OPTIMAL BOUNDS FOR ANCIENT CALORIC FUNCTIONS

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Abstract. For any manifold with polynomial volume growth, we show: The dimension of the space of ancient caloric functions with polynomial growth is bounded by the degree of growth times the dimension of harmonic functions with the same growth. As a consequence, we get a sharp bound for the dimension of ancient caloric functions on any space where Yau’s 1974 conjecture about polynomial growth harmonic functions holds.

0. Introduction

Given a manifold $M$ and a constant $d$, $\mathcal{H}_d(M)$ is the linear space of harmonic functions of polynomial growth at most $d$. Namely, $u \in \mathcal{H}_d(M)$ if $\Delta u = 0$ and for some $p \in M$ and a constant $C_u$ depending on $u$

$$\sup_{B_R(p)} |u| \leq C_u (1 + R)^d \text{ for all } R. \tag{0.1}$$

In 1974, S.T. Yau conjectured that $\mathcal{H}_d(M)$ is finite dimensional for each $d$ when $\text{Ric}_M \geq 0$. The conjecture was settled in [CM2]; see [CM1]–[CM5] for more results. In fact, [CM2]–[CM4] proved finite dimensionality under much weaker assumptions of:

1. A volume doubling bound.
2. A scale-invariant Poincaré inequality or meanvalue inequality.

The natural parabolic generalization is a polynomial growth ancient solution of the heat equation. Given $d > 0$, $u \in \mathcal{P}_d(M)$ if $\partial_t u = \Delta u$ and for some $p \in M$ and a constant $C_u$

$$\sup_{B_R(p) \times [-R^2, 0]} |u| \leq C_u (1 + R)^d \text{ for all } R. \tag{0.2}$$

In her 2006 thesis, [Ca1], [Ca2], Calle initiated the study of dimension bounds for these spaces. They play a fundamental role in geometric flows, see [CM6]. On $\mathbb{R}^n$, these functions are the classical caloric polynomials that generalize the Hermite polynomials.

Our main result is (here $C$ and $d_V$ are any constants):

**Theorem 0.3.** If $\text{Vol}(B_R(p)) \leq C (1 + R)^{d_V}$ for some $p \in M$, all $R > 0$, then for all $k \geq 1$

$$\dim \mathcal{P}_{2k}(M) \leq (k + 1) \dim \mathcal{H}_{2k}(M). \tag{0.4}$$

Combining this with the bound $\dim \mathcal{H}_d(M) \leq C d^{n-1}$ when $\text{Ric}_M \geq 0$ from [CM3] gives:

**Corollary 0.5.** There exists $C = C(n)$ so that if $\text{Ric}_M \geq 0$, then for $d \geq 1$

$$\dim \mathcal{P}_d(M) \leq C d^n. \tag{0.6}$$

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1 For Yau’s 1974 conjecture see: page 117 in [Ya2], problem 48 in [Ya3], Conjecture 2.5 in [Sc], [Ka], [Kz], [DF], Conjecture 1 in [Li1], and problem (1) in [LiTa], amongst others.

2 A volume doubling space with doubling constant $C_D$ has polynomial volume growth of degree $\log_2 C_D$. 


The exponent $n$ in (0.6) is sharp: There is a constant $c$ depending on $n$ so that for $d \geq 1$
\begin{equation}
(0.7)\quad c^{-1}d^n \leq \dim \mathcal{P}_d(\mathbb{R}^n) \leq cd^n.
\end{equation}

Recently, Lin and Zhang, [LZ], proved very interesting related results, adapting the methods of [CM2], [CM4] to get the bound $d^{n+1}$.

An immediate corollary of the parabolic gradient estimate of Li and Yau, [LY], is that if $d < 2$ and $\text{Ric} \geq 0$, then $\mathcal{P}_d(M) = \mathcal{H}_d(M)$ consists only of harmonic functions of polynomial growth. In particular, $\mathcal{P}_d(M) = \{\text{Constant functions}\}$ for $d < 1$ and, moreover, $\dim \mathcal{P}_1(M) \leq n+1$, by Li and Tam, [LiTa], with equality if and only if $M = \mathbb{R}^n$ by [ChCM].

The exponent $n-1$ is also sharp in the bound for $\dim \mathcal{H}_d$ when $\text{Ric}_M \geq 0$. However, as in Weyl’s asymptotic formula, the coefficient of $d^{n-1}$ can be related to the volume, [CM3]:
\begin{equation}
(0.8)\quad \dim \mathcal{H}_d(M) \leq C_n V_M d^{n-1} + o(d^{n-1}).
\end{equation}

- $V_M$ is the “asymptotic volume ratio” $\lim_{r \to \infty} \text{Vol}(B_r)/r^n$.
- $o(d^{n-1})$ is a function of $d$ with $\lim_{d \to \infty} o(d^{n-1})/d^{n-1} = 0$.

