ON $C^{1,\beta}$ DENSITY OF METRICS WITHOUT INVARIANT GRAPHS

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Abstract. We show that given any $C^\infty$ Riemannian structure $(T^2, g)$ in the two torus, $\epsilon > 0$ and $\beta \in (0, \frac{1}{3})$, there exists a $C^\infty$ Riemannian metric $\bar{g}$ with no continuous Lagrangian invariant graphs that is $\epsilon$-$C^{1,\beta}$ close to $g$. The main idea of the proof is inspired in the work of V. Bangert who introduced caps from smoothed cone type $C^1$ small perturbations of metrics with non-positive curvature to get conjugate points. Our new contribution to the subject is to show that positive curvature cone type small perturbations are “less singular” than non-positive curvature cone type perturbations. Positive curvature geometry allows us to get better estimates for the variation of the $C^1$ norm of the singular cone in a neighborhood of its vertex.

1. Introduction. The theory developed by Kolmogorov, Arnold and Moser to show the persistence, under $C^k$ small perturbations, of integrable Hamiltonians on the torus and, under $C^k$ small perturbations, of integrable exact twist maps of the annulus, for $k$ large enough, is one of the landmarks of classical mechanics and mathematical physics. Hermann ([2]) showed that actually $C^3$ small perturbations of exact twist maps preserve the existence of invariant curves with prescribed diophantine rotation number. Moreover, Hermann shows that the $C^3$ class is sharp: given any rotation number, there exist $C^{2,\beta}$ small perturbations with $\beta \in (0, 1)$ of exact twist maps which do not have invariant curves with this rotation number. The diophantine condition proves to be necessary for the persistence problem since J. Mather ([5]) showed that the nonexistence of Liouville invariant curves of $C^\infty$ exact twist maps is $C^\infty$ dense in this category of maps. The question that arises naturally from these results is the following: what is the largest $\alpha \geq 0$ such that $C^{1,\alpha}$ arbitrarily small perturbations of an integrable Hamiltonian do not have any

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Continuous Lagrangian invariant graph? A continuous Lagrangian graph $S$ of a Lagrangian $L : TM \to \mathbb{R}$ is a $C^0$ submanifold of an energy level of $L$ such that the canonical projection restricted to $S$ is a homeomorphism onto $M$ (see Section 2 for more details).

Many interesting answers to this question appeared in the literature in the last 40 years. Takens ([11]) showed that $C^1$ close to any exact twist map there exists one with no invariant curves in the interior of the annulus. Hermann ([3]) considered the nonexistence of $C^1$ invariant Lagrangian graphs of Tonelli Hamiltonians, and showed that for any $\beta \in (0, 1)$ and $C^{d+1-\beta}$ close to any Hamiltonian in the $d$-torus there exists one without $C^1$ Lagrangian invariant graphs. Notice however that continuous Lagrangian invariant graphs are just Lipschitz in general, so Hermann’s result does not imply the nonexistence of continuous Lagrangian invariant graphs. Regarding specific families of Hamiltonians like geodesic flows, Bangert ([1]) came out with a simple and beautiful idea to show that there exists $C^1$ small perturbations of a flat metric in $T^2$ without Lagrangian invariant graphs. Such perturbations are constructed from smoothing cones in singular metrics on $T^2$ with negative curvature close to zero. Since $C^1$ small perturbations of the metric are just $C^0$ small perturbations of the geodesic flow the result might seem unsatisfactory from the point of view of twist maps. It is true that the geodesic flow of a flat torus admits local cross sections where the Poincaré return map is a twist map, but this is not possible globally. Moreover, a perturbation of such Poincaré return map as a twist map might not be the Poincaré return map of another geodesic flow, this is very hard to determine in general. The previous two remarks show the relevance of Bangert’s result in the context of integrable geodesic flows. MacKay ([4]) and Ruggiero ([8]) showed that $C^2$ small perturbations of a Riemannian metric create regions in the phase space of the geodesic flow without Lagrangian invariant graphs. The technique is linked to the creation of conjugate points, and it works only locally.

So the best idea so far to obtain small perturbations of geodesic flows in $T^n$ without Lagrangian invariant graphs is Bangert’s idea, based on smoothing singular metrics having cones and proving by direct calculus of variations that no geodesic through the vertex of the cone is a minimizer. This idea was extended and applied by Ruggiero ([10]) to show the $C^1$-density in the family of mechanical Lagrangians in $T^2$ of the nonexistence of Lagrangian invariant graphs in all supercritical energy levels. The goal of this paper is to consider the gap between $C^1$ and $C^2$ perturbations. The main result of the article is the following:

**Theorem 1.1.** Let $g$ be a $C^\infty$, non flat Riemannian metric in the two torus $T^2$. Given $\epsilon > 0$ there is a $C^\infty$ metric $\tilde{g}$ with the properties:

1. $||g - \tilde{g}||_1 < \epsilon$ and $||g - \tilde{g}||_{1,\frac{1}{2}} < C$, where $C > 0$ depends on the inverse of the supremum of the Gaussian curvature of $g$;
2. $\tilde{g}$ admits no continuous field of minimizers.

From Theorem 1.1 we obtain:

**Theorem 1.2.** Let $g$ be any $C^\infty$ metric in the two torus $T^2$. Given $\epsilon > 0$ and $\beta < \frac{1}{3}$ there exists a metric $\tilde{g}$ without continuous field of minimizers such that $||g - \tilde{g}||_{1,\beta} < \epsilon$.

In the Theorem above, the $C^{1,\beta}$ norm is defined for manifolds in general as follows: equip $M$ with a fixed Riemannian metric $g$ and let $G$ be the Sasaki metric on $TM$ associated with $g$. Denote by $||\cdot||^G$ the norm given the Sasaki metric $G$. For $L :
2. Preliminaries. We start with some definitions and notations, we mainly follow [7]. The pair \((M, g)\) will denote a \(C^\infty\) manifold endowed with a \(C^\infty\) Riemannian metric, whose tangent bundle is denoted by \(TM\). \((M, g)\) has a natural length function defined in the set of rectifiable curves and a structure of metric space, we shall assume that such structure is complete. Let \(\tilde{M}\) be the universal covering of \(M\) and let \(\tilde{g}\) be the pullback of \(g\) by the covering map. The pair \((\tilde{M}, \tilde{g})\) is then a \(C^\infty\) Riemannian manifold. The tangent space at \(p \in M\) is denoted by \(T_p M\), and the canonical local coordinates of the tangent space are given by pairs \((p, v)\) where \(p \in M\) and \(v \in T_p M\). Let \(T_1 M\) be the set of unit tangent vectors, or unit tangent bundle, and let \(\pi : TM \rightarrow M\) be the canonical projection: \(\pi(p, v) = p\).

