On the existence of universal series by trigonometric system

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In this paper we prove the following: let \( \omega(t) \) be a continuous function, increasing in \([0, \infty)\) and \( \omega(+0) = 0 \). Then there exists a series of the form

\[
\sum_{k=-\infty}^{\infty} C_k e^{ikx} \quad \text{with} \quad \sum_{k=-\infty}^{\infty} C_k^2 \omega(|C_k|) < \infty, \quad C_{-k} = \overline{C_k},
\]

with the following property: for each \( \varepsilon > 0 \) a weighted function \( \mu(x) \), \( 0 < \mu(x) \leq 1 \), \( |\{x \in [0, 2\pi] : \mu(x) \neq 1\}| < \varepsilon \) can be constructed, so that the series is universal in the weighted space \( L^1_{\mu}[0, 2\pi] \) with respect to rearrangements.

§1. INTRODUCTION

In 1932 F. Riesz (see [1], p. 655) proved that there exists a function \( f_0(x) \in L^1[0, 2\pi] \) so that its Fourier series with respect to the trigonometric system does not converge in \( L^1[0, 2\pi] \). Consequently, there exist functions in the space \( L^1[0, 2\pi] \) that cannot be represented by trigonometric series in the metric of \( L^1 \).

Let \( \mu(x) \) be a measurable on \([0, 2\pi] \) function with \( 0 < \mu(x) \leq 1 \), \( x \in [0, 2\pi] \) and let \( L^1_{\mu}[0, 2\pi] \) be a space of measurable functions \( f(x), \ x \in [0, 2\pi] \) with

\[
\int_0^{2\pi} |f(x)|\mu(x)dx < \infty.
\]

K.Kazarian and R.Zink in [2] proved that there exist a weighted space \( L^1_{\mu}[0, 2\pi] \), such that for every function \( f(x) \) in the space \( L^1_{\mu}[0, 2\pi] \) one can find a trigonometric series \( \sum_{k=-\infty}^{\infty} C_k e^{ikx} \) that converges to \( f(x) \) in the metric of \( L^1_{\mu}[0, 2\pi] \).

Moreover using other construction of weighted function \( \mu(x) \) M.Grigorian proved the following result [3]

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Theorem 1. There exists a trigonometric series of the form
\[ \sum_{k=-\infty}^{\infty} C_k e^{ikx} \] with the following property: such that for any number \( \epsilon > 0 \) a weighted function \( \mu(x) \), 0 < \( \mu(x) \leq 1 \), \( \{|x \in [0, 2\pi] : \mu(x) \neq 1\}| < \epsilon \) can be constructed, so that the series (1.1) is universal in \( L^1_{\mu}[0, 2\pi] \) with respect to subseries (see Definition 3).

Now we present the definitions of universal series:

Definition 1. A functional series
\[ \sum_{k=1}^{\infty} f_k(x), \quad f_k(x) \in L^1_{\mu}[0, 2\pi] \] is said to be universal in weighted spaces \( L^1_{\mu}[0, 2\pi] \) with respect to rearrangements, if for any function \( f(x) \in L^1_{\mu}[0, 2\pi] \) the members of (1.2) can be rearranged so that the obtained series \( \sum_{k=1}^{\infty} f_{\sigma(k)}(x) \) converges to the function \( f(x) \) in the metric \( L^1_{\mu}[0, 2\pi] \), i.e.
\[ \lim_{n \to \infty} \int_0^{2\pi} \left| \sum_{k=1}^{n} f_{\sigma(k)}(x) - f(x) \right| \cdot \mu(x) dx = 0. \]

Definition 2. The series (1.2) is said to be universal in weighted spaces \( L^1_{\mu}[0, 2\pi] \) in the usual sense, if for any function \( f(x) \in L^1_{\mu}[0, 2\pi] \) there exists a growing sequence of natural numbers \( n_k \) such that the sequence of partial sums with numbers \( n_k \) of the series (1.2) converges to the function \( f(x) \) in the metric \( L^1_{\mu}[0, 2\pi] \).

Definition 3. The series (1.2) is said to be universal in weighted spaces \( L^1_{\mu}[0, 2\pi] \) concerning subseries, if for any function \( f(x) \in L^1_{\mu}[0, 2\pi] \) it is possible to choose a partial series \( \sum_{k=1}^{\infty} f_{n_k}(x) \) from (1.2), which converges to the \( f(x) \) in the metric \( L^1_{\mu}[0, 2\pi] \).

The above mentioned definitions are given not in the most general form and only in the generality, in which they will be applied in the present paper.

