TRIANGULINE GALOIS REPRESENTATIONS
AND SCHUR FUNCTORS

ANDREA CONTI

ABSTRACT. Given a $B$-pair $W$ and a Schur functor $S$, we show under some general assumptions that $W$ is trianguline if and only if $S(W)$ is. This is an extension of earlier work of Di Matteo. We derive some consequences on the behavior of local Galois representations under morphisms of Langlands dual groups. We attach to a Schur functor a map between the trianguline deformation spaces defined by Hellmann, and we study congruence loci on the Hecke-Taylor-Wiles varieties constructed by Breuil, Hellmann and Schraen for unitary groups.

CONTENTS

1. Introduction 1
2. Generalities on $B$-pairs 4
3. Generalities on Schur functors 7
4. Schur functors and potentially trianguline $B$-pairs 8
5. Schur functors and trianguline $B$-pairs 11
6. Lifting trianguline representations via isogenies 12
7. Congruence loci on Hecke-Taylor-Wiles varieties 14
7.1. Schur functors on trianguline varieties 19
7.2. Schur functors on Hecke-Taylor-Wiles varieties 21
References 24

1. Introduction

The original motivation for this paper comes from the intention of generalizing a result of [Cont16b] that we recall here very approximately. Fix a prime $p$ larger than 3. Consider an abstract Hecke algebra $\mathcal{H}$ for $GSp_4$, spherical outside $Np$ for a fixed integer $N$ and of Iwahoric level at $p$. Let $\mathcal{E}$ be the $GSp_4$-eigenvariety constructed by Andreatta, Iovita and Pilloni in [AIP15]. There exists a homomorphism $\theta$ from $\mathcal{H}$ to the ring of rigid analytic functions $\mathcal{O}(\mathcal{E})$ over $\mathcal{E}$ interpolating the systems of Hecke eigenvalues of classical Siegel modular forms of tame level $N$ and Iwahoric level at $p$. We denote by $G_K$ the absolute Galois group of a local or global field $K$. In [Cont16b, Section 10.3], two loci on $\mathcal{E}$ are defined:

* the Galois symmetric cube locus, as the locus of points $x$ of $\mathcal{E}$ such that the associated Galois representation $\rho_x : G_\mathbb{Q} \to GSp_4(\mathbb{Q}_p)$ takes values in the image of the symmetric cube representation $\text{Sym}^3 : GL_2(\mathbb{Q}_p) \to GSp_4(\mathbb{Q}_p)$;
* the automorphic symmetric cube locus, as the locus of points $x$ of $\mathcal{E}$ such that the associated system of Hecke eigenvalues $\theta_x$ derives from that of a $p$-adic overconvergent
modular form for GL$_2$ via a suitable morphism of Hecke algebras attached to the symmetric cube representation.

**Theorem 1.1.** [Cont16b, Theorem 10.10] The Galois and automorphic symmetric cube loci on $\mathcal{E}$ coincide.

Thanks to the wide range of reductive algebraic groups for which eigenvarieties are available, one may wish to generalize Theorem 1.1 to other representations than the symmetric cube. We summarize the proof by a chain of implications below, where the property “trianguline” is intended in the sense of $(\varphi, \Gamma)$-modules or $B$-pairs (see Definition 2.4). Consider the following statements:

1. $x$ is a point of the Galois symmetric cube locus on $\mathcal{E}$;
2. the restriction of $\rho_x$ to a decomposition group at $p$ is trianguline;
3. there exists a continuous representation $\rho_y : G_\mathbb{Q} \to \text{GL}_2(\mathbb{Q}_p)$ that satisfies $\rho_x \cong \text{Sym}^3 \rho_y$;
4. the restriction of the representation $\rho_y$ at point (3) to a decomposition group at $p$ is trianguline;
5. the representation $\rho_y$ at point (3) is attached to a $p$-adic overconvergent automorphic form for GL$_2$;
6. the point $x$ belongs to the automorphic symmetric cube locus on $\mathcal{E}$.

Under some technical hypotheses, there are implications (1) $\Rightarrow$ (2)$\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6).

The implication (1) $\Rightarrow$ (2) is an application of [KPX14, Corollary 6.3.13], a very general result that can be applied whenever the eigenvariety under consideration admits a Zariski-dense set of “refined crystalline” points in the sense of [KPX14, Definition 6.4.1] (see for instance the application in [BHS17, Section 2.2]). The arrow (5) $\Rightarrow$ (6) follows immediately from the definition of the symmetric cube morphism of Hecke algebras. On the other hand, the implication (4) $\Rightarrow$ (5) relies on the work of Emerton and Kisin on the overconvergent Fontaine-Mazur conjecture, that is very specific to the case of representations $G_\mathbb{Q}_p \to \text{GL}_2(\mathbb{Q}_p)$. We will need to assume an analogue of this conjecture in the more general cases we are interested in.

The goal of this paper is to fill the diagram (1.1) by generalizing the results (1) $\Rightarrow$ (3) [Cont16a, Lemma 3.11.5] and (2) + (3) $\Rightarrow$ (4) [Cont16a, Proposition 3.10.25] to an algebraic representation $S : \text{GL}_n \to \text{GL}_m$, for arbitrary integers $n$ and $m$, defined by a Schur functor. This generalization is the content of Proposition 6.3 that relies essentially on the main results of the paper, Theorems 1.2 and 5.2. We summarize them in the following statement. Let $S$ be as above, let $K$ be a $p$-adic field and let $\rho : G_K \to \text{GL}_n(\mathbb{Q}_p)$ be a continuous representation.
Theorem 1.2.

* If $S \circ \rho: G_K \to \GL_m(\mathbb{Q}_p)$ is potentially trianguline (see Definition 2.4) then $\rho$ is potentially trianguline (Theorem 4.2).

* If $S \circ \rho: G_K \to \GL_m(\mathbb{Q}_p)$ is trianguline and it admits a triangulation whose parameters are derived from those of $\rho$ (see Remark 4.8), then $\rho$ is trianguline (Theorem 5.2).

Our results are actually more general than Theorem 1.1, in the sense that we work in the category of $B$-pairs that contains as a strict subcategory that of $p$-adic representations of $G_K$. Accordingly, trianguline Galois representations are replaced by triangulable $B$-pairs. We also remark that when the property “trianguline” is replaced by “de Rham”, the analogues of the previous results are proved by Di Matteo in [DiM 13a, Theorem 2.4.2].

In the last section we describe an application of diagram (1.1) to the Hecke-Taylor-Wiles varieties for unitary groups that are studied by Breuil, Hellmann and Schraen in [BHS17]. We summarize our results here.

Let $n$ be an integer such that $p > 2n + 1$. Let $F^+$ be a totally real number field and let $F$ be a totally imaginary quadratic extension of $F^+$. Let $G$ be a unitary group over $F^+$ such that $G$ is split over $F$, compact at infinity and isomorphic to $\GL_n$ at all places in $\Sigma_p$. Let $k$ be a finite field of characteristic $p$ and let $\rho: G_F \to \GL_n(k)$ be a residual representation appearing in a space of automorphic forms for $G$. For a $p$-adic place $v$ of $F$ and a decomposition subgroup $G_{F_v}$ of $G_F$, the couples $(\rho|_{G_{F_v}}, \delta)$ consisting of a trianguline characteristic 0 lift $\rho$ of $\rho|_{G_{F_v}}$ and an $n$-tuple of parameters of a triangulation of $\rho|_{G_{F_v}}$ live naturally on a “trianguline deformation space” $X_{\tri,v}$ [Hel12, Theorem 1.1]. Breuil, Hellmann and Schraen [BHS17] attached to $G$ and to $\rho$ a “Hecke-Taylor-Wiles variety” $X_p(\rho)$ that is strictly related both to the eigenvariety for $G$ and $\rho$ (see for instance [Che11, Definition 2.2]) and to the trianguline deformations spaces $X_{\tri,v}$. In particular there is a natural morphism $X_p(\rho) \to \prod_{v|p} X_{\tri,v}$.

As before, consider an algebraic representation $S: \GL_n \to \GL_m$. We study the morphisms induced by $S$ between the trianguline deformation spaces and the Hecke-Taylor-Wiles varieties attached to two groups $G_n$ and $G_m$ on $F^+$ that are isomorphic to $\GL_n$ and $\GL_m$, respectively, at $p$-adic places. We show the following.

Theorem 1.3.

* For every $p$-adic place $v$ of $F$, the representation $S$ induces morphisms of trianguline deformation spaces $S_{\tri,v}: X_{\tri,v} \to X_{\tri,S \circ \rho}$.

* Assume part of the standard modularity conjectures for $G_n$ and $G_m$, as reformulated in Conjecture 7.4. Then $\prod_{v \in V} S_{\tri}$ induces a morphism $S_{TW}$ fitting in the following commutative diagram:

$$
\begin{array}{ccc}
X_p(\rho) & \longrightarrow & X_{\tri,v} \\
\downarrow S_{TW} & & \downarrow \prod_{v \in \rho} S_{\tri,v} \\
X_p(S \circ \rho) & \longrightarrow & X_{\tri,S \circ \rho}
\end{array}
$$

We define an $S$-congruence locus on $X_p(S \circ \rho)$ as the locus of points whose associated local Galois representation are obtained via $S$ from $\GL_n$-valued representations. By
combining Theorems 1.2 and 1.3, we give a characterization of the $S$-congruence locus that can be seen as a vast generalization of [Cont16b, Theorem 10.10].

**Theorem 1.4.**

* Every point $x$ of the $S$-congruence locus is the “twin” of a point $x'$ in the image of the morphism $S_{TW}$, that is $x$ and $x'$ carry the same set of local Galois representation but possibly different triangulations. (Theorem 7.18).

* The image of $S_{TW}$ consists of the irreducible components of maximal dimension of the $S$-congruence locus (Corollary 7.20).

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**Notation.** In the following $p$ is a fixed prime number. We denote by $\overline{\mathbb{Q}}_p$ an algebraic closure of the field $\mathbb{Q}_p$. We extend the $p$-adic valuation on $\overline{\mathbb{Q}}_p$ to a valuation on $\overline{\mathbb{Q}}_p$ and we let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$ with respect to this choice. By a “$p$-adic field” we will always mean a finite extension of $\mathbb{Q}_p$. Given a local or global field $F$ we denote by $G_F$ its absolute Galois group, equipped with the profinite topology. Throughout the whole text the letters $K$ and $E$ will denote two $p$-adic fields.

For every $n \geq 1$ we write $1_n$ for the $n \times n$ unit matrix.

Let $K$ be a finite extension of $\mathbb{Q}_p$ and let $X$ be a rigid analytic space over $K$. We denote by $O(X)$ the $K$-algebra of rigid analytic functions on $X$, and by $O(X)^\circ$ the $O_K$-subalgebra of functions of norm bounded by 1. We say that $X$ is wide open if there exists an admissible covering $\{X_i\}_{i \in \mathbb{N}}$ of $X$ by affinoid domains $X_i$ such that, for every $i$, $X_i \subset X_{i+1}$ and the map $O(X_{i+1}) \to O(X_i)$ induced by the previous inclusion is compact.

Given a formal scheme Spf $(A)$ over Spf $(O_K)$, we denote by Spf $(A)^\text{rig}$ the rigid analytic space over $K$ attached to Spf $(A)$ by Berthelot’s construction, described for instance in [deJ95, Section 7].

2. **Generalities on $B$-pairs**

We refer to [Ber08] and [DiM13b] for the basic definitions concerning $B$-pairs. Let $B$ be a topological ring equipped with a continuous action of $G_K$. We call semilinear $B$-representation of $G_K$ a free $B$-module of finite rank $M$ equipped with a semilinear action of $G_K$, meaning that $g(bm) = g(b)g(m)$ for every $b \in B$, $m \in M$ and $g \in G_K$. We denote by $\text{Sem}_{G_K}(B)$ the category whose objects are the semilinear $B$-representations of $G_K$ and whose morphisms are the $G_K$-equivariant morphisms of $B$-modules. We call rank of an object of $\text{Sem}_{G_K}(B)$ its rank as a $B$-module.

We denote by $B_{\text{cris}}, B_{\text{st}}, B_{\text{dR}}^+$ and $B_{\text{dR}}$ the rings of periods defined by Fontaine in [Fon94]. Each of these objects is a topological $\mathbb{Q}_p$-algebra carrying a continuous action of $G_K$. We add an index $E$ denote the extension of scalars from $\mathbb{Q}_p$ to $E$. We denote
by $t$ Fontaine’s chosen generator of the maximal ideal of the complete discrete valuation ring $B_{\text{dR}}^+$. We denote by $\varphi$ the Frobenius endomorphism of $B_{\text{cris}}$. We set $B_{e} = B_{\text{cris}}^e$. Let $B_{e,E} = B_{e} \otimes_{\mathbb{Q}_p} E$. We let $G_K$ act on $B_{e,E}$ via its action on the first factor, that is we set $g(b \otimes e) = g(b) \otimes e$ for every $b \in B_{E}$, $e \in E$ and $g \in G_K$.

**Definition 2.1.** A $B_{K}^E$-pair is a pair $(W_e, W_{\text{dR}}^+)$ where:

* $W_e$ is an object of $\text{Sem}_{G_K}(B_{e,E})$;
* $W_{\text{dR}}^+$ is a $G_K$-stable $B_{\text{dR},E}^+$-lattice of $B_{\text{dR}} \otimes B_{e,E} W_e$.

We define the rank of $(W_e, W_{\text{dR}}^+)$ as the common rank of $W_e$ and $W_{\text{dR}}^+$. Given two $B_{K}^E$-pairs $(W_e, W_{\text{dR}}^+)$ and $(W'_e, W'_{\text{dR}}^+)$, a morphism of $B_{K}^E$-pairs $(W_e, W_{\text{dR}}^+) \to (W'_e, W'_{\text{dR}}^+)$ is a pair $(f_e, f_{\text{dR}}^+)$ where:

* $f_e: W_e \to W'_e$ is a morphism in $\text{Sem}_{G_K}(B_{e,E})$;
* $f_{\text{dR}}^+$ is a morphism in $B_{\text{dR},E}^+$-

with the obvious notations, the $B_{\text{dR},E}$-linear morphisms $f_e[1/t], f_{\text{dR}}[1/t] : W_{\text{dR}} \to W'_{\text{dR}}$ coincide.

The objects and morphisms described above define the category of $B_{K}^E$-pairs.

For a $B_{K}^E$-pair $(W_e, W_{\text{dR}}^+)$ we also set $W_{\text{dR}} = B_{\text{dR},E} \otimes B_{e,E} W_e$, that is the same as $B_{\text{dR},E} \otimes B_{e,E} W_e$; it is an object of $\text{Sem}_{G_K}(B_{\text{dR}})$. Given two $B_{K}^E$-pairs $X = (X_e, X_{\text{dR}}^+)$ and $W = (W_e, W_{\text{dR}}^+)$ such that $X_e \subset W_e$ and $W_{\text{dR}}^+ \subset W_{\text{dR}}^+$, we say that $X$ is a saturated sub-$B_{K}^E$-pair of $W$ if the lattice $W_{\text{dR}}^+$ is saturated in $W_{\text{dR}}^+$. The quotient $W/X$ admits a natural structure of $B_{K}^E$-pair if and only if $X$ is a saturated sub-$B_{K}^E$-pair of $W$.

