Branching Brownian motion in an expanding ball and application to the mild obstacle problem

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Abstract

We first consider a $d$-dimensional branching Brownian motion (BBM) evolving in an expanding ball, where the particles are killed at the boundary of the ball, and the expansion is subdiffusive in time. We study the large-time asymptotic behavior of the mass inside the ball, and obtain a large-deviation (LD) result as time tends to infinity on the probability that the mass is atypically small. Then, we consider the problem of BBM among mild Poissonian obstacles, where a random ‘trap field’ in $\mathbb{R}^d$ is created via a Poisson point process. The trap field consists of balls of fixed radius centered at the atoms of the Poisson point process. The mild obstacle rule is that when a particle is inside a trap, it branches at a lower rate, which is allowed to be zero, whereas when outside the trap field it branches at the normal rate. As an application of our LD result on the mass of BBM inside expanding balls, we prove upper bounds on the LD probabilities for the mass of BBM among mild obstacles, which we then use along with the Borel-Cantelli lemma to prove the corresponding strong law of large numbers. Our results are quenched, that is, they hold in almost every environment with respect to the Poisson point process.

Keywords: Branching Brownian motion, killing boundary, large deviations, strong law of large numbers, Poissonian traps, mild obstacles, random environment

2010 MSC: 60J80, 60K37, 60F15, 60F10

1. Introduction

In this work, we first study a branching Brownian motion (BBM) in an expanding ball of fixed center, where the radius of the ball is increasing subdiffusively in time. We suppose that the boundary of the ball is killing in the sense that once a particle of the BBM hits the boundary, it is instantly killed. On this model, in Theorem 1 we obtain a large-deviation result, giving the large-time asymptotic behavior of the probability that the number of particles, i.e., the mass, of the BBM is atypically small in the expanding ball.

Then, we consider the model of BBM among mild Poissonian obstacles. We study the growth of mass of a BBM evolving in a random environment in $\mathbb{R}^d$, which is composed of randomly located spherical traps of fixed radius with centers given by a Poisson point process (PPP). The mild obstacle rule is that when a particle is inside the traps, it branches at a lower rate, which is allowed to be zero, than usual, that is, when it is not in a trap. The mild obstacle problem for BBM was proposed by Engländer in [6], and on a set of full measure with respect to the PPP, a kind
of weak law of large numbers (WLLN) was obtained (see [6, Thm. 1]) for the mass of the process as well as a result on its spatial spread (see [6, Thm. 2]). In Theorem [2] by estimating successive large-deviation probabilities, we improve the WLLN in [6] to the strong law of large numbers (SLLN). We also include the possibility of no branching inside the traps, which was not covered in [6]. An essential ingredient in the proof of Theorem 2 turns out be Theorem 1, that is, the lower tail asymptotics for the mass of BBM in expanding balls. In the mild obstacle problem, a suitable time-dependent clearing (see Definition [1]) in the random environment in \( \mathbb{R}^d \) serves as the expanding ball of Theorem [1].

1.1. Formulation of the problems

We now describe the two sources of randomness in the models, and formulate the problems in a precise way.

1. Branching Brownian motion in an expanding ball: Let \( Z = (Z_t)_{t \geq 0} \) be a strictly dyadic \( d \)-dimensional BBM with branching rate \( \beta > 0 \), where \( t \) represents time. Strictly dyadic means that every time a particle branches it gives precisely two offspring. The process can be described as follows. It starts with a single particle, which performs a Brownian motion (BM) in \( \mathbb{R}^d \) for a random lifetime, at the end of which it dies and simultaneously gives birth to two offspring. Similarly, starting from the position where their parent dies, each offspring particle repeats the same procedure as their parent independently of others and the parent, and the process evolves through time in this way. All particle lifetimes are exponentially distributed with constant parameter \( \beta > 0 \). For each \( t \geq 0 \), \( Z_t \) can be viewed as a finite discrete measure on \( \mathbb{R}^d \), which is supported at the positions of the particles at time \( t \). For \( t \geq 0 \), we use \( |Z_t| \) to denote the total mass of \( Z \) at time \( t \), and occasionally use \( N_t := |Z_t| \). Also, for a Borel set \( A \subseteq \mathbb{R}^d \) and \( t \geq 0 \), we write \( Z_t(A) \) to denote the mass of \( Z \) that fall inside \( A \) at time \( t \).

We also define a BBM with killing at a boundary. For a Borel set \( A \subseteq \mathbb{R}^d \), denote by \( \partial A \) the boundary of \( A \). Consider a family of Borel sets \( B = (B_t)_{t \geq 0} \), which we may view as a single time-dependent Borel set. Let \( Z^B = (Z^B_t)_{t \geq 0} \) be the BBM with killing at \( \partial B \), that is, a standard BBM with particles killed instantly upon hitting the boundary of \( B \). The process \( Z^B \) can be obtained from \( Z \) as follows. For each \( t \geq 0 \), start with \( Z_t \), and delete from it any particle whose ancestral line up to \( t \) has exited \( B_t \) to obtain \( Z^B_t \). This means, \( Z^B_t \) consists of particles of \( Z_t \) whose ancestral lines up to \( t \) have been confined to \( B_t \) over the time period \( [0, t] \), and therefore it can be viewed as a finite discrete measure in \( B_t \).

We denote by \( \Omega \) the sample space for the BBM, and use \( P_x \) and \( E_x \), respectively, to denote the law and corresponding expectation of a BBM starting with a single particle at \( x \in \mathbb{R}^d \). By an abuse of notation, we use \( P_x \) and \( E_x \) also for the BBM with killing at a boundary. For simplicity, we set \( P = P_0 \). Also, we sometimes use

\[ n_t := |Z^B_t| \]

to denote the mass at time \( t \) of a BBM with killing at \( \partial B \).

Consider a radius function \( r : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \lim_{t \to \infty} r(t) = \infty \), which is subdiffusively increasing, that is, \( r(t) = o(\sqrt{t}) \) as \( t \to \infty \). For \( t > 0 \), let \( B_t := B(0, r(t)) \), and \( p_t \) be the probability of confinement to \( B_t \) of a standard BM (starting from the origin) over \( [0, t] \). In the first part of the current work, for a suitably decreasing function \( \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( \lim_{t \to \infty} \gamma(t) = 0 \), we find the asymptotic behavior as \( t \to \infty \) of the LD probability

\[ P \left( |Z^B_t| < \gamma_t p_t e^{rt} \right), \]
where we have set $\gamma_t = \gamma(t)$ for convenience. It is easy to show that $E[n_t] = pt e^{\beta t}$; therefore, since $\lim_{t \to \infty} \gamma(t) = 0$, for large $t$, $\gamma_t pt e^{\beta t}$ is atypically small for the mass of a BBM with killing at $\partial B$.

2. Trap field and mild obstacle problem for BBM: The setting of random obstacles in $\mathbb{R}^d$ is formed as follows. Let $\Pi$ be a Poisson point process (PPP) in $\mathbb{R}^d$ with constant intensity $\nu > 0$, and $(\Omega, \mathbb{P})$ be the corresponding probability space with expectation $\mathbb{E}$. By a *trap* associated to a point $x \in \mathbb{R}^d$, we mean a closed ball of fixed radius $a > 0$ centered at $x$, and by a *trap field*, we mean the random set

$$K = K(\omega) := \bigcup_{x \in \text{supp}(\Pi)} \tilde{B}(x, a),$$

where $\tilde{B}(x, a)$ denotes the closed ball of radius $a$ centered at $x \in \mathbb{R}^d$.

In the second part of the current work, the BBM is assumed to live in $\mathbb{R}^d$, to which $K$ is attached. For $\omega \in \Omega$, we refer to $\mathbb{R}^d$ with $K(\omega)$ attached simply as the random environment $\omega$, and use $P^\omega$ to denote the conditional law of the BBM in the random environment $\omega$. The mild obstacle problem for BBM has the following rule: when a particle of BBM is in $K^c$, it branches at rate $\beta$, whereas when in $K$, it branches at a lower rate $\bar{\beta}$ with $0 \leq \bar{\beta} < \beta$. That is, under the law $P^\omega$, the BBM has a spatially dependent branching rate

$$\beta = \beta(x, \omega) := \beta 1_{K^c}(x) + \bar{\beta} 1_{K}(x).$$

We emphasize that here we allow the possibility of complete suppression of branching in $K$, that is, $\bar{\beta} = 0$, which was not considered in [6]. Our focus is on the total mass of BBM among mild obstacles. We first find upper bounds that are valid for large $t$ on the LD probabilities that the mass is atypically small and atypically large. Then, via Borel-Cantelli arguments, we obtain the corresponding SLLN. The result is valid in almost every environment $\omega$; hence, it is called a *quenched* SLLN.

1.2. History

The study of branching diffusions in restricted domains with absorbing boundaries goes back to Sevast’yanov [18], who studied the survival of such systems in bounded domains in $\mathbb{R}^d$. In [11], Kesten studied a BBM with negative drift in one dimension starting with a single particle at position $x > 0$ in the presence of absorption at the origin. He obtained a survival criterion depending on the drift of the process, and an asymptotic result on the number of particles in a given interval. Later, various further results were obtained on the one-dimensional model with absorption at a one-sided barrier. In [14], considering a BBM starting with a single particle at the origin and with a strong enough negative drift so as to make extinction almost sure, Neveu studied the process $(Z_x)_{x \geq 0}$ formed by the total mass that is frozen upon exiting $((-\infty, -x), x \geq 0)$. Berestycki et al. followed up on Kesten’s model of BBM with absorption at the origin, and in [1] and [2] studied, respectively, the survival probability of the BBM near the critical drift as a function of $x > 0$, and the genealogy of the process. In [8], on the same model, Harris et al. studied the one-sided FKPP travelling-wave equation, and obtained several results on the asymptotic speed of the rightmost particle, the almost sure exponential growth rate of particles having different speeds, and the asymptotic probability of presence of the BBM in the subcritical speed area. Then, in [13], Maillard improved on Neveu’s work in the case where the process goes extinct almost surely, and obtained precise asymptotics on the number of absorbed particles at the linear one-sided barrier. More recently in [9], Harris et al. studied a BBM with drift in a fixed-size interval, that is, a two-sided barriered version of Kesten’s model, and obtained a survival criterion involving a critical width for the interval, and also the asymptotics of the near-critical survival probability. The one-dimensional model involving a BBM with drift and a fixed barrier is equivalent to the model involving a BBM with no drift and a linearly
moving barrier. The first part of the current work gives a large-deviation result in the downward
direction on the population size of a BBM in a subdiffusively expanding (time-dependent) ball in
d dimensions with killing boundary.

