Calculation of two-layer cylinder with application of contact layer model

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Abstract. The article presents the solution of the axisymmetric problem of the stress-strain state of a two-layer hollow thick-walled cylinder. A model of limited length with loaded ends is considered. It is assumed that the interaction of layers is implemented through a contact layer. The contact layer is considered as a transversally anisotropic elastic medium with such parameters that it can be represented as a set of short elastic rods that are not connected to each other and are normally oriented to the contact surface. Such an assumption makes possible to obtain an analytical solution of the problem presented in a closed form. The solution obtained allows us to calculate essentially inhomogeneous stress and strain fields, including stress concentrations, and also satisfies all boundary conditions.

1. Introduction
V. Gadolin, basing on the results of G. Lamé, was the first to obtain a solution of the problem of the stress-strain state of a thick-walled two-layer compound cylinder. However, the solution obtained by him suggests that the cylinder ends are free from loads, and the absence of tangential stresses in the model is assumed. In practice, as a rule, similar structural elements are loaded at the ends by axial forces. The solution for a thick-walled cylinder with loaded ends can be found, for example, in [1, 2, 3]. The exact solution of the problem (obtained from the condition of the ideal contact of the layers) leads to the appearance of infinite tangential stresses at the interface of the layers at the end of the model.

In this paper it is assumed that the interaction of the layers is not ideal, but it is accomplished by means of a contact layer. The contact layer model was first proposed by R. A. Turusov [4 – 7]. The model of a two-layer cylinder with a contact layer is shown in Figure 1.

2. Solution of the problem
To solve this problem, the mathematical apparatus of the theory of elasticity is used, which is presented by:
Differential equations of equilibrium with allowance for the axial symmetry of the model

\[ \frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\varphi}{r} + \frac{\partial \tau_{rz}}{\partial z} = 0; \quad \frac{\partial \tau_{\varphi z}}{\partial r} + \frac{\partial \sigma_\varphi}{\partial z} + \frac{\tau_{\varphi z}}{r} = 0. \]  

(1)

Cauchy strain-displacement relations

\[ \varepsilon_r = \frac{\partial u}{\partial r}; \quad \varepsilon_\varphi = \frac{u}{r}; \quad \varepsilon_z = \frac{\partial w}{\partial z}; \quad \varepsilon_{\varphi z} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}. \]  

(2)

The Law of Hooke

\[ \varepsilon_r = \frac{\sigma_r - \nu(\sigma_\varphi + \sigma_z)}{E} + \varepsilon_\varphi; \quad \varepsilon_\varphi = \frac{\sigma_\varphi - \nu(\sigma_z + \sigma_\varphi)}{E} + \varepsilon_{\varphi \varphi}; \quad \varepsilon_z = \frac{\sigma_z - \nu(\sigma_r + \sigma_\varphi)}{E} + \varepsilon_{\varphi z}. \]  

(3)

The values \( \varepsilon_\psi, i = r, \varphi, z \) are forced deformations (temperature, chemical shrinkage, etc.).

The assumption that the contact layer can be represented as a set of unconnected rods [6] allows us to integrate the equilibrium equations directly (1), since in this case \( E_\varphi = E_z = 0 \) and therefore \( \sigma_\varphi = \sigma_z = 0 \). As a result of integrating, the initial system of equations for the contact layer taking into account the compatibility of the displacements at the contact boundaries with the inner and outer cylinder, we come to a resolving equation with respect to the unknown function \( f_i(z) \):

\[ f_i(z) = \lambda_i \frac{d^2 f_i(z)}{dz^2} + \lambda_{w_i} \frac{du_i(r_i,z)}{dz} - \lambda_{w_i}^{\ast} \frac{dw_i(r_i,z)}{dz} + \lambda_{w_i}^{\ast} \left[ w_i(r_i,z) - w_i(r_j,z) \right]. \]  

(4)

This equation includes the displacements of the inner cylinder and the outer cylinder at the contact boundary \( u_i(r_i,z), u_i(r_j,z), w_i(r_i,z), w_i(r_j,z) \), which will be found further. Here and further, all values related to the contact layer will be marked with a symbol *. 

