TANNAKIAN PROPERTIES OF UNIT FROBENIUS-MODULES

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Abstract. We show that unit $O_{F,X}^\Lambda$-modules of Emerton and Kisin provide an analogue of locally constant sheaves in the context of Böckle–Pink $\Lambda$-crystals. For example they form a tannakian category if the coefficient algebra $\Lambda$ is a field. Our results hold for a big class of coefficient algebras which includes Drinfeld rings, and for arbitrary locally noetherian base schemes.

Introduction

Let $q$ be a prime power, $\mathbb{F}_q$ a field with $q$ elements. For an $\mathbb{F}_q$-scheme $X$, and a commutative $\mathbb{F}_q$-algebra $\Lambda$, Böckle and Pink [1] introduced the category of $\Lambda$-crystals on $X$ which is in many ways analogous to the category of étale constructible sheaves of $\Lambda$-modules. Let us recall the definitions.

Definition 0.1 (Böckle–Pink [1]). Let $X$ be an $\mathbb{F}_q$-scheme, $\Lambda$ a commutative $\mathbb{F}_q$-algebra. Let $F$ be the endomorphism of Spec $\Lambda \times_{\mathbb{F}_q} X$ which acts as identity on Spec $\Lambda$, and as the absolute $q$-Frobenius on $X$. Let $O_X^\Lambda$ denote the structure sheaf of Spec $\Lambda \times_{\mathbb{F}_q} X$.

(1) An $O_{F,X}^\Lambda$-module is a pair $(M, \varphi)$, where $M$ is an $O_X^\Lambda$-module, and $\varphi: F^*M \to M$ a morphism of $O_X^\Lambda$-modules. A morphism of $O_{F,X}^\Lambda$-modules $\alpha: (M, \varphi_M) \to (N, \varphi_N)$ is a morphism $\alpha: M \to N$ of $O_X^\Lambda$-modules, such that $\alpha \circ \varphi_M = \varphi_N \circ F^*\alpha$.

The category of $O_{F,X}^\Lambda$-modules which are quasi-coherent as $O_X^\Lambda$-modules is denoted $\mu(X, \Lambda)$.

(2) An $O_{F,X}^\Lambda$-module $(M, \varphi)$ is called nilpotent if for $n \gg 0$ the composition $\varphi \circ F^*(\varphi) \circ \ldots \circ F^{*n}(\varphi)$ vanishes. A morphism of $O_{F,X}^\Lambda$-modules is called a nil-isomorphism if its kernel, and cokernel are nilpotent.

(3) The category $\text{Crys}(X, \Lambda)$ of $\Lambda$-crystals on $X$ is the localization of the category of $O_{F,X}^\Lambda$-modules which are $O_X^\Lambda$-coherent at the multiplicative system of nil-isomorphisms.

The connection with constructible sheaves is provided by the following result of Böckle and Pink. Let $\text{Sh}(X_{\text{ét}}, \Lambda)$ be the category of étale sheaves of $\Lambda$-modules. Define a functor $\varepsilon: \mu(X, \Lambda) \to \text{Sh}(X_{\text{ét}}, \Lambda)$ by

$$
\varepsilon(M, \varphi)(u: U \to X) = \text{Hom}_{\mu(U, \Lambda)}\left((O_U^\Lambda, 1), (u^*M, u^*\varphi)\right).
$$

Since $\varepsilon$ transforms nil-isomorphisms to isomorphisms ([1] proposition 10.1.7 (b)) one gets a functor $\varepsilon: \text{Crys}(X, \Lambda) \to \text{Sh}(X_{\text{ét}}, \Lambda)$. 


Theorem 0.2 (Böckle–Pink [1] Theorem 10.3.6). Assume that \( \Lambda \) is of finite dimension as \( \mathbb{F}_q \)-vector space, \( X \) is noetherian, and separated over \( \mathbb{F}_q \). The functor \( \varepsilon \) defines an equivalence of \( \text{Crys}(X, \Lambda) \), and the category of constructible étale sheaves of \( \Lambda \)-modules.

In particular \( \varepsilon \) identifies the subcategory \( \text{Loc}(X, \Lambda) \subset \text{Sh}(\text{ét}, \Lambda) \) of locally constant sheaves with a certain subcategory of \( \text{Crys}(X, \Lambda) \). An explicit description of this subcategory is provided by a classical result of Katz which in the language of this text can be stated as follows.

Definition 0.3 (Emerton–Kisin [4] Definition 5.1). An \( \mathcal{O}_{\mathcal{F},X}^\Lambda \)-module \((\mathcal{M}, \varphi)\) is called unit if \( \varphi \) is an isomorphism. The category of unit \( \mathcal{O}_{\mathcal{F},X}^\Lambda \)-modules which are locally of finite type as \( \mathcal{O}_{\mathcal{X}}^\Lambda \)-modules is denoted \( U(X, \Lambda) \).

