Generation of fan-states of radiation field in a cavity

Nguyen Ba An\textsuperscript{a,b,*} and Truong Minh Duc\textsuperscript{c}

\textsuperscript{a}Institute of Physics, P.O. Box 429 Bo Ho, Hanoi 10000, Vietnam
\textsuperscript{b}Faculty of Technology, Vietnam National University, 144 Xuan Thuy, Cau Giay, Hanoi, Vietnam
\textsuperscript{c}Physics Department, Hue University, 32 Lo Loi, Hue, Vietnam

A scheme of generating recently introduced fan-states $|\alpha, 2k\rangle_F$ ($\alpha$ is complex, $k = 1, 2, 3, ...$) is proposed basing on a $\Lambda$-type atom-cavity field interaction. We show that with suitable atomic preparations and measurements a passage of a sequence of $N$ atoms through a cavity may transform an initial field coherent state $|\alpha\rangle$ to a fan-state $|\alpha, 2k\rangle_F$ with $k = 2^{N-2}$.

PACS number(s): 42.50.Dv.

1. Introduction

Nonclassical states are numerous though almost all of them are not \textit{a priori} available in the natural world but need be prepared by appropriate experimental arrangements. Generation schemes prove very important because without them a state could be looked upon as artificial only. For example, the well-known squeezed state formulated theoretically in 1970 \textsuperscript{[1]} remained unphysical until 1985 when it was for the first time observed in the laboratory by means of a four-wave mixing technique \textsuperscript{[2]}. Many nonclassical states are superposed of a finite or infinite, discrete or continuous number of classical states. Superposition states may exhibit various intriguing nonclassical properties such as squeezing, antibunching, oscillatory number distribution, self-splitting, etc. thanks to quantum interferences between their component states even though the latter are purely classical. Of special interest are the Schrödinger cats \textsuperscript{[3]} which are composed of a pair of macroscopically distinguishable states. Best documented types of Schrödinger cats are states of the forms $\Psi \sim |\alpha\rangle + \exp(i\phi)\sqrt{2} |\alpha\rangle$, $\Psi \sim |\alpha\rangle - \exp(i\phi)\sqrt{2} |\alpha\rangle$ (where $|\alpha\rangle$ with $\alpha \in \mathbb{C}$ is the coherent state of radiation field) which can be produced by letting a coherent state propagate through an amplitude-dispersive medium \textsuperscript{[3]}, $\Psi \sim |\alpha\rangle \pm |\alpha\rangle$ which are called even/odd coherent state and can be produced via a good deal of mechanisms (by quantum-nondemolition method \textsuperscript{[4]}, using a Mach-Zehnder interferometer \textsuperscript{[5]}, via Kerr nonlinearity \textsuperscript{[6]}, ...), $\Psi \sim |\alpha\rangle - |\alpha\rangle$ (where $\alpha \in \mathbb{C}$). The proper reason for the existence of the Schrödinger cats is that they are superposed of a finite or infinite, discrete or continuous number of classical states. Superposition states may exhibit various intriguing nonclassical properties such as squeezing, antibunching, oscillatory number distribution, self-splitting, etc.

A scheme of generating recently introduced fan-states $|\alpha, 2k\rangle_F$ ($\alpha$ is complex, $k = 1, 2, 3, ...$) is proposed basing on a $\Lambda$-type atom-cavity field interaction. We show that with suitable atomic preparations and measurements a passage of a sequence of $N$ atoms through a cavity may transform an initial field coherent state $|\alpha\rangle$ to a fan-state $|\alpha, 2k\rangle_F$ with $k = 2^{N-2}$.

PACS number(s): 42.50.Dv.

2. Fan-states: definition

Before defining the fan-state let us first mention what is called $K$-photon coherent states \textsuperscript{[11,12]} for $K = 1, 2, 3, ..., K-1$. These are the $K$ right eigen-states, $|\alpha, K, j\rangle$ with $\alpha \in \mathbb{C}$, $K = 1, 2, ..., j = 0, 1, ..., K-1$, of the operator $a^K$ (the photon annihilation operator)

$$a^K |\alpha, K, j\rangle = a^K |\alpha, K, j\rangle. \quad (1)$$

*Corresponding author. Email: nbaan@netnam.org.vn
As was proved in [43–46], for given $K$ and $j$, the state $|\alpha, K, j\rangle$ can be expressed in terms of $K$ usual coherent states as

$$|\alpha, K, j\rangle = N(r^2, K, j) \sum_{l=0}^{K-1} e^{2i\pi j/K} |\alpha_l\rangle, \quad \alpha_l = a e^{2i\pi l/K}$$

(2)

where $N(r^2, K, j)$ is the normalization factor with $r = |\alpha|$. In the special case of $j = 0$ we have from Eq. (2)

$$|\alpha, K\rangle = N(r^2, K) \sum_{q=0}^{K-1} |\alpha_q\rangle, \quad \alpha_q = a e^{2i\pi q/K}$$

