A RIESZ REPRESENTATION THEOREM FOR LOG-CONCAVE FUNCTIONS

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Abstract. The classic Riesz representation theorem characterizes all linear and increasing functionals on the space $C_c(X)$ of continuous compactly supported functions. A geometric version of this result, which characterizes all linear increasing functionals on the set of convex bodies in $\mathbb{R}^n$, was essentially known to Alexandrov. This was used by Alexandrov to prove the existence of mixed area measures in convex geometry.

In this paper we characterize linear and increasing functionals on the class of log-concave functions on $\mathbb{R}^n$. Here “linear” means linear with respect to the natural addition on log-concave functions which is the sup-convolution. Equivalently, we characterize pointwise-linear and increasing functionals on the class of convex functions. For some choices of the exact class of functions we prove that there are no non-trivial such functionals. For another choice we obtain the expected analogue of the result for convex bodies. And most interestingly, for yet another choice we find a new unexpected family of such functionals.

Finally, we explain the connection between our results and recent work done in convex geometry regarding the surface area measure of a log-concave functions. An application of our results in this direction is also given.

1. Introduction

The Riesz (or Riesz–Markov–Kakutani) representation theorem is the following classic result of functional analysis:

**Theorem 1.1.** Let $X$ be a locally compact Hausdorff space. Let $C_c(X)$ denote the class of all continuous and compactly supported functions $f : X \to \mathbb{R}$. Let $F : C_c(X) \to \mathbb{R}$ be a functional such that:

1. $F$ is linear: $F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$ for all $f, g \in C_c(X)$ and $\alpha, \beta \in \mathbb{R}$.
2. $F$ is increasing: If $f, g \in C_c(X)$ and $f \geq g$ then $F(f) \geq F(g)$.

Then there exists a unique positive Radon measure $\mu$ on $X$ such that $F(f) = \int_X f d\mu$ for all $f \in C_c(X)$.

For the proof one may consult any standard text on measure theory, e.g. Section 7.1 of [13].

In this paper we study geometric forms of the Riesz representation theorem. We start with a well known result about convex bodies. We quickly give the basic definitions here, and refer the reader to [28] or [15] for more information. We denote by $K^n$ the class of all compact convex sets $K \subseteq \mathbb{R}^n$. Given $K, L \in K^n$ their Minkowski addition is defined as

$$K + L = \{x + y : x \in K, y \in L\}.$$
For $K \in \mathcal{K}^n$ and $\lambda > 0$ we set $\lambda \cdot K = \{\lambda x : x \in K\}$. The operations $+$ and $\cdot$ turn $\mathcal{K}^n$ into a cone. Finally, the support function of a convex body $K \in \mathcal{K}^n$ is the function $h_K : S^{n-1} \to \mathbb{R}$ defined by 

$$h_K(\theta) = \max_{x \in K} \langle x, \theta \rangle.$$ 

Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^n$ and $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ denotes the unit sphere.

We can now state a Riesz-type theorem for convex bodies:

**Theorem 1.2.** Let $F : \mathcal{K}^n \to \mathbb{R}$ be a functional such that:

1. $F$ is linear: $F(\alpha K + \beta L) = \alpha F(K) + \beta F(L)$ for all $K, L \in \mathcal{K}^n$ and $\alpha, \beta > 0$.
2. $F$ is increasing: If $K, L \in \mathcal{K}^n$ and $K \supseteq L$ then $F(K) \geq F(L)$.

Then there exists a unique positive and finite Borel measure $\mu$ on $S^{n-1}$ such that $F(K) = \int_{S^{n-1}} h_K d\mu$.

Recall that a finite Borel measure on $S^{n-1}$ or $\mathbb{R}^n$ is automatically Radon (see e.g. Theorems 1.1 and 1.3 of [2]), so we can ignore issues of regularity in this theorem and throughout this paper.

In its stated form Theorem 1.2 appears to be folklore. However, Alexandrov knew and used this result for a specific function $F$, and his proof works in complete generality (see [2] for the original in Russian, and Section 4 of [3] for an English translation). A modern presentation of Alexandrov’s result with essentially the same reasoning can be found as Theorem 4.1 of [15], and a few more references and historical remarks can be found in Note 1 after Section 5.1 of [28].

Since we will use Theorem 1.2 in the sequel we sketch its proof:

**Proof Sketch.** Let $C\left(S^{n-1}\right)$ denote the space of continuous functions on the sphere with its usual supremum norm, and define 

$$E = \{h_K - h_L : K, L \in \mathcal{K}^n\} \subseteq C\left(S^{n-1}\right).$$

It is well known that $E$ contains $C^2\left(S^{n-1}\right)$, the space of twice differentiable functions, so in particular $E$ is dense in $C\left(S^{n-1}\right)$. We now define $\tilde{F} : E \to \mathbb{R}$ by 

$$\tilde{F}(h_K - h_L) = F(K) - F(L).$$

It is easy to check that $\tilde{F}$ is well-defined, linear and increasing. It follows that $\tilde{F}$ is continuous, so it has a unique extension to $C\left(S^{n-1}\right)$ which is again linear and increasing. By Theorem 1.1 there exists a positive Random measure $\mu$ on $S^{n-1}$ such that $\tilde{F}(f) = \int_{S^{n-1}} f d\mu$ for all $f \in C\left(S^{n-1}\right)$. In particular 

$$F(K) = \tilde{F}(h_K) = \int_{S^{n-1}} f d\mu.$$ 

Since $S^{n-1}$ is compact $\mu$ is clearly finite.

Finally, for uniqueness, assume $\int_{S^{n-1}} h_K d\mu = \int_{S^{n-1}} h_K d\tilde{\mu}$ for all $K \in \mathcal{K}^n$. Then $\int_{S^{n-1}} f d\mu = \int_{S^{n-1}} f d\tilde{\mu}$ for every $f \in E$, and therefore for every $f \in C\left(S^{n-1}\right)$. By the uniqueness part of Theorem 1.1 it follows that $\mu = \tilde{\mu}$. \qed
While interesting in its own right, Theorem 1.2 also has applications to convex geometry. To explain the idea, assume \( G : \mathcal{K}^n \to \mathbb{R} \) is some measure of the “size” of convex bodies. We expect \( G \) to be increasing, but not necessarily linear. However, often one can linearize \( G \): we fix \( L \in \mathcal{K}^n \), and define \( F_L : \mathcal{K}^n \to \mathbb{R} \) by

\[
F_L(K) = \lim_{t \to 0^+} \frac{G(L + tK) - G(L)}{t}.
\]

For several natural choices of \( G \) the functions \( F_L \) will be well-defined and linear. Since they are clearly increasing we can apply Theorem 1.2 and conclude that

\[
\lim_{t \to 0^+} \frac{G(L + tK) - G(L)}{t} = \int_{\mathbb{S}^{n-1}} h_K d\mu_L
\]

for a measure \( \mu_L \) which depends on \( G \) and \( L \). Studying the measures \( \mu_L \) can be of great importance.

The simplest possible example is the choice \( G(K) = |K| \), i.e. the volume of \( K \). In this case we have the formula

\[
\lim_{t \to 0^+} \frac{|L + tK| - |L|}{t} = \int_{\mathbb{S}^{n-1}} h_K dS_L,
\]

where \( S_L \) is known as the surface area measure of the body \( L \).

This example can be extended using the theory of mixed volumes. For example, we can take \( G(K) = v_i(K) \), where \( v_i \) is the \( i \)’th intrinsic volume, or more generally

\[
G(K) = V \left( L_1, L_2, \ldots, L_{n-m}, K, K, \ldots, K \right)
\]

for some fixed convex bodies \( L_1, \ldots, L_{n-m} \in \mathcal{K}^n \). In these examples all linearizations will always be of the form \( c \cdot F_{L_1,\ldots,L_{n-1}}(K) = c \cdot V(L_1,\ldots,L_{n-1},K) \) for some constant \( c > 0 \) and some convex bodies \( L_1, \ldots, L_{n-1} \). The measure which represents the functional \( F_{L_1,\ldots,L_{n-1}} \) is denoted by \( S_{L_1,\ldots,L_{n-1}} \) and is known as a mixed area measure. This example of \( F_{L_1,\ldots,L_{n-1}} \) is exactly the one studied by Alexandrov in [2]. More information on mixed volumes and mixed area measures can be found in [28] or [15], but we will not need these notions for the rest of the paper.

As another possible extension one can take \( G(K) = \nu(K) \) where \( \nu \) is a measure on \( \mathbb{R}^n \) with a continuous density. Livshyts proved in [20] that in this case (1.1) still holds, and gave an explicit formula for the measure \( \mu_L \).

We now turn our attention from bodies to functions. A function \( f : \mathbb{R}^n \to [0, \infty) \) which is not identically 0 is called log-concave if for every \( x, y \in \mathbb{R}^n \) and every \( 0 \leq \lambda \leq 1 \) one has

\[
f((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)\lambda.
\]

In other words \( f \) is log-concave if it is of the form \( f = e^{-\varphi} \) where \( \varphi : \mathbb{R}^n \to (-\infty, \infty) \) is a convex function. We denote by \( \text{LC}_n \) the class of upper semi-continuous log-concave functions on \( \mathbb{R}^n \). Similarly we denote by \( \text{Cvx}_n \) the class of all lower semi-continuous convex functions on \( \mathbb{R}^n \) (which are not identically +\( \infty \)). If \( K \in \mathcal{K}^n \) then \( 1_K \in \text{LC}_n \), where \( 1_K \) is the indicator function of \( K \). In this sense we have a natural embedding \( \mathcal{K}^n \hookrightarrow \text{LC}_n \).

It is well understood nowadays that even if one is ultimately only interested in convex bodies, it is extremely useful to also consider log-concave functions and treat them as “generalized convex bodies”. This opens the door to the use of various analytic and probabilistic techniques in convex geometry, and allows us to make progress on previously impenetrable problems. The systematic geometric treatment of log-concave functions originated in the work of Klartag and Milman (16).
Since $\varphi$ expected. In fact, this property essentially characterizes the support map (see Theorem 6 of [5]). Important thing for us is that for every $f, g \in F$, since we now have an addition, we understand what it means for $f, g \in F$.

