Discontinuous nonlocal conservation laws
and related discontinuous ODEs
Existence, Uniqueness, Stability and Regularity

Alexander Keimer\textsuperscript{1,2} and Lukas Pflug\textsuperscript{2,3}\textsuperscript{*}

\textsuperscript{1}Institute of Transportation Studies, UC Berkeley, Sutardja Dai Hall, Berkeley, 94720, California, USA.
\textsuperscript{2}Competence Unit for Scientific Computing, Friedrich-Alexander University Erlangen-Nuremberg (FAU), Martensstr. 5a, Erlangen, 91058, Bavaria, Germany.
\textsuperscript{3}Department of Mathematics, Chair of Applied Mathematics (Continuous Optimization), Friedrich-Alexander University Erlangen-Nuremberg (FAU), Cauerstr. 11, Erlangen, 91058, Bavaria, Germany.

\textsuperscript{*}Corresponding author(s). E-mail(s): keimer@berkeley.edu; lukas.pflug@fau.de;

Abstract

We study nonlocal conservation laws with a discontinuous flux function of regularity $L^\infty(\mathbb{R})$ in the spatial variable and show existence and uniqueness of weak solutions in $C([0,T];L^1_{\text{loc}})$, as well as related maximum principles. We achieve this well-posedness by a proper reformulation in terms of a fixed-point problem. This fixed-point problem itself necessitates the study of existence, uniqueness and stability of a class of discontinuous ordinary differential equations. On the ODE level, we compare the solution type defined here with the well-known Carathéodory and Filippov solutions.

Keywords: Discontinuous ordinary differential equations, stability theory, conservation laws, nonlocal conservation laws, discontinuous velocity, discontinuous flux

MSC Classification: 34A12, 34A36, 35L03, 35L65, 35Q99, 35R09, 45K05
1 Introduction

In this contribution we study nonlocal conservation laws with a discontinuous part of the velocity in space. The discontinuity enters the equation as a multiplicative term and is assumed to be bounded away from zero. The only additional requirement is that it possesses the regularity $L^\infty$. Nonlocal refers to the fact that the flux of the conservation law at a given space-time point not only depends on the solution at this point, but also on a spatial averaging around this position by means of a convolution, in equations

$$q_t + (v(x)V(\gamma * q) q)_x = 0,$$

with discontinuous part of the velocity $v \in L^\infty(\mathbb{R}; \mathbb{R}_{>0})$ for $v \in \mathbb{R}_{>0}$, Lipschitz-continuous part of the velocity $V \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$ and nonlocal weight $\gamma \in BV(\mathbb{R}; \mathbb{R}_{\geq 0})$. For details see Defn. 2.

A variety of results for nonlocal conservation laws have been provided over the last few years [1–53], but only in recent publications [54, 55] has a discontinuity been considered exactly as denoted in Eq. (1). The authors use Entropy methods together with a type of Godunov discretization scheme and a viscosity approximation to demonstrate well-posedness. They also present a maximum principle for a discontinuity with one jump, where the discontinuity is monotonically chosen so that the solution cannot increase. This can be envisioned by considering a nonlocal version of the classical LWR model in traffic ([56]) and assuming that traffic flows to the right. Then, if the discontinuous part of the velocity is monotonically increasing, the velocity is faster after each jump, meaning that no increase of density can appear around the discontinuities.

Describing our approach in more general terms, we consider the discontinuous velocity as an $L^\infty(\mathbb{R})$ function that is positive and bounded away from zero, and we then deal with the dynamics introduced in Eq. (1) and detailed in Defn. 2. We show that weak solutions exist and are unique without an Entropy condition, and present several maximum principles under which the solution exists semi-globally. In contrast to the Diperna Lions ansatz [57], where the existence of ODEs is shown by studying the corresponding (linear) conservation law, we tackle the problem by formulating the characteristics of the conservation law as a fixed-point problem (as we first described in [10] based on the idea proposed in [58]) and dealing with the corresponding discontinuous ODE. This emerging nonlinear discontinuous ODE, then reads as

$$x'(t) = v(x(t)), \lambda(t, x(t))$$

with discontinuous $v \in L^\infty(\mathbb{R}; \mathbb{R}_{>2})$ for $v \in \mathbb{R}_{>0}$ and $\lambda \in L^\infty((0, T); W^{1,\infty}(\mathbb{R}))$ Lipschitz-continuous w.r.t. the spatial variable. For details see Defn. 1.

For the broad theory on discontinuous ODEs and initial value problems we refer the reader to [59–68]. As the solution to the discontinuous ODE is later subject to the aforementioned fixed-point problem, we not only show existence and uniqueness of solutions, but also stability and continuity results. This is not
covered by the established theory of discontinuous ODEs and requires the specific structure of the discontinuous ODE considered here. Having established these stability estimates and results, demonstrating the existence and uniqueness of discontinuous nonlocal conservation laws on small time horizons is straightforward following the approaches adopted in [10]. This is supplemented by an approximation result in a “weak” topology, which ultimately enables us to present different types of maximum principles resulting in semi-global well-posedness.

1.1 Outline

In Section 1, we introduce the problem and compare our results with those in the literature. We conclude the section with some basic definitions in Section 1.2, which specify what we mean by “solutions to the introduced problem class.”

Section 2 is dedicated to the well-posedness and stability properties of solutions to the class of discontinuous ODEs introduced. Having defined what we mean by solutions and stated required assumptions, we then concentrate in Section 2.1 on the existence and uniqueness of solutions and how they compare to Carathéodory and Filippov solutions. For the existence theory for nonlocal conservation laws in Section 3 we require stability of the characteristics with regard to input datum, In Section 2.2 we thus consider the stability of the solutions to the discontinuous ODE with respect to initial datum, Lipschitz velocity and discontinuous velocity in a suitable topology. Section 2.3 considers the regularity of the derivative of solutions of the discontinuous ODE with respect to the initial datum in the topology induced by $\mathcal{C}([0,T]; L^1_{\text{loc}}(\mathbb{R}))$, another important ingredient for the well-posedness of the discontinuous nonlocal conservation law studied later.

In Section 3 we finally study the described class of discontinuous nonlocal conservation laws, beginning by presenting the assumptions on the data involved. In broad terms, for the initial datum and the discontinuous velocity we assume only $L^\infty$ regularity. This is identical to the assumption described for the discontinuous ODE in Section 2. In Section 3.1 we then study the well-posedness of the discontinuous nonlocal conservation law via formulating a fixed-point problem in the Banach space $L^\infty((0,T); L^\infty(\mathbb{R}))$ and using the method of characteristics. We first establish well-posedness of solutions on small time horizons, followed by stability results for the solution with respect to the discontinuous and continuous part of the velocity. We also establish stability for the initial datum in a weak topology, enabling the approximation of solutions by smooth solutions of the corresponding “smoothed” nonlocal conservation law. Under relatively mild additional assumptions on the nonlocal kernel and the Lipschitz-continuous velocity, for nonnegative initial datum we show different versions of maximum principles in Section 3.2. One version states that the $L^\infty$ norm of the solution can only decrease over time providing
the discontinuity is monotonically decreasing, while another only gives uniform upper bounds on the solutions for a general discontinuity. These results also imply the semi-global well-posedness of the solutions.

We conclude the contribution in Section 4 with some open problems.

**Perspective from (local) conservation laws**

From the perspective of approximating local conservation laws by nonlocal conservation laws \([14, 19, 31, 40, 42, 43]\), we consider the nonlocal approximations of the following discontinuous (local) conservation laws:

\[
q_t + \frac{\partial}{\partial x} \left( v(x) \cdot f(q) \right)_x = 0,
\]

with \( f \equiv V \cdot \operatorname{Id} \) for \( V \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) \) and \( v \in L^\infty(\mathbb{R}; \mathbb{R}_{\geq 0}) \), \( v \in \mathbb{R}_{> 0} \). Thus, we are dealing with the nonlocal approximation of a multiplicative discontinuous – in space – velocity field. However, we will not be studying this limiting behaviour in this work.

Discontinuous conservation laws have been considered in terms of questions of existence and uniqueness, and the need to prescribe the proper Entropy condition at the discontinuity in order to single out the proper (and potentially physically reasonable) solution among the infinite number of weak solutions. A vast number of papers on these topics have been published. For the sake of brevity, we refer the reader to \([69–87]\) and note that this list is by no means exhaustive.

**Simplified results covered by the developed theory**

The results obtained can also be applied to special cases of nonlocal conservation law, i.e. nonlocal dynamics with Lipschitz continuous velocity function (setting \( v \equiv 1 \))

\[
q_t + \left( V(\gamma \ast q) \right)_x = 0.
\]

This case (including source terms on the right hand side) has been intensively studied in \([10]\) and, indeed, we recover the same results also obtained in Section 3. Thus, the theory proposed here generalizes the results presented in \([10]\).

Discontinuous linear conservation laws represent another specific case. Choosing \( V \equiv 1 \) we have

\[
q_t + \left( v(x)q \right)_x = 0
\]

and enriching this with a Lipschitz-continuous (in space) velocity \( \lambda: \Omega_T \rightarrow \) (this is covered by our later analysis on discontinuous ODEs in Section 2), for
the Cauchy problem we obtain
\[ q_t + \left( v(x) \lambda(t, x) q \right)_x = 0. \]
This is supplemented by an initial condition in \( L^\infty \) that there is a unique weak solution. Surprisingly, linear conservation laws with discontinuous velocities have not been considered intensively. We refer the reader to [88], where the author studies
\[ \rho_t + \left( f(t, x) \rho \right)_x = 0 \]
with
- \( f \) continuous and nonnegative,
- \( f \) of such a form that the solutions to the corresponding ODEs do not blow up in finite time (for instance assuming that \( f \) can grow at most linearly with regard to the spatial variable),
- the sets of points where \( f \) is zero are somewhat “nice” (see [88, (A1)-(A3), p. 3138]).

However, this setup differs from our considered class of equations as we allow \( L^\infty \) regularity and have no sign restrictions for the Lipschitz-part. [89] obtains results for velocities of regularity \( L^\infty \) with the additional assumption that \( \text{div}(f) \in L^\infty \). The second assumption is weak for multi-D equations as considered in that publication. However, in the scalar case this assumption boils down to a Lipschitz-continuous velocity field, so the presented result can be seen as a generalization in the 1D case.

The multi-D case is also considered in [90, 91] where the velocity field is assumed to be in \( BV \) or admits other Sobolev regularity.

For the characteristics, [92] uses Filippov solutions [60] (see [92, Eq. 2.5 and Eq. 2.6]) and considers the transport equation (not the conservation law) with a one-sided Lipschitz-continuous velocity processing unique solutions when assuming a continuous initial datum due to the uniqueness of backward characteristics in the sense of Filippov [59]. Similar results are obtained in [93]. Finally, [94] considers again the multi-D case and states conditions on the vector field for existence and uniqueness of solutions. Thereby, the vector field is assumed to be continuous and for existence and uniqueness a “weakened” Lipschitz-condition based on the modulus of continuity is required. Solutions are thought of in the space of signed Borel measures.

1.2 Basic definitions

In this section, we rigorously state the problems that we will tackle. Starting with the discontinuous IVP, the problem reads as:

**Definition 1** (Discontinuous IVP). Let \( T \in \mathbb{R}_{>0} \) and \( v \in \mathbb{R}_{>0} \) be given. For a discontinuous \( v \in L^\infty(\mathbb{R}; \mathbb{R}_{\geq v}) \), a smooth \( \lambda \in L^\infty((0, T); W^{1,\infty}(\mathbb{R})) \) and
$x_0 \in \mathbb{R}$, we consider the following discontinuous IVP
\[
  x'(t) = v(x(t))\lambda(t,x(t)), \quad t \in [0,T] \\
  x(0) = x_0.
\] (3)

Thereby, $v$ represents the **discontinuous part of the velocity**, $\lambda$ the **Lipschitz continuous part of the velocity** and $x_0$ the initial value.

As outlined above, the existence, uniqueness and regularity of solutions to the discontinuous IVP are strongly related to the existence and uniqueness of solutions to the following nonlocal conservation law:

**Definition 2** (The discontinuous (in space) nonlocal conservation law). Let $T \in \mathbb{R}_{>0}$ be given and $\Omega_T \coloneqq (0,T) \times \mathbb{R}$. For $q : \Omega_T \rightarrow \mathbb{R}$, **initial datum** $q_0 \in L^\infty(\mathbb{R})$, Lipschitz-continuous velocity $V \in W^{1,\infty}(\mathbb{R})$, **discontinuous part of the velocity** $v \in L^\infty(\mathbb{R};\mathbb{R}_{>0})$ with $v \in \mathbb{R}_{>0}$ and **nonlocal weight** $\gamma \in BV(\mathbb{R};\mathbb{R}_{>0})$, we call the following Cauchy problem
\[
  q_t(t,x) + \partial_x \left( v(x)V(\gamma q(t,\cdot))(x))q(t,x) \right) = 0 \quad (t,x) \in \Omega_T \\
  q(0,x) = q_0(x) \quad x \in \mathbb{R}
\]
a **discontinuous nonlocal conservation law**.

The stated results can naturally be extended, as outlined in the following **Rmk. 1**.

**Remark 1** (Generalizations – Extensions). For the sake of a type of “completeness or generality” of the developed theory, we mention that the results established in this work can be extended to general nonlocal terms and explicitly space- and time-dependent velocity functions, as well as balance laws, i.e., it is also possible to obtain the well-posedness of the more general **discontinuous nonlocal balance law** in $(t,x) \in \Omega_T$
\[
  q_t(t,x) + \partial_x \left( v(x)\tilde{V}(t,x,\mathcal{W}[q,\tilde{\gamma}](t,x))q(t,x) \right) = h(t,x,q(t,x),\mathcal{W}[q,\tilde{\gamma}](t,x)) \\
  q(0,x) = q_0(x) \\
  \mathcal{W}[q,\tilde{\gamma}] := \int_\mathbb{R} \tilde{\gamma}(t,x,y)q(t,y)\,dx,
\]
with $\tilde{V} : [0,T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ also Lipschitz in the explicit spatial variable, $\tilde{\gamma} : [0,T] \times \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}$ Lipschitz in the second component and TV in the third component (compare [10, 13, 39]), and $h : \Omega_T \times \mathbb{R}^2 \rightarrow \mathbb{R}$ Lipschitz in the third and fourth component and of corresponding regularity in $(t,x)$. We do not go into details here. For smooth kernels, it is even possible to extend results to measure-valued solutions (for measure-valued initial datum) similarly to [46, 52].
Discontinuous nonlocal conservation laws and related discontinuous ODEs

For both problem classes Definitions 1 and 2, we will present proper definitions of solutions in Definitions 3 and 6 and demonstrate the existence and uniqueness in Theorems 2.1 and 3.1. It is worth underlining once more that in particular for the discontinuous nonlocal conservation law no Entropy condition is required to obtain uniqueness of weak solutions. This has already been proven in [10] for Lipschitz-continuous velocities.

2 Existence, uniqueness and stability of the discontinuous IVP

In this section, we study the existence, uniqueness and stability (with regard to all input parameters and functions) of the discontinuous ODE introduced in Defn. 1. Let us first recall the assumptions on the involved datum in the following Asm. 1.

Assumption 1 (Involved datum). For a $T \in \mathbb{R}_{>0}$ denoting the considered time horizon, we assume

Discontinuous part: $v \in L^\infty(\mathbb{R};\mathbb{R}_{\geq})$ with $\underline{v} \in \mathbb{R}_{>0},$

Lipschitz-continuous part: $\lambda \in L^\infty((0,T) ; W^{1,\infty}(\mathbb{R})).$

For the considered class of discontinuous initial value problems in Defn. 1, we must first define what we mean by a solution. This becomes clear when recalling that $v \circ x$ is not necessarily measurable for $x \in W^{1,\infty}((0,T))$ since $x$ could be locally constant and, as a $L^\infty$ function, $v$ does not possess significantly “good representatives” with respect to the Lebesgue measure. However, due to the positive lower bound on $v$ and its time-independence, we can divide the strong form of solution by $v$ and by integration obtain the following integral definition of a solution:

Definition 3 (Solutions for Defn. 1). For $x_0 \in \mathbb{R}$ and the data as in Asm. 1, a solution to the discontinuous IVP in Defn. 1 is defined as a function $x \in C([0,T])$ such that

$$\int_{x_0}^{x(t)} \frac{1}{v(y)} \, dy = \int_0^t \lambda(s, x(s)) \, ds, \quad \forall t \in [0,T].$$

A solution is denoted by $X[v, \lambda](x_0; \cdot),\,$ with $x_0$ indicating the considered initial datum at time $t = 0, v$ the discontinuous part of the velocity and $\lambda$ the Lipschitz-continuous part.

Remark 2 (Reasonability of Defn. 3). The definition of solutions in Eq. (4) is more usable than the “classical” Carathéodory introduced later. It enables the existence – and later also stability properties – to be tackled without
prescribing additional regularity assumptions on \( x \) (such as measurability of \( v(x(\cdot)) \)). Compare in particular with Defn. 4.

The introduced notation \( \mathcal{X}[v,\lambda](x_0;\cdot) \) is later justified in Section 2.1, where we prove existence and uniqueness of solutions. ■

### 2.1 Existence/Uniqueness of solutions and their relation to “classical” Carathéodory and Filippov solutions

In the following Thm. 2.1, we prove the existence and uniqueness of solutions by decomposing the problem into two problems that possess ”nicer” properties and can be studied separately:

**Theorem 2.1** (Existence and uniqueness of solution in Defn. 3). Let \( T \in \mathbb{R}_{>0} \) be given and Asm. 1 hold. Then, in the sense of Defn. 3 there exists a unique solution

\[
\mathcal{X}[v,\lambda](x_0;\cdot) \in W^{1,\infty}_c((0,T)).
\]

In addition, defining the following surrogate expression

\[
Z[v](x_0;\cdot) := \int_{x_0}^{x} \frac{1}{v(s)} \, ds \quad \text{on } \mathbb{R},
\]

it holds that \( \mathbb{R} \ni x \mapsto Z[v](x_0;x) \) is invertible \( \forall (x_0,v) \in \mathbb{R} \times L^{\infty}(\mathbb{R};\mathbb{R}_{\geq 0}) \) and for the inverse we write \( Z[v]^{-1}(x_0;\cdot) : \mathbb{R} \to \mathbb{R} \).

Finally, calling \( C[\lambda,Z[v](x_0;\cdot)] \) the solution \( c \) of the integral equation

\[
c(t) = \int_0^t \lambda(s,Z[v]^{-1}(x_0;c(s))) \, ds, \quad \forall t \in [0,T],
\]

the identity “\( \mathcal{X} \equiv Z^{-1} \circ C \)” holds – in full notation –

\[
\mathcal{X}[v,\lambda](x_0;\cdot) \equiv Z[v]^{-1}(x_0;C[\lambda,Z[v](x_0;\cdot)](\cdot)) \quad \text{on } [0,T].
\]

**Proof** Equation (5) is well defined and by construction \( Z[v](x_0;\cdot) \in W^{1,\infty}_c(\mathbb{R}) \) so that

\[
\frac{1}{|v|_{L^{\infty}(\mathbb{R})}} \leq \partial_x Z[v](x_0;x) \leq \frac{1}{\underline{v}} \quad \forall x \in \mathbb{R}
\]

and thus \( \partial_{x} Z[v](x_0;\cdot) \in L^{\infty}(\mathbb{R}) \). Additionally, as \( x \mapsto Z[v](x_0;x) \) is strictly monotone, the inverse mapping \( Z[v]^{-1}(x_0;\cdot) \) is well defined and, thanks to Eq. (8),

\[
Z[v]^{-1}(x_0;\cdot) \in W^{1,\infty}_c(\mathbb{R}) : \quad \partial_{x} Z[v]^{-1}(x_0;\cdot) \in L^{\infty}(\mathbb{R}).
\]

Next, considering the definition of \( C[\lambda,Z[v](x_0;\cdot)] \) in Eq. (6) and the fact that \( \lambda \in L^{\infty}((0,T);W^{1,\infty}(\mathbb{R})) \), the composition \( x \mapsto \lambda(s,Z[v]^{-1}(x_0;x)) \), is – thanks to the previous estimates – globally Lipschitz-continuous for each \( (s,x_0) \in (0,T) \times \mathbb{R} \) and thus, there exists a unique Carathéodory solution (see for instance [95]) \( C[\lambda,Z[v](x_0;\cdot)] \in W^{1,\infty}_c((0,T)) \).
We now need to check whether the $\mathcal{X}[v, \lambda](x_0; \cdot) \in W^{1,\infty}((0, T))$ as in Eq. (7) indeed satisfies Defn. 3. We have by the very definition of $\mathcal{X}[v, \lambda](x_0; \cdot) \forall t \in [0, T]$

\[ C[\lambda, \mathcal{Z}[v](x_0; \cdot)](t) = \int_{0}^{t} \lambda(s, \mathcal{Z}[v]^{-1}(x_0; c(s))) \, ds = \int_{0}^{t} \lambda(s, \mathcal{X}[v, \lambda](x_0; s)) \, ds \]

and, as $\mathcal{Z}[v](x_0; \cdot) \circ \mathcal{X}[v, \lambda](x_0; \cdot) \equiv C[\lambda, \mathcal{Z}[v](x_0; \cdot)]$, by assumption

\[ \mathcal{Z}[v](x_0; \mathcal{X}[v, \lambda](x_0; t)) = \int_{0}^{x_0} \mathcal{X}[v, \lambda](x_0; t) \frac{1}{v(s)} \, ds, \]

which is the definition of a solution in Defn. 3. This demonstrates the existence of solutions. For the uniqueness, assume that we have two solutions $\mathcal{X}, \tilde{\mathcal{X}} \in C([0, T])$ satisfying Defn. 3. Then, the difference satisfies

\[ \int_{X(t)}^{\tilde{X}(t)} \frac{1}{v(z)} \, dz = \int_{0}^{t} \lambda(s, \mathcal{X}(s)) - \lambda(s, \tilde{\mathcal{X}}(s)) \, ds \quad \forall t \in [0, T] \]

and, as $\lambda \in L^{\infty}((0, T) ; W^{1,\infty}(\mathbb{R}))$, we obtain

\[ \frac{|X(t) - \tilde{X}(t)|}{|v|_{L^{\infty}(0, T)}} \leq \left| \int_{X(t)}^{\tilde{X}(t)} \frac{1}{v(z)} \, dz \right| = \left| \int_{0}^{t} \lambda(s, \mathcal{X}(s)) - \lambda(s, \tilde{\mathcal{X}}(s)) \, ds \right| \leq \| \partial_{2}\lambda \|_{L^{\infty}((0, T) ; L^{\infty}(\mathbb{R}))} \int_{0}^{t} |\mathcal{X}(s) - \tilde{\mathcal{X}}(s)| \, ds. \]

Applying Grönwall’s inequality [96, Chapter I, III Gronwall’s inequality] yields

\[ |x(s) - \tilde{x}(s)| = 0, \quad \forall s \in [0, t], \]

thus the two solutions must be identical. This concludes the proof. 

