SU(3) Knot Solitons: Hopfions in the F₂ Skyrme-Faddeev-Niemi model

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We discuss the existence of the knot solitons (Hopfions) of the SU(3) Skyrme-Faddeev-Niemi model which can be viewed as an effective theory of the SU(3) Yang-Mills theory or the SU(3) anti-ferromagnetic Heisenberg model. We examine with two different types of ansatz; one is a trivially embedded SU(2) configuration and another is a nonembedding configuration which can be generated by the Bücklind transformation. The resulting Euler-Lagrange equations for both ansatz share common structure with the equation of motion of the known CP¹ Skyrmie-Faddeev-Niemi model. The quantum nature of the solutions are examined by the collective coordinate zero-mode quantization method.

I. INTRODUCTION

It is of great importance to consider SU(3) generalizations of the O(3) NLσ model, because they may possibly play a crucial role in a relevant limit of fundamental theories like low energy limit of the SU(3) pure Yang-Mills theory or continuum limit of the SU(3) Heisenberg models. The main achievement of the present paper is that we successfully construct novel soliton solutions which are called as Hopfions on the flag manifold F₂ = SU(3)/U(1)². Hopfions are topological solitons with knotted structure characterized by a Hopf invariant. It has been pointed out that the knotted structures appear in various branches of physics: QCD [11–3], BEC [4, 5], SU(3) Yang-Mills theory or continuum limit of the (3+1)-dimensional Minkowski space-time, i.e., the scalar field theory defined by the Lagrangian density

\[ \mathcal{L} = M^2 \partial_\mu \vec{n} \cdot \partial^\mu \vec{n} - \frac{1}{2} \varepsilon^{abc} \partial_\mu n^a \partial^\mu n^b \]  

where \( M \) has dimension of mass, \( \varepsilon \) is dimensionless coupling constant and \( \vec{n} \) is a three components vector with unit length, i.e., \( \vec{n} \cdot \vec{n} = 1 \). The second term of the right-hand side in (1), the Skyrme term, was introduced by Faddeev [5] so as to make the theory meet the Derrick’s criteria for the existence of stable soliton solutions. The solutions of toroidal shape which are possessed of a lower Hopf number \( H = 1, 2 \) were firstly found under an axial symmetric ansatz by Gladikowski and Hellmund [9], and also by Faddeev and Niemi [1]. After that, the higher charge Hopfions including the form of twisted tori, linked loops and knots were constructed by means of full 3D energy minimization [10–13].

Faddeev and Niemi discussed in detail that, by means of the Cho-Faddeev-Niemi-Shabanov decomposition, the SFN model [1] can be derived as an effective theory which describes the confinement phase of the SU(2) pure Yang-Mills theory [2]. From the point of view, the Hopfions are considered as a natural candidate of glueballs which can be interpreted as closed fluxtubes. The model is sometimes referred to as the CP¹ SFN model, based on a formula that the Lagrangian can be described in terms of a complex scalar field via the stereographic projection \( S^2 \rightarrow CP¹ \), i.e.,

\[ \vec{n} = \frac{1}{\Delta} (u + u^*, \, -i(u - u^*), \, |u|^2 - 1) \]  

where \( u \) is a complex scalar field and \( \Delta = 1 + |u|^2 \). For finite energy configuration, the field \( \vec{n} \) has to approach a constant vector at space infinity. This makes the points at the infinity to be identical and the space \( R³ \) is compactified to \( S^3 \). The field \( \vec{n} \) defines a mapping \( S^3 \rightarrow S^2 \) and field configurations are characterized by an integer, Hopf invariant, the element of \( \pi_3(S^2) = Z \). Since the invariant is nonlocal, an integral form of the invariant cannot be written by \( \vec{n} \) nor \( u \), and in order to define it, we need introduce the complex vector \( \vec{Z} = (\vec{Z}_0, \vec{Z}_1)² \) with \( |\vec{Z}|² = 1 \) which satisfies \( u \equiv \vec{Z}_1/\vec{Z}_0 \). Then, the Hopf invariant can be defined as

\[ H_{CP¹} = \frac{1}{4\pi²} \int A \wedge \partial A, \quad A = i\vec{Z} \wedge d\vec{Z}. \]  

