ON THE DEFORMATION OF KÄHLER METRICS IN THE PRESENCE OF CLOSED CONFORMAL VECTOR FIELDS

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ABSTRACT. In this paper we show that if a Kähler manifold is furnished with a closed conformal vector field, then, in suitable open subsets, its Kähler structure can be deformed into a different one. This deformation can be applied to the whole class of Kähler manifolds that arise as Riemannian cones over Sasaki manifolds, and gives an infinite family of new examples of Kähler manifolds, in each complex dimension greater than 2. Moreover, if we deform the Kähler metric of the cone over an Einstein-Sasaki manifold, then we get an Einstein-Kähler manifold, and a particular class of such deformations presents the same phenomenon of sectional holomorphic curvature decay that takes place when we deform the metric of the complex Euclidean space into that of the complex hyperbolic space. Two infinite families of such deformations arise by taking any of the two infinite families of Einstein-Sasaki manifolds constructed by Cvetic, Lu, Page and Pope.

1. INTRODUCTION

In classical literature (cf. [7], for example), one usually constructs the metric of the $n$–dimensional complex hyperbolic space $\mathbb{CH}^n$ by explicitly giving its expression in canonical complex coordinates. In this paper, we show that this metric arises as a particular example of a general deformation of the metric of a Kähler manifold $(M^n, J, g, \nabla)$, furnished with a closed conformal vector field $\xi \in \mathfrak{X}(M)$ (i.e., one such that $\nabla_X \xi = \psi X$, for all $X \in \mathfrak{X}(M)$), and that this deformation always gives, as result, another Kähler manifold $(\tilde{M}^n, J, \tilde{g}, \tilde{\nabla})$.

More precisely, if $(M, J, g = \langle \cdot, \cdot \rangle)$ is a Kähler manifold, $\xi \in \mathfrak{X}(M)$ is a closed conformal vector field on $M$ such that $g(\xi, \xi) := |\xi|^2 < c < +\infty$ on $M$ and $\mu = (c - |\xi|^2)^{-1}$, we prove in Theorem 2.1 that

$$\tilde{g} = \mu g + \mu^2 (\theta_\xi^2 + \theta_\xi^2)$$

defines another Kähler metric on $(M, \langle, \rangle)$. Moreover, in Proposition 2.3 we establish the completeness of $\tilde{g}$, provided $\xi$ and its conformal factor $\psi$ (i.e., the function $\psi \in C^\infty(M)$ such that $\nabla_X \xi = \psi X$, for all $X \in \mathfrak{X}(M)$) satisfy a reasonable set of conditions.

We then proceed to compare the geometries of $(M, J, g)$ and $(M, J, \tilde{g})$. More precisely, after relating the Levi-Civita connections of $g$ and $\tilde{g}$ in Proposition 3.1 we compare their holomorphic sectional curvatures in Theorem 3.2. However, for

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our purposes, a more interesting result is provided by Corollary 3.3, where we prove that, if \( \sup_M |\xi|^2 < c \) and \( X \in T_p M \) is unitary with respect to \( g \), then

\[
(1.1) \quad \hat{K}(X) \leq cK(X) + 2c \text{Ric}(\hat{\xi}) - 4\psi^2.
\]

Here, \( K(X) \) and \( \hat{K}(X) \) respectively denote the holomorphic sectional curvatures of \((M, g)\) and \((M, \tilde{g})\) along the holomorphic plane of \( T_p M \) generated by \( X \), and \( \text{Ric}(\hat{\xi}) \) denotes the Ricci curvature of \((M, g)\) in the direction of \( \xi \).

A class or Riemannian manifolds possessing closed conformal vector fields it that of warped product manifolds \( \tilde{M} = M \times_f N \), i.e., the product manifold \( M = I \times N \), of an open interval \( I \subset \mathbb{R} \) and a Riemannian manifold \((N, g_N)\), furnished with the Riemannian metric

\[
g = \pi_I^*dt^2 + (f \circ \pi_I)^2 \pi_N^*g_N,
\]

where \( \pi_I \) and \( \pi_N \) denote the canonical projections of \( M \) onto \( I \) and \( N \), respectively. In this case, \( \xi = (f \circ \pi_I)\partial_t \) is closed and conformal, with conformal factor \( \psi = f' \circ \pi_I \).

If \( N^{2m-1} \) is a Sasaki manifold with contact structure \( \theta \in \Omega^1(N) \), its Riemannian cone \( \tilde{M}^{2m} = (0, +\infty) \times_f N \) is a Kähler manifold with Kähler form \( \omega = 2tdt \wedge \theta + t^2d\theta \), such that \( \xi = t\partial_t \) and \( \psi = 1 \). Now, let \( c > 0 \) be given. By applying the above mentioned results to \( M = (0, \sqrt{c}) \times_f N \), we get new examples of Kähler manifolds. We also show, in Theorem 4.2, that these examples are, in fact, Einstein-Kähler.

More particularly, if \( M = (0, 1) \times_f N \) and \( N^{2m-1} \) is an Einstein-Sasaki manifold, it follows from (1.1) that

\[
\hat{K}(X) \leq K(X) - 4.
\]

Therefore, the phenomenon of sectional holomorphic curvature decay, which takes place when we pass from \( \mathbb{C}^n \) to \( \mathbb{CH}^n \), persists in a great many cases.

In view of the above discussion, together with the two infinite families of examples of Einstein-Sasaki manifolds constructed by Cvetic, Lu, Page and Pope in [2] (one in dimension 5 and the other in each odd dimension greater than 5), we conclude that the generic example we have been presenting actually generates two infinite families of new examples of complete Kähler manifolds, one in complex dimension 3 and the other in each complex dimension greater than 3.

Back to the general setting of a Kähler manifold \((M, J, g)\) furnished with a closed conformal vector field \( \xi \), we show, in Theorem 4.2, that the leaves of the integrable distribution \( \langle \xi \rangle \perp \) (restricted to the open set of nonsingular points of \( \xi \)) have parallel mean curvature vector in \((M, \tilde{g})\). Still in this context, our last result is Theorem 4.3, which proves that \((M, g)\) is locally isometric to the Riemannian cone over a Sasaki manifold; however, since no linear combination of \( \xi \) and \( J\xi \) (with smooth coefficients) is closed and conformal with respect to \( \tilde{g} \), it follows that \((M, J, \tilde{g})\) is, most likely, not isometric to the Riemannian cone over a Sasaki manifold.

Finally, for the reader’s convenience, we have collected some lengthy, though elementary, tensor computations in Section 5.

2. Deforming Kähler structures

In the sequel, we let \((M^n, J, g)\) be an \( n \)-dimensional Hermitian manifold, where \( n \) stands for its complex dimension. We also let \( \omega \in \Omega^2(M) \) denote the corresponding Kähler form, so that \( \omega(X, Y) = \langle JX, Y \rangle \), for all \( X, Y \in \mathfrak{X}(M) \). It is a standard
fact (cf. Chapter IX of [7], for instance) that $M$ is a Kähler manifold if and only if $J$ is parallel with respect to the Levi-Civita connection $\nabla$ of $g$, i.e., if and only if
\[
\nabla J = 0.
\]

Whenever convenient, we write $g = \langle \cdot, \cdot \rangle$ and let $| \cdot |$ denote the corresponding norm. Also, for $X \in \mathfrak{X}(M)$, we let $\theta_X$ denote the 1–form metrically dual to $X$, i.e., such that $\theta_X(Y) = \langle X, Y \rangle$, for $Y \in \mathfrak{X}(M)$; we also let $\theta^2_X$ denote the symmetrization of $\theta_X \otimes \theta_X$, i.e., the covariant symmetric 2–tensor field on $M$ such that
\[
\theta^2_X(Y, Z) = \theta_X(Y) \theta_X(Z),
\]
for $Y, Z \in \mathfrak{X}(M)$.

The following result shows how to construct, out of $g$, a new Kähler metric on $(M, J)$. We recall that a vector field $\xi$ on $M$ is closed conformal if there exists a smooth function $\psi$ on $M$ (said to be the conformal factor of $\xi$), such that
\[
\nabla_X \xi = \psi X,
\]
for all $X \in \mathfrak{X}(M)$. If $\xi$ is closed conformal, then straightforward computation (with the aid of Koszul’s formula for exterior differentiation) shows that $\theta_\xi$ is a closed 1–form.

**Theorem 2.1.** Let $(M, J, g = \langle \cdot, \cdot \rangle)$ be a Kähler manifold with Levi-Civita connection $\nabla$, and $\xi \in \mathfrak{X}(M)$ be a closed conformal vector field on $M$. If $|\xi|^2 < c$ on $M$, for some positive constant $c$, and $\mu = (c - |\xi|^2)^{-1}$, then the covariant symmetric 2–tensor field
\[
\tilde{g} = \mu g + \mu^2 (\theta^2_\xi + \theta^2_J\xi)
\]
defines another Kähler metric on $(M, J)$.

**Proof.** The 2–tensor $\tilde{g}$ is clearly positive definite, and thus defines a Riemannian metric on $M$. On the other hand, for $X, Y \in \mathfrak{X}(M)$, the Hermitian character of $\langle \cdot, \cdot \rangle$ gives
\[
\tilde{g}(JX, JY) = \mu \langle JX, JY \rangle + \mu^2 (\theta^2_J(X, Y) + \theta^2_\xi(X, Y))
\]
\[
= \mu \langle X, Y \rangle + \mu^2 (\langle J\xi, JX \rangle \langle J\xi, JY \rangle + \langle J_\xi, JX \rangle \langle J_\xi, JY \rangle)
\]
\[
= \mu \langle X, Y \rangle + \mu^2 (\langle J_\xi, J_\xi X \rangle \langle J_\xi, J_\xi Y \rangle + \langle J_\xi, X \rangle \langle J_\xi, Y \rangle)
\]
\[
= \mu \langle X, Y \rangle + \mu^2 (\langle J_\xi, X \rangle \langle J_\xi, Y \rangle + \langle J_\xi, X \rangle \langle J_\xi, Y \rangle)
\]
\[
= \tilde{g}(X, Y),
\]
so that $\tilde{g}$ is also Hermitian with respect to $J$.

Next, let $\tilde{\omega}$ be the Kähler form of $\tilde{\omega}$. For $X, Y \in \mathfrak{X}(M)$, we have
\[
\tilde{\omega}(X, Y) = \tilde{g}(JX, Y) = \mu \langle JX, Y \rangle + \mu^2 (\theta^2_J(X, Y) + \theta^2_\xi(X, Y))
\]
\[
= \mu \omega(X, Y) + \mu^2 (\langle J_\xi, JX \rangle \langle J_\xi, Y \rangle + \langle J_\xi, JX \rangle \langle J_\xi, Y \rangle)
\]
\[
= \mu \omega(X, Y) + \mu^2 (\langle J_\xi, J_\xi X \rangle \langle J_\xi, J_\xi Y \rangle + \langle J_\xi, X \rangle \langle J_\xi, Y \rangle)
\]
\[
= \mu \omega(X, Y) + \mu^2 (\langle J_\xi, X \rangle \langle J_\xi, Y \rangle + \langle J_\xi, X \rangle \langle J_\xi, Y \rangle)
\]
\[
= \mu \omega(X, Y) + \mu^2 (\theta_\xi \wedge \theta_J\xi)(X, Y)
\]
and, hence,
\[
\tilde{\omega} = \mu \omega + \mu^2 \theta_\xi \wedge \theta_J\xi.
\]
Since $\omega$ and $\theta_\xi$ are closed (the first one being the Kähler form of a Kähler manifold and the second one being metrically dual to a closed conformal vector field), it follows that

\[ d\tilde{\omega} = d\mu \wedge \omega + 2\mu d\mu \wedge \theta_\xi \wedge \theta J_\xi - \mu^2 \theta_\xi \wedge d\theta J_\xi. \]

Letting $\psi$ be the conformal factor of $\xi$ and $X \in \mathfrak{X}(M)$, it follows that

\[ d\mu(X) = X(\mu) = (c - |\xi|^2)^{-2} X(\xi, \xi) \]
\[ = 2\mu^2 (\nabla_X \xi, \xi) = 2\mu^2 (\psi X, \xi) \]
\[ = 2\psi \mu^2 \theta_\xi(X) \]

and, hence,

\[ d\mu = 2\psi \mu^2 \theta_\xi. \]

Therefore, for $X,Y \in \mathfrak{X}(M)$, Koszul’s formula and (2.1) give

\[ d\theta J_\xi(X,Y) = X(\theta J_\xi(Y)) - Y(\theta J_\xi(X)) - \theta J_\xi([X,Y]) \]
\[ = X(J_\xi Y) - Y(J_\xi X) - \langle J_\xi, [X,Y] \rangle \]
\[ = \langle \nabla_X J_\xi, Y \rangle - \langle \nabla_Y J_\xi, X \rangle \]
\[ = \langle J \nabla_X \xi, Y \rangle - \langle J \nabla_Y \xi, X \rangle \]
\[ = \langle J(\psi X), Y \rangle - \langle J(\psi Y), X \rangle \]
\[ = 2\psi \langle JX, Y \rangle = 2\psi \omega(X,Y), \]

i.e., $d\theta J_\xi = 2\psi \omega$. Substituting the above expressions for $d\mu e d\theta J_\xi$ in (2.3) and taking into account that $\theta_\xi \wedge \theta_\xi = 0$, we finally get

\[ d\tilde{\omega} = 2\psi \mu^2 \theta_\xi \wedge \omega + 4\psi \mu^2 \theta_\xi \wedge \theta_\xi \wedge \theta J_\xi - \mu^2 \theta_\xi \wedge (2\psi \omega) = 0. \]

Remark 2.2. The process of deformation of $g$ into $\tilde{g}$, given by the previous result, is irreversible, in the sense that (as one can easily check with the aid of the result of Proposition 3.1) $\xi$ is no longer closed conformal with respect to $\tilde{g}$.

Our next result gives a reasonable set of conditions under which $(M, \tilde{g})$ is a complete Riemannian manifold.

