THE CLASSICAL LIMIT OF W–ALGEBRAS

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ABSTRACT

We define and compute explicitly the classical limit of the realizations of $W_n$ appearing as hamiltonian structures of generalized KdV hierarchies. The classical limit is obtained by taking the commutative limit of the ring of pseudodifferential operators. These algebras—denoted $w_n$—have free field realizations in which the generators are given by the elementary symmetric polynomials in the free fields. We compute the algebras explicitly and we show that they are all reductions of a new algebra $w_{\text{KP}}$, which is proposed as the universal classical $W$-algebra for the $w_n$ series. As a deformation of this algebra we also obtain $w_{1+\infty}$, the classical limit of $W_{1+\infty}$.

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Introduction

It is by now well known that classical realizations of $W$-algebras appear naturally as Poisson brackets of integrable hierarchies of Lax type. For example, the Virasoro algebra is realized as the Magri bracket for the KdV hierarchy [1], and the Zamolodchikov-Fateev-Lykyanov $W_n$ algebras as the second Gel’fand–Dickey bracket for the generalized $n^{th}$-order KdV hierarchy [2]. Although the $W$-algebras appearing in this fashion are often unfortunately (mis)named “classical” $W$-algebras, we prefer to refer to these algebras as classical realizations of $W$-algebras and reserve the name classical $W$-algebras to the nonlinear extensions of Diff $S^1$ which will be the topic of this paper.

We will obtain classical $W$-algebras by taking the classical limit of the second Gel’fand–Dickey bracket: these are the classical limits of the $W_n$-algebras and we denote them by $w_n$. We can think of this classical limit as a suitable contraction of the Gel’fand–Dickey algebra, but more fundamentally as the algebra which is induced after taking the classical (i.e., commutative) limit of the relevant algebraic structures in the Lax formalism, namely the ring of pseudodifferential operators.

The classical $W$-algebras we will obtain are, in a way, the simplest examples of $W$-algebras. The algebra $w_n$ is a nonlinear extension of Diff $S^1$ by primary fields of weights $3, 4, \ldots, n$. These algebras are moreover easy to write down explicitly and have the advantage that the natural fields generating the algebra are already primaries. They therefore seem to be unique candidates to investigate questions (e.g., $W$-geometry) whose answers seem to be obscured by the more complicated quantum or classical realizations of these algebras.

The purpose of this paper is to announce results whose proofs will be given in a forthcoming publication [3] containing also numerous extensions of these results; particularly in the supersymmetric arena. The plan of the paper is the following. After briefly reviewing the emergence of $W$-algebras in the Lax formalism, we define their classical limit by first taking the classical limit of the ring of pseudodifferential operators, and then using a classical version of the Kupershmidt–Wilson theorem [4][5]. In this context, this theorem states that $w_n$ is (a reduction of) the Poisson algebra of symmetric polynomials in $n$ free fields. We then write the algebras explicitly and show that the basis fields in which we write the algebra are already primaries. We then show that all these algebras are reductions of $w_{KP}$: the classical limit of $W_{KP}$—the “natural” second hamiltonian structure of the KP hierarchy [6]. Perhaps this needs some clarification. In [6] it was shown that the KP hierarchy is bihamiltonian: the second hamiltonian structure begin given by the natural generalization of the Adler map to the space of KP operators and the first structure being given by deforming this map as in [4]. However the KP hierarchy has an infinite number
of inequivalent bihamiltonian structures indexed by the natural numbers, of which the one in [6] is to some extent the natural one. The existence of this infinite number of bihamiltonian structures was proven in [7] but appears to have been first known to Radul [8]. We end the paper with a brief note on the universal properties of $w_{\text{KP}}$, its deformation to $w_{1+\infty}$, and some concluding remarks.

