Localized Majorana-like modes in a number conserving setting: 
An exactly solvable model

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(Dated: 17th April 2015)

In this letter we present, in a number conserving framework, a model of interacting fermions in a 
two-wire geometry supporting non-local zero-energy Majorana-like edge excitations. The model has 
an exactly solvable line, on varying the density of fermions, described by a topologically non-trivial 
ground state wave-function. Away from the exactly solvable line we study the system by means 
of the numerical density matrix renormalization group. We characterize its topological properties, 
establish the presence of a gap in its single particle spectrum while the Hamiltonian is gapless, 
and compute the correlations between the edge modes as well as the superfluid correlations. The 
topological phase covers a sizeable portion of the phase diagram, the solvable line being one of its 
boundaries.

Introduction — Large part of the enormous attention devoted in the last years to topological superconductors 
obeys to the exotic quasiparticles such as Majorana modes, which localize at their boundaries (edges, vortices, ...).1 2 
and play a key role in several robust quantum information protocols.3 Kitaev’s p-wave superconducting quantum wire1 
provides a minimal setting showcasing all the key aspects of topological states of matter in fermionic systems. The existence of a so-called “sweet point” supporting an exact and easy-to-handle 
analytical solution puts this model at the heart of our understanding of systems supporting Majorana modes. 
Various implementations in solid state5 6 and ultracold atoms7 via proximity to superconducting or superfluid 
reservoirs have been proposed, and experimental signatures of edge modes were reported8.

Kitaev’s model is an effective mean-field model and its Hamiltonian does not commute with the particle number operator. Considerable activity has been devoted to understanding how this scenario evolves in a number-conserving setting 9 13. This effort is motivated both 
by the fundamental interest in observing a topological parity-symmetry breaking while a U(1) symmetry is intact, 
and by an experimental perspective, as in several setups (e.g. ultracold atoms) number conservation is naturally present. It was realised that a simple way to promote particle number conservation to a symmetry of the model, while keeping the edge state physics intact, was to consider at least two coupled wires rather than a single one 9 11. However, since attractive interactions are pivotal to generate superconducting order in the canonical ensemble, one usually faces a complex interacting many-body problem. Therefore, approximations 
such as bosonization 9 11, or numerical approaches 12 were invoked. An exactly solvable model of a topological superconductor in a number conserving setting, which would directly complement Kitaev’s scenario, is missing, although recent work pointed out an exactly solvable, number conserving model analogous to a non-local variant of the Kitaev chain13.

In this letter we present an exactly solvable model of a topological superconductor which supports exotic Majorana-like quasiparticles at its ends and retains the fermionic number as a well-defined quantum number. The construction of the Hamiltonian with local two-body interactions and of its ground state draws inspiration from recent work on dissipative state preparation for ultracold atomic fermions14 16, here applied to spinless fermions in a two-wire geometry. The solution entails explicit ground state wave-functions, which feature all the main qualitative properties highlighted so far for this class of models, with the advantage of being easy-to-handle.

In particular, we establish the following key features: i) The existence of one/two degenerate ground states depending on the periodic/open boundaries with a two-fold degenerate entanglement spectrum; ii) the presence of exponentially localized, symmetry-protected edge states associated to this degeneracy; iii) the exponential decay of the single-fermion two-point correlations, even if the Hamiltonian is gapless with collective bosonic modes with quadratic dispersion; iv) p-wave superconducting correlations which saturate at large distance. A closely related phenomenology was pointed out in recent approximate analytical11 17 15 and numerical12 18 studies.

By tuning the ratio of interaction vs. kinetic energy of our model, we can explore its properties outside the exactly-solvable line. The full phase diagram (Fig. 1) is obtained by means of density matrix renormalization group (DMRG) calculations. The exactly solvable line appears at λ = 1 and is found to stand between a stable topological phase and a phase-separated state. This finding is rationalized by a relation to the ferromagnetic XXZ chain.
The model — We begin by recapitulating some properties of the Kitaev chain, whose Hamiltonian reads \[ \hat{H}_K = \sum_j \left[ -J \hat{a}_j^\dagger \hat{a}_{j+1} + \Delta \hat{a}_j^\dagger \hat{a}_{j+1} + \text{H.c.} - \mu (\hat{n}_j - 1/2) \right]. \]

Here, \( J > 0 \) denotes the hopping amplitude, \( \mu \) and \( \Delta \) the chemical potential and the superconducting gap, respectively; \( \hat{a}_j^\dagger \) are fermionic annihilation (creation) operators on site \( j \), and \( \hat{n}_j = \hat{a}_j^\dagger \hat{a}_j \). This model has i) two density-driven phase transitions from finite densities to the empty and full states at \( |\mu| = 2J \) (for \( \Delta \neq 0 \)), and ii) a transition driven by the competition of kinetic and interaction energy (responsible for pairing) at \( \Delta/J = 0 \) (for \( |\mu| < 2J \)).

For \( |\mu| < 2J \) and \( \Delta \neq 0 \), the ground state is unique for periodic boundary conditions, but twofold degenerate for open geometry, hosting localized zero-energy Majorana modes. This topological phase is symmetry protected by total fermionic parity \( \hat{P} = (-1)^N \), where \( N = \sum_j \hat{n}_j \).

Let us focus on the so-called “sweet point”, namely \( \mu = 0 \) and \( \Delta = J > 0 \) and real, which enjoys the property \( \hat{H}_K = -(J/2) \sum_j \hat{\epsilon}_j \hat{\xi}_j \) with \( \hat{\xi}_j = C_j^\dagger \hat{a}_j + A_j \), \( C_j^\dagger = \hat{a}_j^\dagger + \hat{\xi}_j^\dagger \) and \( A_j = \hat{a}_j - \hat{\xi}_j + 1 \) (\( \hat{\xi}_L \) is defined identifying \( L + 1 \equiv 1 \)). For open geometry, the two ground states with \( L \) sites satisfy \( \hat{\xi}_j |\psi\rangle = 0 \), for \( 1 \leq j \leq L - 1 \), and can be written \[ |\psi\rangle = \mathcal{N}_{\epsilon(0)}^{-1/2} \sum_n (-1)^n \sum_{\{\hat{a}_{2n+1}\}} |\hat{a}_{2n+1}\rangle . \]

Figure 1. Phase diagram for the number and local parity conserving two-wire model \[ \hat{H}_K \] as a function of \( \lambda \) and filling \( \nu = N/2L \) obtained through DMRG simulations. The exactly solvable topological line is at \( \lambda = 1 \) (another, trivially solvable non-topological line is at \( \lambda = 0 \)). For \( \lambda > 1 \), the system undergoes phase separation (see the density profile \( \langle \hat{n}_i^\alpha \rangle \) in the inset). For \( 0 < \lambda < 1 \) and \( \nu \neq 0, 1 \), the system is in a homogeneous topological phase (see inset). The phase diagram is symmetric with respect to half filling \( \nu = 1/2 \) due to particle-hole symmetry of \( \hat{H}_K \).