Proposition 2.3. Under the hypotheses of Theorem 2.1, suppose that the conformal factor $\psi$ of $\xi$ is bounded and does not vanish inside a compact subset of $M$. If $|\xi| : M \to [0, +\infty)$ is proper and such that $\sup_M |\xi| = c$, then $(M, \tilde{g})$ is complete.

Proof. Let $\tilde{\ell}(\cdot)$ denote length with respect to $\tilde{g}$. By the divergent curve lemma (cf. exercise 7.5 of [4]), it suffices to show that, if a smooth curve $\gamma : [0, +\infty) \to M$ escapes from all compact subsets of $M$, then $\tilde{\ell}(\gamma) = +\infty$. To this end, start by observing that

\[ \tilde{g}(v,v) = \mu g(v,v) + \mu^2 (|\xi|^2 + \langle J_\xi, v \rangle^2) \geq (\mu |\xi|)^2. \]
Let $K \subset M$ be a compact set such that $\psi \neq 0$ on $K^c$, and $t_0 > 0$ such that $\gamma(t) \not\in K$ for $t > t_0$. If $\sup_M |\psi| = \alpha < +\infty$, then
\[
\ell(\gamma_{[0,t]}) \geq \int_{t_0}^{t} \frac{1}{\psi(\gamma(s))} \cdot \frac{1}{c - |\xi(\gamma(s))|} \left| \langle \xi(\gamma(s)), \nabla_{\gamma'}(\xi) \rangle \right| ds
\]
\[
\geq \frac{1}{2\alpha} \int_{t_0}^{t} \frac{1}{c - |\xi(\gamma(s))|} \left( \log(c - |\xi(\gamma(t_0))|) - \log(c - |\xi(\gamma(t))|) \right) ds
\]
\[
= \frac{1}{2\alpha} \left( \log(c - |\xi(\gamma(t_0))|) - \log(c - |\xi(\gamma(t))|) \right).
\]
Let $\epsilon > 0$ be given. Since $|\xi|^2$ is proper, $|\xi|^2 < c$ and $\sup_M |\xi|^2 = c$, there exists a compact subset $L_\epsilon$ of $M$ such that $|\xi|^2 > c - \epsilon$ in $L_\epsilon$. Since $\gamma$ is divergent, there exists $t_\epsilon > t_0$ such that $\gamma(t) \in L_\epsilon$ for $t > t_\epsilon$. Hence, for $t > t_\epsilon$, the above computations give
\[
\ell(\gamma_{[0,t]}) \geq \frac{1}{2\alpha} \left( \log(c - |\xi(\gamma(t_0))|) - \log(c - |\xi(\gamma(t))|) \right),
\]
so that $\ell(\gamma) = \lim_{t \to +\infty} \ell(\gamma_{[0,t]}) = +\infty$. \[\square\]

**Example 2.4.** In the complex Euclidean $n$–space $\mathbb{C}^n$, let $J$ be the standard quasi-complex structure, $g = \langle \cdot, \cdot \rangle$ be the standard metric and $\mathbb{B}^n = \{z \in \mathbb{C}^n; |z| < 1\}$. Since the vector field $\xi(p) = p$ is closed and conformal, we can furnish $\mathbb{B}^n$ with a Kähler metric $\hat{g}$, such that
\[
\hat{g} = \frac{1}{1 - |\xi|^2} \langle \cdot, \cdot \rangle + \frac{1}{(1 - |\xi|^2)^2} (\theta_1^2 + \theta_2^2) \cdot
\]
An immediate application of the previous proposition readily gives the completeness of $(\mathbb{B}^n, \hat{g})$. Therefore, the formula of Theorem 3.2 for the holomorphic sectional curvature of $(\mathbb{B}^n, J, \hat{g})$, together with Hawley-Igusa theorem on the classification of complete, simply connected, Kähler manifolds of constant holomorphic sectional curvature (cf. [5] and [6]), shows that $(\mathbb{B}^n, J, \hat{g})$ is nothing but the complex hyperbolic space $\mathbb{CH}^n$.

**Example 2.5.** Let $I \subset \mathbb{R}$ be an open interval with the standard metric $dt^2$, $f : I \to (0, +\infty)$ a smooth function and $(N^{n-1}, g_N)$ an $(n - 1)$–dimensional Riemannian manifold. The warped product $M^n = I \times f N^{n-1}$ (with warping function $f$) is the product manifold $I \times N$, furnished with the Riemannian metric
\[
g = \pi^* dt^2 + (f \circ \pi)^2 \pi_N^* g_N,
\]
s where $\pi_I$ and $\pi_N$ denote the canonical projections of $M$ onto $I$ and $N$, respectively. It is a classical result (cf. Chapter 7 of [8]) that the vector field $\xi = (f \circ \pi) \partial_t$ is closed and conformal, with conformal factor $\psi = f' \circ \pi_I$.

Now, let $f(t) = t$ for $t > 0$ and $N^{2m-1}$ be a Sasaki manifold with contact structure $\theta \in \Omega^1(N)$. It is a well known fact (cf. [8], for instance) that the Riemannian cone $M^{2m} = (0, +\infty) \times t N$ over $N$ is a Kähler manifold with Kähler form $\omega = 2dt \wedge \theta + t^2 \theta$ (note that $\omega$ determines the quasi-complex structure $J$ of $M$). As in the previous example, for a given $c > 0$ we can apply Theorem 3.2 to $M = (0, \sqrt{c}) \times t N$. Theorem 3.3 will show that $M$ is also Einstein.
If $c = 1$ and $N^{2n-1}$ is an Einstein-Sasaki manifold (i.e., a Sasaki manifold whose metric is also Einstein, so that its Riemannian cone is Ricci-flat), then Corollary 3.3 assures that, upon deforming the Kähler metric of $M$, the holomorphic curvature of $M$ decays.

On the other hand, in [2] the authors constructed an infinite family of 5—dimensional Einstein-Sasaki manifolds, with isometry groups all isomorphic to $U(1) \times U(1) \times U(1)$, as well as another infinite family of Einstein-Sasaki manifolds, in each odd dimension $2m - 1 \geq 5$ and with isometry group $U(1)^m$. Therefore, the generic example we have been presenting actually generates two infinite families of new Kähler manifolds, one in complex dimension 3 and the other in each complex dimension $n \geq 3$. Finally, we stress that the isometry groups of the examples of Cvetic-Lu-Page-Pope, together with Hawley-Igusa theorem, guarantee that these new families of Kähler manifolds are not isometric either to the complex Euclidean space, nor to the complex hyperbolic space.

3. ON GEOMETRIC INVARIANTS OF $(M^n, J, \tilde{g})$

In this section (cf. Theorem 3.2), we relate the holomorphic sectional curvatures of $(M, J, g)$ and $(M, J, \tilde{g})$. To this end, we need first to relate the corresponding Levi-Civita connections, and we do so in the coming result, whose proof can be found in Section 5. Along this section, we write $|\xi|^2$ for $g(\xi, \xi)$.

**Proposition 3.1.** Let $(M, J, g)$ be a Kähler manifold, $\xi \in \mathfrak{X}(M)$ be a closed conformal vector field such that $\text{sup}_M |\xi|^2 < c$, and $\tilde{g}$ be the Kähler metric on $(M, J)$ given as in Theorem 2.7. For $X \in \mathfrak{X}(M)$, let $\nabla$ and $\tilde{\nabla}$ respectively denote the Levi-Civita connections of $g$ and $\tilde{g}$. Then,

\[(3.1) \quad \tilde{\nabla}_X Y = \nabla_X Y + \psi \mu \{\langle \xi, X \rangle Y + \langle \xi, Y \rangle X + \langle J\xi, X \rangle JY + \langle J\xi, Y \rangle JX\},\]

where $\psi$ is the conformal factor of $\xi$.

Our next result relates the holomorphic sectional curvatures of $g = \langle \cdot, \cdot \rangle$ and $\tilde{g}$. For its proof, see Section 5.

**Theorem 3.2.** Let $(M, J, g)$ be a Kähler manifold, $\xi \in \mathfrak{X}(M)$ be a closed conformal vector field such that $\text{sup}_M |\xi|^2 < c$, and $\tilde{g}$ be the Kähler metric on $(M, J)$ given as in Theorem 2.7. For $X \in T_p M$ unitary with respect to $g$, we have

\[
\tilde{K}(X) = \frac{1}{\tilde{g}(X, X)} \{ \mu K(X) + \mu^2 \text{Ric}(\tilde{\xi}) \langle X, \xi \rangle^2 + \langle X, J\xi \rangle^2 \} + \frac{2 \mu \text{Ric}(\tilde{\xi}) \langle X, \xi \rangle^2 + \langle X, J\xi \rangle^2}{\tilde{g}(X, X)} - 4 \psi^2,
\]

where $K(X)$ and $\tilde{K}(X)$ respectively denote the holomorphic sectional curvatures of $(M, J, g)$ and $(M, J, \tilde{g})$ with respect to $X$, $\text{Ric}(\tilde{\xi})$ the Ricci curvature of $g$ in the direction of $\tilde{\xi}$ (taken as 0 if $\tilde{\xi}(p) = 0$) and $\psi$ denotes the conformal factor of $\xi$.

The corollary below extends, to general deformations, the phenomenon of holomorphic curvature decay that occurs when we pass from $\mathbb{C}^n$ to $\mathbb{C}H^n$.

**Corollary 3.3.** Let $(M, J, g)$ be a Kähler manifold, $\xi \in \mathfrak{X}(M)$ be a closed conformal vector field such that $\text{sup}_M |\xi|^2 < c$, and $\tilde{g}$ be the Kähler metric on $(M, J)$ given as in Theorem 2.7. For $X \in T_p M$ unitary with respect to $g$, we have:
(a) If $\mathbf{X} \perp \xi$, $\mathbf{J}\xi$, then $\tilde{K}(X) \leq cK(X) - 4\psi^2$.

(b) In general, $\tilde{K}(X) \leq cK(X) + 2c\text{Ric}(\hat{\xi}) - 4\psi^2$.

**Proof.** If $\mathbf{X} \perp \xi$, $\mathbf{J}\xi$, then $\tilde{g}(X, X) = \mu$. Therefore, our previous result gives

$$\tilde{K}(X) = (c - |\xi|^2)K(X) - 4\psi^2 \leq cK(X) - 4\psi^2.$$ 

For a general $X \in T_pM$ unitary, let $A = \langle X, \xi \rangle^2 + \langle X, J\xi \rangle^2$ and write

$$\tilde{K}(X) = \frac{\mu}{\tilde{g}(X, X)^2} \cdot K(X) + \frac{(\mu^2 + 2\mu\tilde{g}(X, X))A}{\tilde{g}(X, X)^2} \cdot \text{Ric}(\hat{\xi}) - 4\psi^2,$$

For the first summand, note that

$$1 + \mu A = \frac{c - (|\xi|^2 - \langle X, \xi \rangle^2 - \langle JX, \xi \rangle^2)}{c - |\xi|^2} \geq \frac{c - |\xi|^2}{c - |\xi|^2} = 1,$$

where, to get the inequality, we used orthonormal expansion at $T_pM$ with respect to a Hermitian basis containing $X$ and $JX$. Hence,

$$\frac{\mu}{\tilde{g}(X, X)^2} = \frac{1}{\mu(1 + \mu A)^2} \leq \frac{1}{\mu} \leq c.$$

For the second summand, substituting $\tilde{g}(X, X) = \mu + \mu^2A$ we get

$$\frac{(\mu^2 + 2\mu\tilde{g}(X, X))A}{\tilde{g}(X, X)^2} = \frac{(3 + 2\mu A)A}{(1 + \mu A)^2} = \frac{1}{\mu} \cdot \frac{3y + 2y^2}{1 + 2y + y^2},$$

where $y = \mu A$. It now suffices to observe that $\frac{1}{\mu} \leq c$ and (since $y \geq 0$)

$$\frac{3y + 2y^2}{1 + 2y + y^2} = 2 - \frac{1}{y + 1} = \frac{1}{(y + 1)^2} < 2.$$

We end this section by proving that, if $(N, g_N)$ is an Einstein-Sasaki manifold, $c > 0$ and $M = (0, \sqrt{c}) \times lN$, then, upon deforming the metric of $M$ as in Theorem 2.1, we get an Einstein-Kähler manifold. We begin with the following proposition, whose proof is quite similar to that of Theorem 3.2 and will be omitted.

**Proposition 3.4.** Let $(M, J, g)$ be a Kähler manifold, $\xi \in \mathbf{X}(M)$ be a closed conformal vector field such that $\sup_M |\xi|^2 < c$, and $\tilde{g}$ be the Kähler metric on $(M, J)$ given as in Theorem 2.1. Also, let $R$ and $\tilde{R}$ respectively denote the curvature tensors of $g$ and $\tilde{g}$. Then, for $X, Y, Z \in \mathbf{X}(M)$, we have

$$\tilde{R}(X, Y)Z = R(X, Y)Z - \text{Ric}(\hat{\xi})\mu(\langle X, \xi \rangle \langle Z, \xi \rangle Y + \langle X, \xi \rangle \langle Y, J\xi \rangle JZ + \langle X, \xi \rangle \langle Z, J\xi \rangle JY - Y, \xi \rangle \langle X, \xi \rangle JZ - \langle Y, \xi \rangle \langle X, \xi \rangle JZ - \langle Y, \xi \rangle \langle JX, \xi \rangle JZ)$$

$$+ \psi^2\mu(\langle X, \xi \rangle \langle Z, \xi \rangle Y + \langle X, \xi \rangle \langle JY, Z) JZ + \langle JX, Z \rangle JY - \langle Y, Z \rangle JX - \langle JY, Z \rangle JX$$

$$+ \psi^2\mu^2(\langle X, \xi \rangle \langle Z, \xi \rangle Y - \langle Y, \xi \rangle \langle Z, \xi \rangle X + \langle X, \xi \rangle \langle Z, J\xi \rangle JY$$

$$- \langle Y, J\xi \rangle \langle X, \xi \rangle JZ + \langle X, J\xi \rangle \langle JX, \xi \rangle JY - \langle Y, J\xi \rangle \langle Z, J\xi \rangle JX$$

$$- \langle X, J\xi \rangle \langle Z, \xi \rangle JY - (X, \xi) \langle JY, Z \rangle) JX$$

$$+ 2\langle X, \xi \rangle \langle Y, \xi \rangle JZ - 2\langle Y, \xi \rangle \langle X, \xi \rangle JZ)$$

where $\psi$ is the conformal factor of $\xi$. 