The classical limit of the Lax formalism implicit in our considerations yields integrable hierarchies known as the “dispersionless KdV hierarchies” which are reductions of the Khokhlov-Zabolotskaya or dispersionless KP hierarchy. These hierarchies have been studied in [9] and [10]. Particular solutions of the dispersionless Lax equations have been shown in [11] to correspond to the perturbed chiral ring in topological minimal models [12]. The analogs to the $\tau$-functions for these models should therefore obey $w_n$ constraints, obtained as reductions of $w_{\text{KP}}$ constraints as was done in [13] for the matrix model formulation of $2d$-quantum gravity.

**W-algebras from Lax Operators**

In this section we briefly describe how $W$ algebras appear as Poisson structures in the space of Lax operators. Brevity demands that we omit a review of the formalism, which can be found in the literature in various amounts of detail—see, for example, [14].

Let $L = \partial^n + \sum_{i=0}^{n-1} u_i \partial^i$ be a differential operator and $X = \sum_{i=0}^{n-1} \partial^{-i-1} x_i$ be a pseudodifferential operator. The Adler map $J$, defined by [15]

$$J(X) \equiv (LX)_+ L - L(XL)_+ = L(XL)_- -(LX)_-L,$$

sends $X$ linearly to a differential operator of order at most $n-1$. Therefore we can write

$$J(X) = \sum_{i,j=0}^{n-1} (J_{ij} \cdot x_j) \partial^i,$$

which defines differential operators $J_{ij}$. Gel’fand and Dickey [2] proved that the $J_{ij}$ define a Poisson bracket by

$$\{u_i(x), u_j(y)\} = -J_{ij} \cdot \delta(x - y),$$

where the operator $J_{ij}$ is taken at the point $x$. This bracket is known as the second Gel’fand–Dickey bracket and is the second hamiltonian structure for the $n^{th}$-order generalized KdV hierarchy—the hierarchy of isospectral flows of the Lax operator $L$. 

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A more conceptual proof of the fact that the bracket (3) is Poisson derives from the Kupershmidt–Wilson theorem [4][5]. This theorem states the following. Let us factorize the Lax operator as follows:

\[ L = \partial^n + \sum_{i=0}^{n-1} u_i \partial^i = (\partial + \phi_1)(\partial + \phi_2) \cdots (\partial + \phi_n) . \]  

(4)

This way the \( \{u_i\} \) are given as differential polynomials of the \( \{\phi_i\} \). If we then define

\[ \{\phi_i(x), \phi_j(y)\} = \delta_{ij} \delta'(x - y) , \]  

(5)

and compute the induced bracket of the \( \{u_i(\phi)\} \), we find precisely (3).

We can define another Poisson bracket as a deformation of (3). If we shift

\( L \mapsto L + \lambda \) by a constant parameter, the Adler map develops a linear term

\[ J \mapsto J + \lambda J_\infty , \]  

(6)

where

\[ J_\infty(X) = -[L, X]_+ \]

is now a differential operator of order at most \( n - 2 \) and can therefore be written as

\[ J_\infty(X) = \sum_{i,j=0}^{n-2} \left(J^{(\infty)}_{ij} \cdot x_j\right) \partial^i . \]  

(7)

One can then define a Poisson bracket—the first Gel’fand–Dickey bracket—by

\[ \{u_i(x), u_j(y)\}_\infty = -J^{(\infty)}_{ij} \cdot \delta(x - y) , \]  

(8)

where, again, the differential operators \( J^{(\infty)}_{ij} \) are taken at the point \( x \). This is the first Hamiltonian structure for the isospectral flows of \( L \).