Theorem 0.4 (Katz [5] Proposition 4.1.1). Assumptions as in theorem 0.2.

1. The natural functor \( U(X, \Lambda) \to \text{Crys}(X, \Lambda) \) is fully faithful.

2. The functor \( \varepsilon \) identifies the subcategories \( U(X, \Lambda) \) and \( \text{Loc}(X, \Lambda) \).

The definition of \( \Lambda \)-crystals makes sense even if \( \Lambda \) is not finite over \( \mathbb{F}_q \). The interest in infinite coefficient algebras \( \Lambda \) comes from a construction of Drinfeld (see e.g. section 3.5 of [1]) which produces an \( \mathcal{O}_{\mathcal{F},X}^\Lambda \)-module, and a fortiori a \( \Lambda \)-crystal out of a sufficiently good Drinfeld module \( \varphi: \Lambda \to \text{End}_{\mathbb{F}_q}(\mathcal{L}), \mathcal{L} \) a line bundle on \( X \). So one has a natural source of \( \Lambda \)-crystals which have arithmetic significance.

For infinite \( \Lambda \) the connection with étale sheaves breaks down: the functor \( \varepsilon \) may assign a zero sheaf to a nonzero crystal. Nevertheless Böckle and Pink show that under mild technical assumptions on \( X \), and \( \Lambda \), the crystals retain many properties one would expect from constructible sheaves. Crystals form an abelian category. Some of the Grothendieck six operations are defined for them: pullback, proper pushforward, tensor product, and extension by zero. These operations are related in the same way as they are for constructible sheaves, and the equivalence of theorem 0.2 is compatible with them. The pullback is an exact operation, and a crystal is zero if and only if its pullbacks to all points \( x \to X \) are.

Thus one can view \( \Lambda \)-crystals as an extension, or a modification of the notion of a constructible sheaf. From such a point of view it is natural to ask which kind of crystals is the analogue of locally constant sheaves? In this paper we purport to show that the answer is unit modules.

Remark 0.5. The part (1) of theorem 0.4 holds for infinite \( \Lambda \) as well provided one restricts to unit modules which are \( \mathcal{O}_{\mathcal{X}}^\Lambda \)-coherent. So unit modules are indeed a particular kind of crystals. Since we will not use this property we only indicate a proof. A cohomological computation in the spirit of [7] section 4 shows that if \( \mathcal{M} \) is a unit module, and \( \mathcal{N} \) is nilpotent then \( \text{Ext}^1(\mathcal{M}, \mathcal{N}) = 0 \) in \( \mu(X, \Lambda) \). Since \( \text{Hom}(\mathcal{M}, \mathcal{N}) \) also vanishes it follows that a nil-isomorphism to a unit module is necessarily a split epimorphism whence the result.
If $\Lambda$ is finite-dimensional over $\mathbb{F}_q$, $X$ is connected, and locally noetherian then a choice of a geometric point $\mathfrak{p} \to X$ identifies $\text{Loc}(X, \Lambda)$ with the category of $\pi^\text{\acute{e}t}_1(X, \mathfrak{p})$-representations in $\Lambda$-modules. In particular if $\Lambda$ is a finite field extension of $\mathbb{F}_q$ then $\text{Loc}(X, \Lambda)$ is a tannakian $\Lambda$-linear category. It is this last property which we take as the characteristic property of local systems. For infinite $\Lambda$ the connection of $U(X, \Lambda)$ with the étale fundamental group is lost. Nevertheless in the spirit of Böckle and Pink [1] we prove the following:

**Theorem 3.6.** Suppose that $\Lambda$ is a field, $X$ is connected, and locally noetherian. The category $U(X, \Lambda)$ is $\Lambda$-linear tannakian with respect to a rigid monoidal structure inherited from $\text{Crys}(X, \Lambda)$.

It seems that no good notion of a tannakian $\Lambda$-linear category is known in the case when $\Lambda$ is not a field. Still in this case we demonstrate that under mild technical assumptions on $\Lambda$, and $X$, unit modules retain some important tannakian properties which local systems have. Unit modules form an abelian category (theorem 3.1), and the pullback functors on them are faithful (theorem 3.2). A nice feature of these results is that they hold not only for regular but for arbitrary locally noetherian schemes $X$.

In order to deduce these theorems we study structural properties of locally finitely generated $\mathcal{O}^\Lambda_X$-modules which admit a unit module structure. Our approach was inspired by the following folklore lemma: if a $\mathcal{D}$-module is $\mathcal{O}_X$-coherent then it is locally free. It has a counterpart in the setting of Emerton–Kisin:

**Proposition 0.6** (Emerton–Kisin [3] proposition 6.9.3). Assume that $\Lambda$ is finite-dimensional as $\mathbb{F}_q$-vector space, $X$ is locally noetherian. If a unit $\mathcal{O}^\Lambda_{F,X}$-module is $\mathcal{O}_X$-coherent then it is locally free.

