INTERSECTION NUMBERS OF GEODESIC CURVES IN A SURFACE

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Abstract. For a compact surface $X$ with negative curvature we show that the tails of the distribution $i(\alpha, \beta)/l(\alpha)l(\beta)$ are bounded by a decreasing exponential function (here, $\alpha$ and $\beta$ are closed geodesics of $X$, $i(\alpha, \beta)$ denotes their intersection number and $l(\cdot)$ is the hyperbolic length function of $X$.) As a consequence we find the normalized average of the intersection numbers of pairs of closed geodesics. In addition, we prove that the size of the set of geodesics of length $T$ whose self-intersection number is not close to $T^2/(2\pi^2(g-1))$ decrease also exponentially fast as $T \to \infty$, where $g$ is the genus of $X$. As a corollary, we obtain a result of S. Lalley which states that most closed geodesics of length $T$ have roughly $T^2/(2\pi^2(g-1))$ self-intersections, for $T$ large.

1. Introduction

Let $X$ be a compact hyperbolic surface, $G$ be the set of closed geodesics of $X$, $G_t$ be the subset of $G$ consisting of the geodesics whose length is at most $t$ and $N(t)$ be the number of elements of $G_t$. It is a classical result of G. Margulis (see [11, §6, Theorem 5]) that the number $N(t)$ satisfies the asymptotic formula $N(t) \sim e^{t/t}$, i.e., the ratio of the two sides converges to one, as $t \to \infty$.

For a pair of geodesics $\alpha$ and $\beta$ of $X$ we denote by $i(\alpha, \beta)$ the (geometric) intersection number of $\alpha$ and $\beta$, that is, the number of points of intersection of $\alpha$ and $\beta$. In particular, $i(\gamma) := i(\gamma, \gamma)$ is the number of self-intersections of the geodesic $\gamma$.

These numbers have been of interest to many researchers and here are some of the most relevant results so far achieved. S. Lalley showed in [10, Theorem 1] that for $T$ large enough, the number of self-intersections of most of the closed geodesics of length $T$ is $T^2/(2\pi^2(g-1))$, where $g$ is the genus of $X$. Later, M. Pollicot and R. Sharp generalized this result to self-intersections of closed geodesics with an angle in a given interval (see [13, Theorem 1].) And recently, M. Chas and S. Lalley in [8] proved that if a class of a geodesic is chosen at random from among all classes of $m$ letters, then the distribution of the self-intersection numbers approaches the Gaussian distribution, for $m$ “large enough”. Furthermore, M. Chas and S. Lalley also proved in [9] that for a certain constant $\kappa > 0$ the random variable $(N_T - \kappa T^2)/T$ has a limit distribution as $T \to \infty$, where $N_T$ is the number of self-intersections of a closed geodesic of $X$ of length $\leq T$ randomly chosen.

In this paper we prove that the tails of the distribution $i(\alpha, \beta)/l(\alpha)l(\beta)$ are bounded by a decreasing exponential function.

Theorem 1. Let $\epsilon > 0$. There exists $\eta > 0$ such that

$$\frac{1}{N(s)N(t)} \# \left\{ (\alpha, \beta) \in G_s \times G_t : \left| \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} - \frac{1}{2\pi^2(g-1)} \right| > \epsilon \right\} = O(e^{-\eta \min\{s,t\}}),$$

as $s, t \to \infty$. 
Theorem 1 allows us to show that the normalized average of pairs of closed geodesics of lengths at most $s$ and $t$ is asymptotically equal to $1/(2\pi^2(g-1))$, as these lengths become arbitrarily large.

**Theorem 2.**
$$
\frac{1}{N(s)N(t)} \sum_{(\alpha, \beta) \in \mathcal{G}_s \times \mathcal{G}_t} \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} \sim \frac{1}{2\pi^2(g-1)}, \text{ as } s, t \to \infty.
$$

In a similar way, we prove that the size ($\nu_L$-measure) of the set of geodesics of length $T$ whose self-intersection number is not close to $T^2/(2\pi^2(g-1))$ decrease also exponentially fast as $T \to \infty$. For a geodesic (not necessarily closed) $\gamma$, let denote by $\gamma^T$ the restriction of $\gamma$ to its first segment of length $T$.