Notice that \( u_{n-1} \) is central relative to the first Gel’fand–Dickey bracket, which implies that \( u_{n-1} \) does not evolve according to the Lax flows. It is therefore consistent to impose the constraint \( u_{n-1} = 0 \). This can be done for free in the first Gel’fand–Dickey bracket, but it requires the Dirac bracket for the second. Hence we define

\[ J^{(0)}_{ij} \equiv J_{ij} - J_{i,n-1} J_{n-1,n-1}^{-1} J_{n-1,j} , \]  

(9)

which, despite the potential nonlocality present in \( J_{n-1,n-1}^{-1} \), is a differential operator. It is the reduced second Gel’fand–Dickey bracket,

\[ \{u_i(x), u_j(y)\}_0 = -J^{(0)}_{ij} \cdot \delta(x - y) , \]  

(10)

which provides a classical realization of \( W_n \). Indeed, for \( n = 2 \), \( T = u_0 \) obeys the Virasoro algebra; and, for \( n = 3 \), \( T = u_1 \) and \( W = u_0 - \frac{1}{2} u_1' \) obey Zamolodchikov’s \( W_3 \). In general, the \( \{u_i\} \) can be redefined to \( \{T, U_3, U_4, \ldots, U_n\} \) in
such a way that \( T \) obeys the Virasoro algebra and each \( U_i \) is a primary field of weight \( i \) [16].

**Classical Limits**

The classical realization of \( W_n \) defined by the reduced second Gel’fand–Dickey bracket (10) is encoded in the Adler map (1), which necessitates for its definition only the algebraic structures present in the ring \( R \) of pseudodifferential operators (ΨDO’s). The classical limit \( w_n \) of \( W_n \) will be obtained by mimicking the definition of the Adler map in the classical limit of \( R \). This ring \( R \) is, in fact, an algebra and hence, in particular, a vector space. Taking the classical limit consists of endowing this underlying vector space with the structure of a Poisson algebra, in such a way that commutators go into Poisson brackets. It is a general fact (see, for example, [17]) that the classical limit exists whenever, as in the case at hand, the ring \( R \) is filtered—the Poisson structure being defined on the associated graded object. However, rather than appealing to the general theory, we will define the classical limit directly and explicitly.

To every ΨDO \( A \) we associate its symbol—a formal Laurent series—as follows. We first write \( A \) with all \( \partial \)'s to the right: \( A = \sum_{i \leq N} a_i \partial^i \). (Each \( A \) has a unique expression of this form.) Its symbol is then the formal Laurent series in \( \xi^{-1} \) given by

\[
\tilde{A} = \sum_{i \leq N} a_i \xi^i .
\]  

(11)

Symbols have a commutative multiplication given by multiplying the Laurent series; but one can define a composition law \( \circ \) which recovers the multiplication law in \( R \). In other words,

\[
\tilde{A} \circ \tilde{B} = \tilde{AB} ,
\]  

(12)

where \( AB \) means the usual product of ΨDO’s. Let \( \tilde{A} \) and \( \tilde{B} \) be pseudodifferential symbols. Then their composition is easily shown to be given by

\[
\tilde{A} \circ \tilde{B} = \sum_{k \geq 0} \frac{1}{k!} \frac{\partial^k \tilde{A}}{\partial^k \xi} \frac{\partial^k \tilde{B}}{\partial^k x} .
\]  

(13)

For example, \( \xi \circ a = a \xi + a' \) which recovers the basic Leibniz rule: \( \partial a = a \partial + a' \).

To define the classical limit we introduce a formal parameter \( \hbar \) in the composition law

\[
\tilde{A} \circ \tilde{B} = \sum_{k \geq 0} \frac{\hbar^k}{k!} \frac{\partial^k \tilde{A}}{\partial^k \xi} \frac{\partial^k \tilde{B}}{\partial^k x} .
\]  

(14)

interpolating from the (commutative) multiplication of symbols for \( \hbar = 0 \).
to the (noncommutative) multiplication of ΨDO’s for ℏ = 1. The classical limit of any structure is obtained by introducing the parameter ℏ via (14) and keeping only the lowest term in its ℏ expansion. Therefore, the classical limit of ◦ is simply the commutative multiplication of symbols; hence the name commutative limit.

