Anomalous Diffusion with Periodical Initial Conditions on Interval with Reflecting Edges

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1 Introduction

In [1] a mathematical model of an anomalous diffusion in a space was suggested. This model originates in an investigation of processes in complex systems with variable structure: glasses, liquid crystals, biopolymers, proteins and etc.

In this model a coordinate of a particle has stable distribution (not normal one). As a result a density of its distribution function satisfies an analog of diffusion equation in which second derivative by coordinate is replaced by partial derivative.

In this paper the anomalous diffusion with periodic initial conditions on an interval with reflecting edges is considered. Such problem is important for example in technical mechanics for an analysis of fuel mixing in straight flow engine [2] too. As A.A. Borovkov suggested [3] special probability methods are developed to analyze such a diffusion.

Main results of the paper may be represented as follows. Suppose that $T_n(a, r, k)$ is a characteristic time of the anomalous diffusion with a parameter $a$, $0 < a \leq 2$ on $k$-dimension interval $[-r, r]^k$ with $n$-periodic (by each coordinate) initial conditions then

$$T_1(a, r, 1) = T_1(a, 1, 1)r^a, \quad \frac{T_1(a, r, 1)T_1(2, 1, 1)}{T_1(a, 1, 1)T_1(2, r, 1)} = r^{a-2}, \quad T_n(a, 1, 1) = T_1(a, 1, 1)/n^a.$$ 

So the anomalous diffusion on the interval $[-r, r]$ for $r > 1$ works faster and for $r < 1$ - slower than normal one. If $k > 1$ then it is possible to choose $n$-periodical (by each coordinate) initial conditions so that for the normal diffusion

$$T_n(2, 1, k) = T_1(2, 1, 1)/kn^2.$$ 

2 Model of Anomalous Diffusion on Straight Line

Suppose that $y(t), \quad t \geq 0$ is homogenous random process with independent increments and an initial condition $y(0) = 0$. The difference $y(t) - y(\tau), \quad t > \tau \geq 0$ has symmetric
stable distribution on the straight line \((-\infty, \infty)\) with a parameter \(a, \ 0 < a \leq 2\) and a characteristic function

\[ M \exp(iu[y(t) - y(\tau)]) = \exp(-(t - \tau)|u|^a). \]

In \([1]\) it was shown that distribution density \(p_t = p_t(u)\) of the process \(y(t)\) in a moment \(t > 0\) is the generalized solution of the following differential equation with fractional derivatives

\[ \left( \frac{\partial}{\partial t} - \frac{\partial^a}{\partial y^a} \right) p_t(y) = \delta(y)\delta(t). \]

Here \(\partial^a p_t(y)/\partial y^a\) is a–fractional derivative of the function \(p_t(y), \ 0 < a \leq 2; \ \delta(y), \delta(t)\) are delta-functions by variables \(y, t\) accordingly.

3 Construction of Anomalous Diffusion Model on the Interval \([-1, 1]\).

Each realization of random process \(y(t), \ t \geq 0\) may be considered as a curve \(\Gamma\) on the plane \((y, t)\) and may be represented in parametrical form: \(y = y(\tau), \ t = t(\tau) = \tau, \ \tau \geq 0.\) Suppose that the curve \(\Gamma\) is reflected from the lines \(y = 1, \ y = -1.\) For this aim represent the plane \((y, t)\) as a transparent and infinitely thin sheet of paper with the curve \(\Gamma.\) Bend this sheet of the paper along the lines \(y = \pm 1, \ y = \pm 3, \ldots\) into transparent strip \(-1 \leq y \leq 1\) with fragments of initial curve \(\Gamma.\) As a result \(\Gamma\) is transformed into the curve \(\gamma: y = Y(\tau), \ t = t(\tau) = \tau.\) Analogously to \([1]\) the random process \(Y(t), \ t \geq 0\) may be considered as a model of the anomalous diffusion on the interval \([-1, 1]\) with reflecting edges. It is clear that if the curve \(\Gamma\) coincides with some straight line then the curve \(\gamma\) is constructed in accordance with the law of geometrical optics: falling angle equals to reflecting angle.