In order to compute the Ricci tensor of \((M, J, \tilde{g})\), fix \(p \in M\) and take a Hermitian frame field \((e_1, Je_1, \ldots, e_{n-1}, Je_{n-1}, e_n = \tilde{\xi}, Je_n = J\tilde{\xi})\) on \((M, g)\), defined on a neighborhood \(U\) of \(p\). If we set
\[
\tilde{e}_j = \left\{ \begin{array}{ll}
e_j/\sqrt{\mu_j}, & \text{for } 1 \leq j < n \\
e_n/\sqrt{\mu_n}, & \text{for } j = n, \end{array} \right.
\]
an easy computation shows that \((\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{n-1}, \tilde{e}_n, J\tilde{e}_n)\) is a Hermitian frame field on \((M, \tilde{g})\), defined on \(U\). Then, for \(X, Y \in \frak{X}(M)\) and if \(\text{Ric}\) denotes the Ricci tensor of \((M, \tilde{g})\), we have
\[
\tilde{\text{Ric}}(X, Y) = \sum_{i=1}^{n}\{\tilde{g}(\tilde{R}(X, \tilde{e}_i)\tilde{e}_i, Y) + \tilde{g}(\tilde{R}(X, \tilde{e}_i)J\tilde{e}_i, Y)\}
\]
\[
= \sum_{i=1}^{n-1}\mu^{-1}\{\tilde{g}(\tilde{R}(X, e_i)e_i, Y) + \tilde{g}(\tilde{R}(X, Je_i)Je_i, Y)\} + c^{-1}\mu^{-2}\{\tilde{g}(\tilde{R}(X, \tilde{\xi})\tilde{\xi}, Y) + \tilde{g}(\tilde{R}(X, J\tilde{\xi})J\tilde{\xi}, Y)\}.
\]

Now, a quite long but straightforward computation (cf. Section 5), using the formula of Proposition 3.4 gives
\[
\tilde{\text{Ric}}(X, Y) = \text{Ric}(X, Y) - c^{-1}[(\text{R}(X, \xi)\xi, Y) + (\text{R}(X, J\xi)J\xi, Y)] + c^{-1}\text{Ric}(\xi)(\xi, X)\xi, Y) + (\xi, Y)\text{Ric}(X, J\xi)Y, Y) + 2\mu|\xi|^2(\langle X, Y, \xi, Y, J\xi \rangle + \langle X, J\xi, Y, Y, \xi \rangle)
\]
\[
+ \mu\langle Y, \xi, \text{Ric}(X, \xi) + \mu\langle Y, J\xi, \text{Ric}(X, \xi) - c^{-1}\mu[(\text{R}(X, \xi)\xi, J\xi, Y, J\xi) + (\text{R}(X, J\xi)J\xi, Y, \xi)] - 2(n + 1)\psi^2\mu[(\langle X, Y, \xi, Y, J\xi \rangle + \langle X, J\xi, Y, Y, \xi \rangle]
\]
\]
\[
(3.4)
\]

At this point, we specialize our discussion to the case of the Riemannian cone over a Sasaki manifold \((N, g_N)\) of dimension \(2n - 1 \geq 5\), so that, for a given \(c > 0\), \(M^{2n} = (0, \sqrt{c}) \times N^{2n-1}\) is a Kähler manifold. Recall that, in such a case, \(\xi = t\tilde{\theta}\) and \(\psi = 1\). If we let \(R_N\) denote the curvature operator of \((N, g_N)\), then Proposition 1.2 of [9] gives
\[
R_N(X, J\xi)Y = g_N(J\xi, Y)X - g_N(X, Y)J\xi, \quad \text{for all } X, Y \in \frak{X}(N).
\]
We are now in position to state and prove the following result.

**Theorem 3.5.** In the notations of the previous paragraph, if \((N, g_N)\) is Einstein-Sasaki, then \((M, J, \tilde{g})\) is an Einstein-Kähler manifold.

**Proof.** We already know that \((M, J, \tilde{g})\) is a Kähler manifold. On the other hand, as we pointed out in Example 2.3, if \((N, g_N)\) is Einstein-Sasaki, then \((M, g)\) is Ricci flat. Therefore, it follows from (3.4) that, for \(X, Y \in \frak{X}(M)\),
\[
\tilde{\text{Ric}}(X, Y) = -c^{-1}[(\text{R}(X, \xi)\xi, Y) + (\text{R}(X, J\xi)J\xi, Y)] - c^{-1}\mu[(\text{R}(X, \xi)\xi, J\xi, Y, J\xi) + (\text{R}(X, J\xi)J\xi, Y, \xi)] - 2(n + 1)\psi^2\mu[(\langle X, Y, \xi, Y, J\xi \rangle + \langle X, J\xi, Y, Y, \xi \rangle]
\]
\[
\]
\[
(3.5)
\]

If \(X = \xi\), Proposition 7.42(4) of [8] gives \(\text{R}(\xi, J\xi)J\xi = 0\) and, hence,
\[
\tilde{\text{Ric}}(\xi, Y) = -2(n + 1)\mu[\langle \xi, Y, Y, \xi \rangle + \mu(\xi, Y)] = -2(n + 1)\tilde{g}(\xi, Y).
\]
Now, assume that $(X, \xi) = 0$. Invoking Proposition 7.42(2) of [8], we get $R(X, \xi)\xi = 0$; moreover, item (3) of that result, together with (3.5) and the fact that the curvature convention in [8] is opposite to ours give $R(X, J\xi)J\xi = 0$.

Substituting these relations in the formula for $\tilde{\text{Ric}}(X, Y)$, we arrive at

$$\tilde{\text{Ric}}(X, Y) = -2(n + 1)\mu\langle X, Y \rangle + \mu\langle X, J\xi \rangle\langle Y, J\xi \rangle = -2(n + 1)\tilde{g}(X, Y).$$

$\square$

4. Some additional results

Let $(M, g)$ be a Riemannian manifold possessing a nontrivial closed conformal vector field $\xi$. In the open set $\Omega$ of $M$ where $\xi \neq 0$ (it can be shown (cf. [10]) that $\Omega$ consists of isolated points), let $\mathcal{D}$ be the distribution generated by $\xi$ and $J\xi$. It is always integrable, for, by (2.1) and the conformal character of $\xi$, $\mathcal{D}$ is always integrable, for, by (2.1) and the conformal character of $\xi$,

$$[\xi, J\xi] = \nabla_\xi J\xi - J\nabla_\xi \xi = J\nabla_\xi \xi - \psi J\xi = 0.$$  

Moreover, if $\Sigma^2$ is a leaf of $\mathcal{D}$ and we let $\mathcal{N}$ denote the Nijenhuis tensor of $\Sigma$, it is immediate to see that $\mathcal{N} = 0$, so that $\Sigma$ is a complex curve in $(M^n, J, g)$, and the restriction of $g$ to $\Sigma$ (which we shall also call $g$) is Kähler.

With respect to $g$, $\Sigma$ is a totally geodesic surface in $M$; in fact, if we let $\alpha$ denote its second fundamental form and $(\cdot)^\perp$ the orthogonal projection onto $T(\Sigma, g)^\perp$, then

$$\alpha(\xi, J\xi) = \langle \nabla_\xi J\xi \rangle^\perp = \langle J\nabla_\xi \xi \rangle^\perp = \langle (\psi J\xi) \rangle^\perp = 0.$$  

With respect to the restriction of $\tilde{g}$ (which we shall also call $\tilde{g}$), $\Sigma$ is also a complex curve in $(M^n, J, \tilde{g})$, so (cf. Chapter IX of [7]) it is at least a minimal surface of $(M, J, \tilde{g})$. Actually, the following result is true.

**Proposition 4.1.** $(\Sigma, \tilde{g})$ is totally geodesic in $(M, \tilde{g})$.

**Proof.** If $\tilde{\alpha}$ denotes the second fundamental form of $(\Sigma, \tilde{g})$ in $(M, \tilde{g})$ and $(\cdot)^\perp$ the orthogonal projection onto $T(\Sigma, \tilde{g})^\perp$, then

$$\tilde{\alpha}(\xi, J\xi) = \langle \tilde{\nabla}_\xi J\xi \rangle^\perp = \tilde{\nabla}_\xi J\xi - \tilde{g}(\tilde{\nabla}_\xi J\xi, \xi)\frac{\xi}{\tilde{g}(\xi, \xi)} - \tilde{g}(\tilde{\nabla}_\xi J\xi, J\xi)\frac{J\xi}{\tilde{g}(J\xi, J\xi)}.$$

Now, the formula of Proposition 3.1 together with (2.1), gives

$$\tilde{\nabla}_\xi J\xi = \nabla_\xi J\xi + 2\psi \mu\langle \xi, \xi \rangle J\xi = J\nabla_\xi \xi + 2\psi \mu|\xi|^2 J\xi = \psi(1 + 2\mu|\xi|^2) J\xi.$$  

from which we get $\tilde{g}(\tilde{\nabla}_\xi J\xi, \xi) = 0$ and $\tilde{g}(\tilde{\nabla}_\xi J\xi, J\xi) = \psi(1 + 2\mu|\xi|^2) \tilde{g}(J\xi, J\xi)$. Substituting these expressions in the formula for $\tilde{\alpha}(\xi, J\xi)$, it easily follows that it is identically zero. $\square$

Again in $\Omega$, consider the distribution $(\xi)^\perp$, generated by the vector fields orthogonal to $\xi$ with respect to $g$. According to [11], this is also integrable, with leaves totally umbilic in $(M^n, g)$. In fact, if we let $N^{2n-1}$ be such a leaf and $\alpha$ its second fundamental form with respect to $g$, then

$$\alpha(X, Y) = -\psi\langle X, Y \rangle \frac{\xi}{|\xi|^2},$$  

for all $X, Y \in \mathfrak{X}(M)$.  

Since \( \langle J\xi, \xi \rangle = 0 \), we have \( J\xi \in \mathfrak{X}(N) \). In fact, \( J\xi \) is a Killing vector field in \( N \), for, if \( X, Y \in \mathfrak{X}(N) \), \((\cdot)\top\) denotes orthogonal projection onto \( T(N, g) \) and we recall the expression for the Levi-Civita connection of \((N, g)\) (cf. Chapter 6 of [4]), then

\[
\langle (\nabla_X J\xi)^\top, Y \rangle + \langle X, (\nabla_Y J\xi)^\top \rangle = \langle \nabla_X J\xi, Y \rangle + \langle X, \nabla_Y J\xi \rangle
\]

\[
= \langle J\nabla_X \xi, Y \rangle + \langle X, J\nabla_Y \xi \rangle
\]

\[
= \psi(\langle JX, Y \rangle + \langle X, JY \rangle) = 0,
\]

so that Killing’s equation is satisfied.

The next result shows what happens to \( N \), viewed as a hypersurface of \((M, \tilde{g})\).

**Theorem 4.2.** In the above notations, we have that:

(a) \( J\xi \) is a Killing vector field in \((N, \tilde{g})\).

(b) If \( \tilde{\alpha} \) denotes the second fundamental form and \( \tilde{\nu} \) the (normalized) mean curvature vector of the inclusion of \((N, g)\) into \((M, \tilde{g})\), then:

(i) \( \tilde{\alpha}(X, Y) = -\psi(\langle X, Y \rangle + 2\mu\langle J\xi, X \rangle \langle J\xi, Y \rangle) \frac{\tilde{\nu}}{\langle \xi, \xi \rangle} \), for \( X, Y \in \mathfrak{X}(N) \).

(ii) \((2n - 1)\tilde{\nu} = -\frac{\psi(1 + \epsilon \langle \xi, \xi \rangle)}{c_{\epsilon e} \langle \xi, \xi \rangle} \xi \). In particular, \((N, \tilde{g})\) has parallel mean curvature vector in \((M, \tilde{g})\).

**Proof.**

(a) For \( X, Y \in \mathfrak{X}(N) \), it follows from the expression for the Levi-Civita connection of \((N, \tilde{g})\) that

\[
\tilde{g}((\tilde{\nabla}_X J\xi)^\top, Y) = \tilde{g}(\tilde{\nabla}_X J\xi, Y)
\]

\[
= \mu \langle \tilde{\nabla}_X J\xi, Y \rangle + \mu^2(\langle \tilde{\nabla}_X J\xi, \xi \rangle \langle Y, \xi \rangle + \langle \tilde{\nabla}_X J\xi, J\xi \rangle \langle Y, J\xi \rangle).
\]

Now, Proposition 5.1 (2.1) and the conformal character of \( \xi \) give

\[
\tilde{\nabla}_X J\xi = \psi(\langle JX, \mu(\langle \xi, X \rangle J\xi - \langle J\xi, X \rangle \xi + |\xi|^2 JX \rangle, Y \rangle)
\]

and, hence,

\[
\tilde{g}((\tilde{\nabla}_X J\xi)^\top, Y) = \psi \mu(\langle JX, \mu(\langle \xi, X \rangle J\xi - \langle J\xi, X \rangle \xi + |\xi|^2 JX \rangle, Y \rangle
\]

\[
+ \psi \mu^2(\langle JX, \mu(-\langle J\xi, X \rangle \xi + |\xi|^2 JX \rangle, \xi \rangle \langle Y, \xi \rangle
\]

\[
+ \psi \mu^2(\langle JX, \mu(\langle \xi, X \rangle J\xi + |\xi|^2 JX \rangle, J\xi \rangle \langle Y, J\xi \rangle
\]

\[
= \psi \mu(1 + \mu |\xi|^2) \langle JX, Y \rangle + \psi \mu^2(\langle \xi, X \rangle \langle J\xi, Y \rangle - \langle J\xi, X \rangle \langle \xi, Y \rangle)
\]

\[
+ \psi \mu^2(1 + 2\mu |\xi|^2) \langle \langle \xi, X \rangle \langle J\xi, Y \rangle - \langle J\xi, X \rangle \langle \xi, Y \rangle \rangle).
\]

Changing the occurences of \( X \) and \( Y \) in the above expression, it readily follows that

\[
\tilde{g}((\tilde{\nabla}_X J\xi)^\top, Y) + \tilde{g}(X, (\tilde{\nabla}_Y J\xi)^\top) = 0,
\]

and Killing’s equation is satisfied once more.

