Solving the Pricing Problem in a Generic Branch-and-Price Algorithm using Zero-Suppressed Binary Decision Diagrams

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Abstract

Branch-and-price algorithms combine a branch-and-bound search with an exponentially-sized LP formulation that must be solved via column generation. Unfortunately, the standard branching rules used in branch-and-bound for integer programming interfere with the structure of the column generation routine; therefore, most such algorithms employ alternate branching rules to circumvent this difficulty. This paper shows how a zero-suppressed binary decision diagram (ZDD) can be used to solve the pricing problem for a generic branch-and-price algorithm, even in the presence of constraints imposed by branching decisions. This approach facilitates a much more direct solution method, and can improve convergence of the column generation subroutine. Computational results are presented showing the effectiveness of this method for the graph coloring problem.

1 Introduction

Branch-and-price algorithms are of increasing interest in many areas of operations research, including assignment and scheduling problems (Savelsbergh, 1997; Maenhout and Vanhoucke, 2010), vehicle routing problems (Fukasawa et al., 2006), graph coloring (Mehrotra and Trick, 1996; Malaguti et al., 2011), multicommodity flow problems (Barnhart et al., 2000), and cutting stock problems (Vance, 1998; Pisinger and Sigurd, 2007), among others. These algorithms combine a branch-and-bound search together with a tight linear programming relaxation having an exponential number of variables (such a formulation can be derived, for example, by the Dantzig-Wolfe decomposition method described in Dantzig and Wolfe, 1960). This LP relaxation, called the master problem, is used to produce good bounds that are used to prune suboptimal regions of the search space. Because the LP relaxation is too large to be stored in memory, it must be solved via column generation.

Let $S$ be the set of variables for the LP relaxation; each of these variables is associated with a column of the master problem’s constraint matrix (thus, the constraint matrix has an exponential number of columns). In the column generation method, a related LP called the restricted master
problem (RMP) is built using a (small) subset $S' \subseteq S$ of variables. The RMP can be solved efficiently by standard linear programming techniques; however, the solution to the RMP is not necessarily optimal for the master problem. Therefore, a subroutine called the pricing problem must be called to either produce a variable in $S \setminus S'$ that may improve the objective value of the RMP, or provide a guarantee that no such variable exists. If an improving variable is found, it is added to $S'$, and the RMP is re-optimized. New columns are iteratively added to $S'$ until the pricing problem reports that no (potentially) improving variables exist, at which point column generation is terminated and the solution to the RMP is provably optimal for the master problem.

In this paper, the pricing problem is assumed to be a weighted binary combinatorial optimization problem which is characterized by a family of “valid” subsets of some universe; in a slight abuse of notation, solutions to the pricing problem are interchangeably referred to as “variables”, “columns”, or “subsets”, where the meaning is clear from context. The weights associated with the pricing problem are usually the optimal dual prices for the current solution to the RMP. Thus, the pricing problem returns a new variable with negative reduced cost if one exists; if such a variable exists, these variables may improve the value of the RMP (Bertsimas and Tsitsiklis, 1997). From this perspective, the pricing problem is a separation oracle for the dual of the RMP, since new variables for the RMP correspond to additional constraints in the dual.

Since the pricing problem is itself often NP-hard, and must be solved exactly, solving it is typically the most computationally-intensive part of a branch-and-price procedure. Moreover, when combined with the standard integer branching scheme used in most branch-and-bound algorithms, the structure of the pricing problem is destroyed (Barnhart et al., 1998). In such a branching scheme, a variable $x_i$ with fractional value $\alpha$ is selected at a subproblem in the search tree, and two children are created with additional bounding constraints $x_i \leq \lfloor \alpha \rfloor$ and $x_i \geq \lceil \alpha \rceil$ (when all variables are binary, this is called $0-1$ branching). However, imposing these constraints changes the structure of the dual problem, which in turn means that a different separation oracle must be queried at each subproblem in order to generate new columns.

In effect, the pricing problem at these subproblems no longer seeks a variable of minimum reduced cost; it now must produce a variable with minimum reduced cost that respects the current branching decisions. This problem, called the constrained pricing problem, is much harder than the regular pricing problem, and is often related to the $k^{th}$-shortest-path problem, which is
well-known to be a challenging NP-hard problem (Garey and Johnson, 1979).

Moreover, many branch-and-price formulations have an inherent asymmetry due to the large number of variables in the formulation. This asymmetry can lead to extremely lopsided search trees if standard integer branching techniques are used. For example, in a problem with many covering constraints (of the form $\sum_i x_i \geq b$), fixing a variable to zero may not induce much change in the LP relaxation, but fixing a variable to 1 may immediately satisfy many constraints. Thus long paths in the search tree can exist where many variables are fixed to 0 but no progress towards a solution is made.

Therefore, most branch-and-price algorithms employ specialized branching rules or other techniques to avoid eliminating the pricing problem structure, as well as to maintain a more balanced search tree. For example, some branching rules modify the problem structure at each subproblem in the search tree (e.g., the graph coloring rule of Mehrotra and Trick, 1996); others branch on original (non-reformulated) problem variables, or problem constraints (Vanderbeck, 2011). A related scheme by Morrison et al. (2013a) uses a modified branching scheme called wide branching, which does not wholly eliminate calls to the constrained pricing problem, but restructures the search tree in an attempt to reduce the number of such calls.

An alternate approach, called robust branch-and-cut-and-price (BCP), eliminates calls to the constrained pricing problem by further modifying the RMP so that branching restrictions can be added without interfering with the pricing problem structure (de Aragão and Uchoa, 2003). This approach introduces additional variables and constraints into the RMP to form a linear program called the explicit master, which has the same objective value as the RMP. Furthermore, branching decisions made by the algorithm can be communicated to the pricing problem by imposing constraints on the reduced cost values in the dual of the explicit master. This approach has been used successfully in many variants of the capacitated vehicle routing problem (Pessoa et al., 2008; Fukasawa et al., 2006), as well as related problems such as the capacitated minimum spanning tree problem (Uchoa et al., 2008).

However, no algorithm in the literature has described a way to perform branch-and-price without using techniques like robust BCP or alternative branching rules, which often come at the expense of ease of implementation and less-direct (global) solution methods. Alternate branching rules do not allow variables to be directly fixed to values, but rely on problem structure to implicitly fix
variables. Similarly, the robust branch-and-cut-and-price methods require the solution of a larger LP at each subproblem, and again use implicit methods to fix variables. The wide branching approach allows variables to be fixed explicitly, but to obtain good performance, it requires the derivation of a problem-specific branching rule.

Therefore, the primary contribution of this research is to establish an efficient method for solving the pricing problem in a generic branch-and-price algorithm that is directly compatible with the standard integer branching scheme. Two algorithmic ideas enable this result: the first is to use a data structure called a zero-suppressed binary decision diagram (ZDD) to compactly store all valid solutions to the pricing problem. A linear-time algorithm is presented which adds restrictions to the ZDD to prohibit previously-generated columns from being produced a second time, which allows the constrained pricing problem to be solved at every iteration of column generation. The second idea combines the above solution procedure with the cyclic best-first search (CBFS) strategy to overcome the lopsided search trees that can result when using standard integer branching with exponentially-sized problems. Computational results are presented for the graph coloring problem showing nearly order-of-magnitude improvements in solution time for some instances when using these two ideas, together with a proof of optimality for several previously unsolved instances.

