9}{x-7} - \frac{16}{x-9} - \cdots.$$  

For all nonnegative integers $n$, one has

$$\frac{\pi s(e^{x/(s+1)})}{e^x} - w_n(x) \sim \frac{(n!)^2}{x^{2n+1}} (x \to \infty).$$

Moreover, $w_n(x)$ is the unique Padé approximant of $\frac{\pi s(e^{x/(s+1)})}{e^x}$ at $x = \infty$ of order $[n-1, n]$, and the $w_n(x)$ for all nonnegative integers $n$ are precisely the best rational approximations of the function $\frac{\pi s(e^{x/(s+1)})}{e^x}$.

Note that, since

$$\sum_{n \leq x} n^s = \frac{x^{s+1}}{s+1} + O(x^s) \ (x \to \infty)$$

for all $s \in \mathbb{C}$ with Re($s$) $> -1$, the asymptotic continued fraction expansions in the theorem can be re-expressed as

$$\sum_{p \leq x} p^s \sim \frac{\log x}{1 - \frac{(s+1) \log x}{1}} - \frac{1}{1 - \frac{(s+1) \log x}{1}} \cdots (x \to \infty).$$

and

$$\sum_{n \leq x} n^s \sim \frac{s + 1}{(s+1) \log x - 1 - (s+1) \log x - 3 - (s+1) \log x - 5 - (s+1) \log x - 7 - \cdots (x \to \infty).}$$

For the boundary case $s = -1$, note that

$$\sum_{p \leq x} \frac{1}{p} = M + \log \log x + O((\log x)^t) \ (x \to \infty)$$

for all $t \in \mathbb{R}$, where

$$M = \lim_{x \to \infty} \left( \sum_{p \leq x} \frac{1}{p} - \log \log x \right) = 0.2614972128476427837554 \ldots$$

is the Meissel–Mertens constant. In Section 2, we also noted that the uniform distribution on $[-1, 0]$ has Cauchy transform $\log(1 + 1/z)$ with expansions (1.1) and (1.2). Using this, we obtain the following.

**Proposition 3.5.** For all real numbers $a > 1$, one has the asymptotic continued fraction expansions

$$\sum_{a^{x} < p \leq a^{x+1}} \frac{1}{p} \sim \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{4}{5} + \frac{9}{6} + \frac{9}{7} + \frac{16}{8} + \frac{16}{9} + \cdots (x \to \infty)$$
For all real numbers

\[
\sum_{x<p\leq ax} \frac{1}{p} \sim \frac{1}{\log x} \frac{1}{\log x} \frac{1}{\log x} \frac{1}{\log x} \frac{1}{\log x} \frac{1}{\log x} \frac{4}{\log x} \frac{4}{\log x} \frac{9}{\log x} \frac{9}{\log x} \frac{16}{\log x} \frac{16}{\log x} \ldots \ (x \to \infty).
\]

Moreover, for all \( x > 0 \), the first continued fraction converges to \( \log \left( 1 + \frac{1}{x} \right) \), while, for all \( x > 1 \), the second continued fraction converges to \( \log \left( 1 + \frac{1}{\log x} \right) = \log \log(ax) - \log \log x \).

**Proof.** Let \( t = \log a > 0 \). By (1.2), one has the asymptotic expansion

\[
\log(z + t) - \log z = \log \left( 1 + \frac{t}{z} \right) \sim \frac{t}{z} \frac{t}{z} \frac{t}{z} \frac{t}{z} \frac{4t}{z} \frac{4t}{z} \frac{9t}{z} \frac{9t}{z} \frac{16t}{z} \frac{16t}{z} \ldots (z \to \infty),
\]

and therefore, letting \( z = \log x \), one has the asymptotic expansion

\[
\log \log(ax) - \log \log x \sim \frac{1}{\log x} \frac{1}{\log x} \frac{1}{\log x} \frac{1}{\log x} \frac{4}{\log x} \frac{4}{\log x} \frac{9}{\log x} \frac{9}{\log x} \frac{16}{\log x} \frac{16}{\log x} \ldots (x \to \infty).
\]

By (3.1), one has

\[
\sum_{x<p\leq ax} \frac{1}{p} = \sum_{p\leq ax} \frac{1}{p} - \sum_{p\leq x} \frac{1}{p} = \log \log(ax) - \log \log x + o((\log x)^4) \ (x \to \infty)
\]

for all \( t \in \mathbb{R} \). Therefore the function \( \sum_{x<p\leq ax} \frac{1}{p} \) has the same asymptotic expansion as \( \log \log(ax) - \log \log x \).

