The loop-coproduct spectral sequences.

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Abstract
Let $M$ be a closed oriented $d$-dimensional manifold and let $LM$ be the space of free loops on $M$. In this paper, we give a geometrical interpretation of the loop-coproduct and we study its compatibility with the Serre spectral sequence associated to the fibration $\Omega M \to LM \xrightarrow{ev(0)} M$. Then, we show that the spectral sequence associated to the free loop fibration $LN \to LX \to LM$ of some Serre fibration $N \to X \to M$ is a spectral sequence of Frobenius algebra.

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Introduction.
Let $M$ be a 1-connected closed oriented $d$-manifold and let $LM$ be the Hilbert manifold homotopically equivalent to $M^{S^1}$ the space of free loops of $M$ [1]. We denote by $ev(0)$ (respectively $ev(1/2)$) the evaluation map at 0 (respectively 1/2).

$ev(0) : LM \to M \quad \gamma \mapsto \gamma(0)$
$ev(1/2) : LM \to M \quad \gamma \mapsto \gamma(1/2)$

We put

$L_{1/2}M = \{ \gamma \in LM \mid ev(1/2)(\gamma) = ev(0)(\gamma) \}$
and
\[ \Omega_{1/2}M = \{ \gamma \in \Omega M \quad / \quad ev(1/2) = * \} \]
where * denotes the base point of M. Then we have the pull back diagrams:

\[ \begin{array}{ccc}
L_{1/2}M & \xrightarrow{j} & LM \\
\downarrow{ev(1/2)} & & \downarrow{ev(0) \times ev(1/2)} \\
M & \xrightarrow{\Delta} & M \times M
\end{array} \]

\[ \begin{array}{ccc}
\Omega_{1/2}M & \xrightarrow{i} & \Omega M \\
\downarrow{ev(1/2)} & & \downarrow{ev(1/2)} \\
\ast & \xrightarrow{i} & M
\end{array} \]

where \( \Delta \) is the diagonal embedding and \( i \) the canonical inclusion of the base point of M. This diagrams define the d-codimensionnal embeddings \( j \) and \( i \).

We need again the following notations:
\[ LM \times_M LM = \{ (\alpha, \beta) \in LM \times LM \quad / \quad \alpha(0) = \beta(0) \} \]

We have the following commutative pull-back diagram:

\[ \begin{array}{ccc}
LM \times_M LM & \xrightarrow{\Delta} & LM \times LM \\
\downarrow{ev_\infty} & & \downarrow{ev(0) \times ev(0)} \\
M & \xrightarrow{\Delta} & M \times M
\end{array} \]

where \( ev_\infty \) denotes the evaluation at 0:
\[ LM \times_M LM \to M \quad (\alpha, \beta) \mapsto \alpha(0) = \beta(0) \]

We define
\[ \gamma : LM \times_M LM \to L_{1/2}M \]
\[ (\alpha, \beta) \mapsto \alpha \ast \beta \]
with \( \alpha \ast \beta = \alpha(2t) \) if \( t \in [0, 1/2] \) and \( \beta(2t-1) \) if \( t \in [1/2, 1] \) the reparametrisation of a couple of same basepoint loops. We remark that \( \gamma \) is an homeomorphism and we have
\[ \gamma^{-1} : L_{1/2}M \to LM \times_M LM \]
\[ \eta \mapsto (\eta(t/2), \eta(1/2 + t/2)) \]
We denote \( \gamma_\omega \) (resp. \( \gamma^{-1}_\omega \)) the restriction of \( \gamma \) (resp. \( \gamma^{-1} \)) to \( \Omega M \times \Omega M \) (resp. \( \Omega_{1/2}M \)). We denote by \( \text{comp} : LM \times_M LM \to LM \) the composition of free loops and \( \text{comp}_\omega \) its restriction to pointed loops. Remark that \( \text{comp} = j \circ \gamma \) for free loops and \( \text{comp}_\omega = i \circ \gamma_\omega \) for pointed loops.
Assume that $k$ is a fixed ring. The following homology groups are assumed with coefficients in $k$. M.Chas and D.Sullivan have constructed a product

$$P : H_*(LM \times LM) \to H_{*-d}(LM)$$

called the loop product such that the desuspended homology of $LM$ namely $H_*(LM) =: H_{*+d}(LM)$ is a commutative graded algebra. With our notations, $P =: \text{comp} \circ \Delta$.

