Spectral analysis and rational decay rates of strong solutions to a fluid-structure PDE system

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Abstract

In this paper, we consider the problem of obtaining rational decay for a particular time-evolving fluid-structure model, the type of which has been considered in Chueshov and Ryzhkova (2013). In particular, this partial differential equation (PDE) system is composed of a three-dimensional Stokes flow which evolves within a three dimensional cavity. Moreover, on a (fixed) portion of the cavity wall, Ω say, a fourth order plate equation is invoked so as to describe the displacements along Ω. Contact between these respective fluid and structure dynamics is established through the boundary interface Ω. Our main result of decay is as follows: The PDE solutions of this fluid-structure PDE, corresponding to smooth initial data, decay at the rate of \(O(1/t)\). Our method of proof hinges upon the appropriate invocation of a relatively recent resolvent criterion for rational decays for linear \(C_0\)-semigroups.

1 Introduction

1.1 The mathematical model: functional setting, main result

In this paper we focus on the problem of deriving rational rates of uniform decay for a fluid-structure partial differential equation (PDE) system; this model has appeared repeatedly in the literature, in one form or another. (See e.g., [9], [8], [3].) The composite systems of PDE describes the interactions of a viscous, incompressible fluid within a three-dimensional bounded domain \(\mathcal{O}\) (the cavity) with an elastic dynamics along boundary interface \(\Omega\). More precisely, let the walled cavity within which the fluid evolves be denoted as \(\mathcal{O}\), a bounded subset of \(\mathbb{R}^3\). This bounded set \(\mathcal{O}\) will have sufficiently smooth boundary \(\partial\mathcal{O}\), with \(\partial\mathcal{O} = \bar{\Omega} \cup \bar{S}\), and \(\Omega \cap S = \emptyset\). In particular, \(\partial\mathcal{O}\) has the following specific spatial configuration:

\[\Omega \subset \{x = (x_1, x_2, 0)\}, \quad S \subset \{x = (x_1, x_2, x_3) : x_3 \leq 0\};\]

see, e.g., the picture below.

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*The research of G. Avalos was partially supported by the NSF Grants DMS-0908476 and DMS-1211232. The research of F. Bucci was partially supported by the Italian MIUR under the PRIN 2009KNZ5FK Project (Metodi di viscosità, geometrìci e di controllo per modelli diffusivi nonlineari), by the GDRE (Groupement De Recherche Européen) CONEDP (Control of PDEs), and also by the Università degli Studi di Firenze under the Project Calcolo delle variazioni e teoria del controllo.*
In consequence, if \( \nu(x) \) denotes the exterior unit normal vector to \( \partial \Omega \), then
\[
\nu|_{\Omega} = [0, 0, 1].
\] (1.1)

With respect then to this geometry and with “rotational inertia parameter” \( \rho \geq 0 \), the PDE model is as follows, in solution variables \( w(x, t) \) and \( u(x, t) = [u^1(x, t), u^2(x, t), u^3(x, t)] \):
\[
\begin{align*}
  w_{tt} - \rho \Delta w_{tt} + \Delta^2 w &= p|_{\Omega} 
  &\quad \text{in } \Omega \times (0, T) \quad (1.2a) \\
  w &= \frac{\partial w}{\partial \nu} = 0 
  &\quad \text{on } \partial \Omega \quad (1.2b) \\
  u_t - \Delta u + \nabla p &= 0 
  &\quad \text{in } O \times (0, T) \quad (1.2c) \\
  \text{div}(u) &= 0 
  &\quad \text{in } O \times (0, T) \quad (1.2d) \\
  u &= 0 
  &\quad \text{on } S \quad (1.2e) \\
  u &= [u^1, u^2, u^3] = [0, 0, w_t] 
  &\quad \text{on } \Omega, \quad (1.2f)
\end{align*}
\]

with initial conditions
\[
[w(0), w_1(0), u(0)] = [w_0, w_1, u_0] \in H_\rho.
\] (1.3)

Here, the space of initial data \( H_\rho \) is defined as follows: Let
\[
\mathcal{H}_\ell = \left\{ f \in L^2(O) : \text{div}(f) = 0; \ f \cdot \nu|_S = 0 \right\};
\] (1.4)

and
\[
V_\rho = \begin{cases} 
L^2(O)/\mathbb{R} & \text{if } \rho = 0 \\
H^2_0(\Omega) \cap L^2(\Omega)/\mathbb{R} & \text{if } \rho > 0.
\end{cases}
\]

Therewith, we then set
\[
H_\rho = \left\{ [\omega_0, \omega_1, f] \in [H^2_0(\Omega) \cap L^2(\Omega)/\mathbb{R}] \times V_\rho \times \mathcal{H}_\ell, \right. \\
&\quad \left. \text{with } f \cdot \nu|_\Omega = [0, 0, f^3] \cdot [0, 0, 1] = \omega_1 \right\}.
\] (1.5)

As thus presented, the fluid PDE component of this fluid-structure dynamics consists of a three dimensional incompressible Stokes flow which evolves within the walled cavity \( O \), in solution variables \( u(x, t) \) and \( p(x, t) \), with \( u \) being the fluid velocity and \( p \) the pressure constraint (see (1.2c)-(1.2d)). As for the structural component: on the cavity wall portion \( \Omega \) a fourth order plate equation of either Kirchhoff (\( \rho > 0 \)) or Euler-Bernoulli (\( \rho = 0 \)) type is invoked to describe the displacements along \( \Omega \); clamped boundary conditions are in place on \( \partial \Omega \) (see (1.2a)-(1.2b)).

In addition, we note that for the fluid PDE component, the no-slip boundary condition is in play only on the wall \( S \) of the fluid container (see (1.2e)). In particular, there is a matching of velocities on \( \Omega \), by way of accomplishing the coupling between the respective fluid
and structure components; see (1.2f). Moreover, the disparate dynamics are coupled via the Dirichlet boundary trace of the pressure; in particular, pressure variable \( p \) appears as a forcing term in the \( \Omega \)-plate equation (1.2a). We should also state that in general, fluid-structure PDE models with “fixed boundary interface” \( \Omega \) are physically relevant when operating under the assumption that these cavity wall displacements are small relative to the scale of the geometry; see [10].

If one performs a simple energy method, which would commence, by multiplying structural PDE (1.2a) by \( w_t \) and fluid PDE (1.2c) by \( u \), and subsequently integrate in time and space, one would find an underlying dissipation of energy which governs the fluid-structure system. This dissipation comes solely from the gradient of the fluid component \( u \). Given this fluid dissipation which propagates onto the entire fluid-structure PDE, an investigation here of stability properties for this coupled system would seem to be appropriate.

We proceed to write down an abstract realization of the fluid structure PDE (1.2)-(1.3). To this end, let \( A_D : \mathcal{D}(A_D) \subset L^2(\Omega) \to L^2(\Omega) \) be given by

\[
A_D g = -\Delta g, \quad \mathcal{D}(A_D) = H^2(\Omega) \cap H^1_0(\Omega).
\]

If we subsequently make the denotation for all \( \rho \geq 0 \),

\[
P_\rho = I + \rho A_D, \quad \mathcal{D}(P_\rho) = \begin{cases} L^2(\Omega) & \text{if } \rho = 0 \\ D(A_D) & \text{if } \rho > 0 \end{cases},
\]

then the mechanical PDE component (1.2a) can be written as

\[
P_\rho w_t + \Delta^2 w = p|_{\Omega} \quad \text{on } (0,T).
\]

Using that

\[
\mathcal{D}(P_\rho^{1/2}) = \begin{cases} L^2(\Omega) & \text{if } \rho = 0 \\ H^1_0(\Omega) & \text{if } \rho > 0 \end{cases},
\]

(see [12]), then we can endow the Hilbert space \( \mathbf{H}_\rho \) with norm-inducing inner product

\[
(\omega_0, \omega_1, f, [\tilde{\omega}_0, \tilde{\omega}_1, \tilde{f}])_{\mathbf{H}_\rho} = (\Delta \omega_0, \Delta \tilde{\omega}_0|_{\Omega} + (P_\rho^{1/2} \omega_1, P_\rho^{1/2} \tilde{\omega}_1)|_{\Omega} + (f, \tilde{f})_\mathcal{O},
\]

where \((\cdot, \cdot)_\mathcal{O}\) and \((\cdot, \cdot)_\mathcal{O}\) are the \( L^2 \)-inner products on their respective geometries.

