GROUPOIDS AND POISSON SIGMA MODELS WITH BOUNDARY

IVAN CONTRERAS AND ALBERTO S. CATTANEO

Abstract. This note gives an overview on the construction of symplectic groupoids as reduced phase spaces of Poisson sigma models and its generalization in the infinite dimensional setting (before reduction).

1. Introduction

In [2], it was proven that the reduced phase space of the Poisson sigma model under certain boundary conditions and assuming it is a smooth manifold, has the structure of a symplectic groupoid and it integrates the cotangent bundle of a given Poisson manifold $M$. This is a particular instance of the problem of integration of Lie algebroids, a generalized version of the Lie third theorem [1]. The general question can be stated as

- Is there a Lie groupoid $(G, M)$ such that its infinitesimal version corresponds to a given Lie algebroid $(A, M)$?

For the case where $A = T^*M$ and $M$ is a Poisson manifold the answer is not positive in general, as there are topological obstructions encoded in what they are called the monodromy groups [12]. A Poisson manifold is called integrable if such Lie groupoid $G$ exists. The properties of $G$ are of special interest in Poisson geometry, since it is possible to equip $G$ with a symplectic structure $\omega$ compatible with the multiplication map in such a way that $G$ is a symplectic realization for $(M, \Pi)$.

For the integrable case, the symplectic groupoid integrating a given Poisson manifold $(M, \Pi)$ is constructed explicitly in [2], as the phase space modulo gauge equivalence of the Poisson Sigma model (PSM), a 2-dimensional field theory.

In a more recent perspective (see [8, 9]), the study of the phase space before reduction plays a crucial role. This allows dealing with with nonintegrable Poisson structure, for which the reduced phase space is singular, on an equal footing as the integrable ones. This new approach differs from the stacky perspective of Zhu and Tseng (see [10]) and seems to be better adapted to symplectic geometry and to quantization.

In order to include this construction in the more general setting, where the reduced phase space can be singular and more general source spaces are allowed, the study of the phase space before reduction plays a crucial role.

In a more recent perspective, in order to include this construction in the more general setting, where the reduced phase space can be singular and more general
source spaces are allowed, the study of the phase space before reduction plays a crucial role.

In a paper in preparation [5], we introduce a more general version of a symplectic groupoid, called \textit{relational symplectic groupoid}. In the case at hand, it corresponds to an infinite dimensional weakly symplectic manifold equipped with structure morphisms (canonical relations, i.e. immersed Lagrangian submanifolds) compatible with the Poisson structure of $M$. In this work, we prove that

1. For any Poisson manifold $M$ (integrable or not), the relational symplectic groupoid always exists.
2. In the integrable case, the associated relational symplectic groupoid is equipped with locally embedded Lagrangian submanifold.
3. Conjecturally, given a regular relational symplectic groupoid $G$ over $M$ (a particular type of object that admits symplectic reduction), there exists a unique Poisson structure $\Pi$ on $M$ such that the symplectic structure $\omega$ on $G$ and $\Pi$ are compatible. This is still work in progress.

This paper is an overview of this construction and is organized as follows. Section 2 is a brief introduction to the Poisson sigma model and its reduced phase space. Section 3 deals with the version before reduction of the phase space and the introduction of the relational symplectic groupoid. An interesting issue concerning this construction is the treatment of non integrable Poisson manifolds: even if the reduction does not exists as a smooth manifold, the relational symplectic groupoid always exists. One natural question arising at this point is:

- Can there be a finite dimensional relational symplectic groupoid equivalent to the infinite dimensional one for an arbitrary Poisson manifold?

The answer to this question is work in progress and it will be treated in a subsequent paper.

Another aspect, which will be explored later, is the connection between the relational construction and the Poisson Sigma model with branes, where the boundary conditions are understood as choices of coisotropic submanifolds of the Poisson manifold. The relational symplectic groupoid seems to admit the existence of branes and would explain in full generality the idea of dual pairs in the Poisson sigma model with boundary [3, 4].

