Caricature of Hydrodynamics for Lattice Dynamics

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Abstract

The lattice dynamics in \( \mathbb{Z}^d, \ d \geq 1 \), is considered. The initial data are supposed to be random function. We introduce the family of initial measures \( \{\mu_{\varepsilon}, \varepsilon > 0\} \) depending on a small scaling parameter \( \varepsilon \). We assume that the measures \( \mu_{\varepsilon} \) are locally homogeneous for space translations of order much less than \( \varepsilon^{-1} \) and nonhomogeneous for translations of order \( \varepsilon^{-1} \). Moreover, the covariance of \( \mu_{\varepsilon} \) decreases with distance uniformly in \( \varepsilon \). Given \( \tau \in \mathbb{R} \setminus 0, \ r \in \mathbb{R}^d, \) and \( \kappa > 0 \), we consider the distributions of random solution in the time moments \( t = \tau/\varepsilon^\kappa \) and at lattice points close to \( [r/\varepsilon] \in \mathbb{Z}^d \). The main goal is to study the asymptotics of these distributions as \( \varepsilon \to 0 \) and derive the limit hydrodynamic equations of the Euler or Navier-Stokes type. The similar results are obtained for lattice dynamics in the half-space \( \mathbb{Z}^d_+ \).

Key words and phrases: harmonic crystals, random initial data, covariance matrices, weak convergence of measures, Gaussian measures, hydrodynamic limit, hydrodynamic space-time scaling, energy transport equation, Euler and Navier–Stokes equations

\footnote{Supported partly by research grant of RFBR (09-01-00288)}
1 Introduction

One of the central problems in nonequilibrium statistical physics is the derivation of hydrodynamic equations of fluid from the microscopic Hamilton dynamics. The main fluid equations are the Euler and Navier–Stokes equations. The idea that the Euler equation of fluid dynamics could be derived from microscopic dynamics goes back to Morrrey [22]. A systematic explanation of some basic ideas and first results are presented in the survey papers by De Masi, Ianiro, Pellegrinotti, and Presutti [3], by Dobrushin, Sinai and Sukhov [5], and by Spohn [23].

One approach is to derive the Euler and Navier–Stokes equations from the Boltzmann equation (see, for example, [4]). However, the Boltzmann equation is not a microscopic model, it should itself be derived as a scaling limit of a more basic model. An alternate approach is to study the hydrodynamic behaviour of some simplified or idealized models of interacting particles. For models of stochastic dynamics, the results were obtained by Yau et al. (see, for example, [18, 20, 21], the survey paper by Fritz [19] and the bibliography there). Deterministic models, where only initial realizations can be random, are more difficult to study in a hydrodynamic framework. The first results were obtained by Dobrushin et al. for the one-dimensional hard rods [1], and for the one-dimensional oscillators on the lattice [6–8]. The main purpose of these models is to show how hydrodynamic behavior arises.

The present work continues the papers [14, 17], where as the model the harmonic crystals in $\mathbb{Z}^d$ are considered. In the harmonic approximation, the crystal is characterized by the displacements $u(z,t) \in \mathbb{R}^n$, $z \in \mathbb{Z}^d$, of the crystal atoms from their equilibrium positions. The field $u(z,t)$ is governed by a discrete wave equation. The harmonic crystals can be considered as an extension of the model of the infinite chain of one-dimensional harmonic oscillators to the many dimensional case.

The derivation of hydrodynamic equations is connected with the problem of convergence to an equilibrium measure. For harmonic crystals, such convergence was proved in [11, 12, 15]. We outline this result. The initial data are assumed to be random function with a distribution $\mu_0$. We suppose that the measure $\mu_0$ has zero mean value, a finite mean energy density and satisfies the mixing condition. The distribution $\mu_t$ of the random solution $u(\cdot, t)$ in the time moment $t \in \mathbb{R}$ is studied. Then the limit

$$\lim_{t \to \infty} \mu_t = \mu_\infty$$

is established, where $\mu_\infty$ is an equilibrium Gaussian measure. For one-dimensional chain of harmonic oscillators, this result has been proved by Boldrighini et al. [2]. The convergence to equilibrium distribution was proved also for systems described by partial differential equations [9, 10], for the crystal coupled to the scalar field [13], and for the Klein-Gordon equation coupled to a particle [16].

To derive the hydrodynamic equations we apply the special so-called hydrodynamic limit procedure in which the notions of hydrodynamic limit and hydrodynamic space–time rescalings play a central role. Namely, we introduce a small scale parameter $\epsilon > 0$ giving the relation between the microscopic and macroscopic space–time scales and consider the family of the initial measures $\{\mu_0^\epsilon, \epsilon > 0\}$ which satisfies some conditions (see conditions $\mathbf{V1}$ and $\mathbf{V2}$ in Section 2.2 below). In particular, we assume that (i) the measures $\mu_0^\epsilon$ are locally
homogeneous for space translations of order much less than \( \varepsilon^{-1} \) and nonhomogeneous for translations of order \( \varepsilon^{-1} \); (ii) the covariance of \( \mu_0^\varepsilon \) vanishes with distance enough quickly and uniformly in \( \varepsilon \).

To deduce the Euler equation we use (see [14]) a hyperbolic (or Euler) scaling, i.e., the microscopic time and space variables are \( t = \tau / \varepsilon \) and \( z = r / \varepsilon \), where the macroscopic time and space variables are denoted by \( \tau \) and \( r \), respectively. Given nonzero \( \tau \in \mathbb{R} \) and \( r \in \mathbb{R}^d \), we study the distribution \( \mu_{\tau,r}^\varepsilon \) of the random solution \( u(z,t) \) at the space points close to \( [r/\varepsilon] \in \mathbb{Z}^d \) and in time moments \( \tau/\varepsilon \). Then the limit is established,

\[
\lim_{\varepsilon \to 0} \mu_{\tau/r\varepsilon}^\varepsilon = \mu_{\tau,r}^G,
\]

where \( \mu_{\tau,r}^G \) is a Gaussian measure (see Theorem 3.3 in [14] for harmonic crystals in the whole space \( \mathbb{Z}^d \) and Theorem 2.15 in [17] for harmonic crystals in the half-space \( \mathbb{Z}^d_+ \)). In particular, we derive the explicit formulas for covariance matrix \( q_{\tau,r}(z - z') \) of the limit measure \( \mu_{\tau,r}^G \). These formulas allow us to conclude that in Fourier transform the matrix function \( \hat{q}_{\tau,r}(\theta) \), \( \theta \in \mathbb{T}^d \), evolves according to the following equation:

\[
\partial_r \hat{f}(\tau, r; \theta) = i C(\theta) \nabla \Omega(\theta) \cdot \nabla_r \hat{f}(\tau, r; \theta), \quad r \in \mathbb{R}^d, \quad \tau > 0, \quad (1.3)
\]

where \( C(\theta) = \left( \begin{array}{cc} 0 & \Omega^{-1}(\theta) \\ -\Omega(\theta) & 0 \end{array} \right) \), and, roughly, \( \Omega(\theta) \) is the dispersion relation of the harmonic crystal (for details, see Section 3.1 below). Here and below \( \partial_r \) denotes partial differentiation with respect to a time \( \tau \), \( \nabla_r \) is the gradient of \( f \) with respect to \( r \in \mathbb{R}^d \), "\( \cdot \)" stands for the standard Euclidean scalar product in \( \mathbb{R}^d \) (or in \( \mathbb{R}^n \)). The equation (1.3) should be considered as the analog of the Euler equation for our model. In [6], the similar equation was deduced in the case \( d = n = 1 \). These results were extended to the harmonic crystals in the entire space \( \mathbb{Z}^d \), \( d \geq 1 \), (see [14]) and in the half-space \( \mathbb{Z}^d_+ = \{z \in \mathbb{Z}^d : z_1 > 0\} \), see [17].

The main result of the given paper is the derivation of the equation for the "next approximation" to the Euler equation (1.3). To obtain the additional term of order \( \varepsilon \) in (1.3), we use a diffusive (or parabolic) scaling, that is we study the distributions of solution \( u(z,t) \) in time moments of order \( \tau/\varepsilon^2 \). After the appropriate change of variables we derive the equation of the Navier–Stokes type

\[
\partial_r \hat{f}(\tau, r; \theta) = i C(\theta) \left( \nabla \omega(\theta) \cdot \nabla_r \hat{f}(\tau, r; \theta) + \frac{i \varepsilon}{2} \text{tr} \left[ \nabla^2 \Omega(\theta) \cdot \nabla^2 \hat{f}(\tau, r; \theta) \right] \right). \quad (1.4)
\]

Here and below \( \nabla_r^2 f \) stands for the matrix of second partial derivatives of \( f \) with respect to \( r \), "\( \text{tr} \)" stands for trace. The precise statement of the result see in Theorem 3.6 and Corollary 3.7 below. In the case \( d = n = 1 \), these results have been obtained in the work of Dobrushin et al [7]. Therefore, in the proof (see Section 6) we use an approach of [7] and tools of [11, 14] developed for harmonic crystals in any dimension.

In Section 3.3 we use a scaling of the form \( t = \tau/\varepsilon^k \), with \( k \geq 2 \), \( z = r/\varepsilon \), and derive the "corrections" of the higher order (i.e. of the order \( \varepsilon^k \) with \( k \geq 2 \)) to equation (1.3).

Second part of the paper is devoted to the study of the harmonic crystals in the half-space \( \mathbb{Z}^d_+ = \{z \in \mathbb{Z}^d : z_1 > 0\} \), with zero boundary condition. For such model, we also derive the limiting "hydrodynamic" equations (of the Euler and Navier–Stokes types).
The paper is organized as follows. In Section 2 we introduce the model, impose the main conditions on harmonic potentials and initial measures $\mu_0^\alpha$ and give the examples of the potentials and measures $\mu_0^\alpha$ satisfying our conditions. In Section 3 the main results are stated. Sections 4 and 5 are devoted to the harmonic crystals in the half-space $\mathbb{Z}_d^+$. Sections 6 and 7 contain the main steps of the proof of results. The technical details of the proof are given in Appendices A–C. Appendix D is devoted to locally conserved quantities.

2 Model I: Harmonic crystals in $\mathbb{Z}^d$

We study the dynamics of the harmonic crystals in $\mathbb{Z}^d$, $d \geq 1$,

$$
\begin{aligned}
\dot{v}(z,t) &= -\sum_{z' \in \mathbb{Z}^d} V(z-z')v(z',t), \quad z \in \mathbb{Z}^d, \quad t \in \mathbb{R}, \\
v(z,0) &= v_0(z), \quad \dot{v}(z,0) = v_1(z), \quad z \in \mathbb{Z}^d.
\end{aligned}
$$

(2.1)

Here $v(z,t) = (v_1(z,t), \ldots, v_n(z,t))$, $v_0(z) = (v_{01}(z), \ldots, v_{0n}(z)) \in \mathbb{R}^n$, and correspondingly for $v_1(z)$, $V(z)$ is the interaction (or force) matrix, $(V_{kl}(z))$, $k,l = 1, \ldots, n$.

Write $X(t) = (X^0(t), X^1(t)) \equiv (v(\cdot,t), \dot{v}(\cdot,t))$ and $X_0 = (X_0^0, X_0^1) \equiv (v_0(\cdot), v_1(\cdot))$. Then (2.1) becomes

$$
\dot{X}(t) = \mathcal{A}X(t), \quad t \in \mathbb{R}, \quad X(0) = X_0.
$$

(2.2)

Here $\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -V & 0 \end{pmatrix}$, where $V$ is a convolution operator with the matrix kernel $V$, $Vv = \sum_{z' \in \mathbb{Z}^d} V(z-z')v(z')$. Formally, (2.2) is a linear Hamiltonian system with the Hamiltonian functional

$$
H(X) = \frac{1}{2} \sum_{z \in \mathbb{Z}^d} |v_1(z)|^2 + \frac{1}{2} \sum_{z,z' \in \mathbb{Z}^d} v_0(z) \cdot V(z-z')v_0(z'), \quad X = (v_0, v_1),
$$

(2.3)

where the kinetic energy is given by the first term, and the potential energy by the second term.

Assume that the initial data $X_0$ for (2.2) belong to the phase space $\mathcal{H}_\alpha$, $\alpha \in \mathbb{R}$, defined below.

**Definition 2.1** $\mathcal{H}_\alpha$ is the Hilbert space of $\mathbb{R}^n \times \mathbb{R}^n$-valued functions of $z \in \mathbb{Z}^d$ endowed with the norm $\|X\|_\alpha^2 = \sum_{z \in \mathbb{Z}^d} |X(z)|^2 (1 + |z|^2)^\alpha < \infty$.

2.1 Conditions on the harmonic potentials

We impose the following conditions $\text{E1}–\text{E6}$ on the matrix $V$.

**E1.** There are positive constants $C$ and $\gamma$ such that $\|V(z)\| \leq Ce^{-\gamma|z|}$ for $z \in \mathbb{Z}^d$, where $\|V(z)\|$ stands for the matrix norm.

**E2.** The matrix $V(z)$ is real and symmetric, i.e., $V_{lk}(-z) = V_{kl}(z) \in \mathbb{R}$, $k,l = 1, \ldots, n$, $z \in \mathbb{Z}^d$. 

3
Let \( \hat{V}(\theta) \) be the Fourier transform of \( V(z) \) with the convention \( \hat{V}(\theta) = \sum_{z \in \mathbb{Z}^d} V(z)e^{iz \cdot \theta} \), \( \theta \in \mathbb{T}^d \), where \( \mathbb{T}^d \) stands for the \( d \)-torus \( \mathbb{R}^d/(2\pi \mathbb{Z})^d \).

Conditions **E1** and **E2** imply that \( \hat{V}(\theta) \) is a real-analytic Hermitian matrix-valued function of \( \theta \in \mathbb{T}^d \).

**E3.** The matrix \( \hat{V}(\theta) \) is non-negative definite for every \( \theta \in \mathbb{T}^d \).

Let us define the Hermitian non-negative definite matrix,

\[
\Omega(\theta) = (\hat{V}(\theta))^{1/2} \geq 0.
\]

The matrix \( \Omega(\theta) \) has the eigenvalues \( 0 \leq \omega_1(\theta) < \omega_2(\theta) < \ldots < \omega_s(\theta) \), \( s \leq n \), and the corresponding spectral projections \( \Pi_\sigma(\theta) \) with multiplicity \( r_\sigma = \text{tr} \Pi_\sigma(\theta) \). The following lemma holds.

**Lemma 2.2** (see [11, Lemma 2.2]). Let conditions **E1** and **E2** hold. Then there exists a closed subset \( \mathcal{C}_* \subset \mathbb{T}^d \) such that the following assertions hold.

(i) The Lebesgue measure of \( \mathcal{C}_* \) is zero.

(ii) The eigenvalue \( \omega_\sigma(\theta) \), \( \sigma = 1, \ldots, s \), has constant multiplicity in \( \mathbb{T}^d \setminus \mathcal{C}_* \).

(iii) The following spectral decomposition holds: \( \Omega(\theta) = \sum_{\sigma=1}^s \omega_\sigma(\theta) \Pi_\sigma(\theta) \), \( \theta \in \mathbb{T}^d \setminus \mathcal{C}_* \), where \( \Pi_\sigma(\theta) \) is a real-analytic function on \( \mathbb{T}^d \setminus \mathcal{C}_* \).

For \( \theta \in \mathbb{T}^d \setminus \mathcal{C}_* \), denote by \( \nabla^2 \omega_\sigma(\theta) \) the matrix of second partial derivatives. The next condition on \( V \) is as follows.

**E4.** The functions \( D_\sigma(\theta) := \det(\nabla^2 \omega_\sigma(\theta)) \) do not vanish identically on \( \mathbb{T}^d \setminus \mathcal{C}_* \), \( \sigma = 1, \ldots, s \).

Let us write

\[
\mathcal{C}_0 = \{ \theta \in \mathbb{T}^d : \det \hat{V}(\theta) = 0 \}, \quad \mathcal{C}_\sigma = \{ \theta \in \mathbb{T}^d \setminus \mathcal{C}_* : D_\sigma(\theta) = 0 \}, \quad \sigma = 1, \ldots, s.
\]

Then the Lebesgue measure of \( \mathcal{C}_\sigma \) vanishes, \( \sigma = 0, 1, \ldots, s \) (see [11, Lemma 2.3]). Usually, the dispersion relations \( \omega_k(\theta) \) satisfying the condition \( \omega_k(0) = 0 \) are called *acoustic*.

**E5.** For each \( \sigma \neq \sigma' \), the identities \( \omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \equiv \text{const}_\pm \) for \( \theta \in \mathbb{T}^d \setminus \mathcal{C}_* \), do not hold with \( \text{const}_\pm \neq 0 \).

This condition holds trivially for \( n = 1 \).

**E6.** \( \|\hat{V}^{-1}(\theta)\| \in L^1(\mathbb{T}^d) \).

If \( \mathcal{C}_0 = \emptyset \), then \( \|\hat{V}^{-1}(\theta)\| \) is bounded, and **E6** evidently holds.

**Remark 2.3** (i) Instead of condition **E1** we may assume that \( |V(z)| \leq C(1 + |z|)^{-N} \) with an \( N > 0 \). However, in this case, we should assume in addition that there exists a set \( \mathcal{K} \subset \mathbb{T}^d \) such that \( \text{mes}(\mathbb{T}^d \setminus \mathcal{K}) = 0 \) and for \( \theta \in \mathcal{K} \), \( \omega_\sigma \in C^3(\mathcal{K}) \), \( \Pi_\sigma \in C(\mathcal{K}) \), \( \omega_\sigma(\theta) \neq 0 \), \( \det(\nabla^2 \omega_\sigma(\theta)) \neq 0 \), \( \nabla \omega_\sigma(\theta) \neq 0 \), \( \sigma = 1, \ldots, s \). Note that if condition **E1** holds, \( \mathcal{K} = \mathbb{T}^d \setminus \mathcal{C} \) with \( \mathcal{C} \) defined in (6.3).
(ii) Conditions \textbf{E1–E6} are satisfied, in particular, in the case of the nearest neighbor crystal (see [11]) in which the interaction matrix \( V(z) = (V_{kl}(z))_{k,l=1}^n \) is of the form

\[
V_{kl}(z) = 0 \text{ for } k \neq l, \quad V_{kk}(z) = \begin{cases} -\gamma_k & \text{for } |z| = 1, \\ 2d\gamma_k + m_k^2 & \text{for } z = 0, \\ 0 & \text{for } |z| \geq 2, \end{cases}
\]

with \( \gamma_k > 0 \) and \( m_k \geq 0 \). In this case, the Hamiltonian functional has a form

\[
H(v_0, v_1) = \frac{1}{2} \sum_{z \in \mathbb{Z}^d} \sum_{k=1}^n \left( |v_{1k}(z)|^2 + m_k^2 |v_{0k}(z)|^2 + \sum_{j=1}^d \gamma_k |v_{0k}(z + e_j) - v_{0k}(z)|^2 \right),
\]

where \( \Delta_L \) stands for the discrete Laplace operator on the lattice \( \mathbb{Z}^d \),

\[
\Delta_L v(z) := \sum_{j=1}^d (v(z + e_j) - 2v(z) + v(z - e_j)).
\]

Therefore, the eigenvalues of \( \Omega(\theta) \) are

\[
\tilde{\omega}_k(\theta) = \sqrt{2\gamma_k(1 - \cos \theta_1) + ... + 2\gamma_k(1 - \cos \theta_d) + m_k^2}, \quad k = 1, ..., n. \tag{2.6}
\]

These eigenvalues still have to be labelled according to magnitude and degeneracy as in Lemma 2.2. Clearly, conditions \textbf{E1–E5} hold, and \( \mathcal{C}_s = \emptyset \). If \( m_k > 0 \) for any \( k \), then the set \( \mathcal{C}_0 \) is empty and condition \textbf{E6} holds automatically. Otherwise, if \( m_k = 0 \) for some \( k \), then \( \mathcal{C}_0 = \{0\} \). In this case, \textbf{E6} is equivalent to the condition \( \omega_k^{-2}(\theta) \in L^1(\mathbb{T}^d) \). Therefore, conditions \textbf{E1–E6} hold if either (i) \( d \geq 3 \) or (ii) \( d = 1, 2 \) and \( m_k > 0 \) for any \( k \).

**Lemma 2.4** (see [11] Proposition 2.5]) Let conditions \textbf{E1} and \textbf{E2} hold, and \( \alpha \in \mathbb{R} \). Then

(i) for any \( X_0 \in \mathcal{H}_\alpha \), there exists a unique solution \( X(t) \in C(\mathbb{R}, \mathcal{H}_\alpha) \) to the Cauchy problem \tag{2.2};

(ii) for any \( t \in \mathbb{R} \), the operator \( U(t) : X_0 \mapsto X(t) \) is continuous on \( \mathcal{H}_\alpha \).

The proof of Lemma 2.4 is based on the following formula for the solution \( X(t) \) of problem \tag{2.2}:

\[
X(t) = \sum_{z' \in \mathbb{Z}^d} \mathcal{G}_t(z - z')X_0(z'), \tag{2.7}
\]

where the function \( \mathcal{G}_t(z) \) has the Fourier representation

\[
\mathcal{G}_t(z) := \mathcal{F}_{\theta \to z}[\exp (\hat{A}(\theta)t)] = (2\pi)^{-d} \int_{\mathbb{T}^d} e^{-iz\theta} \exp (\hat{A}(\theta)t) d\theta \tag{2.8}
\]

with

\[
\hat{A}(\theta) = \begin{pmatrix} 0 & 1 \\ -\hat{V}(\theta) & 0 \end{pmatrix}, \quad \theta \in \mathbb{T}^d. \tag{2.9}
\]
2.2 The family of initial measures

Let \( \varepsilon > 0 \) be a small scale parameter, \( \{ \mu_0^\varepsilon, \varepsilon > 0 \} \) be a family of initial measures. To formulate the main conditions \( \mathbf{V1} \) and \( \mathbf{V2} \) on the covariance of \( \mu_0^\varepsilon \), let us introduce the complex \( 2n \times 2n \) matrix-valued function \( \hat{R}_0(r, z) = (\hat{R}^{ij}_0(r, z))_{i,j=0}^1, \ r \in \mathbb{R}^d, \ z \in \mathbb{Z}^d, \) with the following properties.

**I1.** For every fixed \( r \in \mathbb{R}^d \) and \( i, j = 0, 1, \) the bound holds,

\[
|\hat{R}^{ij}_0(r, z)| \leq C(1 + |z|)^{-\gamma}, \quad z \in \mathbb{Z}^d,
\]

where \( C \) is some positive constant, \( \gamma > d. \)

**I2.** For every fixed \( r \in \mathbb{R}^d, \) \( \hat{R}_0(r, \theta) \) satisfies

\[
\hat{R}_0^{00}(r, \theta) \geq 0, \quad \hat{R}_0^{11}(r, \theta) \geq 0, \quad \hat{R}_0^{01}(r, \theta) = \hat{R}_0^{10}(r, \theta)^*, \quad \theta \in \mathbb{T}^d.
\]

**I3.** For every fixed \( r \in \mathbb{R}^d \) and \( \theta \in \mathbb{T}^d, \) the matrix \( \hat{R}_0(r, \theta) \) is non-negative definite.

**I4.** For every \( \theta \in \mathbb{T}^d, \) \( \hat{R}^{ij}_0(\cdot, \theta) \) are \( C^d \) functions, and the function

\[
r \to \sup_{\theta \in \mathbb{T}^d} \max_{i,j=0,1} \max_{\alpha_1=0,1,...,d} \left| \frac{\partial^{\alpha_1+...+\alpha_d}}{\partial r_{i_1}^{\alpha_1}...\partial r_{d}^{\alpha_d}} \hat{R}^{ij}_0(r, \theta) \right|
\]

is bounded uniformly on bounded sets.

To derive the equation of the Navier–Stokes type we need an additional condition \( \mathbf{I4'} \).