New operators \( \hat{L}_{1,j} = \hat{C}_{a,j}^\dagger \hat{A}_{b,j} + \hat{C}_{b,j}^\dagger \hat{A}_{a,j} \). The Hamiltonian

\[ \hat{H} = \sum_{\alpha=a,b} \sum_{j} \hat{L}_{\alpha,j} \hat{\psi}_{\alpha,j} \]

coincides with the \( \lambda = 1 \) point of the following more general model:

\[ \hat{H}_L = -4 \sum_{j=1}^{L-1} \sum_{\alpha=a,b} \left[ (\hat{a}_{j}^\dagger \hat{a}_{j+1} + \text{H.c.}) - (\hat{n}_{j}^\alpha + \hat{n}_{j+1}^\alpha + \lambda \hat{n}_{j}^\alpha \hat{n}_{j+1}^\alpha) \right] \]

\[ -2\lambda \sum_{j=1}^{L-1} \left[ (\hat{a}_{j}^\dagger \hat{a}_{j+1} + \text{H.c.}) - (\hat{n}_{j}^\alpha + \hat{n}_{j+1}^\alpha) - (\hat{a}_{j}^\dagger \hat{a}_{j+1} + \text{H.c.}) - (\hat{n}_{j}^\alpha + \hat{n}_{j+1}^\alpha) \right] \]

\[ + \hat{a}_{j}^\dagger \hat{a}_{j+1} \hat{b}_{j}^\dagger \hat{b}_{j+1} - 2\lambda \hat{b}_{j}^\dagger \hat{b}_{j+1} \hat{a}_{j+1} + \text{H.c.} \].

\[ \hat{H}_L \] conserves the total particle number \( \hat{N} = \hat{N}_a + \hat{N}_b \) and the local wire parities \( \hat{P}_{a,b} = (-1)^{\hat{N}_{a,b}} \), which act as protecting symmetries for the topological phase. The coupling \( \lambda \) tunes the relative strength of the kinetic and interaction terms similarly to \( \Delta/J \) in \( \hat{H}_K \). Although only \( \lambda = 1 \) is exactly solvable, we will later consider \( \lambda \neq 1 \) to explore the robustness of the analytical results. The phase diagram is anticipated in Fig. 2.

**Exact results for \( \lambda = 1 \)** — For a fixed particle number \( N \) and open boundaries, the ground state of \( \hat{H} \) is twofold degenerate, due to the freedom in choosing the
local parity. For even \( N \), the ground states read
\[
|\psi_L(N)\rangle_{ee} = N_{ee,L,N} \sum_{n=0}^{N/2} \sum_{\{j_{2n} \}} |j_{2n}\rangle_a \otimes |\bar{q}_{N-2n}\rangle_b, \tag{4}
\]
\[
|\psi_L(N)\rangle_{oo} = N_{oo,L,N} \sum_{n=0}^{N/2-1} \sum_{\{j_{2n+1} \}} |j_{2n+1}\rangle_a \otimes |\bar{q}_{N-2n-1}\rangle_b
\]
where \( N_{ee,L,N} = \sum_{n=0}^{N/2} (L_{2n}) (N_{L-2n}) \); \( N_{oo,L,N} = \sum_{n=0}^{N/2-1} (L_{2n+1}) (N_{L-2n-1}) \). These describe the cases of even (ee) or odd (oo) particle numbers in each of the wires. For odd \( N \), the ground states \( |\psi_L(N)\rangle_{oo(oe)} \) with an even (odd) number of particles in either wire take the identical sum structure as above with the normalization \( N_{oo,L,N} \) in both cases. The wave-functions \( |\psi\rangle \) are the unique ground states of the model \([21]\). An interesting interpretation of \( |\psi(L(N))\rangle_{\sigma\sigma'} \) is in terms of number projection of the ground state of two decoupled even-parity Kitaev chains \( |G\rangle = |\psi_e\rangle \otimes |\psi_b\rangle \).

\[
|\psi_L(N)\rangle_{ee} \propto \hat{P}_N |G\rangle; \quad |\psi_L(N)\rangle_{oo} \propto \hat{P}_N \hat{P}_L |G\rangle; \quad |\psi_L(N)\rangle_{oo} \propto \hat{P}_N \hat{p}_L |G\rangle; \quad |\psi_L(N)\rangle_{ee} \propto \hat{P}_N \hat{p}_L |G\rangle \tag{5}
\]
where \( \hat{P}_N \) and \( \hat{p}_L \) are the zero-energy modes of the decoupled Kitaev wires at half filling. This interpretation provides intuition that the two-fold ground-state degeneracy is absent for periodic boundary conditions: since on a circle \( H_K \) has a unique ground state, the ground state of \( \hat{H} \) with \( N \) particles is also unique \([21]\).

Important evidence of a topologically nontrivial bulk state is obtained from the double degeneracy of the entanglement spectrum, which we now compute for one of the wave-functions \([1]\). To this end, we consider the reduced state of \( l \) sites on each wire \( \rho_l = \text{Tr}_{L-l} \langle |\psi_L(N)\rangle_{ee} \langle \psi_L(N)\rangle_{ee} \rangle_{ee} \). Taking the symmetries into account, it can be written in diagonal form as \(21\)
\[
\rho_l = \sum_{N_l=0}^{\min(2l,N)} \chi^{(L,N)}_{(L,N)} |\psi(N_l)\rangle_{\sigma\sigma'} \langle \psi(N_l)|_{\sigma\sigma'} \tag{6}
\]
with the following nonzero eigenvalues: for \( N_l \) even \( \chi^{(L,N)}_{(L,N)} = N_{ee,oo,L,N} N_{ee,oo,L-1,N-N_l} / N_{ee,L,N} \) whereas for \( N_l \) odd \( \chi^{(L,N)}_{(L,N)} = \chi^{(L,N)}_{(L,N)} = \chi^{(L,N)}_{(L,N)} \). In the odd-particle number sector the entanglement spectrum is manifestly twofold degenerate. In the even one, such degeneracy appears in the thermodynamic limit: \( \chi^{(L,N)}_{(L,N)} / \chi^{(L,N)}_{(L,N)} \to 1 \) (see \(21\) and Fig. 2a).