Theorem 1.3. Let $F : \text{LC}_n \rightarrow \mathbb{R}$ be a linear and increasing functional. Then $F(f) = 0$ for all $f \in \text{LC}_n$. 

In this paper we are interested in Riesz-type representation theorems on $\text{LC}_n$. In other words, we will classify functionals $F : \text{LC}_n \rightarrow \mathbb{R}$ which are linear and increasing. Of course, this requires an addition operation on $\text{LC}_n$. The standard choice that appeared already in [10] is the sup-convolution (also known as the Asplund sum), defined by

$$
(f \ast g)(x) = \sup_{y \in \mathbb{R}^n} (f(y)g(x - y)).
$$

Additionally, if $\lambda > 0$ we define the dilation $\lambda \cdot f$ by $(\lambda \cdot f)(x) = f \left( \frac{x}{\lambda} \right)^\lambda$. These operations extend the standard operations on convex bodies under the embedding $\mathbb{K}^n \hookrightarrow \text{LC}_n$ mentioned above.

A technical issue is that for $f, g \in \text{LC}_n$ the sup-convolution $f \ast g$ could be equal to $+\infty$, which we do not allow, or may fail to be upper semi-continuous. The latter problem can be fixed by defining $f \ast g$ to be the closure of the sup-convolution (see e.g. Section 7 of [25]), but the first problem doesn’t have such a solution. These problems disappear for “nice enough” functions as we shall soon see, so in practice this will not cause difficulties anywhere in this paper. We just have to be slightly careful and define a functional $F : \text{LC}_n \rightarrow \mathbb{R}$ to be linear if

$$
F((\alpha \cdot f) \ast (\beta \cdot g)) = \alpha F(f) + \beta F(g)
$$

holds for every $f, g \in \text{LC}_n$ and every $\alpha, \beta > 0$ such that $(\alpha \cdot f) \ast (\beta \cdot g) \in \text{LC}_n$.

Another way to understand the addition operation on $\text{LC}_n$ is by using the support function. For a function $f \in \text{LC}_n$, we define its support function $h_f \in \text{Cvx}_n$ by $h_f = (-\log f)^\star$. Here $\ast$ denotes the classical Legendre transform, i.e.

$$
\varphi^\star(y) = \sup_{x \in \mathbb{R}^n} \langle x, y \rangle - \varphi(x).
$$

This definition also extends the classical one for convex bodies, in the sense that $h_{1_K} = h_K$. The important thing for us is that for every $f, g \in \text{LC}_n$ and $\alpha, \beta > 0$ we have $h_{(\alpha \cdot f) \ast (\beta \cdot g)} = \alpha h_f + \beta h_g$ as expected. In fact, this property essentially characterizes the support map (see Theorem 6 of [5]). Since $\varphi^\star = \varphi$ for every $\varphi \in \text{Cvx}_n$, we see that for every $f \in \text{LC}_n$ we have $f = e^{-h_f^\star}$ and that every $\varphi \in \text{Cvx}_n$ is the support function of a unique $f \in \text{LC}_n$.

Since we now have an addition, we understand what it means for $F : \text{LC}_n^c \rightarrow \mathbb{R}$ to be linear. However, our first attempt at classifying linear and increasing functionals will be very underwhelming:

Some of the earlier developments in this direction can be found in Section 9.5 of [28] and in the survey [21]. The explosion in the field since then makes it impossible to include here a manageable list of references. Instead, let us just mention that studying functionals on log-concave (or convex) functions, i.e. maps $F : \text{LC}_n \rightarrow \mathbb{R}$, is a very active field of research. Usually one studies valuations on this space, i.e. functionals $F$ which satisfy

$$
F(\max(f, g)) + F(\min(f, g)) = F(f) + F(g)
$$

whenever $f, g, \max(f, g) \in \text{LC}_n$. Many results on such valuations including partial classifications were recently found by Colesanti, Ludwig and Mussnig ([8, 9, 10, 11]), by Mussnig ([22, 23]), by Alesker ([1]), and by Knoerr ([17, 18]).

Additionally, if $\lambda > 0$ we define the dilation $\lambda \cdot f$ by $(\lambda \cdot f)(x) = f \left( \frac{x}{\lambda} \right)^\lambda$. These operations extend the standard operations on convex bodies under the embedding $\mathbb{K}^n \hookrightarrow \text{LC}_n$ mentioned above.

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Theorem 1.3. Let $F : \text{LC}_n \rightarrow \mathbb{R}$ be a linear and increasing functional. Then $F(f) = 0$ for all $f \in \text{LC}_n$. 

and in the proof of the functional Santaló inequality by Artstein-Avidan, Klartag and Milman ([4]), even though in retrospect one can interpret older results in this language. For example, the Prékopa–Leindler inequality ([24], [19]) from the 70’s can be interpreted as a functional version of the Brunn–Minkowski inequality.
In fact, the same theorem is true without the assumption that $F$ is increasing! The reason for such a disappointing result is that the class of functions we are considering is too large. In Theorem 1.2 we did not work with all closed convex sets, but only with compact sets. In the same way we need to impose some kind of “compactness” criterion on our log-concave functions. One natural attempt is to work with coercive functions:

**Definition 1.4.** A convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is called coercive if $\lim_{|x| \to \infty} \varphi(x) = \infty$. A log-concave function $f = e^{-\varphi}$ is called coercive if $\varphi$ is coercive. We denote the classes of coercive convex and log-concave functions by $\text{Cvx}_n^c$ and $\text{LC}_n^c$ respectively.

Coercive log-concave functions are very well-behaved. For example, every $f \in \text{LC}_n^c$ satisfies $\int f < \infty$, and in fact decays exponentially fast as $|x| \to \infty$. Moreover, for $f, g \in \text{LC}_n^c$ the sup-convolution $f \ast g$ is also in $\text{LC}_n^c$, and in particular is always finite and upper semi-continuous (see e.g. Lemma 2.3 of [14]). However, it turns out that the situation for $\text{LC}_n^c$ is not much better than the situation for $\text{LC}_n$:

**Theorem 1.5.** Let $F : \text{LC}_n^c \to \mathbb{R}$ be a linear and increasing functional. Then there exists $c \geq 0$ such that $F(f) = c \cdot h_f(0)$ for all $f \in \text{LC}_n$. Instead, the correct class of functions one should consider is the following:

**Definition 1.6.** A convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ is called super-coercive if $\lim_{|x| \to \infty} \frac{\varphi(x)}{|x|} = \infty$. A log-concave function $f = e^{-\varphi}$ is called super-coercive if $\varphi$ is super-coercive. We denote the classes of super-coercive convex and log-concave functions by $\text{Cvx}_n^{sc}$ and $\text{LC}_n^{sc}$ respectively.

To understand in which sense functions $f \in \text{LC}_n^{sc}$ are “compact”, note that a closed convex set $K \subseteq \mathbb{R}^n$ is compact if and only if $h_K < \infty$ everywhere on $S^{n-1}$. In the same way, it is well known and not difficult to prove that $f \in \text{LC}_n$ is super-coercive if and only if $h_f < \infty$ everywhere on $\mathbb{R}^n$. This also shows that $\text{LC}_n^{sc}$ is closed under sup-convolution. The class of super-coercive convex functions is the one used by Colesanti, Ludwig and Mussnig in [11] to prove a Hadwiger type theorem. We will now see that on this class the Riesz representation theorem is much more interesting:

**Theorem 1.7.** Let $F : \text{LC}_n^{sc} \to \mathbb{R}$ be a linear and increasing functional. Then there exists a unique positive and finite Borel measure $\mu$ on $\mathbb{R}^n$ with compact support such that $F(f) = \int_{\mathbb{R}^n} h_f \, d\mu$.

Theorem 1.7 seems to be the exact analogue of Theorem 1.2, and the best one could hope for. However, as we will see in Section 6 there are useful applications where we want to allow $F$ to attain the value $+\infty$. In this case one can prove an interesting Riesz-type theorem on the entire family $\text{LC}_n$. In fact, one obtains not only the family of functionals from Theorem 1.7 but also a new surprising family of functionals:

**Theorem 1.8.** Let $F : \text{LC}_n \to (-\infty, \infty]$ be a linear and increasing functional. Assume further that:

1. There exists a function $f_0 \in \text{LC}_n$ such that $\int f_0 > 0$ and $F(f_0) < \infty$.
2. For every sequence $\{f_i\}_{i=1}^\infty \subseteq \text{LC}_n$ such that $f_i \uparrow f \in \text{LC}_n$ we have $F(f_i) \to F(f)$.  
Then there exists a unique positive and finite Borel measure $\mu$ on $\mathbb{R}^n$ with a finite first moment, and a unique positive and finite Borel measure $\nu$ on $\mathbb{S}^{n-1}$, such that

$$F(f) = \int_{\mathbb{R}^n} h_f \, d\mu + \int_{\mathbb{S}^{n-1}} h_{K_f} \, d\nu.$$  

Conversely, every functional of the form (1.2) satisfies the assumptions of the theorem.

To explain the notation used in the theorem, $K_f = \{x \in \mathbb{R}^n : f(x) > 0\}$ is the support of $f$. This is a closed convex set, so one may indeed consider its support function $h_{K_f} : \mathbb{S}^{n-1} \to (-\infty, \infty]$. The notation $f_i \uparrow f$ simply means that $f_1 \leq f_2 \leq f_3 \leq \cdots$ and $\lim_{i \to \infty} f_i(x) = f(x)$ for all $x$.

The extra assumptions in Theorem 1.8 are not just an artifact of our proof. We will see how removing any of the two assumptions creates more linear and increasing functionals that are usually not so interesting to consider. In practice it is usually easy to verify that these extra conditions are satisfied in specific applications. As will be clear from the proof, the same theorem holds if $\text{LC}_n$ is replaced with $\text{LC}_c$ or $\text{LC}_{sc}$.

The rest of this paper is organized as follows: First, in Section 2 we give the short proofs of Theorems 1.3 and 1.5. This can be done directly, without appealing to the classical Riesz theorem. In Section 3 we prove Theorem 1.7. In Section 4 we prove some simple results above the behavior of functions $\varphi \in \text{Cvx}_n$ “at infinity”. These results will be needed for the proof of Theorem 1.8 in Section 5. Finally in Section 6 we connect our main theorems to recent works about the surface area measures of log-concave functions. We show that these works provide important examples of linear functionals on $\text{LC}_n$, and give an application of our results to this theory.