The second part of the current work is about a BBM among mild obstacles in $\mathbb{R}^d$. The study
of branching diffusions among random obstacles in $\mathbb{R}^d$ goes back to Engl"ander [3], who studied a
BBM among hard Poissonian obstacles in the case of a uniform field, and obtained the asymptotic
probability of survival for the system as $t \to \infty$ in $d \geq 2$. In the hard obstacle model, the process
is killed instantly when a particle hits the traps. In search for an extension to $d = 1$, Engl"ander
and den Hollander [4] then studied the more interesting case where the trap intensity was radially
decaying in a particular way in $d \geq 2$, and yielding uniform intensity in $d = 1$, so as to give rise
to a phase transition in the survival probability and the optimal survival strategy of the system.

In both [3] and [4], the branching rule was taken as strictly dyadic, and the main result was the
exponential asymptotic decay rate of the annealed survival probability as $t \to \infty$; in addition, in [4], several optimal survival strategies were proved. For a BBM with a generic branching law,
denote by $p_0$ the probability that a particle gives no offspring at the end of its lifetime. In [15], the
work in [3] was extended to a BBM with a generic branching law, including the case where $p_0 > 0$.
Likewise in [16], the work in [4] on the radially decaying trap field was extended to a BBM with
a generic branching law, with the possibility of $p_0 > 0$. Recently in [17], conditioning the BBM
on the event of survival from hard Poissonian obstacles, "Oz and Engl"ander proved several optimal
survival strategies in the annealed environment, with particular emphasis on the population size.
We refer the reader to [5] for a survey, and to [7] for a detailed treatment on the topic of BBM
among random obstacles, and to [12] for a related problem where a critical BBM that is killed at a
small rate inside the traps (such traps are called soft obstacles) is studied. We repeat that the mild
obstacle model studied in the current work was proposed by Engl"ander in [6], and it is the partial
aim of this work to improve the WLLN therein for the population size of the BBM to the SLLN.

Notation: We use $c, c_0, c_1, \ldots$ as generic positive constants, whose values may change from line
to line. If we wish to emphasize the dependence of $c$ on a parameter $p$, then we write $c(p)$. We
denote by $f : A \to B$ a function $f$ from a set $A$ to a set $B$. For two functions $f, g : \mathbb{R}^d_+ \to \mathbb{R}^d_+$,
we write $g(t) = o(f(t))$ if $g(t)/f(t) \to 0$ as $t \to \infty$. Also, for a generic function $g : \mathbb{R}^d_+ \to \mathbb{R}^d_+$, we
occasionally set $g_t = g(t)$ for notational convenience. We use $\mathbb{N}$ as the set of positive integers. For
$x \in \mathbb{R}^d$, we use $|x|$ to denote its Euclidean norm; also, for a generic finite set $S$, we use $|S|$ to denote
its cardinality. We use $B(x, a)$ to denote the open ball of radius $a > 0$ centered at $x \in \mathbb{R}^d$. For an
event $A$, we use $A^c$ to denote its complement, and $1_A$ its indicator function.
We denote by $X = (X(t))_{t \geq 0}$ a generic standard Brownian motion (BM) in $d$-dimensions, and
use $P_x$ and $E_x$, respectively, as the law of $X$ started at position $x \in \mathbb{R}^d$, and the corresponding
expectation.

Outline: The rest of the paper is organized as follows. In Section 2 we present our main results.
Section 3 contains several introductory results, which serve as preparation for the proofs of Theo-
rem 1 and Theorem 2. In Section 4 and Section 5, we present, respectively, the proofs of Theorem 1
and Theorem 2.

2. Main Results

The first main result gives the large-time asymptotic behavior of the probability that the mass
of BBM inside a subdiffusively expanding ball $B = (B_t)_{t \geq 0}$ with killing at the boundary of the ball,
is atypically small. A subdiffusive expansion means that the ball is expanding slower than the rate
at which a typical BM moves away from the origin, which means for large $t$ it would be a ‘rare
event for the BM to be confined in $B_t$. For a generic standard Brownian motion $X = (X(t))_{t \geq 0}$ and a Borel set $A \subseteq \mathbb{R}^d$, define $\sigma_A = \inf\{s \geq 0 : X(s) \notin A\}$ to be the first exit time of $X$ out of $A$.

**Theorem 1** (LD for mass of BBM in an expanding ball). Let $r : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $r(t) \to \infty$ as $t \to \infty$ and $r(t) = o(\sqrt{t})$. Also, let $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $\gamma(t) = e^{-\kappa r(t)}$, where $\kappa > 0$ is a constant. For $t > 0$, set $B_t = B(0, r(t))$, $p_t = P_0(\sigma_{B_t} \geq t)$, and $n_t = |Z_t^B|$.

If $0 < \kappa \leq \sqrt{\beta}/2$, then
\[
\lim_{t \to \infty} \frac{1}{r(t)} \log P(n_t < \gamma_t p_t e^{\beta t}) = -\kappa, \tag{1}
\]
and if $\kappa > \sqrt{\beta}/2$, then
\[
-(\kappa \wedge \sqrt{2\beta}) \leq \liminf_{t \to \infty} \frac{1}{r(t)} \log P(n_t < \gamma_t p_t e^{\beta t}) \leq -\sqrt{\beta}/2, \tag{2}
\]
where
\[
-(\kappa \wedge 2\beta) \leq \liminf_{t \to \infty} \frac{1}{r(t)} \log P(n_t < \gamma_t p_t e^{\beta t}) \leq -(\kappa \wedge \sqrt{2\beta})
\]
and
\[
\limsup_{t \to \infty} \frac{1}{r(t)} \log P(n_t < \gamma_t p_t e^{\beta t}) \leq -\beta/2,
\]
where we use $a \wedge b$ to denote the minimum of the numbers $a$ and $b$.

**Remark 1.** The reason we call $P(n_t < \gamma_t p_t e^{\beta t})$ with $\gamma_t = e^{-\kappa r(t)}$ a large-deviation (LD) probability is that with this choice of $\gamma_t$, both $P(n_t < \gamma_t p_t e^{\beta t})$ and $P(n_t = 0)$ decay as $e^{-c r(t)}$ to the leading order for large $t$, where the values of the constant $c > 0$ may differ.

Indeed, start with
\[
P(n_t < \gamma_t p_t e^{\beta t}) \geq P(n_t = 0).
\]
One way to realize $\{n_t = 0\}$ is to completely suppress the branching and move the initial particle out of $B_t := B(0, r(t))$ over $[0, kr(t)]$, where $k > 0$ is a constant. The probability of realizing this joint strategy is
\[
\exp \left[-\beta kr(t) - \frac{r(t)}{2k} (1 + o(1))\right], \tag{4}
\]
where the second term in the exponent comes from Proposition A along with Brownian scaling.

To minimize the absolute value of the exponent in (4), set $\beta kr(t) = r(t)/(2k)$, which yields $k = 1/(\sqrt{2\beta})$. With this choice of $k$, we arrive at
\[
P(n_t = 0) \geq \exp \left[-\sqrt{2\beta} r(t)(1 + o(1))\right]. \tag{5}
\]

**Remark 2.** In Theorem 1, we have only considered $\gamma$ with $\gamma_t = e^{-kr(t)}$ for the following reason. It can be shown that if $\gamma_t \to 0$ as $t \to \infty$, then for all large $t$, $P(n_t < \gamma_t p_t e^{\beta t}) \geq \delta \gamma_t$ for some $0 < \delta < 1$ (see the proof of the lower bound of Theorem 1 in Section 4). Hence, when $\gamma_t$ is decaying sufficiently slowly so as to satisfy $(\log \gamma_t)/r(t) \to 0$ as $t \to \infty$, due to (1) and (3), the event $\{n_t < \gamma_t p_t e^{\beta t}\}$ would not be an LD event.

Our second main result is a quenched SLLN for the total mass of BBM among mild Poissonian obstacles in $\mathbb{R}^d$. Recall that $(\Omega, \mathbb{P})$ is the probability space for the PPP that creates the random environment, and $N_t := |Z_t^d|$. We now introduce further notation. Let $\lambda_{d, r}$ be the principal Dirichlet eigenvalue of $-\frac{1}{2} \Delta$ on $B(0, r)$ in $d$ dimensions. Write $\lambda_d := \lambda_{d, 1}$, and let $\omega_d$ be the volume of the $d$-dimensional unit ball. For $d \geq 1$ and $\nu > 0$, define the constant
\[
c(d, \nu) := \lambda_d \left(\frac{d}{\nu \omega_d}\right)^{-2/d}. \tag{6}
\]
Theorem 2 (Quenched SLLN for BBM among mild obstacles). On a set of full $\mathbb{P}$-measure,

$$
\lim_{t \to \infty} (\log t)^{2/d} \left( \frac{\log N_t}{t} - \beta \right) = -c(d,v) \quad \text{P$\omega$-a.s.}
$$

(7)

Remark 3. Note that the branching rate $\bar{\beta}$ in the trap field $K$ does not appear in the formula.

Remark 4. It was shown in [6] that on a set of full $\mathbb{P}$-measure,

$$
E_{\omega}[N_t] = \exp \left[ \beta t - c(d,v) \frac{t}{(\log t)^{2/d}} (1 + o(1)) \right].
$$

(8)

Theorem 2 is called a SLLN for BBM among mild obstacles, because it says that with $\mathbb{P}_{\omega}$-probability one, the total mass of BBM among mild obstacles grows as its expectation as $t \to \infty$. The reason why it is called a quenched SLLN is that it holds on a set of full $\mathbb{P}$-measure.

3. Preparations

In this section, we present introductory results that serve as preparations for the proofs of the main theorems. The first two results are standard in the theory of Brownian motion. Proposition A is on the large-time asymptotic probability of atypically large Brownian displacements. For a proof, see for example [16, Lemma 5].

**Proposition A** (Linear Brownian displacements). For $k > 0$,

$$
P_0 \left( \sup_{0 \leq s \leq t} |X(s)| > kt \right) = \exp \left[ -\frac{k^2 t}{2} (1 + o(1)) \right].
$$

The following is a standard result on the large-time Brownian confinement in balls, and for instance can be deduced from [7, Prop. 1.6], along with the scaling $\lambda_{d,r} = \lambda_d / r^2$. Recall that $\sigma_A = \inf\{s \geq 0 : X(s) \not\in A\}$ denotes the first exit time of $X$ out of $A$.

**Proposition B** (Brownian confinement in small balls). For $t > 0$, let $B_t = B(0, r(t))$, where $r : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $r(t) \to \infty$ as $t \to \infty$ and $r(t) = o(\sqrt{t})$. Then, as $t \to \infty$,

$$
P_0 (\sigma_{B_t} \geq t) = \exp \left[ -\frac{\lambda_d t}{r^2(t)} (1 + o(1)) \right].
$$

The following result is well-known in the theory of branching processes. For a proof, see for example [10, Sect. 8.11].

