Robust Proximity Search for Balls using Sublinear Space

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Abstract

Given a set of \( n \) disjoint balls \( b_1, \ldots, b_n \) in \( \mathbb{R}^d \), we provide a data-structure, of near linear size, that can answer \((1 \pm \varepsilon)\)-approximate \( k \)th-nearest neighbor queries in \( O(\log n + 1/\varepsilon^d) \) time, where \( k \) and \( \varepsilon \) are provided at query time. If \( k \) and \( \varepsilon \) are provided in advance, we provide a data-structure to answer such queries, that requires (roughly) \( O(n/k) \) space; that is, the data-structure has sublinear space requirement if \( k \) is sufficiently large.

1. Introduction

The nearest neighbor problem is a fundamental problem in Computer Science [SDI06, Cha08, AI08, Cla06]. Here, one is given a set of points \( P \), and given a query point \( q \) one needs to output the nearest point in \( P \) to \( q \). While there is a trivial \( O(n) \) algorithm for this problem, typically the set of data points is fixed, while different queries keep arriving. Thus, one can use preprocessing to facilitate a faster query. Applications of nearest neighbor search include pattern recognition [FH49, CH67], self-organizing maps [Koh01], information retrieval [SWY75], vector compression [GG91], computational statistics [DW82], clustering [DHS01], data mining, learning, and many others. If one is interested in guaranteed performance and near linear space, there is no known way to solve this problem efficiently (i.e., logarithmic query time) for dimension \( d > 2 \), while using near linear space for the data-structure.

Approximate Nearest Neighbor (ANN). In light of the above, major effort has been devoted to develop approximation algorithms for nearest neighbor search [AMN+98, IM98, KOR00, SDI06, Cha08, AI08, Cla06, HIM12]. In the \((1 + \varepsilon)\)-approximate nearest neighbor problem, one is additionally given an approximation parameter \( \varepsilon > 0 \) and one is required to find a point \( u \in P \) such that \( d(q, u) \leq (1 + \varepsilon)d(q, P) \). In \( d \) dimensional Euclidean space, one can answer ANN queries, in \( O(\log n + 1/\varepsilon^{d-1}) \) time using linear space [AMN+98, Har11]. Unfortunately, the constant hidden in the \( O \) notation is exponential in the dimension (and this is true for all bounds mentioned in this paper), and specifically because of the \( 1/\varepsilon^{d-1} \) in the query time, this approach is only efficient in low dimensions. Interestingly, for this data-structure, the approximation parameter \( \varepsilon \) need not be specified during the construction, and one can provide it during the query. An alternative approach is to use Approximate Voronoi Diagrams (AVD),

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introduced by Har-Peled [Har01], which is a partition of space into regions of low total complexity, with a representative point for each region, that is an ANN for any point in the region. In particular, Har-Peled showed that there is such a decomposition of size $O\left(\frac{n}{\varepsilon^d} \log^2 n\right)$, see also [HIM12]. This allows ANN queries to be answered in $O(\log n)$ time. Arya and Malamatos [AM02] showed how to build AVDs of linear complexity (i.e., $O(n/\varepsilon^d)$). Their construction uses WSPD (Well Separated Pair Decomposition) [CK95]. Further trade-offs between query time and space usage for AVDs were studied by Arya et al. [AMM09].

$k$-nearest neighbors. A more general problem is the $k$-nearest neighbors problem where one is interested in finding the $k$ points in $P$ nearest to the query point $q$. This is widely used in classification, where the majority label is used to label the query point. A restricted version is to find only the $k$th-nearest neighbor. This problem and its approximate version have been considered in [AMM05, HK12].

Sublinear space, summarizing data and $(k, \varepsilon)$-ANN. Recently, the authors [HK12] showed that one can compute a $(k, \varepsilon)$-AVD that $(1 + \varepsilon)$-approximates the distance the $k$th nearest neighbor, and surprisingly, requires $\tilde{O}(n/k)$ space; that is, sublinear space if $k$ is sufficiently large. For example, for the case $k = \Omega(\sqrt{n})$, which is of interest in practice, the space required is only $\tilde{O}(\sqrt{n})$. Such ANN is of interest when ones is worried that there is noise in the data, and thus one is interested in the distance to the $k$th NN which is sadistically more robust and noise resistant. Alternatively, one can think about such data-structures as enabling one to summarize the data in a way that still facilitates meaningful proximity queries.

This paper. In this paper we consider a generalization of the $k$th-nearest neighbor problem. Here, we are given a set of $n$ disjoint balls in $\mathbb{R}^d$ and we want to preprocess them, so that given a query point we can find approximately the $k$th closest ball. The distance of a query point to a ball is defined as the distance to its boundary if the point is outside the ball or 0 otherwise. Clearly, this problem is a generalization of the $k$th-nearest neighbor problem by viewing points as balls of radius 0. Algorithms for the $k$th-nearest neighbor for points, do not extend in a straightforward manner to this problem because the distance function is no longer a metric. Indeed, there can be two very far off points both very close to a single ball, and thus the triangle inequality does not hold. The problem of finding the closest ball can also be modeled as a problem of approximating the minimization diagram of a set of functions; here, a function would correspond to the distance from one of the given balls. There has been some recent work by the authors on this topic, see [HK13], where a fairly general class of functions admits a near-linear sized data structure permitting a logarithmic time query for the problem of approximating the minimization diagram. However, the problem that we consider in this paper does not fall under the framework of [HK13], due to technical reasons, even for $k = 1$. Also, we are interested in the more general $k$th-nearest neighbor problem.

Our results.

In this paper, we first show how preprocess the set of balls into a data structure requiring space $O(n)$, in $O(n \log n)$ time, so that given a query point $q$, a number $1 \leq k \leq n$ and $\varepsilon > 0$, one can compute a $(1 \pm \varepsilon)$-approximate $k$th closest ball in time $O(\log n + \varepsilon^{-d})$. If both $k$ and $\varepsilon$ are available during preprocessing, one can preprocess the balls into a $(k, \varepsilon)$-AVD, using $O\left(\frac{n}{k \varepsilon^d} \log(1/\varepsilon)\right)$ space, so that given a query point $q$, a $(k, \varepsilon)$-ANN closest ball can be computed, in $O(\log(n/k) + \log(1/\varepsilon))$ time.
Plan of attack, and highlights. Our plan of attack, is to try and extend our previous work [HK12] to the new more general setup. Since we are dealing with balls instead of points, the task is more challenging, and we do it in stages:

(A) **Approximate range counting on balls.** Given a set of disjoint balls, and a query ball, we want to count the number of input balls that intersect it, while allowing an approximation only for the query ball. This is somewhat more challenging than approximate range counting, as done by Arya and Mount [AM00], as some of the balls intersecting the query ball might be significantly larger. The this end, we build a data-structure that enable us to quickly count exactly the large input balls that intersect the query ball. This is described in Section 3.

(B) **Linear space \((k, \varepsilon)-\text{ANN on balls.}\)** Given a query point, we compute its distance to the \(i\)th nearest center, for \(i = k - c_d, \ldots, k\), where \(c_d\) is some constant that depends on the dimension. Next, we argue that either one of these distances is the required approximate distance (and this can be verified using the approximate range counting data-structure from above), or alternatively, the distance is determined by “huge” balls that have radius significantly larger than the desired distance. As such, we extract the at most \(c_d\) large balls that might be relevant, add their distance to the query point to the set of candidate distance, and search these distances. This yields a constant approximation to the \(k\)th \text{ANN}, and converting it to \((1 + \varepsilon)\)-approximation is easy using our tools. This is described in Section 4.

(C) **Quorum clustering for balls.** Somewhat oversimplifying things, the basic strategy in the previous work, was to find the point achieving the global minimum of the \(k\)th \text{ANN} distance function, approximate the function correctly in a region around this point, remove the points that define the minimum from the data-set and repeat. This was facilitated by finding the smallest ball that contains \(k\) input points. For balls, it is not clear how to find the smallest ball that intersects \(k\) balls, remove these balls, and repeat this process, and do it efficiently. Furthermore, it is no longer true that one can remove these \(k\) balls, as some of them might be huge. Instead, conceptually, we remove only the small balls (the exact details of what we do are more involved, and require significantly more care) from these \(k\) balls. Furthermore, instead of using the smallest ball intersecting \(k\) balls, we use the smallest ball containing \(k - c_d\) centers of the balls, and expand it so that it intersects \(k\) balls. We then repeat this mining process till all centers are excavated. Surprisingly, since the quorum clustering is done on the centers and not on the balls, we are able to implement this process efficiently, and furthermore, we can argue that it yields a meaningful quorum clustering for the input balls. This is described in Section 5.

