Simulation of majority vote disturbed by power-law noise

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Abstract. Simulations are reported on the Ising two-dimensional ferromagnet in the presence of a special kind of noise. The noise spectrum $P(n)$ follows a power law, where $P(n)$ is the probability of flipping $n$ randomly selected spins at each timestep. This is introduced to mimic the self-organized criticality as a model influence of a complex environment. We reproduced the phase transition in a way that is similar to the 1992 model of de Oliveira. Above some value of the noise amplitude the magnetization tends to zero; otherwise it remains constant after some relaxation. Information of the initial spin orientation remains preserved to some extent by short-range spin–spin correlations. The distribution of the times between flips is exponential. The results are discussed as a step towards modelling of social systems.

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1. Introduction

Ising magnets and their variants have been simulated for opinion dynamics [1, 2], urban segregation [3], economics [4], language change [5, 6], etc, sometimes even at zero temperature. In the latter case, each spin = ±1 aligns in the direction of the majority of its neighbours, with sequential updating as needed in most Ising simulations. If the neighbourhood is evenly divided we orient the spin randomly (Glauber kinetics). Starting from a random distribution one does not always end up with all spins parallel; strip domains may form at zero temperature [7].

Noise can also be introduced into this model in a different way from the traditional temperature and Boltzmann probabilities. At each iteration of an $L \times L$ square lattice with four neighbours for each site, besides the above majority vote, we select a spin $n$ times randomly and flip it: this creates random sequential ‘noise’ [8, 9]; the same spin may be selected and flipped several times. The probability distribution function $P(n)$ of these numbers $n$ is taken as a power law,

$$P(n) \propto 1/n^\alpha.$$  \hfill (1)

To get this distribution, we determined random numbers $r$, homogeneously distributed between zero and one, and then took

$$n = TL^2r^{1/(1-\alpha)},$$ \hfill (2a)

$$n = T\exp(r\ln(L^2))$$ \hfill (2b)

for $\alpha$ smaller than and equal to one, respectively, with $1 \leq n \leq L^2$. Here the proportionality factor $T$ determines the amplitude of the noise, analogously to the temperature in Boltzmann statistics.

In section 2 we motivate this noise, in section 3 we present the results, and we discuss them in section 4.
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2. Motivation

Power-law decays as a function of space or time are known to appear at critical points of second-order phase transitions. For example, the average numbers of clusters containing $s$ sites each in random percolation theory decay with a power of $s$ for large $s$, if one adjusts the concentration to the percolation threshold. And for the Ising model, the same is true for suitably defined clusters if the temperature is adjusted to the critical temperature. If in the latter case we start with all spins parallel, the magnetization decays with a power of the time.

For self-organized criticality, there is no need to adjust one continuously varying parameter to a special critical value; the system develops critical power laws automatically. This self-organization is seen for example in the size distribution of avalanches caused by dropping a single sand grain onto the Bak sand-pile model; this avalanche distribution converges towards a power law \[10,11\]. The power-law behaviour in a social environment has been reported in \[12\].

One can also reverse the situation by putting in a power law from the beginning and observing the system behaviour; then its environment is supposed to be in the self-organized critical state. It has been known for a long time that a power law as a function of distance in the exchange interactions may influence the critical exponents of the Ising model. We also know that an Ising model, put onto a Barabási–Albert network with a power law in the degree distribution, has a size dependence very different from that on the square lattice. For random walks between disordered binding sites, Saxton \[13\] stated: ‘if a particle diffuses in a suitable infinite hierarchy of binding sites, diffusion is well known to be anomalous at all times’ (meaning an unusual exponent relating time to distance).

Thus we take here the majority-vote model of de Oliveira \[8\] and modify its noise such that the noise obeys a power law. Thus at each iteration the whole system first undergoes the zero-temperature Glauber dynamics, and then the $n$ spins randomly selected by equation (1) are flipped.

Since our simulations show that the magnetization nearly always remains positive when we start with all spins up, we will also try correlated noise by flipping whole plaquettes instead of single spins. These plaquettes make it easier to destroy the initial ordering.

3. Results

First we look at the special case $\alpha = 1$ of equation (2b). Figure 1 shows the summed magnetization $\sum_i S_i$ versus time to relax exponentially towards zero at $T \approx 1$, when we started with all spins up. For smaller $T$ the magnetization remains positive. Figure 2 shows its normalized averages over the second half of the simulation. However, these averages over many samples and/or time steps do not tell the full story. Figure 3 shows how the system ‘wants’ to recover towards larger magnetization but crashes back to small values if a particularly large noise $n$ happened. (Pictures at $T = 1$ show medium-size domains of up spins; thus if the noise flips a single spin, that spin mostly reverts in the next time step to its old orientation.)

