Contactomorphisms of the sphere without translated points

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We construct a contactomorphism of $(S^{2n-1}, \alpha_{\text{std}})$ which does not have any translated points, providing a negative answer to a conjecture posed in San13.

1. Introduction

Let $(Y^{2n-1}, \alpha)$ be a contact manifold with a choice of contact form $\alpha$. Recall that this means that $\alpha$ is a 1-form so that $\alpha \wedge d\alpha^{n-1}$ is a volume form. A contactomorphism is a diffeomorphism $\varphi : Y \to Y$ with the property that $\varphi^* \alpha = e^g \alpha$ for some smooth function $g : Y \to \mathbb{R}$. The function $g$ is reasonably called the scaling factor; indeed, we easily compute:

\[
\varphi^*(\alpha \wedge d\alpha^{n-1}) = e^{ng} \alpha \wedge d\alpha^{n-1},
\]

i.e., $e^{ng}$ governs the change in volume due to $\varphi$.

A choice of contact form also selects a special vector field $R$ called the Reeb field, characterized by the equations $\alpha(R) = 1$ and $d\alpha(R, -) = 0$.

We recall the following notion from San12 and San13. Given a contactomorphism $\varphi$, a point $p \in Y$ is called a translated point provided that $g(p) = 0$ and $\varphi(p)$ lies on the Reeb flow line passing through $p$.

In San13, the author conjectures that every contactomorphism $\varphi$ contact isotopic to the identity of a compact contact manifold $Y$ (with any choice of form $\alpha$) has at least one translated point. The goal of the present document is to give counterexamples to this conjecture on $S^{2n-1}$ with the standard contact form $\alpha_{\text{std}}$, for $n > 1$. The main result we will prove is:

**Theorem 1.** Let $n > 1$. There exist contactomorphisms $\varphi : S^{2n-1} \to S^{2n-1}$ contact isotopic to the identity which do not have translated points for the contact form $\alpha_{\text{std}}$.

\footnote{Strictly speaking, our definition selects only the co-orientation preserving contactomorphisms.}
The proof is given in §3 and §4 below. We recall the definition of standard contact form $\alpha_{\text{std}}$ in §2.

**Remark 1.** For the case $Y = (S^1, \alpha)$, every (orientation preserving) diffeomorphism $\varphi$ is a contactomorphism, and the Reeb vector field for $\alpha$ is a nowhere-zero vector field and hence the any two points on $Y$ can be joined by a Reeb flow line. That $\varphi$ has mapping degree 1 implies that $\int \varphi^* \alpha = \int e^g \alpha = \int \alpha$. This identity implies the existence of at least two points satisfying $g = 0$. Thus every contactomorphism of $S^1$ has translated points (for any contact form).

**Remark 2.** Sandon’s conjecture has been proved in multiple cases. In [AFM15, MN18], it is proved when the contact form $\alpha$ is hypertight in the sense that it admits no contractible Reeb orbits. In [She14] (using [AF10]’s work on leaf-wise intersection points), and [Oh21, Oh22], the existence of translated points is proved under a smallness assumption on the oscillation norm $\int_0^1 \max(\lvert H_t \rvert) - \min(\lvert H_t \rvert) dt$. In [AM13], the authors prove Sandon’s conjecture for the boundary of Liouville domains $X$ with non-vanishing Rabinowitz Floer homology, and [MU19] proves the conjecture when the symplectic homology of $X$ is infinite dimensional. The papers [San13, GKPS17, All22b] establish versions of Sandon’s conjecture for (non-spherical) lens spaces ($\mathbb{RP}^{2n-1}$ is a special case), using a generating functions approach and ideas from [The98, Giv90]. This is further supported by the work of [AK22], where certain Rabinowitz Floer homology groups of lens spaces are shown to be non-zero. The work of [All22a] establishes Sandon’s conjecture for a restricted class of contactomorphisms on unit tangent bundles using a variational approach (namely, those contactomorphisms of $STM$ which are lifts of diffeomorphisms of the base manifold $M$ homotopic to the identity).

