The Series Solution to the Metric
of Stationary Vacuum with Axisymmetry

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The multipole moments method is not only an aid to understand the deformation of the space-time, but also an effective tool to solve the approximate solutions of the Einstein field equation. However, The usual multipole moments are recursively defined by a sequence of symmetric and trace-free tensors, which are inconvenient for practical resolution. In this paper, we develop a simple procedure to generate the series solutions, and propose a method to identify the free parameters by taking the Schwarzschild metric as a standard ruler. Some well known examples are analyzed and compared with the series solutions.

Keywords: stationary metric, multipole moments, asymptotically flat

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I. INTRODUCTION

The asymptotically flat vacuum solutions with axisymmetry play an important role in general relativity. To understand the global structure of the metric caused by stars, we have the exact solution families of Schwarzschild, Curzon and Kerr metrics and some of their extensions to the electrovacuum solutions such as Reissner-Nordström and Kerr-Newman metrics [1]-[4]. After more than ten years of work, Manko et al. got an axisymmetric solution with five free parameters[5]. This solution is suitable for a rapidly rotating neutron star, in which the strong magnetic field and the oblate shape should be taken into account[6].

When getting a realistic exact solution becomes more and more difficult, the effective approximate method is a good alternation. The multipole moments expansion was introduced

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to interpret the physical meaning of the solutions to the Einstein equation. Geroch defined a recursive relation of multipole moments for the static asymptotically flat vacuum with axisymmetry by means of the timelike Killing vector[7, 8]. The norm of the Killing vector satisfies the Laplace equation in the 3-dimensional hypersurface orthogonal to the Killing vector, which is reduced from the Einstein equation. For the asymptotically flat space-time, the norm of the Killing vector and the metric is analytic near the spatial infinity, so the solution can be expanded in series of $r^{-1}$ with multipole moments. The multipole moments can be recursively defined by a sequence of symmetric and trace-free tensors. This idea was generalized to the stationary case by Hansen in [9], where two different sets of multipole moments are employed. One is the mass moments and the other is the angular momentum moments. The two sets of moments can be generated from two potential functions defined by the norm and the twist of the time-like Killing vector.

The formalism proposed in [8, 9] is in the covariant form. Some theorems are proved based on this formalism. In [10], Xanthopoulos proved that a stationary space-time is static if and only if all of its current moments vanish, and static space-time is flat if and only if all of its mass moments vanish. Beig-Simon [11] and Kundu[12] proved that, two spacetimes with the same multipole moments have the same space-time geometry at large radii, where the multipole expansion of the metric converges. This formalism was realized in the Weyl-Lewis-Papapetrou canonical metric[13, 14, 15].

An another formalism of multipole moments for realistic calculation of slowly changing systems or precisely stationary ones was reviewed and developed by Thorne in [16]. This formalism is more manifest for practical resolution, which is associated with a special kind of coordinate system called ‘Asymptotical Cartesian and Mass Centered’(ACMC) coordinate system. In such ACMC-N coordinate system, the metric can be expressed by multipole moments expansion, and the coefficients of $r^{-(l+1)}(l \leq N)$ terms are linearly combinations of the spherical harmonics of order $l, l-1, \cdots, 0$. The de Donder coordinate system belongs to ACMC-$\infty$. The comparison and the relation between the two formalisms were discussed in [17]. This formalism in de Donder coordinate system and quotient harmonic one was developed into an effective tool for solving the approximate global solutions to the Einstein equation[18]-[20].

However, most of the previous works concerned mainly theoretical aspect of the multipole moments, which are very inconvenient for practical resolution of the Einstein’s field equation.
Besides, the interpretation of the physical meanings of the parameters is still a general problem in general relativity\[21, 22\], although the primary motivation of Geroch to introduce the multipole moments was to clarify this problem.

In this paper, we consider how to effectively solve the series solutions to the stationary, asymptotically flat vacuum with axisymmetry. This case is of important physical significant. At first, we show that the canonical form of Weyl-Lewis-Papapetrou coordinate system is an ACMC-∞ one. Then we solve the series solution straightforwardly in this coordinate system. To understand the meanings of the free parameters in the solution, we set the Schwarzschild solution as a standard ruler. By comparison with this solution, we find that the bigger the absolute value of the dimensionless free coefficients, the larger the deformation and convection of the gravitating source. Noticing its generality, the series solution is an effective alternation to the exact one, and is helpful to understand the abundant structure of the space-time.

II. THE SERIES SOLUTION IN WEYL-LEWIS-PAPAPETROU METRIC

To describe the axisymmetric space-time, the canonical form of the Weyl-Lewis-Papapetrou metric is the simplest one\[1\]. The line element is equivalent to

\[ ds^2 = U(dt + Wd\varphi)^2 - V(d\rho^2 + dz^2) - U^{-1}\rho^2d\varphi^2, \]  

(2.1)

where \((t, \rho, z, \varphi)\) is the coordinates of the geometrical meaning near the cylindrical coordinates in Minkowski space. \((U, V, W)\) only depend on \((\rho, z)\). In this paper we take \(G = c = 1\) as units. Since the following calculations take the dimensionless form, the units will not be confused. The canonical form (2.1) has an important property, that is,

**Lemma 1.** Except for the translation \(t = \tilde{t} + t_0, \varphi = \tilde{\varphi} + \varphi_0\) related to two Killing vectors \((\partial_t, \partial_\varphi)\) and \(z = \pm(\tilde{z} + z_0)\), the form of metric (2.1) removes other uncertainty caused by coordinate condition.

This property can be checked as follows. The transformation \((\rho, z) \rightarrow (\tilde{\rho}, \tilde{z})\) keeping \(V(d\rho^2 + dz^2) = \tilde{V}(d\tilde{\rho}^2 + d\tilde{z}^2)\) is a conformal transformation

\[ \rho = \Re[f(\tilde{\rho} \pm \tilde{z}i)], \quad z = \Im[f(\tilde{\rho} \pm \tilde{z}i)], \]  

(2.2)

where \(f\) is any given analytic function. But the restrictions \(\rho^2d\varphi^2 = \tilde{\rho}^2d\varphi^2\) and \(\rho \geq 0\) give \(f = \tilde{\rho} \pm (\tilde{z} + z_0)i\), then we have \(z = \pm(\tilde{z} + z_0)\).
Except for the axial symmetry, the celestial body in equilibrium and the related stationary metric still have another top-bottom reflection symmetry, which means the metric will be the even functions of \( z \) by setting the origin (\( \rho = 0, z = 0 \)) at the mass center. In what follows we only discuss this case, that is, \((U, V, W)\) are even functions of \( z \), although the discussion is also valid in the case including odd terms of \( z \). So if we set the direction of \( z \), the relation between coordinate and metric is completely fixed.

