Controllability of impulse controlled systems of heat equations coupled by constant matrices

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Abstract

This paper studies the approximate and null controllability for impulse controlled systems of heat equations coupled by a pair \((A, B)\) of constant matrices. We present a necessary and sufficient condition for the approximate controllability, which is exactly Kalman’s controllability rank condition of \((A, B)\). We prove that when such a system is approximately controllable, the approximate controllability over an interval \([0, T]\) can be realized by adding controls at arbitrary \(n\) different control instants \(0 < \tau_1 < \tau_2 < \cdots < \tau_n < T\), provided that \(\tau_n - \tau_1 < d_A\), where \(d_A \triangleq \min\{\pi/|\text{Im}\lambda| : \lambda \in \sigma(A)\}\). We also show that in general, such systems are not null controllable.

Keywords Impulse control, approximate controllability, null controllability, systems of heat equations

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1 Introduction

In this paper, we will study the null controllability and the approximate controllability for some impulse controlled systems of heat equations coupled by constant matrices. Impulse control belongs to a class of important controls and has wide applications. In many cases impulse control is an interesting alternative to deal with systems that cannot be acted on by means of continuous control inputs, for instance, relevant control for acting

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on a population of bacteria should be impulsive, so that the density of the bactericide may change instantaneously; indeed continuous control would be enhance drug resistance of bacteria (see [29] and [32]). Another application of impulse control in reality can be explained as follows: In materials science, quenching is the rapid cooling of a workpiece to obtain certain material properties. A type of heat treating, quenching prevents undesired low-temperature processes, such as phase transformations, from occurring. In ancient, a sequence of intermittent quenching is widely used in swordsmanship. We can regard such a quenching as an impulse control. Besides, there are many applications of impulse control theory to nanoelectronics (see Chapter 11 in [32]).

To introduce our controlled system, some notations are given in order. Let \( \Omega \subset \mathbb{R}^N \) (with \( N \in \mathbb{N}^+ \equiv \{1, 2, \ldots \} \)) be a bounded domain with a \( C^2 \) boundary \( \partial \Omega \). Let \( \omega \subset \Omega \) be an open and nonempty subset with its characteristic function \( \chi_\omega \). Write \( \mathbb{R}^+ \triangleq (0, +\infty) \). Let \( A \) and \( B \) be respectively \( n \times n \) and \( n \times m \) (with \( n, m \in \mathbb{N}^+ \)) real matrices, which are treated as linear operators from \( \mathbb{R}^n \) and \( \mathbb{R}^m \) to \( \mathbb{R}^n \) respectively. Write \( \Delta \triangleq \text{diag}\{\Delta, \ldots, \Delta\} \) (where there are \( n \) Lapalacians). Define

\[
A \triangleq \Delta - A \quad \text{with} \quad D(A) \triangleq H^2(\Omega; \mathbb{R}^n) \cap H^1_0(\Omega; \mathbb{R}^n). \tag{1.1}
\]

(Namely, for each \( z = (z_1, \ldots, z_n)^T \in D(A) \), with \( z_i \in H^2(\Omega; \mathbb{R}) \cap H^1_0(\Omega; \mathbb{R}) \), \( i = 1, \ldots, n \), we define \( A z \triangleq \Delta(z_1, \ldots, z_n)^T - A(z_1, \ldots, z_n)^T \).) One can easily check that \( A \) generates a \( C_0 \) -semigroup \( \{e^{At}\}_{t \geq 0} \) over \( L^2(\Omega; \mathbb{R}^n) \). We treat \( \chi_\omega \) as a linear and bounded operator on \( L^2(\Omega; \mathbb{R}^n) \) in the following manner: For each \( z = (z_1, \ldots, z_n)^T \in L^2(\Omega; \mathbb{R}^n) \) (where \( z_k \in L^2(\Omega; \mathbb{R}) \), \( k = 1, \ldots, n \)), we define that \( \chi_\omega z \triangleq (\chi_\omega z_1, \ldots, \chi_\omega z_n)^T \).

Consider the following impulse controlled system of heat equations:

\[
\begin{align*}
\partial_t y(t) - Ay(t) & = 0, \quad t \in \mathbb{R}^+ \setminus \{\tau_k\}_{k=1}^p, \\
y(\tau_k) - y(\tau_k−) & = \chi_\omega B u_k, \quad k = 1, 2, \ldots, p, \\
y(0) & = y_0 \in L^2(\Omega; \mathbb{R}^n).
\end{align*} \tag{1.2}
\]

Here, \( p \in \mathbb{N}^+; \ 0 < \tau_1 < \cdots < \tau_p < \infty \), which are called control instants; \( u_k = (u_{k1}, \ldots, u_{km}), k = 1, \ldots, p, \) are taken from \( L^2(\Omega; \mathbb{R}^m) \) and called impulse controls; \( y(\tau_k−) \) denotes the left limit at \( t = \tau_k \) for the function \( y \). One can easily check that the equation (1.2) is well-posed. Write \( y(\cdot; y_0, \{\tau_k\}_{k=1}^p; \{u_k\}_{k=1}^p) \) for the unique solution of (1.2). It is clear that

\[
y(t; y_0, \{\tau_k\}_{k=1}^p; \{u_k\}_{k=1}^p) = e^{At} y_0 + \sum_{1 \leq k \leq p, \tau_k \leq t} e^{A(t-\tau_k)} \chi_\omega B u_k, \quad t \geq 0. \tag{1.3}
\]

Throughout this paper, \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) denote the usual norm and inner product of \( L^2(\Omega; \mathbb{R}^n) \), respectively; For each \( C \in \mathbb{R}^{n \times n} \), we define

\[
d_C \triangleq \min \left\{ \pi/|\text{Im}\lambda| : \lambda \in \sigma(C) \right\}, \tag{1.4}
\]
where $\sigma(C)$ denotes the spectrum of $C$ and in the above definition we agree that $\frac{1}{0} = +\infty$. (We mention that $d_C$ takes value $+\infty$ in the case that $C$ has only real eigenvalues); $A^*$, $B^*$ and $A^*$ stand for the adjoint operators of $A$, $B$ and $A$, respectively.

**Definition 1.1.** (i) Let $T > 0$. The system (1.2) is said to be null controllable over $[0, T]$, if there is $p \in \mathbb{N}^+$, $\{\tau_k\}_{k=1}^p \subset (0, T)$ (with $\tau_1 < \cdots < \tau_p$) so that for each $y_0 \in L^2(\Omega; \mathbb{R}^n)$, there is $\{u_k\}_{k=1}^p \subset L^2(\Omega; \mathbb{R}^m)$ satisfying that

$$y(T; y_0, \{\tau_k\}_{k=1}^p, \{u_k\}_{k=1}^p) = 0 \text{ in } L^2(\Omega; \mathbb{R}^n). \quad (1.5)$$

(ii) The system (1.2) is said to be null controllable, if for each $T > 0$, it is null controllable over $[0, T]$.

**Definition 1.2.** (i) Let $T > 0$. The system (1.2) is said to be approximately controllable over $[0, T]$, if there is $p \in \mathbb{N}^+$, $\{\tau_k\}_{k=1}^p \subset (0, T)$ (with $\tau_1 < \cdots < \tau_p$) so that for any $\epsilon > 0$ and any $y_0, y_1 \in L^2(\Omega; \mathbb{R}^n)$, there is $\{u_k\}_{k=1}^p \subset L^2(\Omega; \mathbb{R}^m)$ satisfying that

$$\|y(T; y_0, \{\tau_k\}_{k=1}^p, \{u_k\}_{k=1}^p) - y_1\| \leq \epsilon. \quad (1.6)$$

(ii) The system (1.2) is said to be approximately controllable, if for each $T > 0$, it is approximately controllable over $[0, T]$.

(iii) We say that the approximate controllability of (1.2) over $[0, T]$ (with $T > 0$) can be realized at $\{\tau_k\}_{k=1}^p$ (with $p \in \mathbb{N}^+$ and $0 < \tau_1 < \cdots < \tau_p < T$), if for any $\epsilon > 0$ and any $y_0, y_1 \in L^2(\Omega; \mathbb{R}^n)$, there is $\{u_k\}_{k=1}^p \subset L^2(\Omega; \mathbb{R}^m)$ satisfying that

$$\|y(T; y_0, \{\tau_k\}_{k=1}^p, \{u_k\}_{k=1}^p) - y_1\| \leq \epsilon. \quad (1.7)$$

Recall that Kalman’s controllability rank condition for a pair $(A, B)$ (in $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$) is as follows:

$$\text{rank } (B, AB, A^2B, \ldots, A^{n-1}B) = n. \quad (1.8)$$

The main results of this paper are presented by the following two theorems. The first one concerns with the null controllability for the system (1.2), while the second one is about the approximate controllability for the system (1.2).

**Theorem 1.3.** The following conclusions are true:

(i) When $\Omega \setminus \overline{\omega} \neq \emptyset$ (where $\overline{\omega}$ is the closure of $\omega$ in $\mathbb{R}^N$), the system (1.2) is not null controllable over $[0, T]$ for any $T > 0$.

(ii) When $\omega = \Omega$, the system (1.2) is null controllable if and only if $(A, B)$ satisfies Kalman’s controllability rank condition (1.8).
\textbf{Theorem 1.4.} The following conclusions are true:

(i) The system (1.2) is approximately controllable if and only if the pair \((A, B)\) satisfies Kalman’s controllability rank condition (1.8).

(ii) Suppose that the pair \((A, B)\) satisfies Kalman’s controllability rank condition (1.8). Then for each \(T > 0\), the approximate controllability of the system (1.2) over \([0, T]\) can be realized at any sequence \(\{\tau_k\}_{k=1}^n\) with \(0 < \tau_1 < \cdots < \tau_n < T\) and with \(\tau_n - \tau_1 < d_A\) (given by (1.4), where \(C = A\)).

Several notes are given in order.

(a) From Theorem 1.3, we see that the system (1.2) does not hold the null controllability except for the special case when the control region \(\omega\) is the whole physical domain \(\Omega\). Thus, for the system (1.2), the approximate controllability is the most likely outcome for us. Fortunately, Theorem 1.4 provides a criterion on the approximate controllability for (1.2). It is exactly Kalman’s controllability rank condition (1.8).

For single impulse controlled heat equation, i.e., \(n = 1\), the approximate controllability can be easily obtained by the qualitative unique continuation at one time point for heat equations (see, for instance, [19] for such unique continuation). Moreover, in this case, the approximate controllability can be realized at only one control instant. In [25] and [28] (see also [4], [8] and [26]), a quantitative version for such unique continuation was built up. Such a quantitative version leads to not only the approximate controllability but also the approximate null controllability with a cost (see [27]).

For the impulse controlled system (1.2), we have not found any result on the controllability in past publications.

(b) The exact controllability was studied in [21] (see also [32]) for the following impulse controlled linear time-invariant ODE:

\[
\begin{cases}
\frac{d}{dt} z = A z, & t \in \mathbb{R}^+ \setminus \{\tau_k\}_{k=1}^p, \\
z(\tau_k) = z(\tau_k^-) + B u_k, & k = 1, 2, \ldots, p,
\end{cases}
\tag{1.9}
\]

where \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, p \in \mathbb{N}^+, \{u_k\}_{k=1}^p \subset \mathbb{R}^m\) and \(\{\tau_k\}_{k=1}^p \subset \mathbb{R}^+\) is an increasing sequence. Let us first recall the following definition of the exact controllability for this system (see [32, Definition 2.3.1]): For each \(T > 0\) and each \(z_0, z_1 \in \mathbb{R}^n\), there exists \(p \in \mathbb{N}^+, \{\tau_k\}_{k=1}^p \) and \(\{u_k\}_{k=1}^p \) so that the corresponding solution of (1.9) drives \(z_0\) at \(t = 0\) to \(z_1\) at \(t = T\). We say that the exact controllability for (1.9) over \([0, T]\) can be realized at \(\{\tau_k\}_{k=1}^p \subset (0, T)\), if for any \(z_0, z_1 \in \mathbb{R}^n\), there exists \(\{u_k\}_{k=1}^p \) so that the corresponding solution of (1.9) drives \(z_0\) at \(t = 0\) to \(z_1\)
at $t = T$. It was obtained in [21, Theorem 1] (see also [32, Theorem 2.3.2]) that the pair $(A,B)$ satisfies Kalman’s controllability rank condition (1.8) if and only if the exact controllability holds. Furthermore, the author in [21] claimed that when $(A,B)$ satisfies (1.8), the number of control instants can be taken as the smallest integer which is bigger than or equals to $n/m$. Unfortunately, we do not understand the proof of this part. (More precisely, we do not understand the argument from Lines 8-9 on Page 83 in [21].)