The remainder of this paper is organized as follows: Section 2 defines the ZDD data structure and shows how it can be used to solve the pricing problem in a branch-and-price algorithm (Section 2.1). This section also presents the ZDD restriction algorithm, which can be used to add restrictions to the pricing problem solution space (Section 2.2). Next, Section 3 describes the cyclic best-first search strategy and how it is used to mitigate the effects of lopsided search trees. In Section 4, the graph coloring problem is defined and computational results are given for this problem. Finally, Section 5 outlines several future research directions for this technique.

2 Zero-Suppressed Binary Decision Diagrams

A zero-suppressed binary decision diagram (Minato 1993) is an extension of the binary decision diagram (BDD) data structure proposed by Lee (1959) and Akers (1978). A BDD is a directed acyclic graph that compactly encodes a binary function. Previously, BDDs have been used in circuit design and verification, as well as a number of formal logic applications (Bryant 1992).
More recently, BDDs have been used in a number of different optimization applications: Bergman et al. (2012a) explore different variable orderings for BDDs used to characterize the independent sets of a graph, and Hadžić and Hooker (2008) add weights to the edges of a BDD to perform post-optimality analysis in a discrete optimization setting. Finally, Cire et al. (2012) and Bergman et al. (2012b) describe how to use BDDs to compute upper and lower bounds to prune subproblems in a branch-and-bound algorithm.

Despite their success in these related areas, BDDs and ZDDs have not appeared in conjunction with branch-and-price in the literature before. Behle and Eisenbrand (2007) give a method for using BDDs to enumerate vertices and facets of 0/1 polyhedra (which can be viewed as solving the pricing problem for a problem which has been reformulated via Dantzig-Wolfe decomposition), but they do not extend this result to the branch-and-price setting. Additionally, Behle (2007) uses BDDs to generate valid inequalities in a branch-and-cut algorithm to perform row generation instead of column generation.

However, the use of decision diagrams together with branch-and-price algorithms can provide substantial benefits to algorithm performance. This is because decision diagrams yield a way to compactly store all the columns even for an exponentially-sized integer program. Note that column generation techniques must still be used to solve the RMP, because the columns encoded in the ZDD cannot be operated on directly by the LP solver. Nonetheless, since the LP solver has (implicit) access to all columns, the pricing problem can be solved exactly at every iteration of column generation, which may improve the convergence of the column generation procedure. In contrast, most branch-and-price solvers terminate the pricing problem solver as soon as a column with “sufficiently negative” reduced cost is found, due to the difficulty of solving the pricing problem. Moreover, as shown in Section 2.2, the set of valid pricing problem solutions can be modified in place, allowing branch-and-price algorithms using ZDDs to employ standard integer branching methods.

### 2.1 Definitions and Notation

A ZDD is a modified version of a BDD that removes some nodes from the data structure to reduce its size. ZDDs are most useful when the binary function it encodes is “sparse” in the sense that there are relatively few valid solutions to the function compared to the number of invalid solutions. Minato
observed that many combinatorial optimization problems have the sparsity characteristic; thus, ZDDs are likely to be more useful in a branch-and-price setting than ordinary BDDs.

Formally, a ZDD $Z$ is defined as follows. Let $\mathcal{E}$ be an ordered set of $n$ elements $(e_1, e_2, ..., e_n)$; then $Z$ is a directed acyclic graph satisfying the following properties:

1. There are two special nodes in $Z$ (denoted 1 and 0), called the **true** node and **false** node, respectively. Additionally, there is exactly one “highest” node in the topological ordering of $Z$, called the **root** of $Z$, and denoted $r$.

2. Every node $a \in Z - \{1, 0\}$ has two outgoing edges, a **high edge** and a **low edge**, which point to the **high child** and **low child**, respectively. The high (low) child of $a$ is denoted $\text{hi}(a)$ ($\text{lo}(a)$). The true and false nodes have no outgoing edges. The **indegree** of $a$, denoted $\delta^{-}(a)$, is the number of incoming edges to $a$; thus, $\delta^{-}(r) = 0$.

3. Every node $a \in Z - \{1, 0\}$ is associated with some element $e_i \in \mathcal{E}$; the index of the associated element for $a$ is given by $\text{var}(a)$, that is, $\text{var}(a) = i$. By convention, $\text{var}(1) = \text{var}(0) = n + 1$. Finally, if $\text{var}(a) = i$, then $\text{var}(\text{hi}(a)) > i$ and $\text{var}(\text{lo}(a)) > i$.

4. No $a \in Z$ has $\text{hi}(a) = 0$ (this property is not satisfied by ordinary BDDs).

Any set $A \subseteq \mathcal{E}$ induces a path $P_A$ from the root of $Z$ to either 1 or 0, in the following manner: starting at the root of $Z$, if $a$ is the current node on the path, the next node along the path is $\text{hi}(a)$ if $e_{\text{var}(a)} \in A$, and $\text{lo}(a)$ otherwise. The **output** of $Z$ on $A$, denoted $Z(A)$, is the last node along this path, which must be either 1 or 0. If $Z(A) = 1(0)$, then $Z$ accepts (rejects) $A$. Note that it is not required for $\text{var}(b) = \text{var}(a) + 1$ when $b$ is a child of $a$; in the case when $\text{var}(b) > \text{var}(a) + 1$, the edge is called a **long edge**, and when an induced path includes such an edge, if $\{e_{\text{var}(a)+1}, e_{\text{var}(a)+2}, ..., e_{\text{var}(b)-1}\} \cap A \neq \emptyset$, then $Z$ rejects $A$. Finally, a ZDD characterizes a family of sets $\mathcal{F} \subseteq 2^E$ (denoted $Z_\mathcal{F}$) if $Z$ accepts all sets in $\mathcal{F}$, and rejects all sets not in $\mathcal{F}$ (see Figure 1).

For an arbitrary family $\mathcal{F}$ and an arbitrary vertex ordering, the size of $Z_\mathcal{F}$ (that is, the number of nodes and edges in the graph, denoted $|Z_\mathcal{F}|$) may be exponential in $n$. However, Bryant (1986) shows that for any fixed variable ordering, every boolean function has a unique smallest BDD characterizing it. This result extends to ZDDs by observing that membership in $\mathcal{F}$ can be defined
Figure 1: Let $\mathcal{E} = (e_1, e_2, e_3, e_4)$, and $\mathcal{F} = \{\emptyset, \{e_1, e_2\}, \{e_3, e_4\}, \{e_1, e_2, e_3, e_4\}\}$ (Andersen 1997). Solid lines represent high edges, and dashed lines represent low edges; all edges are directed downwards. Grey nodes indicate whether $Z_\mathcal{F}$ accepts $A$.

(a) The unique smallest ZDD characterizing $\mathcal{F}$ for the given variable ordering. (b) The induced path corresponding to the set $A = \{e_3, e_4\}$. (c) The induced path corresponding to the set $A = \{e_3\}$. (d) The ZDD does not accept $A = \{e_1, e_2, e_4\}$ since the long edge skips $e_4$, but $e_4 \in A$.

as a boolean function. One way to construct the unique smallest ZDD characterizing $\mathcal{F}$ is to first construct the BDD for $\mathcal{F}$’s indicator function, and then iteratively delete nodes whose high edge points to 0, connecting the low edge to the node’s parent. Alternately, there exists a recursive algorithm to construct $Z_\mathcal{F}$ directly (Knuth 2008).

