A TOPOLOGICAL FINITUDE RESULT FOR OPERAD GROUPS

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Abstract. We proof a theorem which implies that planar or symmetric or braided operads with transformations satisfying some conditions yield operad groups of type $F^\infty$. This unifies and extends existing proofs that certain Thompson-like groups are of type $F^\infty$.

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1. Introduction

In [19] the author proposed to study fundamental groups of categories naturally associated to operads. This class of groups, called operad groups, contains a lot of Thompson-like groups already existent in the literature. Among them are the so-called diagram or picture groups [12], various groups of piecewise linear homeomorphisms of the unit interval [17], groups acting on ultrametric spaces via local similarities [13], higher dimensional Thompson groups $nV^2$ and the braided Thompson group $BV$ [3].

A recurrent theme in the study of these groups are topological finiteness properties, most notably property $F^\infty$ which means that the group admits a classifying space with compact skeleta. The proof of this property is very similar in each case, going back to a method of Brown, the Brown criterion [4], and a technique of Bestvina and Brady, the discrete Morse Lemma for affine complexes [1]. This program has been conducted in all the above mentioned classes of groups: for diagram or picture groups in [7, 8], for the piecewise linear homeomorphisms in [17], for local similarity groups in [9], for the higher dimensional Thompson groups in [10] and for the braided Thompson group in [5].
The aim of this article is to use the ideas which have been developed in the above cited articles, imitate this kind of proof for the class of operad groups and thus unify and conceptualize the results previously obtained in these articles. The side effect of this is that it also naturally extends the class of groups to which the proof can be applied. To apply the theorem to a given Thompson-like group, one has to find the operadic structure underlying the group. Then one has to check whether this operad satisfies certain finiteness conditions. In a lot of cases, the proofs of these conditions are either trivial or straightforward.

In [19, Section 6] we already performed this program for a restricted class of operads. Here, we extend this to operads which might contain symmetries, braidings and invertible degree 1 operations. To cope with these additional isomorphisms in \( S(O) \), we have to pass to the universal covering category and mod out these isomorphisms using the techniques from Section 4. We then obtain a category on which the operad group acts and we can apply the Brown criterion which we recall in Section 3. We then filter this category and show that the descending links are highly connected following the proof in [19, Section 6]. Showing that the core is highly connected is notably more complex in this case. This is done in Section 2 in a more general setting.

1.1. Prerequisites. The present article is based on Sections 2 to 5 of [19]. Section 6 is not required but recommended.

1.2. Notation and Conventions. When \( f : A \to B \) and \( g : B \to C \) are two composable arrows, we prefer the notation \( f \circ g \) or \( fg \) for the composite \( A \to C \) instead of the usual notation \( g \circ f \). To comply with this convention, we sometimes use the notation \( x \triangleright f \) instead of \( f(x) \) for the evaluation of \( f \) at \( x \).

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2. Three types of arc complexes

Let \( d \in \{1, 2, 3\} \) and \( C \) be a set of colors. Let \( X = (c_1, \ldots, c_n) \) be a word in the colors of \( C \). An archetype consists of a unique identifier together with a word in the colors of \( C \) of length at least 2. Let \( A \) be a set of archetypes. To this data, we will associate a simplicial complex \( \mathcal{AC}_d(C, A; X) \).

Consider the points \( 1, \ldots, n \in \mathbb{R} \) and embed them into \( \mathbb{R}^d \) via the first component embedding \( \mathbb{R} \to \mathbb{R}^d \). Color these points with the colors in the word \( X \) (i.e. the point \( i \) is colored with the color \( c_i \)) and call them nodes. Denote the set of nodes by \( N \). A link is the image of an embedding \( \gamma : [0, 1] \to \mathbb{R}^d \) such that \( \gamma(0) \) and \( \gamma(1) \) are nodes. Note that a link may contain more than two nodes. Two links connecting the same set of nodes are equivalent if there is an isotopy of \( \mathbb{R}^d \setminus N \) which takes one link to the other. An equivalence class of links is called an arc. Note that in the case \( d = 1 \), arcs and links are the same since each arc is represented by a unique link. We say that two arcs are disjoint if there are representing links which are disjoint. In the cases \( d = 2, 3 \), we can choose representing links of a collection of arcs such that the links are in minimal position:

**Lemma 2.1.** Assume \( d = 2 \) or \( d = 3 \). Let \( a_0, \ldots, a_k \) be arcs with \( a_i \neq a_j \) for each \( i \neq j \). Then there are representing links \( \alpha_0, \ldots, \alpha_k \) such that \( |\alpha_i \cap \alpha_j| \) is finite and minimal for each \( i \neq j \).
Proof. In the case \( d = 3 \), we can always find representing links which only intersect at nodes, if at all. The case \( d = 2 \) is a bit more complicated. We use the ideas from [5, Lemma 3.2]: Consider the nodes as punctures in the plane \( \mathbb{R}^2 \). Then we can find a hyperbolic metric on that punctured plane. Now define \( \alpha_i \) to be the geodesic within the class \( a_i \).

A link connecting a set of nodes \( M \) is called admissible if there is an isotopy of \( \mathbb{R}^d \setminus M \) taking the link into the image of the first component embedding \( \mathbb{R} \to \mathbb{R}^d \).

In the case \( d = 1 \), this is vacuous. In the case \( d = 2 \), this implies in particular that, when travelling the link starting from the lowest node, the nodes are visited in ascending order. This last property is even equivalent to being admissible in the case \( d = 3 \). An arc is called admissible if one and consequently all of its links are admissible. Now label an admissible arc with the identifier of an archetype in \( A \).

We require that the word formed by the colors of the connected nodes (in ascending order) equals the color word of the archetype. Call such a labelled admissible arc an archetypal arc.

The vertices of \( \mathcal{AC}_d(C, A; X) \) are the archetypal arcs. Two vertices are joined by an edge if the corresponding arcs are disjoint. This determines the complex as a flag complex. A \( k \)-simplex is therefore a set of \( k + 1 \) pairwise disjoint archetypal arcs. We call this an archetypal arc system. It follows from Lemma 2.1 above that if \( \{a_0, ..., a_k\} \) is an archetypal arc system, then we always find representing links \( \alpha_i \) of \( a_i \) such that the \( \alpha_i \) are pairwise disjoint. The following are examples of 2-simplices in the cases \( d = 1, 2, 3 \) (where we have omitted the labels on the arcs).

The following are non-examples of simplices in the case \( d = 2 \). In the first diagram, the two arcs are not disjoint and in the second diagram, the arc is not admissible. However, the second diagram would represent an admissible arc in the case \( d = 3 \).

**Definition 2.2.** Let \( C \) be a set of colors and \( A \) a set of archetypes. A word in the colors of \( C \) is called reduced if it admits no archetypal arc on it. The set of archetypes \( A \) is called tame if the length of all reduced words is bounded from above. The length of an archetype is the length of its color word. The set of archetypes \( A \) is of finite type if the length of all archetypes is bounded from above.

**Theorem 2.3.** Let \( d \in \{1, 2, 3\} \). Let \( C \) be a set of colors and \( A \) a system of archetypes. Assume that \( A \) is tame and of finite type. Let \( m_r \) be the smallest natural number greater than the length of any reduced color word and \( m_a \) be the
length of the longest archetype in $A$. Define
\[ \nu_{\kappa}(l) := \left\lfloor \frac{l - m_r}{\kappa} \right\rfloor - 1 \]
Let $X$ be a word in the colors of $C$ and denote by $lX$ the length of $X$. Then the complex $\mathcal{AC}_d(C, A; X)$ is $\nu_{\kappa_d}(lX)$-connected where
\[ \kappa_1 := 2m_a + m_r \]
\[ \kappa_2 := 2m_a - 1 \]
\[ \kappa_3 := 2m_a - 1 \]

For the proof in the case $d = 2$ we have to pass to a slightly larger class of complexes: Instead of $\mathbb{R}^2$ we consider links and arcs in the punctured plane $S = \mathbb{R}^2 \setminus \{p_1, ..., p_l\}$ with finitely many punctures $p_i \in \mathbb{R}^2$ disjoint from the nodes. Here, we define two links connecting the same set of nodes to be equivalent if they differ by an isotopy of $S \setminus N$ and a link connecting a set of nodes $M$ to be admissible if there is an isotopy of $\mathbb{R}^2 \setminus M$ taking the link into the image of the first component embedding $\mathbb{R} \to \mathbb{R}^2$. Note that in the latter case, we require an isotopy of $\mathbb{R}^2 \setminus M$ and not of $S \setminus M$, i.e. we allow the links to be pulled over punctures. We denote the corresponding complex of archetypal arc systems again by $\mathcal{AC}_d(C, A; X)$, supressing the additional data of punctures since, as we will see, the punctures do not affect the connectivity of the complexes. In other words, Theorem 2.3 is also valid for this larger class of complexes.

2.1. Proof of the connectivity theorem. The proof essentially consists of slightly modified ideas from [5, Subsection 3.3].

We induct over the length $lX$ of $X$. The induction start is $lX \geq m_r$. This implies that $X$ is not reduced and thus admits an archetypal arc on it. It follows that $\mathcal{AC}_d(C, A; X)$ is non-empty, i.e. $(−1)$-connected. For the induction step, assume $lX \geq m_r + \kappa_d$. We look at the cases $d = 1, 2, 3$ separately, starting with the case $d = 2$ since it is the hardest one.

2.1.1. The two-dimensional case. Choose a vertex of $\mathcal{AC}_2 := \mathcal{AC}_2(C, A; X)$ represented by an archetypal arc $b$. Let $v_1 < ... < v_t$ be the nodes connected by $b$. Let $\mathcal{AC}_2^0$ be the full subcomplex of $\mathcal{AC}_2$ spanned by the archetypal arcs which do not meet the nodes $v_i$.

We want to estimate the connectivity of the pair $(\mathcal{AC}_2, \mathcal{AC}_2^0)$ using the Morse method for simplicial complexes (see e.g. [19, Section 5]). Let $a$ be an archetypal arc. Define
\[ s_i(a) := \begin{cases} 1 & \text{if } a \text{ meets } v_i \\ 0 & \text{else} \end{cases} \]
for each $i = 1, ..., t$. Now set
\[ h(a) := (s_1(a), ..., s_t(a)) \]
Note that the right side is a sequence of $t$ numbers in $\{0, 1\}$. Interpret these sequences as binary numbers read from left to right and order them accordingly. For example, 000 corresponds to 0, 100 corresponds to 1, 010 corresponds to 2, 110 corresponds to 3 and so on. Then $h$ is a Morse function building up $\mathcal{AC}_2$ from $\mathcal{AC}_2^0$ since archetypal arcs with $h$-value equal to $(0, ..., 0)$ are exactly the archetypal arcs in $\mathcal{AC}_2^0$ and two archetypal arcs with the same $h$-value different from $(0, ..., 0)$ are not connected by an edge.

We want to inspect the descending links with respect to this Morse function $h$. Let $a$ be an archetypal arc with Morse height greater than $(0, ..., 0)$. Find the greatest $\tau \in \{1, ..., t\}$ such that $s_\tau(a) = 1$. It is not hard to prove that $lk_\tau(a)$ is
the full subcomplex of $\mathcal{AC}_2$ spanned by archetypal arcs disjoint from $a$ and not meeting any $v_i$ with $i > \tau$. Let $X'$ be the color word which is obtained from $X$ by removing the colors corresponding to nodes which are contained in $a$ and to the nodes $v_i$ with $i > \tau$. Then we see that $lk_2(a)$ is isomorphic to $\mathcal{AC}_2(C, A; X')$ with an additional puncture corresponding to $a$ and further additional punctures corresponding to the nodes $v_i$ with $i > \tau$. By induction, it follows that $lk_2(a)$ is $\nu_{k_2}(lX')$-connected. Denote by $\lfloor a \rfloor$ the length of $a$, i.e. the number of nodes it meets. Then we can estimate

$$\begin{align*}
lk' &= lX - (la + t - \tau) \\
&= lX - la - t + \tau \\
&\geq lX - ma - t + \tau \\
&\geq lX - ma - t + 1 \\
&\geq lX - 2ma + 1
\end{align*}$$

Thus, $lk_1(a)$ is $\nu_{k_2}(lX - 2ma + 1)$-connected. Consequently, by the Morse method, the connectivity of the pair $(\mathcal{AC}_2, \mathcal{AC}_2^0)$ is $\nu_{k_2}(lX - 2ma + 1) + 1 = \nu_{k_2}(lX)$ because of $k_2 = 2ma - 1$.

The second step of the proof consists of showing that the inclusion $\iota: \mathcal{AC}_2^0 \hookrightarrow \mathcal{AC}_2$ induces the trivial map in $\pi_m$ for $m \leq \nu_{k_2}(lX)$. It then follows from the long exact homotopy sequence of the pair $(\mathcal{AC}_2, \mathcal{AC}_2^0)$ that $\mathcal{AC}_2$ is $\nu_{k_2}(lX)$-connected which completes the proof in the case $d = 2$.

Let $\varphi: S^m \rightarrow \mathcal{AC}_2^0$ be a map with $m \leq \nu_{k_2}(lX)$. We have to show that $\psi := \varphi \circ \iota: S^m \rightarrow \mathcal{AC}_2$ is homotopic to a constant map. Think of $S^m$ as the boundary of a $(m+1)$-simplex. By simplicial approximation [14, Theorem 3.4.8] we can subdivide $S^m$ further and homotope $\varphi$ to a simplicial map. So we will assume in the following that $\varphi$ is simplicial. Next, we want to apply [5] Lemma 3.9 in order to subdivide $S^m$ further and homotope $\psi$ to a simplexwise injective map. This means that whenever vertices $v \neq w$ in $S^m$ are joined by an edge, then $\psi(v) \neq \psi(w)$. To apply the lemma, we have to show that the link of every $k$-simplex $\sigma$ in $\mathcal{AC}_2$ is $(m - 2k - 2)$-connected. So let $a_0, \ldots, a_{k_2}$ be pairwise disjoint archetypal arcs representing a $k$-simplex $\sigma$. The link of this simplex is the full subcomplex spanned by the archetypal arcs which are disjoint from every $a_i$. Deleting every color corresponding to nodes which are contained in one of the $a_i$ from $X$, we obtain a color word $X'$ and it is easy to see that the link of $\sigma$ is isomorphic to $\mathcal{AC}_2(C, A; X')$ with one additional puncture for each $a_i$. By induction, we obtain that $lk(\sigma)$ is $\nu_{k_2}(lX')$-connected. We have the estimate $lX' \geq lX - (k + 1)m_a$ and thus

$$\begin{align*}
\nu_{k_2}(lX') &\geq \nu_{k_2}(lX - (k + 1)m_a) \\
&\geq \left| \frac{lX - (k + 1)m_a - m_r}{2m_a - 1} \right| - 1 \\
&= \left| \frac{lX - m_r - (k + 1)m_a}{2m_a - 1} \right| - 1 \\
&\geq \left| \frac{lX - m_r - (2k + 2)(2m_a - 1)}{2m_a - 1} \right| - 1 \\
&\geq m - 2k - 2
\end{align*}$$

So the hypothesis of the lemma is satisfied and we will assume in the following that $\psi$ is simplexwise injective.
We now want to show that $\psi$ can be homotoped so that the image is contained in the star of $b$. Since the star of a vertex is always contractible, this will finish the proof. We will homotope $\psi$ by moving single vertices of $S^m$ step by step, eventually landing in the star of $\beta$. Consider the vertices $a_1, \ldots, a_l$ of $\psi(S^m)$ which do not yet lie in the star of $b$, i.e. which are not disjoint to $b$. Choose representing links $\alpha_i$ of $a_i$ and $\beta$ of $b$ such that the system of links $(\beta, \alpha_1, \ldots, \alpha_l)$ is in minimal position as in Lemma 2.1. Note the little subtlety that archetypal arcs may have the same underlying arc but are different because they have different labels. In this case, homotope them a little bit so that they intersect only at nodes. Note also that each $\alpha_i$ intersects $\beta$, but not at nodes since each $a_i$ comes from $\mathcal{A}_2$.

