The scientific interest in the analytical solution of the quantum Rabi model is due to the widespread use of this simple model in quantum optics, quantum computing, cavity QED, and nanoelectromechanical systems. This interest is related to the need for the theoretical description of the interaction of a two-level system with a quantum oscillator in the case when the rotating wave approximation fails. In this Letter, we present an approach to the exact diagonalization of the quantum Rabi Hamiltonian. This approach is based on the properties of the Pauli operators and allows us to readily solve the stationary Schrödinger equation for a two-level system. First, we demonstrate the applicability of the approach to the Jaynes-Cummings Hamiltonian to get the well-known solution. Then, we obtain the eigenvalues and eigenstates for the quantum Rabi Hamiltonian using the proposed approach. It is shown that the obtained eigenstates can be represented in the basis of the eigenstates of the Jaynes-Cummings Hamiltonian.

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The model of the interaction of a two-level system (TLS) with a single-mode field (SMF) considered by Rabi [1] still remains at the center of the theoretical research due to its widespread use in quantum optics, quantum computing, cavity QED, and circuit QED. Such a situation is related, on the one hand, with the need to take into account the quantum nature of the field, and, on the other hand, with the impossibility of utilizing the rotating wave approximation (RWA) in some of today's topical physical problems [2]. Thus, these problems consider the quantum Rabi model (QRM) [3] in the ultra-strong and deep-strong regimes of the light-matter interaction [4-6]. In addition, this simple model attracts researchers by a wide range of physical conditions to be taking into account. For instance, the extensions of the QRM include the cases of multiple TLSs [7, 8], multiple field modes [9, 10], dissipative effects [11], nonadiabatic effects [12, 13], nonclassical states [14], the Stark coupling [15, 16], and the chiral light-matter interaction [17], to name a few.

Analytical expressions for eigenvalues and eigenstates of the QRM, despite its simplicity, have been obtained using the different approximate treatments [18, 19]. The most prominent approximation that allows an exact solution is the RWA in the framework of the Jaynes-Cummings model (JCM) [20]. In the RWA, the counter-rotating terms (CRTs) of the interaction Hamiltonian are neglected. To take into account the CRTs the generalized RWA [4], perturbative [21, 22], adiabatic [23], variational [24, 25], and numerical [26] approaches have been proposed. Recently, the exact solution for the eigenvalue problem of the QRM has been presented by D. Braak [27]. Braak’s solution is determined by the use of the Heun functions in the Bargmann-Fock space. The same results were obtained using the Bogoliubov operators within extended coherent states [28]. It should be stressed that the aforementioned approximate solutions involve expressions for eigenvalues and eigenstates, which are simpler than in the case considered in Ref. [27, 28]. In addition, the problem of determining the exact eigenstates of the QRM cannot be addressed within the Bargmann representation approach.

In this Letter, we present an approach to find the eigenvalues and eigenstates of the QRM Hamiltonian. The approach utilizes the properties of the Pauli operators to diagonalize the Hamiltonian which involves the interaction of a TLS with an SMF. First, we implement this approach for determining the well-known solution for the stationary eigenproblem of the JCM Hamiltonian. Then, the eigenvalues and eigenstates of the QRM are obtained by the use of the presented approach. Finally, we compare the results obtained with the previous findings.

To present the approach, we consider the JCM with the Hamiltonian ($\hbar = 1$)

$$\hat{H}_{JCM} = \omega \left( \hat{b}^\dagger \hat{b} + \frac{1}{2} \right) \hat{\sigma}_0 + \frac{\omega_0}{2} \hat{\sigma}_x + g_0 \left( \hat{\sigma}_x \hat{b} + \hat{b}^\dagger \hat{\sigma}_x \right).$$  \hspace{1cm} (1)

