We present some results conjectured by Bildhauer, Fuchs, and Weickert who investigated analytical aspects of coupled variational models with applications to mathematical imaging. We focus on variants of linear growth, which require a treatment within the framework of relaxation theory and convex analysis. We establish existence and regularity of (dual-) solutions.

1 Introduction

This paper takes up ideas from [1]–[3], where certain variational problems from the field of image analysis have been studied. The general application background is given by the task to retrieve a digital grey-scale image, i.e., a real-valued function mapping every point (which can in this context be thought of as an infinitely small “pixel“) of a plane domain $\Omega \subset \mathbb{R}^2$ to a grey-value between 0 (indicating a black point) and 1 (indicating a white point), from a flawed observation $f : \Omega - D \to [0, 1]$. Here, in our understanding the term “flawed“ includes the phenomena of a statistical distortion (called “noise“) as well as the missing of some parts of the data, which means that $f$ is only defined outside a subset $D \subset \Omega$ (the “deficiency set“). A well established approach to the solution of this problem (called “pure denoising“ if only the first type of data corruption is considered and “inpainting“ for the second type) consists in minimizing a suitable energy that penalizes fluctuations of the data. In the fundamental work [4], it was proposed to consider the variational problem

$$\int_{\Omega} |\nabla u| + \frac{\lambda}{2} \int_{\Omega - D} (u - f)^2 \, dx \to \min \text{ in } BV(\Omega)$$

in the space of functions of bounded variation (cf. [5] or [6] for details on this function space). $BV$-functions are very well-suited for modeling objects like images since they are allowed to have jump-type discontinuities which can reflect edges and sharp contours. On the other hand, they are still in some sense regular enough to allow a treatment with analytical methods. However,
the model (1.1) has some drawbacks. First, the total variation $\int_{\Omega} |\nabla u|$ is neither strictly convex nor differentiable in its argument, hence unsuited to a treatment with PDE-methods. Second, numerical simulations show that the solutions of (1.1) are frequently afflicted with the so called “staircasing effect” (cf., for example, [7]), which becomes manifest in piecewise constant regions of the minimizing function (resembling a staircase). The first problem has been circumvented in [8] by using the concept of convex functions of a measure (cf. [9]). More precisely, the quantity $\int_{\Omega} |\nabla u|$ is being replaced with $\int_{\Omega} F(\nabla u)$ for a smooth strictly convex function $F : \mathbb{R}^2 \to [0, \infty)$ of linear growth which approximates $|.|$, for example, $F(\xi) = \sqrt{\epsilon^2 + |\xi|^2} - \epsilon$.

For avoiding the staircasing effect, one could raise the order of differentiability, i.e., consider the problem

$$\int_{\Omega} |\nabla^2 u| + \frac{\lambda}{2} \int_{\Omega - D} (u - f)^2 dx \to \min \text{ in } BV^2(\Omega),$$

(1.2)

where $\nabla^2 u$ is the Hessian matrix and $BV^2(\Omega)$ denotes the set of all functions $u \in L^1(\Omega)$ such that the weak gradient $\nabla u$ is a $BV$-function. The undesirable effects are now shifted to the first derivative of the solution and therefore become less evident. Analytical properties of these models were studied in [10]. Higher order models are computationally more difficult to handle as they lead to partial differential equations of at least fourth order. That is why in [1], a different approach has been pursued. There, in place of (1.2), a coupled problem has been considered which is obtained by introducing a vector-valued variable $v$, serving as a substitute for the gradient of $u$. To be more precise, the idea is to study the problem

$$(u, v) \mapsto \alpha \int_{\Omega} |\nabla v| + \beta \int_{\Omega} |\nabla u - v \cdot \mathcal{L}^n| + \frac{\lambda}{2} \int_{\Omega - D} (u - f)^2 dx \to \min$$

(1.3)

in $BV(\Omega) \times BV(\Omega, \mathbb{R}^n)$. Minimizing the middle term, the so called “coupling term,” entails $v \approx \nabla u$ and hence $\nabla v \approx \nabla^2 u$. Solutions of (1.3) serve as an approximation to solutions of (1.2) with the advantage that the associated system of differential equations is merely of second order. Of course, there is plenty of different choices of densities other than $| \cdot |$ in the leading as well as in the coupling term of (1.3) and actually, in [1], various constellations of power and superlinear growth were considered. Here, we focus on the case where both leading and coupling terms are of linear growth. Regularity properties of minimizers of functionals of this type were conjectured in [1, Remark 6.4].

Let $F : \mathbb{R}^{2 \times 2} \to \mathbb{R}$ and $G : \mathbb{R}^2 \to \mathbb{R}$ be strictly convex and satisfy the following assumptions. Assume that $\Omega$ is a Lipschitz domain in $\mathbb{R}^2$, $D$ is a measurable subset such that $\Omega - D \neq \emptyset$, and $f : \Omega - D \to \mathbb{R}$ is bounded and measurable. We assume that

(F1) $F \in C^2(\mathbb{R}^{2 \times 2})$, $F(-p) = F(p)$, $F(0) = 0$, $DF(0) = 0$, and $|DF| \leq c$,

(F2) $0 < D^2F(p)(q, q) \leq \frac{1}{1 + |p|^2} |q|^2$ for all $p, q \in \mathbb{R}^{2 \times 2}$,

(F3) $c_1|p| - c_2 \leq F(p)$ for some $c_1 > 0$, $c_2 \in \mathbb{R}$ and all $p \in \mathbb{R}^{2 \times 2}$.

(G1) $G \in C^2(\mathbb{R}^2)$, $G(-y) = G(y)$, $G(0) = 0$, $DG(0) = 0$, and $|DG| \leq c$,
(G2) $0 < D^2 G(y)(x, x) \leq c \frac{1}{1 + |y|} |x|^2$ for all $x, y \in \mathbb{R}^2$,

(G3) $c_1 |y| - c_2 \leq G(y)$ for some $c_1 > 0$, $c_2 \in \mathbb{R}$ and all $y \in \mathbb{R}^2$,

where $c$ denotes a generic positive constant.

We define

$$V := W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2).$$

(1.4)

Then the underlying problem reads

$$E(u, v) := \alpha \int_{\Omega} F(\nabla v) dx + \beta \int_{\Omega} G(\nabla u - v) dx + \int_{\Omega - D} (u - f)^2 dx \to \min \text{ in } V,$$

where $\alpha$ and $\beta$ are positive parameters which control the balance between the leading term

$$\int_{\Omega} F(\nabla v) dx$$

and the coupling term

$$\int_{\Omega} G(\nabla u - v) dx.$$

**Remark 1.1.** By an iteration procedure, the coupling method can be used to reduce functionals of any order higher than two to a problem involving only first order derivatives.

Of course, we cannot expect the solvability of Problem $(\mathcal{P})$ in the nonreflexive space $V$ in general. Therefore, we have to pursue the approach from [2] and [3], which means to consider suitably relaxed variants of the above problem. The first method is the relaxation of Problem $(\mathcal{P})$ in the space $BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$ by using the concept of convex functions of a measure (cf. [9]). Therefore, we replace $E$ with the functional

$$\tilde{E}(u, v) = \alpha \int_{\Omega} F(\nabla v) + \beta \int_{\Omega} G(\nabla u - v \cdot \mathcal{L}^2) + \int_{\Omega - D} (u - f)^2 dx$$

and look for solutions to the problem

$$\tilde{E}(u, v) \to \min \text{ in } BV(\Omega) \times BV(\Omega, \mathbb{R}^2),$$

(\tilde{\mathcal{P}})

where for finite Radon measures $\mu \in \mathcal{M}(\Omega, \mathbb{R}^2)$ and $\nu \in \mathcal{M}(\Omega, \mathbb{R}^{2 \times 2})$ we declare (cf. [9, formula (1.4)])

$$\int_{\Omega} F(\nu) := \int_{\Omega} F(\nu^\alpha) dx + \int_{\Omega} F^\infty \left( \frac{d\nu^s}{d|\nu^s|} \right) d|\nu^s|$$

and

$$\int_{\Omega} G(\mu) := \int_{\Omega} G(\mu^s) dx + \int_{\Omega} G^\infty \left( \frac{d\mu^s}{d|\mu^s|} \right) d|\mu^s|. $$

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We denote by $\mu = \mu^a \mathcal{L}^2 + \mu^s$ the Lebesgue decomposition of $\mu$. We set
\[
F^\infty(p) := \lim_{t \to \infty} \frac{F(tp)}{t} \quad \forall \ p \in \mathbb{R}^{2 \times 2},
\]
\[
G^\infty(y) := \lim_{t \to \infty} \frac{G(ty)}{t} \quad \forall \ y \in \mathbb{R}^2.
\]
We note that the Lebesgue decomposition of the measure $\nabla u - v \cdot \mathcal{L}^2$ is given by
\[
\nabla u - v \cdot \mathcal{L}^2 = (\nabla^a u - v) \cdot \mathcal{L}^2 + \nabla^s u,
\]
where for $u \in BV(\Omega)$ we denote by $\nabla^a u$, $\nabla^s u$ the absolutely continuous part and the singular part respectively of the gradient measure with respect to $\mathcal{L}^2$. As in [1, Theorem 5.1], we can prove the following assertion.

**Theorem 1.1.** Under the above general assumptions regarding $\Omega$, $D$, $f$, $F$, and $G$, the following assertions hold.

(a) Problem $(\mathcal{P})$ has at least one solution $(u, v) \in BV(\Omega) \times BV(\Omega, \mathbb{R}^2)$.

(b) If $(u, v)$ and $(\tilde{u}, \tilde{v})$ are two distinct solutions to Problem $(\mathcal{P})$, then
\[
\begin{align*}
\nabla^a u - v &= \nabla^a \tilde{u} - \tilde{v} \ a.e. \ on \ \Omega - D, \\
\nabla^s v &= \nabla^s \tilde{v} \ a.e. \ on \ \Omega.
\end{align*}
\]

In particular, if $D = \emptyset$, i.e., in the case of pure denoising, the solution to Problem $(\mathcal{P})$ is unique.

(c) The set $\mathcal{M}$ of all solutions of Problem $(\mathcal{P})$ coincides with the set of all $L^1(\Omega) \times L^1(\Omega, \mathbb{R}^2)$-cluster points of $E$-minimizing sequences in $W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)$. If $E$ admits a minimizer $(u, v)$ in the Sobolev class $V$, then $\mathcal{M} = \{(u, v)\}$.