(D) **Sublinear space \((k, \varepsilon)-\text{AVD on balls.}\)** Now, equipped with the new quorum clustering of the balls, we can build a \((k, \varepsilon)\)-\text{AVD} for the balls. Surprisingly, the construction now follows [HK12] in a straightforward fashion. The resulting data-structure uses \(\tilde{O}(n/k)\) space, and answer \((k, \varepsilon)\)-\text{ANN} queries in \(O(\log n)\) time.

Paper organization. In Section 2 we define the problem, list some assumptions, and introduce notations. In Section 3 we set up some basic data-structures to answer approximate range counting queries for balls. In Section 4 we present the data-structure, query algorithm and proof of correctness for our data-structure which can compute \((1 \pm \varepsilon)\)-approximate \(k\)th-nearest neighbors of a query point when \(k, \varepsilon\) are only provided during query time. In Section 6 we present the \((k, \varepsilon)\)-\text{AVD} construction. We conclude in Section 7.
2. Problem definition and notation

Input. We are given a set of disjoint\(^1\) balls \(\mathcal{B} = \{b_1, \ldots, b_n\}\), where \(b_i = b(c_i, r_i)\), for \(i = 1, \ldots, n\). Here \(b(c, r) \subseteq \mathbb{R}^d\) denotes the (closed) ball with center \(c\) and radius \(r \geq 0\). Additionally, we are given an approximation parameter \(\varepsilon \in (0, 1)\). For a point \(q \in \mathbb{R}^d\), the distance of \(q\) to a ball \(b = b(c, r)\) is \(d(q, b) = \max\left(\|q - c\| - r, 0\right)\).

Observation 2.1. For two balls \(b_1 \subseteq b_2 \subseteq \mathbb{R}^d\), and any point \(q \in \mathbb{R}^d\), we have \(d(q, b_1) \geq d(q, b_2)\).

The \(k\)th-nearest neighbor distance of \(q\) to \(\mathcal{B}\), denoted by \(d_k(q, \mathcal{B})\), is the \(k\)th smallest number in \(d(q, b_1), \ldots, d(q, b_n)\). Similarly, for a given set of points \(\mathcal{P}\), \(d_k(q, \mathcal{P})\) denotes the \(k\)th-nearest neighbor distance of \(q\) to \(\mathcal{P}\).

Problem definition. We aim to build a data-structure to answer \((1 \pm \varepsilon)\)-approximate \(k\)th-nearest neighbor (i.e., \((k, \varepsilon)\)-ANN) queries, where for any query point \(q \in \mathbb{R}^d\) one needs to output a ball \(b \in \mathcal{B}\) such that, \((1 - \varepsilon)d_k(q, \mathcal{B}) \leq d(q, b) \leq (1 + \varepsilon)d_k(q, \mathcal{B})\). There are different variants depending on whether \(\varepsilon\) and \(k\) are provided with the query or in advance.

Notations. We use cube to denote a square \((d = 2)\), a cube \((d = 3)\), or a hypercube \((d > 3)\), as the dimension might be.

Observation 2.2. For any set of balls \(\mathcal{B}\), the function \(d_k(q, \mathcal{B})\) is a 1-Lipschitz function; that is, for any two points \(u, v\), we have that \(d_k(u, \mathcal{B}) \leq d_k(v, \mathcal{B}) + \|u - v\|\).

Assumption 2.3. We assume all the balls are contained inside the cube \(\left[1/2 - \delta, 1/2 + \delta\right]^d\), which can be ensured by translation and scaling (which preserves order of distances), where \(\delta = \varepsilon/4\). As such, we can ignore queries outside the unit cube \([0, 1]^d\), as any input ball is a valid answer in this case.

For a real positive number \(x\) and a point \(p = (p_1, \ldots, p_d) \in \mathbb{R}^d\), define \(G_x(p)\) to be the grid point \((\lfloor p_1/x \rfloor x, \ldots, \lfloor p_d/x \rfloor x)\). The number \(x\) is the width or sidelength of the grid \(G_x\). The mapping \(G_x\) partitions \(\mathbb{R}^d\) into cubes that are called grid cells.

Definition 2.4. A cube is a canonical cube if it is contained inside the unit cube \(U = [0, 1]^d\), it is a cell in a grid \(G_r\), and \(r\) is a power of two (i.e., it might correspond to a node in a quadtree having \([0, 1]^d\) as its root cell). We will refer to such a grid \(G_r\) as a canonical grid. Note, that all the cells corresponding to nodes of a compressed quadtree are canonical.

Definition 2.5. Given a set \(b \subseteq \mathbb{R}^d\), and a parameter \(\delta > 0\), let \(G_\infty(b, \delta)\) denote the set of canonical grid cells of sidelength \(2^{-\lceil\log_2(\text{diam}(b)/\sqrt{d})\rceil}\), that intersect \(b\), where \(\text{diam}(b) = \max_{p, u \in b} \|p - u\|\) denotes the diameter of \(b\). Clearly, the diameter of any grid cell of \(G_\infty(b, \delta)\), is at most \(\delta\text{diam}(b)\). Let \(G_\infty(b) = G_\infty(b, 1)\). It is easy to verify that \(|G_\infty(b)| = O(1)\). The set \(G_\infty(b)\) is the grid approximation to \(b\).

Let \(\mathcal{B}\) be a family of balls in \(\mathbb{R}^d\). Given a set \(X \subseteq \mathbb{R}^d\), let

\[\mathcal{B}(X) = \{b \in \mathcal{B} \mid b \cap X \neq \emptyset\}\]

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\(^1\)Our data-structure and algorithm work for the more general case where the balls are interior disjoint, where we define the interior of a “point ball”, i.e., a ball of radius 0, as the point itself. This is not the usual topological definition.
denote the set of all balls in $B$ that intersect $X$.

For two compact sets $X, Y \subseteq \mathbb{R}^d$, $X \preceq Y$ if and only if $\text{diam}(X) \leq \text{diam}(Y)$. For a set $X$ and a set of balls $B$, let $B_\leq(X) = \{b \in B \mid b \cap X \neq \emptyset \text{ and } b \preceq X\}$. Let $c_d$ denote the maximum number of pairwise disjoint balls of radius at least $r$, that may intersect a given ball of radius $r$ in $\mathbb{R}^d$. It can be verified that $2 \leq c_d \leq 3^d$ for all $d$. Clearly, we have $|B_\leq(b)| \leq c_d$ for any ball $b$.

**Definition 2.6.** For a parameter $\delta \geq 0$, a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is $\delta$-**monotonic**, if for every $x \geq 0$, $f(x/(1 + \delta)) \leq f(x)$.

### 3. Approximate range counting for balls

**Data-structure 3.1.** For a given set of balls $B = \{b_1, \ldots, b_n\}$ in $\mathbb{R}^d$, we build the following data-structure, that is useful in performing several of the tasks at hand.

(A) **Store balls in a (compressed) quadtree.** For $i = 1, 2, \ldots, n$, let $G_i = G_\approx(b_i)$, and let $G = \bigcup_{i=1}^n G_i$ denote the union of these cells. Let $T$ be a compressed quadtree decomposition of $[0, 1]^d$, such that all the cells of $G$ are cells of $T$. We preprocess $T$ to answer point location queries. This takes $O(n \log n)$ time [Har11].

(B) **Compute list of “large” balls intersecting each cell.** For each node $u$ of $T$, there is a list of balls registered with it. Formally, **register** a ball $b_i$ with all the cells of $G_i$. Clearly, each ball is registered with $O(1)$ cells, and it is easy to see that each cell has $O(1)$ balls registered with it, since the balls are disjoint.

Next, for a cell $\square$ in $T$ we compute a list storing $B_\geq(\square)$, and these balls are **associated** with this cell. These lists are computed in a top-down manner. To this end, propagate from a node $u$ its list $B_\geq(\square)$ (which we assume is already computed) down to its children. For a node receiving such a list, it scans it, and keep only the balls that intersect its cell (adding to this list the balls already registered with this cell). For a node $\nu \in T$, let $B_\nu$ be this list.

(C) **Build compressed quadtree on centers of balls.** Let $C$ be the set of centers of the balls of $B$. Build, in $O(n \log n)$ time, a compressed quadtree $T_C$ storing $C$.

(D) **ANN for centers of balls.** Build a data structure $D$, for answering $2$-approximate $k$-nearest neighbor distances on $C$, the set of centers of the balls, see [HK12], where $k$ and $\varepsilon$ are provided with the query. The data structure $D$, returns a point $c \in C$ such that, $d_k(q, C) \leq d(q, c) \leq 2d_k(q, C)$.