(3)

where we have identified $|\alpha, K\rangle$ with $|\alpha, K, 0\rangle$ and $N(r^2, K)$ with $N(r^2, K, 0)$. Unlike states (2), states (3) have all the components $|\alpha_q\rangle$ equally weighted. Thus they are invariant with respect to the rotation $\alpha \to \alpha e^{2i\pi/K}$. With use of the scalar product

$$\langle \beta | \alpha \rangle = \exp \left[ \frac{1}{2} (2\beta^* \alpha - |\alpha|^2 - |\beta|^2) \right]$$

(4)

of coherent states we are able to derive the normalization factor $N(r^2, K)$ for an arbitrary $K$ in the explicit form

$$N(r^2, K) = \left\{ K + 2 \sum_{q=1}^{K-1} q \cos \left[ r^2 \sin(2\pi q/K) \right] \exp \left[ r^2 \cos(2\pi q/K) - 1 \right] \right\}^{-1/2}.$$  

(5)

It is easy to verify [40] that, while the state $|\alpha, K\rangle$ with $K$ odd is neither even nor odd (in the sense of a $\pi$-rotation $\alpha \to -\alpha$), it is even for $K$ even. We shall be interested in even $K$, i.e. $K = 2k$ with $k = 1, 2, \ldots$ and define the fan-state $|\alpha, 2k\rangle_F$ characterized by $k$ in terms of a linear superposition of states $|\alpha, 2k\rangle$ as

$$|\alpha, 2k\rangle_F = N_F(r^2, k) \sum_{p=0}^{2k-1} |\alpha_p, 2k\rangle, \quad \alpha_p = a e^{i\pi p/2k}$$

(6)

where $N_F(r^2, k)$, the normalization factor for the fan-state, is found to be

$$N_F(r^2, k) = \frac{N(r^2, 4k)}{k N(r^2, 2k)}$$

(7)

with the $N'$s given by Eq. (3). The name “fan” comes from the orientation in complex plane of the radius-vectors $\alpha_p = a e^{i\pi p/2k}$ that looks like an open paper fan (see Fig. 2 in [40]). In Figs. 1 and 2 we display the Q-function [17] associated with the fan-state,

$$Q_{k, \alpha}(\xi) = \frac{1}{\pi} |\langle \xi | \alpha, 2k\rangle_F|^2,$$

(8)

and its corresponding contour plot versus $x = \Re(\xi)$ and $y = \Im(\xi)$ for two sets of $k$ and $\alpha$. As seen from these figures, there are peaks and fringes between them. The fringes result from interferences between the peaks and may trigger intriguing nonclassical effects such as presence of simultaneous squeezing along $2k$ directions, as investigated in detail in [41].

3. Hamiltonian and evolution operator

In [27] a method based on a Raman-type (A-configuration) interaction between three-level atoms and radiation field in a lossless cavity was outlined to produce quantum superpositions of coherent states of the field. There was however a shortcoming in [27] (see later) and the treatment there was not delicate enough for a specific purpose. Here we consider again the model in greater detail aiming specifically at preparing the fan-state defined above. We start from the same effective interaction Hamiltonian for the atom-field system as in [27]

$$H_{int} = -\lambda a^+ a W$$

(9)

where the operator $W$ is given by
In Eqs. (9) and (10) \( a \) (\( a^+ \)) stands for the cavity field annihilation (creation) operator, \( \lambda \) for the effective coupling constant and \( |\rangle \) (\( +\rangle \)) for the ground (first excited) state of an atom (the second excited state \( |u\rangle \) of the atoms is inactive). Note that in deriving Eq. (9) the distance from the uppermost level \( |u\rangle \) of the three-level atom to its \(|\pm\rangle\)-levels was assumed well detuned from the cavity mode to adiabatically eliminate \(|u\rangle \) and the matrix elements of the two transitions \(|u\rangle \leftrightarrow |\pm\rangle \) were set equal for simplicity. It is easy to verify that

\[
W^l = 2^{l-1}W
\]  

(11)

for \( l = 1, 2, 3, \ldots \) and, hence,

\[
(H_{int})^l = \frac{1}{2}(-2\lambda a^+ a)^l W
\]  

(12)

leading to the evolution operator \( U(t) \) of the form

\[
U(t) = \exp(-iH_{int}t) = \sum_{l=0}^{\infty} \frac{(-iH_{int})^l}{l!} = 1 + \sum_{l=1}^{\infty} \frac{(-iH_{int})^l}{l!}
\]

\[
= 1 + \frac{1}{2}W \sum_{l=1}^{\infty} \frac{(2i\lambda a^+ a)^l}{l!} - 1
\]

\[
= 1 + \frac{W}{2} \left( e^{2i\lambda a^+ a} - 1 \right).
\]  

(13)

Introducing the dimensionless time \( \tau = 2\lambda t \) gives

\[
U(\tau) = 1 + \frac{W}{2} \left( e^{i\tau a^+ a} - 1 \right).
\]  

(14)