We extend this proposition to the case of infinite $\Lambda$ in two ways. One of them works for arbitrary $X$, and $\Lambda$.

**Theorem 2.3.** If a unit $\mathcal{O}^\Lambda_{F,X}$-module is locally of finite presentation as an $\mathcal{O}_X$-module then it is $\mathcal{O}_X$-flat.

The main difficulty in the proof of this theorem is that when $\Lambda$ is infinite unit modules are almost never of finite type as $\mathcal{O}_X$-modules. Another extension of proposition 0.6 is specific for the case when $\Lambda$ is a field:

**Proposition 3.5.** Assume that $\Lambda$ is a field, $X$ is locally noetherian. Let $\mathcal{M}$ be a unit $\mathcal{O}^\Lambda_{F,X}$-module. If $\mathcal{M}$ is locally of finite type as an $\mathcal{O}_X$-module then it is a unit $\mathcal{O}_X$-module. Another extension of proposition 0.6 is specific for the case when $\Lambda$ is a field:

**Proposition 3.1.** Assume that $\Lambda$ is a field, $X$ is locally noetherian. Let $\mathcal{M}$ be a unit $\mathcal{O}^\Lambda_{F,X}$-module. If $\mathcal{M}$ is locally of finite type as an $\mathcal{O}_X$-module then it is a unit $\mathcal{O}_X$-module.

This proposition follows easily from the next result which we call the invariant closed subscheme theorem:

**Theorem 1.1.** Assume that $X$ is connected, and locally noetherian. Let $Z \subset \text{Spec} \Lambda \times \mathbb{F}_q X$ be a closed subscheme. If $F^{-1}Z = Z$ as a subscheme then $Z = Z_0 \times \mathbb{F}_q X$ where $Z_0$ is the scheme-theoretic image of $Z$ in $\text{Spec} \Lambda$. 

Its proof is based on the proof of lemma 4.6.1 from [1].

We leave several important questions unanswered. We do not know how to describe the Tannaka-dual groups (or gerbes) of the tannakian categories which we construct. How do they change when the coefficient algebra varies? If one fixes the base scheme $X$, and varies $\Lambda$ then our unit modules form a stack. It is tempting to call it a stack of tannakian categories. Is there a kind of Tannaka-dual object behind the whole stack?

**Remark 0.7.** The notion of unit modules comes from the work of Emerton and Kisin [4]. However we use unit modules in a way which is different from [4]. Emerton and Kisin construct a characteristic $p > 0$ version of the Riemann–Hilbert correspondence. In their work, $\mathcal{O}^{\Lambda}_{\mathcal{F},X}$-modules are analogues of $\mathcal{D}$-modules, and unit modules play the role of holonomic $\mathcal{D}$-modules. In this text we work not on the $\mathcal{D}$-module side of Emerton–Kisin but on the side of constructible sheaves, i.e. $\Lambda$-crystals.

**Notation and conventions.** Throughout the text $\Lambda$ denotes a commutative $\mathbb{F}_q$-algebra which we will use as the coefficients for unit modules. For an $\mathbb{F}_q$-scheme $X$ the symbol $\mathcal{F}$ indicates the endomorphism of $\text{Spec} \Lambda \times_{\mathbb{F}_q} X$ which acts as identity on $\text{Spec} \Lambda$, and as the absolute $q$-Frobenius on $X$. We make no general assumptions on the coefficient algebra $\Lambda$, and the base scheme $X$. In each theorem we carefully state the precise assumptions on $\Lambda$, and $X$ under which it holds.

Following [4] we denote $\mu(X, \Lambda)$ the category of $\mathcal{O}^{\Lambda}_{\mathcal{F},X}$-modules whose underlying $\mathcal{O}^{\Lambda}_X$-module is quasi-coherent.

$U(X, \Lambda)$ stands for the subcategory of $\mu(X, \Lambda)$ consisting of unit modules whose underlying $\mathcal{O}^{\Lambda}_X$-modules are locally of finite type. We do not use the adjective “coherent” since in the situation of interest for us it may well happen that $\text{Spec} \Lambda \times_{\mathbb{F}_q} X$ is not a locally noetherian scheme.

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1. The invariant closed subscheme theorem

**Theorem 1.1.** Let $X$ be a connected locally noetherian scheme over $\mathbb{F}_q$, $\Lambda$ an $\mathbb{F}_q$-algebra, and $Z \subset \text{Spec} \Lambda \times_{\mathbb{F}_q} X$ a closed subscheme. If $F^{-1}Z = Z$ then $Z = Z_0 \times_{\mathbb{F}_q} X$ where $Z_0$ is the scheme-theoretic image of $Z$ under the projection $\text{Spec} \Lambda \times_{\mathbb{F}_q} X \to \text{Spec} \Lambda$.

