Considerations Concerning the QCD Corrections to $\Delta \rho$

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ABSTRACT

Using recent results of Avdeev et al. and an expansion for $\mu_t/m_t(M_t)$ ($M_t$ is the pole mass and $\mu_t \equiv m_t(\mu_t)$), it is shown that when $\Delta \rho$ is expressed in terms of $m_t^2(M_t)$, the QCD correction is only $(2 - 3) \times 10^{-3}$ in the NLO approximation. As a consequence, in terms of $M_t^2$ the correction to $\Delta \rho$ is almost entirely contained in $m_t^2(M_t)/M_t^2$, a pure QCD effect. The latter is studied using various optimization procedures, and the results compared with the expansion proposed by Avdeev et al.. Implications for electroweak physics are discussed. Threshold effects are analyzed on the basis of a simple sum rule.
The question of the QCD corrections to $\Delta \rho$ has been the subject of several recent studies [1-4]. In particular, Avdeev, Fleischer, Mikhailov, and Tarasov carried out a complete three-loop calculation of $O(\alpha \alpha_s^2)$. Their result, obtained in the limit $m_b \to 0$, can be expressed to good accuracy in the form

$$ (\Delta \rho)_f = \frac{3G_\mu M_t^2}{8\sqrt{2}\pi^2} [1 + \delta_{QCD}], $$

where $f$ denotes the fermionic contributions, $M_t$ is the pole mass, and

$$ \delta_{QCD} = -2.860 \frac{\alpha_s(M_t)}{\pi} - 10.55 \left( \frac{\alpha_s(M_t)}{\pi} \right)^2. $$

is the QCD correction. As an illustration, for $M_t = 200\text{GeV}$, Eq.(2) gives $\delta_{QCD} = -0.0961 - 0.0119 = -0.1080$. Thus, the $O(\alpha_s^2)$ term in Eq.(2) is reasonably small and leads to an enhancement of $0.0119/0.0961 = 12.4\%$ of the leading QCD result. Moreover, if $\alpha_s(M_t)$ in Eq.(2) is expressed in terms of $\alpha_s(\mu)$ and the resulting series is truncated in $O(\alpha_s^2(\mu))$, the $\mu$-dependence of $\delta_{QCD}(\mu)$ is relatively mild. For example, for $M_t = 200\text{GeV}$ and $0.1 \leq \mu/M_t \leq 1$, we find a variation $\leq 5.8 \times 10^{-3}$, which amounts to $5.4\%$ of the total QCD correction or $49\%$ of the $O(\alpha_s^2)$ term.

In spite of these facts, it is worth noting that the expansion in Eq.(2) involves large and increasing coefficients, a feature that is frequently an indication of significant higher order terms. Furthermore, the arguments of Ref.[2] suggest that there are at least two scales in this problem: one, of $O(M_t)$, associated with the correction to the electroweak amplitude, and another one, much smaller, related to contributions to the pole mass $M_t$ involving small gluon momenta. It should also be observed that the $\mu$-sensitivity of the truncated series depends on the chosen interval, and becomes sharper for $\mu/M_t < 0.1$. For these reasons, it is a good idea to find alternative expressions for $\delta_{QCD}$ that separate the two scales, and at the same time involve terms of $O(\alpha_s^2)$ with small coefficients. A simple way of implementing this idea has been outlined in Ref.[3]. One expresses first $(\Delta \rho)_f$ in terms of $\hat{m}_t(M_t)$, the running mass evaluated at the pole mass, and then relates $\hat{m}_t(M_t)$ to $M_t$ by optimizing the expansion of $M_t/\hat{m}_t(M_t)$, which is known through $O(\alpha_s^3)$ [5]. As shown below, the arguments of Ref.[3] can be significantly refined using the new results of Ref.[4].