4. Fan-states: generation

Consider a cavity containing initially at \( \tau = \tau_0 = 0 \) the radiation field in a coherent state \( |\alpha \rangle \). We then send the atoms one by one through the cavity. The \( n^{th} \) \((n = 1, 2, \ldots) \) atom prepared in an entangled state \( |F_n\rangle = \xi_n |\rangle \rangle + \eta_n +\rangle \rangle \) enters the cavity at time \( \tau = \sum_{j=1}^{n-1} \tau_j \), spends a duration of \( \tau_n \) in interaction with the cavity field and is detected on its going out from the cavity at time \( \tau = \sum_{j=1}^{n} \tau_j \) in the state \( |S_n\rangle \) which is either \( |\rangle \rangle \) or \( +\rangle \rangle \). Within the model under consideration (the \( \Lambda \)-type configuration) the uppermost level is always empty so that \( \xi_n \) and \( \eta_n \) should satisfy the condition \( |\xi_n|^2 + |\eta_n|^2 = 1 \). The shortcoming in [27] is that the authors anticipated \( |F_n\rangle = \varepsilon_n |\rangle \rangle + +\rangle \rangle \) for which only \( \varepsilon_n = 0 \), \( \forall n \) are allowed and therefore any argumentations made in [27] on a choice of \( \varepsilon_n \) would be nonsensical! The state of the radiation field left inside the cavity after the detection of the \( N^{th} \) atom is

\[
\Phi_{\{F_1, \ldots, F_N\} |S_1, \ldots, S_N\rangle} (\tau_1 + \ldots + \tau_N) = U_{\{F_1, \ldots, F_{N-1}\} |S_1, \ldots, S_{N-1}\rangle} (\tau_1 + \ldots + \tau_{N-1})
\]

\[
= U_{\{F_1, \ldots, F_N\} |S_{N-1}\rangle} U_{\{F_1, \ldots, F_{N-2}\} |S_{N-2}\rangle} (\tau_1 + \ldots + \tau_{N-2})
\]

\[
= \ldots
\]

\[
= U_{\{F_1, \ldots, F_N\} |S_{N-1}\rangle} U_{\{F_1, \ldots, F_2\} |S_2\rangle} (\tau_1) \Phi_{\{F_1\} |S_1\rangle} (\tau_1)
\]

\[
= U_{\{F_1, \ldots, F_N\} |S_{N-1}\rangle} U_{\{F_1, \ldots, F_2\} |S_2\rangle} U_{\{F_1\} |S_1\rangle} (\tau_1) |\alpha\rangle
\]  

(15)

where

\[
U_{\{F_1, \ldots, F_N\} |S_1\rangle} (\tau_1) = \langle s_1 | U(\tau_1) |F_1\rangle
\]  

(16)

with \( s_n = \pm 1 \) if \( S_n = |\pm\rangle \) and \( U(\tau_n) \) given by Eq. (14). Making use of Eqs. (16) and (14) in Eq. (15) yields

\[
\Phi_{\{F_1, \ldots, F_N\} |S_1, \ldots, S_N\rangle} (\tau_1 + \ldots + \tau_N) = \frac{1}{2^N} \left( \prod_{j=1}^{N} s_j (\eta_j - \xi_j) \right) |\alpha\rangle
\]  

(16)
generating this superposition at time \( \tau = \sum_{j=1}^{N} \tau_j \) is

\[
P_{(F_1)\ldots(F_N)}(\tau_1 + \ldots + \tau_N) = \left| \Phi_{|S_1\rangle\ldots|S_N\rangle}^{(F_1)\ldots(F_N)}(\tau_1 + \tau_2)\ldots\Phi_{|S_1\rangle\ldots|S_N\rangle}^{(F_1)\ldots(F_N)}(\tau_1 + \ldots + \tau_N) \right|^2
\]

(18)

because the success of having \( \Phi_{|S_1\rangle\ldots|S_N\rangle}^{(F_1)\ldots(F_N)} \) at \( \tau = \tau_1 + \ldots + \tau_N \) requires the success of having \( \Phi_{|S_1\rangle\ldots|S_{N-1}\rangle}^{(F_1)\ldots(F_{N-1})} \) at \( \tau = \tau_1 + \ldots + \tau_{N-1} \) which in turns requires the success of having \( \Phi_{|S_1\rangle\ldots|S_{N-2}\rangle}^{(F_1)\ldots(F_{N-2})} \) at \( \tau = \tau_1 + \ldots + \tau_{N-2} \) and so on. Specially, if each of the atoms is prepared at its entrance and detected at its exit in the same state, i.e. \( |F_j\rangle \equiv |S_j\rangle \), then (17) simplifies to

\[
\Phi_{|S_1\rangle\ldots|S_N\rangle}^{(F_1)\ldots(F_N)}(\tau_1 + \ldots + \tau_N) = \frac{1}{2^N} \left\{ |\alpha\rangle + \sum_{L=1}^{N-1} \left[ \sum_{L \geq 1}^{N} \left| \alpha e^{i(\tau_1+\ldots+\tau_L)} \right| \right] + \left| \alpha e^{i(\tau_1+\ldots+\tau_N)} \right| \right\}
\]