The magnetization for $T$ slightly above its critical value gets closer to zero if we let the noise flip small plaquettes of size $b \times b$ instead of merely one spin. Figure 4 compares...
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Figure 1. Summed magnetizations versus time for $0.3 \leq T \leq 1, \alpha = 1$.

Figure 2. Normalized magnetizations, averaged over 100 samples and over $500 < t \leq 1000$. The open and full squares for $t = 10000$ correspond to $L = 501$ and 1001, respectively.

$b = 1$ (previous method) with $b = 3$ and 5. Understandably, the critical value of $T$ is shifted downwards if each noise event flips a whole plaquette, as also seen in figure 4.

All figures so far referred to $\alpha = 1$; for $\alpha = 0$ and 0.5 the results were similar, see figure 5. If for $\alpha = 1$ the noise is used with a probability 0.1 only, the magnetizations returned to one, up to $T = 100$ (not shown). Up to now our simulations started with all spins up. If instead we start with spins randomly oriented up or down and average over the absolute value of the magnetization, then figure 6 shows a spontaneous magnetization to emerge after sufficiently long times for low $T$.

We also determined the histograms of the decision times. These are the time intervals between two consecutive spin flips (or opinion changes). Figure 7 shows exponential distributions: for higher $T$ the decay is faster than for lower $T$, without evidence for critical slowing down. This kind of decay is due to the noise.

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Figure 3. Complicated time dependence is seen when looking at one sample only.

Figure 4. Comparison of the standard case (+) with flipping plaquettes of size $3 \times 3$ (×) and $5 \times 5$ (*). The $T$ axis is logarithmic.

As is appropriate for random noise, the $n$ spins to be selected according to equation (2) were selected randomly, and since $n \propto L^2$ often the same spin was selected and flipped repeatedly. Figure 8 shows the alternative choice of always selecting $n$ different spins. Then order is already destroyed at lower $T$. We do not, however, regard such noise correlated over the whole lattice as realistic.

4. Discussion

In a first approximation, we look at the effect of a single set of $n$ flips. Then we observe two characteristic times of the dynamics. The first one is connected to the restoring of the ferromagnetic ordering. It is relatively short if the magnetization is far from zero; however, it can be arbitrarily long in the opposite case. The second time is due to the average
lifetime of the ordered phase with a given macroscopic direction of the magnetization. This time is expected to be proportional to $\ln(L)$; sooner or later the value of the variable $n$ happens to be of the order of magnitude of the whole system. If the magnetization at that time step happens to be large enough, most of the flipped spins were $+1$; therefore in this case the magnetization change is particularly sharp. This can be observed in an example of the time evolution of $m$, shown in figure 4. As for each spin the probability of flipping is $x = n/N$, in the general case the variation of the magnetization is expected to follow the approximated rule for one iteration ($n$ noise events):

$$m = [n(+) - n(-)]/N \rightarrow [(1 - x)n(+) - xn(+)-(1 - x)n(-) + xn(-)]/N$$

$$= [(1 - 2x)(n(+) - n(-))]/N = (1 - 2x)m,$$

where $n(\pm)$ is the number of spins with orientation $\pm 1$. In this equation spins of both orientation are assumed to flip with the same probabilities. As we see, the state $m = 0$ is the fixed point of the transformation.

More detailed inspection reveals that the two processes indicated above cannot be treated as independent. The exchange-mediated restoring of the ordered phase is slowed down even by small reductions of $m$ caused by the noise. If the noise intensity is small enough, some kind of dynamic equilibrium can be observed between the noise-induced reduction of $m$ and the increase of $m$ due to the exchange. Although it is difficult to speak about stationary processes, the observed time average in a not-too-long period of time can be compared to the partially ordered ferromagnetic system in temperature lower than its Curie temperature. In this sense, the results allow us to state that the investigated phase transition also exists in the presence of the noise $1/n$. As a by-product of the model we obtain the fact that any ordered phase has its finite lifetime.

Now we speculate about possible social implications of these results. Several physical concepts usually applied in sociophysics have no direct counterparts in social systems; one

Figure 5. Different $\alpha$, no plaquettes: normalized magnetizations, averaged over 100 samples and over $500 < t \leq 1000$. (The isolated circle refers to $5000 < t < 10000$.) Left curves $L = 501 (+), 1001 (\times), 2001 (*)$ for $\alpha = 0$; right curves (open squares, full squares, full circles) the same for $\alpha = 0.5$.  