4 Analytical Representation of Reflected Diffusion Process and Its Properties

Additionally to geometrical representation of the process \(Y(t), \ t \geq 0\) consider its analytical representation by the functions \(f: R \rightarrow [-2, 2), \ g: [-2, 2) \rightarrow [-1, 1],\)

\[ f(u) = (u + 2)/\text{mod} \ 4 - 2, \]

\[ g(u) = \begin{cases} 
  u, & -1 \leq u \leq 1, \\
  2 - u, & 1 < u < 2, \\
  -2 - u, & -2 \leq u < -1, 
\end{cases} \]

\[ Y(t) = g(f(y(t))). \tag{4.1} \]

Here \(u/\text{mod} \ A = A\{u/A\}, \ A > 0; \ \{z\}\) is the fractional part of a real number \(z.\) It is known from group theory that for \(u, v \in R, \ A > 0\)

\[ (u/\text{mod} \ A)/\text{mod} \ A = u/\text{mod} \ A, \ (u + v)/\text{mod} \ A = (u/\text{mod} \ A + v/\text{mod} \ A)/\text{mod} \ A. \tag{4.2} \]
Define on the half-interval \([-2, 2]\) binary operation “⊕” and unary operation of an inverse “⊖”:

\[
    u \oplus v = f(u + v), \quad \ominus u = f(-u), \quad u, v \in [-2, 2).
\]

Prove that these operations on the half-interval \([-2, 2]\) form commutative group \(C\) with the summation \(\oplus\) and the inverse \(\ominus\) and the unit \(0\):

\[
    u \oplus v = v \oplus u, \quad u \oplus (\ominus u) = 0, \quad (u \oplus v) \oplus w = u \oplus (v \oplus w), \quad u, v, w \in C.
\]

Really using (4.2) obtain

\[
    u \oplus v = f(u + v) = (u + v + 2) \mod 4 - 2 = (v + u + 2) \mod 4 - 2 = f(v + u) = v \oplus u,
\]

\[
    u \oplus (\ominus u) = (u + (\ominus u) + 2) \mod 4 - 2 = (u + (-u + 2) \mod 4 - 2 + 2) \mod 4 - 2 = (u - u + 2) \mod 4 - 2 = f(0) = 0,
\]

\[
    (u \oplus v) \oplus w) = [(f(u + v) + w + 2) \mod 4 - 2 = [(u + v + 2) \mod 4 - 2 + w + 2] \mod 4 - 2 = (u + v + w + 2) \mod 4 - 2 = [u + (v + w + 2) \mod 4 - 2 + 2] \mod 4 - 2 = (u + v + w + 2) \mod 4 - 2 = u \oplus (v \oplus w).
\]

Prove that \(f : (R, +) \rightarrow (C, \oplus)\) is a homomorphism of the additive group \(R\) of real numbers onto the group \(C\):

\[
    f(u + v) = f(u) \oplus f(v), \quad f(-u) = \ominus f(u).
\]

Using (4.2) obtain

\[
    f(u) \oplus f(v) = f(f(u) + f(v)) = [(u + 2) \mod 4 - 2 + (v + 2) \mod 4 - 2 + 2] \mod 4 - 2 = (u + v + 2) \mod 4 - 2 = f(u + v),
\]

the equality \(f(-u) = \ominus f(u)\) is a corollary of the inverse \(\ominus\) definition.

## 5 Reflection Formula for Diffusion Process

Denote by \(p_t = p_t(u), \pi_t = \pi_t(u), P_t = P_t(u)\) distribution densities of random variables (r.v.) \(y(t), Y(t), f(y(t))\) accordingly. Using the formula (4.1) and the graphic of the function \(f\) find:

\[
    P_t(u) = \sum_{v : f(v) = u} p_t(v) = \sum_{k=-\infty}^{\infty} p_t(u - 4k), \quad u \in [-2, 2); \quad P_t(0) = 0, \quad u \notin [-2, 2). \tag{5.1}
\]

It is easy to prove from [5] chapt. 17, §6, lemma 1] that the row \(\text{(5.1)}\) converges.

