THREE-VERTEX PRIME GRAPHS AND REALITY OF TREES

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Abstract. We continue the study of prime simple modules for quantum affine algebras from the perspective of $q$-factorization graphs. In this paper we establish several properties related to simple modules whose $q$-factorization graphs are afforded by trees. The two most important of them are proved for type $A$. The first completes the classification of the prime simple modules with three $q$-factors by giving a precise criterion for the primality of a 3-vertex line which is not totally ordered. Using a very special case of this criterion, we then show that a simple module whose $q$-factorization graph is afforded by an arbitrary tree is real. Indeed, the proof of the latter works for all types, provided the aforementioned special case is settled in general.

1. Introduction

This paper is a continuation of [10], where we introduced the combinatorial notion of a $q$-factorization graph intended as a tool to study and express results related to the classification of prime simple modules for quantum affine algebras. While the main result of [10] focused on graphs whose vertex sets are totally ordered, the present paper goes in the opposite direction. Most of this paper is dedicated to proving a first step towards describing a set of necessary and sufficient conditions for the primality of simple modules whose $q$-factorization graph is afforded by a tree. The only type of tree which is also totally ordered are the monotonic lines

\[ \circ \rightarrow \circ \rightarrow \cdots \]

which are prime by [10, Theorem 3.5.4]. This first step is Theorem 2.4.6 which completes the proposed goal in the case that the underlying simple Lie algebra is of type $A$ and the $q$-factorization graph has three vertices. Its proof occupies about half of this paper. The present argument can be used to study all other types, but it heavily explores the precise description of the sets $R_{1,3}^{r,s}$ which, by definition, encode the answer for graphs with two vertices (their definition is reviewed in Section 2.2). Hence, the study of other types using similar arguments requires a case-by-case analysis.

In the seminal paper [7], Hernandez and Leclerc showed that the finite-dimensional representation theory of quantum affine algebras and cluster algebras are deeply interconnected. In particular, the notion of cluster variables gave rise to the notion of real prime simple modules. A real module is a simple module whose tensor square is also simple. Thus, beside classifying the prime simple modules, another important task is that of classifying the real ones. As an application of a very particular case of Theorem 2.4.6 we prove Theorem 2.4.8 that shows that every simple module whose $q$-factorization graph is afforded by a tree is real. This particular case (Corollary 2.4.9) is expected to be true for all types and can possibly be proved in a simpler manner with different techniques than those employed here (we have used no cluster algebra argument and made very rudimentary use of $q$-characters). If that is done, the present proof of Theorem 2.4.8 works, exactly as written here, for all types.

Beside these two main results, we also prove a few general criteria for deciding the reducibility or simplicity of a tensor product of two simple modules for which the $q$-factorization graph of the product of the two Drinfeld polynomials is a tree, such as Propositions 2.4.3 and 2.4.4. We also give a few illustrative examples of the usability of the several criteria we have established here as well as in [10, Section 4]. For instance, it was shown in [10] that every connected subgraph of a prime tree is also prime. The converse of this is false and, in Example 2.5.1, we give an example

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of a non-prime tree with four vertices, all of whose proper connected subgraphs are prime. On the other hand, in Example 2.5.2 we give an example of a q-factorization graph afforded by a 4-cycle which is prime but contains connected three-vertex subgraphs which are not prime. Note that we are able to check the primality of the graphs in these examples using our set of criteria, even though they are not covered by the main results from [10] nor by the ones from the present paper.

The paper is organized as follows. In Section 2 we review the basic terminology and notation, state the main results, and develop the aforementioned examples. In Section 3 we review some further background and prove Propositions 2.4.3 and 2.4.4. Section 4 is dedicated to the proofs of Theorems 2.4.6 and 2.4.8 starting with a brief review of short exact sequences describing tensor products of fundamental modules for type A in Section 4.1 (the only part of the paper using q-characters). Section 4.2 prepares the ground for the proof of Theorem 2.4.6 starting from a broader setup (for all types) by discussing a few initial steps for obtaining reducibility criteria for the tensor product of a Kirillov-Reshetikhin module with a general simple module. Lemmas 4.2.1 and 4.2.2 give preliminary answers in this direction. The lengthy Section 4.3 contains the proof of Theorem 2.4.6, while Theorem 2.4.8 is proved in the final subsection.

2. Basic Background and Main Results

Throughout the paper, let \( \mathbb{C} \) and \( \mathbb{Z} \) denote the sets of complex numbers and integers, respectively. Let also \( \mathbb{Z}_{\geq m}, \mathbb{Z}_{< m} \), etc. denote the obvious subsets of \( \mathbb{Z} \). Given a ring \( \mathbb{A} \), the underlying multiplicative group of units is denoted by \( \mathbb{A}^\times \). The symbol \( \cong \) means “isomorphic to”. We shall use the symbol \( \diamond \) to mark the end of remarks, examples, and statements of results whose proofs are postponed. The symbol \( \square \) will mark the end of proofs as well as of statements whose proofs are omitted.

2.1. Quantum Algebras and Their Finite-Dimensional Representations. Since this paper should be regarded as a continuation of [10], in order to avoid excessive repetition of background, we kindly ask the reader to refer to Sections 2.3, 2.4, and 3.1 of that paper. Let us fix some additional notation. If \( J \subseteq I \), we let \( w_0^{J} \) denote the longest element of the Weyl group of the diagram subalgebra \( g_J \subseteq g \), regarded as a subgroup of \( \mathcal{W} \) in the obvious way. Moreover, we introduce the following notation which will be often used in the main proof of this paper. Given \( i, j, k \in I \), set

\[
d_{i,j}^k = \frac{d(k,i) + d(k,j) - d(i,j)}{2}
\]

and note

\[
d_{i,j}^k = \begin{cases} 
0, & \text{if } k \in [i,j]; \\
d(k,i), & \text{if } i \in [k,j]; \\
d(k,j), & \text{if } j \in [k,i]. 
\end{cases}
\]

Also,

\[
d_{i,j}^k \leq \min\{d(k,i), d(k,j)\} \quad \text{and} \quad d_{i,j}^k + d_{k,j}^k = d(k,i).
\]

2.2. Simple Prime Modules and q-Factors. Although this subsection is contained in [10, Section 3.3], we chose to partially reproduce it here for easy of referencing later on.

Given \((i,r), (j,s) \in I \times \mathbb{Z}_{>0}\), there exists a finite set \( \mathcal{R}_{i,j}^{r,s} \subseteq \mathbb{Z}_{>0} \) such that

\[
L_q(\omega_{i,a,r}) \otimes L_q(\omega_{j,b,s}) \text{ is reducible} \iff \frac{a}{b} = q^{|m|} \text{ with } |m| \in \mathcal{R}_{i,j}^{r,s}.
\]

Moreover, in that case,

\[
L_q(\omega_{i,a,r}) \otimes L_q(\omega_{j,b,s}) \text{ is highest-} \ell \text{-weight} \iff m > 0.
\]

If \( g \) is of type A and \( i, j \in I, r, s \in \mathbb{Z}_{>0} \), we have

\[
\mathcal{R}_{i,j}^{r,s} = \{ r + s + d(i,j) - 2p : -d([i,j], \partial I) \leq p < \min\{r,s\} \}.
\]
Given a connected subdiagram $J$ such that $[i, j] \subseteq J$, let $R_{i, j, J}^{r, s}$ be determined by
\[
V_q((\omega_{i, a, r}, J)) \otimes V_q((\omega_{j, b, s}, J)) \text{ is reducible } \iff \frac{a}{b} = q^n \text{ with } |m| \in R_{i, j, J}^{r, s}.
\]
Corollary 3.2.4 below implies
\[
(2.2.4) \quad R_{i, j, J}^{r, s} \subseteq R_{i, j, K}^{r, s} \quad \text{if} \quad J \subseteq K.
\]
Also, setting
\[
(2.2.5) \quad R_i^{r, s} = R_{i, i, \{1\}},
\]
we have
\[
(2.2.6) \quad R_i^{r, s} = \{d_i(r + s - 2p) : 0 \leq p < \min\{r, s\}\} \text{ for every } i \in I, \ r, s \in \mathbb{Z}_{>0}.
\]

**Proposition 2.2.1 ([10] Proposition 3.3.4).** If $\pi, \omega \in \mathcal{P}^+$ are such that $L_q(\pi) \otimes L_q(\omega)$ is simple, then they have dissociate $q$-factorizations. \(\square\)

### 2.3. Factorization Graphs

We again ask the reader to refer to Sections 2.1 and 2.2 of [10] for the basic notation regarding graph theory in order to avoid excessive repetition of background. For easy of referencing, we recall the main parts of [10], Sections 2.5 and 3.4, where the notion of $q$-factorization graphs was introduced.

A pre-factorization graph is a directed graph $G$ equipped with three maps
\[
c : V \to I, \quad \lambda : V \to \mathbb{Z}_{>0}, \quad \epsilon : A \to \mathbb{Z}_{>0},
\]
called the coloring, the weight, and the exponent, respectively, and $\epsilon$ satisfies the following compatibility condition:
\[
(2.3.1) \quad \epsilon_{\rho} = \epsilon_{\rho'} \quad \text{for all} \quad \rho, \rho' \in \mathcal{P}_{v,v'}, v, v' \in V.
\]
Here, $\mathcal{P}_{v,v'}$ is the set of paths from $v$ to $v'$ and
\[
\epsilon_{\rho} = \sum_{j=1}^{m} s_j \epsilon(a_j),
\]
where $\rho = e_1 \cdots e_m$ has signature $\sigma_{\rho} = (s_1, \ldots, s_m)$ and $e_j$ is the edge associated to the arrow $a_j$. Set also
\[
(2.3.2) \quad \mathcal{P}_G^+ = \{\rho \in \mathcal{P}_G : \epsilon_{\rho} > 0\} \quad \text{and} \quad \mathcal{P}_G^- = \{\rho \in \mathcal{P}_G : \epsilon_{\rho} < 0\}.
\]
The structures maps of a pre-factorization graph can be locally represented by a picture of the form $\overset{r}{i} \overset{m}{\longrightarrow} \overset{s}{j}$ where $i$ and $j$ are the colors at the corresponding vertices, $r$ and $s$ are their associated weights and $m$ is the exponent associated to the given arrow. Condition (2.3.1) implies a pre-factorization graph contains no oriented cycles. In particular, the set $A$ induces a partial order on $V$ by the transitive extension of the strict relation
\[
h_a < t_a \quad \text{for} \quad a \in A.
\]
Recall (2.2.1) and (2.2.5). A pre-factorization graph $G$ is said to be a $q$-factorization graph if, for every $i \in I$,
\[
(2.3.3) \quad v, v' \in V_i, \ \rho \in \mathcal{P}_{v,v'} \quad \Rightarrow \quad \epsilon_{\rho} \notin \mathcal{P}_i^{\lambda(v), \lambda(v')}
\]
and
\[
(2.3.4) \quad \rho \in \mathcal{P}_{v,v'} \cap \mathcal{P}_G^+ \quad \text{with} \quad \epsilon_{\rho} \in \mathcal{P}_i^{\lambda(v), \lambda(v')} \quad \Rightarrow \quad \rho = \rho_{v', v} \in A.
\]
We refer to a pre-factorization graph satisfying (2.3.4) as a pseudo $q$-factorization graph.

If $G$ is a connected pre-factorization graph, for each choice of $(v_0, a) \in V \times \mathbb{R}^+$, we can associate a Drinfeld polynomial by
\[
(2.3.5) \quad \pi_{G,v_0,a} = \prod_{v \in V} \omega_{c(v), a_v, \lambda(v)},
\]
where $a_{\nu\upsilon} = a$ and $a_{\nu} = aq^{\rho}$ if $\rho \in \mathcal{P}_{\upsilon,\nu}$. Condition (2.3.1) guarantees this is well-defined. Conversely, any pseudo q-factorization of a Drinfeld polynomial $\pi$ gives rise to a pseudo q-factorization graph which is a q-factorization graph if and only if it is the q-factorization of $\pi$. The vertex set $\mathcal{V}$ is the multiset of (pseudo) q-factors, the coloring is determined by

$$c^{-1}(\{i\}) := \mathcal{V}_i := \{\omega \in \mathcal{V} : \text{supp}(\omega) = \{i\}\}, \quad i \in I,$$

and the weight map $\lambda : \mathcal{V} \to \mathbb{Z}_{>0}$ is defined by

$$(2.3.6) \quad \lambda(\omega) = \text{wt}(\omega)(h_i) \quad \text{for all} \quad \omega \in \mathcal{V}_i.$$ 

In particular,

$$(2.3.7) \quad \sum_{i \in I} \sum_{\omega \in \mathcal{V}_i} \lambda(\omega)\omega_i = \text{wt}(\pi).$$

The set of arrows $A = A(\pi)$ is defined as the set of ordered pairs of q-factors, say $(\omega_{i,a,r}, \omega_{j,b,s})$, such that

$$(2.3.8) \quad a = bq^m \quad \text{for some} \quad m \in \mathcal{R}_{i,j}^r.$$ 

This is equivalent to saying that $L_q(\omega_{i,a,r}) \otimes L_q(\omega_{j,b,s})$ is reducible and highest-$\ell$-weight. In the case of the actual q-factorization, we necessarily have $m \notin \mathcal{R}_{i,j}^r$ when $i = j$. The value of the exponent $\epsilon : A \to \mathbb{Z}_{>0}$ at an arrow satisfying (2.3.8) is set to be $m$. We refer to $G$ as a pseudo q-factorization graph over $\pi$. In the case this construction was performed using the q-factorization of $\pi$, then $G$ is called the q-factorization graph of $\pi$ and it is denoted by $G(\pi)$.

**Proposition 2.3.1** ([10] Proposition 3.4.1). Let $\pi \in \mathcal{P}^+$. If $G_1, \ldots, G_k$ are the connected components of $G(\pi)$ and $\pi^{(j)} \in \mathcal{P}^+$, $1 \leq j \leq k$, are such that $\pi = \prod_{j=1}^k \pi^{(j)}$ and $G_j = G(\pi^{(j)})$, then

$$L_q(\pi) \cong L_q(\pi^{(1)}) \otimes \cdots \otimes L_q(\pi^{(k)}).$$

In particular, $G(\pi)$ is connected if $L_q(\pi)$ is prime. \hfill $\square$

The converse of the above proposition is not true. In fact, Theorem 2.4.6 below implies that, if $q$ is of type $A_2$ and $r = 2$ in (2.3.9), the given graph is not prime. On the other hand, the same theorem implies this graph is prime if $r \geq 3$.

$$(2.3.9) \quad \begin{array}{c}
\frac{r}{1} \xrightarrow{r+1} \frac{2}{2} \xleftarrow{4} \frac{1}{1} \\
\end{array}$$

Note that, for $r = 1$, this is only a pre-factorization graph, but the corresponding simple module is prime and its q-factorization graph is:

$$\begin{array}{c}
\frac{2}{2} \xleftarrow{3} \frac{2}{1} \\
\end{array}$$

The simple modules associated to these graphs belong to the Hernandez-Leclerc category $\mathcal{C}_{r+1}$.

Recall also the following notions of duality for pre-factorization graphs. Given a graph $G$, we denote by $G^-$ the graph obtained from $G$ by reversing all the arrows and keeping the rest of structure of (pre)-factorization graph. Then, $G^-$ is a factorization graph as well, which we refer to as the arrow-dual of $G$. Similarly, the graph $G^*$, called the color-dual of $G$, obtained by changing the coloring according to the rule $i \rightarrow i^*$ for all $i \in I$, is a factorization graph. Moreover,

$$(2.3.10) \quad \pi_{G^-,v,a}^{-1} = \pi_{G,v,a}^* \quad \text{and} \quad \pi_{G^*,v,a} = \pi_{G,v,a}^*.$$ 

2.4. **Statement of the Main Results.** Throughout this section, we let $G = G(\pi) = (\mathcal{V}, A)$ be a q-factorization graph. We say $G$ is prime if $L_q(\pi)$ is prime. We start by recalling:

**Proposition 2.4.1** ([10] Proposition 3.5.2]). Suppose $G$ is prime and $\#\mathcal{V} > 1$. Then, for every $v \in \partial G$, $G_{\mathcal{V}\setminus\{v\}}$ is also prime. \hfill $\square$

Together with elementary combinatorial properties of trees, Proposition 2.4.1 implies:

**Corollary 2.4.2.** If $G$ is a prime tree, every of its proper connected subgraphs are prime. \hfill $\square$
This corollary is false for general $q$-factorization graphs as we show in Example 2.5.2. Its converse is false even for trees as Example 2.5.1 shows. The following proposition, which can also be regarded as a stronger version of this corollary, provides a criterion for checking if a tree is prime by studying its 3-vertex subgraphs. It will be proved in Section 3.4.

**Proposition 2.4.3.** Suppose $\pi, \pi' \in \mathcal{P}^+$ have dissociate $q$-factorizations and that $G(\pi \pi')$ is a tree. Suppose further that the unique arrow connecting $G(\pi)$ and $G(\pi')$ in $G(\pi \pi')$ is of the form $(\omega, \omega')$ with $\omega \in G(\pi), \omega' \in G(\pi')$, and that there exists $\tilde{\omega} \in \mathcal{P}^+$ such that one of the following conditions holds:

1. $(\omega, \tilde{\omega}) \in A_G$ and $L_q(\tilde{\omega} \omega) \otimes L_q(\omega')$ is simple;
2. $(\tilde{\omega}, \omega') \in A_G$ and $L_q(\tilde{\omega} \omega') \otimes L_q(\omega)$ is simple.