An interesting insight is provided by \( O_j \equiv \langle \psi_L(N)\rangle_{oo} \hat{a}_j \hat{b}_j \psi_L(N)\rangle_{ee} \), where \( \hat{a}_j \hat{b}_j \) is the only single-site operator which commutes with \( \hat{N} \) and changes the local parities \( \hat{P}_a, \hat{b} \). Since \( \hat{P}_a, \hat{b} \) is not a fundamental fermionic symmetry for the two-wire geometry (the situation is comparable to two close Kitaev wires in this respect), local operators can locally distinguish the two degenerate ground states. The calculation of such matrix elements leads to a lengthy combinatorial expression \(21\) and is shown in Fig. 2b. We interpret the exponential decay of \( \rho_l \) into the bulk as a clear signature of localized edge modes with support in this region only. At half filling, the edge state mode function has maximal localization around the first and last site. Away from half filling however, the number projection leads to an increased localization length of the edge states. In the thermodynamic limit, this length diverges as the filling \( \nu \to 0, 1 \), indicating a topological phase transition. We emphasize that this exponential behavior is different from \(9\) \(10\), reporting algebraic localization of the edge states, but similar to \(11\) \(15\). Non-local correlations of localized edge states are another clear indication of topological order and can be proven via \( \langle \hat{a}_i \hat{a}_{i+j} \rangle \). Fig. 2b shows edge-edge non-local correlations, as well as an exponential decay into the bulk of the system (analytical expression is given in \(21\)).

The nature of the edge states and the impact of number conservation can be made even more concrete by a braiding gedanken experiment, where observable effects are obtained via interferometry \(22\) \(23\). We consider a system with even particle number—cf. first line of Eq. (5)—and take four Majorana fermions, \( \tilde{\gamma}_1, \tilde{\gamma}_{2L}, \tilde{\gamma}_{1}, \tilde{\gamma}_{2L} \), fulfilling the Clifford algebra and exactly localized at the edges (anticipating half filling). Braiding of Majorana fermions \( \tilde{\gamma}_i, \tilde{\gamma}_j \) is exerted by the unitary transformation

\[
\hat{U}_{\text{braiding}} = \exp \left( i \int_0^L d\gamma \tilde{\gamma}_i \tilde{\gamma}_j \right)
\]
\( \hat{B}_{ij} = \exp\left( \frac{i}{\lambda} \gamma_i \gamma_j \right) \), which exchanges the Majorana modes and generates a relative sign. To make consequences of this phase observable, we initialize the even particle number system in the state \( |\Psi\rangle = \frac{1}{\sqrt{2}} (|\psi_L(N)\rangle_{ee} + |\psi_L(N)\rangle_{oo}) \), and apply the braiding operation \( \hat{B} = \exp\left( \frac{i}{\lambda} \gamma_i \gamma_j \right) \). To distinguish initial and braided wavefunctions \( |\Psi\rangle, \hat{B}|\Psi\rangle \) in an occupation number measurement, we perform a second unitary transformation \( \hat{B}_{12} = \exp\left( \frac{i}{\lambda} \gamma_i \gamma_j \right) \) on both, and then determine \( n_a = \langle \ell_L^a | \ell_L^a \rangle \) and \( n_b = \langle \ell_L^b | \ell_L^b \rangle \) on the wires. In the first case, we get \( n_{a,b} = 1/2 \) (equal probabilities for \( n_{a,b} = 0 \) or \( n_{a,b} = 1 \)). In contrast, for the braided state, at half filling we measure \( n_a = 3/4, n_b = 1/4 \). This has to be compared to the analogous result \( n_a = 1, n_b = 0 \) in the number non-conserving case; the difference can be directly attributed to the absence of number non-conserving correlators in our problem. The non-local nature of the single-particle correlations witnessed here, however, parallels the number non-conserving case \([21]\).

Furthermore, the Hamiltonian is gapless and hosts long wavelength collective bosonic excitations, while the single fermion excitations experience a finite gap. This is a crucial property of the ground state; the absence of gapless fermion modes in the bulk ensures the robustness of the zero energy edge modes, in analogy to non-interacting topologically non-trivial systems. The gapped nature of single fermion excitations is established via the exponential decay of the fermionic two-point function, e.g. \( \langle \hat{a}_i \hat{a}_j \rangle \).

Again, the resulting formula is a lengthy combinatorial expression \([21]\), evaluated numerically for very large sizes and plotted in Fig. 2b. For \( \nu \to 0, 1 \), the correlation length diverges, indicating the vanishing of the fermion gap and a thermodynamic, density-driven phase transition accompanying the topological phase transition at these points, in full analogy to the Kitaev chain.

On the other hand, the analysis of the superfluid correlations demonstrates the existence of gapless modes. More precisely, we obtain

\[
\langle \hat{a}_i^\dagger \hat{a}_{i+1} \hat{a}_{j+1} \hat{a}_j \rangle \sim \nu^2 (1 - \nu)^2.
\]  

(7)

Similar expressions hold for cross-correlations between the wires. The constancy of Eq. (7) hints at the absence of bosonic modes with linear dispersion, which would lead to algebraic decay. A DMRG analysis of the excitation spectrum of \( \hat{H} \) for system sizes up to \( L = 144 \) demonstrates a vanishing of its gap \( \sim L^{-2} \) (Fig. 2d). This indicates the presence of collective excitations with quadratic dispersion. Further support to this statement is provided from the fact that Eq. (3) without the wire coupling term reduces to the XXZ model at the border of its ferromagnetic phase, which hosts quadratically dispersing spin waves, \( \omega \sim q^2 \). This dispersion, with dynamic exponent \( z = 2 \), gives rise to an effective phase space dimension \( d_{\text{eff}} = z + 1 = 3 \) at zero temperature, explaining the constancy of superfluid correlations due to the absence of a divergence in the soft mode correlators. This finding is special for \( \lambda = 1 \).

**Numerical results** — To further explore the status of these results, we now move to the full model \( \hat{H}_3 \) away from the solvable line \( \lambda = 1 \). The study is performed with DMRG on systems with sizes up to \( L = 240 \) and open boundary conditions.

We first establish the absence of a topological phase for \( \lambda > 1 \). The density profile, shown in the inset of Fig. 1 for \( \lambda = 0.5 \) and \( \lambda = 1.01 \), displays a clear phase-separation tendency. Analogous data are obtained for other values of \( \nu, \lambda \) (see dark crosses in Fig. 1).

