The superradiant instability regime of the spinning Kerr black hole

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Spinning Kerr black holes are known to be superradiantly unstable to massive scalar perturbations. We here prove that the instability regime of the composed Kerr-black-hole-massive-scalar-field system is bounded from above by the dimensionless inequality

\[ M\mu < m \cdot \sqrt{\frac{2(1+\gamma)(1-\sqrt{1-\gamma^2})}{4\gamma^2}}, \]

where \( \{\mu, m\} \) are respectively the proper mass and azimuthal harmonic index of the scalar field and \( \gamma \equiv r_-/r_+ \) is the dimensionless ratio between the horizon radii of the black hole. It is further shown that this analytically derived upper bound on the superradiant instability regime of the spinning Kerr black hole agrees with recent numerical computations of the instability resonance spectrum.

I. INTRODUCTION

The intriguing physical mechanism of superradiance [1–3] allows an incident bosonic wave field to extract rotational energy from a spinning Kerr black hole. In particular, a scalar field mode of azimuthal harmonic index \( m \) can be amplified (that is, can gain energy) as it scatters off a Kerr black hole if its proper frequency \( \omega_{\text{field}} \) lies in the bounded regime [1–4]

\[ 0 < \omega_{\text{field}} < m\Omega_H, \]  

where

\[ \Omega_H = \frac{a}{r_+^2 + a^2} \]  

is the angular velocity of the spinning Kerr black hole (Here \( a \) and \( r_+ \) are respectively the angular momentum per unit mass and the outer horizon-radius of the Kerr black hole).

What is even more remarkable is the fact that the rate of energy extraction from the spinning Kerr black hole in the superradiant regime (1) can grow exponentially in time if the scattered scalar wave field, which is used to extract the black-hole rotational energy, is prevented from radiating its energy to infinity. Interestingly, the Klein-Gordon wave equation for a scalar field of mass \( \mu \) [8–11] in the Kerr black-hole spacetime is governed by an effective binding potential [see Eqs. (18) and (19) below] which provides a natural confinement mechanism that prevents low frequency field modes in the regime

\[ 0 < \omega_{\text{field}} < \mu \]  

from escaping to infinity. Scalar field modes which respect the inequalities (1) and (3) in the rotating Kerr black-hole spacetime can grow exponentially over time [3], thus leading to the formation of a composed Kerr-black-hole-massive-scalar-field bomb [12, 13].

The boundary between stable (\( \omega > m\Omega_H \)) and unstable (\( \omega < m\Omega_H \)) composed Kerr-black-hole-massive-scalar-field systems is marked by the presence of stationary field configurations whose orbital frequencies are in resonance with the angular velocity \( \Omega_H \) of the spinning black hole [2, 10]. Specifically, for a given value of the field azimuthal harmonic index \( m \), these marginally-stable (stationary) bound-state field configurations are characterized by the resonance relation [9, 10]

\[ \omega_{\text{field}} = \omega_c \equiv m\Omega_H, \]  

where \( \omega_c \) is the critical (threshold) frequency for superradiant scattering in the Kerr black-hole spacetime.

It was previously proved [14, 15] that, for a scalar field of proper mass \( \mu \) interacting with a spinning Kerr black hole of angular velocity \( \Omega_H \), the inequality

\[ \mu < \sqrt{2} \cdot m\Omega_H \]  

provides an upper bound on the domain of existence of stationary Kerr-black-hole-massive-scalar-field configurations. Since these stationary (marginally-stable) field configurations mark the boundary between stable and unstable Kerr-massive-scalar-field systems, the relation (5) also provides an upper bound on the superradiant instability regime of the composed Kerr-black-hole-massive-scalar-field system.
The main goal of the present paper is to derive a stronger upper bound on the superradiant instability regime of the spinning Kerr black-hole spacetime \cite{16}. In particular, below we shall show that the binding potential well, which is required in order to support the stationary (marginally-stable) scalar field configurations \cite{14} in the rotating Kerr black-hole spacetime, exists only in a restricted regime $\mu/m_\Omega < F(\gamma)$ \cite{17} of the black-hole-field physical parameters. Since this inequality sets an upper bound on the domain of existence of these marginally-stable (stationary \cite{16}) field configurations in the rotating Kerr black-hole spacetime, it also sets an upper bound on the superradiant instability regime of the composed Kerr-black-hole-massive-scalar-field system.

