Gribov Ambiguity and Degenerate Systems

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Outline

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Gribov Ambiguity

- Generating functional for Yang-Mills Theory

\[ Z = \int DA \, e^{-iS} \]

- Action

\[ S = -\frac{1}{4} \int d^4 x \, \text{tr} \left[ F^{\mu \nu} F_{\mu \nu} \right] \]

where \( F^{\mu \nu} \) the field strength associated to \( A_\mu = A^a_\mu T_a \)

- To avoid overcounting we must fix the gauge \( G^a [A_\mu] = 0 \)
The restriction is carried out using the Fadeev-Popov method

\[ Z = \int DA \delta(G^a [A_\mu]) \det\mathcal{M} e^{iS}, \quad \mathcal{M}^a_b(x, y) = \frac{\delta G^a [A^g_\mu (x)]}{\delta \alpha^b (y)} \]

Coulomb gauge does not fix the gauge completely \(\implies\) Gribov copies \([\text{Gribov (1978)}]\)

Same for all gauge fixing conditions \([\text{Singer(1978)}]\).
The condition for this to happen is

\[ G^a \left[ g^{-1} A_\mu g + g^{-1} \partial_\mu g \right] = 0, \quad g \neq 1 \]

Infinitesimal gauge transformations, \( \delta A_\mu = D_\mu \alpha \)

\[ G^a \left[ \left( A_\mu + D_\mu \alpha \right) \right] = 0 \]

\[ \implies \int d^4y M^a_{\mu \nu} (x, y) \alpha^b (y) = 0 \]

Infinitesimal Gribov copies \( \rightarrow \) zero modes of the Faddeev-Popov operator

The functional integral \( Z \) is ill-defined
Gribov proposed to restrict the path integral to the Gribov region

\[ C_0 \equiv \{ A_\mu, G^a[A_\mu] = 0 \mid \det \mathcal{M} > 0 \} \]

\( C_0 \) is bounded and convex \([\text{van Baal (1992)}]\)

All orbits intersect the Gribov region \([\text{Dell’Antonio, Zwanziger (1991)}]\)
Gribov Ambiguity

- The restriction can be implemented in the form

\[ Z_G = \mathcal{N} \int DA \delta (\partial^\mu A_\mu) \det (\mathcal{M}) \exp (-S_{YM}) \mathcal{V} (C_0) \]

- The factor \( \mathcal{V} (C_0) \) ensures integration only over \( C_0 \).

- Gluon propagator is modified: \( D^{ab}_{\mu\nu} (q) = \delta^{ab} g_0^2 \frac{q^2}{q^4 + \gamma^4} \left( \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \).

  [Gribov (1978)]

- Imaginary poles \( \rightarrow \) gluons are not in the spectrum \( \rightarrow \) Confinement

- Studies at finite temperature show a critical \( T \) for which imaginary poles disappear [Canfora, Pais, Salgado-Rebolledo (2014)]

- Restriction to the Gribov horizon can be properly implemented to match with lattice results [Sorella et al (2008)]
Degenerate Systems

- Hamiltonian Systems $\rightarrow$ Symplectic geometry
- Symplectic manifold $= (M, \Omega)$
  \[ \Omega = dA \]
- First order action
  \[ L = A_A \dot{z}^A - H \]
- Poisson Bracket $= \text{Inverse of } \Omega$
  \[ \{ z^A, z^B \} = \Omega^{AB} \]
- Euler-Lagrange equations
  \[ \Omega_{AB} \dot{z}^A = \partial_B H \]
- $\det \Omega \neq 0 \Longrightarrow \text{Regular systems}$
Degenerate Systems

- \( \text{det} \Omega = 0 \) with fixed rank \( \implies \) Local Symmetries
- \( \text{det} \Omega = 0 \) and non-constant rank \( \implies \) Degenerate systems.

\[ \Omega_{AB} \dot{z}^A = \partial_B H \]

- Degeneracy surfaces \( \Sigma = \{ z \in \Gamma / \text{det} \Omega = 0 \} \)

- Divide phase space into dynamical disconnected regions [Saavedra, Troncoso, Zanelli (2001)]

- The measure for the Hilbert space vanishes at the degeneracy surfaces [de Michelli, Zanelli (2012)]
Gribov Ambiguity as Degeneracy

- Consider a system with a finite number of degrees of freedom and a local symmetry.

\[ S = \int dt \, L(x) \]
\[ \delta S = 0 \quad \text{for some } \delta x \]

- Local symmetry \(\rightarrow\) constraints.
- In the Hamiltonian formalism there are primary constraints

\[ \varphi_m(x) \approx 0 \]

- Dirac Formalism: Preservation in time of these can lead to secondary constraints, tertiary constraints, etc.
Gribov Ambiguity as Degeneracy

- They can be classified in first and second class
  
  \[ \varphi_M = (\phi_i, \gamma_\alpha) \]

- First class constraints = generators of the local symmetries
- Second class constraints can be eliminated by implementing Dirac brackets
  
  \[ \{ F, G \}^* = \{ F, G \} - \{ F, \gamma_\alpha \} C^{\alpha\beta} \{ \gamma_\beta, G \} \]
  where
  
  \[ C^{\alpha\beta} = \{ \gamma_\alpha, \gamma_\beta \} \]

