Simulation of separated flow past an inclined and normal plates by a discrete vortex method

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Abstract. Unsteady separated flow past a normal or an inclined flat plate is simulated using the discrete vortex method. The plate is replaced by attached discrete vortices and the separated flow motion behind the plate is modeled by free discrete vortices moving with a local velocity. The intensities of attached vortices at each moment of time are determined from the condition of impermeability of the streamlined body surface, as well as the condition that the total intensity of the attached and free vortices is equal to zero. A formula is introduced for determining the pressure coefficient in the position of the attached vortices, and it is obtained from the pressure in an unsteady-state flow by Cauchy-Lagrange function. The forces exerted on the plate at each instant of time determined from the generalized Blasius theorem.

The calculations have shown, that by choosing an integration step and a quantity of the plate breakdowns, it is possible to obtain a satisfactory convergence with the experimental results on the normal force coefficient. But this results in some discrepancy in the Strouhal number of vortex shedding, especially at the small angles of attack. Replacement of the plate by the attached vortices leads to a passage of the free vortices through it, which leads to a discontinuous change in its aerodynamic characteristics. To avoid the latter, it is necessary to introduce a condition, for example, to exclude a normal velocity component of the free vortex near the plate. The introduction of the initial disturbance affects to some extent a subsequent development of the vortices and, thus, the dependence of the normal force coefficient on a time period.

Key words: discrete vortex method, flow past a flat plate, Strouhal number.

1 Introduction

The discrete vortex calculation method makes it possible to model detached flows-around of the various bodies by an ideal incompressible medium. The given method makes easier the solution of such problem due to linearity of the continuity equation and the possibility of using the flow superpositions. The development and wide use of the discrete vortex method is well described in a review by T. Sarpkaya [1] and Leonard [2]. In the numerous works of S.M. Belotserkovsky [2-5] this method for various problems of flow-around is described in detail.

Carrying out any calculations by the discrete vortex method in a general way is as follows. The contour of the streamlined body is replaced by attached discrete vortices, and the separated flow motion behind the body is modeled by free discrete vortices moving with a local velocity of the medium. The intensities of attached vortices at each moment of time are determined from the condition of impermeability of the streamlined body surface, as well as the condition that the total intensity of the attached and free vortices is equal to zero.

As a rule, during a flow-around of the flat plates, their surfaces are replaced by a continuous vortex sheet with the use of conformal mapping of the planes [6-8]. The intensities of attached vortices are defined as a half of the squared velocity at the separation points, and the positions of the vortex
vanishing points are determined from the Kutta condition, i.e. from the condition of finiteness of velocity at the edges of the plate.

In the work of S.M. Belotserkovsky a flow-around of the plate with its replacement by the attached vortices is considered. The intensities of the breaking-loose vortex are found from the impermeability conditions at the control points. A formula is introduced for determining the pressure coefficient in the position of the attached vortices, and it is obtained from the pressure in an unsteady-state flow by Cauchy-Lagrange function.

In the present work, a flow-around of the plate is modeled according to the method described by S.M. Belotserkovsky. Determination of the aerodynamic force coefficient was made by an impulse method, according to the formula given in the article of Bryson [9] and according to the formula of S.M. Belotserkovsky, obtained from Cauchy-Lagrange function.

The discrete vortex method require less computing power than classical computational fluid dynamics and therefore remains actual for engineering calculations. Various modifications of the discrete vortex method can well predict unsteady separated flows behind bluff bodies [10-16]. In this study, results are generated for angles of attack \( \alpha = 90^\circ, 60^\circ, 40^\circ \) for comparison with available experimental and calculated results. The method is applicable to other angles as well.

2 Materials and methods

2.1 Calculation methodology

Let the \( a \)-wide plate (figure 1) move in an ideal incompressible medium with the velocity \( V_\infty \) at an angle of attack \( \alpha \). It is required to determine the force coefficients acting on the plate by an impulse method for the subsequent calculation of the wind load.