From perspective of control instants, the main difference of [21, Theorem 1] from our result in (ii) of Theorem 1.4 is as follows: The author in [21, Theorem 1] only got the existence of control instants $\{\tau_k\}_{k=1}^p \subset (0, T)$ at which the exact controllability of (1.9) over $[0, T]$ can be realized, but did not know positions of these control instants. In our Theorem 1.4, the approximate controllability of the system (1.2) over $[0, T]$ can be realized at any increasing control instants $\{\tau_k\}_{k=1}^n \subset (0, T)$, provided that $\tau_n - \tau_1 < d_A$ (which is given by (1.4) with $C = A$). Moreover, we showed, via Example 5.2, that for some $(A, B) \in \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 1}$ with (1.8), if an increasing sequence $\{\tau_k\}_{k=1}^2 \subset (0, T)$ satisfies that $\tau_2 - \tau_1 = d_A$, then the approximate controllability of the system (1.2) over $[0, T]$ cannot be realized at $\{\tau_k\}_{k=1}^2$. Hence, this condition has certain rationality. Unfortunately, we don’t know what happen if $\tau_n - \tau_1 > d_A$.

Though our result on control instants improves greatly that in [21, Theorem 1], some idea in our proof of this result is borrowed from the proof of [21, Theorem 1].

(c) Two interesting questions are as follows: First, given $(A, B)$ with (1.8), what is the least number $p$ of control instants in $(0, T)$ so that the approximate controllability for the system (1.2) over $[0, T]$ can be realized at $p$ control instants? Second, what can we say about the approximate controllability for the system (1.2) where elements of $A$ and $B$ are functions of space variable $x \in \Omega$? So far, we are not able to answer these questions.

For the first question, Example 5.2 shows that for some $(A, B) \in \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 1}$, with (1.8), the approximate controllability for (1.2) over $[0, T]$, with $T > 0$, cannot be realized at a single control instant $\tau \in (0, T)$.

(d) There have been many studies on the the approximate controllability, null controllability and the unique continuation for parabolic equations with controls (or observations) on intervals. Here we would like to mention the following publications and the references therein: [1, 2, 3, 4, 6, 7, 9, 10, 11, 12, 13, 17, 19, 20, 22, 23, 26, 30, 36, 38, 39]. About works on impulsive controlled systems, we would like to mention the references [5, 14, 15, 16, 18, 21, 24, 31, 32, 33, 34, 35, 37] and the references therein.

The rest of the paper is organized as follows: Section 2 proves an important property.
Section 3 provides some preliminaries. Section 4 proves Theorem 1.3. Section 5 shows Theorem 1.4.

2 Controllability for impulse controlled ODEs

In this section, we will study some properties on the exact controllability of the system (1.9). Recall the note (b) in Section 1 for the definition of the exact controllability of (1.9) given in [32, Definition 2.3.1]. Two main theorems will be introduced.

The first main result of this section is the next Theorem 2.1, which is one of the bases to prove Theorem 1.4.

**Theorem 2.1.** Let \((A,B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}\) satisfy (1.8). Let \(d_A\) be given by (1.4) with \(C = A\). Then for each increasing sequence \(\{\tau_k\}_{k=1}^n \subset \mathbb{R}\) with \(\tau_n - \tau_1 < d_A\), it stands that
\[
\text{rank} \left( e^{A\tau_1}B, \ldots, e^{A\tau_n}B \right) = n. \tag{2.1}
\]

The second main result in this section is the following Theorem 2.2, which will not be used in the proofs of our main results of the current paper. However, it is independently interesting. We would like to mention that the result in Theorem 2.2 was claimed, without proof, in [21, Theorem 1] (also in [32, Theorem 2.3.2]). (See the proof of [21, Theorem 1] or [32, Theorem 2.3.2].)

**Theorem 2.2.** Let \((A,B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}\) and \(T > 0\). Let \(\{\tau_k\}_{k=1}^p \subset (0,T)\), with \(p \in \mathbb{N}^+\), be an increasing sequence. Then the exact controllability for (1.9) over \([0,T]\) can be realized at \(\{\tau_k\}_{k=1}^p\) if and only if
\[
\text{rank} \left( e^{A(T-\tau_1)}B, \ldots, e^{A(T-\tau_p)}B \right) = n. \tag{2.2}
\]

The proofs of the above two theorems will be given later. The following result is a direct consequence of the above two theorems:

**Corollary 2.3.** Let \((A,B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}\) satisfy (1.8). Let \(T > 0\). Then for each increasing sequence \(\{\tau_k\}_{k=1}^n \subset (0,T)\), with \(\tau_n - \tau_1 < d_A\), the exact controllability for (1.9) over \([0,T]\) can be realized at \(\{\tau_k\}_{k=1}^n\).

The next Example 2.4 explains the rationality of the condition that \(\tau_n - \tau_1 < d_A\) in Theorem 2.1.

**Example 2.4.** We will present a pair \((A,B) \in \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 1}\) with (1.8) so that (2.1) is not true for any \(\tau_1\) and \(\tau_2\), with \(\tau_2 - \tau_1 = d_A\). For this purpose, we let
\[
A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} c \\ d \end{pmatrix} \tag{2.3}
\]
with
\[ a, b, c, d \in \mathbb{R} \text{ and } b(c^2 + d^2) \neq 0. \tag{2.4} \]

One can directly check that
\[ d_A = \frac{\pi}{|b|} \text{ and } \text{rank} (B, AB) = 2 \text{ (since } |(B, AB)| = b(c^2 + d^2) \neq 0). \]

Hence, \((A, B)\) satisfies (1.8). Meanwhile, we can easily verify that
\[ e^{At} = \begin{pmatrix} e^{at} \cos bt & -e^{at} \sin bt \\ e^{at} \sin bt & e^{at} \cos bt \end{pmatrix}, \quad t \in \mathbb{R}. \]

Thus, we have that for any \(\tau_1, \tau_2 \in \mathbb{R},\)
\[ |(e^{A\tau_1}B, e^{A\tau_2}B)| = e^{a(\tau_1 + \tau_2)}(c^2 + d^2) \sin b(\tau_2 - \tau_1). \tag{2.5} \]

By (2.4) and (2.5), we see that
\[ \text{rank} (e^{A\tau_1}B, e^{A\tau_2}B) = 2 \iff b(\tau_2 - \tau_1) \notin \{k\pi : k \in \mathbb{N}\}. \]

Therefore, when \(\tau_1, \tau_2 \in \mathbb{R}\) with \(\tau_2 - \tau_1 = \frac{\pi}{|b|},\) we have that
\[ \text{rank} (e^{A\tau_1}B, e^{A\tau_2}B) < 2. \]

Besides, this example shows that for each \(T > 0,\) the exact controllability for the system (1.9) (with \((A, B)\) given by (2.3)) over \([0, T]\) cannot be realized at any single control instant. Indeed, this follows at once from Theorem 2.2 and the fact that \(\text{rank} (e^{A\tau}B) < 2\) for all \(\tau \in \mathbb{R}.\)

Remark 2.5. (i) It was obtained in [21, Theorem 1] (see also [32, Theorem 2.3.2]) that if \((A, B)\) satisfies (1.8), then for each \(T > 0,\) there is \(p \in \mathbb{N}^+\) and an increasing sequence \(\{\tau_k\}_{k=1}^p \subset (0, T)\) so that the exact controllability for (1.9) over \([0, T]\) can be realized at \(\{\tau_k\}_{k=1}^p.\) However, [21, Theorem 1] does not provide the positions of \(\{\tau_k\}_{k=1}^p.\)

Our Corollary 2.3 shows that if \((A, B)\) satisfies (1.8), then for each \(T > 0,\) the exact controllability for (1.9) over \([0, T]\) can be realized at any increasing sequence \(\{\tau_k\}_{k=1}^\infty \subset (0, T),\) with \(\tau_n - \tau_1 < d_A.\) And our Example 2.4 shows the rationality of the condition that \(\tau_n - \tau_1 < d_A.\) Though our Corollary 2.3 improves a lot [21, Theorem 1] from perspective of control instants, some important idea in [21, Theorem 1] helps us greatly.

The author in [21] further claimed that (see [21, Theorem 1], or [32, Theorem 2.3.2]) when \((A, B)\) satisfies (1.8), the number of control instants can be taken as the smallest integer which is bigger than or equals to \(n/m.\) Unfortunately, we do not understand the
proof of this part. More precisely, we do not understand the argument from Lines 8-9 on Page 83 in [21].

(ii) It deserves to mention several facts on the exact controllability for \((1.9)\) where \((A,B)\) satisfies \((1.8)\). Fact one: In general, for an arbitrarily fixed \(T > 0\), we cannot arbitrarily take an increasing sequence \(\{\tau_k\}_{k=1}^n \subset (0,T)\) so that the exact controllability for \((1.9)\) over \([0,T]\) can be realized at any increasing sequence \(\{\tau_k\}_{k=1}^n \subset (0,T)\) (see Example 2.4). Fact two: If all eigenvalues of \(A\) are real, then for each \(T > 0\), the exact controllability for \((1.9)\) over \([0,T]\) can be realized at any \(\{\tau_k\}_{k=1}^n \subset (0,T)\). Indeed, in this case, we have that \(d_A = \infty\) (see \((1.4)\)). Fact three: When \(T \leq d_A\), the exact controllability for \((1.9)\) over \([0,T]\) can be realized at any increasing sequence \(\{\tau_k\}_{k=1}^n \subset (0,T)\).

To prove Theorem 2.1, we need two lemmas. The first lemma presents a kind of decomposition for some high order ordinary differential operators.

**Lemma 2.6.** Let \(C \in \mathbb{R}^{n \times n}\). Let \(g(\lambda) = \lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i\), \(\lambda \in \mathbb{C}\) be the characteristic polynomial of the matrix \(C\). Let \(d_C\) be given by \((1.4)\). Then given \(t_0 \in \mathbb{R}\), there is \(\{\phi_i\}_{i=1}^n \subset C^\infty((t_0 - \frac{d_C}{2}, t_0 + \frac{d_C}{2}); \mathbb{R})\) so that for each \(h \in C^\infty((t_0 - \frac{d_C}{2}, t_0 + \frac{d_C}{2}); \mathbb{R})\),

\[
g(\frac{d}{dt})h = (e^{-\phi_1} \circ \frac{d}{dt} \circ e^{\phi_1}) \circ \cdots \circ (e^{-\phi_n} \circ \frac{d}{dt} \circ e^{\phi_n})h. \tag{2.6}
\]

Here, each function \(e^{\phi_i}\) (with \(i = 1, \ldots, n\)) is regarded as the operator \(h \mapsto e^{\phi_i}h\) and the notation “ \(\circ\)” denotes the composition of operators.

**Proof.** Arbitrarily fix \(t_0 \in \mathbb{R}\). We first claim the following two facts:

- **(O1)** For each \(a \in \mathbb{R}\),

\[
\frac{d}{dt} - a = e^{at} \circ \frac{d}{dt} \circ e^{-at}. \tag{2.7}
\]

- **(O2)** For any \(b, c \in \mathbb{R}\) with \(c \neq 0\), there are \(\varphi_1\) and \(\varphi_2\) in \(C^\infty((t_0 - \frac{\pi}{2|c|}, t_0 + \frac{\pi}{2|c|}); \mathbb{R})\) so that for each \(h \in C^\infty((t_0 - \frac{\pi}{2|c|}, t_0 + \frac{\pi}{2|c|}); \mathbb{R})\),

\[
\left[ \frac{d^2}{dt^2} - 2b \frac{d}{dt} + (b^2 + c^2) \right] h = (e^{-\varphi_1} \circ \frac{d}{dt} \circ e^{\varphi_1}) \circ (e^{-\varphi_2} \circ \frac{d}{dt} \circ e^{\varphi_2})h. \tag{2.8}
\]

The fact **(O1)** can be directly checked. To prove **(O2)**, we define two functions by

\[
\varphi_1(t) \triangleq \int_{t_0}^t \left[ -b - c \tan c(\tau - t_0) \right] dt, \quad t \in \left(t_0 - \frac{\pi}{2|c|}, t_0 + \frac{\pi}{2|c|}\right),
\]

\[
\varphi_2(t) \triangleq \int_{t_0}^t \left[ -b + c \tan c(\tau - t_0) \right] dt, \quad t \in \left(t_0 - \frac{\pi}{2|c|}, t_0 + \frac{\pi}{2|c|}\right).
\]

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It is clear that \( \varphi_1 \) and \( \varphi_2 \) are in \( C^\infty((t_0 - \frac{\pi}{2|\beta|}, t_0 + \frac{\pi}{2|\beta|}); \mathbb{R}) \). By direct computations, we see that for each \( t \in (t_0 - \frac{\pi}{2|\beta|}, t_0 + \frac{\pi}{2|\beta|}) \),

\[
\varphi'_1(t) + \varphi'_2(t) = -2b \quad \text{and} \quad \varphi'_1(t)\varphi'_2(t) + \varphi''_2(t) = b^2 + e^2. \tag{2.9}
\]

Meanwhile, one can easily check that for each \( t \in (t_0 - \frac{\pi}{2|\beta|}, t_0 + \frac{\pi}{2|\beta|}); \mathbb{R} \) and each \( t \in (t_0 - \frac{\pi}{2|\beta|}, t_0 + \frac{\pi}{2|\beta|}) \),

\[
\left( e^{-\varphi_1} \circ \frac{d}{dt} \circ e^{\varphi_1} \right) \circ \left( e^{-\varphi_2} \circ \frac{d}{dt} \circ e^{\varphi_2} \right) h(t) = \frac{d^2}{dt^2} h(t) + (\varphi'_1(t) + \varphi'_2(t)) \frac{d}{dt} h(t) + (\varphi'_1(t)\varphi'_2(t) + \varphi''_2(t)) h(t). \tag{2.10}
\]

Now, (2.8) follows from (2.9) and (2.10), i.e., the fact (O2) is true.