Every Riemannian metric defines a Lagrangian \(L : TM \rightarrow \mathbb{R}\) given by \(L(p, v) = \frac{1}{2}g_p(v, v)\). where \(g_p\) is the Riemannian metric restricted to \(T_p M\). A smooth curve \(\gamma : (a, b) \rightarrow M\) is called a geodesic if it is a solution of the Euler-Lagrange equation. Geodesics have constant speed so to study geodesics it is enough to consider unit speed geodesics. The Riemannian manifold \((M, g)\) is complete as a metric space if and only if every solution \(\gamma(t)\) of the Euler-Lagrange equation is defined for every \(t \in \mathbb{R}\).

Geodesics are local minimizers: given a point \(p \in \text{there exists } \varepsilon > 0\) such that every geodesic \(\gamma : [a, b] \rightarrow M\) whose length is less than \(\varepsilon\) is the curve of minimal length joining \(\gamma(a)\) and \(\gamma(b)\). In this case the distance from \(\gamma(t)\) to \(\gamma(s)\) is the length of \(\gamma[t, s]\) for every \(t, s \in [a, b]\). When any lift \(\tilde{\gamma} : \mathbb{R} \rightarrow \tilde{M}\) of \(\gamma : \mathbb{R} \rightarrow M\) by the covering map satisfies that the length of \(\tilde{\gamma}[t, s]\) is the distance from \(\tilde{\gamma}(t)\) to \(\tilde{\gamma}(s)\), the geodesic \(\gamma\) is called a global minimizer.

The geodesic flow \(\phi_t : T_1 M \rightarrow T_1 M\) is the one parameter family of diffeomorphisms given by \(\phi_t(p, v) = (\gamma(p, v)(t), \gamma'(p, v)(t))\) where \(\gamma(p, v)(t)\) is the geodesic satisfying the initial conditions \(\gamma(p, v)(0) = p, \gamma'(p, v)(0) = v\). \(\gamma(p, v)(t)\) is uniquely determined by its initial conditions since the Euler-Lagrange equation is a second order ordinary differential equation. We say that a submanifold \(S\) of \(TM\) is invariant by the geodesic flow if for every \(\theta \in S\) the orbit of \(\theta\) by the geodesic flow is contained in \(S\).

One of the main features of the geodesic flow (as well as every Euler-Lagrange flow) is the existence of a natural symplectic two form \(\Omega\) in \(TM \times TM\) that is invariant by the action of the geodesic flow, endowing \(TM\) with a symplectic structure. A \(C^k\) submanifold \(S \subset TM\) is called Lagrangian if its dimension is half the dimension of \(M\) and if the form \(\Omega\) restricted to the tangent space of \(S\) vanishes. If the dimension of \(M\) is two, every smooth surface in \(T_1 M\) that is invariant by the geodesic flow is Lagrangian.

A \(C^k\) submanifold \(S \subset TM\) is called a graph if \(\pi : S \rightarrow M\) is a homeomorphism. The submanifold \(S\) is represented in canonical coordinates by \((p, \pi^{-1}(p)) = \ldots\)
(p, X(p)), p ∈ M. The submanifold S is invariant if and only if the vector field X gives rise to a continuous flow in M whose integral orbits are geodesics. Manifolds with symmetries like, for instance, tori of revolution, totally integrable systems and the 3 sphere have invariant Lagrangian graphs (in the case of the 3 sphere the Hopf fibration is a well known example), and by KAM theory small $C^3$ perturbations of integrable systems possess a lot of invariant Lagrangian graphs as well. As stated in the introduction, a continuous graph with constant energy will be called a continuous Lagrangian graph. The continuity does not imply in general the existence of the tangent space of the graph, so the canonical symplectic form of the geodesic flow might not be defined in the graph. But since smooth invariant Lagrangian graphs have constant energy, our definition of continuous Lagrangian graph can be regarded as a natural extension of the notion of smooth Lagrangian graph.

It is well known (see for instance [6]) that a continuous flow in M whose orbits are geodesics is at least Lipschitz continuous. This allows us to state the following result that can be generalized to convex, superlinear Lagrangians (Tonelli Lagrangians) in the two torus.

**Proposition 1.** Let $(T^2, g)$ be a $C^\infty$ Riemannian structure on the two torus $T^2$ and let $S \subset T_1 T^2$ be a continuous graph that is invariant by the geodesic flow. Then, if $S = \{(p, X(p)), p \in T^2\}$ the orbits of X are global minimizers.

**Proof.** This is well known in calculus of variations, we give a sketch of proof for the sake of completeness.

The fact that the vector field $X$ is Lipschitz continuous implies that it is uniquely integrable. Let $X^\perp$ be a unit vector field of vectors which are perpendicular to $X$ and such that $(X, X^\perp)$ has the canonical orientation. $X^\perp$ is Lipschitz and uniquely integrable as well, and the orbits of its flow are always perpendicular to the orbits of $X$ which are geodesics.

Let us lift the vector fields $X$, $X^\perp$ to a pair of orthogonal vector fields $\tilde{X}$, $\tilde{X}^\perp$ in the tangent space of $T^2 = \mathbb{R}^2$. The orbits of $\tilde{X}$ are geodesics of $(\tilde{M}, \tilde{g})$ which are always perpendicular to the orbits of $\tilde{X}^\perp$. Let $\Gamma$ be the foliation of geodesics given by the flow of $\tilde{X}$, and let $\Gamma^\perp$ be the foliation given by the orbits of $\tilde{X}^\perp$. This pair of foliations define a global coordinate system in $\mathbb{R}^2$ and by the first variation formula, all the leaves of $\Gamma^\perp$ are equidistant. Namely, the distance from a leaf $\Gamma^\perp(x)$ to a leaf $\Gamma^\perp(y)$ containing two points $x, y$ respectively is just the length of the subset of any geodesic in the foliation $\Gamma$ contained in the strip bounded by $\Gamma^\perp(x)$ and $\Gamma^\perp(y)$. From this follows easily that the geodesics in the foliation $\Gamma$ are global minimizers and hence, the orbits of $X$ are global minimizers.