**Remark 3.** Sandon’s work [San13] claimed to show that every contactomorphism on $(S^{2n-1}, \alpha_{\text{std}})$ had a translated point. However, [GD20] has clarified the situation by pointing out a gap in the original argument, and he gives a detailed proof of a restricted statement, see [GD20 Theorem 1.2].

**Remark 4.** Sandon’s conjecture is straightforward to verify in the case when $\varphi_t$ is an autonomous family of contactomorphisms generated by a Hamiltonian $H$ which is constant on Reeb flow lines (i.e., $dH(R) = 0$). In this case, any critical point of $H$ is a translated point of $\varphi_t$, for all $t$.

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2Here $H_t = \alpha(X_t)$ is the contact Hamiltonian associated to the infinitesimal generator $X_t$ of a path of contactomorphisms $\varphi_t$.
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2. The standard contact form on the sphere

Consider $S^{2n-1}$ as the unit sphere in $\mathbb{R}^{2n}$, and recall that

$$\alpha_{\text{std}} = \sum_{i=1}^{n}(x_1dy_i - y_1dx_i)$$

defines the standard contact form on $S^{2n-1}$. It is readily checkable that

$$R = \sum_{i=1}^{n}x_i\partial_{y_i} - y_i\partial_{x_i} = J\sum_{i=1}^{n}(x_i\partial_{x_i} + y_i\partial_{y_i})$$

defines the Reeb flow. In particular, Reeb flow lines are given by $z_i(t) = e^{it}z_i(0)$, i.e., the orbits of the Reeb vector field are the fibers of the Hopf fibration.

3. Proof of the main result

Here is the sketch of the argument proving Theorem 1. First observe that if $p, q$ are points so that $q$ does not lie on the Reeb flow line through $p$, then we can find open sets $U_p, U_q$ so that no Reeb flow line passes through $U_p$ and $U_q$. Indeed, this follows from the Hausdorffness of the space of Reeb flow lines $CP^{n-1} = S^{2n-1}/S^1$ (this is a rather special property of the standard contact form).

Introduce the following notation: given a contactomorphism $\varphi$ with scaling factor $g$, let $\Sigma_\varphi$ be the set $\{g = 0\}$. The existence of a translated point implies the existence of a Reeb flow line joining $\Sigma_\varphi$ and $\varphi(\Sigma_\varphi)$.

Our strategy is simple: construct $\varphi$ so that $\Sigma_\varphi \subset U_p$ while $\varphi(\Sigma_\varphi) \subset U_q$; clearly $\varphi$ will have no translated points.

Definition 5. For the purposes of the argument, let us say that a contactomorphism of a compact manifold $\varphi: Y \to Y$ has the focal property for $(p, q)$ provided the following hold:
(i) \( p, q \) are fixed points of \( \varphi \), \((\varphi^*\alpha)_p > \alpha_p \), and \((\varphi^*\alpha)_q < \alpha_q \).

(ii) \( q \) has arbitrarily small neighbourhoods \( U \) satisfying \( \varphi(U) \subset U \) (attracting).

(iii) denoting \( \varphi_n = \varphi \circ \cdots \circ \varphi \), \( q \) has arbitrarily small neighbourhoods \( U \) satisfying \( \varphi(U) \subset U \).

Lemma 2. Let \( \varphi : Y \to Y \) be a contactomorphism satisfying the focal property for \((p, q)\). Denote by \( \Sigma_n \) the set of points \( z \) so that \((\varphi^n \alpha)_z = \alpha_z \). Then

\[
\lim_{n \to \infty} \text{dist}(\Sigma_n, p) + \text{dist}(\varphi_n(\Sigma_n), q) = 0,
\]

i.e., \( \Sigma_n \) eventually enters arbitrarily small neighbourhoods of \( p \) and \( \varphi_n(\Sigma_n) \) eventually enters arbitrarily small neighbourhoods of \( q \).

