DIRECTED PATHS ON A TREE: COLORING, MULTICUT AND KERNEL

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ABSTRACT. In a paper published in Journal of Combinatorial Theory, Series B (1986), Monma and Wei propose an extensive study of the intersection graph of paths on a tree. They explore this notion by varying the notion of intersection: the paths are respectively considered to be the sets of their vertices and the sets of their edges, and the trees may or may not be directed. Their main results are a characterization of these graphs in term of their clique tree and a unified recognition algorithm. Related optimization problems are also presented.

In the present note, we are interested by the case when the tree is directed, and the paths are considered to be the set of their arcs. An application is the question of wavelength assignment in optimal network. In their article, Monma and Wei show that in this case the intersection graph is a perfect graph and they give polynomial combinatorial algorithms to solve the minimum coloring and the maximum stable problems, with the help of the clique tree representation (Tarjan’s method). The maximum clique problem is immediate. They leave the problem of finding a minimum clique cover, which is here nothing else than the minimum multicut of the dipaths, as an open question.

Costa, Létocart and Roupin in a survey about multicut and integer multiflow noted that the maximum stable, which is a maximum set of arc-disjoint dipaths, and the minimum clique cover, which is the minimum set of arcs intersecting all dipaths (minimum multicut), are integers solutions of a linear program with totally unimodular matrices, and hence can be solved polynomially. For the maximum stable problem (or maximum set of arc-disjoint dipaths), there is a simple polynomial algorithm found by Garg, Vazirani and Yannakakis.

In the present paper, we present faster algorithms for solving the minimum coloring and the minimum clique cover problems (minimum multicut), and maybe simpler, since we work directly with the dipaths on the tree and avoid both the clique decomposition and the linear programming formulation. They both run in \(O(np)\) time, where \(n\) is the number of vertices of the tree and \(p\) the number of paths. Another result is a polynomial algorithm computing a kernel in the intersection graph, when its edges are oriented in a clique-acyclic way. Indeed, such a kernel exists for any perfect graph by a theorem of Boros and Gurvich. Such algorithms computing kernels are known only for few classes of perfect graphs.

1. Introduction

Consider a directed tree \(T = (V,A)\) (with \(n\) vertices) and a collection \(\mathcal{P}\) of dipaths in this tree. The intersection graph is \(I(\mathcal{P},T)\) is a graph whose vertex set is \(\mathcal{P}\) and where two vertices are connected by an edge if the corresponding dipaths share a common arc. Monma et Wei proved that \(I(\mathcal{P},T)\) is then a perfect graph, and explained how such a graph can be efficiently recognized [14]. Actually, the purpose of their paper is an extensive study of the intersection graph by considering various situations: the tree could be directed or undirected, and the paths could be identified with their vertex set or their edge set. Note for instance than when the tree is not directed, and the path identified with their vertex sets, the intersection graph is chordal [10] [8] [18], and hence also perfect.

Once a graph is known to be perfect, min-max formulas hold and corresponding optimization problems can be solved polynomially, at least with the ellipsoids (for the general method, see the book of Grötschel, Lovász and Schrijver [9]). Monma and Wei gave in their paper combinatorial algorithms for these problems, except for the minimum clique cover problem. By a theorem of
Boros and Gurvich \cite{4}, conjectured by Berge and Duchet \cite{1}, we know also that if the edges of a perfect graph are oriented in a clique-acyclic manner, then there is a kernel.

In this paper, we address the problem of finding a minimum coloring, a minimum clique cover of $I(\mathcal{P}, T)$ and a kernel in any clique-acyclic orientation of the edges of $I(\mathcal{P}, T)$. A coloring of $I(\mathcal{P}, T)$ is a color assignment to the dipaths is such a way that two dipaths sharing a common arc get distinct colors. The color assignment problem is often interpreted as a wavelength assignment on an optical fiber (see \cite{3, 2} for further studies from this point of view). A clique cover of $I(\mathcal{P}, T)$ is a set of arcs intersecting all dipaths. Such a set is a multicut of $T$ considered with the sink and the source of each dipath. A clique-acyclic orientation of $I(\mathcal{P}, T)$ corresponds to a reflexive and antisymmetric binary relation $\mathcal{R}$ on $\mathcal{P}$ inducing a total order $\preceq_a$ on each subset of dipaths of $\mathcal{P}$ sharing a common arc $a$. A kernel is a collection $\mathcal{K}$ of arc-disjoint dipaths of $\mathcal{P}$ such that if $P \in \mathcal{P} \setminus \mathcal{K}$, then there is $Q \in \mathcal{K}$ and an arc $a$ such that $Q \succeq_a P$.

