LENGTH DERIVATIVE OF THE GENERATING SERIES OF WALKS
CONFINED IN THE QUARTER PLANE

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Abstract. In the present paper, we use difference Galois theory to study the nature of the
generating series counting walks in the quarter plane. These series are trivariate formal power
series $Q(x, y, t)$ that count the number of discrete paths confined in the first quadrant of
the plane with a fixed directions set. While the variables $x$ and $y$ are associated to the
ending point of the path, the variable $t$ encodes its length. In this paper, we prove that if
$Q(x, y, t)$ does not satisfy any algebraic differential relations with respect to $x$ or $y$, it does
not satisfy any algebraic differential relations with respect to the parameter $t$. Combined with
[BBMR16, DHRS18, DHRS17], we are able to characterize the $t$-differential transcendence of
the generating series for any unweighted walk.

Contents

Introduction 1
1. The walks in the quarter plane 5
2. Generating functions for walks, genus zero case 9
3. Generating functions of walks, genus one case 14
Appendix A. Non archimedean estimates 22
Appendix B. Tate curves and their normal forms 25
Appendix C. Difference Galois theory 29
Appendix D. Meromorphic functions on a Tate curve and their derivations 33
References 42

Introduction

A walk in the quarter plane is a path between integral points of $\mathbb{Z}^2_{\geq 0}$ whose successive steps
belong to a fixed set of directions. One adds a probabilistic flavor to these objects by attaching
to the direction set a probability measure called the weights. If the measure is equidistributed,
we say that the walks is unweighted. For such a set of directions and weights, one denotes by
$q_{i,j,k}$ the probability for the walk confined in the quadrant $\mathbb{Z}^2_{\geq 0}$ to reach the position $(i, j)$ from
the initial position $(0, 0)$ after $k$ steps; and by $Q(x, y, t) = \sum_{i,j,k=0}^{\infty} q_{i,j,k} x^i y^j t^k$ the associated
generating series. As detailed in [KP11], the algebraic nature of this series encodes the numerical
complexity of the counting sequence $(q_{i,j,k})$.

The combinatoric study of discrete walks is a vivid topic and encounters many approaches:
via probabilistic methods, via combinatoric classification, via computer algebra and “Guess and

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Proofs”, via analytic study and boundary value problems and more recently via difference Galois theory and algebraic geometry. Among the 256 unweighted walks in the quarter plane \( \mathbb{Z}^2_{\geq 0} \) whose directions set is a subset of \( \{-1, 0, 1\}^2 \), Bousquet-Mélou and Mishna proved in [BMM10] that up to considering symmetries and withdraw trivial and one dimensional cases, only 79 cases remained. The figure 1 classifies the set of directions into four groups depending on the algebraic nature of the series.

- **Algebraic cases**: the series \( Q(x, y, t) \) satisfies a non trivial polynomial relation with coefficients in \( \mathbb{Q}(x, y, t) \).
- **Holonomic cases**: the series \( Q(x, y, t) \) is transcendent and holonomic, i.e. satisfies a non trivial linear differential equation in coefficients in \( \mathbb{Q}(x, y, t) \) in each of the three derivations.
- **Differentially algebraic cases**: the series \( Q(x, y, t) \) is nonholonomic and differentially algebraic, i.e. satisfies a non trivial polynomial differential equation in coefficients in \( \mathbb{Q} \) in each of the three derivations.
- **Differentially transcendent**: the series is not differentially algebraic with respect to the derivation \( \frac{d}{dx} \) and the derivations \( \frac{d}{dy} \).

This classification combines almost a decade of results and finds its foundation in the seminal paper of Bousquet-Mélou and Mishna. In [BMM10], the authors attached to any walk a **Kernel curve**, that is an algebraic curve of genus zero or one and a group of automorphisms of this curve called the group of the walk. Then, they conjectured that the group of the walk is finite if and only if the generating series of the walk is holonomic. Among the 23 finite group cases, Bousquet-Mélou and Mishna proved that 22 were holonomic (and even algebraic in two cases). The last case, namely Gessel walk, who lead to an algebraic generating series, was considered by Bostan, van Hoeij and Kauers in [BvHK10] (see also [FR10]). Among the 56 walks with infinite group, 51 have a Kernel curve which is an elliptic curve of genus one and 5 have a Kernel curve of genus zero. Following the analytic study of stationary process initiated in [FIM99], Kurkova and Rashel proposed the first systematic approach of the problem in the case of an infinite group and a genus one Kernel curve. Using an analytic parametrization of the genus one Kernel curves, they were able to uniformize the generating series so that it satisfies a linear discrete equation with respect to the dynamic induced by the group of the walk. They were then able to conclude to the non holonomy of the series in the infinite group cases by producing an infinite number of singularities propagating a single singularity with the group action. In [MM14], Melczer and Mishna employed a similar strategy called **iterated Kernel method** to prove that the generating
The question of differential algebraicity was first considered by Bernardi, Bousquet-Mélou and Raschel. For 9 non holonomic cases, they produced a closed form of the generating series using Tutte invariants and the notion of decoupling functions. These closed forms allowed them to prove that in the 9 cases, the generating series was differentially algebraic (see [BBMR16]). However, the study of differential transcendence seemed out of reach of the analytic proofs. Indeed the class of differentially algebraic functions is a very wild class from the analytic point of view and a differential algebraic function might have an infinite number of singularities. Recently, Roques, Singer and the authors of this paper introduced a new approach based on difference Galois theory and algebraic geometry that allowed them to characterize the differential algebraicity of the series in terms of an orbit configuration of certain points of the Kernel curve with respect to the group action (see [DHRS18, DHRS17]). This criteria allowed them to prove that all but 9 of the unweighted walks attached to a genus one curve with infinite group were differentially transcendental with respect to the x and y-variables, and reprove independently from [BBMR16], that the last 9 cases were differentially algebraic with respect to the x and y-variables.

The aim of the present paper is to give a full picture of the differential behavior of the generating series by focusing on the t-derivation. We say that \( Q(x, y, t) \) is \((\frac{d}{dx}, \frac{d}{dt})\)-differentially algebraic over \( \mathbb{Q} \) if the series \( \frac{d}{dy} Q(x, y, t) \) are algebraically dependent over \( \mathbb{Q} \). We say that it is \((\frac{d}{dx}, \frac{d}{dt})\)-differentially transcendental otherwise. Obviously if \( Q(x, y, t) \) is differentially algebraic with respect to one of the derivations \( \frac{d}{dx} \) or \( \frac{d}{dt} \) over \( \mathbb{Q} \), it is \((\frac{d}{dx}, \frac{d}{dt})\)-differentially algebraic over \( \mathbb{Q} \) but the converse does not hold a priori. We define the notion of differential algebraicity and differential transcendence with respect to other derivations similarly, see Definition C.5 for more details. The main result of this paper is as follows:

**Theorem 1.** The trivariate generating series \( Q(x, y, t) \) is \((\frac{d}{dx}, \frac{d}{dt})\)-differentially transcendental (resp.\((\frac{d}{dy}, \frac{d}{dt})\)-differentially transcendental) over \( \mathbb{Q} \) for any non degenerate walk\(^1\) with genus zero Kernel curve, and all but 9 \(^2\) unweighted walks with genus one Kernel curve.

Theorem 1 generalizes some of results obtained by Melczer and Mishna for walks with genus zero Kernel curves but unfortunately does not allow to retrieve the non holonomy of the excursion series \( Q(1,1,t) \) obtained in [MM14, MR09]. Theorem 1 can be deduced the combination of Theorem 2 below with the \( \frac{d}{dx} \) (resp \( \frac{d}{dy} \))-differential transcendence results of [DHRS18, DHRS17].

**Theorem 2.** For any non degenerate walk with infinite group of the walk, if the generating series \( Q(x, y, t) \) is \((\frac{d}{dx}, \frac{d}{dt})\)-differentially algebraic (resp. \( \frac{d}{dy} \)-differentially algebraic) over \( \mathbb{Q} \) then it is \( \frac{d}{dx} \) (resp. \( \frac{d}{dy} \))-differentially algebraic over \( \mathbb{Q} \).

The general strategy of the proof of Theorem 2 is similar to the one of [DHRS18, DHRS17] but its implementation required many new ideas. The first major difference with any of the articles quoted above is that we work in a non archimedean setting. Indeed, until then, the Kernel curve that is defined over \( \mathbb{Q}[t] \) had always been considered as an algebraic curve over \( \mathbb{C} \) by specializing \( t \). Such specialization does not allow to study the \( t \)-dependencies of the generating series. Here, we choose to uniformize the Kernel curve over a complete algebraically closed extension \( C \) of the valued field \( \mathbb{Q}(t) \). In particular, in the genus one case, we prove that the Kernel curve is analytically isomorphic to the Tate curve \( C^*/q^C \) as defined in [Roq70]. This analytification is the ultrametric analogue of the well known uniformization of an elliptic curve over \( \mathbb{C} \) by the quotient

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\(^1\)See Definition 1.3

\(^2\)These 9 walks correspond to the nine differentially algebraic cases of Figure 1.
of \( \mathbb{C} \) by a lattice. However, over non archimedean fields, such an uniformization requires that the \( J \)-invariant of the elliptic curve is of modulus strictly greater than 1. Surprisingly, this condition is fulfilled by any genus one Kernel curve. Then, via some technical non-archimedean estimates, we are able to prove the ultrametric analogue of [DR17]. More precisely, one can continue the specialization of the generating series \( Q(x,0,t) \) (resp. \( Q(0,y,t) \)) on the Kernel curve as a meromorphic function \( F^1(s,t) \) (resp. \( F^2(s,t) \)) over \( C^* \) satisfying
\[
F^1(\mathbf{q}s,t) = F^1(s,t) + b_1(s) \\
F^2(\mathbf{q}s,t) = F^2(s,t) + b_2(s),
\]
where \( \mathbf{q} \in C^* \) and the \( b_i(s) \) belong to \( C(s) \) in the genus zero case and to \( C_q \), the field of rational functions over \( C^*/q^2 \), in the genus one case. This uniformization procedure allows to reduce the question of the differential algebraicity of the series to the study of the differential algebraic relations satisfied by the auxiliary functions \( F^i(s,t) \) that satisfy linear \( q \)-difference equations.

The Galois theory developed in [HS08] gives Galoisian criteria in terms of the coefficients of a linear difference equation to compute the differential algebraic relations satisfied by the solutions with respect to any set of derivations commuting with the difference operator. Noting that the derivation \( \partial_s = s \frac{\partial}{\partial s} \) commutes with the operator \( \sigma_q \) that maps any meromorphic function \( g(s) \) onto \( g(qs) \), the authors of [DHRS18, DHRS17] applied these Galoisian criteria to deduce the differential transcendence of their archimedean auxiliary function with respect to \( s \). However, the derivation \( \partial_t = t \frac{\partial}{\partial t} \) does not commute with \( \sigma_q \). Our second main contribution is to introduce a convenient Galoisian framework for the \( t \)-derivation. We introduce the derivation \( \Delta_{q,t} = \partial_q(t)\ell_q(s)\partial_s + \partial_t \) where \( \ell_q \) is the so called \( q \)-logarithm that is an element of \( \text{Mer}(C^*) \), the field of meromorphic functions on \( C^* \), see \S 2.3 for more details, satisfying \( \sigma_q(\ell_q) = \ell_q + 1 \). We denote by \( C_q = C \in \text{Mer}(C^*) \) the compositum of the fields \( C_q \) and \( C_q \). The derivation \( \Delta_{q,t} \) commutes with \( \sigma_q \) and stabilizes the fields \( C_q(s) \) and \( C_q^*C_q(\ell_t, \ell_q) \). Using Galoisian criteria, we find:

**Theorem 3.** Let \( i \in \{1, 2\} \). If \( F^i(s,t) \) is \( (\partial_s, \partial_t) \)-differentially algebraic then there exist \( c_{k,j} \in C_q \) not all zero such that
\[
\sum_{0 \leq k \leq \ell_1, 0 \leq j \leq \ell_2} c_{k,j}\Delta_{q,t}^k(b_i(s)) = \sigma_q(g) - g,
\]
for some \( g \in C_q(s) \) (resp. \( C_qC_q(\ell_t, \ell_q) \)) in the genus zero case (resp. genus one case).

We call such an equation a telescoping equation. For \( \ell_2 = 0 \) and \( g \) a rational function, there exist some algorithms to test whether a rational function \( b_i \) admits a telescoping equation or not see for instance [Abr95, CS12]. Apparently, (0.1) seems out of reach of these algorithmic methods. However, using the transcendence properties of the \( q \)-logarithm, we were able to perform some descent procedure to deduce from (0.1) a simple telescoping equation involving only the derivation \( \partial_s \). This leads us to the following result:

**Theorem 4.** Let \( i \in \{1, 2\} \). If \( F^i(s,t) \) is \( (\partial_s, \partial_t) \)-differentially algebraic then there exist \( c_k \in C \) not all zero such that
\[
\sum_{\ell_1} c_k \partial_s^k(b_i(s)) = \sigma_q(g) - g,
\]
for some \( g \in C(s) \) (resp. \( C_q \)) in the genus zero case (resp. genus one case).

Since [DHRS17] and [DHRS18] prove that there is no such relation for any of the 56 walks with infinite group, except the nine differentially algebraic cases of Figure 1, this allowed us to conclude the proof of Theorem 1.

The paper is organized as follows. In Section 1 we consider some reminders and notations of walks in the quarter plane. In Section 2 we treat walks with genus zero Kernel curve, while
in Section 3 the genus one case is treated. Since this paper combines many different fields, non
archimedean uniformization, combinatoric, Galois theory, we choose to postpone many technical
intermediate results in the appendices. This should allow the reader to understand the articula-
tion of our proof of Sections 2 and 3 in three steps without being lost in too many technicalities.
These three steps are the uniformization of the Kernel and the construction of a linear difference
equation, the Galoisian criteria, and finally, the resolution of telescoping problems. Appendix A
is devoted to the non-archimedean estimates that we used in the uniformization procedure. Ap-
pendix B contains some reminders on special functions on Tate curves and their normal forms.
Appendix C proves the Galoisian criteria mentioned above. Finally, Appendix D studies the
transcendence properties of special functions on Tate curves that will be used for the descent of
our telescoping equations.

1. The walks in the quarter plane

The goal of this section is to introduce some basic properties about walks in the quarter plane.
In §1.2, we attach to the walk a Kernel curve, which is an algebraic curve defined over \( \mathbb{Q}[t] \). This
curve has been intensively studied as an algebraic curve over \( \mathbb{C} \) and to consider the Kernel curve over a
suitable valued field extension. This forces us to leave the archimedean setting of the
field of complex numbers and to consider the Kernel curve over a suitable valued field extension
of \( \mathbb{Q}(t) \) endowed with the valuation at 0.

1.1. The walks. Let \((d_{i,j})_{(i,j)\in\{0,\pm1\}^2}\) be a family of elements of \( \mathbb{Q}\cap[0,1] \) such that \( \sum_{i,j} d_{i,j} = 1 \).
We consider the walk \( W \) in the quarter plane \( \mathbb{Z}_{\geq 0}^2 \) satisfying the following properties:

- it starts at \((0,0)\),
- it has steps in \( D \subset \{ \ \downarrow, \ \uparrow, \ \rightarrow, \ \leftarrow \ \} \) – these steps will be identified with pairs
  \((i,j)\in\{0,\pm1\}^2\setminus\{(0,0)\}\),
- it goes to the direction \((i,j)\in\{0,\pm1\}^2\setminus\{(0,0)\}\) (resp. stays at the same position) with
  probability \( d_{i,j} \) (resp. \( d_{0,0} \)).

The \( d_{i,j} \) are called the weights of the walk. This walk is unweighted if \( d_{0,0} = 0 \) and if the nonzero
\( d_{i,j} \) all have the same value.

For any \((i,j)\in\mathbb{Z}_{\geq 0}^2 \) and any \( k \in \mathbb{Z}_{\geq 0} \), we let \( q_{i,j,k} \in [0,1] \) be the probability for the walk
confined in the quadrant \( \mathbb{Z}_{\geq 0}^2 \) to reach the position \((i,j)\) from the initial position \((0,0)\) after \( k \) steps. We introduce the corresponding trivariate generating series

\[
Q(x,y,t) := \sum_{i,j,k \geq 0} q_{i,j,k} x^i y^j t^k.
\]

Remark 1.1. For simplicity, we assume that the weights \( d_{i,j} \in \mathbb{Q} \) and that \( t \in \mathbb{R} \) is transcendent
over \( \mathbb{Q} \). However, we would like to mention that any of the arguments and statements below will
hold with arbitrary real weights and replacing the field \( \mathbb{Q} \) with the field \( \mathbb{Q}(d_{i,j}) \).

The Kernel polynomial of the walk is defined by

\[
K(x,y,t) := xy(1 - tS(x,y))
\]

where

\[
S(x,y) = \sum_{(i,j)\in\{0,\pm1\}^2} d_{i,j} x^i y^j
\]

(1.2)

\[
= A_{-1}(x) \frac{y}{t} + A_0(x) + A_1(x)y
\]

\[
= B_{-1}(y) \frac{x}{t} + B_0(y) + B_1(y)x,
\]
and \( A_i(x) \in x^{-1} \mathbb{Q}[x] \), \( B_i(y) \in y^{-1} \mathbb{Q}[y] \).

By [DHRS17, Lemma 1.1], see also [BMM10, Lemma 4], the generating series \( Q(x, y, t) \) satisfies the following functional equation:

\[
K(x, y, t)Q(x, y, t) = xy + F^1(x, t) + F^2(y, t) + td_{-1, -1}Q(0, 0, t),
\]

where

\[
F^1(x, t) := K(x, 0, t)Q(x, 0, t), \quad F^2(y, t) := K(0, y, t)Q(0, y, t).
\]

**Remark 1.2.** We shall often use the following argument of symmetry between \( x \) and \( y \). Exchanging \( x \) and \( y \) in the Kernel equation amounts to consider the Kernel of a walk \( W' \) with set of directions \( D' := \{(i, j) \text{ such that } (j, i) \in D\} \) and weights \( d'_{i, j} := d_{j, i} \).

1.2. **The Kernel Curve.** The Kernel polynomial has coefficient in \( \mathbb{Q}(t) \) which is not algebraically closed. Following [DVH12], we consider the function field \( \mathbb{Q}(t) \) as a valued differential field, that is a field equipped with a derivation \( \partial_t = t \frac{d}{dt} \) and a valuation \( v_0 \), the valuation at zero. Let us fix once for all \( \alpha \in \mathbb{R} \) such that \( 0 < \alpha < 1 \). For any \( f \in \mathbb{Q}(t) \), we define the norm of \( f \) as \( |f| = \alpha^{|v_0(f)|} \). The triple \( (\mathbb{Q}(t), \partial_t = t \frac{d}{dt}, v_0) \) is a valued differential field with a small derivation, that is, such that \( \partial_t(m) \subset m \), where \( m \) is the maximal ideal of the valuation ring, see [AvdDvdH17, Chapter 6]. By [AvdDvdH17, Proposition 6.2.1], any extension of \( v_0 \) to the algebraic closure \( \overline{\mathbb{Q}(t)} \) of \( \mathbb{Q}(t) \) endows \( (\overline{\mathbb{Q}(t)}, \partial_t) \) with a structure of a valued differential field with a small derivation. Since \( \partial_t \) is continuous on \( \overline{\mathbb{Q}(t)} \), it extends to a small derivation of the completion \( C \) of \( \mathbb{Q}(t) \) with respect to \( | \cdot | \), see [AvdDvdH17, Corollary 4.4.12].

By construction \( (C, | \cdot |, \partial_t) \) is an extension of \( (\mathbb{Q}(t), | \cdot |, \partial_t) \) which is both complete and algebraically closed, see [Rob00, Chapter 3].

We need to discard some degenerate cases. Following [FIM99], we have the following definition.

**Definition 1.3.** A walk is called **degenerate** if one of the following holds:

- \( K(x, y, t) \) is reducible as an element of the polynomial ring \( C[x, y] \),
- \( K(x, y, t) \) the has \( x \)-degree less than or equal to 1,
- \( K(x, y, t) \) the has \( y \)-degree less than or equal to 1.

**Remark 1.4.** In [DHRS17], the authors specialize the variable \( t \) as a transcendental complex number. Then, they study the Kernel curve as a complex algebraic curve in \( \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \). In this work, we shall use any algebraic geometric result of [DHRS17] by appealing to Lefschetz Principle : every true statement about a variety over \( \mathbb{C} \) is true for a variety over an algebraically closed field of characteristic zero.

The following lemma is the analogue of [FIM99, Lemma 2.3.2] in our setting. It gives very simple conditions on \( D \) to decide whether a walk is degenerate or not.

**Proposition 1.5** (Proposition 4.2 in [DHRS17]). A walk is degenerate if and only if at least one of the following holds:

1. There exists \( i \in \{-1, 1\} \) such that \( d_{i, -1} = d_{i, 0} = d_{i, 1} = 0 \). This corresponds to walks with steps supported in one of the following configurations

   \[
   \begin{array}{c}
   \text{\includegraphics{image1}}
   \end{array}
   \]

2. There exists \( j \in \{-1, 1\} \) such that \( d_{-1, j} = d_{0, j} = d_{1, j} = 0 \). This corresponds to walks with steps supported in one of the following configurations

   \[
   \begin{array}{c}
   \text{\includegraphics{image2}}
   \end{array}
   \]
(3) All the weights are zero except maybe \( \{d_{1,1}, d_{0,0}, d_{-1,-1}\} \) or \( \{d_{-1,1}, d_{0,0}, d_{1,-1}\} \). This corresponds to walks with steps supported in one of the following configurations

\[
\begin{array}{c}
\cdots \\
\cdot \\
\cdot \\
\cdot \\
\cdots \\
\end{array}
\]

From now on, we shall always assume that the walk under consideration is non degenerate.

Note that we only discard one dimensional problems as explained in [BMM10].