Then, $L_q(\pi) \otimes L_q(\pi')$ is simple, as well.

The following criterion for determining whether a tree is prime will follow as consequence of Lemma 3.5.2.

**Proposition 2.4.4.** If $G$ is a tree and $L_q(\omega)^* \otimes L_q(\omega')$ is simple for any pair of non-adjacent vertices $\omega$ and $\omega'$ of $G$, then $L_q(\pi)$ is prime.

Let us recall the main result of [10].

**Theorem 2.4.5 ([10] Theorem 3.5.5).** If $g$ is of type $A$, every totally ordered $q$-factorization graph is prime.

As we have already pointed out in (2.3.9), even for type $A_2$, it is not true that every tree with three vertices is prime. There are just three possibilities of connected graphs with three vertices: alternating lines, monotonic lines, and a triangles. The latter two are prime by Theorem 2.4.3. If $G$ is an alternating line, say

$$
\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{align*}
$$

we have the following characterization for type $A$ proved in Section 4.3.

**Theorem 2.4.6.** Assume $g$ is of type $A$ and let $G$ be an alternating line as above. For $j = 1, 2$, let also $I_j \subseteq I$ be the minimal connected subdiagram containing $[i, i_j]$ such that $m_j \in R_{i, i_j, I_j}$ and let $j'$ be such that $\{j, j'\} = \{1, 2\}$. Then, $G$ is not prime if and only if there exists $j \in \{1, 2\}$ such that

- $i_{j'} \in I_j$,
- $m_{j'} \in R_{i, i_{j'}, I_j}$,
- $|m_j - m_{j'} - \delta_{I_j}| \in R_{i, i_{j'}, I_j, I_j}$,

and

$$
\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\end{array}
\end{align*}
$$

The present proof of Theorem 2.4.6 utilizes the precise description of the sets $R_{i, i_j}$ and, hence, in order to extend it to other types, it requires a case by case analysis. The proof consists of showing that the cut obtained by isolating the $i_j$-colored vertex, $j = 1, 2$, is simple if and only if the stated conditions are satisfied.

An important concept related to the connection of finite-dimensional representations of quantum affine algebras with the theory of cluster algebras is that of a real simple module, that is a simple module whose tensor square is simple. We shall say $G$ is real if $L_q(\pi)$ is real.

**Conjecture 2.4.7.** Every tree is real.

The last of our main results is:

**Theorem 2.4.8.** Conjecture 2.4.7 is true if $g$ is of type $A$. 


The only reason the proof of this theorem requires the assumption that \( g \) is of type \( A \) is that it uses the following corollary of Theorem 2.4.6 proved in Section 4.1. Hence, Conjecture 2.4.7 is proved if this corollary holds for all types, which is expected to be true.

**Corollary 2.4.9.** If \( g \) is of type \( A \), \( i, j \in I, a \in \mathbb{F}^\times \), \( r, s \in \mathbb{Z}_{>0} \), and \( m \in \mathcal{R}_{i,j} \), the module \( L_q(\omega_{i,a,r}) \otimes L_q(\omega_{j,a,q^m,s}) \) is simple.

### 2.5. Examples.

**Example 2.5.1.** We give a counterexample to the converse of Corollary 2.4.2. More precisely, we give an example of a non-prime tree all of whose proper connected subgraphs are prime. Let \( g \) be of type \( A_n, n \geq 3 \), and consider \( \pi = \omega_{1,a,2}\omega_{2,a,q^5}\omega_{3,a,q^6,3}\omega_{3,a,q^8} \), where \( a \in \mathbb{F}^\times \), whose \( q \)-factorization graph is

\[
\begin{array}{ccccccc}
1 & 3 & 3 & 1 & 2 & 4 & 2 \\
3 & 1 & 2 & 4 & 2 & 1 & 3
\end{array}
\]

Theorems 2.4.5 and 2.4.6 imply the two connected subgraphs with three vertices are prime. We can be more precise regarding the subgraph which is an alternating line using the results of Section 4.3: the tensor products

\[
L_q(\omega_{3,a,q^6,3}\omega_{1,a,2}) \otimes L_q(\omega_{2,a,q^5,2}) \quad \text{and} \quad L_q(\omega_{2,a,q^5,2}\omega_{1,a,2}) \otimes L_q(\omega_{3,a,q^6,3})
\]

are both reducible. Indeed, the first one does not satisfy the first condition in (2.4.6) while the second (which is not highest-\( \ell \)-weight by the more precise analysis of Section 4.3) does not satisfy the third condition. However, we will see that \( G(\pi) \) is not prime by checking that

\[
V = L_q(\omega_{3,a,q^6,3}\omega_{1,a,2}) \otimes L_q(\omega_{3,a,q^6,3})
\]

is simple.

The argument that follows utilizes results reviewed in Section 3.3. In order to show \( V \) is simple, since the tensor product in the opposite order is highest-\( \ell \)-weight by (2.2.2) and Corollary 3.3.3 Proposition 3.3.2 implies it suffices to show \( V \) is highest-\( \ell \)-weight as well. To do that, consider

\[
W = L_q(\omega_{3,a,q^6,3}\omega_{1,a,2}) \otimes L_q(\omega_{3,a,q^6,3}) \otimes L_q(\omega_{3,a,q^5,2}).
\]

Let

\[
W_{1,2} = L_q(\omega_{3,a,q^6,3}\omega_{2,a,q^5}\omega_{1,a,2}) \otimes L_q(\omega_{3,a,q^6}),
\]

\[
W_{1,3} = L_q(\omega_{3,a,q^6,3}\omega_{2,a,q^5}\omega_{1,a,2}) \otimes L_q(\omega_{3,a,q^5,2}),
\]

\[
W_{2,3} = L_q(\omega_{3,a,q^6}) \otimes L_q(\omega_{3,a,q^5,2}).
\]

Notice \( W_{1,3} \) is highest-\( \ell \)-weight. Therefore, we have epimorphisms

\[
W_{1,3} \twoheadrightarrow L_q(\omega_{3,a,q^6,3}) = L_q(\omega_{3,a,q^6,3})
\]

and

\[
W = L_q(\omega_{3,a,q^6,3}\omega_{2,a,q^5}\omega_{1,a,2}) \otimes W_{1,3} \twoheadrightarrow L_q(\omega_{3,a,q^6,3}\omega_{2,a,q^5}\omega_{1,a,2}) \otimes L_q(\omega_{3,a,q^6,3}) = V.
\]

So, it suffices check that \( W \) is highest-\( \ell \)-weight. In its turn, by Theorem 3.3.3 it is enough to show that \( W_{1,2} \) and \( W_{1,3} \) are highest-\( \ell \)-weight.

To see that \( W_{1,2} \) is highest-\( \ell \)-weight, consider

\[
\tilde{W}_{1,2} = L_q(\omega_{3,a,q^6,3}\omega_{2,a,q^5}) \otimes L_q(\omega_{1,a,2}) \otimes L_q(\omega_{3,a,q^6})
\]

and notice \( L_q(\omega_{3,a,q^6,3}\omega_{2,a,q^5}) \otimes L_q(\omega_{1,a,2}) \) is highest-\( \ell \)-weight by Lemma 3.3.4. Therefore, there exists an epimorphism \( \tilde{W}_{1,2} \twoheadrightarrow W_{1,2} \) and we are left to show that \( \tilde{W}_{1,2} \) is highest-\( \ell \)-weight. Notice that \( 7 = 8 - 1 \notin \mathcal{R}_{1,3} \). Thus, \( L_q(\omega_{1,a,2}) \otimes L_q(\omega_{3,a,q^6}) \) is simple (and highest-\( \ell \)-weight). In its turn, \( L_q(\omega_{3,a,q^6,3}\omega_{2,a,q^5}) \otimes L_q(\omega_{3,a,q^6}) \) is simple (and highest-\( \ell \)-weight) by Theorem 2.4.6. Theorem 3.3.3 then implies \( W_{1,2} \) is highest-\( \ell \)-weight as desired.

To prove that \( W_{1,3} \) is highest-\( \ell \)-weight, consider

\[
\tilde{W}_{1,3} = L_q(\omega_{3,a,q^6}) \otimes L_q(\omega_{2,a,q^5}\omega_{1,a,2}) \otimes L_q(\omega_{3,a,q^5,2})
\]

and notice \( L_q(\omega_{3,a,q^6}) \otimes L_q(\omega_{2,a,q^5}\omega_{1,a,2}) \) is highest-\( \ell \)-weight by Lemma 3.3.4. Therefore, there exists an epimorphism \( \tilde{W}_{1,3} \twoheadrightarrow W_{1,3} \) and it suffices to show that \( \tilde{W}_{1,3} \) is highest-\( \ell \)-weight. Since
$L_q(\omega_{3,aq^2}) \otimes L_q(\omega_{3,aq^2},2)$ is clearly highest-$\ell$-weight and $L_q(\omega_{2,aq^2},\omega_{1,aq}) \otimes L_q(\omega_{3,aq^2},2)$ is simple by Theorem 2.4.6, the claim follows from Theorem 3.3.3.

**Example 2.5.2.** We now give an example showing the hypothesis that $G$ is a tree in Corollary 2.4.2 is essential. Let $\mathfrak{g}$ be of type $A_2$, $a \in \mathbb{F}^\times$, and consider $\pi = \omega_{1,aq^7,2} \omega_{1,aq^3,2} \omega_{2,aq^4,2} \omega_{2,aq^2}$, whose $q$-factorization graph is

![](image)

By Theorem 2.4.6

(2.5.1) $L_q(\omega_{2,aq^4,2} \omega_{1,a}) \otimes L_q(\omega_{2,aq^3})$ and $L_q(\omega_{2,aq^3}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^4,2})$ are reducible, while

(2.5.2) $L_q(\omega_{2,aq^4,2} \omega_{1,a}) \otimes L_q(\omega_{2,aq^3})$ and $L_q(\omega_{2,aq^3}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^4,2})$ are simple,

showing that not every proper connected subgraph is prime. Let us check that $L_q(\pi)$ is prime. This is equivalent to showing that the following tensor products are reducible:

1. $L_q(\omega_{1,aq^7,2}) \otimes L_q(\omega_{2,aq^4,2} \omega_{2,aq^3} \omega_{1,a})$;
2. $L_q(\omega_{1,aq^7,2} \omega_{2,aq^3} \omega_{1,a}) \otimes L_q(\omega_{2,aq^4,2})$;
3. $L_q(\omega_{2,aq^3}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^4,2} \omega_{1,a})$;
4. $L_q(\omega_{1,a}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^4,2} \omega_{2,aq^3})$;
5. $L_q(\omega_{1,aq^7,2} \omega_{2,aq^3} \omega_{1,a}) \otimes L_q(\omega_{2,aq^4,2} \omega_{2,aq^3})$;
6. $L_q(\omega_{2,aq^4,2} \omega_{1,a}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^4,2})$;
7. $L_q(\omega_{2,aq^4,2} \omega_{1,a}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^3})$.

1. It follows from (2.5.2) that we have an isomorphism

   $L_q(\omega_{1,aq^7,2}) \otimes L_q(\omega_{2,aq^4,2} \omega_{2,aq^3} \omega_{1,a}) \cong L_q(\omega_{1,aq^7,2}) \otimes L_q(\omega_{2,aq^3}) \otimes L_q(\omega_{2,aq^4,2} \omega_{1,a})$,

   which is reducible because $L_q(\omega_{1,aq^7,2}) \otimes L_q(\omega_{2,aq^3})$ is reducible since $4 = 7 - 3 < 1$.

2. We claim the given tensor product is not highest-$\ell$-weight and, hence, reducible. Indeed, if this were not the case, \cite{10} Proposition 5.2.2 would imply that $L_q(\omega_{2,aq^3} \omega_{1,a}) \otimes L_q(\omega_{2,aq^4,2})$ is highest-$\ell$-weight, as well. Since $4 > \min\{3,0\}$, $L_q(\omega_{2,aq^3},2) \otimes L_q(\omega_{2,aq^3} \omega_{1,a})$ is highest-$\ell$-weight by (2.2.2) and Corollary 3.3.4 Proposition 3.3.4 would then imply $L_q(\omega_{2,aq^3} \omega_{1,a}) \otimes L_q(\omega_{2,aq^4,2})$ is simple, contradicting (2.5.1).

3. Similarly to the previous case, $L_q(\omega_{2,aq^3}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^4,2} \omega_{1,a})$ is not highest-$\ell$-weight, otherwise, \cite{10} Proposition 5.2.2 would imply $L_q(\omega_{2,aq^3}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^4,2})$ were highest-$\ell$-weight, and then simple, contradicting (2.5.1).

4. Similarly to case (1), we have an isomorphism

   $L_q(\omega_{1,a}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^4,2} \omega_{2,aq^3}) \cong L_q(\omega_{1,a}) \otimes L_q(\omega_{2,aq^4,2}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^3})$,

   and we are done since $L_q(\omega_{1,a}) \otimes L_q(\omega_{2,aq^4,2})$ is reducible.

5. In this case we have

   $L_q(\omega_{1,aq^7,2} \omega_{1,a}) \otimes L_q(\omega_{2,aq^4,2} \omega_{2,aq^3}) \cong L_q(\omega_{1,a}) \otimes L_q(\omega_{2,aq^4,2}) \otimes L_q(\omega_{2,aq^3})$,

   and $L_q(\omega_{1,a}) \otimes L_q(\omega_{2,aq^4,2})$ is reducible.

6. If $L_q(\omega_{2,aq^3} \omega_{1,a}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^4,2})$ were highest-$\ell$-weight, then \cite{10} Corollary 4.3.2 would imply that $L_q(\omega_{2,aq^3} \omega_{1,a}) \otimes L_q(\omega_{2,aq^4,2})$ is highest-$\ell$-weight as well and, hence, simple, yielding a contradiction as before.

7. To see that $L_q(\omega_{2,aq^4,2} \omega_{1,a}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^3})$ is reducible, notice that $(\omega_{1,aq^7,2} \omega_{2,aq^3})^* = \omega_{2,aq^4,2} \omega_{1,a}$. Therefore, we have an epimorphism

   $L_q(\omega_{2,aq^4,2} \omega_{1,a}) \otimes L_q(\omega_{1,aq^7,2} \omega_{2,aq^3}) \rightarrow \mathbb{F}$.
and, hence, \( L_q(\omega_2, aq^1, 2\omega_1, a) \otimes L_q(\omega_1, aq^7, 2\omega_2, aq^3) \) is reducible.

Let us also mention a few examples related to the problem of determining the reality of certain modules.

**Example 2.5.3.** The converse of Conjecture 2.4.7 is not true, i.e., there are real modules whose \( q \)-factorizations graphs are not trees. Indeed, if \( G = G(\pi) \) is real, then \( G' = G(\pi^2) \) is real but, it is neither prime nor a tree. For instance, if \( G \) is of the form

\[
\begin{array}{ccc}
  r & \overset{m}{\rightarrow} & j \\
  i & \overset{m}{\rightarrow} & i \\
  & \overset{m}{\rightarrow} & s \\
  & \overset{m}{\rightarrow} & j \\
  1 & \overset{3}{\leftrightarrow} & 3 & \overset{3}{\rightarrow} & 1 \\
  2 & \overset{3}{\rightarrow} & 3 & \overset{3}{\rightarrow} & 1 \\
  2 & \overset{3}{\rightarrow} & 3 & \overset{3}{\rightarrow} & 1 \\
  2 & \overset{3}{\rightarrow} & 3 & \overset{3}{\rightarrow} & 1 \\
  2 & \overset{3}{\rightarrow} & 3 & \overset{3}{\rightarrow} & 1 \\
\end{array}
\]

is real, then \( G' = G(\pi^2) \) is real but, it is neither prime nor a tree. For instance, if \( G \) is of the form

\[
\begin{array}{ccc}
  r_i & \overset{m}{\rightarrow} & j \\
  & \overset{m}{\rightarrow} & s \\
  2 & \overset{4}{\rightarrow} & 2 \\
  1 & \overset{4}{\rightarrow} & 2 \\
  1 & \overset{4}{\rightarrow} & 2 \\
  1 & \overset{4}{\rightarrow} & 2 \\
\end{array}
\]

**Example 2.5.4.** In contrast to the real graph \( G' \) in the previous example, it was shown in [7, Section 13.6] that the following graph for type \( A_4 \) is not real:

\[
\begin{array}{ccc}
  1 & \overset{3}{\leftrightarrow} & 2 \\
  3 & \overset{3}{\rightarrow} & 3 \\
  3 & \overset{3}{\rightarrow} & 3 \\
  3 & \overset{3}{\rightarrow} & 3 \\
  3 & \overset{3}{\rightarrow} & 3 \\
\end{array}
\]

In fact, this graph makes sense for type \( A_n, n \geq 3 \) and is not real (cf. [1, Example 8.7]). In [1, Example 8.6] it was shown the following graph is not real for type \( A_n, n \geq 2 \):

\[
\begin{array}{ccc}
  2 & \overset{4}{\rightarrow} & 2 \\
  4 & \overset{4}{\rightarrow} & 4 \\
  4 & \overset{4}{\rightarrow} & 4 \\
  4 & \overset{4}{\rightarrow} & 4 \\
\end{array}
\]

**3. Further Background and First Proofs**

### 3.1. Hopf Algebra Facts

For easy of referencing, we partially reproduce [10, Section 2.6].

