THE MIXED HODGE STRUCTURE ON THE FUNDAMENTAL GROUP OF HYPERELLIPTIC CURVES AND HIGHER CYCLES

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Abstract. In this paper we give a geometrical interpretation of an extension of mixed Hodge structures (MHS) obtained from the canonical MHS on the group ring of the fundamental group of a hyperelliptic curve modulo the fourth power of its augmentation ideal. We show that the class of this extension coincides with the regulator image of a canonical higher cycle in a hyperelliptic jacobian. This higher cycle was introduced and studied by Collino.

0. Introduction

For any pointed variety \((X,p)\) there is a canonical mixed Hodge structure (MHS) on the group ring of its fundamental group modulo the \((k + 1)\)-th powers of its augmentation ideal \(J_p\). It was J.Morgan [18] who first constructed such a MHS for smooth varieties. R.Hain [11] reformulated and extended the theory using Chen’s De Rham homotopy theory.

In the case of a pointed curve \((C,p)\) this MHS contains various kinds of geometric information. For instance, the first of these MHS which goes beyond cohomology, that is the MHS on \(J_p/J^3_p\), determines in general the holomorphic type of \((C,p)\) as was proved by M.Pulte [22, and [10]]. In the course of his proof and using the work of B. Harris [9], Pulte showed that the MHS on \(J_p/J^3_p\) defines an extension class \(m_p \in \text{Ext}^1_{\text{MHS}}(H^3(JC,\mathbb{Z})_{\text{prim}},\mathbb{Z}) \cong F^2H^3(JC,\mathbb{C})^*_{\text{prim}}/H_3(JC,\mathbb{C})_{\text{prim}}\) (where \(H^*(JC,\mathbb{C})_{\text{prim}}\) denotes the primitive cohomology) such that \(2m_p\) equals the Abel-Jacobi image in the primitive intermediate jacobian of the so-called Ceresa cycle \(C_p - C^- p \subset JC\).

In this paper we seek geometrical interpretations of the MHS on \(J_p/J^4_p\). In general the MHS on it has length 2 and it is an extension of \(J_p/J^3_p\). To start this investigation we restrict ourselves to hyperelliptic curves. In fact for a hyperelliptic curve with Weierstraß point \(p\) the short exact sequence \(0 \to J^2_p/J^3_p \to J_p/J^3_p \to J_p/J^2_p \to 0\) splits as extension of MHS mod torsion, corresponding to the fact that in this case \(C_p = C^- p\) (cf. Prop.2.1). Hence there is a natural way to transform parts of the MHS on \(J_p/J^4_p\) into extensions of MHS’s that are purely built up from the cohomology. As in Pulte’s theorem on \(J_p/J^3_p\), this allows us to compare these extensions with known geometrical data. Here we give an interpretation of one of these extensions in terms of the regulator image \(\text{reg}(Z)\), where \(Z\) is a canonical higher cycle \(Z\) constructed by Collino [1] on the jacobian of a hyperelliptic curve with two fixed Weierstraß points \(q_1\) and \(q_2\). It is not difficult to check that the other subextensions can be built from \(J_p/J^3_p\). As explained in [1] 1.1, \(\text{reg}(Z)\) is \(J_2(JC)_{\text{prim}} := F^1H^2(JC,\mathbb{C})^*_{\text{prim}}/H^2(JC,\mathbb{C})_{\text{prim}}\) can be interpreted as a degeneration of the Abel-Jacobi image of the Ceresa cycle for the singular curve obtained from \(C\) by gluing \(q_1\) and \(q_2\). The extension class \(Pe \in \text{Ext}^1_{\text{MHS}}(H^2(JC)_{\text{prim}},\mathbb{Z}) \cong J_2(JC)_{\text{prim}}\) that we compare with \(\text{reg}(Z)\) is in fact obtained from the MHS on the group ring of the fundamental group of the punctured curves \(C - \{q_1\}\) and \(C - \{q_2\}\), modulo the fourth power of the augmentation ideal. The main result (Th. 2.1) is the equality:

\[ Pe = (2g + 1)\text{reg}(Z). \]
In section 1 we recall briefly the definition of regulator map and we write the regulator image of the higher cycle $Z$ in terms of integrals and iterated integrals on the curve. In section 2 we construct the extension $P_e$ and state Theorem 2.1. In section 3, using properties of iterated integrals and extension theory, we prove the theorem. The proof is a very explicit computation of both terms of the equality on a basis of $F^1H^2(JC,\mathbb{C})_{prim}$. In section 4 and 5 we extend these constructions to families. In particular in section 4 we provide an alternative proof of the result of Collino that $reg(Z)$ is not zero for general hyperelliptic $C$, by showing that the homomorphism of fundamental groups induced by the normal function extending $reg(Z)$ is not trivial (Cor.4.2). Notice that the non triviality of $reg(Z)$ implies that $Z$ is an indecomposable cycle. Finally in section 5 we study the homomorphisms of fundamental groups induced by normal functions associated to the extensions of Sec.2.

Acknowledgments: I wish to thank A.Collino, R.Kaenders and E.Looijenga for very useful discussions and suggestions.

1. The regulator map for hyperelliptic jacobians

The main goal of this section is to write down the regulator image of the higher cycle $Z$ constructed by Collino on the jacobian of a hyperelliptic curve in terms of (iterated) integrals on the curve. First we recall the definition of the regulator and the construction of $Z$.

1.1. The regulator. Let $X$ be a smooth projective variety of dimension $n$. Let $CH^n(X,1)$ be the first higher Chow group. (For the general theory of higher Chow groups and regulator maps we refer to [3] and [1]). An element in $CH^n(X,1)$ is defined by the ‘cycle’ $A := \sum_i(C_i, f_i)$, where $C_i$ is an irreducible curve on $X$ and $f_i$ is a rational function on $C_i$, such that $\sum_i [div(f_i)] = 0$ on $X$ (see [3], 1.3 pag.10 for the definition of the 0-cycle $[div(f_i)]$). Denote by $[0, \infty]$ the positive real axis on $\mathbb{P}^1$ and let $\gamma_i := \mu_i f_i^{-1}([0, \infty])$, where $\mu_i : \tilde{C}_i \to C_i$ is the resolution of singularities of the curve $C_i$ and $f_i$ is viewed as a map $f_i : \tilde{C}_i \to \mathbb{P}^1$. The condition $\sum_i [div(f_i)] = 0$ implies that the 1-chain $\sum \gamma_i$ is a 1-cycle and in fact up to torsion it is a boundary. Suppose $H_1(X,\mathbb{Z})$ has no torsion (otherwise the construction has to be done over $\mathbb{Q}$), the Bloch-Beilinson regulator map in this particular case is defined in the following way:

$$\text{reg} : CH^n(X,1) \to J_2(X) := \frac{(F^1H^2(X,\mathbb{C}))^*}{H_2(X,\mathbb{Z}(1))}, \quad A = \sum_i(C_i, f_i) \mapsto \text{reg}(A)$$

where $\text{reg}(A)$ is the class of the current:

$$\alpha \mapsto \sum_i \int_{C_i - \text{sing} C_i} \log(f_i)\alpha + 2\pi i \int_D \alpha$$

with $\alpha$ a closed 2-form on $X$ whose cohomology class is in $F^1H^2(X,\mathbb{C})$ and $D$ a 2-chain such that $\partial D = \sum \gamma_i$. An analogous definition can be given using instead the primitive cohomology:

$$\text{reg} : CH^n(X,1) \to J_2(X)_{prim} := \frac{F^1H^2(X,\mathbb{C})_{prim}}{H_2(X,\mathbb{Z}(1))_{prim}}$$

1.2. The Collino cycle. Our variety $X$ will be the jacobian $JC$ of a hyperelliptic curve $C$ of genus $g$. We will follow [3] to construct a canonical higher cycle, $Z := (C_1, h_1) + (C_2, h_2)$, as follows. For Weierstraß points $q_1$ and $q_2$, let $h$ be a 2 to 1 morphism

$$h : C \to \mathbb{P}^1, \quad h(q_1) = 0, \quad h(q_2) = \infty.$$

Call $C_s$ (for $s = 1,2$) the image of $C$ via the Abel Jacobi map:

$$\mu_{qs} : C \to JC, \quad x \mapsto x - q_s, \quad C_s := \mu_{qs}(C).$$
If we think of $\mu_{qs}$ as a biholomorphism onto its image we can define $h_s : C_s \rightarrow \mathbb{P}^1$, $h_s := h \circ \mu_{qs}^{-1}$. The rational functions $h_1$ and $h_2$ on $C_1$ and $C_2$ satisfy $\text{div}(h_1) = 2\sigma - 2O = -\text{div}(h_2)$, where $O$ is the origin of $JC$ and $\sigma := q_2 - q_1 \in JC$. The point $\sigma$ is 2 torsion, since $q_1$ and $q_2$ are Weierstraß points. Thus $Z = (C_1, h_1) + (C_2, h_2) \in CH^g(JC, 1)$ and Collino proved that $\text{reg}(Z)$ is not zero, for general $C$.

1.3. **Basic properties of iterated integrals.** We recall the definition of iterated integrals. Suppose $X$ is a smooth manifold. Given $\gamma : [0, 1] \rightarrow X$ a piece-wise smooth path in $X$, and smooth 1 forms $\omega_1, \omega_2, \omega_3$ on $X$ on $X$ we define an iterated integral by:

$$\int_{\gamma} \omega_1 \omega_2 \omega_3 := \int_{0}^{1} \int_{0}^{t_1} \int_{0}^{t_2} f(t_1, t_2, t_3) dt_3 dt_2 dt_1$$

where $\gamma^{i} = f_j(t) dt$.

For an introduction to the subject we refer to [10]. Here we only collect some properties of iterated integral that we need in the sequel and that are easy to check from the definition:

**Lemma 1.1.** If $\omega_1$ and $\omega_2$ are smooth 1-forms, $df$ is a smooth exact 1-form and $\alpha$ and $\beta$ are piece-wise smooth paths in $X$ with $\alpha(1) = \beta(0)$ then:

1. $\int_{\alpha \beta} \omega_1 \omega_2 = \int_{\alpha} \omega_1 \omega_2 + \int_{\beta} \omega_1 \omega_2 + \int_{\alpha} \omega_1 \beta \omega_2$
2. $\int_{\alpha} \omega_1 \omega_2 + \int_{\beta} \omega_2 \omega_1 = \int_{\alpha} \omega_1 \int_{\beta} \omega_2$
3. $\int_{\alpha} df \omega_1 = \int_{\alpha} f \omega_1 - f(\alpha(0)) \int_{\alpha} \omega_1$
4. $\int_{\alpha} \omega_1 df = f(\alpha(1)) \int_{\alpha} \omega_1 - \int_{\alpha} f \omega_1$.