Another well established approach towards the relaxation of Problem $(\mathcal{P})$ is that via convex duality [cf. [3], [11], or [12]]. As in [3], we pass to the dual formulation via Lagrangians. To simplify our notation, we introduce the linear operator
\[
\Lambda : V \to Y := L^1(\Omega, \mathbb{R}^2) \times L^1(\Omega, \mathbb{R}^{2 \times 2}), \quad \mathbf{u} = (u, v) \mapsto (\nabla u - v, \nabla v)
\] 
and the function $\mathcal{F} : \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 2} \to \mathbb{R}$, $(y, p) \mapsto \alpha F(p) + \beta G(y)$. We observe that Problem $(\mathcal{P})$ can be written for short as
\[
E(\mathbf{u}) = \int_{\Omega} \mathcal{F}(\Lambda \mathbf{u}) dx + \int_{\Omega - D} (u - f)^2 dx \to \min,
\]
where $\mathbf{u} = (u, v) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)$. By means of this representation, it is easy to see how to apply the results from [13, Remarks 4.1 and 4.2] in order to obtain the problem in duality to Problem $(\mathcal{P})$. First, for $\mathbf{w} = (u, v) \in V$ and $\mathbf{y} = (x, \lambda) \in Y^* = L^\infty(\Omega, \mathbb{R}^2) \times L^\infty(\Omega, \mathbb{R}^{2 \times 2})$ we define the associated Lagrangian through
\[
\ell(\mathbf{w}, \mathbf{y}) := \int_{\Omega} \Lambda(\mathbf{w}) \odot \mathbf{y} dx - \int_{\Omega} \mathcal{F}^*(\mathbf{y}) dx + \int_{\Omega - D} (u - f)^2 dx,
\]
where for \((x, p), (y, q) \in \mathbb{R}^2 \times \mathbb{R}^{2 \times 2}\) we set \((x, p) \odot (y, q) := x \cdot y + p : q\), with “\(\cdot\)” and “\(:\)" denoting the standard scalar products on \(\mathbb{R}^2\) and \(\mathbb{R}^{2 \times 2}\) respectively. Furthermore, \(\mathcal{F}^*\) is the convex dual to the function \(\mathcal{F}\) which, by [13, Remark 4.3, p. 61], can be split into
\[
\mathcal{F}^*(\xi, \lambda) = \alpha F^*(\lambda) + \beta G^*(\zeta).
\]
Hence we can write (1.7) as
\[
\ell(w, y) = \int_{\Omega} \nabla v : \lambda - \alpha F^*(\lambda) + (\nabla u - v) \cdot \xi - \beta G^*(\zeta) \, dx + \int_{\Omega - D} (u - f)^2 \, dx
\]
and find (cf. [13, p. 56])
\[
E(w) = \sup_{y \in V^*} \ell(w, y).
\]
The dual functional \(R : Y^* \rightarrow [0, \infty]\) is now defined by
\[
R(y) := \inf_{w \in V} \ell(w, y) \tag{1.8}
\]
and the dual problem consists in maximizing \(R\), i.e.,
\[
R \rightarrow \max \text{ in } L^\infty(\Omega, \mathbb{R}^2) \times L^\infty(\Omega, \mathbb{R}^{2 \times 2}). \tag{\(\mathcal{P}^*\)}
\]

**Theorem 1.2.** Under the above general assumptions regarding \(\Omega, D, f, F,\) and \(G\), the following assertions hold.

(a) Problem \((\mathcal{P}^*)\) has at least one solution \((\rho, \sigma) \in L^\infty(\Omega, \mathbb{R}^2) \times L^\infty(\Omega, \mathbb{R}^{2 \times 2})\).

(b) Problems \((\mathcal{P})\) and \((\mathcal{P}^*)\) are related via the so called “inf–sup” relation \(\inf_{w \in V} E(w) = \sup_{y \in Y^*} R(y)\), i.e., there is no duality gap.

(c) Let \((u, v) \in BV(\Omega) \times BV(\Omega, \mathbb{R}^2)\) be a solution to the relaxed problem \((\mathcal{P})\). Then
\[
(\rho, \sigma) = D\mathcal{F}(\Lambda^a(u, v)) = \beta DG(\nabla^a u - v) \oplus \alpha DF(\nabla^a v) \text{ a.e. on } \Omega, \tag{1.9}
\]
where \(\Lambda^a(u, v) := (\nabla^a u - v, \nabla^a v)\). In particular, the solution to the dual problem is unique by Theorem 1.1 (b).

**Remark 1.2.** (i) Theorems 1.1 and 1.2 are valid in arbitrary dimensions \(n \geq 2\).

(ii) Both results remain true if we replace the quantity \(\int_{\Omega - D} (u - f)^2 \, dx\) with a more general data term \(\int_{\Omega - D} \Phi(|u - f|) \, dx\), where \(\Phi : [0, \infty) \rightarrow [0, \infty)\) is a strictly convex, increasing, and differentiable function, and consider the problem
\[
\tilde{E}_\Phi(u, v) = \alpha \int_{\Omega} F(\nabla v) + \beta \int_{\Omega} G(\nabla u - v : \mathbb{L}^2) + \int_{\Omega - D} \Phi(|u - f|) \, dx \rightarrow \min
\]
\[
\text{in }BV(\Omega) \times BV(\Omega, \mathbb{R}^2). \tag{\(\mathcal{P}_\Phi\)}
\]
In order to obtain more regular minimizers, we need to refine our assumptions on $F$ and $G$. In fact, previous work in this regard (cf., for example, [8], [2], [3] for more recent results) indicate that the correct framework for establishing "classical" solvability (i.e., in a Sobolev space) of our primal problem ($\mathcal{P}$) is the concept of "$\mu$-ellipticity." This is to say that we replace the rather general ellipticity condition (F2) with the stronger assumption
\[
c_1 \frac{1}{(1 + |p|^{\mu})} |q|^2 \leq D^2 F(p, q) \leq c_2 \frac{1}{1 + |p|^{\mu}} |q|^2
\]
and (G2) is replaced with
\[
c_1 \frac{1}{(1 + |y|)^{\nu}} |x|^2 \leq D^2 G(y, x) \leq c_2 \frac{1}{1 + |y|^{\nu}} |x|^2
\]
for some $c_1, c_2 > 0$, $\mu \in (1, \infty)$ and for all $p, q \in \mathbb{R}^{2 \times 2}$ and $x, y \in \mathbb{R}^2$. We furthermore have to distinguish the case of pure denoising $D = \emptyset$ from the general case. Then the following assertion holds.

**Theorem 1.3.** In addition to the above general assumptions on $\Omega$ and $f$, we assume that $D = \emptyset$ (pure denoising), $F$ satisfies (F1), (F2)', (F3), and $G$ satisfies (G1), (G2)', (G3) for the parameters
\[
(\mu, \nu) \in \left(1, \frac{3}{2}\right) \times (1, 2).
\]
Then Problem ($\mathcal{P}$) has a unique solution $(u, v)$ in the Sobolev class $V = W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)$. Moreover, $(u, v) \in W^{1,p}_{\text{loc}}(\Omega) \times W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^2)$ for every $p \in [1, \infty)$.

**Remark 1.3.** (i) The uniqueness of a possible Sobolev-minimizer follows from Theorem 1.1 (c).

(ii) In the case $D \neq \emptyset$, we were not able to prove the above result in full generality. However, if we replace the quadratic error term $\int_{\Omega} |u - f|^2 dx$ with $\int_{\Omega} \omega(|u - f|) dx$ for a convex differentiable increasing function $\omega : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ of linear growth and (1.10) with the condition
\[
(\mu, \nu) \in \left(1, \frac{3}{2}\right) \times \left(1, \frac{3}{2}\right),
\]
then Theorem 1.3 holds for the generalized solution of this modified problem even for $D \neq \emptyset$. Of course, this only concerns the regularity of $u$ and the assumptions of Theorem 1.3 are sufficient to establish $v \in W^{1,1}(\Omega, \mathbb{R}^2)$ for arbitrary $D$.

For regularity results in terms of classical differentiability, we need to put further restraints on our density functions. As our coupled model resembles a vector-valued situation, it is natural to impose a structure Uhlenbeck type condition on $F$ in addition to Assumptions (F1), (F2)', and (F3), which means that we consider functions of the special form
\[
F(p) = g(|p|^2)
\]
with a convex increasing function $g : [0, \infty) \to [0, \infty)$ which is at least of class $C^2$. Again we restrict ourselves to the case of pure denoising ($D = \emptyset$). Then we have the following assertion.
Theorem 1.4. In addition to the above general assumptions on \(\Omega\) and \(f\), we assume that \(D = \emptyset\), \(F\) satisfies (F1), (F2)', (F3), (F4), and \(G\) satisfies (G1), (G2)', (G3) with parameters \(\mu\) and \(\nu\) satisfying (1.10). Let \((u, v)\) be the \(W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)\)-minimizer from Theorem 1.3. Then there is an open set \(\Omega_0 \subset \Omega\) of full measure, i.e.,

\[
\mathcal{L}^2(\Omega - \Omega_0) = 0,
\]

such that \((u, v) \in C^{1,\alpha}(\Omega_0) \times C^{1,\beta}(\Omega_0, \mathbb{R}^2)\) for all \((\alpha, \beta) \in (0, 1) \times (0, 1)\). For the set \(\Omega - \Omega_0\) of possible singularities

\[
\mathcal{H}^\epsilon \cdot \dim(\Omega - \Omega_0) = 0,
\]

which means that the \(\epsilon\)-dimensional Hausdorff measure \(\mathcal{H}^\epsilon(\Omega_0)\) is zero for every \(\epsilon > 0\).

Remark 1.4. For \(D \neq \emptyset\) the statement of Theorem 1.4 still holds for the modified problem from Remark 1.3 (ii) and with (1.10)' instead of (1.10).

Remark 1.5. In contrast to Theorems 1.1 and 1.2, the statements of Theorems 1.3 and 1.4 crucially depend on the assumption \(\Omega \subset \mathbb{R}^2\).

2 Relaxation in \(BV\). Proof of Theorem 1.1

As in [11] or [10], a key tool in the proof of Theorem 1.1 is the following density result (cf. [11, Lemma 2.2] or [10, Theorem 1.1] for a generalization to higher orders).

Lemma 2.1. Let the above general assumptions on \(\Omega\) and \(D\) hold. For a given \((u, v) \in BV(\Omega) \times BV(\Omega, \mathbb{R}^2)\) there is a sequence \((\varphi_n, \psi_n) \subset C^\infty(\Omega) \times C^\infty(\Omega, \mathbb{R}^2)\) such that

\[
\varphi_n \to u \text{ in } L^2(\Omega),
\]

\[
\psi_n \to v \text{ in } L^2(\Omega, \mathbb{R}^2),
\]

\[
\int_{\Omega} \sqrt{1 + |\nabla \psi_n|^2} dx \to \int_{\Omega} \sqrt{1 + |\nabla v|^2},
\]

\[
\int_{\Omega} \sqrt{1 + |\nabla \varphi_n - \psi_n|^2} dx \to \int_{\Omega} \sqrt{1 + |\nabla u - v|^2}.
\]

