On the behavior of homogeneous, isotropic and stationary turbulence

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Abstract

The recent development of a statistical model for incompressible Navier-Stokes (NS) fluids based on inverse kinetic theory (IKT, 2004-2008) poses the problem of searching for particular realizations of the theory which may be relevant for the statistical description of turbulence and in particular for the so-called homogeneous, isotropic and stationary turbulence (HIST). Here the problem is set in terms of the 1-point velocity probability density function (PDF) which determines a complete IKT-statistical model for NS fluids. This raises the interesting question of identifying the statistical assumptions under which a Gaussian PDF can be achieved in such a context. In this paper it is proven that for the IKT statistical model, HIST requires necessarily that $f_1$ must be SIED (namely stationary, isotropic and everywhere-defined). This implies, in turn, that the functional form of the PDF is uniquely prescribed at all times. In particular, it is found that necessarily the PDF must coincide with an isotropic Gaussian distribution. The conclusion is relevant for the investigation of the so-called homogenous, isotropic and stationary turbulence.

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I. INTRODUCTION: HYDRODYNAMIC TURBULENCE

The turbulence problem of hydrodynamics is made difficult by the fact that there does not exist a definition of the turbulent state that is universally accepted. The concept of fluid state is nevertheless well defined in fluid dynamics, as it follows from the theory of continuum media. In particular, it is prescribed by an appropriate set of suitably smooth real functions \( \{ Z_i, i = 1, n \} \) denoted as fluid fields, which must be physically realizable, i.e., identified with physical observables. The fluid fields by assumption satisfy a well-posed initial-boundary value problem represented by a set of PDEs denoted as fluid equations. In the theory of continua this means that they are necessarily described by strong solutions (i.e., they are defined and are least continuous everywhere in the existence domain). The same type of requirement is also manifestly imposed by comparison with experimental observations.

Despite these premises, a precise mathematical definition of the concept of "turbulent state" is still missing. In experiments on isolated incompressible viscous fluids, turbulence is typically associated with the manifestation of fluid motion in which the state of the fluid is "turbulent", i.e., is in some sense random and decays in time until the fluid comes ultimately to a state of rest. In the past at least three different views have been adopted in this regard.

The first one is the mathematical viewpoint represented by the Leray theory of turbulence (see Stewart, 1988 [1]) according to which "...turbulence is a fundamentally different problem from smooth flow", which cannot be described in term of strong solutions. This conjecture actually lead Leray (1931, [2-4]) to introduce the concept of weak solutions, i.e., solutions which are not defined, and are not continuous, everywhere in the existence domain. Although appealing and extremely fruitful for its mathematical implications the theory is manifestly un-physical and therefore should be rejected.

The second one is the so-called deterministic theory of turbulence, for which turbulence should be produced, instead, by the occurrence of a strange attractor (Ruelle-Takens theory; Berge, 1984 [5]). According to this view, there should exist a classical dynamical system, characterizing in some sense the time evolution of the fluid fields, which should exhibit a chaotic behavior on a suitable invariant (hyper-)surface. The basic manifestation of chaos for such a dynamical system would be the occurrence of an infinite cascade of quasi-periodic phenomena in which infinitely many periods are sequentially generated in order to give the
appearance of randomness (Hopf-Landau theory; Landau 1959 [6]).

However, the third, and most popular approach, is probably the statistical theory of turbulence, which historically can be referred primarily to the work of Kolmogorov (Kolmogorov, 1941 [7]) and Hopf (Hopf, 1950/51 [10]). The statistical treatment of fluids usually adopted for turbulent flows (which may be invoked, however, to describe also regular flows) consists, instead, in the introduction of appropriate axiomatic approaches denoted as statistical models \( \{ f, \Gamma \} \).

Their construction involves, besides the specification of the phase space \( \Gamma \) and the probability density function (PDF) \( f \), the identification of the functional class to which \( f \) must belong, denoted as \( \{ f \} \). As a consequence, the complete set of fluid fields \( \{ Z \} \), or only a proper subset as in the case of the Monin-Lundgren approach [11, 12], are expressed in terms of suitable functionals (called moments) of \( f \).

### A. Stochastic representation of turbulence

A basic aspect of the statistical description is the introduction of an ensemble average operator acting on the possible realizations of the fluid. A possible definition of such operator may be achieved by representing the fluid fields in terms of hidden variables [17, 19]. By definition they denote a suitable set of independent variables \( \alpha = \{ \alpha_i, i = 1, k \} \in V_\alpha \subseteq \mathbb{R}^k \), with \( k \geq 1 \), which cannot be known deterministically, i.e., are not observable. In the context of turbulence theory these variables are necessarily stochastic. This means that they are characterized by a suitable stochastic probability density \( g \) defined on \( V_\alpha \) (see definitions and related discussion in the Appendix, Subsection B), while the ensemble average \( \langle \cdot \rangle \) can be identified with the stochastic-averaging \( \langle \cdot \rangle_\alpha \) defined by Eq.(50) [see Appendix A]. Hence, for turbulent flows the fluid fields - together with the PDF \( f_1 \) and the vector field \( F(x, t, \alpha) \) - can be assumed to admit a representation of the form [17, 19]

\[
\begin{align*}
\{ Z \} &= \{ Z(r, t, \alpha) \} \\
 f_1 &= f_1(r, u, t, \alpha) \\
 F &= F(x, t, \alpha)
\end{align*}
\]

(1)

to be defined in terms of a set of hidden variables \( \alpha \) and a stochastic model \( \{ g, V_\alpha \} \) (see again Subsection B in the Appendix). Hence, \( \{ Z \} \), \( f_1 \) and \( F(x, t, \alpha) \) are necessarily non-
observable. Nevertheless, if we assume that the fluid fields \{Z\} are uniquely-prescribed ordinary functions of \((x,t,\alpha)\) defined for all \((x,t,\alpha) \in \Gamma \times I \times V_\alpha\), it follows that they can still be considered conditional observables (see Appendix, Subsection A). Similar conclusions apply to \(f_1\), and to the vector field \(F(x,t,\alpha)\) as well.