Now look at the intersection point $p$ of one of the $\alpha_i$ with $\beta$ which is closest to $v_1$ along $\beta$. Write $\alpha$ for the link which intersects $\beta$ at this point and $a$ for the corresponding arc.

Choose a vertex $x$ in $S^m$ which maps to $a$ via $\psi$. Define another link $\alpha'$ as follows: Let $j$ be such that the intersection point $p$ lies on the segment of $\beta$ connecting $v_j$ with $v_{j+1}$. Denote by $w < w'$ the nodes such that $p$ lies on the segment of $\alpha$ connecting $w$ with $w'$. Now push this segment of $\alpha$ along $\beta$ over the node $v_j$ such that $\alpha$ and $\alpha'$ bound a disk whose interior does not contain any puncture or node other than $v_j$.

Note that $\alpha'$ is still admissible. Denote by $a'$ the arc with link $\alpha'$ and the same label as $a$. Our goal is now to homotope $\psi$ to a simplicial map $\psi'$ such that $\psi'(x) = a'$ and $\psi'(y) = \psi(y)$ for all other vertices $y$. Iterating this procedure often enough, we arrive at a map $\psi^*$ homotopic to $\psi$ such that $\psi^*(y) \in \text{st}(b)$ for each vertex $y$. For example, the next step would be to move $x$ to the vertex $\alpha''$:

By simplexwise injectivity, no vertex of $lk(x)$ is mapped to $a$. Furthermore, a vertex of $\mathcal{A}_2$ in the image of $\psi$ disjoint to $a$ must also be disjoint to $a'$ because we have chosen $\alpha$ such that no other $\alpha_i$ intersects $\beta$ between $p$ and $v_1$. From these observations, it follows that

$$\psi(lk(x)) \subset lk(a) \cap lk(a')$$

This inclusion enables us to define a simplicial map $\psi': S^m \to \mathcal{A}_2$ with $\psi'(x) = a'$ and $\psi'(y) = \psi(y)$ for all other vertices $y$. Let $X'$ be the color word obtained from $X$ by removing all colors corresponding to nodes which are contained in $a$ or to
the node $v_j$. Then $lk(a) \cap lk(a')$ is isomorphic to $\mathcal{A}_2(C, A; X')$ with an additional puncture corresponding to the disk bounded by $\alpha \cup \alpha'$. Thus, by induction, it is $\nu_{\kappa_2}(lX')$-connected. We have the estimate $lX' \geq lX - m_a - 1$ and therefore

$$
\nu_{\kappa_2}(lX') \geq \nu_{\kappa_2}(lX - m_a - 1)
= \left\lfloor \frac{lX - m_a - 1 - m_r}{2m_a - 1} \right\rfloor - 1
= \frac{lX - m_r - m_a + 1}{2m_a - 1} - 1
\geq \frac{lX - m_r - 2m_a - 1}{2m_a - 1} - 1
= \nu_{\kappa_2}(lX) - 1
\geq m - 1
$$

Since $lk(x)$ is an $(m - 1)$-sphere, this connectivity bound for $lk(a) \cap lk(a')$ implies that the map $\psi_{lk(x)} : lk(x) \to lk(a) \cap lk(a')$ can be extended to the star $st(x)$ of $x$ which is an $m$-disk. So we obtain a map $\vartheta : st(x) \to lk(a) \cap lk(a')$ coinciding with $\psi$ on the boundary $lk(x)$. We can now homotope $\psi_{st(x)} \rel lk(x)$ to $\vartheta$ within $st(a)$ and further to $\psi'$ within $st(a')$. This finishes the proof of the theorem in the case $d = 2$.

2.1.2. The three-dimensional case. Choose an archetypal arc $b$ connecting the nodes $v_1 < \ldots < v_t$ and let $\mathcal{A}_3^0$ be the full subcomplex of $\mathcal{A}_3 := \mathcal{A}_3(C, A; X)$ spanned by the archetypal arcs which do not meet the nodes $v_i$.

With a very similar Morse argument as in the case $d = 2$ above, we can show that the pair $(\mathcal{A}_3, \mathcal{A}_3^0)$ is $\nu_{\kappa_3}(lX)$-connected.

Again, the second step consists of showing that the inclusion $\iota : \mathcal{A}_3^0 \to \mathcal{A}_3$ induces the trivial map in $\pi_m$ for $m \leq \nu_{\kappa_3}(lX)$. This is much easier in the case $d = 3$: Let $\varphi : S^m \to \mathcal{A}_3^0$ be a map and assume without loss of generality that it is simplicial. But then the map $\psi := \varphi \circ \iota : S^m \to \mathcal{A}_3$ already lies in the star $st(b)$ of $b$ since an archetypal arc not meeting any of the nodes $v_i$ is already disjoint to $b$. Consequently, $\psi$ can be homotoped to a constant map and this concludes the proof in the case $d = 3$.

2.1.3. The one-dimensional case. Choose an archetypal arc $b$ connecting the nodes $v_1 < \ldots < v_t$ such that the color word formed by the first $r$ nodes $w < v_1$ is reduced. Let $\mathcal{A}_1^0$ be the full subcomplex of $\mathcal{A}_1 := \mathcal{A}_1(C, A; X)$ spanned by the archetypal arcs which do not meet the nodes $v_i$. This condition is equivalent to not meeting any nodes $w \leq v_1$. These are simply the first $s$ nodes $w_1 < \ldots < w_s$ where $s = r + t$. In other words, $w_i$ is the point $i \in \mathbb{R}$ colored with the color $c_i$ from $X$.

For each archetypal arc $a$ not contained in $\mathcal{A}_1^0$ there exists a unique $1 \leq q \leq s$ such that $a$ meets $w_q$ but not $w_1, \ldots, w_{q-1}$. Denote this number by $h(a)$. Then $h$ is a Morse function building up $\mathcal{A}_1$ from $\mathcal{A}_1^0$. So let $a$ be such an archetypal arc. Let $X'$ be the color word obtained from $X$ by removing all colors corresponding to the nodes contained in $a$ and to the nodes $w_1, \ldots, w_{h(a)}-1$. Then the descending link $lk(a)$ is isomorphic to $\mathcal{A}_1(C, A; X')$ and by induction, it is $\nu_{\kappa_1}(lX')$-connected. We can estimate

$$
lX' = lX - (la + h(a) - 1)
= lX - la - h(a) + 1
\geq lX - m_a - h(a) + 1
\geq lX - m_a - (m_a + m_r - 1) + 1
$$
Thus, $lk_1(a)$ is $\nu_{\kappa_1}(lX - 2m_a - m_r)$-connected. Consequently, by the Morse method, the connectivity of the pair $(\mathcal{A}_1^0, \mathcal{A}_1)$ is

$$\nu_{\kappa_1}(lX - 2m_a - m_r) + 1 = \nu_{\kappa_1}(lX)$$

because of $\kappa_1 = 2m_a + m_r$.

Just as in the case $d = 3$, one can show that the inclusion $\iota: \mathcal{A}_1^0 \to \mathcal{A}_1$ induces the trivial map in $\pi_m$ for $m \leq \nu_{\kappa_1}(lX)$. This proves the theorem in the case $d = 1$.

**Remark 2.4.** The method used in the proof of [19, Proposition 6.12] yields the better connectivity $\nu_{\kappa}(lX)$ with $\kappa = m_a + m_r - 1$.

### 3. The Brown criterion

In this section, we want to recall a special case of the Brown criterion [3]. We give a different, more geometric proof using a construction of Lück [13, Lemma 4.1]. Let $\Gamma$ be a discrete group and $X$ be a $\Gamma$-CW-complex. Let $I_n$ be the indexing set for the equivariant $n$-dimensional cells. For $i \in I_n$ denote by $H_i$ the isotropy group of the $i$'th cell. When choosing explicit $H_i$-CW-complex models $EH_i$ for the universal covers of the classifying spaces of the isotropy groups $H_i$, then one can construct a free $\Gamma$-CW-complex $F(X)$ which is (non-equivariantly) homotopy equivalent to $X$.

The space $F(X)$ is the colimit over a nested sequence of spaces $F_n$ with

$$F_0 = \prod_{i \in I_0} \Gamma \times H_i EH_i$$

and

$$\prod_{i \in I_n}(\Gamma \times H_i EH_i) \times S^{n-1} \longrightarrow F_{n-1}$$

$$\prod_{i \in I_n}(\Gamma \times H_i EH_i) \times D^n \longrightarrow F_n$$

is a $\Gamma$-pushout for $n \geq 1$. In other words, the equivariant cell $\Gamma/H_i \times D^n$ in $X$ is replaced by the $\Gamma$-CW-complex $(\Gamma \times H_i EH_i) \times D^n$. More details can be found in the proofs of [13, Lemma 4.1 and Theorem 3.1]. The crucial observation is that if $X$ is of finite type (i.e. there are only finitely many equivariant cells in each dimension) and if each chosen $EH_i$ is of finite type, then also $F(X)$ is of finite type.

**Theorem 3.1.** Let $\Gamma$ be a discrete group and $X$ be a (not necessarily equivariantly) contractible $\Gamma$-CW-complex with isotropy groups of type $F_\infty$. Assume we have a filtration $(X_n)_{n \in \mathbb{N}}$ of $X$ such that each $X_n$ is a $\Gamma$-CW-subcomplex of finite type and such that the connectivity of the pairs $(X_n, X_{n-1})$ tends to infinity as $n \to \infty$. Then $\Gamma$ is of type $F_\infty$.

**Proof.** Choose models $EH_i$ of finite type. Using these models we can construct the blow-ups $F(X)$ and $F(X_n)$ for each $n$. These are free $\Gamma$-CW-complexes, each $F(X_n)$ is of finite type and $(F(X_n))_{n \in \mathbb{N}}$ is a filtration of $F(X)$. Furthermore, each $F(X_n)$ is homotopy equivalent to $X_n$ and $F(X)$ is homotopy equivalent to $X$. We have a commutative diagram

$$\cdots \longrightarrow X_n \longrightarrow X_{n+1} \longrightarrow \cdots$$

$$\cdots \longrightarrow F(X_n) \longrightarrow F(X_{n+1}) \longrightarrow \cdots$$
where the vertical arrows are homotopy equivalences and the horizontal arrows are inclusions. The naturality of the long exact sequence of a pair of spaces and the five lemma imply

$$\pi_k(F(X_{n+1}), F(X_n)) \cong \pi_k(X_{n+1}, X_n)$$

Thus, also the connectivity of the pairs \((F(X_{n+1}), F(X_n))\) tends to infinity. These considerations imply that we can, without loss of generality, assume that the action of \(\Gamma\) on \(X\) is free.

Under this assumption, we now prove that \(\Gamma\) is of type \(F_\infty\). Set \(Y := X/\Gamma\) and \(Y_n := X_n/\Gamma\) and let \(y\) be some point in \(Y_0\). We have \(\pi_1(Y, y) \cong \Gamma\) since we assumed that \(\Gamma\) acts freely on \(X\). The projections \(p_n : X_n \rightarrow X_n/\Gamma = Y_n\) and \(p : X \rightarrow X/\Gamma = Y\) are covering maps, i.e. fiber bundles with discrete fibers.

Applying the long exact homotopy sequence of a fibration and using that \(Y\) is aspherical, i.e. all higher homotopy groups of \(Y\) vanish. Furthermore, we have (see e.g. [18, Theorem 4.6])

$$\pi_k(Y_{n+1}, Y_n) \cong \pi_k(X_{n+1}, X_n)$$

Consequently, the connectivity of the pairs \((Y_{n+1}, Y_n)\) goes to infinity as well. Spelled out, this means

$$\forall m \exists_n \forall_{k \geq n} (Y_{k+1}, Y_k) \text{ is } m\text{-connected}$$

Note that for a triple of spaces \(B \subset A \subset Z\), if \((A, B)\) and \((Z, A)\) are \(k\)-connected, then also \((Z, B)\) is \(k\)-connected. This follows from the long exact homotopy sequence of that triple. Furthermore, if \(Z_1 \subset Z_2 \subset \ldots\) is a nested sequence of spaces with colimit \(Z\) and each \((Z_{i+1}, Z_i)\) is \(k\)-connected, then also \((Z, Z_1)\) is \(k\)-connected. This follows from the previous remark and the fact that each map \(K \rightarrow Z\) from a compact space \(K\) (e.g. \(K = S^n\)) factors through some \(Z_i\). Applying this observation to our situation above, we obtain

$$\forall m \exists_n \forall_{k \geq n} (Y, Y_k) \text{ is } m\text{-connected}$$

and this means

$$\forall m \exists_n \forall_{k \geq n} \forall_{i \leq m} \pi_i(Y, Y_k) = 0$$

Using the long exact homotopy sequence of the pair \((Y, Y_k)\), this is equivalent to

$$\forall m \exists_n \forall_{k \geq n} (\forall_{i < m} \pi_i(Y_k, y) \cong \pi_i(Y, y) \text{ and } \pi_m(Y_k, y) \rightarrow \pi_m(Y, y))$$

For some fixed \(m\), we now know that there is a \(k\) large enough such that

$$\pi_1(Y_k, y) \cong \pi_1(Y, y) \cong \Gamma$$

$$\pi_2(Y_k, y) \cong \pi_2(Y, y) = 0$$

$$\vdots$$

$$\pi_m(Y_k, y) \cong \pi_m(Y, y) = 0$$

We now can take that \(Y_k\) (which has only finitely many cells in each dimension by assumption) and attach cells in dimension \(m+2\) and higher to kill all the homotopy groups above dimension \(m\). Call this space \(Y_k^+\). The isomorphisms above then tell us that \(Y_k^+\) is a classifying space for \(\Gamma\) with compact \((m+1)\)-skeleton. Thus, \(Y_k^+\) is a witness that \(\Gamma\) is of type \(F_{m+1}\). Since \(m\) was arbitrary, it follows that \(\Gamma\) is of type \(F_\infty\) (see e.g. [11, Proposition 7.2.2]).
4. Smashing isomorphisms in categories

Recall that a connected groupoid is equivalent, as a category, to any of its automorphism groups. Consequently, a connected groupoid is contractible if and only if its automorphism groups are trivial. This is the case if and only if there is exactly one isomorphism between any two objects.