To any walk \( \mathcal{W} \), we attach a curve \( E \), called the Kernel curve, that is defined as the zero set in \( \mathbf{P}^1(C) \times \mathbf{P}^1(C) \) of the following homogeneous polynomial

\[
\tilde{K}(x_0, x_1, y_0, y_1, t) = x_0x_1y_0y_1 - t \sum_{i,j=0}^{2} d_{i-1,j-1}x_0^ix_1^jy_0^iy_1^j = x_1^2y_1^2K\left(\frac{x_0}{x_1}, \frac{y_0}{y_1}, t\right).
\]

Let us write \( \tilde{K}(x_0, x_1, y_0, y_1, t) = \sum_{i,j=0}^{2} A_{i,j}x_0^ix_1^jy_0^iy_1^j \) where \( A_{i,j} = -td_{i-1,j-1} \) if \( (i,j) \neq (1,1) \) and \( A_{1,1} = 1 - td_{0,0} \). The partial discriminants of \( \tilde{K}(x_0, x_1, y_0, y_1, t) \) are defined as the discriminants of the second degree homogeneous polynomials \( y \mapsto \tilde{K}(x_0, x_1, y, y_1, t) \) and \( x \mapsto \tilde{K}(x, x_1, y_0, y_1, t) \), respectively, i.e.

\[
\Delta_x(x_0, x_1) = \left(\sum_{i=0}^{2} x_0^ix_1^{2-i}A_{i,1}\right)^2 - 4 \left(\sum_{i=0}^{2} x_0^ix_1^{2-i}A_{i,0}\right) \times \left(\sum_{i=0}^{2} x_0^ix_1^{2-i}A_{i,2}\right)
\]

and

\[
\Delta_y(y_0, y_1) = \left(\sum_{j=0}^{2} y_0^jy_1^{2-j}A_{1,j}\right)^2 - 4 \left(\sum_{j=0}^{2} y_0^jy_1^{2-j}A_{0,j}\right) \times \left(\sum_{j=0}^{2} y_0^jy_1^{2-j}A_{2,j}\right).
\]

They are homogeneous polynomials of degree 4 and we will attach to them some Eisenstein invariants. More precisely, following [Dui10, §2.3.5], we define:

**Definition 1.6.** For any homogeneous polynomial

\[
f(x_0, x_1) = a_0x_1^4 + 4a_1x_0x_1^3 + 6a_2x_0^2x_1^2 + 4a_3x_0^3x_1 + a_4x_0^4 \in C[x_0, x_1],
\]

we define the Eisenstein invariants of \( f(x_0, x_1) \) as

- \( D(f) = a_0a_4 + 3a_2^2 - 4a_1a_3 \)
- \( E(f) = a_0a_3^2 + a_1^3a_4 - a_0a_2a_4 - 2a_1a_2a_3 + a_2^3 \)
- \( F(f) = 27E(f)^2 - D(f)^3. \)

Since \( C \) is algebraically closed and of characteristic zero, we can apply [Dui10, §2.4] to the Kernel curve. The following proposition characterizes the smoothness of the Kernel curve in terms of the invariants \( F(\Delta_x), F(\Delta_y) \).

**Proposition 1.7** (Proposition 2.4.3 in [Dui10] and Lemma 4.4 in [DHR17]). The following statements are equivalent

- The Kernel curve \( E \) is smooth, i.e. it has no singular point;
- \( F(\Delta_x) \neq 0; \)
- \( F(\Delta_y) \neq 0. \)

Furthermore, if \( E \) is smooth then it is an elliptic curve with \( J \)-invariant given by

\[
J(E) = 12^3 \frac{D(\Delta_y)^3}{-F(\Delta_y)} \in C.
\]

Otherwise, if \( E \) is non degenerate and singular, \( E \) has a unique singular point and is a genus zero curve.
We define the genus of the walk $W$ as the genus of the associated Kernel curve $E$. We recall the results obtained in [DHRS17, Section 4] that classify all the direction sets $D$ attached to a genus zero Kernel.

**Theorem 1.8.** Any non degenerate walk $W$ with $E$ of genus zero arise from the following 4 sets of steps:

- \[ \begin{array}{c}
  \text{}\vline \hline
  \text{\hspace{1cm}}
  \text{\hspace{1cm}}
  \hline
  \text{\hspace{1cm}}
  \text{\hspace{1cm}}
  \hline
  \end{array} \]

Otherwise, for any other non degenerate walk $W$, the Kernel curve $E$ is an elliptic curve.

**Remark 1.9.** The walks corresponding to the fourth configuration never enter the quarter-plane. As described in [BMM10, Section 2.1], if we consider walks corresponding to the second and third configurations we are in the situation where one of the quarter plane constraints implies the other. So the only interesting genus zero configuration is the set of steps contained in

- \[ \begin{array}{c}
  \text{\hspace{1cm}}
  \text{\hspace{1cm}}
  \text{\hspace{1cm}}
  \text{\hspace{1cm}}
  \text{\hspace{1cm}}
  \end{array} \]

Note that due to Proposition 1.5, the anti diagonal directions have non zero attached weights.

Moreover, by Theorem 1.8, combined with Proposition 1.5, the non degenerate walks of genus one are the walks where there are no three consecutive directions with weight zero. Or equivalently, this corresponds to the situation where the set of directions is not included in any half plane.

Thanks to Theorem 1.8, one is able to reduce our study to two cases depending on the genus of the Kernel curve attached to a non degenerate walk. The following lemma proves that when the Kernel curve is of genus one, its $J$-invariant has modulus strictly greater than 1. This will allow us later on to use the theory of Tate curves in order to analytically uniformize the Kernel curve.

**Lemma 1.10.** When $E$ is smooth, the invariant $J(E) \in \mathbb{Q}(t)$ is such that $|J(E)| > 1$, where $| \cdot |$ denotes the valuation of $(C, | \cdot |)$.

**Proof.** At $t = 0$, $\Delta_y(y_0, y_1)$ reduces to $y_0^2y_1^2$. This proves that the reduction of $D(\Delta_y)$ (resp. $E(\Delta_y)$) at $t = 0$ is $\frac{1}{12}$ (resp. $\frac{1}{12}$). Then, one concludes that $F(\Delta_y)$ vanishes for $t = 0$. Then, by Proposition 1.7, $J(E) \in \mathbb{Q}(t)$ has a strictly negative valuation at $t = 0$. Thus, $|J(E)| > 1$. \qed

Introduce

\begin{equation}
\mathcal{D}(x) := \Delta_x(x, 1) = \sum_{j=0}^{4} \alpha_j x^j \quad \text{and} \quad \mathcal{E}(y) := \Delta_y(y, 1) = \sum_{j=0}^{4} \beta_j y^j.
\end{equation}

More precisely, we have

\begin{align*}
\alpha_4 &= \left(d_{1,0}^2 - 4d_{1,1}d_{-1,1}\right) t^2 \\
\alpha_3 &= 2t^2 d_{1,0}d_{0,0} - 2t d_{1,0} - 4t^2 (d_{0,1}d_{1,-1} + d_{1,1}d_{0,-1}) \\
\alpha_2 &= 1 + t^2 d_{0,0}^2 + 2t^2 d_{-1,0}d_{1,0} - 4t^2 (d_{-1,1}d_{1,-1} + d_{0,1}d_{0,-1} + d_{1,1}d_{-1,-1}) - 2td_{0,0} \\
\alpha_1 &= 2t^2 d_{-1,0}d_{0,0} - 2t d_{-1,0} - 4t^2 (d_{-1,1}d_{0,-1} + d_{0,1}d_{-1,-1}) \\
\alpha_0 &= \left(d_{1,0}^2 + 4d_{-1,1}d_{-1,-1}\right) t^2.
\end{align*}

\begin{align*}
\beta_4 &= \left(d_{0,1}^2 - 4d_{1,1}d_{-1,1}\right) t^2 \\
\beta_3 &= 2t^2 d_{0,1}d_{0,0} - 2td_{0,1} - 4t^2 (d_{1,0}d_{-1,1} + d_{1,1}d_{0,-1}) \\
\beta_2 &= 1 + t^2 d_{0,0}^2 + 2t^2 d_{-1,0}d_{0,1} - 4t^2 (d_{-1,1}d_{-1,-1} + d_{1,0}d_{0,-1} + d_{1,1}d_{-1,-1}) - 2td_{0,0} \\
\beta_1 &= 2t^2 d_{0,0}d_{-1,0} - 2td_{0,-1} - 4t^2 (d_{1,-1}d_{0,-1} + d_{1,0}d_{-1,-1}) \\
\beta_0 &= \left(d_{0,-1}^2 - 4d_{1,-1}d_{-1,-1}\right) t^2.
\end{align*}
Figure 2. The maps $\iota_1, \iota_2$ restricted to the kernel curve $E$

1.3. The automorphism of the walk. Following [BMM10, Section 3] or [KY15, Section 3], we introduce the involutive birational transformations of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ given by

$$i_1(x, y) = \left( x, \frac{A_1(x)}{A_1(x)}y \right) \text{ and } i_2(x, y) = \left( \frac{B_1(y)}{B_1(y)}x, y \right),$$

see §1.1 for the significance of the $A_i, B_i$’s.

They induce birational maps $\iota_1, \iota_2 : E \rightarrow E$ given by

$$\iota_1([x_0 : x_1], [y_0 : y_1]) = \left( \left[x_0 : x_1\right], \left[ \frac{A_1(x_0)}{A_1(x_0)}x_1 \right] \right),$$
and

$$\iota_2([x_0 : x_1], [y_0 : y_1]) = \left( \left[ \frac{B_1(y_0)}{B_1(y_0)}y_1 \right], [y_0 : y_1] \right).$$

Note that $\iota_1$ and $\iota_2$ are nothing but the vertical and horizontal switches of $E$, see Figure 2, i.e. for any $P = (x, y) \in E$, we have

$$\{P, \iota_1(P)\} = E \cap \{x\} \times \mathbb{P}^1(\mathbb{C}) \text{ and } \{P, \iota_2(P)\} = E \cap (\mathbb{P}^1(\mathbb{C}) \times \{y\}).$$

**Proposition 1.11** (Proposition 4.12 in [DHRS17]). The two involutive birational maps $\iota_1, \iota_2 : E \rightarrow E$ are actually involutive automorphisms of $E$.

The automorphism of the walk $\sigma$ is defined by

$$\sigma = \iota_2 \circ \iota_1.$$

The following holds

**Lemma 1.12** (Lemma 4.14 in [DHRS17]). Let $P \in E$. The following statements are equivalent.

- $P$ is fixed by $\sigma$;
- $P$ is fixed by $\iota_1$ and $\iota_2$;
- $P$ is the only singular point of $E$ that is of genus zero.

2. Generating functions for walks, genus zero case

In this section, we fix a non degenerate walk $\mathcal{W}$ with genus zero Kernel curve. Following Remark 1.9, after eliminating duplications arising from trivial cases and the interchange of $x$ and $y$, the walk $\mathcal{W}$ arises from the following 5 sets of steps:

In this section, we shall prove the following theorem:
Theorem 2.1. For any non degenerate genus zero walk, the generating series \( Q(x, y, t) \) is \( (\frac{d}{dP}, \frac{d}{dQ}, \frac{d}{dt}) \) (resp. \( (\frac{d}{dx}, \frac{d}{dt}) \))-differentially transcendent over \( \mathbb{Q} \).

This is Theorems 1 for genus zero walks. Theorem 2.1 implies that the series are \( (\frac{d}{dx}, \frac{d}{dt}) \) (resp. \( (\frac{d}{dy}, \frac{d}{dt}) \))-differentially transcendent over any field extension \( K[Q] \) that contains only \( (\frac{d}{dx}, \frac{d}{dt}) \) (resp. \( (\frac{d}{dy}, \frac{d}{dt}) \)) differentially algebraic elements over \( \mathbb{Q} \), see [Kol73, Proposition 8, Page 101]. For instance, the series are \( (\frac{d}{dx}, \frac{d}{dt}) \) (resp. \( (\frac{d}{dy}, \frac{d}{dt}) \))-differentially transcendent over \( \mathbb{C}(x, y, t) \).

As detailed in the introduction, our proof has three major steps: the reduction to the study of the differential transcendence of an auxiliary function satisfying a simple \( q \)-difference equation via the uniformization of the Kernel (see §2.1 and §2.2), the use of difference Galois theory to interpret the differential relations of the auxiliary function in terms of the existence of a telescoping relation, and the proof that there is no such telescoping relation in the genus zero case (see §2.3).

2.1. Uniformization of the Kernel curve. For notations, we refer to §1 and especially to (1.5) for the definition of the \( \alpha_i, \beta_i \). Note that in the genus zero case, \( \alpha_0 = \alpha_1 = \beta_0 = \beta_1 = 0 \). Remind that the walk is non degenerate, so that \( d_{1,-1}d_{-1,1} \neq 0 \). Furthermore,

\[
-1 + d_{0,0}t \pm \sqrt{(1 - d_{0,0}t)^2 - 4d_{-1,1}d_{1,-1}t^2} \neq 0.
\]

We have the following result of uniformization of the genus zero curve \( E \):

Proposition 2.2 (Propositions 1.4 in [DHR17]). There exist \( \lambda \in \mathbb{C}^* \) and a parameterization \( \phi : \mathbb{P}^1(C) \to E \) with

\[
\phi(s) = (x(s), y(s)) = \left( \frac{4\alpha_2}{\sqrt{\alpha_4^2 - 4\alpha_2\alpha_4(s + \frac{1}{2})} - 2\alpha_3}, \frac{4\beta_2}{\sqrt{\beta_4^2 - 4\beta_2\beta_4(s + \frac{1}{2})} - 2\beta_3} \right),
\]

such that

- \( \phi : \mathbb{P}^1(C) \setminus \{0, \infty\} \to E \setminus \{(0, 0)\} \) is a bijection and \( \phi^{-1}((0, 0)) = \{0, \infty\} \);
- The automorphisms \( \iota_1, \iota_2, \sigma \) of \( E \) induce automorphisms \( \iota_1, \iota_2, \sigma_q \) of \( \mathbb{P}^1(C) \) via \( \phi \) that satisfy \( \iota_1(s) = \frac{1}{s}, \iota_2(s) = \frac{a}{s}, \sigma_q(s) = q^s \), with \( \lambda^2 = q \in \{q, q^{-1}\} \) with

\[
q = \frac{-1 + d_{0,0}t - \sqrt{(1 - d_{0,0}t)^2 - 4d_{-1,1}d_{1,-1}t^2}}{-1 + d_{0,0}t + \sqrt{(1 - d_{0,0}t)^2 - 4d_{-1,1}d_{1,-1}t^2}} \in C^*.
\]

Thus, we have the commutative diagrams

\[
\begin{array}{ccc}
E & \xrightarrow{\iota_\phi} & E \\
\downarrow{\phi} & & \downarrow{\phi} \\
\mathbb{P}^1(C) & \xrightarrow{\iota_q} & \mathbb{P}^1(C) \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
E & \xrightarrow{\sigma} & E \\
\downarrow{\phi} & & \downarrow{\phi} \\
\mathbb{P}^1(C) & \xrightarrow{\sigma_q} & \mathbb{P}^1(C) \\
\end{array}
\]

The following estimate on the norm of \( \tilde{q} \) holds:

Lemma 2.3. We have \( |\tilde{q}| > 1 \).

Proof. Since \( \tilde{q} \) is algebraic over \( \mathbb{Q}(t) \), it admits an expansion as a Puiseux series. It is then easily seen that its valuation is negative, i.e. \( |\tilde{q}| > 1 \). \( \square \)
2.2. Meromorphic continuation of the generating series. In this paragraph, we combine the functional equation (1.3) with the uniformization of the Kernel obtained above to meromorphically continue the generating series.

Since \(|t| < 1\), for any \((x, y) \in \mathbb{P}^1(C) \times \mathbb{P}^1(C)\) such that \(|x|, |y| \leq 1\), the generating series \(Q(x, y, t)\) as well as \(F^1(x, t), F^2(y, t)\) converge and satisfy

\[
K(x, y, t)Q(x, y, t) = xy + F^1(x, t) + F^2(y, t) + td_{-1, -1}Q(0, 0, t).
\]

We claim that there exist two positive real numbers \(c_0, c_\infty\) such that \(\phi\) maps the disks \(U_0 = \{s \in \mathbb{P}^1(C) ||s| < c_0\}\) and \(U_\infty = \{s \in \mathbb{P}^1(C) ||s| > c_\infty\}\) in the domain of convergence of the series, that is, \(\{(x, y) \in E \text{ such that } |x|, |y| \leq 1\}\). Indeed, the \(\alpha_i\) and \(\beta_i\) are of norm smaller or equal to 1 and \(|\alpha_2| = 1\) (see (1.5)). Thus, if \(|s| < \min(1, |\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}|)\), then

\[
|x(s)| = \frac{4\alpha_3 s}{\sqrt{\alpha_3^2 - 4\alpha_2\alpha(s^2 + 1) - 2\alpha_3 s}} = \frac{|4\alpha_3 s|}{|\sqrt{\alpha_3^2 - 4\alpha_2\alpha_4}|} < 1.
\]

An analogous reasoning for \(y(s)\) shows that when \(|s|\) is sufficiently small, we have \(|x(s)|, |y(s)| \leq 1\). Similarly, one can prove that when \(|s|\) is sufficiently big, one has \(|x(s)|, |y(s)| \leq 1\). This proves our claim.

We set \(\bar{F}^1(s) = F^1(x(s), t)\) and \(\bar{F}^2(s) = F^2(y(s), t)\), the composition of the generating series with the parametrization \(\phi\). These functions are defined on \(U_0 \cup U_\infty\). Evaluating (2.1) for \((x, y) = (x(s), y(s))\), one finds

\[
0 = x(s)y(s) + \bar{F}^1(s) + \bar{F}^2(s) + td_{-1, -1}Q(0, 0, t).
\]

The following lemma, shows that one can use the above equation to meromorphically continue the functions \(\bar{F}^i(s)\) on \(C\) so that they satisfy a \(q\)-difference equation.

**Lemma 2.4.** For \(i = 1, 2\), the restriction of the function \(\bar{F}^i(s)\) to \(U_0\) can be continued to a meromorphic function \(\tilde{F}^i(s)\) on \(C\) such that

\[
\tilde{F}^1(qs) - \bar{F}^1(s) = b_1 = (x(qs) - x(s))y(qs)
\]

and

\[
\tilde{F}^2(qs) - \bar{F}^2(s) = b_2 = (y(qs) - y(s)x(s)).
\]

**Proof.** We just give a sketch of this proof that follows the lines of [DHRS17, §2.1]. Since \(\hat{t}_1(s) = \frac{1}{s}\) and \(\hat{t}_2(s) = \frac{q}{s}\), we can assume up to restrict the disks \(U_0\) and \(U_\infty\) that \(\hat{t}_1(U_0) \subset U_\infty\). Then one can specialize (2.2) at any \(s \in U_0\) sufficiently close to zero so that \(qs \in U_0\)

\[
0 = x(s)y(s) + \bar{F}^1(s) + \bar{F}^2(s) + td_{-1, -1}Q(0, 0, t)
\]

and

\[
0 = x(\hat{t}_1(s))y(\hat{t}_1(s)) + \bar{F}^1(\hat{t}_1(s)) + \bar{F}^2(\hat{t}_1(s)) + td_{-1, -1}Q(0, 0, t).
\]

Using the invariance of \(x(s)\) (resp. \(y(s)\)) with respect to \(\hat{t}_1\) (resp. \(\hat{t}_2\)), the second equation is

\[
0 = x(s)y(qs) + \bar{F}^1(s) + \bar{F}^2(qs) + td_{-1, -1}Q(0, 0, t).
\]

Subtracting this last equation to the first, we find

\[
\tilde{F}^2(qs) - \tilde{F}^2(s) = (y(qs) - y(s)x(s).
\]

Since by Lemma 2.3, the norm of \(\tilde{q}\) is strictly greater than one, the norm of \(|q|\) is distinct from 1. This allows us to use (2.3) to meromorphically continue \(\tilde{F}^2\) on \(C\) so that it satisfies everywhere (2.3). The proof for \(\tilde{F}^1\) is similar.

Note that there is a priori no reason why, in the neighborhood of \(\infty\), the function \(\tilde{F}^i(s)\), with \(i = 1, 2\), should coincides with \(\tilde{F}^i(s)\).
2.3. Differential transcendence in the genus zero case. Theorem 2.1 is symmetrical in $x$ and $y$ so that we shall only prove the differential transcendence of $Q(x, y, t)$ with respect to $(\frac{d}{dx}, \frac{d}{dy})$ over $\mathbb{Q}$. Moreover, one can easily note that if the generating series $Q(x, 0, t)$ is $(\frac{d}{dx}, \frac{d}{dt})$-transcendent over $\mathbb{Q}$, the same holds for $Q(x, y, t)$, Indeed, any non trivial algebraic relation between the derivatives of $Q(x, y, t)$ with respect to $\frac{d}{dx}$ and $\frac{d}{dt}$ with constant complex coefficients specializes at $y = 0$ into a non trivial differential algebraic relations for $Q(x, 0, t)$ in the derivatives $\frac{d}{dx}$ and $\frac{d}{dt}$. Thus, Theorem 2.1 reduces to prove that $Q(x, 0, t)$ is $(\frac{d}{dx}, \frac{d}{dt})$-differentially transcendental over $\mathbb{Q}$.

We recall that any holomorphic function $f$ on $\mathbb{C}^*$ can be represented by an everywhere convergent Laurent series $\sum_{n \in \mathbb{Z}} a_n s^n$ with $a_n \in \mathbb{C}$. Moreover any non-zero meromorphic function on $\mathbb{C}^*$ can be written as $\frac{f}{x}$ such that the holomorphic functions $g$ and $\hat{h}$ have no common zeros. We shall denote by $\text{Mer}(\mathbb{C}^*)$ the field of meromorphic functions over $\mathbb{C}^*$. As in §D.1, we denote by $\sigma_q$ the $q$-difference operator that maps a meromorphic function $g(s)$ onto $g(qs)$, by $C_q$ the field of meromorphic functions fixed by $\sigma_q$, by $\ell_q \in \text{Mer}(\mathbb{C}^*)$ the $q$-logarithm that satisfies $\sigma_q(\ell_q) = \ell_q + 1$. Since the derivation $\partial_t = t \frac{d}{dt}$ of $\text{Mer}(\mathbb{C}^*)$ does not commute with $\sigma_q$, we introduce the following derivation $\Delta_{t,q} = \partial_t(q)\ell_q(s)\partial_s + \partial_t$ with $\partial_t = \frac{d}{dt}$. By Lemma D.2, the derivations $\partial_s$ and $\Delta_{t,q}$ commute with $\sigma_q$. The following lemma relates the $(\frac{d}{dx}, \frac{d}{dt})$-transcendence of $Q(x, 0, t)$ to the differential transcendence of the auxiliary function $\hat{F}(s)$ with respect to $(\partial_s, \Delta_{t,q})$.

**Lemma 2.5.** If the generating series is $Q(x, 0, t)$ is $(\frac{d}{dx}, \frac{d}{dt})$-differentially algebraic over $\mathbb{Q}$ then $\hat{F}(s)$ is $(\partial_s, \Delta_{t,q})$-differentially algebraic over $\hat{K} = C_q(s, \ell_q(s))$.

**Proof.** Suppose that the generating series is $Q(x, 0, t)$ is $(\frac{d}{dx}, \frac{d}{dt})$-differentially algebraic over $\mathbb{Q}$.

Let $\phi : \mathbb{P}^1(C) \to E, s \mapsto (x(s), y(s))$ denotes the uniformization of $E$ as in Proposition 2.2. Since $F^1(x, t)$ is the product of $Q(x, 0, t)$ by the polynomial $K(x, 0, t) \in \mathbb{Q}[x, t]$, the function $F^1(x, t)$ is $(\frac{d}{dx}, \frac{d}{dt})$-differentially algebraic over $\mathbb{Q}$. Remember that $\hat{F}(s) = F^1(x(s), t)$ for $s \in U_0$.