Given a Hopf algebra \( \mathcal{H} \) over \( \mathbb{F} \), its category \( \mathcal{C} \) of finite-dimensional representations is an abelian monoidal category and we denote the (right) dual of a module \( V \) by \( V^* \). More precisely, the action of \( \mathcal{H} \) on \( V^* \) is given by

\[
(3.1.1) \quad (hf)(v) = f(S(h)v) \quad \text{for} \quad h \in \mathcal{H}, f \in V^*, v \in V.
\]

It is well known that

\[
\text{Hom}_\mathcal{H}(\mathbb{F}, V \otimes V^*), \quad \text{Hom}_\mathcal{H}(V^* \otimes V, \mathbb{F}) \neq 0.
\]

If the antipode is invertible, the notion of left dual module is obtained by replacing \( S \) by \( S^{-1} \) in \( 3.1.1 \). The left dual of \( V \) will be denoted by \( \ast V \) and we have

\[
\ast (V^*) = (\ast V)^* = V.
\]

Given \( \mathcal{H} \)-modules \( V_1, V_2, V_3 \), we have

\[
(3.1.2) \quad \text{Hom}_\mathcal{C}(V_1 \otimes V_2, V_3) \cong \text{Hom}_\mathcal{C}(V_1, V_3 \otimes V_2^*), \quad \text{Hom}_\mathcal{C}(V_1, V_2 \otimes V_3) \cong \text{Hom}_\mathcal{C}(V_2^* \otimes V_1, V_3),
\]

and

\[
(3.1.3) \quad (V_1 \otimes V_2)^* \cong V_2^* \otimes V_1^*.
\]
Also, given a short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$, we can consider the following short exact sequence

\begin{equation}
0 \to V_3^* \to V_2^* \to V_1^* \to 0.
\end{equation}

**Lemma 3.1.1.** Let $V_1, V_2, V_3, L_1, L_2$ be $\mathcal{H}$-modules and assume $V_2$ is simple. If

$$
\varphi_1 : L_1 \to V_1 \otimes V_2 \quad \text{and} \quad \varphi_2 : V_2 \otimes V_3 \to L_2
$$

are nonzero homomorphisms, the composition

$$
L_1 \otimes V_3 \xrightarrow{\varphi_1 \otimes \text{id}_{V_3}} V_1 \otimes V_2 \otimes V_3 \xrightarrow{\text{id}_{V_1} \otimes \varphi_2} V_1 \otimes L_2
$$

does not vanish. Similarly, if

$$
\varphi_1 : V_1 \otimes V_2 \to L_1 \quad \text{and} \quad \varphi_2 : L_2 \to V_2 \otimes V_3
$$

are nonzero homomorphisms, the composition

$$
V_1 \otimes L_2 \xrightarrow{\text{id}_{V_1} \otimes \varphi_2} V_1 \otimes V_2 \otimes V_3 \xrightarrow{\varphi_1 \otimes \text{id}_{V_3}} L_1 \otimes V_3
$$

does not vanish. \qed

### 3.2. Tensor Products and Diagram Subalgebras

The following result will be used in a crucial moment during the proof of Theorem 2.4.6. A slightly modified argument which avoids the use of Theorem 3.2.1 is given in [12].

**Theorem 3.2.1 ([8]).** Let $M$ and $N$ be finite-dimensional simple $U_q(\mathfrak{g})$-modules. Assume further that $M$ is real. Then $M \otimes N$ has a simple socle and a simple head. Moreover, if $\text{soc}(M \otimes N)$ and $\text{hd}(M \otimes N)$ are isomorphic, then $M \otimes N$ is simple. \qed

If $V$ is a highest-$\ell$-weight module with highest-$\ell$-weight vector $v$ and $J \subseteq I$, we let $V_J$ denote the $U_q(\mathfrak{g})_J$-submodule of $L_q(\pi)$ generated by $v$. Evidently, if $\pi$ is the highest-$\ell$-weight of $V$, then $V_J$ is highest-$\ell$-weight with highest $\ell$-weight $\pi_J$. Moreover, we have the following well-known facts:

\begin{equation}
V_J = \bigoplus_{\eta \in Q_J^+} V_{\text{wt}(\pi) - \eta} = \bigoplus_{\eta \in Q_J^+} V_{\text{wt}(\pi) + \eta}.
\end{equation}

**Lemma 3.2.2.** If $V$ is simple, so is $V_J$. \qed

Since $U_q(\mathfrak{g})_J$ is not a sub-coalgebra of $U_q(\mathfrak{g})$, if $M$ and $N$ are $U_q(\mathfrak{g})_J$-submodules of $U_q(\mathfrak{g})$-modules $V$ and $W$, respectively, it is in general not true that $M \otimes N$ is a $U_q(\mathfrak{g})_J$-submodule of $V \otimes W$. Recalling that we have an algebra isomorphism $U_q(\mathfrak{g})_J \cong U_q(\mathfrak{g})_J$, we shall denote by $M \otimes_J N$ the $U_q(\mathfrak{g})_J$-module obtained by using the coalgebra structure from $U_q(\mathfrak{g})_J$. The next result describes a special situation on which $M \otimes N$ is a submodule isomorphic to $M \otimes_J N$.

**Proposition 3.2.3 ([8 Proposition 2.2]).** Let $V$ and $W$ be finite-dimensional highest-$\ell$-weight modules with highest $\ell$-weights $\pi, \varpi \in P^+$, respectively, and let $J \subseteq I$ be a connected subdiagram. Then, $V_J \otimes W_J$ is a $U_q(\mathfrak{g})_J$-submodule of $V \otimes W$ isomorphic to $V_J \otimes_J W_J$ via the identity map. \qed

**Corollary 3.2.4.** In the notation of Proposition 3.2.3, if $V \otimes W$ is highest-$\ell$-weight, so is $V_J \otimes W_J$. Moreover, if $V \otimes W$ is simple, so is $V_J \otimes W_J$. \qed

**Lemma 3.2.5** (cf. [2 Lemma 5.4]). Let $U, V, W$ be finite-dimensional highest-$\ell$-weight modules with highest $\ell$-weights $\omega, \pi, \varpi \in P^+$, respectively, and let $J \subseteq I$ be a connected subdiagram. If

\begin{equation}
\text{wt}(\pi) + \text{wt}(\varpi) - \text{wt}(\omega) \in Q_J^+,
\end{equation}

restriction induces natural linear maps $\text{Hom}_{U_q(\mathfrak{g})}(V \otimes W, U) \to \text{Hom}_{U_q(\mathfrak{g})}(V_J \otimes W_J, U_J)$ and $\text{Hom}_{U_q(\mathfrak{g})}(U, V \otimes W) \to \text{Hom}_{U_q(\mathfrak{g})}(U_J, V_J \otimes W_J)$. Moreover, the latter is injective and the former is injective if $U$ is simple.
(3.2.3) \[ V_J \otimes W_J = \bigoplus_{\eta \in Q^+_J} (V \otimes W)_{\mathrm{wt}(\pi)+\mathrm{wt}(\varpi) - \eta}. \]

Let \( f \in \text{Hom}_{U(\mathfrak{g})}(V \otimes W, U) \) and denote by \( f_J \) its restriction to \( V_J \otimes W_J \). It follows from (3.2.2), (3.2.1), and (3.2.3) that the image of \( f_J \) lies in \( U_J \). Thus, we have a natural linear map \( \phi : \text{Hom}_{U(\mathfrak{g})}(V \otimes W, U) \rightarrow \text{Hom}_{U(\mathfrak{g})}(V_J \otimes W_J, U_J), f \mapsto f_J \). If \( U \) is simple and \( f \neq 0 \), then the image of \( f \) contains a highest-\( \ell \)-weight vector for \( U \), which, by (3.2.1) and (3.2.2), must be in the image of \( f_J \). It follows that \( f_J \neq 0 \), showing \( \phi \) is injective.

Now let \( f \in \text{Hom}_{U(\mathfrak{g})}(U, V \otimes W) \) and denote by \( f_J \) its restriction to \( U_J \). It follows from (3.2.2), (3.2.1), and (3.2.3) that the image of \( f_J \) lies in \( V_J \otimes W_J \) and, hence, we have a natural linear map \( \phi : \text{Hom}_{U(\mathfrak{g})}(U, V \otimes W) \rightarrow \text{Hom}_{U(\mathfrak{g})}(U_J, V_J \otimes W_J), f \mapsto f_J \). Let \( u \) be a highest-\( \ell \)-weight vector for \( U \). Then, \( f \) is completely determined by its value on \( u \). In particular, \( f_J = 0 \) if and only if \( f = 0 \) showing \( \phi \) is injective.

3.3. Highest-\( \ell \)-weight Tensor Products. We record some results from [10] which will be used in the proof of Theorem 2.4.6.

**Lemma 3.3.1** ([10] Lemma 4.1.2). Let \( \pi, \pi' \in \mathcal{P}^+, V = L_q(\pi) \otimes L_q(\pi') \), and \( W = L_q(\pi') \otimes L_q(\pi) \).

(a) \( V \) contains a submodule isomorphic to \( L_q(\varpi), \varpi \in \mathcal{P}^+ \), if and only there exists an epimorphism \( W \rightarrow L_q(\varpi) \).

(b) If \( W \) is highest-\( \ell \)-weight, the submodule of \( V \) generated by its top weight space is simple.

(c) If \( V \) is not highest-\( \ell \)-weight, there exists an epimorphism \( V \rightarrow L_q(\varpi) \) for some \( \varpi \in \mathcal{P}^+ \) such that \( \varpi \leq \pi \pi' \).

**Proposition 3.3.2** ([10] Corollary 4.1.4]). Let \( \pi, \varpi \in \mathcal{P}^+ \). Then, \( L_q(\pi) \otimes L_q(\varpi) \) is simple if and only if both \( L_q(\pi) \otimes L_q(\varpi) \) and \( L_q(\varpi) \otimes L_q(\pi) \) are highest-\( \ell \)-weight.

The “if” part of the following theorem is the main result of [10] and a proof of the converse can be found in [10] Theorem 4.1.5.

**Theorem 3.3.3.** Let \( S_1, \ldots, S_m \) be simple \( U_q(\mathfrak{g}) \)-modules. Then, \( S_1 \otimes \cdots \otimes S_m \) is highest-\( \ell \)-weight if and only if \( S_i \otimes S_j \) is highest-\( \ell \)-weight for all \( 1 \leq i < j \leq m \).

The following corollary ([10] Corollary 4.1.6]) was prominently used in [10] and will be even more frequently utilized here.

**Corollary 3.3.4.** Given \( \pi, \bar{\pi} \in \mathcal{P}^+ \), \( L_q(\pi) \otimes L_q(\bar{\pi}) \) is highest-\( \ell \)-weight if there exist \( \pi^{(k)} \in \mathcal{P}^+, 1 \leq k \leq m, \bar{\pi}^{(k)} \in \mathcal{P}^+, 1 \leq k \leq \bar{m} \), such that \( \pi = \prod_{k=1}^{m} \pi^{(k)} \), \( \bar{\pi} = \prod_{k=1}^{\bar{m}} \bar{\pi}^{(k)} \),

and the following tensor products are highest-\( \ell \)-weight:

\[ L_q(\pi^{(k)}) \otimes L_q(\pi^{(l)}), L_q(\bar{\pi}^{(k)}) \otimes L_q(\bar{\pi}^{(l)}) \text{ for } k < l, \text{ and } L_q(\pi^{(k)}) \otimes L_q(\bar{\pi}^{(l)}) \text{ for all } k, l. \]

Moreover, if all these tensor products are irreducible, then so is \( L_q(\pi) \otimes L_q(\bar{\pi}) \).

Although the following two corollaries of Corollary 3.3.4 will not be used here, they reveal interesting properties of pseudo \( q \)-factorization graphs afforded by trees. Since trees are the main topic of the present paper, we find it fit to include them here for record keeping. Let us use the following terminology. If \( G \) and \( G' \) are pseudo \( q \)-factorization graphs over \( \pi \) and \( \bar{\pi} \), respectively, we shall say the ordered pair \( (G, G') \) is highest-\( \ell \)-weight if so is \( L_q(\pi) \otimes L_q(\bar{\pi}) \). We shall say \( G \otimes G' \) is highest-\( \ell \)-weight if either \( (G, G') \) or \( (G', G) \) is highest-\( \ell \)-weight.

**Corollary 3.3.5.** Suppose \( G \) is a pseudo \( q \)-factorization graph and \( (G', G'') \) is a connected cut of \( G \). If \( G \) is a tree, then \( G' \otimes G'' \) is highest-\( \ell \)-weight.
Proof. Since $G$ is a tree and both $G'$ and $G''$ are connected, there exist unique $v' \in G'$ and $v'' \in G''$ such that $d(v', v'') = 1$ (Lemma 3.5.1). In particular, 
\[ w' \in V_{G'}, \ w'' \in V_{G''}, \ (w', w'') \neq (v', v'') \Rightarrow L_q(w') \otimes L_q(w'') \text{ is simple.} \]

If $(v', v'') \in \mathcal{A}_G$, we have $L_q(v') \otimes L_q(v'')$ is highest-$\ell$-weight and, hence, Corollary 3.3.4 implies so is $(G', G'')$. Otherwise, $(v'', v') \in \mathcal{A}_G$ and it follows that $(G'', G')$ is highest-$\ell$-weight. \hfill $\square$

Having the last proof in mind, the following is immediate from Proposition 3.3.2.

Corollary 3.3.6. Suppose $G$ is a pseudo $q$-factorization and $(G', G'')$ is a connected cut of $G$. If $G$ is a tree and $v' \in G', v'' \in G''$ are such that $(v', v'') \in \mathcal{A}_G$, then $G' \otimes G''$ is simple if and only if $(G'', G')$ is highest-$\ell$-weight. \hfill $\square$

The last result from [10] we shall need is:

**Proposition 3.3.7** ([10], Corollary 4.3.3). Let $\pi', \pi'' \in \mathcal{P}^+$ with dissociate factorizations and $\pi = \pi' \pi''$. Let also $G = G(\pi), G' = G(\pi'), G'' = G(\pi'')$, and suppose $\omega', \omega'' \in \mathcal{P}^+$ satisfy 
\[ \omega' \text{ is a source in } G', \quad \omega'' \text{ is a sink in } G'', \quad \text{and } (\omega'', \omega') \in \mathcal{A}_G. \]

Then, $L_q(\pi') \otimes L_q(\pi'')$ is not highest-$\ell$-weight. \hfill $\square$

### 3.4. Some Reducibility Criteria

Recall the main result of [5] implies the determination of the simplicity of tensor products can be reduced to that of two-fold tensor products.

**Theorem 3.4.1.** If $S_1, \ldots, S_n$ are simple $U_q(\mathfrak{g})$-modules, the tensor product 
\[ S_1 \otimes \cdots \otimes S_n \]

is simple if, and only if, $S_i \otimes S_j$ is simple for all $1 \leq i < j \leq n$. \hfill $\square$

Next, we prove a couple of criteria for deciding if a tensor product is reducible which will be used in the proofs of Proposition 2.4.3 and Theorem 2.4.6.

**Proposition 3.4.2.** Suppose $\pi, \pi', \pi'' \in \mathcal{P}^+$ are such that the tensor products 
\[ V = L_q(\pi) \otimes L_q(\pi') \quad \text{and} \quad V' = L_q(\pi') \otimes L_q(\pi'') \]

are highest-$\ell$-weight. Then, $L_q(\pi) \otimes L_q(\pi' \pi'')$ is reducible provided $V$ is reducible and similarly for $L_q(\pi \pi') \otimes L_q(\pi'')$ if $V'$ is reducible.

**Proof.** The second claim follows from the first by duality arguments, so we focus on the first. Since $V$ is reducible, there exists $\lambda \in \mathcal{P}^+$ for which there exists a non-surjective monomorphism $f : L_q(\lambda) \hookrightarrow V$. In particular, we must have 
\[ (3.4.1) \quad \text{wt}(\lambda) < \text{wt}(\pi) + \text{wt}(\pi'). \]

We also consider the monomorphism: 
\[ f \otimes \text{id}_{L_q(\pi'')} : L_q(\lambda) \otimes L_q(\pi'') \hookrightarrow L_q(\pi) \otimes L_q(\pi') \otimes L_q(\pi''). \]

On the other hand, since $V'$ is highest-$\ell$-weight, we have epimorphisms 
\[ g : V' \twoheadrightarrow L_q(\pi' \pi'') \]

and 
\[ \text{id}_{L_q(\pi)} \otimes g : L_q(\pi) \otimes L_q(\pi') \otimes L_q(\pi'') \twoheadrightarrow L_q(\pi) \otimes L_q(\pi' \pi'') := W. \]

By Lemma 3.1.1, the composition 
\[ \varphi : L_q(\lambda) \otimes L_q(\pi') \xrightarrow{f \otimes \text{id}_{L_q(\pi'')}} L_q(\pi) \otimes L_q(\pi') \otimes L_q(\pi'') \xrightarrow{\text{id}_{L_q(\pi)} \otimes g} W \]

is non-zero. Therefore, if $W$ were simple, $\varphi$ would be surjective. However, by (3.4.1), we have 
\[ \text{wt}(\lambda) + \text{wt}(\pi'') < \text{wt}(\pi) + \text{wt}(\pi') + \text{wt}(\pi''), \]

yielding a contradiction. Thus, $W$ is reducible as claimed. \hfill $\square$
Proposition 3.4.3. Let $\pi, \pi', \pi'' \in \mathcal{P}^+$ such that the tensor products

$$L_q(\pi) \otimes L_q(\pi') \quad \text{and} \quad L_q(\pi'') \otimes L_q(\pi')$$

are highest-\(\ell\)-weight and

$$L_q(\pi'') \otimes L_q(\pi)^*$$

is simple. Then, if $L_q(\pi) \otimes L_q(\pi')$ is reducible, so is

$$L_q(\pi) \otimes L_q(\pi'') \otimes L_q(\pi').$$

Similarly, if

$$L_q(\pi') \otimes L_q(\pi) \quad \text{and} \quad L_q(\pi') \otimes L_q(\pi'')$$

are highest-\(\ell\)-weight, with the former reducible, and $L_q(\pi'')^* \otimes L_q(\pi)$ is simple, then $L_q(\pi) \otimes L_q(\pi'' \otimes L_q(\pi))$ is reducible.