The form (2.1) is not convenient for the following calculation, so we make a polar coordinate transformation

\[
\rho = r \sin \theta, \quad z = r \cos \theta,
\]

then the line element (2.1) becomes

\[
ds^2 = U(dt + Wd\varphi)^2 - V(dr^2 + r^2d\theta^2) - U^{-1}r^2 \sin^2 \theta d\varphi^2.
\]

Calculating the Ricci tensor \( R_{\mu\nu} \), we get the independent components among the Einstein equations for the axisymmetric vacuum[1]

\[
\frac{\partial^2 U}{\rho^2} + 2\frac{\partial_r U}{\rho} + \cot \theta \frac{\partial_\theta U}{\rho^2} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} - \frac{|\nabla U|^2}{U} + \frac{U^3|\nabla W|^2}{r^2 \sin^2 \theta} \equiv -2V R_{tt} = 0,
\]

\[
\frac{\partial^2 W}{\rho^2} - \frac{\cot \theta}{r^2} \partial_\theta W + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2} + \frac{2}{U} (\nabla U \cdot \nabla W) \equiv \frac{2V}{U} (W R_{tt} + R_{t\varphi}) = 0,
\]

where \( \nabla = (\partial_r, r^{-1}\partial_\theta) \). And then \( V \) can be integrated from

\[
\frac{\partial_r V}{V} + \frac{\partial_\theta U}{U} - \frac{r \sin^2 \theta}{2U^2} \left( \cot \theta \frac{\partial_r U}{\partial_\theta U} \right) + \frac{U^2}{2r} \left( \cot \theta \frac{\partial_r U}{\partial_\theta U} \right) \equiv -\frac{1}{r} \sin^2 \theta (r^2 R_{rr} - R_{\theta\theta}) - 2 \cos \theta \sin \theta R_{r\theta} = 0,
\]

\[
\frac{\partial_\theta V}{V} + \frac{\partial_\theta U}{U} + \frac{r^2 \sin^2 \theta}{2U^2} \left( \cot \theta (\partial_r U)^2 \right) \equiv \cos \theta \sin \theta (r^2 R_{rr} - R_{\theta\theta}) - 2r \sin^2 \theta R_{r\theta} = 0.
\]

Substituting \( R_{\mu\nu} = \kappa (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\alpha}_\alpha) \) into the above equations, we get the dynamical equations for interior metric, which can also be solved by series.
The solutions to (2.5)-(2.8) of most physical interests are the asymptotically flat cases, that is, the solution satisfies the boundary conditions at $r \to \infty$

$$U \to 1 - \frac{2m}{r} + O(r^{-2}), \quad W \to \frac{4J}{r} \sin^2 \theta + O(r^{-2}), \quad V \to 1 + \frac{2m}{r} + O(r^{-2}),$$

(2.9)
in which $m$ is the total gravitational mass, and $J$ the angular momentum of the source. (2.9) means $r = \infty$ is the regular point of the solution. On the other hand, the main part of derivatives in (2.5) and (2.6) are the linear elliptic operators. So the vacuum solutions $(U, W)$ are analytic functions in the neighborhood of $r = \infty$[11, 12], and can be expressed as Taylor series of $r^{-1}$, then we generally have

$$U = 1 - \frac{R_s}{r} + \frac{1}{r^2} \sum_{n=0}^{\infty} \frac{\tilde{A}_n}{r^n}, \quad W = \sum_{n=1}^{\infty} \frac{\tilde{B}_n}{r^n},$$

(2.10)

where $(R_s, \tilde{A}_n, \tilde{B}_n)$ are functions of $\theta$ to be determined. For the regular metric, $(U, V)$ are bounded even functions of $z$, or equivalently, $(R_s, \tilde{A}_n, \tilde{B}_n)$ are bounded functions of $\cos^2 \theta$.

Substituting (2.10) into (2.5), and then expand it into Taylor series of $r^{-1}$, we can get the equations for the coefficients

$$R''_s + \cot \theta R'_s = 0,$$

$$\tilde{A}'_0 + \cot \theta \tilde{A}'_0 + 2\tilde{A}_0 = (R'_s)^2 + R'^2_s.$$  

(2.11)  

(2.12)  

The bounded solution of reflecting symmetry with respect to $z = 0$ or $\theta = \frac{\pi}{2}$, we call it **regular solution** in what follows, is given by

$$R_s = 2m, \quad \tilde{A}_0 = \frac{1}{2} R^2_s,$$

(2.13)

where $m$ is a constant, which means the total mass of the source. $m = 0$ corresponds to the Papapetrou class of solutions or flat space-time. The case of $m < 0$ is unphysical. So only the case with positive mass $m > 0$ has the most physical interest, and we only discuss this case in what follows. Obviously, $R_s$ is the corresponding Schwarzschild radius with dimension of length.

By virtue of the excellent structure of (2.5) and (2.6), we have the following results.

**Theorem 2.** For the regular solution (2.10) with starting value (2.13), it takes the following form

$$U = 1 - \frac{R_s}{r} + \frac{1}{2} \left( \frac{R_s}{r} \right)^2 + \left( \frac{R_s}{r} \right)^2 \sum_{n=1}^{\infty} \tilde{A}_n \left( \frac{R_s}{r} \right)^n,$$

$$W = R_s \sum_{n=1}^{\infty} \tilde{B}_n \left( \frac{R_s}{r} \right)^n,$$

(2.14)  

(2.15)
in which
\[
\tilde{A}_n = \sum_{k=0}^{N_n} \tilde{a}_{nk} \cos(2k\theta), \quad \tilde{B}_n = \sum_{k=1}^{N_n} \tilde{b}_{nk} \cos(2k\theta) - 1, \tag{2.16}
\]
where \((\tilde{a}_{nk}, \tilde{b}_{nk})\) are dimensionless constants to be determined, and \(N_n = \left\lfloor \frac{n+1}{2} \right\rfloor\) stands for the integer part of \(\frac{1}{2}(n+1)\), namely, \(N_{2k-1} = N_{2k} = k\).

