The Large Sieve Inequality for Quadratic Polynomial Amplitudes

by

Gyan Prakash and D.S. Ramana

1. Introduction

An important requirement in the context of inequalities of the large sieve type is to obtain estimates for the sum $\sum_{x \in X} \left| \sum_{i \in I} a_i e(xy_i) \right|^2$, where $X$ is a well-spaced set of real numbers, $I$ is a finite set, $\{a_i\}_{i \in I}$ are complex numbers and $\{y_i\}_{i \in I}$ is a sparse subsequence of the integers.

Basic examples of sparse sequences of integers are provided by the sequence of values of polynomials of degree $\geq 2$ with integer coefficients. The present article is concerned with the case when polynomial is of degree 2. Indeed, in a recent note, Liangyi Zhao [4], showed, by an elegant application of the double large sieve inequality of Bombieri and Iwaniec, that one has the estimate given below, which we state with the following notation.

When $Q$ is a real number $\geq 1$, the Farey series of order $Q$ is the sequence of rational numbers in $(0,1)$ whose denominators, when expressed lowest form, do not exceed $Q$. Given a finite sequence of complex numbers $\{a_i\}_{i \in I}$, indexed by a finite set $I$, we write $\|a\|^2$ to denote $\sum_{i \in I} |a_i|^2$.

**Theorem (L. Zhao).—** Let $Q$ be a real number $\geq 1$ and suppose that $F(Q)$ is the Farey sequence of order $Q$ and that $P(T) = c_0 T^2 + c_1 T + c_2$ is a polynomial in $\mathbb{R}[T]$ with $c_0 \neq 0$, $c_1/c_0 = p/q \in \mathbb{Q}$, $c_1 > 0$ and $(p,q) = 1$. When $\{a_i\}_{M < i \leq M + N}$ is a sequence of complex numbers indexed by the integers in the interval $(M, M + N)$, where $M$ and $N$ are integers with $N \geq 1$, we have

$$\sum_{x \in F(Q)} \left| \sum_{M < i \leq M + N} a_i e(xP(i)) \right|^2 \ll (Q^2 + Q \sqrt{c_0} N(|M| + 2N + p/q + 1) \Pi \|a\|^2,$$

where the implied constant depends on $\varepsilon$ alone and

$$\Pi = \left( \frac{q}{c_0} + 1 \right)^{1/2 + \varepsilon} [Nq(|M| + N) + |p| + q/c_0]^\varepsilon.$$

Zhao has shown in [4], Section 3, page 4 that one may devise examples of $P(T)$ and the sequence $\{a_i\}$ for which the left hand side of (1) is $\gg (NQ^2)\|a\|^2$, thereby showing that (1) is essentially the best possible.

There, however, remains the question of determining the extent to which the dependence of the right hand side of (1) on $M$ and the coefficients of $P(T)$ may be reduced, given that the trivial estimate for the left hand side of (1) is independent of these parameters. In the present
article we combine Zhao’s method with an interpolation argument due to Heath-Brown [3] to show that at least a modest improvement upon the above theorem along these lines is certainly possible. More precisely, the corollary to Theorem 1 of Section 3 below shows that, under the same hypotheses as in the above theorem, (1) holds with the factor II replaced with

\[ \Pi' = \pi \left( \frac{2q}{c_0} + 1 \right)^{\frac{1}{2}} \sup_{1 \leq n \leq 144N^4} r(n), \]

and the \( \ll \) in (1) replaced by \( \leq \). Here \( r(n) \), for an integer \( n \), is the number of integer points on \((x, y)\) satisfying \( x^2 + y^2 = n \).

The reader will note with interest that S. Baier has already shown in [1] that an analog of our conclusion recorded in the corollary to Theorem 1 of Section 3 holds for all quadratic polynomials with real coefficients when one replaces \( \|a\|^2 \) with \( \operatorname{Card}(I) \sup_{i \in I} |a_i|^2 \). Moreover, this work of Baier also investigates what might be expected to hold for higher degree polynomials in this context.

