The aim of this paper is to initiate a study of geometric divisors of Brill-Noether type on the moduli space $\overline{S}_g$ of spin curves of genus $g$. The moduli space $\overline{S}_g$ is a compactification the parameter space $S_g$ of pairs $[C, \eta]$, consisting of a smooth genus $g$ curve $C$ and a theta-characteristic $\eta \in \text{Pic}^{g-1}(C)$, see [C]. The study of the birational properties of $\overline{S}_g$ as well as other moduli spaces of curves with level structure has received an impetus in recent years, see [BV], [FL], [F2], [Lud], to mention only a few results. Using syzygy divisors, it has been proved in [FL] that the Prym moduli space $\overline{R}_g := \overline{M}_g(B\mathbb{Z}_2)$ classifying curves of genus $g$ together with a point of order 2 in the Jacobian variety, is a variety of general type for $g \geq 13$ and $g \neq 15$. The moduli space $\overline{S}^+_g$ of even spin curves of genus $g$ is known to be of general type for $g > 8$, uniruled for $g < 8$, see [F2], whereas the Kodaira dimension of $\overline{S}^{-}_g$ is equal to zero, [FV]. This was the first example of a naturally defined moduli space of curves of genus $g \geq 2$, having intermediate Kodaira dimension. An application of the main construction of this paper, gives a new way of computing the class of the divisor $\Theta_{\text{null}}$ of vanishing theta-nulls on $\overline{S}^+_g$, reproving thus the main result of [F2].

Virtually all attempts to show that a certain moduli space $M_{g,n}$ is of general type, rely on the calculation of certain effective divisors $D \subset M_{g,n}$ enjoying extremality properties in their effective cones $\text{Eff}(M_{g,n})$, so that the canonical class $K_{M_{g,n}}$ lies in the cone spanned by $[D]$, boundary classes $\delta_i$, tautological classes $\lambda, \psi_1, \ldots, \psi_n$, and possible other effective geometric classes. Examples of such a program being carried out, can be found in [EH2], [HM]-for the case of Brill-Noether divisors on $M_g$ consisting of curves with a $g^r_d$ when $\rho(g, r, d) = -1$, [Log]-where pointed Brill-Noether divisors on $M_{g,n}$ are studied, and [FL]-for the case of koszul divisors on $M_g$, which provide counterexamples to the Slope Conjecture on $M_g$. A natural question is what the analogous geometric divisors on the spin moduli space of curves $\overline{S}_g$ should be?

In this paper we propose a construction for spin Brill-Noether divisors on both spaces $\overline{S}^+_g$ and $\overline{S}^-_g$, defined in terms of the relative position of theta-characteristics with respect to difference varieties on Jacobians. Precisely, we fix integers $r, s \geq 1$ such that $d := rs + r \equiv 0$ mod 2, and then set $g := rs + s$. One can write $d = 2i$. By standard Brill-Noether theory, a general curve $[C] \in M_g$ carries a finite number of (necessarily complete and base point free) linear series $g^r_d$. One considers the following loci of spin curves (both odd and even)

$$U^r_{g,d} := \{[C, \eta] \in \overline{S}^+_g : \exists L \in W^r_d(C) \text{ such that } \eta \otimes L^\vee \in C_{g-i-1} - C_i\}.$$

Thus $U^r_{g,d}$ consists of spin curves such that the embedded curve $C \left[\eta \otimes L^\vee\right] \mathbb{P}^{d-1}$ admits an $i$-secant $(i - 2)$-plane. We shall prove that for $s \geq 2$, the locus $U^r_{g,d}$ is always a divisor on

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$S^+_g$, and we find a formula for the class of its compactification in $\overline{S}^+_g$. For simplicity, we display this formula in the introduction only in the case $r = 1$, when $g \equiv 2 \pmod{4}$.

**Theorem 0.1.** We fix an integer $a \geq 1$ and set $g := 4a + 2$. The locus

$$U^1_{4a+2,2a+2} := \{(C, \eta) \in S_{4a+2}^+ : \exists L \in W_{2a+2}(C) \text{ such that } \eta \otimes L^r \subset C_{3a} - C_{a+1} \}$$

is an effective divisor and the class of its compactification in $\overline{S}^+_g$ is given by

$$\overline{U}^1_{4a+2,2a+2} \equiv \left( \frac{4a}{a} \right) \left( \frac{4a + 2}{2a} \right) \frac{a + 2}{8(2a+1)(4a + 1)} \left( 192a^3 + 736a^2 + 692a + 184 \right) \lambda - 
\left( 32a^3 + 104a^2 + 82a + 19 \right) a_0 - \left( 64a^3 + 176a^2 + 148a + 36 \right) \beta_0 - \cdots \right) \in \text{Pic}(\overline{S}^+_g).
$$

To specialize further, in Theorem 0.1 we set $a = 1$, and find the class of (the closure of) the locus of spin curves $[C, \eta] \in S^+_2$, such that there exists a pencil $L \in W_3^1(C)$ for which the linear series $C \to \mathbb{P}^1$ is not very ample:

$$\overline{U}^1_{6,4} \equiv 451\lambda - \frac{237}{4} a_0 - 106\beta_0 - \cdots \in \text{Pic}(\overline{S}^+_2).$$

The case $s = 1$, when necessarily $L = K_C \subset W_{2g-2}^1(C)$, produces a divisor only on $S^+_g$, and we recover in this way the main calculation from [F2], used to prove that $S^+_g$ is a variety of general type for $g > 8$. We recall that $\Theta_{\text{null}} := \{(C, \eta) \in S_g^+ : H^0(C, \eta) \neq 0\}$ denotes the divisor of vanishing theta-nulls.

**Theorem 0.2.** Let $\pi : \overline{S}^+_g \to \overline{M}_g$ be the ramified covering which forgets the spin structure. For $g \geq 3$, one has the following equality $\overline{U}^{g-1}_{g,2g-2} = 2 \cdot \Theta_{\text{null}}$ of codimension 1-cycles on the open subvariety $\pi^{-1}(\overline{M}_g \cup \Delta_0)$ of $\overline{S}^+_g$. Moreover, there is an equality of classes

$$\overline{U}^{g-1}_{g,2g-2} \equiv 2 \cdot \Theta_{\text{null}} \equiv \frac{1}{2} \lambda - \frac{1}{8} a_0 - 0 \cdot \beta_0 - \cdots \in \text{Pic}(\overline{S}^+_g).$$

We remark once more, the low slope of the divisor $\Theta_{\text{null}}$. No similar divisor with such remarkable class is known to exist on $\overline{M}_g$. In Section 4, we present a third way of calculating the class $[\Theta_{\text{null}}]$, by rephrasing the condition that a curve $C$ have a vanishing theta-null $\eta$, if and only if, for a pencil $A$ on $C$ of minimal degree, the multiplication map of sections

$$H^0(C, A) \otimes H^0(C, A \otimes \eta) \to H^0(C, A^2 \otimes \eta)$$

is not an isomorphism. For $[C] \in \overline{M}_g$ sufficiently general, we note that

$$\dim H^0(C, A) \otimes H^0(C, A \otimes \eta) = \dim H^0(C, A^2 \otimes \eta).$$

In this way, $\Theta_{\text{null}}$ appears as the push-forward of a degeneracy locus of a morphism between vector bundles of the same rank defined over a Hurwitz stack of coverings. To compute the push-forward of tautological classes from a Hurwitz stack, we use the techniques developed in [F1] and [Kh].

In the last section of the paper, we study the divisor $\overline{V}_{g,1}$ on the universal curve $\overline{M}_{g,1}$, which consists of points in the support of odd theta-characteristics. This divisor, somewhat similar to the divisor $\overline{W}_g$ of Weierstrass points on $\overline{M}_{g,1}$, cf. [Cu], should be of some importance in the study of the birational geometry of $\overline{M}_{g,1}$:
The class of the compactification in $\overline{M}_{g,1}$ of the effective divisor
\[ \Theta_{g,1} := \{[C, q] \in M_{g,1} : q \in \text{supp(}\eta) \text{ for some } [C, \eta] \in S^{-}_g \} \]
is given by the following formula:
\[ \Theta_{g,1} \equiv 2^{g-3} \left( (2^g-1) \left( \lambda + 2\psi \right) - 2^{g-3} \delta_{\text{irr}} - (2^g-2)\delta_1 - \sum_{i=1}^{g-1} (2^i+1)(2^g-1)\delta_i \right) \in \text{Pic}(\overline{M}_{g,1}). \]

When $g = 2$, the divisor $\Theta_2$ specializes to the divisor of Weierstrass points:
\[ \Theta_{2,1} = W_2 := \{[C, q] \in M_{2,1} : q \in C \text{ is a Weierstrass point} \}. \]

If we use Mumford’s formula $\lambda = \delta_0/10 + \delta_1/5 \in \text{Pic}(\overline{M}_2)$, Theorem 0.3 reads
\[ \Theta_{2,1} \equiv \frac{3}{2} \lambda + 3\psi - \frac{1}{4} \delta_{\text{irr}} - \frac{3}{2} \delta_1 = -\lambda + 3\psi - \delta_1 \in \text{Pic}(\overline{M}_{2,1}), \]
that is, we recover the formula for the class of the Weierstrass divisor on $\overline{M}_{2,1}$, cf. \([EH2]\).

