SOME NEW RESULT ON STRONG CONVERGENCE OF FEJÉR MEANS WITH RESPECT TO VILENKIN SYSTEMS

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Abstract. In this paper we discuss and prove some new strong convergence theorems for partial sums and Fejér means with respect to the Vilenkin system.

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1. INTRODUCTION

Concerning definitions and notations used in this introductions we refer to Sections 2.

It is well-known (for details see e.g. [1, 11, 16]) that Vilenkin system forms not basis in the space $L_1(G_m)$. Moreover, there is a function in the martingale Hardy space $H_1(G_m)$, such that the partial sums of $f$ are not bounded in $L_1(G_m)$-norm. However (for details see e.g. [?]), for all $p > 0$ and $f \in H_p$, there exists an absolute constant $c_p$ such that

$$\|S_M f\|_p \leq c_p \|f\|_{H_p}.$$  \hspace{1cm} (1)

In Gát [6] (see also Simon [17]) the following strong convergence result was obtained for all $f \in H_1(G_m)$:

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0,$$

It follow that

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f\|_1}{k} \leq \|f\|_{H_1}, \quad n = 2, 3, ...$$

In [36] was proved that for any $f \in H_1$ there exists an absolute constant $c$, such that

$$\sup_{n \in \mathbb{N}} \frac{1}{n \log n} \sum_{k=1}^{n} \|S_k f\|_1 \leq \|f\|_{H_1}, \quad n = 2, 3, ...$$

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Moreover for every nondecreasing function \( \varphi : \mathbb{N}_+ \to [1, \infty) \) satisfying the condition
\[
\lim_{n \to \infty} \frac{\log n}{\varphi_n} = +\infty.
\]
there exists a function \( f \in H_1 \), such that
\[
\sup_{n \in \mathbb{N}} \frac{1}{n\varphi_n} \sum_{k=1}^{n} \| S_k f \|_1 = \infty.
\]

For the Vilenkin system Simon [18] proved that there is an absolute constant \( c_p \), depending only on \( p \), such that
\[
\sum_{k=1}^{\infty} \frac{\| S_k f \|_p^p}{k^{2-p}} \leq c_p \| f \|_{H_p}^p,
\]
for all \( f \in H_p(G_m) \), where \( 0 < p < 1 \). In [25] was proved that for any nondecreasing function \( \Phi : \mathbb{N} \to [1, \infty) \), satisfying the condition \( \lim_{n \to \infty} \Phi(n) = +\infty \), there exists a martingale \( f \in H_p(G_m) \), such that
\[
\sum_{k=1}^{\infty} \frac{\| S_k f \|_p^p \Phi(k)}{k^{2-p}} = \infty, \text{ for } 0 < p < 1.
\]

Strong convergence theorems of two-dimensional partial sums was investigate by Weisz [10], Goginava [9], Gogoladze [10], Tephnadze [23, 27], (see also [13]).

Weisz [41] considered the norm convergence of Fejér means of Walsh-Fourier series and proved the following:

**Theorem W1 (Weisz).** Let \( p > 1/2 \) and \( f \in H_p \). Then
\[
\| \sigma_k f \|_p \leq c_p \| f \|_{H_p}.
\]

Moreover, Weisz [41] (see also [15]) also proved that for all \( p > 0 \) and \( f \in H_p \), there exists an absolute constant \( c_p \) such that
\[
\| \sigma_M f \|_p \leq c_p \| f \|_{H_p}.
\]

Theorem W1 implies that
\[
\frac{1}{n^{2p-1}} \sum_{k=1}^{n} \left( \frac{\| S_k f \|_p^p}{k^{2-2p}} \right) \leq c_p \| f \|_{H_p}^p, \quad (1/2 < p < \infty).
\]

If Theorem W1 should hold for \( 0 < p \leq 1/2 \), then we would have
\[
\sum_{k=1}^{\infty} \frac{\| S_k f \|_p^p}{k^{2-2p}} \leq c_p \| f \|_{H_p}^p, \quad (0 < p < 1/2)
\]
\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{\| S_k f \|_{H_{1/2}}^{1/2}}{k} \leq c \| f \|_{H_{1/2}}^{1/2} \quad n = 2, 3, \ldots
\]
and

\[
\frac{1}{n} \sum_{k=1}^{n} \|\sigma_k f\|_{1/2}^{1/2} \leq c \|f\|_{H_{1/2}}^{1/2}.
\]

However, in [21] (see also [3, 4] and [28, 29, 30, 31, 32]) it was proved that the assumption \( p > 1/2 \) in Theorem W1 is essential. In particular, there exists a martingale \( f \in H_{1/2} \) such that

\[
\sup_{n \in \mathbb{N}} \|\sigma_n f\|_{1/2} = +\infty.
\]

In [5] (see also [26]) it was proved that (4) and (5) hold though Theorem W1 is not true for \( 0 < p \leq 1/2 \).