B. The Navier-Stokes dynamical system

It is, however, generally agreed that a common important property should characterize all NS fluids, either regular or turbulent: this is related to the existence of the phase-space dynamical system, to be denoted as NS dynamical system, which advances in time the complete set of fluid fields defining the fluid state.

In other words, should such a dynamical system actually exist, it would permit to cast the complete set of fluid equations in terms of an equivalent (and possibly infinite) set of ordinary differential equations which define the dynamical system itself. For contemporary science the determination of such a dynamical system represents not simply an intellectual challenge, but a fundamental prerequisite for the proper formulation of all phenomenological theories which are based on the description of these fluids, and hence both for the deterministic and statistical approaches to turbulence. These involve, for example, the understanding of the phase-space Lagrangian description of fluids [13, 14] relevant to determine tracer-particle dynamics [15, 16] as well as the time-evolution of scalar and tensor fields in turbulent flows, the search of exact(or approximate) kinetic closure conditions for statistical models (such as in the case of the Monin-Lundgren hierarchy [11, 12]), the investigation of stochastic models [17, 18] able to reproduce phenomenological data (such as the two-point velocity increments PDFs [8, 9]), the theoretical prediction of multi-point velocity probability densities, all essential ingredients in fluid dynamics and in applied sciences.

Surprisingly, although phase-space descriptions of incompressible fluids described by the incompressible NS equations (INSE) have been around for a long time, starting from the historical work of Hopf (see also Hopf, 1952 [10]), Edwards (Edwards, 1964 [26]) and Rosen (Rosen, 1971 [27]), until recently [24] the problem [of the search of the NS dynamical system] has remained unsolved. Its solution for the incompressible NS equations (see also Refs. [19, 25] for its extension to quantum and magneto fluids) is based on the construction of a statistical model \(\{f_1, \Gamma\}\) for the 1-point PDF \(f_1\) which is required to obey a Liouville
equation and whose moments determine - via suitable velocity-moments - the complete set of fluid fields which describe the state of the fluid. As indicated elsewhere [17], the approach can be extended also to the statistical treatment of turbulence theory.

II. MOTIVATIONS

An unsolved problem in the statistical theory of turbulence concerns the so-called homogeneous, isotropic and stationary turbulence (HIST) arising in incompressible Navier-Stokes (NS) fluids. This concerns in particular the determination of the form of the 1-point probability density function (PDF), $f_1$, occurring in the presence of HIST, which should uniquely determine, in turn, the statistical model $\{f, \Gamma\}$.

According to some authors (see in particular Batchelor [20]) this is predicted as almost-Gaussian, while others [21, 22] have pointed that the tails of the PDF might exhibit a strongly non-Gaussian behavior. Despite recent attempts at possible a theoretical explanation [23], still missing is a definite answer to the question whether a generalized behavior of this type should actually be expected or not. Apart insufficient experimental evidence, a major difficulty is represented by the lack of a consistent theoretical description of the PDF in the presence of HIST, permitting a rigorous definite answer to this question. This raises the interesting question whether, in some suitable setting, i.e., for appropriate statistical models, the problem can actually be solved. Being a subject of major importance - not only in fluid dynamics but also in statistical mechanics and, as we intend to prove, in kinetic theory - the issue deserves a careful investigation. The goal of this paper is to pose the problem in the framework of the complete inverse kinetic theory (IKT) approach developed by Tessarotto et al. [24] for incompressible NS fluids (see in particular [17, 19]).

Here we intend to prove that based solely on IKT the appropriate form of the 1-point PDF in the presence of HIST can actually be uniquely established for these fluids, based on suitable statistical assumptions stemming from the requirement of existence of HIST and appropriate initial conditions and the requirement that the initial 1-point PDF is determined imposing PEM (principle of entropy maximization, Jaynes, 1957 [30]; see also related discussion in Ref. [25]). This is shown to be described by a probability density $f_1$, defined on the restricted phase-space $\Gamma = \Omega \times U$, with $\Omega$ and $U$ denoting respectively the configuration space of the fluid [to be identified with a bounded subset of the Euclidean space $\mathbb{R}^3$] and the
Euclidean velocity space $U \subseteq \mathbb{R}^3$, which fulfills the following properties (#1-8):

1. (Property #1) it depends explicitly on the fluid fields $\{Z(r,t,\alpha)\} = \{\mathbf{V}(r,t,\alpha), p_1(r,t,\alpha)\}$, namely is of the type $f_1 = f_1(r, \mathbf{v}, t, \alpha)$, with $\mathbf{V}(r,t,\alpha)$ denoting the fluid velocity and $p_1(r,t,\alpha)$ the kinetic pressure. In particular $p_1(r,t,\alpha)$ is defined as the strictly positive function $p_1(r,t,\alpha) = p(r,t,\alpha) + p_0(t,\alpha) + \phi(r,t,\alpha)$, with $p(r,t,\alpha), p_0(t,\alpha)$ and $\phi(r,t,\alpha)$ representing respectively the fluid pressure, the (strictly-positive) pseudo-pressure and the (possible) potential associated to the conservative volume force density acting on the fluid;

