The pre-WDVV ring of physics and its topology

MARGARET A. READDY

Dedicated to Louis Billera in honor of his 60th Birthday

Abstract

We show how a simplicial complex arising from the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations of string theory is the Whitehouse complex. Using discrete Morse theory, we give an elementary proof that the Whitehouse complex $\Delta_n$ is homotopy equivalent to a wedge of $(n-2)!$ spheres of dimension $n-4$. We also verify the Cohen-Macaulay property. Additionally, recurrences are given for the face enumeration of the complex and the Hilbert series of the associated pre-WDVV ring.

1 Introduction

The moduli space of smooth $n$-pointed stable curves of genus $g$, denoted $\overline{M}_{g,n}$, was introduced by Deligne, Mumford and Knudsen [6, 16, 21] to give a natural compactification of Mumford’s [20] moduli space of nonsingular curves of genus $g$. A new construction of the genus 0 case is due to Keel [14] using blowups. Keel also gives the presentation for the generators and relations of the cohomology ring of $\overline{M}_{0,n}$.

The Associativity Equations in physics, also known as the WDVV Equations [18, Section 0.2], are a system of partial differential equations due to Witten, Dijkgraaf, Verlinde and Verlinde [7, 9, 34]. Their solutions are generating functions $\Phi$ having coefficients which encode “potential” or “free energy” and are related to understanding quantum gravity via the topological quantum theory approach [33].

In [17] Kontsevich and Manin determine the potential function $\Phi$ in the case of Fano manifolds. Underlying their work is the need to construct Gromov-Witten classes. These are linear maps between the cohomology of a projective algebraic manifold and the cohomology of the moduli space $\overline{M}_{g,n}$. Kontsevich and Manin develop a cohomological field theory in terms of the Gromov-Witten class language and the language of operads. From this theory and Keel’s presentation, they derive all the linear relations between homology classes of boundary strata of any codimension [17, Sections 6 and 7]. Since Keel’s presentation and the splitting axiom for the Gromov-Witten classes [18, equations (0.3) and (0.4)]) imply the WDVV equations (see [17] for details), it is natural to refer to the cohomology ring given by Keel’s presentation as the WDVV ring.

In a previous paper, the author studied an analogue of the WDVV ring [22]. This ring, known as the Losev-Manin ring, arises instead from the Commutativity Equations of physics [19]. Like

*To appear in The Ramanujan Journal, 10, No. 2 (2005), 269–281. 2000 Mathematics Subject Classification: Primary: 13F55, Secondary: 05E99 and 55P15. Keywords: WDVV equations, Keel’s presentation, moduli space, pre-WDVV complex, Morse matching, Cohen-Macaulay, Whitehouse complex.
the WDVV ring, the defining ideal for the Losev-Manin ring is composed of linear and quadratic relations. Since the quadratic relations determine the interesting behavior of the Losev-Manin ring, it is again natural to take a closer look at the quadratic relations defining the WDVV ring. We call this new ring the pre-WDVV ring.

In this paper we show the pre-WDVV ring can be realized as the Stanley-Reisner ring of the Whitehouse complex $\Delta_n$. This is a well-studied complex which goes back to work of Boardman [4]. See Section 2 for further references. As a result, many of our results are rediscoveries, but with simpler proofs. Since our overall goal is to find a natural combinatorial object corresponding to the minimal generators of the WDVV ring, we expect our current approach to better lead to its understanding.

In Section 3 we study the links of faces in the Whitehouse complex via a forest representation of the faces. We then describe five injective maps which map $\Delta_n$ into two blocks understanding the minimal generators of the WDVV ring, we expect our current approach to better lead to its simplification.

Finally, for $\Delta$ and $\Gamma$ two simplicial complexes having disjoint vertex sets $V$ and $W$, the join of $\Delta$ and $\Gamma$, denoted $\Delta \ast \Gamma$, is the simplicial complex on the vertex set $V \cup W$ consisting of the faces $\Delta \ast \Gamma = \{F \cup G : F \in \Delta \text{ and } G \in \Gamma\}$.

For completeness, we give the physicist’s definition of the WDVV ring [15, Section 0.10] based on Keel’s presentation [14, Theorem 1]. Let $n \geq 3$ and let $k$ be a field of characteristic zero. We say $\sigma$ is a stable 2-partition of $\{1, \ldots, n\}$ if $\sigma$ is an unordered partition of the elements $\{1, \ldots, n\}$ into two blocks $\sigma = S_1/S_2$ with $|S_i| \geq 2$. Denote by $P_n$ the set of stable 2-partitions of $\{1, \ldots, n\}$. For each $\sigma \in P_n$, the element $x_\sigma$ corresponds to a cohomology class of $H^*(\overline{M}_{0,n})$, that is, the cohomology ring of the moduli space of $n$-pointed stable curves of genus zero. Throughout it will be convenient for fixed $\sigma$ to think of $x_\sigma$ as simply an indeterminate. For $\sigma = S_1/S_2$ and $\tau = T_1/T_2$ from the index set $P_n$, let $a(\sigma, \tau)$ be the number of nonempty pairwise distinct sets among $S_i \cap T_j$, where $1 \leq i, j \leq 2$. Define the ideal $I_n$ in the polynomial ring $k[x_\sigma : \sigma \in P_n]$ by:

1. (linear relations) For $i, j, k, l$ distinct:

$$R_{ijkl} = \sum_{ijk\ell} x_\sigma - \sum_{kj\tau\ell} x_\tau,$$

2 Definitions and notation

Throughout we will assume familiarity with basic poset terminology and combinatorial concepts. An excellent reference for the uninitiated is Stanley’s text [25].

Let $V$ be a finite vertex set. A simplicial complex $\Delta$ is a collection of subsets of $V$ called faces satisfying $\emptyset \in V$ and $\{v\} \in \Delta$ for all $v \in V$, and if $F \subseteq G \in \Delta$ then $F \in \Delta$. The dimension of a face $F \in \Delta$ is given by $|F| - 1$. A face $F$ in $\Delta$ is a facet if there is no face in $\Delta$ strictly containing $F$, while a simplicial complex is pure if all the facets have the same dimension. The link of a face $F$ in the complex $\Delta$ is defined to be $\text{link}_\Delta(F) = \{G \subseteq V : F \cap G = \emptyset \text{ and } F \cup G \in \Delta\}$. Finally, for $\Delta$ and $\Gamma$ two simplicial complexes having disjoint vertex sets $V$ and $W$, the join of $\Delta$ and $\Gamma$, denoted $\Delta \ast \Gamma$, is the simplicial complex on the vertex set $V \cup W$ consisting of the faces $\Delta \ast \Gamma = \{F \cup G : F \in \Delta \text{ and } G \in \Gamma\}$.
where the summand \( ij\sigma kl \) means to sum over all stable 2-compositions \( \sigma = S_1/S_2 \) with the elements \( i, j \in S_1 \) and the elements \( k, l \in S_2 \).

2. (quadratic relations) For each pair \( \sigma \) and \( \tau \) with \( a(\sigma, \tau) = 4 \),

\[
x_\sigma \cdot x_\tau \tag{2.2}
\]

The WDVV ring (or Associativity ring) \( W_n \) is defined to be the quotient ring \( W_n = k[x_\sigma : \sigma \in P_n]/I_n \).

Rather than working with the complex associated with the the entire WDVV ring, we now consider the pre-Associativity or pre-WDVV ring defined by taking the WDVV ring modulo the quadratic relations only. More formally, the pre-WDVV ring \( R_n \) is defined to be \( R_n = k[x_\sigma : \sigma \in P_n]/J_n \), where \( J_n \) is the ideal generated by the quadratic relations (2.2) of \( I_n \).

Keel’s presentation of the cohomology ring of the moduli space \( \overline{M}_{0,n} \) has twice as many variables as he instead indexes the cohomology classes by subsets \( S \) of \( \{1, \ldots, n\} \). However, Keel makes the further requirement that \( x_S = x_{\overline{S}} \), where \( \overline{S} \) is the complement of \( S \) taken with respect to \( \{1, \ldots, n\} \), making his variables correspond to stable 2-partitions. Hence, it is natural to think of the variables \( x_\sigma \) for \( \sigma \in P_n \) to be indexed instead by subsets \( S \subseteq \{2, \ldots, n\} \) having cardinality satisfying \( 2 \leq |S| \leq n-2 \). See [14, Theorem 1].

Based upon Keel’s presentation, we next construct a simplicial complex intimately related to the pre-WDVV ring. Fix an integer \( n \geq 3 \). Let \( \Delta_n \) be the simplicial complex with vertex set \( V_n = \{S \subseteq \{2, \ldots, n\} : 2 \leq |S| \leq n-2\} \). A subset \( F \subseteq V_n \) is a face if for all \( S, T \in F \) either \( S \subseteq T \), \( S \supseteq T \) or \( S \cap T = \emptyset \).

Observe that the complex \( \Delta_n \) is described by its minimal non-faces, and that each minimal non-face has cardinality 2. Also, a set \( S \) in the vertex set of \( \Delta_n \) corresponds to a stable 2-partition \( S/\overline{S} \), where the complement is taken with respect to the set \( \{1, \ldots, n\} \). Hence the condition of being a face \( F \) corresponds to \( a(\sigma, \tau) < 4 \) for all \( S, T \in F \) where \( \sigma = S/\overline{S} \) and \( \tau = T/\overline{T} \).