**Theorem 3.** For every $\epsilon > 0$ there exists $\eta > 0$ such that
$$
\nu_L(\{v \in T_1(X) : |i(\gamma_v^T)/T^2 - 1/(2\pi^2(g-1))| \geq \epsilon\}) = O(e^{-\eta T}).
$$

As a consequence we obtain the following result that was proven by Lalley in [10, Theorem 1].

**Corollary 1 (Lalley).** For every $\epsilon > 0$
$$
\lim_{T \to \infty} \frac{1}{N(T)} \#\{\gamma \in \mathcal{G}_T : |i(\gamma)/T^2 - 1/(2\pi^2(g-1))| < \epsilon\} = 1.
$$

The outline of this paper is the following. Section 2 is the collection of definitions and results needed in the demonstrations of Theorems 1 and 2, and Section 3 contains the proofs of these theorems as well as the proofs of Theorem 3 and Corollary 1. For detailed explanation of all the concepts (or different approach) used in this article, please see [3], [4], [5] and [13].

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## 2. Preliminaries

### 2.1. Tangent Bundles

Let $T_1(X) = \{(x, v) | x \in X, v \in T_x(X), \|v\| = 1\}$ be the unit tangent bundle of $X$, $\Phi = \{\phi^t\}$ be the geodesic flow of $T_1(X)$ and $\mathcal{F}$ be the foliation of $T_1(X)$ by $\Phi$-orbits. Note that the $\Phi$-orbits are the geodesics of $X$. For $v = (x, v) \in T_1(X)$, let $\gamma_v$ denote the geodesic such that $\gamma_v(0) = x$ and $\dot{\gamma}_v(0) = v$.

Consider $\mathcal{E} = \{(x, v, w) : (x, v), (x, w) \in T_1(X), u \neq v\}$. Let $p_1, p_2 : \mathcal{E} \to T_1(X)$ be defined by $p_1((x, v, w)) = (x, v)$ and $p_2((x, v, w)) = (x, w)$, respectively.

Denote by $\mathcal{P} = \mathcal{P}(T_1(X))$ the set of the $\Phi$-invariant probability measures in $T_1(X)$ equipped with the weak*-topology. Let $h(\mu)$ denote the measure theoretic entropy of $\Phi$ with respect to $\mu \in \mathcal{P}$ and $h := \max_{\mu \in \mathcal{P}} h(\mu)$ (please see [5] §4.3 for definitions). For a compact hyperbolic surface there is a unique $\Phi$-invariant probability measure in $T_1(X)$ with maximum entropy $h = 1$, which is the normalized Riemannian measure also called the (normalized) Liouville measure, denoted by $\nu_L$. In our case, this measure coincides with $\mu_{BM}$, the Bowen-Margulis probability measure. (see [12], Proposition 10.)

We will use the characterization of $\nu_L$ given in [5, Theorem 20.1.3]. For $\gamma \in \mathcal{G}$, let $\zeta_\gamma := \int_0^{l(\gamma)} \delta_{\phi^s v}ds$, with $v \in \gamma$ and $\delta_y$ denoting the probability measure with support $\{y\}$.\]
Theorem 4.

\[ \nu_L = \mu_{BM} = \lim_{t \to \infty} \frac{1}{N(t)} \sum_{\gamma \in \mathcal{G}_t} \zeta_{\gamma}/(t) \]

2.2. Current Measures and Intersection Form.

For each \( \Phi \)-invariant finite measure \( \mu \) of \( T_1(X) \) (not necessarily a probability measure) there exists an associated transverse measure to \( \mathcal{F} \), which we denote by \( \widetilde{\mu} \). The set of all of these transverse measures equipped with the weak*-topology is known as the space of current measures of \( X \) and we denote it by \( \mathcal{C} \). Each \( \mu \in \mathcal{C} \) is normalized by the requirement that (locally) \( \mu = \widetilde{\mu} \times dt \), where \( dt \) is the one-dimensional Lebesgue measure along orbits in \( \mathcal{F} \).