2. Preliminaries

2.1. \( \delta \)-Spaced Sets. — Let \( \delta > 0 \). A \( \delta \)-spaced set of real numbers is a finite set \( \mathcal{X} \) of distinct real numbers containing at least two elements and such that \( |x - x'| \geq \delta \), whenever \( x \) and \( x' \) are distinct elements of \( \mathcal{X} \). Let \( \mathcal{X} \) be a \( \delta \)-spaced set and \( \epsilon > 0 \). We then set \( S(\epsilon, x) = \operatorname{Card}(x' \in \mathcal{X} \mid |x - x'| \leq \epsilon) \). We have

\[
S(\epsilon, x) \leq 1 + \frac{2\epsilon}{\delta} \quad \text{for all } x \text{ in } \mathcal{X}.
\]

2.2. The function \( \phi \). — Let \( \phi(t) \) denote the characteristic function of the interval \([-1/2, 1/2]\] and for each \( \epsilon > 0 \), set \( \phi_\epsilon(t) = \phi(t/2\epsilon) \). Thus \( \phi_\epsilon(t) \) is the characteristic function of the interval \([-\epsilon, \epsilon]\). On setting \( \sin t/t \) to 1 when \( t = 0 \) we have

\[
\widehat{\phi_\epsilon}(t) = 2\epsilon \widehat{\phi}(2\epsilon t) = 2\epsilon \left( \frac{\sin 2\pi \epsilon t}{2\pi \epsilon t} \right).
\]

The classical inequality \( 2/\pi \leq \sin t/t \leq 1 \) for \( t \) in \([0, \pi/2]\) then implies the inequality

\[
\frac{1}{2\epsilon} \leq \frac{1}{\phi_\epsilon(t)} \leq \frac{\pi}{4\epsilon},
\]

for all \( t \) with \( \epsilon t \) in \([-1/4, 1/4]\).

2.3. A Simple Majorisation Principle. — Let \( I \) be a finite set, \( \{y_i\}_{i \in I} \) be a sequence of integers, \( \{a_i\}_{i \in I} \) a sequence of complex numbers and \( \{b_i\}_{i \in I} \) a sequence of positive real numbers. When \( |a_i| \leq b_i \) for each \( i \) in \( I \), the triangle inequality gives
(4) \[ \int_0^1 \sum_{i \in I} |a_i e(t y_i)|^2 dt = \left| \sum_{(i,j) \in I \times I} a_i a_j \delta_{ij} \right| \leq \sum_{(i,j) \in I \times I} b_i b_j \delta_{ij} = \int_0^1 \sum_{i \in I} b_i e(t y_i)|^2 dt , \]

where \( \delta_{ij} = 1 \) when \( y_i = y_j \) and 0 otherwise.

3. COUNTING INTEGER POINTS ON CIRCLES

When \( a \) and \( b \) are integers we write \((a,b) = 1\) to mean that either one of \(a,b\) is 1 and the other 0 or that \(a\) and \(b\) are both distinct from 0 and are coprime.

Proposition 1. — Let \( H \) be a real number \( \geq 1 \), \( m \) be a rational number and let \( c_i \), \( 1 \leq i \leq 3 \), be integers with \( c_1 \neq 0 \), \( c_3 \geq 0 \), \( (c_1, c_2) = 1 \). Suppose that there are at least three integer points \((x,y)\) satisfying \(|x|,|y| \leq H\) and lying on the circle

\[(1) \quad (c_1 X - c_2)^2 + (c_1 Y - mc_2)^2 = c_3 . \]

If \( m = \frac{p}{q} \) with \( q > 0 \) we then have

\[(2) \quad |c_1| \leq 4q(1 + |m|)H, \quad |c_2| \leq 2qH^2 \quad \text{and} \quad c_3 \leq 36q^2(1 + |m|)^2H^4 . \]

Proof. — We follow the method of proof of Theorem 4, page 564 in [3]. Suppose that \( p_i = (x_i, y_i) \), \( 1 \leq i \leq 3 \), are three integer points on \((1)\). Since the relation \((1)\) is the same as

\[(3) \quad c_1^2(X^2 + Y^2) - 2c_1c_2(X + mY) + (1 + m^2)c_2^2 - c_3 = 0 , \]

we obtain the following relation of matrices on setting \((X,Y) = (x_i, y_i)\) in \((3)\) for \( 1 \leq i \leq 3 .\)

\[(4) \begin{pmatrix} x_1^2 + y_1^2 & x_1 + my_1 & 1 \\ x_2^2 + y_2^2 & x_2 + my_2 & 1 \\ x_3^2 + y_3^2 & x_3 + my_3 & 1 \end{pmatrix} \begin{pmatrix} c_1^2 \\ -2c_1c_2 \\ (1 + m^2)c_2^2 - c_3 \end{pmatrix} = 0 . \]

