Closed-form Solutions for the Lucas-Uzawa model: Unique or Multiple

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Abstract

Naz and Chaudhry [3] established multiple closed-form solutions for the basic Lucas-Uzawa model. According to Boucekkine and Ruiz-Tamarit [1] and Chilarescu [2] unique closed-form solutions exist for the basic Lucas-Uzawa model. We equate expressions for variables \( h(t) \) and \( u(t) \). We provide here condition for the unique closed-form solution and proposed an open question for evaluation of integral in closed-form. A similar analysis is carried out for the Lucas-Uzawa model with logarithmic utility preferences.

1 Introduction

The following model is discussed for the closed form solutions by Boucekkine and Ruiz-Tamarit [1], Chilarescu [2] and Naz and Chaudhry [3] for fairly general values of parameters. The representative agent’s utility function is defined as

\[
Max_{c,u} \int_0^\infty \frac{e^{1-\sigma} - 1 - e^{-\rho t}}{1 - \sigma} dt, \quad \sigma \neq 1
\]  

subject to the constraints of physical capital and human capital:

\[
\dot{k}(t) = \gamma k^{1-\beta} u^{1-\beta} h^{1-\beta} - \pi k - c, \quad k_0 = k(0)
\]

\[
\dot{h}(t) = \delta (1 - u) h, \quad h_0 = h(0).
\]

Recently, Bethmann [5] developed a stylized version of the two sector Lucas-Uzawa model with logarithmic utility preferences and solved the model by dynamic programming technique. Chilarescu and Sipos [6] derived closed-form solutions for the variables in the model proposed by Bethmann in terms of numerically computable functions involving integrals. Chaudhry and Naz [7] derived multiple closed-form solutions for this model. The representative agent’s utility function is defined as

\[
Max_{c,u} \int_0^\infty e^{-\rho t} \ln(c) dt,
\]  

subject to the constraints of physical capital and human capital:

\[
\dot{k}(t) = A k^{\alpha} (uh)^{1-\alpha} - c, \quad k_0 = k(0),
\]

\[
\dot{h}(t) = \delta (1 - u) h, \quad h_0 = h(0),
\]
where $\rho > 0$ is the discount factor, $\alpha$ is the elasticity of output with respect to physical capital, $A > 0$ is the level of technology in the goods sector, $\delta > 0$ is the level of technology in the education sector, $k$ is physical capital, $h$ is human capital, $c$ is per capita consumption and $u$ is the fraction of labor allocated to the production of physical capital.

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The following closed-form solution derived via two first integrals $I_1$ and $I_2$ is given in equation (3.21) on page 474 of Naz and Chaudhry [3]:

\begin{align*}
c(t) &= c_0 z_0^{\beta} e^{-\frac{(\alpha-\delta)}{\sigma} t} z_0^{-\beta}, \\
k(t) &= \left(\frac{k_0}{c_0 z_0^\beta} - F(t)\right) c_0 z_0^{\beta} z(t)^{-1} e^{-\frac{(\delta+\pi-\pi\beta)}{\sigma} t}, \\
h(t) &= \frac{h_0}{z_0 [\sigma c_0 z_0^{\beta-1} - (\rho + \pi - \pi\sigma) k_0 z_0^{\beta-1} + \beta \gamma (1 - \sigma) k_0] [\sigma c_0 z_0^{\beta} e^{-\frac{(\rho-\delta)}{\sigma} t} z_0^{-\beta+\beta} \\
+ (\beta \gamma (1 - \sigma) - (\rho + \pi - \pi\sigma) z_0^{\beta-1}) (\frac{k_0}{c_0 z_0^\beta} - F(t)) c_0 z_0^{\beta} e^{-\frac{(\delta+\pi-\pi\beta)}{\sigma} t}],} \\
u(t) &= \frac{u_0}{k_0} [\sigma c_0 z_0^{\beta-1} - (\rho + \pi - \pi\sigma) k_0 z_0^{\beta-1} + \beta \gamma (1 - \sigma) k_0] \\
\times \left[\beta \gamma (1 - \sigma) - (\rho + \pi - \pi\sigma) z_0^{\beta-1} (\frac{k_0}{c_0 z_0^\beta} - F(t)) + \sigma z_0^{\beta-1} e^{-\frac{(\delta+\pi-\pi\beta)}{\sigma} t}\right], \\
\lambda(t) &= c_0^{-\sigma} z_0^{-\beta} e^{(\rho-\delta) t} z_0^\beta, \\
\mu(t) &= c_1 e^{(\rho-\delta) t},
\end{align*}
where