1.4. In the next lemmas we show that for $\text{reg}(Z)$ the expression $\int_{D}$ in [12] can be written as an iterated integral. First we fix some notations. Let $[0, \infty)$ be the positive real axis in $\mathbb{P}^1$, $\gamma := h^{-1}([0, \infty])$, with $h$ as in [14]. Since $h$ is a covering of degree 2 outside the set of Weierstraß points we can write $\gamma$ as a union of two paths $\gamma = \gamma^+ + \gamma^-$ where $\gamma^+$ and $\gamma^-$ live in different sheets of $h$ and have in common just the Weierstraß points in $\gamma$. We fix the diffeomorphism $[0, 1] \rightarrow [0, \infty] \subset \mathbb{P}^1$, $t \mapsto \frac{t}{1-t}$ and a $C^\infty$ parametrization of $\gamma^\pm$ compatible with it, i.e.:

$$\gamma^\pm : [0, 1] \rightarrow C \quad t \mapsto \gamma^\pm(t) \quad \text{with} \quad h(\gamma^\pm(t)) = \frac{t}{1-t} \in [0, \infty] \subset \mathbb{P}^1$$

1.5. **Remark.** We also consider, for $s = 1, 2$, $\gamma_s = \mu_{qs}(\gamma)$, written as $\gamma_s = \gamma^+_s + \gamma^-_s$ with $\gamma^+_s = \mu_{qs}(\gamma^+_s([0, 1]))$. Hence $\gamma^+_s$ are paths with $C^\infty$ parameterizations $\mu_s \circ \gamma^+_s$.

In the next lemma we show that $\gamma_1 + \gamma_2$ is the boundary of a 2 chain $D$ in $JC$ which is the sum of two parameterized disks $D^+$ and $D^-$.  

**Lemma 1.2.** Let $D^\pm \subset JC$ be the images of

$$F^\pm : [0, 1] \times [0, 1] \rightarrow JC, \quad (t, s) \mapsto \gamma^\pm \left(1 - \frac{t(1-s)}{1-s(t-1)}\right) - \gamma^\pm(1-t),$$

then $D^\pm$ has boundary $\partial D^\pm$ with parameterization $\gamma^+_1 \gamma^-_2$ and hence $\partial D = \partial(D^+ + D^-) = \gamma^+_1 + \gamma^-_2$.

**Proof.** Restrict the map $F^+$ to the boundary of $[0, 1] \times [0, 1]$:

- $s = 0$, \quad $\{\gamma^+(1-t) - \gamma^+(1-t)\} = O$
- $t = 1$, \quad $\{\gamma^+(s) - q_1\} = \gamma^+_s \subset C_1$
- $s = 1$, \quad $\{q_2 - \gamma^+(1-t)\} = \{\gamma^-(1-t) - q_2\} = (\gamma^-_2)^{-1} \subset C_2$
- $t = 0$, \quad $\{q_2 - q_2\} = O$

so the oriented boundary of $D^+$ is $\partial D^+ = \gamma^+_1 + \gamma^-_2$. (Note that $\gamma^+_1(0) = O = \gamma^-_2(1)$ and $\gamma^+_1(1) = q_2 - q_1 = \sigma = \gamma^-_2(0)$.) The same computation yields $\partial D^- = \gamma^+_1 + \gamma^-_2$. \qed
Lemma 1.3. Let $\phi, \psi$ be closed 1-forms with $\psi$ of type 1,0 on $J(C)$. Then, for $D^\pm$ as in the previous lemma,

$$\int_{D^\pm} \phi \wedge \psi = \int_{\gamma_1^\pm} \phi \psi - \int_{\gamma_2^\pm} \psi \phi,$$

where $\int \phi \psi$ denotes the iterated integral.

Proof. Every closed form on a disk is exact, so $\phi|_{D^\pm} = d\rho^\pm$, then by Stokes theorem:

$$\int_{D^\pm} \phi \wedge \psi = \int_{D^\pm} d(\rho^\pm \psi) = \int_{\partial D^\pm} \rho^\pm \psi = \int_{\gamma_1^\pm \gamma_2^\pm} \rho^\pm \psi.$$

Moreover by Lemma 1.1 (3), choosing $\rho^\pm(0) = 0$:

$$\int_{D^\pm} \phi \wedge \psi = \int_{\gamma_1^\pm} \rho^\pm \psi = \int_{\gamma_2^\pm} \phi \psi$$

and by Lemma 1.1 (1)

$$= \int_{\gamma_1^\pm} \phi \psi + \int_{\gamma_2^\pm} \phi \psi + \int_{\gamma_1^\pm} \phi \int_{\gamma_2^\pm} \psi.$$

Now note that $\int_{\gamma_1^\pm} \phi + \int_{\gamma_2^\pm} \phi = 0$ since $\gamma_1^\pm \gamma_2^\pm = \partial D^\pm$ is homotopically trivial, so

$$\int_{D^\pm} \phi \wedge \psi = \int_{\gamma_1^\pm} \phi \psi + \int_{\gamma_2^\pm} \phi \psi - \int_{\gamma_1^\pm} \phi \int_{\gamma_2^\pm} \psi = \int_{\gamma_1^\pm} \phi \psi - \int_{\gamma_2^\pm} \psi \phi$$

where the last equality follows from Lemma 1.1 (2).

We are ready to compute $\text{reg}(Z)$. It is enough to do it on harmonic forms:

Theorem 1.1. Let $\phi$ and $\psi$ be harmonic 1-forms on $J(C)$ with $\psi$ of type 1,0 and denote in the same way the corresponding 1-forms on $C$. Then:

$$\text{reg}(Z)(\phi \wedge \psi) = 2 \int_{C-\gamma} \log(h) \phi \wedge \psi + 2\pi i \int_{\gamma} (\phi \psi - \psi \phi).$$

Proof. From Lemma 1.3 $\int_{D} \phi \wedge \psi = \int_{\gamma_1} \phi \psi - \int_{\gamma_2} \psi \phi$. Now use the fact that the harmonic forms on $JC$ are translation invariant.

1.6. Remark. The right hand side of the equality of Thm. 1.1 can be computed more generally for $\phi$ and $\psi$ closed 1-form, since it is zero if one of them is exact.

2. Extensions

The theory of iterated integrals for pointed Riemann surfaces $(C, p)$ and pointed punctured ones $(C - \{q\}, p)$ describes explicitly the canonical MHS on the quotients $J_p/J^k_p$ and $J_{q,p}/J^k_{q,p}$, where $J_p := \ker(\epsilon : Z\pi_1(C, p) \to Z)$ and $J_{q,p} := \ker(\epsilon : Z\pi_1(C - \{q\}, p) \to Z)$ (we refer to [10] and in particular for punctured curves to [17]). The weight filtrations on the duals are given, for $l \leq k$, by $W_l(J_p/J^k_p)^* := (J_p/J^{l+1}_p)^*$ and $W_l(J_{q,p}/J^k_{q,p})^* := (J_{q,p}/J^{l+1}_{q,p})^*$. The graded factors have the following identifications:

$$(J_p^l/J^{l+1}_p)^* \simeq Q_l(C) := \bigcap_{i=0}^{i-l-2} (\otimes^i H^1(C, Z)) \otimes Q_2(C) \otimes (\otimes^{l-2-i} H^1(C, Z))$$

where $Q_2 := \ker(\cup : H^1(C, Z) \otimes H^1(C, Z) \to H^2(C, Z))$, and

$$(J_{q,p}^l/J^{l+1}_{q,p})^* \simeq \otimes^i H^1(C - \{q\}, Z) \simeq \otimes^i H^1(C, Z).$$
The Hodge filtration is given by Chen’s $\pi_1$-De Rham -Theorem (cf. [10]). This MHS is compatible with the natural extensions:

\begin{equation}
(2.3) \quad h^k_p : 0 \to (J_p/J_p^{k-1})^* \to (J_p/J_p^k)^* \to Q_{k-1}(C) \to 0.
\end{equation}

\begin{equation}
(2.4) \quad h^k_{q,p} : 0 \to (J_{q,p}/J_{q,p}^{k-1})^* \to (J_{q,p}/J_{q,p}^k)^* \to \otimes^{k-1}H^1(C,\mathbb{Z}) \to 0.
\end{equation}

2.1. Hyperelliptic case. Let now $C$ be a hyperelliptic curve with hyperelliptic involution $i$. It holds:

**Proposition 2.1.** Let $C$ be a hyperelliptic curve and let $p$ and $q$ be Weierstraß points. Then the extensions classes $h^3_p \in Ext_{MHS}(Q_2(C), H^1(C,\mathbb{Z}))$ and $h^3_{p,q} \in Ext_{MHS}(\otimes^2 H^1(C,\mathbb{Z}), H^3(C,\mathbb{Z}))$ are 2-torsion, i.e. $2h^3_p = 0$ and $2h^3_{q,p} = 0$.

**Proof.** The hyperelliptic involution $i$ induces an automorphism of $(J_p/J_p^3)^*$ such that $h^3_p \sim i(h^3_p) = -h^3_p$. Hence $2h^3_p = 0$ in $Ext_{MHS}(Q_2, H^1)$. The same argument holds for $h^3_{q,p}$.

Let $p$, $q_1$ and $q_2$ be Weierstraß points on $C$. Fix a 2 to 1 map: $h : C \to \mathbb{P}^1$ such that $h(q_1) = 0$ and $h(q_2) = \infty$ as in the construction of the Collino cycle $Z$. Moreover fix a set of loops $\{\alpha_l\}$ ($l = 1, \ldots, 2g$) on $C$ with basepoint $p$ whose homotopy classes give a system of generators of $\pi_1(C, p)$ with the relation: $\prod_k [\alpha_k, \alpha_{g+k}]$. We choose all $\alpha_l$ not passing through $q_1$ and $q_2$ so they define also a system of generators of the groups $\pi_1(C - \{q_s\}, p), (s = 1, 2)$. Let $\{A_l\}, (A_l := [\alpha_l], \in H_1(C,\mathbb{Z}))$ be the associated symplectic basis and $\{dx_l\}$ the dual basis of $H^1(C,\mathbb{Z})$. We will identify $dx_l$ with the corresponding harmonic 1-form. From now on we denote $H^1(C,\mathbb{Z})$ by $H^1$.

**Proposition 2.2.** The linear map $r^3_{3,2} : (J_{q,p}/J_{q,p}^3)^* \to H^1$, which is the dual of the linear map defined by: $A_l \mapsto (\alpha_l - i_*\alpha_l) \mod(J_{q,p}^3)$, is a morphism of MHS.

**Proof.** The space $H_1(C,\mathbb{Q})$ can be identified with the eigenspace of the eigenvalue $-1$ of the involution $i_* \in Aut((J_{q,p}/J_{q,p}^3)\mathbb{Q})$.

In order to define the extension class $Pe$ of the main theorem, we need also the following natural morphisms of MHS:

1. The monomorphism given by tensoring with the polarization $\Omega$:

\begin{equation}
J_\Omega = \otimes \Omega : H^1(-1) \to \otimes^3 H^1
\end{equation}

where $H^1(-1)$ denotes $H^1$ twisted by the Tate Hodge structure $\mathbb{Z}(-1)$.

2. The surjection:

$\Pi : \otimes^2 H^1 \to \mathbb{Z}(-1),$

given by the cup product $\otimes^2 H^1 \to H^2$ composed with the isomorphism $H^2(C,\mathbb{Z}) \simeq \mathbb{Z}(-1)$ defined by the integration over $C$.