The complex \( \Delta_n \) coincides with Boardman’s space of fully grown \( n \)-trees [4]. This space was rediscovered by Whitehouse in her thesis [32] and independently by Culler and Vogtmann [5]. In the literature it is commonly referred to as the Whitehouse complex. The Whitehouse complex also has connections with phylogenetic trees. See the work of Billera, Holmes and Vogtmann [2].

Recall for a finite simplicial complex \( \Delta \) with vertex set \( V \) the Stanley-Reisner ring \( k[\Delta] \) is defined to be \( k[\Delta] = k[x_v : v \in V]/I(\Delta) \), where \( I(\Delta) \) is the ideal generated by the non-faces of the complex \( \Delta \); see [26]. Using this terminology, we immediately have the following result.

**Proposition 2.1** The pre-WDVV ring \( R_n \) is the Stanley-Reisner ring of the Whitehouse complex \( \Delta_n \), that is, \( R_n = k[\Delta_n] \).

## 3 Facial structure

We now proceed with a more formal study of the Whitehouse complex \( \Delta_n \) which will lead to understanding its topology. In particular, as new results we derive recurrences for its face \( f \)-vector and \( h \)-vector.
Figure 1: The forest representation of the face $F = \{\{2, 3, 4\}, \{5, 6\}, \{5, 6, 7\}\}$ in $\Delta_8$.

Figure 2: The maps $A$ and $B$ applied to the face $F = \{\{2, 3, 4\}, \{5, 6\}, \{5, 6, 7\}\}$ in $\Delta_8$.

Observe the complex $\Delta_3$ consists of the empty set. It will be convenient to view this complex as a $(-1)$-dimensional sphere. The complex $\Delta_4$ consists of 3 isolated vertices, while the complex $\Delta_5$ is the Peterson graph. Note the two last examples are homotopy equivalent to the wedge of two 0-dimensional spheres, respectively, the wedge of six 1-dimensional spheres.

Each face $F$ of $\Delta_n$ can be described by a forest. The leaves of the forest are $2, \ldots, n$. The internal nodes are the elements of the face $F$. The cover relation of the forests is defined as follows: For $S, T \in F$, we say $S$ covers $T$ if $S \supset T$ and there is no $U \in F$ such that $S \supset U \supset T$. Moreover, $S \in F$ covers a leaf $i \in \{2, \ldots, n\}$ if there is no $U \in F$ such that $i \in U \subset S$. (Here we use $\subset$ and $\supset$ to mean strict containment and reverse containment, respectively.) See Figure 1 for the example $F = \{\{2, 3, 4\}, \{5, 6\}, \{5, 6, 7\}\}$. From the forest representation it is straightforward to see that the complex $\Delta_n$ is pure of dimension $n - 4$.

We have the following results about the links of faces in the complex $\Delta_n$.

**Lemma 3.1** Let $T$ be a vertex of the Whitehouse complex $\Delta_n$. Then

$$\text{link}_{\Delta_n}(T) \cong \Delta_{|T|+1} \ast \Delta_{n-|T|+1},$$

where $\ast$ denotes the join operation.
Figure 3: The maps $C$ and $D$ applied to the face $F = \{\{2, 3, 4\}, \{5, 6\}, \{5, 6, 7\}\}$ and set $S = \{5, 6, 7\}$ in $\Delta_8$.

**Proof:** First observe that for $T$ a vertex of $\Delta_n$, the link of $T$ consists of all faces that contain $T$. Each face in the link$_{\Delta_n}(T)$ can be described as a forest on the elements $\{2, \ldots, n\}$. In a given forest, the element $T$ is an internal node. One can choose the structure beneath $T$ in the forest in $\Delta_{|T|+1}$ ways. To build the rest of the forest, treat the tree built by $T$ as a leaf. Then one sees there are $n - 1 - |T| + 1$ elements available. The resulting structure has the form $\Delta_{n-|T|+1}$. Hence, link$_{\Delta_n}(T) \cong \Delta_{|T|+1} \ast \Delta_{n-|T|+1}$. $\square$

Iterating Lemma 3.1 gives the following more general result.

**Proposition 3.2** Let $F$ be a face of the Whitehouse complex $\Delta_n$. Then

$$\text{link}_{\Delta_n}(F) \cong \Delta_{c+1} \ast \prod_{T \in F} \Delta_{c(T)+1},$$

where $\prod$ denotes taking the join operation $\ast$ among the factors, $c$ is the number of components in the forest representation of $F$, and $c(T)$ is the number of children the node $T$ has in the forest representation of $F$.