The proof of this theorem is based on the proof of lemma 4.6.1 from [1].
Remark 1.2. When $X$ is not locally noetherian theorem [2] fails even if $\Lambda = \mathbb{F}_q$. For an example let $R$ be the $q$-perfection of $\mathbb{F}_q[[t]]$. The ring $R$ is a local domain with maximal ideal $m$ generated by $\{t^n\}_{n \in \mathbb{Z}}$. Since $F$ maps this system of generators to itself the ideal $m$ is $F$-invariant, and defines a closed $F$-invariant subscheme of $\text{Spec } R$ which is different from $\text{Spec } R$, and $\varnothing$.

Proof of theorem [2]. (1) Assume that $X$ is the spectrum of a local noetherian ring $R$ with maximal ideal $m$. Let $I \subset \Lambda \otimes_{\mathbb{F}_q} R$ be the ideal of $Z$. The ideal $I$ is $F$-invariant in the sense that it is generated by $F(I)$. Consider the ideal $I \cap \Lambda$ where $\Lambda$ is viewed as a subring of $\Lambda \otimes_{\mathbb{F}_q} R$ via the coprojection. Our goal is to show that the inclusion $(I \cap \Lambda) \otimes_{\mathbb{F}_q} R \subset I$ is an equality.

It is enough to consider the case $I \cap \Lambda = 0$. Indeed, let $\Lambda_0 = \Lambda/(I \cap \Lambda)$, and let $J$ be the image of $I$ in $\Lambda_0 \otimes_{\mathbb{F}_q} R$. Then $J$ is $F$-invariant, and $J \cap \Lambda_0 = 0$, so the theorem implies that $J = 0$, i.e. $I = (I \cap \Lambda) \otimes_{\mathbb{F}_q} R$.

In order to deduce the equality $I = 0$ from $I \cap \Lambda = 0$ we only need to prove the inclusion $I \subset \Lambda \otimes_{\mathbb{F}_q} m$ since $F$-invariance of $I$ will then imply the inclusion $I \subset \Lambda \otimes_{\mathbb{F}_q} \cap_{n \geq 0} m^n = 0$. Choose a basis $\{\lambda_s\}_{s \in S}$ of $\Lambda$ as an $\mathbb{F}_q$-vector space. It induces a basis of $\Lambda \otimes_{\mathbb{F}_q} R$ as an $R$-module. In this basis $F$ acts by raising coordinates to $q$-th powers.

Suppose that there are elements $f \in I$ which have a coordinate belonging to $R^\times$. Among them pick one which has minimal number of nonzero coordinates, say $f = r_1 \lambda_1 + r_2 \lambda_2 + \ldots + r_n \lambda_n$ with $r_1 \in R^\times$, and all $r_i \neq 0$. Dividing $r_1$ out we may assume that the coordinate of $f$ at $\lambda_1$ is 1. The element $F(f) - f \in I$ has coefficient zero at $\lambda_1$, so by minimality of $n$ the elements $r_i^q - r_i$ belong to $m$. Since $r_i^q - r_i = \prod_{\alpha \in \mathbb{F}_q}(r_i - \alpha)$, and since $m$ is prime we conclude that for each $i$ there exist $\alpha_i \in \mathbb{F}_q$, $m_i \in m$ such that $r_i = \alpha_i + m_i$.

Next consider the set $I_1 \subset I$ of elements having the form $\lambda_1 + (\alpha_2 + m_2)\lambda_2 + \ldots + (\alpha_n + m_n)\lambda_n$ for some $\alpha_i \in \mathbb{F}_q$, $m_i \in m$. Pick an element $f \in I_1$, and let $i \in \{1 \ldots n\}$ be the maximal index with the property that $m_j = 0$ for all $j \leq i$. Consider the element $F(f) - f \in I$. Its coordinates at $\lambda_j$, $j \leq i$, are zero, and the coordinate at $\lambda_{i+1}$ is $m_{i+1}^q - m_{i+1}$ which is equal to $m_{i+1}$ up to a unit. Dividing by this unit, and subtracting the result from $f$ we obtain an element of $I_1$, which has $m_j = 0$ for all $j \leq i + 1$. Repeating this process we see that $I_1$ contains an element of $\Lambda$. As all elements of $I_1$ are nonzero this contradicts the assumption $\Lambda \cap I = 0$. Hence $I \subset \Lambda \otimes_{\mathbb{F}_q} m$ as we want.

(2) Let $X$ be a connected locally noetherian scheme. Pick a point $x \in X$. According to the step (1) the base change of $Z$ to $\text{Spec } \Lambda \times_{\mathbb{F}_q} \text{Spec } \mathcal{O}_{X,x}$ has the form $Z_x \times_{\mathbb{F}_q} \text{Spec } \mathcal{O}_{X,x}$ for a certain closed subscheme $Z_x \subset \text{Spec } \Lambda$.