2. The degenerate cases

Each of our main theorems (Theorems 1.3, 1.5, 1.7, and 1.8) can be reformulated in the language of convex functions. For example, Theorem 1.3 is equivalent to the following:

**Theorem 1.3**. Let $F : \text{Cvx}_n \to \mathbb{R}$ be linear and increasing. Then $F(\varphi) = 0$ for all $\varphi \in \text{Cvx}_n$.

Here and everywhere else in the paper, a functional $F$ defined on a domain $D \subseteq \text{Cvx}_n$ is called linear if it is linear with respect to the usual pointwise addition: $F(\alpha \varphi + \beta \psi) = \alpha F(\varphi) + \beta F(\psi)$ for all $\varphi, \psi \in D$ and all $\alpha, \beta > 0$ such that $\alpha \varphi + \beta \psi \in D$.

We now explain this equivalence by first proving Theorem 1.3 and then using it to prove Theorem 1.3:

**Proof of Theorem 1.3**. Write $c = F(1)$ where $1$ denotes the constant function. For every $\varphi \in \text{Cvx}_n$ and every $p \in \mathbb{R}^n$ such that $\varphi(p) < \infty$ we have

$$\varphi + 1^\infty_{\{p\}} = \varphi(p) \cdot 1 + 1^\infty_{\{p\}},$$

where

$$1^\infty_K(x) = \begin{cases} 0 & x \in K \\ +\infty & \text{Otherwise} \end{cases}$$

denotes the convex indicator function. By linearity of $F$ we then have

$$F(\varphi) + F\left(1^\infty_{\{p\}}\right) = \varphi(p) \cdot F(1) + F\left(1^\infty_{\{p\}}\right),$$

and hence

$$F(\varphi) + F\left(1^\infty_{\{p\}}\right) = F\left(\varphi + 1^\infty_{\{p\}}\right).$$

Finally, for every $\varphi \in \text{Cvx}_n$ we have

$$F(\varphi) + F\left(1^\infty_{\{p\}}\right) = F\left(\varphi + 1^\infty_{\{p\}}\right) = 0$$

since $\varphi + 1^\infty_{\{p\}} \in \text{Cvx}_n$. Thus $F(\varphi) = 0$ for all $\varphi \in \text{Cvx}_n$. This completes the proof of Theorem 1.3.
or \( F(\varphi) = c \cdot \varphi(p) \).

If we now take for example \( \varphi(x) = |x| \) and fix \( \theta \in S^{n-1} \) we get that \( c = c \cdot \varphi(\theta) = F(\varphi) = c \cdot \varphi(2\theta) = 2c \). Hence \( c = 0 \), so \( F(\varphi) = 0 \) for all \( \varphi \in \text{Cvx}_n \). \( \square \)

As was mentioned in the introduction, we actually never use in the proof the fact that \( F \) is increasing. Therefore there are no non-trivial linear functions \( F: \text{Cvx}_n \to \mathbb{R} \).

**Proof of Theorem 1.3** Assume \( F: \text{LC}_n \to \mathbb{R} \) is linear and increasing. Define \( G: \text{Cvx}_n \to \mathbb{R} \) by \( G(\varphi) = F(e^{-\varphi}) \). Since \( F \) is increasing and the Legendre transform is order reversing, \( G \) is also increasing. Moreover

\[
G(\alpha \varphi + \beta \psi) = F(e^{-(\alpha \varphi + \beta \psi)^+}) = F((\alpha \cdot e^{-\varphi}) \ast (\beta \cdot e^{-\psi})) = \alpha F(e^{-\varphi}) + \beta F(e^{-\psi}) = \alpha G(\varphi) + \beta G(\psi)
\]

so \( G \) is linear (to avoid confusion, recall that the notation \( \cdot \) in expressions like \( \alpha \cdot e^{-\varphi} \) does not refer to the pointwise multiplication but to the dilation defined in the introduction).

From Theorem 1.3 we deduce that \( G \equiv 0 \). Therefore for every \( f \in \text{LC}_n \) we have \( F(f) = F(e^{-h_f}) = G(h_f) = 0 \). \( \square \)

For Theorem 1.5 the argument is similar. The set of support functions \( \{ h_f : f \in \text{LC}_n^c \} \) is easily seen to be

\[
\tilde{\text{Cvx}}_n = \{ \varphi \in \text{Cvx}_n : \varphi \text{ is finite in a neighborhood of } 0 \}
\]

Therefore Theorem 1.5 will be equivalent to the following:

**Theorem 1.5**. Let \( F: \tilde{\text{Cvx}}_n \to \mathbb{R} \) be linear and increasing. Then there exists \( c \geq 0 \) such that \( F(\varphi) = c \cdot \varphi(0) \) for all \( \varphi \in \text{Cvx}_n \).

**Proof.** Write \( c = F(1) \), and fix \( \varphi \in \text{Cvx}_n \). Since \( \varphi \) is convex and finite in a neighborhood of 0 it is continuous at 0. Therefore given \( \varepsilon > 0 \) there exists \( r > 0 \) such that \( |\varphi(x) - \varphi(0)| < \varepsilon \) for \( x \in rB^2_n \), the Euclidean Ball of radius \( r \) centered at 0. It follows that

\[
(|\varepsilon| \cdot 1 + 1^\infty_{rB^2_n}) = \varphi + 1^\infty_{rB^2_n} \leq (|\varepsilon| + 1^\infty_{rB^2_n}) \cdot 1 + 1^\infty_{rB^2_n}.
\]

Using the linearity and monotonicity of \( F \) we see that

\[
(\varphi(0) - \varepsilon) \cdot 1 + 1^\infty_{rB^2_n} \leq \varphi + 1^\infty_{rB^2_n} \leq (\varphi(0) + \varepsilon) \cdot 1 + 1^\infty_{rB^2_n}.
\]

or \( |F(\varphi) - c \varphi(0)| \leq c \varepsilon \). Since this is true for all \( \varepsilon > 0 \) we conclude that \( F(\varphi) = c \cdot \varphi(0) \) as we wanted. \( \square \)

Theorem 1.5 follows from 1.5 in exactly the same way Theorem 1.3 follows from 1.3, so we will not repeat the argument.
In this section we prove Theorem \ref{1.7}. As was explained in Section \ref{2} the fact that \( f \) is super-coercive is equivalent to \( h_f \) being everywhere finite. Hence we define:

**Definition 3.1.** We denote by \( \text{Cvx}_n^F \) the class of all convex functions \( \varphi \in \text{Cvx}_n \) such that \( \varphi(x) < \infty \) for all \( x \in \mathbb{R}^n \).

Just like in Section \ref{2} Theorem \ref{1.7} is an immediate corollary of the following Riesz type theorem for \( \text{Cvx}_n^F \):

**Theorem 1.7.** Let \( F : \text{Cvx}_n^F \to \mathbb{R} \) be linear and increasing. Then there exists a unique positive and finite Borel measure \( \mu \) on \( \mathbb{R}^n \) with compact support such that \( F(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu \).

Towards the proof we consider the space of functions

\[ E = \{ \varphi + f : \varphi \in \text{Cvx}_n^F \text{ and } f \in C_c(\mathbb{R}^n) \} . \]

The first main step in the proof of Theorem \ref{1.7} is the following result which extends functionals from \( \text{Cvx}_n^F \) to \( E \):

**Lemma 3.2.** Let \( F : \text{Cvx}_n^F \to \mathbb{R} \) be a linear and increasing functional. Then \( F \) can be extended to a functional \( F : E \to \mathbb{R} \) which is again linear and increasing.

**Proof.** Our first step is to extend \( F \) to the smaller space

\[ \bar{E} = \{ \varphi + f : \varphi \in \text{Cvx}_n^F \text{ and } f \in C^2(\mathbb{R}^n) \} , \]

where \( C^2(\mathbb{R}^n) \) denotes the \( C^2 \)-smooth compactly supported functions.

Towards this goal we define \( \rho_a \in \text{Cvx}_n^F \) for every \( a > 0 \) by

\[ \rho_a(x) = \begin{cases} a \frac{|x|^2}{2} & |x| \leq a \\ a^2 |x| - a^2 & |x| > a. \end{cases} \]

We claim that if \( f \in C^2_c(\mathbb{R}^n) \) then \( f + \rho_a \in \text{Cvx}_n^F \) for large enough \( a \). Indeed, assume \( f \) is supported on \( B(0,r) \), an open ball of radius \( r > 0 \) around the origin. Since the Hessian \( \nabla^2 f \) is continuous and compactly supported there exists \( m > 0 \) such that \( \nabla^2 f \succeq -m \cdot \text{Id} \) in the sense of positive definite matrices. Choose \( a > \max(r,m) + 1 \). Then for every \( x \in B(0,a) \) we have

\[ \nabla^2 (f + \rho_a)(x) = \nabla^2 f(x) + \nabla^2 \rho_a(x) = \nabla^2 f(x) + a \cdot \text{Id} \succeq (a-m) \cdot \text{Id} \succeq \text{Id}, \]

so \( f + \rho_a \) is convex in a neighborhood of \( x \). If on the other hand \( x \notin B(0,r+\frac{1}{2}) \) then \( f + \rho_a = \rho_a \) in a neighborhood of \( x \). It follows that \( f + \rho_a \) is convex in a neighborhood of every point of \( \mathbb{R}^n \), so it is convex.