(19)

in which all the components are equally weighted. Next, if the velocities of the atoms are selected such that \( \tau_j = \pi / 2^{j-1} \) then at time \( \tau = \sum_{j=1}^{N} \tau_j = (2 - 2^{-N})\pi \) the state (13) becomes

\[
\Phi_{|S_1\rangle\ldots|S_N\rangle}^{(2 - 2^{-N})\pi} = \frac{|\alpha, 2^{N-1}\rangle}{2^{N}N(r^2, 2^N)}
\]

(20)

implying that the fan-state characterized by \( k = 2^{N-2} \) can be produced by sending \( N = 2 + \log_2(k) \) atoms (properly prepared, velocity-selected and detected as described above) through the cavity. The total probability of obtaining such fan-states is

\[
P_{2^{N-2}} = \prod_{n=1}^{N} \frac{1}{2^{2n}N(r^2, 2^n)}
\]

(21)

For clarity we now demonstrate the proposed scheme for generating the fan-state with \( k = 1 \). Obviously, the number of atoms to be sent is \( N = 2 \). The initial state of the atom-field system is factorized as \( \Psi(0) = |F_1\rangle \otimes |\alpha\rangle \). Due to the atom-field interaction inside the cavity, at time \( \tau_1 \) the atom-field system becomes entangled whose state is described by

\[
\Psi(\tau_1) = U(\tau_1) |F_1\rangle \otimes |\alpha\rangle = \frac{1}{2} \left\{ \left[ (\xi_1 - \eta_1) |\alpha\rangle + (\xi_1 + \eta_1) |\alpha e^{i\tau_1}\rangle \right] |-\rangle \\
+ \left[ (\eta_1 - \xi_1) |\alpha\rangle + (\xi_1 + \eta_1) |\alpha e^{i\tau_1}\rangle \right] |+\rangle \right\}
\]

(22)

which projects the field on the state

\[
\Phi_{|-\rangle}^{(F_1)}(\tau_1) = \frac{1}{2} \left[ (\xi_1 - \eta_1) |\alpha\rangle + (\xi_1 + \eta_1) |\alpha e^{i\tau_1}\rangle \right]
\]

(23)

if the atom is detected in the ground state \( |S_1\rangle = |-\rangle \), or

\[
\Phi_{|+\rangle}^{(F_1)}(\tau_1) = \frac{1}{2} \left[ (\eta_1 - \xi_1) |\alpha\rangle + (\xi_1 + \eta_1) |\alpha e^{i\tau_1}\rangle \right]
\]

(24)

if the atom is detected in the excited state \( |S_1\rangle = |+\rangle \). The two formulae (23) and (24) can be unified into a single one as
\[ \Phi_{|S_1\rangle}^{(F_1)}(\tau_1) = \frac{1}{2} \left[ s_1 (\eta_1 - \xi_1) |\alpha\rangle + (\eta_1 + \xi_1) |\alpha e^{i\tau_1}\rangle \right] = U_{|S_1\rangle}^{(F_1)}(\tau_1) |\alpha\rangle \]  

(25)

with \( U_{|S_1\rangle}^{(F_1)} \) defined by Eq. (10). The probability of finding states (25) at \( \tau_1 \) is

\[ P_{|S_1\rangle}^{(F_1)}(\tau_1) = \frac{1}{2} \left\{ 1 + s_1 e^{r^2(1-\cos \tau_1)} \left[ (|\eta_1|^2 - |\xi_1|^2) \cos (r^2 \sin \tau_1) - 2 \Im (\xi_1 \eta_1) \sin (r^2 \sin \tau_1) \right] \right\}. \]

(26)

States (25) generally represent superpositions of two arbitrary coherent states whose weights are controlled by \( \xi_1, \eta_1 \) and whose relative phases by the interaction duration \( \tau_1 \). If we prepare the atom initially in the ground state \( |F_1\rangle = |\alpha\rangle \), i.e. \( \eta_1 = 0 (\xi_1 = 1) \), we success in having the field states

\[ \Phi_{|\pm\rangle}^{(-)}(\tau_1) = \frac{1}{2} \left( |\alpha\rangle + |\alpha e^{i\tau_1}\rangle \right), \]

(27)

\[ \Phi_{|\pm\rangle}^{(+)\langle-\rangle}(\tau_1) = \frac{1}{2} \left( |\alpha\rangle + |\alpha e^{i\tau_1}\rangle \right) \]

(28)

with the probabilities

\[ P_{|\pm\rangle}^{(\pm\langle\mp\rangle)}(\tau_1) = \frac{1}{2} \left[ 1 \mp e^{r^2(1-\cos \tau_1)} \cos (r^2 \sin \tau_1) \right]. \]

(29)

Alternatively, if we prepare the atom initially in the excited state \( |F_1\rangle = |\alpha\rangle \), i.e. \( \eta_1 = 1 (\xi_1 = 0) \), we success in having the field states

\[ \Phi_{|\pm\rangle}^{(+)}(\tau_1) = \frac{1}{2} \left( -|\alpha\rangle + |\alpha e^{i\tau_1}\rangle \right), \]