Moreover, in [5] it was proved that if \( 0 < p < 1/2 \) and \( \Phi : \mathbb{N}_+ \to [1, \infty) \) be any nondecreasing function satisfying condition

\[
\lim_{k \to \infty} \frac{k^{2-2p} \Phi_k}{\Phi} = \infty,
\]

then there exists a martingale \( f \in H_p \), such that

\[
\sum_{m=1}^{\infty} \frac{\|\sigma_m f\|_p_{weak-L_p}^{p}}{\Phi_m} = \infty.
\]

On the other hand, for the Walsh system (6) does not hold (see [26]). In particular, it was proved that there exists a martingale \( f \in H_{1/2} \), such that

\[
\sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{m=1}^{n} \|\sigma_m f\|_{1/2}^{1/2} = \infty.
\]

In this paper we prove more general result for bounded Vilenkin system. In special case we also obtain (7).

This paper is organized as follows: in order not to disturb our discussions later on some definitions and notations are presented in Section 2. For the proofs of the main results we need some auxiliary Lemmas, some of them are new and of independent interest. These results are presented in Section 3. The main result with proof is given in Section 4.

2. Definitions and Notations

Let \( \mathbb{N}_+ \) denote the set of the positive integers, \( \mathbb{N} := \mathbb{N}_+ \cup \{0\} \).

Let \( m := (m_0, m_1, \ldots) \) denote a sequence of the positive integers not less than 2.

Denote by

\[
Z_{m_k} := \{0, 1, \ldots, m_k - 1\}
\]

the additive group of integers modulo \( m_k \).

Define the group \( G_m \) as the complete direct product of the group \( Z_{m_j} \) with the product of the discrete topologies of \( Z_{m_j} \)’s.

The direct product \( \mu \) of the measures

\[
\mu_k (\{j\}) := 1/m_k \quad (j \in Z_{m_k})
\]
is the Haar measure on $G_m$ with $\mu(G_m) = 1$.

If $\sup_{n \in \mathbb{N}} m_n < \infty$, then we call $G_m$ a bounded Vilenkin group. If the generating sequence $m$ is not bounded then $G_m$ is said to be an unbounded Vilenkin group. In this paper we discuss bounded Vilenkin groups only.

The elements of $G_m$ are represented by sequences 
\[ x := (x_0, x_1, \ldots, x_k, \ldots) \quad (x_k \in \mathbb{Z}_{m_k}) . \]

It is easy to give a base for the neighbourhood of $G_m$ namely 
\[ I_0(x) := G_m, \]
and 
\[ I_n(x) := \{ y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1} \} \quad (x \in G_m, \ n \in \mathbb{N}) \]

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \setminus I_n$.

Let 
\[ e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G_m \quad (n \in \mathbb{N}) . \]

If we define the so-called generalized number system based on $m$ in the following way:
\[ M_0 := 1, \quad M_{k+1} := m_k M_k, \quad (k \in \mathbb{N}) \]
then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_k$, where $n_j \in \mathbb{Z}_{m_j}$, $(j \in \mathbb{N})$ and only a finite number of $n_j$'s differ from zero. Let $|n| := \max \{ j \in \mathbb{N}; n_j \neq 0 \}$.

For the natural number $n = \sum_{j=1}^{\infty} n_j M_j$, we define 
\[ \delta_j = \text{sign} n_j = \text{sign} (\oplus n_j), \quad \delta_j^* = |\oplus n_j - 1| \delta_j, \]
where $\oplus$ is the inverse operation for $a_k \oplus b_k = (a_k + b_k) \mod m_k$.