Proof. The second statement follows from the first by duality arguments. To shorten notation for proving the first, set

$$V = L_q(\pi) \otimes L_q(\pi'), \quad U = L_q(\pi'') \otimes L_q(\pi'), \quad \text{and} \quad M = L_q(\pi'') \otimes L_q(\pi)^*.$$

As in the proof of Proposition 3.4.2 we consider the monomorphism $f$ and (3.4.1) remains valid.

By (3.1.2), we have a non-zero homomorphism

$$g : L_q(\pi)^* \otimes L_q(\lambda) \to L_q(\pi')$$

which is surjective, since $L_q(\pi')$ is simple. Therefore, we also have an epimorphism

$$\text{id}_{L_q(\pi')^*} \otimes g : L_q(\pi'') \otimes L_q(\pi)^* \otimes L_q(\lambda) \to L_q(\pi'') \otimes L_q(\pi').$$

In its turn, since $U$ is highest-\(\ell\)-weight, we have an epimorphism

$$g' : U \to L_q(\pi'' \otimes L_q(\pi))$$

and the following composition is obviously nonzero (hence, surjective):

$$L_q(\pi'') \otimes L_q(\pi)^* \otimes L_q(\lambda) \to L_q(\pi'' \otimes L_q(\pi)) \to L_q(\pi'') \otimes L_q(\pi') \to L_q(\pi'') \otimes L_q(\pi').$$

Since $M$ is simple, we have isomorphisms

$$h : L_q(\pi)^* \otimes L_q(\pi'') \to M$$

and

$$h \otimes \text{id}_{L_q(\lambda)} : L_q(\pi)^* \otimes L_q(\pi'') \otimes L_q(\lambda) \to M \otimes L_q(\lambda).$$

It follows that the following composition is an epimorphism:

$$L_q(\pi)^* \otimes L_q(\pi'') \otimes L_q(\lambda) \to L_q(\pi'') \otimes L_q(\pi)^* \otimes L_q(\lambda) \to L_q(\pi'') \otimes L_q(\pi').$$

By (3.1.2), we obtain a non-zero homomorphism

$$\varphi : L_q(\pi'') \otimes L_q(\lambda) \to L_q(\pi) \otimes L_q(\pi'').$$

The same argument used at the end of the proof of Proposition 3.4.2 using (3.4.1) completes the proof here as well. \hfill \Box

We end this subsection with:

Proof of Proposition 2.4.3. Since $\omega' \prec \omega$, it follows from Corollary 3.3.4 that $L_q(\pi) \otimes L_q(\pi')$ is highest-\(\ell\)-weight. Therefore, by Proposition 3.3.2 it suffices to prove that $L_q(\pi') \otimes L_q(\pi)$ is also highest-\(\ell\)-weight. Since the proofs of both situations are similar, we only write down the proof in the case condition (i) is satisfied. Thus, $G(\pi \pi')$ contains the following subgraph

$$\varpi \leftarrow \omega \rightarrow \omega'.$$

To shorten notation, write $G = G(\pi), G' = G(\pi'), \mathcal{V} = \mathcal{V}_G$, and $\mathcal{V}' = \mathcal{V}_G'$. Consider the partition

$$\mathcal{V} = \mathcal{V}_+ \cup \{\omega, \varpi\} \cup \mathcal{V}_-$$

where, for each $\varpi \in \mathcal{V}_G \setminus \{\omega, \varpi\}$, we have

$$\varpi \in \mathcal{V}_+ \iff \text{either } [\varpi, \omega] \cap A_\omega \neq \emptyset \text{ or } [\varpi, \varpi] \cap A_\varpi \neq \emptyset \text{ and } \omega \notin [\varpi, \varpi].$$
Thus,
\[ \varpi \in \mathcal{V}_- \iff \text{either } [\varpi, \tilde{\omega}] \cap A^{-1}_{\tilde{\omega}} \neq \emptyset \text{ or } [\varpi, \omega] \cap A^{-1}_{\omega} \neq \emptyset \text{ and } \tilde{\omega} \notin [\varpi, \omega]. \]

Consider also the partition \( \mathcal{V}' = \mathcal{V}_+ \cup \{ \omega' \} \cup \mathcal{V}_- \) with
\[ \varpi \in \mathcal{V}_+ \iff [\varpi, \omega'] \cap A^{-1}_{\omega'} \neq \emptyset \quad \text{and} \quad \varpi \in \mathcal{V}_- \iff [\varpi, \omega'] \cap A^{-1}_{\omega'} \neq \emptyset. \]

Let \( q : \mathcal{V} \to \mathbb{N} \) be a highest-\( \ell \)-weight. Evidently, all \( q \)-factors of \( \pi \) are not adjacent to any \( q \)-factor of \( \pi' \) and, therefore,
\[ L_q(\pi_\pm) \otimes L_q(\varpi) \text{ is highest-} \ell \text{-weight for } \varpi \in \{ \pi_+, \pi_-, \omega' \}. \]

Similarly,
\[ L_q(\varpi) \otimes L_q(\pi'_\pm) \text{ is highest-} \ell \text{-weight for } \varpi \in \{ \pi_+, \pi_-, \omega', \tilde{\omega} \}. \]

By assumption of condition (i), \( L_q(\varpi) \otimes L_q(\omega') \) is simple. In light of Theorem 3.3.3 it remains to check that
\[ L_q(\pi_+) \otimes L_q(\pi_-) \quad \text{and} \quad L_q(\pi'_+) \otimes L_q(\pi'_-) \text{ are highest-} \ell \text{-weight.} \]
This follows by observing that \( G(\pi_+) \) and \( G(\pi_-) \) belong to different connected components of \( G(\pi_+, \pi_-) \) and similarly for \( G(\pi'_+, \pi'_-) \) (cf. [10, Lemma 2.2.2(c)].

3.5. Proof of Proposition 2.4.4. Begin by recalling the following elementary property of trees.

**Lemma 3.5.1.** Assume \( G \) is a tree. If \( \mathcal{V}_1 \cup \mathcal{V}_2 \) is a nontrivial partition of \( \mathcal{V}_G \) such that \( G_{\mathcal{V}_i} \) is connected for \( i = 1, 2 \), there exists unique \( (v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2 \) such that \( d(v_1, v_2) = 1 \). 

Proposition 2.4.4 can be easily deduced from the next lemma together with Lemma 3.5.1.

**Lemma 3.5.2.** Let \( \pi, \pi' \in \mathcal{P}^+ \) have dissociate \( q \)-factorizations and set \( G = G(\pi), G' = G(\pi') \). Assume \( G \) and \( G' \) are simply linked by \( (\omega, \omega') \) in \( G \otimes G' \) with \( \omega \in \mathcal{V}_G \) and \( \omega' \in \mathcal{V}_{G'} \) and that
\[ L_q(\varpi)^* \otimes L_q(\omega') \text{ is simple for all } \varpi \in \mathcal{V}_G, \omega' \in \mathcal{V}_{G'} \text{ such that } \varpi \neq \omega \text{ and } \varpi' \neq \omega'. \]

Then, \( L_q(\pi) \otimes L_q(\pi') \) is reducible.

**Proof.** We will prove by induction on the number \( N \geq 2 \) of vertices of \( G(\pi \pi') \) that there exists a nonzero homomorphism
\[ f : V \to L_q(\pi) \otimes L_q(\pi') \]
for some finite-dimensional \( U_q(\mathfrak{g}) \)-module \( V \) such that
\[ \lambda < \text{wt}(\pi) + \text{wt}(\pi') \quad \text{for all } \lambda \in \text{wt}(V). \]

This obviously implies \( L_q(\pi) \otimes L_q(\pi') \) is reducible.
The assumption that \( G \) and \( G' \) are simply linked together with Corollary \( 3.3.4 \) implies \( L_q(\pi) \otimes L_q(\pi') \) is highest-\( \ell \)-weight. If \( N = 2 \), then \( \pi = \omega, \pi' = \omega' \) and, since \((\omega, \omega') \in A_G(\pi \pi')\), \( L_q(\pi) \otimes L_q(\pi') \) is a reducible highest-\( \ell \)-weight module, showing that inductions starts.

For \( N \geq 2 \), either \( G \) or \( G' \) has an extra vertex. In light of \( 2.3.10 \), up to replacing \( G(\pi \pi') \) by its arrow-dual, we can assume without loss of generality that \( G \) has an extra vertex. Moreover, since each connected component of \( G \) has a sink and a source, we can choose either a sink or a source among the extra vertices. Henceforth, let \( \varpi \) be such a choice.

Quite clearly, after replacing \( \pi \) with \( \pi \varpi^{-1} \), the hypotheses of the lemma remain valid. Thus, the inductive hypothesis implies that there exists a non-zero homomorphism
\[
h : W \rightarrow L_q(\pi \varpi^{-1}) \otimes L_q(\pi'),
\]
where \( W \) is a finite-dimensional module such that
\[
(3.5.2) \quad \lambda < \text{wt}(\pi) - \text{wt}(\varpi) + \text{wt}(\pi')
\]
for all \( \lambda \in \text{wt}(W) \).

Assume first that \( \varpi \) is a source in \( G \). In that case, \( L_q(\varpi) \otimes L_q(\pi \varpi^{-1}) \) is highest-\( \ell \)-weight by Corollary \( 3.3.4 \) and we have an epimorphism
\[
p : L_q(\varpi) \otimes L_q(\pi \varpi^{-1}) \twoheadrightarrow L_q(\pi).
\]
Let \( V = L_q(\varpi) \otimes W \) and let \( f \) be the composition
\[
L_q(\varpi) \otimes W \xrightarrow{id_{L_q(\varpi)} \otimes h} L_q(\varpi) \otimes L_q(\pi \varpi^{-1}) \otimes L_q(\pi') \xrightarrow{p \otimes id_{L_q(\pi')}} L_q(\pi) \otimes L_q(\pi').
\]
Lemma \( 3.1.1 \) implies \( f \) is non-zero. Since \( \text{ch}(V) = \text{ch}(L_q(\varpi)) \text{ch}(W) \), we have
\[
\mu \in \text{wt}(V) \quad \Rightarrow \quad \mu = \nu + \lambda \quad \text{with} \quad \nu \leq \text{wt}(\varpi), \lambda \in \text{wt}(W).
\]
Thus, \( (3.5.1) \) follows from \( (3.5.2) \).

Finally, if \( \varpi \) is a sink in \( G \), we set \( V = W \otimes L_q(\varpi) \). Evidently, \( (3.5.1) \) follows from \( (3.5.2) \) as before and, therefore, we need to show that there exists a non-zero map \( f : V \rightarrow L_q(\pi) \otimes L_q(\pi') \).

Corollary \( 3.3.4 \) implies \( L_q(\pi(1) \varpi^{-1}) \otimes L_q(\varpi) \) is highest-\( \ell \)-weight and, hence, we have an epimorphism
\[
p : L_q(\pi \varpi^{-1}) \otimes L_q(\varpi) \twoheadrightarrow L_q(\pi).
\]
Since \( h \neq 0 \), \( (3.1.2) \) implies we have a non-zero homomorphism
\[
g : W \otimes *L_q(\pi') \rightarrow L_q(\pi \varpi^{-1}).
\]
In particular, since \( L_q(\pi \varpi^{-1}) \) is simple, \( g \) is surjective and, hence, so are
\[
g \otimes \text{id}_{L_q(\varpi)} : W \otimes *L_q(\pi') \otimes L_q(\varpi) \rightarrow L_q(\pi \varpi^{-1}) \otimes L_q(\varpi)
\]
and the composition
\[
W \otimes *L_q(\pi') \otimes L_q(\varpi) \xrightarrow{g \otimes \text{id}_{L_q(\varpi)}} L_q(\pi \varpi^{-1}) \otimes L_q(\varpi) \xrightarrow{p} L_q(\pi).
\]
Since \( \varpi \neq \omega \), it follows from our assumptions that \(*L_q(\pi') \otimes L_q(\varpi)\) is simple and we have an isomorphism
\[
\varphi : V \otimes *L_q(\pi') = W \otimes L_q(\varpi) \otimes *L_q(\pi') \rightarrow W \otimes *L_q(\pi') \otimes L_q(\varpi).
\]
Thus, the composition
\[
V \otimes *L_q(\pi') \xrightarrow{\varphi} W \otimes *L_q(\pi') \otimes L_q(\varpi) \xrightarrow{g \otimes \text{id}_{L_q(\varpi)}} L_q(\pi \varpi^{-1}) \otimes L_q(\varpi) \xrightarrow{p} L_q(\pi)
\]
is surjective. Finally, applying \( (3.1.2) \) again we obtain a non-zero homomorphism
\[
V \rightarrow L_q(\pi) \otimes L_q(\pi')
\]
as desired. \( \square \)
4. Three-Vertex Prime Graphs

This section is dedicated to the proofs of Theorems 2.4.6 and 2.4.8. In fact, most of the section is concerned with the former, while the latter is treated in the final (short) subsection.

4.1. Tensor Product of Fundamental Modules. Assume \( g \) is of type \( A \) and identify \( I \) with the set \( \{1, \ldots, n\} \) in such a way that \( d(i, j) = |j - i| \) for all \( i, j \in I \) (there are two options). Fix \( i, j \in I, a \in F^X, r, s \in \mathbb{Z}_{>0}, m \in R_{i,j}^{r,s} \), and consider

\[
V = L_q(\omega_{j,aq^m,s}) \otimes L_q(\omega_{i,a,r}).
\]

One can consider the problem of characterizing the simple factors of \( V \). For the purpose of proving Theorem 2.4.6, we need the case \( r = s = 1 \) only, in which case

\[
(4.1.1) \quad m = 2 + d(i, j) - 2p \quad \text{for some} \quad -d([i, j], \partial I) \leq p \leq 0.
\]

Letting \( i_- = \min\{i, j\}, i_+ = \max\{i, j\} \), and \( J \) the minimal connected subdiagram such that \( m \in R_{i,j}^{1,1} \), we have

\[
J = [i_- + p, i_+ - p].
\]

Evidently, if \( L_q(\mu) \) is such a simple factor, we must have

\[
\mu \in D := \text{wt}_\ell(V) \cap \mathcal{P}^+.
\]

The \( q \)-character of the fundamental modules are well-known and can be described in terms of column tableaux (see [4, Theorems 3.8 and 3.10], [11, Corollary 7.6 and Remark 7.4 (i)], and references therein). We partially review this approach.

