Lower Bounds for the Minimum Mean-Square Error via Neural Network-based Estimation

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Abstract—The minimum mean-square error (MMSE) achievable by optimal estimation of a random variable \( Y \in \mathbb{R} \) given another random variable \( X \in \mathbb{R}^d \) is of much interest in a variety of statistical contexts. In this paper we propose two estimators for the MMSE, one based on a two-layer neural network and the other on a special three-layer neural network. We derive lower bounds for the MMSE based on the proposed estimators and the Barron constant of an appropriate function of the conditional expectation of \( Y \) given \( X \). Furthermore, we derive a general upper bound for the Barron constant that, when \( X \in \mathbb{R} \) is post-processed by the additive Gaussian mechanism, produces order optimal estimates in the large noise regime.

I. INTRODUCTION

The minimum mean-square error (MMSE) achievable by optimal estimation of a random variable given another one plays a key role in statistics and communications [2], [3], and it is closely related to fundamental information-theoretic concepts [4]. More recently, the MMSE has also found application in the context of data privacy, where it was proposed as an average measure of information leakage [5], [6].

There exist multiple MMSE estimation techniques, including those relying on linear [2], kernel-based [7] and polynomial [8], [9] approximations of the conditional expectation. Given the availability of software libraries to implement neural networks, there is a surge of neural network-based estimation methods for several information measures, see, e.g., [10]–[12]. We adhere to this paradigm and analyze neural-network-based estimators for the MMSE.

In this work, we propose two neural network-based estimators for the MMSE in estimating \( Y \in \mathbb{R} \) given \( X \in \mathbb{R}^d \). These estimators are the minimum empirical square loss attained by a two-layer neural network and a special three-layer neural network, respectively. For bounded \( X \) and \( Y \), we derive a lower bound for the MMSE based on the former estimator and the Barron constant of the conditional expectation of \( Y \) given \( X \). Furthermore, in the particular case of binary \( Y \), we derive another lower bound for the MMSE based on the latter estimator and the Barron constant of the log-likelihood ratio of the conditional densities of \( X \) given \( Y \). In order to make these bounds effective, we derive a general upper bound for the Barron constant that, when \( X \in \mathbb{R} \) is post-processed by the additive Gaussian mechanism, produces order optimal estimates in the large noise regime. Overall, we develop an effective machinery to obtain theoretical lower bounds for the MMSE.

In addition to the aforementioned works, the problem addressed in this paper is related to other fundamental problems in information theory and statistics. In a broad sense, our work belongs to the line of research dedicated to the estimation of information measures, see, e.g., [13]–[15] and references therein. In the special case when \( Y = X \), our setting is closely related to the problem of estimation in Gaussian channels as studied in [16], [17], [18]. Indeed, the problem considered in this work generalizes the aforementioned problem by considering finite samples and \( Y \neq X \). Our study of the additive Gaussian mechanism (Gaussian channel) is also inspired by the data privacy literature, where this post-processing technique appears frequently, see, e.g., [19], [20] and references therein. At a technical level, our starting point is Barron’s approximation theorem [21]. While there is a variety of works extending Barron’s result, see, e.g., [12], [22]–[24], most of them are asymptotic analyses in which the Barron constant is a fixed, yet unknown quantity. In contrast, in the present paper we show that this constant can be effectively controlled in the presence of the additive Gaussian mechanism. To the best of the authors’ knowledge, this is the first time that a quantitative analysis of the Barron constant is performed.

The rest of the paper is organized as follows. In the remainder of this section we recall some common notation used through this paper. In Section II we introduce our proposed estimators, recall Barron’s approximation theorem and present the additive Gaussian mechanism. We derive lower bounds for the MMSE upon the proposed estimators in Section III. In Section IV we derive a general bound for the Barron constant which, in Section V yields order optimal estimates in the presence of the additive Gaussian mechanism. We provide a summary and some final remarks in Section VI.

Notation. We let \((\Omega, \mathcal{F}, \mathbb{P})\) be the underlying probability space and \(\mathbb{E}\) be the corresponding expectation. We denote by \(1_E\) the indicator function of any set \(E \in \mathcal{F}\). We let \(\text{Unif}(\mathcal{U})\) be the uniform distribution over \(\mathcal{U}\). If \(f : \mathbb{R}^d \to \mathbb{R}\) is a probability density function, we let \(\text{Supp}(f)\) be its support. If \(p \in [0,1]\), we let \(\bar{p} = 1 - p\). For \(u,v \in \mathbb{R}^d\), we let \(u \cdot v = u_1v_1 + \cdots + u_dv_d\) and \(|u| = \sqrt{u \cdot u}\). Unless otherwise stated, we let \(\|\cdot\|_p\) be the \(p\)-norm in \(L^p(\mathbb{R})\) or \(L^p(\mathbb{R}^d)\), depending on the context. We say that a function \(f : \mathbb{R}^d \to \mathbb{R}\) is \(\rho\)-Lipschitz if \(\rho\) 

\( |f(u) - f(v)| \leq \rho |u - v| \) for all \( u, v \in \mathbb{R}^d \). Also, we say that \( f \) is of class \( C^m \) if it has continuous partial derivatives of order up to \( m \). We write \( f(z) \sim g(z) \) to denote that \( f(z)/g(z) \to 1 \) as \( z \to \infty \). We let \( \tanh : \mathbb{R} \to (-1, 1) \) be the hyperbolic tangent function. Finally, recall that the gamma function is the mapping determined by \( \Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx \) for \( z > 0 \).

**II. Problem Setting and Preliminaries**

In this section we review preliminary material on the minimum mean square error (MMSE), the function approximation capabilities of two-layer neural networks, and a data post-processing technique known as the additive Gaussian mechanism. Also, we introduce two neural network-based MMSE estimators that are used to derive theoretical lower bounds for the MMSE in Section III.

### A. Minimum Mean Square Error (MMSE)

Given random variables \( X \in \mathbb{R}^d \) and \( Y \in \mathbb{R} \), the minimum mean square error in estimating \( Y \) given \( X \) is defined as

\[
\text{mmse}(Y|X) := \inf_{h : \mathbb{R}^d \to \mathbb{R}} \mathbb{E} \left[ (Y - h(X))^2 \right], \tag{1}
\]

where the infimum is taken over all (Borel) measurable functions \( h : \mathbb{R}^d \to \mathbb{R} \). The infimum in (1) is attained by the conditional expectation of \( Y \) given \( X \), i.e.,

\[
\text{mmse}(Y|X) = \mathbb{E} \left[ (Y - \mathbb{E}[Y|X])^2 \right], \tag{2}
\]

where \( \eta(X) \triangleq \mathbb{E}[Y|X] \). Note that if \( Y \) is independent of \( X \), then \( \text{mmse}(Y|X) = 0 \). Also, note that if \( X \) and \( Y \) are independent, then the MMSE is maximal and \( \text{mmse}(Y|X) = \mathbb{E} \left[ (Y - \mathbb{E}[Y])^2 \right] \).

We would like to point out that, when \( Y \) is binary, the MMSE serves as a lower bound for the probability of error. Specifically, if \( Y \in \{\pm 1\} \), then

\[
P_{\text{error}}(Y|X) = \inf_{h : \mathbb{R}^d \to \{\pm 1\}} \mathbb{E} \left[ I_{Y \neq h(X)} \right] \tag{3}
\]

\[
= \inf_{h : \mathbb{R}^d \to \{\pm 1\}} \frac{\mathbb{E} \left[ (Y - h(X))^2 \right]}{4} \geq \frac{1}{4} \mathbb{E} \left[ (Y - h(X))^2 \right] \geq \frac{1}{4} \text{mmse}(Y|X). \tag{4}
\]

Thus, for binary \( Y \), any lower bound for \( \text{mmse}(Y|X) \) gives rise to a lower bound for \( P_{\text{error}}(Y|X) \). This is particularly relevant in the context of data privacy, where probability of correctly guessing \( (1 - P_{\text{error}}) \) has been used as a measure of information leakage [23, 26].

### B. Neural Network-based MMSE Estimation

A sigmoidal function \( \phi : \mathbb{R} \to [-1, 1] \) is a (measurable) function such that

\[
\lim_{z \to -\infty} \phi(z) = -1 \quad \text{and} \quad \lim_{z \to \infty} \phi(z) = 1. \tag{7}
\]

Observe that we are assuming that \( |\phi(z)| \leq 1 \) for all \( z \in \mathbb{R} \). Let \( \mathcal{H}_k^\phi \) be the hypothesis class associated with a two-layer neural network of size \( k \) with activation function \( \phi \). More specifically, \( \mathcal{H}_k^\phi \) is the set of all functions \( h : \mathbb{R}^d \to \mathbb{R} \) of the form

\[
h(x) = c_0 + \sum_{l=1}^k c_l \phi(a_l \cdot x + b_l), \tag{8}
\]

where \( a_l \in \mathbb{R}^d \) and \( b_l, c_l \in \mathbb{R} \). In this work we propose the following neural network-based estimator for the MMSE of \( Y \) given \( X \). Given a random sample \( \{(X_1, Y_1), \ldots, (X_n, Y_n)\} \), we define

\[
\text{mmse}_{k,n}(Y|X) := \inf_{h \in \mathcal{H}_k^\phi} \frac{1}{n} \sum_{i=1}^n (Y_i - h(X_i))^2, \tag{9}
\]

i.e., \( \text{mmse}_{k,n}(Y|X) \) is the minimum square loss attained by a two-layer neural network. Observe that, optimization matters aside, \( \text{mmse}_{k,n}(Y|X) \) can be obtained from the sample using a device capable of implementing a two-layer neural network of size \( k \). In this paper we take an information-theoretic perspective and assume infinite computational power. Specifically, we assume that \( \text{mmse}_{k,n}(Y|X) \) can be computed exactly [8].

Our goal is to establish a (probabilistic) bound of the form

\[
\text{mmse}_{k,n}(Y|X) - \epsilon_{k,n} \leq \text{mmse}(Y|X), \tag{10}
\]

where \( \epsilon_{k,n} \) is a positive number depending on the sample size \( n \) and the neural network size \( k \). In Section III we establish such a bound and, in addition, we derive an analogous result for a 3-layer neural network whose last layer is a single non-trainable node with \( \tanh \) activation function. Specifically, we replace \( \mathcal{H}_k^\phi \) by \( \tanh \circ \mathcal{H}_k^\phi \), i.e., the family of functions of the form \( \tanh \circ h \) with \( h \in \mathcal{H}_k^\phi \). In this case, the relevant MMSE estimator is the defined as

\[
\text{mmse}^*_k(Y|X) := \inf_{h \in \mathcal{H}_k^\phi} \frac{1}{n} \sum_{i=1}^n (Y_i - \tanh(h(X_i)))^2. \tag{11}
\]

Observe that, by definition, \( \text{mmse}(Y|X) \) is the minimum expected square loss attained by any measurable function. Hence, the bound in (10) differs from classical statistical learning results (e.g., Rademacher complexity bounds) for which the expected loss is minimized over the hypothesis class \( \mathcal{H}_k^\phi \). In particular, we have to consider the so-called approximation error, which could be estimated via the function approximation theorem of Barron [21].

### C. Barron’s Theorem

Let \( B \subset \mathbb{R}^d \) be a bounded set such that \( 0 \in B \). We define \( \Gamma_B \) as the set of all functions \( h : B \to \mathbb{R} \) admitting an integral representation of the form

\[
h(x) = h(0) + \int_{\mathbb{R}^d} (e^{i\omega \cdot x} - 1) \hat{H}(d\omega), \tag{12}
\]

2From an applied perspective, this assumption is not trivial to guarantee. Indeed, it is known that training neural networks to optimality is a computationally difficult problem, see, e.g., [22, 23].
for some complex-valued measure $\hat{H}$ such that $\int |\omega||\hat{H}|(d\omega)$ is finite. Observe that, as pointed out by Barron \cite[Sec. III]{barron1991}, the right hand side of (12) defines an extension of $h$ to $\mathbb{R}^d$. However, it is important to remark that such an extension might not be unique as there might be multiple complex-valued measures $\hat{H}$ satisfying (12).

Given $h \in \Gamma_B$, its Barron constant $C_h$ is defined as

\[
C_h := \inf_{\hat{H}} \int_{\mathbb{R}^d} |\omega||\hat{H}|(d\omega),
\]

where the infimum is over all complex-valued measures $\hat{H}$ satisfying (12) and

\[
|\omega|_B := \sup_{x \in B} |\omega \cdot x|.
\]

To the best of the authors’ knowledge, there is no known method to compute $C_h$ given an arbitrary $h \in \Gamma_B$. However, in practice, we can take any complex-valued measure $\hat{H}$ satisfying (12) and use it to evaluate the bound

\[
C_h \leq \text{rad}(B) \int_{\mathbb{R}^d} |\omega||\hat{H}|(d\omega),
\]

where $\text{rad}(B) := \sup_{x \in B} |x|$.

Under mild assumptions, the Barron constant could be related to the Fourier transform. Recall that, for a function $h \in L^1(\mathbb{R}^d)$, its Fourier transform $\hat{h} : \mathbb{R}^d \to \mathbb{C}$ is defined as

\[
\hat{h}(\omega) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)e^{-i\omega \cdot x} dx.
\]

If, in addition, $\hat{h} \in L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} |\hat{h}(\omega)| d\omega$ is finite, then the Fourier inversion theorem implies that $\hat{h}|_B \in \Gamma_B$ and

\[
C_h \leq \frac{\text{rad}(B)}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |\omega||\hat{h}(\omega)|(d\omega).
\]

The following proposition establishes, in a quantitative manner, the universal approximating capabilities of two-layer neural networks. Observe that the statement below is a translation of Barron’s original formulation \cite[Theorem 1]{barron1991} to the case of sigmoidal functions as defined in Section IB.

