NORM INFLATION FOR INCOMPRESSIBLE MAGNETO-HYDRODYNAMIC SYSTEM IN $\dot{B}_{\infty}^{-1,\infty}$

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ABSTRACT. Based on the construction of Bourgain and Pavlović [1], we demonstrate that the solutions to the Cauchy problem for the three dimensional incompressible magneto-hydrodynamics (MHD) system can develop different types of norm inflations in $\dot{B}_{\infty}^{-1,\infty}$. Particularly the magnetic field can develop norm inflation in short time even when the velocity remains small and vice versa. Efforts are made to present a very expository development of the ingenious construction of Bourgain and Pavlović in [1].

KEY WORDS: magneto-hydrodynamic system; norm inflation; $\dot{B}_{\infty}^{-1,\infty}$; plane waves; interactions of plane
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1. Introduction

In this paper we consider the three dimensional incompressible magneto-hydrodynamics (MHD) system:

(1.1)

\[
\begin{align*}
    u_t - \Delta u + u \cdot \nabla u - b \cdot \nabla b + \nabla p &= 0, \\
    b_t - \Delta b + u \cdot \nabla b - b \cdot \nabla u &= 0, \\
    \nabla \cdot u &= 0, \\
    \nabla \cdot b &= 0,
\end{align*}
\]

with the initial conditions

(1.2)

\[
\begin{align*}
    u(x, 0) &= u_0(x), \\
    b(x, 0) &= b_0(x),
\end{align*}
\]

where $x \in \mathbb{R}^3$, $t \geq 0$, $u$ is the fluid velocity, $b$ is the magnetic field. The initial data $u_0$ and $b_0$ are divergence free. When the magnetic field $b(x, t)$ vanishes, incompressible MHD system is just incompressible Navier-Stokes equations. The solutions to MHD system also share the same scaling property of solutions to Navier-Stokes equations, that is,

\[
    u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad b_\lambda(x, t) = \lambda b(\lambda x, \lambda^2 t), \quad p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t)
\]

solve the MHD system (1.1) with initial data

\[
    u_{0\lambda} = \lambda u_0(\lambda x), \quad b_{0\lambda} = \lambda b_0(\lambda x),
\]

if $u(x, t)$ and $b(x, t)$ solve the MHD system (1.1) with the initial data $u_0(x)$ and $b_0(x)$. The spaces that are invariant under the above scaling are called the critical spaces. Examples of critical spaces in three dimension are

\[
\dot{H}^{\frac{1}{2}} \hookrightarrow L^3 \hookrightarrow BMO^{-1} \hookrightarrow \dot{B}_{\infty}^{-1,\infty}.
\]

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The study of Navier-Stokes equations as well as of MHD system in critical spaces has been one of the focus of the research activities since the initial work of Kato [5]. In case of Navier-Stokes equations, on one hand, in 2001, Koch and Tataru [6] were able to establish global well-posedness of Navier-Stokes equations with small initial data in the space $BMO^{-1}$, on the other hand, recently, Bourgain and Pavlović [1] showed the ill-posedness for Navier-Stokes equations in $\dot{B}^{-1,\infty}_{\infty}$. More precisely, Bourgain and Pavlović constructed some arbitrarily small initial data in $\dot{B}^{-1,\infty}_{\infty}$ and produced the so-called norm inflation in the sense that the solution becomes arbitrarily large in $\dot{B}^{-1,\infty}_{\infty}$ after an arbitrarily short time.

In a recent work [8], Miao, Yuan and Zhang proved the existence of a global mild solution in $BMO^{-1}$ for small initial data and uniqueness of such solution in $C([0,\infty); BMO^{-1})$. It is then an interesting problem to study the solutions to MHD system with initial data in $\dot{B}^{-1,\infty}_{\infty}$. In this paper, we discuss different cases of norm inflation phenomena for the MHD system in $\dot{B}^{-1,\infty}_{\infty}$. We construct arbitrarily small initial data $(u_0, b_0)$ in $\dot{B}^{-1,\infty}_{\infty} \times \dot{B}^{-1,\infty}_{\infty}$. This data when evolved in time through the MHD system give raise to "norm inflation" in $\dot{B}^{-1,\infty}_{\infty}$ for the corresponding solutions $(u, b)$. One particularly interesting scenario is that the magnetic field $b$ shows norm inflation while the velocity $u$ remains small. Namely, we show that

**Theorem 1.1.** For any $\delta > 0$ there exists a solution $(u, b, p)$ to the MHD system (1.1) with data $u_0 \in S$ and $b_0 \in S$ which satisfy

$$
\|u(0)\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \delta, \quad \|b(0)\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \delta,
$$

such that for some $0 < T < \delta$

$$
\|b(T)\|_{\dot{B}^{-1,\infty}_{\infty}} \gtrsim \frac{1}{\delta}
$$

but for any $0 < t < T < \delta$

$$
\|u(t)\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \delta.
$$

**Remark 1.2.** We refer the reader to the beginning of section of Preliminaries for the definition of the symbol $\lesssim$.

Our proof follows the methods introduced by Bourgain and Pavlović in [1]. Efforts are made to give a very expository development of the ingenious ideas of Bourgain and Pavlović in [1].

We now recall some auxiliary concepts related to plane waves, which are necessary in the sequel:

- The “diffusion” of a plane wave $v \sin(k \cdot x)$ in $R^3$ is given by
  $$
e^{\Delta t} v \sin(k \cdot x) = e^{-|k|^2 t} v \sin(k \cdot x)
$$
  Thus the magnitude of the diffusion of a plane wave dies down in time in the scale that is measured by the square of the magnitude of the wave vector $k$.

- It is easy to see that $u = b = e^{-|k|^2 t} v \sin(k \cdot x)$ solve MHD system when the wave vector $k$ is orthogonal to the amplitude vector $v$.

- The nonlinear interaction of two such diffusions in MHD system can be captured, and it only produces a slower diffusion if the two wave vectors are close.
We note that these observations are the basis of the original argument of Bourgain and Pavlović in [1]. We will use them to construct a combination of such “diffusions” with least nonlinear interactions yet producing enough slower “diffusions” to cause the norm inflation in short time.

Remark 1.3. It is interesting to observe that even though the initial velocity is zero the velocity can be triggered to develop norm inflation while the magnetic field stays under control. More precisely we can show that, for any $\delta > 0$ there exists a solution $(u, b, p)$ to the MHD system (1.1) with vanishing initial velocity and some $b_0 \in \mathcal{S}$ which satisfies that

$$\|b(0)\|_{\dot{B}^{-1,\infty}} \lesssim \delta$$

and that for some $0 < T < \delta$

$$\|u(T)\|_{\dot{B}^{-1,\infty}} \gtrsim 1/\delta,$$

while for all $0 < t < T < \delta$

$$\|b(t)\|_{B^{-1,\infty}} \lesssim \delta.$$

Remark 1.4. We also note that due to the interaction between the velocity and the magnetic field, if initially they are the same, they may restrain each other from norm inflations.

In our paper, we present our results in $\mathbb{T}^3$. But, as pointed out in [1], the proof can be modified to $\mathbb{R}^3$.