2. (Property #2) $f_1$ is a velocity probability density, i.e., it is normalized in velocity space so that

$$\int_U d^3\mathbf{v} f_1(r,\mathbf{v},t,\alpha) = 1; \quad (2)$$

3. (Property #3) $f_1$ is strictly positive in the velocity space $U$;

4. (Property #4) $f_1$ Galilei invariant in velocity space, namely it is invariant with respect to a transformation of the form:

$$\begin{align*}
\mathbf{v} \\
\mathbf{V}(r,t,\alpha)
\end{align*} \rightarrow \begin{align*}
\mathbf{v} + \mathbf{V}_o \\
\mathbf{V}(r,t,\alpha) + \mathbf{V}_o
\end{align*} \quad (3)$$

with $\mathbf{V}_o$ such that $\mathbf{v} + \mathbf{V}_o \in U$. As a consequence, $f_1$ is of the form

$$f_1 = f_1(r,\mathbf{u},t,\alpha), \quad (4)$$

with $\mathbf{u} \equiv \mathbf{u}(r,t,\alpha) = \mathbf{v} - \mathbf{V}(r,t,\alpha)$ denoting the relative velocity, namely is homogeneous in the velocity space $U$;

5. (Property #5) the velocity space $U$ coincides with $\mathbb{R}^3$. $f_1$ fulfilling this property is said everywhere defined (in $U \equiv \mathbb{R}^3$).

6. (Property #6) $f_1$ is stationary, namely it can depend on time only via the fluid fields:

$$f_1 = f_1(r,\mathbf{u},\alpha), \quad (5)$$

7. (Property #7) $f_1$ is isotropic in velocity space, i.e., it is of the form

$$f_1 = f_1(r,|\mathbf{u}|,\alpha); \quad (6)$$
8. (Property #8) $f_1$ is a Gaussian distribution of the form

$$f_M(r, |u|, p_1(r, t, \alpha)) = \frac{1}{\pi^{3/2} v_{thp}^3(r, t, \alpha)} \exp \left\{-\frac{u^2}{v_{thp}^2(r, t, \alpha)}\right\},$$  \hspace{1cm} (7)$$

with $v_{thp}(r, t, \alpha) = \sqrt{2p_1(r, t, \alpha)/\rho_o}$ denoting the thermal velocity associated to the kinetic pressure.

III. HOMOGENEOUS, ISOTROPIC AND STATIONARY TURBULENCE

In fluid dynamics two types of descriptions of the fluid, respectively denoted deterministic and stochastic, can be distinguished, in which the fluid fields describing the state of the fluid are treated respectively as deterministic or stochastic functions. In both cases the fluid fields $\{Z\} \equiv \{Z_i, i = 1, n\}$ are considered, in a suitable existence domain, suitably smooth strong solutions of the fluid equations. In the so-called statistical theory of turbulence, historically referred primarily to the work of Kolmogorov (Kolmogorov, 1941 [7]) and Hopf (Hopf, 1950/51 [10]), turbulence is intended as the characteristic property of the fluid in which the fluid fields $\{Z\}$ can only be prescribed in a statistical sense. This implies that they are necessarily stochastic functions of the form $\{Z\} = \{Z(r, t, \alpha)\}$ characterized by a stochastic PDF $g(r, t, \alpha)$ defined for all $(r, t) \in \Omega \times I$ and $\alpha = \{\alpha_i, i = 1, k\} \in V_\alpha \subseteq \mathbb{R}^k$ (with $k \geq 1$), with $\alpha$ suitable stochastic variables independent of $r, t$. Thus, introducing the stochastic-averaging operator $\langle \cdot \rangle_\alpha = \int_{V_\alpha} d\alpha g(r, t, \alpha) \cdot$, acting on an arbitrary integrable function it follows that the fluid fields $Z_i(r, t, \alpha)$ (for $i = 1, n$) can always be represented in terms of their stochastic decompositions $Z_i = \langle Z_i \rangle_\alpha + \delta Z_i$, with $\langle Z_i \rangle_\alpha$ denoting their stochastic averages.

A widespread conjecture is that turbulence, at least in special circumstances [usually ascribed to the so-called "fully developed" turbulence (FDT)], should be characterized by certain universal properties. These concern, in particular, the concept of homogeneous, isotropic and stationary turbulence (HIST). Its definition (see also Refs. [28, 29]) is related to the assumed properties of the operator $\langle \cdot \rangle_\alpha$ and of the velocity increments

$$dV_i \equiv V_i(r_1, t) - V_i(r, t),$$  \hspace{1cm} (8)$$

which are assumed to be defined for arbitrary displacements

$$dr = r_1 - r.$$  \hspace{1cm} (9)
such that both \( r \) and \( r_1 \) belong to \( \Omega \).