1.1. Definitions. Let $G = (V, E)$ be a graph. A stable set of $G$ is a subset $S \subseteq V$ such that no edge of $G$ connect two vertices of $S$. The stable set number is the maximum size of stable set of $G$ and is denoted by $\alpha(G)$.

A clique of $G$ is a set of vertices any two of which are adjacent. The clique number is the maximum size of a clique and is denoted by $\omega(G)$.

A coloring of $G$ is a color assignment on the vertices of $G$ such that two vertices connected by an edge get different colors (or equivalently, a partition of $V$ into stable sets). The minimum number of colors in a coloring of $G$ is called the coloring number and is denoted by $\chi(G)$.

A clique cover of $G$ is a partition of $V$ into cliques. The minimum number of cliques in a clique cover of $G$ is called the clique cover number of $G$ and is denoted by $\bar{\chi}(G)$.

The following relation are immediate (recall that $\bar{G}$ denotes the complementary graph of $G$, in which edges become non-edges and conversely):

$$\alpha(G) = \omega(\bar{G}), \quad \chi(G) = \bar{\chi}(\bar{G}), \quad \alpha(G) \leq \bar{\chi}(G), \quad \omega(G) \leq \chi(G).$$

$G$ is said to be perfect if $\chi(H) = \omega(H)$ for all subgraphs $H$ of $G$, or, equivalently, if $\alpha(H) = \bar{\chi}(H)$ for all subgraphs $H$ of $G$, since according to the well-known theorem of Lovász \cite{11}, $G$ is perfect if and only if $\bar{G}$ is perfect.

A cycle of length $s$ in an oriented graph is a sequence of arcs $(v_1, v_2), (v_2, v_3), \ldots, (v_s, v_1)$. A clique-acyclic orientation of $G$ is an orientation $\bar{G}$ of the edges of $G$ in such a way that each clique becomes acyclic. Said differently, there is no cycle of length 3 in the graph $\bar{G}$.

A kernel in an oriented graph $D = (V, A)$ is a stable subset $K \subseteq V$ such that for each $v \in V \setminus K$, there is a $w \in K$ such that $(v, w) \in A$.

1.2. The intersection graph $I(\mathcal{P}, T)$ as a perfect graph. Monma and Wei proved that $I(\mathcal{P}, T)$ is a perfect graph. In our context, $\alpha(I(\mathcal{P}, T))$ is the maximum number of ar-disjoint dipaths, $\omega(I(\mathcal{P}, T))$ is the maximum number of dipaths passing through a common arc, $\chi(I(\mathcal{P}, T))$ is the minimum colors that can be assigned to the dipaths in such a way that two dipaths sharing a common arc get different colors, and $\bar{\chi}(I(\mathcal{P}, T))$ is the minimum number of arcs intersecting all dipaths of $\mathcal{P}$, that is a minimum multicut. The equality $\alpha = \bar{\chi}$ says that the maximum number of arc-disjoint dipaths equals the minimum multicut, and the equality $\omega = \chi$ says that the maximum number of dipaths passing through a common arc is equal to the minimum colors that can be assigned to the dipaths in such a way that two dipaths sharing a common arc get different colors (see Figure \cite{1} for an illustration).

All these value can be computed in polynomial time with the ellipsoids, as we have said. Monma and Wei gave combinatorial algorithms for the computation of a minimum coloring and a maximum stable set of $I(\mathcal{P}, T)$. Since they developed a clique decomposition for their recognition algorithm, a classical method of Tarjan \cite{17} allows to solve this problem polynomially (and recursively). For the
maximum clique, it is straightforward, just check each arc one after each other (time complexity $O(n)$). The minimum clique cover problem was left as an open question. Costa, Létocart and Roupin in their survey about multicut and integer multiflow [5] noted that the maximum stable set and the minimum clique cover problems admit a linear programming formulation with totally unimodular matrices\(^1\), and hence can be solved polynomially (for instance with the ellipsoids, or the interior-points method). For the maximum stable set problem, the best algorithm is that of Garg, Vazirani and Yannakakis [7].

By the theorem of Boros ang Gurvich already cited, we know that in any clique-acyclic orientation of $I(\mathcal{P}, T)$, there is a kernel. Any clique-acyclic orientation of $I(\mathcal{P}, T)$ induces a reflexive and antisymmetric binary relation $\mathcal{R}$ on $\mathcal{P}$ that is a total order $\preceq_a$ when restricted to a subset of dipaths sharing a common arc $\preceq_a$, and conversely. See Figure 2 for an illustration.

The question whether it is possible to compute in polynomial time a kernel in a clique-acyclic orientation of any perfect graph is an open question. Algorithms are known for few classes of perfect graphs: chordal graphs (straightforward), bipartite graphs [15], line graphs of bipartite graphs (as noted by Maffray [12], it corresponds to the celebrated stable marriage theorem, used below, by Gale and Shapley [6]).

