FAMILIES OF SHORT CYCLES ON RIEMANNIAN SURFACES

YEVGENY LIOKUMOVICH

Abstract. Let $M$ be a closed Riemannian surface of genus $g$. We construct a family of 1-cycles on $M$ that represents a non-trivial element of the $k$’th homology group of the space of cycles and such that the mass of each cycle is bounded above by $C \max \{\sqrt{k}, \sqrt{g}\} \sqrt{\text{Area}(M)}$. This result is optimal up to a multiplicative constant.

0.1. Introduction. Let $M$ be a closed Riemannian 2-dimensional manifold and let $Z_1(M, \mathbb{Z}_2)$ denote the space of mod 2 flat 1-cycles in $M$. Let $Z_1^0$ denote the connected component of $Z_1(M, \mathbb{Z}_2)$ consisting of all null-homologous cycles in $M$. It follows from the work of Almgren [1] that $Z_1^0$ is weakly homotopy equivalent to the Eilenberg-MacLane space $K(\mathbb{Z}_2, 1) \simeq \mathbb{RP}^\infty$. We say that a family of cycles $f : \mathbb{RP}^k \to Z_1^0$ is a $k$-sweepout if it represents the non-zero element of the $k$’th homology group $H_k(Z_1^0, \mathbb{Z}_2) \simeq \mathbb{Z}_2$.

Here is the main result of this paper.

Theorem 0.1. Let $M$ be a 2-dimensional closed Riemannian manifold of genus $g$. For each $k$ there exists a $k$-sweepout $Z_k = \{z_t\}_{t \in \mathbb{RP}^k}$ of $M$, such that for each $t \in \mathbb{RP}^k$ the mass of $z_t$ is bounded above by $1600 \max \{\sqrt{k}, \sqrt{g}\} \sqrt{\text{Area}(M)}$.

$k$-sweepouts have been studied by Gromov in [9], [11] and [12] and by Guth in [14]. More recently, in [21] Coda Marques and Neves used $k$-sweepouts to prove existence of infinitely many minimal hypersurfaces in manifolds of positive Ricci curvature. In [6] Glynn-Adey and the author obtained upper bounds for volumes of these hypersurfaces.

In the case of surfaces Balacheff and Sabourau [2] constructed a sweepout of $M$ by 1-cycles of mass bounded by $C \sqrt{(g+1)\text{Area}(M)}$. This corresponds to the case $k = 1$ of Theorem 0.1. Different proofs of their result, improving the value of an upper bound for the constant $C$, were given in [20], [6]. The proof of Balacheff and Sabourau relies on the estimate of Li and Yau [19] for the first eigenvalue of the Laplacian. In this paper we give an elementary construction of $k$-sweepouts using only the thin-thick decomposition of hyperbolic surfaces and the length-area method.
The upper bound in Theorem 0.1 is optimal up to a constant. Brooks constructed examples of closed hyperbolic surfaces of arbitrarily large genus such that any 1-sweepout of $\Sigma_g$ must contain a cycle of mass greater than $c\sqrt{g}$ for some $c > 0$. On the other hand, Gromov showed in [9] that a $k$-sweepout of the round $n$-sphere by $(n-1)$-cycles must contain a cycle of mass greater than $ck^{\frac{1}{n}}$ for a constant $c > 0$. To prove this Gromov observed that if $\{U_i\}$ is a collection of $k$ disjoint measurable subsets in $M$ and $z_t$ is a $k$-sweepout, then there will be a cycle $z_t$ that separates each $U_i$ into two subsets of equal area. Gromov’s arguments were generalized and extended by Guth in [15]. In that paper Guth proves nearly optimal lower and upper bounds for all homology classes of the space of mod 2 $m$-dimensional cycles on the $n$-dimensional round sphere.

In [9] Gromov suggested that finding bounds on the maximal mass of a cycle in an optimal $k$-sweepout can be thought of as a non-linear analogue of the spectral problem on $M$. Arguments in our paper, especially the use of the length-area method, were inspired by and are similar to the estimates for the eigenvalues of the Laplace operator on Riemannian manifolds in the works of Hersch [17], Yau [23], Yang and Yau [22], Korevaar [18], Gromov [10], Grigoryan, Netrusov and Yau [7], Colbois and Maertens [4], and Hassannezhad [16].

Acknowledgements. I am grateful to Misha Gromov for explaining the connection between $k$-sweepouts and spectral problems and for suggesting methods of Hersch [17] and Korevaar [18] for the kind of problems considered in this paper. I would like to thank my advisers Alexander Nabutovsky and Regina Rotman for many very valuable discussions and for important comments on the first draft of this paper.