**Proposition 1 (Theorem 1, \cite{barron1991}).** Let $B \subset \mathbb{R}^d$ be a bounded set containing 0. For every $h \in \Gamma_B$ and every probability distribution $P$ over $B$, there exists $h_k \in \mathcal{H}_k^B$ such that

\[
\int_B |h_k(x) - h(x)|^2 P(dx) \leq \frac{(2C_h)^2}{k}.
\]

Furthermore, the coefficients of $h_k$ may be restricted to satisfy $|c_0| \leq |h(0)| + C_h$ and $\sum_{i=1}^k |c_i| \leq C_h$.

**D. Additive Gaussian Mechanism**

To motivate the forthcoming applications, consider the following. Assume that $X \in \mathbb{R}^d$ and $Y \in \{\pm 1\}$ represent some features of an individual, e.g., $X$ could be salary and $Y$ gender. Due to privacy concerns, a data analyst might not be able to observe $X$ but a sanitized version of it. In this case, the MMSE

\[
X_i^\sigma := X_i + \sigma Z_i,
\]

where $Z_1, \ldots, Z_n$ are i.i.d. standard Gaussian vectors. Since the Gaussian distribution has unbounded support, it is often convenient to further process extreme values of the random variables $X_i^\sigma$. We consider two processing techniques.

1) **Extreme Values Truncation:** Let $B \subset \mathbb{R}^d$ be a bounded set. Extreme values truncation is the data processing technique that discards all samples with $X_i^\sigma$ outside the set $B$. Let $f_x^\sigma$ be the conditional density of $X^\sigma$ after truncation given $Y = \pm 1$.

It is straightforward to verify that

\[
f_x^\sigma(x) = \frac{(f_{\pm} \ast K_\sigma)(x)}{\mathbb{P}(X^\sigma \in B|Y = \pm 1)},
\]

where $f_{\pm}$ is conditional density of $X$ given $Y = \pm 1$, $\ast$ is the convolution operator, and $K_\sigma$ is the density of $\sigma Z$.

2) **Extreme Values Randomization:** Let $B \subset \mathbb{R}^d$ be a bounded set. Extreme values randomization is the data processing technique that replaces each $X_i^\sigma$ outside the set $B$ and replaces it with a random value on $B$. As before, let $f_x^\sigma$ be the conditional density of $X^\sigma$ after randomization given $Y = \pm 1$.

It is straightforward to verify that

\[
f_x^\sigma(x) = \left[ (f_{\pm} \ast K_\sigma)(x) + \frac{\mathbb{P}(X^\sigma \notin B|Y = \pm 1)}{\text{vol}(B)} \right] \mathbb{1}_{x \in B},
\]

where $\text{vol}(B)$ denotes the volume of $B$ w.r.t. the Lebesgue measure on $\mathbb{R}^d$.

Under mild assumptions on $f_{\pm}$, e.g., bounded and compactly supported, both (20) and (21) define non-negative smooth functions on the interior of $B$. If $B$ is closed, a routine application of Whitney’s extension theorem \cite{whitney1935} shows that, for any smooth function $h : \mathbb{R} \to \mathbb{R}$, the function $h \circ (f_x^\sigma / f_x^\sigma)$ can be extended to a rapidly-decreasing smooth function over $\mathbb{R}^d$ \cite[Ch. 7]{singer2011} and, in particular, $h \circ (f_x^\sigma / f_x^\sigma)$ belongs to $\Gamma_B$.

### III. MMSE LOWER BOUNDS

In this section we provide lower bounds for $\text{mmse}(Y|X)$ based on $\text{mmse}_{k,n}(Y|X)$ and $\text{mmse}_{k,n}(Y|X)$, as envisioned in (10).

**A. 2-Layer Neural Networks**

The following theorem establishes a lower bound for the MMSE in estimating $Y$ given $X$ based on the estimator $\text{mmse}_{k,n}$, as defined in (9), and the Barron constant of the conditional expectation of $Y$ given $X$.

**Theorem 1.** Let $k, n \in \mathbb{N}$ and $B \subset \mathbb{R}^d$ be a bounded set containing 0. If $Y \in [-1, 1]$, $X$ is supported on $B$, and the

\[3\text{Observe that a sigmoidal function } \phi : \mathbb{R} \to [0, 1] \text{ in the sense of Barron \cite[Sec. I]{barron1991} can be converted into a sigmoidal function in the sense of Section II-B by means of the transformation } \phi \mapsto 2(\phi - 1/2).\]

\[4\text{Observe that this technique potentially reduces the sample size, although, the reduction is negligible when } B \text{ is large. In any case, for ease of notation, we let } n \text{ denote the effective sample size after truncation.}\]

\[5\text{In fact, } f_x^\sigma \text{ are bounded away from } 0 \text{ on } B.\]
conditional expectation $\eta(x) := \mathbb{E}[Y|X = x]$ belongs to $\Gamma_B$, then, with probability at least $1 - \delta$,  
\[ \text{mmse}_{k,n}(Y|X) - \epsilon_{k,n,\delta} \leq \text{mmse}(Y|X), \]  
(22)  
where  
\[ \epsilon_{k,n,\delta} = 2\left(1 + C_n^2\right)\sqrt{\frac{2\log(1/\delta)}{n}} + \frac{4C_n^2}{k} + 8C_n^2\frac{\sqrt{k}}{\sqrt{k}}. \]  
(23)  

Proof. For ease of notation, we define  
\[ \Delta := \text{mmse}_{k,n}(Y|X) - \text{mmse}(Y|X). \]  
(24)  
Also, we define $L(h) := \mathbb{E}[(Y - h(X))^2]$ and  
\[ \hat{L}_n(h) := \frac{1}{n} \sum_{i=1}^n (Y_i - h(X_i))^2. \]  
(25)  
Recall that the infimum defining $\text{mmse}(Y|X)$ is attained by the conditional expectation $\eta$, see (2). Thus, we have that  
\[ \Delta = \inf_{h \in \mathcal{H}_k^0} \hat{L}_n(h) - \inf_{h \text{ meas.}} L(h), \]  
(26)  
\[ = \inf_{h \in \mathcal{H}_k^0} \hat{L}_n(h) - L(\eta). \]  
(27)  
Since $\eta \in \Gamma_B$ by assumption, Barron’s theorem (Proposition 1) implies that there exists $\eta_k \in \mathcal{H}_k^0$ such that  
\[ \|\eta_k - \eta\|_2 \leq 2C_n^2 \frac{\sqrt{k}}{\sqrt{k}}, \]  
(28)  
where $\|\|_2$ is the 2-norm w.r.t. the distribution of $X$, i.e.,  
\[ \|h\|_2^2 = \int_B |h(x)|^2 P_X(dx). \]  
(29)  
Furthermore, if we let  
\[ \eta_k(x) = c_0 + \sum_{i=1}^k c_i \phi(a_i \cdot x + b_i), \]  
(30)  
the coefficients $c_0, c_1, \ldots, c_k$ can be restricted to satisfy that  
\[ c_0 \leq |\eta(0)| + C_n \]  
and $\sum_{i=1}^k |c_i| \leq C_n$. Observe that, by (27),  
\[ -\Delta \leq \hat{L}_n(\eta_k) - L(\eta_k) - L(\eta). \]  
(31)  
Since $Y \in [-1, 1]$, we have that $|\eta(x)| \leq 1$ for all $x \in B$. Recall that, by assumption, $\phi(z) \in [-1, 1]$ for all $z \in \mathbb{R}$. Thus, by our choice of the coefficients $c_0, c_1, \ldots, c_k$ in (30),  
\[ |\eta_k(x)| \leq 1 + 2C_n, \quad x \in \mathbb{R}^d. \]  
(32)  
Therefore, $(Y_i - \eta_k(X_i))^2 \leq 4(1 + C_n)^2$ for all $i \in \{1, \ldots, n\}$. As a result, a routine application of Hoeffding’s inequality (31) implies that, with probability at least $1 - \delta$,  
\[ \hat{L}_n(\eta_k) - L(\eta_k) \leq 2(1 + C_n^2)\sqrt{\frac{2\log(1/\delta)}{n}}. \]  
(33)  
It is straightforward to verify that  
\[ L(\eta_k) - L(\eta) = \|\eta_k - \eta\|_2^2 + 2\mathbb{E}[(\eta(X) - \eta_k(X))(Y - \eta(X))]. \]  
(34)  
Recall that $|\eta(x)| \leq 1$ for all $x \in B$. Therefore, an application of the Cauchy–Schwarz inequality implies that  
\[ |L(\eta_k) - L(\eta)| \leq \|\eta_k - \eta\|_2(4 + \|\eta_k - \eta\|_2). \]  
(35)  
By plugging (28) in (33), we conclude that  
\[ |L(\eta_k) - L(\eta)| \leq 2C_n^2 \frac{\sqrt{k}}{\sqrt{k}} + 4C_n^2 + \frac{8C_n^2\sqrt{k}}{\sqrt{k}}. \]  
(36)  
The theorem follows by plugging (31) and (36) in (31). \square

Regarding the assumptions of the previous theorem, it is important to remark that, in general, it might be non-trivial to verify that the conditional expectation $\eta$ belongs to $\Gamma_B$. Nonetheless, as shown in Section V, this assumption is automatically satisfied when data is post-processed by the additive Gaussian mechanism introduced in Section II.D.

Note that $\epsilon_{k,n,\delta}$, as defined in (23), is non-increasing in $k$. Since $\mathcal{H}_k^0 \subseteq \mathcal{H}_{k+1}^0$ for all $k \in \mathbb{N}$, $\text{mmse}_{k,n}(Y|X)$ is also non-increasing in $k$. As a result, the lower bound in (22) does not necessarily improve by making $k$ larger. Indeed, it is known that if $k$ is large enough then $\text{mmse}_{k,n}(Y|X)$ is equal to 0, see, e.g., (32), which makes the lower bound in (22) trivial. Together with the fact that the minimization defining $\text{mmse}_{k,n}(Y|X)$ becomes harder as $k$ increases, the previous observations reveal the non-trivial nature of finding the value of $k$ that produces the best numerical results.

### B. A Special 3-Layer Neural Network

The following theorem establishes a lower bound for the MMSE in estimating $Y$ given $X$ based on the estimator $\text{mmse}_{k,n}^*$, as defined in (11), and the Barron constant of the log-likelihood ratio defined in (37) below. Observe that, unlike Theorem 1 the following theorem requires $Y$ to be binary.

**Theorem 2.** Let $k, n \in \mathbb{N}$ and $B \subset \mathbb{R}^d$ be a bounded set containing 0. Assume that $Y \in \{-1, 1\}$, $X$ is supported on $B$, and the conditional density of $X$ given $Y = \pm 1$, denoted by $f_{\pm}$, is positive on $B$. Let $p := \mathbb{P}(Y = 1) = \theta(x) := \frac{1}{2} \log \left( \frac{pf_+(x)}{pf_-(x)} \right), \quad x \in B. \)  
(37)  
If $\theta$ belongs to $\Gamma_B$, then, with probability at least $1 - \delta$,  
\[ \text{mmse}_{k,n}^*(Y|X) - \epsilon_{k,n,\delta}^* \leq \text{mmse}(Y|X), \]  
(38)  
where  
\[ \epsilon_{k,n,\delta}^* = 2\sqrt{\frac{2\log(1/\delta)}{n}} + 4C_k^2 \frac{\sqrt{k}}{\sqrt{k}} + 8C_n^2 \frac{\sqrt{k}}{\sqrt{k}}. \]  
(39)  

Proof. For ease of notation, we define  
\[ \Delta^* := \text{mmse}_{k,n}^*(Y|X) - \text{mmse}(Y|X). \]  
(40)  
Also, we define $L^*(h) := \mathbb{E}[(Y - \tanh(h(X)))^2]$ and  
\[ \hat{L}_n^*(h) := \frac{1}{n} \sum_{i=1}^n (Y_i - \tanh(h(X_i)))^2. \]  
(41)  
Recall that the infimum defining $\text{mmse}(Y|X)$ is attained by the conditional expectation $\eta$, see (2). A straightforward computation shows that, for every $x \in B$,  
\[ \eta(x) = \frac{pf_+(x) - pf_-(x)}{pf_+(x) + pf_-(x)} \]  
(42)  
\[ = \operatorname{tanh} \left( \frac{1}{2} \log \left( \frac{pf_+(x)}{pf_-(x)} \right) \right) \]  
(43)  
\[ = \operatorname{tanh}(\theta(x)). \]  
(44)
Therefore, we have that

\[
\text{mmse}(Y|X) = \mathbb{E}[(Y - \eta(X))^2] = \mathbb{E}[(Y - \tanh(\theta(X)))^2] = L^*(\theta),
\]

and, as a result,

\[
\Delta^* = \inf_{h \in \mathcal{H}_k^0} \hat{L}^*_n(h) - L^*(\theta).
\]