In [4], R.Cohen and V.Godin have constructed a coproduct

$$\Phi : H_*(LM) \to H_{*-d}(LM \times LM).$$

Since $\tilde{j} : L_{1/2}M \to LM$ is a smooth finite codimensionnal embedding of Hilbert manifolds, we can define $\tilde{j} ! : H_*(LM) \to H_*(L_{1/2}M)$ (see [3], [5] or [6] for details). Furthermore, considering $\gamma$ as a 0-codimensionnal smooth embedding, we can also consider

$$\gamma ! : H_*(L_{1/2}M) \to H_*(LM \times LM).$$

Remark: The coproduct on based loop space $\text{comp}_\omega$ is the coproduct defined by Sullivan in [9].

The remaining of this paper consist to to prove the following results:

**Proposition 1:** Let $h : X \to Y$ a diffeomorphism between smooth Hilbert closed connected manifolds without boundary. We have: $h ! = h_*^{-1}$

**Corollary:** $\gamma ! = \gamma_*^{-1}$ and $\gamma_\omega ! = \gamma_\omega_*^{-1}$

So that we can define $\text{comp}_\gamma = : \gamma ! \circ \tilde{j} ! = \gamma_*^{-1} \circ \tilde{j} !$ (by Proposition 1). As the same, we define $\text{comp}_\omega = : \gamma_\omega ! \circ \tilde{i} ! = \gamma_\omega_*^{-1} \circ \tilde{i} !$. From the definition of $\Phi$, we obtain immediately:

**Theorem 1:** According to the preceding notations, $\Phi = \tilde{\Delta} \circ \text{comp}_\gamma$.

Remark that the loop-coproduct $\Phi = \tilde{\Delta} \circ \text{comp}_\gamma$ can be thinned as Poincaré-dual of the loop-product $P = \text{comp} \circ \Delta$.

Now, let us consider $\{E^*_*, [ev(0)]\}$ the Serre spectral sequence associated to the fibration $\Omega M \to LM \xrightarrow{ev(0)} M$.

**Theorem 2:** The spectral sequence $\{E^*_*, [ev(0)]\}$ is comultiplicative and converges to the coalgebra $(H_*(LM), \Phi)$. At the $E^2$-level, $E^2(\Phi) = \Delta_* \otimes \text{comp}_\omega$.

Theorem 2 explain the interest to compute $\text{comp}_\omega$ in order to do some computations of $\Phi$. So we have:

**Theorem 3:**

The pointed loop-coproduct $\text{comp}_\omega$ is zero.

Theorem 2 and Theorem 3 proves that the loop-coproduct induced on the Serre spectral sequence associated to the fibration $\Omega M \to LM \xrightarrow{ev(0)} M$ vanishes at the $E^\infty$-level. In [2], the computation of this loop-coproduct is done for $H_*(LCP^n; \mathbb{Q})$, using rational homotopy theory. The two authors proves that
in this case, the loop-coproduct is non-zero. This indicates that the extension issues in our spectral sequence are not trivial.

Now, let \( N \to X \xrightarrow{p} M \) be a locally trivial Serre fibration satisfying hypothesis of proposition 1 of \( \mathbb{6} \) namely:

a) \( N \) (respectively \( M \)) is a finite dimensional smooth closed oriented manifold of dimension \( n \) (respectively \( m \)),

b) \( M \) is a connected space and \( \pi_1(M) \) acts trivially on \( H_*(N) \).