**Corollary 3.6.** For all real numbers \( a > b > 0 \), one has the asymptotic continued fraction expansion

\[
\sum_{bx<p\leq ax} \frac{1}{p} \sim \frac{1}{\log a/b(x)} \frac{1}{\log a/b(x)} \frac{1}{\log a/b(x)} \frac{1}{\log a/b(x)} \frac{4}{\log a/b(x)} \frac{4}{\log a/b(x)} \frac{9}{\log a/b(x)} \frac{9}{\log a/b(x)} \frac{16}{\log a/b(x)} \frac{16}{\log a/b(x)} \ldots \ (x \to \infty).
\]

It is clear that the expansion (1.1) can be rewritten in the form

\[
\log \left( 1 + \frac{1}{z} \right) = \frac{1}{z} \frac{1}{z} \frac{1}{z} \frac{2}{z} \frac{2}{z} \frac{3}{z} \frac{3}{z} \frac{4}{z} \frac{4}{z} \ldots, \quad z \in \mathbb{C}\backslash[-1, 0].
\]

Thus we also have the following.

**Corollary 3.7.** For all real numbers \( a > 1 \), one has the asymptotic continued fraction expansions

\[
\sum_{ax^2<p\leq ax+1} \frac{1}{p} \sim \frac{1}{x} \frac{1}{x} \frac{1}{x} \frac{2}{x} \frac{2}{x} \frac{3}{x} \frac{3}{x} \frac{4}{x} \frac{4}{x} \ldots (x \to \infty)
\]

and

\[
\sum_{x<p\leq ax} \frac{1}{p} \sim \frac{1}{\log x} \frac{1}{\log x} \frac{1}{\log x} \frac{2}{\log x} \frac{3}{\log x} \frac{4}{\log x} \frac{4}{\log x} \ldots (x \to \infty).
\]

16
It is noteworthy that the asymptotic expansion of the function \( \sum_{a^x < p \leq a^{x+1}} \frac{1}{p} \) with respect to the asymptotic sequence \( \left\{ \frac{1}{x^n} \right\} \) does not depend on \( a \).

Now, the uniform distribution on \([-1, 1]\) has Cauchy transform

\[
\log \left( \frac{z + 1}{z - 1} \right) = \frac{2}{z} - \frac{1}{3z} - \frac{4}{5z} - \frac{9}{7z} - \frac{16}{9z} - \frac{25}{11z} - \frac{36}{13z} - \cdots, \quad z \in \mathbb{C}\setminus[-1,1]. \tag{3.2}
\]

From this we obtain the following.

**Proposition 3.8.** For all real numbers \( a > 1 \), one has the asymptotic continued fraction expansion

\[
\sum_{a^{-1}x < p \leq ax} \frac{1}{p} \sim \frac{2}{\log_a x} \frac{1}{3\log_a x} \frac{4}{5\log_a x} \frac{9}{7\log_a x} \frac{16}{9\log_a x} \frac{25}{11\log_a x} \cdots \quad (x \to \infty).
\]

**Corollary 3.9.** For all real numbers \( a > 1 \), one has the asymptotic continued fraction expansions

\[
\sum_{a^x < p \leq a^{x+1}} \frac{1}{p} \sim \frac{2}{2x + 1} \frac{1}{6x + 3} \frac{4}{10x + 5} \frac{9}{14x + 7} \frac{16}{18x + 9} \cdots \quad (x \to \infty)
\]

and

\[
\sum_{x < p \leq ax} \frac{1}{p} \sim \frac{2}{2\log_a x + 1} \frac{1}{6\log_a x + 3} \frac{4}{10\log_a x + 5} \frac{9}{14\log_a x + 7} \cdots \quad (x \to \infty).
\]

### 3.3 Functions related to Mertens’ theorems

Like the Meissel–Mertens constant \( M \), the constant

\[
H = -\sum_p \left( \frac{1}{p} + \log \left( 1 - \frac{1}{p} \right) \right)
\]

\[
= \sum_p \left( \frac{1}{2p^2} + \frac{1}{3p^3} + \frac{1}{4p^4} + \cdots \right)
\]

\[
= \sum_{n=2}^{\infty} \frac{P(n)}{n}
\]

\[
= 0.3157184519\ldots,
\]

where

\[
P(s) = \sum_p \frac{1}{p^s}, \quad \text{Re}(s) > 1
\]

is the prime zeta function, encodes information about the primes. Since

\[
\log \zeta(s) = \sum_{n=1}^{\infty} \frac{P(ns)}{n}, \quad \text{Re}(s) > 1
\]
(which is an immediate consequence of the Euler product representation of $\zeta(s)$), one has

$$H = \lim_{x \to 1^+} (\log \zeta(x) - P(x)) = \lim_{x \to 1^+} \left( \log \frac{1}{x - 1} - P(x) \right).$$

The following estimates are well known for $s = 1$.