Given a $B_{K}^E$-pair $W$ and finite extensions $L/K$ and $F/E$, we can define a $B_{K}^E$-pair as $(F \otimes_E W)|_{G_L}$, with the obvious notations. Given a property $?$, we say that $W$ has $?$ potentially if there is a finite extension $L/K$ such that $W|_{G_L}$ has $?$. 

**Definition 2.2.** Let $? \in \{\text{cris}, \text{st}, \text{dR}\}$. We say that a $B_{K}^E$-pair $(W_e, W_{\text{dR}}^+)$ is (potentially) $B_{?,E}$-admissible if $B_{?,E} \otimes_{B_{e,E}} W_e$ is (potentially) trivial in $\text{Sem}_{G_K}(B_{?,E})$. We call $(W_e, W_{\text{dR}}^+)$ (potentially) crystalline, semi-stable, de Rham if it is (potentially) admissible for the corresponding ring.

Recall that the properties of being de Rham and potentially semi-stable are equivalent for a $B_{K}^E$-pair by a result of Berger [Ber08, Théorème 2.3.5].

Let $\text{Rep}_{G_K}(E)$ be the category of continuous, $E$-linear, finite-dimensional representation $V$ of $G_K$. For an object $V$ of $\text{Rep}_{G_K}(E)$ we denote by $W(V)$ the $B_{K}^E$-pair $(B_{e,E} \otimes_E V, B_{\text{dR}}^+ \otimes_E V)$. The rank of $W(V)$ is equal to the $E$-rank of $V$. Given two objects $V$, $V'$ of $\text{Rep}_{G_K}(E)$ and a morphism $f : V \to V'$, we define a morphism $W(f) : W(V) \to W(V')$ by $B_{e,E}$-linearly extending $f$ to the first element of $W(V)$ and $B_{\text{dR}}^+$ linearly to the second. The functor $W(\cdot)$ defined this way is fully faithful and identifies $\text{Rep}_{G_K}(E)$ with a full tensor subcategory of the category of $B_{K}^E$-pairs. This is [Ber08, Théorème 3.2.3] when $E = \mathbb{Q}_p$ and an immediate consequence of it for general $E$.

For $? \in \{\text{cris}, \text{st}, \text{dR}\}$, an object $V$ of $\text{Rep}_{G_K}(E)$ is (potentially) $B_?$-admissible in the sense of Fontaine if and only if the $B_{K}^E$-pair $W(V)$ is (potentially) $B_{?,E}$-admissible in the sense of Definition 2.2. This means that $V$ is (potentially) crystalline, semi-stable, de Rham if and only if $W(V)$ is.

Let $R$ be a ring equipped with an action of $G_K$. A character $\eta : G_K \to R^\times$ is $G_K$-equivariant (for the action of $G_K$ on itself by conjugation) if and only if it takes values
in \((R^\times)^{G_K}\), where we use the standard notation for the elements fixed by a group action. For a character \(\eta: G_K \to (R^\times)^{G_K}\), we denote by \(R(\eta)\) the a free \(R\)-module of rank 1 with a distinguished generator \(e_\eta\), equipped with the semilinear action \(g_\eta(\cdot)\) of \(G_K\) defined by 
\[
g_\eta(x \cdot e_\eta) = \eta(g) g(x) \cdot e_\eta \quad \text{for every } x \in R, \ g \in G_K \text{ and the action } g(\cdot) \text{ of } G_K \text{ on } R.
\]

For an object \(M\) of \(\text{Sem}_{G_K}(R)\), we define a new object of this category as \(M(\eta) = M \otimes_R R(\eta)\). For every \(\eta\) as before, we define a \(B_k^E\)-pair \(B(\eta)\) of rank one as \((B_{e,E}(\eta), B_{dR}^+(\eta))\). For a \(B_k^E\)-pair \(W\), we define a new \(B_k^E\)-pair as \(W(\eta) = W \otimes B(\eta)\).

**Definition 2.3.** Let \(F\) be a field.

- An object \(M\) of \(\text{Sem}_{G_K}(F)\) is split triangulable if there exists a filtration \(0 = M_0 \subset M_1 \subset \ldots \subset M_n = M\) such that, for \(1 \leq i \leq n, \ M_i\) is a rank \(i\) subobject of \(M\) in \(\text{Sem}_{G_K}(F)\). If moreover \(M_i/M_{i-1} \cong F(\eta_i)\) for \(1 \leq i \leq n\) and characters \(\eta_i: G_K \to (F^\times)^{G_K}\), then we say that \(M\) is split triangulable by characters \(\eta_1, \eta_2, \ldots, \eta_n\).

An object \(M\) of \(\text{Sem}_{G_K}(F)\) is triangulable if there is a finite extension \(F'\) of \(F\) such that \(F' \otimes_F M\) is triangulable as an object of \(\text{Sem}_{G_K}(F')\).

Let \(M\) be an object of some rank \(n\) in \(\text{Sem}_{G_K}(F)\) and let \((e_1, e_2, \ldots, e_n)\) be a \(F\)-basis of \(M\). To every \(g \in G_K\) we attach the only matrix \((a_{ij}^g)_{i,j} \in \text{GL}_n(B)\) satisfying 
\[
g(e_i) = \sum_j a_{ij}^g e_j \quad \text{for every } i \in \{1, 2, \ldots, n\} \quad \text{(we leave implicit the dependence of } (a_{ij}^g)_{i,j} \text{ on the choice of a basis)}.
\]

- the object \(M\) is trivial if and only if it admits a basis with respect to which the matrix \((a_{ij}^g)_{i,j}\) is the identity for every \(g \in G_K\);

- the object \(M\) is triangulable if and only if it admits a basis with respect to which the matrix \((a_{ij}^g)_{i,j}\) is upper triangular for every \(g \in G_K\);

**Definition 2.4.**

- A \(B_k^E\)-pair \(W\) is split triangulable if there exists a filtration \(0 = W_0 \subset W_1 \subset \ldots \subset W_n = W\) such that, for \(1 \leq i \leq n\), \(W_i\) is a rank \(i\) sub-\(B_k^E\)-pair of \(W\). If moreover \(W_i/W_{i-1} \cong B_k^E(\eta_i)\) for \(1 \leq i \leq n\) and characters \(\eta_i: G_K \to (F^\times)^{G_K}\), then we say that \(W\) is split triangulable with ordered parameters \(\eta_1, \eta_2, \ldots, \eta_n\).

- A \(B_k^E\)-pair \(W\) is triangulable if there is a finite extension \(F\) of \(E\) such that the \(B_k^E\) pair \(F \otimes_E W\) is split triangulable.

- An object \(V\) of \(\text{Rep}_{G_K}(E)\) is (potentially, split) trianguline if \(W(V)\) is (potentially, split) triangulable.

The ring \(B_{e,E}\) is a principal ideal domain by [DiM13], Proposition 1.2.2]. Let \(F_E\) be its fraction field, equipped with the action of \(G_K\) induced by that on \(B_{E,e}\). For every finite extension \(L\) of \(K\), we have \(B_{dR,E}^L = F_E^L = L\).

The following lemma gives equivalent conditions for a \(B_k^E\)-pair to be triangulable.

**Lemma 2.5.** [DiM13], Corollary 2.2.2] Let \(W = (W_e, W_{dR}^+)\) be a \(B_k^E\)-pair. The following conditions are equivalent:

1. the \(B_k^E\)-pair \(W\) is split triangulable;
2. the semilinear \(B_{e,E}\)-representation \(W_e\) of \(G_K\) is split triangulable;
3. the semilinear \(F_E\)-representation \(F_E \otimes_{B_{e,E}} W_e\) of \(G_K\) is split triangulable.

In particular the same equivalence holds if we replace “split triangulable” by “triangulable”.

6
Thanks to the lemma we will be able to study the triangularity of a $B_{K}^{E}$-pair $W$ by studying the triangularity of a $F_{E}$-representation of $G_{K}$.

3. Generalities on Schur functors

Let $q$ be a positive integer. Consider a decreasing integer partition $\underline{u}$ of $q$, that is a tuple $\underline{u} = (u_1, u_2, \ldots, u_r)$, for some $r \geq 1$, satisfying $u_1 \geq u_2 \geq \ldots \geq u_r \geq 1$ and $u_1 + u_2 + \ldots + u_r = q$. In the following we will refer to $\underline{u}$ simply as a partition, leaving the other properties implicit. If $q$ and $r$ are not specified, we will write $r(u)$ for the length of $\underline{u}$ and $q(\underline{u})$ for the sum $u_1 + u_2 + \ldots + u_r(u)$. Note that in [DiM13a, Section 1.4] the quantity $r(u)$ is defined in a slightly different way.

We recall that to a partition $\underline{u} = (u_1, u_2, \ldots, u_r)$ we can attach a Young diagram whose columns have lengths $u_1, u_2, \ldots, u_r$. A Young tableau of shape $\underline{u}$ is a filling of the Young diagram attached to $\underline{u}$ with positive integers. We call a Young tableau semistandard if its entries are strictly increasing from top to bottom along each column, and weakly increasing along each row from left to right.

Let $R$ be a commutative ring with unity. To a partition $\underline{u}$ we can attach in the standard way a Schur functor $S_{\underline{u}}$ from the category of $R$-modules to itself; we leave the dependence on $R$ implicit. We refer to [FH91, Lecture 6] for the definition of $S_{\underline{u}}$. We recall that $S_{\underline{u}} = (0)$ if $r(\underline{u}) > n$. If $M$ is a free $R$-module of rank $n$, then the $R$-module $S_{\underline{u}}(M)$ is also free and we denote by $r_{\underline{u}}(n)$ its rank.

Let $\underline{u}$ be a partition. We write $q = q(\underline{u})$ in this paragraph. Let $\text{Std}_{n}(R)$ denote a free $R$-module of rank $n$ equipped with an ordered basis $(e_{i})_{1 \leq i \leq n}$ and with the standard representation of $\text{GL}_{n}(R)$ with respect to this basis. The elementary tensors $e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_q}$ form an $R$-basis of the tensor power $\text{Std}_{n}(R)^{\otimes q}$. We make this basis into an ordered basis via the lexicographic order; we denote it by $(\otimes^{q} e_{i})$. We let $g \in \text{GL}_{n}(R)$ act on an elementary tensor $e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_q} \in \text{Std}_{n}(R)^{\otimes q}$ by $g(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_q}) = g(e_{i_1}) \otimes g(e_{i_2}) \otimes \ldots \otimes g(e_{i_q})$, and we extend this action $R$-linearly to $\text{Std}_{n}(R)^{\otimes q}$. The $R$-module $S_{\underline{u}} \text{Std}_{n}(R)$ is a free $R$-module of rank $r_{\underline{u}}(n)$ and a $\text{GL}_{n}(R)$-stable direct summand of the $R$-module $\text{Std}_{n}(R)^{\otimes q}$. More precisely, it is a direct summand of $\bigotimes_{i=1}^{r_{\underline{u}}}(\Lambda^{u_{i}} \text{Std}_{n}(R))$. We use the ordered basis $(e_{i})_{1 \leq i \leq n}$ to define an ordered basis of $S_{\underline{u}} \text{Std}_{n}(R)$ in the following way. Let $T$ be the set of semistandard Young tableaux of shape $\underline{u}$ taking all their entries in the set $\{1, 2, \ldots, n\}$. Let $T \in T$. Denote by $i_{a,b}^{T}$ the integer at the intersection of the $a$-th column from the top and the $b$-th row from the left. Let $e_{T}$ be the element of $\bigotimes_{i=1}^{r_{\underline{u}}}(\Lambda^{u_{i}} \text{Std}_{n}(R))$ given by

$$(e_{i_{1,1}} \land e_{i_{1,2}} \land \ldots \land e_{i_{1,n}}) \otimes (e_{i_{2,1}} \land e_{i_{2,2}} \land \ldots \land e_{i_{2,n}}) \otimes \ldots \otimes (e_{i_{r_{\underline{u}},1}} \land e_{i_{r_{\underline{u}},2}} \land \ldots \land e_{i_{r_{\underline{u}},n}}).$$

Let $[e_{T}]$ be the projection of $e_{T}$ to the quotient $S_{\underline{u}} \text{Std}_{n}(R)$ of $\bigotimes_{i=1}^{r_{\underline{u}}}(\Lambda^{u_{i}} \text{Std}_{n}(R))$. We give the set $T$ the following total ordering: given $T, T' \in T$ we say that $T < T'$ if, proceeding in the lexicographic order, the first value of $(a, b)$ such that $i_{a,b}^{T} \neq i_{a,b}^{T'}$ satisfies $i_{a,b}^{T'} < i_{a,b}^{T}$. When $T$ varies over the ordered set $T$, the string $(e_{T})_{T \in T}$ forms an ordered $R$-basis of $S_{\underline{u}} \text{Std}_{n}(R)$ [FH91, Problem 6.15]. In the following, when we speak of the basis $(e_{T})_{T \in T}$ we always leave implicit the fact that it depends on the choice of an ordered basis $(e_{i})_{i \in \{1, \ldots, n\}}$ of $V$. 


We denote by $S_n^u(R)$ the representation $GL_n(R) \to GL_{r_{\mu}(n)}$ defining the action of $GL_n(R)$ on $S^u \text{Std}_n(R)$ with respect to the ordered $R$-basis $(e_T)_{T \in \mathbb{T}}$.

Our choice of ordering on $\mathbb{T}$ is explained by the following remark.

**Remark 3.1.** For every positive integer $d$, let $B_d$ denote the Borel subgroup of $GL_d(R)$ consisting of upper triangular matrices and let $T_d$ denote the torus of diagonal matrices in $GL_d(R)$. An easy check shows that:

- the images of $T_n$ and $B_n$ via the representation $S_n^u$ are contained in $T_{r_{\mu}(n)}$ and $B_{r_{\mu}(n)}$, respectively;
- the preimages of $T_{r_{\mu}(n)}$ and $B_{r_{\mu}(n)}$ via the representation $S_n^u$ are contained in $T_n$ and $B_n$, respectively.

The properties on Borel subgroups depend on our chosen ordering of the $R$-basis of $S^u \text{Std}_n(R)$. Note that such a choice may be unimportant when working with the linear representation $S_n^u$, but it becomes significant when we move to semilinear representations.

**Remark 3.2.** Let $\mu = (u_1, u_2, \ldots, u_d)$ be any partition. For a positive integer $k$, let $\mu + k$ denote the partition $(u_1 + k, u_2 + k, \ldots, u_d + k)$. Then $S_n^{\mu + k} = S_n^\mu \otimes \det^k$.

When the characteristic of $R$ is zero, the representation $S_n^u(R)$ is irreducible. When $R = \mathbb{C}$ the morphism $S_n^u$ defines the $\mathbb{C}$-representation of $GL_n(\mathbb{C})$ of highest weight $\mu$, that is unique up to isomorphism. For arbitrary $R$, if $r(\mu) = n$ and $u_1 = u_2 = \ldots = u_n = u$ for a positive integer $u$ then the representation $S_n^u(R)$ is $\det^u$.