In the following, we show that the unique solution of the discontinuous IVP in Defn. 1 in the sense of Defn. 3 is also a “classical” Carathéodory solution in the following sense:

**Definition 4** (Carathéodory solutions for Defn. 1). Let Asm. 1 hold. Then, for the initial datum $x_0 \in \mathbb{R}$ we call a function $\mathcal{X} \in C([0, T])$ a Carathéodory solution for Defn. 1 iff $t \mapsto v(\mathcal{X}(t))\lambda(t, \mathcal{X}(t))$ is Lebesque measurable and

\[ \mathcal{X}(t) = x_0 + \int_{0}^{t} v(\mathcal{X}(s))\lambda(s, \mathcal{X}(s)) \, ds, \quad \forall t \in [0, T]. \tag{9} \]

Notice the difference to the usual definition of a Carathéodory solution, where the measurability of the integrand is given by construction (either by being continuous or having a right hand side which is strictly bounded away from zero so that solutions are strictly monotone).

**Lemma 2.1** (Equivalence Carathéodory solution and solutions as in Defn. 3). There exists a unique Carathéodory solution as in Defn. 4 iff there exists a unique solution as in Defn. 3.
Discontinuous nonlocal conservation laws and related discontinuous ODEs

Proof We start by showing that solutions in the sense of Defn. 3 are also Carathéodory solutions as in Defn. 4. Recalling the steps of the proof in Thm. 2.1, we find that

$$Z[v]^{-1}(x_0; s) = x_0 + \int_0^s v(Z[v]^{-1}(x_0; u)) \, du \quad \forall s \in \mathbb{R}. \quad (10)$$

This identity is a direct consequence of the following manipulation for $s \in \mathbb{R}$ (for the sake of briefer notation we write $Z$, $Z^{-1}$ and suppress the dependencies on $v$ and $x_0$):

$$Z^{-1}(Z(s)) = x_0 + \int_0^s \frac{1}{v(x)} \, dx \, v(Z^{-1}(u)) \, du. \quad (11)$$

Substituting $Z^{-1}(u) = w$ (which is possible according to Eq. (8))

$$w = x_0 + \int_0^s v(w) Z'(w) \, dv = x_0 + \int_0^s dv = s. \quad (12)$$

As this holds for all $s \in [0, T]$, we have proven that $Z[v]^{-1}$ indeed satisfies Eq. (10).

To show that $\mathcal{X}$, which is constructed via Thm. 2.1, satisfies Eq. (9), we next apply $Z^{-1}$ to Eq. (4) and obtain

$$Z^{-1}\left(\int_0^t \lambda(s, \mathcal{X}(s)) \, ds\right) = x_0 + \int_0^t \lambda(s, \mathcal{X}(s)) \, ds \, v(Z^{-1}(u)) \, du \quad (13)$$

$$= x_0 + \int_{\mathcal{C}(0)}^{\mathcal{C}(t)} v(Z^{-1}(u)) \, du. \quad (14)$$

Substituting $u = \mathcal{C}(\tau)$ for $\tau \in \mathbb{R}$ chosen accordingly

$$\mathcal{X} = x_0 + \int_0^t v(Z^{-1}(\mathcal{C}(\tau))) \mathcal{C}'(\tau) \, d\tau \quad (15)$$

$$= x_0 + \int_0^t v(\mathcal{X}(\tau)) \lambda(\tau, \mathcal{X}(\tau)) \, d\tau. \quad (16)$$

As the left hand side of Eq. (4) is given by $Z \circ \mathcal{X}$ in terms of $Z$ (as defined in Eq. (5)), by applying $Z^{-1}$ we obtain

$$Z^{-1}(Z(\mathcal{X}(t))) = \mathcal{X}(t), \quad \forall t \in [0, T]. \quad (17)$$

Thus, if $\mathcal{X}$ is a solution in the sense of Defn. 3, it is also a Carathéodory solution. For showing the equivalence we mention that all of the previous manipulations are equivalent transforms, and we can start with the identity Eq. (9) and go backwards in the presented proof. □

Finally, we want to close the gap to Filippov solutions [59, 60]. We first define what we mean by Filippov solutions, sticking with [60, 2: Definition of the solution]:

Definition 5 (Filippov solution for a differential equation). We call a function $\mathcal{X} \in W^{1,1}(0, T)$ for $T \in \mathbb{R}_{>0}$ a Filippov solution of the discontinuous initial value problem in Defn. 1 iff

$$\mathcal{X}(0) = x_0$$

$$\mathcal{X}'(t) \in K[f(t, \cdot)](\mathcal{X}(t)), \quad \forall t \in [0, T] \text{ a.e.}$$
with \( f = v \cdot \lambda \), \( K \) being defined as
\[
K[f(t, \cdot)] := \bigcap_{\delta \in \mathbb{R}_{>0}} \bigcap_{\mu(N) = 0, N \subset \mathbb{R}} \text{conv} \left\{ f(t, B_\delta(\mathcal{X}(t)) \setminus N) \right\}, \tag{18}
\]
where \( \mu \) denotes the Lebesgue measure on \( \mathbb{R} \).

Given this definition, we have the existence of Filippov solutions according to the following

**Theorem 2.2** (Existence of solutions). Given Asm. 1, there exists a Filippov solution as in Defn. 5.

**Proof** This is a direct consequence of [60, Theorem 4, Theorem], recalling that \( v \cdot \lambda \) is essentially measurable and essentially bounded and thus satisfies **Condition B** as well as the boundedness of any solutions on any finite time horizon. \( \square \)

In the next Lem. 2.2, we make a connection between the solutions in Defn. 3 and general Filippov solutions.

**Lemma 2.2** (Relation of Defn. 3 to Filippov solutions). Solutions in the sense of Defn. 3 are Filippov solutions as defined in Defn. 5.

**Proof** According to Thm. 2.1, the solution \( x \in C([0, T]) \) to Defn. 3 exists and it is both unique and Lipschitz. Thus, \( x \in W^{1,1}((0, T)) \). As Defn. 3 is invariant with regard to the choice of representative of \( f \) in the Lebesgue measure, we can choose \( \tilde{f} \) as follows (\( \varepsilon \in \mathbb{R}_{>0} \)):
\[
\tilde{f}(t, \cdot) \equiv \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{+\varepsilon} f(t, y) \, dy
\]
for \( t \in [0, T] \) a.e.. According to the Lebesgue differentiation theorem [97, 3.21 The Lebesgue Differentiation Theorem], we have \( f \equiv \tilde{f} \) a.e.. As it also holds that
\[
\bigcap_{\mu(N) = 0, N \subset \mathbb{R}} \text{conv} \tilde{f}(t, B_\varepsilon(x(t)) \setminus N) = \left[ \text{ess-} \inf_{y \in B_\varepsilon(x)} \tilde{f}(t, y), \text{ess-} \sup_{y \in B_\varepsilon(x)} \tilde{f}(t, y) \right], \tag{19}
\]
we can estimate uniformly in \( \varepsilon \in \mathbb{R}_{>0} \) for \( x \in \mathbb{R} \)
\[
\frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \tilde{f}(t, y) \, dy
\]
and obtain
\[
\mathcal{X}'(t) \in K[\tilde{f}(t, \cdot)](\mathcal{X}(t)) \quad \forall t \in [0, T] \text{ a.e.},
\]
with \( K \) as in Eq. (18). This is the definition of a Filippov solution, concluding the proof. \( \square \)
However, uniqueness results for Filippov solutions are only presented for right hand sides of specific structure ([60, Theorem 10] (too strong for our setup)) or for autonomous right hand sides [64] where the famous Osgood condition plays a crucial role [65]. We detail this in the following for our setup, but need to restrict ourselves to the fully autonomous case (as this is where [64] is applicable). We thus assume that the Lipschitz part of the velocity, i.e. \( \lambda \), does not explicitly depend on time.

**Theorem 2.3** (Uniqueness of Filippov solutions for scalar autonomous discontinuous ODEs as presented in [64]). Let Asm. 1 hold. Moreover, let \( \exists \lambda \in W^{1,\infty}(\mathbb{R}) : \lambda(t,\cdot) \equiv \tilde{\lambda} \) on \( \mathbb{R} \) (i.e., the discontinuous IVP is autonomous) and let

\[
\left\{ x \in \mathbb{R} : \tilde{\lambda}(x) \not= 0 \text{ and } v \text{ discontinuous at } x \right\}
\]

have Lebesgue measure zero. Then, the Filippov solution to the discontinuous IVP in Defn. 1 is unique.

**Proof** We take advantage of the result in [64] and repeat what is stated there: Recall the definition of \( K \) in Eq. (18) and assume that

- the set

\[
\left\{ x \in \mathbb{R} : 0 \not\in K[v \cdot \tilde{\lambda}](x) \text{ and } v \cdot \tilde{\lambda} \text{ discontinuous at } x \right\} \subset \mathbb{R}
\]

has Lebesgue measure zero,
- for every \( x \in \mathbb{R} \) with \( 0 \in K[v \cdot \tilde{\lambda}](x) \), the function

\[
g[v \cdot \tilde{\lambda}]: \begin{cases} \mathbb{R} &\mapsto \mathbb{R} \\ z &\mapsto \lim_{\delta \to 0} \text{ess- sup}_{y \in B_\delta(x)} \left( (v \cdot \tilde{\lambda})(y + z) \text{sgn}(z) \right)^+ \end{cases}
\]

is an Osgood function, i.e. \( g[v \cdot \tilde{\lambda}] \) is non-negative, Borel measurable, and for a \( \delta \in \mathbb{R}_{>0} \) satisfies

\[
\int_{-\delta}^{0} \frac{1}{g[v \cdot \tilde{\lambda}](u)} \, du = \infty = \int_{0}^{\delta} \frac{1}{g[v \cdot \tilde{\lambda}](u)} \, du \quad \forall \delta \in (0, \delta]. \tag{22}
\]

Then there exists a unique Filippov solution to the considered discontinuous IVP.

However, both conditions are satisfied as we will detail in the following:

- The first point is satisfied by construction as \( f \equiv v \cdot \tilde{\lambda} \), \( \tilde{\lambda} \) is Lipschitz and \( v \geq v \).
- For the second point, we first recall that the definition of solution does not vary with respect to the representative (here \( f \)) in the Lebesgue-measure.
Instead of $f \equiv v \cdot \tilde{\lambda}$, we can therefore choose a Borel measurable function $\hat{v} \in L^\infty(\mathbb{R}; \mathbb{R}_+)$, with 

$$v(x) = \hat{v}(x), \quad x \in \mathbb{R} \text{ a.e.}$$

(For this, use Lusin’s theorem [97, 44. Lusin’s Theorem, p.64] to approximate $v$ by a continuous function up to a set of arbitrarily small Lebesgue measure). Then, the corresponding function $g[\hat{v} \cdot \tilde{\lambda}]$ defined in Eq. (21) is by construction non-negative and Borel measurable. We now prove the so-called Osgood condition in Eq. (22). To accomplish this, we estimate as follows for $\tilde{\delta} \in (0, \delta)$ and $z \in \mathbb{R}$:

$$\text{ess-} \sup_{y \in B_{\tilde{\delta}}(x)} ((v \cdot \tilde{\lambda})(y + z)\text{sgn}(z))^+ \leq \|\hat{v}\|_{L^\infty(\mathbb{R})} \text{ess-} \sup_{y \in B_{\tilde{\delta}}(x)} |\tilde{\lambda}(x) + \int_x^{y+z} \tilde{\lambda}'(s) \, ds|.$$

$\lambda(x) = 0$ as we are in the case $0 \in K[v \cdot \tilde{\lambda}(x)]$ and $\hat{v} \geq v$

$$\leq \|\hat{v}\|_{L^\infty(\mathbb{R})} |z + \delta| \|\tilde{\lambda}'\|_{L^\infty(\mathbb{R})}.$$

Letting $\delta \to 0$, for $z \in \mathbb{R}$ we obtain

$$g[\hat{v} \cdot \tilde{\lambda}](z) \leq \|\hat{v}\|_{L^\infty(\mathbb{R})} \|\tilde{\lambda}'\|_{L^\infty(\mathbb{R})} |z|,$$

from which Eq. (22) follows. This concludes the proof.

The previous Thm. 2.3 has made a connection between our discontinuous IVP and Filippov theory, and has established the necessary uniqueness for the fully autonomous IVP. However, it does not directly apply to the general non-autonomous case. More importantly, although continuous dependency of the solution with regard to the input datum might be obtained, we require rather strong stability or continuity results, which can be obtained with our definition of solution Defn. 3 by taking advantage of the surrogate system in Thm. 2.1. This is detailed in the next section.

### 2.2 Stability of the solutions with respect to initial datum and velocities

In this section, we deal with the stability of the discontinuous IVP introduced in Defn. 1. To this end, we use the surrogate system introduced in Thm. 2.1 and study its components Eqs. (5) and (6) in detail.

**Proposition 2.1** (Auxiliary stability results). Given Asm. 1 and $T \in \mathbb{R}_{>0}$, the surrogate ODEs defined in Eq. (5) and Eq. (6) are stable with respect to the
initial datum, smooth velocity \( \lambda \) and discontinuous velocity \( v \), i.e., the following stability results hold: For any \( Z \)

\[
\forall (u, x_0, \bar{x}_0, v, \bar{v}) \in \mathbb{R}^3 \times L^\infty(\mathbb{R} ; \mathbb{R}_2^k)^2 : \\
|Z[v_0](x_0 ; u) - Z[\bar{v}_0](\bar{x}_0 ; u)| \\
\leq \frac{1}{u} |x_0 - \bar{x}_0| + \frac{1}{u} \|v - \bar{v}\|_1^1((\min\{x_0, \bar{x}_0, u\}, \max\{x_0, \bar{x}_0, u\}))
\]

(23)

and for \( C \) and \( \forall t \in [0, T] \)

\[
\forall (\lambda, \bar{\lambda}, x_0, \bar{x}_0, v, \bar{v}) \in L^\infty((0, T) ; W^{1,\infty}(\mathbb{R}))^2 \times \mathbb{R}^2 \times L^\infty(\mathbb{R} ; \mathbb{R}_2^k)^2 \\
|C[\lambda, Z[v_0](x_0 ; \cdot)](t) - C[\bar{\lambda}, Z[\bar{v}_0](\bar{x}_0 ; \cdot)](t)| \\
\leq e^{t \|v\|_{L^\infty(\mathbb{R})}^2} \int_0^t \|\lambda(s, \cdot) - \bar{\lambda}(s, \cdot)\|_{L^\infty(X(x_0, v, \lambda))} \, ds \\
+ e^{t \|v\|_{L^\infty(\mathbb{R})}^2} t \|\bar{v}\|_{L^\infty(\mathbb{R})}^2 \|Z[v](x_0 ; \cdot) - Z[\bar{v}](\bar{x}_0 ; \cdot)\|_{L^\infty(X(x_0, v, \lambda))}
\]

(24)

with

\[
C := \max \{ \|\lambda\|_{L^\infty((0,T) ; L^\infty(\mathbb{R}))}, \|\bar{\lambda}\|_{L^\infty((0,T) ; L^\infty(\mathbb{R}))} \} \\
C_2 := \max \{ \|\partial_2 \lambda\|_{L^\infty((0,T) ; L^\infty(\mathbb{R}))}, \|\partial_2 \bar{\lambda}\|_{L^\infty((0,T) ; L^\infty(\mathbb{R}))} \} \\
X(x_0, v, \lambda) := x_0 + T \|v\|_{L^\infty(\mathbb{R})} \|\lambda\|_{L^\infty((0,T) ; L^\infty(\mathbb{R}))} \cdot (-1, 1) \subset \mathbb{R}.
\]

Proof We start by proving Eq. (23). To achieve this, recall the definition of \( Z \) in Eq. (5). From this definition, we can make the following estimate for \( u \in \mathbb{R}^2 \):

\[
|Z[v_0](x_0 ; u) - Z[\bar{v}_0](\bar{x}_0 ; u)| = \left| \int_{x_0}^{u} \frac{1}{v(s)} \, ds - \int_{\bar{x}_0}^{\bar{u}} \frac{1}{\bar{v}(s)} \, ds \right| \\
\leq \left| \int_{x_0}^{\bar{x}_0} \frac{1}{u} \, ds \right| + \left| \int_{\min\{x_0, \bar{x}_0, u\}}^{\max\{x_0, \bar{x}_0, u\}} \frac{v(s) - \bar{v}(s)}{v(s) \bar{v}(s)} \, ds \right| \\
\leq \frac{1}{u} |x_0 - \bar{x}_0| + \frac{1}{u^2} \|v - \bar{v}\|_1^1((\min\{x_0, \bar{x}_0, u\}, \max\{x_0, \bar{x}_0, u\})) \\
\]

This proves the first claim. For the second, namely the estimate in Eq. (24), we first show that

\[
\frac{1}{\|v\|_{L^\infty(\mathbb{R})}} |x - \bar{x}| \leq |Z[v](x_0 ; x) - Z[v](x_0 ; \bar{x})| \quad \forall x, \bar{x} \in \mathbb{R}^2.
\]

(27)

Recalling again Eq. (5) we end up with

\[
|Z[v](x_0 ; x) - Z[v](x_0 ; \bar{x})| = \left| \int_{\bar{x}}^{x} \frac{1}{v(s)} \, ds \right| \geq \left( \max\{x, \bar{x}\} - \min\{x, \bar{x}\} \right) \frac{1}{\|v\|_{L^\infty(\mathbb{R})}},
\]

which is exactly Eq. (27).