3. The monomorphism $\iota : \wedge^2 H^1 \to H^1 \otimes H^1$, $\phi \wedge \psi \mapsto \phi \otimes \psi - \psi \otimes \phi$.

Notice that the map $\Pi \circ \iota$ can be identified with the integration $\int_C : \wedge^2 H^1 \to \mathbb{Z}$ over $C$ and it’s in this form that we shall often write it. Set

$\wedge^2 H^1_{prim} := \ker \Pi \circ \iota = \ker \int_C$.
2.2. Main result. Let

\[ e_s \in \text{Ext}_{\text{MHS}}(\otimes^3 H^1, H^1) \]

(for \( s = 1, 2 \)) be the extension class obtained by pushing forward \( h^4_{q_s, p} \) along \( r^s_{3, 2} \). The pull back of \( e_2 - e_1 \) along \( J_\Omega \) defines the extension class

\[ e_\Omega \in \text{Ext}_{\text{MHS}}(H^1(-1), H^1). \]

By tensoring by \( H^1 \) on the left and then by pushing it down along \( \Pi \) one obtains

\[ \tilde{e} \in \text{Ext}_{\text{MHS}}(\otimes^2 H^1(-1), \mathbb{Z}(-1)) \simeq \text{Ext}_{\text{MHS}}(\otimes^2 H^1, \mathbb{Z}). \]

The pull back along the monomorphism \( \iota : \wedge^2 H^1 \to \otimes^2 H^1 \) defines

\[ e \in \text{Ext}_{\text{MHS}}(\wedge^2 H^1, \mathbb{Z}). \]

Finally the pullback along \( \text{Ker} \int_C \hookrightarrow \wedge^2 H^1 \) defines

\[ Pe \in \text{Ext}_{\text{MHS}}(\text{Ker} \int_C, \mathbb{Z}). \]

The isomorphism \( \wedge^2 H^1 \simeq H^2(JC, \mathbb{Z}) \) and the standard theory of separated extensions of MHS (see [4]), that we briefly recall in the next section, tell us that we can identify:

\[ \text{Ext}_{\text{MHS}}(\wedge^2 H^1, \mathbb{Z}) \simeq \frac{\text{Hom}(H^2(JC, \mathbb{C}), \mathbb{C})}{F^0 + \text{Hom}(H^2(JC, \mathbb{Z}), \mathbb{Z})} \simeq J_2(JC). \]

Moreover the identification \( \wedge^2 H^1_{\text{prim}} \) with \( H^2(JC)_{\text{prim}} \) gives the isomorphism

\[ \text{Ext}_{\text{MHS}}(\wedge^2 H^1_{\text{prim}}, \mathbb{Z}) \simeq J_2(JC)_{\text{prim}}. \]

The main result is:

**Theorem 2.1.** Let \( C \) be a hyperelliptic curve and let \( q_1, q_2 \) and \( p \) be Weierstraß points. Let \( h : C \to \mathbb{P}^1 \) a 2:1 map with \( h(q_1) = 0 \) and \( h(q_2) = \infty \), then

\[ (2.7) \quad e = (2g + 1) \left( \text{reg}(Z) + \text{log}(h(p)) \right) \int_C \in J_2(JC), \]

which implies

\[ (2.8) \quad Pe = (2g + 1)\text{reg}(Z) \in J_2(JC)_{\text{prim}}. \]

3. Carlson’s representatives and proof of 2.1

To a MHS \( V \) whose weights are all negative, can be associated the intermediate jacobian:

\[ (3.1) \quad J(V) := \frac{V_C}{F^0 V_C + V_Z}. \]

An extension of MHS \( 0 \to A \to H \to B \to 0 \) is called separated if the minimal non zero weight of \( B \) is bigger then the maximal non zero weight of \( A \). Thus in particular \( \text{Hom}(B, A) \) has all negative weights. Carlson’s theory of separated extensions of MHS defines the isomorphism (see [4])

\[ \text{Ext}_{\text{MHS}}(B, A) \simeq J(\text{Hom}(B, A)), \]

which associates to the class of \( 0 \to A \to H \to B \to 0 \) the class of the composed map \( r_Z \circ s_F \in \text{Hom}(B_C, A_C) \) (called the Carlson representative), where \( s_F \in \text{Hom}(B_C, H_C) \) is a section preserving the Hodge filtration and \( r_Z \in \text{Hom}(H, A) \) is a retraction of \( \mathbb{Z} \) modules. In this section we first describe explicitly the Carlson representatives of the extensions classes introduced in Sect.2, then we manipulate these expressions using just basic properties of integrals over Riemann surfaces and iterated integrals and at the end we prove Thm.2.1.
3.1. The extension class \( h_{q_s,p}^4 \in Ext_{MHS}(\otimes^3 H^1, (J_{q_s,p}/J_{q_s,p}^3)^*) \). Via Carlson theory, \( h_{q_s,p}^4 \), defined in \( \ref{2.4} \), corresponds to the class of \( (r^s_{4,3}) \circ s^4_F \) defined in the following way. The linear map

\[
r^s_{4,3} : (J_{q_s,p}/J_{q_s,p}^4)^* \rightarrow (J_{q_s,p}/J_{q_s,p}^3)^*
\]

is the dual of a linear map defined by fixing a basis in \( J_{q_s,p}/J_{q_s,p}^3 \) and lifting the elements of this basis to independent elements in \( J_{q_s,p}/J_{q_s,p}^4 \). The only condition required is that the chosen basis contains the elements \((\alpha_l - 1) mod J_{q_s,p}^3\) which are lifted to \((\alpha_l - 1) mod J_{q_s,p}^4\).

The section preserving the Hodge filtration \( s^4_F : \otimes^3 H^1 \rightarrow (J_{q_s,p}/J_{q_s,p}^4)^* \) is provided by Chen theory:

\[
(3.2) \quad s^4_F(dx \otimes dx_m \otimes dx_n) = \int dx_l dx_m dx_n + dx_l \mu_{mn,q_s} + \mu_{lm,q_s} dx_n + \mu_{lmm,q_s}
\]

where \( \mu_{lm,q_s}, \mu_{mn,q_s} \) and \( \mu_{lmm,q_s} \) are smooth, logarithmic forms on \( C - \{q_s\} \) satisfying:

\[
(3.3) \quad dx_l \wedge dx_m + d\mu_{lm,q_s} = 0, \quad dx_m \wedge dx_n + d\mu_{mn,q_s} = 0,
\]

\[
(3.4) \quad dx_l \wedge \mu_{mn,q_s} + \mu_{lm,q_s} \wedge dx_n + d\mu_{lmm,q_s} = 0.
\]

3.2. The extension class \( e_s \in Ext_{MHS}(\otimes^3 H^1, H^1) \) \((s = 1, 2)\). Since \( e_s \) is obtained by pushing forward along \( r^s_{3,2} \), it can be identified as the class of the map

\[
G_s := (r^s_{3,2} \circ r^s_{4,3}) \circ s^4_F \in Hom(\otimes^3 H^1, H^1).
\]

Notice that the map \( r^s_{3,2} \circ r^s_{4,3} : (J_{q_s,p}/J_{q_s,p}^4)^* \rightarrow H^1 \) is in fact the dual of the linear map given by \( A_l \mapsto (\alpha_l - i_s \alpha_l) mod J_{q_s,p}^4 \) and it is not longer a morphism of MHS as it is \( r^s_{3,2} \).

3.3. The extension class \( e_\Omega \in Ext_{MHS}(H^1(-1), H^1) \). The extension \( e_\Omega \) is obtained by pulling back \( e_2 - e_1 \) along \( J_\Omega \), hence it is represented by:

\[
G := (G_2 - G_1) \circ J_\Omega \in Hom(H^1(-1), H^1).
\]

3.4. In the next proposition we compute \( G \) explicitly on the basis chosen in \( \ref{2.4} \). Recall that \( \Omega \) can be written in coordinates as

\[
\Omega = \sum_k (dx_k \otimes dx_{g+k} - dx_{g+k} \otimes dx_k) \in \otimes^2 H^1.
\]

3.5. Assumption. Notice that we can choose solutions of \( \ref{3.3} \) and \( \ref{3.4} \) satisfying the properties listed below (for \( s = 1, 2)\):

1. \( \mu_{ml,q_s} = -\mu_{tm,q_s} \);
2. for \(|l - m| \neq g\), \( \mu_{tm,q_s} \) is smooth on \( C \) and orthogonal to all harmonic forms, i.e. \( \mu_{tm,q_s} \wedge dx_n \) is exact;
3. \( \mu_{i(g+i),q_2} \) has logarithmic singularity on \( q_2 \) with residue 1;
4. for \(|l - m| \neq g\), \( \mu_{lm,q_1} = \mu_{lm,q_2} \);
5. \( \mu_{i(g+i),q_1} = \mu_{i(g+i),q_2} + dh/2h \). Notice that \( \mu_{i(g+i),q_1} \) has a pole of order 1 on \( q_1 \) with residue 1;
6. \( i^*\mu_{tm,q_s} = \mu_{tm,q_s}; i^*\mu_{lmm,q_s} = -\mu_{lmm,q_s} \).

Proposition 3.1. With the choices done in Assumption \( \ref{3.3} \), a map \( G : H^1(-1) \rightarrow H^1 \), whose class defines \( e_\Omega \), is given by:

\[
(3.5) \quad G(dx_l)(A_m) = \int_{\alpha_m} [(2g + 1)(\log(h) - \log(h(p)))dx_l - 2W(dx_l)]
\]
where

\[(3.6)\quad W(dx_l) := \sum_{k=1}^{g}\{ (\mu_{k(g+k),q_2} - \mu_{k(g+k),q_1}) - (\mu_{l(g+k),q_2} - \mu_{l(g+k),q_1}) \}.\]

\[\text{Proof.}\] From the definition of \(J\) and \(s^A_F\) and by the equality \(\mu_{(g+k)q_2} = -\mu_{(g+k)q_1}\) of Assumption 3.5 (1):

\[G(dx_l)(A_{m}) = \int_{(\alpha_m - i_{\alpha_m})} \left[ \sum_{k} \left\{ 2dx_l(\mu_{k(g+k),q_2} - \mu_{k(g+k),q_1}) + (\mu_{k,q_2} - \mu_{k,q_1})dx_{g+k} - (\mu_{l(g+k),q_2} - \mu_{l(g+k),q_1})dx_k \right\} + W(dx_l) \right].\]

By Assumption 3.5 (4), setting \(i = l\) if \(l \leq g\) and \(i = l - g\) if \(l > g\):

\[G(dx_l)(A_{m}) = \int_{(\alpha_m - i_{\alpha_m})} \left[ 2dx_l \sum_{k} (\mu_{k(g+k),q_2} - \mu_{k(g+k),q_1}) - (\mu_{l(g+l),q_2} - \mu_{l(g+l),q_1})dx_l + W(dx_l) \right]\]

and by Assumption 3.5 (5):

\[G(dx_l)(A_{m}) = -\int_{(\alpha_m - i_{\alpha_m})} \left[ gdx_l (dh/h) - \frac{1}{2}(dh/h)dx_l + W(dx_l) \right].\]