**Proposition C** (Distribution of mass in branching systems). For a strictly dyadic continuous-time branching process $N = (N_t)_{t \geq 0}$ with constant branching rate $\beta > 0$, the probability distribution at time $t$ is given by

$$
P(N_t = k) = e^{-\beta t} (1 - e^{-\beta t})^{k-1}, \quad k \geq 1,
$$

from which it follows that

$$
P(N_t > k) = (1 - e^{-\beta t})^k.
$$

(9)

We now focus on the model of Poissonian traps in $\mathbb{R}^d$. Recall that a random environment in $\mathbb{R}^d$ is created via a PPP, called $\Pi$, with

$$
K := \bigcup_{x_i \in \text{supp}(\Pi)} \bar{B}(x_i, a)
$$
being the trap field attached to \( \mathbb{R}^d \).

**Definition 1.** A clearing in the random environment \( \omega \) is a trap-free region in \( \mathbb{R}^d \), that is, \( A \subseteq \mathbb{R}^d \) is a clearing if \( A \subseteq K^c \). By a clearing of radius \( r \), we mean a ball of radius \( r \) which is a clearing.

The following result is Lemma 4.5.2 in [19].

**Proposition D.** Let

\[
R_0 = R_0(d, \nu) := \left( \frac{d}{\nu \omega_d} \right)^{1/d} = \sqrt{\frac{\lambda_d}{\nu(d, \nu)}}.
\]

Then, on a set of full \( \mathbb{P} \)-measure, there exists \( \ell_0 > 0 \) such that for each \( \ell \geq \ell_0 \) the cube \((-\ell, \ell)^d\) contains a clearing of radius

\[
R_\ell := R_0(\log \ell)^{1/d} - (\log \log \ell)^2, \quad \ell > 1.
\]

We now prove a somewhat stronger version of Proposition D, which will be needed in the proof of the lower bound of Theorem 2 (see Section 5). For a Borel set \( B \) and \( x \in \mathbb{R}^d \), we define their sum in the sense of sum of sets as \( x + B := \{ x + y : y \in B \} \).

**Lemma 1** (Almost sure clearings). Let \( n \in \mathbb{N} \) and \( a \in \mathbb{R}_+ \) be fixed, and for \( \ell > 0 \) let \( x_1, \ldots, x_{\ell n} \) be any set of \( \ell n \) points in \( \mathbb{R}^d \). Define the cubes \( C_{j, \ell} = x_j + (-\ell, \ell)^d \), \( 1 \leq j \leq \ell n \). Then, on a set of full \( \mathbb{P} \)-measure, there exists \( \ell_0 > 0 \) such that for each \( \ell \geq \ell_0 \), each of \( C_{1, \ell}, C_{2, \ell}, \ldots, C_{\ell n, \ell} \) contains a clearing of radius \( R_\ell + a \), where \( R_\ell \) is as in (11).

**Proof.** Let \( x_1, x_2, \ldots \) be a sequence of points in \( \mathbb{R}^d \), and \( C_{j, \ell} := x_j + (-\ell, \ell)^d \) for \( j = 1, 2, \ldots \). For \( k \geq 0 \), let \( A_{\ell, k} \) be the event that there is a clearing of radius \( R_\ell + k \) in each \( C_{1, \ell}, C_{2, \ell}, \ldots, C_{\ell n, \ell} \). Also, for \( k \geq 0 \), define

\[
E_{\ell, k} = \{ (-\ell, \ell)^d \text{ contains a clearing of radius } R_\ell + k \}.
\]

Due to the homogeneity of the PPP, it is clear that for all \( x \in \mathbb{R}^d \) and \( k > 0 \),

\[
\mathbb{P} \left( x + (-\ell, \ell)^d \text{ contains a clearing of radius } R_\ell + k \right) = \mathbb{P}(E_{\ell, k}).
\]

Then, the union bound gives

\[
\mathbb{P}(A_{\ell, k}^c) \leq \lfloor (2\ell)^n \rfloor \mathbb{P}(E_{\ell, k}^c). \quad (12)
\]

We now estimate \( \mathbb{P}(E_{\ell, k}^c) \). Partition \((-\ell, \ell)^d\) into smaller cubes of side length \( 2(R_\ell + k) \). Then, a ball of radius \( R_\ell + k \) can be inscribed in each smaller cube, and we can bound \( \mathbb{P}(E_{\ell, k}^c) \) from above as

\[
\mathbb{P}(E_{\ell, k}^c) \leq \left[ 1 - e^{-\nu \omega_d (R_\ell + k)^d} \right]^{\lfloor \ell/(R_\ell + k) \rfloor^d} \leq \exp \left[ - \left( \frac{\ell}{R_\ell + k} \right)^d e^{-\nu \omega_d (R_\ell + k)^d} \right], \quad (13)
\]

where we have used the estimate \( 1 + x \leq e^x \). Let

\[
\alpha_\ell := \left( \frac{\ell}{R_\ell + k} \right)^d e^{-\nu \omega_d (R_\ell + k)^d}.
\]
Then, using (11), and that $\log \frac{\ell}{(R_\ell + k)} \geq \log \frac{\ell}{2(R_\ell + k)}$, we obtain

$$
\log \alpha \geq d \log \ell - d \log [2(R_0(\log \ell)^{1/d} - (\log \log \ell)^2 + k)] - \nu \omega d_0 [R_0(\log \ell)^{1/d} - (\log \log \ell)^2 + k]^d \\
= d \log \ell - d \log [2(R_0(\log \ell)^{1/d} - (\log \log \ell)^2 + k)] - \nu \omega d_0 R_0^d \log \ell \left[ 1 - \frac{(\log \log \ell)^2 - k}{R_0(\log \ell)^{1/d}} \right]^d \\
\geq d \log \ell - d \log 2 - d \log [R_0(\log \ell)^{1/d} - (\log \log \ell)^2 + k] \\
- d \log \ell + \frac{d^2}{R_0}(\log \ell)^{1-1/d}[\alpha (\log \log \ell)^2 - k] - \frac{d^2}{2R_0^2}(\log \ell)^{1-2/d}[\alpha (\log \log \ell)^2 - k]^2 \\
\geq \frac{1}{2R_0}(\log \log \ell)^2
$$

(14)

for all large $\ell$, where we have used in the first inequality that $R_0^d = d/(\nu \omega d)$, and that $(1 - x)^n \leq 1 - xn + (xn)^2/2$ for $x \geq 0$. It follows from (13) and (14) that for a given $k > 0$, for all large $\ell$,

$$
P(E_{\ell,k}^c) \leq e^{-\alpha \ell} \leq e^{-\exp[(\log \log \ell)^2/(2R_0)]}.
$$

Take $\ell = 2^m$ with $m \in \mathbb{N}$. Then, for all large $m$, $e^{-\alpha (2^m)} \leq \exp\left(-m \frac{\log m}{2R_0}\right)$, which along with (12) and (14) implies

$$
\sum_{m=1}^{\infty} P(A_{2^m,k}^c) \leq c(m_0) + \sum_{m=m_0}^{\infty} [(2^m+1)^n] \exp\left(-m \frac{\log m}{2R_0}\right) < c(m_0) + \sum_{m=m_0}^{\infty} [(2^n)^{m+1}] e^{-m^2} < \infty,
$$

where $c(m_0)$ is a constant that depends on $m_0$. Applying Borel-Cantelli lemma to the cubes $(-2^m, 2^m)^d$, we conclude that with $P$-probability one, only finitely many $A_{2^m,k}^c$ occur. That is, $P(\Omega_0) = 1$, where

$$
\Omega_0 = \{\omega : \exists m_0 \forall m \geq m_0, \text{ each } C_{1,2^m,\ldots,C_{(2^m)^n},2^m} \text{ has a clearing of radius } R_{(2^m)} + k\}. \quad (15)
$$

Let $\omega_0 \in \Omega_0$, and $m_0$ be as in (15). If we choose $k \geq a$, then to complete the proof, it suffices to show that in the environment $\omega_0$ for each $m \geq m_0$ and $2^m \leq \ell \leq 2^{m+1}$, each $C_{1,\ell,\ldots,C_{(2^n)^n},\ell}$ contains a clearing of radius $R_{(2^n)} + a$. Let $\ell \geq 2^{m_0}$ so that $2^m \leq \ell \leq 2^{m_1}$ for some $m \geq m_0$. Fix this integer $m$. Observe that

$$
R_{(2^{m+1})} - R_{(2^m)} \leq R_0(\log 2)^{1/d} \left[(m+1)^{1/d} - m^{1/d}\right] \leq R_0 \log 2.
$$

Choose $k = R_0 \log 2 + a$ (so far the choice of $k > 0$ was arbitrary). Then, since $R_\ell$ is increasing in $\ell$ for large $\ell$, we have

$$
R_\ell + a \leq R_{(2^{m+1})} + a \leq R_{(2^m)} + R_0 \log 2 + a = R_{(2^m)} + k.
$$

Furthermore,

$$
[\ell^n] \leq [(2^{m+1})^n]. \quad (17)
$$

Then, setting $\ell_0 = 2^{m_0}$, (15), (16) and (17) imply that for $\ell \geq \ell_0$, each of $C_{1,\ell,\ldots,C_{(2^n)^n},\ell}$ contains a clearing of radius $R_{(2^n)} + a$. This completes the proof since the choice of $\omega_0 \in \Omega_0$ was arbitrary and $P(\Omega_0) = 1$. \qed
4. Proof of Theorem 1

Recall that by assumption \( r : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfies \( r(t) \to \infty \) as \( t \to \infty \) and \( r(t) = o(\sqrt{t}) \). Also, we set \( B_t = B(0, r(t)) \) for \( t \geq 0 \). Throughout this section, we will use that the law of \( |Z^B_t| \) is the same as the law of number of particles of \( Z \) which are present at \( t \) and whose ancestral lines over \( [0, t] \) have been confined to \( B_t \).