We introduce five maps $A, B, C, D, E : \Delta_n \to \Delta_{n+1}$ which together map $\Delta_n$ onto $\Delta_{n+1}$. For a face $F \in \Delta_n$ define

$$A(F) = F \quad \text{and} \quad B(F) = F \cup \{2, \ldots, n\}.$$ 

For $S \in F$ define

$$C(F, S) = \{T \cup \{n+1\} : S \subseteq T, T \in F\} \cup \{T : S \not\subseteq T, T \in F\}$$

and

$$D(F, S) = C(F, S) \cup \{S\}.$$ 

Finally, for $2 \leq i \leq n$ define

$$E(F, i) = \{\{i, n+1\}\} \cup \{T \cup \{n+1\} : i \in T, T \in F\} \cup \{T : i \not\in T, T \in F\}.$$
Informally speaking, the \( A \) map adds the element \( n + 1 \) to the set diagram of the face \( F \) whereas the \( B \) map creates a new set consisting of the entire set diagram of the face \( F \) and then adds the element \( n + 1 \). The \( C \) map selects a set \( S \) from the face \( F \) and adds the element \( n + 1 \), whereas the \( D \) map selects a set \( S \) from the face \( F \) and creates a new set consisting of the set \( S \) and the element \( n + 1 \). The \( E \) map replaces an element \( i \) with the set consisting of \( i \) and \( n + 1 \).

For a face of dimension \( i \) the maps \( A \) and \( C \) leave the dimension of \( F \) unchanged while the maps \( B \), \( D \) and \( E \) each increase the dimension by one.

It is easily seen that all of these maps are injective and that \( \Delta_{n+1} \) is a disjoint union of their images. See Figures 2, 3 and 4 for an example of each of these maps.

Observe that for fixed \( i \) the image of \( E(F,i) \) as \( F \) runs over all faces \( F \) in the complex \( \Delta_n \) is the link of the vertex \( \{i, n + 1\} \) in \( \Delta_{n+1} \) which is isomorphic to \( \Delta_n \). Hence we can decompose the face poset of the complex \( \Delta_{n+1} \) into the images of the five maps \( A, B, C, D \) and \( E \). In turn, the image of \( E \) further decomposes into \( n - 1 \) copies of the face poset of \( \Delta_n \). It is straightforward to see each of the \( n - 1 \) copies of the \( \Delta_n \) in the face poset of \( \Delta_{n+1} \) is an upper order ideal.

From the maps we have just defined, we can give a recurrence for the number of faces in the Whitehouse complex.

**Theorem 3.3** Let \( f_{n,i} \) denote the number of faces in the Whitehouse complex \( \Delta_n \) having dimension \( i \). Then \( f_{n,i} \) satisfies the recursion

\[
f_{n,i} = (i + 2) \cdot f_{n-1,i} + (n + i - 1) \cdot f_{n-1,i-1},
\]

where \(-1 \leq i \leq n - 4\) and \( f_{n,-1} = 1 \).

**Proof:** Easily \( f_{n,-1} = 1 \) for all \( n \geq 3 \), as there is one \((-1)\)-dimensional face in \( \Delta_n \), namely the empty set. To show the recurrence, note that a face having dimension \( i \) has \( i + 1 \) sets occurring in its set representation. The maps \( A, B, C, D \) and \( E \) each either leave the number of sets unchanged or increase the number by one. Since they map \( \Delta_{n-1} \) bijectively onto \( \Delta_n \), we can build all the \( i \)-dimensional faces in \( \Delta_n \). First, we take a face from \( \Delta_{n-1} \) that has \( i + 1 \) sets. The \( A \) map simply adds the element \( n \) in 1 way, while the \( C \) map adds it in \( i + 1 \) ways. Overall we have constructed \((i + 2) \cdot f_{n-1,i}\) new faces in \( \Delta_n \). The other way to build an \( i \)-dimensional face is to take a face from \( \Delta_{n-1} \) having \( i \) sets and add the element \( n \) in such a way to
\[ (1 - t)^{n-3} \cdot \mathcal{H}(R_n) \]

| \( n \) | \( (1 - t)^{n-3} \cdot \mathcal{H}(R_n) \) |
|---|---|
| 3 | 1 |
| 4 | 1 + 2t |
| 5 | 1 + 8t + 6t^2 |
| 6 | 1 + 22t + 58t^2 + 24t^3 |
| 7 | 1 + 52t + 328t^2 + 444t^3 + 120t^4 |
| 8 | 1 + 114t + 1452t^2 + 4400t^3 + 3708t^4 + 720t^5 |

Table 1: The Hilbert series of the pre-WDVV ring \( R_n \) for \( 3 \leq n \leq 8 \).

increase the number of sets. The \( B \) map does this in 1 way, the \( D \) map in number of sets ways, that is, \( i \) ways, and the \( E \) map does this in \( n - 2 \) ways. Hence we have constructed \( (n+i-1) \cdot f_{n-1,i-1} \) new faces in \( \Delta_n \), and the recurrence holds. \( \square \)

**Corollary 3.4** The Whitehouse complex \( \Delta_n \) is pure of dimension \( n - 4 \) with \( (2n - 5)!! = (2n - 5) \cdot (2n - 7) \cdot \cdots \cdot 3 \cdot 1 \) facets.