1.3. Results. In the present work, we show how to find a minimum coloring and a minimum clique cover of $I(\mathcal{P}, T)$ directly on the tree $T$, neither using clique decompositions nor using linear programming. We get a time complexity of $O(n|\mathcal{P}|)$ for both the minimum coloring and the minimum clique cover. It is faster that all previous known algorithms. We also give an $O(n|\mathcal{P}|^2)$ algorithm that computes a kernel in any clique-acyclic orientation of $I(\mathcal{P}, T)$. It provides a new class of perfect graphs for which the problem of finding a kernel is known to be polynomial.

1.4. Plan. Section 2 explains the basic observation on which the present paper is based. If the tree is a star, all problems can be reformulated in terms of a bipartite graph: coloring of the path becomes coloring of the edges, arc-disjoint paths become matching, and a kernel becomes a stable

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\(^1\)actually, network matrices, but the authors do not seem to have noticed it.
matching. A tree can be then seen as a collection of intricate bipartite graphs that can be processed in a consistent way.

Section 3 is devoted to the minimum coloring problem (minimum coloring of the dipaths in such a way that two dipaths of $P$ sharing at least a common arc get distinct colors), Section 4 to the minimum clique cover (minimum multicut) and Section 5 to the kernel.

2. Starting observation: stars and bipartite graphs

Let $G = (V(G), A(G))$ be an oriented star, and denote by $v$ the central vertex (a star is a tree whose diameter is equal to 2). Denote by $u_1, \ldots, u_s$ the other vertices. Consider a set $P$ of dipaths on this star. We will show that any of the problems we are interested in can be encoded on a bipartite graph.

Adding vertices $u'_1, \ldots$, we can assume that all dipaths have length 2. Build the bipartite graph $B$ whose vertices are the $u_i$ and the $u'_i$ (vertices with outgoing arcs on one side, vertices with ingoing arcs on the other side) and whose edges connect vertices in the same dipath (with multiplicities). Hence there is a one-to-one correspondence between dipaths of $G$ and edges of $B$. See Figure 3 for an illustration of the construction.

Color the dipaths in $G$ is equivalent to color the edges in $B$. Find a set of arc-disjoint dipaths in $G$ is equivalent to find a matching in $B$. Finally, find a set of arc-disjoint dipaths in $G$ that provides a kernel in the intersection graph $I(P, G)$ is equivalent to find a stable matching in $B$. Given a graph $G = (V(G), E(G))$, and a total order $\preceq_v$ on the set of edges incident to $v$, for each $v \in V(G)$, a matching $M$ is said to be stable if for any edge $e \in E(G) \setminus M$, there is an edge $f \in M$ and a vertex $v$ (incident to both $e$ and $f$) such that $f \succeq_v e$.

Note that the construction of $B$ can be done in linear time.

Remark: The graphs $I(P, T)$ are the common generalization of the bipartite graphs (see above) and of the interval graphs (obvious: the underlying tree is then simply a path).

Figure 2. A kernel in the arc-intersection graph of the dipaths.
3. Minimum Coloring

The strategy to color the dipaths is simple. Choose any vertex \( v \) of \( T \). Consider the set of paths \( \mathcal{P}_v \) passing through \( v \). Coloring the paths is equivalent of coloring the edges of a bipartite multigraph as explained in the previous section. It can be done in \( O(\deg(v) |\mathcal{P}_v|) \), as explained by Schrijver in his book [16] (Theorem 20.13, p.333). Once these paths are colored, start again with one of the neighbors of \( v \), consider the set of paths passing through this neighbor \( w \), color the paths of the associated bipartite multigraph without changing the colors of the paths already colored (those passing through the arc \( vw \)), in \( O(\deg(w) |\mathcal{P}_w|) \). Start again, with one of the neighbors of \( v \) or \( w \), etc. Stop when all vertices have been processed. The tree structure of \( T \) explains why no color conflict can happen. The whole complexity is then in \( O(n |\mathcal{P}|) \) since \( \sum_{v \in V} \deg(v) = 2n \).

**Theorem 1.** Let \( T = (V,A) \) be a directed tree and \( \mathcal{P} \) a collection of dipaths in this tree. The minimum number of colors that can be assigned to the dipaths of \( \mathcal{P} \) such that two dipaths sharing a common arc get distinct color is equal to the maximum number of dipaths passing through a common arc, and such a minimum coloring can be found in \( O(n |\mathcal{P}|) \), where \( n \) is the number of vertices of \( T \).

The first part of the assertion, which concerns the equality between the chromatic and clique numbers of \( I(\mathcal{P}, T) \), was already proved by Monma and Wei, of course. The algorithm that we propose below provides actually another (and algorithmic) proof.