**Proposition 3.10.** For all $s \in \mathbb{R}$ not equal to 0 or a prime, and for all $t \in \mathbb{R}$, one has the following.

1. \[-\frac{1}{s} \sum_{p \leq x} \log \left( 1 - \frac{s}{p} \right) = G(s) + \log \log x + o((\log x)^t) \quad (x \to \infty), \]

   where

   $$G(s) = -\lim_{x \to \infty} \left( \frac{1}{s} \sum_{p \leq x} \log \left( 1 - \frac{s}{p} \right) + \log \log x \right),$$

   and where $G(1) = \gamma$.

2. \[-\sum_{p \leq x} \left( \frac{1}{p} + \frac{1}{s} \log \left( 1 - \frac{s}{p} \right) \right) = sH(s) + o((\log x)^t) \quad (x \to \infty), \]

   where

   $$H(s) = -\frac{1}{s} \sum_{p} \left( \frac{1}{p} + \frac{1}{s} \log \left( 1 - \frac{s}{p} \right) \right),$$

   and where $H(1) = H$.

3. \[\prod_{p \leq x} \left( 1 - \frac{s}{p} \right)^{-1/s} = e^{G(s)} \log x + o((\log x)^t) \quad (x \to \infty).\]

4. \[\prod_{p \leq x} \left( 1 - \frac{s}{p} \right)^{-1} = e^{sG(s)} (\log x)^s + o((\log x)^t) \quad (x \to \infty).\]

**Proof.** We prove (1), from which the other statements readily follow. Since $s$ is not zero or a prime, the sum $\sum_{p \leq x} \log \left( 1 - \frac{s}{p} \right)$ is finite for all $x > 0$. Let $N = \max(2, |s| + 1)$. From the series expansion

$$\log \left( 1 - \frac{s}{t} \right) = -\sum_{k=1}^{\infty} \frac{s^k}{kt^k}, \quad |t| > |s| \quad (3.3)$$

it follows that

$$\log \left( 1 - \frac{s}{t} \right) = -\frac{s}{t} + O \left( \frac{1}{t^2} \right) \quad (t \to \infty).$$
It follows that the function \( F(u) = \log \left( 1 - \frac{s}{u} \right) \) satisfies the three necessary hypotheses of Landau’s theorem \([9, p. 201–203]\), and therefore one has

\[
\sum_{p \leq x} \log \left( 1 - \frac{s}{p} \right) = A(s) + \int_{N}^{x} \frac{\log \left( 1 - \frac{s}{t} \right)}{\log t} dt + O \left( \left( \log x \right)^{u} \right) \quad (x \to \infty)
\]

for all \( u \in \mathbb{R} \), for some constant \( A(s) \) depending on \( s \). Now, since \( |t| > |s| \) for all \( t \geq N \), from (3.3) it follows that

\[
\int_{N}^{x} \frac{\log \left( 1 - \frac{s}{t} \right)}{\log t} dt = B(s) - s \log \log x - \sum_{k=1}^{\infty} \frac{s^{k+1}}{k+1} \text{li}(x^{-k})
\]

for some constant \( B(s) \) depending on \( s \). But also

\[
0 < - \text{li}(1/x) < \frac{1}{x \log x},
\]

for all \( x > 1 \) and therefore

\[
\left| \sum_{k=1}^{\infty} \frac{s^{k+1}}{k+1} \text{li}(x^{-k}) \right| \leq \sum_{k=1}^{\infty} \frac{|s|^{k+1}}{k+1} \text{li}(x^{-k}) < \sum_{k=1}^{\infty} \frac{|s|^{k+1}}{k(k+1)x^{k} \log x} = O \left( \frac{1}{x \log x} \right) \quad (x \to \infty)
\]