Calling $\mu_t$ the solution of $\hat{m}_t(\mu) = \mu$ and using Eq.(19) of Ref.[4], $(\Delta \rho)_f$ can be written in the form

$$ (\Delta \rho)_f = \frac{3G_\mu \mu_t^2}{8\sqrt{2}\pi^2} [1 + \delta_{QCD}^{\overline{MS}}], $$

$$ \delta_{QCD}^{\overline{MS}} = -0.19325 \frac{\alpha_s(\mu_t)}{\pi} + 0.07111 \left( \frac{\alpha_s(\mu_t)}{\pi} \right)^2. $$

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We see that the convergence pattern of Eq.(4) is very nice, with very small and decreasing coefficients and alternating signs. For this reason we will assume that the terms of $O(\alpha_s^3)$ and higher are negligible and evaluate Eq.(4) with $\mu_t \rightarrow M_t$ as this introduces only a small change of $O(\alpha_s^3)$. This argument is further strengthened in Appendix A, where we discuss the potential contribution of threshold effects to perturbative expansions, such as Eq.(4), on the basis of a simple sum rule.

In order to express $(\Delta \rho_f)$ in terms of $\hat{m}_t(M_t)$, we use the following next-to-leading order (NLO) expansion, derived in Appendix B:

$$\frac{\mu_t}{\hat{m}_t(M_t)} = 1 + \frac{8}{3} \left( \frac{\alpha_s(M_t)}{\pi} \right)^2 + [36.33 - 2.45 n_f] \left( \frac{\alpha_s(M_t)}{\pi} \right)^3,$$

(5)

where $n_f = 5$ is the number of light flavors. Writing

$$(\Delta \rho)_f = \frac{3G_F \hat{m}_t^2(M_t)}{8\sqrt{2}\pi^2} [1 + \Delta_{QCD}],$$

(6)

it is possible to combine Eq.(3-5) in a single expansion

$$\Delta_{QCD} = -0.19325 \frac{\alpha_s(M_t)}{\pi} + C \left( \frac{\alpha_s(M_t)}{\pi} \right)^2.$$  

(7)

In fact, Eq.(4) was proposed in Ref.[3] before the results of Ref.[4] were known, and it was argued, on the basis of convergence assumptions, that $|C| \leq 6$. From Eqs.(3-5) we see that $C = 5.40$, consistent with the arguments of Ref.[3]. However, with $C = 5.40$ the two terms in Eq.(7) nearly cancel and its convergence properties become problematic. In order to evaluate $\Delta_{QCD}$, it is better to employ the separate expansions in Eqs.(3,4) and Eq.(5). In the latter we retain the relatively large $O((\hat{\alpha}_s/\pi)^3)$ term in order to control the scale of the leading contribution. Applying the Brodsky-Lepage-Mackenzie (BLM) optimization \cite{6} to Eq.(5), we have

$$\frac{\mu_t}{\hat{m}_t(M_t)} = 1 + \frac{8}{3} \left( \frac{\alpha_s(0.252M_t)}{\pi} \right)^2 - 4.10 \left( \frac{\alpha_s(0.252M_t)}{\pi} \right)^3 \text{(BLM)}.$$  

(8)

For $M_t = 200$GeV, Eqs.(5) and (8) give $\mu_t/\hat{m}_t(M_t) = 1.00392$ and 1.00425, respectively (in this paper $\alpha_s(\mu)$ is evaluated with a 3-loop $\beta$-function for $n_f = 5$ light flavors, normalized such that $\alpha_s(m_Z) = 0.118$, with $m_Z = 91.19$GeV). We have verified that the evaluations of $\mu_t/\hat{m}_t(M_t)$ by the principle of minimal sensitivity (PMS)\cite{7} and the method of fastest apparent convergence (FAC)\cite{8} are very close to the BLM result (the difference is $\leq 5 \times 10^{-6}$). The correction $\Delta_{QCD}$ in Eq.(7) is obtained from the relation

$$1 + \Delta_{QCD} = \left( \frac{\mu_t}{\hat{m}_t(M_t)} \right)^2 \left[ 1 + \delta_{\overline{MS}}^{QCD} \right].$$  

(9)
where $\delta_{QCD}^{MS}$ and $\mu_t/\hat{m}_t(M_t)$ are evaluated via Eq.(4) (with $\mu_t \rightarrow M_t$) and Eq.(8), respectively.