Note that the choice of ordering on the elements of $\mathcal{E}$ is important; Bryant (1986) shows examples where different variable orderings yield BDDs of dramatically different sizes for the same function. In fact, it is NP-hard to determine the variable ordering for any arbitrary boolean function that will yield the smallest BDD (Bollig and Wegener 1996). These results apply for ZDDs as well; nevertheless, the use of heuristic variable orderings often results in tractably-sized ZDDs in practical applications.

To see how ZDDs can be used to solve the unconstrained pricing problem in a branch-and-price algorithm, let $\mathcal{F}$ be the family of solutions to the pricing problem. Then, using the technique of Hadžić and Hooker (2008), assign weights to the edges of $Z_\mathcal{F}$ and compute the longest path or shortest path in $Z_\mathcal{F}$ from the root to 1, depending on whether the pricing problem is a maximization or minimization problem. Specifically, let $(\pi_1, \pi_2, ..., \pi_n)$ be a weight vector for the elements of $\mathcal{E}$; set the weight of edge $(a, b) \in Z_\mathcal{F}$ to $\pi_{\text{var}(a)}$ if $b = \text{hi}(a)$, and 0 otherwise. Then, finding the
longest or shortest path with respect to \{\pi\} from the root of \(Z_F\) to 1 can be found in \(O(|Z_F|)\) time using dynamic programming \cite{Sedgewick2011}. The resulting path corresponds to the optimal solution to the pricing problem (see Figure 2).

Figure 2: The ZDD from Figure 1a with weights given by the objective function \(\max[-e_1 - e_2 + 2e_3 + e_4]\); the bold path corresponds to the maximum-weight valid set, that is \(\{e_3, e_4\}\). Weights not shown are 0.

### 2.2 The ZDD Restriction Algorithm

In order to use standard integer branching methods in a branch-and-price algorithm it is necessary to solve the constrained pricing problem. Recall that this problem seeks a new variable of minimum reduced cost that respects all branching decisions made at the subproblem. Note that it is sufficient to generate a new variable that does not appear in the pool \(S'\) for the RMP; to see this, observe that if any variable in \(S'\) has negative reduced cost, then the current solution to the RMP is not optimal. Therefore, in this section, an algorithm is presented to add restrictions to a ZDD characterizing the pricing problem so that any time a new column is generated and added to \(S'\), it can be immediately restricted from ever being generated as a solution to the pricing problem again. In this way, the ZDD will actually solve the constrained pricing problem at each iteration of the algorithm.

Let \(\mathcal{F}\) be the family of valid solutions to the pricing problem, where each \(A \in \mathcal{F}\) is a subset of \(\mathcal{E} = \{e_1, e_2, ..., e_n\}\), and let \(Z_F\) be the ZDD characterizing \(\mathcal{F}\). The restriction algorithm for
ZDDs, called **RestrictSet**, takes as input a set \( A \in \mathcal{F} \), and builds a new ZDD \( Z_{\mathcal{F}'} \) that accepts \( \mathcal{F}' = \mathcal{F} - A \). The key feature of the ZDD restriction algorithm that makes it effective in practice is that it operates in \( O(n) \) time, and it increases the size of \( Z_{\mathcal{F}} \) by at most \( n \) nodes and \( 2n \) edges (and often by much less).

Intuitively, the **RestrictSet** algorithm identifies the path \( P_A \) in \( Z_{\mathcal{F}} \) corresponding to the set \( A \), and updates this path so that it ends at the false node instead of the true node. However, if there exists \( A' \neq A \) such that \( P_A \) and \( P_A' \) overlap, this update could also restrict \( A' \). Therefore, **RestrictSet** duplicates the portion of \( P_A \) that could overlap with some other root-to-1 path, and sets the endpoint of the duplicate path to 0. This ensures that no additional sets are restricted by the algorithm. The first node on \( P_A \) with indegree greater than one, referred to as the **split node**, is the first node with some potential overlap; thus it, and all subsequent nodes, are duplicated.

Pseudocode for the restriction algorithm is given in Algorithm 1; this algorithm makes use of a function called \( Z_{\mathcal{F}}.\text{insert}(i, a_1, a_2) \), which takes as input an index \( i \in \{1, 2, \ldots, n\} \) and pointers to two pre-existing nodes \( a_1, a_2 \in Z_{\mathcal{F}} \). The function inserts a new node into \( Z_{\mathcal{F}} \) associated with element \( e_i \), with low edge pointing to \( a_1 \) and high edge pointing to \( a_2 \), and returns a pointer to the newly-inserted node. It also updates the indegrees of the high and low children. \( Z_{\mathcal{F}}.\text{insert} \) can be implemented in (average) constant time (see Andersen [1997] for details).

The following theorem establishes the correctness of the **RestrictSet** algorithm and proves the claims made previously about its time and space complexity behavior; an example of the **RestrictSet** applied to the ZDD in Figure 1a is given in Figure 3.

**Theorem 1.** Given a ZDD \( Z_{\mathcal{F}} \) describing a family of subsets \( \mathcal{F} \) of an ordered set \( \mathcal{E} \) with \( n \) elements, together with a set \( A \in \mathcal{F} \), the **RestrictSet** algorithm modifies \( Z_{\mathcal{F}} \) in \( O(n) \) time to produce a new ZDD called \( Z_{\mathcal{F}'} \) such that \( \mathcal{F}' = \mathcal{F} - A \). Furthermore, \( |Z_{\mathcal{F}'}| \leq |Z_{\mathcal{F}}| + 3n \).

**Proof.** First, note that **RestrictSet** visits each node along \( P_A \) at most twice, and \( P_A \) has at most \( n \) nodes. Furthermore, the algorithm performs a constant amount of work for each visited node. Thus the running time of **RestrictSet** is \( O(n) \). Also, since node \( c \) is at most the root of \( Z_{\mathcal{F}} \), at most \( |\mathcal{E}| \) nodes are added to \( Z_{\mathcal{F}} \) to form \( Z_{\mathcal{F}'} \), and each new node has two outgoing edges.