\[
F(t) = \int_0^t z(\tau) \frac{\sigma - \beta}{\sigma - \rho} e^{-\frac{(\delta + \pi - \pi) t}{\sigma}} dt,
\]

\[
z(t) = \frac{z^* z_0}{\left[(z^* - z_0) e^{\frac{(1-\beta)(1+\delta)}{\beta} t} + z_0^{1-\beta}\right]^{1-\beta}}, \quad \text{(5)}
\]

\[
\lim_{t \to \infty} F(t) = \frac{k_0}{c_0 z_0^{\frac{\beta}{\sigma}}},
\]

\[
\rho < \delta < \rho + \delta \sigma, \quad \frac{\delta + \pi - \pi \beta}{\beta} - \frac{\delta - \rho}{\sigma} > 0,
\]

\[
c_0 z_0^{\frac{\beta}{\sigma}} = \left( \frac{c_1 \delta}{(1-\beta)\gamma} \right)^{-\frac{1}{\beta}},
\]

\[
\frac{\gamma(1-\beta)(\rho - \delta + \delta \sigma)}{\delta}
\]

\[
= \frac{u_0}{k_0} [\sigma c_0 z_0^{\beta-1} - (\rho + \pi - \pi \sigma) k_0 z_0^{\beta-1} + \beta \gamma (1-\sigma) k_0],
\]

\[
z^* = \left( \frac{\beta \gamma}{\delta + \pi} \right)^{\frac{1}{\beta - 1}}.
\]

The following closed-form solution via one first integral \(I_1\) is given in equation (4.6) on page 476 of Naz and Chaudhry [3]:

\[
c(t) = c_0 z_0^{\frac{\beta}{\sigma}} e^{-\frac{(\sigma - \beta) t}{\sigma}} z^* - \beta,
\]

\[
k(t) = \left( \frac{k_0}{c_0 z_0^{\frac{\beta}{\sigma}}} - F(t) \right) c_0 z_0^{\frac{\beta}{\sigma}} z(t)^{-1} e^{-\frac{(\sigma - \beta) t}{\sigma}},
\]

\[
h(t) = \left[ \left( \frac{(\delta + \pi)(1-\beta)}{\beta} \right) \frac{k_0}{c_0 z_0^{\frac{\beta}{\sigma}}} + \frac{\delta u_0 k_0}{c_0 z_0^{\frac{\beta}{\sigma}}} - \delta u_0 G(t) \right] e^{-\frac{(\delta + \pi)(1-\beta) t}{\beta}}
\]

\[
- \delta u_0 \left( \frac{k_0}{c_0 z_0^{\frac{\beta}{\sigma}}} - F(t) \right) \times \frac{c_0 z_0^{\frac{\beta}{\sigma}}}{(\delta + \pi)(1-\beta) u_0} e^{-\frac{(\delta + \pi)(1-\beta) t}{\beta}},
\]

\[
u(t) = \frac{(\delta + \pi)(1-\beta)}{\beta} u_0 \left[ \frac{k_0}{c_0 z_0^{\frac{\beta}{\sigma}}} - F(t) \right] \left( \frac{(\delta + \pi)(1-\beta)}{\beta} u_0 \right] - \delta u_0 \left( \frac{k_0}{c_0 z_0^{\frac{\beta}{\sigma}}} - F(t) \right)
\]

\[
\mu(t) = c_1 e^{(\rho - \delta) t}
\]

where
\(\rho < \delta < \rho + \delta \sigma, \frac{\delta + \pi - \pi \beta}{\beta} - \frac{\delta - \rho}{\sigma} > 0,\)

\[
F(t) = \int_0^t z(t) \frac{\alpha - \beta}{\pi} e^{-\frac{(\delta + \pi - \pi \beta)}{\pi} \cdot \frac{\delta - \rho}{\sigma} t} dt, \\
G(t) = \int_0^t z(t) \frac{\alpha - \beta}{\pi} e^{-\frac{(\delta + \pi - \pi \beta)}{\pi} \cdot \frac{\delta - \rho}{\sigma} t} dt,
\]

\(z(t) = \frac{z^* z_0}{[(z^* 1 - \beta - z_0^{-1 - \beta}) e^{-\frac{1 - \beta}{\beta} (\delta + \pi) t} + z_0^{-1 - \beta}]^{\frac{1}{1 - \beta}}},\)

\[
c_0 \frac{\beta}{z_0} = \left(\frac{c_1}{(1 - \beta) \gamma}\right)^{-\frac{1}{1 - \beta}}, \\
\lim_{t \to \infty} F(t) = \frac{k_0}{c_0 z_0^{\frac{1}{1 - \beta}}},
\]

\[
\lim_{t \to \infty} \left[ \frac{(\delta + \pi) (1 - \beta)}{\beta} + \delta u_0 \frac{k_0}{c_0 z_0^{\frac{1}{1 - \beta}}} - \delta u_0 G(t) \right] = 0,
\]

\[
\lim_{t \to \infty} G(t) = \frac{(\delta + \pi) (1 - \beta)}{\beta} + \frac{\delta u_0}{\delta u_0} \lim_{t \to \infty} F(t), \\
z^* = \left(\frac{\beta \gamma}{\delta + \pi}\right)^{\frac{1}{1 - \beta}}.
\]