When $g = 3$, the condition $[C, q] \in \Theta_{3,1}$ states that the point $q \in C$ lies on one of the 28 bitangent lines of the canonically embedded curve $C \to \mathbb{P}^2$.

**Corollary 0.4.** The class of the compactification in $\overline{M}_{3,1}$ of the bitangent locus
\[ \Theta_{3,1} := \{[C, q] \in M_{3,1} : q \text{ lies on a bitangent of } C \} \]
is equal to $\Theta_{3,1} \equiv 7\lambda + 14\psi - 2\delta_{\text{irr}} - 5\delta_1 - 5\delta_2 \in \text{Pic}(\overline{M}_{3,1})$.

If $p : \overline{M}_{g,1} \to \overline{M}_g$ is the map forgetting the marked point, we note the equality
\[ D_3 \equiv p^*(\overline{M}_{3,2}^1) + 2 : \overline{W}_3 + 2\psi \in \text{Pic}(\overline{M}_{3,1}), \]
where $\overline{W}_3 \equiv -\lambda + 6\psi - 3\delta_1 - \delta_2$ is the divisor of Weierstrass points on $\overline{M}_{3,1}$. Since the class $\psi \in \text{Pic}(\overline{M}_{3,1})$ is big and nef, it follows that $\Theta_{3,1}$ (unlike the divisor $\Theta_{2,1} \in \text{Pic}(\overline{M}_{2,1})$), lies in the interior of the cone of effective divisors $\text{Eff}(\overline{M}_{3,1})$, or it other words, it is big. In particular, it cannot be contracted by a rational map $\overline{M}_{3,1} \to X$ to any projective variety $X$. This phenomenon extends to all higher genera:

**Corollary 0.5.** For every $g \geq 3$, the divisor $\Theta_{g,1} \in \text{Eff}(\overline{M}_{g,1})$ is big.

It is not known whether the Weierstrass divisor $\overline{W}_g$ lies on the boundary of the effective cone $\text{Eff}(\overline{M}_{g,1})$ for $g$ sufficiently large.

1. **Generalities about $\overline{S}_g$**

As usual, we follow that the convention that if $\mathbf{M}$ is a Deligne-Mumford stack, then $M$ denotes its associated coarse moduli space. We first recall basic facts about Cornalba’s stack of stable spin curves $\pi : \overline{S}_g \to \overline{M}_g$, see \([C], [F2], [Luc]\) for details and other basic properties. If $X$ is a nodal curve, a smooth rational component $R \subset X$ is said to be exceptional if $\#(R \cap \overline{X} - R) = 2$. The curve $X$ is said to be quasi-stable if $\#(R \cap \overline{X} - R) \geq 2$ for any smooth rational component $R \subset X$, and moreover, any two exceptional components of $X$ are disjoint. A quasi-stable curve is obtained from a stable curve by possibly inserting a rational curve at each of its nodes. We denote by $[\text{st}(X)] \in \overline{M}_g$ the stable model of the quasi-stable curve $X$. 

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Definition 1.1. A spin curve of genus $g$ consists of a triple $(X, \eta, \beta)$, where $X$ is a genus $g$ quasi-stable curve, $\eta \in \text{Pic}^{g-1}(X)$ is a line bundle of degree $g - 1$ such that $\eta_R = \mathcal{O}_R(1)$ for every exceptional component $R \subset X$, and $\beta : \eta^{\otimes 2} \to \omega_X$ is a sheaf homomorphism which is generically non-zero along each non-exceptional component of $X$.

Stable spin curves of genus $g$ form a smooth Deligne-Mumford stack $\overline{S}_g$ which splits into two connected components $\overline{S}_g^+$ and $\overline{S}_g^-$, according to the parity of $h^0(X, \eta)$. Let $f : C \to \overline{S}_g$ be the universal family of spin curves of genus $g$. In particular, for every point $[X, \eta, \beta] \in \overline{S}_g$, there is an isomorphism between $f^{-1}([X, \eta, \beta])$ and the quasi-stable curve $X$. There exists a (universal) spin line bundle $\mathcal{P} \in \text{Pic}(C)$ of relative degree $g - 1$, as well as a morphism of $\mathcal{O}_C$-modules $B : \mathcal{P}^{\otimes 2} \to \omega_f$ having the property that $\mathcal{P}|_{f^{-1}([X, \eta, \beta])} = \eta$ and $B|_{f^{-1}([X, \eta, \beta])} = \beta : \eta^{\otimes 2} \to \omega_X$, for all spin curves $[X, \eta, \beta] \in \overline{S}_g$. Throughout we use the canonical isomorphism $\text{Pic}(\overline{S}_g)^{\mathbb{Q}} \cong \text{Pic}(\overline{S}_g)^{\mathbb{Q}}$ and we make little distinction between line bundles on the stack and the corresponding moduli space.

1.1. The boundary divisors of $\overline{S}_g$.

We discuss the structure of the boundary divisors of $\overline{S}_g$ and concentrate on the case of $\overline{S}_g^+$, the differences compared to the situation on $\overline{S}_g^-$ being minor. We describe the pull-backs of the boundary divisors $\Delta_i \subset \overline{M}_g$ under the map $\pi$. First we fix an integer $1 \leq i \leq [g/2]$ and let $[X, \eta, \beta] \in \pi^{-1}([C \cup_y D])$, where $[C, y] \in \mathcal{M}_{i,1}$ and $[D, y] \in \mathcal{M}_{g-i,1}$. For degree reasons, then $X = C \cup_{y_1} R \cup_{y_2} D$, where $R$ is an exceptional component such that $C \cap R = \{y_1\}$ and $D \cap R = \{y_2\}$. Furthermore $\eta = (\eta_C, \eta_D, \eta_R = \mathcal{O}_R(1)) \in \text{Pic}^{g-1}(X)$, where $\eta_C^{\otimes 2} = K_C$ and $\eta_D^{\otimes 2} = K_D$. The theta-characteristics $\eta_C$ and $\eta_D$ have the same parity in the case of $\overline{S}_g^+$ (and opposite parities for $\overline{S}_g^-$). One denotes by $A_i \subset \overline{S}_g^+$ the closure of the locus corresponding to pairs of pointed spin curves

$$[[C, y, \eta_C], [D, y, \eta_D]] \in \mathcal{S}_{i,1}^+ \times \mathcal{S}_{g-i,1}^-$$

and by $B_i \subset \overline{S}_g^+$ the closure of the locus corresponding to pairs

$$[[C, y, \eta_C], [D, y, \eta_D]] \in \mathcal{S}_{i,1}^- \times \mathcal{S}_{g-i,1}^+.$$

If $\alpha := [A_i]$, $\beta_i := [B_i] \in \text{Pic}(\overline{S}_g^+)$, we have the relation $\pi^*(\delta_i) = \alpha_i + \beta_i$.

Next, we describe $\pi^*(\delta_0)$ and pick a stable spin curve $[X, \eta, \beta]$ such that $\text{st}(X) = C_{yy} := C/y \sim q$, with $[C, y, q] \in \mathcal{M}_{g-1,2}$. There are two possibilities depending on whether $X$ possesses an exceptional component or not. If $X = C_{yy}$ and $\eta_C := \nu^*(\eta)$ where $\nu : C \to X$ denotes the normalization map, then $\eta_C^{\otimes 2} = K_C(y + q)$. For each choice of $\eta_C \in \text{Pic}^{g-1}(C)$ as above, there is precisely one choice of gluing the fibres $\eta_C(y)$ and $\eta_C(q)$ such that $h^0(X, \eta) \equiv 0 \mod 2$. We denote by $A_0$ the closure in $\overline{S}_g^+$ of the locus of points $[C_{yy}, \eta_C \in \text{Pic}^{g-1}(C), \eta_C^{\otimes 2} = K_C(y + q)]$ as above.