We define functions $v$ and $v^*$ by
\[ v(n) = \sum_{j=0}^{\infty} |\delta_{j+1} - \delta_j| + \delta_0, \quad v^*(n) = \sum_{j=0}^{\infty} \delta_j^* . \]

The $n$-th Lebesgue constant is defined in the following way
\[ L_n = \|D_n\|_1. \]

The norm (or quasi norm) of the space $L_p(G_m)$ is defined by
\[ \|f\|_p := \left( \int_{G_m} |f(x)|^p \, d\mu(x) \right)^{1/p} \quad (0 < p < \infty) . \]

The space $\text{weak} - L_p(G_m)$ consists of all measurable functions $f$ for which
\[ \|f\|_{\text{weak} - L_p(G_m)} := \sup_{\lambda > 0} \lambda^p \mu \{ f > \lambda \} < +\infty . \]

Next, we introduce on $G_m$ an orthonormal system which is called the Vilenkin system.
At first define the complex valued function $r_k(x) : G_m \rightarrow \mathbb{C}$, the generalized Rademacher functions, as

$$r_k(x) := \exp \left( \frac{2\pi i x_k}{m_k} \right) \quad \left( i^2 = -1, \ x \in G_m, \ k \in \mathbb{N} \right).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_n^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specially, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L^2(G_m)$ (for details see e.g. [1, 16, 37]).

If $f \in L^1(G_m)$ then we can define Fourier coefficients, partial sums of the Fourier series, Fejér means, Dirichlet and Fejér kernels with respect to the Vilenkin system in the usual manner:

$$\hat{f}(k) := \int_{G_m} f(x) \psi_k(x) \, d\mu \quad (k \in \mathbb{N});$$

$$S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k \quad (n \in \mathbb{N}_+, \ S_0 f := 0);$$

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f \quad (n \in \mathbb{N}_+);$$

$$D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbb{N}_+);$$

$$K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k \quad (n \in \mathbb{N}_+).$$

Recall that (for details see e.g. [11] and [13])

$$D_M n(x) = \begin{cases} M_n & x \in I_n \\ 0 & x \notin I_n \end{cases}$$

and

$$D_{s_n M_n} = D_{M_n} \sum_{k=0}^{s_n-1} \psi_k M_n = D_{M_n} \sum_{k=0}^{s_n-1} r_n^k, \quad 1 \leq s_n \leq m_n - 1.$$

The $\sigma$-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ will be denoted by $F_n (n \in \mathbb{N})$. Denote by $f = (f_n, \ n \in \mathbb{N})$ a martingale with respect to $F_n (n \in \mathbb{N})$ (for details see e.g. [33]). The maximal function of a martingale $f$ is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f_n|.$$
In the case \( f \in L_1(G_m) \), the maximal functions are also be given by
\[
f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \mu(u) \right|.
\]

For \( 0 < p < \infty \) the Hardy martingale spaces \( H_p(G_m) \) consist of all martingales for which
\[
\|f\|_{H_p} := \|f^*\|_p < \infty.
\]

If \( f \in L_1(G_m) \), then it is easy to show that the sequence \( (S_{M_n}f : n \in \mathbb{N}) \) is a martingale. If \( f = (f_n, \ n \in \mathbb{N}) \) is martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:
\[
\hat{f}(i) := \lim_{k \to \infty} \int_{G_m} f_k(x) \overline{\psi_i(x)} \, d\mu(x).
\]
The Vilenkin-Fourier coefficients of \( f \in L_1(G_m) \) are the same as those of the martingale \((S_{M_n}f : n \in \mathbb{N})\) obtained from \( f \).

A bounded measurable function \( a \) is \( p \)-atom, if there exist an interval \( I \), such that
\[
\int_I ad\mu = 0, \quad \|a\|_\infty \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.
\]

3. Auxiliary Lemmas

**Lemma 1.** \([38, 39]\) A martingale \( f = (f_n, \ n \in \mathbb{N}) \) is in \( H_p(0 < p \leq 1) \) if and only if there exist a sequence \( (a_k, k \in \mathbb{N}) \) of \( p \)-atoms and a sequence \( (\mu_k, k \in \mathbb{N}) \) of real numbers such that, for every \( n \in \mathbb{N} \),
\[
\sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f_n, \quad \text{a.e.}
\]
where
\[
\sum_{k=0}^{\infty} |\mu_k|^p < \infty.
\]
Moreover,
\[
\|f\|_{H_p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}
\]
where the infimum is taken over all decomposition of \( f \) of the form \((10)\).