II. DESCRIPTION OF THE SYSTEM

We shall study the dynamics of a massive scalar field $\Psi$ which is linearly coupled to a spinning Kerr black hole. The black-hole spacetime is described by the line element \cite{5, 6}

$$ds^2 = -\frac{\Delta}{\rho^2}(dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} \left[ a dt - (r^2 + a^2) d\phi \right]^2 ,$$

(6)

where $(t, r, \theta, \phi)$ are the Boyer-Lindquist coordinates, \{M, a\} are the mass and angular momentum per unit mass of the black hole, and

$$\Delta \equiv r^2 - 2Mr + a^2 ; \quad \rho^2 \equiv r^2 + a^2 \cos^2 \theta .$$

(7)

The zeros of $\Delta$,

$$r_{\pm} = M \pm \sqrt{M^2 - a^2} ,$$

(8)

are the (outer and inner) horizon radii of the spinning black hole.

The dynamics of a linearized scalar field $\Psi$ of proper mass $\mu$ in the black-hole spacetime is governed by the Klein-Gordon wave equation

$$(\nabla^\nu \nabla_\nu - \mu^2)\Psi = 0 .$$

(9)

One can decompose the eigenfunction $\Psi$ of the massive scalar field in the form \cite{18}

$$\Psi(t, r, \omega, \theta, \phi) = \sum_{l,m} e^{im\phi} S_{lm}(\theta; m, a\sqrt{\mu^2 - \omega^2}) R_{lm}(r; M, a, \mu, \omega) e^{-i\omega t} .$$

(10)

Substituting (10) into the Klein-Gordon wave equation (9), one finds that the angular function $S_{lm}$ satisfies the angular equation \cite{19–24}

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS_{lm}}{d\theta} \right) + \left[ K_{lm} + a^2 (\mu^2 - \omega^2) \sin^2 \theta - \frac{m^2}{\sin^2 \theta} \right] S_{lm} = 0 .$$

(11)

Demanding the angular functions to be regular at the two poles $\theta = 0$ and $\theta = \pi$, one finds that the differential equation (11) is characterized by a discrete set $\{K_{lm}\}$ of angular eigenvalues (see \cite{23, 27} and references therein). Below we shall use the fact that the characteristic eigenvalues of the angular equation (11) are bounded from below by the relation \cite{27, 28}

$$K_{lm} \geq m^2 - a^2 (\mu^2 - \omega^2) .$$

(12)

The radial function $R_{lm}$ satisfies the radial equation \cite{19, 20}

$$\Delta \frac{d}{dr} \left( \frac{dR_{lm}}{dr} \right) + \left\{ \omega(r^2 + a^2) - ma^2 + \Delta [2ma\omega - \mu^2 (r^2 + a^2) - K_{lm}] \right\} R_{lm} = 0 .$$

(13)

It is worth noting that the angular eigenvalues $\{K_{lm}\}$ couple equation (13) for the radial eigenfunctions to equation (11) for the angular eigenfunctions \cite{29}. The radial equation (13) should be supplemented by the physical boundary condition of purely ingoing waves (as measured by a comoving observer) at the horizon of the black hole \cite{8, 10}:

$$R_{lm} \sim e^{-i(\omega - m\Omega)y} \quad \text{for} \quad r \to r_+ \quad (y \to -\infty) ,$$

(14)
where the radial coordinate $y$ is determined by the relation $dy = (r^2/\Delta)dr$ [see Eq. (17) below]. In addition, the asymptotic (large-$r$) behavior

$$R_{lm} \sim \frac{1}{r} e^{-\sqrt{\mu^2-\omega^2}y} \quad \text{for} \quad r \to \infty \quad (y \to \infty)$$

(15)
of the radial eigenfunction, together with the characteristic inequality [8], guarantee that the external bound-state configurations of the massive scalar fields are characterized by spatially decaying (bounded) radial eigenfunctions at asymptotic infinity.