(30)

\[ \Phi_{|\pm\rangle}^{(\mp\langle\pm\rangle)}(\tau_1) = \frac{1}{2} \left( |\alpha\rangle + |\alpha e^{i\tau_1}\rangle \right) \]

(31)

with the probabilities

\[ P_{|\pm\rangle}^{(\pm\langle\mp\rangle)}(\tau_1) = \frac{1}{2} \left[ 1 \mp e^{r^2(1-\cos \tau_1)} \cos (r^2 \sin \tau_1) \right]. \]

(32)

From Eqs. (29) and (31) we realize that

\[ P_{|\pm\rangle}^{(\pm\langle\mp\rangle)}(\tau_1) = P_{|\pm\rangle}^{(\mp\langle\pm\rangle)}(\tau_1) \text{ and } P_{|\pm\rangle}^{(\pm\langle\mp\rangle)}(\tau_1) = P_{|\pm\rangle}^{(\mp\langle\pm\rangle)}(\tau_1). \]

(33)

Figure 4 plots \( P_{|\pm\rangle}^{(\pm\langle\mp\rangle)}(\tau_1) \) versus \( \tau_1 \) for several values of \( r \). Transparently, at \( \tau_1 = \pi \) (or 3\( \pi \), 5\( \pi \),...), the states \( \Phi_{|\pm\rangle}^{(\pm\langle\mp\rangle)} \) (\( \Phi_{|\mp\rangle}^{(\pm\langle\mp\rangle)} \)) are proportional to an even (odd) coherent state. Even coherent states are necessary precursors for generating fan-states in next steps.

To generate fan-states with \( k = 1 \) sending a second atom at time \( \tau_1 \) is needed. Let the initial state of the second atom be \( |F_2\rangle \). After an interaction duration \( \tau_2 \), when the second atom is detected at its exit in state \( |S_2\rangle \), the field will be projected on the state

\[ \Phi_{|S_1\rangle|S_2\rangle}^{(F_1|F_2\rangle}(\tau_1 + \tau_2) = \frac{1}{4} \left[ s_1 s_2 (\eta_1 - \xi_1) (\eta_2 - \xi_2) |\alpha\rangle + s_2 (\eta_1 + \xi_1) (\eta_2 - \xi_2) |\alpha e^{i\tau_1}\rangle + s_1 (\eta_1 - \xi_1) (\eta_2 + \xi_2) |\alpha e^{i\tau_2}\rangle + (\eta_1 + \xi_1) (\eta_2 + \xi_2) |\alpha e^{i(\tau_1 + \tau_2)}\rangle \right] \]

(34)

which can be formulated in the form

\[ \Phi_{|S_1\rangle|S_2\rangle}^{(F_1|F_2\rangle}(\tau_1 + \tau_2) = U_{|S_2\rangle}^{(F_2\rangle}(\tau_2) \Phi_{|S_1\rangle}^{(F_1\rangle}(\tau_1). \]

(35)
The final probability of having $\Phi^{F_1,F_2}_{S_1,S_2}(\tau_1 + \tau_2)$ is

$$P^{F_1,F_2}_{S_1,S_2}(\tau_1 + \tau_2) = \left| \Phi^{F_1}_{S_1} (\tau_1) \Phi^{F_2}_{S_1} (\tau_1 + \tau_2) \right|^2.$$  

(36)

For $\tau_1 = \tau_2$ the state (43) reduces to a superposition of three coherent states whose weights are always different. Nevertheless, for $\tau_1 \neq \tau_2$, it is superposed of four coherent states whose weights may be made equal by suitably preparing the atomic states at the entrance to the cavity. Four particular situations are of interest.

i) Both the first and second atoms are prepared at their entrance in the excited state $|F_1\rangle = |F_2\rangle = |+\rangle$, i.e. $\eta_1 = \eta_2 = 1$ ($\xi_1 = \xi_2 = 0$), leading to

$$\Phi^{[+]|+}_{[+]|+} (\tau_1 + \tau_2) = \frac{1}{4} \left( |\alpha\rangle + |\alpha e^{i\tau_1}\rangle + |\alpha e^{i\tau_2}\rangle + |\alpha e^{i(\tau_1+\tau_2)}\rangle \right),$$

(37)

$$\Phi^{[+]|+}_{[+]|+} (\tau_1 + \tau_2) = \frac{1}{4} \left( |\alpha\rangle - |\alpha e^{i\tau_1}\rangle - |\alpha e^{i\tau_2}\rangle - |\alpha e^{i(\tau_1+\tau_2)}\rangle \right),$$

(38)

$$\Phi^{[+]|+}_{[+]|+} (\tau_1 + \tau_2) = \frac{1}{4} \left( -|\alpha\rangle + |\alpha e^{i\tau_1}\rangle - |\alpha e^{i\tau_2}\rangle + |\alpha e^{i(\tau_1+\tau_2)}\rangle \right),$$

(39)

$$\Phi^{[+]|+}_{[+]|+} (\tau_1 + \tau_2) = \frac{1}{4} \left( |\alpha\rangle - |\alpha e^{i\tau_1}\rangle - |\alpha e^{i\tau_2}\rangle - |\alpha e^{i(\tau_1+\tau_2)}\rangle \right).$$

