POLYDENDRIFORM STRUCTURE ON FACES OF HYPERGRAPH POLYTOPES

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Abstract. We extend the works of Loday-Ronco and Burgunder-Ronco on the tridendriform decomposition of the shuffle product on the faces of associahedra and permutohedra, to other families of nestohedra, including simplices, hypercubes and yet other less known families. We also extend the shuffle product to take more than two arguments, and define accordingly a new algebraic structure, that we call polydendriform, from which the original tridendriform equations can be crisply synthesised.

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1. Introduction

In 1998 Loday-Ronco introduced a Hopf algebra on the linear span of rooted planar binary trees [LR98]. This Hopf algebra is closely related to the Malvenuto-Reutenauer Hopf algebra on permutations [MR95]. Planar binary trees and permutations label the vertices of two well-known families of polytopes: associahedra and permutohedra. The associative products of these Hopf algebras were then extended to associative products on all faces of these polytopes labeled respectively by planar trees and surjections by Loday-Ronco [LR02] and Burgunder-Ronco [BR10]. More precisely, Loday-Ronco introduced an associative product \( \ast \) on planar trees as a
shown in [Cha00].

words, introduced as such in [HNT08]. They label the faces of permutohedra, as on example, we have $g_n$ for some max($\{1,\ldots,n\}$)

surjective. For arbitrary $h$, we look for all surjections $\phi : \{1,\ldots,n\} \to \{1,\ldots,m\}$, where $\phi$ is the unique increasing bijection $\text{Im}(h) \to |\text{Im}(h)|$. For example, we have $\text{std}(1, 4, 3, 4) = (1, 3, 2, 3)$. Surjections are also known as packed words, introduced as such in [HNT08]. They label the faces of permutohedra, as shown in [Cha00].

If $f : \{m_1\} \to \{n_1\}$ and $g : \{m_2\} \to \{n_2\}$ are surjections, we look for all surjections $(h, k)$ such that $\text{std}(h) = f$ and $\text{std}(k) = g$. Note that we have then $\text{Im}(h, k) = \{1, \ldots, n\}$, for some $\max\{n_1, n_2\} \leq n \leq n_1 + n_2$. Below, we do this for $f := (1, 2, 1)$ and $g := (2, 1)$, underlining the maximum elements of $h$ and of $k$.

- $n = 2$: (1, 2, 1, 2, 1)

Associahedra and permutohedra are instances of polytopes called hypergraph polytopes [DP11] which are obtained by truncating some faces of simplices, and are also known as nestohedra [Pos09]. The description of faces of hypergraph polytopes in terms of tree structures described in [COI19] gives an adapted frame to extend Loday-Ronco and Burgunder-Ronco to other families of polytopes. The main result in this paper is more precisely a two-fold extension of this result. We first extend the notion of tridendriform algebras to so-called polydendriform algebras, and prove that we can endow the faces of the polytopes associated with some families of hypergraphs with a polydendriform algebra structure. The underlying associative product coincides with the associative product defined by Ronco [Ron12] on graph associahedra [CD06], which are a special type of hypergraph polytopes where the associated hypergraphs have only hyperedges of cardinality two. Our results apply also to other families of hypergraph polytopes such as simplices and hypercubes.

The article is organized as follows. In §2 we explain in detail the case of the permutohedra studied by Burgunder-Ronco, and motivate and recall Burgunder-Ronco’s notion of $q$-tridendriform, i.e. algebras with operations $\prec$, $\succ$ and $\cdot$ satisfying the same equations as the tridendriform one, but with the associated (associative) product being now defined as $*_{q} = \prec + \succ + q \cdot$ for an arbitrary $q \in \mathbb{K}$, where $\mathbb{K}$ is the ambient field. In §3 we recall the necessary notions of hypergraph polytopes and constructs. In §4 we introduce our condition for a family of polytopes to have a polydendriform algebra structure. We first define a so-called “strict” condition that makes it possible to define $q$-tridendriform algebras. We then define a weaker condition called “semi-strict” which allows us to deal with a wider class of polytopes but for which only $*_{-1}$ is associative.

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2. Prologue

We recall Burgunder-Ronco’s shuffle product on the faces of permutohedra [BR10]. We set $[n] = \{1, \ldots, n\}$, and identify a function $f : [n] \to X$ with the sequence $(f(1), \ldots, f(n))$.

By surjection, we mean a function $f : [m] \to [n]$ (for some $m, n \geq 1$) that is surjective. For arbitrary $h : [m] \to [n]$, we can build a surjection $\text{std}(h) := \phi \circ h : [m] \to |\text{Im}(h)|$, where $\phi$ is the unique increasing bijection $\text{Im}(h) \to |\text{Im}(h)|$. For example, we have $\text{std}(1, 4, 3, 4) = (1, 3, 2, 3)$. Surjections are also known as packed words, introduced as such in [HNT08]. They label the faces of permutohedra, as shown in [Cha00].

If $f : [m_1] \to [n_1]$ and $g : [m_2] \to [n_2]$ are surjections, we look for all surjections $(h, k)$ such that $\text{std}(h) = f$ and $\text{std}(k) = g$. Note that we have then $\text{Im}(h, k) = \{1, \ldots, n\}$, for some $\max(n_1, n_2) \leq n \leq n_1 + n_2$. Below, we do this for $f := (1, 2, 1)$ and $g := (2, 1)$, underlining the maximum elements of $h$ and of $k$. 

- $n = 2$: (1, 2, 1, 2, 1)
We collect those pairs in the following formal sums:

\[ f ≺ g := (2, 3, 2, 1) + (1, 3, 1, 2) + (1, 4, 1, 3) + (2, 3, 2, 3, 1) + (3, 4, 3, 2, 1) \]

\( (\text{max}(h) > \text{max}(k)) \)

\[ f \cdot g := (1, 2, 1, 2, 1) + (1, 3, 1, 3, 2) + (2, 3, 2, 3, 1) \]

\( (\text{max}(h) = \text{max}(k)) \)

\[ f ≻ g := (1, 2, 1, 3, 1) + (1, 2, 1, 3, 2) + (1, 2, 1, 4, 3) + (1, 3, 1, 4, 2) + (2, 3, 2, 4, 1) \]

\( (\text{max}(h) < \text{max}(k)) \)

\[ f * g := (f ≺ g) + (f \cdot g) + (f ≻ g). \]

The operations \( ≺, \cdot \) and \( ≻ \) satisfy the following tridendriform equations

\[ (≺ *) \ (a ≺ b) ≺ c = a ≺ (b * c) \]

\[ (≻ *) \ (a ≻ b) ≺ c = a ≻ (b * c) \]

\[ (\ast ≻) \ (a \ast b) ≻ c = a ≻ (b \ast c) \]

\[ (\cdot ≻) \ (a \cdot b) ≻ c = a \cdot (b \cdot c) \]

\[ (\cdot ≺) \ (a \cdot b) ≺ c = a \cdot (b \cdot c), \quad \text{and the operation} \ \ast \ \text{is associative.} \]

The tridendriform structure was first recognised and defined by Loday and Ronco [LR04] on Schröder trees, i.e., planar trees without unary nodes. We will denote them as \( \bullet \ (T_1, \ldots, T_n) \), for \( n \neq 1 \), where \( T_1, \ldots, T_n \) are themselves Schröder trees. The tree with only one leaf is then \( \bullet () \). Schröder trees with at least two leaves leave the faces of associahedra. The three tridendriform operations are defined as follows (with the convention that \( \bullet () S = S = S \ast \bullet () \)):

\[ \bullet (S_1, \ldots, S_n) ≺ T := \bullet (S_1, \ldots, S_{n-1}, S_n * T) \]

\[ S ≻ \bullet (T_1, \ldots, T_n) := \bullet (S * T_1, T_2, \ldots, T_n) \]

\[ \bullet (S_1, \ldots, S_m) \ast \bullet (T_1, \ldots, T_n) := \bullet (S_1, \ldots, S_{m-1}, S_m * T_1, T_2, \ldots, T_n). \]

Associahedra and permutohedra are examples of hypergraph polytopes, also known as nestohedra. Our goal is to define, in this more general framework, an associative product, with associated decomposition, instantiating to these two examples and more.

We close this section by studying the relation between tridendriform structures and associativity more closely. Burgunder and Ronco [BR10] have introduced a variation of tridendriform algebras, called \( q \)-tridendriform algebras (for \( q \in \mathbb{R} \), or more generally \( q \in k \) for some field \( k \)), where the equations are the same as above, except that now the operation \( \cdot \) is weighted, i.e., \( a * b \) is redefined as \( a \prec b + q(a \cdot b) + (a \succ b) \). This is justified by the following proposition.

**Proposition 1.** Setting \( a * b := \lambda_1 (a \prec b) + \lambda_2 (a \cdot b) + \lambda_3 (a \succ b) \), if the tridendriform equations are satisfied (with this definition of \( \ast \)), then \( \ast \) is associative if \( \lambda_1 = \lambda_3 = 1 \).
Proof. We match
\[
\lambda_1 \lambda_1 (a \prec b) \prec c + \lambda_1 \lambda_2 (a \cdot b) \prec c + \lambda_1 \lambda_3 (a \succ b) \prec c \\
\phantom{+} + \lambda_2 \lambda_1 (a \prec b) \cdot c + \lambda_2 \lambda_2 (a \cdot b) \cdot c + \lambda_2 \lambda_3 (a \succ b) \cdot c \\
\phantom{+} + \lambda_3 (a \cdot b) \succ c
\]
with
\[
\lambda_1 a \prec (b \ast c) \\
\phantom{+} + \lambda_2 \lambda_1 a \cdot (b \prec c) + \lambda_2 \lambda_2 a \cdot (b \cdot c) + \lambda_2 \lambda_3 a \cdot (b \succ c) \\
\phantom{+} + \lambda_3 \lambda_1 a \succ (b \prec c) + \lambda_3 \lambda_2 a \succ (b \cdot c) + \lambda_3 \lambda_3 a \succ (b \succ c)
\]
using \(\prec \ast\) (resp. \(\ast \succ\), \(\prec \cdot \succ\)) and the assumption \(\lambda_1 = 1\) (resp. \(\lambda_3 = 1\).

\(\lambda_1 = \lambda_3\).