### III. THE EFFECTIVE BINDING POTENTIAL OF THE COMPOSED KERR-BLACK-HOLE-MASSIVE-SCALAR-FIELD SYSTEM

Our main goal is to obtain an upper bound on the domain of existence of the stationary (marginally-stable) Kerr-black-hole-massive-scalar-field configurations [16]. To this end, it proves useful to transform the radial equation (13) into a Schrödinger-like wave equation. Substituting

$$\psi = rR$$

and

$$dy = \frac{r^2}{\Delta}dr$$

into the radial equation (13), one obtains the Schrödinger-like wave equation

$$\frac{d^2\psi}{dy^2} - V(y)\psi = 0,$$

(18)

where the effective potential which governs the radial equation (18) is given by

$$V = V(r; \omega, M, a, \mu, l, m) = \frac{2\Delta}{r^6}(Mr - a^2) + \frac{\Delta}{r^4}[K_{lm} - 2ma\omega + \mu^2(r^2 + a^2)] - \frac{1}{r^4}[(\omega(r^2 + a^2) + ma)^2].$$

(19)

Note that this radial potential is characterized by the asymptotic properties [see Eqs. (2), (4), and (19)]

$$V(r = r_+; \omega = \omega_c, M, a, \mu, l, m) = 0$$

(20)

and

$$V(r = r_+; \omega = \omega_c, M, a, \mu, l, m) \to \mu^2 - \omega_c^2 > 0$$

(21)
at the black-hole horizon and at spatial infinity, respectively.

In the next section we shall analyze the spatial properties of the effective radial potential (19) for the stationary bound-state configurations of the massive scalar fields in the rotating Kerr black-hole spacetime (these marginally stable field configurations are characterized by the critical (threshold) superradiant frequency [4]). In particular, we shall show that the requirement that the effective radial potential (19) has the form of a binding potential well outside the black-hole horizon sets an upper bound on the mass of the explosive scalar field in the superradiant regime [1].

### IV. AN UPPER BOUND ON THE MASS OF THE EXPLOSIVE SCALAR FIELD

In the present section we shall study the spatial behavior of the effective radial potential (19) which governs the interaction of the stationary (marginally-stable) massive scalar configurations with the rotating Kerr black hole. Substituting the characteristic resonant frequency (4) of the stationary scalar fields into the expression (19) for the effective radial potential, and using the inequality (12) for the eigenvalues of the angular equation (11), one obtains the inequality

$$V(r) \geq m^2 \cdot \frac{r - r_+}{r^3(r_+^2 + a^2)^2} \left[\beta a^2 r^2 - a^2(2M + \beta r_-)r + 2Mr_+^2\right] + \frac{2\Delta}{r^6}(Mr - a^2)$$

(22)
for the effective radial potential which characterizes the stationary Kerr-black-hole-massive-scalar-field configurations, where the dimensionless parameter $\beta > 0$ \cite{31} is defined by the relation

$$\mu^2 = (1 + \beta) \cdot \omega^2 . \quad (23)$$

Furthermore, substituting the inequality $Mr - a^2 \geq 0 \cite{32}$ into (22), one finds the lower bound

$$V(r) \geq m^2 \cdot \frac{r - r_+}{r(r_+ + a)^2} \left[ \beta a^2 r^2 - a^2(2M + \beta r_+)r + 2Mr^3_+ \right] \quad (24)$$
on the effective radial potential.

A necessary condition for the existence of the stationary (marginally-stable) bound-state scalar configurations in the rotating Kerr black-hole spacetime is provided by the requirement that the effective radial potential \cite{19}, which characterizes the composed Kerr-black-hole-massive-scalar-field system, has the form of a binding potential well. In particular, taking cognizance of the property \cite{20} which characterizes the effective radial potential of the composed black-hole-field system, one concludes that the inequality

$$V(r) \leq 0 \quad \text{for} \quad r \in [r^-_b, r^+_b] \quad (25)$$

provides a necessary condition for the existence of stationary bound-state scalar configurations in the Kerr black-hole spacetime. The inequality (25) reflects the fact that, in order to be able to support stationary bound-state scalar configurations outside the black-hole horizon, the effective radial potential \cite{19} of the composed black-hole-field system must have the form of a binding potential well in some interval $r_+ \leq r^-_b \leq r \leq r^+_b$ outside the black-hole horizon.