From the equalities:

\[\int_{i_{\alpha_m}} [gdx_l (dh/h) - \frac{1}{2}(dh/h)dx_l] = -\int_{\alpha_m} [gdx_l (dh/h) - \frac{1}{2}(dh/h)dx_l]\]

coming from \(i^*dx_l = -dx_l, i^*dh/h = dh/h\) and,

\[\int_{i_{\alpha_m}} W(dx_l) = -\int_{\alpha_m} W(dx_l),\]

coming from \(i^*W(dx_l) = -W(dx_l)\) (see Assumption 3.5 (6)) it follows that:

\[(3.7)\quad G(dx_l)(A_{m}) = \int_{\alpha_m} [-2gdx_l (dh/h) - (dh/h)dx_l - 2W(dx_l)].\]

On \(C - \gamma, dh/h\) is an exact form. Hence if \(\alpha_m \cap \gamma = \phi\), then the statement follows from Lemma 1.1 (3) and (4). If \(\alpha_m \cap \gamma \neq \phi\), then the computation has to be done on a path lifting \(\alpha_m\) on a covering of \(C\) where \(dh/h\) is exact. But the difference between it and the expression 3.5 is given by a multiple of \(2\pi i \int_{\alpha_m} dx_l\). Hence it defines an element in \(Hom_{\mathbb{Z}}(H^1(-1), H^1)\), which is trivial in \(J(Hom(H^1(-1), H^1))\). \(\square\)

3.6. The extension class \(\tilde{e} \in Ext_{MHS}(\otimes^2 H^1, \mathbb{Z})\). The extension \(\tilde{e}\) is constructed by tensoring \(e_g - e_1\) by \(H^1\) on the left and pushing forward along \(\Pi\) so it can be identified with the class of the map:

\[(3.8)\quad F := \Pi \circ (id \times G) \in (\otimes^2 H^1)^*.\]

**Proposition 3.2.** Let \(F\) be the map defined in 3.3. We have

\[(3.9)\quad F(dx_m \otimes dx_l) = c(m) \int_{\alpha_{\sigma(m)}} [(2g + 1)(\log(h) - \log(h(p)))dx_l - 2W(dx_l)]\]

where \(\sigma(m) = g + m\) and \(c(m) = 1\) if \(m \leq g\), \(\sigma(m) = m - g\) and \(c(m) = -1\) if \(m > g\).
Proof. The map $\Pi : \otimes^2 H^1 \to \mathbb{Z}$ (cf. [2, 11.(2)]) can be written, with our choice of basis, as:

$$\Pi(v \otimes w) = \sum_{k=1}^{g} [v(A_k)w(A_{g+k}) - v(A_{g+k})w(A_k)].$$

Hence

$$F(dx_m \otimes dx_l) = \Pi \circ (id \otimes G)(dx_m \otimes dx_l) =$$

$$\sum_k [dx_m(A_k)G(dx_l)(A_{g+k}) - dx_m(A_{g+k})G(dx_l)(A_k)]$$

and finally, by 3.3 and by Lemma [11](3)

$$= c(m)G(dx_l)(A_{\sigma(m)})$$

$$= c(m) \int_{\alpha_{\sigma(m)}} (2g + 1)(\log(h) - \log(h(p)))dx_l - 2W(dx_l).$$

The next proposition is a fundamental step toward the equality of Th.3.1. It provides an identification of the key integrals over $C$ with iterated integrals along paths. Let $\gamma := h^{-1}([0, \infty])$ as in Sect.1.

**Proposition 3.3.** Let $\alpha$ be a simple smooth loop on $C$ transverse to $\gamma$. Let $\phi$, $\psi$ and $\varpi$ be 1-forms such that $\phi$, $\psi$ and $(\log(h)\psi + \varpi)$ are closed and the cohomology class of $\phi$ is the Poincaré dual of $[\alpha]$. Then:

$$\int_{\alpha} (\log(h)\psi + \varpi) = \int_{C-\gamma} \phi \land (\log(h)\psi + \varpi) + 2\pi i \int_{C} \phi \psi.$$  

Proof. Denote by $\eta$ a closed 1-form in the same cohomology class of $\phi$ with compact support on a tubular neighborhood of $\alpha$. Hence $\phi = \eta + df$ where $df$ is an exact 1-form. Thus $df \land (\log(h)\psi + \varpi) = d(f(\log(h)\psi + \varpi))$ is an exact form on $C - \gamma$ or, equivalently, on the Riemann surface with boundary obtained from $C$ cutting along $\gamma$. Hence by Stokes theorem and by taking the difference of the determinations for $\log$ at the boundary, we have:

$$\int_{C-\gamma} df \land (\log(h)\psi + \varpi) = -2\pi i \int_{\gamma} f \psi.$$  

Choose $f$ such that $f(p_0) = 0$, with $p_0$ the basis point of $\alpha$, so

$$\int_{\gamma} f \psi = \int_{\gamma} (f - f(p_0)) \psi = \int_{\gamma} df \psi.$$  

To compute $\int_{C} \eta \land (\log(h)\psi + \varpi)$ we recall that the class of $\eta$ is the Poincaré dual of the class $\alpha$. We give a more explicit construction of such $\eta$ with support on a tubular neighborhood $D = D^+ \cup D^-$ of $\alpha$. Following for example [11, 11.3.3] let $G$ be a $C^\infty$ function on $C - \alpha$, which is the constant 1 on a smaller strip $D_0 \subset D^-$ and 0 on $C - D^-$. Then take $\eta$ equal to $dG$ in $D - \alpha$ and 0 otherwise. We distinguish two cases. First suppose that $\alpha$ doesn’t intersect $\gamma$. Then we can take $D \cap \gamma = \phi$, so $\log(h)\psi + \varpi$ is a closed form well defined on the support of $\eta$, and:

$$\int_{C-\gamma} \eta \land (\log(h)\psi + \varpi) = \int_{\alpha} (\log(h)\psi + \varpi).$$  

Moreover since in this case $\phi|_{\gamma} = df|_{\gamma}$, adding [3.10] and [3.11] gives the result. Suppose now that $\alpha$ intersects $\gamma$. Notice that now $\log(h)$ is not well defined on $D$ and we need to compute the integral on disjoint union of rectangles $D''$ obtained by cutting $D$ along $D \cap \gamma$. Applying Stokes’ theorem:
\[
\int_{C-\gamma} \eta \wedge (\log(h)\psi + \varpi) = \int_{D^\gamma} \eta \wedge (\log(h)\psi + \varpi)
\]
\[
= \int_{\alpha} (\log(h)\psi + \varpi) - 2\pi i \int_{\gamma \cap D^\gamma} dG\psi,
\]
since \(G\) is 0 outside \(D_0\), and \(\eta = dG\) on \(C - \alpha\), we obtain
\[
= \int_{\alpha} (\log(h)\psi + \varpi) - 2\pi i \int_{\gamma} \eta\psi.
\]
To conclude we add the equalities \eqref{3.10} and \eqref{3.12} recalling that \(\phi|_{\gamma} = \eta|_{\gamma} + df|_{\gamma}\).

\[\square\]

**Corollary 3.1.** Choosing as \(\alpha_m\) simple smooth loops transverse to \(\gamma\), we get:
\[
\int_{C-\gamma} dx_m \wedge \left( (\log(h)dx_i - \frac{2}{2g+1}W(dx_i)) + 2\pi i \int_{\gamma} dx_m dx_i =
\right.
\]
\[
\left. c(m) \int_{\alpha_{\sigma(m)}} (\log(h)dx_i - \frac{2}{2g+1}W(dx_i)) \right) \tag{3.13}
\]
where \(\sigma(m) = g + m\) and \(c(m) = 1\) if \(m \leq g\), \(\sigma(m) = m - g\) and \(c(m) = -1\) if \(m > g\).

**Proof.** The 1-form \(\log(h)dx_i - \frac{2}{2g+1}W(dx_i)\) is closed and the class of \(dx_m\) is \(c(m)\) times the Poincaré dual of the class of \(\alpha_{\sigma(m)}\). \(\square\)

**Corollary 3.2.**
\[
F(dx_m \otimes dx_l) = (2g+1) \left[ \int_{C-\gamma} dx_m \wedge \left( (\log(h) - \log(h(p)))dx_i - \frac{2}{2g+1}W(dx_i) \right) + 2\pi i \int_{\gamma} dx_m dx_l \right].
\]

**Proof.** This follows from Prop. 3.2 and Cor. 3.1. \(\square\)

Since we have the isomorphism
\[
J((\otimes^2 H^1)^*) \simeq \frac{F^1((\otimes^2 H^1_C)^*)}{(\otimes^2 H^1)^*},
\]
in order to determine \(F\) it is enough to compute it on elements in \(F^1((\otimes^2 H^1_C)^*)\), namely linear combinations of \(dx_l \otimes dz_i\) and \(dz_i \otimes dx_l\), where \(\{dz_i\}_{i=1,...,g}\) is a basis of \(H^{1,0}(X)\). We choose such a basis to satisfy the condition \(\int_{\alpha_i} dz_j = \delta_{ij}\). Hence
\[
dz_i = dx_i + \sum_{j=1}^g Z_{ij} dx_{g+j} \quad \text{with} \quad Z_{ij} := \int_{\alpha_{g+i}} dz_j.
\]

**Proposition 3.4.** The map \(F\) evaluated on elements \(dz_i \otimes dx_l\) gives:
\[
F(dz_i \otimes dx_l) = (2g+1) \left[ \int_C (\log(h) - \log(h(p)))dz_i \wedge dx_l + 2\pi i \int_{\gamma} dz_i dx_l \right].
\]

**Proof.** This follows from Cor. 3.2 using the linearity of \(F\) and the fact that \(dz_i \wedge W(dx_l) = 0\) for reasons of type. \(\square\)

In order to compute \(F(dx_l \otimes dz_i)\) we prove the following:
Lemma 3.1. With a suitable choices of the $\mu_{k\mu n, q_s}$, we have:

$$W(dz_i) := W(dx_i) + \sum_{j=1}^{g} Z_{ij} W(dx_{g+j}) = 0.$$ 

Proof. We set

$$\mu_{k(g+k), q_s} = R_{k(g+k), q_s} + S_{k(g+k), q_s}, \quad \mu_{l(g+k), k, q_s} = R_{l(g+k)k, q_s} + S_{l(g+k)k, q_s},$$

where the $R_{k(g+k), q_s}$ and the $R_{l(g+k)k, q_s}$ satisfy

(3.15) \quad $dx_l \wedge \mu_{k(g+k), q_s} + dR_{k(g+k), q_s} = 0,$ \quad $dx_l \wedge \mu_{l(g+k)k, q_s} + dR_{l(g+k)k, q_s} = 0$

while the $S_{k(g+k), q_s}$ and the $S_{l(g+k)k, q_s}$ satisfy

(3.16) \quad $\mu_{k, q_s} \wedge dx_{g+k} + dS_{k(g+k), q_s} = 0,$ \quad $\mu_{l(g+k), q_s} \wedge dx_k + dS_{l(g+k)k, q_s} = 0.$