(b) An easy computation shows that \( \xi \) is orthogonal to \((N, \tilde{g})\). Therefore, if \( X, Y \in \mathfrak{X}(N) \) and we let \((\cdot)\perp\) denote orthogonal projection onto \( T(N, \tilde{g})\), then

\[
\tilde{\alpha}(X, Y) = (\tilde{\nabla}_X Y)\perp = \tilde{g}(\tilde{\nabla}_X Y, \xi) \frac{\xi}{\tilde{g}(\xi, \xi)}.
\]

We now observe that

\[
\tilde{\nabla}_X Y = \nabla_X Y + \psi \mu(\langle J\xi, X \rangle JY + \langle J\xi, Y \rangle JX),
\]
so that the definition of $\tilde{g}$ gives
\[
\tilde{g}(\tilde{\nabla}_X Y, \xi) = \mu(\tilde{\nabla}_X Y, \xi) + \mu^2(\langle \tilde{\nabla}_X Y, \xi \rangle \langle \xi, \xi \rangle + \langle \tilde{\nabla}_X J\xi, J\xi \rangle) \\
= \mu(1 + \mu|x|^2)(\tilde{\nabla}_X Y, \xi) \\
= c\mu^2(\langle \tilde{\nabla}_X Y, \xi \rangle + \psi\mu(\langle J\xi, X \rangle JY + \langle J\xi, Y \rangle JX), \xi) \\
= c\mu^2(\langle \tilde{\nabla}_X Y, \xi \rangle - 2\psi\mu(\langle J\xi, X \rangle JY)) \\
= -c\psi\mu^2(\langle X, Y \rangle + 2\mu(\langle J\xi, X \rangle JY)).
\]

The formula for $\tilde{\alpha}(X, Y)$ in (i) follows easily from the above computations, together with the fact that
\[
(4.1) \quad \tilde{g}(\xi, \xi) = \mu|x|^2(1 + \mu|x|^2) = c\mu^2|x|^2.
\]

In order to prove (ii), let $p \in N$ and $(e_1, J)e_1, \ldots, e_{n-1}, J)e_{n-1}, e_n = \hat{\xi}, J)e_n = \hat{\xi}$ be a Hermitian frame field on $(M, g)$, defined on a neighborhood $U$ of $p$ and adapted to $(N, g)$. Take $(\hat{e}_1, \hat{J)e}_1, \ldots, \hat{e}_{n-1}, \hat{J)e}_{n-1}, \hat{J)e}_n)$ as in (3.2), so that it is a Hermitian frame field on $(M, \tilde{g})$, defined on $U$ and adapted to $(N, \tilde{g})$. Therefore,
\[
(2n - 1)\tilde{H} = \frac{1}{\mu} \sum_{j=1}^{n-1} (\tilde{\alpha}(e_j, e_j) + \tilde{\alpha}(J)e_j, J)e_j) + \frac{1}{c\mu^2|x|^4}\tilde{\alpha}(\hat{\xi}, \hat{\xi}).
\]

To get the first part of (ii), it now suffices to apply the formula of item (i), for $\tilde{\alpha}(X, Y)$. Since $N$ has codimension one in $M$, the parallelism of $\tilde{H}$ is equivalent to the constancy of its norm with respect to $\tilde{g}$. This, in turn, follows from (4.1), as well as (cf. [11]) the facts that $|\xi|^2, \psi$ and $\mu$ are constant on $N$. \hfill \Box

Continue to take $N$ as above (i.e., to be a leaf of $|\xi|^\perp$ in $\Omega$), and let $p \in N$. It is a standard fact (cf. [11], for instance) that there exist an open neighborhood $U \subset \Omega$ of $p$, an open interval $I \subset \mathbb{R}$ and a smooth function $f : I \to (0, +\infty)$ such that $U$ is isometric to the warped product $I \times_f N$. Moreover, under this isometry, the vector field $\xi$ corresponds to $(f \circ \pi_I)\partial_t$, where $\partial_t$ stands for the horizontal lift of the canonical vector field on $\mathbb{R}$.

From now on, we shall write $M = I \times_f N$ and $\xi = (f \circ \pi_I)\partial_t$. In such a case, if $\psi$ stands for the conformal factor of $\xi$ then, as we pointed out before, one has $\psi = f' \circ \pi_I$. Our aim is to prove the following result.

**Theorem 4.3.** Let $(M, J, g)$ be a Kähler manifold possessing a closed conformal vector field $\xi$, with conformal factor $\psi$. Let $p \in M$ be such that $\xi(p) \neq 0$, $\psi(p) \neq 0$ and the restriction of $J\xi$ to the leaf of $|\xi|^\perp$ that passes through $p$ is not parallel on it. Then, on a sufficiently small neighborhood of $p$, $M$ is isometric to a frustum of the Riemannian cone over a Sasaki manifold.

**Proof.** By the preceding discussion, we can think of $M$ as $M = I \times_f N$, for some Riemannian manifold $(N, g_N)$ and some smooth function $f : I \to (0, +\infty)$, with $f' \neq 0$ on $I$. Whenever there is no danger of confusion, we shall write $\xi = f\partial_t$ and $\psi = f'$ (instead of $f \circ \pi_I$ and $f' \circ \pi_I$, respectively).

If $\nabla$ and $D$ respectively denote the Levi-Civita connection of $M$ and $N$, it follows from (2.4) that $\nabla_X J\xi = J\nabla_X \xi$, for all $X \in \mathfrak{X}(N)$. Therefore, Proposition 7.35 of [8] gives
\[
D_X J\xi - g(J\xi, X) \frac{f}{f} \cdot f\partial_t = J(\psi X).
\]
By the definition of the warped product metric, we get

\[(4.2) \quad JX = -g_N(J\xi, X)\xi + \frac{1}{\psi} D_X J\xi.\]

On the other hand, one also has \(\nabla_{\xi} JX = J\nabla_{\xi} X\), for all \(X \in \mathfrak{X}(N)\). Hence, by the previous relation, and invoking Proposition 7.35 of [8] again, we get

\[\nabla_{\xi} \left( -g_N(J\xi, X)\xi + \frac{1}{\psi} D_X J\xi \right) = J\nabla_X \xi = \psi JX.\]

Since \(g_N(J\xi, X)\) is constant along the integral curves of \(\xi\), we get from the above that

\[-g_N(J\xi, X)\nabla_{\xi} \xi + \xi \left( \frac{1}{\psi} \right) D_X J\xi = J\nabla_X \xi = \psi JX.\]

Substituting (4.2) into the above, together with \(\nabla_{\xi} D_X J\xi = \nabla D_X J\xi = \psi JX\) (which is gotten by Proposition 7.35 of [8], once more), after performing some cancellations we arrive at

\[\xi \left( \frac{1}{\psi} \right) D_X J\xi = 0,\]

for all \(X \in \mathfrak{X}(N)\).

Since \(J\xi\) is not parallel along \(N\), by shrinking \(I\), if necessary, we can suppose that \(D_X J\xi \neq 0\) on \(M\). Therefore, \(\xi \left( \frac{1}{\psi} \right) = 0\) on \(I\), so that \(f'' = 0\) on \(I\). If \(f(t) = at + b\), then (\(\approx\) means “is isometric to”)

\[M = I \times_{at+b} (N, g_N) \approx I \times_{t+\frac{b}{a}} (N, a^2 g_N) \approx I' \times_{t} (N, a^2 g_N),\]

where \(I' = \{x - \frac{b}{a}; x \in I\}\). Finally, since \(M\) is Kähler, it follows that \((N, a^2 g_N)\) is Sasaki. \(\square\)

**Remark 4.4.** With the help of (3.1), it can easily be shown that there is no smooth function \(\phi : M \to \mathbb{R}\) such that the vector field \(\tilde{\xi} = \phi \xi\) is closed and conformal with respect to \(\tilde{g}\). Therefore, if \(M = I \times f N\) is Kähler (so that \(\xi = f \partial_t\) and we apply Theorem 2.1 to \(M\) with its original metric \(g\), then \(N\) will not be Sasaki in the metric induced from \(\tilde{g}\).

In fact, (3.1) also allows one to show that no nontrivial linear combination (with smooth coefficients) of the vector fields \(\xi\) and \(J\xi\) is closed and conformal with respect to \(\tilde{g}\). Therefore, \((M, J, \tilde{g})\) is Kähler but doesn’t seem to be the Riemannian cone over a Sasaki manifold.

## 5. Some Computations

In this section we collect some lengthy, though elementary, tensor computations used in the text.

**Proof of Proposition 3.1.** As in the proof of Theorem 2.1 let \(g = \langle \cdot, \cdot \rangle\) and \(\cdot\) be the corresponding norm. On the one hand, we have, for \(X, Y, Z \in \mathfrak{X}(M)\),

\[2\tilde{g}(\tilde{\nabla}_X Y, Z) = 2\mu(\tilde{\nabla}_X Y, Z) + 2\mu^2(\theta_{\xi}^2 + \theta_{J\xi}^2)(\nabla_X Y, Z)\]

\[= 2\mu(\tilde{\nabla}_X Y, Z) + 2\mu^2(\langle \xi, \tilde{\nabla}_X Y \rangle \langle J\xi, Z \rangle + \langle J\xi, \tilde{\nabla}_X Y \rangle \langle \xi, Z \rangle)\]

\[= \langle 2\mu \tilde{\nabla}_X Y + 2\mu^2(\tilde{\nabla}_X Y, \xi)\xi + 2\mu^2(\tilde{\nabla}_X Y, J\xi) J\xi, Z \rangle.\]
On the other, it follows from Koszul’s formula that

\begin{equation}
2\tilde{g}(\tilde{\nabla}_X Y, Z) = X(\tilde{g}(Y, Z)) + Y(\tilde{g}(Z, X)) - Z(\tilde{g}(X, Y))
\end{equation}

\begin{equation}
- \tilde{g}(X, [Y, Z]) + \tilde{g}(Y, [Z, X]) + \tilde{g}(Z, [X, Y]).
\end{equation}

We compute each summand of the right hand expression above:

\begin{align*}
X(\tilde{g}(Y, Z)) &= X(\mu(Y, Z) + \mu^2(\theta_\xi^2(Y, Z) + \theta_\zeta^2(Y, Z))) \\
&= X(\mu(Y, Z) + \mu X(Y, Z) + \mu^2(\langle \xi, Y \rangle \langle \xi, Z \rangle + \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle) \\
&+ \mu^2 X(\langle \xi, Y \rangle \langle \xi, Z \rangle + \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle).
\end{align*}

If we now take into account (2.4), the closed conformal character of \( \xi \) and (2.1), we get

\begin{align*}
X(\tilde{g}(Y, Z)) &= 2\psi\mu^2(\langle \xi, X \rangle \langle Y, Z \rangle + \mu X(Y, Z) \\
&+ 2\mu \cdot 2\psi \mu^2(\langle X, Y \rangle(\langle \xi, Y \rangle \langle \xi, Z \rangle + \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle)) \\
&+ \mu^2(\psi(\langle X, Y \rangle \langle \xi, Z \rangle + \langle \xi, \nabla_X Y \rangle \langle \xi, Z \rangle) \\
&+ \mu^2(\psi(\langle \xi, Y \rangle \langle X, \nabla_Z Y \rangle + \langle \xi, Y \rangle \langle \xi, \nabla_Z Z \rangle)) \\
&+ \mu^2(\psi(\langle J_\xi, Y \rangle \langle J_\xi, X \rangle \langle J_\xi, Z \rangle + \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle) \\
&+ \mu^2(\psi(\langle J_\xi, Y \rangle \langle J_\xi, X \rangle \langle J_\xi, Z \rangle + \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle)).
\end{align*}

Also,

\begin{equation}
\tilde{g}([X, Y], Z) = \mu([\langle X, Y \rangle, Z] + \mu(\theta_{\xi}^2 + \theta_{\zeta}^2)([X, Y], Z)
\end{equation}

\begin{equation}
= \mu([\langle X, Y \rangle, Z] + \mu^2(\langle [X, Y], [X, Y] \rangle(\langle \xi, Z \rangle + \langle J_\xi, [X, Y] \rangle(\langle \xi, Z \rangle).
\end{equation}