**I4'.** \( \hat{R}_0(\cdot, \theta) \in L^1(\mathbb{R}^d), \ \forall \theta \in \mathbb{T}^d, \) and there exist constants \( C > 0 \) and \( N > d + 3 \) such that

\[
\sup_{\theta \in \mathbb{T}^d} |\hat{R}_0(s, \theta)| \leq C(1 + |s|)^{-N}, \quad s \in \mathbb{R}^d.
\]

Here by \( \hat{R}_0(s, \theta) \) we denote the Fourier transform of \( \hat{R}_0(r, \theta) \) with respect to \( r: \)

\[
\hat{R}_0(s, \theta) = F_{r \rightarrow s}[\hat{R}_0(r, \theta)] = \int_{\mathbb{R}^d} e^{is\cdot r} \hat{R}_0(r, \theta) \, dr, \quad s \in \mathbb{R}^d, \ \theta \in \mathbb{T}^d.
\]

This condition could be weakened (see condition D' in [7] for the case \( d = n = 1 \)). Instead of \( \mathbf{I4'} \) we may assume that for each \( \theta \in \mathbb{T}^d \) the function \( \hat{R}_0(\cdot, \theta) \) admits the representation

\[
\hat{R}_0(r, \theta) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-is\cdot r} \mu(\theta, ds),
\]

where \( \mu(\theta, ds) \) is some Borel (complex-valued) measure on \( \mathbb{R}^d, \) depending on the parameter \( \theta \in \mathbb{T}^d. \) Let \( |\mu|(\theta, \cdot), \ \theta \in \mathbb{T}^d, \) be the total variation of the measure \( \mu(\theta, \cdot). \) This means (see [26 §29]) that for any measurable set \( E \subset \mathbb{R}^d, \) \( |\mu|(\theta, E) = \sup \{ \int f(s)\mu(\theta, ds) \} \), where supremum is taken over all measurable functions \( f \) such that \( |f| \leq 1. \) We assume that there exist constants \( C > 0 \) and \( \delta > d^2/2 + 2d + 1 \) such that for any \( n \in \mathbb{N}, \)

\[
\sup_{\theta \in \mathbb{T}^d} |\mu|(\theta, K_n) \leq C(1 + n)^{-\delta}, \quad \text{where} \quad K_n = \{ s \in \mathbb{R}^d : n \leq |s| < n + 1 \}.
\]
Denote by $\mathbb{E}_0^\varepsilon$ expectation with respect to the measure $\mu_0^\varepsilon$, and by $Q_{\varepsilon}^{ij}(z, z') = \mathbb{E}_0^\varepsilon(X^i(z) \otimes X^j(z'))$, $z, z' \in \mathbb{Z}^d$, $i, j = 0, 1$, the correlation functions of $\mu_0^\varepsilon$. Assume that $\mathbb{E}_0^\varepsilon(X(z)) = 0$ and functions $Q_{\varepsilon}^{ij}(z, z')$ satisfy the following conditions $V1$ and $V2$.

$V1$. For any $\varepsilon > 0$, $z, z' \in \mathbb{Z}^d$,
\[
|Q_{\varepsilon}^{ij}(z, z') - R_0^{ij}(z, z')| \leq C \min \left[ (1 + |z - z'|)^{-\gamma}, \varepsilon |z - z'| \right], \tag{2.15}
\]
with the constants $C$, $\gamma$ as in (2.10).

$V2$. For any $\varepsilon > 0$ and all $z, z' \in \mathbb{Z}^d$, $i, j = 0, 1$, $|Q_{\varepsilon}^{ij}(z, z')| \leq C(1 + |z - z'|)^{-\gamma}$ with the constants $C$ and $\gamma$ as in (2.10).

Remarks (i) Condition $V1$ can be formulated in the another form (see [6, 14]). Namely, for any $\varepsilon > 0$ there exists an even integer $N_\varepsilon$ such that
a) for all $M \in \mathbb{R}^d$ and $z, z' \in I_M$,
\[
|Q_{\varepsilon}^{ij}(z, z') - R_0^{ij}(\varepsilon M, z - z')| \leq C \min[(1 + |z - z'|)^{-\gamma}, \varepsilon N_\varepsilon],
\]
where $C$, $\gamma$ are the constants from (2.10), and $I_M$ is the cube centered at the point $M$ with edge length $N_\varepsilon$, $I_M = \{z = (z_1, \ldots, z_d) \in \mathbb{Z}^d : |z_j - M_j| \leq N_\varepsilon/2, M = (M_1, \ldots, M_d)\};$

b) $N_\varepsilon \sim \varepsilon^{-\beta}$ as $\varepsilon \to 0$, with some $\beta \in (1/2, 1)$.

The formulation of $V1$ in the form (2.15) is more natural and convenient for our proof.

(ii) By conditions $V1$ and $V2$, $\sum_{p \in \mathbb{Z}^d} e^{i \theta_p} Q_{\varepsilon}^{ij}(\epsilon r + p/2, \epsilon r - p/2) \to R_0^{ij}(r, \theta)$ as $\varepsilon \to 0$ uniformly in $r \in \mathbb{R}^d$ and $\theta \in \mathbb{T}^d$. Here and below $[x] = ([x_1], \ldots, [x_d])$ for $x \in \mathbb{R}^d$ and $[x_i]$ stands for the integer part of $x_i \in \mathbb{R}^1$, $i = 1, \ldots, d$.

2.3 Example of initial measures $\mu_0^\varepsilon$

We construct Gaussian initial measures $\mu_0^\varepsilon$ satisfying conditions $V1$ and $V2$. At first, we introduce matrix-valued functions $q_{0}^{ij}(z), z \in \mathbb{Z}^d$, such that $q_{0}^{ij}(z) = 0$ for $i \neq j$, and
\[
q_0^{ii}(\theta) = F_{z \to \theta}[q_0^{ii}(z)] \in L^1(\mathbb{T}^d), \quad q_0^{ii}(\theta) \geq 0, \quad i = 0, 1.
\]

Next, we set
\[
R_0^{ij}(r, z) = T(r) q_0^{ij}(z), \quad r \in \mathbb{R}^d, \quad z \in \mathbb{Z}^d, \tag{2.16}
\]
where $T \in C^d(\mathbb{R}^d)$, $T(r) \geq 0$, $\sup \sup_{r \in \mathbb{R}^d, |s| \leq d} |D^a T(r)| \leq C < \infty$, $|F_{r \to s}[T(r)]| \leq C(1 + |s|)^{-N}$ with an $N > d$. Finally, we put
\[
Q_{\varepsilon}^{ij}(z, z') = \sqrt{T(\varepsilon z)T(\varepsilon z')} q_0^{ij}(z - z'), \quad z, z' \in \mathbb{Z}^d, \quad i, j = 0, 1. \tag{2.17}
\]

By the Minlos theorem, for any $\varepsilon > 0$, there exists a Borel Gaussian measure $\mu_0^\varepsilon$ on $\mathcal{H}_\alpha$, $\alpha < -d/2$, with the correlation functions $Q_{\varepsilon}^{ij}(z, z')$, because
\[
\mathbb{E}_0^\varepsilon(\|X\|_{\alpha}^2) = \sum_{z \in \mathbb{Z}^d} (1 + |z|^2)^\alpha \text{tr}[q_0^{00}(z, z) + Q_{\varepsilon}^{11}(z, z)].
\]
\[
\begin{align*}
&= \sum_{z \in \mathbb{Z}^d} (1 + |z|^2)^n T(z) \operatorname{tr}[q_0^{00}(0) + q_0^{11}(0)] \\
&\leq C(\alpha, d) \operatorname{tr} \int_{\mathbb{T}^d} (q_0^{00}(\theta) + q_0^{11}(\theta)) \, d\theta \leq C_1 < \infty.
\end{align*}
\]

Let us assume that there exist constants \( C > 0 \) and \( \gamma > d \) such that
\[
|q_0^{ij}(z)| \leq C(1 + |z|)^{-\gamma}, \quad z \in \mathbb{Z}^d.
\]

Then \( Q_\varepsilon^{ij}(z, z') \) satisfy the conditions \( \textbf{V1} \) and \( \textbf{V2} \).

**Definition 2.5** Formally, Gibbs measure \( g \) is \( g(dX) = \frac{1}{Z} e^{-\frac{\beta}{2} \sum_x H(x)} \prod_x dX(z) \), where \( H(X) \) is defined in (2.3), \( \beta = T^{-1}, \, T \geq 0 \) is the corresponding absolute temperature. We define the Gibbs measure \( g \) on \( \mathcal{H}_\alpha \), \( \alpha < -d/2 \), as the Gaussian measure with the correlation matrices defined by their Fourier transform as \( \hat{q}^{00}(\theta) = TV^{-1}(\theta), \, \hat{q}^{11}(\theta) = TI \), \( \hat{q}^{01}(\theta) = \hat{q}^{10}(\theta) = 0 \), where \( I \) stands for the unit matrix in \( \mathbb{R}^n \).

Let \( \hat{q}^{00}(\theta) = \hat{V}^{-1}(\theta), \, \hat{q}^{11}(\theta) = I, \, \theta \in \mathbb{T}^d \). Then the functions \( R_\varepsilon^{ij}(r, z) \) defined in (2.16) are correlation matrices of the Gibbs measures \( g_r, \, r \in \mathbb{R}^d \), with \( \beta = 1/T(r) \). If we assume, in addition, that \( C_0 = 0 \), i.e. \( \det \hat{V}(\theta) \neq 0 \), \( \forall \theta \in \mathbb{T}^d \), then the bound (2.18) holds, and \( Q_\varepsilon^{ij}(z, z') \) defined in (2.17) satisfy the conditions \( \textbf{V1} \) and \( \textbf{V2} \).

### 3 Main results

**Definition 3.1** (i) \( \mu_\varepsilon \) is a Borel probability measure on \( \mathcal{H}_\alpha \) which gives the distribution of the random solution \( X(t), \mu_\varepsilon(B) = \mu_0(U(-t)B), \) where \( B \in \mathcal{B}(\mathcal{H}_\alpha) \) and \( t \in \mathbb{R} \).

(ii) The correlation functions of the measure \( \mu_\varepsilon \) are defined by
\[
Q_\varepsilon^{ij}(z, z') = E^\varepsilon I(X^i(z) \otimes X^j(z')) = E^\varepsilon(X^i(z, t) \otimes X^j(z', t)), \quad i, j = 0, 1, \quad z, z' \in \mathbb{Z}^d,
\]
where \( E^\varepsilon \) stands for expectation with respect to the measure \( \mu_\varepsilon \), and \( X^i(z, t) \) are the components of the random solution \( X(t) = (X^0(\cdot, t), X^1(\cdot, t)) \) to problem (2.3).

(iii) Let \( T_h \), \( h \in \mathbb{Z}^d \), be the group of space translations: \( T_h X(z) = X(z-h), \, z \in \mathbb{Z}^d \). For \( \tau \neq 0 \), \( r \in \mathbb{R}^d \), and \( \kappa > 0 \), the measures \( \mu_{\varepsilon/\kappa, r/\varepsilon} \) are defined by the rule
\[
\mu_{\varepsilon/\kappa, r/\varepsilon}(B) = \mu_{\varepsilon/\kappa}(T_{r/\varepsilon}B), \quad \text{where } B \in \mathcal{B}(\mathcal{H}_\alpha).
\]

**Remarks** (i) In addition to conditions \( \textbf{V1} \) and \( \textbf{V2} \), let us assume that the measures \( \mu_0^\varepsilon \) satisfy the mixing condition. To formulate this condition, denote by \( \sigma(A), \, A \subset \mathbb{Z}^d \), the \( \sigma \)-algebra on \( \mathcal{H}_\alpha \) generated by \( X_0(z) \) with \( z \in \mathcal{A} \). Define the Ibragimov mixing coefficients of the probability measure \( \mu_0^\varepsilon \) on \( \mathcal{H}_\alpha \) by the rule
\[
\varphi_\varepsilon(r) = \sup_{A, B \subset \mathbb{Z}^d, \operatorname{dist}(A, B) \geq r} \frac{\sup_{A \in \sigma(A), \, B \in \sigma(B), \mu_0^\varepsilon(B) > 0} |\mu_0^\varepsilon(A \cap B) - \mu_0^\varepsilon(A) \mu_0^\varepsilon(B)|}{\mu_0^\varepsilon(B)}.
\]
A measure $\mu_0^\varepsilon$ is said to satisfy the strong uniform Ibragimov mixing condition if $\varphi_\varepsilon(r) \to 0$ as $r \to \infty$. Moreover, we assume that $\forall r \in \mathbb{R}^d$, $\sup_{\varepsilon>0} \varphi_\varepsilon(r) \leq C(1 + r)^{-2\gamma}$, with the constant $\gamma$ as in [2,10]. Note that the last bound on $\varphi_\varepsilon(r)$ implies condition $V_2$.

Then for $\tau \neq 0$, $r \in \mathbb{R}^d$, the measures $\mu_{\tau/\varepsilon, r/\varepsilon}^\varepsilon$ converge weakly to a limit measure $\mu_{\tau, r}^G$ on the space $\mathcal{H}_\alpha$, $\alpha < -d/2$. By definition, this means that

$$\lim_{\varepsilon \to 0} \int f(X) \mu_{\tau/\varepsilon, r/\varepsilon}^\varepsilon(dX) = \int f(X) \mu_{\tau, r}^G(dX)$$

for any bounded continuous functional $f$ on $\mathcal{H}_\alpha$. Moreover, the limit measure $\mu_{\tau, r}^G$ is a Gaussian measure on $\mathcal{H}_\alpha$ (see Theorem 3.3 in [14]).

(ii) Let $\kappa < 1$. Then for $\tau \neq 0$, $r \in \mathbb{R}^d$, the following limit holds, $\mu_{\tau/\varepsilon, r/\varepsilon}^\varepsilon \to \mu_r$ as $\varepsilon \to 0$ in the sense of weak convergence on $\mathcal{H}_\alpha$, $\alpha < -d/2$. Moreover, the limit measure $\mu_r$ is Gaussian, its correlation matrix does not depend on $\tau$ and has a form (in Fourier transform)

$$\frac{1}{2} \sum_{\sigma=1}^s \Pi_\sigma(\theta) \left( \hat{R}_0(r, \theta) + C_\sigma(\theta) \hat{R}_0(r, \theta) C_\sigma^*(\theta) \right) \Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus C_s,$$

where $C_\sigma(\theta)$ is defined in [34].

### 3.1 Equation of Euler type

In this subsection we put $\kappa = 1$. Let us introduce the matrix $q_{r, r}(z), z \in \mathbb{Z}^d, \tau \in \mathbb{R}$, $r \in \mathbb{R}^d$. In Fourier space,

$$\hat{q}_{r, r}(\theta) = \frac{1}{4} \sum_{\sigma=1}^s \Pi_\sigma(\theta) \left[ \sum_{\pm} (I \pm iC_\sigma(\theta)) \hat{R}_0(r \pm \nabla \omega_\sigma(\theta) \tau, \theta)(I \mp iC_\sigma^*(\theta)) \right] \Pi_\sigma(\theta)$$

$$= \sum_{\sigma=1}^s \Pi_\sigma(\theta) \left( M_+^\sigma(\tau, r; \theta) + M_-^\sigma(\tau, r; \theta) \right) \Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus C_s,$$

where $\Pi_\sigma(\theta)$ is the spectral projection introduced in Lemma 2.2 (iii),

$$M_+^\sigma(\tau, r; \theta) = \frac{1}{2} \left( R_+^\sigma(\tau, r; \theta) + C_\sigma(\theta) R_+^\sigma(\tau, r; \theta) C_\sigma^*(\theta) \right),$$

$$M_-^\sigma(\tau, r; \theta) = \frac{1}{2} \left( C_\sigma(\theta) R_-^\sigma(\tau, r; \theta) - R_-^\sigma(\tau, r; \theta) C_\sigma^*(\theta) \right),$$

$$C_\sigma(\theta) = \begin{pmatrix} 0 & \omega^{-1}_\sigma(\theta) \\ -\omega_\sigma(\theta) & 0 \end{pmatrix}, \quad \sigma = 1, \ldots, s,$$

$$R_\pm^\sigma(\tau, r; \theta) = \frac{1}{2} \left( \hat{R}_0(r + \nabla \omega_\sigma(\theta) \tau, \theta) \pm \hat{R}_0(r - \nabla \omega_\sigma(\theta) \tau, \theta) \right).$$

**Theorem 3.2** (see [14], Theorem 4.1) Let the conditions $V_1$, $V_2$ and $E_1$–$E_6$ hold. Then for any $r \in \mathbb{R}^d, z, z' \in \mathbb{Z}^d, \tau \neq 0$, the correlation functions of measures $\mu_{\tau/\varepsilon, r/\varepsilon}^\varepsilon$ converge to a limit,

$$\lim_{\varepsilon \to 0} Q_{\varepsilon, \tau/\varepsilon}([r/\varepsilon] + z, [r/\varepsilon] + z') = q_{r, r}(z - z').$$
Corollary 3.3 Write $f_\sigma(\tau, r; \theta) = \Pi_\sigma(\theta)q_{\tau, r}(\theta)\Pi_\sigma(\theta)$, $\sigma = 1, \ldots, s$. Then the function $f_\sigma(\tau, r; \theta)$ satisfies the "hydrodynamic" Euler type equation:

$$
\partial_r f_\sigma(\tau, r; \theta) = iC_\sigma(\theta)\nabla\omega_\sigma(\theta) \cdot \nabla f_\sigma(\tau, r; \theta), \quad r \in \mathbb{R}^d, \quad \tau > 0, \quad (3.6)
$$

$$
f_\sigma(\tau, r; \theta)|_{\tau=0} = \frac{1}{2}\Pi_\sigma(\theta)\left(\hat{R}_0(r, \theta) + C_\sigma(\theta)\hat{R}_0(r, \theta)C_\sigma^*(\theta)\right)\Pi_\sigma(\theta). \quad (3.7)
$$

Remarks

(i) Note that $\hat{W}_{\tau, r}(\theta) = (\hat{q}_{\tau, r}(\theta))_{ij}^{0, 0}$ satisfies the equilibrium condition, i.e.,

$$
\hat{q}_{\tau, r}(\theta) = \Omega^2(\theta)q_{\tau, r}^{0, 0}(\theta), \quad \hat{q}_{\tau, r}(\theta)^* = -\hat{q}_{\tau, r}(\theta). \quad (3.10)
$$

Moreover, $\hat{q}_{\tau, r}(\theta)^\ast = \hat{q}_{\tau, r}(\theta) \geq 0$, $\hat{q}_{\tau, r}(\theta)^* = \hat{q}_{\tau, r}(\theta)^\ast$.

(ii) In the case when $\kappa \in [0, 1/2]$, the correlation matrices of measures $\mu_{\tau, r}(z - z')$ converge as $\varepsilon \to 0$ to the same matrices $q_{\tau, r}(z - z')$ as in Theorem 3.2. This result can be proved by the similar way as Theorem 3.2.

Theorem 3.2 and Corollary 3.3 can be rewritten in the terms of the Wigner matrices. Write $(\mathcal{V}^k v)(x) = F_0^{-1}[\mathcal{V}^k v(\theta)]$, $k \in \mathbb{R}$, and introduce the complex-valued field

$$
a(x) = \frac{1}{\sqrt{2}}(\mathcal{V}^{1/4}v_0(x) + i\mathcal{V}^{-1/4}v_1(x)) \in \mathbb{C}^n, \quad x \in \mathbb{Z}^d, \quad (3.8)
$$

with complex conjugate field $a(x)^* = (1/\sqrt{2})\left(\mathcal{V}^{1/4}v_0(x) - i\mathcal{V}^{-1/4}v_1(x)\right)$. Define the scaled $n \times n$ Wigner matrix as

$$
W^\varepsilon(\tau, r; \theta) = \sum_{y \in \mathbb{Z}^d} e^{y+y}E_{\tau, r}^\varepsilon(a^*([r/\varepsilon + y/2]) \otimes a([r/\varepsilon - y/2])). \quad (3.9)
$$

By properties of $\mu_0^\varepsilon$, the following limit exists

$$
\lim_{\varepsilon \to 0} W^\varepsilon(0, r; \theta) = \frac{1}{2}\left(\Omega^{1/2}\hat{R}_0^{0, 0}(r, \theta)\Omega^{1/2} + \Omega^{-1/2}\hat{R}_0^{1, 0}(r, \theta)\Omega^{-1/2}
\right.
$$

$$
+ i\Omega^{1/2}\hat{R}_0^{0, 1}(r, \theta)\Omega^{-1/2} - i\Omega^{-1/2}\hat{R}_0^{1, 0}(r, \theta)\Omega^{1/2}) = W(0, r; \theta). \quad (3.10)
$$

Theorem 3.5 (see [17, Theorem 3.2]) Let the conditions V1, V2 and E1–E6 hold. Then for any $r \in \mathbb{R}^d$ and $\tau \neq 0$ the following limit exists

$$
\lim_{\varepsilon \to 0} W^\varepsilon(\tau, r; \theta) = W^\varepsilon(\tau, r; \theta) \quad \text{(in the sense of distributions),}
$$

where

$$
W^\varepsilon(\tau, r; \theta) = \sum_{\sigma=1}^s \Pi_\sigma(\theta)W(0, r - \tau\nabla\omega_\sigma(\theta); \theta)\Pi_\sigma(\theta). \quad (3.11)
$$

Remarks

(i) The limit correlation matrices $q_{\tau, r}(\theta) = (q_{\tau, r}(\theta))_{ij}^{0, 0}$ are expressed by $W^\varepsilon(\tau, r; \theta)$ as

$$
\Omega(\theta)q_{\tau, r}(\theta) = \Omega(\theta)^{-1}q_{\tau, r}(\theta) = \frac{1}{2}(W^\varepsilon(\tau, r; \theta) + W^\varepsilon(\tau, r; \theta)^*),
$$

$$
q_{\tau, r}(\theta)^* = -\frac{i}{2}(W^\varepsilon(\tau, r; \theta) - W^\varepsilon(\tau, r; \theta)^*). \quad (3.10)
$$

(ii) The matrix $f_\sigma(\tau, r; \theta) \equiv \Pi_\sigma(\theta)W^\varepsilon(\tau, r; \theta)\Pi_\sigma(\theta)$, $\sigma = 1, \ldots, s$, satisfies the energy transport equation (see [13, p.656])

$$
\partial_r f_\sigma(\tau, r; \theta) + \nabla\omega_\sigma(\theta) \cdot \nabla f_\sigma(\tau, r; \theta) = 0, \quad \tau > 0, \quad r \in \mathbb{R}^d,
$$

with the initial condition $f_\sigma(\tau, r; \theta)|_{\tau=0} = \Pi_\sigma(\theta)W(0, r; \theta)\Pi_\sigma(\theta)$, $r \in \mathbb{R}^d$. 

10
3.2 Navier–Stokes equation

In this subsection we study the behaviour (as \( \varepsilon \to 0 \)) of the correlation functions of \( \mu_{\tau/\varepsilon^2,r/\varepsilon}^\varepsilon \) for \( \kappa = 2 \), and obtain the next term of the decomposition in \( \varepsilon \) to equation (3.6). Let us introduce the matrix \( \hat{q}_{t,r}(z) \), \( z \in \mathbb{Z}^d \), \( r \in \mathbb{R}^d \), \( \tau \neq 0 \), \( \varepsilon > 0 \), which has the following form (in the Fourier transform)

\[
\hat{q}_{t,r}(\theta) = \frac{1}{4} \sum_{\sigma=1}^{s} \Pi_{\sigma}(\theta) \left[ \sum_{\pm} (I \pm iC_{\sigma}(\theta))A_{\varepsilon,\sigma}^\pm(\tau,r;\theta)(I \mp iC_{\sigma}^*(\theta))\right] \Pi_{\sigma}(\theta), \quad \theta \in \mathbb{T}^d \setminus \mathcal{C}, \quad (3.12)
\]

where matrices \( C_{\sigma}(\theta) \) are defined in (3.4),

\[
A_{\varepsilon,\sigma}^\pm(\tau,r;\theta) = \int_{\mathbb{R}^d} \hat{R}_0(r \pm \nabla \omega_{\sigma}(\theta)\tau/\varepsilon - x,\theta)K_{\sigma}^\pm(\tau,x,\theta) \, dx, \quad (3.13)
\]

\[
K_{\sigma}^\pm(\tau,x,\theta) := F_{y \to x}^{-1}[e^{\mp i(\tau/2)\omega(\theta)x}] = F_{y \to x}^{-1}[\hat{q}_{t,r}(\theta)] \quad (3.14)
\]

(see also formula (6.25)).

**Theorem 3.6** Let conditions I1–I4', V1, V2 and E1–E6 hold. Then for any \( \tau \neq 0 \), \( r \in \mathbb{R}^d \), \( z,z' \in \mathbb{Z}^d \), the correlation functions of measures \( \mu_{\tau/\varepsilon^2,r/\varepsilon}^\varepsilon \) have the following asymptotics

\[
\lim_{\varepsilon \to 0} \left( Q_{\varepsilon,r/\varepsilon}(\lfloor r/\varepsilon \rfloor + z,\lfloor r/\varepsilon \rfloor + z') - q_{\varepsilon,r}(z - z') \right) = 0, \quad (3.15)
\]

where the matrix \( q_{\varepsilon,r}(z) = F_{\theta \to z}^{-1}[\hat{q}_{t,r}(\theta)] \) is defined by (3.14).