We now extend \( F \) to \( \bar{E} \) by setting

\[ F(\varphi + f) = F(\varphi) + F(f + \rho_a) - F(\rho_a) \]

for some \( a > 0 \) such that \( f + \rho_a \in \text{Cvx}_n^F \). To see that this is well-defined, fix \( \varphi_1, \varphi_2 \in \text{Cvx}_n^F \), \( f_1, f_2 \in C^2_c(\mathbb{R}^n) \) and \( a, b > 0 \) such that \( \varphi_1 + f_1 = \varphi_2 + f_2 \) and \( f_1 + \rho_a, f_2 + \rho_b \in \text{Cvx}_n^F \). Then by linearity of \( F \) on \( \text{Cvx}_n^F \) we have

\[ F(\varphi_1) + F(f_1 + \rho_a) + F(\rho_b) \leq F(\varphi_2 + f_2 + \rho_b + \rho_a), \]

\[ F(\varphi_2) + F(f_2 + \rho_b) + F(\rho_a), \]

\[ F(\varphi_1) + F(f_1 + \rho_a) + F(\rho_b) = F(\varphi_1 + f_1 + \rho_a + \rho_b) = F(\varphi_2 + f_2 + \rho_b + \rho_a) = F(\varphi_2) + F(f_2 + \rho_b) + F(\rho_a) \]
so indeed
\[ F(\varphi_1) + F(f_1 + \rho_a) - F(\rho_a) = F(\varphi_2) + F(f_2 + \rho_b) - F(\rho_b). \]

Next, to show that \( F \) is linear on \( \bar{E} \), fix \( \varphi_1, \varphi_2 \in \text{Cvx}_{n}^{F} \), \( f_1, f_2 \in C_{c}^{2}(\mathbb{R}^{n}) \) and \( \alpha, \beta > 0 \). Choose \( a > 0 \) such that \( f_1 + \rho_a, f_2 + \rho_a, \alpha f_1 + \beta f_2 + \rho_a \in \text{Cvx}_{n}^{F} \). Then using the linearity of \( F \) on \( \text{Cvx}_{n}^{F} \) we can compute:
\[
F(\alpha(\varphi_1 + f_1) + \beta (\varphi_2 + f_2)) = F(\alpha(\varphi_1 + \beta \varphi_2) + (\alpha f_1 + \beta f_2))
= F(\alpha \varphi_1 + \beta \varphi_2) + F(\alpha f_1 + \beta f_2 + \rho_a) = F(\alpha f_1 + \beta f_2 + \rho_a + \alpha \rho_a + \beta \rho_a) - (\alpha + \beta + 1)F(\rho_a)
= F(\alpha \varphi_1 + \beta \varphi_2) + F(\alpha (f_1 + \rho_a) + \beta (f_2 + \rho_a)) - (\alpha + \beta)F(\rho_a)
= \alpha F(\varphi_1) + F(f_1 + \rho_a) - (\alpha + \beta)F(\varphi_2) + F(f_2 + \rho_a) - F(\rho_a)
= \alpha F(\varphi_1 + f_1) + \beta F(\varphi_2 + f_2),
\]
which shows that \( F \) is linear on \( \bar{E} \).

Similarly we show that \( F \) is increasing on \( \bar{E} \): If \( \varphi_1 + f_1 \leq \varphi_2 + f_2 \) we can choose \( a > 0 \) such that \( f_1 + \rho_a, f_2 + \rho_a \in \text{Cvx}_{n}^{F} \) and then
\[
F(\varphi_1 + f_1) = F(\varphi_1) + F(f_1 + \rho_a) - F(\rho_a) = F(\varphi_1 + f_1 + \rho_a) - F(\rho_a)
\leq F(\varphi_2 + f_2 + \rho_a) - F(\rho_a) = F(\varphi_2 + f_2).
\]

Our next step is to extend \( F \) from \( \bar{E} \) to \( E \). Since \( F \) is linear and increasing it is also continuous with respect to the supremum norm, in the sense that
\[
|F(\varphi_1 + f_1) - F(\varphi_2 + f_2)| \leq \|\varphi_1 + f_1 - (\varphi_2 + f_2)\|_{\infty} \cdot F(1).
\]
Here \( 1 \in \text{Cvx}_{n}^{F} \) denotes the constant function. Of course the right hand side may be equal to \(+\infty\), in which case the claim is trivial. Since \( \bar{E} \) is dense in \( E \) it follows that \( F \) can be uniquely extended to a continuous linear functional on \( E \).

It only remains to show that \( F \) is increasing on \( E \). To this end note that for every \( f \in C_{c}(\mathbb{R}^{n}) \) one can find a sequence \( \{g_i\}_{i=1}^{\infty} \subseteq C_{c}^{2}(\mathbb{R}^{n}) \) such that \( g_i \rightarrow f \) uniformly and \( g_i \leq f \) for all \( i \) (or \( g_i \leq f \) for all \( i \)). Assume now that \( \varphi_1 + f_1, \varphi_2 + f_2 \in E \) and \( \varphi_1 + f_1 \leq \varphi_2 + f_2 \). Choose a sequence \( \{g_i\}_{i=1}^{\infty} \subseteq C_{c}^{2}(\mathbb{R}^{n}) \) approximating \( f_1 \) from below and a sequence \( \{h_i\}_{i=1}^{\infty} \subseteq C_{c}^{2}(\mathbb{R}^{n}) \) approximating \( f_2 \) from above. Then
\[
F(\varphi_1 + f_1) = \lim_{i \rightarrow \infty} F(\varphi_1 + g_i) \leq \lim_{i \rightarrow \infty} F(\varphi_2 + h_i) = F(\varphi_2 + f_2),
\]
finishing the proof. \( \square \)

The reader may wonder about the choice of the functions \( \rho_a \) in the proof above. It appears that a simpler choice such as \( \rho_a(x) = a \left\| x \right\|^2 \) would work just as well and slightly simplify the proof. This is correct, but in Section 5 we will claim that the proof above can also serve as proof of Lemma 5.2 and there such simpler choices will not be possible.

To proceed we will also need the following lemma about fast growing convex function. Similar statements have undoubtedly appeared in the literature before, but as we were unable to find a suitable reference we provide the proof:

**Lemma 3.3.**

1. For every \( \varphi \in \text{Cvx}_{n}^{F} \) there exists \( \psi \in \text{Cvx}_{n}^{F} \) such that \( \psi \geq 0 \) and \( \lim_{|x| \rightarrow \infty} \frac{\psi(x)}{\varphi(x)} = +\infty \).
(2) Let $\mu$ be a positive Borel measure on $\mathbb{R}^n$ which is not compactly supported. Then there exists $\varphi \in \text{Cvx}_n^F$ with $\int_{\mathbb{R}^n} \varphi \, d\mu = +\infty$.

**Proof.** Both parts of the lemma rely on the same principle: If $\{a_k\}_{k=1}^\infty$ is an arbitrary sequence of real numbers, then one can find an increasing convex function $\rho : [0, \infty) \to [0, \infty)$ such that $\rho(k) > a_k$ for all $k \in \mathbb{N}$.

To prove this principle define a sequence $\{b_k\}_{k=0}^\infty$ inductively by setting $b_0 = 1$ and

$$b_{k+1} = \max \{2b_k, a_{k+1}\} + 1$$

for $k \geq 1$. Clearly $b_k > \max \{a_k, 0\}$ for all $k \geq 1$, and since

$$\frac{b_{k+1} + b_{k-1}}{2} \geq \frac{b_{k+1}}{2} > b_k$$

the sequence $\{b_k\}_{k=0}^\infty$ is convex and increasing. Define $\rho : [0, \infty) \to [0, \infty)$ by setting $\rho(k) = b_k$ and extending $\rho$ to be linear on any interval of the form $[k, k+1]$. Then $\rho$ is the required function.

We can now prove the two parts of the lemma:

(1) Given $\varphi \in \text{Cvx}_n^F$ we choose $\rho : [0, \infty) \to [0, \infty)$ to be convex and increasing and satisfy

$$\rho(k) \geq k \cdot \max \{\varphi(x) : |x| \leq k + 1\}$$

for all $k \in \mathbb{N}$. Define $\psi \in \text{Cvx}_n^F$ by $\psi(x) = \rho(|x|)$. Then for every $x \in \mathbb{R}^n$ such that $k \leq |x| \leq k + 1$ we have

$$\frac{\psi(x)}{\varphi(x)} \geq \frac{\rho(k)}{\varphi(x)} \geq \frac{k \cdot \varphi(x)}{\varphi(x)} = k,$$

so $\lim_{|x| \to \infty} \frac{\psi(x)}{\varphi(x)} = +\infty$.

(2) Given $\mu$ we set $a_k = \mu(\{x \in \mathbb{R}^n : k \leq |x| < k + 1\})$ and $I = \{k \in \mathbb{N} : a_k > 0\}$. Since $\mu$ is not compactly supported, $I$ must be infinite. We choose $\rho : [0, \infty) \to [0, \infty)$ to satisfy

$$\rho(k) \geq \begin{cases} \frac{1}{a_k} & \text{if } a_k > 0 \\ 0 & \text{otherwise}, \end{cases}$$

and define $\varphi \in \text{Cvx}_n^F$ by $\varphi(x) = \rho(|x|)$. Then

$$\int \varphi \, d\mu \geq \sum_{k=1}^\infty \int_{k \leq |x| < k+1} \varphi \, d\mu \geq \sum_{k=1}^\infty a_k \rho(k) \geq \sum_{k \in I} 1 = +\infty,$$

finishing the proof.

\[\square\]

We can now prove Theorem 1.7.

**Proof of Theorem 1.7.** Assume $F : \text{Cvx}_n^F \to \mathbb{R}$ is linear and increasing. By Lemma 3.2 $F$ can be extended to a linear increasing functional on $E$ which we also denote by $F$. In particular the restriction $F|_{C_c(\mathbb{R}^n)}$ satisfies the assumptions of the classical Riesz theorem (Theorem 1.1). Hence there exists a unique positive Radon measure $\mu$ on $\mathbb{R}^n$ such that $F(f) = \int f \, d\mu$ for all $f \in C_c(\mathbb{R}^n)$. Our main goal is to show that the same formula holds for every $\varphi \in \text{Cvx}_n^F$. 

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Fix \( \varphi \in \text{Cvx}_n^F \) and assume first that \( \varphi \geq 0 \). For every \( R > 0 \) let \( \eta_R \in C_c(\mathbb{R}^n) \) be a function such that \( 0 \leq \eta_R \leq 1 \), \( \eta_R \equiv 1 \) on the ball \( B(0, R) \), and \( \eta_R \) is compactly supported. Since \( \varphi \geq \varphi \eta_R \) we have
\[
F(\varphi) \geq F(\varphi \eta_R) = \int_{\mathbb{R}^n} \varphi \eta_R \, d\mu \geq \int_{B(0,R)} \varphi \, d\mu,
\]
where the middle equality holds since \( \varphi \eta_R \in C_c(\mathbb{R}^n) \). Letting \( R \to \infty \) and using the monotone convergence theorem we see that \( F(\varphi) \geq \int \varphi \, d\mu \).

