Properties of magnetized neutral mesons within a full RPA evaluation

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We consider the two flavor Nambu–Jona-Lasinio model within the RPA framework to evaluate the masses of the $\sigma$ and $\pi^0$ mesons and the $\pi^0$ decay constant in the presence of a magnetic field at vanishing temperatures and baryonic densities. The present work extends other RPA applications by fully considering the external momenta which enter the integrals representing the magnetized polarization tensor. We employ a field independent regularization scheme so that more accurate results can be obtained in the evaluation of physical quantities containing pionic contributions. As we show, this technical improvement generates results which agree well with those produced by lattice simulations and chiral perturbation theory. Our method may also proves to be useful in future evaluations of quantities such as the shear viscosity and the equation of state of magnetized quark matter with mesonic contributions.

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I. INTRODUCTION

The study of strongly interacting magnetized matter has been receiving much attention recently due to the fact that this type of matter may be produced in peripheral heavy ion collisions [1] apart from possibly being present in magnetars [2]. In both situations the magnitude of the magnetic fields is huge and may reach about $\sim 10^{19} \, G$ and $\sim 10^{18} \, G$ in each case respectively. As far as heavy ion-collisions are concerned the presence of a strong magnetic field most certainly plays a role despite the fact that, in principle, the field intensity should decrease very rapidly lasting for about 1-2 fm/c only [1]. The possibility that this short time interval may [3] or may not [4] be affected by conductivity remains under dispute. Current theoretical investigations analyze, eg, the influence of the magnetic field in the quantum chromodynamics (QCD) phase diagram and related quantities such as the order parameters for the chiral and deconfinement transitions as well as the equation of state to be used in stellar modelling. To perform evaluations one usually employs lattice QCD or model approximations. The first approach is limited by the notorious sign problem which prevents its application at finite densities. Within the second approach one usually considers some effective theory, such as the Nambu–Jona-Lasinio model (NJL), in the framework of a given approximation such as the traditional mean field approximation (MFA) which is used in most applications. A summary of results covering these topics can be found in recent reviews [5]. On the other hand, the analysis of other important quantities such as those related to mesonic modes has received less attention in the literature despite their phenomenological importance. It is important to note that even the properties of neutral mesons may be affected by the external magnetic field produced in the earliest phase of heavy-ion collisions due to their quark content. Usually, the evaluation of mesonic observables, such as masses and decay constants, is more cumbersome than the MFA approximation evaluation of the equation of state (EoS) for example. The reason is that the momentum flow within the integrals representing quarkionic loops may depend on the external momentum, $k$, so that the sum over Matsubara’s frequencies and/or Landau levels become more cumbersome. One could then use a further approximation by considering $I(k^2) \approx I(0)$ which is the approach considered in an early evaluation of mesonic properties in the presence of a background magnetic field [6] where the authors have considered the NJL model in the RPA framework performing the evaluations with Schwinger’s proper time approach. The same model has been recently considered within the Ritus formalism in Ref. [7] where the authors have adopted the lowest Landau level (LLL) approximation in order to regularize their integrals. However, this procedure generates unwanted tachyonic instabilities at low temperatures and strong magnetic fields. Another powerful tool which has been considered in the evaluation of pionic observables is chiral perturbation theory (ChPT) [8]. Finally, we point out that the mass of magnetized pions has also been evaluated using lattice QCD (LQCD) simulations [9]. In the present work we use the RPA, within a magnetic field independent regularization scheme (MFIR) [10, 11], to fully evaluate the magnetized polarization tensor without any further approximations in order to describe the phenomenology of magnetized pions in a more accurate fashion. To the best of our knowledge this technically non trivial task, which requires the sum over Landau levels contributing to momentum dependent loops, has not been carried out before. Here, we employ the (full) RPA method to evaluate the $\sigma$ meson mass ($m_\sigma$) as well as the $\pi^0$ meson mass ($m_{\pi^0}$), decay constant ($f_{\pi^0}$), and its coupling to quarks ($g_{\sigma^aq}$). In particular, our numerical results for $m_{\pi^0}$ and $f_{\pi^0}$ agree well, respectively, with LQCD and ChPT predictions. It is