This corollary can be found in [24] and [30].

Recall the \( h \)-vector of a \((d-1)\)-dimensional simplicial complex \( \Delta \) is the vector \((h_0, \ldots, h_d)\) where \( h_k = \sum_{i=0}^{k} (-1)^{k-i} (d-i) \cdot f_{d-i-1} \). The Hilbert series of the Stanley-Reisner ring of \( \Delta \) is thus given by \( \mathcal{H}(k[\Delta]) = (h_0 + h_1 \cdot t + \cdots + h_d \cdot t^d) / (1 - t)^d \). See [24]. Using the expression for the \( h \)-vector in terms of the \( f \)-vector and Theorem 3.3, we obtain the following identity.

**Corollary 3.5** The \( h \)-vector of the Whitehouse complex \( \Delta_n \) satisfies the recurrence

\[ h_{n,k} = (k+1) \cdot h_{n-1,k} + (2n - k - 5) \cdot h_{n-1,k-1}, \]

where \( 0 \leq k \leq n - 3 \).

The Hilbert series of the pre-WDVV ring \( R_n \) for \( 3 \leq n \leq 8 \) (and thus the \( h \)-vector values for the Whitehouse complex \( \Delta_n \)) are displayed in Table 1.

### 4 Morse matching and the topology of \( \Delta_n \)

R. Forman devised a discrete version of Morse theory to study the topology of simplicial complexes, and more generally, CW-complexes. We give a brief overview of this. More details can be found in [10]. We then describe a Morse matching of the Whitehouse complex and verify its homotopy type.

Let \( P \) be an arbitrary poset. Begin by orienting all of the edges in the Hasse diagram of \( P \), that is, the cover relations of \( P \), downwards. Next, form a matching \( M \) on the elements of \( P \). Reverse the orientation of the edges in the matching to be upwards. Such a matching \( M \) is a Morse matching if the resulting directed graph is acyclic. An unmatched element of \( P \) is called critical.

For a simplicial complex \( \Delta \) the face poset \( P \) is the poset formed by taking the faces of the complex as elements and ordering them by inclusion. The face poset is ranked with the rank of an
element \( x \in P \) given by \( i \) if \( |x| = i \). In the case we do not wish to include the empty face in the face poset, we denote the resulting face poset by \( P \).

The following is Forman’s result \([10]\).

**Theorem 4.1** For a simplicial complex \( \Delta \) with face poset \( P \), let \( M \) be a Morse matching of \( P \). For \( i \geq 0 \) let \( u_i \) denote the number of critical \( i \)-dimensional simplices. Then \( \Delta \) is homotopy equivalent to a CW-complex consisting of \( u_i \) \( i \)-cells, where \( i \geq 0 \).

As it will be convenient for us to include the empty face in our matching, we will use the following corollary of Forman’s theorem.

**Corollary 4.2** Let \( \Delta \) be a simplicial complex having Morse matching on the face poset \( P \) with exactly \( m \) critical \( k \)-dimensional simplices. Then \( \Delta \) is homotopy equivalent to a wedge of \( m \) \( k \)-dimensional spheres.

There is a straightforward criterion to determine when a matching is a Morse matching.

**Lemma 4.3** For a ranked poset \( P \) to determine a given matching \( M \) is a Morse matching, it is enough to verify the Morse condition on the elements that are in the matching \( M \) from adjacent ranks in \( P \).

**Proof:** We first show it is enough to check the Morse acyclicity condition on elements from ranks \( i \) and \( i + 1 \) in the poset \( P \) for \( i \) fixed. As it is impossible to construct a cycle composed entirely of edges oriented downward, any cycle in \( P \) must include at least two elements from the matching \( M \). So suppose \( x \) and \( y \) are two elements from the matching of ranks \( i \) and \( i + 1 \) with the edge oriented from \( x \) to \( y \). If the edge \( x \to y \) is in a cycle, then the edge \( y \to z \) following this one cannot have \( \rho(z) = i + 1 \) (and hence, pointing upwards) since the element \( y \) is already matched. Similarly, if the edge \( x \to y \) is in a cycle, then the edge \( w \to x \) proceeding this one cannot have \( \rho(w) = i - 1 \) since \( x \) is already matched. Hence we have shown if there is a cycle in \( P \), all the elements are from adjacent ranks. Finally, if we have a cycle in \( P \) on adjacent ranks, the edges in the cycle alternate pointing upwards and downwards, implying every other edge is from the matching \( M \) and hence all the elements in the cycle are elements of \( M \). \( \square \)

What follows is a result which will be helpful when we construct a Morse matching of the Whitehouse complex.

**Lemma 4.4** Let \( P \) be a ranked poset which is the disjoint union of a lower order ideal \( L \) and an upper order ideal \( U \). If \( M_1 \) is a Morse matching of \( L \) and \( M_2 \) is a Morse matching of \( U \), then \( M_1 \cup M_2 \) is a Morse matching of \( P \).

