A new proof of the Larman–Rogers upper bound for the chromatic number of the Euclidean space

Roman Prosanov

Université de Fribourg, Chemin du Musée 23, CH-1700 Fribourg, Switzerland
Moscow Institute of Physics and Technology, Institutsky per. 9, 141700, Dolgoprudny, Russia
Technische Universität Wien, Wiedner Hauptstraße 8-10, 1040 Wien, Austria

Keywords:
Chromatic number
Geometric covering

1. Introduction

The chromatic number $\chi(\mathbb{R}^n)$ of the Euclidean space $\mathbb{R}^n$ is the smallest number of colors sufficient for coloring all points of the space in such a way that any two points at the distance 1 have different colors. This problem was initially posed by Nelson for $n = 2$ (see the history of this problem in [2,19,21,22,27]).

The exact value of $\chi(\mathbb{R}^n)$ is not known even in the planar case. The best known bounds are

$$5 \leq \chi(\mathbb{R}^2) \leq 7.$$  

See [27] for the upper bound and [6] for the lower one. Bounding the chromatic numbers of Euclidean spaces of small dimension attracts constant attention of researchers. Some recent results were obtained, e.g., in [1], [4] and [5]. It is interesting that if one considers the product of the Euclidean plane with an arbitrarily small square, then the lower bound can be improved. This is investigated in [10]. In the case of growing $n$ we have

$$(1.239 + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n.$$  

The lower bound is due to Raigorodskii [18] and the upper bound is due to Larman and Rogers [12].

The proof of Larman and Rogers is based on a hard theorem due to Butler [3] and on a result of Erdős and Rogers about coverings of $\mathbb{R}^n$ with translates of a convex body. In this paper we present a new proof that does not use neither of them. Instead we adapt the approach developed by Marton Naszódi in [14]. It connects geometrical covering problems with coverings of finite hypergraphs. An advantage of this approach in contrast to the previous one is that it could be turned into an algorithmic one. Indeed, the original paper of Erdős and Rogers used probabilistic arguments. Therefore, the same is true for the Larman and Rogers proof. We will rely only on a theorem by Johnson, Lovász and Stein that establishes a connection between the fractional covering number of a hypergraph and its integral covering number. The proofs of this

E-mail address: rprosanov@mail.ru.

1 The author is supported in part by the grant from Russian Federation President Program of Support for Leading Scientific Schools 6760.2018.1.

Published in "Discrete Applied Mathematics 276(): 115–120, 2020" which should be cited to refer to this work.
Theorem 1. We have
\[ \chi(\mathbb{R}^n) \leq (1 + \gamma(K, k))^n \left[ n \ln n + n \ln \ln n + 2 \ln k + 2n \left( 1 + \ln(2 \gamma(K, k)) \right) \right]. \]

In particular, if \( K_n \) is a sequence of bodies and \( k_n \) is a sequence of positive numbers such that for some absolute constant \( c \) we have \( k_n \leq n^c \), then
\[ \chi(\mathbb{R}^n) \leq (1 + \gamma(K_n, k_n) + o(1))^n. \]
2.2. Preliminaries with fractional coverings

Let \( Z \) be a set, \( \mathcal{F} \) be a family of its subsets, and \( Y \subseteq Z \). By the covering number \( \tau(Y, \mathcal{F}) \) denote the minimal cardinality of a family \( \mathcal{H} \subseteq \mathcal{F} \) such that \( Y \) is covered by the union of all sets \( F \in \mathcal{H} \).