Let us compose a system of the linear algebraic equations \( AX = B \) to determine the intensities of the attached vortices \( \Gamma_j \) from the impermeability condition at the control points, i.e. at \( \nu_\infty = 0 \). The velocity matrix \( A \) (1) from the attached vortices is filled in once. The velocity induced by the \( j \)-th attached vortex at the \( i \)-th control point

\[
A = (a_{ij}) = \frac{1}{2\pi(x^T - x_j)},
\]

The column matrix \( X = (\Gamma_j) \) represents the unknown intensities of the attached vortices. Let us compose a column matrix \( B \), the free terms of which are known and represent a projection of the velocity from the incoming flow and free vortices onto the normal at each control point. The complex velocity (2) induced by free vortices \( \gamma_j \) at the \( i \)-th control point, is written as

\[
v_i = \text{conj} \left[ \sum_{j=2}^{m} \frac{-ig_j}{2\pi(x^T - x_j)} \right],
\]
then
\[ B = (b_j) = -V \cdot \sin \alpha - \ln(v_i). \tag{3} \]

The resulting matrix \( A \) has the dimension \( k \times (k-1) \) (since the number of attached vortices is 1 more than the control points), to solve it, it is necessary to introduce an additional line to the system \( AX = B \):
\[ \sum_{j=0}^{k} a_{ij} \Gamma_j + \sum_{\gamma=2}^{m} g_{\gamma} = 0 \tag{4} \]

This condition follows from the theorem on the constancy of circulation along a contour that encompasses the attached vortices on a plate and the detached ones.

Having solved the resulting system:
\[ a_{00} \Gamma_0 + a_{01} \Gamma_1 + \ldots + a_{0k} \Gamma_k = b_0, \]
\[ a_{10} \Gamma_0 + a_{11} \Gamma_1 + \ldots + a_{1k} \Gamma_k = b_1, \]
\[ \vdots \]
\[ a_{k-1} \Gamma_0 + a_{k-1} \Gamma_1 + \ldots + a_{k-1} \Gamma_k = b_{k-1}, \]
\[ \Gamma_0 + \Gamma_1 + \ldots + \Gamma_k = -\sum g_{\gamma} \tag{5} \]

by the Gauss method, we find the intensities \( \Gamma_j \) of the attached vortices replacing the plate. According to the Chaplygin-Zhukovsky [3] hypothesis, the velocities at the edges of the plate should be finite. This is ensured by a tangential downstream of the flow from the edges, i.e. at each calculated moment of time, the outermost attached vortices are taken as the free ones. This allows one to find circulations of the free (detached) vortices at each calculated instant of time.

To determine a position of the free vortices at the next instant of time it is necessary to solve the system of ordinary differential equations of motion of these vortices:
\[ \frac{dz_{\gamma}}{dt} = v_{\gamma}. \tag{6} \]

These equations (6) are solved by the Euler method, while ensuring a numerical diffusion of the vortices:
\[ z_{\gamma}^{i+\Delta t} = z_{\gamma}^{i} + v_{\gamma} \cdot \Delta t, \tag{7} \]

where \( \Delta t \) is a time step, \( v_{\gamma} \) is a velocity of a free vortex, it is defined as the conjugate derivative of the complex flow potential \( F(z) \):
\[ F(z) = -\frac{k}{2\pi} \sum_{j=0}^{k} \ln(z - x_j) - \sum_{\gamma=2}^{m} \frac{ig_{\gamma}(z - z_{\gamma})}{2\pi} + V_{\gamma}e^{-i\alpha}, \quad v_{\gamma} = \text{conjugate}\left( \frac{dF}{dz} \right)_{z = z_{\gamma}}. \tag{8} \]

After determining a new position of the free vortex, the right-hand side (3) of the equation system (5) is re-formed, new intensities of the attached vortices are found and according to (7) their subsequent positions are determined. Thus, the expressions (3), (5) and (7) form a closed cycle of the numerical method of discrete vortices.