**Proof.** Substituting (2.14) and (2.15) into (2.5) and (2.6) and expand them in series, we get the following equations
\[
\begin{align*}
\tilde{A}_1' + \cot \theta \tilde{A}_1' + 2 \cdot 3\tilde{A}_1 &= -1, \tag{2.17} \\
\tilde{A}_2' + \cot \theta \tilde{A}_2' + 3 \cdot 4\tilde{A}_2 &= -\frac{1}{2} - 6\tilde{A}_1 - \frac{1}{\sin^2 \theta} \left( (\tilde{B}_1')^2 + \tilde{B}_1^2 \right), \tag{2.18} \\
\tilde{B}_1'' - \cot \theta \tilde{B}_1' + 1 \cdot 2\tilde{B}_1 &= 0, \tag{2.19} \\
\tilde{B}_2'' - \cot \theta \tilde{B}_2' + 2 \cdot 3\tilde{B}_2 &= 2\tilde{B}_1. \tag{2.20}
\end{align*}
\]
Solve the equations, we find that the regular solutions take the forms (2.16). We check the succeeding equations by mathematical induction.

Assume the forms (2.16) hold for \(n = 2N - 1\) and \(n = 2N\). Since the equations (2.5) and (2.6) are homogeneous with respect to \(r\), and the highest derivatives with respect to \(r\) is linear. Calculation shows that the equations for \(n = 2N + 1\) and \(n = 2N + 2\) take the following form
\[
\begin{align*}
\tilde{A}_{2N+1}'' + \cot \theta \tilde{A}_{2N+1}' + (2N + 2)(2N + 3)\tilde{A}_{2N+1} &= P_1 - 4(N + 1)\tilde{A}_{2N}, \tag{2.21} \\
\tilde{A}_{2N+2}'' + \cot \theta \tilde{A}_{2N+2}' + (2N + 3)(2N + 4)\tilde{A}_{2N+2} &= P_2 - 2(2N + 3)\tilde{A}_{2N+1}, \tag{2.22} \\
-4\tilde{b}_{11}[2 \cot \theta \tilde{B}_{2N+1}' + (2N + 1)\tilde{B}_{2N+1}] &= P_2 - 2(2N + 3)\tilde{A}_{2N+1}, \tag{2.23} \\
\tilde{B}_{2N+1}' - \cot \theta \tilde{B}_{2N+1}' + (2N + 1)(2N + 2)\tilde{B}_{2N+1} &= Q_1 + 4N\tilde{B}_{2N}, \tag{2.24}
\end{align*}
\]
where \((P_k, Q_k)\) are polynomials determined by functions \((\tilde{A}_{2j-1}, \tilde{A}_{2j}, \tilde{B}_{2j-1}, \tilde{B}_{2j}), (j \leq N)\) and their derivatives, which all take the forms (2.16). Since \(\cos(2n\theta) = \frac{1}{2}(e^{2n\theta} + e^{-2n\theta})\), under the constraint of the index of \(\left(\frac{B_r}{r}\right)^n\), we find
\[
\begin{align*}
P_k &= \sum_{n=0}^{N+1} p_{kn} \cos(2n\theta), \quad Q_k = \sum_{n=0}^{N+1} q_{kn} \cos(2n\theta), \tag{2.25}
\end{align*}
\]
with constants \((p_{kn}, q_{kn})\). Substituting (2.25) into (2.21)-(2.24) and solving the solutions with finite and symmetrical conditions, the solutions \((\tilde{A}_{2N+2}, \tilde{B}_{2N+1})\) have the forms (2.16),
but \((\tilde{A}_{2N+1}, \tilde{B}_{2N+2})\) will introduce extra terms similar to the second kind Legendre functions, which include the following factor

\[
F(\theta) = \ln \left(\frac{1 + \cos \theta}{1 - \cos \theta}\right) \sin \theta \cos \theta. \tag{2.26}
\]

\(F(\theta)\) also satisfies the boundary condition and symmetry. Nevertheless, the terms including this factor will soon be cancelled at the next step of resolution. Then we prove the regular solutions always take the forms (2.16).

(2.16) means that the Weyl-Lewis-Papapetrou coordinate system (2.1) is an ACMC-\(\infty\) one. (2.16) can be transformed into the Legendre polynomials, but the Legendre polynomials are less convenient to calculate the nonlinear terms than the trigonometric functions. If the solutions of \((U, W)\) is determined, \(V\) can be solved only by (2.7) with boundary condition \(V(\infty, \theta) = 1\). Then the stationary axisymmetrical metric of vacuum is determined in principle. We determine the coefficients of the series in the next section.

III. COEFFICIENTS OF THE SERIES AND EXAMPLES

Similarly to determine the coefficients of the special functions by recursion, the theorem 2 implies that, the resolution of (2.5) and (2.6) can be transformed into a problem of solving coefficients by recursive relation. For convenience of calculation, we make transformation \(U = \exp \left(\frac{u}{2}\right)\), then (2.5) and (2.6) become

\[
r^2 \partial_r^2 u + 2r \partial_r u + \cot \theta \partial_\theta u + \partial_\theta^2 u + \frac{2e^u}{r^2 \sin^2 \theta} |r^2 (\partial_r W)^2 + (\partial_\theta W)^2| = 0, \tag{3.1}
\]

\[
r^2 \partial_r^2 W - \cot \theta \partial_\theta W + \partial_\theta^2 W + (r^2 \partial_r u \partial_r W + \partial_\theta u \partial_\theta W) = 0. \tag{3.2}
\]

and (2.14)-(2.16) become

\[
u = \sum_{n=1}^{\infty} A_n \left(\frac{R_s}{r}\right)^n, \quad W = R_s \sum_{n=1}^{\infty} B_n \left(\frac{R_s}{r}\right)^n, \tag{3.3}
\]

\[
A_n = \sum_{k=0}^{N_n-1} a_{nk} \cos(2k\theta), \quad B_n = \sum_{k=1}^{N_n} b_{nk} [\cos(2k\theta) - 1], \tag{3.4}
\]

where \(N_n = \left\lfloor \frac{n+1}{2} \right\rfloor\).