Consider the foliations \( \mathcal{F}_1 = p_1^{-1}(\mathcal{F}) \) and \( \mathcal{F}_2 = p_2^{-1}(\mathcal{F}) \) of \( T_1(X) \), for \( p_1 \) and \( p_2 \) as defined in Section 2.1. Furthermore, for \( \mu, \nu \in \mathcal{C} \), define \( \widehat{\mu}_1 = p_1^{-1}(\mu) \) and \( \widehat{\nu}_2 = p_2^{-1}(\nu) \). Note that the new measures \( \widehat{\mu}_1 \) and \( \widehat{\nu}_2 \) are transverse to \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), respectively. The intersection form of \( \mu \) and \( \nu \), denoted by \( \iota(\mu, \nu) \), is defined as the total mass of \( \mathcal{E} \) with respect to the product measure \( \widehat{\mu}_1 \times \widehat{\nu}_2 \), that is, \( \iota(\mu, \nu) = \int_{\mathcal{E}} d(\widehat{\mu}_1 \times \widehat{\nu}_2) = (\mu_1 \times \nu_2)(\mathcal{E}) \). In addition, for \( \mu \in \mathcal{C} \) define \( \ell(\mu) \), by \( \ell(\mu) = \iota(\mu, \nu_0) \).

Now, for every \( \gamma \in \mathcal{G} \) there exists a unique \( \Phi \)-invariant measure \( \mu^\gamma \) of total mass \( l(\gamma) \), where \( l(\gamma) \) is the length of the geodesic \( \gamma \) with respect to the hyperbolic metric of \( X \). This measure \( \mu^\gamma \) is supported on the orbit of \( \gamma \). Let \( \widetilde{\mu}^\gamma \) denote the corresponding transverse measure to the orbit foliation \( \mathcal{F} \). These type of current measures are simply finite sums of Dirac measures on the quotient of pairs of closed geodesics on \( \tilde{X} \) (the universal cover of \( X \) consisting of lifts of \( \gamma \) and its \( \pi_1(X) \)-images (see [2], §1)).

By identifying the current measure \( \widetilde{\mu}^\gamma \) with the corresponding closed geodesic \( \gamma \) \( \mathcal{F} \). Bonahon showed the following properties of the Liouville current \( \nu_L \) as well as the properties of the functions \( \iota \) and \( \ell \) which we will use later on this paper (see [3], Propositions 14 and 15, and [2], Proposition 4.5.).

Theorem 5 (Bonahon). Consider the functions \( \iota \) and \( \ell \) as defined above. Then

1. The function \( \iota \) is a continuous extension of the intersection number function on \( \mathcal{G} \times \mathcal{G} \). In particular, \( \iota(\mu^\alpha, \mu^\beta) = \iota(\alpha, \beta) \). The Liouville current satisfies \( \iota(\nu_L^\gamma) = \pi^2 |\chi(X)| = 2\pi^2(g - 1) \).

2. The function \( \ell \) is a continuous extension of the length function on \( \mathcal{G} \). In particular, if \( \gamma \) is a closed geodesic of \( X \) then \( \ell(\nu_L^\gamma) = l(\gamma) \) where \( l(\gamma) \) is the length of \( \gamma \). The Liouville current satisfies \( \ell(\nu_L^\gamma) = \pi^2 |\chi(X)| = 2\pi^2(g - 1) \).

3. Proofs of the Main Results.

The proofs of Theorems 1 and 3 are basically based on the large deviation results proven by Yuri Kifer in [6], Theorem 3.4, and [7], Theorem 2.1.

For \( t > 0 \) and \( \nu \in T_1(X) \) consider the following measure \( \zeta^\nu := \int_0^t \delta_{\Phi^t\nu} ds \). In particular, \( \zeta^\nu(T_\gamma) = \zeta_{\gamma \nu} \), for \( \gamma \nu \in \mathcal{G} \).