Thus $W(dx_i)$ becomes:

$$W(dx_i) = WR(dx_i) + WS(dx_i)$$

with

$$WR(dx_i) = \sum_{k} \{(R_{k(g+k), q_2} - R_{l(g+k), q_1}) - (R_{l(g+k)k, q_2} - R_{l(g+k)k, q_1})\},$$

$$WS(dx_i) = \sum_{k} \{(S_{k(g+k), q_2} - S_{l(g+k), q_1}) - (S_{l(g+k)k, q_2} - S_{l(g+k)k, q_1})\}.$$

Now we claim that:

3.7. Claim. After a suitable choice of $R_{\mu}$ and $S_{\mu}$,

1) \quad $WR(dz_i) := WR(dx_i) + \sum_{j} Z_{ij} WR(dx_{g+j}) = 0$

2) \quad $WS(dz_i) := WS(dx_i) + \sum_{j} Z_{ij} WS(dx_{g+j}) = 0.$

The Lemma follows directly by (3.7) which we prove below. \hfill \Box 

Proof. (of Claim 3.7) Point 1): We choose $R_{l(g+k)k, q_s} := -R_{l(g+k)k, q_s}$, so

$$WR(dx_i) = 2 \sum_{k} \{(R_{k(g+k), q_2} - R_{l(g+k), q_1})\}.$$

The thesis follows once we fix any $R_{(g+j)k(g+k), q_s}$ satisfying condition 3.13 and define:

$$R_{k(g+k), q_s} := -\sum_{j} Z_{ij} R_{(g+j)k(g+k), q_s}.$$

Point 2): Since by Assumption 3.3 for $|m-k| \neq g, \mu_{mk, q_2} = \mu_{mk, q_1}$, we can choose $S_{mk(g+k), q_2} = S_{mk(g+k), q_1}$ and for the same argument for $|m-(g+k)| \neq g, S_{m(g+k)k, q_2} = S_{m(g+k)k, q_1}$. Thus

$$WS(dx_i) = S_{i(g+i), q_1} - S_{i(g+i), q_2} \quad WS(dx_{g+j}) = S_{(g+j)(g+j), q_2} - S_{(g+j)(g+j), q_1}.$$

For all $j \leq g$, fix $S_{(g+j)(g+j), q_s}$ ($s = 1, 2$) and $S_{(g+j)(g+j), q_1}$. To get the result it enough to set:

$$S_{i(g+i), q_2} := S_{i(g+i), q_1} + \sum_{j} Z_{ij} (S_{(g+j)(g+j), q_2} - S_{(g+j)(g+j), q_1}).$$

\hfill \Box 

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Remark. 3.9. Proof. (of Thm. 2.1) We compare the explicit expressions in Prop. 3.6 and Thm. 1.1. The extension classes 3.8. Proposition 3.5. The map $F$ evaluated on elements $dx_l \otimes dz_i$ gives:

1) $F(dx_l \otimes dz_i) = (2g + 1)c(l) \int_{\alpha_{\sigma(l)}} ((\log(h) - \log(h(p)))dz_i$

where $\sigma(l) = g + l$ and $c(l) = 1$ if $l \leq g$, $\sigma(l) = l - g$ and $c(l) = -1$ if $l > g$.

2) $F(dx_l \otimes dz_i) = (2g + 1) \left[ \int_C (\log(h) - \log(h(p)))dx_l \wedge dz_i - 2\pi i \int_\gamma dx_idz_i \right].$

Proof. First notice that by Prop. 3.2 and using the linearity of $F$ we have that

$F(dx_l \otimes dz_i) = (2g + 1)c(l) \int_{\alpha_{\sigma(l)}} [(\log(h) - \log(h(p)))dz_i - 2W(dz_i)].$

Then Point 1) follows directly from Prop. 3.3 (using the linearity of $F$) and Lemma 3.1. Point 2) follows from:

$c(l) \int_{\alpha_{\sigma(l)}} \log(h)dz_i = \int_C dx_l \wedge \log(h)dz_i + 2\pi i \int_\gamma dx_idz_i$

which is the equality of Prop. 3.3 with $\phi = c(l)dx_l$, $\psi = dz_i$ and $\varpi = 0$. □

3.8. The extension classes $e \in Ext_{MHS}(\wedge^2 H^1, \mathbb{Z})$ and $Pe \in Ext_{MHS}(\wedge^2 H^1_{prim}, \mathbb{Z})$. Carlson representatives of $e$ and $Pe$ are simply $F \circ \iota \in (\wedge^2 H^1)^*$ and $F \circ \iota_{\wedge^2 H^1_{prim}} \in (\wedge^2 H^1_{prim})^*$. The final step towards the proof of Th. 2.1 is the following proposition in which we compute $F \circ \iota$ on the elements $dx_l \wedge dz_i$ of the basis of $F^1 \wedge^2 H^1$.

Proposition 3.6. The map $F \circ \iota$ whose class defines $e \in Ext_{MHS}(\wedge^2 H^1, \mathbb{Z})$ is given by:

$(3.17) \quad F \circ \iota(dx_l \wedge dz_i) = F(dx_l \otimes dz_i) - F(dz_i \otimes dx_l)$

$= (2g + 1) \left[ 2 \int_C \log(h)dx_l \wedge dz_i + 2\pi i \int_\gamma (dx_idz_i - dz_idx_l) + 2\log(h(p)) \int_C dx_l \wedge dz_i \right].$

Proof. This follows from Prop. 3.4 and Prop. 3.5. □

Now the main result follows immediately:

Proof. (of Thm. 2.1) We compare the explicit expressions in Prop. 3.6 and Thm. 1.1. □

3.9. Remark. Notice that using Thm. 1.1 and Prop. 3.3 reg$(Z)$ can be computed in a simple way in terms of iterated integrals along paths on $C$, namely we have:

$\text{reg}(Z)(dx_m \wedge dz_i) = 2c(m) \left[ \int_{\alpha_{\sigma(m)}} (dh/h)dz_i + \log(h(p)) \int_{\alpha_{\sigma(m)}} dz_i \right],$

where $\sigma(m) = g + m$ and $c(m) = 1$ if $m \leq g$, $\sigma(m) = m - g$ and $c(m) = -1$ if $m > g$.
4. The normal function defined by the regulator

In this section we extend the construction of \( \text{reg}(Z) \) to families. In this setting we construct a normal function on a fine moduli space of hyperelliptic curves with Weierstraß points. We show that the homomorphism between fundamental groups induced by such a normal function is not trivial. This provides an alternative proof (Cor. 4.2) of the result of Collino that \( \text{reg}(Z) \) is not zero for general hyperelliptic curves. The method of proof of Collino was to show that the associated infinitesimal invariant of a normal function extending \( \text{reg}(Z) \) was not zero. For all the theory related to moduli spaces of curves and mapping class groups we refer to \[14\].

4.1. Mapping class group. Fix a compact orientable surface \( S \) of genus \( g \) together with \( n \) distinct points \( x_1, \ldots, x_n \). The mapping class group \( \Gamma^g_n \) is the group of isotopy classes of orientation preserving diffeomorphisms of \( S \) that fix each of the chosen points (for \( n = 0 \) we will drop the apex). A classically known system of generators of \( \Gamma^g_n \) is given by the Dehn twists \( D_a \) of simple closed curves \( a \subset S \). The mapping class group \( \Gamma^g_n \) has a natural representation

\[
\rho : \Gamma^g_n \rightarrow Sp(H_1(S, \mathbb{Z})) \simeq Sp_g(\mathbb{Z})
\]

(4.1)

given by the action of \( \Gamma^g_n \) on the first homology group of the surface \( S \) and the kernel of the representation \( \rho \) is, by definition, the Torelli group \( \text{Tor}^g_n \). The Dehn twists \( D_a \), with \( a \) such that \( S - a \) is not connected ("bounding curve") and the products \( D_aD_b^{-1} \), with \( a \) and \( b \) not disconnecting \( S \) but such that \( S - \{a, b\} \) is not connected ("bounding pair") generate the Torelli group \( \text{Tor}_g \) for \( g \geq 3 \).

Moreover fix an embedding of \( S \) in \( E^3 \) as in the figure below:

![Figure 1](image)

The rotation of 180 degrees around the \( y \)-axis defines a topological involution \( i \) (called hyperelliptic involution) on \( S \) with quotient homeomorphic to the 2-sphere \( S^2 \). Let

\( h_S : S \rightarrow S^2 \)

be the corresponding 2 to 1 map with \( 2g + 2 \) fixed points.

Denote by \( \Gamma^H_g \) the subgroup of \( \Gamma^g_g \) generated by those elements which can be represented by fiber preserving diffeomorphisms with respect to \( h_S \). The group \( \Gamma^H_g \) is isomorphic to the group of fiber preserving diffeomorphisms of \( S \) modulo fiber preserving isotopies (for example by Th.1 of \[3\]). Moreover a diffeomorphism of \( S \) that is isotopic to identity and preserves the fibers of \( h_S \) is isotopic to identity through fiber preserving diffeomorphisms (cf. Th.2 of \[3\]). From this it follows that \( \Gamma^H_g \) is an extension of the mapping class group of \( S^2 \) with \( 2g + 2 \) marked points by \( \mathbb{Z}/2\mathbb{Z} \). Set:

\[
\text{Tor}^H_g := \text{Tor}_g \cap \Gamma^H_g
\]

4.2. Moduli of curves. The mapping class group is related to the moduli space of curves via Teichmüller theory. The Teichmüller space is the space of complex structures on \( S \) up to isotopies that fix \( \{x_1, \ldots, x_n\} \) pointwise. It is a contractible complex manifold of dimension \( 3g - 3 + n \) on which \( \Gamma^g_n \) acts properly discontinuously with quotient analytically isomorphic to
the moduli space $\mathcal{M}_g^n$ of $n$-marked smooth projective curves of genus $g$. The action of $Tor^n_g$ is free and the quotient $\mathcal{T}^n_g := Tor^n_g \setminus \mathcal{X}^n_g$, called the Torelli space, is the moduli space of $n$-pointed smooth projective curves $C$ of genus $g$ with a fixed symplectic basis of homology.

There are the natural projections:

$$q : \mathcal{X}^n_g \longrightarrow \mathcal{T}^n_g, \quad p_T : \mathcal{T}^n_g \longrightarrow \mathcal{M}_g^n, \quad p_X = p_T \circ q : \mathcal{X}^n_g \longrightarrow \mathcal{M}_g^n.$$

4.3. **Moduli of hyperelliptic curves.** Let $H_g \subset \mathcal{M}_g$ be the moduli space of hyperelliptic curves of genus $g$. Let $\tilde{H}$ be a connected component of $p^{-1}(H_g)$ and $H := q(\tilde{H})$ the corresponding connected component of $p_T^{-1}(H_g)$. They are complex submanifolds of dimension $2g - 1$ of $\mathcal{X}_g$ and $\mathcal{T}_g$ respectively. The group $\Gamma^H_g$ is the orbifold fundamental group of the hyperelliptic locus $H_g$ and $Tor^H_g$ is the fundamental group of $H$. Let

$$\pi : \mathcal{C} \longrightarrow H, \quad \text{and} \quad \pi_{\mathcal{J}} : \mathcal{JC} \rightarrow H$$

be the universal families of curves and jacobians on $H$.