Since \( \theta \in \Gamma_B \) by assumption, Barron’s theorem (Proposition \[1\]) implies that there exists \( \theta_k \in \mathcal{H}_k^0 \) such that

\[
\|\theta_k - \theta\|_2 \leq \frac{2C_\theta}{\sqrt{k}},
\]

where \( \| \cdot \|_2 \) is the 2-norm w.r.t. the distribution of \( X \). Furthermore, if we let

\[
\theta_k(x) = c_0 + \sum_{i=1}^k c_i \phi(a_i \cdot x + b_i),
\]

the coefficients \( c_0, c_1, \ldots, c_k \) can be restricted to satisfy that \( c_0 \leq |\theta(0)| + C_\theta \) and \( \sum_{i=1}^k |c_i| \leq C_\theta \). Observe that, by (48),

\[
\Delta \leq \hat{L}^*_n(\theta_k) - L^*(\theta_k) + L^*(\theta_k) - L^*(\theta).
\]

Since \( Y \in \{\pm 1\} \) and \( \tanh : \mathbb{R} \to (-1, 1) \), it is immediate to verify that \( (Y_i - \tanh(\theta_k(X_i)))^2 \leq 4 \) for all \( i \in \{1, \ldots, n\} \). As a result, a routine application of Hoeffding’s inequality [3] Sec. 4.2] implies that, with probability at least \( 1 - \delta \),

\[
\hat{L}^*_n(\theta_k) - L^*(\theta_k) \leq 2\sqrt{\frac{2\log(1/\delta)}{n}}.
\]

It is straightforward to verify that

\[
L^*(\theta_k) - L^*(\theta) = \|\tanh \circ \theta_k - \tanh \circ \theta\|_2^2 + 2\mathbb{E}[(\tanh(\theta(X)) - \tanh(\theta_k(X))) \times (Y - \tanh(\theta(X))].
\]

Thus, the Cauchy–Schwarz inequality leads to

\[
|L^*(\theta_k) - L^*(\theta)| \leq \|\tanh \circ \theta_k - \tanh \circ \theta\|_2 \times (4 + \|\tanh \circ \theta_k - \tanh \circ \theta\|_2).
\]

Since \( \tanh \) is a 1-Lipschitz function, it can be verified that \( \|\tanh \circ \theta_k - \tanh \circ \theta\|_2 \leq \|\theta_k - \theta\|_2 \). Therefore, we have that

\[
|L^*(\theta_k) - L^*(\theta)| \leq \|\theta_k - \theta\|_2 (4 + \|\theta_k - \theta\|_2).
\]

By plugging (49) in (53), we conclude that

\[
|L^*(\theta_k) - L^*(\theta)| \leq \frac{2C_\theta}{\sqrt{k}} \left(4 + \frac{2C_\theta}{\sqrt{k}}\right).
\]

The theorem follows by plugging (52) and (56) in (51). \( \square \)

As with Theorem \[1\], the hypotheses of the previous theorem are automatically satisfied when data is post-processed by the additive Gaussian mechanism introduced in Section \[II-D\] We discuss this claim in detail in Section \[V\].

Observe that, as established in (44), the conditional expectation \( \eta \) and the log-likelihood ratio \( \theta \) satisfy that \( \eta = \tanh \circ \theta \). In view of this relation, the choice of the 3-layer neural network defining \( \text{mmse}_{k,n}(Y|X) \) becomes evident: the non-trainable node in the third layer implements \( \tanh \) while the second layer approximates \( \theta \).

IV. A General Bound for the Barron Constant

Theorem \[I\] establishes a lower bound for \( \text{mmse}(Y|X) \) based on the estimator \( \text{mmse}_{k,n}(Y|X) \) and the Barron constant \( C_\eta \) of the conditional expectation of \( Y \) given \( X \). While \( \text{mmse}_{k,n}(Y|X) \) can be computed from the sample, providing estimates for the Barron constant \( C_\eta \) might be challenging for two reasons: (i) the conditional expectation of \( Y \) given \( X \) depends on the distribution of \( X \) and \( Y \), which is typically unavailable in practice, and (ii) the Barron constant \( C_\eta \) is defined in terms of the Fourier transform of \( \eta \), which makes its computation unfeasible in most cases. (A similar remark applies, \textit{mutatis mutandis}, to Theorem \[2\].) In this section we provide some results that alleviate the second issue; the discussion of the first issue is left for the following section.

A. 1-Dimensional Bound

In this section we focus on a special family of real-valued functions of a real variable whose Barron’s constant admits a relatively tractable representation.

Let \( h : \mathbb{R} \to \mathbb{R} \) be a differentiable function. If \( h' \in L^1(\mathbb{R}) \) and \( \hat{h}' \in L^1(\mathbb{R}) \), the Fourier inversion theorem implies that

\[
h'(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{h}'(\omega)e^{i\omega x}d\omega.
\]

As pointed out by Barron [21, Appendix], (67) implies that \( h|_B \) belongs to \( \Gamma_B \) for every bounded set \( B \) containing \( 0 \) and

\[
C_{h|_B} \leq \frac{\text{rad}(B)}{\sqrt{2\pi}} \int_{\mathbb{R}} |\hat{h}'(\omega)|d\omega.
\]

Thus, by abuse of notation, we define

\[
C_h := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\hat{h}'(\omega)|d\omega,
\]

whenever \( h : \mathbb{R} \to \mathbb{R} \) satisfies that \( h', \hat{h}' \in L^1(\mathbb{R}) \).

Theorem 3. Let \( h : \mathbb{R} \to \mathbb{R} \) be a thrice differentiable function. If \( h', \hat{h}' \in L^1(\mathbb{R}) \) and vanish at infinity, then

\[
C_h \leq \frac{2\sqrt{2}}{\sqrt{\pi}} \left(1 + \log \left(\frac{\sqrt{|h' \hat{h}'|}}{|h''|_1}\right)\right) \|h''|_1.
\]

Proof. Let \( 0 < \lambda_1 < \lambda_2 \). We split the integral in (59) as

\[
I := \int_{-\lambda_1}^{\lambda_1} |\hat{h}'(\omega)|d\omega,

II := \left( \int_{-\lambda_2}^{\lambda_2} + \int_{-\lambda_1}^{-\lambda_2} \right) |\hat{h}'(\omega)|d\omega,

III := \left( \int_{\lambda_2}^{\infty} + \int_{-\infty}^{-\lambda_2} \right) |\hat{h}'(\omega)|d\omega.
\]

First, observe that

\[
I \leq 2\|\hat{h}'\|_\infty \lambda_1 \leq 2\|h'\|_1 \lambda_1,
\]

(64)
where we applied the inequality $\|\hat{h}'\|_\infty \leq \|h'\|_1$. Since $h'$ vanishes at infinity and $h'' \in L^1(\mathbb{R})$, $\hat{h}'(\omega) = i\omega \hat{h}'(\omega)$ for every $\omega \in \mathbb{R}$. Thus, we have that

$$II = \left( \int_{\lambda_1}^{\lambda_2} + \int_{-\infty}^{-\lambda_2} \right) \frac{1}{\omega} |\hat{h}'(\omega)|d\omega \leq 2 \|\hat{h}'\|_\infty \|\log(\lambda_2/\lambda_1)\|_1.$$ 

Similarly, $\hat{h}''(\omega) = (i\omega)^2 \hat{h}'(\omega)$ for every $\omega \in \mathbb{R}$ and

$$III = \left( \int_{\lambda_2}^{\infty} + \int_{-\infty}^{-\lambda_2} \right) \frac{1}{\omega^2} |\hat{h}'(\omega)|d\omega \leq 2 \|\hat{h}'\|_1 \|\log(\lambda_2/\lambda_1)\|_1.$$ 

By plugging (64), (67) and (68) in (59), we conclude that

$$C_h \leq \sqrt{\frac{2}{\pi}} \left( \|h'\|_1 \lambda_1 + \|h''\|_1 \log(\lambda_2/\lambda_1) + \|h'''\|_1 \lambda_2 \right).$$ 

By taking $\lambda_1 = \|h''\|_1 / \|h'''\|_1$ and $\lambda_2 = \|h'''\|_1 / \|h''\|_1$, the result follows.

Observe that if we let $\lambda_1 = \lambda_2 = \sqrt{\|h'''\|_1/\|h''\|_1}$ in (69), we obtain

$$C_h \leq \frac{2\sqrt{2}}{\sqrt{\pi}} \sqrt{\|h'\|_1 \|h'''\|_1}.$$ 

This bound is generalized for functions $h : \mathbb{R}^d \to \mathbb{R}$ in Theorem 4 below. Observe that while the bound in (70) is simpler than the one provided in Theorem 3, it is typically weaker in applications. Since Theorem 3 will be applied to the conditional expectation $\eta$ and the log-likelihood ratio $\theta$, we need to compute the derivatives of these functions. The following lemma provides useful expressions for the first three derivatives of $\eta$ in the case when $Y$ is binary. Recall that if $f_\pm : B \to \mathbb{R}$ is the conditional density of $X$ given $Y = \pm 1$ and $p = \mathbb{P}(Y = 1)$, then the conditional expectation of $Y$ given $X$ is given by

$$\eta(x) = \frac{pf_+(x) - pf_-(x)}{pf_+(x) + pf_-(x)}.$$ 

**Lemma 1.** If $\eta$ is defined as in (71), then

$$\eta' = g_+ g_- - g_+ g_- \frac{g_+ - g_-}{(g_+ + g_-)^2},$$

$$\eta'' = 2g_+ g_- g_- + g_+ g_- g_- - 2g_+ g_+ g_- + g_+ g_- g_- \frac{g_+ + g_-}{(g_+ + g_-)^2},$$

$$\eta''' = 2g_+ g_- g_- + g_+ g_- g_- - 4g_+ g_+ g_- + g_+ g_- g_+ g_- - 3g_+ g_- g_+ g_- \frac{g_+ + g_-}{(g_+ + g_-)^2}.$$ 

where $g_+ = pf_+$ and $g_- = pf_-$.  

**Proof.** Equation (72) follows easily from the (71). By the quotient rule $\left(\frac{h_1}{h_2}\right)' = \frac{h_1'h_2 - h_1h_2'}{h_2^2}$, (73) follows from (72).  

Using similar arguments, (74) follows from (73).  

Similarly, the following lemma provides useful expressions for the first three derivatives of the log-likelihood ratio

$$\theta(x) = \frac{1}{2} \log \left( \frac{p f_+(x)}{p f_-(x)} \right).$$ 

**Lemma 2.** If $\theta$ is defined as in (75), then

$$2\theta' = \frac{g_+}{g_+ - g_-},$$

$$2\theta'' = \left[ \frac{g_+ g_-}{g_+ + g_-} - \left( \frac{g_+}{g_+ + g_-} \right)^2 \right] - \left[ \frac{g_+ g_-}{g_+ - g_-} - \left( \frac{g_+}{g_+ - g_-} \right)^2 \right],$$

$$2\theta''' = \left[ \frac{g_+ g_-}{g_+ + g_-} - 3 \left( \frac{g_+ g_-}{g_+ - g_-} \right)^2 \right] - \left[ \frac{g_+ g_-}{g_+ - g_-} - 3 \left( \frac{g_+ g_-}{g_+ - g_-} \right)^2 \right],$$

where $g_+ = pf_+$ and $g_- = pf_-$.  

**Proof.** The identities in (76) – (78) follow from the quotient rule and the logarithmic derivative $(\log h)' = h'/h$.  

**B. $d$-Dimensional Extension**

In this section we focus on a special family of real-valued functions of $d$-real variables whose Barron’s constant admits a relatively tractable representation. Let $h : \mathbb{R}^d \to \mathbb{R}$ be a differentiable function. For ease of notation, we let $h_{x_j} := \frac{\partial}{\partial x_j} h$ and $h_{x_j} \in L^1(\mathbb{R}^d)$ for every $j \in \{1, \ldots, d\}$, the Fourier inversion theorem implies that

$$\nabla h(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \nabla \hat{h}(\omega)e^{i\omega \cdot x}d\omega,$$ 

where $\nabla h = (h_{x_1}, \ldots, h_{x_d})$ and $\nabla \hat{h} = (\hat{h}_{x_1}, \ldots, \hat{h}_{x_d})$. As pointed out by Barron (21 Appendix), (79) implies that $h|_B$ belongs to $\Gamma_B$ for every bounded set $B$ containing 0 and

$$C_h|_B \leq \frac{\text{rad}(B)}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |\nabla \hat{h}(\omega)|d\omega.$$ 

Thus, by abuse of notation, we define

$$C_h := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |\nabla \hat{h}(\omega)|d\omega,$$ 

whenever $h : \mathbb{R}^d \to \mathbb{R}$ satisfies that $h_{x_j}, \hat{h}_{x_j} \in L^1(\mathbb{R}^d)$ for every $j \in \{1, \ldots, d\}$.

**Theorem 4.** Let $h : \mathbb{R}^d \to \mathbb{R}$ be a function of class $C^{d+2}$. If the partial derivatives of $h$ of order up to $d+2$ belong to $L^1(\mathbb{R}^d)$ and vanish at infinity, then

$$C_h \leq A_d N_1^{1/(d+1)} N_2^{d/(d+1)},$$ 

where $A_d = \frac{d+1}{d^{d/2} \Gamma(2d/2+1)}$, $N_1^2 = \sum_{j=1}^d \|h_{x_j}\|_1^2$ and

$$N_2^2 = \sum_{j=1}^d \left( \sum_{j'=1}^d \left\| \frac{\partial^{d+1}}{\partial x_j^{d+1}} h_{x_j} \right\|_1^2 \right).$$
\textbf{Proof.} For \( \lambda > 0 \), let \( B_\lambda \) be the \( d \)-dimensional ball of radius \( \lambda \). We split the integral in (81) as
\[
I := \int_{B_\lambda} |\nabla h(\omega)| d\omega \quad \text{and} \quad II := \int_{B_\lambda^c} |\nabla h(\omega)| d\omega.
\]
Observe that, for every \( \omega \in \mathbb{R}^d \),
\[
|\nabla h(\omega)|^2 = \sum_{j=1}^d |\tilde{h}_{x_j}(\omega)|^2 \leq \sum_{j=1}^d \|h_{x_j}\|^2 =: N^2_1,
\]
where we applied the inequality \( \|h_{x_j}\|_\infty \leq \|h_{x_j}\|_1 \). Therefore, (88) and (89) imply that
\[
I \leq N_1 \text{vol}(B_\lambda) = \frac{\pi^{d/2} N_1 \lambda^d}{\Gamma(d/2 + 1)},
\]
as \( \text{vol}(B_1) = \pi^{d/2}/\Gamma(d/2 + 1) \).