2. Preliminaries

2.1. Notation. We denote by $A \lesssim B$ an estimate of the form $A \leq CB$ with some constant $C$, and by $A \sim B$ an estimate of the form $C_1 B \leq A \leq C_2 B$ with some constants $C_1, C_2$. For completeness we recall the defining norms for the Besov space $\dot{B}^{-1, \infty}$ and the $BMO^{-1}$ space

$$\|f\|_{\dot{B}^{-1, \infty}} = \sup_{t > 0} t^{1/2} \|e^{t \Delta} f\|_{L^\infty}.$$  

$$\|f\|_{BMO^{-1}} = \sup_{x_0 \in \mathbb{R}^3, R > 0} \left( \frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0, \sqrt{R})} |e^{t \Delta} f(y)|^2 \, dy \, dt \right)^{1/2}.$$  

We will also work with the so-called inhomogeneous Besov space $B^{-1, \infty}_\infty$ with the norm

$$\|f\|_{B^{-1, \infty}_\infty} = \sup_{0 < t < 1} \sqrt{t} \|e^{t \Delta} f\|_{L^\infty}.$$  

Clearly

$$\|f\|_{B^{-1, \infty}_\infty} \leq \|f\|_{\dot{B}^{-1, \infty}}.$$  

and,

$$\|f\|_{B^{-1, \infty}_\infty} \leq \|f\|_{L^\infty},$$  

since $\|e^{t \Delta} f\|_{L^\infty} \leq \|f\|_{L^\infty}$. 

2.2. The well-posedness result of the incompressible MHD system in $BMO^{-1}$. We recall the well-posedness result of C. Miao, B. Yuan and B. Zhang in $BMO^{-1}$ in [S]. For this we introduce the spaces $X_T$ and the corresponding norm.

**Definition 2.1.** Let $u(x, t)$ be a measurable function on $\mathbb{R}^3 \times [0, T)$ for $T > 0$ and let

\begin{align}
\|u(\cdot, t)\|_{X_T} &= \sup_{0 \leq t < T} t^{1/2} \|u(\cdot, t)\|_{L^\infty} \\
&\quad + \sup_{x_0 \in \mathbb{R}^3, 0 < R < T} \left( \frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0, \sqrt{R})} |u(y, t)|^2 dy \, dt \right)^{\frac{1}{2}}.
\end{align}

Then the space-time space $X_T$ is defined by

$$X_T = \{ f(x, t) \in L^2(0, T; L^2(\mathbb{R}^3)) : \|f\|_{X_T} < \infty \}.$$ 

It is worth to mention that, for each $t \in (0, T]$, 

$$\|f(\cdot, t)\|_{L^\infty} \leq \frac{1}{\sqrt{t}} \|f\|_{X_T}.$$ 

In [S], Miao, Yuan and Zhang proved the following existence theorem:

**Theorem 2.2.** (Miao, Yuan and Zhang) Let $(u_0(x), b_0(x)) \in BMO^{-1} \times BMO^{-1}$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then, there exists a positive constant $\varepsilon$ such that if $\|(u_0, b_0)\|_{BMO^{-1}} < \varepsilon$ then the MHD system has a unique global mild solution $(u(x, t), b(x, t)) \in X_T \times X_T$ satisfying that $\|(u(x, t), b(x, t))\|_{X_T \times X_T} \leq 2\varepsilon$ for all $T > 0$.

2.3. Bilinear operators. Let $\mathbb{P}$ denote the projection on divergence-free vector fields, which acts on a function $\phi$ as 

$$\mathbb{P}(\phi) = \phi + \nabla \cdot (-\nabla)^{-1} \text{div} \phi.$$ 

As shown in [6][8] the bilinear operator

$$\mathcal{B}(u, v) = \int_0^t e^{(t-\tau)\Delta} \mathbb{P}(u \cdot \nabla v) \, d\tau,$$

maps $X_T \times X_T$ into $X_T$ continuously. More precisely we have,

$$\|\mathcal{B}(u, v)\|_{X_T} \lesssim \|u\|_{X_T} \|v\|_{X_T}.$$ 

2.4. Rewriting the MHD system. Following ideas from [1] we rewrite the MHD system [11] introducing the expression

\begin{align}
u &= e^{t\Delta} u_0 - u_1 + y \\
b &= e^{t\Delta} b_0 - b_1 + z
\end{align}

where

\begin{align}
u_1(x, t) &= \mathcal{B}(e^{t\Delta} u_0(x), e^{t\Delta} u_0(x)) - \mathcal{B}(e^{t\Delta} b_0(x), e^{t\Delta} b_0(x)), \\
b_1(x, t) &= \mathcal{B}(e^{t\Delta} u_0(x), e^{t\Delta} b_0(x)) - \mathcal{B}(e^{t\Delta} b_0(x), e^{t\Delta} u_0(x)),
\end{align}

An easy calculation shows that

\begin{align}
y_t - \Delta y + G_0 + G_1 + G_2 &= 0, \\
z_t - \Delta z + K_0 + K_1 + K_2 &= 0,
\end{align}
where
\[ G_0 = \mathbb{P}[(e^{t\Delta}u_0 \cdot \nabla)u_1 + (u_1 \cdot \nabla)e^{t\Delta}u_0 + (u_1 \cdot \nabla)u_1] \]

\[ = \mathbb{P}[(e^{t\Delta}b_0 \cdot \nabla)b_1 + (b_1 \cdot \nabla)e^{t\Delta}b_0 + (b_1 \cdot \nabla)b_1] \]

\[ G_1 = \mathbb{P}[(e^{t\Delta}u_0 \cdot \nabla)y + (u_1 \cdot \nabla)y + (y \cdot \nabla)e^{t\Delta}u_0 + (y \cdot \nabla)u_1] \]

\[ = \mathbb{P}[(e^{t\Delta}b_0 \cdot \nabla)z + (b_1 \cdot \nabla)z + (z \cdot \nabla)e^{t\Delta}b_0 + (z \cdot \nabla)b_1] \]

\[ G_2 = \mathbb{P}[(y \cdot \nabla)y] - \mathbb{P}[(z \cdot \nabla)z] \]

and
\[ K_0 = \mathbb{P}[(e^{t\Delta}u_0 \cdot \nabla)b_1 + (u_1 \cdot \nabla)e^{t\Delta}b_0 + (u_1 \cdot \nabla)b_1] \]

\[ = \mathbb{P}[(e^{t\Delta}b_0 \cdot \nabla)u_1 + (b_1 \cdot \nabla)e^{t\Delta}u_0 + (b_1 \cdot \nabla)u_1] \]

\[ K_1 = \mathbb{P}[(e^{t\Delta}u_0 \cdot \nabla)z + (u_1 \cdot \nabla)z + (y \cdot \nabla)e^{t\Delta}b_0 + (y \cdot \nabla)b_1] \]

\[ = \mathbb{P}[(e^{t\Delta}b_0 \cdot \nabla)y + (b_1 \cdot \nabla)y + (z \cdot \nabla)e^{t\Delta}u_0 + (z \cdot \nabla)u_1] \]

\[ K_2 = \mathbb{P}[(y \cdot \nabla)z] - \mathbb{P}[(z \cdot \nabla)y]. \]

Here \(G_0\) and \(K_0\) are constants, \(G_1\) and \(K_1\) are linear, while \(G_2\) and \(K_2\) are quadratic in terms of \(y\) and \(z\).

**Remark 2.3.** Note that although the second equation in the MHD system has no pressure, since \(u\) and \(b\) are both divergence free, the term \(u \cdot \nabla b - b \cdot \nabla u\) is automatically divergence free. Hence the projector \(\mathbb{P}\) acting on this term does not change the second equation and hence we can write \(b_1\) and \(K_i\)’s as described above.

### 3. Interactions of Plane Waves

In this section we show how the diffusions of plane waves interact in MHD system. These interactions are the basis for the constructions of initial data which will evolve into the different cases of velocity and magnetic field norm inflations.