By using atomic decomposition of \( f \in H_p \) martingales, we can derive a counterexample, which play a central role to prove sharpness of our main results and it will be used several times in the paper:
Lemma 2. \[19\] Let \( n \in \mathbb{N} \) and \( 1 \leq s_n \leq m_n - 1 \). Then

\[ s_n M_n K_{s_n M_n} = \sum_{i=0}^{s_n-1} \left( \sum_{l=0}^{i-1} r_{n}^l \right) M_n D_{M_n} + \left( \sum_{l=0}^{s_n-1} r_{n}^l \right) M_n K_{M_n} \]

and

\[ |s_n M_n K_{s_n M_n} (x)| \geq \frac{M_n^2}{2 \pi}, \quad \text{for} \quad x \in I_{n+1} (e_n - 1 + e_{n}). \]

Moreover, if \( x \in I_t / I_{t+1}, \ x - x_t e_t \notin I_n \) and \( n > t \), then

(11) \[ K_{s_n M_n} (x) = 0. \]

Lemma 3. \[5\] Let \( n = \sum_{i=1}^{r} s_{n_i} M_{n_i}, \) where \( n_{n_1} > n_{n_2} > \cdots > n_{n_r} \geq 0 \) and \( 1 \leq s_{n_i} < m_{n_i} \) for all \( 1 \leq i \leq r \) as well as \( n^{(k)} = n - \sum_{i=1}^{k} s_{n_i} M_{n_i}, \) where \( 0 < k \leq r. \) Then

\[ nK_n = \sum_{k=1}^{r} \left( \prod_{j=1}^{k-1} r_{n_j}^{s_{n_j}} \right) s_{n_k} M_{n_k} K_{s_{n_k} M_{n_k}} \]

\[ + \sum_{k=1}^{r-1} \left( \prod_{j=1}^{k-1} r_{n_j}^{s_{n_j}} \right) n^{(k)} D_{s_{n_k} M_{n_k}}. \]

Lemma 4. Let

\[ n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} n_k M_k, \]

where \( 0 \leq l_1 \leq m_1 \leq l_2 - 2 < l_2 \leq m_2 \leq \ldots \leq l_s - 2 < l_s \leq m_s. \)

Then

\[ n |K_n (x)| \geq c M_{l_i}^2, \quad \text{for} \quad x \in I_{l_i+1} (e_{l_i} - 1 + e_{l_i}), \]

where \( \lambda = \sup_{n \in \mathbb{N}} m_n \) and \( c \) is an absolute constant.

Proof. Let \( x \in I_{l_i+1} (e_{l_i} - 1 + e_{l_i}). \) By combining (11) in Lemma 2 (8) and \( 9 \) we obtain that

\[ D_{l_i} = 0 \]

and

\[ D_{s_{n_k} M_{s_{n_k}}} = K_{s_{n_k} M_{s_{n_k}}} = 0, \quad s_{n_k} > l_i. \]

Since \( s_{n_1} > s_{n_2} > \cdots > s_{n_r} \geq 0 \) we find that

\[ n^{(k)} = n - \sum_{i=1}^{k} s_{n_i} M_{n_i} = \sum_{i=k+1}^{s} s_{n_i} M_{n_i} \]

\[ \leq \sum_{i=0}^{n_{k+1}} (m_i - 1) M_i = m_{n_{k+1}} M_{n_{k+1}} - 1 \leq M_{n_k}. \]
According to Lemma 3 we have that

\[ n |K_n| \geq |s_{l_i} M_{l_i} K_{s_{l_i} M_{l_i}}| \]

\[ - \sum_{r=1}^{i-1} \sum_{k=l_r}^{m_r} |s_k M_k K_{s_k M_k}| \]

\[ - \sum_{r=1}^{i-1} \sum_{k=l_r}^{m_r} |M_k D_{s_k M_k}| \]

\[ = I_1 - I_2 - I_3. \]

Let \( x \in I_{l_i+1} (e_{l_i-1} + e_{l_i}) \) and \( 1 \leq s_{l_i} \leq m_{l_i} - 1 \). By using Lemma 2 we get that

\[ I_1 = |s_{l_i} M_{l_i} K_{s_{l_i} M_{l_i}}| \geq \frac{M_{l_i}^2}{2\pi} \geq \frac{2M_{l_i}^2}{9}. \]