Proof. Bear in mind that \( Y \) is assumed compact. Let \( U_p, U_q \) be arbitrary (small) open sets around \( p, q \) respectively. It suffices to show that \( \Sigma_n \subset U_p \) and \( \varphi_n(\Sigma_n) \subset U_q \) for \( n \) sufficiently large. Let \( g \) be the scaling factor for \( \varphi \), i.e., \( \varphi^* \alpha = e^g \alpha \), and let \( g_n \) be the scaling factor for \( \varphi_n := \varphi \circ \cdots \circ \varphi \).

The focal property (i) implies that \( g(p) > 0 \) and \( g(q) < 0 \). A straightforward computation establishes that:

\[
g_n = g + g \circ \varphi + \cdots + g \circ \varphi_{n-1}.
\]

It is clear that \( g \) is a bounded function, so pick some \( M > \sup_z |g(z)| \).

Shrinking \( U_p, U_q \) if necessary, and using the focal properties for \( \varphi \), we may suppose that:

(a) \( \varphi(U_q) \subset U_q \),

(b) \( g > \delta \) on \( U_p \) and \( g < -\delta \) on \( U_q \) for some \( \delta > 0 \),

(c) \( \varphi_N(Y - U_p) \subset U_q \) for some \( N \in \mathbb{N} \).

We will refer to the constants \( N, M, \delta \) in the subsequent arguments.

Suppose that \( z \in \Sigma_n \), and \( z \notin U_p \). Then \( \varphi_k(z) \in U_q \) for all \( k \geq N \). In particular, \( g(\varphi_k(z)) < -\delta \) for \( k \geq N \). We then estimate:

\[
0 = g_n(z) = g(z) + \cdots + g(\varphi_{N-1}(z)) + g(\varphi_N(z)) + \cdots + g(\varphi_{n-1}(z))
\]

\[
< NM - (n - N)\delta.
\]
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Thus for \( n \) sufficiently large we have a contradiction, and so eventually we must have \( \Sigma_n \subset U_p \).

On the other hand, suppose that \( z \in \Sigma_n \) but \( \varphi_n(z) \notin U_q \). Thanks to the previous part, we may assume that \( z \in U_p \). Let \( k \) be the smallest integer so that \( \varphi_k(z) \notin U_p \). We then estimate:

\[
0 = g(z) + g(\varphi(z)) + \cdots + g(\varphi_{k-1}(z)) + g(\varphi_k(z)) + \cdots + g(\varphi_{n-1}(z))
\]

\[> k \delta - (n - k)M.\]

Rearranging yields:

\[
k < \frac{n}{\delta/M + 1}.
\]

For \( n \) sufficiently large (depending only on \( \delta, M \), and not on \( z \)), we have

\[
\frac{n}{\delta/M + 1} \leq n - N,
\]

Thus \( k < n - N \). Since \( \varphi_k(z) \notin U_p \) implies \( \varphi_{k+N}(z) \in U_q \), we thus has \( \varphi_n(z) \in U_q \). Since \( z \) was arbitrary, it follows that \( \varphi_n(\Sigma_n) \subset U_q \) for \( n \) sufficiently large, as desired.

**Corollary 3.** If there exists a contactomorphism \( \varphi : S^{2n-1} \to S^{2n-1} \) which has the focal property for \((p, q)\), and \( q \) does not lie on the Hopf circle through \( p \), then a sufficiently large iterate of \( \varphi \) will have no translated points.

**Proof.** As explained above, we can find open sets \( U_p, U_q \) around \( p, q \), respectively, so that no Reeb orbit passes through \( U_p \) and \( U_q \). Since \( \varphi \) has the focal property, Lemma 2 guarantees that eventually \( \Sigma_n \subset U_p \) and \( \varphi_n(\Sigma_n) \subset U_q \). Thus there are no Reeb flow lines joining \( \Sigma_n \) to \( \varphi_n(\Sigma_n) \), so the iterate \( \varphi_n \) has no translated points, as desired.