We note here, as it was in [8], the necessity for imposing that wave initial displacement and velocity each have zero mean average. To see this: invoking the boundary condition (1.2e)-(1.2f) and the fact that the normal vector \( \nu \) coincides with \([0,0,1]\) on \( \Omega \), we have then by Green’s formula, that for all \( t \geq 0 \),

\[
\int_\Omega w_t(t) \, dx = \int_\Omega u_3(t) \, dx = \int_{\partial \Omega} u(t) \cdot \nu \, d\sigma = 0.
\]

And so we have necessarily,

\[
\int_\Omega w(t) \, dx = \int_\Omega w_0 \, dx \quad \text{for all } t \geq 0.
\]

This accounts for the choice of the structural finite energy space components for \( \mathbf{H}_\rho \) in (1.5).

Well-posedness of the initial/boundary value problem (1.2)-(1.3) has been fully discussed in [3] for both cases \( \rho > 0 \) and \( \rho = 0 \). The proof of well-posedness provided therein hinges upon
demonstrating the existence of a modeling semigroup \( \{e^{A_t}\}_{t \geq 0} \subset \mathcal{L}(H_\rho) \), for appropriate generator \( A : H_\rho \to H_\rho \). Subsequently, by means of this family, the solution to (1.2)-(1.3), for initial data \([w_0, w_1, u_0] \in H_\rho \), will then of course be given via the relation

\[
\begin{bmatrix} w(t) \\ w_t(t) \\ u(t) \end{bmatrix} = e^{A_t} \begin{bmatrix} w_0 \\ w_1 \\ u_0 \end{bmatrix} \in C([0,T];H_\rho) .
\] (1.9)

We recall here that the particular choice here of generator \( A_\rho : H_\rho \to H_\rho \) is dictated by the following consideration, whose proof is given for the reader’s convenience.

**Lemma 1.1.** If \( p(t) \) is a viable pressure variable for (1.2)-(1.3), then pointwise in time \( p(t) \) necessarily satisfies the following boundary value problem, for \([w(t), u(t)] \) “smooth enough”:

\[
\begin{align*}
\Delta p &= 0 & \text{in } & \mathcal{O} \\
\frac{\partial p}{\partial n} &= \nabla u \cdot \nu & \text{on } & S \\
\frac{\partial p}{\partial n} + P^{-1}_\rho p &= -\rho \Delta^2 w + \Delta u^3 |_{\Omega} & \text{on } & \Omega .
\end{align*}
\] (1.10)

**Proof.** To show that \( p \) is harmonic in \( \Omega \), we take the divergence of both sides of (1.2c) and use the divergence free condition in (1.2d). Moreover, dotting both sides of (1.2c) with the unit normal vector \( \nu \), and then subsequently taking the resulting trace on \( S \) will yield the boundary condition in (1.10) that pertains to \( S \). (Implicitly, we are also using the fact that \( u = 0 \) on \( S \).)

Finally, we consider the particular geometry which is in play (where \( \nu = (0,0,1) \) on \( \Omega \)). Using the equation (1.2a) and the boundary condition (1.2d), we have on \( \Omega \):

\[
P^{-1}_\rho \Delta^2 w = -w_{tt} + P^{-1}_\rho p |_{\Omega}
\]

\[
= -\frac{d}{dt}(0,0,w_t) \cdot \nu + P^{-1}_\rho p |_{\Omega}
\]

\[
= -u_t \cdot \nabla p |_{\Omega} + P^{-1}_\rho p |_{\Omega}
\]

\[
= -[\Delta u \cdot \nu] |_{\Omega} + \frac{\partial p}{\partial \nu} |_{\Omega} + P^{-1}_\rho p |_{\Omega} ,
\]

which gives the boundary condition in (1.10) that pertains to \( \Omega \).

The boundary value problem (BVP) (1.10) can be solved through the agency of appropriate harmonic extensions from the boundary of \( \mathcal{O} \), that are the “Robin-Neumann” maps \( R_\rho \) and \( \tilde{R}_\rho \) defined by

\[
R_\rho g = f \iff \Delta f = 0 \text{ in } \mathcal{O} , \quad \frac{\partial f}{\partial \nu} + P^{-1}_\rho f |_{\Omega} = g , \quad \frac{\partial f}{\partial \nu} = 0 \text{ on } S ;
\]

\[
\tilde{R}_\rho g = f \iff \Delta f = 0 \text{ in } \mathcal{O} , \quad \frac{\partial f}{\partial \nu} + P^{-1}_\rho f |_{\Omega} = 0 , \quad \frac{\partial f}{\partial \nu} = g \text{ on } S .
\]

It is well known that for all real \( s \),

\[
R_\rho \in \mathcal{L}(H^s(\Omega),H^{s+3/2}(\mathcal{O})); \quad \tilde{R}_\rho \in \mathcal{L}(H^s(S),H^{s+3/2}(\mathcal{O})) ;
\] (1.11)

see, e.g., [19]. (We are also using implicitly the fact that \( P^{-1}_\rho \) is positive definite, self-adjoint on \( \Omega \).)

Therewith, the pressure variable \( p(t) \), as necessarily the solution of (1.10), can be written pointwise in time as

\[
p(t) = G_{\rho,1} w(t) + G_{\rho,2} u(t) ,
\] (1.12)
where $G_{\rho,1}$ and $G_{\rho,2}$ are defined as follow:

\begin{align}
G_{\rho,1}w &= R_{\rho}(P_{\rho}^{-1}\Delta^2 w); \\
G_{\rho,2}u &= R_{\rho}(\Delta u^2|_{\Omega}) + \tilde{R}_{\rho}(\Delta u \cdot \nu|_{S}).
\end{align}

These relations suggest the following choice for the generator $A_{\rho} : H_{\rho} \to H_{\rho}$. We set

\begin{equation}
A_{\rho} = \begin{bmatrix}
0 & I & 0 \\
-P_{\rho}^{-1}\Delta^2 + P_{\rho}^{-1}G_{\rho,1}|_{\Omega} & P_{\rho}^{-1}G_{\rho,2}|_{\Omega} \\
-\nabla G_{\rho,1} & 0 & \Delta - \nabla G_{\rho,2}
\end{bmatrix}
\end{equation}

with domain

\[ \mathcal{D}(A_{\rho}) = \begin{\{w_1, w_2, u\} \in H_{\rho} : w_1 \in H^3(\Omega) \cap H^2_0(\Omega), w_2 \in H^2_0(\Omega), u \in H^2(\mathcal{O}) ; \]
\[ u = 0 \text{ on } S, \quad u = (0, 0, w_2) \text{ on } \Omega, \]
\[ \Delta u \cdot \nu \in H^{-1/2}(\Omega), \text{ and hence } G_{\rho,1}w_1 + G_{\rho,2}u \in H^1(\mathcal{O}) \} . \]

assuming $\rho$ is positive, whereas more precisely the first membership in (1.16) is as follows:

\[ w_1 \in \mathcal{S}_{\rho} := \begin{cases}
H^3(\Omega) \cap H^2_0(\Omega) & \rho = 0 \\
H^3(\Omega) \cap H^2_0(\Omega) & \rho > 0.
\end{cases} \]

Thus, we remind the reader that well-posedness for the dynamics governed by the operator $A_{\rho}$, when $\rho = 0$ (i.e., when the elastic equation is the Euler-Bernoulli one), was originally established in [8], by using Galerkin approximations. A novel proof of well-posedness pertaining to both cases $\rho = 0$ and $\rho > 0$, based upon the classical Lumer-Phillips Theorem as well as on a clever use of the Babuska-Brezzi Theorem (see, e.g., [17, p. 116]) has been recently given in [3]. The corresponding statement is as follows.

**Theorem 1.2 (Well-posedness [3]).** The operator $A_{\rho} : H_{\rho} \to H_{\rho}$ defined by (1.15)-(1.16) generates a $C_0$-semigroup of contractions $\{e^{tA_{\rho}}\}_{t \geq 0}$ on $H_{\rho}$. Thus, given $[w_0, w_1, u_0] \in H_{\rho}$, the weak solution $[w, w_t, u] \in C([0, T]; H_{\rho})$ of (1.2)-(1.3) is given by (1.16).