The vector field $X$ is usually called a field of minimizers. The article is devoted to get examples of metrics in $T^2$ without fields of minimizers, which implies by the previous proposition that such metrics cannot have continuous invariant graphs in $T_1 M$. We shall explore the idea of smoothing metrics with conic singularities introduced by V. Bangert with further developments by R. Ruggiero.

3. **Trying to smoothen flat singular cones.** Let us remind briefly the previous versions of singular cones and caps to motivate the main subject of the article. In Ruggiero [9], flat small cones of revolution generated by rotating straight segments in the $x, z$ plane, for $x, z \geq 0$, around the $z$ axis were used to show that the set of Riemannian metrics in the two torus with no minimizing graphs is dense in the $C^1$ topology. The proof follows essentially from two facts. First of all, segments
crossing the $z$ axis with negative slope have the property that minimizing geodesics
joining antipodal points in the boundary of the cone generated by the segment avoid
an open neighborhood $V$ of the vertex of the cone. The radius of $V$ depends on the
slope of the generating segment. If the slope is small, it is easy to smoothen the
cone inside $V$ in a way that the variation of the $C^1$ jets of the smoothed cap is small.
Secondly, any Riemannian metric in the two torus has a point where the curvature
vanishes, so we can perturb the metric in the $C^2$ topology to make the curvature
constant equal to zero in an open neighborhood $W$ of this point and glue the flat
part of the smoothed cap to get a $C^\infty$ surface with an open neighborhood avoided
by minimizers.

The above argument is enough to get a $C^1$ perturbation of the metric without
minimizing graphs. However, a finer topology for the perturbation needs a more
careful look at the rate of the variation of the $C^1$ jets of the singular cone inside $V$
and the size of $V$. Let us give some of the details of the above construction.

Let $\epsilon > 0$, $0 < a < \epsilon$, and let $C_{\epsilon,a}$ be the flat singular surface of revolution
generated by the segment $c(x) = (x,0,-a(x-\epsilon))$, $x \in [0,\epsilon]$, turning around the
vertical axis. Let us notice some elementary properties of $C_{\epsilon,a}$.

1. We have $c(0) = (0,0,\epsilon a)$, $c'(0) = (1,0,-a)$ is constant depending on $a$, so the
   surface of revolution is smooth at $x = 0$ if and only if $a = 0$.
2. The perimeter of the circle bounding $C_{\epsilon,a}$ is $2\pi\epsilon$ of course.
3. The radius of $C_{\epsilon,a}$ is $\tau = \epsilon \sqrt{1 + a^2}$.

The singular surface $C_{\epsilon,a}$ is then isometric to a orbifold type flat cone $\Sigma_{\theta(\epsilon,a)}$
obtained by removing a conic sector

$$S_{\theta(\epsilon,a)} = \{(r,\theta), r \leq \epsilon, \theta \in (0,\theta(\epsilon,a))\},$$

where $(r,\theta)$ are polar coordinates in the plane, and next identifying the segments
in the boundary of $S_{\theta(\epsilon,a)}$. The expression of $\theta(\epsilon,a)$ in terms of $\epsilon,a$ follows from
plane trigonometry. The perimeter of the flat cone $\Sigma_{\theta(\epsilon,a)}$ is $(2\pi - \theta(\epsilon,a))\tau$ and has
to coincide with the perimeter of $C_{\epsilon,a}$. So we have

$$(2\pi - \theta(\epsilon,a))\tau = 2\pi\epsilon$$

which yields

$$\theta(\epsilon,a) = 2\pi a^2 \frac{1}{(\sqrt{1 + a^2})(1 + \sqrt{1 + a^2})} \approx \pi a^2$$

for $a$ small enough.

With the expression of $\theta(\epsilon,a)$ we can estimate the radius of a neighborhood $V$ of
the singularity of $\Sigma_{\theta(\epsilon,a)}$ that is avoided by minimizers. Indeed, two antipodal points
$p,q$ in the boundary of $\Sigma_{\theta(\epsilon,a)}$ are joined by a straight segment $[p,q]$ in a way that
the geodesic rays from the singularity to $p$ and $q$ form an angle $\alpha(\epsilon,a) = \pi - \theta(\epsilon,a)$.
So the radius of our neighborhood $V$ can be estimated by the distance $\delta(\epsilon,a)$ from
the singularity to $[p,q]$. By trigonometry of right triangles,

$$\delta(\epsilon,a) = \tau \cos\left(\frac{1}{2}\alpha(\epsilon,a)\right) = \tau \sin\left(\frac{1}{2}\theta(\epsilon,a)\right).$$

The Taylor expansion of $\sin(x)$ for $x$ small gives

$$\delta(\epsilon,a) = \frac{1}{2} \tau \theta(\epsilon,a) + O(\theta(\epsilon,a)^3) = \frac{1}{2} \epsilon \sqrt{1 + a^2} \pi a^2 + O(a^3).$$
So we conclude that, up to higher order terms, the distance $\delta(\epsilon, a)$ is

$$\delta(\epsilon, a) = \frac{1}{2} \epsilon \pi a^2.$$

The first derivative of the generating function $f(x) = -a(x - \epsilon)$, $x \in [0, \epsilon]$, $f(x) = a(x + \epsilon)$ for $x \in [-\epsilon, 0]$, is $f'(x) = -a$ if $x > 0$, $f'(x) = a$ if $x < 0$. And we have an interval of size of order $\epsilon a^2$ to try to improve the regularity of $f'(x)$, namely, to glue $f'(x)$ with a more regular function $g_{\epsilon,a}$ in an interval $[-\rho, \rho] \subset [-\delta(\epsilon,a), \delta(\epsilon,a)]$, in a way that the variation of the first derivatives of the new function is as small as possible.

So let us take $\rho = \frac{1}{2} \delta(\epsilon,a)$, and consider the function $g_{\epsilon,a} : (-\epsilon,\epsilon) \to \mathbb{R}$ given by

$$g_{\epsilon,a}(x) = -\sqrt{\frac{4}{\epsilon \pi}} \sqrt{x}$$

for $x \geq 0$ and

$$g_{\epsilon,a}(x) = \sqrt{\frac{4}{\epsilon \pi}} \sqrt{|x|}$$

for $x \leq 0$. It is easy to check that

1. $g_{\epsilon,a}(\frac{1}{2} \delta(\epsilon,a)) = g_{\epsilon,a}(\frac{1}{4} \epsilon \pi a^2) = -a$,
2. $g_{\epsilon,a}(-\frac{1}{2} \delta(\epsilon,a)) = g_{\epsilon,a}(-\frac{1}{4} \epsilon \pi a^2) = a$,
3. The function $g_{\epsilon,a}$ is Hölder with exponent $\frac{1}{2}$ and constant $L_\epsilon = L(\frac{1}{\sqrt{a}})$.