In this paper, we construct Hopfions in a generalization of the SFN model to the case of SU(3), the gauge group of QCD. In the SU(N + 1) case where \( N \geq 2 \), there are several possibilities for the field decomposition associated to dynamical symmetry breaking patterns. In particular, the two options, the maximal case \( SU(N + 1) \rightarrow U(1)² \) [21] and the minimal case \( SU(3) \rightarrow U(2) \) [15–16], have been well studied. According to the options, the SFN type models on the relevant target spaces \( F_N = SU(N+1)/U(1)² \) and \( CN^N = SU(N+1)/U(N) \) were proposed in [4] and [17] respectively. Note that \( CP¹ = F_1 = SU(2)/U(1) \) is equivalent to \( S^2 \), the target space of the standard SFN model. Unfortunately, the \( CN^N \) model cannot possess knot solitons as a static stable solution because of the relevant homotopy group being trivial, i.e., \( \pi_3(CP^N) = 0 \) for \( N \geq 2 \). In \( 2(N + 1) \) dimensional space-time, however, higher dimensional Hopfions associated to \( \pi_{2N+1}(CP^N) \) are discussed [18]. In addition, if \( N \) is odd, 3D time-dependent non-topological solitons, Q-balls and Q-shells, are obtained in the \( CN^N \) model with V-shaped potential [19].

Contrary to the case of \( CP² \), the third homotopy group of the flag manifold is nontrivial, i.e., \( \pi_3(F_2) = Z \). Thus it is expected that there exist Hopfions in the \( F_2 \) SFN model, which is composed of the \( F_2 \) nonlinear \( \sigma \)-model and quadratic terms in derivatives. The main purpose of the present paper is to confirm the existence of the \( F_2 \) Hopfion and understand their detailed structure. It is recently found that the 2-dimensional \( F_2 \) nonlinear \( \sigma \)-model possesses vortex-like solutions (2D instantons) both of embedding type [20] and of genuine (nonembedding) type [21–22]. The Hopfions considered in this paper may be the vortices with knot structure. Note that in [21] the so-called Kalb-Ramond field is introduced with specific coefficients so that the model gets integrable [23–24]. Though the Kalb-Ramond field naturally appears for some continuum limit of the SU(3) anti-ferromagnetic spin chain, for the moment we do not consider the field. The reason is that if one derives nonlinear \( \sigma \)-model from other fundamental theories, it is not so clear whether the field can naturally appear. The genuine solutions are constructed by the \( CP^2 \) Dim-
Falent to the flag manifold \( SU \) and the left global Kostant (KK) symplectic forms. The Lagrangian (5) is invariant un-
ter the Maurer-Cartan form which is given in terms of only the off-diagonal components of
\( F \) as QCD.

The fundamental degrees of freedom of \( F_2 \) nonlinear \( \sigma \)-models can be given by the \( su(3) \) valued fields, called color direction fields in the context of QCD, defined as
\[
\nu_a = U_h A^1 a, \quad a = 1, 2
\]
where \( U \) is an element of \( SU(3) \) and the matrices \( h_a \) are the Cartan generators in \( su(3) \). The \( F_2 \) SFN model is defined by the Lagrangian density (3+1)-dimensional particle-like configurations evidently evade the Derrick’s approach to constant matrices at space infinity, so that the space
\[
F_a = \text{Tr} \left( \dot{A}^1 B \right) \text{ for } A, B \in su(3). \quad (4)
\]
The second rank tensors are defined as
\[
F_{\mu \nu} = -\frac{i}{2} \sum_{a=1}^{2} \left[ [\nu_a, [\dot{\nu}_a, \dot{\nu}_a]] \right]
\]
and the 2-forms \( F^a = \frac{i}{2} F_{\mu \nu} dx^\mu \wedge dx^\nu \) are called the Kirillov-
Kostant (KK) symplectic forms. The Lagrangian (5) is invariant under the left global \( SU(3) \) transformation such that \( U \rightarrow g U \), \( g \in SU(3) \), and the local \( U(1)^2 \) transformation such that \( U \rightarrow U k \), \( k \in U(1)^2 \).

According to the symmetries, one can understand that the target space of this model is the coset space \( SU(3)/U(1)^2 \) equivalent to the flag manifold \( F_2 \). For simplicity, we employ the length unit \( M e^{-1} \) and the energy unit \( 4 M / e \). Then the static energy functional associated with (5) is given by
\[
E = \int d^3 x \left\{ \frac{1}{2} \sum_{a=1}^{2} \left[ \partial_a \nu_a, \partial_a \nu_a \right] + F_{\mu \nu}^a F_{\mu \nu}^a \right\}.
\]
Since the energy consists of both quadratic and quartic term, three dimen-
sional particle-like configurations evidently evade the Derrick’s no-go theorem.