In the present work the long-time behaviour, as $t \to +\infty$, of the linear dynamics described by (1.2) is addressed, with focus on the more challenging case $\rho > 0$ (the elastic equation is the Kirchhoff one). When $\rho = 0$, uniform (exponential) stability of finite energy solutions holds true; this issue has been dealt with in [8], by using Lyapunov function arguments (in the time domain). A different proof of the aforesaid result has been subsequently given in [2], with a proof geared rather toward establishing the necessary resolvent estimates in the frequency domain. The very same ‘frequency domain perspective’ enables us to infer that in the case $\rho > 0$, a weaker notion of uniform decay will prevail for the fluid-structure PDE (1.2)-(1.3). In particular, the main result of this paper is the following stability result pertaining to strong solutions, which provides sharp polynomial rates of decay.

**Theorem 1.3 (Main result: Rational decay rates).** Let the rotational inertia parameter $\rho$ be positive in (1.2a). Then for initial data $[w_0, w_1, u_0] \in D(A_{\rho})$, the corresponding solution $[w, w_t, u] \in C([0, T]; D(A_{\rho}))$ of (1.2)-(1.3) satisfies the following decay rate for time $t$ large enough:

\[ \|[w(t), w_t(t), u(t)]\|_{H_{\rho}} \leq \frac{C}{t} \|[w_0, w_1, u_0]\|_{D(A_{\rho})}. \]

In what follows, rotational parameter $\rho$ will be positive always.
1.2 Background and further remarks

We should note that in principle, one might attempt to derive the rational decay estimate (1.17) by an analysis in the “time domain”; the associated energy method is in principle abstractly outlined in [18, Theorem 3.2.2, p. 43]. However, the details of proof in the time domain, at least from our vantage point, would seem to be quite daunting, if it can be done at all, in the t-domain.

(We note that the time domain approach—along energy methods, combined with interpolation techniques—underlines as well the recent work [15], which provides a novel criterion to derive the decay rates of the solutions to evolutionary PDE systems, whose range of application includes certain coupled PDE systems of hyperbolic type.)

Instead, we choose to operate here in the “frequency domain”, by invoking an energy method with respect to a formally ‘Laplace transformed’ version of the system (1.2)-(1.3), with an ultimate view of invoking the sharp resolvent criterion in [7] (see a penultimate version of this resolvent criterion in [20]). Such a frequency domain approach was previously invoked in [4], by way of establishing rational decays for a wholly different fluid-structure PDE model. We should state at the outset that one advantage which the frequency domain approach enjoys over the time domain approach, is that the former eventually allows for a adequate treatment of the pressure variable (as it appears as a forcing term in the Ω-plate equation). In particular, upon formally taking the Laplace Transform of (1.2)-(1.3), so as to obtain a corresponding static fluid-structure system with frequency domain parameter β (see (3.4) below), one can then attempt to invoke classic Stokes Theory for (static) incompressible fluid flows.

Alternatively, if one were working directly with the time evolving fluid-structure system (1.2)-(1.3), by way of analyzing the pressure term \( p(x, t) \) on \( \Omega \), it seems likely to us that there would be the necessity of microlocalizing the fluid-structure system in order to obtain the required \emph{a priori} estimates. Besides being quite technical in its own right, such a pseudo-differential approach might even be ultimately unavailing, inasmuch as there would be the issue of keeping a close track of the time dependent constants which would surely accumulate in the course of such a microlocal analysis. Hence, we are drawn instead to a frequency domain approach which would ultimately invoke the resolvent criterion Theorem 2.1 below.

We should also state that uniform stability results for higher dimensional coupled PDE models (namely, involving equations on \( n \)-dimensional manifolds, with \( n > 1 \)) which are attained via a frequency domain approach are largely not available in the literature; see e.g., [4], [5]. We recall the recent polynomial decay result obtained in [13] for a complicated Mindlin-Timoshenko plate model, which also depends upon a frequency domain approach and an argument by contradiction, with a view of invoking the aforesaid resolvent criterion in [7]; see also [14]. In general, those few higher dimensional frequency domain results which are available typically invoke an argument by contradiction, in the style of [20], by way of establishing the requisite resolvent estimate in Theorem 2.1 below. By contrast, in the present paper we \emph{explicitly} generate the necessary frequency domain estimates; see also [4] and [5].

2 Spectral analysis

In order to establish the sharp estimate of the decay rates for the solutions of the PDE system, we will use a recent and powerful \emph{frequency domain} criterion by A. Borichev and Y. Tomilov, which for the readers’ convenience is recorded below.

**Theorem 2.1** ([7]). \( \text{Let } (T(t))_{t \geq 0} \text{ be a bounded } C_0 \text{-semigroup on a Hilbert space } H \text{ with gen-} \)
erator $A$, such that $i\mathbb{R} \subset \rho(A)$. Then, for fixed $\alpha > 0$ the following are equivalent:

\begin{align}
(i) \quad R(is; A) &= O(|s|^\alpha), \quad |s| \to \infty; \quad \tag{2.1}
(ii) \quad \|T(t)x\|_H &= o(t^{-1/\alpha})\|x\|_{D(A)}, \quad t \to +\infty.
\end{align}

To apply the above result, we preliminary need to show that the imaginary axis belongs to the resolvent set of the dynamics operator $A$. The present Section is entirely devoted to this objective.

2.1 $\lambda = 0$ is in the resolvent set $\rho(A)$

We begin our analysis by showing that the dynamics operator $A$ is \textit{boundedly invertible} on the state space $H$; the corresponding statement is given separately. In this connection, we will need the following trace regularity result, which is readily established; see e.g., Proposition 2 of [3].

**Proposition 2.2.** Suppose a function $\mu \in L^2(\mathcal{O})$ and pair $(\varrho, h) \in H^1(\Omega) \times L^2(\mathcal{O})$ satisfy the relation

\begin{equation}
-\Delta \mu + \nabla \varrho = h, \quad \tag{2.2}
\end{equation}

where $\text{div}(\mu) = \text{div}(h) = 0$. Then $\Delta \mu \cdot \nu|_{\partial \mathcal{O}} \in H^{-\frac{1}{2}}(\partial \mathcal{O})$, with the estimate

\begin{equation}
\|\Delta \mu \cdot \nu|_{\partial \mathcal{O}}\|_{H^{-\frac{1}{2}}(\partial \mathcal{O})} \leq C \left[\|\varrho\|_{H^1(\mathcal{O})} + \|h\|_{L^2(\mathcal{O})}\right]. \quad \tag{2.3}
\end{equation}

**Proposition 2.3.** The generator $A : D(A) \subset H_\rho \to H_\rho$ is \textit{boundedly invertible} on $H_\rho$. Namely, $\lambda = 0$ is in the resolvent set of $A$.

**Proof.** Given data $[\omega_1^*, \omega_2^*, \mu^*] \in H_\rho$, we look for $[\omega_1, \omega_2, \mu] \in D(A)$ which solves

\begin{equation}
A \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\mu
\end{bmatrix} = \begin{bmatrix}
\omega_1^* \\
\omega_2^* \\
\mu^*
\end{bmatrix}. \quad \tag{2.4}
\end{equation}

To this end, we must search $[\omega_1, \omega_2, \mu]$ in $D(A)$ and $\pi_0 \in H^1(\mathcal{O})$ which solve

\begin{align}
\omega_2 &= \omega_1^* & \text{in } \Omega \quad \tag{2.5a}
\rho^{-1} \Delta \omega_1 - \rho^{-1} \pi_0 &= -\omega_2^* & \text{in } \Omega \quad \tag{2.5b}
\frac{\partial \omega_1}{\partial n} &= 0 & \text{on } \partial \Omega \quad \tag{2.5c}
\Delta \mu - \nabla \pi_0 &= \mu^* & \text{in } \mathcal{O} \quad \tag{2.5d}
\text{div} \mu &= 0 & \text{in } \mathcal{O} \quad \tag{2.5e}
\mu &= 0 & \text{on } S \quad \tag{2.5f}
\mu &= (0, 0, \omega_2) & \text{on } \Omega. \quad \tag{2.5g}
\end{align}

Moreover, we must justify that the pressure variable $\pi_0$ is given by the expression

\begin{equation}
\pi_0 = G_1 \omega_1 + G_1 \mu, \quad \tag{2.6}
\end{equation}

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where $G_i$ denote, in short, the linear operators $G_{\rho,i}$, $i = 1, 2$ defined by (1.13) and (1.14), respectively (in line with the appearance of $A_\rho$ in (1.15)).