- Quantization \(\rightarrow\) fix the gauge \(\rightarrow\) extra constraints \(G_i\) such that first class constraints become second class.

  \[ \gamma_I = (\phi_i, G_j) \]

- Defining Dirac brackets we can set all the constraints to zero strongly.
Proper gauge fixing:

1. Accessibility
2. Complete gauge fixation [Henneaux, Teitelboim (1992)]

Dirac brackets $\longrightarrow$ Symplectic structure of the reduced phase space.

$$\{ y^a, y^b \}^* = \Omega_{red}^{ab}$$

$$\Omega_{red} = \frac{1}{2} \Omega_{red}^{ab} dy^a \wedge dy^b$$

We can redefine the Dirac matrix by defining $\gamma_I \rightarrow \tilde{\gamma}_I = V_{IJ} \gamma_J$

$$\tilde{C} = V^T CV = \begin{pmatrix} & & & & 1 \\ & & & \cdots & \\ & & 1 & & \\ & \cdots & & & \\ -1 & & & & \end{pmatrix}$$
In other words we use new coordinates $z^A = (\bar{\gamma}_I, y^a)$.

Implementing the constraints strongly, the path integral in Hamiltonian form is

$$Z = \int D\gamma e^{iS} = \int D\gamma \prod I \delta (\bar{\gamma}_I) e^{iS}$$

Turning back to the old variables

$$Z = N \int Dx \prod I \delta (\gamma_I) \det \{ G_i, \phi_j \} e^{iS}$$

$\det \{ G_i, \phi_j \}$ is identified with the Faddeev-Popov determinant and

$$\mathcal{M}_{ij} = \{ G_i, \phi_j \}$$

If the system has Gribov ambiguity then

$$\det \{ G_i, \phi_j \} = 0 \text{ at the Gribov horizon}$$
Gribov Ambiguity as Degeneracy

- Dirac matrix

\[ C_{IJ} = \{\gamma_I, \gamma_J\} = \begin{pmatrix} \{G_i, G_j\} & M_{ij} \\ -M_{ij} & \{\phi_i, \phi_j\} \end{pmatrix}. \]

- Therefore \( \det C \approx (\det M)^2 \)

- In the new coordinates

\[ \{z^A, z^B\} = \begin{pmatrix} \{y^a, y^b\} \\ 0 \\ 0 \end{pmatrix} C_{IJ} \]

\[ \det \Omega^{-1} = \det \Omega_{red}^{-1} (\det M)^2 \]

- \( \Omega \) regular \( \implies \det \Omega_{red}^{-1} \) blows up at the Gribov horizon

\[ \implies \det \Omega_{red} = 0 \] at the Gribov horizon

- **Theorem:** In the presence of Gribov ambiguity the reduced system is degenerate [Canfora, de Michelli, Salgado-Rebolledo, Zanelli (2015)].
FLPR Model

- Solvable model [Friedberg, Lee, Pang, Ren (1995)].

\[ L = \frac{1}{2} \left( (\dot{x} + \alpha yq)^2 + (\dot{y} - \alpha xq)^2 + (\dot{z} - q)^2 \right) - V(\rho) \]

- Canonical momenta

\[
\begin{align*}
p_x &= \frac{\partial L}{\partial \dot{x}} = \dot{x} + \alpha yq, \\
p_y &= \frac{\partial L}{\partial \dot{y}} = \dot{y} - \alpha xq, \\
p_z &= \frac{\partial L}{\partial \dot{z}} = \dot{z} - q, \\
p_q &= \frac{\partial L}{\partial \dot{q}} = 0
\end{align*}
\]

- First class constraints

\[ \varphi = p_q \approx 0 \]

\[ \phi = p_z + \alpha (xp_y - yp_x) \approx 0 \]
\( \phi \) generates helicoidal orbits 
\[ \delta_\phi(x, y, z, q) = \epsilon(t)(-\alpha y, \alpha x, 1, 0) \]

- Gauge condition

\[ G = z - \lambda x \approx 0 \]

- \( G \) presents Gribov Ambiguity 
\[ \mathcal{M} = \{ G, \phi \} = 1 + \alpha \lambda y \]
The pair $G, \phi$ is second class everywhere, except at the Gribov horizon

$$\Xi = \{(x, p_x, y, p_y, z, p_z) \in \Gamma | M = 0\}$$
FLPR Model

- Second class constraints \( \{G, \phi\} \)

\[
\begin{align*}
\gamma_I : \quad & \gamma_1 = G = z - \lambda x, \\
& \gamma_2 = \phi = p_z + \alpha (xp_y - yp_x)
\end{align*}
\]

- Setting constraints strongly equal to zero \( \rightarrow z \) and \( p_z \) eliminated from phase space

- Dirac matrix

\[
C_{ij} = \begin{pmatrix}
0 & M \\
-M & 0
\end{pmatrix}
\]