We reformulate the energy functional (7) into a more tractable form which is given in terms of only the off-diagonal components of the Maurer-Cartan form \( U^1 \partial_a U \). To perform the reformulation, we decompose the Maurer-Cartan form in terms of the \( SU(3) \) Cartan-Weyl basis as
\[
U^1 \partial_a U = i A^\mu_\alpha h_a + i J^\mu_\alpha e_p.
\]
where we use the basis of the form
\[
h_1 = \frac{1}{\sqrt{2}} \lambda_3, \quad h_2 = \frac{1}{\sqrt{2}} \lambda_8, \quad e_{\pm 1} = \frac{1}{2} (\lambda_1 \pm i \lambda_2), \quad e_{\pm 2} = \frac{1}{2} (\lambda_4 \mp i \lambda_5), \quad e_{\pm 3} = \frac{1}{2} (\lambda_6 \pm i \lambda_7).
\]
Since the basis are orthonormalized, one can write
\[
A^\mu_\alpha = -i \left< h_a, U^1 \partial_a U \right>, \quad J^\mu_\alpha = -i \left< e_p, U^1 \partial_a U \right>.
\]
where \( A^\mu_\alpha \) are real and \( J^\mu_\alpha \) is \( J^\mu_\alpha = \left( J^\mu_\alpha \right)^* \). Notice that the KK forms can be written as \( F^{\mu \nu} = 4 A^\mu \) where \( A^\mu = A^\mu_a dx^a \), so that the 2-
forms are closed. Under the gauge transformation \( U \rightarrow U k \) with \( k = \exp (i \theta^a h_a) \), \( A^\mu_a \) transforms as a gauge field and \( J^\mu_\alpha \) a charged particle, i.e.,
\[
A^\mu_a \rightarrow A^\mu_a + \partial_a \theta^p, \quad J^\mu_\alpha \rightarrow J^\mu_\alpha e^{\theta^p \alpha_\beta_\gamma} \gamma^\beta_{\gamma}. \quad (9)
\]
where \( \alpha_{\beta_\gamma} \) is a component of the root vector corresponding to \( e_p \) and \( J^{\mu_\alpha} = J^{\mu_\alpha} \gamma^p \). Now the root vectors are given by
\[
\alpha^1 = \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right), \quad \alpha^2 = \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right), \quad \alpha^3 = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right), \quad (10)
\]
with \( \alpha^p = -\alpha^p \) for \( p = 1, 2, 3 \).

It is known that the quadratic term in (3), the nonlinear \( \sigma \)-model, can be written by only the off-diagonal components \( J^\mu_\alpha \). In addition, one can write the KK forms in terms of the off-diagonal components as \( F^{\mu \nu} = -i \sum_p \alpha^p \gamma^p \wedge J^{\mu_\alpha} \). Thus the energy of a static configura-
tion can be written in terms of the off-diagonal components as
\[
E = \int d^3 x \left\{ J^\mu_\alpha J^{\mu_\alpha} - \frac{1}{4} \left[ J^\mu_\alpha, J^{\mu_\alpha} \right] - J^\mu_\alpha J^{\mu_\alpha} \right\}.
\]
where \( q \) is a mod 3 number, \( q \equiv q + 3 \) (mod 3). Note that \( J^\mu_\alpha J^{\mu_\alpha} = J^\mu_\alpha J^{\mu_\alpha} - J^\mu_\alpha J^{\mu_\alpha} \) is purely imaginary, and then the energy
functional is positive definite. It is worth to note that, similar to the \( CP^1 \) case, the energy functional (12) can be interpreted as a gauge fixing functional for a nonlinear maximal Abelian gauge, without making the Abelian subgroup components fixed.