**Proof.** The proof works exactly as described in the beginning of the section. For a vertex \( v \) of \( T \), define \( B_v := (X,E) \) the bipartite graph that corresponds to \( v \) and its neighbors. The dipaths considered on this star are the dipaths induced by \( \mathcal{P} \).

We can color the dipaths while coloring the edges of the bipartite graphs \( B_v \), taking for \( v \) the vertices of \( T \) one after the other, with the additional requirement that a new vertex must be the neighbor of a vertex already processed. The fact that \( T \) is a tree ensures then that when a new
bipartite graph $B_u$ is considered, the edges already colored emanate all from a common vertex (the vertex already processed). Since the color of the dipaths already colored is never modified, one keeps an admissible coloring along the algorithm.

At the end, one has an admissible coloring, and the number of colors is equal to the maximum degree of the $B_u$, which is exactly the maximum number of dipaths passing through a common arc. The whole complexity is $O\left(\sum_{v \in V} \deg(v)|P_v|\right) \leq O\left(\sum_{v \in V} \deg(v)|P|\right) = O(n|P|).$ \hfill \Box

4. MINIMUM MULTICUT

A theorem of Größel, Lovász and Schrijver explains how to derive a minimum clique cover in a perfect graph once we know how to compute a maximum clique and a maximum stable set. This method would provide an $O(n \times DP)$ complexity for our problem, where $DP$ is the complexity for finding a maximum set of arc-disjoint dipaths. The best algorithm for this last purpose is the one proposed by Garg, Vazirani and Yannakakis [7], and whose complexity is $O(n)$ for finding a maximum set of arc-disjoint dipaths. The best algorithm for this last purpose is the one proposed by Garg, Vazirani and Yannakakis [7], and whose complexity is $O(n^{1/2}|P|)$ at best (the exact complexity is not easy to determine). Actually, their algorithm is written for undirected trees, in which case the minimum multicut problem is NP-hard. Nevertheless, one would get with this approach a $O(n^{3/2}|P|)$ time algorithm at best for the minimum clique cover of $I(P,T)$.

We show in this section how to adapt the algorithm of Garg, Vazirani and Yannakakis in order to give simultaneously the maximum stable set and the minimum clique cover of $I(P,T)$, that is the maximum set of arc-disjoint dipaths and a minimum set of arcs intersecting all dipaths of $P$, i.e. a minimum multicut. Our approach improves the complexity, since the algorithm for the arc-disjoint dipaths problem is applied only once, and we finally get a $O(n|P|)$ time algorithm. Before proving this, we need to prove a technical lemma about vertex-cover in bipartite graph.

Lemma 1. Let $G = (V,E)$ be a bipartite graph, of color classes $U$ and $W$, and suppose that there is a vertex $x \in W$ of $G$ contained in every maximum matching. Suppose given a maximum matching $M$. Then one can find in linear time a minimum vertex cover $C$ having the following properties:

(i): $x \in C$ and
(ii): each edge $e$ incident to $x$ and contained in no maximum matching has its other endpoint contained in $C$, too.

Proof. Let $M$ be a maximum matching. One can assume that there is a neighbor $y$ of $x$ that is not covered by $M$; indeed, if not, add a new vertex, call it $y$, and add an edge between $y$ and $x$. One cannot improve the cardinality of the maximum matching with this new edge $yx$, since, otherwise, this would mean that $x$ is not in all maximum matching of $G$. Hence a minimum cover of this new graph is a minimum vertex cover of the former one.

Make the classical construction of the directed graph $D_M$ (see [16]) by orienting each edge $e = uw$ of $G$ (with $u \in U$ and $w \in W$) as follows:

- if $e \in M$, then orient $e$ from $u$ to $w$.
- if $e \notin M$, then orient $e$ from $u$ to $w$.

Let $U_M$ and $W_M$ be the sets of vertices in $U$ and $W$ (respectively) missed by $M$, and define $R_M$ to be the set of vertices reachable in $D_M$ from $U_M$. So $R_M \cap W_M = \emptyset$. Then each edge $uw$ in $M$ is either contained in $R_M$ or disjoint from $R_M$ (that is $u \in R_M \Leftrightarrow w \in R_M$). Moreover, no edge of $G$ connects $U \cap R_M$ and $W \setminus R_M$, as no arc of $D_M$ leaves $R_M$. So $C := (U \setminus R_M) \cup (W \cap R_M)$ is a vertex cover of $G$. Since $C$ is disjoint from $U_M \cup W_M$ and since no edge in $M$ is contained in $C$, one has $|C| \leq |M|$. Therefore, $C$ is a minimum-size vertex cover and contains $x$.