This new program might be useful for quantization as well. Using ideas from geometric quantization, what is expected as the quantization of the relational symplectic groupoid is an algebra with a special element, which fails to be a unit, but whose action is a projector in such a way that on the image of the projector we obtain a true unital algebra. Deformation quantization of a Poisson manifold could be interpreted in this way.

2. PSM and its reduced phase space.

We consider the following data

1. A compact surface $\Sigma$, possibly with boundary, called the source space.
2. A finite dimensional Poisson manifold $(M, \Pi)$, called the target space. Recall that a bivector field $\Pi \in \Gamma(TM \wedge TM)$ is called Poisson if the the bracket $\{,\} : \mathcal{C}^\infty(M) \otimes \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, defined by

\[ \{f, g\} = \Pi(df, dg) \]
GROUPOIDS AND POISSON SIGMA MODELS WITH BOUNDARY

is a Lie bracket and it satisfies the Leibniz identity
\[ \{ f, gh \} = g\{ f, h \} + h\{ f, g \}, \forall f, g, h \in \mathcal{C}^\infty(M). \]

In local coordinates, the condition of a bivector \( \Pi \) to be Poisson reads as follows
\[(1) \quad \Pi^{sr}(x)(\partial_r)\Pi^{lk}(x) + \Pi^{kr}(x)(\partial_r)\Pi^{sl}(x) + \Pi^{lr}(x)(\partial_r)\Pi^{ks}(x) = 0,\]
that is, the vanishing condition for the Schouten-Nijenhuis bracket of \( \Pi \).

The space of fields for this theory is denoted with \( \Phi \) and corresponds to the space of vector bundle morphisms between \( T\Sigma \) and \( T^*M \). This space can be parametrized by the pair \((X, \eta)\), where \( X \) is a \( C^{k+1} \)-map from \( \Sigma \) to \( M \) and \( \eta \in \Gamma^k(\Sigma, T^*\Sigma \otimes X^*T^*M) \).

On \( \Phi \), the following first order action is defined:
\[ S(X, \eta) := \int_\Sigma \eta \wedge dX + \frac{1}{2} \Pi^#(X)\eta \wedge \eta, \]
where \( \Pi^# \) is the map from \( T^*M \rightarrow TM \) induced from the Poisson bivector \( \Pi \), the integrand, called the Lagrangian, will be denoted by \( L \). Associated to this action, the corresponding variational problem \( \delta S = 0 \) induces the following space
\[ EL = \{ \text{Solutions of the Euler-Lagrange equations} \} \subset \Phi, \]
where, using integration by parts
\[ \delta S = \int_\Sigma \frac{\delta L}{\delta X} \delta X + \frac{\delta L}{\delta \eta} \delta \eta + \text{boundary terms}. \]

The partial variations correspond to:
\[(2) \quad \frac{\delta L}{\delta X} = dX + \Pi^#(X)\eta = 0 \]
\[(3) \quad \frac{\delta L}{\delta \eta} = d\eta + \frac{1}{2} \partial\Pi^#(X)\eta \wedge \eta = 0. \]

Now, if we restrict to the boundary, the general space of boundary fields corresponds to
\[ \Phi_\partial := \{ \text{vector bundle morphisms between } T(\partial \Sigma) \text{ and } T^*M \}. \]

Following [11], \( \Phi_\partial \) is endowed with a symplectic form and a surjective submersion \( p: \Phi \rightarrow \Phi_\partial \). We define
\[ L_\Sigma := p(EL) \]

Finally, we define \( C_\Pi \) as the set of fields in \( \Phi_\partial \) which can be completed to a field in \( L_{\Sigma'} \), with \( \Sigma' := \partial \Sigma \times [0, \varepsilon] \), for some \( \varepsilon \).
It turns out that \( \Phi_\partial \) can be identified with \( T^*(PM) \), the cotangent bundle of the path space on \( M \) and that
\[ C_\Pi := \{(X, \eta) | dX = \pi^#(X)\eta, X : \partial \Sigma \rightarrow M, \eta \in \Gamma(T^*I \otimes X^*(T^*M))\}. \]