3. Hypergraph Polytopes

A hypergraph is given by a set \(H\) of vertices (the carrier), and a subset \(\mathcal{H} \subseteq \mathcal{P}(H)\) such that \(\bigcup \mathcal{H} = H\). The elements of \(\mathcal{H}\) are called the \textit{hyperedges} of \(\mathcal{H}\). We always assume that \(\mathcal{H}\) is \textit{atomic}, by which we mean that \(\{x\} \in \mathcal{H}\), for all \(x \in H\). Identifying \(x\) with \(\{x\}\), \(H\) can be seen as the set of hyperedges of cardinality 1, also called \textit{vertices}. We shall use the convention to give the same name to the hypergraph and to its carrier, the former being the bold version of the latter. A hyperedge of cardinality 2 is called an \textit{edge}. Note that any ordinary graph \((V, E)\) can be viewed as the atomic hypergraph \(\{\{v\} \mid v \in V\} \cup \{e \mid e \in E\}\) (with no hyperedges of cardinality \(\geq 3\)).

If \(\mathcal{H}\) is a hypergraph, and if \(X \subseteq H\), we set \(\mathcal{H}_X := \{Z \mid Z \in \mathcal{H} \text{ and } Z \subseteq X\}\), and \(\mathcal{H} \setminus X = \mathcal{H}_{H \setminus X}\). We say that \(\mathcal{H}\) is \textit{connected} if there is no non-trivial partition \(H = X_1 \cup X_2\) such that \(\mathcal{H} = \mathcal{H}_{X_1} \cup \mathcal{H}_{X_2}\), and that \(X \subseteq H\) is connected in \(\mathcal{H}\) if \(\mathcal{H}_X\) is connected. For each finite hypergraph there exists a partition \(H = X_1 \cup \ldots \cup X_m\) such that each \(\mathcal{H}_{X_i}\) is connected and \(\mathcal{H} = \bigcup (\mathcal{H}_{X_i})\). The \(\mathcal{H}_{X_i}\)'s are the \textit{connected components} of \(\mathcal{H}\). The notation \(\mathcal{H}, X \rightsquigarrow \mathcal{H}_1, \ldots, \mathcal{H}_n\) will mean that \(\mathcal{H}_1, \ldots, \mathcal{H}_n\) are the connected components of \(\mathcal{H} \setminus X\).

Dosen and Petrić [DP1] have proposed the following insightful reading of the data of a finite connected hypergraph \(\mathcal{H}\) as a truncated simplex: the elements of \(H\) are identified with the facets (i.e. codimension 1 faces) of the \(|H| - 1\)-dimensional simplex, and each \(\emptyset \subseteq X \subseteq H, \ |X| \geq 2\), such that \(\mathcal{H}_X\) is connected designates the intersection of the facets in \(X\) as a face to be truncated. The obtained polytopes, called \textit{hypergraph polytopes}, extend the construction of graph associahedra [CD06, Zel06], and are equivalent to nestohedra, as introduced by Postnikov [Pos09]. Moreover, the faces of the polytope obtained by performing all the prescribed truncations are labeled by non-planar trees whose nodes are decorated by non-empty subsets of \(H\), called \textit{constructs}, whose recursive definition we give next using a syntax introduced in [COI19].
There are as many tubes in the tubing as nodes in the construct. T

node decorated by \( H \) that edge, one gets a covering construct.

hypergraph polytope through the process of contracting tree edges: by contracting
\( H \) of \( H \) (written simply \( = \) root(\( Y \)).

The formalism of constructs allows us to view the inclusion of faces of a
hypergraph polytope through the process of contracting tree edges: by contracting
an edge of a construct and merging the decorations of the two nodes related by
that edge, one gets a covering construct.

Simplices are “encoded” as the hypergraphs
\[
S^X = \{ \{ x \} \mid x \in X \} \cup \{ \{ X \} \}
\]
(no truncation prescribed). The constructs have the form \( Y(\{ y \}, \ldots, \{ y \}) \) where
\( \emptyset \subseteq Y \subseteq X \) and \( y \) ranges over \( H \setminus Y \), and are therefore isomorphic to multipointed
sets.

In order to illustrate how the hypergraph structure dictates truncations, con-
derider the hypergraph \( H = \{ \{ x \}, \{ y \}, \{ z \}, \{ y, z \}, \{ x, y, z \} \} \), obtained from \( S^{x,y,z} \)
by adding the edge \( \{ y, z \} \). The construct \( \{ x \}(\{ y \}, \{ z \}) : S^{x,y,z} \) is not a
construct of \( H \), since \( H(y,z) \) is connected. Instead, \( H \) features 3 new constructs:
\( \{ x \}(\{ y \}, \{ z \}) \), \( \{ x \}(\{ z \}, \{ y \}) \) and \( \{ x \}(\{ y \}, \{ z \}) \), encoding two vertices and one edge,
obtained by truncating the vertex \( \{ x \}(\{ y \}, \{ z \}) \) of \( S^{x,y,z} \).

As a slightly more involved example, we show in Figure 3.1 the polytope encoded
by the hypergraph \( H = \{ \{ x \}, \{ y \}, \{ u \}, \{ v \}, \{ x, y \}, \{ x, u \}, \{ x, v \}, \{ u, v \}, \{ x, u, v \} \} \),
obtained from the tetrahedron by truncating three of its vertices and four of its
edges. We also “zoom in” into the square obtained by the truncation prescribed
by \( \{ u, v \} \) and label its four 1-dimensional and four 0-dimensional faces by the
appropriate constructs of \( H \).

We recover associahedra and permutohedra as linear and complete graphs, re-
respectively:

\[
K^X = \{ \{ x_1 \}, \ldots, \{ x_n \}, \{ x_1, x_2 \}, \ldots, \{ x_{n-1}, x_n \}, \{ x_1 \ldots x_n \} \},
\]
\[
P^X = \{ \{ x_1 \}, \ldots, \{ x_n \}, \{ x_1, \ldots, x_n \} \} \cup \{ \{ x_i, x_j \} \mid 1 \leq i \neq j \leq n \},
\]

\footnote{Constructs are in one-to-one correspondence with tubings as defined in \cite{CDC06}: for a given
construct \( T \), each tube of the associated tubing is given by a node of \( T \) and all its descendance.
There are as many tubes in the tubing as nodes in the construct.}
The labeling enabling to identify planar trees with constructs of the associahedra is obtained as a generalization of the one for binary search trees: given a planar tree with root of arity $p+1$, the root is labeled by $x_1, \ldots, x_p$ and each subtree $T_0, \ldots, T_p$ is labeled recursively in such a way that the condition $\max T_i < x_{i+1} < \min T_{i+1}$ for any $0 \leq i \leq p-1$ is satisfied. For permutahedra, note that we can arrange the data of a surjection $f : [m] \to [n]$ as the linear construct $f^{-1}(1)(f^{-1}(2)(\ldots(f^{-1}(n))))$ of height $n$.

Another example which will be developed in the next section is the friezohedron. The associated hypergraphs are:

$$F^X = \{ \{x_i, x_j\} \mid \forall i, j, |x_i - x_j| \leq 2 \}$$

for $X = \{x_1, \ldots, x_n\}$. The name friezohedron comes from the shape of hypergraphs for $X$ sufficiently large, as illustrated in Figure 3.3, where the associated polytope in dimension 3 is also drawn. We don’t have at the time of writing a “simple” combinatorial interpretation of constructs associated to friezohedra. We give the number of constructs with $k$ vertices and $n$ labels (for low values of $k, n$) in Figure 3.2.

| $n$ | 1   | 2   | 3   | 4 | 5 | Sum over $k$ |
|-----|-----|-----|-----|---|---|--------------|
| 1   | 1   |     |     |   |   | 1            |
| 2   | 1   | 1   |     |   |   | 2            |
| 3   | 1   | 6   | 6   |   |   | 13           |
| 4   | 1   | 13  | 33  | 22|   | 69           |
| 5   | 1   | 25  | 119 | 188| 94| 427          |

**Figure 3.2.** Number of constructs with $k$ vertices of the friezohedron on $n$ vertices.

**Figure 3.3.** Top: Hypergraph and truncated simplex associated with the friezohedron on 4 vertices. Bottom: Friezohedron on 10 vertices.
As a final example in this section, for a finite ordered set \( X = \{x_1 < \cdots < x_n\} \), consider the hypergraph

\[
C_X = \{\{x_j \mid 1 \leq j \leq i\} \mid 1 \leq i \leq n\}.
\]

It is not difficult to see that the constructs of \( C_X \) are in one-to-one correspondence with the set of words of length \( n \) over the alphabet \( \{+, -, \bullet\} \) starting with +, and hence decorate the faces of an \((n - 1)\)-dimensional cube. More examples of truncations and constructs are to be found in [DPT11, COI19], and also below.

4. Anthropological shuffle product

In our main section, we unify the above mentioned works of Burgunder, Loday and Ronco into the notion of anthropological shuffle product of constructs. Towards achieving this goal, in §4.1 we introduce a general anthropological framework based on the formalism of hypergraph polytopes of \( \mathbb{P} \) which will serve as “carrier” of the polydendriform structure that we define by recursion in §4.2, and we show that associahedra and permutohedra fit in the framework, as well as other families of polytopes. In §4.3, we give an alternative non-recursive definition of the associated associative product. We illustrate the notions introduced with the example of friezohedra. In §4.4, we further enlarge the anthropological framework to cover more examples.

4.1. Anthropological framework: teams, clans and delegations. We first specify a collection (or universe) \( \mathcal{U} \) of hypergraphs.