1. **Definition - HIST**

Turbulence is said *homogeneous, isotropic and stationary* if:

- **HIST Requirement \#1:** the stochastic-averaging operator \( \langle Z_i \rangle_\alpha \) commutes with all the differential operators appearing in the fluid equations (namely, for the NS equations, this means it must commute with the operators \( \frac{\partial}{\partial t}, \nabla \) and \( \nabla^2 \));

- **HIST Requirement \#2:** for all \( n \in \mathbb{N}_0 \) the *structure functions* - i.e., the stochastic-averages of \( \{dV_i\}^n \), with \( S_i^{(n)}(r,dr,t) \equiv \langle \{dV_i\}^n \rangle_\alpha \) - are respectively:
  2\text{a)} independent of \( r \), namely for all \( i = 1, 2, 3 \)
  
  \[
  S_i^{(n)} = S_i^{(n)}(dr,t) \tag{10}
  \]
  *(homogeneous turbulence)*;

  2\text{b)} independent of the directions of \( dr \) and \( V \), hence for all \( i = 1, 2, 3 \):
  
  \[
  S_i^{(n)} = S_i^{(n)}(r,l,t) \tag{11}
  \]
  *(isotropic turbulence)*, where \( l = |dr| \) is the magnitude of the displacement \( (9) \);

  2\text{c)} independent of \( t \), hence for all \( i = 1, 2, 3 \):
  
  \[
  S_i^{(n)} = S_i^{(n)}(r,dr) \tag{12}
  \]
  *(stationary turbulence)*.

Thus, for all \( n \in \mathbb{N}_0 \) and \( i = 1, 2, 3 \), HIST is by assumption characterized by structure functions of the form

\[
S_i^{(n)} = S^{(n)}(l),
\]

(13) i.e., depending solely on the magnitude of the displacement \( (9) \).
IV. THE IKT-STATISTICAL MODEL FOR TURBULENT FLUIDS

Starting point for the statistical treatment of turbulence in NS fluids in the IKT approach is the introduction of a statistical model \( \{ f_1, \Gamma \} \) (to be denoted as IKT-statistical model) for the INSE problem. The corresponding fluid fields are \( \{ \rho_o, \mathbf{V}(r, t, \alpha), p(r, t, \alpha), S_T \} \) with \( \rho_o \) and \( S_T \) to be identified, respectively, with the constant mass density and the constant thermodynamic entropy. In the following the fluid fields are required to be: 1) they are strong solutions of the INSE problem in \( \overline{\Omega} \times I \times V_\alpha \) with bounded configuration space \( \overline{\Omega} \) (internal domain); 2) global solutions, i.e., defined for all \( t \in I \equiv \mathbb{R} \).

The construction of \( \{ f_1, \Gamma \} \) involves the definition of a suitable PDF \( f_1 \) defined on a phase space \( \Gamma \), denoted as 1-point velocity PDF, which permits the representation, via a suitable mapping \( \{ f_1, \Gamma \} \Rightarrow \{ Z \} \), \( \{ Z \} \) denoting the complete set of the fluid fields \( \{ Z \} \equiv \{ Z_i, i = 1, n \} \) defining the state of the fluid. In particular by assumption \( \Gamma \) is identified with the restricted phase-space \( \Gamma = \Omega \times U \times V_\alpha \) [with closure \( \overline{\Gamma} = \overline{\Omega} \times U \times V_\alpha \)]; furthermore, the 1-point velocity PDF:

1. is taken of the general form \( f_1(t) \equiv f_1(\mathbf{x}, t, \alpha) \), with \( \mathbf{x} = (r, \mathbf{v}) \), where \( r \in \overline{\Omega} \) and \( \mathbf{v} \in U \subseteq \mathbb{R}^3 \) (with \( U \) defined as the open subset of \( \mathbb{R}^3 \) spanned by \( \mathbf{v} \) on which \( f_1 > 0 \)). In addition \( f_1 \) is by assumption Galilei invariant and hence invariant w.r. to (3). It follows that \( f_1 \) is necessarily homogeneous in velocity space, namely of the form (4);

2. determines in terms of suitable moments the complete set of the fluid fields \( \{ Z \} \) which define the state of the fluid. This requires that the fluid fields are determined by the velocity moments

\[
\int_U d^3 \mathbf{v} G(\mathbf{x}, t, \alpha) f_1(\mathbf{x}, t, \alpha) = \mathbf{V}(r, t, \alpha), p_1(r, t, \alpha),
\]

defined respectively for the weight-functions \( G(\mathbf{x}, t, \alpha) = \mathbf{v} \cdot \rho_o \mathbf{u}^2 / 3 \), whereas \( S_T \) is identified with the Boltzmann-Shannon entropy, i.e., the phase-space moment

\[
S(f_1(t)) \equiv - \int f d\mathbf{x} f_1(\mathbf{x}, t, \alpha) \ln f_1(\mathbf{x}, t, \alpha),
\]

by imposing that \( \forall t \in I \equiv \mathbb{R} \) the constraint

\[
S_T = S(f_1(t)).
\]
The set of equations (14) and (15), denoted as correspondence principle, are assumed do be fulfilled identically in the whole existence domain of the fluid fields $\mathbf{V}(\mathbf{r}, t, \alpha), p_1(\mathbf{r}, t, \alpha)$ and $S_T$. This implies manifestly that $f_1(\mathbf{x}, t, \alpha)$ must be defined and strictly positive on $\overline{\Gamma} = \overline{\Omega} \times U \times V_\alpha$. The time-evolution of $f_1(\mathbf{x}, t, \alpha)$ is then uniquely determined by the flow (stochastic N-S dynamical system)

$$T_{t_o,t} : \mathbf{x}_o \rightarrow \mathbf{x}(t) = T_{t_o,t} \mathbf{x}_o,$$

with $T_{t_o,t}$ the corresponding evolution operator generated by the initial value problem

$$\frac{d}{dt}\mathbf{x} = \mathbf{X}(\mathbf{x}, t, \alpha)$$
$$\mathbf{x}(t_o) = \mathbf{x}_o.$$