Define auxiliary function

\[
    \bar{P}_t(u) = \sum_{k=-\infty}^{\infty} p_t(u - 4k), \quad -\infty < u < \infty.
\]
The function $P_t(u)$ coincides with $P_t(u)$ for $u \in [-2, 2)$ and has the period 4 and is symmetric. So from the formula (5.1) and the graphic of the function $g$ obtain

$$\pi_t(u) = P_t(u) + P_t(u + 2) = \sum_{k=-\infty}^{\infty} p_t(u - 2k), \ u \in [-1, 1], \ \pi_t(u) = 0, \ u \not\in [-1, 1]. \quad (5.2)$$

Suppose that r.v. $z$ is distributed on $[-1, 1]$, its density $\mu(v)$ has continuous derivative,

$$\frac{d\mu(v)}{dv} = 0, \ v = \pm 1$$

and r.v. $z$ doesn't depend on random process $y(t)$ then the formula (5.1) allows to calculate distribution density $\pi_z(t,u)$ of the random process $Z(t) = g(f(y(t) - 1 + z)), \ t \geq 0$ in the moment $t > 0$ :

$$\pi_z(t,u) = \int_{-1}^{1} (P_t(u-v) + P_t(u + 2 + v))\mu(v)dv. \quad (5.3)$$

Remark that the formulas (5.2), (5.3) which give distribution of diffusion reflected process are analogous to reflection method formulas which give solution of wave equation for finite string with fixed edges [4, chapt. III, §13, points 5, 6].

### 6 Convergence of Reflected Diffusion Process Distribution to Uniform Distribution

As $f$ is the homomorphism of the additive group $R$ onto the group $C$, then for $0 < t_1 < t_2$

$$f(y(t_2)) = f(y(t_1)) \oplus f(y(t_2) - y(t_1)).$$

So random process $f(y(t))$, $t \geq 0$ with independent and stable distributed increments is homogenous Markov process with state set $[-2, 2)$ and symmetric distribution density of a transition from the state $u$ into the state $v$ during the time $t$

$$q_t(u,v) = P_t(v - u), \ -2 \leq u, v < 2.$$

Denote $Q_t = \inf\{q_t(u,v), \ -2 \leq u, v < 2\}$ then from (5.1) obtain

$$0 < Q_t < \frac{1}{4}, \ t > 0. \quad (6.1)$$

Suppose that

$$P = P(u) = \begin{cases} \frac{1}{4}, & u \in [-2, 2), \\ 0, & u \not\in [-2, 2), \end{cases} \quad (6.2)$$

$$\pi = \pi(u) = \begin{cases} \frac{1}{2}, & u \in [-1, 1], \\ 0, & u \not\in [-1, 1]. \end{cases}$$

For function $\varphi$ on $(-\infty, \infty)$ define the norm $\|\varphi\| = \sup\{\varphi(u), -\infty < u < \infty\}$ and denote $[z]$ the integer part of a real number $z$. 

4
Lemma 1. For arbitrary $h > 0$, $t \geq h$:

$$||\pi_t - \pi|| \leq 2(1 - 4Q_h)^{k-1}||P_h - P||, \ k = \lfloor t/h \rfloor.$$  \hfill (6.3)

*Proof.* As transition density $q_t(u, v)$ is symmetric then the formula (6.2) gives

$$\int_{-2}^{2} P(v)q_t(v, u)dv = \frac{1}{4} \int_{-2}^{2} q_t(u, v)dv = P(u), \ -2 \leq u < 2.$$  \hfill (6.4)

If $\Delta q_t = q_t - Q_t \geq 0$ then the formulas (6.1), (6.4) give

$$||P_{t+1} - P|| = \sup \left( \left| \int_{-2}^{2} P(v)q_t(v, u)dv - \int_{-2}^{2} P(v)q_t(v, u)dv \right|, \ -2 \leq u < 2 \right) =$$

$$= \sup \left( \left| \int_{-2}^{2} P(v)\Delta q_t(v, u)dv - \int_{-2}^{2} P(v)\Delta q_t(v, u)dv \right|, \ -2 \leq u < 2 \right)$$

and for $t \geq 0$

$$||P_{t+1} - P|| \leq ||P_1 - P|| \sup \left( \left| \int_{-2}^{2} \Delta q_t(v, u)dv, \ -2 \leq u < 2 \right| \right) = ||P_1 - P||(1 - 4Q_t).$$  \hfill (6.5)