Therefore, the above reasoning suggests a natural way to regularize the singular cone $C_{\epsilon,a}$: to glue $f'$ with $g_{\epsilon,a}$ by means of a $C^\infty$ symmetric bump function with support in the interval $[-\frac{1}{2} \delta(\epsilon,a), \frac{1}{2} \delta(\epsilon)]$ that is equal to 1 in the interval $[-\frac{1}{2} \delta(\epsilon,a), \frac{1}{2} \delta(\epsilon,a)]$ for instance. Let $\tilde{C}_{\epsilon,a}$ be the new cone generated by the glued functions. It is not difficult to show that we can chose the bump function such that the first jets of $\tilde{C}_{\epsilon,a}$ are Hölder with exponent $\frac{1}{2}$ and constant $L_\epsilon$ close to $L_\epsilon$. We can finally smoothen $\tilde{C}_{\epsilon,a}$ near the singularity to get a $C^\infty$ cone keeping the same geometric properties. In particular, we get a $C^\infty$ cone without fields of minimizers.

The big problem with the above construction is that the Hölder constant of the first jets of the cone $\tilde{C}_{\epsilon,a}$ is proportional to the inverse of the square root of the radius $\sqrt{1 + a^2}$ of the cone. Indeed, to get a metric $(T^2,g)$ without fields of minimizers we glue a cone like the previous one in a small neighborhood of $(T^2,g)$ with vanishing curvature (which there always exists after a $C^2$ perturbation of $g$). So if the injectivity radius of $(T^2,g)$ is small, the size of the cone must be small and hence, we cannot approach $(T^2,g)$ in the $C^{1,\frac{1}{2}}$ topology by metrics like the above ones. Moreover, we can check that the dependence of the Hölder constant on the size of the cone makes it impossible to get a small $C^{1,\beta}$ perturbation of $(T^2,g)$ for any $\beta < \frac{1}{2}$, by gluing our cones to $(T^2,g)$. Therefore, it is hopeless to get $k > 1$ such that $d_{C^k}(g,g)$ is small by applying the above procedure.

The purpose of the forthcoming sections is to construct a family of cones with positive curvature where the $C^2$ jets have Hölder exponent $\frac{1}{4}$ with uniformly bounded Hölder constant independent of the size of the cone. With this in hand, given any $\beta < \frac{1}{3}$, we shall be able to approach any Riemannian structure $(T^2,g)$ by one without fields of minimizers in the $C^{1,\beta}$ topology.

4. **Positive curvature cones in surfaces of revolution.** The goal of the section is to try to apply the same scheme of the previous section to singular surfaces of revolution with positive curvature. The procedure does not extend straightforwardly
and poses some new technical difficulties. However, at the end we shall get singular cones which are more “smoothable” than the flat cones of the previous section.

To begin with the analysis, let us remark that no surface of revolution is a singular cone of positive constant curvature. Simply because the classification of surfaces of revolution of positive constant curvature yields that such surfaces must be convex subsets of a round sphere invariant by rotations. So here we have the first important difference with respect to flat singular cones, that leads us to make some changes in the strategy of the last section.

Let \( a > 0 \) be a small number and \( r > 0 \). Consider the family of spherical cones given by the revolution around the \( z \)-axis of the arc of circle \( \alpha_a(x) = (x, 0, f_a(x)) \), where

\[
f_a(x) = \left( r^2 - (x + a)^2 \right)^{\frac{1}{2}}
\]

and \( x \in [0, r_n] \), with \( r_n \) satisfying \( \frac{a}{r_n} = \frac{1}{n} \) for some large \( n \) (the parameters \( a, r_n, n \) will be chosen more precisely during the proof of the main theorem). Let us denote by \( C_{a,r,n} \) this family of cones. Observe that \( C_{0,r,n} \) is a disk in the sphere of radius \( r \), and the radius \( \rho_{a,r,n} \) of the ball \( B_{\rho_{a,r,n}}(p) \subset C_{a,r,n} \) around \( p = (0, 0, \sqrt{r^2 - a^2}) \) is given by the formula

\[
\rho_{a,r,n} = \int_0^{r_n} \| \alpha_a'(x) \| \, dx.
\]

The following technical, elementary Lemma will be used in the sequel.

**Lemma 4.1.** The cone \( C_{a,r,n} \) satisfies the following geometric properties:

1. The derivative of \( \alpha_a(x) \) at \( x = 0 \) is
   \[
   \alpha'_a(0) = (1, 0, -\frac{a}{\sqrt{r^2 - a^2}}).
   \]

2. The radius \( r_n \) and the radius \( \rho_{a,r,n} \) are related by the following formula:
   \[
   \rho_{a,r,n} = r(\arcsin(\frac{r_n + a}{r}) - \arcsin(\frac{a}{r})).
   \]

3. The curvature of \( C_{a,r,n} \) at every point of the parallel containing \( \alpha_a(x), x \in [0, r_n] \), is
   \[
   K(x) = \frac{1}{r^2}(1 + \frac{a}{x}).
   \]

The proof of the Lemma is straightforward from the geometry of \( C_{a,r,n} \), it is a surface of revolution with one singularity where the curvature is not defined. Notice that the curvature is always greater than \( \frac{1}{r^2} \), and if \( x = r_n \) then we get

\[
K(r_n) = \frac{1}{r^2}(1 + \frac{1}{n}) = \frac{1}{\tau_n^2}
\]

where \( \tau_n = \frac{r}{\sqrt{1 + \frac{1}{n}}} \). So the curvature at \( r_n \) coincides with the curvature of a round sphere centered at \((0, 0, 0)\) with radius \( \tau_n \), slightly smaller than \( r \).

As we mentioned before, the cone \( C_{a,r,n} \) does not have constant curvature and thus cannot be identified with a orbifold-type cone of constant positive curvature. We would like to remind that this identification was crucial in Section 2 in order to get an estimate of the rate of variation of the \( C^1 \) jets of the cone compared to the size of the cone. This estimate would give us a good hint of how to smoothen the singular cone in a neighborhood of the singularity to produce a perturbation of a
smooth metric in some $C^{1,\beta}$ topology. A procedure to get around this problem is proposed in the next section.