For the reverse inequality, we use Lemma 3.3 and choose \( \psi \in \text{Cvx}_n^F \) such that \( \psi \geq 0 \) and \( \lim_{|x| \to \infty} \frac{\psi(x)}{\varphi(x)} = +\infty \). This implies that for every \( R > 0 \) the function
\[
\zeta_R(x) = \max \{ R \varphi(x) - \psi(x), 0 \}
\]
is compactly supported. Since \( R \varphi - \psi \leq \zeta_R \leq R \varphi \) we have \( \varphi \leq \frac{1}{R} (\psi + \zeta_R) \) and both sides are in \( E \). By linearity and monotonicity of \( F \) we have
\[
F(\varphi) = F\left(\frac{\psi + \zeta_R}{R}\right) \leq \frac{1}{R} F(\psi) + \frac{1}{R} F(\zeta_R) = \frac{1}{R} F(\psi) + \frac{1}{R} \int \zeta_R \, d\mu
\]
\[
= \frac{1}{R} F(\psi) + \frac{1}{R} \int R \varphi \, d\mu = \frac{1}{R} F(\psi) + \int \varphi \, d\mu.
\]
Letting \( R \to \infty \) we conclude that \( F(\varphi) \leq \int \varphi \, d\mu \), finishing the proof in the case \( \varphi \geq 0 \).

If \( \varphi \in \text{Cvx}_n^F \) is not necessarily positive, we use the fact that for every \( \varphi \in \text{Cvx}_n^F \) there exists \( \alpha, \beta > 0 \) such that \( \varphi(x) \geq -\alpha |x| - \beta \) for all \( x \). Since \( \varphi + \alpha |x| + \beta \geq 0 \) we have
\[
F(\varphi) = F(\varphi + \alpha |x| + \beta) - F(\alpha |x| + \beta)
\]
\[
= \int (\varphi + \alpha |x| + \beta) \, d\mu - \int (\alpha |x| + \beta) \, d\mu = \int \varphi \, d\mu
\]
finishing the proof that \( F(\varphi) = \int \varphi \, d\mu \) for all \( \varphi \in \text{Cvx}_n^F \).

In particular we have \( \int \, d\mu = F(1) < \infty \), so \( \mu \) must be finite. Moreover, since \( F(\varphi) < \infty \) for all \( \varphi \in \text{Cvx}_n^F \) it follows from Lemma 3.3 that \( \mu \) is compactly supported.

It only remains to show that \( \mu \) is unique. Assume \( \int \varphi \, d\mu_1 = \int \varphi \, d\mu_2 \) for all \( \varphi \in \text{Cvx}_n^F \). As we saw in the proof of Lemma 3.2 every \( f \in C^2_\text{c}(\mathbb{R}^n) \) can be written as a difference \( f = \varphi_1 - \varphi_2 \) for \( \varphi_1, \varphi_2 \in \text{Cvx}_n^F \). Therefore \( \int f \, d\mu_1 = \int f \, d\mu_2 \) for all \( f \in C^2_\text{c}(\mathbb{R}^n) \), and by approximation the same holds for every \( f \in C_\text{c}(\mathbb{R}^n) \). It then follows from the classical Riesz theorem that \( \mu_1 = \mu_2 \). \( \square \)

The proof of Theorem 1.7 from Theorem 1.7 works in the usual way, so we will not repeat the argument.

4. Behavior at infinity of convex functions

We now turn our attention to Theorem 1.8. In this section we collect some properties of convex functions that we will need for the proof. As before, we want to restate the theorem in the language of pointwise linear functionals on convex functions. However, Theorem 1.8 involves not only \( h_f \) but also \( h_{K_f} \), where \( K_f \) is the support of \( f \). For this reason we need to know how to recover \( h_{K_f} \) from \( h_f \):
Definition 4.1. Given \( \varphi \in \text{Cvx}_n \) we define \( \varphi : \mathbb{S}^{n-1} \to (-\infty, \infty] \) by

\[
\varphi(\theta) = \lim_{\lambda \to \infty} \frac{\varphi(p + \lambda \theta)}{\lambda},
\]

where \( p \in \mathbb{R}^n \) is an arbitrary point such that \( \varphi(p) < \infty \).

Proposition 4.2. The limit in definition (4.1) exists and is independent of \( p \). Moreover, for every \( f \in \text{LC}_n \) we have \( \mathcal{H}_f = \mathcal{H}_f \).

Proof. Given \( \varphi \in \text{Cvx}_n \) we define \( f \in \text{LC}_n \) by \( f = e^{-\varphi^*} \) so that \( \mathcal{H}_f = \varphi \). Fix \( \theta \in \mathbb{S}^{n-1} \) and fix a point \( x \in \mathbb{R}^n \) with \( f(x) > 0 \), which means that \( \varphi^*(x) < \infty \). Since \( \varphi = \varphi^{**} \) we conclude that for every \( \lambda > 0 \) we have

\[
\langle x, \theta \rangle = \frac{\langle x, p + \lambda \theta \rangle - \varphi^*(x) + \varphi^*(x) - \langle x, p \rangle}{\lambda} \leq \frac{\varphi(p + \lambda \theta) + \varphi^*(x) - \langle x, p \rangle}{\lambda},
\]

and therefore

\[
\langle x, \theta \rangle \leq \liminf_{\lambda \to \infty} \frac{\varphi(p + \lambda \theta) + \varphi^*(x) - \langle x, p \rangle}{\lambda} = \liminf_{\lambda \to \infty} \frac{\varphi(p + \lambda \theta)}{\lambda}.
\]

Hence we have

\[
\mathcal{H}_{Kf}(\theta) = \sup_{x : f(x) > 0} \langle x, \theta \rangle \leq \liminf_{\lambda \to \infty} \frac{\varphi(p + \lambda \theta)}{\lambda}.
\]

Conversely, since \( \varphi(p) < \infty \) for every \( y \in \mathbb{R}^n \) we have

\[
\varphi^*(y) = \sup_{x \in \mathbb{R}^n} [\langle x, y \rangle - \varphi(x)] \geq \langle p, y \rangle - \varphi(p).
\]

Hence for every \( \lambda > 0 \) we obtain

\[
\frac{\varphi(p + \lambda \theta)}{\lambda} = \frac{1}{\lambda} \cdot \sup_{y \in \mathbb{R}^n} [\langle y, p + \lambda \theta \rangle - \varphi^*(y)] = \frac{1}{\lambda} \cdot \sup_{y \in Kf} [\langle y, p + \lambda \theta \rangle - \varphi^*(y)] \leq \frac{1}{\lambda} \sup_{y \in Kf} [\langle y, p + \lambda \theta \rangle - \langle p, y \rangle + \varphi(p)] = \frac{\varphi(p)}{\lambda} + \mathcal{H}_{Kf}(\theta).
\]

Therefore \( \limsup_{\lambda \to \infty} \frac{\varphi(p + \lambda \theta)}{\lambda} \leq \mathcal{H}_{Kf}(\theta) \). Together it follows that

\[
\mathcal{H}_{Kf}(\theta) = \lim_{\lambda \to \infty} \frac{\varphi(p + \lambda \theta)}{\lambda},
\]

which in particular shows that the limit exists and is independent of \( p \in \mathbb{R}^n \). \( \square \)

We can now restate Theorem 1.8 in the language of convex functions:

Theorem 1.8*. Let \( F : \text{Cvx}_n \to (-\infty, \infty] \) be a linear and increasing functional. Assume further that:

1. There exists a function \( \varphi_0 \in \text{Cvx}_n \) with \( \int e^{-\varphi_0} > 0 \) and \( F(\varphi_0) < \infty \).
2. For every \( \{\varphi_i\}_{i=1}^{\infty} \subset \text{Cvx}_n \) such that \( \varphi_i \downarrow \varphi^* \) we have \( F(\varphi_i) \to F(\varphi) \).
Then there exists a unique finite Borel measure $\mu$ on $\mathbb{R}^n$ with a finite first moment, and a unique finite Borel measure $\nu$ on $\mathbb{S}^{n-1}$, such that

$$F(\varphi) = \int_{\mathbb{R}^n} \varphi \, d\mu + \int_{\mathbb{S}^{n-1}} \varphi \, d\nu$$

for all $\varphi \in \text{Cvx}_n$. Conversely, every functional of the form (4.2) satisfies the assumptions of the theorem.

The equivalence of Theorems 1.8 and 1.8* is proved in the usual way, so we will not repeat the argument again. Instead we will start by proving the “conversely” part of Theorem 1.8*. It is obvious that the map $\varphi \mapsto \varphi$ is linear and increasing, and hence every $F$ of the form (4.2) is linear and increasing. Condition 1 of Theorem 1.8* is also simple to check: We take $\varphi_0(x) = |x|$ and observe that $\varphi_0^* = 1_{\mathbb{B}^2_2}^\infty$ so $\int e^{-\varphi_0^*} = |B_2^2| > 0$. Moreover the assumptions on $\mu$ and $\nu$ guarantee that

$$F(\varphi_0) = \int_{\mathbb{R}^n} |x| \, d\mu + \int_{\mathbb{S}^{n-1}} d\nu$$

is finite.

All that remains is to check condition 2 of Theorem 1.8*. We do so in the following proposition:

**Proposition 4.3.** If $\{\varphi_i\}_{i=1}^\infty$, $\varphi \in \text{Cvx}_n$ and $\varphi_i^* \downarrow \varphi^*$ then $\varphi_i \rightharpoonup \varphi$ and $\varphi_i^* \rightharpoonup \varphi^*$. Hence $F(\varphi_i) \to F(\varphi)$ for every functional $F$ of the form (4.2).

**Proof.** Since $\varphi_i^* \downarrow \varphi^*$ obviously $\varphi_i^* \geq \varphi^*$ for all $i$, so $\varphi_i \leq \varphi$ for all $i$.