Finally, focusing on Eq. (24), we recall the definition of \( C \) in Eq. (6) and thus estimate for \( t \in [0, T] \)

\[
|C[\lambda, Z[v](x_0 ; \cdot)](t) - C[\bar{\lambda}, Z[\bar{v}](\bar{x}_0 ; \cdot)](t)| \\
\]
For the term in Eq. (28), we get the following estimate by substitution:
\[
= \left| \int_0^t \lambda(s, Z[v]^{-1}(x_0; c[\lambda, Z[v]](x_0;))(s)) \right| \\
= \left| \int_0^t \lambda(s, Z[\bar{v}]^{-1}(\bar{x}_0; c[\bar{\lambda}, Z[\bar{v}]](\bar{x}_0;))(s)) \right| ds \\
\leq \mathcal{L}_2 \int_0^t \left| Z[v]^{-1}(x_0; c[\lambda, Z[v]](x_0;))(s) - Z[\bar{v}]^{-1}(\bar{x}_0; c[\bar{\lambda}, Z[\bar{v}]](\bar{x}_0;))(s) \right| ds \\
+ \int_0^t \| \lambda(s, \cdot) - \bar{\lambda}(s, \cdot) \|_{L^\infty(X(x_0, v, \lambda))} ds \\
\leq \mathcal{L}_2 \int_0^t \left| Z[v]^{-1}(x_0; c[\lambda, Z[v]](x_0;))(s) - Z[\bar{v}]^{-1}(\bar{x}_0; c[\bar{\lambda}, Z[\bar{v}]](\bar{x}_0;))(s) \right| ds \\
+ \int_0^t \| \lambda(s, \cdot) - \bar{\lambda}(s, \cdot) \|_{L^\infty(X(x_0, v, \lambda))} ds. \\
\tag{28}
\]
For the term in Eq. (29), we now apply Eq. (27). As \( \|v\|_{L^\infty(\mathbb{R})} \) is then an upper bound on the derivative of \( Z[v]^{-1} \) (recall the estimate in Eq. (8)), we obtain
\[
\|v\|_{L^\infty(\mathbb{R})} \int_0^t [C[\lambda, Z[v](x_0;)](s) - C[\bar{\lambda}, Z[\bar{v}](\bar{x}_0;)](s)] ds. \\
\tag{29}
\]
For the term in Eq. (28), we get the following estimate by substitution:
\[
\|v\|_{L^\infty(\mathbb{R})} \int_0^t [C[\lambda, Z[v](x_0;)](s) - C[\bar{\lambda}, Z[\bar{v}](\bar{x}_0;)](s)] ds. \\
\tag{29}
\]
For the term in Eq. (28), there is then an upper bound on the derivative of \( Z[v]^{-1} \) when recalling once more Eq. (8)
\[
\leq \mathcal{L}_2 t \| \bar{v} \|_{L^\infty(\mathbb{R})} \| Z[\bar{v}](\bar{x}_0; -) - Z[v](x_0; -) \|_{L^\infty(X(x_0, v, \lambda))}. \\
\]
Applying once more Eq. (27), as well as \( \|v\|_{L^\infty(\mathbb{R})} \), there is then an upper bound on the derivative of \( Z[v]^{-1} \) when recalling once more Eq. (8)
\[
\|v\|_{L^\infty(\mathbb{R})} \int_0^t [C[\lambda, Z[v](x_0;)](s) - C[\bar{\lambda}, Z[\bar{v}](\bar{x}_0;)](s)] ds. \\
\tag{29}
\]
The claimed inequality in (24) then follows by applying Grönwall’s inequality [96, Chapter I, III Gronwall’s inequality], concluding the proof.
Having obtained the previous stability results on the “surrogate system”,
we can apply these results to obtain the stability of solutions to the original
discontinuous ODE in Defn. 1 in the sense of Defn. 3:

**Theorem 2.4** (Stability of solutions for initial datum and velocities). For

\((x_0, \bar{x}_0) \in \mathbb{R}^2, \ (v, \bar{v}) \in L^\infty(\mathbb{R}; \mathbb{R}^-_2)^2 \) and \((\lambda, \bar{\lambda}) \in L^\infty((0, T); W^{1, \infty}(\mathbb{R}))\), the fol-
lowing stability result holds for the corresponding solutions \(\mathcal{X}\) as in Defn. 3 for all \(t \in [0, T]\):

\[
\begin{align*}
|\mathcal{X}[v, \lambda](x_0; t) - \mathcal{X}[\bar{v}, \bar{\lambda}](\bar{x}_0; t)| & \\
\leq & \|v\|_{L^\infty(\mathbb{R})} e^t \|v\|_{L^\infty(\mathbb{R})} \mathcal{L}_2 \int_0^t \|\lambda(s, \cdot) - \bar{\lambda}(s, \cdot)\|_{L^\infty(X(x_0, v, \lambda))} ds \\
& + \left(\|v\|_{L^\infty(\mathbb{R})} e^t \|v\|_{L^\infty(\mathbb{R})} \mathcal{L}_2 + 1\right) \frac{\|v\|_{L^\infty(\mathbb{R})}}{\|v\|_{L^\infty(\mathbb{R})}} \left(\|x_0 - \bar{x}_0\| + \frac{1}{\mathcal{L}} \|v - \bar{v}\| L(Y(x_0, \bar{x}_0, v, \mathcal{L}))\right)
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{L}_2 & := \max \left\{\|\partial_2 \lambda\|_{L^\infty((0, T); L^\infty(\mathbb{R}))}, \|\bar{\partial}_2 \bar{\lambda}\|_{L^\infty((0, T); L^\infty(\mathbb{R}))}\right\} \\
\mathcal{L} & := \max \left\{\|\lambda\|_{L^\infty((0, T); L^\infty(\mathbb{R}))}, \|\bar{\lambda}\|_{L^\infty((0, T); L^\infty(\mathbb{R}))}\right\}
\end{align*}
\]

\(Y(x_0, \bar{x}_0, v, \mathcal{L}) := \left(\min\{x_0 - T\|v\|_{L^\infty(\mathbb{R})} \mathcal{L}, \bar{x}_0\}, \max\{x_0 + T\|v\|_{L^\infty(\mathbb{R})} \mathcal{L}, \bar{x}_0\}\right)\)

\(X(x_0, v, \lambda) := x_0 + T\|v\|_{L^\infty(\mathbb{R})} \mathcal{L}_{(0, T); L^\infty(\mathbb{R}))} \cdot (-1, 1) \subset \mathbb{R}\)

**Proof** This is a direct application of the previous stability results for the surrogate
system of ODEs in Prop. 2.1. We detail the required steps in the following and start
by estimating for \(t \in [0, T]\) using Eq. (7)

\[
\begin{align*}
|\mathcal{X}[v, \lambda](x_0; t) - \mathcal{X}[\bar{v}, \bar{\lambda}](\bar{x}_0; t)| & \\
= & \left|\mathcal{Z}[v]^{-1}(x_0; \mathcal{C}[\lambda, \mathcal{Z}[v](x_0; \cdot)](t)) - \mathcal{Z}[\bar{v}]^{-1}(\bar{x}_0; \mathcal{C}[\bar{\lambda}, \mathcal{Z}[\bar{v}](\bar{x}_0; \cdot)](t))\right| \\
\leq & \left|\mathcal{Z}[v]^{-1}(x_0; \mathcal{C}[\lambda, \mathcal{Z}[v](x_0; \cdot)](t)) - \mathcal{Z}[v]^{-1}(x_0; \mathcal{C}[\bar{\lambda}, \mathcal{Z}[\bar{v}](\bar{x}_0; \cdot)](t))\right| \\
& + \left|\mathcal{Z}[v]^{-1}(x_0; \mathcal{C}[\bar{\lambda}, \mathcal{Z}[\bar{v}](\bar{x}_0; \cdot)](t)) - \mathcal{Z}[\bar{v}]^{-1}(\bar{x}_0; \mathcal{C}[\bar{\lambda}, \mathcal{Z}[\bar{v}](\bar{x}_0; \cdot)](t))\right|
\end{align*}
\]

applying Eq. (27) so that we can estimate the derivative of the inverse of \(\mathcal{Z}[v](x_0; \cdot)\) by \(\|v\|_{L^\infty(\mathbb{R})}\),

\[
\begin{align*}
\leq & \|v\|_{L^\infty(\mathbb{R})} \mathcal{C}[\lambda, \mathcal{Z}[v](x_0; \cdot)](t) - \mathcal{C}[\bar{\lambda}, \mathcal{Z}[\bar{v}](\bar{x}_0; \cdot)](t) \\
& + \left|\mathcal{Z}[v]^{-1}(x_0; \mathcal{C}[\bar{\lambda}, \mathcal{Z}[\bar{v}](\bar{x}_0; \cdot)](t)) - \mathcal{Z}[\bar{v}]^{-1}(\bar{x}_0; \mathcal{C}[\bar{\lambda}, \mathcal{Z}[\bar{v}](\bar{x}_0; \cdot)](t))\right|
\end{align*}
\]

Focusing on the second term, we have with \(y := \mathcal{Z}[v]^{-1}(x_0; \mathcal{C}[\bar{\lambda}, \mathcal{Z}[\bar{v}](\bar{x}_0; \cdot)](t))\) and thus \(\mathcal{Z}[v](x_0; y) = \mathcal{C}[\bar{\lambda}, \mathcal{Z}[\bar{v}](\bar{x}_0; \cdot)](t)\)

\[
(37) = \left|y - \mathcal{Z}[v]^{-1}(\bar{x}_0; \mathcal{Z}[v](x_0; y))\right|
\]

Applying \(y = \mathcal{Z}[\bar{v}]^{-1}(\bar{x}_0; \mathcal{Z}[\bar{v}](\bar{x}_0; y))\)

\[
= \left|\mathcal{Z}[\bar{v}]^{-1}(\bar{x}_0; \mathcal{Z}[\bar{v}](\bar{x}_0; y)) - \mathcal{Z}[\bar{v}]^{-1}(\bar{x}_0; \mathcal{Z}[v](x_0; y))\right|
\]

(39)
Again using Eq. (27) to obtain a Lipschitz-estimate from above for the inverse mapping

\[ \| \tilde{v} \|_{L^\infty(\mathbb{R})} |\mathcal{Z}[\tilde{v}](\tilde{x}_0 ; y) - \mathcal{Z}[v](x_0 ; y)|. \]  

(40)

Next, estimating \( y \) we have by the definitions of \( \mathcal{Z}[v] \) and \( \mathcal{C}[\cdot, \cdot] \) as in Eqs. (5) and (6) (and by Eq. (27) to once more obtain an upper bound on the Lipschitz constant of \( \mathcal{Z}[v]^{-1} \))

\[ |y - x_0| = |\mathcal{Z}[v]^{-1}(x_0 ; \mathcal{C}[\tilde{\lambda}, \mathcal{Z}[\tilde{v}](\tilde{x}_0 ; \cdot)](t)) - \mathcal{Z}[v]^{-1}(x_0 ; 0)| \]

\[ \leq \| v \|_{L^\infty(\mathbb{R})} |\mathcal{C}[\tilde{\lambda}, \mathcal{Z}[\tilde{v}](\tilde{x}_0 ; \cdot)](t)| \leq T \| v \|_{L^\infty(\mathbb{R})} \| \tilde{\lambda} \|_{L^\infty((0, T) \times L^\infty(\mathbb{R}))}. \]

In the last estimate we have used the identity for \( \mathcal{C} \) Eq. (6), which implies the upper bounds used. Altogether, this implies that \( y \in X(x_0, v, \tilde{\lambda}) \) with \( X \) as in Eq. (25). Continuing the estimate, we have

\[ (37) \leq (40) = \| \tilde{v} \|_{L^\infty(\mathbb{R})} |\mathcal{Z}[\tilde{v}](\tilde{x}_0 ; y) - \mathcal{Z}[v](x_0 ; y)| \]

\[ \leq \| \tilde{v} \|_{L^\infty(\mathbb{R})} |\mathcal{Z}[\tilde{v}](\tilde{x}_0 ; \cdot) - \mathcal{Z}[v](x_0 ; \cdot)| \| L^\infty(X(x_0, v, \tilde{\lambda})) \]  

(41)

and applying the stability estimate in \( \mathcal{Z} \) in Prop. 2.1

\[ \leq \| \tilde{v} \|_{L^\infty(\mathbb{R})} \left( \frac{1}{2} |x_0 - \tilde{x}_0| + \frac{1}{2^T} \| v - \tilde{v} \|_{L^1(Y(x_0, \tilde{x}_0, v, L))} \right), \]

with \( Y \) as defined in Eq. (33).

Continuing the original estimate, we take advantage of the stability in \( \mathcal{C} \) in Eq. (24) and have

\[ (35) \leq \| v \|_{L^\infty(\mathbb{R})} e^{T \| v \|_{L^\infty(\mathbb{R})} L_2} \int_0^T \| \lambda(s, \cdot) - \tilde{\lambda}(s, \cdot) \|_{L^\infty(X(x_0, v, \lambda))} \, ds \]

\[ + \| v \|_{L^\infty(\mathbb{R})} e^{T \| v \|_{L^\infty(\mathbb{R})} L_2} t \| \tilde{v} \|_{L^\infty(\mathbb{R})} L_2 \| \mathcal{Z}[v](x_0 ; \cdot) - \mathcal{Z}[\tilde{v}](\tilde{x}_0 ; \cdot) \|_{L^\infty(X(x_0, v, \lambda))} \]

\[ + \| \tilde{v} \|_{L^\infty(\mathbb{R})} \left( \frac{1}{2} |x_0 - \tilde{x}_0| + \frac{1}{2^T} \| v - \tilde{v} \|_{L^1(Y(x_0, \tilde{x}_0, v, L))} \right). \]

Using the stability estimate for \( \mathcal{Z}[\cdot, \cdot] \) in Eq. (23)

\[ \leq \| v \|_{L^\infty(\mathbb{R})} e^{T \| v \|_{L^\infty(\mathbb{R})} L_2} \left( \int_0^T \| \lambda(s, \cdot) - \tilde{\lambda}(s, \cdot) \|_{L^\infty(X(x_0, v, \lambda))} \, ds \right) \]

\[ + t \| \tilde{v} \|_{L^\infty(\mathbb{R})} L_2 \left( \frac{1}{2} |x_0 - \tilde{x}_0| + \frac{1}{2^T} \| v - \tilde{v} \|_{L^1(Y(x_0, \tilde{x}_0, v, L))} \right) \]

\[ = \| v \|_{L^\infty(\mathbb{R})} e^{T \| v \|_{L^\infty(\mathbb{R})} L_2} \int_0^T \| \lambda(s, \cdot) - \tilde{\lambda}(s, \cdot) \|_{L^\infty(X(x_0, v, \lambda))} \, ds \]

\[ + \left( \| v \|_{L^\infty(\mathbb{R})} e^{T \| v \|_{L^\infty(\mathbb{R})} L_2} t L_2 + 1 \right) \frac{\| \tilde{v} \|_{L^\infty(\mathbb{R})}}{2} \left( \| x_0 - \tilde{x}_0 \| + \frac{1}{2^T} \| v - \tilde{v} \|_{L^1(Y(x_0, \tilde{x}_0, v, L))} \right). \]

This is indeed the claimed estimate and thus the proof is concluded.

The previous result in Thm. 2.4 gives Lipschitz-continuity for the solution with regard to the initial datum. As the existence of an explicit solution formula for smooth velocities \((v, \lambda)\) for \( \partial_3 \mathcal{X} \) is important for several later results, we detail it in the following:
Remark 3 (An “explicit” formula for $\partial_{x_0} \mathcal{X}[v, \lambda](x_0; \cdot)$ if $v$ is smooth). Let Asm. 1 hold and also $v \in C^1(\mathbb{R})$ and $\lambda \in C^1([0, T] \times \mathbb{R})$. Then we have for $(x_0, t) \in \mathbb{R} \times (0, T)$ a solution formula for the derivative of $\mathcal{X}[v, \lambda](x_0, \cdot)$ with regard to $x_0 \in \mathbb{R}$, namely for $t \in [0, T]$

$$\partial_3 \mathcal{X}[v, \lambda](x_0; t) = \frac{v(\mathcal{X}[v, \lambda](x_0; t))}{v(x_0)} e^{\int_0^t \partial_2 \lambda(s, \mathcal{X}[v, \lambda](x_0; s)) \partial_3 \mathcal{X}[v, \lambda](x_0; s) \, ds}. \quad (42)$$

To show this, we can differentiate through the integral form of the – now – continuously differentiable IVP in Defn. 1. We thus take the derivative of $X$ with regard to $x_0$ and have – following the Carathéodory solution in Defn. 4 – for $t \in [0, T]$

$$\partial_3 \mathcal{X}[v, \lambda](x_0; t) = 1 + \int_0^t \left( v'(\mathcal{X}[v, \lambda](x_0; s)) \lambda(s, \mathcal{X}[v, \lambda](x_0; s)) + v(\mathcal{X}[v, \lambda](x_0; s)) \partial_2 \lambda(s, \mathcal{X}[v, \lambda](x_0; s)) \right) \cdot \partial_3 \mathcal{X}[v, \lambda](x_0; s) \, ds, \quad (43)$$

which we can explicitly solve to obtain

$$\partial_3 \mathcal{X}[v, \lambda](x_0; t) = e^{\int_0^t a(s) + v(\mathcal{X}[v, \lambda](x_0; s)) \partial_2 \lambda(s, \mathcal{X}[v, \lambda](x_0; s)) \, ds}$$

with $a(s) = v'(\mathcal{X}[v, \lambda](x_0; s)) \lambda(s, \mathcal{X}[v, \lambda](x_0; s))$. Focusing on the first factor we can write

$$e^{\int_0^t a(s) \, ds} = e^{\int_0^t v'(\mathcal{X}[v, \lambda](x_0; s)) \lambda(s, \mathcal{X}[v, \lambda](x_0; s)) \, ds}$$

and by Defn. 4, i.e. $\partial_4 \mathcal{X}[v, \lambda](\cdot; \cdot) = v(\mathcal{X}[v, \lambda](\cdot; \cdot)) \lambda(\cdot, \mathcal{X}[v, \lambda](\cdot; \cdot))$

$$= e^{\int_0^t v'(\mathcal{X}[v, \lambda](x_0; s)) \lambda(s, \mathcal{X}[v, \lambda](x_0; s)) \, ds}$$

$$= e^{\left[ \ln \left( v(\mathcal{X}[v, \lambda](x_0; s)) \right) \right]_{s=0}^{s=t}} = \frac{v(\mathcal{X}[v, \lambda](x_0; t))}{v(x_0)}.$$

In the latter manipulation, we also employed the fact that, according to Defn. 4, $\mathcal{X}[v, \lambda](x_0; 0) = x_0$. Together with Eq. (43), the solution formula for $\partial_3 \mathcal{X}$ in Eq. (42) follows. Although the solution formula in Eq. (42) does not require any regularity on the derivative of $v$, there is a problem in its interpretation if $\partial_2 \lambda$ and $v$ are not continuous in the time integration in the exponent. For instance, assume that the characteristic curve $\mathcal{X}[v, \lambda](x_0; \cdot)$ is constant over a small time horizon. It is not clear how to interpret the integral in Eq. (42) over this specific time horizon.

To obtain an improved estimate for $\partial_3 \mathcal{X}$, we take advantage of Rmk. 3 for smooth datum $(v, \lambda)$ and use an approximation argument. This is performed in the following Cor. 2.1:
Corollary 2.1 (Improved bounds on the Lipschitz constant of \( \mathcal{X} \)). Let velocities \( v, \lambda \) satisfy Asm. 1. Then, the solution of the discontinuous IVP as in Defn. 3 satisfies \( \forall (t, x_0, \tilde{x}_0) \in [0, T] \times \mathbb{R}^2 \)

\[
\frac{e^{-tL_2\|v\|_{L^\infty(\mathbb{R})}}}{\|v\|_{L^\infty(\mathbb{R})}} \leq \left| \mathcal{X}[v, \lambda](x_0; t) - \mathcal{X}[v, \lambda](\tilde{x}_0; t) \right| \leq \left| \mathcal{X}[v, \lambda](x_0; t) - \mathcal{X}[v, \lambda](x_0; t) \right|
\]

(44)

with \( L_2 := \|\partial_2 \lambda\|_{L^\infty((0, T), L^\infty(\mathbb{R}))} \).

Proof We recall the surrogate system in Eqs. (5) and (6), use the approximation result in Cor. 2.2 with \( v_\varepsilon, \lambda_\varepsilon \) smoothed, and assume w.l.o.g. \( x_0 > \tilde{x}_0 \). Then, we can estimate for \( t \in [0, T] \)

\[
\mathcal{X}[v, \lambda](x_0; t) - \mathcal{X}[v, \lambda](\tilde{x}_0; t) \geq \mathcal{X}[v_\varepsilon, \lambda_\varepsilon](x_0; t) - \mathcal{X}[v_\varepsilon, \lambda_\varepsilon](\tilde{x}_0; t) - \mathcal{X}[v_\varepsilon, \lambda_\varepsilon](x_0; t) - \mathcal{X}[v_\varepsilon, \lambda_\varepsilon](\tilde{x}_0; t).
\]

According to Cor. 2.2, the last two terms in the latter estimate vanish if \( \varepsilon \to 0 \). So we only focus on the first term and continue – for given \( \varepsilon \in \mathbb{R}_{>0} \) – the estimate as follows

\[
\mathcal{X}[v_\varepsilon, \lambda_\varepsilon](x_0; t) - \mathcal{X}[v_\varepsilon, \lambda_\varepsilon](\tilde{x}_0; t) \geq \inf_{x \in \mathbb{R}} \partial_3 \mathcal{X}[v_\varepsilon, \lambda_\varepsilon](x; t)(x_0 - \tilde{x}_0).
\]

(45)

As \( \mathcal{X}[v_\varepsilon, \lambda_\varepsilon] \) is the classical solution of an ODE with a smooth right hand side in space, the “explicit” solution formula for \( \partial_3 \mathcal{X} \) in Rmk. 3, namely Eq. (42), applies and we have

\[
(45) \geq (x_0 - \tilde{x}_0) \inf_{y \in \mathbb{R}} \frac{v_\varepsilon(\mathcal{X}[v_\varepsilon, \lambda_\varepsilon](y; x))}{v_\varepsilon(y)} e^{\int_0^t \partial_2 \lambda_\varepsilon(u, \mathcal{X}[v_\varepsilon, \lambda_\varepsilon](y; u)) v_\varepsilon(\mathcal{X}[v_\varepsilon, \lambda_\varepsilon](y; u)) du}
\]

Here we have also used the fact that it holds by construction

\[
\|\partial_2 \lambda_\varepsilon\|_{L^\infty((0, T), L^\infty(\mathbb{R}))} \leq \|\partial_2 \lambda\|_{L^\infty((0, T), L^\infty(\mathbb{R}))} \quad \forall \varepsilon_\varepsilon(x) \leq \|v\|_{L^\infty(\mathbb{R})} \forall x \in \mathbb{R} \text{ a.e.}
\]

Letting \( \varepsilon \to 0 \) we obtain the lower bound when also taking into account the previous arguments on the approximation. For the upper bound, almost the same arguments can be made. \( \square \)

In the following example, we illustrate the obtained result by means of numerics.

Example 1 (Numerical illustration of the results for \( \mathcal{X} \) and \( \partial_3 \mathcal{X} \)). We consider the following data

\[
v \equiv \text{sgn}(\sin(\pi x^{-1})) + 2, \quad \lambda(\cdot, *) \in \{1, 1 - *, \cos(2\pi \cdot)\}, \quad x_0 \in \{-1, -0.5, 0\}
\]

and solve the discontinuous initial value problem in Defn. 1 by an explicit Runge-Kutta scheme [98, 99] with adaptive time stepping. As can be seen, the numerical approximations are highly accurate as the blue circles (which represent the exact value at the considered time) match the blue line. In the
different cases, the impact of the discontinuities that accumulate at \( x = 0 \) can be observed. The pictures on the right illustrate the finite difference approximation of \( \partial \mathcal{X} \). The solid lines represent the derived bounds, and the dashed lines are the named numerical approximations. Clearly, these bounds are satisfied. It is worth mentioning that the green lines represent the bounds for the blue and yellow cases and that these estimates are somewhat sharp.