For \( \omega \in \mathbb{R}^d \), the generalized mean inequality asserts that
\[
\sqrt[d]{\frac{|\omega|^2 + \cdots + |\omega_d|^2}{d}} \leq \frac{d}{d} \sqrt[d]{\sum_{j=1}^d |\omega_j||d+1|^2},
\]
which in turn leads to
\[
|\omega|_{d+1} \leq d(1-d)/2 \sum_{j=1}^d |\omega_j|_{d+1}.
\]
For ease of notation, we define \( \partial_{x_j}^{d+1} := \partial_{\omega_j}^{d+1} \). Since the partial derivatives of \( h \) of order up to \( d+2 \) belong to \( L^1(\mathbb{R}^d) \) and vanish at infinity, then, for every \( \omega \in \mathbb{R}^d \),
\[
\partial_{x_j}^{d+1} \tilde{h}_{x_j}(\omega) = (i \omega_j)^{d+1} \tilde{h}_{x_j}(\omega).
\]
Therefore, (88) and (89) imply that
\[
|\omega|_{d+1} \tilde{h}_{x_j}(\omega)| \leq d(1-d)/2 \sum_{j=1}^d |\omega_j|_{d+1} \tilde{h}_{x_j}(\omega)|
\]
\[
= d(1-d)/2 \sum_{j=1}^d |\partial_{x_j}^{d+1} \tilde{h}_{x_j}(\omega)|
\]
\[
\leq d(1-d)/2 \sum_{j=1}^d \|\partial_{x_j}^{d+1} \tilde{h}_{x_j}\|_1,
\]
where we applied the inequality \( \|\partial_{x_j}^{d+1} \tilde{h}_{x_j}\|_\infty \leq \|\partial_{x_j}^{d+1} \tilde{h}_{x_j}\|_1 \).

Alternatively, we have that
\[
\tilde{h}_{x_j}(\omega) \leq \frac{d(1-d)/2}{|\omega|_{d+1} \sum_{j=1}^d |\partial_{x_j}^{d+1} \tilde{h}_{x_j}|_1}.
\]
As a result, we obtain that
\[
|\nabla h(\omega)| = \left( \sum_{j=1}^d |\tilde{h}_{x_j}(\omega)|^2 \right)^{1/2}
\]
\[
\leq \frac{d(1-d)/2}{|\omega|_{d+1} \sum_{j=1}^d |\partial_{x_j}^{d+1} \tilde{h}_{x_j}|_1} \left( \sum_{j=1}^d \|\partial_{x_j}^{d+1} \tilde{h}_{x_j}\|_1 \right)^{1/2}
\]
\[
= \frac{d(1-d)/2}{|\omega|_{d+1} N_2}.
\]
Since \( \omega \rightarrow |\omega|_{d+1} \) is a radial function, we have that
\[
II \leq \int_{B_\lambda} \frac{1}{\lambda (d-1)}|\nabla h(\omega)| d\omega
\]
\[
= \frac{d(d-1)/2 N_2}{\Gamma(d/2 + 1)} \int_{\lambda}^{\gamma} \frac{1}{\lambda} d\lambda
\]
\[
= \frac{\pi^{d/2} (d+1)/2 N_2}{\Gamma(d/2 + 1)} \int_{\lambda}^{\gamma} \frac{1}{\lambda} d\lambda
\]
\[
= \frac{\pi^{d/2} (d+1)/2 N_2}{\Gamma(d/2 + 1)} \lambda.
\]
By plugging (86) and (100) in (81), we conclude that
\[
C_h \leq \frac{1}{2d/2/\gamma^{d+1}} \left( N_1 \lambda^{d} + \frac{d(d+1)}{\lambda} \right).
\]
By taking \( \lambda = 1/|\omega|_{d+1} \), the result follows. \([\blacksquare]\)

Note that (82) generalizes (70), as it only involves derivatives of order 1 and \( d + 2 \). While it is possible to establish bounds that more closely resemble Theorem 3 (e.g., by using derivatives of order 1, \( d + 1 \) and \( d + 2 \)), they are more convoluted than (82) and add little practical value.

Theorem 4 might not be straightforward to apply as it heavily depends on the higher order partial derivatives of \( h \). Furthermore, since \( d_{N_d+1} \sim 1 \) and \( \Gamma(z + 1) \sim \sqrt{2\pi z} \left( \frac{z}{e} \right)^z \), we can show that
\[
A_d \sim \sqrt{\frac{\pi^d}{d!}}.
\]
In particular, the bound in (82) has an exponential dependency on the dimension. Despite the negative nature of this observation, it is indeed natural in view of a similar comment made by Barron in [11, Sec. IX-9].

\section{Additive Gaussian Mechanism}

In this section we consider the situation in which \( Y \) is binary and \( X \in \mathbb{R} \) is post-processed by the additive Gaussian mechanism introduced in Section II-D. Specifically, we assume that \( X \) is post-processed to produce a new random variable
\[
X^\sigma := X + \sigma Z,
\]
where \( \sigma > 0 \) and \( Z \) is a standard Gaussian random variable independent of \( X \) and \( Y \). Also, we assume that the random variable \( X^\sigma \) is further processed to remove extreme values, giving rise to a random variable \( X^\sigma \). Specifically, we consider extreme values truncation and extreme values randomization as introduced in Sections II-D1 and II-D2 respectively. Given the different nature of these two processing techniques, below we provide estimates for the Barron constant of (i) the conditional expectation under truncation and (ii) the log-likelihood ratio under randomization.

\subsection{Extreme Values Truncation}

Consider extreme values truncation with \( B = [-r, r] \) for some \( r > 0 \). As before, we let \( p := P(Y = 1) \) and \( f^u_{x,y} \) be
the conditional density of $\tilde{X}^\sigma$ given $Y = \pm 1$. The conditional expectation of $Y$ given $\tilde{X}^\sigma$ is equal to

$$\tilde{\eta}^\sigma(x) = \frac{p f^\sigma_\pm(x) - \tilde{p} f^\sigma_\pm(x)}{p f^\sigma_+(x) + \tilde{p} f^\sigma_-(x)} 1_{|x| \leq r}$$

(104)

$$= \tanh \left( \frac{1}{2} \log \left( \frac{p f^\sigma_+(x)}{p f^\sigma_-(x)} \right) \right) 1_{|x| \leq r}.$$  

(105)

As established in (20),

$$f^\sigma_\pm(x) = \frac{(f_\pm * K_\sigma)(x)}{\mathbb{P}(\{|X^\sigma| \leq r|Y = \pm 1\})} 1_{|x| \leq r},$$  

(106)

where $f_\pm$ is the conditional density of $X$ given $Y = \pm 1$ and $K_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$, $x \in \mathbb{R}$.

(107)

From (106), we conclude that $x \mapsto \frac{p f^\sigma_+(x)}{p f^\sigma_-(x)}$ is a non-negative smooth function over $[-r, r]$ and, as a result, $\tilde{\eta}^\sigma$ is a smooth function over the same domain as well. As pointed out by Barron [21 Sec. IX], this implies that $\tilde{\eta}^\sigma$ belongs to $\Gamma_B$ and, in particular, $C_{\tilde{\eta}^\sigma}$ is finite. Our goal is to find a tractable, yet useful, upper bound for $C_{\tilde{\eta}^\sigma}$.

As discussed in [15] and [38], a first step in order to find an upper bound for the Barron constant of $\tilde{\eta}^\sigma$ is to find a function, say $\eta^\sigma$, such that $\eta^\sigma$ is defined over $\mathbb{R}$ and $\tilde{\eta}^\sigma = \eta^\sigma|_B$. In this situation, we have that

$$C_{\tilde{\eta}^\sigma} \leq \frac{r}{\sqrt{2\pi}} \int_\mathbb{R} |\omega| |\tilde{\eta}^\sigma(\omega)| d\omega = r C_{\eta^\sigma}.$$  

(108)

Motivated by (104) and (106), we define $\eta^\sigma : \mathbb{R} \to \mathbb{R}$ by

$$\eta^\sigma(x) = \frac{\lambda_+ f^\sigma_+(x) - \lambda_- f^\sigma_-(x)}{\lambda_+ f^\sigma_+(x) + \lambda_- f^\sigma_-(x)},$$  

(109)

where $f^\sigma_\pm = f_\pm * K_\sigma$ and

$$\lambda_\pm = \frac{1}{\mathbb{P}(|X^\sigma| \leq r|Y = \pm 1)}.$$  

(110)

Observe that, by large deviations arguments, $\lambda_\pm$ can be estimated with relatively high precision as it only depends on the probabilities of the events $\{Y = \pm 1\}$ and $\{|X^\sigma| \leq r\}$ given $\{Y = \pm 1\}$. Furthermore, it can be shown that $\lim_{\sigma \to \infty} \lambda_\pm = \frac{1}{p}$, making the estimation of $\lambda_\pm$ unnecessary for large $\sigma$.

To gain some intuition about the behavior of the Barron constant as a function of $\sigma$, in the next proposition we compute $C_{\eta^\sigma}$ in a simple case.

**Proposition 2.** If $Y \sim \text{Unif}(\{\pm 1\})$ and $X = Y$, then

$$C_{\eta^\sigma} = \frac{1}{\sigma^2}, \quad \sigma > 0.$$  

(111)

**Proof.** By symmetry, we have that $\lambda_+ = \lambda_-$. Thus, by (109),

$$\eta^\sigma(x) = \frac{f^\sigma_+(x) - f^\sigma_-(x)}{f^\sigma_+(x) + f^\sigma_-(x)}.$$  

(112)

A direct computation shows that

$$f^\sigma_\pm(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x \mp \eta^\sigma)^2/2\sigma^2}.$$  

(113)

Therefore, for all $x \in \mathbb{R},$

$$\eta^\sigma(x) = \tanh \left( \frac{x}{\sigma^2} \right).$$  

(114)

Observe that $(\eta^\sigma)'(x) = \frac{1}{\sigma^2} \sech^2 \left( \frac{x}{\sigma^2} \right)$. Using contour integration, it can be verified that

$$\sech^2(\omega) = \frac{\sqrt{2}}{\sqrt{\pi}} \omega \csch \left( \frac{\sqrt{2}}{\sqrt{\pi}} \omega \right).$$  

(115)

In particular, $(\eta^\sigma)'(\eta^\sigma) = \frac{1}{\sqrt{2\pi/\sigma^4}} \sech^2 \left( \frac{\sqrt{2}}{\sqrt{\pi}} \omega \right)$.

(116)

By (115), we have that $(\eta^\sigma)'(\eta^\sigma)$ is non-negative for all $\omega \in \mathbb{R}$. Therefore, by the Fourier inversion theorem, (116) implies that $C_{\eta^\sigma} = \frac{1}{\sigma^2} \sech^2(0)$.  

The next theorem provides an upper bound for the Barron constant of $\eta^\sigma$, as defined in (109), under minimal assumptions on the distribution of $X$.

**Theorem 5.** If $\text{Supp}(f_\pm) \subset [-1, 1]$, then, for every $\sigma > 0$, $C_{\eta^\sigma} \leq \frac{2\sqrt{2}}{\sqrt{\pi}} + \frac{16\sqrt{2 M^\sigma}}{\sqrt{\pi\sigma^4}} \left( 1 + \frac{1}{2} \log \left( \frac{M^\sigma}{\sigma^2} \right) \right),$  

where

$$M^\alpha := \int_\mathbb{R} |x|^\alpha \lambda_+ f^\sigma_+(x) \lambda_- f^\sigma_-(x) \left( \frac{\lambda_+ f^\sigma_+(x) + \lambda_- f^\sigma_-(x)}{\lambda_+ f^\sigma_+(x) - \lambda_- f^\sigma_-(x)} \right)^2 dx,$$  

(118)

$$M^\sigma := M_0^\sigma \left( 64 M_0^\sigma + 176 M_0^\sigma + (136 + 48\sigma^2) M_0^\sigma \right).$$  

(119)

Furthermore, if $\sqrt{8e M_0^\sigma} \leq \sigma$, then

$$C_{\eta^\sigma} \leq \frac{16\sqrt{2 M^\sigma}}{\sqrt{\pi\sigma^4}} \left( 1 + \frac{1}{2} \log \left( \frac{M^\sigma}{M_0^\sigma} + \frac{3 M^\sigma}{M_0^\sigma} + 3 + \sigma^2 \right) \right).$$  

(120)

The proof of the previous theorem, which can be found in Appendix A, relies on Theorem 3 and careful estimates of $\eta^\sigma$ and its derivatives. Specifically, it exploits the cancellations that occur between the terms in the numerators of (72) – (74).