for all \( x > 1 \). Thus we have

\[
\int_{N}^{x} \frac{\log \left( 1 - \frac{s}{t} \right)}{\log t} dt = B(s) - s \log \log x + O \left( \frac{1}{x \log x} \right) \quad (x \to \infty)
\]

and therefore

\[
\sum_{p \leq x} \log \left( 1 - \frac{s}{p} \right) = A(s) + B(s) - s \log \log x + O \left( \frac{1}{x \log x} \right) + O \left( \left( \log x \right)^{u} \right) \quad (x \to \infty)
\]

\[
= -sG(s) - s \log \log x + O \left( \left( \log x \right)^{u} \right) \quad (x \to \infty),
\]

where \( G(s) = -\frac{1}{s}(A(s) - B(s)) \). By Mertens’ third theorem, we know that \( G(1) = \gamma \). \( \square \)

Note that

\[
H(s) = -\frac{1}{s} \sum_{p} \left( \frac{1}{p} + \frac{1}{s} \log \left( 1 - \frac{s}{p} \right) \right)
\]

\[
= \sum_{p} \left( \frac{1}{2p^{2}} + \frac{s}{3p^{3}} + \frac{s^{2}}{4p^{4}} + \cdots \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{P(n+2)}{n+2} s^{n},
\]

provided that the given series converges absolutely. In fact, for any \( r \geq 0 \) the sequence \( \frac{P(n+1)}{n+1} r^{n} \) converges monotonically to 0 if and only if \( r \leq 2 \), so the radius of convergence of the series \( \sum_{n=0}^{\infty} \frac{P(n+2)}{n+2} s^{n} \) for \( s \in \mathbb{C} \) is 2, and the series converges on the entire disk \( |s| \leq 2 \) except at \( s = 2 \).
**Corollary 3.11.** One has the following.

1. For all \( s \in \mathbb{R} \) not equal to 0 or a prime, one has
   \[
   G(s) = M + sH(s).
   \]
2. \( G(0) := \lim_{s \to 0} G(s) = M. \)
3. \( H(0) := \lim_{s \to 0} H(s) = \frac{1}{2} P(2) = G'(0) = 0.2261237100205 \ldots \)
4. One has Maclaurin series expansions
   \[
   H(s) = \sum_{n=0}^{\infty} \frac{P(n + 2)}{n + 2} s^n
   \]
   and
   \[
   G(s) = M + \sum_{n=1}^{\infty} \frac{P(n + 1)}{n + 1} s^n
   \]
   valid for all \( s \in \mathbb{R} \) with \( |s| \leq 2 \) except \( s = 2 \), and both series converge for all \( s \in \mathbb{C} \) with \( |s| \leq 2 \) except \( s = 2 \).
5. \( \gamma = G(1) = M + H. \)
6. \( H = H(1) = G(1) - G(0). \)
7. \( G^{(n)}(0) = \frac{n!}{n+1} P(n + 1) = nH^{(n-1)}(0) \) for all \( n \geq 1. \)
8. \( H^{(n)}(0) = \frac{n!}{n+2} P(n + 2) \) for all \( n \geq 0. \)

Note that equation \( \gamma = M + H \) is a well-known relationship between the constants \( \gamma, M, \) and \( H. \) By the corollary, the function \( G(s) \) continuously deforms the constant \( M \) to the constant \( \gamma = M + H \) over the interval \([0, 1]\) and extends uniquely to the analytic function \( M + \sum_{n=1}^{\infty} \frac{P(n+1)}{n+1} s^n \) on the closed disk \( |s| \leq 2 \) minus \( s = 2. \) An approximation of the graph of \( G(s) \) on \([-2, 2]\) by the first 400 terms of its Maclaurin series, is provided in Figure 4.

![Figure 4: Approximation of G(s) on [-2, 2) by the first 400 terms of its Maclaurin series](image-url)
By statements (1) and (2) of Proposition 3.10 for all \( a > 1 \) and all \( s \in \mathbb{R} \) not equal to 0 or a prime, the function

\[
\log \prod_{x < p \leq ax} \left( 1 - \frac{s}{p} \right)^{-1/s} = -\frac{1}{s} \sum_{x < p \leq ax} \log \left( 1 - \frac{s}{p} \right)
\]

has the same asymptotic expansions as the function \( \sum_{x < p \leq ax} \frac{1}{p} \). We may combine this with the results in the previous section as follows.