Given \( a \in F^X, \) let \( \begin{array}{|c|}
\hline
i_1 \\
\vdots \\
1 \\
\hline
\end{array} \\
\begin{array}{c}
| \\
\hline
s \\
\hline
| \\
\begin{array}{|c|}
\hline
i_k \\
\hline
\end{array}
\end{array} \\
\end{array} \\
\begin{array}{c}
| \\
\hline
s \\
\hline
| \\
\begin{array}{|c|}
\hline
i_1 \\
\hline
\end{array}
\end{array} \\
\begin{array}{c}
| \\
\hline
s \\
\hline
| \\
\begin{array}{|c|}
\hline
i_k \\
\hline
\end{array}
\end{array} \\
\end{array} \\
\begin{array}{c}
| \\
\hline
s \\
\hline
| \\
\begin{array}{|c|}
\hline
i_1 \\
\hline
\end{array}
\end{array} \\
\begin{array}{c}
| \\
\hline
s \\
\hline
| \\
\begin{array}{|c|}
\hline
i_k \\
\hline
\end{array}
\end{array} \\
\end{array} \\
\begin{array}{c}
| \\
\hline
s \\
\hline
| \\
\begin{array}{|c|}
\hline
i_1 \\
\hline
\end{array}
\end{array} \\
\begin{array}{c}
| \\
\hline
s \\
\hline
| \\
\begin{array}{|c|}
\hline
i_k \\
\hline
\end{array}
\end{array} \\
\end{array}
\]}

and the support of the box having content \( i_j \) is \( s + 2(k - j) \). The \( \ell \)-weight associated to such tableau \( T \) is

\[
\omega^T = \prod_{j=1}^{k} \begin{array}{|c|}
\hline
i_j \\
\hline
\end{array}^{s+2(k-j)}.
\]

In particular, if \( i_j = j \) for all \( 1 \leq j \leq k \), \( \omega^T = \omega_{k,aq^{k-1}} \). Let \( \text{Tab}_{i,s} \) denote the set of all semi-standard column tableaux of height \( i \) supported at \( 1-i \). Then,

\[
(4.1.2) \quad qch(L_q(\omega_{i,a})) = \sum_{T \in \text{Tab}_{i,s}} \omega^T.
\]

In particular,

\[
\dim(L_q(\omega_{i,a})) = 1 \quad \text{for all} \quad \omega \in \text{wt}_\ell(L_q(\omega_{i,a})).
\]

Proceeding as in [9, Lemma 3.3.1], one easily checks that every element of \( D \) is of the form

\[
(4.1.3) \quad \nu \omega_{j,aq^m} \quad \text{with} \quad \nu \in \text{wt}_\ell(L_q(\omega_{i,a}))
\]
and, therefore, we must have $\nu = \omega^T$ for some $T \in \text{Tab}_{i,s}$ having at most one gap. Note $T$ has exactly one gap if and only if it is of the form

$$T_{k,l} := \begin{array}{c}
1 \\
\vdots \\
k \\
l+1 \\
l+i-k \\
l+i-k+1 \\
\vdots \\
1-i \\
\end{array} \quad \text{with } 0 \leq k < \min\{i,l\}, \ l \leq n - i + k + 1.$$

Note also

$$\omega^{T_{k,l}} = \omega_1^{-1} \omega_{i,aq^i-i-2k} \omega_{k,qa^{i-k}} \omega_{i+l-k,qa^{l-k}}.$$

Letting $J_{k,l} = [k + 1, l + i - k - 1]$, one can easily check that (cf. [9] Eq. (2.2.11))

$$\omega_i - \text{wt}(\omega^{T_{k,l}}) \in Q^+_{J_{k,l}}.$$

Evidently, for $\nu = \omega_{i,a}, L_q(\nu \omega_{j,qa}^m)$ is a simple factor. On the other hand, if $\nu$ is as in (4.1.4), then $\nu \omega_{j,qa}^m \in \mathcal{P}^+$ if and only if $l = j, i + l - 2k = m$, and, hence,

$$k = p - 1 + i, \quad J_{k,l} = J,$$

and

$$D = \{\omega_{i,a} \omega_{j,qa}^m, \ \omega_{i+p-1,qa^{1-p-i-1}} \omega_{i+1-p,qa^{1-p+j-i-1}}\}.$$

Moreover,

$$L_q(\omega_{i+p-1,qa^{1-p+i-1}}) \otimes L_q(\omega_{i+1-p,qa^{1-p+j-i-1}}) \text{ is simple.}$$

Indeed, to check this, it suffices to show that

$$d(i,j) = |(1 - p + j - i - 1) - (1 - p + i - i)| \notin R_{1+p-1,i+1+p}^{1,1}.$$

The elements of $R_{1+p-1,i+1+p}^{1,1}$ are of the form $2 + d(i_1 + p - 1, i_1 + 1 - p) - 2p'$ with $p' \leq 0$. Since

$$d(i_1 + p - 1 - 1 + 1 - p) = 2(1 - p) + d(i, j) > d(i, j),$$

(4.1.7) follows. In summary, we have proved there exists a short exact sequence (see also [9] and references therein).

$$0 \rightarrow L_q(\omega_{i+p-1,qa^{1-p+i-1}}) \otimes L_q(\omega_{i+1-p,qa^{1-p+j-i-1}}) \rightarrow V \rightarrow L_q(\omega_{i,a} \omega_{j,qa}^m) \rightarrow 0.$$

## 4.2. Singleton Cuts

Any nontrivial cut of a $q$-factorization graph with three vertices has one of its subgraphs being a singleton. More generally, if [10] Conjecture 3.5.1 is true, the relevance of such cuts for the theory is evident. Thus, let us start from this broader setup.

Given $\pi \in \mathcal{P}^+, i \in I, a \in \mathbb{F}^\times, \text{ and } r \in \mathbb{Z}_{>0}$, let

$$V = L_q(\omega_{i,a,r}) \otimes L_q(\pi).$$

The goal is to describe conditions on the given parameters for deciding whether $V$ is simple or not. By Corollary 3.3.4 if $L_q(\omega) \otimes L_q(\omega_{i,a,r})$ is simple for every $q$-factor $\omega$ of $\pi$, then $V$ is simple, but this is certainly not a necessary condition. In any case, we can assume there exists $j \in I, m, s \in \mathbb{Z}, \ s > 0$, such that $\omega_{j,qa^m,s}$ is a $q$-factor of $\pi$ and $|m| \in R_{i,j}^{r,s}$. We can also assume $\pi$ and $\omega_{i,a,r}$ have dissociated $q$-factorizations, otherwise Proposition 2.2.3 implies $V$ is reducible. Evidently,

$$L_q(\pi) \cong L_q(\omega_{j,qa^m,s}) \otimes L_q(\omega_{j,qa^m,s}^{-1}) \Rightarrow \ V \text{ is reducible.}$$

Thus, we shall assume $L_q(\omega_{j,qa^m,s}) \otimes L_q(\pi \omega_{j,qa^m,s}^{-1})$ is reducible.

**Lemma 4.2.1.** Let $J$ be a connected subdiagram such that $[i, j] \subseteq J$ and $|m| \in R_{i,j}^{s,j}$. Let $G(\pi)$ be the connected component of $G(\pi, j)$ containing $(\omega_{j,qa^m,s})$. If $L_q((\omega_{i,a,r}, j)) \otimes L_q(\pi)$ is reducible, so is $V$. In particular, this is the case if $G(\pi)$ is a singleton.
Proof. Write $\pi J = \omega \omega'$. The assumption on $G(\omega)$ together with Corollary 3.3.4 implies $L_q(\pi J) \cong L_q(\omega) \otimes L_q(\omega')$ and, hence,

$$L_q((\omega_{i,a,r})_J) \otimes L_q(\pi J) \cong L_q((\omega_{i,a,r})_J) \otimes L_q(\omega) \otimes L_q(\omega'),$$

is reducible. The first conclusion now follows from Corollary 3.2.3. Note $G(\omega)$ is a singleton if and only if $\omega = (\omega_{i,aq^m,s})_J$. Thus, the second conclusion follows from the first and the assumption on $m$. □

The next lemma is immediate from the previous.

Lemma 4.2.2. Let $J$ be a connected subdiagram such that $[i, j] \subseteq J$ and $|m| \in R^{s,s'}$. If $L_q((\omega_{i,aq^m,s})_J) \otimes L_q(\omega')$ is simple for every $q$-factor $\omega'$ of $\pi$ adjacent to $\omega_{i,aq^m,s}$ in $G(\pi)$, then $V$ is reducible. In particular, if $V$ is simple, there exist $j' \in J$, $s' \in \mathbb{Z}_{>0}$, and $m' \in \pm R^{s,s'}_{j,j'}$ such that $\omega_{j',aq^m-m',s'}$ is a $q$-factor of $\pi \omega_{i,aq^m,s}^{-1}$.

Henceforth, we assume $\pi$ has two $q$-factors, say

$$\pi = \omega_{j,aq^m,s} \omega_{j',b,s}.$$

Note condition (i) of the above lemma becomes equivalent to $j' \neq j$, since we are assuming $L_q(\omega_{j,aq^m,s}) \otimes L_q(\omega_{j',b,s})$ is reducible. This assumption also implies we must have

$$aq^m = bq^{m'} \quad \text{with} \quad |m'| \in R^{s,s'}.$$

We have the following possibilities for $G := G(\pi \omega_{i,a,r})$: a line (directed or alternating) with $\omega_{i,a,r} \in \partial G$ or a triangle (with $\omega_{i,a,r}$ extremal or not). So, the pictures, up to arrow duality, are:

1. \[
    \begin{array}{c}
    i \quad \overrightarrow{r} \quad j \quad \overrightarrow{m} \quad j' \quad \overrightarrow{s'}
    \end{array}
\] (which implies $m > 0$ and $m' - m \notin R^{r,s'}$);

2. \[
    \begin{array}{c}
    i \quad \overrightarrow{r} \quad j \quad \overrightarrow{m} \quad j' \quad \overrightarrow{s'}
    \end{array}
\] (which implies $m > 0$ and $|m - m'| \notin R^{r,s'}$);

3. \[
    \begin{array}{c}
    s \quad \overrightarrow{i} \quad j \quad \overrightarrow{m''} \quad s' \quad \overrightarrow{j'}
    \end{array}
\] (which implies $m' = m'' = m \in R^{r,s'}$);

4. \[
    \begin{array}{c}
    s \quad \overrightarrow{i} \quad j \quad \overrightarrow{m'} \quad s' \quad \overrightarrow{j'}
    \end{array}
\] or \[
    \begin{array}{c}
    s \quad \overrightarrow{i} \quad j \quad \overrightarrow{m'} \quad s' \quad \overrightarrow{j'}
    \end{array}
\] (with $m > 0$ in the first case and $m < 0$ in the second).

We chose to draw the pictures corresponding to the case $m' > 0$. The cases with $m' < 0$ are obtained by arrow duality (which preserves the property of the studied tensor product being simple or not). In the last two cases, the arrow duality interchanges the roles of $j$ and $j'$ in the following arguments. The reducibility of $V$ if $G$ is as in (1) or (4) follows from Proposition 3.3.7, while case (3) follows from Theorem 2.4.5 for type $A$. The next subsection is dedicated to the study of case (2), thus proving Theorem 2.4.6.

4.3. The proof of Theorem 2.4.6. We now suppose $G$ is as in (2) of the previous subsection. Let $J \subseteq I$ denote a minimal connected subdiagram satisfying

$$i, j \in J \quad \text{and} \quad m \in R^{r,s}_{i,j,J}.$$ 

One easily checks using the characterization of the sets $R^{r,s}_{i,j,J}$ in (2.2.3) that, if $g$ is of type $A$, $J$ is unique. In that case, we will show that $V$ is simple if and only if

$$j' \in J, \quad m' \in R^{s,s'}_{j,j',J}, \quad |m - m' - h_j| \in R^{r,s'}_{w_0(i),j',J}.$$
and

\[(4.3.2)\quad m - m' + 1 \notin \mathbb{R}^{r-1,s'}_{i,j'} \quad \text{if} \; r > 1.\]

We begin by showing that \((4.3.1)\) holds if \(V\) is simple and \(g\) is simply laced. The first two conditions are immediate from Lemma \((1.2.2)\). For the last condition, assume \(j' \in J, m' \in \mathbb{R}^{s',s'}_{j,j'}, \) consider

\[N = L_q(\omega_{j,aq^m,s}) \otimes L_q(\omega_{i,a,r}) \quad \text{and} \quad N' = L_q(\omega_{j,aq^m,s}) \otimes L_q(\omega_{j',aq^m-m',s'}),\]

and note that \(N_J\) and \(N'_J\) are highest-\(\ell\)-weight and reducible. Thus, by the second part of Proposition \((3.4.3)\), \(V_J\) is reducible provided

\[L_q((\omega_{i,a,r})_J) \otimes L_q((\omega_{j',aq^m-m',s'})_J)^*\]

is simple or, equivalently,

\[|m - m' - \tilde{h}_J| \notin \mathbb{R}^{r,s'}_i,\quad \text{for some} \; \tilde{h}_J \in \mathbb{R}^{r,s'}_i, \; j', J.\]

For type \(A\), we shall see below that

\[|m - m' - \tilde{h}_J| = m' - m + \tilde{h}_J\]

and \((4.3.2)\) is equivalent to

\[(4.3.3)\quad m + r \leq m' + s' + d(i,j').\]

Henceforth, we assume \((4.3.1)\) holds and \(g\) is of type \(A\). In that case, we show \(V\) is simple if and only \((4.3.2)\) holds. Note

\[V' := L_q(\pi) \otimes L_q(\omega_{i,a,r})\]

is highest-\(\ell\)-weight. Therefore, in light of Proposition \((3.3.2)\),

\[(4.3.4)\quad V \text{ is simple} \iff V \text{ is highest-}\ell\text{-weight.}\]

Since \(g\) is of type \(A\), \((2.2.3)\) implies

\[(4.3.5)\quad m = r + s + d(i,j) - 2p \quad \text{for some} \quad -d([i,j], \partial I) \leq p < \min\{r,s\}\]

and

\[(4.3.6)\quad m' = s + s' + d(j,j') - 2p' \quad \text{for some} \quad -d([j,j'], \partial I) \leq p' < \min\{s,s'\}.\]

Moreover, \(m \notin \mathbb{R}^{r,s'}_{i,j,[i,j]}\) if and only if \(p < 0\) and similarly for \(m'\).

The proof will proceed by separate analyzes of several cases and subcases. The main splitting is according the sign of the parameter \(p\) defined in \((4.3.5)\). The proof for \(p \leq 0\) is in Section \((4.3.2)\) while the case \(p > 0\) is treated in Section \((4.3.3)\). The knowledge of the case \(p \leq 0\) is used in Section \((4.3.3)\), so the proof of the latter is not independent of that of the former. To prove that \(V\) is simple if \((4.3.2)\) holds, in most cases we will prove that \(V\) is highest-\(\ell\)-weight as an application of Theorem \((3.3.3)\). This will not be the case only for \(p \leq 0\) and \(r = 1\), which requires a more sophisticated argument which uses \((4.1.5)\), \((4.1.7)\), and, in one crucial moment, Theorem \((3.2.1)\). However, this case is used in the proof of other cases with \(p \leq 0\), as well as for the proof of the converse. So, indirectly, the whole proof depends on this more complicated argument for this particular subcase. To prove the converse, i.e., that \(V\) is reducible if \((4.3.2)\) does not hold, we use Corollary \((3.2.4)\) and Propositions \((3.4.2)\) and \((3.4.3)\) for \(p \leq 0\), while, for \(p > 0\), we show \(V\) is not highest-\(\ell\)-weight by contradiction using Theorem \((3.3.3)\).
4.3.1. Technical Rephrasing of the Conditions. We start by describing detailed conditions on the parameters $i, j, j', r, s, s', p,$ and $p'$ arising from (4.3.1) and (4.3.2). Recall (2.1.1) and that, since $g$ is of type $A$, we have $d(i, \partial J) = d(j, \partial J)$, which implies

$$w_0^j(i) = j.$$  

If $p'$ is given by (4.3.6), we have

$$m - m' = \pm (r + s' + d(i, j') - 2p_\pm)$$

where

$$p_+ = s' - p' + p + d_{i,j}^{j'}$$

and

$$p_- = r - p + p' + d_{j,j'}.$$  

Note

$$|m - m'| = r + s' + d(i, j') - 2p_+ \quad \iff \quad 2p_+ \leq r + s' + d(i, j') \quad \iff \quad m \geq m';$$

and

$$|m - m'| = r + s' + d(i, j') - 2p_- \quad \iff \quad 2p_- \leq r + s' + d(i, j') \quad \iff \quad m \leq m'.$$

Thus, the condition $|m - m'| \notin \mathcal{R}_{i,j'}^{s,s'}$, arising from the assumption that $G$ is as in (2) of Section 4.2 can be rephrased as

$$p_+ < -d([i, j'], \partial I) \quad \text{or} \quad p_+ \geq \min\{r, s', \} \quad \text{if} \quad m \geq m';$$

and

$$p_- < -d([i, j'], \partial I) \quad \text{or} \quad p_- \geq \min\{r, s', \} \quad \text{if} \quad m \leq m'.$$

In what follows, any statement about $p_+$ will carry implicitly the assumption that $m \geq m'$ and similarly for $p_-$.  

**Case 1.** If $p \leq 0$, we have

$$d(i, \partial J) = d(j, \partial J) = -p$$

and the first two requirements in (4.3.1) imply

$$-p' \leq d([j, j'], \partial J) = d([i, j'], \partial J) = -p - d_{i,j}^{j'}.$$  

Indeed, one easily checks the equalities follow from (4.3.10) and the condition $j' \in J$. The inequality is immediate from the definition of $\mathcal{R}_{i,j'}^{s,s'}$ and the second condition in (4.3.1). Note (4.3.11) implies

$$m - m' = r - s' + 2(p' - p) + d(i, j) - d(j', j) \geq r - s' + d(i, j').$$

Thus,

$$m \leq m' \quad \Rightarrow \quad r \leq s' - d(i, j') \leq s'.$$

Let us see that (4.3.11) also implies

$$p_\pm \geq \min\{r, s', \} \quad \text{and} \quad p_+ \leq s'.$$

In the case of $p_-$, (4.3.11) implies

$$p_- = r + p' + d_{j,j'} - p \geq r + d_{j,j'} + d_{j,j'} \geq r.$$  

For $p_+$, (4.3.11) implies

$$p_+ = s' - p' + p + d_{i,j}^{j'} \leq s'.$$  

It remains to check the first inequality for $p_+$. By (4.3.9), it suffices to show we cannot have $p_+ < -d([i, j'], \partial I)$. Indeed, if this were the case, we would have

$$s' - p' + d_{i,j}^{j'} = p_+ < -d([i, j'], \partial I) \leq -d([i, j'], \partial J) = p + d_{i,j}^{j'},$$

which yields a contradiction since $p' < s'$. In particular, it follows from (4.3.13) that

$$r \geq s' \quad \Rightarrow \quad p_+ = s'.$$

Next, we check the third condition in (4.3.1) is equivalent to

$$|m - m' - \bar{h}_{i,j}| = -(m - m' - \bar{h}_{i,j}) \in \mathcal{R}_{j,j'}^{s,s'}.$$
To do that, it suffices to check
\[ m - m' - \hat{h}_{ij} \notin \mathbb{R}_{i,j}^{r,s'}. \]
If \( m - m' \leq 0 \), this is obvious since \( m - m' - \hat{h}_{ij} < 0 \). Otherwise,
\[ m - m' - \hat{h}_{ij} = r + s' + d(i,j') - 2p_+ - \hat{h}_{ij} = r + s' + d(j,j') - 2(p_+ - p + d_{i,j'}^j + 1) \]
and we are done since
\[ p_+ - p + d_{i,j'}^j + 1 \geq \min\{r, s'\} - 0 + 0 + 1 > \min\{r, s'\}. \]
Let us see (4.3.15) implies
\[ (4.3.16) \quad p' \leq \min\{r, s'\} - r \leq 0. \]
Indeed,
\[ -(m - m' - \hat{h}_{ij}) = r + s' + d(j,j') - 2(r + p' - 1), \]
and the claim follows from the description of \( \mathbb{R}_{i,j}^{r,s'}. \) This completes the information we needed to extract from (4.3.1).

