Analog black holes in flowing dielectrics

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Abstract

We show that a flowing dielectric medium with a linear response to an external electric field can be used to generate an analog geometry that has many of the formal properties of a Schwarzschild black hole for light rays, in spite of birefringence. The surface gravity of this analog black hole has a contribution that depends only on the dielectric properties of the fluid (in addition to the usual term dependent on the acceleration). This term may be give a hint to a new mechanism to increase the temperature of Hawking radiation.

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I. INTRODUCTION

In recent years, there has been a growing interest in models that mimic in the laboratory some features of gravitation [1]. The actual realization of these models relies on systems that are very different in nature: ordinary nonviscous fluids, superfluids, flowing and nonflowing dielectrics, non-linear electromagnetic in vacuum, and Bose-Einstein condensates (see [2] for a complete list of references). The basic feature shared by these systems is that the behaviour of the fluctuations around a background solution is governed by an “effective metric”. More precisely, the particles associated to the perturbations do not follow geodesics of the background spacetime but of a Lorentzian geometry described by the effective metric, which depends on the background solution. It is important to notice that only some kinematical aspects of general relativity can be imitated by this method [3], but not its dynamical features (see however [2, 4]).

This analogy has permitted the simulation of several configurations of the gravitational field, such as wormholes and closed space-like curves for photons [5, 6]. Particular attention has been paid to analog black holes, because these would emit Hawking radiation, as was shown first by Unruh in the case of dumb black holes [7]. It is the prospect of observing this radiation (thus testing the hypothesis that the thermal emission is independent of the physics at arbitrarily short wavelengths [7]) that motivates the quest for a realization of analog black holes in the laboratory. Let us emphasize that the actual observation of the radiation is a difficult task from the point of view of the experiment, if only because of the extremely low temperatures involved. In the case of a quasi one-dimensional flow of a Bose-Einstein condensate for instance, the temperature of the radiation would be around 70 nK, which is comparable but lower than the temperature needed form the condensate [8].

In this paper we would like to explore the possibilities offered in the construction of analog black holes for photons by nonlinear flowing dielectrics [23]. Let us remark that, to our knowledge, all the articles that analyze different aspects of black holes in dielectric fluids [13] published up to date are devoted to the case of a constant permittivity tensor. Here instead we would like to present a new static and spherically symmetric analog black hole, generated by a flowing isotropic dielectric that depends on an applied electric field. As we shall see, the radius of the horizon and the temperature of this black hole depend on three parameters (the zeroth order permittivity, the charge that generates the external field, and the linear susceptibility) instead of depending only on the zeroth order permittivity. Another feature of this black hole is that there is a new term in the surface gravity (and hence in the temperature of Hawking radiation), in addition to the usual term proportional to the acceleration of the fluid. This new term depends exclusively on the dielectric properties of the fluid, and it points to a new mechanism to get Hawking radiation with temperature higher than that reported up to date.

We shall begin with the examination of the issue of photon propagation in a nonlinear medium in the next Section. We anticipate that photons experience birefringence, i.e. there are two effective metrics associated to this medium. A photon “sees” one of these metrics or the other depending on its polarization. Although it may be argued that birefringence spoils the whole effective geometry idea, we shall show in Section [11] that photons see one and the same black hole, independently of their polarization. In Section [14] we shall analyze in detail the case of a medium with a linear dependence in the external field. Some features of the motion of photons in this black hole are also exhibited in this section for the special case of a constant velocity flow, and it will be shown that there is a simple relation between the two effective
metrics. In Section VI we calculate the effective surface gravity of these black holes for an arbitrary velocity profile, and display the appearance of the new term. We close in Section VII with a discussion of the results.