5. Comparison with a cone of constant curvature. In order to find and measure a neighborhood of the singularity of $C_{a,r,n}$ that is avoided by minimizing geodesics joining points in its boundary, we shall compare (i.e., applying CAT comparison theory) the geometry of $C_{a,r,n}$ with the geometry of a cone of constant curvature $K(r_n)$. Namely, let $\Sigma(\theta)$ be the orbifold obtained by cutting off a conic sector $S(\theta)$ of the sphere $x^2 + y^2 + z^2 = \frac{1}{K^2}$ of angle $\theta > 0$ (small) with vertices at $p_0 = (0, 0, \frac{1}{n^2})$, and $(0, 0, -\frac{1}{n^2})$, bounded by two half meridians, and then identifying the boundary of $S(\theta)$ with a single half meridian.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{In the right, the curve connecting $x_1$ and $x_2$ is a minimizing geodesic of length $2\rho$ that not intersect a neighborhood of the singularity.}
\end{figure}

Lemma 5.1. Let $K = K(r_n)$ where $r_n = an$. There exist $\theta_{a,n} > 0$, $L = L(r,n) > 0$, $\epsilon > 0$, such that the orbifold $\Sigma(\theta_{a,n})$ has the following properties:

1. The sphere which bounds the ball $B_{\rho_{a,r,n}}(p_0) \subset \Sigma(\theta_{a,n})$ centered at $p_0$ has length $2\pi r_n$, which is the length of the sphere in the boundary of $C_{a,r,n}$.
2. For every $r_n \leq \epsilon$ we have,
   \[ \theta_{a,n} > L(r,n)r_n^2. \]
3. The constant $L(r,n)$ satisfies
   \[ K(r_n) - L(r,n) = \frac{1}{r^2} \left( \frac{n-1}{n^2} \right). \]

Proof. To simplify the notation, let us call by $\theta$ the angle $\theta_{a,n}$, and by $\rho$ the number $\rho_{a,r,n}$. Let $S_K$ be the sphere centered at $(0,0,0)$ with curvature $K$. First of all, notice that we can express the perimeter $l$ of the sphere in the boundary of $B_{\rho_{a,r,n}}(p_0)$ by
   \[ l = \int_0^{2\pi - \theta} \frac{1}{\sqrt{K}} \sin(\sqrt{K}\rho) ds = (2\pi - \theta) \frac{1}{\sqrt{K}} \sin(\sqrt{K}\rho), \]
since the singular surface $\Sigma(\theta)$ is locally isometric to a round sphere of curvature $K$ outside the singular point. The function
   \[ f(t) = \frac{1}{\sqrt{K}} \sin(\sqrt{K}t) \]
is the solution of the Jacobi equation $J''(t) + K J(t) = 0$ with $J(0) = 0$ and $J'(0) = 1$. It describes the variation by unit geodesics given by $\exp_{p_0}(tv(s))$, where $v(s)$ is an arc length parametrization of the unit circle in $T_{p_0}S\kappa$.

By the assumptions in the Lemma, $l = 2\pi r_n$, which implies

$$\theta = 2\pi(1 - \frac{\sqrt{K}r_n}{\sin(\sqrt{K}\rho)})$$

$$= \frac{2\pi}{\rho}(\rho - \frac{\sqrt{K}\rho}{\sin(\sqrt{K}\rho)}r_n).$$

By Lemma 4.1 item (ii), the expression of $\rho$ in terms of $r_n$ is

$$\rho = \rho_{n,r,n} = r(\arcsin\left(\frac{r_n + \frac{a}{r}}{r}\right) - \arcsin\left(\frac{a}{r}\right)).$$

Expanding the Taylor series of $f(x) = \arcsin(x)$ around 0 we get

$$f(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + O(x^7),$$

which yields

$$\rho = r_n(1 + \frac{1}{6r^2}(3a^2 + 3ar_n + r_n^2) + O(\frac{r_n^5}{r^4})).$$

(1)

Since

$$\frac{x}{\sin(x)} = 1 - \frac{x^2}{6} + O(x^4) = 1 + \frac{x^2}{6} + O(x^4),$$

we have

$$\frac{\sqrt{K}\rho}{\sin(\sqrt{K}\rho)} = 1 + \frac{K\rho^2}{6} + O(K^2\rho^4)$$

$$= 1 + \frac{K\rho^2}{6}(1 + \frac{1}{6r^2}(3a^2 + 3ar_n + r_n^2))$$

$$+ O(\frac{r_n^5}{r^4})^2 + O(\frac{r_n^4}{r^4})$$

$$= 1 + \frac{Ky_n^2}{6} + O(\frac{r_n^4}{r^4}).$$

Since $2r_n \geq \rho \geq r_n$, replacing the above estimate in the expression of $\theta$ we get

$$\theta = \frac{2\pi}{\rho}\left(\rho - r_n(1 + \frac{K\rho^2}{6} + O(\frac{r_n^4}{r^4}))\right)$$

$$\geq \pi\left(\frac{1}{6r^2}(3a^2 + 3ar_n + r_n^2) - \frac{Ky_n^2}{6} + O(\frac{r_n^4}{r^4})\right).$$

Now, by Lemma 4.1 item (3), $K(r_n) = \frac{1}{r^2}(1 + \frac{1}{n^2})$, which implies

$$\theta \geq \pi\left(\frac{a^2 + r_n^2}{2r^2} - \frac{r_n^2}{6ar^2} + O(\frac{r_n^4}{r^4})\right)$$

$$\geq \pi\left(\frac{r_n^2}{2r^2}(1 + \frac{1}{n^2}) - \frac{r_n^2}{6ar^2} + O(\frac{r_n^4}{r^4})\right).$$

So there exists $\epsilon > 0$ such that for every $|r_n| < \epsilon$ we have

$$\theta > (1 + \frac{1}{n^2})\frac{\pi}{4r^2}r_n^2 = Lr_n^2,$$

as we claimed in item (2).
item (3) is just a consequence of the expressions of $K(r_n)$ and $L = L(r, n)$. 

6. Estimates of the size of balls around the vertex avoided by minimizers. We shall use the estimates of the previous section to get the most important estimate for the proof of the main theorem: the maximal size of a ball around the vertex of the cone, in terms of the angle $\theta$, that is avoided by minimizers. The following Lemma states that minimizing geodesics in $B_{\rho_n}(p_0)$ joining two antipodal points in the boundary sphere must avoid an open neighborhood of $p_0$ whose radius depends on $\theta_{a,n}$.