Theorem 6 (Kifer). For any closed \( K \subset \mathcal{P} \),

\[ \lim_{t \to \infty} \frac{1}{t} \log \nu_L(\{ \nu \in T_1(X) : \zeta^\nu/T \in K \}) \leq - \inf \{ 1 - h(\nu) : \nu \in K \} \]
For $v \in T_1(X)$, let $\gamma_v$ be the geodesic such that $v \in \gamma_v$. And let $v_\gamma = (x, v) \in T_1(X)$ be a vector of the geodesic $\gamma$, i.e., $\gamma(t) = x$ and $\dot{\gamma}(t) = v$, for some $t \in \mathbb{R}$.

**Proof of Theorem 3** Consider the continuous function

$$f : \mathcal{P} \rightarrow \mathbb{R}$$

$$\nu \mapsto i(\tilde{\nu})/\ell(\tilde{\nu})^2.$$  

Let $\epsilon > 0$ and take $K := \{\nu \in \mathcal{P} : |f(\nu) - f(\nu_L)| \geq \epsilon\}$. Note that $K$ is a closed set because of the continuity of $f$, hence, $K$ is compact since $\mathcal{P}$ is compact.

By Theorem 6,

$$\lim_{T \to \infty} \frac{1}{T} \log \nu_L(\{v \in T_1(X) : \gamma_v \in K\}) \leq -\inf \{1 - h(\nu) : \nu \in K\}.$$  

Thus,

$$\nu_L(\{v \in T_1(X) : |i(\gamma_v)/T^2 - 1/(2\pi^2(g-1))| \geq \epsilon\})$$

$$= \nu_L(\{v \in T_1(X) : \gamma_v \in K\}) = O(e^{-\eta T}),$$

where $\eta := \inf \{1 - h(\nu) : \nu \in K\}$. Since $\nu_L$ is the unique measure with maximum entropy $h(\nu_L) = 1$ and $\nu_L \not\in K$, we always have $\eta > 0$. □

**Proof of Corollary 1** Let $T, \epsilon > 0$. Consider the set

$$\mathcal{O}(T, \epsilon) := \{\gamma \in \mathcal{G}_T : |i(\gamma)/T^2 - 1/(2\pi^2(g-1))| \geq \epsilon\}.$$  

Since $\mathcal{O}(T, \epsilon) = \mathcal{G}_T \setminus \{\gamma \in \mathcal{G}_T : |i(\gamma)/T^2 - 1/(2\pi^2(g-1))| < \epsilon\}$, it is enough to prove that

$$\lim_{T \to \infty} \frac{1}{N(T)} \# \mathcal{O}(T, \epsilon) = 0.$$  

Let $\epsilon > 0$. By Theorem 3 there is $R_1 := R_1(\epsilon)$ such that for every $T > R_1$,

$$\frac{1}{N(t)} \sum_{\gamma \in \mathcal{G}_T} \frac{\zeta_\gamma}{l(\gamma)}(\mathcal{O}(T, \epsilon)) < \nu_L(\mathcal{O}(T, \epsilon)) + \epsilon/2.$$  

And, by Theorem 3 there exist $\eta := \eta(\epsilon), C := C(\epsilon), R_2 := R_2(\epsilon) > 0$ such that for $t \geq R_2$,

$$\nu_L(\mathcal{O}(t, \epsilon)) \leq Ce^{-\eta t} < \epsilon/2.$$  

Thus, taking $R = \max\{R_1, R_2\}$, we get for $T > R$ that

$$\frac{1}{N(T)} \# \mathcal{O}(T, \epsilon) = \frac{1}{N(T)} \sum_{\gamma \in \mathcal{G}_T} \frac{\zeta_\gamma}{l(\gamma)}(\mathcal{O}(T, \epsilon)) \leq Ce^{-\eta T} + \epsilon/2 < \epsilon.$$  

Since $\epsilon$ was arbitrarily chosen, we conclude

$$\lim_{T \to \infty} \frac{1}{N(T)} \# \mathcal{O}(T, \epsilon) = 0$$  

and the result of the corollary. □
Another deviation result by Y. Kifer given in [7, Theorem 2.1], similar to Theorem 6, states that the portion of “irregular” geodesics vanishes exponentially fast.