Now suppose that $R$ carries an action of $G_K$ and that $M$ is an object of rank $n$ in $\text{Sem}_{G_K}(R)$. We choose an ordered $R$-basis $(e_i)_{1 \leq i \leq n}$. We let $g \in G_K$ act on $M^{\otimes q}$ by setting $g(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_q}) = g(e_{i_1} \otimes g(e_{i_2}) \otimes \ldots \otimes g(e_{i_q}))$ for every elementary tensor $e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_q}$, and extending semilinearly. This action leaves the $R$-submodule $S_n^\mu(M)$ stable, making it into an object of $\text{Sem}_{G_K}(R)$ that we still denote by $S_n^\mu(M)$. Note that the action of $G_K$ on $M^{\otimes q}$, hence on $S_n^\mu(M)$, depends on our choice of basis for $M$ and for $M^{\otimes q}$.

**Remark 3.3.** With notations as before, we identify $M$ with $\text{Std}_n(R)$ via the chosen ordered basis $(e_i)_{1 \leq i \leq n}$ and we give $S_n^\mu(M)$ the ordered basis $(e_T)_{T \in \mathbb{T}}$ constructed in this section. For $g \in G_K$, let $A^p$ be the matrix in $GL_n(R)$ describing the action of $g$ on the basis $(e_i)_{1 \leq i \leq n}$ of $M$. Then $g$ acts on the basis $(e_T)_{T \in \mathbb{T}}$ of $S_n^\mu(M)$ via the matrix $S_n^\mu(A^p)$.

We recall a standard result.

**Lemma 3.4.** [Hun86, Theorem 1] Let $n$ be a positive integer. Let $\mu$ be a partition that satisfies (△). Then the morphism $GL_n \to GL_n/\ker S_n^\mu$ is an isogeny. The kernel of $S_n^\mu$ is the algebraic group $\mu_q$ of $q$-th roots of unity, embedded in $GL_n$ via $\mu \mapsto \mu 1_n$.

When choosing a partition $\mu$ we often make the following assumption:

$$(*) \quad \text{neither (a) } r(\mu) > n \text{ nor (b) } u_1 = u_2 = \ldots = u_{r(\mu)}.$$ 

When (a) holds the representation $S_n^\mu$ is trivial, and when (b) holds it is a power of the determinant. We bear no interest in these two cases: for these representations the conclusions of Theorems 4.2 and 5.2 are trivally false, if $n > 1$, or trivally true, if $n = 1$. Note that (△) coincides with the assumption on the partition in [DiM13a, Sections 2.4, 3.3], even if the hypothesis there is phrased differently.
Let $\mathbf{u}$ be a partition and let $W = (W_e, W_{dR}^+)$ be a $B_K^F$-pair. As in [DiM13b, Section 1.4], we define a $B_K^F$-pair $S^u(W)$ as $(S^u(W_e), S^u(W_{dR}^+))$. If $V$ is an object of $\text{Rep}_{G_K}(E)$, then $S^u(W(V)) \cong W(S^u(V))$ in the category of $B_K^F$-pairs.

4. Schur functors and potentially trianguline $B$-pairs

We recall a result of Di Matteo.

**Theorem 4.1.** Let $\mathbf{u}$ be a partition satisfying (iii). Let $W$ be a $B_K^F$-pair.

1. If $W$ is de Rham then $S^u(W)$ is de Rham.
2. If $S^u(W)$ is de Rham, then there exists a finite extension $F/E$ and a character $\mu : G_K \to F^\times$ such that $W(\mu)$ is de Rham.
3. Statements (1) and (2) hold if we replace “de Rham” with “crystalline”.

Part (1) follows by a straightforward semilinear algebra computation. Parts (2) and (3) are Theorems 2.4.2 and 3.3.2 of [DiM13a], respectively.

The goal of this section is to prove an analogue of Theorem 4.1(1,2), with “de Rham” replaced by “potentially triangulable”. Our result is the following.

**Theorem 4.2.** Let $\mathbf{u}$ be a partition satisfying (iii). Let $W$ be a $B_K^F$-pair. Then:

1. If $W$ is split triangulable of parameter $\delta$ then $S^u(W)$ is split triangulable of parameter $S^u \circ \delta$;
2. If $S^u(W)$ is split triangulable, then $W$ is potentially split triangulable.

By specializing the theorem to the case where $W = W(V)$ for an $E$-linear representation $V$ of $G_K$, we obtain the following corollary.

**Corollary 4.3.** Let $\mathbf{u}$ be a partition satisfying (iii) and let $V$ be an object of $\text{Rep}_{G_K}(E)$. Then:

1. If $V$ is split trianguline of parameter $\delta$, then $S^u(V)$ is split trianguline of parameter $S^u \circ \delta$;
2. If $S^u(V)$ is split trianguline, then $V$ is potentially trianguline.

**Proof.** (of Theorem 4.2) Let $\mathbf{u}$ and $W$ be as in the statement of the theorem. Note that $F_E \otimes_{B_{dR}} S^u W_e = S^u(F_E \otimes_{B_{dR}} W_e)$ in $\text{Sem}_{G_K}(F_E)$. By the implication (1) $\implies$ (3) in Lemma 2.3, the object $S^u(F_E \otimes_{B_{dR}} W_e)$ is triangulable in $\text{Sem}_{G_K}(F_E)$. On the other hand, the implication (3) $\implies$ (1) gives that if $F_E \otimes_{B_{dR}} W_e$ is triangulable then the $B_K^F$-pair $W$ is triangulable. We reduce Theorem 4.2 to a statement about semilinear representations of $G_K$. More precisely, the theorem follows by specializing the next lemma to the representation $V = F_E \otimes_{B_{dR}} W_e$.

Let $V$ be an object of rank $n$ in $\text{Sem}_{G_K}(F_E)$.

**Lemma 4.4.**

1. If the action of $G_K$ on $V$ is triangulable, then the action of $G_K$ on $S^u(V)$ is triangulable.
2. If the action of $G_K$ on $S^u(V)$ is triangulable, then the action of $G$ on $S^u(V)$ is potentially triangulable.
We prove part (1). Let \((e_i)_{i \in \{1, \ldots, n\}}\) be a \(F_E\)-basis of \(V\) in which \(G_K\) acts via upper triangular matrices. For \(g \in G_K\) let \(\sigma_g \in \text{GL}_n(F_E)\) be the matrix describing the action of \(g\) on \(V\) in the basis \((e_i)_{i \in \{1, \ldots, n\}}\). Then the matrix describing the action of \(g\) on \(S^m(V)\) in the basis \((e_T)_{T \subseteq T}\) is \(S^{\mu}_n \sigma_g\). Since the matrix \(\sigma_g\) is upper triangular, the matrix \(S^{\mu}_n \sigma_g\) is also upper triangular by Remark 3.1 as desired.

We prove part (2). We will need to apply a few results that we state here.

**Lemma 4.5.** [DiM13a, Lemma 3.1.1] Let \(0 \to X' \to X \to X'' \to 0\) be an exact sequence in \(\text{Sem}_{G_K}(F_E)\). Then \(X\) is triangulable if and only if \(X'\) and \(X''\) are.

**Lemma 4.6.** [DiM13a, Theorem 3.2.2] If \(X\) and \(Y\) are two objects in \(\text{Sem}_{G_K}(F_E)\) such that \(X \otimes Y\) is triangulable, then \(X\) and \(Y\) are potentially triangulable.

**Lemma 4.7.** [Del82, Proposition 3.1] Let \(F\) be a field of characteristic zero, \(G\) a reductive algebraic group over \(F\) and let \(S: G \to \text{GL}(V)\) be a finite-dimensional faithful representation of \(G\) over \(F\). Then every irreducible, finite-dimensional \(F\)-linear representation of \(G\) appears as a direct factor of \(S^{\otimes k}\) for some positive integer \(k\).

Let \(u\) be a partition of an integer \(q\) such that \(u\) satisfies (1) and assume that \(S^u(V)\) is triangulable. Let \(\underline{u}\) be another partition of the same integer, also satisfying (1). We show that \(S^\underline{u}(V)\) is also triangulable. To this purpose we apply Lemma 4.7 to the reductive algebraic group \(G\) over \(F_E\) defined as \(\text{GL}_n/\ker S^u\). We make this precise. The \(F_E\)-module \(S^u(V)\), equipped with the action of \(G\) given by \(S^u: G \to \text{GL}(S^u(V))\), is a faithful representation of \(G\). By Lemma 4.4 the kernels of \(S^u\) and \(S^\underline{u}\) coincide the group \(\mu_q\) embedded diagonally in \(\text{GL}_n\). In particular \(G = \text{GL}_n/\ker S^u\), so that \(S^u(V)\), equipped with the action of \(G\) given by \(S^u: G \to \text{GL}(S^u(V))\), is a faithful representation of \(G\). By Lemma 4.7 the irreducible representation \(S^u(V)\) of \(G\) appears as a subquotient of \((S^u(V))^{\otimes k}\) for some positive integer \(k\). Since \(S^u(V)\) is triangulable by assumption, its tensor power \((S^u(V))^{\otimes k}\) is also triangulable. In particular its direct factor \(S^\underline{u}(V)\) is triangulable by Lemma 4.5.

Now consider the object \(\text{Sym}^{q-1} V \otimes V\) in \(\text{Sem}_{G_K}(F_E)\). By the Littlewood-Richardson rule (see for instance [FH91, Appendix 8]) the \(F_E\)-linear representation \(\text{Sym}^{q-1} V \otimes V\) of \(\text{GL}_n\) can be decomposed in a direct sum of irreducible representations of the form \(S^\underline{u}(V)\), where \(\underline{u}\) is a partition of \(q\). By the result of the previous paragraph, every direct factor \(S^\underline{u}(V)\) is triangulable as an object of \(\text{Sem}_{G_K}(F_E)\). We deduce that \(\text{Sym}^{q-1} V \otimes V\) is triangulable by Lemma 4.5. Now Lemma 4.6 gives that \(V\) is potentially triangulable, as desired. This concludes the proof of Lemma 4.4(2), hence of Theorem 4.2.

**Remark 4.8.**

1. Suppose that \(W\) is triangulable with respect to ordered parameters \(\eta_i: G_K \to E^\times, 1 \leq i \leq n\). Let \(\tilde{\eta}: G_K \to \text{GL}_n(E)\) be the representation sending \(g \in G_K\) to the diagonal matrix \((\eta_1(g)), \ldots, \eta_n(g))\). By Remark 3.4, the representation \(S^\mu_n\) maps the diagonal torus of \(\text{GL}_n(E)\) to the diagonal torus of \(\text{GL}_{\tau_\mu(n)}(E)\). In particular the representation \(S^\mu_n \circ \tilde{\eta}: G_K \to \text{GL}_{\tau_\mu(n)}(E)\) maps an element \(g \in G_K\) to a diagonal matrix that we write as \((\eta_T(g))_{T \subseteq T}\) for the usual index set \(T\) of an \(E\)-basis of \(S^\mu\text{Std}_n(E)\) and characters \(\eta_T: G_K \to E^\times\). Then it is immediate to see that the \(B_K^E\)-pair \(S^\mu(W)\) is triangulable with respect to the ordered parameters \((\eta_T)_{T \subseteq T}\).

2. Each diagonal entry of \(S^\mu_n \circ \tilde{\eta}\) is expressed by a monomial of degree \(q\) in the variables \(\eta_1, \ldots, \eta_n\).
We adapt the notion of strict triangulinity from [KPX14, Definition 6.3.1] to the context of \( B \)-pairs.

**Definition 5.1.** Let \( W \) be a \( B^K_\mathcal{E} \)-pair of rank \( n \) admitting a triangulation \( \{ W_i \}_{0 \leq i \leq n} \) with ordered parameters \( \eta : G_K \to K^\times \), \( 1 \leq i \leq n \). We say that \( W \) is strictly triangulable with respect to the given ordered parameters if, for \( 1 \leq i \leq n \), \( W_i \) is the only saturated, rank \( i \) sub-\( B^K_\mathcal{E} \)-pair of \( W \) that contains \( W_{i-1} \) and satisfies \( W_i/W_{i-1} \cong B(\eta_i) \). We call an \( E \)-linear representation \( V \) of \( G_K \) strictly triangulable if \( W(V) \) is strictly triangulable.

We refer to Remark 7.2 for a condition on the parameter \((\eta_i)_{1 \leq i \leq n}\) that guarantees that a \( B^K_\mathcal{E} \) pair admitting a triangulation of this parameter is strictly trianguline.

Note that the property of being strictly trianguline with respect to some parameters depends on the choice of an ordering of the parameters. If \( L \) is a finite extension of \( K \) and \( W \) is a \( B^K_\mathcal{E} \)-pair that is strictly trianguline with respect to ordered parameters \((\eta_i)_{1 \leq i \leq n}\), then \( W|_{GL} \) is strictly trianguline with respect to the ordered parameters \((\eta_i|_{GL})_{1 \leq i \leq n}\).

Under a strict triangulinity assumption, we can improve Theorem 4.2 in the following way. Let \( \mathcal{L} \) be a partition satisfying (13). Let \( W = (W_e, W^+_d, W^+_{idr}) \) be a \( B^K_\mathcal{E} \)-pair such that \( S^w(W) \) is triangulable. Let \( L \) be a finite extension of \( K \) such that \( W^+_d|_{GL} \) is triangulable. It exists by Theorem 4.2. Let \((\eta_i)_{1 \leq i \leq n}\) be the ordered set of parameters of the triangulation.

Let \((\eta_T)_{T \in \mathcal{T}}\) be the ordered set of parameters defined in Remark 4.8.

**Theorem 5.2.** If \( S^w(W) \) is strictly triangulable with respect to the ordered parameters \((\eta_T|_{GL})_{T \in \mathcal{T}}\) then \( W \) is split triangulable.

*Proof.* As usual we can restrict ourselves to the split triangulable case. Let \( \{ W_i \}_{1 \leq i \leq n} \) be a triangulation of \( W^+_d|_{GL} \) and write \( W_i = (W_i,e, W^+_{idr}) \). Also write \( W_i/W_{i-1} = (W_i/W_{i-1}|_c, W_i/W_{i-1}|_{idr}) \). For every \( i \in \{1, \ldots, n\} \) let \( \tau_i^e \) and \( \tau_{idr}^i \) be generators of \( (W_i/W_{i-1})^e \) over \( B_{e,E} \) and of \( (W_i/W_{i-1})^+_{idr} \) over \( B^+_{idr,E} \), respectively. Lift them to elements of \( e_i^e \) and \( e_{idr}^i \) of \( W_i,e \) and \( W^+_{idr} \), so that:

- \((e_i^e)_{1 \leq i \leq n}\) is an ordered \( B_{e,E} \)-basis of \( W_e \) such that the set \((e_{j,e})_{1 \leq j \leq i}\) generates \( W_i^e \) for \( 1 \leq i \leq n \),
- \((e_{idr}^i)_{1 \leq i \leq n}\) is an ordered \( B^+_{idr,E} \)-basis of \( W^+_{idr} \) such that the set \((e_{j,idr}^j)_{1 \leq j \leq i}\) generates \( W^+_{idr} \) for \( 1 \leq i \leq n \).