Chilarescu [2] derived same solution given on page 113 in Theorem 1 by classical approach and utilized numerical simulations to evaluate functions \(F(t)\) and \(G(t)\). Boucekkine and Ruiz-Tamarit [1] derived a similar solution and they expressed unknown functions similar to \(F(t)\) and \(G(t)\) in terms of Hypergeometric functions. Naz and Chaudhry [3] claimed that in closed-form solutions (5) and (6) the expressions for the variables \(c(t), k(t)\) are same but expressions for the variables \(h(t)\) and \(u(t)\) are different. Thus closed-form solution (5) is different from closed-form solution (6).

The uniqueness of solution discussed by Boucekkine and Ruiz-Tamarit [1], Chilarescu [2] indicates that the expressions for variables \(h(t)\) and \(u(t)\) in closed-form (5) and (6) should be same. We equate expression for \(h(t)\) and \(u(t)\) in (5) and (6), after simplifications, we obtain following expression for unknown function \(G(t)\) in terms of \(F(t)\):

\[
G(t) = F_* - (F_* - F(t)) e^{\frac{(\delta + \pi)(1 - \beta)}{\beta} t} + \frac{(\delta + \pi)(1 - \beta)}{\beta} F_* e^{\frac{(\delta + \pi)(1 - \beta)}{\beta} t} - \frac{\gamma (1 - \beta) \rho - \delta (1 - \sigma)}{\gamma (1 - \beta) (\rho - \delta (1 - \sigma))} \\
\times \left[ \sigma z^* \frac{\beta}{\pi} e^{-\frac{(\delta + \pi)(1 - \beta)}{\beta} t} + \left(\gamma \beta (1 - \sigma) - (\rho + \pi - \pi \sigma) z(t)^{\beta - 1}\right) (F_* - F(t)) \right],
\]

\[(7)\]
provided following condition holds
\[
\frac{\gamma (1 - \beta) (\rho - \delta + \delta \sigma)}{\delta} = \frac{u_0}{k_0} [\sigma c_0 z_0^{\beta - 1} - (\rho + \pi - \pi \sigma) k_0 z_0^{\beta - 1} + \beta \gamma (1 - \sigma) k_0], \quad (8)
\]
where \( F_* = \frac{k_0}{c_0 z_0^{\beta - 1}} \). It is important to mention here that condition (8) arises systematically for the closed-form solution (5).

In (6) the expression for \( G(t) \) is
\[
G(t) = \int_0^t z(s) e^{-\zeta s} ds, \quad \zeta = \left( \frac{\delta + \pi (1 - \beta)}{\beta} - \frac{\delta - \rho}{\sigma} \right) - \frac{(\delta + \pi) (1 - \beta)}{\beta} \quad (9)
\]

From (7) and (9), we deduce that
\[
\int_0^t z(s) e^{-\zeta s} ds = F_* - (F_* - F(t)) e^{\frac{(\delta + \pi) (1 - \beta) t}{\beta}} - \frac{e^{\frac{(\delta + \pi) (1 - \beta) t}{\beta}}}{\gamma (1 - \beta) [\rho - \delta (1 - \sigma)]} \times \left[ \sigma z^{\beta - \frac{2}{\beta}} e^{-(\delta + \pi) (1 - \beta) t} + \left( \gamma \beta (1 - \sigma) - (\rho + \pi - \pi \sigma) t (1 - \beta) \right) (F_* - F(t)) \right], \quad (10)
\]
provided condition (8) holds. If one can prove (10) as true only for that case then the expressions for the variables \( h(t) \) and \( u(t) \) in closed-form (5) and (6) will be same. Thus (5) and (6) provided by Naz and Chaudhry [3] takes same form. This is consistent with Chilarescu [2] and Boucekkine and Ruiz-Tamarit [1].

If \( G(t) \) is different from (10) then multiple closed-form solutions exist for the Lucas-Uzawa model for fairly general values of parameters. It is an open question to prove (10) in closed-form and not numerically.

