Synthesis and Upper Bound of Schmidt Rank of Bipartite Controlled-Unitary Gates

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Quantum circuit model is the most popular paradigm for implementing complex quantum computation. Based on Cartan decomposition, it is shown $2(N-1)$ generalized controlled-X (GCX) gates, 6 single-qubit rotations about the y- and z-axes, and $N+5$ single-partite y- and z-rotation-types which are defined in this paper are sufficient to simulate a controlled-unitary gate $U_{cu(Q\otimes N)}$ with $A$ controlling on $C^{2}\otimes C^{N}$. In the scenario of the unitary gate $U_{cd(M\otimes N)}$ with $M \geq 3$ that is locally equivalent to a diagonal unitary on $C^{M} \otimes C^{N}$, 2$M(N-1)$ GCX gates and 2$M(N-1) + 10$ single-partite y- and z-rotation-types are required to simulate it. The quantum circuit for implementing $U_{cu(Q\otimes N)}$ and $U_{cd(M\otimes N)}$ are presented. Furthermore, it is found that $U_{cu(Q\otimes N)}$ with $A$ controlling has Schmidt rank two, and in other cases the diagonalized form of the target unitaries can be expanded in terms of specific simple types of product unitary operators.

1. Introduction

Tremendous progress has been made in quantum computation[1–4] and communication[5–9] in recent years. Unitary operations play a central role in many quantum information processing tasks: quantum circuits,[10–16] quantum algorithm,[17] creating quantum entangled states,[18–21] quantum state fusion,[22,23] cryptography,[24] entanglement purification and concentration,[25–27] etc. There are local and nonlocal unitary operations. The local unitary operations, known as the tensor product operators locally acting on subsystems, can be deterministically implemented by local operations and classical communication (LOCC).[28] Unfortunately, the local unitary operations alone cannot create entanglement. The nonlocal unitary operations, which can not be implemented by LOCC, have a more complex structure and play a more powerful role than local unitary operations in quantum information processing.[29–31] So far, the properties and implementations of nonlocal unitary operations, even the simplest and the most popular controlled-unitary operations, are still far from complete. Hence it is interesting and important to find simple ways to implement nonlocal unitary operations.

There are two ways to study complex unitary operations. One approach is to decompose a unitary operation into a sum of product operations: $U = \sum_{j} A_{j} \otimes B_{j} \otimes C_{j} \otimes \ldots \otimes R_{j}$, where $A_{j}, B_{j},$ etc. are linearly independent local operations acting on respective parties. The smallest possible $m$ is defined as Schmidt rank $\text{Sch}(U)$,[32] which can be used for quantifying the nonlocality of $U$ defined by $K_{\text{LOCC}}(U) \equiv \log_{2}[\text{Sch}(U)]$,[33–35] providing a sufficient condition for when $U$ is a controlled-unitary operation,[36–42] and optimizing the synthesis of quantum computation[43] and quantum transistors.[40] Another common approach, called synthesis or quantum circuit, is to factorize $U$ into fewer achievable simple local and nonlocal operations from a universal library, which may simplify such physical implementations: $U = X_{1}X_{2}\ldots X_{n}$. The complexity of the quantum circuit is characterized by assessing the number of entangled operations involved in the implementation. Quantum circuit is the dominant paradigm for implementing efficiently complex quantum computation.