Let $G(x, t)$ be any bivariate function converging on $|x|, |y| \leq 1$ and let us denote for short $\partial_s$ (resp. $\partial_t$) the derivations $\frac{d}{dx}$ (resp. $\frac{d}{dt}$). We denote by $\hat{G}(s) = G(x(s), t)$. We note that $(\partial_s \hat{G}(s)) = \partial_s(x(s))(\partial_s \hat{G})(x(s), t)$ and

$$\partial_t(\hat{G}(s)) = (\partial_t G)(x(s), t) + \partial_t(x(s))(\partial_s \hat{G})(x(s), t) = (\partial_t G)(x(s), t) + c \partial_s(\hat{G}(s))$$

where $c = \frac{\partial_t(x(s))}{\partial_s(x(s))} \in \hat{K}$ because $x(s) \in \hat{K}$ and $\hat{K}$ is stable by $\partial_s, \Delta_{t,q}$, see Lemma D.5, and thereby by $\partial_t = \Delta_{t,q} - \partial_t(q)\ell_q(s)\partial_s$. An easy induction proves that

$$(\partial_s^n \hat{G})(x(s), t) = \partial_s^n(\hat{G}(s)) + \sum_{i \leq n, j < n} b_{i,j} \partial_t^i \partial_s^j(\hat{G}(s)),$$

where $b_{i,j} \in \hat{K}$. By Lemma D.2, we have $\partial_s \Delta_{t,q} = \Delta_{t,q}\partial_s = f \partial_s$ for some $f \in \partial_t(q)\ell_q(s)\partial_s$. An easy induction proves that

$$(\partial_t^n \hat{G})(x(s), t) = \Delta_{t,q}^n(\hat{G}(s)) + \sum_{i \leq 2n, j < n} d_{i,j} \Delta_{t,q}^j \partial_t^i(\hat{G}(s)),$$

for $d_{i,j} \in \hat{K}$. Moreover, an easy induction shows that, for any $n \in \mathbb{N}^*$, we have

$$(\partial_s^n \hat{G})(x(s), t) = \frac{1}{\partial_s(x(s))^m} \partial_s^m(\tilde{G}(s)) + \sum_{i=1}^{m-1} a_i \partial_t^i(\tilde{G}(s)),$$
where \( a_i \in \tilde{K} \). Applying (2.5) with \( G \) replaced by \( \partial^m \varphi \), we find that for every \( m, n \in \mathbb{N} \),
\[
(\partial^m \varphi)(x, t) = \Delta^m_{i,j}((\partial^m \varphi)(x(t), t)) + \sum_{i \leq 2n, j < n} d_{i,j} \Delta^i_{i,j}(\partial^m \varphi)(x(t), t)).
\]
Combining this equation with (2.6), we conclude that
\[
(\partial^m \varphi)(x, t) = \frac{1}{\partial_s(x(t))^n} \Delta^m_{i,j}(\partial^m \varphi)(s) + \sum_{i \leq 2n+m, j < n} r_{i,j} \Delta^i_{i,j}(\partial^m \varphi)(s)),
\]
for \( r_{i,j} \in \tilde{K} \).

Applying the computations above to \( G = F^1(x, t) \), we find that any non trivial polynomial equation in the derivatives \( \partial^m \varphi F^1(x, t) \) over \( \mathbb{Q} \) yields to a non trivial polynomial equation over \( \tilde{K} \) between the derivatives \( \Delta^m_{i,j}(\partial^m \varphi)(s) \). 

Thus, we have reduced the proof of Theorem 2.1 to the following proposition:

**Proposition 2.6.** The function \( \tilde{F}^1(s) \) is \( (\partial_s, \Delta, \partial_s) \)-differentially transcendental over \( \tilde{K} \).

**Proof.** Suppose to the contrary that \( \tilde{F}^1(s) \) is \( (\partial_s, \Delta, \partial_s) \)-differentially algebraic over \( \tilde{K} \). By Lemma 2.4, the meromorphic function \( \tilde{F}^1(s) \) satisfies \( \tilde{F}^1(qs) - \tilde{F}^1(s) = b_1 = (x(qs) - x(s))y(qs) \) with \( b_1(s) \in C(s) \subset C_q(s) \). Thus, by Proposition D.13 and Corollary D.14 with \( K = C_q(s) \), there exist \( m \in \mathbb{N} \), \( d_0, \ldots, d_m \in C_q \) not all zero and \( h \in C_q(s) \) such that
\[
(2.7) \quad d_0 b_1 + d_1 \partial_s(b_1) + \cdots + d_m \partial^m_s(b_1) = \sigma_q(h) - h.
\]
Considering a \( C \)-basis \( (e_\beta)_{\beta \in B} \) of \( C(s) \). This basis gives rise to a \( C_q \)-basis of \( C_q(s) \). Now, decompose the \( d_k \)'s and \( h \) over \( (e_\beta)_{\beta \in B} \). Since \( b_1(s) \in C(s) \), it is easily seen that (2.7) amounts into a collection of polynomial equations with coefficients in \( C \) that should satisfy the coefficients of the \( d_k \)'s and \( h \) with respect to the basis \( (e_\beta)_{\beta \in B} \). Since this collection has a non zero solution in \( C_q \), we can conclude using the fact that \( C \) is algebraically closed that it has a non zero solution in \( C \). Finally, we have shown that there exist \( c_k \in C \) not all zero and \( g \in C(s) \) such that
\[
\sum_k c_k \partial^k_s(b_1) = \sigma_q(g) - g.
\]
By [HS08, Lemma 6.4] there exists \( f \in C(s) \) and \( c \in C \), such that
\[
\tilde{F}^1(qs) - \tilde{F}^1(s) = b_1 = \sigma_q(f) - f + c.
\]
Since \( \tilde{F}^1 \) is meromorphic at \( s = 0 \), we deduce \( c = 0 \). Finally, we have shown that there exist \( f \in C(s) \) such that
\[
(2.8) \quad b_1 = \sigma_q(f) - f.
\]

Using the uniformization \( \phi \), the relation (2.8) corresponds to an equation in the function field of the Kernel curve \( E \). More precisely, denoting by \( C(E) \) the field \( C(x, y) \) of rational functions over \( E \), by \( \sigma \) the action induced by \( \sigma \) on \( C(E) \), the equation (2.8) is equivalent to
\[
(2.9) \quad (\sigma(x) - x)\sigma(y) = \sigma(f) - f,
\]
where \( \tilde{f} \in C(x, y) \) is the rational function corresponding to \( f \) via \( \phi \). The coefficients of \( \tilde{f} \) as a rational function over \( E \) belong to a finitely generated extension \( F \) of \( \mathbb{Q}(t) \).

Since \( C \) is algebraically closed, there exists a \( \mathbb{Q} \)-embedding \( \psi \) of \( F \) into \( C \) that maps \( t \) onto a transcendent complex number. Since \( \sigma \) and \( E \) are defined over \( \mathbb{Q}(t) \), one can apply \( \psi \) to (2.9) to find
\[
(\pi(x) - x)\pi(y) = \pi(f) - f,
\]
where \( \mathcal{J} \in \mathbb{C}(x,y) = \mathbb{C}(E) \) the field of rational functions of the complex algebraic curve \( E \) defined by the Kernel polynomial and \( \sigma \) the automorphism of \( \mathbb{C}(E) \) induced by the automorphism of the walk in \( E \). In [DHRS17, §3.2], the authors prove that there is no such equation. This concludes the proof by contradiction.

3. Generating functions of walks, genus one case

In this section we consider the situation where the Kernel curve \( E \) is an elliptic curve. By Remark 1.9, this corresponds to the case where the set of directions is not included in any half plane. In this section, we work under the assumption that the group of the walk is infinite. The case of a finite group of the walk is treated in [DR17] where the authors prove that the uniformization of the generating series is an elliptic function over an elliptic curve isogeneous to the Kernel curve. This allows them to prove that for any genus one Kernel curve, the generating series is holonomic with respect to any of the two variables \( x \), \( y \).

Our strategy is very similar to the genus zero situation. However, the uniformization procedure in the genus one case is entirely new. Indeed, previous works such as [FIM99, KR12, DR17] relied on the uniformization of elliptic curves over \( \mathbb{C} \) by a fundamental parallelogram of periods. Over non archimedean fields, there might be a lack of non trivial lattices so that one has to use their multiplicative analogues \( C^*/q\mathbb{Z} \), the so called Tate curves (see [Roq70] for more details). This multiplicative uniformization allows us to continue the generating series as meromorphic functions \( \tilde{F}^i(s) \) satisfying

\[
\tilde{F}^i(qs) - \tilde{F}^i(s) = b_i(s),
\]

for some \( q \in C^* \) and \( b_i(s) \in C_q \), the field of \( q \)-periodic meromorphic functions over \( C^* \). This process that is detailed in §3.1, 3.2 and 3.3 has many advantages. Though technical, it is much more simple than the uniformization by a fundamental parallelogram of periods since we only have to deal with one generator of the fundamental group of the elliptic curve, precisely the loop around the origin in \( C^* \). Moreover, it gives a unified framework to study the genus zero and one case, namely, the Galois theory of \( q \)-difference equations. This is the content of §3.4 where we apply the Galois criteria of Appendix C to reduce the differential algebraicity of the generating series to the existence of a telescoper. Finally, we show how one can apply the results of [DHRS18] to our context in order to conclude that there is no such telescoper for all but 9 of the unweighted walks of genus one.

3.1. Uniformization of the Kernel curve. Let us fix a non degenerate genus one walk. By Lemma 1.10, the norm of the \( J \)-invariant \( J(E) \) of the Kernel curve is such that \( |J(E)| > 1 \). By Proposition B.3, there exists \( q \in C \) such that \( 0 < |q| < 1 \) and \( J(E) = J(E_q) \), where \( E_q \) is the elliptic curve attached to the Tate curve \( C^*/q\mathbb{Z} \), see Proposition 3.1. Note that by Lemmas B.6 and B.8, \( J(E_q) = \frac{1}{q^k} \). The analytic isomorphism between \( C^*/q\mathbb{Z} \) and \( E_q \) is given by special functions that have their origins in the theory of Jacobi \( q \)-theta functions. In order to describe the uniformization of the Kernel curve \( E \), one needs to explicit the algebraic isomorphism between \( E \) and \( E_q \) due to the equality of their \( J \)-invariants. This is not completely obvious since \( E_q \) is given by its Tate normal form in \( \mathbb{P}^2 \), i.e. by an equation of the form

\[
Y^2 + XY = X^3 + BX + C.
\]

We postpone many intermediate results to the appendix B and we state directly the uniformization of the Kernel curve by \( C^* \).

Following [Roq70, Page 28], we set \( s_k = \sum_{n \geq 0} \frac{n^k q^n}{1-q^n} \in C \) for \( k \geq 1 \). The following proposition introduces the analogue of the Weierstrass \( \wp \) function for Tate curves.

**Proposition 3.1.** The series

\[
\wp(q, s) = \frac{1}{q} + \sum_{k=1}^{\infty} s_k q^k.
\]
\begin{itemize}
  \item $X(s) = \sum_{n \in \mathbb{Z}} \frac{q^n}{(1-q^n)^2} - 2s_1$;
  \item $Y(s) = \sum_{n \in \mathbb{Z}} \frac{q^n}{(1-q^n)^2} + s_1$;
\end{itemize}

are $q$-periodic functions that are meromorphic over $C^\ast$. Furthermore $X(s) = X(1/s)$, and $X(s)$ has a pole of order 2 at any element of the form $q^2$. Moreover, the analytic map

$$\pi : C^\ast \to \mathbb{P}^2(C),$$

$s \mapsto [X(s) : Y(s) : 1]$ is onto and his image is $E_q$, the elliptic curve defined by the following Tate normal form

$$(3.1) \quad Y^2 + XY = X^3 + BX + \tilde{C}$$

where $B = -5s_3$ and $\tilde{C} = -\frac{1}{12}(5s_3 + 7s_3)$. \hfill \Box

**Proof.** This is [FvdP04, Theorem 5.1.4, Corollary 5.1.5, and Theorem 5.1.10].

In the notation of Section 1.2, set $D(x) := \Delta_2(x,1)$ and let us write the Kernel curve $K(x,y,t) = \tilde{A}_0(x) + \tilde{A}_1(x)y + \tilde{A}_2(x)y^2 = \tilde{B}_0(y) + \tilde{B}_1(y)x + \tilde{B}_2(y)x^2$ with $\tilde{A}_i(x) \in C[x]$ and $\tilde{B}_i(y) \in C[y]$. For $i \geq 1$, let $D^{(i)}$ denote the $i$-th derivative with respect to $x$ of $D(x)$. Then, we find the following parametrization for the Kernel curve:

**Theorem 3.2.** There exist a root $a \in C$ of $D(x)$ such that $|a|, |D^{(2)}(a) - 2|, |D^{(i)}(a)| < 1$ for $i = 3, 4$ and $|q|^{1/2} < |D^{(1)}(a)| < 1$ and $u \in C^\ast$ with $|u| = 1$ such that the analytic map $\phi$ below is surjective

$$\phi : C^\ast \to E,$$

$s \mapsto (\pi(s), \overline{y}(s)),$

with

$$(3.2) \quad \begin{align*}
\pi(s) &= a + \frac{D^{(1)}(a)}{u^2X(s) + \frac{u^2}{12} - \frac{D^{(2)}(a)}{6}}, \\
\overline{y}(s) &= \frac{D^{(1)}(a)(2u^2Y(s) + u^2X(s))}{2u^2X(s) + \frac{u^2}{12} - \frac{D^{(2)}(a)}{6}} - \tilde{A}_1 \left( a + \frac{D^{(1)}(a)}{u^2X(s) + \frac{u^2}{12} - \frac{D^{(2)}(a)}{6}} \right) \\
&\quad - \frac{2\tilde{A}_2}{u^2X(s) + \frac{u^2}{12} - \frac{D^{(2)}(a)}{6}} \left( a + \frac{D^{(1)}(a)}{u^2X(s) + \frac{u^2}{12} - \frac{D^{(2)}(a)}{6}} \right).
\end{align*}$$

**Proof.** Lemma A.1 and Lemma B.8 guaranty the existence of $a$. By Proposition B.5, the application $w_E$

$$E_1 \quad \begin{array}{c}
[x:y:1] \rightarrow E \subset \mathbb{P}^1(C) \times \mathbb{P}^1(C) \\
\end{array}$$

where

$$\begin{align*}
\pi &= a + \frac{D^{(1)}(a)}{x_1 - \frac{D^{(2)}(a)}{6}}, \\
\overline{y} &= \frac{D^{(1)}(a)y_1}{2(x_1 - \frac{D^{(2)}(a)}{6})} - \tilde{A}_1 \left( a + \frac{D^{(1)}(a)}{x_1 - \frac{D^{(2)}(a)}{6}} \right) \\
&\quad - \frac{2\tilde{A}_2}{x_1 - \frac{D^{(2)}(a)}{6}} \left( a + \frac{D^{(1)}(a)}{x_1 - \frac{D^{(2)}(a)}{6}} \right),
\end{align*}$$

is an isomorphism of the elliptic curves $E_1 \subset \mathbb{P}^2(C)$ given by its Weierstrass equation $y_1^2 = 4x_1^3 - g_2x_1 - g_3$. By Lemma B.6, we find that the application $w_T$

$$E_q \quad \begin{array}{c}
[X:Y:1] \rightarrow [X + \frac{1}{2} : 2Y + X : 1] \\
\end{array}$$

induces an isomorphism between $E_q$ and the Weierstrass curve $\tilde{E}_1$ given by $y^2 = 4x^3 - h_2x - h_3$. By Lemma B.7, there exists $u \in C^\ast$ such that

$$\psi : \tilde{E}_1 \quad \begin{array}{c}
[x:y:1] \rightarrow [u^2x : u^3y : 1] \\
\end{array}$$

induces an isomorphism of elliptic curve. To conclude,
we set \( \phi = w_E \circ \psi \circ w_F \circ \pi \) where \( \pi \) is the uniformization of \( E_q \) by \( C^* \) given in Proposition 3.1. The norm estimate on \( u \) is Lemma B.8.

\[ \square \]

Remark 3.3. • The conditions on \( a \) are crucial to guaranty the meromorphic continuation of the generating series (see the proof of Lemma 3.7).

• The arguments of symmetry between \( x \) and \( y \) of Remark 1.2 allow us to construct another uniformization of \( E \) as follows. Denoting by \( \mathcal{E}(y) := \Delta_y(y, 1) \). One can prove that there exists a root \( b \in C^* \) of \( \mathcal{E} \) such that \(|b|, |\mathcal{E}^{(2)}(b) - 2|, |\mathcal{E}^{(1)}(b)| < 1 \) for \( i = 3, 4 \) and \(|q|^{1/2} < |\mathcal{E}^{(1)}(b)| < 1 \) and \( v \in C^* \) with \(|v| = 1 \) such that the analytic map \( \psi \) below is surjective

\[ \begin{align*}
\psi : & C^* \to E, \\
& s \mapsto (\varphi(s), \vartheta(s)),
\end{align*} \]

such that \( \vartheta(s) = b + \frac{\mathcal{E}^{(1)}(b)}{v^3 X(s)^{1/2} - \frac{\mathcal{E}^{(2)}(b)}{6}} \) (see [DR17, Page 21] for similar arguments).

3.2. The group of the walk. The goal of the following proposition is to explicit automorphisms of \( C^* \) that induce via \( \phi \) the automorphisms \( \sigma_{t_1}, t_2 \) of \( E \).

Proposition 3.4. There exists \( q \in C^* \) such that the automorphism of \( C^* \) defined by \( \sigma_q : s \mapsto q s \) induces via \( \phi \) the automorphism \( \sigma \). Similarly, the involutions \( \overline{i}_1, \overline{i}_2 \) of \( C^* \), that are defined by \( \overline{i}_1(s) = 1/s \) and \( \overline{i}_2(s) = q/s \), induce via \( \phi \) the automorphisms \( i_1, i_2 \).

In other words, we have the commutative diagrams

\[ \begin{align*}
E & \xrightarrow{\overline{i}_k} E \\
\phi & \downarrow \quad \phi \\
C^* & \xrightarrow{i_k} C^*
\end{align*} \quad \begin{align*}
E & \xrightarrow{\sigma} E \\
\phi & \downarrow \quad \phi \\
C^* & \xrightarrow{\sigma_q} C^*
\end{align*} \]

Proof. The automorphism \( \sigma \) of the Kernel corresponds to the addition by a prescribed point \( \Omega \in E(C) \). Let \( \pi : C^* \to E_q \) be the surjective map defined in Proposition 3.1. By [FvdP04, Exercise 5.1.9], the map \( \pi \) is a group isomorphism between the multiplicative group \( C^* \) and the Mordell-Weil group \( E_q(C) \) of \( E_q \). Moreover, since \( E_q \) and \( E \) are elliptic curves, any isomorphism between \( E_q \) and \( E \) is a group morphism. This proves that \( \phi : C^* \to E \) is a group morphism. Then, there exists \( q \in C^* \) only determined modulo \( q^2 \) such that the pullback of \( \sigma \) to \( C^* \) is the automorphism \( \sigma_q \). This proves the first statement.

Let us denote by \( \overline{i}_1, \overline{i}_2 \) some automorphisms of \( C^* \), obtained by pulling back to \( C^* \) via \( \phi \) the automorphisms \( i_1, i_2 \) of \( E \). The automorphisms \( \overline{i}_1, \overline{i}_2 \) are uniquely determined up to multiplication by some power of \( q \). The automorphisms of \( C^* \) are of the form \( s \mapsto l s^{k+1} \) with \( l \in C^* \). Note that \( \varpi(q^2) = a \), and \( (a, -B(a)) \in E \) is fixed by \( i_1 \). Indeed, by construction \( D(a) = 0 \). This proves that \( \overline{i}_1(1) \in q^2 \) and since \( i_1 \) is not the identity, up to change our choice for \( \overline{i}_1 \), we find \( \overline{i}_1(s) = 1/s \). The expression of \( \overline{i}_2 \) follows with \( \sigma = i_2 \circ \overline{i}_1 \).

\[ \square \]

Remark 3.5. • The choice of the element \( q \) is unique up to multiplication by \( q^2 \). Since \( |q| \neq 1 \), we can choose \( q \) such that \(|q|^{1/2} \leq |q| < |q|^{-1/2} \).

• Pursuing the symmetry arguments of Remark 3.3, we easily not that Proposition 3.4 has a straightforward analogue when one replaces \( \phi \) by \( \psi \) and one exchange \( \overline{i}_1 \) and \( \overline{i}_2 \).

Lemma 3.6. The automorphism \( \sigma \) has infinite order if and only if \( q \) and \( q^2 \) are multiplicatively independent, that is, there are no \( r, l \in \mathbb{Z} \setminus (0, 0) \) such that \( q^r = q^l \).

Proof. This is obvious. Note that multiplicatively independent is sometimes replaced in the literature by non commensurable (see [Roq70, §6]).

\[ \square \]
3.3. Meromorphic continuation. In the sequel, we shall prove that the functions
\( F^1(x, t) := K(x, 0, t)Q(x, 0, t) \) and \( F^2(y, t) := K(0, y, t)Q(0, y, t) \) can be meromorphically
continued to \( \mathbb{C}^* \). First, let us follow the ideas initiated in [FIM99] and we note
that since \( |t| < 1 \), the series \( F^1(x, t) \) and \( F^2(y, t) \) converge on the affinoid subset
\( U = \{(x, y) \in E \subset \mathbb{P}^1(C) \times \mathbb{P}^1(C) | |x| \leq 1, |y| \leq 1 \} \) of \( E \). With Lemma A.3, \( U \neq \emptyset \). On \( U \),
we have
\[
0 = xy + F^1(x, t) + F^2(y, t) + td_{-1,-1}Q(0, 0, t).
\]

Let us set \( U_x = \{(x, y) \in E \subset \mathbb{P}^1(C) \times \mathbb{P}^1(C) | |x| \leq 1 \} \). Note that \( F^1(x, t) \) is analytic on \( U_x \).
We continue \( F^2(y, t) \) on \( U_x \) by setting
\[
F^2(y, t) = -xy - F^1(x, t) - td_{-1,-1}Q(0, 0, t).
\]

Composing \( F^1(x, t) \) with the surjective map
\[
\phi : \mathbb{C}^* \to E, \quad s \mapsto (\overline{\tau}(s), \overline{\tau}(s)),
\]
we define the functions \( \hat{F}^1(s) = F^1(\tau(s), t) \) and \( \hat{F}^2(s) = F^2(\overline{\tau}(s), t) \) for any
\[
s \in \phi^{-1}(U_x) \cap \{ s \in \mathbb{C}^* | |s| \in [|q|^{1/2}, |q|^{-1/2}] \} =: \mathcal{U}_x.
\]

The goal of the following Lemma is to prove that \( \mathcal{U}_x \) is an annulus whose size is large enough
in order to continue the functions \( \hat{F}^1, \hat{F}^2 \), see Figure 3.

**Lemma 3.7.** Let \( |s| \in [|q|^{1/2}, |q|^{-1/2}] \). The following holds,
- If \( |s| \in |\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(1)}(a)|^{-1} \], then \( |\tau(s)| < 1 \).
- If \( |s| = |\mathfrak{D}^{(1)}(a)|^{\pm 1} \), then \( |\tau(s)| = 1 \).
- Otherwise \(|\tau(s)| > 1 \).