In particular, denoting by $J(t) = \left| \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_o} \right| = \exp \left\{ \int_{t_o}^t dt' \frac{\partial}{\partial \mathbf{x}(t')} \cdot \mathbf{X}(\mathbf{x}(t'), t', \alpha) \right\}$ the Jacobian determinant of the flow the time-advanced PDF $f_1(\mathbf{x}(t), t, \alpha)$ is uniquely determined requiring that for all $(\mathbf{x}_o, \alpha, t_o, t) \in \overline{\Gamma} \times I \times I$ it satisfies the Lagrangian inverse kinetic equation (IKE)

$$f_1(t) = J(t)f_1(t_o).$$

In particular, as shown elsewhere [19], the initial PDF $f_1(t_o) \equiv f(\mathbf{x}, t_o, \alpha)$ can be assumed to be a strictly positive smooth PDF of the form

$$f_1(t_o) = \langle f_1(t_o) \rangle_\Omega \frac{h(t_o)}{\langle h(t_o) \rangle_\Omega},$$

with $h(t_o) \equiv h(\mathbf{x}_o, t_o, \alpha)$ to be determined and $\langle f_1(t_o) \rangle_\Omega$ subject to the constraint

$$\langle f_1(t_o) \rangle_\Omega = \tilde{f}_1^{(freq)}(t_o, \mathbf{v}_o, \alpha)$$

(physical realizability condition). Here $\tilde{f}_1^{(freq)}(t_o, \mathbf{v}_o, \alpha)$ denotes a suitable continuous velocity-frequency density, uniquely associated to the initial fluid velocity $\mathbf{V}(\mathbf{r}_o, t_o, \alpha)$. Furthermore, $\langle f_1(t) \rangle_\Omega$ denotes the configuration-space average (defined at at time $t$), i.e.,

$$\langle f_1(\mathbf{r}, \mathbf{v}, t, \alpha) \rangle_\Omega \equiv \frac{1}{\mu(\Omega)} \int_{\Omega} d^3 \mathbf{r} f_1(\mathbf{r}, \mathbf{v}, t, \alpha),$$

while $\mu(\Omega) = \int d^3 \mathbf{r} > 0$ is the canonical measure of $\Omega$. As a consequence $h(t_o)$, can be uniquely determined imposing that $f_1(\mathbf{x}_o, t_o, \alpha)$ satisfies PEM [30], namely the variational
principle \( \delta S(f_1(t)) = 0 \) subject to the constrains \( (20) \) and \( (21) \), taking the form of a generalized Gaussian distribution

\[
h(t_o) = \exp \left\{ -1 - \lambda_0(r_o,t_o,\alpha) - \lambda_{2i}(r_o,t_o,\alpha)u_i^2(r_o,t_o,\alpha) \right\} . \tag{23}\]

Here, \( \lambda_0(r_o,t_o,\alpha) \) and \( \lambda_{2i}(r_o,t_o,\alpha) \) (for \( i = 1, 2, 3 \)) denote suitable Lagrange multipliers to be determined imposing the moment equations \( (14) \) and \( (15) \), together with the constraint \( (16) \). Furthermore, \( h(t_o) \) can be shown to take the form of an isotropic Gaussian distribution, i.e.,

\[
h_o(t) = \exp \left\{ -1 - \lambda_0(r,t,\alpha) - \lambda_2(r,t,\alpha)u^2(r,t,\alpha) \right\} \tag{24}\]

[see related discussion the Appendix].

The equivalence theorem pointed out elsewhere \[31\] between the INSE problem and the NS dynamical system \( (17) \) then warrants the validity of the correspondence principle, i.e., that the moments prescribed by equations \( (14) \) and \( (15) \) actually define a strong solution of the INSE problem.

V. CONNECTION WITH HIST - SIED IKT-STATISTICAL MODELS

Here we are interested in determining a particular subclass of IKT-statistical models \( \{f_1, \Gamma\} \) which may fulfill at least some of the properties which characterize HIST for NS fluids. Let us now analyze the consequences placed by (the assumption) of HIST. First, let us require that - consistent with the requirement of FDT - the velocity space \( U \) [on which \( f_1 \) is defined and strictly positive] coincides with \( \mathbb{R}^3 \) and hence \( f_1 \) is everywhere defined. Second, we notice that the constraints imposed by Eq.\( (13) \) on arbitrary structure functions \( S_i^{(n)} \) can generally be satisfied only if:

- \( f_1(t) \) if stationary in the sense \( (5) \);

- \( f_1(t) \) isotropic in velocity space (since no preferred direction in velocity space can exist in such a case).