Combining (0.8) with Theorem 0.3 gives $\dim \mathcal{P}_d(M) \leq C_n V_M d^n + o(d^n)$ when $\text{Ric}_M \geq 0$.

An interesting feature of these dimension estimates is that they follow from “rough” properties of $M$ and are therefore surprisingly stable under perturbation. For instance, [CM4] proves finite dimensionality of $\mathcal{H}_d$ for manifolds with a volume doubling and a Poincaré inequality, so we also get finite dimensionality for $\mathcal{P}_d$ on these spaces. Unlike a Ricci curvature bound, these properties are stable under bi–Lipschitz transformations. Moreover, these properties make sense also for discrete spaces, vastly extending the theory and methods out of the continuous world. Recently Kleiner, [K], (see also Shalom-Tao, [ST], [T1], [T2]) used, in part, this in his new proof of an important and foundational result in geometric group theory, originally due to Gromov, [G]. We expect that the proof of Theorem 0.3 extends to many discrete spaces, allowing a wide range of applications.

1. Ancient Solutions of the Heat Equation

The next lemma gives a reverse Poincaré inequality for the heat equation.

Lemma 1.1. There is a universal constant $c$ so that if $u_t = \Delta u$, then
\begin{equation}
(1.2)\quad r^2 \int_{B_{r^2} \times [-\frac{r^2}{100},0]} |\nabla u|^2 + r^4 \int_{B_{r^2} \times [-\frac{r^2}{100},0]} u_t^2 \leq c \int_{B_r \times [-\frac{r^2}{2},0]} u^2.
\end{equation}

Proof. Let $Q_R$ denote $B_{R} \times [-R^2,0]$ and $\psi$ be a cutoff function on $M$. Since $u_t = \Delta u$, integration by parts and the absorbing inequality $4ab \leq a^2 + 4b^2$ give
\begin{equation}
(1.3)\quad \partial_t \int u^2 \psi^2 = 2 \int u \psi \Delta u = -2 \int |\nabla u|^2 \psi^2 - 4 \int u \psi \langle \nabla \psi, \nabla u \rangle \leq -\int |\nabla u|^2 \psi^2 + 4 \int u^2 |\nabla \psi|^2.
\end{equation}

Integrating this in time from $-R^2$ to 0 gives
\begin{equation}
(1.4)\quad \int_{t=0} u^2 \psi^2 - \int_{t=-R^2} u^2 \psi^2 \leq \int_{-R^2}^0 \left( -\int |\nabla u|^2 \psi^2 + 4 \int u^2 |\nabla \psi|^2 \right) dt.
\end{equation}
In particular, we get
\begin{equation}
\int_{-R^2}^{0} \int |\nabla u|^2 \psi^2 \, dt \leq \int_{t=-R^2}^{t=0} u^2 \psi^2 \, dt + 4 \int_{-R^2}^{0} \int u^2 |\nabla \psi|^2 \, dt.
\end{equation}
Let $|\psi| \leq 1$ be one on $B_{R/2}$, have support in $B_R$, and satisfy $|\nabla \psi| \leq 2/R$, so we get
\begin{equation}
\int_{Q_{R/2}} |\nabla u|^2 \leq \int_{B_R \times \{-R^2\}} u^2 + \frac{16}{R^2} \int_{Q_R} u^2.
\end{equation}
Next, we argue similarly to get a bound on $u_t^2$. Namely, differentiating, then integrating by parts and using that $u_t = \Delta u$ gives
\begin{equation}
\partial_t \int |\nabla u|^2 \psi^2 = 2 \int \langle \nabla u, \nabla u_t \rangle \psi^2 = -2 \int u_t^2 \psi^2 - 4 \int u_t \psi \langle \nabla u, \nabla \psi \rangle
\end{equation}
\begin{equation}
\leq - \int u_t^2 \psi^2 + 4 \int |\nabla u|^2 |\nabla \psi|^2.
\end{equation}
Integrating (1.7) in time from $-R^2$ to 0 gives
\begin{equation}
\int_{t=0}^{t=-R^2} |\nabla u|^2 \psi^2 - \int_{t=-R^2}^{0} |\nabla u|^2 \psi^2 \leq \int_{-R^2}^{0} \left( - \int u_t^2 \psi^2 + 4 \int |\nabla u|^2 |\nabla \psi|^2 \right) \, dt.
\end{equation}
Letting $\psi$ be as above, we conclude that
\begin{equation}
\int_{Q_{R/2}} u_t^2 \leq \frac{16}{R^2} \int_{Q_R} |\nabla u|^2 + \int_{B_R \times \{-R^2\}} |\nabla u|^2.
\end{equation}
Next, choose some $r_1 \in [4r/5, r]$ with
\begin{equation}
\int_{B_r \times \{-r_1^2\}} u^2 \leq \frac{25}{9r^2} \int_{-r_2}^{0} \left( \int_{B_r} u^2 \right) \, dt = \frac{25}{9r^2} \int_{Q_r} u^2.
\end{equation}
Applying (1.6) with $R = r_1$ and using the bound (1.10) at $r_1$ gives
\begin{equation}
\int_{Q_{2r}} |\nabla u|^2 \leq \int_{Q_{3r}} |\nabla u|^2 \leq \int_{B_{r_1} \times \{-r_1^2\}} u^2 + \frac{16}{r_1^2} \int_{Q_{r_1}} u^2 \leq \frac{20}{r_1^2} \int_{Q_r} u^2.
\end{equation}
For simplicity, $c$ is a constant independent of everything that can change from line to line. It follows from (1.11) that there must exist some $\rho \in [r/5, 2r/5]$ so that
\begin{equation}
\int_{B_{2r} \times \{-\rho^2\}} |\nabla u|^2 \leq \frac{25}{3r^2} \int_{-\frac{1}{2}r^2}^{0} \left( \int_{B_{2r}} |\nabla u|^2 \right) \, dt = \frac{25}{3r^2} \int_{Q_{2r}} |\nabla u|^2 \leq \frac{c}{r^4} \int_{Q_r} u^2.
\end{equation}
Now applying (1.9) with $R = \rho$ and using (1.11) and (1.12) gives
\begin{equation}
\int_{Q_{r/2}} u^2 \leq \frac{16}{\rho^2} \int_{Q_r} |\nabla u|^2 + \int_{B_{\rho} \times \{-\rho^2\}} |\nabla u|^2 \leq \frac{c}{r^4} \int_{Q_r} u^2.
\end{equation}