**Theorem 7 (Kifer).** Let $U$ be an open neighborhood of $\nu_L$ in $\mathcal{P}$. Then, there exists $\eta > 0$ such that

$$
\frac{1}{N(t)} \# \{ \gamma \in G_t : \mu^\gamma/l(\gamma) \notin U \} = O(e^{-\eta t}),
$$

as $t \to \infty$. Moreover, $\eta = \inf_{\nu \in U} \{ 1 - h(\nu) \}$.

Theorems 5 and 7 imply our main result Theorem 11.

**Proof of Theorem 11** Let $s, t, \epsilon > 0$ with $s \leq t$ and consider the following function

$$
\eta : \mathcal{P} \times \mathcal{P} \to \mathbb{R}
\quad (\mu, \nu) \mapsto \eta(\mu, \nu) = \frac{1}{(\mu^a/\ell(\mu)) - 1/(2\pi^2(g - 1))}.
$$

The function $\eta$ is continuous because it is the quotient of two continuous functions, the intersection form function (by Theorem 5) and the function that assigns to a pair of measures the product of the lengths of their associated currents. As a result, the set $Z = \eta^{-1}(1/(2\pi^2(g - 1)) - \epsilon, 1/(2\pi^2(g - 1)) + \epsilon)$, the preimage of the ball of radius $\epsilon$ centered at $h(\nu_L) = e(\nu_L)^2 = 1/(2\pi^2(g - 1))$, is an open subset of $\mathcal{P} \times \mathcal{P}$.

Let $\mathcal{W}_{s,t} = \{(\alpha, \beta) \in G_s \times G_t : |(\mu^a \times \mu^b)/(\alpha)l(\beta) - 1/(2\pi^2(g - 1))| < \epsilon \}.$

Note that $\mathcal{W}_{s,t} = \{(\alpha, \beta) \in G_s \times G_t : \mu^a/\ell(\alpha), \mu^b/\ell(\beta) \in Z \}.$

Since $Z$ is an open set of the product topology of $\mathcal{P} \times \mathcal{P}$, there exist $U, V \subseteq \mathcal{P}$ open neighborhoods of $\nu_L$ such that $U \times V \subseteq \mathcal{Z}$.

Consider the sets $U_s := \{ \gamma \in G_s : \mu^\gamma/l(\gamma) \in U \}$ and $V_t := \{ \gamma \in G_t : \mu^\gamma/l(\gamma) \in V \}.$ Then, $U_s \times V_t \subseteq \mathcal{W}_{s,t}.$

By Theorem 7, there exist $\eta_1 := \eta_1(U), \eta_2 := \eta_2(V), T_1, T_2, C_1, C_2 > 0$ such that for $s \geq T_1$ and $t \geq T_2$,

$$
\frac{\# G_s \setminus U_s}{N(s)} \leq C_1 e^{-\eta_1 s} \quad \text{and} \quad \frac{\# G_t \setminus V_t}{N(t)} \leq C_2 e^{-\eta_2 t}.
$$

Thus, taking $T = \max\{T_1, T_2\},$ $C = C_1 + C_2 + C_1 C_2$ and $\eta = \min\{\eta_1, \eta_2\},$ we get

$$
\frac{\# (G_s \times G_t) \setminus \mathcal{W}_{s,t}}{N(s)N(t)} \leq \frac{\# G_s \setminus U_s \times \# G_t \setminus V_t}{N(s)N(t)} + \frac{\# G_s \setminus U_s \times \# G_t}{N(s)N(t)} + \frac{\# G_t \setminus V_t \times \# U_s}{N(s)N(t)}
$$

$$
\leq \frac{C_1 C_2 + C_1}{e^{\eta_1 s} e^{\eta_2 t}} + \frac{C_2}{e^{\eta_2 t}} \leq \frac{C}{e^{\eta t}}
$$

whenever $T \leq s \leq t.$ Hence, we conclude the result of the theorem. \(\square\)

The other key point for the proof of Theorem 2 is to find an adequate bound for the intersection numbers of the pairs of closed geodesics of $X$. This was achieved by Basmajian in [11, Theorem 1.2]. Here we give a different bound with a different proof. This bound can also be deduced by Basmajian’s techniques.