4.1. Proof of the lower bound

Suppose that
\[
\gamma_t = e^{-\kappa r(t)}, \quad \text{where} \quad \kappa > 0.
\]

Consider the joint strategy of suppressing the branching over \([0, f(t)]\), and then letting the BBM evolve ‘normally’ in the remaining interval \([f(t), t]\). To be precise, recall that \( n_t := |Z^B_t| \), \( \sigma_A \) denotes the first exit time out of \( A \), and \( p_t := \mathbb{P}_0(\sigma_{B_t} \geq t) \); and let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfy \( f(t) = o(t) \). For \( t > 0 \) define the events
\[
A_t = \{N_{f(t)} = 1\}, \quad E_t = \{n_t < \gamma_t e^{\beta t}\}.
\]

Estimate
\[
P(E_t) \geq P(E_t \cap A_t) = P(E_t \mid A_t)P(A_t). \tag{18}
\]

We will show that \( P(E_t \mid A_t) \) tends to a constant smaller than one as \( t \to \infty \) for suitable \( f \). Let \((Y_1(s))_{0 \leq s \leq \tau_1}\) be the path of the initial particle, where \( \tau_1 \) denotes the particle’s lifetime. Conditional on \( A_t \), it is clear that \( \tau_1 \geq f(t) \), and that \( n_t = 0 \) if \( Y_1(z) \notin B_t \) for some \( 0 \leq z \leq f(t) \). Next, for \( t > 0 \) define
\[
D_t = \{Y_1(z) \in B_t \quad \forall \quad 0 \leq z \leq f(t)\}.
\]

Then,
\[
E[n_t \mid A_t] = E[n_t \mathbb{1}_{D_t} \mid A_t] + E[n_t \mathbb{1}_{D_t^c} \mid A_t] = E[n_t \mid A_t, D_t]P(D_t \mid A_t), \tag{19}
\]

where the second term on the right-hand side vanishes. Write
\[
E[n_t \mid A_t, D_t] = \int_{B_t} E[n_t \mid A_t, D_t, Y_1(f(t)) = y] P(Y_1(f(t)) \in dy \mid A_t, D_t). \tag{20}
\]

Define
\[
\tilde{p}^{(t)}(x, s, dy) := \mathbb{P}_x(X(s) \in dy \mid X(z) \in B_t \quad \forall \quad 0 \leq z \leq s) \quad \text{and} \quad p_{s,x}^t := \mathbb{P}_x(\sigma_{B_t} \geq s). \tag{21}
\]

Applying the Markov property of a standard BM at time \( s \) with \( 0 < s < t \) gives
\[
p_t = p_{s,0}^t \int_{B_t} p_{t-s,y}^t \tilde{p}^{(t)}(0, s, dy). \tag{22}
\]

Furthermore, it follows from (19) and (20) that
\[
E[n_t \mid A_t] = p_{f(t),0}^t \int_{B_t} E[n_t \mid A_t, D_t, Y_1(f(t)) = y] \tilde{p}^{(t)}(0, f(t), dy). \tag{23}
\]

Now apply the Markov property of BBM at time \( f(t) \), and use the many-to-one lemma (see for instance [1, Lemma 1.6]) to obtain
\[
E[n_t \mid A_t, D_t, Y_1(f(t)) = y] = p_{f(t),y}^t e^{\beta(t-f(t))}, \quad y \in B_t. \tag{24}
\]
Using (22) with $s$ therein replaced by $f(t)$, it then follows from (22), (23) and (24) that

$$E[n_t \mid A_t] = e^{\beta(t-f(t))}p_{f(t),0}^t \int_{B_t} p^t_{t-f(t),y} \tilde{P}^{(t)}(0,f(t),dy) = e^{\beta(t-f(t))}p_t.$$  

Then, by the Markov inequality,

$$P(E_t^c \mid A_t) \leq \frac{E[n_t \mid A_t]}{\gamma_t p_t e^{\beta t}} = \gamma_t^{-1}e^{-\beta f(t)}.$$  

Choose $f(t) = -(1/\beta)\log((1-\delta)\gamma_t)$, where $0 < \delta < 1$. With this choice of $f$, (25) implies that $P(E_t \mid A_t) \geq \delta$. Then, noting that $P(A_t) = e^{-\beta f(t)}$, it follows from (18) that

$$P(E_t) \geq \delta e^{-\beta f(t)} = \delta(1-\delta)\gamma_t = e^{-\kappa r(t)(1+o(1))}.$$  

This, along with (19), proves (2), and the lower bound in (1).

### 4.2. Proof of the upper bound

For the proof of the upper bound, we follow a method that is based on Chebyshev’s inequality, similar to the proof of [1] Thm. 1. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $g(t) \to 0$ as $t \to \infty$. Later, we will choose $g_t := g(t)$ in a precise way. For $t \geq 0$, let $N_t = |Z_t|$ as before, and estimate

$$P(. \mid A_t) = P\left(. \mid N_t > e^{\beta t}g_t\right) + P\left(N_t \leq e^{\beta t}g_t\right).$$  

We first bound the second term on the right-hand side of (26) from above. It follows from (23) that $P(N_t \leq k) = 1 - (1 - e^{-\beta t})^k \leq ke^{-\beta t}$ for $k \geq 1$. Set $k = \lfloor e^{\beta t}g_t \rfloor$ to obtain

$$P\left(N_t \leq e^{\beta t}g_t\right) = P\left(N_t \leq \lfloor e^{\beta t}g_t \rfloor \right) \leq \lfloor e^{\beta t}g_t \rfloor e^{-\beta t} \leq g_t.$$  

Next, for $t > 0$ define

$$\tilde{P}_t(\cdot) = P(\cdot \mid N_t > e^{\beta t}g_t),$$  

and let $\tilde{E}_t$ be the corresponding expectation. We now bound the first term on the right-hand side of (26) from above. Let $\mathcal{N}_t$ denote the set of particles of $Z$ that are alive at time $t$. For $u \in \mathcal{N}_t$, let $(Y_u(s))_{0 \leq s \leq t}$ denote the ancestral line up to $t$ of particle $u$. By the ancestral line up to $t$ of a particle present at time $t$, we mean the continuous trajectory traversed up to $t$ by the particle, concatenated with the trajectories of all its ancestors including the one traversed by the initial particle. Note that $(Y_u(s))_{0 \leq s \leq t}$ is identically distributed as a Brownian path $(X(s))_{0 \leq s \leq t}$ for each $u \in \mathcal{N}_t$. Let us pick randomly, independent of their genealogy and position, $\lfloor e^{\beta t}g_t \rfloor$ particles from $\mathcal{N}_t$. Note that this is possible under $\tilde{P}_t(\cdot)$. Denote this collection of particles by $\mathcal{M}_t$, set $M_t := |\mathcal{M}_t|$, and define

$$\hat{n}_t = \sum_{u \in \mathcal{M}_t} \mathbb{1}_{A_u},$$  

where $A_u = \{Y_u(s) \in B_t \forall 0 \leq s \leq t\}$. Observe that $\hat{n}_t$ counts, out of $\mathcal{M}_t$, the particles whose ancestral lines are confined to $B_t$ over $[0,t]$. Since the collection $\mathcal{M}_t$ is chosen independently of the motion process, each particle $u$ in $\mathcal{M}_t$ has an ancestral line $(Y_u(s))_{0 \leq s \leq t}$ that is Brownian. Then, since the branching and motion mechanisms are independent of each other, the many-to-one lemma
implies that for $t > 0$,
\[
\bar{E}_t [\hat{n}_t] = p_t M_t = p_t [e^{\beta t} g_t],
\]
where $p_t$ is as before the probability of confinement of a standard BM to $B_t$ over $[0, t]$. It is clear that $\hat{n}_t \leq n_t$. At this point, choose $g$ such that $g_t \geq \gamma_t$ for all $t > 0$. Then, using Chebyshev’s inequality, it follows from (26), (27), and (28) that
\[
P(n_t < \gamma_t p_t e^{\beta t}) \leq \bar{P}_t \left( \hat{n}_t < \gamma_t p_t e^{\beta t} \right) + g_t
\]
\[
\leq \bar{P}_t \left( |\hat{n}_t - \bar{E}_t [\hat{n}_t]| > (g_t - \gamma_t) p_t e^{\beta t} \right) + g_t
\]
\[
\leq \frac{\bar{\text{Var}}_t (\hat{n}_t)}{(g_t - \gamma_t)^2 p_t^2 e^{2\beta t}} + g_t,
\]
where $\bar{\text{Var}}_t$ denotes the variance associated to $\bar{P}_t$. In the rest of the proof, we estimate $\bar{\text{Var}}_t (\hat{n}_t)$.

Let $P$ be the probability under which the pair $(i, j)$ is chosen uniformly at random among the $M_t (M_t - 1)$ possible pairs in $M_t$, and let $E$ be the corresponding expectation. Also, for a generic Brownian motion $X$, let $Var$ denote its variance, and let $A = \{X(s) \in B_t \forall 0 \leq s \leq t\}$. Then,
\[
\bar{\text{Var}}_t (\hat{n}_t) = \text{Var}_t \left( \sum_{u \in M_t} 1_{A_u} \right)
\]
\[
= M_t \text{Var} (1_A) + \sum_{1 \leq i \neq j \leq M_t} \text{Cov}_t (1_{A_i}, 1_{A_j})
\]
\[
= M_t (p_t - p_t^2) + M_t (M_t - 1) \sum_{1 \leq i \neq j \leq M_t} \text{Cov}_t (1_{A_i}, 1_{A_j})
\]
\[
\leq g_t e^{\beta t} (p_t - p_t^2) + g_t e^{\beta t} (p_t e^{\beta t} - 1) \left[ (E \otimes \bar{P}_t) (A_i \cap A_j) - p_t^2 \right].
\]

Let $Q^{(t)}$ be the distribution of the splitting time of the most recent common ancestor of $i$th and $j$th particles under $E \otimes \bar{P}_t$. Applying the Markov property at this splitting time, we obtain
\[
(E \otimes \bar{P}_t) (A_i \cap A_j) = p_t \int_0^t \int_{B_t} p_{t-s,x}^{(0)} p_{s,0}^{(t)} (0, s, dx) Q^{(t)} (ds),
\]
where $p_{t-s,x}^{(0)}$ and $p_{s,0}^{(t)}$ are as defined in (21). Set $p_{s}^t = p_{s,0}^{(t)}$. Then, it follows from (22) and (31) that
\[
(E \otimes \bar{P}_t) (A_i \cap A_j) = p_t^2 \int_0^t \frac{1}{p_{s}^t} Q^{(t)} (ds).
\]

For $t > 0$ define
\[
J_t := \int_0^t \frac{1}{p_{s}^t} Q^{(t)} (ds).
\]

Then, by (30) and (32), we have
\[
\bar{\text{Var}}_t (\hat{n}_t) \leq g_t p_t e^{\beta t} + g_t^2 p_t^2 e^{2\beta t} (J_t - 1).
\]

It is clear that $J_t - 1 \geq 0$. Next, we bound $J_t - 1$ from above.