The values that determine the stress-strain state of the contact layer are expressed in terms of the function \( f_i(z) \) as follows:
\[
\tau^*_r (r, z) = \frac{f_1 (z)}{r} ; \quad \sigma^*_r = \left\{ \frac{r_i - r_e}{r \ln (r_i/r_e)} - 1 \right\} \frac{df_1 (z)}{dz} + \frac{E^* \left[ u_i (r, z) - u_e (r, z) \right]}{r \ln (r_i/r_e)} ; \\
\quad u^* (r, z) = a^*_f \frac{df_1 (z)}{dz} + c^*_i u_i (r, z) - c^*_2 u_e (r, z) ; \\
\quad w^* = -b^*_f \frac{d^2 f_1 (z)}{dz^2} - b^*_2 \frac{du_i (r, z)}{dz} + b^*_3 \frac{du_e (r, z)}{dz} + c^*_i w_i (r, z) - c^*_2 w_e (r, z),
\]

in which
\[
a^*_j = \frac{r_i \ln (r/r_e) - r_e \ln (r/r_i) - r \ln (r_i/r_e)}{E^* \ln (r_i/r_e)} ; \quad b^*_j = \psi^*_j + \frac{\psi^*_i (r_e) \ln (r/r_i) - \psi^*_i (r_i) \ln (r/r_e)}{\ln (r_i/r_e)} ; \\
b^*_2 = \psi^*_1 + \frac{\psi^*_1 (r_e) \ln (r/r_i) - \psi^*_1 (r_i) \ln (r/r_e)}{\ln (r_i/r_e)} ; \quad b^*_3 = \psi^*_2 + \frac{\psi^*_2 (r_e) \ln (r/r_i) - \psi^*_2 (r_i) \ln (r/r_e)}{\ln (r_i/r_e)} ; \\
c^*_1 = \frac{\ln (r/r_i)}{\ln (r_i/r_e)} ; \quad c^*_2 = \frac{\ln (r_i/r_e)}{\ln (r_i/r_e)} ; \quad \psi^*_i = \frac{r r_i \left[ \ln (r/r_e) - 1 \right] - r_e r_i \left[ \ln (r/r_i) - 1 \right] - \frac{r^2}{2} \ln (r_i/r_e)}{E^* \ln (r_i/r_e)} ; \\
\psi^*_2 = \frac{r \left[ \ln (r/r_e) - 1 \right] - \frac{r^2}{2} \ln (r_i/r_e)}{\ln (r_i/r_e)}.
\]

Further, it is required to determine the displacements and stresses in the inner and outer cylinders. For this purpose, we introduce the assumption that the stresses \( \sigma_z \) are distributed uniformly over the thickness of these layers.

Basing on this assumption, we consider the equilibrium of an infinitesimal element of the model with respect to the internal forces acting along the axis \( z \). To do this, we replace the contact layer with the stresses acting in it.

![Figure 2. The equilibrium of an infinitesimal element of the model.](image-url)

As a result, according to (5) we obtain
\[
\frac{d \sigma^*_{z,l}}{dz} = -\frac{2 f_1 (z)}{r_i^2 - r_e^2} \frac{d \sigma_{z,l}}{dz} = \frac{2 f_1 (z)}{r_i^2 - r_e^2}. \tag{6}
\]

Using (6) we integrate the second equation (1) taking into account the continuity of the tangential stresses at the interface with the contact layer and the first equation (5).
\[ \tau_{r,1} = f_1(z) \frac{r^2 - \rho_0^2}{r(r_2^2 - r_0^2)}; \quad \tau_{r,2} = -f_1(z) \frac{r^2 - \rho_0^2}{r(r_2^2 - r_0^2)}. \]  

(7)

Basing on the formulae (2) and (3), we obtain a resolving equation for the radial stresses in the inner and outer cylinders

\[ \frac{\partial^2 \sigma_{r,i}}{\partial r^2} + 3 \frac{\partial \sigma_{r,i}}{\partial r} + r \frac{\partial^2 \tau_{r,i}}{\partial r \partial z} + (\nu_i + 2) \frac{\partial \tau_{r,i}}{\partial z} = 0. \]  

(8)

Substituting in (8) the expressions (7), considering the boundary conditions and the continuity of the radial stresses within the model, we obtain

\[ \sigma_{r,i} = \psi_{f,i} \frac{df_i(z)}{dz} - \psi_{u,i} \left[u_i(r_i, z) - u_2(r_i, z)\right] + \psi_{p,i} p_{in}(z); \]  

(9)