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important to remark that by establishing a reliable technique to evaluate the magnetized polarization tensor our work also represents an important step towards the complete evaluation of pionic contributions to the equation of state describing magnetized quark matter among other possible future applications. The pseudo-scalar polarization loop was also treated in the RPA approximation in Ref.\,\cite{12} using a similar approach to the one adopted in the present paper. Since the motivation of the former work was to propose a mechanism to explain the inverse magnetic catalysis, no systematic study of neutral mesons properties under strong magnetic fields was performed. In fact, one of the major goals of that study was to analyze the role of the $\pi^0$ in producing an effect opposed to the magnetic catalysis. Then, a rather complex expression for the pion propagator was written down in terms of Languerre polynomials and integrals which would certainly be inconvenient for further generalizations. Also, an approximate version of the pion propagator as an expansion for low momentum was obtained. Next, in order to justify the performed approximations, the full expression was evaluated numerically confirming the accuracy of the approximated expression. Here, on the other hand, one of our main contributions is to provide a simple, and exact, analytical expression for the polarization tensor which, although similar to Eq.(10) of Ref.\,\cite{12}, can be used without any restriction. As Ref.\,\cite{12} shows, understanding the hadron properties under strong magnetic fields is important to understand the QCD phase diagram. In this sense, the present work adds to the theoretical effort by providing an elegant and accurate analytical treatment of the magnetized polarization tensor. The work is organized as follows. In the next section we present the two flavor NJL model in the presence of a magnetic field and evaluate the neutral pion mass and decay constant. In Sec. III we compare our numerical results with ChPT and LQCD. Our conclusions are presented in Sec IV. For completeness, an appendix containing technical details is also included.

II. GENERAL FORMALISM

In the presence of an electromagnetic field the two-flavor NJL model can be described by

$$\mathcal{L} = \bar{\psi} \left( i\hat{D} - \hat{m} \right) \psi + G \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2 \right] - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad (1)$$

where $A^\mu$, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ are respectively the electromagnetic gauge and tensor fields, $G$ represents the coupling constant, $\vec{\tau}$ are isospin Pauli matrices, $Q$ is the diagonal quark charge matrix, $Q=\text{diag}(q_u=2e/3, q_d=-e/3)$, $D^\mu = (i\partial^\mu - QA^\mu)$ is the covariant derivative, $\psi$ is the quark fermion field, and $\hat{m}$ represents the bare quark mass matrix,

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}, \quad \hat{m} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}. \quad (2)$$

In the present work we consider the NJL model within the mean field approximation (MFA), which is obtained through a linearization of the $\mathcal{L}$ interaction term disregarding quadratic fluctuations. Taking into account that the pseudo-scalar condensate vanishes due to parity considerations, one obtains \cite{14}:

$$\mathcal{L} = \bar{\psi} \left( i\hat{D} - M \right) \psi + G \langle \bar{\psi}\psi \rangle^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad (3)$$

where the constituent quark mass is defined by

$$M = m - 2G \langle \bar{\psi}\psi \rangle. \quad (4)$$

We consider here $m = m_u = m_d$ and choose the Landau gauge, $A^\mu = \delta_{\mu2}x_1B$, which satisfies $\nabla \cdot \vec{A} = 0$ and $\nabla \times \vec{A} = \vec{B} = Be_3$, i.e., resulting in a constant magnetic field in the z-direction.

The $\pi^0$ pole mass calculation in the presence of strong magnetic fields is performed generalizing the derivation of Refs.\,\cite{12,10}. The pion and sigma mesons are associated respectively to scalar and pseudoscalar collective modes. In the following, in order to point out the new features that arise because of the presence of the external magnetic field, we briefly discuss the main steps involved in the $\pi^0$ mass calculation within the two flavor NJL model. The $T$-matrix for the scattering of pairs of quarks, $(q_1q_2) \rightarrow (q_1'q_2')$, can be calculated by solving the Bethe-Salpeter equation in the ladder or random phase approximation (RPA). Being mainly interested in pionic degrees of freedom we start by considering the pseudoscalar channel within the RPA which formally consists in summing the geometric series diagrammatically represented in Fig.\,\cite{11}. The left hand side of the equality in Fig.\,\cite{11} can be calculated by representing the quark-pion interaction with the following term \cite{14}:

$$\mathcal{L}_{\pi qq} = ig_{\pi qq} \bar{\psi} \gamma_5 \vec{\tau} \vec{\pi} \psi, \quad (5)$$

where $\vec{\pi}$ stands for the pion field while $g_{\pi qq}$ represents the coupling strength between pions and quarks. The right hand side of Fig.\,\cite{11} is calculated using the NJL model and selecting the quantum numbers associated to the neutral pion, $\pi^0$. Then, one can easily show that the effective interaction is given by the relation:

$$\left(i g_{\pi qq}^2\right)^2 iD_{\pi^0}(k^2) = \frac{2iG}{1 - 2GH_{\pi^0}(k^2)}, \quad (6)$$