Recall that $r(t)$ is a distance scale. For $k > 0$, we will use $kr(t)$ as a time scale. Note that for large $t$ it is atypical for a BM starting at the origin to escape $B_t = B(0, r(t))$ over $[0, kr(t)]$. For
large \( t \), split \( J_t \) up as

\[
J_t = \int_0^{kr(t)} \frac{1}{p_s^t} Q(t)(ds) + \int_{kr(t)}^t \frac{1}{p_s} Q(t)(ds),
\]

and define

\[
J_t^{(1)} := \int_0^{kr(t)} \frac{1}{p_s^t} Q(t)(ds), \quad J_t^{(2)} := \int_{kr(t)}^t \frac{1}{p_s} Q(t)(ds).
\]

It then follows from (34) and (35) that

from Proposition A,

\[
g(s) \text{ its density function, which we denote by } \frac{1}{p_s}, \text{ from (36) and (37), we have }
\]

We first bound \( J_t^{(1)} - 1 \) from above. Observe that \( p_s^t \) is nonincreasing in \( s \), and estimate

\[
J_t^{(1)} = \int_0^{kr(t)} \frac{1}{p_s^t} Q(t)(ds) \leq \frac{1}{p_{kr(t)}}.
\]  

From Proposition A,

\[
1 - p_{kr(t)}^t = \exp \left[ -\frac{r(t)}{2k} (1 + o(1)) \right].
\]

It then follows from (34) and (35) that

\[
J_t^{(1)} - 1 \leq \exp \left[ -\frac{r(t)}{2k} (1 + o(1)) \right] = \exp \left[ -\frac{r(t)}{2k} (1 + o(1)) \right].
\]  

To bound \( J_t^{(2)} \) from above, we will use the following fact on the distribution \( Q(t) \) from [6, Prop. 5]: \( Q(t) \) is absolutely continuous with respect to the Lebesgue measure, which we denote by \( ds \), and its density function, which we denote by \( g(t) \), satisfies

\[
\exists C > 0, s_0 > 0 \text{ such that } \forall s \geq s_0, g(t)(s) \leq Ce^{-\beta s}.
\]

Since \( r(t) = o(\sqrt{t}) \) by assumption, this implies that for all large \( t \) we have \( r(t) \leq t \), which implies \( 1/p_s^t \leq 1/p_{r(t)}^t \). Here, \( p_{r(t)}^t = P_0(\sigma_{B_{r(t)}} \geq s) \) with \( B_{r(t)} = B(0, r(t)) \) in accordance with previous notation. Then, since \( r(t) \to \infty \) as \( t \to \infty \), for all large \( t \) and for \( kr(t) \leq s \leq t \),

\[
\frac{1}{p_s} \leq \frac{1}{p_{r(t)}^t} = \exp \left[ \frac{\lambda ds}{r^2(r(t))} (1 + o(1)) \right] \leq \exp \left[ \frac{2\lambda ds}{r^2(r(t))} \right],
\]

where we have used Proposition B. Then, we continue with

\[
J_t^{(2)} = \int_{kr(t)}^t \frac{1}{p_s} Q(t)(ds) \leq \int_{kr(t)}^t \exp \left[ \frac{2\lambda ds}{r^2(r(t))} \right] Cse^{-\beta s}ds
\]

\[
\leq C \int_{kr(t)}^\infty s \exp \left[ -s \left( \beta - \frac{2\lambda}{r^2(r(t))} \right) \right] ds
\]

\[
\leq \exp \left[ -\beta kr(t)(1 + o(1)) \right],
\]

where we have used integration by parts. From (36) and (37), we have

\[
J_t - 1 = J_t^{(1)} - 1 + J_t^{(2)} \leq \exp \left[ -\frac{r(t)}{2k} (1 + o(1)) \right] + \exp \left[ -\beta kr(t)(1 + o(1)) \right].
\]
To optimize the smallest absolute exponent on the right-hand side of (38), choose \( k \) so that 
\[
\beta kr(t) = \frac{r(t)}{2^k}.
\]
This yields \( k = \frac{1}{\sqrt{2^\beta}} \). With this choice of \( k \), we have
\[
J_t - 1 \leq \exp\left(-\sqrt{\beta/2} r(t)(1 + o(1))\right).
\]
It then follows from (29) and (33) that
\[
P(n_t < \gamma_t p_t e^{\beta t}) \leq \frac{2}{\gamma_t p_t} e^{-\beta t} + 4 e^{-\sqrt{\beta/2} r(t)(1 + o(1))} + g_t.
\] (39)

By assumption, \( \gamma_t = e^{-\kappa r(t)} \) with \( \kappa > 0 \). Choose \( g_t = 2 \gamma_t \). Then, we can continue (39) with
\[
P(n_t < \gamma_t p_t e^{\beta t}) \leq \exp\left[\frac{-\beta}{\sqrt{2}} r(t)(1 + o(1))\right] + 4 e^{-\sqrt{\beta/2} r(t)(1 + o(1))} + g_t.
\] (40)

Using Proposition B, and the assumptions that \( r(t) \to \infty \) and \( r(t) = o(\sqrt{t}) \) as \( t \to \infty \), we have
\[
\gamma_t p_t = e^{-\kappa r(t)} = \exp[O(t)].
\]
Then, using that \( g_t = 2 \gamma_t = 2 e^{-\kappa r(t)} \), it follows from (40) that
\[
P(n_t < \gamma_t p_t e^{\beta t}) \leq \begin{cases} 
\exp\left[O(t)\right], & \kappa > \sqrt{\beta/2} \\end{cases}
\]
This completes the proof of (3) and the upper bound of (1).

5. Proof of Theorem 2

5.1. Proof of the upper bound

The following upper bound was proved in [6, Section 6.1] via a first moment argument, using (8) and the Markov inequality. On a set of full \( \mathbb{P} \)-measure, say \( \Omega_0 \), for any \( \varepsilon > 0 \),
\[
P^\omega \left( (\log t)^{2/d} \left( \frac{\log N_t}{t} - \beta \right) + c(d,v) > \varepsilon \right) \leq \exp\left[ -\varepsilon t (\log t)^{-2/d} + o\left( t (\log t)^{-2/d} \right) \right].
\] (41)

To pass from (41) to the upper bound of the corresponding SLLN, we use a standard Borel-Cantelli argument. Recall that \( \hat{\Omega} \) is the sample space for the BBM. For \( t > 0 \), define
\[
Y_t := (\log t)^{2/d} \left( \frac{\log N_t}{t} - \beta \right),
\]
and let
\[
\hat{\Omega}_0 := \{ \omega \in \hat{\Omega} : \forall \varepsilon > 0 \ \exists t_0 = t_0(\omega) \text{ such that } \forall t \geq t_0, Y_t \leq -c(d,v) + \varepsilon \}.
\]
Let \( \omega \in \Omega_0 \). We will show that \( P^\omega(\hat{\Omega}_0) = 1 \). For \( n \in \mathbb{N} \), define
\[
A_n := \{ Y_n > -c(d,v) + \varepsilon \}.
\]
By (41), there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), \( P(\omega) \leq e^{-c(\epsilon)n(lg n)^{-2/d}} \). Then,

\[
\sum_{n=1}^{\infty} P(\omega) = c + \sum_{n=n_0}^{\infty} P(\omega) \leq c + \sum_{n=n_0}^{\infty} e^{-c(\epsilon)n(lg n)^{-2/d}} < \infty.
\]

By the Borel-Cantelli lemma, it follows that \( P(\omega) \) occurs i.o. Then,

\[
\sum_{n=1}^{\infty} P(\omega) = c + \sum_{n=n_0}^{\infty} P(\omega) \leq c + \sum_{n=n_0}^{\infty} e^{-c(\epsilon)n(lg n)^{-2/d}} < \infty.
\]

By the Borel-Cantelli lemma, it follows that \( P(\omega) \) occurs i.o., where i.o. stands for infinitely often. Choosing \( \epsilon = 1/k \), this implies that for each \( k \geq 1 \), we have

\[
P(\omega) = 1 \quad \text{for each } k \geq 1,
\]

and then use this upper bound along the Borel-Cantelli lemma to pass to the corresponding SLLN.

5.2. Proof of the lower bound

Let \( \epsilon > 0 \). We will find an upper bound for

\[
P(\omega) \left( \left( \frac{\log N_t}{t} - \beta \right) + c(d,v) < -\epsilon \right) = P(\omega) \left( N_t < \exp \left[ t \left( \frac{\beta - c(d,v) + \epsilon}{(\log t)^{2/d}} \right) \right] \right)
\]

that is valid for large \( t \) on a set of full \( \mathbb{P} \)-measure, and then use this upper bound along with the Borel-Cantelli lemma to pass to the corresponding SLLN.

The proof is split into four parts for better readability. The first three parts are based on a bootstrap argument, where in part one, we find an upper bound on \( P(\omega) < e^{\delta t} \) for \( 0 < \delta < \beta \), and then use this upper bound in parts two and three to find a similar upper bound on (42). We mainly follow the proof strategy given in [6]. We significantly improve the first and third parts of the corresponding proof in [6] in order to extend the WLLN therein to SLLN, where the extra work is due to finding rates of decay to zero for the relevant probabilities as \( t \to \infty \) as opposed to merely showing that they tend to zero.

In the first part of the proof, we use probabilistic arguments alone, including Theorem 1, in contrast to the partial differential equations (PDE) approach used in [6]. The main challenge is due to the possibility of \( \beta = 0 \) (no branching inside the traps), which makes it difficult to show that even in the presence of mild obstacles the system produces exponentially many particles with 'high' probability. We emphasize that the case \( \beta = 0 \) was not covered in [6], and the PDE approach used therein exploits the condition \( \beta > 0 \). The second part of the proof is similar to that in [6]; here, with minor further work, we find the rate of convergence to zero of the probability of the relevant unlikely event. The third part of the proof is an application of Theorem 1 where we argue that with 'high' probability sufficiently many particles are produced in a certain expanding clearing in \( \mathbb{R}^d \), which exists in almost every environment. The fourth part of the proof uses a Borel-Cantelli argument along with the upper bound on (42) from part three to obtain the lower bound of the SLLN in (7).

**Part 1: Upper bound on exponentially few total mass**

In the first part of the proof, we will find an upper bound for

\[
P(\omega) < e^{\delta t} \quad \text{with } 0 < \delta < \beta,
\]

that is valid for large \( t \) on a set of full \( \mathbb{P} \)-measure. The argument will be based on the following lemma of independent interest, which is on the hitting probability of a standard BM to clearings.
of a certain size. Recall that \( X = (X(t))_{t \geq 0} \) denotes a standard BM in \( d \) dimensions, and \( P_x \) is the law of \( X \) started at position \( x \in \mathbb{R}^d \). Also, recall the definition of \( R_0 \) from \( \text{[1]} \).