#### 3.1. Diffusion of a plane wave.

As a first step, we consider the same initial data for velocity and magnetic using one single plane wave. Suppose \(k \in \mathbb{R}^3, v \in S^2\) and \(k \cdot v = 0\). Let

\[ u_0 = b_0 = v \cos(k \cdot x). \]

Then \(\nabla \cdot u_0 = 0\), and \(\nabla \cdot b_0 = 0\). and

\[ e^{t\Delta}v \cos(k \cdot x) = e^{-|k|^2t}v \cos(k \cdot x). \]

In fact the “diffusions” \((e^{t\Delta}v \cos(k \cdot x), e^{t\Delta}v \cos(k \cdot x))\) of a plane wave solve the MHD system with vanishing pressure. It is important to notice that

- \(\|v \cos(k \cdot x)\|_{B^2_{-1, \infty}} \lesssim \frac{1}{|k|}\),
- \(\|e^{t\Delta}v \cos(k \cdot x)\|_{X_T} \lesssim \frac{1}{|k|}\),

which says that the size of a plane wave in the space \(B^2_{-1, \infty}\) is reciprocal to the magnitude of its wave vector, and in \(X_T\) it is bounded by this same reciprocal.
3.2. Interaction of plane waves. Now we consider different plane wave initial data for the velocity and magnetic. Suppose $k_i \in \mathbb{R}^3$, $v_i \in S^2$ and $k_i \cdot v_i = 0$, for $i = 1, 2$. Let

\[
u_0 = \cos(k_1 \cdot x)v_1, \]
\[b_0 = \cos(k_2 \cdot x)v_2.\]

Using the decomposition given in Section 2.4,
\[u = e^{t\Delta}u_0, \quad b = e^{t\Delta}b_0 - b_1 + z\]
solve the MHD system with vanishing pressure. To simplify our calculations we assume that
\[k_2 \cdot v_1 = 0, \quad \text{and} \quad k_1 \cdot v_2 = \frac{1}{2},\]
which eliminates the term $e^{t\Delta}u_0 \cdot \nabla(e^{t\Delta}b_0)$ and gives
\[e^{t\Delta}b_0 \cdot \nabla(e^{t\Delta}u_0) = -e^{-(|k_1|^2 + |k_2|^2)t}v_1 \sin(k_1 \cdot x)\cos(k_2 \cdot x)(k_1 \cdot v_2)
= -\frac{1}{4}e^{-(|k_1|^2 + |k_2|^2)t}v_1(\sin((k_1 - k_2) \cdot x) + \sin((k_1 + k_2) \cdot x)).\]

Hence
\[b_1 = \frac{1}{4}v_1 \sin((k_1 - k_2) \cdot x) \int_0^t e^{-(|k_1|^2 + |k_2|^2)\tau}e^{-|k_1 - k_2|^2(t-\tau)}d\tau
+ \frac{1}{4}v_1 \sin((k_1 + k_2) \cdot x) \int_0^t e^{-(|k_1|^2 + |k_2|^2)\tau}e^{-|k_1 + k_2|^2(t-\tau)}d\tau
= b_{1,0} + b_{1,1},\]
where
\[b_{1,0} = \frac{1}{4}v_1 \sin((k_1 - k_2) \cdot x) \frac{-e^{-(|k_1|^2 + |k_2|^2)t} + e^{-|k_1 - k_2|^2t}}{2k_1 \cdot k_2}\]
and
\[b_{1,1} = \frac{1}{4}v_1 \sin((k_1 + k_2) \cdot x) \frac{e^{-(|k_1|^2 + |k_2|^2)t} - e^{-|k_1 + k_2|^2t}}{2k_1 \cdot k_2}.\]

Therefore, if we can manage to control $z$ in the light of the continuity of the bilinear operator $B$ in $X_T$, then the interaction of two plane waves are small in $B_{ \infty}^{-1, \infty}$ if neither the sum nor the difference of their wave vectors is small in magnitude. In the mean time, the interaction is sizable in $B_{ \infty}^{-1, \infty}$ if either the sum or the difference of their wave vectors is small in magnitude.

4. Proof of theorem

In this section we will follow the idea from [1] to construct initial data to produce norm inflation for solutions to MHD system. From the discussions in the previous sections we know that the interaction of two plane waves is not enough to show the norm inflation. We need to build interactions of more plane waves. The construction in [1] depends on the rather sophisticated choices of plane waves. We will use a similar scheme.
4.1. Construction of initial data for the MHD system. For a fixed small number $\delta > 0$ we will specify later following initial data:

\begin{equation}
    u_0 = \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} |k_s| v_s \cos(k_s \cdot x)
\end{equation}

and

\begin{equation}
    b_0 = \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} |k'_s| v'_s \cos(k'_s \cdot x).
\end{equation}

We expect for each $s$, the interaction of the two plane waves $v_s \cos(k_s \cdot x)$ and $v'_s \cos(k'_s \cdot x)$ is sizable in $\dot{B}^{-1,\infty}_\infty$; while the interactions of plane waves of different $s$ is small, as demonstrated in Section 3.2. Hence

- Wave vectors: The wave vectors $k_s \in \mathbb{R}^3$ are parallel to a given vector $k_0 \in \mathbb{R}^3$. The modulo $|k_0|$ will be taken to be large, depending on $Q$. The magnitude of $k_s$ is defined by,

\begin{equation}
    |k_s| = 2^s |k_0||k_{s-1}|, \quad s = 1, 2, 3, ..., r.
\end{equation}

The wave vectors $k'_s \in \mathbb{R}^3$ is defined by

\begin{equation}
    k_s - k'_s = \eta
\end{equation}

for a given vector $\eta \in \mathbb{S}^2$.

- Amplitude vectors: The amplitude vectors $v_s, v'_s \in \mathbb{S}^2$ satisfy

\begin{equation}
    k_s \cdot v_s = k'_s \cdot v'_s = 0
\end{equation}

to ensure the initial data are divergence free.

- Auxiliary assumptions: We also require that

\begin{equation}
    \eta \cdot v_s = 0, \quad \eta \cdot v'_s = \frac{1}{2}
\end{equation}

to simplify our calculations. In fact we will choose $v_s = v$ a fixed vector.

We first point out the following simple facts to further motivate the choices of the magnitudes of $k_s$.

**Lemma 4.1.**

\begin{equation}
    \sum_{l<s} |k_l| \sim |k_{s-1}| \quad \text{and} \quad \sum_{l<s} |k'_l| \sim |k'_{s-1}|
\end{equation}

\begin{equation}
    \sum_{s=1}^{r} |k_s| e^{-|k_s|^2 t} \lesssim \frac{1}{\sqrt{t}} \quad \text{and} \quad \sum_{s=1}^{r} |k'_s| e^{-|k'_s|^2 t} \lesssim \frac{1}{\sqrt{t}}
\end{equation}

\begin{equation}
    v_i \cdot k_j = v_i \cdot k'_j = v_i \cdot \eta = 0, \quad \forall \quad i, j = 1, 2, \ldots, r.
\end{equation}

\begin{equation}
    \sum_{i=1}^{r} |k_i| e^{-\frac{|k_i|^2}{160 t}} \lesssim 1, \quad \text{and} \quad \sum_{i=1}^{r} |k'_i| e^{-\frac{|k'_i|^2}{160 t}} \lesssim 1.
\end{equation}
Proof of Lemma: By the definition (4.16), it is clear that $|k_{L-1}| < \frac{1}{2}|k_l|$, which easily implies the first statement. For second statement, again due to the definition (4.16), we know that $|k_s| \sim |k_s| - |k_{s-1}|$. Thus,

$$\sum_{s=1}^{r} |k_s| e^{-|k_s|^2 t} \sim \sum_{s=1}^{r} (|k_s| - |k_{s-1}|) e^{-|k_s|^2 t},$$

while the later one can be considered as a finite Riemann summation of the function $e^{-x^2}$. Therefore

$$\sum_{s=1}^{r} |k_s| e^{-|k_s|^2 t} \lesssim \int_0^{\infty} e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-x^2} d(x \sqrt{t}) = \frac{\sqrt{\pi}}{2 \sqrt{t}}.$$

For the third statement, we note that, $k_s$ are parallel to the given vector $k_0$ for all $s$ and $v_s = v$ is a fixed vector by the above choice. Hence, by (4.18)

$$v_i \cdot k_j = 0, \quad \text{for all } i, j = 1, 2, ..., r.$$  

On the other hand side, from (4.17) and (4.19), we have

$$v_i \cdot k_j' = v_i \cdot (k_j - \eta) = v_i \cdot k_j - v_i \cdot \eta = 0.$$  

The forth statement in the Lemma follows from the second one provided an appropriate choice of $k_0$. It completes the proof of Lemma.