For \( \lambda < 1 \), simulations support the existence of a homogeneous phase (Fig. 1). Note that \( \lambda = 0 \) is a free-fermion point trivially non-topological. For \( \lambda \neq 0 \) we observe: i) two quasi-degenerate ground states with different relative parity and same particle numbers, ii) degenerate entanglement spectrum, iii) a gap closing as \( L^{-1} \) for fixed parity, iv) exponentially decaying single-fermion correlations, v) power-law decaying superfluid correlators. Plots in Fig. 3 display our numerical results. Simulations at lower filling \( \nu \to 0 \) and small \( \lambda \) are more demanding, owing to the increasing correlation length of the system.

The numerics is consistent with the phase diagram in Fig. 1, exhibiting a topological phase delimited by three trivial lines at \( \lambda = 0, \nu = 0 \) and \( \nu = 1 \) and an inhom-
geneous non-topological phase for $\lambda > 1$. The exactly solvable topological line at $\lambda = 1$ serves as a boundary; the fixed-$\nu$ phase diagram is reminiscent of the ferromagnetic transition in the XXZ model.

Conclusions — We presented an exactly solvable two-wire fermionic model which conserves the number of particles and features Majorana-like exotic quasiparticles at the edges. Our results can be a valuable guideline to understand topological edge states in number conserving systems. For example, the replacement $\hat{a}_i \rightarrow \hat{c}_{i, \uparrow}, \hat{b}_i \rightarrow \hat{c}_{i, \downarrow}$ results in a one-dimensional spinful Hubbard Hamiltonian without continuous spin rotation, but time reversal symmetry. The resulting model with an exactly solvable line belongs to the class of time reversal invariant topological superconductors [24], analyzed in a number conserving setting recently [18], with edge modes protected by the latter symmetry. Moreover, exactly solvable number conserving models can be constructed in arbitrary dimension.

During the final step of preparation, we became aware of similar results obtained by Lang and Büchler [25].

Acknowledgments — We thank E. Altman, E. Berg, J. C. Budich, M. Burrello, M. Dalmonte and G. Ortiz for enlightening discussions. F.I. acknowledges financial support by the Brazilian agencies FAPEMIG, CNPq, and INCT- IQ (National Institute of Science and Technology for Quantum Information). S.D. acknowledges support via the START Grant No. Y 581-N16 and the German Research Foundation through ZUK 64. R.F. acknowledges financial support from the EU integrated project SIQS and from Italian MIUR via PRIN Project 2010LLKJBX. D.R. and L.M. acknowledge the Italian MIUR through FIRB project RBFR12NLNA. L.M. was supported by Regione Toscana POR FSE 2007-2013.

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Supplemental Material to
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In this Supplemental Material we provide additional information about some details of the analytical results for the exactly solvable two-wire topological system which have been omitted from the main text.

**TWO-WIRE GROUND STATE**

In this section we show that the wave-functions $|\psi_L(N)\rangle_{ee(oo)}$ in Eq. (4) of the main text are the only ground states of the two-wire Hamiltonian $\hat{H}_{L=1}$. Our proof actively constructs all of the zero-energy eigenstates of the Hamiltonian, which are the lowest-energy states because $\hat{H}_{L=1} \geq 0$. Such states are obtained projecting the grand-canonical ground state of two decoupled Kitaev chains onto a given particle-number sector.

Let us first consider only the operators $\{\hat{L}_{a,j}, \hat{L}_{a,j}\}$ and the corresponding parent Hamiltonian $\hat{H}_{ab} = \sum_{a,b} \hat{L}_{a,j}^\dagger \hat{L}_{a,j}$ which corresponds to two decoupled single-wires. We know that the ground states of each single wire are given by $\hat{P}_a^N |\psi^a_a\rangle$. Hamiltonian $\hat{H}_{ab}$ thus has a ground space spanned by

$$\{ \hat{P}_a^N \hat{P}_b^{n-N-n} |\psi^a_a\rangle \otimes |\psi^b_b\rangle \}_{n=0, (\sigma, \sigma')=e(o)}$$

(S1)

and $(\sigma, \sigma')$ take the values $(e, e)$ and $(o, o)$ when $N$ is even and $(e, o)$ and $(o, e)$ when $N$ is odd.

An important relation holds because $\hat{L}_i^\dagger |\psi^a_a\rangle \otimes |\psi^b_b\rangle = 0$. Upon the insertion of the identity operator $\sum_{n,n'=0}^L \hat{P}_n^a \hat{P}_n^b$ we get

$$\sum_{n,n'}^L (\hat{C}_{a,j}^\dagger \hat{P}_{n-1}^a + \hat{A}_{a,j} \hat{P}_{n+1}^a) \hat{P}_n^b |\psi^a_a\rangle \otimes |\psi^b_b\rangle = 0.$$  

(S2)

Each of the elements in the above sum must vanish independently because of orthogonality, and the important relation

$$\hat{C}_{a,j}^\dagger \hat{P}_{n-1}^a \hat{P}_n^b |\psi^a_a\rangle \otimes |\psi^b_b\rangle = -\hat{A}_{a,j} \hat{P}_{n+1}^a \hat{P}_n^b |\psi^a_a\rangle \otimes |\psi^b_b\rangle, \quad \forall n, n'$$

(S3)

is derived (the same holds for the b wire).

Let us now compute the $N$-fermions state such that $\hat{H} |\phi_N\rangle = 0$. In general, $|\phi_N\rangle$ must be in the kernel of $\hat{H}_{ab}$:

$$|\phi_N\rangle = \sum_{n=0}^N \sum_{\sigma, \sigma'} x_{n, \sigma, \sigma'} \hat{P}_n^a \hat{P}_n^b |\psi^a_a\rangle \otimes |\psi^b_b\rangle ; \quad \sum_{n, \sigma, \sigma'} |x_{n, \sigma, \sigma'}|^2 = 1.$$  

(S4)

Imposing now that $\hat{L}_{I,j} |\phi_N\rangle = \sum \hat{L}_{I,j}^\dagger |\phi_N\rangle = 0$, we obtain that:

$$\hat{L}_{I,j} |\phi_N\rangle = (\hat{C}_{a,j}^\dagger \hat{A}_{b,j} + \hat{C}_{b,j}^\dagger \hat{A}_{a,j}) \sum_{n=0}^N \sum_{\sigma, \sigma'} x_{n, \sigma, \sigma'} \hat{P}_n^a \hat{P}_n^b |\psi^a_a\rangle \otimes |\psi^b_b\rangle = 0$$

$$= \hat{C}_{a,j}^\dagger \hat{A}_{b,j} |\phi_N\rangle - \hat{C}_{a,j}^\dagger \hat{A}_{b,j} \sum_{n, \sigma, \sigma'} x_{n, \sigma, \sigma'} \hat{P}_n^a \hat{P}_{n-2}^b |\psi^a\rangle \otimes |\psi^b\rangle$$

$$= \hat{C}_{a,j}^\dagger \hat{A}_{b,j} \sum_{\sigma, \sigma'} (x_{n, \sigma, \sigma'} - x_{n+2, \sigma, \sigma'}) \hat{P}_n^a \hat{P}_{n-2}^b |\psi^a\rangle \otimes |\psi^b\rangle = 0$$