Let $C$ be a category and $G \subset C$ a subcategory which is a disjoint union of contractible groupoids. We define the quotient category $C/G$ as follows: The objects of $C/G$ are equivalence classes of objects of $C$ where we say that $X \sim Y$ are equivalent if there is an isomorphism $X \to Y$ in $G$. Note that such an isomorphism is unique since each component of $G$ is contractible. We define $\text{Hom}_{C/G}([X],[Y]) := \{ A \to B \in C \mid A \in [X], B \in [Y] \}/\sim$ where two elements $(A \to B) \sim (A' \to B')$ in the set are defined to be equivalent if the diagram

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
| & \downarrow & | \\
G & \ni & G \\
| & \downarrow & | \\
A' & \longrightarrow & B'
\end{array}
\]

commutes. Let $[\alpha: A \to B]$ and $[\beta: C \to D]$ be two composable arrows, i.e. $[B] = [C]$, then there is a unique isomorphism $\gamma: B \to C$ in $G$ and one defines $[\alpha: A \to B] * [\beta: C \to D] := [\alpha \gamma \beta: A \to D]$

Set $\text{id}_{[X]} = [\text{id}_X]$. One easily checks that $C/G$ is a well defined category.

**Remark 4.1.** Observe that if $X \to Y$ is an arrow in $C/G$ and representatives $X$ and $Y$ have been chosen for $X$ and $Y$, then there is a unique arrow $X \to Y$ representing $X \to Y$.

**Remark 4.2.** Let $X$ and $Y$ be two objects in $C/G$. Fix some object $X_0$ representing $X$. Then the arrows $X \to Y$ in $C/G$ are in one to one correspondence with arrows $X_0 \to Y$ in $C$ modulo isomorphisms in $G$ on the right. Likewise, if we fix some object $Y_0$ representing $Y$, then arrows $X \to Y$ in $C/G$ are in one to one correspondence with arrows $X \to Y_0$ in $C$ modulo isomorphisms in $G$ on the left.

**Proposition 4.3.** The canonical projection $p: C \to C/G$ is a homotopy equivalence.

**Proof.** For each object in $C/G$ choose a representing object in $C$. This defines a functor $f: C/G \to C$ by noting that for any arrow $\alpha$ in $C/G$ and any choice of representing objects $A$ and $B$ for the domain and codomain of $\alpha$ respectively, there is a unique arrow $A \to B$ representing $\alpha$. We have $f \circ p = \text{id}_{C/G}$ and we will show that $f$ is a homotopy equivalence. It then follows that also $p$ is a homotopy equivalence.

Let $Y$ be an object in $C$. We show that $Y \downarrow f$ has an initial object and thus is contractible. It then follows from Quillen’s Theorem A [19 Theorem 2.2] that $f$ is a homotopy equivalence. There is a unique isomorphism $\gamma: Y \to f([Y])$ in $G$ and the data $([Y], \gamma)$ is an object in $Y \downarrow f$. We want to show that it is initial. Let $([Z], \alpha)$ be another object in $Y \downarrow f$. Then $\mu := [\gamma^{-1} \alpha]: [Y] \to [Z]$ gives an arrow
(\([Y],\gamma\) \rightarrow ([Z],\alpha)) because \(f(\mu) = \gamma^{-1}\alpha\) and thus the triangle

\[
\begin{array}{c}
f([Y]) \\
\downarrow \gamma \\
Y \end{array}
\begin{array}{c}
f(\mu) \\
\downarrow \alpha \\
f([Z])
\end{array}
\]

commutes. Furthermore, since \(\gamma\) is an isomorphism, \(f(\mu)\) is the only arrow such that the triangle commutes. Since \(f\) is faithful, also \(\mu\) is the unique arrow such that the triangle commutes and thus represents the unique arrow \((\([Y],\gamma\) \rightarrow ([Z],\alpha))\). □

5. Statement of the main theorem

**Definition 5.1.** A planar resp. symmetric resp. braided operad \(O\) is called a **planar** resp. symmetric resp. braided operad with transformations if the category \(I(O)\) formed by the set \(C\) of colors as objects and the degree 1 operations as arrows is a groupoid.

For such an operad, a transformation is an arrow in \(S(O)\) of the form \([\sigma,X]\) where \(\sigma\) is an identity (planar case) or a colored permutation (symmetric case) or a colored braiding (braided case) and \(X = (X_1,...,X_n)\) is a sequence of operations of degree 1. The transformations form a groupoid which we call \(T(O)\).

**Definition 5.2.** Let \(O\) be a planar resp. symmetric resp. braided operad with transformations. We say that the degree 1 operations are well-behaved if the following conditions hold:

- [\(\mathcal{W}_1\)] If \(\theta\) is a higher degree operation (i.e. the degree is at least 2) and \(\alpha\) a non-identity degree 1 operation, then \(\theta * \alpha \neq \theta\).
- [\(\mathcal{W}_2\)] The groupoid \(I(O)\) is of type \(F^\infty_\infty\), i.e. the automorphism groups are of type \(F^\infty_\infty\). Here, type \(F^\infty_\infty\) means that the group and all of its subgroups are of type \(F^\infty_\infty\).
- [\(\mathcal{W}_3\)] For each higher degree operation \(\theta\) there exist finitely many operations \(\psi_1,...,\psi_k\) such that the following holds: For each degree 1 operation \(\alpha\) with \(\theta * \alpha\) defined, there exists a transformation \(\gamma\) and an \(i\) with \(\theta * \alpha = \gamma * \psi_i\).

Note that the conditions \([\mathcal{W}_2]\) and \([\mathcal{W}_3]\) are automatically satisfied if there are only finitely many degree 1 operations.

We say that two operations \(\theta_1\) and \(\theta_2\) are transformation equivalent if there is a transformation \(\alpha\) such that \(\theta_2 = \alpha * \theta_1\). We denote by \(\mathcal{T}(O)\) the set of equivalence classes of operations modulo transformation. Note that two transformation equivalent operations have the same degree. Thus, we also have a notion of degree for elements in \(\mathcal{T}(O)\). We define a partial order on the set \(\mathcal{T}(O)\) as follows: Write \(\Theta_1 \leq \Theta_2\) if there is an operation \(\theta_1\) with \(\theta_1 = \Theta_2\) and operations \(\psi_1,...,\psi_n\) such that \((\psi_1,...,\psi_n) * \theta_1 \in \Theta_2\). Then, for every \(\theta_1\) with \(\theta_1 = \Theta_1\) there are operations \(\psi_1,...,\psi_n\) such that \((\psi_1,...,\psi_n) * \theta_1 \in \Theta_2\). It is not hard to prove that this relation is indeed a partial order. Note that the degree function on \(\mathcal{T}(O)\) strictly respects that order relation which means

\[\Theta_1 < \Theta_2 \implies \deg(\Theta_1) < \deg(\Theta_2)\]

**Definition 5.3.** Let \(O\) be a (symmetric/braided) operad with transformations. A higher degree transformation class \(\Theta\) (i.e. the degree is at least 2) is called very
elementary if there is no other higher degree transformation class \( \Psi \) with \( \Psi < \Theta \). Denote the set of very elementary classes by \( VE \).

Let \( \Theta, \Theta_1, \ldots, \Theta_k \in TC(O) \) be (not necessarily distinct) transformation classes. We say that \( \Theta \) is decomposable into the classes \( \Theta_i \) if we find operations \( \theta_i \in \Theta_i \) for \( i = 1, \ldots, k \) which can be partially composed (see [19, Remark 3.2]) in a certain way to an operation in \( \Theta \). It can be shown that any higher degree class in \( TC(O) \) decomposes into very elementary classes.

**Definition 5.4.** Let \( O \) be a (symmetric/braided) operad with transformation. We define a subset \( E_i \subset TC(O) \) for \( i \in \{0, 1, 2, \ldots\} \) inductively. Set \( E_0 := VE \). Now assume \( E_i \) has been constructed. If \( |E_i| \leq 1 \), set \( E_{i+1} = \emptyset \). Else, for each pair \( \Theta_1, \Theta_2 \) of elements in \( E_i \) with \( \Theta_1 \neq \Theta_2 \), define the set \( M_{i+1}(\Theta_1, \Theta_2) \) as follows: If \( \Theta_1 < \Theta_2 \) or \( \Theta_1 > \Theta_2 \), define \( M_{i+1}(\Theta_1, \Theta_2) \) to be the one element set consisting of the element \( \max\{\Theta_1, \Theta_2\} \). Else, define it to be the set of all the minimal elements in the set

\[
\{ \Theta \in TC(O) \mid \Theta_i < \Theta \text{ for } i = 1, 2 \}
\]

Now set

\[
E_{i+1} = \bigcup_{\Theta_1 \neq \Theta_2 \in E_i} M_{i+1}(\Theta_1, \Theta_2)
\]

and then \( E = \bigcup_{i=1}^{\infty} E_i \).

An operation in \( O \) is called (very) elementary if it is contained in a (very) elementary transformation class. We will call the elementary but not very elementary classes resp. operations strictly elementary.

**Definition 5.5.** \( O \) is finitely generated if \( VE \) is finite. It is of finite type if \( E \) is finite.

**Definition 5.6.** Let \( O \) be a (symmetric/braided) operad with transformations. An object \( X \) in \( S(O) \) is called reduced if no non-transformation arrow in \( S(O) \) has \( X \) as its domain. We call \( O \) color-tame if there are only finitely many colors and if the degree of all reduced objects is bounded from above.

Note that if \( O \) is monochromatic and there exists at least one operation of degree greater than 1, then it is automatically color-tame.

**Theorem 5.7.** Let \( O \) be a planar or symmetric or braided operad with transformations. Assume that \( O \) is color-tame. Assume further that at least one of the following holds:

A) \( O \) is finitely generated free and there are only finitely many operations of degree 1.

B) \( O \) is of finite type, satisfies the right cancellative calculus of fractions and the degree 1 operations are well-behaved.

Then for every object \( X \) in \( S(O) \) the operad group \( \pi_1(O, X) \) is of type \( F_\infty \).

If we start with a set \( C \) of colors, for each \( n \geq 2 \) and \( (n+1) \)-tuple \( (a_1, \ldots, a_n; b) \) of colors a certain set of operations \( S(\{a_1, \ldots, a_n; b\}) \) and a groupoid \( G \) with objects in one to one correspondence with the colors in \( C \), then we may form the free (symmetric/braided) operad with transformations \( O \) having \( C \) as its set of colors and such that \( S(\{a_1, \ldots, a_n; b\}) \subset O(\{a_1, \ldots, a_n; b\}) \) as well as \( I(O) = G \). An operad \( O \) as in the theorem is called free if it is isomorphic to such a free (symmetric/braided) operad with transformations.

**Conjecture 5.8.** The requirement color-tameness can be weakened to having only finitely many colors.
The next three sections will be devoted to the proof of this theorem. We will assume without loss of generality that the category $S := S(O)$ is connected. Otherwise, we can pass to the connected component of the object $X$. Furthermore, we abbreviate $\Gamma := \pi_1(O, X)$.

### 6. A contractible complex

As usual, the strategy to proof Theorem 5.7 is to apply the Brown criterion to a suitable contractible complex on which the group in question acts. In our case, this will be the following category: Consider the universal covering category $\mathcal{U} := U_X(S)$ of $S$ based at $X$ (see [19, Subsection 2.1]). In Lemmas 6.1 and 6.3 below we will show that the subgroupoid $G$ of $\mathcal{U}$ consisting of the transformations in $S$ (lifted to $\mathcal{U}$) is a disjoint union of contractible groupoids. We therefore can apply Section 4 and consider the quotient category $\mathcal{U}/G$. Recall that $\Gamma$ acts on $\mathcal{U}$ which is encoded in a functor $\Gamma \to \text{CAT}$ sending the unique object of $\Gamma$ to $\mathcal{U}$. One can easily see that this functor induces a functor $\Gamma \to \text{CAT}$ sending the unique object to $\mathcal{U}/G$. In other words, $\Gamma$ also acts on $\mathcal{U}/G$. More concretely, an arrow $f: X \to X$ in $\Gamma$ acts on an object $[g: X \to Y]$ of $\mathcal{U}/G$ from the right via $[g] \cdot f := [f^{-1}g]$. Moreover, we will see in Propositions 6.2 and 6.4 below that $\mathcal{U}$ and therefore $\mathcal{U}/G$ is contractible. So the first condition in the Brown criterion will be satisfied.

To prove the above claims, we start with case B) of Theorem 5.7.

**Lemma 6.1.** Let $O$ be a (symmetric/braided) operad with transformations. If $O$ satisfies the calculus of fractions and condition $[W_1]$ in Definition 5.2 is satisfied, then the transformations in the universal covering $\mathcal{U}_X(S)$ of $S := S(O)$ based at any object $X$ form a disjoint union of contractible groupoids.

**Proof.** We have to show that each transformation in $\mathcal{U}$ with the same domain and codomain is trivial. In other words, we have to show that each null-homotopic transformation $\gamma: Y \to Y$ in $S$ is trivial. So assume that $\gamma: Y \to Y$ is such a null-homotopic transformation in $S$. This means that we have a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\gamma} & Y \\
\downarrow{\text{id}_Y} & & \downarrow{\text{id}_Y} \\
Y & \xrightarrow{\gamma} & Y \\
\end{array}
$$

in $S$. This implies the existence of an arrow $\alpha$ with $\alpha \gamma = \alpha$. By precomposing with a suitable colored permutation or colored braiding, we can assume that $\alpha$ is just a sequence of operations. By pulling the operations of $\alpha$ through the permutation resp. braiding part of $\gamma$, one can see that the latter must be trivial. Then, using $[W_1]$, one deduces that also the degree 1 operations in $\gamma$ must be trivial. \qed

**Proposition 6.2.** If $O$ is a (symmetric/braided) operad satisfying the calculus of fractions, then the universal covering $\mathcal{U}_X(S)$ of $S := S(O)$ based at any object $X$ is contractible.

**Proof.** This follows from [19, Corollary 2.8]. \qed

Now we turn to case A) of Theorem 5.7. So let $O$ be a free (symmetric/braided) operad with transformations which is generated by a set $C$ of colors, a set $S$ of higher degree operations and a groupoid $I$ of degree 1 operations. It is very helpful to view arrows in $\pi_1(S)$, i.e. paths in $S$ modulo homotopy, as string diagrams (see
Subsection 2.6): Every arrow in $\pi_1(S)$ can be represented by a string diagram with higher degree operations in $S$ or degree 1 operations in $I$ as transistors and with strings labelled with the colors in $C$. The transistors may point in any direction (left or right). String diagrams where the transistors only point to the right (from the domain to the codomain) represent arrows in $S$. Note that we have the following homotopy
\[
\bullet \xrightarrow{\gamma} \bullet \sim \bullet \xleftarrow{\gamma^{-1}} \bullet \xrightarrow{\gamma} \bullet \sim \bullet \xrightarrow{\gamma^{-1}} \gamma \sim \bullet \xrightarrow{\gamma^{-1}} \bullet
\]
for isomorphisms $\gamma$ in categories. Consequently, we can assume that all degree 1 operations point to the right and that no degree 1 operation in the diagram is connected to another degree 1 operation.