0.2. Outline of the proof. Let $M$ be a surface of area 1. Suppose we can cover $M$ by $k$ sets $U_i$ with piecewise smooth boundary and disjoint interiors, each of area $\sim \frac{1}{k}$, and such that the boundary length of each set is $\sim \frac{1}{\sqrt{k}}$. Assume furthermore that for each $U_i$ there exists a 1-sweepout of $U_i$ by cycles of length at most $\sim \frac{1}{\sqrt{k}}$. We can now sweep out all of $M$ as follows. First we sweep out $U_1$, starting on a 0-cycle and ending on the boundary of $U_1$. We hold cycle $\partial U_1$ fixed and start adding to it a sweepout of $U_2$ and so on. Eventually cycles in the boundaries of $U_i$’s will overlap and cancel out.

Denote this sweepout of $M$ by $z_t$ and consider a cycle $z = \sum_{i=1}^{k} z_{t_i}$, where $\{t_i\}$ are $k$ different moments of time. Each $z_{t_i}$ can be decomposed into two parts: one that lies in $\bigcup \partial U_i$ and one that is contained in only one of the sets $U_i$ and has mass at most $\sim \frac{1}{\sqrt{k}}$. Since the cycles are mod 2, the parts that lie in $\bigcup \partial U_i$ will cancel out, so mass$(z) \lesssim \sqrt{k}$. There exists a $k$-sweepout of $M$ that consists of cycles like $z$ and therefore satisfies the desired upper bound.
The idea described above was successfully used by Gromov and Guth to bound volumes of \(k\)-sweepouts in various contexts.

If \(M\) is topologically a 2-sphere then one can find a covering of \(M\) by \(k\) sets as described above. But if the surface has genus greater than \(k\) then it may happen that every collection of \(k\) open sets of approximately equal areas that cover \(M\) have large length of the boundary and some of these open sets do not admit a sweepout by short cycles.

Instead we will first cover \(M\) by \(\sim g\) ‘good regions’ \(V_i\) (where \(g\) is the genus). These regions can have arbitrary areas, but they have the following nice properties:

1. There exists a sweepout of \(V_i\) by cycles of length at most \(\sim \sqrt{\text{Area}(V_i)}\)
2. We can subdivide \(V_i\) into \(m\) (where \(m\) is any positive integer) subsets of approximately equal areas, such that the length of the union of their boundaries is at most \(\sim m\sqrt{\text{Area}(V_i)} + l(\partial V_i)\)

So for our purposes these good regions are as good as subsets of the sphere. We will then subdivide them into subsets of the right area.

To obtain these good regions we use uniformization theorem and the length-area method. By uniformization theorem a surface of genus \(g \geq 2\) is conformally equivalent to a hyperbolic surface. R. Buser used thin-thick decomposition to construct a tessellation of a hyperbolic surface by polygons of approximately equal areas with some special properties. The thin part of the surface in this tessellation is covered by long and narrow rectangles and the thick part is covered by triangles that are close to equilateral triangles. For us the most important thing about this tessellations is that every polygon contains at most \(c\) other polygons in its \(1/2\)-neighbourhood. Our good regions will be those that are covered by at most \(c\) polygons from this tessellation.

To control lengths of the boundaries of good regions we observe that if a family of concentric geodesic circles (i.e. level sets of the distance function) on the hyperbolic surface (conformal to our surface \(M\)) covers a set of small area, when measured with the original (non-hyperbolic) metric, then some of these circles must be short in the original metric. This is a classical observation sometimes called the length-area method (see Section 0.4). We use it to find short cycles on \(M\) in \(1/2\)-neighbourhood of a polygon from the hyperbolic tessellation. Actually, the length of the boundary of each individual good region in our construction may still be comparatively long, but the total length of the union of their boundaries will be at most \(\sim \max\{\sqrt{g}, \sqrt{k}\}\). This is still sufficient to bound lengths of \(k\)-sweepouts using the argument described above.

Here’s the plan of the paper. In Section 0.3 we define \(k\)-sweepouts and a technical notion of monotone sweepouts. These sweepouts have a nice property that it is easy to glue two short monotone sweepouts of adjacent regions into a short monotone
sweepout of their union. In Section 0.4 we use the length-area method to prove a key lemma for finding subsets of $M$ with small length of the boundary. In Section 0.5 we describe Buser’s tessellation $\mathcal{T}$ of a hyperbolic surface by quadrilaterals and triangles. In Section 0.6 we describe Guth’s construction of sweepouts of open subsets of $\mathbb{R}^2$. We use this result as the base of induction in the proof that a subset of $M$ of very small area admits a sweepout by short cycles. In Section 0.7 we prove that if a subset $U$ of $M$ can be covered by at most 40 elements of $\mathcal{T}$, then it admits a sweepout by cycles of length at most $\sqrt{\text{Area}(U)}$. In Section 0.8 we construct a covering of $M$ by sets that are contained in at most 40 elements of $\mathcal{T}$ and have area at most $\frac{\text{Area}(M)}{k}$ and finish the proof of the theorem.

0.3. Preliminaries. For the definition of the space of mod 2 cycles with flat metric we refer the reader to [5] or a concise description in [2, Section 2], which will be sufficient for our purposes.