In this paper we consider a question on existence of series by trigonometric system universal in weighted \( L^1_{\mu}[0, 2\pi] \) spaces with respect to rearrangements.

Note, that many papers are devoted (see [3]- [10]) to the question on existence of various types of universal series in the sense of convergence almost everywhere and on a measure.

Here we will give those results which directly concern to the Theorems, proved in this paper.

The first usual universal in the sense of convergence almost everywhere trigonometric series were constructed by D.E.Menshov [4] and V.Ya.Kozlov [5].
The series of the form
\[ \frac{1}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \] (1.3)
was constructed just by them such that for any measurable on \([0, 2\pi]\) function \(f(x)\) there exists the growing sequence of natural numbers \(n_k\) such that the series (1.3) having the sequence of partial sums with numbers \(n_k\) converges to \(f(x)\) almost everywhere on \([0, 2\pi]\).

Note here, that in this result, when \(f(x) \in L^1_{[0, 2\pi]}\), it is impossible to replace convergence almost everywhere by convergence in the metric \(L^1_{[0, 2\pi]}\).

This result was distributed by A.A. Talalian on arbitrary orthonormal complete systems (see [6]). He also established (see [7]), that if \(\{\phi_n(x)\}_{n=1}^{\infty}\) - the normalized basis of space \(L^p_{[0,1]}\), \(p > 1\), then there exists a series of the form
\[ \sum_{k=1}^{\infty} a_k \phi_k(x), \quad a_k \to 0. \] (1.4)
which has property: for any measurable function \(f(x)\) the members of series (1.4) can be rearranged so that the again received series converge on a measure on \([0,1]\) to \(f(x)\).

W. Orlicz [8] observed the fact that there exist functional series that are universal with respect to rearrangements in the sense of a.e. convergence in the class of a.e. finite measurable functions.

It is also useful to note that even Riemann proved that every convergent numerical series which is not absolutely convergent is universal with respect to rearrangements in the class of all real numbers.

In [9] it is proved the following result:

**Theorem 2.** Let \(\{\beta_k\}_{k=0}^{\infty}\) be a sequence of positive numbers with \(\lim_{k \to \infty} \beta_k = 0\). There exists a trigonometric series of the form
\[ \sum_{k=-\infty}^{\infty} C_k e^{ikx} \quad \text{with} \quad \sum_{k=-\infty}^{\infty} |C_k| |\beta_k| < \infty, \quad C_{-k} = \overline{C_k} \] (1.5)
with the following property: such that for any number \(\epsilon > 0\) a weighted function \(\mu(x), 0 < \mu(x) \leq 1, |\{x \in [0, 2\pi] : \mu(x) \neq 1\}| < \epsilon\) can be constructed, so that the series (1.5) is universal in \(L^1_{\mu}[0, 2\pi]\) with respect to rearrangements (in the usual sense).

In this paper we prove the following results.

**Theorem 3.** Let \(\omega(t)\) be a continuous function, increasing in \([0, \infty)\) and \(\omega(+0) = 0\). Then there exists a series of the form
\[ \sum_{k=-\infty}^{\infty} C_k e^{ikx} \quad \text{with} \quad \sum_{k=-\infty}^{\infty} C_k^2 \omega(|C_k|) < \infty, \quad C_{-k} = \overline{C_k}, \] (1.6)
with the following property: for each \( \varepsilon > 0 \) a weighted function \( \mu(x), 0 < \mu(x) \leq 1, |\{x \in [0, 2\pi] : \mu(x) \neq 1\}| < \varepsilon \) can be constructed, so that the series (1.6) is universal in the weighted space \( L^1_\mu[0, 2\pi] \) with respect to rearrangements.

Analogous of this Theorem for Walsh system was proved by author in [10].

**Remark.** Using the proofs of Theorem 3 we can construct the series of the form (1.6) which are universal in the weighted space \( L^1_\mu[0, 2\pi] \) with respect simultaneously to rearrangements as well as to subseries.