It is easy to see that

\[ \sum_{s=0}^{k} n_s^2 M_s^2 \leq \sum_{s=0}^{k} (m_s - 1)^2 M_s^2 \]

\[ \leq \sum_{s=0}^{k} M_{s+1}^2 - 2 \sum_{s=0}^{k} M_s M_{s+1} + \sum_{s=0}^{k} M_s^2 \]

\[ = M_{k+1}^2 + 2 \sum_{s=0}^{k} M_s^2 - 2 \sum_{s=0}^{k} M_{s+1} M_s - M_0^2 \]

\[ \leq M_{k+1}^2 - 1. \]

and

\[ \sum_{s=0}^{k} n_s M_s \leq \sum_{s=0}^{k} (m_s - 1) M_s = m_k M_k - m_0 M_0 \leq M_{k+1} - 2. \]

Since \( m_{i-1} \leq l_i - 2 \) if we use the estimates above, then we obtain that

\[ I_2 \leq \sum_{s=0}^{l_i-2} |n_s M_s K_{n_s M_s} (x)| \leq \sum_{s=0}^{l_i-2} n_s M_s \frac{(n_s M_s + 1)}{2} \]

\[ \leq \frac{(m_{l_i-2} - 1) M_{l_i-2}}{2} \sum_{s=0}^{l_i-2} (n_s M_s + 1) \]
\[ \frac{(m_{l_i-2} - 1) M_{l_i-2} M_{l_i-1}}{2} + \frac{(m_{l_i-2} - 1) M_{l_i-2} l_i}{2} \leq \frac{M_{l_i-1}^2}{2} - \frac{M_{l_i-2} M_{l_i-1}}{2} + M_{l_i-1} l_i. \]

For \( I_3 \) we have that
\[
I_3 \leq \sum_{k=0}^{l_i-2} |M_k D_{n_k M_k} (x)| \leq \sum_{k=0}^{l_i-2} n_k M_k^2
\]
\[
\leq M_{l_i-2} \sum_{k=0}^{l_i-2} n_k M_k \leq M_{l_i-1} M_{l_i-2} - 2M_{l_i-2}.
\]

By combining (12)-(13) we have that
\[
n |K_n (x)| \geq I_1 - I_2 - I_3
\]
\[
\geq \frac{M_{l_i}^2}{2\pi} + \frac{3}{2} + 2M_{l_i-2}
\]
\[
- \frac{M_{l_i-1} M_{l_i-2}}{2} - \frac{M_{l_i-1}^2}{2} - M_{l_i-1} l_i
\]
\[
\geq \frac{M_{l_i}^2}{2\pi} - \frac{M_{l_i}^2}{16} - \frac{M_{l_i}^2}{8} + \frac{7}{2} - M_{l_i-1} l_i
\]
\[
\geq \frac{2M_{l_i}^2}{9} - \frac{3M_{l_i}^2}{16} + \frac{7}{2} - M_{l_i-1} l_i
\]
\[
\geq \frac{M_{l_i}^2}{144} - M_{l_i-1} l_i.
\]

Suppose that \( l_i \geq 4 \). Then
\[
n |K_n (x)| \geq \frac{M_{l_i}^2}{36} - \frac{M_{l_i}}{4} \geq \frac{M_{l_i}^2}{36} - \frac{M_{l_i}^2}{64} \geq \frac{5M_{l_i}^2}{36 \cdot 16} \geq \frac{M_{l_i}^2}{144}.
\]

The proof is complete. \( \square \)

4. The Main Result

Our main result reads:

**Theorem 1.** a) Let \( f \in H_{1/2} \). Then there exists an absolute constant \( c \), such that
\[
\sup_{n \in \mathbb{N}} \frac{1}{n \log n} \sum_{k=1}^{n} \| \sigma_k f \|_{H_{1/2}}^{1/2} \leq c \| f \|_{H_{1/2}}^{1/2} \quad n = 2, 3, ...
\]

b) Let \( \varphi : \mathbb{N}_+ \to [1, \infty) \) be a nondecreasing function satisfying the condition
\[
\lim_{n \to \infty} \frac{\log n}{\varphi_n} = +\infty.
\]
Then there exists a function \( f \in H_{1/2} \), such that
\[
\sup_{n \in \mathbb{N}^+} \frac{1}{n} \sum_{k=1}^{n} \| \sigma_k f \|_{1/2}^{1/2} = \infty.
\]

**Corollary 1.** There exists a martingale \( f \in H_{1/2} \), such that
\[
\sup_{n \in \mathbb{N}^+} \frac{1}{n} \sum_{k=1}^{n} \| \sigma_k f \|_{1/2}^{1/2} = \infty.
\]