One of the central problems in quantum computing is to minimize the number of one- and two-partite gates required to implement a desired quantum gate. Utilizing Cartan decomposition,[44] Vatan et al.[45] designed a controlled-NOT (CNOT)-optimized general two-qubit quantum circuit in 2004, Shende et al.[46] presented the highest known lower bound on asymptotic CNOT cost required to implement an unstructured $n$-qubit quantum computing, Di and Wei[47,48] synthesized universal multiple-valued quantum circuits. Using higher-dimensional Hilbert spaces, Lanyon et al.[11] reduced the cost of a Toffoli gate from six CNOTs to three CNOTs in 2009, Li et al.[49] further optimized the $n$-qubit universal quantum circuit, Liu and Wei[14] decreased the complexity of a Fredkin gate to three entangling gates in 2020. Nowadays, many works have been devoted to multiple-valued quantum circuit.[47,48] Nonetheless, the synthesis of multi-valued quantum gates is still far from complete, and multiple-valued unitary operations still are open problems.
Bipartite controlled-unitary operation is one of the most easily understood, extensively studied, and widely used quantum operations, such as CNOT gate. In this paper, we study the synthesis and possible Schmidt rank of the controlled-unitary operation on bipartite Hilbert space. By analogy with single-qubit rotations, we give the concepts of single-partite $y$- and $z$-rotation-types in higher-dimensional system, and present the synthesis of arbitrary single-partite unitary gate. Then, utilizing Cartan decomposition technique, we present a program for synthesizing a bipartite controlled-unitary gate with $A$ controlling on $C^2 \otimes C^N$ with $2(N-1)$ generalized controlled-$X$ gates (GCX), six single-qubit rotations about the $y$- and $z$-axes, and $N+5$ single-partite $y$- and $z$-rotation-types in the worst case. And generally, the gate on $C^M \otimes C^N (M \geq 3)$ locally equivalent to a diagonal gate, which is the subset of controlled-unitaries with $A$ or $B$ controlling, was synthesized by $2M(N-1)$ GCX gates and $2M(N-1) + 10$ single-partite $y$- and $z$-rotation-types in the worst case. Furthermore, the possible Schmidt rank of the bipartite controlled-unitary operation is presented in detail. The results indicate that upper bound of Schmidt rank of the bipartite controlled-unitary operation with $A$ controlling on $C^2 \otimes C^N$ is $2N$, and the one on $C^2 \otimes C^2$ is two. In the scenario of the unitary operation on $C^M \otimes C^N$ locally equivalent to a diagonal unitary, the upper bound is $MN$.

2. Cartan Decomposition of Lie Group $U(N)$

Cartan decomposition technique is extremely valuable tool employed to obtain the possible Schmidt rank and design compact quantum circuits.\cite{33,45,50,51} Cartan decomposition of real semi-simple Lie algebra $\mathfrak{g}$ is defined as

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$$

(1)

The Lie subalgebra $\mathfrak{l}$ and the complement subspace $\mathfrak{p} = \mathfrak{l}^\perp$ satisfying the commutation relations

$$[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}, \quad [\mathfrak{l}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{l}$$

(2)

Let $G$ be a compact Lie group with a real semi-simple Lie algebra $\mathfrak{g}$, then

$$G = K_1 \cdot A \cdot K_2$$

(3)

with $K_1, K_2 \in C^d$ and $A \in \mathfrak{a}$ for the Lie group exponential. $\mathfrak{l}$ is the maximal Abelian subalgebra of $\mathfrak{g}$,$\mathfrak{l}$ contained in $\mathfrak{p}$.

It is well known that the generators of the Lie algebra $u(2)$ can be given by

$$u(2) = \text{span}\{\sigma_x, \sigma_y, \sigma_z, I_2\}$$

(4)

Here

$$\sigma_x = T^{(2,1)}_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = T^{(2,2)}_{12} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_z = T^{(2,3)}_{12} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(5)

The basis for $u(3)$ takes the form

$$T^{(3,1)}_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^{(3,1)}_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$T^{(3,1)}_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T^{(3,2)}_{12} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^{(3,2)}_{13} = \begin{pmatrix} 0 & 0 & -i \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T^{(3,2)}_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$T^{(3,3)}_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^{(3,3)}_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

(6)

Let $I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(7)

$$T^{(N,1)}_{ab} = \begin{pmatrix} 1, & c = a, d = b \\ 1, & c = b, d = a \\ 0, & \text{others} \end{pmatrix}$$

(8)

$$T^{(N,2)}_{ab} = \begin{pmatrix} -i, & c = a, d = b \\ i, & c = b, d = a \\ 0, & \text{others} \end{pmatrix}$$

(9)

The basis for $u(N)$ is spanned by $N^2$ matrices $\{ T^{(N,1)}_{ab}, T^{(N,2)}_{ab}, T^{(N,1)}_{1a}, I_N \}$ ($a < b$), where $I_N$ is the $N \times N$ identity matrix, $T^{(N,1)}_{ab}$, and $T^{(N,3)}_{1a}$ are defined as

$$T^{(N,3)}_{1a} = \begin{pmatrix} 1, & c = d = 1 \\ -1, & c = d = a, \quad 2 \leq a \leq N \\ 0, & \text{others} \end{pmatrix}$$

The subscript $a$ and $b$ are order index of the basis. The $c$ and $d$ represent the cth row and dth column in matrix $T^{(N,1)}_{ab}$, $T^{(N,2)}_{ab}$, and $T^{(N,3)}_{1a}$. The $b, c, d \in [1, 2, \ldots, N]$, and $N$ represents the $N$ dimensional system.