In [1] Almgren constructed maps from homotopy groups of the integral cycle space $\pi_k(\mathbb{Z}(M^n, \mathbb{Z}); 0)$ to homology groups of the manifold $H_{k+m}(M^n, \mathbb{Z})$ and proved that these maps are isomorphisms for all non-negative integers $k$ and $m$. Almgren’s proof works for $\mathbb{Z}_2$ coefficients as well. For a surface $M$ we have an isomorphism $\pi_k(\mathbb{Z}_2(M, \mathbb{Z}_2); 0) \cong H_{k+1}(M, \mathbb{Z}_2)$. Since homology groups of $M$ are zero for $k > 1$, the connected component $Z_1^n$ of $Z_1(M, \mathbb{Z}_2)$, $0 \in Z_1^n$, is weakly homotopy equivalent to the Eilenberg-MacLane space $K(\mathbb{Z}_2, 1) \simeq \mathbb{RP}^\infty$.

For a surface $M$ Almgren’s map $F_A : \pi_1(\mathbb{Z}_1(M, \mathbb{Z}_2), 0) \to H_2(M, \mathbb{Z}_2)$ is defined as follows. Consider a loop $z_t : S^1 \to Z_1(M, \mathbb{Z}_2)$ representing some class of the fundamental group and pick a fine subdivision $\{t_1, \ldots, t_n\}$ of $S^1$. For each $t_i$ cycle $z_{t_i}$ can be approximated by a cycle that consists of a finite collection of Lipschitz circles. If $c_i$ and $c_{i+1}$ are two such approximations of $z_{t_i}$ and $z_{t_{i+1}}$, respectively, we can find an area minimizing chain $A_i$ with $\partial A_i = c_i - c_{i+1}$. We can then assemble chains $A_i$ into a 2-cycle that represents an element of $H_2(M, \mathbb{Z}_2)$. It turns out that if the subdivision and approximations are fine enough then the 2-cycle will represent the same element in the homology independent of the particular subdivision and approximations.

We say that $\{z_t\}_{t \in \mathbb{R}^1}$ is a sweepout (or 1-sweepout) of $M$ if loop $\{z_t\}$ is non-contractible in $Z_1(M, \mathbb{Z}_2)$, i.e. $F_A([z_t]) \neq 0$. More generally, we say that $\{z_t\}_{t \in \mathbb{R}^k}$ is a $k$-sweepout if it represents the non-zero element of $H_k(Z_1^n) \cong \mathbb{Z}_2$. The ring structure of $H^*(Z_1^n, \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]$, where $\alpha$ is the non-zero class of $H^1(Z_1^n, \mathbb{Z}_2)$, provides a useful criterion for when a family is a $k$-sweepout. We have that map $f : \mathbb{RP}^k \to Z_1^n$ is a $k$-sweepout if and only if the pull-back $f^*(\alpha^k) \neq 0$.

We will frequently need to consider sweepouts of manifolds with boundary. In this case we consider the space of cycles relative to the boundary and all definitions above carry over to this setting.
The 1-sweepouts that we construct in this paper are nicer than an arbitrary 1-sweepout. After a small perturbation different cycles in it will not intersect each other and one can turn them into level sets of a function $f : M \setminus \mathbb{R}$. We summarize this in the following definition.

**Definition 0.2.** Let $M$ be a Riemannian surface (possibly with boundary). Let $\text{int}(M)$ denote the interior of $M$. We say that $z_t$ is a monotone sweepout if $z_t$ is a sweepout of $M$ and for each $t$ cycle $z_t$ can be represented by a finite collection of points and piecewise smooth simple closed curves, which satisfy the following condition. There exists a family of nested subsets $A_t \subset M$, $A_{t'} \subset A_t$ for all $t' < t$, such that $z_t$ contains $\partial A_t \setminus \partial M$ and is contained in $\partial A_t$.

Since the cycles are nested and they can be glued into the fundamental class of $M$, it follows that $A_0$ is collection of points and $A_1$ is all of $M$. Below we use this property to concatenate sweepouts of two adjacent regions.

**Lemma 0.3.** Let $M$ be a Riemannian surface, possibly with boundary, and let $\gamma$ be a relative 1-cycle composed of finitely many piecewise smooth closed curves that have not self-intersections or pairwise intersections and separate $M$ into $M_1$ and $M_2$. Suppose there exist monotone sweepouts of $M_1$ and $M_2$ of length at most $L$. Then there exists a monotone sweepout of $M$ by cycles $z_t$, such that we can decompose $z_t$ as a sum of 1-chains $z^1_t + z^2_t$, where $l(z^1_t) \leq L + \epsilon$ and $z^2_t$ is contained in $\gamma$.

**Proof.** By definition of a monotone sweepout for each $i = 1, 2$ there exists a family $A^i_t$ of nested sets with $\text{int}(M_i) \cap \partial A^i_t = z^i_t \subset \partial A^i_t$. After a small perturbation that keeps $A^i_t$’s nested and increases lengths of cycles by at most $\epsilon$ we can assume that $\partial A^i_t$ will intersect $\gamma$ in a (possibly empty) finite collection of arcs and closed curves $I^i_t$ with $I^i_t \subset I^i_{t'}$ if $t < t'$.

