Cohomological constraint to deformations of compact Kähler manifolds

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Abstract
We prove that for every compact Kähler manifold \( X \) the cup product
\[
H^*(X, T_X) \otimes H^*(X, \Omega_X^*) \to H^*(X, \Omega_X^{*-1})
\]
can be lifted to an \( L_\infty \)-morphism from the Kodaira-Spencer differential graded Lie algebra to the suspension of the space of linear endomorphisms of the singular cohomology of \( X \). As a consequence we get an algebraic proof of the principle “obstructions to deformations of compact Kähler manifolds annihilate ambient cohomology”.
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Introduction
In this paper we give an algebraic proof of the principle “obstructions to deformations of compact Kähler manifolds annihilate ambient cohomology” recently proved, in a different way, by Herb Clemens [4] and Ziv Ran [18].

Let \( X \) be a fixed compact Kähler manifold of dimension \( n \) and consider the graded vector space \( M_X = \text{Hom}_C(H^*(X, \mathbb{C}), H^*(X, \mathbb{C})) \) of linear endomorphisms of the singular cohomology of \( X \). The Hodge decomposition gives natural isomorphisms
\[
M_X = \bigoplus_i M_i^X, \quad M_i^X = \bigoplus_{r+s=p+q+i} \text{Hom}_C(H^p(\Omega_q^X), H^r(\Omega_s^X))
\]
and the composition of the cup product and the contraction operator \( T_X \otimes \Omega_X^p \to \Omega_X^{p-1} \) gives natural linear maps
\[
\theta_p : H^p(X, T_X) \to \bigoplus_{r,s} \text{Hom}_C(H^r(\Omega_s^X), H^{r+p}(\Omega_s^{*-1}_X)) \subset M[-1]^p = M_{X}^p.
\]
The Dolbeaut’s complex of the holomorphic tangent bundle \( T_X \)
\[
KS_X = \bigoplus_p KS_X^p, \quad KS_X^p = \Gamma(X, \mathcal{A}^{0,p}(T_X))
\]
has a natural structure of differential graded Lie algebra (DGLA), \([3], [8], [11, 3.4.1]\), called the Kodaira-Spencer algebra of \( X \). By Dolbeaut’s theorem \( H^*(KS_X) = H^*(X, T_X) \) and then the maps \( \theta_i \) give a morphism of graded vector spaces \( \theta : H^*(KS_X) \to M[-1] \). This morphism is generally nontrivial: consider for instance a Calabi-Yau manifold where the map \( \theta_p \) induces an isomorphism \( H^p(X, T_X) = \text{Hom}_C(H^0(\Omega^p_X), H^p(\Omega_X^{*-1})) \).

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Theorem A. In the above notation, consider \( M[-1]_X \) as a differential graded Lie algebra with trivial differential and trivial bracket. Every choice of a Kähler metric on \( X \) induces a canonical lifting of \( \theta \) to an \( L_\infty \)-morphism from \( K\mathcal{S}_X \) to \( M[-1]_X \).

The above theorem, together some standard and purely formal results in Schlessinger’s theory, gives immediate applications to the study of deformations of \( X \). In fact the deformations of \( X \) are governed by the Kodaira-Spencer differential graded Lie algebra \( K\mathcal{S}_X \) and every \( L_\infty \)-morphism between DGLAs induces a natural transformation between the associated deformation functors. The triviality of the DGLA structure on \( M[-1]_X \) allows to prove easily the following:

Corollary B. Let \( f: \mathcal{Y} \to \mathcal{B} \) be the semiuniversal deformation of a compact Kähler manifold \( Y \) and let \( X \xrightarrow{\pi} Y \) be a finite unramified covering. For every \( p \geq 0 \) denote by \( \alpha_p \), the composite linear map

\[
\alpha_p: H^p(Y, T_Y) \xrightarrow{\pi^*} H^p(X, T_X) \xrightarrow{\theta_p} \bigoplus_{r,s} \text{Hom}_\mathbb{C}(H^r(\Omega^s_X), H^{r+p}(\Omega^{s-1}_X)).
\]

Then:

1. If \( \alpha_1 \) is injective then \( f: \mathcal{Y} \to \mathcal{B} \) is universal.

2. There exists a morphisms of complex analytic singularities \( q: (H^1(Y, T_Y), 0) \to (\ker \alpha_2, 0) \) such that \( \mathcal{B} \) is isomorphic to \( q^{-1}(0) \). In particular if \( \alpha_2 \) is injective then \( \mathcal{B} \) is smooth.

As an example, if \( Y \) is a projective manifold with torsion canonical bundle and \( \pi: X \to Y \) is the canonical covering, then all the maps \( \alpha_p \) are injective.