For \( g \in GL_K \), let \( A^g_{e} \) be the matrix in \( GL_n(B_{e,E}) \) defining the action of \( g \) on the basis \((e_i)_{1 \leq i \leq n}\). If \( g \in GL_L \), the matrix \( A^g_{e} \) is upper triangular since \( \{ W_i,e \}_{1 \leq i \leq n} \) is a triangulation of \( W|_{GL} \) in \( \text{Sem}_{GL}(B_{e,E}) \). Now let \((e_T,e)_{T \in \mathcal{T}}\) be the ordered \( B_{e,E} \)-basis of \( S^w(W|_{GL}) \) constructed from \((e_i)_{1 \leq i \leq n}\) as in Section 3. For \( T \in \mathcal{T} \), let \( W_{T,e} \) be the subobject of \( S^w(W_e) \) generated over \( B_{e,E} \) by the set \((e_U)_{U \subseteq T, U \in \mathcal{T}}\). The set \( \{ W_{T,e} \}_{T \in \mathcal{T}} \) defines a triangulation of \( S^w(W_e) \) by the characters \((\eta_T|_{GL})_{T \in \mathcal{T}}\). By Remark 3.3 the matrix describing the action of \( g \in G_K \) on the basis \((e_T)_{T \in \mathcal{T}}\) is \( S^w(A^g_{e}) \).

By repeating the reasoning of the preceding paragraph starting with the basis \((e_{idr}^i)_{1 \leq i \leq n}\) of \( W^+_{idr} \), we obtain an ordered basis \((e^+_{idr})_{T \in \mathcal{T}} \) of \( S^w(W^+_{idr}) \) and we define a triangulation \( \{ W^+_{T,idr} \}_{T \in \mathcal{T}} \) of \( S^w(W^+_{idr}) \). We let \( A^g_{idr} \) be the element of \( GL_n(B^+_{idr,E}) \) defining the action of \( g \in G_K \) on the basis \((e_i)_{1 \leq i \leq n}\), so that \( S^w(A^g_{idr}) \) defines the action of \( g \) on the basis \((e_T)_{T \in \mathcal{T}}\). The matrix \( A^g_{idr} \) is upper triangular for \( g \in GL_L \). We deduce from the
construction of our bases that the couple \((W_{T,e}, W_{T,\text{dR}}^+)\) defines a \(B^P_K\)-pair \(W_T\) for every \(T \in T\), and that the set \(\{W_T\}_{T \in T}\) is a triangulation of \(\mathcal{S}(W)\) with respect to the ordered parameters \((\eta_T|_{G_L})_{T \in T}\).

Now let \((X_T)_{T \in T}\) be a triangulation of \(\mathcal{S}(W)\) with respect to the ordered parameters \((\eta_T)_{T \in T}\). The restriction \((X_T|_{G_L})_{T \in T}\) is a triangulation of \(\mathcal{S}(W|_{G_L})\) with respect to the ordered parameters \((\eta_T|_{G_L})_{T \in T}\). By assumption \(\mathcal{S}(W|_{G_L})\) is strictly split triangulable with respect to the aforementioned parameters, so the triangulations \(X_T|_{G_L}\) and \(W_T\) must coincide. In particular the filtration \(\{W_T\}_{T \in T}\) is stable under every \(g \in G_K\) (not only \(g \in G_L\)) since \(X_T\) is a triangulation of \(\mathcal{S}(W)\). This implies that the matrices \(\mathcal{S}(\mathcal{A}(g)^e)\) and \(\mathcal{S}(\mathcal{A}(g)^{\text{dR}})^{\pm}\) are upper triangular. Then Remark 4.8 implies that \(\mathcal{A}(g)^e\) and \(\mathcal{A}(g)^{\text{dR}}\) are upper triangular for every \(g \in G_K\), hence that \(\{W_i\}_{1 \leq i \leq n}\) is a triangulation of \(W\). □

**Definition 5.3.** We say that an \(E\)-linear representation \(V\) of \(G_K\) is strictly trianguline with ordered parameters \((\delta_i)_{1 \leq i \leq n}\) if the \(B\)-pair \(W(V)\) is strictly triangulable with the same ordered parameters.

If \(V\) is refined trianguline in the sense of [KPX14, Definition 6.4.1], then \(V\) is strictly trianguline by [KPX14, Lemma 6.4.2]. However the condition of refined triangulinity is too strong for our purposes since it implies potential semi-stability.

By combining Theorems 4.2 and 5.2 we obtain the following corollary.

**Corollary 5.4.** Let \(\mu\) be a partition satisfying \((\dagger)\). Let \(V\) be an \(E\)-representation of \(G_K\). Then \(V\) is strictly trianguline with ordered parameters \((\delta_i)_{1 \leq i \leq n}\) if and only if \(\mathcal{S}W\) is strictly trianguline with ordered parameters \((\delta_T)_{T \in T}\) constructed from \((\delta_i)_{1 \leq i \leq n}\) as in Remark 4.8.

6. Lifting trianguline representations via isogenies

Let \(G\) be a reductive group over \(\mathbb{Q}\). In light of Theorem 5.2, the following definition makes sense.

**Definition 6.1.** We say that a continuous representation \(\rho: G_{\mathbb{Q}_p} \to G(\mathbb{Q}_p)\) is strictly trianguline if there exists a homomorphism \(\mu: G_m \to G\) over \(\mathbb{Q}_p\) satisfying the following property: for every algebraic representation \(S: G \to \text{GL}_n\) defined over \(\mathbb{Q}_p\), the representation \(S \circ \rho: G_{\mathbb{Q}_p} \to \text{GL}_n(\mathbb{Q}_p)\) is strictly trianguline with respect to the \(n\)-tuple of parameters \(S \circ \mu(\mathbb{Q}_p): \mathbb{Q}_p^\times \to \text{GL}_n(\mathbb{Q}_p)\). In this case we also say that \(\rho\) is trianguline with parameter \(\mu\).

In this section \(L\) is a number field, \(H\) and \(H'\) be two algebraic groups and \(\pi: H' \to H\) is an isogeny. Given a continuous representation \(\rho: G_L \to H(\mathbb{Q}_p)\) with some prescribed local properties, we can investigate whether there exists a representation \(\rho': G_L \to H'(\mathbb{Q}_p)\), with the same local properties, satisfying \(\pi \circ \rho' \cong \rho\). When the required local properties are:

1. unramifiedness outside of a finite set of places containing the places above \(p\);
2. a \(p\)-adic Hodge theoretic property at \(p\), taken from the set \{Hodge Tate, de Rham, semi-stable, crystalline\};
the lifting problem is studied and solved in [Win95]. The same question is studied in depth in the Ph.D. thesis of Hoang Duc [Hoa13], that includes a treatment of the problem of minimizing the set of ramification primes of the lift. In this section we study the analogue of the problem above when (2) is replaced by the property that \( \rho \) is strictly trianguline at \( p \), in the special case where the isogeny \( \pi \) is of the form \( H' \to \mathbb{S}H' \) for a partition \( u \) satisfying (1).

The following result guarantees that a lift always exists in our situation. It is a corollary of a theorem of Tate [Tat77, Théorème 4].

**Proposition 6.2.** Suppose that \( \ker \pi \) is a torus. Let \( \rho: G_L \to H(\mathbb{Q}_p) \) be a continuous representation. There exists a representation satisfying (2) in the corollary.

**Proposition 6.3.** Keep the notations and hypotheses of Proposition 6.2 and suppose that:

1. \( \rho \) is unramified outside of a finite set of places;
2. \( \rho \) is strictly trianguline at the places of \( L \) above \( p \);
3. the trianguline parameter \( \mu: \mathbb{G}_m \to H \) of \( \rho \) admits a lift \( \mu': \mathbb{G}_m \to H'(\mathbb{Q}_p) \).

Let \( \rho': G_{\mathbb{Q}_p} \to H'(\mathbb{Q}_p) \) be a representation lifting \( \rho \). Then \( \rho' \) is unramified outside of a finite set of places and strictly trianguline at \( p \) with parameter \( \mu' \).

Assumption (3) in the corollary is an analogue of the condition on the lifting of the Hodge-Tate parameter in [Win95, Théorème 1.1.3]. Note that the existence of \( \rho' \) is guaranteed by Proposition 6.2.

**Proof.** By a result of Conrad [Conr11, Lemma 5.2 and Proposition 5.3] the representation \( \rho' \) is unramified outside of a finite set of places. Let \( v \) be a place of \( L \) above \( p \) and let \( \rho_v \) and \( \rho'_v \) be the restrictions of \( \rho \) and \( \rho' \), respectively, to a decomposition group at \( v \). Let \( q \) be the order of \( \ker \pi \). Let \( u \) be a partition of \( q \) satisfying (1). Let \( S': H' \to GL_n \) be a representation. We need to show that \( S' \circ \rho'_\mu \), strictly trianguline of parameters \( S' \circ \rho'_\mu \). By Lemma 3.4, the kernel of the representation \( S|^n \circ S': H' \to GL_{r_{2(n)}} \) contains \( (S')^{-1}(\mu \cdot 1_n) \), that in turn contains \( \ker \pi \) since \( \pi \) is a central isogeny with kernel of order \( q \). In particular, \( S'^n \circ S' \) factors as \( S \circ \pi \) for a representation \( S: H \to GL_{r_{2(n)}} \). By composing with \( \rho'_v \), we obtain

\[ S^n \circ S' \circ \rho'_v \cong S \circ \pi \circ \rho'_v \cong S \circ \rho_v, \]

where the last equivalence comes from the definition of \( \rho' \). The representation \( S \circ \rho_v \) is trianguline of parameters \( S \circ \mu \) because \( \rho_v \) is strictly trianguline of trianguline parameter \( \mu \) by assumption. From the equivalence (6.1) we deduce that \( S^n \circ S' \circ \rho'_v \) is trianguline of parameter \( S \circ \mu \). Now \( \mu = \pi \circ \mu' \) for the parameter \( \mu' \) provided by condition (3) in the statement, so the trianguline parameter of \( S^n \circ S' \circ \rho'_v \) is \( S \circ \pi \circ \mu' \), that is \( S^n \circ S' \circ \mu' \) by definition of \( S \). Thanks to Corollary 3.4, we conclude that the representation \( S' \circ \rho'_v \) is strictly trianguline of parameters \( S' \circ \mu' \) because the representation \( S^n \circ S' \circ \rho'_v \) is strictly trianguline of parameters \( S^n \circ S' \circ \mu' \). Since this is true for every choice of \( S' \), we obtain the thesis.

Let \( \rho: G_L \to G(E) \) be a continuous representation. We denote by \( \overline{\rho} \) the Zariski-closure over \( E \) of the image of \( \rho \). The following is a corollary of Propositions 6.2 and 6.3.
Corollary 6.4. Let $H$ be a reductive group over $\mathbb{Q}$ equipped with a central isogeny $\pi: H \to \text{Im}\rho$ over $E$. Suppose that:

1. $\rho$ is unramified outside of a finite set of places;
2. $\rho$ is strictly trianguline at the places of $L$ above $p$;
3. the trianguline parameter $\mu: \mathbb{G}_m \to \text{Im}\rho$ of $\rho$ admits a lift $\mu': \mathbb{G}_m \to H(E)$.

Then there exists a continuous representation $\rho': G_L \to H'(E)$ that lifts $\rho$ via $\pi$ and is:

* unramified outside of a finite set of places;
* strictly trianguline at $p$ with parameter $\mu'$.

7. Congruence loci on Hecke-Taylor-Wiles varieties

We apply the results of the previous section to the study of certain $p$-adic Langlands lifts as maps between the “Hecke-Taylor-Wiles varieties” studied by Breuil, Hellmann and Schraen [BHS17]. We begin by recalling some definitions from there.

In the following $n$ is a positive integer and $p$ a prime number larger than $2n + 1$. Let $F^+$ be a totally real number field and let $F$ be a totally imaginary quadratic extension of $F^+$. We denote by $\Sigma_p$ the set of places of $F^+$ above $p$. Let $G$ be a unitary group over $F^+$ such that $G$ is split over $F$, compact at infinity and isomorphic to $\text{GL}_n$ at all places in $\Sigma_p$. Let $L$ be a $p$-adic field with ring of integers $\mathcal{O}_L$ and residue field $k_L$. We say that a representation $\rho: G_F \to \text{GL}_n(\mathcal{O}_L)$ is of automorphic origin if it is attached to a maximal ideal of a Hecke algebra acting on the space of $L$-valued automorphic forms for $G$ of some fixed tame level $\mathbb{A}_{F^+}^\infty$. We denote by $\Sigma$ the support of $\mathbb{A}_{F^+}^\infty$, so that $\mathfrak{p}$ will be unramified outside $\Sigma \cup \Sigma_p$. Fix a representation $\rho$ of automorphic origin. Let $\mathfrak{p}: G_F \to \text{GL}_n(k_L)$ be the reduction of $\rho$ modulo the maximal ideal of $\mathcal{O}_L$. Let $\zeta_p$ be a $p$-th root of unity; we assume that $\mathfrak{p}[G_F(\zeta_p)]$ is absolutely irreducible. For every place $v$ of $F^+$ above $p$, we denote by $\tilde{v}$ a place of $F$ above $v$. Up to enlarging $L$, we can assume that for every $v$ the field $F_{\tilde{v}}$ admits $[F_{\tilde{v}}: \mathbb{Q}_p]$ homomorphisms to $L$. For every $v$ let $G_{F_{\tilde{v}}}$ be a decomposition group for $F$ at $\tilde{v}$ and set $\mathfrak{f}_{\tilde{v}}$ be the restriction of $\mathfrak{f}$ to $G_{F_{\tilde{v}}}$.

Let $\mathcal{C}_{k_L}$ be the category of local Artinian $k_L$-algebras with residue field $k_L$. Let $W(k_L)$ be the ring of Witt vectors of $k_L$ and let $L_0$ be its field of fractions. Let $v \in \Sigma$. Let $D_{F_{\tilde{v}}}: \mathcal{C}_{k_L} \to \text{Sets}$ be the functor that associates with $A \in \mathcal{C}_{k_L}$ the set of continuous representations $\rho': G_{F_{\tilde{v}}} \to \text{GL}_n(A)$ such that $\pi_A \circ \rho' = \mathfrak{p}_{\tilde{v}}$, where $\pi_A$ denotes reduction of the coefficients modulo the maximal ideal of $A$. By Schlessinger’s criterion the functor $D_{F_{\tilde{v}}}$ is represented by a complete local Noetherian $W(k_L)$-algebra $R(\mathfrak{f}_{\tilde{v}})$ with residue field $k_L$. We call framed deformation space for $\mathfrak{f}_{\tilde{v}}$ and denote by $\mathfrak{X}_{F_{\tilde{v}}}^o$ the rigid analytic space $\text{Spf}(R(\mathfrak{f}_{\tilde{v}})^o)^{rig} \times_{L_0} L$ over $L$. We also set $\mathfrak{X}_{\mathfrak{f}_{\tilde{v}}}^o \cong \prod_{v \in \Sigma} \mathfrak{X}_{\pi_{\tilde{v}}}^o$ and $\mathfrak{X}_{\mathfrak{f}_{\tilde{v}}}^o \cong \prod_{v \in \Sigma} \mathfrak{X}_{\pi_{\tilde{v}}}^o$.