Proof. First, we note that the existence of a sequence \((\psi_n)\) with the properties (2.2) and (2.3) directly follows from [11, Lemma 2.2] (note that due to \(\Omega \subset \mathbb{R}^2\), we have \(u \in L^2(\Omega), v \in L^2(\Omega, \mathbb{R}^2)\) by the embedding theorems). Let us define a linear differential operator \(S : C^\infty(\Omega) \times C^\infty(\Omega, \mathbb{R}^2) \to C^\infty(\Omega, \mathbb{R}^2)\), \((\eta, \varphi) \mapsto \nabla \eta - \varphi\) with constant coefficients. The operator \(S\) is of local type in the sense of [9, p. 688], and we can quote Theorem 2.2 from [9] (cf. also Remark 2.1 therein) to conclude that there is a sequence \((\varphi_n, \bar{\psi}_n) \subset C^\infty(\Omega) \times C^\infty(\Omega, \mathbb{R}^2)\) such that

\[
(\varphi_n, \bar{\psi}_n) \to (u, v) \text{ in } L^2(\Omega) \times L^2(\Omega, \mathbb{R}^2),
\]

\[
\int_{\Omega} \sqrt{1 + |S(\varphi_n, \bar{\psi}_n)|^2} dx \to \int_{\Omega} \sqrt{1 + |S(u, v)|^2} = \int_{\Omega} \sqrt{1 + |\nabla u - v|^2}.
\]
Furthermore,
\[
\left| \int_\Omega \sqrt{1 + |S(\varphi_n, \tilde{\psi}_n)|^2} \, dx - \int_\Omega \sqrt{1 + |S(\varphi_n, \psi_n)|^2} \, dx \right| \leq c \int_\Omega |\tilde{\psi}_n - \psi_n| \, dx \to 0
\]
since \( \tilde{\psi}_n, \psi_n \to v \) in \( L^1(\Omega, \mathbb{R}^2) \). Together with (2.5), this proves that \((\varphi_n, \psi_n)\) is a sequence as claimed in the lemma. \( \square \)

**Proof of Theorem 1.1.** (a) Let \((u_k, v_k) \in BV(\Omega) \times BV(\Omega, \mathbb{R}^2)\) denote an \( E \) minimizing sequence. By Lemma (2.1) in combination with the Reshetnyak continuity theorem (cf., for example, [14, Proposition 2.2]), we can assume that \((u_k, v_k) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)\). Thanks to the linear growth of \( F \) and \( G \) it is clear that there are constants \( M_1, M_2, M_3 > 0 \) such that

\[
\sup_{k \in \mathbb{N}} \int_\Omega |\nabla v_k| \, dx \leq M_1,
\]
\[
\sup_{k \in \mathbb{N}} \int_\Omega |\nabla u_k - v_k| \, dx \leq M_2,
\]
\[
\sup_{k \in \mathbb{N}} \int_{\Omega-D} |u_k| \, dx \leq M_3.
\]

We need the following version of the Poincaré inequality (cf. [1, Lemma 4.2]).

**Lemma 2.2.** Let \( \Omega \subset \mathbb{R}^2 \) be a Lipschitz domain, and let \( \rho \in C^1_0(\Omega) \) be such that
\[
\int_\Omega \rho \, dx = 1.
\]
Then there is a constant \( c > 0 \) depending only on \( \Omega \) and such that for any \( u \in W^{1,1}(\Omega) \)
\[
\left\| u - \int_\Omega \rho \, dx \right\|_{1,\Omega} \leq c \left\| \nabla u \right\|_{1,\Omega}.
\]

Now, we choose \( \rho \) as in the lemma and such that \( \text{spt} (\rho) \subset \Omega - D \) (note that \( \Omega - \overline{D} \neq \emptyset \) by our assumption). Then we have

\[
\sup_{k \in \mathbb{N}} \left| \int_\Omega \rho \nabla u_k \, dx - \int_\Omega \rho v_k \, dx \right| \leq \|\rho\|_\infty \sup_{k \in \mathbb{N}} \int_\Omega |\nabla u_k - v_k| \, dx \leq \|\rho\|_\infty M_2. \tag{2.9}
\]

Furthermore,
\[
\left| \int_\Omega \rho \nabla u_k \, dx \right| = \left| \int_\Omega \nabla \rho u_k \, dx \right| \leq \|\nabla \rho\|_\infty \int_{\Omega-D} |u_k| \, dx \leq \|\nabla \rho\|_\infty M_3. \tag{2.10}
\]
Thus, from (2.9) and (2.10) we conclude

$$\sup_{k \in \mathbb{N}} \left| \int_{\Omega} \rho v_k \, dx \right| < \infty$$

and (2.6), together with Lemma 2.2, implies \(\sup_{k \in \mathbb{N}} \|v_k\|_{1,1;\Omega} < \infty\). But then the boundedness of \(v_k\) in \(L^1(\Omega, \mathbb{R}^2)\) along with (2.7) and (2.8) implies (by another application of the Poincaré inequality) \(\sup_{k \in \mathbb{N}} \|u_k\|_{1,1;\Omega} < \infty\), so that \((u_k, v_k)\) is bounded in \(BV(\Omega) \times BV(\Omega, \mathbb{R}^2)\). By the \(BV\)-compactness theorem (cf. [5, Theorem 3.23]), there exists \((u, v) \in BV(\Omega) \times BV(\Omega, \mathbb{R}^2)\) such that \((u_k, v_k) \to (u, v)\) in \(L^1(\Omega) \times L^1(\Omega, \mathbb{R}^2)\) and almost everywhere. The fact that \((u, v)\) is indeed \(\tilde{E}\)-minimal immediately follows since the relaxation \(\tilde{E}\) is lower semicontinuous with respect to the \(L^1\)-convergence by definition (cf. [5, Remark 5.46]).

(b) The statements of part (b) are a mere consequence of the strict convexity of the functions \(F, G\) and the data fitting term \(|u - f|^2\).

The fact that every \(\tilde{E}\)-minimizer is indeed the \(L^1\)-limit of an \(E\)-minimizing sequence in the Sobolev class \(V\) follows from Lemma (2.1) together with \(\tilde{E}_V = E\). Every such limit in \(BV\) minimizes \(\tilde{E}\) because of the above-mentioned lower semicontinuity property of the relaxation. It remains to prove that \((\tilde{\mathcal{P}})\) has a unique solution if \(\mathcal{M} \cap V \neq \emptyset\). Therefore, we assume that \((u, v) \in V\) minimizes \(\tilde{E}\). Let \((\tilde{u}, \tilde{v})\) be another element of \(\mathcal{M}\). From \(\tilde{E}(u, v) = E(u, v) = \tilde{E}(\tilde{u}, \tilde{v})\) and part (b) we infer

$$\int_{\Omega} F^{\infty} \left( \frac{\nabla^s \tilde{v}}{|\nabla^s \tilde{v}|} \right) d|\nabla^s \tilde{u}| + \int_{\Omega} G^{\infty} \left( \frac{\nabla^s \tilde{u}}{|\nabla^s \tilde{u}|} \right) d|\nabla^s \tilde{u}| = 0$$

and thus \(\nabla^s \tilde{v} = 0\) and \(\nabla^s \tilde{u} = 0\), which means \((\tilde{u}, \tilde{v}) \in V\). But then b) implies \(\nabla \tilde{v} = \nabla v\) and thereby \(v = \tilde{v} + c\). Further it follows from \(\nabla u - v = \nabla \tilde{u} - \tilde{v}\) that \(\nabla u = \nabla \tilde{u} + c\), i.e.,

\[u(x) = \tilde{u}(x) + c \cdot x + b\]

for some \(b, c \in \mathbb{R}^2\). Finally, \(u = \tilde{u}\) on \(\Omega - D\) along with \(\mathcal{L}^2(\Omega - D) > 0\) requires \(b = c = 0\), hence \(u = \tilde{u}\) and \(v = \tilde{v}\).

\[\square\]

3 Duality. Proof of Theorem 1.2

Since our arguments follow the ideas in [11] very closely, the reader will hopefully approve our attempt to give a rather condensed outline of the proof of Theorem 1.2, referring to [11] or [12] for details.

The proof relies on a suitable approximation of Problem \((\mathcal{P})\) through a family of regularizations. To be precise, for \(\delta \in (0, 1)\) we look at the problem

\[E_\delta(u, v) := \frac{\delta}{2} \int_\Omega |\nabla u|^2 + |\nabla v|^2 \, dx + E(u, v) \to \min \text{ in } W^{1,2}(\Omega) \times W^{1,2}(\Omega, \mathbb{R}^2). \quad (\mathcal{P}_\delta)\]

**Lemma 3.1.** Under the above general assumptions on \(\Omega, f, F, \) and \(G\), the following assertions hold.

(a) For any \(\delta \in (0, 1)\) Problem \((\mathcal{P}_\delta)\) admits a unique solution \(u_\delta = (u_\delta, v_\delta)\) in the space \(W^{1,2}(\Omega) \times W^{1,2}(\Omega, \mathbb{R}^2)\).
(b) The family of \( u_\delta \) satisfies the following conditions:

\[
\sup_{\delta \in (0,1)} \int_\Omega |\nabla u_\delta|^2 + |\nabla v_\delta|^2 dx < \infty, \tag{3.1}
\]

\[
\sup_{\delta \in (0,1)} \int_{\Omega-D} |u_\delta|^2 dx < \infty. \tag{3.2}
\]

\[
u_\delta \in W^{2,2}_{\text{loc}}(\Omega) \times W^{2,2}_{\text{loc}}(\Omega, \mathbb{R}^2) \tag{3.3}
\]

(not necessarily uniformly with respect to \( \delta \)).

**Proof.** (a) Let \( \delta \in (0,1) \) be fixed. Quoting standard results concerning the weak lower semicontinuity of convex functionals on Sobolev spaces, to prove the existence of a minimizer by the direct method, it suffices to show that any \( E_\delta \) minimzing sequence is bounded in \( W^{1,2}(\Omega) \times W^{1,2}(\Omega, \mathbb{R}^2) \). So let us fix \( \delta \in (0,1) \) and denote by \( u_k = (u_k, v_k) \), \( k \in \mathbb{N} \) such a minimizing sequence. By \( E_\delta(u_k) \leq E_\delta(0,0) = E(0,0) \), it is clear that \( |\nabla v_k| \) and \( |\nabla u_k| \) are bounded in \( L^2(\Omega) \). Furthermore, \( f - u_k \) is bounded in \( L^2(\Omega - D) \). By the Poincaré inequality, we have

\[
\int_\Omega \left| u_k(x) - \overline{u_k(x)} \right|^2 dx \leq c \int_\Omega |\nabla u_k|^2 dx,
\]

where

\[
\overline{u_k(x)} := \int_{\Omega-D} u_k(t) dt.
\]

We infer that \( u_k \) is bounded in \( W^{1,2}(\Omega) \). But then

\[
\int_\Omega G(\nabla u_k - v_k) dx \leq E(0,0)
\]

implies that also \( |v_k| \) is bounded in \( L^1(\Omega) \) and another application of the Poincaré inequality yields the boundedness of \( v_k \) in \( W^{1,2}(\Omega, \mathbb{R}^2) \).

(b) The statement immediately follows from \( E_\delta(u_k) \leq E_\delta(0,0) = E(0,0) \).

(c) Let \( \delta \in (0,1) \) be fixed. The proof of this statement is a standard application of the difference quotient technique to the quadratic variational problems

\[
E_\delta(u, v_\delta) \to \min \text{ in } W^{1,2}(\Omega), \text{ with } v_\delta \text{ fixed},
\]

\[
E_\delta(u_\delta, v) \to \min \text{ in } W^{1,2}(\Omega, \mathbb{R}^2), \text{ with } u_\delta \text{ fixed}
\]

respectively.