Probably the main interesting aspect of Theorem A is that it gives a concrete construction of a morphism whose existence is predicted by the general philosophy of extended deformation theory.

Roughly speaking, to every deformation problem over a field of characteristic 0, it is associated a differential graded Lie algebra \( L \), unique up to quasiisomorphism, and a formal pointed quasismooth dg-manifold \( M \) quasiisomorphic to \( L \) as \( L_\infty \)-algebra. The differential graded Lie algebra \( L \) governs the deformation problem via the solutions Maurer-Cartan modulo gauge action and the truncation in degree 0 of \( M \) is the classical moduli space (cf. [15], Section 2 of [3] and references therein).

Moreover, according to this general philosophy, every natural morphism between moduli spaces (e.g. the period map from deformations of a compact Kähler manifold to deformations of its Hodge decomposition) should extend to a morphism of their extended moduli spaces and therefore induces an \( L_\infty \)-morphism between the associated differential graded Lie algebras.

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Notation

For every holomorphic vector bundle \( E \) on a complex manifold we denote by \( \mathcal{A}^{p,q}(E) \) the sheaf of differential \((p,q)\)-forms with coefficients in \( E \).

For every vector space \( V \) and every linear functional \( \alpha: V \to \mathbb{C} \) we denote by \( \alpha \vdash: \bigwedge^k V \to \bigwedge^{k-1} V \) the contraction operator

\[
\alpha \vdash (v_1 \wedge \ldots \wedge v_k) = \sum_{i=1}^k (-1)^{i-1} \alpha(v_i) v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_k.
\]
We point out for later use that \( \alpha \) is a derivation of degree \(-1\) of the graded algebra \((\Lambda^* V, \wedge)\).

We denote by \( \Sigma_m \) the symmetric group of permutations of the set \( \{1, 2, \ldots, m\} \) and, for every \( 0 \leq p \leq m \) by \( S(p, m - p) \subset \Sigma_m \) the set of unshuffles of type \( (p, m - p) \). By definition \( \sigma \in S(p, m - p) \) if and only if \( \sigma_1 < \sigma_2 < \ldots < \sigma_p \) and \( \sigma_{p+1} < \sigma_{p+2} < \ldots < \sigma_m \).

1 \textbf{ } \( L_\infty \)-morphisms

Let \( V = \oplus V^i \) be a \( \mathbb{Z} \)-graded vector space, for every integer \( n \) we denote by \( V[n] = \oplus V^i \) the graded vector space where \( V[n]^i = V^{n+i} \). The space \( V[-1] \) is also called the suspension of \( V \) and \( V[1] \) the unsuspension.

The graded \( m \)-th symmetric power of \( V \) is denoted by \( \bigodot^m V \). If \( \sigma \in \Sigma_m \) and \( a_1, \ldots, a_m \in V \) are homogeneous elements, the Koszul sign \( \epsilon(V, \sigma; a_1, \ldots, a_m) = \pm 1 \) is defined by the rule

\[
\epsilon(V, \sigma; a_1, \ldots, a_m) = \sigma_1 \odot \cdots \odot a_{\sigma_m} = \epsilon(V, \sigma; a_1, \ldots, a_m) a_1 \odot \cdots \odot a_m \in \bigodot^m V.
\]

For simplicity of notation we write \( \epsilon(V, \sigma) \) when the elements \( a_1, \ldots, a_m \) are clear from the context. If \( a \in V \) is homogeneous we denote by \( \deg(a, V) \) its degree; we also write \( \deg(a, V) = \pi \) when there is no ambiguity about \( V \). Note that \( \deg(a, V[n]) = \deg(a, V) - n \).

We denote by \( C(V) \) the reduced graded symmetric coalgebra generated by \( V[1] \); more precisely it is the graded vector space

\[
C(V) = \overline{S}(V[1]) = \bigoplus_{m=1}^{\infty} \bigodot^m V[1] \end{align*}
\]

endowed with the coproduct \( \Delta: C(V) \to C(V) \otimes C(V) \), \( \Delta(a) = 0 \) for every \( a \in V[1] \) and

\[
\Delta(a_1 \odot \cdots \odot a_m) = \sum_{r=1}^{m-1} \sum_{\sigma \in S(r, m-r)} \epsilon(V[1], \sigma)(a_{\sigma_1} \odot \cdots \odot a_{\sigma_r}) \odot (a_{\sigma_{r+1}} \odot \cdots \odot a_{\sigma_m})
\]

for every \( a_1, \ldots, a_m \in V[1], m \geq 2 \).