Let $R_{\mathfrak{f}_{\tilde{v}}}^o$ be the maximal reduced and $p$-torsion free quotient of $R_{\mathfrak{f}_{\tilde{v}}}^o$. Let $R_{\mathfrak{f}_{\tilde{v}}}^{\text{loc}} = \hat{\bigotimes}_{v \in \Sigma} R_{\mathfrak{f}_{\tilde{v}}}^o$. Let $g$ be a positive integer. We define formal series rings $R_\infty = R_{\mathfrak{f}_{\tilde{v}}}^{\text{loc}}[[x_1, \ldots, x_g]]$ and $S_\infty = \mathcal{O}_L[[y_1, \ldots, y_g]]$, where $g' = \frac{g + [F^+: \mathbb{Q}]n^2}{2} + |\Sigma_p|n^2$. We suppose from now on that $g$ satisfies the conditions in [BHS17, Théorème 3.4] and we give $R_\infty$ an $S_\infty$-algebra structure via the morphism given by loc. cit..

Fix an isomorphism of algebraic groups $G \times_{F^+} F \cong \text{GL}_{n,F}$. We set $G_p = \prod_{v \in \Sigma} G(F^+_v)$. The choice of the place $\tilde{v}$ of $F$ determines an isomorphism $i_{\tilde{v}}: G(F^+_v) \cong \text{GL}_n(F_{\tilde{v}})$. We
denote by $B_v$, $T_v$ and $N_v$ the inverse images via $i_v$ of the subgroups of upper triangular, diagonal and upper unipotent matrices in $GL_n(F_{v})$, respectively. We denote by $K_v$ the inverse image of $GL_n(O_{F_v})$ via $i_v$; it is a compact open subgroup of $G(R^+_v)$. We define $B_p = \prod_{v \in S_p} B_v$ and analogously $T_p$, $N_p$ and $K_p$. We also set $B_p^0 = B_p \cap K_p$, $N_p^0 = N_p \cap K_p$ and $T_p^0 = T_p \cap K_p$. We denote by $\hat{T}_p$ and $\hat{T}_p^0$ the rigid analytic spaces over $Q_p$ parametrizing the continuous characters of $T_p$ and $T_p^0$, and we set $\hat{T}_{p,L} = \hat{T}_p \times_{Q_p} L$, $\hat{T}_{p,0}^0 = \hat{T}_p^0 \times_{Q_p} L$.

Using the Taylor-Wiles-Kisin systems constructed in [Car+16], Breuil, Hellmann and Schraen construct a Hecke-Taylor-Wiles variety $X_p(\overline{\mathfrak{p}})$ (see [BHS17, Section 3] and Définition 3.6 there). It is a reduced rigid analytic space over $L$, defined as a closed subspace of $\text{Spf}(R_\infty)^{\text{rig}} \times L \hat{T}_{p,L}$ by means of Emerton’s Jacquet functor. We set $W_{\infty} = \text{Spf}(S_\infty)^{\text{rig}} \times \hat{T}_{p,L}$ and we call it the weight space. We define a weight morphism $\omega_X: X_p(\overline{\mathfrak{p}}) \to W_{\infty}$ as the composition of the inclusion $X_p(\overline{\mathfrak{p}}) \hookrightarrow \text{Spf}(R_\infty)^{\text{rig}} \times \hat{T}_{p,L}$ and the morphism $\text{Spf}(S_\infty)^{\text{rig}} \times \hat{T}_{p,L} \to \text{Spf}(S_\infty)^{\text{rig}} \times \hat{T}_{p,L}$ given by the $S_\infty$-algebra structure of $R_\infty$ and the restriction of characters $\hat{T}_{p,L} \to \hat{T}_{p,0}^0$. If $\mathcal{E}(U^p)\overline{\mathfrak{p}}$ denotes the $\overline{\mathfrak{p}}$ part of the eigenvariety for $G$ (as defined for instance in [Che11, Definition 2.2] when $n = 3$), equipped with a weight morphism $\omega: \mathcal{E}(U^p)\overline{\mathfrak{p}} \to \hat{T}(O_{F^+_v} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, then there is a commutative diagram of reduced rigid analytic spaces

$$
\begin{array}{ccc}
\mathcal{E}(U^p)\overline{\mathfrak{p}} & \xrightarrow{\omega} & X_p(\overline{\mathfrak{p}}) \\
\downarrow & & \downarrow \omega_X \\
\hat{T}(O_{F^+_v} \otimes_{\mathbb{Z}} \mathbb{Z}_p) & \to & \hat{T}(O_{F^+_v} \otimes_{\mathbb{Z}} \mathbb{Z}_p) \times X_{S_{\infty}}
\end{array}
$$

where the lower horizontal arrow is defined by the inclusion of a closed point in $X_{S_{\infty}}$.

**Remark 7.1.** We recall that:

1. the weight morphism $\omega_X$ is flat and finite locally on the domain $[BHS17, \text{Proposition 3.10}]$;
2. both $W_{\infty}$ and $X_p(\overline{\mathfrak{p}})$ are equidimensional of dimension $g + |F^+: \mathbb{Q}| + n_1^2$; this is clear for $W_{\infty}$ and it is given by $[BHS17, \text{Corollary 3.11}]$ for $X_p(\overline{\mathfrak{p}})$;
3. the image of an irreducible component of $X_p(\overline{\mathfrak{p}})$ via $\omega_X$ is Zariski-open in $W_{\infty}$.

We refer to $[BHS17, \text{Définition 3.14}]$ for the definition of a classical point of $X_p(\overline{\mathfrak{p}})$.

We denote by $\mathcal{F}_v$ and $W_v$ the rigid analytic spaces over $\mathbb{Q}_p$ parametrizing continuous characters $F^+_v \times \to \mathbb{C}_p$ and $O_{F^+_v} \to \mathbb{C}_p$, respectively. Restriction of characters induces a morphism of rigid analytic spaces $\mathcal{F}_v \to W_v$. Let $\mathcal{F}_{v,L} = \mathcal{F}_v \times_{\mathbb{Q}_p} L$ and $W_{v,L} = W_v \times_{\mathbb{Q}_p} L$. The space $\mathcal{F}_{v,L}$ is a product of unit balls over $L$ of dimension $\dim(F^+_v) + n_1 + n$, so $W_{v,L}$ is a product of unit balls of dimension $\dim(F^+_v) + n_1 + n - 1$. We denote by $\mathcal{F}_{v,\text{reg}}$ the complement in $\mathcal{F}_{v,L}$ of the set of $L$-points

$$
\{ x \mapsto x^{-1} \}_{x \in \text{Hom}(F^+_v, L)} \cup \{ x \mapsto x^{i+1} | x^{F^+_v} \}_{x \in \text{Hom}(F^+_v, L)}
$$

and by $\mathcal{F}_{v,\text{reg}}^{n, L}$ the Zariski-open subset of $\mathcal{F}_{v,L}$ consisting of the points $(\delta_1, \ldots, \delta_n)$ satisfying $\delta_i \delta_j^{-1} \in \mathcal{F}_{v,\text{reg}}$ for all $i, j$ with $i \neq j$. We denote by $W_{v,\text{reg}}^{n, L}$ the image of $\mathcal{F}_{v,\text{reg}}^{n, L}$ in $W_{v,L}$.

Let $U_{\mathcal{F}_{v,\text{reg}}}^{(n)}(\overline{\mathfrak{p}})$ be the set of couples $(r, \overline{\mathfrak{p}})$ where:

* $r: G_{F_v} \to \text{GL}_n(O_L)$ is a framed trianguline lift of $\overline{\mathfrak{p}}$;
* $\overline{\mathfrak{p}} \in \mathcal{F}_{v,\text{reg}}^{n, L}$ is an ordered set of parameters of a triangulation of $r$. 

15
Let $X_{\tri}^{\square}(\mathfrak{p}_v)$ be the smallest closed rigid analytic subspace of $X_{\tri}^{\square}_{\mathfrak{p}_v} \times \mathcal{T}^n_{v,L}$ containing the $U_{\tri}(\mathfrak{p}_v)^{\text{reg}}$.

**Remark 7.2.** By [HS16, Lemma 2.12], the points of $U_{\tri}(\mathfrak{p}_v)^{\text{reg}}$ are strictly trianguline in the sense of Definition 5.1.

By composing the inclusion $X_{\tri}(\mathfrak{p}_v) \hookrightarrow X_{\tri}^{\square}_{\mathfrak{p}_v} \times \mathcal{T}^n_{v,L}$ with the second projection and then with the morphism $\mathcal{T}^n_{v,L} \rightarrow \mathcal{W}^n_{v,L}$ given by the restriction of characters we obtain a weight morphism $\omega_v: X_{\tri}(\mathfrak{p}_v) \rightarrow \mathcal{W}^n_{v,L}$. We set $X_{\tri}(\mathfrak{p}_v) = \prod_{v \in \Sigma_p} X_{\tri}(\mathfrak{p}_v)$, that is a closed rigid analytic subvariety of $X_{\tri}^{\square}_{\mathfrak{p}_v} \times \hat{T}_{p,L}$. This space is equipped with a weight morphism $\omega_v: X_{\tri}(\mathfrak{p}_v) \rightarrow \prod_{v \in \Sigma_p} \mathcal{W}_{v,L}$ defined by $\prod_{v \in \Sigma_p} \omega_v$.

We denote by $\hat{U}$ the open rigid analytic disc of radius 1 over $L$, that is the rigid analytic space $\text{Spf}(\mathcal{O}_L[[T]])^{\text{rig}}$. For every $v \in \Sigma_p$, let $\iota_{F_v^+}$ be the automorphism of $\hat{T}(F_v^+)$ defined by

$$
\iota_{F_v^+}((\delta_1, \ldots, \delta_n) = \delta_{B_v} \cdot (\delta_1, \ldots, \delta_i \cdot (\varepsilon \circ \text{rec}_K)^{i-1}, \ldots, \delta_n \cdot (\varepsilon \circ \text{rec}_K)^{n-1}),
$$

where $\delta_{B_v}$ is the modular character of $B_v(F_v^+)$, that is $\delta_{B_v} = |\cdot|_{F_v^+}^{n-1} \cdot |\cdot|_{F_v^+}^{n-3} \cdot |\cdot|_{F_v^+}^{-n}$. Let $\iota$ be the automorphism

$$
\text{Id}_{X_{\tri}^{\square}_{\mathfrak{p}_v}} \times (\iota_{F_v^+})_{v \in \Sigma_p}: X_{\tri}^{\square}_{\mathfrak{p}_v} \times \hat{T}_{p,L} \rightarrow X_{\tri}^{\square}_{\mathfrak{p}_v} \times \hat{T}_{p,L}.
$$

By [BHS17, Théorème 3.20], there is an embedding

$$
X_{p}(\mathfrak{p}) \hookrightarrow X_{\tri}^{\square}_{\mathfrak{p}_v} \times \mathcal{T}_{v,L}^{\square} \times \hat{U}^g \times \hat{T}_{p,L}
$$

that identifies $X_{p}(\mathfrak{p})$ with a union of irreducible components of $\iota(X_{\tri}(\mathfrak{p}_v)) \times X_{\tri}^{\square}_{\mathfrak{p}_v} \times \hat{U}^g$. For every $v \in \Sigma_p$, we denote by $\text{res}_v$ the morphism $X_{p}(\mathfrak{p}) \rightarrow X_{\tri}(\mathfrak{p}_v)$ defined as the composition of the inclusion $X_{p}(\mathfrak{p}) \hookrightarrow \iota(X_{\tri}(\mathfrak{p}_v)) \times X_{\tri}^{\square}_{\mathfrak{p}_v} \times \hat{U}^g$ with the projection to $\iota(X_{\tri}(\mathfrak{p}_v))$ and the automorphism $\iota^{-1}$. We also set $\text{res}_p = \prod_{v \in \Sigma_p} \text{res}_v: X_{p}(\mathfrak{p}) \rightarrow X_{\tri}(\mathfrak{p}_v)$.

If $v \in \Sigma - \Sigma_p$, we denote by $\text{res}_v$ the morphism $X_{p}(\mathfrak{p}) \rightarrow X_{\tri}^{\square}_{\mathfrak{p}_v}$ obtained by composing the inclusion $X_{p}(\mathfrak{p}) \hookrightarrow \iota(X_{\tri}(\mathfrak{p}_v)) \times X_{\tri}^{\square}_{\mathfrak{p}_v} \times \hat{U}^g$ with the projection to $X_{\tri}^{\square}_{\mathfrak{p}_v}$.

As in [BHS17, Définition 3.21], we call an irreducible component $X$ of $X_{\tri}^{\square}_{\mathfrak{p}_v}$ automorphic if there exists an irreducible component $X'$ of $X_{\tri}^{\square}_{\mathfrak{p}_v}$ such that $\iota(X) \times X' \times \hat{U}^g$ is an irreducible component of $X_{p}(\mathfrak{p})$.

Let $\varpi_{\mathfrak{p}}$ be a uniformizer of $F_{\mathfrak{p}}$. For $i \in \{1, \ldots, n\}$ we denote by $\beta'_i: \mathbb{G}_{m, F_{\mathfrak{p}}}$ into the $i$-th entry of $T_{F_{\mathfrak{p}}}$ the cocharacter embedding $\mathbb{G}_m$ into the $i$-th entry of $T_{F_{\mathfrak{p}}}$. We set $\beta_i = \prod_{j \leq i} \beta'_i$ and $\gamma_{\varpi, i} = \beta_i(\varpi)$. Recall that we denote by $\delta_{B_v}: B_v(F_v^+) \rightarrow \mathbb{Q}^\times$ the modular character of $B_v$. For $i \in \{1, \ldots, n\}$, the map $(x, \hat{\delta}) \rightarrow \delta \beta_{B_v}^{-1}(\beta_i(\gamma_{\varpi, i}))$ defines an analytic function on $X_{\tri}(\mathfrak{p}_{n, \varpi})$.

**Definition 7.3.** Given $i \in \{1, \ldots, n\}$ and a point $(x, \hat{\delta})$ of $X_{\tri}(\mathfrak{p}_{n, \varpi})$, we set $s_l(x, \hat{\delta}) = v_{F_v^+}((\delta \beta_{B_v}^{-1}(\gamma_{\varpi, i})))$ and we call it the $i$-th slope of $(x, \hat{\delta})$.

Since the functions $\delta \beta_{B_v}^{-1}(\gamma_{\varpi, i})$ are analytic on $X_{\tri}(\mathfrak{p}_{n, \varpi})$, the slopes $s_l: X_{\tri}(\mathfrak{p}_{n, \varpi}) \rightarrow \mathbb{R}^\times$ are locally constant functions.