The universe associated with the friezohedra is the set of all connected hypergraphs on a finite set \( V \subseteq \mathbb{N} \) with all possible edges of type \( \{i, i+1\} \) or \( \{i, i+2\} \). In other words, \( \mathcal{U}_V \) is the set of hypergraphs \( F^X \), such that \( X \in \mathbb{N} \) and \( \{i, i+1\} \subseteq \mathbb{N}\setminus X \) implies that either \( i+1 < \min X \) or \( i > \max X \), with \( F^X \) defined by:

\[
F^X = X \cup \{\{x_i, x_j\} \mid \forall i, j, |x_i - x_j| \leq 2\} \tag{4.1}
\]

A preteam is a pair \( \tau = (\{H_a \mid a \in A\}, H) \) of a finite set \( \{H_a \in \mathcal{U} \mid a \in A\} \) of hypergraphs (for some indexing set \( A \)) and a hypergraph \( H \in \mathcal{U} \), such that the \( H_a \)'s are mutually disjoint and \( H = \bigcup_{a \in A} H_a \). We call \( H \) and the \( H_a \)'s the coordinating hypergraph and the participating hypergraphs, respectively. This notion is illustrated on Figure 4.4. Considering associahedra, a preteam is obtained as a pair \( (\{K^{I_1}, \ldots, K^{I_n}\}, K) \), where \( \{I_1, \ldots, I_n\} \) is a partition of the integer interval \( J \) into integer intervals. Considering permutohedra, a preteam is obtained as a pair \( (\{K^{A_1}, \ldots, K^{A_n}\}, K) \), where \( \{A_1, \ldots, A_n\} \) is any partition of \( J \). Considering friezeohedra, a preteam is obtained as a pair \( (\{F^{A_1}, \ldots, F^{A_n}\}, F) \) where \( \{A_1, \ldots, A_n\} \) is a partition of \( J \) and for each \( A_i \), \( x, x+1 \notin A_i \) implies that either \( x+1 < \min A_i \) or \( x > \max A_i \).

A preteam is called a strict team if for each choice of a subset \( \emptyset \neq B \subseteq A \) and of a subset \( \emptyset \neq X_b \subseteq H_b \) for each \( b \in B \), inducing the decompositions \( H_b = H_b^{(B)} \cup H_b^{(B\setminus b)} \), we have that, for each \( a \in A_b := (A\setminus B) \cup \{(b, i) \mid b \in B, 1 \leq i \leq n_b\} \), \( H_a \in \mathcal{U} \) and \( H_a \) is included in a connected component of \( H \setminus (\bigcup_{b \in B} X_b) \). Preteams associated respectively with associahedra, permutohedra and friezeohedra are strict. Preteams associated with simplices and hypercubes are not strict, as some \( H_a \) are not included in a connected component of \( H \setminus (\bigcup_{b \in B} X_b) \) but completely disconnected: these last examples fit in the formalism of semi-strict teams introduced in the last section.
Figure 4.4. a) A preteam $\tau = (\{H_{a_1}, H_{a_2}, H_{a_3}, H_{a_4}\}, H)$ is represented as a cobordism whose upper and lower boundary disks feature the participating and coordinating hypergraphs, respectively. b) For $X_{a_1} \subseteq H_{a_1}$ and $X_{a_3} \subseteq H_{a_3}$, the decompositions $H_{a_1} \setminus X_{a_1} \rightsquigarrow H_{(a_1,1)}, H_{(a_1,2)}, H_{(a_1,3)}, H_{a_3} \setminus X_{a_3} \rightsquigarrow H_{(a_2,1)}$ and $H_{(X_{a_1} \cup X_{a_2})} \rightsquigarrow H_1, H_2, H_3$ are represented by “embeddings of little disks into big disks”, in such a way that the little disks represent the corresponding connected components, and their complements in big disks are the removed sets. This allows to visualise the induced teams $\tau, X_{a_1} \cup X_{a_2} \rightsquigarrow \tau_1, \tau_2, \tau_3$ as cobordisms in the interior of $\tau$.

Lemma 1. A preteam $(\{H_a \mid a \in A\}, H)$ is strict if, for all $a \in A$ and $e \in H_a$, $e$ is connected in $H$. Also, in the above definition of strict team, it holds that $H_{\tilde{a}}$ is connected in $H$, for all $\tilde{a} \in \tilde{A}$.

Proof. We shall prove the equivalence $(1) \iff (2) \iff (3)$, where $(1)$ is the definition of strict team given above, $(2)$ is the characterisation claimed in the statement, and $(3)$ is the definition of team above enhanced with the additional property claimed in the statement.

- $(1) \implies (2)$. If $(\{H_a \mid a \in A\}, H)$ is a strict team in the sense of the definition given above, then, in particular, for each $a \in A$ and $e \in H_a$, taking $B = A$, $X_b = H_b$ for $b \neq a$ and $X_a = (H_a \setminus e)$, we get that $H_a, X_a \rightsquigarrow e$, and hence that $e$ is included in a connected component $K$ of $H \setminus (\bigcup_{b \in B} X_b)$. But for our choice of $B$, we have $H \setminus (\bigcup_{b \in B} X_b) = e$, hence this forces $K = e$, and a fortiori $e$ is connected in $H$.

- $(2) \implies (3)$. Let $\tilde{a} \in \tilde{A}$ and $\tilde{e} \in H_{\tilde{a}}$. Then a fortiori $\tilde{e} \in H_{\pi(a)}$, where $\pi : \tilde{A} \to A$ is defined by $\pi(\tilde{a}) = \tilde{a}$ if $\tilde{a} \in A \setminus B$ and $\pi(b, i) = b$. Since we assume $(2)$, we have that $\tilde{e}$ is connected in $H$. Thus, all hyperedges of $H_{\tilde{a}}$ are connected in $H$. By standard connectedness arguments, this, together with the fact that $H_{\tilde{a}}$ is connected, implies that $H_{\tilde{a}}$ is connected in $H$: informally, every path of hyperedges of $H_{\tilde{a}}$ witnessing the connexity of $H_{\tilde{a}}$, for arbitrary chosen vertices in $H_{\tilde{a}}$, can be turned into a path of hyperedges of $H$ witnessing the connexity of $H_{\tilde{a}}$ in $H$ for the same chosen vertices.

- $(3) \implies (1)$. Obvious.
Note that, for each $\emptyset \neq B \subseteq A$ and the choice of $\emptyset \neq X_b \subseteq H_b$ for each $b \in B$, the structure of a strict team $\tau$ implies the existence of a surjective function
\[ \varphi^B_\tau(X_b|b \in B) : A \to \{1, \ldots, n_B\} \] (written $\varphi^B_\tau$ for short),
which associates to $\tilde{a} \in A$ the index of the connected component of $H_b \setminus \bigcup_{b \in B} X_b$ that contains $H_a$. By Lemma II this determines preteams
\[ \tau_i = (\{H_a | \tilde{a} \in A$ and $\varphi^B_\tau(\tilde{a}) = i\}, H^B_a) \quad (1 \leq i \leq n_B). \]

We summarise this by the notation $\tau, \bigcup_{b \in B} X_b \supseteq A \supseteq \tau_1, \ldots, \tau_n$.

A strict clan is a set $\Xi$ of strict teams such that, for each team $\tau_i \in \Xi$, and each situation $\tau, \bigcup_{b \in B} X_b \supseteq \tau_1, \ldots, \tau_n$ as above, we have that $\tau_i \in \Xi$ for all $i$. In order to ease the understanding of the decomposition $\tau_i : \bigcup_{b \in B} X_b \supseteq \tau_1, \ldots, \tau_n$, in Figure 4.4 we suggest an interpretation of preteams and strict teams in terms of cobordisms. The set of all (strict) teams in the universes of associahedra, permutohedra ans friezohedra, respectively, form strict clans.

Let us fix a strict clan $\Xi$, and some $q \in \mathbb{R}$ (our product will be parameterised by $q$, cf. end of §2). A $\Xi$-delegation (or delegation for short) is a pair $\delta = (\{C_a : H_a | a \in A\}, H)$, such that $\tau := (\{H_a | a \in A\}, H) \in \Xi$. We say that $\tau$ is the support of $\delta$, and that $C_a$ is the construct of $\delta$ at position $a$. Observe that, for $\emptyset \neq B \subseteq A$ and $A$ as above, assuming that $X_a$ is the root vertex of $C_a$ for each $a \in A$, there is a canonical association of a construct $C_a$ to each $\tilde{a} \in A$, which gives rise to delegations
\[ \delta^B_{\tilde{a}} = (\{C_a : H_a | \tilde{a} \in A$ and $\varphi^B_\tau(\tilde{a}) = i\}, H^B_a), \quad (4.2) \]
for $1 \leq i \leq n_B$. More precisely, for $b \in B$, we set $C_b = X_b(C_{(b,1)}, \ldots, C_{(b,n_b)})$ with $C_{(b,i)} : H_{(b,i)}$. We summarise this by the notation $\delta, \bigcup_{b \in B} X_b \supseteq \delta^B_1, \ldots, \delta^B_{n_B}$.

4.2. Anthropological shuffle product of delegations of strict clans. We now define the anthropological shuffle product $*(\delta)$, for a $\Xi$-delegation $\delta$, where $\Xi$ is a strict clan. Until [4.4] we shall omit the adjective “strict” for brevity, but its presence is understood. A linear construct of a hypergraph $H$ is an element of the vector space spanned by all the constructs of $H$. We shall denote linear constructs with bold capital letters, e.g., $C = \sum_{i \in I} \lambda_i C_i$, where $C_i : H$, for each $i \in I$, and the notation $C : H$ will mean that $C$ is a linear construct of $H$. We take the convention to denote a linear construct of the form $\sum_{i \in I} \lambda_i X(C_1, \ldots, C_j, \ldots, C_n)$ by $X(C_1, \ldots, \sum_{i \in I} \lambda_i C^j_1, \ldots, C_n)$. A rooted linear construct is a linear construct of the form $C = X(\{C_a | a \in A\}$, and we write root$(C) = X$.

The anthropological shuffle product (or anthropological product) of a delegation $\delta = (\{C_a : H_a | a \in A\}, H)$, with root$(C_a) = X_a$ for all $a \in A$, is the linear construct of $H$ defined recursively as follows (with $\delta^B_1, \ldots, \delta^B_{n_B}$ as in [4.2]):
\[ * (\delta) = \sum_{\emptyset \subseteq B \subseteq A} q^{|B| - 1} *_B (\delta), \quad \text{where } *_B (\delta) = \bigcup_{b \in B} (X_b)(* (\delta^B_1), \ldots, *(\delta^B_{n_B})). \quad (4.3) \]
This shuffle product applied to associahedra and permutohedra are exactly the ones introduced by Burgunder and Ronco in [BR10].