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§2. BASIC LEMMA

**Lemma.** Let \( \omega(t) \) be a continuous function, increasing in \([0, \infty)\) and \( \omega(+0) = 0 \). Then for any given numbers \( 0 < \varepsilon < \frac{1}{2}, N_0 > 2 \) and a step function

\[
f(x) = \sum_{s=1}^{q} \gamma_s \cdot \chi_{\Delta_s}(x),
\]

where \( \Delta_s \) is an interval of the form \( \Delta_s^{(i)} = \left[ \frac{i-1}{2^m}, \frac{i}{2^m} \right] \), \( 1 \leq i \leq 2^m \), there exists a measurable set \( E \subset [0, 2\pi] \) and a polynomial \( P(x) \) of the form

\[
P(x) = \sum_{N_0 \leq |k| < N} C_k e^{ikx}
\]

which satisfy the conditions:

1. \( |E| > 2\pi - \varepsilon \),
2. \( \int_E |P(x) - f(x)| dx < \varepsilon \),
3. \( \sum_{N_0 \leq |k| < N} |C_k|^2 \cdot \omega(|C_k|) < \varepsilon \), \( C_{-k} = \overline{C_k} \),
4. \( \max_{N_0 \leq m < N} \left[ \int_{\varepsilon} \left| \sum_{N_0 \leq |k| \leq m} C_k e^{ikx} \right| dx \right] < \varepsilon + \int_{\varepsilon} |f(x)| dx \),

for every measurable subset \( e \) of \( E \).

**Proof of Lemma.** Let \( 0 < \varepsilon < \frac{1}{2} \) be an arbitrary number. For any positive number \( \eta \) with

\[
\eta < \frac{\varepsilon^2}{4} \cdot \left[ \int_0^{2\pi} f^2(x) dx \right]^{-1},
\]

(2.2)
by definition of function \( \omega(t) \), there exists a positive number \( \delta < \epsilon \) so that for any \( t, 0 < t < \delta \) we have
\[
\omega(t) < \omega(\delta) < \eta. \tag{2.3}
\]

Without restriction of generality, we assume that
\[
\max_{1 \leq s \leq q} \frac{4}{\epsilon} \left| \gamma_s \right| \cdot \sqrt{|\Delta_s|} < \min \left\{ \frac{\epsilon}{2}, \delta \right\}, \ s = 1, 2, \ldots, q. \tag{2.4}
\]

Set
\[
g(x) = \begin{cases} 
1, & \text{if } x \in [0, 2\pi] \setminus \left[ \frac{\epsilon \pi}{2}, \frac{3\epsilon \pi}{2} \right]; \\
1 - \frac{2}{\epsilon}, & \text{if } x \in \left[ \frac{\epsilon \pi}{2}, \frac{3\epsilon \pi}{2} \right].
\end{cases} \tag{2.5}
\]
we choose natural numbers \( \nu_1 \) and \( N_1 \) so large that the following inequalities be satisfied:
\[
\frac{1}{2\pi} \left| \int_0^{2\pi} g_1(t) e^{-ikt} dt \right| < \frac{\epsilon}{16 \cdot \sqrt{N_0}}, \ |k| < N_0, \tag{2.6}
\]
where
\[
g_1(x) = \gamma_1 \cdot g(\nu_1 \cdot x) \cdot \chi_{\Delta_1}(x). \tag{2.7}
\]
(By \( \chi_E(x) \) we denote the characteristic function of the set \( E \).)

We put
\[
E_1 = \{ x \in \Delta_s : g_s(x) = \gamma_s \}, \tag{2.8}
\]
By (2.5), (2.7) and (2.8) we have
\[
|E_1| > 2\pi \cdot (1 - \epsilon) \cdot |\Delta_1|; \ g_1(x) = 0, \ x \notin \Delta_1, \tag{2.9}
\]
\[
\int_0^{2\pi} g_1^2(x) dx < \frac{2}{\epsilon} \cdot |\gamma_1|^2 \cdot |\Delta_1|. \tag{2.10}
\]

Since the trigonometric system \( \{ e^{ikx} \}_{k=-\infty}^{\infty} \) is complete in \( L^2[0,2\pi] \), we can choose a natural number \( N_1 > N_0 \) so large that
\[
\int_0^{2\pi} \left| \sum_{0 \leq |k| < N_1} C_k^{(1)} e^{ikx} - g_1(x) \right| dx \leq \frac{\epsilon}{8}, \tag{2.11}
\]
where
\[
C_k^{(1)} = \frac{1}{2\pi} \int_0^{2\pi} g_1(t) e^{-ikt} dt.
\]
Hence by (2.6) and (2.7) we obtain
\[
\int_0^{2\pi} \left| \sum_{\substack{0 \leq |k| < N_1 \\text{max} |N_0 \leq |k| < N_1}} C_k^{(1)} e^{ikx} - g_1(x) \right| dx \leq \frac{\epsilon}{8} \left[ \sum_{0 \leq |k| < N_0} |C_k^{(1)}|^2 \right]^\frac{1}{2} < \frac{\epsilon}{4}.
\]
Now assume that the numbers $\nu_1 < \nu_2 < \ldots \nu_{s-1}$, $N_1 < N_2 < \ldots < N_{s-1}$, functions $g_1(x), g_2(x), \ldots, g_{s-1}(x)$ and the sets $E_1, E_2, \ldots, E_{s-1}$ are defined.