(E) **Answering approximate range searching for the centers of balls.**

Given a query ball $b_q = b(q, x)$ and a parameter $\delta > 0$, one can, using $T_C$, report (approximately), in $O(\log n + 1/\delta^d)$ time, the points in $b_q \cap C$. Specifically, the query process computes $O(1/\delta^d)$ sets of points, such that their union $X$, has the property that $b_q \cap C \subseteq X \subseteq (1 + \delta)b_q \cap C$, where $(1 + \delta)b_q$ is the scaling of $b_q$ by a factor of $1 + \delta$ around its center. Indeed, compute the set $G_\approx(b_q)$, and then using cell queries in $T_C$ compute the corresponding cells (this takes $O(\log n)$ time). Now, descend to the relevant level of the quadtree to all the cells of the right size, that intersect $b_q$. Clearly, the union of points stored in their subtrees are the desired set. This takes overall $O(\log n + 1/\delta^d)$ time.

A similar data-structure for approximate range searching is provided by Arya and Mount [AM00], and our description above is provided for the sake of completeness.

Overall, it takes $O(n \log n)$ time to build this data-structure.
3.1. Approximate range counting among balls

We need the ability to answer approximate range counting queries on a set of balls. Specifically, given a set of balls $\mathcal{B}$, and a query ball $b$, the target is to compute the size of the set $b \cap \mathcal{B} = \{ b' \in \mathcal{B} \mid b \cap b' \neq \emptyset \}$. To make this query computationally fast, we allow an approximation. More precisely, for a ball $b$ a set $\tilde{b}$ is a $(1+\delta)$-ball of $b$, if $b \subseteq \tilde{b} \subseteq (1+\delta)b$, where $(1+\delta)b$ is the $(1+\delta)$-scaling of $b$ around its center. The purpose here, given a query ball $b$, is to compute the size of the set $b \cap \mathcal{B}$ for some $(1+\delta)$-ball $\tilde{b}$ of $b$. It turns out that this is challenging, as the query ball can be approximated, but the balls in $\mathcal{B}$ remain the same. This is to prevent the undesired situation where a “giant” ball is a valid answer for any ANN query.

3.1.1. Some useful tools

**Lemma 3.2.** Given a compressed quadtree $T$ of size $n$, a convex set $X$, and a parameter $\delta > 0$, one can compute the set of nodes in $T$, that realizes $G_{\approx}(X, \delta)$ (see Definition 2.5), in $O\left(\log n + 1/\delta^d\right)$ time. Specifically, this output a set $X_N$ of nodes, of size $O\left(1/\delta^d\right)$, such that their cells intersect $G_{\approx}(X, \delta)$, and their parents cell diameter is larger than $\delta \text{diam}(X)$. Note that the cells in $X_N$ might be significantly larger if they are leaves of $T$.

**Proof:** Let $G_{\approx} = G_{\approx}(X, 1)$ be the grid approximation to $X$. Using cell queries on the compressed quadtree, one can compute the cells of $T$ that corresponds to these canonical cells. Specifically, for each cube $\square \in G_{\approx}(X)$, the query either returns a node for which this is its cell, or it returns a compressed edge of the quadtree; that is, two cells (one is a parent of the other), such that $\square$ is contained in of them and contains the other. Such a cell query takes $O(\log n)$ time [Har11]. This returns $O(1)$ nodes in $T$ such that their cells cover $G_{\approx}(X)$.

Now, traverse down the compressed quadtree starting from these nodes and collect all the nodes of the quadtree that are relevant. Clearly, one has to go at most $O(\log 1/\delta)$ levels down the quadtree to get these nodes, and this takes $O(1/\delta^d)$ time overall. $\blacksquare$

**Lemma 3.3.** Let $X$ be any convex set in $\mathbb{R}^d$, and let $\delta > 0$ be a parameter. Using DS$_3.1$, one can compute, in $O\left(\log n + 1/\delta^d\right)$ time, all the balls of $\mathcal{B}$ that intersects $X$, with diameter $\geq \delta \text{diam}(X)$.

**Proof:** We compute the cells of the quadtree realizing $G_{\approx}(X, \delta)$ using Lemma 3.2. Now, from each such cell (and its parent), we extract the list of large balls intersecting it (there are $O(1/\delta^d)$ such nodes, and the size of each such list is $O(1)$). Next we check for each such ball if it intersects $X$ and if its diameter is at least $\delta \text{diam}(X)$. We return the list of all such balls. $\blacksquare$

3.1.2. Answering a query

Given a query ball $b_q = b(q, x)$, and an approximation parameter $\delta > 0$, our purpose is to compute a number $N$, such that $|B(b(q, x))| \leq N \leq |B(b(q, (1+\delta)x))|$.

The query algorithm works as follows:

(A) Using Lemma 3.3, compute a set $X$ of all the balls that intersect $b_q$ and are of radius $\geq \delta x/4$.

(B) Using the DS$_3.1$, compute $O(1/\delta^d)$ cells of $T_C$ that corresponds to $G_{\approx}(b_q(1+\delta/4), \delta/4)$. Let $N'$ be the total number of points in $C$ stored in these nodes.

(C) The quantity $N' + |X|$ is almost the desired quantity, except that we might be counting some of the balls of $X$ twice. To this end, let $N''$ be the number of balls in $X$ with centers in $G_{\approx}(b_q(1+\delta/4), \delta/4)$.

(D) Let $N \leftarrow N' + |X| - N''$. Return $N$. 

6
Correctness. We only sketch the proof, as the proof is straightforward. Indeed, the union of the cells of $G_\infty(b_q(1 + \delta/4), \delta/4)$ contains $b(q, x(1 + \delta/4))$ and is contained in $b(q, (1 + \delta)x)$. All the balls with radius smaller than $\delta x/4$ and intersecting $b(q, x)$ have their centers in cells of $G_\infty(b_q(1 + \delta/4), \delta/4)$, and their number is computed correctly. Similarly, the “large” balls are computed correctly. The last stage ensures we do not over-count by 1 each large ball that also has its center in $G_\infty(b_q(1 + \delta/4), \delta/4)$. It is also easy to check that $|B(b(q, x))| \leq N \leq |B(b(q, x(1 + \delta))|$, and the same result can be used at $x/(1 + \delta)$.

Query running time. Computing all the cells of $G_\infty(b_q(1 + \delta/4), \delta/4)$ takes $O(\log n + 1/\delta^d)$ time. Computing the “large” balls takes $O\left(\log n + 1/\delta^d\right)$ time. Checking for each large ball if it is already counted by the “small” balls takes $O(1/\delta^d)$ by using a grid.

Result. The above implies the following.

Lemma 3.4 (rangeCount). Given a set $B$ of $n$ balls in $\mathbb{R}^d$, it can be preprocessed, in $O(n \log n)$ time, into a data structure of size $O(n)$, such that given a query ball $b(q, x)$ and approximation parameter $\delta > 0$, the query algorithm rangeCount$(q, x, \delta)$ returns, in $O(\log n + 1/\delta^d)$ time, a number $N$ satisfying the following:

(A) $N \leq |B(b(q, (1 + \delta)x))|$,  
(B) $|B(b(q, x))| \leq N$, and  
(C) for a query ball $b(q, x)$ and $\delta$, the number $N$ is $\delta$-monotonic as a function of $x$, see Definition 2.6.

4. Answering k-ANN queries among balls

4.1. Computing a constant factor approximation to $d_k(q, B)$

Lemma 4.1. Let $B$ be a set of disjoint balls in $\mathbb{R}^d$, and consider a ball $b = b(q, r)$ that intersects at least $k$ balls of $B$. Then, among the $k$ nearest neighbors of $q$ from $B$, there are at least $\max(0, k - c_d)$ balls of radius at most $r$. The centers of all these balls are in $b(q, 2r)$.

Proof: Consider the $k$ nearest neighbors of $q$ from $B$. Any such ball that has its center outside $b(q, 2r)$, has radius at least $r$, since it intersects $b = b(q, r)$. Since the number of balls that are of radius at least $r$ and intersecting $b$ is bounded by $c_d$, there must be at least $\max(0, k - c_d)$ balls among the $k$ nearest neighbors, each having radius less than $r$. Now, $b(q, 2r)$ will contain the centers of all such balls. ■

Corollary 4.2. Let $\gamma = \min(k, c_d)$. Then, $d_{k-\gamma}(q, C)/2 \leq d_k(q, B)$.

Idea. The basic observation is that we only need a rough approximation to the right radius, as using approximate range counting (i.e., Lemma 3.4), one can get improve the approximation.