**Proof:** By Lemma 4.3 without loss of generality we may assume all the elements we consider here are those elements from \( M_1 \cup M_2 \) of ranks \( i \) and \( i + 1 \), where \( i \geq 1 \). As there is no element of \( U \) matched with an element of \( L \), all the edges from rank \( i + 1 \) elements in \( U \) to rank \( i \) elements of \( L \) are oriented downwards. Furthermore, as \( U \) is an upper order ideal, there is no edge between any
rank \( i + 1 \) element in \( L \) to a rank \( i \) element of \( U \). Hence, it is impossible to construct a cycle on the elements of \( M_1 \cup M_2 \), implying the matching \( M_1 \cup M_2 \) is indeed a Morse matching of \( P \). \( \square \)

We now describe a Morse matching on the face poset of the Whitehouse complex.

**Proposition 4.5** Let \( Q \) be the face poset given by the images of the maps \( A, B, C \) and \( D \) applied to the Whitehouse complex \( \Delta_n \). Let \( M \) be the matching described by orienting the edges from \( A(F) \) to \( B(F) \) and \( C(F, S) \) to \( D(F, S) \), where \( S \subseteq F \) and \( F \) ranges over all faces of the Whitehouse complex \( \Delta_n \). Then \( M \) is a Morse matching on \( Q \).

**Proof:** Assume on the contrary that we can find a cycle between rank \( i \) and \( i + 1 \) elements of \( Q \). The edges oriented from a rank \( i \) to rank \( i + 1 \) elements in this cycle are Morse matched edges. In terms of the forest representation of a face, such an edge corresponds to the element \( n + 1 \) being moved up one level higher in the tree. (Here we are thinking of the leaves in the forest as being the lowest level.) In terms of the maps, such an edge corresponds to \( C(F, S) \) and \( D(F, S) \) for some face \( F \) and subset \( S \subseteq F \) in \( \Delta_n \). The final move to raise the element \( n + 1 \) corresponds to the maps \( A(F) \) and \( B(F) \). However, the path we have created cannot be continued, and more importantly, cannot be completed to form a cycle, since the element \( n + 1 \) cannot be moved any higher. Hence we cannot construct a cycle on the elements of \( Q \), so the matching \( M \) described is a Morse matching. \( \square \)

**Corollary 4.6** The complex \( Q \) described in Proposition 4.5 is contractible.

We are now ready to prove our main result. For other proofs, see [24, 27, 30].

**Theorem 4.7** The Whitehouse complex \( \Delta_n \) is homotopy equivalent to a wedge of \((n - 2)!\) spheres of dimension \( n - 4 \).

**Proof:** We proceed by induction on the dimension \( n \). For \( n = 3 \), the complex \( \Delta_3 \) consists solely of the empty set, so there is the trivial empty Morse matching. This complex is homotopy equivalent to one \((-1)\)-dimensional sphere.

Begin to construct a Morse matching on the face poset \( P \) of \( \Delta_{n+1} \) by first orienting the edges in the face poset \( Q = \text{Im}(A(\Delta_n)) \cup \text{Im}(B(\Delta_n)) \cup \text{Im}(C(\Delta_n)) \cup \text{Im}(D(\Delta_n)) \) as described in Proposition 4.5. The remainder of the face poset of \( \Delta_{n+1} \) is \text{Im}(E(\Delta_n)). Recall the image of \( E \) applied to \( \Delta_n \) is isomorphic to \((n - 1)\) copies of \( \Delta_n \). By induction, we have a Morse matching in each of these \((n - 1)\) copies of \( \Delta_n \). Each copy of \( \Delta_n \) is an upper order ideal in the face poset \( P \). Hence Lemma 4.4 applies, so we have a constructed a Morse matching on the face poset of \( \Delta_{n+1} \). In each of the \((n - 1)\) copies of \( \Delta_n \) there are \((n - 2)!\) critical elements. Moreover, all the critical elements are facets of dimension \((n - 3)\). Thus, by Corollary 4.2 the complex \( \Delta_{n+1} \) is homotopy equivalent to a wedge of \((n - 1)!\) spheres each having dimension \((n - 3)\). \( \square \)
5 The Cohen-Macaulay property

Recall that a simplicial complex $\Delta$ is Cohen-Macaulay if the associated Stanley-Reisner ring $k[\Delta]$ is Cohen-Macaulay [26]. In combinatorial commutative algebra Cohen-Macaulay complexes have some very nice enumerative properties. Reisner’s Criterion [23, 26] gives a characterization of Cohen-Macaulay simplicial complexes in terms of reduced homology of their links.