Let \( M \) denote the \(3 \times 3\) matrix and \( c\) the vector \((c_1^2, -2c_1c_2, (1 + m^2)c_2^2 - c_3)\) in \(\mathbb{Q}^3\) on the left hand side of \((4)\). Since \( c_1 \neq 0 \), we have \( c \neq 0 \) and hence \( \det(M) = 0 \). Let \( a = (a_1, a_2, a_3) \) be a solution distinct from 0 to \( Ma = 0 \). Then the points \( p_i \) lie on the conic

\[(5) \quad a_1(X^2 + Y^2) + a_2(X + mY) + a_3 = 0 . \]

Since the points \( p_i \) do not all lie on a line we must have \( a_1 \neq 0 \). Then \((5)\) is an affine circle which intersects the affine circle \((1)\) at the three points \( p_i \). Since distinct affine circles intersect at no more than 2 points, we have \( a = \alpha c \), for some complex number \( \alpha \) which must necessarily be an element of \(\mathbb{Q} \). Thus the rank of the matrix \( M \) over \(\mathbb{Q} \) is 2.
Suppose now that the rows \( i \) and \( j \) of \( M \), with \( i > j \), are linearly independent over \( \mathbb{Q} \). Then \( a = (a_1, a_2, a_3) \), where the \( a_i \) are given by the relations

\[
(6) \quad a_1 = q \det \begin{pmatrix} x_i + my_i & 1 \\ x_j + my_j & 1 \end{pmatrix}, \quad a_2 = -q \det \begin{pmatrix} x_i^2 + y_i^2 & 1 \\ x_j^2 + y_j^2 & 1 \end{pmatrix}, \quad a_3 = q \det \begin{pmatrix} x_i^2 + y_i^2 & x_i + my_i \\ x_j^2 + y_j^2 & x_j + my_j \end{pmatrix},
\]

satisfies \( Ma = 0 \). Since \( a \neq 0 \) and there is an \( \alpha \) in \( \mathbb{Q} \) such that \( \alpha c = a \), we have

\[
(7) \quad \frac{a_2}{a_1} = -\frac{2c_2}{c_1}, \quad \frac{a_3}{a_1} = \frac{(1 + m^2)c_2^2 - c_3}{c_1^2},
\]

from which we deduce the following relations on setting \( k = -2a_1/c_1 \neq 0 \).

\[
(8) \quad kc_1 = -2a_1, \quad kc_2 = a_2, \quad \text{and} \quad k^2c_3 = (1 + m^2)a_2^2 - 4a_1a_3.
\]

Since the \( a_i \) are integers and \((c_1, c_2) = 1\), the relation \( kc_2 = a_2 \) shows that \( c_1 \) divides \(-2a_1\) or that \( k \) is an integer. Moreover, the triangle inequality applied to the relations in (6) gives

\[
(9) \quad |a_1| \leq 2q(1 + |m|)H, \quad |a_2| \leq 2qH^2, \quad \text{and} \quad |a_3| \leq 4q(1 + |m|)H^3.
\]

We now obtain (2) on combining (8) with (9) using the triangle inequality and \(|k| \geq 1\).

When \( n \) is an integer \( \geq 0 \), \( r(n) \) denotes the number of integer points \((x, y)\) satisfying \( x^2+y^2 = n \).

**Corollary 1.** — Let \( H \) be a real number \( \geq 1 \) and let \( c_i, 1 \leq i \leq 3, \) be integers with \( c_1 \neq 0 \). The number of integer points \((x, y)\) satisfying \(|x| \leq H, |y| \leq H \) and lying on \((c_1X-c_2)^2+(c_1Y-c_2)^2 = c_3 \) does not exceed \( \sup_{1 \leq n \leq 144H^4} r(n) \).