Substituting (3.3) and (3.4) into the original equation (3.1) and (3.2), and expanding them in series, we get the relations among the parameters \((a_{kn}, b_{kn})\). (see appendix). In each term, the parameters \((a_{kn}, b_{kn})\) with the larger indexes always take linear form with
bigger coefficients, which correspond to the linear Laplace-like operator in (3.1) and (3.2). So it is easy to solve the coefficients. However, the equation are underdetermined for the parameters \((a_{kn}, b_{kn})\), and we have the following two sequences of free parameters. Let

\[
a_{10} = -2, \quad a_{31} = m_1, \quad a_{52} = m_2, \quad a_{2k+1,k} = m_k, \tag{3.5}
\]

\[
b_{11} = -\frac{1}{2}w_1, \quad b_{32} = -\frac{1}{4}w_2, \quad b_{2k-1,k} = -\frac{1}{2k}w_k, \quad (k = 1, 2, 3, \ldots). \tag{3.6}
\]

where \(\{m_k, w_k\}\) are dimensionless numbers determined by the multipole moments of the energy-momentum tensor distribution of the source, \(a_{10} = -2\) is determined by (2.9), and the factor \(\frac{1}{2k}\) is introduced to scale the coefficients in the recursive relation. The free parameters (3.5) and (3.6) correspond to ‘null data’ in papers [23, 24], but the definitions are different. The coefficients can be recursively expressed as the polynomials of the free parameters (see appendix). Solving the coefficients, we get the series solutions of the metric

\[
u = -2\frac{R_s}{r} + \frac{1}{3}m_1 (1 + 3 \cos 2\theta) \left(\frac{R_s}{r}\right)^3 - \frac{1}{2}w_1 (1 + \cos 2\theta) \left(\frac{R_s}{r}\right)^4 + [m_2 \cos 4\theta + \left(\frac{2}{7}w_1^2 + \frac{4}{7}m_2\right) \cos 2\theta + \frac{8}{35}w_1^2 + \frac{9}{35}m_2] \left(\frac{R_s}{r}\right)^5 + O(r^{-6}), \tag{3.7}
\]

\[
U = 1 - \frac{R_s}{r} + \frac{1}{2} \left(\frac{R_s}{r}\right)^2 + \left(\frac{1}{2}m_1 \cos 2\theta + \frac{1}{6}w_1 - \frac{1}{6}m_1\right) \left(\frac{R_s}{r}\right)^4 +
\]

\[
\left[\left(-\frac{1}{4}w_1^2 - \frac{1}{2}m_1\right) \cos 2\theta + \left(\frac{1}{24} - \frac{1}{4}w_1^2 - \frac{1}{2}m_1\right) \left(\frac{R_s}{r}\right)^4 + O(r^{-5})\right], \tag{3.8}
\]

\[
W = R_s \sin^2 \theta \left\{w_1 \left(\frac{R_s}{r}\right) + \frac{w_1}{2} \left(\frac{R_s}{r}\right)^2 + \left(w_2 \cos 2\theta + \frac{3}{5}w_2 + \frac{1}{5}w_1\right) \left(\frac{R_s}{r}\right)^3 +
\]

\[
\left[\left(\frac{3}{4}w_2 + \frac{1}{8}m_1 w_1\right) \cos 2\theta + \frac{1}{15}w_1 + \frac{5}{24}m_1 w_1 + \frac{9}{20}w_2 \right] \left(\frac{R_s}{r}\right)^4 + O(r^{-5})\right\}, \tag{3.9}
\]

\[
V = 1 + \frac{R_s}{r} + \left(\frac{1}{8} \cos 2\theta + \frac{3}{8}\right) \left(\frac{R_s}{r}\right)^2 + \left[\left(-\frac{1}{2}m_1 + \frac{1}{8}\right) \cos 2\theta - \frac{1}{6}m_1 + \frac{1}{24}\right] \left(\frac{R_s}{r}\right)^3 +
\]

\[
\left[\left(\frac{9}{64}w_1^2 - \frac{5}{32}m_1 + \frac{1}{256}\right) \cos 4\theta + \left(-\frac{3}{8}m_1 + \frac{3}{16}w_1^2 + \frac{3}{64}\right) \cos 2\theta +
\left(\frac{11}{64}w_1^2 - \frac{7}{768} - \frac{13}{96}m_1\right) \left(\frac{R_s}{r}\right)^4 + O(r^{-5})\right]. \tag{3.10}
\]

Comparing (3.8)-(3.9) with (2.9), we learn the physical meaning of \(R_s\) and \(w_1\),

\[
R_s = 2m, \quad J = \frac{1}{4}w_1 R_s^2 = w_1 m^2. \tag{3.11}
\]

That is, \(R_s\) corresponds to the Schwarzschild radius, \(w_1\) to the angular momentum.
Now we compare the series solution with some exact metrics, i.e. the Curzon, Schwarzschild and Kerr solutions. The Curzon metric is diagonal,

\[ U = \exp \left( -\frac{R_s}{r} \right), \quad V = \exp \left( \frac{R_s}{r} - \frac{1}{4} \left( \frac{R_s}{r} \right)^2 \sin^2 \theta \right), \quad W = 0. \] (3.12)

Expanding (3.12) into Taylor series and then comparing them with (3.8)-(3.9), we find that all free parameters vanish

\[ m_k = w_k \equiv 0, \quad (\forall k). \] (3.13)

Evidently, such results is caused by the coordinate system.