Therefore, in order to prove Theorem 1, it suffices to construct a contactomorphism contact isotopic to the identity which satisfies the focal property

\[3\]To prove that \( \varphi_n(\Sigma_n) \subset U_q \) for \( n \) sufficiently large, we can also observe that \( \varphi^{-1} \) has the focal property for \((q, p)\), and that:

\[((\varphi^{-1}_n)^* \alpha)_{\varphi_n(z)} = \alpha_{\varphi_n(z)} \iff \alpha_z = (\varphi_n^* \alpha)_z,\]

i.e., \( \varphi_n(\Sigma_n) = \{(\varphi^{-1}_n)^* \alpha = \alpha\} \). Thus the second part of the proof is a consequence the first half. It is not hard to show that \( \varphi^{-1} \) has the focal property for \((q, p)\), although one needs to come up with a slightly clever choice of neighbourhood basis at \( p \) to establish property (ii) for the inverse.
for \((q, p)\) with \(q\) disjoint from the Reeb orbit through \(p\). We perform this construction in the next section.

4. Constructing contactomorphisms with the focal property

First we observe that the focal property is preserved under conjugation:

**Lemma 4.** If \(\varphi\) has the focal property for \((q, p)\), and \(\sigma\) is any contactomorphism, then \(\sigma \circ \varphi \circ \sigma^{-1}\) has the focal property for \((\sigma(q), \sigma(p))\).

**Proof.** Let \(h\) be the scaling factor for \(\sigma\) and \(g\) the scaling factor for \(\varphi\). Then the scaling factor of \(\sigma \circ \varphi \circ \sigma^{-1}\) equals \(h \circ \varphi \circ \sigma^{-1} + g \circ \sigma^{-1} - h \circ \sigma^{-1}\). Using this formula, and the fact that \(p, q\) are fixed points for \(\varphi\), focal property (i) with \((\sigma(q), \sigma(p))\) is easily established for the conjugated contactomorphism. The focal properties (ii) and (iii) are straightforward to check, and are left to the reader. □

Now let \(p, q\) be two points on \(S^{2n-1}\), so that \(q\) is not on the Reeb flow line through \(p\) (this forces \(n > 1\)). Since the contactomorphism group acts 2-transitively, the existence of a focal contactomorphism for any other pair \((Q, P)\) implies the existence of a focal contactomorphism for \((q, p)\). The following explicit formula proves the existence of a focal contactomorphism for a specific pair \((Q, P)\).

**Proposition 5 (see Remark 8.2 in [EKP06]).** Let \(a \in (0, 1)\) and for \((z_1, \ldots, z_n) \in \mathbb{C}^n\) consider the mapping:

\[
\varphi(z) = \left( \frac{1 + a^2}{1 - a^2}, \frac{2az_2}{1 + a^2}, \ldots, \frac{2az_n}{1 + a^2} \right).
\]

Then \(\varphi(S^{2n-1}) \subset S^{2n-1}\), and \(\varphi\) induces a contactomorphism of \(S^{2n-1}\), contact isotopic to the identity, which is focal for \(P = (-1, 0, \ldots, 0)\) and \(Q = (1, 0, \ldots, 0)\).