(i) The Plate Velocity. From (2.5a), the velocity component $\omega_2$ is immediately resolved.

(ii) The Fluid Velocity. We next consider the Stokes system (2.5d)-(2.5g). From (2.5a) and (2.5g) it follows that $\mu|_{\partial O}$ satisfies

$$\int_{\partial O} \mu \cdot \nu \, d\sigma = \int_\Omega [0, 0, \mu_3]^T \cdot \nu \, d\sigma = \int_\Omega \omega_*^2 \, d\sigma = 0 \quad (2.7)$$

(as $[\omega_*^1, \omega_*^2, \mu^*] \in H_\rho$). Since this compatibility condition is satisfied and data $\{\mu^*, \omega_*^1\} \in L^2(\Omega) \times H^2_0(\Omega)$, we can find a unique (fluid and pressure) pair $(\mu, q_0) \in [H^2(\Omega) \cap H^f] \times H^1(\Omega)/R$ which solves

$$\begin{align*}
\Delta \mu - \nabla q_0 &= \mu^* \quad \text{in } \Omega \\
\text{div}(\mu) &= 0 \quad \text{in } \Omega \\
\mu &= (0, 0, \omega_*^1) \quad \text{on } \Omega, \quad \mu = 0 \quad \text{on } S.
\end{align*} \quad (2.8)$$

Moreover, one has the estimate

$$\|\mu\|_{H^2(\Omega) \cap H^f} + \|q_0\|_{H^1(\Omega)/R} \leq C \left[ \|\mu^*\|_{H^f} + \|\omega_*^1\|_{H^2(\Omega)} \right]; \quad (2.9)$$

see e.g., [21, Proposition 2.3, p. 25].

(iii) The Mechanical Displacement. Subsequently, we consider the equations (2.5b)-(2.5c) pertaining to the (plate) component $\omega_1$. By classical elliptic theory there exists a solution $\tilde{\omega}_1 \in H^3(\Omega) \cap H^2_0(\Omega)$ to the boundary value problem

$$\begin{align*}
\Delta^2 \tilde{\omega}_1 &= q_0|_{\Omega} - P_\rho \omega_*^2 \quad \text{in } \Omega \\
\tilde{\omega}_1 &= \frac{\partial \omega}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
\end{align*} \quad (2.10)$$

where $q_0$ is the (pressure) variable in (2.8a); moreover, the following estimate holds true:

$$\|\tilde{\omega}_1\|_{H^3(\Omega) \cap H^2_0(\Omega)} \leq C \|q_0\|_{\Omega} + C_\rho \|\omega_*^2\|_{H^{-1}(\Omega)} \leq C \|q_0\|_{\Omega} + C \|\omega_*^1\|_{H^2(\Omega)} \leq C \|\omega_*^1, \omega_*^2, \mu^*\|_{H^2}, \quad (2.11)$$

(In the last inequality we have also invoked Sobolev trace theory and (2.9)).

Now if, as in [8], we let $P$ denote the orthogonal projection of $H^2_0(\Omega)$ onto $H^2_0(\Omega) \cap L^2(\Omega)/R$ (orthogonal with respect to the inner product $[\omega, \tilde{\omega}] \to (\Delta \omega, \Delta \tilde{\omega})_{\Omega}$), then one can readily show that its orthogonal complement $I - P$ can be characterized as

$$(I - P)H^2_0(\Omega) = \text{span} \left\{ \varpi : \Delta^2 \varpi = 0 \text{ in } \Omega, \varpi = \varpi_{\nu} = 0 \text{ on } \partial \Omega \right\}; \quad (2.12)$$

see [8] Remark 2.1, p. 6].

With these projections, we then set

$$\omega_1 := P \tilde{\omega}_1, \quad \pi_0 := q_0 - \Delta^2 (I - P) \tilde{\omega}_1; \quad (2.13)$$

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therefore, by (2.10) and \( \hat{\omega}_1 = P\hat{\omega}_1 + (I - P)\hat{\omega}_1 \), we will have that \( \omega_1 \) solves (2.5a)–(2.5c). (And of course since \( \pi_0 \) and \( q_0 \) differ only by a constant, then the pair \( (\mu, \pi_0) \) also solves (2.5d)–(2.5g).)

Thus, in view of elliptic theory, (2.9) and (2.11), we obtain the estimate

\[
\|\omega_1\|_{H^2(\Omega)} + \|q_0\|_{L^2(\Omega)} + \|\omega_2\|_{H^1(\Omega)} + \|\omega_3\|_{H^1(\Omega)} \\
\leq C \left( \|\Delta^2(I - P)\hat{\omega}_1\|_{L^2(\Omega)} + \|q_0\|_{H^1(\Omega)} + \|\omega_2\|_{H^1(\Omega)} \right)
\]

(2.14)

where implicitly we are also using the fact that \( \Delta^2(I - P) \in L^2(\Omega, \mathbb{R}) \), by the Closed Graph Theorem.

(v) Resolution of the Pressure. At this point we invoke Proposition 2.2 and (2.9) to have the following trace regularity for the fluid velocity in (2.8):

\[
\|\Delta \mu \cdot \nu\|_{H^{-1/2}(\partial \Omega)} \leq C \left( \|q_0\|_{H^1(\Omega)} + \|\mu^*\|_{L^2(\Omega)} \right)
\]

(2.15)

In consequence, the pressure variable \( \pi_0 \) of problem (2.5a)–(2.5g)—given explicitly in (2.13)—solves a fortiori

\[
\begin{align*}
\Delta \pi_0 &= 0 & \text{in } \Omega \\
\frac{\partial \pi_0}{\partial \nu} &= \Delta \mu \cdot \nu|_{S} & \text{on } S \\
\frac{\partial \pi_0}{\partial \nu} + P^{-1}_\rho \pi_0 &= P^{-1}_\rho \Delta \omega_1 + \Delta \mu^*|_{\Omega} & \text{on } \Omega
\end{align*}
\]

(2.16)

We justify the previous assertion. Applying the divergence operator to both sides of (2.5d) and using \( \text{div}_\omega = \text{div}_\mu = 0 \), we obtain that \( \pi_0 \) is harmonic in \( \Omega \). Next, dotting both sides of (2.5d) with respect to the normal vector, and subsequently taking the boundary trace on the portion \( S \), we get the corresponding boundary condition in (2.16). (Implicitly we are also using \( \mu^* \cdot \nu|_S = 0 \), as \( [\omega_1^*, \omega_2^*, \mu^*] \in H_\rho \).

Finally, since \( \mu^* \cdot \nu|_\Omega = \omega_2^* \), as \( [\omega_1^*, \omega_2^*, \mu^*] \in H_\rho \), from (2.16) it follows that

\[
P^{-1}_\rho \pi_0|_{\Omega} = \omega_2^* + P^{-1}_\rho \Delta \omega_1 = \Delta \mu \cdot \nu|_{\Omega} - \nabla \pi_0 \cdot \nu|_{\Omega} + P^{-1}_\rho \Delta \omega_1,
\]

which gives the boundary condition on \( \Omega \).

Necessarily then, the pressure term must be given by the expression

\[
\pi_0 = G_1 \omega_1 + G_2 \mu \in H^1(\Omega),
\]

(2.17)

with the well-definition of the right hand side ensured by (2.16).