The formulas (6.4), (6.5) allow to obtain by induction

$$||P_k - P|| \leq (1 - 4Q_1)^{k-1}||P_1 - P||, \ k \geq 1.$$  \hfill (6.6)

Using the formulas (6.1), (6.6) it is easy to generalize the inequality (6.6) onto arbitrary $t \geq 1$

$$||P_t - P|| \leq (1 - 4Q_1)^{k-1}||P_1 - P||, \ k = \lfloor t \rfloor.$$  \hfill (6.7)

The formulas (5.2), (6.7) give

$$||t - \pi|| \leq 2||P_t - P|| \leq 2(1 - 4Q_1)^{k-1}||P_1 - P||, \ k = \lfloor t \rfloor, \ t \geq 1.$$  \hfill (6.8)

The inequality (6.8) may be rewritten for arbitrary positive $h$ in the form (6.3).

\section*{7 Diffusion on Interval $[-r, \ r]$}

Suppose that $m > 0$, $r = m^{-1/a}$ and consider Markov process $Y_r(t), t \geq 0$:

$$Y_r(t) = rg(f(y(t)/r)).$$

The process $Y_r(t), t \geq 0$ is obtained from the process $y(t), t \geq 0$ by reflections from edges of the interval $[-r, \ r]$. Denote by $\pi_{t\cdot r} = \pi_{t\cdot r}(u)$ distribution density of r.v. $Y_r(t)$. It is obvious that $\pi_{t\cdot 1} = \pi_t$. Introduce normalized r.v.

$$W_{t\cdot r} = \frac{Y_r(t)}{r} = g(f(y(t)/r)), \ t \geq 0$$

with distribution density $\psi_{t\cdot r}(u) = r\pi_{t\cdot r}(ru)$. 

5
Lemma 2. For $K = [tm]$, $tm \geq 1$

$$||\pi_{t, r}(u) - \pi(u/r)/r|| \leq \frac{2(1 - 4Q_1)^{K-1}||P_1 - P||}{r}. \quad (7.1)$$

Proof. The definition of stable distribution (see [5, chapt. 17, §5]) gives that for $t > 0$ r.v.’s $y(tm)$, $y(t)/r$ coincide by distribution. So r.v.’s $W_{t, r}$, $g(y(tm))$ coincide by distribution too and from (6.8)

$$||\psi_{t, r} - \pi|| = ||\pi_{tm} - \pi|| \leq 2(1 - 4Q_1)^{K-1}||P_1 - P||, \ K = [tm], \ tm \geq 1.$$

The formula (7.1) is proved. □

The formula (7.1) may be rewritten conditionally as

$$T_1(a, r, 1) = T_1(a, 1, 1)r^a. \quad (7.2)$$

8 Numerical Experiment

Here analytical results of last paragraphs are compared with results of a numerical experiment. In this experiment a closeness of probability densities $\pi_t$, $\pi$ is investigated for different $t > 0$. For this aim we imitate independent r.v.’s which coincide with r.v. $Y(t)$ and so with r.v. $g(f(t^{1/a}y(1)))$ by distribution. R.v. $y(1)$ is imitated approximately by normalized sum

$$\hat{y}(1) = \frac{v_1 + \ldots + v_N}{N^{1/a}}$$

of independent identically distributed r.v. $v_1, \ldots, v_N$,

$$P(v_1 > t) = \frac{t^{-a}}{2}, \ P(v_1 < -t) = \frac{|t|^{-a}}{2}, \ t > 1.$$

Using $M$ independent realizations of r.v. $g(f(t^{1/a}\hat{y}(1)))$ it is possible to calculate frequences $S_j(t)$, $j = 0, \ldots, 9$ of these realizations scorings into the sets

$$\left[-1 + \frac{2j}{10}, -1 + \frac{2(j+1)}{10}\right), \ j = 0, \ldots, 8, \ \left[-1 + \frac{2j}{10}, 1\right], \ j = 9.$$

We calculate quantities

$$S(t) = 10M \sum_{j=0}^{9} \left(S_j(t) - \frac{1}{10}\right)^2$$

which characterize (analogously to $\chi^2$-statistics) deviations of $Y(t)$ distribution densities from uniform density for different $t > 0$. 