(S5)

since $\hat{C}_{a,j}^\dagger \hat{A}_{b,j} \hat{P}_n^a \hat{P}_{n-2}^b |\psi^a\rangle \otimes |\psi^b\rangle \neq 0$. Such a relation uniquely defines a ground state for a fixed local parity (even-even, odd-odd, even-odd, odd-even), and thus, for each fixed particle number $N$, there is a double degeneracy related to distinct wire parities. Indeed, a general ground state for $2N$ particles is given by

$$|\phi_{2N}\rangle \propto \sum_n \left[ w_0 \hat{P}_n^a \hat{P}_{2(N-n)}^b |\psi^a_e\rangle \otimes |\psi^b_e\rangle + w_1 \hat{P}_{2(n+1)}^a \hat{P}_{2N-(2n+1)}^b |\psi^a_o\rangle \otimes |\psi^b_o\rangle \right],$$

(S6)
and is parametrized by the complex coefficients \( w_0 \) and \( w_1 \).

Alternatively, we can consider the ground states of two decoupled even parity Kitaev chains at half filling \( \mu = 0 \) and \( \Delta = J \) on a circle with odd number of sites (no edges), \( |G\rangle = |\psi_e^a\rangle \otimes |\psi_e^b\rangle \) and on an open system of the same length,

\[
\{ |G\rangle, \hat{\rho}_L^a |G\rangle, \hat{\rho}_L^b |G\rangle, \hat{\rho}_L^{a\dagger} \hat{\rho}_L^b |G\rangle \},
\]

which are related to the edge Majorana fermions \( \hat{\ell}_L^\alpha = \hat{\gamma}_L^\alpha + i \hat{\gamma}_L^\beta \), where \( \{ \hat{\gamma}_L^\alpha, \hat{\gamma}_L^\beta \} = 2\delta_{ij}\delta_{\alpha\beta}, \hat{\gamma}_L^{2j-1} = i(\hat{\alpha}_j - \hat{\alpha}_j^\dagger) \), and \( \gamma_{L+1}^\alpha = \hat{\alpha}_j + \hat{\alpha}_j^\dagger \), for \( \alpha = a, b \). In this writing, we are exploiting the highly non-generic properties of Kitaev’s wire ‘sweet point’, namely that the ground state of a closed wire (\( L \) odd) coincides with the ground state with even parity of an open wire \( |\psi_e^a\rangle (L \) odd).

Thus, the ground states for the two wires number conserving Hamiltonian, as analysed in this section, are described by

\[
|\phi_N\rangle \subset \text{span} \left\{ \hat{P}_N |G\rangle, \hat{P}_N \hat{\rho}_L^{a\dagger} \hat{\rho}_L^b |G\rangle \right\} \quad \text{for } N \text{ even},
\]

\[
|\phi_N\rangle \subset \text{span} \left\{ \hat{P}_N \hat{\rho}_L^a |G\rangle, \hat{P}_N \hat{\rho}_L^{b\dagger} |G\rangle \right\} \quad \text{for } N \text{ odd}.
\]

**ENTANGLEMENT SPECTRUM**

In this section we provide the detailed derivation for the entanglement spectrum, presented in the main text. We consider the reduced state of \( l \) sites on each wire \( \rho_l = \text{Tr}_{(L-l)} \left[ |\psi_L(N)\rangle_{ee} \langle \psi_L(N)\rangle_{ee} \right] \) (in the following expression identity operators on the first \( l \) sites are omitted):

\[
\rho_l = \sum_{\{\bar{J}_{m}\}, \{\bar{q}_{m}\}} \langle \bar{J}_{m}\rangle |\bar{J}_{m}\rangle_{a} \otimes \langle \bar{q}_{m}\rangle_{b} |\psi_L(N)\rangle_{ee} \langle \psi_L(N)\rangle_{ee} |\bar{J}_{m}\rangle_{a} \otimes |\bar{q}_{m}\rangle_{b}.
\]

Notice now that for \( N_1 \) even,

\[
\langle \bar{J}_{2m(2m+1)} \rangle_{a} \otimes \langle \bar{q}_{N-N_1-2m(2m+1)} \rangle_{b} |\psi_L(N)\rangle_{ee} = \sqrt{\frac{N_{ee,oo,l,N_1}}{N_{ee,l,N}}} |\psi_l(N_1)\rangle_{ee,oo} \quad \text{(S7)},
\]

which, as we see, does not depend on the specific \( \bar{J} \) or \( \bar{q} \). A similar relation exists for \( N_1 \) odd,

\[
\langle \bar{J}_{2m(2m+1)} \rangle_{a} \otimes \langle \bar{q}_{N-N_1-2m(2m+1)} \rangle_{b} |\psi_L(N)\rangle_{ee} = \sqrt{\frac{N_{ee,oo,l,N_1}}{N_{ee,l,N}}} |\psi_l(N_1)\rangle_{oo,ee} \quad \text{(S8)}.
\]

Summing up such terms, we obtain the reduced state in diagonal form and its eigenvalues, as given in the main text.

The demonstration for the double degeneracy in the entanglement spectrum in the limit of large \( l \) and \( L-l \) (i.e., large lattices and bipartitions not close to its edges), is related to the fact that, in this limit, \( N_{ee,l,N_1} \sim N_{oo,l,N_1} \). Even if we do not have an explicit analytical proof of the previous relation, numerical tests in several regimes corroborate this intuitive result.

From the eigenvalues computed in this section, we can also compute the entanglement entropy of the block matrices, and see how it scales with the size of the block. We see in Fig. S11 a behavior typical of a gapless Hamiltonian, which does not scale as an area law.

**EDGE MODES**

As discussed in the main text, in order to directly characterize the localization length of the edge modes we may compute the local parity breaking perturbation \( \langle g_2 | a_i^\dagger b_i | g_1 \rangle \), where \( |g_{1(2)}\rangle \) correspond to the two ground states for fixed \( N \) in Eq. 1. The task of computing these observables reduces merely to counting the suitable configurations, by looking at the ground states as given in Eq. 1. Let us consider, for simplicity, \( N = \text{even} \) number of particles, and \( |g_{1(2)}\rangle \) the even-even (odd-odd) local parity ground state.