Assume now that \( V \) has a structure of differential graded Lie algebra with differential \( d \) and bracket \([, ,]\), then the linear map

\[
Q: \bigodot^2 V[1] \to V[1], \quad Q(a \odot b) = (-1)^{\deg(a, V[1])}[a, b]
\]

has degree 1 and the map \( \delta: C(V) \to C(V) \) defined by

\[
\delta(a_1 \odot \cdots \odot a_m) = \sum_{\sigma \in S(1, m-1)} \epsilon(V[1], \sigma; a_1, \ldots, a_m) da_{\sigma_1} \odot a_{\sigma_2} \odot \cdots \odot a_{\sigma_m}
\]

\[
+ \sum_{\sigma \in S(2, m-2)} \epsilon(V[1], \sigma; a_1, \ldots, a_m) Q(a_{\sigma_1} \odot a_{\sigma_2}) \odot a_{\sigma_3} \odot \cdots \odot a_{\sigma_m}
\]

is a codifferential of degree 1 on the coalgebra \( C(V) \). The differential graded coalgebra \((C(V), \delta)\) is called the \( L_\infty \)-algebra associated to the DGLA \((V, d, [\ ,\ ,])\).

By definition, an \( L_\infty \)-morphism between two DGLA \( V, V' \) is a morphism of differential graded coalgebras \( \Theta: (C(V), \delta) \to (C(V'), \delta') \).

It is easy to check that if \( f: V \to V' \) is a morphism of differential graded Lie algebras then the linear map

\[
(C(V), \delta) \to (C(V'), \delta'), \quad a_1 \odot \cdots \odot a_m \to f(a_1) \odot \cdots \odot f(a_m)
\]
is an $L_{\infty}$-morphism. We refer to [1], [2], [3], [4] for the general theory of $L_{\infty}$-morphisms.

In this paper we are interested only in the particular and simple case when $V'$ has trivial differential and trivial bracket: under these assumption $\delta' = 0$ and there exists a bijection between the set of $L_{\infty}$-morphism $\Theta: (C(V), \delta) \to (C(V'), 0)$ and the set of morphisms of graded vector spaces $F: C(V) \to V'[1]$ such that $F \circ \delta = 0$. The bijection is given by the formulas

$$
F = p_1 \circ \Theta, \quad p_1: C(V') \to \bigodot^1 V'[1] = V'[1] \quad \text{the projection}
$$

$\Theta = \sum_{m=1}^{\infty} \frac{1}{m!} F^{\otimes m} \circ \Delta_{C(V)}^{m-1}: C(V) \to C(V')$

where $F^{\otimes m}$ is the composition of $F^{\otimes m}: \bigotimes^n C(V) \to \bigotimes^n (V'[1])$ with the projection onto the symmetric product $\bigotimes^n (V'[1]) \to \bigotimes^n (V'[1])$.

Let $F_1: V[1] \to V'[1]$ the composition of $F$ with the inclusion $V[1] \to C(V)$. Just to explain the statement of Theorem A we observe that the condition $F \circ \delta = 0$ implies $F_1 \circ d = 0$ and then $F_1$ induce a map in cohomology $\theta: H^*(V) \to H^*(V') = V'$.

## 2 Proof of Theorem A

Let $X$ be a complex manifold of dimension $n$; consider the graded vector space $L = \oplus L^p$, where $L^p = \Gamma(X, A^{p,0+1}(T_X))$, $-1 \leq p \leq n-1$, and two linear maps of degree +1, $d: L \to L$, $Q: \bigotimes L \to L$ defined in the following way: if $z_1, \ldots, z_n$ are local holomorphic coordinates, then

$$
d \left( \frac{\partial}{\partial z_i} \right) = (\partial \phi) \frac{\partial}{\partial z_i}, \quad \phi \in A^{0,*}.
$$

If $I, J$ are ordered subsets of $\{1, \ldots, n\}$, $a = f \partial^{z_I} \frac{\partial}{\partial z_i}$, $b = g \partial^{z_J} \frac{\partial}{\partial z_j}$, $f, g \in A^{0,0}$ then

$$
Q(a \circ b) = (-1)^{\overline{\pi}} \partial^{z_I} \wedge \partial^{z_J} \left( f \frac{\partial g}{\partial z_i} \frac{\partial}{\partial z_j} - g \frac{\partial f}{\partial z_j} \frac{\partial}{\partial z_i} \right), \quad \overline{\pi} = \deg(a, L).
$$

The equation (1), with $L$ in place of $V[1]$, gives a codifferential $\delta$ of degree 1 on $\overline{S}(L)$ and the differential graded coalgebra $(\overline{S}(L), \delta)$ is exactly the $L_{\infty}$-algebra associated to the Kodaira-Spencer DGLA $KS_X$.