Next we calculate the norm of our initial data.

**Lemma 4.2.** For $u_0$ and $b_0$ given in (4.14) (4.15) we have

$$\|u_0\|_{\dot{B}^{-1,\infty}} \lesssim \frac{Q}{\sqrt{T}}, \quad \|b_0\|_{\dot{B}^{-1,\infty}} \lesssim \frac{Q}{\sqrt{T}}.$$  

**Proof of Lemma:** For the given initial data $u_0$, we have that, due to (3.13)

$$e^{\tau \Delta} u_0 = \frac{Q}{\sqrt{T}} \sum_{s=1}^{r} |k_s| v_s \cos(k_s \cdot x) e^{-|k_s|^2 \tau}.$$  

Hence by Lemma 4.1

$$\|u_0\|_{\dot{B}^{-1,\infty}} \sim \frac{Q}{\sqrt{T}} \sup_{\tau > 0} \sqrt{T} \sum_{s=1}^{r} |k_s| e^{-|k_s|^2 \tau}$$

$$\lesssim \frac{Q}{\sqrt{T}},$$

The bound for $\|b_0\|_{\dot{B}^{-1,\infty}}$ follows in the similar way.

**Lemma 4.3.** For $u_0$ and $b_0$ given in (4.14) (4.15) we have

$$\|e^{t \Delta} u_0\|_{X_T} \lesssim Q, \quad \|e^{t \Delta} b_0\|_{X_T} \lesssim Q.$$  

**Proof of Lemma:** We only need to verify one of the two.

$$\|e^{t \Delta} u_0\|_{X_T} \lesssim \frac{Q}{\sqrt{T}} \left( 1 + \sup_{\tau \in [0,T]} \left( \int_0^{\tau} \left( \sum_{i=1}^{r} |k_i| e^{-|k_i|^2 \tau} \right)^2 d\tau \right)^{\frac{1}{2}} \right),$$

where

$$\left( \sum_{i=1}^{r} |k_i| e^{-|k_i|^2 \tau} \right)^2 \lesssim \sum_{i=1}^{r} |k_i|^2 e^{-2|k_i|^2 \tau} + 2 \sum_{i=1}^{r} |k_i| e^{-|k_i|^2 \tau} \sum_{j < i} \sum_{i=1}^{r} |k_j| \lesssim \sum_{i=1}^{r} |k_i|^2 e^{-|k_i|^2 \tau}$$

□
Then a straight calculation shows

\[ \int_0^t \left( \sum_{i=1}^r |k_i|^2 e^{-|k_i|^2\tau} \right) d\tau \lesssim \sum_{i=1}^r (1 - e^{-|k_i|^2}) \lesssim r, \]

which implies

\[ \| e^{t\Delta} u_0 \|_{X_T} \lesssim \frac{Q}{\sqrt{r}} + Q. \]

This completes the proof of the Lemma.

Finally we make a note that

**Lemma 4.4.** For \( t \in [0, +\infty) \),

\[ \| e^{t\Delta} u_0 \|_{\dot{B}^{-1,\infty}_\infty} \lesssim \frac{Q}{\sqrt{r}} e^{-|k_0|^2 t}, \quad \| e^{t\Delta} b_0 \|_{\dot{B}^{-1,\infty}_\infty} \lesssim \frac{Q}{\sqrt{r}} e^{-|k_0|^2 t}. \]

### 4.2. Analysis of \( u_1 \)

As demonstrated in Section 3.2 we consider the decomposition

\[ u = e^{t\Delta} u_0 - u_1 + y \]
\[ b = e^{t\Delta} b_0 - b_1 + z \]

We want to handle the \( u_1 \) first. Recall the definition (2.10)

\[ u_1 = B(e^{t\Delta} u_0, e^{t\Delta} u_0) - B(e^{t\Delta} b_0, e^{t\Delta} b_0) \]

By our discussions in Section 3.2 the interactions should be small. By the fact that \( v_i, k_j = 0 \) it is immediately seen that

\[ e^{t\Delta} u_0 \cdot \nabla e^{t\Delta} u_0 = 0. \]

Then a straight calculation shows

\[ e^{t\Delta} b_0 \cdot \nabla e^{t\Delta} b_0 = -\frac{Q^2}{2r} \sum_{i,j=1}^r |k_i'| |k_j'| e^{-||k_i'|^2+|k_j'|^2} t (v_i' \cdot k_j') v_j' \cos(k_j' \cdot x) \sin(k_j' \cdot x) \]
\[ = -\frac{Q^2}{2r} \sum_{i,j=1}^r |k_i'| |k_j'| e^{-||k_i'|^2+|k_j'|^2} t (v_i' \cdot k_j') v_j' \sin(k_i' + k_j') \cdot x \]
\[ - \frac{Q^2}{2r} \sum_{i,j=1}^r |k_i'| |k_j'| e^{-||k_i'|^2+|k_j'|^2} t (v_i' \cdot k_j') v_j' \sin(k_j' - k_i') \cdot x \]

and

\[ \mathbb{P} e^{t\Delta} b_0 \cdot \nabla e^{t\Delta} b_0 = \frac{Q^2}{2r} \sum_{i,j=1}^r |k_i'| |k_j'| e^{-||k_i'|^2+|k_j'|^2} t (v_i' \cdot k_j') u_j \sin(k_j' + k_i') \cdot x \]
\[ - \frac{Q^2}{2r} \sum_{i,j=1}^r |k_i'| |k_j'| e^{-||k_i'|^2+|k_j'|^2} t (v_i' \cdot k_j') w_j \sin(k_j' - k_i') \cdot x \]
\[ = E_1 + E_2, \]

where \( u_j \) is the projection of \( v_i' \) to the orthogonal to \( k_j' + k_i' \) and \( w_j \) is the projection of \( v_j' \) to the orthogonal to \( k_j' - k_i' \). Hence

\[ B((e^{t\Delta} b_0, e^{t\Delta} b_0)) = \int_0^t e^{(t-\tau)\Delta} E_1 d\tau + \int_0^t e^{(t-\tau)\Delta} E_2 d\tau = F_1 + F_2. \]
We work for $F_1$ in the following. $F_2$ will be handled similarly and so the detail is omitted.

$$F_1 = \frac{Q^2}{2\tau} \sum_{i,j=1}^{r} |k'_i| |k'_j| (u'_i \cdot k'_j) u_j \sin(k'_i + k'_j) \cdot e^{-((|k'_i|^2 + |k'_j|^2) \tau t - e^{-|k'_i + k'_j|^2 \tau t})}.$$ 