When dealing with the associativity of the anthropological product in Theorem I below, we shall have to take anthropological products of (delegations made of) linear constructs, which is not a problem, as the above definitions of $*$, $*_B$ of course extend by linearity (with the notion of delegation accordingly extended to linear
the following lemmas show two situations in which the linear extension of $\ast_B$ still satisfies its “defining” equation \[10\] (now a property!). To see the need for such lemmas, note that the definitions of the delegations $\delta^B_i$ do depend on the root of the constructs $C_b$ ($b \in B$), which no longer exists if $C_b$ is a linear construct that is not rooted.

**Lemma 2.** Let $\langle \{ H_a \mid a \in A \}, H \rangle$ be a strict team, and suppose that we are given rooted linear constructs $C_a$ for each $a \in A$ with root $X_a$, forming a delegation $\delta$ (in the extended sense). Let $\emptyset \subset B \subseteq A$ and let $X_B = \bigcup_{b \in B} X_b$. Then we have $\ast_B(\delta) = \left( \bigcup_{b \in B} X_b \right) (\ast(\delta^B_1), \ldots, \ast(\delta^B_{n_B}))$, with the same definition of the teams $\delta^B_i$ as above.

**Proof.** We first notice that we can indeed still define $\delta^B_i$ as before, since the only information used on constructs are their roots. Let us assume for simplicity that only one of the $C_a$, say $C_{b_0}$, is a rooted linear construct, all the others being plain constructs, and that $b_0 \in B$, as Lemma \[3\] will a fortiori cover the case where $b_0 \notin B$. We shall also assume for simplicity that $C_{b_0} = X_{b_0} \cup \{ C_{b_0,i} \mid 1 \leq i \leq n_{b_0} \}$ only one of the $C_{b_0,i}$, say $C_{b_0,i_0} = \bigcup_{k \in K} \lambda_k C_{b_0,i_0,k}$, is a linear construct, all the others being plain constructs (and we write then $C_a = C_a$ for $a \neq b_0$ and $C_{b_0,i} = C_{b_0,i}$ for $i \neq i_0$). Then, by “outward” linearity, we can write $C_{b_0} = \bigcup_{k \in K} \lambda_k C_{b_0,k}$, where $C_{b_0,k} = X_{b_0} \cup \{ C_{b_0,i} \mid i \neq i_0 \}$.

By definition, we have

\[ \ast_B(\{ C_a \mid a \in A \backslash \{ b_0 \} \}) \bigcup \{ C_{b_0} \}, H \rangle = \bigcup_{k \in K} \lambda_k X_k \ast(\{ \delta^B_k \}) \bigcup \{ C_{b_0,k} \}, H \rangle, \]  

where for all $j \neq j_0 = \varphi(\tau(b_0, i_0))$, all $\delta^B_j$ are equal to $\delta^B_{j_0}$, and where the $\delta^B_{j_0}$ differ only in one (and the same) position (the one indexed by $(b_0, i_0)$), filled with $C_{b_0,i_0,k}$. Then we conclude by applying “inward” linearity.

**Lemma 3.** Let $\langle \{ H_a \mid a \in A \}, H \rangle$ be a strict team, and let $a_0 \in A$, and suppose that we are given constructs $C_a : H_a$ with root $X_a$ for all $a \neq a_0$, and a linear construct $C_{a_0}$. Let $B \subseteq A \backslash \{ a_0 \}$, and let $X_B = \bigcup_{b \in B} X_b$. Then we have $\ast_B(\tau) = \left( \bigcup_{b \in B} X_b \right) (\ast(\tau^B_1), \ldots, \ast(\tau^B_{n_B}))$, with the same definition of the teams $\tau^B_i$ as above.

**Proof.** The proof goes like in Lemma \[2\]. The only difference is that, under the assumption that $a_0 \notin B$, no information at all is required on $C_{a_0} = \bigcup_{k \in K} \lambda_k C_{a_0,k}$.

So far, we have a magmatic unbiased notion of product, by which we mean that the product can take more than two arguments. We seek unbiased associativity. A clan $\Xi$ is **associative** if for all $\tau = \{ H_a \mid a \in A \}, H \in \Xi$, $a_0 \in A$ and $\tau' = \{ H_{a_0,a'} \mid a' \in A' \}, H_{a_0} \in \Xi$, we have $\tau'' := \{ H_a \mid a \in A \backslash \{ a_0 \} \} \cup \{ H_{a_0,a'} \mid a' \in A' \}, H \in \Xi$. We shall refer to $\tau''$ as the grafting of $\tau'$ to $\tau$ along $a_0$. The following theorem establishes the associativity of the anthropological product for strict associative clans.

**Theorem 1.** Let $\Xi$ be an associative clan, and suppose that $\tau = \{ H_a \mid a \in A \}, H \in \Xi$, $a_0 \in A$, and $\tau' = \{ H_{a_0,a'} \mid a' \in A' \}, H_{a_0} \in \Xi$, and that we are given constructs $C_a : H_a$ for all $a \in A \backslash \{ a_0 \}$ and constructs $C_{(a_0,a')} : H_{(a_0,a')}$. 
for all all \(a' \in A'\). Taking \(\tau''\) to be the grafting of \(\tau'\) to \(\tau\) along \(a_0\) and setting \(A'' := (A \setminus \{a_0\}) \cup \{(a_0, a') \mid a' \in A'\}\), denote the corresponding delegations by \(\delta'' = (\tau'', \{C_{a''} \mid a'' \in A''\})\) and \(\delta' = (\tau', \{C_{(a_0, a')} \mid a' \in A'\})\). We then have that, for each \(\emptyset \neq B'' \subseteq A''\), the following polydendriform equation holds:

\[
\ast_{B''}(\delta'') = \begin{cases} 
\ast_{B''}^\ast \{C_a \mid a \in A \setminus \{a_0\}\} \cup \{\ast^\ast(\delta')\}, & \text{if } B'' \subseteq A \setminus \{a_0\} \\
\ast_B \{C_a \mid a \in A \setminus \{a_0\}\} \cup \{\ast_{B'}(\delta')\}, & \text{if } B'' \not\subseteq A \setminus \{a_0\}
\end{cases}
\]

where \(B = (B'' \cap (A \setminus \{a_0\})) \cup \{a_0\}\), \(B' = \{a' \in A' \mid (a_0, a') \in B''\}\) and the superscripts record the respective support teams. Moreover, the polydendriform equation implies the following associativity equation:

\[
\ast^\ast(\delta'') = \ast^\ast(\{C_a \mid a \in A \setminus \{a_0\}\} \cup \{\ast^\ast(\delta')\}).
\]

**Remark 1.** The polydendriform equations can be drawn as:

\[
\begin{align*}
\ast_{B''}(\delta'') & = \begin{cases}
\ast_{B''}^\ast \{C_a \mid a \in A \setminus \{a_0\}\} \cup \{\ast^\ast(\delta')\}, & \text{if } B'' \subseteq A \setminus \{a_0\} \\
\ast_B \{C_a \mid a \in A \setminus \{a_0\}\} \cup \{\ast_{B'}(\delta')\}, & \text{if } B'' \not\subseteq A \setminus \{a_0\}
\end{cases} \\
\ast^\ast(\delta'') & = \ast^\ast(\{C_a \mid a \in A \setminus \{a_0\}\} \cup \{\ast^\ast(\delta')\}).
\end{align*}
\]

**Proof.** We set \(\delta = (\tau, \{C_a \mid a \in A \setminus \{a_0\}\} \cup \{\ast^\ast(\delta')\})\). We first show the polydendriform equation. We proceed by induction on \(|H|\). Figure 4.5 will help the reader to visualise the notations introduced in case (2) of the proof. Denote, for each \(a'' \in A''\), \(X_{a''} := \text{root}(C_{a''})\). By definition of the operation \(\ast_{B''}\), supposing that \(H, X_{B''} \leadsto H_{1,B''}, \ldots, H_{n_{B''},B''}\), where \(X_{B''} = \bigcup_{v'' \in B''} X_{v''}\), we have that

\[
\ast_{B''}(\delta'') = X_{B''}(\ast((\delta'')_1^{B''}), \ldots, \ast((\delta'')_{n_{B''}}^{B''})),
\]

where, for \(1 \leq i \leq n_{B''}\),

\[
(\delta'')_i^{B''} = (\{C_{\tilde{a}''} : H_{\tilde{a}''} \mid \tilde{a}'' \in \tilde{A}''\text{ and } \varphi^{B''}_{\tilde{a}''}(\tilde{a}'') = i\}, H_i^{B''})
\]

with the indexing set

\[
\tilde{A}'' := A'' \setminus B'' \cup \{(b'', q) \mid b'' \in B'' \text{ and } 1 \leq q \leq n_{B''}\}
\]

arising from the decompositions \(H_{v''}, X_{v''} \leadsto H_{(v'', 1)}, \ldots, H_{(v'', n_{v''})}, b'' \in B''\). We examine the two cases of the statement in turn.

1. If \(B'' \subseteq A \setminus \{a_0\}\), then, setting \(C_{a_0} := \ast^\ast(\delta')\), we have (using Lemma 3):

\[
\ast_{B''}(\delta) = X_{B''}(\ast((\delta'')_1^{B''}), \ldots, \ast((\delta'')_{n_{B''}}^{B''})),
\]

where, for \(1 \leq l \leq n_{B''}\),

\[
\delta_l^{B''} = (\{C_{\tilde{a}} : H_{\tilde{a}} \mid \tilde{a} \in \tilde{A}\text{ and } \varphi^{B''}_\tilde{a}(\tilde{a}) = l\}, H_l^{B''})
\]

with the indexing set

\[
\tilde{A} := A \setminus B'' \cup \{(b, p) \mid b \in B'' \text{ and } 1 \leq p \leq n_b\}
\]
arising from the decompositions $H_b, X_b \leadsto H(b,1), \ldots, H(b,n_b)$, $b \in B''$. Then, establishing $\ast_{B''}^\tau(\delta'') = *_{B''}(\delta')$ amounts to showing that $\ast((\delta'')^B') = \ast(\delta''_l)$, for all $1 \leq l \leq n_{B''}$.