That is, \( \mathcal{U}_x = |\mathfrak{D}^{(1)}(a)|, |\mathfrak{D}^{(1)}(a)|^{-1} \], see Figure 3.

**Figure 3.** The plain circles correspond to \( |s| = |q|^{\pm 1/2} \). The dashed circles
 correspond to \( |\tau(s)| = 1 \).
Lemma 3.9. The following hold

- $|q| \neq 1$;
- Moreover, up to replace $q$ by some convenient $q^2$-multiple, the following holds:
- Assume that either $d_{-1,1} = 0$ or $d_{-1,1} \neq 0$. Then,
  \[ \bigcup_{t \in \mathbb{Z}} \sigma_q^t(U_x) = C^*. \]
- Assume that either $d_{1,-1} = 0$ or $d_{-1,1} \neq 0$. Then,
  \[ \bigcup_{t \in \mathbb{Z}} \sigma_q^t(U_y) = C^*. \]

**Proof.** We start by proving that $|q| \neq 1$. By Remark 3.5, one can choose $q$ so that we have $|q|^{1/2} \leq |q| < |q|^{-1/2}$. By construction, $\pi(1) = a$. Let $b \in \mathbb{P}^1(C)$ such that $(a,b) \in E$. Since $\nu_1(a,b) = (a,b)$ we have $\nu_2(a,b) \neq (a,b)$ by Lemma 1.12. So let $a' \neq a \in \mathbb{P}^1(C)$ such that $\sigma(a,b) = (a',b)$. By construction, $\pi(q) = a'$. Since by Lemma 3.7, $|\pi(s)| < 1$ for $|s| = 1$, it suffices to prove that $|\pi(q)| = |a'| \geq 1$ to conclude that $|q| \neq 1$.

Remind that $K(x,y,t) = \tilde{A}_{-1}(x) + \tilde{A}_0(x)y + \tilde{A}_1(x)y^2 = \tilde{B}_{-1}(y) + \tilde{B}_0(y)x + \tilde{B}_1(y)x^2$ with $\tilde{A}_i(x) \in C[x]$ and $\tilde{B}_i(y) \in C[y]$. With $\nu_1(a,b) = (a,b)$ and the formulas in §1.3,

\[ b^2 = \frac{A_{-1}(a)}{A_1(a)} = \frac{\tilde{A}_{-1}(a)}{\tilde{A}_1(a)}. \]

Let $\nu$ be the valuation at zero of $\frac{\tilde{A}_{-1}(X)}{\tilde{A}_1(X)} \in C(X)$. Lemma A.2 with $|a| < 1$ gives $|b|^2 = |a|^\nu$. Note that $\tilde{A}_{\pm 1} = xA_{\pm 1}$ are polynomial of degree at most two, so the integer $\nu$ belongs to $\{-2,-1,0,1,2\}$. We have

\[ (3.4) \]

We will prove that $|a'| \geq 1$ with a case by case study on the values of $\nu$.

Remember that

\[ \begin{align*}
\tilde{A}_{-1} &= d_{-1,-1} + d_{0,-1}x + d_{1,-1}x^2 \\
\tilde{A}_1 &= d_{-1,1} + d_{0,1}x + d_{1,1}x^2 \\
\tilde{B}_{-1} &= d_{-1,-1} + d_{-1,0}y + d_{-1,1}y^2 \\
\tilde{B}_1 &= d_{1,-1} + d_{1,0}y + d_{1,1}y^2.
\end{align*} \]

**Case $\nu \geq 1$.** Then, $|b| = |a|^{\nu/2} < 1$. Combining (3.4) and Lemma A.2, we find $|a||a'| = |b|^l$ with $l$ the valuation at zero of $\frac{\tilde{B}_{-1}(X)}{\tilde{B}_1(X)}$. This gives $|a'| = |a|^{\nu/2-1}$. Since $l \in \{-2,\ldots,2\}$ and $\nu \in \{1,2\}$, we get $-3 \leq l\nu/2 - 1 \leq 1$. If $l\nu/2 - 1 = 1$ then $\nu$ must be equal to 2 and by (3.5), we must have $d_{0,-1} = 0$ and $d_{1,-1} \neq 0$. By Remark 1.9, we must have $d_{-1,0}d_{1,1} \neq 0$ so that $l = 1$ and $l\nu/2 - 1 \neq 0$. A contradiction. Then, $l\nu/2 - 1 \leq 0$ and $|a'| \geq 1$.

**Case $\nu = 0$.** Then, $|b| = 1$. With Lemma A.3 and $|a| < 1$, we obtain $|a'| > 1$.

**Case $\nu < 1$.** Then, $|b| = |a|^{\nu/2} > 1$. Combining (3.4) and Lemma A.2, we find $|a'| = |a|^{\nu/2-1}$ where $l \in \{-2,\ldots,2\}$ is the valuation at infinity of $\frac{\tilde{B}_{-1}(X)}{\tilde{B}_1(X)}$. Since $l \in \{-2,\ldots,2\}$ and $\nu \in \{-1,-2\}$, we get $1 \geq l\nu/2 - 1 \geq -3$. If $l\nu/2 - 1 = 1$ then $\nu = -2$ and by (3.5), we must have $d_{0,-1} = 0$ and $d_{1,-1} \neq 0$. By Remark 1.9, we must have $d_{-1,0}d_{1,1} \neq 0$ so that $l = -1$ and $l\nu/2 - 1 \leq 0$. A contradiction. Then, $l\nu/2 - 1 \leq 0$ and $|a'| \geq 1$.

Assume that either $d_{-1,1} = 0$ or $d_{1,-1} \neq 0$ and let us prove

\[ \bigcup_{t \in \mathbb{Z}} \sigma_q^t(U_x) = C^*. \]

By Lemma A.4, let $(a_0,b_0) \in E$ such that $|a_0| = 1$ and $\sigma(a_0,b_0) = (a_1,b_1)$ with $|a_1| \leq 1$. By Lemma 3.7, let $s_0 \in C^*$ with $|s_0| = |D^{(1)}(a)|^{\pm 1}$ such that $\pi(s_0) = a_0$. Since $|q|^{1/2} \leq |q| < |q|^{-1/2}$.
and $|q|^{1/2} < |\mathcal{D}^{(1)}(a)| < 1$, we find that $|q| < |q_{s_0}| < |q|^{-1/2}$. Since $|\sigma(q_{s_0})| = |a_1| \leq 1$, we conclude by Lemma 3.7 that

- either $|q_{s_0}| \in U_x = |\mathcal{D}^{(1)}(a)|^{1/2}$. We now conclude that we have $U_x \cap \sigma_q(U_x) = |\mathcal{D}^{(1)}(a)|^{1/2}$ and $\sigma_q([\mathcal{D}^{(1)}(a)|^{1/2}]) \neq \emptyset$. Since $|q| \neq 1$, we deduce

$$\bigcup_{\ell \in \mathbb{Z}} \sigma_q(U_x) = \mathbb{C}^*.$$  

- or $|q_{s_0}| \in |q||\mathcal{D}^{(1)}(a)|, |q||\mathcal{D}^{(1)}(a)|^{-1}$. Replacing $q$ by $q/q$ allows to conclude.
- or $|q_{s_0}| \in |q|^{-1}\mathcal{D}^{(1)}(a), |q|^{-1}|\mathcal{D}^{(1)}(a)|^{-1}$. Replacing $q$ by $q/q$ allows to conclude.

The last statement concerning $\mathcal{U}_q$ comes from Lemma A.4 and Remark 3.8.

Accordingly to Lemma 3.9, we denote by $\widetilde{F}^i(s)$ the functions

- $F^i(\phi(s), t)$ for $s \in U_x$ if $d_{-1, 1} = 0$
- and $F^i(\psi(s), t)$ for $s \in U_y$ if $d_{-1, 1} \neq 0$.

Theorem 3.10 below shows that one can meromorphically continue the functions $\widetilde{F}^i(s)$ on $\mathbb{C}^*$ so that they satisfy non homogeneous rank 1 linear $\mathbb{q}$-difference equations.

**Theorem 3.10.** For $i = 1, 2$, the function $\widetilde{F}^i(s)$ can be continued as meromorphic function on $\mathbb{C}^*$ such that

$$\widetilde{F}^1(qs) - \widetilde{F}^1(s) = b_1$$

and

$$\widetilde{F}^2(qs) - \widetilde{F}^2(s) = b_2,$$

where $b_1 = (x(qs) - x(s)y(qs))$ and $b_2 = (y(qs) - y(s))x(s)$ are two elements of $C_q = C(E)$.

**Proof.** The proof is completely similar to the proof of Lemma 2.4 and relies on the fact that either the $\mathbb{q}$-orbit of $U_x$ or $U_y$ covers $\mathbb{C}^*$.

3.4. **Differential transcendence.** Applying the Galoisian criteria of Appendix C to the functional equations obtained in Theorem 3.10, we find:

**Theorem 3.11.** Assume that the walk is non degenerate of genus one with an infinite group. If $Q(x, y, t)$ is $(\frac{d}{dx}, \frac{d}{dt})$-differentially algebraic over $\mathbb{Q}$ then there exist $c_0, \ldots, c_n \in C$ not all zero and $h \in C_q$ such that

$$c_0b_1 + c_1\partial_h(b_1) + \cdots + c_n\partial^n_h(b_1) = \sigma_q(h) - h.$$  

A symmetrical result holds for $Q(0, y, t)$ replacing $b_1$ by $b_2$.

**Proof.** Note that since the group of the walk is of infinite order, the automorphism $\sigma$ is of infinite order. Therefore by Lemma 3.6 the elements $\mathbb{q}$ and $\mathbb{q}$ defined in Proposition 3.4 are multiplicatively independent. Suppose that $Q(x, y, t)$ is $(\frac{d}{dx}, \frac{d}{dt})$-differentially algebraic over $\mathbb{Q}$. Then $Q(x, 0, t)$ is $(\frac{d}{dx}, \frac{d}{dt})$-differentially algebraic over $\mathbb{Q}$. Let $\tilde{F}^1(s)$ be the function

- $F^1(\phi(s), t)$ for $s \in U_x$ if $d_{-1, 1} = 0$
- and $F^1(\psi(s), t)$ for $s \in U_y$ if $d_{-1, 1} \neq 0$.

We remind that we note $\mathcal{Mer}(\mathbb{C}^*)$ the field of meromorphic functions over $\mathbb{C}^*$ (see §2.3 for a precise definition) and denote by $C_x, C_q \subset \mathcal{Mer}(\mathbb{C}^*)$ the compositum of fields. We claim that $\tilde{F}^1(s)$ is $(\partial_s, \partial_{t, q})$-differentially algebraic over $C_{x, q}(\ell_q, \ell_q)$. Let us prove this claim when $d_{-1, 1} = 0$, the proof when $d_{-1, 1} \neq 0$ being similar. Accordingly to the definition of $\tilde{F}^i(s)$, we
denote by \(\pi(s)\) the first coordinate of the parametrization \(\phi\). Reasoning as in Lemma 2.5, one can show that, for \(n, m \in \mathbb{N}\), one has
\[
\left(\partial_t^n \partial_x^m F^1(\pi(s), t)\right) = \frac{1}{\partial_t(\pi(s))^{m}} \Delta^m \partial_x^m (\tilde{F}^1(s)) + \sum_{i \leq 2n + m, j < n} r_{i,j} \Delta^i \partial_x^i (\tilde{F}^1(s)),
\]
where \(r_{i,j} \in C_q(\ell_q(\pi(x), \partial_x^k (\pi(s))), \ldots)\). Since by construction \(\pi(s) \in C_q\), Lemma D.5 proves that \(\partial_x^k (\pi(s)) \in C_q(\ell_q)\) for any \(k, t\) so that \(C_q(\ell_q(\pi(x), \partial_x^k (\pi(s))), \ldots) \subset C_q C_q(\ell_q, \ell_q)\). This proves that any non trivial polynomial relation between the \(x,t\)-derivatives of \(Q(x,0,t)\) yields to a non trivial polynomial relation between the derivatives of \(\tilde{F}^1(s)\) with respect to \(\partial_t\) and \(\Delta_{s,t}q\) over \(C_q C_q(\ell_q, \ell_q)\). This proves the claim. By Theorem 3.10, the function \(\tilde{F}^1(s)\) satisfies \(\tilde{F}^1(q) = b_1(s) + b_1(s) \in C_q C_q(\ell_q, \ell_q)\). Since \(\tilde{F}^1(s)\) is \((\partial_{s}, \Delta_{s,t}q)\)-differentially algebraic over \(C_q C_q(\ell_q, \ell_q)\), Proposition D.13 and Corollary D.14 imply that there exist \(m \in \mathbb{N}\) and \(d_0, \ldots, d_m \in C_q\) not all zero and \(g \in C_q C_q(\ell_q)\) such that
\[
d_0b_1 + d_1 \partial_s(b_1) + \cdots + d_m \partial_s^m (b_1) = \sigma_q(g) - g.
\]
Since \(b_1 \in C_q\), Lemma D.12 proves that there exist \(c_0, \ldots, c_n \in C\) not all zero and \(h \in C_q\) such that
\[
c_0b_1 + c_1 \partial_s(b_1) + \cdots + c_n \partial_s^n (b_1) = \sigma_q(h) - h.
\]
This concludes the proof. □

The following Corollary completes the proof of Theorem 2 for walks with genus one curves.

**Theorem 3.12.** For any non degenerate walk with a genus one Kernel curve and infinite group, the following statements are equivalent:

1. The series \(Q(x,y,t)\) is \((\frac{d}{dx}, \frac{d}{dy})\)-differentially algebraic over \(Q\).
2. There exist \(c_0, \ldots, c_n \in C\) not all zero and \(h \in C_q\) such that
   \[
c_0b_1 + c_1 \partial_s(b_1) + \cdots + c_n \partial_s^n (b_1) = \sigma_q(h) - h.
   \]
3. The series \(Q(x,y,t)\) is \(\frac{d}{dx}\)-differentially algebraic over \(C\).

**Remark 3.13.** Similarly, we may prove that the following statements are equivalent:

1. The series \(Q(x,y,t)\) is \((\frac{d}{dy}, \frac{d}{dx})\)-differentially algebraic over \(Q\).
2. There exist \(c_0, \ldots, c_n \in C\) not all zero and \(h \in C_q\) such that
   \[
c_0b_2 + c_1 \partial_s(b_2) + \cdots + c_n \partial_s^n (b_2) = \sigma_q(h) - h.
   \]
3. The series \(Q(x,y,t)\) is \(\frac{d}{dy}\)-differentially algebraic over \(C\).

**Proof.** Note that since the group is infinite, the automorphism \(\sigma\) is of infinite order. Therefore by Lemma 3.6 the elements \(q\) and \(g\) defined in Proposition 3.4 are multiplicatively independent.

(1) implies (2) is Theorem 3.11. Let us assume that there exist \(c_0, \ldots, c_n \in C\) not all zero and \(h \in C_q\) such that
\[
c_0b_1 + c_1 \partial_s(b_1) + \cdots + c_n \partial_s^n (b_1) = \sigma_q(h) - h.
\]
Combining (3.7) with the functional equation satisfied by \(\tilde{F}^1(s)\) and using the commutativity of \(\sigma_q\) and \(\partial_s\), one finds that
\[
\sigma_q \left[ c_0 \tilde{F}^1(s) + \cdots + c_n \partial_s^n (\tilde{F}^1(s)) - h \right] = c_0 \tilde{F}^1(s) + \cdots + c_n \partial_s^n (\tilde{F}^1(s)) - h.
\]
This means that there exists \(g \in C_q\) such that
\[
c_0 \tilde{F}^1(s) + \cdots + c_n \partial_s^n (\tilde{F}^1(s)) - h = g.
\]
One finds that \( \tilde{F}^1(s) \) is \( \partial_s \)-differentially algebraic over \( C_q \). Reasoning as in Lemma 2.5, one proves that there exists a non trivial algebraic relation with coefficients in \( C_q \) between the functions \( \partial_s^{m} F^1 \) evaluated in \( (\pi(s), t) \). Note that any element of \( C_q = C(\pi(s), \pi(s)) \) is algebraic over \( C(\pi(s)) \). One concludes that the functions \( \partial_s^{m} F^1 \) evaluated in \( (\pi(s), t) \) are algebraic over \( C(\pi(s)) \). This proves that \( F^1(x, t) = K(x, 0, t)Q(x, 0, t) \) is \( \frac{d}{dx} \)-differentially algebraic over \( C(x) \). Then \( Q(x, 0, t) \) is \( \frac{d}{dx^2} \)-differentially algebraic over \( C(x) \) and therefore over \( \mathbb{Q} \) because any rational fraction is differentially algebraic over \( \mathbb{Q} \). Of course it is differentially algebraic over \( \mathbb{C} \), and this completes the proof of (2) implies (3). Statement (3) implies obviously (1).

As a corollary, one finds:

**Corollary 3.14.** For all but 9 of the non degenerate unweighted walks with genus one Kernel curve, the generating series \( Q(x, y, t) \) is \( \left( \frac{d}{dx}, \frac{d}{dy} \right) \)-transcendent over \( \mathbb{Q} \) (see Figure 1). If the walk is non degenerate of genus one with an infinite group and at least one of the following situation holds:

- \( d_{2,0}^2 - 4d_{1,1}d_{1,-1} \) is not a square in \( \mathbb{Q} \);
- \( d_{2,1}^2 - 4d_{1,1}d_{-1,1} \) is not a square in \( \mathbb{Q} \);
- \( d_{1,1} = 0 \), \( d_{1,0}d_{0,1} \neq 0 \) and there are no \( \mathbb{Q} \) points of \( E \) fixed by \( t_1 \) or \( t_2 \);
- \( d_{1,1} = d_{1,0} = 0 \), \( d_{0,1} \neq 0 \);
- \( d_{1,1} = d_{0,1} = 0 \), \( d_{1,0} \neq 0 \);

then, \( Q(x, y, t) \) is \( \left( \frac{d}{dx}, \frac{d}{dy} \right) \) and \( \left( \frac{d}{dx}, \frac{d}{dt} \right) \)-transcendent over \( \mathbb{Q} \).

**Proof of Corollary 3.14.** By Theorem 3.12, it is sufficient to prove that the the generating series \( Q(x, y, t) \) is \( \frac{d}{dx} \)-transcendent over \( \mathbb{Q} \). This is the main result of [DHRS18, Section 5] for all but 9 of the non degenerate walks with genus one Kernel curve and of [DR17, Section 3.2] for the weighted cases above.

**Appendix A. Non archemedean estimates**

In this section, we state and prove some non archimedean estimates that allow us to uniformize the Kernel curve.

**A.1. Discriminants of the Kernel equation.** Next Lemma gives useful properties on the roots of the discriminants.

**Lemma A.1.** If the Kernel curve of the walk is a genus one curve then:

- all the roots of \( \Delta_4(x_0, x_1) \) in \( \mathbb{P}^1(C) \) are simple;
- the discriminant \( \mathcal{D}(x) := \Delta_4(x, 1) \) has a root \( a \in C \) such that \( |a| < 1 \), \( |\mathcal{D}(2)(a) - 2| < 1 \), and \( |\mathcal{D}(1)(a)|, |\mathcal{D}(3)(a)|, |\mathcal{D}(4)(a)| < 1 \) where \( \mathcal{D}(i) \) denote the \( i \)-th derivative with respect to \( x \) of \( \mathcal{D}(x) \).

A symmetric statement holds for \( \Delta_4(y_0, y_1) \) by replacing \( \mathcal{D} \) by \( \mathcal{E} \).

**Proof.** The first assertion is [DHRS17, Lemma 4.4]. First, let us prove the existence of a root \( a \in C \) of \( \mathcal{D}(x) \) such that \( |a| < 1 \). Let us first assume that \( a_4 \neq 0 \). Suppose to the contrary that all the roots of \( \mathcal{D}(x) \) have a norm greater than or equal to 1. The product of these roots equals

\[
\frac{a_0}{a_4} = \frac{t^2(d_{2,1,0}^2 - 4d_{1,-1,1}d_{1,1,1})}{t^2(d_{2,0}^2 - 4d_{1,1,1}d_{1,1,1})}.
\]

If \( a_0 = 0 \) then taking \( a = 0 \) yields a contradiction. If \( a_0a_4 \neq 0 \), we conclude that \( |\frac{a_0}{a_4}| = 1 \) so that each of the roots must have norm equal to 1. Then, considering the symmetric functions
of the roots of $\mathcal{D}(x)$, we conclude that for any $i = 0, \ldots, 3$, the element $\frac{a_i}{\alpha_i}$ has a norm smaller than or equal to 1. If one considers

$$\frac{\alpha_2}{\alpha_4} = \frac{-4d_{-1,-1}d_{1,1}t^2 - 4d_{0,0}d_0t^2 - 4d_{1,0}d_{1,0}t^2 + 2d_0d_0t^2 - 2td_0t + 1}{t^2(d_{1,0}^2 - 4d_{-1,-1}d_{1,1})},$$

it is easily seen that this element has norm strictly greater than 1. A contradiction.

Assume now that $\alpha_4 = 0$. Since the roots of $\Delta_4(x_0, x_1)$ in $\mathbb{P}^1(C)$ are simple, $\alpha_3 \neq 0$. Suppose to the contrary that all the roots of $\mathcal{D}(x)$ have a norm greater than or equal to 1. The product of these roots equals

$$\frac{-\alpha_2}{\alpha_3} = \frac{-t^2(d_{1,0}^2 - 4d_{-1,-1}d_{1,1})}{2t^2d_1d_0d_0t - 2td_1t - 4t^2(d_{1,0}d_{1,1} + d_{1,1}d_{1,0})}.$$

If $\alpha_0 = 0$ then taking $a = 0$ yields to a contradiction. Assume that $\alpha_0 \alpha_3 \neq 0$. We conclude that $|\frac{\alpha_2}{\alpha_3}| \leq 1$ so that each of the roots must have norm equal to 1. The symmetric function $\frac{\alpha_2}{\alpha_3}$ should have norm smaller or equal to 1. But it is easily seen that

$$-\frac{\alpha_2}{\alpha_3} = \frac{-4d_{-1,-1}d_{1,1}t^2 - 4d_{0,0}d_0t^2 - 4d_{1,0}d_{1,0}t^2 + 2d_0d_0t^2 - 2td_0t + 1}{2t^2d_1d_0d_0t - 2td_1t - 4t^2(d_{1,0}d_{1,1} + d_{1,1}d_{1,0})},$$

has a norm strictly bigger than 1. A contradiction again.

So let us consider a root $a \in C$ with $|a| < 1$. Let us prove that we automatically have $|\mathcal{D}^{(2)}(a) - 2| < 1$, and $|\mathcal{D}^{(1)}(a)|, |\mathcal{D}^{(3)}(a)|, |\mathcal{D}^{(4)}(a)| < 1$. Since $a, \alpha_1, \alpha_3, \alpha_4$ have norm smaller than 1 and $|\alpha_2 - 1| < 1$, we deduce the result with

- $\mathcal{D}^{(1)}(a) = \alpha_1 + 2\alpha_2a + 3\alpha_3a^2 + 4\alpha_4a^3$;
- $\mathcal{D}^{(2)}(a) = 2\alpha_2 + 6\alpha_3a + 12\alpha_4a^2$;
- $\mathcal{D}^{(3)}(a) = 6\alpha_3 + 24\alpha_4a$;
- $\mathcal{D}^{(4)}(a) = 24\alpha_4$.