Then, we state the following theorem:

**Theorem 4.** Under the above hypothesis, the loop-coproduct \( \Phi_X \) induces on the Serre spectral sequence associated to the fibration \( LN \to LX \xrightarrow{p} LM \) a structure of coalgebra with a coproduct of degree \( -(m+n) = \dim X \). The tensor product of coalgebra \( \langle H_*(N), \Phi_N \rangle \otimes \langle H_*(B), \Phi_B \rangle \) is a sub-coalgebra of \( E^2[p] \).

Let us recall the following theorem of \( \mathbb{6} \):

**Theorem B.** Under hypothesis of Proposition 1, the \((m,n)\)-regraded Serre spectral sequence \( \{ E^r[Lp] \}_{r \geq 0} \) of the Serre fibration \( LN \to LX \xrightarrow{p} LM \) is a multiplicative spectral sequence which converges to the algebra \( \mathbb{H}_*(LX) \). Moreover the tensor product of graded algebras : \( \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN) \) is a subalgebra of \( E^2[Lp] \). In particular if \( H_*(LM) \) is torsion free then \( E^2[p] = \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN) \).

As a immediate byproduct of theorem 4 and of this theorem B, we get:

**Corollary.** The regraded Serre spectral sequence associated to the fibration \( LN \to LX \xrightarrow{p} LM \) is a spectral sequence of Frobenius algebra. Moreover the tensor product of graded Frobenius algebras : \( \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN) \), is a sub-Frobenius algebra of \( E^2[Lp] \).

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The paper is organized as follows:

1) Definition of the loop coproduct
2) Proof of Proposition 1
3) Proof of Theorem 1
4) Proof of Theorem 2
5) Proof of Theorem 3
6) Proof of Theorem 4

**1. Definition of the loop coproduct.**

Following \( \mathbb{1} \) and \( \mathbb{2} \) the loop product (respectively the loop coproduct) is a particular case of an "operation”

\[
\mu_\Sigma : H_*(LM)^{\otimes p} \to H_*(LM)^{\otimes q}
\]
with \( p = 2, q = 1 \) (respectively \( p = 1, q = 2 \)) that will define a positive boundary TQFT. Here \( \Sigma \) denotes an oriented surface of genus 0 with a fixed parametrization of the \( p + q \) boundary components:

\[
(S^1)\coprod^p \in \partial_{in} \Sigma \hookrightarrow \Sigma \leftarrow \partial_{out} \Sigma \overset{out}{\longrightarrow} (S^1)\coprod^q
\]

Applying the functor \( \text{Map}(-, M) \) on gets the diagram:

\[
(1) \quad (LM)^{\times p} \overset{\in} {\longrightarrow} \text{Map}(\Sigma, M) \overset{out}{\longrightarrow} (LM)^{\times q}
\]

Diagram (1) is homotopically equivalent to the diagram

\[
(2) \quad (LM)^{\times p} \overset{\rho_{in}} {\longrightarrow} \text{Map}(c, M) \overset{\rho_{out}} {\longrightarrow} (LM)^{\times q}
\]

where \( c \) denotes a reduced Sullivan chord diagram with marking which is associated to \( \Sigma \) [4]-§2.

In order to define the loop coproduct we restrict to the case \( p = 1 \) and \( q = 2 \). Then \( \Sigma \) is the oriented surface of genus 0 with one incoming and two outgoing components of the boundary. The associated Sullivan chord diagram with markings is determined by the pushout diagram

\[
\begin{array}{c}
\ast \coprod \ast \\
\downarrow & \downarrow \\
S^1 & c
\end{array}
\]