To prove that \( Z_{\mathcal{F}'} \) has the desired properties, let \( a'_1, a'_2, \ldots, a'_l \) be the nodes added to \( Z_{\mathcal{F}} \) in lines 18-19 (Algorithm 1), in increasing order of depth. Consider some set \( A' \subseteq \mathcal{E} \); if \( A' = A \), the path
Algorithm 1: restricting ZDD

input: A ZDD $Z_{F}$ and a characteristic vector $(\alpha_1, \alpha_2, ..., \alpha_n)$ for a set $A \in F$

output: A modified ZDD $Z_{F'}$ such that $F' = F - A$

1 \langle \langle Find the first node on $P_A$ with indegree higher than 1 \rangle \rangle
2 \hspace{1em} a = \text{root}(Z_F); \ b = -1
3 \textbf{while} $\delta^{-}(a) < 2$ and $a \not\in \{1, 0\}$:
4 \hspace{1em} $i = \text{var}(a); \ b = a$
5 \hspace{1em} \text{if} $\alpha_i = 1$: \hspace{1em} $a = \text{hi}(a)$
6 \hspace{1em} \text{else}: \hspace{1em} $a = \text{lo}(a)$
7 \langle \langle The node $c$ is the “split” node, and $b$ is its parent \rangle \rangle
8 \hspace{1em} $c = a$
9 \langle \langle Make copies of all remaining nodes on $P_A$ and point to 0 \rangle \rangle
10 \hspace{1em} \text{list} = ()
11 \textbf{while} $a \not\in \{1, 0\}$:
12 \hspace{1em} \text{list}.\text{append}(a)
13 \hspace{1em} \text{if} $\alpha_{\text{var}(a)} = 1$: \hspace{1em} $a = \text{hi}(a)$
14 \hspace{1em} \text{else}: \hspace{1em} $a = \text{lo}(a)$
15 \hspace{1em} $a' = 0$
16 \langle \langle Insert the duplicated nodes into $Z_F$ \rangle \rangle
17 \hspace{1em} \textbf{for each} $a \in \text{list}$ (in reverse order):
18 \hspace{1em} \text{if} $\alpha_{\text{var}(a)} = 1$: \hspace{1em} $a' = Z_F.\text{insert}(\text{var}(a), \text{lo}(a), a')$
19 \hspace{1em} \text{else}: \hspace{1em} $a' = Z_F.\text{insert}(\text{var}(a), a', \text{hi}(a))$
20 \langle \langle Point the correct edge of the parent node $b$ to the root of the duplicated path \rangle \rangle
21 \hspace{1em} \textbf{if} $\alpha_{\text{var}(b)} = 1$: \hspace{1em} $\text{hi}(b) = a'$
22 \hspace{1em} \text{else}: \hspace{1em} $\text{lo}(b) = a'$
23 \hspace{1em} \text{return } Z_F$

from the root of $Z_{F'}$ to the bottom of the ZDD is the same as the path from the root of $Z_F$ up to the parent $b$ of the split node $c$. By construction, the next node visited in $Z_{F'}$ is $a'_1$ (lines 21 and 22, Algorithm 1). Then, the remainder of the path in $Z_{F'}$ follows the added nodes; at each $a'_i$, the high and low children are constructed to agree with the values of $A$. Finally, the last node along this path is 0 (line 15, Algorithm 1), so $Z_{F'}(A') = 0$.

Furthermore, if $A' \neq A$, then consider the first index $i$ where the characteristic vectors of $A$ and $A'$ differ; if $i < \text{var}(c)$, then the modifications to $Z_{F'}$ have no effect on whether $A'$ is accepted, since the only nodes added to $Z_{F'}$ appear at depths greater than or equal to that of $c$. However, if $i \geq \text{var}(c)$, the path will follow along the newly added nodes $a'_1, a'_2, ..., a'_j$, where $\text{var}(a'_j) = i$. At this point, $A$ and $A'$ differ, and by construction, the path returns to the original node in $Z_F$ and never returns to the newly-added nodes. Therefore, $Z_{F'}(A') = Z_F(A')$, as desired.
To reduce the size of $Z_{F'}$, a check can be performed to see if the high and low edges of newly-inserted nodes both point to 0; in this case, the node is suppressed (see Figure 3c). Finally, note that the ZDD produced by the RestrictSet is no longer necessarily minimal with respect to $F'$. In particular, in the worst case, if all $2^n$ subsets of $E$ are restricted, the size of the ZDD can grow by $O(n2^n)$ nodes. However, in this case, the resultant ZDD is $Z_{\emptyset}$, which can be described with only two nodes. In the event that the ZDD becomes too large, a reduction algorithm can be periodically called that searches for duplicate nodes in $Z_F$ that can be merged.

Figure 3: The result of applying the RestrictSet algorithm to $Z_F$ from Figure 1a with $A = \{e_1,e_2,e_3,e_4\}$. The final ZDD accepts $F' = \{\emptyset, \{e_1,e_2\}, \{e_3,e_4\}\}$.

Using the RestrictSet procedure, a branch-and-price algorithm can be developed that uses traditional integer branching. This branch-and-price algorithm first builds a ZDD characterizing all valid solutions to the pricing problem; in the worst case, this may take exponential time, but dynamic programming or memoization techniques can be used to speed up the construction. The ZDD is then used to produce new variables at every iteration of column generation, which correspond to solutions of the constrained pricing problem. Once a new set (or variable) has been generated, RestrictSet is called to prohibit that column from being generated again at a later
time. The ZDD is therefore guaranteed to produce the optimal solution to the pricing problem at each stage, and since in most cases $n \ll |Z_F|$, the increase in size of the ZDD over the course of the branch-and-price search is small. Hence, the time needed to solve the pricing problem does not significantly increase over the course of the algorithm. Pseudocode for the resulting branch-and-price search is given in Algorithm 2.

**Algorithm 2: Branch-and-Price with ZDDs**

1. Construct $Z_F$, where $F$ is the set of valid columns
2. Compute an initial pool $S'$ of columns for the RMP
3. for each $A \in S'$: $Z_F = \text{RestrictSet}(Z_F, A)$
4. Initialize the branch-and-price search tree $T$
5. **⟨⟨ Main branch-and-price loop ⟩⟩**
6. while $T$ has an unexplored subproblem:
   7. Select a subproblem $s$ that has not been explored
   8. Generate children of $s$ according to branching rule
   9. for each child of $s$:
      10. **⟨⟨ Column generation loop ⟩⟩**
      11. while $\exists$ a variable in $S \setminus S'$ with negative reduced cost:
      12. Use $Z_F$ to generate a variable $A \in S \setminus S'$ with negative reduced cost
      13. $\text{RestrictSet}(Z_F, A)$
      14. Add $A$ to $S'$ and re-optimize the RMP
      15. Apply pruning rules to delete child, or insert child into $T$
     16. Mark $s$ as explored
   17. return the best solution found in $T$

3 Cyclic Best-First Search

As described in Section 1, when using standard integer branching in a branch-and-price setting, the structure of the search tree can become extremely unbalanced. In particular, long chains of assignments that make no progress towards a solution exist, which (if explored) can dramatically increase the search time. Moreover, in many cases these long chains appear more promising than shorter chains which progress towards a solution. For instance, in a problem with many covering constraints of the form $\sum_i x_i \geq b$, setting a variable $x_i = 0$ generally does not change the lower bound much, nor does it restrict the solution considerably since many other unfixed variables could also satisfy the constraint.

Therefore, standard search strategies such as depth-first search (DFS) or best-first search (BFS)
do not perform particularly well in this setting. If DFS gets unlucky, it can start exploring some long chain early in the process which does not improve the incumbent solution but requires a large amount of search time. On the other hand, a strategy like BFS which relies on the lower bound to perform node selection will also perform poorly, since the lower bounds along the long branches will often be smaller than lower bounds in other parts of the tree.

Historically, the iterative deepening depth-first search (IDFS) strategy [Korf 1985] has been used in such settings to prevent the exploration of long chains that do not make progress towards better incumbents. However, this paper proposes the use of the cyclic best-first search (CBFS) strategy as an alternative search strategy that enables the use of lower bound information during subproblem selection. The CBFS strategy, originally proposed by Kao et al. [2009] (and called distributed best-first search in this paper), has since been used successfully in a number of additional settings including two different scheduling problems [Morrison et al. 2013b; Sewell et al. 2012].