In the other direction, fix $x \in \mathbb{R}^n$ with $\varphi(x) < \infty$ and fix $\varepsilon > 0$. Choose $y_\varepsilon \in \mathbb{R}^n$ such that

$$\varphi(x) = \sup_{y \in \mathbb{R}^n} \langle x, y \rangle \leq \langle x, y_\varepsilon \rangle - \varphi^*(y_\varepsilon) + \varepsilon.$$

Then for every $i \geq 1$ we have

$$\varphi_i(x) \geq \langle x, y_\varepsilon \rangle - \varphi_i^*(y_\varepsilon) \geq (\varphi(x) + \varphi^*(y_\varepsilon) - \varepsilon) - \varphi_i^*(y_\varepsilon)$$

$$= \varphi(x) + (\varphi^*(y_\varepsilon) - \varphi_i^*(y_\varepsilon)) - \varepsilon.$$

Letting $i \to \infty$ and using the fact that $\varphi_i^*(y_\varepsilon) \to \varphi^*(y_\varepsilon)$ we see that $\lim_{i \to \infty} \varphi_i(x) \geq \varphi(x) - \varepsilon$. As $\varepsilon > 0$ was arbitrary we conclude that $\varphi_i(x) \rightharpoonup \varphi(x)$. The case $\varphi(x) = \infty$ is handled similarly.

We now prove that $\varphi_i^* \rightharpoonup \varphi^*$. We fix a point $p \in \mathbb{R}^n$ such that $\varphi(p) < \infty$, and therefore $\varphi_i(p) < \varphi(p) < \infty$ for all $i$. Then we can write

$$\varphi_i(\theta) = \lim_{\lambda \to \infty} \frac{\varphi_i(p + \lambda \theta)}{\lambda} = \lim_{\lambda \to \infty} \frac{\varphi(p + \lambda \theta) - \varphi(p)}{\lambda} = \sup_{\lambda > 0} \frac{\varphi_i(p + \lambda \theta) - \varphi_i(p)}{\lambda},$$

where the last equality holds by the convexity of $\varphi_i$. The same of course holds for $\varphi_i$ instead of $\varphi_i$.

Therefore we have

$$\lim_{i \to \infty} \varphi_i(\theta) = \lim_{i \in N} \sup_{\lambda > 0} \frac{\varphi_i(p + \lambda \theta) - \varphi_i(p)}{\lambda} = \sup_{\lambda > 0} \frac{\varphi(p + \lambda \theta) - \varphi(p)}{\lambda} = \varphi(\theta).$$

Since we clearly have $\varphi_i^* \leq \varphi^*$ for all $i$, we conclude that indeed $\varphi_i^* \rightharpoonup \varphi^*$.

Finally, the monotone convergence theorem implies that every functional $F$ of the form (4.2) must satisfy $F(\varphi_i) \to F(\varphi)$. $\square$
For most of the proof of Theorem 1.8 we will not work with the full class Cvx_n. Instead, we will work with the following class:

**Definition 4.4.** A function $\varphi \in \text{Cvx}_n$ is of linear growth if there exists constants $A_\varphi, B_\varphi > 0$ such that $\varphi(x) \leq A_\varphi |x| + B_\varphi$ for all $x \in \mathbb{R}^n$. We denote the class of all convex functions of linear growth by Cvx$^\ell_n$.

If $\varphi \in \text{Cvx}_n^\ell$, then clearly $\varphi$ is everywhere finite. In fact a little more is true:

**Proposition 4.5.** Assume $\varphi \in \text{Cvx}_n^\ell$. Then the limit $\hat{\varphi}(\theta) = \lim_{\lambda \to \infty} \frac{\varphi(\lambda \theta)}{\lambda}$ exists uniformly in $\theta \in S^{n-1}$.

**Proof.** The main point here is that a convex function of linear growth is (globally) Lipschitz. To see this, fix $x, y \in \mathbb{R}^n$. For $t > 0$ we define $z_t = y + t(y - x)$ so that $y \in [x, z_t]$. By convexity we then have

$$\frac{\varphi(y) - \varphi(x)}{|y - x|} \leq \frac{\varphi(z_t) - \varphi(x)}{|z_t - x|} \leq \frac{(A_\varphi |z_t| + B_\varphi) - \varphi(x)}{|z_t - x|} \to_{\lambda \to \infty} A_\varphi,$$

so $\varphi$ is $A_\varphi$-Lipschitz. It follows that the family of functions $\varphi_\lambda : S^{n-1} \to \mathbb{R}$ defined by $\varphi_\lambda(\theta) = \frac{\varphi(\lambda \theta)}{\lambda}$ are all Lipschitz on $S^{n-1}$ with a uniform constant $A_\varphi$. In particular the family $\{\varphi_\lambda\}_{\lambda > 0}$ is an equicontinuous, which implies that the convergence $\varphi_\lambda \to \hat{\varphi}$ is uniform. \qed

Finally, we will need the following approximation lemma:

**Lemma 4.6.** For every $\varphi \in \text{Cvx}_n$ there exist a sequence $\{\varphi_k\}_{k=1}^\infty \subseteq \text{Cvx}_n^\ell$ such that $\varphi_k^* \downarrow \varphi^*$.

**Proof.** We define

$$\varphi_k(x) = \inf_{y \in \mathbb{R}^n} [\varphi(y) + k |x - y|].$$

Since $\varphi_k(x) \leq \varphi(0) + k |x|$ it follows that every $\varphi_k$ is of linear growth. A computation shows that $\varphi_k^* = \varphi^* + 1^{\infty}_{kB^n_2}$, so we indeed have $\varphi_k^* \downarrow \varphi^*$. \qed

### 5. Non-Finite linear functionals

We now prove Theorem 1.8

**Lemma 5.1.** Assume $F : \text{Cvx}_n \to (-\infty, \infty]$ satisfies the assumptions of Theorem 1.8 Then $F(\varphi) < \infty$ for every $\varphi$ of linear growth.

**Proof.** We are given that there exists a function $\varphi_0 \in \text{Cvx}_n$ such that $\int e^{-\varphi_0} > 0$ and $F(\varphi_0) < \infty$. If $0$ denotes the constant function 0 then $F(\varphi_0) = F(0 + \varphi_0) = F(0) + F(\varphi_0)$, so $F(0) = 0$. For every affine function $\ell$ we have $\pm \ell \in \text{Cvx}_n$ and $F(\ell) + F(-\ell) = F(0) = 0$, so $F(\ell) < \infty$. \hfill 14
Since $\int e^{-\varphi_0^*} > 0$ we can find a ball $B(p, r)$ such that $\varphi_0^*$ is finite and bounded on $B(p, r)$, say $\varphi_0^* < m$. Then
\[
\varphi_0(x) = \sup_{y \in \mathbb{R}^n} [\langle x, y \rangle - \varphi_0^*(y)] \geq \left\langle x, p + r \frac{x}{2|x|} \right\rangle - \varphi_0^* \left( p + r \frac{x}{2|x|} \right)
\geq \langle x, p \rangle + \frac{r}{2} |x| - m.
\]
It follows that
\[
F(|x|) = \frac{2}{r} \left( F \left( \frac{r}{2} |x| + \langle x, p \rangle - m \right) + F \left( - \langle x, p \rangle + m \right) \right)
\leq \frac{2}{r} \left( F(\varphi_0) + F \left( - \langle x, p \rangle + m \right) \right) < \infty.
\]
Finally, for every $\varphi \in \text{Cvx}_n^\ell$ we have
\[
F(\varphi) \leq F \left( A_{\varphi} |x| + B_{\varphi} \right) = A_{\varphi} F(|x|) + F\left( B_{\varphi} \right) < \infty
\]
as we claimed. $\square$

We now proceed in a way similar to the proof of Theorem 1.7. We consider the space
\[
E_{\ell} = \left\{ \varphi + f : \varphi \in \text{Cvx}_n^\ell \text{ and } f \in C_c(\mathbb{R}^n) \right\},
\]
and claim the following extension result:

**Lemma 5.2.** Let $F : \text{Cvx}_n^\ell \to \mathbb{R}$ be a linear and increasing functional. Then $F$ can be extended to a functional $F : E_{\ell} \to \mathbb{R}$ which is again linear and increasing.

The proof of Lemma 5.2 is identical to the proof of Lemma 3.2: just replace $\text{Cvx}_n^F$ by $\text{Cvx}_n^\ell$ everywhere in the proof. It should now be clear why we chose in the proof to work the functions $\rho_a$ (which satisfy $\rho_a \in \text{Cvx}_n^\ell$) and not with simpler functions like $a \frac{|x|^2}{2}$ (which do not).

We are ready to prove a Riesz type theorem for the space $\text{Cvx}_n^\ell$:

**Theorem 5.3.** Let $F : \text{Cvx}_n^\ell \to \mathbb{R}$ be a linear and increasing functional. Then there exists a unique finite Borel measure $\mu$ on $\mathbb{R}^n$ with a finite first moment, and a unique finite Borel measure $\nu$ on $\mathbb{S}^{n-1}$, such that
\[
F(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu + \int_{\mathbb{S}^{n-1}} \varphi d\nu
\]
for all $\varphi \in \text{Cvx}_n$.