The Poisson structure on symbols is defined as the classical limit of the commutator—namely,\(^1\)

\[ \tilde{[A, B]} = \lim_{\hbar \to 0} \hbar^{-1} [\tilde{A}, \tilde{B}] , \]  

which can be written explicitly with the help of (14) as

\[ \tilde{[A, B]} = \frac{\partial \tilde{A}}{\partial \xi} \frac{\partial \tilde{B}}{\partial x} - \frac{\partial \tilde{A}}{\partial x} \frac{\partial \tilde{B}}{\partial \xi} . \]  

One recognizes this at once as the standard Poisson bracket on a two-dimensional phase space with canonical coordinates \((x, \xi)\).

We must now take the classical limit of the Adler map (1). The Adler map can be rewritten as follows:

\[ J(X) = [L, X]_+ L - [L, (XL)_+] , \]  

which makes its classical limit obvious. Indeed, if—dropping tildes—\(L = \xi^n + \sum_{i=0}^{n-1} u_i \xi^i\) and \(X = \sum_{i=0}^{n-1} x_i \xi^{-i-1}\) are symbols, the classical limit of the Adler map \(J(X)\) is given by

\[ J_{\text{cl}}(X) = [L, X]_+ L - [L, (XL)_+] = [L, (XL)_-] - [L, X]_- L . \]

It is clear from the above expression that \(J_{\text{cl}}(X)\) is a polynomial in \(\xi\) of order at most \(n - 1\) and moreover that it is linear in \(X\). This allows us to write

\[ J_{\text{cl}}(X) = \sum_{i,j=0}^{n-1} \left( \tilde{J}_{ij} \cdot x_j \right) \xi^i , \]  

where the \(\tilde{J}_{ij}\) are differential operators which are now at most of order one. The \(\tilde{J}_{ij}\) can be used to define Poisson brackets

\[ \{u_i(x), u_j(y)\}^{\text{cl}} = -\tilde{J}_{ij} \cdot \delta(x - y) , \]

as before.

\(^1\) We use \([,]\) to denote the Poisson bracket to avoid confusion with the Poisson bracket defining the classical \(W\)-algebras.
It is easy to show that this bracket is indeed a Poisson bracket. One can see this from the fact that this is a contraction of the original Adler map and that the Jacobi identity is satisfied order by order in $\hbar$; or, alternatively, one can easily modify Dickey’s proof [5] of the Kupershmidt–Wilson theorem [4] to prove the following. If we formally factorize the symbol $L = \xi^n + \sum_{i=0}^{n-1} u_i \xi^i = \prod_{i=1}^{n}(\xi + \phi_i)$, we find that $u_i = \sigma_{n-i}(\phi)$—the elementary symmetric functions of the $\{\phi_i\}$. If we then define the Poisson bracket of the $\{\phi_i\}$ as

$$\{\phi_i(x), \phi_j(y)\} = \delta_{ij}\delta'(x - y),$$

and we compute the induced bracket on the $u_i$, we find precisely the bracket (20).

We can also find the classical limit of the first Gel’fand–Dickey bracket by shifting $L \mapsto L + \lambda$ in the classical Adler map $J_{\ell}$ to obtain

$$J_{\ell}^{(\infty)}(X) = -[L, X]_+. \quad (22)$$

To obtain the classical limit, $w_n$, of $W_n$ we perform the reduction $u_{n-1} = 0$ in the bracket (20). This defines the $w_n$ algebra

$$\{u_i(x), u_j(y)\}^{\ell} = \tilde{J}_{ij}^{(0)} \cdot \delta(x - y), \quad (23)$$

where $\tilde{J}_{ij}^{(0)}$ are differential operators given by

$$\tilde{J}_{ij}^{(0)} = \tilde{J}_{ij} - \tilde{J}_{i,n-1}\tilde{J}_{n-1,n-1}\tilde{J}_{n-1,j}. \quad (24)$$