Next, since \( g(\cdot) \) is a polynomial with real coefficients (which implies that \( g(\lambda_0) = 0 \) if and only if \( g(\lambda_0) = 0 \)), we can write all of its roots (i.e., the solutions of \( g(\lambda) = 0 \)) in the following manner:

\[
\alpha_1, \ldots, \alpha_{n_1} \text{ (real),} \quad \beta_1, \overline{\beta}_1, \ldots, \beta_{n_2}, \overline{\beta}_{n_2} \text{ (non-real), \quad with \quad } n_1 + 2n_2 = n.
\]

(Here, it is allowed that the multiplicity of \( \alpha_i \) (or \( \beta_j \)) is bigger than 1. i.e., it may happen that \( \alpha_1 = \alpha_2 \) (or \( \beta_1 = \beta_2 \)).) There are only three possibilities on \( n_1 \): (i) \( n_1 = 0 \); (ii) \( n_1 = n \); (iii) \( 1 \leq n_1 < n \).

Case (i): We have that \( n_1 = 0 \) and \( n = 2n_2 \). Then

\[
\frac{d}{dt} = \prod_{i=1}^{n_2} \left( \frac{d}{dt} - \beta_i \right) \left( \frac{d}{dt} - \overline{\beta}_i \right) = \prod_{i=1}^{n_2} \left( \frac{d^2}{dt^2} - 2\text{Re} \beta_i \frac{d}{dt} + |\beta_i|^2 \right). \tag{2.11}
\]

For each \( i \in \{1, \ldots, n_2\} \), we can apply the fact (O2), where \( b = \text{Re} \beta_i \) and \( c = \text{Im} \beta_i \), to find \( \phi_{i_1} \) and \( \phi_{i_2} \) in \( C^\infty((t_0 - \frac{\pi}{2|\beta_i|}, t_0 + \frac{\pi}{2|\beta_i|}); \mathbb{R}) \) so that for each \( h \in C^\infty((t_0 - \frac{\pi}{2|\beta_i|}, t_0 + \frac{\pi}{2|\beta_i|}); \mathbb{R}) \),

\[
\left[ \frac{d^2}{dt^2} - 2\text{Re} \beta_i \frac{d}{dt} + |\beta_i|^2 \right] h = \left[ (e^{-\phi_{i_1}} \circ \frac{d}{dt} \circ e^{\phi_{i_1}}) \circ (e^{-\phi_{i_2}} \circ \frac{d}{dt} \circ e^{\phi_{i_2}}) \right] h. \tag{2.12}
\]

Meanwhile, from the definition of \( d_C \) (see (1.4)), it follows that \( d_C \leq \frac{\pi}{|\text{Im} \beta_i|} \) for each \( i \in \{1, \ldots, n_2\} \). This, along with (2.11) and (2.12), leads to (2.6) in Case (i).

Case (ii): We have that \( n_1 = n \) and \( n_2 = 0 \). Then

\[
g\left( \frac{d}{dt} \right) = \prod_{i=1}^{n} \left( \frac{d}{dt} - \alpha_i \right). \tag{2.13}
\]
For each $i \in \{1, \ldots, n\}$, we can apply the fact (O1), where $a = \alpha_i$, to find $\phi_i \in C^\infty(\mathbb{R}; \mathbb{R})$ so that
\[
\frac{d}{dt} - \alpha_i = e^{-\phi_i} \circ \frac{d}{dt} \circ e^{\phi_i}.
\]
This, along with (2.13), leads to (2.6) in Case (ii).

Case (iii): We have that $n_1 \geq 1$ and $n_2 \geq 1$. Then
\[
g\left(\frac{d}{dt}\right) = \left[ \prod_{i=1}^{n_1} \left(\frac{d}{dt} - \alpha_i\right) \right] \left[ \prod_{j=1}^{n_2} \left(\frac{d}{dt} - \beta_j\right) \right]
\]
\[
= \left[ \prod_{i=1}^{n_1} \left(\frac{d}{dt} - \alpha_i\right) \right] \left[ \prod_{j=1}^{n_2} \left(\frac{d^2}{dt^2} - 2\text{Re} \, \beta_j \frac{d}{dt} + |\beta_j|^2\right) \right].
\]
(2.14)

For each $i \in \{1, \ldots, n_1\}$ and each $j \in \{1, \ldots, n_2\}$, we can apply respectively the facts (O1) where $a = \alpha_i$ and (O2) where $b = \text{Re} \, \beta_j$ and $c = \text{Im} \, \beta_j$, to find that $\phi_i \in C^\infty(\mathbb{R}; \mathbb{R})$ and $\phi_{j_1}, \phi_{j_2} \in C^\infty\left((t_0 - \frac{\pi}{2\text{Im} \, \beta_j}, t_0 + \frac{\pi}{2\text{Im} \, \beta_j}); \mathbb{R}\right)$, so that for each $h \in C^\infty\left((t_0 - \frac{\pi}{2\text{Im} \, \beta_j}, t_0 + \frac{\pi}{2\text{Im} \, \beta_j}); \mathbb{R}\right)$,
\[
\left[ \frac{d^2}{dt^2} - 2\text{Re} \, \beta_j \frac{d}{dt} + |\beta_j|^2 \right] h = \left[ (e^{-\phi_{j_1}} \circ \frac{d}{dt} \circ e^{\phi_{j_1}}) \circ (e^{-\phi_{j_2}} \circ \frac{d}{dt} \circ e^{\phi_{j_2}}) \right] h.
\]
(2.15)

Meanwhile, from the definition of $d_C$ (see (1.4)), it follows that $d_C \leq \frac{\pi}{|\text{Im} \, \beta_j|}$ for each $j \in \{1, \ldots, n_2\}$. This, along with (2.14) and (2.15), leads to (2.6) in Case (iii).

In summary, we end the proof of Lemma 2.6. \hfill \Box

The following lemma presents a kind of unique continuation for some high order ordinary differential equations.

**Lemma 2.7.** Let $C \in \mathbb{R}^{n \times n}$. Let $g(\lambda) = \lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i$, $\lambda \in \mathbb{C}$ be the characteristic polynomial of the matrix $C$. Let $d_C$ be given by (1.4). Suppose that $f \in C^\infty(\mathbb{R}; \mathbb{R})$ satisfies that
\[
g\left(\frac{d}{dt}\right) f = 0 \text{ over } \mathbb{R} \text{ and } f(\tau_k) = 0, \ \forall \ k \in \{1, \ldots, n\},
\]
(2.16)
for some increasing sequence $\{\tau_k\}_{k=1}^n \subset \mathbb{R}$, with $\tau_n - \tau_1 < d_C$. Then
\[
f \equiv 0 \text{ over } \mathbb{R}.
\]
(2.17)

**Proof.** Let $f$ (in $C^\infty(\mathbb{R}; \mathbb{R})$) and an increasing sequence $\{\tau_i\}_{i=1}^n$ (in $\mathbb{R}$), with $\tau_n - \tau_1 < d_C$, satisfy (2.16). We aim to show that $f$ satisfy (2.17). For this purpose, we arbitrarily fix $\hat{t}_0 \in \mathbb{R}$ so that
\[
\{\tau_i\}_{i=1}^n \subset \left(\hat{t}_0 - \frac{d_C}{2}, \hat{t}_0 + \frac{d_C}{2}\right).
\]
(2.18)
(Such \( \hat{t}_0 \) exists because \( \tau_n - \tau_1 < d_C \).) According to Lemma 2.6, there is a sequence 
\[ \{\phi_i\}_{i=1}^{n-k} \] in \( C^\infty((\hat{t}_0 - \frac{d_C}{2}, \hat{t}_0 + \frac{d_C}{2}); \mathbb{R}) \) so that for each \( h \in C^\infty((\hat{t}_0 - \frac{d_C}{2}, \hat{t}_0 + \frac{d_C}{2}); \mathbb{R}) \),
\[
g(\frac{d}{dt}h) = (e^{-\phi_n} \circ \frac{d}{dt} \circ e^{\phi_n}) \circ \ldots \circ (e^{-\phi_1} \circ \frac{d}{dt} \circ e^{\phi_1})h.
\] (2.19)

When \( n = 1 \), we have that \( g(\frac{d}{dt}) = \frac{d}{dt} + a_0 \), which, along with (2.16), yields that 
\[ f'(t) + a_0 f(t) = 0 \] for all \( t \in \mathbb{R} \); and that \( f(\tau_1) = 0 \). Hence, \( f \equiv 0 \), i.e., (2.17) is true in the case that \( n = 1 \).

We now show (2.17) for the case when \( n \geq 2 \). The proof is organized as two steps.

**Step 1.** We show that for each \( k \in \{1, \ldots, n-1\} \), there is an increasing sequence 
\[ \{\xi_{k,j}\}_{j=1}^{n-k} \subset (\hat{t}_0 - \frac{d_C}{2}, \hat{t}_0 + \frac{d_C}{2}) \] so that
\[
\left[(e^{-\phi_k} \circ \frac{d}{dt} \circ e^{\phi_k}) \circ \ldots \circ (e^{-\phi_1} \circ \frac{d}{dt} \circ e^{\phi_1}) f\right](\xi_{k,j}) = 0, \quad \forall j \in \{1, \ldots, n-k\}.
\] (2.20)

First, we show (2.20) with \( k = 1 \). By the second equality in (2.16), we find that
\[
(e^{\phi_1} f)(\tau_i) = 0, \quad \forall i \in \{1, \ldots, n\}.
\]
From this and the mean value theorem, we deduce that there exists an increasing sequence 
\[ \{\xi_{1,j}\}_{j=1}^{n-1} \subset (\tau_1, \tau_n) \] so that
\[
\left[\frac{d}{dt}(e^{\phi_1} f)\right](\xi_{1,j}) = 0, \quad \forall j \in \{1, \ldots, n-1\},
\] from which, it follows that
\[
[e^{-\phi_1} \frac{d}{dt}(e^{\phi_1} f)](\xi_{1,j}) = 0, \quad \forall j \in \{1, \ldots, n-1\}.
\]

Since \( (\tau_1, \tau_n) \subset (\hat{t}_0 - \frac{d_C}{2}, \hat{t}_0 + \frac{d_C}{2}) \) (see (2.18)), the above yields (2.20) for \( k = 1 \).

Next, we will show (2.20) for each \( k \in \{1, \ldots, n-1\} \). Since we are in the case that \( n \geq 2 \), there are only two possibilities on \( n \): either \( n = 2 \) or \( n \geq 3 \). In the first case that \( n = 2 \), we have that \( n-1 = 1 \). Then (2.20) has been proved since \( k \) can only take 1 now.

In the second case \( n \geq 3 \), we will show (2.20) by using mathematical induction. We already have (2.20) with \( k = 1 \). Suppose that (2.20) holds for \( k = m \), with \( m < n-1 \). That is, there exists an increasing sequence 
\[ \{\xi_{m,j}\}_{j=1}^{n-m} \subset (\hat{t}_0 - \frac{d_C}{2}, \hat{t}_0 + \frac{d_C}{2}) \] so that
\[
\left[(e^{-\phi_m} \circ \frac{d}{dt} \circ e^{\phi_m}) \circ \ldots \circ (e^{-\phi_1} \circ \frac{d}{dt} \circ e^{\phi_1}) f\right](\xi_{m,j}) = 0, \quad \forall j \in \{1, \ldots, n-m\}.
\] (2.21)
We aim to prove (2.20) with \( k = m + 1 \). For this purpose, we set
\[
q_m(t) \triangleq (e^{-\phi_m} \circ \frac{d}{dt} \circ e^{\phi_m}) \circ \cdots \circ (e^{-\phi_1} \circ \frac{d}{dt} \circ e^{\phi_1}) f(t), \quad t \in \left( \hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2} \right). \tag{2.22}
\]
From (2.22) and (2.21), it follows that
\[
e^{\phi_{m+1}} q_m(\xi_{m,j}) = 0, \quad \forall \ j \in \{1, \ldots, n - m\}.
\]
By this and the mean value theorem, we find that there exists an increasing sequence
\[
\{ \xi_{m+1,j} \}_{j=1}^{n-m-1} \subset (\hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2})
\]
so that
\[
\frac{d}{dt}(e^{\phi_{m+1}} q_m)(\xi_{m+1,j}) = 0, \quad \forall \ j \in \{1, \ldots, n - m - 1\},
\]
from which, it follows that
\[
e^{-\phi_{m+1}} \frac{d}{dt}(e^{\phi_{m+1}} q_m)(\xi_{m+1,j}) = 0, \quad \forall \ j \in \{1, \ldots, n - m - 1\}.
\]
This, along with (2.22), leads to (2.20) with \( k = m + 1 \). In summary, we conclude that (2.20) is true. This ends the proof of Step 1.