Furthermore the following proposition holds

**Proposition 1.** [2]. The space \( C_\Pi \) is a coisotropic submanifold of \( \Phi_\partial \).

In fact, the converse of this proposition also holds in the following sense. If we define \( S(X, \eta) \) and \( C_\Pi \) in the same way as before, without assuming that \( \Pi \) satisfies Equation (1) it can be proven that
Proposition 2. If $C_\Pi$ is a coisotropic submanifold of $\Phi_\partial$, then $\Pi$ is a Poisson bivector field.

The geometric interpretation of the Poisson sigma model will lead us to the connection between Lie algebroids and Lie groupoids in Poisson geometry. First we need some definitions.

A pair $(A, \rho)$, where $A$ is a vector bundle over $M$ and $\rho$ (called the anchor map) is a vector bundle morphism from $A$ to $TM$ is called a Lie algebroid if

1. There is Lie bracket $[,]_A$ on $\Gamma(A)$ such that the induced map $\rho^*: \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism.
2. Leibniz identity:

$$[X, fY]_A = f[X, Y] + \rho_\ast(X)(f)Y, \forall X, Y \in \Gamma(A), f \in C^\infty(M).$$

Lie algebras, Lie algebra bundles and tangent bundles appear as natural examples of Lie algebroids. For our purpose, the cotangent bundle of a Poisson manifold $T^*M$, where $[,]_{T^*M}$ is the Koszul bracket for 1-forms, that is defined for exact forms by

$$[df, dg] := d\{f, g\}, \forall f, g \in C^\infty(M),$$

whereas for general forms it is recovered by Leibnitz and the anchor map given by $\Pi^\#: T^*M \to TM$, is a central example of Lie algebroids. To define a morphism of Lie algebroids we consider the complex $\Lambda^\bullet A^*$, where $A^*$ is the dual bundle and a differential $\delta_A$ is defined by the rules

1. $\delta_A f := \rho^*df, \forall f \in C^\infty(M)$.
2. $\langle \delta_A \alpha, X \wedge Y \rangle := -\langle \alpha, [X, Y]_A \rangle + \langle \delta(\alpha, X), Y \rangle - \langle \delta(\alpha, Y), X \rangle, \forall X, Y \in \Gamma(A), \alpha \in \Gamma(A^*),$

where $\langle , \rangle$ is the natural pairing between $\Gamma(A)$ and $\Gamma(A^*)$.

A vector bundle morphism $\varphi: A \to B$ is a Lie algebroid morphism if $\delta_A \varphi^* = \varphi^* \delta_B$.

This condition written down in local coordinates gives rise to some PDE’s the anchor maps and the structure functions for $\gamma(A)$ and $\Gamma(B)$ should satisfy. In particular, for the case of Poisson manifolds, $C_\Pi$ corresponds to the space of Lie algebroid morphisms between $T(\partial \Sigma)$ and $T^*M$ where the Lie algebroid structure on the left is given by the Lie bracket of vector fields on $T(\partial \Sigma)$ with identity anchor map and on the right is the one induced by the Poisson structure on $M$.

As it was mentioned before, it can be proven that this space, is a coisotropic submanifold of $T^*PM$. Its symplectic reduction, i.e. the space of leaves of its characteristic foliation, called the reduced phase space of the PSM, when is smooth, has a particular feature, it is a symplectic groupoid over $M$ [2]. More precisely, a groupoid is a small category with invertible morphisms. When the spaces of objects and morphisms are smooth manifolds, a Lie groupoid over $M$, denoted by $G\rightrightarrows M$, can be rephrased as the following data\footnote{$G \times_{(s,t)} G$ is a smooth manifold whenever $s$ (or $t$) is a surjective submersion.}:

$$G \times_{(s,t)} G \xrightarrow{\mu} G \xrightarrow{t} G \xrightarrow{s} M$$

where $s$, $t$, $\iota$, $\mu$ and $\varepsilon$ denote the source, target, inverse, multiplication and unit
map respectively, such that the following axioms hold (denoting \(G_{(x,y)} \coloneqq s^{-1}(x) \cap t^{-1}(y)\)):