For example, let $S = \{\theta, \psi\}$ with $\theta, \psi$ operations of degree 2, $|C| = 1$ and $I = \mathbb{Z}/2\mathbb{Z}$ the group with 2 elements. Then a string diagram representing an arrow $1 \to 5$ in $\pi_1(S)$ could look like the following one:

We can delete all the dipoles formed by higher degree operations in $S$ from the diagram and afterwards combine all the degree 1 operations in $I$ which then are connected by strings:

Repeating this procedure, we end up with a unique (up to isotopy in the appropriate dimension) string diagram which contains no dipoles, where all degree 1 operations point to the right and where no two degree 1 operations are connected by strings. We call this the reduced string diagram representing the given arrow in $\pi_1(S)$. The above diagram already is in reduced form.

An arrow in $\pi_1(S)$ is trivial if and only if its reduced string diagram is the trivial one (i.e. contains no operations and no non-trivial permutations or braidings). Equivalently, two arrows in $\pi_1(S)$ are equal if and only if their reduced string diagrams are equal (up to isotopy in the appropriate dimension). In particular, two arrows in $S$ are homotopic if and only if they are equal. This implies the following:

**Lemma 6.3.** Let $O$ be a (symmetric/braided) operad with transformations. If $O$ is free, then the universal covering $U_X(S)$ of $S := S(O)$ based at any object $X$ is a poset. In particular, the transformations in $U$ form a disjoint union of contractible groupoids.

A similar result as in the next proposition (with different terminology) can be found in [6, Theorem 4.4].
Proposition 6.4. Let $\mathcal{O}$ be a (symmetric/braided) operad with transformations. If $\mathcal{O}$ is free, then the universal covering $U_X(S)$ of $S := S(\mathcal{O})$ based at any object $X$ is contractible.

Proof. As usual, assume that $S$ is connected, else we pass to the connected component of $X$. Denote by $\mathcal{G}$ the subgroupoid of $U := U_X(S)$ consisting of all the transformations. Then, as pointed out above (Lemma 6.3), we can form the quotient $U/\mathcal{G}$ which is homotopy equivalent to $U$. We will show that $U/\mathcal{G}$ is contractible to prove the proposition.

We do so by introducing a Morse function on $U/\mathcal{G}$ and apply the Morse method for categories [19, Section 5]. Let $S$ be a set of higher degree operations which generates $\mathcal{O}$ freely (together with a groupoid of degree 1 operations). An object in $U$ is a path from $X$ to some other object modulo homotopy. As explained above, such an object is represented by a unique reduced string diagram having operations in $S$ or operations in $I(\mathcal{O})$ as transistors. An object in $U/\mathcal{G}$ is an object in $U$ modulo transformation. They are thus also represented by reduced string diagrams but two such diagrams are considered equal if they differ by a permutation or a braiding and by degree 1 operations in $I(\mathcal{O})$ on the right. For a reduced string diagram $D$, denote by $\#_i(D)$ the number of higher degree operations in $S$ pointing to the left and by $\#_r(D)$ the number of higher degree operations pointing to the right. These numbers are of course invariant under attaching transformations on the right, so we get well-defined numbers $\#_i(A)$ and $\#_r(A)$ for an object $A$ in $U/\mathcal{G}$. Now define

$$f(A) := (\#_i(A), \#_r(A))$$

on the objects of $U/\mathcal{G}$ and order the pairs on the the right lexicographically. This is a Morse function on $U/\mathcal{G}$ but not on $U$ and that’s why we have to pass to the quotient $U/\mathcal{G}$. For example, the following reduced string diagram

has Morse height $(5, 1)$. There is exactly one object with Morse height $(0, 0)$. Thus, we are building up $U/\mathcal{G}$ from a point. If we prove that each descending link $lk_i(A)$ is contractible, it follows that $U/\mathcal{G}$ is contractible.

So let $A$ be an object in $U/\mathcal{G}$ represented by a reduced diagram as above. Assume $f(A) > (0, 0)$, so there must be at least one operation of $S$ in $A$. We call such a higher degree operation in $A$ exposed if there is no other higher degree operation connected to that operation on the right. For example, the thick bordered operations in the above diagram are the exposed operations. Assume that there is at least one exposed operation pointing to the left (in the above diagram we have two). In this case, we will show that $lk_i(A)$ is contractible. In case there are only right pointing exposed operations, we observe that $lk_j(A)$ is contractible using similar arguments. Since $A$ is an object in the quotient $U/\mathcal{G}$, we can assume that there are no permutations or braidings and no degree 1 operations acting on the exposed operations from the right:
Fix such a representative of $A$ and denote by $Y$ the codomain of that representative. Arrows $A \to B$ in $U/G$ are in one to one correspondence with arrows $Y \to Z$ in $S$ modulo transformations acting from the right (Remark 4.2). We will represent these as reduced string diagrams with transistors in $S$ or $I(O)$ which all point to the right.

We now want to show that $lk \downarrow (A)$ is homotopy equivalent to a $(k-1)$-simplex where $k$ is the number of left pointing exposed arrows in $A$. The proof now proceeds in a very similar fashion as in the last two paragraphs of the proof of [19, Theorem 6.8]: First consider the full subcategory $Z$ in $lk\downarrow (A)$ spanned by the classes $\beta$ modulo right transformations where $\beta: Y \to Z$ is a tensor product of identities and higher degree operations in $S$ which all form dipoles with the left pointing exposed operations. This subposet apparently realizes a $(k-1)$-simplex with top face represented by the class $[\beta_0]$ where $\beta_0$ contains all the left pointing exposed operations.

The proof is finished by showing that the inclusion $Z \to lk\downarrow (A)$ is a homotopy equivalence using Quillen’s Theorem A [19, Theorem 2.2]: An object in $lk\downarrow (A)$ is an arrow $\beta: Y \to Z$ modulo right transformations such that at least one left pointing exposed operation forms a dipole with one of the operations in the string diagram representation of $\beta$. Let $\gamma: Y \to Z'$ be the arrow which is obtained from $\beta$ by removing all operations in the string diagram representation of $\beta$ which do not form dipoles with the left pointing exposed operations. In $lk\downarrow (A)$, there is exactly one arrow $[\gamma] \to [\beta]$, adding the removed operations, which represents an object in the comma category $Z\downarrow [\beta]$. This object is the terminal object in $Z\downarrow [\beta]$ and thus the inclusion $Z \to lk\downarrow (A)$ is a homotopy equivalence by Quillen’s Theorem A. □

7. Isotropy groups

We continue to verify the conditions in the Brown criterion for the action of $\Gamma$ on $U/G$. In this section we prove that $U/G$ is indeed a $\Gamma$-CW-complex by showing that group elements fixing a cell already fix these pointwise. Furthermore, we show that cell stabilizers are of type $F_\infty$. In the following, we abbreviate $\mathcal{T} := \mathcal{T}(O)$ and $\mathcal{I} := \mathcal{I}(O)$.

Lemma 7.1. The groupoid $\mathcal{T}$ formed by the transformations in $S$ is of type $F_\infty$, i.e. all the components of $\mathcal{T}$ are equivalent to groups of type $F_\infty$.

Proof. The groupoid $\mathcal{I}$ formed by the degree 1 operations is of type $F_\infty$ because it is a finite groupoid in case A) of the theorem and even of type $F_{\infty}^+$ by condition [22] in case B) of the theorem. The groupoid $\mathcal{T}$ is $\mathcal{Mon}(\mathcal{I})$ in the planar case, $\mathcal{Sym}(\mathcal{I})$ in the symmetric case and $\mathcal{Braid}(\mathcal{I})$ in the braided case (see [19, Subsection 2.6] for the definitions of these categories).

Choose a color in each component of $\mathcal{I}$. Let $Y$ be an object in $\mathcal{T}$. We can assume without loss of generality that $Y$ decomposes as a tensor product of chosen colors: $Y = c_1 \otimes ... \otimes c_k$. In the planar case we have

$$\text{Aut}_{\mathcal{Mon}(\mathcal{I})}(Y) = \text{Aut}_{\mathcal{T}}(c_1) \times ... \times \text{Aut}_{\mathcal{T}}(c_k)$$
and the claim follows because the $\text{Aut}_\mathcal{I}(c_i)$ are of type $F_\infty$. For the symmetric and braided case, first assume that all the colors $c_i$ are equal to one chosen color $c$. In the symmetric case, we then have

$$\text{Aut}_{\text{sym}}(\mathcal{I})(Y) = S_k \ltimes \text{Aut}_\mathcal{I}(c)^k$$

where $S_k$, the symmetric group on $k$ strands, acts by permutation of the factors.

More precisely, we have the group homomorphism

$$\varphi: S_k \rightarrow \text{Aut}(G^k) \quad \sigma \mapsto [(g_1, \ldots, g_k) \mapsto (g_{1\sigma^{-1}}, \ldots, g_{k\sigma^{-1}})]$$

which gives a right action of $S_k$ on $G^k$ by the definition $g \cdot \sigma = \varphi(\sigma g \varphi)$. The multiplication in the semidirect product $S_k \ltimes G^k$ is then given by

$$(\sigma, g) \cdot (\sigma', g') := (\sigma \sigma', (g \cdot \sigma') * g')$$

Since $S_k$ is a finite group, it is also of type $F_\infty$. Since semidirect products of type $F_\infty$ groups are of type $F_\infty$ [11 Exercise 1 on page 176 and Proposition 7.2.2], it follows that $\text{Aut}_{\text{sym}}(\mathcal{I})(Y)$ is of type $F_\infty$. In the braided case we have

$$\text{Aut}_{\text{braid}}(\mathcal{I})(Y) = B_k \ltimes \text{Aut}_\mathcal{I}(c)^k$$

where $B_k$, the braid group on $k$ strands, acts via permutation of the factors through the projection $B_k \rightarrow S_k$. The braid groups $B_k$ are of type $F_\infty$ [10 Theorem A]. As above, it follows that $\text{Aut}_{\text{braid}}(\mathcal{I})(Y)$ is of type $F_\infty$.

Remains to handle the case where not all the colors $c_i$ lie in the same component of $\mathcal{I}$. Denote by $B_k'$ the finite index subgroup of $B_k$ consisting of the elements $\sigma$ with the property $e_{c_i} = c_i$. Since different $c_i$ are not connected by an isomorphism in $\mathcal{I}$, we now have

$$\text{Aut}_{\text{braid}}(\mathcal{I})(Y) = B_k' \ltimes (\text{Aut}_\mathcal{I}(c_1) \times \ldots \times \text{Aut}_\mathcal{I}(c_k))$$

where $B_k'$ still acts by permuting the factors. This action is well-defined due to the definition of $B_k'$. Recall that a group is of type $F_\infty$ if and only if a finite index subgroup is of type $F_\infty$ [11 Corollary 7.2.4]. It follows that $B_k'$ and thus $\text{Aut}_{\text{braid}}(\mathcal{I})(Y)$ is of type $F_\infty$. The symmetric case can be treated similarly. \qed

**Lemma 7.2.** Let $\mathcal{P}$ be an object in $\mathcal{U}/\mathcal{G}$. Then the stabilizer subgroup $\text{Stab}_\mathcal{T}(\mathcal{P})$ is of type $F_\infty$.

**Proof.** Fix an arrow $p: X \rightarrow Y$ in $\pi_1(\mathcal{S})$ which represents the object $\mathcal{P}$ in $\mathcal{U}/\mathcal{G}$, i.e. $[p] = \mathcal{P}$. Let $\gamma \in \Gamma$ fix the point $\mathcal{P}$. This means $[\gamma^{-1}p] = [p] \cdot \gamma = [p]$. It follows that there is some transformation $t: Y \rightarrow Y$ such that $\gamma^{-1}pt = pt$. This is equivalent to $p^{-1}\gamma p = t^{-1}$ from which follows that $p^{-1}\gamma p$ is an element in $\text{Aut}_\mathcal{T}(Y)$. Conversely, for $\tau$ a transformation in $\text{Aut}_\mathcal{T}(Y)$, the element $p\tau p^{-1}$ is contained in $\text{Stab}_\mathcal{T}(\mathcal{P})$. Thus, the map

$$\text{Stab}_\mathcal{T}(\mathcal{P}) \rightarrow \text{Aut}_\mathcal{T}(Y) \quad \gamma \mapsto p^{-1} \gamma p$$

is an isomorphism with inverse given by $\gamma \mapsto p \tau p^{-1}$. Since $\text{Aut}_\mathcal{T}(Y)$ is of type $F_\infty$ by the previous lemma, the claim follows. Note that this isomorphism depends on the choice of $p$. However, two such choices differ by a transformation $\tau$ and the two corresponding isomorphisms differ by conjugation with $\tau$. \qed

In the following, the degree of an object $\mathcal{P}$ in $\mathcal{U}/\mathcal{G}$ is the degree of the object $Y$ when $p: X \rightarrow Y$ is an arrow in $\pi_1(\mathcal{S})$ representing $\mathcal{P}$.

**Lemma 7.3.** Consider a cell $\sigma$ in the geometric realization of $\mathcal{U}/\mathcal{G}$ and $\gamma \in \Gamma$. Then the following are equivalent:

- The element $\gamma$ fixes $\sigma$ set-wise.
- The element $\gamma$ fixed $\sigma$ point-wise.
- The element $\gamma$ fixes all the vertices of $\sigma$. 

In particular, $\mathcal{U}/\mathcal{G}$ is indeed a $\Gamma$-CW-complex.

Proof. A non-degenerate cell in the geometric realization of $\mathcal{U}/\mathcal{G}$ is a sequence of composable, non-trivial arrows in $\mathcal{U}/\mathcal{G}$

$$\mathcal{P}_0 \overset{\epsilon_0}{\longrightarrow} \mathcal{P}_1 \overset{\epsilon_1}{\longrightarrow} \cdots \overset{\epsilon_{k-1}}{\longrightarrow} \mathcal{P}_k$$

Note that the degree of the objects $\mathcal{P}_i$ decreases strictly because the arrows are assumed to be non-trivial (i.e. not represented by transformations). Assume $\gamma \in \Gamma$ permutes the vertices $\mathcal{P}_i$ in this cell non-trivially. Then there must be $j > 0$ such that $\mathcal{P}_j \cdot \gamma = \mathcal{P}_{j'}$ and $\mathcal{P}_{j-1} \cdot \gamma = \mathcal{P}_{j''}$ with $j' < j''$. The arrow $\epsilon_{j-1} \cdot \gamma\colon \mathcal{P}_{j''} \to \mathcal{P}_{j'}$ is non-trivial and thus $\deg(\mathcal{P}_{j''}) > \deg(\mathcal{P}_{j'})$, a contradiction. Thus, the vertices in the cell must already be fixed point-wise. Furthermore, we claim that in this case $\epsilon_i \cdot \gamma = \epsilon_i$ for each $i$. This follows directly from the definition of the action. This proves the lemma. $\square$

We say that two operations $\theta_1$ and $\theta_2$ are two-sided transformation equivalent if there are transformations $\alpha, \gamma$ such that $\theta_2 = \alpha \ast \theta_1 \ast \gamma$.

Proposition 7.4. The stabilizer subgroups of cells are of type $F_\infty$.