We omit the proof of this theorem since it can be proved by the similar technique as Theorem 5.6, below.

**Remark.** Note that \( \hat{q}_{t,r}(\theta) \) satisfies the equilibrium condition (see Remarks 3.3 (i)).

Set \( \tau = \varepsilon t \). It follows from formulas (3.12)–(3.14) that

\[
A_{\varepsilon,\sigma}^\pm(\varepsilon t,r;\theta)|_{\varepsilon=0} = \hat{R}_0(r \pm \nabla \omega_{\sigma}(\theta)t,\theta).
\]

Hence, \( \hat{q}_{t,t}(\theta)|_{\varepsilon=0} = \hat{q}_{t,r}(\theta) \), where \( \hat{q}_{t,r}(\theta) \) is defined in (3.1). Denote

\[
\nabla^k \omega_{\sigma}(\theta) \cdot \nabla f(r) := \sum_{i_1,\ldots,i_k=1}^d \frac{\partial^k \omega_{\sigma}(\theta)}{\partial \theta_{i_1} \cdots \partial \theta_{i_k}} \frac{\partial^k f}{\partial r_{i_1} \cdots \partial r_{i_k}}, \quad k \in \mathbb{N}. \quad (3.16)
\]

**Corollary 3.7** Let \( r \in \mathbb{R}^d \), \( t \in \mathbb{R} \), \( \theta \in \mathbb{T}^d \setminus \mathcal{C} \). It follows from formulas (3.12)–(3.14) that for each \( \sigma = 1,\ldots,s \), the matrix-valued function \( f_{\sigma}^\varepsilon(t,r;\theta) \equiv \Pi_{\sigma}(\theta)\hat{q}_{t,t}(\theta)\Pi_{\sigma}(\theta) \) evolves according to the following (Navier–Stokes type) equation

\[
\partial_t f_{\sigma}^\varepsilon(t,r;\theta) = iC_{\sigma}(\theta) \left( \nabla \omega_{\sigma}(\theta) \cdot \nabla f + \frac{i\varepsilon}{2} \nabla^2 \omega_{\sigma}(\theta) \cdot \nabla^2 f \right) f_{\sigma}^\varepsilon(t,r;\theta), \quad t \in \mathbb{R}, \quad r \in \mathbb{R}^d, \quad (3.17)
\]

with the initial condition (3.7).
3.3 Corrections of the higher order

To obtain the "corrections" of order $\varepsilon^2$, $\varepsilon^3$, ... to equation (3.17), it is necessary to study the behavior (as $\varepsilon \to 0$) of the correlation functions of measures $\mu_{r/\varepsilon,x/\varepsilon}^\varepsilon$ with $\kappa > 2$.

**Theorem 3.8** Let $\kappa \geq 2$, $r \in \mathbb{R}^d$, $\tau > 0$. Then $\forall z, z' \in \mathbb{Z}^d$,

$$Q_{\varepsilon, r/\varepsilon}([r/\varepsilon] + z, [r/\varepsilon] + z') - q_{r, r}^\varepsilon([r/\varepsilon] + z - z') \to 0, \quad \varepsilon \to 0.$$  

The matrix $q_{r, r}^\varepsilon(z)$ ($k = 2, 3, \ldots$) has the following form (in the Fourier transform)

$$q_{r, r}^\varepsilon(k) = \frac{1}{4} \sum_{\sigma = 1}^s \Pi_\sigma(\theta) \left[ \sum_{\pm} \left( I + iC_\sigma(\theta) \right) A_{\varepsilon, \sigma}^{\pm,k}(\tau, r; \theta)(I \mp iC_\sigma^*(\theta)) \right] \Pi_\sigma(\theta),$$

where

$$A_{\varepsilon, \sigma}^{\pm,k}(\tau, r; \theta) = \int_{\mathbb{R}^d} \hat{R}_0(r \pm \nabla_\sigma(\theta) r/\varepsilon^{k-1} - x, \theta) K_{\varepsilon, \sigma}^{\pm,k}(\tau, x, \theta) dx, \quad \theta \in \mathbb{T}^d \setminus C,$$

$$K_{\varepsilon, \sigma}^{\pm,k}(\tau, x, \theta) = F_{y \to x}^{-1} \left[ \exp \left\{ \mp i \frac{\tau}{\varepsilon^{k-2}} \left( \frac{\nabla^2_\sigma(\theta) \cdot y_2^2}{2!} + \cdots + \frac{\nabla^k_\sigma(\theta) \cdot y_k^k}{k!} \right) \right\} \right], \quad x, y \in \mathbb{R}^d.$$

Here, by definition,

$$\nabla^k_\sigma(\theta) \cdot y := \sum_{i_1, \ldots, i_k = 1}^d \frac{\partial^k_\sigma(\theta)}{\partial \theta_{i_1} \cdots \partial \theta_{i_k}} y_{i_1} \cdots y_{i_k}, \quad y = (y_1, \ldots, y_d) \in \mathbb{R}^d.$$

Note that the matrix $q_{r, r}^{\varepsilon,k}(\theta)$ coincides with $q_{r, r}^\varepsilon(\theta)$ from formula (3.12). The proof of Theorem 3.8 are similar to the proof of Theorem 3.6.

Set $r = \varepsilon^{k-1} t$, $k \geq 2$. Then $\partial_\tau A_{\varepsilon, \sigma}^{\pm,k}(\varepsilon^{k-1} t, r; \theta) = \pm P_{\varepsilon}^k(\theta, \partial_\tau) A_{\varepsilon, \sigma}^{\pm,k}(\varepsilon^{k-1} t, r; \theta)$, where

$$P_{\varepsilon}^k(\theta, \partial_\tau) := \sum_{p=1}^k \frac{(i\varepsilon)^{p-1}}{p!} \nabla^p_\sigma(\theta) \cdot \nabla^p_\tau$$

(see notation (3.10)). Therefore, the matrix $\Pi_\sigma(\theta) q_{r, r}^{\varepsilon,k}(\theta) \Pi_\sigma(\theta)$ (denote its by $f_{\varepsilon}(t, r; \theta)$) is the solution of the following equation

$$\partial_\tau f_{\varepsilon}(t, r; \theta) = iC_\sigma(\theta) P_{\varepsilon}^k(\theta, \partial_\tau) f_{\varepsilon}(t, r; \theta), \quad t > 0, \quad r \in \mathbb{R}^d,$$

with the initial condition (3.17).

Denote by $f_{\varepsilon}^{ij}$, $i, j = 0, 1$, the elements of the matrix $f_{\varepsilon}(t, r; \theta)$. Then $f_{\varepsilon}^{11} = \omega_\sigma^2(\theta) f_{\varepsilon}^{00}$, $f_{\varepsilon}^{10} = -f_{\varepsilon}^{01}$, and the equations (3.18) can be rewritten in the form

$$\partial_\tau f_{\varepsilon}^{00} = \left( -i/\omega_\sigma(\theta) \right) P_{\varepsilon}^k(\theta, \partial_\tau) f_{\varepsilon}^{01}, \quad \partial_\tau f_{\varepsilon}^{01} = i\omega_\sigma(\theta) P_{\varepsilon}^k(\theta, \partial_\tau) f_{\varepsilon}^{00}.$$
4 Model II: Harmonic crystals in the half-space $\mathbb{Z}^d_+$

We study the dynamics of the harmonic crystals in $\mathbb{Z}^d_+, d \geq 1$,

$$
\ddot{u}(z, t) = -\sum_{z' \in \mathbb{Z}^d_+} (V(z - z') - V(z - \tilde{z}')) u(z', t), \quad z \in \mathbb{Z}^d_+, \quad t \in \mathbb{R},
$$

(4.1)

with zero boundary condition,

$$
u(z, t)|_{z_1=0} = 0,
$$

(4.2)

and with the initial data

$$
u(z, 0) = u_0(z), \quad \dot{u}(z, 0) = u_1(z), \quad z \in \mathbb{Z}^d_+.
$$

(4.3)

Here $\mathbb{Z}^d_+ = \{ z \in \mathbb{Z}^d : z_1 > 0 \}$. For convenience, we assume that $u_0(z) = u_1(z) = 0$ for $z_1 = 0$.

Write $Y(t) = (Y^0(t), Y^1(t)) \equiv (u(\cdot, t), \dot{u}(\cdot, t))$ and $Y_0 = (Y^0_0, Y^1_0) \equiv (u_0(\cdot), u_1(\cdot))$. Then (4.1)-(4.3) becomes the evolution equation

$$
\dot{Y}(t) = A_+ Y(t), \quad t \in \mathbb{R}, \quad z \in \mathbb{Z}^d_+, \quad Y^0(t)|_{z_1=0} = 0, \quad Y(0) = Y_0.
$$

(4.4)

Here $A_+ = \begin{pmatrix} 0 & 1 \\ -V_+ & 0 \end{pmatrix}$ with $V_+ u(z) := \sum_{z' \in \mathbb{Z}^d_+} (V(z - z') - V(z - \tilde{z}')) u(z')$.

Let us assume that

$$
V(z) = V(\tilde{z}), \quad \text{where} \quad \tilde{z} = (-z_1, \tilde{z}), \quad \tilde{z} = (z_2, \ldots, z_d) \in \mathbb{Z}^{d-1}.
$$

(4.5)

Then the solution to problem (4.1) can be represented as the restriction of the solution to the Cauchy problem (2.1) with odd initial data on the half-space. More exactly, assume that the initial data $X_0(z)$ form an odd function with respect to $z_1 \in \mathbb{Z}^1$, i.e., let $X_0(z) = -X_0(\tilde{z})$. Then the solution $v(z, t)$ of (2.1) is also an odd function with respect to $z_1 \in \mathbb{Z}^1$. Restrict the solution $v(z, t)$ to the domain $\mathbb{Z}^d_+$ and set $u(z, t) = v(z, t)|_{z_1 \geq 0}$. Then $u(z, t)$ is the solution to problem (4.1) with the initial data $Y_0(z) = X_0(z)|_{z_1 \geq 0}$.

Assume that the initial data $Y_0$ for (4.1) belong to the phase space $\mathcal{H}_{\alpha,+}$, $\alpha \in \mathbb{R}$, defined below.

**Definition 4.1** $\mathcal{H}_{\alpha,+}$ is the Hilbert space of $\mathbb{R}^n \times \mathbb{R}^n$-valued functions of $z \in \mathbb{Z}^d_+$ endowed with the norm $\|Y\|^2_{\alpha,+} = \sum_{z \in \mathbb{Z}^d_+} |Y(z)|^2(1 + |z|^2)^\alpha < \infty$.

In addition, it is assumed that the initial data vanish ($Y_0 = 0$) at $z_1 = 0$.

**Lemma 4.2** (see [15, Corollary 2.4]) Let conditions E1 and E2 hold. Choose some $\alpha \in \mathbb{R}$. Then (i) for any $Y_0 \in \mathcal{H}_{\alpha,+}$, there exists a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{H}_{\alpha,+})$ to the mixed problem (4.1);

(ii) the operator $U_+(t) : Y_0 \mapsto Y(t)$ is continuous on $\mathcal{H}_{\alpha,+}$.

Lemma 4.2 follows from Lemma 2.4 since the solution $Y(t)$ of (4.1) admits the representation

$$
Y(t) = \sum_{z' \in \mathbb{Z}^d_+} G_{t,+}(z, z') Y_0(z'), \quad z \in \mathbb{Z}^d_+, \quad t \geq 0,
$$

(4.6)

where $G_{t,+}(z, z') := G_t(z - z') - G_t(z - \tilde{z}')$.

(4.7)
4.1 The family of the initial measures

Let us introduce the complex $2n \times 2n$ matrix-valued function $R(r, x, y) = (R^{ij}(r, x, y))_{i,j=0}^1$, $r \in \mathbb{R}^d$, $x, y \in \mathbb{Z}_+^d$, with the following properties (a)-(d).

(a) $R(r, x, y) = 0$ for $x_1 = 0$ or $y_1 = 0$. The $n \times n$ matrix-valued functions $R^{ij}(r, x, y)$ have the form

$$R^{ij}(r, x, y) = \mathbf{R}^{ij}(r, x_1, y_1, \bar{x} - \bar{y}), \quad \text{where } x = (x_1, \bar{x}), \quad y = (y_1, \bar{y}), \quad i, j = 0, 1.$$ 

Moreover, we assume that for every fixed $r \in \mathbb{R}^d$, $R^{ij}(r, x, y)$ satisfies conditions I1–I4 (see Section 2.2).

(b) For every fixed $r \in \mathbb{R}^d$ and $i, j = 0, 1$, the bound holds,

$$|R^{ij}(r, x, y)| \leq C(1 + |x - y|)^{-\gamma}, \quad x, y \in \mathbb{Z}_+^d,$$

with the same constants $C$ and $\gamma$ as in (2.10).

(c) For every fixed $r \in \mathbb{R}^d$, the matrix-valued function $R(r, x, y)$ satisfies

$$R^{ij}(r, x, y) = (R^{ij}(r, y, x))^T, \quad x, y \in \mathbb{Z}_+^d.$$ 

(d) For every $x, y \in \mathbb{Z}_+^d$, $R^{ij}(\cdot, x, y), i, j = 0, 1$, are $C^1$ functions.

To derive the equation of the Navier-Stokes type we need the additional condition (d’)

(d’) Let $d > 1$. For every fixed $x, y \in \mathbb{Z}_+^d$,

$$\sup_{\bar{r} = (r_2, \ldots, r_d) \in \mathbb{R}^{d-1}} |R(r_1, \bar{r}, x, y) - R(0, \bar{r}, x, y)| \to 0 \quad \text{as } r_1 \to 0.$$ 

Moreover, we assume that for every fixed $x, y \in \mathbb{Z}_+^d$,

$$|\tilde{R}(0, \bar{s}, x, y)| \leq C(1 + |\bar{s}|)^{-N}, \quad \bar{s} \in \mathbb{R}^{d-1},$$

with $N > d + 2$. Here by $\tilde{R}(0, \bar{s}, x, y)$ we denote the Fourier transform of $R(0, \bar{r}, x, y)$ w.r.t. $\bar{r} \in \mathbb{R}^{d-1}$,

$$\tilde{R}(0, \bar{s}, x, y) = \int_{\mathbb{R}^{d-1}} e^{i\bar{s}\bar{r}} R(0, \bar{r}, x, y) d\bar{r}, \quad \bar{s} \in \mathbb{R}^{d-1}.$$ 

Let $\{\mu_0^\varepsilon, \varepsilon > 0\}$ be a family of initial measures on $\mathcal{H}_{\alpha,+}$ and $E_0^\varepsilon$ stand for expectation with respect to the measure $\mu_0^\varepsilon$. Assume that $E_0^\varepsilon(Y(x)) = 0$ and define the covariance $Q^ij(x, x') = E_0^\varepsilon(Y^i(x) \otimes Y^j(x'))$, $x, x' \in \mathbb{Z}_+^d$, $i, j = 0, 1$.

The family of measures $\{\mu_0^\varepsilon, \varepsilon > 0\}$ satisfies the following conditions V1’ and V2’.

V1’. For any $\varepsilon > 0$, $x, x' \in \mathbb{Z}_+^d$,

$$|Q^ij_\varepsilon(x, x') - R^{ij}(\varepsilon x, x, x')| \leq C \min[(1 + |x - x'|)^{-\gamma}, \varepsilon |x - y|],$$

with the constants $C$ and $\gamma$ as in (4.9).

V2’. For any $\varepsilon > 0$ and all $x, x' \in \mathbb{Z}_+^d$, $i, j = 0, 1$, $|Q^ij_\varepsilon(x, x')| \leq C(1 + |x - x'|)^{-\gamma}$ with the constants $C$, $\gamma$ as in (4.9).
5 Main results in the half-space

Definition 5.1 (i) \( \mu^t_\varepsilon \) is a Borel probability measure on \( \mathcal{H}_{\alpha,+} \) which gives the distribution of \( Y(t) \), \( \mu^t_\varepsilon(B) = \mu^\varepsilon_0(U_\varepsilon(-t)B) \), where \( B \in \mathcal{B}(\mathcal{H}_{\alpha,+}) \) and \( t \in \mathbb{R} \).
(ii) The correlation functions of the measure \( \mu^t_\varepsilon \) are defined by

\[
Q^{ij}_{\varepsilon,t}(x,y) = \int (Y^i(x) \otimes Y^j(y)) \mu^t_\varepsilon(dY) = \mathbb{E}^\varepsilon_0(Y^i(x,t) \otimes Y^j(y,t)), \quad i,j = 0,1, \quad x,y \in \mathbb{Z}^d_+.
\]

Here \( Y^i(x,t) \) are the components of the random solution \( Y(t) = (Y^0(\cdot,t),Y^1(\cdot,t)) \) to the problem (4.3).

5.1 Euler limit

At first, we introduce the matrix \( g_{\tau,r}(z) \), \( z \in \mathbb{Z}^d \), \( r \in \mathbb{R}^d \), \( \tau \neq 0 \), by the Fourier transform,

\[
\hat{g}_{\tau,r}(\theta) = \frac{1}{4} \sum_{\sigma=1}^s \Pi_\sigma(\theta)[\sum_{\pm} (I \pm iC_\sigma(\theta))\hat{R}_0(r \pm \nabla \omega_\sigma(\theta)\tau,\theta)\chi^\pm_{\tau,r1}(\theta)(I \mp iC_\sigma^*(\theta))]\Pi_\sigma(\theta),
\]

(cf (3.1)), where \( \theta \in \mathbb{T}^d \setminus C_s \),

\[
\chi^\pm_{\tau,r1}(\theta) = (1 + \text{sign}(r_1 \pm \partial_1 \omega_\sigma(\theta)\tau))/2. \tag{5.1}
\]

Theorem 5.2 (see Theorem 2.10 in [17]) Let conditions (a) – (d), V1’, V2’ and E1–E6 hold. Then for any \( \tau \neq 0 \), \( r \in \mathbb{R}^d \) with \( r_1 \geq 0 \), the correlation functions of \( \mu_{\tau/\varepsilon,r/\varepsilon} \) converge to a limit,

\[
\lim_{\varepsilon \to 0} Q_{\varepsilon,\tau/\varepsilon}(r/\varepsilon + z_\varepsilon,r/\varepsilon + z'_\varepsilon) = Q_{\tau,r}(z,z'). \tag{5.2}
\]

Here \( z,z' \in \mathbb{Z}^d \) if \( r_1 > 0 \), \( z,z' \in \mathbb{Z}^d_+ \) if \( r_1 = 0 \),

\[
Q_{\tau,r}(z,z') = \begin{cases} q_{\tau,r}(z-z') + g_{\tau,r}(\bar{z}-\bar{z'}), & \text{if } r_1 > 0, \\ g_{\tau,r}(z-\bar{z'}) - g_{\tau,r}(\bar{z}-\bar{z'}) + g_{\tau,r}(\bar{z}-\bar{z'}), & \text{if } r_1 = 0, \end{cases} \tag{5.3}
\]

where \( \bar{r} := (-r_1,1), \bar{r} = (r_2,\ldots,r_d), \) and the matrix \( g_{\tau,r}(z) \) is defined above.

Corollary 5.3 Let \( r \in \mathbb{R}^d_+ \equiv \{ r \in \mathbb{R}^d : r_1 > 0 \} \) and \( \tau > 0 \). For each \( \sigma = 1,\ldots,s \), the \( \sigma \)-band of \( q_{\tau,r}(\theta) \), \( \theta \in \mathbb{T}^d \setminus C_s \), satisfies the following "hydrodynamic" (Euler type) equation:

\[
\partial_\tau f_\sigma(\tau,r;\theta) = iC_\sigma(\theta)\nabla \omega_\sigma(\theta) \cdot \nabla_r f_\sigma(\tau,r;\theta), \quad r \in \mathbb{R}^d_+, \quad \tau > 0,
\]

with the initial condition (3.7) (if \( \tau = 0 \)) and with the boundary condition (if \( r_1 = 0 \)) expressed by \( \hat{R}_0 \). In particular, if \( \hat{R}_0(r,\theta) = \hat{R}_0(\tau,\theta) \), the boundary condition has a form

\[
f_{\sigma}\big|_{r_1=0} = \Pi_\sigma(\theta)\frac{1}{2} \left( P^\sigma_+ + C_\sigma(\theta)P^\sigma_0C^*_\sigma(\theta) + iC_\sigma(\theta)P^- - iP^- C^*_\sigma(\theta) \right)\Pi_\sigma(\theta), \quad \bar{r} = \mathbb{R}^{d-1}, \quad \tau > 0,
\]

where \( P^\sigma_\pm \) stands for the \( 2n \times 2n \) matrix-valued function,

\[
P^\sigma_\pm = \frac{1}{2} \left( \hat{R}_0(\partial_1 \omega_\sigma(\theta)|\tau,\bar{r} + \nabla \omega_\sigma(\theta)\tau,\theta) \pm \hat{R}_0(\partial_1 \omega_\sigma(\theta)|\tau,\bar{r} - \nabla \omega_\sigma(\theta)\tau,\theta) \right).
\]
Introduce the complex-valued field (cf. (3.8))

\[ a_+(x) = \frac{1}{\sqrt{2}} \left( (\nabla_{\pm}^{1/4} u_0)(x) + i(\nabla_{\pm}^{-1/4} u_1)(x) \right) \in \mathbb{C}^n, \quad x \in \mathbb{Z}^d, \tag{5.4} \]

where

\[ (\nabla_{\pm}^k u)(x) := \sum_{z \in \mathbb{Z}^d_+} (V^k(x - z) - V^k(x - \bar{z}))u(z), \quad \text{with} \quad V^k(z) := F_{\theta \to z}^{-1} \left( \hat{V}^k(\theta) \right). \]

Let us introduce the scaled \( n \times n \) Wigner matrix (cf. (3.9))

\[ W^\varepsilon_+(\tau, r; \theta) = \sum_{\sigma \in \mathbb{Z}^d} e^{i\theta \cdot \varepsilon} \mathbb{E}_{\tau / \varepsilon} \left( a_+^*([r/\varepsilon + y/2]) \otimes a_+([r/\varepsilon - y/2]) \right), \quad r \in \mathbb{R}^d. \]

where \( a_+(x) \) is given in (5.4). By conditions \( V1' \) and \( V2' \), for fixed \( r \in \mathbb{R}^d_+ \),

\[ \lim_{\varepsilon \to 0} W^\varepsilon_+(0, r; \theta) = W(0, r; \theta), \]

uniformly on \( \theta \in \mathbb{T}^d \setminus \mathcal{C}_* \), where \( W(0, r; \theta) \) is defined in (3.10).

We also define the limit Wigner matrix as follows (cf. (3.11))

\[ W^P_+(\tau, r; \theta) = \left\{ \begin{array}{ll}
\sum_{\sigma = 1}^s \Pi_\sigma(\theta)W(0, r - \tau \nabla \omega_\sigma(\theta); \theta)\Pi_\sigma(\theta), & \text{if } r_1 > \tau \partial_1 \omega_\sigma(\theta), \\
\sum_{\sigma = 1}^s \Pi_\sigma(\theta)W(0, -r_1 + \tau \partial_1 \omega_\sigma(\theta), \bar{r} - \tau \nabla_\theta \omega_\sigma(\theta); \bar{\theta})\Pi_\sigma(\theta), & \text{if } r_1 < \tau \partial_1 \omega_\sigma(\theta),
\end{array} \right. \tag{5.5} \]

where \( \bar{\theta} = \theta_1, \theta_2, \ldots, \theta_d \), \( \nabla_\theta \omega_\sigma(\theta) = (\partial_2 \omega_\sigma(\theta), \ldots, \partial_d \omega_\sigma(\theta)) \), \( \partial_k = \partial / \partial \theta_k \).