(A.1) \(s \circ \varepsilon = t \circ \varepsilon = \text{id}_M\)

(A.2) If \(g \in G_{(x,y)}\) and \(h \in G_{(y,z)}\) then \(\mu(g,h) \in G_{(x,z)}\)

(A.3) \(\mu(\varepsilon \circ s \times \text{id}_G) = \mu(\text{id}_G \times \varepsilon \circ t) = \text{id}_G\)

(A.4) \(\mu(\text{id}_G \times i) = \varepsilon \circ t\)

(A.5) \(\mu(i \times \text{id}_G) = \varepsilon \circ s\)

(A.6) \(\mu(\mu \times \text{id}_G) = \mu(\text{id}_G \times \mu)\).

A Lie groupoid is called \textit{symplectic} if there exists a symplectic structure \(\omega\) on \(G\) such that \(Gr_\mu := \{(a, b, c) \in G^3 | c = \mu(a, b)\}\) is a lagrangian submanifold of \(G \times G \times G\), where \(\overline{G}\) denotes the sign reversed symplectic structure on \(G\). Finally, we can state the following

\[\textbf{Theorem 1.} [2]. \text{The symplectic reduction } C_{\Pi} \text{ of } C_{\Pi} \text{ (the space of leaves of the characteristic foliation), if it is smooth, is a symplectic groupoid over } M.\]

The smoothness of the reduced phase space has particular interest. In [12], the necessary and sufficient conditions for integrability of Lie algebroids, i.e. whether a Lie groupoid such that its \textit{infinitesimal version} corresponds to a given Lie algebroid exists, are stated. In [13], these conditions have been further specialized to the Poisson case. It turns out that the reduced phase space of the PSM coincides with the space of equivalent classes of what are called \(\mathcal{A}-\)paths modulo \(\mathcal{A}-\)homotopy [12], with \(\mathcal{A} = T^*M\).

3. The version before reduction.

The main motivation for introducing the relational groupoid construction is the following. In general, the leaf space of a characteristic foliation is not a smooth finite dimensional manifold and in this particular situation, the smoothness of the space of reduced boundary fields is controlled by the integrability conditions stated in [12]. In this paper, we define a groupoid object in the extended symplectic category, where the objects are symplectic manifolds, possibly infinite dimensional, and the morphisms are immersed Lagrangian submanifolds. It is important to remark here that this is extended category is not properly a category! (The composition of morphisms is not smooth in general). However, for our construction, the corresponding morphisms will be composable.

We restrict ourselves to the case when \(C\) is the sometimes called \textit{Extended Symplectic Category}, denoted by \(\text{Sym}^{\text{Ext}}\) and defined as follows:

\[\textbf{Definition 1.} \text{Sym}^{\text{Ext}} \text{ is a category in which the objects are symplectic manifolds and the morphisms are immersed canonical relations.} \]

Recall that \(L : \mathcal{M} \rightarrow \mathcal{N}\) is an immersed canonical relation between two symplectic manifolds \(\mathcal{M}\) and \(\mathcal{N}\) by definition if \(L\) is an immersed Lagrangian submanifold of \(\mathcal{M} \times \mathcal{N}\). \(\text{Sym}^{\text{Ext}}\) carries an involution \(\dagger : (\text{Sym}^{\text{Ext}})^{\text{op}} \rightarrow \text{Sym}^{\text{Ext}}\) that is the identity in objects and in morphisms, for \(f : A \rightarrow B\), \(f^\dagger := \{(b, a) \in B \times A | (a, b) \in f\}\).
This category extends the usual symplectic category in the sense that the symplectomorphisms can be thought in terms of canonical relations.