Let $x, x' \in X$ be points such that $x$ is a specialization of $x'$. There exists a discrete valuation ring $R$ and a morphism $f : \text{Spec } R \to X$ mapping the generic point of $R$ to $x'$, and the closed point to $x$ [Stacks 054F]. Step (1) shows that the base change of $Z$ to $\text{Spec } \Lambda \times_{\mathbb{F}_q} \text{Spec } R$ along $1_\Lambda \times_{\mathbb{F}_q} f$ has the form $Z_f \times_{\mathbb{F}_q} \text{Spec } R$ for a certain $Z_f \subset \text{Spec } \Lambda$. Therefore $Z_x = Z_f = Z_{x'}$. Since every two points of $X$ can be connected by a chain of generalizations, and specializations, we see that $Z_x = Z_0$. 


is independent of $x \in X$. As the ideal sheaves of $Z$, and $Z_0 \times_{F_q} X$ have the same stalks at every point of $\text{Spec} \Lambda \times_{F_q} X$ these closed subschemes are equal. □

We will use theorem 1.1 through the following corollary:

**Lemma 1.3.** Let $X$ be a connected locally noetherian scheme over $F_q$, $\Lambda$ an $F_q$-algebra, $\mathcal{M}$ a unit $O_{F,X}^\Lambda$-module which is locally of finite type as an $O_{X}^\Lambda$-module. Let $Z_n(\mathcal{M}) \subset \text{Spec} \Lambda \times_{F_q} X$ be the closed subscheme defined by the $n$-th Fitting ideal sheaf of $\mathcal{M}$ as an $O_{X}^\Lambda$-module. The subscheme $Z_n(\mathcal{M})$ has the form $Z \times_{F_q} X$ for a certain closed subscheme $Z \subset \text{Spec} \Lambda$.

**Proof.** $F$-invariance of $\mathcal{M}$ implies $F$-invariance of $Z_n(\mathcal{M})$, whence the result. □

**Remark 1.4.** Here are two more easy but interesting corollaries of theorem 1.1. Let $X$ be a connected locally noetherian scheme over $F_q$, $\Lambda$ an $F_q$-algebra, and $\mathcal{M}$ a unit $O_{F,X}^\Lambda$-module which is locally of finite type as an $O_{X}^\Lambda$-module.

(1) If for every maximal ideal $m \subset \Lambda$ the restriction of $\mathcal{M}$ to the closed subscheme $\text{Spec} \Lambda/m \times_{F_q} X \subset \text{Spec} \Lambda \times_{F_q} X$ is zero then $\mathcal{M} = 0$.

(2) If a fiber of $\mathcal{M}$ over a point $x \in X$ is locally free of rank $r$ as an $O_{k(x)}^\Lambda$-module then $\mathcal{M}$ is locally free of rank $r$ as an $O_X^\Lambda$-module.

What makes (1) a nontrivial statement is the fact that in general there are closed subsets of $\text{Spec} \Lambda \times_{F_q} X$ whose image in $\text{Spec} \Lambda$ misses all the closed points.

2. Flatness

In this section we will prove that under suitable finiteness assumptions every unit $O_{F,X}^\Lambda$-module is $O_X$-flat (theorem 2.3). We will deduce this theorem from the following lemma.

**Lemma 2.1.** Let $R$ be an artinian local ring over $F_q$, $m$ its maximal ideal, $k$ the residue field.

(1) Equip $k$ with the structure of an $F_q$-algebra via the composite arrow $F_q \to R \to k$. Every splitting $k \to R$ of the quotient map $R \to k$ provided by Cohen structure theorem is a morphism of $F_q$-algebras.

(2) Fix a splitting as in (1). Let $\Lambda$ be an $F_q$-algebra. If $M$ is a unit $O_{F,R}^\Lambda$-module which is of finite presentation as an $O_{R}^\Lambda$-module then $M \cong (M/m) \otimes_k R$ over $O_{R}^\Lambda$.

**Remark 2.2.** Let $U_{pf}(X, \Lambda) \subset U(X, \Lambda)$ be the subcategory of modules which are of finite presentation as $O_{X}^\Lambda$-modules. In fact under the assumptions of lemma 2.1 (2) the reduction functor $- \otimes_R R/m: U_{pf}(R, \Lambda) \to U_{pf}(k, \Lambda)$ is an equivalence of categories, its inverse being $- \otimes_k R$. One can say that local artinian rings have a henselian property with respect to unit modules. We do not know if other types of rings exhibit such a property.
Since $R$ is local it follows that $z^{2-1} \in \{0, 1\}$. By assumption $z$ is not a unit, so $z = z \cdot z^{2-1} = 0$.