**Theorem 3.12.** Let \( a > 1 \). One has the following asymptotic expansions.

1. \[
\sum_{a^x < p \leq a^{x+1}} \frac{1}{p} \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{nx^n} \quad (x \to \infty).
\]

2. \[
\sum_{a^x < p \leq a^{x+1}} \frac{1}{p} \sim \frac{1}{1 + 2 + 3 + \cdots + 4 + 5 + 6 + 7 + 8 + 9 + \cdots} \quad (x \to \infty).
\]

3. \[
\sum_{a^x < p \leq a^{x+1}} \frac{1}{p} \sim \frac{1}{1 + 2 + 3 + 2 + 5 + 2 + 7 + 2 + 9 + \cdots} \quad (x \to \infty).
\]

4. \[
\sum_{a^x < p \leq a^{x+1}} \frac{1}{p} \sim \frac{2}{2x + 1} - \frac{1}{6x + 3} - \frac{4}{10x + 5} - \frac{9}{14x + 7} - \frac{16}{18x + 9} \cdots \quad (x \to \infty).
\]

5. \[
\sum_{x < p \leq ax} \frac{1}{p} \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(\log a)^n x^n} \quad (x \to \infty).
\]

6. \[
\sum_{x < p \leq ax} \frac{1}{p} \sim \frac{\log a}{1 + 2 + 3 + \cdots + 4 + 5 + 6 + 7 + 8 + 9 + \cdots} \quad (x \to \infty)
\]

7. \[
\sum_{x < p \leq ax} \frac{1}{p} \sim \frac{\log a}{1 + 2 + 3 + 2 + 5 + 2 + 7 + 2 + 9 + \cdots} \quad (x \to \infty).
\]

8. \[
\sum_{x < p \leq ax} \frac{1}{p} \sim \frac{2}{2 \log a + 1} - \frac{1}{6 \log a + 3} - \frac{4}{10 \log a + 5} - \frac{9}{14 \log a + 7} \cdots \quad (x \to \infty).
\]

Let \( s \in \mathbb{R} \) be nonzero and not equal to a prime. Then the asymptotic continued fraction expansions in (1)–(4) also hold for the functions \( \log(1 + \frac{1}{x}) \) and

\[
\log \prod_{a^x < p \leq a^{x+1}} \left( 1 - \frac{s}{p} \right)^{-1/s} = -\frac{1}{s} \sum_{a^x < p \leq a^{x+1}} \log \left( 1 - \frac{s}{p} \right),
\]

while the continued fraction expansions in (5)–(8) also hold for the functions \( \log(1 + \frac{1}{\log a x}) = \log \log(ax) - \log \log x \) and

\[
\log \prod_{x < p \leq ax} \left( 1 - \frac{s}{p} \right)^{-1/s} = -\frac{1}{s} \sum_{x < p \leq ax} \log \left( 1 - \frac{s}{p} \right).
\]
We may rewrite the asymptotic expansion (3.2) as the asymptotic Jacobi continued fraction expansion
\[
\log \left( \frac{z + 1}{z - 1} \right) = \frac{2}{z - z - z - z - z - z - z - z - z - \cdots}, \quad z \in \mathbb{C} \setminus [-1, 1], \quad (3.4)
\]
so, substituting \( z = 2x + 1 \), the asymptotic expansion (4) of the theorem can be rewritten as the asymptotic Jacobi continued fraction expansion
\[
\sum_{a^2 < p \leq a^2 + 1} \frac{1}{p} \sim \frac{1}{x + 1/2} - \frac{1/4 - 3}{x + 1/2} - \frac{4/4 - 15}{x + 1/2} - \frac{9/4 - 35}{x + 1/2} - \frac{16/4 - 63}{x + 1/2} - \cdots (x \to \infty).
\]
It is known that the denominator in the \( n \)th approximant of the Jacobi continued fraction in (3.4) is the \( n \)th Legendre polynomial \( P_n(z) \). It follows that the denominator in the \( n \)th approximant of the continued fraction in the expansion of \( \sum_{a^2 < p \leq a^2 + 1} \frac{1}{p} \) above is the integer polynomial \( \widetilde{P}_n(x) = P_n(2x + 1) \). It is known that these polynomials are given explicitly by
\[
\widetilde{P}_n(x) = P_n(2x + 1) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k
\]
for all \( n \). Now, applying [3, Theorem 2.4], we obtain the following.