This search strategy can be thought of as a hybrid algorithm between DFS and BFS; the algorithm uses a measure-of-best function $\mu$ to select the next subproblem to explore (as in BFS), but repeatedly cycles through a set of labeled contours (i.e., a collection of subproblems), selecting one subproblem from each contour to explore before advancing to the next contour. The cyclic behavior can be thought of as a variant of backtracking in DFS. The contour labels are simply taken from $\mathbb{N}_0$ as a way to order the set of contours. For example, the levels of the search tree provide a natural contour definition, where subproblems are grouped by their distance from the root of the tree (see Figure 4a).

Let $C_i$ denote the contour with label $i$; Once a subproblem has been explored from $C_i$, CBFS chooses the next subproblem for exploration from $C_{i+p}$, where $p = \min\{p' \in \mathbb{Z}^+ \mid C_{i+p'} \neq \emptyset\}$, and index addition is done modulo $K$ (the largest contour label currently in use). The subproblem chosen from this contour is one that minimizes the measure-of-best function $\mu$. In contrast, BFS

| Algorithm 3: CBFS |
|-------------------|
| 1 Let $i$ be the label of the contour containing the last explored subproblem |
| 2 if $\exists C_j \neq \emptyset$ with $j > i$: |
| 3 Let $j$ be the first index larger than $i$ of a non-empty contour |
| 4 else: |
| 5 Let $j$ be the first index in $\{0, 1, \ldots, i\}$ of a non-empty contour |
| 6 return $s \in \arg\min_{s' \in C_j} \mu(s')$ |
always chooses the subproblem with the best (global) value of $\mu$ to explore. Pseudocode for the CBFS strategy is given in Algorithm 3; this code is called in Line 7 of Algorithm 2 to select a new subproblem for exploration.

Previous implementations of CBFS have only used the depth-based contour definition; however, using this contour definition tends does not produce better performance than DFS or BFS, for the same reason that BFS performs poorly. In fact, CBFS can produce worse performance than DFS in some cases, if DFS gets lucky and finds a good incumbent early. The key insight provided here is that other, more complicated, contour definitions are possible which may dramatically improve the search process for branch-and-price algorithms.

In particular, consider the following contour definition (called the **positive assignment** definition), which assigns a subproblem to contour $\ell$ if and only if there have been $\ell$ branching decisions made of the form $x_i = 1$ (called a positive assignment). Using this contour definition significantly restructures the order in which subproblems are selected for exploration (see Figure 4b). In particular, using this definition prohibits the immediate exploration of a child subproblem which assigns $x_i = 0$, even if the lower bound at this child is better than the lower bound at the $x_i = 1$ child. In this way, the search is weighted towards exploration of subproblems that make positive assignments, which serves to counterbalance the effects of a lopsided search tree.
4 Computational Results for Graph Coloring

A branch-and-price algorithm for the graph coloring problem was implemented using a ZDD to solve the pricing problem, together with the CBFS strategy for subproblem exploration, and computational experiments were run on a subset of the instances from the DIMACS graph coloring challenge [Johnson and Trick, 1996; Trick, 2005]. This section describes some implementation details for this program, called B&P+ZDD, as well as discussing the results of these experiments and a comparison to the best algorithms in the literature.

4.1 The Graph Coloring Problem

The graph coloring problem is a classic NP-hard problem [Garey and Johnson, 1979]; given a graph $G = (V, E)$, the objective is to find a minimum proper coloring of vertices (i.e., a coloring in which no adjacent vertices share colors). The chromatic number $\chi$ of $G$ is the minimum number of colors required in any proper coloring. For a vertex $v$, the neighborhood of $v$, denoted by $N(v)$, is the set of vertices adjacent to $v$. For a subset $S \subseteq V$, the induced subgraph $G[S]$ is the subgraph of $G$ with vertex set $S$ that has an edge between $u, v \in S$ if and only if $(u, v)$ is an edge in $G$. A set $S \subseteq V$ is an independent set if $G[S]$ has no edges, and $S$ is a maximal independent set if there is no vertex $v \in V \setminus S$ such that $S + v$ is independent. For any set of vertices $S$, a vertex $v$ is covered if $v \in S$ or $v \in N(S)$. Finally, a set $S \subseteq V$ is a clique if all pairs of vertices in $S$ are adjacent.

A standard IP formulation for the graph coloring problem creates binary variables $x_{vj}$ for each vertex $v \in V$ and each color $j \in \{1, 2, \ldots, \chi\}$, where $\chi$ is any upper bound on the chromatic number. Additional binary variables $r_j$ are introduced for each $j \in \{1, 2, \ldots, \chi\}$. In this formulation, setting $x_{vj} = 1$ assigns color $j$ to vertex $v$, and setting $r_j = 1$ indicates that color $j$ is assigned to some vertex in the graph. Using these variables, the following IP encodes the graph coloring problem:
\[
\text{minimize } \sum_{j=1}^{\chi} r_j \\
\text{subject to } \sum_{j=1}^{\chi} x_{vj} = 1 \ \forall \ v \in V \\
x_{uj} + x_{vj} \leq r_j \ \forall \ (u, v) \in E, j \in \{1, 2, ..., \chi\} \\
x_{vj}, r_j \in \{0, 1\} \ \forall \ v \in V, j \in \{1, 2, ..., \chi\}. 
\]

(1)

Here, the first set of constraints ensures that every vertex receives exactly one color, and the second set of constraints ensures that adjacent vertices are assigned different colors. However, as noted by Mehrotra and Trick (1996), this formulation has two weaknesses: first, it contains many symmetric solutions which arise by permuting the colors used in a solution. These symmetries lead to many redundant subtrees in a branch-and-bound search algorithm, which cannot be easily detected and substantially slow the search process. Secondly, the linear programming bound of (1) is often quite weak, which does not allow for much pruning to occur in the search tree. Therefore, Mehrotra and Trick (1996) propose the following integer program with an exponential number of variables:

\[
\begin{align*}
\text{Minimize} & \sum_{S \in \mathcal{S}} x_S \\
\text{Subject to} & \sum_{S \ni v \in S} x_S \geq 1 \ \forall \ v \in V \\
& x_S \in \{0, 1\}.
\end{align*}
\]

(2)

In this formulation, \( \mathcal{S} \) is the family of maximal independent sets in \( G \); since any proper coloring can be viewed as a partition of \( V \) into independent sets, this is equivalent to searching for the smallest coloring. The binary variables \( x_S \) indicate whether the maximal independent set \( S \) is used in the coloring, and the constraints ensure that each vertex in the graph appears in some color class. This formulation eliminates the symmetry inherent in (1), as well as producing much tighter bounds for many instances.

However, since there are potentially an exponential number of maximal independent sets in \( G \), it is generally infeasible to keep all variables in memory. Thus, to solve the LP relaxation of (2), column generation techniques must be employed. The pricing problem for the graph coloring
The problem as formulated in \((2)\) is a maximum-weight maximal independent set problem, where the weights on the vertices are given by the values of the dual variables of the RMP. If a maximal independent set \(S\) with weight larger than 1 is found, then variable \(x_S\) has negative reduced cost, which means that \(x_S\) is a candidate to improve the solution value of the RMP and can be added to \(S'\).