The following conjecture of Breuil, Hellmann and Schraen can be seen as an analogue of the overconvergent Fontaine-Mazur conjecture that Kisin and Emerton proved for the group $\text{GL}_2/\mathbb{Q}$.
Conjecture 7.4. [BHS17, Conjecture 3.22] An irreducible component of $\prod_{v \mid p} X^{\square}_{\tri}(\overline{\mathcal{F}}_v)$ is automorphic if and only if its intersection with $\prod_{v \mid p} U^{\square}_{\tri}(\overline{\mathcal{F}}_v)_{\text{reg}}$ contains a crystalline point.

Given a $\overline{\mathbb{Q}}_p$-point $x$ of $X_p(\overline{\mathcal{F}})$ and a $p$-adic place $v$ of $F$, we denote by $\rho_x: G_{F_v} \to \text{GL}_n(\overline{\mathbb{Q}}_p)$ and $\Delta_{x,v} \in \hat{T}_{v,L}$ the $p$-adic Galois representation and the triangule parameter at $v$, respectively, associated with $x$ by the immersion $X_p(\overline{\mathcal{F}}) \hookrightarrow \mathcal{X}_{\overline{\mathbb{Q}}_p} \times \mathcal{X}_{\overline{\mathbb{Q}}_p} \times \mathcal{U}_x \times \hat{T}_{p,L}$.

**Definition 7.5.** We call strictly trianguline locus of $X_p(\overline{\mathcal{F}})$ and denote by $X_p(\overline{\mathcal{F}})^{\text{str}}$ the locus of $\overline{\mathbb{Q}}_p$-points $x$ of $X_p(\overline{\mathcal{F}})$ such that $\rho_{x,v}$ is strictly trianguline for every $p$-adic place $v$ of $F$.

The strictly trianguline locus is Zariski-open in $X_p(\overline{\mathcal{F}})$. In particular it admits a unique structure of rigid $\overline{\mathbb{Q}}_p$-analytic subspace of $X_p(\overline{\mathcal{F}})$.

**Lemma 7.6.**

1. There exists an $n$-dimensional pseudocharacter $T_{\square_{\overline{\mathbb{Q}}_p}}^{\square_{\overline{\mathbb{Q}}_p}}: G_{F_v} \to \mathcal{O}^\diamond(X^{\square}_{\tri}(\overline{\mathcal{F}}))$ with the property that, for every $\overline{\mathbb{Q}}_p$-point $x$ of $X^{\square}_{\tri}(\overline{\mathcal{F}})$, the specialization of $T_{\square_{\overline{\mathbb{Q}}_p}}^{\square_{\overline{\mathbb{Q}}_p}}$ at $x$ is the trace pseudocharacter attached to $\rho_x$.

2. There exists an $n$-dimensional pseudocharacter $T_{\square_{\overline{\mathbb{Q}}_p}}^{\square_{\overline{\mathbb{Q}}_p}}: G_{F_v} \to \mathcal{O}^\diamond(X_p(\overline{\mathcal{F}}))$ with the property that, for every $\overline{\mathbb{Q}}_p$-point $x$ of $X_p(\overline{\mathcal{F}})$, the specialization of $T_{\square_{\overline{\mathbb{Q}}_p}}^{\square_{\overline{\mathbb{Q}}_p}}$ at $x$ is the trace pseudocharacter attached to $\rho_{x,v}$.

3. Let $U$ be a connected wide open subdomain of $X_p(\overline{\mathcal{F}})$ such that $\omega_{X|U}: U \to \omega_X(U)$ is a finite morphism. Then the pseudocharacter $T$ can be lifted over $U$ to a continuous representation $\rho_U: G_{F_v} \to \text{GL}_n(\mathcal{O}^\diamond(U))$, and for every $\overline{\mathbb{Q}}_p$-point $x$ of $U$ the specialization of $\rho_U$ at $x$ is equivalent to the representation $\rho_x$.

**Proof.** Part (1) follows from the argument in [BC09, Proposition 7.5.4], using the fact that $X^{\square}_{\tri}(\overline{\mathcal{F}})$ is wide open (or nested in the terminology of loc. cit.). The pseudocharacter of part (2) is obtained by composing with the morphism $\mathcal{O}^\diamond(i(X^{\square}_{\tri}(\overline{\mathcal{F}}))) \to \mathcal{O}^\diamond(X_p(\overline{\mathcal{F}}))$ induced by $\text{res}_{\overline{\mathbb{Q}}_p}: X_p(\overline{\mathcal{F}}) \to X^{\square}_{\tri}(\overline{\mathcal{F}})$.

The ring of analytic functions of norm bounded by 1 on a wide open separated rigid analytic space is profinite by [BC09, Lemma 7.2.11(2)], hence the ring $\mathcal{O}^\diamond(U)$ admits all of the previous properties; it is moreover local and Noetherian since $U$ is connected and admits a finite morphism to $\omega_X(U)$. Then part (3) follows from a classical theorem of Nyssen and Rouquier [Nys96, Rom96, Corollary 5.2].

Note that by [BHS17, Proposition 3.10] $X_p(\overline{\mathcal{F}}_{\omega_m})$ admits an admissible covering by domains of the form required in part (2) of the above lemma.

Let $(x, \delta)$ be a point of $X^{\square}_{\tri}(\overline{\mathcal{F}}_{\omega,m})$ such that the representation $\rho_x$ is crystalline and non-critical. Let $(\varphi_1, \ldots, \varphi_n)$ be the eigenvalues of the Frobenius automorphism of $D_{\text{cris}}(\rho_x)$, with the ordering given by the triangulation of parameter $\delta$, and let $(k_1, \ldots, k_n)$ be the Hodge-Tate weights of $\rho_x$, in decreasing order. Since $(x, \delta)$ is non-critical, [BHS17, Proposition 3.15] gives $\varphi_i = p^{k_i} \delta_i(\gamma_0)$ for every $i$. In particular $v_p(\delta_i(\gamma_0)) = v_p(\varphi_i) - k_i$, so

$$sl_i(x, \delta) = v_p(\varphi_i) - v_p(\delta_i(\gamma_0)) - k_i.$$ (7.2)
Now let \((x, \tilde{\delta}')\) be a point of \(\mathcal{X}_{\tri}(\mathcal{F}_{x, \tilde{x}})\) corresponding to a different triangulation \(\rho_x\). Since we are working over the strict trianguline locus, the triangulation of parameter \(\delta'\) induces an ordering \((\varphi_{\sigma(1)}, \ldots, \varphi_{\sigma(n)})\) of the eigenvalues of the crystalline Frobenius for some non-trivial permutation \(\sigma\) of \(\{1, \ldots, n\}\). For every \(i\), Equation (7.2) gives

\[
sl_i(x, \delta') = v_p(\varphi_{\sigma(i)}) - v_p(\delta_B(\gamma_{\tilde{\delta}})) - k_i = sl_{\sigma(i)}(x, \tilde{\delta}) + k_{\sigma(i)} - k_i.
\]

**Definition 7.7.** We call family of representations of \(G_{\mathbb{F}_p}\) the datum \(\mathcal{F} = (A, \rho_{\mathcal{F}})\) of:

* an affinoid \(L\)-algebra \(A\) that is a domain,
* a continuous representation \(\rho_{\mathcal{F}} : G_{\mathbb{F}_p} \to \text{GL}_n(A)\), such that the set of points \(x\) such that \(\rho_{\mathcal{F}}\) specializes to a crystalline representation at \(x\) is Zariski-dense and accumulation in \(\text{Sp}(A)\). We call support of the family the affinoid space \(\text{Sp}(A)\) and dimension of the family the \(L\)-dimension of \(\text{Sp}(A)\).

**Definition 7.8.** Let \(\mathcal{F} = (\text{Sp}(A), \rho_A)\) be a family of representations of \(G_{\mathbb{F}_p}\). We say that \(\mathcal{F}\) appears on \(\mathcal{X}_{\tri}(\mathcal{F}_{x, \tilde{x}})\) if there exists an embedding \(\alpha : \text{Sp}(A) \to \mathcal{X}_{\tri}(\mathcal{F}_{x, \tilde{x}})\) such that the specialization of \(\rho_{\mathcal{F}}\) at \(x \in \text{Sp}(A)\) is isomorphic to \(\rho_{\alpha(x)}\). We say that \(\mathcal{F}\) appears \(j\) times on \(\mathcal{X}_{\tri}(\mathcal{F}_{x, \tilde{x}})\) if there are \(j\) distinct morphisms \(\alpha\) satisfying the previous condition.

**Definition 7.9.** We say that the \(i\)-th weight of \(\mathcal{F}\) is constant if there is an integer \(k\) such that the \(i\)-th Hodge-Tate weight of every crystalline specialization of \(\rho_{\mathcal{F}}\) is \(k\).

We say that \(\mathcal{F}\) has \(r\) constant weights if there exist \(r\) distinct indices \(i\) in the set \(\{1, \ldots, d\}\) with the property that the \(i\)-th weight of \(\mathcal{F}\) is constant.

**Remark 7.10.** Let \(\mathcal{F}\) be a family with \(r\) constant weights. If \(\mathcal{F}\) appears on \(\mathcal{X}_{\tri}(\mathcal{F}_{x, \tilde{x}})\), then its codimension in \(\mathcal{X}_{\tri}(\mathcal{F}_{x, \tilde{x}})\) is at least the \(L\)-codimension \(r[F_{x, 2} : \mathbb{Q}_p]\) of \(\dim_L \mathcal{W}_{v,L}^{m-r}\) in \(\dim_L \mathcal{W}_{v,L}^m\), since only \(n-r\) weights are allowed to vary.

**Proposition 7.11.** Assume that \(\mathcal{F}\) has exactly \(r\) constant weights. Then \(\mathcal{F}\) appears at most \(r\) times on \(\mathcal{X}_{\tri}(\mathcal{F}_{x})\).

**Proof.** Let \(I_{\text{st}}^\mathcal{F}\) be the set of indices of the constant weights of \(\mathcal{F}\). Let \(\ell\) be the number of appearances of the family \(\mathcal{F}\) on \(\mathcal{X}_{\tri}(\mathcal{F}_{x, \tilde{x}})\). This means that there exist \(\ell\) families \(\mathcal{F}_1, \ldots, \mathcal{F}_\ell\) that satisfy \(\rho_{\mathcal{F}_i} \cong \rho_{\mathcal{F}}\) for \(i = 1, \ldots, \ell\). By the local constancy of the slopes, up to restricting the family \(\mathcal{F}\) to a family of the same dimension we can suppose that all the slopes \(s_{\mathcal{F}_i}\), \(1 \leq i \leq n\), are constant on each of the families \(\mathcal{F}_1, \ldots, \mathcal{F}_\ell\). Given a point \(\rho_x\) of the family \(\mathcal{F}\), we denote by \((\rho_x, \delta_{x,1}), \ldots, (\rho_x, \delta_{x,\ell})\) the \(\ell\) triangulations of \(\rho_x\) appearing in the families \(\mathcal{F}_1, \ldots, \mathcal{F}_\ell\). Consider two crystalline points \(\rho_x, \rho_y\) in the family \(\mathcal{F}\) such that both \(x\) and \(y\) appear with \(\ell\) distinct triangulations on the families \(\mathcal{F}_1, \ldots, \mathcal{F}_\ell\). This is possible since the rigid analytic subspaces of \(\mathcal{X}_{\tri}(\mathcal{F}_{x, \tilde{x}})\) supporting the families \(\mathcal{F}_1, \ldots, \mathcal{F}_\ell\) pairwise intersect in a Zariski-closed subspace, while the set of crystalline points is Zariski-dense in each of them since it is Zariski-dense in \(\mathcal{F}\). We denote the triangulations of \(x\) and \(y\) on the family \(\mathcal{F}_j\) by \((x, \delta_{x,j}), (y, \delta_{y,j})\), respectively. Let \((k_{1,x}, \ldots, k_{n,x})\) and \((k_{1,y}, \ldots, k_{n,y})\) be the decreasing \(n\)-tuples of Hodge-Tate weights of \(\rho_x\) and \(\rho_y\), respectively. Up to modifying our choice of the point \(y\) respecting the previous condition, we can suppose that

\[
(7.4) \quad k_{11,x} - k_{12,x} = k_{i_1', y} - k_{i_2', y} \iff i_1, i_2, i_1', i_2' \in I_{\text{st}}^\mathcal{F}.
\]
Indeed, the points $y$ satisfying the first condition accumulate at $x$, so we can choose a point $y$ as $p$-adically close to $x$ as we want making the right hand side of Equation (7.3) arbitrarily large.

Let $i \in \{1, \ldots, n\}$ and $j_1, j_2 \in \{1, \ldots, \ell\}$ with $j_1 \neq j_2$. The triangulation of $\rho_x$ of parameter $\Delta_{j_1}$ induces an ordering on the eigenvalues of the Frobenius automorphism of $D_{\text{cris}}(\rho_x)$. The triangulation of parameter $\Delta_{j_2}$ gives a new ordering, obtained by acting on the indices of the previous one with a permutation $\sigma_{j_2}^{j_1}$ of $\{1, \ldots, n\}$. In the same way we define a permutation $\sigma_{j_2}^{j_1}$ attached to the change of triangulations on $\rho_y$. By Equation (7.3) we have

$$sl_i(x, \Delta_{j_2,x}) = sl_{\sigma_{j_2}^{j_1}}(i)(x, \Delta_{j_1,x}) + k_{\sigma_{j_2}^{j_1}}(i,x) - k_{i,x},$$

$$sl_i(y, \Delta_{j_2,y}) = sl_{\sigma_{j_2}^{j_1}}(i)(y, \Delta_{j_1,y}) + k_{\sigma_{j_2}^{j_1}}(i,y) - k_{i,y}.$$ 

Now the slopes are constant on the families $\mathcal{F}_{j_1}$ and $\mathcal{F}_{j_2}$, so we deduce that

$$k_{\sigma_{j_2}^{j_1}}(i,x) - k_{i,x} = k_{\sigma_{j_2}^{j_1}}(i,y) - k_{i,y}.$$ 

Then condition (7.4) implies that either

$$i, \sigma_{j_2}^{j_1}(i) \in I^{\text{st}}_{\mathcal{F}}$$

or

$$i = \sigma_{j_2}^{j_1}(i) = \sigma_{j_1,y}^{j_1}(i).$$

Therefore $\sigma_{j_2}^{j_1}$ acts trivially on $\{1, \ldots, n\} - I^{\text{st}}_{\mathcal{F}}$, which implies that there are at most $r!$ distinct choices for $\sigma_{j_2}^{j_1}$. Since distinct triangulations of $\rho_x$ determine distinct values of $\sigma_{j_2}^{j_1}$, we conclude that $\ell \leq r!$. \qed

### 7.1. Schur functors on trianguline varieties.

Let $\underline{u}$ be a partition satisfying (1), let $q = q(\underline{u})$ and let $m = r, \underline{u}(n)$. Suppose that $p > 2m + 1$. Let $S : \text{GL}_n \to \text{GL}_m$ be the representation of $\text{GL}_n$ of highest weight $\underline{u}$. We omit the reference to $\underline{u}$ since the partition will be fixed throughout the rest of the text.