If $X = C \cup_{(y,q)} R$, where $R$ is an exceptional component, then $\eta_C := \eta \otimes \mathcal{O}_C$ is a theta-characteristic on $C$. Since $H^0(X, \omega) \cong H^0(C, \omega_C)$, it follows that $[C, \eta_C] \in \mathcal{S}_{g-1}^+$. We denote by $B_0 \subset \overline{S}_g^+$ the closure of the locus of points

$$[[C \cup_{(y,q)} R, \eta_C \in \sqrt{K_C}, \eta_R = \mathcal{O}_R(1)] \in \overline{S}_g^+.$$
A local analysis carried out in [C], shows that $B_0$ is the branch locus of $\pi$ and the ramification is simple. If $\alpha_0 = [A_0] \in \text{Pic}(S_g^+) \subset \text{Pic}(\Sigma_g^+)$ and $\beta_0 = [B_0] \in \text{Pic}(\Sigma_g^+)$, we have the relation
\begin{equation}
\pi^*(\delta_0) = \alpha_0 + 2\beta_0.
\end{equation}

2. DIFFERENCE VARIETIES AND THETA-CHARACTERISTICS

We describe a way of calculating the class of a series of effective divisors on both moduli spaces $\Sigma_g^-$ and $\Sigma_g^+$, defined in terms of the relative position of a theta-characteristic with respect to the divisorial difference varieties in the Jacobian of a curve. These loci, which should be thought of as divisors of Brill-Noether type on characteristic with respect to the divisorial difference varieties in the Jacobian of a curve.

These loci, which should be thought of as divisors of Brill-Noether type on characteristic with respect to the divisorial difference varieties in the Jacobian of a curve.

The next result from [FMP] provides a scheme-theoretic identification of divisors on the Jacobian variety
\begin{equation}
C_{g-i-1} - C_i = \Theta_{\wedge^iQ_C} \subset \text{Pic}^{g-2i-1}(C),
\end{equation}
where the right-hand side denotes the Raynaud locus \[\mathcal{R}\] defined in terms of the relative position of a theta-characteristic with respect to the divisorial difference varieties in the Jacobian of a curve.

The non-vanishing $H^0(C, \wedge^iQ_C \otimes \xi) \neq 0$ for all line bundles $\xi = \mathcal{O}_C(D - E)$, where $D \in C_{g-i-1}$ and $E \in C_i$, is a consequence of the thrust of [FMP] that the reverse inclusion $\Theta_{\wedge^iQ_C} \subset C_{g-i-1} - C_i$ also holds. Moreover, identification (2) shows that, somewhat similarly to Riemann’s Singularity Theorem, the product $C_{g-i-1} \times C_i$ can be thought of as a canonical desingularization of the generalized theta-divisor $\Theta_{\wedge^iQ_C}$.

We fix integers $r, s > 0$ and set $d := rs + r, g := rs + s$, therefore the Brill-Noether number $\rho(g, r, d) = 0$. We assume moreover that $d \equiv 0 \mod 2$, that is, either $r$ is even or $s$ is odd, and write $d = 2i$. We define the following loci in the spin moduli space $\Sigma_g^+$:
\begin{equation}
U_{g,d} := \{ [C, \eta] \in \Sigma_g^+ \mid \exists L \in W_d^i(C) \text{ such that } \eta \otimes L^\vee \in C_{g-i-1} - C_i \}.
\end{equation}

Using (2), the condition $[C, \eta] \in U_{g,d}$ can be rewritten in a determinantal way as,
\begin{equation}
H^0(C, \wedge^iM_{K_C} \otimes \eta \otimes L) \neq 0.
\end{equation}
Tensoring by $\eta \otimes L$ the exact sequence coming from the definition of $M_{K_C}$, namely
\[
0 \rightarrow \wedge^i M_{K_C} \rightarrow \wedge^i H^0(C, K_C) \otimes \mathcal{O}_C \rightarrow \wedge^{i-1} M_{K_C} \otimes K_C \rightarrow 0,
\]
then taking global sections and finally using that $M_{K_C}$ (hence all of its exterior powers) are semi-stable vector bundles, we find that $[C, \eta] \in \mathcal{U}_{g,d}$ if and only if the map
\[
(3) \quad \phi(C, \eta, L) : \wedge^i H^0(C, K_C) \otimes H^0(C, \eta \otimes L) \rightarrow H^0(C, \wedge^{i-1} M_{K_C} \otimes K_C \otimes \eta \otimes L)
\]
is not an isomorphism for a certain $L \in W_d^r(C)$. Since $\mu(\wedge^{i-1} M_{K_C} \otimes K_C \otimes \eta \otimes L) \geq 2g-1$ and $\wedge^{i-1} M_{K_C}$ is a semi-stable vector bundle on $C$, it follows that
\[
h^0(C, \wedge^{i-1} M_{K_C} \otimes K_C \otimes \eta \otimes L) = \chi(C, \wedge^{i-1} M_{K_C} \otimes K_C \otimes \eta \otimes L) = \binom{g}{i} d.
\]
We assume that $h^1(C, \eta \otimes L) = 0$. This condition is satisfied outside a locus of $S_g^+$ of codimension at least 2; if $H^1(C, \eta \otimes L) \neq 0$, then $H^1(C, K_C \otimes L^{(12)}) \neq 0$, in particular the Petri map
\[
\mu_0(C, L) : H^0(C, K_C) \otimes H^0(C, K_C \otimes L) \rightarrow H^0(C, K_C)
\]
is not injective. Then $h^0(C, L \otimes \eta) = d$ and we note that $\phi(C, \eta, L)$ is a map between vector spaces of the same rank. This obviously suggests a determinantal presentation of $\mathcal{U}_{g,d}$ as the (push-forward of) a degeneracy locus between vector bundles of the same rank. In what follows we extend this presentation over a partial compactification of $S_g^+$. We refer to [FH] Section 2 for a similar calculation over the Prym moduli stack $\overline{R}_g$.

We denote by $M_g^0 \subset M_g$ the open substack classifying curves $[C] \in M_g$ such that $W_{d-1}^r(C) = \emptyset$, $W_{d+1}^r(C) = \emptyset$ and moreover $H^1(C, L \otimes \eta) = 0$, for every $L \in W_d^r(C)$ and each odd-theta characteristic $\eta \in \text{Pic}^{g-1}(C)$. From general Brill-Noether theory one knows that $\text{codim}(M_g - M_g^0, M_g) \geq 2$. Then we define $\overline{\Delta}_g \subset \Delta_g$ to be the open substack consisting of 1-nodal stable curves $[C_{yq} := C/y \sim q]$, where $[C] \in M_{g-1}$ is a curve satisfying the Brill-Noether theorem and $y, q \in C$. We then set $\overline{M}_g^0 := M_g^0 \cup \overline{\Delta}_g$, hence $\overline{M}_g^0 \subset \overline{M}_g$ and then $S_g^0 = \pi^{-1}(\overline{M}_g^0) = (S_g^0)^+ \cup (S_g^0)^-$. Following [EH], [FH], we consider the proper Deligne-Mumford stack
\[
\sigma_0 : \mathcal{G}_d^r \rightarrow \overline{M}_g^0
\]
classifying pairs $[C, L]$ with $[C] \in \overline{M}_g^0$ and $L \in W_d^r(C)$. For any curve $[C] \in \overline{M}_g^0$ and $L \in W_d^r(C)$, we have that $h^0(C, L) = r+1$, that is, $\mathcal{G}_d^r$ parameterizes only complete linear series. For a point $[C_{yq} := C/y \sim q] \in \overline{\Delta}_g$, we have the identification
\[
\sigma_0^{-1}[C_{yq}] = \{ L \in W_d^r(C) : h^0(C, L \otimes \mathcal{O}_C(-y-q)) = r \},
\]
that is, we view linear series on singular curves as linear series on the normalization such that the divisor of the nodes imposes only one condition. We denote by $f_{d, g}^r : \mathcal{G}_{g, d}^r := \overline{M}_{g, 1} \times \overline{M}_g \mathcal{G}_d^r \rightarrow \mathcal{G}_d^r$ the pull-back of the universal curve $p : \overline{M}_{g, 1} \rightarrow \overline{M}_g$ to $\mathcal{G}_d^r$. Once we have chosen a Poincaré bundle $L$ on $\mathcal{G}_{g, d}^r$, we can form the three codimension 1 tautological classes in $A^1(\mathcal{G}_d^r)$:
\[
(4) \quad a := (f_{d, g}^r)_*(c_1(L)^2), \quad b := (f_{d, g}^r)_*(c_1(L) \cdot c_1(\omega_{f_d})), \quad c := (f_{d, g}^r)_*(c_1(\omega_{f_d})^2) = (\sigma_0)^*(\kappa_1(\overline{M}_g^0)).
\]
The dependence on $a, b, c$ on the choice of $L$ is discussed in both [2] and [FL]. We introduce the stack of $g_d$'s on spin curves

$$\sigma : \mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0) := \mathcal{E}_d^r \times_{\mathcal{M}_g^0} \mathcal{E}_d^r \to \mathcal{S}_g^0$$

and then the corresponding universal spin curve over the $g_d$ parameter space

$$f' : \mathcal{C}_d^r := C \times_{\mathcal{S}_g^0} \mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0) \to \mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0).$$