By the fact that $k'_i \cdot v'_i = 0$ and since the function $\frac{1-e^{-x}}{x}$ is bounded for $x > 0$, we have

$$|e^{\tau \Delta} F_1| \lesssim \frac{Q^2}{r} \sum_{i,j=1}^{r} |k'_i|^2 |k'_j|^2 t e^{-(|k'_i|^2 + |k'_j|^2) \tau t} e^{-|k'_i + k'_j|^2 \tau t}$$

and

$$|F_1| \lesssim \frac{Q^2}{r} \sum_{i,j=1}^{r} |k'_i|^2 |k'_j|^2 t e^{-(|k'_i|^2 + |k'_j|^2) \tau t}.$$ 

Hence

$$|F_1| \lesssim \frac{Q^2}{r} \sum_{i,j=1}^{r} |k'_i|^2 |k'_j|^2 t e^{-(|k'_i|^2 + |k'_j|^2) \tau t} \frac{Q^2}{r} \sum_{j=1}^{r} |k'_{j-1}| e^{-\frac{1}{2} |k'_j|^2 \tau t}$$

and

$$|e^{\tau \Delta} F_1| \lesssim \frac{Q^2}{r} \sum_{j=1}^{r} |k'_{j-1}| e^{-\frac{1}{2} |k'_j|^2 \tau t} e^{-\frac{1}{2} |k'_j|^2 \tau t},$$

where we use the fact that $xe^{-x}$ is bounded for $x > 0$ and Lemma 4.1.

$$\|F_1\|_{X_T} \lesssim \frac{Q^2}{r} \sum_{j=1}^{r} |k'_{j-1}| \sup_{t \in [0,T]} \sqrt{t} e^{-\frac{1}{2} |k'_j|^2 t}$$

$$+ \sup_{t \in [0,T]} \left( \int_0^t \left( \sum_{j=1}^{r} |k'_{j-1}| e^{-\frac{1}{2} |k'_j|^2 r} \right)^2 dr \right)^{\frac{1}{2}}$$

where

$$\sum_{j=1}^{r} |k'_{j-1}| \sup_{t \in [0,T]} \sqrt{t} e^{-\frac{1}{2} |k'_j|^2 t} \lesssim \sum_{j=1}^{r} \frac{|k'_{j-1}|}{|k'_j|} \lesssim 1$$

and

$$\sup_{t \in [0,T]} \left( \int_0^t \left( \sum_{j=1}^{r} |k'_{j-1}| e^{-\frac{1}{2} |k'_j|^2 r} \right)^2 dr \right)^{\frac{1}{2}} \lesssim \left( \sum_{j=1}^{r} \frac{|k'_{j-1}|}{|k'_j|} \right)^{\frac{1}{2}} \lesssim 1$$

by the same argument as used in the proof of Lemma 4.3. And similarly

$$\|F_1\|_{B^{-1}_{2,\infty}} = \sup_{\tau > 0} \sqrt{\tau} \|e^{\tau \Delta} F_1\|_{L^\infty} \lesssim \frac{Q^2}{r} \sum_{j=1}^{r} \frac{|k'_{j-1}|}{|k'_j|} \lesssim \frac{Q^2}{r}.$$ 

Therefore we conclude that

**Lemma 4.5.**

$$\|u_1\|_{B^{-1}_{2,\infty}} \lesssim \frac{Q^2}{r}, \quad \|u_1\|_{X_T} \lesssim \frac{Q^2}{r}.$$ 

**Proof of Lemma:** $F_2$ can be handled just like what we did with $F_1$. 

4.3. Analysis of $b_1$. First we recall that from (2.11)
$$b_1(x,t) = B(e^{t\Delta}u_0(x), e^{t\Delta}b_0(x)) - B(e^{t\Delta}b_0(x), e^{t\Delta}u_0(x)).$$
Similar to the calculations in the previous section, first, due to the fact that $v_i \cdot k^\prime_j = 0$, we have
$$e^{	au \Delta}u_0 \cdot \nabla e^{	au \Delta}b_0 = 0.$$
And
$$e^{	au \Delta}b_0 \cdot \nabla e^{	au \Delta}u_0 = -\frac{Q^2}{2r} \sum_{i,j=1}^{r} |k_i||k_j| e^{-(|k_i|^2+|k_j|^2)t} (v_i' \cdot k_j)v_j \sin(k_j + k^\prime_j) \cdot x$$
$$- \frac{Q^2}{2r} \sum_{i\neq j}^{r} |k_i||k_j| e^{-(|k_i|^2+|k_j|^2)t} (v_i' \cdot k_j)v_j \sin(k_j - k^\prime_j) \cdot x$$
$$- \frac{Q^2}{2r} \sum_{i=1}^{r} |k_i||k^\prime_i| e^{-(|k_i|^2+|k^\prime_i|^2)t} (v_i' \cdot k_i)v_i \sin(k_i - k^\prime_i) \cdot x.$$
We will write
$$e^{	au \Delta}u_0 \cdot \nabla e^{	au \Delta}b_0 - e^{	au \Delta}b_0 \cdot \nabla e^{	au \Delta}u_0 = \tilde{b}_{1,0} + \tilde{b}_{1,1},$$
where
$$\tilde{b}_{1,0} = \frac{Q^2}{2r} \sum_{i=1}^{r} |k_i||k^\prime_i| e^{-(|k_i|^2+|k^\prime_i|^2)t} \sin(k_i - k^\prime_i) \cdot x (v_i' \cdot k_i)v_i$$
$$= \frac{Q^2}{4r} \sin(\eta \cdot x) \sum_{i=1}^{r} |k_i||k^\prime_i| e^{-(|k_i|^2+|k^\prime_i|^2)t} v_i,$$
due to our choices of wave vectors and amplitude vectors in Section 4.1. We then set
$$\tilde{b}_{1,1} = \tilde{d}_{1,1} + \tilde{e}_{1,1},$$
where
$$\tilde{d}_{1,1} = \frac{Q^2}{2r} \sum_{i=1}^{r} |k_i||k^\prime_i| e^{-(|k_i|^2+|k^\prime_i|^2)t} \sin(k_i + k^\prime_i) \cdot x (v_i' \cdot k_i)v_i$$
$$= \frac{Q^2}{4r} \sum_{i=1}^{r} |k_i||k^\prime_i| e^{-(|k_i|^2+|k^\prime_i|^2)t} v_i \sin(k_i + k^\prime_i) \cdot x.$$
By the choices of $k^\prime_i$, which behaves more or less like $k_i$ for each $i$ when $|k_0|$ is very large, we conclude that $\tilde{e}_{1,1}$ can be handled just like what we did for $E_1$ in the previous section. We then have
$$d_{1,1} = \int_0^t e^{\Delta(t-\tau)} \mathcal{P} \tilde{d}_{1,1}(\tau) d\tau$$
$$= \frac{Q^2}{4r} \sum_{i=1}^{r} |k_i||k^\prime_i| v_i \sin(k_i + k^\prime_i) \cdot x \frac{e^{-|k_i|^2-|k^\prime_i|^2t} - e^{-|k_i+k^\prime_i|^2t}}{|k_i+k^\prime_i|^2 - (|k_i|^2 + |k^\prime_i|^2)}$$
which gives
$$|d_{1,1}| \lesssim \frac{Q^2}{r} \sum_{i=1}^{r} |k_i|^2 t e^{-\frac{1}{2}|k_i|^2t} \lesssim \frac{Q^2}{r} \sum_{i=1}^{r} e^{-\frac{1}{2}|k_i|^2t}.$$
Lemma 4.6. we may conclude that

\[ |e^{\Delta d_{1,1}}| \lesssim \frac{Q^2}{r} \sum_{i=1}^{r} e^{-\frac{1}{2}|k_i|^2} e^{-\frac{1}{2}|k_i|^2 r}. \]