1 Our results are expressed in Gaussian natural units where \(1 \text{GeV}^2 = 1.44 \times 10^{13} \, G \) and \(e^2 = 1/\sqrt{137}\).
where the pseudo-scalar polarization loop reads:

\[
\frac{1}{i} \Pi_{ps}(k^2) = - \sum_{q=u,d} \int \frac{d^4 p}{(2\pi)^4} Tr[i\gamma_5 iS_q(p + k/2) i\gamma_5 iS_q(p - k/2)].
\]  

(7)

Proceeding in an analogous way one obtains

\[
\frac{1}{i} \Pi_s(k^2) = - \sum_{q=u,d} \int \frac{d^4 p}{(2\pi)^4} Tr[iS_q(p + k/2) iS_q(p - k/2)],
\]  

(8)

for the scalar channel. In the evaluation of the polarization loops one needs the propagators representing mesons, \(D_{\pi^0}(k^2)\), as well as quarks, \(S_q(k^2)\), as we discuss next.

\[
\begin{array}{c}
\text{FIG. 1: Diagrammatic representation of the RPA approximation.}
\end{array}
\]

Since \(\pi^0\) is uncharged, \(D_{\pi^0}(k^2)\), in Eq. (9) represents the usual \(\pi^0\)-meson propagator:

\[
D_{\pi^0}(k^2) = \frac{1}{k^2 - m_{\pi^0}^2}.
\]  

(9)

At the same time, the (dressed) quark propagator is defined as

\[
iS_q(x, x') = \langle 0 | T [\bar{\psi}_q(x)\gamma_\mu \psi_q(x')] | 0 \rangle, \text{ } q = u, d,
\]  

(10)

where \(T\) is the time ordering operator. Using the standard quantum field theory procedure, i.e., expanding the fermion field operator, \(\psi_q(x)\), in a basis of states of the Dirac equation in a constant magnetic field, one obtains at zero temperature in coordinate space [17, 18]:

\[
S_q(x, x') = e^{i\Phi_q(x, x')} \sum_{n=0}^{\infty} S_{q,n}(x - x'), \text{ } q = u, d,
\]  

(11)

the above propagator is given by the product of a gauge dependent factor \(\Phi_q(x, x')\), called Schwinger phase, times a translational invariant term. The Schwinger phase can be written in the Landau gauge as:

\[
\Phi_q(x, x') = Q_q \int_x^{x'} dy \mu A^\mu(y),
\]  

(12)

where the integral should be performed along a straight line connecting \(x\) and \(x'\). Explicitly, one has:

\[
\Phi_q(x, x') = \frac{Q_q B}{2} (x^1 + x'^1) (x^2 - x'^2),
\]  

(13)

The translation invariant part of the propagator is given by:

\[
S_{q,n}(Z) = \frac{\beta_q}{2\pi} e^{\frac{\beta_q}{4} Z^2} \int \frac{d^2 p_{\parallel}}{(2\pi)^2} \frac{1}{p_{\parallel}^2 - M^2 - 2\beta_q n} \times \left\{ \left[ (p\gamma_\parallel) + M \right] \left[ \Pi_+ L_n \left( \frac{\beta_q}{2} Z_\perp^2 \right) + \Pi_- L_n-1 \left( \frac{\beta_q}{2} Z_\perp^2 \right) \right] + 2im \left( \frac{Z_\parallel}{Z_\perp^2} \right) \right\}.
\]  

(14)

In the above expression \(Z = x - x', Z_\parallel^2 = (Z_0^2 - Z_3^2), Z_\perp^2 = (Z_1^2 + Z_2^2), \Pi_\pm = \frac{1}{2}(\mathbb{1} \pm i\gamma^1 \gamma^2), \beta_q = |q| B\) and \(L_n(x)\) represent the Laguerre polynomials.