If we act with the \( \hat{V}_j = \hat{a}_j^\dagger \hat{b}_j \) operator on the even-even ground state, the only non-null configurations remaining are those which have a particle at site \( b_j \), and a hole at site \( a_j \). Due to the anticommutation relations, each of these
configurations will have a phase \((-1)^{(n_R^L + n_L^b)}\), where \(n_R^L = \sum_{r=j+1}^{L} n_r^a\) is the number of particles located at the right of the \(j\)th site in the \(a\)-wire, and \(n_L^b = \sum_{r=1}^{j-1} n_r^b\) the number of particles located at the left of the \(j\)th site in the \(b\)-wire. These phases describe the parity of the configuration on the segments \([j+1, L]\) for the \(a\)-wire and \([1, j-1]\) for the \(b\)-wire. Since the ground state corresponds to all possible configurations, depending on the \(j\)th site under analysis, and the total \(N\) number of particles, we see that such \(n_R^L\) number of particles varies from a minimum value equal to \(\max(0, N-j)\) particles (where the remaining side contains all the \(N\) particle, or it is completely filled) and a maximum value equal to \(\min(N, L-j)\) (where the right side contains all the \(N\) particles, or it is completely filled). Analogous reasoning follows for \(n_L^b\). Realizing then the expectation value of these configurations with the odd-odd ground state, it will be simply related to the number of such configurations weighted by its corresponding phases.

In a more detail, we have the following. The total number of configurations for the even-even state, is given by

\[
\sum_n \left( \frac{L}{2n-1} \right) \left( \frac{L}{N-2n} \right),
\]

where those which have a particle at site \(b_j\), and a hole at site \(a_j\), correspond to

\[
\sum_n \left( \frac{L}{2n-1} \right) \left( \frac{L}{N-2n} \right) = \sum_n \left( \frac{L}{n_R^b} \right) \left( \frac{L}{n_L^a} \right) \left( \frac{j-1}{2n-1-n_R^b} \right) \left( \frac{j-1}{N-2n-n_R^b} \right).
\]

Imposing then their respective phases \((-1)^{(n_R^L + n_L^b)}\), we have that

\[
\langle g_2 | \hat{a}_j^\dagger \hat{b}_j | g_1 \rangle = \frac{1}{\sqrt{N_{ee,L,N}N_{oo,L,N}}} \sum_{n,n_R^b,n_L^a} (-1)^{(n_R^L + n_L^b)} \left( \frac{j-1}{n_R^b} \right) \left( \frac{L-j}{2n-1-n_R^b} \right) \left( \frac{L-j}{n_R^a} \right) \left( \frac{j-1}{N-2n-n_R^a} \right).
\]

Following the same steps as above, it is not hard to see that for a general single-particle operator we have

\[
\langle g_2 | \hat{a}_j^\dagger \hat{b}_r | g_1 \rangle = \frac{1}{\sqrt{N_{ee,L,N}N_{oo,L,N}}} \sum_{n,n_R^b,n_L^a} (-1)^{(n_R^L + n_L^b)} \left( \frac{j-1}{n_R^b} \right) \left( \frac{L-j}{2n-1-n_R^b} \right) \left( \frac{L-j}{n_R^a} \right) \left( \frac{j-1}{N-2n-n_R^a} \right).
\]
for \((L + r) - j > 1\).

In particular, the edge-edge correlations can be reduced to a simple expression in the thermodynamic limit. These correlations are given by

\[
\langle g_2 | \hat{a}^+_L \hat{b}_L | g_1 \rangle = \frac{1}{\sqrt{\mathcal{N}_{ee,L,N}}} \sum_n (-1)^{(N-1)} \frac{L - 1}{2n - 1} \frac{L - 1}{N - 2n},
\]

(S13)

\[
\langle g_2 | \hat{a}^+_L \hat{b}_1 | g_1 \rangle = \frac{1}{\sqrt{\mathcal{N}_{ee,L,N}}} \sum_n \frac{L - 1}{2n - 1} \frac{L - 1}{N - 2n},
\]

(S14)

Using the Chu-Vandermonde identity: \(\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}\), for non-negative integer \(m, n, r\), we obtain in the limit of large lattices,

\[
\sum_n \binom{L - 1}{2n - 1} \binom{L - 1}{N - 2n} = \frac{1}{2} \binom{2L - 2}{N - 1},
\]

(S15)

and \(\mathcal{N}_{ee,oo,L,N} \approx \frac{1}{2} \binom{2L}{N}\). Thus, the edge-edge correlations are

\[
\langle g_2 | \hat{a}^+_1 \hat{b}_L | g_1 \rangle \approx \frac{\nu(1 - \nu)}{1 - \nu^2} \nu(1 - \nu),
\]

(S16)

and similarly for \(\langle g_2 | \hat{a}^+_1 \hat{b}_1 | g_1 \rangle\). Note that, if \(N\) is odd, we would have a minus sign in the above correlation, \(\langle g_2 | \hat{a}^+_1 \hat{b}_L | g_1 \rangle \xrightarrow{L \to \infty} -\nu(1 - \nu)\), due to the overall phase \((-1)^{(N-1)}\) in Eq. (S13).

**SINGLE-PARTICLE AND SUPERFLUID CORRELATIONS**

In a general way, any ground state observable can be computed as in the above section through a simple counting of the suitable configurations. In this section we evaluate the single particle correlations \(\langle \hat{a}^+_j \hat{a}_{j+r} \rangle\), as well as the superfluid correlations \(\langle \hat{a}^+_j \hat{a}_{j+1} \hat{a}_{j+1} \hat{a}_{j} \rangle\). We skip unnecessary details and focus mainly on the presentation of the final results. We only consider ground states with even-even or odd-odd local parities because the even-odd and odd-even cases are mathematically equivalent to the even-even one.