In particular, \( c \simeq S^1 \vee S^1 \), \( S^1 \overset{in}{\rightarrow} c \) is homotopic to the folding map \( \nabla : S^1 \to S^1 \vee S^1 \) and \( S^1 \coprod S^1 \overset{out}{\rightarrow} c \) is homotopic to the natural projection \( S^1 \coprod S^1 \to S^1 \vee S^1 \). Diagram (2) then reduces to a commutative diagram of fibrations:

\[
\begin{array}{ccc}
LM \times LM & \overset{\delta_{out}} {\longrightarrow} & LM \times M \\
\downarrow & & \downarrow ev(0) \times ev(0)
\end{array}
\]

\[
\begin{array}{ccc}
LM \times M & \overset{\delta_{in}} {\longrightarrow} & LM \\
\downarrow ev_\infty & & \downarrow ev(0) \times ev(1/2)
\end{array}
\]

\[
\begin{array}{ccc}
M \times M & \overset{\Delta} {\longrightarrow} & M \times M
\end{array}
\]

By definition (see [4]), \( \Phi = \delta_{out} \circ \delta_{in}! \).

2. **Proof of Theorem 1**

Geometrically, \( \delta_{out} \) is the inclusion of composable loops in \( LM \times LM \), namely \( \Delta \). The other map, \( \delta_{in} \), can be thought as the inclusion of \( \text{map}(8, M) \) in \( LM \). Then this coproduct \( \phi \) can be understood as follows: the space \( L_{1/2}M \) is the space of decomposable loops which embeds in \( LM \). This embedding
is $d$-codimensional so that we can define the shriek map of the embedding. Then, we decompose the loops of the space of decomposable loops.

The top line of the preceding diagram can be decomposed as follows:

\[(\ast) \quad LM \times LM \xrightarrow{\Delta \circ \delta_{\text{out}}} LM \times M \xrightarrow{\delta} L_{1,2}M \xrightarrow{\tilde{\gamma}} LM.\]

Observe that $\delta_{\text{in}} = \tilde{\gamma} \circ \gamma = \text{comp}$ then $\Phi = \delta_{\text{out}*} \circ \delta_{\text{in}} = \tilde{\Delta} \circ \text{comp}$. This proves Theorem 1.

\[\square\]

3. Proof of Proposition 1

By definition, $h_1$ is given by the following composition (see [6]):

\[H_*(Y) \xrightarrow{\text{inc}_*} H_*(Y, Y-h(Y)) \xrightarrow{\text{exc}} H_*(\text{Tube} \, h/\partial \text{Tube} \, h) \xrightarrow{\text{exp}} H_*(\nu_Y, \partial \nu_Y) \xrightarrow{\pi_*} H_*(X)\]

where $\text{inc}_*$ denotes the inclusion of pair, $\text{exc}$ is the excision isomorphism, $\text{exp}$ is the homeomorphism given by the exponential between the tubular neighbourhood of the embedding and its normal bundle and $\pi_* (\tau \cap -)$, where $\pi$ denotes the projection of the normal bundle and $\tau \in H^*(\nu_Y, \partial \nu_Y)$ is the Thom class of the embedding, is the Thom isomorphism. In the case we consider, the normal bundle of $h$ is $\nu_Y := \ast \to Y \xrightarrow{h^{-1}} X$. Then, $\text{Tube} \, h = Y = E(\nu_Y)$ and $\partial \text{Tube} \, h = \emptyset = \partial E(\nu_Y)$ so that three first application of the definition of $h_1$ are identity. We remark that the projection of the normal bundle $\nu_Y$ is only $h^{-1}$ and that $\tau$ lies in $H_0(\nu_Y, \partial \nu_Y) = H_0(Y)$. Then, the Thom isomorphism is only $h_*^{-1}$, this achieve the proof Proposition 1.

\[\square\]

4. Proof of Theorem 2

For the reader convenience, we recall here the definition of a fiber embedding of [6] and the main result of [6].