From a direct comparison of Eqs (6) and (9), one sees that under the present approximation the mass of the \(\pi^0\)-meson can be associated to the root of the equation:

\[
1 - 2G\Pi_{ps}(k^2)|_{k^2 = m_{\pi^0}^2} = 0.
\]  

(15)

Substituting the quark propagator in the pseudoscalar polarization loop, Eq. (7), one obtains:

\[
\frac{1}{i} \Pi_{ps}(k^2) = - \sum_{q=u,d} \sum_{n,m=0}^{\infty} \int d^4 x (x - x') e^{-ik(x-x')} \times Tr[i\gamma_5 iS_{q,n}(x-x') i\gamma_5 iS_{q,m}(x'-x')] e^{i\Phi_q(x,x')} e^{i\Phi_q(x',x')},
\]  

(16)

Since the Schwinger phases cancel out, \(e^{i\Phi_q(x,x')} e^{i\Phi_q(x',x')} = 1\), the Fourier transform can be applied to the last expression yielding:

\[
\frac{1}{i} \Pi_{ps}(k^2) = - \sum_{q=u,d} \sum_{n,m=0}^{\infty} \int \frac{d^4 p}{(2\pi)^4} \times Tr[i\gamma_5 iS_{q,n}(p + k/2) i\gamma_5 iS_{q,m}(p - k/2)].
\]  

(17)

Substituting the quark propagator, computing the traces, integrating over the Laguerre polynomials and performing a partial fraction decomposition, one obtains, after a rather long calculation, the expression:

\[
\begin{align*}
\frac{1}{i} \Pi_{ps}(k^2) &= \sum_{q=u,d} \sum_{n=0}^{\infty} 2g_n\beta_q N_c \\
&\times \left( \int \frac{d^2 p_{\parallel}}{(2\pi)^3} \frac{1}{p_{\parallel}^2 - M^2 - 2\beta_q n} \right.
\end{align*}
\]  

(18)

\[
\left. - \int \frac{d^2 p_{\parallel}}{(2\pi)^3} \frac{(k_{\parallel}^2/2)}{(p_{\parallel}^2 - M^2 - 2\beta_q n)((p+k)^2 - M^2 - 2\beta_q n)} \right),
\]

where \(g_n = 2 - \delta_{n0}\), \(p_{\parallel} = p_0 - p_3\), and \(k_{\parallel} = k_0 - k_3\).
In the last expression the second integral can be rewritten using the Feynman integration trick and making an appropriate change of variables \[19, 20\]. Furthermore, the integral over \( p_0 \) can be easily performed through a Wick rotation, resulting in the following expression for the polarizability loop:

\[
\frac{1}{i} \Pi_{ps}(k^2) = - \sum_{q=u,d} \sum_{n=0}^{\infty} 2g_n \beta_q N_c \left( i \int \frac{dp_3}{(2\pi)^3} \frac{\pi}{\sqrt{p_3^2 + M^2 + 2\beta_q n}} + \frac{k_\parallel^2}{2(2\pi)^3} I_{q,n}(k^2) \right),
\]

(19)

where the integral \( I_{q,n}(k^2) \) is defined as:

\[
I_{q,n}(k^2) = \frac{i\pi}{2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp_3 \frac{1}{[p_3^2 + M^2(k_n)] + 2\beta_q n}^3/2,
\]

(20)

and \( M^2(k_n) = M^2 - x(1-x)(k_n^2) \) \[19\].

The polarization loop expression, Eq. (19), can be further simplified if we consider the mass gap expression which be obtained from Eq. (1) as follows:

\[
M = m - 2G(\bar{\psi}\psi) = m + 2G \lim_{\vec{x} \to \vec{x}'} \lim_{\nu' \to \nu} \sum_{q=u,d} \text{Tr}[iS_q(x,x')].
\]

(21)

The last equation can be calculated using the quark propagator, Eq. (11), and noticing that the Schwinger phase corresponds to \( \Phi_q(x,x') = 1 \). Performing such calculations one then finds the usual gap equation \[11\]:

\[
\frac{M - m}{2MG} = \sum_{q=u,d} \sum_{n=0}^{\infty} 2g_n \beta_q N_c \int_{-\infty}^{\infty} \frac{dp_3}{(2\pi)^3} \frac{\pi}{\sqrt{p_3^2 + M^2 + 2\beta_q n}},
\]

(22)



where \( N_c=3 \). Thus, substituting the last expression in Eq. (19) the loop polarization becomes:

\[
\frac{1}{i} \Pi_{ps}(k^2) = -i \left( \frac{M - m}{2MG} \right) - \sum_{q=u,d} \beta_q N_c \left( \sum_{n=0}^{\infty} \frac{g_n I_{q,n}(k^2)}{(2\pi)^3} \right).
\]

(23)

Therefore, from Eq. (15), the \( \pi^0 \) mass can be written as:

\[
m_{\pi^0}^2(B) = - \frac{m}{M(B)} \left( \frac{2\pi^3}{M(B)} \right) \sum_{q=u,d} i2G\beta_q N_c \sum_{n=0}^{\infty} g_n I_{q,n}(m_{\pi^0}^2).
\]