**Proof.** — Let \( N(H) \) be the number of integer points satisfying the conditions of the corollary. We assume \( N(H) \geq 1 \) and set \( d = c_1 \) when \( c_2 = 0 \) and \( d = \text{g.c.d.}(c_1, c_2) \) otherwise. Then \( d^2 \) divides \( c_3 \). Let \( c_1/d = c_1^*, c_2/d = c_2^* \) and \( c_3/d^2 = c_3^* \). Then \( N(H) \) is the same as the number of integer points \((x, y)\) satisfying \(|x| \leq H, |y| \leq H \) and lying on \((c_1^*X-c_2^*)^2+(c_1^*Y-c_2^*)^2 = c_3^* \).

Plainly, \( N(H) \leq r(c_3^*) \). Since \( c_1^* \neq 0, c_3^* \geq 0 \) and \((c_1^*, c_2^*) = 1\), Proposition 1 applied with \( m = 1, q = 1 \) implies that either \( N(H) \leq 2 \) or \( c_3^* \leq 144H^4 \) so that \( N(H) \leq \sup(2, \sup_{1 \leq n \leq 144H^4} r(n)) \), from which the corollary follows on noting that \( \sup_{1 \leq n \leq 144H^4} r(n) \geq 2 \) when \( H \geq 1 \).

**Corollary 2.** — Let \( I \) be a real interval of length \( H \geq 1 \). For all quadratic polynomials \( P(T) \) in \( \mathbb{Z}[T] \) and all integers \( k \) the number of integer points \((x, y)\) in \( I \times I \) lying on \( P(X) + P(Y) = k \) does not exceed \( \sup_{1 \leq n \leq 144H^4} r(n) \).

**Proof.** — Suppose that \( P(T) = a_0T^2 + a_1T + a_2 \), with the \( a_i, 0 \leq i \leq 2, \) integers and \( a_0 \neq 0 \) and let \( x_0 \) be an integer in \( I \). On completing the square and rearranging the terms we see that \( P(x) + P(y) = k \) is equivalent to
finite set, \( \text{(2)} \) we deduce that \( \text{(1)} \),

\[
(2a_0(x - x_0) + 2a_0x_0 + a_1)^2 + (2a_0(y - x_0) + 2a_0x_0 + a_1)^2 = 4a_0(k - 2a_2) + 2a_1^2,
\]

for any point \( (x, y) \) in the real plane. Let us set \( c_1 = 2a_0, c_2 = -(2a_0x_0 + a_1) \) and write \( c_3 \) to denote the right hand side of \( \text{(10)} \). Since \( x_0 \in I \), we have \( |x - x_0| \leq H \) and \( |y - x_0| \leq H \) for all \( (x, y) \in I \times I \) and the number of integer points satisfying the conditions of the corollary does not exceed the number of integer points \( (x, y) \) satisfying \( |x| \leq H, |y| \leq H \) and lying on \( (c_1X - c_2)^2 + (c_1Y - c_2)^2 = c_3 \), so that the corollary follows from Corollary 1.

4. A VARIANT OF THE DOUBLE LARGE SIEVE INEQUALITY

The following lemma is the essence of the double large sieve inequality, modified slightly for our purpose. The proof follows pages 88 to 90, [2] closely.

**Lemma 1.** — Let \( \mathcal{X} \) be a \( \delta \)-spaced set of real numbers lying in the interval \([-P, P]\). Further, let \( I \) be a finite set, \( \{y_i\}_{i \in I} \) a sequence of integers and \( \{a_i\}_{i \in I} \) a sequence of complex numbers. When \( T \) is a real number such that \( |y_i| \leq T \) for all \( i \) in \( I \) we have the inequality

\[
\left| \sum_{x \in \mathcal{X}} f(x) \right| \leq \pi \left( \text{Card}(\mathcal{X}) T + \frac{\text{Card}(\mathcal{X})}{\delta} \right)^{1/2} (P + 2)^{1/2} \left( \int_0^1 |f^*(t)|^2 dt \right)^{1/2},
\]

where \( f(t) = \sum_{i \in I} a_i e(t y_i) \) and \( f^*(t) = \sum_{i \in I} |a_i| e(t y_i) \).

**Proof.** — We have \( \text{Card}(\mathcal{X}) \geq 2 \) and that \( \mathcal{X} \) is contained in \([-P, P]\). Therefore \( \text{Card}(\mathcal{X}) - 1 \geq \text{Card}(\mathcal{X})/2 \) and, using (1) of 2.1, \( \text{Card}(\mathcal{X}) - 1 \leq 2P/\delta \). These relations show that \( \text{(1)} \) holds when all the \( y_i \) are 0.