We note that $f'$ is a family of quasi-stable curves carrying at the same time a spin structure as well as a $g_d$. Just like in [FL], the boundary divisors of $\mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0)$ are denoted by the same symbols, that is, one sets $A'_0 := \sigma^*(A'_0)$ and $B'_0 := \sigma^*(B'_0)$ and then

$$\alpha_0 := [A'_0], \quad \beta_0 := [B'_0] \in A^1(\mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0)).$$

We observe that two tautological line bundles live on $\mathcal{C}_d^r$, namely the pull-back of the universal spin bundle $P_d^r \in \text{Pic}(\mathcal{C}_d^r)$ and a Poincaré bundle $L \in \text{Pic}(\mathcal{C}_d^r)$ singling out the $g_d$'s, that is, $L_{|f' := X, \eta, \beta, L} = L \in W_d^r(\mathcal{C})$, for each point $[X, \eta, \beta, L] \in \mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0)$. Naturally, one also has the classes $a, b, c \in A^1(\mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0))$ defined by the formulas (4).

The following result is easy to prove and we skip details:

**Proposition 2.1.** We denote by $f' : \mathcal{C}_d^r \to \mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0)$ the universal quasi-stable spin curve and by $P_d^r \in \text{Pic}(\mathcal{C}_d^r)$ the universal spin bundle of relative degree $g - 1$. One has the following formulas in $A^1(\mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0))$:

(i) $f'_*(c_1(\omega_{f'})) \cdot c_1(P_d^r) = \frac{1}{2}c$,

(ii) $f'_*(c_1(P_d^r)^2) = \frac{1}{4}c - \frac{1}{2}\beta_0$,

(iii) $f'_*(c_1(\mathcal{L}) \cdot c_1(P_d^r)) = \frac{1}{2}b$.

We determine the class of a compactification of $\mathcal{U}_{g,b}^r$ by pushing-forward a codimension 1 degeneracy locus via the map $\sigma : \mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0) \to \mathcal{S}_g^0$. To that end, we define a sequence of tautological vector bundles on $\mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0)$: First, for $l \geq 0$ we set

$$\mathcal{A}_{0,l} := f'_*(\mathcal{L} \otimes \omega_{f'}^l \otimes P_d^r).$$

It is easy to verify that $R^1 f'_*(\mathcal{L} \otimes \omega_{f'}^l \otimes P_d^r) = 0$, hence $\mathcal{A}_{0,l}$ is locally free over $\mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0)$ of rank equal to $h^0(X, L \otimes \omega_{X}^l \otimes \eta) = l(2g - 2) + d$. Next we introduce the global Lazarsfeld vector bundle $\mathcal{M}$ over $\mathcal{C}_d^r$ by the exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow (f')^*(f'_* \omega_{f'}) \longrightarrow \omega_{f'} \longrightarrow 0,$$

and then for all integers $a, j \geq 1$ we define the sheaf over $\mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0)$

$$\mathcal{A}_{a,j} := f'_*(\wedge^a \mathcal{M} \otimes \omega_{f'}^j \otimes \mathcal{L} \otimes P_d^r).$$

In a way similar to [FL] Proposition 2.5 one shows that $R^1 f'_*(\wedge^a \mathcal{M} \otimes \omega_{f'}^{i-a} \otimes \mathcal{L} \otimes P_d^r) = 0$, therefore by Grauert's theorem $\mathcal{A}_{a,i-a}$ is a vector bundle over $\mathcal{E}_d^r(\mathcal{S}_g^0/\mathcal{M}_g^0)$ of rank

$$\text{rk}(\mathcal{A}_{a,i-a}) = \chi(X, \wedge^a M_{\omega_X} \otimes \omega_{X}^{i-a} \otimes L \otimes \eta) = 2(i-a)g \binom{g-1}{a}.$$
Furthermore, for all $1 \leq a \leq i - 1$, the vector bundles $\mathcal{A}_{a,i-a}$ sit in exact sequences
\begin{equation}
0 \longrightarrow \mathcal{A}_{a,i-a} \longrightarrow \wedge^a f'_s(\omega_f) \otimes \mathcal{A}_{0,i-a} \longrightarrow \mathcal{A}_{a-1,i-a+1} \longrightarrow 0,
\end{equation}
where the right exactness boils down to showing that $H^1(X, \wedge^a M_X \otimes \omega_X^{i-a} \otimes \eta \otimes L) = 0$ for all $[X, \eta, \beta, L] \in \mathcal{G}_g(S_g^0/M_g^0)$.

We denote as usual $E := f'_s(\omega_f)$ the Hodge bundle over $\mathcal{G}_d(S_g^0/M_g^0)$ and then note that there exists a vector bundle map
\begin{equation}
\phi : \wedge^i E \otimes \mathcal{A}_{0,0} \rightarrow \mathcal{A}_{i-1,1}
\end{equation}
between vector bundles of the same rank over $\mathcal{G}_d(S_g^0/M_g^0)$. For $[C, \eta, L] \in \sigma^{-1}(M_g^0)$ the fibre of this morphism is precisely the map $\phi(C, \eta, L)$ defined by (3).

**Theorem 2.2.** The vector bundle morphism $\phi : \wedge^i E \otimes \mathcal{A}_{0,0} \rightarrow \mathcal{A}_{i-1,1}$ is generically non-degenerate over $\mathcal{G}_d(S_g^0/M_g^0)$. It follows that $U_{g,d}$ is an effective divisor over $S^+_g$ for all $s \geq 1$, and over $S_g^-$ as well for $s \geq 2$.

**Proof.** We specialize $C$ to a hyperelliptic curve, and denote by $A \in W^2_d(C)$ the hyperelliptic involution. The Lazarsfeld bundle splits into a sum of line bundles $Q_C \cong A^{\oplus(g-1)}$, therefore the condition $H^0(C, \wedge^i M_C \otimes \eta \otimes L) = 0$ translates into $H^0(C, \eta \otimes A^{\oplus i} \otimes L^v) = 0$. Suppose that $h^0(C, \eta \otimes A^{\oplus i} \otimes L^v) \geq 1$ for any $L = A^{\oplus r} \otimes \mathcal{O}_C(x_1 + \cdots + x_{d-2r}) \in W^2_d(C)$, where the $x_1, \ldots, x_{d-2r} \in C$ are arbitrarily chosen points. This implies that $h^0(C, \eta \otimes A^{\oplus(i-r)}) \leq d - 2r + 1$. Any theta-characteristic on $C$ is of the form
\[ \eta = A^{\oplus m} \otimes \mathcal{O}_C(p_1 + \cdots + p_{g-2m-1}), \]
where $1 \leq m \leq (g-1)/2$ and $p_1, \ldots, p_{g-2m-1} \in C$ are Weierstrass points. Choosing a theta-characteristic on $C$ for which $m \leq i - r - 1$ (which can be done in all cases except on $S_g^-$ when $i = r$), we obtain that $h^0(C, \eta \otimes A^{\oplus(i-r)}) \leq d - 2r$, a contradiction. 