Finally, we collect: the fluid variable \( \mu \) as the solution to (2.5) with the estimate (2.9), the respective structure and pressure variables \( \omega_1, \omega_2 \) and \( \pi_0 \) given by (2.5a), (2.13) along with the estimate (2.14) (and where \( \hat{\omega}_1 \) is defined by (2.10)); (2.17) characterizes the pressure \( \pi_0 \) in terms of the variables \( \omega_1 \) and \( \mu \). This shows that the solution of (2.5) actually belongs to \( D(A_\rho) \). In short, \( 0 \in \rho(A_\rho) \), which concludes the proof.
2.2 $\lambda = i\beta$ is in the resolvent set $\rho(A)$

Let us recall the expression of the dynamics operator semigroup $A_\rho$ in \((1.15)\). In straightforward fashion, one can then compute the associated adjoint operator $A_\rho^* : D(A_\rho) \subset H_\rho \to H_\rho$ to be

\[
A_\rho^* = \begin{bmatrix}
0 & -I & 0 \\
\Delta^{-1} \nabla G_{\rho,1} & 0 & \Delta^{-1} \nabla G_{\rho,2} \\
\nabla G_{\rho,1} & P_{\rho,1}^{-1} & \nabla G_{\rho,2} \Omega
\end{bmatrix},
\]

with $D(A_\rho^*) = D(A_\rho)$. The above operator will be utilized in the proof of the following result.

**Proposition 2.4.** Let $\sigma(A_\rho)$ be the spectrum of the dynamics operator $A_\rho$ defined by \((1.15)-(1.16)\). Then $i\mathbb{R} \cap \sigma(A_\rho) = \emptyset$.

**Proof.** Let $\sigma_p(A_\rho)$, $\sigma_c(A_\rho)$, $\sigma_r(A_\rho)$ denote—respectively—the point, continuous and residual spectrum of the operator $A_\rho$.

1. (Point spectrum) We aim at showing that $i\mathbb{R} \cap \sigma_p(A_\rho) = \emptyset$. Given $\beta \in \mathbb{R} \setminus \{0\}$, we consider the equation

\[
A_\rho \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\mu
\end{bmatrix} = i\beta \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\mu
\end{bmatrix}
\]

for some $[\omega_1, \omega_2, \mu] \in D(A_\rho)$. Moreover, we set

\[
\pi_0 \equiv \rho_{\rho,1}(\omega_1) + \rho_{\rho,2}(u).
\]

Taking the inner product of both sides, and subsequently integrating by parts, then it follows

\[
i\beta \left\| \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\mu
\end{bmatrix} \right\|_{H_\rho}^2 = \left( A_\rho \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\mu
\end{bmatrix}, \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\mu
\end{bmatrix} \right)
\]

\[
= (\Delta \omega_2, \Delta \omega_1)_\Omega + (-\Delta^2 \omega_1 + \pi_0|_{\Omega} \omega_2)_\Omega + (\Delta \mu - \nabla \pi_0, \mu)_\Omega
\]

\[
= (\Delta \omega_2, \Delta \omega_1)_\Omega + (\nabla \Delta \omega_1, \nabla \omega_2)_\Omega + (\pi_0|_{\Omega} (0, 0, 1), (\mu^1, \mu^2, \omega_2))_\Omega
\]

\[
- \nabla \mu, \nabla \mu)_\Omega + \left( \frac{\partial \mu}{\partial \nu}_\Omega \right) - (\pi_0 \nu, \mu)_\Omega
\]

\[
= (\Delta \omega_2, \Delta \omega_1)_\Omega - (\Delta \omega_1, \Delta \omega_2)_\Omega - (\nabla \mu, \nabla \mu)_\Omega
\]

\[
+ \left( \begin{bmatrix}
\partial_{x^1} \mu^1 \\
\partial_{x^2} \mu^2 \\
\partial_{x^3} \mu^3
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
\mu^3
\end{bmatrix} \right)_\Omega,
\]

or

\[
i\beta \left\| \begin{bmatrix}
\omega_1 \\
\omega_2 \\
\mu
\end{bmatrix} \right\|_{H_\rho}^2 = -\|\nabla \mu\|_{\Omega}^2 - 2i \text{Im}(\Delta \omega_1, \Delta \omega_2)_\Omega,
\]

whence we obtain

\[
\mu = 0 \quad \text{in} \ \mathcal{O}.
\]

In turn, the boundary condition $\mu = (0, 0, \omega_2)$ on $\Omega$, intrinsic to elements of $\mathcal{D}(A_\rho)$, yields as well

\[
\omega_2 = 0 \quad \text{in} \ \Omega.
\]
And further in turn, the first component relation in (2.19), combined with the appearance of \( \mathcal{A}_\rho \) in (1.15), yield \( i \beta \omega_1 = \omega_2 \). Hence for \( \beta \neq 0 \),

\[
\omega_1 = 0 \quad \text{in } \Omega. \tag{2.25}
\]

The relations (2.23), (2.24) and (2.25) give the conclusion that \( i \beta \) is not an eigenvalue of \( \mathcal{A}_\rho \).

2. (Residual spectrum) We aim at showing that \( i \mathbb{R} \cap \sigma_i(\mathcal{A}_\rho) = \emptyset \). Given \( \beta \in \mathbb{R} \setminus \{0\} \), if \( i \beta \) is in the residual spectrum of \( \mathcal{A}_\rho \), then necessarily \( i \beta \) is in the point spectrum of \( \mathcal{A}_\rho^* : D(\mathcal{A}_\rho^*) \subset \mathbf{H}_\rho \to \mathbf{H}_\rho \); see e.g., [11] p. 127]. In this case, given the appearance and the domain of \( \mathcal{A}_\rho^* \) in (2.18), we proceed verbatim along the lines of Step 1. to deduce that \( i \mathbb{R} \cap \sigma_i(\mathcal{A}_\rho) = \emptyset \).

3. (Continuous spectrum) This is by far the most challenging part of the proof. To make the inference that \( i \mathbb{R} \) has empty intersection with the continuous spectrum, it is enough to show that \( i \mathbb{R} \) does not intersect with the approximate spectrum; see e.g., [11] p. 128].

To this end, let \( \beta \in \mathbb{R} \setminus \{0\} \) be given. If \( i \beta \) is in the approximate spectrum of \( \mathcal{A}_\rho \), then by definition there exists a sequence

\[
\left\{ \begin{bmatrix} \omega_{1,n} \\ \omega_{2,n} \\ \mu_n \end{bmatrix} \right\}_{n=1}^{\infty} \subset D(\mathcal{A}_\rho) \text{ such that for all } n:\ |\begin{bmatrix} \omega_{1,n} \\ \omega_{2,n} \\ \mu_n \end{bmatrix}||_{\mathbf{H}_\rho} = 1
\]

and \( \begin{bmatrix} \omega^*_{1,n} \\ \omega^*_{2,n} \\ \mu^*_n \end{bmatrix} = (i \beta - \mathcal{A}_\rho) \begin{bmatrix} \omega_{1,n} \\ \omega_{2,n} \\ \mu_n \end{bmatrix} \) satisfies \( |\begin{bmatrix} \omega^*_{1,n} \\ \omega^*_{2,n} \\ \mu^*_n \end{bmatrix}||_{\mathbf{H}_\rho} < \frac{1}{n} \). \tag{2.26}

We consider therewith the relation

\[
(i \beta - \mathcal{A}_\rho) \begin{bmatrix} \omega_{1,n} \\ \omega_{2,n} \\ \mu_n \end{bmatrix} = \begin{bmatrix} \omega^*_{1,n} \\ \omega^*_{2,n} \\ \mu^*_n \end{bmatrix}. \tag{2.27}
\]

In PDE terms, each \([\omega_{1,n}, \omega_{2,n}, \mu_n]\) satisfies the following problem:

\[
\begin{align*}
i \beta \omega_{1,n} - \omega_{2,n} &= \omega^*_{1,n} \quad \text{in } \Omega \\
i \beta \omega_{2,n} + P^* P \Delta \omega_{1,n} - P^* P_n &= \omega^*_{2,n} \quad \text{in } \Omega \\
\omega_{1,n} |_{\Omega} = \frac{\partial \omega_{1,n}}{\partial \nu} |_{\partial \Omega} &= 0 \quad \text{on } \partial \Omega \\
i \beta \mu_n - \Delta \mu_n + \nabla p_n &= \mu^*_n \quad \text{in } \mathcal{O} \\
\text{div}(\mu_n) &= 0 \quad \text{in } \mathcal{O} \\
\mu_n &= 0 \quad \text{on } S \\
\mu_n &= (0,0, \omega_{2,n}) \quad \text{on } \Omega,
\end{align*}
\tag{2.28}
\]

where for each \( n \), the associated pressure term is given by

\[
p_n = G_1 \omega_{1,n} + G_2 \mu_n. \tag{2.29}
\]