Let $x_i$ denote the distance of $q$ to the $i$th closest center in $C$. Let $d_k = d_k(q, B)$. Let $i$ be the minimum index, such that $d_k \leq x_i$. Since $d_k \leq x_k$, it must be that $i \leq k$. There are several possibilities:

(A) If $i \leq k - c_d$ (i.e., $d_k \leq x_{k-c_d}$) then, by Lemma 4.1, the ball $b(q, 2d_k)$ contains at least $k - c_d$ centers. As such, $d_k < x_{k-c_d} \leq 2d_k$, and $x_{k-c_d}$ is a good approximation to $d_k$.

(B) If $i > k - c_d$, and $d_k \leq 4x_{i-1}$, then $x_{i-1}$ is the desired approximation.

(C) If $i > k - c_d$, and $d_k \geq x_i/4$, then $x_i$ is the desired approximation.
(D) Otherwise, it must be that \( i > k - c_d \), and \( x_{i-1}/4 \leq d_k \leq x_i/4 \). Let \( b_j = b(c_j, r_j) \) be the \( j \)th closest ball to \( q \), for \( j = 1, \ldots, k \). It must be that \( b_i, \ldots, b_k \) are much larger than \( b(q, d_k) \). But then, the balls \( b_i, \ldots, b_k \) must intersect \( b(q, x_i/2) \), and their radius is at least \( x_i/2 \). We can easily compute these big balls using \( DS_{3.1}(B) \), and the number of centers of the small balls close to query, and then compute \( d_k \) exactly.

**Preprocessing.** We build \( DS_{3.1} \) in \( O(n \log n) \) time.

**Notations.** For \( x \geq 0 \), let \( N(x) \) denote the number of balls in \( B \) that intersect \( b(q, x) \); that is \( N(x) = |\{ b \in B \mid b \cap b(q, x) \neq \emptyset \}| \), and \( C(x) \) denote the number of centers in \( b(q, x) \), i.e., \( C(x) = |C \cap b(q, x)| \). Also, let \( \#(x) \) denote the 2-approximation to the number of balls of \( B \) intersecting \( b(q, x) \), as computed by Lemma 3.4; that is \( N(x) \leq \#(x) \leq N(2x) \).

**Answering a query.** We are given a query point \( q \in \mathbb{R}^d \) and a number \( k \).

Using \( DS_{3.1} \), compute a 2-approximation for the smallest ball containing \( k - i \) centers of \( B \), for \( i = 0, \ldots, \gamma \), where \( \gamma = \min(k, c_d) \), and let \( r_{k-i} \) be this radius. That is, for \( i = 0, \ldots, \gamma \), we have \( C(r_{k-i}/2) \leq k - i \leq C(r_{k-i}) \). For \( i = 0, \ldots, \gamma \), compute \( N_{k-i} = \#(r_{k-i}) \) (Lemma 3.4).

Let \( \alpha \) be the maximum index such that \( N_{k-\alpha} \geq k \). Clearly, \( \alpha \) is well defined as \( N_k \geq k \). The algorithm is executed in the following steps.

(A) If \( \alpha = \gamma \) we return \( 2r_{k-\gamma} \).

(B) If \( \#(r_{k-\alpha}/4) < k \), then we return \( 2r_{k-\alpha} \).

(C) Otherwise, compute all the balls of \( B \) that are of radius at least \( r_{k-\alpha}/4 \) and intersect the ball \( b(q, r_{k-\alpha}/4) \), using \( DS_{3.1}(B) \). For each such ball \( b \), compute the distance \( \zeta = d(q, b) \) of \( q \) to it. Return \( 2\zeta \) for the minimum such number \( \zeta \) such that \( \#(\zeta) \geq k \).

**Lemma 4.3.** Given a set of \( n \) disjoint balls \( B \) in \( \mathbb{R}^d \), one can preprocess them, in \( O(n \log n) \) time, into a data structure of size \( O(n) \), such that given a query point \( q \in \mathbb{R}^d \), and a number \( k \), one can compute, in \( O(\log n) \) time, a number \( x \) such that, \( x/4 \leq d_k(q, B) \leq 4x \).

**Proof:** The data-structure and query algorithm are described above. We next prove correctness. To prove that (A) returns the correct answer observe that under the given assumptions,

\[
r_{k-\gamma}/4 \leq d_{k-\gamma}(q, C)/2 \leq d_k(q, B) \leq 2r_{k-\gamma},
\]

where the second inequality follows from Corollary 4.2, and the third inequality follows as \( N(2r_{k-\gamma}) \geq \#(r_{k-\gamma}) \geq k \), while \( d_k(q, B) \) is the smallest number \( x \) such that \( N(x) \geq k \).

For (B) observe that we have that \( N(r_{k-\gamma}/4) \leq \#(r_{k-\gamma}/4) < k \) and as such we have \( r_{k-\gamma}/4 < d_k(q, B) \). But by assumption, \( \#(r_{k-\gamma}) \geq k \) and so \( N(2r_{k-\gamma}) \geq \#(r_{k-\gamma}) \geq k \), thus \( d_k(q, B) \leq 2r_{k-\gamma} \).

For (C), first observe that \( \alpha < \gamma \) as the algorithm did not return in (A). Since \( \alpha \) is the maximum index such that \( \#(r_{k-\alpha}) \geq k \), so \( N(r_{k-\alpha}/4) \leq \#(r_{k-\alpha}/4) < k \) implying, \( r_{k-\alpha}/4 < d_k(q, B) \). Also, \( d_k(q, B) \leq r_{k-\alpha}/4 \), as the algorithm did not return in (B). Now the ball \( b(q, r_{k-\alpha}/4) \) contains at least \( k - \alpha - 1 \) centers from \( C \), but it does not contain \( k - \alpha \) centers. Indeed, otherwise we would have \( d_{k-\alpha}(q, C) \leq r_{k-\alpha}/4 \) and so \( r_{k-\alpha} \leq 2d_{k-\alpha}(q, C) \leq 2r_{k-\alpha}/4 \), but on the other hand \( r_{k-\alpha} < d_k(q, B) \leq r_{k-\alpha}/4 \), which would be a contradiction. Similarly, there is no center of any ball whose distance from \( q \) is in the range \( (r_{k-\alpha}/4, r_{k-\alpha}/2) \) otherwise we would have that \( d_{k-\alpha}(q, C) < r_{k-\alpha}/2 \) and this would mean that \( r_{k-\alpha} \leq 2d_{k-\alpha}(q, C) < r_{k-\alpha} \), a contradiction. Now, the center of the \( k \)th closest ball is clearly more than \( r_{k-\alpha}/2 \) away from \( q \). As such its distance from \( q \) is at least \( r_{k-\alpha}/2 \). Since \( d_k(q, B) \leq r_{k-\alpha}/4 \) it
follows that the $k$th closest ball intersects $b(q, r_{k-\alpha}/4)$ and moreover, its radius is at least $r_{k-\alpha}/4$. Since we compute all such balls in (C), we do encounter the $k$th closest ball. It is easy to see that in this case we return a number $\zeta$ satisfying, $\zeta/2 \leq d_k(q, B) \leq 2\zeta$.

As for the running time, notice that we need to use the algorithm of Lemma 3.4 $O(1)$ times, each iteration taking time $O(\log n)$. After this we need another $O(\log n)$ time for the invocation of the algorithm in Lemma 3.3. As such, the total query time is $O(\log n)$.

4.1.1. Refining the approximation of $d_k(q, B)$

**Lemma 4.4.** Given a set $B$ of $n$ balls in $\mathbb{R}^d$, it can be preprocessed, in $O(n \log n)$ time, into a data structure of size $O(n)$. Given a query point $q$, numbers $k, x$, and an approximation parameter $\varepsilon > 0$, such that $x/4 \leq d_k(q, B) \leq 4x$, one can find a ball $b \in B$ such that, $(1-\varepsilon)d_k(q, B) \leq d(q, b) \leq (1+\varepsilon)d_k(q, B)$, in $O(\log n + 1/\varepsilon^d)$ time.