If $\text{Der}^p(A^{*,*}, A^{*,*})$ denotes the vector space of $\mathbb{C}$-derivations of degree $p$ of the sheaf of graded algebras $(A^{*,*}, \wedge)$, where the degree of a $(p, q)$-form is $p + q$ (note that $\partial, \overline{\partial} \in \text{Der}^1(A^{*,*}, A^{*,*})$), then we can define a morphism of graded vector spaces

$$
L \longrightarrow \text{Der}^* (A^{*,*}, A^{*,*}) = \bigoplus_p \text{Der}^p (A^{*,*}, A^{*,*}), \quad a \to \widehat{a}
$$

given in local coordinates by

$$
\widehat{\phi} \frac{\partial}{\partial z_i} (\eta) = \phi \wedge \left( \frac{\partial}{\partial z_i} \mid \eta \right).
$$

If $\overline{\pi} = p$ then $\widehat{a}$ is a bihomogeneous derivation of bidegree $(-1, p+1)$: in particular $\widehat{a}(A^{0,*}) = 0$. 

4
Lemma 2.1. If [ , ] denotes the standard bracket on \( \text{Der}^*(A^{*+}, A^{*-}) \), then for every pair of homogeneous \( a, b \in L \) we have:

1. \( \widehat{da} = [\partial, \widehat{a}] = \partial a - (-1)^{|\partial|}\widehat{a}\partial \).
2. \( Q(a \odot b) = -[[\partial, \widehat{a}], \widehat{b}] = (-1)^{|\partial|}\widehat{a}\partial \widehat{b} + (-1)^{|\partial|+|\widehat{b}|}\partial \widehat{a} \pm \partial \widehat{b} \pm \widehat{a}\partial \).

Proof. By linearity we may assume \( a = f dz_i \frac{\partial}{\partial z_i}, b = gdz_j \frac{\partial}{\partial z_j} \), \( f, g \in \mathcal{A}^{0,0} \). Moreover all the four expressions are derivations vanishing on the subalgebra \( \mathcal{A}^{0,*} \) and therefore it is sufficient to check the above equalities when computed on the \( dz_i \)'s; since \( \partial dz_i = \partial \overline{dz}_i = \overline{a} \overline{b} dz_i = 0 \), the computation becomes straightforward and it is left to the reader.

Remark. The apparent asymmetry in the right hand side of Item 2 of the above lemma is easily understood: in fact \( [\partial, \widehat{a}] = 0 \) and then by Jacobi identity

\[
0 = [\partial, [\partial, \widehat{b}]] = [[\partial, \widehat{a}], \widehat{b}] - (-1)^{|\partial|}[\partial, \widehat{b}](\partial \widehat{a}).
\]

Assume now that \( X \) is compact Kähler, fix a Kähler metric on \( X \) and denote by: \( \mathcal{A}^{p,q} = \Gamma(X, A^{p,q}) \) the vector space of global \( (p,q) \)-forms, \( \overline{\partial}: \mathcal{A}^{p,q} \to \mathcal{A}^{p,q-1} \) the adjoint operator of \( \overline{\partial} \), \( \Delta_{\overline{\partial}} = \overline{\partial} \overline{\partial}^* + \overline{\partial}^* \overline{\partial} \) the \( \overline{\partial} \)-Laplacian, \( G_{\overline{\partial}} \) the associated Green operator, \( \mathcal{H} \subset \mathcal{A}^{*,*} \) the graded vector space of harmonic forms, \( i: \mathcal{H} \to \mathcal{A}^{*,*} \) the inclusion and \( h = Id - \Delta_{\overline{\partial}} G_{\overline{\partial}} = Id - G_{\overline{\partial}} \Delta_{\overline{\partial}}: A^{*,*} \to \mathcal{H} \) the harmonic projector.

We identify the graded vector space \( M_X \) with the space of endomorphisms of harmonic forms \( \text{Hom}_{KS}(\mathcal{H}, \mathcal{H}) \). We also denote by \( N = \text{Hom}_{KS}(A^{*+}, A^{*+}) \) the graded associative algebra of linear endomorphisms of the space of global differential forms on \( X \).

For notational simplicity we identify \( \text{Der}^*(A^{*+}, A^{*+}) \) with its image into \( N \).