The formulation presented in \((2)\) is used in most state-of-the-art exact solvers for the graph coloring problem. These solvers often use a branching rule called edge branching to avoid destruction of the pricing problem. Edge branching selects two non-adjacent, uncolored vertices in \(G\) and creates two branches, one in which the vertices are linked by an edge, and one in which they are merged together \((\text{Mehrotra and Trick, 1996})\). However, in this paper, a ZDD is built characterizing the family of maximal independent sets of \(G\), and is used with a standard 0–1 branching method (called vertex branching by \(\text{Malaguti et al., 2011}\)).

### 4.2 Constructing the Independent Set ZDD

To build a ZDD for the pricing problem of formulation \((2)\), fix an ordering \(\{v_1, v_2, ..., v_n\}\) on the vertices of the graph, and for some vertex \(u \in V\), let \(u - 1\) be the vertex immediately preceding \(u\) in this ordering (in this setting, the vertex set \(V\) plays the role of \(E\)). This section describes a recursive algorithm for building the minimal ZDD (with respect to this vertex ordering) that characterizes \(\mathcal{F} = \{A \mid A\text{ is a maximal independent set in }G\}\).

The maximal independent set ZDD is stored as a lookup table; if \(a\) is an index in this table, the lookup table stores \(\text{var}(a), \text{lo}(a), \text{hi}(a),\) and \(\delta^- (a)\). In addition, to facilitate the merging of isomorphic regions of the ZDD, a reverse lookup table is stored that maps a tuple \((i, b, c)\) to an index \(a\) such that \(\text{var}(a) = i, \text{lo}(a) = b,\) and \(\text{hi}(a) = c\), if such an \(a\) exists in \(Z_{\mathcal{F}}\). This reverse table is implemented as a hash table which allows for average constant insertion and lookup time.

A recursive construction algorithm for \(Z_{\mathcal{F}}\), called \texttt{MakeIndSetZDD}, can be formulated following the general approach given in \(\text{Knuth, 2008}\). At each stage, a set \(U\) of \(k\) vertices and an index \(i \leq k + 1\) is given as input to \texttt{MakeIndSetZDD}; \(U\) is the set of uncovered vertices for some (not necessarily maximal) independent set \(R \subseteq \{v_1, v_2, ..., u_i - 1\}\). Here, vertices \(\{u_i, u_{i+i}, ..., u_k\}\) can still be added to this (hypothetical) set \(R\). To construct the ZDD node corresponding to \(U\) and \(i\), the high child \(b_h\) and low child \(b_l\) must first be constructed. To do this, vertex \(u_i\) and all its
neighbors are removed from \( U \) to form a set \( U_H \), and \( h \) is set to the index (in \( U_H \)) of the first vertex appearing after \( u_i \) in the vertex ordering, or \(|U_H| + 1\), if no such vertex exists. Then, to compute \( b_h \), \texttt{MakeIndSetZDD}(\( U_H, h \)) is called; this mimics the addition of vertex \( u_i \) to an independent set \( R \) at the current node. Conversely, to compute the low child, \texttt{MakeIndSetZDD}(\( U, i+1 \)) is called, which forbids vertex \( u_i \) from being used in \( R \). To ensure minimality of \( \mathcal{Z}_F \), before a node corresponding to some set \( U \) and index \( i \) is inserted, the algorithm checks to see if any node \( a \) exists in the ZDD with \( \text{var}(a) = i \), \( \text{hi}(a) = b_h \), and \( \text{lo}(a) = b_\ell \). If such a node exists, the index of that node is returned; otherwise a new node is inserted.

In the base case, \( i = k + 1 \) (that is, no more vertices can be added to an independent set from the current node). If \( U \) is empty, all vertices in \( G \) are covered by an independent set, so the algorithm returns 1; if \( U \) is not empty, there is some uncovered vertex in \( U \), so the algorithm returns 0 (in fact, a tighter base case can be developed by observing that if there is some vertex in \( \{u_1, u_2, ..., u_i-1\} \) that is not adjacent to any vertex in \( \{u_i, u_{i+1}, ..., u_k\} \), it is impossible to build a maximal independent set from the current ZDD node).

Pseudocode for \texttt{MakeIndSetZDD} is given in Algorithm 4; to build \( \mathcal{Z}_F \), \texttt{MakeIndSetZDD}(\( V, v_1 \)) is called. An example of running \texttt{MakeIndSetZDD} on the graph in Figure 5 is given in Figure 6.

**Algorithm 4: MakeIndSetZDD(\( U, i \))**

| Input: A set \( U = \{u_1, u_2, ..., u_k\} \) of uncovered vertices such that \( u_j < u_{j+1} \) with respect to the vertex ordering on \( V \), and a “current index” \( i \) |
| Output: The root node of a ZDD characterizing all the maximal independent sets in \( G[U] \) that can be formed with vertices in \( \{u_i, u_{i+1}, ..., u_k\} \) |
| 1 if \( G[U] \) cannot be covered by taking all vertices in \( \{u_i, u_{i+1}, ..., u_k\} \): return 0 |
| 2 if \( U == \emptyset \): return 1 |
| 3 \( U_H = U - u_i - N(u_i) \) \( \langle \langle \) Use vertex \( u_i \); remove it and its neighbors from \( U \) \( \rangle \rangle \) |
| 4 \( h = \min\{j \mid u_j > u_i \text{ and } u_j \in U_H\} \) or \( |U_H| + 1 \) if no such \( j \) exists |
| 5 \( b_h = \texttt{MakeIndSetZDD}(U_H, h) \) |
| 6 \( b_\ell = \texttt{MakeIndSetZDD}(U, i+1) \) |
| 7 if \( b_h == 0 \): return \( b_\ell \) |
| 8 \( \langle \langle \) Use reverse lookup table \( \rangle \rangle \) |
| 9 if \( \exists a \in \mathcal{Z}_F \) s.t. \( \text{var}(a) = i, \text{lo}(a) = b_\ell \), and \( \text{hi}(a) = b_h \): return \( a \) |
| 10 else: return \( \mathcal{Z}_F.\text{insert}(i, b_\ell, b_h) \) |

Note that \texttt{MakeIndSetZDD} does not actually maintain the vertices that are used in a current independent set \( R \) during the ZDD construction. It is sufficient to maintain a list of vertices that
are left uncovered by some independent set, since many independent sets may yield the same set of uncovered vertices.

A number of different ordering heuristics can be applied to the vertex set of $G$ to derive ZDDs of varying size. The rule that was empirically found to produce the smallest ZDDs is the maximal path decomposition rule, which computes a set of paths $P_1, P_2, \ldots, P_q$ such that $P_i$ is maximal in $G[V - \bigcup_{j=1}^{i-1} P_j]$. The vertices are then ordered as

$$v_1^1, v_2^1, \ldots, v_1^l_1, v_2^2, \ldots, v_2^l_2, v_1^q, v_2^q, \ldots, v_q^q,$$

where $v_i^j$ is the $i^{th}$ vertex along the path $P_j$, and $l_j$ is the length of path $P_j$. Morrison et al. (2013c) show that the number of nodes associated with the $k^{th}$ vertex in this ordering is bounded by the $k^{th}$ Fibonacci number $F_k$.