**Proof.** First, by Lemma 5.2 we can extend $F$ to the space $E_{\ell}$. Applying the classical Riesz theorem to the restriction $F|_{C_c(\mathbb{R}^n)}$, we conclude that there exists a Random measure $\mu$ on $\mathbb{R}^n$ such that $F(f) = \int_{\mathbb{R}^n} f d\mu$ for all $f \in C_c(\mathbb{R}^n)$. The same argument as in Theorem 1.7 shows that $F(\varphi) \geq \int_{\mathbb{R}^n} \varphi d\mu$ for all non-negative $\varphi \in \text{Cvx}_n^\ell$ (we will essentially repeat the argument in the next paragraph). In particular
\[
\int d\mu \leq F(1) < \infty, \quad \int |x| d\mu \leq F(|x|) < \infty,
\]
which implies that $\mu$ is finite with a finite first moment.
Define $G : \text{Cvx}_n^\ell \to \mathbb{R}$ by

$$G(\varphi) = F(\varphi) - \int_{\mathbb{R}^n} \varphi \, d\mu.$$ 

Note that $G$ is indeed always finite, since $\mu$ is finite with finite first moment and $\varphi$ is of linear growth. We will need two properties of the functional $G$. First, we claim that $G$ is itself linear and increasing. Indeed, $G$ is clearly linear. To show that $G$ is increasing we fix two functions $\varphi_1, \varphi_2 \in \text{Cvx}_n^\ell$ such that $\varphi_1 \leq \varphi_2$. For every $R > 0$ let $\eta_R \in C_c(\mathbb{R}^n)$ be a function such that $0 \leq \eta_R \leq 1$, $\eta_R \equiv 1$ on the ball $B(0, R)$, and $\eta_R$ is compactly supported. Since $f \cdot (\varphi_2 - \varphi_1) \in C_c(\mathbb{R}^n)$ and $f \cdot (\varphi_2 - \varphi_1) + \varphi_1 \leq \varphi_2$ we conclude that

$$F(\varphi_2) \geq F(f(\varphi_2 - \varphi_1)) + F(\varphi_1) = \int_{\mathbb{R}^n} f(\varphi_2 - \varphi_1) \, d\mu + F(\varphi_1)$$

$$\geq \int_{B(0, R)} (\varphi_2 - \varphi_1) \, d\mu + F(\varphi_1).$$

Letting $R \to \infty$ and using the monotone convergence theorem we conclude that

$$F(\varphi_2) \geq \int_{\mathbb{R}^n} (\varphi_2 - \varphi_1) \, d\mu + F(\varphi_1),$$

or $G(\varphi_2) \geq G(\varphi_1)$.

Next, we want to prove that $G(\varphi)$ depends only on $\varphi$. In other words, we want to prove that if $\varphi_1 = \varphi_2$ then $G(\varphi_1) = G(\varphi_2)$. We first show it under the extra assumption that $\varphi_1 \leq \varphi_2$.

Under this assumption we already proved that $G(\varphi_1) \leq G(\varphi_2)$. For the reverse inequality, we again fix $R > 0$ and define

$$\zeta_R(x) = \max(0, R(\varphi_2(x) - \varphi_1(x)) - |x|).$$

We claim that $\zeta_R \in C_c(\mathbb{R}^n)$. Since it is obviously continuous, we just need to show it is compactly supported. By our assumption we have

$$\lim_{\lambda \to \infty} \frac{\varphi_1(\lambda \theta)}{\lambda} = \frac{\varphi_1(\theta)}{\lambda} = \frac{\varphi_2(\lambda \theta)}{\lambda},$$

and by Proposition 4.5 these limits are uniform in $\theta \in S^{n-1}$. It follows that

$$\lim_{\lambda \to \infty} \frac{\varphi_2(\lambda \theta) - \varphi_1(\lambda \theta)}{\lambda} = 0$$

uniformly in $\theta \in S^{n-1}$, or equivalently $\varphi_2(x) - \varphi_1(x) = o(|x|)$ as $|x| \to \infty$. This shows that $\zeta_R$ is compactly supported.

We can bound $\zeta_R$ from above and from below. From above, it is clear that $\zeta_R \leq R(\varphi_2 - \varphi_1)$. From below, we have $\zeta_R + |x| + R\varphi_1 \geq R\varphi_2$. It follows that

$$F(\varphi_2) = \frac{1}{R} F(R\varphi_2) \leq \frac{1}{R} F(\zeta_R + |x| + R\varphi_1)$$

$$= \frac{1}{R} \left( F(\zeta_R) + F(|x|) + R \cdot F(\varphi_1) \right)$$

$$= \frac{1}{R} \left( \int \zeta_R \, d\mu + F(|x|) + R \cdot F(\varphi_1) \right)$$

$$\leq \int (\varphi_2 - \varphi_1) \, d\mu + \frac{1}{R} F(|x|) + F(\varphi_1).$$

Letting $R \to \infty$ we see that indeed $G(\varphi_2) \leq G(\varphi_1)$.
This concludes the proof that $G(\varphi_1) = G(\varphi_2)$ under the assumptions $\varphi_1 = \varphi_2$ and $\varphi_1 \leq \varphi_2$. But we can now get rid of this second assumption: If $\varphi_1 = \varphi_2$ then we also have \[
abla_{(\varphi_1, \varphi_2)} = \nabla_{(\varphi_1, \varphi_2)} = \nabla_1 = \nabla_2,
\] and since $\varphi_1, \varphi_2 \leq \nabla_{(\varphi_1, \varphi_2)}$ it follows that $G(\varphi_1) = G(\nabla_{(\varphi_1, \varphi_2)}) = G(\varphi_2)$. Hence $G(\varphi)$ depends only on $\nabla_1$.

Consider now the spaces \[
E_1 = \{h_K : K \in K_n\}
\]
\[
E_2 = \{\nabla_1 : \varphi \in C^{n\ell}\}.
\]
We have $E_1 \subseteq E_2 \subseteq C(S^{n-1})$ since every support function $h_K : S^{n-1} \rightarrow \mathbb{R}$ can also be though of as a 1-homogeneous convex function on $\mathbb{R}^n$, and under this identification we have $\nabla_1 = h_K$. Note that we can define $\overline{G} : E_2 \rightarrow \mathbb{R}$ by $\overline{G}(\nabla_1) = G(\varphi)$, since we just proved that this definition does not depend on the choice of $\varphi$.

Since the restriction $\overline{G} \big|_{E_1} : E_1 \rightarrow \mathbb{R}$ is clearly linear and increasing, Theorem 1.2 implies that exists a finite Borel measure $\mu$ on $S^{n-1}$ such that
\[
\overline{G}(h_1) = \int_{S^{n-1}} h_K d\mu.
\]
Moreover, in the proof of Theorem 1.2 we saw that $\overline{G} \big|_{E_1}$ has a unique extension to $C(S^{n-1})$ which is linear and increasing, and by inspecting the proof we see that an extension to $E_2$ also has to be unique. But $\overline{G}$ itself is such an extension, and so is the map $\nabla_1 \mapsto \int_{S^{n-1}} \nabla_1 d\mu$. Therefore they must coincide, which means that
\[
G(\varphi) = \overline{G}(\nabla_1) = \int_{S^{n-1}} \nabla_1 d\mu
\]
for all $\varphi \in C^{n\ell}$. But then
\[
F(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu + G(\varphi) = \int_{\mathbb{R}^n} \varphi d\mu + \int_{S^{n-1}} \nabla_1 d\mu,
\]
finishing the proof of existence.

Finally, we need to show that $\mu$ and $\nu$ are uniquely defined. Towards this goal assume that
\[
\int_{\mathbb{R}^n} \varphi d\mu_1 + \int_{S^{n-1}} \nabla_1 d\nu_1 = \int_{\mathbb{R}^n} \varphi d\mu_2 + \int_{S^{n-1}} \nabla_1 d\nu_2
\]
for all $\varphi \in C^{n\ell}$. Fix $K \in K_n$ with $0 \in K$ and for $R > 0$ define $\nabla_R = \max \{h_K - R, 0\} \in C^{n\ell}$. Then $\nabla_R = h_K$, so
\[
\int_{\mathbb{R}^n} \varphi_R d\mu_1 + \int_{S^{n-1}} h_K d\nu_1 = \int_{\mathbb{R}^n} \varphi_R d\mu_2 + \int_{S^{n-1}} h_K d\nu_2
\]
for all $R > 0$. But clearly $\nabla_R \rightarrow 0$ pointwise as $R \rightarrow \infty$ so by the monotone convergence theorem it follows that $\int_{S^{n-1}} h_K d\nu_1 = \int_{S^{n-1}} h_K d\nu_2$. The same will also be true without the assumption that $0 \in K$, since for every $K \in K_n$ and every $r > 0$ we have $h_K = h_{K + rB_2^n} - h_{rB_2^n}$, and for large enough $r > 0$ we do have $0 \in K + rB_2^n$. It now follows from the uniqueness part of Theorem 1.2 that $\nu_1 = \nu_2$. We can then repeat the argument from Theorem 1.7 to show that also $\mu_1 = \mu_2$. 

From here Theorem 1.8 (and Theorem 1.8) is a simple corollary:
Proof of Theorem 1.8
Given $F : \text{Cvx}_n \to (-\infty, \infty)$, the restriction $F|_{\text{Cvx}_n^\ell}$ satisfies the assumptions of Theorem 5.3 (we need Lemma 5.1 here to know that $F|_{\text{Cvx}_n^\ell}$ is always finite).

From Theorem 5.3 there exists $\mu$ and $\nu$ of the required form such that
$$F(\varphi) = \int_{\mathbb{R}^n} \varphi \, d\mu + \int_{S^{n-1}} \varphi \, d\nu$$
for all $\varphi \in \text{Cvx}_n^\ell$. For an arbitrary $\varphi \in \text{Cvx}_n$ we use Lemma 4.6 and choose a sequence $\{\varphi_k\}_{k=1}^\infty \subseteq \text{Cvx}_n^\ell$ such that $\varphi_k \downarrow \varphi^*$. We then have
$$F(\varphi) = \lim_{k \to \infty} F(\varphi_k) = \lim_{k \to \infty} \left( \int_{\mathbb{R}^n} \varphi_k \, d\mu + \int_{S^{n-1}} \varphi_k \, d\nu \right)$$
$$= \int_{\mathbb{R}^n} \varphi \, d\mu + \int_{S^{n-1}} \varphi \, d\nu,$$
where we used Proposition 4.3 in the last equality.

Uniqueness of $\mu$ and $\nu$ is obvious since by Theorem 5.3 even the restriction $F|_{\text{Cvx}_n^\ell}$ determines $\mu$ and $\nu$ uniquely.

We conclude this section with two examples that show the extra conditions imposed in Theorem 1.8 are indeed necessary:

Example 5.4. Define $F : \text{LC}_n \to (-\infty, \infty]$ by
$$F(f) = \begin{cases} 0 & \text{If } f = c \cdot 1_{\{p\}} \text{ for } c \geq 0 \text{ and } p \in \mathbb{R}^n \\ +\infty & \text{Otherwise.} \end{cases}$$
$F$ is linear and increasing, but is not of the form (1.2). The reason of course is that $F$ is essentially "lower dimensional", i.e. $F(f) = \infty$ for every $f \in \text{LC}_n$ with $\int f > 0$.