Based on Equations (1)–(9), it is easy to find that the Cartan decomposition of Lie algebra $u(N)$ may have the form $u(N) = \mathfrak{l}(N) \oplus \mathfrak{p}(N)$, where

$$\mathfrak{l}(N) = \text{span}\{ T^{(N,2)}_{ab}, I_N \},$$

(10)
about the qubit rotations about the $\tilde{\vartheta}$ and $\delta$-$\vartheta$.

Then, the associated Cartan decomposition of group $U(N)$ is of the form

$$U(N) = e^{i\vartheta_{ij}} \cdot M_i \cdot e(\theta_{ij}T_{i,j} + \vartheta_{ij}T_{i,j}) \cdot M_j,$$

with $M_i, M_j \in e^{i\vartheta_{ij}T_{i,j}}$. For instance, any 2D unitary operator can be expanded into

$$U_{ij} = e^{i\vartheta_{ij}} \cdot e^{i\vartheta_{i1}T_{1,j}} \cdot e^{i\vartheta_{i2}T_{2,j}} \cdot e^{i\vartheta_{i3}T_{3,j}}.$$

Any 3D unitary operator can be expanded into

$$U_{i,j} = e^{i\vartheta_{ij}} \cdot e^{i\vartheta_{i1}T_{1,j}} \cdot e^{i\vartheta_{i2}T_{2,j}} \cdot e^{i\vartheta_{i3}T_{3,j}} \cdot e^{i\vartheta_{i4}T_{4,j}} \cdot e^{i\vartheta_{i5}T_{5,j}}.$$

Local single-qubit gates $e^{i\vartheta_{ij}}$ are single-qubit rotations about the $y$- and $z$-axes, respectively. That is, two single-qubit rotations about the $y$- and $z$-axes in one single-qubit rotation about the $z$-axes are equivalent to simulate a generic single-qubit gate. The global phase factor $e^{i\vartheta_{ij}}$ is missing here because it is irrelevant for quantum information processing. The matrices in the set $\{T_{i,j}\}$ ($i < j$) do not commute with each other.

**Definition:** Similar to the definition of single-qubit rotations, we define

$$\{e(\theta_{ij}T_{i,j} + \vartheta_{ij}T_{i,j} + \ldots + \vartheta_{i,j-1,j}T_{i,j-1})\}$$

as single-partite $y$-rotation-type, and

$$\{e(\theta_{ij}T_{i,j} + \vartheta_{ij}T_{i,j} + \ldots + \vartheta_{i,j,n}T_{i,j,n})\}$$

as single-partite $z$-rotation-type. When $N = 2$, they are compatible with single-qubit rotations about the $y$- and $z$-axes.

Based on Equations (10)–(13), one can see that arbitrary single-partite multi-valued unitary $U_{i,j}$ can be implemented by two single-partite $y$-rotation-types and one $z$-rotation-type. Each $y$-rotation-type and $z$-rotation-type has $N(N-1)/2$ and $N - 1$ free parameters, respectively. We omit the overall phase change $e^{i\vartheta_{ij}}$.

3. **Synthesis and Quantum Schmidt Rank of Controlled-Unitary Gate with A Controlling**

It is known that, if the controlled-unitary gate $U_{\text{cnot}}$ on $C^2 \otimes C^N$ is controlled from A side, then $\exists U_A, V_A$ unitaries such that

$$U_{\text{cnot}} = (U_A \otimes I_B) \cdot \left( \sum_{i=0}^{M-1} \alpha_i (i) \otimes U_B \right) \cdot (V_A \otimes I_B).$$