**Fig. 1** Left: Numerical approximation of the solutions of the discontinuous IVP for \( v \equiv \text{sgn}(\sin(\pi \star^{-1})) + 2 \) and \( \lambda \equiv 1 \) (blue) \( \lambda(\cdot, \cdot) \equiv 1 - \star \) (red) and \( \lambda(\cdot, \cdot) \equiv \cos(2\pi \cdot) \) (yellow), as well as initial datum \( x_0 \in \{-1, -0.5, 0\} \) (dash-dotted, dashed and solid respectively), over time. The blue circles denote analytical solutions for the autonomous and piecewise constant case, i.e. \( \lambda \equiv 1 \), showing the high accuracy of the numerical approximation. Right: Finite difference analog to Cor. 2.1 and the bounds derived therein (solid green lines for the blue and yellow case).

Though not a classical stability result, the result presented in the following Rmk. 4 will enable us later in Section 2 to obtain a more general approximation result when measuring the differences in the discontinuous velocities in a weak topology.

**Remark 4** (Stability of the solution for \( v \) close in a weak topology). Given the assumptions of Thm. 2.4, the difference in the characteristics can actually be estimated for \( t \in [0, T] \) as

\[
\begin{align*}
&\left| \mathcal{X}[v, \lambda](x_0; t) - \mathcal{X}[\bar{v}, \bar{\lambda}](\bar{x}_0; t) \right| \\
&\leq \|v\|_{L^\infty(\mathbb{R})} e^{t\|v\|_{L^\infty(\mathbb{R})} \mathcal{L}_2} \int_0^t \|\lambda(s, \cdot) - \bar{\lambda}(s, \cdot)\|_{L^\infty(\mathcal{X}(x_0, v, \lambda))} \, ds \\
&+ \left( \|v\|_{L^\infty(\mathbb{R})} e^{t\|v\|_{L^\infty(\mathbb{R})} \mathcal{L}_2} + 1 \right) \|\bar{v}\|_{L^\infty(\mathbb{R})} \frac{1}{2} \|x_0 - \bar{x}_0\| \\
&+ \left( \|v\|_{L^\infty(\mathbb{R})} e^{t\|v\|_{L^\infty(\mathbb{R})} \mathcal{L}_2} + 1 \right) \|\bar{v}\|_{L^\infty(\mathbb{R})} \frac{1}{2} \left| \int_{Y(x_0, \bar{x}_0, v, \mathcal{L})} \left| v(s) - \bar{v}(s) \right| \, ds \right|,
\end{align*}
\]

(46)
with \( Y(x_0, \tilde{x}_0, v, \mathcal{L}) , X(x_0, v, \lambda) \) as in Eq. (33). The proof consists of improving the estimate Eq. (26) and using this estimate in Thm. 2.4, in particular after Eq. (41).

This result is significantly stronger than the result in Thm. 2.4 as the term \( v - \tilde{v} \) goes into the estimate integrated and not in "norm difference". The estimate is not as canonical as a classical \( L^1 \)-norm estimate, but is required — particularly in Section 3 in Thm. 3.2, a stability result with regard to the input datum. 

The previous stability result in Thm. 2.4 and Rmk. 4 enables us to approximate the discontinuous IVP by a sequence of continuous IVPs:

**Corollary 2.2** (Approximating the discontinuous IVP by a smooth IVP). Let \( v \in L^\infty(\mathbb{R} ; \mathbb{R} \geq L) \) and \( \lambda \in L^\infty((0, T) ; W^{1,\infty}(\mathbb{R})) \) be given. Take

\[
\begin{align*}
\{ v_\varepsilon := \phi_\varepsilon \ast v \}_{\varepsilon \in \mathbb{R}_0} \subset C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R} ; \mathbb{R}_{\geq L}), \\
\{ \lambda_\varepsilon := \phi_\varepsilon \ast \lambda(t, \cdot) \}_{\varepsilon \in \mathbb{R}_0} \subset C^\infty(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}) \quad \forall t \in [0, T] \quad a.e.
\end{align*}
\]

with \( \{ \phi_\varepsilon \}_{\varepsilon \in \mathbb{R}_0} \subset C^\infty(\mathbb{R}) \) the standard mollifier as in [100, Remark C.18, ii]. Then, for the solutions \( \mathcal{X} \) to the corresponding discontinuous IVP as in Defn. 3, it holds

\[
\lim_{\varepsilon \to 0} \| \mathcal{X}[v, \lambda] - \mathcal{X}[v_\varepsilon, \lambda_\varepsilon] \|_{L^\infty((0, T) ; L^\infty(\mathbb{R}))} = 0.
\]

**Proof** Thanks to the used mollifier we have that

\[
\limsup_{\varepsilon \to 0} \int_0^y v_\varepsilon(x) - v(x) \, dx = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \| \lambda_\varepsilon - \lambda \|_{L^\infty((0, T) ; L^\infty(\mathbb{R}))}
\]

(for the precise result see [100, Theorem 13.9 & Remark 13.11]). However, the first approximation result needs to be detailed. Let \( y \in \mathbb{R} \) be given. We can estimate for \( \varepsilon \in \mathbb{R}_0 \)

\[
\begin{align*}
&\left| \int_0^y (v \ast \phi_\varepsilon)(x) - v(x) \, dx \right|
= \left| \int_{-\varepsilon}^{\varepsilon} \phi_\varepsilon(z) \left( \int_z^{y+z} v(x) \, dx - \int_0^y v(x) \, dx \right) \, dz \right|
\leq 2 \int_{-\varepsilon}^{\varepsilon} |\phi_\varepsilon(z)| \| v \|_{L^\infty(\mathbb{R})} \, dz \leq 2 \varepsilon \| v \|_{L^\infty(\mathbb{R})}.
\end{align*}
\]

As this is uniform in \( y \) we indeed obtain the claim in Eq. (47). Thus, we can recall Rmk. 4 and obtain for given \( x_0 \in \mathbb{R} \) and \( \tilde{x}_0 = x_0 \) and \( t \in [0, T] \)

\[
\begin{align*}
&\left| \mathcal{X}[v, \lambda](x_0 ; t) - \mathcal{X}[v_\varepsilon, \lambda_\varepsilon](x_0 ; t) \right|
\leq \| v \|_{L^\infty(\mathbb{R})} e^{t \| v \|_{L^\infty(\mathbb{R})} L^2} \int_0^t \left\| \lambda(s, \cdot) - \lambda_\varepsilon(s, \cdot) \right\|_{L^\infty(X(x_0, v, \lambda))} \, ds
\end{align*}
\]

(48)

\[
\begin{align*}
+ \left( \| v \|_{L^\infty(\mathbb{R})} e^{t \| v \|_{L^\infty(\mathbb{R})} L^2} t L^2 + 1 \right) \frac{|v_\varepsilon|_{L^\infty(\mathbb{R})}}{\varepsilon} \left| \int_{Y(x_0, x_0, v, \mathcal{L})} v(s) - \tilde{v}_\varepsilon(s) \, ds \right|
\end{align*}
\]

with

\[
Y(x_0, x_0, v, \mathcal{L}) = x_0 + T \| v \|_{L^\infty(\mathbb{R})} \mathcal{L}(-1, 1)
\]
\( X(x_0, v, \lambda) = x_0 + T\|v\|_{L^\infty(\mathbb{R})}\|\lambda\|_{L^\infty((0,T); L^\infty(\mathbb{R}))}\cdot (-1, 1) \subset \mathbb{R} \)

as in Eq. (34). Making this uniform in \( x_0 \) and \( t \), we indeed obtain
\[
\|X[v, \lambda] - X[v_\varepsilon, \lambda_\varepsilon]\|_{L^\infty((0,T); L^\infty(\mathbb{R}))} \\
\leq \|v\|_{L^\infty(\mathbb{R})} e^T \|v\|_{L^\infty(\mathbb{R})} 2 T \|\lambda - \lambda_\varepsilon\|_{L^\infty((0,T); L^\infty(\mathbb{R})))} \\
+ \left( \|v\|_{L^\infty(\mathbb{R})} e^T \|v\|_{L^\infty(\mathbb{R})} 2 T 2 + 1 \right) \frac{\|v\|_{L^\infty(\mathbb{R})}}{\varepsilon} \sup_{y \in \mathbb{R}} \left| \int_{y-T}^{y+T} \int_{y-T} \int_{y-2T} v(s) - \tilde{v}_\varepsilon(s) \, ds \right|.
\]

For \( \varepsilon \to 0 \), the last two terms go to zero thanks to Eq. (56). Thus, we have shown the claimed approximation result. \( \square \)

In the following remark we explain why the result in Thm. 2.4 needs to be strengthened by a regularity result connecting the dependency of \( \mathcal{X} \) with the initial datum and the considered time.

**Remark 5** (\( \partial_y \mathcal{X}[v, \lambda](\cdot, t) \) as a function). Thanks to Thm. 2.4, the mapping \( y \mapsto \partial_y \mathcal{X}[v, \lambda](y, t) \) is well-defined for each \( t \in [0, T] \) in the sense that we have by Thm. 2.4 that \( \mathcal{X}[v, \lambda](y, t) \) is Lipschitz with regard to \( y \). Thus \( \forall t \in [0, T] \) it holds that
\[
\mathcal{X}[v, \lambda](\cdot, t) \in W_{loc}^{1, \infty}(\mathbb{R}) : \partial_y \mathcal{X}[v, \lambda](\cdot, t) \in L^\infty(\mathbb{R})
\]

(also compare Rmk. 3).

However, the regularity discussed in Rmk. 5 does not tell us anything about how the solution changes over time. As we later require a continuity in time, the most natural choice in space for this continuity to hold is \( L^1_{loc} \). Then, the continuity can be proven under no additional assumptions on the discontinuous velocity. This is detailed and shown in the Section 2.3, and in particular in Prop. 2.2.

### 2.3 Time-continuity/properties of the derivative of the solutions with respect to the initial datum in \( L^1_{loc}(\mathbb{R}) \)

As previously mentioned, in this section we study the regularity of \( \partial_y \mathcal{X} \) in time when measuring space in \( L^1_{loc}(\mathbb{R}) \). This is particularly important for our later analysis of the discontinuous nonlocal conservation law in Defn. 2 (see Section 3). Before detailing the claimed continuity, we require a convergence result for a composition of functions in \( L^1_{loc}(\mathbb{R}) \) with locally Lipschitz-continuous functions.

**Lemma 2.3** (Convergence of a composition of sequences of functions). Let \( \{f_\varepsilon\}_{\varepsilon \in \mathbb{R}_{>0}} \subset L^1_{loc}(\mathbb{R}) \ni f \) be given, as well as \( \{g_\varepsilon\}_{\varepsilon \in \mathbb{R}_{>0}} \subset W_{loc}^{1,\infty}(\mathbb{R}) \ni g \) so that \( \exists C \in \mathbb{R}_{>0} \forall \varepsilon \in \mathbb{R}_{>0} : C \leq g_\varepsilon \). Then, it holds that
\[
\lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_{L^1_{loc}(\mathbb{R})} = 0 = \lim_{\varepsilon \to 0} \|g_\varepsilon - g\|_{L^\infty(\mathbb{R})} \implies \lim_{\varepsilon \to 0} \|f_\varepsilon \circ g_\varepsilon - f \circ g\|_{L^1_{loc}(\mathbb{R})} = 0. \quad (49)
\]
Proof Let \( X \subset \mathbb{R} \) be open and bounded. With \( \{ \psi_{\delta} \}_{\delta \in \mathbb{R}_{>0}} \subset C^\infty(\mathbb{R}) \) we estimate the standard mollifier as in [100, Remark C.18, ii] as follows:

\[
\| f_\varepsilon \circ g_\varepsilon - f \circ g \|_{L^1_{\text{loc}}(X)} \\
\leq \| f_\varepsilon \circ g_\varepsilon - f \circ g_\varepsilon \|_{L^1_{\text{loc}}(X)} + \| f \circ g_\varepsilon - (f \ast \psi_\delta) \circ g_\varepsilon \|_{L^1_{\text{loc}}(X)} \\
+ \| (f \ast \psi_\delta) \circ g_\varepsilon - (f \ast \psi_\delta) \circ g \|_{L^1_{\text{loc}}(X)} + \| (f \ast \psi_\delta) \circ g - f \circ g \|_{L^1_{\text{loc}}(X)}.
\]

Letting \( \varepsilon \to 0 \), the first term (due to the uniform Lipschitz bound of \( g_\varepsilon \) from below) and the third term by the dominated convergence [97, 2.24 The Dominated Convergence Theorem] converge to zero. The second and fourth term vanish for \( \delta \to 0 \) as \( \psi_\delta \) was a standard mollifier. \( \square \)

In the following Prop. 2.2, we present the main result of this section, the continuity of \( \partial_3 X \) in time when measuring space in \( L^1_{\text{loc}} \). To this end, we take advantage of the derived solutions formula in Rmk. 3 for smooth velocities and the stability of solutions \( X \) with regard to different velocities (see Thm. 2.4).

**Proposition 2.2** (Stability spatial derivative of \( X \) in time assuming \( TV_{\text{loc}} \) regularity in \( v \)). Given \( x_0 \in \mathbb{R} \) and \((v, \lambda) \in L^\infty(\mathbb{R}; \mathbb{R}_{2\mathbb{Z}}) \times L^\infty((0,T); W^{1,\infty}(\mathbb{R})) \) as in Asm. 1, then, for \( X[v, \lambda](x_0, \cdot) \) as in Defn. 3, it holds that

\[
[0, T] \ni t \mapsto \left( \mathbb{R} \ni y \mapsto \partial_y X[v, \lambda](y, t) \right) \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}))
\]

i.e.

\[
\lim_{t \to \tilde{t}} \left\| \partial_3 X[v, \lambda](\cdot; t) - \partial_3 X[v, \lambda](\cdot; \tilde{t}) \right\|_{L^1_{\text{loc}}(\mathbb{R})} = 0.
\]

In addition, if we have

\[
v \in TV_{\text{loc}}(\mathbb{R}) \iff \| v \|_{TV(\Omega)} < \infty \quad \forall \Omega \subset \mathbb{R} \text{ compact},
\]

we obtain Lipschitz-continuity in time, i.e., the following estimate holds \( \forall (t, \tilde{t}) \in [0, T]^2 \) and \( \forall X \subset \mathbb{R} \) open and bounded:

\[
\| \partial_3 X[v, \lambda](\cdot; t) - \partial_3 X[v, \lambda](\cdot; \tilde{t}) \|_{L^1(X)} \leq \frac{|t - \tilde{t}| \| v \|_{L^\infty(\mathbb{R})}^2 \| \mathcal{L}_2 \| | \mathcal{L} \| | v \|_{TV(X)} | v \|_{L^\infty(\mathbb{R})} | v \|_{L^\infty(\mathbb{R})} | v \|_{TV(X)} | v \|_{L^\infty(\mathbb{R})} | v \|_{TV(X)} }{2} \left( \mathcal{L}_2 \| X \|_{L^\infty(\mathbb{R})} + \frac{|v|_{L^\infty(\mathbb{R})}^2 | \mathcal{L} \| | v \|_{TV(X)} | v \|_{L^\infty(\mathbb{R})} | v \|_{TV(X)} | v \|_{L^\infty(\mathbb{R})} | v \|_{TV(X)} }{2} \right),
\]

with

\[
\mathcal{L} := \| \lambda \|_{L^\infty((0,T); L^\infty(\mathbb{R}))}; \quad \mathcal{L}_2 := \| \partial_2 \lambda \|_{L^\infty((0,T); L^\infty(\mathbb{R}))}.
\]

**Proof** We show the claim by a finite difference approximation in the initial value and approximate the discontinuous velocity \( v \) by a smoothed velocity, as well as the Lipschitz-continuous velocity \( \lambda \) by a smoothed velocity. Now, let \((\varepsilon, h) \in \mathbb{R}_{>0}^2 \) be
given, in the notation suppress the dependency of $x$ with regard to the velocities, i.e., for now write (recall Defn. 3)
\[ x \equiv \mathcal{X}[v, \lambda] \quad \text{and} \quad x_\varepsilon \equiv \mathcal{X}[v_\varepsilon, \lambda_\varepsilon] \quad \text{on} \ \mathbb{R} \times (0, T). \]

Consider for $X \subset \mathbb{R}$ compact and $(t, \tilde{t}) \in [0, T]^2$:
\[
\| \partial_1 x(\cdot; t) - \partial_1 x(\cdot; \tilde{t}) \|_{L^1(X)} = \int_X \lim_{h \to 0} \left| \frac{x(z+h; \tilde{t}) - x(z; \tilde{t})}{h} - \frac{x(z+h; t) - x(z; t)}{h} \right| \, dz \tag{51}
\]
\[
= \lim_{h \to 0} \int_X \left| \frac{x(z+h; \tilde{t}) - x(z; \tilde{t})}{h} - \frac{x(z+h; t) - x(z; t)}{h} \right| \, dz, \tag{52}
\]
where changing the order of the limit with the integration follows by the dominated convergence theorem [97, 2.24 The Dominated Convergence Theorem]. Adding several zeros – the smoothed version of the previous terms – we estimate
\[
\text{(52) } \leq \lim_{h \to 0} \frac{1}{h} \left( \| x(\cdot + h; \tilde{t}) - x_\varepsilon(\cdot + h; \tilde{t}) \|_{L^1(X)} + \| x(\cdot; \tilde{t}) - x_\varepsilon(\cdot; \tilde{t}) \|_{L^1(X)} \right) \tag{53}
\]
\[
+ \| x(\cdot + h; t) - x_\varepsilon(\cdot + h; t) \|_{L^1(X)} + \| x(\cdot; t) - x_\varepsilon(\cdot; t) \|_{L^1(X)} \tag{54}
\]
\[
+ \lim_{h \to 0} \int_X \left| \frac{x_\varepsilon(z+h; \tilde{t}) - x_\varepsilon(z; \tilde{t})}{h} - \frac{x_\varepsilon(z+h; t) - x_\varepsilon(z; t)}{h} \right| \, dz. \tag{55}
\]

Concentrating on the last term, we have for $h \in \mathbb{R}_{>0}$, $h \leq 1$ and by defining $g_\varepsilon(\cdot, \cdot) := \partial_2 \lambda_\varepsilon(\cdot, \cdot) v_\varepsilon(\cdot)$ fixed:
\[
\int_X \left| \frac{x_\varepsilon(z+h; \tilde{t}) - x_\varepsilon(z; \tilde{t})}{h} - \frac{x_\varepsilon(z+h; t) - x_\varepsilon(z; t)}{h} \right| \, dz \tag{56}
\]
\[
\overset{(42)}{=} \frac{1}{h} \int_X \left[ \int_0^h \frac{v_\varepsilon(x_\varepsilon(z+s; t))}{v_\varepsilon(z+s)} \frac{g_\varepsilon(u, u_\varepsilon(z+s; u))}{v_\varepsilon(z+s)} \, du \right] \, ds \, dz \tag{57}
\]
\[
\leq \frac{1}{hv} \int_X \int_0^h \left| v_\varepsilon(x_\varepsilon(z+s; t)) \frac{g_\varepsilon(u, u_\varepsilon(z+s; u))}{v_\varepsilon(z+s)} \, du \right| \, ds \, dz,
\]
where the last two estimates used the identity in Rmk. 3 for smooth data and the fact that $v \leq v_\varepsilon(\cdot) \forall x \in \mathbb{R}_{>0}, \forall x \in \mathbb{R}$ a.e. Again recalling that it holds
\[
\| v_\varepsilon \|_{L^\infty(\mathbb{R})} \leq \| v \|_{L^\infty(\mathbb{R})} \quad \text{as well as} \quad \partial_2 \lambda_\varepsilon \|_{L^\infty((0, T), L^\infty(\mathbb{R}))} \leq \partial_2 \lambda \|_{L^\infty((0, T), L^\infty(\mathbb{R}))}
\]
and once more adding zeros yields
\[
\text{(57) } \leq \frac{\| v \|_{L^\infty(\mathbb{R})} e^T \| \partial_2 \lambda \|_{L^\infty(\mathbb{R})} \| v \|_{L^\infty(\mathbb{R})} \int_X \int_0^h \int_0^t \, \frac{g_\varepsilon(u, u_\varepsilon(z+s; u))}{v_\varepsilon(z+s)} \, du \, ds \, dz \tag{58}
\]
\[
+ \frac{e^T \| \partial_2 \lambda \|_{L^\infty(\mathbb{R})} \| v \|_{L^\infty(\mathbb{R})} \| \partial_2 \lambda \|_{L^\infty(\mathbb{R})} \| X \|_{L^\infty(\mathbb{R})}}{h v} \int_X \int_0^h \left| v_\varepsilon(x_\varepsilon(z+s; t)) - v_\varepsilon(x_\varepsilon(z+s; \tilde{t})) \right| \, ds \, dz \tag{59}
\]
\[
\leq \frac{\| v \|_{L^\infty(\mathbb{R})} e^T \| \partial_2 \lambda \|_{L^\infty(\mathbb{R})} \| v \|_{L^\infty(\mathbb{R})} \| \partial_2 \lambda \|_{L^\infty(\mathbb{R})} \| X \|_{L^\infty(\mathbb{R})}}{2 h v} \int_X \int_0^h \left| v_\varepsilon(x_\varepsilon(z+s; t)) - v_\varepsilon(x_\varepsilon(z+s; \tilde{t})) \right| \, ds \, dz \tag{60}
\]
\[
\leq \frac{\| v \|_{L^\infty(\mathbb{R})} e^T \| \partial_2 \lambda \|_{L^\infty(\mathbb{R})} \| v \|_{L^\infty(\mathbb{R})} \| \partial_2 \lambda \|_{L^\infty(\mathbb{R})} \| X \|_{L^\infty(\mathbb{R})}}{2 h v} \int_X \int_0^h \left| v_\varepsilon(x_\varepsilon(z+s; t)) - v_\varepsilon(x_\varepsilon(z+s; \tilde{t})) \right| \, ds \, dz \tag{61}
\]
for which the ODE solution

\[ \lim_{\varepsilon \to 0} \int_{\tilde{X}(h)} |v_\varepsilon(x_\varepsilon(z; t)) - v_\varepsilon(x_\varepsilon(z; \tilde{t}))| \, dz = \int_{\tilde{X}(h)} |v(x(z; t)) - v(x(z; \tilde{t}))| \, dz. \]

It is worth noting that this last term is well-defined as \( v(x(z; t)) \) is integrated over \( z \), for which the ODE solution \( x = \mathcal{X}[v, \lambda] \) is strictly monotone according to Cor. 2.1.