**Theorem 5.4** (see Theorem 2.12 in [17]) Let conditions \( V1' \), \( V2' \) and \( E1 - E6 \) hold. Then for any \( r \in \mathbb{R}^d_+ \) and \( \tau > 0 \), the following limit exists in the sense of distributions,

\[ \lim_{\varepsilon \to 0} W^\varepsilon_+(\tau, r; \theta) = W^P_+(\tau, r; \theta). \tag{5.6} \]

**Corollary 5.5** (see Corollary 2.13 in [17]) Denote by \( W^P_\sigma(\tau, r; \theta) \), \( \sigma = 1, \ldots, s \), the \( \sigma \)-th band of the Wigner function \( W^P_+(\tau, r; \theta) \). Then, for any fixed \( \sigma = 1, \ldots, s \), \( W^P_\sigma \) is a solution of the "energy transport" equation

\[ \partial_\tau W^P_\sigma(\tau, r; \theta) + \nabla \omega_\sigma(\theta) \cdot \nabla W^P_\sigma(\tau, r; \theta) = 0, \quad \tau > 0, \quad r \in \mathbb{R}^d, \]

with the initial and boundary conditions

\[ W^P_\sigma(\tau, r; \theta)|_{\tau=0} = W_\sigma(0, r; \theta), \quad r \in \mathbb{R}^d_+, \]

\[ W^P_\sigma(\tau, r; \theta)|_{r_1=0} = b_\sigma(\tau, \bar{r}; \theta), \quad \bar{r} \in \mathbb{R}^{d-1}, \quad \tau > 0. \tag{5.7} \]

Here \( W_\sigma(0, r; \theta) \) is the \( \sigma \)-th band of the initial Wigner matrix \( W(0, r; \theta) \) (see (3.10)),

\[ b_\sigma(\tau, \bar{r}; \theta) := \left\{ \begin{array}{ll}
W_\sigma(0, -\tau \partial_1 \omega_\sigma(\theta), \bar{r} - \tau \nabla_\theta \omega_\sigma(\theta); \theta), & \text{if } \partial_1 \omega_\sigma(\theta) < 0, \\
W_\sigma(0, \tau \partial_1 \omega_\sigma(\theta), \bar{r} - \tau \nabla_\theta \omega_\sigma(\theta); \theta), & \text{if } \partial_1 \omega_\sigma(\theta) > 0.
\end{array} \right. \]

In particular, if we assume that \( \hat{R}_0(r; \theta) = R_0(r; \theta) \) for \( r \in \mathbb{R}^d, \theta \in \mathbb{T}^d \), then the boundary condition (5.7) can be rewritten in the form

\[ W^P_\sigma(\tau, r; \theta)|_{r_1=0} = W_\sigma(0, \tau |\partial_1 \omega_\sigma(\theta)|, \bar{r} - \tau \nabla_\theta \omega_\sigma(\theta); \theta), \quad \bar{r} \in \mathbb{R}^{d-1}, \quad \tau > 0. \]
Remark (see [17] Theorem 2.15)] Let the measures \( \mu_0^\varepsilon \) satisfy the mixing condition (of the Rosenblatt or Ibragimov type). Then for \( \tau \neq 0, \ r \in \mathbb{R}^d \) with \( r_1 \geq 0 \), in the sense of weak convergence on \( \mathcal{H}_{a+} \), \( \lim_{\varepsilon \to 0} \mu_{\tau/r; \varepsilon}^\varepsilon = \mu_{\tau/r}^G \). The measure \( \mu_{\tau/r}^G \) is a Gaussian measure on \( \mathcal{H}_{a+} \), which is invariant under the time translation \( U_\pm(t) \). \( \mu_{\tau/r}^G \) has mean zero and covariance \( Q_{\tau/r}(z, z') \) defined by (5.3).

### 5.2 Second approximation

In this subsection we treat the main result of this paper. Let us introduce the matrix \( g_{\tau/r}^\varepsilon(z) \), \( z \in \mathbb{Z}^d, \ r \in \mathbb{R}^d, \ \tau \neq 0, \ \varepsilon > 0 \), which has the following form in the Fourier transform (cf (3.12))

\[
\hat{g}_{\tau/r}^\varepsilon(\theta) = \frac{1}{4} \sum_{\sigma=1}^4 \Pi_\sigma(\theta) \left[ \sum_{\pm} (I \pm iC_\sigma(\theta)) A_{\varepsilon, \sigma}^\pm(\tau, r; \theta)(I \mp iC_\sigma^*(\theta)) \right] \Pi_\sigma(\theta), \quad \theta \in \mathbb{T}^d \setminus \mathcal{C}, \quad (5.8)
\]

where the matrices \( C_\sigma(\theta) \) are defined in (3.14),

\[
A_{\varepsilon, \sigma}^\pm(\tau, r; \theta) = A_{\varepsilon, \sigma}^\pm(\tau, r; \theta) \chi_{\tau, r}^\pm(\theta),
\]

with the matrix-valued functions \( A_{\varepsilon, \sigma}^\pm(\tau, r; \theta) \) from (3.13) and \( \chi_{\tau, r}^\pm \) from (5.1).

**Theorem 5.6** Let conditions (a)–(d'), V1', V2' and E1–E6 hold. Then for any \( \tau \neq 0 \) and \( r \in \mathbb{R}^d \) with \( r_1 \geq 0 \), the correlation functions of measures \( \mu_{\tau/r, \varepsilon}^\varepsilon \) have the following asymptotics

\[
\lim_{\varepsilon \to 0} \left( Q_{\tau/r, \varepsilon}^\varepsilon([r/\varepsilon] + z, [r/\varepsilon] + z') - Q_{\tau/r}^\varepsilon(z, z') \right) = 0,
\]

(5.9)

where \( z, z' \in \mathbb{Z}^d \) if \( r_1 > 0 \), \( z, z' \in \mathbb{Z}_+^d \) if \( r_1 = 0 \),

\[
Q_{\tau/r}^\varepsilon(z, z') = \begin{cases} q_{\tau/r}^\varepsilon(z - z') + g_{\tau/r}^\varepsilon(\tilde{z} - \tilde{z'}), & \text{if } r_1 > 0, \\ g_{\tau/r}^\varepsilon(z - z') - g_{\tau/r}^\varepsilon(\tilde{z} - \tilde{z'}) + g_{\tau/r}^\varepsilon(\tilde{z} - \tilde{z'}) & \text{if } r_1 = 0, \end{cases}
\]

\[
g_{\tau/r}^\varepsilon(\tilde{z}) = F_{\tau/r}^{-1} g_{\tau/r}^\varepsilon(\theta) \quad \text{and} \quad \hat{g}_{\tau/r}^\varepsilon(\theta) \text{ defined by (5.8)}.
\]

**Remark.** The matrices \( \hat{q}_{\tau/r}(\theta) \) and \( \hat{q}_{\tau/r}^\varepsilon(\theta) \) satisfy the equilibrium condition (see Remarks 3.4 (i)).

Set \( \tau = \varepsilon t \). In this case, \( A_{\varepsilon, \sigma}^\pm(\varepsilon t, r; \theta) \rvert_{\varepsilon=0} = \hat{R}_0(r \pm \nabla \omega_\sigma(\theta)t, \theta) \chi_{t, r_1}^\pm(\theta) \), and, then, \( \hat{g}_{\tau/r}^\varepsilon(\theta) \rvert_{\varepsilon=0} = \hat{q}_{\tau/r}(\theta) \). Therefore, for \( r_1 > 0 \), \( \hat{q}_{\tau/r}^\varepsilon(\theta) \rvert_{\varepsilon=0} = \hat{q}_{\tau/r}(\theta) \), where \( \hat{q}_{\tau/r}(\theta) = \hat{F}_{t \to \theta}[q_{\tau/r}(\varepsilon t)] \) and \( q_{\tau/r}(\varepsilon t) \) from (5.5).

**Corollary 5.7** Let \( r_1 > 0 \) and \( t > 0 \). Then for each \( \sigma = 1, \ldots, s \), the matrix-valued function \( F_\sigma \equiv F_\sigma^\varepsilon(t, r; \theta) = \Pi_\sigma(\theta) \hat{q}_{\tau/r}^\varepsilon(\theta) \Pi_\sigma(\theta), \ \theta \in \mathbb{T}^d \setminus \mathcal{C}, \) evolves according to the following mixing problem

\[
\partial_t F_\sigma = iC_\sigma(\theta) \left( \nabla \omega_\sigma(\theta) \cdot \nabla F_\sigma + \frac{i}{2} \nabla^2 \omega_\sigma(\theta) \cdot \nabla^2 F_\sigma \right), \quad r_1 > 0, \quad t > 0,
\]

\[
F_\sigma \rvert_{t=0} = \frac{1}{2} \Pi_\sigma(\theta) \left( \hat{R}_0(r, \theta) + C_\sigma(\theta) \hat{R}_0(r, \theta) C_\sigma^*(\theta) \right) \Pi_\sigma(\theta), \quad r_1 > 0,
\]

\[
F_\sigma \rvert_{t=0} = \frac{1}{4} \sum_{\pm} \Pi_\sigma(\theta) (I \pm iC_\sigma(\theta)) A_{\varepsilon, \sigma}^\pm(\varepsilon t, r; \theta) \rvert_{r_1=0} (I \mp iC_\sigma^*(\theta)) \Pi_\sigma(\theta), \quad t > 0.
\]
6 Convergence of correlation functions

6.1 Bounds for initial covariance

Definition 6.1 By \( \ell^p \equiv \ell^p(\mathbb{Z}^d) \otimes \mathbb{R}^n \) (by \( \ell^p_+ \equiv \ell^p((\mathbb{Z}_+^d) \otimes \mathbb{R}^n) \)), where \( p \geq 1 \) and \( n \geq 1 \), denote the space of sequences \( f(z) = (f_1(z), \ldots, f_n(z)) \) endowed with norm \( \|f\|_{\ell^p} = \left( \sum_{z \in \mathbb{Z}^d} |f(z)|^p \right)^{1/p} \), respectively, \( \|f\|_{\ell^p_+} = \left( \sum_{z \in \mathbb{Z}_+^d} |f(z)|^p \right)^{1/p} \).

The following lemma follows from condition V2.

Lemma 6.2 Let condition V2 hold. Then, for \( i, j = 0, 1 \), the following bounds hold:

\[
\sum_{z \in \mathbb{Z}^d} |Q^{ij}_\varepsilon(z, z')| \leq C < \infty \quad \text{for all} \quad z \in \mathbb{Z}_+^d,
\]

\[
\sum_{z \in \mathbb{Z}_+^d} |Q^{ij}_\varepsilon(z, z')| \leq C < \infty \quad \text{for all} \quad z' \in \mathbb{Z}_+^d.
\]

Here the constant \( C \) does not depend on \( z, z' \in \mathbb{Z}_+^d \) and \( \varepsilon > 0 \).

Corollary 6.3 By the Shur lemma, it follows from Lemma 6.2 that

\[
|\langle Q_\varepsilon(z, z'), \Phi(z) \otimes \Psi(z') \rangle| \leq C \|\Phi\|_{\ell^q_\varepsilon} \|\Psi\|_{\ell^p_\varepsilon}, \quad \text{for any} \quad \Phi, \Psi \in \ell^q_\varepsilon,
\]

where the constant \( C \) does not depend on \( \varepsilon > 0 \).

6.2 Stationary phase method

By (2.8) and (2.9) we see that \( \hat{G}_t(\theta) \) is of the form

\[
\hat{G}_t(\theta) = \begin{pmatrix} \cos \Omega t & \sin \Omega t & \Omega^{-1} \\ -\sin \Omega t & \cos \Omega t & 0 \end{pmatrix},
\]

where \( \Omega = \Omega(\theta) \) is the Hermitian matrix defined by (2.4). Hence, by Lemma 2.2 (iii), by formulas \( \cos \omega_\sigma(\theta)t = (e^{i\omega_\sigma t} + e^{-i\omega_\sigma t})/2 \), \( \sin \omega_\sigma(\theta)t = (e^{i\omega_\sigma t} - e^{-i\omega_\sigma t})/(2i) \) and by (3.4), the matrix \( \mathcal{G}_t(x) \) can be rewritten in the form

\[
\mathcal{G}_t(x) = \sum_{\pm, \sigma = 1}^{s} \int e^{-ix \cdot \theta} e^{\pm i\omega_\sigma(\theta)/2} e^{\mp i \sigma(\theta) t} c_\sigma(\theta) d\theta,
\]

where \( c_\sigma(\theta) = \Pi_\sigma(\theta)(I \mp iC_\sigma(\theta))/2 \). We are going to apply the stationary phase arguments to the integral (6.2) which require a smoothness in \( \theta \). Then we have to choose certain smooth branches of the functions \( c_\sigma(\theta) \) and \( \omega_\sigma(\theta) \) and cut off all singularities. First, introduce the critical set as

\[
\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_s \cup \Bigcup_{\sigma = 1}^{s} \bigcup_{i = 1}^{d} \left\{ \theta \in \mathbb{T}^d \setminus \mathcal{C}_s : \frac{\partial^2 \omega_\sigma(\theta)}{\partial \theta_i^2} = 0 \right\} \cup \left\{ \theta \in \mathbb{T}^d \setminus \mathcal{C}_s : \frac{\partial \omega_\sigma(\theta)}{\partial \theta_i} = 0 \right\},
\]

(6.3)
with $C_\sigma$ as in Lemma 2.2 and sets $C_0$ and $C_\sigma$ defined by (2.5). Obviously, $\text{mes} C = 0$ (see Lemma 2.2, 2.3). Secondly, fix an $\delta > 0$ and choose a finite partition of unity,

$$f(\theta) + g(\theta) = 1, \quad g(\theta) = \sum_{k=1}^{K} g_k(\theta), \quad \theta \in \mathbb{T}^d,$$

(6.4)

where $f, g_k$ are non-negative functions in $C_0^\infty(\mathbb{T}^d)$, and

$$\text{supp } f \subset \{ \theta \in \mathbb{T}^d : \text{dist}(\theta, \mathcal{C}) < \delta \}, \quad \text{supp } g_k \subset \{ \theta \in \mathbb{T}^d : \text{dist}(\theta, \mathcal{C}) \geq \delta/2 \}. \quad (6.5)$$

Then we represent $G_t(x)$ in the form $G_t(x) = G^1_t(x) + G^2_t(x)$, where

$$G^1_t(x) = F_{\theta \to x}^{-1}[f(\theta) \hat{G}_t(\theta)], \quad G^2_t(x) = F_{\theta \to x}^{-1}[g(\theta) \hat{G}_t(\theta)]. \quad (6.6)$$

By Lemma 2.2 and the compactness arguments, we can choose the supports of $g_k$ so small that the eigenvalues $\omega(\theta)$ and the amplitudes $c(\theta)$ are real-analytic functions inside the supports $g_k$ for every $k$. (We do not label the functions by the index $k$ to simplify the notation.) The Parseval identity, (6.1), and condition $E6$ imply

$$\|G^1_t(\cdot)\|_{L^2}^2 = C \int_{\mathbb{T}^d} |\hat{G}_t(\theta)|^2 |f(\theta)|^2 d\theta \leq C \int_{\text{dist}(\theta, \mathcal{C}) < \delta} |\hat{G}_t(\theta)|^2 d\theta \to 0 \quad \text{as } \delta \to 0,$$

$$\|G^2_t(\cdot)\|_{L^2}^2 \leq C \int_{\mathbb{T}^d} |\hat{G}_t(\theta)|^2 d\theta \leq C_1 < \infty,$$

(6.7)

uniformly in $t \in \mathbb{R}$. For the function $G^2_t(x)$, the following lemma holds.

**Lemma 6.4** (see [14, Lemma 4.2]) Let conditions $E1$-E4 and E6 hold. Then the following bounds hold.

(i) $\sup_{x \in \mathbb{Z}^d} |G^2_t(x)| \leq C t^{-d/2}$.

(ii) For any $p > 0$, there exist $C_p, \gamma_g > 0$ such that $|G^2_t(x)| \leq C_p(|t| + |x| + 1)^{-p}$ for $|x| \geq \gamma_g t$.

### 6.3 Proof of Theorem 5.6

The representation (6.6) yields

$$Q_{\varepsilon, t}(z, z') = \mathbb{E}_0^d(Y(z, t) \otimes Y(z', t)) = \sum_{x, y \in \mathbb{Z}^d} G_{t, \varepsilon}^2(x, y) G^{2, t}(z, x) Q_{\varepsilon, t}(z, y) G^{2, t}(y, z'), \quad z, z' \in \mathbb{Z}_+^d,$$

for any $t \in \mathbb{R}$. It follows from condition (4.5) and from formulas (2.8) and (2.9) that $G_t(z) = G_t(\bar{z})$ with $\bar{z} = (-z_1, z_2, \ldots, z_d)$. In this case, by (1.7), the covariance $Q_{\varepsilon, t}(z, z')$ can be decomposed into the sum of fourth terms,

$$Q_{\varepsilon, t}(z, z') = S_{\varepsilon, t}(z, z') - S_{\varepsilon, t}(\bar{z}, z') - S_{\varepsilon, t}(z, \bar{z}') + S_{\varepsilon, t}(\bar{z}, \bar{z}'), \quad z, z' \in \mathbb{Z}_+^d,$$

where

$$S_{\varepsilon, t}(z, z') := \sum_{x, y \in \mathbb{Z}_+^d} G_t(z - x) Q_{\varepsilon, t}(x, y) G_{t}(z' - y)^T.$$
Proposition 6.5 Let \( r \in \mathbb{R}^d \) and \( z, z' \in \mathbb{Z}^d \). Then
\[
\lim_{\varepsilon \to +0} (S_{\varepsilon, r/\varepsilon}([r/\varepsilon] + z, [r/\varepsilon] + z') - g_{\varepsilon, r}(z - z')) = 0, \tag{6.8}
\]
where \( g_{\varepsilon, r}(z) \) is defined by (5.8).

This proposition implies Theorem 5.6. Indeed, let \( r_1 = 0 \). Then \( \bar{r}/\varepsilon = r/\varepsilon \), and for \( z, z' \in \mathbb{Z}^d_+ \),
\[
Q_{\varepsilon, \bar{r}}([r/\varepsilon] + z, [r/\varepsilon] + z') = S_{\varepsilon, \bar{r}}([r/\varepsilon] + z, [r/\varepsilon] + z') - S_{\varepsilon, \bar{r}}([r/\varepsilon] + \bar{z}, [r/\varepsilon] + z')
\]
Therefore, convergence (6.8) implies (5.9).

It can be proved similarly to Proposition 6.5.

Proof of Proposition 6.5 Let us denote
\[
Q_{\varepsilon}(x, y) = \begin{cases} Q_{\varepsilon}(x, y) & \text{for } x, y \in \mathbb{Z}^d_+, \\ 0 & \text{otherwise} \end{cases}
\]
The partition (6.4), Corollary 6.3 and the bound (6.7) yield
\[
S_{\varepsilon, \bar{r}}(z, z') = \sum_{x, y \in \mathbb{Z}^d} G_{\varepsilon}^2(z - x)Q_{\varepsilon}(x, y)G_{\varepsilon}^2(z' - y)^T + o(1),
\]
where \( G_{\varepsilon}^2 \) is defined in (6.6), \( o(1) \to 0 \) as \( \delta \to 0 \) uniformly in \( t \in \mathbb{R} \) and \( z, z' \in \mathbb{Z}^d \). In particular, setting \( t = \tau/\varepsilon^2 \), \( z = [r/\varepsilon] + l \) and \( z' = [r/\varepsilon] + p \) we obtain
\[
S_{\varepsilon, \tau/\varepsilon^2}([r/\varepsilon] + l, [r/\varepsilon] + p) = \sum_{x, y \in \mathbb{Z}^d} G_{\tau/\varepsilon^2}^2([r/\varepsilon] + l - x)Q_{\varepsilon}(x, y)G_{\tau/\varepsilon^2}^2([r/\varepsilon] + p - y)^T + o(1)
\]
\[
= \sum_{x, y \in \mathbb{Z}^d} G_{\tau/\varepsilon^2}^2(l + x)Q_{\varepsilon}([r/\varepsilon] - x, [r/\varepsilon] - y)G_{\tau/\varepsilon^2}^2(p + y)^T + o(1). \tag{6.9}
\]
Let \( c = \gamma_g + \max(|l|, |p|) \) with \( \gamma_g \) from Lemma 6.4. Then Lemma 6.4(ii) and condition \textbf{V2} imply that the series in (6.9) can be taken over \( x, y \in [-c\tau/\varepsilon^2, c\tau/\varepsilon^2]^d \).

By definition, the function \( \bar{R} \) is equal to
\[
\bar{R}(r, x, y) = \begin{cases} R(r, x, y) & \text{if } x, y \in \mathbb{Z}^d_+, \\ 0 & \text{otherwise} \end{cases}
\]
The asymptotics of \( S_{\varepsilon, \tau/\varepsilon^2} \) is not changed when we replace \( Q_{\varepsilon}([r/\varepsilon] - x, [r/\varepsilon] - y) \) in the r.h.s. of (6.9) by \( \bar{R}(\varepsilon[r/\varepsilon] - \varepsilon x, [r/\varepsilon] - x, [r/\varepsilon] - y) \), i.e.,
\[
S_{\varepsilon, \tau/\varepsilon^2}([r/\varepsilon] + l, [r/\varepsilon] + p) = \sum_{x, y \in [-c\tau/\varepsilon^2, c\tau/\varepsilon^2]^d} G_{\tau/\varepsilon^2}^2(l + x)\bar{R}(\varepsilon[r/\varepsilon] - \varepsilon x, [r/\varepsilon] - x, [r/\varepsilon] - y)G_{\tau/\varepsilon^2}^2(p + y)^T + o(1). \tag{6.10}
\]
Indeed, by Lemma 6.4 (i) and condition (4.12),
\[
\left| \sum_{x, y \in [-r, r]^{d}} G^g_{r/\varepsilon^2}(l + x) \left( \bar{Q}_\varepsilon ([r/\varepsilon] - x, [r/\varepsilon] - y) - \bar{R}(\ldots) \right) G^g_{r/\varepsilon^2}(p + y)^T \right| \\
\leq C \sum_{y \in \mathbb{Z}^d} \min([1 + |x - y|]^{-\gamma}, \varepsilon b|x - y|),
\]
where the last series is order of $\varepsilon^{(\gamma-d)/(\gamma+1)}$. This goes to zero as $\varepsilon \to 0$, since $\gamma > d$. Using Lemma 6.4 and properties of $\bar{R}$, we can take the series in (6.10) over $x, y \in \mathbb{Z}^d$. Hence,
\[
S_{\varepsilon, \tau/\varepsilon^2}([r/\varepsilon] + l, [r/\varepsilon] + p) = \sum_{x, y \in \mathbb{Z}^d} G^g_{r/\varepsilon^2}(l + x) \bar{R}(\ldots) G^g_{r/\varepsilon^2}(p + y)^T + o(1), \varepsilon \to 0. \quad (6.11)
\]
Let us split the function $\bar{R}$ into the following three matrix functions:
\[
R^+(r, x, y) := \frac{1}{2} R_0(r, x - y), \quad (6.12) \\
R^-(r, x, y) := \frac{1}{2} R_0(r, x - y) \text{sign}(y_1), \quad (6.13) \\
R^0(r, x, y) := \bar{R}(r, x, y) - R^+(r, x, y) - R^-(r, x, y). \quad (6.14)
\]
Next, introduce the matrices
\[
S^a_{\varepsilon, \tau/\varepsilon^2} \equiv S^a_{\varepsilon, \tau/\varepsilon^2}([r/\varepsilon] + l, [r/\varepsilon] + p) = \sum_{x, y \in \mathbb{Z}^d} G^g_{r/\varepsilon^2}(l + x) \bar{R}^a(\varepsilon [r/\varepsilon] - \varepsilon x, [r/\varepsilon] - x, [r/\varepsilon] - y) G^g_{r/\varepsilon^2}(p + y)^T, \quad (6.15)
\]
for each $a = \{+,-,0\}$ and split $S_{\varepsilon, \tau/\varepsilon^2}$ into three terms, $S_{\varepsilon, \tau/\varepsilon^2} = S^+_{\varepsilon, \tau/\varepsilon^2} + S^-_{\varepsilon, \tau/\varepsilon^2} + S^0_{\varepsilon, \tau/\varepsilon^2}$. The convergence (6.8) results now from the following three Lemmas 6.6–6.8 since (see (5.8)) $g^\varepsilon_{r, r}(z) = (1/2)q^\varepsilon_{r, r}(z) + (1/2)f^\varepsilon_{r, r}(z)$, where $q^\varepsilon_{r, r}(z)$ is defined by (3.12) and $f^\varepsilon_{r, r}(z)$ is defined in Lemma 6.7.

**Lemma 6.6** $\lim_{\varepsilon \to 0}(S^+_{\varepsilon, \tau/\varepsilon^2}([r/\varepsilon] + l, [r/\varepsilon] + p) - (1/2)q^\varepsilon_{r, r}(l - p)) = 0$, $l, p \in \mathbb{Z}^d$, where $q^\varepsilon_{r, r}(l)$ is defined by (3.12).

**Lemma 6.7** $\lim_{\varepsilon \to 0}(S^-_{\varepsilon, \tau/\varepsilon^2}([r/\varepsilon] + l, [r/\varepsilon] + p) - (1/2)f^\varepsilon_{r, r}(l - p)) = 0$, $l, p \in \mathbb{Z}^d$.