Let $\pi'' : \tilde{A}'' \rightarrow A''$ and $\pi : \tilde{A} \rightarrow A$ be the obvious projections (cf. proof of Lemma 1). Then it is readily seen (remembering that $H(a_0,a') \subseteq H_{a_0}$) that $(\pi'')^{-1}(A\setminus\{a_0\}) = \tilde{A}'' \cap \tilde{A} = \pi^{-1}(A\setminus\{a_0\})$ and

$$\varphi_{\tau}^B|_{\tilde{A}'' \cap \tilde{A}} = \varphi_{\tau}^B|_{\tilde{A}'' \cap \tilde{A}} \quad \text{and} \quad \varphi_{\tau}^B(a_0) = \varphi_{\tau}^B(a_0, a'),$$

for all $a' \in A'$. It follows that for $l \neq \varphi_{\tau}^B(a_0) := l_0$ we have that $(\delta'')^B_l = \delta''^B_l$, while (remembering the definition of $C_{a_0}$) the equality $\ast((\delta'')^B_l) = \ast(\delta''^B_l)$ follows by induction on $H_{a_0}$.

Figure 4.5. Illustration of associativity via cobordisms. We have: $A = \{a_0, \ldots, a_3\}$, $A' = \{b_0, \ldots, b_2\}$, $A'' = \{a_1, \ldots, a_3, (a_0, b_0), (a_0, b_1), (a_0, b_2)\}$, $B'' = \{a_2, (a_0, b_1)\}$, $B' = b_1$ and $B = a_2, a_0$. 
(2) For \( B'' \not\subseteq A\{a_0\}, \) let us denote \( B := (B'' \cap (A \setminus \{a_0\})) \cup \{a_0\} \) and \( B' := \{a' \in A \mid \{a_0, a'\} \in B''\}. \) Denote \( X_{B'} := \bigcup_{b' \in B'} X_{\{a_0, b'\}} \) and suppose that \( H_{a_0}, X_{B'} \leadsto (H_{a_0})_{B'}^1, \ldots, (H_{a_0})_{m_{B'}}^1. \) We have by definition

\[
*_{B'} (\delta') = X_{B'}(*((\delta'))_{B'}^1, \ldots, *((\delta'))_{m_{B'}}^1),
\]

where, for \( 1 \leq j \leq m_{B'}, \)

\[
(\delta')_{B'}^j = (\{C_a : H_{a_j}^1 \mid a_j \in \tilde{A} \text{ and } \varphi_{a_j}^B (a_j) = j\}, (H_{a_0})_{B'}^j)
\]

with the indexing set

\[
\tilde{A} := \{(a_0, a') \mid a' \in A \setminus B' \}
\]

\( \cup \{(a_0, b', k) \mid b' \in B' \text{ and } 1 \leq k \leq n_{B'}\} \)

arising from the decompositions \( H_{a_0, b'}, X_{a_0, b'} \leadsto H_{a_0, b', 1}, \ldots, H_{a_0, b', n_{B'}}, \) \( b' \in B'. \) Setting \( C_{a_0} := X_{B'}(*((\delta'))_{B'}^1, \ldots, *((\delta'))_{m_{B'}}^1) \), the equality that we aim to prove displays as

\[
*_{B''} (\delta'') = *_{B'} (\{C_a \mid a \in A \setminus \{a_0\}\} \cup \{C_{a_0}^B\}).
\]  \[(4.5)\]

Furthermore, by setting \( X_{a_0} := X_{B'} \) and \( X_B := \bigcup_{b \in B} X_b, \) we can write

\[
X_{B''} = (\bigcup_{b \in B \setminus \{a_0\}} X_b) \bigcup \{X_{B'}\} = (\bigcup_{b \in B \setminus \{a_0\}} X_b) \bigcup \{X_{a_0}\} = X_B.
\]

We can then transform \((4.5)\) (applying Lemma \[2\]) to the right-hand side into

\[
X_B(*((\delta''))_{B''}^1, \ldots, *((\delta''))_{n_{B''}}^1) = X_B(*((\delta')_{B'}^1, \ldots, (\delta')_{n_{B'}}^1)),
\]  \[(4.6)\]

where \( H, X_B \leadsto H^1_B, \ldots, H_{n_{B'}}^1, n_B = n_{B''}, \) and for \( 1 \leq l \leq n_B, \)

\[
\delta_l^B = (\{C_a : H_a \mid \tilde{a} \in \tilde{A} \text{ and } \varphi_a^B (\tilde{a}) = l\}, H_l^B)
\]

with the indexing set

\[
\tilde{A} := A \setminus B \cup \{(b, p) \mid b \in B \text{ and } 1 \leq p \leq n_b\}
\]

arising from the decomposition \( H_b, X_b \leadsto H_{(b, 1)}, \ldots, H_{(b, n_b)} \), \( b \in B, \) and where

\[
C_{\tilde{a}} = \begin{cases} C_a, & \text{ if } \tilde{a} \in (A \setminus B), \\ *((\delta')_{B'})^p, & \text{ if } \tilde{a} = (a_0, p), \\ C_{(b, p)}, & \text{ if } \tilde{a} = (b, p) \quad (b \in B \setminus \{a_0\}). \end{cases}
\]

Now, since \( X_{B''} = X_B, \) we can suppose, without loss of generality, that \( H_i^{B''} = H_i^B, \) for all \( 1 \leq i \leq n_B = n_{B''}. \) Therefore, it remains to show that

\[
*_{((\delta''))_{B''}^i} = *_{(\delta_l^B)}.
\]

Observe that, since

\[
H_{(a_0, 1)}, \ldots, H_{(a_0, n_{a_0})} \leadsto H_{a_0} \setminus X_{a_0} = H_{a_0} \setminus X_{B'} \leadsto (H_{a_0})_{B'}^1, \ldots, (H_{a_0})_{m_{B'}}^1,
\]

we have that \( n_{a_0} = m_{B'}, \) and we can assume (without loss of generality) that \( H_{(a_0, p)} = (H_{a_0})_{B'}^p, \) for each \( 1 \leq p \leq n_{a_0}. \) Simple inspection (and standard argumentation with connected components) yields

\[
\pi^{n_{a_0} - 1}(A \setminus \{a_0\}) = \tilde{A} \cap \tilde{A} = \pi^{-1}(A \setminus \{a_0\})
\]

\[
\pi^{n_{a_0} - 1}(a_0) = \tilde{A},
\]
where $\pi'' : \tilde{A}' \to A''$ and $\pi : \tilde{A} \to A$ are the obvious projections, and

$$\varphi''_{\tilde{A}'\backslash A} = \varphi''_{\tilde{A}\backslash A'} \quad \text{and} \quad \varphi''_{\tilde{A}'\backslash A} = \varphi''_{\tilde{A}'\backslash A''} \circ \varphi''_{\tilde{A}'\backslash A}.$$ \hspace{1cm} (4.7)

where $\varphi''_{\tilde{A}'\backslash A} (j) := \varphi''_{\tilde{A}'\backslash A} (\{a_0, j\})$, for each $1 \leq j \leq n_{a_0}$. We note that, thanks to (4.7), $\ast ((\delta''_{B''})_i)$ and $\ast (\delta''_i)$ look like this, respectively:

$$\ast ((\delta''_{B''})_i) = \ast (\ldots, C_y, \ldots, \ldots, \ldots),$$

with $B, B'$ determined from $B''$ as specified in the statement, and on the other hand (expanding the second summand by linearity):

$$\ast \tau (\delta) = \sum_{\emptyset \subseteq B \subseteq A \backslash \{a_0\}} \ast B \ast (\delta) + \sum_{\emptyset \subseteq B \subseteq A \backslash \{a_0\}} \sum_{\emptyset \subseteq B' \subseteq A'} q^{\mid B'' \mid - 1} \ast B \ast (\delta_{B'})$$

and we conclude by applying induction to each $H''_i$. We shall now relate the polydendriform structure to the tridendriform one, by showing that the former implies (and can be considered as the unbiased version of) the latter, in the ordered anthropological framework. We first define the notion of ordered (strict) universe, preteam, team and clan. We suppose given an ordered set, say $\mathbb{Z}$. For $X_1, X_2 \subseteq \mathbb{Z}$, we write $X_1 < X_2$ if $\max (X_1) < \min (X_2)$. An ordered universe is a universe $\mathcal{U}$ such that, for all $H \in \mathcal{U}$, $H \subseteq \mathbb{Z}$, and such that all decompositions $H, X \twoheadrightarrow H_1, \ldots, H_p$ can be indexed in such a way that $H_i < H_{i+1}$ for all $i$. An ordered preteam is a pair $((H_1, \ldots, H_p), \mathcal{H})$ such that $((H_1, \ldots, H_p), \mathcal{H})$ is a preteam and such that $H_1 < \cdots < H_p$. Ordered teams are teams whose underlying preteam is ordered. Note that when $\tau$ is ordered, if $\tau \in B \subseteq X_0$, then each $\tau_i$ is ordered (to see this, one uses the assumption that $\mathcal{U}$ is ordered). An ordered clan is a clan whose teams are all ordered.

**Remark 2.** One could formulate the polydendriform structure as an algebra over a coloured operad, where the colours are hypergraphs, the operations are teams, and the carrier of the algebra for the colour $\mathcal{H}$ is the set of constructs of $\mathcal{H}$.
Let $\Xi$ be an ordered associative clan. Suppose that we have

$$\{(H_1, H_2), (H_2, H_3), (H_3, H_1), ((H_1, H_2), H_3), ((H_1, H_3), H_2), ((H_1, H_3), H_2)\} \in \Xi.$$  

Denote by $\tau''_1$ the grafting of $((H_1, H_2), H_3)$ to $((H_1, H_3), H_2)$ along $1'$, and by $\tau''_2$ the grafting of $((H_2, H_3), H_1)$ to $((H_1, H_2), H_3)$ along $2'$. Note that the above teams are all of the (generic) form $((H_i, H_r), K)$. We write

$$\prec := *_{\{l\}} \cdot := *_{\{l, r\}} \succ := *_{\{r\}}$$

**Proposition 2.** In the ordered setting, the tridendriform equations follow from the polydendriform one. More precisely, Loday-Ronco’s seven equations (as listed in section 2) correspond to choosing $B''$ to be $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2, 3\}$, $\{2, 3\}$, $\{1, 3\}$, $\{1, 2\}$, respectively.