The core of the proof of Theorem 1.2 now consists in a careful analysis of the convergence behavior of \( u_\delta \) as \( \delta \) approaches zero. Our claim is that (at least for a subsequence) \( u_\delta \) converges in \( L^1(\Omega) \times L^1(\Omega, \mathbb{R}^2) \) towards a solution of the relaxed problem (\( \tilde{P} \)), and that

\[
\sigma_\delta = (\rho_\delta, \sigma_\delta) := \delta \nabla u_\delta + D.F(\Lambda u_\delta) = [\delta \nabla u_\delta + \beta DG(\nabla u_\delta - v_\delta)] \oplus [\delta \nabla v_\delta + \alpha DF(\nabla v_\delta)] \tag{3.4}
\]

converges weakly in \( L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R}^{2 \times 2}) \) towards a solution of the dual problem (\( \mathcal{P}^* \)).
Lemma 3.2. The family \( \mathbf{u}_\delta \) is uniformly bounded in the space \( V = W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2) \). In particular, there is null-sequence \( \delta \downarrow 0 \) and a function \( \mathbf{u} = (\mathbf{u}, \mathbf{v}) \in BV(\Omega) \times BV(\Omega, \mathbb{R}^2) \) such that \( \mathbf{u}_\delta \to \mathbf{u} \) in \( L^1(\Omega) \times L^1(\Omega, \mathbb{R}^2) \) and almost everywhere as \( \delta \downarrow 0 \).

Proof. We start with the observation that due to \( E_\delta(\mathbf{u}_\delta) \leq E_\delta(0,0) = E(0,0) \) and the linear growth of \( F \) and \( G \) we have the following bounds:

\[
\sup_{\delta \in (0,1)} \int_\Omega |\nabla v_\delta| \, dx \leq M'_1, \tag{3.5}
\]

\[
\sup_{\delta \in (0,1)} \int_\Omega |\nabla u_\delta - v_\delta| \, dx \leq M'_2, \tag{3.6}
\]

\[
\sup_{\delta \in (0,1)} \int_{\Omega - D} |\nabla u_\delta|^2 \, dx \leq M'_3 \tag{3.7}
\]

for constants \( M'_1, M'_2, M'_3 > 0 \). From here on, we can repeat the arguments from the proof of Lemma 2.2 to conclude the boundedness of \( (u_\delta, v_\delta) \) in \( W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2) \). The claimed convergence is then seen to be a consequence of the BV-compactness theorem. \( \square \)

The next step in the proof of Theorem 1.2 is to show that the function \( \mathbf{u} \) from Lemma 3.2 in fact minimizes \( \bar{E} \). Let us fix a null-sequence \( \delta \downarrow 0 \) as in Lemma 3.2. By (3.1),

\[
\delta \nabla \mathbf{u}_\delta = (\delta \nabla u_\delta, \delta \nabla v_\delta) \to 0 \quad \text{in} \quad L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R}^{2 \times 2}) \tag{3.8}
\]

and, since \( |D\mathcal{F}| \) is bounded,

\[
\sup_{\delta \in (0,1)} \int_\Omega |\sigma_\delta|^2 \, dx < \infty. \tag{3.9}
\]

Thus, there exists \( \sigma \in L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R}^{2 \times 2}) \) such that \( \sigma_\delta \to \sigma \) in \( L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R}^{2 \times 2}) \) as \( \delta \downarrow 0 \) at least for another subsequence. Furthermore, setting \( \tau_\delta := D\mathcal{F}(\Lambda \mathbf{u}_\delta) \), we can assume that there exists \( \tau \in L^\infty(\Omega, \mathbb{R}^2) \times L^\infty(\Omega, \mathbb{R}^{2 \times 2}) \) such that

\[
\tau_\delta \rightharpoonup \tau. \tag{3.10}
\]

By \( \sigma_\delta = \delta \nabla \mathbf{u}_\delta + \tau_\delta \) and (3.8),

\[
\sigma = \tau. \tag{3.11}
\]

By the minimality with respect to \( E_\delta \), \( \mathbf{u}_\delta \) satisfies the weak Euler–Lagrange equation

\[
\delta \int_\Omega \nabla \mathbf{u}_\delta \otimes \nabla \varphi \, dx + \int_\Omega D\mathcal{F}(\Lambda \mathbf{u}_\delta) \otimes \Lambda \varphi \, dx + 2 \int_{\Omega - D} (u - f) \varphi \, dx = 0 \tag{EL}
\]

\[
\forall \varphi = (\varphi, \psi) \in W^{1,2}(\Omega, \mathbb{R}) \times W^{1,2}(\Omega, \mathbb{R}^2).
\]

This can be decoupled into the two equations

\[
\int_\Omega DF_\delta(\nabla v_\delta) : \nabla \psi \, dx - \beta \int_\Omega DG(\nabla u_\delta - v_\delta) \cdot \psi \, dx = 0 \quad \forall \psi \in W^{1,2}(\Omega, \mathbb{R}^2), \tag{EL1}
\]

\[
\int_\Omega \nabla G(\nabla u_\delta - v_\delta) \cdot \nabla \psi \, dx - \int_\Omega \nabla F_\delta(\nabla v_\delta) \cdot \nabla \psi \, dx = 0 \quad \forall \psi \in W^{1,2}(\Omega, \mathbb{R}^2), \tag{EL2}
\]

\[
\varepsilon \int_\Omega \frac{1}{2} |\nabla v_\delta|^2 \, dx - \beta \int_\Omega \frac{1}{2} |\nabla u_\delta - v_\delta|^2 \, dx \leq \varepsilon \int_\Omega (\nabla \phi \cdot \nabla v_\delta - (u - f) \phi) \, dx \quad \forall \phi \in \mathcal{C}_c^\infty(\Omega, \mathbb{R}).
\]
where \( F_\delta(p) := \frac{\delta}{2} |p|^2 + \alpha F(p) \) for all \( p \in \mathbb{R}^{2 \times 2} \), and

\[
0 = \delta \int_\Omega \nabla u_\delta \cdot \nabla \varphi \, dx + \beta \int_\Omega DG(\nabla u_\delta - v_\delta) \cdot \nabla \varphi \, dx + 2 \int_{\Omega-D} (u_\delta - f) \varphi \, dx \quad \text{(EL2)}
\]

for all \( \varphi \in W^{1,2}(\Omega) \). Using the same arguments as in [11, Section 4] and (EL) along with the duality relation (cf. [13, Proposition 5.1]) \( \mathcal{F}(\Lambda u_\delta) = \tau \circ \Lambda u_\delta - \mathcal{F}^*(\tau) \), we can establish the formula

\[
E_\delta(u_\delta) = -\frac{\delta}{2} \int_\Omega |\nabla u_\delta|^2 \, dx - \int_\Omega \mathcal{F}^*(\tau) \, dx - \int_{\Omega-D} u_\delta^2 \, dx + \int_{\Omega-D} f^2 \, dx
\]

\[
\leq -\int_\Omega \mathcal{F}^*(\tau) \, dx - \int_{\Omega-D} u_\delta^2 \, dx + \int_{\Omega-D} f^2 \, dx.
\]

(3.12)

Note that from the definition of \( R \) (cf. (1.8)) it is clear that

\[
\sup_{y \in V^*} R(y) \leq \inf_{w \in V} E(w),
\]

and we observe that

\[
\inf_{w \in V} E(w) \leq E(u_\delta) \leq E_\delta(u_\delta).
\]

Hence, applying \( \limsup \) to both sides of (3.12), we get

\[
\sup_{y \in V^*} R(y) \leq \inf_{w \in V} E(w) \leq -\int_\Omega \mathcal{F}^*(\tau) \, dx - \int_{\Omega-D} \overline{w}^2 \, dx + \int_{\Omega-D} f^2 \, dx,
\]

(3.13)

where we used the convexity of \( \mathcal{F}^* \) and the Fatou lemma. Following [11, Section 4] (with \( F \) replaced by \( \mathcal{F} \) and \( \nabla \) replaced by \( \Lambda \)), we furthermore obtain

\[
R(\tau) = -\int_\Omega \mathcal{F}^*(\tau) \, dx + \inf_{w=(w_1,w_2) \in V} \left[ \int_{\Omega-D} (\overline{w} - w_1)^2 + f^2 - \overline{w}^2 \, dx \right].
\]

Since the term in the brackets is obviously minimal for \( w_1 = \overline{w} \), we infer from (3.13) the equation

\[
\sup_{y \in V^*} R(y) = R(\tau) = \inf_{w \in V} E(w),
\]

i.e., the inf-sup relation. Further, we see that \( \tau = \sigma \) maximizes the dual functional and therefore (3.12) implies that (a subsequence of) \( u_\delta \) is in fact an \( E \) minimizing sequence. Parts (a) and (b) of Theorem 1.2 are thus proved.

For part (c) we claim that it is enough to revise the steps in the proof of Theorem 1.3 in [11] with \( F \) replaced by \( \mathcal{F} \) and \( \nabla^a \) replaced by \( \Lambda^a \). Therefore, we omit details.
4 Sobolev Solutions. Proof of Theorem 1.3

Under the assumptions of Theorem 1.3, let \( u_\delta = (u_\delta, v_\delta) \) be the \( \tilde{E} \)-minimizing sequence as constructed in the previous section. Our proof mainly relies on the following lemma:

**Lemma 4.1.** Under the assumptions of Theorem 1.3,

\[
\varphi_\delta := (1 + |\nabla v_\delta|)^{1 - \frac{\mu}{2}},
\]

\[
\bar{\varphi}_\delta := (1 + |\nabla u_\delta|)^{1 - \frac{\nu}{2}}
\]

are uniformly bounded in \( W^{1,2}_{\text{loc}}(\Omega) \).

**Proof.** Let us prove (4.1). Throughout the proof, we use the summation convention with respect to the index \( i \in \{1, 2\} \) and denote by \( c \) a generic constant. We start with the discussion of the quantity \( \varphi_\delta \). First, the uniform boundedness of \( \varphi_\delta \) in \( L^2_{\text{loc}}(\Omega) \) is clear since we assume \( \mu > 1 \) and \( v_\delta \) is uniformly bounded in \( W^{1,1}(\Omega, \mathbb{R}^2) \) by Lemma 3.2. Choosing \( \partial_i \psi \) instead of \( \psi \) in the Euler equation (EL1) and integrating by parts, we get

\[
0 = \int_{\Omega} D^2 F_\delta(\nabla v_\delta)(\partial_i \nabla v_\delta, \nabla \psi) dx + \beta \int_{\Omega} DG(\nabla u_\delta - v_\delta) \cdot \partial_i \psi dx,
\]

which, by approximation, holds for all \( \psi \in \tilde{W}^{1,2}(\Omega, \mathbb{R}^2) \). Let \( x_0 \in \Omega \) be some point and \( R > 0 \) such that \( B_{2R}(x_0) \subset \Omega \). We choose \( \psi = \eta^2 \partial_i v_\delta \), where \( \eta \in C^1_0(\Omega) \) is such that

\[
\text{spt}(\eta) \subset B_{2R}(x_0), \quad 0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B_R(x_0), \quad |\nabla \eta| \leq \frac{c}{R}.
\]