Now, consider the edge $e$ as in the statement of the lemma. The other endpoint of $e$ cannot be in $R_M$ since otherwise there is an $M$-alternating path starting at $U_M$ and containing $e$ – one might then switch the edges on this path if it is not starting at a new vertex $y$ or one might switch the edges on a cycle containing $e$ and $x$ if it is starting at a new vertex $y$, and in both cases $e$ would
be in a maximum matching of the starting graph. Hence $e$ is necessarily between $U \setminus R_M$ and $x \in W \cap R_M$. Both endpoints of $e$ are in $C$. 

Theorem 2. Let $T = (V, A)$ be a directed tree with $n$ vertices and $P$ be a collection of dipaths of this tree. The minimum number of arcs intersecting all dipaths of $P$ and the maximum number of arc-disjoint dipaths in $P$ are equal, and such a minimum set of arcs can be found in $O(n|P|)$, where $n$ is the number of vertices.

Again, the first part of the assertion, which says that $\alpha = \bar{\chi}$ for $I(P, T)$ was already proved by Monma and Wei. Here, we have a new and algorithmic proof.

Proof. The algorithm we describe now follows the scheme proposed by Garg, Vazirani and Yannakakis. 

Root the tree at a vertex $r$. Denote by $Q_v$ all dipaths that contain the arc $uv$ where $u$ is the father of $v$ in the rooted tree, and by $T_v$ the subtree rooted at $v$. Now consider the subproblem of finding the maximal number of arc-disjoint dipaths entirely contained in $T_v$, and denote by $\alpha_v$ this maximal number. Define $B_v$ to be the set of bad dipaths, that is, those dipaths $P$ of $Q_v$ such that, when $P$ is selected, the maximum number of arc-disjoint dipaths entirely contained in $T_v$ and arc-disjoint from $P$ is not $\alpha_v$. Said differently, for any maximum subset of arc-disjoint dipaths in $T_v$, there is a dipath in this subset that shares a common arc with $P$. By convention, we set $Q_r := \emptyset$ and $B_r := \emptyset$.

The algorithm works in two pass, an upward one and a downward one. During the upward pass, all sets $B_v$ are determined. During the downward pass, using the sets $B_v$, a set $S$ of arc-disjoint dipaths, and a set $C$ of arcs are updated at each vertex $v$. These two sets are such that, at the end, $|S| = |C|$ and each dipath of $P$ contains at least an arc of $C$ (it is the multicut we are looking for).

Upward pass: Process first the leaves. Clearly, $B_v = \emptyset$. Then choose a vertex $v$ whose sons have all been processed. Denote by $u$ its father. Take $P$ in $Q_u$. If $v$ is an endpoint of $P$, then clearly $P$ is not in $B_v$. If $v$ is not an endpoint of $P$, then consider the star $G_v$ induced by $v$ and its neighbors, and consider on this star the dipath induced by the non-bad dipaths, i.e. dipaths of $\left(\bigcup_w \text{ is a son of } v \ Q_w\right) \setminus \left(\bigcup_w \text{ is a son of } v \ B_w\right)$. According to Section 2 the question of disjoint paths in $G_v$ is in one-to-one correspondence with the question of matching in a bipartite graph $B$, in which path $P$ induces an edge $e_P = uv_P$, for some vertex $w_P$ that is a son of $v$. Then, $P$ is in $B_v$ if and only if there is no maximum matching in $B - u$ that avoids this $w_P$. The other elements of $B_v$ are the dipaths of $Q_v$ that are already in a $B_w$, for $w$ a son of $v$. Hence, it is easy to determine $B_v$. It can be computed in linear time once the matching is computed.

See Figure 1 for an illustration. When the upward pass is over, $B_v$ is known for each vertex $v$. We run now the downward pass.

Downward pass: We process first $r$. Consider the star centered at $r$ and the dipaths induced by $\left(\bigcup_w \text{ is a son of } r \ Q_w\right) \setminus \left(\bigcup_w \text{ is a son of } r \ B_w\right)$. Find a maximum matching $M$ in the corresponding bipartite graph (Section 2) and a minimum cover $C$. Define $S$ to be the dipaths corresponding to $M$ and $C$ to be the arcs corresponding to $C$. We have clearly $|S| = |C|$.

Now, take a vertex $v$ whose father $u$ has been processed. Consider the star centered at $v$ with the dipaths induced by $\left(\bigcup_w \text{ is a son of } v \ Q_w\right) \setminus \left(\bigcup_w \text{ is a son of } v \ B_w\right) \cup Q_v$. Denote by $B$ the bipartite graph that corresponds. Two situations may occur:

- One of these dipaths has already been fixed in $S$: Denote by $f$ the edge in $B$ that corresponds to the already fixed dipath. Find a maximum matching $M$ that uses $f$ and a minimum cover $C$ provided by Lemma 1. Add to $S$ all dipaths that correspond to edges in $M$. It makes $|M| - 1$ new dipaths. Add to $C$ all arcs $vw$ with $w \in C$ except arc $vu$. It makes $|M| - 1 = |C| - 1$ new arcs. $S$ and $C$ are increased by the same quantity. Each
Theorem 3 \(a\) \(P\) on the set of dipaths \(P\) for each vertex \(v\) is here played by vertex \(u\).