Taking cognizance of Eqs. (24) and (25), one finds the characteristic inequality

$$\beta a^2 r^2 - a^2(2M + \beta r_+)r + 2Mr^3_+ \leq 0 \quad (26)$$
in the interval $r \in [r^-_b, r^+_b]$ which characterizes the binding potential well outside the black-hole horizon. The zeros of the quadratic function on the l.h.s of (26) are given by

$$r^\pm_b = \frac{a(2M + \beta r_+) \pm \sqrt{a^2(2M + \beta r_+)^2 - 8\beta Mr^3_+}}{2\beta a} . \quad (27)$$
The requirements $r^\pm_b \in \mathbb{R}$ with $r_+ \leq r^-_b \leq r^+_b$ yield the relation

$$a^2 r_+^2 \cdot \beta^2 + 4M(a^2 r_+ - 2r^3_+) \cdot \beta + 4M^2a^2 > 0 \quad (28)$$
as a necessary condition for the validity of the inequality \cite{20}. From (28) one finds the upper bound

$$\beta \leq \frac{2M \left[2r^2_+ - r^-_+ - 2r_+ (r^2_+ - r^2_-)^{1/2} \right]}{r^3_+} \quad (29)$$
on the dimensionless quantity $\beta$.

Finally, taking cognizance of Eqs. (4), (23), and (29), and defining the dimensionless ratio

$$\gamma \equiv \frac{r_-}{r_+} \quad (30)$$
between the horizon radii of the spinning Kerr black hole, one finds the upper bound

$$\mu < \mathcal{F}(\gamma) \cdot m\Omega_H \quad (31)$$
on the scalar mass of the stationary (marginally-stable) bound-state field configurations, where the dimensionless function $\mathcal{F} = \mathcal{F}(\gamma)$ is given by

$$\mathcal{F}(\gamma) = \sqrt{\frac{2(1 + \gamma)(1 - \sqrt{1 - \gamma^2}) - \gamma^2}{\gamma^3}} . \quad (32)$$

It is worth emphasizing again that the stationary bound-state field configurations \cite{41} mark the onset of the superradiant instabilities in the composed Kerr-black-hole-massive-scalar-field system. Thus, the analytically derived upper bound (31) on the domain of existence of these stationary (marginally-stable) black-hole-field configurations also provides an upper bound on the superradiant instability regime of the composed Kerr-black-hole-massive-scalar-field system \cite{33}.
V. NUMERICAL CONFIRMATION

It is of physical interest to test the validity of the analytically derived upper bound (31) on the superradiant instability regime of the composed Kerr-black-hole-massive-scalar-field system. As emphasized earlier, the boundary between stable and unstable composed black-hole-field systems is marked by the stationary (marginally-stable) black-hole-field configurations studied in [9, 10]. In particular, the scalar field masses \( \mu = \mu(m, \Omega_H) \) which correspond to these stationary (marginally-stable) composed Kerr-black-hole-massive-scalar-field configurations were computed numerically in [10].

In Table I we present the dimensionless ratio \( \mu_{\text{numerical}}/\mu_{\text{bound}} \), where \( \mu_{\text{numerical}} \) is the numerically computed [10] field masses which mark the onset of the superradiant instabilities in the composed Kerr-black-hole-massive-scalar-field system, and \( \mu_{\text{bound}} \) is the analytically derived upper bound (31) on the superradiant instability regime of the composed black-hole-field system. One finds from Table I that the superradiant instability regime of the composed Kerr-black-hole-massive-scalar-field system is characterized by the relation \( \mu_{\text{numerical}}/\mu_{\text{bound}} < 1 \), in agreement with the analytically derived upper bound (31).

| \( s \equiv a/M \) | \( F(s) \) | \( \mu(l=m=1)/\mu_{\text{bound}} \) | \( \mu(l=m=10)/\mu_{\text{bound}} \) |
|---|---|---|---|
| 0.1 | 1.00031 | 0.99977 | 0.99940 |
| 0.2 | 1.00129 | 0.99903 | 0.99967 |
| 0.3 | 1.00301 | 0.99774 | 0.99948 |
| 0.4 | 1.00567 | 0.99573 | 0.99901 |
| 0.5 | 1.00960 | 0.99276 | 0.99776 |
| 0.6 | 1.01541 | 0.98835 | 0.99715 |
| 0.7 | 1.02437 | 0.98163 | 0.99938 |
| 0.8 | 1.03955 | 0.96995 | 0.99276 |
| 0.9 | 1.07168 | 0.94694 | 0.98676 |
| 0.95 | 1.11039 | 0.91878 | 0.97942 |
| 0.99 | 1.21646 | 0.84910 | 0.96165 |
| 0.999 | 1.37370 | 0.76084 | 0.94280 |