If \( Z \) is finite, then the pair \( (Z, \mathcal{F}) \) is a finite hypergraph. In this case a fractional covering of \( Y \) by \( \mathcal{F} \) is a function \( v : \mathcal{F} \to [0; +\infty) \) such that for all \( y \in Y \) we have
\[
\sum_{F \in \mathcal{F} : y \in F} v(F) \geq 1.
\]

Define the fractional covering number of \( Y \):
\[
\tau^*(Y, \mathcal{F}) = \inf \{ \sum_{F \in \mathcal{F}} v(F) : v \text{ is a fractional covering of } Y \text{ by } \mathcal{F} \}.
\]

The following theorem establishes a connection between \( \tau \) and \( \tau^* \):

**Theorem 2** ([9,13,28]). Suppose \( Z \) is a finite set and \( \mathcal{F} \subseteq 2^Z \), then
\[
\tau(Z, \mathcal{F}) < \left( 1 + \ln \left( \max_{F \in \mathcal{F}} |F| \right) \right) \tau^*(Z, \mathcal{F}).
\]

2.3. Proof of Theorem 1

In what follows, all distances are calculated with respect to the norm, determined by \( K \). For \( 0 < \mu < 1 \) define
\[
\mu \Psi = \bigcup_{\psi \in \Psi} (\mu(x - \psi_x) + \psi_x).
\]

Fix \( \varepsilon > 0 \). Choose a pair \( (\Phi, \Psi) \) such that
\[
\gamma(K, \Phi, \Psi) < \gamma(K, k) + \varepsilon.
\]

Let \( \alpha, \beta \) be numbers such that
\[
\gamma = \beta/\alpha < \gamma(K, \Phi, \Psi) + \varepsilon
\]
and for all \( x \in \Phi \) we have
\[
\alpha K + x \subset \text{int}(\psi_x),
\]
\[
\psi_x \subset \text{int}(\beta K + x).
\]

Since \( \psi_x \) is contained in \( \text{int}(\beta K + x) \), we see that the diameter of \( \psi_x \) is strictly less than \( 2\beta \). Let
\[
\mu = \alpha/(\alpha + \beta).
\]

Then for all \( x \) the polytope \( \psi_x \) does not contain a pair of points at distance \( 2\beta \mu \).

We show that for all \( x, y \in \Phi, x \neq y \), the distance between \( \mu \psi_x \) and \( \mu \psi_y \) is greater than \( 2\beta \mu \). It is sufficient to consider only such polytopes that share a common face in some dimension. Since \( \psi_x \) and \( \psi_y \) are convex and share some \( k \)-dimensional face, there is a hyperplane containing this face and separating \( \psi_x \) and \( \psi_y \). Let \( l_x \) and \( l_y \) be the distances from \( x \) and \( y \) to this hyperplane. The distance between \( \mu \psi_x \) and \( \mu \psi_y \) is greater than the distance between the images of this hyperplane under homothety with center \( x \) and homothety with center \( y \). Since \( \alpha K + x \subset \text{int}(\psi_x) \), this distance is
\[
(l_x + l_y)(1 - \mu) > 2\alpha(1 - \mu) = 2\beta \mu.
\]

Therefore, the set \( \mu \Psi \) does not contain a pair of points at the distance \( 2\beta \mu \) and we can color it with one color.

Next, we cover \( \mathbb{R}^n \) by the copies of \( \mu \Psi \). This set is a disjoint union of several convex bodies. Hence, typical covering results [like in [7]] cannot be applied to it. Now we show how to overcome this difficulty.

Let \( \Omega \) be the base lattice of \( \Phi \). Consider the torus \( T^n = \mathbb{R}^n/\Omega \). Let \( \check{x}_i \) be the projections onto \( T^n \) of the translation vectors \( x_i \) of the lattice \( \Omega \) in the multilattice \( \Phi \) and \( \hat{X} \) be their union. The tiling \( \Psi \) is periodical over the lattice \( \Omega \), hence we can define its projection \( \check{\Psi} \), which is a tiling of \( T^n \) associated to the set \( \hat{X} \). For \( \check{x}_i \in \hat{X} \) and \( \delta : 0 < \delta \leq 1 \) we denote by \( \delta \check{\psi}_i \) the projection of \( \delta (\check{x}_i - x_i) + x_i \) to \( T^n \).

We will cover \( T^n \) by less than
\[
(1 + \gamma)^n (n \ln n + n \ln \ln n + 2 \ln k + 2n(1 + \ln 2 \gamma))
\]
translates of \( \mu \check{\Psi} \).