Lemma 6.1. Let $\rho = \rho_{a,r,n}$. There exists $\epsilon_1 > 0$ such that for any $0 < \theta < \epsilon_1$ and $0 < \sigma < \frac{\rho \theta}{3}$, the geodesics in $B_{\rho}(p_0)$ joining two antipodal points in its boundary do not meet $B_{\sigma}(p_0)$.

Proof. The proof is just trigonometry and Alexandrov-Toponogov comparison Theorem. Consider an Euclidean geodesic triangle $(\sigma_1, \sigma_2, \sigma_3)$ such that $\ell(\sigma_1) = \ell(\sigma_2) = \rho$ and the angle at the vertex $\bar{p} = \sigma_1(\rho) = \sigma_2(0)$ given by $\angle (-\sigma_1'(\rho), \sigma_2'(0)) = \pi - \theta$.

If $d_0$ is the Euclidean distance, then

$$d_0(\bar{p}, \sigma_3) = \rho \sin\left(\frac{\theta}{2}\right).$$

By the Toponogov’s Theorem, the distance from $p_0$ to any minimizing geodesic in $B_{\rho}(p_0)$ connecting two antipodal points in its boundary is greater than $d_0(\bar{p}, \sigma_3) = \rho \sin\left(\frac{\theta}{2}\right)$. Choosing $\epsilon_1 > 0$ small enough, we have that, for $\theta < \epsilon_1$,

$$\frac{\rho \theta}{3} \leq \rho \sin\left(\frac{\theta}{2}\right).$$

(2)

Corollary 1. The minimizing geodesics joining two antipodal points in $C_{a,r,n}$ do not intersect the ball $B_{\sigma}(p_0) \subset C_{a,r,n}$ of radius

$$\sigma = M r_n^3$$

where $M = M(r, n) = (1 + \frac{1}{n}) \frac{\pi}{4n}$. 

Proof. By equation (1) in the proof of Lemma 3.1 we have that

$$\rho = \rho_{a,r,n} \geq r_n.$$ 

Therefore, the corollary follows by combining Lemmas 5.1, 6.1 and Alexandrov-Toponogov comparison Theorem.

Remark. The estimate in Corollary 1 is the second main difference with respect to the results of Section 2 for the flat case. The constant $M(r, n)$ in the formula of the radius $\sigma$ of the neighborhood avoided by minimizers does not depend on the size of the singular cone. This constant will be crucial for the proof of the main theorem.

7. Proof of Theorem 1.1. Consider the spherical cone $\Phi^* : B_{r_n}(0) \subset \mathbb{R}^2 \to \mathbb{R}^3$, where $\Phi^*(x, y) = (x, y, f_a(\sqrt{x^2 + y^2}))$. Define the singular metric $g^*_a$ on $B_{r_n}(0)$ by

$$g^*_a = \Phi^* g_{eucl},$$

where $g_{eucl}$ is the standard Euclidean metric on $\mathbb{R}^3$. In the same way, the metric of the sphere of radius $r$ is given by $g^r = \Phi^*_0 g_{eucl}$. The main idea of the proof of Theorem 1.1 is to compare the variation of the slope of $f_a$ with the radius of the ball around the vertex of the cone that is avoided by minimizers.
Lemma 7.1. Let $b > 0$ and suppose that $r_n \geq b$ for every $n$. There exists a sequence of $C^1$ metrics $g_{a,r,n}$ on $B_r(0)$, such that

1. $g_{a,r,n}$ has no continuous field of minimizers;
2. $g_{a,r,n}$ converges to $g^*$ in a disk containing $B_b(0)$ in the $C^1$ topology;
3. for $a > 0$ sufficiently small,
   \[ \|g_{a,r,n} - g^*\| < M(r,n)^{-\frac{2}{3}}, \]
   where the constant $M(r,n)$ was defined in Corollary 1.

Proof. Let $r_\sigma$ be given implicitly by $\sigma = \int_0^{r_\sigma} ||\alpha'_a(s)||\, ds$.

From Lemma 6.1 it is clear that we can choose $\omega > 0$ such that $B_{\sigma+\omega}(p)$ still does not meet any minimizer joining antipodal points in $C_{a,r,n}$. For simplicity, let us denote $M = M(r,n)$.

Consider the bump function $\beta$ such that $\beta(t) = 1$ for $[0,r_\sigma]$ and $\beta(t) = 0$ for $t \in [r_{\sigma+\omega},r_n]$.

Define $\lambda_a : [0,r_n] \to \mathbb{R}$ by
   \[ \lambda_a(x) = -\beta(x)M^{-\frac{1}{3}}x^{1/3} + (1 - \beta(x))f'_a(x), \]
   \[ \Lambda_a(x) = \sqrt{r^2 - (r_n + a)^2} - \int_0^{r_n} \lambda_a(s)\, ds + \int_0^x \lambda_a(l)\, dl. \]

The metric $g_{a,r,n} = \Psi^*_ag_{eucl}$, where $\Psi_a(x,y) = (x,y,\lambda_a(\sqrt{x^2 + y^2}))$, is $C^1$ in $B_{r_n}(0)$ and $C^\infty$ in $B_{r_n}(0) \setminus \{0\}$.

Take $b > 0$ and $r_n = an \geq b$ for every $n \in \mathbb{N}$. Then, keeping $r$ fixed and letting $a \to 0^+$, we have that $n \to \infty$, and the metrics $g_{a,r,n}$ converge in the $C^1$ topology to the spherical metric $g^*$ in a disk of radius at least $b$. Because $f_a \to f_0$, $r_\sigma \to 0$ and $\lambda_a \to f'_0$. The convergence of the derivatives happens because $|\lambda_a|$ is bounded above by $\max\{r_n,|f'_a|\}$. More precisely, since, by Corollary 1, $M_{r_n}^3 = \sigma \geq r_\sigma$, on $[0,r_\sigma]$, we have that
   \[ \lambda_a(t) \geq \lambda_a(r_\sigma) = -M^{-1/3}r_\sigma \geq -r_n. \]

The Hölder norm of $\lambda_a - f'_a$ is given by
   \[ ||\lambda_a - f'_a|| \frac{1}{3} = \sup_{x \neq y; x,y \in [0,r_n]} \frac{|\lambda_a(x) - \lambda_a(y)|}{|x-y|^{\frac{1}{3}}} \leq M^{-\frac{1}{3}}, \]

Outside the neighborhood $B_{r_{\sigma+\omega}}(0)$, the surface $(B_{r_n}(0),g_{a,r,n})$ is isometric to the cone $C_{a,r,n}$. Therefore, a minimizing geodesic of length $2r_n$ does not intersect $B_{r_n}(0)$ so such a geodesic does not meet the origin. This yields that $g_{a,r,n}$ cannot have a continuous field of minimizers.

\[ \square \]

Lemma 7.2. Given $b > 0$, there exists a family $\tilde{g}_{a,r,n}$ of $C^\infty$ metrics on $B_r(0)$ for $r_n \geq b$ satisfying properties 1,2 and 3 of Lemma 7.1.