- None of these dipaths has already been fixed in \(S\): Delete from \(B\) all edges induced by the dipaths in \(Q_v \setminus B_v\). Find a maximum matching \(M\) containing no edge induced by a path in \(B_v\) (it is possible by definition of \(B_v\)) and a minimum cover \(C\) in \(B\). Add to \(S\) all dipaths that correspond to edges in \(M\). It makes \(|M|\) new dipaths. Add to \(\mathcal{C}\) all arcs that correspond to vertices of \(C\). It makes \(|M| = |C|\) new arcs. \(S\) and \(\mathcal{C}\) are increased by the same quantity.

Each dipath in \((\bigcup_v w \text{ is a son of } v \ Q_w) \setminus (\bigcup_v w \text{ is a son of } v \ B_w)\) but not in \(Q_v \setminus B_v\) contains at least one of these new arcs.

At the end, we get a set of arc-disjoint dipaths \(S\) and a set of arcs \(\mathcal{C}\) of same cardinality. It remains to prove that each dipath contains at least one arc of \(\mathcal{C}\). It is enough to show that for each dipath \(P\), there is a vertex \(v\) such that either

- we have simultaneously \(P \in Q_w \setminus B_w\) and \(P \in B_v\) where \(w\) is a son of \(v\) or
- we have simultaneously \(P \in Q_w \setminus B_w\) and \(v\) an endpoint of \(P\), where \(w\) is a son of \(v\) or
- we have simultaneously \(P \in Q_w \setminus B_w\) and \(P \in Q_w' \setminus B_w'\) with \(w\) and \(w'\) sons of \(v\).

Indeed we have seen that such dipaths always contain at least one arc of \(\mathcal{C}\). Such a vertex \(v\) exists: either there is vertex \(v'\) such that \(P \in B_{v'}\), and then take \(v\) the farthest vertex of \(P\) from \(r\) that is such that \(P \in B_v\), or define \(v\) to be the nearest vertex of \(P\) from \(r\).

The whole complexity is \(O(u|P|)\), since at each vertex \(v\), the complexity is \(O(\deg(v)|P_v|)\) where \(P_v\) denotes the set of dipaths passing through \(v\). \(\square\)

5. Kernel

In this section, we assume that we have a reflexive and antisymmetric binary relation \(R\) defined on the set of dipaths \(P\) that induces for each arc \(a\) a total order \(\preceq_a\) on the set of dipaths containing \(a\).

Before stating and proving the main theorem of this section, we state two theorems about stable matchings in bipartite graphs.

**Theorem 3 (Stable marriages theorem, [6]).** Let \(B = (V(B), E(B))\) be a bipartite graph with a total order \(\preceq_v\) on the set of edges incident to \(v\) for each vertex \(v \in V(B)\). Then there is stable matching in \(B\) and it can be computed in \(O(|V(B)||E(B)|)\).

**Theorem 4 ([13]).** Any two distinct stable matchings cover the same set of vertices.

The algorithm is based on the following lemma, which derives directly from Theorem 4.

**Lemma 2.** Let \(B = (V(B), E(B))\) be a bipartite graph with a total order \(\preceq_v\) on the set of edges incident to \(v\) for each vertex \(v \in V(B)\). Choose a vertex \(u\). Define \(U \subseteq \delta(u)\) to be the set of edges \(e \in \delta(u)\) that are in no stable matching of \((B \setminus \delta(u)) \cup \{e\}\). Then for all \(f \in \delta(u) \setminus U\) and all \(U' \subseteq U\), the edge \(f\) is in all stable matchings of \((B \setminus \delta(u)) \cup \{f\} \cup U'\).