Next, for \( r > 1 \), we check that (4.3.2) is equivalent to
\[ (4.3.17) \quad r \leq s'. \]
If \( m < m' \), then (4.3.2) is obvious and (4.3.17) follows from (4.3.12). Otherwise, we have
\[ m - m' + 1 = (r - 1) + s' + d(i,j') - 2(p_+ - 1), \]
and it lies in \( \mathbb{R}_{i,j'}^{r-1,s'} \) if and only if \(-d([i,j'], \partial J) \leq p_+ - 1 < \min\{r - 1, s'\}\). The first inequality in (4.3.13) implies this is the same as
\[ \min\{r, s'\} - 1 \leq p_+ - 1 < \min\{r - 1, s'\}, \]
which is clearly impossible if \( r \leq s' \). For \( r > s' \), (4.3.14) implies \( p_+ = s' \) and, hence, \( m - m' + 1 \in \mathbb{R}_{i,j'}^{r-1,s'} \).

Finally, let us check that (4.3.2) is equivalent to (4.3.3). Using (4.3.5) and (4.3.6), one easily checks the latter is equivalent to
\[ (4.3.19) \quad r - s' \leq p - p' + d_{i,j'}^{j'}. \]
This, together with (4.3.11), clearly implies (4.3.17) and, hence, (4.3.2). Conversely, if (4.3.17) holds, then (4.3.3) is obvious if \( m \leq m' \). Otherwise, (4.3.19) follows from the first inequality in (4.3.13). In particular, (4.3.19) holds if \( r = 1 \).

**Case 2.** If \( p \geq 0 \), then
\[ (4.3.20) \quad J = [i, j] \quad \text{and} \quad \hat{h}_{ij} = d(i,j) + 2 \]
and the first two conditions in (4.3.11) immediately imply
\[ (4.3.21) \quad j' \in [i, j] \quad \text{and} \quad p' \geq 0. \]
Observe also that the first claim in (4.3.13) remains valid. Indeed, in light of (4.3.9), it suffices to show \( p_+ > 0 \). Since, by definition, \( p < r \) and \( p' < s' \), we have
\[ p_- = (r - p) + p' + d_{i,j'}^{j'} > 0 \quad \text{and} \quad p_+ = (s' - p') + p + d_{i,j'}^{j'} > 0. \]
We also observe that (4.3.13) remains valid. As before, this is obvious if \( m - m' \leq 0 \). Otherwise,
\[ m - m' - \hat{h}_{ij} = r + s' + d(i,j') - 2p_+ - \hat{h}_{ij} = r + s' + d(j,j') - 2(p_+ + d(j,j') + 1) \]
and we are done since
\[ p_+ + d(j,j') + 1 > p_+ \geq \min\{r, s'\}. \]
Using (4.3.7), we also have
\[ -(m - m' - \hat{h}_{ij}) = r + s' + d(j,j') - 2(r - p + p' - 1). \]
Thus, (4.3.15) is equivalent to
\[ (4.3.22) \quad 0 \leq r - p + p' - 1 < \min\{r, s'\}, \]
which, in particular, implies
\[(4.3.23)\quad p \geq p' \quad \text{as well as} \quad r \leq s' + p - p' \quad \text{if} \quad r \geq s'.\]

Next, we check that \[(4.3.2)\] is equivalent to
\[(4.3.24)\quad r \leq s' \quad \text{or} \quad p \neq p'.\]

Indeed, if \(m < m'\), then \[(4.3.2)\] is obvious and, if \(r > s'\), the second part of \[(4.3.23)\] implies \(p \neq p'\), showing \[(4.3.24)\] holds. Otherwise, studying \[(4.3.18)\], we see
\[
m - m' + 1 \in R_{i,j',j}^{r-1,s'} \iff r > s' \quad \text{and} \quad p_+ = s'.
\]

Now \[(4.3.7)\] implies the condition \(p_+ = s'\) is equivalent to \(p = p'\), completing the proof.

Finally, let us check that \[(4.3.2)\] is equivalent to \[(4.3.3)\] or, equivalently, to \[(4.3.19)\] which, in this case, reduces to
\[(4.3.25)\quad r - s' \leq p - p'.\]

Evidently, \[(4.3.25)\] implies \[(4.3.24)\]. Conversely, \[(4.3.25)\] clearly holds if \(r \leq s'\) (which includes the case \(r = 1\)). Otherwise, if \(p \neq p'\), \[(4.3.25)\] follows from the second statement in \[(4.3.23)\].

4.3.2. The Proof - Case 1. It follows from Section 4.3.1 that, for \(p \leq 0\), \[(4.3.2)\] is equivalent to \(r \leq s'\). The case \(r \leq s'\) will be treated in Step 1 below while Step 2 will deal with the case \(r > s'\).

We remark that Step 1 with \(r = 1\) will be used in the proof of Step 2.

**Step 1.** Assume \(r \leq s'\). We proceed by induction on \(r\). The proofs of the cases \(r = 1\) and \(r > 1\) can be read independently. The argument for the latter is simpler and only evokes the simplicity of \(V\) in the case \(r = 1\) with no partial use of its proof.

**Base of induction.** We will now treat the case \(r = 1\). However, we will prepare the argument without this restriction and set \(r = 1\) later on. We will also use a subinduction on \(s\), which clearly starts for \(s = 0\), since \(m - m' \notin R_{i,j}^{r',s'}\). Thus, we assume \(s \geq 1\).

If \(V\) were reducible, so would be \(V'\) and therefore, there would exist \(\lambda \in P^+\) such that
\[(4.3.26)\quad \lambda < \pi \omega_{i,a,r}.
\]

and nonzero homomorphisms
\[(4.3.27)\quad L_0(\lambda) \to V' \quad \text{and} \quad V \to L_0(\lambda).
\]

Since \(L_0(\omega_{j',b,s'}) \otimes L_0(\omega_{i,a,r})\) is simple, \[(4.1.3)\] implies there exists \(\nu \in P^+\) such that
\[(4.3.28)\quad \lambda = \nu \omega_{j',aq^m-b',s'} \quad \text{and} \quad L_0(\nu) \hookrightarrow T_{r,s}^{m_1} := L_0(\omega_{j',aq^m,s}) \otimes L_0(\omega_{i,a,r}).
\]

In fact, \(L_0(\nu)\) is the socle of \(T_{r,s}^{m_1}\) and, since \(m > 0\), we have \(\nu < \omega_{i,a,r} \omega_{j',aq^m,s}\).

Setting \(\pi = \pi(\omega_{j',aq^{m+1}},-1) = \omega_{j',aq^{m-1},s-1} \omega_{j',aq^m-b',s'}\), note also that we have an inclusion
\[(4.3.29)\quad L_0(\pi) \hookrightarrow L_0(\pi) \otimes L_0(\omega_{j',aq^{m+1}}),
\]

which can be combined with the above one to obtain
\[
L_0(\lambda) \hookrightarrow L_0(\pi) \otimes T_{r,s}^{m+s-1}.
\]

By \[(3.1.2)\], this gives rise to a nonzero map
\[
L_0(\pi)^* \otimes L_0(\lambda) \to T_{r,s}^{m+s-1}.
\]

Let \(L_0(\mu)\) be a simple quotient of the image of this map. Thus, we get an epimorphism
\[
L_0(\mu)^* \otimes L_0(\lambda) \to L_0(\mu)
\]

and, using \[(3.1.2)\] again, an inclusion
\[(4.3.30)\quad L_0(\lambda) \hookrightarrow T' := L_0(\pi) \otimes L_0(\mu).
\]

Since \(L_0(\mu)\) is a simple factor of \(T_{r,s}^{m+s-1}\), a description of the simple factors of \(T_{r,s}^{m+s-1}\), which is done in Section \(4.1\) for \(r = 1\), allows us to study the possibilities for \(\mu\) to reach a contradiction.
For instance, if \( \mu = \omega_{j,aq^{m+s-1}i,a,r} \), we claim
\[
L_q(\mu) \otimes L_q(\pi) \quad \text{is highest-} \ell\text{-weight.}
\]
Using this and Lemma 3.3.1 it follows from (4.3.30) that
\[
(4.3.31) \quad \lambda = \mu \pi = \pi \omega_{i,a,r},
\]
contradicting (4.3.26). Thus, we can assume
\[
(4.3.32) \quad \mu < \omega_{j,aq^{m+s-1}i,a,r}.
\]
To prove the claim, note
\[
m + s - 1 \geq m - 1 \geq m - m'.
\]
Thus, by Corollary 3.3.4 we are left to check that
\[
\tilde{V} := L_q(\omega_{i,a,r}) \otimes L_q(\pi) \quad \text{is highest-} \ell\text{-weight.}
\]
We proceed by induction on \( s \) to show it is actually simple. The case \( s = 1 \) is immediate from the assumption \( |m - m'| \notin \mathcal{R}_{i,j}^{r,s'} \). Otherwise, we need to check whether \( \tilde{V} \) satisfies all the assumptions we have made on \( V \) or not, starting by checking that its \( q \)-factorization graph is as in (2) of Section 4.2. We have replaced \( m \) by \( m - 1 \) and \( s \) by \( s - 1 \), so the condition required on the graph rephrases as
\[
m - 1 \in \mathcal{R}_{i,j}^{r,s-1}, \quad (m - 1) - (m - m') \in \mathcal{R}_{j,j'}^{s-1,s'}, \quad \text{and} \quad |m - m'| \notin \mathcal{R}_{i,j}^{r,s'}.
\]
Evidently, the last of these is satisfied. Note
\[
m - 1 = r + (s - 1) + d(i, j) - 2p
\]
and, since we are assuming \(-d([i, j], \partial I) \leq p \leq 0\), the first condition is verified. The checking of the second is similar since we know \( p' \leq 0 \) by (4.3.16). The inductive argument is completed by checking that \( \tilde{V} \) satisfies the corresponding version of (4.3.1), since (4.3.17) is obviously satisfied. All conditions are easily checked after noting that \( J \) is also the minimal subdiagram satisfying
\[
i, j \in J \quad \text{and} \quad m - 1 \in \mathcal{R}_{i,j,j}. \]
Indeed, if \( J' \subsetneq J \), we have
\[
d([i, j], \partial J') < d([i, j], \partial J) \leq -p.
\]
But then,
\[
m - 1 = r + (s - 1) + d(i, j) - 2p \notin \mathcal{R}_{i,j,j}^{r,s-1}.
\]
This completes the proof that we can assume (4.3.32).

Note (4.3.32) implies \( T_{r,s-1}^{m+s} \) is reducible and, hence,
\[
(4.3.33) \quad m + s - 1 = r + 1 + d(i, j) - 2(p - s + 1) \in \mathcal{R}_{i,j}^{1,1}.
\]
We now set \( r = 1 \). To simplify the use of the formulas from Section 4.1, we assume, without loss of generality, that \( i \geq j \). Recall \( T_{1,1}^{m+s-1} \) has length 2 by (4.1.7) and (4.1.8). Using (4.3.33) in (4.1.8), we have
\[
(4.3.34) \quad \mu = \omega_{j+p-s,aq^{s-p}+d(i, j)\ i+s-p,aq^{s-p}+}. \quad \text{We claim that}
\]
\[
(4.3.35) \quad T' \cong L_q(\lambda).
\]
In particular, the first equality in (4.3.31) remains valid and (4.3.27) implies there exists a monomorphism \( T' \to V' \). Let \( K \) be the minimal connected subdiagram such that \( m - s = 1 \in \mathcal{R}_{i,j,K}^{1,1} \). It follows from (4.3.33) and (4.1.6) that
\[
K = [j + p - s + 1, i - p + s - 1] \quad \text{and, hence,} \quad \mu_K = 1.
\]
Moreover, it follows from the first equality in (4.3.31), (4.1.5), and (4.1.6) that
\[
\omega_i + \operatorname{wt}(\pi) - \operatorname{wt}(\lambda) = \omega_i + \operatorname{wt}(\pi) - (\operatorname{wt}(\mu) + \operatorname{wt}(\pi)) = \omega_i + (\operatorname{wt}(\pi) - \operatorname{wt}(\pi)) - \operatorname{wt}(\mu)
\]
\[
= \omega_i + \omega_j - \operatorname{wt}(\mu) \in Q_K^+. \]
Therefore, Lemma 3.2.5 implies we also have a monomorphism
\[ L_q(\pi_K) \to L_q(\omega_{i,a}) K \]
An application of (3.1.2) for the subalgebra \( U_q(\tilde{\mathfrak{g}})_K \) implies we have an epimorphism
\[ L_q(\pi_K) \otimes L_q(\omega_{i,a}) K^s \to L_q(\pi_K) \]
Since \( h^+_K = (i - p + s - 1) - (j + p - s + 1) + 2 = m + s - 1 \) and \( w^+_0(i) = j \), this is the same as
\[ L_q(\pi_K) \otimes L_q(\omega_{j,aq^{m+1-1}})_K \to L_q(\pi_K) \]
Combining with (4.1.29), it follows that this map is an isomorphism. However, let us see that
\[ L_q(\pi_j) \otimes L_q(\omega_{j,aq^{m+1-1}})_j \cong L_q(\omega_{j,aq^{m-1}})_j \otimes L_q(\omega_{j,aq^{m+1-1}})_j \]
which is reducible since \( \omega_{j,aq^{m-1}} = \omega_{j,aq^{m},s} \), reaching the desired contradiction. Indeed, if \( j' \neq j \) the isomorphism is clear since \( \omega_{j',aq^{m-1},s} = 1 \). Otherwise, if \( j' = j \), the isomorphism follows because \( \omega_{j',aq^{m-1},s} \) and \( \omega_{j,aq^{m},s} \) are \( q \)-factors of \( \pi \), or equivalently, \( m' \notin R^{s,s'} \).
Indeed, the isomorphism is not true if and only if
\[ m' - 1 = (m - 1) - (m - m') \in R^{s-1,s'}, \]
which implies \( m' \notin R^{s,s'} \).

It remains to prove (4.3.35). By (4.1.7) and Theorem 3.4.1, it suffices to show that
(4.3.36)
\[ L_q(\omega_{j,aq^{m-1},s},\omega_{j,aq^{m+1-1},s-1}) \otimes L_q(\omega_{j,aq^{m+1-1},s}) \]
and
\[ L_q(\omega_{j,aq^{m+1-1},s},\omega_{j,aq^{m+1-1},s-1}) \otimes L_q(\omega_{i+1,s-p,q^{s-p}}) \]
are simple. By Corollary 3.3.1, this follows, respectively, if we check.
(4.3.37) \[ |m - m' - (s - p + d(i, j))| \notin R^{1,s'}_{j+p-s-j'}, \quad |m - 1 - (s - p + d(i, j))| \notin R^{1,s-1}_{j+p-s-j}, \]
(4.3.38) \[ |m - m' - (s - p)| \notin R^{1,s'}_{i+s-p-j}, \quad \text{and} \quad |m - 1 - (s - p)| \notin R^{1,s-1}_{i+s-p-j}, \]
The second condition in each of these lines is vacuous if \( s = 1 \).

We start with the first condition in (4.3.37). If \( m - m' - (s - p) \geq 0 \), then \( m \geq m' \) and (4.3.7) implies
\[ m - m' - (s - p) = 1 + s' + d(i, j') - 2p_+ - (s - p) \]
\[ = 1 + s' + d(i + s - p, j') - 2(p_+ - d_{i+s-p,j'} + s - p). \]
Thus, it suffices to check that \( p_+ - d_{i+s-p,j'} + s - p \geq 1 \). Indeed, using (2.1.2) we get
\[ p_+ - d_{i+s-p,j'} + s - p \geq 1 = \min\{1, s'\}. \]
and (4.3.13) completes the checking. If \( m - m' - (s - p) < 0 \), we have\(^1\)
\[ -(m - m' - (s - p)) = 1 + s' + d(i + s - p, j') - 2(p_+ - d_{i+s-p,j'}). \]
But,
\[ p_+ - d_{i+s-p,j'} \geq p_- - d(i, j') \geq (1 - p + p' + d_{i,j'}) - d(i, j') \]
\[ = 1 - p + p' - d_{i,j} \geq 1, \]
completing the proof of the first condition in (4.3.37). As for the second, note
\[ m - 1 - s + p \Rightarrow d(i, j) = 1 + (s - 1) + d(i + s - p, j) - 2s. \]
The middle expression shows this number is nonnegative and we are done since \( s \geq \min\{1, s - 1\}. \)

\(^1\)Here we are using \( p_- \) to shorten notation without making any assumption on the sign of \( m - m' \).
Next, consider the first condition in \((4.3.37)\). By \((4.3.5)\) and \((4.3.6)\), we have
\[
m - m' - (s - p + d(i, j)) = 1 - s - s' + 2p' - p - d(j, j')
\]
\[
= 1 + s' + d(j', j + p - s) - 2(s' - p' + 1 + d)
\]
where
\[
2d = d(j, j') + (j' - j) + 2(s - 1) \geq 0.
\]
Since \(p' \leq 0\) by \((4.3.16)\), it follows that \(s' - p' + 1 + d \geq 1\) and, hence,
\[
m - m' - (1 - p + d(i, j)) \notin \mathcal{R}_{j + p - 1, j'}^{1, s'}.
\]

On the other hand,
\[
-(m - m' - (s - p + d(i, j))) = s + s' - 1 - 2p' + p + d(j, j')
\]
\[
= 1 + s' + d(j', j + p - s) - 2(p' - p + 1 - d)
\]
where \(2d = d(j, j') + (j' - j) \geq 0\). By \((4.3.11)\), we have
\[
p' - p + 1 - d \geq 1 + d'_{i,j} - d = \frac{1}{2}(d(i, j') + j' - i) \geq 1 = \min\{1, s'\},
\]
thus completing the proof of the first condition in \((4.3.37)\). As for the second,
\[
m - 1 - (s - p + d(i, j)) \equiv -p = 1 + (s - 1) + d(j + p - s, j) - 2s.
\]

The middle expression shows this number is nonnegative and we are done since \(s \geq \min\{1, s - 1\}\).