Step 2. We show (2.17).

We first claim that for each \( k \in \{1, \ldots, n\} \),
\[
(e^{-\phi_k} \circ \frac{d}{dt} \circ e^{\phi_k}) \circ \cdots \circ (e^{-\phi_1} \circ \frac{d}{dt} \circ e^{\phi_1}) f \equiv 0 \text{ over } \left( \hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2} \right). \tag{2.23}
\]
(We will only use (2.23) with \( k = 1 \) later.) By contradiction, we suppose that (2.23) was not true for some \( \bar{k} \in \{1, \ldots, n\} \). Then we would have that
\[
S \neq \emptyset, \quad \text{where } S \triangleq \{ \bar{k} \in \{1, \ldots, n\} \ : \ (2.23), \text{ with } k = \bar{k}, \text{ fails} \}. \tag{2.24}
\]
By (2.16) and (2.19), we see that (2.23), with \( k = n \), is true. Hence, \( n \notin S \). This, along with (2.24), yields that
\[
\hat{k}_1 \triangleq \max_{k \in S} \bar{k} < n. \tag{2.25}
\]
By (2.25) and the definition of \( S \) (see (2.24)), we find that (2.23), with \( k = \hat{k}_1 + 1(\leq n) \), is true. That is,
\[
(e^{-\phi_{k_1+1}} \circ \frac{d}{dt} \circ e^{\phi_{k_1+1}}) \circ \cdots \circ (e^{-\phi_1} \circ \frac{d}{dt} \circ e^{\phi_1}) f \equiv 0 \text{ over } \left( \hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2} \right). \tag{2.26}
\]
Define a function $\hat{f}$ in the following manner:

$$\hat{f}(t) \triangleq (e^{-\phi_{k_1}} \circ \frac{d}{dt} \circ e^{\phi_{k_1}}) \circ \cdots \circ (e^{-\phi_1} \circ \frac{d}{dt} \circ e^{\phi_1}) f(t), \quad t \in \left(\hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2}\right).$$  \hspace{1cm} (2.27)

By (2.27) and (2.26), we find that

$$(e^{-\phi_{k_1+1}} \circ \frac{d}{dt} \circ e^{\phi_{k_1+1}}) \hat{f}(t) \equiv 0 \quad \text{for all} \quad t \in \left(\hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2}\right),$$

from which, it follows that

$$\frac{d}{dt}(e^{\phi_{k_1+1}(t)} \hat{f}(t)) \equiv 0 \quad \text{for all} \quad t \in \left(\hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2}\right).$$

This implies that

$$e^{\phi_{k_1+1}(t)} \hat{f}(t) \equiv \text{const} \quad \text{for all} \quad t \in \left(\hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2}\right).$$ \hspace{1cm} (2.28)

Meanwhile, by (2.27), (2.25) and Step 1 (where $k = \hat{k}_1$), we get that $\hat{f}(\hat{\tau}) = 0$ for some $\hat{\tau} \in (\hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2})$. This, along with (2.28), indicates that

$$\hat{f} \equiv 0 \quad \text{over} \quad \left(\hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2}\right).$$

This, along with (2.27), leads to (2.23) with $k = \hat{k}_1$. By (2.23), with $k = \hat{k}_1$, and by the definition of $S$ (see (2.24)), we find that $\hat{k}_1 \notin S$, which contradicts the definition of $\hat{k}_1$ (see (2.25)). Therefore, (2.23) holds for all $k \in \{1, \ldots, n\}$.

Finally, by (2.23), with $k = 1$, we have that

$$(e^{\phi_1} \circ \frac{d}{dt} \circ e^{\phi_1}) f \equiv 0 \quad \text{over} \quad \left(\hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2}\right),$$

from which, it follows that

$$e^{\phi_1} f \equiv \text{const} \quad \text{over} \quad \left(\hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2}\right).$$

This, along with the second equality in (2.16), indicates that

$$f \equiv 0 \quad \text{over} \quad \left(\hat{t}_0 - \frac{dC}{2}, \hat{t}_0 + \frac{dC}{2}\right).$$  \hspace{1cm} (2.29)

At same time, since $f$ is a solution to the equation: $g\left(\frac{d}{dt}\right)f = 0$, we see that $f$ is analytic over $\mathbb{R}$. This, along with (2.29), leads to (2.17).

In summary, we end the proof of Lemma 2.7. \hfill \Box
Now we are on the position to prove Theorem 2.1.

**Proof of Theorem 2.1.** When \( n = 1 \), one can easily check, from (1.8), that \( \text{rank } B = 1 \). Then we find that for each \( t > 0 \), \( \text{rank } e^{At}B = \text{rank } B = 1 \), which leads to (2.1) for the case that \( n = 1 \).

We now show (2.1) for the case that \( n > 1 \). Let \( g(\lambda) = \lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i, \lambda \in \mathbb{C} \), be the characteristic polynomial of the matrix \( A \). Write

\[
\hat{A} = \begin{pmatrix}
0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & \cdots & 0 & -a_1 \\
0 & 1 & \cdots & 0 & -a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -a_{n-1}
\end{pmatrix}.
\] (2.30)

Consider the equation:

\[
\frac{d}{dt} \begin{pmatrix}
f_0(t) \\
\vdots \\
f_{n-1}(t)
\end{pmatrix} = \hat{A} \begin{pmatrix}
f_0(t) \\
\vdots \\
f_{n-1}(t)
\end{pmatrix}, \quad t \in \mathbb{R},
\] (2.31)

with the initial value condition:

\[
(f_0(0), f_1(0), \ldots, f_n(0))^\top = (1, 0, \ldots, 0)^\top \triangleq e_1.
\] (2.32)

Let \( (\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{n-1})^\top \in C^\infty(\mathbb{R}^1; \mathbb{R}^n) \) be the solution of (2.31)-(2.32).

The rest proof of this theorem is divided into the following three steps:

**Step 1. We show that**

\[
e^{At} = \sum_{i=0}^{n-1} \hat{f}_i(t) A^i \quad \text{for all } t \in \mathbb{R}.
\] (2.33)

Define a matrix-valued function in the following manner:

\[
\Phi(t) \triangleq \sum_{i=0}^{n-1} \hat{f}_i(t) A^i \quad \text{for all } t \in \mathbb{R}.
\] (2.34)

By (2.34), (2.31) and (2.30), we find that for all \( t \in \mathbb{R} \),

\[
\frac{d}{dt} \Phi(t) = \sum_{i=0}^{n-1} \frac{d}{dt} \hat{f}_i(t) A^i = -a_0 \hat{f}_{n-1}(t) + \sum_{i=1}^{n-1} (\hat{f}_{i-1}(t) - a_i \hat{f}_{n-1}(t)) A^i
\]

\[
= \sum_{i=1}^{n-1} \hat{f}_{i-1}(t) A^i - \left( \sum_{i=0}^{n-1} a_i A^i \right) \hat{f}_{n-1}(t).
\] (2.35)
Since \( g(\lambda) = \lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i \), \( \lambda \in \mathbb{C} \), is the characteristic polynomial of the matrix \( A \), it follows by the Cayley-Hamilton Theorem, (2.35) and (2.34) that for each \( t \in \mathbb{R} \),

\[
\frac{d}{dt} \Phi(t) = \sum_{i=1}^{n-1} \hat{f}_{i-1}(t) A^i - (g(A) - A^n) \hat{f}_{n-1}(t) = \sum_{i=1}^{n} \hat{f}_{i-1}(t) A^i
\]

\[
= A \sum_{i=1}^{n} \hat{f}_{i-1}(t) A^{i-1} = A \sum_{i=0}^{n-1} \hat{f}_{i}(t) A^{i} = A \Phi(t). \tag{2.36}
\]

Meanwhile, by (2.34) and (2.32), we find that

\[
\Phi(0) = I. \tag{2.37}
\]

Now, (2.33) follows from (2.36), (2.37) and (2.34). This ends the proof of Step 1.

**Step 2.** We prove that for each increasing sequence \( \{\tau_k\}_{k=1}^{n} \subset \mathbb{R} \), with \( \tau_n - \tau_1 < d_A \),

\[
\text{rank} \begin{pmatrix} \hat{f}_0(\tau_1) & \cdots & \hat{f}_0(\tau_n) \\ \vdots & \ddots & \vdots \\ \hat{f}_{n-1}(\tau_1) & \cdots & \hat{f}_{n-1}(\tau_n) \end{pmatrix} = n. \tag{2.38}
\]

 Arbitrarily take an increasing sequence \( \{\tau_k\}_{k=1}^{n} \subset \mathbb{R} \) so that

\[
\tau_n - \tau_1 < d_A. \tag{2.39}
\]

To show (2.38), it suffices to prove that the following equation has a unique solution \( x = 0 \) in \( \mathbb{R}^n \):

\[
x^T \begin{pmatrix} \hat{f}_0(\tau_1) & \cdots & \hat{f}_0(\tau_n) \\ \vdots & \ddots & \vdots \\ \hat{f}_{n-1}(\tau_1) & \cdots & \hat{f}_{n-1}(\tau_n) \end{pmatrix} = 0. \tag{2.40}
\]

For this purpose, we let \( x \in \mathbb{R}^n \) be a solution to (2.40). Since \( (\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_{n-1})^T \) solves (2.31)-(2.32), we find from (2.40) that

\[
x^T e^{\hat{A}_{\tau} e_1} = x^T \begin{pmatrix} \hat{f}_0(\tau_k) \\ \vdots \\ \hat{f}_{n-1}(\tau_k) \end{pmatrix} = 0, \quad \forall k \in \{1, \ldots, n\}. \tag{2.41}
\]

Let \( \hat{h} \triangleq (\hat{h}_0, \cdots, \hat{h}_{n-1})^T \in C^\infty(\mathbb{R}; \mathbb{R}^n) \) satisfy that

\[
\left\{ \begin{array}{l}
\frac{d}{dt} \hat{h}(t) = \hat{A}^T \hat{h}(t), \quad t \in \mathbb{R}, \\
\hat{h}(0) = x.
\end{array} \right. \tag{2.42}
\]
Since \( g(\lambda) = \lambda^n + \sum_{i=0}^{n-1} a_i \lambda^i, \lambda \in \mathbb{C} \), it follows by (2.42) and (2.30) that
\[
\hat{h} = (\hat{h}_0, \frac{d}{dt} \hat{h}_0, \cdots, \frac{d^{n-1}}{dt^{n-1}} \hat{h}_0)^\top \quad \text{and} \quad g(\frac{d}{dt}) \hat{h}_0 = 0.
\]  
(2.43)

Meanwhile, by (2.42) and (2.41), we get that for each \( k \in \{1, \ldots, n\} \),
\[
\hat{h}_0(\tau_k) = \langle \hat{h}(\tau_k), e_1 \rangle_{\mathbb{R}^n} = \langle e^{\hat{A}^\top \tau_k} x, e_1 \rangle_{\mathbb{R}^n} = x^\top e^{\hat{A}^\tau_k} e_1 = 0.
\]  
(2.44)

Since \( g(\cdot) \) is the characteristic polynomial of the matrix \( A \), by the second equality in (2.43), (2.44) and (2.39), we can apply Lemma 2.7, to obtain that \( \hat{h}_0 \equiv 0 \) over \( \mathbb{R} \). This, together with the first equality in (2.43), yields that \( \hat{h} \equiv 0 \) over \( \mathbb{R} \), from which, as well as (2.42), it follows that \( x = 0 \), i.e., (2.40) has a unique solution \( x = 0 \). Hence, (2.38) is true.

Step 3. We prove (2.1).