TABLE I: The superradiant instability regime of the composed Kerr-black-hole-massive-scalar-field system (the rotating black-hole bomb). We present the dimensionless ratio \( \mu_{\text{numerical}}/\mu_{\text{bound}} \), where \( \mu_{\text{numerical}} \) is the numerically computed [10] field masses which mark the onset of the superradiant instabilities in the composed Kerr-black-hole-massive-scalar-field system, and \( \mu_{\text{bound}} \) is the analytically derived upper bound on the superradiant instability regime of the composed system as given by Eq. (31). The data presented are for the cases \( l = m = 1 \) and \( l = m = 10 \). One finds that the superradiant instability regime of the composed Kerr-black-hole-massive-scalar-field system is characterized by the relation \( \mu_{\text{numerical}}/\mu_{\text{bound}} < 1 \), in agreement with the analytically derived upper bound (31).

VI. SUMMARY

In this paper we have explored the superradiant instability regime of the spinning Kerr black hole to massive scalar perturbations. In particular, we have shown that the binding potential well, which is required in order to support the stationary [16] bound-state scalar field configurations (4) in the rotating black-hole spacetime, exists only in the restricted regime [see Eqs. (31) and (32)]

\[
M \mu < m \cdot \sqrt{\frac{2(1 + \gamma)(1 - \sqrt{1 - \gamma^2}) - \gamma^2}{4 \gamma^2}} \quad ; \quad \gamma \equiv r_-/r_+ .
\]

(33)

The dimensionless inequality (33) sets an upper bound on the domain of existence of the stationary bound-state scalar field configurations (4) in the rotating Kerr black-hole spacetime. Since these marginally-stable (stationary) bound-state scalar field configurations (34) mark the boundary between stable (\( \omega > m \Omega_H \)) and unstable (\( \omega < m \Omega_H \)) field configurations in the Kerr black-hole spacetime, the analytically derived inequality (33) also provides an upper bound on the superradiant instability regime (35) of the composed Kerr-black-hole-massive-scalar-field system (36).
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It is worth noting that the lower bound (12) on the angular eigenvalues can be saturated in the eikonal \( l = m \gg 1 \) limit, in which case one finds the simple relation \( K_{mn} = m^2[1 + O(m^{-1})] - a^2(\mu^2 - \omega^2) \) [28].

Note, in particular, that the eigenvalues \( \{K_{lm}\} \) which characterize the angular equation (11) also appear in the effective potential of the radial Klein-Gordon equation (13).

Note that the coordinate \( r \in [0, \infty] \) is mapped into \( y \in [-\infty, +\infty] \) by the radial transformation (14).

This relation follows from the series of inequalities \( Mr - a^2 \geq Mr_+ - a^2 = r_+(M - r_-) \geq 0 \) for \( r \geq r_+ \).

It is interesting to note that the newly derived upper bound (31) on the mass of the explosive scalar field is stronger than the bound (33) in the entire regime \( a \lesssim 0.999481 \).

It is worth emphasizing again that these stationary (marginally-stable) scalar field configurations are characterized by the marginal (critical) frequency \( \omega_{\text{field}} = \omega_c \equiv m\Omega H \) for superradiant scattering in the Kerr black-hole spacetime [see Eq. (4)].

That is, the inequality (33) provides a necessary condition for the development of the superradiant instabilities in the composed Kerr-black-hole-massive-scalar-field system.

It is worth noting that the expression on the r.h.s of (33) not only bounds the superradiant instability regime of the composed Kerr-black-hole-massive-scalar-field system, but it also predicts with a fairly good accuracy the exact (numerically computed) scalar field masses which mark the onset of these superradiant instabilities. For instance, from the data presented in Table I alone learns that, for the \( l = m = 1 \) mode, the expression on the r.h.s of (33) agrees with the numerically computed scalar field masses which mark the onset of the superradiant instabilities to within ~ 5% in the regime \( a/M \lesssim 0.9 \). The agreement becomes even better for the higher scalar modes. For instance, for the \( l = m = 10 \) mode, the expression on the r.h.s of (33) agrees with the numerically computed scalar field masses which mark the onset of the superradiant instabilities to within ~ 2% in the regime \( a/M \lesssim 0.95 \).