For finiteness of the energy functional the fields \( \nu_a \) should appro-
ach to constant matrices at space infinity, so that the space \( \mathbb{R}^4 \) is topo-
logically compactified to \( S^3 \) and the fields \( \nu_a \) define a map \( \mathbb{R} \rightarrow S^3 = SU(3)/U(1)^2 \) . Consequently, the finite energy configura-
tions can be characterized by elements of the homotopy group \( \pi_3 \left( SU(3)/U(1)^2 \right) = \mathbb{Z} \). The corresponding topological charge, Hopf invariant, is given by
\[
H_{F_3} = \frac{1}{8 \pi^2} \int d^3 x \left\{ \epsilon^{ijk} \left( A^i_k F^2_j - \Gamma \right) \right\}.
\]
where
\[
\Gamma = \frac{-i}{8 \pi^2} \int d^3 x \epsilon^{ijk} \left( J^i_k J^j_l \delta_{ij} - J^i_k J^j_l J^k_j \right).
\]
The Hopf invariant (13) is nonlocal since \( A^i_k \) cannot be written in terms of the fields \( \nu_a \), and therefore (13) does not possess the local \( U(1)^2 \) symmetry. Moreover, in general, the exact constitution, i.e., the Abelian Chern-Simons (CS) terms or \( \Gamma \), is non-topological. The Hopf invariant can be constructed by means of the Novikov’s procedure (14) via the isomorphism between \( \pi_3 \left( SU(3)/U(1)^2 \right) \) and \( \pi_3 \left( SU(3) \right) \) which indicates \( H_{F_3} = \pi_3 \left( SU(3) \right) \).
SU(3). Note that since the winding number is equivalent to the Chern-Simons term for the SU(3) flat connection \( U^1 \mu_U \), even the SU(3) case the Hopf term is given by the non-Abelian Chern-Simons term as discussed in the SU(2) case \([20,22]\).

For non-symmetric manifold, unlike Hermitian symmetric spaces, the Kähler form \( \lambda \) is not closed, i.e., \( \lambda \wedge \lambda \neq 0 \), in general. For the flag manifold, the Kähler form can be defined as

\[
\lambda = \frac{i}{2\pi} \sum_{p=1}^{3} B_p J^p \wedge J^{-p}
\]

where the coefficients \( B_p \) are real constants \([23]\). The so-called skew torsion \( T = d \lambda \) is given by the form

\[
T = \frac{1}{2\pi} \sum_p B_p \left( J^1 \wedge J^2 \wedge J^3 + J^{-1} \wedge J^{-2} \wedge J^{-3} \right).
\]

Under the local \( U(1)^2 \) transformation, the Kähler form is invariant and the torsion too. Note that in the 2-dim case \( \chi \) is given by the form

\[
\chi = (\text{constant}) \Delta_{J^1} + \Delta_{J^2} + \Delta_{J^3}.
\]

### III. PARAMETRIZATION

In order to make the analysis transparent, let us parametrize the \( SU(3) \) matrix \( U \) in terms of complex scalar fields which equivalent to the local coordinates of the target space \( F_2 \). As a result of the Iwasawa decomposition (see e.g. \([24]\)), one can construct the \( SU(3) \) matrix from a triangular matrix in \( SL(3, \mathbb{C}) \). Therefore we begin the parametrization with the \( 3 \times 3 \) lower triangular matrix with the determinant of unity:

\[
X = \begin{pmatrix}
\chi_1 & 0 & 0 \\
\chi_2 & \chi_4 & 0 \\
\chi_3 & \chi_5 & (\chi_1 \chi_4)^{-1}
\end{pmatrix} \in SL(3, \mathbb{C}).
\]

where \( \chi_i \) are complex functions with \( \chi_1 \) and \( \chi_4 \) being finite. In the 2-dim \( F_2 \) nonlinear \( \sigma \)-model, the solutions of the Euler-Lagrange equation make the Kähler form topological, and then the torsion disappears. By analogy, in this paper we consider a class of configuration that satisfies the torsion-free condition \( T = 0 \).

### IV. EQUATION OF MOTION AND HOPFIONS

Firstly we derive the formal Euler-Lagrange equation, and then implement two classes of configuration that satisfy the torsion-free condition \( T = 0 \). The Euler-Lagrange equation is equivalent to the conservation of the Noether current \( J_\mu \) associated with the global \( SU(3) \) transformation, i.e., \( \partial_\mu J^\mu = 0 \). The current takes the form

\[
J_\mu = \sum_{a=1}^2 \left( m_a, \partial_\mu m_a \right) - \sum_{b=1}^2 F^a_{\mu \nu} \left( m_a, \partial_\nu m_b \right).
\]