**Proof of Theorem 0.1.** To compute the class of the degeneracy locus of $\phi$ we use repeatedly the exact sequence (5). We write the following identities in $A^1(\mathcal{G}_d(S_g^0/M_g^0))$:
\[ c_1(A_{i-1,1} - \wedge^i E \otimes \mathcal{A}_{0,0}) = \sum_{l=0}^i (-1)^{i-l-1} c_1(\wedge^{i-l} E \otimes \mathcal{A}_{0,i}) = \]
\[ = \sum_{l=0}^i (-1)^{i-l} \left[ (\begin{array}{c} 2l(g-1) + d \\ i-l \end{array}) \left( \begin{array}{c} g-1 \\ i-l \end{array} \right) c_1(E) + \left( \begin{array}{c} g \\ i-l \end{array} \right) c_1(\mathcal{A}_{0,i}) \right]. \]

Using Proposition 2.1 one can show via the Grothendieck-Riemann-Roch formula applied to $f'_s : C'_d \rightarrow \mathcal{G}_d(S_g^0/M_g^0)$ that one has that
\[ c_1(A_{0,i}) = \lambda + \left( \frac{l^2 - 1}{8} \right) c + \frac{1}{2} a + l b - \frac{1}{4} \beta_0 \in A^1(\mathcal{G}_d(S_g^0/M_g^0)). \]

To determine $\sigma_*(c_1(A_{i-1,1} - \wedge^i E)) \in A^1(S_g^0)$ we use [F1],[KH]: If $N := \deg(\sigma) = \#(W^2_d(C))$
denotes the number of $g^r_2$'s on a general curve $[C] \in \mathcal{M}_g$, then there exists a precisely described choice of a Poincaré bundle on $\mathcal{E}_{g,d}$ such that the push-forwards of the tautological classes on $\mathcal{G}^r_0(S_g^0/M_g^0)$ are given as follows (cf. [FL], [Kh] and especially [FL], Section 2, for the similar argument in the Prym case):

$$\sigma_*(a) = \frac{dN}{(g-1)(g-2)} \left( (gd - 2g^2 + 8d - 8g + 4)\lambda + \frac{1}{6}(2g^2 - gd + 3g - 4d - 2)(\alpha_0 + 2\beta_0) \right)$$

and

$$\sigma_*(b) = \frac{dN}{2g-2} \left( 12\lambda - \alpha_0 - 2\beta_0 \right) \in A^1(\mathcal{G}^r_0(S_g^0/M_g^0)).$$

One notes that $c_1(A_{i-1} - \Lambda \otimes \mathcal{A}_{0,0}) \in A^1(\mathcal{G}^r_0(S_g^0/M_g^0))$ does not depend of the Poincaré bundle. Using the previous formulas, after some arithmetic, one computes the class of

$$\bar{dN} = 220g_2 \alpha_a + 328g_a + 246r_a + 84r^2 a^2 + 3984g_a^2 + 1080r^2 a^2 + 528ra + 192r^4 a^4 + 384r^4 a^3 + 768r^3 a^4 + 960r^2 a^4 + 240r^4 a^2 + 384ra^4,$$

$$\bar{\alpha}_0 = 220r_a^2 + 536a^2 + 324a^4 + 74r^3 a + 160r^2 a + 268r^3 a^2 + 110r^2 a - 3r^2 - 12a^2$$

and

$$\bar{\beta}_0 = 96ra + 64r^4 a + 16r^4 a + 416ra^2 + 928r^3 a^3 + 448r^3 a^2 + 208r^2 a + 608r^3 a^2 + 256r^3 a^4 + 112r^3 a + 80r^4 a^2 + 320r^2 a^4 + 128r^4 a^2 + 464r^3 a^2 + 816r^2 a^2.$$

These formulas, though unwieldy, carry a great deal of information about $S_g$. In the simplest case, $s = 1$ (that is, $a = 0$) and $r = g - 1$, then necessarily $L = K_C \in W_{2g-2}(C)$ and the condition $\eta - K_C \in -C_{g-1}$ is equivalent to $H^0(C, \eta) \neq 0$. In this way we recover the theta-null divisor $\Theta_{\text{null}}$ on $S_g^+$, or more precisely also taking into account multiplicities [F2],

$$U_{g,2g-2}^{2-1} = 2 \cdot \Theta_{\text{null}}.$$

At the same time, on $S_g^+$ one does not get a divisor at all. In particular, we find that

$$\tilde{U}_{g,2g-2}^{3-1} \equiv 2 \cdot \Theta_{\text{null}} \equiv \frac{1}{2} \lambda - \frac{1}{8} \alpha_0 - 0 \cdot \beta_0 - \cdots \in \text{Pic}(S_g^+).$$

Another interesting case is when $r = 2$, hence $g = 3s$, $L \in W_{2s+2}(C)$ and the condition $\eta \otimes L^2 \in C_{2s-2} - C_{s+1}$ is equivalent to requiring that the embedded curve $C \frac{[\eta \otimes L]}{P_{2s+1}}$ has an $(s + 1)$-secant $(s - 1)$-plane:
Theorem 2.3. For $g = 3s, d = 2s + 2$, the class of the closure in $\overline{S}_g^+$ of the effective divisor

$$\mathcal{U}_{g,d}^2 := \{ [C, \eta] \in \overline{S}_g^{+} : \exists L \in W_{2s+2}^{2}(C) \text{ such that } \eta \otimes L^\vee \in C_{2s-2} - C_{s+1} \}$$

is given by the formula in $\text{Pic}(\overline{S}_g^+)$:

$$\mathcal{U}_{g,d}^2 \equiv \left( \frac{g}{s+2} \right) \left( \frac{g}{s,s,s} \right) \frac{1}{24g(g-1)^2(g-2)(s+1)^2} \left( 4(216s^4 + 513s^3 - 348s^2 - 387s + 18) \lambda - (144s^4 + 225s^3 - 268s^2 - 99s + 10) \alpha_0 - (288s^4 + 288s^3 + 320s^2 + 32) \beta_0 - \cdots \right).$$

For instance, for $g = 9$, we obtain the class of the closure of the locus spin curves $[C, \eta] \in \overline{S}_g^{+}$, for which there exists a net $L \in W_{9}^{2}(C)$ such that $\eta \otimes L^\vee \in C_4 - C_4$:

$$\mathcal{U}_{9,8}^2 \equiv 235 \cdot 35 \left( \frac{36}{5} \lambda - \alpha_0 - \frac{428}{235} \beta_0 - \cdots \right) \in \text{Pic}(\overline{S}_g^+).$$

3. THE CLASS OF $\overline{\Theta}_{\text{null}}$ ON $\overline{S}_g^+$: AN ALTERNATIVE PROOF USING THE HURWITZ STACK

We present an alternative way of computing the class of the divisor $[\overline{\Theta}_{\text{null}}]$ (in even genus), as the push-forward of a determinantal cycle on a Hurwitz scheme of degree $k$ coverings of genus $g$ curves. We set

$$g = 2k - 2, \quad r = 1, \quad d = k,$$

hence $\rho(g, 1, k) = 0$, and use the notation from the previous section. In particular, we have the proper morphism $\sigma_0 : \mathfrak{S}_k^1 \to \overline{M}_g^0$ from the Hurwitz stack of $g^1$'s, and the universal spin curve over the Hurwitz stack

$$f' : C_k^+ := C \times_{\overline{S}_g^+} \mathfrak{S}_k^1(\overline{S}_g^0/\overline{M}_g^0) \to \mathfrak{S}_k^1(\overline{S}_g^0/\overline{M}_g^0).$$

Once more, we introduce a number of vector bundles over $\mathfrak{S}_k^1(\overline{S}_g^0/\overline{M}_g^0)$: First, we set $\mathcal{H} := f'_* (\mathcal{L})$. By Grauert’s theorem, $\mathcal{H}$ is a vector bundle of rank $2$ over $\mathfrak{S}_k^1(\overline{S}_g^0/\overline{M}_g^0)$, having fibre $\mathcal{H}[X, \eta, \beta, L] = H^0(X, L)$, where $L \in W_k^1(X)$. Then for $j \geq 1$ we define

$$B_j := f'_*(\mathcal{L}^{\otimes j} \otimes \mathcal{P}_k^1).$$

Since $R^1 f'_*(\mathcal{L}^{\otimes j} \otimes \mathcal{P}_k^1) = 0$, we find that $B_j$ is a vector bundle over $\mathfrak{S}_k^1(\overline{S}_g^0/\overline{M}_g^0)$ of rank equal to $h^0(X, L^{\otimes j} \otimes \eta) = kj$.