Example 5.5. Without the weak continuity assumption of Theorem 1.8 one can create less explicit counterexamples. For example, fix a free ultrafilter $U$ on the natural numbers $\mathbb{N}$, a direction $\theta \in S^{n-1}$ and a number $\alpha > 1$ and consider the functional $F : \text{LC}_n \to (-\infty, \infty]$ defined by
$$F(f) = \lim_{U} \frac{h_f(n\theta)}{n^\alpha}.$$  
Since ultralimits are linear and increasing, it follows that $F$ is also linear and increasing. However, $F$ is not of the form (1.2) as it is not continuous: For every $f \in \text{LC}_n$ with $h_f \in \text{Cvx}_n^\ell$ we have $F(f) = 0$, but $F$ is not identically 0.

6. Surface area measures

As was explained in the introduction, linear functionals appear naturally in convex geometry as the derivatives of more general functionals. For convex bodies, the simplest choice of such a functional was the volume, i.e. $G(K) = |K|$. For log-concave functions, the simplest choice is to consider the functional $G : \text{LC}_n \to [0, \infty]$ defined by $G(f) = \int f$. We now take its derivative:

Definition 6.1. For $f, g \in \text{LC}_n$ such that $0 < \int g < \infty$, we define
$$\delta(g, f) = \lim_{t \to 0^+} \frac{\int (g * (t \cdot f)) - \int g}{t}.$$
This first variation of the integral was first systematically studied by Colesanti and Fragalà in [7], who showed that the limit in the definition exists and belongs to \((-\infty, \infty]\). If we now consider the functional \(F_g : LC_n \to (-\infty, \infty]\) defined by \(F_g(f) = \delta(g, f)\), it is clear from the definition that \(F_g\) is increasing. In the case of convex bodies we have the theory of mixed volumes available to us, which implies that functionals analogous to \(F_g\) are also linear. For log-concave functions there is no such theory, so while we expect \(F_g\) to be linear as a directional derivative, this is not obvious. For this reason one cannot use a result such as Theorem 1.8 directly. However, Theorem 1.8 still gives us a clue for the type of formulas we expect. Under technical conditions such a formula was proved in [7]. We now describe this result.

**Definition 6.2.** Fix \(g \in LC_n\) with \(0 < \int g < \infty\) and write \(\psi = -\log g\). We define the Borel measure \(\mu_g\) on \(\mathbb{R}^n\) by \((\nabla \psi)\sharp (gdx)\), where \(\sharp\) denotes the push-forward. Similarly the measure \(\nu_g\) is the Borel measure on \(S^{n-1}\) defined by \(\nu_g = (nK_g)^\sharp (gH^{n-1}|_{\partial K_g})\).

Here \(nK_g : \partial K_g \to S^{n-1}\) is the Gauss map (mapping every point \(x \in \partial K_g\) to the normal to \(K_g\) at the point \(x\)), and \(H^{n-1}\) denotes the \((n-1)\)-Hausdorff measure.

Note that we need no regularity assumptions on \(g \in LC_n\) in order to define \(\mu_g\) and \(\nu_g\). Indeed, it is well-known \(\nabla \psi\) exists (Lebesgue) almost everywhere on \(K_g\), so the push-forward is always well defined. Similarly, the Gauss map \(nK_g\) is always defined \(H^{n-1}\)-almost everywhere on \(\partial K_g\). However, the theorem of [7] does require significant regularity assumptions on our functions.

For the statement of the theorem, let us say that a function \(f \in LC_n\) is **sufficiently regular** if:

1. The support \(K_f\) of \(f\) is a \(C^2\) smooth convex body with everywhere positive Gauss curvature.
2. The function \(\psi = -\log f\) is continuous in \(K_f\), \(C^2\)-smooth in the interior of \(K_f\), and has a strictly positive-definite Hessian.
3. \(\lim_{x \to \partial K_f} |\nabla \psi(x)| = \infty\).

We can now state:

**Theorem 6.3** (Colesanti–Fragalà). Assume \(f, g \in LC_n\) are sufficiently regular. Assume further that \(h_g - c \cdot h_f\) is convex for some \(c > 0\). Then

\[
\delta(g, f) = \int_{\mathbb{R}^n} h_f d\mu_g + \int_{S^{n-1}} h_{K_f} d\nu_g.
\]

From this theorem we see that functionals of the form (1.2) do appear “in nature”. This explains why we are interested in results like Theorem 1.8 and are not satisfied with the simpler Theorem 1.7.

The technical assumptions in Theorem 6.3 are known to be non-optimal. Proving the same result for every \(f, g \in LC_n\) is an interesting problem outside the scope of this paper. However, we do understand completely the case \(\nu_g \equiv 0\):

**Theorem 6.4** ([20]). Fix \(g \in LC_n\) such that \(0 < \int g < \infty\). Then the following are equivalent:

1. \(g\) is essentially continuous, i.e. the set \(\{x \in \mathbb{R}^n : g\text{ is not continuous at }x\}\) has zero \(H^{n-1}\) measure.
2. For every \(f \in LC_n\) one has \(\delta(g, f) = \int_{\mathbb{R}^n} h_f d\mu_g\).
To better understand the notion of essential continuity, note that as a log-concave function $g$ is automatically continuous outside of $\partial K_g$. Moreover, since $g$ is upper semi-continuous it is easy to check that $g$ is continuous at a point $x \in \partial K_g$ if and only if $g(x) = 0$. Therefore $g$ is essentially continuous if and only if $g$ vanishes $\mathcal{H}^{n-1}$-almost everywhere on $\partial K_g$. Equivalently, $g$ is essentially continuous if and only if for every $x \in \partial K_g$, $g(x) = 0$.

The importance of essential continuity in the context of surface area measures was first realized by Cordero-Erausquin and Klartag in [12]. In this paper the authors studied the measure $\mu_g$ from Definition 6.2 in a different language: If $g = e^{-\psi}$ then $\mu_g$ is called there the moment measure of $\psi$.

The main result of [12] is the following theorem which characterizes measures of the form $\mu_g$:

**Theorem 6.5 (Cordero-Erausquin, Klartag).** Let $\mu$ be a Borel measure on $\mathbb{R}^n$. Then $\mu = \mu_g$ for an essentially continuous function $g \in \text{LC}_n$ if and only if $\mu$ satisfies the following properties:

1. $0 < \mu(\mathbb{R}^n) < \infty$.
2. $\mu$ has a finite first moment, and $\int_{\mathbb{R}^n} x d\mu = 0$.
3. $\mu$ is not concentrated on any lower dimensional subspace (i.e. for every proper linear subspace $H$ of $\mathbb{R}^n$ we have $\mu(H) < \mu(\mathbb{R}^n)$).

Santambrogio gave in [27] an alternative proof of Theorem 6.5 using methods of optimal transportation.

Our Theorem 1.7 can be combined with the known results mentioned above in order to characterize linear increasing functionals as the first variation of the integral. Here is an example of such a result:

**Theorem 6.6.** Let $F : \text{LC}_{n}^{\text{sc}} \to \mathbb{R}$ be a linear and increasing functional. Assume further that:

1. $F(1_{\{p\}}) = 0$ for every $p \in \mathbb{R}^n$.
2. For every line $\ell$ through the origin $F\left(e^{-\frac{1}{2}|x|^2}1_{\ell}\right) \neq 0$.

Then there exists an essentially continuous log-concave function $g \in \text{LC}_n$ such that $F(f) = \delta(g,f)$ for all $f \in \text{LC}_{n}^{\text{sc}}$.

Of course, the choice of the function $e^{-\frac{1}{2}|x|^2}1_{\ell}$ is rather arbitrary. As will be clear from the proof, this is a non-degeneracy condition meant to exclude “lower dimensional” examples.

**Proof.** Since $F$ is linear and increasing, by Theorem 1.7 there exists a finite compactly supported Borel measure $\mu$ such that $F(f) = \int_{\mathbb{R}^n} h_f d\mu$ for all $f \in \text{LC}_{n}^{\text{sc}}$.

For every $p \in \mathbb{R}^n$ we have

$$\left< \int_{\mathbb{R}^n} x d\mu, p \right> = \int_{\mathbb{R}^n} (x,p) d\mu = \int_{\mathbb{R}^n} h_{1_{\{p\}}} d\mu = F(1_{\{p\}}) = 0$$

by our assumptions. Hence $\int x d\mu = 0$.

Next, fix a hyperplane $H \subseteq \mathbb{R}^n$ and define $\ell = H^\perp$. Note that if $f = e^{-\frac{1}{2}|x|^2}1_{\ell}$ then

$$h_f(x) = \sup_{y \in \ell} \left[ \langle x, y \rangle - \frac{1}{2} |y|^2 \right] = \frac{1}{2} |\text{Pr}_\ell x|^2,$$
where $\text{Pr}_\ell$ denotes the orthogonal projection onto $\ell$. In particular $h_f \equiv 0$ on $H$. Therefore

$$\int_{\mathbb{R}^n} h_f \, d\mu = F(f) \neq 0 = \int_H h_f \, d\mu,$$

so $\mu$ is not supported on $H$.

We see that $\mu$ satisfies all the assumptions of Theorem 6.5, so there exists an essentially continuous $g \in \mathcal{L}C_n$ such that $\mu = \mu_g$. But then by Theorem 6.4 we have for every $f \in \mathcal{L}C_n$

$$F(f) = \int_{\mathbb{R}^n} h_f \, d\mu = \int_{\mathbb{R}^n} h_f \, d\mu_g = \delta(g, f),$$

proving the result. □

A natural question is whether this result can be extended to functionals $F : \mathcal{L}C_n \to (-\infty, \infty]$. However, in order to prove such a result one first needs to solve the following problem:

**Problem 6.7.** Fix a finite Borel measure $\mu$ on $\mathbb{R}^n$ and a finite Borel measure $\nu$ on $S^{n-1}$. Under what conditions on $\mu$ and $\nu$ can one find a function $g \in \mathcal{L}C_n$ with $\mu_g = \mu$ and $\nu_g = \nu$?

This is a very natural question, but it is much beyond the scope of this paper and is better left for future research.

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