**Proof.** The \((n+1) \times (n+1)\) matrix acting on \(\mathbb{C}^{n+1}\):

\[
M_a = \frac{1}{2a} \begin{bmatrix}
1 + a^2 & 1 - a^2 & 0 \\
1 - a^2 & 1 + a^2 & 0 \\
0 & 0 & 2a1_{(n-1)(n-1)}
\end{bmatrix}
\]

preserves the quadratic form \(q = -u_0\bar{u}_0 + u_1\bar{u}_1 + \cdots + u_n\bar{u}_n\), i.e., \(M_a\) lies in the group \(U(n, 1)\). Projectivizing via \(z_i = u_i/u_0\), we see that the quotient group \(PU(n, 1) = U(n, 1)/S^1\) acts by biholomorphisms of the unit ball in \(\mathbb{C}^n\) (where \(q < 0\)) and extends smoothly to the unit sphere (where \(q = 0\)). Thus
PU(n, 1) acts on $S^{2n-1}$ by contactomorphisms, since the contact distribution is the distribution of complex tangencies $TS^{2n-1} \cap JT S^{2n-1}$. Our formula for $\varphi$ is given by the action of $M_n$, and hence $\varphi$ is a contactomorphism. Moreover $U(n, 1)$ (and hence $PU(n, 1)$) is a connected Lie group, and so $\varphi$ is contact isotopic to the identity. It remains only to verify the focal properties. We see that $\varphi(z) = z$ if and only if $z_1 = \pm 1$ (hence $z_2 = \cdots = z_n = 0$), and so $P, Q$ are the only fixed points of $\varphi$.

The tangent space to $S^{2n-1}$ at $P, Q$ is equal to $i\mathbb{R} \oplus \mathbb{C}^{n-1}$, and, if $1_{i\mathbb{R}}, 1_{\mathbb{C}^{n-1}}$ denote the projections onto these subspaces (which are the characteristic line and the contact hyperplane, respectively), we can write the derivatives as:

$$d\varphi_P = c_P^2 1_{i\mathbb{R}} + c_P 1_{\mathbb{C}^{n-1}} \quad \text{where} \quad c_P = 1/a > 1,$$

$$d\varphi_Q = c_Q^2 1_{i\mathbb{R}} + c_Q 1_{\mathbb{C}^{n-1}} \quad \text{where} \quad c_Q = a < 1.$$ 

The focal property (i) follows immediately. Focal property (ii) follows by comparing $\varphi$ with its derivative at $Q$. Indeed, for $\epsilon > 0$ small, one can find a small neighborhood of $Q$ (identified with an open set in $T_Q S^{2n-1}$), so that:

$$|\varphi(x) - Q| \leq |d\varphi_Q(x - Q)| + |\varphi(x) - \varphi(Q) - d\varphi_Q(x - Q)| \leq (a + \epsilon) |x - Q|.$$ 

Thus the open sets $U$ given by $|x - Q| \leq r$ satisfy $\varphi(U) \subset U$ if $\epsilon$ and $r$ are sufficiently small.

Finally, recalling that focal property (iii) states $z \neq P \implies \lim_n \varphi_n(z) = Q$, we argue as follows: the sequence $\varphi_n(z)$ must have its limit points contained in the fixed point set $\{P, Q\}$. Moreover, if $\varphi_n(z)$ has $Q$ as a limit point, then, by the attracting property, $\varphi_n(z)$ must converge to $Q$. Therefore if $\varphi_n(z)$ does not converge to $Q$ it must converge to $P$. However, since the derivative at $P$ is expanding, it is clear that $\varphi_n(z) \neq P$ cannot converge to $P$. This completes the proof.

Thus we conclude the existence of a focal contactomorphism for the chosen pair $(p, q)$; simply take the explicit formula from Proposition 5, and

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4 Proof of connectedness: by an explicit argument, one can deform the columns of any $U(n, 1)$ matrix to ensure the first column is $e_0$. Then the other columns form a unitary basis for $\{0\} \times \mathbb{C}^n$. Thus everything in $U(n, 1)$ can be joined to an embedded copy of $U(n)$, which is connected.

5 One can also see directly that $\varphi \to \id$ as $a \to 1$.

6 This formula for the derivative is related to the rescaling contactomorphism $(x, y, z) \mapsto (cx, cy, c^2 z)$. See [EKP06, pp. 1743] for further details.
conjugate by a contactomorphism $\sigma$ which takes $P = (-1,0,\ldots,0)$ to $p$ and $Q = (1,0,\ldots,0)$ to $q$. Applying Corollary 3 then completes the proof of Theorem 1.