Similar computations to those above yield corresponding formulae to the remaining summands on the right hand of (2.2). Substituting all of these analogous formulae, together with (2.3) and (2.4) in (5.2), we obtain

\begin{align*}
2\tilde{g}(\tilde{\nabla}_X Y, Z) &= 2\psi\mu^2(\langle \xi, X \rangle \langle Y, Z \rangle + \langle \xi, Y \rangle \langle X, Z \rangle - \langle \xi, Z \rangle \langle X, Y \rangle) \\
&+ \mu(\langle X, Y \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle) \\
&+ 4\psi\mu^3(\langle \xi, X \rangle \langle \xi, Y \rangle \langle \xi, Z \rangle + \langle \xi, X \rangle \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle + \langle \xi, Y \rangle \langle \xi, \nabla_X Y \rangle \langle \xi, Z \rangle) \\
&+ 4\psi^3(\langle \xi, Y \rangle \langle J_\xi, X \rangle \langle J_\xi, Z \rangle = \langle \xi, Z \rangle \langle \xi, \nabla_X Y \rangle \langle \xi, Z \rangle - \langle \xi, Z \rangle \langle J_\xi, X \rangle \langle J_\xi, Y \rangle) \\
&+ \mu^2(\psi(\langle X, Y \rangle \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle + \langle \xi, \nabla_X Y \rangle \langle \xi, Z \rangle + \psi(\langle X, Y \rangle \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle)) \\
&+ \mu^2(\langle \xi, Y \rangle \langle \xi, \nabla_X Z \rangle + \psi(\langle J_\xi, X \rangle \langle J_\xi, Z \rangle + \langle J_\xi, \nabla_X Y \rangle \langle J_\xi, Z \rangle) \\
&+ \mu^2(\langle \xi, Y \rangle \langle \xi, \nabla_X Z \rangle + \psi(\langle J_\xi, X \rangle \langle J_\xi, Z \rangle + \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle)) \\
&+ \mu^2(\langle \xi, Y \rangle \langle \xi, \nabla_X Z \rangle - \psi(\langle J_\xi, X \rangle \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle + \psi(\langle J_\xi, X \rangle \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle)) \\
&+ \mu^2(\langle \xi, Y \rangle \langle \xi, \nabla_X Z \rangle - \psi(\langle J_\xi, X \rangle \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle + \psi(\langle J_\xi, X \rangle \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle)) \\
&- \mu^2(\psi(\langle X, Y \rangle \langle Z, X \rangle + \langle \xi, Y \rangle \langle \xi, \nabla_Z X \rangle + \psi(\langle J_\xi, Y \rangle \langle J_\xi, Z \rangle \langle J_\xi, X \rangle)) \\
&- \mu^2(\langle J_\xi, \nabla_Y Z \rangle \langle J_\xi, X \rangle + \psi(\langle J_\xi, Y \rangle \langle J_\xi, X \rangle \langle J_\xi, Z \rangle + \langle J_\xi, Y \rangle \langle J_\xi, Z \rangle \langle J_\xi, X \rangle) \\
&+ \mu(\langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle) \\
&+ \mu^2(\langle \xi, [X, Y] \rangle \langle \xi, Z \rangle + \langle J_\xi, [X, Y] \rangle \langle J_\xi, Z \rangle - \langle \xi, [Y, Z] \rangle \langle \xi, X \rangle) \\
&- \mu^2(\langle J_\xi, [X, Y] \rangle \langle J_\xi, X \rangle - \langle \xi, [Z, X] \rangle \langle \xi, Y \rangle - \langle J_\xi, [Z, X] \rangle \langle J_\xi, Y \rangle).
Making some cancellations, the above expression turns into
\[
2\tilde{y}(\tilde{\nabla}X, Z) = \mu(X(Y, Z) + Y(X, Z) - Z(X, Y))
+ \mu((\langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle)
+ 2\psi\mu^2\langle \xi, X \rangle\langle Y, Z \rangle + 2\psi^2\langle \xi, Y \rangle\langle X, Z \rangle
\]
\[
+ 4\psi\mu^3(\langle \xi, X \rangle\langle \xi, Y \rangle\langle \xi, Z \rangle + \langle \xi, X \rangle\langle J\xi, Y \rangle\langle J\xi, Z \rangle)
+ 4\psi\mu^3(\langle \langle J\xi, X \rangle\langle \xi, Y \rangle\langle J\xi, Z \rangle - \langle J\xi, X \rangle\langle J\xi, Y \rangle\langle J\xi, Z \rangle)
+ \mu^2(2\langle \xi, \nabla_X Y \rangle\langle \xi, Z \rangle + 2\langle J\xi, \nabla_X Y \rangle\langle J\xi, Z \rangle)
+ \mu^2(2\psi\langle J\xi, Y \rangle\langle J\xi, X \rangle\langle J\xi, Z \rangle).
\]

Formula (5.5) can be written as
\[
2\tilde{y}(\tilde{\nabla}X, Z) = \langle 2\mu\nabla_X Y + 2\psi\mu^2(\langle \xi, X \rangle Y + \langle \xi, Y \rangle X)
+ 4\psi\mu^3(\langle \xi, X \rangle\langle \xi, Y \rangle\xi + \langle \xi, X \rangle\langle J\xi, Y \rangle\langle J\xi, \xi \rangle)
+ 4\psi\mu^3(\langle J\xi, X \rangle\langle \xi, Y \rangle J\xi - \langle J\xi, X \rangle\langle J\xi, Y \rangle\xi)
+ 2\mu^2(\langle \nabla_X Y, \xi \rangle\xi + \langle \nabla_X Y, J\xi \rangle J\xi)
+ 2\mu^2(\psi\langle J\xi, Y \rangle J\xi + \psi\langle J\xi, X \rangle J\xi), Z \rangle.
\]

Comparing (5.5) with (5.6), we get
\[
2\mu\nabla_X Y + 2\mu^2(\nabla_X Y, \xi)\xi + 2\mu^2(\nabla_X Y, J\xi) J\xi =
= 2\mu\nabla_X Y + 2\psi\mu^2(\langle \xi, X \rangle Y + \langle \xi, Y \rangle X)
+ 4\psi\mu^3(\langle \xi, X \rangle\langle \xi, Y \rangle\xi + \langle \xi, X \rangle\langle J\xi, Y \rangle J\xi + \langle J\xi, X \rangle\langle \xi, Y \rangle\xi - \langle J\xi, X \rangle\langle J\xi, Y \rangle\xi)
+ 2\mu^2(\langle \nabla_X Y, \xi \rangle\xi + \langle \nabla_X Y, J\xi \rangle J\xi + \psi\langle J\xi, Y \rangle J\xi + \psi\langle J\xi, X \rangle J\xi).
\]

In particular,
\[
\nabla_X Y - \nabla_X Y + \mu(\nabla_X Y - \nabla_X Y, \xi)\xi + \mu(\nabla_X Y - \nabla_X Y, J\xi) J\xi =
= \psi\mu(\langle \xi, X \rangle Y + \langle \xi, Y \rangle X + \langle J\xi, Y \rangle J\xi + \langle J\xi, X \rangle J\xi)
+ 2\mu^2(\langle \xi, X \rangle\langle \xi, Y \rangle - \langle J\xi, X \rangle\langle J\xi, Y \rangle)\xi
+ 2\mu^2(\langle \xi, X \rangle\langle J\xi, Y \rangle + \langle J\xi, X \rangle\langle \xi, Y \rangle) J\xi.
\]

If we let \( W = \nabla_X Y - \nabla_X Y \) and \( F(X, Y) \) be the summand on the right side of the equality in (5.7) divided by \( \mu \), we obtain
\[
W + \mu\langle W, \xi \rangle\xi + \mu\langle W, J\xi \rangle J\xi = \mu F(X, Y).
\]

Taking the inner product of (5.8) with \( \xi \) and \( J\xi \), respectively, we find
\[
\begin{cases}
\langle W, \xi \rangle (1 + \mu\langle \xi, \xi \rangle) = \mu F(X, Y, \xi), \\
\langle W, J\xi \rangle (1 + \mu\langle \xi, \xi \rangle) = \mu F(X, Y, J\xi).
\end{cases}
\]

Now, since \( 1 + \mu\langle \xi, \xi \rangle = c\mu \), it follows that
\[
\langle W, \xi \rangle = c^{-1} \langle F(X, Y), \xi \rangle \text{ and } \langle W, J\xi \rangle = c^{-1} \langle F(X, Y), J\xi \rangle.
\]

Hence, (5.8) provides
\[
W = \mu(F(X, Y) - c^{-1} F(X, Y), \xi)\xi - c^{-1} (F(X, Y), J\xi) J\xi.
\]
Finally, since $\langle J\xi, \xi \rangle = 0$, we get
\[
\langle F(X,Y), \xi \rangle \xi = \psi(\langle \xi, X \rangle \langle Y, \xi \rangle + \langle \xi, Y \rangle \langle X, \xi \rangle + \langle J\xi, X \rangle \langle JY, \xi \rangle
+ 2\mu(\langle \xi, X \rangle \langle Y, \xi \rangle - \langle J\xi, X \rangle \langle JY, \xi \rangle)
= 2\psi(\langle \xi, X \rangle \langle Y, \xi \rangle(1 + \mu(\xi, \xi)) - \langle J\xi, X \rangle \langle JY, \xi \rangle(1 + \mu(\xi, \xi)))
= 2\psi\mu(\langle \xi, X \rangle \langle Y, \xi \rangle - \langle J\xi, X \rangle \langle JY, \xi \rangle)
\]
and, similarly,
\[
\langle F(X,Y), J\xi \rangle J\xi = 2\psi\mu\langle \xi, X \rangle \langle J\xi, Y \rangle + \langle \xi, Y \rangle \langle J\xi, X \rangle
\]
Therefore, up to a constant, $\langle F(X,Y), \xi \rangle \xi$ and $\langle F(X,Y), J\xi \rangle J\xi$ are the components of $F(X,Y)$ in the directions $\xi$ and $J\xi$, respectively, so that (5.7) and (5.9) yield
\[
\tilde{\nabla} X - \nabla X = W = \psi\mu(\langle \xi, X \rangle \langle Y, \xi \rangle + \langle \xi, Y \rangle \langle X, \xi \rangle + \langle J\xi, Y \rangle \langle JX, \xi \rangle + \langle J\xi, X \rangle \langle JY, \xi \rangle)
\]
□

Before we can prove Theorem 3.2, we need a few more preliminaries. First of all, the computations leading to (2.3) give
\[
X(\mu) = 2\psi\mu^2\langle X, \xi \rangle \text{ and } JX(\mu) = -2\psi\mu^2\langle X, J\xi \rangle.
\]
Second, the closed conformal condition readily gives that $\psi = \frac{1}{2n}\text{div} \xi$, and Lemma 1 of [11] gives
\[
|\xi|^2\nabla(\text{div} \xi) = -2n\text{Ric}(\xi)\xi,
\]
where $\nabla(\text{div} \xi)$ stands for the gradient of the divergence of $\xi$ with respect to $g$ and $\text{Ric}(\xi)$ for the normalized Ricci curvature of $(M, g)$ in the direction of $\xi$. Thus, the above expression gives, at each point where $\xi \neq 0$,
\[
\nabla \psi = -\text{Ric}(\hat{\xi})\xi,
\]
where $\hat{\xi} = \frac{\xi}{|\xi|}$. Therefore, at each such a point,
\[
X(\psi) = \langle X, \nabla \psi \rangle = -\text{Ric}(\hat{\xi})\langle X, \xi \rangle
\]
and, analogously,
\[
JX(\psi) = -\text{Ric}(\hat{\xi})\langle JX, \xi \rangle = \text{Ric}(\hat{\xi})\langle J\xi, X \rangle
\]
Also, if $\langle X, X \rangle = 1$, then, since $1 + \mu(\xi, \xi) = c\mu$, we get
\[
\tilde{g}(X, \xi) = \mu(X, \xi) + \mu^2(\langle X, \xi \rangle \langle \xi, \xi \rangle + \langle X, J\xi \rangle \langle J\xi, \xi \rangle)
= \mu(X, \xi)(1 + \mu(\xi, \xi))
= c\mu^2(X, \xi),
\]
\[
\tilde{g}(X, J\xi) = -\tilde{g}(JX, \xi) = -c\mu^2(\langle JX, \xi \rangle = c\mu^2(X, J\xi)
\]
and
\[
\hat{g}(X, X) = \mu(X, X) + \mu^2(\langle X, \xi \rangle^2 + \langle X, J\xi \rangle^2)
= \mu + \mu^2(\langle X, \xi \rangle^2 + \langle X, J\xi \rangle^2).
\]
We shall also need to use Hermitian geodesic frames, in the sense of the following lemma.
Lemma 5.1. If \((M^n, J, g)\) is a Kähler manifold and \(p \in M\), then there exist a neighborhood \(U \subset M\) of \(p\) and a Hermitian frame field in \(U\) which is geodesic at \(p\).

Proof. Take a normal ball \(U \subset M\) centered at \(p\) and a Hermitian basis \((e_1, J_p e_1, \ldots, e_n, J_p e_n)\) for \(T_p M\). By parallel transporting such vectors along the geodesic rays on \(U\) departing from \(p\), we get (cf. exercise 3.7 of [1]) an orthonormal frame field \((e_1, e_1', \ldots, e_n, e_n')\) in \(U\), which is geodesic at \(p\). We assert that \(e_k' = Je_k\) in \(U\), for \(1 \leq k \leq n\). In fact, given \(q \in U\), take the radial geodesic \(\gamma : [0, 1] \to U\), such that \(\gamma(0) = p\) and \(\gamma(1) = q\); it follows from (2.1) that

\[
\frac{D}{dt} Je_k = \nabla_{\gamma'} Je_k = J \nabla_{\gamma'} e_k = 0,
\]

so that \(Je_k\) is parallel along \(\gamma\). However, since \(e_k'(p) = J_p e_k = (Je_k)(p)\), uniqueness of parallel transport gives \(e_k' = Je_k\) along \(\gamma\), so that \(e_k' = Je_k\) at \(q\). \(\square\)

We are finally in position to state and prove Theorem 3.2.