(2) Let $A$ be a matrix which defines a presentation of $M$ as an $\Lambda \otimes_{\mathbb{F}_q} R$-module, $F^n(A)$ the matrix obtained from $A$ by applying $F$ componentwise $n$ times. $F^n(A)$ defines a presentation of $F^n M$, and hence of $M$. Since $R$ is localartinian the coefficients of $F^n(A)$ lie in the subring $\Lambda \otimes_{\mathbb{F}_q} k \subset \Lambda \otimes_{\mathbb{F}_q} R$ for $n$ big enough, whence the claim.

\begin{proof}[Proof of lemma 2.1] (1) Let $x \in \mathbb{F}_q \subset R$, and pick $y \in k \subset R$ which reduces to the same element as $x$ in $k$. Set $z = x - y$. Then $z^q = z$, so $(z^{2-1})^2 = z^q \cdot z^{2-2} = z^{2-1}$.

Theorem 2.3. Let $X$ be a scheme over $\mathbb{F}_q$, $\Lambda$ an $\mathbb{F}_q$-algebra. If $(\mathcal{M}, \varphi)$ is a unit $\mathcal{O}_{F,X}$-module which is locally of finite presentation as an $\mathcal{O}_{\hat{X}}$-module then $\mathcal{M}$ is flat over $\mathcal{O}_X$.

Remark 2.4. We expect that a stronger result is true. Namely, if $X = \text{Spec } R$ is a localnoetherian $\mathbb{F}_q$-algebra, $\Lambda$ an arbitrary $\mathbb{F}_q$-algebra, and $M$ a unit $\mathcal{O}_{F,X}$-module of finite presentation as an $\mathcal{O}_{\hat{X}}$-module then $M$ is free as an $R$-module.

Even in the case $\Lambda = \mathbb{F}_q$ theorem 2.3 fails if the module $\mathcal{M}$ is not of finite presentation over $\mathcal{O}_{\hat{X}}$. For an example let $R$ be the ring of remark 1.2, $\mathfrak{m}$ its maximal ideal. $F$-invariance of $\mathfrak{m}$ implies that $R/\mathfrak{m}$ has a structure of a unit $\mathcal{O}_{F,R}$-module, and this module is not $R$-flat.

\begin{proof}[Proof of theorem 2.3] The question being local on $X$ it is enough to consider the case $X = \text{Spec } R$. Let $M$ be the $\Lambda \otimes_{\mathbb{F}_q} R$-module of corresponding to $\mathcal{M}$.

(1) Assume that $R$ is noetherian, and $\Lambda$ is of finite type over $\mathbb{F}_q$. Let $N_1 \to N_2$ be an inclusion of $R$-modules of finite type, $K = \ker(M \otimes_R (N_1 \to N_2))$. We want to show that $K = 0$. As $K$ is of finite type over $S = \Lambda \otimes_{\mathbb{F}_q} R$ it is enough to show that $K \otimes_R k(p) = 0$ for every prime $p \in \text{Spec } R$.

So we may assume $R$ is local. Let $\mathfrak{m}$ be its maximal ideal, $\hat{S}$ the completion of $S$ at $\mathfrak{m}$, and $\hat{M} = M \otimes_S \hat{S}$. For every ideal $I \subset R$ the module $I \otimes_R \hat{M}$ is $\mathfrak{m}$-adically separated as it is the completion of $I \otimes_R M$. Moreover for all $n > 0$ the $R/\mathfrak{m}^n$-module $\hat{M}/\mathfrak{m}^n = M/\mathfrak{m}^n$ is free by lemma 2.1. Hence $\hat{M}$ is $R$-flat by theorem 1 of \cite[ch. III, §5]{[2]}. So $K \otimes_R \hat{S} = 0$, and $K/\mathfrak{m} = 0$.

(2) Let $\Lambda$, $R$ be arbitrary. By \cite{[Stacks 05N7]} there exist finitely generated $\mathbb{F}_q$-subalgebras $\Lambda_0 \subset \Lambda$, $R_0 \subset R$, a finitely presented $\Lambda_0 \otimes_{\mathbb{F}_q} R_0$-module $M_0$, and morphisms $f : F^* M_0 \to M_0$, $g : M_0 \to F^* M_0$ such that $M \cong \Lambda \otimes_{\Lambda_0} M_0 \otimes_{R_0} R$, and under this isomorphism the structure morphism $\varphi : F^* M \to M$ is the base extension of $f$, $\varphi^{-1}$ is the base extension of $g$, and moreover $fg = 1, gf = 1$. Thus $(M_0, f)$ is a unit $\mathcal{O}_{F,R_0}$-module.

Represent $\Lambda$ as a filtered colimit of finitely generated $\Lambda_0$-subalgebras $\Lambda_i$. If $I \subset R$ is an ideal then $$\Lambda \otimes_{\Lambda_0} M_0 \otimes_{R_0} (I \to R) = \text{colim}_i \left( \Lambda_i \otimes_{\Lambda_0} M_0 \otimes_{R_0} (I \to R) \right)$$
The module $\Lambda_i \otimes_{\Lambda_0} M_0$ is $R_0$-flat by (1), so the morphism on the left hand side being a filtered colimit of injections is an injection itself. \qed
3. Tannakian properties

**Theorem 3.1.** Let $X$ be a locally noetherian scheme over $\mathbb{F}_q$, $\Lambda$ an algebra essentially of finite type over $\mathbb{F}_q$. The category $U(X, \Lambda)$ is closed under kernels, and cokernels in $\mu(X, \Lambda)$. In particular, it is an abelian category.