Here \( p_{in}; p_{ex} \) is the radial load applied to the inner and outer sides of the model, respectively,

\[ \psi_{f,i} = \frac{(\nu_i + 1) \left[r_i^2 - 4 \rho_0^2 \ln(r_i)\right] + 2 r_i^2 + (\nu_i + 1) \left[r_i^2 - 4 \rho_0^2 \ln(r_i)\right] + 2 \rho_i^2 \left(r_i^2 - r_0^2\right)}{8 \left(r_0^2 - r_i^2\right)} + r \left[\frac{r_i^2 \left(r_i^2 - r_0^2\right)}{r_0^2 - r_i^2} - \frac{r_i^2 \ln(r_i / r_0)}{r_0^2 - r_i^2} - 1\right]; \]

\[ \psi_{u,i} = \frac{E' r_i \left(r_i^2 - r_0^2\right)}{r_i^2 \left(r_i^2 - r_0^2\right) \ln(r_i / r_0)}; \quad \psi_{u,2} = \frac{E' r_2 \left(r_2^2 - r_0^2\right)}{r_2^2 \left(r_2^2 - r_0^2\right) \ln(r_2 / r_0)}; \quad \psi_{p,i} = \frac{\rho_i^2 \left(r_i^2 - r_0^2\right)}{r_i^2 \left(r_i^2 - r_0^2\right) \ln(r_i / r_0)}; \quad \psi_{p,2} = \frac{\rho_2^2 \left(r_2^2 - r_0^2\right)}{r_2^2 \left(r_2^2 - r_0^2\right) \ln(r_2 / r_0)}. \]

The circumferential stresses are determined from the radial and tangential stresses

\[ \sigma_{\rho,i} = \frac{\partial (r \sigma_{r,i})}{\partial r} + r \frac{\partial \tau_{r,i}}{\partial z}. \]  

(10)

Further, from the equations (2) considering the expressions (7), (9) and (10) we determine the unknown displacements \( u_i(r_i, z), u_2(r_i, z), w_i(r_i, z), w_2(r_i, z) \).

\[ \frac{d^2 u_i(r_i, z)}{dz^2} = a_{i,1} \frac{d^2 f_i(z)}{dz^2} + a_{i,2} f_i(z); \quad \frac{d^2 u_2(r_i, z)}{dz^2} = a_{2,1} \frac{d^2 f_i(z)}{dz^2} + a_{2,2} f_i(z); \]  

\[ \frac{d^2 w_1(r_i, z)}{dz^2} = b_{1,1} \frac{d^2 f_i(z)}{dz^2} + b_{1,2} f_i(z); \quad \frac{d^2 w_2(r_i, z)}{dz^2} = b_{2,1} \frac{d^2 f_i(z)}{dz^2} + b_{2,2} f_i(z), \]  

(11)

in which

\[ a_{i,1} = \frac{r_i (X_i - 1)}{E_i (X_i - X_2 + 1)} A_i - \frac{r_i (X_2 - 1)}{E_i (X_i - X_2 + 1)} A_i; \quad a_{i,2} = -\frac{2}{X_i - X_2 + 1} \frac{r_i (X_i - 1)}{E_i (r_i^2 - r_0^2) + E_i (r_i^2 - r_0^2)}; \]

\[ a_{2,1} = \frac{r_i (X_i - X_2 + 1)}{E_i (X_i - X_2 + 1)} A_i - \frac{r_i (X_2 - 1)}{E_i (X_i - X_2 + 1)} A_i; \quad a_{2,2} = -\frac{2}{X_i - X_2 + 1} \frac{r_i (X_i - X_2 + 1)}{E_i (r_i^2 - r_0^2) + E_i (r_i^2 - r_0^2)}; \]
\[ A_1 = r_1 \frac{d\psi_{f,1}(r_1)}{dr} + (1 - v_1)\psi_{f,1}(r_1) + 1; \quad A_2 = r_2 \frac{d\psi_{f,2}(r_2)}{dr} + (1 - v_2)\psi_{f,2}(r_2) + 1; \]

\[ b_{1,1}(r) = -\frac{v_1}{E_1} \left[ \frac{1}{r \, dr} \left( \psi_{f,1}(r) r^2 - \psi_{u,1}(r) r^2 (a_{1,1} - a_{2,1}) \right) + \frac{r_1^2 - r_0^2}{r_1^2 - r_0} \right]; \]

\[ b_{1,2}(r) = \frac{1}{E_1} \left[ -\frac{2}{r^2 - r_0^2} + \frac{v_1 (a_{1,2} - a_{2,2})}{r} \frac{d\psi_{u,1}(r) r^2}{dr} \right]; \]

\[ b_{2,1}(r) = -\frac{v_2}{E_2} \left[ \frac{1}{r \, dr} \left( \psi_{f,2}(r) r^2 - \psi_{u,2}(r) r^2 (a_{1,1} - a_{2,1}) \right) - \frac{r_2^2 - r_0^2}{r_2^2 - r_0} \right]; \]

\[ b_{2,2}(r) = \frac{1}{E_2} \left[ -\frac{2}{r^2 - r_0^2} + \frac{v_1 (a_{1,2} - a_{2,2})}{r} \frac{d\psi_{u,1}(r) r^2}{dr} \right]. \]