Proof of Theorem 3.2. Extend \(X\) to a smooth vector field around \(p\). If \(\tilde{R}\) denotes the curvature tensor of \((M, \tilde{g})\), the holomorphic sectional curvature of \((M, \tilde{g})\) with respect to \(X\) is given by

\[
\tilde{K}(X) = \frac{\tilde{g}(\tilde{R}(X, JX)JX, X)}{\tilde{g}(X, X)\tilde{g}(JX, JX) - \tilde{g}(X, JX)^2} = \frac{1}{\tilde{g}(X, X)^2} \tilde{g}(\tilde{\nabla}_X \tilde{\nabla}_X JX - \tilde{\nabla}_X \tilde{\nabla}_X JX - \tilde{\nabla}_{\{X, JX\}} JX, X) = \frac{1}{\tilde{g}(X, X)^2} (X(\tilde{g}(\tilde{\nabla}_X JX, X)) - \tilde{g}(\tilde{\nabla}_X JX, \tilde{\nabla}_X X) - JX(\tilde{g}(\tilde{\nabla}_X JX, X)) + \tilde{g}(\tilde{\nabla}_X JX, \tilde{\nabla}_X X) - \tilde{g}(\tilde{\nabla}_{\{X, JX\}} JX, X)).
\]

In order to simplify the notation, from now on we call

\[
(5.13) \quad \alpha = 2\psi\mu(X, \xi) \quad \text{and} \quad \beta = 2\psi\mu(X, J\xi).
\]

It easily follows from (5.11) that

\[
\tilde{\nabla}_X X = \nabla_X X + 2\psi\mu(X, \xi)X + \langle X, J\xi\rangle JX = \nabla_X X + \alpha X + \beta JX,
\]

\[
\tilde{\nabla}_X JX = \nabla_X JX + J(\nabla_X X + \alpha X + \beta JX) = \nabla_X JX + \alpha JX - \beta X,
\]

\[
\tilde{\nabla}_{\{X, JX\}} JX = \nabla_{\{X, JX\}} JX + \psi\mu(JX, J\xi)X + \langle X, J\xi\rangle JX + \langle JX, J\xi\rangle JX + \langle X, J\xi\rangle J^2 X = \nabla_{\{X, JX\}} JX + 2\psi\mu(-\langle X, J\xi\rangle X + \langle X, \xi\rangle JX) = \nabla_{\{X, JX\}} JX - \beta X + \alpha JX
\]

and, finally

\[
\tilde{\nabla}_{\{X, JX\}} JX = \nabla_{\{X, JX\}} JX = \nabla_{\{X, JX\}} X - \beta X + \alpha JX = \nabla_{\{X, JX\}} JX - \beta X + \alpha JX.
\]

To the subsequent computations, we observe that, since the value of \(\tilde{K}(X)\) at \(p\) depends only on the value of \(X\) at \(p\), we can assume \((\nabla_v X)(p) = 0\) and \((\nabla_v JX)(p) = 0\) for all \(v \in T_p M\) (just apply Lemma 5.1 to take a Hermitian geodesic frame \((e_1, Je_1, \ldots, e_n, Je_n)\) around \(p\), such that \(e_1(p) = X_p\). Hence,
So that, with the aid of the previous computations

\[
\hat{K}(X) = \frac{1}{g(X, X)^2} \left\{ \begin{array}{l}
X(\hat{g}(\hat{\nabla}_J X, X))
- \hat{g}(\hat{\nabla}_J X - \alpha X - \beta JX) - JX(\hat{g}(\hat{\nabla}_J X, X))
+ \hat{g}(\hat{\nabla}_J X, \hat{\nabla}_J X)
\end{array} \right\},
\]

so that, with the aid of the previous computations

\[
\hat{K}(X) = \frac{1}{g(X, X)^2} \left\{ \begin{array}{l}
X(\hat{g}(\hat{\nabla}_J X, X))
- \hat{g}(\hat{\nabla}_J X - \alpha X - \beta JX - \alpha X + \beta JX)
- JX(\hat{g}(\hat{\nabla}_J X, X))
+ \hat{g}(\hat{\nabla}_J X, \hat{\nabla}_J X)
\end{array} \right\},
\]

Thus,

\[
\hat{K}(X) = \frac{1}{g(X, X)^2} \left\{ \begin{array}{l}
X(\hat{g}(\hat{\nabla}_J X, X))
- \hat{g}(\hat{\nabla}_J X - \alpha X - \beta JX - \alpha X + \beta JX)
- JX(\hat{g}(\hat{\nabla}_J X, X))
+ \hat{g}(\hat{\nabla}_J X, \hat{\nabla}_J X)
\end{array} \right\},
\]

Let us compute separately each one of three terms of \((5.14)\), replacing \(\alpha\) and \(\beta\) by \(\mu\) when necessary:

\[
I = X(\mu)\langle \hat{\nabla}_J X, X \rangle + \mu X \langle \hat{\nabla}_J X, J X \rangle
+ X(\mu^2)\langle \langle \hat{\nabla}_J X, X \rangle \langle \hat{\nabla}_J X, \xi \rangle \rangle
+ \hat{\nabla}_J X(\mu)\langle \hat{\nabla}_J X, J \xi \rangle \langle X, J \xi \rangle
- JX(\mu)\langle \hat{\nabla}_J X, X \rangle - \mu JX \langle \hat{\nabla}_J X, X \rangle
- JX(\mu^2)\langle \langle \hat{\nabla}_J X, \xi \rangle \rangle
- \mu JX(\mu^2)\langle \langle \hat{\nabla}_J X, J \xi \rangle \rangle.
\]
Recalling that $[X, JX] = 0$ at $p$, we get at this point

$$
I = \mu\langle \nabla_X \nabla_JX, JX, X \rangle + \mu^2 \langle \nabla_X \nabla_JX, JX, JX \rangle - \mu \langle \nabla_JX \nabla_X JX, X \rangle
+ \mu^2 \langle \nabla_X \nabla_JX, JX, JX \rangle - \mu \langle \nabla_JX \nabla_X JX, JX \rangle
- \mu^2 \langle \nabla_JX \nabla_X JX, JX \rangle
= \mu \langle R(X, JX) JX, X \rangle + \mu^2 \langle R(X, JX) JX, JX \rangle
+ \mu^2 \langle R(X, JX) JX, JX \rangle
$$

(5.15)

We now turn our attention to the computation of $II$.

$$
II = 2X(\psi \mu(X, \xi)) + 2((2\psi \mu(X, \xi))^2 + (2\psi \mu(X, J\xi))^2) + 2JX(\psi \mu(X, J\xi))
= 2X(\psi \mu(X, \xi) - 2\psi \mu(X, \xi) - 2\psi \mu(X, \xi) + 8\psi^2 \mu^2(\xi, J\xi) + \langle X, J\xi \rangle^2
+ 2JX(\psi \mu(X, J\xi) + 2\psi JX(\mu(X, J\xi) + 2\psi X, \nabla_JX J\xi).
$$

By using (5.11), we get

$$
II = -2X(\psi \mu(X, \xi) - 4\psi \mu(X, \xi)^2 - 2\psi \mu(X, X) + 8\psi^2 \mu^2(\xi, J\xi) + \langle X, J\xi \rangle^2
+ 2JX(\psi \mu(X, J\xi) - 4\psi \mu(X, J\xi) + 2\psi \mu(X, J\xi)
- 2JX(X, \mu(X, J\xi) + 4\psi JX(\mu(X, J\xi) + 2\psi \mu(X, \nabla_JX J\xi).
$$

If $p \in M$ is such that $\xi(p) \neq 0$, it follows from (5.11) and (5.12) that

$$
II = 2\mu \text{Ric}(\hat{\xi})\langle X, J\xi \rangle^2 + 2\mu \text{Ric}(\hat{\xi})\langle X, J\xi \rangle^2
- 8\psi^2 \mu + 4\psi^2 \mu \langle X, J\xi \rangle^2
+ 2\mu \text{Ric}(\hat{\xi})\langle X, J\xi \rangle^2
+ 4\psi^2 \hat{g}(X, X) - 8\psi^2 \mu.
$$

(5.16)

If $\xi(p) = 0$, take a sequence $(p_j)_{j \geq 1}$ in $M$, converging to $p$ and such that $\xi(p_j) \neq 0$ (this can always be done, for the zeroes of a (non-trivial) closed conformal vector field are isolated – cf. Lemma 1 of [11]). Since $\langle \nabla \psi \rangle(p_j) = -\text{Ric}(\hat{\xi}(p_j))\xi(p_j)$, we get

$$
\nabla \psi(p) = \lim_j \nabla \psi(p_j) = -\lim_j \text{Ric}(\hat{\xi}(p_j)) \frac{\xi(p_j)}{\xi(p_j)} \langle \xi(p_j) \rangle \rightarrow 0.
$$

Therefore, if we take $\text{Ric}(\hat{\xi}) = 0$ at $p$, we have (5.10) valid in all cases.

In order to deal with $III$, we recall once more that, without loss of generality, we can assume $\langle \nabla v \rangle(p) = 0$ for all $v \in T_p M$. Then, at the point $p$ and for $Y \in \mathfrak{X}(M)$, we get

$$
Y(\hat{g}(X, X) = Y(\mu) + Y(\mu)^2(\langle X, \xi \rangle^2 + \langle X, J\xi \rangle^2) + \mu^2(Y(\langle X, \xi \rangle^2 + Y(\langle X, J\xi \rangle^2)
= Y(\mu) + 2\mu Y(\mu)(\langle X, \xi \rangle^2 + \langle X, J\xi \rangle^2)
+ 2\mu^2(\langle X, \xi \rangle Y(\langle X, \xi \rangle) + \langle X, J\xi \rangle Y(\langle X, J\xi \rangle))
= 2\psi \mu^2(Y(\xi) + 4\psi \mu^2(Y(\xi) + \langle X, \xi \rangle^2 + \langle X, J\xi \rangle^2)
+ 2\mu^2(\langle X, \xi \rangle \langle X, \psi Y \rangle + \langle X, J\xi \rangle Y(\langle X, J\psi Y \rangle)
$$
where we have used the closed conformal character of $\xi$ and (5.10) in the last equality. It follows that
\[
Y(\tilde{g}(X, X)) = 4\psi\mu \langle Y, \xi \rangle (\mu + \mu^2 (\langle X, \xi \rangle^2 + (X, J\xi)^2)) - 2\psi\mu^2 \langle Y, \xi \rangle
+ 2\psi\mu^2 (\langle X, \xi \rangle \langle X, Y \rangle + \langle X, J\xi \rangle Y(\langle X, JY \rangle))
= 4\psi\mu \langle Y, \xi \rangle \tilde{g}(X, X) + 2\psi\mu^2 (\langle X, \xi \rangle \langle X, Y \rangle + \langle X, J\xi \rangle Y(\langle X, JY \rangle))
- 2\psi\mu^2 \langle Y, \xi \rangle.
\]

In particular,
\[
III = -2\psi\mu \langle X, \xi \rangle \{4\psi\mu \langle X, \xi \rangle \tilde{g}(X, X) + 2\psi\mu^2 (\langle X, \xi \rangle - 2\psi\mu^2 \langle X, \xi \rangle)
+ 2\psi\mu \langle X, J\xi \rangle \{4\psi\mu \langle JX, \xi \rangle \tilde{g}(X, X) + 2\psi\mu^2 (\langle X, J\xi \rangle - 2\psi\mu^2 \langle JX, \xi \rangle)
= -8\psi^2 \mu^2 (\langle X, \xi \rangle^2 + \langle X, J\xi \rangle^2) \tilde{g}(X, X).
\]

Substituting (5.15), (5.16) and (5.17) in (5.14), we obtain
\[
\tilde{K}(X) = \frac{1}{\tilde{g}(X, X)}^2 \{\mu K(X) + \mu^2 \langle R(X, JX)JX, \xi \rangle \langle X, \xi \rangle
+ \mu^2 \langle R(X, JX)JX, J\xi \rangle \langle X, J\xi \rangle
+ 2\mu \text{Ric}(\hat{\xi}) (\langle X, \xi \rangle^2 + \langle X, J\xi \rangle^2) - 8\psi^2 \mu \tilde{g}(X, X)
- 8\psi^2 \mu^2 (\langle X, \xi \rangle^2 + \langle X, J\xi \rangle^2) \tilde{g}(X, X)
= \frac{1}{\tilde{g}(X, X)}^2 \{\mu K(X) + \mu^2 \langle R(X, JX)JX, \xi \rangle \langle X, \xi \rangle
+ \mu^2 \langle R(X, JX)JX, J\xi \rangle \langle X, J\xi \rangle
+ 2\mu \text{Ric}(\hat{\xi}) (\langle X, \xi \rangle^2 + \langle X, J\xi \rangle^2) \}
\tilde{g}(X, X) - 4\psi^2.
\]

Now, by invoking Lemma 1 of [11] once more, we have
\[
\langle R(X, JX)JX, \xi \rangle = -\langle R(X, JX)JX, \xi \rangle
= -\text{Ric}(\hat{\xi}) (\langle JX, \xi \rangle \langle JX \rangle - \langle X, \xi \rangle \langle JX, JX \rangle)
= \text{Ric}(\hat{\xi}) \langle X, \xi \rangle
\]
and, similarly,
\[
\langle R(X, JX)JX, J\xi \rangle = \text{Ric}(\hat{\xi}) \langle X, J\xi \rangle.
\]

Substituting these formulae in the last expression above for $\tilde{K}(X)$, we finally arrive at the formula displayed in the statement of the theorem. \[\square\]

We end this section deriving the relation between the Ricci tensors of $g$ and $\tilde{g}$. 