II. PROPAGATING MODES

In this section we shall study the propagation of photons with different polarizations in a nonlinear medium (see Ref. 13 for details and notation). Let us define first the antisymmetric tensors $F_{\mu\nu}$ and $P_{\mu\nu}$ representing the electromagnetic field. They can be expressed in terms of the strengths $(E, H)$ and the excitations $(D, B)$ of the electric and magnetic fields as

\[
F_{\mu\nu} = v_{\mu}E_{\nu} - v_{\nu}E_{\mu} - \eta_{\mu\nu}^{\alpha\beta}v_{\alpha}B_{\beta},
\]

\[
P_{\mu\nu} = v_{\mu}D_{\nu} - v_{\nu}D_{\mu} - \eta_{\mu\nu}^{\alpha\beta}v_{\alpha}H_{\beta},
\]

where $v_{\mu}$ represents the 4-velocity of an arbitrary observer (which we will take later as comoving with the fluid). The Levi-Civita tensor introduced above is defined in such way that $\eta_{0123} = +1$ in Cartesian coordinates. Since the electric and magnetic fields are spacelike vectors, the notation $E^{\alpha}E_{\alpha} = -E^{2}$, $H^{\alpha}H_{\alpha} = -H^{2}$ will be used. We will consider here media with properties determined only by the tensors $\epsilon_{\alpha\beta}$ and $\mu_{\alpha\beta}$ (i.e. media with null magneto-electric tensor), which relate the electromagnetic excitations to the field strengths by the constitutive laws,

\[
D_{\alpha} = \epsilon_{\alpha\beta}(E, H)E_{\beta}, \quad B_{\alpha} = \mu_{\alpha\beta}(E, H)H_{\beta}. \tag{1}
\]

In order to get the effective metric, we shall use Hadamard’s method 10. The discontinuity of a function $J$ through a surface $\Sigma$ will be represented by the symbol $[J]_{\Sigma}$, and its definition is

\[
[J]_{\Sigma} \equiv \lim_{\delta \to 0^{+}}(J|_{\Sigma+\delta} - J|_{\Sigma-\delta}).
\]

By taking the discontinuity of the field equations $*F^{\mu\nu,\nu} = 0$ and $P^{\mu\nu,\nu} = 0$, and assuming that

\[
\epsilon^{\mu\beta} = \epsilon(E)(\eta^{\mu\beta} - v^{\mu}v^{\beta}), \tag{2}
\]

\[
\mu^{\mu\beta} = \mu_{0}(\eta^{\mu\beta} - v^{\mu}v^{\beta}), \tag{3}
\]

with $\mu_{0} = \text{const.}$, we get

\[
\epsilon(k.e) - \frac{\epsilon'}{E}(E.e)(k.E) = 0 \tag{4}
\]

\[
\mu_{0}(k.h) = 0 \tag{5}
\]

\[
\epsilon(k.v)\epsilon^{\mu} - \frac{\epsilon'}{E}E^{\alpha}e_{\alpha}(k.v)E^{\mu} + \eta^{\mu\nu\alpha\beta}k_{\nu}v_{\alpha}h_{\beta} = 0 \tag{6}
\]

\[
\mu_{0}(k.v)h^{\mu} - \eta^{\mu\nu\alpha\beta}k_{\nu}v_{\alpha}e_{\beta} = 0 \tag{7}
\]

where $k^{\mu}$ is the wave propagation vector, $\epsilon'$ is the derivative of $\epsilon$ w.r.t. $E$, and

\[
[E_{\mu},] = e_{\mu}k_{\lambda}, \quad [H_{\mu},] = h_{\mu}k_{\lambda}. \tag{8}
\]

Note in particular that Eqn. (4) shows that the vectors $k^{\mu}$ and $e^{\mu}$ are not always orthogonal, as would be the case if $\epsilon'$ was zero.

Substituting Eqn. (4) in (7), we get

\[
Z^{\mu\beta}e_{\beta} = 0 \tag{9}
\]

where the matrix $Z$ is given by

\[
Z^{\mu\beta} = [k^{2} + (k.v)^{2}(\mu_{0}\epsilon - 1)] \eta^{\mu\beta} - \mu_{0} \frac{\epsilon'}{E}(k.v)^{2}E^{\mu}E^{\beta} + (v.k)(v^{\mu}k^{\beta} + k^{\mu}v^{\beta}) - [\epsilon_{0}(k.v) + k^{2}] v^{\mu}v^{\beta} - k^{\mu}k^{\beta}. \tag{10}
\]

Non-trivial solutions of Eqn. (5) can be found only for cases in which $\det[Z^{\mu\beta}] = 0$ (this condition is a generalization of the well-known Fresnel equation 11).