(24)

In the present work, the divergent integrals such as \( I_{q,n} \) are regularized by a sharp noncovariant cutoff, \( \Lambda \), within the MFIR scheme. This method, which has been reported in Ref. \[11\], follows the steps of the dimensional regularization prescription of QCD, performing a sum over all Landau levels in the vacuum term. This allows to isolate the divergencies into a term that has the form of the zero magnetic field vacuum energy and that can be regularized in the standard fashion. We point out that originally such a method was proposed in Ref. \[10\] but in terms of the proper-time formulation. Recently, the MFIR has been successfully used in the case of magnetized color superconducting cold matter \[21\] where its advantages, such as the avoidance of unphysical oscillations, are fully discussed.

Then, following the techniques illustrated in Refs \[10,11\] which consider the Hurwitz-Zeta function integral representation, the sum over Landau levels can be performed and the regularized integral reads (see appendix):

\[
\sum_{n=0}^{\infty} g_n I_{q,n}(k^2 = m_{\pi^0}^2) = \frac{i\pi}{\beta_q} \int_{0}^{1} dx \left( -\psi \left( \frac{M^2(m_{\pi^0})}{2\beta_q} + 1 \right) + \frac{\beta_q}{\psi \left( \frac{M^2(m_{\pi^0})}{2\beta_q} \right)} \right) - \frac{2}{\sqrt{\Lambda^2 + M^2(m_{\pi^0})}} \sinh^{-1} \left( \frac{\Lambda}{\sqrt{\Lambda^2 + M^2(m_{\pi^0})}} \right),
\]

(25)

where \( \psi \) is the digamma function. The \( \sigma \)-meson mass, \( m_\sigma \), can be obtained in a completely analogous fashion by calculating the scalar polarization loop, Eq. (8), yielding:

\[
m_{\sigma}^2(B) = 4M^2(B) + m_{\pi^0}^2(B).
\]

(26)

We next discuss the calculation of the pion decay constant, \( f_{\pi^0} \), which implies the evaluation of following matrix element \[10\]:

\[
\langle 0 | J_{\gamma_5}^\pi | \pi^1 \rangle,
\]

(27)

that is equivalent to:

\[
if_{\pi^0}^2 \delta_{ij} = \sum_{q=u,d} \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[ ig_{\gamma_5 \mu} \gamma_5 \frac{1}{2} i S_q(p + \frac{1}{2} k) g_{\pi qq} \gamma_5 \gamma_j \gamma_5 i S_q(p + \frac{1}{2} k) \right].
\]

(28)

After evaluating the trace one obtains:

\[
f_{\pi^0}^2(B) = -i \sum_{u,d} \frac{\beta_q}{(2\pi)^3} N_c M^2 \sum_{n=0}^{\infty} g_n I_{q,n}(0),
\]

(29)

where \( I_{q,n}(0) \approx I_{q,n}(m_{\pi^0}^2) \) is the integral given in Eq. (23). Multiplying the pion mass, Eq. (24), by the pion decay
constant, Eq. (29), one obtains:

\[ m_{\pi^0}^2(B) f_{\pi^0}^2(B) = \frac{m M(B)}{2G}. \]  

(30)

Now, Eq. (30) can be used in order to eliminate the coupling constant, \(G\), of the last equation leading to the Gell-Mann–Oakes–Renner (GOR) relation in a magnetic medium

\[ m_{\pi^0}^2(B) f_{\pi^0}^2(B) = m \langle \bar{\psi} \psi \rangle(B). \]  

(31)