**Inductive Argument.** For \(r > 1\) we will show \(V\) is highest-\(\ell\)-weight which completes the proof by \((4.3.4)\). To prove this, it suffices to show that
\[
L_q(\omega_{i, aq, r - 1}) \otimes L_q(\omega_{i, aq^{1-r}}) \otimes L_q(\pi) \text{ is highest-\(\ell\)-weight}.
\]

Since the tensor product of the first two factors is highest-\(\ell\)-weight, Theorem \(3.3.3\) implies it suffices to show
\[
(4.3.39) \quad L_q(\omega_{i, aq^{1-r}}) \otimes L_q(\pi) \text{ and } L_q(\omega_{i, aq, r - 1}) \otimes L_q(\pi) \text{ are highest-\(\ell\)-weight}.
\]
The remainder of the argument will show that these tensor products are actually simple, thus completing Step 1.

By Corollary \(3.3.4\), \(2.2.1\), and \(2.2.2\), the first claim in \((4.3.39)\) follows if
\[
(4.3.40) \quad m - (1 - r) \notin \mathcal{R}_{i,j}^{1,s} \quad \text{and} \quad (m - m') - (1 - r) \notin \mathcal{R}_{j,j'}^{1,s'}.
\]

For the second number, note that, if \(m - m' \leq 0\), then
\[
m - m' - (1 - r) \leq r - 1 < r + 1 + d(i, j') = \min\mathcal{R}_{i,j'}^{1,s'}.
\]

Thus, we can assume \(m > m'\) in which case we have
\[
(4.3.41) \quad (m - m') - (1 - r) = r + s' + d(i, j') - 2p' - (1 - r) = 1 + s' + d(i, j') - 2(p' + r - 1).
\]

Since \(r \leq s'\), \((4.3.13)\) implies \(p' + r - 1 \geq 1 = \min\{1, s'\}\), thus proving the second claim in \((4.3.40)\).

For the first claim in \((4.3.40)\), we have
\[
0 \leq m - (1 - r) = 1 + s + d(i, j) - 2(p + r - 1).
\]

If \(p - r + 1 < -d(i, j, \partial I)\), \((4.3.40)\) is proved. Otherwise, instead of checking \((4.3.40)\), we assume its first claim fails and use the induction hypothesis to prove directly that the first tensor product in \((4.3.39)\) is simple.

Note we have \(a\) replaced by \(aq^{1-r}\) and \(r\) replaced by \(1\), so \(m - (1 - r)\) plays the role that \(m\) did while \(p - r + 1\) plays the role that \(p\) did. In particular, the initial assumption of Section \(4.3.2\) is satisfied. Let us check whether the associated \(q\)-factorization graph is as in (2) of Section \(4.2\) i.e., if
\[
(4.3.42) \quad m - (1 - r) \in \mathcal{R}_{i,j}^{1,s} \quad \text{m' } \in \mathcal{R}_{j,j'}^{s,s'}, \quad \text{and} \quad |m - m' - (1 - r)| \notin \mathcal{R}_{i,j'}^{1,s'}.
\]
The second condition is obvious and we have already partially proved the third in (4.3.40), remaining to consider the case \(|m - m' - (1 - r)| = -(m - m' - (1 - r))\). In this case we necessarily have \(m \leq m'\) and, thus,

\[-(m - m' - (1 - r)) = r + s' + d(i, j') - 2p_- + (1 - r) = 1 + s' + d(i, j') - 2p_-\]

and the proof is completed using (4.3.13). After this, note that if the first condition on (4.3.42) fails, the first tensor product in (4.3.39) is simple by Corollary 3.3.4. In this case we do not need to use the induction hypothesis on \(r\).

Otherwise, the inductive argument on \(r\) follows if we check the corresponding version of (4.3.1) is satisfied. More precisely,

(4.3.43) \[j' \in J', \quad m' \in \mathcal{R}^{s', s}_{i, j', J'}, \quad |m - (1 - r) - m' - \bar{h}_{J'}| \in \mathcal{R}^{1, s'}_{i, j', J'},\]

where \(J'\) is the minimal subdigram satisfying

\[i, j \in J' \quad \text{and} \quad m - (1 - r) \in \mathcal{R}^{1, s}_{i, j, J'}\]

Since \(p - r + 1 \leq 0\), \(J'\) is characterized by

\[d(i, \partial J') = d(j, \partial J') = -(p - r + 1) \quad \text{and} \quad \bar{h}_{J'} = d(i, j) - 2(p - r).\]

It then follows from (4.3.10) that \(J \subseteq J'\) and this immediately implies the first two conditions in (4.3.43). Finally, using (4.3.5), (4.3.6), and (4.3.10), one easily checks that

\[-(m - (1 - r) - m' - \bar{h}_{J'}) = 1 + s' + d(j, j') - 2p'.\]

Since \(p' \leq 0\) by (4.3.16), the third condition in (4.3.43) follows if we check

\[p' \geq -d([j, j'], \partial J').\]

But this is immediate from the second condition in (4.3.43).

To prove the second claim in (4.3.39), we again use the induction hypothesis on \(r\) to show the corresponding module is simple. We again start by checking whether the \(q\)-factorization graph is as in (2) of Section 4.2 which this time rephrases as

(4.3.44) \[m - 1 \in \mathcal{R}^{r-1, s}_{i, j, J'}, \quad m' \in \mathcal{R}^{s', s}_{i, j', J'}, \quad \text{and} \quad |m - m' - 1| \notin \mathcal{R}^{r-1, s'}_{i, j', J'}.\]

The second condition is obvious and, since

\[m - 1 = (r - 1) + s + d(i, j) - 2p\]

and we are assuming \(p \leq 0\), the first condition and the initial assumption of Section 4.3.2 are also obvious. For the third, if \(m > m'\), we have

\[0 \leq m - m' - 1 = (r - 1) + s' + d(i, j') - 2p_+ \quad \text{and} \quad p_+ \geq r > r - 1 = \min\{r - 1, s'\}\]

where we used (4.3.13) and the hypothesis \(r \leq s'\) to obtain the inequalities. If \(m \leq m'\), then

\[0 < -(m - m' - 1) = (r - 1) + s' + d(i, j') - 2(p_+ - 1).\]

Now,

\[p_+ - 1 = r - p + p' + d_{j, j'} - 1 \geq r + d_{i, j} + d_{j, J'} - 1 \geq r - 1 = \min\{r - 1, s'\}.\]

To complete the inductive argument, since \(r - 1 < s'\) and, hence, we remain under the assumption of Step 2, it remains to check the corresponding version of (4.3.1), which reads

(4.3.45) \[j' \in J', \quad m' \in \mathcal{R}^{s', s}_{i, j', J'}, \quad |m - 1 - m' - \bar{h}_{J'}| \in \mathcal{R}^{r-1, s'}_{i, j', J'},\]

where \(J'\) is the minimal subdigram satisfying

\[i, j \in J' \quad \text{and} \quad m - 1 \in \mathcal{R}^{r-1, s}_{i, j, J'}\]

It is immediate that \(J' = J\) and, therefore, it remains to check the third condition in (4.3.45). Using (4.3.5), (4.3.6), and (4.3.10), one easily checks that

\[-(m - 1 - m' - \bar{h}_{J}) = (r - 1) + s' + d(j, j') - 2(p' + r - 2).\]
and, therefore, we need to check
\[-d([j, j'], \partial J) \leq p' + r - 2 \leq r - 1 = \min\{r - 1, s'\}.

The first inequality is clear from (4.3.11) since \(r > 1\), while the second is clear from (4.3.10).

**Step 2.** We now show \(V\) is not simple if \(r > s'\) or, equivalently, if
\[(4.3.46)\]
\[m - m' + 1 \in \mathcal{P}_{i,j',j}^{s-s'}.
\]
By Corollary 3.2.4 it suffices to show that \(V_{j'}\) is reducible while (4.3.46) implies
\[L_{q}((\omega_{i, aq^{m-m'}s'})_{J}) \otimes L_{q}((\omega_{i, aq^{r-1}})_{J})
\]
is reducible and highest-\(\ell\)-weight. Proposition 3.4.3 and Corollary 3.3.3 then imply
\[L_1 := L_{q}(\pi_{J}) \otimes L_{q}((\omega_{i, aq^{r-1}})_{J})
\]
is also reducible and highest-\(\ell\)-weight. On the other hand, obviously so is
\[L_2 := L_{q}((\omega_{i, aq^{r-1}})_{J}) \otimes L_{q}((\omega_{i, aq^{r-1}})_{J}).
\]
By Proposition 3.4.3 we are done if we show
\[(4.3.47)\]
\[N := L_{q}((\omega_{i, aq^{r-1}})_{J}) \otimes L_{q}(\pi_{J})^{*}
\]
is simple.

To do this, we will check the \(q\)-factorization graph of \(N\) is as in (2) of Section 4.2 and that the corresponding version of (4.3.1) holds. Then, (4.3.47) follows from the \(r = 1\) case which has been shown already.

Set
\[k = w_{0}^{i}(j')\quad \text{and} \quad L_{q}(\pi) \cong L_{q}(\pi_{J}).
\]
We start by checking that the \(q\)-factorization graph of \(N\) is
\[(4.3.48)\]
\[\begin{array}{ccc}
1 & \xrightarrow{\tilde{m}} & s' \\
\tilde{m} & \xleftarrow{k} & s
\end{array}
\]
where \(\tilde{m} = r - 1 - m + m' + \tilde{h}_{J}\).

Recall \(i = w_{0}^{i}(j)\), and, therefore,
\[(4.3.49)\]
\[d(i, k) = d(j, j'), \quad d([j, j'], \partial J) = d([i, k], \partial J),
\]
and
\[\pi' = \omega_{i, aq^{m-h_{J}, s'}} \omega_{k, aq^{m-h_{J}, s'}}. \quad \text{and} \quad m' = (m - \tilde{h}_{J}) - (m - m' - \tilde{h}_{J}) \in \mathcal{P}_{i,k}^{s-s'}.
\]
Evidently,
\[(4.3.48)\]
\[\{m - \tilde{h}_{J} \leq (r - 1)\} \notin \mathcal{P}_{i,i}^{1,s'} \quad \text{and} \quad \tilde{m} \in \mathcal{P}_{i,k}^{1,s'.}
\]
The first of these follows since
\[0 \leq 1 + s + d(i, i) - 2 \cdot 1 = s - 1 = (m - \tilde{h}_{J}) - (r - 1) \quad \text{and} \quad 1 \geq \min\{1, s\},
\]
where we used (4.3.5) and (4.3.10) in the second equality. As for the second, one easily checks that
\[(4.3.50)\]
\[\tilde{m} = (r - 1) - (m - m' - \tilde{h}_{J}) = 1 + s' + d(j, j') - 2p' = 1 + s' + d(i, k) - 2p'
\]
and we know
\[\begin{cases}
-d([i, k], \partial J) = -d([j, j'], \partial J) \leq p' \\
\text{(4.3.10)}
\end{cases}
\]
\[\leq 0.
\]
Now, let us write down the corresponding version of (4.3.1). From (4.3.50), the minimal subdiagram \(J''\) such that
\[i, k \in J' \quad \text{and} \quad \tilde{m} \in \mathcal{P}_{i,k,j'}^{1,s'}
\]
is characterized by the property
\[d(i, \partial J') = d(k, \partial J') = -p' \quad \text{and} \quad \bar{h}_{J'} = d(i, k) - 2p' + 2.
\]
Thus, the corresponding version of (4.3.1) reads
\[(4.3.51)\]
\[i \in J', \quad m' \in \mathcal{P}_{i,k,j'}^{s,s'}, \quad |\tilde{m} - m' - \bar{h}_{J'}| \in \mathcal{P}_{k,i,j'}^{1,s'.}
\]
The first condition is obvious and the second follows from the characterization of $J'$, (4.3.49), and (4.3.6). Finally, using (4.3.50) as well, we see that
\[ m' + h_{J'} - \tilde{m} = 1 + s + d(i,k), \]
which proves the third condition in (4.3.51).

4.3.3. The Proof - Case 2. In light of (4.3.4) and (4.3.23), we need to show that $V$ is highest-$\ell$-weight if and only if $r \leq s'$ or $r > s'$ and $p \neq p'$. In Steps 1 and 2 below, we prove that $V$ is highest-$\ell$-weight if $r \leq s'$ or $r > s'$ and $p \neq p'$, respectively. In Step 3, we show $V$ is not highest-$\ell$-weight if $r > s'$ and $p = p'$. The main tool in all the steps is Theorem 3.3.3 although previously proved cases are used as well.

Step 1. Assume $r \leq s'$. We shall use Theorem 3.3.3 several times in order to show $V$ is highest-$\ell$-weight, which completes the proof by (4.3.4). Let
\[ \tilde{\pi} = \pi_{j',aqm-m'+s'-p',g'}^{-1} = \omega_{j,aq^m,s}^{j'} \omega_{j',aqm-m'-p',s'-p'}. \]
To conclude that $V$ is highest-$\ell$-weight, we will show that the following tensor products are highest-$\ell$-weight:
\[ L_q(\omega_{j',aqm-m'+s'-p',g'}) \otimes L_q(\tilde{\pi}), \]  
\[ L_q(\omega_{i,a,r}) \otimes L_q(\omega_{j',aqm-m'+s'-p',g'}), \]
and
\[ L_q(\omega_{i,a,r}) \otimes L_q(\tilde{\pi}). \]
Together with Theorem 3.3.3 this implies
\[ L_q(\omega_{i,a,r}) \otimes L_q(\omega_{j',aqm-m'+s'-p',g'}) \otimes L_q(\tilde{\pi}) \]
is also highest-$\ell$-weight. Moreover, (4.3.52) implies we have a surjective map
\[ L_q(\omega_{i,a,r}) \otimes L_q(\omega_{j',aqm-m'+s'-p',g'}) \otimes L_q(\tilde{\pi}) \rightarrow V, \]
showing $V$ is highest-$\ell$-weight.

To prove (4.3.52) is highest-$\ell$-weight we use Lemma 3.3.4 which says it suffices to show
\[ L_q(\omega_{j',aqm-m'+s'-p',g'}) \otimes L_q(\omega_{j',aqm-m'-p',s'-p'}) \]
and
\[ L_q(\omega_{j',aqm-m'+s'-p',g'}) \otimes L_q(\omega_{j,aq^m,s}) \]
are highest-$\ell$-weight. The first is clear since $s' - p' > 0$. Proving the second is equivalent to checking that
\[ m - (m - m' + s' - p') \notin \mathcal{R}^{s',p'}_{j,j'} \]
if $p' > 0$. By (4.3.6) this number is equal to
\[ s + d(j,j') - p' = s + p' + d(j,j') - 2p' \]
and $p' \geq \min\{s, p'\} = p'$.

Showing that (4.3.53) is highest-$\ell$-weight is equivalent to checking that
\[ m - m' + s' - p' \notin \mathcal{R}^{s',p'}_{i,j}, \]
if $p' > 0$. Using (4.3.5) and (4.3.6), we see this number is equal to
\[ r + d(i,j) - 2p - d(j,j') + p' = r + p' + d(i,j') - 2(p + d_{i,j}). \]

Since (4.3.23) implies $p + d_{i,j} \geq p' \geq \min\{r, p'\}$, the checking is completed.

To prove (4.3.54) is highest-$\ell$-weight, we will show
\[ L_q(\omega_{i,aq^r,p}) \otimes L_q(\tilde{\pi}) \]  
and
\[ L_q(\omega_{i,aq^r,r-p}) \otimes L_q(\tilde{\pi}) \]
are also highest-$\ell$-weight. Assuming this and noting that
\[ L_q(\omega_{i,aq^r,p}) \otimes L_q(\omega_{i,aq^r,r-p}) \]
is highest-\(\ell\)-weight and maps onto \(L_q(\omega_{i,a,r})\), Theorem 3.3.3 implies
\[
L_q(\omega_{i,aq^{-r}-p,p}) \otimes L_q(\omega_{i,aq^{-r}-p,p}) \otimes L_q(\pi)
\]
is highest-\(\ell\)-weight and maps onto (4.3.54), thus completing the proof.