**Proof.** As a sanity check, we first note that there are $2^3 - 1 = 7$ non-empty subsets of $\{1, 2, 3\}$. We check the equation $\prec \cdot$. Let $S : H_1, T : H_2, U : H_3$. We have

$$*_{\{1, 2, 3\}}(S, T, U) = *_{\{1\}}(\prec_{\{1\}}(S, T, U) = ((S \cdot T) \cdot U)$$

and

$$*_{\{2, 3\}}(S, T, U) = *_{\{2\}}(S, T, U) = ((S \cdot (T, U))) = S \prec (T \cdot U).$$

Note that all tridendriform equations follow from the second case of the polydendriform equation, except $\prec \cdot$ and $\cdot \succ$ (for which we use the first case, and which are the only tridendriform equations involving $\ast$).

We end the section by giving examples of ordered associative clans that encompass associahedra and permutohedra. Fix a (possibly infinite) hypergraph $K$, and let $U_K$ be the universe consisting of all hypergraphs $K_X$, such that $X \subseteq K$ is non-empty and finite, and $K_X$ is connected. In Proposition 3 we shall characterise universes arising in this way from some hypergraph $K$.

The $K$-restrictohedron is the set $\Xi_K$ of all pairs $\{K_V | a \in A\}$, $K_V$ where $V \subseteq K$, $\{V_a\}_{a \in A}$ forms a partition of $V$, and the hypergraphs $K_{V_a}$ and $K_V$ are all in $U_K$. We can restrict this to an ordered setting if $K$ is order-friendly, meaning that $K \subseteq Z$ and that the connected components $K_{V_1}, \ldots, K_{V_n}$ of $K_V$, for any finite $V \subseteq Z$ such that $K_V$ is not connected, can be indexed in such a way that $V_i < V_{i+1}$ for all $i$.

**Proposition 3.** For all $K$, $\Xi_K$ is an associative clan. If $K$ is order-friendly, then the restriction of $\Xi_K$ (still denoted by $\Xi_K$) to its ordered preteams is an ordered associative clan.

**Proof.** We first note that every preteam $\{K_V | a \in A\}$, $K_V$ satisfies $\bigcup_{a \in A} K_{V_a} \subseteq K_V$ by definition, and hence, by Lemma 1, is a fortiori a strict team. Next, if (in the notation of Section 4) $\tau \setminus \bigcup_{b \in B} X_b \supseteq \tau_1, \ldots, \tau_n$, we have to prove that $\tau_i \in \Xi_K$ for all $i$. This follows from the fact that, for any $V$ and $W \subseteq V$, $(H_V)\setminus W = H_V\setminus W$ and that, for all $X \subseteq K$, the connected components of $K_X$ are all the form $K_Y$ for some $Y \subseteq X$. Finally, the clan is associative since $\Xi_K$ includes all “possible” preteams, in the sense that for any $X \subseteq K$ and any partition $\{X_a | a \in A\}$ of $X$, we have $\{K_{X_a} | a \in A\}, K_X \in \Xi_K$ if and only if $K_X$ and $K_{X_a}$ (for all $a \in A$) are connected.

Suppose now that $K$ is moreover order-friendly. Then it is immediate that $U_K$ is ordered. Since we limit ourselves to ordered preteams of the form $((H_1, \ldots, H_n), K_V)$ with $H_i = K_{V_i}$ and $V_i < V_{i+1}$ for all $i$, and since $K$ is order-friendly, then for all
For all \( k \geq 1 \in \mathbb{N} \cup \{ \infty \} \), the following graph is order-friendly:

\[
\Gamma_k := \{ \{a\} \mid a \in \mathbb{Z} \} \cup \{ \{a,a+l\} \mid a \in \mathbb{Z}, l \in \mathbb{N}, 1 \leq l \leq k \}.
\]

**Proof.** A subset \( V \subseteq \mathbb{Z} \) is not connected in \( \Gamma_k \) if and only if there is a set \( X \) of at least \( k \) consecutive integers in \( \lfloor \min(V); \max(V) \rfloor \), which does not intersect \( V \). If \( X_1, \ldots, X_p \) are the sets of such maximal sequences of consecutive integers, then the interval \( [\min(V), \max(V)] \) in \( \mathbb{Z} \) is the union of consecutive intervals \( I_0, X_1, I_1, \ldots, X_p, I_p \), and the connected components of \( \Gamma^k \) are \( (\Gamma^k)_{V \cap I_0}, \ldots, (\Gamma^k)_{V \cap I_p} \). Then \( (V \cap I_j) < (V \cap I_{j+1}) \) follows a fortiori from \( I_j < I_{j+1} \).

By Propositions 4 and 3 we get an induced associative ordered clan \( \Xi_{\Gamma^k} \) and hence a polydendriform and tridendriform structure for each \( k \). In the extreme cases \( k = 1 \) and \( k = \infty \), we have \( \Gamma_1^k = P^X \) (for \( X \) interval of \( \mathbb{Z} \)) and \( \Gamma_\infty^X = P^X \) (for finite \( X \subseteq \mathbb{Z} \)), respectively. The teams are of the form \( \{\{\Gamma_{X_1}, \ldots, X_p, \Gamma_{X_p}\}, \Gamma_{\bigcup X_i}\} \) (where the \( X_i \)‘s are adjacent intervals) and \( \{\{\Gamma_{X_1}, \ldots, X_p, \Gamma_{X_p}\}, \Gamma_{\bigcup X_i}\} \) (where \( X_i < X_{i+1} \) for all \( i < p \)), respectively. It is quite straightforward to check that the respective tridendriform structures obtained by instantiating Theorem 1 and Proposition 2 to \( \Xi_{\Gamma^1} \) and \( \Xi_{\Gamma^\infty} \) are those of \( \Xi_{P^X} \) thus fulfilling our unifying goal, with a whole infinity of examples sitting “in the middle”.

**Proposition 5.** A universe \( \mathcal{U} \) is of the form \( \mathcal{U}_K \), for some hypergraph \( K \), if and only if it satisfies the following four conditions:

1. For any hypergraphs \( H_1 \) and \( H_2 \) in \( \mathcal{U} \), if \( H_1 = H_2 \), then \( H_1 = H_2 \).
2. If \( H \in \mathcal{U} \) and \( e \in H \), if \( G \in \mathcal{U} \) is such that \( e \subseteq G \), then \( e \in G \).
3. If \( H \in \mathcal{U} \), and if \( X \subseteq H \) is such that \( H_X \) is connected, then there exists \( G \in \mathcal{U} \) such that \( G = X \).
4. If \( K, L \in \mathcal{U} \) are such that \( K \cap L \) is non-empty, then there exists \( H \in \mathcal{U} \) such that \( K, L \subseteq H \).

**Proof.** We first check that any universe of the form \( \mathcal{U}_K \) satisfies the conditions in the statement. Condition (1) is immediate. Conditions (2), (3) and (4) follow immediately from the observations that, by definition, for arbitrary \( X \), we have \( e \in K_X \) if and only if \( e \in K \) and \( e \subseteq X \), that \( (K_H)_X = K_X \), and that the union of two connected sets with a non-empty intersection is connected.

Conversely, suppose that \( \mathcal{U} \) satisfies the four conditions of the statement. We set \( K = \bigcup \{H \mid H \in \mathcal{U}\} \). We shall show the following two properties, which (together with (1)) imply immediately that \( \mathcal{U} = \mathcal{U}_K \).

(a) If \( X \) is a finite set such that there exists a hypergraph \( H \) such that \( H = X \) and \( H \in \mathcal{U} \), then there exists a hypergraph \( H' \in \mathcal{U}_K \) such that \( H' = X \) and \( H \subseteq H' \).

(b) If \( X \) is a finite set such that there exists a hypergraph \( H' \) such that \( H' = X \) and \( H' \in \mathcal{U}_K \), then there exists a hypergraph \( H \in \mathcal{U} \) such that \( H = X \) and \( H' \subseteq H \).
For (a), we note that \( H \subseteq K \) by definition of \( K \), hence \( H \subseteq K_H \), so we can set \( H' := K_H \), noticing that \( K_H \) is connected since it contains a connected hypergraph (namely \( H \)) with the same set of vertices.

We now proceed to prove (b). By definition of \( \mathcal{U}_K \), the assumptions of (b) can be rephrased as saying that \( H' = K_X \) is connected. Also, by definition of \( K \), for each \( e \in K_X \), there exists a hypergraph \( H_e \in \mathcal{U} \) such that \( e \in H_e \). So we have \( K_X \subseteq \bigcup \{ H_e \mid e \in K_X \} \), this union being finite since \( X \) is. Suppose that \( X \) has more than one hyperedge and pick \( e_0 \in K_X \). We claim that there exists \( e_1 \in K_X \) such that \( H_{e_0} \cap H_{e_1} \) is non-empty. If it were not the case, then \( K_X \) would be the disjoint union of \( K_X \cap H_{e_0} \) and of \( K_X \cap \{ H_e \mid e \neq e_0 \} \), which would contradict the connectedness of \( K_X \). We can thus replace \( \{ H_e \mid e \in K_X \} \) by \( \{ H_e \mid e \neq e_0, e_1 \} \cup \{ H_{e_1} \} \), where \( H_{e_1} \in \mathcal{U} \) is obtained from \( H_{e_0} \) and \( H_{e_1} \) by applying (4). By iterating this, we obtain a hypergraph \( H' \in \mathcal{U} \) such that \( K_X \subseteq H' \).

Note that we can write this as well as \( K_X \subseteq H' X \), and that, as above, we have that the connectedness of \( K_X \) implies the connectedness of \( H' X \).

Our next (independent) observation is that in the presence of (2), condition (3) can be reinforced as follows. If \( H \in \mathcal{U} \) and if \( X \subseteq H \) is such that \( H_X \) is connected, then there exists \( G \in \mathcal{U} \) such that \( G = X \) and \( H_X \subseteq G \). Indeed, let \( G \) be obtained by applying (3), and let \( e \in H_X \); then this latter assumption reads as \( e \subseteq G \), and hence \( e \in G \) by (2).

Coming back to the proof of (b), we can apply the reinforced version of (3) to deduce the existence of a hypergraph \( H \in \mathcal{U} \) such that \( H = X \) and \( H' X \subseteq H \). We thus have \( K_X \subseteq H' X \subseteq H \), which concludes the proof.