Proof. In the following, we restrict ourselves to the braided case. The planar and symmetric cases are similar and simpler.

We first choose a color in each connected component of $\mathcal{I}$. Next, we choose an operation in each two-sided transformation class such that the output of the chosen operation is a chosen color.

Consider a cell as in the proof of Lemma 7.3. Let $p_k\colon X \to Y_k$ be a representing path of $\mathcal{P}_k$ such that $Y_k = c_1 \otimes \ldots \otimes c_l$ is a tensor product of chosen colors. In the proofs of Lemmas 7.1 and 7.2 we have seen that there is an isomorphism

$$\varphi\colon \text{Stab}_\mathcal{F}(\mathcal{P}_k) \to B'_1 \ltimes \left( \text{Aut}_\mathcal{I}(c_1) \times \ldots \times \text{Aut}_\mathcal{I}(c_l) \right)$$

Choose some $\mathcal{P}_i =: \mathcal{P}$ different from $\mathcal{P}_k$ and observe the arrow $\epsilon\colon \mathcal{P} \to \mathcal{P}_k$ which is the composition of the $\epsilon_j$ in between. Choose a representing path $p\colon X \to Y$ of $\mathcal{P}$. Then there is exactly one arrow $\epsilon\colon Y \to Y_k$ representing $\epsilon$. One can compose $p$ and $p_k$ with transformations $\eta$ and $\lambda$ such that $\lambda\colon Y_k \to Y_k$ is a tensor product of degree 1 operations $\lambda_i\colon c_i \to c_i$ and $\epsilon$ is a tensor product of chosen operations. Write $p_k' = p_k\lambda$ for the new representative of $\mathcal{P}_k$. To $p_k'$ corresponds another isomorphism

$$\varphi'\colon \text{Stab}_\mathcal{F}(\mathcal{P}_k) \to B'_1 \ltimes \left( \text{Aut}_\mathcal{I}(c_1) \times \ldots \times \text{Aut}_\mathcal{I}(c_l) \right)$$

which differs from $\varphi$ by conjugation with $\lambda$. Denote the new representative $p\eta$ of $\mathcal{P}$ again by $p$.

Now let $\gamma \in \text{Stab}_\mathcal{F}(\mathcal{P}_k)$. Then $\gamma$ fixes also $\mathcal{P}$, i.e. $\mathcal{P} \cdot \gamma = \mathcal{P}$, if and only if

$$[p_k'e^{-1}] = [p] = [p] \cdot \gamma$$

$$= [\gamma^{-1}p_k'e^{-1}]$$

$$= [\gamma^{-1}p_k]$$

$$= [\gamma^{-1}p_k'e^{-1}p_k^{-1}]$$

$$= [p_k't^{-1}_\gamma e^{-1}]$$

where we have set $t_\gamma := p_k'^{-1} \gamma p_k'$, an element in the image of the isomorphism $\varphi'$. Therefore, we have to identify all such $t_\gamma$ which satisfy this equation. In other words, we look for all $t_\gamma$ such that there is a transformation $\tau$ with

$$et_\gamma = \tau e$$

Roughly speaking, we look for all $t_\gamma$ which can be pulled through $e$ from the codomain to the domain.
For better readability, we assume without loss of generality that the colors \( c_i \) are all equal to one color \( c \). In particular, the codomains of \( \varphi \) and \( \varphi' \) are of the form \( B_l \times \text{Aut}_I(c)^l \). Then write \( e = \theta_1 \otimes \ldots \otimes \theta_l \) where the \( \theta_i \) are chosen operations with codomain the chosen color \( c \). Define \( H_i \) to be the subgroup of \( \text{Aut}_I(c) \) consisting of elements \( h \) which can be pulled through the operation \( \theta_i \), i.e. there exists a transformation \( \tau \) with \( \theta_i \h = \tau \theta_i \). Furthermore, let \( B^*_l \) be the finite index subgroup of \( B_l \) consisting of the elements \( \sigma \) with the property \( \theta_i \sigma \theta_i = \theta_i \). Denote by \( \text{Stab}_I(P, P_k) \) the subgroup of \( \text{Stab}_I(P_k) \) which also fixes \( P \). Then the isomorphism \( \varphi' \) restricts to an isomorphism

\[
\varphi'_P : \text{Stab}_I(P, P_k) \to B^*_l \times (H_1 \times \ldots \times H_l) =: \Lambda_P
\]

where the subgroup \( B^*_l \) still acts via permutation of the factors and this is well-defined due to the definition of \( B^*_l \). The proof of this is straightforward and uses the fact that two two-sided transformation equivalent \( \theta_i \) must be equal.

Recall that \( \varphi' \) differs from \( \varphi \) by conjugation with \( \lambda \). So the image of \( \text{Stab}_I(P, P_k) \) under \( \varphi \) is its image under \( \varphi' \) conjugated with \( \lambda \). More precisely, \( \varphi \) restricts to an isomorphism

\[
\varphi_P : \text{Stab}_I(P, P_k) \to \lambda \Lambda_P \lambda^{-1} =: \Omega_P
\]

Consider the pure braid group \( P_l \) which is a finite index subgroup of \( B_l \). It is also a finite index subgroup of \( B^*_l \). Recall that we have \( \lambda = \lambda_1 \otimes \ldots \otimes \lambda_l \) where the \( \lambda_i \) are degree 1 operations. We have

\[
\lambda(P_l \times (H_1 \times \ldots \times H_l)) \lambda^{-1} = \lambda(P_l \times (H_1 \times \ldots \times H_l)) \lambda^{-1}
\]

\[
= P_l \times (\lambda_1 H_1 \lambda_1^{-1} \times \ldots \times \lambda_l H_l \lambda_l^{-1})
\]

\[
= P_l \times (H^P_1 \times \ldots \times H^P_l)
\]

where \( H^P_l := \lambda_1 H_1 \lambda_1^{-1} \) is isomorphic to \( H_i \). This is a finite index subgroup of \( \Omega_P \).

Remains to consider the case when \( \gamma \in \text{Stab}_I(P, P_k) \) fixes more than one additional vertex \( P_i \). For this we have to show that the intersection

\[
\Omega_{P_0} \cap \ldots \cap \Omega_{P_{k-1}} \subset B_l \times \text{Aut}_I(c)^l
\]

is of type \( F_\infty \). For better readability, we assume without loss of generality that \( k = 2 \). Then the last statement is equivalent to

\[
(P_l \times (H^P_1 \times \ldots \times H^P_0)) \cap (P_l \times (H^P_1 \times \ldots \times H^P_0)) = P_l \times (H^P_1 \times \ldots \times H^P_0)
\]

being of type \( F_\infty \) since it is a finite index subgroup. This is true because \( P_l \) is of type \( F_\infty \) and the groups \( H^P_1 \cap H^P_0 \) are subgroups of \( \text{Aut}_I(c) \) which is finite in case A) of the theorem and of type \( F_\infty^+ \) by [29,2] in case B) of the theorem. This completes the proof of the proposition. \( \Box \)

8. Finite type filtration

To apply the Brown criterion to the \( \Gamma \)-CW-complex \( \mathcal{U}/\mathcal{G} \), we need a filtration by \( \Gamma \)-CW-subcomplexes \( \langle \mathcal{U}/\mathcal{G} \rangle_n \), which are of finite type. Recall that the degree function on \( S \) induce degree functions on \( \mathcal{U} \) and \( \mathcal{U}/\mathcal{G} \). Define \( S_n \) resp. \( \mathcal{U}_n \) resp. \( \langle \mathcal{U}/\mathcal{G} \rangle_n \) to be the full subcategories spanned by the objects of degree at most \( n \). Note that we have \( \mathcal{U}_n/\mathcal{G} = \langle \mathcal{U}/\mathcal{G} \rangle_n \). In the following, we want to show that \( \langle \mathcal{U}/\mathcal{G} \rangle_n \) only has finitely many \( \Gamma \)-equivariant cells in each dimension.

Choose one operation in each very elementary transformation class and denote the resulting set of operations by \( S' \). By the assumptions in Theorem 5.7, \( S' \) is a finite set. We extend \( S' \) to a finite set \( S \) of operations in the following way:
A) Let $S$ be the set of operations of the form $(\gamma_1, \ldots, \gamma_k) * \theta$ where $\theta \in S'$ of degree $k$ and $\gamma_1, \ldots, \gamma_k$ are degree 1 operations.

B) Choose for each $\theta \in S'$ a finite set $S_0$ of operations as in [20] and set $S = S' \cup \bigcup_{\theta \in S'} S_0$.

Denote by $\Omega$ the set of all identity operations together with all operations of degree $1$ which proves that there are only finitely many $\Gamma$-equivariant cells in $(\mathcal{U}/G)_n$. Again, the set $\Lambda$ is finite.

We claim that there is a surjective function

$$\Lambda^p \twoheadrightarrow \{p\text{-cells in } (\mathcal{U}/G)_n\}/\Gamma$$

which proves that there are only finitely many $\Gamma$-equivariant cells in $(\mathcal{U}/G)_n$. Let $(e_0, \ldots, e_{p-1}) \in \Lambda^p$. Choose a path $p_0: X \to \text{dom}(e_0)$. Define paths $p_k: X \to \text{dom}(e_k)$ by the composite $p_k := p_0 e_0 \ldots e_{k-1}$. The $p_i$ represent objects $P_i$ and the $e_i$ represent arrows $e_i: P_i \to P_{i+1}$ in $(\mathcal{U}/G)_n$. Thus, the sequence $e_0, \ldots, e_{p-1}$ gives a $p$-cell in $(\mathcal{U}/G)_n$. This $p$-cell surely depends on the choice of $p_0$ but two such choices give equivalent $p$-cells modulo the action of $\Gamma$. So we get a well-defined function as above.

Remains to show that this function is indeed surjective. Consider a $p$-cell in $(\mathcal{U}/G)_n$ in the form of a string

$$P_0 \xrightarrow{e_0} P_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{p-1}} P_p$$

of composable arrows in $(\mathcal{U}/G)_n$. For each $P_i$ we can choose representatives $P_i$ in $\mathcal{U}_n$. Then each $e_i$ is represented by a unique arrow $e_i: P_i \to P_{i+1}$ in $\mathcal{S}_n$. We now want to change these representatives so that each $e_i$ lies in $\Lambda$.

Start with the last arrow $e_{p-1} = [\sigma, \Theta]$. Let $S''$ be the set of operations of the form $\tau * \theta$ where $\tau$ is a transformation and $\theta \in S'$. In other words, $S''$ is the union of all the very elementary transformation classes. Each higher degree operation $\theta$ in the sequence $\Theta$ can be written, up to transformation equivalence, as a partial composition of operations in $S''$ (see the remarks after Definition 5.3). It follows that $\theta = s * \psi$ where $s$ is a transformation and $\psi$ is an operation decomposable into operations of the form $(\gamma_1, \ldots, \gamma_k) * \xi$ with $\xi \in S'$ and $\gamma_i$ of degree 1. Note that, in case A) of the theorem, $\psi$ is already decomposable into operations of $S$. In case B) of the theorem, using [20], we can further pull the degree 1 operations to the domain of $\theta$, starting with the rightmost degree 1 operations, and obtain $\theta = s * \psi$ where $s$ is a transformation and $\psi$ is an operation decomposable into operations of $S$. In any case, we now have $e_{p-1} = \tau * \Psi$ where $\tau$ is some transformation and $\Psi$ is simply a tensor product of identities or operations decomposable into operations of $S$. By changing the representatives $P_{p-1}$, $e_{p-1}$ and $e_{p-2}$ in their respective classes modulo the subgroupoid $\mathcal{G}$, we can assume that $\tau = \text{id}_{P_{p-1}}$ and thus that $e_{p-1}$ lies in $\Lambda$. We can now repeat this argument with $e_{p-2}$ and then with $e_{p-3}$ and so forth until we have changed each $e_i$ to lie in $\Lambda$. This proves surjectivity.

9. Connectivity of the filtration

It remains to show the connectivity statement in Brown’s criterion, i.e. we have to show that the connectivity of the pair $(\mathcal{U}/G)_n, (\mathcal{U}/G)_{n-1})$ tends to infinity as $n \to \infty$. To show this, we apply the Morse method for categories [19, Section 5].

The degree function on $\mathcal{U}/G$ is a Morse function and the corresponding filtration is exactly $(\mathcal{U}/G)_n$. Thus, we have to prove that the connectivity of the descending link $lk_{\lambda}(K)$ tends to infinity as the degree of the object $K$ tends to infinity. Note that the descending up link $\overline{lk}_{\lambda}(K)$ is always empty, so we have $lk_{\lambda}(K) = \overline{lk}_{\lambda}(K)$. 

Definition 9.1. An arrow \([\sigma, \Theta]\) in \(S\) is called (very) elementary if it is not a transformation and every higher degree operation in \(\Theta\) is (very) elementary. An arrow in \(U\) is called (very) elementary if the corresponding arrow in \(S\) is (very) elementary.

We want to promote these definitions to arrows in \(U/\mathcal{G}\). For this, we need the following lemma:

Lemma 9.2. Let \(\theta\) be a higher degree operation and \(\gamma\) be a degree 1 operation. Then the transformation class \([\theta]\) is (very) elementary if and only if the class \([\theta] \* \gamma := [\theta \* \gamma]\) is (very) elementary. In particular, the operation \(\theta\) is (very) elementary if and only if \(\theta \* \gamma\) is (very) elementary.

Proof. The main observation is that if \(\Theta, \Theta'\) are two transformation classes, then \(\Theta < \Theta'\) holds if and only if \(\Theta + \gamma < \Theta' + \gamma\) holds. This implies that \(\Theta \in V E\) if and only if \(\Theta + \gamma \in V E\) or, in other words, \(V E \* \gamma = V E\). Recall the notation in Definition 5.3. Then assuming \(E_i \* \gamma = E_i\) for \(i \geq 0\), one can show \(E_{i+1} \* \gamma = E_{i+1}\). It follows \(\Theta \in \mathcal{G}\) if and only if \(\Theta + \gamma \in \mathcal{G}\).

Definition 9.3. An arrow in \(U/\mathcal{G}\) is called (very) elementary if one and therefore all representing arrows are (very) elementary.

The data of an object in \(\text{lk}_k(K)\) consists of an object \(Y\) in \(U/\mathcal{G}\) with \(\deg(Y) < \deg(K)\) and an arrow \(\alpha: K \to Y\) in \(U/\mathcal{G}\). Now we define \(\text{Core}(K)\) to be the full subcategory of \(\text{lk}_k(K)\) spanned by the objects \((Y, \alpha)\) where \(\alpha\) is a very elementary arrow. Denote by \(\text{Corona}(K)\) the full subcategory of \(\text{lk}_k(K)\) spanned by the objects \((Y, \alpha)\) with \(\alpha\) an elementary arrow. So we have

\[
\text{Core}(K) \subset \text{Corona}(K) \subset \text{lk}_k(K)
\]

and we will study the connectivity of these spaces successively.

9.1. The core. In this subsection, we adopt the normal form point of view of [19, Subsection 3.1]: Arrows in \(S\) are always represented by a unique pair \((\sigma, \Theta)\) such that \(\sigma^{-1}\) is unpermuted resp. unbraided on the domains of the operations in the sequence \(\Theta\).