**Lemma 4.1.** The family $\pi : \mathcal{C} \longrightarrow H$ has $2g + 2$ sections $\tilde{q}_i : H \rightarrow \mathcal{C}$ corresponding to the Weierstraß points sets. In particular we ask the first two sections to satisfy $h_1(\tilde{q}_1(t)) = 0$ and $h_1(\tilde{q}_2(t)) = \infty$ and we denote the third one by $\tilde{p}$.

**Proof.** The set $W_t$ of Weierstraß points of $C_t := \pi^{-1}(t)$ defines a covering $W \rightarrow H$ of degree $2g + 2$. The lemma states that $W$ has $2g + 2$ connected components defining the sections $\tilde{q}_i$.

Consider the universal family of theta characteristics:

$$S_g := \{ L \in \text{Pic}(\mathcal{C}_T/T_g), \ L^2 \simeq \omega_{\pi_T} \} \longrightarrow \mathcal{T}_g.$$  

The fiber $S_t$ over $t \in \mathcal{T}_g$ is the space of the theta-characteristics over $C_t$. The set $S_t$ can be identified with the set $Q$ of all the quadratic forms $H_1(C_t, \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$ compatible with the intersection product. Since $Tor_g$ acts trivially on the first homology group, we have that:

$$\lambda : S_g \longrightarrow Q \times \mathcal{T}_g$$

is an isomorphism. From the natural inclusion

$$W \hookrightarrow S|_{\mathcal{T}_g}, \quad \tilde{q}_i(t) \mapsto \mathcal{O}((g - 1)\tilde{q}_i(t))$$

and the isomorphism $\lambda$, it follows that $W$ is the union of $2g + 2$ connected components which define the sections.

\[\square\]

4.4. **The normal function extending $\text{reg}(\mathbb{Z})$.** Consider on $H$ the variations of Hodge structures $R^2\pi_{\mathcal{J},*}\mathbb{Z} \simeq \wedge^2 R^1\pi_*\mathbb{Z}$ and $P_2 \simeq \text{ker}(\wedge^2 R^1\pi_*\mathbb{Z} \rightarrow R^2\pi_*\mathbb{Z})$ which extend the second and second primitive cohomology of the jacobian. Associated to them there are the families of intermediate jacobians:

$$J_2 := (F^1R^2\pi_{\mathcal{J},*}\mathbb{C})/R^2\pi_{\mathcal{J},*}\mathbb{Z} \rightarrow H, \quad J_{2\text{prim}} := (F^1P_2\mathbb{C})/P_2^* \rightarrow H,$$

extending $J_2(\mathcal{J})$ and $J_2(JC)_{\text{prim}}$ (cf. [1.1, 1.3]).

Since $Tor_g$ acts trivially on $H^1$ the local system $R^1\pi_*\mathbb{Z}$, the fibrations $\mathcal{JC}$, $J_2$ and $J_{2\text{prim}}$ are topologically trivial on $H$. Let

$$p_{J_2} : J_2 \rightarrow J_2(J\mathcal{C}), \quad p_{J_{2\text{prim}}} : J_{2\text{prim}} \rightarrow J_2(J\mathcal{C})_{\text{prim}}$$

be the projections onto the fibers.

By Lemma 4.1 $H$ is a fine moduli space for hyperelliptic curves with marked Weierstraß points, hence the Collino cycle extends to a family of higher cycles $Z$ on the associated family
of jacobians. The construction of the regulator images of any fiber $Z_t$ of $Z$ extends to normal functions $R_Z$ and $r_Z$ i.e. to holomorphic sections of $J_2$ and $J_{2\text{prim}}$:

$$
\begin{array}{ccc}
R_Z & \rightarrow & J_2 \\
\downarrow & & \downarrow \\
r_Z & \rightarrow & J_{2\text{prim}} \\
\end{array}
$$

4.5. **Remark.** The construction of these normal functions could have been done (as Collino does) on a finite covering of $H_g$ given for example by the moduli space of hyperelliptic curves with a convenient level structure. The reason why we work on $H$ is that, using the projections to the fibers $p_{J_2}$ and $p_{J_{2\text{prim}}}$, we can forget about the $Sp_g$-contribution to monodromy as in the following.

4.6. **The induced homomorphism.** We are interested in the homomorphism of fundamental groups induced by the compositions $p_{J_2} \circ R_Z$ and $p_{J_{2\text{prim}}} \circ r_Z$. To prove that $\text{reg}(Z)$ is not zero for a general hyperelliptic it is enough to prove that

$$(p_{J_{2\text{prim}}} \circ r_Z)_*: \pi_1(H) \rightarrow H_2(JC)_{\text{prim}}$$

is not trivial.

In order to prove this we compute the image of the class of loops $\lambda_d$ in $H$ based at $[C]$, that correspond to a Dehn twist $D_d$ of a bounding curve $d$ on $C$, invariant with respect to the hyperelliptic involution (thus $D_d \in \text{Tor}^H_g$). The loop $\lambda_d$ lifts to a path $\tilde{\lambda}_d : [0,1] \rightarrow \tilde{H}$ with a parametrization such that $\tilde{\lambda}_d(0) = [C]$ and $\tilde{\lambda}_d(1) = [D_dC]$ in $\tilde{H}$. Restrict the universal family of curves to $\tilde{\lambda}_d$:

$$C_{X|\lambda_d} =: C_{\tilde{\lambda}_d} \supseteq C_t$$

$$\pi_{X|\lambda} \downarrow \quad \downarrow$$

For any $t$ consider the holomorphic map $h_t : C_t \rightarrow \mathbb{P}^1$ such that $\tilde{q}_1(\tilde{\lambda}(t)) = h_t^{-1}(0)$ and $\tilde{q}_2(\tilde{\lambda}(t)) = h_t^{-1}(\infty)$, corresponding to the topological quotient $h_S : S \rightarrow S^2$. In particular $h_0 = h$ and $h_1 = h \circ D_d =: h_d$. Set $\gamma_t := h_t^{-1}([0,\infty])$ and $\gamma_d := \gamma_1$.

The expression of $\text{reg}(Z)$ obtained in Th. 4.4 (see Remark 4.6) allows us to lift the normal function $R_Z$ along $\tilde{\lambda}_d$ to a section of $(\mathcal{F}^1 \wedge^2 R^1\pi_{X|\lambda*d})^*$ in the following way. For all $t \in [0,1]$, let $\phi_t$ and $\psi_t$ be closed 1-forms on $C_t$, with $\psi_t$ of type $(1,0)$. It is enough to define the lifting of $R_Z|_{\tilde{\lambda}_d}$ along $\tilde{\lambda}_d$ as

$$\tilde{R}_{\tilde{\lambda}_d} : [0,1] \rightarrow (\mathcal{F}^1 \wedge^2 R^1\pi_{X|\lambda*d})^*$$

on the classes of $\phi_t \wedge \psi_t$:

$$\tilde{R}_{\tilde{\lambda}_d}(t)([\phi_t \wedge \psi_t]) = 2 \int_{C_t-\gamma_t} \log(h_t)\phi_t \wedge \psi_t + 2\pi i \int_{\gamma_t} (\phi_t \psi_t - \psi_t \phi_t).$$

By covering theory we have:

$$(p_{J_2} \circ R_Z)_*([\lambda_d]) = (1/2\pi)(\tilde{R}_{\tilde{\lambda}_d}(1) - \tilde{R}_{\tilde{\lambda}_d}(0)) \in H_2(JC, Z).$$

**Proposition 4.1.** Let $\lambda_d$ be a loop in $H$ with basis point $[C]$, whose homotopy class corresponds to the Dehn twist of a bounding curve $d$ invariant for the hyperelliptic involution and splitting $C$ in a component $S_1$ containing $q_1$ and a component $S_2$ containing $q_2$. Let $\phi$ and $\psi$ be closed 1-forms on $C$, with $\psi$ of type $(1,0)$, then

$$(p_{J_2} \circ R_Z)_*([\lambda_d])([\phi \wedge \psi]) = 4 < [\phi]_{S_2}, [\psi]_{S_2} >_{S_2}$$

where $<,>_{S_2}$ is the intersection form on $S_2$. 

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Proof. Notice that, since $D_d$ acts trivially in homology and by Remark 1.6, we can choose $\phi_1 = \phi_0 := \phi$ and $\psi_1 = \psi_0 := \psi$. Thus

$$\tilde{R}_{\lambda_d}(1) - \tilde{R}_{\lambda_d}(0)(\phi \land \psi) = 2 \left( \int_{C_{d-\gamma_d}} \log(h_d) \phi \land \psi - \int_{C-\gamma} \log(h) \phi \land \psi \right) + 2\pi i \left( \int_{\gamma_d} (\phi \psi - \phi \phi) - \int_{\gamma} (\phi \psi - \phi \phi) \right).$$

The proposition follows then from the following two equalities:

\begin{align*}
\int_{C_{d-\gamma_d}} \log(h_d) \phi \land \psi - \int_{C-\gamma} \log(h) \phi \land \psi &= 4\pi i < [\phi], [\psi] >_{S_2} - 4\pi i \int_{\gamma} \phi \psi. \\
\int_{\gamma_d} (\phi \psi - \phi \phi) - \int_{\gamma} (\phi \psi - \phi \phi) &= 4 \int_{\gamma} \phi \psi.
\end{align*}

First we prove (4.2). By the assumptions, the separating curve $d$ is the preimage of a curve $d_0$ on $\mathbb{P}^1$ whose complement is the union of two open disks that are neighborhoods of 0 and $\infty$ respectively. The surface with boundary $S_1$ is the preimage of a disk containing 0 and $S_2$ is the preimage of a disk containing $\infty$. We assume that $d_0$ intersects $[0, \infty]$ transversely in a single point. Thus the Dehn twist $D_{d_0}$ carries $[0, \infty]$ to $[0, \infty] + d_0$. The square $D_{d_0}^2$ lifts to the Dehn twist $D_d$. Its effect on the integral is via the multivaluedness of the logarithm: this will change by $4\pi i$ on $S_2$. Then the difference between the 2 integrals is $4\pi i \int_{S_2} \phi \land \psi$. To calculate this, first observe that since the boundary curve $d$ is null homologous, the restriction of $\phi$ to $d$ is exact. Let $\rho$ be a smooth function $\rho$ on $C$ such that $\phi_0 = \phi - d\rho$ vanishes on a neighborhood of $d$ and such that $\rho$ is zero in the point $d \cap \gamma^+$. Clearly $\phi_0$ defines the same homology class as $\phi$. Its restriction to $S_2$ vanishes near the boundary so, by the De Rham theorem:

$$\int_{S_2} \phi_0 \land \psi = < [\phi], [\psi] >_{S_2}.$$

It remains to compute

$$\int_{S_2} d\rho \land \psi.$$

If $d$ is orientated as the boundary of $S_1$, then Stokes' theorem implies that this is equal to $- \int_d \rho \psi$, which is, by definition the iterated integral and Lem.1.1.(3), $- \int_d \phi \psi$.