**Theorem 5.1** A pure simplicial complex $\Delta$ is Cohen-Macaulay if and only if for all faces $F \in \Delta$ and for all $i < \dim(\text{link}(F))$ we have $\tilde{H}_i(\text{link}(F); k) = 0$.

We have the following result about the topology of the links of the faces in the Whitehouse complex.

**Theorem 5.2** For $F$ a face of the Whitehouse complex $\Delta_n$, $\text{link}_{\Delta_n}(F)$ is a wedge of $(c-1)! \cdot \prod_{T \in F} (c(T) - 1)!$ spheres of dimension $n - 4 - |F|$, where $c$ is the number of components in the forest representation of $F$ and $c(T)$ is the number of children the node $T$ has in the forest representation of $F$.

**Proof:** Let $(S^n)^k$ denote the wedge of $k$-dimensional spheres. The free join of an $n$-dimensional sphere with an $m$-dimensional sphere satisfies $S^n \ast S^m \cong S^{n+m+1}$. Additionally, the wedge and free join operations are distributive over simplicial complexes, that is, $(X \ast Y) \ast Z \cong (X \ast (Y \ast Z))$. For a proof of this fact, see [8, Lemma 3.14]. Hence it follows that $(S^n)^k \ast (S^m)^l \cong (S^{n+m+1})^{k \cdot l}$.

But $c + \sum_{T \in F} c(T) = n - 1 + |F|$. Hence

$$\text{link}_{\Delta_n}(F) \cong \left(S^{n-4-|F|}\right)^{(c-1)!} \prod_{T \in F} (c(T) - 1)! \quad .$$

From Reisner’s Criterion and Theorem 5.2 we have the following immediate result. This can also be found in work of Robinson-Whitehouse, Sundaram and Vogtmann [24, 27, 30].

**Theorem 5.3** The Whitehouse complex $\Delta_n$ is Cohen-Macaulay and hence the pre-WDVV ring $R_n$ is Cohen-Macaulay.

Trappmann and Ziegler [28] prove shellability of the $k$-tree complex. This is Hanlon’s generalization of the Whitehouse tree complex corresponding to the case $k = 2$. In unpublished work, Wachs independently determined shellability of the Whitehouse complex using a different shelling order. As shellability implies Cohen-Macaulayness, this gives another proof of Theorem 5.3.
Table 2: The Hilbert series of the WDVV ring $W_n$ for $3 \leq n \leq 8$.

| $n$ | $\mathcal{H}(W_n)$ |
|-----|-------------------|
| 3   | 1                 |
| 4   | $1 + t$           |
| 5   | $1 + 5t + t^2$    |
| 6   | $1 + 16t + 16t^2 + t^3$ |
| 7   | $1 + 42t + 127t^2 + 42t^3 + t^4$ |
| 8   | $1 + 99t + 715t^2 + 715t^3 + 99t^4 + t^5$ |

6 Concluding remarks

The author is currently studying the Hilbert series of the WDVV ring. The first few values are given in Table 2. The symmetry follows immediately from Poincaré duality and the fact the moduli space $\overline{M}_{g,n}$ is smooth and compact. It would be interesting to find a natural combinatorial object corresponding to these values.

Vic Reiner has asked if the WDVV ring and the pre-WDVV rings are Koszul. Evidence for this is that both rings are defined by quadratic relations and the reciprocal of the Hilbert series for each ring has alternating coefficients. From a result of Fröberg [11], it follows that the Stanley-Reisner ring of a simplicial complex with minimal non-faces having cardinality 2 is Koszul. This latter result applies to the pre-WDVV ring.

**Theorem 6.1** The pre-WDVV ring is Koszul.

It remains to determine if either the pre-WDVV ring or the WDVV ring are Gorenstein.

The complex of not 1-connected graphs on $(n + 1)$ vertices is also homotopy equivalent to a wedge of $(n - 2)!$ spheres of dimension $n - 4$; see [12, 29]. Although the Whitehouse complex $\Delta_n$ and this complex are homotopy equivalent, they are not the same. By Corollary 3.4 the pre-WDVV complex is pure, while the the cardinality of a facet in the complex of not 1-connected graphs on $(n + 1)$ vertices ranges from $\lfloor \frac{n^2}{4} \rfloor$ to $\binom{n}{2}$.

7 Acknowledgements

The author would like to thank Vic Reiner for pointing out references and suggesting the Koszul question, and Louis Billera, Richard Ehrenborg, Arkady Kholodenko and Michelle Wachs for valuable comments and references.