We need the following lemma.
Lemma 1. Fix \(0 < \delta < 1\). Let \(F\) and \(F'\) be the families of translates of the sets \(\mu\hat{Y}\) and \(\mu(1 - \delta)\hat{Y}\) by all points of \(T^n\). Suppose \(\Lambda \subset T^n\) is a finite point set of maximal cardinality such that \(\frac{\alpha\mu\delta}{2}K + \Lambda\) is a packing of the bodies \(\frac{\alpha\mu\delta}{2}K\). Then \(\tau(T^n, F) \leq \tau(\Lambda, F')\).

Proof. Since the cardinality of \(\Lambda\) is maximal, then \(\alpha\mu\delta K + \Lambda\) is a covering of \(T^n\).

Let \(Y = \{y_j, j = 1, \ldots, m\} \subset T^n\) be a point set such that \(\mu(1 - \delta)\hat{Y} + Y\) covers \(\Lambda\). We show that \(\mu\hat{Y} + Y\) covers \(T^n\).

Let \(t \in T^n\) be an arbitrary point. Since \(\alpha\mu\delta K + \Lambda\) is a covering of \(T^n\), then there exists \(\lambda \in \Lambda\) such that \(\alpha\mu\delta K + \lambda\) contains \(t\). There also exists \(i\) such that

\[
\lambda \in ((\mu(1 - \delta)\hat{y}_i + y_j))
\]

for some \(i\). Since for all \(i\) we have \(aK \subset \text{int}(\hat{y}_i - \tilde{y}_i)\), we obtain

\[
t \in \mu\delta\hat{y}_i - \tilde{y}_i + \lambda \subset ((\mu\delta\hat{y}_i - \tilde{y}_i) + (\mu(1 - \delta)\hat{y}_i - \tilde{y}_i)) + \tilde{y}_i + y_j \subset \mu\hat{y}_i - \tilde{y}_i + \tilde{y}_i + y_j = \mu\hat{y}_i + y_j.
\]

The proof is complete. \(\square\)

Consider \(F\) and \(F'\) and \(\Lambda\) as in the notation of Lemma 1. Define

\[
\mathcal{E} = \{\Lambda \cap F : F \in F'\}.
\]

Then \((\Lambda, \mathcal{E})\) is a finite hypergraph and \(\tau(\Lambda, F') = \tau(\Lambda, \mathcal{E})\).

From Lemma 1 and Theorem 2 it follows that

\[
\tau(T^n, F) \leq \tau(\Lambda, \mathcal{E}) \leq \left(1 + \ln \left(\max_{\mathcal{E}} |\mathcal{E}|\right)\right) \tau^*(\Lambda, \mathcal{E}).
\]

We want to bound \(\tau^*(\Lambda, \mathcal{E})\). By \(\sigma\) denote the usual measure on \(T^n\) induced by the Lebesgue measure on \(\mathbb{R}^n\) and scaled in such a way that \(\sigma(T^n) = 1\).

For \(\lambda \in \Lambda\) and \(E \in \mathcal{E}\) define

\[
S(\lambda) = \{t \in T^n : \lambda \in \mu(1 - \delta)\hat{Y} + t\},
\]

\[
S(E) = \{t \in T^n : E = \Lambda \cap (\mu(1 - \delta)\hat{Y} + t)\}.
\]

The sets \(S(\lambda), S(E)\) are measurable. Moreover, for every \(\lambda \in \Lambda\),

\[
\sigma(S(\lambda)) = \sigma(\mu(1 - \delta)\hat{Y}),
\]

\[
\sigma(S(E)) = \sum_{\lambda \in E} \sigma(S(\lambda)).
\]

\[
\sum_{\lambda \in E} \sigma(S(\lambda)) = \sigma(T^n) = 1.
\]

Define \(v : \mathcal{E} \to [0, +\infty)\) as

\[
v(E) = \frac{\sigma(S(E))}{\sigma(\mu(1 - \delta)\hat{Y})}.
\]

Then it is a fractional covering of \(\Lambda\) by \(\mathcal{E}\) and

\[
\tau^*(\Lambda, \mathcal{E}) \leq \sum_{\mathcal{E} \in \mathcal{E}} v(E) = \frac{1}{\sigma(\mu(1 - \delta)\hat{Y})}.
\]

Now we bound \(\max_{\mathcal{E}} |\mathcal{E}| = \max_{\mathcal{E} \in \mathcal{F} \subset F} |\Lambda \cap F'|\).