Note that the bounds in the previous theorem only depend on the moment-like quantities $M^\sigma$, as defined in (118). As we show below, in some canonical situations $M^\sigma = O(\sigma^{2(1+\alpha)})$ as $\sigma \to \infty$. Therefore, in the large noise regime ($\sigma \gg 1$),

$$C_{\eta^\sigma} \leq O \left( \frac{\log(\sigma)}{\sigma^2} \right).$$  

(121)

In view of Proposition 2, we conclude that the previous bound is order optimal up to logarithmic factors. Below we also show that in some situations $M^\sigma = O(1)$ as $\sigma \to 0^+$. Therefore, in the small noise regime ($\sigma \ll 1$),

$$C_{\eta^\sigma} \leq O \left( \frac{\log(1/\sigma)}{\sigma^4} \right).$$  

(122)

Although the order optimal of this bound is unclear, it is by no means trivial. Observe that, as $\sigma \to 0^+$, $\eta^\sigma$ converges pointwise to $\eta$ which in principle might have an unbounded Barron constant. Thus, (122) shows that even if $C_{\eta^\sigma}$ diverges to infinity as $\sigma \to 0^+$, it does it polynomially in $1/\sigma$. 

We end this section providing an upper bound for the moment-like quantities \( M_α^σ \) under different structural properties of the support of \( f_± \). In the next proposition, we do so in the case where the supports of \( f_+ \) and \( f_- \) are well-separated by some margin.

**Proposition 3.** Let \( M_α^σ \) be the quantities defined in (118). If there exist \( γ \in (0, 1) \) such that \( \text{Supp}(f_+) \subset [γ, 1] \) and \( \text{Supp}(f_-) \subset [-1, -γ] \), then, for every \( σ > 0 \),

\[
M_0^γ \leq 2 + \frac{σ^2 λ_+^2 + λ_-^2}{2γ} e^{-2γ/σ^2}, \quad (123)
\]
\[
M_1^γ \leq 2 + \left( \frac{σ^4}{4γ^2} + \frac{σ^2}{2γ}, \frac{λ_+^2 + λ_-^2}{λ_+ λ_-} e^{-2γ/σ^2} \right), \quad (124)
\]
\[
M_2^γ \leq 2 + \left( \frac{σ^6}{4γ^3} + \frac{σ^4}{2γ^2} + \frac{σ^2}{2γ}, \frac{λ_+^2 + λ_-^2}{λ_+ λ_-} e^{-2γ/σ^2} \right). \quad (125)
\]

In particular, \( M_α^σ = O(σ^{2(1+α)}) \) as \( σ → ∞ \) and \( M_α^σ = O(1) \) as \( σ → 0^+ \).

**Proof.** See Appendix [B].

In the next proposition we provide upper bounds for \( M_α^σ \) in the case where the supports of \( f_± \) overlap but extreme values determine the value of \( Y \), i.e., there exists \( γ_0 \) such that \( X > γ_0 \) then \( Y = 1 \), and if \( X < -γ_0 \) then \( Y = -1 \).

**Proposition 4.** Let \( M_α^σ \) be the quantities defined in (118). If there exist \( γ_0 \in (0, 1) \) such that

\[
[γ_0, 1] \subset \text{Supp}(f_+) \subset (-γ_0, 1], \quad (126)
\]
\[
[-1, -γ_0] \subset \text{Supp}(f_-) \subset [-1, γ_0), \quad (127)
\]

then, for every \( γ \in (γ_0, 1) \) and \( σ > 0 \),

\[
M_0^γ \leq 2 + \frac{σ^2}{γ_0^2} \Lambda, \quad (128)
\]
\[
M_1^γ \leq 2 + \left( \frac{σ^4}{(γ_0^2 - γ)^2} + \frac{σ^2}{γ_0 - γ} \right) \Lambda, \quad (129)
\]
\[
M_2^γ \leq 2 + \left( \frac{σ^6}{(γ_0^2 - γ)^2} + \frac{σ^4}{(γ_0 - γ)^2} + \frac{σ^2}{γ - γ_0^2} \right) \Lambda, \quad (130)
\]

where \( \Lambda = \frac{δ_+ λ_+^2 + δ_- λ_-^2}{δ_+ λ_+ δ_- λ_-} \).

\[
δ_+ = \int_{γ}^{1} f_+(s)ds \quad \text{and} \quad δ_- = \int_{-1}^{-γ} f_+(s)ds. \quad (131)
\]

In particular, \( M_α^σ = O(σ^{2(1+α)}) \) as \( σ → ∞ \) and \( M_α^σ = O(1) \) as \( σ → 0^+ \).

**Proof.** See Appendix [C].

**B. Extreme Values Randomization**

Consider extreme values randomization with \( B = [-r, r] \) for some \( r > 0 \). As before, we let \( p := P( Y = 1) \) and \( f_{±}^σ \) be the conditional density of \( X^σ \) given \( Y = ±1 \). Recall the definition of the log-likelihood function

\[
\tilde{θ}(x) = \frac{1}{2} \log \left( \frac{p_{f_+}^σ(x)}{p_{f_-}^σ(x)} \right) \mathbb{1}_{|x| ≤ r}, \quad (132)
\]

As established in (21),

\[
f_{±}^σ(x) = \left( (f_± + K_σ)(x) + \mathbb{P}( |X^σ| > r | Y = ±1 ) \right) \mathbb{1}_{|x| ≤ r}, \quad (133)
\]

where \( f_± \) is the conditional density of \( X \) given \( Y = ±1 \) and

\[
K_σ(x) = \frac{1}{\sqrt{2πσ^2}} e^{-x^2/2σ^2}, \quad x ∈ ℝ. \quad (134)
\]

From (133), we conclude that \( x → \frac{pf_+^σ(x)}{pf_-^σ(x)} \) is a positive smooth function over \([ r, r] \) and, as a result, \( \tilde{θ}^σ \) is a smooth function over the same domain as well. As pointed out by Barron [21 Sec. IX], this implies that \( \tilde{θ}^σ \) belongs to \( G_B \) and, in particular, \( C_{θ^σ} \) is finite. Our goal is to find a tractable, yet useful, upper bound for \( C_{θ^σ} \).

As discussed in (1) and (58), a first step in order to find an upper bound for the Barron constant of \( θ^σ \) is to find a function, say \( θ^σ \), such that \( θ^σ \) is defined over \( ℝ \) and \( θ^σ = θ^σ_B \). In this situation, we have that

\[
C_{θ^σ} ≤ \frac{r}{\sqrt{2π}} \int_{ℝ} |ω|^2 |\tilde{θ}^σ(ω)| |dω| =: r C_{θ^σ}. \quad (135)
\]

Motivated by (132) and (133), we define \( θ^σ : ℝ → ℝ \) by

\[
θ^σ(x) = \frac{1}{2} \log \left( \frac{pf_+^σ(x) + λ_+}{pf_-^σ(x) + λ_-} \right), \quad (136)
\]

where \( f_± = f_± + K_σ \) and

\[
λ_± = \mathbb{P}( |X^σ| > r | Y = ±1 ). \quad (137)
\]

Observe that, by large deviations arguments, \( λ_± \) can be estimated with relatively high precision as it only depends on the probabilities of the events \{ \( Y = ±1 \) \} and \{ \( |X^σ| > r \) \} given \{ \( Y = ±1 \) \}. Furthermore, it can be shown that \( \lim_{σ → ∞} λ_{±} = \frac{1}{2r} \), making the estimation of \( λ_± \) unnecessary for large \( σ \).

The next theorem provides an upper bound for the Barron constant of \( θ^σ \), as defined in (136), under minimal assumptions on the distribution of \( X \).

**Theorem 6.** If \( f_± \) is a probability density function, then, for every \( σ > 0 \),

\[
C_{θ^σ} ≤ \frac{2\sqrt{2}}{ε√π} + \frac{2\sqrt{2}}{√π} \left( 1 + \frac{1}{2} \log \left( N_1^σ N_2^σ \right) \right) N_2^σ. \quad (138)
\]

where

\[
N_1^σ := \frac{Λ_1}{\sqrt{2πσ}}, \quad (139)
\]
\[
N_2^σ := \frac{Λ_1}{σ^2} + \frac{Λ_2}{8\sqrt{πσ^3}}, \quad (140)
\]
\[
N_3^σ := \frac{5Λ_1}{2\sqrt{2πσ}} + \frac{3\sqrt{5}Λ_2}{8√2πσ^4} + \frac{√2Λ_3}{9\sqrt{πσ^5}}, \quad (141)
\]

with \( Λ_α = \frac{p^α}{λ_+} + \frac{p^α}{λ_-} \). Moreover, if \( N_2^σ ≤ 1/ε \), then

\[
C_{θ^σ} ≤ \frac{2\sqrt{2}}{√π} \left( 1 + \frac{1}{2} \log \left( N_1^σ N_3^σ / (N_2^σ)^2 \right) \right) N_2^σ. \quad (142)
\]
The proof of the previous theorem, which can be found in Appendix D, relies on Theorem and careful estimates of the $L^1$-norms of the derivatives of $\theta^\sigma$.

Note that the previous theorem does not assume anything about $f_\pm$ apart from its existence. Furthermore, (142) implies that, in the large noise regime ($\sigma \gg 1$),

$$C_{\theta^\sigma} = O \left( \frac{1}{\sigma^2} \right).$$  \hfill (143)

By [21] Sec. IX-14, it can be verified that $C_{\theta^\sigma} = 1/\sigma^2$ in the context of Proposition 2 with $r = \infty$. As a result, we conclude that the previous bound is in fact order optimal. Similarly, (142) implies that, in the small noise regime ($\sigma \ll 1$),

$$C_{\theta^\sigma} = O \left( \frac{\log(1/\sigma)}{\sigma^3} \right).$$  \hfill (144)

As with (122), it is unclear if the previous bound is order optimal.

VI. SUMMARY AND FINAL REMARKS

In this paper, we proposed two neural network-based estimators for $\text{mmse}(Y|X)$, the minimum mean square error in estimating $Y \in \mathbb{R}$ given $X \in \mathbb{R}^d$. The first estimator, denoted by $\text{mmse}_{k,n}(Y|X)$, uses a two-layer neural network of size $k$; while the second estimator, denoted by $\text{mmse}_{k,n}^*(Y|X)$, uses a 3-layer neural network whose last layer is a single non-trainable node with tanh activation function. For bounded $X$ and $Y$, we derived a lower bound for $\text{mmse}(Y|X)$ based on $\text{mmse}_{k,n}(Y|X)$ and the Barron constant of the conditional expectation of $Y$ given $X$. Furthermore, in the particular case of binary $Y$, we derived another lower bound for $\text{mmse}(Y|X)$ based on $\text{mmse}_{k,n}^*(Y|X)$ and the Barron constant of the log-likelihood ratio of the conditional densities of $X$ given $Y = \pm 1$. Finding meaningful estimates for the Barron constant is challenging since (i) the underlying conditional expectation/log-likelihood ratio is rarely available in practice and (ii) the Barron constant is defined in terms of the Fourier transform of this conditional expectation/log-likelihood ratio.

To alleviate the second issue, we established an upper bound for the Barron constant of a function $h: \mathbb{R} \rightarrow \mathbb{R}$ based on the $L^1$-norms of its derivatives. We further generalized this result to multivariate functions $h: \mathbb{R}^d \rightarrow \mathbb{R}$, although the complexity of the result and the bound itself increase dramatically with $d$. In addition, we showed that one can circumvent the first issue in applications where the additive Gaussian mechanism is used. In such applications, our estimates for the Barron constant are order optimal in the large noise regime. Overall, we developed an effective machinery to obtain theoretical lower bounds for the MMSE which, at a technical level, required to perform a non-trivial quantitative analysis of the Barron constant.

While we only considered shallow neural networks, there are fundamental obstructions in trying to generalize the present work to deep neural networks.

- From a function approximation perspective, at the moment there is no analogue of Barron’s theorem for deep neural networks.
- From a computational perspective, the optimization landscape of deep neural networks is significantly more complex than its shallow counterpart, see, e.g., [34]–[36]. As a result, it is harder to guarantee that a deep neural network has been trained to optimality, which is essential for the estimators proposed in this paper.
- As mentioned at the end of Section H-A, if $k$ is sufficiently large then our lower bounds for the MMSE become trivial due to overfitting, i.e., $\text{mmse}_{k,n}(Y|X)$ being equal to 0. Given the astonishing expressive power of deep neural networks, see, e.g., [37], [38], they seem likely to produce trivial lower bounds for the MMSE.

Overall, generalizing the present work to deep neural networks is highly non-trivial and, at the same, it is unclear if it will provide significantly better results.

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APPENDIX A

PROOF OF THEOREM 5

For each $x \in \mathbb{R}$, we define

$$g^\pm_k(x) := \lambda_\pm(f_\pm \ast K_\sigma)(x).$$  \hfill (145)

Observe that, with this notation,

$$\eta^\sigma(x) = g^\sigma_\pm(x) - g^\sigma_\mp(x) = \frac{g^\sigma_\mp(x) - g^\sigma_\mp(x)}{g^\sigma_\pm(x) + g^\sigma_\mp(x)}.$$  \hfill (146)

The following simple lemma provides useful expressions for the derivatives of $g^\pm_k$.