6
| $t$ | $S(t)$ for $a=1.9$ | $S(t)$ for $a=1.95$ | $S(t)$ for $a=1.99$ |
|-----|-----------------|-----------------|-----------------|
| 0.01 | 9169.31         | 7867.49         | 6584.26         |
| 0.02 | 4055.39         | 3053.09         | 2537.75         |
| 0.03 | 2200.99         | 1298.15         | 1017.17         |
| 0.04 | 1061.13         | 609.86          | 347.35          |
| 0.05 | 488.14          | 289.15          | 185.43          |
| 0.06 | 220.69          | 177.32          | 122.32          |
| 0.07 | 135.03          | 52.97           | 39.26           |
| 0.08 | 102.79          | 39.37           | 13.26           |
| 0.09 | 54.89           | 14.55           | 12.78           |

Table 1

Results of these calculations represented in the table 1 show how fast do distributions of $Y(t)$ converge to uniform if the time $t$ increases. Qualitative coincidence of numerical experiment results with the formula (6.3) is demonstrated.

9 Diffusion with Periodical Initial Conditions

Diffusion process with periodical initial conditions origines for example in fuel mixing at straight flow engine [2]. To model this process take natural $n$ and define markov process

$$Z_n(t) = g(f(y(t) − 1 + z_n)), \ t \geq 0$$

where r.v. $z_n$ has uniform distribution on finite set

$$I_n = \left\{ s = \frac{2k + 1}{n} : k = 0, 1, \ldots, n − 1 \right\}$$

and $z_n, y(t)$ are independent. Then random process $Z_n(t), \ t \geq 0$ may be considered as anomalous diffusion on interval $[-1, 1]$ but with periodical initial conditions

$$P(Z_n(0) = −1 + s) = \frac{1}{n}, \ s \in I_n. \quad (9.1)$$

Denote by $\Pi_{t, n} = \Pi_{t, n}(u), \ P_{t, n} = P_{t, n}(u)$ distribution densities of r.v.'s $Z_n(t), \ f(y(t) − 1 + z_n), \ t > 0$.

**Lemma 3.** The following formula is true

$$\Pi_{t, n}(u) = \frac{1}{n} \sum_{k=0}^{n-1} \pi_{t, 1/n} \left(u + 1 - \frac{2k + 1}{n}\right). \quad (9.2)$$

**Proof.** Analogously to the formulas (5.1) – (5.3) obtain

$$P_{t, n}(u) = \frac{1}{n} \sum_{s \in I_n} P_t(u + 1 - s) = \frac{1}{n} \sum_{s \in I_n} \sum_{k=−\infty}^{\infty} p_t(u - 4k + 1 - s), \ -2 \leq u < 2,$$
\[ \Pi_{t, n}(u) = \sum_{v: g(v) = u} P_{t, n}(v) = \frac{1}{n} \sum_{s \in I_n} (\bar{P}_t(u + 1 - s) + \bar{P}_t(u + 1 + s)), \quad -1 \leq u \leq 1 \]

then

\[ \Pi_{t, n}(u) = \frac{1}{n} \sum_{k=\infty}^{\infty} p_t \left( u - \frac{2k}{n} \right), \quad -1 \leq u \leq 1. \] (9.3)

The formula (9.3) leads to

\[ \Pi_{t, n}(u) = \Pi_{t, n} \left( u + \frac{2}{n} \right), \quad -1 \leq u \leq 1 - \frac{2}{n}. \] (9.4)

Calculate now the function

\[ \Pi_{t, n}(u), \quad -1 \leq u < -1 + \frac{2}{n}. \]