Proof. Let $r_\delta < r_\sigma$. Define the function $\gamma_a : [0,r_n] \to \mathbb{R}$ by
   \[ \gamma_a(x) = -\alpha(x)M^{-\frac{1}{3}}r_\delta^{-2/3}x - (1 - \alpha(x))\lambda_a(x), \]
   where $\alpha$ is a bump function which is equal to one on $[0,r_\delta - \epsilon]$ and equal to zero on $[r_\delta + \epsilon,r_\sigma]$. Observe that $r_\delta$ can be chosen as small as we want. So, since the first
part of the expression of $\gamma_a$ is linear, choosing $r_\delta$ small, the $\frac{1}{q}$-Hölder norm of $\gamma_a$ is bounded by the norm of $\lambda_a$. Finally, define $\Gamma_a : [0, r_n] \to \mathbb{R}$ by

$$\Gamma_a(x) = \sqrt{r^2 - (r_n + a)^2} - \int_x^{r_n} \gamma_a(l) \, dl.$$  

The desired family of metrics is given

$$\bar{g}_{a,r,n} = \bar{\Psi}_a g_{\text{euc}}$$

where $\bar{\Psi}_a(x, y) = (x, y, \Gamma_a(\sqrt{x^2 + y^2}))$. As we have seen before, this family satisfies property 3 of 7.1. The other two properties follow from the fact that the metrics $\bar{g}_{a,r,n}$ are equal to $g_{a,r,n}$ outside a small neighborhood around the vertex, which is inside de neighborhood avoided by the geodesics and the fact that $r_\delta < r_\sigma$ and $r_\sigma \to 0$ as $a \to 0$. 

To finish the proof of Theorem 1.1 we glue the smoothed cone constructed above to a perturbation of a metric in the two torus with constant positive curvature in a small neighborhood.

So let $g$ be any Riemannian $C^2$ metric on $T^2$. If $(T^2, g)$ is not flat, by the Gauss-Bonnet Theorem there exists $p \in T^2$ such that the sectional curvature $K(p)$ is positive. Then, given $\epsilon > 0$, there exists a metric $(T^2, g_0)$ in an $\epsilon-$neighborhood of $(T^2, g)$ in the $C^2$ topology such that the sectional curvature of $g_0$ satisfies

$$K_0(q) = K(p)$$ 

for every $q$ in a geodesic ball $(B_{\rho + \delta}(p), g_0) \subset (T^2, g_0)$ of $\rho$ of radius $\rho + \delta$.

Now we can use the preceding construction. Let $r > 0$, $a > 0$, $n > 0$ such that $r_n = an$ as in Lemma 3.1. Let $\tau_n = \frac{r}{\sqrt{1 + \frac{\rho}{n^2}}}$, let $C_{a,r,n}$ be the singular surface of revolution constructed in Lemma 3.1. Let us choose $r$ and $n$ such that

$$K(p) = \frac{1}{\tau_n^2},$$

then choose $a$ small enough such that $\rho = r_n + \delta'$ for some $\delta' > 0$ (that without loss of generality can be assumed to be equal do $\delta$). Then Lemma 3.1 (3) tells us that the curvature of the surface of revolution $C_{a,r,n}$ at the parallel with radius $r_n$ is $K(p)$.

Since on $(B_{\rho + \delta}(p), g_0)$ we have constant positive curvature $K(p)$, by Cartan’s Theorem, for $r_n + \delta = \rho$ there is an isometry

$$F : B_{r_n + 2\delta}(0) \subset \mathbb{R}^2 \to B_{\rho + \delta}(p),$$

$B_{r_n + 2\delta}(0) \subset \mathbb{R}^2$ furnished with the pulled back metric from the sphere of radius $\tau_n$ in $\mathbb{R}^3$.

Now, let us consider the metrics $\bar{g}_{a,r,n}$ defined on $B_{r_n + \delta}$ which were obtained in Lemma 7.2. Consider $(B_{\rho + \delta}(p), F_\ast \bar{g}_{a,r,n})$ the push-forward by $F$ of the metric $\bar{g}_{a,r,n}$ to $B_{\rho + \delta}(p) \subset T^2$. Since the metrics $\bar{g}_{a,r,n}$ are isometric to the surfaces $C_{a,r,n}$ outside a very small neighborhood of 0, the curvature of $(B_{\rho + \delta}(p), F_\ast \bar{g}_{a,r,n})$ at the sphere $S_{r_n}$ - in the boundary of $B_{r_n}(p)$ - is equal to $K(p)$.

Let $\Delta$ be a bump function in $B_{\rho + \delta}(p)$ such that $\Delta(q) = 1$ in $(B_{\rho_n}(p), F_\ast \bar{g}_{a,r,n})$, and $\Delta(q) = 0$ outside $B_{\rho}(p)$.

Let us consider the metric $(T^2, G_{a,r,n})$, defined as follows:

$$G_{a,r,n} = \Delta(F_\ast \bar{g}_{a,r,n}) + (1 - \Delta)g_0.$$
Notice that in the annulus $B_p(0) \setminus B_r(0)$ the metrics $g_0$ and $F_\ast \bar{g}_{a,r,n}$ are $C^2$ close to each other. Then Lemma 7.1 tells us that the $C^{1,\bar{x}}$-norm of $G_{a,r,n}$ is bounded above by $M^{-\frac{1}{4}}$, where $M$ is the constant defined in Corollary 1, and

$$||g - G_{a,r,n}||_{1,\bar{x}} < M^{-\frac{1}{4}}.$$  

Since the curvature $K(p)$ is positive and $n$ can be chosen large, the radius $r$ in the construction of the cones $C_{a,r,n}$ is practically $\frac{1}{\sqrt{K(p)}}$. So by item (3) in Lemma 3.1, the constant $M = M(r,n)$ is comparable to $K(p)$. This proves Theorem 1.1.