**Definition 2.** A relational symplectic groupoid is a triple $(\mathcal{G}, L, I)$ where

- $\mathcal{G}$ is a symplectic manifold (possibly infinite dimensional, in this case is a weak symplectic manifold).
- $L$ is an immersed Lagrangian submaifold of $\mathcal{G}^3$.
- $I$ is an antisymplectomorphism of $\mathcal{G}$

satisfying the following axioms

- **A.1** $L$ is cyclically symmetric (i.e. if $(x, y, z) \in L$, then $(y, z, x) \in L$)
- **A.2** $I$ is an involution (i.e. $I^2 = Id$).

**Notation** $L$ is a canonical relation $\mathcal{G} \times \mathcal{G} \not\rightarrow \bar{\mathcal{G}}$ and will be denoted by $L_{rel}$.

Since the graph of $I$ is a Lagrangian submanifold of $\mathcal{G} \times \mathcal{G}$, $I$ is a canonical relation $\mathcal{G} \not\rightarrow \mathcal{G}$ and will be denoted by $I_{rel}$.

$L$ and $I$ can be regarded as well as canonical relations $\bar{\mathcal{G}} \times \bar{\mathcal{G}} \not\rightarrow \mathcal{G}$ and $\mathcal{G} \not\rightarrow \bar{\mathcal{G}}$ respectively and will be denoted by $\bar{L}_{rel}$ and $\bar{I}_{rel}$.

The transposition $T : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$

\[(x, y) \mapsto (y, x)\]

induces canonical relations

$T_{rel} : \mathcal{G} \times \mathcal{G} \not\rightarrow \mathcal{G} \times \mathcal{G}$

and $\bar{T}_{rel} : \bar{\mathcal{G}} \times \bar{\mathcal{G}} \not\rightarrow \bar{\mathcal{G}} \times \bar{\mathcal{G}}$.

The identity map $Id : \mathcal{G} \rightarrow \mathcal{G}$ as a relation will be denoted by $Id_{rel} : \mathcal{G} \not\rightarrow \mathcal{G}$ and by $\bar{Id}_{rel} : \bar{\mathcal{G}} \not\rightarrow \bar{\mathcal{G}}$.

- **A.3** $I_{rel} \circ L_{rel} = L_{rel} \circ T_{rel} \circ (I_{rel} \times I_{rel}) : \mathcal{G} \not\rightarrow \mathcal{G}$.

**Remark 1:** Since $I$ and $T$ are diffeomorphisms, both sides of the equality correspond to immersed Lagrangian submanifolds.

Define

$L_3 := I_{rel} \circ L_{rel} : \mathcal{G} \times \mathcal{G} \not\rightarrow \mathcal{G}$.

As a corollary of the previous axioms we get that

**Corollary 1.** $\bar{T}_{rel} \circ L_3 = L_3 \circ \bar{T}_{rel} \circ (\bar{L}_{rel} \times \bar{L}_{rel})$.

- **A.4** $L_3 \circ (L_3 \times Id) = L_3 \circ (Id \times L_3) : \mathcal{G}^3 \not\rightarrow \mathcal{G}$ is an immersed Lagrangian submanifold.

The fact that the composition is Lagrangian follows from the fact that, since $I$ is an antisymplectomorphism, its graph is Lagrangian, therefore $L_3$ is Lagrangian, and so $(Id \times L_3)$ and $(L_3 \times Id)$. The graph of the map $I$, as a relation $* \not\rightarrow \mathcal{G} \times \mathcal{G}$ will be denoted by $L_I$.

- **A.5** $L_3 \circ L_I$ is an immersed Lagrangian submanifold of $\mathcal{G}$.

**Remark 2:** It can be proven that Lagrangianity in these cases is automatic if we start with a finite dimensional symplectic manifold $\mathcal{G}$.