**Lemma 3.** If $j \in \{0, 1, 2, 3\}$, then, for every $x \in \mathbb{R}$,

$$\frac{d^j g^\pm_k}{dx^j}(x) = \int_{\mathbb{R}} \lambda_\pm f_\pm(s) P^\pm_j(x - s) K_\sigma(x - s) ds,$$  \hfill (147)

where $P^\pm_0(x) = 1$, $P^\pm_1(x) = -x/\sigma^2$, $P^\pm_2(x) = (x^2 - \sigma^2)/\sigma^4$, and $P^\pm_3(x) = -(x^3 - 3\sigma^2 x)/\sigma^6$.

**Proof.** It can be verified that, for each $j \in \{0, 1, 2, 3\}$,

$$\frac{d^j K_\sigma^{(j)}}{dx^j}(x) = P^\sigma_j(x) K_\sigma(x).$$  \hfill (148)

Observe that $g^{\pm}_k = (\lambda_\pm f_\pm) \ast K_\sigma$. Therefore, the lemma follows from the general formula $\frac{d^j}{dx^j}(h_1 \ast h_2) = h_1 \ast \frac{d^j}{dx^j} h_2$.

In order to avoid cumbersome notation, we omit the superscript $\sigma$ when there is no risk of confusion, e.g., $g^\pm_k$ is written as $g^\pm$ and $\eta^\sigma$ is written as $\eta$. In order to simplify our calculations, we introduce the following notation.

Lee et al. [23] have an important effort in this direction, although their results depend on a specific decomposition of the target function.
Definition 1. For each $n \in \mathbb{N}$, we define

$$K_{\delta_1, \ldots, \delta_n}(s_1, \ldots, s_n; x) := \prod_{i=1}^{n} \lambda_{\delta_i} f_{\delta_i}(s_i) K_\sigma(x - s_i),$$

(149)

where $\delta_1, \ldots, \delta_n \in \{\pm\}$, $s_1, \ldots, s_n \in \mathbb{R}$ and $x \in \mathbb{R}$. Also, for $j_1, \ldots, j_n \in \{0, 1, 2, 3\}$, we define

$$P_{j_1, \ldots, j_n}(s_1, \ldots, s_n; x) := \prod_{i=1}^{n} P_{j_i}(x - s_i).$$

(150)

With the above notation, Lemma 3 implies that for every $n \in \mathbb{N}, j_1, \ldots, j_n \in \{0, 1, 2, 3\}$ and $x \in \mathbb{R}$,

$$\int_{\mathbb{R}^n} g^{(j)}(x) = \int_{\mathbb{R}^n} P_{j_1, \ldots, j_n}(s; x) K_{\delta_1, \ldots, \delta_n}(s; x)ds,$$

(151)

where $s = (s_1, \ldots, s_n)$ and $ds = ds_1 \cdots ds_n$. In particular, by taking $j_i = 0$ for all $i \in [n]$,

$$\int_{\mathbb{R}^n} g_i(x) = \int_{\mathbb{R}^n} K_{\delta_1, \ldots, \delta_n}(s; x)ds.$$

(152)

Finally, observe that for every $x \in \mathbb{R}$,

$$\text{Supp}(K_{\delta_1, \ldots, \delta_n} ; x) = \text{Supp}(f_{\delta_1}) \cdots \text{Supp}(f_{\delta_n}).$$

(153)

Now we derive a pointwise bound for $\eta'$. Lemma 4. If $\text{Supp}(f_{\pm}) \subset [-1, 1]$, then, for all $x \in \mathbb{R}$,

$$|\eta'(x)| \leq \frac{4}{\sigma^2} \frac{g_+(x)g_-(x)}{(g_+(x) + g_-(x))^2}.$$

(154)

Proof. In Lemma 1 we prove that for all $x \in \mathbb{R}$,

$$\eta'(x) = \frac{I(x)}{(g_+(x) + g_-(x))^2},$$

(155)

where

$$I(x) := 2 \left( g'_+(x)g_-(x) - g_+(x)g'_-(x) \right).$$

(156)

The integral formula in (151) and (153) imply that

$$g'_+(x)g_-(x) = \int_{-1}^{1} \int_{-1}^{1} P_{0,0}(s;x) K_{+, -}(s;x)ds,$$

(157)

\textbf{Mutatis mutandis}, we have that

$$g_+(x)g'_-(x) = \int_{-1}^{1} \int_{-1}^{1} P_{0,0}(s;x) K_{+, -}(s;x)ds.$$

(158)

Thus, we have that

$$I(x) = \int_{-1}^{1} \int_{-1}^{1} Q(s;x) K_{+, -}(s;x)ds,$$

(159)

where $Q(s;x) = 2P_{0,0}(s;x) - 2P_{0,1}(s;x)$. By the definition of $P_{j_1, \ldots, j_n}$ in (150),

$$2P_{0,0}(s;x) = \frac{-2}{\sigma^2} x + \frac{2s_1}{\sigma^2},$$

(160)

$$-2P_{0,1}(s;x) = \frac{2}{\sigma^2} x - \frac{2s_2}{\sigma^2}.$$

(161)

As a result, $Q(s_1, s_2; x) = \frac{2(s_1 - s_2)}{\sigma^2}$ and (159) becomes

$$I(x) = \int_{-1}^{1} \int_{-1}^{1} \frac{2(s_1 - s_2)}{\sigma^2} K_{+, -}(s_1, s_2; x)ds_1ds_2.$$
\[ P_{3,0}(s_1, s_2; x) = \frac{-x^3 + 3s_1 x^2 + 3(\sigma^2 - s_1^2)x + s_1 (s_1^2 - 3\sigma^2)}{\sigma^6} \]  
\[ P_{2,1}(s_1, s_2; t) = \frac{-x^3 + (2s_1 + s_2)x^2 + (\sigma^2 - s_1^2 - 2s_1 s_2)x + s_2 (s_1^2 - \sigma^2)}{\sigma^6} \]  
\[ -P_{1,2}(s_1, s_2; t) = \frac{x^3 - (s_1 + 2s_2)x^2 - (\sigma^2 - 2s_1 s_2 - s_2^2)x - s_1 (s_2^2 - \sigma^2)}{\sigma^6} \]  
\[ -P_{0,3}(s_1, s_2; x) = \frac{x^3 - 3s_2 x^2 - 3(\sigma^2 - s_2^2)x - s_2 (s_2^2 - 3\sigma^2)}{\sigma^6} \]  
where the equality follows from \([152]\). Mutatis mutandis, it can be shown that

\[ I_2(x) = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} Q_2(s; x) K_{+,-,-}(s; x) ds, \]  
where

\[ Q_2(s; x) = \frac{2(s_2^2 - s_1^2 - 2s_1 s_2 + 2s_2 s_3)}{\sigma^4}. \]  
As before, \((183)\) and \((184)\) imply that, for all \(x \in \mathbb{R}\),

\[ |I_2(x)| \leq \frac{8}{\sigma^4} g_+(x) g_-(x) g_-(x). \]  
Since \(\eta''(x) = \frac{I_1(x) + I_2(x)}{(g_+(x) + g_-(x))^2}\), \((182)\) and \((183)\) imply that

\[ |\eta''(x)| \leq \frac{8}{\sigma^4} g_+(x) g_-(x), \]  
as required.

Finally, we establish an upper bound for \(\eta''\) akin to those in the previous lemmas.

**Lemma 6.** If \(\text{Supp}(f_{\pm}) \subset [-1, 1]\), then, for all \(x \in \mathbb{R}\),

\[ |\eta''(x)| \leq \frac{16x^2 + 44|x| + 34 + 12\sigma^2}{\sigma^6} \frac{g_+(x) g_-(x)}{(g_+(x) + g_-(x))^2}. \]  

**Proof.** In Lemma 1, we prove that

\[ \eta''(x) = \frac{1}{(g_+ + g_-)^2} - 2\eta' g''_+ + g''_- - 3\eta'' g_+ + g_+ - g_- \]  
where

\[ I = 2(g''_+(g_- - g'_- g''_- - g'_- g''_-) + g''_- g_+ g_- - g''_+ g_- + g''_+ g_-). \]  
The integral formula in \((151)\) implies that, for all \(x \in \mathbb{R}\),

\[ I(x) = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} Q(s; x) K_{+,-,-}(s; x) ds, \]  
where \(Q = P_{3,0} + P_{2,1} - P_{1,2} - P_{0,3}\). By the definition of \(P_{j_1, \ldots, j_n}\) in \((130)\) and equations \((165)\) - \((168)\), we conclude that, for all \(x \in \mathbb{R}\),

\[ Q(s_1, s_2; x) = \frac{4(s_1 - s_2) x^2 + 4(s_2^2 - s_1^2) x}{\sigma^6} \]  
\[ + \frac{(s_1 - s_2)(s_1 + s_2)^2 - 2(s_1 - s_2)\sigma^2}{\sigma^6}. \]  
In particular, for \(s_1, s_2 \in [-1, 1]\),

\[ |Q(s; x)| \leq \frac{8x^2 + 4|x| + 2 + 4\sigma^2}{\sigma^6}. \]  
Therefore, \((190)\) implies that

\[ |I(x)| \leq \frac{8x^2 + 4|x| + 2 + 4\sigma^2}{\sigma^6} \int_{-1}^{1} \int_{-1}^{1} K_{+,-,-}(s; t) ds \]  
\[ = \frac{8x^2 + 4|x| + 2 + 4\sigma^2}{\sigma^6} g_+(x) g_-(x), \]  
where the equality follows from \((152)\). Lemma 3 and the fact that \(\text{Supp}(f_{\pm}) \subset [-1, 1]\) imply that

\[ |g'_{\pm}(x)| \leq \frac{x^2 + 2|x| + 1 + \sigma^2}{\sigma^4} g_{\pm}(x). \]  
As a result, we obtain that

\[ |2\eta'(x) g''_+(x) + g''_-(x) g_+(x) + g_-(x)| \leq \frac{2x^2 + 4|x| + 2 + 2\sigma^2}{\sigma^4} |\eta''(x)|. \]  
Therefore, by Lemma 4

\[ |2\eta'(x) g''_+(x) g''_-(x) + g'_+(x) g'_-(x) g_+(x) + g_-(x)| \leq \frac{8x^2 + 16|x| + 8 + 8\sigma^2}{\sigma^4} \times \frac{g_+(x) g_-(x)}{(g_+(x) + g_-(x))^2}. \]  
Lemma 3 and the fact that \(\text{Supp}(f_{\pm}) \subset [-1, 1]\) imply that

\[ |g'_{\pm}(x)| \leq \frac{|x| + 1}{\sigma^2} g_{\pm}(x). \]  
Therefore, we conclude that

\[ |3\eta''(x) g''_+(x) + g''_-(x) g_+(x) + g_-(x)| \leq \frac{3|x| + 1}{\sigma^2} |\eta''(x)| \]  
\[ \leq \frac{24(|x| + 1)}{\sigma^6} g_+(x) g_-(x), \]  
where the last inequality follows from Lemma 5. By plugging \((194)\), \((197)\) and \((200)\) in \((188)\), the result follows.

It is possible to obtain slightly better constants than those in \((187)\) by avoiding the use of Lemmas 4 and 5. However, the complexity of the proof increases considerably, and the benefit is marginal given that the \(L^1\)-norm of \(\eta''\) only appears inside a logarithm.
Proof of Theorem 5. By Theorem 3 we have that
\[ C_\eta \leq \frac{2\sqrt{2}}{\sqrt{\pi}} \left( 1 + \frac{1}{2} \log \left( \frac{M}{\sigma^4} \right) \right) \sqrt{\frac{M}{\sigma^6}} \sqrt{\log(\|\eta''\|_1)} \|\eta''\|_1. \]
(201)
Recall the definition of \( M_\eta \) in (118). By Lemmas 4–6 we have that
\[ \|\eta'\|_1 \leq \frac{4M_0}{\sigma^2}, \]
(202)
\[ \|\eta''\|_1 \leq \frac{8M_0}{\sigma^4}, \]
(203)
\[ \|\eta''\|_1 \leq \frac{16M_2 + 44M_1 + (34 + 12\sigma^2)M_0}{\sigma^6}. \]
(204)
As a result, for all \( \sigma > 0 \),
\[ C_\eta \leq \frac{16\sqrt{2}M_0}{\sqrt{\pi}\sigma^4} \left( 1 + \frac{1}{2} \log \left( \frac{M}{\sigma^4} \right) \right) \sqrt{\frac{M}{\sigma^6}} \sqrt{\log(\|\eta''\|_1)} \|\eta''\|_1, \]
(205)
where
\[ M := 64M_2M_0 + 176M_1M_0 + (136 + 48\sigma^2)M_0^2. \]
(206)
Since \( -\log(z) \leq 1/e \) for all \( z \in [0, \infty) \), (201) implies that, for all \( \sigma > 0 \),
\[ C_\eta \leq \frac{2\sqrt{2}}{\sqrt{\pi}} + \frac{16\sqrt{2}M_0}{\sqrt{\pi}\sigma^4} \left( 1 + \frac{1}{2} \log \left( \frac{M}{\sigma^4} \right) \right) \sqrt{\frac{M}{\sigma^6}} \sqrt{\log(\|\eta''\|_1)} \|\eta''\|_1. \]
(207)
It is straightforward to verify that \( z \rightarrow -\log(z) \) is increasing over \([0, 1/e]\). Thus, if \( \sqrt{8eM_0} \leq \sigma \), (205) implies that
\[ C_\eta \leq \frac{16\sqrt{2}M_0}{\sqrt{\pi}\sigma^4} \left( 1 + \frac{1}{2} \log \left( \frac{M}{\sigma^4} \right) \right) \sqrt{\frac{M}{\sigma^6}} \sqrt{\log(\|\eta''\|_1)} \|\eta''\|_1. \]
(208)
After some manipulations, (120) follows. \( \square \)

APPENDIX B

Proof of Proposition 3
Recall that, for each \( x \in \mathbb{R} \),
\[ g^+_{\pm}(x) := \lambda_{\pm}(f_{\pm} * K_\gamma)(x). \]
(209)
The next lemma provides upper and lower bounds for \( g^+_{\pm}(x) \) under the assumptions of Proposition 3.