**Definition** We define a fiber embedding $(f, f^B)$ as a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
p \downarrow & & \downarrow p' \\
B & \xrightarrow{f^B} & B'
\end{array}
\]

where

\[
\begin{array}{l}
\text{a) } X, X', B \text{ and } B' \text{ are connected Hilbert manifolds without boundary} \\
\text{b) } f \text{ (respectively } f^B) \text{ is a smooth embedding of finite codimension } k_X \text{ (respectively } k_B) \\
\text{c) } p \text{ and } p' \text{ are locally trivial fibrations} \\
\text{d) } \text{ for some } b \in B \text{ the induced map} \\
\quad f^F : F := p^{-1}(b) \to p'^{-1}(f(b)) := F' \\
\quad \text{ is an embedding of finite codimension } k_F \\
\text{e) } \text{ embeddings } f, f^B \text{ and } f^F \text{ admit Thom classes.}
\end{array}
\]
First part of the main result. Let \( f : X \to X' \) be a fiber embedding as above. For each \( n \geq 0 \) there exist filtrations \( \{ 0 \} \subset F_0C_n(X) \subset F_1C_n(X) \subset \ldots \subset F_nC_n(X) = C_n(X) \) and a chain representative \( f_i : C_\ast(X) \to C_\ast-k\chi(X) \) of \( f : H_\ast(X') \to H_\ast-k\chi(X) \) satisfying: \( f_i(F,C_n(X')) \subset F_{s-k\beta}C_{s-k\chi}(X') \).

Let \( \{ E^r[p]\}_{r \geq 0} \) and \( \{ E^r[p']\}_{r \geq 0} \) be the spectral sequence induced by the above filtration. Second part of the main result. The chain map \( f \) induces a homomorphism of bidegree \( (-k\beta,-k\delta) \) between the associated spectral sequences \( \{ E^r(f_i) \} : \{ E^r(p') \}_{r \geq 0} \to \{ E^r(p) \}_{r \geq 0} \). There exists a chain representative \( f^B : C_\ast(B') \to C_{s-k\delta}B(B) \) (respectively \( f^F : C_\ast(F') \to C_{s-k\delta}F(F) \)) of \( f_\delta^B : H_\ast(B') \to H_{s-k\beta}B(B) \) (respectively of \( f_\delta^F : H_\ast(F') \to H_{s-k\beta}F(F) \)) such that

\[
\{ E^2(f_i) \} = H_\ast(f^B; H_\ast(f^F)) : E^2_{s,t}(p) = H_\ast(B'; H_\ast(F')) \to E^2_{s-k\beta,t-k\delta}(p) = H_{s-k\beta}(B; H_{t-k\delta}(F)),
\]

where \( H(\ast) \) denote the usual system of local coefficients. These spectral sequences are the Serre spectral sequences of the fibration.

We use the description of the loop coproduct \( \Phi \) given in the proof of Theorem 1. Thus we obtain the following commutative diagram of fibrations, where the central column is the composite \((\ast)\):

\[
\begin{array}{cccccc}
\Omega M & \longrightarrow & LM & \xrightarrow{ev(0)} & M \\
\downarrow i & & \downarrow j & & \downarrow = \\
\Omega_{1/2} M & \longrightarrow & L_{1/2} M & \xrightarrow{ev(0)} & M \\
\downarrow \gamma^{-1} & & \downarrow \gamma^{-1} & & \downarrow = \\
\Omega M \times \Omega M & \longrightarrow & LM \times M & \xrightarrow{ev_{\infty}} & M \\
\downarrow = & & \downarrow \Delta & & \downarrow \\
\Omega M \times \Omega M & \longrightarrow & LM \times LM & \xrightarrow{\phi(0) \times ev(0)} & M \times M
\end{array}
\]