**Lemma 2** (Hitting probability of BM to large clearings). Let \( r : \mathbb{R}_+ \to \mathbb{R}_+ \) be such that

\[
r(t) = \frac{R_0}{3} \left( \frac{1}{6d} \right)^{1/d} \left( \log t \right)^{1/d}, \quad t > 1.
\]

For \( \omega \in \Omega \) and \( t > 0 \), define

\[
\Phi_t^\omega = \{ x \in \mathbb{R}^d : B(x, r(t)) \subseteq K^c(\omega) \}.
\]

Let \( P_x^\omega \) be the conditional law of \( X \) started at position \( x \in \mathbb{R}^d \) in the random environment \( \omega \). Then, there exists \( \Omega_1 \subseteq \Omega \) with \( P(\Omega_1) = 1 \) such that for every \( \omega \in \Omega_1 \), there exists \( t_0 = t_0(\omega) \) such that for all \( t \geq t_0 \),

\[
P_0^\omega \left( \left( \bigcup_{0 \leq s \leq t} X(s) \right) \cap \Phi_t^\omega = \emptyset \right) \leq e^{-t^{1/3}}.
\]

**Proof.** Introduce a time scale \( h(t) \), and two different space scales \( \rho(t) \) and \( r(t) \), as follows. Let \( h, \rho, r : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfy:

(a) \( \lim_{t \to \infty} h(t) = \infty \) and \( h(t) = o(t) \),

(b) \( \lim_{t \to \infty} \rho(t) = \infty \), \( \rho(t) = o(\sqrt{h(t)}) \) and \( \rho^n(t) = t/h(t) \) for some \( n \geq 2 \), \( n \in \mathbb{N} \),

(c) \( r(t) = \frac{R_0}{3}(\log \rho(t))^{1/d} \) for \( t > t_0 \),

where \( t_0 \) is large enough. Later, we will choose \( h(t) \) and \( \rho(t) \), and hence \( r(t) \), in a precise way so that \( r(t) \) will be as in \( \text{[43]} \). For notational convenience\(^1\) we suppose that \( t/h(t) \) is an integer. Split \( [0, t] \) into \( t/h(t) \) pieces as

\[
[0, h(t)], [h(t), 2h(t)], \ldots, [t - h(t), t].
\]

Let \( x_1, \ldots, x_{t/h(t)} \) be any set of \( t/h(t) \) points in \( \mathbb{R}^d \). For \( j = 1, 2, \ldots, t/h(t) \), define the intervals \( I_{j,t} \) and the balls \( B_{j,t} \), respectively, as

\[
I_{j,t} = [(j - 1)h(t), jh(t)], \quad B_{j,t} = B(x_j, \rho(t)).
\]

The balls \( (B_{j,t} : 1 \leq j \leq t/h(t)) \) need not be disjoint.

First, we establish a suitable almost-sure environment in \( \mathbb{R}^d \) with sufficient concentration of 'large' clearings. Let \( \Omega_1 \subseteq \Omega \) consist of environments \( \omega \) with the property that there exists \( t_0 = t_0(\omega) \) such that for all \( t \geq t_0 \) there exists a clearing of radius \( 2r(t) \) inside each \( B_{j,t} \), \( 1 \leq j \leq t/h(t) \). By assumption, \( t/h(t) = o(t) \) for some \( n \geq 2 \). Also, it follows from \( \text{[1]} \) and (c) that \( 2r(t) \leq R_\rho(t) \) for all large \( t \). Then, since \( \rho(t) \to \infty \) as \( t \to \infty \), Lemma \( \text{[1]} \) implies that \( P(\Omega_1) = 1 \).

We call \( x \in \mathbb{R}^d \) a good point for \( \omega \in \Omega \) at time \( t \) if \( B(x, r(t)) \) is a clearing (see Definition \( \text{[1]} \) in the random environment \( \omega \). That is,

\[
\Phi_t^\omega = \{ x \in \mathbb{R}^d : B(x, r(t)) \subseteq K^c(\omega) \}
\]

is the set of good points associated to the pair \( (\omega, t) \). Now choose

\[
x_j = X((j - 1)h(t)), \quad 1 \leq j \leq t/h(t),
\]

\(^1\)We would like to avoid the floor function in notation.
and define $\Omega_1$ as above. We know that $\mathbb{P}(\Omega_1) = 1$. In what follows, we will use $\Omega_1$ as the almost-sure, i.e., quenched, environment for the $BBM$. We now estimate the conditional probability that $X$ does not hit $\Phi^\omega_t$ up to a large time $t$ given that $\omega \in \Omega_1$.

Let $\omega \in \Omega_1$, and choose $t$ large enough so that $t \geq t_0(\omega)$, where $t_0(\omega)$ is as above. Then, for each $1 \leq j \leq t/h(t)$, $B_{j,t} = B(x_j, \rho(t))$ contains a clearing of radius $2r(t)$, hence a ball of radius $r(t)$, say $B_{j,t}$, that is entirely contained in $\Phi^\omega_t$. That is, $B_{j,t} \subseteq \Phi^\omega_t \cap B_{j,t}$. For $f \in C[0,t]$ and $0 \leq a < b \leq t$, let $f_{[a,b]} = \{f(s) : a \leq s \leq b\}$. Then, for $t > 1$, define the events

$$E_{j,t} = \{X_{I_{j,t}} \cap \Phi^\omega_t = \emptyset\} \text{ for } 1 \leq j \leq t/h(t), \quad E_t := \bigcap_{j=1}^{t/h(t)} E_{j,t}.$$ 

In words, $E_t$ is the event that $X$ does not hit a good point associated to $(\omega, t)$ over $[0, t]$, i.e.,

$$E_t = \{(\cap_{0 \leq s \leq t} X(s)) \cap \Phi^\omega_t = \emptyset\}.$$ 

Next, for $t > 1$, define the sets $\Sigma_t$ and the events $F_t$ as

$$\Sigma_t := \{j \in \mathbb{N} : j \leq t/h(t), X \text{ exits } B_{j,t} \text{ over } I_{j,t}\}, \quad F_t := \{|\Sigma_t| \leq t/(2h(t))\}.$$ 

Equivalently, $F_t$ is the event that the number of intervals $I_{j,t}$ where $X$ is confined to $B_{j,t}$ over $I_{j,t}$ is greater than or equal to $t/(2h(t))$ out of a total of $t/h(t)$ intervals. Recall that $\rho(t)$ is the radius of the ball $B_{j,t}$ and $h(t)$ is the length of the time interval $I_{j,t}$. Then, since a BM typically moves a distance of order $\sqrt{s}$ over a time period of length $s$ and since $\rho(t) = o(\sqrt{h(t)})$, $F_t$ is an unlikely event for large $t$. Set $P^\omega = P_0$. We estimate $P^\omega(E_t)$ as

$$P^\omega(E_t) \leq P^\omega(F_t) + P^\omega(F_t^c). \quad (44)$$

Let $q_0(t)$ be the probability that $X$ stays inside $B_{j,t}$ over the period $I_{j,t}$, and $q_1(t)$ be the probability that it doesn’t hit $B_{j,t}$ conditional on exiting $B_{j,t}$ over $I_{j,t}$. Recall that $x_j = X((j-1)h(t))$ by choice, and $B_{j,t} \supseteq B_{j,t}$ is a ball of radius $r(t)$. If $X$ is conditioned to exit $B_{j,t} = B(x_j, \rho(t))$ over $I_{j,t} = [(j-1)h(t), jh(t)]$, over this same period it must also exit $B(x_j, \tilde{r}(t))$, where $\tilde{r}(t)$ is the distance between $x_j$ and the center of $B_{j,t}$. Therefore, since the Brownian exit distribution out of a ball centered at the starting point has rotational invariance (even under the conditioning), by comparing the surface area of the $\tilde{r}(t)$-ball that intersects $B_{j,t}$ to the total surface area of the $\tilde{r}(t)$-ball, and since $\tilde{r}(t) \leq \rho(t)$ for each $t > 1$, we obtain

$$q_1(t) \leq 1 - \frac{\kappa_d r^{d-1}(t)}{\rho^{d-1}(t)} \quad \text{for all } t > 1,$$

where $\kappa_d$ is a constant that only depends on the dimension $d$. Then, apply the Markov property of the Brownian path $(X(s))_{0 \leq s \leq t}$ at times $h(t), 2h(t), \ldots, t - h(t)$ to continue the estimate in (44) as

$$P^\omega(F_t) = \sum_{j=0}^{t/h(t)} \binom{t/h(t)}{j} (1 - q_0(t))^j (q_0(t))^{t/h(t) - j} \leq 2^{h(t)} (q_0(t))^{t/h(t)}, \quad (45)$$

and

$$P^\omega(E_t \mid F_t^c) \leq \left[1 - \frac{\kappa_d r^{d-1}(t)}{\rho^{d-1}(t)} \right] \frac{t}{2h(t)} \leq \exp \left[-\kappa_d \left(\frac{r(t)}{\rho(t)}\right)^{d-1} \frac{t}{2h(t)}\right], \quad (46)$$
We now choose \( h \) where the last inequality follows since \( 1/3 < 1/2 - \frac{d-1}{6d} \). Hence, we reach the following conclusion. There exists \( \Omega_1 \subseteq \Omega \) with \( \mathbb{P}(\Omega_1) = 1 \) such that \( \forall \omega \in \Omega_1, \exists t_0 = t_0(\omega) \) such that \( \forall t \geq t_0 \),

\[
\mathbb{P}^\omega ((\cup_{0 \leq s \leq t} X(s)) \cap \Phi_t^\omega = \emptyset) \leq e^{-t^{1/3}}.
\]

Next, we use Lemma 2 to complete the first part of the proof of the upper bound of Theorem 2. Recall that \( 0 \leq \delta < \beta \). Choose \( \alpha \) such that \( 0 < \alpha < 1 - \delta/\beta \). Split the interval \([0, t]\) into two pieces as \([0, \alpha t]\) and \([\alpha t, t]\). We argue that with ‘high’ probability, the BBM hits a good point, say \( z_0 \in \mathbb{R}^d \), associated to \( (\omega, \alpha t) \) over \([0, \alpha t]\), and then the sub-BBM emanating from the particle that hits \( z_0 \) produces at least \( e^{\delta t} \) particles over \([\alpha t, t]\) inside \( B(z_0, r(\alpha t)) \).