If we factorize the current as \( J_\mu = W B_\mu W^† \), the equations of motion can be written as

\[
\partial_\mu B^\mu + W^† \partial_\mu W B^\mu = 0.
\]

We again decompose the Maurer-Cartan form \( W^† \partial_\mu W \) as

\[
W^† \partial_\mu W = i C^a \mu h_a + i K^p_\mu e_p
\]

The composite vector fields \( C^a_\mu \) is a gauge transformed version of \( A_\mu \) and \( K^p_\mu \) is of \( J^p_\mu \). Then, one finds \( B_\mu \) consists of the only off-diagonal components as

\[
B_\mu = i \sum_p \left( K^p_\mu - i \sum_{a=1}^2 \alpha^a_\mu F^a_{\mu \nu} K^{\nu p} \right) e_p.
\]
For short notation, we introduce $R^\mu_\nu = \sum_\alpha \alpha_\alpha C^\alpha_\mu \nu$ and $G_{\mu\nu} = \sum_\alpha \alpha_\alpha F^\alpha_{\mu\nu}$. Then, the equation (24) can explicitly be written as

$$\partial^\mu (K^\mu_{\nu} - i G^\mu_{\nu} K^{\nu} u) + i R^\mu_{\nu} (K^\mu_{\nu} - i G^\mu_{\nu} K^{\nu}) + G^{\mu\nu} K^\nu u^{-1} K^\nu_{\nu+1} = 0 \quad (27)$$

for all $\nu \equiv 1, 2, 3$ (mod 3) and their complex conjugations.

Since the equations (27) are very complicated and highly nonlinear, we shall introduce two different ansatzes that satisfy the torsion-free condition $T = 0$ as mentioned earlier.

### A. Trivial $C^P_1$ reduction

The first class is a trivially embedded configuration an $F_1 = C^P_1$ Hopfion into $F_2$ space. It can be implemented by imposing two of the three scalar fields trivial. Here we set $u_1 = u_3 = 0$ and also write $u_2 = u$ without loss of generality. Then, the complex vectors $Z_a$ are written by the function $u(x)$ as

$$ Z_A = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} 1 \\ 0 \\ u \end{pmatrix}, \quad Z_B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad Z_C = \frac{1}{\sqrt{\Delta}} \begin{pmatrix} -u^* \\ 0 \\ 1 \end{pmatrix} \quad (28) $$

where $\Delta = 1 + |u|^2$. The currents $K^\mu_{\nu}$ are given by

$$ K^1_{\mu} = K^2_{\mu} = 0, \quad K^2_{\mu} = \frac{i}{\Delta} \partial_\mu u \quad (29) $$

and the skew torsion $T$ is thus vanished. It is directly to see that the equations of motion (27) for $q \equiv 1, 3$ are automatically satisfied and that for $q \equiv 2$ reduces to

$$ \partial^\nu [\partial_\mu u - i G_{\mu\nu} \partial^\nu u] + (i R_{\mu\nu} - \partial_\mu \log \Delta) (\partial^\nu u - i G_{\mu\nu} \partial_\nu u) = 0 \quad (30) $$

where for convenience we introduced $R_{\mu\nu}$ and $G_{\mu\nu}$ as

$$ R_{\mu\nu} = \frac{i}{\Delta} \begin{pmatrix} u^* \partial_\mu u - u \partial_\mu u^* \end{pmatrix}, \quad G_{\mu\nu} = -\frac{2i}{\Delta} \begin{pmatrix} \partial_\mu u \partial_\nu u^* - \partial_\nu u \partial_\mu u^* \end{pmatrix}. \quad (31) $$

The static energy for the configuration (28) is given by

$$ E_{\text{test}} = \int d^4x \left( \frac{\partial u \partial u^*}{\Delta^2} - \frac{(\partial_\mu u \partial_\nu u^* - \partial_\nu u \partial_\mu u^*)^2}{2 \Delta^4} \right) \quad (33) $$

Both the equation of motion (30) and the energy (33) are exactly same as those of the $C^P_1$ SFN model (1). Further, by a definition $Z_0 = e^{i\varphi}/\sqrt{\Delta}$, $Z_1 = u e^{i\varphi}/\sqrt{\Delta}$ with $u_1 = 0$, the $F_2$ Hopf invariant (33) coincides with the $C^P_1$ version (3), i.e.,

$$ H_{\text{test}} = \frac{1}{4e^2} \int A \wedge dA, \quad A = i\hat{\mathcal{F}} d\hat{\mathcal{F}}, \quad \hat{\mathcal{F}} = \begin{pmatrix} Z_0 \\ Z_1 \end{pmatrix}. \quad (34) $$

Form of the static energy (33) and the Hopf charge (34) are perfectly same as the $C^P_1$ model. Apparently this coincidence is induced because the trivial embedding configuration (28) corresponds to a $C^P_1$ submanifold in $F_2$. Next we examine another class of configuration which may exhibit more nontrivial nature and see what happens for the equations and others.