Recalling all the previous estimates starting from Eq. (51), for \( h \to 0 \) we have:

\[
\| \partial_1 x(\cdot; t) - \partial_1 x(\cdot; \tilde{t}) \|_{L^1(\tilde{X})} \leq \frac{e^{T|\partial_2 \lambda(\varepsilon)|}|v|_{L^\infty(\mathbb{R})}}{2} \| \varepsilon \|_{L^\infty(\mathbb{R})} \| \partial_2 \lambda \|_{L^\infty(\mathbb{R})} \| v \|_{L^\infty(\mathbb{R})} |X| |t - \tilde{t}| \]

\[ + \frac{e^{T|\partial_2 \lambda(\varepsilon)|}|v|_{L^\infty(\mathbb{R})}}{2} \int_{\tilde{X}(h)} |v_\varepsilon(x_\varepsilon(z; t)) - v_\varepsilon(x_\varepsilon(z; \tilde{t}))| \, dz, \]

with

\[ \tilde{X}(h) := (0, h) + X \subset \mathbb{R}. \]

Letting \( \varepsilon \to 0 \) the first term remains the same, while by Lem. 2.3 the second yields

\[
\lim_{\varepsilon \to 0} \int_{\tilde{X}(h)} |v_\varepsilon(x_\varepsilon(z; t)) - v_\varepsilon(x_\varepsilon(z; \tilde{t}))| \, dz = \int_{\tilde{X}(h)} |v(x(z; t)) - v(x(z; \tilde{t}))| \, dz.
\]

Now, distinguish the two considered cases:

\( v \notin TV_{\text{loc}}(\mathbb{R}) \): Then, when letting \( t \to \tilde{t} \), the first term in Eq. (58) converges to zero. This is also the case for the second term by Lem. 2.3 as we have \( \| x(\cdot; t) - x(\cdot; \tilde{t}) \|_{L^\infty(\mathbb{R})} \to 0 \) for \( t \to \tilde{t} \) and \( \partial_1 x \) bounded away from zero by Cor. 2.1.

\( v \in TV_{\text{loc}}(\mathbb{R}) \): Since we then want to obtain the Lipschitz-continuity in time, we need to study the second term in Eq. (58) in more detail. To this end, we reformulate as follows and use the standard mollifier for \( v_\varepsilon \) to have

\[
\int_{\tilde{X}} |v(x(z; t)) - v(x(z; \tilde{t}))| \, dz 
\]

\[
\leq \int_{\tilde{X}} |v(x(z; t)) - v_\varepsilon(x(z; t))| \, dz + \int_{\tilde{X}} |v_\varepsilon(x(z; t)) - v_\varepsilon(x(z; \tilde{t}))| \, dz 
\]

\[
+ \int_{\tilde{X}} |v_\varepsilon(x(z; \tilde{t})) - v(x(z; \tilde{t}))| \, dz. 
\]

Focusing only on the second term

\[
\int_{\tilde{X}} |v_\varepsilon(x(z; t)) - v_\varepsilon(x(z; \tilde{t}))| \, dz 
\]

\[
\leq \int_{\tilde{X}} \int_{x(z; t)}^{x(z; \tilde{t})} |v_\varepsilon'(y)| \, dy \, dz 
\]

\[
\leq \| x^{-1}(\cdot; t) - x^{-1}(\cdot; \tilde{t}) \|_{L^\infty(\mathbb{R}; \mathbb{R})} \int_{x(z; \tilde{t})}^{x(z; t)} |v_\varepsilon'(y)| \, dy, 
\]

with \( x^{-1} \) the inverse of \( x \) w.r.t. its first argument. This argument exists and is unique according to Cor. 2.1.

\[
\leq \frac{\| v \|_{L^\infty(\mathbb{R})}}{2} e^{Tc_2 \| v \|_{L^\infty(\mathbb{R})}} \| x(\cdot; \tilde{t}) - x(\cdot; t) \|_{L^\infty(\mathbb{R})} \int_{x(z; \tilde{t})}^{x(z; t)} |v_\varepsilon'(y)| \, dy 
\]

\[
+ \frac{\| v \|_{L^\infty(\mathbb{R})}}{2} \| \lambda \|_{L^\infty(\mathbb{R}; \mathbb{R})} e^{Tc_2 \| v \|_{L^\infty(\mathbb{R})}} |t - \tilde{t}| \int_{t \leq s \leq \tilde{t}} |x(z; s)| |v_\varepsilon'(y)| \, dy 
\]
Discontinuous nonlocal conservation laws and related discontinuous ODEs

\[
\begin{aligned}
\varepsilon \to 0 & \quad \frac{\|v\|^2_{L^\infty(\mathbb{R})}}{\varepsilon} \lambda |X|_{L^\infty((0,T);L^\infty(\mathbb{R}))} T L_2 \|v\|_{L^\infty(\mathbb{R})} |t - \tilde{t}| \|v\|_{TV} (X + T \|v\|_{L^\infty(\mathbb{R})} \mathcal{L}(-1,1)) \\
& \leq \frac{\|v\|^2_{L^\infty(\mathbb{R})}}{\varepsilon} \lambda |X|_{L^\infty((0,T);L^\infty(\mathbb{R}))} T L_2 \|v\|_{L^\infty(\mathbb{R})} |t - \tilde{t}| \|v\|_{TV} (x + \tau \|v\|_{L^\infty(\mathbb{R})} \mathcal{L}(-1,1)).
\end{aligned}
\]

Using this in the estimate in Eq. (58) yields the claim. \( \square \)

The previous statement used an approximation result to derive the required regularity of \( \partial_3 \mathcal{X} \). However, at least for \( v \in TV(\mathbb{R}) \) it is possible to obtain this directly with the surrogate system introduced in Thm. 2.1.

**Corollary 2.3** (An alternative proof of Prop. 2.2 using Thm. 2.1). Given \( x_0 \in \mathbb{R} \) and \( (v, \lambda) \in L^\infty(\mathbb{R};\mathbb{R}_{\geq 0}) \times L^\infty((0,T);W^{1,\infty}(\mathbb{R})) \) as in Asm. 1, assume in addition

\[ v \in TV_{loc}(\mathbb{R}). \]

Then, we have for \( \mathcal{X}[v, \lambda](x_0 ; \cdot) \) as in Defn. 3 that \( \forall (t, \tilde{t}) \in [0,T]^2 \)

\[
\|\partial_3 \mathcal{X}[v, \lambda](\cdot ; t) - \partial_3 \mathcal{X}[v, \lambda](\cdot ; \tilde{t})\|_{L^1(X)} \\
\leq \frac{\|v\|^2_{L^\infty(\mathbb{R})}}{\varepsilon} L_2 |X| |t - \tilde{t}| e^{\varepsilon T L_2 \|v\|_{L^\infty(\mathbb{R})}} \\
+ \left( 1 + t \|v\|_{L^\infty(\mathbb{R})} \right) L_2 \exp \left( t \|v\|_{L^\infty(\mathbb{R})} L_2 \right) \frac{\|v\|^2_{L^\infty(\mathbb{R})}}{\varepsilon} L |t - \tilde{t}| \|v\|_{TV}(X + \tau \|v\|_{L^\infty(\mathbb{R})} \mathcal{L}(-1,1)),
\]

with

\[
\mathcal{L} := \|\lambda\|_{L^\infty((0,T);L^\infty(\mathbb{R}))}, \quad L_2 := \|\partial_2 \lambda\|_{L^\infty((0,T);L^\infty(\mathbb{R}))}.
\]

In particular, it holds that

\[ [0,T] \ni t \mapsto \left( \mathbb{R} \ni y \mapsto \partial_y \mathcal{X}[v, \lambda](y, t) \right) \in C([0,T];L^1_{loc}(\mathbb{R})). \]

**Proof** This time, we prove the claim (with slightly different bounds) for discontinuous \( v \in TV_{loc}(\mathbb{R}) \) by taking advantage of the surrogate system in Thm. 2.1. Following the first steps in the previous proof of Prop. 2.2, we now only smooth the discontinuous velocity and use the notation

\[ x := \mathcal{X}[v, \lambda] \quad \text{and} \quad x_\varepsilon := \mathcal{X}[v_\varepsilon, \lambda_\varepsilon] \quad \text{on} \ \mathbb{R} \times (0,T). \]

Then, it holds that

\[
\int_X \left| \frac{x_\varepsilon(z + h ; t) - x_\varepsilon(z + h ; \tilde{t})}{h} - \frac{x_\varepsilon(z; t) - x_\varepsilon(z; \tilde{t})}{h} \right| \, dz \tag{56}.
\]

Using that the solution is a Carathéodory solution as in Lem. 2.1

\[
\frac{1}{h} \int_X \int_t^{\tilde{t}} \left| \int_s^{\tilde{t}} v_\varepsilon(x_\varepsilon(z + h ; s)) \lambda(s, x_\varepsilon(z + h ; s)) \\
- v_\varepsilon(x; \varepsilon(z ; s)) \lambda(s, x_\varepsilon(z ; s)) \, ds \right| \, dz \tag{60}.
\]
and applying the Fundamental theorem of integration
\[
\frac{1}{h} \int_X \left| \int_t^{\tilde{t}} \int_{x_\varepsilon(z + h; s)} x_\varepsilon(z; s) v'_{\varepsilon}(y) \lambda(s, y) + v_\varepsilon(y) \partial_2 \lambda(s, y) \, dy \, ds \right| \, dz
\]
\[
\leq \frac{1}{h} \int_X \left| \int_t^{\tilde{t}} \int_{x_\varepsilon(z + h; s)} x_\varepsilon(z; s) |v'_{\varepsilon}(y) \lambda(s, y)| \, dy \, ds \right| \, dz
\]
\[
+ \frac{1}{h} \int_X \left| \int_t^{\tilde{t}} \int_{x_\varepsilon(z + h; s)} x_\varepsilon(z; s) \left| v_\varepsilon(y) \partial_2 \lambda(s, y) \right| \, dy \, ds \right| \, dz.
\]
(61)
The second term in the previous estimate, Eq. (61), is estimated as follows
\[
\frac{1}{h} \int_X \left| \int_t^{\tilde{t}} \int_{x_\varepsilon(z + h; s)} x_\varepsilon(z; s) |v_\varepsilon(y) \partial_2 \lambda(s, y) \, dy \, ds \right| \, dz
\]
\[
\leq \frac{|v_\varepsilon|_{L^\infty(\mathbb{R})}}{h} \partial_2 \mathcal{Z}[\varepsilon, \mathcal{Z}[\varepsilon](x_\varepsilon(z; s))|X|\|t - \tilde{t}\|_{L^2_v} v|_{L^\infty(\mathbb{R})}}
\]
By applying the stability estimate in Cor. 2.1 and recalling that \( |v_\varepsilon|_{L^\infty(\mathbb{R})} \leq \|v\|_{L^\infty(\mathbb{R})} \), we obtain
\[
\leq \frac{|v_\varepsilon|_{L^\infty(\mathbb{R})}}{h} \mathcal{L}_2 |X|\|t - \tilde{t}\|_{L^2_v} v|_{L^\infty(\mathbb{R})}
\]
Clearly, for \( t \to \tilde{t} \) the previous term converges to zero uniformly in \( h \) and \( \varepsilon \).

The first term in Eq. (61) is more involved. However, thanks to the smoothing of \( v \) by \( v_\varepsilon \), for now we can take advantage of this higher regularity and have
\[
\frac{1}{h} \int_X \left| \int_t^{\tilde{t}} \int_{x_\varepsilon(0 + h; s)} x_\varepsilon(0; s) v'_{\varepsilon}(y) \lambda(s, y) \, dy \, ds \right| \, dx_0.
\]
(62)
Using Eqs. (5) and (6) and Eq. (7): \( x_\varepsilon(x_0; \cdot) = \mathcal{Z}[v_\varepsilon]^{-1}(x_0; \mathcal{C}[\lambda, \mathcal{Z}[v_\varepsilon](x_0; \cdot))(\cdot) \)
\[
\frac{1}{h} \int_X \left| \int_t^{\tilde{t}} \int_{x_\varepsilon(0 + h; s)} x_\varepsilon(0; s) v'_{\varepsilon}(y) \lambda(s, y) \, dy \, ds \right| \, dx_0.
\]
(63)
Substituting \( y = \mathcal{Z}[v_\varepsilon]^{-1}(x_0; u) \Rightarrow u = \mathcal{Z}[v_\varepsilon](x_0; y) \), we have for the derivative \( \frac{d}{du} \mathcal{Z}[v_\varepsilon]^{-1}(x_0; u) = v_\varepsilon(\mathcal{Z}[v_\varepsilon]^{-1}(x_0; u)) \). Thus
\[
\frac{1}{h} \int_X \left| \int_t^{\tilde{t}} \int_{\mathcal{C}[\lambda, \mathcal{Z}[v_\varepsilon](x_0; \cdot))a(u, s, x_0) \, du \, ds \right| \, dx_0,
\]
(64)
with
\[
a(u, s, x_0) = v'_{\varepsilon}(\mathcal{Z}[v_\varepsilon]^{-1}(x_0; u)) \lambda(s, \mathcal{Z}[v_\varepsilon]^{-1}(x_0; u)) \mathcal{Z}[v_\varepsilon]^{-1}(x_0; u)).
\]
(65)
Recalling that for \( x \in \mathbb{R} \) we can estimate
\[
\mathcal{Z}[v_\varepsilon](x_0; \mathcal{Z}[v_\varepsilon]^{-1}(x_0 + h; x)) = \mathcal{Z}[v_\varepsilon](x_0; \mathcal{Z}[v_\varepsilon]^{-1}(x_0 + h; x))
\]
(66)
and by Eq. (23) in Prop. 2.1
\[
\leq x + \frac{h}{2},
\]
(69)
In addition, by Eq. (24) together with Eq. (23) in Prop. 2.1, we have for $s \in [0, T]$
\[
C[\lambda, Z[v_e](x_0 + h; \cdot)](t) - C[\lambda, Z[v_e](x_0; \cdot)](t)
\leq t \left\| v_e \right\|_{L^\infty(R)} L_2 \exp \left( t \left| v_e \right|_{L^\infty(R)} L_2 \right) h \leq t \left| v_e \right|_{L^\infty(R)} L_2 \exp \left( t \left| v \right|_{L^\infty(R)} L_2 \right) h = C(v, L_2, t) h.
\]  
(70)

Continuing the previous estimate, assuming without loss of regularity that $\tilde{t} \geq t$ yields
\[
\left\| v \right\|_{L^\infty(R)} L_2 \sup y \left| a(u, u, x, 0) \right| du ds dx_0 \leq \frac{1}{h} \int_X \int_{\tilde{t}} t \int_{\tilde{t}} C[\lambda, Z[v_e](x_0 + h; \cdot)](s) + \frac{h}{2} \left| v \right|_{L^\infty(R)} L_2 \exp \left( t \left| v \right|_{L^\infty(R)} L_2 \right) h \leq \frac{1}{h} \int_X \int_{\tilde{t}} t \int_{\tilde{t}} C[\lambda, Z[v_e](x_0; \cdot)](s) + \frac{h}{2} \left( 1 + C(v, L_2, s) \right) \left| a(u, x_0, x) \right| du ds dx_0.
\]  
(71)

By Eq. (6) \[
\left| a(u, x_0, x) \right| \leq T \left| \lambda \right|_{L^\infty((0, T) \times L^\infty(R))} \forall (x_0, s) \in \mathbb{R} \times (0, T)
\]
\[
\left\| v \right\|_{L^\infty(R)} L_2 \sup y \left| a(u, u, x, 0) \right| du ds dx_0 \leq \frac{1}{h} \int_X \int_{\tilde{t}} t \int_{\tilde{t}} C[\lambda, Z[v_e](x_0; \cdot)](s) + \frac{h}{2} \left( 1 + C(v, L_2, s) \right) \left| a(u, x_0, x) \right| du ds dx_0 \leq \frac{1}{h} \int_X \int_{\tilde{t}} t \int_{\tilde{t}} C[\lambda, Z[v_e](x_0; \cdot)](s) + \frac{h}{2} \left( 1 + C(v, L_2, s) \right) \left| a(u, x_0, x) \right| du ds dx_0.
\]  
(72)

By Eq. (6) \[
\left| a(u, x_0, x) \right| \leq T \left| \lambda \right|_{L^\infty((0, T) \times L^\infty(R))} \forall (x_0, s) \in \mathbb{R} \times (0, T)
\]
\[
\left\| v \right\|_{L^\infty(R)} L_2 \sup y \left| a(u, u, x, 0) \right| du ds dx_0 \leq \frac{1}{h} \int_X \int_{\tilde{t}} t \int_{\tilde{t}} C[\lambda, Z[v_e](x_0; \cdot)](s) + \frac{h}{2} \left( 1 + C(v, L_2, s) \right) \left| a(u, x_0, x) \right| du ds dx_0 \leq \frac{1}{h} \int_X \int_{\tilde{t}} t \int_{\tilde{t}} C[\lambda, Z[v_e](x_0; \cdot)](s) + \frac{h}{2} \left( 1 + C(v, L_2, s) \right) \left| a(u, x_0, x) \right| du ds dx_0.
\]  
(73)

By Eq. (6) \[
\left| a(u, x_0, x) \right| \leq T \left| \lambda \right|_{L^\infty((0, T) \times L^\infty(R))} \forall (x_0, s) \in \mathbb{R} \times (0, T)
\]
\[
\left\| v \right\|_{L^\infty(R)} L_2 \sup y \left| a(u, u, x, 0) \right| du ds dx_0 \leq \frac{1}{h} \int_X \int_{\tilde{t}} t \int_{\tilde{t}} C[\lambda, Z[v_e](x_0; \cdot)](s) + \frac{h}{2} \left( 1 + C(v, L_2, s) \right) \left| a(u, x_0, x) \right| du ds dx_0 \leq \frac{1}{h} \int_X \int_{\tilde{t}} t \int_{\tilde{t}} C[\lambda, Z[v_e](x_0; \cdot)](s) + \frac{h}{2} \left( 1 + C(v, L_2, s) \right) \left| a(u, x_0, x) \right| du ds dx_0.
\]  
(74)

In the previous estimate we have used the substitution
\[
y = Z[v_e]^{-1}(x_0; u) \Rightarrow Z[v_e](x_0; y) = u
\]
\[
\Rightarrow \frac{d}{dy} Z[v_e](x_0; y) = \frac{1}{v_e(y)} - \frac{1}{v_e(x_0(y))} = 0 \Rightarrow \left| \frac{d}{dy} x_0(y) \right| \leq \frac{\left| v_e \right|_{L^\infty(R)}}{v}.
\]

Thus, for $x \in X$ and $u \in Z(v, L_2, T, h)$ we can estimate
\[
\left| Z[v]^{-1}(x; u) - x \right| = \left| Z[v]^{-1}(x; u) - Z[v]^{-1}(x; 0) \right| \leq \left| v \right|_{L^\infty(R)} L T + \frac{h}{2} \left( 1 + C(v, L_2, T) \right).
\]

Consequently for $h \rightarrow 0$
\[
Z[v_e]^{-1}(X; u) \subset X + \left| v \right|_{L^\infty(R)} L T(-1, 1).
\]

Using this to further estimate in Eq. (76), we have
\[
(76) \leq \left( 1 + C(v, L_2, T) \right) \left| v_e \right|_{L^\infty(R)} L \left| \tilde{t} - t \right| \left| v_e \right| TV(X + \left| v \right|_{L^\infty(R)} L(-1, 1)))
\]
\[
\leq \left( 1 + C(v, L_2, T) \right) \left| v \right|_{L^\infty(R)} L \left| \tilde{t} - t \right| \left| v \right| TV(X + \left| v \right|_{L^\infty(R)} L(-1, 1)).
\]

As the terms are all bounded, we can let $\tilde{t} \rightarrow t$ and obtain – together with the previous estimates – the claimed continuity in time.
3 Analysis of discontinuous nonlocal conservation laws

In this section, we leverage the theory established in Section 2 to obtain existence and uniqueness of weak solutions for the following class of nonlocal conservation laws with discontinuous (in space) velocity (as stated in Defn. 2).

First, we state the assumptions on the involved datum:

**Assumption 2** (Input datum – discontinuous nonlocal conservation law). For \( T \in \mathbb{R}_+ \) it holds that

- \( q_0 \in L^\infty(\mathbb{R}) \)
- \( \gamma \in BV(\mathbb{R};\mathbb{R}^\geq) \) with \( \| \cdot \|_{BV(\mathbb{R})} := \| \cdot \|_{L^1(\mathbb{R})} + | \cdot |_{TV(\mathbb{R})} \).
- \( V \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) \)
- \( v \in L^\infty(\mathbb{R};\mathbb{R}^\geq) \) for a \( v \in \mathbb{R}^\geq \).

As can be seen, the assumptions on \( V \) and \( v \) are identical to those for the discontinuous IVP in Defn. 1 (compare with Asm. 1) and are not restrictive. The assumptions on the initial datum \( q_0 \in L^\infty(\mathbb{R}) \) are relatively standard in the theory of conservation laws and the assumptions on the nonlocal kernel \( \gamma \) are also minimal (compare in particular with [41]).