Table 1 shows that $\delta_{QCD}^{MS}$ and $\Delta_{QCD}$ are indeed small corrections for $M_t \geq 130$GeV. In particular, $\Delta_{QCD} = (2 - 3) \times 10^{-3}$, depending on $M_t$. This conforms with the idea that $\hat{m}_t(M_t)$ is a good expansion parameter in the sense that the associated QCD corrections are small [3]. Furthermore, in the $(\Delta \rho)_f$ case this occurs to a rather remarkable degree. However, it is not a good approximation to retain only the $O(\alpha_s)$ correction in Eq.(6), as it is nearly cancelled by higher order contributions. On the other hand, the expansion in Eq.(4) is very well behaved. In this sense, $\mu_t$ is a better expansion parameter than $\hat{m}_t(M_t)$, which may reflect the fact that it makes no reference to the pole mass $M_t$.

Following the strategy outlined in Ref.[3], $\delta_{QCD}$ in Eq.(1) can be obtained from the relation

$$1 + \delta_{QCD} = \left(\frac{\hat{m}_t(M_t)}{M_t}\right)^2 \left[1 + \Delta_{QCD}\right].$$

(10)

For $M_t/\hat{m}_t(M_t)$ we have the well-known expansion [5]

$$\frac{M_t}{\hat{m}_t(M_t)} = 1 + \frac{4}{3} \frac{\alpha_s(M_t)}{\pi} + (16.11 - 1.04n_f) \left(\frac{\alpha_s(M_t)}{\pi}\right)^2,$$

(11)

where the masses of the light quarks have been neglected. As $M_t/\hat{m}_t(M_t)$ plays a central role in our final result, we record the expressions obtained by applying the three optimization methods to the r.h.s. of Eq.(11)

$$\frac{M_t}{\hat{m}_t(M_t)} = 1 + \frac{4}{3} \frac{\alpha_s(\mu^*)}{\pi} \left(\frac{\alpha_s(\mu^*)}{\pi}\right)^2 \quad (BLM),$$

(12)

$$\frac{M_t}{\hat{m}_t(M_t)} = 1 + \frac{4}{3} \frac{\alpha_s(\mu^{**})}{\pi} - 0.84 \left(\frac{\alpha_s(\mu^{**})}{\pi}\right)^2 \quad (PMS),$$

(13)

$$\frac{M_t}{\hat{m}_t(M_t)} = 1 + \frac{4}{3} \frac{\alpha_s(\mu^{***})}{\pi} \quad (FAC),$$

(14)

where $\mu^* = 0.0963M_t$, $\mu^{**} = 0.1004M_t$, $\mu^{***} = 0.1183M_t$. We note that Eqs.(12-14) involve coefficients of $O(1)$ and similar scales. For $M_t = 200$GeV, Eqs.(12-14) give $M_t/\hat{m}_t(M_t) = 1.06305, 1.06304, and 1.06299$, respectively. For $M_t = 174$GeV, the corresponding values are 1.06479, 1.06478, 1.06472. Finally, for $M_t = 130$GeV, we

[1] The usefulness of $\mu_t$ as an expansion parameter has also been emphasized in Ref.[8].
have 1.06877, 1.06876, 1.06869. Thus, the three approaches give remarkably consistent results. In contrast, the expansion in Eq.(11), which involves a large second order coefficient, gives, for $M_t = (200, 174, 130)\text{GeV}$, 1.0571, 1.0584, 1.0614, respectively, which are $(0.6 - 0.7)\%$ smaller. One must conclude that the coefficients of the unknown terms of $O((\alpha_s/\pi)^3)$ and higher in Eq.(11) and or Eqs.(12-14) are large. For instance, if the truncated BLM expansion in Eq.(12) were exact, the coefficient of the $(\alpha_s(M_t)/\pi)^3$ and $(\alpha_s(M_t)/\pi)^4$ terms in Eq.(11) would be $\approx 104$ and $1,041$, respectively. In the following we employ the optimized expression for $M_t/\hat{m}_t(M_t)$, which, for definiteness, we identify with Eq.(12).