That is, up to local unitarities, $U_{\text{cnot}}$ with A controlling is equivalent to

$$U_{\text{cnot}} = \sum_{i=0}^{M-1} \alpha_i (i) \otimes U_B.$$

Using the well-known commutation relations

$$[A \otimes B, C \otimes D] = [A, C] \otimes (B \times D) + [C \times A] \otimes [B, D],$$

one can verify that the Cartan decomposition of $U_{\text{cnot}}$ has the form

$$U_{\text{cnot}} = \sum_{i=0}^{M-1} \alpha_i (i) \otimes U_B.$$

As a consequence, $U_{\text{cnot}}$ with A controlling can be decomposed as

$$U_{\text{cnot}} = (U_A \otimes U_B) \cdot \wedge \left( \Delta \otimes N \right) \cdot \left( V_A \otimes V_B \right)$$

where $\wedge \left( \Delta \otimes N \right) = e^{i\vartheta_{ij}T_{i,j}}$.

Therefore, two CNOT gates together with $13 = 3 \times 4 + 1$ single-qubit rotations about the $y$- and $z$-axes are sufficient to simulate a controlled-unitary $U_{\text{cnot}}$ with A controlling.

Based on Equation (22), we may expand $\wedge \left( \Delta \otimes N \right)$ as

$$\wedge \left( \Delta \otimes N \right) = \lambda \left( \Delta \otimes N \right)$$

where $\lambda \left( \Delta \otimes N \right) = e^{i\vartheta_{ij}T_{i,j}}$.

3.1. **Synthesis and Schmidt Rank of Controlled Unitary Gate with A Controlling**

When $U_{\text{cnot}}$ with A controlling is acting on $C^2 \otimes C^2$, it can be decomposed as

$$U_{\text{cnot}} = (U_A \otimes U_B) e^{i\vartheta_{ij}T_{i,j}} \cdot (V_A \otimes V_B)$$

where $V_A, V_B$ can be expanded in the form of $e^{i\vartheta_{ij}T_{i,j}} \cdot \lambda \left( \Delta \otimes N \right)$, that is, each one is the product of three single-qubit rotations about the $y$- and $z$-axes, see Equation (14). CNOT represents a controlled-NOT gate, and it is given by the matrix

$$\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}$$

Therefore, as shown in Figure 1, two CNOT gates together with $13 = 3 \times 4 + 1$ single-qubit rotations about the $y$- and $z$-axes are sufficient to simulate a controlled-unitary $U_{\text{cnot}}$ with A controlling.

Based on Equation (22), we may expand $\wedge \left( \Delta \otimes N \right)$ as

$$\wedge \left( \Delta \otimes N \right) = \lambda \left( \Delta \otimes N \right)$$

where $\lambda \left( \Delta \otimes N \right) = e^{i\vartheta_{ij}T_{i,j}}$. Therefore, $U_{\text{cnot}}$ with A controlling only has Schmidt rank 2 when $\vartheta_{ij} \neq 0$.
That is to say, a $U_{\text{cu}(2\otimes3)}$ with A controlling can be synthesized by four GCX gates, $6 = 3 \times 2$ single-qubit rotations about the $y$- and $z$-axes, and $8 = 3 \times 2 + 2$ single-partite $y$- and $z$-rotation-types in the worst case, see Figure 2. Note that each above $y$-rotation-type and $z$-rotation-type has three and two free parameters, respectively, see Equation (15). By the same argument as that made for $C^2 \otimes C^4$, we can find that as shown in Figure 3, the quantum circuit for implementing $U_{\text{cu}(2\otimes4)}$ with A controlling contains six GCX gates, $6 = 3 \times 2$ single-qubit rotations about the $y$- and $z$-axes, and $9 = 3 \times 2 + 3$ single-partite $y$- and $z$-rotation-types.

3.2. Synthesis and Schmidt Rank of $U_{\text{cu}(2\otimes3)}$ with A Controlling

When $U_{\text{cu}}$ with A controlling is acting on $C^2 \otimes C^3$, it can be decomposed as

$$U_{\text{cu}(2\otimes3)} = (U_A \otimes U_B) e^{i(\theta_1 \sigma_x \otimes \sigma_z^{(1)} + \theta_2 \sigma_y \otimes \sigma_z^{(1)})} (V_A \otimes V_B)$$  \hspace{1cm} (25)