Recall that \(\frac{\alpha\mu\delta}{2}K + \Lambda\) is a packing of bodies \(\frac{\alpha\mu\delta}{2}K\). If \(\lambda \in F' = \mu(1 - \delta)\hat{Y} + t\), then \(\lambda \in \mu(1 - \delta)\hat{y}_i + t\) for some \(i\).

Since \(aK \subset \text{int}(\hat{y}_i - \tilde{y}_i)\), we have

\[
\frac{\alpha\mu\delta}{2}K + \lambda \leq \left(\frac{\delta}{2}\hat{y}_i - \tilde{y}_i\right) + (\mu(1 - \delta)\hat{y}_i - \tilde{y}_i) + \tilde{y}_i + t \subset \mu(1 - \delta)\hat{y}_i + t \subset \mu\hat{y}_i + t.
\]

Now we can compare the volumes and get the bound for \(|\Lambda \cap F'|\). Since every \(\hat{y}_i\) is contained in \(\beta K + \tilde{y}_i\), we get \(\text{vol}(\hat{y}_i) \leq \beta^n \text{vol}(K)\) and

\[
\frac{\alpha\mu\delta}{2}K + \lambda \leq k\mu^n(\beta^n K) \leq \sum_{i=1}^{k} \frac{\text{vol}(\mu\hat{y}_i)}{\text{vol}(\frac{\alpha\mu\delta}{2}K)} \leq \frac{k\mu^n\beta^n \text{vol}(K)}{\mu^n(\beta^n K)} = k(2\gamma/\delta)^n.
\]

Finally, we obtain

\[
\tau(T^n, F) \leq \left(1 + \ln \left(\max_{F \subset F'} |\Lambda \cap F'|\right)\right) \tau^*(T^n, F') \leq \frac{1 + n\ln(2\gamma/\delta) + \ln k(1 + \gamma)^n}{(1 - \delta)^n}.
\]
Now we take $\delta = \frac{1}{n \ln n}$ and use (for arbitrary large $n$)
\[
\left(1 - \frac{1}{2n \ln n}\right)^{n} \leqslant \exp\left(\frac{1}{\ln n}\right) \leqslant 1 + \frac{2}{\ln n}.
\]
Thus we have
\[
\lambda([R^n]) \leqslant \tau(T^n, \mathcal{F}) \leqslant (1 + \gamma)^{n}\left(1 + \frac{2}{\ln n}\right) + 2 n + 2n(1 + \ln(2n(1 + \ln 2)))
\]
\[
\leqslant (1 + \gamma)^{n}(\ln n + 2ln k + 2n + 2n\ln(2\gamma))\]
\[
\leqslant (1 + \gamma)(K, k) + \epsilon)^{n} \ln(2 \ln n).\]

This inequality holds for every $\epsilon > 0$. This completes the proof of Theorem 1.

### 3. Chromatic number for the Euclidean metric

In the paper [12], Larman and Rogers proved that for a Euclidean ball $B^n$, the lattice tiling parameter $\gamma(B^n, 1) \leqslant 2 + o(1)$ as $n \to \infty$. They used a theorem due to Butler [3]. We need some notation to state the Butler result.

Let $K = K + \Omega$ be a system of translates of $K$ by the vectors of the lattice $\Omega$, $\xi_1 = \xi_1(K)$ be the infimum of the positive numbers $\xi$ such that the system $\xi K$ is a covering of $\mathbb{R}^n$, and $\xi_2 = \xi_2(K)$ be the supremum of the positive numbers $\xi$ such that $\xi K$ is a packing in $\mathbb{R}^n$.