Table 2 displays the values of $\delta_{QCD}$ obtained from Eq.(11), using Eq.(12) and our previous determination of $\Delta_{QCD}$ (Eq.(1)), and compares them with those derived from Eq.(2). In order to show the effect of the higher order contributions (H.O.C.), we also exhibit the fractional enhancement of the total QCD correction over the conventional $O(\alpha_s)$ result (first term in Eq.(2)).

As pointed out by Kniehl [11], the various evaluations of QCD corrections can be conveniently interpreted as effective re-scalings of the $O(\alpha_s)$ result. In fact, we find that the two evaluations of $\delta_{QCD}$ in Table 2 can be very well represented by the simple formulas

$$\delta_{QCD} = -2.86 \frac{\alpha_s(0.444M_t)}{\pi} \ [\text{Eq.(2)}], \tag{15}$$

$$\delta_{QCD} = -2.86 \frac{\alpha_s(0.323M_t)}{\pi} \ [\text{Eq.(10)}]. \tag{16}$$

The effective scales have been chosen so that Eqs.(15,16) reproduce the values in Table 2 with an error of at most 1 or 2 times $10^{-4}$.

From Table 2 we see that, for $130\text{GeV} \leq M_t \leq 220\text{GeV}$, the evaluation of $|\delta_{QCD}|$ based on Eq.(11) is $(5.1 - 6.5) \times 10^{-3}$ larger than the results from Eq.(2). This difference amounts to $\approx 5\%$ of the total QCD correction. On the other hand, the last two columns in Table 2 show that the H.O.C. are $\approx 45\%$ larger in the evaluation based on Eq.(11).

As the two calculations coincide through terms of $O((\alpha_s(M_t)/\pi)^2)$, one must conclude that the coefficients of the $O((\alpha_s/\pi)^3)$ and higher terms in Eq.(2) and or Eq.(11) are large. For instance, if Eq.(11) (with $\Delta_{QCD}$ and $M_t/\hat{m}_t(M_t)$ evaluated via Eqs.(1) and (12)) were exact, the coefficient of the neglected $(\alpha_s(M_t)/\pi)^3$ term in Eq.(2) would be $\approx -93.4$. Of course, it could happen that the exact answer lies between the two evaluations, in which case the $O(\alpha_s^3)$ terms would be smaller. However, because of its pattern of large and increasing coefficients, we think that this problem is more likely to occur in Eq.(2). It is also useful to note that if one applies the optimization procedures to Eq.(2), the results become closer to those from Eq.(11). Specifically, we have

\[\text{For a recent application of the PMS and FAC approaches to estimate higher order coefficients in other cases, see Ref.[11].}\]
\[ \delta_{QCD} = -2.86 \frac{\alpha_s(0.154M_t)}{\pi} + 9.99 \left( \frac{\alpha_s(0.154M_t)}{\pi} \right)^2 \]  
(Eq.(2); BML),  
(17)

\[ \delta_{QCD} = -2.86 \frac{\alpha_s(0.324M_t)}{\pi} + 1.80 \left( \frac{\alpha_s(0.324M_t)}{\pi} \right)^2 \]  
(Eq.(2); PMS),  
(18)

\[ \delta_{QCD} = -2.86 \frac{\alpha_s(0.382M_t)}{\pi} \]  
(Eq.(2); FAC).  
(19)

For \( M_t = 200\text{GeV} \), Eqs. (17-19) give 0.1085, 0.11044, 0.11037, respectively. We see that the difference between Eqs.(18,19) and Eq.(10) is \( \approx 45\% \) smaller than that between Eq.(2) and Eq.(10). It has also been pointed out that in most NLO QCD calculations, it is a good approximation to retain only the leading term, evaluated at the BLM scale [2,7]. From Eq.(17) we see that this is not the case for \((\Delta \rho)_f\), as the coefficient of the residual \( O\left((\alpha_s/\pi)^2\right) \) term, 9.99, is quite large.