As a consequence, in the presence of HIST, \( f_1 \) is necessarily stationary, isotropic and everywhere defined (SIED). In the following we shall denote as SIED the IKT-statistical models \( \{f_1, \Gamma\} \) which fulfill these requirements. In such a case it is immediate to reach the following result [which proves Properties #1-8]:

11
THM.1 - Characteristic property of SIED \{f_1, \Gamma\}

For SIED IKT-statistical models \{f_1, \Gamma\} there results identically in \(\Gamma \times I\):

\[
h_o(t) = f_M(r, |u(r,t,\alpha)|, p_1(r,t,\alpha)), \quad (25)
\]

\[
\frac{\langle f_1(t) \rangle_\Omega}{\langle h(t) \rangle_\Omega} = 1, \quad (26)
\]

where \(f_M(r, |u(r,t,\alpha)|, p_1(r,t,\alpha))\) is the Gaussian 1-point PDF \(7\) and \(h_o(t)\) the Gaussian distribution \(24\). In particular, \(\lambda(r,t,\alpha)\) and \(\lambda_2(r,t,\alpha)\) (for \(i = 1, 2, 3\)) denote suitable Lagrange multipliers to be determined imposing the moment equations \(14\) and \(15\), together with the constraint \(16\).

PROOF In fact, first, since \(f_1(t)\) is stationary for all \(t \in I \equiv \mathbb{R}\), \(f_1(t)\) is necessarily determined by PEM, subject to the constrains \(20\) and \(21\). Therefore for all \(t \in I \equiv \mathbb{R}\), \(f_1(t)\) has the form

\[
f_1(t) = \langle f_1(t) \rangle_\Omega \frac{h(t)}{\langle h(t) \rangle_\Omega}, \quad (27)
\]

with \(h(t) \equiv h(r,v,t,\alpha)\) of the type \(23\). Furthermore, since \(\{f_1, \Gamma\}\) is by assumption isotropic and everywhere-defined (so that \(U \equiv \mathbb{R}^3\)) it follows necessarily for all \((v,t) \in \mathbb{R}^3 \times I\) that

\[
\frac{\langle f_1(t) \rangle_\Omega}{\langle h(t) \rangle_\Omega} = c(t,\alpha) > 0, \quad (28)
\]

\[
h(t) = \exp \{-1 - \lambda_o(r,t,\alpha) - \lambda_2(r,t,\alpha)u^2\} \equiv h_o(t). \quad (29)
\]

Here \(c(t,\alpha), \lambda_o(r,t,\alpha)\) and \(\lambda_2(r,t,\alpha)\) denote, respectively, a suitable strictly positive function of time and two Lagrange multipliers defined so that there results identically in \(\overline{\Omega} \times I\)

\[
\int_{\mathbb{R}^3} d^3v f_1(t) = 1, \quad (30)
\]

\[
\int_{\mathbb{R}^3} d^3v \frac{\rho_0 u^2}{3} f_1(t) = p_1(r,t,\alpha), \quad (31)
\]

while the pseudo-pressure \(p_0(t,\alpha)\) must be determined by imposing the constraint \(16\). Hence, it is always possible to set \(c(t,\alpha) = 1\), so that \(26\) is identically satisfied. As a consequence, it follows identically that \(h_o(t) \equiv f_M(r, |u(r,t,\alpha)|, p_1(r,t,\alpha))\), which proves Eq.\(25\) too. Q.E.D.
VI. CONCLUSIONS

In this paper properties of the IKT-statistical model, \( \{ f_1, \Gamma \} \), defined in terms of the 1-point PDF \( f_1 \) for a turbulent fluid obeying the INSE problem, have been investigated.

In particular we have proven that for the IKT statistical model \( \{ f_1, \Gamma \} \):

- the assumption of HIST requires that \( f_1 \) must necessarily be SIED, i.e., stationary and isotropic and everywhere-defined in velocity space;
- the requirement of stationarity implies that PEM must hold identically for all \( t \in I \equiv \mathbb{R} \). As a consequence the functional form of the function \( h(t) \) remains uniquely determined.

Main result is the proof here achieved (THM.1) that the requirement of \( f_1 \) to be SIED implies necessarily that \( f_1 \) must coincides identically with an isotropic Gaussian distribution.

The conclusion is relevant for the investigation of the so-called homogenous, isotropic and stationary turbulence.

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VIII. APPENDIX: THE MATHEMATICAL DESCRIPTION OF INCOMPRESSIBLE NS FLUIDS

In fluid dynamics the state of an arbitrary fluid system is assumed to be defined everywhere in a suitable extended configuration domain \( \Omega \times I \) [\( \Omega \) denoting the configuration space and \( I \subseteq \mathbb{R} \) the time axis] by an appropriate set of suitably smooth functions \( \{ Z \} \), denoted as fluid fields, and by a well-posed set of PDEs, denoted as fluid equations, of which the former are solutions. The fluid fields are by assumption functions of the observables \( (r, t) \), with \( r \) and \( t \) spanning respectively the sets \( \Omega \) and \( I \), namely smooth real functions. Therefore, they are also strong solutions of the fluid equations. In particular, this means
that they are required to be at least continuous in all points of the closed set $\overline{\Omega} \times I$, with $\overline{\Omega} = \Omega \cup \partial \Omega$ closure of $\Omega$. In the remainder we shall require, for definiteness, that:

1. $\Omega$ (configuration domain) is a bounded subset of the Euclidean space $E^3$ on $\mathbb{R}^3$;

2. $I$ (time axis) is identified, when appropriate, either with a bounded interval, i.e., $I=]t_0, t_1[ \subseteq \mathbb{R}$, or with the real axis $\mathbb{R}$;