Proposition 3.1. If $a, b, c$ are the codimension $1$ tautological classes on $\mathfrak{S}_k^1(\overline{S}_g^0/\overline{M}_g^0)$ defined by $\mathcal{A}$, then for all $j \geq 1$ one has the following formula in $A^1(\mathfrak{S}_k^1(\overline{S}_g^0/\overline{M}_g^0))$:

$$c_1(B_j) = \lambda - \frac{1}{8} c + \frac{j^2}{2} a - \frac{j}{4} b - \frac{1}{4} \beta_0.$$

Proof. We apply Grothendieck-Riemann-Roch to the morphism $f' : C_k^+ \to \mathfrak{S}_k^1(\overline{S}_g^0/\overline{M}_g^0)$:

$$c_1(B_j) = c_1(f'_*(\mathcal{L}^{\otimes j} \otimes \mathcal{P}_k^1)) =$$

$$= f'_* \left[ \left( 1 + c_1(\mathcal{L}^{\otimes j} \otimes \mathcal{P}_k^1) \right) \left( 1 + \frac{c_2(\mathcal{L}^{\otimes j} \otimes \mathcal{P}_k^1)}{2} \right) \left( 1 - \frac{c_1(\omega f')}{2} + \frac{c_2(\omega f')}{12} + [\text{Sing}(f')] \right) \right].$$
where \( \text{Sing}(f') \subset X_1^1 \) denotes the codimension 2 singular locus of the morphism \( f' \), therefore \( f'_*[\text{Sing}(f')] = \alpha_0 + 2\beta_0 \). We then use Mumford’s formula [HM] pulled back from \( \overline{M}_g^0 \) to \( \mathcal{G}_k^1(\overline{M}_g^0) \), to write that
\[
\kappa_1 = f'_*(c_1^2(\omega_{f'})) = 12\lambda - (\alpha_0 + 2\beta_0)
\]
and then note that \( f'_*(c_1(L) \cdot c_1(\mathcal{P}^1_k)) = 0 \) (the restriction of \( L \) to the exceptional divisor of \( f' : C_k^1 \to \mathcal{G}_k^1(\overline{M}_g^0) \) is trivial). Similarly, we note that \( f'_*(c_1(\omega_{f'}) \cdot c_1(\mathcal{P}^1_k)) = \epsilon/2 \).

Finally, we write that \( f'_*(c_1^2(\mathcal{P}^1_k)) = \epsilon/4 - \beta_0/2 \). \( \square \)

For \( j \geq 1 \) there are natural vector bundle morphisms over \( \mathcal{G}_k^1(\overline{M}_g^0) \)
\[
\chi_j : \mathcal{H} \otimes \mathcal{B}_j \to \mathcal{B}_{j+1}.
\]
Over a point \([C, \eta, L] \in \mathcal{S}_g^+ \times \mathcal{M}_g \mathcal{G}_k^1 \) corresponding to an even theta-characteristic \( \eta_C \) and a pencil \( L \in W^1_k(C) \), the morphism \( \chi_j \) is given by multiplications of global sections
\[
\chi_j[C, \eta, L] : H^0(C, L) \otimes H^0(C, \mathcal{L}^{\otimes j} \otimes \eta_C) \to H^0(C, \mathcal{L}^{\otimes (j+1)} \otimes \eta_C).
\]
In particular, \( \chi_1 : \mathcal{H} \otimes \mathcal{B}_1 \to \mathcal{B}_2 \) is a morphism between vector bundles of the same rank. From the base point free pencil trick, the degeneration locus \( Z_1(\chi_1) \) is (set-theoretically) equal to the inverse image \( \sigma^{-1}(\mathcal{S}_g^+ \cap (\overline{\mathcal{S}}_g^0)^+) \).

**Theorem 3.2.** We fix \( g = 2k - 2 \). The vector bundle morphism \( \chi_1 : \mathcal{H} \otimes \mathcal{B}_1 \to \mathcal{B}_2 \) defined over \( \mathcal{G}_k^1(\overline{M}_g^0) \) is generically non-degenerate and we have the following formula for the class of its degeneracy locus:
\[
[Z_1(\chi_1)] = c_1(\mathcal{B}_2 - \mathcal{H} \otimes \mathcal{B}_1) = \frac{1}{2}\lambda - \frac{1}{8}\alpha_0 + a - kc_1(\mathcal{H}) \in A^1(\mathcal{G}_k^1(\overline{M}_g^0)).
\]
The class of the push-forward \( \sigma_*[Z_1(\chi_1)] \) to \( \mathcal{S}_g^+ \) is given by the formula:
\[
\sigma_*(c_1(\mathcal{B}_2 - \mathcal{H} \otimes \mathcal{B}_1)) \equiv \frac{(2k-2)!}{k!(k-1)!} \left( \frac{1}{2}\lambda - \frac{1}{8}\alpha_0 - 0 \cdot \beta_0 \right) \equiv \frac{2(2k-2)!}{k!(k-1)!} \overline{\mathcal{S}}_{\Delta} \subset (\overline{\mathcal{S}}_g^0)^+ \in \text{Pic}(\mathcal{S}_g^+).
\]

**Proof.** The first part follows directly from Theorem 3.1. To determine the push-forward of codimension 1 tautological classes to \( (\overline{\mathcal{S}}_g^0)^+ \), we use again [F1, Kh]: One writes the following relations in \( A^1((\overline{\mathcal{S}}_g^0)^+) = A^1((\overline{\mathcal{S}}_g^0)^+) \):
\[
\sigma_*(a) = \text{deg}(\mathcal{G}_k^1/\overline{M}_g^0) \left( -3\frac{k}{2k-3} \lambda + \frac{k^2}{2(2k-3)}(\alpha_0 + 2\beta_0) \right),
\]
\[
\sigma_*(b) = \text{deg}(\mathcal{G}_k^1/\overline{M}_g^0) \left( \frac{6k}{2k-3} \lambda - \frac{k}{2(2k-3)}(\alpha_0 + 2\beta_0) \right),
\]
and
\[
\sigma_*(c_1(\mathcal{H})) = \text{deg}(\mathcal{G}_k^1/\overline{M}_g^0) \left( -3\frac{k+1}{2k-3} \lambda + \frac{k}{2(2k-3)}(\alpha_0 + 2\beta_0) \right),
\]
where
\[
N := \text{deg}(\mathcal{G}_k^1/\overline{M}_g^0) = \frac{(2k-2)!}{k!(k-1)!}
\]
denotes the Catalan number of linear series \( q_1^g \) on a general curve of genus \( 2k - 2 \). This yields yet another proof of the main result from [F2], in the sense that we compute the class of the divisor \( \Theta_{\text{null}} \) of vanishing theta-nulls:

\[
\sigma_s(Z_1(\chi_1)) = \deg(\mathcal{G}^s_k/\mathcal{M}_g^0) \left( \frac{1}{2} \lambda - \frac{1}{8} \alpha_0 \right) \equiv 2 \deg(\mathcal{G}^s_k/\mathcal{M}_g^0) [\Theta_{\text{null}}]_{[(S^0_g)^+]}.
\]

\( \Box \)

**Remark 3.3.** The multiplicity 2 appearing in the expression of \( \sigma_s(Z_1(\chi_1)) \) is justified by the fact that \( \dim \ker(\chi_1(t)) = h^0(C, \eta) \) for every \([C, \eta, L] \in \sigma^{-1}((S^0_g)^+)\). This of course is always an even number. Thus we have the equality cycles

\[
Z_1(\chi_1) = Z_2(\chi_1) = \{ t \in \mathcal{G}^s_k(\mathcal{M}_g^0) : \text{co-rank}(\phi_1(t)) \geq 2 \},
\]

that is \( \chi_1 \) degenerates in codimension 1 with corank 2, and \( Z_1(\chi_1) \) is an everywhere non-reduced scheme.