Part of this research was done while the author was a Visiting Member at the Institute of Advanced Study in Princeton during June 2002 and a Visiting Professor at the University of Minnesota in August 2002. This research was partially supported by a Summer 2002 University of Kentucky Faculty Fellowship.
References

[1] E. Babson, A. Björner, S. Linusson, J. Shareshian and V. Welker, Complexes of not i-connected graphs, Topology 38 (1999), 271–299.
[2] L. Billera, S. Holmes and K. Vogtmann, Geometry of the space of phylogenetic trees, Adv. in Appl. Math. 27 (2001), 733-767.
[3] A. Björner and V. Welker, The homology of “k-equal” manifolds and related partition lattices, Adv. Math. 110 (1995), 277–313.
[4] J. M. Boardman, Homotopy structures and the language of trees, Proc. Sympos. Pure Math. 22 (1971), 37–58.
[5] M. Culler and K. Vogtmann, Moduli of graphs and automorphisms of free groups, Invent. Math. 84 (1986), 91–119.
[6] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, I.H.E.S. Publ. Math. 36 (1969), 75–109.
[7] R. Dijkgraaf, E. Verlinde and H. Verlinde, Topological strings in d < 1, Nucl. Phys. B 352 (1991), 59–86.
[8] X. Dong, Topology of bounded degree graph complexes, J. Algebra 262 (2003), 287–312.
[9] B. Dubrovin, Integrable systems in topological field theory, Nucl. Phys. B 379 (1992), 627–689.
[10] R. Forman, Morse theory for cell complexes, Adv. Math. 134 (1998), 90–145.
[11] R. Fröberg, Determination of a class of Poincaré series, Math. Scand. 37 (1975), 29–39.
[12] P. Hanlon, Otter’s method and the homology of homeomorphically irreducible k-trees, J. Combin. Theory Ser. A 74 (1996), 301–320.
[13] P. Hanlon and M. Wachs, On Lie k-algebras, Adv. Math. 113 (1995), 206–236.
[14] S. Keel, Intersection theory of moduli space of stable n-pointed curves of genus zero, Trans. Amer. Math. Soc. 330 (1992), 545–574.
[15] B. Kind and P. Kleinschmidt, Schälbare Cohen-Macaulay-Komplexe und ihre Parametrisierung, Math. A. 167 (1979), 173–179.
[16] F. Knudsen, Projectivity of the moduli space of stable curves. II, Math. Scand. 52 (1983), 1225–1265.
[17] M. Kontsevich and Y. Manin, Gromov-Witten classes, quantum cohomology and enumerative geometry, Comm. Math. Phys. 164:3 (1994), 525–562.
[18] M. Kontsevich and Y. Manin, Quantum cohomology of a product (with Appendix by R. Kaufmann), Inv. Math. 124, f. 1-3 (1996), 313–339.
[19] A. Losev and Y. Manin, New modular spaces of pointed curves and pencils of flat connections, Michigan Math. J. 48 (2000), 443–472.
[20] D. Mumford, “Geometric invariant theory,” Springer-Verlag, Berlin-New York, 1965.
[21] D. Mumford, Stability of projective varieties, L’Ens. Math. 23 (1977), 39–110.
[22] M. Readdy, The Yuri Manin ring and its B_n-analogue, Adv. in Appl. Math 26 (2001), 154–167.
[23] G. Reisner, Cohen-Macaulay quotients of polynomial rings, Adv. Math 21 (1976), 30–49.
[24] A. Robinson and S. Whitehouse, The tree representation of \Sigma_n+1, J. Pure Appl. Algebra 111 (1996), 245–253.
[25] R. P. Stanley, “Enumerative Combinatorics, Vol. I,” Wadsworth and Brooks/Cole, Pacific Grove, CA, 1986.
[26] R. P. Stanley, “Combinatorics and commutative algebra, second edition,” Birkhäuser Boston, Inc., Boston, MA, 1996.

[27] S. Sundaram, Homotopy of non-modular partitions and the Whitehouse module, J. Algebraic Combin. 9 (1999), 251–269.

[28] H. Trappmann and G. Ziegler, Shellability of complexes of trees, J. Combin. Theory Ser. A 82 (1998), 168–178.

[29] V. A. Vassiliev, Complexes of connected graphs, in The Gelfand Mathematical Seminars, 1990–1992, L. Corwin et al, eds., Birkhäuser Boston, Boston, MA, 1993, 223–235.

[30] K. Vogtmann, Local structure of some Out(F_n)-complexes, Proc. Edinburgh Math. Soc. (2) 33 (1990), 367–379.

[31] M. Wachs, personal communication.

[32] S. Whitehouse, “T-(Co)homology of commutative algebras and some related representations of the symmetric groups,” Doctoral dissertation, University of Warwick, 1994.

[33] E. Witten, On the structure of the topological phase of two-dimensional gravity, Nucl. Phys. B 340 (1990), 281–332.

[34] E. Witten, Two-dimensional gravity and intersection theory on moduli space, Surveys in differential geometry (Cambridge, MA, 1990), 243–310, Lehigh Univ., Bethlehem, PA, 1991.

Margaret A. Readdy
Department of Mathematics
University of Kentucky
Lexington, KY 40506-0027
readdy@ms.uky.edu