Setting \( \tau = G_{\overline{\partial}} \overline{\partial} \partial \in N^0 \) we have by Kähler identities (cf. \([10], [21]\)):

\[
\partial \overline{\partial} = \partial h = h \tau = h \tau = \partial \overline{\partial} = \tau \overline{\partial} = 0.
\]

\[
[\partial, \overline{\partial}] = [\partial, G_{\overline{\partial}}] = [\overline{\partial}, G_{\overline{\partial}}] = 0, \quad [\overline{\partial}, \tau] = \overline{\partial} G_{\overline{\partial}} \overline{\partial} - G_{\overline{\partial}} \overline{\partial} \overline{\partial} = G_{\overline{\partial}} \Delta_{\overline{\partial}} \overline{\partial} = \partial.
\]

We introduce the morphism

\[
F_1: L \to M_X, \quad F_1(a) = h \widehat{a}^i.
\]

We note that \( F_1 \) is a morphism of complexes, in fact \( F_1(da) = h \widehat{a}^i = h(\overline{\partial} \widehat{a} \pm \overline{a} \partial) = 0 \). Next we define, for every \( m \geq 2 \), the morphisms of graded vector spaces

\[
f_m: \otimes^m L \to M_X, \quad F_m: \bigodot^m L \to M_X, \quad F = \sum_{m=1}^{\infty} F_m: \mathcal{S}(L) \to M_X,
\]

\[
f_m(a_1 \odot a_2 \odot \ldots \odot a_m) = h \widehat{a_1} \tau \widehat{a_2} \tau \widehat{a_3} \ldots \tau \widehat{a_m_i}.
\]

\[
F_m(a_1 \odot a_2 \odot \ldots \odot a_m) = \sum_{\sigma \in \Sigma_m} \epsilon(L, \sigma; a_1, \ldots, a_m) f_m(a_{\sigma_1} \odot \ldots \odot a_{\sigma_m}).
\]

Theorem 2.2. In the above notation \( F \circ \delta = 0 \) and therefore

\[
\Theta = \sum_{m=1}^{\infty} \frac{1}{m!} F_m \circ \Delta_{\overline{\partial}}^{m-1}: (C(KS_X), \delta) \to (C(M[-1]^1), 0)
\]

is an \( L_\infty \)-morphism with linear term \( F_1 \).
Proof We need to prove that for every \( m \geq 2 \) and \( a_1, \ldots, a_m \in L \) we have

\[
F_m \left( \sum_{\sigma \in S(1, m-1)} \epsilon(L, \sigma) da_{\sigma_1} \odot a_{\sigma_2} \odot \ldots \odot a_{\sigma_m} \right) =

= -F_{m-1} \left( \sum_{\sigma \in S(2, m-2)} \epsilon(L, \sigma) Q(a_{\sigma_1} \odot a_{\sigma_2}) \odot a_{\sigma_3} \odot \ldots \odot a_{\sigma_m} \right),
\]

where \( \epsilon(L, \sigma) = \epsilon(L, \sigma; a_1, \ldots, a_m) \).

It is convenient to introduce the auxiliary operators \( q: \bigotimes^2 L \to N[1], q(a \otimes b) = (-1)^{\overline{a} \overline{b}} \hat{a} \hat{b} \).

On the other hand

\[
g_m(a_1 \otimes \ldots \otimes a_m) = - \sum_{i=0}^{m-2} (-1)^{\overline{a} \overline{i} + \overline{i} \overline{a} + \overline{a} \overline{a}} h_{\overline{a} \overline{i}} \tau \ldots \hat{a}_i \tau a_{\overline{i}+1} \hat{\tau} \overline{a}_{\overline{i}+1} \tau \ldots \tau \overline{a}_m i.
\]

Since for every choice of operators \( \alpha = h, \tau \) and \( \beta = \sigma, i \) and every \( a, b \in L \) we have

\[
\alpha Q(a \otimes b) \beta = \alpha ((-1)^{\overline{a} \overline{b}} \hat{a} \hat{b} + (-1)^{\overline{a} \overline{b} + \overline{b} \overline{a}} \beta) = \alpha (q(a \otimes b) + (-1)^{\overline{a} \overline{b}} q(b \otimes a)) \beta,
\]

a straightforward computation about symmetrization and unshuffles gives

\[
\sum_{\sigma \in \Sigma_m} \epsilon(L, \sigma) g_m(a_{\sigma_1} \otimes \ldots \otimes a_{\sigma_m}) = -F_{m-1} \left( \sum_{\sigma \in S(2, m-2)} \epsilon(L, \sigma) Q(a_{\sigma_1} \odot a_{\sigma_2}) \odot a_{\sigma_3} \odot \ldots \odot a_{\sigma_m} \right).
\]

On the other hand

\[
f_m \left( \sum_{i=0}^{m-1} (-1)^{\overline{a} \overline{i} + \overline{i} \overline{a} + \overline{a} \overline{a}} a_1 \otimes \ldots \otimes a_i \otimes da_{i+1} \otimes \ldots \otimes a_m \right) =