### 4.3 Initialization and Preprocessing

To reduce the size of problem instances, B&P+ZDD uses a standard preprocessing technique: a search is done to find a large clique $C$ in the graph, and any vertex $v \in V$ with degree less than $|C|$ is removed. Since a valid coloring for $G$ must use at least $|C|$ colors, at least one color exists in any proper coloring that is not assigned to any neighbor of $v$; thus, any proper coloring of $G - v$ can be extended to $G$ without increasing the number of colors used (Méndez-Díaz and Zabala, 2006).

A branch-and-bound search is employed in a heuristic manner to find an initial large clique. The clique $C$ can also be used to prove optimality – if a proper coloring of $G$ is found that uses exactly $|C|$ colors, this coloring must be optimal.

To initialize B&P+ZDD, a starting pool of independent sets needs to be generated. A modified
Figure 6: A visualization of the steps taken by MakeIndSetZDD to build $Z_F$ for the example graph given in Figure 5. Grey nodes and edges have been visited by a recursive call, but are not yet stored in the ZDD. Black nodes and edges have been stored in the ZDD’s lookup table. Bold elements have been inserted in the most recent step of the algorithm; nodes are labeled in order of insertion. The set listed to the right of each node is the set $U$ for that recursive call; the notation $[n]$ denotes the set $\{1, 2, ..., n\}$.

(a) Two recursive calls are made; using vertices $v_1$ and $v_3$ leaves $U$ empty, so 1 is returned. No nodes have been inserted into the ZDD at this point.

(b) If $v_1$ is used and $v_3$ is not used, $v_4$ must be used. Node $a$ is the first node inserted in the ZDD.

(c) Both children of $b$, the high branch of the root, have been computed, so $b$ is inserted into the ZDD.

(d) If $v_1$ is not used in an independent set, and $v_2$ is, $v_4$ must also be used to ensure maximality. Node $a'$ is computed as the high child, but is not inserted because it is a duplicate of node $a$.

(e) If $v_3$ is the first vertex used in an independent set, $v_5$ must also be used to ensure maximality.

(f) Some vertex in $\{v_1, v_2, v_3\}$ must be used in any maximal independent set of $G$, so $lo(d) = 0$. All branches are now complete and the algorithm terminates.
version of the initialization procedure described in [Malaguti et al. (2008)] is used for this purpose. Their algorithm employs a 2-phase approach to find good initial solutions. In the first phase, a genetic algorithm combined with a local search rule searches for valid $k$-colorings of the graph for some input parameter $k$. If a valid $k$-coloring is found, then the procedure is iteratively called with successively smaller values of $k$ until a user-specified time limit is reached. The second phase takes the best solution found in phase 1 and applies a covering heuristic to improve the solution further. B&P+ZDD uses a similar procedure to generate its initial pool of independent sets for the RMP, which only runs the first phase of the algorithm described by [Malaguti et al. (2008)].

Any column generated by the initialization routine can be added to the initial pool $S'$ for the RMP. However, the initialization procedure often generates a large number of sets; thus, it is necessary to reduce the size of the initial pool. To this end, the RMP is solved once with only the sets used by the best available coloring to get initial dual prices. Only the generated sets with a price above 0.8 are included in $S'$. This rule includes all sets with negative or close-to-negative reduced cost in $S'$, since these sets are more likely to improve upon the LP solution to the RMP in early stages of the search.

### 4.4 Results from the DIMACS Database

B&P+ZDD was implemented in C++ and used CPLEX 12.5 with default settings to solve the RMP; all computational experiments described in this section were performed on a desktop machine with an Intel Core i7-930 2.8GHz quad-core processor and 12 GB of available memory. The branch-and-price algorithm utilized only a single processor core; however, CPLEX operates in parallel by default. All times reported here are aggregated over all cores. For the sake of comparison with the results obtained with the MMT algorithm, the $dfmax$ benchmark program was run on the r500.5 instance provided by [Trick (2005)]. The computer used for these experiments took 6.60s CPU time to solve this benchmark instance.

Comparisons were made against four different branch-and-price algorithms available in the literature. First, [Malaguti et al. (2011)] give an exact algorithm for the graph coloring problem that uses an improved initialization heuristic, together with extensive computational results. These results were performed using standard 0–1 branching instead of edge branching. Secondly, [Gualandi and Malucelli (2012)] describe a branch-and-price solver for graph coloring that uses constraint program-
Morrison et al. (2013a) provide extensive computational results using a wide branching technique, which modifies the standard 0 – 1 branching rule to allow multiple children to be generated from a subproblem in the search tree. Finally, Held et al. (2012) provide a method for computing a numerically safe lower bound for graph coloring, which they embed inside a branch-and-price solver. Using this algorithm, they are able to prove new lower bounds for a number of unsolved instances.

| Instance | Name of the tested instance |
|----------|-----------------------------|
|       |                            |

| n       | Number of vertices in the instance |
|---------|-----------------------------------|
| m       | Number of edges in the instance    |
| χ       | Chromatic number of the instance, if known |
| LB      | Lower bound found by B&P+ZDD     |
| UB      | Best solution found by B&P+ZDD   |
| tZ      | Time needed to construct the maximal independent set ZDD |
| t       | Time spent in the branch-and-price phase of the algorithm |
| (Z_f – Z_i)/Z_i | Percent change in size of the ZDD over the course of the algorithm |
| t_MMT   | Adjusted time to verify optimality by Malaguti et al. (2011) |
| t_wide  | Adjusted time to verify optimality by the wide branching solver of Morrison et al. (2013a) |
| t_CP-BnP | Adjusted time to verify optimality by the branch-and-price solver of Gualandi and Malucelli (2012) |

Table 1: Notation used for computational results data (Table 2).

Experiments were run on 40 instances from the DIMACS instance database (Trick, 2005). Experiments were not run on easy instances (those for which the lower bound at the root is sufficient to prove optimality), since these instances do not demonstrate the effectiveness of the ZDD data structure for solving the pricing problem in the presence of branching constraints. The remaining instances were chosen to span a range of difficulty, including ones that are easily solved to optimality by all algorithms in the literature, and others for which no algorithm has yet been able to verify optimality. In addition, experiments were run on 7 additional instances taken from Gualandi and Malucelli (2012).