4.3. A non-recursive definition of the anthropological product. In this subsection, we give an equivalent, non-recursive, definition of the anthropological product, directly inspired from \cite{Ron12}. Let \( H, K \) be two connected hypergraphs such that \( H \subseteq K \) and such that, for all \( e \in H \), \( e \) is connected in \( K \). This entails in particular that \( H \) is connected in \( K \). Let \( S = X(S_1, \ldots, S_n) \) be a construct of \( K \), with \( S_i := K_i \) where \( K, X \rightsquigarrow K_1, \ldots, K_n \). Then we define a construct \( S|_H \) of \( H \) as follows. We distinguish two cases:

- if \( X \cap H = \emptyset \), then there is a unique \( j \) such that \( H \subseteq K_j \), and we set
  \[
  S|_H = (S_j)|_H.
  \]

- if \( X \cap H \neq \emptyset \), let \( H, (X \cap H) \rightsquigarrow H_1, \ldots, H_p \). This determines a function
  \( \varphi^{H,K}_X : \{1, \ldots, p\} \to \{1, \ldots, n\} \), and we set
  \[
  S|_H = (X \cap H)(\ldots, (S_{\varphi^{H,K}_X(i)})|_{H_i}, \ldots).
  \]

That \( S|_H \) is indeed a construct of \( H \) is easily seen by induction.

In the next lemma, we give a simpler (but more “mysterious”) alternative description of \( S|_H \) in terms of tubings. Recall the following notations from \cite{CO19}. For every node \( Y \) of \( S \), we denote by \( \uparrow_S(Y) \) (or simply \( \uparrow(Y) \)) the union of the labels of the descendants of \( Y \) in \( S \) (all the way to the leaves), including \( Y \). By definition of constructs, \( \uparrow_S(Y) \) is always connected in \( K \). We then associate with
• The compartments with red/blue border are the connected components of $K \backslash X$.
• The compartments with green/red/blue border are the connected components of $H \backslash X$.
• In this example, we have $(\varphi_X^H K)^{-1}(j) = \{2, 3\}$.
• The small yellow compartments with orange/green borders feature the tubes in $\psi(t_H)$.
• while those additionally marked with a dot are the tubes in $\psi(t_{H_2})$.

Figure 4.6. Illustration of the proof of Lemma 4

$S$ the following set of connected subsets, or tubing$^3$(cf. footnote$^1$):
$$\psi(S) = \{\uparrow(Y) \mid Y \text{ is a (label of a) node of } S\}.$$ Alternately, the function $\psi$ is defined recursively by
$$\psi(X(S_1, \ldots, S_n)) = \{K\} \bigcup \bigcup_{i=1, \ldots, n} \psi(S_i)) .$$

We need one definition (adapted to the setting of hypergraphs from $^{[Ron12]}$). With each $t$ connected in $K$ ($t$ is also called a tube), we associate a construct $t_H$ as follows (note the heterogeneous nature of this definition: we go from tubes to constructs):

• If $H \subseteq t$, then we set $t_H = H$;
• if $H \backslash t \neq \emptyset$ yielding $H, (H \backslash t) \rightsquigarrow H_1, \ldots, H_k$, we set $t_H = (H \backslash t)(H_1, \ldots, H_k)$.

This definition can be seen as an instantiation of our definition of $S_{H}$: more precisely, we can coerce a tube $t$ of $K$ to a construct $(K \backslash t)(t) : K$, and we have $t_H = ((K \backslash t)(t))_{H}$.

The following lemma asserts that $S_{H}$, viewed as a tubing, is entirely determined by the restrictions of the tubes of $S$, thus providing a non-recursive definition for this restriction operation.

Lemma 4. For $H, K$ and $S$ as above, we have $\psi(S_{H}) = \bigcup \{\psi(t_H) \mid t \in \psi(S)\}$.

Proof. (Sketch) Let $S = X(S_1, \ldots, S_n)$ and $K \neq t \in \psi(S)$, i.e. $t \in \psi(S_j)$ for some $j$. Then the statement follows from the observation (illustrated in Figure 4.6) that, with the notation introduced above:
$$\psi(t_H) = \bigcup \{\psi(t_{H_i}) \mid \varphi_X^{H \cdot K}(i) = j\} \quad (j, t \in \psi(S_j) \text{ fixed, } i \text{ varying}).$$

$^2$We refer to $^{[COI19]}$Proposition 2 for an exact characterisation of inductively defined constructs as tubings. We just note here that the function $\psi$ defined above provides a bijection from constructs to tubings.
Indeed, by definition of $\psi$, we have on one hand that $(\bigcup \{\psi(t_H) \mid t \in \psi(S)\}) \setminus \{H\}$ is the union of the sets $(\bigcup \{\psi(t_H) \mid t \in \psi(S_j)\})$, indexed by $1 \leq j \leq n$. On the other hand, applying induction, we have that $\psi(S_H) \setminus \{H\}$ is the union of the sets $\psi(t_H)$, for $1 \leq i \leq p$ and $t \in \psi(S(X)_{r_X}^{K}(i))$, which we can repackage as a union indexed by $j$ (gathering all $i$ such that $\varphi_X^{H,K}(i) = j$). We then conclude by the observation.

In particular, via the characterisation of tubings as constructs, the lemma says that the definition in terms of tubings given in [Ron12] returns indeed a tubing.

We now come back to the promised alternative definition of the anthropological product. Let $\tau = (\{H_a \mid a \in A\}, H)$ be a team and $U = X(U_1, \ldots, U_n)$ be a construct of $H$. We associate with $U$ a “measure” $\mu^\tau(U)$ as follows (with the notation of §1). We set $B = \{b \in A \mid X \cap H_b \neq \emptyset\}$ and $X_b = X \cap H_b$ for each $b \in B$ (so that $n = n_B$), and we set

$$
\mu^\tau(U) = (|B| - 1) + \sum_{1 \leq i \leq n_B} \mu^\tau(U_i).
$$

The following proposition gives a non-inductive characterisation of our product $\ast$.

**Proposition 6.** Let $\delta = (\{C_a : H_a \mid a \in A\}, H)$ be a delegation of support $\tau$. Then we have:

$$
\ast(\delta) = \sum_{U : H \text{ and } \forall a \in A \cup U \setminus H_a = C_a} \mu^\tau(U),
$$

and for each $\emptyset \neq B \subseteq A$, we have that $q^{\mu^\tau(U)} | \ast_B(\delta) |$ is the summand of the above sum where $U$ is further constrained to be such that root$(U) = X_B$.

**Proof.** (Sketch) We use the same notations as above. By unfolding the definition of $U = X(U_1, \ldots, U_n)$, with $X = X_B$, the constraints on $U$ boil down to the constraints (for each $i$) $(U_i)_{H_a} = C_a$ for all $a \in A$ such that $\varphi_B^a(\tilde{a}) = i$. This entails that, taking the right-hand side of the equality and its summands in the statement as a definition of $\ast$ and $\ast_B$, and noticing that

$$
q^{\mu^\tau(U)} X_B(U_1, \ldots, U_n) = q^{\mu^\tau(U)} X_B(U_1, \ldots, U_i, \ldots)
$$

these definitions satisfy the equation $\ast_B(\delta) = (\bigcup_{b \in B} X_b)(\ast_B^1(\delta), \ldots, \ast_B^{n_B})$. ■

We note that the non-recursive definition leads to another proof of the polydendriform equation and of associativity — that is technically very simple but geometrically less appealing than the one we gave in §12, based on the observation, say for $(\{H_1, H_2, H_3\}, H)$, $(\{H_1, H_2\}, H_12)$, $(\{H_2, H_3\}, H)$, and

$$
\delta = (\{S : H_1, T : H_2, U : H_3\}, H),
$$

that the data of $V : H$ such that $V_{H_1} = S$, $V_{H_2} = T$ and $V_{H_3} = U$ is equivalent to the data of $V : H$ and $W : H_{12}$ such that $W_{H_1} = S$, $W_{H_2} = T$, $V_{H_{12}} = W$, and $V_{H_{12}} = U$.\(^3\)

\(^3\)In turn, this observation relies on the composability of restrictions, i.e., one can prove that $(V_{H_{12}})_{H_1} = V_{H_1}$. 
4.4. Extending the anthropological framework. In this section, we enlarge the coverage of our formalism of teams and clans, and we adapt the anthropological product accordingly, in order to cover other families of polytopes like simplices or hypercubes.

A preteam $\tau = (\{H_a \mid a \in A\}, H)$ is called a semi-strict team if for each choice of a subset $\emptyset \neq B \subseteq A$ and of a subset $\emptyset \neq X_b \subseteq H_b$ for each $b \in B$, we have that, for each $\tilde{a} \in A$,

1. $H_{\tilde{a}}$ is included in a connected component of $H \setminus (\bigcup_{b \in B} X_b)$, or
2. $|H_{\tilde{a}}| \geq 2$, and, for all $x \in H_{\tilde{a}}$, $\{x\}$ is a connected component of $H \setminus (\bigcup_{b \in B} X_b)$,

where $A$ is as in §4.1. Let us denote with $\tilde{A}_d$ the set of elements $\tilde{a}$ of $A$ such that case (2) applies. We define $\tilde{A}$ by removing from $\tilde{A}$ all elements $\tilde{a}$ of $\tilde{A}_d$ and replacing them by the elements of $H_{\tilde{a}}$ (thus expressing the atomisation of $H_{\tilde{a}}$), for all $\tilde{a} \in \tilde{A}_d$, i.e., $\tilde{A} := (\tilde{A} \setminus \tilde{A}_d) + \cup_{\tilde{a} \in \tilde{A}_d} H_{\tilde{a}}$. The whole situation determines a partition $\tilde{A} = \tilde{A}_1 \cup \ldots \cup \tilde{A}_{n_B}$, and $n_B$ preteams $\tau_i = (\{H_{\tilde{a}} \mid \tilde{a} \in \tilde{A}_i\}, H)$, where $H_{\tilde{a}}$ is defined on the new elements $\tilde{a} \in \cup_{\tilde{a} \in \tilde{A}_d} H_{\tilde{a}}$ as $H_{\tilde{a}} = \{\{\tilde{a}\}\}$. We still use the notation $\tau_i \cup_{b \in B} X_b \sim \tau_i \ldots \tau_{n_B}$. The definition of clan is unchanged, except that a clan now consists of semi-strict teams and not of strict teams. The definition of the anthropological product is adapted as follows. We assign a construct $C_{\tau}$ of $H_{\tau}$ for all $\tilde{a} \in \tilde{A}$, via the following adjustment with respect to the strict case: if $\tilde{a}$ is an element of $H_{\tilde{a}}$ for some $\tilde{a} \in \tilde{A}_d$, then we set $C_{\tau} = \{\tilde{a}\}$, and we finish as in the strict case: the assignment determines delegations $\delta_i^B$ ($1 \leq i \leq n_B$), and we define the product exactly as in [4.3], but setting $q = -1$ (see below).