Let us suppose that atleast one of the \( y_i \) is distinct from 0. Since the \( y_i \) are integers, we have \( T \geq 1 \). We then set \( \epsilon = 1/4T \) and note that \( \epsilon \leq 1 \). Since \( \phi_\epsilon(t) = \phi_\epsilon(-t) \), for any real number \( x \) we have the relation

\[
\hat{\phi}_\epsilon(xy) = \int_0^1 \phi_\epsilon(t - x)e(ty) \, dt.
\]

From \( \text{(2)} \) we deduce that

\[
\sum_{x \in \mathcal{X}} f(x) = \sum_{x \in \mathcal{X}} \sum_{i \in I} a_i e(x y_i) = \int_0^1 \left( \sum_{x \in \mathcal{X}} \phi_\epsilon(t - x) \right) \left( \sum_{i \in I} \frac{a_i}{\phi_\epsilon(y_i)} e(t y_i) \right) \, dt.
\]

Since \( \mathcal{X} \) is a subset of \([-P, P]\) and \( \epsilon \leq 1 \), it follows that \( \sum_{x \in \mathcal{X}} \phi_\epsilon(t - x) \) vanishes outside the interval \([-P - 2, P + 2]\). Let \( \chi_P(t) \) denote the characteristic function of this interval. The Cauchy-Schwarz inequality then gives

\[
\left| \sum_{x \in \mathcal{X}} f(x) \right| \leq \left\| \sum_{x \in \mathcal{X}} \phi_\epsilon(t - x) \right\|_2 \left\| \chi_P(t) \sum_{i \in I} \frac{a_i}{\phi_\epsilon(y_i)} e(t y_i) \right\|_2.
\]

5
We have that $0 \leq \phi_{e} * \phi_{e}(t) \leq 2\epsilon$ for all $t \in \mathbb{R}$ and that the support of $\phi_{e} * \phi_{e}$ is $[-2\epsilon, 2\epsilon]$. These remarks together with $\phi_{e}(t) = \phi_{e}(-t)$ imply

$$\left(5\right) \left\| \sum_{x \in \mathcal{X}} \phi_{e}(t-x) \right\|_{2}^{2} = \sum_{(x,x') \in \mathcal{X} \times \mathcal{X}} \phi_{e} * \phi_{e}(x-x') \leq 2\epsilon \sum_{x \in \mathcal{X}} S(2\epsilon, x) \leq 2\epsilon \text{Card}(\mathcal{X}) \left(1 + \frac{4\epsilon}{\delta}\right),$$

where the last inequality follows from (1) of (2.1). Turning to the second term on the right hand side, we note that since the $y_{i}$ are integers, $e(ty_{i})$ is periodic of period 1 for each $i \in I$. Thus

$$\left(6\right) \int_{-\lfloor P \rfloor - 2}^{\lfloor P \rfloor + 2} \left| \sum_{i \in I} \frac{a_{i}}{\phi_{e}(y_{i})} e(ty_{i}) \right|^{2} dt = 2(\lfloor P \rfloor + 2) \int_{0}^{1} \left| \sum_{i \in I} \frac{a_{i}}{\phi_{e}(y_{i})} e(ty_{i}) \right|^{2} dt$$

Recalling that $T = 1/4\epsilon$ and that the $y_{i}$ lie in the interval $[-T, T]$ we obtain $|a_{i}/\hat{\phi}_{e}(y_{i})| \leq \pi |a_{i}|/4\epsilon$ for all $i \in I$ on using (3) of (2.2). Using (6) and the majorisation principle (2.3) we then conclude that

$$\left(7\right) \left\| \chi_{P}(t) \sum_{i \in I} \frac{a_{i}}{\phi_{e}(y_{i})} e(ty_{i}) \right\|_{2}^{2} \leq 2 \left( \frac{\pi}{4\epsilon} \right)^{2} (P + 2) \int_{0}^{1} |f^{*}(t)|^{2} dt.$$  

The lemma now follows on combining (7) with (4) and (5).