III. NUMERICAL RESULTS

Let us now perform numerical evaluations in order to compare our results with those from other applications. In this work we choose the following parametrization set \([14]\): \(\Lambda = 664.3\,\text{MeV}, m = 5.0\,\text{MeV},\) and \(G \Lambda^2 = 2.06\) which reproduce \(f_0^0 = 92.4\,\text{MeV}, m_\pi^0 = 135.0\,\text{MeV},\) and \(\langle \bar{q}q \rangle^{1/3} = -250.8\,\text{MeV}.\) Fig 2 compares our result for \(f_{\pi^0}\) with the one obtained, with the \(\bar{q}q\) formalism in Ref. [22] for \(eB\) up to 1 GeV². For very weak field values the figure shows that both approximations predict an opposite behavior with the \(\bar{q}q\) formalism predicting an initial decrease of \(f_0^0\). Then, for fields higher than about 0.05 GeV², both approximations predict that \(f_0^0\) increases with \(B\) with the full RPA predicting a less dramatic increase. In order to settle the qualitative disagreement occurring within the weak field range we compare, in Fig. 3 both results with the ChPT predictions for low \(eB\) values \([8]\). The figure shows a good agreement between the full RPA and ChPT while the \(\bar{q}q\) formalism predicts an opposite behavior. We remark that in the evaluation of \(f_{\pi^0}\) one always considers the polarization integral \(I(k^2 = 0)\) so that our result for this quantity agrees with the ones obtained in early evaluations \([16]\). Next, in Fig. 4 we compare the results generated by the RPA with, and without, the further approximation \(I(k^2) \approx I(0)\) for the mass pole. A visible, although very small, difference is observed at intermediate \(eB\) values while at high field intensity, when only the LLL contribute, the results coincide. It is important to note that in Ref. [16], where the RPA with \(I(k^2 = 0)\) has been used, the pionic mass seems to be almost insensitive to \(B\) but that is due to the fact of using an inconsistent regularization procedure in the purely magnetic sector (in that work all the magnetic dependence enters via the effective quark mass only). In Fig 5 our full RPA results are compared with the LQCD predictions of Ref. [9] showing a rather good agreement up to about 0.5 GeV² whereas for intermediate field values the maximum difference is not higher than about 15% at intermediate values dropping again for high values \((eB \geq 2\,\text{GeV}^2)\). For lower field strength values, our predictions can also be compared with the ChPT available in Ref. [8] as Fig. 6 shows. The agreement is also good for this range where LQCD results are not available. Now, having evaluated \(f_{\pi^0}, m_{\pi^0}\) as well as the quark condensate, \(\langle \bar{\psi} \psi \rangle\), one may investigate how the GOR relation, given by Eq. (31), holds in the presence of a magnetic field. Fig 7 shows the behavior of this quantity within the RPA formalism using the complete polarization integral and the approximated one \((I(k^2) \approx I(0))\). The figure shows that after dropping from the \(B = 0\) value by about only 1% the full RPA result remains quite stable when \(B\) increases in agreement with the analytical predictions by Agasian and Shushpanov [23]. On the other hand, the results obtained with the approximated integral appear to be less stable at high \(B\) values.

Concerning the \(\sigma\) meson mass our results, shown in Fig. 8 confirm the steady increase observed in Ref. [16]. Finally, Fig. 9 shows how the meson-quark coupling, \(g_{\pi^0\bar{q}q}\), slightly decreases up to \(eB \approx 0.2\,\text{GeV}^2\) and then steadily increases at higher field values.

![FIG. 2: Normalized pion decay constant \(f_{\pi^0}^2(B)/f_{\pi^0}^2(0)\) evaluated within the full RPA and the \(\bar{q}q\) formalism of Ref. [22] up to \(eB = 1\,\text{GeV}^2\).](image)

![FIG. 3: Normalized pion decay constant \(f_{\pi^0}^2(B)/f_{\pi^0}^2(0)\) evaluated within the full RPA, the \(\bar{q}q\) formalism [22], and ChPT [8] up to \(eB = 0.1\,\text{GeV}^2\).](image)
FIG. 4: Normalized neutral meson mass, $m_\pi^0(B)/m_\pi^0(0)$, evaluated with the full RPA and ChPT [8].

FIG. 5: Normalized neutral meson mass, $m_\pi^0(B)/m_\pi^0(0)$, evaluated with the full RPA and LQCD [9]. Within the latter, the lines are shown just to guide the eye.

FIG. 6: Normalized neutral meson mass, $m_\pi^0(B)/m_\pi^0(0)$, evaluated with the full RPA and ChPT [8].

FIG. 7: Normalized Gell-Mann-Oakes-Renner relation, $\text{GOR}(B)/\text{GOR}(0)$, as given by Eq. (31), evaluated with the full RPA using the complete polarization integral as well as the approximation $I(m_\pi^2) \approx I(0)$.

FIG. 8: Normalized $\sigma$ meson mass, $m_\sigma(B)/m_\sigma(0)$, evaluated with the full RPA formalism developed in this work.

FIG. 9: Normalized meson-quark coupling, $g_{e\pi qq}^\phi(B)/g_{e\pi qq}(0)$, evaluated with the full RPA formalism.