Eqn. (5) can be solved by expanding $e_{\alpha}$ as a linear combination of the four linearly independent vectors $v_{\nu}$, $E_{\nu}$, $k_{\nu}$, and $\eta_{\alpha\beta\mu\nu}v^{\alpha}E^{\beta}k^{\mu}$ (the particular case in which the vectors $v_{\nu}$, $E_{\nu}$ and $k_{\nu}$ are coplanar will be examined below). That is,

\[
e_{\nu} = \alpha E_{\nu} + \beta \eta_{\alpha\beta\mu\nu}v^{\alpha}E^{\beta}k^{\mu} + \gamma k_{\nu} + \delta v_{\nu}. \tag{11}
\]

Notice that taking the discontinuity of $E_{\alpha}^{\nu}$ we can show that $(e.v) = 0$. This restriction imposes a relation between the coefficients of Eqn. (11):

\[
\delta = -\gamma(k.v)
\]

With the expression given in Eqn. (11), Eqn. (5) reads

\[
\alpha \left[ k^{2} - (1 - \mu_{0}(\epsilon E'))(k.v)^{2} \right] - \gamma \left[ \mu_{0}(k.v) \frac{1}{E} \epsilon'(E.E)k_{\alpha} \right] = 0
\]

\[
\alpha E^{\mu}k_{\mu} + \gamma(1 - \mu_{0}\epsilon)(k.v)^{2} + \delta(k.v) = 0
\]

\[
\alpha(k.v)E^{\mu}k_{\mu} + \gamma(k.v)k^{2} + \delta [k^{2} + \mu_{0}(k.v)^{2}] = 0
\]

\[
\beta [k^{2} - (1 - \mu_{0}\epsilon)(k.v)^{2}] = 0
\]

The solution of this system results in the following dispersion relations:

\[
k_{-}^{2} = (k.v)^{2} \left[ 1 - \mu_{0}(\epsilon E') \right] + \frac{1}{\epsilon E} \epsilon' E^{\alpha}E^{\beta}k_{\alpha}k_{\beta}, \tag{11}
\]

\[
k_{+}^{2} = (k.v)^{2} \left[ 1 - \mu_{0}(\epsilon E) \right] \tag{12}
\]

They correspond to the propagation modes

\[
e_{\nu}^{-} = \rho^{-} \left\{ \mu_{0}(k.v)^{2}E_{\nu} + E^{\alpha}k_{\alpha}[k_{\nu} - (k.v)u_{\nu}] \right\}. \tag{13}
\]

\[
e_{\nu}^{+} = \rho^{+} \eta_{\alpha\lambda\nu\mu}v^{\alpha}E^{\beta}k^{\mu}, \tag{14}
\]

where $\rho^{-}$ and $\rho^{+}$ are arbitrary constants. The labels “+” and “−” refer to the ordinary and extraordinary rays, respectively. Eqns. (11) and (12) govern the propagation of
photons in the medium characterized by $\mu = \mu_0 = \text{const.}$, and $\epsilon = \epsilon(E)$. They can be rewritten as $g_{\mu
u}^{\text{eff}} k_{\mu} k_{\nu} = 0$, where we have defined the effective geometries

$$g^{(-)}_{\mu\nu} = \eta_{\mu\nu} - [1 - \mu_0 (\epsilon E)] v^\mu v^\nu - \frac{1}{\epsilon E} \epsilon' E^\mu E^\nu$$
$$g^{(+)}_{\mu\nu} = \eta_{\mu\nu} - [1 - \mu_0 \epsilon] v^\mu v^\nu.$$  \hspace{1cm} (15)
$$g^{(+)}_{\mu\nu} = \eta_{\mu\nu} - [1 - \mu_0 \epsilon] v^\mu v^\nu.$$  \hspace{1cm} (16)

The metric given by Eqn. (15) was derived in [15], while the second metric very much resembles the metric derived by Gordon [16]. The difference is that in the case under consideration, $\epsilon$ is a function of the modulus of the external electric field, while Gordon worked with a constant permeability.