**Corollary 1.14.** If $\text{Vol}(B_R) \leq C (1+R)^{d_V}$ and $u \in \mathcal{P}_d(M)$, then $\partial_k^4 u \equiv 0$ for $4k > 2d + d_V + 2$. 
Proof. Since the metric on $M$ is constant in time, $\partial_t - \Delta$ commutes with $\partial_x$ and, thus, $(\partial_t - \Delta) \partial_x^j u = 0$ for every $j$. Let $Q_r$ denote $B_R \times [-R^2, 0]$. Applying Lemma 1.1 to $u$ on $Q_r$ for some $r$, then to $u_t$ on $Q_r$, etc., we get a constant $c_k$ depending just on $k$ so that

$$\int_{Q_r_{t=0}} |\partial^k_t u|^2 \leq \frac{c_k}{r^{4k}} \int_{Q_r} u^2 \leq \frac{c_k}{r^{4k}} r^2 \text{Vol}(B_r) \sup_{Q_r} u^2 \leq C c_k r^{2-4k} (1 + r)^{2d + d_V}.$$  

Since $4k > 2d + d_V + 2$, the right-hand side goes to zero as $r \to \infty$, giving the corollary. \qed

Proof of Theorem 4.3. Choose an integer $m$ with $4m > 2k + d_V + 2$. Corollary 1.4 gives that $\partial^m_t u = 0$ for any $u \in \mathcal{P}_{2k}(M)$. Thus, any $u \in \mathcal{P}_{2k}(M)$ can be written as

$$u = p_0 + t p_1 + \cdots + t^{m-1} p_{m-1},$$

where each $p_j$ is a function on $M$. Moreover, using the growth bound $u \in \mathcal{P}_{2k}(M)$ for $t$ large and $x$ fixed, we see that $p_j \equiv 0$ for any $j > k$. (See theorem 1.2 in [1Z] for a similar decomposition under more restrictive hypotheses.)

We will show next that the functions $p_j$ grow at most polynomially of degree $d$. Fix distinct values $-1 < t_1 < t_2 < \cdots < t_k < t_{k+1} = 0$. We claim that the $k + 1$-vectors

$$(1, t_i, t_i^2, \ldots, t_i^k)$$

are linearly independent in $\mathbb{R}^{k+1}$ for $i = 1, \ldots, k + 1$. If this was not the case, then there would be some (non-trivial) $(a_0, \ldots, a_k) \in \mathbb{R}^{k+1}$ that is orthogonal to all of them. But this means that there would be $k + 1$ distinct roots to the degree $k$ polynomial