Hence it is even easier to handle \( d_{1,1} \) than to handle \( F_1 \) in the previous section. Now, if denote

\[ b_{1,1} = \int_0^t e^{\Delta(t-\tau)} \mathcal{P} \tilde{b}_{1,1}(\cdot, \tau) d\tau, \]

we may conclude that

**Lemma 4.6.**

\[ \|b_{1,1}\|_{\dot{B}^1_{-1,\infty}} \lesssim \frac{Q^2}{r}, \quad \|b_{1,1}\|_{X_T} \lesssim \frac{Q^2}{r}. \]

The focus is now on

\[ b_{1,0} = \int_0^t e^{\Delta(t-\tau)} \mathcal{P} \tilde{b}_{1,0}(\cdot, \tau) d\tau \]

\[ = \frac{Q^2}{4r} e^{-t} \sin(\eta \cdot x) v \sum_{i=1}^{r} |k_i||k_i'| \int_0^t e^{1-((|k_i|^2+|k_i'|^2) |\eta| + 2|k_i|^2) r} d\tau, \]

\[ = \frac{Q^2}{4r} \sin(\eta \cdot x) v \sum_{i=1}^{r} |k_i||k_i'| e^{-t} \frac{e^{-((|k_i|^2+|k_i'|^2) t)} \cdot (|k_i|^2 + 2|k_i'|^2 - 1)} \]

since \( v_i = v \) is fixed. Therefore we have

**Lemma 4.7.** Suppose \( \frac{1}{|k_i|} \ll T \ll 1 \). Then

\[ \|b_{1,0}(\cdot, T)\|_{\dot{B}^1_{-1,\infty}} = \sup_{\tau \in (0,1)} \sqrt{r} \|e^{\Delta} b_{1,0}\|_{L^\infty} \sim Q^2, \quad \|b_{1,0}\|_{X_T} \lesssim \sqrt{r} Q^2. \]

**Proof of Lemma:** By (4.31), it follows that

\[ b_{1,0} \sim Q^2 \sin(\eta \cdot x) v, \]

for \( \frac{1}{|k_i|} \ll T \ll 1 \). Indeed, \( T \ll 1 \) insures \( e^{-t} \sim 1 \), for \( t \leq T; \) \( \frac{1}{|k_i|^2} \ll T \) insures \( e^{-((|k_i|^2+|k_i'|^2) t)} \sim 0 \). And, since \( |k_i| \) is very large, \( |k_s| \sim |k_i'| \) for every \( s \) by (4.17).

Thus,

\[ \|b_{1,0}\|_{\dot{B}^1_{-1,\infty}} \sim Q^2 \sup_{0 < t < 1} \sqrt{t} \|e^{\Delta} \sin(\eta \cdot x)\|_{L^\infty} \]

\[ \sim Q^2 \sup_{0 < t < 1} \sqrt{t} e^{-|\eta|^2 t} \]

\[ \sim Q^2. \]

And

\[ \|b_{1,0}\|_{X_T} \sim Q^2 \sup_{0 < t < T} \sqrt{r} \|\sin(\eta \cdot x)\|_{L^\infty} \]

\[ + Q^2 \sup_{x_0, 0 < R < T} \left( \frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0, \sqrt{R})} |\sin(\eta \cdot x)|^2 dx dt \right)^{\frac{1}{2}} \]

\[ \lesssim \sqrt{r} Q^2. \]
4.4. **Analysis of** $y$ **and** $z$. **In this section we analyze the parts** $y$ **and** $z **of the solution. The idea is to control** $y$ **and** $z **using the boundedness of the bilinear operator** $B **in the space** $X_T$. **Naively one would hope that nonlinear terms turn out to be even smaller. But the trouble is at the linear term** $G_1$ **and** $K_1 **since**

$$
\| e^{t\Delta}u_0 \|_{X_T} \lesssim Q, \quad \text{and} \quad \| e^{t\Delta}b_0 \|_{X_T} \lesssim Q,
$$

by Lemma 4.3, which is the best we can have. The problem is the plane waves should not be lumped together in one single time scale. Plane waves that have much bigger wave vectors diffuse much quicker. Therefore it makes sense to analyze how $y$ and $z$ evolve in different time scales and see how different plane waves contribute. In [1], Bourgain and Pavlović very skillfully designed time steps to group appropriately the plane waves. We will use the same idea. We now introduce the time step division as used in [1]. Let

$$
0 < T_1 < T_2 < \cdots < T_\beta,
$$

where $\beta = Q^3$ and

$$
T_\alpha = |k_{r_\alpha}|^{-2}, \quad r_\alpha = r - \alpha Q^{-3} r, \quad \alpha = 1, 2, \ldots.
$$

In particular, $r_\beta = 0$ and $T_\beta = |k_0|^{-2}$. The following are the key estimates for the time step design in [1].

**Lemma 4.8.** Suppose that $r$ is sufficiently large for a fixed large number $Q$. Then

$$
\| (e^{t\Delta}u_0)\chi_{[T_\alpha,T_{\alpha+1}]}(t) \|_{X_{T_\alpha+1}} \lesssim Q^{-1/2}, \quad \| (e^{t\Delta}b_0)\chi_{[T_\alpha,T_{\alpha+1}]}(t) \|_{X_{T_\alpha+1}} \lesssim Q^{-1/2}.
$$

**Proof of Lemma:** The proof is the same as in [1]. For the convenience of the reader we outline the proof showing how the design of the time steps. First use the decomposition

$$
(e^{t\Delta}u_0)\chi_{[T_\alpha,T_{\alpha+1}]}(t) \approx L_1 + L_2 + L_3,
$$

where

$$
L_1 = \frac{Q}{\sqrt{r}} \sum_{s < r_{\alpha+1}} |k_s|v_s \cos(k_s \cdot x)e^{-|k_s|^2 t} \chi_{[T_\alpha,T_{\alpha+1}]}(t)
$$

$$
L_2 = \frac{Q}{\sqrt{r}} \sum_{s = r_{\alpha+1}} \sum_{r_\alpha < s \leq r} |k_s|v_s \cos(k_s \cdot x)e^{-|k_s|^2 t} \chi_{[T_\alpha,T_{\alpha+1}]}(t)
$$

$$
L_3 = \frac{Q}{\sqrt{r}} \sum_{r_\alpha < s \leq r} |k_s|v_s \cos(k_s \cdot x)e^{-|k_s|^2 t} \chi_{[T_\alpha,T_{\alpha+1}]}(t).
$$

The first group are those plane waves whose sizes are small. Provided $\frac{Q}{\sqrt{r}} \lesssim Q^{-\frac{1}{2}}$ we have

$$
\| L_1 \|_{X_{T_{\alpha+1}}} \lesssim \frac{Q}{\sqrt{r}} \sqrt{T_{\alpha+1}|k_{T_{\alpha+1}-1}|} + \frac{Q}{\sqrt{r}}(T_{\alpha+1}|k_{T_{\alpha+1}-1}|^2)^{\frac{1}{2}} \lesssim \frac{Q}{\sqrt{r}} \lesssim Q^{-\frac{1}{2}},
$$
The second group is the group of active plane waves in the time scale \([T_\alpha, T_{\alpha+1}]\). But, by the design, the number of plane waves in this group is small.