**Proof:** We are going to use the same data-structure as Lemma 3.4, for the query ball $b_q = b(q, 4x(1+\varepsilon))$. We compute all large balls of $B$ that intersect $b_q$. Here a large ball is a ball of radius $> x\varepsilon$, and a ball of radius at most $x\varepsilon$ is considered to be a small ball. Consider the $O(1/\varepsilon^d)$ grid cells of $G_s(b_q, \varepsilon/16)$. In $O(1/\varepsilon^d)$ time we can record the number of centers of large balls inside any such cell. Clearly, any small ball that intersects $b(q, 4x)$ has its center in some cell of $G_s(b_q, \varepsilon/16)$. We use the quadtree $T_C$ to find out exactly the number of centers, $N_C$, of small balls in each cell $\Box$ of $G_s(b_q, \varepsilon/16)$, by finding the total number of centers using $T_C$, and decreasing this by the count of centers of large balls in that cell. This can be done in time $O(\log n + 1/\varepsilon^d)$. We pick an arbitrary point in $\Box$, and assign it weight $N_\Box$, and treat it as representing all the small balls in this grid cell - clearly, this introduces an error of size $\leq \varepsilon x$ in the distance of such a ball from $q$, and as such we can ignore it in our argument. In the end of this snapping process, we have $O(1/\varepsilon^d)$ weighted points, and $O(1/\varepsilon^d)$ large balls. We know the distance of the query point from each one of these points/balls. This results in $O(1/\varepsilon^d)$ weighted distances, and we want the smallest $\ell$, such that the total weight of the distances $\leq \ell$ is at least $k$. This can be done by weighted median selection in linear time in the number of distances, which is $O(1/\varepsilon^d)$. Once we get the required point we can output any ball $b$ corresponding to the point. Clearly, $b$ satisfies the required conditions.

4.1.2. The result

**Theorem 4.5.** Given a set of $n$ disjoint balls $B$ in $\mathbb{R}^d$, one can preprocess them in time $O(n \log n)$ into a data structure of size $O(n)$, such that given a query point $q \in \mathbb{R}^d$, a number $k$ with $1 \leq k \leq n$ and $\varepsilon > 0$, one can find in time $O(\log n + \varepsilon^{-d})$ a ball $b \in B$, such that, $(1-\varepsilon)d_k(q, B) \leq d(q, b) \leq (1+\varepsilon)d_k(q, B)$.

5. Quorum clustering

We are given a set $B$ of $n$ disjoint balls in $\mathbb{R}^d$, and we describe how to compute quorum clustering for them quickly.

Let $\xi$ be some constant. Let $B_0 = \emptyset$. For $i = 1, \ldots, m$, let $R_i = B \setminus (\bigcup_{j=0}^{i-1} B_j)$, and let $\Lambda_i = b(w_i, x_i)$ be any ball that satisfies,

(A) $\Lambda_i$ contains $\min(k - c_d, |R_i|)$ balls of $R_i$ completely inside it,

(B) $\Lambda_i$ intersects at least $k$ balls of $B$, and

(C) the radius of $\Lambda_i$ is at most $\xi$ times the radius of the smallest ball satisfying the above conditions.
Next, we remove any \( k - c_d \) balls that are contained in \( \Lambda_i \) from \( R_i \) to get the set \( R_{i+1} \). We repeat this process till all balls are extracted. Notice that at each step \( i \) we only require that the \( \Lambda_i \) intersects \( k \) balls of \( B \) (and not \( R_i \)), but that it must contain \( k - c_d \) balls from \( R_i \). Also, the last quorum ball may contain fewer balls. The balls \( \Lambda_1, \ldots, \Lambda_m \), are the resulting \textit{\( \xi \)-approximate quorum clustering}.

## 5.1. Computing an approximate quorum clustering

### Definition 5.1.

For a set \( P \) of \( n \) points in \( \mathbb{R}^d \), and an integer \( \ell \), with \( 1 \leq \ell \leq n \), let \( r_{opt}(P, \ell) \) denote the radius of the smallest ball which contains at least \( \ell \) points from \( P \), i.e., \( r_{opt}(P, \ell) = \min_{q \in \mathbb{R}^d} d_q(P) \).

Similarly, for a set \( R \) of \( n \) balls in \( \mathbb{R}^d \), and an integer \( \ell \), with \( 1 \leq \ell \leq n \), let \( R_{opt}(R, \ell) \) denote the radius of the smallest ball which completely contains at least \( \ell \) balls from \( R \).

### Lemma 5.2 ([HK12]).

Given a set \( P \) of \( n \) points in \( \mathbb{R}^d \) and integer \( \ell \), with \( 1 \leq \ell \leq n \), one can compute, in \( O(n \log n) \) time, a sequence of \( \lceil n/\ell \rceil \) balls, \( o_1 = b(u_1, \psi_1), \ldots, o_{\lceil n/\ell \rceil} = b(u_{\lfloor n/\ell \rfloor}, \psi_{\lfloor n/\ell \rfloor}) \), such that, for all \( i, 1 \leq i \leq \lceil n/\ell \rceil \), we have

(A) For every ball \( o_i \), there is an associated subset \( P_i \) of \( \min(\ell, |Q_i|) \) points of \( Q_i = P \setminus (P_1 \cup \ldots \cup P_{i-1}) \), that it covers.

(B) The ball \( o_i = b(u_i, \psi_i) \) is a 2-approximation to the smallest ball covering \( \min(\ell, |Q_i|) \) points in \( Q_i \); that is, \( \psi_i/2 \leq \min_{o_i \in \bigcup_{\ell \geq i} \pi(Q_i)} \) points in \( Q_i \).

### The algorithm.

The algorithm to construct an approximate quorum clustering is as follows. We use the algorithm of Lemma 5.2 with the set of points \( P = C \), and \( \ell = k - c_d \) to get a list of \( m = \lceil n/(k - c_d) \rceil \) balls \( o_1 = b(u_1, \psi_1), \ldots, o_m = b(u_m, \psi_m) \), satisfying the conditions of Lemma 5.2. Next we use the algorithm of Theorem 4.5, to compute \((k, \varepsilon)\)\-ANN distances from the centers \( u_1, \ldots, u_m \), to the balls of \( B \).

Thus, we get numbers \( \gamma_i \) satisfying, \((1/2)\mathbf{d}_k(u_i, B) \leq \gamma_i \leq (3/2)\mathbf{d}_k(u_i, B)\). Let \( \zeta_i = \max(2\gamma_i, 3\psi_i) \), for \( i = 1, \ldots, m \). Sort \( \zeta_1, \ldots, \zeta_m \) (we assume for the sake of simplicity of exposition that \( \zeta_m \) being the radius of the last cluster is the largest number). Suppose the sorted order is the permutation \( \pi \) of \( \{1, \ldots, m \} \) (by assumption \( \pi(m) = m \)). We output the balls \( \Lambda_i = b(u_{\pi(i)}, \zeta_{\pi(i)}) \), for \( i = 1, \ldots, m \), as the approximate quorum clustering.

### 5.1.1. Correctness

### Lemma 5.3.

Let \( B = \{b_1, \ldots, b_n\} \) be a set of \( n \) disjoint balls, where \( b_i = b(c_i, r_i) \), for \( i = 1, \ldots, n \). Let \( C = \{c_1, \ldots, c_n\} \) be the set of centers of these balls. Let \( b = b(c, r) \) be any ball that contains at least \( \ell \) centers from \( C \), for some \( 2 \leq \ell \leq n \). Then \( b(c, 3r) \) contains the \( \ell \) balls that correspond to those centers.

### Proof:

Without loss of generality suppose \( b \) contains the \( \ell \) centers \( c_1, \ldots, c_\ell \), from \( C \). Now consider any index \( i \) with \( 1 \leq i \leq \ell \), and consider any \( j \neq i \), which exists as \( \ell \geq 2 \) by assumption. Since \( b(c, r) \) contains both \( c_i \) and \( c_j \), \( 2r \geq ||c_i - c_j|| \) by the triangle inequality. On the other hand, as the balls \( b_i \) and \( b_j \) are disjoint we have that \( ||c_i - c_j|| \geq r_i + r_j \geq r_i \). It follows that \( r_i \leq 2r \) for all \( 1 \leq i \leq \ell \). As such the ball \( b(c, 3r) \) must contain the entire ball \( b_i \), and thus it contains all the \( \ell \) balls \( b_1, \ldots, b_\ell \), corresponding to the centers.

### Lemma 5.4.

Let \( B = \{b_1 = b(c_1, r_1), \ldots, b_n = b(c_n, r_n)\} \) be a set of \( n \) disjoint balls in \( \mathbb{R}^d \). Let \( C = \{c_1, \ldots, c_n\} \) be the corresponding set of centers, and let \( \ell \) be an integer with \( 2 \leq \ell \leq n \). Then, \( r_{opt}(C, \ell) \leq R_{opt}(B, \ell) \leq 3r_{opt}(C, \ell) \).
Proof: The first inequality follows since the ball realizing the optimal covering of \( \ell \) balls, clearly contains their centers as well, and therefore \( \ell \) points from \( \mathcal{C} \). To see the second inequality, consider the ball \( b = b(c, r) \) realizing \( r_{\text{opt}}(\mathcal{C}, \ell) \), and use Lemma 5.3 on it. This implies \( R_{\text{opt}}(\mathcal{B}, \ell) \leq 3r_{\text{opt}}(\mathcal{C}, \ell) \).