8. Proof of Theorem 1.2. First of all, if needed, we approach $(T^2, g)$ by a non-flat metric in the $C^\infty$ topology. Actually, if $(T^2, g)$ is flat we shall need the following more precise statement.

Lemma 8.1. Let $(T^2, g)$ be a flat metric in $T^2$. Given $\epsilon > 0$, $b > 0$ less than the injectivity radius of $(T^2, g)$, $p \in T^2$, there exists $0 < 2\rho < b$ and a $C^\infty$ metric $g_{a,b}$ that is $\sqrt{\epsilon}$-close to $g$ in the $C^2$ topology with constant curvature equal to $\epsilon$ in the ball $B_{2\rho}(p)$.

Proof. Let $a, b$ positive numbers and let $f_{a,b} : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f_{a,b}(x,y) = \frac{1}{6}a(x^2 + y^2 - b^2)^3 = \frac{1}{6}a(r^2 - b^2)^3,$$

in polar coordinates $r = \sqrt{x^2 + y^2}$. The following observations are elementary.

1. The graph of $f_{a,b}$ is invariant by rotations around the vertical axis in $\mathbb{R}^3$.
2. $\frac{\partial}{\partial \rho} f_{a,b}(x,y) = \frac{1}{6}a(x^2 - b^2)^2$,
3. $\frac{\partial}{\partial \theta} f_{a,b}(x,y) = \frac{2}{3}a x y (r^2 - b^2)$,
4. $\frac{\partial^2}{\partial \rho^2} f_{a,b}(x,y) = \frac{1}{6}a(r^2 - b^2) (4x^2 + r^2 - b^2)$,
5. $\frac{\partial^2}{\partial \rho \partial \theta} f_{a,b}(x,y) = \frac{1}{6}a x y (r^2 - b^2)$,
6. $\frac{\partial^2}{\partial \theta^2} f_{a,b}(x,y) = \frac{1}{6}a(r^2 - b^2) (4y^2 + r^2 - b^2)$,
7. If $b < 1$ then the $C^2$ norm of $f_{a,b}$ is bounded above by $\frac{5}{6}ab^4$ in the disk $r \leq b$,
8. The function $f_{a,b}$ and its first and second partial derivatives vanish at $r = b$.

The curvature of the graph of $f_{a,b}$ is

$$K(x,y) = \frac{\det \text{Hess}(f_{a,b}(x,y))}{\left(1 + \left(\frac{\partial}{\partial \rho} f_{a,b}(x,y)\right)^2 + \left(\frac{\partial}{\partial \theta} f_{a,b}(x,y)\right)^2\right)^2}.$$  

where $\text{Hess}(f_{a,b}(x,y))$ is the Hessian matrix of $f_{a,b}(x,y)$. Calculating the curvature at $(0,0, f_{a,b}(0,0))$ we get

$$K(0,0, f_{a,b}(0,0)) = a^2 b^8.$$  

Moreover, since the graph is a surface of revolution we have a complete profile of the curvature by considering its values at the points of the graph where $y = 0$. At these points we get

$$K(x,0, f_{a,b}(x,0)) = \frac{a^2(x^2 - b^2)^3(5x^2 - b^2)}{(1 + (\partial_{\rho} f_{a,b}(x,0))^2)^2}.$$  

From this equation we get that for every $r \leq b$ the curvature is bounded above by $4a^2 b^8$. Let $S_{a,b}$ be the graph of the restriction of $f_{a,b}$ to the disk $r \leq b$.

Given $\epsilon > 0$ let us choose $\sqrt{\epsilon} = ab^4$, $b < 1$. By items 1 to 8 above we have that
1. The $C^2$ norm of $f_{a,b}$ in the disk $r \leq b$ is bounded above by $\sqrt{\epsilon}$.
2. The curvature of $S_{a,b}$ at the origin is $\epsilon$.
3. Since $b < 1$ can be chosen arbitrarily small while keeping $\sqrt{\epsilon} = ab^4$, the radius of $S_{a,b}$ can be arbitrarily small as well.

Thus, given any flat metric $(T^2, g)$ (viewed as the quotient of the plane by a lattice) and $b$ small enough we can glue $S_{a,b}$ and the metric $g$ along the boundary of the disk $D$. Let $\bar{g}$ be this metric, by construction the metric $\bar{g}$ is of class $C^2$ at the boundary of $D_b$, and by standard density theorems in functional analysis we can arbitrarily approach $\bar{g}$ in the $C^2$ topology by a $C^\infty$ metric $\hat{g}$ in $T^2$ whose curvature at some point $p$ is as close as $\epsilon$ as we wish. Now, a last perturbation of $\hat{g}$ in the $C^2$ topology makes the curvature equal to $\epsilon$ around a small neighborhood of $p$ of radius depending on $b$.

Now we can apply Theorem 1.1. So given $\beta \in (0, \frac{1}{3})$ we can write

$$ ||G_a|_x - G_a|_y ||_{1, \frac{1}{3}} \leq M^{-\frac{1}{3}} \frac{d(x, y)^{\frac{1}{3} - \beta}}{d(x, y)^{\beta}} d(x, y)^{\beta}. $$

Since $\frac{d(x, y)^{\frac{1}{3} - \beta}}{d(x, y)^{\beta}} = d(x, y)^{\frac{1}{3} - \beta}$ then for $d(x, y)$ small this quantity is small.

Clearly, $||G_a|_x - G_a|_y ||_{1, \frac{1}{3}}$ attains its largest values at the support of the cone-type perturbation, namely, in the neighborhood of radius $\rho + \delta$. Now, observe that a $\epsilon$-$C^2$ perturbation supported in an open ball, which attains the value $\epsilon$ of the $C^2$ norm at the center of the ball, can be supported in arbitrarily small neighborhoods by Lemma 8.1, so we can choose $\rho + \delta$ arbitrarily small while $K_0(q) \geq \epsilon$ in the $\rho + \delta$-ball centered at $p$. By item (3) in Lemma 5.1, by choosing $n$ very large in the construction of the cones $C_{a,r,n}$ we have that $K(p)$ is commensurable to $M$. So to get the estimate in Corollary 1.2, given $\beta < \frac{1}{3}$ very close to $\frac{1}{3}$ it is enough to choose $\rho + \delta$ such that

$$ K(p)^{-\frac{1}{3}} d(x, y)^{\frac{1}{3} - \beta} \leq \epsilon $$

for every $d(x, y) \leq \rho + \delta$.

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