Then (EL1)' reads

\[
0 = \int_{\Omega} D^2 F_\delta(\nabla v_\delta)(\partial_i \nabla v_\delta, \nabla (\eta^2 \partial_i v_\delta)) dx + \beta \int_{\Omega} DG(\nabla u_\delta - v_\delta) \cdot \partial_i (\eta^2 \partial_i v_\delta) dx
\]

which can be expanded to

\[
0 = \int_{\Omega} D^2 F_\delta(\nabla v_\delta)(\partial_i \nabla v_\delta, \partial_i \nabla v_\delta) \eta^2 dx + 2 \int_{\Omega} D^2 F_\delta(\nabla v_\delta)(\eta \partial_i \nabla v_\delta, \nabla \eta \otimes \partial_i v_\delta) dx
\]

\[
= T_1
\]

\[
+ \beta \int_{\Omega} DG(\nabla u_\delta - v_\delta) \cdot \partial_i (\eta^2 \partial_i v_\delta) dx.
\]

We define \( \Theta_\delta := D^2 F_\delta(\nabla v_\delta)(\partial_i \nabla v_\delta, \partial_i \nabla v_\delta)^{\frac{1}{2}} \) and can write (4.4) as

\[
\int_{\Omega} \Theta_\delta^2 \eta^2 dx = -T_1 - T_2.
\]

Recalling (F2)', we see that the first claim of Lemma 4.1 follows via a uniform estimate of the integral \( \int_{\Omega} \Theta_\delta^2 \eta^2 dx \) on the left-hand side of (4.5). So, let us have a look at the quantity \( T_1 \)

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first. Applying the Cauchy–Schwarz inequality to the bilinear form $D^2 F_\delta(\nabla v_\delta)(\cdot, \cdot)$ and then the Young inequality we obtain ($\epsilon > 0$ is arbitrary)

$$|T_1| \leq \epsilon \int_\Omega (\Theta^2)_{\delta} \eta^2 dx + \epsilon^{-1} \int_\Omega D^2 F_\delta(\nabla v_\delta)(\nabla \eta \otimes \partial_i v_\delta, \nabla \eta \otimes \partial_i v_\delta) dx. \quad (4.6)$$

Choosing $\epsilon = \frac{1}{2}$, the first summand can be absorbed on the right-hand side of (4.5), whereas to the second summand we apply the estimate $(F2)'$ as well as Lemma 3.1 b) and Lemma 3.2 with the result

$$\int_\Omega D^2 F_\delta(\nabla v_\delta)(\nabla \eta \otimes \partial_i v_\delta, \nabla \eta \otimes \partial_i v_\delta) dx \leq c \frac{1}{R^2} + c \frac{1}{1 + |\nabla v_\delta|^2} |\nabla v_\delta|^2 dx = c(R). \quad (4.7)$$

Hence we have shown

$$\int_\Omega \Theta_\delta^2 \eta^2 dx \leq c(R) - T_2,$$

and it remains to estimate $T_2$. Therefore, we notice that due to our assumption (G1) on the function $G$, we have $DG(\nabla u - v_\delta) \in L^\infty(\Omega, \mathbb{R}^2)$ uniformly and therefore

$$|T_2| \leq c \int_\Omega |\partial_i (\eta^2 \partial_i v_\delta)| dx \leq c \int_\Omega |\nabla \eta| |\nabla v_\delta| dx + c \int_\Omega \eta^2 |\nabla^2 v_\delta| dx \leq c(R) + c \int_\Omega \eta^2 |\nabla^2 v_\delta| dx =: c(R) + T_3.$$ 

For the quantity $T_3$ we observe

$$T_3 = \int_\Omega \eta^2 (1 + |\nabla v_\delta|)^\frac{\mu}{2} \frac{|\nabla^2 v_\delta|}{(1 + |\nabla v_\delta|)^\frac{\mu}{2}} dx,$$

which, using the Young inequality can be estimated through

$$T_3 \leq \epsilon \int_\Omega \eta^2 \frac{|\nabla^2 v_\delta|^2}{(1 + |\nabla v_\delta|)^\mu} dx + \epsilon^{-1} \int_\Omega \eta^2 (1 + |\nabla v_\delta|)^\mu dx, \quad (4.8)$$

where $\epsilon > 0$ is arbitrary. Therefore, $(F2)'$ implies

$$T_3 \leq c \epsilon \int_\Omega \Theta_\delta^2 \eta^2 dx + \epsilon^{-1} \int_\Omega \eta^2 (1 + |\nabla v_\delta|)^\mu dx =: c \epsilon \int_\Omega \eta^2 \Theta_\delta^2 dx + T_4.$$

Choosing $\epsilon$ small enough, we can absorb the first term on the left-hand side of (4.5). Setting

$$\omega_\delta := (1 + |\nabla v_\delta|)^\frac{\mu}{2}, \quad (4.9)$$

we can write

$$T_4 = \int_\Omega (\eta \omega_\delta)^2 dx.$$
Observing the relation $\omega_\delta = \frac{4\mu - 4}{2 - \mu}(1 - \frac{\mu}{2}) = 2\mu - 2 < 1$, which enables us to apply the Hölder inequality

$$\int_{B_{2R}(x_0)} \varphi_\delta^{\frac{4\mu - 4}{2 - \mu}} dx \leq \pi^s R^{2s} \left( \int_\Omega \nabla v_\delta |dx|^{-\frac{2\mu - 2}{\mu}} \right)^{\frac{\mu}{\mu - 2}} \leq c R^{2s}$$

(4.10)

where $s = 3 - 2\mu > 0$ and the constant $c$ is independent of $R$. Combining our estimates of $T_1$ and $T_2$ with (4.5), we arrive at

$$(1 - c R^{2s}) \int_\Omega \eta^2 \Theta_\delta^2 dx \leq c(R).$$

(4.11)

Thus, for radii $R < R_0$ and $R_0$ such that $c R_0^{2s} < 1$, we have the uniform estimate

$$\int_\Omega \eta^2 \Theta_\delta^2 dx \leq c(R_0).$$

(4.12)

Claim (4.1) of Lemma 4.1 now follows from a covering argument. 

As a consequence of (4.1) and the Sobolev embedding theorem (recall $n = 2$), we have

$$\nabla v_\delta \in L^p_{\text{loc}}(\Omega, \mathbb{R}^{2 \times 2}) \quad \forall \ p \in [1, \infty) \ \text{uniformly with respect to } \delta.$$  

(4.13)

In particular,

$$v_\delta \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^{2 \times 2}) \ \text{uniformly with respect to } \delta.$$  

(4.14)

Furthermore, it follows (at least after passing to a suitable subsequence $\delta \downarrow 0$) that $\nabla v_\delta$ has a weak $L^p_{\text{loc}}(\Omega, \mathbb{R}^2)$-limit for some $p > 1$ and, since $v_\delta \rightharpoonup v$ in $L^1(\Omega, \mathbb{R}^2)$ and almost everywhere, we infer $v \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^2)$. Eventually, $v \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2) \cap BV(\Omega, \mathbb{R}^2)$ which is a subset of $W^{1,1}(\Omega, \mathbb{R}^2)$.

Let us now turn to the corresponding quantity $\tilde{v}_\delta$ involving $u_\delta$. We start with the Euler equation (EL2) (keep in mind that $D = \emptyset$ in the setting of Theorem 1.3), where we choose $\varphi = \partial_1(\eta^2 \partial_1 u_\delta)$ for some $\eta \in C^1_0(\Omega_0)$ satisfying the set of conditions from (4.3). Writing $G_\delta(x) := \delta |x|^2 + \beta G(x)$ for $x \in \mathbb{R}^2$, (EL2) reads after an integration by parts:

$$\int_\Omega D^2 G_\delta(\nabla v_\delta - v_\delta)(\partial_1 \nabla v_\delta, \nabla (\eta^2 \partial_1 u_\delta)) dx - \beta \int_\Omega D^2 G(\nabla u_\delta - v_\delta)(\partial_1 v_\delta, \nabla (\eta^2 \partial_1 u_\delta)) dx$$

$$- 2 \int_\Omega (u_\delta - f) \partial_1 (\eta^2 \partial_1 u_\delta) dx = 0.$$  

(4.15)
Now we make the definition $\tilde{\Theta}_\delta := D^2G_\delta(\nabla u_\delta - v_\delta)(\partial_t \nabla u_\delta, \partial_t \nabla u_\delta)^{1/2}$ due to which we can write (4.15) as

$$
\int_\Omega \eta^2 \tilde{\Theta}_\delta^2 \, dx = -2 \int_\Omega D^2G_\delta(\nabla u_\delta - v_\delta)(\eta \partial_t \nabla u_\delta, \nabla \eta \partial_t u_\delta) \, dx
+ \beta \int_\Omega D^2G(\nabla u_\delta - v_\delta)(\eta \partial_t v_\delta, \eta \partial_t u_\delta) \, dx
+ 2\beta \int_\Omega D^2G(\nabla u_\delta - v_\delta)(\eta \partial_t v_\delta, \nabla \eta \partial_t u_\delta) \, dx + 2 \int_\Omega (u_\delta - f) \partial_t (\eta^2 \partial_t u_\delta) \, dx
=: T'_1 + T'_2 + T'_3 + T'_4.
$$

(4.16)

First, we note that due to (G2)'

$$
\eta^2 \tilde{\Theta}_\delta^2 \geq c_1 \frac{1}{(1 + |\nabla u_\delta - v_\delta|)^\nu} |\eta \nabla^2 u_\delta|^2 \geq c_1 \frac{1}{(1 + |\nabla u_\delta| + |v_\delta|)^\nu} |\eta \nabla^2 u_\delta|^2
\geq (4.14) \frac{1}{(1 + |\nabla u_\delta|)^\nu} |\eta \nabla^2 u_\delta|^2 = c_\nu^2 |\nabla \varphi_\delta|^2.
$$

(4.17)

Hence, by (4.16) and our choice of $\eta$, we have

$$
\int_{B_R(x_0)} |\nabla \varphi_\delta|^2 \, dx \leq c(T'_1 + T'_2 + T'_3 + T'_4).
$$

In the Integral $T'_1$, we first apply the Cauchy–Schwarz inequality to the bilinear form $D^2G_\delta(\nabla u_\delta - v_\delta)(\cdot, \cdot)$, followed by the Young inequality to obtain

$$
|T'_1| \leq \epsilon \int_\Omega \eta^2 \tilde{\Theta}_\delta^2 \, dx + c(\epsilon) \int_\Omega D^2G(\nabla u - v_\delta)(\nabla \eta \partial_t u_\delta, \nabla \eta \partial_t u_\delta) \, dx,
$$

where $\epsilon > 0$ is arbitrarily small. The first summand can be absorbed on the left-hand side of (4.16). For the second term we consider the set $\Sigma := \{x \in B_{2R}(x_0) : |\nabla u_\delta| \leq |v_\delta| + 1\}$. We observe that

$$
\int_\Omega D^2G(\nabla u_\delta - v_\delta)(\nabla \eta \partial_t u_\delta, \nabla \eta \partial_t u_\delta) \, dx
\leq \int_\Sigma \frac{1}{2} |D^2G||\nabla \eta|^2 |\nabla u_\delta|^2 \, dx + \int_{B_{2R}(x_0) - \Sigma} D^2G(\nabla u_\delta - v_\delta)(\nabla \eta \partial_t u_\delta, \nabla \eta \partial_t u_\delta) \, dx
\overset{(G2)' \& (4.14)}{\leq} c \left( \frac{1}{R^2} + \frac{1}{R^2} \int_{B_{2R}(x_0) - \Sigma} \frac{1}{1 + |\nabla u_\delta - v_\delta|} |\nabla u_\delta|^2 \, dx \right)
\leq c \left( \frac{1}{R^2} + \frac{1}{R^2} \int_{B_{2R}(x_0) - \Sigma} \frac{1}{1 + |\nabla u_\delta| - |v_\delta|} |\nabla u_\delta|^2 \, dx \right) \leq c(R).
$$
To the quantity $T'_2$ we apply the Cauchy–Schwarz inequality and then the Young inequality with the following result:

$$|T'_2| \leq \epsilon \int \eta^2 \tilde{\Theta}_3^2 dx + c(\epsilon) \int \Omega D^2 G(\nabla u_{\delta} - v_{\delta})(\eta \partial_i v_{\delta}, \eta \partial_i v_{\delta}) dx.$$ 