The statement for $\Delta_y(y_0, y_1)$ is symmetrical and we omit its proof. \qed

### A.2. Automorphisms of the Kernel on the unit disk.

In this section, we study the action of the group of the walk on the product of the unit disks in $\mathbb{P}^1(C) \times \mathbb{P}^1(C)$. This product is the fundamental domain of convergence of the generating series.

First, we prove an elementary lemma concerning non-archimedean estimates.

**Lemma A.2.** Let $f \in C(X)$ be a non zero rational function with coefficients in $C$ and let $a \in \mathbb{P}^1(C)$. Let $\nu$ (resp. $d$) be the valuation at $X = 0$ (resp. $\infty$) of $f$ with the convention that $\nu = +\infty$, $d = -\infty$ if $f = 0$. The following holds:

- if $|a| < 1$, then $|f(a)| = |a|^{\nu}$;
- if $|a| > 1$, then $|f(a)| = |a|^{d}$.

**Proof of Lemma A.2.** Let us prove the first case, the second being completely symmetrical. Let $f(X) = \sum_{i+j \geq 2} c_{ij}X^iY^j$ with $c_{ij}d_{ij} \neq 0$. Note that $|a^k| < |a^l|$ if $k > l$. Then

$$|f(a)| = \frac{|\sum_{i+j \geq 2} c_{ij}a^i|}{|\sum_{i+j \geq 2} d_{ij}a^j|} = |a|^{\nu_1 - \nu_2} = |a|^\nu.$$ 

\qed

The following Lemma explains how the fundamental involutions permute the interior and the exterior of the fundamental domain of convergence. We define the norm of an element $b = [b_0 : b_1] \in \mathbb{P}^1(C)$ as follows: if $b_1 \neq 0$, we set $|b| = |b_0| |b_1|$ and $||1 : 0|| = \infty$ by convention.

**Lemma A.3.** For any non-degenerate walk, the following holds.
(1) For any \( a \in C \) with \(|a| = 1\), there exist \( b_\pm \in \mathbb{P}^1(C) \) with \(|b_-| < 1\), and \(|b_+| > 1\), such that \( K(a, b_\pm, t) = 0 \).

(2) For any \( b \in C \) with \(|b| = 1\), there exist \( a_\pm \in \mathbb{P}^1(C) \) with \(|a_-| < 1\), and \(|a_+| > 1\), such that \( K(a_\pm, b, t) = 0 \).

**Proof.** See [DR17, Section 1.3] for a similar result in the situation where \( C \) is replaced by \( C \).

The statements are symmetrical, so we shall only give a proof for the first one. Since \( C \) is algebraically closed and the walk is non degenerate, Proposition 1.5 implies that \( K(x, y, t) \) is of degree 2 in \( y \). Then, for any \( a \in C \), there are two elements \( b_\pm \in \mathbb{P}^1(C) \) such that \( K(a, b_\pm, t) = 0 \).

Assume that \(|a| = 1\) and write

\[
K(a, y, t) = t\alpha + \beta y + t\gamma y^2
\]

where

\[
\alpha = -\sum_{i=-1}^1 d_{i-1}a^{i+1}; \\
\beta = a - t \sum_{i=-1}^1 d_0a^{i+1}; \\
\gamma = -\sum_{i=-1}^1 d_{i+1}a^{i+1}.
\]

Since \(|a| = 1\), we find \(|\beta| = 1\), \(|\alpha|, |\gamma| \leq 1\). First let us remark that there is no point \((a, b) \in E\) such that \(|a| = |b| = 1\). Indeed if \(|b| \leq 1\), the equality \(|\beta b| = |t(\alpha + \gamma b^2)|\) implies \(|b| < 1\). Now let us remark that, if \( K(a, b, t) = 0 \) then

\[
(a.2) \quad \text{if } |b| < 1 \text{ then } |\alpha t| = |\beta b + t\gamma b^2| = |\beta b| \text{ which gives } |b| = |\alpha t|;
\]

\[
(a.3) \quad \text{if } |b| > 1 \text{ then } \frac{|b|}{|b|} < 1 \text{ and we find } |t\gamma| = \left|\frac{t\alpha + \beta b}{|b|^2} \right| = \left|\frac{\beta}{b} \right| = \frac{1}{|b|}.
\]

Now, we are ready to prove the lemma. Using \( K(a, b_\pm, t) = 0 \), we find

\[
b_\pm b_\mp = \frac{\alpha}{\gamma},
\]

with the convention that if \( \gamma = 0 \) then \( b_+ \) is \([1 : 0]\). In that case, \( b_\mp = \frac{\alpha}{\gamma} \) has a norm smaller than 1. This concludes the proof in the case \( \gamma = 0 \). Let us now assume that \( \gamma \neq 0 \). Then, since by the above \(|b_+|, |b_-| \neq 1\), we just need to discard the cases \(|b_+| < 1 \text{ and } |b_-| < 1\) or \(|b_+| > 1 \text{ and } |b_-| > 1\). First, assume that \( \alpha \neq 0 \). Suppose to the contrary that \(|b_+| < 1 \text{ and } |b_-| < 1\). By (A.2), \( |b_+| = |b_-| = |\alpha t| \) which gives

\[
|b_+b_-| = |\alpha|^2 = \frac{|\alpha|}{|\gamma|}
\]

that is \(|\alpha^2| = \frac{1}{|\gamma|} \geq 1\). Since \(|\alpha^2| < 1\), we find a contradiction. Suppose to the contrary that \(|b_+| > 1 \text{ and } |b_-| > 1\). By (A.3), \( |b_+| = |b_-| = \frac{1}{|\gamma|} \) which gives

\[
|b_+b_-| = \frac{1}{|\gamma|^2} = \frac{|\alpha|}{|\gamma|}
\]

that is \(|\alpha^2| = \frac{1}{|\gamma|^2} \geq 1\). Once again a contradiction. Finally if \( \alpha = 0 \), then one of the root is zero, say \( b_- = 0 \), and \( |b_+| = \frac{|\beta b|}{|\gamma|^2} > 1 \), which concludes the proof. \( \square \)

Finally, Lemma A.4 explains how the fundamental domain and its image by \( \sigma \) have a non empty intersection. It will be therefore crucial to continue the generating series on the whole \( C^* \) by using its functional equation with respect to \( \sigma \).

**Lemma A.4.** The following holds:

- if \( d_{-1,1} = 0 \) or \( d_{1,-1} \neq 0 \) there exists \((a, b) \in E\) with \(|a| = 1\) such that \( \sigma(a, b) = (a', b') \) with \(|a'| \leq 1\).
• if \(d_{1,-1} = 0\) or \(d_{-1,1} \neq 0\) there exists \((a, b) \in E\) with \(|b| = 1\) such that \(\sigma(a, b) = (a', b')\) with \(|b'| \leq 1\).

**Proof of Lemma A.4.** Via the symmetry between \(x\) and \(y\) mentioned in Remark 1.2, we see that it is enough to prove the first statement of Lemma A.4.

Let \(a \in \mathbb{P}^1(C)\) such that \(|a| = 1\). By Lemma A.3, there exist \(b_+ \in \mathbb{P}^1(C)\) with \(|b_+| > 1\) and \(b_- \in C\) with \(|b_-| < 1\) such that \((a, b_+) \in E\). Let \(B_i\) as in (1.2) and let \(\nu\) (resp \(d\)) be the valuation at 0 (resp \(\infty\)) of the rational fraction \(\frac{B_{-1}(y)}{B_1(y)} = \frac{\sum_{j=-1}^{d} a_j b^j}{\sum_{j=1}^{d-1} a_j b^j} \in C(y)\) (note that \(B_1\) is not identically zero by Proposition 1.5). We claim that either \(\nu \geq 0\) or \(d \leq 0\). If \(d_{1,-1} \neq 0\) then \(\nu \geq 0\). If \(d_{-1,1} = 0\) then either \(d \leq 0\) or \(d = 1\). In the latter situation, we must have \(d_{1,1} = d_{1,0} = 0\) and \(d_{-1,0} \neq 0\). Since the walk is not degenerate, we must have \(d_{1,-1} \neq 0\) by Proposition 1.5. In that case, \(\nu \geq 0\). This proves the claim.

Let \(a_+, a_- \in \mathbb{P}^1(C)\) such that \(t_2(a, b_+) = (a_+, b_+)\) and \(t_2(a, b_-) = (a_-, b_-)\). This gives

\[(A.5)\quad a_+ = \frac{B_{-1}(b_+)}{B_1(b_+)a} \quad \text{and} \quad a_- = \frac{B_{-1}(b_-)}{B_1(b_-)a}.
\]

Since \(\sigma(a, b_-) = (a_-, b_+)\) (resp \(\sigma(a, b_+) = (a_+, b_-)\)), it is enough to prove that either \(a_+\) or \(a_-\) has norm smaller or equal to 1. If \(d \leq 0\), we combine (A.5), Lemma A.2 and \(|b| > 1\) to find \(|aa_+| = |a_+| = |b_+|^d \leq 1\). If \(\nu \geq 0\), we combine (A.5), Lemma A.2 and \(|b_-| < 1\) to find \(|aa_-| = |a_-| = |b_-| \leq 1\). This ends the proof. \(\square\)

**Appendix B. Tate curves and their normal forms**

Let \((C, |\cdot|)\) be a complete non archimedean algebraically closed valued field of zero characteristic and let \(q \in C\) such that \(0 < |q| < 1\). In this section, we recall some of the basic properties of elliptic curves over non archimedean fields. As mentioned in the introduction, the classical notion of lattices is here replaced by its multiplicative analogue \(q^Z\) and the quotient of \(C\) by a lattice of periods by the *rigid analytic space* corresponding to the naive quotient of the multiplicative group \(C^\ast\) by \(q^Z\). In that context, the uniformization of an elliptic curve requires some technical assumptions on its \(J\)-invariant (see Proposition B.3). This proposition is the only strong result of rigid analytic geometry that we will use in this paper. Therefore, we will not introduce this theory and will refer the interested reader to [FvdP04]. In the sequel, we just recall briefly the algebraic geometrical and special functions aspects of Tate curves.

**B.1. Special functions on a Tate curve.** We recall that any holomorphic function \(f\) on \(C^\ast\) can be represented by an everywhere convergent Laurent series \(\sum_{n \in \mathbb{Z}} a_n q^n\) with \(a_n \in C\). Moreover any non-zero meromorphic function on \(C^\ast\) can be written as \(\frac{f}{q^N}\) such that the holomorphic functions \(g\) and \(h\) have no common zeros. We shall denote by \(\mathcal{Mer}(C^\ast)\) the field of meromorphic functions over \(C^\ast\).

**Proposition B.1.** For any \(q \in C^\ast\) such that \(0 < |q| < 1\), we define the series

- \(X(s) = \sum_{n \in \mathbb{Z}} \frac{q^n}{(1-q^n s^2)} = 2s_1\);
- \(Y(s) = \sum_{n \in \mathbb{Z}} \frac{(q^n s^2)^2}{(1-q^n s^2)^2} + s_1\).

They are \(q\)-periodic functions that are meromorphic over \(C^\ast\). Furthermore \(X(s) = X(1/s)\), and \(X(s)\) has a pole of order 2 at any element of the form \(q^2\). The field \(C_0\) of meromorphic functions over \(C^\ast\) that are \(q\)-periodic coincides with the field generated over \(C\) by \(X(s)\) and \(Y(s)\).

Moreover, the analytic map

\[\pi : C^\ast \to \mathbb{P}^2(C),\]

\[s \mapsto [X(s) : Y(s) : 1]\]
is onto and his image is \( E_q \subset \mathbb{P}^2 \), the elliptic curve defined as the zero set of

\[
Y^2 + XY = X^3 + BX + C
\]

where \( B = -5s_3 \) and \( C = -\frac{1}{12}(5s_3 + 7s_5) \) and \( s_k = \sum_{n>0} \frac{n^k q^n}{1-q^n} \in C \) for \( k \geq 1 \).

**Proof.** This is Theorem 5.1.4, Corollary 5.1.5, and Theorem 5.1.10 in [FvdP04]. \( \square \)

**Remark B.2.** If \( k \) is a complete non archimedian sub-valued field of \( C \) and \( q \in k \), every result quoted above still holds over \( k \).

The analytification of the elliptic curve \( E_q \) is isomorphic to the Tate curve, that is the rigid analytic space corresponding to the naive quotient of \( C^*/q^2 \). The curve \( E_q \) is therefore a “canonical” elliptic curve. A natural question is "Given an elliptic curve \( E \) defined over \( C \), is there a \( q \) such that \( E \) is isomorphic to \( E_q \)?" The answer is positive under certain assumption on the \( J \)-invariant \( J(E) \) of \( E \).

**Proposition B.3** (Theorem 5.1.18 in [FvdP04]). Let \( E \) be an elliptic curve over \( C \) of modulus \( J(E) \) such that \( |J(E)| > 1 \). Then, there exists \( q \in C \) such that \( 0 < |q| < 1 \) and such that \( E \) is isomorphic to the elliptic curve \( E_q \).

The functions \( X(s) \) and \( Y(s) \) are the building blocks of the uniformization of the Kernel curve. Since we need to understand what is the pullback of the fundamental domain of convergence of the generating series via this uniformization, we prove some basic properties on the norm of \( X(s) \).

Remind that \( X(s) = X(1/s) \) and \( X(qs) = X(s) \) so it suffices to study \( |X(s)| \) for \( |q|^{1/2} \leq |s| \leq 1 \). The following study follows the arguments of [Sil94, §V.4].

**Lemma B.4.** Let \( s \in C^* \). The following holds:

- If \( |q|^{1/2} < |s| < 1 \), then \( |X(s)| = |s| \).
- If \( |s| = 1 \), then \( |X(s)| \geq 1 \).
- If \( |s| = |q|^{1/2} \), then \( |X(s)| \leq |s| \).

**Proof.** Since \( X(s) \) has a pole in \( s = 1 \) we may further assume that \( s \neq 1 \). Let us rewrite \( X(s) \):

\[
X(s) = \frac{s}{(1-s)^2} + \sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2 \frac{q^n}{1-q^n}.
\]

This means that we have

\[
|X(s)| \leq \max \left( \left| \frac{s}{(1-s)^2} \right|, \sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2 \frac{q^n}{1-q^n} \right),
\]

with equality when \( |s| = \left| \frac{s}{(1-s)^2} \right| = \sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2 \frac{q^n}{1-q^n} \). Let us consider \( s \in C^* \setminus \{1\} \) with \( |q|^{1/2} \leq |s| \leq 1 \). Using \( |q| < 1 \) we find that for every \( n \geq 1 \), \( |q^n s| \leq |q| |s| < 1 \). This shows that the norm of \( q^n s \) is strictly smaller than 1, and then \( \left| \frac{q^n s}{(1-q^n s)^2} \right| = |q^n s|^{-1} < |s| \). On the other hand, \( |q^n| < |q| < |s| \) and \( \left| \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} \right| < |s| \). Finally, when \( |q|^{1/2} < |s| \), \( |q^n s^{-1}| \leq |q s^{-1}| < |q^{-1/2}| < |s| \) and therefore \( \left| \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} \right| = |q^n s^{-1}| < |s| \). This proves that for any \( s \in \mathbb{P}^1(C) \) such that \( |q|^{1/2} < |s| \leq 1 \), we have

\[
\sum_{n>0} \frac{q^n s}{(1-q^n s)^2} + \frac{q^n s^{-1}}{(1-q^n s^{-1})^2} - 2 \frac{q^n}{1-q^n} < |s|.
\]
When, \(|q|^{1/2} = |s|\) and \(n \geq 2\), the inequality holds \(|q^n s^{-1}| \leq |q^2 s^{-1}| = |q^2 q^{-1/2}| < |s|\), and therefore we find \(|q^n s^{-1} - y^{-1} + q^{-1} q^{-1/2} < |s|\). Moreover, if \(|q|^{1/2} = |s|\) then \(|q^{-1}| = |q|^{-1/2} = |s|\) and therefore \(|q^n s^{-1} - y^{-1} = |q^{-1} q^{-1/2} < |s|\). This gives

(B.3) \[\sum_{n > 0} \left( \frac{q^n s}{(1 - q^n s)^2} + \frac{q^n s^{-1}}{(1 - q^n s^{-1})^2} - \frac{2 q^n}{1 - q^n} \right) = |s|\]

It remains to consider the term \(|\frac{s}{(1 - s)^2}|\). Assume further that \(|s| \neq 1\). Then, \(|s| < 1\) and we have \(|\frac{s}{(1 - s)^2}| = |s|\). Combining with (B.1), (B.2) and (B.3) respectively, we obtain the result when \(|q|^{1/2} < |s| < 1\) and \(|q|^{1/2} = |s| < 1\) respectively.

Assume now that \(|s| = 1\) and \(s \neq 1\). Then \(|1 - s| \leq 1\). Then, \(|\frac{s}{(1 - s)^2}| \geq |s| = 1\), which, combined with (B.1) and (B.2) shows the result. □

B.2. Tate and Weierstrass normal forms. In [DR17], the authors generalize the results of [KR12] and attach a Weierstrass normal form to the Kernel curve. The following proposition proves that, with some care, their result passes to a non archimedean framework.

Denoting \(K(x, y, t) = A_0(x) + A_1(x) y + A_2(x) y^2\) and \(B_0(x) + B_1(x) y + B_2(x) y^2\) with \(A_i(x) \in C[x]\) and \(B_i(y) \in C[y]\), one finds:

**Proposition B.5.** Let \(a \in C\) be as in Lemma A.1. Let \(E_1\) be the elliptic curve defined by the Weierstrass equation

(B.4) \[y_1^2 = 4x_1^3 - g_2 x_1 - g_3,\]

with

(B.5) \[g_2 = -\frac{\mathcal{D}^{(2)}(a)^2}{3} - 2\frac{\mathcal{D}^{(1)}(a)\mathcal{D}^{(3)}(a)}{3}, \quad g_3 = -\frac{\mathcal{D}^{(2)}(a)^3}{27} + \frac{\mathcal{D}^{(1)}(a)\mathcal{D}^{(2)}(a)\mathcal{D}^{(3)}(a)}{9} - \frac{\mathcal{D}^{(1)}(a)^2\mathcal{D}^{(4)}(a)}{6}.\]

Then, the rational map

\[E_1 : [x_1 : y_1 : 1] \mapsto E \subset \mathbb{P}^1(C) \times \mathbb{P}^1(C)\]

where

\[\tau = a + \frac{\mathcal{D}^{(1)}(a)}{x_1 - \frac{\mathcal{D}^{(2)}(a)}{6}} \quad \text{and} \quad \bar{y} = \frac{\mathcal{D}^{(1)}(a) y_0}{2(x_1 - \frac{\mathcal{D}^{(2)}(a)}{6})^2} - \frac{\bar{A}_1}{2\bar{A}_2} \left( a + \frac{\mathcal{D}^{(1)}(a)}{x_1 - \frac{\mathcal{D}^{(2)}(a)}{6}} \right),\]

is an isomorphism of elliptic curves that sends the point \(\mathcal{O} = [1 : 0 : 0]\) of the Weierstrass form to the point \(\left(a, -\frac{\bar{A}_1}{2\bar{A}_2}\right) \in E\).

**Proof.** This is the same proof as in [DR17, Proposition 18]. Note that there is only on configuration here since we have chosen a root of the discriminant \(|a| < 1\) which can not be infinity. □

We recall that the modulus of the elliptic curve given in the Weierstrass form \(E : y^2 = 4x^3 - g_2 x - g_3\) equals to \(J(E) = 12³ \left(\frac{g_2}{g_2 - 27g_3}\right)\). Moreover, it is proved in Lemma 1.10 that the \(J\)-invariant \(J(E) = J(E_1)\) has modulus strictly greater than 1. By Proposition B.3 there exists \(q \in C^*\) such that \(0 < |q| < 1\) and \(E_1\) is isomorphic to \(E_q\). In order to explicit...
Proof. Lemma B.8. Lemma B.7. follows from $\Delta$ is an isomorphism of elliptic curves. Moreover, the following holds

$$X = x - \frac{1}{12}$$

onto the Weierstrass equation

$$Y^2 + XY = X^3 + BX + C$$

where $h_2 = \frac{1}{12} + 20s_3$ and $h_3 = \frac{1}{12} + \frac{5}{3}s_5$. The $J$-invariant of $E_q$ is given by

$$J(E_q) = \frac{12^3h_3^2}{\Delta_q}, \text{ where } \Delta_q = h_2^3 - 27h_3^2.$$ (B.6)

As detailed above, the elliptic curves $E_1$ and $E_q$ are isomorphic. The following lemma gives the form of an explicit isomorphism between these two curves.

Lemma B.7. There exists $u \in C^*$ such that the following map

$$E_q \to E_1,$$

$$(X, Y) \mapsto (u^2(X + \frac{1}{12}), u^3(2Y + X))$$
is an isomorphism of elliptic curves. Moreover, the following holds

- $h_2 = \frac{22}{u^2}$ and $h_3 = \frac{23}{u^2}$;
- $\Delta_q = \frac{27}{u^2}$ where $\Delta_1$ and $\Delta_q$ denote the discriminants of the Weierstrass equations of $E_1$ and $E_q$ respectively.

Proof. Let $y^2 = 4x^3 - h_2x - h_3$ be the Weierstrass normal form of $E_q$ as in Lemma B.6. From [Sil09, Proposition 3.1, Chapter III], we deduce that any isomorphism between the elliptic curves $E_1$ and $E_q$ is as follows $x_1 = u^2x + \alpha$ and $y_1 = u^3y + \beta u^2x + \gamma$ with $u \in C^*$, $\alpha, \beta, \gamma \in C$. Since both equations are in Weierstrass normal form, we necessarily have $\alpha = \beta = \gamma = 0$. Combining this remark with Lemma B.6 proves the first point. From $y_1^2 = 4x_1^3 - g_2x_1 - g_3$, we substitute $x_1, y_1$ by $x, y$ to find

$$u^6y^2 = 4u^6x^3 - g_2u^2x - g_3.$$ (4)

Dividing the both sides by $u^6$ we find $h_2 = \frac{22}{u^2}$ and $h_3 = \frac{23}{u^2}$. The assertion on the discriminant follows from $\Delta_q = h_2^3 - 27h_3^2$ and $\Delta_1 = g_2^3 - 27g_3^2$. □

The lemma below gives some precise estimate for the norms of $\Delta_q = h_2^3 - 27h_3^2$ and $\Delta_1 = g_2^3 - 27g_3^2$, the discriminants of the elliptic curves $E_q, E_1$ and the element $u$ defined in Lemma B.7.