Multiplying both parts of the expression (2.27) by \([\omega_{1,n}, \omega_{2,n}, \mu_n]\) and integrating by parts gives

\[
|\nabla \mu_n|^2_{\mathcal{O}} = \left( |\begin{bmatrix} \omega^*_{1,n} \\ \omega^*_{2,n} \\ \mu^*_n \end{bmatrix} |_{\mathbf{H}_\rho} , |\begin{bmatrix} \omega_{1,n} \\ \omega_{2,n} \\ \mu_n \end{bmatrix} |_{\mathbf{H}_\rho} \right) - i \beta |\begin{bmatrix} \omega_{1,n} \\ \omega_{2,n} \\ \mu_n \end{bmatrix}||_{\mathbf{H}_\rho}^2 - 2i \text{Im}(\Delta \omega_{1,n} , \Delta \omega_{2,n})_{\Omega}.
\]

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We have then from (2.26) that
\[ \mu_n \to 0 \quad \text{(strongly) in } H^1(\Omega). \] (2.30)

In turn, from the boundary condition (2.28g) and the Sobolev Embedding Theorem, we have
\[ \|\omega_{2,n}\|_{H^{1/2}(\Omega)} = \|\mu_n^3\|_{H^{1/2}(\Omega)} \leq C \|\mu_n\|_{H^1(\Omega)}, \]
whence
\[ \omega_{2,n} \to 0 \quad \text{(strongly) in } H^{1/2}(\Omega). \] (2.31)

At this point, we invoke the unique decomposition
\[ p_n = c_n + q_n, \] (2.32)
where for each \( n \),
\[ c_n = \text{constant}; \quad q_n \in L^2(\Omega)/\mathbb{R}. \] (2.33)

Then, from the known regularity for Stokes flow—see, e.g., estimate (2.46) in [21, p. 23]—we have from (2.28d)-(2.28f)
\[ \|q_n\|_{L^2(\Omega)/\mathbb{R}} \leq C (\|i\beta \mu_n\|_{L^2(\Omega)} + \|\mu_n\|_{H^{1/2}(\partial\Omega)} + \|\mu_n^*\|_{L^2(\Omega)}) \]
\[ \leq C_\beta (\|\mu_n\|_{H^1(\Omega)} + \|\mu_n^*\|_{L^2(\Omega)}) \] (2.34)
whence we obtain from (2.30) and (2.26),
\[ q_n \to 0 \quad \text{strongly in } L^2(\Omega). \] (2.35)

Moreover, since each \( q_n \) is harmonic a fortiori, we have available the boundary trace estimate
\[ \|q_n|_{\partial\Omega}\|_{H^{-1/2}(\partial\Omega)} \leq C \|q_n\|_{L^2(\Omega)} \]
\[ \leq C_\beta (\|\mu_n\|_{H^1(\Omega)} + \|\mu_n^*\|_{H^1(\Omega)}) \] (2.36)
(see e.g., [6, Proposition 1]); in attaining the second estimate we have also used (2.34); appealing again to (2.30) and (2.26) we then have
\[ q_n|_{\partial\Omega} \to 0 \quad \text{strongly in } H^{-1/2}(\Omega). \] (2.37)

Now using the decomposition (2.32) in the structural equation (2.28b), we have for all \( n \),
\[ c_n = -q_n|_\Omega + \Delta^2 \omega_{1,n} + i\beta P_\rho \omega_{2,n} - P_\rho \omega_{2,n}^*, \]
and so a measurement in the \( H^{-2}(\Omega) \)-topology gives
\[ c_n \|1\|_{H^{-2}(\Omega)} = \| -q_n|_\Omega + \Delta^2 \omega_{1,n} + i\beta P_\rho \omega_{2,n} - P_\rho \omega_{2,n}^*\|_{H^{-2}(\Omega)} \]
\[ \leq C_\beta (\|q_n|_\Omega\|_{H^{-1/2}(\Omega)} + \|\omega_{1,n}\|_{H^2(\Omega)} + \|\omega_{2,n}\|_{L^2(\Omega)} + \|\omega_{2,n}^*\|_{L^2(\Omega/\rho^{1/2})}). \] (2.38)

Combining (2.26), (2.37) and (2.31) with (2.38) we achieve the conclusion that
\[ \{c_n\}_{n \geq 1} \text{ is uniformly bounded in } n. \]

Hence, there is a subsequence of constants—still denoted as \( \{c_n\}_{n \geq 1} \)—which satisfies for some \( \tilde{c} \)
\[ c_n \to \tilde{c} \quad \text{(strongly) in } \mathbb{C}. \] (2.39)
We now turn our attention to the mechanical system (2.28b)–(2.28c), that is
\[
\begin{cases}
\Delta^2 \omega_1, n = p_n |_{\Omega} - i \beta P_\rho \omega_2, n + P_\rho \omega^*, n \\
\omega_1, n = \frac{\partial \omega_1, n}{\partial \nu} = 0
\end{cases}
\]
in \( \Omega \)
and on \( \partial \Omega \).

By way of looking at this sequence of boundary value problems, let us invoke the realization 
\( A \) of the bilaplacian operator, defined by
\[
A \varphi := \Delta^2 \varphi, \varphi \in \mathcal{D}(A) = H^4(\Omega) \cap H^2_0(\Omega).
\]
Then we have abstractly
\[
A \omega_1, n = c_n + q_n |_{\Omega} - i \beta P_\rho \omega_2, n + P_\rho \omega^*, n \in \mathcal{D}(A^{1/2})^*,
\]
where \( \mathcal{D}(A^{1/2}) = H^2_0(\Omega) \).

Applying the inverse \( A^{-1} \in L(L^2(\Omega), \mathcal{D}(A)) \) to both sides of the above equality gives
\[
\omega_1, n = A^{-1} c_n + A^{-1} (q_n |_{\Omega} - i \beta P_\rho \omega_2, n + P_\rho \omega^*, n) \in \mathcal{D}(A^{1/2}).
\]
(2.40)

Subsequently we can then pass to the limit in (2.40) (meanwhile using (2.39), (2.37), (2.31) and (2.26)) so as to have
\[
\tilde{\omega} = \lim_{n \to \infty} \omega_1, n = \lim_{n \to \infty} A^{-1} c_n = A^{-1} \tilde{c}.
\]
(2.41)

Thus, this (structural) limit must satisfy
\[
\Delta^2 \tilde{\omega} = \tilde{c} \quad \text{in } \Omega, \quad \tilde{\omega} = \frac{\partial \tilde{\omega}}{\partial \nu} = 0 \quad \text{on } \partial \Omega.
\]
(2.42)

Now since \( \omega_1, n \in H^2_0(\Omega) \cap \frac{L^2(\Omega)}{\mathbb{R}} \) for every \( n \), then so is strong limit \( \tilde{\omega} \). But from (2.42) and the characterization (2.12), we have also that \( \tilde{\omega} \in \left[ H^2_0(\Omega) \cap \frac{L^2(\Omega)}{\mathbb{R}} \right]^\perp \). Thus,
\[
\lim_{n \to \infty} \omega_1, n = 0.
\]
(2.43)

Finally, from (2.28b),
\[
\omega_2, n = i \beta \omega_1, n - \omega^*, n;
\]
whence we obtain with (2.26) and (2.43),
\[
\lim_{n \to \infty} \omega_2, n = 0 \text{ in } \mathcal{D}(P^{1/2}).
\]
(2.44)

The limits in (2.43) and (2.44), combined with the one in (2.30), now contradict the fact from (2.26) that
\[
\|[\omega_1, n, \omega_2, n, \mu_n]||_{H_\rho} = 1 \quad \forall n.
\]

Since \( \beta \in \mathbb{R} \setminus \{0\} \) was arbitrary, we conclude that the approximate spectrum of \( A_\rho \) does not intersect with \( i\mathbb{R} \).