**Single particle correlations:**

\[
\langle \psi_{ee,oo} | \hat{a}^+_j \hat{a}_{j+r} | \psi_{ee,oo} \rangle = \begin{cases} 
\frac{1}{\mathcal{N}_{ee,oo,L,N}} \sum_{n, n^a_{(j,r)}} (-1)^{n^a_{(j,r)}} \binom{L}{2n(2n + 1)} \binom{r - 1}{n^a_{(j,r)}} \binom{L - r - 1}{N - 2n(2n + 1) - 1 - n^a_{(j,r)}}, & \text{if } r > 1; \\
\frac{1}{\mathcal{N}_{ee,oo,L,N}} \sum_n \binom{L}{2n(2n + 1)} \binom{L - r - 1}{N - 2n(2n + 1) - 1}, & \text{if } r \leq 1.
\end{cases}
\]

(S17)

where \(n^a_{(j,r)} = \sum_{i=1}^{r-1} n^a_{j+i}\) is the number of particles between the sites \(j\) and \(j + r\), which varies from a minimum of zero (where all the particles are in the b-wire), to a maximum value equal to \(\min(N - 1, r - 1)\) (where all the remaining \(N - 1\) particles lie between these sites, or it is completely filled).
Superfluid correlations:

$$\langle \psi_{ee(oo)} | \hat{a}_i^\dagger \hat{a}_{i+1}^\dagger \hat{a}_j \hat{a}_{j+1} | \psi_{ee(oo)} \rangle = \frac{1}{N_{ee(oo)}(L, N)} \sum_n \begin{pmatrix} L \\ 2n \end{pmatrix} \begin{pmatrix} L-4 \\ N-2n-2 \end{pmatrix}, \quad i + 1 < j. \tag{S18}$$

Using the Chu-Vandermonde identity, we obtain in the limit of large lattices,

$$\sum_n \begin{pmatrix} L \\ 2n \end{pmatrix} \begin{pmatrix} L-4 \\ N-2n-2 \end{pmatrix} \approx \frac{1}{2} \begin{pmatrix} 2L-4 \\ N-2 \end{pmatrix}, \tag{S19}$$

and $N_{ee(oo)}(L, N) \approx \frac{1}{2} (\frac{2L}{L})^N$. Thus, the superfluid correlations is

$$\langle \psi_{ee(oo)} | \hat{a}_i^\dagger \hat{a}_{i+1}^\dagger \hat{a}_j \hat{a}_{j+1} | \psi_{ee(oo)} \rangle \approx \frac{2^4 \nu^2 (1-\nu)^2 L^4 + O(L^3)}{2^4 L^4 + O(L^3)} \xrightarrow{L \to \infty} \nu^2 (1-\nu)^2. \tag{S20}$$

**BRAIDING**

Following [22, 23] we perform a Gedanken experiment that explicitly demonstrates the non-trivial behavior of the localized edge states under our number-conserving setting. To this end, we assume that the two wires host edge Majorana fermions, $\gamma_1^a, \gamma_2^a, \gamma_1^b, \gamma_2^b$, for the extreme left and right Majorana fermions of the $a$ and $b$ wires; indeed we can expect exactly localized states at half filling. We then study the behavior of these degrees of freedom under unitary, non-abelian braiding operations described by $B_{ij} = \exp(\frac{i}{4} \gamma_1^i \gamma_1^j) = \frac{1}{\sqrt{2}} (1 + \gamma_3^i \gamma_3^j)$ for the two Majorana modes $i, j$.

The four real Majoranas correspond to two complex fermions $\hat{\ell}_L^a = (\gamma_1^a + i \gamma_2^a)/2$, and $\hat{\ell}_L^b = (\gamma_1^b + i \gamma_2^b)/2$, where $\{\gamma_1^a, \gamma_1^b\} = 2\delta_{ij} \delta_{ij}$, $\tilde{\gamma}_2^a = i(\tilde{\gamma}_2^a - \tilde{\gamma}_1^a)$, and $\tilde{\gamma}_2^b = \tilde{\gamma}_1^b + \tilde{\gamma}_2^b$, for $\alpha = a, b$. We now prepare an initial state for the system as,

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\psi_L(N)\rangle_{ee} + |\psi_L(N)\rangle_{oo}). \tag{S21}$$

If we braid the Majoranas $\gamma_1^a$, and $\gamma_2^a$, the initial state $|\Psi\rangle$ transforms into $|\Phi\rangle = \hat{B} |\Psi\rangle$, where $\hat{B} = \frac{1}{\sqrt{2}} (1 + \gamma_1^a \gamma_2^a)$.

The states $|\Psi\rangle$ and $|\Phi\rangle$ can be distinguished after making a change of basis via $B_{12} = \frac{1}{\sqrt{2}} (1 + \gamma_1^a \gamma_2^a)$, and subsequent measurement of the number of fermions $n_a = \langle \hat{\ell}_L^a | \hat{\ell}_L^a \rangle$ and $n_b = \langle \hat{\ell}_L^b \hat{\ell}_L^b \rangle$ on the wires. The final states are,

$$|F_\Phi\rangle = \hat{B}_{12} \hat{B} |\Psi\rangle, \quad \tag{S22}$$

$$|F_\Psi\rangle = |\Psi\rangle, \quad \tag{S23}$$

and we will then compute,

$$\langle F_\Phi | \hat{\ell}_L^a \hat{\ell}_L^b | F_\Phi \rangle = \langle \Psi | (\hat{B}_{12} \hat{B})^\dagger \hat{\ell}_L^a \hat{\ell}_L^b (\hat{B}_{12} \hat{B}) | \Psi \rangle, \quad \tag{S24}$$

$$\langle F_\Phi | \hat{\ell}_L^a \hat{\ell}_L^b | F_\Psi \rangle = \langle \Psi | (\hat{B}_{12} \hat{B})^\dagger \hat{\ell}_L^a \hat{\ell}_L^b (\hat{B}_{12} \hat{B}) | \Psi \rangle, \quad \tag{S25}$$

$$\langle F_\Psi | \hat{\ell}_L^a \hat{\ell}_L^b | F_\Phi \rangle = \langle \Psi | \hat{B}_{12} \hat{B} \hat{\ell}_L^a \hat{\ell}_L^b | \Psi \rangle, \quad \tag{S26}$$

$$\langle F_\Psi | \hat{\ell}_L^a \hat{\ell}_L^b | F_\Psi \rangle = \langle \Psi | \hat{B}_{12} \hat{B} \hat{\ell}_L^a \hat{\ell}_L^b | \Psi \rangle. \quad \tag{S27}$$