IV. CONCLUSIONS

We have used the RPA without any further approximations to evaluate quantities related to magnetized neutral mesons. In this way, the momentum dependence of the integral representing the polarization function has been fully taken into account in order to render more accurate results. After performing the summation over the Landau levels we have obtained results regularized in a
consistent way, through the use of a magnetic field independent regularization scheme, which is a welcome feature since the polarization function contributes to physical quantities such as the mass pole and decay constants. Our numerical results for these observables are in good agreement with other powerful methods such as ChPT and LQCD. It is also important to remark that, within magnetized hadronic matter, the more cumbersome evaluations related to mesonic contributions have received less attention than the contributions due to quarks which stem from the uncomplicated zero external momentum Green’s functions such as the ones considered in the evaluation of quantities like the (mean field) equation of state or quark condensates. Therefore, by presenting the complete evaluation of the magnetized polarization tensor, our work provides a framework for investigations where this quantity plays a crucial role as, e.g., the equation of state with pionic terms \[12\] and shear viscosity \[13\] among others. The method can also be readily extended to accommodate charged mesons and thermal effects.

**Appendix A: Regularization procedure**

In this appendix we present some technical details in order to obtain our main analytical result, i.e., Eq. (25). This procedure is based on the magnetic field independent regularization scheme \[10,\,11\]. We start with:

\[
\sum_{n=0}^{\infty} g_n I_{q,n}(k^2) = \sum_{n=0}^{\infty} g_n \int d^2p_\parallel \frac{1}{[p_\parallel^2 - M^2 - 2\beta_q n^2](p + k_\parallel^2)^2 - 2\beta_q n^2}.
\]

The last integral can be rewritten using Feynman’s parametrization \[13,\,20\]

\[
\frac{1}{AB} = \int_0^1 dx \frac{1}{[x A + (1-x)B]^2},
\]

making a change of variable \((p_\parallel \to p_\parallel + x k_\parallel)\) and defining \(\mathcal{M}^2(k_\parallel) = M^2 - x(1-x)k_\parallel^2\). Thus, eq. (A1), becomes:

\[
\sum_{n=0}^{\infty} g_n I_{q,n}(k^2) = \int_0^1 dx \sum_{n=0}^{\infty} g_n \int d^2p_\parallel \frac{1}{[p_\parallel^2 - \mathcal{M}^2(k_\parallel)^2 - 2\beta_q n^2]}
\]

\[
= \frac{i\pi}{2(2\beta_q)^{3/2}} \int_0^1 dx \sum_{n=0}^{\infty} g_n \int_{-\infty}^{\infty} dp_3 \frac{1}{\left[p_3^2 + \mathcal{M}^2(k_\parallel)^2 + n\right]^{3/2}}, \tag{A2}
\]

where in the last expression the \(p_0\)-integral was performed through a Wick rotation and for convenience \(2(\beta_q)^{3/2}\) was factored out. Now, the summation over \(n\) in eq. (A2) can be done by using the Riemann-Hurwitz zeta function,

\[
\zeta(z, x) = \sum_{n=0}^{\infty} \frac{1}{(x + n)^z}, \tag{A3}
\]

and its integral representation \[24\]

\[
\int_0^\infty dy y^{z-1} \exp[-\kappa y] \coth(\alpha y) = \Gamma[z] \left(2^{1-z} \alpha^{-z} \zeta(z, \frac{\kappa}{2\alpha}) - \kappa^{-z}\right), \tag{A4}
\]

with the identification:

\[
\alpha = |Q_q| B \equiv \beta_q, \quad \kappa = \mathcal{M}^2(k_\parallel) + p_3^2, \quad z = \frac{3}{2}. \tag{A5}
\]

After some algebraic manipulations one obtains

\[
I_t = \sum_{n=0}^{\infty} g_n I_{q,n} = i\pi \int_0^1 dx \int_0^\infty dy \exp(-\mathcal{M}^2 y) \coth(\beta_q y). \tag{A6}
\]

Expanding \(\coth(\beta_q y)\) as a power series in \(\beta_q y\), we find for \(I_t\):

\[
I_t = i\pi \int_0^\infty dx \int_0^\infty dy \exp(-\mathcal{M}^2 y) \left(\frac{1}{\beta_q y} + \frac{\beta_q y}{3} + \ldots\right). \tag{A7}
\]

Now, for convenience we define:

\[
I_a = i\pi \int_0^1 dx \int_0^\infty dy \exp(-\mathcal{M}^2 y) \frac{1}{\beta_q y}. \tag{A8}
\]