We remark that \( (\tilde{j}, id) \) is a fiber embedding in the sense of [3], more precisely:

a) \( LM, L_{1/2}M, M \) are connected Hilbert manifolds without boundary.
b) \( j \) (resp. \( id \)) is a smooth embedding of finite codimension \( d \) (resp. \( 0 \)).
c) \( ev(0) \) and \( ev(1/2) \) are locally trivial fibrations (\( ev(1/2) \) is locally trivial since it is homeomorphic to \( ev_\infty \) which is locally trivial by lemma 3 of [2]).
d) \( i \) is an embedding of finite codimension \( d \).
e) \( j, id \) and \( i \) admit Thom classes. If we denote \( \tau \) the Thom class of \( j \) and \( u : \Omega M \to LM \) the canonical embedding, then, the Thom class of \( i \) is \( u^*(\tau) \), [2].
Then, we apply the main result of [6] to prove that \( \tilde{j}! \) induces a morphism of spectral sequences:

\[
E_{s,*}^*(\tilde{j}!): E_{s,*}^*[ev(0)] \to E_{s-d,*}^*[ev(1/2)].
\]

By naturality of the Serre spectral sequences, \( \gamma^{-1} \) and \( \tilde{\Delta} \) induce a morphism of spectral sequences:

\[
E_{s,*}^*(\gamma^{-1}): E_{s-d,*}^*[ev(1/2)] \to E_{s-d,*}^*[ev_{\infty}]
\]

and:

\[
E_{s,*}^*(\tilde{\Delta}): E_{s-d,*}^*[ev_{\infty}] \to E_{s-d,*}^*[ev(0) \times ev(0)].
\]

Then, composing this morphisms, we define a coproduct

\[
E^*(\Phi): E_{s,*}^*[ev(0)] \to E_{s-d,*}^*[ev(0) \times ev(0)]
\]

induced by \( \Phi \). On the base of the fibration \( ev(0) \), \( \Phi \) induces \( \Delta^* \). On the fiber, \( \Phi \) induces \( comp_{\omega!} \) such that at the \( E^2 \)-level, \( E^2(\Phi) = \Delta^* \otimes comp_{\omega!} \).

\[\square\]

5. Proof of Theorem 3

Consider the following pull-back diagram:

\[
\begin{array}{ccc}
\Omega_{1/2}M & \xrightarrow{=} & \Omega_{1/2}M \\
\downarrow & & \downarrow \\
\Omega_{1/2}M & \xrightarrow{i} & \Omega M \\
\downarrow & & \downarrow_{ev(1/2)} \\
\ast & \xrightarrow{i} & M
\end{array}
\]

where \( i \) and \( \tilde{i} \) are defined in the introduction. From Lemma 3 of [3], this diagram is a fiber embedding thus \( \tilde{i}! \) induces a morphism of Serre spectral sequences: \( E^*(\tilde{i}) : E_{s,*}^*[ev(2)] \to E_{s-d,*}^*[ev(1)] \) (resp. \( E_{s,*}^*[ev(1)] \) (resp. \( E_{s,*}^*[ev(2)] \)) denotes the Serre spectral sequence associated to the left fibration of the above diagram (resp. the Serre spectral sequence associated to the right fibration of the above diagram). Moreover, \( \tilde{i}_* \) induces a morphism of Serre spectral sequences [3]:

\[
E^*(\tilde{i}_*) : E_{s,*}^*[ev(1)] \to E_{s,*}^*[ev(2)].
\]

We know that \( \tilde{i}_* \circ \gamma_* = comp_* \) is onto (\( comp_* \) has a unit). Since \( \gamma_* \) is an isomorphism, this proves that \( \tilde{i} \) is onto. The spectral sequence \( E_{s,*}^*[ev(1)] \) collapses at the \( E^2 \)-level, it has only one column at this level:

\[
E_{s,*}^*[1] \simeq H_*(\Omega_{1/2}M), \ n \geq 2.
\]
Then, since $i_\ast$ is onto, $E^\infty(\tilde{i}_\ast)$ is onto. Moreover, $E^2(\tilde{i}_\ast) = i_\ast \otimes \text{id}$ thus $\text{Im}(E^2(\tilde{i}_\ast)) = E^2_{0, \ast}(2)$ so that $\text{Im}(E^2(\tilde{i}_\ast)) \subset E^\infty_{0, \ast}(2)$ that is why $E^\infty_{\ast, \ast}(2) = \text{Im}(E^\infty(\tilde{i}_\ast)) \subset E^\infty_{0, \ast}(2)$.