Again, we absorb the first term on the left-hand side of (4.16) and the second term is bounded by (4.13). Combining the arguments for $T'_1$ and $T'_2$, we can estimate $T'_3$ by

$$|T'_3| \leq \frac{c}{R^2}$$

and (4.16) reads

$$\int \Omega \eta^2 \tilde{\Theta}_3^2 dx \leq c \left(1 + \frac{1}{R^2}\right) + |T'_4|.$$  

(4.18)

It remains to give a bound for $T'_4$. An integration by parts yields

$$T'_4 = - \int \Omega \eta^2 |\nabla u_{\delta}|^2 dx - \int \Omega f \partial_i (\eta^2 \partial_i \nabla u_{\delta}) dx.$$ 

The Dirichlet integral can be moved to the left-hand side of (4.18), so that

$$\int \Omega \eta^2 \tilde{\Theta}_3^2 dx + \int \Omega \eta^2 |\nabla u_{\delta}|^2 dx \leq c(R) + \int \Omega f ||\partial_i (\eta^2 \partial_i \nabla u_{\delta})|| dx.$$ 

(4.19)

Note that by our assumptions we have $f \in L^\infty(\Omega)$ and thus (4.19) together with Lemma 3.2 implies

$$\int \Omega \eta^2 \tilde{\Theta}_3^2 dx + \int \Omega \eta^2 |\nabla u_{\delta}|^2 dx \leq c(R) + c \int \Omega \eta^2 |\nabla^2 u_{\delta}| dx.$$ 

The nonconstant term on the right-hand side can now be estimated just like the corresponding term $T_3$ in (4.8), which yields

$$\int \Omega \eta^2 |\nabla^2 u_{\delta}| dx \leq c \epsilon \int \Omega \eta^2 \tilde{\Theta}_3^2 dx + c \epsilon^{-1} \int \Omega \eta^2 (1 + |\nabla u_{\delta}|)^\mu dx.$$ 

For $\epsilon$ small enough, the first term can be absorbed on the left-hand side of (4.19) and to the second term we apply the Young inequality once again (making use of $\mu < 2$) which results in

$$\int \Omega \eta^2 (1 + |\nabla u_{\delta}|)^\mu dx \leq \epsilon \int \Omega \eta^2 |\nabla u| dx + c(R),$$

and the Dirichlet integral can be absorbed on the left-hand side of (4.19) (provided $\epsilon$ is chosen small enough). Then claim (4.2) follows from (4.18) and (4.17). By the Sobolev embedding theorem, (4.2) yields

$$\nabla u_{\delta} \in L^p_{loc}(\Omega) \text{ for any } p \in [1, \infty) \text{ and uniform with respect to } \delta,$$ 

(4.20)

which allows us to infer $u \in W^{1,1}(\Omega)$ and $u \in W^{1,p}_{loc}(\Omega)$. It even follows from (4.20):

$$u_{\delta} \in L^\infty_{loc}(\Omega) \text{ uniform with respect to } \delta.$$ 

(4.21)
Remark 4.1. If \( D \neq \emptyset \), we cannot readily perform an integration by parts to estimate the crucial quantity
\[
T'_4 := \int_{\Omega - D} (u_\delta - f) \partial_i (\eta^2 \partial_i u_\delta) \, dx.
\]
However, switching to an error term of linear growth as proposed in Remark 1.3 turns \( T'_4 \) into
\[
T''_4 := \int_{\Omega - D} \omega' (|u_\delta - f|) \frac{u_\delta - f}{|u_\delta - f|} \partial_i (\eta^2 \partial_i u_\delta) \, dx
\]
and, since \( |\omega'| \) is bounded, we can estimate
\[
|T''_4| \leq c \int_{\Omega - D} |\partial_i (\eta^2 \partial_i u_\delta)| \, dx \leq c(R) + c \int_{\Omega} \eta^2 |\nabla^2 u_\delta| \, dx.
\]
Now, employing the same arguments that have been used for the term \( T_3 \), we can establish \( u \in W^{1,1}(\Omega) \) even for \( D \neq \emptyset \).

5 Hölder Continuity. Proof of Theorem 1.4

Our proof follows the ideas in [2] which are based on the results of [15] and [16]. An essential condition for the application of these techniques is the validity of following lemma:

**Lemma 5.1.** Under the assumptions of Lemma 4.1, for any \( s \in (1, 2) \)
\[
v_\delta \text{ is uniformly bounded in } W^{2,s}_{\text{loc}}(\Omega, \mathbb{R}^2), \tag{5.1}
\]
\[
u \delta \text{ is uniformly bounded in } W^{2,s}_{\text{loc}}(\Omega). \tag{5.2}
\]
Moreover,
\[
\omega_\delta := (1 + |\nabla v_\delta|)^{\frac{s}{2}} \text{ is uniformly bounded in } W^{1,2}_{\text{loc}}(\Omega), \tag{5.3}
\]
\[
\tilde{\omega}_\delta := (1 + |\nabla u_\delta|)^{\frac{s}{2}} \text{ is uniformly bounded in } W^{1,2}_{\text{loc}}(\Omega). \tag{5.4}
\]

**Proof.** Let us prove (5.1). Recalling the definition of \( \varphi_\delta \) from Lemma 4.1 as well as inequality (F2)', we see that the uniform boundedness of \( \nabla \varphi_\delta \) in \( L^2_{\text{loc}}(\Omega, \mathbb{R}^2) \), which is obtained from (4.13), implies that for any compact subset \( \Omega^* \subseteq \Omega \), there is a constant \( c(\Omega^*) > 0 \) (independent of \( \delta \)) such that
\[
\int_{\Omega^*} \frac{|\nabla^2 \varphi_\delta|^2}{(1 + |\nabla \varphi_\delta|)^{\mu}} \, dx \leq c(\Omega^*).
\]
Let now \( s \in (1, 2) \) be arbitrary. We can write
\[
\int_{\Omega^*} |\nabla^2 \varphi_\delta|^s \, dx = \int_{\Omega^*} \left( \frac{|\nabla^2 \varphi_\delta|^2}{(1 + |\nabla \varphi_\delta|)^{\mu}} \right)^{\frac{s}{2}} (1 + |\nabla \varphi_\delta|)^{\mu \frac{s}{2}} \, dx
\]
and an application of the Hölder inequality yields

\[
\int_{\Omega^*} |\nabla^2 v_\delta|^s dx \leq \left( \int_{\Omega^*} \frac{|\nabla^2 v_\delta|^2}{(1 + |\nabla v_\delta|)^{s}} dx \right)^{\frac{s}{2}} \left( \int_{\Omega^*} \frac{1}{(1 + |\nabla v_\delta|)^{s}} dx \right)^{\frac{2-s}{2}},
\]

so that (5.1) follows from (4.1) and (4.13). The same argument works for (5.2).

We continue with (5.3). Choose \(B_{2R}(x_0) \subset \Omega\) and \(\eta\) according to (4.3).

Setting \(\Gamma_\delta := 1 + |\nabla v_\delta|^2\), we observe

\[
\int_{B_{2R}(x_0)} |\nabla \omega_\delta|^2 dx = \int_{B_{2R}(x_0)} (1 + |\nabla v_\delta|)^{-2} |\nabla^2 v_\delta|^2 dx
\]

\[= \int_{B_{2R}(x_0)} (1 + |\nabla v_\delta|)^{-2} |\nabla v_\delta|^2 (1 + |\nabla v_\delta|)^{2\mu - 2} dx
\]

\[\leq c \int_{B_{2R}(x_0)} \eta^2 D^2 F_\delta(\nabla v_\delta)(\partial_i \nabla v_\delta, \partial_i \nabla v_\delta) \Gamma_\delta^{\mu - 1} dx.
\]

Choosing \(\psi = \partial_i (\eta^2 \partial_i v_\delta \Gamma_\delta^{\mu - 1})\) in (EL1) yields

\[
\int_{B_{2R}(x_0)} \eta^2 D^2 F_\delta(\nabla v_\delta)(\partial_i \nabla v_\delta, \partial_i \nabla v_\delta) \Gamma_\delta^{\mu - 1} dx
\]

\[= -\beta \int_{B_{2R}(x_0)} D G(\nabla v_\delta - v_\delta) \cdot (\partial_i (\eta^2 \partial_i v_\delta \Gamma_\delta^{\mu - 1}) dx - \int_{B_{2R}(x_0)} D^2 F_\delta(\nabla v_\delta)(\partial_i \nabla v_\delta, \partial_i v_\delta \otimes \nabla \eta \Gamma_\delta^{\mu - 1}) \Gamma_\delta^{\mu - 1} dx
\]

\[= \int_{B_{2R}(x_0)} D^2 F_\delta(\nabla v_\delta)(\partial_i \nabla v_\delta, \partial_i v_\delta \otimes \nabla \nabla v_\delta \Gamma_\delta^{\mu - 1}) \eta^2 dx =: -I_1 - I_2 - I_3.
\]

We start with the term \(I_2\). Applying the Cauchy–Schwarz inequality to the bilinear form \(D^2 F_\delta(\cdot, \cdot)\) and then the Young inequality yields

\[|I_2| \leq \frac{1}{2} \int_{B_{2R}(x_0)} D^2 F_\delta(\nabla v_\delta)(\partial_i \nabla v_\delta, \partial_i \nabla v_\delta) \eta^2 dx
\]

\[+ \frac{1}{2} \int_{B_{2R}(x_0)} D^2 F_\delta(\nabla v_\delta)(\partial_i v_\delta \otimes \nabla \eta, \partial_i v_\delta \otimes \nabla \eta) \Gamma_\delta^{2\mu - 2} dx,
\]

\[\leq \frac{c}{F^2} \left[ \int_{B_{2R}(x_0)} \Theta_\delta^2 \eta^2 dx + \int_{B_{2R}(x_0)} \delta |\nabla v_\delta|^2 \Gamma_\delta^{2\mu - 2} + \frac{1}{1 + |\nabla v_\delta|} |\nabla v_\delta|^2 \Gamma_\delta^{2\mu - 2} dx \right]
\]

and this term is bounded due to (4.12) and (4.13). Let us continue with \(I_3\). At this point, we make use of the structure condition ( F4) which enables us to write (cf. calculation in [17, p. 62])

\[D^2 F_\delta(\nabla v_\delta)(\partial_i \nabla v_\delta, \partial_i v_\delta \otimes \nabla \nabla v_\delta \Gamma_\delta^{\mu - 1}) \eta^2 = \frac{1}{2} D^2 F_\delta(\nabla v_\delta)(e_i \otimes \nabla |\nabla v_\delta|^2, e_i \otimes \nabla |\nabla v_\delta|^2) \Gamma_\delta^{\mu - 2} > 0,
\]
where \( e_i \) denotes the canonical basis of \( \mathbb{R}^2 \). Hence we can just neglect the term \( I_1 \) and it remains to give a bound on the quantity \( I_1 \). By the boundedness of \( DG(\nabla u_\delta - v_\delta) \), we have

\[
|I_1| \leq c \int_{B_{2R}(x_0)} |\partial_1(\eta^2 \partial_1 v_\delta \Gamma_\delta^{-1})| dx \leq c \left[ \int_{B_{2R}(x_0)} |\nabla v_\delta| \Gamma_\delta^{-1} dx + \int_{B_{2R}(x_0)} \eta^2 |\nabla^2 v_\delta| \Gamma_\delta^{-1} dx \right]. \tag{5.5}
\]