Lemma B.8. The following holds

- $|\Delta_q| = |q|$, with $|h_2 - \frac{1}{12}| = |q|$ and $|h_3 - \left(-\frac{1}{7}\right)| = |q|$;
- $|\Delta_1| = |q|$ with $|g_2 - \frac{4}{3}| < 1$, $|g_3 - \left(-\frac{5}{27}\right)| < 1$;
- $|u| = 1$;
- $|\Omega^{(1)}(a)| \leq |q|^{1/2}, 1|.$

Proof. Following [Roq70, Pages 29-30], we find $|\Delta_q| = |q|, |s_3| = |q| = |s_5|$. Combining the latter norm estimates with Lemma B.6, we find $|h_2 - \frac{1}{12}| = |q|$ and $|h_3 - \left(-\frac{1}{7}\right)| = |q|$. Let us prove the second point. It follows from (1.5) that $|1 - \alpha_i| < 1$ and $|\alpha_i| < 1$ for $i = 0, 1, 3, 4$. By Lemma A.1, $|\Omega^{(1)}(a)|, |\Omega^{(3)}(a)|, |\Omega^{(4)}(a)| < 1, |\Omega^{(2)}(a) - 2| < 1$. Combining these norm estimates with (B.5), we find $|g_2 - \frac{4}{3}| < 1$, $|g_3 - \left(-\frac{5}{27}\right)| < 1$. Since $|J(E_1)| = |J(E_q)| = \frac{|12^3s_2|}{|\Delta_q^2|}$ and $|g_2| = |h_2| = 1$, we find $|\Delta_q| = |\Delta_1| = |q|$. By Lemma B.7, $\Delta_q = \frac{27}{u^2}$, and then
Let us prove the last point. Let us expand $\Delta_1 = g_2^2 - 27g_3^2$ with the expression of $g_2, g_3$ given in (B.5):

$$\Delta_1 = \left(\frac{\mathcal{D}^{(2)}(a)^2}{3} - 2\frac{\mathcal{D}^{(1)}(a)\mathcal{D}^{(3)}(a)}{3} - 27\sqrt[3]{\frac{\mathcal{D}^{(2)}(a)^2 + \mathcal{D}^{(1)}(a)\mathcal{D}^{(3)}(a) - \mathcal{D}^{(1)}(a)^2\mathcal{D}^{(4)}(a)}{6}}\right)^3.$$ 

By Lemma B.8 again, we find

$$\mathcal{B.5}$$

with

$$\omega = \frac{\mathcal{D}^{(2)}(a)^2}{3} + \mathcal{D}^{(1)}(a)\omega', \quad g_3 = \frac{-\mathcal{D}^{(2)}(a)^3}{27} + \mathcal{D}^{(1)}(a)\omega'',$$

where $|\omega|, |\omega'| < 1$. This proves that

$$\frac{g_3}{g_2} = \frac{-\mathcal{D}^{(2)}(a)}{9} + \mathcal{D}^{(1)}(a)\omega''$$

with $|\omega''| < 1$. Then, we find

$$|\frac{u^2}{12} - \mathcal{D}^{(2)}(a)/6| = \frac{u^2}{12} + 3\frac{g_3}{2g_2} - 3\frac{g_3}{2g_2} - \mathcal{D}^{(2)}(a)/6 \leq \max \left(\frac{u^2}{12} + \frac{3g_3}{2g_2}, \frac{3}{2}\mathcal{D}^{(1)}(a)\omega''\right).$$

Finally, with the norm estimate of Lemma B.8, it is sufficient to show that $|\frac{u^2}{12} + \frac{3g_3}{2g_2}| < |q|$. By Lemma B.7, we have $\frac{u^2}{12} = \frac{g_2 h_2}{12h_3}$. By Lemma B.8, $|h_2 - \frac{1}{12}| = |q|$ and $|h_3 - (-\frac{1}{63})| = |q|$. Then, by Lemma B.8 again, we find

$$|h_2 + 18h_3| = \left|\left(h_2 - \frac{1}{12}\right) + 18 \left(h_3 - \left(-\frac{1}{63}\right)\right)\right| \leq \max \left(\left|\frac{1}{12} + 18h_3\right|, \left|\frac{1}{12} + 18h_3\right|\right) \leq |q|.$$

□

**Appendix C. Difference Galois theory**

In this section, we establish some criteria to guaranty the transcendance of functions satisfying a difference equation of order 1. These criteria are based on the Galois theory of difference fields as developed in [vdPS97] but generalizes some of the existing results in the literature, for instance the assumption on algebraically closed field of constants (see for instance Theorem C.8).

The algebraic framework of this section is difference algebra and more precisely the notion of difference fields. A difference field is a pair $(K, \sigma)$ where $K$ is a field and $\sigma$ is an automorphism of $K$. The $\sigma$-constants $K^\sigma$ of $(K, \sigma)$ are the elements $f \in K$ such that $\sigma(f) = f$. An extension
Proof. Let \( \exists g \in f \). Lemma C.3. Let \( \exists x \in L \). The following statements are equivalent

1. \( x \) is algebraic over \( K^\sigma \);
2. there exists \( r \in \mathbb{N}^* \) such that \( \sigma^r(x) = x \).

**Proof.** Assume that \( x \) is algebraic over \( K^\sigma \). Then, \( \sigma \) induces a permutation on the set of roots of the minimal polynomial of \( x \) over \( K^\sigma \). Thus, there exists \( r \in \mathbb{N}^* \) such that \( \sigma^r(x) = x \). Conversely, if there exists \( r \in \mathbb{N}^* \) such that \( \sigma^r(x) = x \), the polynomial \( P(X) = \prod_{i=0}^{n-1}(X - \sigma^i(x)) \in L[X] \) is fixed by \( \sigma \) and thereby \( P(X) \in L^\sigma[X] = K^\sigma[X] \). Since \( P(x) = 0 \), we have proved that \( x \) is algebraic over \( K^\sigma \). \( \square \)

**Lemma C.2.** Let \( (K, \sigma) \subset (L, \sigma) \) be an extension of difference fields such that \( L^\sigma = K^\sigma \). Let \( f \in L \) and \( 0 \neq c \in K \), such that \( \sigma(f) = f + c \). The following statements are equivalent

1. \( f \in K \);
2. \( f \) is algebraic over \( K \);
3. There exists \( \alpha \in K \) such that \( \sigma(\alpha) = \alpha + c \).

Moreover, let \( K \) be the algebraic closure of \( K \) endowed with a structure of \( \sigma \)-field extension of \( K \). If \( f \) is transcendental over \( K \) then for all \( \alpha \in K \), \( i, j \in \mathbb{Z} \) such that \( i \neq j \), the elements \( \sigma^i(f - \alpha f) = f - (\sigma^i(\alpha) - c + \sigma(c) - \cdots - \sigma^{i-1}(c)) \) and \( \sigma(f) - \alpha f = f - (\sigma^j(\alpha) - c - \sigma(c) - \cdots - \sigma^{j-1}(c)) \) are distinct.

**Proof.** Let us prove the first part of the proposition. The first statement implies trivially the second one. Assume that \( f \) is algebraic over \( K \) and let \( P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in K[X] \) be its minimal polynomial over \( K \). Note that \( n \neq 0 \). Using \( \sigma(f) = f + c \) and \( P(f) = 0 \), we find that \( \sigma(P(f)) - P(f) = 0 = (n + \sigma(a_{n-1}) - a_{n-1})f^{n-1} + b_{n-2}f^{n-2} + \cdots + b_0 \) with \( b_i \in K \) for \( i = 0, \ldots, n-2 \) by minimality of \( P(X) \), we find that \( \sigma(a_{n-1}) - a_{n-1} = -nc \) with \( a_{n-1} \in K \).

Then, \( \sigma(\alpha) - \alpha f = c \) with \( \alpha = \frac{a_{n-1}}{n} \in K \). We have shown that the second statement implies the third. Finally, if there exists \( \alpha \in K \) such that \( \sigma(\alpha) = \alpha + c \). With \( \sigma(f) - f = c \), we find that \( \sigma(\alpha - f) = -f \). This gives that \( f - \alpha \in L^\sigma = K^\sigma \) and the element \( f \) belongs to \( K \).

Now, let us assume that \( f \) is transcendental over \( K \). Suppose to the contrary that there exist \( \alpha \in K \) and \( i > j \in \mathbb{Z} \) such that \( \sigma^i(f - \alpha) = \sigma^j(f - \alpha) \). Using the functional equation satisfied by \( f \), the latter equality gives \( \sigma^i(\beta) = \gamma \) where \( r = i - j > 0, \beta = \sigma^i(\alpha) \) and \( \gamma = \sigma^{i-1}(c) + \cdots + \sigma(c) \). Since \( \alpha \) is algebraic over \( K \), the same holds for \( \beta \). Let \( P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in K[X] \) \( K \) be the minimal polynomial of \( \beta \) over \( K \) Using the fact that \( \sigma^i(\beta) = \gamma \) and the minimality of \( P \), we conclude, as above, that \( \sigma^i(a_{n-1}) - a_{n-1} = -n\gamma \), that is \( \sigma^i(\beta) - \beta = \gamma \) where \( \beta = \frac{a_{n-1}}{n} \in K \). Combining this equality with \( \sigma^i(\sigma^j(f)) = \sigma^i(f) = \gamma \), we find that \( \hat{\beta} - \sigma^j(f) \in L \) is fixed by \( \sigma^i \). By Lemma C.1, this means that \( f \in K^\sigma \), which yields to \( f \) algebraic over \( K \). A contradiction. \( \square \)

**Lemma C.3.** Let \( (K, \sigma) \subset (L, \sigma) \) be an extension of difference fields such that \( L^\sigma = K^\sigma \). Let \( f \in L \) and \( 0 \neq c \in K \), such that \( \sigma(f) = f + c \). Assume that \( f \) is transcendental over \( K \). If there exists \( g \in K(f) \) such that \( \sigma(g) = g \in K[f] \), then \( g \in K[f] \).

**Proof.** Let \( K \) be an algebraic closure of \( K \), endowed with a structure of \( \sigma \)-field extension of \( K \). Since \( f \) is transcendental over \( K \), we can write a partial fraction decomposition of \( g \in K(f) \). Let
If there is no confusion, we shall omit the subscripts $K$, $\sigma$ to the framework of the genus zero and genus one Kernel curve. In the two examples, $\Delta$ is the largest integer such that there exists $\alpha$, $\beta$ be the largest integer such that there exists $\alpha$, $\beta$.

\section*{C.2. Differential transcendence criteria.}

In this section, a $(\sigma, \partial, \Delta)$-field $K$ is a difference field $(K, \sigma)$ endowed with two derivations $\partial, \Delta$ commuting with $\sigma$ such that $\partial \Delta - \Delta \partial = cK \partial$ with $c_K \in K^\sigma$. We assume that $\partial$ is non trivial on $K$, that is, it is not the zero derivation. The element $c_K$ has to be considered as part of the data of the notion of $(\sigma, \partial, \Delta)$-field.

An extension of $(\sigma, \partial, \Delta)$-fields is an inclusion of two $(\sigma, \partial, \Delta)$-fields $(K, \sigma, \partial, \Delta_K) \subset (L, \sigma, \partial, \Delta_L)$ such that

\begin{itemize}
  \item $K \subset L$ is a field extension;
  \item $\sigma_K, \partial_K, \Delta_K$ are the restrictions of $\sigma_L, \partial_L, \Delta_L$ to $K$;
  \item $c_K = c_L$.
\end{itemize}

If there is no confusion, we shall omit the subscripts $K$, $L$. If $\sigma$ is the identity, we shall speak of $(\partial, \Delta)$-fields, $(\partial, \Delta)$-fields extension for short.

\subsection*{Example C.4.}

As proved in \S 6, the following fields are $(\sigma, \partial, \Delta)$-fields, that correspond respectively to the framework of the genus zero and genus one Kernel curve. In the two examples, we have $\Delta_{q,t} = \partial_t(q)\ell_q(s)\partial_s + \partial_t; \text{ where } \ell_q$ is the so called $q$-logarithm, that is an element of $\text{Mer}(C^*)$ satisfying $\sigma_q(\ell_q) = \ell_q + 1$, and $c_K = \partial_t(q)\partial_s(\ell_q) \in C_q$.

- Let $q \in C^*$ with $|q| \neq 1$. Then, let us consider $(C_q(s, \ell_q), \sigma_q, \partial_s, \Delta_{t,q}) \subset (\text{Mer}(C^*), \sigma_q, \partial_s, \Delta_{t,q})$.

- Let $q$ and $q'$ two elements of $C^*$ such that $|q|, |q'| \neq 1$, that are multiplicatively independent, that is, there are no $r, l \in \mathbb{Z}^2 \setminus \{0, 0\}$ such that $q^r = q'^l$. Since $C_q \subset \text{Mer}(C^*)$ and $C_{q'} \subset \text{Mer}(C^*)$, we may consider $C_q, C_{q'} \subset \text{Mer}(C^*)$, the compositum of fields $C_q, C_{q'} \subset \text{Mer}(C^*)$. Then, let us consider $(C_q, C_{q'}, \ell_q, \ell_{q'}, \sigma_q, \partial_s, \Delta_{t,q}) \subset (\text{Mer}(C^*), \sigma_{q'}, \partial_s, \Delta_{t,q})$.

\subsection*{Definition C.5.}

Let $(K, \partial, \Delta) \subset (L, \partial, \Delta)$. An element $f \in L$ is said to be $(\partial, \Delta)$-differentially algebraic over $K$ if there exists $N \in \mathbb{N}$, such that the elements

\begin{itemize}
  \item $\partial^i(f)$ for $i \leq N$ are algebraically dependent over $K$ if $\Delta$ is a $K$-multiple of $\partial$;
  \item $\partial^i \Delta^j(f)$ for $i, j \leq N$ are algebraically dependent over $K$ otherwise.
\end{itemize}

Otherwise, we will say that $f$ is $(\partial, \Delta)$-transcendent over $K$.

\subsection*{Remark C.6.}

Note that since $\partial \Delta - \Delta \partial = c \partial$ with $c \in K^\sigma \subset K$, the $(\partial, \Delta)$-field extension of $K$ generated by some element $f \in L$ coincides with the field extension of $K$ generated by the set $\{\partial^i \Delta^j(f), i, j \in \mathbb{N}\}$.

The following lemma will be crucial in many arguments:

\subsection*{Lemma C.7.}

If $K \subset M$ is a $\sigma$-field extension such that $M^\sigma = K$ and $K \subset L$ is a $\sigma$-field extension with $L^\sigma = L$. Then $M$ and $L$ are linearly disjoint over $K$. 

Proof. Let $c_1, \ldots, c_r \in L$ be $K$-linearly independent elements, that become dependent over $M$. Up to a permutation of the $c_i$’s, a minimal linear relation among these elements over $M$ has the following form
\begin{equation}
(c.1) \quad c_1 + \sum_{i=2}^{r} \lambda_i c_i = 0,
\end{equation}
with $\lambda_i \in M$ for $i = 2, \ldots, r$. Computing $\sigma((c.1)) - (c.1)$, we find
\begin{equation*}
\sum_{i=2}^{r} (\sigma(\lambda_i) - \lambda_i) c_i = 0.
\end{equation*}
By minimality, $\sigma(\lambda_i) = \lambda_i$ and $\lambda_i \in M^\sigma = K$. By $K$-linear independence of the $c_i$, we find that $\lambda_i = 0$ for $i = 2, \ldots, r$ and then $c_1 = 0$. A contradiction.

The following statement, whose proof is due to Michael Singer, is a version of an old theorem of Ostrowski [Ost46, Kol68] and its proof follows the lines of the proof of [DHRS18, Proposition 3.6]. In this last paper, it was assumed that $K^\sigma$ is algebraically closed which is not the case in this article. One could use the powerful scheme-theoretic tools developed in [OW15] to prove the result in our more general setting. Instead we will argue in a more elementary way to reduce Theorem C.8 to the case where $K^\sigma$ is algebraically closed. To do this we will use results from [DHRS18] which already have been useful in studying generating series of walks.

Theorem C.8. Let $(K, \sigma, \partial, \Delta)$ be a $(\sigma, \partial, \Delta)$-field such that $K^\sigma$ is relatively algebraically closed in $K$, that is there are no proper algebraic extension of $K^\sigma$ inside $K$. Let $(L, \sigma, \partial, \Delta)$ be a $(\sigma, \partial, \Delta)$-ring extension of $(K, \sigma, \partial, \Delta)$. Let $f \in L$ and $b \in K$ such that $\sigma(f) = f + b$. If $f$ is $(\partial, \Delta)$-differentially algebraic over $K$ then there exist $\ell_1, \ell_2 \in \mathbb{N}, c_{i,j} \in K^\sigma$ not all zero and $g \in K$ such that
\begin{equation}
(C.2) \quad \sum_{0 \leq i \leq \ell_1, \ 0 \leq j \leq \ell_2} c_{i,j} \partial^i \Delta^j(b) = \sigma(g) - g.
\end{equation}
Furthermore, we may take $\ell_2 = 0$ in the case where $\partial$ and $\Delta$ are $K$-linearly dependent.

The proof of this result depends on results from the Galois theory of linear difference equations and we will refer to [DHRS18, Appendix A] and the references given there for relevant facts from this theory. Let $(K, \sigma)$ be a difference field and consider the system of difference equations
\begin{equation}
(C.3) \quad \sigma(y_0) - y_0 = b_0, \ldots, \sigma(y_n) - y_n = b_n, \quad b_0, \ldots, b_n \in K.
\end{equation}
Let us see (C.3) as a system $\sigma(Y) = AY$, where $A \in \text{GL}_{2(n+1)}(K)$ is a diagonal bloc matrix $A = \text{Diag}(A_0, \ldots, A_n)$ with $A_i = \begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix}$ which correspond to the equation $\sigma(y_i) - y_i = b_i$. A Picard-Vessiot extension for $\sigma(Y) = AY$ is a difference ring extension $(R, \sigma)$ of $(K, \sigma)$ such that:
- there exists $U \in \text{GL}_{2n+2}(K)$ such that $\sigma(U) = AU$;
- $R$ is generated as a $K$-algebra by the entries of $U$ and $\det(U)^{-1}$;
- $R$ is a simple difference ring, that is, the $\sigma$-ideals of $R$ are $\{0\}$ and $R$.

We will need the following result.

Lemma C.9 (Proposition A.9 in [DHRS18]). Assume that $(K, \sigma)$ is a difference field with $K^\sigma$ algebraically closed. Let $R$ be the Picard-Vessiot extension for the system (C.3) and $z_0, \ldots, z_n \in R$ be solutions of this system. If $z_0, \ldots, z_n$ are algebraically dependent over $K$, then there exist $c_i \in K^\sigma$, not all zero, and $g \in K$ such that
\begin{equation*}
c_0 b_0 + \ldots + c_n b_n = \sigma(g) - g.
\end{equation*}
Before proving Theorem C.8, we give a slight generalization of Lemma C.9.

**Lemma C.10.** Let $(K, \sigma)$ be a difference field with $K^\sigma$ relatively algebraically closed in $K$ and let $b_0, \ldots, b_n$ be some elements in $K$. Let $(L, \sigma)$ be a $\sigma$-ring extension of $(K, \sigma)$. Let $z_0, \ldots, z_n \in L$ be solutions of $\sigma(z_i) - z_i = b_i$. If $z_0, \ldots, z_n$ are algebraically dependent over $K$, then there exist $c_i \in K^\sigma$, not all zero, and $g \in K$ such that

$$c_0 b_0 + \cdots + c_n b_n = \sigma(g) - g.$$

**Proof.** Let $K$ be the algebraic closure of $K^\sigma$. We extend $\sigma$ to be the identity on $K^3$. Under the assumption that $K^\sigma$ is relatively algebraically closed, the ring $\tilde{K} = K \otimes_{K^\sigma} K$ is an integral domain and in fact is a field. We have $\tilde{K}^\sigma = K$. Let $\tilde{L} = L \otimes_{K^\sigma} K$. We then have a natural inclusion of $\tilde{K} \subset \tilde{L}$. Let $S = \tilde{K}[z_0, \ldots, z_n] \subset \tilde{L}$. It is easily seen that $S$ is a $\sigma$-ring extension of $\tilde{K}$. Let $I$ be a maximal difference ideal in $S$ and let $R = S/I$. For each $r = 0, \ldots, n$, let $u_r$ be the image of $z_r$ in $R$. Since $\tilde{K}^\sigma = K$ is algebraically closed and $R$ is a simple difference ring, we have that $R$ is a Picard-Vessiot ring for the system associated to $\sigma(y_r) - y_r = b_r$, $r = 0, \ldots, n$, over $\tilde{K}$. The elements $u_0, \ldots, u_n$ are algebraically dependent over $K$ and solutions of $\sigma(y_r) - y_r = b_r, r = 0, \ldots, n$. Lemma C.9 proves that there exist $c_i \in K$, not all zero, and $g \in \tilde{K}$ such that

$$\sum_{0 \leq i \leq n} c_i b_i = \sigma(g) - g.$$

Let $\{d_r\} \subset K$ be a $K^\sigma$-basis of $K$. By Lemma C.7, it is also a $K$-basis of $\tilde{K}$. We may write each $c_i$ and $g$ as

$$c_i = \sum_r c_{i,r} d_r$$

and

$$g = \sum_r g_r d_r$$

for some $c_{i,r} \in K^\sigma$ and $g_r \in K$. Since not all the $c_i$ are zero, there exists $r$ such that $c_{i,r}$ are not all zero. For this $r$, we have

$$\sum_{i \leq n} c_{i,r} b_i = \sigma(g_r) - g_r.$$

This yields the conclusion of the proof. \hfill \square

**Proof of Theorem C.8.** Assuming that $f$ is $(\partial, \Delta)$-differentially algebraic over $K$, there is some finite set $\{\partial^{j_0} \Delta^{j_0}(f), \ldots, \partial^{j_n} \Delta^{j_n}(f)\} \subset L$ of elements that are algebraically dependent over $K$. Note that $j_k = 0$ for all $k$ if $\Delta$ is $K$-linearly dependent from $\partial$. Since $\sigma$ commutes with $\Delta$ and $\partial$, we have for all $r = 0, \ldots, n$,

$$\sigma(\partial^{j_r} \Delta^{j_r}(f)) = \partial^{j_r} \Delta^{j_r}(f) = \partial^{j_r} \Delta^{j_r}(b).$$

To conclude it remains to apply Lemma C.10 with $z_r = \partial^{j_r} \Delta^{j_r}(f)$ and $b_r = \partial^{j_r} \Delta^{j_r}(b)$ for $r = 0, \ldots, n$. \hfill \square

**Appendix D. Meromorphic functions on a Tate curve and their derivations**

**D.1. Non archimedean elliptic functions.** Let $q \in C^*$ such that $|q| \neq 1$. Let us denote by $\sigma_q$ the automorphism of $Mer(C^*)$ by $\sigma_q(f(s)) = f(qs)$. We denote by $C_q$ the field of meromorphic functions fixed by $\sigma_q$. By Proposition B.1, it is the field of meromorphic functions on the Tate curve $E_q$ or $E_{1/q}$, depending whether $|q| < 1$ or $|q| > 1$. In this section, we consider as in [DVH12, §2] a derivation of these functions that encode their $t$-dependencies and commute with $\sigma_q$.\footnote{On the other hand, there is no unique procedure to extend a field automorphism of $K^\sigma$ to the algebraic closure $K$. Indeed, these extensions are controlled by the Galois group of the field $K$ over $K^\sigma$.}
The fact that $\partial_s = s \frac{d}{dt}$ acts on $\mathcal{Mer}(C^*)$, and its commutation with $\sigma_q$ is straightforward. Unfortunately, the $t$-derivative of $q$ may be non trivial, implying a more complicated commutation rule between $\partial_t = t \frac{d}{dt}$ and $\sigma_q$. More precisely, we have

$$\partial_s \circ \sigma_q = \sigma_q \circ \partial_s; \quad \partial_t \circ \sigma_q = \partial_t(q) \sigma_q \circ \partial_s + \sigma_q \circ \partial_t.$$ 

The following holds:

**Lemma D.1.** The $\partial_s$-constants $\mathcal{Mer}(C^*)^{\partial_s} = \{ f \in \mathcal{Mer}(C^*) | \partial_s(f) = 0 \}$ of $\mathcal{Mer}(C^*)$ are precisely the constant functions $C^*$.