Define $A_t = A^1_t$, for $t \in [0, \frac{1}{2}]$ and $A_t = A^1_t \cup A^2_{\frac{1}{2} + t}$, for $t \in (\frac{1}{2}, 1]$. We define sweepout $z_t = \partial A_t \cap \text{int}(M).$ For $t \leq \frac{1}{2}$ each cycle $z_t$ can be decomposed into a chain that is contained in $z^1_{\frac{1}{2}}$, and a chain $I^1_t \subset \gamma$. For $t > \frac{1}{2}$ cycle $z_t$ can be decomposed into a chain that is contained in $z^2_{\frac{1}{2} + t}$ and a chain $\gamma \setminus I^2_t$. □

0.4. **Length-area method.** Given a closed Riemannian surface $(M, h)$ by uniformization theorem there exists a conformal diffeomorphism $\phi : (M, h) \to (M, h_0)$ from $(M, h)$ to a surface of constant curvature $(M, h_0)$. This conformal equivalence will play a key role in our construction of parametric sweepouts. For a subset $U \subset M$ we will write $\mu_0(U)$ to denote its area with respect to metric $h_0$ and $\mu(U)$ to denote its area with respect to $h$. Similarly, we will write $d(x, y)$, $B(x, r)$ and $\nabla$ to denote distance function, closed metric ball of radius $r$ about $x$, and gradient with respect to $h$ and we let $d^0(x, y)$, $B^0(x, r)$, $\nabla^0$ denote the corresponding quantities with respect to $h_0$. 
A key tool in this paper is an old technique sometimes called the length-area method (see, for example, page 4 of [8]). It is based on the observation that the $n$’th power of the absolute value of the gradient of a function (where $n$ is the dimension of the space) times the volume element is a conformal invariant. Using this observation and coarea formula we can obtain the following lemma, which will be used throughout the paper.

Let $N^0_r(U)$ denote the set $\{x \in M : d^0(x, U) < r\}$

**Lemma 0.4.** For any $U \subset M$ and $r > 0$ there exists an $r'$, $0 < r' < r$, such that

$$l(\partial N^0_{r'}(U)) \leq \frac{\sqrt{\mu_0(N^0_r(U))}}{r} \sqrt{\mu(N^0_r(U))}.$$  

**Proof.** Let $f(x) = d^0(x, U)$. By Rademacher’s theorem $f$ is differentiable almost everywhere. By coarea formula we have

$$\int_{t=0}^{r} \mu(f^{-1}(t)) dt = \int_{N^0_r(U)} |\nabla f| d\mu$$  

By Cauchy-Schwartz inequality this quantity can be bounded by

$$\left( \int_{N^0_r(U)} |\nabla f|^2 d\mu \right)^{1/2} \mu_0(N^0_r(U))^{1/2}$$  

We observe that $|\nabla f|^2 dV$ is a conformal invariant, so

$$\int_{N^0_r(U)} |\nabla f|^2 d\mu = \int_{N^0_r(U)} |\nabla^0 f|^2 d\mu_0 = \mu_0(N^0_r(U))$$

We conclude that one of the level sets of function $f$ satisfies the desired bound. □

0.5. **Tessellations of hyperbolic surfaces.** We use the following tessellation of a Riemann surface due to Buser.

**Proposition 0.5.** *(Buser)* Let $\Sigma$ be a closed hyperbolic surface. There exists a tessellation of $\Sigma$ into polygons $T = T_1 \cup T_2$ with the following properties:

1. $T_1$ is a collection of triangles with sidelengths between $\log(2)$ and $2\log(2)$ and areas between 0.19 and 0.55.
2. $T_2$ is a collection of quadrilaterals (see figure [0.5]) with three right angles and one angle $\phi > \pi/3$. The sidelengths satisfy the following relations: $a \leq \log(2)/2$, $\log(2)/2 \leq c \leq 0.45$ and $b \geq d \geq 0.57$. The area of each quadrilateral is between 0.26 and 0.34.
3. For each polygon $T \in T$ the 1/2-neighbourhood of $T$ is contained in at most 40 polygons of $T$.  

The construction of Buser ([3], p.116-121) relies on the thin-thick decomposition of $\Sigma$. Let $\beta_1,...,\beta_k$ be the set of all simple closed geodesics of length $\leq \log(2)$ and let $w_i = \arcsinh(\frac{1}{\sinh(\frac{1}{2}||\beta_i||)}) > 1$. Then the tubular neighbourhood of $\beta_i$, $C_i = \{ p \in \Sigma | d(p, \beta_i) \leq w_i \}$ is isometric to the cylinder $[-w_i, w_i] \times S^1$ with the Riemannian metric $ds^2 = d\rho^2 + |\beta_i|^2 \cosh^2(\rho)dt^2$. Moreover, the cylinders $C_i$ are disjoint.