The first term in the brackets is bounded by (4.13). For the second one we note that an application of the Young inequality yields

\[
\int_{B_{2R}(x_0)} \eta^2 |\nabla^2 v_\delta| \Gamma_\delta dx \leq c \left[ \int_{B_{2R}(x_0)} \eta^2 \frac{|\nabla^2 v_\delta|^2}{(1 + |\nabla v_\delta|)^\mu} dx + \int_{B_{2R}(x_0)} \eta^2 \Gamma_\delta^{\frac{\mu}{\mu - 2}} dx \right]
\]

\[
\leq c \left[ \int_{\Omega} \Theta_\delta^2 \eta^2 dx + \int_{B_{2R}(x_0)} \Gamma_\delta^{\frac{\mu}{\mu - 2}} dx \right]
\]

and this is bounded due to (4.12) and (4.13). Thus, (5.3) follows. For (5.4), we only note that this follows from a similar argument and thereby finish the proof of Lemma 5.1. \( \square \)

We continue with the proof of Theorem 1.4. In the differentiated Euler-Lagrange equation (EL1)', we now consider \( \psi = \eta^2 (\partial_1 v_\delta - \overline{\partial_1 v_\delta}) \), where we set

\[
\overline{\partial_1 v_\delta} := \int_{\Omega} \partial_1 v_\delta dx
\]

and \( \eta \in C^1_0(\Omega) \) is chosen according to (4.3). Denoting by \( T \) the annulus \( B_{2R}(x_0) - B_R(x_0) \) (remember \( \eta \equiv \text{const. outside } T \)), (EL1)' reads

\[
0 = \int_{\Omega} \Theta_\delta^2 \eta^2 dx + 2 \int_T D^2 F_\delta(\nabla v_\delta)(\partial_1 \nabla v_\delta, \nabla \eta \otimes (\partial_1 v_\delta - \overline{\partial_1 v_\delta})) \eta dx
\]

\[
+ \beta \int_{\Omega} \left\{ DG(\nabla u_\delta - v_\delta) \cdot \partial_1 (\eta^2 (\partial_1 v_\delta - \overline{\partial_1 v_\delta})) \right\} dx \in L^\infty(\Omega, \mathbb{R}^2)
\]

and we infer that for some constant \( c > 0 \) independent of \( \eta \)

\[
\int_{\Omega} \Theta_\delta^2 \eta^2 dx \leq c \left[ \int_T |D^2 F_\delta(\nabla v_\delta)(\partial_1 \nabla v_\delta, \nabla \eta \otimes (\partial_1 v_\delta - \overline{\partial_1 v_\delta}))| dx \right.
\]

\[
+ \beta \int_{\Omega} |\partial_1 (\eta^2 (\partial_1 v_\delta - \overline{\partial_1 v_\delta}))| dx \right] = c[S_1 + S_2]. \tag{5.6}
\]

In \( S_1 \), we apply the Cauchy–Schwarz inequality to the bilinear form \( D^2 F_\delta(\cdot, \cdot) \) and obtain

\[
S_1 \leq \int_T D^2 F_\delta(v_\delta)(\partial_1 \nabla v_\delta, \partial_1 \nabla v_\delta)^2 D^2 F_\delta(\nabla v_\delta)(\nabla \eta \otimes (\partial_1 v_\delta - \overline{\partial_1 v_\delta}), \nabla \eta \otimes (\partial_1 v_\delta - \overline{\partial_1 v_\delta}))^2 dx.
\]

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Applying the Hölder inequality yields

\[
S_1 \leq \left[ \int_T \Theta_\delta^2 \, dx \right]^{1/2} \left[ \int_T D^2 F_\delta(\nabla v_\delta)(\nabla \eta \otimes (\partial_i v_\delta - \partial_i \bar{v}_\delta), \nabla \eta \otimes (\partial_i v_\delta - \partial_i \bar{v}_\delta)) \, dx \right]^{1/2}
\]

\[
\leq c \left[ \int_T \Theta_\delta^2 \, dx \right]^{1/2} \left[ \int_T |D^2 F_\delta(\nabla v_\delta)||\nabla \eta|^2|\partial_i v_\delta - \partial_i \bar{v}_\delta|^2 \, dx \right]^{1/2}
\]

and, since $|D^2 F_\delta|$ is bounded, we arrive at

\[
S_1 \leq \frac{c}{R} \left[ \int_T \Theta_\delta^2 \, dx \right]^{1/2} \left[ \int_T |\partial_i v_\delta - \partial_i \bar{v}_\delta|^2 \, dx \right]^{1/2} \leq \frac{c}{R} \left[ \int_T \Theta_\delta^2 \, dx \right] \int_T |\nabla^2 v_\delta| \, dx, \quad (5.7)
\]

where the Sobolev–Poincaré inequality was used at the last step. Next, by (F2)', we have

\[
(1 + |\nabla v_\delta|)^{-\frac{\alpha}{2}} |\nabla^2 v_\delta| \leq c \Theta_\delta
\]

and thus

\[
|\nabla^2 v_\delta| \leq c \Theta_\delta (1 + |\nabla v_\delta|)^{\frac{\alpha}{2}} = c \Theta_\delta \omega_\delta \quad (5.8)
\]

(with $\omega_\delta$ as in Lemma 5.1). Consequently, it follows from (5.7) that

\[
S_1 \leq \frac{c}{R} \left[ \int_T \Theta_\delta^2 \, dx \right]^{1/2} \int_T |\nabla^2 v_\delta| \, dx \leq \frac{c}{R} \left[ \int_T \Theta_\delta^2 \, dx \right]^{1/2} \int_T \Theta_\delta \omega_\delta \, dx. \quad (5.9)
\]

The term $S_2$ can be treated in the same way as the corresponding quantity in [2, p. 164] with the result

\[
S_2 \leq \int_T \Theta_\delta \omega_\delta \, dx + \int_\Omega \eta^2 \Theta_\delta \omega_\delta \, dx.
\]

The estimates of $S_1$ and $S_2$ together with (5.6) now establish the crucial inequality (3.17) from [2] in our setting:

\[
\int_{B_{2R}(x_0)} \Theta_\delta^2 \, dx \leq \frac{c}{R} \left[ \int_T \Theta_\delta^2 \, dx + R^2 \right]^{1/2} \int_T \Theta_\delta \omega_\delta \, dx + c \int_{B_{2R}(x_0)} \eta^2 \Theta_\delta \omega_\delta \, dx, \quad (5.10)
\]

which holds for all radii $0 < R < R_0$ and all points $x_0 \in \Omega$ such that $B_{2R_0}(x_0) \subseteq \Omega$ with a constant $c$ only depending on $R_0$. To the last term, we apply the Young inequality and the Hölder inequality to get

\[
\int_{B_{2R}(x_0)} \eta^2 \Theta_\delta \omega_\delta \, dx \leq \frac{1}{2} \int_{B_{2R}(x_0)} \Theta_\delta^2 \eta^2 \, dx + cR^{2\frac{1}{q}},
\]
where due to (4.13), the exponent $q$ can be chosen from $(1, \infty)$. We fix $\gamma < 1$ and thus obtain:

$$
\int_{B(R(x_0))} \Theta_\delta^2 \, dx \leq \frac{c}{R} \left[ \int_{T} \Theta_\delta^2 \, dx + R^2 \right]^{\frac{1}{2}} \int_{T} \Theta_\delta \omega_\delta \, dx + c R^\gamma.
$$

(5.11)

As it is explained in detail in [2, p. 164], this inequality suffices to deduce the following growth estimate for the quantity $\Theta_\delta$:

$$
\int_{B(R(x_0))} \Theta_\delta^2 \, dx \leq \frac{1}{\ln \left( \frac{1}{R} \right)} \text{ for all } t \geq 1,
$$

(5.12)

for all balls $B_R(x_0)$ as above with $0 < R < R_0$ and with a local constant $c$ only depending on $R_0$. Since for $\sigma_\delta = DF_\delta (\nabla v_\delta)$ (cf. (3.4))

$$
|\nabla \sigma_\delta|^2 = \partial_v \sigma_\delta \cdot \partial_i \sigma_\delta = D^2 F_\delta (\nabla v_\delta) (\partial_i \nabla v_\delta, \partial_i \sigma_\delta)
\leq (D^2 F_\delta (\nabla v_\delta) (\partial_i \nabla v_\delta, \partial_i \nabla v_\delta))^{1/2} (D^2 F_\delta (\nabla v_\delta) (\partial_i \sigma_\delta, \partial_i \sigma_\delta))^{1/2} \leq c \Theta_\delta |\nabla \sigma_\delta|
$$

(5.13)

and, consequently, $|\nabla \sigma_\delta| \leq c \Theta_\delta$, the estimate (5.12) implies

$$
\int_{B(R(x_0))} |\nabla \sigma_\delta|^2 \, dx \leq c \frac{1}{\ln \left( \frac{1}{R} \right)} \text{ for } t \geq 1.
$$

Along with Lemma 4.1 and (5.3), this is enough to infer the continuity of $\sigma_\delta$ on every ball $B_R(x_0)$ with $R < R_0$ from the results in [15, p. 287] (cf. also [18, Lemmas 6 and 7]), with the modulus of continuity

$$
\sup_{x, y \in B_R(x_0)} |\sigma_\delta(x) - \sigma_\delta(y)| \leq K |\ln R|^{1 - \frac{s}{s'}}.
$$

(5.14)

with a constant $K = K(R_0)$. The uniform boundedness of $\sigma_\delta$ in $L^2(\Omega, \mathbb{R}^{2 \times 2})$ (cf. (3.9)) along with (5.14) now implies

$$
\sup_{\delta \in (0,1)} \|\sigma_\delta\|_{L^\infty(B_R(x_0))} < \infty.
$$

(5.15)