The force, acting on the plate at each instant of time, can be determined by the theorem on the change in the impulse of force [9], [17]:
\[ X + iY = \frac{d}{dt} \left( \sum_{j=1}^{k-1} \Gamma_j \cdot x_j + \frac{\Gamma_0 \cdot x_0}{\Delta t} + \frac{\Gamma_k \cdot x_k}{\Delta t} \right) - ip \sum_{j=1}^{k-1} \Gamma_j \cdot \hat{s}_j \tag{9} \]

The first term in the brackets is a force acting on the attached vortex due to a change in its intensity over the interval of time \( \Delta t \). After each separation of the outermost vortices to the left and to the right of the plate, we place new vortices in their place to ensure the Chaplygin-Zhukovsky condition. Therefore, the force from the newly introduced vortices at the end of the interval \( \Delta t \) is taken into account by the expressions \( \Gamma_0 \cdot x_0/\Delta t \) and \( \Gamma_k \cdot x_k/\Delta t \) in (9). According to figure 1, the origin of coordinates coincides with a position of the outermost vortex on the left (\( x_0 = 0 \)) and \( \Gamma_0 \cdot x_0/\Delta t \) will not affect the force. The last expression in (9) takes into account the force, due to the presence of velocity at the points of location of the attached vortices:
Introducing the following dimensionless variables $x' = x/a$, $t' = V_\infty t/a$, $\Gamma' = \Gamma/(V_\infty a)$ to expression (9), the force coefficients take the form:

$$c_x + ic_y = \frac{2(X + iY)}{pV_\infty^2 a} = i\left(\sum_{j=1}^{k-1} \Gamma_j \cdot x'_j - \sum_{j=1}^{k-1} \Gamma'_j \cdot x'_j + \frac{\Gamma_k}{\Delta'}\right).$$  \hspace{1cm} (11)

The aerodynamic force coefficients can also be obtained by integrating the pressure coefficient using the Cauchy-Lagrange formula:

$$c_p = 1 - \frac{u^2}{V_\infty^2} - \frac{2}{V_\infty^2} \frac{\partial \phi}{\partial t}.$$  \hspace{1cm} (12)

Let us replace the plate with a continuous vortex layer and calculate the load at a certain point $s$ (figure 1):

$$\Delta c_p = c_{p-} - c_{p+} = -\frac{u_2^2}{V_\infty^2} + \frac{u_1^2}{V_\infty^2} - \frac{2}{V_\infty^2} \left(\frac{\partial \phi_-}{\partial t} - \frac{\partial \phi_+}{\partial t}\right).$$  \hspace{1cm} (13)

Let us convert $\Delta c_p$ according to [3]:

$$\frac{1}{V_\infty^2}(u_2^2 - u_1^2) = \frac{1}{V_\infty^2}(u_+ - u_+)(u_+ + u_-).$$  \hspace{1cm} (14)

The velocity circulation in the vicinity of the point $s$ is equal to the intensity of the attached vortex:

$$\Gamma_j = \int_{V_\infty} \omega dx = (u_- - u_+)\Delta.$$  \hspace{1cm} (15)

Velocity in the vortex layer:

$$(u_+ + u_-)/2 = u.$$  \hspace{1cm} (16)

The circulation along the contour, crossing the vortex layer at a certain point $s$ and covering all the vortices on the right, is equal to the difference of potentials (figure 1):

$$\Gamma = \phi_+ - \phi_- = \sum_{j=1}^{k-1} \Gamma_j.$$  \hspace{1cm} (17)

Using the obtained expressions (15)-(17) the pressure coefficient at the location of the attached vortex is reduced to the form:

$$\Delta c_p = -\frac{1}{V_\infty^2} \frac{\Gamma_j}{\Delta t} + 2 \frac{\Delta \Gamma + \Gamma_k}{\Delta t}.$$  \hspace{1cm} (18)

Or in the dimensionless variables ($u' = u/V_\infty$):

$$\Delta c_p = -2 \Gamma_j u'(k-1) + 2(\Delta \Gamma' + \Gamma_k')/\Delta t'.$$  \hspace{1cm} (19)

Summing $\Delta c_p$ along the length of the plate and taking into account that the edges of the plate end with the control points, we obtain a normal force coefficient:

$$c_y = \frac{N}{a} \sum_{j=1}^{k-1} \Delta c_{p_j} = 2 \sum_{j=1}^{k-1} \left(-\Gamma_j u' + \frac{\Delta \Gamma' + \Gamma_k'}{\Delta t(k-1)}\right).$$  \hspace{1cm} (20)

Taking into account the vortex core does not practically affect the flow pattern and the normal force coefficient. As the calculations show, the plate is often broken through by the free discrete vortices. The detached discrete vortex comes close to the attached vortices and, under their influence, appears on the windward side of the plate. This leads to an incorrect calculation of the acting forces. To eliminate bursts of forces, a normal component of the vortex velocity is considered to be zero if less than $\Delta l = a/k$ from the vortex to the plate.