**Explicit Structure of $w_n$**

We now compute the $w_n$ algebras explicitly. Let $L = \xi^n + \sum_{i=0}^{n-1} u_i \xi^i$ and $X = \sum_{i=0}^{n-1} x_i \xi^{-i-1}$ be symbols. The classical limit of the second Gel’fand–Dickey bracket is given by equation (20) where the differential operators $\tilde{J}_{ij}$ are given by (18) and (19). A reasonably easy computation yields the following $\tilde{J}_{ij}$:

$$\begin{align*}
\tilde{J}_{n-1,n-1} &= -n\partial, \\
\tilde{J}_{i,n-1} &= -(i + 1)u_{i+1}\partial, \\
\tilde{J}_{n-1,j} &= -(j + 1)\partial u_{j+1}, \\
\tilde{J}_{ij} &= (n - j - 1)\partial u_{i+2+j-n} + (n - i - 1)u_{i+2+j-n}\partial \\
&\quad + \sum_{l=j+2}^{n-1} [(l - i - 1)u_{i+j+2-l}\partial u_l + (l - j - 1)u_l\partial u_{i+j+2-l}] \\
&\quad - (i + 1)u_{i+1}\partial u_{j+1},
\end{align*} \quad (25)$$

where $i, j = 0, 1, \ldots, n - 2$ and with the proviso that $u_{i<0} = 0$ is the above formulas.
After imposing the constraint \( u_{n-1} = 0 \), the induced bracket is given by (23) where the differential operators \( \tilde{J}_{ij}^{(0)} \) are given by (24). An easy computation yields the following \( \tilde{J}_{ij}^{(0)} \):

\[
\tilde{J}_{ij}^{(0)} = (n - j - 1)\partial u_{i+2+j} + (n - i - 1)u_{i+2+j-n}\partial
+ \sum_{l=j+2}^{n-1} [(l - i - 1)u_{i+j+2-l}\partial u_{l} + (l - j - 1)u_{l}\partial u_{i+j+2-l}]
+ \frac{(i+1)(j+1-n)}{n}u_{i+1}\partial u_{j+1},
\]

(26)

where now both \( i, j = 0, 1, \ldots, n-2 \) and with the proviso that \( u_{n-1} = u_{i<0} = 0 \). In particular, if \( j = n - 2 \) we find

\[
\tilde{J}_{i,n-2}^{(0)} = (n - i)u_{i}\partial + u'_{i},
\]

(27)

which identifies \( u_{n-2} \) as a generator of a \( \text{Diff} S^1 \) subalgebra and each \( u_{i} \) as a \( \text{Diff} S^1 \) tensor of weight \( n - i \). Therefore, the natural basis for \( w_{n} \)—namely, the coefficients in the Lax symbol—are already primary fields.

We can easily work out the first two examples \( n = 2 \) and \( n = 3 \). For \( n = 2 \) we only have one field, \( T = u_0 \), and the algebra is clearly \( \text{Diff} S^1 \):

\[
\{T(x) , T(y)\}^c_{\ell} = - \left[ 2T(x)\partial + T'(x) \right] \cdot \delta(x - y).
\]

(28)

For \( n = 3 \) we define \( T = u_1 \) and \( W = u_0 \). Then from (26) we find (28) together with

\[
\{W(x) , T(y)\}^c_{\ell} = - \left[ 3W(x)\partial + W'(x) \right] \cdot \delta(x - y),
\]

and

\[
\{W(x) , W(y)\}^c_{\ell} = - \left[ \frac{2}{3}T(x)\partial T(x) \right] \cdot \delta(x - y),
\]

(29)

which is \( w_{3} \).
Reduction from $w_{\text{KP}}$ to $w_n$

The space of Lax operators $L = \partial^n + \sum_{i=0}^{n-1} u_i \partial^i$ is naturally a subspace of the space $M^{(n)}$ of all ΨDO’s of the form $\partial^n + \sum_{i \leq n-1} u_i \partial^i$. The Adler map (1) naturally extends to $M^{(n)}$ by the same formula, but where $L$ is now understood as a ΨDO in $M^{(n)}$ and $X = \sum_{i \leq n-1} \partial^{-i-1} x_i$. The expression for $J(X)$ tells us that it is a ΨDO of order at most $n - 1$, whence we can write it as

$$J(X) = \sum_{i,j \leq n-1} (\Omega_{ij} \cdot x_j) \partial^i,$$

where the $\Omega_{ij}$ are differential operators defining a Poisson bracket as in (3). For reasons that will appear clear below, we call this resulting algebra $W^{(n)}_{\text{KP}}$.