Furthermore, (5.14) yields the equicontinuity of $\sigma_\delta$ on any compact subset $\Omega_* \Subset \Omega$, such that $B_{2R_0}(x) \subset \Omega$ for all $x \in \Omega_*$. An application of the Arzelà–Ascoli compactness theorem thus gives the existence of a continuous function $\sigma$ such that $\sigma_\delta \to \sigma$ locally uniformly, at least for a subsequence $\delta \downarrow 0$. By (5.1), we can in addition assume $\nabla v_\delta \to \nabla v$ almost everywhere, where $(u, v) \in W^{1,1}(\Omega) \times W^{1,1}(\Omega, \mathbb{R}^2)$ is the unique $E$-minimizer. Hence

$$
DF(\nabla v(x)) = \sigma(x)
$$

(5.16)

for almost all $x \in \Omega$. By the inverse function theorem and (F2)', $\text{Im}(DF)$ is an open set. In particular, $\sigma^{-1}(\text{Im}(DF))$ is open and for every point $x_0 \in \Omega$, for which (5.16) holds, there is a small ball $B_\epsilon(x_0) \subset \mathbb{R}^2$ such that $\sigma(x) \in \text{Im}(DF)$ for all $x \in B_\epsilon(x_0)$. Hence $DF^{-1}(\sigma)$ is a continuous representative of $\nabla v$ on $B_\epsilon(x_0)$. But due to the continuity of $\sigma$ (and the Lipschitz continuity of $DF$), (5.16) particularly holds for all Lebesgue points of $\nabla v$. Identifying $\nabla v$
with its Lebesgue point representative, we thus obtain from (5.16) by inversion a continuous representative of $\nabla v$ on the set:

$$\Omega_0 := \left\{ x \in \Omega : \lim_{r \downarrow 0} \int_{B_r(x)} \nabla v dx \text{ exists in } \mathbb{R}^{2 \times 2} \right\},$$

which alongside is proved to be open. In particular, $\nabla v \in L^\infty_{\text{loc}}(\Omega_0, \mathbb{R}^{2 \times 2})$ and we can therefore argue just like in [1, p. 76] to deduce the Hölder continuity of $v$ on $\Omega_0$ from the hole-filling technique applied to the inequality (5.11). That $\Omega - \Omega_0$ does indeed have Hausdorff-dimension 0 is a consequence of Theorem 2.1 on p. 100 of [19] and $v \in W^{2,s}_{\text{loc}}(\Omega, \mathbb{R}^2)$, $s \in [1, 2)$.

We now come to the corresponding statements concerning $u$. In the following calculations, we restrict ourselves to the open subset $\Omega_0 \subset \Omega$ on which we have already established local Hölder continuity of $v$. We introduce a new sequence $(\tilde{u}_\delta)$ of $\delta$-regularizers which solve

$$\beta \int_{\Omega_0} G(\nabla w - v) dx + \int_{\Omega_0} (w - f)^2 dx + \frac{\delta}{2} \int_{\Omega_0} |\nabla w|^2 dx \rightarrow \min \text{ in } W^{1,2}(\Omega),$$

where $v$ is the Hölder continuous minimizer from above. We note that due to

$$E(u, v) \leq \alpha \int_{\Omega} F(\nabla v) dx + \beta \int_{\Omega} G(\nabla \tilde{u}_\delta - v) dx + \int_{\Omega} (\tilde{u}_\delta - f)^2 dx
\leq \alpha \int_{\Omega} F(\nabla v) dx + \beta \int_{\Omega} G(\nabla \tilde{u}_\delta - v) dx + \int_{\Omega} (\tilde{u}_\delta - f)^2 dx + \frac{\delta}{2} \int_{\Omega} |\nabla \tilde{u}_\delta|^2 dx
\leq \alpha \int_{\Omega} F(\nabla v) dx + \beta \int_{\Omega} G(\nabla u_{\delta} - v) dx + \int_{\Omega} (u_{\delta} - f)^2 dx + \frac{\delta}{2} \int_{\Omega} |\nabla u_{\delta}|^2 dx \overset{\delta \downarrow 0}{\rightarrow} E(u, v),$$

the sequence $(\tilde{u}_\delta, v)$ is $E$-minimizing and Theorem 1.1 c) implies that $\tilde{u}_\delta \to u$ in $L^1(\Omega)$ and almost everywhere (at least for a subsequence $\delta \downarrow 0$). Moreover, we can verify the properties from Lemmas 4.1 and 5.1 for the sequence $(\tilde{u}_\delta)$.

**Lemma 5.2.**

$$\tilde{u}_\delta \in W^{2,s}_{\text{loc}}(\Omega_0) \forall \ s \in (1, 2),$$

$$\tilde{\varphi}_\delta := (1 + |\nabla \tilde{u}_\delta|)^{1 - \frac{1}{2}} \in W^{1,2}_{\text{loc}}(\Omega_0),$$

$$\tilde{G}_\delta := D^2 G(\nabla \tilde{u}_\delta - v)(\partial_i \nabla \tilde{u}_\delta, \partial_i \nabla \tilde{u}_\delta)^{\frac{1}{2}} \in W^{1,2}_{\text{loc}}(\Omega_0),$$

$$\tilde{\omega}_\delta := (1 + |\nabla \tilde{u}_\delta|)^{\frac{1}{2}} \in W^{1,2}_{\text{loc}}(\Omega_0)$$

uniformly with respect to the parameter $\delta$.

The lemma can be proved just like the corresponding results from Lemmas 4.1 and 5.1. Continuing with the proof of Theorem 1.4, we find that $\tilde{u}_\delta$ satisfies the Euler equation

$$\delta \int_{\Omega_0} \partial_i \nabla \tilde{u}_\delta \cdot \nabla \varphi dx + \beta \int_{\Omega_0} D^2 G(\nabla \tilde{u}_\delta - v)(\partial_i \nabla \tilde{u}_\delta - \partial_i v, \nabla \varphi) dx - \int_{\Omega_0} (\tilde{u}_\delta - f)^2 \partial_i \varphi = 0$$
for all $\varphi \in \dot{W}^{1,2}(\Omega_0)$. By the choice $\varphi := \eta^2(\partial_t \tilde{u}_\delta - \partial_t \bar{u}_\delta)$ (now with $\eta \in C^\infty(\Omega_0)$ and the properties (4.3)), we have

$$\int_{\Omega_0} \tilde{\Theta}_3^2 \eta^2 dx = -\beta \int_{\Omega_0} \nabla G(\nabla \tilde{u}_\delta - v)(\partial_t \nabla \tilde{u}_\delta, \nabla \eta^2 \otimes (\partial_t \tilde{u}_\delta - \partial_t \bar{u}_\delta))dx$$

$$+ \int_{\Omega_0} (\bar{u}_\delta - f)\partial_t(\eta^2(\partial_t \tilde{u}_\delta - \partial_t \bar{u}_\delta))dx + \beta \int_{\Omega_0} \nabla G(\nabla \tilde{u}_\delta - v)(\partial_t v, \partial_t \nabla \tilde{u}_\delta)\eta^2 dx$$

$$+ \beta \int_{\Omega_0} \nabla G(\nabla \tilde{u}_\delta - v)(\partial_t v, \nabla \eta^2 \otimes (\partial_t \tilde{u}_\delta - \partial_t \bar{u}_\delta))dx =: \tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3 + \tilde{S}_4.$$ 

We see that the terms $\tilde{S}_1$ and $\tilde{S}_2$ can be treated like the above quantities $S_1$ and $S_2$. We apply the Cauchy–Schwarz inequality and the Young inequality to $\tilde{S}_3$:

$$|\tilde{S}_3| \leq c \int_{\Omega_0} \tilde{\Theta}_3^2 \eta^2 dx + c(\epsilon) \int_{\Omega_0} |\nabla v|^2 \eta^2 dx \leq \epsilon \int_{\Omega_0} \tilde{\Theta}_3^2 \eta^2 dx + cR^2.$$

Arguing as in the case of $S_1$, we obtain the following estimate for $\tilde{S}_4$:

$$|\tilde{S}_4| \leq c \int_{B_{2R}(x_0)} |\nabla v||\nabla \tilde{u}_\delta - \nabla \bar{u}_\delta| dx \leq c \int_{B_{2R}(x_0)} |\nabla v|^2 dx \left( \int_{T} |\nabla \tilde{u}_\delta - \nabla \bar{u}_\delta|^2 dx \right)^{\frac{1}{2}}$$

$$\leq c \int_{T} |\nabla^2 \tilde{u}_\delta| dx \leq c \int_{T} \tilde{\Theta}_3 \tilde{\omega}_\delta dx.$$

Altogether, this suffices to prove the estimate

$$\int_{B_{2R}(x_0)} \tilde{\Theta}_3^2 dx \leq c \frac{1}{R} \left( \int_{T} \Theta_3^2 dx + R^2 \right)^{\frac{1}{2}} \int_{T} \tilde{\Theta}_3 \tilde{\omega}_\delta dx + cR^\gamma$$

for $\gamma \in (0, 2)$, which implies

$$\int_{B_R(x_0)} \tilde{\Theta}_3^2 dx \leq c \frac{1}{\ln \left( \frac{1}{R} \right)} \quad \forall \ t \geq 1.$$

From this point on, we can repeat the arguments used for deriving the Hölder continuity of $v$. However, one should note that we have to replace $\sigma_\delta$ with $\rho_\delta := DG_\delta(\nabla \tilde{u}_\delta - v)$. Then, as in (5.13), we have

$$\int_{B_R(x_0)} |\nabla \rho_\delta|^2 dx = \int_{B_R(x_0)} (\partial_t \rho_\delta \cdot \partial_t \bar{\rho}_\delta) dx = \int_{B_R(x_0)} \nabla G(\nabla \tilde{u}_\delta - v)(\partial_t \nabla \tilde{u}_\delta - \partial_t v, \partial_t \rho_\delta) dx$$

$$\leq c \left( \left( \int_{B_R(x_0)} \tilde{\Theta}_3^2 dx \right)^{\frac{1}{2}} + \left( \int_{B_R(x_0)} |\nabla v|^2 dx \right)^{\frac{1}{2}} \right) \left( \int_{B_R(x_0)} |\nabla \rho_\delta|^2 dx \right)^{\frac{1}{2}}.$$
Therefore,
\[
\left( \int_{B_R(x_0)} | \nabla \rho_\delta |^2 \, dx \right)^{\frac{1}{2}} \leq c \left( \int_{B_R(x_0)} \hat{\Theta}_\delta^2 \, dx \right)^{\frac{1}{2}} + cR.
\]

Since \( \lim_{R \to 0} \ln(1/R) \frac{t}{2} R = 0 \) for all \( t \geq 1 \), we can neglect the additional term \( cR \) and conclude
\[
\left( \int_{B_R(x_0)} | \nabla \rho_\delta |^2 \, dx \right)^{\frac{1}{2}} \leq c \frac{1}{\ln \left( \frac{1}{R} \right)^{\frac{1}{2}}} \forall t \geq 1.
\]

Arguing as for \( v \), we get an open subset \( \tilde{\Omega}_0 \subset \Omega_0 \) such that \( \Omega_0 - \tilde{\Omega}_0 \) has Hausdorff dimension 0 and \( (u, v) \in C^{1,\alpha}(\tilde{\Omega}_0) \times C^{1,\beta}(\tilde{\Omega}_0, \mathbb{R}^2) \) for every pair \( (\alpha, \beta) \in (0, 1) \times (0, 1) \). Theorem 1.4 is proved.

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Submitted on January 26, 2017

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