We prove now equality (4.3). The Dehn twist $D_d$ carries $\gamma$ to $\gamma_d := \gamma + 2d$, with the chosen orientation of $d$. Then, by Lem.1.1.(1) and (2) and the fact that $\int_d \phi = \int_d \psi = 0$:

$$\int_{\gamma_d} (\phi \psi - \phi \phi) - \int_{\gamma} (\phi \psi - \phi \phi) = 2 \int_d (\phi \psi - \phi \phi) = 4 \int_d \phi \psi.$$

\[\square\]

Corollary 4.1. Keep the notation of Prop.4.1. Choose a symplectic basis $\{A_k, A_{g+k}\}_{k \leq g}$ of $H_1(C, \mathbb{Z})$ such that $\{A_k, A_{g+k}\}_{k \leq g_1}$ is a symplectic basis for $H_1(S_1, \mathbb{Z})$ and $\{A_k, A_{g+k}\}_{g_1 < k \leq g}$ is a symplectic basis for $H_1(S_2, \mathbb{Z})$, then

$$(p_{J_2} \circ R_Z)_*([\lambda_d]) = 4 \sum_{k = g_1 + 1}^g A_k \land A_{g+k}.$$
Proof. The intersection form on $S_2$, seen as an element in $\wedge^2 H^1(C, \mathbb{Z})$ is, up to constant:

$$<, >_{S_2} = \sum_{k=g_1+1}^g A_k \wedge A_{g+k}$$

4.7. Remark. Notice that $H^2(JC, \mathbb{Z})_{prim}^* \simeq H_2(JC, \mathbb{Z})_{prim} \simeq \wedge^2 H_1(C, \mathbb{Z})/ < \omega >$, where $\omega := \sum_{k=1}^g A_k \wedge A_{g+k}$ is the dual of the polarization $\Omega$.

Corollary 4.2. With the notation of Cor.4.1, we have:

$$(p_{J_{2prim} \circ r_Z})_*(\lambda_d) = 4 \sum_{k=g_1+1}^g A_k \wedge A_{g+k} \mod < \omega > .$$

In particular $(p_{J_{2prim} \circ r_Z})_*$ is not trivial and hence $\text{reg}(Z)$ is not zero for general hyperelliptic.

Proof. It follows directly from the previous corollary, since $r_Z$ is the composition of $R_Z$ with the natural projection $J_2 \to J_{2prim}$.

5. Monodromy of the extensions

5.1. The goal of this section is to extend the extension classes of Sect.3, to normal functions on the moduli space $H$ and to study the induced homomorphism on fundamental groups. We compare the homomorphism induced by the normal function extending $Pe$ with the one induced by $R_Z$ (see Cor.5.1).

5.2. The normal functions extending $e_s$ and $Pe$. Any finite dimensional $Sp_g(\mathbb{Z})$ representation $V$ has a natural Hodge structure which can be extended to a variation of Hodge structures $\mathcal{V}$ on any fine moduli space of curves. Moreover if $V$ has negative weight, its intermediate jacobian $J(V)$ (see 3.1) can be extended to a corresponding intermediate jacobians fibration

$$J(\mathcal{V}) := \frac{V_C}{\mathcal{J}^0 + \mathcal{V}} .$$

We will restrict ourselves to the case in which this moduli space is $H$. Since $\text{Tor}_g^H$ acts trivially on homology the associated bundles of intermediate jacobians are topologically trivial. We shall denote by

$$p_V : J(\mathcal{V}) \to J(V)$$

the projection onto the fiber.

We consider the case $V = \text{Hom}(\otimes^3 H^1, H^1)$ and denote by $\mathcal{E}_s$ ($s = 1, 2$) and $\mathcal{PE}$ the normal functions which associate to the curve $C$ the extensions $e_s$ and $Pe$ fitting in the following commutative diagram:

$$
\begin{array}{ccc}
J(\mathcal{V}) & \xrightarrow{\phi} & J_{2prim} \\
\downarrow & & \downarrow \\
H & \xrightarrow{id} & H
\end{array}
$$

and

$$
\begin{array}{ccc}
H & \xrightarrow{id} & H & \xrightarrow{id} & H & \xleftarrow{id} & H.
\end{array}
$$
5.3. Monodromy of $\mathcal{E}_s$. The pairs of sections $(\tilde{q}_s, \tilde{p})$ ($s = 1, 2$), of Lemma 3.1 define two different inclusions:

$$j_{q,p} : H \hookrightarrow T_g^2.$$  

We will show that the homomorphism

$$(p_V \circ \mathcal{E}_s)_* : \text{Tor}_g^H \rightarrow \text{Hom}(\otimes^3 H^1, H^1)) \simeq \text{Hom}(H_1, \otimes^3 H_1)$$

factorizes via $(j_{q,p})_*$ and a natural homomorphism associated to the action of the mapping class group on $J_{q,p}/J_{q,p}$ described in the following. Let $\pi := \pi_1(C - \{q\}, p)$ be the fundamental group of the punctured curve $C - \{q\}$ in $p$. The mapping class group $\Gamma^2_g$ acts naturally on $\pi$ and on its lower central series of $\pi$. The action on $\pi$ induces a natural action on the $J$-adic quotients of the group algebra of the fundamental group ($J = J_{q,p}$ in the notation of Sec.3) that we will briefly illustrate (see [13], [16]).

Let $\{\pi^{(k)}\} (k \geq 1)$ be the lower central series of $\pi$, namely $\pi^{(1)} := \pi$ and $\pi^{(k+1)} = [\pi^{(k)}, \pi]$, with nilpotent quotients

$$N_k := \pi/\pi^{(k)}$$

fitting in the sequence $0 \rightarrow L_k \rightarrow N_{k+1} \rightarrow N_k \rightarrow 0$.

The action of $\Gamma^2_g$ on $\pi$ induces an action on the quotients $N_k$:

$$(5.2) \quad \rho_k : \Gamma^2_g \rightarrow \text{Aut}(N_k).$$

Notice that $N_2 = H_1$, so $\rho_2 = \rho$ and $\ker \rho_2 = \text{Tor}_g^2$. The action of $\Gamma^2_g$ on $L_k$ factorizes through $\rho$, thus $L_k$ is a $Sp_g$-module. More precisely $\otimes_k L_k$ is the free Lie algebra (over $\mathbb{Z}$). In particular $L_k$ is torsion free and there are the following identifications:

$$L_2 \simeq \wedge^2 H_1, \quad L_3 \simeq (\wedge^2 H_1 \otimes H_1)/\wedge^3 H_1.$$

Johnson’s homomorphisms $\tau_k$ in the punctured case are:

$$(5.3) \quad \tau_k : \ker \rho_k \rightarrow \text{Hom}(H_1, L_k) \quad f \mapsto [\overline{\pi} \mapsto f(n)n^{-1} \mod \pi^{(k+1)}]$$

where $n$ is any lifting of $\overline{n}$ to $N_{k+1}$. By construction $\ker \rho_{k+1} = \ker \tau_k$.

**Lemma 5.1.** The inclusion $j : \pi \hookrightarrow J$, $\alpha \mapsto \alpha - 1$ satisfies $j(\pi^{(k)}) \subset J^k$ and induces an injective homomorphism $j_k : L_k \rightarrow J^k/J^{k+1}$.

**Proof.** The inclusion $j(\pi^{(k)}) \subset J^k$ can be checked by induction and it implies that $\pi^{(k)} \subset \pi^k_j$ where $\pi^k_j := \{\alpha \in \pi | \alpha - 1 \in J^k\}$. Moreover the group $\pi/\pi^k_j$ has no torsion and $\pi^k_j/\pi^{(k)}$ is a torsion group, hence $j_k : L_k \otimes_{\mathbb{Z}} \mathbb{Z} \rightarrow J^k/J^{k+1} \otimes_{\mathbb{Z}} \mathbb{Z}$ and, since $L_k$ has no torsion, $j_k : L_k \hookrightarrow J^k/J^{k+1}$. \hfill $\square$

We define analogues of Johnson’s homomorphisms associated to the action on $J$-adic quotients:

**Lemma 5.2.** The homomorphism

$$\tau_{k,j} : \ker \rho_k \rightarrow \text{Hom}(H_1, J^k/J^{k+1}), \quad \tau_{k,j} : f \mapsto \{[\alpha] \mapsto f(\alpha) - \alpha \mod J^{k+1}_p\}$$

is well defined and

$$\tau_{k,j} = H(j_k) \circ \tau_k,$$

where $H(j_k) : \text{Hom}(H_1, L_k) \rightarrow \text{Hom}(H_1, J^k/J^{k+1})$, $H(j_k) : f \mapsto j_k \circ f$.

**Proof.** By Lem. 5.1 $\ker \rho_k \subset \ker(\Gamma^2_g \rightarrow \text{Aut}(\pi/\pi^k_j)) = \ker(\Gamma^2_g \rightarrow \text{GL}(J/J^k))$ hence $\tau_{k,j}$ is well defined. Moreover $\tau_{k,j}(f)(a) = j_k \tau_k(f)(a)$ where $f \in \Gamma^1_g(k)$, $a \in H$, because:

$$f(n) = (f(n)n^{-1} - 1)n = f(n)n^{-1} - 1 \mod J^{k+1}.$$  

\hfill $\square$
Lemma 5.3. The homomorphism \((j_{q,p})_*\) satisfies:

\[
(j_{q,p})_*(\text{Tor}^H_g) \subset \ker \rho_3 = \ker \tau_2.
\]

Proof. Since the hyperelliptic involution acts as \(-I\) on \(H_1\), it also acts as \(-I\) on \(\text{Hom}(H_1, L_2) \simeq \text{Hom}(H_1, \wedge^2 H_1)\) while it acts trivially on \((j_{q,p})_*(\text{Tor}^H_g)\).

\[\square\]

Lemma 5.4. The following equality holds:

\[ (p_V \circ \mathcal{E}_*) = \tau_{3J} \circ (j_{q,p})_* \]

Proof. The homomorphism \((p_V \circ \mathcal{E}_*)\) is determined by the action of \((j_{q,p})_*(\text{Tor}^H_g) \subset \Gamma^2_g\) on \(J_{q,p}/J^4_{q,p}\). Since, by the previous lemma, this action is trivial on \(N_3\), it defines in fact \(\tau_{3J}\).

\[\square\]

5.4. Monodromy of \(\mathcal{P}\mathcal{E}\). We want to compute \((p_{J_{2\text{prim}}} \circ \mathcal{P}\mathcal{E})_*\) on the Dehn twist \(D_d\) in \(\text{Tor}^H_g\) of a simple curve \(d\) separating \(q_1\) and \(q_2\) exactly as in Sect.4. We first compute \((p_V \circ (\mathcal{E}_2 - \mathcal{E}_1))_*\):

Proposition 5.1. Let \(D_d \in \text{Tor}^H_g\) be the Dehn twist of the simple curve \(d\) separating \(q_1\) and \(q_2\) and such that \(p\) is in the same component of \(q_1\). It holds

\[ (p_V \circ (\mathcal{E}_1 - \mathcal{E}_2))_*(D_d) = \begin{cases} \omega \wedge A_k & \text{if } g + 1 \leq l \leq g + 1 \leq l \leq 2g \\ 0 & \text{otherwise} \end{cases} \]

where \(\omega := \sum_{k=1}^{g} A_k \wedge A_{g+k}\) is the dual of the polarization \(\Omega\).