(40)

ii) The first (second) atom is prepared at its entrance in its excited $|F_1\rangle = |+\rangle$, i.e. $\eta_1 = 1$ ($\xi_1 = 0$) (ground $|F_2\rangle = |\rangle$, i.e. $\eta_2 = 0$ ($\xi_2 = 1$)) state, leading to

$$\Phi^{[+]|+}_{[+]|+} (\tau_1 + \tau_2) = \frac{1}{4} \left( -|\alpha\rangle - |\alpha e^{i\tau_1}\rangle + |\alpha e^{i\tau_2}\rangle + |\alpha e^{i(\tau_1+\tau_2)}\rangle \right),$$

(41)

$$\Phi^{[+]|+}_{[+]|+} (\tau_1 + \tau_2) = \frac{1}{4} \left( |\alpha\rangle + |\alpha e^{i\tau_1}\rangle - |\alpha e^{i\tau_2}\rangle - |\alpha e^{i(\tau_1+\tau_2)}\rangle \right),$$

(42)

$$\Phi^{[+]|+}_{[+]|+} (\tau_1 + \tau_2) = \frac{1}{4} \left( -|\alpha\rangle + |\alpha e^{i\tau_1}\rangle - |\alpha e^{i\tau_2}\rangle + |\alpha e^{i(\tau_1+\tau_2)}\rangle \right),$$

(43)

$$\Phi^{[+]|+}_{[+]|+} (\tau_1 + \tau_2) = \frac{1}{4} \left( |\alpha\rangle - |\alpha e^{i\tau_1}\rangle + |\alpha e^{i\tau_2}\rangle - |\alpha e^{i(\tau_1+\tau_2)}\rangle \right).$$

(44)

iii) The first (second) atom is prepared at its entrance in its ground $|F_1\rangle = |\rangle$, i.e. $\eta_1 = 0$ ($\xi_1 = 1$) (excited $|F_2\rangle = |+\rangle$, i.e. $\eta_2 = 1$ ($\xi_2 = 0$)) state, leading to

$$\Phi^{[+]|+}_{[+]|+} (\tau_1 + \tau_2) = \frac{1}{4} \left( |\alpha\rangle - |\alpha e^{i\tau_1}\rangle - |\alpha e^{i\tau_2}\rangle + |\alpha e^{i(\tau_1+\tau_2)}\rangle \right),$$

(45)

$$\Phi^{[+]|+}_{[+]|+} (\tau_1 + \tau_2) = \frac{1}{4} \left( |\alpha\rangle - |\alpha e^{i\tau_1}\rangle - |\alpha e^{i\tau_2}\rangle + |\alpha e^{i(\tau_1+\tau_2)}\rangle \right),$$

(46)

$$\Phi^{[+]|+}_{[+]|+} (\tau_1 + \tau_2) = \frac{1}{4} \left( -|\alpha\rangle + |\alpha e^{i\tau_1}\rangle + |\alpha e^{i\tau_2}\rangle + |\alpha e^{i(\tau_1+\tau_2)}\rangle \right),$$

(47)

$$\Phi^{[+]|+}_{[+]|+} (\tau_1 + \tau_2) = \frac{1}{4} \left( -|\alpha\rangle - |\alpha e^{i\tau_1}\rangle + |\alpha e^{i\tau_2}\rangle - |\alpha e^{i(\tau_1+\tau_2)}\rangle \right).$$

(48)

iv) Both the first and second atoms are prepared at their entrance in their ground state $|F_1\rangle = |F_2\rangle = |\rangle$, i.e. $\eta_1 = \eta_2 = 0$ ($\xi_1 = \xi_2 = 1$), leading to
\[ \Phi_{\{\pm\}+}(\tau_1 + \tau_2) = \frac{1}{4} \left( |\alpha\rangle - |ae^{i\tau_1}\rangle - |ae^{i\tau_2}\rangle + |ae^{i(\tau_1+\tau_2)}\rangle \right), \]  
(49)

\[ \Phi_{\{\pm\}+}(\tau_1 + \tau_2) = \frac{1}{4} \left( -|\alpha\rangle + |ae^{i\tau_1}\rangle - |ae^{i\tau_2}\rangle + |ae^{i(\tau_1+\tau_2)}\rangle \right), \]  
(50)

\[ \Phi_{\{\pm\}+}(\tau_1 + \tau_2) = \frac{1}{4} \left( -|\alpha\rangle - |ae^{i\tau_1}\rangle + |ae^{i\tau_2}\rangle + |ae^{i(\tau_1+\tau_2)}\rangle \right), \]  
(51)

\[ \Phi_{\{\pm\}+}(\tau_1 + \tau_2) = \frac{1}{4} \left( |\alpha\rangle + |ae^{i\tau_1}\rangle + |ae^{i\tau_2}\rangle + |ae^{i(\tau_1+\tau_2)}\rangle \right). \]  
(52)