To prove the first tensor product in (4.3.55) is highest-\(\ell\)-weight we use Lemma 3.3.4 which says it suffices to show
\[
L_q(\omega_{i,aq^{-r}-p,p}) \otimes L_q(\omega_{j,aq^{-m},s}) \quad \text{and} \quad L_q(\omega_{i,aq^{-r}-p,p}) \otimes L_q(\omega_{j',aq^{-m'-r'},s'-p'})
\]
are highest-\(\ell\)-weight. For the first, note
\[
m - (r - p) = s - p + d(i, j) = p + s + d(i, j) - 2p
\]
and \(p \geq \min\{p, s\} = p\), showing \(m - (r - p) \notin \mathcal{R}_{i,j}^{p,s}\) as desired. As for the second, we have
\[
m - m' - p' - (r - p) = -s' + d(i, j) - d(j', j) + p' - p
\]
(4.3.21) \[= p + (s' - p') + d(i, j') - 2(p + s' - p').\]
Since \(p + s' - p' \geq \min\{p, s', p'\}\), this number is not in \(\mathcal{R}_{i,j'}^{p,s'-p'}\), as desired.

It remains to prove that the second tensor product in (4.3.55) is highest-\(\ell\)-weight. We will actually prove it is simple by showing that it is a module in the conditions studied in Step 2 of Section 4.3.2. We begin by checking that its \(q\)-factorization graph is
\[
\begin{array}{c}
\text{\(r-p\)}  \\
\text{\(i\)}  \\
\downarrow  \\
\text{\(j\)}  \\
\text{\(s\)}  \\
\downarrow  \\
\text{\(j'\)}  \\
\uparrow  \\
\text{\(p\)}  \\
\text{\(m\)}  \\
\downarrow  \\
\text{\(m'\)}  \\
\text{\(s'\)}  \\
\uparrow  \\
\text{\(p'\)}  \\
\end{array}
\]
where \(\tilde{m} = m + p\) and \(\tilde{m}' = m' + p'\).

Or, equivalently,
\[
(4.3.56) \quad \tilde{m} \in \mathcal{R}_{i,j}^{r-p,s}, \quad \tilde{m}' \in \mathcal{R}_{j,j'}^{s'-p'}, \quad \text{and} \quad |\tilde{m} - \tilde{m}'| \notin \mathcal{R}_{i,j}^{r-p,s'-p'}.
\]
The first two are clear since
\[
\tilde{m} = (r - p) + s + d(i, j) - 2\tilde{p}, \quad \tilde{m}' = s + (s' - p') + d(j, j') - 2\tilde{p}', \quad \text{where} \quad \tilde{p} = \tilde{p}' = 0.
\]
For the third, note
\[
\tilde{m} - \tilde{m}' = m - m' + p - p'.
\]
Therefore, by (4.3.23), if \(\tilde{m} \leq \tilde{m}'\), we also have \(m \leq m'\) and we get
\[
|\tilde{m} - \tilde{m}'| = -(\tilde{m} - \tilde{m}') = -(m - m') + p - p' \geq r + s' + d(i, j') - 2p_- + p + p' = (r - p) + (s' - p') + d(i, j') - 2(p_+ - p').
\]
Using (4.3.13) and the assumption \(r \leq s'\), we get
\[
p_- - p' \geq \min\{r, s'\} - p' = r - p' \geq r - p \geq \min\{r - p, s' - p'\},
\]
which completes the proof of (4.3.56) in this case. If \(\tilde{m} > \tilde{m}'\), then using (4.3.5) and (4.3.6) we get
\[
|\tilde{m} - \tilde{m}'| = \tilde{m} - \tilde{m}' = m - m' + p - p' = (r - p) + (s' - p') + d(i, j') - 2(s' - p' + d_{i,j'}^j).
\]
Since
\[
s' - p' + d_{i,j'}^j \geq s' - p' \geq \min\{r - p, s' - p'\},
\]
the proof of (4.3.56) is complete.

The remaining conditions required in Step 2 of Section 4.3.2 are now easily checked. Indeed, since we have already seen that \(\tilde{p} = \tilde{p}' = 0\), it remains to check that \(r - p \leq s' - p'\) which is clear from (4.3.23) and the assumption \(r \leq s'\). Thus, the second tensor product in (4.3.55) is simple as claimed.

Step 2. We now show that \(V\) is highest-\(\ell\)-weight if \(r > s'\) and \(p > p'\). Note this follows if we show
\[
W := L_q(\omega_{i,aq^{-r-p'},p-p'}) \otimes L_q(\omega_{i,aq^{-r-p'},p-p'}) \otimes L_q(\pi)
\]
is highest-\(\ell\)-weight.
Indeed, since \( r - p + p' > p' - p \) (because \( p > p' \)), the tensor product of the first two factors is \( \text{highest}-\ell\text{-weight} \) and has \( L_q(\omega_{i,a,r}) \) as irreducible quotient. Thus, if we show \( W \) is \( \text{highest}-\ell\text{-weight} \) the natural map \( W \to V \) implies that so is \( V \). Using Theorem \[ 3.3.3 \] it now suffices to show

\[
L_q(\omega_{i,a,q-p+p',p-p'}) \otimes L_q(\pi) \quad \text{and} \quad L_q(\omega_{i,a,q-p',p-(p-p')}) \otimes L_q(\pi)
\]
are \( \text{highest}-\ell\text{-weight} \).

For the first of these tensor products, Corollary \[ 3.3.4 \] implies it suffices to check

\[
m - (r - p + p') \not\in \mathcal{R}_{i,j}^{s,p,s'} \quad \text{and} \quad (m - m') - (r - p + p') \not\in \mathcal{R}_{i,j}^{p-p',s'}.
\]

Indeed,

\[
m - (r - p + p') = s + d(i, j) - p - p' = (p - p') + s + d(i, j) - 2p
\]
and \( p \geq p - p' \geq \min\{p - p', s\} \), while, using \( (4.3.6) \) as well, we get

\[
(m - m') - (r - p + p') = d(i, j) - d(j, j') - p + p' - s' = (p - p') + s' + d(i, j') - 2(p - p' + s' + d_{i,j}')
\]
and \( p - p' + s' + d_{i,j}' \geq \min\{p - p', s'\} \).

We will now check that the second tensor product in \[ (4.3.57) \] satisfies the conditions either of Step 2 or of Step 2 from Section \[ (4.3.2) \]. This implies it is simple, thus completing the proof. Hence, we need to check that its \( q\)-factorization graph is

\[
\tilde{r} \rightarrow i \rightarrow m \rightarrow \cdots \rightarrow j \rightarrow s' \rightarrow j'
\]
where \( \tilde{m} = m + p - p' \) and \( \tilde{r} = r - (p - p') \), or, equivalently,

\[
\tilde{m} \in \mathcal{R}_{i,j}^{s,s'} \quad \text{and} \quad |\tilde{m} - m'| \not\in \mathcal{R}_{i,j}^{s,s'}.
\]

The other requirement to satisfy either one of Steps 2 is

\[
\tilde{r} \leq s',
\]
which follows from \[ (4.3.22) \]. For the first number in \[ (4.3.58) \], we have

\[
\tilde{m} \underset{13.5}{=} r + s + d(i, j) - p - p' = (r - (p - p')) + s + d(i, j) - 2p.
\]
We also know \( 0 \leq p' \leq s' \) and \( p < r \) which implies \( p' < r - (p - p') \), as desired\(^2\). For the second, assume first \( \tilde{m} \leq m' \) which implies \( m \leq m' \) since \( p > p \). Then,

\[
|m - m'| = -(m - m') - (p - p') \underset{4.3.5}{=} \tilde{r} + s' + d(i, j') - 2p
\]
and

\[
p_\rightarrow \geq \min\{r, s'\} = s' \geq \min\{s', \tilde{r}\}.
\]

On the other hand, if \( \tilde{m} > m', \) \[ (4.3.5) \] and \[ (4.3.6) \] imply

\[
|m - m'| = m - m' + p - p' = r - s' + d(i, j) - d(j, j') - p + p' \quad = (r - (p - p')) + s' + d(i,j') - 2(s' + d_{i,j}')
\]
and \( s' + d_{i,j}' \geq s' \geq \min\{s', \tilde{r}\} \).

**Step 3.** Finally, we show that \( V \) is not \( \text{highest}-\ell\text{-weight} \) if \( r > s' \) and \( p = p' \). We proceed by contradiction. More precisely, we will show using Theorem \[ 3.3.3 \] that, if \( V \) were \( \text{highest}-\ell\text{-weight} \), so would be

\[
T := L_q(\omega_{i,a,r}) \otimes L_q(\omega_{j,a,q-p+s'-p+q-p'}).\n\]

However, \( m - (s' - p) \in \mathcal{R}_{i,j}^{s+s'-p} \) yielding a contradiction. Indeed,

\[
m - (s' - p) \underset{4.3.5}{=} r + (s + s' - p) + d(i,j) - 2s'
\]

\(^2\)Note this last computation shows that we are in the conditions of Step 2 from Section \[ (4.3.2) \] if and only if \( p' = 0 \).
Indeed, \( m \) and \( s' < \min\{r, s + s' - p\} \) since \( s' < r \) and \( p < s \).

In order to show \( T \) is highest-\( \ell \)-weight, it suffices to show this is true for

\[
T \otimes L_q(\omega_{j', aq^{m - m', s'}}).
\]

Set

\[
\tilde{\pi} = \omega_{j, aq^{m'(s' - p), s + s' - p}} \omega_{j', aq^{m', s'}} = \omega_{j, aq^m, s' - p} \pi,
\]

where

\[
\tilde{m} = m - (s + s' - p).
\]

We claim

\[
L_q(\tilde{\pi}) \cong L_q(\omega_{j, aq^{m'(s' - p), s + s' - p}}) \otimes L_q(\omega_{j', aq^{m', s'}}),
\]

which implies

\[
T \otimes L_q(\omega_{j', aq^{m - m', s'}}) \cong L_q(\omega_{i, a, r}) \otimes L_q(\tilde{\pi}),
\]

and, hence, it suffices to show

\[
\tilde{V} := L_q(\omega_{i, a, r}) \otimes L_q(\tilde{\pi}) \text{ is highest-} \ell \text{-weight.}
\]

Finally, in order to show this, it suffices to show

\[
W \quad \text{and} \quad L_q(\pi) \otimes L_q(\omega_{j, aq^{\tilde{m}, s' - p}}) \quad \text{are highest-} \ell \text{-weight}
\]

where

\[
W = L_q(\omega_{i, a, r}) \otimes L_q(\pi) \otimes L_q(\omega_{j, aq^{\tilde{m}, s' - p}}) = V \otimes L_q(\omega_{j, aq^m, s' - p}).
\]

Indeed, if this is true, we obtain a natural epimorphism \( W \to \tilde{V} \), showing \( \tilde{V} \) is highest-\( \ell \)-weight.

Note that, assuming \( V \) were highest-\( \ell \)-weight, Theorem 3.3.3 implies the first claim in (4.3.60) would follow from the second together with

\[
L_q(\omega_{i, a, r}) \otimes L_q(\omega_{j, aq^{\tilde{m}, s' - p}}) \quad \text{is highest-} \ell \text{-weight.}
\]

To check the latter, note

\[
\tilde{m} = r - s' - p + d(i, j) = r + (s' - p) + d(i, j) - 2s'
\]

and, since \( r > s' \), \( s' \geq s' - p = \min\{r, s' - p\} \), so \( \tilde{m} \notin \mathcal{R}^{r, s' - p}_{i, j} \) as desire. As for the former, Corollary 3.3.4 implies it suffices to check

\[
\tilde{m} - m \notin \mathcal{R}^{s + s' - p}_{j, j'} \quad \text{and} \quad \tilde{m} - (m - m') \notin \mathcal{R}^{s' - p, s'}_{j, j'}.
\]

The first is clear since \( \tilde{m} \leq m \), while

\[
\tilde{m} - (m - m') = m' - (s + s' - p) = d(j, j') - 2p' + p = d(j, j') - p
\]

\[
= (s' - p) + s' + d(j, j') - 2s',
\]

and \( s' \geq \min\{s', s' - p\} \).

It remains to check (4.3.59), i.e.,

\[
|m - (s' - p) - (m - m')| = |m' - s' + p| \notin \mathcal{R}^{s + s' - p, s'}. \]

Indeed, \( m' - s' + p = s + d(j, j') - p \geq 0 \) since \( p < s \), so

\[
|m' - s' + p| = s + d(j, j') - p = (s + s' - p) + s' + d(j, j') - 2s',
\]

and \( s' \geq \min\{s', s + s' - p\} = s' \).
4.4. Reality of Trees. We start with:

Proof of Corollary 2.4.9. Letting \( \pi = (\omega_{i,a,r})^2 \omega_{j,aq^m,s} \), we see that \( G(\pi) \) is

\[
\begin{array}{ccc}
  r & \xleftarrow{m} & s \\
  i & \quad \quad & j \\
  m & \xrightarrow{r} & i
\end{array}
\]

Therefore, we need to check that (4.3.1) and (4.3.2) hold under the given assumptions, i.e., \( j' = i, s' = r, m' = m \). The former becomes

\[
\hat{h}_{ij} \in \mathcal{R}_t^{r,1} = \{ 2r + d(i,j) - 2k : -d([i,j], \partial J) \leq k < r \}.
\]

If \( p \) is given by (4.3.3), we have seen in (4.3.10) and (4.3.20) that

\[
\hat{h}_{ij} = 2 + d(i,j) - 2 \min\{0, p\}.
\]

Thus, letting \( k = r - 1 + \min\{0, p\} \), we have

\[
-d([i,j], \partial J) \leq \min\{0, p\} \leq k < r
\]

which shows (4.3.1) holds.

On the other hand, (4.3.2) becomes

\[
1 \notin \mathcal{R}_t^{r-1} = \{ 2r - 1 - 2k : -d(i, \partial J) \leq k < r - 1 \} \quad \text{if} \quad r > 1.
\]

Since the unique solution of the equation \( 2r - 1 - 2k = 1 \) is \( k = r - 1 \), we are done. \( \Box \)

Finally:

Proof of Theorem 2.4.8. We proceed by induction on the number \( N \) of \( q \)-factors of \( \pi \), which clearly starts if \( N = 1 \). Thus, assume \( N > 1 \) and recall it suffices to show \( L_q(\pi) \otimes L_q(\pi) \) is highest-\( \ell \)-weight by Proposition 3.3.2.

Choose \( \omega_{i,a,r} \in \partial G \), let \( \omega_{j,aq^m,s} \) be the unique vertex adjacent to \( \omega_{i,a,r} \), and write

\[
\pi = \omega'_{i,a,r} \omega_{j,aq^m,s} \quad \text{and} \quad \varpi = \omega'_{j,aq^m,s}.
\]

In particular, \( m \in \mathcal{R}_t^{r,s} \) and \( L_q(\omega_{i,a,r}) \otimes L_q(\omega) \) is simple for every \( q \)-factor \( \omega \) of \( \varpi' \). Corollary 3.3.4 then implies \( L_q(\omega_{i,a,r}) \otimes L_q(\omega') \) is simple. Note also that

\[
\pi^2 = \pi \omega_{i,a,r} \varpi
\]

and, since \( G \) is a tree, so is \( G(\varpi) \) by Lemma 3.5.1. Hence, the induction hypothesis applies to \( \varpi \).

Without loss of generality, assume \( m > 0 \), so that \( L_q(\omega_{i,a,r}) \otimes L_q(\omega_{j,aq^m,s}) \) is highest-\( \ell \)-weight. We claim

(4.4.1) \( L_q(\pi) \otimes L_q(\omega_{i,a,r}), \ L_q(\pi) \otimes L_q(\varpi), \ \text{and} \ L_q(\omega_{i,a,r}) \otimes L_q(\varpi) \)

are highest-\( \ell \)-weight. Together with Theorem 3.3.3 this implies

\[
L_q(\pi) \otimes L_q(\omega_{i,a,r}) \otimes L_q(\varpi)
\]

is highest-\( \ell \)-weight and has \( L_q(\pi) \otimes L_q(\pi) \) as a quotient, thus completing the proof.

For the last tensor product in (4.4.1), it follows from the observations we have already made that \( L_q(\omega_{i,a,r}) \otimes L_q(\omega) \) is highest-\( \ell \)-weight for every \( q \)-factor \( \omega \) of \( \varpi \). Thus, the claim follows from Corollary 3.3.4 in this case. Moreover, since \( L_q(\varpi) \) is real by the induction hypothesis, this also implies that

\[
L_q(\omega_{i,a,r}) \otimes L_q(\varpi) \otimes L_q(\varpi)
\]

is highest-\( \ell \)-weight and has the middle tensor product in (4.4.1) as a quotient. Thus, it remains to prove the claim for the first tensor product.

To do that, let \( \varpi_+ \) be the product of all \( q \)-factors \( \omega \) of \( \varpi' \) such that \( \omega > \omega_{j,aq^m,s} \) and let \( \varpi_- \) be such that

\[
\pi = \varpi_+ \omega_{i,a,r} \omega_{j,aq^m,s} \varpi_-.
\]

Consider

\[
W = L_q(\varpi_+) \otimes L_q(\omega_{i,a,r} \omega_{j,aq^m,s}) \otimes L_q(\varpi-) \otimes L_q(\omega_{i,a,r})
\]
and note that, except for
\[ L_q(\omega_{i,a,r} \omega_{j,aq^m,s}) \otimes L_q(\omega_{i,a,r}), \]
all the others 2-fold ordered tensor products in the definition of \( W \) are highest-\( \ell \)-weight by construction. On the other hand, this latter tensor product is simple by Corollary 2.4.9. Therefore, \( W \) is highest-\( \ell \)-weight and has the first tensor product in (4.4.1) as a quotient. This completes the proof. \( \square \)

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