We find that each $e^{i(\theta_1 \sigma_x \otimes \sigma_z^{(1)})}$ can be synthesized by

$$e^{i(\theta_1 \sigma_x \otimes \sigma_z^{(1)})} = \text{GCX}(1 \rightarrow X^{(0,\alpha-1)}) \cdot e^{i\theta_1 \sigma_x \otimes \sigma_z^{(1)}} \cdot \text{GCX}(1 \rightarrow X^{(0,\alpha-1)})$$  \hspace{1cm} (26)

Here the universal generalized controlled-X gate,\(^\text{[47,48]}\) GCX($m \rightarrow X^{(j)}$), implements the operation

$$X^{(j)} = |i\rangle \langle j| + |j\rangle \langle i| + \sum_{k \neq i,j} |k\rangle \langle k|$$  \hspace{1cm} (27)

on the target particle if and only if the control particle is in the state $|m\rangle$, and has no effect otherwise. Hence, Equation (25) can be rewritten as

$$U_{\text{cu}(2\otimes3)} = (U_A \otimes U_B) \cdot \text{GCX}(1 \rightarrow X^{(01)}) \cdot e^{i\theta_1 \sigma_x \otimes \sigma_z^{(12)}} \cdot \text{GCX}(1 \rightarrow X^{(02)}) \cdot e^{i\theta_2 \sigma_x \otimes \sigma_z^{(13)}} \cdot \text{GCX}(1 \rightarrow X^{(03)}) \cdot (V_A \otimes V_B).$$  \hspace{1cm} (28)

We note that there are $N-1$ items in the exponential $e^{i\theta_1 \sigma_x \otimes \sigma_z^{(12)} + \cdots + \theta_{N-1} \sigma_x \otimes \sigma_z^{(1N)}}$, and each of them can be synthesized by two GCX gates and one single-partite $z$-rotation-type. Hence, the quantum circuit for implementing $U_{\text{cu}(2\otimes N)}$ with A controlling contains $2N-2$ GCX gates, and one single-partite $y$-rotation-type. Hence, the quantum circuit for implementing $U_{\text{cu}(2\otimes4)}$ with A controlling contains six GCX gates, $6 = 3 \times 2$ single-qubit rotations about the $y$- and $z$-axes, and $9 = 3 \times 2 + 3$ single-partite $y$- and $z$-rotation-types.
Based on Equation (30), we may expand \( \wedge(\triangle_{2\otimes\eta}) \) as
\[
\wedge(\triangle_{2\otimes\eta}) = \gamma_k I_{2N} + \delta_k \sigma_z \otimes I_N + \sum_{j=2}^{N-1} (\gamma_{kj} I_2 \otimes T_{1k}^{(N,3)} + \delta_{kj} \sigma_z \otimes T_{1k}^{(N,3)})
\]
(31)

Here parameters \( \gamma_k \) and \( \delta_k \) can be expressed by following
\[
\gamma_k = \begin{cases} 
\frac{1}{N}[c_{\theta_1 + \theta_2 + \ldots + \theta_{N-1}}] + \sum_{n=1}^{N-1} c_{\theta_n} & k = 1, \\
\frac{1}{N}[c_{\theta_1 + \theta_2 + \ldots + \theta_{N-1}}] + \sum_{n=1}^{N-1} c_{\theta_n} - N \epsilon_{\theta_{N-1}} & k = 2, \ldots, N-1
\end{cases}
\]
(32)

\[
\delta_k = \begin{cases} 
\frac{1}{N}[s_{\theta_1 + \theta_2 + \ldots + \theta_{N-1}}] - \sum_{n=1}^{N-1} s_{\theta_n} & k = 1, \\
\frac{1}{N}[s_{\theta_1 + \theta_2 + \ldots + \theta_{N-1}}] + \sum_{n=1}^{N-1} s_{\theta_n} + N s_{\theta_{N-1}} & k = 2, \ldots, N-1
\end{cases}
\]
(33)

Therefore, the Schmidt rank of a \( \mathcal{U}_{cd(2\otimes\eta)} \) with A controlling is no more than \( 2N \).