**Lemma 7.12.**

1. For every $v \in \Sigma$, there exists a closed morphism of rigid analytic spaces

$$S_v : \mathcal{X}_{\rho_v}^{\square} \to \mathcal{X}_{S \circ \rho_v}^{\square}$$

that maps $\rho_x$ to $S \circ \rho_x$.

2. For every $v \in \Sigma_p$, there exists a closed morphism of rigid analytic spaces

$$S_{\text{tri}, v} : \mathcal{X}_{\text{tri}}^{\square}(\mathcal{F}_v) \to \mathcal{X}_{\text{tri}}^{\square}(S \circ \mathcal{F}_v)$$

that maps $(\rho_x, \Delta)$ to $(S \circ \rho_x, S \circ \Delta)$.

**Proof.** For part (2), we show that the map

$$\mathcal{X}_{\rho_v}^{\square} \times \mathcal{F}_v^n \to \mathcal{X}_{S \circ \rho_v}^{\square} \times \mathcal{F}_v^n$$

$$(r, (\delta_i)_{1 \leq i \leq n}) \mapsto (S \circ r, S \circ \Delta).$$

induces a morphism of rigid analytic spaces. It is sufficient to show that this is true separately for the maps $\mathcal{X}_{\rho_v}^{\square} \to \mathcal{X}_{S \circ \rho_v}^{\square}$ and $\mathcal{F}_v^n \to \mathcal{F}_v^n$. This will also give part (1) for every $v \in \Sigma$, since only the first of the two maps appears there. The second map is analytic.
because it is polynomial. We show that the first map is also analytic. By definition the spaces \( X\varpi \) and \( X_S \circ \varpi \) represent the framed deformation functors \( D\varpi \) and \( D_S \circ \varpi \), respectively. The representation \( S \) defines a morphism of functors \( S\varpi : D\varpi \rightarrow D_S \circ \varpi \) for an object \( A \) of \( C_{k_1} \) and an object \( \rho : G_F \rightarrow GL_n(A) \) of \( D\varpi \). We define \( S\varpi (\rho) \) as \( S \circ \rho \). Since both \( D\varpi \) and \( D_S \circ \varpi \) are representable, \( S\varpi \) induces a morphism of representing spaces \( x\varpi \rightarrow x_S \circ \varpi \).

It follows from Corollary \( \text{[4.3]}(1) \) that the morphism defined in the previous paragraph maps \( U_{\text{tri}}(\varpi)_{\text{reg}} \) to \( U_{\text{tri}}(S \circ \varpi)_{\text{reg}} \), hence induces a morphism between the Zariski-closures \( X\varpi \) and \( X_S(S \circ \varpi) \) of the two sets. \( \Box \)

We set \( S_p = \prod_{v \in \Sigma_p} S_v, S^n = \prod_{v \in \Sigma - \Sigma_p} S_v \) and \( S_{\text{tri.p}} = \prod_{v \in \Sigma_p} S_{\text{tri.v}} \).

A weight \( \kappa \in W^p_L(\Lambda) \) is an \( n \)-tuple \( (\kappa_1, \ldots, \kappa_n) \) of characters \( O^\times_{F_v} \rightarrow L^\times \). We define a character \( \eta : G_m \rightarrow T_n(\Lambda) \) as \( \prod_{1 \leq i \leq n} \kappa_i \circ \beta'_{n,i} \), recalling that \( \beta'_{n,i} \) is the embedding of \( G_m \) into the \( i \)-th diagonal entry of \( T_n \). Then the composition \( S \circ \eta \) is a character \( G_m \rightarrow T_m(\Lambda) \) that we can write as \( \rho = \prod_{1 \leq i \leq n} (S \circ \kappa_i) \circ \beta'_{m,i} \) for an \( m \)-tuple \( (S \circ \kappa_1, \ldots, S \circ \kappa_n) \) of characters \( O^\times_{F_v} \rightarrow L^\times \). This \( m \)-tuple defines an element of \( W^p_L \) that we denote by \( S \circ \kappa \).

We call \( S \)-weight space and denote by \( W^S_L \) the image of the closed morphism of rigid analytic spaces

\[
\begin{align*}
S_W : W^p_L & \rightarrow W^m_L \\
\kappa & \mapsto S \circ \kappa.
\end{align*}
\]

The space \( W^S_L \) is a Zariski-closed subspace of \( W^m_L \). Since \( S : T_n(\Lambda) \rightarrow T_m(\Lambda) \) is an isogeny, \( W^S_L \) is equidimensional of dimension \( n|F_v^+ : \mathbb{Q}_p| + n - 1 \). It contains an accumulation and Zariski-dense subset of algebraic weights, for instance the images via \( S_W \) of the algebraic weights in \( W^m_L \).

We add subscripts \( m \) and \( n \) to distinguish the tori of \( GL_n \) and \( GL_m \) and their spaces of characters. Let \( (x, \hat{x}) \) be a point of \( X_{\text{tri}}(\varpi) \) and let \( \kappa \in \hat{T}_{n,v,L} \) be its weight. Then \( (S \circ x, S \circ \hat{x}) \) defines a point of weight \( S \circ \kappa \in \hat{T}_{m,v,L} \). For \( n_0 \in \mathbb{Z}_{\geq 1} \) and an element \( k = (k_{i,\tau}) \in (\mathbb{Z}^n_{\geq 0})_{\text{Hom}(F_v,L)} \), we also denote by \( x^k \) the element of \( W^m_{n_0,L} \) defined by \( x \mapsto \prod_{1 \leq i \leq n_0, \tau \in \text{Hom}(F_v,L)} x_{k_{i,\tau}} \). When \( n_0 = n \), we denote by \( S \circ k = ((S \circ k)_{i,\tau}) \) the unique element of \( (\mathbb{Z}^n_{\geq 0})_{\text{Hom}(F_v,L)} \) such that \( S \circ x^k = x^S \circ k \). With an abuse of notation, we will also write \( k \) for the weight \( x^k \). If \( k \in (\mathbb{Z}^n_{\geq 0})_{\text{Hom}(F_v,L)} \) and \( (k_{i,\tau}) \) is decreasing for fixed \( \tau \), we call \( k \) a classical weight.

**Lemma 7.13.** Let \( \tau \in \text{Hom}(F_v,L) \). Let \( (x, \hat{x}) \) be a point of \( X_{\text{tri}}(\varpi) \) of classical weight \( k \in W^n_{v,L}(\varpi) \). Then:

1. the slopes of \( S(x, \hat{x}) \) satisfy \( \max_{1 \leq i \leq m} |s_i, s(x, \hat{x})| \leq q \cdot \max_{1 \leq i \leq n} |s_i, x, \hat{x}| ; \)
2. if \( \kappa = (x_{k_1}, \ldots, x_{k_n}) \) with \( (k_{1,\tau}, \ldots, k_{n,\tau}) \) decreasing, then:
   - (i) \( \max_{1 \leq i \leq m} |S(k)_{i,\tau}| \leq q \cdot \max_{1 \leq i \leq n} |k_{i,\tau}| ; \)
   - (ii) \( \min_{1 \leq i \leq m} |S(k)_{i,\tau} - S(k)_{i+1,\tau}| \geq \min_{1 \leq i \leq n} |k_{i,\tau} - k_{i+1,\tau}| ; \)
   - (iii) \( \min_{1 \leq i \leq m} |S(k)_{i-1,\tau} - 2S(k)_{i,\tau} + S(k)_{i+1,\tau}| \geq \min_{1 \leq i \leq n} |k_{i-1,\tau} - 2k_{i,\tau} + k_{i+1,\tau}| . \)

**Proof.** Statements (1) and (2.i) are an immediate consequence of Remark \( \text{[4.3]}(2) \). Statements (2.ii-iii) follow from the fact that \( (k_{i,\tau})_{1 \leq i \leq n} \) is a decreasing \( n \)-tuple and that the ordered \( n \)-tuple of exponents of \( (k_{1,\tau}, \ldots, k_{n,\tau}) \) in the monomial defining \( (S \circ k)_{i+1,\tau} \) is
strictly smaller than that of the monomial defining \((S \cdot k)_{i, \tau}\) with respect to the lexicographic order.

7.2. **Schur functors on Hecke-Taylor-Wiles varieties.** We add subscripts \(m\) and \(n\) to most objects depending on the dimension of the relevant Galois representations. Let \(\overline{\rho}_m : G_F \rightarrow \text{GL}_m(\overline{\mathbb{Q}}_p)\) be a residual representation of automorphic origin. Let \(H\) be an algebraic subgroup of \(\text{GL}_m\).

**Definition 7.14.** We call \(H\)-congruence locus on the eigenvariety \(X_p(\overline{\rho}_m, p)\) the locus of its \(\overline{\mathbb{Q}}_p\)-points \(x\) satisfying \(\text{Im} \rho_{x,v} \subset H(\overline{\mathbb{Q}}_p)\) for every \(v \in \Sigma\). We denote it by \(X_p(\overline{\rho}_m)^H\).

When \(H = S(\text{GL}_n(\overline{\mathbb{Q}}_p))\) with the above notations, we also call \(X_p(\overline{\rho}_m)^H\) the \(S\)-congruence locus and we denote it by \(X_p(\overline{\rho}_m)^S\).

**Remark 7.15.** Consider the morphism \(\omega_{m,v} \circ \text{res}_{m,v} : X_p(\overline{\rho}_m) \rightarrow \mathcal{W}_L^n\). Under this map, the image of \(X_p(\overline{\rho}_m)^S\) is contained in \(\mathcal{W}_L^n\).

**Lemma 7.16.** Assume that \(\overline{\rho}_m\) is absolutely irreducible. The \(H\)-congruence locus is Zariski-closed in \(X_p(\overline{\rho}_m)(\overline{\mathbb{Q}}_p)\). In particular it admits a unique structure of reduced rigid \(\overline{\mathbb{Q}}_p\)-analytic subspace of \(X_p(\overline{\rho}_m)\).

**Proof.** Let \(\{U_i\}_{i \in \mathbb{N}}\) be an admissible covering of \(X_p\) in wide open subdomains with the properties required by Lemma 7.6(2). It is sufficient to show that the intersection of the \(H\)-congruence locus with each of the \(U_i\)’s is Zariski-closed. Let \(i \in \mathbb{N}\). By Lemma 7.6(2) there is a continuous representation \(\rho_{U_i} : G_F \rightarrow \text{GL}_m(\mathcal{O}(U_i))\) that specializes to \(\rho_x\) for every \(\overline{\mathbb{Q}}_p\)-point \(x\) of \(U_i\). Let \(\{f_j\}_{j \in I}\) be a set of equations for the subgroup \(H(\overline{\mathbb{Q}}_p)\) of \(G(\overline{\mathbb{Q}}_p)\). Then the \(H\)-congruence locus intersects \(U_i\) on the zero-locus of the ideal of \(\mathcal{O}^\circ(U_i)\) generated by \(\{f_j(\rho_{U_i}(g))\}_{j \in I, g \in G_0}\) for a finite number of these generators will be sufficient since \(\mathcal{O}^\circ(U_i)\) is Noetherian), hence on a Zariski-closed subspace of \(U_i\). □

In what follows we assume that \(\overline{\rho}_m\) satisfies the condition:

\[ \text{(Type S)} \]

there exists a continuous representation \(\overline{\rho}_n : G_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)\)

of automorphic origin such that \(\overline{\rho}_m = S \circ \overline{\rho}_n\).

If \(\text{[Type S]}\) is not true, the \(S\)-congruence locus on \(X_p(\overline{\rho}_m)\) is clearly empty. We fix from now on a representation \(\overline{\rho}_n\) satisfying assumption \(\text{[Type S]}\).

We set \(X_p(\overline{\rho}_m)^{S, \text{str}} = X_p(\overline{\rho}_m)^S \cap X_p(\overline{\rho}_m)^{\text{str}}\).

**Proposition 7.17.** Suppose that the “only if” part of Conjecture \([\ell, \Delta]\) is true for \(U(m)\). Then there exists a morphism of rigid analytic spaces \(S_{TW} : X_p(\overline{\rho}_n) \rightarrow X_p(\overline{\rho}_m)\) fitting in the commutative diagram

\[
\begin{array}{ccc}
X_p(\overline{\rho}_n) & \xrightarrow{\text{res}_p} & X_{\text{tri}}(\overline{\rho}_{n,m}) \times X_{\text{tri}}^\square \\
\downarrow{S_{TW}} & & \downarrow{S_{\text{tri}}} \\
X_p(\overline{\rho}_m) & \xrightarrow{\text{res}_p} & X_{\text{tri}}(\overline{\rho}_{m,p}) \times X_{\text{tri}}^\square
\end{array}
\]

**Proof.** Consider an irreducible component \(I_n\) of \(X_p(\overline{\rho}_n)\). By [BHS17, Théorème 3.18] the component \(I_n\) contains a classical \(\overline{\mathbb{Q}}_p\)-point \(x\). The representation \(\rho_{x,v}\) is crystalline for every \(v \in \Sigma_p\). Let \(y_{\text{tri}}\) denote the \(\overline{\mathbb{Q}}_p\)-point \(S_{\text{tri}} \circ \prod v(x)\) of \(X_{\text{tri}}(\overline{\rho}_{n,m})\). The representation \(\rho_{y_{\text{tri}},v}\) is isomorphic to \(S \circ \rho_{x,v}\) at every \(v \in \Sigma_p\) by definition of \(S_{\text{tri}}\). In particular it
is crystalline by Theorem 4.1(3). By the “only if” part of Conjecture 7.4 for $U(m)$, the component $I_m$ is in the image of $\text{res}_\Sigma$.

Conditionally on a part of Conjecture 7.4, we give a description of the space $X_p(\overline{\rho}_m)^{S,\text{str}}$. This result is an analogue of [Cont16b, Theorem 10.10].

In the following $g_m$ is the integer given by [BHS17, Theorem 3.4] for the representation $\overline{\rho}_m$. We denote by $\text{res}_\Sigma$ the morphism $X_p(\overline{\rho}_m)^{S,\text{str}} \to X_{\text{tri}}(\overline{\rho}_{n,p}) \times X_{\text{tri}}(\overline{\rho}_{m,p})$ given by the embedding (1) composed with the projection onto the first two factors, and we use the same notation when $m$ is replace by $n$.

**Theorem 7.18.** Suppose that the “if” part of Conjecture 7.4 is true for $U(n)$. Let $x$ be a $\overline{\mathbb{Q}}_p$-point of $X_p(\overline{\rho}_m)^{S,\text{str}}$ such that:

(1) the embedding

$$X_p(\overline{\rho}_m)^{\text{str}} \to \iota(X_{\text{tri}}(\overline{\rho}_{n,p})) \times X_{\text{tri}}(\overline{\rho}_{m,p})$$

is a local isomorphism at $x$;

(2) the trianguline parameters $(\mu_{x,v})_{v \in \Sigma_p} \in \mathcal{T}_{v,L}(\overline{\mathbb{Q}}_p)$ of $(\rho_{x,v})_{v \in \Sigma_p}$, lift to cocharacters $(\mu_{x,v,n})_{v \in \Sigma_p} \in \mathcal{T}_m(\overline{\mathbb{Q}}_p)$ via the isogeny $S: \mathcal{T}_m \to \mathcal{T}_{v,L}$.