Here, $\omega_0$ is the energy difference between the ground $|\varphi_0\rangle$ and excited $|\varphi_1\rangle$ TLS states; $\hat{\sigma}_x$ and $\hat{\sigma}_z$ are the Pauli operators (matrices); $\hat{\sigma}_0$ is the identity operator which is explicitly pointed out; $\hat{b}$ and $\hat{b}^\dagger$ are the annihilation and creation operators of the SMF of the frequency $\omega$; $g_0$ is the coupling strength of the TLS-SMF interaction. Introducing the detuning parameter, $\Delta = \omega_0 - \omega$, we can rewrite Eq. (1) in the matrix form

$$\hat{H}_{JCM} = \begin{pmatrix} \omega \hat{b}^\dagger \hat{b} + \frac{\Delta}{2} & g_0 \hat{b} \\ g_0 \hat{b}^\dagger & \omega \hat{b}^\dagger \hat{b} - \Delta \frac{1}{2} \end{pmatrix}.$$  \hspace{1cm} (2)
Then, we assume that the Hamiltonian (2) can be diagonalized and presented as
\[
\hat{H}_{\text{SMF}} = \omega \sigma_z^i + \Omega \sigma_x^i,
\]
\[
\sigma_z^i = \frac{\Delta}{2\Omega} \sigma_z^i + \frac{g_0}{\Omega} (\sigma_x^i \hat{b}^+ \hat{b} + \sigma_y^i \hat{b}^+ \hat{b} \sigma_z^i),
\]
\[
\sigma_x^i = \frac{\hat{\sigma}_x + \sigma_x^i \hat{b}^+ \hat{b} + \sigma_y^i \hat{b}^+ \hat{b} \sigma_z^i}{2}.
\]

In Eq. (3), we introduce the new spin operators which include the annihilation and creation operators of the SMF. That is similar to the parity operator approach presented in Ref. [5]. In addition, we assume that \( \sigma_z^i \) possesses the Pauli operator property which ensures the unitarity of the transformation to a new basis
\[
\sigma_z^{i^2} = \sigma_z^i,
\]
which we can use to find the eigenvalues for the \( \sigma_z^i \) operator as it can be done for the case of the \( \sigma_z \)-representation [29]. The squared operator looks like
\[
\sigma_z^{i^2} = \left\{ \frac{\Delta^2}{4\Omega^2} + \frac{g_0^2}{\Omega^2} \hat{b}^+ \hat{b} \right\} \left| \phi_i \right\rangle \langle \phi_i \right| + \left\{ \frac{\Delta^2}{4\Omega^2} + \frac{g_0^2}{\Omega^2} \hat{b}^+ \hat{b} \right\} \left| \phi_i \right\rangle \langle \phi_i \right|.
\]

Thus, we can see from Eq. (5) that the equation
\[
\langle \psi | \sigma_z^{i^2} | \psi \rangle = 1,
\]
is satisfied for the states of the operator \( \sigma_z^i \)
\[
\left| \psi \right\rangle = \sum_{m} (c_{1,m} | \phi_1, m \rangle + c_{2,m} | \phi_2, m + 1 \rangle),
\]
if the condition is met
\[
\Delta^2 + \frac{4g_0^2}{\Omega^2} (m + 1) = 1.
\]

Here, \( c_{1,m} \) denotes the probability amplitude for the TLS state \( i \) with the \( m \) quanta of the SMF. In Eq. (7), we present the eigenstates of the operator \( \sigma_z^i \) as the linear superposition of the bare states (BSs) hence, we have defined the BS basis which is associated with the eigenstates of the operator \( \sigma_z^i \), and have found the eigenvalues \( \Omega \), which we rewrite as
\[
2\Omega_{m+1} = 2\Delta^2 + 4g_0^2 (m + 1).
\]

Now we consider the problem for the eigenstates of the operator \( \sigma_z^i \) for a certain number of the SMF quanta \( m \)
\[
\sigma_z^i | \psi, m \rangle = | \psi, m \rangle.
\]

To this end, we utilize the expressions for the operators and states determined in the BS basis and rewrite Eq. (10) in the following form
\[
\begin{aligned}
\left\{ \frac{\Delta}{2\Omega_{m+1}} \sigma_z^1 + \frac{g_0}{\Omega_{m+1}} \left( \sigma_x^1 \hat{b}^+ \hat{b} + \sigma_y^1 \hat{b}^+ \hat{b} \sigma_z^1 \right) \right\} \left. \right| \psi, m \rangle + & \\
\times (c_{1,m} | \phi_1, m \rangle + c_{2,m+1} | \phi_2, m + 1 \rangle) &= c_{1,m} | \phi_1, m \rangle + c_{2,m+1} | \phi_2, m + 1 \rangle.
\end{aligned}
\]