We have proved that $E^\infty_{\ast, \ast}(2) = E^\infty_{0, \ast}(2)$.

But $E^2(\tilde{\imath}_1) = \imath_1 \otimes \text{id}$ and $\tilde{\imath}_1$ is non zero only in degree $d$ with values in degree 0 ($\imath_1$ send the fundamental class of $H_d(M)$ on a generator of $H_0(M)$ and is zero elsewhere). Then, at the $E^2$-level of $E^*_{\ast, \ast}(2)$, only the column of abscisse $d$ has a non zero image by $E^2(\tilde{\imath}_1)$. Consequently, at the aboutment, only $E^\infty_{d, \ast}(2)$ has a non zero image by $E^\infty(\tilde{\imath}_1)$. We have shown that $E^\infty_{d, \ast}(2) = 0$ thus $\text{Im}(E^\infty(\tilde{\imath}_1)) = 0$.

\[ \square \]

6. Proof of Theorem 4

**Notations.** In this section, we use the same notations as in the introduction but we add a subscript to indicate which manifold we refer (for example, $\Phi_X$ denotes the loop-coproduct on $H_\ast(LX)$).

First, we need the following lemma:

**Lemma.** The commutative diagram $\begin{array}{c} LN \ar[r] & LX \ar[r] & LM \\ L_{1/2}N \ar[u] & L_{1/2}X \ar[u] & L_{1/2}M \ar[u] \end{array}$ is a fiber embedding.

Proof of the Lemma. We check that the diagram verify all the properties of the definition of a fiber embedding. The point a) of the definition is verified in proof of Theorem 2. The same holds for the point b) with $i_X$ and $i_M$ smooth embedding of codimension $x = m + n$ and $m$. The point c) comes from the fact that the fibration $LN \times N LN \ar[r] & LX \times_X LX \ar[r]^{Lp} & LM \times_M LM$ is locally trivial (see [4, 3.3]) and since it is homeomorphic to the fibration $L_{1/2}N \ar[r] & L_{1/2}X \ar[r] & L_{1/2}M$, this last fibration is locally trivial. The point d) comes directly from the diagram of the Lemma. Since their exists Thom class for the embedding $L_{1/2}M \hookrightarrow LM$ (see proof of theorem 2 part 4), the embeddings $L_{1/2}N \hookrightarrow LN$, $L_{1/2}X \hookrightarrow LX$ and $L_{1/2}M \hookrightarrow LM$ admit Thom class. This proves the point e).

\[ \square \]
Proof of Theorem 4: we consider the following commutative diagram:

\[
\begin{align*}
LN & \longrightarrow LX \longrightarrow LM \\
\downarrow j_N & \downarrow j_X \downarrow j_M \\
L_{1/2}N & \longrightarrow L_{1/2}X \longrightarrow L_{1/2}M \\
\downarrow h_N & \downarrow h_X \downarrow h_M \\
LN \times NLN & \longrightarrow LX \times XLX \longrightarrow LM \times MLM \\
\downarrow \Delta_N & \downarrow \Delta_X \downarrow \Delta_M \\
LN \times LN & \longrightarrow LX \times XLX \longrightarrow LM \times MLM
\end{align*}
\]

It comes from the Lemma and the main result of [6] that \(j_X!\) induces a morphism of spectral sequence of degree \(-x\) so that \(\Phi_X = \Delta_{X*} \circ h_{X*} \circ i_X!\) induces a morphism of spectral sequence \(E^*[\Phi_X]\). This morphism induces on the homology of \(M\) \(\Phi_M = \Delta_{M*} \circ h_{M*} \circ j_M!\) and on the homology of \(N\) \(\Phi_N = \Delta_{N*} \circ h_{N*} \circ j_N!\). This achieve the proof of Theorem 4.

\[\square\]

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