4. Synthesis and Quantum Schmidt Rank of the Gate Locally Equivalent to Diagonal Unitary on \( C^M \otimes C^N \)

It is well known that, if \( \mathcal{U}_{cd(M\otimes\eta)} \) is a unitary gate locally equivalent to a diagonal unitary on \( C^N \otimes C^N \), then it can be viewed as a controlled unitary controlled from A side or B side, and \( \exists U_A, U_B, V_A, V_B \) unitaries such that
\[
\mathcal{U}_{cd(M\otimes\eta)} = (U_A \otimes U_B) \cdot \left( \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} |i\rangle_{A} \otimes |j\rangle_{B} \cdot \{V_A \otimes V_B\} \right)
\]
(34)

Here we use the symbol “cd” in the subscript, rather than simply using \( d \), in order to distinguish them from the diagonal unitaries \( \sum_{k=1}^{d_1} \sum_{l=1}^{d_2} |i\rangle_{A} \otimes |j\rangle_{B} \). For the same unitary of this type, either party may act as the control, or the choice would not affect the decomposition.

As \( \sum_{j=1}^{d_2} |i\rangle_{A} \otimes |j\rangle_{B} \in \expansion{1}_{1_1} \otimes 1_{1_1} \), \( 1_{1_2} \otimes 1_{1_2} \), \( 1_{1_3} \otimes 1_{1_3} \), \( I_{1_4} \otimes I_{1_4} \), \( T_{2_3} \otimes T_{2_3} \), \( T_{2_3} \otimes T_{2_3} \), \( T_{2_3} \otimes I_{2_3} \) and the matrices in the set \( \{1_{1_1} \otimes \sigma_{y_{1_1}}, 1_{1_2} \otimes I_{1_1}, 1_{1_2} \otimes T_{1_1}, 1_{1_2} \otimes T_{1_1}, 1_{1_2} \otimes T_{1_1} \} \) do not commute with each other, local single-partite z-rotation-type
\[ e^{\exp[1_u \otimes T^{(N,N)}_{1_a}]} \text{ will be “absorbed” by their neighboring gate } U_b \text{ or } V_b, \text{ and } e^{\exp[1_u \otimes T^{(N,N)}_{1_a}]} \text{ will be “absorbed” by } U_A \text{ or } V_A. \text{ Then } U_{c\text{d}(M@N)} \text{ can be rewritten as} \]

\[
U_{c\text{d}(M@N)} = (U_A \otimes U_b) \cdot \Lambda(\Delta_{M@N}^{(1)}) \cdot (V_A \otimes V_{\beta}) \tag{35}
\]

where \( \Lambda(\Delta_{M@N}^{(1)}) = e^{\exp[1_u \otimes T^{(N,N)}_{1_a}]} \) with \( A \in \{2, 3, \ldots, M\} \text{ and } a \in \{2, 3, \ldots, N\} \).

4.1. Synthesis and Schmidt Rank of \( U_{c\text{d}(M@N)} \) on \( C_3 \otimes C_3 \)

When the \( U_{c\text{d}(M@N)} \) is acting on \( C_3 \otimes C_3 \), it can be decomposed as

\[
U_{c\text{d}(M@N)} = (U_A \otimes U_b) \cdot e^{[\theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a}]} \cdot (V_A \otimes V_{\beta}) \tag{36}
\]

One can verify that each \( e^{[\theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a}]} \) can be synthesized by

\[
e^{[\theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a}]} = \Lambda(0 \rightarrow e^{[\theta T^{(N,N)}_{1_a}]}) \cdot \Lambda(\tilde{a} \rightarrow 1 \rightarrow e^{[\theta T^{(N,N)}_{1_a}]}) \tag{37}
\]

where \( \Lambda(m \rightarrow e^{[\theta T^{(N,N)}_{1_a}]}) \) is to implement the operation \( e^{[\theta T^{(N,N)}_{1_a}]} \) on the target particle if and only if the control particle is in the state \( |m\rangle \), and has no effect otherwise. Given verification, \( \Lambda(m \rightarrow e^{[\theta T^{(N,N)}_{1_a}]}) \) can be simulated by two GCX gates and two single-partite \( x \)-rotation-types as described in Figure 5a, that is,

\[
\Lambda(m \rightarrow e^{[\theta T^{(N,N)}_{1_a}]}) = e^{[\theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a}]} \cdot \text{GCX}(m \rightarrow X^{(0,0,1)})
\]

\[\quad \cdot e^{[-i \theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a}]} \cdot \text{GCX}(m \rightarrow X^{(0,0,1)}) \tag{38}
\]