The left-hand side of Eq. (11) can be readily transformed to the form
\[
\begin{aligned}
\left( u_{m+1}^2 - \nu_{m+1}^2 \right) (c_{1,m} | \phi_1, m \rangle - c_{2,m+1} | \phi_2, m + 1 \rangle) + & \\
2u_{m+1} \nu_{m+1} \left( c_{2,m+1} | \phi_1, m \rangle + c_{1,m} | \phi_2, m + 1 \rangle \right) &= 0.
\end{aligned}
\]

where the following denotations are presented
\[
\begin{aligned}
u_{m+1} &= \frac{1}{2} \left( 1 + \frac{\Delta}{2\Omega_{m+1}} \right), & u_{m+1} &= \frac{1}{2} \left( 1 - \frac{\Delta}{2\Omega_{m+1}} \right), \\
\Delta &= \frac{4g_0^2}{\Omega_{m+1}}, & 2s_{m+1} u_{m+1} &= \frac{g_0^2 \sqrt{m}}{2\Omega_{m+1}}.
\end{aligned}
\]

After some algebraic calculations, Eq. (11) takes the form
\[
\begin{aligned}
&c_{1,m} \left( u_{m+1} \xi_{m} + s_{m+1} \xi_{m+1} \right) - sv_{m+1} \xi_{m} \right) \left( c_{2,m+1} | \phi_1, m \rangle + c_{1,m} | \phi_2, m + 1 \rangle \right) = c_{1,m} | \phi_1, m \rangle + c_{2,m+1} | \phi_2, m + 1 \rangle.
\end{aligned}
\]

Hence, the eigenstates of the operator \( \sigma_z^i \) can be defined as
\[
\begin{aligned}
\left| \psi_1, m \right\rangle &= u_{m+1} \left| \phi_1, m \right\rangle - s_{m+1} \left| \phi_2, m \right\rangle, & \left| \psi_2, m \right\rangle &= s_{m+1} \left| \phi_1, m \right\rangle + u_{m+1} \left| \phi_2, m \right\rangle.
\end{aligned}
\]

Here, the eigenstates are presented in the basis of the dressed states (DSs)
\[
\begin{aligned}
\left| \xi_1, m \right\rangle &= u_{m+1} \left| \phi_1, m \right\rangle + s_{m+1} \left| \phi_2, m \right\rangle, & \left| \xi_2, m \right\rangle &= s_{m+1} \left| \phi_1, m \right\rangle - u_{m+1} \left| \phi_2, m \right\rangle.
\end{aligned}
\]

It should be stressed that Eqs. (15) and (16) determine the states for the whole system consisting of an SMF and a TLS and cannot be factorized. Only BS can be presented as factorized TLS and SMF states.

The DSs are the eigenstates of the Hamiltonian (3), which can be readily proved by the direct substitution of these states into the eigenproblem equation. As a result, the eigenvalues of the Hamiltonian (3) are defined as
\[
\hat{H}_{\text{SMF}} | \xi_1, m \rangle = \left( \omega [m + 1] + \Omega_{m+1} \right) | \xi_1, m \rangle,
\]
\[
\hat{H}_{\text{SMF}} | \xi_2, m \rangle = \left( \omega [m + 1] - \Omega_{m+1} \right) | \xi_2, m \rangle.
\]
To get the Eq. (17), we use the expressions for the operators and states determined in the BS basis and the obvious algebraic expression that stems from Eq. (13), \( u_n^2 + v_n^2 = 1 \). In addition, we want to emphasize that the probability amplitudes, \( c_{n,m} \), do not play any role in obtaining the solution.

Thus, we implement the diagonalization approach for the JCM Hamiltonian (2) using the property of a Pauli operator and obtain the well-known solution \([30]\). It should be noted that, in the \( \hat{\sigma}_z \)-representation, the operator \( \hat{\sigma}_z \) is the energy exchange integral and commutes with the JCM Hamiltonian \([30]\). In the \( \hat{\sigma}_z \)-representation, the states (15) are also the eigenstates of the JCM Hamiltonian, which is diagonal in this representation. We prove this statement for the states \( |\psi_{i,2}, m\rangle \)

\[
\hat{H}_{\text{JCM}} |\psi_i, m\rangle = \omega (m + 1) |\psi_i, m\rangle + \Omega_{m,i} |\psi_{i}, m\rangle \\
= \left[ \omega (m + 1) + \Omega_{m,i} \right] |\psi_i, m\rangle, \\
\hat{H}_{\text{JCM}} |\psi_j, m\rangle = \omega (m + 1) |\psi_j, m\rangle - \Omega_{m,i} |\psi_{j}, m\rangle \\
= \left[ \omega (m + 1) - \Omega_{m,i} \right] |\psi_j, m\rangle,
\]

Here, we treat the states with the different signs before the state \( |\xi_2, m\rangle \) as the same states \([29]\) since the probability to find the system in the certain DS is determined by the squared probability amplitude.