**Proof.** Take \(f \in \delta(u) \setminus U\) and \(U' \subseteq U\). Let \(M\) be a stable matching of \((B \setminus \delta(u)) \cup \{f\} \cup U'\). None of the edge \(e\) in \(U'\) can belong to \(M\), otherwise \(M\) would be a stable matching of \((B \setminus \delta(u)) \cup \{e\}\), which contradicts the definition of \(U\). Thus \(M\) is a stable matching of \((B \setminus \delta(u)) \cup \{f\}\). By definition, \(f\) is in a stable matching of \((B \setminus \delta(u)) \cup \{f\}\), hence there is a stable matching of \((B \setminus \delta(u)) \cup \{f\}\) for which \(u\) is covered. According to Theorem 4 all stable matchings of \((B \setminus \delta(u)) \cup \{f\}\) cover \(u\), and thus, \(M\) covers \(u\). As the only edge that can cover \(u\) in \(M\) is \(f\), we have \(f \in M\). \(\square\)

We are now in position to state and prove the main theorem of this section.
Figure 4. We are in the upward pass of the algorithm for multicut. Assume that
\( P_1 \in Q_{w_2} \setminus B_{w_2} \), \( P_2 \in (Q_{w_1} \cup Q_{w_3}) \setminus (B_{w_2} \cup B_{w_3}) \), and \( P_3 \in Q_{w_1} \setminus B_{w_1} \). Then \( P_1 \in Q_v \setminus B_v \) and \( P_3 \in B_v \).

**Theorem 5.** Let \( T = (V, A) \) be a directed tree with \( n \) vertices and \( \mathcal{P} \) be a collection of dipaths of this tree. Assume that there is a reflexive and antisymmetric binary relation on \( \mathcal{P} \) inducing for each arc \( a \in A \) a total order \( \preceq_a \) on the set of dipaths containing this arc. Then it is possible to compute in \( O(n|\mathcal{P}|^2) \) a subset \( \mathcal{K} \) of arc-disjoint dipaths of \( \mathcal{P} \) such that whenever there is a dipath \( Q \) in \( \mathcal{P} \setminus \mathcal{K} \), then there are an arc \( a \) and a dipath \( P \) in \( \mathcal{K} \) such that \( a \in Q \) and \( P \preceq_a Q \) (said differently, \( \mathcal{K} \) is a kernel of \( I(\mathcal{P}, T) \)).

The existence of such a \( \mathcal{K} \) is already ensured by the theorem of Boros and Gurvich. Here we have moreover a polynomial algorithm that computes this kernel.

**Proof.** As before, we root the tree at a particular vertex \( r \) and denote by \( Q_v \) all dipaths that contain the arc \( uv \) where \( u \) is the father of \( v \) in the rooted tree, and by \( T_v \) the subtree rooted at \( v \).

The algorithm works in the same spirit than the algorithm of previous section. Instead of having “bad dipaths”, we will define *uninteresting dipaths* \( \mathcal{U}_v \), which are such that, roughly speaking, when selected, they make impossible the existence of a kernel in the subtree \( T_v \). By convention, we set \( Q_r := \emptyset \) and \( \mathcal{U}_r := \emptyset \).

As before, the algorithm works in two passes, an upward one and a downward one.

**Upward pass:** If \( v \) is a leave, define \( \mathcal{U}_v := \emptyset \). Now, take a vertex \( v \) for which \( \mathcal{U}_w \) has already be defined, for each son \( w \) of \( v \). Denote by \( u \) its father. Let \( B = (V(B), E(B)) \) be the bipartite
graph that corresponds to the star centered at \( v \) and all its neighbors, with the dipaths induced by \((\bigcup_{w} w \text{ is a son of } v) Q_w \setminus (\bigcup_{w} w \text{ is a son of } v) U_w\). For each \( e \in \delta(u) \), test if \( e \) is in a stable matching of \((B \setminus \delta(u)) \cup \{e\}\) (use Lemma \ref{lem:stable_matching}). Then \( P \) is in \( U_v \) if the corresponding edge \( e \) is not in a stable matching of \((B \setminus \delta(u)) \cup \{e\}\). The other elements of \( U_v \) are those dipaths of \( Q_v \) already in a \( U_w \), for \( w \) a son of \( v \).

When the upward pass is finished, we have computed \( U_v \) for all vertices \( v \) of the tree \( T \).

**Downward pass:** We process first \( r \). Consider the star centered at \( r \) and the dipaths induced by \((\bigcup_{w} w \text{ is a son of } r) Q_w \setminus (\bigcup_{w} w \text{ is a son of } r) U_w\). Find a stable matching in the corresponding bipartite graph. It induces a set \( K \) of dipaths.

Now, take a vertex \( v \) whose father \( u \) has been processed. Consider the star centered at \( v \) with the dipaths induced by \((\bigcup_{w} w \text{ is a son of } v) Q_w \setminus (\bigcup_{w} w \text{ is a son of } v) U_w \cup Q_v\). Denote by \( B \) the bipartite graph that corresponds. Two situations can occur:

- One of these dipaths has already been fixed in \( K \): Denote by \( f \) the edge in \( B \) that corresponds to the already fixed dipath. Delete from \( B \) all other edges that correspond to a dipath in \( Q_v \setminus U_v \). Compute a stable matching in \( B \) (which necessarily contains \( f \) by definition of \( U_v \) and according to Lemma \ref{lem:stable_matching}). Add to \( K \) the dipaths whose corresponding edges are in the stable matching.
- None of these dipaths has already been fixed in \( K \): Delete from \( B \) all edges that correspond to a dipath in \( Q_v \setminus U_v \). Compute a stable matching in \( B \). Add to \( K \) the dipaths whose corresponding edges are in the stable matching. By definition of \( U_v \), no dipath using arc \( uv \) is added to \( K \).