There is a simple but important point that should be stressed: once it is recognized that \( \delta_{QCD} \) is almost entirely contained in the first factor of Eq.(11) (and this follows from Eq.(4) and (8), which are quite robust), it becomes clear that its magnitude and precision are largely controlled by the value of \((M_t/\hat{m}_t(M_t))^2\) and the accuracy within which it can be calculated. We emphasize that this is a pure QCD, rather than electroweak effect. We have also pointed out that there is a significant difference between the evaluations of this correction via Eq.(11) or by means of Eqs.(12-14). In this paper we have taken the point of view that \((M_t/\hat{m}_t(M_t))^2\) can be best evaluated at present by means of the optimized expansions of Eqs.(12-14). In particular, we have noted that the coefficients in these equations are \( O(1) \) and that the values derived from the three optimization procedures remarkably consistent. In fact, it would be very interesting to check this approach by an explicit evaluation of the unknown third order term in Eq.(11).

As shown in Table 2, the formulation of this paper, based in Eqs.(3,11,12) and summarized in the effective formula of Eq.(10), leads, for \( M_t = 200\text{GeV} \), to an enhancement of 18.0% relative to the conventional \( O(\alpha_s) \) calculation. This modification corresponds to an additional contribution \(-2.2 \times 10^{-4}\) to \((\Delta \rho)_f\) and \(+0.76 \times 10^{-3}\) to \(\Delta r\), and shifts the predicted values of \( M_t \) and \( m_W \) by \(+1.9\text{GeV}\) and \(-13\text{MeV}\), respectively. It can be compared with an enhancement of 12.4% from Eq.(2), or 14.9% from the optimized expressions of Eq.(15) or Eq.(13). Although the H.O.C. obtained from Eq.(2) and Eq.(10) are significantly different, the effects of this difference on electroweak physics, namely variations of 0.6GeV in \( M_t \) and 4MeV in \( m_W \), are rather small. As mentioned before, it is possible that the exact answer lies between the two
calculations. Although we prefer the evaluation based on Eq.(10), the difference with Eq.(2) is interesting because it may be used as an estimate of the theoretical error due to uncalculated higher order terms. We recall that there is at present an additional uncertainty of $\approx 5\%$ due to the $\pm 0.006$ error in $\alpha_s(m_Z)$\[3\]. Following the discussion of Ref.[3], it is also easy to see how the approach in this paper can be applied to other electroweak amplitudes proportional to $M_t^2$, such as those present in $Z^0 \rightarrow b\bar{b}$.

Finally, it is instructive to compare the above results with some of the estimates made before the complete $O(\alpha_s^2)$ contribution became known. Although conceptually very different, the dispersive approach as implemented in the last paper of Ref.[1], gives, for $130\text{GeV} \leq M_t \leq 220\text{GeV}$, results very close numerically to those of Ref.[2]. The latter approximates the relevant Feynman diagrams by the $O(\alpha_s)$ contribution evaluated at the BLM scale (first term of Eq.(17)). For $M_t = 200\text{GeV}$, both estimates led approximately to a 34\% enhancement of the $O(\alpha_s)$ contribution, a result of the same sign but clearly larger than the conclusions from either Eq.(2) or Eq.(10). On the other hand, our previous estimate \[(26 \pm 6\%)\] \[3\], is roughly consistent with Eq.(10), but larger than the values from Eq.(2).

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APPENDIX A

We briefly discuss threshold effects on the basis of a simple sum rule that follows as a corollary from the arguments of Ref.[12]. In that work the operator product expansion \[13\] is applied to investigate the validity of the dispersion relations [D.R.] for vacuum polarization functions previously proposed by Chang et al. and Kniehl and Sirlin \[11\], to all orders in perturbation theory. We call $\Pi^{V,A}_{\mu\nu}(q,m_1,m_2)$ the vacuum polarization tensors associated with the vector and axial vector currents attached to pairs of quarks, such as $t\bar{t}$, $t\bar{b}$, $b\bar{b}$, endowed with masses $m_1$ and $m_2$. Decomposing

$$
\Pi^{V,A}_{\mu\nu}(q,m_1,m_2) = \Pi^{V,A}(q^2,m_1,m_2)g_{\mu\nu} + \lambda^{V,A}(q^2,m_1,m_2)q_{\mu}q_{\nu},
$$

(A.1)

we have the sum rule

$$
\int_{s'}^{\infty} ds' \quad \text{Im} \quad \sum_{i=1}^{2} \frac{1}{2} \left( \lambda^V(s',m_i,m_i) + \lambda^A(s',m_i,m_i) \right)
$$