We use the classical definition of weak solutions as follows

**Definition 6** (Weak solution). For the initial datum \( q_0 \in L^\infty(\mathbb{R}) \), \( q \in C([0,T];L^1_{\text{loc}}(\mathbb{R})) \cap L^\infty((0,T);L^\infty(\mathbb{R})) \) with datum as in Asm. 2 is called a weak solution of Defn. 2 iff \( \forall \phi \in C^1_c((0,42) \times \mathbb{R}) \) and it holds that

\[
\int_{\Omega_T} q(t,x) \left( \phi_t(t,x) + v(x)V(\gamma * q(t,\cdot))(x) \phi_x(t,x) \right) \, dx \, dt
+ \int_{\mathbb{R}} \phi(0,x) q_0(x) \, dx = 0.
\]

### 3.1 Well-posedness of solutions

In this section, we will establish the existence and uniqueness of the weak solution to discontinuous nonlocal conservation laws. We start with existence for sufficiently small time horizons and use a reformulation in terms of a fixed-point problem. Such a reformulation has been used in various contributions dealing with nonlocal conservation laws, including [2, 5, 10, 12, 13, 18, 19, 22, 46, 58, 101].

**Theorem 3.1** (Existence/uniqueness, weak solutions, small time horizon). There exists a time horizon \( T \in \mathbb{R}_+ \) such that the nonlocal, discontinuous conservation law in Defn. 2 admits a unique weak solution (as in Defn. 6)

\[
q \in C([0,T];L^1_{\text{loc}}(\mathbb{R})) \cap L^\infty((0,T);L^\infty(\mathbb{R})).
\]
The solution can be stated as
\[ q(t,x) = q_0(\xi_w(t,x;0)) \partial_2 \xi_w(t,x;0), \quad (t,x) \in \Omega_T \]  
(77)
where \( \xi_w : \Omega_T \times [0,T] \rightarrow \mathbb{R} \) is the unique solution of the IVP as in Defn. 3 for \((t,x) \in \Omega_T\)

\[
\partial_2 \xi_w(t,x;\tau) = v(\xi_w(t,x;\tau))V(w(\tau,\xi(t,x;\tau))), \quad \tau \in (0,T)
\]
(78)
and \( w \) is the solution of the fixed-point equation in \( L^\infty((0,T);L^\infty(\mathbb{R})) \)
\[
w(t,x) = \int_\mathbb{R} \gamma(x - \xi_w(0,y;t))q_0(y) \, dy, \quad (t,x) \in \Omega_T.
\]

Proof Define the fixed-point mapping
\[
F : \begin{cases} L^\infty((0,T);W^{1,\infty}(\mathbb{R})) & \to L^\infty((0,T);W^{1,\infty}(\mathbb{R})) \\
w(t,x) & \mapsto \left( (t,x) \mapsto \int_\mathbb{R} \gamma(x - \xi_w(0,y;t))q_0(y) \, dy \right)
\end{cases}
\]
(79)
with \( \xi_w \) the characteristics as defined in Eq. (78). Let us first look into the well-posedness of these characteristics. Given that \( w \in L^\infty((0,T);W^{1,\infty}(\mathbb{R})) \) and recalling Asm. 2, we can invoke Thm. 2.1 to demonstrate that \( \xi_w \) is uniquely determined by \( w \). Next, we show that \( F \) is a fixed-point mapping on the proper subset of \( L^\infty((0,T);W^{1,\infty}(\mathbb{R})) \). To this end, define
\[
M := 42 \gamma \| \xi_w \|_{L^1(\mathbb{R})} \| q_0 \|_{L^\infty(\mathbb{R})} \frac{\|v\|_{L^\infty(\mathbb{R})}}{2},
\]
(80)
\[
M' := 42 \gamma \| v \|_{TV(\mathbb{R})} \| q_0 \|_{L^\infty(\mathbb{R})} \frac{\|v\|_{L^\infty(\mathbb{R})}}{2},
\]
\[
\Omega_M^M(T) := \left\{ w \in L^\infty((0,T);W^{1,\infty}(\mathbb{R})) : \| w \|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \leq M \right\} \cap \{ \partial_2 w \|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \leq M' \}.
\]
(81)
Self-mapping: Taking \( w \in \Omega_M^M(T_1) \) for \( T_1 \in (0,T] \), we estimate for \( t \in (0,T_1] \)
\[
|F[w]|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \leq \| q_0 \|_{L^\infty(\mathbb{R})} \gamma \| \xi_w \|_{L^1(\mathbb{R})} \| \partial_2 \xi_w(t,\cdot;0) \|_{L^\infty(\mathbb{R})}
\]
\[
\leq \| q_0 \|_{L^\infty(\mathbb{R})} \gamma \| \xi_w \|_{L^1(\mathbb{R})} \frac{\|v\|_{L^\infty(\mathbb{R})}}{2} e^{T_1 \| v \|_{L^\infty(\mathbb{R})} |V'|_{L^\infty((-M,M))} M'}. 
\]
(82)
Here we have used the stability estimate of the IVP in Cor. 2.1 to uniformly estimate the spatial derivative of the characteristics. Thus, \( |F[w]|_{L^\infty((0,T_1);L^\infty(\mathbb{R}))} \leq M \) holds if
\[
e^{T_1 \| v \|_{L^\infty(\mathbb{R})} |V'|_{L^\infty((-M,M))} M'} \leq 42.
\]
We then pick the maximal \( T_1 \in (0,T] \) satisfying this inequality.
Next, consider the spatial derivative of \( F \) on the time interval \( T_2 \in (0,T_1] \) and this time choose \( w \in \Omega_M^M(T_2) \). Analogous to the previous estimate, we estimate for \( t \in (0,T_2] \)
\[
\| \partial_2 F[w] \|_{L^\infty((0,T);L^\infty(\mathbb{R}))} \leq \| q_0 \|_{L^\infty(\mathbb{R})} \| v \|_{TV(\mathbb{R})} \| \partial_2 \xi_w(t,\cdot;0) \|_{L^\infty(\mathbb{R})}
\]
\[ Eq. \ (44) \leq \| q_0 \|_{L^\infty(\mathbb{R})} \gamma_{TV(\mathbb{R})} \frac{\| v \|_{L^\infty(\mathbb{R})}}{\beta} e^{t \gamma_{L^\infty(\mathbb{R})}} |V'|_{L^\infty((-M,M))} M'. \]

Here we have used the stability estimate of the IVP in Cor. 2.1 to uniformly estimate the spatial derivative of the characteristics. Thus, \( \| \partial_2 F[w] \|_{L^\infty((0,T_x); L^\infty(\mathbb{R}))} \leq M' \) holds if
\[ e^{T_2 \gamma_{L^\infty(\mathbb{R})}} |V'|_{L^\infty((-M,M))} M' \leq 42. \] (83)
As this is identical to the condition in Eq. (82), we can indeed pick \( T_2 = T_1 \) as our considered time horizon. Based on the previous estimates, we thus have a self-mapping on the considered time horizon, i.e.
\[ F\left( \Omega^M_M(T_1) \right) \subseteq \Omega^M_M(T_1). \]

**Contraction:** Next, we show that the mapping \( F \) is a contraction for a yet to be determined \( T_3 \in (0, T_1) \) in \( L^\infty((0,T_3); L^\infty(\mathbb{R})) \). To this end, take \( w, \tilde{w} \in \Omega^M_M(T_3) \) and estimate for \( (t, x) \in [0, T_1] \times \mathbb{R} \)
\[ |F[w](t, x) - F[\tilde{w}](t, x)| \leq \| q_0 \|_{L^\infty(\mathbb{R})} \int_\mathbb{R} |\gamma(x - \xi_{\tilde{w}}(0, y; t)) - \gamma(x - \xi_{\tilde{w}}(0, y; t))| \ dy \]
\[ \leq \| q_0 \|_{L^\infty(\mathbb{R})} \gamma_{TV(\mathbb{R})} \| v \|_{L^\infty(\mathbb{R})} \| \xi_w - \xi_{\tilde{w}} \|_{L^\infty((0, t) \times \mathbb{R})} e^{t \gamma_{L^\infty(\mathbb{R})}} |V'|_{L^\infty((-M,M))} M'. \] (84)
In the last estimate we have used the following:

1. For \( f \in TV(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and diffeomorphisms \( g, h \) it holds (for the proof see for instance [20, Lemma 2.4])
\[ \| f \circ g - f \circ h \|_{L^1(\mathbb{R})} \leq \| f \|_{TV(\mathbb{R})} \| g^{-1} - h^{-1} \|_{L^\infty(\mathbb{R})}. \]

It also holds for the characteristics that
\[ \xi(t, \xi(t, x; t); \tau) = x \quad \forall(t, x, \tau) \in \Omega_T \times (0, T), \] (85)
meaning that the inverse of the mapping \( x \mapsto \xi(t, x; \tau) \) is the mapping \( x \mapsto \xi(t, x; t) \). This can be shown by approximating \( \xi \) by \( \xi_\varepsilon \) with smooth \( v_\varepsilon, \lambda_\varepsilon \) and the claim that for \( \varepsilon \in \mathbb{R}_{>0} \) it holds that
\[ \xi_\varepsilon(t, \xi_\varepsilon(t, x; t); \tau) = x \quad \forall x \in \mathbb{R}, \ (t, \tau) \in [0, T]. \]

However, this result was carried out in [10, Lemma 2.6 Item 1]. As we have the strong convergence of \( \xi_\varepsilon \) to \( \xi \) by Thm. 2.4, this carries over to Eq. (85).

2. We have identified in Thm. 2.4 that
\[ \lambda \equiv V(w), \ \tilde{\lambda} \equiv V(\tilde{w}) \text{ and} \]
\[ \mathcal{L}_2 := \max \left\{ \| \partial_2 \lambda \|_{L^\infty((0,T_3); L^\infty(\mathbb{R}))}, \| \partial_2 \tilde{\lambda} \|_{L^\infty((0,T_3); L^\infty(\mathbb{R}))} \right\} \]
\[ \leq \| V' \|_{L^\infty((-M,M))} \cdot \max \left\{ \| \partial_2 w \|_{L^\infty((0,T_3); L^\infty(\mathbb{R}))}, \| \partial_2 \tilde{w} \|_{L^\infty((0,T_3); L^\infty(\mathbb{R}))} \right\} \]
\[ \leq \| V' \|_{L^\infty((-M,M))} M'. \]
where we have used the fact that $w, \tilde{w} \in \Omega^M_M(T_3)$ and in particular Eq. (81).

To obtain an estimate of $\|\xi w - \xi \tilde{w}\|_{L^\infty((0,T_3)\times\mathbb{R}^d)}$ in terms of $\|w - \tilde{w}\|_{L^\infty((0,T_3)\times\mathbb{R}^d)}$, we can again take advantage of Thm. 2.4, which yields

$$
(30) \quad M' \|w - \tilde{w}\|_{L^\infty((0,T_3)\times\mathbb{R}^d)} \leq M' \|w - \tilde{w}\|_{L^\infty((0,T_3)\times\mathbb{R}^d)} \leq M' \|v\|_{L^\infty(\mathbb{R})} e^{t\|v\|_{L^\infty(\mathbb{R})}} \|V'(\xi w)\|_{L^\infty(\mathbb{R})} M' \int_0^t \|V(w(s,\cdot)) - V(\tilde{w}(s,\cdot))\|_{L^\infty(\mathbb{R})} ds
$$

Reconnecting to Eq. (84), we thus have for small enough time $T_3 \in (0,T_1]$ (recall that in the previous estimate the right hand side consists of constants except for the term $\|w - \tilde{w}\|_{L^\infty((0,T_3)\times\mathbb{R}^d)}$)

$$
\|F[w] - F[\tilde{w}]\|_{L^\infty((0,T_3)\times\mathbb{R}^d)} \leq \frac{1}{2} \|w - \tilde{w}\|_{L^\infty((0,T_3)\times\mathbb{R}^d)},
$$

i.e., $F$ is a contraction in $L^\infty((0,T_3); L^\infty(\mathbb{R}))$.

**Concluding the fixed-point argument:** As $M, M' \in \mathbb{R}_{>0}$ are fixed, we have proven $F$ to be a self-mapping on $\Omega^M_M(T_3)$, and $\Omega^M_M(T_3)$ – thanks to the uniform bound $M$ on the functions and $M'$ on their spatial derivatives – is closed in the topology induced by $L^\infty((0,T_3); L^\infty(\mathbb{R}))$, we can apply Banach’s fixed-point theorem [102, Theorem 1.a] and obtain

$$
\exists! \, w^* \in \Omega^M_M(T_3) : F[w^*] \equiv w^* \text{ on } (0,T_3) \times \mathbb{R}. \tag{86}
$$

**Constructing a solution to the conservation law:** Having obtained the existence and uniqueness of $w^*$ as a fixed-point on a small time horizon, we use the method of characteristics as carried out in [10, Theorem 2.20] to state the solution as

$$
q(t,x) = q_0(\xi w^*(t,x;0)) \partial_2 \xi w^*(t,x;0), \quad (t,x) \in (0,T_3) \times \mathbb{R}. \tag{87}
$$

Note that due to Thm. 2.4, $x \mapsto \xi w^*(t,x;0)$ is Lipschitz-continuous and strictly monotone increasing by Cor. 2.1. By Prop. 2.2 $\partial_2 \xi w^*(\cdot,\cdot;0) \in C((0,T_3]; L^1_{loc}(\mathbb{R}))$ so that $q \in C([0,T_3]; L^1_{loc}(\mathbb{R}))$. The fact that $q \in L^\infty((0,T_3); L^\infty(\mathbb{R}))$ is a direct consequence of Eq. (77). It can easily be checked that $q$ as in Eq. (87) is a solution by plugging it into the definition of weak solutions in Defn. 6 and applying the substitution rule.

The uniqueness of solutions is more involved, but ultimately only an adaptation of the proof in [58] and adjusted in [10, Theorem 3.2]. Therefore, we only sketch the idea: In a first step we show by a proper choice of test functions in Defn. 6 that each weak solution can be stated in the form of Eq. (87) with a proper nonlocal term $w$. The next step is then to show that for the thus constructed solution, the nonlocal term satisfies the same fixed-point mapping as introduced in Eq. (79). However, as this mapping has a unique fixed-point as we have proven previously, we have shown the uniqueness and are done.

The previous result only demonstrates the existence of solutions on a small time horizon. Given the later Asm. 3, we can show that even for general discontinuities in $v$, i.e., $v \in L^\infty(\mathbb{R}; L^\infty_{\#})$, the solution remains bounded on every finite time horizon. This is a key ingredient for extending the solution from small time in Thm. 3.1 to arbitrary times.
To prove a weakened form of a maximum principle, we require the solutions to be smooth. Consequently, we first introduce the following (weak) stability result:

**Theorem 3.2** (Weak stability of \( q \) w.r.t. discontinuous velocity \( v \), velocity \( V \) and initial datum \( q_0 \)). Let Asm. 2 hold. Denote by

\[
\{ v_\varepsilon \}_{\varepsilon \in R_{>0}} \subset C^\infty (R), \quad \{ V_\varepsilon \}_{\varepsilon \in R_{>0}} \subset C^\infty (R) \quad \text{and} \quad \{ q_0, \varepsilon \}_{\varepsilon \in R_{>0}}
\]

the mollified versions of \( v, V, q_0 \) convoluted with the standard mollifier outlined as in Cor. 2.2. Denote by \( T^* \in (0, T] \) the minimal time horizon of existence for the solution \( q, \{ q_\varepsilon \}_{\varepsilon \in R_{>0}} \) (where \( q \) is the solution to initial datum \( q_0 \), discontinuous velocity \( v \) and Lipschitz velocity \( V \) and \( q_\varepsilon \) the solution to initial datum \( q_0, \varepsilon \), discontinuous velocity \( v_\varepsilon \) and Lipschitz velocity \( V_\varepsilon \)) as guaranteed in Thm. 3.1. Then, it holds that

\[
\forall g \in C_c (R) : \lim_{\varepsilon \to 0} \max_{t \in [0, T^*]} \left| \int_R (q(t, x) - q_\varepsilon (t, x)) g(x) \, dx \right| = 0. \tag{88}
\]

**Proof** We start by showing that such a time horizon \( T^* \) exists uniformly in \( \varepsilon \). Recalling the proof of Thm. 3.1, the properties of the standard mollifier for each \( \varepsilon \in R_{>0} \) enable us to define the upper bounds on the nonlocal term as in Eq. (80)

\[
M_\varepsilon := 42 \| \gamma \|_{L^1 (R)} \| q_{0, \varepsilon} \|_{L^\infty (R)} \frac{\| v_\varepsilon \|_{L^\infty (R)}}{\varepsilon} \leq 42 \| \gamma \|_{L^1 (R)} \| q_0 \|_{L^\infty (R)} \frac{\| v \|_{L^\infty (R)}}{\varepsilon} =: M,
\]

\[
M'_\varepsilon := 42 \| \gamma \|_{TV (R)} \| q_{0, \varepsilon} \|_{L^\infty (R)} \frac{\| v_\varepsilon \|_{L^\infty (R)}}{\varepsilon} \leq 42 \| \gamma \|_{TV (R)} \| q_0 \|_{L^\infty (R)} \frac{\| v \|_{L^\infty (R)}}{\varepsilon} =: M'.
\]

However, this means that we can take as upper bounds uniformly \( M, M' \). Looking into the self-mapping condition in Eq. (82), it then reads in our case

\[
\exp \left( T_1 \| v \|_{L^\infty (R)} \| V' \|_{L^\infty ((-M, M))^M} \right) \leq 42,
\]

which can also be replaced by the stronger form

\[
\exp \left( T_1 \| v \|_{L^\infty (R)} \| V' \|_{L^\infty ((-M, M))^M} \right) \leq 42.
\]

Now choosing \( T_1 \) to satisfy the previous inequality this is by construction \( \varepsilon \) invariant. The identical argument can be made for the estimate in Eq. (82), so that for the chosen \( T_1 \) the mapping \( F \) in Eq. (79) is a self-mapping on \( \Omega_M^M (T_1) \), as in Eq. (81). So the only point that remains is to check whether the fixed-point mapping is also a contraction uniformly in \( \varepsilon \in R_{>0} \) for a small time horizon. Recollecting the contraction estimate starting in Eq. (84), we have for \( T_2 \in (0, T_1] \)

\[
\| F[w] - F[\tilde{w}] \|_{L^\infty ((0, T_2); L^\infty (R))} \leq \| v_\varepsilon \|_{L^\infty (R)} T_2 \| V' \|_{L^\infty ((-M, M))^M} T_2 \| v \|_{L^\infty (R)} \| V'' \|_{L^\infty ((-M, M))^M} T_2 \| w - \tilde{w} \|_{L^\infty ((0, T_2); L^\infty (R))}
\]

\[
\leq \| v \|_{L^\infty (R)} T_2 \| V' \|_{L^\infty ((-M, M))^M} M' T_2 \| v \|_{L^\infty (R)} \| V'' \|_{L^\infty ((-M, M))^M} M' \| w - \tilde{w} \|_{L^\infty ((0, T_2); L^\infty (R))}.
\]

Again choosing \( T_2 \in (0, T_1] \) so that

\[
\| v \|_{L^\infty (R)} T_2 \| V' \|_{L^\infty ((-M, M))^M} M' T_2 \| v \|_{L^\infty (R)} \| V'' \|_{L^\infty ((-M, M))^M} M' \leq \frac{1}{2}
\]
is invariant on \( \varepsilon \in \mathbb{R}_{>0} \), we can set \( T^* := T_2 \in \mathbb{R}_{>0} \) and have found the time horizon on which the existence of solutions is guaranteed for all \( \varepsilon \in \mathbb{R}_{>0} \) simultaneously.