= \sum_{i=0}^{m-1} (-1)^{\overline{a} \overline{i} + \overline{i} \overline{a} + \overline{a} \overline{a}} h_{\overline{a} \overline{i}} \ldots \hat{a}_i \tau(d \overline{a}_{i+1} - (-1)^{\overline{a} \overline{i} + \overline{i} \overline{a} + \overline{a} \overline{a}} \hat{d}) \tau \ldots \tau \overline{a}_m i

= \sum_{i=0}^{m-2} (-1)^{\overline{a} \overline{i} + \overline{i} \overline{a} + \overline{a} \overline{a}} h_{\overline{a} \overline{i}} \ldots \hat{a}_i \tau(-(-1)^{\overline{a} \overline{i} + \overline{i} \overline{a} + \overline{a} \overline{a}} \hat{d} \overline{a}_{i+1} \tau \overline{a}_{i+1} + (-1)^{\overline{a} \overline{i} + \overline{i} \overline{a} + \overline{a} \overline{a}} \hat{d} \overline{a}_{i+1} \tau \overline{a}_{i+1} \tau \overline{a}_m i

= g_m(a_1 \otimes \ldots \otimes a_m).
\]

Taking the symmetrization of this equality we get

\[
\sum_{\sigma \in \Sigma_m} \epsilon(L, \sigma) g_m(a_{\sigma_1} \otimes \ldots \otimes a_{\sigma_m}) = F_m \left( \sum_{\sigma \in S(1, m-1)} \epsilon(L, \sigma) da_{\sigma_1} \odot a_{\sigma_2} \odot \ldots \odot a_{\sigma_m} \right).
\]

Since it is clear that \( F_1 \) is a morphism of complexes inducing the morphism \( \theta \) in cohomology, Theorem \( \Box \) is proved.

Remark. If \( X \) is a Calabi-Yau manifold with holomorphic volume form \( \Omega \), then the composition of \( F \) with the evaluation at \( \Omega \) induces an \( L_\infty \)-morphism \( C(KS_X) \to C(\mathcal{H}[n-1]) \). For every \( m \geq 2 \), \( \text{ev}_\Omega \circ F_m: \bigotimes^m L \to \mathcal{H}[n] \) vanishes on \( \bigotimes^m \{ a \in L \mid \partial(a + \Omega) = 0 \} \).

The following corollary gives a formality criterion:

**Corollary 2.3.** In the notation of introduction, if \( \theta: H^*(X, T_X) \to M[-1]^X \) is injective, then \( KS_X \) is \( L_\infty \)-quasiiisomorphic to an abelian differential graded Lie algebra.
2.11 of \[19\].

Therefore also \(t_r\) relative to the small extension primary obstruction map (cf. \[19, 2.15\]) of

\[
\text{Art}
\]

\[
\text{Def}
\]

\[
\text{Set}
\]

Let \(\text{Art}\) be the category of local Artinian \(C\)-algebras \((A, m_A)\) with residue field \(A/m_A = \mathbb{C}\). Following \[19\], by a functor of Artin rings we intend a covariant functor \(\mathcal{F}: \text{Art} \rightarrow \text{Set}\) such that \(\mathcal{F}(\mathbb{C}) = \{0\}\) is a set of cardinality 1.

With the term Schlessinger’s condition we mean one of the four conditions \((H_1), \ldots, (H_4)\) described in Theorem 2.11 of \[19\].

**Lemma 3.1.** Let \(\alpha: \mathcal{F} \rightarrow \mathcal{G}\) be a natural transformation of functors of Artin rings; if \(\mathcal{F}\) satisfies Schlessinger’s conditions \((H_1)\) and \((H_2)\), \(\mathcal{G}\) is prorepresentable and \(\alpha: t_{\mathcal{F}} \rightarrow t_{\mathcal{G}}\) is injective, then also \(\mathcal{F}\) is prorepresentable.

**Proof** Since \(\mathcal{G}\) is prorepresentable its tangent space \(t_{\mathcal{G}}\) is finite dimensional and then the same holds for \(t_{\mathcal{F}}\). Moreover for every small extension \(0 \rightarrow J \rightarrow A \rightarrow B \rightarrow 0\) there exists a natural transitive free action (cf. \[19\]) of \(t_{\mathcal{G}} \otimes J\) on the nonempty fibres of \(\mathcal{G}(A) \rightarrow \mathcal{G}(B)\).