We take sufficiently large natural numbers $\nu_s > \nu_{s-1}$ and $N_s > N_{s-1}$ to satisfy

$$\frac{1}{2\pi} \left| \int_0^{2\pi} g_s(t) e^{-ikt} dt \right| < \frac{\varepsilon}{16 \cdot \sqrt{N_{s-1}}}, \quad 1 \leq s \leq q, \quad |k| < N_{s-1}, \quad (2.12)$$

$$\int_0^{2\pi} \left| \sum_{0 \leq |k| < N_s} C^{(s)}_k e^{ikx} - g_s(x) \right| dx \leq \frac{\varepsilon}{4s+1}, \quad (2.13)$$

where

$$g_s(x) = \gamma_s \cdot g(\nu_s \cdot x) \cdot \chi_{\Delta_s}(x), \quad C^{(s)}_k = \frac{1}{2\pi} \int_0^{2\pi} g_s(t) e^{-ikt} dt. \quad (2.14)$$

Set

$$E_s = \{ x \in \Delta_s : g_s(x) = \gamma_s \}, \quad (2.15)$$

Using the above arguments (see (2.9)-(2.11)), we conclude that the function $g_s(x)$ and the set $E_s$ satisfy the conditions:

$$|E_s| > 2\pi \cdot (1 - \varepsilon) \cdot |\Delta_s|; \quad g_s(x) = 0, \quad x \notin \Delta_s, \quad (2.16)$$

$$\int_0^{2\pi} g_s^2(x) dx < \frac{2}{\varepsilon} \cdot |\gamma_s|^2 \cdot |\Delta_s|. \quad (2.17)$$

$$\int_0^{2\pi} \left| \sum_{N_{s-1} \leq |k| < N_s} C^{(s)}_k e^{ikx} - g_1(x) \right| dx < \frac{\varepsilon}{2s+1}. \quad (2.18)$$

Thus, by induction we can define natural numbers $\nu_1 < \nu_2 < \ldots \nu_q$, $N_1 < N_2 < \ldots < N_q$, functions $g_1(x), g_2(x), \ldots, g_q(x)$ and sets $E_1, E_2, \ldots, E_q$ such that conditions (2.16)-(2.18) are satisfied for all $s, \quad 1 \leq s \leq q$.

We define a set $E$ and a polynomial $P(x)$ as follows:

$$E = \bigcup_{s=1}^q E_s, \quad (2.19)$$

$$P(x) = \sum_{N_0 \leq |k| < N} C_k e^{ikx} = \sum_{s=1}^q \left[ \sum_{N_{s-1} \leq |k| < N_s} C^{(s)}_k e^{ikx} \right], \quad (2.20)$$

where
\[ C_k = C_k^{(s)} \text{ for } N_{s-1} \leq |k| < N_s, \quad s = 1, 2, \ldots, q, \quad C_{-k} = \overline{C_k} \quad N = N_q - 1. \quad (2.21) \]

By Bessel’s inequality and (2.5), (2.14) for all \( s \in [1, q] \) we get

\[ \left[ \sum_{N_{s-1} \leq |k| < N_s} |C_k^{(s)}|^2 \right]^{\frac{1}{2}} \leq \left[ \int_0^{2\pi} g_s^2(x)dx \right]^{\frac{1}{2}} \leq \frac{2}{\sqrt{\epsilon}} \cdot |\gamma_s| \cdot \sqrt{|\Delta_s|}, \quad s = 1, 2, \ldots, q. \quad (2.22) \]

From (2.5), (2.14) and (2.15) it follows that

\[ |E| > 2\pi - \epsilon. \]

Taking relations (2.1), (2.5), (2.12), (2.14), (2.18) - (2.21) we obtain

\[ \int_E |P(x) - f(x)|dx \leq \sum_{s=1}^{q} \left[ \int_E \left| \sum_{N_{s-1} \leq |k| < N_s} C_k^{(s)} e^{ikx} - g_s(x) \right|dx \right] < \epsilon. \]

By (2.4), (2.21) and (2.22) for any \( k \in [N_0, N] \) we have

\[ |C_k| \leq \max_{1 \leq s \leq q} \left[ \frac{2}{\sqrt{\epsilon}} \cdot |\gamma_s| \cdot \sqrt{|\Delta_s|} \right] < \delta. \]