We say that two operations \(\theta_1\) and \(\theta_2\) are right transformation equivalent if there is a transformation \(\gamma\) such that \(\theta_2 = \theta_1 \* \gamma\).

The object \(K\) in \(U/\mathcal{G}\) is a class of objects in \(U\) modulo transformations. Fix some representing object \(K\). Then the objects in \(\text{Core}(K)\) are in one to one correspondence with pairs \((Y, a)\) where \(Y\) is an object in \(U\) with \(\deg(Y) < \deg(K)\) and \(a: K \to Y\) is a very elementary arrow in \(U\) modulo transformations on the codomain (compare with Remark 12). Choose one operation in each right transformation equivalence class and denote the resulting set of operations by \(R\). We choose the identity for a class of degree 1 operations so that the degree 1 operations in \(R\) are identities. Now define a very elementary \(R\)-arrow to be a very elementary arrow \((\sigma, \Theta)\) in \(S\) such that the operations in \(\Theta\) are elements of \(R\). Thus, \(\Theta\) is a tensor product of identities and at least one very elementary operation lying in \(R\).

This notion of very elementary \(R\)-arrows can be lifted to arrows in \(U\). Now the objects in \(\text{Core}(K)\) are in one to one correspondence with pairs \((Y, a)\) where \(Y\) is an object with \(\deg(Y) < \deg(K)\) and \(a: K \to Y\) is

- (planar case) a very elementary \(R\)-arrow
- (symmetric case) a very elementary \(R\)-arrow modulo colored permutations on the codomain
- (braided case) a very elementary \(R\)-arrow modulo colored braidings on the codomain
The equivalence relation modulo braidings on the codomain is called “dangling” in [5] because these objects may be visualized as a braiding where some strands at one end are connected by very elementary operations in \( R \), called “feet”, and these are allowed to dangle freely (see [5, Figure 9]).

Now let \( C \) be the set of colors of the operad \( O \). We define a set of archetypes \( A \) as follows: For each operation in \( R \) form an archetype with identifier this operation and with color word the domain of that operation. The object \( K \in \mathcal{U} \) is a path of arrows in \( \mathcal{S} \) modulo homotopy. It starts at the color word \( X \) and ends at some other color word \( T \). Consider the simplicial complex \( \mathcal{AC}_d(C, A; T) \) from Section 2. It can be seen as a poset of simplices with an arrow from a simplex \( \sigma \) to another simplex \( \sigma' \) if and only if \( \sigma \) is a face of \( \sigma' \).

**Proposition 9.4.** The category \( \text{Core}(K) \) is a poset and isomorphic, as a poset, to \( \mathcal{AC}_d(C, A; T) \) where \( d = 1 \) in the planar case, \( d = 2 \) in the braided case and \( d = 3 \) in the symmetric case.

**Proof.** We restrict our attention to the braided case, i.e. \( d = 2 \). The other two cases are much simpler.

Let \( \Lambda \) be an object of \( \text{Core}(K) \) in the form of a very elementary \( R \)-arrow \( K \to Y \) modulo dangling. Fix some very elementary \( R \)-arrow \( \lambda \) representing this class with the property that the colored braiding of that arrow is unbraided not only on the sets of strands connected to single operations but also on the set of strands connected to identity operations. Then arrows in \( \text{Core}(K) \) with domain \( \Lambda \) are in one to one correspondence with very elementary \( R \)-arrows \( \alpha \) in \( \mathcal{U} \), modulo dangling, such that the very elementary operations of \( \alpha \) only connect to identity operations of \( \lambda \) in the composition \( \lambda \ast \alpha \) (since compositions of very elementary operations are not very elementary any more). The following diagram, in which the gray triangles are identity operations, illustrates such a situation:

This observation implies that, if \( \alpha \) and \( \alpha' \) are two such very elementary \( R \)-arrows and are not equivalent modulo dangling, then also the composites \( \lambda \ast \alpha \) and \( \lambda \ast \alpha' \) are not equivalent modulo dangling. It follows that \( \text{Core}(K) \) is a poset.

The above considerations yield the following interpretation of the poset structure: We have \( \Lambda \to \Lambda' \) if and only if there is a very elementary \( R \)-arrow \( \lambda \) representing the dangling class \( \Lambda \) such that adding very elementary operations of \( R \) to loose strands of \( \lambda \) (i.e. strands connected to identity operations) gives a very elementary \( R \)-arrow representing the dangling class \( \Lambda' \).

We will consider an isomorphism of posets

\[
\text{comb}: \text{Core}(K) \to \mathcal{AC}_2(C, A; T)
\]

called “combing” as in [5 Section 4] and its inverse

\[
\text{weave}: \mathcal{AC}_2(C, A; T) \to \text{Core}(K)
\]
which we call “weaving”.

Start with an object \( \Lambda \) in Core(\( \mathcal{K} \)). As above, it is a very elementary \( R \)-arrow in normal form modulo dangling. Thus, it is represented by a colored braid with unbraided strands connected with very elementary operations in \( R \). Think of the domain of the braid as being fixed on the line \( L_1 := \{(x,0,1) \mid x \in \mathbb{R}\} \subset \mathbb{R}^3 \), the codomain as being fixed on the line \( L_0 := \{(x,0,0) \mid x \in \mathbb{R}\} \subset \mathbb{R}^3 \) and visualize the operations as straight lines in \( L_0 \) connecting the ends of the corresponding strands. Now “combing straight” the braid means moving around the ends of the braid in the plane \( P := \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3 \) such that the whole braid becomes unbraided. The segments representing the operations get deformed in \( P_0 \) this way and in fact become the archetypal arcs in \( \text{comb}(\Lambda) \). They are admissible because the braid was required to be unbraided on the domains of the operations. This process is visualized by [5, Figure 17]. Note that combing does not depend on the representative under dangling, so it is a well-defined map on the objects of Core(\( \mathcal{K} \)). It also respects the poset structures, so it is a map of posets.

Conversely, start with an archetypal arc system \( \mathcal{A} \). This is a priori embedded in \( \mathbb{R}^2 \) but embed it in \( \mathbb{R}^3 \) via the embedding \( \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3 \). Connect the nodes of the archetypal arc system with the line \( L_1 \) by straight lines parallel to the third component axis. The process of weaving first tries to separate the archetypal arcs by moving the nodes in the plane \( P \). Here, being separate means being separated by a straight line in \( P \) parallel to the second component axis. Also, the set of nodes which are not contained in an arc should be separated from the arcs. By doing these moves, the vertical lines connecting the nodes with the line \( L_1 \) become braided in a certain way. The separation process is always possible but the resulting braid is not unique (think of two nodes connected by an arc and turn around the arc several times). To make the resulting braid unique (up to dangling), we additionally require that the subbraid determined by an archetypal arc never becomes braided during the separation process. This can be achieved for example by the following additional movement rule: The nodes of an archetypal arc always have to stay on the same line \( L \) in \( P \) parallel to the first component axis. This line \( L \) may move up and down and the nodes of the archetypal arc may move left and right on \( L \) but they must never cross each other on \( L \). Then, when the archetypal arcs are separated from each other and from the isolated nodes, the property admissible of the archetypal arcs ensures that they can be homotoped to straight lines lying in \( L_0 \). The following figure visualizes this process:

Replacing the archetypal arcs by the identifier operations of the corresponding archetype yields a representative of \( \text{weave}(\mathcal{A}) \) and the class modulo dangling does not depend on the weaving process. Thus, we get a well-defined map on the objects of \( \mathcal{AC}_2(C, A; T) \). It also respects the poset structures, so it is a map of posets. □
It follows from the finiteness of $VE$ that the set of archetypes $A$ is of finite type and from the color-tameness of $\mathcal{O}$ that it is tame. More precisely, let $m_V$ be the largest degree of very elementary classes and $m_C$ be the smallest natural number greater than the degree of every reduced object in $S$. Then we can set $m_\alpha = m_V$ and $m_r = m_C$ in Theorem 9.3. We thus get the following

**Corollary 9.5.** Core($K$) is $\nu_d(\deg K)$-connected where

$$
\nu_1(l) := \left\lfloor \frac{l - m_C}{2m_V + m_C} \right\rfloor - 1
$$

$$
\nu_2(l) := \left\lfloor \frac{l - m_C}{2m_V - 1} \right\rfloor - 1
$$

$$
\nu_3(l) := \left\lfloor \frac{l - m_C}{2m_V - 1} \right\rfloor - 1
$$

Here, $d = 1$ corresponds to the planar case, $d = 2$ to the braided case and $d = 3$ to the symmetric case.

9.2. The corona. First note that $VE = E$ and therefore Core($K$) = Corona($K$) in case A) of the main theorem. So this subsection only applies to case B).

We build up Corona($K$) from Core($K$) using again the Morse method for categories. We then get a connectivity result for the corona from the connectivity result for the core. The idea is attributed to [10].

We assumed $\mathcal{O}$ to be of finite type, i.e. the set of elementary classes $E$ is finite. Let $m_E$ be the largest degree of elementary classes. An object in Corona($K$) is a pair $(Y, \alpha: K \to Y)$ where $\deg(Y) < \deg(K)$ and $\alpha$ is an elementary arrow in $U/G$. For $2 \leq k \leq m_E$ denote by $\#_{se}^k(\alpha)$ the number of strictly elementary operations of degree $k$ in any representative of $\alpha$. Define

$$
f((Y, \alpha)) := (\#_{se}^m(\alpha), \#_{se}^{m-1}(\alpha), ..., \#_{se}(\alpha), \deg(Y))
$$

Order the values of $f$ lexicographically. Then $f$ becomes a Morse function building up Corona($K$) from Core($K$). Define

$$
\mu_1(l) := \left\lfloor \frac{l - m_C}{2m_V + m_C + m_E} \right\rfloor - 2
$$

$$
\mu_2(l) := \left\lfloor \frac{l - m_C}{2m_V + m_E} \right\rfloor - 1
$$

$$
\mu_3(l) := \left\lfloor \frac{l - m_C}{2m_V + m_E} \right\rfloor - 1
$$

**Proposition 9.6.** For each object $(Y, \alpha)$ in Corona($K$) which is not an object in Core($K$), the descending link $lk_1(Y, \alpha)$ with respect to the Morse function $f$ above is $\mu_d(\deg K)$-connected.

From [19, Theorem 5.6] we get that Core($K$) and Corona($K$) share the same homotopy groups up to dimension $\mu_d(\deg K)$. We already know that Core($K$) is $\nu_d(\deg K)$-connected. Furthermore, we have $\nu_d(l) \geq \mu_d(l)$. Consequently, we get the following

**Corollary 9.7.** Corona($K$) is $\mu_d(\deg K)$-connected. In particular, its connectivity tends to infinity as $\deg(K) \to \infty$.

In the rest of this subsection, we give a proof of Proposition 9.6. We distinguish between two sorts of objects $(Y, \alpha)$ in Corona($K$) which are not objects in Core($K$): Such an object is called mixed if there is at least one very elementary operation in $\alpha$. It is called pure if there is no very elementary operation in $\alpha$. 
Lemma 9.8. Let $(\mathcal{Y}, \alpha)$ be mixed. Then $\overline{\text{lk}}_4(\mathcal{Y}, \alpha)$ and therefore $\text{lk}_4(\mathcal{Y}, \alpha)$ is contractible. In particular, Proposition 9.6 is true for mixed objects.

Proof. The data of an object in $\overline{\text{lk}}_4(\mathcal{Y}, \alpha)$ is $\Omega = ((\mathcal{L}, \beta_1), \beta_2)$ where $\mathcal{L}$ is an object in $U/G$, $\beta_1$ is an elementary arrow in $U/G$, $\beta_2$ is an arrow in $U/G$ such that $\beta_1 \beta_2 = \alpha$ and $(\mathcal{L}, \beta_1)$ forms an object in Corona($\mathcal{K}$) of strictly smaller Morse height than $(\mathcal{Y}, \alpha)$. Let $\Omega' = ((\mathcal{L}', \beta_1'), \beta_2')$ be another such object. An arrow $\Omega \rightarrow \Omega'$ is represented by an arrow $\delta: \mathcal{L} \rightarrow \mathcal{L}'$ such that $\beta_1 \delta = \beta_1'$ and $\delta \beta_2' = \beta_2$.

Choosing representatives for the objects and arrows in the diagram, it easily follows from the right cancellation property of $\mathcal{S}$ that $\overline{\text{lk}}_4(\mathcal{Y}, \alpha)$ is a poset.

Choose representatives $K$ and $Y$ of $\mathcal{K}$ and $\mathcal{Y}$. Then $\alpha$ is represented by a unique arrow $a: K \rightarrow Y$. We can choose $K$ such that $a$ is a tensor product of higher degree operations and identities. Let $a^v: K \rightarrow Y^v$ be the arrow obtained from $a$ by replacing all strictly elementary operations $\theta$ with $\deg(\theta)$ identity operations. Let $a^se$ be the arrow obtained from $a$ by replacing all very elementary operations by one identity operation each. An example of $a, a^v, a^se$ is pictured below. There, a white triangle is a placeholder for a strictly elementary operation. A black triangle indicates a very elementary operation. A straight horizontal line represents an identity operation.

Set $Y^v := [Y^v]$ and $a^v := [a^v]$ as well as $a^se := [a^se]$. Then $(Y^v, a^v)$ is an object in Core($\mathcal{K}$) and $a^se$ represents an arrow $(Y^v, a^v) \rightarrow (Y, \alpha)$ in Corona($\mathcal{K}$). Moreover, the pair $\Xi := ((Y^v, a^v), a^se)$ is an object in $\text{lk}_4(\mathcal{Y}, \alpha)$.

Let $\Omega = ((\mathcal{L}, \beta_1), \beta_2)$ be an object in $\overline{\text{lk}}_4(\mathcal{Y}, \alpha)$. We define another object $F(\Omega) = ((\mathcal{M}, \gamma_1), \gamma_2)$ of $\overline{\text{lk}}_4(\mathcal{Y}, \alpha)$ as follows: Choose a representative $L$ of $\mathcal{L}$ such that $\beta_2$ is represented by $b_2: L \rightarrow Y$ which is a tensor product of identities and higher degree operations. Then $\beta_1$ is represented by $b_1: K \rightarrow L$. Note that $b_1 b_2 = a$. Think of $b_1$ as splitting higher degree operations of $a$ into operations of smaller degree and of $b_2$ as merging them back to their original form. Now define the arrows $g_1: K \rightarrow M$ and $g_2: M \rightarrow Y$ to be the same splitting of $a$ with the only exception that no very elementary operation of $a$ is split. An example fitting to the example above is pictured below. There, a gray triangle is a placeholder for an elementary operation or a degree 1 operation, a blue triangle can be any operation and a dot on a straight horizontal line indicates a possibly non-trivial degree 1 operation.
Now set $\mathcal{M} := [M]$ and $\gamma_1 := [g_1]$ as well as $\gamma_2 = [g_2]$.