Denote $\xi(K) = \xi_1(K)/\xi_2(K)$. Consider $\gamma(K) = \inf_{\xi} \xi(K)$, where the infimum is over the set of all lattices in $\mathbb{R}^n$. By $DK$ denote the difference body of $K$, i.e. $DK = \{x - y : x, y \in K\}$.

**Theorem 3 (Butler, [3]).** Let $K$ be a bounded convex body in $\mathbb{R}^n$, then there exists an absolute constant $c$ such that
\[
\gamma(K) \leqslant \left[\frac{\text{vol}(DK)}{\text{vol}(K)} n^{\log_2(\ln n) + c}\right]^{1/n}.
\]
If $K$ is centrally symmetric, then we get $\gamma(K) \leqslant 2 + o(1)$. It is easy to see that if $K = B^n$, then $\gamma(B^n, 1) \leqslant \gamma(B^n)$. Indeed, let $\Omega$ be a lattice such that $\xi(K) < \gamma(B^n) + \epsilon$ and $\Psi$ be a Voronoi tiling which corresponds to $\Omega$. Then for all $x \in \Omega$ we get
\[
K + x = \psi_x \subset \xi(K)K + x.
\]
Since $\epsilon$ is arbitrary close to zero, we obtain our inequality. Unfortunately, if $K$ is not a Euclidean ball, then Voronoi polytopes might be non-convex and the locus of the points that have equal distances to a pair of given points might have nonzero measure. Therefore, the problem of bounding $\gamma(K, 1)$ becomes much harder.

The proof of Theorem 3 is quite nontrivial. But the problem of bounding of our generalized tiling parameter instead of the lattice one is much easier. First, we show that for some $k$, $\gamma(B^n, k) \leqslant 2$.

Let $\Omega$ be a lattice such that $K = B^n + \Omega$ is a packing. We claim that there is some multilattice $\Phi$ with the base lattice $\Omega$ such that $B^n + \Phi$ is a packing and $2k = 2B^n + \Phi$ is a covering. By $T^n$ denote the torus $\mathbb{R}^n/\Omega$. Choose a set $Y = \{yi\}$ of the maximal cardinality such that
\[
(B^n + y_i) \cap (B^n + y_j) = \varnothing, \quad \forall i \neq j.
\]
For all $x \in T^n$ there exists $i$ such that $\|x - y_i\|_K < 2$. Otherwise, $(B^n + x) \cap (B^n + y_j) = \varnothing$ which implies that $Y$ does not have the maximal cardinality. Therefore, we have proved that $\bigcup_i 2B^n + y_i$ covers $T^n$. Hence, we can take the multilattice $\Phi = \Omega + Y$.

Now associate to it the Voronoi tiling $\psi$ of the point set $\Omega + Y$. Then in turn
\[
\gamma(B^n, k) \leqslant \gamma(B^n, \Phi, \psi) \leqslant 2.
\]

Now we supply an upper bound on $k$. Let $B^n$ be inscribed into a cube $C$ with the side length 2. The edges of $C$ generate a lattice $\Omega$ in $\mathbb{R}^n$ such that $K = B^n + \Omega$ is a packing. We can bound $k$ using volumes
\[
k \leqslant \frac{\text{vol}(C)}{\text{vol}(B^n)} \leqslant n^m.
\]
Finally, we can apply Theorem 1 and get the upper bound for (1).
Acknowledgments

The author is grateful to A. M. Raigorodskii and A. B. Kupavskii for their constant attention to this work and for useful remarks.