Let us assume now that $|q| > 1$ and let us consider the Jacobi Theta function

$$\theta_q(s) = \sum_{n \in \mathbb{Z}} q^{-n(n+1)/2} s^n \in \mathcal{Mer}(C^*).$$

This is a meromorphic function on $C^*$ and it satisfies the the $q$-difference equation

$$\theta_q(qs) = s \theta_q(s).$$

We also consider the logarithmic derivative $\ell_q(s) = \frac{\partial_s(qs)}{q_s} \in \mathcal{Mer}(C^*)$ so that $\ell_q(qs) = \ell_q(s) + 1$. If $|q| < 1$ then $-\ell_{1/q} \in \mathcal{Mer}(C^*)$ is solution of $\sigma_q(-\ell_{1/q}) = -\ell_{1/q} + 1$. Abusing notations, we still note $\ell_q$ this function when $|q| < 1$. Then, for every $|q| \neq 1$, we have defined $\ell_q(s) \in \mathcal{Mer}(C^*)$ such that $\sigma_q(\ell_q) = \ell_q + 1$. Next Lemma introduces a twisted $t$-derivative that commutes with $\sigma_q$.

**Lemma D.2** (Lemma 2.1 in [DVH12]). The following derivations of $\mathcal{Mer}(C^*)$

$$\begin{cases} \partial_s \\ \Delta_{t,q} = \partial_t(q) \ell_q(s) \partial_s + \partial_t, \end{cases}$$

commute with $\sigma_q$. Moreover, we have

$$\partial_s \Delta_{t,q} - \Delta_{t,q} \partial_s = \partial_t(q) \partial_s(\ell_q) \partial_s,$$

where $\partial_t(q) \partial_s(\ell_q) \in C_q$.

**Remark D.3.** Note that since $\partial_s, \Delta_{t,q}$ commute with $\sigma_q$, we can derive the equation $\sigma_q(\ell_q) = \ell_q + 1$ to get that $\sigma_q(\partial_s(\ell_q)) = \partial_s(\ell_q)$ and $\sigma_q(\Delta_{t,q}(\ell_q)) = \Delta_{t,q}(\ell_q)$. We then have $\partial_t(\ell_q), \Delta_{t,q}(\ell_q) \in C_q$.

The link with the iterates of $\Delta_{t,q}$ and the derivatives $\partial_s, \partial_t$ is now made in the following lemma.

**Lemma D.4.** For any $i \in \mathbb{N}$, there exist $c_{j,k,l} \in C_q$ such that

$$\Delta_{t,q}^i = (\partial_t(q) \ell_q)^i \partial_s^i + \sum_{k=0}^{i-1} \sum_{j=0}^{k} \sum_{l=0}^{i-k} c_{j,k,l} (\ell_q)^j \partial_s^k \partial_t^l.$$ 

**Proof.** Let us prove the result by induction on $i$. For $i = 1$, this comes from the fact that $\Delta_{t,q} = \partial_t(q) \ell_q \partial_s + \partial_t$. Let us fix $i \in \mathbb{N}$ and assume that the result holds for $i$. We find

$$\Delta_{t,q}^{i+1} = (\partial_t(q) \ell_q)^i \partial_s^i \partial_t + \sum_{k=0}^{i-1} \sum_{j=0}^{k} \sum_{l=0}^{i-k} c_{j,k,l} (\ell_q)^j \partial_s^k \partial_t^l,$$

that is

$$\Delta_{t,q}^{i+1} = (\partial_t(q) \ell_q)^i \partial_s^i \partial_t + \sum_{k=0}^{i-1} \sum_{j=0}^{k} \sum_{l=0}^{i-k} c_{j,k,l} (\ell_q)^j \partial_s^k \partial_t^l.$$
Note that the commutation of $\sigma_q$ with $\Delta_{t,q}$ implies that $C_q$ is stabilized by $\Delta_{t,q}$. Since by Remark D.3, $\Delta_{t,q}(q)$ belongs to $C_q$, we get that for any integer $j$, any $\tilde{c} \in C_q$, we have $\Delta_{t,q}(\tilde{c}(t,q)^j) = \Delta_{t,q}(\tilde{c})t(q)^j + c\tilde{c}(t,q)^{j-1}$ where $c = j\Delta_{t,q}(q) \in C_q$. Therefore, with $\Delta_{t,q}(\tilde{c}) \in C_q$, we find that $\Delta_{t,q}(\tilde{c}(t,q)^j) \in C_q[t]$ is of degree at most $j$. With $\partial_t(q), c_{j,k,l} \in C_q$, this ends the proof.

D.2. Transcendence properties. The goal of this subsection is to prove some transcendence properties of $q$-logarithm in order to perform some descent procedure on telescopes. More precisely, we need to prove that the assumptions (H1) to (H3) of Proposition D.13 are satisfied for the fields $C_q(s)$ and $C_q, C_q'(t,q,\ell_q)$ for $q$ and $\ell_q$ two multiplicatively independent elements of $\mathbb{C}$ with $|q| \neq 1, |q| \neq 1$. We recall that $q$ and $\ell_q$ are multiplicatively independent if there are no non-zero elements $a, b \in \mathbb{Z} \setminus \{0\}$ such that $aq = b\ell_q$. From now, let us assume that $q$ and $\ell_q$ are multiplicatively independent. Remind that $C_q, C_q' \subset \operatorname{Mer}(\mathbb{C})$ is the compositum of fields and $\ell_q \in \operatorname{Mer}(\mathbb{C})$ is a solution of $\sigma_q(y) = y + 1$. We start with the assumption (H1).

Lemma D.5. The following holds:

1. The field $C_q(s, \ell_q)$ is stabilized by $\sigma_q$, $\partial_s$, and $\Delta_{t,q}$. The field $C_q(s)$ is stabilized by $\sigma_q$, $\partial_s$, and $\Delta_{t,q}$. The field $C_q(s)$ is stable by $\partial_s$, $\partial_t$.

2. The field $C_q, C_q'(t,q,\ell_q)$ is stabilized by $\sigma_q$, $\partial_s$, and $\Delta_{t,q}$. The field $C_q, C_q'(t,q,\ell_q)$ is stabilized by $\sigma_q$ and $\partial_s$. The field $C_q'(t,q,\ell_q)$ is stable by $\partial_s$, $\partial_t$.

In particular, (H1) of Proposition D.13 is satisfied for $K = C_q(s)$ and $K = C_q, C_q'(t,q,\ell_q)$.

Proof. (1) Since $\sigma_q(\ell_q) = \ell_q + 1$, we easily see that $C_q(s, \ell_q), C_q(s)$ are stabilized by $\sigma_q$. Since $\sigma_q$ commutes with $\partial_t$ and $\Delta_{t,q}$, the field $C_q$ is stabilized by $\partial_t$ and $\Delta_{t,q}$. It is now clear that $C_q$ is stabilized by $\partial_t$ and $\Delta_{t,q}(C_q(s)) \subset C_q(s, \ell_q)$. By Remark D.3, $\Delta_{t,q}(\ell_q), \partial_t(\ell_q) \in C_q$. Combining the last assertions, we obtain the result for $C_q(s, \ell_q)$. Finally, the field $C_q(s)$ is stable by $\partial_t$, $\partial_s$, since $C$ is stable by $\partial_t$, $\partial_s$, and $\partial_t(s) = s$, $\partial_s(s) = 0$.

(2) Let us prove that $C_q'(t,q,\ell_q)$ is stabilized by $\sigma_q$. Using $\sigma_q(\ell_q) = \ell_q + 1$ and the commutation between $\sigma_q$ and $\partial_s$, we find that $\sigma_q(\ell_q) - \ell_q \in C_q$. Similarly, $\sigma_q(C_q'(t,q,\ell_q)) \subset C_q'(t,q,\ell_q)$, proving that $C_q'(t,q,\ell_q)$ is stabilized by $\sigma_q$. Using $\partial_t(C_q'(t,q,\ell_q)) \subset C_q$ and $\partial_s(\ell_q) \in C_q$, we find that the field $C_q, C_q'(t,q,\ell_q)$ is stabilized by $\sigma_q$ and $\partial_s$.

Let us now consider the field $C_q, C_q'(t,q,\ell_q)$. The field $C_q'(t,q,\ell_q)$ is clearly stable by $\sigma_q$. From what precedes, $C_q'(t,q,\ell_q)$ is stable by $\sigma_q$ and, therefore, $C_q, C_q'(t,q,\ell_q)$ is stable by $\sigma_q$. The same arguments than those used in (1), prove that $\Delta_{t,q}(C_q'(t,q,\ell_q)) \subset C_q, C_q'(t,q,\ell_q)$ and $\partial_s(C_q'(t,q,\ell_q)) \subset C_q'(t,q,\ell_q)$. It remains to prove that $\Delta_{t,q}(C_q'(t,q,\ell_q)) \subset C_q, C_q'(t,q,\ell_q)$. We note that $\partial_t(q)\ell_q \partial_s + \partial_t(q)\ell_q \partial_s = \Delta_{t,q}(q)\ell_q - \partial_t(q)\ell_q \partial_s$. Since $C_q$ is stabilized by $\Delta_{t,q}$ and $\partial_t$, we find that $\Delta_{t,q}(C_q) \subset C_q, C_q'(t,q,\ell_q)$. Moreover, since $\partial_t(q)\ell_q \partial_s(\ell_q)$ belong to $C_q$, see Remark D.3, we find that $\Delta_{t,q}(\ell_q) \subset C_q, C_q'(t,q,\ell_q)$. We have shown the inclusion $\Delta_{t,q}(C_q'(t,q,\ell_q)) \subset C_q, C_q'(t,q,\ell_q)$. This concludes the proof for $C_q, C_q'(t,q,\ell_q)$.

Let us now consider $C_q'(t,q,\ell_q)$. By Remark D.3 and $\partial_t = \Delta_{t,q} - \partial_t(q)\ell_q \partial_s$, we find that the inclusion holds $\partial_t(q)\ell_q \partial_s \partial_t(\ell_q) \partial_s \partial_s \Delta_{t,q}$. With $\partial_t = \Delta_{t,q} - \partial_t(q)\ell_q \partial_s$, it follows that $\partial_t(C_q) \subset C_q'(t,q,\ell_q)$. Finally, we obtain that the field $C_q'(t,q,\ell_q)$ is stable by $\partial_s$, $\partial_t$.

Remind that $q$ and $\ell_q$ are multiplicatively independent.

Lemma D.6. The elements of $C_q$ invariant by $\sigma_q$ are in $C$. The elements of $C_q$ invariant by $\sigma_q$ are in $C$.

Proof. The two statements are symmetrical, so let us only prove the first one. Let $f$ be an element of $C_q$ that is $\sigma_q$-invariant. Suppose to the contrary that $f$ is not constant. Then $f$ has a
Let \( \partial \) be a non zero pole. Since \( \sigma_q(f) = f \), the multiplication by \( q \) induces a permutation of the poles of \( f \) modulo \( q \). Since the set of poles modulo \( q \) is a finite set, there exists \( m \in \mathbb{N} \) such that \( q^m c = q^d c \) for some \( d \in \mathbb{Z} \). A contradiction with the fact that \( q \) and \( q \) are multiplicatively independent.

\[ \text{Lemma D.7.} \quad \text{The following statements hold:} \]

1. The fields \( C_q \) and \( C_q \) are linearly disjoint over \( C \).
2. For all \( \alpha \in C_q, C_q, \sigma_q(\alpha) \neq \alpha + 1 \) and \( \sigma_q(\alpha) \neq \alpha + 1 \).
3. For all \( \alpha \in C_q(s), \sigma_q(\alpha) \neq \alpha + 1 \).

\[ \text{Proof.} \quad \text{(1) This is Lemmas D.6 and C.7 with } K = C, M = C_q \text{ and } L = C_q, \sigma = \sigma_q. \]

(2) Suppose to the contrary that there exists \( \alpha \in C_q, C_q \), such that \( \sigma_q(\alpha) = \alpha + 1 \). Since \( C_q \) is by Proposition B.1, the field of meromorphic functions over a Tate curve, there exist \( x, y \in C_q \) such that \( x \) is transcendent over \( C, y \) algebraic of degree 2 over \( C(x) \) and \( C_q = C(x, y) \).

Since \( C_q \) is linearly disjoint from \( C_q \) over \( C \), the field \( C_q, C_q \) equals \( C_q(x, y) \) and there are \( P(X), Q(X) \in C_q(X) \) such that \( \alpha = P(x)y + Q(x) \). Since \( x, y \) are fixed by \( \sigma_q \) and \( y \) is of degree 2 over \( C_q(x) \), we deduce from \( \sigma_q(\alpha) = \alpha + 1 \) that \( P^{\sigma_q}(x) = P(x) \) and \( Q^{\sigma_q}(x) - Q(x) = 1 \) where \( P^{\sigma_q}(X) \) (resp. \( Q^{\sigma_q}(X) \)) denotes the fraction obtained from \( P(X) \) (resp. \( Q(X) \)) by applying \( \sigma_q \) to the coefficients. Let \( \overline{C_q} \) be some algebraic closure of \( C_q \) and let us write \( Q(X) = \frac{x^2}{x} + \cdots + \frac{x^2}{x} + R(X) \) with \( R \in \overline{C_q}(X) \) with no pole at \( X = 0 \). Then, since \( x \) is transcendent over \( \overline{C_q} \) and fixed by \( \sigma_q \)

\[ Q^{\sigma_q}(x) - Q(x) = 1 = \frac{\partial_q(c_r) - c_r}{x^r} + \cdots + \frac{\partial_q(c_1) - c_1}{x} + R^{\sigma_q}(x) - R(x). \]

Using the transcendence of \( x \) over \( \overline{C_q} \), we find that \( \alpha = \sigma_q(\beta) - \beta \) for \( \beta = R(0) \in \overline{C_q} \). By Lemma C.1, there exists \( r \in \mathbb{N}^* \) such that \( \sigma_q^r(\beta) = \beta \). Deriving \( 1 = \sigma_q(\beta) - \beta \) and \( \sigma_q^r(\beta) - \beta \) with respect to \( \partial_q \), we conclude that \( \sigma_q(\beta) \in C_q \cap C_q \). Note that \( q \) and \( q \) are multiplicatively independent. By Lemma D.6, we can apply \( q \) replaced by \( q \), we get \( \partial_q(\beta) \in C_q \) which leads to \( \beta = cs + d \) for some \( c, d \in C \). A contradiction with \( 1 = \sigma_q(\beta) - \beta \). The proof for \( q \) is similar.

(3) Let \( \alpha \in C_q(s) \). Using the partial fraction decomposition of \( \alpha \) in \( \overline{C_q}(s) \), the fact that \( \sigma_q(s) = q \) and the transcendence of \( s \) over \( C_q \), one can easily see that \( \sigma_q(\alpha) - \alpha \neq 1 \).

\[ \text{Lemma D.8.} \quad \text{The following holds:} \]

1. the function \( \ell_q \) (resp. \( \ell_q \)) is transcendent over \( C_q, C_q \);
2. the function \( \ell_q \) is transcendent over \( C_q(s) \). In particular, (H3) of Proposition D.13 is satisfied for \( K = C_q(s) \).

\[ \text{Proof.} \quad \text{(1) Since } \sigma_q(\ell_q) = \ell_q + 1 \text{ and } C_q \subset (C_q, C_q)^{s_0} \subset \text{Mer}(C^*)^{s_0} = C_q, \text{ we can apply Lemma C.2 and find that } \ell_q \text{ is algebraic over } C_q, C_q \text{ if and only if there exists } \alpha \in C_q, C_q \text{ such that } \sigma_q(\alpha) = \alpha + 1. \text{ We conclude by Lemma D.7. The proof for } \ell_q \text{ is symmetrical.} \]

(2) Since \( \sigma_q(\ell_q) = \ell_q + 1 \) and \( C_q \subset (C_q(s))^{s_0} \subset \text{Mer}(C^*)^{s_0} = C_q, \text{ we can apply Lemma C.2 and find that } \ell_q \text{ is algebraic over } C_q(s) \text{ if and only if there exists } \alpha \in C_q(s) \text{ such that } \sigma_q(\alpha) = \alpha + 1. \text{ We again conclude by Lemma D.7.} \]

\[ \text{Lemma D.9.} \quad \text{The following holds:} \]

1. Let \( f \in C_q \). If there exists \( \alpha \in C_q, C_q \) satisfying \( \sigma_q(\alpha) = \alpha = f \), then, there exists \( \beta \in C_q \) such that \( \sigma_q(\beta) - \beta = f \);
2. Let \( f \in C_q, C_q \). If there exists \( \alpha \in C_q, C_q(\ell_q) \) satisfying \( \sigma_q(\alpha) = \alpha = f \), then, there exist \( \tilde{a} \in C_q, \tilde{b} \in C_q, C_q \) such that \( \sigma_q(\tilde{a} \ell_q + \tilde{b}) - (\tilde{a} \ell_q + \tilde{b}) = f \).
Proof. (1) Analogously to the proof of Lemma D.7, let us write \( \alpha = P(x)y + Q(x) \) for \( P(X), Q(X) \in C_q(X) \) and \( C_q = C(x, y) \). Reasoning as in the proof of Lemma D.7, we find that \( Q^a(x) - Q(x) = f \). Since \( x \) is transcendent over \( C_q \), we conclude as in Lemma D.7 that there is \( \beta \in \mathbb{C}_q \) for some \( \mathbb{C}_q \) algebraic closure of \( C_q \) such that \( \sigma_q(\beta) - \beta = f \). Since by Lemma D.6, \( C_q^b = C_q^a = C \), Lemma C.2 implies that there exists \( \beta \in C_q \) such that \( \sigma_q(\beta) - \beta = f \).

(2) First of all, let us note that since \( \sigma_q \) and \( \sigma_q \) commute, there exists \( d \in C_q \) such that

\[
\sigma_q(\ell_q) = \ell_q + d.
\]

By Lemma D.8, the function \( \ell_q \) is transcendent over \( C_q C_q \). This implies that \( \ell_q \not\in C_q \) and then \( d \neq 0 \). Since \( C_q C_q (\ell_q)^a = C_q = \text{Mer}(C^a)^a = C_q C_q^a = C_q \), Lemma C.3, applied to \( \sigma_q(\ell_q) = \ell_q + d \), implies that there exists \( P \in C_q C_q[X] \) such that

\[
f = \sigma_q(P(\ell_q)) - P(\ell_q).
\]

Now, let us write \( P(X) = \sum_{k=0}^N a_k X^k \) with \( a_k \in C_q C_q \), and \( N \) minimal. We find

\[
f = (\sigma_q(a_N) - a_N) \ell_q^N + (\sigma_q(a_{N-1}) - a_{N-1} + N d a_q(a_N)) \ell_q^{N-1} + \text{terms of order less than } N - 1.
\]

We conclude in view of (D.2) that if \( N = 0 \) we are done by setting \( \bar{a} = 0 \) and \( \bar{b} = a_N \). Let us now assume that \( N > 0 \). Then, by minimality of \( N \), \( \sigma_q(a_N) = a_N \). We claim that \( \sigma_q(a_{N-1}) - a_{N-1} + N d a_q(a_N) = \sigma_q(a_{N-1}) - a_{N-1} + N d a_N \neq 0 \). To the contrary, \( \sigma_q(a_{N-1}) - a_{N-1} - N d a_N \) implies \( \sigma_q(\frac{a_{N-1}}{a_N} + N \ell_q) = \frac{a_{N-1}}{a_N} + N \ell_q \) and \( \frac{a_{N-1}}{a_N} + N \ell_q \in C_q \), contradicting the transcendent of \( \ell_q \) over \( C_q C_q \), see Lemma D.8. This proves the claim. If \( N > 1 \), then (D.2) with \( \sigma_q(a_N) = a_N \) and \( \sigma_q(a_{N-1}) - a_{N-1} + N d a_N \neq 0 \), would give an equation of order \( N - 1 \) which would contradicts the transcendent of \( \ell_q \) over \( C_q C_q \). This proves that \( N = 1 \) and \( f = \sigma_q(a_1 \ell_q + a_0) - (a_1 \ell_q + a_0) \) for some \( a_1 \in C_q, a_0 \in C_q C_q \).}

**Lemma D.10.** The function \( \ell_q \) is transcendent over \( C_q C_q(\ell_q) \). In particular, the assumption (H3) of Proposition D.13 holds for \( K = C_q C_q(\ell_q) \).

**Proof.** By Lemma C.2, the function \( \ell_q \) is algebraic over \( C_q C_q(\ell_q) \) if and only if we have \( \ell_q \in C_q C_q(\ell_q) \). Suppose to the contrary that \( \ell_q \not\in C_q C_q(\ell_q) \). Since \( 1 = \sigma_q(\ell_q) - \ell_q \), we conclude by Lemma D.9 that there exist \( \bar{a} \in C_q, \bar{b} \in C_q C_q \) such that \( 1 = \sigma_q(\bar{a} \ell_q + \bar{b}) - (\bar{a} \ell_q + \bar{b}) \).

Combining this equation with \( \sigma_q(\ell_q) - \ell_q = 1 \), we find that \( \sigma_q(\ell_q) - \ell_q = \sigma_q(\bar{a} \ell_q + \bar{b}) - (\bar{a} \ell_q + \bar{b}) \), proving that \( \sigma_q(\bar{a} \ell_q + \bar{b} - \ell_q) = \bar{a} \ell_q + \bar{b} - \ell_q \in C_q \). Then, there exists \( b_1 \in C_q C_q \) such that

\[
\ell_q = \bar{a} \ell_q + \bar{b}.
\]

Deriving (D.3) with respect to \( \partial_s \), we find

\[
\partial_s(\ell_q) = \partial_s(\bar{a}) \ell_q + \bar{a} \partial_s(\ell_q) + \partial_s(\bar{b}).
\]

Since \( \partial_s(\ell_q), \partial_s(\ell_q), \partial_s(\bar{a}), \partial_s(\bar{b}) \in C_q C_q \) (we use Remark D.3, and the fact that \( C_q C_q \) are stabilized by \( \partial_s \), in virtue of the commutation between \( \partial_s \) and \( \sigma_q, \sigma_q \)), and by Lemma D.8 \( \ell_q \) is transcendent over the latter field, we conclude that \( \partial_s(\bar{a}) = 0 \) and therefore \( \bar{a} \in C \). In particular it belongs to \( C_q \) and \( C_q \). Using \( 1 = \sigma_q(\bar{a} \ell_q + \bar{b}) - (\bar{a} \ell_q + \bar{b}) \), we find

\[
1 - \bar{a} d = \sigma_q(\bar{b}) - \bar{b},
\]

where \( d = \sigma_q(\ell_q) - \ell_q \in C_q \), see (D.1). Since \( 1 - \bar{a} d \in C_q \), we conclude by Lemma D.9, that there exists \( b_2 \in C_q \) such that \( 1 - \bar{a} d = \sigma_q(b_2) - b_2 \). Replacing the left hand side gives

\[
\sigma_q(\ell_q) - \ell_q = \sigma_q(\bar{a} \ell_q + \bar{b}) = \sigma_q(b_2) - b_2.
\]
This shows that \( \ell_q - \tilde{a}\ell_q - \tilde{b}_2 \in C_q \) and then, there exists \( c \in C_q \) such that \( \ell_q + c = \tilde{a}\ell_q + \tilde{b}_2 \).