**Theorem 1.** — Let $\mathcal{X}$ be a $\delta$-spaced set of real numbers lying in the interval $[-P, P]$. When $I$ is a finite set, $\mathcal{Y} = \{y_{i}\}_{i \in I}$ is a sequence of integers and $\{a_{i}\}_{i \in I}$ are complex numbers we have the inequality

$$\left(8\right) \sum_{x \in \mathcal{X}} \left| \sum_{i \in I} a_{i} e(xy_{i}) \right|^{2} \leq \pi \left( \text{Card}(\mathcal{X}) \Delta(\mathcal{Y}) + \frac{\text{Card}(\mathcal{X})}{\delta} \right)^{1/2} (P + 2)^{1/2} \text{sup}_{k} A_{y}^{1/2}(k) \|a\|^{2},$$

where $\Delta(\mathcal{Y})$ denotes $\text{sup}_{(i,j) \in I} |y_{i} - y_{j}|$, $A_{y}(k)$, for each integer $k$, denotes the number of $(i, j) \in I \times I$ such that $y_{i} + y_{j} = k$ and $\|a\|^{2}$ denotes $\sum_{i \in I} |a_{i}|^{2}$.

**Proof.** — Following the principle of Zhao [4], we apply the preceding lemma with $f(t)$ replaced by

$$\left(9\right) g(t) = \sum_{(i,j) \in I \times I} a_{i}a_{j}e(t(y_{i} - y_{j})) = |f(t)|^{2}$$

and with $T = \text{sup}_{(i,j) \in I \times I} |y_{i} - y_{j}| = \Delta(\mathcal{Y})$. The theorem follows on noting that $g^{*}(t) = |f^{*}(t)|^{2}$ and using the estimate for $\int_{0}^{1} |g^{*}(t)|^{2} dt = \int_{0}^{1} |f^{*}(t)|^{4} dt i \leq \text{sup}_{k} A_{y}(k) \|a\|^{4}$.

**Corollary 1.** — Let $Q$ be a real number $\geq 1$ and suppose that $\mathcal{F}(Q)$ is the Farey sequence of order $Q$ and that $P(T) = c_{0}T^{2} + c_{1}T + c_{2}$ is a polynomial in $\mathbb{R}[T]$ with $c_{0} \neq 0$, $c_{1}/c_{0} = p/q \in \mathbb{Q}$ and $(p, q) = 1$. When $\{a_{i}\}_{M \leq i \leq M + N}$ is a sequence of complex numbers indexed by the integers in the interval $(M, M + N]$, where $M$ and $N$ are integers with $N \geq 1$, we have

$$\left(10\right) \sum_{x \in \mathcal{F}(Q)} \left| \sum_{M \leq i \leq M + N} a_{i} e(xP(i)) \right|^{2} \leq (Q^{2} + Q \sqrt{c_{0}N(|M| + 2N + \frac{|P|}{q} + 1)}) \Pi \|a\|^{2},$$
where

$$\Pi = \pi \left( \frac{2q}{c_0} + 1 \right)^{\frac{1}{2}} \sup_{1 \leq n \leq 144N^4} r(n).$$

Here $r(n)$ is the number of pairs of integers $(x, y)$ such that $x^2 + y^2 = n$.

**Proof.** — We may assume $c_0 > 0$. We set $\alpha = \frac{c_0}{q}$ so that $P(T) = \alpha(qT^2 + pT + c_2)$. We take $I$ to be the set of integers in the interval $(M, M + N]$,

$$y_i = qt^2 + pi, \quad X = \alpha F(Q)$$

We may then set $\delta = \frac{\alpha}{q}$, Card$(X) \leq Q^2$ and $P = \alpha$. We obtain Corollary 1 on applying Theorem 1 to the above data and taking into account that

$$(12) \quad \Delta(Y) \leq |q|N(2N + |M| + 1) + |p|N \leq qN(|M| + 2N + |\frac{p}{q}| + 1).$$

and that, by Corollary 2 to Proposition 1 of Section 3, we have $A_Y(k) \leq \sup_{1 \leq n \leq 144N^4} r(n)$, for all integers $k$.

**References**

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*Harish-Chandra Research Institute,*

*Chhatnag Road, Jhunsi,*

*Allahabad - 211 019, India.*

*email : gyan@mri.ernet.in, suri@mri.ernet.in*