**Corollary 3.13.** Let \( a > 1 \), and let \( s \in \mathbb{R} \) be nonzero and not equal to a prime. Let \( f(x) \) denote any of the three functions \( \log(1 + 1/x), \sum_{a^2 < p \leq a^2 + 1} \frac{1}{p}, \) and \( \log \prod_{a^2 < p \leq a^2 + 1} (1 - s/p)^{-1/s} \). Then \( f(x) \) has the asymptotic Jacobi continued fraction expansion
\[
f(x) \sim \frac{2}{2x + 1} - \frac{1}{6x + 3} - \frac{4}{10x + 5} - \frac{9}{14x + 7} - \frac{16}{18x + 9} - \cdots (x \to \infty),
\]
The best rational approximations of the function \( f(x) \) are precisely the approximants \( w_n(x) \) of the given continued fraction for \( n \geq 0 \), which converge to \( \log(1 + 1/x) \) for all \( x \in \mathbb{C} \setminus [-1, 0] \) as \( n \to \infty \). Moreover, one has
\[
f(x) - w_n(x) \sim \frac{c_n}{x^{2n+1}} (x \to \infty)
\]
for all \( n \geq 0 \), where \( c_0 = 1 \) and
\[
c_n = \frac{1}{2^{2n}} \prod_{k=1}^{n} \frac{k^2}{4k^2 - 1} = \frac{1}{(2n + 2)(2n + 1)(2n - 1)}
\]
for all \( n \geq 1 \). Furthermore, one has
\[
w_n(x) = \sum_{k=1}^{n} \frac{c_{k-1}}{P_k(2x + 1)P_{k-1}(2x + 1)}
\]
for all \( n \geq 0 \), where \( P_n(x) \) denotes the \( n \)th Legendre polynomial and
\[
P_n(2x + 1) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k
\]
for all \( n \geq 0 \).
Remark 3.14. From [11 [1.14]], one deduces that the numerator \( \hat{R}_n(x) \) of \( w_n(x) = \frac{\hat{R}_n(x)}{\hat{P}_n(x)} \) is

\[
\hat{R}_n(x) = \sum_{k=1}^{n} a_{n,k} x^{k-1},
\]

where

\[
a_{n,k} = \sum_{j=k}^{n} \frac{(-1)^{j-k}}{j-k+1} \binom{n+j}{j} \left( \binom{n+k}{2k} \right) F_3(1, 1, k-n, n+k+1; 2, k+1, k+1; 1)
\]

for all \( n \geq 1 \) and \( 1 \leq k \leq n \).

3.4 \( \pi(ax) - \pi(bx) \) for \( a > b > 1 \)

Let \( s < t \) be real numbers. Consider the measure \( \mu \) on \([s, t]\) of density \( e^{-u} du \). The \( n \)th moment of \( \mu \) is

\[
m_n(\mu) = \int_s^t u^n e^{-u} du = \int_s^\infty u^n e^{-u} du - \int_t^\infty u^n e^{-u} du = e^{-s} r_n(s) - e^{-t} r_n(t),
\]

where \( r_n(X) = \sum_{k=1}^{n} \frac{n!}{k!} X^k \in \mathbb{Z}[X] \). Moreover, one has the asymptotic expansion

\[
\frac{\text{li}(e^{x-s}) - \text{li}(e^{x-t})}{e^x} \sim \sum_{n=0}^{\infty} \frac{m_n(\mu)}{x^{n+1}} (x \to \infty),
\]

and the same expansion holds for the function \( \frac{\pi(e^{x-s}) - \pi(e^{x-t})}{e^x} \). The Stieltjes transform of \( \mu \) is

\[
S_\mu(z) = -e^{-z} \left( E_1(-z + s) - E_1(-z + t) \right), \quad z \in \mathbb{C} \setminus [s, t],
\]

and one has

\[
S_\mu(x) = -e^{-x} \left( E_1(-x + s) - E_1(-x + t) \right) = \frac{\text{li}(e^{x-s}) - \text{li}(e^{x-t})}{e^x}, \quad x \in \mathbb{R} \setminus [s, t].
\]