Let $L_1 := L_3 \circ L_I : * \not\rightarrow \mathcal{G}$. From the definitions above we get the following

\[4\text{it means that the induced map } T\mathcal{G} \rightarrow T^*\mathcal{G} \text{ is injective.}\]
Corollary 2. \( \overline{I_{\text{rel}}} \circ L_1 = L_1 \),
that is equivalent to \( I(L_1) = \overline{L_1} \)
where \( L_1 \) is regarded as an immersed Lagrangian submanifold of \( \mathcal{G} \).

- **A.6**
  \( L_3 \circ (L_1 \times L_1) = L_1 \).
- **A.7** \( L_3 \circ (L_1 \times \text{Id}) \) is an immersed Lagrangian submanifold of \( \mathcal{G} \times \mathcal{G} \).

We define
\[
L_2 := L_3 \circ (L_1 \times \text{Id}) : \mathcal{G} \to \mathcal{G}.
\]

Corollary 3.
\[
L_2 := L_3 \circ (\text{Id} \times L_1).
\]

Corollary 4. \( L_2 \) leaves invariant \( L_1, L_2 \) and \( L_3 \), i.e.
\[
L_2 \circ L_1 = L_1 \\
L_2 \circ L_2 = L_2 \\
L_2 \circ L_3 = L_3.
\]

Corollary 5.
\[
\overline{I_{\text{rel}}} \circ L_2 = \overline{L_2} \circ \overline{I_{\text{rel}}} \text{ and } L_2^* = L_2.
\]

The next set of axioms defines a particular type of relational symplectic groupoids, in which the relation \( L_2 \) plays the role of an equivalence relation and it allows to study the case of symplectic reductions.

**Definition 3.** A relational symplectic groupoid \( (\mathcal{G}, L, I) \) is called **regular** if the following axioms are satisfied. Consider \( \mathcal{G} \) as a coisotropic relation \( * \hookrightarrow \mathcal{G} \) denoted by \( \mathcal{G}_{\text{rel}} \).

- **A.8** \( L_2 \circ \mathcal{G}_{\text{rel}} \) is an immersed coisotropic relation.

**Remark 3:** Again in this case, the fact that this is a coisotropic relation follows automatically in the finite dimensional setting.

Corollary 6. Setting \( C := L_2 \circ \mathcal{G}_{\text{rel}} \) the following corollary holds.

1. \( C^* = \mathcal{G}^* \circ L_2 \)
2. \( L_2 \) defines an equivalence relation on \( C \).
3. This equivalence relation is the same as the one given by the characteristic foliation on \( C \).

**A.9** The reduction \( L_1 = L_1 / L_2 \) is a finite dimensional smooth manifold.

**A.10** \( S := \{ (c, [l]) \in C \times M : \exists l \in [l], g \in \mathcal{G} | (l, c, g) \in L_3 \} \) is an immersed submanifold of \( \mathcal{G} \times M \).

Corollary 7.
\[
T := \{ (c, [l]) \in C \times M : \exists l \in [l], g \in \mathcal{G} | (c, l, g) \in L_3 \}
\]
is an immersed submanifold of \( \mathcal{G} \times M \).
The following conjectures (this is part of work in progress) give a connection between the symplectic structure on \( G \) and Poisson structures on \( M \).

**Conjecture 1.** Let \((G, L, I)\) be a regular relational symplectic groupoid. Then,

1. There exists a unique Poisson structure on \( M \) such that \( S \) is coisotropic in \( G \times M \).
2. This is also the unique Poisson structure on \( M \) such that \( T \) is coisotropic in \( G \times M \).

**Conjecture 2.** Assume \( G := C/L_2 \) is smooth. Then \( G \) is a symplectic groupoid on \( M \) with structure maps \( s := S/L_2, t := T/L_2, \mu := L_{rel}/L_2, \iota = I, \varepsilon = L_1/L_2 \).