The above observables may be also rewritten as follows,

$$\langle \hat{B}_{12} \hat{B} \hat{\ell}_L^a \hat{\ell}_L^b (\hat{B}_{12} \hat{B}) = \frac{1}{2} [1 + (\hat{a}_L^b \hat{b}_1 + \hat{a}_L \hat{b}_1 + \text{H.c.})], \quad \tag{S28}$$

$$\langle \hat{B}_{12} \hat{B} \hat{\ell}_L^a \hat{\ell}_L^b (\hat{B}_{12} \hat{B}) = \frac{1}{2} [1 - (\hat{a}_L^b \hat{b}_1 + \hat{a}_L \hat{b}_1 + \text{H.c.})], \quad \tag{S29}$$

$$\langle \hat{B}_{12} \hat{B} \hat{\ell}_L^a \hat{\ell}_L^b (\hat{B}_{12} \hat{B}) = \frac{1}{2} [1 - i(\hat{a}_L^b \hat{b}_1 - \hat{a}_L \hat{b}_1 - \text{H.c.})], \quad \tag{S30}$$

$$\langle \hat{B}_{12} \hat{B} \hat{\ell}_L^a \hat{\ell}_L^b (\hat{B}_{12} \hat{B}) = \frac{1}{2} [1 + i(\hat{a}_L^b \hat{b}_1 + \hat{a}_L \hat{b}_1 - \text{H.c.})], \quad \tag{S31}$$
where \( \hat{a}_j, \hat{b}_j \) are the real space fermionic operators on the lattice. The effects of a fixed particle number can be made transparent here. To this end, we write the projections of the observables onto fixed number subspaces,

\[
\hat{P}_N(\hat{B}_{12}\hat{B}) | \hat{b}_L\rangle = \frac{1}{2} \left[ | \hat{a}_L^\dagger \hat{b}_1 + \text{H.c.} \rangle \right],
\]

\[
\hat{P}_N(\hat{B}_{12}\hat{B}) | \hat{b}_L\rangle = \frac{1}{2} \left[ | \hat{a}_L^\dagger \hat{b}_L + \text{H.c.} \rangle \right],
\]

\[
\hat{P}_N \hat{B}_{12}^\dagger \hat{b}_L \hat{B}_{12}^\dagger \hat{P}_N = \frac{1}{2} \left[ | \hat{a}_L^\dagger \hat{b}_1 - \text{H.c.} \rangle \right],
\]

\[
\hat{P}_N \hat{B}_{12}^\dagger \hat{b}_L \hat{B}_{12}^\dagger \hat{P}_N = \frac{1}{2} \left[ | \hat{a}_L^\dagger \hat{b}_L - \text{H.c.} \rangle \right].
\]

On the one hand, this reduces the amount of contributing terms, compared to an analogous computation in a number non-conserving setting; on the other hand, the non-locality of the observable is preserved. This absence of number non-conserving terms reduces the efficiency of braiding as we see from the expectation values for the braided state,

\[
\langle F_\phi | \hat{b}_L^\dagger \hat{b}_L | F_\phi \rangle = \frac{1}{4} \sum_{\sigma, \sigma' = (e, o)} \left[ \langle I \rangle_{\sigma\sigma, \sigma'\sigma'} + \langle \hat{a}_L^\dagger \hat{b}_1 \rangle_{\sigma\sigma, \sigma'\sigma'} + \langle \hat{a}_L^\dagger \hat{b}_1 \rangle_{\sigma\sigma, \sigma'\sigma'}^* \right]
\]

\[
= \frac{1}{2} \left[ 1 + \text{Re} \langle \hat{a}_L^\dagger \hat{b}_1 \rangle_{ee, oo} + \text{Re} \langle \hat{a}_L^\dagger \hat{b}_1 \rangle_{oo, ee} \right]
\]

\[
L \rightarrow \infty \frac{1}{2} \left[ 1 + 2 \nu (1 - \nu) \right],
\]

\[
\langle F_\phi | \hat{b}_L^\dagger \hat{b}_L | F_\phi \rangle = \frac{1}{4} \sum_{\sigma, \sigma' = (e, o)} \left[ \langle I \rangle_{\sigma\sigma, \sigma'\sigma'} - \langle \hat{a}_L^\dagger \hat{b}_L \rangle_{\sigma\sigma, \sigma'\sigma'} - \langle \hat{a}_L^\dagger \hat{b}_L \rangle_{\sigma\sigma, \sigma'\sigma'}^* \right]
\]

\[
= \frac{1}{2} \left[ 1 - \text{Re} \langle \hat{a}_L^\dagger \hat{b}_L \rangle_{ee, oo} - \text{Re} \langle \hat{a}_L^\dagger \hat{b}_L \rangle_{oo, ee} \right]
\]

\[
L \rightarrow \infty \frac{1}{2} \left[ 1 - (1-N^{-1}) 2 \nu (1 - \nu) \right] = \frac{1}{2} [1 - 2 \nu (1 - \nu)].
\]

where \( \langle \hat{O} \rangle_{\sigma\sigma, \sigma'\sigma'} = \langle \psi(N) | \sigma \hat{O} | \psi(N) \rangle_{\sigma'\sigma'} \). Note that, due to the factor \((-1)^{(N-1)}\) in the above equation (Eq. (S25)), we have that \( \langle F_\phi | \hat{b}_L^\dagger \hat{b}_L | F_\phi \rangle = \langle F_\phi | \hat{b}_L^\dagger \hat{b}_L | F_\phi \rangle \) if we were using \( N \) odd number of particles. The optimal outcome at half filling \( n_a = 3/4, n_b = 1/4 \), has to be compared to the corresponding occupations \( n_a = 1, n_b = 0 \) in the number non-conserving, half filled case. Nevertheless, the non-local properties of the wave-function are captured in an observable way here.

In contrast, for the non-braided wave-function \( | \Psi \rangle \) we have,

\[
\langle F_\phi | \hat{b}_L^\dagger \hat{b}_L | F_\phi \rangle = \frac{1}{4} \sum_{\sigma, \sigma' = (e, o)} \left[ \langle I \rangle_{\sigma\sigma, \sigma'\sigma'} - i \left( \langle \hat{a}_L^\dagger \hat{b}_1 \rangle_{\sigma\sigma, \sigma'\sigma'} - \langle \hat{a}_L^\dagger \hat{b}_1 \rangle_{\sigma\sigma, \sigma'\sigma'}^* \right) \right] = \frac{1}{2}.
\]

\[
\langle F_\phi | \hat{b}_L^\dagger \hat{b}_L | F_\phi \rangle = \frac{1}{4} \sum_{\sigma, \sigma' = (e, o)} \left[ \langle I \rangle_{\sigma\sigma, \sigma'\sigma'} + i \left( \langle \hat{a}_L^\dagger \hat{b}_L \rangle_{\sigma\sigma, \sigma'\sigma'} - \langle \hat{a}_L^\dagger \hat{b}_L \rangle_{\sigma\sigma, \sigma'\sigma'}^* \right) \right] = \frac{1}{2}.
\]

In summary, while for the unbraided state \( | F_\phi \rangle \) there is an equal probability of finding either 0 or 1 as expected, for \( | F_\phi \rangle \) the effects of braiding are present and strongest in the number measurement at half filling, where our ansatz for the braiding matrix with exactly localized Majorana operators is optimal. Nevertheless, number projection precludes the ideal signal obtained without number conservation.