Here $f^\varepsilon_{r, r}(l)$ stands for the matrix which is defined similarly to $q^\varepsilon_{r, r}(l)$ but with the matrix $A^{\pm}_{r, \theta}(\tau, r; \theta) \text{sign}(r_1 \pm \partial_1 \omega_\varepsilon(\tau/\varepsilon))$ instead of $A^{\pm}_{r, \theta}(\tau, r; \theta)$ in the r.h.s. of (3.12).

**Lemma 6.8** $\lim_{\varepsilon \to 0} S^0_{\varepsilon, \tau/\varepsilon^2}([r/\varepsilon] + l, [r/\varepsilon] + p) = 0$, $l, p \in \mathbb{Z}^d$.

The proofs of Lemmas 6.7 and 6.8 see in Appendices A and B, resp.

**Proof of Lemma 6.6** Step (i): By (6.12) and (6.15), the function $S^+_{\varepsilon, \tau/\varepsilon^2}$ can be represented as
\[
S^+_{\varepsilon, \tau/\varepsilon^2} = \frac{1}{2} \sum_{x \in \mathbb{Z}^d} G^g_{r/\varepsilon^2}(l + x) \sum_{y \in \mathbb{Z}^d} R_0(\varepsilon [r/\varepsilon] - \varepsilon x, y - x) G^g_{r/\varepsilon^2}(p + y)^T.
\]
Using the Fourier transform and the Parseval identity we rewrite \( S^+_{\varepsilon, \tau/\varepsilon^2} \) as

\[
S^+_{\varepsilon, \tau/\varepsilon^2} = (2\pi)^{-2d} \frac{1}{2} \sum_{x \in \mathbb{Z}^d} e^{ix \cdot \theta} \int_{\mathbb{T}^d} e^{-i x \cdot \theta + i x \cdot \theta} \hat{\mathcal{G}}_{\varepsilon, \tau/\varepsilon^2}^g(\theta) \tilde{\mathcal{R}}_0(\varepsilon [r/\varepsilon] - \varepsilon x, \theta) \tilde{\mathcal{G}}_{\varepsilon, \tau/\varepsilon^2}^g(\theta) d\theta d\theta'.
\] (6.16)

Change variables in (6.16): \((\theta, \theta') \rightarrow (\theta, \varphi)\), \(\varphi = \theta - \theta'\) and rewrite \( S^+_{\varepsilon, \tau/\varepsilon^2} \) in the form

\[
S^+_{\varepsilon, \tau/\varepsilon^2} = (2\pi)^{-d} \frac{1}{2} \int_{\mathbb{T}^d} e^{-i(l-\varphi) \theta} I_{\varepsilon}(\theta) \tilde{\mathcal{G}}_{\varepsilon, \tau/\varepsilon^2}^g(\theta) d\theta,
\] (6.17)

where \( I_{\varepsilon}(\theta) \) stands for the matrix-valued function,

\[
I_{\varepsilon}(\theta) = \sum_{x \in \mathbb{Z}^d} (2\pi)^{-d} \int_{\mathbb{T}^d} e^{i(l+x) \cdot \varphi} \tilde{\mathcal{G}}_{\varepsilon, \tau/\varepsilon^2}^g(\theta - \varphi) d\varphi \tilde{\mathcal{R}}_0(\varepsilon [r/\varepsilon] - \varepsilon x, \theta).
\] (6.18)

**Step (ii):** For simplicity of the proof, let us assume that \( \tilde{\mathcal{R}}_0(r, \theta) \) satisfies condition (2.11). Under the more weakened condition (2.14) the proof is given in Appendix C.

We apply the Poisson summation formula (see, for example, [24]) and obtain

\[
\sum_{x \in \mathbb{Z}^d} e^{ix \cdot \varphi} \tilde{\mathcal{R}}_0(\varepsilon [r/\varepsilon] - \varepsilon x, \theta) = \varepsilon^{-d} \sum_{n \in \mathbb{Z}^d} \tilde{\mathcal{R}}_0(-\varphi/\varepsilon - 2\pi n/\varepsilon, \theta) e^{i[r/\varepsilon] \varphi},
\] (6.19)

where \( \tilde{\mathcal{R}}_0(\cdot, \theta) \) stands for the Fourier transform of \( \tilde{\mathcal{R}}_0(\cdot, \theta) \) (see (2.12)). We substitute the last expression to (6.18) and obtain

\[
I_{\varepsilon}(\theta) = (2\pi\varepsilon)^{-d} \int_{\mathbb{T}^d} e^{i(l+[r/\varepsilon]) \cdot \varphi} \tilde{\mathcal{G}}_{\varepsilon, \tau/\varepsilon^2}^g(\theta - \varphi) \left( \sum_{n \in \mathbb{Z}^d} \tilde{\mathcal{R}}_0(-\varphi/\varepsilon - 2\pi n/\varepsilon, \theta) \right) d\varphi + o(1).
\] (6.20)

Condition I4’ implies that

\[
\varepsilon^{-d} \sum_{n \neq 0} \tilde{\mathcal{R}}_0(-\varphi/\varepsilon - 2\pi n/\varepsilon, \theta) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0
\] (6.21)

uniformly in \( \theta \in \mathbb{T}^d, \varphi \in [-\pi, \pi]^d \). Hence (6.20) and (6.21) yield

\[
I_{\varepsilon}(\theta) = (2\pi\varepsilon)^{-d} \int_{[-\pi, \pi]^d} e^{i[l+[r/\varepsilon]] \cdot \varphi} \tilde{\mathcal{G}}_{\varepsilon, \tau/\varepsilon^2}^g(\theta - \varphi) \tilde{\mathcal{R}}_0(-\varphi/\varepsilon, \theta) d\varphi + o(1).
\]

**Step (iii):** Change variables \( \varphi \rightarrow -\varepsilon \varphi \) in the last integral and obtain

\[
I_{\varepsilon}(\theta) = (2\pi)^{-d} \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} e^{-i\varepsilon \varphi(l+[r/\varepsilon])} \tilde{\mathcal{G}}_{\varepsilon, \tau/\varepsilon^2}^g(\theta + \varepsilon \varphi) \tilde{\mathcal{R}}_0(\varphi, \theta) d\varphi + o(1).
\]

Now we use the following representation for \( \tilde{\mathcal{G}}_{\varepsilon}^g(\theta) \) (see (6.22)):

\[
\tilde{\mathcal{G}}_{\varepsilon}^g(\theta) = \sum_{\sigma=1}^s \sum_{\pm} e^{\pm i\omega_{\sigma}(\theta) t} h^\pm_{\sigma}(\theta), \quad \text{where} \quad h^\pm_{\sigma}(\theta) = g(\theta) c^\pm_{\sigma}(\theta) = \frac{g(\theta) I \pm i C_{\sigma}(\theta)}{2} \Pi_{\sigma}(\theta),
\] (6.22)
and rewrite $I_\varepsilon(\theta)$ in the form

$$
I_\varepsilon(\theta) = (2\pi)^{-d} \int_{\sigma,\pm[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-i\varepsilon\varphi \cdot (l+|r|/\varepsilon)} e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2} h_\sigma^\pm(\theta+\varepsilon\varphi) \tilde{R}_0(\varphi, \theta) \, d\varphi + o(1)
$$

$$
= (2\pi)^{-d} \int_{\sigma,\pm[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-i\varphi \cdot r} e^{\pm i\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2} \tilde{R}_0(\varphi, \theta) \, d\varphi + o(1),
$$

since $e^{-i\varepsilon\varphi \cdot t} - 1 = O(\varepsilon)$ and $e^{-i\varphi \cdot |r|/\varepsilon} - e^{-i\varphi \cdot r} = O(\varepsilon)$ by (2.11).

**Step (iv):** Let us take the Taylor sum representation for $\omega_\sigma(\theta+\varepsilon\varphi)$:

$$
\omega_\sigma(\theta+\varepsilon\varphi)\tau/\varepsilon^2 = \omega_\sigma(\theta)\tau/\varepsilon^2 + \varphi \cdot \nabla \omega_\sigma(\theta)\tau/\varepsilon + (\tau/2)\varphi \cdot H_\sigma(\theta)\varphi + O(\varepsilon|\varphi|^3),
$$

where $H_\sigma(\theta)$ denotes the matrix $\nabla^2 \omega_\sigma(\theta)$. Hence, by (2.11),

$$
I_\varepsilon(\theta) = \sum_{\sigma,\pm} h_\sigma^\pm(\theta) e^{\pm i\omega_\sigma(\theta)\tau/\varepsilon^2} A_{\varepsilon,\sigma}^\pm(\tau, r; \theta) + o(1), \quad \varepsilon \to 0. \tag{6.24}
$$

Here (see formulas (3.13) and (3.14))

$$
A_{\varepsilon,\sigma}^\pm(\tau, r; \theta) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\varphi \cdot (r+\nabla \omega_\sigma(\theta)\tau/\varepsilon)} e^{\pm i(\tau/2)\varphi \cdot H_\sigma(\theta)\varphi} \tilde{R}_0(\varphi, \theta) \, d\varphi
$$

$$
= \int_{\mathbb{R}^d} \tilde{R}_0(\varphi, \theta) \, d\varphi = F_{\varphi \to x} \left[ e^{\pm i(\tau/2)\varphi \cdot H_\sigma(\theta)\varphi} \tilde{R}_0(\varphi, \theta) \right]
$$

and $K_\sigma^\pm(\tau, x, \theta)$ ($\tau > 0$, $x \in \mathbb{R}^d$, $\theta \in \mathbb{T}^d \setminus C$) stands for the following matrix-valued function

$$
K_\sigma^\pm(\tau, x, \theta) = F_{\varphi \to x} \left[ e^{\pm i(\tau/2)\varphi \cdot H_\sigma(\theta)\varphi} \right] = e^{\pm i\pi s/4} e^{\mp i\pi/2 \sqrt{\det H_\sigma(\theta)}} \left( \frac{2\pi \tau)^{d/2}}{\sqrt{\det H_\sigma(\theta)}} \right), \tag{6.25}
$$

where $s$ denotes the signature of the matrix $H_\sigma(\theta)$. By definition, the signature of the non-degenerate symmetric matrix $A$ means the signature of the quadratic form with this matrix (or the difference between the sums of positive and negative eigenvalues of $A$). Formula (6.25) follows from the following equality (see, for example, Ramanujan’s integrals in [24, section 7.5] or [25, §3]): for any $a > 0$,

$$
\int_{-\infty}^{+\infty} e^{-ix\varphi \pm ia\varphi^2} \, d\varphi = \sqrt{\frac{\pi}{a}} e^{\pm i\pi/4} e^{\mp i\pi x^2/(4a)}, \quad x \in \mathbb{R}^1.
$$

**Step (v):** We substitute (6.24) in (6.17), apply the decomposition (6.22) to $\tilde{G}_{\tau/\varepsilon^2}(\theta)^*$ and obtain

$$
S_{\varepsilon,\tau/\varepsilon^2}^+ = (2\pi)^{-d/2} \int_{\mathbb{T}^d} e^{-i(l-p)\theta} \left[ \sum_{\sigma,\pm} h_\sigma^\pm(\theta) e^{\pm i\omega_\sigma(\theta)\tau/\varepsilon^2} A_{\varepsilon,\sigma}^\pm(\tau, r; \theta) \right]
$$

$$
\times \left( \sum_{\sigma',\pm} e^{\pm i\omega_{\sigma'}(\theta)\tau/\varepsilon^2} h_{\sigma'}^\pm(\theta)^* \right) \, d\theta + o(1), \quad \varepsilon \to 0. \tag{6.26}
$$
Therefore, to find the asymptotics of $S^\pm_{\varepsilon, \tau/\varepsilon^2}$ it suffices to study the behaviour as $\varepsilon \to 0$ of the following integrals

$$I_{\sigma\sigma'}^\pm(\varepsilon) \equiv (2\pi)^{-d-1} \int_{\mathbb{T}^d} e^{-i(l-p)\theta} e^{i(\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta))\tau/\varepsilon^2} h^-_\sigma(\theta) A^-_{\varepsilon, \sigma}(\tau, r; \theta) h^+_\sigma'(\theta)^* d\theta, \quad (6.27)$$

$\sigma, \sigma' = 1, \ldots, s$. Note that $\sup_{\theta \in \mathbb{T}^d} |A^-_{\varepsilon, \sigma}(\tau, r; \theta)| \leq C < \infty$ by condition $I4'$. Hence, the function $h^-_\sigma(\theta) A^-_{\varepsilon, \sigma}(\tau, r; \theta) h^+_\sigma'(\theta)^* \in L^1(\mathbb{T}^d)$ by condition $E6$. Therefore, the oscillatory integrals with $\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \not\equiv \text{const}_\pm$ vanish as $\varepsilon \to 0$ by the Lebesgue–Riemann theorem. Furthermore, the identities $\omega_\sigma(\theta) \pm \omega_{\sigma'}(\theta) \equiv \text{const}_\pm$ in the exponent of $(6.27)$ with $\text{const}_\pm \neq 0$ are impossible by condition $E5$. Hence, only the integrals with $\omega_\sigma(\theta) - \omega_{\sigma'}(\theta) \equiv 0$ contribute to the integral $(6.27)$ since $\omega_\sigma(\theta) + \omega_{\sigma'}(\theta) \equiv 0$ would imply $\omega_\sigma(\theta) \equiv \omega_{\sigma'}(\theta) \equiv 0$ which is impossible by $E4$. Therefore, if $\sigma \neq \sigma'$, $I_{\sigma\sigma'}^\pm(\varepsilon) = o(1)$ as $\varepsilon \to 0$. If $\sigma = \sigma'$, $I_{\sigma\sigma}^\pm(\varepsilon) = o(1)$,

$$I^-_{\sigma\sigma}(\varepsilon) = (2\pi)^{-d-1} \int_{\mathbb{T}^d} e^{-i(l-p)\theta} h^-_\sigma(\theta) A^-_{\varepsilon, \sigma}(\tau, r; \theta) h^+_\sigma(\theta)^* d\theta + o(1), \quad \varepsilon \to 0. \quad (6.28)$$

Finally, using $(6.26)$ and the equalities $h^+_\sigma(\theta) = g(\theta) \Pi_{\sigma}(\theta)(I - iC_{\sigma}(\theta))/2$ and $(3.12)$, we obtain

$$S^+_{\varepsilon, \tau/\varepsilon^2} = (2\pi)^{-d-1} \int_{\mathbb{T}^d} e^{-i(l-p)\theta} \sum_{\sigma = 1}^s \sum_{\pm} h^\pm_\sigma(\theta) A^\pm_{\varepsilon, \sigma}(\tau, r; \theta) h^\mp_\sigma(\theta)^* d\theta + o(1) \quad (6.28)$$

$$= \frac{1}{2} q^\varepsilon_{\tau, r}(l - p) + o(1), \quad \varepsilon \to 0. \quad \blacksquare$$

### 7 Convergence of Wigner matrices

Here we prove Theorem $5.4$. Theorem $5.6$ implies that for any fixed $r \in \mathbb{R}^d$, $\tau \neq 0$, and $y \in (2\mathbb{Z})^d$, the following convergence holds,

$$\lim_{\varepsilon \to 0} \mathbb{E}^\varepsilon_{\tau/\varepsilon^2}(a_+^*(\lfloor r/\varepsilon \rfloor + y/2) \otimes a_+^*(\lfloor r/\varepsilon \rfloor - y/2)) = W^p_+(\tau, r; y), \quad (7.1)$$

where in the Fourier space one has

$$\hat{W}^p_+(\tau, r; \theta) = \frac{1}{2} \left( \Omega^{1/2} q^0_{r, r}(\theta) \Omega^{1/2} + \Omega^{-1/2} q^{11}_{r, r}(\theta) \Omega^{-1/2} + i \Omega^{1/2} q^{01}_{r, r}(\theta) \Omega^{-1/2} - i \Omega^{-1/2} q^{10}_{r, r}(\theta) \Omega^{1/2} \right) = W^p_+(\tau, r; \theta), \quad r \in \mathbb{R}^d, \quad (7.2)$$

by formulas $(5.5)$ and $(5.3)$. Therefore, convergence $(5.6)$ follows from $(7.1)$, $(7.2)$ and the following bound:

$$\sup_{\varepsilon > 0} \sup_{t \in \mathbb{R}} \sup_{z, z' \in \mathbb{Z}^d} \left| \mathbb{E}^\varepsilon_{\tau/\varepsilon^2}(a_+^*(z) \otimes a_+^*(z')) \right| \leq C < \infty.$$

The proof of this bound follows from Lemma $7.1$. 

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Lemma 7.1 Let conditions V2' and E1–E3, E6 hold and let $\alpha < -d/2$. Then the following bound holds: $\sup_{\varepsilon > 0} \sup_{t \in \mathbb{R}} \sup_{z, z' \in \mathbb{Z}_+^d} \|Q_{\varepsilon,t}(z,z')\| \leq C < \infty$.

Proof. The representation (4.6) gives
\[
Q_{\varepsilon,t}^{ij}(z, z') = E_0^i \left( Y_i(z,t) \otimes Y_j(z',t) \right) = \sum_{y \in \mathbb{Z}_+^d} \sum_{k,l=0,1} G_{t,+}^{ik}(z,y) Q_{t,+}^{kl}(y,y') G_{t,0}^{ji}(z',y')
\]
where $\Phi_i(y,t)$ is given by
\[
\Phi_i(y,t) = (\hat{G}_{t,+}^{0i}(z,y), \hat{G}_{t,+}^{1i}(z,y))
= (G_t^{0}(z-y) - G_t^{0}(z-\bar{y}), G_t^{1}(z-y) - G_t^{1}(z-\bar{y})), \quad i = 0, 1.
\]
The Parseval identity, formula (6.1), and condition E6 imply that
\[
\|\Phi_i(\cdot,t)\|_l^2 = (2\pi)^{-d} \int_{\mathbb{T}^d} |\hat{\Phi}_i(\theta,t)|^2 d\theta \leq C \int_{\mathbb{T}^d} \left( |\hat{G}_t^{0i}(\theta)|^2 + |\hat{G}_t^{1i}(\theta)|^2 \right) d\theta \leq C_0 < \infty,
\]
where the constant $C_0$ does not depend on $z \in \mathbb{Z}_+^d$ and $t \in \mathbb{R}$. Corollary 6.3 gives now
\[
|Q_{\varepsilon,t}^{ij}(z, z')| = |\langle Q_{\varepsilon}(y,y'), \Phi_i(y,t) \otimes \Phi_j(y',t) \rangle| \leq C \|\Phi_i(\cdot,t)\|_l^2 \|\Phi_j(\cdot,t)\|_l^2 \leq C_1 < \infty,
\]
where the constant $C_1$ does not depend on $z, z' \in \mathbb{Z}_+^d$, $t \in \mathbb{R}$, and $\varepsilon > 0$.

Appendix A: Proof of Lemma 6.7

By (6.13) and (6.15), the function $S_{-\varepsilon,\tau/\varepsilon^2}$ can be represented as
\[
S_{-\varepsilon,\tau/\varepsilon^2} = \frac{1}{2} \sum_{x \in \mathbb{Z}_+^d} G_{\tau/\varepsilon^2}^g(l+x) \sum_{y \in \mathbb{Z}_+^d} R_0(\kappa_{\varepsilon,x}, y-x) \text{sign}(r_1/\varepsilon - y_1) G_{\tau/\varepsilon^2}^g(p+y)^T,
\]
where, by definition, $\kappa_{\varepsilon,x} = \varepsilon[r/\varepsilon] - \varepsilon x \in \mathbb{R}^d$. The Parseval equality yields
\[
\sum_{y \in \mathbb{Z}_+^d} \text{sign}(r_1/\varepsilon - y_1) R_0(\kappa_{\varepsilon,x}, y-x) G_{\tau/\varepsilon^2}^g(p+y)^T = (2\pi)^{-d} \int_{\mathbb{T}^d} F_{y \rightarrow \theta} \left[ \text{sign}(r_1/\varepsilon - y_1) R_0(\kappa_{\varepsilon,x}, y-x) \right] \overline{F_{y \rightarrow \theta} \left[ G_{\tau/\varepsilon^2}^g(p+y)^T \right]} \, d\theta.
\]
Note that
\[
F_{y \rightarrow \theta} \left[ \text{sign}(r_1/\varepsilon - y_1) R_0(\kappa_{\varepsilon,x}, y-x) \right] = (2\pi)^{-d} F_{y \rightarrow \theta} \left[ \text{sign}(r_1/\varepsilon - y_1) \right] \ast F_{y \rightarrow \theta} \left[ R_0(\kappa_{\varepsilon,x}, y-x) \right],
\]
where \( F_{y \to \theta} [\hat{R}_0(\kappa_{\varepsilon,x}, y - x)] = e^{i\varepsilon \theta} \hat{R}_0(\kappa_{\varepsilon,x}, \theta), \)
\[
F_{y \to \theta} \left[ \text{sign}(\lceil \varepsilon r_1 \rceil - y_1) \right] = -i (2\pi)^d-1 \delta(\bar{\theta}) \text{PV} \left( \frac{1}{\text{tg}(\theta_1/2)} \right) e^{i\varepsilon r_1/\theta_1},
\]
\( \theta = (\theta_1, \bar{\theta}), \quad \bar{\theta} = (\theta_2, \ldots, \theta_d), \) and \( \text{PV} \) stands for the Cauchy principal part. Hence,
\[
S_{\varepsilon, \tau/\varepsilon^2}^- = -\frac{i}{2}(2\pi)^{d-1} \sum_{x \in \mathbb{Z}^d} G_{\tau/\varepsilon^2}(l + x) \int_{\mathbb{T}^d, (1 + \varepsilon r_1)} \left( \text{PV} \int_{\mathbb{T}^1} e^{i\varepsilon r_1/\varepsilon} \hat{R}_0(\kappa_{\varepsilon,x}, z, \bar{\theta}) dz \times \right.
\times e^{i\varepsilon \theta} \hat{G}_{\tau/\varepsilon^2}(\theta)^* e^{ip \theta} d\theta
\]
\[
= -\frac{i}{2}(2\pi)^{-2d-1} \sum_{x \in \mathbb{Z}^d} \int_{\mathbb{T}^d} \left( e^{-i(l+\varepsilon) \theta^\prime} \hat{G}_{\tau/\varepsilon^2}(\theta^\prime) \times \right.
\times \text{PV} \int_{\mathbb{T}^1} e^{i\varepsilon r_1/\varepsilon} \hat{R}_0(\kappa_{\varepsilon,x}, z, \bar{\theta}) dz e^{i\varepsilon \theta} \hat{G}_{\tau/\varepsilon^2}(\theta)^* e^{ip \theta} d\theta d\theta^\prime.
\]
We change variables \( \theta_1 \to \phi = \theta_1 - z, \) then denote \( z = \theta_1, \) and change variables \( \theta^\prime = \varphi = \theta - \theta^\prime. \) Therefore,
\[
S_{\varepsilon, \tau/\varepsilon^2}^- = (2\pi)^{-d} \frac{1}{4\pi i} \int_{\mathbb{T}^d} e^{-i(l-p) \theta} I_\varepsilon(\theta) P_\varepsilon(\theta) d\theta, \tag{A.1}
\]
where \( I_\varepsilon(\theta) \) is defined by (6.18), and \( P_\varepsilon(\theta) \) stands for the matrix-valued function
\[
P_\varepsilon(\theta) = \text{PV} \int_{\mathbb{T}^1} e^{i\varepsilon r_1/\varepsilon} \hat{G}_{\tau/\varepsilon^2}(\theta + \phi, \bar{\theta})^* d\phi, \quad \theta = (\theta_1, \bar{\theta}).
\]
Applying the partition of unity (6.4), (6.5), and formula (6.22), we rewrite \( P_\varepsilon(\theta) \) in the form
\[
P_\varepsilon(\theta) = \sum_{\sigma=1}^{s} \sum_{\text{PV}} \int_{\mathbb{T}^1} e^{i\varepsilon r_1/\varepsilon} e^{i\omega_\sigma(\theta_1, \phi, \bar{\theta}, \tau/\varepsilon^2) g(\theta_1, \phi, \bar{\theta}) c^{\tau}_\sigma(\theta_1 + \phi, \bar{\theta})^* d\phi,
\]
where \( c^{\tau}_\sigma(\theta)^* = \Pi_\sigma(\theta)(I \mp i C^{\tau}_\sigma(\theta))/2. \) Let us fix \( r_1 \in \mathbb{R}, \tau \neq 0. \) Since \( \partial_1 \omega_\sigma(\theta) \neq 0 \) for \( \theta \in \text{supp} g, \) we can choose \( \varepsilon_g \equiv \varepsilon_g(r_1, \tau) > 0 \) such that for all \( \varepsilon < \varepsilon_g \) and \( \theta \in \text{supp} g, \)
\( \partial_1 \omega_\sigma(\theta) \neq \pm \varepsilon r_1/\tau. \)

**Lemma A.1** Let us fix \( r_1 \in \mathbb{R} \) and \( \tau \neq 0. \) Then
(i) \( \sup_{\theta \in \mathbb{T}^d} \sup_{\varepsilon > 0} |P_\varepsilon(\theta)| < \infty. \)
(ii) Let \( r_1 \pm \partial_1 \omega_\sigma(\theta) \tau/\varepsilon \neq 0 \) for all \( \theta \in \text{supp} g \) and \( \varepsilon \in (0, \varepsilon_g). \) Then
\[
P_\varepsilon(\theta) - 2\pi i \sum_{\sigma=1}^{s} \sum_{\pm} e^{\pm i\omega_\sigma(\theta) \tau/\varepsilon^2} g(r_1 \pm \partial_1 \omega_\sigma(\theta) \tau/\varepsilon) g(\theta) c^{\tau}_\sigma(\theta)^* \to 0 \quad \text{as} \quad \varepsilon \to +0.
\]

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Lemma A.1 can be proved by using the technique of [12, Lemma 8.3] or of [2, Proposition A.4 (i), (ii)]. The proof of this lemma is based on the following well-known assertion:

\[
\lim_{\lambda \to +\infty} \left( \text{PV} \int_{-\pi}^{\pi} \frac{e^{i\lambda \omega(z)} \chi(z)}{z} \, dz - \pi i e^{i\lambda \omega(0)} \chi(0) \text{sign} \omega'(0) \right) = 0,
\]

where \(\chi \in C^1\), \(\omega \in C^2\), and \(\omega'(0) \neq 0\).