In each collar $C_i$, Buser defines two isometric annular regions, which he calls trigons. One boundary component of the trigon is the closed geodesic $\beta_i$ and the other boundary component consists of two geodesic arcs of equal length. The endpoints of these geodesic arcs lie at a distance $w_i - \log(2)/2$ from $\beta_i$. Each trigon can be subdivided into four isometric quadrilaterals as on Figure 0.5. These quadrilaterals have three right angles. A computation then yields the desired bounds on the sidelengths and the fourth angle. We define $\mathcal{T}_2$ to be the collection of all such quadrilaterals (eight in each collar).

In the remaining (thick) part of $\Sigma$ the injectivity radius at a point $x$ is bounded from below by $\min\{\log(2), d(x, V_2)\}$, where $V$ denotes the set of vertices of quadrilaterals in $\mathcal{T}_2$. Buser considers a maximal set of points at pairwise distances at least $\log(2)$. He then defines a geodesic triangulation of the thick part with this set as the set of vertices.

To prove the last assertion we observe that the worst case is when $T$ is a triangle that is not adjacent to any of the quadrilaterals. As computed by Buser, all angles of the triangle are bounded below by $22.6^\circ$. It follows that $1/2$-neighbourhood of $T$ can be covered by less than 40 triangles.

0.6. **Sweepouts of open subsets of $\mathbb{R}^2$.** Our proof of Proposition 0.8 relies on its Euclidean analogue. Namely, we need to know that for any open subset $U$ of
Euclidian plane there exists a sweepout of $U$ by relative cycles of small length. This result was proved by Guth in [13] along with its high dimensional generalizations.

**Theorem 0.6.** (Guth) Let $U \subset \mathbb{R}^2$ be a bounded open subset with piecewise smooth boundary. There exists a monotone sweepout of $U$ by cycles of length $\leq 3\sqrt{\text{Area}(U)}$.

**Proof.** We give an outline of the argument in [13]. The 2-dimensional case is significantly easier than the general inequality obtained by Guth for $k$-dimensional cycles sweeping out an open subset in $\mathbb{R}^n$.

At first one may hope that for some line $l \in \mathbb{R}^2$ the projection of $U$ on $l$ will have short fibers. However, there exist sets in $\mathbb{R}^2$ (known as Besicovitch sets) of arbitrarily small area such that any such projection will contain a fiber of length larger than 1.

Instead of sweeping out $U$ by parallel lines we will use cycles that are mostly contained in the 1-skeleton of a square lattice. Scale $U$ to have area 1. If we consider translates of the unit lattice the total length of the intersection of the lattice with set $U$ will have, on average, length equal to 2. Consider a large square $C_0 = [-N, N]^2$ that contains $U$ and let $l_0 = \partial C_0$. Let $C_1 = C_0 \setminus [-N, -N+1] \times [N-1, N]$. Continue removing unit squares one by one (see Figure 0.6). This way we obtain $N^2$ connected unions of unit squares $C_i$ with boundary in the 1-skeleton of the unit lattice. Observe that one can homotop $\partial C_i$ to $\partial C_{i+1}$ via cycles that are contained in the unit lattice except for a piece of length 1.

This gives rise to a family of nested open sets $A_t$, $A_c = \frac{k}{4N^2} = C_k$, and a homotopy $l_t = \partial A_t = l_t^1 + l_t^2$, where $l_t^1$ is contained in the unit lattice and $l_t^2$ is either empty or an interval of length 1. Defining $z_t = \partial A_t \cap \text{int}(U)$ we obtain a monotone sweepout with the desired length bound.

□
Sweepouts of subsets covered by a small number of polygons. When the genus \( g \) of \( M \) is greater than or equal to 2 we scale \((M, h_0)\) to have constant curvature \(-1\). By Gauss-Bonnet its volume satisfies \( \mu_0(M) = 4\pi(g - 1) \). By Lemma 0.5 there exists a tessellation \( \mathcal{T} \) of \( M \) into polygons.

When \( g \) is equal to 0 or 1 we scale the constant curvature space (sphere, projective plane, torus or a Klein bottle) so that it has volume 1. In this case we set \( \mathcal{T} \) to consist of only one element, the whole space \( M \).

Lemma 0.7. \( \mathcal{T} \) satisfies the following properties:

1. \( \# \mathcal{T} \leq \max\{67(g - 1), 1\} \)
2. Suppose \( \{T_i\}_{i=1}^k \subset \mathcal{T} \), \( k \leq 40 \), and let \( B^0(x, r) \) be any ball and let \( A \) denote the annulus \( B^0(x, \frac{3r}{2}) \setminus B^0(x, r) \). There exists 40 balls \( \{B^0(x_j, r)\} \), such that \( A \cap \bigcup T_i \subset \bigcup B^0(x_j, r) \).

Proof. When genus \( g \leq 1 \) we have \( \# \mathcal{T} = 1 \). It is easy to show that an annulus in the plane \( B(3/2) \setminus B(1) \subset \mathbb{R}^2 \) can be covered by 5 discs of radius 1. A similar covering also works on the round sphere \( S^2 \). We conclude that both properties hold when \( g \leq 1 \).