$$a_0 + a_1 t + \cdots + a_k t^k,$$

which is impossible, and the claim follows. Let $e_j \in \mathbb{R}^{k+1}$ be the standard unit vectors. Since the $(1, t_i, t_i^2, \ldots, t_i^k)$’s span $\mathbb{R}^{k+1}$, we can choose coefficients $b_i^j$ so that for each $j$

$$e_j = \sum_i b_i^j (1, t_i, t_i^2, \ldots, t_i^k).$$

It follows from (1.16) and (1.19) that

$$p_j(x) = \sum_i b_i^j u(x, t_i).$$

Since $u \in \mathcal{P}_{2k}(M)$, (1.20) implies that each $p_j$ is a linear combination of functions that grow polynomially of degree at most $2k$ and, thus, $p_j$ grows polynomially of degree at most $2k$. Since $u$ satisfies the heat equation, it follows that $\Delta p_k = 0$ and

$$\Delta p_j = (j + 1) p_{j+1}.$$  

Thus, we get a linear map $\Psi_0 : \mathcal{P}_{2k}(M) \to \mathcal{H}_{2k}(M)$ given by $\Psi_0(u) = p_k$. Let $\mathcal{K}_0 = \text{Ker}(\Psi_0)$. It follows from this that

$$\dim \mathcal{P}_{2k}(M) \leq \dim \mathcal{K}_0 + \dim \mathcal{H}_{2k}(M).$$

If $u \in \mathcal{K}_0$, then $p_k = 0$ and $\Delta p_{k-1} = 0$, so we get a linear map $\Psi_1 : \mathcal{K}_0 \to \mathcal{H}_{2k}(M)$ given by $\Psi_1(u) = p_{k-1}$. Let $\mathcal{K}_1$ be the kernel of $\Psi_1$ on $\mathcal{K}_0$. It follows as above that

$$\dim \mathcal{K}_0 \leq \dim \mathcal{K}_1 + \dim \mathcal{H}_{2k}(M).$$

Repeating this $k + 1$ times gives the theorem. \qed
Finally, note that our argument actually gives a better constant in the bound for \( \dim \mathcal{P}_d \) since the \( p_j \)'s in (1.16) must grow slower according to the power of \( t \) in front. This refinement does not change the exponent.

**Appendix A. Caloric polynomials**

It is a classical fact that \( \mathcal{P}_d(\mathbb{R}^n) \) consists of caloric polynomials, i.e., polynomials in \( x, t \) that satisfy the heat equation. In this appendix, we compute the dimensions of these spaces.

Given a polynomial in \( x \) and \( t \), define its *parabolic degree* by considering \( t \) to have degree two. Thus, \( x^{m_1}t^{m_2} \) has parabolic degree \( m_1 + 2m_2 \). A polynomial in \( x, t \) is homogeneous if each monomial has the same parabolic degree. Let \( A^n_p \) denote the homogeneous degree \( p \) polynomials on \( \mathbb{R}^n \). The parabolic homogeneous degree \( p \) polynomials \( A^n_p \) are

\[
A^n_p = A^n_p \oplus tA^n_{p-2} \oplus t^2A^n_{p-4} \oplus \ldots
\]

**Lemma A.2.** If \( p \geq n \), then

\[
\frac{1}{(n-1)!} p^{n-1} \leq \dim A^n_p \leq \frac{2^{n-1}}{(n-1)!} p^{n-1}.
\]

**Proof.** To get the upper bound, we use that \( p \geq n \) to get

\[
\dim A^n_p = \frac{(p+n-1)!}{p!(n-1)!} \leq \frac{(2p)^{n-1}}{(n-1)!} \leq \frac{2^{n-1}}{(n-1)!} p^{n-1}.
\]

The lower bound follows similarly since \( \frac{(p+n-1)!}{p!(n-1)!} \geq \frac{p^{n-1}}{(n-1)!} \).

**Lemma A.5.** The dimension of the degree \( p \) homogeneous caloric polynomials is \( \dim A^n_p \).

**Proof.** Observe that \( \partial_t \) and \( \Delta \) map \( A^n_p \) to \( A^n_{p-2} \). Moreover, given any \( u \in A^n_{p-2} \), we have

\[
(\partial_t - \Delta) \left[ t u - \frac{1}{2} t^2 (\partial_t - \Delta) u + \frac{1}{6} t^3 (\partial_t - \Delta)^2 u - \ldots \right] = u.
\]

Therefore, the map \( (\partial_t - \Delta) : A^n_p \to A^n_{p-2} \) is onto. It follows that the dimension of the kernel is \( \dim A^n_p - \dim A^n_{p-2} = \dim A^n_p \), giving the lemma.

The dimension bounds for \( \mathcal{P}_d(\mathbb{R}^n) \) in (0.7) now follow by combining Lemmas A.2 and A.5.

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