From this and (2.3) we get

\[ \omega(|C_k|) < \omega(\delta) < \eta, \quad \forall \ k \in [N_0, N]. \]

Hence by (2.1), (2.2), (2.4) and (2.22) we obtain

\[
\sum_{N_0 \leq |k| < N} |C_k|^2 \omega(|C_k|) < \eta \sum_{s=1}^{q} \left[ \sum_{N_{s-1} \leq |k| < N_s} |C_k^{(s)}|^2 \right] < \eta \cdot \frac{4}{\epsilon} \left[ \int_0^{2\pi} f^2(x)dx \right] < \epsilon.
\]

That is, the statements 1) - 3) of Lemma are satisfied. Now we will check the fulfillment of statement 4) of Lemma 2.

Let \( N_0 \leq m < N \), then for some \( s_0, \ 1 \leq s_0 \leq q, \quad (N_{s_0} \leq m < N_{s_0+1}) \) we will have (see (2.13) and (2.14))

\[ \sum_{N_0 \leq |k| \leq m} C_k e^{ikx} = \sum_{s=1}^{s_0} \left[ \sum_{N_{s-1} \leq |k| < N_s} C_k^{(s)} e^{ikx} \right] + \sum_{N_{s_0} \leq |k| \leq m} C_k^{(s_0+1)} e^{ikx}. \]

Hence and from (2.1), (2.4), (2.5), (2.11) and (2.12) for any measurable set \( e \subset E \) we obtain
\[
\int e^{ikx} \sum_{N_0 \leq |k| \leq m} C_k e^{ikx} \, dx \leq \\
\leq \sum_{s=1}^{s_0} \left[ \int e^{ikx} \sum_{N_{s-1} \leq |k| < N_s} C_k^{(s)} e^{ikx} - g_s(x) \, dx \right] + \\
+ \sum_{s=1}^{s_0} \int e^{ikx} \sum_{N_{s-1} \leq |k| \leq m} C_k^{(s+1)} e^{ikx} \, dx < \\
< \sum_{s=1}^{s_0} \frac{\varepsilon}{2^{s+1}} + \int e^{ikx} \sum_{N_{s-1} \leq |k| \leq m} C_k^{(s+1)} e^{ikx} \, dx < \\
< \int |f(x)| \, dx + \frac{2}{\sqrt{\varepsilon}} \cdot |\gamma_{s_0+1}| \cdot \sqrt{\Delta_{s_0+1}} < \\
< \sum_{s=1}^{s_0} \int |f(x)| \, dx + \varepsilon.
\]

Lemma is proved.

\section*{§3. \textsc{proof of theorem 3}}

Let \( \omega(t) \) be a continuous function, increasing in \([0, \infty) \) and \( \omega(+0) = 0 \) and let

\[
\{ f_n(x) \}_{n=1}^{\infty}, \quad x \in [0, 2\pi]
\]

be a sequence of all step functions, values and constancy interval endpoints of which are rational numbers. Applying Lemma 2 consecutively, we can find a sequence \( \{ E_s \}_{s=1}^{\infty} \) of sets and a sequence of polynomials

\[
P_s(x) = \sum_{N_{s-1} \leq |k| < N_s} C_k^{(s)} e^{ikx},
\]

\( 1 = N_0 < N_1 < \ldots < N_s < \ldots, \quad s = 1, 2, \ldots, \)

which satisfy the conditions:

\[
P_s(x) = f_s(x), \quad x \in E_s \quad (3.3)
\]

\[
|E_s| > 1 - 2^{-2(s+1)}, \quad E_s \subset [0, 2\pi], \quad (3.4)
\]

\[
\sum_{N_{s-1} \leq |k| < N_s} |C_k^{(s)}| \cdot \omega(|C_k^{(s)}|) < 2^{-2s}, \quad C_{-k} = \overline{C_k^{(s)}} \quad (3.5)
\]

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\[
\max_{N_s - 1 \leq p < N_s} \left[ \left| \int_{N_s - 1 \leq |k| < p} e^{ikx} \right| \right] < 2^{-2(s+1)} + \int_{E} |f_s(x)|dx,
\]

(3.6)

for every measurable subset \( e \) of \( E_s \).

Denote
\[
\sum_{k=-\infty}^{\infty} C_k e^{ikx} = \sum_{s=1}^{\infty} \left[ \sum_{N_s - 1 \leq |k| < N_s} C^{(s)}_k e^{ikx} \right],
\]

(3.7)

where \( C_k = C^{(s)}_k \) for \( N_s - 1 \leq |k| < N_s, \ s = 1, 2, \ldots \).