**Proof of Theorem 7.** In [24] was proved that there exists an absolute constant \( c \), such that
\[
\| \sigma_k f \|_{1/2}^{1/2} \leq c \log k \| f \|_{1/2}^{1/2}, \quad k = 1, 2, \ldots
\]
Hence,
\[
\frac{1}{n \log n} \sum_{k=1}^{n} \| \sigma_k f \|_{1/2}^{1/2} \leq \frac{c}{n \log n} \sum_{k=1}^{n} \log k \leq c \frac{\| f \|_{1/2}^{1/2}}{n \log n}, \quad n = 2, 3, \ldots
\]

The proof of part a) is complete.

Under the condition (13) there exists an increasing sequence of the positive integers \( \{ \alpha_k : k \in \mathbb{N} \} \) such that
\[
\lim_{k \to \infty} \log M_{\alpha_k} = +\infty
\]
and
\[
\sum_{k=0}^{\infty} \frac{\varphi_{2M_{\alpha_k}}^{1/2}}{\log^{1/2} M_{\alpha_k}} < c < \infty. \tag{14}
\]

Let \( f = (f_n) \) be martingale, defined by
\[
f_n := \sum_{\{k : 2\alpha_k < n\}} \lambda_k a_k,
\]
where
\[
a_k = M_{\alpha_k} r_{\alpha_k} D_{M_{\alpha_k}} = M_{\alpha_k} (D_{2M_{\alpha_k}} - D_{M_{\alpha_k}})
\]
and
\[
\lambda_k = \frac{\varphi_{2M_{\alpha_k}}}{\log M_{\alpha_k}}.
\]

Since
\[
S_{2A} a_k = \begin{cases} a_k, & \alpha_k < A, \\ 0, & \alpha_k \geq A, \end{cases}
\]
\[
\text{supp}(a_k) = I_{\alpha_k}, \quad \int_{I_{\alpha_k}} a_k d\mu = 0, \quad \| a_k \|_{\infty} \leq M_{\alpha_k}^{2} = \mu(\text{supp} a_k)^{-2},
\]
if we apply Lemma 1 and (13) we conclude that \( f \in H_{1/2} \).
Moreover,

\[
\tilde{f}(j) = \begin{cases} 
M_{\alpha k} \lambda_k, & j \in \{M_{\alpha k}, \ldots, 2M_{\alpha k} - 1\}, \quad k \in \mathbb{N} \\
0, & j \notin \bigcup_{k=1}^{\infty} \{M_{\alpha k}, \ldots, 2M_{\alpha k} - 1\}.
\end{cases}
\]

We have that

\[
\sigma_n f = \frac{1}{n} \sum_{j=0}^{n-1} S_j f + \frac{1}{n} \sum_{j=M_{\alpha k}}^{n-1} S_j f
= I + II.
\]

Let \(M_{\alpha k} \leq j < 2M_{\alpha k}\). Since \(D_{j+M_{\alpha k}} = D_{M_{\alpha k}} + \psi_{M_{\alpha k}} D_j\), when \(j \leq M_{\alpha k}\), if we apply (16) we obtain that

\[
S_j f = S_{M_{\alpha k}} f + \sum_{v=M_{\alpha k}}^{j-1} \tilde{f}(v) \psi_v
= S_{M_{\alpha k}} f + M_{\alpha k} \lambda_k \sum_{v=M_{\alpha k}}^{j-1} \psi_v
= S_{M_{\alpha k}} f + M_{\alpha k} \lambda_k \left( D_j - D_{M_{\alpha k}} \right)
= S_{M_{\alpha k}} f + \lambda_k \psi_{M_{\alpha k}} D_{j-M_{\alpha k}}
\]

According to (18) concerning \(II\) we conclude can that

\[
II = \frac{n - M_{\alpha k}}{n} S_{M_{\alpha k}} f
+ \frac{\lambda_k M_{\alpha k}}{n} \sum_{j=M_{\alpha k}}^{n-1} \psi_{M_{\alpha k}} D_{j-M_{\alpha k}}
= II_1 + II_2.
\]