4. **The Divisor of Points of Odd Theta-Characteristics**

In this section we compute the class of the divisor \( \Theta_{g,1} \). The study of geometric divisors on \( \mathcal{M}_{g,1} \) begins with [Cu], where the locus of Weierstrass points is determined:

\[
\mathcal{W}_g \equiv -\lambda + \left( \frac{g+1}{2} \right) \psi - \sum_{i=1}^{g-1} \left( \frac{g-i+1}{2} \right) \delta_{i;1} \in \text{Pic}(\mathcal{M}_{g,1}).
\]

More generally, if \( \tilde{\alpha} : 0 \leq \alpha_0 \leq \ldots \leq \alpha_r \leq d - r \) is a Schubert index of type \((r, d)\) such that \( \rho(g, r, d) - \sum_{i=0}^{r} \alpha_i = -1 \), one defines the pointed Brill-Noether divisor \( \mathcal{M}_{g,r,d}(\tilde{\alpha}) \) as being the locus of pointed curves \([C, q] \in \mathcal{M}_{g,1} \) possessing a linear series \( l \in G^r_g(C) \) with ramification sequence \( \alpha'(q) \geq \tilde{\alpha} \). It follows from [EH3] that the cone spanned by the pointed Brill-Noether divisors on \( \mathcal{M}_{g,1} \) is 2-dimensional, with generators \([\mathcal{W}_g]\) and the pull-back of the Brill-Noether class from \( \mathcal{M}_g \). Our aim is to analyze the divisor \( \Theta_{g,1} \), whose definition is arguably simpler than that of the divisors \( \mathcal{M}_{g,r,d}(\tilde{\alpha}) \), and which seems to have been overlooked until now. A consequence of the calculation is that (as expected) \([\Theta_{g,1}] \) lies outside the Brill-Noether cone of \( \mathcal{M}_{g,1} \).

We begin by recalling basic facts about divisors on \( \mathcal{M}_{g,1} \). For \( i = 1, \ldots, g - 1 \), the divisor \( \Delta_i \) on \( \mathcal{M}_{g,1} \) is the closure of the locus of pointed curves \([C \cup D, q]\), where \( C \) and \( D \) are smooth curves of genus \( i \) and \( g - i \) respectively, and \( q \in C \). Similarly, \( \Delta_{\text{irr}} \) denotes the closure in \( \mathcal{M}_{g,1} \) of the locus of irreducible 1-pointed stable curves. We set \( \delta_i := [\Delta_i], \delta_{\text{irr}} := [\Delta_{\text{irr}}] \in \text{Pic}(\mathcal{M}_{g,1}) \), and recall that \( \psi \in \text{Pic}(\mathcal{M}_{g,1}) \) is the universal cotangent class. Clearly, \( p^*(\delta_{\text{irr}}) = \delta_{\text{irr}} \) and \( p^*(\delta_i) = \delta_i + \delta_{g-i} \in \text{Pic}(\mathcal{M}_{g,1}) \) for \( 1 \leq i \leq [g/2] \). For \( g \geq 3 \), the group \( \text{Pic}(\mathcal{M}_{g,1}) \) is freely generated by the classes \( \lambda, \psi, \delta_{\text{irr}}, \delta_1, \ldots, \delta_{g-1} \), cf. [AC1]. When \( g = 2 \), the same classes generate \( \text{Pic}(\mathcal{M}_{2,1}) \) subject to the Mumford relation

\[
\lambda = \frac{1}{10} \delta_{\text{irr}} + \frac{1}{5} \delta_1,
\]

expressing that \( \lambda \) is a boundary class. We expand the class \([\Theta_{g,1}] \) in this basis of \( \text{Pic}(\mathcal{M}_{g,1}) \),

\[
[\Theta_{g,1}] = a \lambda + b \psi - b_{\text{irr}} \delta_{\text{irr}} - \sum_{i=1}^{g-1} b_i \delta_i \in \text{Pic}(\mathcal{M}_{g,1}),
\]

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and determine the coefficients in a classical way, by understanding the restriction of $\tilde{\Theta}_{g,1}$ to sufficiently many geometric subvarieties of $\overline{M}_{g,1}$. To ease calculations, we set

$$N_g^- := 2^{g-1}(2^g - 1) \quad \text{and} \quad N_g^+ := 2^{g-1}(2^g + 1),$$

to be the number of odd (respectively even) theta-characteristic on a curve of genus $g$.

We define some test-curves in the boundary of $\overline{M}_{g,1}$. For an integer $2 \leq i \leq g - 1$, we choose general (pointed) curves $[C] \in M_i$ and $[D, x, q] \in M_{g-i+2}$. In particular, we may assume that $x, q \in D$ do not appear in the support of any odd theta-characteristic $\eta_D$ on $D$, and that $h^0(D, \eta_D^i) = 0$, for any even theta-characteristic $\eta_D^i$. By joining $C$ and $D$ at a variable point $x \in C$, we obtain a family of $1$-pointed stable curves

$$F_{g-i} = \{ [C \cup_x D, q] : x \in C \} \subset \Delta_{g-i} \subset \overline{M}_{g,1},$$

where the marked point $q \in D$ is fixed. It is clear that $F_{g-i} \cdot \delta_{g-i} = 2 - 2i$, $F_{g-i} \cdot \lambda = F_{g-i} \cdot \psi = 0$. Moreover, $F_{g-i}$ is disjoint from all the other boundary divisors of $\overline{M}_{g,1}$.

**Proposition 4.1.** For each $2 \leq i \leq g - 1$, one has that $b_{g-i} = N_i^- \cdot N_{g-i}^- / 2$.

**Proof.** We observe that the curve $F_{g-i} \times \overline{M}_{g,1} \overline{\Theta}_{g}$ splits into $N_i^+ \cdot N_{g-i}^+ + N_i^- \cdot N_{g-i}^+$ irreducible components, each isomorphic to $C$, corresponding to a choice of a pair of theta-characteristics of opposite parities on $C$ and $D$ respectively. Let $t \in F_{g-i} \cdot \overline{\Theta}_{g,1}$ be an arbitrary point in the intersection, with underlying stable curve $C \cup_x D$, and spin curves $([C, \eta_C], [D, \eta_D]) \in S_i \times S_{g-i}$ on the two components.

Suppose first that $\eta_C = \eta_C^+$ and $\eta_D = \eta_D^+$, that is, $t$ corresponds to an even theta-characteristic on $C$ and an odd theta-characteristic on $D$. Then there exist non-zero sections $\sigma_C \in H^0(C, \eta_C^+ \otimes O_C((g - i)x))$ and $\sigma_D \in H^0(D, \eta_D^+ \otimes O_D(ix))$ such that

$$(7) \quad \text{ord}_x(\sigma_C) + \text{ord}_x(\sigma_D) \geq g - 1, \quad \text{and} \quad \text{ord}_x(\sigma_D) = 0.$$

In other words, $\sigma_C$ and $\sigma_D$ are the aspects of a limit $\emptyset_{g-1}^0$ on $C \cup_x D$ which vanishes at $q \in D$. Clearly, $\text{ord}_x(\sigma_C) \leq g - i - 1$, hence $\text{div}(\sigma_D) \geq ix + q$, that is, $q \in \text{supp}(\eta_D^+)$. This contradicts the generality assumption on $q \in D$, so this situation does not occur.

Thus, we are left to consider the case $\eta_C = \eta_C^-$ and $\eta_D = \eta_D^+$. We denote again by $\sigma_C \in H^0(C, \eta_C^- \otimes O_C((g - i)x))$ and $\sigma_D \in H^0(D, \eta_D^+ \otimes O_D(ix))$ the sections satisfying the compatibility relations (7). The condition $h^0(D, \eta_D^+ \otimes O_D(x - q)) \geq 1$ defines a correspondence on $D \times D$, cf. [DK], in particular, we can choose the points $x, q \in D$ general enough such that $H^0(D, \eta_D^+ \otimes O_D(x - q)) = 0$. Then $\text{ord}_x(\sigma_D) \leq i - 2$, thus $\text{ord}_x(\sigma_C) \geq g - i + 1$. It follows that we must have equality $\text{ord}_x(\sigma_C) = g - i + 1$, and then, $x \in \text{supp}(\eta_C^-)$. An argument along the lines of [EH3] Lemma 3.4, shows that each of these intersection points has to be counted with multiplicity $1$, thus $F_{g-i} \cdot \overline{\Theta}_{g,1} = \#\text{supp}(\eta_C^-) \cdot N_{g-i}^- \cdot N_{g-i}^-$. We conclude by noting that $(2i - 2)b_{g-i} = F_{g-i} \cdot \overline{\Theta}_{g,1}$. \hfill $\square$

**Proposition 4.2.** The relation $b = N_g^- / 2$ holds.

**Proof.** Having fixed a general curve $[C] \in \overline{M}_g$, by considering the fibre $p^\ast([C])$ inside the universal curve, one writes the identity $(2g - 2)b = p^\ast([C]) \cdot \overline{\Theta}_{g,1} = (g - 1)N_g^-$. \hfill $\square$

We compute the class of the restriction of the divisor $\Theta_{g,1}$ over $M_{g,1}$:

**Proposition 4.3.** One has the equivalence $\Theta_{g,1} \equiv N_g^-(\psi / 2 + \lambda / 4) \in \text{Pic}(M_{g,1})$. 