Let \( \varepsilon \) be an arbitrary positive number. Setting
\[
\Omega_n = \bigcap_{s=n}^{\infty} E_s, \quad n = 1, 2, \ldots;
\]
\[
E = \Omega_{n_0} = \bigcap_{s=n_0}^{\infty} E_s, \quad n_0 = \left\lfloor \log_{1/2} \varepsilon \right\rfloor + 1;
\]

(3.8)

\[
B = \bigcup_{n=n_0}^{\infty} \Omega_n = \Omega_{n_0} \cup \left( \bigcup_{n=n_0+1}^{\infty} \Omega_n \setminus \Omega_{n-1} \right).
\]

It is clear (see (3.4)) that \(|B| = 2\pi\) and \(|E| > 2\pi - \varepsilon\).

We define a function \( \mu(x) \) in the following way:
\[
\mu(x) = \begin{cases} 
1 & \text{for } x \in E \cup \left( [0, 2\pi] \setminus B \right); \\
\mu_n & \text{for } x \in \Omega_n \setminus \Omega_{n-1}, \ n \geq n_0 + 1,
\end{cases}
\]

(3.9)

where
\[
\mu_n = \left[ 2^{2n} \cdot \prod_{s=1}^{n} h_s \right]^{-1}; \quad h_s = \|f_s(x)\|_{C^+} + \max_{N_s - 1 \leq p < N_s} \left\| \sum_{N_s - 1 \leq |k| < p} C^{(s)}_k e^{ikx} \right\|_{C^+ + 1},
\]

(3.10)

where
\[
\|g(x)\|_{C} = \max_{x \in [0, 2\pi]} |g(x)|,
\]

\(g(x)\) is a continuous function on \([0, 2\pi]\).

From (3.5),(3.7)-(3.10) we obtain
\((A) - \mu(x)\) is a measurable function and
\[
0 < \mu(x) \leq 1, \quad |\{x \in [0, 2\pi] : \mu(x) \neq 1\}| < \varepsilon.
\]
(B) \[- \sum_{k=1}^{\infty} |C_k|^2 \cdot \omega(|C_k|) < \infty.\]

Hence, obviously we have

\[
\lim_{k \to \infty} C_k = 0. \tag{3.11}
\]

It follows from (3.8)-(3.10) that for all \( s \geq n_0 \) and \( p \in [N_{s-1}, N_s) \)

\[
\int_{[0,2\pi] \setminus \Omega_s} \left| \sum_{N_{s-1} \leq |k| < p} C_k^{(s)} e^{ikx} \right| \mu(x) dx = \sum_{n=s+1}^{\infty} \left[ \int_{\Omega_n \setminus \Omega_{n-1}} \left| \sum_{N_{s-1} \leq |k| < p} C_k^{(s)} e^{ikx} \right| \mu_n dx \right] \leq \sum_{n=s+1}^{\infty} 2^{-2n} \left[ \int_0^{2\pi} \left| \sum_{N_{s-1} \leq |k| < p} C_k^{(s)} e^{ikx} \right| h_s^{-1} dx \right] \leq \frac{1}{3} 2^{-2s}. \tag{3.12}
\]

By (3.3), (3.8)-(3.10) for all \( s \geq n_0 \) we have

\[
\int_0^1 |P_s(x) - f_s(x)| \mu(x) dx = \int_{\Omega_s} |P_s(x) - f_s(x)| \mu(x) dx + 
\]

\[
+ \int_{[0,2\pi] \setminus \Omega_s} |P_s(x) - f_s(x)| \mu(x) dx = 2^{-2(s+1)} + 
\]

\[
+ \sum_{n=s+1}^{\infty} \left[ \int_{\Omega_n \setminus \Omega_{n-1}} |P_s(x) - f_s(x)| \mu_n dx \right] \leq 2^{-2(s+1)} + 
\]

\[
\leq \sum_{n=s+1}^{\infty} 2^{-2s} \left[ \int_0^{2\pi} \left| f_s(x) + \left| \sum_{N_{s-1} \leq |k| < N_s} C_k^{(s)} e^{ikx} \right| h_s^{-1} dx \right] \right] < 
\]

\[
< 2^{-2(s+1)} + \frac{1}{3} 2^{-2s} < 2^{-2s}. \tag{3.13}
\]