Lemma 5.5. The balls \( \Lambda_1, \ldots, \Lambda_m \) computed above are a 12-approximate quorum clustering of \( \mathcal{B} \).

Proof: Consider the balls \( o_1 = b(u_1, \psi_1), \ldots, o_m = b(u_m, \psi_m) \) computed by the algorithm of Lemma 5.2. Suppose \( \mathcal{C}_i \), for \( i = 1, \ldots, m \), is the set of centers assigned to the balls \( b_i \). That is \( \mathcal{C}_1, \ldots, \mathcal{C}_m \) form a disjoint decomposition of \( \mathcal{C} \), each of size \( k - c_d \) (except for the last set, which might be smaller – a technicality that we ignore for the sake of simplicity of exposition).

For \( i = 1, \ldots, m \), let \( \mathcal{B}_i \) denote the set of balls corresponding to the centers in \( \mathcal{C}_i \). Now while constructing the approximate quorum clusters we are going to assign the set of balls \( B_{\pi(i)} \) for \( i = 1, \ldots, m \), to \( \Lambda_i \). Now, fix a \( i \) with \( 1 \leq i \leq m - 1 \). The balls of \( \bigcup_{j=1}^i B_{\pi(j)} \) have been used up. Consider an optimal ball, i.e., a ball \( b = b(c, r) \) that contains completely \( k - c_d \) balls among \( \bigcup_{j=i+1}^m B_{\pi(j)} \) and intersects \( k \) balls from \( \mathcal{B} \), and is the smallest such possible. Fix some \( k - c_d \) balls from \( \bigcup_{j=i+1}^m B_{\pi(j)} \) that this optimal ball contains. Consider the sets of centers \( \mathcal{C}' \) of these balls. The quorum clusters \( o_{\pi(j)} \) for \( j = i + 1, \ldots, m \), contain all these centers, by construction. Out of these indices, i.e., out of the indices \( \{\pi(i+1), \ldots, \pi(m)\} \), suppose \( p \) is the minimum index such that \( o_p \) contains one of these centers. When \( o_p \) was constructed, i.e., at the \( p \)th iteration of the algorithm of Lemma 5.2, all the centers from \( \mathcal{C}' \) were available. Now since the optimal ball \( b = b(c, r) \) contains \( k - c_d \) available centers too, it follows that \( \psi_p \leq 2r \) since Lemma 5.2 guarantees this. Since \( k - c_d \geq 2 \), by Lemma 5.3, \( b(u_p, 3\psi_p) \) contains the balls of \( B_p \). Moreover, by the Lipschitz property, see Observation 2.2, it follows that \( d_k(u_p, \mathcal{B}) \leq d_k(c, \mathcal{B}) + \|u_p - c\| \leq r + (r + \psi_p) \leq 4r \), where the second last inequality follows as the balls \( b = b(c, r) \) and the ball \( o_p = b(u_p, \psi_p) \) intersect. Therefore, for the index \( p \) we have that, \( d_k(u_p, \mathcal{B}) \leq 2\gamma_p \leq 3d_k(u_p, \mathcal{B}) \leq 12r \), and also that \( 3\psi_p \leq 6r \). As such \( \zeta_p = \max(2\gamma_p, 3\psi_p) \leq 12r \). The index \( \pi(i+1) \) minimizes this quantity among the indices \( \{\pi(i+1), \ldots, \pi(m)\} \) (as we took the sorted order), such as it follows that \( \zeta_{i+1} \leq 12r \).

Lemma 5.6. Given a set \( \mathcal{B} \) of \( n \) disjoint balls in \( \mathbb{R}^d \), such that \( (k - c_d)n \), and a number \( k \) with \( 2c_d < k \leq n \), in \( O(n \log n) \) time, one can output a sequence of \( m = n/(k - c_d) \) balls \( \Lambda_1, \ldots, \Lambda_m \), such that

(A) For each ball \( \Lambda_i \), there is an associated subset \( B_i \) of \( k - c_d \) balls of \( \mathcal{R}_i = \mathcal{B} \setminus (B_1 \cup \ldots \cup B_{i-1}) \), that it completely covers.

(B) The ball \( \Lambda_i \) intersects at least \( k \) balls from \( \mathcal{B} \).

(C) The radius of the ball \( \Lambda_i \) is at most 12 times that of the smallest ball covering \( k - c_d \) balls of \( \mathcal{R}_i \) completely, and intersecting \( k \) balls of \( \mathcal{B} \).

Proof: The correctness was proved in Lemma 5.5. To see the time bound is also easy as the computation time is dominated by the time in Lemma 5.2, which is \( O(n \log n) \).

6. Construction of the sublinear data-structure for \((k, \varepsilon)\)-\(\text{ANN}\)

Here we show how to compute an approximate Voronoi diagram for approximating the \( k \)-th-nearest ball, that takes \( O(n/k) \) space. We assume \( k > 2c_d \) without loss of generality, and we let \( m = \lceil n/(k - c_d) \rceil = O(n/k) \). Here \( k \) and \( \varepsilon \) are prespecified in advance.
6.1. Preliminaries

The following notation was introduced in [HK12]. A ball $b$ of radius $r$ in $\mathbb{R}^d$, centered at a point $c$, can be interpreted as a point in $\mathbb{R}^{d+1}$, denoted by $b' = (c, r)$. For a regular point $p \in \mathbb{R}^d$, its corresponding image under this transformation is the mapped point $p' = (p, 0) \in \mathbb{R}^{d+1}$, i.e., we view it as a ball of radius 0 and use the mapping defined on balls. Given point $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ we will denote its Euclidean norm by $\|u\|$. We will consider a point $u = (u_1, u_2, \ldots, u_{d+1}) \in \mathbb{R}^{d+1}$ to be in the product metric of $\mathbb{R}^d \times \mathbb{R}$ and endowed with the product metric norm

$$\|u\| \oplus = \sqrt{u_1^2 + \cdots + u_d^2 + |u_{d+1}|}.$$ 

It can be verified that the above defines a norm, and for any $u \in \mathbb{R}^{d+1}$ we have $\|u\| \leq \|u\| \oplus \leq \sqrt{2}\|u\|$.

6.1.1. Construction

The input is a set $\mathcal{B}$ of $n$ disjoint balls in $\mathbb{R}^d$, and parameters $k$ and $\varepsilon$.

The construction of the data-structure is similar to the construction of the $k$-th-nearest neighbor data-structure from the authors’ paper [HK12]. We compute, using Lemma 5.6, a $\xi$-approximate quorum clustering of $\mathcal{B}$ with $m = n/(k - c_d) = O(n/k)$ balls, $\Sigma = \{\Lambda_1 = b(w_1, x_1), \ldots, \Lambda_m = b(w_m, x_m)\}$, where $\xi \leq 12$. The algorithm then continues as follows:

(A) Compute an exponential grid around each quorum cluster. Specifically, let

$$\mathcal{I} = \bigcup_{i=1}^{m} \bigcup_{j=0}^{\lfloor \log(32\xi/\varepsilon) \rfloor} \mathcal{G}_{m,j}(b(w_i, 2^jx_i), \frac{\varepsilon}{\zeta_1})$$

be the set of grid cells covering the quorum clusters and their immediate environ, where $\zeta_1$ is a sufficiently large constant (say, $\zeta_1 = 256\xi$).

(B) Intuitively, $\mathcal{I}$ takes care of the region of space immediately next to a quorum cluster. For the other regions of space, we can apply a construction of an approximate Voronoi diagram for the centers of the clusters (the details are somewhat more involved). To this end, lift the quorum clusters into points in $\mathbb{R}^{d+1}$, as follows

$$\Sigma' = \{\Lambda_1', \ldots, \Lambda_m'\},$$

where $\Lambda_i' = (w_i, x_i) \in \mathbb{R}^{d+1}$, for $i = 1, \ldots, m$. Note that all points in $\Sigma'$ belong to $U' = [0, 1]^{d+1}$ by Assumption 2.3. Now build a $(1 + \varepsilon/8)$-AVD for $\Sigma'$ using the algorithm of Arya and Malamatos [AM02], for distances specified by the $\|\cdot\|_\oplus$ norm. The AVD construction provides a list of canonical cubes covering $[0, 1]^{d+1}$ such that in the smallest cube containing the query point, the associated point of $\Sigma'$, is a $(1 + \varepsilon/8)$-ANN to the query point. (Note, that these cubes are not necessarily disjoint. In particular, the smallest cube containing the query point $q$ is the one that determines the assigned approximate nearest neighbor to $q$.)