It is not hard to see that $\Omega \mapsto F(\Omega)$ extends to a functor $\mathcal{K}(\mathcal{Y}, \alpha) \rightarrow \mathcal{K}(\mathcal{Y}, \alpha)$ which means that whenever we have an arrow $\delta: \Omega \rightarrow \Omega'$, then there is an arrow $\Delta: F(\Omega) \rightarrow F(\Omega')$.

We also have arrows $\xi: \Xi \rightarrow F(\Omega)$ and $\iota: \Omega \rightarrow F(\Omega)$.

The arrow $\xi$ is represented by an arrow $x: \mathcal{Y} \rightarrow \mathcal{M}$ which satisfies $a^v x = g_1$ and $x g_2 = a^{se}$. The arrow $\iota$ is represented by $i: L \rightarrow M$ which satisfies $b_1 i = g_1$ and $i g_2 = b_2$. In the example from above, these arrows look as follows:
The claim of the proposition now follows from [19, item iii) in Subsection 2.4 applied to the functor $F$ and the object $\Xi$. \hfill $\Box$

**Lemma 9.9.** Let $(\mathcal{Y}, \alpha)$ be pure. Then $lk_1(\mathcal{Y}, \alpha)$ is $\mu_d(\text{deg} \ K)$-connected and Proposition 7.3 is true for pure objects.

**Proof.** Choose representatives $K$ and $Y$ of $\mathcal{K}$ and $\mathcal{Y}$ such that $\alpha: K \to Y$ representing $\alpha$ is a tensor product of higher degree operations and identities.

First observe the descending up link $\overline{\mathcal{K}}_i(\mathcal{Y}, \alpha)$. An object in $\overline{\mathcal{K}}_i(\mathcal{Y}, \alpha)$ is a pair $((L, \beta_1), \beta_2)$ with $f(L, \beta_1) < f(\mathcal{Y}, \alpha)$ and $\beta_1 \beta_2 = \alpha$. When choosing a representative $L$ of $\mathcal{L}$, we get unique representatives $b_1: K \to L$ of $\beta_1$ and $b_2: L \to Y$ of $\beta_2$ such that $b_1 b_2 = \alpha$. As in the proof of the previous lemma, $b_1$ can be interpreted as splitting higher degree operations of $a$ into operations of smaller degree and $b_2$ as merging them back to their original form. Denote by $A_i$ the full subcategory of $\overline{\mathcal{K}}_i(\mathcal{Y}, \alpha)$ spanned by the objects which only split the $i$'th higher degree operation in $a$. Denote by $n$ the number of higher degree operations in $a$. Observe now that when splitting operations in $a$ one by one, then we can also split all that operations at once. This observation reveals that $\overline{\mathcal{K}}_i(\mathcal{Y}, \alpha) = A_1 \circ \ldots \circ A_n$ is the Grothendieck join of the $A_i$, explained in [19, Subsection 2.7]. Note that the categories $A_i$ are all non-empty since all the higher degree operations in $a$ are elementary but not very elementary and splitting such a strictly elementary operation decreases the Morse height. Thus, $\overline{\mathcal{K}}_i(\mathcal{Y}, \alpha)$ is $(n-2)$-connected.

Now look at the descending down link $lk_i(\mathcal{Y}, \alpha)$. Objects are pairs $((L, \beta_1), \beta_2)$ with $f(L, \beta_1) < f(\mathcal{Y}, \alpha)$ and $\alpha \beta_2 = \beta_1$. When choosing a representative $L$ of $\mathcal{L}$, we get representatives $b_1: K \to L$ of $\beta_1$ and $b_2: Y \to L$ of $\beta_2$ such that $a b_2 = b_1$. Looking at the Morse function $f$ for the corona, one sees that the higher degree operations of $b_2$ must be very elementary operations which only compose with identity operations of $a$. At this point, we have to distinguish between the planar case on the one hand and the braided resp. symmetric case on the other.

We start with the braided resp. symmetric case: The arguments in the proof of Proposition 9.4 reveal that $lk_i(\mathcal{Y}, \alpha)$ is isomorphic to $\mathcal{A}_C(C, A; T')$ where $T'$ is the color word obtained from the codomain of $a$ after deleting the higher degree operations. Denote by $l$ the length of $T'$, i.e. the number of identity operations in $a$. Then we already know that $\mathcal{A}_C(C, A; T')$ is $\nu_d(l)$-connected (compare with Corollary 9.3). Consequently, the connectivity of the descending link $lk_i(\mathcal{Y}, \alpha) = \overline{\mathcal{K}}_i(\mathcal{Y}, \alpha) \ast lk_1(\mathcal{Y}, \alpha)$ is

$$n + \nu_d(l) = n + \left\lfloor \frac{1 - m_C}{2m_V - 1} \right\rfloor - 1$$
$$\geq n + \left\lfloor \frac{\text{deg} \ K - nm_E - m_C}{2m_V - 1} \right\rfloor - 1$$
$$\geq n + \left\lfloor \frac{\text{deg} \ K - nm_E - m_C}{2m_V + m_E} \right\rfloor - 1$$
$$= \left\lfloor \frac{\text{deg} \ K - m_C + 2m_V n}{2m_V + m_E} \right\rfloor - 1$$
$$\geq \left\lfloor \frac{\text{deg} \ K - m_C}{2m_V + m_E} \right\rfloor - 1$$
$$= \mu_d(\text{deg} \ K)$$

where we have used that $nm_E + l \geq \text{deg} \ K$. 

Now we turn to the planar case: An identity component in \( a \) is a maximal subsequence of identity operations. Let \( m \) be the number of identity components and denote by \( l_i \) for \( i = 1, \ldots, m \) the length of the \( i \)'th identity component. Denote by \( l \) the total number of identity operations in \( a \), i.e. the sum of the \( l_i \). Define \( B_i \) to be the full subcategory of \( \text{lk}_i(Y, \alpha) \) spanned by the objects which only add very elementary operations into the \( i \)'th identity component. Observe now that when adding very elementary operations into different identity components one by one, then we can also add all that operations at once. This reveals that \( \text{lk}_i(Y, \alpha) \) is the Grothendieck join of the \( B_i \). Note, though, when inspecting the direction of the arrows in \( \text{lk}_i(Y, \alpha) \), one sees that it is in fact the dual Grothendieck join. So we have

\[
\text{lk}_i(Y, \alpha) = B_1 \cdots B_m
\]

Similarly as in the braided resp. symmetric case, \( B_i \) is isomorphic to \( \text{AC}_1(C, A; T_i) \) where \( T_i \) is the color word obtained from the codomain of \( a \) after deleting all operations except the identity operations of the \( i \)'th identity component. The length of \( T_i \) is \( l_i \). Then we already know that \( \text{AC}_1(C, A; T_i) \) is \( \nu_1(l_i) \)-connected. Therefore, the connectivity of \( \text{lk}_i(Y, \alpha) \) is at least

\[
2m - 2 + \sum_{j=1}^{m} \nu_1(l_i)
\]

Thus, the connectivity of \( \text{lk}_i(Y, \alpha) \) is

\[
 n + 2m - 2 + \sum_{j=1}^{m} \nu_1(l_j) = n + m - 2 + \sum_{j=1}^{m} \left\lfloor \frac{l_j - m}{2m + m} \right\rfloor
\]

\[
\geq n - 2 + \sum_{j=1}^{m} \frac{l_j - m}{2m + m}
\]

\[
= n - 2 + \frac{1 - mm}{2m + m}
\]

\[
\geq n + \frac{\deg K - mmE - (n + 1)m}{2m + m}
\]

\[
\geq n + \frac{\deg K - mmE - mmC - m}{2m + m + E} - 2
\]

\[
= \frac{\deg K - mC + 2mE n}{2m + m + E} - 2
\]

\[
\geq \frac{\deg K - mC + 2mE}{2m + m + E} - 2
\]

\[
\geq \mu_1(\deg K)
\]

where we have used in the fourth step that \( m \leq n + 1 \) and \( mmE + 1 \geq \deg K \).

9.3. The whole link. In this last step, we show that the inclusion \( \text{Corona}(K) \subset \text{lk}_i(K) \) is a homotopy equivalence. It then follows from Corollary 9.7 that the connectivity of \( \text{lk}_i(K) \) tends to infinity as \( \deg(K) \to \infty \) which is what we wanted to show in order to finish the proof of Theorem 5.7. This step is analogous to the reduction to the Stein space of elementary intervals in [17]. We again apply the Morse method for categories to build \( \text{lk}_i(K) \) up from \( \text{Corona}(K) \). The Morse function on objects of \( \text{lk}_i(K) \) which do not lie in \( \text{Corona}(K) \) is given by

\[
f((Y, \alpha)) := -\deg(Y)
\]
We have $\text{lk}_k(\mathcal{Y}, \alpha) = \emptyset$ and thus $\text{lk}_k(\mathcal{Y}, \alpha) = \overline{\mathcal{E}}_k(\mathcal{Y}, \alpha)$ with respect to this Morse function. Similarly as in the proofs of Lemmas 9.8 and 9.9 we obtain

$$\overline{\mathcal{E}}_k(\mathcal{Y}, \alpha) = A_1 \circ \ldots \circ A_n$$

where the $A_i$ are full subcategories of $\overline{\mathcal{E}}_k(\mathcal{Y}, \alpha)$ spanned by the objects which correspond to splitting exactly one of the $n$ higher degree operations in a representative $\alpha$ of $\alpha$. At least one of these operations must be non-elementary since $(\mathcal{Y}, \alpha)$ is not an object in $\text{Corona}(\mathcal{K})$. Without loss of generality, assume that $A_1$ corresponds to such a non-elementary operation. If we show that $A_1$ is contractible, it follows that $\overline{\mathcal{E}}_k(\mathcal{Y}, \alpha)$ is contractible. Thus, we are building $\text{lk}_k(\mathcal{K})$ up from $\text{Corona}(\mathcal{K})$ along contractible descending links and it follows from [19, Theorem 5.6] that the inclusion $\text{Corona}(\mathcal{K}) \subset \overline{\mathcal{E}}_k(\mathcal{K})$ is a homotopy equivalence. That $A_1$ is contractible follows from Proposition 9.12 below. But first we need a general lemma concerning elementary transformation classes:

**Lemma 9.10.** Let $\mathcal{O}$ be a (symmetric/braided) operad with transformations. For each non-elementary transformation class $\Omega$ with $\deg(\Omega) > 1$, the set

$$\Omega := \{ \Theta \in E \mid \Theta \prec \Omega \}$$

has a greatest element.

**Proof.** Recall that $\Theta_1 < \Theta_2$ implies $\deg(\Theta_1) < \deg(\Theta_2)$ for transformation classes $\Theta_1, \Theta_2$. This will be important for the proof.

Let $S$ be any set of transformation classes. Assume that $\Psi$ is a class with $\Sigma < \Psi$ for all $\Sigma \in S$. We claim that there is a minimal operation $\Pi$ in the set $\{ \Theta \in \mathcal{T}(\mathcal{O}) \mid \forall \Sigma \in S \, \Sigma < \Theta \}$ such that $\Pi \leq \Psi$. If $\Psi$ is minimal, then we can set $\Pi = \Psi$. If it is not minimal, there must be another class $\Psi'$ with $\Sigma < \Psi' < \Psi$ for all $\Sigma \in S$. Then $\Psi'$ has strictly smaller degree than $\Psi$. If we repeat this argument with $\Psi'$, we have to end up with a minimal element $\Pi$ at some time, because the degree function is bounded below. This $\Pi$ surely satisfies $\Pi \leq \Psi$.

Now we find the greatest element in $\Omega$. Recall the notation in Definition 9.8. For each $i$, set $E_i^+ = E_i \cap \Omega$. We claim: There exists exactly one $i_0$ such that $|E^+_j| > 1$ for $j < i_0$, $|E^+_j| = 1$ and $E^+_j = \emptyset$ for $j > i_0$ and the unique element in $E^+_i$ is the greatest element in $\Omega$. First, it is clear that $E^+_0 \neq \emptyset$. Note that either all but finitely many of the $E_i$ are empty or the sequence of numbers $d_i := \min\{\deg(\Theta) \mid \Theta \in E_i\}$ tends to infinity. But the degree of all the elements in all the $E^+_i$ is bounded by $\deg(\Omega)$. It follows that in any case there must be a $i_0$ such that $E^+_j = \emptyset$ for all $j > i_0$. Choose the $i_0$ such that it is minimal with respect to this property, i.e. $E^+_i \neq \emptyset$. Assume $|E^+_i| > 1$ and let $\Theta_1 \neq \Theta_2$ be two classes in this set. If $\Theta_1$ and $\Theta_2$ are comparable, e.g. $\Theta_1 < \Theta_2$, then $\Theta_2 \in E^+_i$, a contradiction. Else, write $S = \{\Theta_1, \Theta_2\}$. Recall that $\Theta_1, \Theta_2 < \Omega$. Thus, by the observations in the second paragraph, we find that there must be a minimal $\Pi \in M_{i_0+1}(\Theta_1, \Theta_2)$ with $\Pi \leq \Omega$. Since $\Omega$ is non-elementary, we have indeed $\Pi < \Omega$. By definition, this means $\Pi \in E^+_{i_0+1}$, a contradiction again. So we have indeed $|E^+_i| = 1$. Next, observe that for any $j$, if $E^+_j \neq \emptyset$, then $E^+_{j-1}$ consists of at least two elements. This follows directly from the definitions. Consequently, the same holds for the $E^+_j$ from this. It easily follows $|E^+_j| > 1$ for $j < i_0$.

We now use this to prove that the unique element $\Upsilon$ in $E^+_i$ is the greatest element in $\Omega$, i.e. $\Theta < \Upsilon$ whenever $\Theta \in E$ with $\Theta < \Omega$. Let $\Theta$ be such an element. If $\Theta \neq \Upsilon$, then there must be some $j < i_0$ such that $\Theta \in E^+_j$. There is another element $\Theta'$ in this $E^+_j$. If $\Theta$ and $\Theta'$ are comparable, then set $\Pi = \max\{\Theta_1, \Theta_2\}$. We then have $\Pi \in E^+_{j+1}$ and $\Theta \leq \Pi$. Else, the argument in the second paragraph applied to
$S = \{ \Theta, \Theta' \}$ and $\Psi = \Omega$ shows that there is $\Pi \in E_{j+1}^h$ with $\Theta < \Pi$. If $j + 1 = i_0$, that $\Pi$ must be $\Psi$ and we are done. Else, we repeat this process with $\Pi$ in place of $\Theta$ until we reach level $i_0$. This completes the proof. \qed

In the situation of Theorem 5.7 we can reinterpret the above lemma in terms of the category $U/G$. Note that the right cancellation property for $S$ is an assumption in case B) of the main theorem and a consequence of freeness in case A).

Furthermore, this property is passed on to $U$ and also to $U/G$.