### B. Nontrivial $C^P_1$ reduction

For the trivial embedding (28), we observed that two pairs of the currents $K^\mu_{\nu}$ vanished, i.e., $K^1_{\mu} = K^2_{\mu} = 0$. Here we examine the case of just one pair of the current $K^2_{\mu} = 0$, while $K^{1,2}_{\mu \nu}$ remain finite. This setting automatically satisfies the torsion-free condition. Note that the result is independent for the choice of the components; for a different pair one just repeat the same prescription by permuting the vectors $Z_a$. The condition $K^2_{\mu\nu} = 0$ reads

$$ u_3 \partial_\mu u_1 - \partial_\mu u_2 = 0, \quad \mu = 0, 1, 2, 3 \quad (35) $$

which is satisfied if $u_2$ is a function of $u_1$, i.e. $u_2 = f(u_1)$, and $u_3$ is given by $u_3 = f'(u_1)$ where the prime stands for the derivative in $u_1$. It means the independent field is only $u_1$, so that the Euler-Lagrange equation seems to be an overdetermined system. In order to resolve the overdeterminedness, we consider the case that the Euler-Lagrange equations for $q \equiv 1, 3$ are proportional to each other. It is realized when the ratio $\Delta_1/\Delta_2$ is a constant Note that we leave the equation for $q \equiv 2$ intact because $q \equiv 2$ is now special due to the constraint $K_2^\mu = 0$. By comparing the order of $u_1$ in $\Delta_1$, it implies that

$$ |u_1|^2 = |f'(u_1)|^2, \quad |f|^2 = |u_1 f - f|^2. \quad (36) $$

Since we are not interested in embedding solutions now, we omit the case where $u_1$ is a constant and we get

$$ f(u) = \frac{1}{2} u^2 e^{i\varphi} \quad (37) $$

where $\varphi \in [0, 2\pi]$ is a constant. Note that due to $U(1)$ symmetries the constant $\varphi$ can take an arbitrary value. For simplicity, we choose $\varphi = \pi$ and write $u_1 = \sqrt{2} u$. Then, the triplet vectors can be written as

$$ Z_A = \frac{1}{\Delta} \begin{pmatrix} 1, \sqrt{2} u, u^2 \end{pmatrix}^T, \quad Z_C = \frac{1}{\Delta} \begin{pmatrix} -u^2, \sqrt{2} u^*, 1 \end{pmatrix}^T, \quad Z_B = \frac{1}{\Delta} \begin{pmatrix} -\sqrt{2} u, 1 - |u|^2, -\sqrt{2} u^* \end{pmatrix}^T. \quad (38) $$

It is worth to note that the three vectors are linked by the Bäcklund transformation, i.e.,

$$ Z_B = \frac{P_z Z_A}{|P_z Z_A|}, \quad Z_C = \frac{P_z Z_B}{|P_z Z_B|} \quad (39) $$

where $P_z Z = \partial_\mu Z_a - (Z^\alpha_\mu \partial_\alpha) Z_a$. Such relation between the triplet vector is observed in the nonembedding solutions of the two dimensional $F_2$ nonlinear $\sigma$-model (21) (22). The currents $K^\mu_{\nu}$ are given by the form

$$ K^1_{\mu} = \frac{\sqrt{2}}{\Delta} \partial_\mu u^*, \quad K^2_{\mu} = 0, \quad K^3_{\mu} = -\frac{\sqrt{2}}{\Delta} \partial_\mu u. \quad (40) $$