The subspace of $M^{(n)}$ obtained by demanding that $L$ be differential inherits a Poisson bracket which coincides with the one derived from the original Adler map (1). In terms of hamiltonian reduction, the constraints can be written simply as $L_\mu = 0$. Moreover the constraint $u_{n-1} = 0$ can be imposed before or after the reduction from $M^{(n)}$. In other words, $W_n$ is a hamiltonian reduction of $W^{(n)}_{\text{KP}}$. Taking the classical limit of the previous sentence, we find that $w_n$ is a hamiltonian reduction of the classical limit of $W^{(n)}_{\text{KP}}$. We will see below that this limit is independent of $n$.

Every operator $L$ in $M^{(n)}$ has a unique $n^{th}$ root of the form $\Lambda = \partial + \sum_{i \geq 0} a_i \partial^{-i}$—that is, a KP operator—and, similarly, every KP operator $\Lambda$ can be raised to the $n^{th}$ power to yield a ΨDO in $M^{(n)}$. Therefore the space $M^{(n)}$ is isomorphic to the space $M^{(1)}$ of KP operators. It was show in [6] that in $M^{(1)}$ one can define a Poisson bracket from a generalization of the Adler map

$$J_{\text{KP}}(X) = (\Lambda X)_+ \Lambda - \Lambda (X \Lambda)_+ = \Lambda (X \Lambda)_- - (\Lambda X)_- \Lambda,$$

for $\Lambda \in M^{(1)}$ and $X = \sum_{i \geq 0} \partial^{-i-1} x_i$. The associated Poisson bracket defines a $W$–algebra, called $W_{\text{KP}}$, which was shown to be a hamiltonian structure for the KP hierarchy.

Now, it can be shown that $W^{(n)}_{\text{KP}}$—the extension of the Adler map to $M^{(n)}$—is also a hamiltonian structure for the KP hierarchy, which explains the name. It can be shown by explicit computation of the first few Poisson brackets that these algebras are all different and, in particular, different from $W_{\text{KP}}$. Nevertheless, as we now show, they all have the same classical limit.
The algebra $w_{KP}$ is the Poisson algebra induced by the classical limit of the generalized Adler map (31) which can be written as

$$J_{KP}^{\ell}(X) = [\Lambda, X]_+\Lambda - [\Lambda, (X\Lambda)_+]_+ ,$$

(32)

where $\Lambda = \xi + \sum_{i \geq 0} a_i \xi^{-i}$ and $X = \sum_{i \geq 0} x_i \xi^{i-1}$ are symbols. In the usual manner this defines a Poisson bracket on the variables $\{a_i\}$ which defines the $w_{KP}$ algebra. Now let $\varphi$ denote the map sending $\Lambda$ to its $n$th power $\Lambda^n = \xi^n + \sum_{i \leq n-1} u_i \xi^i$. The $\{u_i\}$ can then be solved as polynomials in the $\{a_i\}$ and this map is in fact invertible, so that the $\{a_i\}$ can be in turn be solved for as polynomials in the $\{u_i\}$. The Poisson bracket of the $\{a_i\}$ then induces a Poisson bracket among the $\{u_i\}$ which defines a classical $W$–algebra. On the other hand, the $\{u_i\}$ inherit another Poisson bracket from the classical limit of the extension of the Adler map to $M(n)$ and it is this bracket which we have seen to reduce to $w_n$. What we will now show is that these two brackets on the $\{u_i\}$ are in fact the same.