Now we utilize the proposed approach to find the eigenvalues and eigenstates of the QRM Hamiltonian, which we present in a matrix form as

\[
\hat{H}_{\text{QRM}} = \begin{pmatrix}
\omega \hat{b}^\dagger \hat{b} + \frac{\Delta}{2} & g_c (\hat{b} + \hat{b}^\dagger) \\
g_c (\hat{b} + \hat{b}^\dagger) & \omega \hat{b}^\dagger \hat{b} - \frac{\Delta}{2}
\end{pmatrix}.
\]

The diagonalized form of the Hamiltonian (19) looks like

\[
\hat{H}_{\text{QRM}}' = \omega \hat{\sigma}_z + \Omega \hat{\sigma}_x^2, \\
\hat{\sigma}_x^2 = \frac{\Delta^2}{4\Omega^2} + \frac{g_c^2}{\Omega^2} \left[ \hat{b}^\dagger \hat{b} + \hat{b} \hat{b}^\dagger \right]
\]

while successive application of the operator twice gives

\[
\hat{\sigma}_x^2 = \left\{ \frac{\Delta^2}{4\Omega^2} + \frac{g_c^2}{\Omega^2} \left[ \hat{b}^\dagger \hat{b} + \hat{b} \hat{b}^\dagger \right] \right\} |\phi_1\rangle \langle \phi_1 | + \left\{ \frac{\Delta^2}{4\Omega^2} + \frac{g_c^2}{\Omega^2} \left[ \hat{b}^\dagger \hat{b} + \hat{b} \hat{b}^\dagger \right] \right\} |\phi_2\rangle \langle \phi_2 |.
\]

In contrast to the case of the JCM, in the considered case, it is possible to use a state presented as the linear superposition of the BSs with the same number of the field quanta

\[
|\eta\rangle = \sum_n \left( c_{1,n} |\phi_1, m\rangle + c_{2,n} |\phi_2, m\rangle \right). 
\]

The equation, which is similar to Eq. (6), allow us to determine the condition for the eigenvalues of the Hamiltonian (20) in the following form

\[
\frac{\Delta^2}{4\Omega^2} + \frac{g_c^2}{\Omega^2} (2m + 1) = 1.
\]

Using the denotation (9), we can write \( \Omega = \Omega_{2m+1} \) and find the following relation

\[
\Omega_{2m+1}^2 = \Omega_{2m+1}^2 + \Omega_m^2 - \frac{\Delta^2}{4},
\]

which we will utilize in the further calculations.

Now the problem for the eigenstates of the operator \( \hat{\sigma}_x \) can be written in the BS basis using Eq. (22)

\[
|\eta\rangle = \sum_n \left( c_{1,n} |\phi_1, m\rangle + c_{2,n} |\phi_2, m\rangle \right).
\]

For further transformation, we use the denotations (13) and take the relation

\[
\frac{\Delta}{\Omega_{2m+1}} = \frac{\Omega_m}{\Omega_{2m+1}} \frac{\Delta}{\Omega_m}.
\]

As a result, we find

\[
c_{1,n} \left( \Omega_{2m+1} |\psi_1, m\rangle + g_c \sqrt{m} |\psi_2, m - 2\rangle \right) \\
+ c_{2,n} \left( \Omega_m |\psi_2, m - 1\rangle + g_c \sqrt{m + 1} |\psi_1, m + 1\rangle \right)
\]

\[
= c_{1,n} |\phi_1, m\rangle + c_{2,n} |\phi_2, m\rangle.
\]
In Eq. (27), we use the expressions (15) that help us to introduce the eigenstate basis of the operator $\hat{\sigma}_z^*$ in the following form

$$\begin{align*}
|\eta^+,m\rangle &= \frac{\Omega_{2n+1}}{\Omega_{2m+1}}|\psi_{1,m}\rangle + \frac{g_n\sqrt{m}}{\Omega_{2n+1}}|\psi_{2,m-2}\rangle, \\
|\eta^-,m\rangle &= \frac{g_n \sqrt{m+1}}{\Omega_{2n+1}}|\psi_{1,m+1}\rangle + \frac{\Omega_{2m+1}}{\Omega_{2n+1}}|\psi_{2,m-1}\rangle. \quad (28)
\end{align*}$$