3 Proof of Theorem 1.3 (Main result)

Here we will utilize Theorem 2.1 (see [7, Theorem 2.4]) in the case currently being considered; namely, \( \rho > 0 \), so that rotational forces are accounted for in the fluid-structure PDE dynamics. By way of using the aforesaid resolvent criterion, we consider arbitrary data \( [\omega_1^*, \omega_2^*, u^*] \in H_\rho \),
we then have

\[
(i\beta - A_p) \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} = \begin{bmatrix} \omega_1^+ \\ \omega_2^+ \\ \mu^+ \end{bmatrix} \in H_\beta. \tag{3.1}
\]

Upon a rearrangement and setting pressure variable

We have then following (static) fluid-structure PDE system:

\[
\alpha
\]

this is the frequency estimate with \(\alpha = 1\).

Using the definition of \(A_\beta \in D(A_\beta) \subseteq H_\beta \rightarrow H_\beta\), this gives

\[
i\beta \omega_1 - \omega_2 = \omega_1^* \quad \text{in } \Omega
\]

\[
i\beta \omega_2 + P_\rho^{-1} \Delta^2 \omega_1 - P_\rho^{-1} G_{\rho,1} \omega_1|_\Omega - P_\rho^{-1} G_{\rho,2} \mu|_\Omega = \omega_2^* \quad \text{in } \Omega
\]

\[
i\beta \mu - \Delta \mu + \nabla \pi = \mu^* \quad \text{in } \mathcal{O}.
\]

Upon a rearrangement and setting pressure variable

\[
\pi \equiv G_{\rho,1} \omega_1 + G_{\rho,2} \mu, \tag{3.3}
\]

we then have

\[
\omega_2 = i\beta \omega_1 - \omega_1^* \quad \text{in } \Omega
\]

\[-\beta^2 \omega_1 - i\beta \omega_1^* + P_\rho^{-1} \Delta^2 \omega_1 - P_\rho^{-1} \pi|_\Omega = \omega_2^* \quad \text{in } \Omega
\]

\[
i\beta \mu - \Delta \mu + \nabla \pi = \mu^* \quad \text{in } \mathcal{O}.
\]

We have then following (static) fluid-structure PDE system:

\[
\omega_2 = i\beta \omega_1 - \omega_1^* \quad \text{in } \Omega \tag{3.4a}
\]

\[-\beta^2 P_\rho \omega_1 + \Delta^2 \omega_1 - \pi|_\Omega = P_\rho \omega_2^* + i\beta P_\rho \omega_1^* \quad \text{in } \Omega \tag{3.4b}
\]

\[
\omega_1|_{\partial \Omega} = \frac{\partial \omega_1}{\partial n}|_{\partial \Omega} = 0 \quad \text{on } \partial \Omega \tag{3.4c}
\]

\[
i\beta \mu - \Delta \mu + \nabla \pi = \mu^* \quad \text{in } \mathcal{O} \tag{3.4d}
\]

\[
\text{div}(\mu) = 0 \quad \text{in } \mathcal{O} \tag{3.4e}
\]

\[
\mu = \mu|_{\partial \mathcal{O}} \tag{3.4f}
\]

Step 1. (An estimate for the fluid gradient) Let us return to the resolvent equation (3.1). It is easily seen that an integration by parts gives the following static dissipation relation:

\[
\left( i\beta - A_p \right) \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} = \begin{bmatrix} \omega_1^+ \\ \omega_2^+ \\ \mu^+ \end{bmatrix} \in H_\beta.
\]

\[
= \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|^2_{H_\beta} + 2 \text{Im}(\Delta \omega_1, \Delta \omega_2)\Omega + \| \nabla \mu \|^2_{\mathcal{O}}:
\]

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inequality and (3.5). In this way, we then obtain

\[ \beta \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{H^s}^2 + 2i \text{Im}(\Delta \omega_1, \Delta \omega_2) + \| \nabla \mu \|_{L^2}^2 = \]
\[ = \left( (i\beta - A_\mu) \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} , \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right)_{H^s}, \]

whence we obtain

\[ \| \nabla \mu \|^2_{L^2(\Omega)} = \text{Re} \left( \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} , \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right)_{H^s}. \] (3.5)

**Step 2. (Control of the \( \beta \)-mechanical displacement in a lower topology)** Using the fluid Dirichlet boundary condition in (3.4b), we have

\[ i\beta \omega_1 = \mu^3 \|_{\Omega} + \omega_1^*. \]

We estimate this expression by invoking in sequence, the Sobolev Embedding Theorem, Poincaré’s inequality and (3.5). In this way, we then obtain

\[ \| \beta \omega_1 \|_{H^{1/2}(\Omega)} \leq \| \mu^3 \|_{\Omega} + \omega_1^* \|_{H^{1/2}(\Omega)} \]
\[ \leq C \left( \| \nabla \mu \|_{L^2(\Omega)} + \| \omega_1^* \|_{H^2(\Omega)} \right) \]
\[ \leq C \left( \sqrt{\text{Re} \left( \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} , \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right)_{H^s}} + \| \omega_1^* \|_{H^2(\Omega)} \right). \] (3.6)

**Step 3. (Control of the mechanical displacement)** We multiply both sides of the mechanical equation in (3.4b) by \( \omega_1 \) and integrate. This gives the relation

\[(\Delta^2 \omega_1, \omega_1)_{L^2(\Omega)} = \beta^2 \| P_{L^{1/2}} \omega_1 \|_{L^2(\Omega)}^2 + (\pi_1, \omega_1)_{\Omega} + (P_{\rho^2} \omega_2^* + i\beta \mu^* \omega_1, \omega_1)_{L^2(\Omega)}. \] (3.7)

(3.1) To handle the first term on the right hand side of (3.7), we invoke Poincaré’s Inequality, thereby obtaining

\[ \beta^2 \| P_{L^{1/2}} \omega_1 \|_{L^2(\Omega)}^2 = \beta^2 \left( \| \omega_1 \|_{L^2(\Omega)}^2 + \rho \| \nabla \omega_1 \|_{L^2(\Omega)}^2 \right) \leq C_\alpha \beta^2 \| \nabla \omega_1 \|_{L^2(\Omega)}^2. \] (3.8)

Now,

\[ \beta^2 \| \nabla \omega_1 \|_{L^2(\Omega)}^2 = \beta (\nabla \omega_1, \beta \nabla \omega_1)_{L^2(\Omega)} = \beta (\nabla \omega_1, \beta \nabla \omega_1)_{H^{1/2}(\Omega), H^{-1/2}(\Omega)} \]
\[ \leq C |\beta| \| \omega_1 \|_{H^{1/2}(\Omega)} \| \beta \omega_1 \|_{H^{1/2}(\Omega)}. \]

Subsequently, interpolating between \( H^2(\Omega) \) and \( H^{1/2}(\Omega) \) with interpolation parameter \( \theta = \frac{1}{3} \) (see e.g., [19]), we obtain