Let us discuss now a particular instance in which the vectors used as a basis in Eqn. (10) are not linearly independent. If we assume that

$$E^\mu = ak^\mu + bv^\mu,$$  \hspace{1cm} (17)
then, by Eqn. (10), the vectors $e^\mu$, $k^\mu$, and $v^\mu$ are coplanar. In this case, the basis chosen in Eqn. (10) is not appropriate. Notice however that if we assume that $e^\mu$ is a combination of vectors that are perpendicular to $k^\mu$, so that $(e.k) = 0$, Eqn. (3) implies that $(E.e) = 0$. The converse is also true: if $(E.e) = 0$, then, from Eqn. (17), $(k.e) = 0$. For this particular case, in which $e^\mu$ is perpendicular to $v^\mu$, $k^\mu$ (and consequently to $E^\mu$), Eqs. (8) and (9) imply that

$$[k^2 + (k.v)^2 (\mu_0 \epsilon - 1)] e^\mu = 0$$

We see then that in the case in which $E^\mu = ak^\mu + bv^\mu$, Fresnel’s equation determines that the polarization of the photons is perpendicular to the direction of propagation and to the velocity of the fluid. Moreover, the motion of these photons is governed by the metric $g^{\mu\nu}_{\text{eff}}$. For instance, if the electric field, the velocity of the fluid, and the direction of propagation are all radial, then the polarization is in the plane perpendicular to the propagation, and the two polarization modes feel the same geometry.

III. THE ANALOG BLACK HOLE

We shall show in this section that the system described by the effective metrics given by Eqs. (15-16) can be used to produce an analog black hole. It will be convenient to rewrite at this point the inverse of the effective metric given by Eqn. (10) using a different notation:

$$g^{(-)}_{\mu\nu} = \eta_{\mu\nu} - \frac{v^\mu v^\nu}{c^2} (1 - f) + \frac{\xi}{1 + \xi} l^\mu l^\nu.$$  \hspace{1cm} (18)

where we have defined the quantities

$$f \equiv \frac{1}{c^2 \mu_0 \epsilon (1 + \xi)}, \quad \xi \equiv \frac{\epsilon' E}{\epsilon}, \quad l^\mu \equiv \frac{E^\mu}{E}.$$  \hspace{1cm} (19)

Note that $\epsilon = \epsilon(E)$. We have introduced here the velocity of light $c$, which was set to 1 before.

Taking a Minkowskian background in spherical coordinates, and

$$v^\mu = (v_0, v_1, 0, 0), \quad E^\mu = (E_0, E_1, 0, 0),$$  \hspace{1cm} (20)
we get for the effective metric described by Eqn. (18),

$$g^{(-)}_{00} = 1 - \frac{v_0^2}{c^2} (1 - f) + \frac{\xi}{1 + \xi} l_0^2,$$  \hspace{1cm} (21)
$$g^{(-)}_{11} = - 1 - \frac{v_1^2}{c^2} (1 - f) + \frac{\xi}{1 + \xi} l_1^2,$$  \hspace{1cm} (22)
and $g^{(-)}_{22}$ and $g^{(-)}_{33}$ as in Minkowski spacetime. The vectors $v^\mu$ and $l^\mu$ satisfy the constraints

$$v_0^2 - v_1^2 = c^2,$$  \hspace{1cm} (23)
$$l_0^2 - l_1^2 = -1,$$  \hspace{1cm} (24)
$$v_0 l_0 - v_1 l_1 = 0.$$  \hspace{1cm} (25)

This system of equations can be solved in terms of $v_1$, and the result is

$$v_0^2 = c^2 + v_1^2,$$  \hspace{1cm} (26)
$$l_0^2 = \frac{v_1^2}{c^2}, \quad l_1^2 = \frac{c^2 + v_1^2}{c^2}.$$  \hspace{1cm} (27)