Suppose \( g \geq 1 \). The first property follows since areas of polygons in \( \mathcal{T} \) are bounded from below by 0.19.

To prove the second property we consider two cases. Suppose \( B(x, r) \) is a ball with \( r \geq 1 \). We can cover every triangle in \( \mathcal{T} \) by a ball of radius \( \log(2) < r \). The remaining points of \( A \cap \bigcup T_i \) lie in quadrilaterals. A quadrilateral \( T \in \mathcal{T} \) can be arbitrarily long, but it has to be narrow: by construction the distance from a point \( x \) on one of its long sides to the other long side is at most 0.45.

Suppose the intersection \( A \cap T \) is nonempty and let \( y \) be a point in \( A \cap T \) that is closest to \( x \), \( d^0(x, y) = d^0(x, A \cap T) \) (recall that \( x \) is the centre of \( B^0(x, r) \)). Then simple trigonometry shows that \( B(y, r) \) contains \( A \cap T \). We conclude that at most 40 balls of radius \( r \) are needed to cover \( A \cap \bigcup T_i \).

Suppose \( r \leq 1 \). In this case we need only 10 balls \( B^0(x_j, r) \) to cover \( A \). This is illustrated on Figure 0.7. Consider two concentric circles \( S_1 \) and \( S_2 \) in the hyperbolic plane of radii \( r \) and \( \frac{3r}{2} \) respectively. Suppose two geodesic rays emanating from \( x \) intersect circles \( S_1 \) and \( S_2 \) at \( A_1 \), \( B_1 \) and \( A_2 \), \( B_2 \). For a correct value of the angle \( \phi(r) \) between two geodesic rays we will have all four points lying on a circle of radius \( r \). Angle \( \phi(r) \) is minimized when \( r = 1 \). We compute \( \phi \geq 31.5^\circ \), so 10 discs will cover the annulus.

Proposition 0.8. Let \( U \subset M \) be an open subset with boundary and suppose there exists \( k \) sets \( T_i \in \mathcal{T} \), \( k \leq 40 \), such that \( U \subset \bigcup T_i \). Then there exists a monotone sweepout \( z_t \) of \( U \), such that \( l(z_t) \leq 466 \sqrt{\mu(U)} \).
We will inductively cut $U$ into smaller pieces until the volume of each piece becomes so small that we can apply Lemma 0.9. We will then use Lemma 0.3 to concatenate these sweepouts into one sweepout.

**Proposition 0.9.** For every $\epsilon > 0$ there exists a $\delta > 0$, such that for every open set $U \subset M$ with $\mu(U) < \delta^3$ there exists a monotone sweepout $z_t$ of $U$ of length $l(z_t) \leq \epsilon$.

**Proof.** Choose $\delta > 0$ be smaller than the injectivity radius and suppose that it is small enough so that for every $x \in M$ and every $r \leq \delta$ the ball $B(x, r)$ with metric $g$ restricted to it is $1.01$-bilipschitz diffeomorphic to a disc of radius $r$ in $\mathbb{R}^2$.

We will show that there exists a monotone sweepout of $U$ by cycles of length $\leq C \log(1/\delta^2)\delta$, where $C$ is a constant that does not depend on $\delta$ (but depends on the volume of $M$). Note that we can make this quantity arbitrarily small by choosing sufficiently small $\delta$.

Let $B$ be a minimal collection of balls of radius $1/2\delta$ covering $U$ and let $k$ denote the number of balls in $B$.

We claim that there exists a monotone sweepout $z_t$ satisfying

$$l(z_t) \leq 500\log(k + 1)\delta$$

We prove equation (1) by induction on $k$. Suppose $k \leq 100$. By coarea inequality for each $B_i \in B$ there exists a concentric ball $B_i' \supset B_i$ of radius $r$, $\delta/2 \geq r \leq \delta$, such that $l(\partial B_i' \cap U) \leq 2\delta$. By Theorem 0.6 there exists a monotone sweepout of $U \cap B_i'$ by cycles of length at most $3\delta$. Let $B_j$ be a different ball in $B$. As for $B_i$ we can find a sweepout of $B_j' \cap (U \setminus B_i')$ for some $B_j' \supset B_j$, such that $B_i$ has radius $\delta$. 

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**Figure 3.** Covering annulus in hyperbolic plane
\[ \leq \delta \] and \( l(\partial B'_i \cap (U \setminus B'_i)) \leq 2\delta \). By Lemma 0.3, there exists a monotone sweepout of \((B'_i \cup B_j) \cap U\) by cycles of length \( \leq 5\delta \). By repeating this step at most 100 times we obtain a monotone sweepout of \( U \) by cycles of length at most \( 203\delta \).

Assume the assertion holds for all \( U \) that can be covered by \( < k \) balls of radius \( \frac{1}{2}\delta \). Let \( k' \) be the smallest integer greater or equal to \( k/100 \) and let \( B \) denote the union of \( k' \) balls in \( B \). By coarea inequality there exists \( r \leq \delta/2 \), s.t. the boundary of the tubular neighbourhood \( \partial(N_r(B) \cap U) \) has length at most \( 2\delta \). Set \( U_1 = N_r(B) \cap U \).