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Proof. We consider the universal pointed spin curve \( \text{pr} : S_{g,1}^- := S_g^- \times_{M_g} M_{g,1} \rightarrow M_{g,1} \). As usual, \( \mathcal{P} \in \text{Pic}(S_{g,1}^-) \) denotes the universal spin bundle, which over the stack \( S_{g,1}^- \), is a root of the dualizing sheaf \( \omega_{\mathcal{P}} \), that is, \( 2c_1(\mathcal{P}) = \text{pr}^*(\psi) \). We introduce the divisor
\[
\mathcal{Z} := \{(C, \eta, q) \in S_{g,1}^- : q \in \text{supp}(\eta) \} \subset S_{g,1}^-,
\]
and clearly \( \Theta_{g,1} := \text{pr}_*(\mathcal{Z}) \). We write \( \mathcal{Z} = c_1(\mathcal{P}) - c_1(\text{pr}^*(\text{pr}_*(\mathcal{P}))) \), and take into account that \( c_1(\text{pr}(\mathcal{P})) = 2c_1(\text{pr}_*(\mathcal{P})) = -\lambda/2 \). The rest follows by applying the projection formula. \( \square \)

In order to determine the remaining coefficients \( b_0, b_1 \), we study the pull-back of \( \Theta_{g,1} \) under the map \( \nu : \mathcal{M}_{1,2} \rightarrow M_{g,1} \), given by \( \nu([E, x, q]) := [C \cup_x E, q] \in M_{g,1} \), where \([C, x] \in M_{g-1,1}\) is a fixed general pointed curve.

On the surface \( \mathcal{M}_{1,2} \), if we denote a general element by \([E, x, q]\), one has the following relations between divisors classes, see [AC2]:
\[
\psi_x = \psi_q, \lambda = \psi_x - \delta_{0;xq}, \delta_{\text{irr}} = 12(\psi_x - \delta_{0;xq}).
\]
We describe the pull-back map \( \nu^* : \text{Pic}(M_{g,1}) \rightarrow \text{Pic}(\mathcal{M}_{1,2}) \) at the level of divisors:
\[
\nu^*(\lambda) = \lambda, \quad \nu^*(\psi) = \psi_q, \quad \nu^*(\delta_{\text{irr}}) = \delta_{\text{irr}}, \quad \nu^*(\delta_1) = -\psi_x, \quad \nu^*(\delta_{g-1}) = \delta_{0;xq}.
\]
By direct calculation, we write \( \nu^*(\Theta_{g,1}) = (a + b - 12b_0 + b_1)\psi_x - (a + b_0 - 12b_0)\delta_{0;xq} \).

We compute \( b_0 \) and \( b_1 \) by describing \( \nu^*(\Theta_{g,1}) \) viewed as an explicit divisor on \( \mathcal{M}_{1,2} \):

**Proposition 4.4.** One has the relation \( \nu^*(\Theta_{g,1}) = N_{g-1}^- \cdot \overline{\mathcal{Z}}_2 \in \text{Pic}(\mathcal{M}_{1,2}) \), where
\[
\overline{\mathcal{Z}}_2 := \{[E, x, q] \in \mathcal{M}_{1,2} : 2x \equiv 2q \}.
\]

**Proof.** We fix an arbitrary point \( t := [C \cup_x E, q] \in \nu^*(\Theta_{g,1}) \). Suppose first that \( E \) is a smooth elliptic curve, that is, \( j(E) \neq \infty \) and \( x \neq q \). Then there exist theta-characteristics of opposite parities \( \eta_C, \eta_E \) on \( C \) and \( E \) respectively, together with non-zero sections
\[
\sigma_C \in H^0(C, \eta_C \otimes O_C(x)) \quad \text{and} \quad \sigma_E \in H^0(E, \eta_E \otimes O_E((g-1)x)),
\]
such that \( \sigma_E(q) = 0 \) and \( \text{ord}_x(\sigma_C) + \text{ord}_x(\sigma_E) \geq g-1 \).

First, we assume that \( \eta_C = \eta_C^+ \) and \( \eta_E = \eta_E^- \), thus, \( \eta_E = O_E \). Since \( H^0(C, \eta_C^+) = 0 \), one obtains that \( \text{ord}_x(\sigma_C) = 0 \), that is \( \text{ord}_x(\sigma_E) = g-1 \), which is impossible, because \( \sigma_E \) must vanish at \( q \) as well. Thus, one is lead to study the remaining case, when \( \eta_C = \eta_C^- \) and \( \eta_E = \eta_E^+ \). Since \( x \notin \text{supp}(\eta_C) \), we obtain \( \text{ord}_x(\sigma_C) \leq 1 \), and then by compatibility, the last inequality becomes equality, while \( \text{ord}_x(\sigma_E) = g-2 \), hence \( \eta_E^+ = O_E(x - q) \), or equivalently, \([E, x, q] \in \overline{\mathcal{Z}}_2 \). The multiplicity \( N_{g-1}^- \) in the expression of \( \nu^*(\Theta_{g,1}) \) comes from the choices for the theta-characteristics \( \eta_C^- \), responsible for the \( C \)-aspect of a limit \( g_{g-1}^- \) on \( C \cup_x E \). It is an easy moduli count to show that the cases when \( j(E) = \infty \), or \([E, x, q] \in \delta_{0;xq} \) (corresponding to the situation when \( x \) and \( q \) coalesce on \( E \)), do not occur generically on a component of \( \nu^*(\Theta_{g,1}) \). \( \square \)

**Proposition 4.5.** \( \overline{\mathcal{Z}}_2 \) is an irreducible divisor on \( \mathcal{M}_{1,2} \) of class \( \overline{\mathcal{Z}}_2 \equiv 3\psi_x \in \text{Pic}(\mathcal{M}_{1,2}) \).

**Proof.** We write \( \overline{\mathcal{Z}}_2 \equiv \alpha \psi_x - \beta \delta_{0;xq} \in \text{Pic}(\mathcal{M}_{1,2}) \), and we need to understand the intersection of \( \overline{\mathcal{Z}}_2 \) with two test curves in \( \mathcal{M}_{1,2} \). First, we fix a general point \([E, q] \in \mathcal{M}_{1,1}\) and consider the family \( E_1 := \{[E, x, q] : x \in E\} \subset M_{1,2} \). Clearly, \( E_1 \cdot \delta_{0;xq} = E_1 \cdot \psi_x = 1 \).

On the other hand \( E_1 \cdot \overline{\mathcal{Z}}_2 \) is a 0-cycle simply supported at the points \( x \in E - \{q\} \) such that \( x - q \in \text{Pic}^0(E)[2] \), that is, \( E_1 \cdot \overline{\mathcal{Z}}_2 = 3 \). This yields the relation \( \alpha - \beta = 3 \).
As a second test curve, we denote by $[L, u, x, q] \in \overline{M}_{0,3}$ the rational 3-pointed rational curve, and define the pencil $R := \{(L \cup u, E_{\lambda}, x, q) : \lambda \in \mathbb{P}^1\} \subset \overline{M}_{1,2}$, where $\{E_{\lambda}\}_{\lambda \in \mathbb{P}^1}$ is a pencil of plane cubic curves. Then $R \cap \overline{\mathcal{F}}_2 = \emptyset$. Since $R \cdot \delta_{\text{irr}} = 12$, we obtain the additional relation $\beta = 0$, which completes the proof. \[\square\]

Putting together Propositions 4.1, 4.3 and 4.5 we obtain the system of equations
\[
a + b_{g-1} - 12b_{\text{irr}} = 0, \quad a - 12b_{\text{irr}} + b + b_1 = 3N_{g-1}^-, \quad a = \frac{1}{4}N_g^-, \quad b = \frac{1}{2}N_g^-, \quad b_1 = \frac{3}{2}N_{g-1}^-.
\]
Thus $b_{\text{irr}} = 2^{2g-6}$ and $b_{g-1} = 2^{g-3}(2^{g-1} + 1)$. This completes the proof of Theorem 0.3.

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