Clip this collection of cubes to the hyperplane $x_{d+1} = 0$ (i.e., throw away cubes that do not have a face on this hyperplane). For a cube $\square$ in this collection, denote by $\text{mn}'(\square)$, the point of $\Sigma'$ assigned to it. Let $\mathcal{S}$ be this resulting set of canonical $d$-dimensional cubes.

(C) Let $\mathcal{W}$ be the space decomposition resulting from overlaying the two collection of cubes, i.e. $\mathcal{I}$ and $\mathcal{S}$. Formally, we compute a compressed quadtree $\mathcal{T}$ that has all the canonical cubes of $\mathcal{I}$ and $\mathcal{S}$ as

---

2That is, intuitively, if the query point falls into one of the grid cells of $\mathcal{I}$, we can answer a query in constant time.
nodes, and \( \mathcal{W} \) is the resulting decomposition of space into cells. One can overlay two compressed quadtrees representing the two sets in linear time [BHTT10, Har11]. Here, a cell associated with a leaf is a canonical cube, and a cell associated with a compressed node is the set difference of two canonical cubes. Each node in this compressed quadtree contains two pointers – to the smallest cell of \( \mathcal{T} \), and to the smallest cell of \( \mathcal{S} \), that contains it. This information can be computed by doing a BFS on the tree.

For each cell \( \square \in \mathcal{W} \) we store the following.

   (I) An arbitrary representative point \( \square_\text{rep} \in \square \).

   (II) The point \( \text{nn}'(\square) \in \mathcal{S}' \) that is associated with the smallest cell of \( \mathcal{S} \) that contains this cell. We also store an arbitrary ball, \( b(\square) \in \mathcal{B} \), that is one of the balls completely inside the cluster specified by \( \text{nn}'(\square) \) – we assume we stored such a ball inside each quorum cluster, when it was computed.

   (III) A number \( \beta_k(\square_\text{rep}) \) that satisfies \( d_k(\square_\text{rep}, \mathcal{B}) \leq \beta_k(\square_\text{rep}) \leq (1 + \varepsilon/4)d_k(\square_\text{rep}, \mathcal{B}) \), and a ball \( \text{nn}_k(\square_\text{rep}) \in \mathcal{B} \) that realizes this distance. In order to compute \( \beta_k(\square_\text{rep}) \) and \( \text{nn}_k(\square_\text{rep}) \) use the data-structure of Section 4, see Theorem 4.5.

6.1.2. Answering a query

Given a query point \( q \), compute the leaf cell (equivalently the smallest cell) in \( \mathcal{W} \) that contains \( q \) by performing a point-location query in the compressed quadtree \( \mathcal{T} \). Let \( \square \) be this cell. Let,

\[
\lambda^* = \min \left( \| q' - \text{nn}'(\square) \|_{\mathcal{B}}, \beta_k(\square_\text{rep}) + \| q - \square_\text{rep} \| \right).
\]  

(6.2)

If \( \text{diam}(\square) \leq (\varepsilon/8)\lambda^* \) we return \( \text{nn}_k(\square_\text{rep}) \) as the approximate \( k \)th-nearest neighbor, else we return \( b(\square) \).

6.2. Correctness

**Lemma 6.1.** The number \( \lambda^* = \min \left( \| q' - \text{nn}'(\square) \|_{\mathcal{B}}, \beta_k(\square_\text{rep}) + \| q - \square_\text{rep} \| \right) \) satisfies, \( d_k(q, \mathcal{B}) \leq \lambda^* \).

**Proof:** This follows by the Lipschitz property, see Observation 2.2. \( \blacksquare \)

**Lemma 6.2.** Let \( \square \in \mathcal{W} \) be any cell containing \( q \). If \( \text{diam}(\square) \leq \varepsilon d_k(q, \mathcal{B})/4 \), then \( \text{nn}_k(\square_\text{rep}) \) is a valid \( (1 \pm \varepsilon) \)-approximate \( k \)th-nearest neighbor of \( q \).

**Proof:** For the point \( \square_\text{rep} \), by Observation 2.2, we have that

\[
d_k(\square_\text{rep}, \mathcal{B}) \leq d_k(q, \mathcal{B}) + \| q - \square_\text{rep} \| \leq d_k(q, \mathcal{B}) + \text{diam}(\square) \leq (1 + \varepsilon/4)d_k(q, \mathcal{B}).
\]

Therefore, the ball \( \text{nn}_k(\square_\text{rep}) \) satisfies

\[
d(\square_\text{rep}, \text{nn}_k(\square_\text{rep})) \leq (1 + \varepsilon/4)d_k(\square_\text{rep}, \mathcal{B}) \leq (1 + \varepsilon/4)^2d_k(q, \mathcal{B}) \leq (1 + 3\varepsilon/4)d_k(q, \mathcal{B}).
\]

As such we have that

\[
d(q, \text{nn}_k(\square_\text{rep})) \leq d(\square_\text{rep}, \text{nn}_k(\square_\text{rep})) + \| q - \square_\text{rep} \| \leq ((1 + 3\varepsilon/4) + \varepsilon/4) d_k(q, \mathcal{B}) \leq (1 + \varepsilon)d_k(q, \mathcal{B}).
\]

Similarly, using the Lipschitz property, we can argue that, \( d(q, \text{nn}_k(\square_\text{rep})) \geq (1 - \varepsilon)d_k(q, \mathcal{B}) \), and therefore we have, \( (1 - \varepsilon)d_k(q, \mathcal{B}) \leq d(q, \text{nn}_k(\square_\text{rep})) \leq (1 + \varepsilon)d_k(q, \mathcal{B}) \), and the required guarantees are satisfied. \( \blacksquare \)
Lemma 6.3. For any point \( q \in \mathbb{R}^d \) there is a quorum ball \( \Lambda_i = b(w_i, x_i) \) such that (A) \( \Lambda_i \) intersects \( b(q, d_k(q, B)) \), (B) \( x_i \leq 3\xi d_k(q, B) \), and (C) \( \|q - w_i\| \leq 4\xi d_k(q, B) \).

Proof: By assumption, \( k > 2c_d \), and so by Lemma 4.1 among the \( k \) nearest neighbor of \( q \), there are \( k - c_d \) balls of radius at most \( d_k(q, B) \). Let \( B' \) denote the set of these balls. Among the indices \( 1, \ldots, m \), let \( i \) be the minimum index such that one of these \( k - c_d \) balls is completely covered by the quorum cluster \( \Lambda_i = b(w_i, x_i) \). Since \( b(q, d_k(q, B)) \) intersects the ball while \( \Lambda_i \) completely contains it, clearly \( \Lambda_i \) intersects \( b(q, d_k(q, B)) \). Now consider the time \( \Lambda_i \) was constructed, i.e., the \( i \)th iteration of the quorum clustering algorithm. At this time, by assumption, all of \( B' \) was available, i.e., none of its balls were assigned to earlier quorum clusters. The ball \( b(q, 3d_k(q, B)) \) contains \( k - c_d \) unused balls and touches \( k \) balls from \( B \), as such the smallest such ball had radius at most \( 3d_k(q, B) \). By the guarantee on quorum clustering, \( x_i \leq 3\xi d_k(q, B) \). As for the last part, as the balls \( b(q, d_k(q, B)) \) and \( \Lambda_i = b(w_i, x_i) \) intersect and \( x_i \leq 3\xi d_k(q, B) \), we have by the triangle inequality that \( \|q - w_i\| \leq (1 + 3\xi)d_k(q, B) \leq 4\xi d_k(q, B) \), as \( \xi \geq 1 \).

Definition 6.4. For a given query point, any quorum cluster that satisfies the conditions of Lemma 6.3 is defined to be an anchor cluster. By Lemma 6.3 an anchor cluster always exists.

Lemma 6.5. Suppose that among the quorum cluster balls \( \Lambda_1, \ldots, \Lambda_m \), there is some ball \( \Lambda_i = b(w_i, x_i) \) which satisfies that \( \|q - w_i\| \leq 8\xi d_k(q, B) \) and \( \varepsilon d_k(q, B) / 4 \leq x_i \leq 8\xi d_k(q, B) \) then the output of the algorithm is correct.