For the Kerr and Schwarzschild solution, under some parameter transformation, in the canonical coordinate system (2.4) it becomes

\[ U = 1 - \frac{2(R_+ + R_- + R_s \cosh \omega)R_s \cosh \omega}{[(R_+ - R_-)^2 + R_s^2] \cosh^2 \omega + 2(R_+ + R_-)R_s \cosh \omega + 4R_+ R_-}, \] (3.14)

\[ V = 1 + \frac{[(R_+ + R_-)^2 + R_s^2] \cosh^2 \omega + 2(R_+ + R_-)R_s \cosh \omega}{4R_+ R_-}, \] (3.15)

\[ W = \frac{[(R_+ - R_-)^2 - R_s^2](R_+ + R_- + R_s \cosh \omega) \sinh \omega}{[(R_+ - R_-)^2 - R_s^2] \cosh^2 \omega + 4R_+ R_-}, \] (3.16)

in which

\[ R_\pm = \sqrt{r^2 \cosh^2 \omega \pm 2rm \cos \theta \cosh \omega + m^2}, \quad R_s = 2m, \quad \tanh \omega = \frac{2J}{m^2}. \] (3.17)

\( \omega = 0 \) corresponds to the Schwarzschild solution in the Weyl-Lewis-Papapetrou coordinate system. The above formalism is introduced for convenience of the following comparison. In contrast the Taylor series of the above functions with (3.8)-(3.9), we get the following sequence of the free parameters,

\[ m_1 = -\frac{1}{8} + \frac{3}{8} \alpha^2, \quad m_2 = -\frac{7}{2^9} + \frac{17}{2^8} \alpha^2 - \frac{5 \cdot 7}{2^9} \alpha^4, \]

\[ m_3 = -\frac{3 \cdot 11}{2^{14}} + \frac{11 \cdot 19}{2^{14}} \alpha^2 - \frac{17 \cdot 23}{2^{14}} \alpha^4 + \frac{3 \cdot 7 \cdot 11}{2^{14}} \alpha^6, \]

\[ m_4 = -\frac{5 \cdot 11 \cdot 13}{2^{21}} + \frac{3 \cdot 5 \cdot 7 \cdot 13}{2^{19}} \alpha^2 - \frac{3 \cdot 5^3 \cdot 19}{2^{20}} \alpha^4 + \frac{59 \cdot 67}{2^{19}} \alpha^6 - \frac{3^2 \cdot 5 \cdot 11 \cdot 13}{2^{21}} \alpha^8, \] (3.18)

\[ w_1 = \frac{1}{2} \alpha, \quad w_2 = \frac{3 \alpha}{32} - \frac{5 \alpha^3}{32}, \quad w_3 = \frac{3^4}{2^{12}} \alpha - \frac{3 \cdot 41}{2^{11}} \alpha^3 + \frac{3^3 \cdot 7}{2^{12}} \alpha^5, \]

\[ w_4 = \frac{11 \cdot 13}{2^{15}} \alpha - \frac{13 \cdot 47}{2^{15}} \alpha^3 + \frac{881}{2^{15}} \alpha^5 - \frac{3 \cdot 11 \cdot 13}{2^{15}} \alpha^7, \quad \ldots \]

where \( \alpha = \tanh \omega < 1 \). The sequences vanish quite fast.
\( \alpha = 0 \) corresponds to the Schwarzschild solution. In this case, the ‘mass moments’ \( m_k \neq 0 \), which is caused by the coordinates. So to understand these data we need a unified standard.

Evidently, the above calculation shows that any stationary metric with axisymmetry is identical with two sequences \( \{m_n, w_n\} \), in which \( \{m_n\} \) is mainly related to the multipole moments of mass density, and \( \{w_n\} \) to the current distribution. On the contrary, for any given sequences \( \{m_n, w_n\} \) with suitable upper bounds, the series is a solution to the Einstein equation of vacuum in the region of its convergence. So the relation between the radius of convergence of the series and the values of free parameters \( \{m_n, w_n\} \) is important. The convergence of the multipole moments serieses were discussed in different context. The conditions of convergence for the static solutions were established in [14, 23, 25], and the conditions for the stationary ones were given in [24, 26]. Although the results ensure the convergence of the above series, how to derive the concrete constraint for free parameters is quite complicated.

In the static case, the radius of convergence may be directly derived from the exact solution. The result is enlighten, so we give some analysis. Since \( W = 0 \), (3.1) becomes Laplace equation, and its general asymptotically flat solution is the Weyl class([2],Ch.18),

\[
    u = \sum_{n=0}^{N} m_n P_n(\cos \theta) \left( \frac{R_s}{r} \right)^{n+1}, \quad (m_0 = -2, \ N \leq \infty),
\]

(3.19)

where \( P_n \) are the Legendre polynomials of \( n \) degree. For any \( N \), the function \( V \) can be also exactly solved from (2.7)[2]. The Curzon solution is the simplest case with \( N = 0 \). In the case \( N = \infty \), we have

**Theorem 3.** For the series solution (3.19) with \( N = \infty \), if the free parameters \( m_n \) satisfy the following condition

\[
    |m_n| \leq C(n + 1)^K \lambda^n,
\]

(3.20)

where \( (C > 0, \lambda > 0, K) \) are given numbers independent of \( n \), then the series solution (3.19) and all its derivatives converge in the region

\[
    r > r_0 \equiv \lambda R_s.
\]

(3.21)
Proof. By the property of the Legendre polynomials, for all \( n \) we have

\[
|P_n| \leq 1, \quad (3.22)
\]

\[
\frac{d}{d\theta} P_n(\cos \theta) = -(2n - 1) \sin \theta P_{n-1}(\cos \theta) + \frac{d}{d\theta} P_{n-2}(\cos \theta). \quad (3.23)
\]

By (3.23) we get

\[
\left| \frac{d}{d\theta} P_n(\cos \theta) \right| \leq (2n - 1) + \left| \frac{d}{d\theta} P_{n-2}(\cos \theta) \right|
\leq (2n - 1) + (2n - 5) + (2n - 9) + \cdots = \frac{1}{2} n(n + 1). \quad (3.24)
\]

Substituting (3.20), (3.22) and (3.24) into (3.19), we find that the series and its derivatives are controlled by,

\[
|u| < C \sum_{n=1}^{\infty} (n + 1)^K \left( \frac{\lambda R_s}{r} \right)^{n+1}, \quad (3.25)
\]

\[
|\partial_r u| < C \sum_{n=1}^{\infty} (n + 1)^{K+1} \left( \frac{\lambda R_s}{r} \right)^{n+1}, \quad (3.26)
\]

\[
|\partial_\theta u| < C \frac{1}{2} \sum_{n=1}^{\infty} (n + 1)^{K+2} \left( \frac{\lambda R_s}{r} \right)^{n+1}. \quad (3.27)
\]

The right hand serieses all converge in the region \( r > r_0 \), so \((u, \partial_r u, \partial_\theta u)\) are absolutely convergent in the region \( r > r_0 \), and uniformly absolutely convergent in the region \( r \geq r_1 \) for any given \( r_1 > r_0 \). Similarly we can check the results for higher order derivatives. Integrating (2.7), we find the radius of convergence of \( V \) is also \( r_0 \). The proof is finished.