Proof. By Lem. 5.4

\[ (p_V \circ (\mathcal{E}_1 - \mathcal{E}_2))_*(D_d) = \tau_{3J}(j_{q_2,p})_*(D_d)(j_{q_1,p})_*(D_d^{-1}). \]

Fixing the isomorphism \(\Gamma^2_g \simeq \text{Aut}^+(\pi_1(C - \{q\}, p))\), the element \((j_{q_2,p})_*(D_d)(j_{q_1,p})_*(D_d^{-1}) \in \text{Tor}^2_g\) can be identified with the element \(D_d D_d^{-1}\) as in Fig.2, where \(d'\) and \(d''\) are homotopic to \(d\) and bound a cylinder containing the missing point \(q\).

\[\text{Figure 2.}\]

The loops \(\alpha_l\), for \(1 \leq l \leq g\) and \(g + 1 \leq l \leq g + g\), can be chosen not to intersect \(d'\) and \(d''\) thus the action of \(D_d D_d^{-1}\) on them is the identity, while on \(\alpha_l\) for \(g + 1 \leq l \leq g\) and \(g + g + 1 \leq l \leq 2g\) the action is given by the conjugation by \(\delta_q\), where \(\delta_q\) bounds a punctured disk around \(q\) and satisfies \(\delta_q = \prod_k [\alpha_k \alpha_{g-k}]\). Hence, for \(1 \leq l \leq g\) and \(g + 1 \leq l \leq g + g\)

\[ \tau_3(D_d D_d^{-1})(A_l) = 0, \]

while for \(g + 1 \leq l \leq g\) and \(g + g + 1 \leq l \leq 2g\)

\[ \tau_3(D_d D_d^{-1})(A_l) = \prod_{k=1}^{g} [\alpha_k \alpha_{g-k}], \alpha_l] \in L_3. \]
Via the identification $L_3 \simeq (\wedge^2 H_1 \otimes H_1)/\wedge^3 H_1$ and the inclusion $J_3 : L_3 \hookrightarrow J^3/J^4 = \otimes^3 H_1$ we get

$$\tau_3(J^3(J^3)^{-1})(A_l) = \left( \sum_{k=1}^{g} A_k \wedge A_{g+k} \right) \wedge A_l = \omega \wedge A_l.$$  

\[
\square
\]

**Corollary 5.1.**

$$(p_{J_{2prim}} \circ \mathcal{P} \mathcal{E})_*(D_d) = 2(2g + 1) \sum_{k=g+1}^{g} A_k \wedge A_{g+k} \mod \omega = (2g + 1)(p_{J_{2prim}} \circ r_Z)_*(D_d).$$

**Proof.** The fiber bundle map $\phi : J(V) \rightarrow J_{2prim}$ of diagram 5.1 restricted to a fiber, is the surjective homomorphism $\Phi : J(V) \simeq Ext_{MHS}(\otimes^3 H^1, H^1) \rightarrow J_2(JC)_{prim} \simeq Ext_{MHS}(\wedge^2 H^1, Z)$ described in Sect.2.2: i.e. it is obtained by pulling back along $J_0$, tensoring by $H^1$ on the left, pushing down along $\Pi$ and pulling back along the inclusion $\wedge^2 H^1_{prim} \hookrightarrow \otimes^3 H^1$. The induced homomorphism between the fundamental groups of the two intermediate jacobians is then:

$$\Phi_* : Hom(H_1, \otimes^3 H_1) \rightarrow \wedge^2 H_{1prim}$$

$$f \mapsto \sum_k A_k \wedge J_1 f(A_{g+k}) - A_{g+k} \wedge J_1 f(A_k) \mod \omega$$

where $J_1 : \otimes^3 H_1 \rightarrow H_1$, $A_1 \otimes A_m \otimes A_n \mapsto \sum_k (\delta_{lk}\delta_{m(g+k)} - \delta_{l(g+k)}\delta_{mk})A_n$ is the dual of tensoring by $\Omega$ from the left (cf. 2.5). The first equality then follows by direct computation using the equality:

$$p_{J_{2prim}*} \circ \phi_* = \Phi_* \circ p_{V*}.$$  

Prop.2.1 and the explicit expression of $\Phi_*$. The second equality is the statement of Cor.4.1. \[
\square
\]

6. Final Remarks

6.1. **An alternative proof of Th 2.1.** The computations of Cor. 5.1 and Cor. 4.1 can be easily improved to show that two normal functions $(2g + 1)r_Z$ and $\mathcal{P} \mathcal{E}$ induce the same homomorphism on fundamental groups, without applying Th 2.1. In fact this last monodromy computation can be used to give an alternative proof of Th 2.1, following the steps in which Hain proved in a different way in [13] the result of Hain-Pulte on $C_p - C_{p'}$ and the extension $J_p/J_{p'}$. To apply the rigidity argument of [13] Cor. 6.4 to our case one just needs to show the following two facts. First of all both $(2g + 1)r_Z$ and $\mathcal{P} \mathcal{E}$ have to define a good VMHS in the sense of Saito [23]. Secondly one has to exhibit a hyperelliptic curve $C$ with Weierstraß points $p_1$ and $q_2$, for which $(2g + 1)reg(Z) = Pe$. This last point could be achieved by taking a hyperelliptic curve with a further involution exchanging $q_1$ and $q_2$ and fixing $p$. For such a curve $reg(Z) = 0$ while $Pe = -Pe$ so $Pe$ is 2-torsion. Then one needs to show that $Pe$ is zero, for example by showing that there are no sections of $J_{2prim} \rightarrow H$ of order 2. The variation defined by $\mathcal{P} \mathcal{E}$ is good since the set of the $J/J^k$ form a good VMHS over any fine pointed moduli space of curves ([14]). About $r_Z$ we can argue as follows. Let $W = \sum_i (V_i, f_i)$ be a cycle in $CH^n(X, 1)$, with $X$ smooth projective of dimension $n$. The image of $reg(W)$ under the map: $J_2(X) = Ext_{MHS}(Z(n-1), H^{2n-2}(X)) \rightarrow Ext_{MHS}(Z(n-1), H^{2n-2}(X)/[V_i])$ is described by the following construction (see for example [23] (0.5) or [21]). Set $U = X - [W]$ and look at the long exact sequence:

$$\cdots \rightarrow H^{2n-2}_W(X) \rightarrow H^{2n-2}(X) \rightarrow H^{2n-2}(U) \rightarrow \cdots$$

The cycle $W$ gives rise to a class in $F^n \cap H^{2n-2}_W(X, Z)$ hence by pullback it gives rise to an extension.
0 \to H^{2n-2}(X)/<[V_i]> \to E \to \mathbb{Z}(n-1) \to 0.

The class of this extension is exactly the image of reg(W). Since E is a sub MHS of the cohomology of the open variety U, when we extend the construction to families we get a "geometric" VMHS, hence it is admissible for Steenbrink and Zucker (cf. [22] and [3]) and then good for Saito. In the case of the Collino cycle $Z = (C_1, h_1) + (C_2, h_2)$, $H^{2g-2}(JC)/<[C_i]>$ = $H^{2g-2}(JC)/<[C]> \simeq H^2(JC)^{prim}$, hence $Ext_{MHS}(\mathbb{Z}(g-1), H^{2g-2}(JC)/<[C_i]> = J(C)^{2prim}$ and the class of $E$ is $reg(Z)$.

6.2. Degeneration. As last remark we illustrate a heuristic argument about the monodromy of the normal function associated to $reg(Z)$. This argument leads to the formula Cor.4.1 without writing $reg(Z)$ in terms of integrals on the curve $C$ as in Th.1.2. The starting point is the idea, explained in Collino ([1],1), of viewing the higher cycle $\tilde{Z}$ as a degeneration of the Ceresa cycle, when two points are identified. More precisely, set $\tilde{C}_1 := \psi_1(C)$ and $\tilde{C}_2 := \psi_2(C)$ where

$$\psi_1 : C \to JC \times \mathbb{P}^1, \quad x \mapsto (x - q_1, h(x))$$

$$\psi_2 : C \to JC \times \mathbb{P}^1, \quad x \mapsto (x - q_2, 1/h(x))$$

and

$$\tilde{Z} := \tilde{C}_1 - \tilde{C}_2 \sim_{hom} 0.$$  

Denote by $\tilde{\Gamma}$ a 3-chain such that $\partial \tilde{\Gamma} = \tilde{Z}$ and let $\alpha \in F^1 H^2(JC)$. From Stokes’ theorem it follows that:

$$\int_{\tilde{\Gamma}} p_1^*(\alpha) \wedge p_2^*(dz/z) \mod H_2(JC, \mathbb{Z}) = reg(Z)(\alpha)$$

where $p_1$ and $p_2$ are the projections to $JC$ and to $\mathbb{P}^1$.

Let $\overline{M}_g$ be the Deligne-Mumford compactification of the moduli space of smooth projective genus $g$ curves. Let $\overline{M}_g \to \overline{M}_{g+1}$ be the map that identifies the 2 marked points, hence it associated to a smooth curve $C$ of genus $g$ an irreducible nodal curve $C'$ of geometric genus $g + 1$. Topologically this curve can be seen as the curve obtained shrinking a loop $d_0$ of a smooth curve $G$ of genus $g + 1$ to a point, as indicated in Fig.3: For $C$ hyperellitic the

![Figure 3](image-url)
(see Theorem 5.1 of [12]). Fix a system of generators \( \{\alpha_k, \alpha_0, \beta_0\}_{1 \leq k \leq 2g} \) on \( \pi_1(G, p) \) satisfying the relation:

\[
\prod_{k=1}^{g_1} [\alpha_k, \alpha_{g+k}][\alpha_0, \beta_0] \prod_{k=g_1+1}^{g} [\alpha_k, \alpha_{g+k}] = 1
\]

and such that \( \alpha_k \) corresponds to generators of \( \pi_1(C') \) and \( \pi_1(C) \) denoted by the same letter.

By Corollary of section 4 of [15] it holds

\[
\tau_2(D_d D_{d_0}^{-1}) = \left( \sum_{k=g_1+1}^{g} [\alpha_k] \wedge [\alpha_{g+k}] \right) \wedge [\beta_0].
\]

The Dehn twist \( D_d \) of \( C \) corresponds to an analogous Dehn twist of \( C' \) and hence to the image in \( C' \) of the product of Dehn twists \( D_d D_{d_0}^{-1} \) of \( G \) in the degeneration from \( G \) to \( C' \). Notice that the class \( [\beta_0] \) is equal to \( [d_0] \) in \( C' \) \( - \) \( \{q_1\} \) and \( \int_{d_0} \gamma_2 dz/z = 4\pi i \), thus we expected

\[
(p_J \circ R_Z)_*(D_d) = \frac{1}{2\pi i} 8\pi i \sum_{k=g_1+1}^{g} A_k \wedge A_{g+k},
\]

as was proved in Cor.4.1.

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