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of them is temperature. In physics, it makes sense to speak about temperature in the case of thermal equilibrium. In simulations, temperature measures the amount of noise; it is relevant to distinguish between the probabilities of states with different energies. In social systems energy is not defined and the equilibrium state is never attained. A model society can be considered to be rather in a self-organized critical state [10, 11] than in any kind of equilibrium [14]–[17]. A question arises, if results of simulations interpreted within the social sciences persist if the thermal noise is substituted by another kind of noise, designed to reflect features of the self-organized criticality. In particular, we have found that some equivalent of the ferro–paramagnetic phase transition persists in the presence of such a noise. The finite lifetime of any ordered phase can be compared to an average time during which a society supports the government formed by a given party. The $1/n$ noise in this case can be compared to a series of scandals which reduce the

Figure 6. Growth of absolute value of magnetization if initially the spins are up or down randomly. Average over 100 samples. The upper part shows various $T$ at fixed $L = 201$, the lower part various $L$ at fixed $T = 0.9$. 
Figure 7. Histograms of times between two spin flips, observed between 500 and 1000 iterations; $\alpha = 1, L = 501, T = 0.3, 0.4, \ldots 1.2$. The steeper the negative slope is, the higher is $T$. Results for $L = 1001$ and 2001 were similar (not shown).

Figure 8. Example for non-random noise where exactly $n$ different spins are flipped.

government reputation. The size of such scandals is not limited from above; examples are at hand.

In sociophysics, the phase transition itself is a physical trait. In social sciences the up–down symmetry does not exist; there is always some bias. Then, the spontaneous symmetry breaking—the ultimate base of the model—again has no social analogue. One could say that in sociology, the opposition is not ‘plus–minus’, but rather ‘something–nothing’, related with society or its institutions. Then maybe we should consider the random spins equal to zero (=non-interacting) or one rather than $\pm 1$. Another physical trait is an interaction; in physics it joins two interacting systems symmetrically, as in the third Newton law, $\text{actio} = -\text{reactio}$. In social systems this symmetry is also absent,
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at least on a local timescale. (One may wonder to what extent the conquest of Latin America had petrified feudal attitude in Spanish elites, making the whole country unable to develop—this would be only one example of a restoring of this symmetry.) These arguments indicate that further improvements of the model picture are needed, with a possible enrichment of the set of tools of statistical physics.

A need for modifications towards reality does not alter the fact that the models are of interest per se. Sociophysics is a part of statistical (theoretical) physics, where we reconstruct the society by combining simple models, step by step. The scale-free $1/n$ noise, suggested by the self-organized criticality, is such a modest step towards this goal.

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References

[1] Liggett T M, 1985 Interacting Particle Systems (New York: Springer)
[2] Sznajd-Weron K and Sznajd J, 2000 Int. J. Mod. Phys. C 11 1157
[3] Schelling T C, 1971 J. Math. Sociol. 1 143
[4] Hohnisch M, Pittnauer S, Solomon S and Stauffer D, 2006 Physica A 345 646
[5] Nettl D, 1999 Lingua 18 119
[6] Wichmann S, Stauffer D, Schulze C and Holman E W, 2007 Preprint 0706.1842
[7] Spirin Y, Krapivsky P L and Redner S, 2001 Phys. Rev. E 63 036118
[8] de Oliveira M J, 1992 J. Stat. Phys. 66 273
[9] Jones F L, 1985 Aust. N. Z. J. Sociol. 21 431
[10] Bak P, Tang C and Wiesenfeld K, 1987 Phys. Rev. Lett. 59 381
[11] Bak P, How nature works. The science of self-organized criticality, 1986 Copernicus (New York: Springer)
[12] Johnson N F, 2006 Am. Phys. Soc. News (November)
  http://www.aps.org/publications/apsnews/200611/backpage.cfm
[13] Saxton M J, 2007 Biophys. J. 92 1178
[14] Weisbuch G and Solomon G, 2003 Handbook of Graphs and Networks ed S Bornholdt and H G Schuster
  (Weinheim: Wiley–VCH) chapter 15
[15] Brunk G G, 2002 J. Theor. Politics 14 195
[16] Aleksiejuk A, Holyst J A and Kossinets G, 2002 Int. J. Mod. Phys. C 13 333
[17] Saloma S, Perez G J, Tapang G, Lim M and Palme-Saloma C, 2003 Proc. Nat. Acad. Sci. 100 11947

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