Then:

(i) there exists a $\overline{\mathbb{Q}}_p$-point $y$ of $X_p(\overline{\rho}_n)^{\text{str}}$ such that $\rho_x = S \circ \rho_y$ and $\mu_y = \mu_{x,n}$;

(ii) every $\overline{\mathbb{Q}}_p$-point $y_\Sigma$ of $X_{\text{tri}}(\overline{\rho}_{n,p}) \times X_{\text{tri}}(\overline{\rho}_{m,p})$ such that $S_p \times S^p(y_{\text{str}}) = \text{res}_\Sigma(x)$ belongs to $\text{res}_\Sigma(X_p(\overline{\rho}_n)^{\text{str}})$.

Note that condition (1) is trivially satisfied for all smooth $\overline{\mathbb{Q}}_p$-points of $X_p(\overline{\rho}_n)^{S,\text{str}}$.

**Proof.** Consider the following diagram:

$$\begin{array}{ccc}
X_p(\overline{\rho}_n)^{\text{str}} & \xrightarrow{\prod_{v \in \Sigma} \text{res}_v} & \iota(X_{\text{tri}}(\overline{\rho}_{n,p})) \times X_{\text{tri}}(\overline{\rho}_{m,p}) \\
\downarrow S_{\text{tri},p} \times S^p & & \downarrow S_{\text{tri},p} \times S^p \\
X_p(\overline{\rho}_m)^{\text{str}} & \xrightarrow{\prod_{v \in \Sigma} \text{res}_v} & \iota(X_{\text{tri}}(\overline{\rho}_{m,p})) \times X_{\text{tri}}(\overline{\rho}_{m,p})
\end{array}$$

Since $x \in X_p(\overline{\rho}_n)^{S,\text{str}}(\overline{\mathbb{Q}}_p)$. By Corollary 5.4 and the assumption on $x$, there exists for every $v \in \Sigma$ a representation $\rho_{y,v}: G_{F_v} \to \text{GL}_n(\overline{\mathbb{Q}}_p)$ such that:

(1) $\rho_{x,v} = S \circ \rho_{y,v}$;

(2) if $v \in \Sigma_p$, then $\rho_{y,v}$ is strictly trianguline of parameter $\mu_{x,v}$.

Let $y_{\text{str}}$ be the point of $\iota(X_{\text{tri}}(\overline{\rho}_{n,p})) \times X_{\text{tri}}(\overline{\rho}_{m,p})$ whose corresponding Galois representations (with parameters at $v \in \Sigma_p$) are $(\rho_{y,v}, \mu_{x,v,n})$ for $v \in \Sigma_p$ and $(\rho_{y,v})$ for $v \in \Sigma - \Sigma_p$. It is clear from the properties (1-3) that $S_{\text{tri}}(y_{\text{str}}) = x_{\text{tri}}$. We will show that $y_{\text{str}}$ is in the image of the map

$$\prod_{v \in \Sigma} \text{res}_v: X_p(\overline{\rho}_n)^{\text{str}} \to \iota(X_{\text{tri}}(\overline{\rho}_{n,p})) \times X_{\text{tri}}(\overline{\rho}_{m,p}) .$$

Every inverse image $y$ of $y_{\text{str}}$ via $\prod_{v \in \Sigma} \text{res}_v$ then satisfies statement (i) in the theorem because of the properties (1,2) above. We will also obtain statement (ii) of the theorem in the following way. If $y_{\Sigma}$ is a $\overline{\mathbb{Q}}_p$-point of $X_{\text{tri}}(\overline{\rho}_{n,p}) \times X_{\text{tri}}(\overline{\rho}_{m,p})$ satisfying condition (1) above, it belongs to $\iota(X_{\text{tri}}(\overline{\rho}_{n,p})) \times X_{\text{tri}}(\overline{\rho}_{m,p})$ by Corollary 5.4 and its trianguline parameters $\mu_{y_{\text{str}}}$ are lifts of the parameters $\mu_{x,v}$. In particular $y_{\Sigma}$ is the image of a $\overline{\mathbb{Q}}_p$-point $y_{\text{str}}$ of $\iota(X_{\text{tri}}(\overline{\rho}_{n,p})) \times X_{\text{tri}}(\overline{\rho}_{m,p})$ that satisfies (1) and (2). By showing that such a point is in the image of $\prod_{v \in \Sigma} \text{res}_v: X_p(\overline{\rho}_n)^{\text{str}} \to \iota(X_{\text{tri}}(\overline{\rho}_{n,p})) \times X_{\text{tri}}(\overline{\rho}_{m,p})$, we obtain (ii).
Consider an irreducible component $J_n$ of $\iota(X_{\text{tri}}(\bar{\mathcal{P}}_{n,p})) \times \mathfrak{X}_{\bar{\mathcal{P}}_{n,p}}$ containing $y_{\text{tri}}$. Let $J_m$ be an irreducible component of $\iota(X_{\text{tri}}(\bar{\mathcal{P}}_{m,p})) \times \mathfrak{X}_{\bar{\mathcal{P}}_{m,p}}$ containing $S_{\text{tri},p} \times S^0(J_m)$. By assumption (1), $J_m$ is contained in the image of $X_p(\bar{\mathcal{P}}_n)$.

The component $J_n$ is a product $J_{n,p} \times \mathfrak{X}_n$ of an irreducible component $J_{n,p}$ of $\iota(X_{\text{tri}}(\bar{\mathcal{P}}_{p,n}))$ and an irreducible component $\mathfrak{X}_n$ of $\mathfrak{X}(\bar{\mathcal{P}})$. We show that the component $J_{n,p}$ is automorphic. This will imply that the point $y_{\text{tri}}$ of $J_{n,p} \times \mathfrak{X}_n$ is in the image of $X_p(\bar{\mathcal{P}}_n)$ via $\prod_{v \in \Sigma} \text{res}_v$, giving the conclusion of the theorem.

By the “if” part of Conjecture [7.4] for $U(n)$, it is sufficient to show that $J_{n,p} \cap U_{\text{tri}}(\bar{\mathcal{P}}_p)$ contains a crystalline point. By Remark [7.13] (3) the image of $J_{n,p}$ in the weight space $\mathcal{W}_L^n$ contains a weight $k \in (\mathbb{Z}^n)^{\text{Hom}(F^+, \mathcal{L})}$. Let $y_{\text{tri},k}$ be a point of $J_{n,p}$ of weight $k$. Consider an affinoid subdomain $U_n$ of $J_{n,p}$ containing both $y_{\text{tri}}$ and $y_{\text{tri},k}$. The absolute values of the slopes $s_l$ are bounded on $U_n$ by a common constant $C$. We claim that there exists a weight $k'$ and a point $y_{\text{tri},k'}$ of $U_n$ such that, for every embedding $\tau: F^+ \to \overline{\mathbb{Q}}_p$:

1. $k_{1,\tau} - k_{2,\tau} > 2q(C + 1)$;
2. $k_{i,\tau} - k_{i+1,\tau} > k_{i-1,\tau} - k_{i,\tau} + q(C + 1)$ for $i \geq 2$.

This is clear because $\omega_X(U_n)$ will contain an affinoid neighborhood of $k$, and we can choose a weight $k'$ arbitrarily $p$-adically close to $k$ satisfying (1) and (2). Let $x'_{\text{tri}} = S_{\text{tri},p} \times S^0(y_{\text{tri},k'})$, which is a point of weight $S \circ k'$ on the component $J_m$. Note that $\rho_{x'_{\text{tri}},v} = S \circ \rho_{y_{\text{tri},k'},v}$ for every $v \in \Sigma$, by definition of $S_{\text{tri},p}$. It follows from assumptions (1) and (2) and Lemma [7.13] 2.ii-iii) that

1. $S \circ k'_{1,\tau} - (S \circ k')_{2,\tau} > 2(C + 1)$;
2. $S \circ k'_{i,\tau} - (S \circ k')_{i+1,\tau} > (S \circ k')_{i-1,\tau} - (S \circ k')_{i,\tau} + C + 1$ for $i \geq 2$.

In particular $x'_{\text{tri}}$ is a classical point by the argument in the proof [BHS17, Théorème 3.18] (it is shown there that the set $W$ appearing in the statement can be taken to be the set of weights satisfying conditions (1S) and (2S)). In particular the representation $\rho_{x'_{\text{tri}},v}$ is crystalline for every $v \in \Sigma$. By the fact that $\rho_{x'_{\text{tri}},v} = S \circ \rho_{y_{\text{tri},k'},v}$ and a result of Di Matteo, the representation $\rho_{y_{\text{tri},k'},v}$ is also crystalline. We deduce that the component $J_{n,p}$ of $X(\bar{\mathcal{P}}_{n,p})$ contains the crystalline point $y_{\text{tri},k'}$, as desired.

Let $X_p(\bar{\mathcal{P}}_m)^{\text{sm}}$ be the locus of smooth points of $X_p(\bar{\mathcal{P}}_m)$.

**Corollary 7.19.** Suppose that the “if” part Conjecture [7.4] is true for $U(n)$ and the “only if” part of the conjecture is true for $U(m)$. Let $\mathcal{F} = (\text{Sp}(A), \rho_A)$ be a family of representations of $G_{F_0}$ appearing in $X_p(\bar{\mathcal{P}}_n)^{\text{sm}}$. Then $\mathcal{F}$ also appears in $S_{\text{TW}}(X_p(\bar{\mathcal{P}}_n))$.

**Proof.** Consider the following commutative diagram:

$$
\begin{array}{ccc}
X_p(\bar{\mathcal{P}}_n)^{\text{str}} & \xrightarrow{\text{res}_\Sigma} & \mathfrak{X}_{\bar{\mathcal{P}}_n} \times \mathfrak{X}_{\bar{\mathcal{P}}_n} \\
\downarrow S_{\text{TW}} & & \downarrow S_p \times S^p \\
X_p(\bar{\mathcal{P}}_m)^{\text{str}} & \xrightarrow{\text{res}_\Sigma} & \mathfrak{X}_{\bar{\mathcal{P}}_m} \times \mathfrak{X}_{\bar{\mathcal{P}}_m}
\end{array}
$$

Suppose that $\mathcal{F}$ appears in $X_p(\bar{\mathcal{P}}_n)^{\text{sm}}$ via $\alpha: \text{Sp}(A) \to X_p(\bar{\mathcal{P}}_n)^{\text{sm}}$. Consider the image $\text{res}_\Sigma(\alpha(\text{Sp}(A)))$ in $\mathfrak{X}_{\bar{\mathcal{P}}_m} \times \mathfrak{X}_{\bar{\mathcal{P}}_m}$ and its inverse image $(S_p \times S^p)^{-1}(\text{res}_\Sigma(\alpha(\text{Sp}(A))))$ in $\mathfrak{X}_{\bar{\mathcal{P}}_m} \times \mathfrak{X}_{\bar{\mathcal{P}}_m}$. By Theorem [7.18(i)], the morphism $S_p \times S^p$ induces a surjection $(S_p \times S^p)^{-1}(\text{res}_\Sigma(\alpha(\text{Sp}(A)))) \to \text{res}_\Sigma(\alpha(\text{Sp}(A)))$. This morphism is also continuous and closed,
so there exists an affinoid subdomain \( \text{Sp}(B) \) of \( \mathbb{X}^\square \times \mathbb{X}^\square \) such that \((S_p \times S^p)(\text{Sp}(B)) = \text{res}_\Sigma(\alpha(\text{Sp}(A))))\). By Theorem [7.18(ii)], the affinoid \( \text{Sp}(B) \) is contained in the image of \( X_p(\mathfrak{f}) \) via \( \text{res}_\Sigma \). Since \( \text{res}_\Sigma \) is a closed embedding, we deduce that there exists an affinoid subdomain \( \text{Sp}(C) \) of \( X_p(\mathfrak{f}) \) such that \( \text{res}_\Sigma(\text{Sp}(C)) = \text{Sp}(B) \). By the commutativity of Diagram (7.3), \( \text{res}_\Sigma(\text{ST}W(\text{Sp}(C))) = \text{res}_\Sigma(\alpha(\text{Sp}(A))) \), which implies that the family of representations of \( G_{F_0} \) carried by \( \text{Sp}(C) \) is the same as the family carried by \( \alpha(\text{Sp}(A)) \), that is that the family \( \mathcal{F} \) appears in \( \text{ST}W(X_p(\mathfrak{f})) \) with support \( \text{ST}W(\text{Sp}(C)) \).

\[ \square \]

**Corollary 7.20.** Suppose that the “if” part Conjecture 7.4 is true for \( U(n) \) and the “only if” part of the conjecture is true for \( U(m) \). Let \( S_p : X_p(\mathfrak{f}) \rightarrow X_p(\mathfrak{f}) \) be the morphism of rigid analytic spaces given by Proposition [7.17]. Then the rigid analytic space \( S_p(X_p(\mathfrak{f})) \cap X_p(\mathfrak{f})^{sm} \) consists of the irreducible components of maximal dimension of \( X_p(\mathfrak{f})^{S} \cap X_p(\mathfrak{f})^{sm} \).

**Proof.** Let \( d \) be the dimension of \( \mathcal{W}_L \). Since \( \omega(X_p(\mathfrak{f})^{S}) \) is Zariski-open in \( \mathcal{W}_L^{S} \) and \( \omega \) is of relative dimension 0, the irreducible components of \( X_p(\mathfrak{f})^{S} \) have dimension at most \( d \). This is also the dimension of every irreducible component \( X_p(\mathfrak{f}) \), hence of every irreducible component of \( S_p(X_p(\mathfrak{f})) \).

Let \( A \) be an affinoid \( \mathbb{Q}_p \)-algebra and let \( \mathcal{F} = (\text{Sp}(A), \rho_A) \) be a family of Galois representations appearing in \( X_p(\mathfrak{f})^{S} \) via \( \alpha : \text{Sp}(A) \rightarrow X_p(\mathfrak{f})^{S} \). Assume that \( \alpha(\text{Sp}(A)) \) is not contained in \( S_p(X_p(\mathfrak{f})) \). By Corollary [7.19] the family \( \mathcal{F} \) also appears in \( S_p(X_p(\mathfrak{f})) \), so it appears at least twice on \( X_p(\mathfrak{f})^{S} \). By Remark [7.10] the family \( \mathcal{F} \) must have some constant weight in \( \mathcal{W}_L^{S} \). If \( \mathcal{F} \) were of dimension \( d \) then \( \omega \circ \alpha(\text{Sp}(A)) \) would be Zariski-open in \( \mathcal{W}_L^{S} \), hence could not have any constant weight in \( \mathcal{W}_L^{S} \). We conclude that the dimension of \( \mathcal{F} \) is strictly smaller than \( d \).

\[ \square \]

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Andrea Conti
MAX-PLANCK-INSTITUT FÜR MATHEMATIK, VIVATSGASSE 7, 53111 BONN, GERMANY
E-mail address: aconti@mpim-bonn.mpg.de