**Remark 6.** It is clear that our construction is related to the contact (non)squeezing problem for domains in the standard sphere. Indeed, the existence of contactomorphisms with the focal property implies that large domains can be squeezed inside of arbitrarily small domains. In [Ulj22], the author proves a non-squeezing property holds for domains in certain non-standard contact spheres, namely the Ustilovsky spheres. As part of the argument, the author shows that the Ustilovsky spheres admit Liouville fillings with infinite dimensional symplectic homology, and hence Sandon’s conjecture holds in this case by the results of [MU19].

**References**

[AF10] Peter Albers and Urs Frauenfelder. Leaf-wise intersections and Rabinowitz Floer homology. *Journal of Topology and Analysis*, 2(1):77–98, 2010.

[AFM15] Peter Albers, Urs Fuchs, and Will J. Merry. Orderability and the Weinstein conjecture. *Compositio Mathematica*, 151(12):2251–2272, 2015.

[AK22] Peter Albers and Jungsoo Kang. Rabinowitz Floer homology of negative line bundles and Floer Gysin sequence. arXiv, 2022.

[All22a] Simon Allais. Morse estimates for translated points on unit tangent bundles. arXiv, 2022.

[All22b] Simon Allais. On the minimal number of translated points in contact lens spaces. *Proceedings of the American Mathematical Society*, 150:2685–2693, 2022.

[AM13] Peter Albers and Will J. Merry. Translated points and Rabinowitz Floer homology. *Journal of Fixed Point Theory and Applications*, 13:201–214, 2013.

[EKP06] Yakov Eliashberg, Sang Seon Kim, and Leonid Polterovich. Geometry of contact transformations and domains: orderability versus squeezing. *Geom. Topol.*, 10(3):1635–1747, 2006.

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[GD20] Aaron Gootjes-Dreesbach. Generating functions in symplectic and contact geometry. Master’s thesis, 2020. Supervised by Dr. Fabian Ziltener.

[Giv90] Alexander B. Givental. Nonlinear generalization of the Maslov index. In Theory of singularities and its applications, Advances in Soviet Mathematics, pages 71–103. American Mathematical Society, 1990.

[GKPS17] Gustavo Granja, Yael Karshon, Milena Pabiniak, and Sheila Sandon. Givental’s non-linear Maslov index on Lens spaces. arXiv, 2017.

[MN18] Matthias Meiwes and Kathrin Naef. Translated points on hypertight contact manifolds. Journal of Topology and Analysis, 2018.

[MU19] Will J. Merry and Igor Uljarević. Maximum principles in symplectic homology. Israel Journal of Mathematics, 229:39–65, 2019.

[Oh21] Yong-Geun Oh. Geometry and analysis of contact instantons and entanglement of Legendrian links I. arXiv, 2021.

[Oh22] Yong-Geun Oh. Evaluation transversality of contact instantons and proof of Shelukhin’s conjecture, 2022.

[San12] Sheila Sandon. On iterated translated points for contactomorphisms of $\mathbb{R}^{2n+1}$ and $\mathbb{R}^{2n} \times S^1$. International Journal of Mathematics, 23(2), 2012.

[San13] Sheila Sandon. A Morse estimate for translated points of contactomorphisms of spheres and projective spaces. Geom. Dedicata, 165:95–110, 2013.

[She14] Egor Shelukhin. The Hofer norm of a contactomorphism. J. Symp. Geom., 15, 11 2014.

[Thé98] David Théret. Rotation numbers of Hamiltonian isotopies in complex projective spaces. Duke Math. J., 94(1):13–27, 1998.

[Ulj22] Igor Uljarević. Selective symplectic homology with applications to contact non-squeezing, 2022.
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