### 2.1 Closed-form solution reported by Naz et al [4]

When
\[
\sigma = \frac{\beta (\rho + \pi)}{2 \pi \beta - \delta + \delta \pi - \pi}
\]
Naz et al [4] provided a closed-form solution under a specific parametric restriction \( \sigma = \frac{\beta (\rho + \pi)}{2 \pi \beta - \delta + \delta \pi - \pi} \) provided \( 2 \pi \beta - \delta + \delta \pi - \pi > 0 \) to ensure that \( \sigma > 0 \). The parametric restriction arises automatically and it was important to mention this solution which at the moment seems purely mathematical solution. It might be interesting for economists to test it empirically and it is an open question to test this empirically.
3 Closed-form solutions for Lucas-Uzawa model with logarithmic utility preferences: Unique or multiple

Chaudhry and Naz [7] provided two sets of closed-form solutions. The first set of closed-form solutions for all variables is

\[
\begin{align*}
    c(t) &= c_0 z_0^\beta e^{(\delta - \rho)t} z^{-\beta}, \\
    k(t) &= c_0 z_0^\beta z(t)^{-1} e^{\frac{\delta - \rho}{c_0} t} \left( \frac{k_0 z_0^{1-\beta}}{c_0} - F(t) \right), \\
    h(t) &= \frac{\rho c_0 h_0}{c_0 - \rho k_0} \left[ 1 - e^{-(\rho - \delta-t)} - z(t)^{-1} e^{\frac{\delta - \rho}{c_0} t} \left( \frac{k_0 z_0^{1-\beta}}{c_0} - F(t) \right) \right], \\
    u(t) &= \frac{u_0 z_0^{\beta - 1} (c_0 - \rho k_0) \left( k_0 z_0^{1-\beta} - F(t) \right)}{k_0 e^{(\delta - \rho - \frac{1}{\beta}) t} - \rho z(t)^{-1} e^{\frac{\delta - \rho}{c_0} t} \left( \frac{k_0 z_0^{1-\beta}}{c_0} - F(t) \right)}, \\
    \mu(t) &= \frac{A(1 - \beta)}{\delta z_0^\beta} e^{(\rho - \delta) t}, \\
    \lambda(t) &= \frac{1}{c_0 z_0^\beta} e^{(\rho - \delta) t} z^\beta, \\
\end{align*}
\]

where

\[
F(t) = \int_0^t z(t)^{1-\beta} e^{(\delta - \rho - \frac{1}{\beta}) t} dt, \quad \lim_{t \to \infty} F(t) = \frac{k_0 z_0^{1-\beta}}{c_0},
\]

\[
\frac{\delta u_0 z_0^\beta}{A(1 - \beta) k_0 z_0} = \frac{\rho}{c_0 - \rho k_0},
\]

\[
z(t) = \frac{z^* z_0}{\left( z_0^{1-\beta} + (z^* 1 - \beta - z_0^{1-\beta}) e^{\left(1 - \beta\right) \frac{\delta A}{\delta}} t \right)^{\frac{1}{1-\beta}}}, \quad z^* = \left( \frac{\beta A}{\delta} \right)^{\frac{1}{1-\beta}}.
\]
The second set of closed-form solutions for all variables as follows:

\[ c(t) = c_0 z_0^{1-\beta} e^{1(\delta-\rho) t}, \]
\[ k(t) = c_0 z_0^{1-\beta} e^{\tilde{\gamma} t} \left( \frac{k_0 z_0^{1-\beta}}{c_0} - F(t) \right), \]
\[ h(t) = \left[ \left( \frac{\gamma}{\beta} - \delta + \delta u_0 \right) \frac{k_0 z_0^{1-\beta}}{c_0} - \delta u_0 G(t) \right] e^{(\delta-\tilde{\gamma}) t} - \delta u_0 \left( \frac{k_0 z_0^{1-\beta}}{c_0} - F(t) \right) \]
\[ u(t) = \frac{\left( \frac{\delta}{\beta} - \delta \right) u_0 \left( \frac{k_0 z_0^{1-\beta}}{c_0} - F(t) \right)}{[\left( \frac{\gamma}{\beta} - \delta + \delta u_0 \right) \frac{k_0 z_0^{1-\beta}}{c_0} - \delta u_0 G(t)] e^{(\delta-\tilde{\gamma}) t} - \delta u_0 \left( \frac{k_0 z_0^{1-\beta}}{c_0} - F(t) \right]}, \]
\[ \mu(t) = \frac{A(1-\beta)}{\delta z_0^{1-\beta}} e^{(\rho-\delta) t}, \quad (12) \]
\[ \lambda(t) = \frac{1}{c_0 z_0^{1-\beta}} e^{(\rho-\delta) t} z^{\beta}. \]