Using the relations obtained in (11), we transform equation (4) by differentiating twice

\[ \frac{d^2 f_1(z)}{dz^2} - 2\omega \frac{d^2 f_1(z)}{dz^2} + \eta^2 f_1(z) = 0, \]

in which

\[ 2\omega = \lambda_{u,1}^* a_{1,2} - \lambda_{u,2}^* a_{1,2} - \lambda_{w}^* \left[ b_{1,1}(r_1) + b_{2,1}(r_2) \right] + 1; \]

\[ \eta^2 = \frac{\lambda_{w}^* \left[ b_{1,2}(r_1) - b_{2,2}(r_2) \right]}{\lambda_{1,1}^* + \lambda_{1,2}^* a_{1,1} - \lambda_{2,1}^* a_{2,1}}. \]

The final resolving system is represented by equations (6) and (12). Six integration constants are determined from six boundary conditions.

3. Calculation example

As an example of a calculation, we consider the problem of the stress-strain state of a polymer fibre reinforced with glass fibre.

A similar model of Kargin-Malinsky [8, 9] is the simplest example of unidirectional reinforced plastic. The choice of this problem is due to the peculiarities of the destruction of this model during the rupture tests. The filming of the process of destruction of this sample demonstrates some regularity. The first break of the reinforcing fibre occurs in a random place. Under further loading, subsequent ruptures occur at some rather small distance from the previous rupture and so until the sample collapses. Near these ruptures, stress concentration zones are observed in the polarized light, which contribute to the reduction of the strength of model samples. It is noteworthy that this phenomenon of successive ruptures is observed in a certain temperature range. At the same time, the strength of reinforced samples, as a rule, is lower than the strength of similar-sized polymer samples without a reinforcing element.

Figure 3 shows the calculation model. It is an element of the experimental model described above, enclosed between two ruptures of the reinforcing element.

The boundary conditions are written below

\[ \tau_{r_t}(r,-l/2) = \tau_{r_t}(r,l/2) = 0; \quad \sigma_{z,1}(-l/2) = \sigma_{z,1}(l/2) = 0; \quad \sigma_{z,2}(-l/2) = \sigma_{z,2}(l/2) = q. \]
For the transition from a hollow cylinder to a solid one, it is necessary to put in the limit \( r_0 \to 0 \) in expressions for \( \psi_{f,1} \) and \( \psi_{u,3} \), and also to equate the internal radial pressure to zero. The remaining values remain unchanged. As a result

\[
\psi_{f,1}(r) = \frac{r - r_0}{r_1 \ln(r_1/r_2)} + \frac{(v_1 + 1)(r_1^2 - r^2)}{8r_1^2} - 1; \quad \psi_{u,3}(r) = -\frac{E^*}{r_1 \ln(r_1/r_2)}.
\]

We will use the following physical, mechanical and geometric parameters of the model for the calculation

\[
r_1 = 0.2\text{mm}; \quad r_3 = 2\text{mm}; \quad l = 10\text{mm}; \quad G^* = 3000\text{MPa}; \\
E_1 = 72000\text{MPa}; \quad E_2 = E_M = [100; 200; 300]\text{MPa}.
\]

The results of the calculation of tangential stresses at the fibre-contact layer interface and the principal stresses \([2]\), calculated by the formula are given below:

\[
\sigma_{1,1} = \frac{\sigma_{f,1} + \sigma_{u,1}}{2} + \sqrt{\left(\frac{\sigma_{f,1}}{2}\right)^2 + \left(\frac{\sigma_{u,1}}{2}\right)^2 \sigma_{f,1}\sigma_{u,1} + \tau_{1,2}^2}.
\]

**Figure 4.** Tangential stresses.

**Figure 5.** Principal stresses.

The presented series of graphs clearly shows the edge effect of stresses in the fibre and on the boundary, arising in a narrow zone. It is worthwhile noting that these stresses turn to zero at the load-free end of the model.

4. **Conclusion**

The analytical solution for the calculation of a two-layer cylinder is obtained, which allows calculating a significantly inhomogeneous stress-strain state, predicts the occurrence of the edge effect and satisfies all the boundary conditions. The calculations presented as an example explain qualitatively the effect of “successive ruptures” discovered experimentally.

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