In the stationary case with \( W \neq 0 \), the situation becomes more complicated due to the nonlinear terms, because \((a_{nk}, b_{nk})\) are polynomials of \((m_j, w_j)\) of \( j \leq \left\lfloor \frac{n}{2} \right\rfloor \). However, from the recursive relations, we find the following inequalities seems true,

\[
|a_{nk}| < CM_n(n + 1)^K, \quad |b_{nk}| < CM_n(n + 1)^K, \quad \forall (n, k), \quad (3.28)
\]

where \((K > 0, C > 0)\) are constants independent of \( n \), and

\[
M_n = \max_{j \leq \left\lfloor \frac{n}{2} \right\rfloor} \{|m_j|, |w_j|, |m_j|^\frac{1}{2}, |w_j|^\frac{1}{2}\}. \quad (3.29)
\]

In principle, the relation (3.28) can be derived from the results in [24, 26]. If (3.28) holds, the series solutions (2.14)-(2.16) will be controlled by

\[
|U| < C_0 \sum_{n=0}^{\infty} (n + 1)^{K+1} \left( \frac{\lambda R_s}{r} \right)^n, \quad |W| < C_1 \sum_{n=0}^{\infty} (n + 1)^{K+1} \left( \frac{\lambda R_s}{r} \right)^{n+1}, \quad (3.30)
\]
where \((C_0, C_1)\) are constants independent of \(n\), and

\[
\lambda = \lim_{n \to \infty} \sqrt[n]{M_n}.
\]

Then (3.30) also imply that the series and their derivatives converge in the region \(r > \lambda R_s\).

IV. INTERPRETATION OF THE MULTIPOLE MOMENTS

From (3.13) and (3.18) we find that, in the canonical form of the Weyl-Lewis-Papapetrou coordinate system (2.4), the Curzon metric has not ‘multipole moment’ (where it means the free parameters), but the Schwarzschild metric has infinite ones. These results are somewhat unnatural and puzzle. Evidently, such results are caused by the coordinate system, although the canonical form (2.4) is the most convenient one to solve the metric. So how to extract the understandable information from the solutions is also an important problem.

Similar to the concepts of point charge and dipole in the electromagnetism, an ideal explanation for the multipole moments should be expressed in the forms of some conserved spatial integrals of the source [21]. However, this ideal is associated with how to define the covariant generalized functions for nonlinear differential equations, which may have not a general solution for the higher order moments. A realistic explanation for these free parameters is to solve them by associating the exterior solution with the interior one, the results will endow the parameters with concrete values [20]. However, the influence of the coordinate system still exists. In [22], the authors suggested two ways of carrying out comparison of approximate and exact solutions: one is calculating the multipole structure of the Ernst complex potentials for the solutions, and the second is to generating approximate solutions from exact ones by expanding the latter in Taylor series with respect to a small parameter.

To interpret the physical meanings of these free parameters, introducing a standard ruler may be a convenient choice. As an approach of first step, we find that the Schwarzschild metric is a good ruler, because its properties are the simplest and have been well understood. By comparing the other solutions with this ruler, we can get some definite and understandable meanings of these free parameters.

The standard exterior Schwarzschild space-time is described by

\[
ds^2 = \left(1 - \frac{R_s}{R}\right) dt^2 - \left(1 - \frac{R_s}{R}\right)^{-1} dR^2 - R^2 d\Theta^2 - R^2 \sin^2 \Theta d\varphi^2.
\]
The transformation between (4.1) and (2.4) reads
\[
   r = \sqrt{R^2 - 2mR + m^2\cos^2\Theta}, \quad \cos\theta = \frac{(R - m)\cos\Theta}{\sqrt{R^2 - 2mR + m^2\cos^2\Theta}}, \quad (4.2)
\]
which is valid in the region \( R > \frac{1}{2}R_s(1 + \sin\Theta) \). In the coordinate system \((t, R, \Theta, \varphi)\), the line element (2.4) becomes
\[
   ds^2 = U(dt + Wd\varphi)^2 - \tilde{V}[dR^2 + R(R - R_s)d\Theta^2] - U^{-1}R(R - R_s)\sin^2\Theta d\varphi^2, \quad (4.3)
\]
\[
   \tilde{V} = \left(1 + \frac{R_s^2\sin^2\Theta}{4R(R - R_s)}\right)V. \quad (4.4)
\]
\((t, R, \Theta, \varphi)\) is also an ACMC-\(\infty\) coordinate system.

For the Curzon solution (3.12), \((U \to 0, V \to \infty)\) corresponds to the surface
\[
   R = \frac{1}{2}R_s(1 + \sin\Theta), \quad \frac{1}{2}R_s \leq R \leq R_s, \quad (4.5)
\]
which is an oblate spheroid. This implies the solution is the analytic extension of the vacuum produced by an ellipsoid. However, by \(\tilde{V}\) in (4.4), we find the solutions are only valid in the region \( R > R_s \). Expanding \((U, \tilde{V})\) in Taylor series, we get the free parameters for Curzon solution in the form of Legendre polynomials \(P_n(\cos\Theta)\),
\[
   U = 1 - \frac{R_s}{R} + \frac{1}{12}P_2\left(\frac{R_s}{R}\right)^3 + \frac{1}{24}P_2\left(\frac{R_s}{R}\right)^4 + \left(\frac{1}{1440}P_0 + \frac{1}{144}P_2 - \frac{1}{160}P_4\right)\left(\frac{R_s}{R}\right)^6 + O(r^{-7}). \quad (4.6)
\]
\[
   \tilde{V} = 1 + \frac{R_s}{R} + \left(\frac{R_s}{R}\right)^2 + \left(P_0 - \frac{1}{12}P_2\right)\left(\frac{R_s}{R}\right)^3 + \left(P_0 - \frac{29}{168}P_2 - \frac{1}{28}P_4\right)\left(\frac{R_s}{R}\right)^4 + \\
   \left(P_0 - \frac{41}{168}P_2 - \frac{57}{560}P_4\right)\left(\frac{R_s}{R}\right)^5 + O(r^{-6}). \quad (4.7)
\]
For the Kerr solution (3.14) and (3.16), the multipole moments in the coordinate system \((t, R, \Theta, \varphi)\) go as follows
\[
   U = 1 - \frac{R_s}{R} + \frac{1}{4}\alpha^2P_2\left(\frac{R_s}{R}\right)^3 + \alpha^2\left(-\frac{1}{24}P_0 + \frac{1}{24}P_2\right)\left(\frac{R_s}{R}\right)^4 + \\
   \alpha^2\left(-\frac{1}{16}\alpha^2 + \frac{1}{140}P_4 - \frac{1}{40}P_0 + \frac{1}{56}P_2\right)\left(\frac{R_s}{R}\right)^5 + O(r^{-7}) + \\
   \alpha^2\left[\left(-\frac{3}{280} - \frac{47}{1120}\alpha^2\right)P_4 + \left(\frac{1}{28}\alpha^2 + \frac{1}{168}P_2\right) + \left(-\frac{1}{60} + \frac{1}{160}\alpha^2\right)P_0\right]\left(\frac{R_s}{R}\right)^6, \quad (4.8)
\]
\[ W = \alpha R_s \sin^2 \Theta \left( \frac{1}{2} \frac{R_s}{R} + \frac{1}{2} \left( \frac{R_s}{R} \right)^2 + \left[ -\frac{5}{24} \alpha^2 P_2 + \left( \frac{1}{2} - \frac{1}{24} \alpha^2 \right) P_0 \right] \left( \frac{R_s}{R} \right)^3 + \right. \]