Therefore also \(t_{\mathcal{F}} \otimes J\) acts without fixed points on \(\mathcal{F}(A)\) and then, according to Theorem 2.11 of \[19\], \(\mathcal{F}\) is prorepresentable. \(\square\)

For every differential graded complex Lie algebra \(K = \oplus K^i\), we denote respectively by \(\text{MC}_K, \text{Def}_K: \text{Art} \rightarrow \text{Set}\) the associated Maurer-Cartan and deformation functors (cf. \[2\], \[3\], \[10\]):

\[
\text{MC}_K(A) = \left\{ a \in K^1 \otimes m_A \mid da + \frac{1}{2}[a, a] = 0 \right\}, \quad \text{Def}_K(A) = \frac{\text{MC}_K(A)}{\exp(K^0 \otimes m_A)}.
\]

The functors \(\text{MC}_K\) and \(\text{Def}_K\) are functors of Artin rings satisfying the Schlessinger’s conditions \((H_1), (H_2)\) (cf. \[10\], \[11\]), the projection \(\text{MC}_K \rightarrow \text{Def}_K\) is smooth and the tangent space \(t_{\text{Def}_K}\) of \(\text{Def}_K\) is naturally isomorphic to \(H^1(K)\).

**Example 3.2.**
1. If \(K\) has trivial bracket and trivial differential then the gauge action is trivial and therefore, for every \((A, m_A) \in \text{Art}\), \(\text{Def}_K(A) = \text{MC}_K(A) = K^1 \otimes m_A\); in particular if \(K^1\) is finite dimensional then \(\text{Def}_K\) is prorepresented by a smooth germ.

2. If \(K = KS_X\) is the Kodaira-Spencer DGLA of a compact complex manifold \(X\) then \(\text{Def}_K\) is isomorphic to the functor \(\text{Def}_X\) of infinitesimal deformations of \(X\) (cf. \[10\]).

The functor \(\text{Def}_K\) has a natural obstruction theory with obstruction space \(H^2(K)\): this means that for every small extension \(\epsilon: 0 \rightarrow J \rightarrow A \xrightarrow{p} B \rightarrow 0\) in the category \(\text{Art}\) it is given an “obstruction map” \(\text{ob}_\epsilon: \text{Def}_K(K^2) \otimes J\) such that an element \(b \in \text{Def}_K(B)\) lifts to \(\text{Def}_K(A)\) if and only if \(\text{ob}_\epsilon(b) = 0\). Moreover all the obstruction maps behave functorially with respect to morphisms of small extensions (cf. e.g. \[10\], \[11\]).

By definition the primary obstruction map is the obstruction map \(q_2 = \text{ob}_\epsilon: H^1(K) \rightarrow H^2(K)\) relative to the small extension

\[
\epsilon: 0 \rightarrow \mathbb{C} \xrightarrow{\epsilon^2} \mathbb{C}[t] \left(\frac{t^3}{(t^3)}\right) \xrightarrow{\mathbb{C}[t]} \mathbb{C}[t] \left(\frac{t^2}{(t^2)}\right) \rightarrow 0.
\]
Concretely, if $b \in MC_K(B)$ and $a \in K^1 \otimes m_A$ is a lifting of $b$, then by the Jacobi identity $h = da + [a, a]/2 \in K^2 \otimes J$ is a cocycle and its cohomology class $ob_e(b) = [h] \in H^2(K) \otimes J$ does not depend from the choice of $a$. It is easy to prove that $ob_e(b) = 0$ if and only if $b$ can be lifted to $MC_K(A)$.

The map $ob_e$ is invariant under the gauge action (this follows from a general result [3, 7.5] but it is also easy to prove directly) and then factors to a map $ob_e : Def_K(B) \to H^2(K) \otimes J$. Since the projection $MC_K \to Def_K$ is smooth, we have that the class of $b$ lifts to $Def_K(A)$ if and only if $ob_e(b) = 0$.

The obstruction space $O_K \subset H^2(K)$ is by definition the vector space generated by the images of the maps $(Id \otimes f) \circ ob_e$, where $f \in \text{Hom}_\mathbb{C}(J, \mathbb{C})$ and $e$ ranges over all small extension in $\text{Art}$.

Remark. If the DGLA $K$ is not formal, it may happen that the primary obstruction map vanishes but $O_K \neq 0$. If $O_K' \subset O_K$ denotes the subspace generated by the obstructions coming from all the curvilinear small extensions

$0 \to C[t] \xrightarrow{t^n} C[t][t^n] \to C[t]/(t^n) \to 0$

then, by the (abstract) $T^1$-lifting theorem [3], $Def_K$ is smooth if and only if $O_K' = 0$ but in general $O_K' \neq O_K$ (cf. [3, 5.7]).