This uses some geometric formalism which is reviewed, for example, in [14]. Roughly the Adler map can be understood as a tensorial map from 1-forms to vector fields. The map $\varphi : M(1) \to M(n)$ then induces an Adler-type map on $M(n)$ from the one on $M(1)$ as follows. If $X$ is any 1-form on $M(n)$, we first pull it back to $M(1)$ via $\varphi^*$. We then apply the map (31) to it yielding a vector field on $M(1)$, which can be then pushed forward to $M(n)$ with $\varphi_*$. Therefore the induced Poisson brackets will be the ones associated to the Adler-type map $\varphi_* \circ J_{KP}^{\ell} \circ \varphi^*$. We now proceed to compute this. A straightforward calculation shows that if $A$ is a tangent vector on $M(1)$ at $\Lambda$, its push-forward is the tangent vector to $M(n)$ at $L = \Lambda^n$ given by $\varphi_* A = n \Lambda^{n-1} A$. Similarly, if $X$ is a 1-form on $M(n)$ at $L$, its pull-back is the 1-form on $M(1)$ at $L^{1/n} = \Lambda$ given by $\varphi^* X = n \Lambda^{n-1} X$. Therefore, the induced hamiltonian map is given by

$$\left( \varphi_* \circ J_{KP}^{\ell} \circ \varphi^* \right)(X) = n^2 \Lambda^{n-1} \left( [\Lambda, \Lambda^{n-1} X]_+ + [\Lambda, (\Lambda^n X)_+]_+ \right)$$

$$= nL[L, X]_+ - n[L, (LX)_+]_+ ,$$

(33)

where we have used repeatedly the fact that $n[\Lambda, \Lambda^{n-1} Z] = n[\Lambda, Z] \Lambda^{n-1} = [L, Z]$ for any $Z$. But—up to the inessential multiplicative factor $n$—we recognize in the right-hand side of (33) the classical limit of the extension to $M(n)$ of the Adler map (1). This concludes the proof.
A Word on Universality

Reductions clearly commute with classical limits. From this we conclude that since $W_n$ is a reduction of $W^{(n)}_\text{KP}$, $w_n$—the classical limit of $W_n$—is a reduction of the classical limit of $W^{(n)}_\text{KP}$, i.e., $w_\text{KP}$. But this limit is $n$-independent, hence all $w_n$ are hamiltonian reductions of $w_\text{KP}$.

In [18] we proved, as an easy corollary of the Kupershmidt-Wilson theorem, that $W_n$ is a hamiltonian reduction of $W_{n+1}$ for any $n$. This allowed us to define a universal $W_n$-algebra as the inverse limit of the $W_n$ under these reductions. Existence and uniqueness then followed from the universal properties of (co)limits. One can repeat the steps in [18] for the $w_n$ algebras, to show that $w_n$ is a hamiltonian reduction of $w_{n+1}$ for every $n$ and therefore define a universal algebra as their colimit. Roughly, this algebra has the property that all $w_n$ algebras can be obtained as hamiltonian reductions from it and that, in some sense, it is the smallest such algebra. The remarks at the beginning of this section thus clearly suggest that $w_\text{KP}$ is the universal (classical) $W$–algebra for the $w_n$ series.

Moreover, we expect it to be the classical limit of the universal $W$–algebra for the $W_n$ series. Or, said differently, that the universal $W$–algebra of [18] is a deformation of $w_\text{KP}$. We have already seen that $w_\text{KP}$ has an infinite number of different deformations $W^{(n)}_\text{KP}$. Since the universal $W$–algebra is unique, it would have to be at most one of them, and the “principle of insufficient reason” would dictate that if it is one of them, it should be $W_\text{KP}$. However, we have not yet been able to exhibit any $W_n$ as a hamiltonian reduction of $W_\text{KP}$.