Arbitrarily take an increasing sequence \( \{\tau_k\}_{k=1}^n \subset \mathbb{R} \) so that
\[
\tau_n - \tau_1 < d_A.
\]  
(2.45)

To prove (2.1), it suffices to show that the following equation has a unique solution \( \alpha = 0 \):
\[
\alpha^\top (e^{A\tau_1} B, e^{A\tau_2} B, \ldots, e^{A\tau_n} B) = 0.
\]  
(2.46)

For this purpose, we let \( \alpha \in \mathbb{R}^n \) satisfy (2.46). Then, by Step 1, we see that for each \( k \in \{1, \ldots, n\} \),
\[
\begin{bmatrix}
B^* \alpha, B^* A^* \alpha, \ldots, B^* (A^*)^{n-1} \alpha
\end{bmatrix}
\begin{pmatrix}
\hat{f}_0(\tau_k) \\
\vdots \\
\hat{f}_{n-1}(\tau_k)
\end{pmatrix}
= \sum_{i=0}^{n-1} \hat{f}(\tau_k) B^*(A^*)^i \alpha
= B^* \sum_{i=0}^{n-1} \hat{f}(\tau_k) A^i \alpha
= B^* e^{A^\tau_k} \alpha
= (\alpha^\top e^{A^\tau_k} B)^\top = 0.
\]

From this, we get that
\[
\begin{bmatrix}
B^* \alpha, B^* A^* \alpha, \ldots, B^* (A^*)^{n-1} \alpha
\end{bmatrix}
\begin{pmatrix}
\hat{f}_0(\tau_1) & \cdots & \hat{f}_0(\tau_n) \\
\vdots & \ddots & \vdots \\
\hat{f}_{n-1}(\tau_1) & \cdots & \hat{f}_{n-1}(\tau_n)
\end{pmatrix}
= 0.
\]

This, along with Step 2 and (2.45), yields that
\[
B^*(A^*)^{k-1} \alpha = 0 \quad \text{for each} \quad k \in \{1, \ldots, n\},
\]
from which, it follows that
\[ \alpha^\top (B, AB, A^2B, \ldots, A^{n-1}B) = (\alpha^\top B, \alpha^\top AB, \alpha^\top A^2B, \ldots, \alpha^\top A^{n-1}B) \]
\[ = \begin{pmatrix} B^\ast \alpha \\ B^\ast A^\ast \alpha \\ \vdots \\ B^\ast (A^\ast)^{n-1} \alpha \end{pmatrix}^\top = 0. \]

Since \((A, B)\) satisfies (1.8), the above yields that \(\alpha = 0\). Thus, (2.46) only has the trivial solution in \(\mathbb{R}^n\). Therefore, (2.1) is true.

In summary, we end the proof of Theorem 2.1.

We are now on the position to prove Theorem 2.2.

Proof of Theorem 2.2. Let \(T > 0\). Let \(\{\tau_k\}_{k=1}^p \subset (0, T)\), with \(p \in \mathbb{N}^+\), be an increasing sequence. First of all, for each \(z_0 \in \mathbb{R}^n\) and \(\{u_k\}_{k=1}^p \subset \mathbb{R}^m\), the solution of (1.9) satisfies

\[ z(T; z_0, \{\tau_k\}_{k=1}^p, \{u_k\}_{k=1}^p) = e^{AT} z_0 + \sum_{1 \leq k \leq p} e^{A(T-\tau_k)} Bu_k, \quad (2.47) \]

where \(z(\cdot; z_0, \{\tau_k\}_{k=1}^p, \{u_k\}_{k=1}^p)\) denotes the solution of (1.9).

We next prove the sufficiency. Assume that (2.2) is true. Arbitrarily fix \(\hat{z}_0, \hat{z}_1 \in \mathbb{R}^n\). By (2.2), we find that there exists \(\{\hat{u}_k\}_{k=1}^p \subset \mathbb{R}^m\) so that

\[ \sum_{k=1}^p e^{A(T-\tau_k)} B \hat{u}_k = \hat{z}_1 - e^{AT} \hat{z}_0. \]

From this and (2.47), we see that

\[ z(T; \hat{z}_0, \{\tau_k\}_{k=1}^p, \{\hat{u}_k\}_{k=1}^p) = \hat{z}_1. \]

Since \(\hat{z}_0, \hat{z}_1\) were arbitrarily taken in \(\mathbb{R}^n\), the above, along with the definition in the note (b) in Section 1, implies that the exact controllability for (1.9) over \([0, T]\) can be realized at \(\{\tau_k\}_{k=1}^p\). This proves the sufficiency.

Finally, we show the necessity. Assume that the exact controllability for (1.9) over \([0, T]\) can be realized at \(\{\tau_k\}_{k=1}^p\). Then, by the definition in the note (b) in Section 1 and (2.47), we get that for each \(z_1 \in \mathbb{R}^n\), there exists \(\{u_k\}_{k=1}^p \subset \mathbb{R}^m\) so that

\[ z_1 = z(T; 0, \{\tau_k\}_{k=1}^p, \{u_k\}_{k=1}^p) = \sum_{1 \leq k \leq p} e^{A(T-\tau_k)} Bu_k \quad (2.48) \]
\[ \in \text{Range } (e^{A(T-\tau_1)} B, \ldots, e^{A(T-\tau_p)} B). \]
This indicates that
\[ \mathbb{R}^n \subset \text{Range} \left( e^{A(T-\tau_1)}B, \ldots, e^{A(T-\tau_n)}B \right). \]
Thus, (2.2) is true, which leads to the necessity.

In summary, we end the proof of this theorem. \qed

3 Unique continuation for system of heat equations

Some connections among \( \{ e^{A t} \} \) \( t \geq 0 \), \( \{ e^{-A t} \} \) \( t \geq 0 \) and \( \{ e^{\Delta t} \} \) \( t \geq 0 \) (which denotes the \( C_0 \)-semigroup generated by \( \Delta \) over \( L^2(\Omega; \mathbb{R}^n) \)) are given in the next Proposition 3.1.

**Proposition 3.1.** The following two equalities hold for all \( t \geq 0 \):
\[ e^{A^* t} = e^{-A^* t} e^{\Delta t} \quad \text{and} \quad e^{A t} = e^{\Delta t} e^{-A t}. \] (3.1)

**Proof.** Arbitrarily fix \( z \in L^2(\Omega; \mathbb{R}^n) \). One can easily check that the function \( t \mapsto e^{\Delta t} z \), \( t > 0 \), belongs to the following space:
\[ C^1((0, \infty); L^2(\Omega; \mathbb{R}^n)) \cap C((0, \infty); H^2(\Omega; \mathbb{R}^n) \cap H^1_0(\Omega; \mathbb{R}^n)). \] (3.2)

Define
\[ h_z(t) \triangleq e^{-A^* t} e^{\Delta t} z, \quad t \geq 0. \] (3.3)
By (3.3) and (3.2), we get that
\[ h_z \in C^1((0, \infty); L^2(\Omega; \mathbb{R}^n)) \cap C((0, \infty); H^2(\Omega; \mathbb{R}^n) \cap H^1_0(\Omega; \mathbb{R}^n)). \] (3.4)
Since \( A^* = \Delta - A^* \) and because \( \Delta \) is commutative with the operator \( A^* \), it follows from (3.3) that for each \( t > 0 \),
\[
\frac{d}{dt} h_z - A^* h_z = \frac{d}{dt} h_z - \Delta h_z + A^* h_z
= \left[ - A^* e^{-A^* t} e^{\Delta t} z + e^{-A^* t} \frac{d}{dt} e^{\Delta t} z \right] - \Delta e^{-A^* t} e^{\Delta t} z + A^* e^{-A^* t} e^{\Delta t} z
= e^{-A^* t} \left[ \frac{d}{dt} e^{\Delta t} z - \Delta e^{\Delta t} z \right] = 0. \] (3.5)
Meanwhile, we observe from (3.3) that \( h_z(0) = z \). This, along with (3.5), yields that
\[ h_z(t) = e^{A^* t} z, \quad t \geq 0, \]
which, together with (3.3), leads to the first equality in (3.1).

By taking the adjoint on both sides of the first equality in (3.1), we obtain the second equality in (3.1). This ends the proof of Proposition 3.1. \qed
The next Proposition 3.2 presents unique continuation property for the system of heat equations. The key to proving this proposition is the use of the unique continuation property at one point in time for the heat equation. This property says that if a solution $y$ of the heat equation with the homogeneous Dirichlet boundary condition satisfies that for some $\tau > 0$, $y(x, \tau) = 0$ for a.e. $x \in \omega$, then $y \equiv 0$. (See, for instance, [19], [26] and [28]).

**Proposition 3.2.** Let $\omega_1$ be an open and nonempty subset of $\Omega$ and let $V$ be a subspace of $\mathbb{R}^n$. Then the following two conclusions are true:

(i) If $z \in L^2(\Omega; \mathbb{R}^n)$ and $\tau > 0$, then $z(x) \in V$ for a.e. $x \in \Omega \iff e^{\Delta \tau} z(x) \in V$ for all $x \in \omega_1$.

(ii) If $z \in L^2(\Omega; \mathbb{R}^n)$ and $\tau > 0$, then $\chi_{\omega_1} e^{A \tau} z = 0$ if and only if $z = 0$.

**Proof.** (i) Arbitrarily fix $z \in L^2(\Omega; \mathbb{R}^n)$ and $\tau > 0$. Let $\Psi(t) \equiv e^{\Delta t} z$ for each $t \geq 0$. Then $\Psi$ satisfies that

$$
\begin{aligned}
\partial_t \Psi - \Delta \Psi &= 0 \quad \text{in } \mathbb{R}^+ \times \Omega,
\Psi &= 0 \quad \text{on } \mathbb{R}^+ \times \partial \Omega,
\Psi(0) &= z \quad \text{in } \Omega.
\end{aligned}
$$

We organize the rest of the proof by the following two steps:

**Step 1.** We show that if $z(x) \in V$ for a.e. $x \in \Omega$, then $e^{\Delta \tau} z(x) \in V$ for all $x \in \omega_1$.

Suppose that $z(x) \in V$ for a.e. $x \in \Omega$. Then

$$
z(x) \in V \quad \text{for a.e. } x \in \Omega. \quad (3.7)
$$

We aim to prove that

$$
e^{\Delta \tau} z(x) \in V \quad \text{for all } x \in \omega_1. \quad (3.8)
$$

Arbitrarily fix $\alpha$ in $V^\perp$ which is the orthogonal complement space of $V$ in $\mathbb{R}^n$. Then define a function $\psi_\alpha : \Omega \times [0, \infty) \to \mathbb{R}$ in the following manner:

$$
\psi_\alpha(x, t) \equiv \langle e^{A t} z(x), \alpha \rangle_{\mathbb{R}^n}, \quad (x, t) \in \Omega \times [0, \infty). \quad (3.9)
$$

By (3.9) and (3.6), one can directly check that $\psi_\alpha$ satisfies that

$$
\begin{aligned}
\partial_t \psi_\alpha - \Delta \psi_\alpha &= 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\
\psi_\alpha &= 0 \quad \text{on } \partial \Omega \times \mathbb{R}^+. \quad (3.10)
\end{aligned}
$$

Meanwhile, since $\alpha \in V^\perp$, it follows from (3.9) and (3.7) that $\psi_\alpha(x, 0) = 0$ for a.e. $x \in \Omega$. This, along with (3.10), yields that $\psi_\alpha \equiv 0$ over $\Omega \times (0, \infty)$, which, together with (3.9), indicates that

$$
\langle e^{A \tau} z(x), \alpha \rangle_{\mathbb{R}^n} = 0 \quad \text{for all } x \in \Omega.
$$
Since $\alpha$ was arbitrarily taken from $V^\perp$, the above leads to (3.8).

**Step 2.** We show that if $e^{\Delta_T}z(x) \in V$ for each $x \in \omega_1$, then $z(x) \in V$ for a.e. $x \in \Omega$

We only need to consider the case that $\dim V < n$, since when $\dim V = n$, the desired result is clearly true. Suppose that

$$e^{\Delta_T}z(x) \in V \text{ for all } x \in \omega_1. \quad (3.11)$$

Arbitrarily take $\alpha \in V^\perp$. Let $\psi_\alpha$ be the function defined by (3.9). Then from (3.9) and (3.11), we see that

$$\psi_\alpha(x, \tau) = 0 \text{ for all } x \in \omega_1. \quad (3.12)$$

Since $\psi_\alpha$ solves the adjoint heat equation with the zero Dirichlet boundary condition (see (3.10)), by (3.10) and (3.12), using the unique continuation property for heat equations (see, for instance, [19], [26] and [28]), we see that $\psi_\alpha(x, 0) = 0$ for a.e. $x \in \Omega$. This, along with (3.9), yields that

$$\langle z(x), \alpha \rangle_{\mathbb{R}^n} = 0 \text{ for a.e. } x \in \Omega.$$

Since $\alpha$ was arbitrarily taken from the finitely dimensional subspace $V^\perp$, the above leads to that $z(x) \in V$ for a.e. $x \in \Omega$.

By the results in Step 1 and Step 2, we see that the conclusion (i) of this proposition is true.

(ii) Arbitrarily fix $z \in L^2(\Omega; \mathbb{R}^n)$ and $\tau > 0$. It is clear that $\chi_{\omega_1}e^{A_T}z = 0$, when $z = 0$. To show the reverse, we suppose that

$$\chi_{\omega_1}e^{A_T}z = 0. \quad (3.13)$$

From (3.13) and the second equality in Proposition 3.1, we see that

$$\chi_{\omega_1}e^{\Delta_T}(e^{-A_T}z) = \chi_{\omega_1}e^{A_T}z = 0. \quad (3.14)$$

By (3.14), we can use the conclusion (i) (in this proposition), with $V = \{0\}$, to get that $e^{-A_T}z(x) = 0$ a.e. $x \in \Omega$. This yields that $z(x) = 0$ for a.e. $x \in \Omega$, since $e^{-A_T}$ is invertible. Hence, the conclusion (ii) is true.