References

[1] L.I. Bogolyubskii, A.M. Raigorodskii, A remark on lower bounds for the chromatic numbers of spaces of small dimension with metrics $\ell_p$ and $\ell_q$, Mat. Zametki 105 (2) (2019) 187–213.
[2] P. Brass, W. Moser, J. Pach, Research problems in discrete geometry, Springer, New York, 2005, p. xiii+499.
[3] G.J. Butler, Simultaneous packing and covering in euclidean space, Proc. Lond. Math. Soc. (3) 25 (1972) 721–735.
[4] D. Cherkashin, A. Kulikov, A. Raigorodskii, On the chromatic numbers of small-dimensional Euclidean spaces, Discrete Appl. Math. 243 (2018) 125–131.
[5] D. Cherkashin, A. Raigorodskii, On the chromatic numbers of small-dimensional spaces, Doklady Math. 95 (1) (2017) 5–6.
[6] A.D.N.J. de Grey, The chromatic number of the plane is at least 5, Geombinatorics 28 (1) (2018) 18–31.
[7] P. Erdős, C.A. Rogers, Covering space with convex bodies, Acta Arith. 7 (1961/1962) 281–285.
[8] Z. Füredi, J.-H. Kang, Covering the $n$-space by convex bodies and its chromatic number, Discrete Math. 308 (19) (2008) 4495–4500.
[9] D.S. Johnson, Approximation algorithms for combinatorial problems, J. Comput. System Sci. 9 (1974) 256–278, Fifth Annual ACM Symposium on the Theory of Computing (Austin, Tex., 1973).
[10] A.Y. Kanel-Belov, V.A. Voronov, D.D. Cherkashin, On the chromatic number of a plane, Algebra i Analiz 29 (5) (2017) 68–89.
[11] A. Kupavskiy, On the chromatic number of $\mathbb{R}^n$ with an arbitrary norm, Discrete Math. 311 (6) (2011) 437–440.
[12] D.C. Larmarn, C.A. Rogers, The realization of distances within sets in Euclidean space, Mathematika 19 (1972) 1–24.
[13] L. Lovász, On the ratio of optimal integral and fractional covers, Discrete Math. 13 (4) (1975) 383–390.
[14] M. Naszódi, On some covering problems in geometry, Proc. Amer. Math. Soc. 144 (8) (2016) 3555–3562.
[15] R. Prosanov, Chromatic numbers of spheres, Discrete Math. 341 (11) (2018) 3123–3133.
[16] R. Prosanov, Upper bounds for the chromatic numbers of Euclidean spaces with forbidden Ramsey sets, Mat. Zametki 103 (2) (2018) 248–257.
[17] R. Prosanov, Counterexamples to Borsuk’s conjecture that have large girth, Mat. Zametki 105 (6) (2019) 890–898.
[18] A.M. Raigorodskii, On the chromatic number of a space, Uspekhi Mat. Nauk 55 (2(332)) (2000) 147–148.
[19] A.M. Raigorodskii, The Borsuk problem and the chromatic numbers of some metric spaces, Uspekhi Mat. Nauk 56 (1(337)) (2001) 107–146.
[20] A.M. Raigorodskii, On the chromatic number of a space with the metric $l_p$, Uspekhi Mat. Nauk 59 (5(359)) (2004) 161–162.
[21] A.M. Raigorodskii, Coloring distance graphs and graphs of diameters, in: Thirty Essays on Geometric Graph Theory, Springer, New York, 2013, pp. 429–460.
[22] A.M. Raigorodskii, Cliques and cycles in distance graphs and graphs of diameters, in: Discrete Geometry and Algebraic Combinatorics, in: Contemp. Math., vol. 625, Amer. Math. Soc., Providence, RI, 2014, pp. 93–109.
[23] A. Sagdeev, Exponentially Ramsey sets, Problems of Information Transmission 54 (4) (2018) 372–396.
[24] A. Sagdeev, An improved Frankl–Rödl theorem and some of its geometric consequences, Problemy Peredachi Informatsii 54 (2) (2018) 45–72.
[25] A. Sagdeev, On the Frankl–Rödl theorem, Izvestiya: Mathematics 82 (6) (2018) 1196–1224.
[26] A. Steiner, The Mathematical Coloring Book, Springer, New York, 2009, p. xxx+607, Mathematics of coloring and the colorful life of its creators, With forewords by Branko Grünbaum, Peter D. Johnson, Jr. and Cecil Rousseau.
[27] S.K. Stein, Two combinatorial covering theorems, J. Comb. Theory A 16 (1974) 391–397.