**Definition 4.** A morphism between relational symplectic groupoids \((G, L_G, I_G)\) and \((H, L_H, I_H)\) is a map \( F \) from \( G \) and \( H \) satisfying the following properties:

1. \( F \) is a Lagrangian subspace of \( G \times \bar{H} \).
2. \( F \circ I_G = I_H \circ F \).
3. \( F^3(L_G) = L_H \).

**Definition 5.** A morphism of relational symplectic groupoids \( F: G \to H \) is called an equivalence if the transpose canonical relation \( F^{op} \) is also a morphism.

**Remark 4:** For our motivational example, it can be proven that

1. Different differentiability degrees (the \( C^k \) – type of the maps \( X \) and \( \eta \)) give rise to equivalent relational symplectic groupoids.
2. For regular relational symplectic groupoids, \( G \) and \( G \) are equivalent.

### 3.1. Examples.

The following are natural examples of relational symplectic groupoids.

**3.1.1. Lie groupoids.** Symplectic groupoids Given a Lie symplectic groupoid \( G \) over \( M \), we can endow it naturally with a relational symplectic structure:

- \( G = G \).
- \( L = \{(g_1, g_2, g_3) | (g_1, g_2) \in G \times (s, t) G, g_3 = \mu(g_1, g_2) \} \).
- \( I = g \mapsto g^{-1}, g \in G \).

**3.1.2. Symplectic manifolds with a given Lagrangian submanifold:** Let \((G, \omega)\) be a symplectic manifold and \( L \) a Lagrangian submanifold of \( G \). We define

- \( G = G \).
- \( L = \mathcal{L} \times \mathcal{L} \times \mathcal{L} \).
- \( I = \{\text{identity of } G\} \).

It is an easy check that this construction satisfies the relational axioms and furthermore

**Proposition 3.** The previous relational symplectic groupoid is equivalent to the zero dimensional symplectic groupoid (a point with zero symplectic structure and empty relations).

**Proof:** It is easy by checking that \( L \) is an equivalence from the zero manifold to \( G \).
3.1.3. Powers of symplectic groupoids: Let us denote $G_{(1)} = G$, $G_{(2)}$ the fiber product $G \times_{(s,t)} G$, $G_{(3)} = G \times_{(s,t)} (G \times_{(s,t)} G)$ and so on. It can be proven the following

**Lemma 1.** [5] Let $G \rightrightarrows M$ be a symplectic groupoid.

1. $G_{(n)}$ is a coisotropic submanifold of $G^n$.
2. The reduced spaces $G_{(n)}$ are symplectomorphic to $G$. Furthermore, there exists a natural symplectic groupoid structure on $G_{(n)}$ coming from the quotient, isomorphic to the groupoid structure on $G$.

We have natural canonical relations $P_{\alpha} : G_{(n)} \to G^n$ defined as:

$$P := \{(x, \alpha, \beta) \in G_{(n)} \mid [\alpha] = [\beta] = x\},$$

satisfying the following relations:

$$P^{op} \circ P = Gr(I_{dG}), P \circ P^{op} = \{(g, h) \in G^n \mid [g] = [h]\}.$$

It can be checked that

**Proposition 4.** $G_{(i)}$ is equivalent to $G_{(j)}, \forall i, j \geq 1$ and the equivalence is given by $P_i \circ P_j^{op}$.

3.1.4. The cotangent bundle of the path space of a Poisson manifold. This is the motivational example for the construction of relational symplectic groupoids. In this case, the coisotropic submanifold $C_{\Pi}$ is equipped with an equivalence relation, called $T^*M$- homotopy [12], and denoted by $\sim$. More precisely, to points of $C_{\Pi}$ are $\sim$- equivalent if they belong to the same leaf of the characteristic foliation of $C_{\Pi}$.