\[ \beta^2 \| \nabla \omega_1 \|_{L^2(\Omega)}^2 \leq C |\beta|^{2/3} \| \beta \|^{1/3} \| \omega_1 \|_{H^{1/2}(\Omega)} \| \beta \omega_1 \|_{H^{1/2}(\Omega)} \]
\[ \leq C |\beta|^{2/3} \left( \| \omega_1 \|_{H^2(\Omega)}^{2/3} \| \beta \omega_1 \|_{H^{1/2}(\Omega)}^{1/3} \right) \| \beta \omega_1 \|_{H^{1/2}(\Omega)} \].
Via Young’s inequality, with conjugate exponents 3 and 3/2, we then have
\[ \beta^2 \| \nabla \omega_1 \|_{L^2(\Omega)}^2 \leq C \| \omega_1 \|_{H^{1/2}(\Omega)}^{2/3} \beta \| \omega_1 \|_{H^{1/2}(\Omega)}^{4/3} \leq \epsilon \| \omega_1 \|_{H^{1/2}(\Omega)}^2 + C_\epsilon \| \omega_1 \|_{H^{1/2}(\Omega)}^2, \]
subsequently reinvoking the estimate (3.8), we so have for \(|\beta| > 1,\)
\[ \beta^2 \| \nabla \omega_1 \|_{L^2(\Omega)}^2 \leq \epsilon \| \omega_1 \|_{H^2(\Omega)} + C_\epsilon \| \beta \| \| \omega_1 \|_{H^{1/2}(\Omega)}^2 \]
with the estimate
\[\nu \bigg|_{\partial \Omega} = 0, \]
and subsequently estimates (3.6), we so have for \(|\beta| > 1,\)
\[ \beta^2 \| \nabla \omega_1 \|_{L^2(\Omega)}^2 \leq 2 \epsilon \left[ \left( \frac{\omega_1}{\omega_2} \right) \mu \right]^2 \| \omega_1 \|_{H^2(\Omega)}^2 + C_\epsilon \| \beta \| \| \omega_1 \|_{H^{1/2}(\Omega)}^2. \]
(3.ii) To handle the second term on the right hand side of (3.8), we observe that since \([\omega_1, \omega_2, \mu] \in H^1_\rho,\) then in particular
\[ \int_\Omega \omega_1 \, dx = 0. \]
In consequence, one has wellposedness of the following boundary value problem (see [21] Proposition 2.2):
\[
\begin{align*}
-\Delta \psi + \nabla q &= 0 & \text{in } \Omega \\
\text{div} \, (\psi) &= 0 & \text{in } \Omega \\
\psi|_{S} &= 0 & \text{on } S \\
\psi|_{\Omega} &= (\psi^1, \psi^2, \psi^3)|_{\Omega} = (0, 0, \omega_1) & \text{on } \Omega,
\end{align*}
\]
with the estimate
\[ \| \nabla \psi \|_{L^2(\Omega)} + \| q \|_{L^2(\Omega)} \leq C \| \omega_1 \|_{H^{1/2}(\Omega)} \]
(implicitly, we are also using Poincaré inequality).

With this solution variable \(\psi\) of (3.11) in hand, we now address the second term on the right hand side of (3.7). Since the normal vector \(\nu\) equals \((0, 0, 1)\) on \(\Omega\) (and as the fluid variable \(\mu\) is divergence free), we have
\[
(\pi|_\Omega, \omega_1)_\Omega = - \left( \frac{\partial \mu}{\partial \nu}, \begin{bmatrix} 0 \\ 0 \\ \omega_1 \end{bmatrix} \right)_{L^2(\Omega)} + \left( \pi|_\Omega \nu, \begin{bmatrix} 0 \\ 0 \\ \omega_1 \end{bmatrix} \right)_{L^2(\Omega)}
\]
\[= - \left( \frac{\partial \mu}{\partial \nu}, \psi \right)_{L^2(\partial \Omega)} + \left( \pi|_\Omega \nu, \psi \right)_{L^2(\partial \Omega)}, \]
after invoking the boundary conditions in (3.11).
The use of Green’s Identities and the Stokes system in (3.11) then gives

\[
\pi_{|\Omega, \omega_1}\Omega = -\left( \frac{\partial \mu}{\partial \nu}, \psi \right)_{L^2(\partial \Omega)} + \left( \pi_{|\Omega, \nu}, \psi \right)_{L^2(\partial \Omega)}
\]

\[
= -\left( \Delta \mu, \psi \right)_{L^2(\Omega)} - \left( \nabla \mu, \nabla \psi \right)_{L^2(\Omega)} + \left( \nabla \pi, \psi \right)_{L^2(\Omega)}
\]

\[
= -i \beta (u, \psi)_{L^2(\Omega)} - \left( \nabla u, \nabla \psi \right)_{L^2(\Omega)} + \left( \nabla^* \psi, \psi \right)_{L^2(\Omega)}.
\]

Estimating this right hand side by means of Poincaré Inequality, we then have for |β| > 1,

\[
|\pi_{|\Omega, \omega_1}\Omega| \leq C|\beta| \|\nabla \psi\|_{L^2(\Omega)} \left( \|\nabla u\|_{L^2(\Omega)} + \|u^*\|_{L^2(\Omega)} \right);
\]

and subsequently refining this inequality by means of (3.14), (3.12) and (3.6), we establish

\[
|\pi_{|\Omega, \omega_1}\Omega| \leq C|\beta| \|\omega_1\|_{H^{1/2}(\Omega)} \left( \left| \text{Re} \left( \begin{bmatrix} \omega_1^1 & \omega_2 \\ \omega_2^2 & \mu^2 \end{bmatrix} \right) \right|_{H_p}^{1/2} \right)
\]

\[
\leq C \left( \left| \text{Re} \left( \begin{bmatrix} \omega_1^1 & \omega_2 \\ \omega_2^2 & \mu^2 \end{bmatrix} \right) \right|_{H_p}^{1/2} \right) + \|u^*\|_{H^2(\Omega)}.
\]

(3.15)

\[
\leq \epsilon \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{H_p}^2 + C_\epsilon \left\| \begin{bmatrix} \omega_1^1 & \omega_2^2 & \mu^2 \end{bmatrix} \right\|_{H_p}^2,
\]

after again using Young’s Inequality.

(3.iii) It remains to handle the third term on the right hand side of (3.7). By way of estimate (3.6) we have readily for |β| > 1

\[
|\left( P_{\mu} \omega_1^2 + i \beta P_{\mu} \omega_1^1, \omega_1 \right)_{L^2(\Omega)}| = \left| (\omega_2^1 + i \beta \omega_1^1, \omega_1)_{L^2(\Omega)} + \rho(\nabla \omega_2^1 + i \beta \omega_1^1, \nabla \omega_1)_{L^2(\Omega)} \right|
\]

\[
\leq C_\rho |\beta| \left( \|\nabla \omega_1\|_{L^2(\Omega)} \left( \|\nabla \omega_1^1\|_{L^2(\Omega)} + \|\nabla \omega_2^1\|_{L^2(\Omega)} \right) \right)
\]

\[
\leq 2 \epsilon \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{H_p}^2 + C_\epsilon |\beta|^2 \left\| \begin{bmatrix} \omega_1^1 & \omega_2^2 & \mu^2 \end{bmatrix} \right\|_{H_p}^2.
\]

(3.16)

after reusing |ab| ≤ c a^2 + c b^2.

Applying (3.10), (3.15), and (3.16) to the right hand side of (3.7), and using the fact that ω_1 satisfies hinged boundary conditions, we then have

\[
\|\Delta \omega_1\|_{L^2(\Omega)}^2 = \left( \Delta^2 \omega_1, \omega_1 \right)_{L^2(\Omega)}
\]

\[
\leq \epsilon (C_\rho + 1 + 2\epsilon) \left\| \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix} \right\|_{H_p}^2 + C_\epsilon |\beta|^2 \left\| \begin{bmatrix} w_1^1 \\ w_2^2 \\ u^* \end{bmatrix} \right\|_{H_p}^2.
\]

(3.17)
Step 4. (Control of the mechanical velocity) Via the resolvent relation \( \omega_2 = i\beta \omega_1 - \omega_1^* \) we have

\[ \|\omega_2\|_{H^1(\Omega)} \leq \|\beta \omega_1\|_{H^1(\Omega)} + \|\omega_1^*\|_{H^1(\Omega)} \leq C \|\beta \nabla \omega_1\|_{L^2(\Omega)} + \|\omega_1^*\|_{H^1(\Omega)}, \]

after again using Poincaré Inequality. Applying (3.9) once more, we attain

\[ \|\omega_2\|_{H^1(\Omega)}^2 \leq \epsilon C \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{H_p}^2 + C \epsilon \|\beta\|^2 \left\| \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{H_p}^2. \]  

(3.18)

To finish the proof of Theorem 1.3 we collect (3.5), (3.17) and (3.18). This gives the following conclusion: the solution of the resolvent equation satisfies, for \(|\beta| > 1\), the estimate

\[ \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{H_p}^2 \leq \epsilon C \left\| \begin{bmatrix} \omega_1 \\ \omega_2 \\ \mu \end{bmatrix} \right\|_{H_p}^2 + C \epsilon |\beta|^2 \left\| \begin{bmatrix} \omega_1^* \\ \omega_2^* \\ \mu^* \end{bmatrix} \right\|_{H_p}^2, \]

which gives the estimate (3.2), for \(\epsilon > 0\) small enough. This concludes the proof of Theorem 1.3.

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