It follows that two (convergent) continued fraction expansions of \( S_\mu(x) \) provide asymptotic continued fraction expansions of both \( \frac{\text{li}(e^{x-s}) - \text{li}(e^{x-t})}{e^x} \) and \( \frac{\pi(e^{x-s}) - \pi(e^{x-t})}{e^x} \) as \( x \to \infty \). These take the form

\[
zS_\mu(z) = c_0(s, t) + \frac{c_1(s, t)}{z + \frac{c_2(s, t)}{1 + \frac{c_3(s, t)}{z + \cdots}}}, \quad z \in \mathbb{C} \setminus [s, t]
\]

and

\[
zS_\mu(z) = a_0(s, t) + \frac{b_1(s, t)}{z + a_1(s, t) + \frac{b_2(s, t)}{z + a_2(s, t) + \frac{b_3(s, t)}{z + \cdots}}}, \quad z \in \mathbb{C} \setminus [s, t],
\]

where using the qd-algorithm [2 Section 6.1] we compute

\[
c_0(s, t) = a_0(s, t) = e^{-s} - e^{-t},
\]

23
\[ c_1(s, t) = b_1(s, t) = e^{-s}(1 + s) - e^{-t}(1 + t), \]
\[ c_2(s, t) = a_1(s, t) = \frac{e^{-s}(2 + 2s + s^2) - e^{-t}(2 + 2t + t^2)}{e^{-s}(1 + s) - e^{-t}(1 + t)}, \]
\[ c_3(s, t) = \frac{e^{-2s}(2 + 4s + s^2) + e^{-2t}(2 + 4t + t^2) - e^{-s-t}g(s, t)}{(e^{-s}(1 + s) - e^{-t}(1 + t))(e^{-s}(2 + 2s + s^2) - e^{-t}(2 + 2t + t^2))}, \]
and
\[ b_2(s, t) = c_2(s, t)c_3(s, t), \]
where \( g(s, t) \) is the symmetric polynomial
\[ g(s, t) = s^3t + st^2 - 2s^2t^2 + s^3 + t^3 - 2s^2t - st^2 - s^2 - t^2 + 4st + 4s + 4t + 4. \]
It follows that one has asymptotic expansions of the form
\[ \frac{\pi(e^{e-s}) - \pi(e^{e-t})}{e^x/x} \sim e^{-s} - e^{-t} + \frac{e^{-s}(1 + s) - e^{-t}(1 + t)}{x} + \frac{c_2(s, t)}{1} + \frac{c_3(s, t)}{x} + \cdots (x \to \infty) \]
and
\[ \frac{\pi(e^{e-s}) - \pi(e^{e-t})}{e^x/x} = e^{-s} - e^{-t} + \frac{e^{-s}(1 + s) - e^{-t}(1 + t)}{x} + \frac{b_2(s, t)}{x + a_2(s, t)} + \frac{b_3(s, t)}{x + a_3(s, t)} + \cdots (x \to \infty), \]
the latter of which gives explicitly the first two best rational approximations of \( \frac{\pi(e^{e-s}) - \pi(e^{e-t})}{e^x/x} \), where the error in the first approximation \( e^{-s} - e^{-t} \) is asymptotic to \( \frac{b_2(s, t)}{x^3} \), while the error in the second approximation is asymptotic to \( \frac{b_1(s, t)b_2(s, t)}{x^3} \).
Under the obvious transformation, for \( a > b > 0 \) the above yields
\[ \frac{\pi(ax) - \pi(bx)}{x/\log x} \sim a - b + \frac{a(1 - \log a) - b(1 - \log b)}{\log x} + \frac{c_2(- \log a, - \log b)}{1} + \frac{c_3(- \log a, - \log b)}{\log x} + \cdots (x \to \infty) \]
and
\[ \frac{\pi(ax) - \pi(bx)}{x/\log x} \sim a - b + \frac{a(1 - \log a) - b(1 - \log b)}{\log x + a_1(- \log a, - \log b)} + \frac{b_2(- \log a, - \log b)}{\log x + a_2(- \log a, - \log b)} + \cdots (x \to \infty), \]
where \( c_2, c_3, a_1, \) and \( b_2 \) are given explicitly as above. (Here, of course, \( \pi(ax) - \pi(bx) \) is the number of primes \( p \) such that \( bx < p \leq ax \).) It also follows that the first two best approximations of \( \frac{\pi(ax) - \pi(bx)}{x/\log x} \) that are rational functions of \( \log x \) are \( a - b \) and
\[ a - b + \frac{a(1 - \log a) - b(1 - \log b)}{\log x + a(2 - 2 \log a + (\log a)^2) - b(2 - 2 \log b + (\log b)^2)} \cdot \frac{a(1 - \log a) - b(1 - \log b)}{a(1 - \log a) - b(1 - \log b)}. \]
For example, for \( a = 2 \) and \( b = 1 \), the second approximation is
\[ \frac{\pi(2x) - \pi(x)}{x/\log x} \sim 1 - \frac{\log 4 - 1}{\log x + \frac{2(\log 2 - 1)^2}{\log 4 - 1}} \approx 1 - \frac{0.38629436111989}{\log x + 0.48749690534099}. \]
For $a = e$ and $b = 1$ (so $s = -1$, $t = 0$, $r_n(s) = D_n$ (the number of derangements of an $n$-element set), and $r_n(t) = n!$) we computed some additional terms:

\[
\frac{\pi(ex) - \pi(x)}{x/\log x} \sim e^{-1} + \frac{-1}{\log x} + \frac{e-2}{\log x + 1} + \frac{e^2-2e-2}{\log x + 1 + \log x + \frac{e^2-2e-2}{\log x + 1 + \frac{8-e+2e-2-e^2}{e^2-2e-2}}}
\]

\[
\sim \frac{1}{\log x} (x \to \infty)
\]

and

\[
\frac{\pi(ex) - \pi(x)}{x/\log x} \sim e^{-1} + \frac{-1}{\log x} + \frac{e-2}{\log x + e - 2} + \frac{e^2-2e-2}{\log x + e - 2 + \frac{8-e+2e-2-e^2}{e^2-2e-2}}
\]

\[
\sim \frac{16e^3-85e^2+104e+24}{(e^2-2e-2)^2} \quad (x \to \infty).
\]

Thus

\[
\frac{\pi(ex) - \pi(x)}{x/\log x} - (e-1) \sim -\frac{1}{\log x} (x \to \infty),
\]

\[
\frac{\pi(ex) - \pi(x)}{x/\log x} - \left( e-1 + \frac{-1}{\log x + e - 2} \right) \sim \frac{-1(2+2e-e^2)}{(\log x)^3} (x \to \infty),
\]

and

\[
\frac{\pi(ex) - \pi(x)}{x/\log x} - \left( e-1 + \frac{-1}{\log x + e - 2 + \frac{e^2-2e-2}{\log x + 8-e+2e-2-e^2}} \right) \sim \frac{16e^3-85e^2+104e+24}{(e^2-2e-2)^2} (x \to \infty).
\]

In particular, the first three best approximations of $\frac{\pi(ex) - \pi(x)}{x/\log x}$ that are rational functions of $\log x$ are $e-1$, $e-1 + \frac{-1}{\log x + e - 2}$, and $e-1 + \frac{-1}{\log x + e - 2 + \frac{e^2-2e-2}{\log x + 8-e+2e-2-e^2}}$.

References

[1] N. I. Akhiezer, The Classical Moment Problem: and Some Related Questions in Analysis, University Mathematical Monographs, Olver & Boyd, London, 1965.

[2] A. A. M. Cuyt, V. Petersen, B. Verdonk, H. Waadeland, and W. B. Jones, Handbook of Continued Fractions for Special Functions, Springer Science & Business Media, 2008.

[3] J. Elliott, Asymptotic expansions of the prime counting function, submitted.

[4] C.-J. de la Vallée Poussin, Recherches analytiques la théorie des nombres premiers, Ann. Soc. scient. Bruxelles 20 (1896) 183–256.

[5] C.-J. de la Vallée Poussin, Sur la fonction Zeta de Riemann et le nombre des nombres premiers inferieur a une limite donnée, C. Mém. Couronnés Acad. Roy. Belgique 59 (1899) 1–74.

[6] GH from MO (mathoverflow user), Asymptotic for $\text{li}(x) - \text{Ri}(x)$, https://mathoverflow.net/q/308631/17218.
[7] J. Hadamard, Sur la distribution des zéros de la fonction \( \zeta(s) \) et ses conséquences arithmétiq"es, Bull. Soc. Math. France 24 (1896) 199–220.

[8] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Fifth Edition, Oxford Science Publications, Oxford University Press, 2002.

[9] E. Landau, Handbuch der Lehre von der Verteilung de Primzahlen, B. G. Teubner, Leipzig, 1909.

[10] H. L. Montgomery and R. C. Vaughan, Multiplicative Number Theory I. Classical Theory, Cambridge Studies in Advanced Mathematics, Vol. 97, Cambridge University Press, 2007.

[11] L. Panaitopol, Some of the properties of the sequence of powers of prime numbers, Rocky Mt. J. Math. 31 (4) (2001) 1407–1415.