\[ \left. \left[ -\frac{7}{16} \alpha^2 P_2 + \left( \frac{1}{2} - \frac{1}{16} \alpha^2 \right) P_0 \right] \left( \frac{R_s}{R} \right)^4 + O(r^{-7}) + \right. \]

\[ \left. \left[ \frac{9}{160} \alpha^4 - \frac{3}{280} \alpha^2 \right] P_4 + \left( \frac{1}{32} \alpha^4 - \frac{113}{168} \alpha^2 \right) P_2 + \frac{1}{2} + \frac{1}{160} \alpha^4 - \frac{1}{15} \alpha^2 \right] \left( \frac{R_s}{R} \right)^5 + \right. \]

\[ \left. \left[ \frac{79}{448} \alpha^4 - \frac{19}{560} \alpha^2 \right] P_4 + \left( \frac{113}{1344} \alpha^4 - \frac{305}{336} \alpha^2 \right) P_2 + \frac{1}{2} + \frac{1}{48} \alpha^4 - \frac{7}{120} \alpha^2 \right] \left( \frac{R_s}{R} \right)^6 \right) \] (4.9)

Comparing the dimensionless coefficients of the term \( P_2(\cos \Theta) \left( \frac{R_s}{R} \right)^3 \) in (4.6) and (4.8), namely, \( \frac{1}{12} \) and \( \frac{1}{4} \alpha^2 \), we get a definite concept for the relative deformations of each space-time. The bigger the absolute value of the free coefficients, the larger the deformation and convection of the gravitating source. In the viewpoint of series, the Kerr solution has not any speciality. However, besides \( (R_s = 2m, w_1 = \frac{1}{2} \alpha = \frac{\omega}{m^2}) \), whether the other coefficients have some relations with the general physical concepts is unclear.

The solutions in the metric (4.3) can be directly solved from the following equations

\[ R(R - R_s) \partial^2_R u + (2R - R_s) \partial_R u + \cot \Theta \partial_\Theta u + \partial^2_\Theta u + \]

\[ \frac{2e^u}{\sin^2 \Theta} \left[ (\partial_R W)^2 + \frac{(\partial_\Theta W)^2}{R(R - R_s)} \right] = 0, \] (4.10)

\[ R(R - R_s) \partial^2_R W - \cot \Theta \partial_\Theta W + \partial^2_\Theta W + \left[ R(R - R_s) \partial_R u \partial_R W + \partial_\Theta u \partial_\Theta W \right] = 0. \] (4.11)

The solution is equivalent to (3.7)-(3.10).

V. DISCUSSION AND CONCLUSION

The above procedure provides a simple but effective method to solve the series solution of stationary and asymptotically flat metric with axisymmetry to any wanted precision. The solution is identical with two sequences of free dimensionless parameters, which correspond to the usual multipole moments. The free parameters are determined by the energy-momentum distribution of gravitating source. For a wide class of given parameters with suitable upper bound, the series solution converges in the region \( r > R_s \). For a normal star, we always have \( r \gg R_s \), so the series provides high precise solutions to the stationary metric with axisymmetry.

To interpret the meanings of free parameters, using dimensionless form and setting up a standard ruler are meaningful. Only compared with a simple ruler, we can distinguish
the differences between the solutions, and get a clear concept of the free parameters. For a
given star, to solve the free parameters by associating the exterior metric with the interior
solution, the results will have concrete physical meanings. However, the matching conditions
on the surface of the star should be carefully discussed. In \[27\], we find \( U \in C^1 \) but \( V \in C^0 \).

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Appendix

Substituting (3.3) and (3.4) into the original equation (3.1) and (3.2), and expanding them in series, we get the relations among the parameters \((a_{kn}, b_{kn})\) as follows.

\[
0 = 22a_{20}\left(\frac{R_s}{r}\right)^2 + (-2a_{3,1} + 6a_{30})\left(\frac{R_s}{r}\right)^3 + \\
(6a_{4,1} + 12b_{11}^2)\cos 2\theta + (-2a_{41} + 12a_{40}) + 20b_{11}^2\left(\frac{R_s}{r}\right)^4 + \\
((-8a_{5,2} + 14a_{5,1}) + 4b_{1,1}(3a_{1,0}b_{1,1} + 4b_{2,1}))\cos 2\theta + \\
(-4a_{5,2} - 2a_{5,1} + 20a_{5,0}) + 4b_{1,1}(5a_{1,0}b_{1,1} + 12b_{2,1})\left(\frac{R_s}{r}\right)^5 + O(r^{-6}), \quad (5.1)
\]