In summary, we end the proof of Proposition 3.2. \hfill $\square$

### 4 Proof of Theorem 1.3

The following formula will be frequently used in the rest of this paper: For each $T > 0$, $y_0 \in L^2(\Omega; \mathbb{R}^n)$, $p \in \mathbb{N}^+$, $\{\tau_k\}_{k=1}^p \subset (0, T)$ (with $\tau_1 < \cdots < \tau_p$), $\{u_k\}_{k=1}^p \subset L^2(\Omega; \mathbb{R}^m)$ and
\( z \in L^2(\Omega; \mathbb{R}^n) \), it stands that
\[
\langle y(T; y_0, \{\tau_k\}_{k=1}^p, \{u_k\}_{k=1}^p), z \rangle = \langle y_0, e^{A^*T}z \rangle + \sum_{k=1}^p \langle u_k, \chi_\omega B^*e^{A^*(T-\tau_k)}z \rangle_{L^2(\Omega; \mathbb{R}^m)}.
\]

The equality (4.1) follows directly from (1.3). Before proving Theorem 1.3, we present three lemmas. The first two lemmas will be used in the proof of Theorem 1.3, while the last one will be used in the proofs of both Theorem 1.3 and Theorem 1.4.

**Lemma 4.1.** There exists a linear map \( C \) from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) so that
\[
BC\alpha = \alpha \quad \text{for each } \alpha \in \text{Range } B.
\]

*Proof.* When \( B = 0 \), (4.2) is trivial. Thus, we can assume, without loss of generality, that \( B \neq 0 \). Take a basis \( \{\alpha_1, \ldots, \alpha_q\} \) of \( \text{Range } B \), where \( q \triangleq \dim \text{Range } B \) (since \( B \neq 0 \)). Then for each \( k \in \{1, \ldots, q\} \), there exists \( \beta_k \in \mathbb{R}^m \) so that
\[
B\beta_k = \alpha_k.
\]

Define a linear map \( \hat{C} \) from \( \text{Range } B \) to \( \mathbb{R}^m \) so that
\[
\hat{C}(\alpha_k) = \beta_k \quad \text{for each } k \in \{1, \ldots, q\}.
\]

We claim that
\[
B\hat{C}(\alpha) = \alpha \quad \text{for each } \alpha \in \text{Range } B.
\]

To this end, we arbitrarily fix \( \alpha \in \text{Range } B \) and write
\[
\alpha = \sum_{k=1}^q a_k\alpha_k \quad \text{with } \{a_k\}_{k=1}^q \subset \mathbb{R}.
\]

It follows from (4.6), (4.4) and (4.3) that
\[
B\hat{C}(\alpha) = \sum_{k=1}^q a_kB\hat{C}(\alpha_k) = \sum_{k=1}^q a_kB\beta_k = \sum_{k=1}^q a_k\alpha_k = \alpha.
\]

Since \( \alpha \) was arbitrarily taken from \( \text{Range } B \), the above leads to (4.5).

Finally, we observe that the map \( \hat{C} \) is defined over \( \text{Range } B \) (see (4.4)). It is easy to extend \( \hat{C} \) to be a linear operator over \( \mathbb{R}^n \), with the property (4.5). This ends the proof of Lemma 4.1. \( \square \)
Lemma 4.2. Let \( V_1, \ldots, V_q \), with \( q \in \mathbb{N}^+ \), be subspaces of \( \mathbb{R}^n \) so that
\[
\sum_{k=1}^{q} V_k = \mathbb{R}^n. \tag{4.7}
\]
Then there are linear operators \( P_1, \ldots, P_q \) (from \( \mathbb{R}^n \) to \( \mathbb{R}^n \)) so that for each \( k \in \{1, \ldots, q\} \), \( P_k \) maps \( \mathbb{R}^n \) into \( V_k \) and so that
\[
\alpha = \sum_{k=1}^{q} P_k \alpha \text{ for each } \alpha \in \mathbb{R}^n. \tag{4.8}
\]
Proof. Take a basis \( \{\alpha_1, \ldots, \alpha_n\} \) of \( \mathbb{R}^n \). By (4.7), we find that for each \( j \in \{1, \ldots, n\} \), there exits \( (\alpha_{j,1}, \alpha_{j,2}, \ldots, \alpha_{j,q}) \in \prod_{k=1}^{q} V_k \) so that
\[
\alpha_j = \sum_{k=1}^{q} \alpha_{j,k}. \tag{4.9}
\]
For each \( k \in \{1, \ldots, q\} \), we define a linear map \( P_k : \mathbb{R}^n \to V_k \) in the following manner:
\[
P_k(\alpha_j) = \alpha_{j,k} \text{ for each } j \in \{1, \ldots, n\}. \tag{4.10}
\]
We claim that the above \( \{P_k\}_{k=1}^{q} \) satisfies (4.8). For this purpose, we arbitrarily take \( \alpha \in \mathbb{R}^n \). Write
\[
\alpha = \sum_{j=1}^{n} a_j \alpha_j \text{ with } \{a_j\}_{j=1}^{n} \subset \mathbb{R}. \tag{4.11}
\]
Then it follows from (4.11), (4.9) and (4.10) that
\[
\alpha = \sum_{j=1}^{n} a_j (\sum_{k=1}^{q} \alpha_{j,k}) = \sum_{k=1}^{q} (\sum_{j=1}^{n} a_j \alpha_{j,k}) = \sum_{k=1}^{q} (\sum_{j=1}^{n} a_j P_k(\alpha_j)) = \sum_{k=1}^{q} P_k(\sum_{j=1}^{n} a_j \alpha_j) = \sum_{k=1}^{q} P_k(\alpha).
\]
Since \( \alpha \) was arbitrarily taken from \( \mathbb{R}^n \), the above leads to (4.8). We end the proof of Lemma 4.2.

Lemma 4.3. Let \( (A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \) satisfies that
\[
\text{rank} (B, AB, A^2B, \ldots, A^{n-1}B) < n. \tag{4.12}
\]
Then there is \( \hat{z} \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\} \) so that
\[
\chi_\omega B^*e^{At}\hat{z} = 0 \text{ for each } t > 0. \tag{4.13}
\]
Proof. We first claim that

\[
\bigcap_{k=0}^{n-1} \ker (B^*(A^*)^k) \neq \{0\}. \tag{4.14}
\]

Indeed, since \((A, B)\) satisfies (4.12), there is \(\alpha \in \mathbb{R}^n \setminus \{0\}\) so that for all \(k \in \{0, \ldots, n-1\}\),

\[
\alpha^\top (B, AB, \ldots, A^{n-1}B) = 0, \quad \text{i.e.,} \quad \alpha^\top A^kB = 0,
\]

from which, it follows that

\[
B^*(A^*)^k \alpha = (\alpha^\top A^kB)^\top = 0 \quad \text{for all} \quad k \in \{0, \ldots, n-1\}.
\]

This implies that \(\alpha \in \ker (B^*(A^*)^k)\) for all \(k \in \{0, \ldots, n-1\}\).

Since \(\alpha \neq 0\), the above leads to (4.14).

Next, we let

\[
V \triangleq \bigcap_{k=0}^\infty \ker (B^*(A^*)^k). \tag{4.15}
\]

By the Cayley-Hamilton theorem, we have that \(V = \bigcap_{k=0}^{n-1} \ker (B^*(A^*)^k)\). Then by (4.14), there is \(\alpha \in \mathbb{R}^n\) so that

\[
\alpha \in V \setminus \{0\}. \tag{4.16}
\]

We define a function \(z_\alpha : \Omega \to \mathbb{R}^n\) in the following manner:

\[
z_\alpha(x) = \alpha \quad \text{a.e.} \quad x \in \Omega. \tag{4.17}
\]

Then by Proposition 3.1 (see the first equality in (3.1)), we find that for each \(t > 0\),

\[
\begin{align*}
\chi_\omega B^* e^{A^t} z_\alpha &= \chi_\omega B^* e^{-A^t} e^{\Delta t} z_\alpha = B^* e^{-A^t} \chi_\omega e^{\Delta t} z_\alpha \\
&= B^* \left( \sum_{k \geq 0} \frac{1}{k!} (-t)^k (A^*)^k \right) \chi_\omega e^{\Delta t} z_\alpha \\
&= \sum_{k \geq 0} \frac{1}{k!} (-t)^k B^* (A^*)^k \chi_\omega e^{\Delta t} z_\alpha. \tag{4.18}
\end{align*}
\]

Meanwhile, since \(z_\alpha(x) \in V\) a.e. \(x \in \Omega\) (see (4.16) and (4.17)), it follows by (i) of Proposition 3.2 (where \(\omega_1 = \omega, \tau = t\) and \(z = z_\alpha\)) that for each \(t > 0\),

\[
\chi_\omega(x) e^{\Delta t} z_\alpha(x) \in V \quad \text{for a.e.} \quad x \in \Omega,
\]

which, together with (4.15), yields that for each \(t > 0\),

\[
B^* (A^*)^k \chi_\omega(x) e^{\Delta t} z_\alpha(x) = 0 \quad \text{for a.e.} \quad x \in \Omega.
\]

The above, along with (4.18), leads to (4.13) with \(\tilde{z} = z_\alpha\). This ends the proof. \(\square\)
We now on the position to prove Theorem 1.3.

Proof of Theorem 1.3. (i) Assume that
\[ \Omega \setminus \overline{\omega} \neq \emptyset. \] (4.19)

By contradiction, we suppose that the system (1.2) were null controllable over \([0, \hat{T}]\) for some \(\hat{T} > 0\). Then, according to (i) of Definition 1.1, for an arbitrarily fixed
\[ \hat{y}_0 \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\}, \] (4.20)
there is \(\hat{p} \in \mathbb{N}^+, \{\hat{\tau}_k\}_{k=1}^{\hat{p}} \subset (0, \hat{T})\) (with \(\hat{\tau}_1 < \cdots < \hat{\tau}_\hat{p}\)), and \(\{\hat{u}_k\}_{k=1}^{\hat{p}} \subset L^2(\Omega; \mathbb{R}^m)\) so that
\[ y(\hat{T}; \hat{y}_0, \{\hat{\tau}_k\}_{k=1}^{\hat{p}}, \{\hat{u}_k\}_{k=1}^{\hat{p}}) = 0. \]

This, along with (1.3), indicates that
\[ 0 = y(\hat{T}; \hat{y}_0, \{\hat{\tau}_k\}_{k=1}^{\hat{p}}, \{\hat{u}_k\}_{k=1}^{\hat{p}}) = e^{A(T-\hat{\tau}_\hat{p})}(e^{A\hat{\tau}_p}\hat{y}_0 + \sum_{k=1}^{\hat{p}} e^{A(\hat{\tau}_p - \hat{\tau}_k)}\chi_\omega B\hat{u}_k). \]

From this and (ii) of Proposition 3.2, where
\[ \omega_1 = \Omega, \tau = T - \hat{\tau}_\hat{p} \quad \text{and} \quad z = e^{A\hat{\tau}_p}\hat{y}_0 + \sum_{k=1}^{\hat{p}} e^{A(\hat{\tau}_p - \hat{\tau}_k)}\chi_\omega B\hat{u}_k, \]
we find that
\[ 0 = e^{A\hat{\tau}_p}\hat{y}_0 + \sum_{k=1}^{\hat{p}} e^{A(\hat{\tau}_p - \hat{\tau}_k)}\chi_\omega B\hat{u}_k \]
\[ = e^{A(\hat{\tau}_p - \hat{\tau}_{\hat{p}-1})} \left( e^{A\hat{\tau}_{\hat{p}-1}}\hat{y}_0 + \sum_{k=1}^{\hat{p}-1} e^{A(\hat{\tau}_{\hat{p}-1} - \hat{\tau}_k)}\chi_\omega B\hat{u}_k \right) + \chi_\omega B\hat{u}_{\hat{p}}. \] (4.21)

Meanwhile, from (4.19), there exists \(B_r(x_0) \subset \Omega\) so that
\[ B_r(x_0) \subset \Omega \setminus \overline{\omega}, \] (4.22)
where \(B_r(x_0)\) denotes the ball in \(\mathbb{R}^N\), centred at \(x_0\) and of radius \(r > 0\). From (4.22), it follows that \(\chi_\omega B\hat{u}_{\hat{p}} = 0\) on \(B_r(x_0)\), which, along with (4.21), yields that
\[ e^{A(\hat{\tau}_p - \hat{\tau}_{\hat{p}-1})} \left( e^{A\hat{\tau}_{\hat{p}-1}}\hat{y}_0 + \sum_{k=1}^{\hat{p}-1} e^{A(\hat{\tau}_{\hat{p}-1} - \hat{\tau}_k)}\chi_\omega B\hat{u}_k \right) = 0 \quad \text{on} \quad B_r(x_0). \]
From this and (ii) of Proposition 3.2, where
\[
\omega_1 = B_r(x_0), \tau = \hat{\tau}_p - \hat{\tau}_{p-1} \quad \text{and} \quad z = e^{A\hat{\tau}_{p-1}}\hat{y}_0 + \sum_{k=1}^{\hat{p}-1} e^{A(\hat{\tau}_{p-1}-\hat{\tau}_k)}\chi_\omega B\hat{u}_k,
\]
we see that
\[
e^{A\hat{\tau}_{p-1}}\hat{y}_0 + \sum_{k=1}^{\hat{p}-1} e^{A(\hat{\tau}_{p-1}-\hat{\tau}_k)}\chi_\omega B\hat{u}_k = 0. \tag{4.23}
\]
Following the same way as that showing (4.23), one can verify that when \(1 \leq q \leq \hat{p} - 1\),
\[
e^{A\hat{\tau}_q}\hat{y}_0 + \sum_{k=1}^{q} e^{A(\hat{\tau}_q-\hat{\tau}_k)}\chi_\omega B\hat{u}_k = 0.
\]
In particular, by taking \(q = 1\) in the above, we obtain that
\[
e^{A\hat{\tau}_1}\hat{y}_0 + \chi_\omega B\hat{u}_1 = 0.
\]
This, combined with (4.22), yields that \(e^{A\hat{\tau}_1}\hat{y}_0 = 0\) over \(B_r(x_0)\). From the above and (ii) of Proposition 3.2, where \(\omega_1 = B_r(x_0), \tau = \hat{\tau}_1\) and \(z = \hat{y}_0\), we find that \(\hat{y}_0 = 0\), which contradicts (4.20). Hence, when \(\omega\) satisfies (4.19), the system (1.2) is not null controllable on \([0, T]\) for any \(T > 0\). This ends the proof of the conclusion (i) of Theorem 1.3.