where

\[ F(t) = \int_0^t z(t)^{1-\beta} e^{1(\delta-\rho-\tilde{\gamma}) t} dt, \quad \lim_{t \to \infty} F(t) = \frac{k_0 z_0^{1-\beta}}{c_0}, \]
\[ G(t) = \int_0^t z(t)^{1-\beta} e^{-\rho t} dt, \quad \lim_{t \to \infty} G(t) = \frac{(\frac{\delta}{\beta} - \delta + \delta u_0) k_0 z_0^{1-\beta}}{\delta u_0 c_0}, \]
\[ z(t) = \frac{z^* z_0}{\left( z_0^{1-\beta} + (z^{1-\beta} - z_0^{1-\beta}) e^{-(1-\beta)\tilde{\gamma} t} \right)^{\frac{1}{1-\beta}}}, \quad z^* = \left( \frac{\beta A}{\delta} \right)^{\frac{1}{\frac{1}{\beta}-1}}. \]

It is not difficult to show that closed-form solution (12) which was derived by utilizing \( I_1 \) is exactly the same as the solution found by Chilarescu and Sipos [6]. Chaudhry and Naz [7] claimed that in closed-form solutions (11) and (12) the expressions for the variables \( c(t), k(t) \) are same but expressions for the variables \( h(t) \) and \( u(t) \) are different. Thus closed-form solution (11) is different from closed-form solution (12).

The uniqueness of solution discussed by Chilarescu and Sipos [6] indicates that the expressions for variables \( h(t) \) and \( u(t) \) in closed-form (11) and (12) should be same. We equate expression for \( h(t) \) and \( u(t) \) in (11) and (12), after simplifications, we obtain following expression for unknown function \( G(t) \) in terms of \( F(t) \):

\[ G(t) = F_* - (F_* - F(t)) e^{(\frac{\delta}{\beta} - \delta) t} + \frac{(\frac{\delta}{\beta} - \delta) F_*}{\delta u_0} \]
\[ - e^{(\frac{\delta}{\beta} - \delta) t} (\frac{\delta}{\beta} - \delta) \]
\[ - \frac{A(1-\beta)\rho}{e^{(\delta-\rho-\tilde{\gamma}) t} - \rho z^{\beta-1}(F_* - F(t))}, \quad (13) \]
provided following condition holds

$$\frac{\delta u_0 z_0}{A(1 - \beta)k_0 z_0} = \frac{\rho}{c_0 - \rho k_0},$$

(14)

where \(F_* = \frac{k_0}{c_0} = \delta\beta\). It is important to mention here that condition (14) arises for the closed-form solution (11).

In (12) the expression for \(G(t)\) is

$$G(t) = \int_0^t z(t)^{(1 - \beta)} e^{-\rho t} dt$$

(15)

From (13) and (15), we deduce that

$$\int_0^t z(t)^{(1 - \beta)} e^{-\rho t} dt = F_* - (F_* - F(t))e^{(\frac{\delta}{\rho} - \delta)t} + \frac{(\frac{\delta}{\rho} - \delta)F_*}{\delta u_0}$$

$$- e^{(\frac{\delta}{\rho} - \delta)t}(\frac{\delta}{\rho} - \delta) \left[ e^{(\delta - \frac{\rho}{\delta})t} - \rho \delta z^{\beta - 1}(F_* - F(t)) \right],$$

(16)

provided condition (13) holds. If one can proof (16) as true only for that case the expressions for the variables \(h(t)\) and \(u(t)\) in closed-form (11) and (12) will be same. Thus (11) and (12) provided by Chaudhry and [7] takes same form. This is consistent with Chilarescu and Sipos [6].

If \(G(t)\) is different from (16) then multiple closed-form solutions exist for the Lucas-Uzawa model for fairly general values of parameters. It is an open question to prove (16) in closed-form and not numerically.

4 Conclusions

Naz and Chaudhry [3] established multiple closed-form solutions for the basic Lucas-Uzawa model. According to Boucekkine and Ruiz-Tamarit [1] and Chilarescu [2] unique closed-form solutions exist for the basic Lucas-Uzawa model. We equated expressions for variables \(h(t)\) and \(u(t)\). We provide here condition for the unique closed-form solution. A similar analysis was carried out for the Lucas-Uzawa model with logarithmic utility preferences. We propose open questions to prove (10) and (16) in closed-form and not numerically. Can one test empirically the closed-form solution reported by Naz et al [4] when \(\sigma = \frac{\beta(\rho + \pi)}{2\pi\beta - \delta + 3\beta - \pi}\)?

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