\[
0 = \left\{(8b_{21} + 2a_{10}b_{11})\left(\frac{R_s}{r}\right)^2 + \left[(16b_{32} + 20b_{31}) + 4a_{20}b_{11} + 4a_{10}b_{21}\right]\left(\frac{R_s}{r}\right)^3 + \\
(32b_{4,2} - 2a_{1,1}b_{1,1} + 12a_{1,0}b_{3,2})\cos 2\theta + (48b_{4,2} + 36b_{4,1}) + \\
6b_{1,1}a_{3,0} - 8a_{3,1}b_{1,1} + 12a_{1,0}b_{3,2} + 8a_{2,1}b_{2,1} + 6a_{1,0}b_{3,1}\left(\frac{R_s}{r}\right)^4 + \\
((48b_{5,3} + 72b_{5,2}) + 16a_{1,0}b_{4,2} + 4a_{3,1}b_{1,1} + 24a_{2,0}b_{3,2})\cos 2\theta + \\
(72b_{5,3} + 88b_{5,2} + 56b_{5,1}) + 8b_{1,1}a_{4,0} + 16a_{1,0}b_{4,2} - 8a_{3,1}b_{2,1} + 24a_{2,0}b_{3,2} + \\
12a_{2,1}b_{3,1} + 12b_{2,1}a_{3,0} - 8a_{4,1}b_{1,1} + 8a_{1,0}b_{4,1}\left(\frac{R_s}{r}\right)^5 + O(r^{-6}) \right\} R_s \sin^2 \theta. \quad (5.2)
\]

The solutions of the parameters are given by the following recursive relations

\[
a_{10} = -2, \quad a_{20} = 0, \quad a_{31} = m_1, \quad a_{30} = \frac{1}{3}m_1, \quad a_{41} = -\frac{1}{2}w_1^2, \quad a_{40} = -\frac{1}{2}w_1^2, \\
a_{52} = m_2, \quad a_{51} = \frac{2}{l}w_1^2 + \frac{4}{l}m_2, \quad a_{50} = \frac{8}{35}w_1^2 + \frac{9}{35}m_2, \\
a_{62} = -\frac{1}{2}w_1w_2, \quad a_{61} = -\frac{1}{10}w_1^2 - \frac{4}{5}w_1w_2, \quad a_{60} = -\frac{1}{10}w_1^2 - \frac{3}{10}w_1w_2, \quad \cdots \\
b_{11} = -\frac{1}{2}w_1, \quad b_{21} = -\frac{1}{4}w_1, \quad b_{32} = -\frac{1}{4}w_2, \quad b_{31} = \frac{1}{5}w_2 - \frac{1}{10}w_1, \\
b_{42} = -\frac{1}{32}m_1w_1 - \frac{3}{16}w_2, \quad b_{41} = -\frac{1}{30}w_1 + \frac{3}{20}w_2 - \frac{1}{24}m_1w_1, \quad (5.3) \\
b_{52} = -\frac{1}{6}w_3, \quad b_{52} = -\frac{1}{12}w_3 + \frac{1}{9}w_3, \quad b_{51} = -\frac{1}{21}m_1w_1 - \frac{1}{105}w_1 + \frac{1}{15}w_2 + \frac{5}{126}w_3 \\
b_{63} = -\frac{5}{36}w_3 - \frac{1}{16}w_1m_2 + \frac{1}{96}m_1w_2, \\
b_{62} = -\frac{1}{168}w_1^3 - \frac{3}{56}w_1m_2 + \frac{1}{240}m_1w_2 + \frac{1}{120}m_1w_1 - \frac{1}{36}w_2 + \frac{5}{54}w_3 \\
b_{61} = \frac{11}{480}m_1w_2 - \frac{1}{420}w_1^3 + \frac{51}{560}w_1m_2 - \frac{1}{35}m_1w_1 + \frac{1}{45}w_2 + \frac{25}{756}w_3 - \frac{1}{420}w_1, \quad \cdots
\]
The solutions to \((4.10)\) and \((4.11)\) in the metric \((4.3)\) are given by

\[
U = 1 - \frac{R_s}{R} + \left( \frac{1}{2} M_1 \cos 2\Theta + \frac{1}{6} M_1 \right) \left( \frac{R_s}{R} \right)^3 + \frac{1}{4} M_1 - \frac{1}{4} \left( \frac{R_s}{R} \right)^2 \cos 2\Theta + \frac{1}{12} M_1 - \frac{1}{4} \left( \frac{R_s}{R} \right)^2 \cos 2\Theta + \frac{1}{6} \left( \frac{R_s}{R} \right) + O(r^{-5}), \tag{5.4}
\]

\[
\tilde{V} = 1 + \frac{R_s}{R} + \left( \frac{R_s}{R} \right)^2 + \left( 1 - \frac{1}{2} M_1 \cos 2\Theta - \frac{1}{6} M_1 \right) \left( \frac{R_s}{R} \right)^3 + O(r^{-5}) + \left( 1 - \frac{37}{96} M_1 + \frac{11}{64} w_1^2 + \left( -\frac{5}{32} M_1 + \frac{9}{64} w_1^2 \right) \cos 4\Theta + \left( -\frac{9}{8} M_1 + \frac{3}{16} w_1^2 \cos 2\Theta \right) \right) \left( \frac{R_s}{R} \right)^4, \tag{5.5}
\]

\[
W = R_s \sin^2 \Theta \left\{ w_1 \frac{R_s}{R} + w_1 \left( \frac{R_s}{R} \right)^2 + O(r^{-5}) + \frac{1}{4} M_1 + \frac{9}{8} w_1^2 \right\} \frac{R_s}{R} + \frac{31}{5} w_2 \left( \frac{R_s}{R} \right)^3 + \frac{1}{8} M_1 - \frac{9}{4} w_2 \cos 2\Theta + \frac{239}{320} w_1 - \frac{5}{24} M_1 w_1 + \frac{27}{20} w_2 \right\} \left( \frac{R_s}{R} \right)^4, \tag{5.6}
\]

where

\[
M_1 = m_1 + \frac{1}{8}, \quad M_2 = m_2 + \frac{7}{29}, \quad M_3 = m_3 + \frac{3 \cdot 11}{214}, \quad \cdots \tag{5.7}
\]

are somewhat ‘pure mass’ multipole moments deducted the influence of the coordinates (see \((3.18)\)). In the case of Schwarzschild metric, we have \((M_k = w_k = 0, \forall k)\).