Hence, Equation (36) can be rewritten as

\[
U_{c\text{d}(M@N)} = (U_A \otimes U_b) \cdot \Lambda(0 \rightarrow e^{[\theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a}]}) \cdot \Lambda(0 \rightarrow e^{[\theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a}]})
\]

\[\quad \cdot \Lambda(1 \rightarrow e^{[-i \theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a}]}) \cdot \Lambda(1 \rightarrow e^{[-i \theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a} \otimes \theta T^{(N,N)}_{1_a}]}) \tag{39}
\]

\[\quad \cdot (V_A \otimes V_{\beta}).\]

Note that the rightmost gate of the circuit of Figure 5a will be “absorbed” by its neighboring gate \( U_b \) of Figure 6, and the leftmost gate of the circuit of Figure 5b will be “absorbed” by its neighboring gate \( V_b \) of Figure 6. Therefore, according to Equations (38) and (39), 12 GCX gates and 22 = 3 × 4 + (12 − 2) single-partite \( y \)- and \( z \)-rotation-types are sufficient to simulate a \( U_{c\text{d}(M@N)} \) locally equivalent to a diagonal unitary as in Figure 6.

Based on Equation (39), we may expand \( \Lambda(\Delta_{3@3}) \) as

\[\Lambda(\Delta_{3@3}) = CI_{3} \otimes I_{3} + \left[ C - \frac{1}{3} (c_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{12}^{(1,3)} \]

\[+ \frac{i}{3} (s_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{12}^{(1,3)} \]

\[+ \left[ C - \frac{1}{3} (c_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{12}^{(1,3)} \]

\[+ \frac{i}{3} (s_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{12}^{(1,3)} \]

\[+ \left[ C - \frac{1}{3} (c_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{13}^{(1,3)} \]

\[+ \frac{i}{3} (s_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{13}^{(1,3)} \]

\[+ \left[ C - \frac{1}{3} (c_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{13}^{(1,3)} \]

\[+ \frac{i}{3} (s_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{13}^{(1,3)} \]

\[+ \left[ C - \frac{1}{3} (c_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{13}^{(1,3)} \]

\[+ \frac{i}{3} (s_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{13}^{(1,3)} \]

\[+ \left[ C - \frac{1}{3} (c_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{13}^{(1,3)} \]

\[+ \frac{i}{3} (s_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{13}^{(1,3)} \]

\[+ \left[ C - \frac{1}{3} (c_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{13}^{(1,3)} \]

\[+ \frac{i}{3} (s_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{13}^{(1,3)} \]

\[+ \left[ C - \frac{1}{3} (c_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{13}^{(1,3)} \]

\[+ \frac{i}{3} (s_{[\theta_{[\theta_{3+a+\theta_4}], c_{\theta_{1} + c_{\theta_{2}}}]}) \right] T_{13}^{(1,3)} \]
where

\[
C = \frac{1}{9} [4c_{(a_1 + b_1 + c_1)} + \sum_{n=1}^{4} c_{(a_1 + b_1 + c_1)} + c_{(b_1 + a_1 + c_1)} + c_{(b_1 + c_1 + a_1)}] + \sum_{n=1}^{4} (s_{(a_1 + b_1 + c_1)} - s_{(a_1 + b_1 + c_1)} - s_{(a_1 + c_1 + b_1)} - s_{(b_1 + a_1 + c_1)}) \tag{41}
\]

Hence, the Schmidt rank of \( U_{cd}(\otimes \otimes) \) is no more than nine.

### 4.2. Synthesis and Schmidt Rank of \( U_{cd}(M \otimes N) \) on \( C^M \otimes C^N \)

When the \( U_{cd} \) is on \( C^M \otimes C^N \), it can be decomposed as

\[
U_{cd}(M \otimes N) = (U_A \otimes U_B) \cdot \prod_{a=2,3,\ldots,N} e^{i(\theta_{1a}^{(M)} \otimes \gamma_{1a}^{(N)})} \cdot (V_A \otimes V_B) \tag{42}
\]

Putting all the pieces together, and taking the trick “absorption” into account, we find that \( 2M(N - 1) \) GCX gates and \( 2M(N - 1) + 10 \) single-partite \( \gamma \) - and \( z \) -rotation-types are sufficient to implement a \( U_{cd}(M \otimes N) \) locally equivalent to a diagonal unitary. It is noted that each \( \gamma \) - and \( z \) -rotation-type has many free parameters in it when \( M, N \) are larger.