For this aim take

\[ u = w - 1 + \frac{1}{n}, \quad -\frac{1}{n} \leq w < \frac{1}{n}. \]

Then in an accordance with (5.2), (9.3)

\[ \Pi_{t, n}(w) = \frac{\pi_{t, 1/n}(w)}{n}, \quad -\frac{1}{n} \leq w < \frac{1}{n}. \] (9.5)

The formulas (9.4), (9.5) lead to the equality (9.2). \[ \square \]

The equality (9.2) means that the diffusion (normal or anomalous) on the set \([-1, 1]\] with periodical initial conditions (9.1) and reflecting edges leads to the same result as a diffusion on isolated (by reflecting edges) subsets

\[ \left[ -1 + \frac{2k + 1}{n} - \frac{1}{n}, -1 + \frac{2k + 3}{n} + \frac{1}{n} \right], \quad k = 0, \ldots, n - 1 \]

of the set \([-1, 1]\).

Remark that the equality is true for each process \(y(t)\) with independent and symmetrically distributed increments.

**Theorem 1.** For \(tn^a \geq 1, \quad L = [tn^a]\):

\[ \|\Pi_{t, n} - \pi\| \leq 2(1 - 4Q_1)^{L-1}\|P_1 - P\|. \] (9.6)

**Proof.** The statement of the theorem is obtained directly from the equality (9.2) and the formula (7.1) in which \(r\) equals to \(1/n). \[ \square \]

The inequality (9.6) may be interpreted as a decreasing of characteristic time into \(n^a\) times in model of anomalous diffusion with \(n\)-periodical initial conditions (9.1):

\[ T_n(a, 1, 1) = T_1(a, 1, 1)/n^a. \]

This result may be easily spread onto general case when r.v. \(Z_n(0)\) has distribution density with continuous derivative \(r_n(u)\) which satisfies periodicity conditions:

\[ r_n(u) = r_n \left( u + \frac{2}{n} \right), \quad -1 \leq u \leq 1 - \frac{2}{n} \]
and symmetry conditions
\[ r_n(-1 + \frac{1}{n} - v) = r_n\left(-1 + \frac{1}{n} + v\right), \quad 0 \leq v \leq \frac{1}{n} \]
and boundary condition
\[ \frac{dr_n(v)}{dv} = 0, \quad v = \pm 1. \]

It is a generalization of [2] results from normal onto anomalous diffusion.

10 Multi-Dimensional Diffusion with Periodic Initial Conditions

Multi-Dimensional Anomalous Diffusion. Results of last paragraph may be generalized onto two-dimensional case if the interval \([-1, 1]\) is replaced by square \([-1, 1] \times [-1, 1]\) and process of anomalous diffusion on square with reflecting boundaries is considered as two independent processes of one dimension anomalous diffusion on the intervals \([-1, 1]\) with reflecting edges. Disintegrate the unit square by rectangular network with \(n^2\) equal squares. Suppose that initial state of two-dimension diffusion process on unit square has uniform distribution on the set of these \(n^2\) squares centers. Then all one-dimension estimates of convergence rate are spread onto two-dimension case.

Denote all one-dimension distributions from last paragraph by indexes characterized numbers of appropriate coordinates \(j = 1, 2\):

\[ P^{(j)} = P^{(j)}(u_j), \quad P_1^{(j)} = P_1^{(j)}(u_j), \quad \pi^{(j)} = \pi^{(j)}(u_j), \quad \Pi_{t, n}^{(j)} = \Pi_{t, n}^{(j)}(u_j), \quad j = 1, 2. \]

In accordance with the model of two-dimension diffusion (as a pair of independent one-dimension diffusion models) obtain:

\[ \pi = \pi(u_1, u_2) = \pi^{(1)}(u_1)\pi^{(2)}(u_2), \quad \Pi_{t, n} = \Pi_{t, n}(u_1, u_2) = \Pi_{t, n}^{(1)}(u_1)\Pi_{t, n}^{(2)}(u_2). \]