We can estimate \(II_2\) as follows:

\[
|II_2| = \left| \frac{\lambda_k M_{\alpha k}}{n} \right| \psi_{M_{\alpha k}} \sum_{j=0}^{n-M_{\alpha k}-1} D_j
= \frac{\lambda_k M_{\alpha k} (n-M_{\alpha k})}{n} |K_{n-M_{\alpha k}}|
\geq \lambda_k (n-M_{\alpha k}) |K_{n-M_{\alpha k}}|.
\]

Let \(n = \sum_{i=1}^{s} \sum_{k=l_i}^{m_i} M_k\), where
\(0 \leq l_1 \leq m_1 \leq l_2 - 2 \leq m_2 \leq \ldots \leq l_s - 2 \leq l_s \leq m_s\).
By applying Lemma \[4\] we get that
\[
|II_2| \geq c\lambda_k \left| (n - M_{\alpha_k}) K_n - M_{\alpha_k} (x) \right| \\
\geq c\lambda_k M_{\alpha_k}^2, \quad \text{for} \ x \in I_{i+1} (e_{i-1} + e_i).
\]

Hence
\[
\int_{G_m} |II_2|^{1/2} \, d\mu \geq \sum_{i=1}^{s-1} \int_{I_{i+1} (e_{i-1} + e_i)} |II_2|^{1/2} \, d\mu \\
\geq c \sum_{i=1}^{s-1} \int_{I_{i+1} (e_{i-1} + e_i)} \lambda_k^{1/2} M_i \, d\mu \\
\geq c\lambda_k^{1/2} (s - 1) \geq c\lambda_k^{1/2} v (n - M_{\alpha_k}).
\]

In view of (11), (3) and (17) we find that
\[
\text{In view of (11), (3) and (17) we find that}
\]

\[
\|I\|^{1/2} = \left\| \frac{M_{\alpha_k}}{n} \sigma M_{\alpha_k} f \right\|^{1/2} \leq \left\| \sigma M_{\alpha_k} f \right\|_{H_{1/2}}^{1/2} \leq c \|f\|_{H_{1/2}}^{1/2}
\]

and
\[
\|II_1\|^{1/2} = \left\| \frac{n - M_{\alpha_k}}{n} S_{M_{\alpha_k}} f \right\|^{1/2} \leq \left\| S_{M_{\alpha_k}} f \right\|_{H_{1/2}}^{1/2} \leq c \|f\|_{H_{1/2}}^{1/2}.
\]

By combining (19), (20) and (21) we get that
\[
\|\sigma_n f\|_{H_{1/2}}^{1/2} \geq \|II_2\|_{H_{1/2}}^{1/2} - \|II_1\|_{H_{1/2}}^{1/2} - \|I\|_{H_{1/2}}^{1/2} \\
\geq c\lambda_k^{1/2} v (n - M_{\alpha_k}) - c \|f\|_{H_{1/2}}^{1/2}.
\]

By using estimates with the above we can conclude that
\[
\frac{1}{n} \sum_{k=1}^{n} \|\sigma_k f\|_{H_{1/2}}^{1/2} \\
\geq \frac{1}{M_{\alpha_k} + 1} \sum_{M_{\alpha_k} \leq l \leq 2 M_{\alpha_k}} \|\sigma_l f\|_{H_{1/2}}^{1/2} \\
\geq \frac{c}{M_{\alpha_k} \varphi_{2M_{\alpha_k}}} \sum_{M_{\alpha_k} \leq l \leq 2 M_{\alpha_k}} \left( \lambda_k^{1/2} v (l - M_{\alpha_k}) - c \|f\|_{H_{1/2}}^{1/2} \right) \\
\geq \frac{c\lambda_k^{1/2}}{M_{\alpha_k} \varphi_{2M_{\alpha_k}}} \sum_{l=1}^{M_{\alpha_k}} v (l) - \frac{c \|f\|_{H_{1/2}}^{1/2}}{M_{\alpha_k} \varphi_{2M_{\alpha_k}}} \sum_{M_{\alpha_k} \leq l \leq 2 M_{\alpha_k}} 1 \\
\geq \frac{c\lambda_k^{1/2}}{M_{\alpha_k} \varphi_{2M_{\alpha_k}}} \sum_{l=1}^{M_{\alpha_k} - 1} v (l) - c \geq c \frac{\log^{1/2} M_{\alpha_k}}{\varphi_{2M_{\alpha_k}}} \rightarrow \infty, \quad \text{as} \ k \rightarrow \infty.
\]

The proof is complete. \[\square\]
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