By Lemma A.1 we can rewrite the r.h.s. of (A.1) in the form

\[
S_{-\epsilon,\tau/\epsilon^2} = (2\pi)^{-d/2} \sum_{\sigma, \pm} \int_{\mathbb{T}^d} e^{-i(l-p) \cdot \theta} I_{\pm}(\theta) e^{\pm i\omega_\sigma(\theta) \tau/\epsilon^2} \text{sign}(r_1 \pm \partial_1 \omega_\sigma(\theta) \tau/\epsilon) h^\pm_\sigma(\theta)^* \, d\theta, \quad (A.2)
\]

where \(h^\pm_\sigma(\theta) = g(\theta) e^{\mp i\omega_\sigma(\theta) \tau/\epsilon^2} \). Substituting (6.24) in (A.2) we obtain

\[
S_{-\epsilon,\tau/\epsilon^2} = (2\pi)^{-d/2} \int_{\mathbb{T}^d} e^{-i(l-p) \cdot \theta} \left( \sum_{\sigma, \pm} h^\pm_\sigma(\theta) e^{\pm i\omega_\sigma(\theta) \tau/\epsilon^2} A^\pm_{\epsilon, \tau/\epsilon^2}(\tau, r; \theta) \right) \times \left( \sum_{\sigma', \pm} e^{\pm i\omega_{\sigma'}(\theta) \tau/\epsilon^2} \text{sign}(r_1 \pm \partial_1 \omega_{\sigma'}(\theta) \tau/\epsilon) h^{\mp}_{\sigma'}(\theta)^* \right) \, d\theta + o(1), \quad \epsilon \to 0. \quad (A.3)
\]

Finally, comparing the r.h.s. of (A.3) and (6.26) we see that the problem of evaluating the limit value of (A.3) is solved by the similar way as in the step (v) of the proof of Lemma 6.6.

\[\blacksquare\]

**Appendix B: Proof of Lemma 6.8**

By (6.15) we write

\[
S^0_{\epsilon,\tau/\epsilon^2} = \sum_{x,y \in \mathbb{Z}^d} G^\epsilon_{\tau/\epsilon^2}(l+x) R^0(\kappa_{\epsilon,x}, [r/\epsilon] - x, [r/\epsilon] - y) G^\epsilon_{\tau/\epsilon^2}(p+y)^T,
\]

where \(\kappa_{\epsilon,x} = \epsilon [r/\epsilon] - \epsilon x\). Change variables \(y \to z = y - x\) and denote the series over \(x\) by \(\Phi_\epsilon(z)\),

\[
\Phi_\epsilon(z) \equiv \Phi_\epsilon(z, \tau, r, l, p) = \sum_{x \in \mathbb{Z}^d} G^\epsilon_{\tau/\epsilon^2}(l+x) R^0(\kappa_{\epsilon,x}, [r/\epsilon] - x, [r/\epsilon] - x - z) G^\epsilon_{\tau/\epsilon^2}(p+x+z)^T. \quad (B.1)
\]

Therefore,

\[
S^0_{\epsilon,\tau/\epsilon^2} = \sum_{z \in \mathbb{Z}^d} \Phi_\epsilon(z). \quad (B.2)
\]

The estimate (4.9) for \(R\) and the notation (6.14) imply the same estimate for \(R^0\),

\[
|R^0(r, x, y)| \leq C(1 + |x - y|)^{-\gamma}, \quad x, y \in \mathbb{Z}^d. \quad (B.3)
\]
Next, the Cauchy–Schwartz inequality yields
\[
\sum_{x \in \mathbb{Z}^d} |G_{\tau/\varepsilon^2}^g(l + x)||G_{\tau/\varepsilon^2}^g(p + x + z)^T| \leq \|G_{\tau/\varepsilon^2}^g(\cdot)\|_{L^2(\mathbb{T}^d)}^2 \leq C(1 + \|\bar{V}^{-1}\|_{L^2(\mathbb{T}^d)}^2).
\] (B.4)
Hence, condition E6 and the estimate (B.3) imply that \(|\Phi_\varepsilon(z)| \leq C(1 + |z|)^{-\gamma}\). Since \(\gamma > d\),
\[
\sum_{x \in \mathbb{Z}^d} |\Phi_\varepsilon(z)| \leq C < \infty,
\]
and the series in (B.2) converges uniformly in \(\varepsilon\) (and also in \(\tau, r, l, p\)). Therefore, it suffices to prove that
\[
\lim_{\varepsilon \to 0} \Phi_\varepsilon(z) = 0 \quad \text{for each} \quad z \in \mathbb{Z}^d.
\] (B.5)
Let us consider the series in (B.1). At first, note that by the property (a) for \(R\) and the definition (6.14), the function \(R_0^0(r, x, y)\) has the form \(R_0^0(r, x, y) = R_0^0(r, x_1, y_1, \bar{x} - \bar{y})\), and \(R_0^0(r, x_1, y_1, \bar{z}) = 0\) for \(y_1 < 0\). Hence,
\[
R_0^0(r, [r_1/\varepsilon] - x_1, [r_1/\varepsilon] - x_1 - z_1, \bar{z}) = 0 \quad \text{for} \quad x_1 > [r_1/\varepsilon] - z_1.
\]
Secondly, it follows from condition (4.8) that \(\forall \delta > 0 \exists K_\delta > 0\) such that for any \(y_1 > K_\delta\) and \(\forall r \in \mathbb{R}^d, |R_0^0(r, y_1, y_1 - z_1, \bar{z})| < \delta\). Hence, \(\forall \delta > 0 \exists M_\delta = \max(K_\delta, z_1) > 0\) such that
\[
\left| \sum_{x \in \mathbb{Z}^d, x < [r_1/\varepsilon] - M_\delta} G_{\tau/\varepsilon^2}^g(l + x)R_0^0(\kappa_{\varepsilon, x}, [r_1/\varepsilon] - x_1, [r_1/\varepsilon] - x_1 - z_1, \bar{z})G_{\tau/\varepsilon^2}^g(p + x + z)^T \right|
\leq \delta \sum_{x \in \mathbb{Z}^d} G_{\tau/\varepsilon^2}^g(l + x)G_{\tau/\varepsilon^2}^g(p + x + z)^T \leq C\delta,
\]
by the bound (B.4). Let us fix \(\delta > 0\). Therefore, it suffices to prove that for each \(l, p, z \in \mathbb{Z}^d\),
\[
\sum_{x_1 \in A_x} \sum_{x \in \mathbb{Z}^{d-1}} G_{\tau/\varepsilon^2}^g(l + x)R_0^0(\kappa_{\varepsilon, x}, [r_1/\varepsilon] - x_1, [r_1/\varepsilon] - x_1 - z_1, \bar{z})G_{\tau/\varepsilon^2}^g(p + x + z)^T \rightarrow 0 \quad \text{as} \quad \varepsilon \to 0,
\] (B.6)
as \(\varepsilon \to 0\), where \(A_x := \{x_1 \in \mathbb{Z}^1 : x_1 \in [r_1/\varepsilon] - M_\delta, [r_1/\varepsilon] - z_1\}\).
At first, note that if \(x_1 \in A_x, [r_1/\varepsilon] - x_1 \in [z_1, M_\delta]\). Denote \(\bar{\kappa}_{\varepsilon, x} \equiv \varepsilon[\bar{r}/\varepsilon] - \varepsilon \bar{x}, \bar{r} \in \mathbb{R}^{d-1}, \bar{x} \in \mathbb{Z}^{d-1}\). Then, by property (2.11) for \(R_0\) and property (4.10) for \(R\),
\[
\sup_{x \in \mathbb{Z}^{d-1}} |R_0^0(\varepsilon[r_1/\varepsilon] - \varepsilon x_1, \bar{\kappa}_{\varepsilon, x}, y_1, y'_1, \bar{z}) - R_0^0(0, \bar{\kappa}_{\varepsilon, x}, y_1, y'_1, \bar{z})| \rightarrow 0, \quad \varepsilon \to 0,
\]
for every \(x_1 \in A_x, y_1, y'_1 \in \mathbb{Z}^1, \) and \(\bar{z} \in \mathbb{Z}^{d-1}\). Hence, by the bound (B.4), we can replace \(\kappa_{\varepsilon, x}\) into \((0, \bar{\kappa}_{\varepsilon, x})\) in the series (B.6).

Further, we change the variable \(x_1 \rightarrow x_1' : x_1' = [r_1/\varepsilon] - x_1\) in (B.6) and denote \(x_1' = x_1, x_\varepsilon = ([r_1/\varepsilon] - x_1, \bar{x})\). Therefore, to derive (B.6) it suffices to check that for any fixed \(l, p, z \in \mathbb{Z}^d, \) and \(x_1 \in [z_1, M_\delta]\),
\[
D_\varepsilon := \sum_{x \in \mathbb{Z}^{d-1}} G_{\tau/\varepsilon^2}^g(l + x_\varepsilon)R_0^0(0, \bar{\kappa}_{\varepsilon, x}, x_1, x_1 - z_1, \bar{z})G_{\tau/\varepsilon^2}^g(p + x_\varepsilon + z)^T \rightarrow 0 \quad \text{as} \quad \varepsilon \to 0.
\]
Let $d = 1$. Then $D_\varepsilon = G^g_{\tau/\varepsilon^2}(l_1 + [r_1/\varepsilon] - x_1)R^0(0, x_1, x_1 - z_1)G^g_{\tau/\varepsilon^2}(p_1 + [r_1/\varepsilon] - x_1 + z_1)^T$ vanishes as $\varepsilon \to 0$. Indeed, applying the inverse Fourier transform and the decomposition \cite{6.22}, we have

$$G^g_{\tau/\varepsilon^2}(l_1 + [r_1/\varepsilon] - x_1) = \frac{1}{2\pi} \int_{T^d} e^{-i(l_1 + [r_1/\varepsilon] - x_1)\theta_1} \hat{G}^g_{\tau/\varepsilon^2} (\theta_1) d\theta_1$$

$$= \frac{1}{2\pi} \sum_{\sigma=1}^{s} \sum_{\tau=1}^{\infty} \int_{T^d} e^{-i(l_1 + [r_1/\varepsilon] - x_1)\theta_1} e^{\pm i\omega_\sigma(\theta_1)\tau/\varepsilon^2} g(\theta_1) c^\varepsilon_\sigma(\theta_1) d\theta_1.$$

The last integral vanishes as $\varepsilon \to 0$, since $\omega'_\sigma(\theta_1) \neq 0$ for all $\theta_1 \in \text{supp} \, g$.

Let $d > 1$. In this case, we put, for simplicity, $R^0(0, \tilde{\kappa}_e, \ldots) = R^0(0, \tilde{\kappa}_e, x_1, x_1 - z_1, \tilde{z})$. Using the Fourier transform we rewrite $D_\varepsilon$ as

$$D_\varepsilon = \sum_{x \in \mathbb{Z}^{d-1}} (2\pi)^{-2d} \int_{T^{2d}} e^{-i(l_1 + x_2)\theta_1} \hat{G}^g_{\tau/\varepsilon^2}(\theta') R^0(0, \tilde{\kappa}_e, \ldots) e^{i(l_2 + x_2 + z_1)\theta_1} \hat{G}^g_{\tau/\varepsilon^2}(\theta^*) d\theta d\theta'.$$

Let us change variables $\theta = (\theta_1, \tilde{\theta})$, $\theta' = (\theta'_1, \tilde{\theta}')$ as follows: $\tilde{\varphi} = \tilde{\theta} - \tilde{\theta}'$, and denote $\theta_1 = \varphi_1$, $\theta'_1 = \theta_1$. Hence,

$$D_\varepsilon = (2\pi)^{-d+1} \int_{T^{d-1}} e^{-i(l - p - \tilde{z})\tilde{\theta}} A_\varepsilon(\tilde{\theta}) B_\varepsilon(\tilde{\theta}) d\tilde{\theta},$$ \hfill (B.7)

where

$$A_\varepsilon(\tilde{\theta}) = \frac{1}{2\pi} \int_{T^d} e^{-i(l_1 + [r_1/\varepsilon] - x_1)\theta_1} \mathcal{I}_\varepsilon(\theta) d\theta_1,$$ \hfill (B.8)

$$B_\varepsilon(\tilde{\theta}) = \frac{1}{2\pi} \int_{T^d} e^{i(p_1 + [r_1/\varepsilon] - x_1 + z_1)\varphi_1} \hat{G}^g_{\tau/\varepsilon^2}(\varphi_1, \tilde{\theta}) d\varphi_1,$$ \hfill (B.9)

$$\mathcal{I}_\varepsilon(\theta) = (2\pi)^{-d+1} \int_{T^{d-1}} e^{i\tilde{\varphi}} \hat{G}^g_{\tau/\varepsilon^2}(\theta_1, \tilde{\theta}) \left( \sum_{x \in \mathbb{Z}^{d-1}} e^{ix\cdot\varphi} R^0(0, \varepsilon[\tilde{r}/\varepsilon] - \varepsilon\tilde{x}, \ldots) \right) d\tilde{\varphi} \cdot$$ \hfill (B.10)

(cf. formula \cite{6.18}). Below we prove the following bound for $\mathcal{I}_\varepsilon(\theta)$ (cf. \cite{6.24}):

$$\mathcal{I}_\varepsilon(\theta) = \sum_{\sigma=1}^{s} \sum_{\pm} h_{\sigma}^{\pm}(\theta) e^{\pm i\omega_\sigma(\tau/\varepsilon^2) A^\pm_{\varepsilon, \sigma}(\tau, \tilde{\varphi}; \tau) + o(1)}, \quad \varepsilon \to 0,$$ \hfill (B.11)

where $h_{\sigma}^{\pm}(\theta)$ is defined in \cite{6.22},

$$A^\pm_{\varepsilon, \sigma}(\tau, \tilde{\varphi}; \tau) = \int_{\mathbb{R}^{d-1}} R^0(0, \tilde{r} \mp \nabla_{\tilde{\varphi}} \omega_\sigma(\theta)\tau/\varepsilon - \tilde{y}, \ldots) \mathcal{K}^\pm_{\sigma}(\tau, \tilde{y}, \theta) d\tilde{y},$$ \hfill (B.12)

$\mathcal{K}^\pm_{\sigma}(\tau, \tilde{y}, \theta)$, $\tau > 0$, $\tilde{y} \in \mathbb{R}^{d-1}$, $\theta \in \mathbb{T}^d \setminus C$, stands for the following matrix-valued function

$$\mathcal{K}^\pm_{\sigma}(\tau, \tilde{y}, \theta) = F_{\tilde{r} \mp \tilde{y}}^{-1} \left[ e^{\pm i(\tau/2)\varphi(\nabla_{\tilde{\varphi}}^2 \omega_\sigma(\theta)t)\varphi} \right] = \frac{e^{\pm i\pi s/4}}{(2\pi \tau)^{(d-1)/2}} \sqrt{\det(\nabla_{\tilde{\varphi}}^2 \omega_\sigma(\theta))}.$$
and \( s \) denotes the signature of the matrix \( (\nabla^2_\theta \omega_\sigma(\theta)) \), \( \theta = (\theta_1, \bar{\theta}) \in T^d \setminus C \) (cf. (6.25)).

To prove (B.11) we first apply the Poisson summation formula (see (6.19)) and obtain

\[
\sum_{\bar{x} \in \mathbb{Z}^{d-1}} e^{i \bar{x} \cdot \bar{r}} R^0(0, \varepsilon \bar{r}/\varepsilon - \varepsilon \bar{x}, \ldots) = e^{-d+1} \sum_{\bar{n} \in \mathbb{Z}^{d-1}} \bar{R}^0(0, -\bar{\varphi}/\varepsilon - 2\pi \bar{n}/\varepsilon, \ldots) e^{i \bar{n}/\varepsilon} \varepsilon, \ldots),
\]

where \( \bar{R}^0(0, \bar{\varphi}, \ldots) \) stands for the Fourier transform \( \bar{R}^0(0, \bar{\varphi}, \ldots) = F_{\bar{r} \rightarrow \bar{\varphi}}[R^0(0, \bar{r}, \ldots)], \bar{r}, \bar{\varphi} \in \mathbb{R}^{d-1}. \) Conditions (2.11) and (4.11) yield

\[
\varepsilon^{-d+1} \sum_{\bar{n} \neq 0} \bar{R}^0(0, -\bar{\varphi}/\varepsilon - 2\pi \bar{n}/\varepsilon, \ldots) \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad (B.13)
\]

uniformly in \( \bar{\varphi} \in [-\pi, \pi]^{d-1}. \) Hence, by (B.10) and (B.13),

\[
\mathcal{I}_\varepsilon(\theta) = (2\pi \varepsilon)^{-d+1} \int_{[-\pi, \pi]^{d-1}} e^{i \bar{\varphi} \cdot (\bar{l} + [\bar{r}/\varepsilon])} \hat{\mathcal{G}}^0_{\bar{r}/\varepsilon}(\theta_1, \bar{\theta} - \bar{\varphi}) \bar{R}^0(0, -\bar{\varphi}/\varepsilon, \ldots) d\bar{\varphi} + o(1)
\]

\[
= (2\pi)^{-d+1} \int_{[-\pi/\varepsilon, \pi/\varepsilon]^{d-1}} e^{-i \bar{\varphi} \cdot (\bar{l} + [\bar{r}/\varepsilon])} \hat{\mathcal{G}}^0_{\bar{r}/\varepsilon}(\theta_1, \bar{\theta} + \varepsilon \bar{\varphi}) \bar{R}^0(0, \bar{\varphi}, \ldots) d\bar{\varphi} + o(1).
\]

Secondly, using the representation (6.22) for \( \hat{\mathcal{G}}^0_{\bar{r}/\varepsilon}(\theta) \) we rewrite \( \mathcal{I}_\varepsilon(\theta) \) in the form

\[
\mathcal{I}_\varepsilon(\theta) = (2\pi)^{-d+1} \sum_{\sigma, \tau, \pm \in [-\pi/\varepsilon, \pi/\varepsilon]^{d-1}} \int e^{i \bar{\varphi} \cdot (\bar{l} + [\bar{r}/\varepsilon])} e^{\pm i \omega_\sigma(\theta_1, \bar{\theta} + \varepsilon \bar{\varphi}) \tau/\varepsilon^2} h^\pm_{\sigma}(\theta_1, \bar{\theta} + \varepsilon \bar{\varphi}) \bar{R}^0(0, \bar{\varphi}, \ldots) d\bar{\varphi} + o(1)
\]

by conditions (2.11) and (4.11). Finally, we use the Taylor sum representation for \( \omega_\sigma(\theta_1, \bar{\theta} + \varepsilon \bar{\varphi}) \) (cf. (6.23)) and obtain the bound (B.11), since

\[
(2\pi)^{-d+1} \int_{\mathbb{R}^{d-1}} e^{-i \bar{\varphi} \cdot (\bar{l} + [\bar{r}/\varepsilon])} e^{\pm i \omega_\sigma(\theta_1, \bar{\theta} + \varepsilon \bar{\varphi}) \tau/\varepsilon^2} \bar{R}^0(0, \bar{\varphi}, \ldots) d\bar{\varphi}
\]

\[
= F_{\bar{\varphi} \rightarrow \bar{\varphi} \cdot (\bar{l} + [\bar{r}/\varepsilon])} \bar{R}^0(0, \bar{\varphi}, \ldots) e^{\pm i (\tau/2) \bar{\varphi} \cdot (\nabla^2_\theta \omega_\sigma(\theta))^2} = A^\pm_{\bar{\varphi}}(\tau, \bar{r}; \theta),
\]

where \( A^\pm_{\bar{\varphi}}(\tau, \bar{r}; \theta) \) is defined in (B.12). The bound (B.11) is proved.

Now we prove that the r.h.s. of (B.7) vanishes as \( \varepsilon \to 0 \), using the Lebesgue dominated convergence theorem. At first, we show that

\[
|A_\varepsilon(\bar{\theta}) B_\varepsilon(\bar{\theta})| \leq v(\bar{\theta}), \quad \forall \varepsilon > 0, \quad \text{where} \quad v(\bar{\theta}) \in L^1(T^{d-1}).
\]

Indeed, by conditions (4.11),

\[
\sup_{\bar{\theta} \in T^d} \sup_{\varepsilon > 0} |A^\pm_{\bar{\varphi}}(\tau, \bar{r}; \theta)| \leq C < \infty.
\]
Since \( h_\sigma^\pm(\theta) = g(\theta)\Pi_\sigma(\theta)(I \mp iC_\sigma(\theta))/2 \), the bound (B.11) yields

\[
|I_\varepsilon(\theta)| \leq \sum_{\sigma=1}^s (C_1\omega_{\sigma}^{-1}(\theta) + C_2\omega_{\sigma}(\theta)) + C_3, \quad \theta \in \mathbb{T}^d \setminus C.
\]

Write \( v(\tilde{\theta}) := \int_{\mathbb{T}^1} (1 + \|\hat{V}^{-1}(\theta)\|) d\theta_1 \), \( \tilde{\theta} \in \mathbb{T}^{d-1} \). Therefore, by (B.8) and (B.9),

\[
|A_\varepsilon(\tilde{\theta})| \leq C \int_{\mathbb{T}^1} |I_\varepsilon(\theta)| d\theta_1 \leq C_1(v(\tilde{\theta}))/2,
\]

\[
|B_\varepsilon(\tilde{\theta})| \leq C \int_{\mathbb{T}^1} |\hat{G}_{r/\varepsilon}^g(\theta)| d\theta_1 \leq C_1(v(\tilde{\theta}))/2,
\]

and, evidently, \( |A_\varepsilon(\tilde{\theta})B_\varepsilon(\tilde{\theta})| \leq C_v(\varepsilon) \), where \( v(\tilde{\theta}) \in L^1(\mathbb{T}^{d-1}) \) by condition E6. Therefore, to prove that the integral in (B.7) tends to zero it suffices to check that \( B_\varepsilon(\tilde{\theta}) \) vanishes as \( \varepsilon \to 0 \) for a.a. \( \tilde{\theta} \in \mathbb{T}^{d-1} \). By the representation (6.22),

\[
B_\varepsilon(\tilde{\theta}) = \frac{1}{2\pi} \sum_{\sigma=1}^s \sum_{\pm} \int_{\mathbb{T}^1} e^{i(p_1/\varepsilon - x_1 + z_1)\phi_1} e^{\pm i\omega_\sigma(\phi_1, \tilde{\theta})r/\varepsilon^2} h_\sigma^\pm(\phi_1, \tilde{\theta})^* d\phi_1.
\]

For fixed \( r_1 \in \mathbb{R}^1, \tau \neq 0 \), and \( p_1, x_1, z_1 \in \mathbb{Z}^1 \), \( \{\phi_1 \in \mathbb{T}^1 : \partial_1\omega_\sigma(\phi_1, \tilde{\theta}) = 0\} = 0 \) for a.a. fixed \( \tilde{\theta} \in \mathbb{T}^{d-1} \). Hence, \( B_\varepsilon(\tilde{\theta}) \) vanishes as \( \varepsilon \to 0 \). It can be proved similarly to the Lebesgue–Riemann theorem.