It should be noted that the orthogonality of the eigenstates (28) is due to the orthogonality of the SMF states with the different numbers of quanta. To find the eigenstates (28) in the DS basis, we can use the relations for $|\psi_{z,\pm}^+,m\rangle$ determined in Eq. (18) and rewrite Eq. (28) as

$$\begin{align*}
|\eta^+,m\rangle &= \frac{\Omega_{2n+1}}{\Omega_{2m+1}}\left(u_{n-1}|\xi^+,m\rangle - sv_{n-1}|\xi^-,m\rangle\right) \\
&\quad + \frac{g_n \sqrt{m}}{\Omega_{2n+1}}\left( sv_{n-1}|\xi^+,m-2\rangle - u_{n-1}|\xi^-,m-2\rangle\right), \\
|\eta^-,m\rangle &= \frac{g_n \sqrt{m+1}}{\Omega_{2n+1}}\left(u_{n+1}|\xi^+,m+1\rangle + sv_{n+1}|\xi^-,m-1\rangle\right) \\
&\quad + \frac{\Omega_{2m+1}}{\Omega_{2n+1}}(sv_{n}|\xi^+,m-1\rangle + u_{n}|\xi^-,m-1\rangle). \quad (29)
\end{align*}$$

In the $\hat{\sigma}_z$-representation, the QRM Hamiltonian (19) and the operator $\hat{\sigma}_z^*$ do not commute in contrast to the JCM case since $[\hat{\sigma}_z, \hat{\sigma}_z^*] \neq 0$. In the $\hat{\sigma}_z^*$-representation, the operators $\hat{H}_{\text{QRM}}^\prime$ and $\hat{\sigma}_z^*$ commute, and the states (28) are also the eigenstates of the Hamiltonian (20). Hence, we can define the Hamiltonian eigenvalues as

$$\begin{align*}
\hat{H}_{\text{QRM}}^\prime|\eta^+,m\rangle &= \left(\omega m + \Omega_{2m+1}\right)|\eta^+,m\rangle + \omega|\eta^-,m\rangle \\
&= \omega(m+1)|\eta^+,m\rangle, \\
\hat{H}_{\text{QRM}}^\prime|\eta^-,m\rangle &= \left[\omega(m+1) - \Omega_{2m+1}\right]|\eta^-,m\rangle + \omega|\eta^+,m\rangle \\
&= \omega(m+2)|\eta^-,m\rangle. \quad (30)
\end{align*}$$

Here, we use the denotation $|\eta,m\rangle = |\eta^+,m\rangle$, which is explained earlier after Eq. (18).

It is convenient to analyze the results obtained in the DS basis using Eqs. (29). These equations show the probability distribution both between the DS manifolds and between the DSSs inside the manifolds. Since the states $|\eta^+,m\rangle$ and $|\eta^-,m\rangle$ are orthogonal, we can find the system only in one of these states. Let us consider the first equation of Eqs. (29). With low values of the coupling strength, $g_n \ll \omega$, the contribution of the $m$-manifold to the $|\eta,m\rangle$ prevails under the contribution of the $(m-2)$-manifold, and we get the JCM results. If $g_n \sim \omega$, the contribution of the $(m-2)$-manifold becomes substantial. The energy spectrum of the QRM Hamiltonian is determined by the Rabi frequency which includes odd numbers of field quanta and consists of the contributions from the $m$ and $(m-1)$ DS manifolds [see Eq. (24)].