Since \( N_r(B) \) is contained in the \( \delta/2 \) neighbourhood of \( B \), it can be covered by at most \( k/10 + 1 \) balls of radius \( \delta/2 \). The set \( U_2 = U \setminus N_r(B) \cap U \) can be covered by \( \frac{99}{100}k \) balls in \( B \). By inductive assumption there exists a monotone sweepout of \( U_i \), \( i = 1, 2 \), by cycles of length \( \leq 500 \log(\frac{99}{100}k)\delta \). By Lemma 0.3 there exists a sweepout of \( U \) by cycles of length at most \( 500 \log(\frac{99}{100}k)\delta + 2\delta < 500 \log(k)\delta \). This completes the proof of equation 1.

Since \( B \) is a minimal covering, balls with the same centers and \( 1/3 \) of the radius are disjoint. In particular, the sum of their volumes is bounded above by \( \mu(M) \). It follows that \( k \leq 12\log(M) \). We conclude that \( l(z_i) \leq C \log(1/\delta^2) \delta \) as desired. \( \square \)

We can now prove Proposition 0.8. Let \( \epsilon < 0.001 \sqrt{\mu(U)} \) be a small number and choose \( \delta(\epsilon) > 0 \) as in Lemma 0.9. We will prove that for any subset \( U' \subset U \) with piecewise smooth boundary there exists a monotone sweepout of \( U' \) by cycles of length \( \leq 466 \sqrt{\mu(U')} \).

The proof proceeds by induction on \( n = \log \frac{\mu(U')}{12} \). When \( \mu(U') \leq \delta^2 \) we are done by Lemma 0.9. Assume the result to be true for all subsets of \( \mu \)-volume \( \leq (\frac{42}{41})^{n-1} \delta^2 \) and consider \( U' \subset U \) with \((\frac{42}{41})^{n-1} < \frac{\mu(U')}{\delta^2} \leq (\frac{42}{41})^n \).

Let \( r \) be the smallest radius, such that \( \mu(B^0(x,r) \cap U') \geq \frac{\mu(U')}{42} \) for some \( x \in M \). By Lemma 0.7, the intersection of the annulus \( B^0(x,3/2r) \setminus B^0(x,r) \) with \( U' \) can be covered by at most 40 balls \( B^0(xj,r) \). For each \( j \) we have \( \mu(B^0(xj,r) \cap U') \leq \frac{\mu(U')}{42} \) since \( B^0(x,r) \) has maximal \( \mu \)-volume for a ball of this radius. It follows that the total \( \mu \)-volume of the set \( A = (B^0(x,3/2r) \setminus B^0(x,r)) \cap U' \) is bounded by \( \frac{40}{42} \mu(U') \). By Lemma 0.4 we can find a relative cycle \( \gamma \subset A \) of length \( \leq 2\sqrt{\mu(U')}^r \sqrt{\mu(U')} \) separating \( U' \) into two regions each having \( \mu \) volume less or equal to \( \frac{41}{42} \mu(U') \). Denote these two regions by \( U_1 \) and \( U_2 \).

Now we derive a bound for the length of \( \gamma \) that does not depend on \( r \). Since \( U' \) can be covered by at most 40 elements of \( \mathcal{T} \) its \( \mu_0 \)-volume is bounded by \( 40 \times 0.55 = 22 \) (recall that 0.55 is the maximal area of an element in \( \mathcal{T} \)). Hence, if \( r > 1.68 \) we obtain that \( l(\gamma) \leq 5.58 \sqrt{\mu(U')} \).
On the other hand, suppose $r \leq 1.68$. In this case we can directly compute (using a formula for the area of a disc in a space of constant curvature $-1$, 0 or 1)

$2\sqrt{\mu(U)} \leq 5.57$.

By inductive assumption both $U_1$ and $U_2$ admit a monotone sweepout with the desired length bound. By Lemma 0.3 there exists a monotone sweepout of $U'$ by cycles of length $\leq 466\sqrt{\frac{41}{45}\mu(U')} + 5.58\sqrt{\mu(U')} + \epsilon \leq 466\sqrt{\mu(U')} + \epsilon$.

This concludes the proof of Proposition 0.8.

0.8. Good covering of $M$.

**Proposition 0.10.** Consider a surface $M$ and let $U \subset M$ be an open subset with piecewise smooth boundary and suppose that it can be covered by $m$ elements of $\mathcal{T}$. Let $k$ be given. Then there exists a collection $U = \{U_i\}$ of at most $m + \max\{m, 41k\}$ sets, such that $\bigcup U_i$ covers $U$, $\mu(U_i \cap U) \leq \frac{\mu(U)}{k}$, each $U_i$ is contained in at most 40 elements of $\mathcal{T}$ and $l(int(U) \cap \bigcup \partial U_i) \leq (94\sqrt{m} + 36\max\{m, 41k\})\sqrt{\mu(U)}$.