Now we can rewrite the metric in terms of $\beta \equiv v_1/c$ [24]. The explicit expression for the metric coefficients is:

$$g^{(-)}_{00} = \frac{1 - \beta^2 (c^2 \mu_0 \epsilon - 1)}{c^2 \mu_0 (\epsilon + \epsilon' E)},$$  \hspace{1cm} (28)
$$g^{(-)}_{01} = \beta \sqrt{1 + \beta^2} \frac{1 - c^2 \mu_0 \epsilon}{c^2 \mu_0 (\epsilon + \epsilon' E)},$$  \hspace{1cm} (29)
$$g^{(-)}_{11} = \frac{\beta^2 - c^2 \mu_0 \epsilon (1 + \beta^2)}{c^2 \mu_0 (\epsilon + \epsilon' E)}.$$  \hspace{1cm} (30)

From Eqn. (28) it is easily seen that, depending on the function $\epsilon(E)$, this metric has a horizon at $r = r_h$, given by the condition $g^{(-)}_{00}(r_h) = 0$ or, equivalently,

$$\left(\frac{c^2 \mu_0 \epsilon - 1}{\beta^2}\right) |_{r_h} = 1.$$  \hspace{1cm} (31)

The metric given above resembles the form of Schwarzschild’s solution in Painlevé-Gullstrand coordinates [18, 19]:

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 \pm 2\sqrt{\frac{2GM}{r}} dr dt - dr^2 - r^2 d\Omega^2.$$  \hspace{1cm} (32)
With the coordinate transformation
\[ dt_F = dt_S + \sqrt{\frac{2GM}{r}} \frac{1}{1 - \frac{2GM}{r}} \, dr, \] (33)
the line element given in Eqn.\((32)\) can be written in Schwarzschild’s coordinates. The “+” sign covers the future horizon and the black hole singularity.

The effective metric given by Eqs.\((28, 30)\) looks like the metric in Eqn.\((31)\). In fact, it can be written in Schwarzschild’s coordinates, with the coordinate change
\[ dt_{FG} = dt_S - \frac{g_{01}(r)}{g_{00}(r)} dr. \] (34)
Using this transformation with the metric coefficients given in Eqs.\((28, 29)\), we get the expression of \(g_{11}^{-}\) in Schwarzschild coordinates:
\[ g_{11}^{-} = -\frac{\epsilon(E)}{(1 - \beta^2[c^2\mu_0\epsilon(E) - 1])(\epsilon(E) + \epsilon(E)')}. \] (35)
Note that \(g_{00}^{-}\) is zero in the new coordinate system, while \(g_{00}^{-}\) is still given by Eqn.\((28)\). Consequently, the position of the horizon does not change, and is still given by Eqn.\((31)\).

Working in Painlevé-Gullstrand coordinates, we have shown that the metric for the “+” polarization describes a Schwarzschild black hole if Eqn.\((31)\) has a solution. Afterwards we have rewritten the “-” metric in the more familiar Schwarzschild coordinates. Let us consider now photons with the other polarization. They “see” the metric given by Eqn.\((10)\), whose inverse is given by:
\[ g^{\mu\nu}_{(+)} = \eta_{\mu\nu} - \frac{v_\mu v_\nu}{c^2} \left( 1 - \frac{1}{c^2\mu_0\epsilon(E)} \right). \] (36)
Using this equation and Eqs.\((26, 27)\) it is straightforward to show that
\[ g^{(+)}_{00} = 1 - (1 + \beta^2) \left( 1 - \frac{1}{c^2\mu_0\epsilon(E)} \right), \] (37)
\[ g^{(+)}_{01} = -\beta \sqrt{1 + \beta^2} \left( 1 - \frac{1}{c^2\mu_0\epsilon(E)} \right), \] (38)
\[ g^{(+)}_{11} = 1 - \beta^2 \left( 1 - \frac{1}{c^2\mu_0\epsilon(E)} \right). \] (39)
This metric also corresponds to a Schwarzschild black hole, for some \(\epsilon(E)\) and \(\beta\). Comparing Eqs.\((28, 30)\) and \((31)\) we see that the horizon of both analog black holes is located at \(r_h\), given by Eqn.\((31)\).