Proof: We have
\[
\frac{32\xi x_i}{\varepsilon} \geq \frac{32\xi (\varepsilon d_k(q, B) / 4)}{\varepsilon} = 8\xi d_k(q, B) \geq \|q - w_i\| .
\]
Thus, by construction, the expanded environ of the quorum cluster \( b(w_i, x_i) \) contains the query point, see Eq. (6.1)p12. Let \( j \) be the smallest non-negative integer such that \( 2^j x_i \geq d(q, w_i) \). We have that, \( 2^j x_i \leq \max(x_i, 2d(q, w_i)) \). As such, if \( \square \) is the smallest cell in \( \mathcal{W} \) containing the query point \( q \), then
\[
\text{diam}(\square) \leq \frac{\varepsilon}{\xi} 2^{j+1} x_i \leq \frac{\varepsilon}{\xi} \max(2x_i, 4d(q, w_i)) \leq \frac{\varepsilon}{\xi} \max(16\xi d_k(q, B), 32\xi d_k(q, B)) \\
\leq \frac{\varepsilon}{8} d_k(q, B) ,
\]
by Eq. (6.1)p12, and if \( \xi \geq 256\xi \). Now, by Lemma 6.1 we have that \( \lambda^* \geq d_k(q, B) \), so \( \text{diam}(\square) \leq (\varepsilon/8)\lambda^* \). Therefore, the algorithm returns \( \text{nn}_k(\square, \text{rep}) \) as the \( (1 \pm \varepsilon) \)-approximate \( k \)th-nearest neighbor, but then by Lemma 6.2 it is a correct answer.

Lemma 6.6. The query algorithm always outputs a correct approximate answer, i.e., the output ball \( b \) satisfies \( (1 - \varepsilon)d_k(q, B) \leq d(q, b) \leq (1 + \varepsilon)d_k(q, B) \).

Proof: Suppose that among the quorum cluster balls \( \Lambda_1 = b(w_1, x_1), \ldots, \Lambda_m = b(w_m, x_m) \), there is some ball \( \Lambda_i \) such that \( \|q - w_i\| \leq 8\xi d_k(q, B) \) and \( (\varepsilon/4)d_k(q, B) \leq x_i \leq 8\xi d_k(q, B) \), then by Lemma 6.5 the algorithm returns a valid approximate answer. Assume this condition is not satisfied. Let the anchor cluster be \( \Lambda = b(w, x) \). Since the anchor cluster satisfies \( \|q - w\| \leq 4d_k(q, B) \) and \( x \leq 3\xi d_k(q, B) \), it must be the case that, \( x < (\varepsilon/4)d_k(q, B) \). Since the anchor cluster intersects \( b(q, d_k(q, B)) \), we have that \( \|q - w\| \leq (1 + \varepsilon/4)d_k(q, B) \). Thus, \( \|q' - \Lambda'\|_{\square} = \|q - w\| + x \leq (1 + \varepsilon/2)d_k(q, B) \). Let \( \square \) be the smallest cell in which \( q \) is located. Now consider the point \( \text{nn}'(\square) \in \Sigma' \). Suppose it corresponds to the cluster \( \Lambda_j \), i.e., \( \Lambda_j = \text{nn}'(\square) \). Since \( \text{nn}'(\square) \) is a \( (1 + \varepsilon/8)\)-ANN to \( q \) among the points of \( \Sigma' \),
\( \Vert q' - \text{nn}'(\square) \Vert_{\Theta} \leq (1 + \varepsilon/8) \Vert q' - \Lambda' \Vert_{\Theta} \leq (1 + \varepsilon/8)(1 + \varepsilon/2) d_k(q, B) \leq (1 + \varepsilon)d_k(q, B) \leq 2d_k(q, B) \leq 8\varepsilon d_k(q, B) \). It follows that, \( \Vert q - w_j \Vert \leq 8\varepsilon d_k(q, B) \), and \( x_j \leq 8\varepsilon d_k(q, B) \). By our assumption, it must be the case that, \( x_j < (\varepsilon/4)d_k(q, B) \). Now, there are two cases. Suppose that, \( \text{diam}(\square) \leq (\varepsilon/8)\lambda^* \). Then, since we have \( \Lambda^* \leq \Vert q' - \text{nn}'(\square) \Vert_{\Theta} \) so \( \lambda^* \leq 2d_k(q, B) \). As such, \( \text{diam}(\square) \leq (\varepsilon/4)d_k(q, B) \). In this case we return \( \text{nn}_k(\square) \) by the algorithm, but the result is correct by Lemma 6.2. On the other hand, if we return \( b(\square) \), it is easy to see that \( d(q, b(\square)) \leq q - w_j \Vert \leq (1 + \varepsilon)d_k(q, B) \). Also, as \( b(\square) \) lies completely inside \( \Lambda_j \) it follows by Observation 2.1, that \( d(q, b(\square)) \geq d(q, \Lambda_j) \geq q - w_j \Vert \geq x_j \geq (q - w_j \Vert + x_j) - 2x_j \geq d_k(q, B) - (\varepsilon/2)d_k(q, B) \geq (1 - \varepsilon/2)d_k(q, B) \), where the second last inequality follows by Lemma 6.1.

6.3. The result

**Theorem 6.7.** Given a set \( B \) of \( n \) disjoint balls in \( \mathbb{R}^d \), a number \( k \), with \( 1 \leq k \leq n \), and \( \varepsilon \in (0, 1) \), one can preprocess \( B \), in \( O\left(n \log n + \frac{n}{k} C_\varepsilon \log n + \frac{n}{k} C'_\varepsilon\right) \) time, where \( C_\varepsilon = O\left(\varepsilon^{-d} \log \varepsilon^{-1}\right) \) and \( C'_\varepsilon = O\left(\varepsilon^{-2d} \log \varepsilon^{-1}\right) \). The space used by the data-structure is \( O\left(C_\varepsilon n/k\right) \). Given a query point \( q \), this data-structure outputs a ball \( b \in B \) in \( O\left(\log \frac{n}{k \varepsilon}\right) \) time, such that \( (1 - \varepsilon) d_k(q, B) \leq d(q, b) \leq (1 + \varepsilon) d_k(q, B) \).

**Proof:** If \( k \leq 2c_d \) then Theorem 4.5 provides the desired result.

For \( k > 2c_d \), the correctness was proved in Lemma 6.6. We only need to bound the construction time and space as well as the query time. Computing the quorum clustering takes time \( O(n \log n) \) by Lemma 5.6. Observe that \( |\mathcal{I}| = O\left(\frac{n}{k \varepsilon^d} \log \frac{1}{\varepsilon}\right) \). From the construction of Arya and Malamatos [AM02], we have \( |\mathcal{S}| = O\left(\frac{n}{k \varepsilon^d} \log \frac{1}{\varepsilon}\right) \) (note, that since we clip the construction to a hyperplane, we get \( 1/\varepsilon^d \) in the bound and not \( 1/\varepsilon^{d+1} \)). A careful implementation of this stage takes time \( O\left(n \log n + |\mathcal{W}| \left(\log n + \frac{1}{\varepsilon^d}\right)\right) \).

Overlaying the two compressed quadtrees representing them takes linear time in their size, that is \( O(|\mathcal{I}| + |\mathcal{S}|) \).

The most expensive step is to perform the \((1 \pm \varepsilon/4)\)-approximate \( k \)-th-nearest neighbor query for each cell in the resulting decomposition of \( \mathcal{W} \), see Eq. (6.2)_{b_{13}} (i.e., computing \( \beta_k(\square_{\text{rep}}) \) for each cell \( \square \in \mathcal{W} \).

Using the data-structure of Section 4 (see Theorem 4.5) each query takes \( O\left(\log n + 1/\varepsilon^d\right) \) time.

\[
O\left(n \log n + |\mathcal{W}| \left(\log n + \frac{1}{\varepsilon^d}\right)\right) = O\left(n \log n + \frac{n}{k \varepsilon^d} \log \frac{1}{\varepsilon} \log n + \frac{n}{k \varepsilon^d} \log \frac{1}{\varepsilon}\right)
\]
time, and this bounds the overall construction time.

The query algorithm is a point location query followed by an \( O(1) \) time computation and takes time \( O\left(\log \left(\frac{n}{k \varepsilon}\right)\right) \).

Note, that the space decomposition generated by Theorem 6.7 can be interpreted as a space decomposition of complexity \( O(C_\varepsilon n/k) \), where every cell has two input balls associated with it, which are the candidates to be the desired \((k, \varepsilon)\)-ANN. That is, Theorem 6.7 compute a \((k, \varepsilon)\)-AVD of the input balls.

7. Conclusions

In this paper, we presented a generalization of the usual \((1 \pm \varepsilon)\)-approximate \( k \)-th-nearest neighbor problem in \( \mathbb{R}^d \), where the input are balls of arbitrary radius, while the query is a point. We first
presented a data structure that takes $O(n)$ space, and the query time is $O(\log n + \varepsilon^{-d})$. Here, both $k$ and $\varepsilon$ could be supplied at query time. Next we presented an $(k, \varepsilon)$-AVD taking $\tilde{O}(n/k)$ space. Thus showing, surprisingly, that the problem can be solved in sublinear space if $k$ is sufficiently large.

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