It implies that the Euler-Lagrange equation (27) for $q \equiv 2$ is automatically satisfied. In addition, we obtain $R^\mu_{\nu} = R^\nu_{\mu}$ and $G^\mu_{\nu} = G^\nu_{\mu}$, then one can easily observe that the equation (27) for both $q \equiv 1, 3$ are reduced to the complex conjugation of (27). To see this, one can use the fact that $R_{\mu\nu}$ and $G_{\mu\nu}$ are real. Now we got somewhat surprising observations; These results clearly mean that it is not only in the trivial embedding case but also in the nonembedding case that all the known Hopfion solutions $u$ in the $C^P_1$ SFN model are also solutions. However, we should remark with emphasis that their structure is perfectly different because of the parametrizations (33) and (28). The configuration (38) possesses the static energy

$$ E_{\text{non}\text{in} \ [u]} = 4 E_{\text{test}}[u]. \quad (41) $$
which is exactly four times greater than \(28\). For the evaluation of the relevant Hopf number, we parametrize \(u = 2\hat{e}_1/2\hat{e}_0, e^{i\theta_4} = 2\hat{e}_2/|Z_0|^2\), and \(\theta_4 = 0\). Then, we obtain \(A^1 = -\sqrt{2}A, A^2 = \sqrt{6}A\), and therefore

\[
H_{\text{mono}} = \frac{1}{\pi^2} \int A \wedge dA = 4H_{\text{ini}}. \tag{42}
\]

Note that \(34\) is given by only the sum of the Abelian CS terms, which now become topological, because the configuration \(28\) satisfies \(\Gamma = 0\) as well as the desron-free condition \(T = 0\).

The \(F_2\) nontrivial Hopfion with \(H_{\text{mono}} = 4n\) (\(n\) is an integer) might be interpreted as a molecule state of some lower charged ones with the Hopf number \(H = n\) such as the trivial Hopfion, sitting on top of each other with no binding energy. It is not true, however. Note that it does not imply there exists no interaction between the constituents and it is not possible to remove one of them others because the form of the Hopf charge \(12\) is defined by the different field variables \(25\), not the \(28\). Such situation has been observed in an \(SU(N)\) Skyrme model \(34\).

V. ISO-SPINNING HOPFIONS

We have seen that the EL eq. in both the trivial embedding and nonembedding ansatz solved by same function \(u\) and their energy functional or Hopf invariant proportional like \(21\) and \(22\). Their structure is clearly different, but there might be someone who considers these solutions have almost same features. Now we shall show that their quantum natures are quite different. In this section, we give a brief analysis to demonstrate noticeable differences in the quantum aspects based on the collective coordinate quantization of the zero modes. We consider an adiabatic iso-rotation associated to the \(SU(3)\) global symmetry, i.e., the time dependent transformation \(m_a(\vec{x}) \rightarrow m_a(t, \vec{x}) = \beta(t)m_a(\vec{x})\beta^\dagger(t)\) where \(\beta(t) \in SU(3)\). The Lagrangian can be written as

\[
L = -E_{cl} + r_0^2 \int d^3x \left[ \text{Tr} \left( \left[ \hat{\beta}^\dagger \beta, m_\alpha \right] \left[ \hat{\beta}^\dagger \beta, m_\alpha \right] \right) + 2F^\alpha_0 F^0_\alpha \right] \tag{43}
\]

where \(E_{cl}\) is the static energy of the Hopfion, the dot denotes the time derivative, i.e., \(\dot{x} = d\beta/dt\), and

\[
F^\alpha_0 = \frac{i}{2} \text{Tr} \left( m_a \left[ \left[ \hat{\beta}^\dagger \beta, m_\alpha \right], \partial_t m_\alpha \right] \right). \tag{44}
\]

The energy collection depends on the scale length unit \(r_0 = (Me)^{-1}\).

In order for the integral in \(43\) to be finite, \(\beta^\dagger \beta\) and \(m_\alpha\) should commute to each other at space infinity. Since the fields \(m_\alpha(\vec{x})\) approach constant elements of \(u(1) \times u(1)\) for \(x\) goes to infinity, \(\beta^\dagger \beta\) should also be in \(u(1) \times u(1)\) and therefore can be written as

\[
\beta^\dagger \beta = \sqrt{2} \left( \frac{\omega_1}{2} h_1 + \frac{\omega_2}{\sqrt{3}} h_2 \right) \tag{45}
\]

where \(\omega_\alpha\) denote the angular velocity in the iso-space. We chose the coefficients in \(45\) consistent with the definition of the \(SU(3)\) Euler angle \(23\).