Deriving this equation with respect to \( \partial_a \), we find (we use \( \partial_a(\tilde{a}) = 0 \))

\[
\partial_a(\ell_q) + \partial_a(c) = \tilde{a}\partial_a(\ell_q) + \partial_a(\tilde{b}_2).
\]

By Remark D.3, the left hand side of the equation belongs to \( C_q \) whereas the right hand side is in \( C_q \). By Lemma D.6, we conclude that \( \partial_a(\ell_q + c) \in C \). This means that there exists \( a_0, b_0 \in C \) such that \( \ell_q = a_0s + b_0 - c \) in contradiction with \( \ell_q \) transcendent over \( C_q(s) \), see Lemma D.8. \( \square \)

We can now prove that our fields satisfy the assumption \((H2)\) of Proposition D.13.

**Lemma D.11.** The following holds:

1. \( C_q \) is relatively algebraically closed in \( C_q(s, \ell_q) \);
2. \( C_q \) is relatively algebraically closed in \( C_q, C_q(\ell_q, \ell_q) \).

In particular, \((H2)\) of Proposition D.13 holds for \( K = C_q(s) \) and \( K = C_q, C_q(\ell_q, \ell_q) \).

**Proof.** (1) The first point is a consequence of transcendence of \( s \) over \( C_q \), and the transcendence of \( \ell_q \) over \( C_q(s) \), see Lemma D.8. Let us prove the second point.

(2) Let us start by proving that \( C_q \) is relatively algebraically closed in \( C_q, C_q \). As in the proof of Lemma D.7, we have \( C_q = C(x, y) \) and \( C_q, C_q = C_q(x, y) \) where \( y \) is of degree 2 over both \( C(x) \) and \( C_q(x) \). Let \( f \in C_q(x, y) \). Then \( f = \tilde{P}(x)\tilde{Q}(x) = \tilde{P}(x)(x - \ell_q)^{-1} \) and \( \tilde{Q}(x) \). We claim that \( \tilde{P}(x) \) and \( \tilde{Q}(x) \) are in \( C(x) \), and therefore that \( f \in C_q \). Let \( \tilde{P}(x) = P_1(x)/P_2(x) \) where \( P_1(x), P_2(x) \in C_q[x] \) are relatively prime and \( P_1(x) \) is monic. We then have that \( \sigma_q(P_1(x))P_2(x) = \sigma_q(P_2(x))P_1(x) \) and consequently \( P_1(x) \) divides \( \sigma_q(P_1(x)) \) (resp. \( \sigma_q(P_1(x)) \) divides \( P_1(x) \)). Since \( P_1(x) \) is monic, \( P_1(x) = \sigma_q(P_1(x)) \) and \( P_2(x) / \sigma_q(P_2(x)) \). This implies that the coefficients of \( P_1(x) \) and \( P_2(x) \) are left fixed by \( \sigma_q \). Note that by assumption, \( q \) and \( q^r \) are multiplicatively independent. Therefore, by Lemma D.6, applied with \( q \) replaced by \( q^r \), the polynomials \( P_1, P_2 \) lie in \( C[X] \). The proof for \( Q \) is similar.

This proves our claim and show that \( f \in C_q \). Then \( C_q \) is relatively algebraically closed in \( C_q, C_q \).

Note that Lemma D.8 implies that \( \ell_q \) is transcendent over \( C_q, C_q \) and Lemma D.10 implies that \( \ell_q \) is transcendent over \( C_q, C_q(\ell_q, \ell_q) \). Therefore \( C_q \) is relatively algebraically closed in \( C_q, C_q(\ell_q, \ell_q) \). \( \square \)

Finally, we prove a lemma that will allows us to descend some telescoping relations on smaller base fields.

**Lemma D.12.** Let \( b \in C_q \) such that there exist \( N \in \mathbb{N} \) and \( c_i \in C_q \) with \( c_N \neq 0 \) and \( g \in C_q, C_q(\ell_q, \ell_q) \) such that

\[
(D.4) \quad \sum_{i=0}^{N} c_i \partial^i_s(b) = \sigma_q(g) - g.
\]

Then, there exist \( m \in \mathbb{N} \) and \( d_0, \ldots, d_m \in C \) not all zero and \( h \in C_q \) such that

\[
d_0b_2 + d_1\partial_s(b_2) + \cdots + d_mb^m_s(b_2) = \sigma_q(h) - h.
\]

**Proof.** First of all note that the left hand side of \((D.4)\) belongs to \( C_q, C_q \). By Lemma D.10, the function \( \ell_q \) is transcendent over \( C_q, C_q(\ell_q) \). By Lemma C.3, \( g \in C_q, C_q(\ell_q)[\ell_q] \). So let us write

\[
g = \sum_{k=0}^{R} \alpha_k \ell^k_q \quad \text{with} \quad \alpha_k \in C_q, C_q(\ell_q), \alpha_R \neq 0.
\]

Claim. There exist \( m \in \mathbb{N}, c'_k \in C_q, c'_m \neq 0, \) and \( \alpha \in C_q, C_q(\ell_q) \) such that

\[
(D.5) \quad \sum_{k=0}^{m} c'_k \partial^k_s(b) = \sigma_q(\alpha) - \alpha.
\]
If \( R = 0 \) the claim is proved. Assume that \( R > 0 \). Then, we have

\[
\sigma_q(g) - g = \ell_q R(\sigma_q(\alpha_R) - \alpha_R)) + \ell_q R^{-1}(\sigma_q(\alpha_{R-1}) - \alpha_{R-1} + R\alpha_R) + P(\ell_q),
\]

where \( P(X) \in C_qC_q(\ell_q)[X] \) is a polynomial of degree smaller than \( R-1 \). Then, comparing (D.6) and (D.4), we find, by transcendence of \( \ell_q \) over \( C_qC_q(\ell_q) \), see Lemma D.10, that \( \sigma_q(\alpha_R) = \alpha_R \).

We then obtain that \( R = 1 \) since otherwise we would deduce from (D.6) an algebraic relation for \( \ell_q \) over \( C_qC_q(\ell_q) \), contradicting Lemma D.10. Thus,

\[
\sum_{i=0}^{N} \frac{C_n}{\alpha_1} \partial^{\alpha_1}_1(b) = \sigma_q \left( \frac{\alpha_0}{\alpha_1} \right) - \frac{\alpha_0}{\alpha_1} + 1.
\]

Remind that \( \alpha_1 \in C_q \) and the latter field is stable by \( \partial_s \) due to the commutation between \( \partial_s \) and \( \sigma_q \). By Lemma D.5, the field \( C_qC_q(\ell_q) \) is stabilized by \( \partial_s \). We can derive (D.7) with respect to \( \partial_s \) and using the commutation between \( \sigma_q \) and \( \partial_s \), we obtain our claim.

**Claim.** There exist \( M \in \mathbb{N} \), \( d_k \in C_q \) and \( d_M \neq 0 \) and \( \beta \in C_qC_q \) such that

\[
\sum_{k=0}^{M} d_k \partial^{\alpha_1}_1(b) = \sigma_q(\beta) - \beta.
\]

Indeed, by Lemma D.9, we can find \( a \in C_q \), \( b \in C_qC_q \) such that

\[
\sum_{k=0}^{m} c_k \partial^k_1(b) = \sigma_q(a\ell_q + b) - (a\ell_q + b).
\]

Either \( a = 0 \) and \( \sum_k c_k \partial^k_1(b) = \sigma_q(b) - (b) \) for some \( b \in C_qC_q \). Or \( a \neq 0 \) and dividing (D.8) by \( a \) and deriving with respect to \( \partial_s \), we find

\[
\sum_{k=0}^{m+1} d_k \partial^k_1(b) = \sigma_q(\partial_s(\ell_q) + \partial_s(b/a)) - (\partial_s(\ell_q) + \partial_s(b/a)),
\]

where the \( d_k \) are in \( C_q \), \( d_{m+1} = \frac{c_n}{a} \neq 0 \). Furthermore, by Remark D.3 and the fact that \( C_q, C_q \), are stable by \( \partial_s \), we find \( \partial_s(\ell_q) + \partial_s(b/a) \in C_qC_q \). This proves the claim.

Now, let us consider an equation of the form

\[
\sum_{k=0}^{M} d_k \partial^k_1(b) = \sigma_q(\beta) - \beta,
\]

with \( \beta \in C_qC_q \), the \( d_k \in C_q \) and \( d_M \neq 0 \), minimal with respect to the maximal order of derivation \( M \) of \( b \). We can write this minimal equation as follows

\[
d_M \partial^M_1(b) + \sum_{k=0}^{M-1} d_k \partial^k_1(b) = \sigma_q(\beta) - \beta,
\]

with \( d_M \in C_q^* \). Then dividing by \( d_M \), we find

\[
\partial^M_1(b) + \sum_{k=0}^{M-1} \frac{d_k}{d_M} \partial^k_1(b) = \sigma_q \left( \frac{\beta}{d_M} \right) - \frac{\beta}{d_M}.
\]
Lemma D.9 shows that there exists $h \in C_q$, we find
\[
\sum_{k=0}^{M-1} (\sigma_q(d_k) - d_k) \partial_s^k(b) = \sigma_q(\beta) - \beta.
\]
By minimality, we find that, for all $k$, the element $d_k \in C_q$ is fixed by $\sigma_q$. This means that $d_k \in C$ by Lemma D.6.

Since $\partial_s^M(b) + \sum_{k=0}^{M-1} d_k \partial_s^k(b) \in C_q$ and $\partial_s^M(b) + \sum_{k=0}^{M-1} d_k \partial_s^k(b) = \sigma_q(\beta) - \beta$ with $\beta \in C_q$, $C_q$, Lemma D.9 shows that there exists $h \in C_q$ such that
\[
\partial_s^M(b) + \sum_{k=0}^{M-1} d_k \partial_s^k(b) = \sigma_q(h) - h.
\]
\[\square\]

D.3. **Difference Galois theory for elliptic function fields.** In this section, we shall apply the results of §C to the specific cases of elliptic function fields. As proved in §D, the following fields are $(\sigma, \partial, \Delta)$-fields.

- Let $q \in C^*$ with $|q| \neq 1$. Then, let us consider
\[
(C_q(s, \ell_q), \sigma_q, \partial_s, \Delta_t, q) \subset (\text{Mer}(C^*), \sigma_q, \partial_s, \Delta_t, q).
\]

- Let $q$ and $q$ two elements of $C^*$ such that $|q|, |q| \neq 1$, that are multiplicatively independent. Let us consider
\[
(C_q, C_q(\ell_q, \ell_q), \sigma_q, \partial_s, \Delta_t, q) \subset (\text{Mer}(C^*), \sigma_q, \partial_s, \Delta_t, q).
\]

In that framework, the criteria obtained in §C to guaranty the $(\partial_s, \Delta_t, q)$-differential transcendence of a solution of a rank one $q$-difference equation can be simplified and some descent arguments prove that the existence of a telescoping relation involving the two derivatives implies the existence of a telescoping relations involving only the derivation $\partial_s$. More precisely, we find the following proposition:

**Proposition D.13.** Let $K \subset \text{Mer}(C^*)$ be a $(\sigma_q, \partial_s)$-field and let us assume that

- $(H_1)$ $L = K(\ell_q)$ is a $(\sigma_q, \partial_s, \Delta_t, q)$-field;
- $(H_2)$ $K^{\sigma_q} = L^{\sigma_q} = C_q$ is relatively algebraically closed in $L$;
- $(H_3)$ $\ell_q$ is transcendent over $K$.

Let $f \in \text{Mer}(C^*)$, that satisfies $\sigma_q(f) = f + b$, for some $b$ that belongs to a subfield of $K$ stable by $\partial_s, \partial_t$.

If $f$ is $(\partial_s, \Delta_t, q)$-differentially algebraic over $L$ then, there exist $m \in \mathbb{N}, d_0, \ldots, d_m \in C_q$ not all zero, and $h \in K$ such that
\[
d_0b_1 + d_1\partial_s(b) + \cdots + d_m\partial_s^m(b) = \sigma_q(h) - h.
\]

**Proof.** Since $f$ is $(\partial_s, \Delta_t, q)$-differentially algebraic over $L$ and $K^{\sigma_q}$ is relatively algebraically closed, Theorem C.8 yields that there exist $M \in \mathbb{N}$, $c_{i,j} \in L^{\sigma_q}$ not all zero, and $g \in L$ such that
\[
\sum_{i,j \leq M} c_{i,j} \partial_s^i \Delta_t^j(q)(b) = \sigma_q(g) - g.
\]

By Lemma D.4, for all $i \in \mathbb{N}$, there exist $c_{j,k,l} \in C_q$ such that
\[
\Delta_t^j(q)(\ell_q)^j \partial_s^i + \sum_{k=0}^{i-1} \sum_{j=0}^{i} \sum_{l=0}^{i} c_{j,k,l} \partial_s^k \partial_t^l.
\]
The left hand side of (D.9) is a polynomial in $\ell_q$ with coefficients in $K$. By Lemma C.3 with (H2) and (H3), we find that $g \in K[\ell_q]$ as well.

Thus, let us write $g = \sum_{k=0}^R \alpha_k \ell_q^k$ with $\alpha_k \in K$ and $\alpha_R \neq 0$. Let

$$N = \max\{j \in \mathbb{N}|\exists i \text{ such that } c_{i,j} \neq 0\}.$$ 

By (D.10), the coefficient of highest degree in $\ell_q$ of the left hand side of (D.9) is

$$\left(\sum_{i \leq M} c_{i,N}(\partial_i(q))^N \partial_s^{N+1}(b)\right) \ell_q^N.$$ 

On the other hand, we have

$$\sigma_q(g) - g = \ell_q^N(\sigma_q(\alpha_R) - \alpha_R)) + \ell_q^{R-1}(\sigma_q(\alpha_{R-1}) - \alpha_{R-1} + R\sigma_q(\alpha_R)) + P(\ell_q),$$

where $P(X) \in K[X]$ is a polynomial of degree strictly smaller than $R - 1$. Then, comparing (D.11) and (D.12), we find that

- either $R < N$ so that

$$\sum_{i \leq M} c_{i,N}(\partial_i(q))^N \partial_s^{N+1}(b) = 0,$$

- either $R = N$ so that

$$\sum_{i \leq M} c_{i,N}(\partial_i(q))^N \partial_s^{N+1}(b) = \sigma_q(\alpha_N) - \alpha_N,$$

- or $R > N$ so that $R > 0$, $0 \neq \alpha_R \in L^a$. We claim that $R = N - 1$. Indeed, $R > N - 1$ implies $\sigma_q(\alpha_R) = \sigma_q(\alpha_{R}), \sigma_q(\alpha_{R-1}) - \alpha_{R-1} + R\alpha_R = 0$ and then $\sigma_q(\frac{\alpha_{R-1}}{\alpha_R}) - \frac{\alpha_{R-1}}{\alpha_R} + R = 0$ with $\frac{\alpha_{R-1}}{\alpha_R} \in K$ in contradiction with Lemma C.2 applied to $f = \ell_q$. Thus, we get $R = N - 1$ and

$$\sum_{i \leq M} c_{i,N}(\partial_i(q))^N \partial_s^{N+1}(b) = \sigma_q(\frac{\alpha_{R-1}}{\alpha_R}) - \frac{\alpha_{R-1}}{\alpha_R} + R.$$ 

For all these cases, note that there exists $i_0$ such that $c_{i_0,N} \neq 0$ by definition of $N$. Since $\partial_s$ commutes with $\sigma_q$, we can derive (D.15) with respect to $\partial_s$ and obtain that in any case, there exists $d_k \in L^{a} = C_q$ not all zero and $h \in K$ such that

$$\sum_{k \leq M+1} d_k \partial_s^k(b) = \sigma_q(h) - h.$$

The results of Appendix D.2 are summarized in the following crucial corollary.

**Corollary D.14.** The assumptions of Proposition D.13 are satisfied for

- Genus zero case: $K = C_q(s)$ and $b \in C(s)$ with $q \in C^*$ such that $|q| \neq 1$;
- Genus one case: $K = C_q.C_q(\ell_q)$ and $b \in C_q(\ell_q)$ with $q, q \in C^*$ such that $|q|, |q| \neq 1$ and $q$ and $q$ are multiplicatively independent.

**Proof.** The fact that the field $K$ and $b$ satisfy the assumptions (Hi) is Lemmas D.5, D.8, D.10, and D.11. \qed
References

[Abr95] S. A. Abramov, *Indefinite sums of rational functions*, Proceedings of the 1995 International Symposium on Symbolic and Algebraic Computation (1995), 303–308.

[AvdDvdH17] Matthias Aschenbrenner, Lou van den Dries, and Joris van der Hoeven, *Asymptotic differential algebra and model theory of transseries*, Annals of Mathematics Studies, vol. 195, Princeton University Press, Princeton, NJ, 2017.

[BBMR16] Olivier Bernardi, Mireille Bousquet-Mélou, and Kilian Raschel, *Counting quadrant walks via Tutte’s invariant method (extended abstract)*, to appear in *Proceedings of FPSAC 2015*, Discrete Math. Theor. Comput. Sci. Proc., 2016.

[BMM10] Mireille Bousquet-Mélou and Marni Mishna, *Walks with small steps in the quarter plane*, Algorithmic probability and combinatorics, Contemp. Math., vol. 520, Amer. Math. Soc., Providence, RI, 2010, pp. 1–39.

[BvHK10] Alin Bostan, Mark van Hoeij, and Manuel Kauers, *The complete generating function for gessel walks is algebraic*, Proc. Amer. Math. Soc. 138 (2010), no. 9, 3063–3078.

[Coh65] Richard M. Cohn, *Difference algebra*, Interscience Publishers John Wiley & Sons, New York-London-Sydney, 1965.

[CS12] Shaoshi Chen and Michael F. Singer, *Residues and telescopers for bivariate rational functions*, Adv. in Appl. Math. 49 (2012), no. 2, 111–133. MR 2946428.

[DHR17] Thomas Dreyfus, Charlotte Hardouin, Julien Roques, and Michael F Singer, *Walks in the quarter plane, genus zero case*, arXiv preprint arXiv:1710.02848 (2017).

[DR17] Thomas Dreyfus and Kilian Raschel, *Differential transcendence & algebraicity criteria for the series counting weighted quadranti walks*, To appear in Publications mathématiques de Besançon (2017).

[Dui10] J. Duistermaat, *Discrete integrable systems: Qrt maps and elliptic surfaces*, Springer Monographs in Mathematics, vol. 304, Springer-Verlag, New York, 2010.

[DVH12] Lucia Di Vizio and Charlotte Hardouin, *Descent for differential Galois theory of difference equations: confluence and q-dependence*, Pacific J. Math. 256 (2012), no. 1, 79–104. MR 2928542.

[FIM99] Guy Fayolle, Roudolf Iasnogorodski, and Vadim Malyshev, *Random walks in the quarter-plane*, Applications of Mathematics (New York), vol. 40, Springer-Verlag, Berlin, 1999, Algebraic methods, boundary value problems and applications. MR 1691900.

[FR10] Guy Fayolle and Kilian Raschel, *On the holonomy or algebraicity of generating functions counting lattice walks in the quarter-plane*, Markov Process. Related Fields 16 (2010), no. 3, 485–496. MR 2759770.

[FvdP04] Jean Fresnel and Marius van der Put, *Rigid analytic geometry and its applications*, Progress in Mathematics, vol. 218, Birkhäuser Boston, Inc., Boston, MA, 2004.

[HS08] Charlotte Hardouin and Michael F. Singer, *Differential Galois theory of linear difference equations*, Math. Ann. 342 (2008), no. 2, 333–377.

[Ko68] E. R. Kolchin, *Algebraic groups and algebraic dependence*, Amer. J. Math. 90 (1968), 1151–1164. MR 0240106.

[Kol73] Ellis Robert Kolchin, *Differential algebra & algebraic groups*, vol. 54, Academic press, 1973.

[KP11] Manuel Kauers and Peter Paule, *The concrete tetrahedron*, Texts and Monographs in Symbolic Computation, SpringerWienNewYork, Vienna, 2011, Symbolic sums, recurrence equations, generating functions, asymptotic estimates.

[KR12] Iriina Kurkova and Kilian Raschel, *On the functions counting walks with small steps in the quarter plane*, Publ. Math. Inst. Hautes Etudes Sci. 116 (2012), 69–114. MR 3090255.

[KY15] Manuel Kauers and Rika Yatchak, *Walks in the quarter plane with multiple steps*, Proceedings of FPSAC 2015, Discrete Math. Theor. Comput. Sci. Proc., Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2015, pp. 25–36.

[MM14] Stephen Melczer and Marni Mishna, *Singularity analysis via the iterated kernel method*, Combin. Probab. Comput. 23 (2014), no. 5, 861–888. MR 3249228.

[MR09] Marni Mishna and Arne Ruzsa, *Two non-holonomic lattice walks in the quarter plane*, Theoret. Comput. Sci. 410 (2009), no. 38–40, 3616–3630. MR 2553316.

[Ost46] Alexandre Ostrowski, *Sur les relations algébriques entre les intégrales indéfinies*, Acta Math. 78 (1946), 315–318. MR 0016764.

[OW15] Alexey Ovchinnikov and Michael Wibmer, *σ-Galois theory of linear difference equations*, Int. Math. Res. Not. IMRN (2015), no. 12, 3962–4018.
[Rob00] Alain M. Robert, *A course in p-adic analysis*, Graduate Texts in Mathematics, vol. 198, Springer-Verlag, New York, 2000.

[Roq70] Peter Roquette, *Analytic theory of elliptic functions over local fields*, no. 1, Vandenhoeck u. Ruprecht, 1970.

[Sil94] Joseph H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994.

[Sil09] Joseph H Silverman, *The arithmetic of elliptic curves*, vol. 106, Springer Science & Business Media, 2009.

[vdPS97] Marius van der Put and Michael F. Singer, *Galois theory of difference equations*, Lecture Notes in Mathematics, vol. 1666, Springer-Verlag, Berlin, 1997.

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