Next, we prove the claimed continuity as stated in Eq. (88). We recall the solution formula in Eq. (77) and obtain for a \( g \in C_{\text{loc}}(\mathbb{R}) \) and \( t \in [0, T^*] \) and for \( \varepsilon \in \mathbb{R}_{>0} \)

\[
\left| \int_{\mathbb{R}} (q(t, x) - q_\varepsilon(t, x)) g(x) \, dx \right|
\]

\[
= \left| \int_{\mathbb{R}} q_\varepsilon(x) \partial_2 \xi \left( t, x, 0 \right) - q_\varepsilon(x) \partial_2 \xi_\varepsilon \left( t, x, 0 \right) \right| \, dx
\]

\[
\leq \left| \int_{\mathbb{R}} q_\varepsilon(x) \partial_2 \xi_\varepsilon \left( t, x, 0 \right) \right| \, dx
\]

\[
\leq \left\| q_\varepsilon \right\|_{L^\infty(\mathbb{R})} \left\| \partial_2 \xi \right\|_{L^\infty(\mathbb{R})} \left\| \partial_2 \xi_\varepsilon \right\|_{L^\infty(\mathbb{R})}
\]

Recalling the bounds on the nonlocal term in Eq. (89) as detailed in Eq. (81) for \( s \in [0, T^*] \), it holds by Rmk. 4 and in particular Eq. (46) together with Thm. 2.4 that

\[
\left\| \xi_\varepsilon \left( t, x, 0 ; \varepsilon \right) - \xi \left( t, x, 0 ; \varepsilon \right) \right\|_{L^\infty(\mathbb{R})}
\]

\[
\leq \left\| v \right\|_{L^\infty(\mathbb{R})} E^\varepsilon \left[ \int_0^{T^*} \left\| V \right\|_{L^\infty(\mathbb{R})} \, ds \right]
\]

\[
\leq \left\| v \right\|_{L^\infty(\mathbb{R})} E^\varepsilon \left[ \int_0^{T^*} \left\| V \right\|_{L^\infty(\mathbb{R})} \, ds \right]
\]

Applying the fixed-point identity Eq. (79) and Eq. (86), we end up with

\[
\int_0^{T^*} \left\| w \right\|_{L^\infty(\mathbb{R})} \, ds
\]

\[
\leq \int_0^{T^*} \left\| \int_0^{T^*} \gamma (t - \varepsilon, w_\varepsilon (0, y ; s)) q_\varepsilon(y) - \gamma (t - \varepsilon, w_\varepsilon (0, y ; s)) q_\varepsilon(y) \, dy \right\|_{L^\infty(\mathbb{R})} \, ds
\]

\[
\leq \int_0^{T^*} \left\| \int_0^{T^*} \gamma (t - \varepsilon, w_\varepsilon (0, y ; s)) (q_\varepsilon(y) - q_\varepsilon(y)) \, dy \right\|_{L^\infty(\mathbb{R})} \, ds
\]

\[
+ \left\| \int_0^{T^*} \gamma (t - \varepsilon, w_\varepsilon (0, y ; s)) \, dy \right\|_{L^\infty(\mathbb{R})} \, ds
\]

\[
\leq \int_0^{T^*} \left\| \int_0^{T^*} \gamma (t - \varepsilon, w_\varepsilon (0, y ; s)) (q_\varepsilon(y) - q_\varepsilon(y)) \, dy \right\|_{L^\infty(\mathbb{R})} \, ds
\]

\[
+ \left\| \int_0^{T^*} \gamma (t - \varepsilon, w_\varepsilon (0, y ; s)) \, dy \right\|_{L^\infty(\mathbb{R})} \, ds
\]
Discontinuous nonlocal conservation laws and related discontinuous ODEs

\[ \leq \int_0^{T^*} \left\| \gamma'(\cdot - \xi_w(0, y; s)) \partial_2 \xi_w(0, y; s) \int_0^y (q_0(z) - q_0, \varepsilon(z)) \, dz \, dy \right\|_{L^\infty(\mathbb{R})} + \| q_0 \|_{L^\infty(\mathbb{R})} \gamma|TV(\mathbb{R}) \int_0^{T^*} \| \xi_w(s, \cdot ; *) - \xi_{\varepsilon, w_\varepsilon}(s, \cdot ; *) \|_{L^\infty((0, T^*); L^\infty(\mathbb{R}))} \, ds \]

In the last estimate we have again used what was described in Item 1 in the proof of Thm. 3.1 and the assumptions on the involved datum Asm. 2, particularly \( \gamma \in \mathcal{BV}(\mathbb{R}) \).

As \( T^* \) was arbitrary (but small enough so that solutions still exist), we can apply Grönwall’s inequality [96, Chapter I, III Grönwall’s inequality] and, recollecting all previous terms, obtain

\[ \| \xi_{\varepsilon, w_\varepsilon} - \xi_w \|_{L^\infty((0, T^*); L^\infty(\mathbb{R}))} \leq \left( \| v \|_{L^\infty(\mathbb{R})} e^{T^*} \sup_{y \in \mathbb{R}} \sup_{s, t} \int_0^{T^*} \left| v(s) - v(\varepsilon(s) \right| \, ds \right) + \| v \|_{L^\infty(\mathbb{R})} e^{T^*} \sup_{y \in \mathbb{R}} \left| v' \right|_{L^\infty((-M, M))} \gamma|TV(\mathbb{R}) \int_0^{T^*} \| q_0(z) - q_0, \varepsilon(z) \|_{L^\infty((0, T^*); L^\infty(\mathbb{R}))} \, dz \right) \]

However, this means that \( \xi_w - \xi_{\varepsilon, w_\varepsilon} \) is small in the uniform topology for \( \varepsilon \in \mathbb{R}_{>0} \) small.

Thanks to the lower bounds on the spatial derivatives on \( \xi_w, \xi_{\varepsilon, w_\varepsilon} \), i.e., thanks to the fact that they are diffeomorphisms in space with a Lipschitz constant from below which is greater than zero (see Cor. 2.1), we can apply Lem. 2.3 on Eq. (90) and obtain the claimed continuity in Eq. (88).

The previous theory enables to have smooth solutions when assuming smooth initial datum and smooth velocities. Even more, we can later use the previous approximation to derive bounds on the smoothed solution (as considered in the following Lem. 3.1). These bounds carry over to the weak solutions. Let us also state that this smoothness of solutions is in line with the regularity results in [10].

**Lemma 3.1 (Smooth solutions for smooth datum).** Let Asm. 2 hold. In addition,

\[ q_0 \in C^\infty(\mathbb{R}), \ v \in C^\infty(\mathbb{R}), \ V \in C^\infty(\mathbb{R}). \]

Then, there exists \( T^* \in \mathbb{R}_{>0} \) so that the weak solution

\[ q \in C([0, T^*]; L^1_{\text{loc}}(\mathbb{R})) \cap L^\infty((0, T^*); L^\infty(\mathbb{R})) \]

in Defn. 6 of the (now continuous) nonlocal conservation law in Defn. 2 is a classical solution and

\[ q \in C^\infty(\Omega_{T^*}). \]
Proof From Thm. 3.1 we know that there exists a solution on an assured time horizon $[0, T^*)$ with a sufficiently small $T^* \in \mathbb{R}_>$. Due to the regularity of the involved functions, we can take advantage of the fixed-point equation in Eq. (79) as follows. As $q_0$ is smooth, the convolution means that the solution of the fixed-point problem is smooth provided the characteristics $\xi_w$ do not destroy regularity. However, $\xi_w$ as in Eq. (78) is – for given $w$ a smooth solution to the fixed-point problem – the solution of an IVP with a smooth right hand side and is thus smooth. This explains why the nonlocal term $w$ is smooth, the characteristics are smooth in each component and finally, looking at the solution formula in Eq. (87), the solution $q$ is also smooth. This solution therefore satisfies the PDE point-wise and is a classical solution. □

Remark 6 (Regularity of solutions). It is possible – similar to the results in [10, Section 5] – to obtain regularity results in $W^{k,p}$ for properly chosen initial datum and velocities and $(k,p) \in \mathbb{N}_0 \times (\mathbb{R}_{\geq 1} \cup \{\infty\})$ instead of $C^\infty$ solutions as in Lem. 3.1. However, we do not go into details as we only require smooth solutions in the following analysis.

### 3.2 Maximum principles

First we will list some assumptions that are particularly interesting for traffic flow modelling. They are inspired by classical maximum principles as laid out in [10, 47]:

**Assumption 3.** In addition to Asm. 2, we assume

- $V' \leq 0$
- $\text{supp}(\gamma) \subset \mathbb{R}_{\geq 0}$
- $\gamma$ monotonically decreasing on $\mathbb{R}_>$
- $q_0 \in L^\infty(\mathbb{R}; \mathbb{R}_{\geq 0})$, i.e., nonnegative.

The assumption that $V$ is monotonically decreasing is very common in traffic flow (compare with the classical LWR model in traffic [103–105]) as it states that the velocity must decrease with higher density. The assumption that $q_0$ is nonnegative and essentially bounded is inspired by interpreting solutions as traffic densities on roads that have limited capacity.

Finally, the assumptions on the kernel $\gamma$ ensure that density further ahead does not impact the nonlocal term as much as density immediately ahead. Traffic density behind generally does not matter. However, new models are emerging that incorporate nudging (looking behind) (see for example [44]). General maximum principles cannot be expected for looking behind nonlocal terms.

**Theorem 3.3 (A maximum principle/uniform bounds).** Let Asm. 3 hold, and consider the following two cases for the weak solution (in the sense of Defn. 6) of the discontinuous nonlocal conservation law in Defn. 2.

**Monotonically increasing $v$:** The weak solution exists for each $T \in \mathbb{R}_{>0}$ and satisfies the classical maximum principle

$$0 \leq q(t,x) \leq \|q_0\|_{L^\infty(\mathbb{R})} \quad \forall (t,x) \in (0,T) \times \mathbb{R} \ a.e. \quad (92)$$
Initial datum $L^1$ integrable and $\gamma$ more regular: In detail, assuming $q_0 \in L^1(\mathbb{R}; \mathbb{R}_{\geq 0}) \cap L^\infty(\mathbb{R}; \mathbb{R}_{\geq 0}) \land \gamma \in W^{1,\infty}(\mathbb{R}_{>0}; \mathbb{R}_{\geq 0}) \cap L^1(\mathbb{R}_{>0}; \mathbb{R}_{\geq 0}) \land \gamma' \leq 0$, the weak solution exists for every $T \in \mathbb{R}_{>0}$ with the following bounds for any $\delta \in \mathbb{R}_{>0}$:

- if $\operatorname{ess-sup}_{s \in X(q_0, \gamma)} V'(s) < 0$ it holds $\forall (t, x) \in \Omega_T$ a.e.
  \[ 0 \leq q(t, x) \leq \max \left\{ \frac{\|q_0\|_{L^\infty(\mathbb{R})}}{v}, \frac{\|V\|_{L^\infty(\mathbb{R})}}{v} - \frac{\|V'\|_{L^\infty(X(q_0, \gamma) + (-\delta, \delta))} \|q_0\|_{L^1(\mathbb{R})}}{\operatorname{ess-sup}_{s \in X(q_0, \gamma) + (-\delta, \delta)} V'(s)} \right\} \]

- if $\operatorname{ess-sup}_{s \in X(q_0, \gamma) + (-\delta, \delta)} V'(s) = 0$ it holds $\forall (t, x) \in \Omega_T$ a.e.
  \[ 0 \leq q(t, x) \leq \frac{\|q_0\|_{L^\infty(\mathbb{R})}}{v} \exp \left( t \frac{\|V\|_{L^\infty(\mathbb{R})}}{v} \|V'\|_{L^\infty(X(q_0, \gamma) + (-\delta, \delta))} \gamma(0) \|q_0\|_{L^1(\mathbb{R})} \right), \]
  with
  \[ X(q_0, \gamma) = (0, \|q_0\|_{L^1(\mathbb{R})} \|\gamma\|_{L^1(\mathbb{R})}) \subset \mathbb{R}. \] (93)

Proof The nonnegativity of the solution immediately follows from the representation of the solution in Thm. 3.1, specifically in Eq. (87). Thus we only need to focus on the upper boundedness in all the presented cases. To this end, approximate $q_0, v, V$ by a smooth $q_{0, \varepsilon}, v_{\varepsilon}, V_{\varepsilon}$ according to Thm. 3.2. As the solutions for $\varepsilon \in \mathbb{R}_{\geq 0}$ are smooth (and thus, classical solutions), we can work on the classical form and have for $(t, x) \in \Omega_T$

\[
\partial_t q_{\varepsilon}(t, x) = -\partial_x \left( v_{\varepsilon}(x) V_{\varepsilon}(W[q_{\varepsilon}](t, x)) q_{\varepsilon}(t, x) \right) \\
= -v'_{\varepsilon}(x) V_{\varepsilon}(W[q_{\varepsilon}](t, x)) q_{\varepsilon}(t, x) - v_{\varepsilon}(x) V_{\varepsilon}(W[q_{\varepsilon}](t, x)) \partial_x q_{\varepsilon}(t, x) \\
- v_{\varepsilon}(x) V_{\varepsilon}(W[q_{\varepsilon}]) \partial_x W[q_{\varepsilon}](t, x) q_{\varepsilon}(t, x).
\]

For $x \in \mathbb{R}$ s.t. $q_{\varepsilon}(t, x)$ this is maximal (and thus $\partial_x q_{\varepsilon}(t, x) = 0$)

\[
\left( -v'_{\varepsilon}(x) V_{\varepsilon}(W[q_{\varepsilon}](t, x)) - v_{\varepsilon}(x) V'_{\varepsilon}(W[q_{\varepsilon}](t, x)) \partial_x W[q_{\varepsilon}](t, x) \right) q_{\varepsilon}(t, x).
\]

Consider now the first case, i.e., assume $v' \geq 0$ and by construction $v'_{\varepsilon} \geq 0$, as well as $V_{\varepsilon} \geq 0$. Then, we obtain with the previous computation at the $x \in \mathbb{R}$ where $q_{\varepsilon}(t, x)$ is maximal

\[
\partial_t q_{\varepsilon}(t, x) \leq -v_{\varepsilon}(x) V'(W[q_{\varepsilon}](t, x)) \partial_x W[q_{\varepsilon}](t, x).
\]

However, thanks to $V'_{\varepsilon} \leq 0$ and $\partial_x W[q_{\varepsilon}](t, x) \leq 0$ for the $(t, x) \in \Omega_T$ where $q_{\varepsilon}(t, x)$ is maximal, the last term is nonpositive. This implies that the maxima can only decrease, i.e.,

\[
q_{\varepsilon}(t, x) \leq \|q_{0, \varepsilon}\|_{L^\infty(\mathbb{R})} \leq \|q_0\|_{L^\infty(\mathbb{R})} \quad \forall (t, x) \in \Omega_T.
\]

According to Thm. 3.2, we have

\[
\forall g \in C_c(\mathbb{R}) : \lim_{\varepsilon \to 0} \max_{t \in [0, T]} \left| \int_{\mathbb{R}} (q(t, x) - q_{\varepsilon}(t, x)) g(x) \, dx \right| = 0
\]
Taking any $x \in \mathbb{R}$ s.t. $\rho_{\varepsilon}(t, x) = \|\rho_{\varepsilon}(t, \cdot)\|_{L^\infty(\mathbb{R})}$, we thus have $\partial_x \rho_{\varepsilon}(t, x) = 0$ and by the previous computations

$$\partial_t \rho_{\varepsilon}(t, x) = -v_{\varepsilon}(x) V'_{\varepsilon}(\mathcal{W}[q_{\varepsilon}](t, x)) \partial_x \mathcal{W}[q_{\varepsilon}](t, x) \rho_{\varepsilon}(t, x),$$

Taking any $x \in \mathbb{R}$ s.t. $\rho_{\varepsilon}(t, x) = \|\rho_{\varepsilon}(t, \cdot)\|_{L^\infty(\mathbb{R})}$, we thus have $\partial_x \rho_{\varepsilon}(t, x) = 0$ and by the previous computations

$$\partial_t \rho_{\varepsilon}(t, x) = -v_{\varepsilon}(x) V'_{\varepsilon}(\mathcal{W}[q_{\varepsilon}](t, x)) \partial_x \mathcal{W}[q_{\varepsilon}](t, x) \rho_{\varepsilon}(t, x).$$

As we have by assumption $\gamma \in W^{1,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$ with $\gamma' \leq 0$, this yields for $x \in \mathbb{R}$ where $\rho_{\varepsilon}(t, x) = \|\rho_{\varepsilon}(t, \cdot)\|_{L^\infty(\mathbb{R})}$,

$$\partial_t \rho_{\varepsilon}(t, x) = v_{\varepsilon}(x) V'_{\varepsilon}(\mathcal{W}[q_{\varepsilon}](t, x)) (\gamma(0) q_{\varepsilon}(t, x)) + \int_{\mathbb{R}_{\geq 0}} \gamma'(y-x) q_{\varepsilon}(t, y) \, dy \rho_{\varepsilon}(t, x) \leq \text{ess-inf} \frac{\|v\|_{L^\infty(\mathbb{R})} V'_{\varepsilon}(s) \gamma(0) \rho_{\varepsilon}^2(t, x)}{s \in (-\varepsilon, 0)} + \|v\|_{L^\infty(\mathbb{R})} V'_{\varepsilon}(s) \gamma(0) \rho_{\varepsilon}(t, x).$$

The latter estimate holds due to $|\gamma|_{TV(\mathbb{R}_{\geq 0})} = \gamma(0)$, as $\gamma' \leq 0$ and $\gamma \in BV(\mathbb{R}_{\geq 0})$. We have also used Young’s convolution inequality [106, Theorem 4.15 (Young)] several times to estimate $\mathcal{W}$ and $\partial_x \mathcal{W}$. In detail, it holds that

$$0 \leq \int_{\mathbb{R}_{\geq 0}} \gamma(x-y) q_{\varepsilon}(t, y) \, dy \leq \|\gamma\|_{L^1(\mathbb{R})} \|q_{\varepsilon}\|_{L^1(\mathbb{R})} = \|\gamma\|_{L^1(\mathbb{R})} \|q_{0, \varepsilon}\|_{L^1(\mathbb{R})} \leq \|\gamma\|_{L^1(\mathbb{R})} \|q_0\|_{L^1(\mathbb{R})};$$

by construction of $q_{0, \varepsilon}$, the conservation of mass of $q$, and the fact that $\gamma$ is monotonically decreasing on $\mathbb{R}_{\geq 0}$. Altogether, we obtain for $(t, x) \in \Omega_T$ a.e.

$$\rho_{\varepsilon}(t, x) \leq \max \left\{ \|v\|_{L^\infty(\mathbb{R})}, \frac{\|v\|_{L^\infty(\mathbb{R})} V'_{\varepsilon}(\mathcal{W}[q_{0, \varepsilon}]) \gamma(0) \rho_{\varepsilon}(t, x)}{\text{ess-inf} \frac{\|v\|_{L^\infty(\mathbb{R})} V'_{\varepsilon}(s) \gamma(0) \rho_{\varepsilon}(t, x)}{s \in (-\varepsilon, 0)}} \right\}. $$

With the identical argument as before, once more using Thm. 3.2, we obtain the stated bounds for $\varepsilon \to 0$ when later recalling that $\rho \equiv v \cdot q$ and $v \equiv q$.

The third case follows immediately when reconnecting to Eq. (94) and noticing that

$$\partial_t \rho_{\varepsilon}(t, x) \leq \|v\|_{L^\infty(\mathbb{R})} V'_{\varepsilon}(s \in (-\varepsilon, 0) \|\gamma\|_{L^1(\mathbb{R})} + \varepsilon) \gamma(0) \rho_{\varepsilon}(t, x),$$

which leads to at most exponential growth of $\|\rho_{\varepsilon}\|_{L^\infty(\mathbb{R})}$.

**Corollary 3.1** (Compatible initial datum). In contradiction to the concluding remarks in [54], we can state that for any given discontinuity $v$, $V' \in L^\infty(\mathbb{R})$ with $\text{ess-sup}_{x \in \mathbb{R}} V'(x) < 0$ and desired upper bound $C \in \mathbb{R}_{\geq 0}$, there exists $q_0 \neq 0$ such that the corresponding solution $q$ to the discontinuous nonlocal conservation law as defined in Defn. 2 satisfies

$$q \leq C \quad \text{on } \Omega_T$$
Discontinuous nonlocal conservation laws and related discontinuous ODEs

for each $T \in \mathbb{R}_{>0}$. This is still possible for the case $\text{ess-sup}_{x \in \mathbb{R}} V'(x) = 0$. However, the time horizon must be fixed.

We illustrate the discontinuous nonlocal conservation law by means of the following

**Example 2** (Some numerical illustrations). In order to visualize the effect of a discontinuity in space (demonstrated via $v$), we consider the following modelling archetypes:

$$q_0 := \frac{1}{2} \chi_{[-0.5,-0.1]}(x), \quad V(\cdot) \equiv 1 - \cdot, \quad \gamma \equiv 10 \chi_{[0,0.1]}, \quad v \in \{1, 1 + \chi_{\mathbb{R}_{>0}}, 1 - \frac{1}{2} \chi_{\mathbb{R}_{>0}}\},$$

which are illustrated in Fig. 2. As can be seen, for $v \equiv 1$ and $v \equiv 1 + \chi_{\mathbb{R}_{>0}}$, i.e. the monotonically increasing cases, the first statement of Fig. 2 applies, and indeed the proposed maximum principle holds. Moreover, a jump downwards is evident at position $x = 0$ in density $q$, when the velocity jumps from 1 to 2 in the second case. This is in line with intuition that an increased speed reduces the density accordingly, in equations roughly

$$\forall t \in [0,T]: \lim_{x \uparrow x_0} q(t,x)v(x)V(W[q](t,x)) = \lim_{x \downarrow x_0} q(t,x)v(x)V(W[q](t,x))$$

In the third case, the velocity is halved at $x = 0$ and the density is doubled. This is – for specific times $t \in \{0,0.5,1\}$ – also illustrated in the bottom row of Fig. 2.

In Fig. 3, the evolution of the maximum of the solution, in equations $\|q(t,\cdot)\|_{L^\infty(\mathbb{R})}$, is illustrated. This reflects our previous remarks. The total variation for the different cases is also shown and as can be seen, it changes significantly when the discontinuity comes into play. Clearly, an upper bound will depend on the total variation of $q_0$ as well as $v$.

4 Conclusions and open problems

In this contribution, we have studied nonlocal conservation laws in $C([0,T]; L^1_{\text{loc}}(\mathbb{R})) \cap L^\infty((0,T); L^\infty(\mathbb{R}))$ with general multiplicative discontinuities ($L^\infty$-type) in space. By employing the method of characteristics and a reformulation as a fixed-point problem, we could instead consider specific discontinuous ODEs, which we studied for existence, uniqueness and stability. The results obtained were then applied to the discontinuous nonlocal conservation law to prove existence and uniqueness of weak solutions on a small time horizon. These results were supplemented by several “maximum principles” guaranteeing the semi-global existence of solutions. We have thus generalized the existing theory on (purely) nonlocal conservation laws to include discontinuities in space, and have proven that Entropy conditions are – once more
Fig. 2 Evolution of the solution $q$ in space-time with discontinuities $v \equiv 1$ (top left), $v \equiv 1 + \chi_{R > 0}$ (top middle) and $v \equiv 1 - \frac{1}{2}\chi_{R > 0}$ (top right). The solutions at time $t = 0$ (solid), $t = 0.5$ (dashed) and $t = 1$ (dash-dotted) are shown in the lower row.