An asymmetry was introduced ($\alpha = 90^\circ$) by displacing the detached vortices from the left edge of the plate downstream in the interval $2 \leq t' \leq 3$ according to the dependence [18]:

$$\Delta t' = 0.01 \left[1 - \cos \left(\frac{t'(5 - 5)}{4}\right)\right].$$  \hspace{1cm} (21)
3 Results and discussions

3.1 Calculation results

The main parameters, affecting the normal-force coefficient and the frequency of vortex separation from the plate, are a time integration step $\Delta t' = 1/n$ and an interval between the attached vortices on the plate $\Delta x' = 1/k$. The methodological calculations have shown that the integration step $\Delta t'$ strongly affects the dynamics of free vortices. The decrease $\Delta t'$ delays the development and separation of the vortices, but at the same time it increases the total intensity of the vortices behind the plate, which leads to an increase in the normal force coefficient $c_y$. An increase in the number of breakdowns of the plate $k$ has little effect on the frequency of vortex separations, but it leads to a slight decrease in $c_y$. This is explained by the fact that, with an increase in the breakdown, the intensities of the outermost vortices on the plate are more evenly distributed and the intensities of the detached vortices become not so significant.

The calculations of a crossflow of the plate for the various parameters $n$ and $k$ have shown that at $n = 12$ and $k = 20$ an average value of the normal force coefficient is $c_y \approx 2.5$, and the Strouhal number is $Sh = 0.175$ (figure 2). As the recent experimental studies [19-20] have shown, $Sh = 0.165$, which is 0.01 less than that obtained by the discrete vortex method in this work. By decreasing the integration step, the Strouhal number can be brought closer to the experimental value, but at the same time, the normal force coefficient of the plate will increase.

The developments of the vortex swirl formation and the detachment from the plate are shown in figure 3 for angle $\alpha = 90^\circ$. The Strouhal number of the vortex shedding $Sh$ can be determined according to the time evolution of those vortex structures. The vortex shedding frequency was calculated using the period of oscillations in $c_y$ when the flow has attained a steady or nearly steady state.

The calculations performed at an angle of attack $\alpha = 60^\circ$ and $40^\circ$ for the same parameters $n$ and $k$, selected for the crossflow ($\alpha = 90^\circ$), have given the following dependences of the normal force coefficient versus non-dimensional time (figures 4, 5).

According to [6] at $\alpha = 60^\circ$ the experimental value of the Strouhal number is $Sh = 0.171$, but according to the calculations it is much larger than $Sh \approx 0.2$. Varying the parameters $k$ and $n$ does not introduce changes in $Sh$.

![Figure 2. Variation of normal force coefficient versus non-dimensional time for $\alpha=90^\circ$.]
Figure 3. Growth of wakes behind a flat plate after the impulsive start from rest. Numbers in the figure denote the non-dimensional time after start ($\alpha=90^\circ$).

Figure 4. Variation of normal force coefficient versus non-dimensional time for $\alpha=60^\circ$. 
4 Conclusion
The above calculations have shown, that by choosing an integration step and a quantity of the plate breakdowns, it is possible to obtain a satisfactory convergence with the experimental results on the normal force coefficient. But this results in some discrepancy in the Strouhal number, especially at the small angles of attack.

Replacement of the plate by the attached vortices leads to a passage of the free vortices through it, which leads to a discontinuous change in its aerodynamic characteristics. To avoid the latter, it is necessary to introduce a condition, for example, to exclude a normal velocity component of the free vortex near the plate.

The force formula obtained by the impulse method turned out to be more sensitive to a motion of the vortices near the plate. The introduction of the initial disturbance affects to some extent a subsequent development of the vortices and, thus, the dependence of the normal force coefficient on a time period.

A periodic change in loads on the plated structures can lead to their destruction. The calculated normal force coefficients for the plate by the discrete vortex method for various maximum contact angles allow us to determine the actual loads for a specific case of the flow-around and calculate the plated structures for strength and fatigue.

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