It follows directly from above that the four component coherent states contribute equally in four situations implied by

\[ \Phi_{\{\pm\}+} = \Phi_{\{\pm\}+} = \Phi_{\{\pm\}+} = \Phi_{\{\pm\}+}. \]  
(53)

These mean that if each of the two atoms is prepared at its entrance and detected at its exit in the same pure state, i.e. \( |F_j\rangle = |S_j\rangle \), then the field reduces to a superposition of four equally weighted coherent states, i.e.

\[ \Phi_{|S_1\rangle|S_2\rangle}(\tau_1 + \tau_2) = \frac{1}{4} \left( |\alpha\rangle + |ae^{i\tau_1}\rangle + |ae^{i\tau_2}\rangle + |ae^{i(\tau_1+\tau_2)}\rangle \right), \]  
(54)

no matter \( |S_{1,2}\rangle \) is \( |\rangle \) or \( |_\rangle \). For atomic velocity selections such that \( \tau_1 = \pi / 2 \) and \( \tau_2 = \pi / 2 \) (\( \pi \)) the superposition \( \Phi\) at \( \tau = \tau_1 + \tau_2 = 3\pi / 2 \) becomes

\[ \Phi_{|S_1\rangle|S_2\rangle}(3\pi / 2) = \frac{1}{4} \left( |\alpha\rangle + |\alpha\rangle + |\alpha\rangle + |\alpha\rangle \right) = \frac{1}{4\sqrt{r^2}} \]  
(55)

which is proportional to the simplest fan-state characterized by \( k = 1 \). The final probability \( P_1 \) of observing the \( k = 1 \) fan-state depends on \( r \),

\[ P_1(r) = \frac{1}{8} \left( 1 + e^{-2r^2} + 2e^{-r^2 \cos r^2} \right) \times \begin{cases} \frac{1}{2} e^{-2r^2} \text{ for } \tau_1 = \pi, \tau_2 = \pi / 2, \\ 1 + e^{-r^2 \cos r^2} \text{ for } \tau_1 = \pi / 2, \tau_2 = \pi, \end{cases} \]  
(56)

as depicted in Fig. 5.

Similarly, we can send a third atom to generate the \( k = 2 \) fan-state with a certain probability. In this way we find out that fan-states with \( k = 2^N - 2 \) (\( N = 2, 3, 4, \ldots \) is the number of atoms to be sent) can be generated provided the atoms be appropriately prepared, velocity-selected and detected as described above. Figure 6 illustrates the \( r \)-dependent probability of finding the fan-state \( |\alpha, 2k\rangle_F \) for several values of \( k \) at time \( \tau = (2 - 2^{1-N}) \pi \) when the duration the \( j \text{th} \) atom spends inside the cavity is controlled so that \( \tau_j = \pi / 2^{j-1} \). This figure shows that the greater the value of \( k \) the smaller the generation probability and a finite probability maintains for all \( k \) in the small-\( r \) domain where field amplitude squeezing is favorable as revealed in \( 40 \).

5. Conclusion

To sum up we have developed further the method proposed in \( 27 \) to generate the fan-state in which multi-directional higher-order amplitude squeezing is possible. We have shown that by sending a sequence of \( N \geq 2 \) atoms through a cavity initially containing radiation field in a coherent state we are able to generate fan-states \( |\alpha, 2k\rangle_F \) with \( k = 2^N - 2 \). For the success each of the atoms should enter and go out from the cavity in the same state (either \( |\rangle \) or \( |\rangle \)). Moreover, the atoms should be velocity-selected so that the \( n \text{th} \) atom spends the duration of \( \tau_n = \pi / 2^{n-1} \) in interaction with the cavity field. Our analysis has also revealed that the probability of success decreases with increasing \( k \) and \( |\alpha| \). Of interest has been the fact that the fan-state generation probability is finite for small values of \( |\alpha| \) for which amplitude squeezing always exists \( 40 \).