We note that there are \( MN - 1 \) independently diagonal bases in \( \mathcal{u}(MN) \), and \( U_{cd}(M \otimes N) \) is equivalent to \( \wedge(\Delta_{M \otimes N}) = \prod_{a=2,3,\ldots,N} e^{i(\theta_{1a}^{(M)} \otimes \gamma_{1a}^{(N)})} \). Hence, \( \wedge(\Delta_{M \otimes N}) \) can be expanded as

\[
\wedge(\Delta_{M \otimes N}) = \varepsilon_{I_{MN}} + \sum_{k=1}^{N-1} \gamma_k I_{IM} \otimes T_{1k}^{(N)} \tag{43}
\]

\[
+ \sum_{j=2}^{M-1} \delta_j T_{ij}^{(M)} \otimes I_N + \sum_{j=2}^{M-1} \sum_{k=2}^{N-1} \tau_{jk} T_{ij}^{(M)} \otimes T_{1k}^{(N)}
\]

Based on Equation (43), one can see that the Schmidt rank of a \( U_{cd}(M \otimes N) \) is no more than \( MN \).

### 5. Conclusion

Quantum circuit is the dominant paradigm for implementing and characterizing complex quantum computation, and the works are mainly focused on two-valued systems. Utilizing Cartan decomposition technique, we have presented compact quantum circuits for implementing controlled-unitary gates with \( A \) controlling on \( C^2 \otimes C^N \) and controlled-diagonal gates on \( C^M \otimes C^N \), respectively. We first showed the Cartan decomposition of Li algebra \( \mathcal{u}(N) \) and reported that an arbitrary single-partite unitary can be implemented by three single-partite \( \gamma \) - and \( z \) -rotation-types which are defined in this paper, and their appearance in decomposition of nonlocal gates is not previously studied, to our best knowledge. Subsequently, we designed compact quantum circuits for implementing controlled-unitary gates with \( A \) controlling on \( C^2 \otimes C^2, C^2 \otimes C^3, \ldots, C^2 \otimes C^N \) in terms of GCX gates and local single-partite rotation-types in detail. The results indicate that \( 2(N - 1) \) GCX gates together with six single-qubit rotations about the \( \gamma \) - and \( z \) -axes, and \( N + 5 \) single-partite \( \gamma \) - and \( z \) -rotation-types are sufficient to implement a controlled-unitary gate with \( A \) controlling on \( C^2 \otimes C^N \). Last, we extended the program to gates on \( C^M \otimes C^N \) which are locally equivalent to diagonal gates, and the quantum circuit is comprised of \( 2M(N - 1) \) GCX gates and \( 2M(N - 1) + 10 \) single-partite \( \gamma \) - and \( z \) -rotation-types.

Based on Cartan decomposition, we have verified that expanding the diagonal form of the unitary is a way to get the upper bound of Schmidt rank of the controlled-unitary gate with \( A \) controlling on \( C^2 \otimes C^2 \) or the gate locally equivalent to a diagonal gate on \( C^M \otimes C^N \), and such upper bound may be not tight. The results showed that the controlled-unitary gate with \( A \) controlling on \( C^2 \otimes C^N \) has Schmidt rank at most \( 2N \), whereas the one on \( C^2 \otimes C^2 \) reaches the lower bound of two, and the gate locally equivalent to a diagonal gate on \( C^M \otimes C^N \) has Schmidt rank at most \( MN \). Although the bounds are not tight in general, the form of our decomposition involves special types of operators: the local operators in each term are either the identity or unitaries in low-dimensional subspaces. This explains that our decomposition in general has more terms than the Schmidt rank.

It is expected that the algorithm and technique employed here to implement controlled-unitary gates with \( A \) controlling on \( C^2 \otimes C^2 \) can be generalized to more general algebraic structures.
C^N and the gates locally equivalent to diagonal gates on C^M ⊗ C^N could be useful for studying arbitrary multi-partite quantum computation in multi-valued systems.

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Keywords

controlled-unitary gates, quantum circuits, quantum Schmidt rank

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