(ii) Suppose that \(\omega = \Omega\). \tag{4.24}

Step 1. We prove the sufficiency.

Assume that \((A, B)\) satisfies Kalman’s controllability rank condition (1.8). We aim to show the null controllability for the system (1.2). To this end, we arbitrarily fix \(T > 0\) and an increasing sequence \(\{\tau_k\}_{k=1}^n \subset (0, T)\), with \(\tau_n - \tau_1 < d_A\) (given by (1.4)). Then by Theorem 2.1 and (1.8), we get that
\[
\text{rank} \left( e^{A\tau_1} B, \ldots, e^{A\tau_n} B \right) = n. \tag{4.25}
\]
For each \(k \in \{1, \ldots, n\}\), we define a subspace \(V_k\) of \(\mathbb{R}^n\) in the following manner:
\[
V_k = \{ e^{A\tau_k} B\alpha \in \mathbb{R}^n : \alpha \in \mathbb{R}^m \}. \tag{4.26}
\]
From (4.25) and (4.26), we see that \(\mathbb{R}^n = \sum_{k=1}^n V_k\). This, together with Lemma 4.2, yields that for each \(k \in \{1, \ldots, n\}\), there is a linear map \(P_k : \mathbb{R}^n \to V_k\) so that
\[
\alpha = \sum_{k=1}^n P_k\alpha \quad \text{for each} \quad \alpha \in \mathbb{R}^n. \tag{4.27}
\]
We now claim that for an arbitrarily fixed $y_0 \in L^2(\Omega; \mathbb{R}^n)$,
\[ e^{\Delta \tau_k} e^{-A\tau_k} P_k y_0(x) \in \text{Range } B \quad \text{a.e. } x \in \Omega, \text{ when } 1 \leq k \leq n. \] (4.28)

For this purpose, we observe from (4.26) that $e^{-A\tau_k} V_k \subset \text{Range } B$ for each $k \in \{1, \ldots, n\}$. This yields that for each $k \in \{1, \ldots, n\}$,
\[ e^{-A\tau_k} P_k y_0(x) \in \text{Range } B \quad \text{for a.e. } x \in \Omega. \] (4.29)

By (4.29) and (i) of Proposition 3.2, where
\[ \omega_1 = \Omega, V = \text{Range } B, \tau = \tau_k \text{ and } z = e^{-A\tau_k} P_k y_0, \]
we are led to (4.28).

Next, according to Lemma 4.1, there exists a linear map $C$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ so that
\[ BC\alpha = \alpha \quad \text{for all } \alpha \in \text{Range } B. \] (4.30)

From (4.28) and (4.30), we see that for each $k \in \{1, \ldots, n\}$,
\[ e^{\Delta \tau_k} e^{-A\tau_k} P_k y_0 + B\hat{u}_k = 0, \] (4.31)

where the control $\hat{u}_k$ is defined by
\[ \hat{u}_k(x) \triangleq -C e^{\Delta \tau_k} e^{-A\tau_k} P_k y_0(x), \quad x \in \Omega. \]

Since $\omega = \Omega$ (see (4.24)), it follows from (1.3), (4.27), Proposition 3.1 (see the second equality in (3.1)) and (4.31) that
\[ y(T; y_0, \{\tau_k\}_{k=1}^n; \{\hat{u}_k\}_{k=1}^n) = e^{AT} y_0 + \sum_{k=1}^{n} e^{A(T-\tau_k)} B\hat{u}_k \]
\[ = e^{AT} \left( \sum_{k=1}^{n} P_k y_0 \right) + \sum_{k=1}^{n} e^{A(T-\tau_k)} B\hat{u}_k \]
\[ = \sum_{k=1}^{n} e^{A(T-\tau_k)} (e^{A\tau_k} P_k y_0 + B\hat{u}_k) \]
\[ = \sum_{k=1}^{n} e^{A(T-\tau_k)} (e^{\Delta \tau_k} e^{-A\tau_k} P_k y_0 + B\hat{u}_k) = 0. \]

Because $T > 0$ and $y_0 \in L^2(\Omega; \mathbb{R}^n)$ were arbitrarily taken, the above, along with (ii) of Definition 1.1, leads to the null controllability for the system (1.2).
Step 2. We show the necessity.
Assume that the system (1.2) is null controllable. We aim to prove that the pair \((A, B)\) satisfies Kalman’s controllability rank condition (1.8). By contradiction, we suppose that

\[
\text{rank } (B, AB, A^2B, \ldots, A^{n-1}B) < n.
\]

(4.32)

Then by Lemma 4.3, there would be \(\hat{z} \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\}\) so that

\[
\chi_\omega B^* e^{A^*t} \hat{z} = 0 \text{ for each } t > 0.
\]

(4.33)

Since the system (1.2) is null controllable, it follows by (ii) of Definition 1.1 that the system (1.2) is null controllable over \([0, T]\) for each \(T > 0\). Then by this and (i) of Definition 1.1, there is \(p \in \mathbb{N}^+\), \(\{\tau_k\}_{k=1}^p \subset (0, 1)\) and \(\{v_k\}_{k=1}^p \subset L^2(\Omega; \mathbb{R}^m)\) so that

\[
0 = y(1; e^{A^*\hat{z}}, \{\tau_k\}_{k=1}^p, \{v_k\}_{k=1}^p) = e^A(e^{A^*\hat{z}}) + y(1; 0, \{\tau_k\}_{k=1}^p, \{v_k\}_{k=1}^p).
\]

(4.34)

By (4.34), (4.1) and (4.33), we get that

\[
\langle e^A(e^{A^*\hat{z}}), \hat{z} \rangle = -\langle y(1; 0, \{\tau_k\}_{k=1}^p, \{v_k\}_{k=1}^p), \hat{z} \rangle = -\sum_{k=1}^p \langle v_k, \chi_\omega B^* e^{A^*(1-\tau_k)} \hat{z} \rangle = 0.
\]

This implies that \(e^{A^*\hat{z}} = 0\), which, combined with Proposition 3.1, shows that

\[
e^\Delta \hat{z} = e^{A^*} e^{-A^*} e^{\Delta} \hat{z} = e^{A^*} e^{A^*} \hat{z} = 0.
\]

From the above and (i) of Proposition 3.2, where

\[
\omega_1 = \Omega, \ V = \{0\}, \ \tau = 1 \text{ and } z = \hat{z},
\]

we find that \(\hat{z} = 0\). This leads to a contradiction, since \(\hat{z}\) is not zero. Hence, \((A, B)\) satisfies Kalman’s controllability rank condition (1.8). This proves the necessity.

In summary, we end the proof of Theorem 1.3.

\[
\square
\]

5 The proof of Theorem 1.4

The key to proving Theorem 1.4 is the following unique continuation property.
Theorem 5.1. Let $T > 0$ and $p \in \mathbb{N}^+$. Let $\{\tau_k\}_{k=1}^p \subset (0, T)$ be an increasing sequence. Then the following two statements are equivalent:

(i) It holds that
$$\text{rank } (e^{A\tau_1}B, \ldots, e^{A\tau_p}B) = n.$$ 

(ii) If $z \in L^2(\Omega; \mathbb{R}^n)$, then
$$\chi_\omega B^* e^{A^* (T-\tau_k)} z = 0 \text{ for all } k \in \{1, \ldots, p\} \implies z = 0 \text{ over } \Omega.$$  

Proof. The proof is divided into the following two steps.

Step 1. We show that (i) $\implies$ (ii).

Suppose that (i) is true. Let $z \in L^2(\Omega; \mathbb{R}^n)$ satisfy that
$$\chi_\omega B^* e^{A^* (T-\tau_k)} z = 0 \text{ for all } k \in \{1, \ldots, p\}. \quad (5.1)$$

It suffices to show that $z = 0$. For this purpose, we first claim that
$$\bigcap_{k=1}^p \ker (B^* e^{-A^* (T-\tau_k)}) = \{0\}. \quad (5.2)$$

Indeed, if $\alpha \in \bigcap_{k=1}^p \ker (B^* e^{-A^* (T-\tau_k)})$ (where $\alpha$ is a column vector in $\mathbb{R}^n$), then
$$B^* e^{-A^* (T-\tau_k)} \alpha = 0 \text{ for all } k \in \{1, \ldots, p\},$$
from which, it follows that
$$\alpha^T e^{-AT} (e^{A\tau_1}B, \ldots, e^{A\tau_p}B) = (\alpha^T e^{-A(T-\tau_1)} B, \ldots, \alpha^T e^{-A(T-\tau_p)} B) = \begin{pmatrix} B^* e^{-A^* (T-\tau_1)} \alpha \\ \vdots \\ B^* e^{-A^* (T-\tau_p)} \alpha \end{pmatrix} = 0.$$  

The above, along with (i), yields that $\alpha^T e^{-AT} = 0$, which leads to that $\alpha = 0$. Hence, (5.2) is true.

Next, by Proposition 3.1 and (5.1), we get that for each $k \in \{1, \ldots, p\}$,
$$B^* e^{-A^* (T-\tau_k)} \chi_\omega e^{\Delta (T-\tau_k)} z = \chi_\omega B^* e^{-A^* (T-\tau_k)} e^{\Delta (T-\tau_k)} z = \chi_\omega B^* e^{-A^* (T-\tau_k)} z = 0.$$ 

This indicates that for each $k \in \{1, \ldots, p\}$,
$$e^{\Delta (T-\tau_k)} z(x) \in \ker (B^* e^{-A^* (T-\tau_k)}) \text{ for a.e. } x \in \omega. \quad (5.3)$$
By (5.3), for each $k \in \{1, \ldots, p\}$, we can apply (i) of Proposition 3.2, where 

$$V = \ker \left( B^* e^{-A^*(T-\tau_k)} \right), \quad \omega_1 = \omega, \quad \tau = T - \tau_k,$$

to obtain that 

$$z(x) \in \bigcap_{k=1}^p \ker \left( B^* e^{-A^*(T-\tau_k)} \right) \text{ for a.e. } x \in \Omega.$$ 

The above, together with (5.2), leads to that $z = 0$. Hence, (ii) holds.

**Step 2. We show that (ii) $\implies$ (i).**

Suppose, by contradiction, that (ii) were true, but (i) did not hold. Then we would have that 

$$\text{rank } (e^{A\tau_1} B, \ldots, e^{A\tau_p} B) < n.$$ 

Thus, we can apply Lemma 4.3 to find $\hat{z} \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\}$ so that $\chi_\omega B^* e^{A^t} \hat{z} = 0$ for each $t > 0$. This, in particular, implies that 

$$\chi_\omega B^* e^{A^t(T-\tau_k)} \hat{z} = 0 \text{ for all } k = 1, \ldots, p. \quad (5.4)$$

From (5.4) and (ii), we see that $\hat{z} = 0$, which leads to a contradiction. Hence, (ii) implies (i).

In summary, we end the proof of Theorem 5.1.