By means of the coordinate change defined by Eqn.\((34)\), we can write this metric in Schwarzschild’s coordinates. The relevant coefficients are given by
\[ g^{(+)}_{00} = \frac{1 + \beta^2(1 - c^2\mu_0\epsilon(E))}{c^2\mu_0\epsilon(E)} \epsilon(E). \] (40)

It is important to stress then that the horizon is located at \(r_h\) given by Eqn.\((31)\) for photons with any polarization. Moreover, the motion of the photons in both geometries will be qualitatively the same, as we shall show below.

### IV. AN EXAMPLE

We have not specified up to now the functions \(\epsilon(E)\) and \(E(r)\) that determine the dependence of the coefficients of the effective metrics with the coordinate \(r\). From now on we assume a linear \(\epsilon(E)\):\(^20\),
\[ \epsilon(E) = \epsilon_0(\chi + \chi^2E(r)), \] (42)
with \(\chi = 1 + \chi^{(1)}\). The nontrivial Maxwell’s equation then reads
\[ (\sqrt{-\gamma} \epsilon(r) F^{01})_{,1} = 0, \] (43)
where \(\gamma\) is the determinant of the flat background metric. Taking into account that
\[ (F^{01})^2 = \frac{E^2}{c^2}, \] (44)
we get as a solution of Eqn.\((41)\) for a point source in a flat background in spherical coordinates
\[ F^{01} = \frac{-\chi \pm \sqrt{\chi^2 + 4\chi^2Q/\epsilon_0r^2}}{2\chi^{(2)}}. \] (45)
Let us consider a particular combination of parameters: \(\chi^{(2)} > 0\), \(Q > 0\) and the “+” sign in front of the square root in \(F^{01}\), in such a way that \(E > 0\) for all \(r\). To get more manageable expressions for the metric, it is convenient to define the function \(\sigma(r)\):
\[ E(r) = \frac{\chi}{2\chi^{(2)}} \sigma(r) \] (46)
where
\[ \sigma(r) = -1 + \frac{1}{r^2} \left( \frac{1}{\sqrt{r^2 + q}} \right) \] (47)
and
\[ q = \frac{4\chi^{(2)}Q}{\epsilon_0\chi^{(2)}}. \] (48)
In terms of \(\sigma\), the metrics take the form
\[ ds^2^{-} = \frac{2 - \beta^2}{2} \left[ \frac{\chi (\sigma(r) + 2)}{(1 + \sigma(r))} d\tau^2 - \frac{2 + \sigma(r)}{|2 - \beta^2 (\chi (\sigma(r) + 2) - 2) (1 + \sigma(r))|} dr^2 - r^2 d\Omega^2, \] (49)
\[ ds^2_{(+)} = \frac{2 - \beta^2\left[ \chi (\sigma(r) + 2) - 2 \right]}{\chi (2 + \sigma(r))} \, dt^2 - \frac{2}{2 + \beta^2\left[ 2 - \chi (\sigma(r) + 2) \right]} \, dr^2 - r^2 d\Omega^2. \]  

Notice that the \((t, r)\) sectors of these metrics are related by the following expression:

\[ ds^2_{(+)} = \Phi(r) \, ds^2_{(-)} \]  

where the conformal factor \(\Phi\) is given by:

\[ \Phi = 2 \frac{1 + \sigma(r)}{2 + \sigma(r)} \]

We shall study next some features of the effective black hole metrics. We would like to remark that up to this point, the velocity of the fluid \(v_1\) is completely arbitrary; it can even be a function of the coordinate \(r\). We shall assume in the following that \(v_1\) is a constant. This assumption, which will be lifted in Sect. V, may seem rather restrictive but it helps to display the main features of the effective metrics in an easy way.

To study the motion of the photons in these geometries, we can use the technique of the effective potential. Standard manipulations (see for instance [20]) show that in the case of a static and spherically symmetric metric, the effective potential is given by

\[ V(r) = \varepsilon^2 \left( 1 + \frac{1}{g_{00}(r) g_{11}(r)} \right) - \frac{L^2}{r^2 g_{11}(r)} \]  

where \(\varepsilon\) is the energy and \(L\) the angular momentum of the photon.