The quantum Lagrangian \(43\) can be written as the quadratic form of the angular velocities

\[
L = -E_{cl} + \frac{1}{2} \omega^T \cdot \omega \tag{46}
\]

where \(\omega^T = (\omega_1, \omega_2)\) and

\[
I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}. \tag{47}
\]

The moment of inertia (m.o.i) are explicitly obtained as following:

- the non-trivial reduction case

\[
I_{11} = 2\pi^2 \int d^3x \frac{1}{\Delta^2} \left[ (10 - 7|u|^2 + 10|u|^4) |u|^2 \right] + 4 \right) (\partial_1 \log \Delta)^2 \right] 
\]

\[
I_{12} = -4\pi^2 \int d^3x \frac{1}{\Delta^4} \left[ (3 - \Delta) (3 - 2\Delta) \right. 
\]

\[
\times \left[ |u|^2 + 4 (\partial_1 \log \Delta)^2 \right] 
\]

\[
I_{13} = 8\pi^2 \int d^3x \frac{1}{\Delta^2} \left[ (2 + |u|^2 + 2|u|^4) |u|^2 \right] + 4 \left( 1 - |u|^2 + |u|^4 \right) (\partial_1 \log \Delta)^2 \right] 
\]

- the trivial reduction case

\[
I_{11} = \frac{I_{12}}{2} = \frac{I_{22}}{4} = 2\pi^2 \int d^3x \frac{|u|^2 + (\partial_1 \log \Delta)^2}{\Delta^2}. \tag{49}
\]

By the Legendre transformation of the Lagrangian \(43\), the Hamiltonian are derived as \(H = \omega_i P_i - L\) with the canonical momentum defined by

\[
P_i = \frac{\partial L}{\partial \dot{\omega}_i} = I_{ij}\omega_j, \quad i, j = 1, 2. \tag{50}
\]

In the nontrivial reduction case, the Hamiltonian are straightforwardly obtained as

\[
H = E_{cl} + \frac{1}{2} \frac{1}{\Delta} \left( I_{12} P_1^2 - 2I_{12} P_1 P_2 + I_{11} P_2^2 \right) \tag{51}
\]

where we used the commutation relation \([P_1, P_2] = 0\) because the operators are associated the Abelian subgroup of \(SU(3)\). The commutation relation and the form of Hamiltonian imply the genuine \(SU(3)\) Hopfion can be possessed of two quantum numbers. On the other hand, in the embedding case, it is allowed to define only one operator because

\[
P_1 = \frac{P_2}{2} = I (\omega_1 + 2\omega_2) \equiv P \tag{52}
\]

where we wrote \(I_{11} = I\). Therefore the Hamiltonian are obtained as

\[
H = E_{cl} + \frac{P^2}{2I}. \tag{53}
\]

Consequently, the Hopfions of the embedding type inherit quantum property of the \(CP^3\) Hopfions, possessing only one quantum number. The quantum property of the two types of the Hopfion solutions seems quite different, at least qualitatively, as a reflection of their symmetries.

VI. CONCLUSION

We have studied Hopfions in the SFN model on the target space \(F_2 = SU(3)/U(1)^2\) which is an \(SU(3)\) generalization of the standard SFN model whose target space is \(CP^1 = SU(2)/U(1)\). By analogy of the 2-dim \(F_2\) nonlinear \(\sigma\)-model, we introduced two classes of configuration which satisfy the torsion-free condition, i.e., the trivial embedding of the \(CP^1\) Hopfions and the \(SU(3)\) genuine one which can be constructed through the Bäcklund transformation.
For both the cases, the Euler-Lagrange equation reduces to that of the $C^P$ SFN model. In addition, though the Hopf invariant is equivalent to the Chern-Simons term for the $SU(3)$ flat connection, it is shown that the invariant is given by the Chern-Simons terms for Abelian components of the flat connection if one substitutes the configuration into the Hopf invariant.

The most important open problem is probably stability of the genuine solutions. Their energy is four times greater than that of the embeddings, comparing the configurations given by the same scalar function. It is also of importance to understand mathematical implication of the torsion-free condition in detail and to confirm whether there exist Hopfions outside the condition.

We examined the quantum aspect of the Hopfions based on the collective coordinate quantization and found that their aspects are quite different for the choice of the ansatz.

For the estimation of the physical spectrum of the glueball, we need to perform more complete analysis of the collective coordinate quantization, including the space rotational modes and also the discussion of their statistical property. The analysis of this subject is now in progress and the results will be reported in a subsequent paper.

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