- Dirac brackets

\[
\begin{align*}
[x, p_x]^{*} &= \frac{1}{M}, \\
[x, y]^{*} &= 0, \\
[x, p_y]^{*} &= 0, \\
[y, p_y]^{*} &= 1, \\
[y, p_x]^{*} &= \frac{\alpha \lambda x}{M}, \\
[p_x, p_y]^{*} &= -\frac{\alpha \lambda p_x}{M}
\end{align*}
\]
FLPR Model

- Reduced symplectic form is

\[
\omega_{ab} = \begin{pmatrix}
0 & -\mathcal{M} & -\alpha \lambda p_x & \alpha \lambda x \\
-\mathcal{M} & 0 & 0 & 0 \\
\alpha \lambda p_x & 0 & 0 & -1 \\
-\alpha \lambda x & 0 & 1 & 0
\end{pmatrix}.
\]

- Closed but degenerates precisely at the Gribov horizon

\[
\text{det}[\omega_{ab}] = \mathcal{M}^2
\]

\[
\Sigma = \{(x, p_x, y, p_y) \in \Gamma_0|\, \Upsilon(u) \equiv \mathcal{M} = 0\}
\]
The degeneracy surface divides phase space into dynamically disconnected regions

\[ C_+ := \{(x, y, z) \mid z - \lambda x = 0, 1 + \alpha \lambda y > 0\} , \]
\[ C_- := \{(x, y, z) \mid z - \lambda x = 0, 1 + \alpha \lambda y < 0\} . \]
Conclusions

- We have studied Gribov ambiguity from a Hamiltonian point of view.
- It has been shown that, for finite dimensional systems, the presence of Gribov copies implies a degeneracy for the reduced phase space.
- We have studied the FLPR model and found the degenerate reduced symplectic form in the presence of a Gribov horizon.
- The degeneracy surface divides phase space into dynamically disconnected regions.
- This suggests that the restriction to the Gribov horizon in QCD is natural.
Future Directions

- To look for explicit degeneracies in the symplectic form for Yang-Mills theories after gauge fixing [WORK IN PROGRESS]
- In Yang-Mills theory the canonical momenta associated to the gauge field $A^a_{\mu}$ is
  \[
  \Pi^\mu_a = \frac{\partial L}{\partial \left( \dot{A}^a_{\mu} \right)} = F^\mu_0.
  \]
- There is a primary constraint
  \[
  \phi^0_0 = \Pi^0_a \approx 0
  \]
- The canonical Hamiltonian is given by
  \[
  H = \int d^3x \left( \dot{A}^a_i \Pi^i_a - L \right) = \int d^3x \left( \mathcal{H}_0 + A^a_i (D_i)_a^b \Pi^i_b \right)
  \]
  where
  \[
  \mathcal{H}_0 = \frac{1}{2} \Pi^i_a \Pi^a_i + \frac{1}{4} F^a_{ij} F^i_a
  \]
Future Directions

- Total hamiltonian
  \[ H_T = H + \int d^3x \mu^a \phi^0_a \]

- Preservation in time of the primary constraint leads to
  \[ \phi_a = - (D_i)_a^b \Pi^i_b \approx 0 \]

- The set \( \{ \phi^0_a, \phi_a \} \) is first class.

- Eliminating \( \phi^0_a \) and \( A^a_0 \) the extended action
  \[ S_E = \int dx^0 \int d^3x (\dot{A}^i_a \Pi^a_i - \mathcal{H}_0 - \lambda^a \phi_a) \]
  is invariant only under the transformations generated by \( \phi_a \)
  \[ \delta A^a_i (x) = \int d^3y \epsilon^b (y) \{ A^a_i (x), \phi_b (y) \} = (D_i)_a^b \epsilon^b (x) \]
The first class constraints satisfy \( \{ \phi_a, \phi_b \} = f^c_{ab} \phi_c \)

To fix the gauge we choose the Coulomb condition \( G^a = \partial^i A^a_i \approx 0 \)

Now the set \( \gamma_A = (\phi_a, G^b) \) is second class

Dirac matrix

\[
C_{AB}(x, y) = \begin{pmatrix}
0 & -\partial^i (D_i)^a_b \delta^3 (x - y) \\
\partial^i (D_i)^a_b \delta^3 (x - y) & 0
\end{pmatrix}
\]

Eigenvalue equation

\[
-\partial^i (\delta^a_b \partial_i + if^a_{cb} A^c_i) \alpha^b = \epsilon (A_i) \alpha^a.
\]
Future Directions

- For vanishing gauge potentials \(-\partial^i \partial_i \alpha^a = \epsilon \alpha^a\) has positive eigenvalues \(\epsilon = p^2\)
- For small enough gauge fields \(A^a_i\) there are only positive eigenvalues
- For sufficiently large gauge fields, a zero mode \(\epsilon = 0\) can appear
- This will be a zero mode of the Dirac Matrix and of the reduced symplectic form
- Set the constraints strongly to zero and evaluate Dirac brackets
- Compute the reduced phase space symplectic form and look for degeneracies
- Generalization for the theory at finite temperature
- In the finite temperature case the degeneracy should disappear at some critical temperature
Thank You!