**Appendix C: Proof of Lemma 6.6 in the general case**

We prove Lemma 6.6 under the weaker condition (2.13)–(2.14) on \( \hat{R}_0(r, \theta) \) than (2.11).

**Step (i)** of the proof is not changed and we derive formulas (6.16)–(6.18). Next step is to apply the Poisson summation formula (6.19). However, in general, we can not apply (6.19) to \( \hat{R}_0(r, \theta) \). Therefore, introduce

\[
\hat{R}_{r,\varepsilon}(y, \theta) = \hat{R}_0(y, \theta)e^{-\varepsilon^2(y-r)^2}, \quad y \in \mathbb{R}^d, \quad \theta \in \mathbb{T}^d, \quad \text{ (C.1)}
\]

with \( \beta \in (d+2, 2(\delta - 1 - d)/d) \) and \( \delta \) from (2.14). The asymptotics of (6.18) is not changed if we replace the function \( \hat{R}_0(y, \theta) \) by \( \hat{R}_{r,\varepsilon}(y, \theta) \) in (6.18). Indeed, consider the following series:

\[
\sum_{x \in \mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i(l+x)\cdot \xi} \hat{G}_{r/\varepsilon}(\theta - \varphi) d\varphi \left( \hat{R}_0(\varepsilon |r/\varepsilon| - \varepsilon x, \theta) - \hat{R}_{r,\varepsilon}(\varepsilon |r/\varepsilon| - \varepsilon x, \theta) \right)
\]

\[
= \sum_{|x| \leq \varepsilon r/\varepsilon^2} \ldots + \sum_{|x| \geq \varepsilon r/\varepsilon^2} \ldots = A_\varepsilon(\theta) + B_\varepsilon(\theta).
\]
and prove that the series $A_\varepsilon(\theta)$ and $B_\varepsilon(\theta)$ vanish as $\varepsilon \to 0$. For the first series $A_\varepsilon(\theta)$, we apply Lemma 6.4 (i), definition (C.1) and property I1 and obtain

$$A_\varepsilon(\theta) \leq C \sum_{|x| \leq cr/\varepsilon^2} \left| \int_{\mathbb{T}^d} e^{i(l+x) \cdot \varphi} \hat{G}_{r/\varepsilon^2}(\theta - \varphi) \, d\varphi \right| \left| \hat{R}_0(\varepsilon[r/\varepsilon] - \varepsilon x, \theta) - \hat{R}_{r,\varepsilon}(\varepsilon[r/\varepsilon] - \varepsilon x, \theta) \right|$$

$$\leq C_1 \sum_{|x| \leq cr/\varepsilon^2} \left| G_{r/\varepsilon^2}(l + x) \varepsilon^2 (\varepsilon|x| + \varepsilon)^2 \right| \leq C \varepsilon^d \varepsilon^{\beta + 2} (\tau/\varepsilon)^{2+d} \to 0, \quad \varepsilon \to 0,$$

uniformly in $\theta$, since $\beta > 2 + d$. For the second series $B_\varepsilon(\theta)$, Lemma 6.4 (ii) yields

$$B_\varepsilon(\theta) \leq C \sum_{|x| \geq cr/\varepsilon^2} \left| \int_{\mathbb{T}^d} e^{i(l+x) \cdot \varphi} \hat{G}_{r/\varepsilon^2}(\theta - \varphi) \, d\varphi \right| \left| \hat{R}_0(\varepsilon[r/\varepsilon] - \varepsilon x, \theta) \right|$$

$$\leq C_1 \sum_{|x| \geq cr/\varepsilon^2} \left| G_{r/\varepsilon^2}(l + x) \right| \to 0, \quad \varepsilon \to 0.$$

Step (ii): Now we apply formula (6.19) to $\sum_{x \in \mathbb{Z}^d} e^{i\varepsilon x \cdot \varphi} \hat{R}_{r,\varepsilon}(\varepsilon[r/\varepsilon] - \varepsilon x, \theta)$. By $\hat{R}_{r,\varepsilon}(s, \theta)$, $s \in \mathbb{R}^d$, $\theta \in \mathbb{T}^d$, we denote the Fourier transform of $\hat{R}_{r,\varepsilon}(y, \theta)$ w.r.t. $y$ (see (2.12)) and obtain

$$I_\varepsilon(\theta) = (2\pi)^{-d} \int_{\mathbb{T}^d} e^{i(l+[r/\varepsilon]) \cdot \varphi} \hat{G}_{r/\varepsilon^2}(\theta - \varphi) \left( \sum_{n \in \mathbb{Z}^d} \hat{R}_{r,\varepsilon}(-\varphi/\varepsilon - 2\pi n/\varepsilon, \theta) \right) \, d\varphi + o(1) \quad (C.2)$$

(cf (6.20)). We check that the contribution of the series with $n \neq 0$ in (C.2) vanishes,

$$\varepsilon^{-d} \sum_{n \in \mathbb{Z}^d, n \neq 0} \hat{R}_{r,\varepsilon}(-\varphi/\varepsilon - 2\pi n/\varepsilon, \theta) \to 0 \quad \text{as} \quad \varepsilon \to 0, \quad (C.3)$$

uniformly in $\theta \in \mathbb{T}^d$, $\varphi \in [-\pi, \pi]^d$ (cf (6.21)). At first, we derive the formula for $\hat{R}_{r,\varepsilon}(\xi, \theta)$:

$$\hat{R}_{r,\varepsilon}(\xi, \theta) = e^{i\varepsilon \xi \cdot (4\pi k)^{-d/2}} \int_{\mathbb{R}^d} e^{-i\varepsilon r \cdot e^{-|s\xi|^2/(4k)} \mu(\theta, ds)}, \quad \xi \in \mathbb{R}^d, \quad \theta \in \mathbb{T}^d, \quad (C.4)$$

where $k := \varepsilon^d$ and $\mu(\theta, ds)$ from (2.13). Indeed, by definition (C.1),

$$\hat{R}_{r,\varepsilon}(\xi, \theta) = \int_{\mathbb{R}^d} e^{i\varepsilon y \cdot k(y-r)^2} \hat{R}_0(y, \theta) \, dy = e^{i\varepsilon \xi \cdot |\xi|^2/(4k)} \int_{\mathbb{R}^d} e^{-k(y-r-i\xi/(2k))^2} \hat{R}_0(y, \theta) \, dy.$$

On the other hand, by definition (2.13), for any $\alpha \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} e^{-k(y-\alpha)^2} \hat{R}_0(y, \theta) \, dy = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-iy \cdot s} e^{-k(y-\alpha)^2} \mu(\theta, ds) \, dy$$

$$= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-is \cdot \alpha} \left( \frac{\pi}{k} \right)^{d/2} e^{-|s|^2/(4k)} \mu(\theta, ds).$$
Hence,
\[ \hat{R}_{r,e}(\xi, \theta) = e^{ir\xi - |\xi|^2/(4k)}(4\pi k)^{-d/2} \int_{\mathbb{R}^d} e^{-is(r+i\xi)/(2k)}e^{-|s|^2/(4k)} \mu(\theta, ds). \]

This implies formula (C.4).

Now we prove (C.3). Let us set \( \xi_{n,e} = -\varphi/e - 2\pi n/e \), where \( |\varphi| \leq \pi \), \( i = 1, \ldots, d \), apply (C.4) and divide the integration into two: \( |s| \leq \pi/(2e) \) and \( |s| \geq \pi/(2e) \):
\[
\varepsilon^{-d} \left| \sum_{n \neq 0} \hat{R}_{r,e}(\xi_{n,e}, \theta) \right| \leq \varepsilon^{-d} \sum_{n \neq 0} (4\pi k)^{-d/2} \left| \int_{\mathbb{R}^d} e^{-isr}e^{-|s-\xi_{n,e}|^2/(4k)} \mu(\theta, ds) \right|
= I_1 + I_2, \tag{C.5}
\]
where
\[
I_1 = (\varepsilon \sqrt{4\pi k})^{-d} \sum_{n \neq 0} \left| \int_{|s| \leq \pi/(2e)} e^{-isr}e^{-|s-\xi_{n,e}|^2/(4k)} \mu(\theta, ds) \right|
I_2 = (\varepsilon \sqrt{4\pi k})^{-d} \sum_{n \neq 0} \left| \int_{|s| \geq \pi/(2e)} e^{-isr}e^{-|s-\xi_{n,e}|^2/(4k)} \mu(\theta, ds) \right|. \tag{C.6}
\]
Let \( |s| \leq \pi/(2e) \). Put \( \xi_{n,e} = -\varphi/e - 2\pi n/e \). Then \( |s_i - \xi_{n_i,e}| \geq |2\pi n_i/e - s_i + \varphi_i/e| \geq (\pi/e)(2|n_i| - 3/2) \) if \( n_i \neq 0 \). Therefore, there exist positive constants \( a, b \) such that
\[
\sum_{n_i \in \mathbb{Z}^d : n_i \neq 0} e^{-(s_i - \xi_{n_i,e})^2/(4k)} \leq \sum_{n_i \neq 0} e^{-\pi^2(2|n_i| - 3/2)^2/(4ke^2)} \leq a e^{-b/(ke^2)} \sqrt{k\varepsilon}, \quad i = 1, \ldots, d.
\]
Hence, there exist positive constants \( a_s \) and \( b_s \) such that
\[
I_1 \sim \sum_{s=0}^{d-1} a_s e^{-b_s/(ke^2)}(\sqrt{k\varepsilon})^{-s} \to 0, \quad \varepsilon \to 0,
\]
since \( k = \varepsilon^\beta \) with \( \beta > 0 \). Let \( |s| \geq \pi/(2e) \). Then
\[
\sum_{n \in \mathbb{Z}^d} e^{-(s-\xi_{n,e})^2/(4k)} = \sum_{n \in \mathbb{Z}^d} e^{-(\pi^2/(ke^2))(n+\varphi/(2\pi)+s\varepsilon/(2\pi))^2} \leq C_1(\varepsilon \sqrt{k})^d + C_2. \tag{C.7}
\]
Using (C.6), (C.7), and condition (2.14), we obtain
\[
I_2 \sim (C_1 + C_2(\varepsilon \sqrt{k})^{-d}) \sup_{\theta \in \mathbb{T}^d} \int_{|s| \geq \pi/(2e)} |\mu(\theta, ds)| \leq C_3 \varepsilon^{\delta - 1 - (\beta/2 + 1)d} \to 0, \quad \varepsilon \to 0,
\]
because \( k = \varepsilon^\beta \) and \( \delta > 1 + (\beta/2 + 1)d \). This completes the proof of the bound (C.3).

**Step (iii):** We apply (C.2), (C.3), change variables \( \varphi \to -\varepsilon \varphi \) and use formula (6.22):
\[
I_\varepsilon(\theta) = (2\pi \varepsilon)^{-d} \int_{[-\pi,\pi]^d} e^{i\varphi(1+|r/e|)} G_{r/e^2}(\theta - \varphi) \hat{R}_{r,e}(-\varphi/e, \theta) d\varphi + o(1)
\]
\[(2\pi)^{-d} \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} e^{-i\varepsilon \varphi(l+\lceil r/\varepsilon \rceil)} \hat{g}_{\tau/\varepsilon^2}^\sigma(\theta + \varepsilon \varphi) \tilde{R}_{\tau, \varepsilon}(\varphi, \theta) d\varphi + o(1) = (2\pi)^{-d} \sum_{\sigma, \pm} \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} e^{-i\varepsilon \varphi(l+\lceil r/\varepsilon \rceil)} e^{\pm i\omega_\sigma(\theta+\varepsilon \varphi) r/\varepsilon^2} h_\sigma^\pm(\theta + \varepsilon \varphi) \tilde{R}_{\tau, \varepsilon}(\varphi, \theta) d\varphi + o(1). \] 

The asymptotics of (C.8) is not changed if we replace \(e^{-i\varepsilon \varphi(l+\lceil r/\varepsilon \rceil)}\) by \(e^{-i\varphi r}\) in the integrand. Indeed, we check that the integral

\[D_\varepsilon(\theta) = \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} \left( e^{-i\varepsilon \varphi(l+\lceil r/\varepsilon \rceil)} - e^{-i\varphi r} \right) e^{\pm i\omega_\sigma(\theta+\varepsilon \varphi) r/\varepsilon^2} h_\sigma^\pm(\theta + \varepsilon \varphi) \tilde{R}_{\tau, \varepsilon}(\varphi, \theta) d\varphi\]

vanishes as \(\varepsilon \to 0\). We apply formula (C.4) and change the order of the integration \(\mu(\theta, ds) d\varphi \to d\varphi \mu(\theta, ds)\):

\[D_\varepsilon(\theta) = \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} \left( e^{-i\varepsilon \varphi(l+\lceil r/\varepsilon \rceil)} - e^{-i\varphi r} \right) e^{\pm i\omega_\sigma(\theta+\varepsilon \varphi) r/\varepsilon^2} h_\sigma^\pm(\theta + \varepsilon \varphi) \tilde{R}_{\tau, \varepsilon}(\varphi, \theta) d\varphi \times \int_{\mathbb{R}^d} \frac{\mu(\theta, ds)}{\varepsilon} d\varphi = \int_{\mathbb{R}^d} e^{-is\cdot r} e^{-|s-\varphi|^2/(4k)} \mu(\theta, ds) d\varphi = \int_{\mathbb{R}^d} e^{-is\cdot r} C_\varepsilon(s, \theta) \mu(\theta, ds), \] 

where \(C_\varepsilon(s, \theta)\) stands for the inner integral in \(d\varphi:\)

\[C_\varepsilon(s, \theta) = \frac{1}{(4\pi k)^{d/2}} \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)} \left( e^{i\varphi(r-\varepsilon\lceil r/\varepsilon \rceil)} - 1 \right) e^{\pm i\omega_\sigma(\theta+\varepsilon \varphi) r/\varepsilon^2} h_\sigma^\pm(\theta + \varepsilon \varphi) d\varphi.\]

Note that

\[|C_\varepsilon(s, \theta)| \leq \varepsilon \sup_{\theta \in \mathbb{T}^d} |h_\sigma^\pm(\theta)||\frac{1}{(4\pi k)^{d/2}} \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)} |\varphi| d\varphi,\]

where

\[\frac{1}{(4\pi k)^{d/2}} \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)} |\varphi| d\varphi \leq \frac{1}{(4\pi k)^{d/2}} \int_{\mathbb{R}^d} e^{-|\varphi|^2/(4k)} (|s| + |\varphi|) d\varphi \leq C_1 |s| + C_2 \sqrt{k}.\]

Hence, by (C.9) and (2.14), \(\sup_{\theta \in \mathbb{T}^d} |D_\varepsilon(\theta)| \leq C \varepsilon.\) Similarly, the asymptotics of (C.8) is not changed if we replace \(h_\sigma^\pm(\theta + \varepsilon \varphi)\) by \(h_\sigma^\pm(\theta)\) in the integrand. Therefore,

\[I_\varepsilon(\theta) = \sum_{\sigma, \pm} h_\sigma^\pm(\theta) N_{\sigma, \sigma}^\pm(\tau, r; \theta) + o(1), \quad \varepsilon \to 0, \] 

where \(N_{\sigma, \sigma}^\pm(\tau, r; \theta)\) stands for the matrix-valued function,

\[N_{\sigma, \sigma}^\pm(\tau, r; \theta) = (2\pi)^{-d} \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} e^{-i\varphi r} e^{\pm i\omega_\sigma(\theta+\varepsilon \varphi) r/\varepsilon^2} \tilde{R}_{\tau, \varepsilon}(\varphi, \theta) d\varphi. \]
Step (iv): Now we apply the Taylor sum representation (cf. (6.23)) and replace $\omega_\sigma(\theta + \varepsilon \varphi)$ by $\omega_\sigma(\theta) + \varepsilon \nabla \omega_\sigma(\theta) \cdot \varphi + (\varepsilon^2/2)\varphi \cdot H_\sigma(\theta) \varphi$. To do it we check that

$$L_\varepsilon(\theta) := \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} e^{-i\varphi \cdot r} \left[ e^{i\omega_\sigma(\theta + \varepsilon \varphi)\tau/\varepsilon^2} - e^{i\omega_\sigma(\theta) + \varepsilon \nabla \omega_\sigma(\theta) \cdot \varphi + (\varepsilon^2/2)\varphi \cdot H_\sigma(\theta) \varphi\tau/\varepsilon^2} \right] \mathcal{R}_{r,\varepsilon}(\varphi, \theta) \, d\varphi \to 0$$

as $\varepsilon \to 0$. As in Step (iii), we use formula (C.4) and change the order of the integration $\mu(\theta, ds) \, d\varphi \to d\varphi \mu(\theta, ds)$. Therefore,

$$L_\varepsilon(\theta) = \int_{\mathbb{R}^d} e^{-i\varphi \cdot r} D_\varepsilon(s, \theta) \, \mu(\theta, ds),$$

where $D_\varepsilon(s, \theta)$ stands for the inner integral in $d\varphi$:

$$D_\varepsilon(s, \theta) = \frac{1}{(4\pi k)^{d/2}} \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)} \left[ e^{i\omega_\sigma(\theta + \varepsilon \varphi)\tau/\varepsilon^2} - e^{i\omega_\sigma(\theta) + \varepsilon \nabla \omega_\sigma(\theta) \cdot \varphi + (\varepsilon^2/2)\varphi \cdot H_\sigma(\theta) \varphi\tau/\varepsilon^2} \right] d\varphi.$$  

Moreover, by condition (2.14),

$$\sup_{\theta \in \mathbb{T}^d} \left| \int_{|s| \geq N} e^{-i\varphi \cdot r} D_\varepsilon(s, \theta) \, \mu(\theta, ds) \right| \leq \sup_{\theta \in \mathbb{T}^d} \int_{|s| \geq N} \left( \frac{2}{(4\pi k)^{d/2}} \int_{\mathbb{R}^d} e^{-|s-\varphi|^2/(4k)} d\varphi \right) |\mu|(|\theta, ds) \leq C(1 + N)^{-\delta + 1} \to 0 \text{ as } N \to \infty,$$

uniformly in $\varepsilon$. Hence, it suffices to prove that for any fixed $N \in \mathbb{N}$,

$$L_\varepsilon^N(\theta) := \int_{|s| \leq N} e^{-i\varphi \cdot r} D_\varepsilon(s, \theta) \, \mu(\theta, ds) \to 0 \text{ as } \varepsilon \to 0. \quad \text{(C.12)}$$

To check (C.12) we estimate the inner integral $D_\varepsilon(s, \theta)$:

$$|D_\varepsilon(s, \theta)| \leq C_\varepsilon \frac{1}{(4\pi k)^{d/2}} \int_{[-\pi/\varepsilon, \pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)} |\varphi|^3 \, d\varphi \leq C_1 \frac{1}{(4\pi k)^{d/2}} \left[ \int_{\mathbb{R}^d} e^{-|\varphi|^2/(4k)} |\varphi|^3 \, d\varphi + \int_{\mathbb{R}^d} e^{-|\varphi|^2/(4k)} \, d\varphi \right] \leq C_2 \varepsilon [k^{3/2} + |s|^3].$$

Hence, for fixed $N$,

$$|L_\varepsilon^N(\theta)| \leq C \varepsilon \int_{|s| \leq N} (\varepsilon^{3\beta/2} + |s|^3) |\mu|(|\theta, ds) \leq C_2(N) \varepsilon,$$

and (C.12) follows. Therefore, sup$_{\theta \in \mathbb{T}^d} |L_\varepsilon(\theta)| \to 0$ as $\varepsilon \to 0$, and

$$N_{\varepsilon, \sigma}^+(\tau, r; \theta) = e^{i\omega_\sigma(\theta)\tau/\varepsilon^2} B_{\varepsilon, \sigma}(\tau, r; \theta) + o(1), \quad \varepsilon \to 0, \quad \text{(C.13)}$$
where, by definition,
\[
B_{\epsilon,\sigma}^\pm(\tau, r; \theta) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\varphi \cdot (\tau + \nabla \omega_\sigma(\theta) \tau / \epsilon)} e^{\pm i(\tau / 2) \varphi \cdot H_\sigma(\theta) \varphi} \hat{R}_{\tau, \epsilon}(\varphi, \theta) \, d\varphi
\]
\[
= \int_{\mathbb{R}^d} \hat{R}_{\tau, \epsilon}(\tau + \nabla \omega_\sigma(\theta) \tau / \epsilon - x, \theta) K^{\pm}_\sigma(\tau, x, \theta) \, dx,
\]
with the function \( K^{\pm}_\sigma(\tau, x, \theta) \) (\( \tau > 0 \), \( x \in \mathbb{R}^d \), \( \theta \in \mathbb{T}^d \setminus C \)) from (6.25).

**Remark.** The integral \( N^{\pm}_{\epsilon, \sigma}(\tau, r; \theta) \) can be rewritten in another form. Namely,
\[
N^{\pm}_{\epsilon, \sigma}(\tau, r; \theta) = e^{\pm i\omega_\sigma(\theta)\tau / \epsilon^2} C^{\pm}_{\epsilon, \sigma}(\tau, r; \theta) + o(1), \quad \epsilon \to 0,
\]
where, by definition,
\[
C^{\pm}_{\epsilon, \sigma}(\tau, r; \theta) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-is \cdot (\tau + \nabla \omega_\sigma(\theta) \tau / \epsilon)} e^{\pm i(\tau / 2) s \cdot H_\sigma(\theta) s} \mu(\theta, ds).
\]

**Proof.** We first substitute formula (C.4) in (C.11) and change the order of the integration \( \mu(\theta, ds) \, d\varphi \to d\varphi \, \mu(\theta, ds) \). Therefore,
\[
N^{\pm}_{\epsilon, \sigma}(\tau, r; \theta) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-is \cdot r} \left( \frac{1}{(4\pi k)^d / 2} \int_{[-\pi/\epsilon, \pi/\epsilon]^d} e^{-|s - \varphi|^2/(4k)} e^{\pm i(\tau / 2) s \cdot H_\sigma(\theta) s} \mu(\theta, ds) \right) \, d\varphi.
\]

Secondly, we replace \( \omega_\sigma(\theta + \epsilon \varphi) \) by \( \omega_\sigma(\theta + \epsilon s) \) in the inner integral. To do this we divide the integration into two: \( |s - \varphi| \geq \epsilon \gamma \) and \( |s - \varphi| \leq \epsilon \gamma \) with a \( \gamma \), \( \gamma \in (1, \beta/2) \). For \( |s - \varphi| \geq \epsilon \gamma \),
\[
\frac{1}{(4\pi k)^d / 2} \int_{[-\pi/\epsilon, \pi/\epsilon]^d, |s - \varphi| > \epsilon \gamma} e^{-|s - \varphi|^2/(4k)} \, d\varphi \leq \frac{e^{-\epsilon \gamma / (4k)}}{(4\pi k)^d / 2} \left( \frac{\pi}{\epsilon} \right) \leq C \frac{e^{-\epsilon \gamma / (4k)}}{\epsilon^{\beta d / 2 + d}} \to 0 \quad (C.18)
\]
as \( \epsilon \to +0 \), since \( k = \epsilon^\beta \) and \( \gamma < \beta/2 \). Note that
\[
\frac{1}{(4\pi k)^d / 2} \int_{[-\pi/\epsilon, \pi/\epsilon]^d} e^{-|s - \varphi|^2/(4k)} \, d\varphi = 1 + o(1), \quad \epsilon \to 0,
\]
where \( k = \epsilon^\beta \). Therefore, for \( |s - \varphi| \leq \epsilon \gamma \), we estimate the difference
\[
\frac{1}{(4\pi k)^d / 2} \int_{[-\pi/\epsilon, \pi/\epsilon]^d} e^{-|s - \varphi|^2/(4k)} (e^{\pm i\omega_\sigma(\theta + \epsilon \varphi) \tau / \epsilon^2} - e^{\pm i\omega_\sigma(\theta + \epsilon s) \tau / \epsilon^2}) \, d\varphi \leq \frac{C}{(4\pi k)^d / 2} \int_{[-\pi/\epsilon, \pi/\epsilon]^d} e^{-|s - \varphi|^2/(4k)} \epsilon^\gamma + 1 \tau / \epsilon^2 \, d\varphi \leq C \epsilon^{\gamma - 1} (1 + o(1)), \quad (C.19)
\]
where the last expression vanishes as $\varepsilon \to 0$ since $\gamma > 1$. By (C.18) and (C.19), the inner integral in (C.17) is
\[
\frac{1}{(4\pi k)^{d/2}} \int_{[-\pi/\varepsilon,\pi/\varepsilon]^d,|s-\varphi|<\varepsilon} e^{-|s-\varphi|^2/(4k)} e^{\pm i\omega_{\tau}/(\varepsilon)} e^{i\varphi/\varepsilon^2} d\varphi + o(1)
\]

\[
= e^{\pm i\omega_{\tau}/(\varepsilon)} \frac{1}{(4\pi k)^{d/2}} \int_{[-\pi/\varepsilon,\pi/\varepsilon]^d,|s-\varphi|<\varepsilon} e^{-|s-\varphi|^2/(4k)} d\varphi + o(1)
\]

\[
= e^{\pm i\omega_{\tau}/(\varepsilon)} \frac{1}{(4\pi k)^{d/2}} \int_{[-\pi/\varepsilon,\pi/\varepsilon]^d} e^{-|s-\varphi|^2/(4k)} d\varphi + o(1)
\]

uniformly in $\theta \in \mathbb{T}^d$. We substitute the last expression in (C.17) and get
\[
N_{\varepsilon,\sigma}^+(\tau, r; \theta) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\omega_{\tau}/(\varepsilon)} e^{i\varphi/\varepsilon^2} \mu(\theta, ds) + o(1), \quad \varepsilon \to 0,
\]

uniformly in $\theta \in \mathbb{T}^d$. Finally, we apply the Taylor sum representation (6.23) to $\omega_{\tau}(\theta + \varepsilon s)$ and obtain (C.15)–(C.16). The integral in (C.16) can be taken over $|s| \leq N$ with some $N \in \mathbb{N}$, by condition (2.14).