Taking relations (3.6), (3.8)-(3.10) and (3.12) into account we obtain that for all \( p \in [N_{s-1}, N_s) \) and \( s \geq n_0 + 1 \)

\[
\int_0^{2\pi} \left| \sum_{N_{s-1} \leq |k| < p} C_k^{(s)} e^{ikx} \right| \mu(x) dx = \]

\[
= \int_{\Omega_s} \left| \sum_{N_{s-1} \leq |k| < p} C_k^{(s)} e^{ikx} \right| \mu(x) dx + 
\]
\[
\int_{\Omega} \left| \sum_{N_{s-1} \leq |k| < p} C_k^{(s)} e^{ikx} \right| \mu(x) dx < \\
< \sum_{n=n_0+1}^s \left( \int_{\Omega_n \setminus \Omega_{n-1}} \left| \sum_{N_{s-1} \leq |k| < p} C_k^{(s)} e^{ikx} \right| dx \right) \cdot \mu_n + \frac{1}{3} 2^{-2s} < \\
< \sum_{n=n_0+1}^s \left( 2^{-2(s+1)} + \int_{\Omega_n \setminus \Omega_{n-1}} |f_s(x)|dx \right) \mu_n + \frac{1}{3} 2^{-2s} = \\
= 2^{-2(s+1)} \cdot \sum_{n=n_0+1}^s \mu_n + \int_{\Omega_s} |f_s(x)|\mu(x) dx + \frac{1}{3} 2^{-2s} < \\
< \int_0^{2\pi} |f_s(x)|\mu(x) dx + 2^{-2s}. \quad (3.14)
\]

Let \( f(x) \in L^1_{\mu}[0, 2\pi] \), i.e. \( \int_0^{2\pi} |f(x)|\mu(x) dx < \infty \).

It is easy to see that we can choose a function \( f_{\nu_1}(x) \) from the sequence (3.1) such that

\[
\int_0^{2\pi} |f(x) - f_{\nu_1}(x)|\mu(x) dx < 2^{-2}, \quad \nu_1 > n_0 + 1. \quad (3.15)
\]

Hence, we have

\[
\int_0^{2\pi} |f_{\nu_1}(x)|\mu(x) dx < 2^{-2} + \int_0^{2\pi} |f(x)|\mu(x) dx. \quad (3.16)
\]

From (2.1), (A), (3.13) and (3.15) we obtain with \( m_1 = 1 \)

\[
\int_0^{2\pi} \left| f(x) - \left[ P_{\nu_1}(x) + C_{m_1} e^{im_1x} \right] \right| \mu(x) dx \leq \\
\leq \int_0^{2\pi} |f(x) - f_{\nu_1}(x)|\mu(x) dx + \\
+ \int_0^{2\pi} |f_{\nu_1}(x) - P_{\nu_1}(x)|\mu(x) dx + \\
+ \int_0^{2\pi} |C_{m_1} e^{im_1x}| \mu(x) dx < 2 \cdot 2^{-2} + 2\pi \cdot |C_{m_1}|. \quad (3.17)
\]

Assume that numbers \( \nu_1 < \nu_2 < \ldots < \nu_{q-1}; m_1 < m_2 < \ldots < m_{q-1} \) are chosen in such a way that the following condition is satisfied:
\[
\int_0^{2\pi} \left| f(x) - \sum_{s=1}^j \left[ P_{\nu_s}(x) + C_{m_s} e^{im_s x} \right] \right| \mu(x)dx < 2 \cdot 2^{-2j} + 2\pi \cdot |C_{m_j}|, \quad 1 \leq j \leq q - 1.
\]

(3.18)

We choose a function \( f_{\nu_q}(x) \) from the sequence (3.1) such that

\[
\int_0^{2\pi} \left| f(x) - \sum_{s=1}^{q-1} \left[ P_{\nu_s}(x) + C_{m_s} e^{im_s x} \right] - f_{\nu_q}(x) \right| \mu(x)dx < 2^{-2q},
\]

(3.19)

where \( \nu_q > \nu_{q-1} \), \( \nu_q > m_{q-1} \). This with (3.18) imply

\[
\int_0^{2\pi} |f_{\nu_q}(x)| \mu(x)dx < 2^{-2q} + 2 \cdot 2^{-2(q-1)} + 2\pi \cdot |C_{m_{q-1}}| = 9 \cdot 2^{-2q} + 2\pi \cdot |C_{m_{q-1}}|.
\]

(3.20)

By (3.13), (3.14) and (3.20) we obtain

\[
\int_0^{2\pi} |f_{\nu_q}(x) - P_{\nu_q}(x)| \mu(x)dx < 2^{-2\nu_q},
\]

(3.21)

\[
P_{\nu_q}(x) = \sum_{N_{\nu_{q-1}} \leq |k| < N_{\nu_q}} C_k^{(\nu_q)} e^{ikx}.
\]