At the end, we get a set \( K \) of arc-disjoint dipaths. It remains to check that whenever a dipath \( P \) of \( \mathcal{P} \) is not in \( K \), there is an arc \( a \in P \), and a dipath \( Q \in K \) such that \( a \in Q \) and \( Q \geq_a P \). It is enough to show that for each dipath \( P \) of \( \mathcal{P} \), there is a vertex \( v \) such that either

- we have simultaneously \( P \in Q_w \setminus U_w \) and \( P \in U_v \) where \( w \) is a son of \( v \) or
- we have simultaneously \( P \in Q_w \setminus U_w \) and \( v \) an endpoint of \( P \), where \( w \) is a son of \( v \) or
- we have simultaneously \( P \in Q_w \setminus U_w \) and \( P \in Q_{w'} \setminus U_{w'} \) where \( w \) and \( w' \) are sons of \( v \).

Indeed such dipaths are edges of a bipartite graph for which a kernel is computed. As in the previous section, such a vertex \( v \) exists: either there is vertex \( v' \) such that \( P \in U_{v'} \), and then take \( v \) the farthest vertex of \( P \) from \( r \) that is such that \( P \in U_v \), or define \( v \) to be the nearest vertex of \( P \) from \( r \).

The whole complexity is \( O(n|\mathcal{P}|^2) \), since at each vertex \( v \), the complexity is \( O(\deg(v)|\mathcal{P}_v|^2) \) where \( \mathcal{P}_v \) denotes the set of dipaths passing through \( v \): we repeat at most \(|\mathcal{P}_v|\) times the computation of a stable matching for each vertex \( v \).

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**References**

1. C. Berge and P. Duchet, *Probleme*, Séminaire MSH (Paris), January 1983.
2. J.-C. Bermond, L. Braud, and D. Coudert, *Traffic grooming on the path*, Theoret. Comput. Sci. 384 (2007), 139–151.
3. J.-C. Bermond, M. Cosnard, D. Coudert, and S. Pérennes, *Optimal solution of the maximum all request path grooming problem*, Proceedings of the Advanced International Conference on Telecommunications, AICT06 (Guadeloupe, France), February 2006.
4. E. Boros and V. Gurvich, *Perfect graphs are kernel solvable*, Discrete Mathematics 159 (1996), 35–55.
5. M.-C. Costa, L. Léotocart, and F. Roupin, *Minimal multicut and maximal integer multflow: A survey*, Eur. J. Op. Res. 162 (2005), 55–69.
6. D. Gale and L.S. Shapley, *College admissions and the stability of marriage*, American Mathematical Monthly (1962), 9–15.
7. N. Garg, V. V. Vazirani, and M. Yannakakis, *Primal-dual approximation algorithms for integral flow and multicut in trees*, Algorithmica 18 (1997), 3–20.
8. F. Gavril, *The intersection graphs of subtrees in trees are exactly the chordal graphs*, J. Combin. Theory Ser. B 16 (1974), 47–56.
9. M. Grötzchel, L. Lovász, and A. Schrijver, *Geometric algorithms and combinatorial optimization*, Springer, Berlin, 1988.
10. G. Gyárfás and J. Lehel, *A helly-type problem in trees*, Combinatorial Theory and its Applications II (P. Erdős and V. Rnyi, A. Sós, eds.), North-Holland, Amsterdam, 1970, pp. 571–584.
11. L. Lovász, *Normal hypergraphs and the perfect graph conjecture*, Discrete Math. 2 (1972), 253–267.
12. F. Maffray, *Kernel in perfect line graph*, Journal of Combinatorial Theory, Series B 55 (1992), 1–8.
13. D.G. McVitie and L.B. Wilson, *Stable marriage assignment for unequal sets*, BIT (1970), 295–309.
14. C. L. Monma and V. K. Wei, *Intersection graphs of paths in a tree*, J. Combin. Theory Ser. B 41 (1986), 141–181.
15. M. Richardson, *Solutions of irreflexive relations*, Annals of Mathematics 58 (1953), 573–590.
16. A. Schrijver, *Combinatorial optimization*, Springer, 2003.
17. R. E. Tarjan, *Decomposition by clique separators*, Discrete Math. 55 (1985), 221–232.
18. J. R. Walter, *Representations of chordal graphs as subtrees of a tree*, J. of Graph Theory (1978), 265–267.

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