The energy spectra of the JCM and the QRM are compared in Fig. 1, where the dependencies of the first three eigenstates on the coupling strength are depicted for both models. It is clearly seen from Fig. 1 that the dependencies for the QRM change faster than for the JCM. However, two QRM eigenvalues coincide with the two JCM eigenvalues, namely the eigenvalues of $|\eta_1,0\rangle$ and $|\eta_1,1\rangle$ states coincide with the eigenvalues of $|\psi_{1,0}\rangle$ and $|\psi_{1,2}\rangle$ states, correspondingly. In addition, the energy differences between the eigenvalues for the $|\eta_1,m\rangle$ states sharply decrease. This situation leads to a degeneracy of the energy spectrum in the case when the coupling constant becomes compared with the SMF frequency.

It should be stressed that in our approach we use the Pauli operator properties to find the diagonalized Hamiltonian. In such an approach, the probability amplitudes, $c_{i,m}$, do not participate in obtaining the results. In contrast to our approach, the previous results [27, 28] are based on the procedure of determining the probability amplitudes of the SMF states, $|m\rangle$, in the parity operator representation. The appropriate equations for these amplitudes can be readily derived from Eq. (25). The eigenvalues obtained in this case are compared with the JCM results in Ref. [27]. This comparison can be generalized to the results obtained in this Letter.

FIG. 1. (Color online) Comparison of the energy spectra of the JCM [dashed (red) lines] and the QRM [solid (blue) lines] for the case of $m = 0, 1, 2$, and $\Delta = 0.4$. 

[Graph showing energy spectra comparison]
In conclusion, we determine the eigenvalues and eigenstates of the QRM in the representation of the z Pauli operator using its unitary dynamics. To demonstrate the applicability of the used approach, we first solve the eigenproblem for the JCM Hamiltonian and obtain the well-known results. Then, we diagonalize the QRM Hamiltonian using the proposed approach. The eigenstates of the QRM are presented in the basis of the JCM eigenstates.

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[16] L. Cong, S. Felicetti, J. Casanova, L. Lamata, E. Solano, and I. Arrazola, Selective interactions in the quantum Rabi model, Phys. Rev. A 101, 032350 (2020).
[17] S. Mahmoodian, Chiral Light-Matter Interaction beyond the Rotating-Wave Approximation, Phys. Rev. Lett. 123, 133603 (2019).
[18] Q. Xie, H. Zhong, M. T. Batchelor, and C. Lee, The quantum Rabi model: solution and dynamics, J. Phys. A: Math. Theor. 50, 113001 (2017).
[19] A. Le Boite, Theoretical Methods for Ultrastrong Light–Matter Interactions, Adv. Quantum Technol. 3, 1900140 (2020).
[20] E. T. Jaynes, F. W. Cummings, Comparison of quantum and semiclassical radiation theories with application to the beam maser, Proc. IEEE 51, 89 (1963).
[21] J. Hausinger, M. Grifoni, Qubit-oscillator system: An analytical treatment of the ultrastrong coupling regime, Phys. Rev. A 82, 062320 (2010).
[22] H. R. Jauslin, S. Guérin, S. Thomas, Quantum averaging for driven systems with resonances, Physica A 279, 432 (2000).
[23] E.K. Irish, J. Gea-Banacloche, I. Martin, K.C. Schwab, Dynamics of a two-level system strongly coupled to a high-frequency quantum oscillator, Phys. Rev. B 72, 195410 (2005).
[24] Y.Y. Zhang, Q.H. Chen, Generalized rotating-wave approximation for the two-qubit quantum Rabi model, Phys. Rev. A 91, 013814 (2015).
[25] N. Rivera, J. Flick, P. Narang, Variational Theory of Nonrelativistic Quantum Electrodynamics, Phys. Rev. Lett. 122, 193603 (2019).
[26] I. D. Feranchuk, L. I. Komarov and A. P. Ulyanenkov, Two-level system in a one-mode quantum field: numerical solution on the basis of the operator method, J. Phys. A: Math. Gen. 29, 4035(1996).
[27] D. Braak, Integrability of the Rabi Model, Phys. Rev. Lett. 107, 100401 (2011).
[28] Q. H. Chen, C. Wang, S. He, T. Liu, and K. L. Wang, Exact solvability of the quantum Rabi model using Bogoliubov operators, Phys. Rev. A 86, 023822 (2012).
[29] W. H. Louisell, Radiation and Noise In Quantum Electronics (McGraw Hill, New York, 1964).
[30] M. Scully and M. Zubairy, Quantum optics (Cambridge University Press, 1997).