**Proposition 1 (Basmajian).** Let $\alpha, \beta \in \mathcal{G}$ and $\varepsilon_0 = \text{inj} X$, the injectivity radius of $X$. Then $i(\alpha, \beta) \leq 4l(\alpha)l(\beta)/\varepsilon_0^2.$
Proof. Let $\alpha : [0, l(\alpha)] \to X$ and $\beta : [0, l(\beta)] \to X$ be closed geodesics of $X$. Let $\bar{\alpha}$ be a subarc of $\alpha$ of length less than $\varepsilon_0/2$ for which $i(\bar{\alpha}, \beta)$ is the largest. Hence, $i(\alpha, \beta) \leq ([l(\alpha)/(\varepsilon_0/2)] + 1)i(\bar{\alpha}, \beta)$. Let $\{x_1, x_2, \ldots, x_n\}$ be the ordered set of points of intersection of $\bar{\alpha}$ and $\beta$, where $\beta^{-1}(x_i) \leq \beta^{-1}(x_{i+1})$, for $1 \leq i \leq n - 1$, and $n = i(\bar{\alpha}, \beta)$. Let $\beta_k$ be the subarc of $\beta$ from $x_k$ to $x_{k+1}$, for $1 \leq k \leq n - 1$ and $\beta_n$ the subarc of $\beta$ joining $x_1$ and $x_n$ which does not intersect $\bar{\alpha}$. Similarly, let $\tilde{\alpha}_k$ be the subarc of $\alpha$ from $x_k$ to $x_{k+1}$, for $1 \leq k \leq n - 1$, and $\tilde{\alpha}_n$ be the subarc joining $x_1$ and $x_n$ not intersecting $\bar{\alpha}$. Consider $\gamma_k$ the concatenation of $\tilde{\alpha}_k$ and $\beta_k$, for $1 \leq k \leq n$. Thus, $\gamma_k$ is an essential loop of $X$, for $1 \leq k \leq n$. Hence, $\varepsilon_0 < l(\gamma_k) = l(\tilde{\alpha}_k) + l(\beta_k) \leq \varepsilon_0/2 + l(\beta_k)$, which implies $\varepsilon_0/2 \leq l(\beta_k)$, for $1 \leq k \leq n$. Consequently, $n < l(\beta)/(\varepsilon_0/2)$. Therefore,

$$i(\alpha, \beta) \leq \left(\frac{l(\alpha)}{\varepsilon_0/2} + 1\right)i(\bar{\alpha}, \beta) \leq \frac{l(\alpha) l(\beta)}{\varepsilon_0/2 \varepsilon_0/2} = \frac{4l(\alpha)l(\beta)}{\varepsilon_0^2}.$$ 

Hence, $\varepsilon_n \leq l(\gamma_k)$ for $1 \leq k \leq n$. Consequently, $n = i(\bar{\alpha}, \beta)$. Let $\gamma = \alpha \cup \beta$ be the concatenation of $\bar{\alpha}$ and $\beta$. Thus, $\gamma$ is an essential loop of $X$.

$\square$

Theorem 2 is a straightforward result of the following result, whose proof is a simple computation which uses Theorem 1.

Theorem 8. \[ \lim_{N(s)N(t) \to \infty} \sum_{(\alpha, \beta) \in \mathcal{G}_s \times \mathcal{G}_t} \frac{\langle \mu^2 \times \mu^2 \rangle(\delta_{\alpha})}{l(\alpha)l(\beta)} \sim \frac{1}{2\pi^2(g-1)} \] as $s, t \to \infty$.

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