A time limit of 10 hours was imposed for all experiments, and the ZDD size was limited to 100 000 000 nodes. The initialization procedure from Section 4.3 was run for 100 seconds for each instance to generate an initial pool; this did not contribute to the 10-hour time limit. Of the 40 instances tested, most were extremely difficult, and could not be solved by any algorithm within
Table 2: Results from computational experiments with B&P+ZDD; cells highlighted in grey show the fastest algorithm. Entries labeled “init” indicate that the initial upper bound equaled the root lower bound; cells labels “oom” indicate that the algorithm ran out of memory.

| Instance       | n  | m  | χ | LB | UB | t | $\frac{Z_f - Z_i}{Z_i}$ | $t_{MMT}$ | $t_{wide}$ | tCP-BnP |
|----------------|----|----|---|----|----|---|--------------------------|-----------|-----------|---------|
| DSJC125.5      | 125| 3891| 17| 16 | 17 | 0.47| 284.92                  | 0.08      | 17019.33  | 225.21  | 14372.75 |
| DSJC250.9      | 250| 27897| 72| 71 | 72 | 0.04| 649.9                   | 0.02      | >10hrs    | >10hrs  | >10hrs  |
| DSJR600.1c     | 500| 121275| 85| 85 | 85 | 0.14| 0.1                   | 0.00      | 272.01    | 1.29    | 0.60    |
| DSJR600.5      | 500| 58862| 122| 122| 122| 689.48| 102.75                  | 0.23      | 322.6     | 6802.28 | >10hrs  |
| 1e450_25c      | 450| 17343| 25| 25 | 25 | 19889.92| oom                    | 0.00      | init      | >10hrs  | -       |
| 1e450_25d      | 450| 17425| 25| 25 | 25 | 16262.45| oom                    | 0.00      | init      | >10hrs  | -       |
| queen9_9       | 81 | 1056| 11| 11 | 11 | 33.1 | 14354.16                | 0.03      | 1759.08   | 19208.45| 21858.5 |
| mycie13        | 11 | 23  | 4 | 3  | 4  | 0     | 0                        | 0.00      | 0         | 0.01    | -       |
| mycie14        | 20 | 71  | 5 | 4  | 5  | 0     | 0.09                     | 0.11      | 111.26    | 0.47    | -       |
| mycie15        | 47 | 236 | 6 | 4  | 6  | 0.01 | 392.21                  | 0.53      | 0         | 3207.63 | -       |
| 1-FullIns_4    | 93 | 593 | 5 | 4  | 5  | 8.76  | 122.58                  | 0.05      | >10hrs    | >10hrs  | -       |
| qg_order30     | 900| 26100| 30| 0  | 0  | >10hrs | -                       | 0.00      | 0.19      | >10hrs  | -       |
| wap06          | 947| 43571| 40| 0  | 0  | >10hrs | -                       | 0.00      | 165       | >10hrs  | -       |
| r250.5         | 250| 14849| 65| 65 | 65 | 14.63 | 7.15                    | 0.94      | 0         | 0       | 6.8     |
| r1000.5        | 1000| 238267| 234| 234| 234| >10hrs | 4690.16                   | 0.01      | 0         | 0       | >10hrs  |
| flat300_28_0   | 300| 21695| 28| 28 | 28 | 144.57| 19883.45               | 0.01      | 0         | 0       | >10hrs  |

The 10-hour time limit. Data for the remaining nineteen instances which could be solved by at least one algorithm are shown in Table 2 and notation used in this table is given in Table 1. To provide the most fair comparison between different computational platforms, all running times are scaled according to the benchmark value of the dfmax utility reported.

B&P+ZDD was able to find and verify optimality for 15 of the 47 instances tested. On average, B&P+ZDD solves problems four times faster than the best previous algorithm in the literature, and in four cases, the improvement in speed is at least an order of magnitude. Additionally, B&P+ZDD is able to verify optimality for three new instances (1-FullIns_4, r1000.5, and flat_300_0) that have not been solved previously by branch-and-price algorithms in the literature (however, in the case of r1000.5, the ZDD construction took longer than 10 hours). One other instance, DSJC250.9, has only been solved by the branch-and-price solver of Held et al. (2012); their algorithm found a solution in 8685 (adjusted) CPU seconds.

It was observed that modifying the initial pool size can dramatically improve the running time of B&P+ZDD; for example, running the initialization procedure for 6100 seconds (the default initialization time limit in [Malaguti et al., 2011] and [Morrison et al., 2013a]) allows B&P+ZDD to solve DSJC125.5 in 31 seconds. Similarly, running the initialization procedure for only 3 seconds allows B&P+ZDD to solve queen9_9 in 2.3 seconds. (this is explained by noting that a large initial
pool can slow down the LP solver for the RMP).

Data were collected regarding the average length of time needed to solve the pricing problem for each instance, as well as the growth in size of the ZDD over the course of the algorithm. The average growth in size of a ZDD for any problem was 14%, with a standard deviation of 27%. In one case, the size of the ZDD nearly doubled, at 93% growth; however, even in this case, the length of time needed to solve the pricing problem was not impacted substantially. In most cases when the ZDD could be fully constructed, the length of time needed to solve one iteration of the pricing problem was under a second.

Finally, there are five instances which were solved substantially faster by the MMT graph coloring solver than by B&P+ZDD; however, four of these instances were solved at the root node by the MMT solver due to a better initialization procedure, and so do not provide a useful comparison against B&P+ZDD. This leaves only one instance (queen11_11) for which some other algorithm substantially outperforms B&P+ZDD; for this instance, the lower bound is equal to the optimal objective value, which means the search can be terminated as soon as an optimal solution is found.

5 Conclusions and Future Work

This paper presents a framework for using standard integer branching in conjunction with branch-and-price algorithms; this framework solves the pricing problem using a zero-suppressed binary decision diagram that is constructed during a preprocessing phase. When new columns are generated, they are restricted from generation by the ZDD a second time; this allows the constrained pricing problem to be solved exactly at every iteration of the algorithm. Using this technique combined with a new contour definition for the cyclic best-first search strategy to counterbalance the resulting lopsided search tree, the standard integer branching scheme can be used in conjunction with a branch-and-price algorithm, which yields a much faster and more direct solution method in many cases. Computational results were presented showing that a branch-and-price algorithm implementation for the graph coloring problem outperforms other branch-and-price graph coloring solvers in the literature, in five cases by one or more orders of magnitude.

A number of future research directions exist for this method; firstly, since ZDDs are a generic method to solve the pricing problem, they can be used in conjunction with other branch-and-price
methods, even if these methods do not require the solution of the constrained pricing problem (for instance, the robust branch-and-price-and-cut algorithm of de Aragão and Uchoa, 2003). In these settings, the ZDD does not need to have restrictions imposed via RestrictSet when a new variable is generated; however, they may still provide benefits, since the ZDD is able to produce a variable of most negative reduced cost at every iteration of column generation. Thus, additional research can be done to study how ZDDs interact with other established branch-and-price methods.

Secondly, research can be performed to determine the best way to use ZDDs when the entire data structure will not fit in memory; one proposed idea uses approximate ZDDs (described in Bergman et al., 2012b) to solve the pricing problem in these settings. An approximate ZDD is a width-constrained ZDD that does not eliminate any valid solutions to the pricing problem, but may accept some inputs that are not valid solutions to the pricing problem. In this setting, a post-generation check can be performed to see if the ZDD produced a set that is a valid solution to the pricing problem; if not, the RestrictSet routine can be called with the erroneous solution to prevent it from being generated again. If the approximate ZDD can be constructed in an appropriate fashion, it is hypothesized that invalid solutions will be generated relatively infrequently, and the algorithm will not suffer much loss of efficiency.

A final important question addresses the removal of restrictions from the pricing problem ZDD; in particular, the variable pool for branch-and-price algorithms may grow quite large over the course of the algorithm. Since the size of this pool directly impacts the solution time for the RMP, and most of the elements of this pool are never used in any optimal solutions, most branch-and-price solvers will prune the pool by deleting variables with very large positive cost. In such a setting, it is necessary to modify $Z_\mathcal{F}$ to allow removed variables to be generated again. Thus, future research should analyze the effects of such a modification.

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