Lemma 7. In the context of Proposition 3 for all \( x \in \mathbb{R} \),
\[ \lambda_{\pm} \min_{s \in [x-1, x-\gamma]} K_\sigma(s) \leq g^+_{\pm}(x) \leq \lambda_{\pm} \max_{s \in [x-1, x-\gamma]} K_\sigma(s), \]
\[ \lambda_{\pm} \min_{s \in [x+\gamma, x+1]} K_\sigma(s) \leq g^-(x) \leq \lambda_{\pm} \max_{s \in [x+\gamma, x+1]} K_\sigma(s). \]

Proof. By assumption \( \text{Supp}(f_+) \subset [\gamma, 1] \), thus
\[ g^+_{\pm}(x) = \lambda_{\pm} \int_{\gamma}^{1} f_+(s) K_\sigma(x-s) ds. \]
(210)
Therefore, for all \( x \in \mathbb{R} \),
\[ \lambda_{\pm} \left( \min_{s \in [\gamma, 1]} K_\sigma(x-s) \right) \int_{\gamma}^{r} f_+(s) ds \leq g^+_{\pm}(x) \leq \lambda_{\pm} \left( \max_{s \in [\gamma, 1]} K_\sigma(x-s) \right) \int_{\gamma}^{1} f_+(s) ds. \]
(211)
Since \( \int_{-\infty}^{\infty} f_+(s) ds = 1 \), the inequalities for \( g^+_\pm \) follow. The inequalities for \( g^- \) are proved mutatis mutandis. \( \square \)

The next lemma provides an upper bound for \( g^+_{\pm}/(g^+_\pm + g^-) \).

Lemma 8. In the context of Proposition 3,
\[ g^+_{\pm}(x) \frac{g^+_{\pm}(x)}{g^+_{\pm}(x) + g^-}(x) \leq \frac{\lambda_{\pm}}{\lambda_{\pm} - \lambda_{\pm} \max_{s \in [x-1, x-\gamma]} K_\sigma(s)}, \]
\[ g^+_{\pm}(x) \frac{g^+_{\pm}(x) + g^-}(x) \leq \frac{\lambda_{\pm}}{\lambda_{\pm} - \lambda_{\pm} \min_{s \in [x+\gamma, x+1]} K_\sigma(s)}. \]

Proof. By Lemma 4 for all \( x \in \mathbb{R} \),
\[ g^+_{\pm}(x) \frac{g^+_{\pm}(x)}{g^+_{\pm}(x) + g^-}(x) \leq \frac{\lambda_{\pm}}{\lambda_{\pm} - \lambda_{\pm} \max_{s \in [x-1, x-\gamma]} K_\sigma(s)}, \]
\[ g^+_{\pm}(x) \frac{g^+_{\pm}(x) + g^-}(x) \leq \frac{\lambda_{\pm}}{\lambda_{\pm} - \lambda_{\pm} \min_{s \in [x+\gamma, x+1]} K_\sigma(s)}. \]
(212)
When \( x < -1 \), we have that
\[ [x-1, x-\gamma], [x+\gamma, x+1] \subset (-\infty, 0). \]
(213)
Since \( K_\sigma \) is increasing on \([-\infty, 0)\),
\[ g^+_{\pm}(x) \frac{g^+_{\pm}(x)}{g^+_{\pm}(x) + g^-}(x) \leq \frac{\lambda_{\pm} K_\sigma(x-\gamma)}{\lambda_{\pm} K_\sigma(x+\gamma)} = \frac{\lambda_{\pm}}{\lambda_{\pm} - \lambda_{\pm} \max_{s \in [x-1, x-\gamma]} K_\sigma(s)}, \]
\[ g^+_{\pm}(x) \frac{g^+_{\pm}(x) + g^-}(x) \leq \frac{\lambda_{\pm}}{\lambda_{\pm} - \lambda_{\pm} \min_{s \in [x+\gamma, x+1]} K_\sigma(s)}. \]
(214)
Mutatis mutandis, it can be shown that, for \( x > 1 \), we have the inequality
\[ g^+_{\pm}(x) \frac{g^+_{\pm}(x)}{g^+_{\pm}(x) + g^-}(x) \leq \frac{\lambda_{\pm}}{\lambda_{\pm} - \lambda_{\pm} \min_{s \in [x+\gamma, x+1]} K_\sigma(s)}. \]
\[ \square \]

Now we are in position to prove Proposition 3.

Proof of Proposition 3
By definition, we have that
\[ M_\sigma^+ = \left( \int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{\infty} \right) |x|^p \frac{g^+_{\pm}(x) g^-(x)}{(g^+_{\pm}(x) + g^-)(x)} dx. \]
(215)
Lemma 8 and the trivial upper bound
\[ \frac{g^+_{\pm}(x)}{g^+_{\pm}(x) + g^-}(x) \leq 1 \]
imply that
\[ M_\sigma^+ \leq 2 + \frac{\lambda^2_{\pm} + \lambda^2}{\lambda_{\pm} - \lambda_{\pm}} \int_{1}^{\infty} x^p \exp \left\{ -2\gamma x/\sigma^2 \right\} dx. \]
(216)
By the formulas,
\[ \int e^{-\beta x} dx = -\frac{e^{-\beta x}}{\beta}, \]
\[ \int xe^{-\beta x} dx = -\frac{e^{-\beta x}}{\beta^2} (\beta x + 1), \]
\[ \int x^2 e^{-\beta x} dx = -\frac{e^{-\beta x}}{\beta^3} (\beta^2 x^2 + 2\beta x + 2), \]
the result follows. \( \square \)

APPENDIX C

Proof of Proposition 4
Recall that, for each \( x \in \mathbb{R} \), we define
\[ g^\pm_{\pm}(x) := \lambda_{\pm}(f_{\pm} * K_\gamma)(x). \]
(220)
The next lemma provides upper and lower bounds for \( g_{\pm}(x) \) under the assumptions of Proposition 4.
Lemma 9. In the context of Proposition 4, for all $x \in \mathbb{R}$,
\[
\lambda_+ \delta_+ \min_{s \in [x-1,x-\gamma]} K_\sigma(s) \leq g_+^\sigma(x) \leq \lambda_+ \max_{s \in [x-1,x+\gamma_0]} K_\sigma(s),
\]
\[
\lambda_- \delta_- \min_{s \in [x+\gamma,x+1]} K_\sigma(s) \leq g_-^\sigma(x) \leq \lambda_- \max_{s \in [x-\gamma_0,x+1]} K_\sigma(s).
\]

Proof. By assumption $\text{Supp}(f_+) \subset [-\gamma_0,1]$, thus
\[
g_+^\sigma(x) = \lambda_+ \int_{-\gamma_0}^1 f_+(s) K_\sigma(x-s) \, ds.
\] (221)
Since $-\gamma_0 < \gamma < 1$, for all $x \in \mathbb{R}$,
\[
g_+^\sigma(x) \geq \lambda_+ \int_{-\gamma_0}^1 f_+(s) K_\sigma(x-s) \, ds
\geq \lambda_+ \int_{-\gamma_0}^1 f_+(s) \min_{s \in [-1,1]} K_\sigma(s) \, ds
= \lambda_+ \delta_+ \min_{s \in [-1,1]} K_\sigma(s).
\] (222)
Similarly, for all $x \in \mathbb{R}$,
\[
g_-^\sigma(x) \leq \lambda_+ \int_{-\gamma_0}^1 f_+(s) \max_{s \in [-1,1]} K_\sigma(s) \, ds
= \lambda_+ \max_{s \in [-1,1]} K_\sigma(s).
\] (223)

The inequalities for $g^\sigma$ are proved mutatis mutandis. \qed

The next lemma provides an upper bound for $g_+^\sigma/(g_+^\sigma + g_-^\sigma)$.

Lemma 10. In the context of Proposition 4,
\[
\frac{g_+^\sigma(x)}{g_+^\sigma(x) + g_-^\sigma(x)} \leq \frac{\lambda_- \max_{s \in [x-1,x+\gamma_0]} K_\sigma(s)}{\lambda_+ \delta_- \min_{s \in [x+\gamma,x+1]} K_\sigma(s)}
\leq \frac{\lambda_- \max_{s \in [x-1,x+\gamma_0]} K_\sigma(s)}{\lambda_+ \delta_+ K_\sigma(x+\gamma_0)} (237)
\]
When $x < -1$, we have that
\[
[x-1,x+\gamma_0], [x+\gamma,x+1] \subset (-\infty,0).
\] (228)
Since $K_\sigma$ is increasing on $(-\infty,0)$,
\[
\frac{g_+^\sigma(x)}{g_+^\sigma(x) + g_-^\sigma(x)} \leq \frac{\lambda_- K_\sigma(x+\gamma_0)}{\lambda_+ \delta_- K_\sigma(x+\gamma)} \leq \frac{\lambda_- e^{-(\gamma-\gamma_0)(|x|-1)/\sigma^2}}{\lambda_+ \delta_- e^{-(\gamma-\gamma_0)(|x|-1)/\sigma^2}}.
\] (229)

Mutatis mutandis, it can be shown that, for $x > 1$, we have the inequality
\[
\frac{g_+^\sigma(x)}{g_+^\sigma(x) + g_-^\sigma(x)} \leq \frac{\lambda_- e^{-(\gamma-\gamma_0)(|x|-1)/\sigma^2}}{\lambda_+ \delta_+ e^{-(\gamma-\gamma_0)(|x|-1)/\sigma^2}}.
\] \qed

Note that the previous integral is the first absolute moment of a Gaussian random variable with mean 0 and variance $\sigma^2$. Therefore, we obtain that
\[
\|K_\sigma\|_1 = \frac{\sqrt{\pi}}{\sqrt{\pi \sigma}}.
\] (240)

Similarly, we have that
\[
\|K_\sigma\|_2 = \frac{1}{2 \sqrt{\pi \sigma^3}} \int_\mathbb{R} x^2 e^{-x^2/2\sigma^2} \, dx.
\] (241)

Note that the previous integral is the second moment of a Gaussian random variable with mean 0 and variance $\sigma^2/2$. Therefore, we obtain that
\[
\|K_\sigma\|_3 = \frac{1}{2 \sqrt{3 \pi \sigma^3}}.
\] (242)
Note that the previous integral is the third absolute moment of a Gaussian random variable with mean 0 and variance $\sigma^2/3$. Therefore, we obtain that
\[
\|K''_\sigma\|_3^3 = \frac{\sqrt{2}}{9\sqrt{\pi}\sigma^3},
\] (244)
as required.

The following lemma provides useful expressions for the $L^1$ and $L^2$-norms of the second derivative of $K_\sigma$.

**Lemma 12.** If $K_\sigma(x) = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$, then $\|K''_\sigma\|_1 \leq \frac{2}{\sigma^2}$ and $\|K''_\sigma\|_2 = \frac{\sqrt{3}}{2\sqrt{2}\sqrt{\pi}\sigma^{5/2}}$.

**Proof.** It can be verified that, for all $x \in \mathbb{R}$,
\[
K''_\sigma(x) = \frac{\sigma^2 - \sigma^2}{\sigma^4} K'(x).
\] (245)
Thus, we have that
\[
\|K''_\sigma\|_1 \leq \int_\mathbb{R} \left( \frac{x^2}{\sigma^4} + \frac{1}{\sigma^2} \right) e^{-x^2/2\sigma^2} \frac{dx}{\sqrt{2\pi\sigma^2}}
\] (246)
\[
= \frac{1}{\sigma^4} \int_\mathbb{R} x^2 e^{-x^2/2\sigma^2} dx + \frac{1}{\sigma^2}.
\] (247)
Note that the last integral is the second moment of a Gaussian random variable with mean 0 and variance $\sigma^2$. Therefore,
\[
\|K''_\sigma\|_1 \leq \frac{2}{\sigma^2}.
\] (248)
Similarly, we have that
\[
\|K''_\sigma\|_2^2 = \int_\mathbb{R} \left( \frac{x^4}{2\sqrt{\pi}\sigma^3} - \frac{x^2}{\sqrt{\pi}\sigma^5} + \frac{1}{2\sqrt{\pi}\sigma^5} \right) e^{-x^2/2\sigma^2} \frac{dx}{\sqrt{2\pi\sigma^2}}
\] (249)
Note that the last integral is determined by the even moments of a Gaussian random variable with mean 0 and variance $\sigma^2/2$. Therefore, we obtain that
\[
\|K''_\sigma\|_2 = \left( \frac{3}{8\sqrt{\pi}\sigma^5} - \frac{1}{2\sqrt{\pi}\sigma^5} + \frac{1}{2\sqrt{\pi}\sigma^5} \right)^{1/2}
\] (250)
\[
= \frac{\sqrt{3}}{2\sqrt{2}\sqrt{\pi}\sigma^{5/2}},
\] (251)
as required.

The following lemma provides useful expressions for the $L^1$-norm of the third derivative of $K_\sigma$.

**Lemma 13.** If $K_\sigma(x) = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$, then $\|K'''_\sigma\|_1 \leq \frac{5\sqrt{2}}{\sqrt{\pi}\sigma^3}$.