We can still define a function $\varphi^B_{\tau}$ from $\tilde{A} \setminus \tilde{A}_d$ to $\{1, \ldots, n_B\}$, which we prefer to see as a partial function from $A$ to $\{1, \ldots, n_B\}$. Abusing notation, we can still write (cf. (4.2)) $\delta_i^B = (\{C_{\tilde{a}} : H_{\tilde{a}} \mid \tilde{a} \in \tilde{A} \text{ and } \varphi^B_{\tau}(\tilde{a}) = i\}, H^B)$, noticing that the participating hypergraphs of $\tau_i$ that are not the hypergraphs $H_{\tilde{a}}$ with $\tilde{a} \in (\varphi^B_{\tau})^{-1}(i)$ are all singleton graphs, so that the sloppy notation above extends in a unique way to the “true” definition of $\delta_i^B$. We shall say that the constructs $C_{\tilde{a}}$, for $\tilde{a} \in \tilde{A}_d$, have been dissolved in $*_{B}(\{H_{\tilde{a}} \mid a \in A\}, H)$, $\{C_a \mid a \in A\}$). Note however that our abuse of notation is not as innocent as it seems, since the convention relies on the fact that a singleton hypergraph $\{\{a\}\}$ admits a unique plain construct $a$. But the same hypergraph admits all $\lambda a$ ($\lambda \in k$) as linear constructs – a fact that is stressed in the following remark.

Remark 3. It follows from the definitions that if $\delta_1$ and $\delta_2$ are delegations of plain constructs having the same support $\tau = (\{H_a \mid a \in A\}, H)$, if $\delta_1$ and $\delta_2$ differ only on one participating hypergraph $H_{a_0}$, if $B$ is a non-empty subset of $A$ such that $a_0 \notin B$ and $\varphi^B_{\tau}(a_0)$ is undefined, then $*_{B}(\delta_1) = *_{B}(\delta_2)$. Moreover, if $\phi$ is a (linear) delegation which coincides with $\delta_1$ and $\delta_2$ on all $a \in A \setminus \{a_0\}$ and has in position $a_0$ a linear construct $\sum_{i \in I} \lambda_i C_i$, then we have $*_{B}(\phi) = (\sum_{i \in I} \lambda_i) *_{B}(\delta_1) = (\sum_{i \in I} \lambda_i) *_{B}(\delta_2)$.

The notion of associative clan is unchanged. The associativity theorem still holds, but only under the assumption $q = -1$. The reason for this restriction stems from Remark 3 and from the following lemma.

Lemma 5. If $q = -1$, then, for any delegation (in the semi-strict setting) $\delta$, the sum of all coefficients in the expansion of $*_{B}(\delta)$ as a linear combination of plain constructs is equal to 1.
Proof. We prove the statement by induction on $|H|$. From the binomial expansion $(1 + x)^n = \sum_{0 \leq i \leq n} \binom{n}{i} x^i$ expressed as $(1 + x)^n = 1 + x(\sum_{1 \leq i \leq n} \binom{n}{i} x^{i-1})$ and instantiated with $x = q = -1$, we readily obtain $\sum_{1 \leq i \leq n} \binom{n}{i} q^{i-1} = 1$. The statement will then follow if we prove that, for each $\emptyset \subset B \subseteq A$, the sum of the coefficients in the expansion of $X_B(*(\delta_1), \ldots, *(\delta_n))$ as a linear combination of plain constructs is equal to 1. But this in turn follows by induction and by multilinearity. \hfill\qed

**Theorem 2.** **Theorem 1** extends to the semi-strict setting for $q = -1$.

Proof. Using the convention above of still defining the anthropological product by appealing to the functions $\varphi_B^*$, the proof of **Theorem 1** goes through, as long as we do not use the totality of these functions. More precisely, the reasoning in case (1) unfolds without change until the equalities (4.4) included, which still hold but have now to be understood in the partial sense, i.e. the left-hand side is defined if and only if the right-hand side is defined, in which case they are equal.

Then two subcases arise:

(1a) If $\varphi_B^{1''} (a_0)$ is defined, then we conclude case (1) by induction as in the proof of **Theorem 1**

(1b) Suppose (new case!) that $\varphi_B^{1''} (a_0)$ is undefined. Let $\varphi'(\delta') = \sum_{i \in I} \lambda_i C_i$. By Lemma 3, we have $\sum_{i \in I} \lambda_i = 1$. Let $\delta_i'$ be the delegation obtained by replacing $\varphi(\delta')$ by $C_i$ in $\delta$. By Remark 3, we have $\varphi_B(\delta_i') = \varphi_B(\delta_i')$ for all $i, j$, and calling $K$ the common value:

$$\varphi_B(\delta) = (\sum_{i \in I} \lambda_i) K = K.$$  

On the other hand, by (4.4), we also have that $\varphi_B^{1''}(a_0, a')$ is undefined (for all $a' \in A'$), and, again, $\varphi^{1''}(\delta''')$ does not depend on the constructs $C_{(a_0, a')}$. Moreover, observing that $\delta$ and $\delta''$ coincide on the indices $a \in A \setminus \{a_0\}$, we get easily that $\varphi_B(\delta''')$ is also equal to the common value $K$, which concludes this new case in the proof of associativity.

Similarly, the reasoning in case (2) unfolds without change until the equalities (4.7) included, which again hold in the partial sense explained above. Let us repeat here the expressions for $\varphi_B^{1''} (a_0, a')$ and for $\varphi_B (\delta''')$ that we wrote at this point of the proof of **Theorem 1**

$$\varphi_B^{1''} (a_0, a') = \varphi_B(\delta^{1''}) = \varphi_B^*(\delta^{1''}) = \sum_{i} \varphi_B(\delta^{1''})(\lambda_i K)$$

where $\lambda_i$ is undefined. Then, by (4.7), we have that also $\varphi_B^{1''} (x)$ is undefined for all $x \in (\varphi_B^*)^{-1}(j)$, and the corresponding $C_x$ (which are plain, as stressed above)
are dissolved in $*_B^r(\bar{\delta}^r)$ and hence do not make their way into $(\bar{\delta}^r)^B$. On the other hand, the linear constructs $*(\ldots, C_x, \ldots)$ (where $x$ ranges over $(\varphi_r^B)^{-1}(j)$ for some $j$ not in the domain of definition of $\varphi^B_r$) appear in $*_B^r(\bar{\delta}^r)$, and are also dissolved. It follows that the same as what we argued about the first expression can be argued about the second one, except for the “trace” left by the constructs $*(\ldots, C_x, \ldots)$ not being plain constructs, which is taken care of by reasoning as in case (1b). Thus also the second expression is still correct, and the proof of Theorem 1 goes through to the end without change.

We finish with examples of semi-strict clans that are not strict. The universe formed by all simplicies $S^X$ (for a finite set $X$) gives rise to the semi-strict clan formed by all preteams of the form $(\{S^X_a \mid a \in A\}, S^X_a)$ (for mutually disjoint $X_a$). That this clan is not strict is easily checked: given a delegation of constructs $C_a$ and $B \subseteq A$, all constructs $C_a$ for $a \in A \setminus B$ are dissolved. The anthropological product instantiates as:

$$*(Y_1(\ldots), Y_2(\ldots), \ldots, Y_n(\ldots)) = \sum_{\emptyset \neq J \subseteq [n]} (\bigcup_{j \in J} Y_j)(\ldots),$$

where $(\ldots)$ is a shortcut for a tuple of singletons. We use this example to illustrate the need to choose $q = -1$ in the semi-strict setting. Take $A = \{a_1, a_2, a_3\}$ and $Y_i \subseteq X_{a_i}$. Then, identifying constructs $Z(\ldots, z, \ldots)$ with their root $Z$, we have $*_{Y_1}(Y_1, Y_2, Y_3) = Y_1$. On the other hand, we have

$$*_{Y_1}(Y_1, *(Y_2, Y_3)) = *_{Y_1}(Y_1, Y_2) + *_{Y_1}(Y_1, Y_2 \cup Y_3) + *_{Y_1}(Y_1, Y_3) = Y_1 + qY_1 + Y_1.$$

Therefore, the two expressions match if and only if $q = -1$.

The universe formed by all hypercubes $C^X$ (for $X = \{x_1 < \cdots < x_n\}$) is ordered and gives rise to the semi-strict clan formed by all preteams of the form $(\{C^X_1, \ldots, C^X_n\}, C^X_1)$, where $\bigcup_{1 \leq i \leq n} X_i$ is endowed with the order in which $X_1, \ldots, X_n$ form successive intervals. In the notation introduced at the end of §3 the tridendriform structure instantiates as follows ($|v|$ stands for the length of $v$):

$$u \prec v = u (|v|)$$

$$u \cdot (v_1 + v_2) = u (|v_1|) \cdot v_2$$

$$u \triangleright (v_1 + v_2) = \begin{cases} (u \ast v_1) + v_2 & (v_1 \neq \epsilon) \\ u + v_2 & (v_1 = \epsilon) \end{cases}.$$

5. FURTHER WORK

This work can be pursued in different ways.

First of all, it remains to be studied whether the resulting tridendriform / polydendriform algebras are free. It would also be of great interest to study the associated polydendriform operad.

Then, one would like to find further restrictions allowing us to add a coproduct and endow the associative algebra with a structure of Hopf algebra. Such structures are known already for associahedra and permutahedra (see [BR10, LR98]).

We also aim at extending the results of [Ron12] to restrictohedra by studying the “flip” order obtained by exchanging the order of elements when seeing constructs as posets.
Finally, in the spirit of [LR98], we would like to address the following question in some generality: given two hypergraphs, one included in the other, what are the relations between the associated polytopes and between the associated algebras?

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