Proof of (3.3). Rewrite (3.3) as

\[
\begin{align*}
\widetilde{\text{Ric}}(X, Y) & = \sum_{i=1}^{n} \{ \hat{g}(\hat{R}(X, \hat{e}_i), Y) + \hat{g}(\hat{R}(X, J\hat{e}_i), J\hat{e}_i, Y) \} \\
& = \sum_{i=1}^{n-1} \mu^{-1} \{ \hat{g}(\hat{R}(X, e_i), Y) + \hat{g}(\hat{R}(X, Je_i), Je_i, Y) \} \\
& \quad + c^{-1} \mu^{-2} \{ \hat{g}(\hat{R}(X, \hat{\xi}), Y) + \hat{g}(\hat{R}(X, J\hat{\xi}), J\hat{\xi}, Y) \}.
\end{align*}
\]

We shall compute separately each term in the equality above.

\[
(5.18) \quad I = \mu \{ \hat{R}(X, e_i)e_i, Y) + \mu^2 \{ (\hat{R}(X, e_i)e_i, \xi)(Y, \xi) + (\hat{R}(X, e_i)e_i, J\xi)(Y, J\xi) \}
\]

\[
(5.19) \quad I_a = \langle R(X, e_i)e_i, Y \rangle - \text{Ric}(\hat{\xi})\mu[\langle \xi, X \rangle \langle \xi, e_i \rangle \langle e_i, Y \rangle + \langle \xi, X \rangle \langle J\xi, e_i \rangle \langle e_i, Y \rangle \\
\quad + \langle \xi, X \rangle \langle J\xi, e_i \rangle \langle Je_i, Y \rangle - \langle \xi, e_i \rangle \langle \xi, e_i \rangle \langle X, Y \rangle \\
\quad - \langle \xi, e_i \rangle \langle J\xi, X \rangle \langle Je_i, Y \rangle - \langle \xi, e_i \rangle \langle J\xi, e_i \rangle \langle JX, Y \rangle \\
\quad = \langle X, Je_i \rangle - \langle X, Je_i \rangle \\
\quad + \psi^2 \mu \langle (X, e_i)\langle e_i, Y \rangle + 2 \langle JX, e_i \rangle \langle Je_i, Y \rangle + \langle JX, e_i \rangle \langle Je_i, Y \rangle \\
\quad = \psi^2 \mu \langle (X, e_i)\langle e_i, Y \rangle + 2 \langle JX, e_i \rangle \langle Je_i, Y \rangle + \langle JX, e_i \rangle \langle Je_i, Y \rangle \\
\quad + \psi^2 \mu \langle (X, e_i)\langle e_i, Y \rangle + 2 \langle JX, e_i \rangle \langle Je_i, Y \rangle + \langle JX, e_i \rangle \langle Je_i, Y \rangle \\
\quad = \psi^2 \mu \langle (X, e_i)\langle e_i, Y \rangle + 2 \langle JX, e_i \rangle \langle Je_i, Y \rangle + \langle JX, e_i \rangle \langle Je_i, Y \rangle \
\quad + \psi^2 \mu \langle (X, e_i)\langle e_i, Y \rangle + 2 \langle JX, e_i \rangle \langle Je_i, Y \rangle + \langle JX, e_i \rangle \langle Je_i, Y \rangle.
\]
\[ I_b = \langle Y, \xi \rangle \{ \langle R(X, e_i) e_i, \xi \rangle - \text{Ric}(\xi) \mu[0] \]
\[ + \psi^2 \mu \langle X, e_i \rangle \langle e_i, \xi \rangle - 3 \langle X, J e_i \rangle \langle \xi, J e_i \rangle \]
\[ + \psi^2 \mu[0] \} . \] (5.20)

\[ I_c = \langle Y, J \xi \rangle \{ \langle R(X, e_i) e_i, J \xi \rangle - \text{Ric}(\xi) \mu[0] \]
\[ + \psi^2 \mu \langle X, e_i \rangle \langle e_i, J \xi \rangle - 3 \langle X, J e_i \rangle \langle \xi, J e_i \rangle \]
\[ + \psi^2 \mu[0] \} . \] (5.21)

where we simply substitute \( Y = \xi \) and \( Y = J \xi \) in the formula \( I_a \) to obtain the parcels of \( I_b \) and \( I_c \), respectively.

Consequently,

\[ I = \mu \langle R(X, e_i) e_i, Y \rangle + \psi^2 \mu^2 \langle X, e_i \rangle \langle Y, e_i \rangle - 3 \langle X, J e_i \rangle \langle Y, J e_i \rangle - \langle X, Y \rangle \]
\[ + \mu^2 \langle R(X, e_i) e_i, \xi \rangle \langle Y, \xi \rangle - \psi^2 \mu^2 \langle X, \xi \rangle \langle Y, \xi \rangle \]
\[ + \mu^2 \langle R(X, e_i) e_i, J \xi \rangle \langle Y, J \xi \rangle - \psi^2 \mu^2 \langle X, J \xi \rangle \langle Y, J \xi \rangle . \] (5.22)

Now, we compute \( II \).

\[ II = \mu \langle \hat{R}(X, J e_i) J e_i, Y \rangle + \mu^2 \{ \langle \hat{R}(X, J e_i) J e_i, \xi \rangle \langle Y, \xi \rangle + \langle \hat{R}(X, J e_i) J e_i, J \xi \rangle \langle Y, J \xi \rangle \} \] (5.23)
\begin{align}
(5.24) 
IIa &= \langle R(X, Je_i)Je_i, Y \rangle - \text{Ric}(\hat{\xi})\mu[\{\xi, X\langle J, Je_i \rangle \langle J, Je_i \rangle \langle J, Je_i \rangle \langle J, Je_i \rangle \langle J, Je_i \rangle \langle J, Je_i \rangle \rangle + \langle \xi, X \rangle \langle J, Je_i \rangle \langle J, Je_i \rangle \rangle + \langle \xi, X \rangle \langle J, Je_i \rangle \langle J, Je_i \rangle \rangle \\
&\quad + \langle \xi, X \rangle \langle J, Je_i \rangle \langle J, Je_i \rangle \rangle - \langle \xi, J, Je_i \rangle \langle J, Je_i \rangle \langle J, Je_i \rangle \rangle - \langle \xi, J, Je_i \rangle \langle J, Je_i \rangle \rangle \\
&\quad + \langle \xi, J, Je_i \rangle \langle J, Je_i \rangle \rangle - \langle \xi, J, Je_i \rangle \langle J, Je_i \rangle \rangle + \langle \xi, J, Je_i \rangle \langle J, Je_i \rangle \rangle \\
&\quad + \psi^2\mu[\langle X, Je_i \rangle \langle J, Je_i \rangle \rangle + 2 \langle J, Je_i \rangle \rangle + \langle J, Je_i \rangle \rangle + \langle J, Je_i \rangle \rangle + \langle J, Je_i \rangle \rangle \\
\quad + \langle \xi, X \rangle \langle J, Je_i \rangle \rangle - \langle \xi, J, Je_i \rangle \rangle - \langle \xi, J, Je_i \rangle \rangle - \langle \xi, J, Je_i \rangle \rangle \\
&\quad - 2 \langle \xi, J, Je_i \rangle \rangle - \langle \xi, J, Je_i \rangle \rangle - \langle \xi, J, Je_i \rangle \rangle - \langle \xi, J, Je_i \rangle \rangle \\
&\quad + \langle J, Je_i \rangle \rangle + \langle \xi, J, Je_i \rangle \rangle + \langle \xi, J, Je_i \rangle \rangle \\
\end{align}

\begin{align}
(5.25) 
IIb &= \langle Y, J\xi \rangle \{\langle R(X, Je_i)Je_i, \xi \rangle - \text{Ric}(\hat{\xi})\mu[0] \\
&\quad + \psi^2\mu[\langle X, Je_i \rangle \rangle - 3 \langle X, e_i \rangle \langle J, Je_i \rangle \rangle \rangle - \langle X, \xi \rangle - \langle J, Je_i \rangle \langle J, Je_i \rangle \rangle \\
&\quad + \psi^2\mu[0] \}.
\end{align}

\begin{align}
(5.26) 
IIc &= \langle Y, J\xi \rangle \{\langle R(X, Je_i)Je_i, J\xi \rangle - \text{Ric}(\hat{\xi})\mu[0] \\
&\quad + \psi^2\mu[\langle X, Je_i \rangle \rangle - 3 \langle X, e_i \rangle \langle J, Je_i \rangle \rangle \rangle - \langle X, J\xi \rangle - \langle J, Je_i \rangle \langle J, Je_i \rangle \rangle \\
&\quad + \psi^2\mu[0] \},
\end{align}

where we simply substitute $Y = \xi$ and $Y = J\xi$ in the parcels of the formula IIa to obtain the parcels of IIb and IIc, respectively.
Thus,

\begin{equation}
(5.27)
III = \mu \langle R(X, J\xi)e_i, J\xi, Y \rangle + \psi^2 \mu^2 \langle \langle X, J\xi \rangle \langle Y, J\xi \rangle - 3\langle X, e_i \rangle \langle Y, e_i \rangle - \langle X, Y \rangle \rangle \\
+ \mu^2 \langle R(X, J\xi)e_i, \xi \rangle \langle Y, \xi \rangle - \psi^2 \mu^3 \langle X, \xi \rangle \langle Y, \xi \rangle \\
+ \mu^2 \langle R(X, J\xi)e_i, J\xi \rangle \langle Y, J\xi \rangle - \psi^2 \mu^3 \langle X, J\xi \rangle \langle Y, J\xi \rangle.
\end{equation}

For now, we get

\begin{equation}
(5.28)
I + II = \mu \{ \langle R(X, e_i)e_i, Y \rangle + \langle R(X, J\xi)e_i, Y \rangle \}
- 2\psi^2 \mu^2 \{ \langle X, e_i \rangle \langle Y, e_i \rangle + \langle X, J\xi \rangle \langle Y, e_i \rangle + \langle X, Y \rangle \}
+ \mu^2 \{ \langle R(X, e_i)e_i + R(X, J\xi)e_i, \xi \rangle \langle Y, \xi \rangle \}
+ \mu^2 \{ \langle R(X, e_i)e_i + R(X, J\xi)e_i, J\xi \rangle \langle Y, J\xi \rangle \}
- 2\psi^2 \mu^3 \{ \langle X, \xi \rangle \langle Y, \xi \rangle + \langle X, J\xi \rangle \langle Y, J\xi \rangle \}.
\end{equation}

The computation of III is slightly similar.

\begin{equation}
(5.29)
III = \mu \langle \tilde{R}(X, \hat{\xi})\hat{\xi}, Y \rangle + \mu^2 \{ \langle \tilde{R}(X, \hat{\xi})\hat{\xi}, \hat{\xi} \rangle \langle Y, \hat{\xi} \rangle + \langle \tilde{R}(X, \hat{\xi})\hat{\xi}, J\xi \rangle \langle Y, J\xi \rangle \}.
\end{equation}

\begin{equation}
(5.30)
IIIa = \langle R(X, \hat{\xi})\hat{\xi}, Y \rangle - \text{Ric}(\hat{\xi})\mu \{ \langle \xi, X \langle \xi, \hat{\xi} \rangle \langle \hat{\xi}, Y \rangle + \langle \xi, X \rangle \langle J\xi, \hat{\xi} \rangle \langle J\xi, Y \rangle \}
+ \langle \xi, X \rangle \langle J\xi, \hat{\xi} \rangle \langle J\xi, Y \rangle - \langle \xi, \hat{\xi} \rangle \langle \hat{\xi}, \hat{\xi} \rangle \langle X, Y \rangle \\
- \langle \xi, \hat{\xi} \rangle \langle J\xi, X \rangle \langle J\xi, Y \rangle - \langle \xi, \hat{\xi} \rangle \langle J\xi, \hat{\xi} \rangle \langle J\xi, Y \rangle \\
+ \psi^2 \mu \{ \langle X, \hat{\xi} \rangle \langle \hat{\xi}, Y \rangle + 2 \langle JX, \hat{\xi} \rangle \langle J\xi, \hat{\xi} \rangle + \langle JX, \hat{\xi} \rangle \langle J\xi, Y \rangle \\
- \langle \hat{\xi}, \hat{\xi} \rangle \langle X, Y \rangle - \langle J\xi, \hat{\xi} \rangle \langle JX, Y \rangle \\
+ \psi^2 \mu^2 \{ \langle \xi, X \langle \xi, \hat{\xi} \rangle \langle \hat{\xi}, Y \rangle + 2 \langle \xi, X \rangle \langle J\xi, \hat{\xi} \rangle \langle J\xi, Y \rangle \\
+ \langle \xi, X \rangle \langle J\xi, \hat{\xi} \rangle \langle J\xi, Y \rangle - \langle \xi, \hat{\xi} \rangle \langle \hat{\xi}, \hat{\xi} \rangle \langle X, Y \rangle \\
- 2 \langle \xi, \hat{\xi} \rangle \langle J\xi, X \rangle \langle J\xi, Y \rangle - \langle \xi, \hat{\xi} \rangle \langle J\xi, \hat{\xi} \rangle \langle J\xi, Y \rangle \\
- \langle J\xi, \hat{\xi} \rangle \langle J\xi, \hat{\xi} \rangle \langle X, Y \rangle - \langle \xi, \hat{\xi} \rangle \langle J\xi, X \rangle \langle J\xi, Y \rangle \\
+ \langle J\xi, \hat{\xi} \rangle \langle J\xi, X \rangle \langle \hat{\xi}, Y \rangle + \langle \xi, \hat{\xi} \rangle \langle J\xi, \hat{\xi} \rangle \langle J\xi, Y \rangle \}. 
\end{equation}
Thus,

\begin{align}
IIIb = \{ \langle Y, J\xi \rangle \{ \langle R(X, \xi)\hat{\xi}, J\xi \rangle - \text{Ric}(\hat{\xi})\mu \langle \xi, X \rangle \langle \xi, \hat{\xi} \rangle \langle \xi, J\xi \rangle \\
- \langle \xi, \hat{\xi} \rangle \langle \xi, \hat{\xi} \rangle \langle X, J\xi \rangle \\
- \langle \xi, \hat{\xi} \rangle \langle J\xi, X \rangle \langle J\xi, J\xi \rangle \}
\end{align}

(5.31)

\[= 0 + \psi^2 \mu \langle X, \hat{\xi} \rangle \langle \xi, \hat{\xi} \rangle + 2 \langle JX, \hat{\xi} \rangle \langle J\xi, \hat{\xi} \rangle + \langle JX, \xi \rangle \langle J\xi, \xi \rangle \\
- \langle \xi, \hat{\xi} \rangle \langle X, J\xi \rangle \]

Thus,

\begin{align}
III = \mu \langle R(X, \hat{\xi})\hat{\xi}, Y \rangle - \text{Ric}(\hat{\xi})\mu^2 [\langle \xi, X \rangle \langle \xi, Y \rangle - |\xi|^2 \langle X, Y \rangle - \langle J\xi, X \rangle \langle J\xi, Y \rangle] \\
+ \psi^2 \mu^2 [\langle X, \hat{\xi} \rangle \langle Y, \hat{\xi} \rangle - 3 \langle X, J\xi \rangle \langle Y, J\xi \rangle - \langle X, Y \rangle] \\
+ \psi^2 \mu^3 [\langle X, \xi \rangle \langle Y, \xi \rangle - 7 \langle X, J\xi \rangle \langle Y, J\xi \rangle - |\xi|^2 \langle X, Y \rangle] \\
+ \mu^2 \langle R(X, \hat{\xi})\hat{\xi}, J\xi \rangle \langle Y, J\xi \rangle - 4 \psi^3 \mu^4 |\xi|^2 \langle X, J\xi \rangle \langle Y, J\xi \rangle \\
+ 2 \text{Ric}(\hat{\xi})\mu^3 |\xi|^2 \langle X, J\xi \rangle \langle Y, J\xi \rangle.
\end{align}

Finally, we compute IV.