(A.1)
\[-\lambda^V(s', m_1, m_2) - \lambda^A(s', m_1, m_2)\] = 0. \hspace{1cm} (A.2)

In order to derive Eq.(A.2), one invokes analitycity and applies Cauchy’s theorem to the appropriate combination of \(\lambda\)'s over a contour with straight lines just above and below the positive real axis, closed by a large circle. The arguments of Ref.\[1\] and the results of Ref.\[3\] then show that the contribution of the large circle vanishes in the limit of infinite radius. Identifying \(m_1 = m_t\), \(m_2 = m_b\), there are large and equal non-relativistic contributions from the threshold region to \(\text{Im}\lambda^V(s', m_t, m_t)\) and \(\text{Im}\lambda^A(s', m_t, m_t)\) (threshold contributions to the other \(\lambda\)'s are suppressed by reduced mass effects).

However, the appropriate D.R.\[4\] involve the integrals \(\int ds' \text{Im}\lambda^A(s', m_t, m_t)\) or \(\int ds' \text{Im}\lambda^V(s', m_t, m_t)\) over all values of \(s'\). It is precisely from the contributions of such integrals in the D.R. that large threshold effects may potentially arise. On the other hand, Eq.(A.2) tells us that \(\int ds' \text{Im}\lambda^A(s', m_t, m_t) + \lambda^A(s', m_t, m_t)\) can be expressed as a linear combination of integrals of the spectral functions \(\text{Im}\lambda^{(i)}(s', m_t, m_b)\) and \(\lambda^A(s', m_t, m_t)\) \(\gamma = V, A\) which do not have significant threshold contributions and involve different channels. The most sensible interpretation of this result is that the large threshold contributions to \(\text{Im}\lambda^V(s', m_t, m_t)\) and \(\lambda^A(s', m_t, m_t)\) cancel against other contributions when the integrals over all \(s'\) values are considered. As it is known that such integrals can be expanded perturbatively in powers of \(\alpha_s\), this cancellation must occur order by order in perturbation theory. In particular, this applies to the expansion in Eq.(4). Similar conclusions were reached in Ref.\[2\] on the basis of different arguments. In summary, although the sum rule in Eq.(A.2) does not provide a rigorous proof of the cancellation of large threshold effects in electroweak amplitudes, it gives strong support to such conclusion when all contributions of a given order are included. On the other hand, it does not reveal the detailed mechanism of the cancellation, an interesting problem which hopefully can be clarified in the future.

**APPENDIX B**

In order to derive Eq.(5), we start with the well-known expression \[13\]

\[
\hat{m}_t(\mu) = m_t^* \left( -\beta_1 \frac{\alpha_s(\mu)}{\pi} \right)^d \left[ 1 + a_1 \frac{\alpha_s(\mu)}{\pi} + a_2 \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 + \ldots \right], \hspace{1cm} (B.1)
\]

where \(m_t^*\) is the RG invariant mass, \(\beta_1 = \frac{n_f}{3} - \frac{11}{2}\), \(d = -2/\beta_1\), \(a_1 = (\beta_2/\beta_1)[2/\beta_1 - \gamma_2/\beta_2]\), \(\gamma_2 = \frac{101}{12} - \frac{5n_f}{18}\) (the other quantities need not be specified for our purposes).

Setting \(\mu = \mu_t \equiv \hat{m}_t(\mu_t)\) in Eq.(B.1) and dividing by \(\hat{m}_t(M_t)\) we have