**Proof.**

Step 1. First we construct a covering of $U$ by sets $V_1, ..., V_m$, such that each $V_i$ is contained in at most 40 polygons of $\mathcal{T}$. The $\mu$-volume of each $V_i$, however, can be equal to anything between 0 and $\mu(U)$.

Let $\mathcal{T}' \subset \mathcal{T}$ be the set of $m$ polygons that cover $U$ and let $T_i \subset \mathcal{T}'$ be such that $\mu(T_i \cap U) \geq \mu(T \cap U)$ for all $T \in T'$. By Proposition 0.5 there are at most 39 polygons neighbouring $T_i$. The intersection of each of them with $U$ has $\mu$-volume less than or equal to $\mu(T_i \cap U)$. By the length-area argument Lemma 0.4 there is a monotone sweepout with the desired length bound. By Lemma 0.3 there exists a monotone sweepout of $U'$ by cycles of length $\leq 466\sqrt{\frac{41}{45}\mu(U')} + 5.58\sqrt{\mu(U')} + \epsilon \leq 466\sqrt{\mu(U')} + \epsilon$.

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This concludes the proof of Proposition 0.8.
use comparison with the constant curvature space, and the case when \( r \) is large \((r > 1.68)\) and use upper bound on the area of 40 polygons. By Lemma 0.4 we conclude that there exists a set \( U^i_1 \subseteq B^0(x, r) \cap V_i \) of volume at most \( \frac{41\mu(U)}{N} \) and with
\[
l(int(V_i) \cap \partial U^i_1) \leq 5.58 \sqrt{\frac{41\mu(U)}{N}}.
\]
Similarly, for each \( j \) we can find subsets \( \partial U^i_j \) with disjoint interiors, \( \mu \)-volume between \( \frac{\mu(U)}{4u} \) and \( \frac{\mu(U)}{N} \) and \( l(\partial U^i_j \cap int(V_i \setminus (U^i_1 \cup \cdots \cup U^i_{j-1}))) \leq 5.58 \sqrt{\frac{41\mu(U)}{N}} \). Observe that \( \mathcal{U}_i = \{U^i_j\} \) has at most \( k_i \) elements.

We can now estimate the total length of the union of the boundaries \( L = l(int(U)) \cap \bigcup_{i,j} \partial U^i_j \leq 58 \sqrt{m\mu(U)} + \sum k_i * 5.58 \sqrt{\frac{41\mu(U)}{N}} \). The second term is bounded by \( 36(\frac{m}{N} + \sqrt{N})\sqrt{\mu(U)} \). We conclude that the total length is bounded by \( (94\sqrt{m} + 36\sqrt{N})\sqrt{\mu(U)} \).

\[ \square \]

0.9. Proof of Theorem 0.1. Now we can prove Theorem 0.1. Let \( \mathcal{T} \) be a tessellation of \( M \) by \((at most)\) \( \max\{1, 67g\} \) polygons as in Lemma 0.7.

By Proposition 0.10 we can cover \( M \) by a collection of sets \( U_i \) each of \( \mu \)-volume at most \( \mu(M)/k \) and contained in at most 40 polygons of \( \mathcal{T} \). Let \( N \) denote the number of sets in this covering.

First we construct a monotone 1-sweepout \( z_t \) of \( M \). By Proposition 0.8 for each \( U_i \) there exists a monotone sweepout of \( U_i \) by cycles \( z^i_t \) of length at most \( 466\sqrt{\frac{\mu(M)}{k}} \).

For \( j/N \leq t \leq (j+1)/N \) we set \( z_t = z^i_{Nt-j} + \sum_{i=1}^{j-1} z^i_i \). This defines a monotone sweepout of \( M \) with the property that each cycle can be written as a sum of chains \( z_t = c_t^1 + c_t^2 \), where \( c_t^1 \) has length at most \( 466\sqrt{\frac{\mu(M)}{k}} \) and \( c_t^2 \) is contained in \( \bigcup \partial U_i \).

Consider truncated symmetric product \( TP^k(S^1) \), i.e. all expressions of the form \( \sum_{i=1}^{k} a_i t_i \), where \( a_i \in \mathbb{Z}_2 \) and \( t_i \in S^1 \). For any 1-sweepout \( z_t \) the family of cycles \( \{\sum_{i=1}^{k} a_i z_t_i\}_{a_i \in TP^k(S^1)} \) is a \( k \)-sweepout of \( M \) (see [14], [6]).

We estimate the mass of each cycle
\[
l(\sum_{i=1}^{k} a_i z_t_i) \leq k \max \{l(z^1_t)\} + l(\bigcup \partial U_i)
\]
\[
\leq (466\sqrt{k} + 94\sqrt{67}\max\{g, 1\} + 36\max\{67g, 41k\})\sqrt{\mu(M)}
\]

In particular, \( l(\sum_{i=1}^{k} a_i z_t_i) \leq 16000 \max\{\sqrt{k}, \sqrt{g}\} \sqrt{\mu(M)} \). This concludes the proof of Theorem 0.1.

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