Fig. 3 The evolution of the maximum (solid) of the solution as well as its total variation (dash-dotted) are visualized. Blue represents the case with $v \equiv v_0 \equiv 1$, orange the case with $v \equiv v_1 \equiv 1 + \chi_{R > 0}$ and, finally, green the case with $v \equiv v_2 \equiv 1 - \frac{1}{2}\chi_{R > 0}$.

– obsolete (compare with [10]) although still used in literature [54, 55]. The established theory sets the stage for several future directions: 1) similar to [40], consideration of the convergence to the local discontinuous conservation law when we let the convolution kernel in the nonlocal part of the velocity converge to a Dirac distribution, 2) the bounded domain case similar to [1], 3) measure-valued solutions similar to [46], assuming that the kernel is in $W^{1,\infty}(\mathbb{R})$, 4) discontinuous (in space) multi-dimensional nonlocal conservation laws.
Discontinuous nonlocal conservation laws and related discontinuous ODEs

Acknowledgement

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 416229255 – SFB 1411.

We would like to thank Deborah Bennett for proofreading this manuscript carefully.

References

[1] Keimer, A., Pflug, L., Spinola, M.: Nonlocal scalar conservation laws on bounded domains and applications in traffic flow. SIAM SIMA 50(6), 6271–6306 (2018)

[2] Coron, J.-M., Wang, Z.: Controllability for a scalar conservation law with nonlocal velocity. Journal of Differential Equations 252(1), 181–201 (2012)

[3] Ackleh, A.S., Deng, K.: Monotone Method for First Order Nonlocal Hyperbolic Initial-boundary Value Problems. Applicable Analysis 67(3), 283–293 (1997). https://doi.org/10.1080/00036819708840612

[4] Gröschel, M., Keimer, A., Leugering, G., Wang, Z.: Regularity Theory and Adjoint Based Optimality Conditions for a Nonlinear Transport Equation with Nonlocal Velocity. SIAM Journal on Control and Optimization 52(4), 2141–2163 (2014)

[5] Gugat, M., Keimer, A., Leugering, G., Wang, Z.: Analysis of a system of nonlocal conservation laws for multi-commodity flow on networks. Networks & Het. Media 10(4), 749–785 (2015). https://doi.org/10.3934/nhm.2015.10.749

[6] Amorim, P.: On a nonlocal hyperbolic conservation law arising from a gradient constraint problem. Bulletin of the Brazilian Mathematical Society, New Series 43(4), 599–614 (2012)

[7] Colombo, R.M., Lécureux-Mercier, M.: Nonlocal crowd dynamics models for several populations. Acta Mathematica Scientia 32(1), 177–196 (2012)

[8] Betancourt, F., Bürger, R., Karlsen, K.H., Tory, E.M.: On nonlocal conservation laws modelling sedimentation. Nonlinearity 24(3), 855 (2011)

[9] Zumbrun, K.: On a nonlocal dispersive equation modeling particle suspensions. Quarterly of Applied Mathematics, 573–600 (1999)

[10] Keimer, A., Pflug, L.: Existence, uniqueness and regularity results on
nonlocal balance laws. Journal of Differential Equations \textbf{263}, 4023–4069 (2017)

[11] Aggarwal, A., Colombo, R.M., Goatin, P.: Nonlocal systems of conservation laws in several space dimensions. SIAM Journal on Numerical Analysis \textbf{53}(2), 963–983 (2015)

[12] Keimer, A., Pflug, L., Spinola, M.: Nonlocal balance laws: Theory of convergence for nondissipative numerical schemes. submitted (2018)

[13] Keimer, A., Pflug, L., Spinola, M.: Existence, uniqueness and regularity of multi-dimensional nonlocal balance laws with damping. Journal of Mathematical Analysis and Applications \textbf{466}(1), 18–55 (2018). \url{https://doi.org/10.1016/j.jmaa.2018.05.013}

[14] Colombo, M., Crippa, G., Spinolo, L.V.: On the singular local limit for conservation laws with nonlocal fluxes. Archive for Rational Mechanics and Analysis volume \textbf{233}, 1131–1167 (2019)

[15] Ngoduy, D., Wilson, R.E.: Multianticipative nonlocal macroscopic traffic model. Computer-Aided Civil and Infrastructure Engineering \textbf{29}(4), 248–263 (2014)

[16] Amorim, P., Colombo, R.M., Teixeira, A.: On the numerical integration of scalar nonlocal conservation laws. ESAIM: Math. Modelling and Numerical Analysis \textbf{49}(1), 19–37 (2015)

[17] Colombo, R.M., Garavello, M., Lécureux-Mercier, M.: A class of nonlocal models for pedestrian traffic. Mathematical Models and Methods in Applied Sciences \textbf{22}(04), 1150023 (2012)

[18] Keimer, A., Pflug, L., Spinola, M.: Nonlocal scalar conservation laws on bounded domains and applications in traffic flow. SIAM SIMA \textbf{50}(6), 6271–6306 (2018)

[19] Keimer, A., Pflug, L.: On approximation of local conservation laws by nonlocal conservation laws. Journal of Mathematical Analysis and Applications \textbf{475}(2), 1927–1955 (2019)

[20] Coron, J.-M., Keimer, A., Pflug, L.: Nonlocal transport equations – existence and uniqueness of solutions and relation to the corresponding conservation laws. SIAM SIMA \textbf{52}(6), 5500–5532 (2020)

[21] Colombo, M., Crippa, G., Spinolo, L.V.: Blow-up of the total variation in the local limit of a nonlocal traffic model. arXiv preprint arXiv:1808.03529 (2018)
[22] Keimer, A., Leugering, G., Sarkar, T.: Analysis of a system of nonlocal balance laws with weighted work in progress. Journal of Hyperbolic Diff. Equations 15(03), 375–406 (2018)

[23] Keimer, A., Singh, M., Veeravalli, T.: Existence and uniqueness results for a class of nonlocal conservation laws by means of a lax–hopf-type solution formula. Journal of Hyperbolic Differential Equations 17(04), 677–705 (2020)

[24] Li, D., Li, T.: Shock formation in a traffic flow model with arrhenius look-ahead dynamics. Networks & Heterogeneous Media 6(4), 681–694 (2011)

[25] Colombo, R.M., Guerra, G.: Hyperbolic balance laws with a non local source. Communications in Partial Differential Equations 32(12), 1917–1939 (2007)

[26] Coron, J., Wang, Z.: Output feedback stabilization for a scalar conservation law with a nonlocal velocity. SIAM Journal on Mathematical Analysis 45(5), 2646–2665 (2013)

[27] Chen, W., Liu, C., Wang, Z.: Global feedback stabilization for a class of nonlocal transport equations: The continuous and discrete case. SIAM Journal on Control and Optimization 55(2), 760–784 (2017)

[28] Baker, G.R., Li, X., Morlet, A.C.: Analytic structure of two 1d-transport equations with nonlocal fluxes. Physica D: Nonlinear Phenomena 91(4), 349–375 (1996). https://doi.org/10.1016/0167-2789(95)00271-5

[29] Piccoli, B., Rossi, F.: Transport equation with nonlocal velocity in Wasserstein spaces: Convergence of numerical schemes. Acta Applicandae Mathematicae 124(1), 73–105 (2013)

[30] Francesco, M.D., Fagioli, S., Radici, E.: Deterministic particle approximation for nonlocal transport equations with nonlinear mobility. Journal of Differential Equations 266(5), 2830–2868 (2019)

[31] Bressan, A., Shen, W.: On traffic flow with nonlocal flux: a relaxation representation. Archive for Rational Mechanics and Analysis volume 237 (2020)

[32] Colombo, M., Crippa, G., Marconi, E., Spinolo, L.V.: Local limit of nonlocal traffic models: convergence results and total variation blow-up (2018)

[33] Colombo, R.M., Marcellini, F.: Nonlocal systems of balance laws in several space dimensions with applications to laser technology. Journal of
Discontinuous nonlocal conservation laws and related discontinuous ODEs

Differential Equations 259(11), 6749–6773 (2015)

[34] Colombo, R.M., Rossi, E.: Nonlocal conservation laws in bounded domains. SIAM Journal on Mathematical Analysis 50(4), 4041–4065 (2018). https://doi.org/10.1137/18M1171783

[35] Chalons, C., Goatin, P., Villada, L.M.: High-order numerical schemes for one-dimensional nonlocal conservation laws. SIAM Journal on Scientific Computing 40(1), 288–305 (2018). https://doi.org/10.1137/16M110825X

[36] Friedrich, J., Kolb, O.: Maximum principle satisfying CWENO schemes for nonlocal conservation laws. SIAM Journal on Scientific Computing 41(2), 973–988 (2019)

[37] Lee, Y.: Thresholds for shock formation in traffic flow models with nonlocal-concave-convex flux. Journal of Differential Equations 266(1), 580–599 (2019)

[38] Ridder, J., Shen, W.: Traveling waves for nonlocal models of traffic flow. Discrete & Continuous Dynamical Systems - A 39, 4001 (2019)

[39] Bayen, A., Keimer, A., Pflug, L., Veeravalli, T.: Modeling multi-lane traffic with moving obstacles by nonlocal balance laws. submitted (2021)

[40] Coclite, G.M., Coron, J.-M., De Nitti, N., Keimer, A., Pflug, L.: A general result on the approximation of local conservation laws by nonlocal conservation laws: The singular limit problem for exponential kernels. arXiv preprint arXiv:2012.13203 (2020)

[41] Coclite, G.M., Nitti, N.D., Keimer, A., Pflug, L.: On existence and uniqueness of weak solutions to nonlocal conservation laws with BV kernels. submitted (2021)

[42] Bressan, A., Shen, W.: Entropy admissibility of the limit solution for a nonlocal model of traffic flow. Communications in Mathematical Sciences 19(5), 1447–1450 (2021)

[43] Colombo, M., Crippa, G., Marconi, E., Spinolo, L.V.: Local limit of nonlocal traffic models: Convergence results and total variation blow-up. Annales de l’Institut Henri Poincaré C, Analyse non linéaire 38(5), 1653–1666 (2021)

[44] Karafyllis, I., Theodosis, D., Papageorgiou, M.: Analysis and control of a non-local pde traffic flow model. International Journal of Control 0(0), 1–19 (2020)
Discontinuous nonlocal conservation laws and related discontinuous ODEs

[45] De Filippis, C., Goatin, P.: The initial–boundary value problem for general non-local scalar conservation laws in one space dimension. Nonlinear Analysis 161, 131–156 (2017)

[46] Crippa, G., Lécureux-Mercier, M.: Existence and uniqueness of measure solutions for a system of continuity equations with non-local flow. Nonlinear Differential Equations and Applications NoDEA 20(3), 523–537 (2013)

[47] Goatin, P., Scialanga, S.: Well-posedness and finite volume approximations of the LWR traffic flow model with non-local velocity. Networks and Heterogeneous Media 11(1), 107–121 (2016)

[48] Chiarello, F.A., Goatin, P.: Global entropy weak solutions for general non-local traffic flow models with anisotropic kernel. ESAIM: Math. Modelling and Numerical Analysis 52(1), 163–180 (2018)

[49] Blandin, S., Goatin, P.: Well-posedness of a conservation law with non-local flux arising in traffic flow modeling. Numerische Mathematik 132(2), 217–241 (2016)

[50] Chiarello, F.A., Goatin, P.: Non-local multi-class traffic flow models. Networks & Heterogeneous Media 14, 371 (2019)

[51] Goatin, P., Rossi, E.-: Well-posedness of IBVP for 1D scalar non-local conservation laws. ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik 99(11) (2019)

[52] Gong, X., Kawski, M.: Weak measure-valued solutions of a nonlinear hyperbolic conservation law. SIAM Journal on Mathematical Analysis 53(4), 4417–4444 (2021)

[53] Kloeden, P.E., Lorenz, T.: Nonlocal multi-scale traffic flow models: analysis beyond vector spaces. Bulletin of Mathematical Sciences 6(3), 453–514 (2016)

[54] Chiarello, F.A., Villada, L.M.: On existence of entropy solutions for 1d nonlocal conservation laws with space discontinuous flux (2021) https://arxiv.org/abs/2103.13362

[55] Chiarello, F.A., Coclite, G.M.: Non-local scalar conservation laws with discontinuous flux (2021) https://arxiv.org/abs/2003.01975

[56] Lighthill, M.J., Whitham, G.B.: On kinematic waves ii. a theory of traffic flow on long crowded roads. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences 229(1178), 317–345
Discontinuous nonlocal conservation laws and related discontinuous ODEs

(1955)

[57] DiPerna, R.J., Lions, P.-L.: Ordinary differential equations, transport theory and sobolev spaces. Inventiones mathematicae 98(3), 511–547 (1989)

[58] Coron, J.-M., Kawski, M., Wang, Z.: Analysis of a conservation law modeling a highly re-entrant manufacturing system. Disc. Contin. Dyn. Syst. Ser. B 14(4), 1337–1359 (2010). https://doi.org/10.3934/dcdsb.2010.14.1337

[59] Filippov, A.F.: Differential equations with discontinuous right-hand side. Matematicheskii sbornik 93(1), 99–128 (1960)

[60] Filippov, A.F.: Differential Equations with Discontinuous Right-hand Sides. Springer, Dordrecht (1988). https://doi.org/10.1007/978-94-015-7793-9

[61] Bressan, A., Shen, W.: Uniqueness for discontinuous ode and conservation laws. Nonlinear analysis 34(5), 637–652 (1998)

[62] Bownds, J.M.: A uniqueness theorem for $y' = f(x,y)$ using a certain factorization of $f$. Journal of Differential Equations 7(2), 227–231 (1970)

[63] Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations. International series in pure and applied mathematics. Tata McGraw-Hill Education, New York (1955)

[64] Fjordholm, U.S.: Sharp uniqueness conditions for one-dimensional, autonomous ordinary differential equations. Comptes Rendus Mathematique 356(9), 916–921 (2018)

[65] Osgood, W.F.: Beweis der Existenz einer Lösung der Differentialgleichung $\frac{dy}{dx} = f(x,y)$ ohne Hinzunahme der Cauchy-Lipschitz’schen Bedingung. Monatshefte für Mathematik und Physik 9(1), 331–345 (1898)

[66] Hu, S.: Differential equations with discontinuous right-hand sides. Journal of Mathematical Analysis and Applications 154(2), 377–390 (1991)

[67] Binding, P.: The differential equation $\dot{x} = f(x)$. Journal of Differential Equations 31(2), 183–199 (1979)

[68] Agarwal, R.P., Lakshmikantham, V.: Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations vol. 6. World Scientific, Singapore (1993)

[69] Garavello, M., Natalini, R., Piccoli, B., Terracina, A.: Conservation laws
with discontinuous flux. Networks & Heterogeneous Media 2(1), 159 (2007)

[70] Adimurthi, Mishra, S., Gowda, G.D.V.: Optimal entropy solutions for conservation laws with discontinuous flux-functions. Journal of Hyperbolic Differential Equations 02(04), 783–837 (2005). https://doi.org/10.1142/S0219891605000622

[71] Bürger, R., Karlsen, K.H., Towers, J.D.: An engquist–osher-type scheme for conservation laws with discontinuous flux adapted to flux connections. SIAM Journal on Numerical Analysis 47(3), 1684–1712 (2009). https://doi.org/10.1137/07069314X

[72] Klausen, R.A., Risebro, N.H.: Stability of conservation laws with discontinuous coefficients. Journal of Differential Equations 157(1), 41–60 (1999). https://doi.org/10.1006/jdeq.1998.3624

[73] Klingenberg, C., Risebro, N.H.: Convex conservation laws with discontinuous coefficients. existence, uniqueness and asymptotic behavior. Communications in Partial Differential Equations 20(11-12), 1959–1990 (1995)

[74] Gimse, T.: Conservation laws with discontinuous flux functions. SIAM Journal on Mathematical Analysis 24(2), 279–289 (1993). https://doi.org/10.1137/0524018

[75] Audusse, E., Perthame, B.: Uniqueness for scalar conservation laws with discontinuous flux via adapted entropies. Proceedings of the Royal Society of Edinburgh: Section A Mathematics 135(2), 253–265 (2005)

[76] Bürger, R., Karlsen, K.H.: Conservation laws with discontinuous flux: a short introduction. Journal of Engineering Mathematics (2008)

[77] Ostrov, D.N.: Solutions of Hamilton–Jacobi equations and scalar conservation laws with discontinuous space–time dependence. Journal of Differential Equations 182(1), 51–77 (2002)

[78] Andreianov, B., Karlsen, K.H., Risebro, N.H.: On vanishing viscosity approximation of conservation laws with discontinuous flux. Networks and Heterogeneous Media 5(3), 617 (2010)

[79] Adimurthi, Jaffré, J., Gowda, G.V.: Godunov-type methods for conservation laws with a flux function discontinuous in space. SIAM Journal on Numerical Analysis 42(1), 179–208 (2004)

[80] Karlsen, K., Klingenberg, C., Risebro, N.: A relaxation scheme for conservation laws with a discontinuous coefficient. Mathematics of
Discontinuous nonlocal conservation laws and related discontinuous ODEs

[81] Karlsen, K.H., Towers, J.D.: Convergence of the lax-friedrichs scheme and stability for conservation laws with a discontinuous space-time dependent flux. Chinese Annals of Mathematics 25(03), 287–318 (2004)

[82] Towers, J.D.: Convergence of a difference scheme for conservation laws with a discontinuous flux. SIAM journal on numerical analysis 38(2), 681–698 (2000)

[83] Towers, J.D.: A difference scheme for conservation laws with a discontinuous flux: the nonconvex case. SIAM journal on numerical analysis 39(4), 1197–1218 (2001)

[84] Adimurthi, Ghoshal, S.S., Dutta, R., Veerappa Gowda, G.D.: Existence and nonexistence of tv bounds for scalar conservation laws with discontinuous flux. Communications on Pure and Applied Mathematics 64(1), 84–115 (2011)

[85] Adimurthi, Mishra, S., Veerappa Gowda, G.D.: Explicit Hopf–Lax type formulas for Hamilton–Jacobi equations and conservation laws with discontinuous coefficients. Journal of Differential Equations 241(1), 1–31 (2007). https://doi.org/10.1016/j.jde.2007.05.039

[86] Mishra, S., Veerappa Gowda, G., et al.: Existence and stability of entropy solutions for a conservation law with discontinuous non-convex fluxes. Networks and Heterogeneous Media 2(1), 127–157 (2007)

[87] Andreianov, B., Karlsen, K.H., Risebro, N.H.: A theory of $L^1$-dissipative solvers for scalar conservation laws with discontinuous flux. Archive for rational mechanics and analysis 201(1), 27–86 (2011)

[88] Crippa, G.: Lagrangian flows and the one-dimensional Peano phenomenon for ODEs. Journal of Differential Equations 250(7), 3135–3149 (2011)

[89] Besson, O., Pousin, J.: Solutions for linear conservation laws with velocity fields in $L^\infty$. Archive for Rational Mechanics and Analysis 186(1), 159–175 (2007)

[90] Ambrosio, L.: Transport equation and cauchy problem for $bv$ vector fields. Inventiones mathematicae 158(2), 227–260 (2004)

[91] Ambrosio, L., Crippa, G.: Continuity equations and ode flows with non-smooth velocity. Proceedings of the Royal Society of Edinburgh Section A: Mathematics 144(6), 1191–1244 (2014)
[92] Petrova, G., Popov, B.: Linear transport equations with discontinuous coefficients. Communications in partial differential equations 24(9-10), 1849–1873 (1999)

[93] Bouchut, F., James, F.: One-dimensional transport equations with discontinuous coefficients. Nonlinear Analysis 32(7), 891 (1998)

[94] Clop, A., Jylhä, H., Mateu, J., Orobitg, J.: Well-posedness for the continuity equation for vector fields with suitable modulus of continuity. Journal of Functional Analysis 276(1), 45–77 (2019)

[95] Coddington, E.A., Levinson, N.: Theory of Ordinary Differential Equations. Tata McGraw-Hill Education, New York (1955)

[96] Walter, W.: Differential and Integral Inequalities. Springer, ??? (1970)

[97] Folland, G.B.: Real Analysis, 2nd edn. Pure and Applied Mathematics (New York), p. 386. John Wiley & Sons Inc., New York (1999). Modern techniques and their applications, A Wiley-Interscience Publication

[98] Runge, C.: Über die numerische Auflösung von Differentialgleichungen. Mathematische Annalen 46(2), 167–178 (1895). https://doi.org/10.1007/bf01446807

[99] Kutta, W.: Beitrag zur näherungsweisen Integration totaler Differentialgleichungen. Z. Math. Phys. 46, 435–453 (1901)

[100] Leoni, G.: A First Course in Sobolev Spaces. Graduate Studies in Mathematics, vol. 105, p. 607. American Mathematical Society, Providence, RI (2009)

[101] Shang, P., Wang, Z.: Analysis and control of a scalar conservation law modeling a highly re-entrant manufacturing system. Journal of Differential Equations 250(2), 949–982 (2011). https://doi.org/10.1016/j.jde.2010.09.003

[102] Zeidler, E.: Applied Functional Analysis: Applications to Mathematical Physics. Applied Mathematical Sciences. Springer, New York (1995)

[103] Lighthill, M.J., Whitham, G.B.: On kinematic waves. i. flood movement in long rivers. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences 229(1178), 281–316 (1955)

[104] Richards, P.I.: Shock waves on the highway. Operations research 4(1), 42–51 (1956)

[105] Greenshields, B., Channing, W., Miller, H., et al.: A study of traffic capacity. In: Highway Research Board Proceedings, vol. 1935 (1935).
Discontinuous nonlocal conservation laws and related discontinuous ODEs

National Research Council (USA), Highway Research Board

[106] Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext, p. 599. Springer, New York (2011)