**Proof.** It can be verified that, for all $x \in \mathbb{R}$,
\[
K'''_\sigma(x) = \frac{x^3 - 3\sigma^2 x}{\sigma^6} K'(x).
\] (252)
Thus, we have that
\[
\|K'''_\sigma\|_1 \leq \int_\mathbb{R} \left( \frac{|x|^3}{\sigma^6} + \frac{3|x|}{\sigma^4} \right) e^{-x^2/2\sigma^2} \frac{dx}{\sqrt{2\pi\sigma^2}}.
\] (253)
Note that the last integral is determined by the absolute moments of a Gaussian random variable with mean 0 and variance $\sigma^2$. Therefore,
\[
\|K'''_\sigma\|_1 \leq \frac{2\sqrt{2}}{\sqrt{\pi}\sigma^3} + \frac{3\sqrt{2}}{\sqrt{\pi}\sigma^3} = \frac{5\sqrt{2}}{\sqrt{\pi}\sigma^3},
\] (254)
as required.

In order to avoid cumbersome notation, we omit the superscript $\sigma$ when there is no risk of confusion, e.g., $g''_{\pm}$ is written as $g_{\pm}$ and $\theta^\sigma$ is written as $\theta$. The following corollary provides an upper bound for the $L^1$-norm of $\theta'$.

**Corollary 1.** If $f_{\pm}$ is a probability density function, then
\[
\|\theta'\|_1 \leq \frac{p_p}{\lambda_+} + \frac{p_-}{\lambda_-} \frac{1}{\sqrt{2\pi}\sigma}.
\] (255)

**Proof.** In Lemma 2 we prove that, for all $x \in \mathbb{R}$,
\[
2\theta'(x) = \frac{g'_+(x)}{g_+(x)} - \frac{g'_-(x)}{g_-(x)}.
\] (256)
By the triangle inequality, we have that
\[
2|\theta'(x)| \leq \left| \frac{g'_+(x)}{g_+(x)} \right| + \left| \frac{g'_-(x)}{g_-(x)} \right|
\] (257)
\[
\leq \left| \frac{g'_+(x)}{g_+(x)} \right| + \left| \frac{g'_-(x)}{g_-(x)} \right|,
\] (258)
where the last inequality follows trivially from (256). Thus,
\[
\|\theta'\|_1 \leq \frac{\|g'_+\|_1}{\lambda_+} + \frac{\|g'_-\|_1}{\lambda_-}.
\] (259)
From (259), it is immediate to see that $g'_+ = p_+(f_+ * K_\sigma)'$. Hence, the formula $(h_1 * h_2)' = h_1 * h_2'$ implies that
\[
\|\theta'\|_1 \leq \frac{p_+\|f_+ * K_\sigma\|_1}{2\lambda_+} + \frac{p_-\|f_- * K_\sigma'\|_1}{2\lambda_-}.
\] (260)
Recall that Young’s convolution inequality establishes that
\[
\|h_1 * h_2\|_1 \leq \|h_1\|_1 \|h_2\|_2,
\] (261)
whenever $1/r_1 + 1/r_2 = 1$. Hence, by taking $r = r_1 = r_2 = 1$,
\[
\|\theta'\|_1 \leq \frac{p_+\|f_+\|_1\|K_\sigma\|_1}{2\lambda_+} + \frac{p_-\|f_-\|_1\|K_\sigma'\|_1}{2\lambda_-}.
\] (262)
Since $f_{\pm}$ is a probability density function, we have that $\|f_{\pm}\|_1 = 1$. Therefore, Lemma 11 implies that
\[
\|\theta'\|_1 \leq \frac{p_+}{\lambda_+} + \frac{p_-}{\lambda_-} \frac{1}{\sqrt{2\pi}\sigma},
\] (263)
as required.

The following corollary provides an upper bound for the $L^1$-norm of $\theta''$.

**Corollary 2.** If $f_{\pm}$ is a probability density function, then
\[
\|\theta''\|_1 \leq \left( \frac{p_+}{\lambda_+} + \frac{p_-}{\lambda_-} \right) \frac{1}{\sigma^2} + \left( \frac{p^2_+}{\lambda^2_+} + \frac{p^2_-}{\lambda^2_-} \right) \frac{1}{8\sqrt{\pi}\sigma^3}.
\] (264)
Proof. In Lemma 2 we prove that, for all \(x \in \mathbb{R}\),
\[
2\theta''(x) = \left[ \frac{g''_\lambda(x)}{g_\lambda(x)} - \left( \frac{g_\lambda(x)}{g_\lambda(x)} \right)^2 \right] - \left[ \frac{g''_\lambda(x)}{g_-^\lambda(x)} - \left( \frac{g_-^\lambda(x)}{g_-^\lambda(x)} \right)^2 \right].
\]
(265)

By the triangle inequality, we have that
\[
2|\theta''(x)| \leq \frac{|g''_\lambda(x)|}{g_\lambda(x)} + \frac{|g''_\lambda(x)|^2}{g_\lambda(x)^2} + \frac{|g''_\lambda(x)|}{g_-^\lambda(x)} + \frac{|g''_\lambda(x)|^2}{g_-^\lambda(x)^2}
\]
(266)
\[
\leq \frac{2}{\lambda_+^*} + \frac{2}{\lambda_-^*},
\]
(267)
where the last inequality follows trivially from (236). Thus,
\[
\|\theta''\|_1 \leq \frac{2}{\lambda_+^*} + \frac{2}{\lambda_-^*}.
\]
(268)

From (236), it is immediate to see that \(g''_\lambda = p_+(f_\sigma + K_\sigma)''\). Hence, the formula \((h_1 * h_2)'' = h_1 * h_2''\) implies that
\[
\|g''_\lambda\|_1 = p_+ \|f_\sigma * K_\sigma\|_1 \leq p_+ \|f_\sigma\|_1 \|K_\sigma\|_1,
\]
(269)
where we applied Young’s convolution inequality (261) with \(r = r_1 = r_2 = 1\). Since \(f_\pm\) is a probability density function, we have that \(\|f_\pm\|_1 = 1\). Therefore, Lemma 12 implies that
\[
\|g''_\lambda\|_1 \leq \frac{2p_+}{\sigma^2}.
\]
(270)

Similarly, we have that
\[
\|g'_{\lambda_\pm} \|^2 \leq \frac{p_+^2}{4\sqrt{\pi \sigma^3}}.
\]
(271)

where we applied Young’s convolution inequality (261) with \(r = r_2 = 2\) and \(r_1 = 1\). Thus, Lemma 11 implies that
\[
\|g'_{\lambda_\pm} \|^2 \leq \frac{p_+^2}{4\sqrt{\pi \sigma^3}}.
\]
(272)

By plugging (270) and (272) in (268), we conclude that
\[
\|\theta''\|_1 \leq \frac{(p_+ + p_-)}{\lambda_+^*} \frac{1}{\sigma^2} + \frac{(p_+^2 + p_-^2)}{\lambda_\pm^*} \frac{1}{\lambda_\pm^*} \frac{1}{\sqrt{8\sqrt{\pi \sigma^3}}},
\]
(273)
as required.

The following corollary provides an upper bound for the \(L^1\)-norm of \(\theta''\).

Corollary 3. If \(f_\pm\) is a probability density function, then
\[
\|\theta''\|_1 \leq \frac{(p_+ + p_-)}{\lambda_+^*} \frac{5}{\sqrt{2\pi \sigma^3}} + \frac{3\sqrt{3}}{8\sqrt{\pi \sigma^4}} \frac{p_+^3}{\lambda_\pm^*} + \frac{p_-^3}{\lambda_\pm^*} \frac{\sqrt{2}}{9\sqrt{\pi \sigma^5}},
\]
(274)

Proof. In Lemma 2 we prove that
\[
2\theta'' = \left[ \frac{g''_\lambda}{g_\lambda} - 3 \frac{g_\lambda g''_\lambda}{g_\lambda^2} + 2 \left( \frac{g_\lambda}{g_\lambda} \right)^3 \right] - \left[ \frac{g''_\lambda}{g_-^\lambda} - 3 \frac{g_\lambda g''_\lambda}{g_-^\lambda} + 2 \left( \frac{g_\lambda}{g_-^\lambda} \right)^3 \right].
\]
(275)

By the triangle inequality, we have that
\[
2|\theta''| \leq \frac{|g''_\lambda|}{g_\lambda} + \frac{3|g_\lambda g''_\lambda|}{g_\lambda^2} + \frac{2|g_\lambda^3|}{g_\lambda^3} \leq \frac{|g''_\lambda|}{g_-^\lambda} + \frac{3|g_\lambda g''_\lambda|}{g_-^\lambda} + \frac{2|g_\lambda^3|}{g_-^\lambda}
\]
(276)
\[
\leq \frac{2|g''_\lambda|}{\lambda_+^*} + \frac{2|g_\lambda^3|}{\lambda_+^*} \frac{2|g_\lambda^3|}{\lambda_\pm^*} + \frac{2|g_\lambda^3|}{\lambda_\pm^*},
\]
(277)
where the last inequality follows trivially from (236). Thus,
\[
\|\theta''\|_1 \leq \frac{2|g''_\lambda|}{2\lambda_+^*} + \frac{3|g_\lambda g''_\lambda|}{2\lambda_+^*} + \frac{2|g_\lambda^3|}{2\lambda_\pm^*}
\]
(278)

From (236), it is immediate to see that \(g''_\lambda = p_+(f_\sigma + K_\sigma)''\). Hence, the formula \((h_1 * h_2)'' = h_1 * h_2''\) implies that
\[
\|g''_\lambda\|_1 = p_+ \|f_\sigma * K_\sigma\|_1 \leq p_+ \|f_\sigma\|_1 \|K_\sigma\|_1,
\]
(279)
where we applied Young’s convolution inequality (261) with \(r = r_1 = r_2 = 1\). Since \(f_\pm\) is a probability density function, we have that \(\|f_\pm\|_1 = 1\). Therefore, Lemma 11 implies that
\[
\|g''_\lambda\|_1 \leq \frac{5\sqrt{2p_+}}{\sqrt{\pi \sigma^3}}.
\]
(280)

By Hölder’s inequality, we observe that
\[
\|g'_{\lambda_\pm} \|_2 \leq \|g'_{\lambda_\pm} \|_2 \leq \|g'_{\lambda_\pm} \|_2 \leq \|g'_{\lambda_\pm} \|_2 \leq \frac{\sqrt{3}p_+}{4\sqrt{2\pi \sigma^3}},
\]
(281)
as before, we have that
\[
\|g'_{\lambda_\pm} \|_2 = \frac{\sqrt{3}p_+}{4\sqrt{2\pi \sigma^3}},
\]
(282)
where we applied Young’s inequality (261) with \(r = r_2 = 2\) and \(r_1 = 1\). Thus, Lemma 12 implies that
\[
\|g'_{\lambda_\pm} \|_2 \leq \frac{\sqrt{3}p_+}{4\sqrt{2\pi \sigma^3}},
\]
(283)

The previous inequality and (272) lead to
\[
\|g'_{\lambda_\pm} \|_1 \leq \frac{\sqrt{3}p_+}{4\sqrt{2\pi \sigma^3}},
\]
(284)

Finally, we have that
\[
\|g'_{\lambda_\pm} \|^3 \leq \frac{\sqrt{3}p_+}{4\sqrt{2\pi \sigma^3}},
\]
(285)

where we applied Young’s convolution inequality (261) with \(r = r_2 = 3\) and \(r_1 = 1\). Thus, Lemma 11 implies that
\[
\|g'_{\lambda_\pm} \|^3 \leq \frac{\sqrt{2p_+^3}}{9\sqrt{\pi \sigma^5}},
\]
(286)

By plugging (280), (284) and (286) in (278), we conclude that
\[
\|\theta''\|_1 \leq \frac{(p_+ + p_-)}{\lambda_+^*} \frac{5}{\sqrt{2\pi \sigma^3}} + \frac{3\sqrt{3}}{8\sqrt{\pi \sigma^4}} \frac{p_+^3}{\lambda_\pm^*} + \frac{p_-^3}{\lambda_\pm^*} \frac{\sqrt{2}}{9\sqrt{\pi \sigma^5}},
\]
(287)
as required.
Now we are in position to prove Theorem 6. 

Proof of Theorem 6: By Theorem 5, we have that

\[
C_\theta \leq \frac{2\sqrt{2}}{\sqrt{\pi}} \left( 1 + \frac{1}{2} \log \left( \frac{\|\theta''\|_1}{\|\theta''\|_1} \right) \right) \|	heta''\|_1. 
\]

(288)

Recall the definition of \( N_\alpha \) in (139) – (141). Corollaries 1–5 imply that \( \|\theta''\|_1 \leq N_\alpha \) for every \( \alpha \in \{1, 2, 3\} \). As a result,

\[
C_\theta \leq \frac{2\sqrt{2}}{\sqrt{\pi}} \left( 1 + \frac{1}{2} \log \left( N_1 N_3 \right) \right) N_2. 
\]

(289)

It is straightforward to verify that \( z \rightarrow -\log(z)z \) is increasing over \([0, 1/e]\). Thus, if \( N_2 \leq 1/e \), (289) implies that

\[
C_\theta \leq \frac{2\sqrt{2}}{\sqrt{\pi}} \left( 1 + \frac{1}{2} \log \left( \frac{N_1 N_3}{N_2} \right) \right). 
\]

(291)

as required. □

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