Deformation to $w_{1+\infty}$

Shifting $\Lambda \mapsto \Lambda + \lambda$ in (32) defines another hamiltonian map

$$J_{\text{KP},\infty}^\ell(X) = -[[\Lambda, X_+]]_-, \quad (34)$$

which defines fundamental Poisson brackets for the coefficients $\{a_i\}$ of $\Lambda$. In fact, if we denote by $\Omega^{(\infty)}_{ij}$ the differential operators defined by

$$J_{\text{KP},\infty}^\ell(X) = \sum_{i,j=0}^\infty \left( \Omega^{(\infty)}_{ij} \cdot x_j \right) \xi^{-i}, \quad (35)$$

for $X = \sum_{i=0}^\infty x_i \xi^i$, the fundamental Poisson bracket of the $\{a_i\}$ associated to the hamiltonian map (34) is given by

$$\{a_i(x), a_j(y)\}_{\text{KP},\infty}^\ell = -\Omega^{(\infty)}_{ij} \cdot \delta(x - y). \quad (36)$$

These differential operators can be computed explicitly. Notice that since only $X_+$ appears, $x_0$ never enters the picture, whence $\Omega^{(\infty)}_{0j} = \Omega^{(\infty)}_{i0} = 0$. 

\[ -11 - \]
Therefore $a_0$ is central and can be dropped without harm. For $i, j \geq 1$ one finds
\[
\Omega_{ij}^{(\infty)} = (j - 1)\partial a_{i+j-2} + (i - 1)a_{i+j-2}\partial .
\] (37)

Notice that $\Omega_{11}^{(\infty)} = 0$ and that
\[
\Omega_{22}^{(\infty)} = \partial a_2 + a_2\partial ,
\] (38)
which means that $a_2$ generates a Diff $S^1$ subalgebra. Moreover,
\[
\Omega_{i2}^{(\infty)} = a'_i + ia_i\partial ,
\] (39)
which identifies $a_i$ as a Diff $S^1$ tensor of weight $i$. In fact, (37) is nothing but the algebra $w_{1+\infty}$ of hamiltonian vector fields on the 2-plane, with coordinates $(x, y)$, which depend polynomially on $y$ [19][20]. Also, the subalgebra generated by $\{a_i\}_{i \geq 2}$ is nothing but $w_\infty$. Therefore we see that $w_{\text{KP}}$ deforms to $w_{1+\infty}$.

Conclusions and Outlook

In this paper we have studied the classical limits of the $W_n$ algebras. We call these algebras $w_n$ and they are nonlinear extensions of the diffeomorphism algebra of the circle, Diff $S^1$, by tensors of weights $3, 4, \ldots, n$. Their construction followed the construction of the realization of $W_n$ as the second hamiltonian structure of the generalized KdV hierarchy. These realizations are defined in terms of the algebraic structure of the ring of pseudodifferential operators and therefore their classical limit is simply obtained by repeating the construction with the commutative limit of this ring: the Poisson algebra of pseudodifferential symbols, which is isomorphic to the Poisson algebra of smooth sections of the cotangent bundle (with the zero section removed) of the circle under the canonical symplectic structure.

These algebras are simplified versions of the $W$-algebras appearing in the Lax formalism; but they contain their essential feature, namely polynomial nonlinearity. We think that they make suitable candidates to begin to understand things $W$, in particular $W$-geometry.

The results of the last section on the reduction from $w_{\text{KP}}$ to $w_n$ beg for the development of the deformation theory of $W$-algebras. In the context in which they are treated here, they are particular examples of Poisson algebras, whereas in the quantum case they are particular examples of associative algebras, both of which count with a reasonably well-developed deformation theory. But it would be interesting to see if the features which characterize $W$-algebras (both classically among Poisson algebras and quantum-mechanically among associative algebras) allow for some simplification of their deformation theory which would make at least their infinitesimal deformations readily computable.
In this letter we have been rather sketchy in the development of the formalism. The details as well as many other results will appear in our forthcoming paper [3], which will also treat the supersymmetric case. There are other issues that deserve some study, e.g., uniqueness of classical $W$-algebras, relation with Lie algebras and with integrable systems, quantization, ... which will be treated in [3] and/or in future work.

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