\[
\|L_2\|_{X^{T_{\alpha+1}}} \lesssim \frac{Q}{\sqrt{r}} + \frac{Q}{\sqrt{r}} (r_\alpha - r_{\alpha+1})^{\frac{1}{2}} \\
= \frac{Q}{\sqrt{r}} + \frac{Q}{\sqrt{r}} (Q^{-3}r)^{\frac{1}{2}} \\
\lesssim Q^{-1/2},
\]

if \(\frac{Q}{\sqrt{r}} \lesssim Q^{-\frac{1}{2}}\). The last group of plane waves are those that have diffused too much and become small in size.

\[
\|L_3\|_{X^{T_{\alpha+1}}} \lesssim \frac{Q}{\sqrt{r}} \sup_{t < T_{\alpha+1}} \sum_{s = r_{\alpha+1}}^r |k_s| \sqrt{te^{-|k_s|^2t}} \\
+ \frac{Q}{\sqrt{r}} \sup_{t < T_{\alpha+1}} \left( \int_0^t \left| \sum_{s = r_{\alpha+1}}^r |k_s|^2 e^{-|k_s|^2t} \chi_{[T_\alpha, T_{\alpha+1}]}(\tau) \right| d\tau \right)^{\frac{1}{2}}
\]

The first supremum of the last line is controlled by the integral

\[
\int_{|k_{r_{\alpha+1}}|}^{|k_r|} \sqrt{te^{-x^2t}} dx = \int_{|k_{r_{\alpha+1}}|\sqrt{T}}^{|k_r|\sqrt{T}} e^{-y^2} dy \\
\leq \int_{|k_{r_{\alpha+1}}|\sqrt{T}}^{|k_r|\sqrt{T}} e^{-y^2} dy \\
\leq e^{-|k_{r_{\alpha+1}}|\sqrt{T}} \leq e^{-|k_{r_{\alpha+1}}|/|k_r|} \ll 1,
\]

where we used the fact that \(|k_{r_{\alpha+1}}|\sqrt{T} > 1\) for \(t \in [T_\alpha, T_{\alpha+1}]\) to the second step and the last step follows from \(4.16\).

The second supremum is controlled by

\[
\left( \sum_{s = r_{\alpha+1}}^r e^{-|k_s|^2T_{\alpha+1}} \right)^{1/2} \leq (r - r_\alpha) e^{-|k_{r_{\alpha+1}}|^2/|k_{r_{\alpha+1}}|^2} \\
\lesssim (Q^{-3}r)e^{-4r^{-\alpha}Q^{-3}}, \ll 1.
\]

In the same way one can show the same estimate for \(b_0\). The proof of Lemma is complete.

\[\square\]

**Lemma 4.9.** For \(T > T_\beta\),

\[(4.36)\quad \|(e^{t\Delta} u_0)\chi_{[T_\beta, T]}(t)\|_{X^T} \lesssim \frac{Q}{\sqrt{r}},\]

\[(4.37)\quad \|(e^{t\Delta} b_0)\chi_{[T_\beta, T]}(t)\|_{X^T} \lesssim \frac{Q}{\sqrt{r}}.\]
Proof of Lemma: From Lemma 4.1 and (4.25), we see that
\[
\| (e^{\Delta u_0}) \chi_{[T_\beta, T]}(t) \|_{X_T} \lesssim \frac{Q}{\sqrt{T}} + \frac{Q}{\sqrt{T}} \sum_{s=1}^{r} |k_s| e^{-\frac{|k_s|^2}{8|u_0|^2}} |(T - T_\beta)^{\frac{1}{2}}
\]
\[
\lesssim \frac{Q}{\sqrt{T}}.
\]
The second one follows in the same way.

Recall the equations for $y$ and $z$ from Section 2.4 that
\[
y_t - \nabla y + G_0 + G_1 + G_2 = 0,
\]
\[
z_t - \nabla z + K_0 + K_1 + K_2 = 0,
\]
Note that, $y(0) = z(0) = 0$. Hence, $t \in [T_\alpha, T_{\alpha+1}]$.

(4.38) \[ y(t) = - \int_0^t e^{(t-\tau)\Delta} G(\tau) d\tau \]
\[ = - \int_0^t e^{(t-\tau)\Delta} G(\tau) \chi_{[0,T_\alpha]}(\tau) d\tau - \int_0^t e^{(t-\tau)\Delta} G(\tau) \chi_{[T_\alpha,T_{\alpha+1}]}(\tau) d\tau, \]
where $G = G_0 + G_1 + G_2$. So we can write
(4.39) \[ \| y \|_{X_{T_{\alpha+1}}} \leq I_1 + I_2 \]
to see how $y$ develop in the time step $[T_\alpha, T_{\alpha+1}]$.

Similarly for $z$, we have
(4.40) \[ \| z \|_{X_{T_{\alpha+1}}} \]
\[ \leq \int_0^t e^{(t-\tau)\Delta} K(\tau) \chi_{[0,T_\alpha]} d\tau \|_{X_{T_{\alpha+1}}} + \int_0^t e^{(t-\tau)\Delta} K(\tau) \chi_{[T_\alpha,T_{\alpha+1}]} d\tau \|_{X_{T_{\alpha+1}}} \]
\[ = J_1 + J_2, \]
where $K = K_0 + K_1 + K_2$. Now we are ready to estimate the increments of $y$ and $z$ during the time scale $[T_\alpha, T_{\alpha+1}]$.

Lemma 4.10. With appropriate choice of $r$ and $T$, we have
(4.41) \[ \| y \|_{X_{T_{\alpha+1}}} + \| z \|_{X_{T_{\alpha+1}}} \lesssim Q^3 \left( \frac{1}{r^2} + \sqrt{T_\beta} \right) + Q(\| y \|_{X_{T_\alpha}} + \| z \|_{X_{T_\alpha}}). \]

Proof of Lemma: Applying bilinear estimate (2.7), estimates in space $X_{T_\alpha}$ from Lemma 4.9 (4.32), (4.30) and (4.29), we have
(4.42) \[ I_1 \lesssim (Q + \frac{Q^2}{r^2} + \| y \|_{X_{T_\alpha}}) \| y \|_{X_{T_\alpha}} \]
\[ + (Q + Q^2 \sqrt{T_\alpha} + \frac{Q^2}{r^2} + \| z \|_{X_{T_\alpha}}) \| z \|_{X_{T_\alpha}} \]
\[ + (Q + \frac{Q^2}{r^2}) \frac{Q^2}{r} \]
\[ + (Q + Q^2 \sqrt{T_\alpha} + \frac{Q^2}{r^2})(Q^2 \sqrt{T_\alpha} + \frac{Q^2}{r}). \]
We next apply the Lemma 4.8 and estimate

\begin{equation}
I_2 \lesssim (Q^{-1/2} + \frac{Q^2}{r} + \|y\|_{X_{T_{n+1}}} \|y\|_{X_{T_{n+1}}})
\end{equation}

+ (Q^{-1/2} + Q^2 \sqrt{T_{n+1}} + \frac{Q^2}{r} + \|z\|_{X_{T_{n+1}}} \|z\|_{X_{T_{n+1}}})

+ (Q^{-1/2} + \frac{Q^2}{r} \frac{Q^2}{r})

+ (Q^{-1/2} + Q^2 \sqrt{T_{n+1}} + \frac{Q^2}{r} + Q^2 \sqrt{T_{n+1}} + \frac{Q^2}{r}).