**Lemma 9.11.** Let $\alpha : K \rightarrow Y$ be a non-elementary arrow in $U/G$ such that $\deg(K) = n > 1$ and $\deg(Y) = 1$. Then there is a unique pair $(\alpha_1, \alpha_2)$ of arrows in $U/G$ (called the maximal elementary factorization of $\alpha$) such that $\alpha_2$ is elementary, $\alpha_1 \alpha_2 = \alpha$ and such that the following universal property is satisfied: Whenever $(\beta_1, \beta_2)$ is another pair with $\beta_2$ elementary and $\beta_1 \beta_2 = \alpha$ (called an elementary factorization of $\alpha$), then there is a unique arrow $\gamma$ with $\alpha_1 \gamma = \beta_1$ and $\gamma \beta_2 = \alpha_2$.

\[ \begin{array}{c}
K \\
\alpha \\
\alpha_1 \\
Q \\
\alpha_2 \\
Y \\
\beta_1 \\
P \\
\beta_2 \\
\gamma \\
\end{array} \]

**Proof.** The uniqueness of $\gamma$ follows at once from the right cancellation property of $U/G$. Then, if also $(\beta_1, \beta_2)$ satisfies the universal property, $\gamma$ has to be an isomorphism. Since the only isomorphisms in $U/G$ are identities, it follows $(\alpha_1, \alpha_2) = (\beta_1, \beta_2)$, i.e. the first uniqueness statement.

Remains to prove existence of such a pair: Choose representatives $K, Y$ of $K, Y$. Then $\alpha$ is represented by a unique arrow $a : K \rightarrow Y$. Note that $a$ is just an operation since $\deg(Y) = 1$. Denote the transformation class of $a$ by $\Omega$. Its degree is $\deg(K) = n > 1$ and it is non-elementary by assumption. Thus, by Lemma 9.10 there is a greatest elementary class $\Theta$ with the property $\Theta < \Omega$. This implies that there is an operation $\theta \in \Theta$ and an arrow $q$ in $S$ such that $q \ast \theta = a$ in $S$. Define $Q := K \ast q$ as an object in $U$ and further $Q' := [Q]$ as an object in $U/G$. The arrows $q : K \rightarrow Q$ resp. $Q' : Q \rightarrow Y$ in $U$ represent arrows $\alpha_1$ resp. $\alpha_2$ in $U/G$ such that $\alpha_1 \alpha_2 = \alpha$ and $\alpha_2$ is elementary.

These two arrows satisfy the universal property: Let $b_1 : K \rightarrow P$ and $b_2 : P \rightarrow Y$ be the representatives of $\beta_1 : K \rightarrow P$ and $\beta_2 : P \rightarrow Y$. Obviously, the transformation class $[b_2]$ of $b_2$ is elementary and satisfies $[b_2] < [a] = \Omega$. Since $\Theta = [\theta]$ is the largest such class, we obtain $[b_2] \leq [\theta]$. This means that there is an arrow $g$ in $S$ such that $g \ast b_2 = \theta$ in $S$. If $g$ is interpreted as an arrow $Q \rightarrow P$ in $U$, then it represents an arrow $\gamma : Q \rightarrow Y$ in $U/G$ which satisfies $\gamma \beta_2 = \alpha_2$. It then follows from the right cancellation property of $U/G$ that also $\alpha_1 \gamma = \beta_1$. \qed

We now turn to the announced proposition which concludes the proof of the main theorem.

**Proposition 9.12.** Let $\alpha : K \rightarrow Y$ be a non-elementary arrow in $U/G$ such that $\deg(K) = n > 1$ and $\deg(Y) = 1$. Let $\mathcal{M}$ be the full subcategory of $K \downarrow (U/G)_{n-1}$ spanned by the objects $(Z, \beta : K \rightarrow Z)$ with $\deg(Z) > 1$ and

$$\mathcal{L} := \mathcal{M} \downarrow (Y, \alpha)$$
Proof. Note that the data of an object of \( L \) is a non-trivial factorization of \( \alpha \), i.e. a pair \((\alpha_1, \alpha_2)\) of arrows in \( U/G \) such that \( \alpha_1 \neq \text{id} \neq \alpha_2 \) and \( \alpha_1 \alpha_2 = \alpha \). An arrow from \((\alpha_1, \alpha_2)\) to \((\beta_1, \beta_2)\) is an arrow \( \gamma \) such that \( \alpha_1 \gamma = \beta_1 \) and \( \gamma \beta_2 = \alpha_2 \). From the right cancellation property of \( U/G \) it follows that \( L \) is a poset.

Apply Lemma 9.11 above to obtain a maximal elementary factorization \((\alpha_1, \alpha_2)\) of \( \alpha \). Note that \((\alpha_1, \alpha_2)\) is an object of \( L \) and the universal property says that this object is initial among the objects \((\beta_1, \beta_2)\) of \( L \) with \( \beta_2 \) elementary.

More generally, for an object \((\epsilon_1, \epsilon_2)\) of \( L \) with \( \epsilon_2 \) non-elementary, we can apply the lemma to obtain a maximal elementary factorization \((\epsilon_1^*, \epsilon_2^*)\) of \( \epsilon_2 \). Then define \( F(\epsilon_1, \epsilon_2) := (\epsilon_1 \epsilon_1^*, \epsilon_2 \epsilon_2^*) \) which is again an object in \( L \). If \( \epsilon_2 \) is already elementary, we set \( \epsilon_1^* = \text{id} \) and \( \epsilon_2^* = \epsilon_2 \) so that \( F(\epsilon_1, \epsilon_2) = (\epsilon_1, \epsilon_2) \).

We claim that \( F \) extends to a functor \( L \to L \). So let \((\epsilon_1, \epsilon_2)\) and \((\beta_1, \beta_2)\) be two objects of \( L \) and \( \gamma : (\epsilon_1, \epsilon_2) \to (\beta_1, \beta_2) \) an arrow in \( L \). We have to show that there is an arrow \( \varphi : F(\epsilon_1, \epsilon_2) \to F(\beta_1, \beta_2) \).

Observe first that if \( \epsilon_1^* = \text{id} \), then \( \gamma \beta_1^* \) is an arrow \( F(\epsilon_1, \epsilon_2) \to F(\beta_1, \beta_2) \) as required. Else, observe that the pair \((\gamma \beta_1^*, \beta_2^*)\) is another elementary factorization of \( \epsilon_2 \). Thus, by the universal property, we get a unique arrow \( \varphi \) such that \( \varphi \beta_2^* = \epsilon_2^* \) and \( \epsilon_1^* \varphi = \gamma \beta_1^* \). This amounts to an arrow \( F(\epsilon_1, \epsilon_2) \to F(\beta_1, \beta_2) \).

Since \( F(\epsilon_1, \epsilon_2) \) is an elementary factorization of \( \alpha \), we get an arrow \((\alpha_1, \alpha_2) \to F(\epsilon_1, \epsilon_2) \) for each object \((\epsilon_1, \epsilon_2)\) in \( L \). Furthermore, \( \epsilon_1^* \) clearly gives an arrow \((\epsilon_1, \epsilon_2) \to F(\epsilon_1, \epsilon_2) \). The claim of the proposition now follows from [19] item iii) in Subsection 2.4 applied to the functor \( F \) and the object \((\alpha_1, \alpha_2)\). \( \square \)

10. Applications

In this section, we want to apply Theorem 5.7 to the examples in [19] Section 4]. In the last two subsections, we briefly discuss further operads to which the main theorem can be applied and whose operad groups are maybe not so well-known as the others.

10.1. Free operads. Case \( A \) of the main theorem handles free planar resp. symmetric resp. braided operads with transformations. Note that the very elementary transformation classes in such an operad are in one to one correspondence with operations of the form \( \theta * \gamma \) where \( \theta \) is a higher degree free generator and \( \gamma \) is a degree 1 operation. Thus, assuming that there are only finitely many degree 1 operations (and consequently only finitely many colors), having only finitely many free generators is the same as being finitely generated in the sense of Definition 5.5.

It then only remains to check that the degree of reduced objects is bounded from above to ensure that the operad is color-tame.

The special case where there are no degree 1 operations besides the identities has been treated in [7]. In [7, Theorem 4.4] it is shown that the planar diagram groups
associated to a finite complete presentation of a finite semigroup are of type $F_\infty$. Assume without loss of generality that one side of each relation in this presentation consists of a single generator only (see the discussion in [19 Subsection 4.1]). The finiteness of the presentation then corresponds to finitely many colors and finitely many free generators in our language. Furthermore, if we have a finite complete presentation of a finite semigroup, then there are only finitely many reduced words (see [7 Subsection 4.2]). This implies color-tameness. Thus, also the conditions of our main theorem are satisfied. In [8, Theorem 1] it is shown that the symmetric diagram groups associated to a finite semigroup presentation with terminating rewrite system having only finitely many reduced objects is of type $F_\infty$. Assume again without loss of generality that one side of each relation in this presentation consists of a single generator only. Then the free symmetric operad corresponding to this presentation satisfies the conditions of our main theorem.

Note that Theorem 5.7 A) extends the results from [7, 8] since it also allows braidings and degree 1 isomorphisms in $O$.

10.2. Cube cutting operads. Consider a planar or symmetric cube cutting operad $O$. Since it is monochromatic and contains at least one higher degree operation, it is color-tame. It satisfies the cancellative calculus of fractions by [19 Proposition 4.1]. So we are in case B) of the main theorem. Since there are no degree 1 operations other than the identity, it remains to check that $O$ is of finite type.

Let $N_j$ be the set of coprime integers corresponding to the cuttings in direction $j \in \{1, \ldots, d\}$. An operation in $O$ can be visualized as a $(N_j)_j$-subdivided unit cube where the subbricks are labelled with numbers from 1 to $d$. A transformation class $\Theta$ of operations can be visualized as a $(N_j)_j$-subdivided unit cube without such labels. We have $\Theta \leq \Theta'$ if and only if $\Theta'$ can be obtained from $\Theta$ by further subdividing the subbricks. For each element $S = (S_1, \ldots, S_d) \subset 2^{N_1} \times \cdots \times 2^{N_d}$ of the product of the power sets such that $S \neq (\emptyset, \ldots, \emptyset)$, there is a transformation class $\Theta_S$ which is obtained by iteratively performing, for each $j \in \{1, \ldots, d\}$ and each $n \in S_j$, an $n$-cut in direction $j$ on every subbrick. The result is independent of the order of the cuts. These classes are exactly the elementary classes. If we have two such classes $\Theta_S$ and $\Theta_{S'}$ with $S \neq S'$, then $\Theta_{S \cup S'}$ is the unique minimal class $\Theta$ with $\Theta_S < \Theta > \Theta_{S'}$. Here, the union $S \cup S'$ is meant to be coordinatewise.

The very elementary classes are the classes $\Theta_S$ where $|S_i| = 1$ for exactly one $i$ and $|S_i| = 0$ for the others. The figure below pictures the elementary operations in the case $d = 2$, $N_1 = \{2\}$ and $N_2 = \{3\}$.

```
E_0
```
```
E_1
```
```
E_2
```
```
E_3
```

Consequently, the cube cutting operads are of finite type and the main theorem implies that the corresponding operad groups are of type $F_\infty$. 

This has been shown before in [10] for the groups $nV$ which correspond to the symmetric cube cutting operads with $d = 2$ and $N_1 = \{2\} = N_2$. Furthermore, the case $d = 1$ has already been treated in [17].

As already remarked in [19, Subsection 4.2], we can modify the cube cutting operads by allowing more than just affine linear maps as components of an operation. For example, we can allow such maps with an isometry of the unit cube precom-posed. We could allow the unit cubes to rotate by an angle of $\pi/2$, $\pi$ or $3\pi/2$ before they are coordinate-wise rescaled to fit into the subbricks of the subdivision. These operads differ from the normal cube cutting operads above by additional invertible degree 1 operations, namely exactly the allowed isometries of the unit cube. These operads still satisfy the cancellative calculus of fractions, are color-tame and of finite type. Since there are only finitely many isometries of the unit cube, properties $[W_2]$ and $[W_3]$ are satisfied. Property $[W_1]$ follows from the left cancellation property. Thus, also these modified cube cutting operads are of type $F_\infty$.

10.3. Local similarity operads. Let $\text{Sim}_X$ be a finite similarity structure on the compact ultrametric space $X$. Choose a ball in each $\text{Sim}_X$-equivalence class so that we obtain an explicit operad $\mathcal{O} := \mathcal{O}(\text{Sim}_X)$ as described in [19, Subsection 4.3]. This operad satisfies the cancellative calculus of fractions [19, Proposition 4.3]. So we are in case B) of the main theorem. Furthermore, it follows that it satisfies $[W_1]$. We assume that $\mathcal{O}$ has only finitely many $\text{Sim}_X$-equivalence classes of balls so that it has only finitely many colors. In this case, by the definition of a finite similarity structure, it has only finitely many degree 1 operations. This implies $[W_2]$ and $[W_3]$.

The transformation classes of operations are in one to one correspondence with chosen balls $B$ together with a subdivision of $B$ into proper subballs. The very elementary transformation classes are in one to one correspondence with chosen balls $B$ together with the subdivision into its maximal proper subballs. There are no elementary classes which are not very elementary, similarly as in the free case. Thus, there are as many elementary classes as there are colors. Since we assumed that there are just finitely many, it follows that $\mathcal{O}$ is of finite type.

In order to apply the main theorem, it remains to check that the degree of reduced color words is bounded from above. This is not true in general and thus has to be checked in each case separately.

This result has also been obtained in [9, Theorem 6.5]. The hypotheses in this theorem consists of demanding that the finite similarity structure possesses only finitely many $\text{Sim}_X$-equivalence classes of balls and of the property rich in simple contractions which is implied by the easier to state property rich in ball contractions. It is not hard to see that the latter property exactly means that the degree of reduced color words is bounded from above.

10.4. Cubes and triangles. We briefly discuss a modification to the cube cutting operads. This modification adds additional colors in form of other geometric objects other than the cube. For example, we could look at the unit cube and a right-angled triangle with catheti of length 1. These two shapes correspond to the two colors of the symmetric operad we want to define. We allow the following very elementary cuttings of the cube into cubes or triangles and of the triangle into triangles:

The very elementary operations are sequences of similarities (i.e. isometries up to a scaling factor) of the cube or the triangle to the components of these subdivisions. The subdivisions in turn correspond exactly to the very elementary transformation
classes. The only elementary class which is not very elementary is the following one:

\[
\begin{array}{c}
\text{X} \\
\text{X} \\
\text{X} \\
\text{X}
\end{array}
\]

It can be checked that the conditions of Theorem 5.7 B) are satisfied.

10.5. **Ribbon Thompson group.** First observe the free braided operad with transformation generated by a single color, the group $\mathbb{Z}$ as groupoid of degree 1 operations and a single binary operation. The components of the corresponding groupoid of transformations are the groups $B_n \ltimes \mathbb{Z}^n$. Think of elements of these groups as ribbons which can braid and twist. A single twist corresponds to a generator in $\mathbb{Z}$. Then we impose the following relation on this operad:

\[
\begin{array}{c}
\text{X} \\
\text{X} \\
\text{X} \\
\text{X}
\end{array} = \begin{array}{c}
\text{X} \\
\text{X} \\
\text{X} \\
\text{X}
\end{array}
\]

The triangle corresponds to the generating binary operation. The operations in this braided operad with transformations are in one to one correspondence with binary trees together with braiding and twisting ribbons attached to the leaves. It can be checked that the conditions of Theorem 5.7 B) are satisfied.

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