3. in the open set $\Omega \times I$ the functions $\{Z\}$, are assumed to be solutions of a closed set of fluid equations. In the case of an incompressible Navier-Stokes fluid the fluid fields are $\{Z\} \equiv \{V, p, S_T\}$ and their fluid equations

$$\rho = \rho_o, \quad (32)$$

$$\nabla \cdot V = 0, \quad (33)$$

$$NV = 0, \quad (34)$$

$$\frac{\partial}{\partial t} S_T = 0, \quad (35)$$

$$Z(r,t_o) = Z_o(r), \quad (36)$$

$$Z(r,t)|_{\partial \Omega} = Z_w(r,t)|_{\partial \Omega}, \quad (37)$$

where Eqs. (32)-(35) denote the incompressible Navier-Stokes equations (INSE) and Eqs. (32)-(37) the corresponding initial-boundary value INSE problem. In particular, Eqs. (32)-(37) are respectively the incompressibility, isochoricity, Navier-Stokes and constant thermodynamic entropy equations and the initial and Dirichlet boundary conditions for $\{Z\}$, with $\{Z_o(r)\}$ and $\{Z_w(r,t)|_{\partial \Omega}\}$ suitably prescribed initial and boundary-value fluid fields, defined respectively at the initial time $t = t_o$ and on the boundary $\partial \Omega$.

4. by assumption, these equations together with appropriate initial and boundary conditions are required to define a well-posed problem with unique strong solution defined everywhere in $\Omega \times I$.

Here the notation as follows. $N$ is the NS nonlinear operator

$$NV = \frac{D}{Dt} V - F_H, \quad (38)$$
with $\frac{D}{Dt}V$ and $F_H$ denoting respectively the Lagrangian fluid acceleration and the total force per unit mass

$$\frac{D}{Dt}V = \frac{\partial}{\partial t}V + V \cdot \nabla V,$$  \hspace{1cm} (39)

$$F_H \equiv -\frac{1}{\rho_o} \nabla p + \frac{1}{\rho_o} f + \nu \nabla^2 V,$$  \hspace{1cm} (40)

while $\rho_o > 0$ and $\nu > 0$ are the constant mass density and the constant kinematic viscosity. In particular, $f$ is the volume force density acting on the fluid, namely which is assumed of the form

$$f = -\nabla \phi(r, t) + f_R,$$  \hspace{1cm} (41)

$\phi(r, t)$ being a suitable scalar potential, so that the first two force terms [in Eq.(40)] can be represented as $-\nabla p + f = -\nabla p_r + f_R$, with

$$p_r(r, t) = p(r, t) - \phi(r, t),$$  \hspace{1cm} (42)

denoting the reduced fluid pressure. As a consequence of Eqs. (32), (33) and (34) it follows that the fluid pressure necessarily satisfies the Poisson equation

$$\nabla^2 p = S,$$  \hspace{1cm} (43)

where the source term $S$ reads

$$S = -\rho_o \nabla \cdot (V \cdot \nabla V) + \nabla \cdot f.$$  \hspace{1cm} (44)

**A. Physical/conditional observables - Hidden variables**

The fluid fields $\{Z\}$ are, by assumption, prescribed smooth real functions of $(r, t) \in \Omega \times I$. In particular, they can be either physical observables or conditional observable, according to the definitions indicated below.

1. **Definition - Physical observable/conditional observable**

A physical observable is an arbitrary real-valued and uniquely-defined smooth real function of $(r, t) \in \Omega \times I$. Hence, as a particular case $(r, t)$ are observable too.
A *conditional observable* is, instead, an arbitrary real-valued and uniquely-defined smooth real function of \((r,t) \in \Omega \times I\) which depends also on non-observable variables and is, as such, an uniquely-prescribed function of the latter ones.

Therefore the functions \(Z_{i}\) can be assumed respectively of the form

\[ Z_{i} \equiv Z_{i}(r,t) \quad (45) \]

or

\[ Z_{i} \equiv Z_{i}(r,t,\alpha), \quad (46) \]

\(\alpha \in V_{\alpha} \subseteq \mathbb{R}^{k}\) (with \(k \geq 1\)) denoting a suitable set of *hidden variables*. In fluid dynamics these are intended as:

2. **Definition - Hidden variables**

A *hidden variable* is an arbitrary real variable which is independent of \((r,t)\) and is not an observable.

B. **Deterministic and stochastic fluid fields**

Hence, fluid fields of the type (46) are manifestly non-observables. However, if in the whole set \(\bar{\Omega} \times I \times V_{\alpha}\), they are uniquely-prescribed functions of \((r,t,\alpha)\) then they are *conditional observables*. Hidden variables can be considered in principle either *deterministic* or as *stochastic variables*, in the sense specified as follows.

1. **Definition - Stochastic variables**

Let \((S,\Sigma,P)\) be a probability space; a measurable function \(\alpha : S \rightarrow V_{\alpha}\), where \(V_{\alpha} \subseteq \mathbb{R}^{k}\), is called *stochastic* (or *random*) variable.