We choose \(r\) sufficiently large and \(T\) appropriately small such that

\begin{equation}
\frac{Q^2}{r} < Q^{-1/2}, \quad Q^2 \sqrt{T} < Q^{-1/2}.
\end{equation}

Hence we combine (4.39), (4.42), and (4.43) to arrive at

\begin{equation}
\|y\|_{X_{T_{n+1}}} \lesssim Q^3 \left(\frac{1}{r} + \sqrt{T_{\beta}}\right) + \|y\|_{X_{T_n}} + \|z\|_{X_{T_n}}
\end{equation}

+ Q^{-1/2}(\|y\|_{X_{T_{n+1}}} + \|z\|_{X_{T_{n+1}}}) + \|y\|_{X_{T_{n+1}}} \|z\|_{X_{T_{n+1}}}.

Similarly we can obtain

\begin{equation}
\|z\|_{X_{T_{n+1}}} \lesssim Q^3 \left(\frac{1}{r} + \sqrt{T_{\beta}}\right) + \|y\|_{X_{T_n}} + \|z\|_{X_{T_n}}
\end{equation}

+ Q^{-1/2}(\|y\|_{X_{T_{n+1}}} + \|z\|_{X_{T_{n+1}}}) + \|y\|_{X_{T_{n+1}}} \|z\|_{X_{T_{n+1}}}.

Therefore, adding (4.45) and (4.46), we have

\begin{equation}
\|y\|_{X_{T_{n+1}}} + \|z\|_{X_{T_{n+1}}} \lesssim Q^3 \left(\frac{1}{r} + \sqrt{T_{\beta}}\right) + \|y\|_{X_{T_n}} + \|z\|_{X_{T_n}}
\end{equation}

+ (\|y\|_{X_{T_{n+1}}} + \|z\|_{X_{T_{n+1}}})^2.

So, for much larger \(r\) and \(|k_0|\), we have \(\|y\|_{X_{T_{n+1}}} + \|z\|_{X_{T_{n+1}}}\) small and

\begin{equation}
\|y\|_{X_{T_{n+1}}} + \|z\|_{X_{T_{n+1}}} \lesssim Q^3 \left(\frac{1}{r} + \sqrt{T_{\beta}}\right) + \|y\|_{X_{T_n}} + \|z\|_{X_{T_n}}.
\end{equation}

By iterating (4.44) it then follows easily that

**Lemma 4.11.**

\begin{equation}
\|y\|_{X_{T_{\beta}}} + \|z\|_{X_{T_{\beta}}} \lesssim Q^{3+2} \left(\frac{1}{r} + \sqrt{T_{\beta}}\right)
\end{equation}

Next for \(T > T_{\beta}\), in the light of Lemma 4.9, one may repeat the argument in the proof of 4.41 and obtain

**Lemma 4.12.** For appropriate choice of \(r\) and \(T\),

\begin{equation}
\|y\|_{X_T} + \|z\|_{X_T} \lesssim 4Q^4 T.
\end{equation}

This implies that,

\begin{equation}
\|y(\cdot,T)\|_{B^{-1,1}_{\infty}} \lesssim \|y(\cdot,T)\|_{L^\infty} \lesssim T^{-1/2} \|y\|_{X_T} \lesssim 4Q^4 \sqrt{T}
\end{equation}

and

\begin{equation}
\|z(\cdot,T)\|_{B^{-1,1}_{\infty}} \lesssim \|z(\cdot,T)\|_{L^\infty} \lesssim T^{-1/2} \|z\|_{X_T} \lesssim 4Q^4 \sqrt{T}.
\end{equation}
4.5. **Finishing the Proof.** Now we are ready to complete the proof of Theorem 1.1. Since (4.30) implies

\[(4.51) \quad \|b_{1,1}(\cdot, T)\|_{B^{-1,1}_{\infty}} \lesssim T^{-\frac{1}{2}} \|b_{1,1}\|_{X_T} \lesssim \frac{Q^2}{r\sqrt{T}},\]

from (2.9) we combine (4.32), (4.51) and (4.50) to obtain that

\[
\|b(\cdot, T) - e^{T\Delta}b_0\|_{B^{-1,1}_{\infty}} \geq \|b(\cdot, T)\|_{B^{-1,1}_{\infty}} - \|b_{1,1}(\cdot, T)\|_{B^{-1,1}_{\infty}} - \|z(\cdot, T)\|_{B^{-1,1}_{\infty}} \quad \geq Q^2 - \|b_{1,1}\|_{L^\infty} - \|z\|_{L^\infty} \geq Q^2(1 - \frac{1}{r\sqrt{T}} - 4Q^2\sqrt{T}).
\]

Therefore, in the light of (4.28),

\[
\|b(T)\|_{B^{-1,1}_{\infty}} \gtrsim Q^2.
\]

On the other hand, from (2.9), we combine (4.28), (4.29), and (4.49) and have, for any \(t \in [0, T]\),

\[(4.52) \quad \|u(\cdot, t)\|_{B^{-1,1}_{\infty}} \lesssim \frac{Q}{\sqrt{T}} + \frac{Q^2}{r} + Q^4\sqrt{T} \quad \text{and remains small in } B^{-1,1}_{\infty}. \]

Thus we proved theorem 1.1.

**Remark 4.13.** We would like to note the following simple chart to indicate how the choices of the several parameters are made.

\[\delta \rightarrow Q \rightarrow T \rightarrow |k_0| \rightarrow |k_s|, \quad Q \rightarrow r.\]

5. **Other scenarios of norm inflations**

In this section, we consider other interesting norm inflation phenomena for MHD system. The essence of the construction introduced by Bourgain and Pavlović in [1] and used in this paper is that the collisions of plane waves with similar but large wave vectors can cause norm inflations in NSE as well as in MHD system. More precisely we see that the collisions in the quadratic terms trigger the norm inflations. Hence we can arrange the initial data to have collisions of plane waves with similar wave vectors either in \(u_1\) or in \(b_1\) to produce various norm inflation modes for MHD system. For example, even the initial velocity is zero, if the initial magnetic field contains enough plane waves to collide, we can produce the scenario where the velocity develops norm inflation in \(\dot{B}_{\infty}^{-1,1}\) while the magnetic field remains small in the space \(B_{\infty}^{-1,1}\). Namely,

**Theorem 5.1.** Let \(u_0 \equiv 0\) and

\[
b_0 = \frac{Q}{\sqrt{T}} \sum_{i=1}^{r} (|k_i|v_i \cos(k_i \cdot x) + |k'_i|v'_i \cos(k'_i \cdot x)).
\]

Then, for any \(\delta > 0\), there exists a solution \((u, b, p)\) to the MHD system (1.1) with initial data \(u_0\) and \(b_0\) as in the above satisfying

\[
\|b(0)\|_{B_{\infty}^{-1,1}} \lesssim \delta.
\]
such that for some $0 < T < \delta$

$$\|u(T)\|_{\dot{B}^{-1, \infty}_\infty} \gtrsim \frac{1}{\delta},$$

while for any $0 < t < T < \delta$

$$\|b(t)\|_{\dot{B}^{-1, \infty}_\infty} \lesssim \delta.$$

The proof will be more or less the same as the proof in [1] in the light of our discussions in the previous section. Another interesting case mentioned in Remark 1.4 is that when the initial velocity and the initial magnetic field are the same, although they both include many plane waves that are to collide, the collisions cancel each other in the evolution of the MHD system and produce no norm inflations.

Remark 5.2. Finally we would like to mention that, in [3], Cheskidov and Shvydkoy introduced a different construction of initial data to prove the ill-posedness of NSE in certain Besov spaces. There are two more works about the ill-posedness results for Navier-Stokes equations by Germain [4] and Yoneda [9].

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