In terms of \(\sigma(r)\), and of the impact parameter \(b^2 = L^2/\varepsilon^2\), the "small" effective potential \(v(r) \equiv V(r)/\varepsilon^2\) for the metric Eqn. (50) in Schwarzschild coordinates can be written as follows:

\[ v^(-)(r) = 1 - \frac{2(1 + \sigma(r))^2}{2 + \sigma(r)} - \frac{b^2 (2 - \beta^2 \sigma(r))(1 + \sigma(r))}{2 + \sigma(r)} \]  

A short calculation shows that \(v^(-)\) is a monotonically decreasing function of \(\beta\). Consequently, we shall choose a convenient value of it, for the sake of illustrating the features of the effective potential. Figures 1 and 2 show the plots of the potential for the \((-)\) metric for several values of the relevant parameters.

The effective potential for the Gordon-like metric can be obtained in the same way. From Eqns. (52) and (51) we get

\[ v^(+)(r) = 1 - \frac{2 + \sigma(r)}{2} + \frac{b^2}{2r^2} [2 - \beta^2 \sigma(r)] \]  

The plots in Figures 3 and 4 show the dependence of \(v^(+)(r)\) on the different parameters.

We see from these plots that, in the case of a constant flux velocity, the shape of the effective potential for both metrics qualitatively agrees with that for photons moving on the geometry of a Schwarzschild black hole (see for instance Ref. [20], pag. 143).

\[ \beta^2(r_h) = \frac{1}{\chi - 1}. \]

V. SURFACE GRAVITY AND TEMPERATURE

Let us now go back to the more general case of \(\beta = \beta(r)\), and calculate the "surface gravity" of our analog black hole. We present first the results for the constant permittivity case. By setting \(\epsilon'(E) \equiv 0\) in the metrics Eqns. (14) and (15), we regain the example of constant index of refraction studied for instance in [13]. It is easy to show from Eqn. (52) that the horizon of the black hole in this case is given by

\[ \beta^2(r_h) = \frac{1}{\chi - 1}. \]
The “surface gravity” of a spherically symmetric analog black hole in Schwarzschild coordinates is given by \[17\]

\[\kappa = \frac{c^2}{2} \lim_{r \to r_h} \frac{g_{00,r}}{\sqrt{|g_{11}|} g_{00}}.\]  

For the metrics Eqns. (15) and (16) with \(\epsilon = \epsilon_0 \bar{\chi}\) and \(r_h\) given by Eqn. (55), the analog surface gravity is

\[\kappa = -\frac{c^2}{2} \frac{1 - \bar{\chi}}{\sqrt{\bar{\chi}}} (\beta^2)'|_{r_h}.\]  

In this expression we can see the influence of the dielectric properties of the fluid (through the constant \(\bar{\chi}\)) and also of its dynamics through the physical acceleration in the radial direction, given by

\[a_r|_{r_h} = \frac{c^2}{2} (\beta^2)'|_{r_h},\]  

for \(\beta^2(r_h) \ll 1\). This acceleration is a quantity that must be determined solving the equations of motion of the fluid \[27\].

Going back the the more general case of a linear permittivity, described by the metrics given by Eqns. (15) and (51), and considering that \(\beta(r_h) \ll 1\), the radius of the horizon is \[28\]:

\[r_h^2 = \frac{9\bar{\chi}^2}{4} \beta^4(r_h).\]  

Using the expressions given above, the result for the surface gravity of the “−” black hole for \(\beta(r_h) \ll 1\) is

\[\kappa(-) = \frac{c^2}{\beta} \left(\frac{1}{\bar{\chi} \sqrt{q}} - \frac{1}{2} (\beta^2')\right)|_{r_h}.\]  