Let \( Y_1 = (Y_1(s))_{s \geq 0} \) be a randomly chosen ancestral line of the BBM in the random environment \( \omega \). Note that even under \( P^\omega \), since branching and motion mechanisms are independent of each other, \( (Y_1(s))_{s \geq 0} \) is identically distributed as a standard Brownian motion. The range (accumulated support) of \( Z \) is the process defined by

\[
R(t) = \bigcup_{0 \leq s \leq t} \text{supp}(Z(s)).
\]

Since \( Y_1 \) is an ancestral line of \( Z \), we have \( \cup_{0 \leq s \leq t} Y_1(s) \subseteq R(t) \) for each \( t \geq 0 \). Then, since \( Y_1 \) is

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2One can choose an ancestral line randomly as follows: start with the path of the initial particle from \( t = 0 \) until it branches, and when it branches, pick one of the two offspring with probability 1/2, and concatenate the previously traced path to the path of the chosen offspring until it, too, branches; repeat indefinitely the procedure of picking an offspring particle with probability 1/2 upon branching and concatenating its path to the previously traced path.
Brownian, Lemma 2 implies that for $0 < \alpha < 1$, on a set of full $\mathbb{P}$-measure, say $\Omega_2$, for all large $t$,

$$P^\omega (R(\alpha t) \cap \Phi_{\alpha t}^\omega = \emptyset) \leq e^{-\alpha^{1/3} t^{1/3}}. \quad (49)$$

Observe that $\{R(\alpha t) \cap \Phi_{\alpha t}^\omega = \emptyset\}$ is the event that $Z$ doesn’t hit a good point associated to $(\omega, \alpha t)$ over $[0, \alpha t]$.

Now let $\tau = \tau(\omega) = \inf\{s > 0 : R(s) \cap \Phi_{\alpha t}^\omega \neq \emptyset\}$ be the first hitting time of $Z$ to $\Phi_{\alpha t}^\omega$. Let $Y_2$ be the ancestral line of $Z$ that first hits $\Phi_{\alpha t}^\omega$, and let $z_0 = Y_2(\tau)$. Conditional on $\tau < \alpha t$, apply the strong Markov property at time $\tau$, and then apply Theorem 1 to the growth inside $B(z_0, r(\alpha t))$ of the sub-BBM initiated by $Y_2$ at time $\tau$. Note that $B_t := B(z_0, r(\alpha t))$ is a clearing in the random environment $\omega$ by definition of $\tau$, $z_0$ and $\Phi_{\alpha t}^\omega$.

In detail, for $u \geq 0$, let $|Z_{[\tau, \tau + u]}|$ denote the mass at time $\tau + u$ of the sub-BBM initiated at position $z_0$ and time $\tau$ by $Y_2$ with killing at $\partial B_t$. Let $s := (1 - \alpha)t, \tilde{r} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be such that

$$\tilde{r}(s) = \frac{R_0}{3} \left( \frac{1}{6d} \right)^{1/d} \left[ \log \left( \frac{\alpha s}{1 - \alpha} \right) \right]^{1/d}$$

for large $s$, and $B_s := B(0, \tilde{r}(s))$. Observe the equality of events $\{\tau \leq \alpha t\} = \{R(\alpha t) \cap \Phi_{\alpha t}^\omega \neq \emptyset\}$, and that $t - \tau \geq (1 - \alpha)t$ conditional on $\tau \leq \alpha t$, and $\tilde{r}(s) = r(\alpha t)$. Then, on $\Omega_2$ with $\mathbb{P}(\Omega_2) = 1$, applying the strong Markov property at $\tau = \tau(\omega)$, and taking $\gamma_s = \exp[-\sqrt{\beta/2} \tilde{r}(s)]$ for instance, Theorem 1 implies that for all large $t$,

$$P^\omega \left( N_t < e^{\delta t} \mid R(\alpha t) \cap \Phi_{\alpha t}^\omega \neq \emptyset \right) \leq P^\omega \left( |Z_{[\tau, \tau + t]}| < e^{-\sqrt{\beta/2} \tilde{r}(s)} e^{-\frac{\lambda ds}{(r(s))^2} (1 + o(1))} e^{\beta s} \right)$$

$$\leq P^\omega \left( |Z_{[\tau, \tau + (1 - \alpha)t]}| < e^{-\sqrt{\beta/2} \tilde{r}(s)} e^{-\frac{\lambda ds}{(r(s))^2} (1 + o(1))} e^{\beta s} \right)$$

$$= P \left( |Z_{[\tau, \tau + (1 - \alpha)t]}| < e^{-\sqrt{\beta/2} \tilde{r}(s)} e^{-\frac{\lambda ds}{(r(s))^2} (1 + o(1))} e^{\beta s} \right)$$

$$= e^{-\sqrt{\beta/2} \tilde{r}(s)(1 + o(1))}, \quad (50)$$

where, in the first inequality, we have used that $\delta t < (1 - \alpha)\beta t = \beta s$ due to the choice $\alpha < 1 - \delta/\beta$, and in the first equality we have used that $\tilde{r}(s) = r(\alpha t)$ and the definition of $r(t)$ from (43), we reach the following conclusion via (49) and (50): on $\Omega_2 \subseteq \Omega$ with $\mathbb{P}(\Omega_2) = 1$,

$$P^\omega \left( N_t < e^{\delta t} \right) \leq e^{-c(\log t)^{1/d}(1 + o(1))}, \quad (51)$$

where $c = c(d, \nu, \beta, \delta) > 0$. (The dependence of $c$ on $\nu$ is through $R_0$, which appears in the definition of $r(t)$; see (10) and (43).) This gives a quenched upper bound on the probability that $N_t = |Z_t|$ is exponentially few, and completes the first part of the proof of the lower bound of Theorem 2.

**Part 2: Time scales within $[0, t]$ and moving a particle into a large clearing**

This part of the proof is not new; it is essentially taken from [3] with minor improvements, where we also estimate the rate of decay to zero as $t \rightarrow \infty$ of the probabilities of the relevant unlikely events as opposed to merely showing that they tend to zero. Introduce two different time scales, $\ell(t)$ and $m(t)$, where $\ell(t) = o(m(t))$ and $m(t) = o(t)$, and split the interval $[0, t]$ into $[0, \ell(t)], [\ell(t), m(t)]$ and $[m(t), t]$. More precisely, let $\ell, m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be two functions satisfying $\ell(t) < m(t) < t$ for all $t > 0$, and
Here, as before, we use $P$ to denote the law of the law of the free \(^{\[24\]}\) BBM with branching rate $\beta$ everywhere. It is a standard result that $E[N_t] = \exp(\beta t)$ (one can deduce this, for example, from Proposition A), and we know from Proposition A that $\mathbb{P}(\sup_{0 \leq s \leq t} |X(s)| > \gamma t) = \exp[-\gamma^2 t/(2(1+o(1))). Moreover, the following stochastic domination is clear: for all $B \subseteq \mathbb{R}^d$ Borel, all $k \in \mathbb{N}$, and $t \geq 0$,

$$P(Z_t(B) < k) \leq P^\omega(Z_t(B) < k) \quad \text{for each } \omega \in \Omega.$$  

Then, taking $B = (B(0, \sqrt{2\beta + \varepsilon \ell(t)}))^c$ and $k = 1$ in \([53]\), and replacing

\(^{\[24\]}\)Here, free means in $\mathbb{R}^d$ without obstacles.
Since \( G_t \cap H_t \subseteq F_t \), we have \( P^\omega(F_t^c) \leq P^\omega(G_t^c) + P^\omega(H_t^c) \), which, in view of (52) and (55) implies that on \( \Omega_2 \),

\[
P^\omega(F_t^c) \leq e^{-c(\log \ell(t))^{1/d}(1+o(1))}, \quad c = c(d, \nu, \beta, \delta) > 0.
\]  

(56)

This means, on a set of full \( \mathbb{P} \)-measure, with ‘high’ \( P^\omega \)-probability, there are at least \( I(t) \) particles in \( B(0, \sqrt{2\beta + \varepsilon \ell(t)}) \) at time \( \ell(t) \) for large \( t \).

Next, we prepare the setting at time \( m(t) \). Recall (11) and define

\[
R(t) = R_{\ell(t)} = R_0[\log \ell(t)]^{1/d} - [\log \log \ell(t)]^2, \quad \text{for } t > e^c.
\]  

(57)

Since \( \lim_{t \to \infty} \ell(t) = \infty \), Lemma (1) implies that on a set of full \( \mathbb{P} \)-measure, say \( \Omega_3 \), there is a clearing \( B(x_0, R(t) + 1) \) such that \( |x_0| \leq \ell(t) \) for all large \( t \). Let \( \omega \in \Omega_2 \cap \Omega_3 \). Conditional on the event \( F_t \), the distance between \( x_0 \) and each of the at least \( I(t) \) many particles in \( B(0, \sqrt{2\beta + \varepsilon \ell(t)}) \) at time \( \ell(t) \) is at most

\[
(1 + \sqrt{2\beta + \varepsilon})\ell(t).
\]

A Brownian particle present at time \( \ell(t) \) in \( B(0, \sqrt{2\beta + \varepsilon \ell(t)}) \) moves to \( B(x_0, 1) \) over \([\ell(t), m(t)]\) with probability at least

\[
q_t = \exp \left[ - \frac{[(1 + \sqrt{2\beta + \varepsilon})\ell(t)]^2}{2[m(t) - \ell(t)]}(1 + o(1)) \right],
\]

which follows from (iii) and (iv), along with the Brownian transition density. Apply the Markov property of the BBM at time \( \ell(t) \), and neglect possible branching of particles over \([\ell(t), m(t)]\) for an upper bound on the probability of \( C_t^c \), where

\[
C_t := \{Z_{m(t)}(B(x_0, 1)) > 0\}.
\]  

(58)

Observe that conditional on the event \( F_t \), the event \( C_t \) is realized if one of the sub-BBMs initiated by each of the at least \( I(t) \) many particles present in \( B(0, \sqrt{2\beta + \varepsilon \ell(t)}) \) at time \( \ell(t) \) contributes a particle to \( B(x_0, 1) \) at time \( m(t) \). Therefore, by the independence of particles present at time \( \ell(t) \), we have

\[
P^\omega(C_t^c \mid F_t) \leq (1 - q_t)^{I(t)} = e^{-q_t I(t)},
\]  

(59)

where we have used the estimate \( 1 + x \leq e^x \). Since (iii) implies that \( \frac{\ell^2(t)}{m(t)} = o(\ell(t)) \), we have

\[
q_t e^{\delta \ell(t)} = \exp[\delta \ell(t)(1 + o(1))].
\]

Then, it follows from (59) that for all large \( t \),

\[
P^\omega(C_t^c \mid F_t) \leq \exp \left[ -e^{\delta \ell(t)(1+o(1))} \right] \leq e^{-\ell^2},
\]  

(60)

where we have used that by choice, \( \ell(t) = t^{1-1/(\log \log t)} \). Note that (60) is superexponentially small in \( t \). Thus far, the value of \( \delta \in (0, \beta) \) was arbitrary. For the rest of the argument, the exact value of \( \delta \) has no importance; therefore let us now fix it as \( \delta = \beta/2 \). Then, it follows from (55), (60), and
assumption (ii) that on $\Omega_0 := \Omega_2 \cap \Omega_3$ with $\mathbb{P}(\Omega_0) = 1$,
\[ P^\omega(C_t^c) \leq P^\omega(C_t^c \mid F_t) + P(F_t^c) \leq e^{-c(\log t)^{1/d}(1+o(1))}, \quad c = c(d, \nu, \beta) > 0. \tag{61} \]

This means, on a set of full $\mathbb{P}$-measure, with ‘high’ $P^\omega$-probability, there is at least one particle of $Z$ inside $B(x_0, 1)$ at time $m(t)$ for large $t$, where $|x_0| \leq \ell(t)$. Let us generically call this particle $v$, and denote by $y_0 := X_v(m(t))$ its position at time $m(t)$.