The only obstruction this theorem has to deal with is the non-exactness of $\mathbb{F}^*$. According to Kunz [6] $\mathbb{F}^*$ is exact if and only if $X$ is regular. So the main interest in theorem 3.1 comes from the fact that it holds for singular base schemes $X$ too.

**Proof of theorem 3.1.** Let $\alpha : (M, \varphi_M) \rightarrow (N, \varphi_N)$ be a morphism of unit modules which is an epimorphism of underlying $O_X^{\Lambda}$-modules, and let $(Q, \varphi_Q)$ be the kernel of $\alpha$ computed in $\mu(X, \Lambda)$. The lower row of the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & Q & \rightarrow & M & \rightarrow & N & \rightarrow & 0 \\
& \downarrow \varphi_Q & & \downarrow \alpha & & \downarrow \varphi_N & & \downarrow \varphi_M & & \downarrow \varphi_{\mathbb{F}^*} & & \downarrow \varphi_{\mathbb{F}^*} & & \downarrow \varphi_{\mathbb{F}^*} & & 0 \\
0 & \rightarrow & \mathbb{F}^*Q & \rightarrow & \mathbb{F}^*M & \rightarrow & \mathbb{F}^*N & \rightarrow & 0
\end{array}
\]

is exact because $N$ is $O_X$-flat by theorem 2.3. Thus $\varphi_Q$ is an isomorphism. Using epi-mono factorization in $\mu(X, \Lambda)$, and the fact that $U(X, \Lambda)$ is closed under cokernels one concludes that $\ker \alpha \in U(X, \Lambda)$ for general $\alpha$. □

If $f : X \rightarrow Y$ is a morphism of schemes then pullback of the underlying $O_X^{\Lambda}$-module defines a functor $f^* : U(X, \Lambda) \rightarrow U(Y, \Lambda)$.

**Theorem 3.2.** Let $f : Y \rightarrow X$ be a morphism of locally noetherian schemes over $\mathbb{F}_q$, $\Lambda$ an algebra essentially of finite type over $\mathbb{F}_q$.

1. The pullback functor $f^* : U(X, \Lambda) \rightarrow U(Y, \Lambda)$ is exact.
2. If $X$ is connected, and $Y \neq \emptyset$ then $f^*$ is faithful, and conservative.

**Proof.** (1) Indeed according to theorem 2.3 the modules in question are $O_X^{\Lambda}$-flat.

(2) If $\mathcal{M} \in U(X, \Lambda)$ then by lemma 1.3 $Z_0(\mathcal{M}) = Z \times_{\mathbb{F}_q} X$ for a certain closed subscheme $Z \subset \text{Spec} \Lambda$. Hence $Z_0(f^*\mathcal{M}) = Z \times_{\mathbb{F}_q} Y$ can be empty if and only if $Z = \emptyset$, i.e. $\mathcal{M} = 0$. □

**Definition 3.3.** Let $X$ be an $\mathbb{F}_q$-scheme, $\Lambda$ an $\mathbb{F}_q$-algebra.

1. The tensor product of two unit modules $(\mathcal{M}, \varphi_M), (\mathcal{N}, \varphi_N) \in U(X, \Lambda)$ is defined as
   \[
   (\mathcal{M}, \varphi_M) \otimes (\mathcal{N}, \varphi_N) = (\mathcal{M} \otimes_{O_X^{\Lambda}} \mathcal{N}, \varphi_M \otimes_{O_X^{\Lambda}} \varphi_N),
   \]
   and similarly for morphisms.
2. The symbol 1 denotes the object $(O_X^{\Lambda}, 1) \in U(X, \Lambda)$.
3. The constraints of associativity, commutativity, and unity for this tensor product are taken from the respective constraints for $O_X^{\Lambda}$-modules.
It is easy to check that the tensor product above provides $U(X, \Lambda)$ with a symmetric monoidal structure.

**Definition 3.4.** Let $X$ be an $\mathbb{F}_q$-scheme, $\Lambda$ an $\mathbb{F}_q$-algebra, and $(\mathcal{M}, \varphi) \in U(X, \Lambda)$. Assume that $\mathcal{M}$ is locally free as an $\mathcal{O}_X^\Lambda$-module.

(1) Let $(\mathcal{M}, \varphi)^\vee$ be the module $(\mathcal{M}^\vee, (\varphi^\vee)^{-1})$ where dualization happens with respect to the $\mathcal{O}_X^\Lambda$-module structure.