(5.33)

\[IV = \mu \langle \hat{\hat{R}}(X, J\xi)\hat{\xi}, Y \rangle + \mu^2 \langle \hat{\hat{R}}(X, J\xi)\hat{\xi}, \xi \rangle \langle Y, \xi \rangle + \langle \hat{\hat{R}}(X, J\xi)\hat{\xi}, J\xi \rangle \langle Y, J\xi \rangle \].
(5.34)

\[ IVa = \langle R(X, J\xi)J\xi, Y \rangle - \text{Ric}(\xi)\mu\langle \xi, J\xi \rangle J\xi, Y \rangle + \langle \xi, X \rangle \langle J\xi, J\xi \rangle J\xi, Y \rangle \\
+ \langle \xi, X \rangle \langle J\xi, J\xi \rangle J\xi, Y \rangle - \langle \xi, J\xi \rangle \langle \xi, J\xi \rangle \langle J\xi, Y \rangle \\
- \langle \xi, J\xi \rangle \langle J\xi, X \rangle J\xi, Y \rangle - \langle \xi, J\xi \rangle \langle J\xi, J\xi \rangle J\xi, Y \rangle \\
+ \psi^2\mu_1(\langle X, J\xi \rangle J\xi, Y \rangle + 2(\langle X, J\xi \rangle J\xi, Y \rangle + \langle JX, J\xi \rangle J\xi, Y \rangle \\
- \langle J\xi, J\xi \rangle X, Y \rangle - \langle J\xi, J\xi \rangle J\xi, Y \rangle \\
+ \psi^2\mu_2(\langle X, J\xi \rangle J\xi, Y \rangle + 2(\langle X, J\xi \rangle J\xi, Y \rangle + \langle JX, J\xi \rangle J\xi, Y \rangle \\
- \langle \xi, J\xi \rangle J\xi, Y \rangle - \langle \xi, J\xi \rangle J\xi, J\xi \rangle J\xi, Y \rangle \\
- \langle J\xi, J\xi \rangle J\xi, J\xi \rangle J\xi, J\xi \rangle J\xi, Y \rangle \\
+ \langle J\xi, J\xi \rangle J\xi, J\xi \rangle J\xi, J\xi \rangle J\xi, J\xi \rangle J\xi, Y \rangle. \]

(5.35)

\[ IVb = \langle Y, \xi \rangle \{ \langle R(X, J\xi)J\xi, \xi \rangle - \text{Ric}(\xi)\mu[-2|\xi|^2\langle \xi, X \rangle] \\
+ \psi^2\mu_1(\langle X, J\xi \rangle J\xi, \xi \rangle - 3\langle X, \xi \rangle J\xi, \xi \rangle - \langle X, \xi \rangle \}
+ \psi^2\mu_2[-3\langle X, \xi \rangle \langle \xi, \xi \rangle - |\xi|^2\langle \xi, X \rangle]. \]

Consequently,

\[ IV = \mu\langle R(X, J\xi)J\xi, Y \rangle + 2\text{Ric}(\xi)\mu^2\langle \xi, X \rangle \langle \xi, Y \rangle \\
+ \psi^2\mu_2(\langle X, J\xi \rangle Y, J\xi \rangle - 3\langle X, \xi \rangle J\xi, \xi \rangle - \langle X, \xi \rangle \rangle \\
+ \psi^2\mu_2(\langle X, J\xi \rangle J\xi, \xi \rangle - 7\langle X, \xi \rangle Y, \xi \rangle - |\xi|^2\langle X, Y \rangle \rangle \\
+ \mu^2\langle R(X, J\xi)J\xi, \xi \rangle \langle Y, \xi \rangle - 4\psi^2\mu_4|\xi|^2\langle X, \xi \rangle Y, \xi \rangle \\
+ 2\text{Ric}(\xi)\mu^3|\xi|^2\langle X, \xi \rangle Y, \xi \rangle. \]
We observe that by definition of the Ricci tensor,

\[ III + IV = \mu [\langle R(X,\hat{\xi}),Y \rangle + \langle R(X,J\hat{\xi}),J\hat{\xi},Y \rangle] \]
\[ + \text{Ric}(\hat{\xi})\mu^2 (\langle \xi, X \rangle \langle \xi, Y \rangle + |\xi|^2 \langle X, Y \rangle + \langle J\xi, X \rangle \langle J\xi, Y \rangle) \]
\[ - 2\psi^2 \mu [\langle X, \hat{\xi} \rangle \langle Y, \hat{\xi} \rangle + \langle X, J\hat{\xi} \rangle \langle J\hat{\xi}, Y \rangle + \langle X, Y \rangle] \]
\[ - 2\psi^2 \mu^3 [3 \langle X, J\xi \rangle \langle Y, J\xi \rangle + 3 \langle X, \xi \rangle \langle Y, \xi \rangle + |\xi|^2 \langle X, Y \rangle] \]
\[ + \mu^2 [(\langle R(X,\hat{\xi}),J\xi \rangle \langle Y, J\xi \rangle + \langle R(X,J\hat{\xi}),J\xi \rangle \langle Y, J\xi \rangle] \]
\[ - 4\psi^2 \mu^4 |\xi|^2 [\langle X, \xi \rangle \langle Y, \xi \rangle + \langle X, J\xi \rangle \langle J\xi, Y \rangle] \]
\[ + 2\text{Ric}(\hat{\xi})\mu^3 |\xi|^2 \langle X, \xi \rangle \langle Y, \xi \rangle + \langle X, J\xi \rangle \langle J\xi, Y \rangle]. \]

(5.37)

This done, let us return to the computation of \( \text{Ric}(X,Y) \). We write

\[ \text{Ric}(X,Y) = \sum_{i=1}^{n-1} \mu^{-1} (I + II) + c^{-1} \mu^{-2} (III + IV). \]

Once more, let us separate the terms. We begin computing the first summand.

\[ \sum_{i=1}^{n-1} \mu^{-1} (I + II) = \sum_{i=1}^{n-1} \{(\langle R(X,e_i)e_i, Y \rangle + \langle R(X,Je_i),Je_i, Y \rangle) \]
\[ - 2\psi^2 \mu \sum_{i=1}^{n-1} [\langle X, e_i \rangle \langle Y, e_i \rangle + \langle X, Je_i \rangle \langle Y, Je_i \rangle + \langle X, Y \rangle] \]
\[ + \mu \langle Y, \xi \rangle \sum_{i=1}^{n-1} [\langle R(X,e_i)e_i, \xi \rangle + \langle R(X,Je_i),Je_i, \xi \rangle] \]
\[ + \mu \langle Y, J\xi \rangle \sum_{i=1}^{n-1} [\langle R(X,e_i)e_i, J\xi \rangle + \langle R(X,Je_i),Je_i, J\xi \rangle] \]
\[ - 2\psi^2 \mu^2 \sum_{i=1}^{n-1} [\langle X, \xi \rangle \langle Y, \xi \rangle + \langle X, J\xi \rangle \langle J\xi, Y \rangle]. \]

(5.38)

We observe that by definition of the Ricci tensor,

\[ \sum_{i=1}^{n-1} \{(\langle R(X,e_i)e_i, Y \rangle + \langle R(X,Je_i),Je_i, Y \rangle) \}
\[ = \text{Ric}(X,Y) - \langle R(X,\hat{\xi}),\hat{\xi}, Y \rangle - \langle R(X,J\hat{\xi}),J\hat{\xi}, Y \rangle, \]
\[ \sum_{i=1}^{n-1} [\langle R(X,e_i)e_i, \xi \rangle + \langle R(X,Je_i),Je_i, \xi \rangle] =
\[ \frac{\text{Ric}(X,\xi)}{\mu} - \langle R(X,\hat{\xi}),\hat{\xi}, \xi \rangle - \langle R(X,J\hat{\xi}),J\hat{\xi}, \xi \rangle. \]
and

\[ \sum_{i=1}^{n-1} [(R(X, e_i)e_i, J\xi) + (R(X, Je_i)Je_i, J\xi)] = 0 \]

\[ = \text{Ric}(X, J\xi) - \langle R(X, \hat{\xi}), J\xi \rangle - \langle R(X, J\hat{\xi})J\xi, J\xi \rangle. \]

Besides, since that

\[ Y = \sum_{i=1}^{n-1} [(Y, e_i)e_i + (Y, Je_i)Je_i] + \langle Y, \hat{\xi} \rangle \hat{\xi} + \langle Y, J\hat{\xi} \rangle J\hat{\xi}, \]

we obtain

\[ \sum_{i=1}^{n-1} [(X, e_i)(Y, e_i) + (X, Je_i)(Y, Je_i) + \langle X, Y \rangle] = \]

\[ = \langle X, \sum_{i=1}^{n-1} [(Y, e_i)e_i + (Y, Je_i)Je_i] + \langle X, Y \rangle \rangle = \]

\[ = n \langle X, Y \rangle - \langle X, \hat{\xi} \rangle \hat{\xi} - \langle X, J\hat{\xi} \rangle J\hat{\xi}. \]

Therefore, the first summand can be written as

\[ \sum_{i=1}^{n-1} \mu^{-1}(I + II) = \text{Ric}(X, Y) - \langle R(X, \hat{\xi}), \hat{\xi}, Y \rangle - \langle R(X, J\hat{\xi})J\hat{\xi}, Y \rangle \]

\[ - 2\psi^2 \mu [n \langle X, Y \rangle - \langle X, \hat{\xi} \rangle \hat{\xi} - \langle X, J\hat{\xi} \rangle \langle Y, J\hat{\xi} \rangle] \]

\[ + \mu (Y, \xi) \langle \text{Ric}(X, \xi) - \langle R(X, J\hat{\xi})J\hat{\xi}, \xi \rangle \rangle \]

\[ + \mu (Y, J\xi) \langle \text{Ric}(X, J\xi) - \langle R(X, \hat{\xi}), J\xi \rangle \rangle \]

\[ - 2\psi^2 \mu^2 (n - 1) \langle [X, \xi] \langle Y, \xi \rangle + \langle X, J\xi \rangle \rangle \]

(5.39)

Now, we turn our attention to the second term of \( \hat{\text{Ric}}(X, Y) \).

(5.40)

\[ c^{-1} \mu^{-2}(III + IV) = c^{-1} \mu^{-1} [(R(X, \hat{\xi})\hat{\xi}, Y) + (R(X, J\hat{\xi})J\hat{\xi}, Y)] \]

\[ + c^{-1} \text{Ric}(\hat{\xi}) [(\xi, X) \langle \xi, Y \rangle + |\xi|^2 \langle X, Y \rangle + \langle J\xi, X \rangle \langle J\xi, Y \rangle] \]

\[ - 2c^{-1} \psi^2 [(X, \hat{\xi}) \langle Y, \hat{\xi} \rangle + \langle X, J\hat{\xi} \rangle \langle Y, J\hat{\xi} \rangle + \langle X, Y \rangle] \]

\[ - 4c^{-1} \psi^2 \mu [3 \langle X, J\xi \rangle \langle Y, J\xi \rangle + 3 \langle X, \xi \rangle \langle Y, \xi \rangle] \]

\[ + c^{-1} [(R(X, \hat{\xi})\hat{\xi}, J\xi) \langle Y, J\xi \rangle + \langle R(X, J\hat{\xi})J\hat{\xi}, \xi \rangle \langle Y, \xi \rangle] \]

\[ - 4c^{-1} \psi^2 \mu^2 |\xi|^2 \langle (X, \xi) \langle Y, \xi \rangle + \langle X, J\xi \rangle \rangle \]

\[ + 2c^{-1} \text{Ric}(\hat{\xi}) \mu |\xi|^2 \langle (X, \xi) \langle Y, \xi \rangle + \langle X, J\xi \rangle \rangle \].

Consequently, putting all together
\[
\widehat{\text{Ric}}(X,Y) = \sum_{i=1}^{n-1} \mu^{-1}(I + II) + c^{-1}\mu^{-2}(III + IV).
\]

\[
= \text{Ric}(X,Y) + c^{-1}\mu^{-1}(1 - c\mu)[(R(X,\tilde{\xi})\tilde{\xi},Y) + (R(X,J\tilde{\xi})J\tilde{\xi},Y)]
+ c^{-1}\text{Ric}(\tilde{\xi})[(\xi, X)\langle \xi, Y \rangle + \langle \xi \rangle^2(X,Y) + (J\xi,X)(J\xi,Y)
+ 2\mu\langle \xi \rangle^2((X,\xi)(Y,\xi) + \langle X, J\xi \rangle(Y, J\xi))]
+ \mu(Y,\xi)\text{Ric}(X,\xi) + \mu(Y,J\xi)\text{Ric}(X,J\xi)
+ c^{-1}(1 - c\mu)[(R(X,\tilde{\xi})\tilde{\xi},J\xi)(Y,J\xi) + (R(X,J\tilde{\xi})J\tilde{\xi},\xi)(Y,\xi)]
+ (X,Y)[2n\psi^2\mu - 2c^{-1}\psi^2 - 2c^{-1}\psi^2\mu\langle \xi \rangle^2]
\]

where the computation in the last line is similar to the previous. \hfill \square

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