---

**Proof of Theorem 1.4.** (i) We divide the proof into the following two steps:

**Step 1. We show the sufficiency.**

Assume that the pair $(A, B)$ satisfies Kalman’s controllability rank condition (1.8). We aim to show the approximate controllability for the system (1.2). To this end, we arbitrarily fix $T > 0$ and an increasing sequence $\{\hat{\tau}_k\}_{k=1}^n \subset (0, T)$ with $\hat{\tau}_n - \hat{\tau}_1 < d_A$ (given by (1.4)). Then by Theorem 2.1, we get that 

$$\text{rank } (e^{A\hat{\tau}_1} B, \ldots, e^{A\hat{\tau}_n} B) = n. \quad (5.5)$$

Define an operator $G_T : (L^2(\Omega; \mathbb{R}^m))^n \to L^2(\Omega; \mathbb{R}^n)$ in the following manner:

$$G_T(\{u_k\}_{k=1}^n) = y(T; 0, \{\hat{\tau}_k\}_{k=1}^n, \{u_k\}_{k=1}^n), \quad \forall \{u_k\}_{k=1}^n \in (L^2(\Omega; \mathbb{R}^m))^n. \quad (5.6)$$

We claim that the range of the map $G$ is dense in $L^2(\Omega; \mathbb{R}^n)$, i.e., 

$$\overline{\text{Range } G_T} = L^2(\Omega; \mathbb{R}^n). \quad (5.7)$$

By contradiction, we suppose that (5.7) were not true. Then there would be 

$$z \in \overline{\text{Range } G_T}^\perp \setminus \{0\}. \quad (5.8)$$

---
By (5.8), (5.6) and (4.1), we find that for all \( \{u_k\}_{k=1}^n \subset L^2(\Omega; \mathbb{R}^n) \),
\[
0 = \langle z, G_T(\{u_k\}_{k=1}^n) \rangle = \langle z, y(T; 0, \{\hat{\tau}_k\}_{k=1}^n, \{u_k\}_{k=1}^n) \rangle
= \sum_{k=1}^n \langle \chi_\omega B^* e^{At}(T-\hat{\tau}_k) z, u_k \rangle_{L^2(\Omega; \mathbb{R}^m)}.
\]
From the above, we see that
\[
\chi_\omega B^* e^{At}(T-\hat{\tau}_k) z = 0, \quad \forall k \in \{1, \ldots, n\},
\]
This, along with (5.5) and Theorem 5.1, shows that \( z = 0 \) over \( \Omega \), which contradicts (5.8). Hence (5.7) is true.

Now, we will use (5.7) to prove the approximate controllability for the system (1.2) over \([0, T]\). For this purpose, we arbitrary take \( y_0, y_1 \in L^2(\Omega; \mathbb{R}^n) \) and \( \varepsilon > 0 \). Then by (5.7), we see that
\[
y_1 - e^{AT} y_0 \in \overline{\text{Range } G_T}.
\]
Thus, there exists \( \{\hat{u}_k\}_{k=1}^n \subset L^2(\Omega; \mathbb{R}^m) \) so that
\[
\|G_T(\{\hat{u}_k\}_{k=1}^n) - (y_1 - e^{AT} y_0)\| \leq \varepsilon.
\]
It follows from (5.6) and (5.9) that
\[
\|y(T; y_0, \{\hat{\tau}_k\}_{k=1}^n, \{\hat{u}_k\}_{k=1}^n) - y_1\| = \|y(T; 0, \{\hat{\tau}_k\}_{k=1}^n, \{\hat{u}_k\}_{k=1}^n) - (y_1 - e^{AT} y_0)\|
= \|G_T(\{\hat{u}_k\}_{k=1}^n) - (y_1 - e^{AT} y_0)\| \leq \varepsilon.
\]
This leads to the approximate controllability for the system (1.2) over \([0, T]\) (see (i) of Definition 1.2). Since \( T > 0 \) was arbitrarily fixed, the approximate controllability for the system (1.2) over \([0, T]\) follows at once (see (ii) of Definition 1.2).

Step 2. We prove the necessity.
Assume that the system (1.2) has the approximate controllability. By contradiction, we suppose that the pair \((A, B)\) did not satisfy Kalman’s controllability rank condition (1.8). Then we would have that
\[
\text{rank } (B, AB, A^2 B, \ldots, A^{n-1} B) < n.
\]
By (5.10), we can use Lemma 4.3 to find \( \tilde{z} \in L^2(\Omega; \mathbb{R}^n) \setminus \{0\} \) so that
\[
\chi_\omega B^* e^{At} \tilde{z} = 0 \quad \text{for all } t > 0.
\]
Meanwhile, according to the approximate controllability of the system (1.2) (see (ii) of the Definition 1.2), there exists \( p \in \mathbb{N}^+ \) and an increasing sequence \( \{\tau_k\}_{k=1}^p \subset (0, 1) \) so that for each \( \varepsilon > 0 \), there is \( \{v_{k,\varepsilon}\}_{k=1}^p \subset L^2(\Omega; \mathbb{R}^m) \) so that
\[
\|y(1; 0, \{\tau_k\}_{k=1}^p, \{v_{k,\varepsilon}\}_{k=1}^p) - \tilde{z}\| \leq \varepsilon.
\]
By (4.1), (5.11), the Cauchy-Schwarz inequality and (5.12), we get that for each \( \varepsilon > 0 \),

\[
\langle \hat{z}, \hat{z} \rangle = \langle y(1; 0, \{ \tau_k \}_{k=1}^p, \{ v_{k, \varepsilon} \}_{k=1}^p), \hat{z} \rangle + \langle \hat{z} - y(1; 0, \{ \tau_k \}_{k=1}^p, \{ v_{k, \varepsilon} \}_{k=1}^p), \hat{z} \rangle
\]

\[
= \sum_{k=1}^p \langle v_{k, \varepsilon}, \chi_\omega B^* e^{A^*(T-\tau_k)} \hat{z} \rangle + \langle \hat{z} - y(1; y_0, \{ \tau_k \}_{k=1}^p, \{ v_{k, \varepsilon} \}_{k=1}^p), \hat{z} \rangle
\]

\[
\leq \| \hat{z} - y(1; y_0, \{ \tau_k \}_{k=1}^p, \{ v_{k, \varepsilon} \}_{k=1}^p) \| \| \hat{z} \| \leq \varepsilon \| \hat{z} \|.
\]

This implies that \( \hat{z} = 0 \), which leads to a contradiction, since \( \hat{z} \neq 0 \). Hence, \((A, B)\) satisfies Kalman’s controllability rank condition (1.8). This proves the necessity.

(ii) The conclusion (ii) has been proved in Step 1 of the proof of the conclusion (i).

In summary, we end the proof of Theorem 1.4

\[\square\]

The next Example 5.2 explains the rationality of the condition that \( \tau_n - \tau_1 < d_A \) in (ii) in Theorem 1.4. Here, \( d_A \) is given by (1.4) where \( C = A \).

Example 5.2. Let \((A, B) \in \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 1}\) be the pair given by (2.3)-(2.4). From Example 2.4, we see that \((A, B)\) satisfies Kalman’s controllability rank condition (1.8) and that \( d_A = \pi/|b| \). We will show that for any \( T > 0 \), any \( \tau_1, \tau_2 \in (0, T) \), with \( \tau_2 - \tau_1 = d_A \), the approximate controllability for the system (1.2) (governed by this pair \((A, B)\)) over \([0, T]\) cannot be realized at \( \{ \tau_k \}_{k=1}^2 \).

To this end, we arbitrarily fix \( T > 0 \), and then fix \( \tau_1, \tau_2 \in (0, T) \), with \( \tau_2 - \tau_1 = d_A \). From Example 2.4, we have that

\[
\text{rank} (e^{A\tau_1} B, e^{A\tau_2} B) < 2.
\]  

(5.13)

Define a subspace of \( V \subset \mathbb{R}^2 \) by

\[
V \triangleq \{ e^{A\tau_1} B \alpha_1 + e^{A\tau_2} B \alpha_2 : \alpha_1, \alpha_2 \in \mathbb{R}^1 \}.
\]  

(5.14)

By (5.13) and (5.14), there is \( \hat{\alpha} \in \mathbb{R}^2 \setminus \{0\} \) so that

\[
\hat{\alpha}^T \beta = 0, \quad \forall \beta \in V.
\]  

(5.15)

Define two functions \( y_0 \) and \( \hat{z}_T \) over \( \Omega \) in the following manner:

\[
y_0(x) \equiv e^{A^T} \hat{z}_T(x) \text{ for all } x \in \Omega \text{ and } \hat{z}_T(x) \equiv e^{A^T} \hat{\alpha} \text{ for all } x \in \Omega.
\]  

(5.16)

We claim that there exists \( \delta(T) > 0 \) so that for any \( \{u_k\}_{k=1}^2 \subset L^2(\Omega; \mathbb{R}^1) \),

\[
\| y(T; y_0, \{ \tau_k \}_{k=1}^2, \{ u_k \}_{k=1}^2) \| \geq \delta(T).
\]  

(5.17)
To this end, we arbitrarily fix \( \{u_k\}_{k=1}^2 \subset L^2(\Omega; \mathbb{R}^1) \). By (5.16) and the second equality in Proposition 3.1, we get that

\[
\langle \hat{z}_T, \sum_{k=1}^2 e^{A(T-\tau_k)} B\chi \omega u_k \rangle = \sum_{k=1}^2 \int_{\Omega} \langle e^{A^T} \hat{\alpha}, e^{A(T-\tau_k)} B\chi \omega u_k(x) \rangle_{\mathbb{R}^1} dx
\]

\[
= \sum_{k=1}^2 \int_{\Omega} \langle \hat{\alpha}, e^{A^T} e^{A(T-\tau_k)} B\chi \omega u_k(x) \rangle_{\mathbb{R}^1} dx. \tag{5.18}
\]

Meanwhile, by (5.14), we see that when \( k \in \{1, 2\} \),

\[
e^{-AT} e^{AT} B\chi \omega u_k(x) \in e^{-AT} (V) \text{ for a.e. } x \in \Omega.
\]

From the above, we can apply (i) of Proposition 3.2 (where \( \omega_1 = \Omega, \tau = T - \tau_k \) and \( z = e^{-AT} e^{AT} B\chi \omega u_k \)) to obtain that

\[
e^{A(T-\tau_k)} e^{-AT} e^{AT} B\chi \omega u_k(x) \in e^{-AT} (V) \text{ for a.e. } x \in \Omega. \tag{5.19}
\]

Now, by (5.19), (5.18) and (5.15), it follows that

\[
\langle \hat{z}_T, \sum_{k=1}^2 e^{A(T-\tau_k)} B\chi \omega u_k \rangle = 0. \tag{5.20}
\]

Then, from (1.3), (5.20), (5.16) and the second equality in Proposition 3.1, we see that

\[
\langle \hat{z}_T, y(T; y_0, \{\tau_k\}_{k=1}^2, \{u_k\}_{k=1}^2) \rangle = \langle \hat{z}_T, e^{AT} y_0 \rangle + \langle \hat{z}_T, \sum_{k=1}^2 e^{A(T-\tau_k)} B\chi \omega u_k \rangle
\]

\[
= \int_{\Omega} \langle \hat{z}_T(x), e^{AT} e^{-AT} y_0(x) \rangle_{\mathbb{R}^1} dx = \int_{\Omega} \langle \hat{z}_T(x), e^{AT} \hat{z}_T(x) \rangle_{\mathbb{R}^1} dx
\]

\[
= \|e^{\frac{T}{2}} \hat{z}_T\|^2. \tag{5.21}
\]

On the other hand, it follows by the Cauchy-Schwarz inequality that

\[
\langle \hat{z}_T, y(T; y_0, \{\tau_k\}_{k=1}^2, \{u_k\}_{k=1}^2) \rangle \leq \|\hat{z}_T\| \|y(T; y_0, \{\tau_k\}_{k=1}^2, \{u_k\}_{k=1}^2)\|.
\]

This, along with (5.21), yields that

\[
\|y(T; y_0, \{\tau_k\}_{k=1}^2, \{u_k\}_{k=1}^2)\| \geq \frac{\|e^{\frac{T}{2}} \hat{z}_T\|^2}{\|\hat{z}_T\|},
\]

which leads to (5.17), since \( \hat{z}_T \neq 0 \) only depends on \( T \) (see (5.16)).
Finally, from (5.17), we see that the approximate controllability for the system (1.2) (governed by this pair \((A,B)\)) over \([0,T]\) cannot be realized at any \(\{\tau_k\}_{k=1}^2 \subset (0,T)\) with \(\tau_2 - \tau_1 = d_A\).

Besides, this example also shows that for each \(T > 0\), the approximate controllability for the system (1.2) (governed by this pair \((A,B)\)) over \([0,T]\) cannot be realized at a single control instant \(\tau \in (0,T)\). Let us explain the reason. Let \(T > 0\). Since \(\text{rank}(e^{A\tau}B) < 2\) for all \(\tau \in (0,T)\) (see Example 2.4), it follows from Theorem 5.1 that there is \(\hat{z} \in L^2(\Omega; \mathbb{R}^2) \setminus \{0\}\) so that

\[
\chi_\omega B^* e^{A^*\tau} \hat{z} = 0 \quad \text{for all } \tau \in (0,T).
\]

(5.22)

On the other hand, one can easily check that the approximate controllability for the system (1.2) over \([0,T]\) can be realized at a single control instant \(\tau \in (0,T)\) if and only if

\[
z \in L^2(\Omega; \mathbb{R}^n) \quad \text{and} \quad \chi_\omega B^* e^{A^*(T-\tau)} z = 0 \implies z = 0.
\]

This, along with (5.22), yields that the approximate controllability for the system (1.2) (governed by this pair \((A,B)\)) over \([0,T]\) cannot be realized at a single control instant \(\tau \in (0,T)\).

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