We get the following relational symplectic groupoid (where $L$ is the restriction to the boundary of the solutions of the Euler-Lagrange equations in the bulk)

$$\mathcal{G} = T^*(PM),$$

$$L = \{(X_1, \eta_1), (X_2, \eta_2), (X_3, \eta_3) \in C^{\infty}_\Pi | (X_1 * X_2, \eta_1 * \eta_2) \sim (X_3 * \eta_3)\}.$$

$I = (X, \eta) \mapsto (\phi^*X, \phi^*\eta)$.

Here $*$ denotes path concatenation and

$$\phi : [0,1] \to [0,1]$$

$$t \mapsto 1 - t$$

**Theorem 2.** [5]. The relational symplectic groupoid $\mathcal{G}$ defined above is regular.

The improvement of Theorem 1 in terms of the relational symplectic groupoids can be summarized as follows. $L_1$ can be understood as the space of $T^*M$- paths that are $T^*M$- homotopy equivalent to the trivial $T^*M$- paths and

$$L_1 := \cup_{x_0 \in M} T^*_{(x,\eta)}(PM) \cap L_1,$$

where $(X, \eta) = \{(X, \eta)|X \equiv X_0, \eta \in \ker \Pi^\#\}$, we can prove the following

**Theorem 3.** [5]. If the Poisson manifold $M$ is integrable, then, there exists a tubular neighborhood of the zero section of $T^*PM$, denoted by $N(\Gamma_0(T^*PM))$ such that $L_1 \cap N(\Gamma_0(T^*PM))$ is an embedded submanifold of $T^*PM$.

**Theorem 4.** [5]. If $M$ is integrable, then $L_1 \cap N(\Gamma_0(T^*PM))$, $L_2 \cap N(\Gamma_0(T^*PM))^2$ and $L_3 \cap N(\Gamma_0(T^*PM))^3$ are embedded Lagrangian submanifolds.
References

[1] J.J. Duistermaat and J.A.C. Kolk, Lie Groups Universitext, Springer, 1999.
[2] A. S. Cattaneo and G. Felder. Poisson sigma models and symplectic groupoids, Progress in Mathematics 198, 61-93, 2001.
[3] A.S. Cattaneo. Coisotropic submanifolds and dual pairs., preprint.
[4] A. S. Cattaneo and G. Felder Coisotropic submanifolds in Poisson geometry and branes in the Poisson sigma model, Lett. Math. Phys. 69. 157-175. 2004.
[5] A.S. Cattaneo and I. Contreras, Relational symplectic groupoids and Poisson sigma models with boundary, in preparation.
[6] A. S. Cattaneo and G. Felder, Coisotropic submanifolds and dual pairs, unpublished.
[7] A. Cattaneo, On the integration of Poisson manifolds, Lie algebroids, and Coisotropic submanifolds, Letters in Mathematical Physics 67: 33-48, 2004.
[8] A. S. Cattaneo, P. Mnev and N. Reshetikhin, Classical BV theories on manifolds with boundaries, math-ph/1201.0290.
[9] A. S. Cattaneo, P. Mnev and N. Reshetikhin, Classical and Quantum Lagrangian Field Theories with Boundary, preprint.
[10] H.H. Tseng, C. Zhu, Integrating Lie algebroids via stacks, Compositio Mathematica, Volume 142 (2006), Issue 01, pp 251-270.
[11] A.S. Cattaneo, P. Mnev and N. Reshetikhin, Classical and Quantum Lagrangian Field Theories with Boundary, Proceedings of the Corfu Summer Institute 2011 School and Workshops on Elementary Particle Physics and Gravity, Corfu, Greece, 2011.
[12] M. Crainic and R. L. Fernandes. Integrability of Lie brackets. Ann. of Math.(2) 157 (2003), 575–620.
[13] M. Crainic and R. L. Fernandes. Integrability of Poisson brackets, parXiv:math/0210152v1
[14] I. Moerdjik and J.Mrcun. Introduction to Foliations and Lie Groupoids, Cambridge studies in advanced mathematics, 91, 2003.
[15] P. Severa, Some title containing the words homotopy and symplectic, e.g. this one, preprint math.SG-0105080.