(2) The evaluation $\text{ev} : (\mathcal{M}, \varphi) \otimes (\mathcal{M}, \varphi)^\vee \to 1$, and coevaluation $\delta : 1 \to (\mathcal{M}, \varphi)^\vee \otimes (\mathcal{M}, \varphi)$ morphisms are lifted from the corresponding morphisms of locally free $\mathcal{O}_X^\Lambda$-modules.

It is straightforward to check that the evaluation, and coevaluation morphisms of (2) are morphisms of $\mathcal{O}_X^\Lambda$-modules. They satisfy identities (2.1.2) of [3] by construction.

In the case when $\Lambda$ is a field unit modules enjoy an important property.

**Proposition 3.5.** Let $X$ be a locally noetherian scheme over $\mathbb{F}_q$, $\Lambda$ a field containing $\mathbb{F}_q$. Every $M \in U(X, \Lambda)$ is a unit $(\Lambda, F)$-crystal ([4], definition 6.9.1), i.e. it is locally free as an $\mathcal{O}_X^\Lambda$-module.

**Proof.** We can assume $X$ is connected. Since $\Lambda$ is a field lemma [1,3] shows that Fitting subschemes $Z_n(M)$ are either empty or of the form Spec $\Lambda \times \mathbb{F}_q X$. Hence $\mathcal{M}$ is locally free [Stacks 07ZD]. □

Thus the dual objects of definition 3.4 exist for every unit module, and $U(X, \Lambda)$ becomes a rigid monoidal category. Also note that under assumptions of the proposition $\mathcal{O}_X^\Lambda$ is coherent in the sense of [Stacks 01BV] so every $\mathcal{M} \in U(X, \Lambda)$ is a coherent $\mathcal{O}_X^\Lambda$-module.

**Theorem 3.6.** Let $X$ be a connected locally noetherian scheme over $\mathbb{F}_q$, $\Lambda$ a field containing $\mathbb{F}_q$. The category $U(X, \Lambda)$ equipped with the tensor structure above is $\Lambda$-linear tannakian, and the inclusion $U(X, \Lambda) \subset \mu(X, \Lambda)$ is exact. If $X(\mathbb{F}_q) \neq \emptyset$ then $U(X, \Lambda)$ is neutral, i.e. has a fiber functor to $\Lambda$-vector spaces.

A nice feature of this theorem is that it places no restriction on the size of $\Lambda$ as an $\mathbb{F}_q$-algebra. One can take $\Lambda = \mathbb{F}_q((t))$ or the completion of its algebraic closure. Note that abelianness of $U(X, \Lambda)$ does not follow directly from theorem 3.1.

**Proof of theorem 3.6.** We will verify that $(U(X, \Lambda), \otimes)$ is a tensor category as defined in [3], section 2. It is then tannakian since pullback to a point $y \in \text{Spec} \Lambda \times \mathbb{F}_q X$ provides a fiber functor to $k(y)$-vector spaces. If $s : \text{Spec} \mathbb{F}_q \to X$ is a section of the structure morphism $X \to \text{Spec} \mathbb{F}_q$ then the pullback functor $s^* : U(X, \Lambda) \to U(\mathbb{F}_q, \Lambda)$ gives rise to a fiber functor with values in $\Lambda$-vector spaces.

The conditions (2.1.1), (2.1.2) of [3] mean precisely that the category in question is rigid symmetric monoidal which we know from proposition 3.5.
(2.1.3). $U(X, \Lambda)$ is closed under cokernels in $\mu(X, \Lambda)$. Therefore in order to check that it is closed with respect to kernels it is enough to consider a morphism $\alpha: (M, \varphi_M) \rightarrow (N, \varphi_N)$ which is an epimorphism of underlying $\mathcal{O}_X^\Lambda$-modules. Let $(Q, \varphi_Q)$ be the kernel of $\alpha$ computed in $\mu(X, \Lambda)$. The modules $M, N$ are locally free by proposition 3.5. Hence they are locally of finite presentation over $\mathcal{O}_X^\Lambda$, and as a consequence $Q$ is locally of finite type [Stacks 0519]. Since $N$ is $\mathcal{O}_X^\Lambda$-flat it follows that $\varphi_Q$ is an isomorphism.

(2.1.4). We need to check that the natural morphism $\Lambda \rightarrow \text{End}(1)$ is bijective. The endomorphism ring in question consists of $f \in \Gamma(\text{Spec } \Lambda \times_{\mathbb{F}_q} X, \mathcal{O}_X^\Lambda)$ which are invariant under $F$. Since $\Gamma(\text{Spec } \Lambda \times_{\mathbb{F}_q} X, \mathcal{O}_X^\Lambda) = \Lambda \otimes_{\mathbb{F}_q} \Gamma(X, \mathcal{O}_X)$ the claim follows from the theory of Artin-Schreier equation. 

□

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