This equation differs from the surface gravity of the case of constant permittivity (Eqn. (57)) by the presence of a new term that does not depend on the acceleration of the fluid. To see where this new term comes from, we can take the derivative of the metric coefficient \(g_{00}\), given by Eqn. (28), with respect to the coordinate \(r\) (see the definition of \(\kappa\), Eqn. (56)). Taking into account Eqn. (12), the result is

\[g_{00,r} = 2 \frac{\beta \beta' (\epsilon(E) + \epsilon(E)' E) (1 - c^2 \mu_0 \epsilon(E)) - d\epsilon(E)}{c^2 \mu_0 (\epsilon(E) + \epsilon(E)' E)^2}.\]  

In this expression, the first term is the generalization of the term corresponding to the case \(\epsilon = \text{const.}\) (compare with Eqn. (57)), which mixes the acceleration of the fluid with its dielectric properties. On the other hand, the second term, which is the new term displayed in Eqn. (59), is related only to the dielectric properties of the fluid.

It is easy to show that these results also apply to the black hole described by the Gordon-like metric. This is not surprising though, because of the conformal relation between the two metrics, given by Eqn. (51) \[22\].

\section{VI. DISCUSSION}

We have shown that a flowing inhomogeneous dielectric that depends linearly on an external electric field generates one effective metric for each polarization state of photons. A particular configuration of the fluid (i.e. pure radial flow) plus an external electric field was shown to be an analog black hole, with a radius that depends on the function \(\epsilon(E)\). The features of the black hole depend on the charge that generates the electric field, on the properties of the dielectric, and on the velocity profile. Note in particular that in the absence of flux \((\beta = 0)\) the metrics do not describe a black hole.

The existence of two metrics reflect the birefringent properties of the medium. Although some claims have been made that birefringent materials spoil from the beginning the idea of an effective metric, we have shown here that for a special configuration, even if two metrics
are present, photons with different polarizations experience the same horizon. Moreover, as seen from the plots of the effective potential, the motion of these photons in the medium will depend on their polarization, but is qualitatively the same for both types of photons.

At first sight it may seem that by choosing an appropriate material and a convenient value of the charge we could obtain a high value of the temperature of the radiation, given by

\[ T \approx \frac{\hbar}{2\pi k_{BC}} \kappa \approx 4 \times 10^{-21} \kappa Ks^2/m, \]

with \( \kappa \) given in Eqn. (59). However, the equation for the surface gravity can be rewritten as

\[ \kappa = c^2 \left( \frac{\beta}{2r} - \beta' \right) \bigg|_{r_h} . \]

We see then that, because \( \beta(r_h) \ll 1 \), the new term appearing in \( \kappa \) is bound to be very small, but of the same order of magnitude of the acceleration term. To summarize, we have shown that a new term emerges in the surface gravity which is produced by the properties of the material and is comparable in magnitude to the acceleration term. This result indicates that it may be worth to study if some media with nonlinear dependence on an external electric field are appropriate to generate analog black holes with Hawking radiation with a higher temperature than that produced up to date with other systems.

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[24] Notice that this definition of \( \beta \) coincides with the usual one for small \( v_1 \).
[25] It should be kept in mind that for the metric in Eqns. (28)-(30) to have \( g_{tt} = -1 \) as in Painlevé-Gullstrand, a simple conformal transformation is needed. Consequently, the metric of the “Schwarzschild black hole” presented below is actually conformal to the Schwarzschild metric.
[26] This type of behaviour is exhibited for instance by electrorheological fluids. See for instance W. Wen, S. Men, and K. Lu, Phys. Rev. E 55, 3015 (1997).
[27] If we set \( \beta \equiv 0 \) in Eqns. (28)-(30), we cease to have a black hole (this situation was analyzed in [4]).
[28] Notice that we cannot take the limit \( q \rightarrow 0 \) in this expression or in any expression in which this one has been used.
[29] The concept of temperature, and indeed that of effective geometry is valid in this context only for photons with wavelengths long compared to the intermolecular spacing in the fluid. For shorter wavelengths, there would be corrections to the propagation dictated by the effective metric. However, results for other systems (such as dumb black holes [4] and Bose-Einstein condensates) suggest that the phenomenon of Hawking radiation is robust (i.e. independent of this “high-energy” physics).