Acknowledgments

This work was in part supported by the National Basic Research Program KT-04.1.2 and the Faculty of Technology of VNU.
[1] D. Stoler, Phys. Rev. D 1, 3217 (1970).
[2] R. E. Slusher et al., Phys. Rev. Lett. 55, 2409 (1985).
[3] E. Schrodinger, Naturwissenschaften 23, 812 (1935).
[4] B. Yurke and D. Stoler, Phys. Rev. Lett. 57, 13 (1986).
[5] M. Brune et al., Phys. Rev. A 45, 5193 (1992).
[6] C. C. Gerry, Opt. Commun. 91, 247 (1992).
[7] C. C. Gerry and E. E. Hach III, Phys. Lett. A 174, 185 (1993).
[8] S-D Du et al., Opt. Commun. 138, 193 (1997).
[9] A. Napoli and A. Messina, J. Opt. B: Quantum Semiclass. Opt. 2, 282 (2000).
[10] R. L. de Matos Filho and W. Vogel, Phys. Rev. Lett. 76, 608 (1996).
[11] S-C. Gou, J. Steinbach and P. L. Knight, Phys. Rev. A 54, 4315 (1996).
[12] C. C. Gerry, Phys. Rev. A 55, 2478 (1997).
[13] B. Roy, Phys. Lett. A 249, 25 (1998).
[14] S. Sivakumar, Phys. Lett. A 250, 257 (1998).
[15] S-B Zheng, Phys. Rev. A 58, 761 (1998).
[16] S-B Zheng, Uer. Phys. J. D 1, 105 (1998).
[17] L-X Li and G-C Guo, J. Opt. B: Quantum Semiclass. Opt. 1,339 (1999).
[18] X-M Liu, Phys. Lett. A 279, 123 (2001).
[19] M. Feng, quant-ph/0103112 (2001).
[20] G. S. Agarwal, R. R. Puri and R. P. Singh, Phys. Rev. A 56, 2249 (1997).
[21] C. C. Gerry and R. Grobe, Phys. Rev. A 56, 2390 (1997).
[22] C. C. Gerry and R. Grobe, Phys. Rev. A 57, 2247 (1998).
[23] E. Solano, R. L. de Matos Filho and N. Zagury, quant-ph/0101056 (2001).
[24] K. Vogel, V. M. Akulin and W. P. Schleich, Phys. Rev. Lett. 71, 1816 (1993).
[25] V. Buzek and P. L. Knight, Progress in Optics XXXIV, 1 (1995).
[26] C. C. Law and J. H. Eberly, Phys. Rev. Lett. 76, 1055 (1996).
[27] S-B Zheng and G-C Guo, Quantum Semiclass. Opt. 9, L45 (1997).
[28] B. Kneer and C. K. Law, Phys. Rev. A 57, 2096 (1998).
[29] H. Moya-Cessa, S. Wallentowitz and W. Vogel, Phys. Rev. A 59, 2920 (1999).
[30] H-W Lee, J. Mod. Opt. 40, 1081 (1993).
[31] S-B Zheng, Opt. Commun. 154, 290 (1998).
[32] J. Janszky, P. Domokos and P. Adam, Phys. Rev. A 48, 2213 (1993).
[33] P. Domokos, J. Janszky, P. Adam and T. Larsen, Quantum Opt. 6, 187 (1994).
[34] M. J. Gagen, Phys. Rev. A 51, 2715 (1995).
[35] S. Szabo, P. Adam, J. Janszky and P. Domokos, Phys. Rev. A 53, 2698 (1996).
[36] R. Ragi, B. Baseia and S. S. Mizrahi, J. Opt. B: Quantum Semiclass. Opt. 2, 299 (2000).
[37] W. D. Jose and S. S. Mizrahi, J. Opt. B: Quantum Semiclass. Opt. 2, 306 (2000).
[38] A. L. S. Silva, W. D. Jose, V. V. Dodonov and S. S. Mizrahi, Phys. Lett. A 282, 235 (2001).
[39] V. I. Manko et al., Physica Scripta 55, 528 (1997).
[40] Nguyen Ba An, Phys. Lett. A 284, 72 (2001).
[41] V. Buzek, I. Jex and T. Quang, J. Mod. Opt. 37, 159 (1990).
[42] J. Sun, J. Wang and C. Wang, Phys. Rev. A 44, 3369 (1991).
[43] J. Sun, J. Wang and C. Wang, Phys. Rev. A 46, 1700 (1992).
[44] X-M Liu, J. Phys. A: Math. Gen. 32, 8685 (1999).
[45] V. Manko et al., Phys. Rev. A 62, 0053407 (2000).
[46] Nguyen Ba An, Chinese J. Phys. 39, in print (2001).
[47] C. L. Melta and E. C. G. Shudarshan, Phys. Rev. 138, B274 (1965).
Figure captions

Fig. 1: Distribution function \( \pi Q_{k,\alpha}(\xi) \) versus \( x = \Re(\xi) \) and \( y = \Im(\xi) \) for \( k = 1 \) and \( |\alpha| = 2 \) (left) and its corresponding contour plot (right).

Fig. 2: As in Fig. 1 but for \( k = 2 \) and \( |\alpha| = 3.5 \).

Fig. 3: The time-dependent probability \( P \equiv P_{|\pm\rangle}^{|\pm\rangle} \) (dashed curves) and \( P_{|\mp\rangle}^{|\mp\rangle} \) (solid curves) for \( r = 0.5 \) (top), \( r = 1.0 \) (middle) and \( r = 5.0 \) (bottom). Here “Time” denotes the dimensionless time \( \tau_1 \) defined in the text.

Fig. 4: The \( r \)-dependent probability \( P_1 \) of obtaining the \( k = 1 \) fan-state for \( \tau_1 = \pi, \tau_2 = \pi/2 \) (solid curve) and \( \tau_1 = \pi/2, \tau_2 = \pi \) (dashed curve).

Fig. 5: The \( r \)-dependent probability \( P_k \) of obtaining the fan-state characterized by \( k = 2^{N-2} \) for \( \tau_j = \pi/2^{j-1} \) with \( j = 1, 2, ..., N \). The used \( k \)-values (\( N \)-values) are \( k = 1, 2, 4 \) and \( 8 \) (\( N = 2, 3, 4 \) and \( 5 \)) downwards.