A stochastic variable \(\alpha\) is called *continuous* if it is endowed with a *stochastic model* \(\{g_{\alpha}, V_{\alpha}\}\), namely a real function \(g_{\alpha}\) (called as *stochastic PDF*) defined on the set \(V_{\alpha}\) and such that:

1) \(g_{\alpha}\) is measurable, non-negative, and of the form

\[ g_{\alpha} = g_{\alpha}(r,t,\cdot); \quad (47) \]
2) if \( A \subseteq V_\alpha \) is an arbitrary Borelian subset of \( V_\alpha \) (written \( A \in \mathcal{B}(V_\alpha) \)), the integral
\[
P_\alpha(A) = \int_A d\mathbf{x} g_\alpha(r, t, \mathbf{x})
\] (48)
exists and is the probability that \( \alpha \in A \); in particular, since \( \alpha \in V_\alpha \), \( g_\alpha \) admits the normalization
\[
\int_{V_\alpha} d\mathbf{x} g_\alpha(r, t, \mathbf{x}) = P_\alpha(V_\alpha) = 1.
\] (49)

The set function \( P_\alpha : \mathcal{B}(V_\alpha) \rightarrow [0, 1] \) defined by (48) is a probability measure and is called distribution (or law) of \( \alpha \). Consequently, if a function \( f : V_\alpha \rightarrow V_f \subseteq \mathbb{R}^m \) is measurable, \( f \) is a stochastic variable too.

Finally define the stochastic-averaging operator \( \langle \cdot \rangle_\alpha \) (see also [17, 19]) as
\[
\langle f \rangle_\alpha = \langle f(\mathbf{y}, \cdot) \rangle_\alpha \equiv \int_{V_\alpha} d\mathbf{x} g_\alpha(r, t, \mathbf{x}) f(\mathbf{y}, \mathbf{x}),
\] (50)
for any \( P_\alpha \)-integrable function \( f(\mathbf{y}, \cdot) : V_\alpha \rightarrow \mathbb{R} \), where the vector \( \mathbf{y} \) is some parameter.

2. **Definition - Homogeneous, stationary stochastic model**

The stochastic model \( \{g_\alpha, V_\alpha\} \) is denoted:

a) **homogeneous** if \( g_\alpha \) is independent of \( r \), namely
\[
g_\alpha = g_\alpha(t, \cdot);
\] (51)

b) **stationary** if \( g_\alpha \) is independent of \( t \), i.e.,
\[
g_\alpha = g_\alpha(r, \cdot).
\] (52)

3. **Definition - Deterministic variables**

Instead, if \( g_\alpha(r, t, \cdot) \) is a **deterministic PDF**, namely it is of the form
\[
g_\alpha(r, t, \mathbf{x}) = \delta^{(k)}(\mathbf{x} - \alpha_\alpha),
\] (53)

\( \delta^{(k)}(\mathbf{x} - \alpha_\alpha) \) denoting the \( k \)-dimensional Dirac delta in the space \( V_\alpha \), the hidden variables \( \alpha \) are denoted as **deterministic**.
Let us now assume that, for a suitable stochastic model \( \{g_\alpha, V_\alpha\} \), with \( g_\alpha \) non-deterministic, the stochastic variables \( Z_i \equiv Z_i(r, t, \alpha) \) and \( f_1(r, v, t, \alpha) \) (where \( Z_i(r, t, \cdot) \) and \( f_1(r, v, t, \cdot) \) are measurable functions) admit everywhere in \( \Omega \times I \) and \( \Gamma \times I \) the stochastic averages \( \langle Z_i \rangle_\alpha \) and \( \langle f_1 \rangle_\alpha \) defined by (50).

Hence, \( Z_i \equiv Z_i(r, t, \alpha) \), \( f_1(r, v, t, \alpha) \) and the mean-field force \( F(f_1) \) [see Sections 2, 3 and 4] admit also the stochastic decompositions

\[
Z_i = \langle Z_i \rangle_\alpha + \delta Z_i, \tag{54}
\]

\[
f_1 = \langle f_1 \rangle_\alpha + \delta f_1, \tag{55}
\]

\[
F(f_1) = \langle F(f_1) \rangle_\alpha + \delta F(f_1). \tag{56}
\]

In particular, unless \( g_\alpha(r, t, \cdot) \) is suitably smooth, it follows that generally \( \langle Z_i \rangle_\alpha, \delta Z_i \) and respectively \( \langle f_1 \rangle_\alpha, \delta f_1 \) may belong to different functional classes with respect to the variables \( (r, t) \).

C. Deterministic and stochastic INSE problems

Therefore, assuming, for definiteness, that all the fluid fields \( Z \), the volume force \( f \) and the initial and boundary conditions, are either deterministic or stochastic variables and both belong to the same functional class, i.e., are suitably smooth w.r. to \( (r, t) \) and \( \alpha \), Eqs. (32)- (37) define respectively a deterministic or stochastic initial-boundary value INSE problem. In both cases we shall assume that it admits a strong solution in \( \Omega \times I \) (or \( \Omega \times I \times V_\alpha \)).

In the first case, which characterizes flows to be denoted as regular, the fluid fields are by assumption physical observables, i.e., uniquely-defined, smooth, real functions of \( (r, t) \in \Omega \times I \) [with \( \Omega \), the configuration space, and \( \Omega \) its closure, to be assumed subsets of the Euclidean space on \( \mathbb{R}^3 \) and \( I \), the time axis, denoting a subset of \( \mathbb{R} \)].

In the second case, characterizing instead turbulent flows, the fluid fields are only conditional observables (see again Subsection A). In this case, besides \( (r, t) \), they may be assumed to depend also on a suitable stochastic variable \( \alpha \), (with \( \alpha \in V_\alpha \) and \( V_\alpha \) subset of \( \mathbb{R}^k \) with \( k \geq 1 \)). Hence they are stochastic variables too.

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