\[
\frac{\mu_t}{\hat{m}_t(M_t)} = \left( \frac{\alpha_s(\mu_t)}{\alpha_s(M_t)} \right)^d \left[ 1 + a_1 \frac{\alpha_s(\mu_t)}{\pi} + a_2 \left( \frac{\alpha_s(\mu_t)}{\pi} \right)^2 + \ldots \right] \left[ 1 + a_1 \frac{\alpha_s(M_t)}{\pi} + a_2 \left( \frac{\alpha_s(M_t)}{\pi} \right)^2 + \ldots \right] \hspace{1cm} (B.2)
\]
In the numerators we expand $\alpha_s(\mu)$ in powers of $\alpha_s(M_t)$. The first factor gives
\[
\left( \frac{\alpha_s(\mu_t)}{\alpha_s(M_t)} \right)^d = 1 + d \frac{\alpha_s(M_t)}{\pi} \ln \left( \frac{\mu_t}{M_t} \right) \left( \beta_1 + \beta_2 \frac{\alpha_s(M_t)}{\pi} \right) + O(\alpha_s^4) \tag{B.3}
\]
where we have used the fact that $\ln(\mu_t/M_t) = O(\alpha_s)$. Similarly, we see that the second factor in Eq.(B.2) becomes $1 + a_1 \beta_1 (\alpha_s(M_t)/\pi)^2 \ln(\mu_t/M_t) + O(\alpha_s^4)$. Combining the two factors, noting that $\beta_1 d = -2$, $\beta_2 d + a_1 \beta_1 = -\gamma_2$, and splitting the logarithm we have
\[
\frac{\mu_t}{\hat{m}_t(M_t)} = 1 - \frac{\alpha_s(M_t)}{\pi} \left[ \ln \left( \frac{\mu_t}{\hat{m}_t(M_t)} \right) + \ln \left( \frac{\hat{m}_t(M_t)}{M_t} \right) \right] \left[ 2 + \gamma_2 \frac{\alpha_s(M_t)}{\pi} \right] + O(\alpha_s^4) \tag{B.4}
\]
Inserting $\mu_t/\hat{m}_t(M_t) = 1 + \sum_{n=1}^{3} x_n (\alpha_s(M_t)/\pi)^n$ in both sides of Eq.(B.4), using Eq.(11) in the second logarithm and matching coefficients, we obtain $x_1 = 0$, $x_2 = 8/3$, $x_3 = 2(16.11 - 1.04 n_f - 8/9) + 4\gamma_2/3 - 2x_2 = 36.33 - 2.45 n_f$, which leads to Eq.(5). This derivation assumes that $\hat{m}_t(\mu)$ evolves according to the $n_f$ light quarks. The difference in $\gamma_2$ and $x_3$ when one employs 6 active quarks, as in Ref.[4], is very small.

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Table 1. The corrections $\delta_{MS}^{\Delta}$ and $\Delta_{QCD}$. The first one is given by Eq.(4) with $\mu_t \rightarrow M_t$, while the second is obtained from Eq.(9), with $\mu_t/\hat{\mu}_t(M_t)$ evaluated according to Eq.(8) ($\alpha_s(m_Z) = 0.118$ is employed).

| $M_t$(GeV) | $10^3\delta_{MS}^{\Delta}$ | $10^3\Delta_{QCD}$ |
|------------|-----------------|-----------------|
| 130        | -6.80           | 2.98            |
| 150        | -6.67           | 2.65            |
| 174        | -6.53           | 2.33            |
| 200        | -6.41           | 2.06            |
| 220        | -6.33           | 1.88            |

Table 2. Comparison of two determinations of $\delta_{QCD}$. The second column is based on Eq.(2) [4]. The third column is based on Eq.(10) with $M_t/\hat{\mu}_t(M_t)$ obtained from Eq.(12) and $\Delta_{QCD}$ evaluated according to Table 1. The fourth and fifth columns give the fractional enhancement over the conventional $O(\alpha_s)$ result (first term of Eq.(2)) due to the inclusion of higher order contributions (H.O.C.).

| $M_t$ (GeV) | $\delta_{QCD}$ (Eq.(2)) | $\delta_{QCD}$ (Eq.(10)) | H.O.C. (Eq.(2))% | H.O.C. (Eq.(10))% |
|-------------|-----------------|-----------------|-----------------|----------------|
| 130         | -0.1154         | -0.1219         | 13.2            | 19.5           |
| 150         | -0.1128         | -0.1189         | 12.9            | 19.0           |
| 174         | -0.1102         | -0.1159         | 12.6            | 18.4           |
| 200         | -0.1080         | -0.1133         | 12.4            | 18.0           |
| 220         | -0.1064         | -0.1115         | 12.2            | 17.6           |