Given two differential graded Lie algebras $K, M$, every $L_\infty$-morphism $\mu : C(K) \to C(M)$ induces a natural transformation $\widetilde{\mu} : Def_K \to Def_M$ (see e.g. [11], [15]). Writing $\mu = \sum_{i \leq j} \mu_{i,j}, \mu_{i,j} : \Omega^i K[1] \to \Omega^j M[1]$, the morphism $\mu_{i,j}$ is a morphism of complexes, $H^1(\mu_{i,j}) : H^1(K) \to H^1(M)$ equals the restriction of $\widetilde{\mu}$ on tangent spaces and $H^2(\mu_{i,j}) : H^2(K) \to H^2(M)$ commutes with $\widetilde{\mu}$ and all the obstruction maps.

**Proposition 3.3.** Let $K$ be a differential graded Lie algebra, $M \oplus M^1$ be a graded vector space considered as a differential graded Lie algebra with trivial bracket and differential and let $\mu = \sum_{i \leq j} \mu_{i,j} : C(K) \to C(M)$ be an $L_\infty$-morphism. Then:

1. If $M^1$ is finite dimensional and $H^1(\mu_{1,1})$ is injective then $Def_K$ is prorepresentable.

2. The obstruction space $O_K$ is contained in the kernel of $H^2(\mu_{1,1}) : H^2(K) \to M^2$.

**Proof** The first part follows immediately from Lemma [3.3]. The second part follows from the fact that all the obstruction maps of the functor $Def_M$ are trivial.

If $X$ is a compact Kähler manifold we have, in the notation of the Introduction and Section 2, for every $A \in \text{Art}$,

$$Def_X(A) = \text{Def}_{KS_X}(A) = \left\{ a \in L^0 \otimes m_A \mid da + \frac{1}{2} Q(a \otimes a) = 0 \right\} / \exp(L^{-1} \otimes m_A),$$

$$\text{Def}_{M[-1]|X}(A) = M^1_X \otimes m_A$$

and the natural transformation $\widetilde{\Theta} : \text{Def}_{KS_X} \to \text{Def}_{M[-1]|X}$ associated to the $L_\infty$-morphism $\Theta$ of Theorem 2.2 is induced by

$$\widetilde{\Theta}(a) = \sum_{m=1}^{\infty} \frac{1}{m!} F_m(a^{\otimes m}) = F(\exp(a) - 1), \quad a \in L^0 \otimes m_A.$$
Corollary 3.4. Let $X$ be a compact Kähler manifold and denote by $O$ the kernel of
$$\theta_2: H^2(X, T_X) \to \bigoplus_{r,s} \operatorname{Hom}_C^r(H^r(\Omega^s_X), H^{r+2}(\Omega^{s-1}_X)).$$
Then for every small extension $e: 0 \to J \to A \to B \to 0$ and every $b \in \operatorname{Def}_X(B)$, the obstruction $ob_e(b)$ belongs to $O \otimes J$.

Proof of Corollary \[3\] We first recall that, if $Y \to B$ is the Kuranishi family of a compact complex manifold $Y$ and $O \subset H^2(Y, T_Y)$ is the subspace generated by all the obstruction to the deformations of $Y$, then the singularity $B$ is analytically isomorphic to $q^{-1}(0)$, where $q: (H^1(Y, T_Y), 0) \to (O, 0)$ is the Kuranishi map.

The pull-back of forms and vector fields give a morphism of differential graded Lie algebras $\pi^*: KS_Y \to KS_X$. The composition of $\pi^*$ with $\Theta$ gives an $L_\infty$-morphism from $KS_Y$ to $M[-1, X]$. It is now sufficient to apply Proposition \[3\].

Example 3.5. Let $Z$ be a projective Calabi-Yau manifold of dimension $n \geq 3$ with $H^2(O_Z) = 0$ and let $\pi: Y \to Z$ be a smooth Galois double cover. Denoting by $D \subset Z$ the branching divisor, $R \subset Y$ the ramification divisor and $\pi_* O_Y = O_Z \oplus O_Z(-L)$ the eigensheaves decomposition we have (cf. \[3\], \[4\]) $O_Y(R) = K_Y = \pi^* O_Z(L)$, $O_Z(D) = O_Z(2L)$, an exact sequence of sheaves over $Y$
$$0 \to T_Y \to \pi^* T_Z \to O_R(2R) \to 0$$
and, for every $i$, $H^i(\pi^* T_Z) = H^i(T_Z) \oplus H^i(T_Z(-L))$, $H^i(O_R(2R)) = H^i(O_D(D))$.

If $L$ is sufficiently ample then $H^1(O_D(D)) = H^2(O_Z) = 0$, $H^2(T_Z(-L)) = 0$ and then $H^2(T_Y)$ injects into $H^2(T_Z)$. Therefore the cup product with the pull-back of the holomorphic volume form of $Z$ is nondegenerate and then $\theta_2: H^2(T_Y) \to M^1$ is injective. Applying Corollary \[3\] (with $X = Y$) we get that $Y$ is unobstructed.

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