Note that for all the observables of interest, e.g., eqs. (23, 24, 29), \(I_t\) always appears as the product \(\beta_q \times I_t\). Hence, we split \(I_t\) in two terms,

\[
I_t = (I_t - I_a) + I_a, \tag{A9}
\]

the first term in parentheses results to be a \(B\)-dependent finite contribution for the observables and the second a \(B\)-independent infinity one, which needs to be regularized. Let us now evaluate explicitly \(I_t - I_a\), which is given by:

\[
I_t - I_a = i\pi \int_0^1 dx \int dy \exp(-\mathcal{M}^2 y) \left(\coth(\beta_q y) - \frac{1}{\beta_q y}\right). \tag{A10}
\]

We can be rearranged the above expression using the gamma function integral representation \[24\]

\[
\frac{\Gamma[z+1]}{\beta_q^{z+1}} = \int_0^\infty dy y^z e^{-\beta_q y}, \tag{A11}
\]

and again the zeta function integral representation, eq. (A3), obtaining:

\[
I_t - I_a = \lim_{\epsilon \to 0} \frac{i\pi}{\beta_q} \int_0^1 dx \left[\Gamma[1 + \epsilon] \left(2^{-\epsilon} \zeta(1 + \epsilon, x_q) - (2x_q)^{-\epsilon-1}\right) - \Gamma(\epsilon) \left(\frac{1}{2x_q}\right)^\epsilon\right]. \tag{A12}
\]
where \( x_q = M^4/(2\beta_q) \). The limit \( \epsilon \to 0 \) may be obtained using \( a^{-\epsilon} \approx 1 - \epsilon \ln a \) and the following properties\[24\]:

\[
\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon), \quad \Gamma(\epsilon + 1) = \epsilon \Gamma(\epsilon), \quad \zeta(1 + \epsilon, a) = \left( \frac{1}{\epsilon} - \psi(a) \right),
\]

where in the last expression \( \psi(x) \) is the digamma function\[24, 25\] and \( \gamma_E \) the Euler-Mascheroni constant. Thus, one obtains:

\[
I_t - I_a = \frac{i\pi}{\beta_q} \int_0^1 dx \left( -\Psi(x_q) - \ln 2 - (2x_q)^{-1} + \ln 2x_q \right).
\]

Finally, we have just to evaluate \( I_a \), eq.\ref{eq:A8}. Starting from the gamma function representation, eq.\ref{eq:A11}, one obtains:

\[
I_a = \frac{i\pi}{\beta_q} \lim_{\epsilon \to 0} \int_0^1 dx \int_0^\infty dy \exp(-M^2 y) y^{-1+\epsilon}
= \frac{i\pi}{\beta_q} \lim_{\epsilon \to 0} \int_0^1 dx \frac{\Gamma(\epsilon)}{(M^2)^\epsilon},
\]

Here we use the integral representation of the Beta function\[24\]

\[
\int_0^\infty dx x^{\mu-1}(1 + x^2)^{\nu-1} = \frac{1}{2} B\left( \frac{\mu}{2}, \frac{1}{2} - \frac{\nu}{2} \right).
\]

and the property

\[
B(x, y) = \frac{\Gamma[x] \Gamma[y]}{\Gamma[x + y]}.
\]
to write:

\[
\int_0^\infty \frac{p^2 dp}{(p^2 + M^2)^{3/2}} = \lim_{\epsilon \to 0} \frac{1}{2(M^2)^{\epsilon}} B\left( \frac{3}{2}, \epsilon \right) = \lim_{\epsilon \to 0} \frac{\Gamma[\epsilon]}{2(M^2)^{\epsilon}}.
\]

Thus, comparing this result with eq.\ref{eq:A16}, we conclude that the usual vacuum term of the NJL model without external magnetic field is recovered and again a regularization is necessary. We choose to use the three-momentum noncovariant cutoff scheme. In this case, the \( I_a \) integral can be written as

\[
I_a = -2\frac{i\pi}{\beta_q} \int_0^1 dx \left[ \frac{\Lambda}{\sqrt{\Lambda^2 + M^2}} - \sinh^{-1} \frac{\Lambda}{M} \right].
\]

Finally, considering the above results one obtains

\[
I_t = \frac{i\pi}{\beta_q} \int_0^1 dx \left[ -\Psi(x_q + 1) + \frac{1}{2} x_q^{-1} + \ln (x_q) - 2 \left( \frac{\Lambda}{\sqrt{\Lambda^2 + M^2}} - \sinh^{-1} \frac{\Lambda}{M} \right) \right],
\]

which leads to Eq.\ref{eq:25}.

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