**Part 3: BBM in the large expanding clearing**

Let $\omega \in \Omega_0$, and recall that $B(x_0, R(t) + 1)$ is a clearing in $\omega$. In this part of the proof, we will work under the law $P^\omega(\cdot \mid C_t)$, where $C_t$ was defined in (58). Conditional on the event $C_t$, we show that $B_t = B(y_0, R(t))$ is a large enough expanding clearing in $\omega$ in which the BBM can produce sufficiently many particles. In particular, we study the evolution of the sub-BBM initiated by $v$ at time $m(t)$ over the period $[m(t), t]$ within the expanding clearing $B_t$. Denote this sub-BBM by $\hat{Z}$. We will use $P^\omega_x$ for the law of a BBM started with a single particle at $x \in \mathbb{R}^d$ in the random environment $\omega$. For $t > 0$ and $\varepsilon > 0$, define
\[ A_{t, \varepsilon} := \left\{ N_t < \exp \left[ t \left( \beta - \frac{c(d, \nu) + \varepsilon}{(\log t)^{2/d}} \right) \right] \right\}. \tag{62} \]

Recall that our goal (see (62)) is to find a suitable upper bound on $P^\omega(A_{t, \varepsilon})$.

Define $\hat{R} : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\hat{R}(t - m(t)) = R(t)$ for all large $t$. (By the choice of $m(t)$, $t - m(t)$ is increasing on $t \geq t_0$ for some $t_0 > 0$. Therefore, $t_1 - m(t_1) = t_2 - m(t_2)$ implies that $t_1 = t_2$ for $t_1 \wedge t_2 \geq t_0$). Next, let $s := t - m(t)$, $\hat{B}_s := B(y_0, \hat{R}(s))$ and
\[ p_s := P_{y_0}(\sigma_{B_s} \geq s) = P_0(\sigma_{B_{(0, \hat{R}(s))}} \geq s), \]

where, as before, $P_x$ denotes the law of a standard BM started at $x \in \mathbb{R}^d$. By the Markov property of $Z$ applied at time $m(t)$, $\hat{Z}$ is a BBM started with a single particle at $y_0$. Noting that $\hat{B}_s$ is a clearing in $\omega$ for all large $s$, and taking $\gamma_s = \exp[-\sqrt{\beta/2} \hat{R}(s)]$, Theorem 1 implies that
\[ P^\omega \left( |Z_s| < e^{-\sqrt{\beta/2} \hat{R}(s)} p_s e^{\beta s} \mid C_t \right) \leq P_{y_0}^\omega \left( |Z_s| < e^{-\sqrt{\beta/2} \hat{R}(s)} p_s e^{\beta s} \right) \]
\[ = \exp \left[ -\sqrt{\beta/2} \hat{R}(s)(1 + o(1)) \right]. \tag{63} \]

By Proposition B, (62), and since $\hat{R}(s) = R(t)$ and $c(d, \nu) = \lambda_d/R^2_0$,
\[ p_s = \exp \left[ -\frac{\lambda_d s}{R^2(s)} (1 + o(1)) \right] = \exp \left[ -\frac{c(d, \nu)(t - m(t))}{(\log \ell(t))^{2/d}} (1 + o(1)) \right]. \tag{64} \]

For two functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, use $f(t) \sim g(t)$ to express that $\lim_{t \to \infty} f(t)/g(t) = 1$. Then, it follows from assumptions (ii) and (v) that
\[ \frac{t - m(t)}{(\log \ell(t))^{2/d}} \sim \frac{t}{(\log t)^{2/d}}, \]

by which, we can continue (62) with
\[ p_s = \exp \left[ -\frac{c(d, \nu)t}{(\log t)^{2/d}} (1 + o(1)) \right]. \tag{65} \]
Furthermore, using that \( s = t - m(t) \), we have for any \( \varepsilon > 0 \),
\[
\exp \left[ t \left( \beta - \frac{c(d, v) + \varepsilon}{(\log t)^{2/d}} \right) \right] = \exp \left[ \beta s + \beta m(t) - \frac{(c(d, v) + \varepsilon)t}{(\log t)^{2/d}} \right] \leq e^{-\sqrt{2} R(s) p_s e^{\beta s}}
\]
for all large \( t \), where we have used (65), assumption (v), and that \( \hat{R}(s) = R(t) = o(t(\log t)^{-2/d}) \) in passing to the inequality. It is clear that \( N_t \geq |\hat{Z}_{t-m(t)}| = |\hat{Z}_s| \). Then, it follows from (63) that for all large \( t \),
\[
P^\omega \left( N_t < \exp \left[ t \left( \beta - \frac{c(d, v) + \varepsilon}{(\log t)^{2/d}} \right) \right] | C_t \right) \leq P^\omega \left( |\hat{Z}_s| < e^{-\sqrt{2} R(s) p_s e^{\beta s}} | C_t \right) \leq \exp \left[ -\sqrt{2} R(s) (1 + o(1)) \right].
\]

Finally, using assumption (ii), (57) along with \( \hat{R}(s) = R(t) \), and (61), we reach the following conclusion. On \( \Omega_0 \), which is a set of full \( \mathbb{P} \)-measure, for any \( \varepsilon > 0 \),
\[
P^\omega \left( N_t < \exp \left[ t \left( \beta - \frac{c(d, v) + \varepsilon}{(\log t)^{2/d}} \right) \right] \right) \leq \exp \left[ -c(\log t)^{1/d} (1 + o(1)) \right], \quad (66)
\]
where \( c = c(d, v, \beta) > 0 \). In the next part of the proof, we will exploit the fact that the right-hand side of (66) does not depend on \( \varepsilon \).

**Part 4: Borel-Cantelli argument**

We will show that on a set of full \( \mathbb{P} \)-measure, for any \( \varepsilon > 0 \),
\[
\liminf_{t \to \infty} \left( \frac{\log N_t}{t} - \beta \right) \geq -\left( \frac{c(d, v) + \varepsilon}{(\log t)^{2/d}} \right) \quad (67)
\]
Recall the definition of \( A_{t, \varepsilon} \) from (62). It follows from (66) that there exists \( c = c(d, v, \beta) / 2 \), independent of \( \varepsilon \), such that on \( \Omega_0 \), for all large \( t \),
\[
P^\omega \left( A_{t, \varepsilon/2} \right) \leq e^{-c(\log t)^{1/d}}. \quad (68)
\]
Define the function \( f : \mathbb{N} \to \mathbb{R}_+ \) by
\[
f(k) = \exp \left[ \left( \frac{2}{c} \right)^{\frac{d}{\log k^d}} \right]. \quad (69)
\]
Take \( \omega \in \Omega_0 \). By the choice of \( f(k) \) and (68), there exist constants \( c_0 > 0 \) and \( k_0 > 0 \) such that
\[
\sum_{k=1}^{\infty} P^\omega \left( A_{f(k), \varepsilon/2} \right) \leq c_0 + \sum_{k=k_0}^{\infty} e^{-c(\log f(k))^{1/d}} = c_0 + \sum_{k=k_0}^{\infty} \frac{1}{k^{1/2}} < \infty.
\]
Then, by the Borel-Cantelli lemma, on a set of full \( P^\omega \)-measure, only finitely many events \( A_{f(k), \varepsilon/2} \) occur. That is, for each \( \omega \in \Omega_0 \) there exists \( \hat{\Omega}_0 \subseteq \hat{\Omega} \) such that
\[
P^\omega (\hat{\Omega}_0) = 1, \quad \hat{\Omega}_0 = \left\{ \pi \in \hat{\Omega} : \exists k \geq k_0 \ n_{f(k)} \geq \exp \left[ f(k) \left( \beta - \frac{c(d, v) + \varepsilon/2}{(\log f(k))^{2/d}} \right) \right] \right\}. \quad (70)
\]
To prove (67), it suffices to show that for each \( \bar{w} \in \hat{\Omega}_0 \),
\[
N_s \geq \exp \left[ s \left( \beta - \frac{c(d,v) + \varepsilon}{(\log s)^{2/d}} \right) \right], \quad f(k) < s < f(k+1)
\tag{71}
\]
for all large \( k \). Indeed, (70) and (71) would together imply (67) on \( \Omega_0 \). Observe that \( N_s \) is \( P^\omega \)-almost surely increasing in \( s \), and the right-hand side of (71) is also increasing in \( s \) for all large \( s \). Therefore, to prove (71), it suffices to show that for all large \( k \),
\[
N_{f(k)} \geq \exp \left[ f(k+1) \left( \beta - \frac{c(d,v) + \varepsilon}{(\log f(k+1))^{2/d}} \right) \right].
\tag{72}
\]
Next, we control \( f(k+1) - f(k) \). Using (69), it can be shown that as \( k \to \infty \),
\[
f(k+1) - f(k) \sim d \left( \frac{2}{c} \right)^d t \frac{(\log t)^{d-1}}{k}.
\tag{73}
\]
and that if we set \( t = f(k) \), then
\[
k = \exp \left[ \frac{c}{2} (\log t)^{1/d} \right].
\tag{74}
\]
Then, setting \( g(t) = f(k+1) - f(k) \), it follows from (73) and (74) that
\[
g(t) \sim d \left( \frac{2}{c} \right)^d t \frac{(c/2)^{d-1}(\log t)^{(d-1)/d}}{\exp \left[ \frac{c}{2} (\log t)^{1/d} \right]}.
\tag{75}
\]
Using (57) and (75), it can be shown that
\[
\lim_{t \to \infty} \frac{g(t)R^2(t)}{t} = 0.
\]
This implies that for all large \( k \),
\[
f(k+1) - f(k) \leq \frac{\varepsilon f(k)}{2\beta(\log f(k))^{2/d}},
\]
which further implies that
\[
\beta f(k) - \frac{c(d,v) + \varepsilon}{(\log f(k))^{2/d}} f(k) \geq \beta f(k+1) - \frac{c(d,v) + \varepsilon}{(\log f(k+1))^{2/d}} f(k+1)
\tag{76}
\]
since \( f(z)/(\log f(z))^{2/d} \) is increasing for large \( z \). As (76) implies (72) on \( \hat{\Omega}_0 \), this proves (67), and hence completes the proof of the lower bound of Theorem 2.

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