From the inequality (9.6) for \(L = [tn^{1/a}]\), \(tn^{1/a} \geq 1, \quad j = 1, 2\)

\[ ||\Pi_{t, n}^{(j)} - \pi^{(j)}|| \leq 2(1 - 4Q_j)^{-1}\|P_1^{(j)} - P^{(j)}\| = \Delta^{(j)}_{t, n}, \quad . \quad (10.1) \]

Define the norm \(||\Phi|| = \sup\{\Phi(u_1, u_2)|, \quad -\infty < u_1, u_2 < \infty\}\) of the function \(\Phi\) on a plane and put \(\Delta_{t, n} = \Delta^{(1)}_{t, n} = \Delta^{(2)}_{t, n}\).

Theorem 2. For \(L = [tn^{1/a}]\), \(tn^{1/a} \geq 1\)

\[ ||\Pi_{t, n} - \pi|| \leq \Delta_{t, n}(1 + \Delta_{t, n}). \quad (10.2) \]

Proof. From triangle inequality obtain

\[ ||\Pi_{t, n} - \pi|| = ||\Pi_{t, n}^{(1)}\Pi_{t, n}^{(2)} - \pi^{(1)}\pi^{(2)}|| \leq ||\Pi_{t, n}^{(1)}\Pi_{t, n}^{(2)} - \pi^{(1)}\Pi_{t, n}^{(2)}|| + ||\pi^{(1)}\Pi_{t, n}^{(2)} - \pi^{(1)}\pi^{(2)}|| \leq \frac{1}{2}||\Pi_{t, n}^{(1)}||||\Pi_{t, n}^{(2)} - \pi^{(1)}|| + ||\pi^{(1)}\Pi_{t, n}^{(2)} - \pi^{(2)}|| + \frac{1}{2} ||\Pi_{t, n}^{(2)} - \pi^{(2)}|| \]

Using the formula (10.1) obtain the inequality (10.2). \(\square\)
As $\Delta t, n \to 0, \ t \to \infty$ so convergence rate of two-dimension case (10.2) is analogous to one-dimension case (9.6). This result may be easily spread onto multi-dimension case. Specifics of this multi-dimension diffusion model is that as one-dimension diffusion components so their initial conditions are independent.

**Multi-Dimensional Normal Diffusion.** Consider $k$-dimension normal diffusion model in which $\Pi_{t, n}(y_1, \ldots, y_k)$ is distribution density in the moment $t$ at the point $(y_1, \ldots, y_k) \in [-1, 1]^k$:

$$
\left( \frac{\partial}{\partial t} - \sum_{j=1}^{k} \frac{\partial^2}{\partial y_j^2} \right) \Pi_{t, n}(y_1, \ldots, y_k) = 0,
$$

$$
\frac{\partial}{\partial y_j} \Pi_{t, n}(\pm1, \ldots, \pm1, y_j, \pm1, \ldots, \pm1) = 0, \ -1 \leq y_j \leq 1, \ j = 1, \ldots, k,
$$

$$
\Pi_{0, n}(y_1, \ldots, y_k) = \frac{1}{2^k} + \sum_{j_1, \ldots, j_k=1}^{m} a(j_1, \ldots, j_k) \prod_{r=1}^{k} \cos \pi n_j y_r > 0, \ m < \infty.
$$

Then

$$
\Pi_{t, n}(y_1, \ldots, y_k) = \frac{1}{2^k} + \sum_{j_1, \ldots, j_k=1}^{m} a(j_1, \ldots, j_k) \prod_{r=1}^{k} \cos \pi n_j y_r \exp \left( -\pi^2 n^2 t \sum_{j=1}^{m} k_j^2 \right)
$$

$|a(1, \ldots, 1)| > 0$

$$
||\Pi_{t, n}(y_1, \ldots, y_k) - \frac{1}{2^k}|| \sim |a(1, \ldots, 1)| \exp(-\pi^2 kn^2 t), \ t \to \infty.
$$

So it is possible to choose $n$-periodic (by each coordinate) initial conditions that

$$
T_n(2, 1, k) = T_1(2, 1, 1)/kn^2.
$$

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