(3.22)

Denote

\[
m_q = \min \left\{ n \in N : n \notin \left\{ \{k\}_{k=N_{\nu_{q-1}}}^{N_{\nu_q-1}} \right\} \cup \left\{ m_{s}^{q-1} \right\} \right\}.
\]

(3.23)

From (2.1), (A), (3.19) and (3.21) we have

\[
\int_0^{2\pi} \left| f(x) - \sum_{s=1}^{q} \left[ P_{\nu_s}(x) + C_{m_s} e^{im_s x} \right] \right| \mu(x)dx \leq
\]

\[
\leq \int_0^{2\pi} \left| \left( f(x) - \sum_{s=1}^{q-1} \left[ P_{\nu_s}(x) + C_{m_s} e^{im_s x} \right] \right) - f_{\nu_q}(x) \right| \mu(x)dx +
\]

\[
+ \int_0^{2\pi} \left| f_{\nu_q}(x) - P_{\nu_q}(x) \right| \mu(x)dx +
\]

\[
+ \int_0^{2\pi} \left| f(x) - \sum_{s=1}^{q} \left[ P_{\nu_s}(x) + C_{m_s} e^{im_s x} \right] \right| \mu(x)dx \leq
\]

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Thus, by induction we on $q$ can choose from series (3.7) a sequence of members

$$C_m e^{imx}, \quad q = 1, 2, ...,$$

and a sequence of polynomials

$$P_{\nu q}(x) = \sum_{N_{\nu q}-1 \leq |k| < N_{\nu q}} C_k^{(\nu q)} e^{ikx}, \quad N_{\nu q-1} > N_{\nu q-1}, \quad q = 1, 2, ... \quad (3.25)$$

such that conditions (3.22) - (3.24) are satisfied for all $q \geq 1$.

Taking account the choice of $P_{\nu q}(x)$ and $C_m e^{imx}$ (see (3.23) and (3.25)) we conclude that the series

$$\sum_{q=1}^{\infty} \left[ \sum_{N_{\nu q}-1 \leq |k| < N_{\nu q}} C_k^{(\nu q)} e^{ikx} + C_m e^{ix} \right]$$

is obtained from the series (3.7) by rearrangement of members. Denote this series by $\sum C_{\sigma(k)} e^{i\sigma(k)x}$.

It follows from (3.11), (3.22) and (3.24) that the series $\sum C_{\sigma(k)} e^{i\sigma(k)x}$ converges to the function $f(x)$ in the metric $L^1_{\mu}[0,2\pi]$, i.e. the series (3.7) is universal with respect to rearrangements (see Definition 1).

This completes the proof of Theorem 4.

REFERENCES

[1] N.K.Bary, Trigonometric series, Nauka, Moscow,1961; English trans. in Pergamon Press, Oxford, 1964.

[2] K.S.Kazarian, R.Zink ,Some ramifications of a theorem of Boas and Pollard concerning the completion of a set of functions in $L^2$, Trans.Amer.Math.Soc., v.349, n.11, p. 4367-4383.

[3] M. G. Grigorian "On the representation of functions by orthogonal series in weighted $L^p$ spaces, Studia. Math. 134(3)1999, 211-237.

[4] D.E.Menshov, On the partial sums of trigonometric series, Mat. Sb. 20(1947), 197-238 [in russian].

[5] V.Ya.Kozlov, On the complete systems of orthogonal functions, Mat. Sb. 26(1950), 351-364 [in russian].
[6] A.A. Talalian, On the convergence almost everywhere the subsequence of partial sums of general orthogonal series, Izv. Ak. Nauk Arm. SSR ser. Math. 10(1957), 17-34 [in russian].

[7] A.A. Talalian, On the universal series with respect to rearrangements, Izv. AN. SSSR ser. Math. 24(1960), 567-604 [in russian].

[8] W. Orlicz, Uber die unabhängigung von der Anordnung fast überall konvergenten Reihen, Bull. de l’Academie Polonaise des Sciences, 81 (1927), p. 117-125.

[9] S.A. Episkoposian, M.G. Grigorian, "On universal trigonometric series in weighted spaces \( L^1_\mu [0, 2\pi] \)”, East Journal on Approximations, 1999, v.5, n.4, 483-492.

[10] S.A. Episkoposian, "On the existence of universal series by Walsh system”, Izvestiya Natsionalnoi Akademii Nauk Armenii, English trans. in: Journal of Contemporary Mathematical Analysis, 2003, v. 38, n.4, p.25-40.

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