Step (v): We substitute $I_{\varepsilon}(\theta)$ of the form (C.10) with $N_{\varepsilon,\sigma}^+(\tau, r; \theta)$ from (C.13) (or (C.15)) in the r.h.s. of (6.17). Applying the decomposition (6.22) to $\hat{G}_{\tau/\varepsilon^2}(\theta)\ast$, we obtain (6.26), with $B_{\varepsilon,\sigma}^+$ (or $C_{\varepsilon,\sigma}^+$, respectively) instead of $A_{\varepsilon,\sigma}^+$. It remains to study the behaviour (as $\varepsilon \to 0$) of integrals of the form (6.27) with $B_{\varepsilon,\sigma}^+$ (or $C_{\varepsilon,\sigma}^+$, resp.) instead of $A_{\varepsilon,\sigma}^+$.

Let $\omega_{\sigma}(\theta) \pm \omega_{\sigma}(\theta) \neq \text{const}_\pm$. In this case, the oscillatory integrals
\[
I_{\sigma,\sigma}^\pm(\varepsilon) \equiv (2\pi)^{-d} \frac{1}{2} \int_{\mathbb{T}^d} e^{-i(l-p)\theta} e^{i(\omega_{\sigma}(\theta)\pm \omega_{\sigma}(\theta))\tau/\varepsilon^2} h_{\sigma}^-(\theta) C_{\varepsilon,\sigma}^-(\tau, r; \theta) h_{\sigma}^+(\theta)^\ast \, d\theta
\]

vanish as $\varepsilon \to 0$, since $\sup_{\varepsilon > 0, \theta \in T^d} \left| C_{\varepsilon,\sigma}^-(\tau, r; \theta) \right| \leq C < \infty$, and hence, $h_{\sigma}^-(\theta) C_{\varepsilon,\sigma}^-(\tau, r; \theta) h_{\sigma}^+(\theta)^\ast \in L^1(\mathbb{T}^d)$. The identities $\omega_{\sigma}(\theta) \pm \omega_{\sigma}(\theta) \equiv \text{const}_\pm$ in the exponent of (C.20) with $\text{const}_\pm \neq 0$ are impossible by condition E5. The identity $\omega_{\sigma}(\theta) + \omega_{\sigma}(\theta) \equiv 0$ implies $\omega_{\sigma}(\theta) \equiv \omega_{\sigma}(\theta) \equiv 0$ which is impossible by E4. Therefore, if $\sigma \neq \sigma'$, $I_{\sigma,\sigma}^\pm(\varepsilon) = o(1)$ as $\varepsilon \to 0$. If $\sigma = \sigma'$, $I_{\sigma,\sigma}^\pm(\varepsilon) = o(1)$, and only $I_{\sigma,\sigma}^\pm(\varepsilon)$ contributes in the limit. Hence, (cf. (6.28))
\[
S_{\varepsilon,\tau/\varepsilon^2}^+ = (2\pi)^{-d} \frac{1}{2} \int_{\mathbb{T}^d} e^{-i(l-p)\theta} \sum_{\sigma,\pm} h_{\sigma}^+(\theta) B_{\varepsilon,\sigma}^+(\tau, r; \theta) h_{\sigma}^+(\theta)^\ast \, d\theta + o(1), \quad \varepsilon \to 0.
\]

Step (vi): To obtain (6.28), it remains to replace $B_{\varepsilon,\sigma}^+$ by $A_{\varepsilon,\sigma}^+$ in (C.21) (or $\hat{R}_{r,\varepsilon}$ by $\hat{R}_0$ in (C.14)). We check that the difference $B_{\varepsilon,\sigma}^--A_{\varepsilon,\sigma}^+$ vanishes as $\varepsilon \to 0$ (for the difference $B_{\varepsilon,\sigma}^+-A_{\varepsilon,\sigma}^+$ the proof is similar). Indeed, by (3.13), (C.1), (C.14), and (6.25),
\[
A_{\varepsilon,\sigma}^+-B_{\varepsilon,\sigma}^- = C(\theta) \int_{\mathbb{R}^d} \hat{R}_0(r - \nabla \omega_{\sigma}(\theta)\tau/\varepsilon - x, \theta) \left(1 - e^{-i\beta(\nabla \omega_{\sigma}(\theta)\tau/\varepsilon + x)}\right) e^{-i(2\pi)\kappa H_{\sigma}^-(\theta)^\ast} \, dx,
\]

37
with $C(\theta) = e^{i\pi s/(2\pi \tau)^{-d/2}} \det H_\sigma(\theta)^{-1/2}$. Write $c = 2|\tau| \max_{\theta \in T^d,a} |\nabla \omega_\sigma(\theta)|$. We divide the integration into two: $|x + \nabla \omega_\sigma(\theta)\tau/\varepsilon| \leq c/\varepsilon$ and $|x + \nabla \omega_\sigma(\theta)\tau/\varepsilon| \geq c/\varepsilon$. For $|x + \nabla \omega_\sigma(\theta)\tau/\varepsilon| \leq c/\varepsilon$,

$$\left| \int \frac{\hat{R}_0(r - \nabla \omega_\sigma(\theta)\tau/\varepsilon - x, \theta)(1 - e^{-e \beta(\nabla \omega_\sigma(\theta)\tau/\varepsilon + x)^2})e^{-i/(2\tau)x}H_\sigma^{-1}(\theta)x}{|x + \nabla \omega_\sigma(\theta)\tau/\varepsilon|} dx \right| \leq C \int |e^{\beta |x + \nabla \omega_\sigma(\theta)\tau/\varepsilon|^2} dx \leq C_1e^{\beta - 2d} \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

since $\beta > d + 2$. For $|x + \nabla \omega_\sigma(\theta)\tau/\varepsilon| \geq c/\varepsilon$, we apply the integration by parts in every variable $x_i$, $i = 1, \ldots, d$. For simplicity, let $d = 1$ and $h$ stand for $-1/(2\tau)H_\sigma^{-1}(\theta)$. Therefore,

$$\int_{c/\varepsilon - \omega'_\sigma(\theta)\tau/\varepsilon}^{+\infty} \hat{R}_0(r - \omega'_\sigma(\theta)\tau/\varepsilon - x, \theta)(1 - e^{-\omega'_\sigma(\theta)\tau/\varepsilon + x}x)e^{ihx^2} dx$$

$$= \frac{1}{2ih} \left[ -\hat{R}_0(r - c/\varepsilon, \theta)(1 - e^{-c/\varepsilon})e^{ihx^2} x \right]_{x = (c - \omega'_\sigma(\theta)r)/\varepsilon}^{+\infty} + \int_{(c - \omega'_\sigma(\theta)r)/\varepsilon}^{+\infty} \partial_r \hat{R}_0(r - \omega'_\sigma(\theta)\tau/\varepsilon - x, \theta)(1 - e^{-\omega'_\sigma(\theta)\tau/\varepsilon + x}x)e^{ihx^2} x dx$$

$$-\varepsilon^2 \int_{(c - \omega'_\sigma(\theta)r)/\varepsilon}^{+\infty} \hat{R}_0(r - \omega'_\sigma(\theta)\tau/\varepsilon - x, \theta)(\omega'_\sigma(\theta)\tau/\varepsilon + x)e^{-\omega'_\sigma(\theta)\tau/\varepsilon + x}e^{ihx^2} x^2 dx$$

$$+ \int_{(c - \omega'_\sigma(\theta)r)/\varepsilon}^{+\infty} \hat{R}_0(r - \omega'_\sigma(\theta)\tau/\varepsilon - x, \theta)e^{-\omega'_\sigma(\theta)\tau/\varepsilon + x}e^{ihx^2} x^2 dx$$

$$= I_1 + I_2 + I_3 + I_4.$$

The terms $I_1$ and $I_4$ tend to zero as $\varepsilon \rightarrow 0$, since $|\hat{R}_0| \leq C < \infty$ by condition I1. Note that $|x| \geq c/(2\varepsilon)$ if $|x + \omega'_\sigma(\theta)\tau/\varepsilon| \geq c/\varepsilon$. Hence,

$$|I_3| \leq C_1\varepsilon^\beta \int_{(c - \omega'_\sigma(\theta)r)/\varepsilon}^{+\infty} \frac{(\omega'_\sigma(\theta)\tau/\varepsilon + x)e^{-\omega'_\sigma(\theta)\tau/\varepsilon + x}e^{ihx^2} x}{|x|} dx$$

$$\leq C_2\varepsilon^{\beta + 1} \int_c^{+\infty} ye^{-c^2} dy = C_3 \varepsilon e^{-\beta(c/\varepsilon)^2} \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

To prove that the contribution of the integral $I_2$ vanishes, we repeat the integration by parts and use the bound $\sup_{r \in \mathbb{R}^d, \theta \in T^d} |\partial_r^k \hat{R}_0(r, \theta)| \leq C < \infty$, where $k \in [0, d^2/2 + 2d]$. This bound follows from condition (2.13)–(2.14).
Appendix D: Local conservation law

Let \( v(x, t) \) be a solution of (2.1) with the finite energy. The local energy in the point \( x \in \mathbb{Z}^d \) is defined as

\[
E(x, t) := \frac{1}{2} \left\{ |\dot{v}(x, t)|^2 + \sum_{y \in \mathbb{Z}^d} v(x, t) \cdot V(x - y)v(y, t) \right\}.
\]

We derive formally the expression for the energy current of the finite energy solutions \( v(x, t) \). For the half-space \( \Omega_k := \{ x \in \mathbb{Z}^d : x = (x_1, \ldots, x_d), x_k \leq 0 \}, \ k = 1, \ldots, d \), we define the energy in the region \( \Omega_k \) as

\[
E_{\Omega_k}(t) := \frac{1}{2} \sum_{x \in \Omega_k} \left\{ |\dot{v}(x, t)|^2 + \sum_{y \in \mathbb{Z}^d} v(x, t) \cdot V(x - y)v(y, t) \right\}.
\]

By formal calculation, using Eqn (2.1) we obtain \( \dot{E}_{\Omega_k}(t) = -\sum_{x' \in \mathbb{Z}^d, x'_k = 0} j_k(x', t) \), where \( j_k(x', t) \) stands for the energy current density in the direction \( e_k = (\delta_{1k}, \ldots, \delta_{dk}), \ k = 1, \ldots, d, \ x' \in \mathbb{Z}^d \) with \( x'_k = 0 \),

\[
j_k(x', t) := -\frac{1}{2} \sum_{y' \in \mathbb{Z}^d : y'_k = 0} \left\{ \sum_{m \geq 1, p \leq 0} \dot{v}(x' + me_k, t) \cdot V(x' + me_k - y' - pe_k)v(y' + pe_k, t) \right. \\
- \sum_{m \leq 0, p \geq 1} \dot{v}(x' + me_k, t) \cdot V(x' + me_k - y' - pe_k)v(y' + pe_k, t) \right\}.
\]

(D.1)

Now let \( v(x, t) \) be the random solution to (2.1) with the initial measure \( \mu_0^\varepsilon \) satisfying V1 and V2. Therefore, for any \( x \in \mathbb{Z}^d \), \( \tau \in \mathbb{R} \setminus 0 \), and \( r \in \mathbb{R}^d \), the average energy is

\[
\mathbb{E}_0[^E(x + [r/\varepsilon], \tau/\varepsilon)] = \mathbb{E}[^E(x + [r/\varepsilon], 0)] \\
= \frac{1}{2} \text{tr} \left[ Q_{\varepsilon, \tau/\varepsilon}^{11}(r/\varepsilon + x, [r/\varepsilon] + x) + \sum_{y \in \mathbb{Z}^d} Q_{\varepsilon, \tau/\varepsilon}^{00}(r/\varepsilon + x, [r/\varepsilon] + y) \cdot V^T(x - y) \right],
\]

by condition E1 and the uniform bound for the correlation functions of the measure \( \mu_{\tau/\varepsilon}^\varepsilon \) (see [14 Lemma 5.1] or Lemma [7.1]):

\[
\sup_{\varepsilon > 0} \sup_{i,j=0,1} \sup_{z, z' \in \mathbb{Z}^d} \|Q_{\varepsilon, \tau/\varepsilon}^{ij}(z, z')\| \leq C < \infty.
\]

(D.2)

It follows from Theorem 3.2 that for any \( \tau \neq 0, r \in \mathbb{R}^d \), \( \mathbb{E}_0[^E(x + [r/\varepsilon], \tau/\varepsilon)] \to e(\tau, r) \) as \( \varepsilon \to 0 \), where

\[
e(\tau, r) = \frac{1}{2} \text{tr} \left[ q_{\tau, \tau}^{11}(0) + \sum_{x \in \mathbb{Z}^d} q_{\tau, \tau}^{00}(x) V^T(x) \right] \\
= \frac{1}{2}(2\pi)^{-d} \text{tr} \int_{\mathbb{T}^d} \left( \hat{q}_{\tau, \tau}^{11}(\theta) + \hat{q}_{\tau, \tau}^{00}(\theta) \hat{V}^*(\theta) \right) d\theta = (2\pi)^{-d} \text{tr} \int_{\mathbb{T}^d} \hat{q}_{\tau, \tau}^{11}(\theta) d\theta.
\]

(D.3)

The last equality follows from condition E2 and Remarks 3.4 (i).
Similarly, by (D.1) we obtain
\[
\mathbb{E}_0^\varepsilon[j_k(x',t)] = \frac{1}{2} \sum_{y' \in \mathbb{Z}^d; y_k' = 0} \left( \sum_{m \leq -1, p \geq 0} \text{tr} \left[ Q_{x,t}^{10}(x' + me_k, y' + pe_k) \cdot V^T(x' - y' + (m-p)e_k) \right] 
- \sum_{m \geq 1, p \leq 0} \text{tr} \left[ Q_{x,t}^{10}(x' + me_k, y' + pe_k) \cdot V^T(x' - y' + (m-p)e_k) \right] \right).
\]
Therefore, by Theorem 3.2 the following limit exists, \( \lim_{\varepsilon \to 0} \mathbb{E}_0^\varepsilon[j_k(x + [r/\varepsilon], \tau/\varepsilon)] = j_k(\tau, r) \) for any \( \tau \neq 0, r \in \mathbb{R}^d, k = 1, \ldots, d \). Here
\[
j_k(\tau, r) = \frac{1}{2} \sum_{y' \in \mathbb{Z}^d; y_k' = 0} \left( \sum_{m \leq -1, p \geq 0} \text{tr} \left[ q_{10}^{10}(x' - y' + (m-p)e_k)V^T(x' - y' + (m-p)e_k) \right] 
- \sum_{m \geq 1, p \leq 0} \text{tr} \left[ q_{10}^{10}(x' - y' + (m-p)e_k)V^T(x' - y' + (m-p)e_k) \right] \right).
\]
Write \( x' - y' =: z' \in \mathbb{Z}^d \) with \( z_k' = 0 \), \( m - p := s \in \mathbb{Z}^1 \) and change the order of the summation in the series. Therefore,
\[
j_k(\tau, r) = -\frac{1}{2} \sum_{z' \in \mathbb{Z}^d; z_k' = 0} \sum_{s \in \mathbb{Z}^1} \text{tr}[q_{10}^{10}(z' + se_k)V^T(z' + se_k)]s = -\frac{1}{2} \sum_{z \in \mathbb{Z}^d} \text{tr}[q_{10}^{10}(z)z_kV^T(z)] - \frac{i}{2}(2\pi)^{-d} \int_{\mathbb{T}^d} \hat{q}_{\tau,r}^{10}(\theta)\partial_k \hat{V}(\theta) d\theta, \quad k = 1, \ldots, d. \tag{D.4}
\]
Write \( j(\tau, r) = (j_1(\tau, r), \ldots, j_d(\tau, r)) \). Finally, (D.3), (D.4) and Corollary 3.3 yield
\[
\partial_\tau e(\tau, r) + \nabla_r \cdot j(\tau, r) = 0, \quad \tau \in \mathbb{R}, \quad r \in \mathbb{R}^d.
\]
Remark. In Section 2.3, we give the example of the "local equilibrium" initial measures \( \mu_0^\varepsilon \). Namely, let \( \hat{q}_{ij}^\varepsilon(z) \) be the correlation functions of the Gibbs measure \( g \) (see Definition 2.5) with \( \beta = 1 \), i.e., \( \hat{q}_{00}^0(\theta) = \hat{V}^{-1}(\theta) \), \( \hat{q}_{11}^0(\theta) = I \), \( \hat{q}_{01}^0(\theta) = \hat{q}_{10}^0(\theta) = 0 \). Put \( \hat{R}_0(r, \theta) = T(r)q_0(\theta) \), where the function \( T(r), r \in \mathbb{R}^d \), is defined in Section 2.3. Moreover, let \( \mu_0^\varepsilon, \varepsilon > 0 \), be Gaussian measures with the correlation functions \( Q_{i}^\varepsilon(z, z') \) defined in (2.17). Then conditions V1 and V2 hold. In this case, the limit correlation matrices \( \hat{q}_{\tau,r}^{ij}(\theta) \) have a form
\[
\hat{q}_{11}^{\tau}(\theta) = \hat{V}(\theta)\hat{q}_{00}^{\tau}(\theta) = \sum_{\sigma = 1}^s T^\sigma_+(\tau, r; \theta)\Pi_\sigma(\theta),
\hat{q}_{11}^{\tau}(\theta) = \hat{V}(\theta)\hat{q}_{00}^{\tau}(\theta) = \sum_{\sigma = 1}^s T^\sigma_-(\tau, r; \theta)\Pi_\sigma(\theta),
\]
where, by definition, \( T^\sigma_+(\tau, r; \theta) := (1/2)(T(r + \nabla \omega_\sigma(\theta)\tau) \pm T(r - \nabla \omega_\sigma(\theta)\tau)) \). Therefore,
\[
e(\tau, r) = (2\pi)^{-d} \sum_{\sigma = 1}^s \int_{\mathbb{T}^d} T^\sigma_+(\tau, r; \theta) \text{tr} \Pi_\sigma(\theta) d\theta,
\hat{j}(\tau, r) = -(2\pi)^{-d} \sum_{\sigma = 1}^s \int_{\mathbb{T}^d} T^\sigma_-(\tau, r; \theta) \nabla \omega_\sigma(\theta) \text{tr} \Pi_\sigma(\theta) d\theta.
\]
In [6], the "locally conserved" quantities were studied in the case when \( d = 1 \). Let us consider these quantities in many-dimensional case. At first, introduce the following matrix-valued functions 

\[
E(z + h, z; X_0) := \frac{1}{2} \left( v_1(z + h) \otimes v_1(z) + v_0(z + h) \otimes \sum_{z' \in \mathbb{Z}^d} V(z - z') v_0(z') \right),
\]

\[
A(z + h, z; X_0) := \frac{1}{2} \left( v_0(z + h) \otimes v_1(z) - v_1(z + h) \otimes v_0(z) \right), \quad z, h \in \mathbb{Z}^d, \quad X_0 = (v_0, v_1).
\]

The "locally conserved" quantities \( X_h^\varepsilon(\varphi, X_0), Y_h^\varepsilon(\varphi, X_0) \) are defined as follows. For \( \varphi \in C_0^1(\mathbb{R}^d) \), we set 

\[
X_h^\varepsilon(\varphi, X_0) := \varepsilon^d \sum_{z \in \mathbb{Z}^d} \varphi(\varepsilon z) E(z + h, z; X_0), \quad Y_h^\varepsilon(\varphi, X_0) := \varepsilon^d \sum_{z \in \mathbb{Z}^d} \varphi(\varepsilon z) A(z + h, z; X_0), \quad h \in \mathbb{Z}^d, \quad \varepsilon > 0.
\]

**Theorem D.1** Let conditions I1–I4, V1, V2 hold. Then for any \( \varphi \in C_0^1(\mathbb{R}^d) \), \( h \in \mathbb{Z}^d \), \( \tau \neq 0 \), there exist the limits 

\[
\lim_{\varepsilon \to 0} \mathbb{E}_{\varepsilon/\tau}^\varepsilon [X_h^\varepsilon(\varphi, \cdot)] = E(\varphi; \tau, h), \quad \lim_{\varepsilon \to 0} \mathbb{E}_{\varepsilon/\tau}^\varepsilon [Y_h^\varepsilon(\varphi, \cdot)] = A(\varphi; \tau, h).
\]

The matrices \( E(\varphi; \tau, h) \) and \( A(\varphi; \tau, h) \) are given by their Fourier transform in the following way. Write 

\[
\hat{E}(\varphi; \tau, \theta) := \sum_{h \in \mathbb{Z}^d} e^{ih \cdot \theta} E(\varphi; \tau, h), \quad \hat{A}(\varphi; \tau, \theta) := \sum_{h \in \mathbb{Z}^d} e^{ih \cdot \theta} A(\varphi; \tau, h), \quad \theta \in \mathbb{T}^d.
\]

Then

\[
\hat{E}(\varphi; \tau, \theta) = \frac{1}{2} \int_{\mathbb{R}^d} \varphi(r) \left( \hat{q}_{t, r, r}^{11}(\theta) + \hat{q}_{t, r, r}^{00}(\theta) \hat{V}(\theta) \right) dr = \int_{\mathbb{R}^d} \varphi(r) \hat{q}_{t, r, r}^{11}(\theta) \hat{V}(\theta) dr, \quad (D.5)
\]

\[
\hat{A}(\varphi; \tau, \theta) = \frac{1}{2} \int_{\mathbb{R}^d} \varphi(r) \left( \hat{q}_{t, r, r}^{01}(\theta) - \hat{q}_{t, r, r}^{10}(\theta) \right) dr = \int_{\mathbb{R}^d} \varphi(r) \hat{q}_{t, r, r}^{01}(\theta) \hat{V}(\theta) dr, \quad (D.6)
\]

where \( \hat{q}_{t, r, r}^{ij}(\theta) \) are defined in (3.7).

The proof of Theorem D.1 is based on Theorem 3.2 and the bound (D.2).

**Remark** It follows from (D.5), (D.6) and Corollary 3.3 that \( \hat{E} \) and \( \hat{A} \) satisfy the equations

\[
\partial_r \hat{E}(\varphi; \tau, \theta) = i\omega(\theta) \int_{\mathbb{R}^d} \nabla \omega(\theta) \cdot \nabla \varphi(r) \hat{q}_{t, r, r}^{01}(\theta) dr = i\omega(\theta) \nabla \omega(\theta) \cdot \hat{A}(\nabla \varphi; \tau, \theta),
\]

\[
\partial_r \hat{A}(\varphi; \tau, \theta) = -i\omega^{-1}(\theta) \int_{\mathbb{R}^d} \nabla \omega(\theta) \cdot \nabla \varphi(r) \hat{q}_{t, r, r}^{11}(\theta) dr = -i\omega^{-1}(\theta) \nabla \omega(\theta) \cdot \hat{E}(\nabla \varphi; \tau, \theta).
\]
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