On the existence of global solutions to equations for mixtures of compressible viscous fluids

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Abstract. We prove existence of global generalized solutions for initial boundary value problem which describes the three-dimensional unsteady barotropic motions of binary mixtures of compressible viscous fluids in a bounded domain with impermeable boundary.

1. Introduction
Assume that a binary mixture of viscous gases occupies a bounded domain \(\Omega \subset R^3\) in the Euclidean space of points \(x = (x_1, x_2, x_3)\) with boundary of class \(C^{2+\nu}\). A motion of the mixture is characterized by the velocity fields \(\vec{u}^{(i)}(t, x)\), the densities \(\rho_i(t, x)\) and also by pressures \(p_i(t, x)\) of the mixture components \((i = 1, 2)\), which satisfy the following equations [1]

\[
\frac{\partial}{\partial t}(\rho_i \vec{u}^{(i)}) + \text{div}(\rho_i \vec{u}^{(i)} \otimes \vec{u}^{(i)}) + \nabla p_i = \text{div} \sigma^{(i)} + \vec{J}^{(i)} \quad \text{in} \; Q_T = (0, T) \times \Omega, \tag{1}
\]

\[
\frac{\partial}{\partial t}(\rho_i) + \text{div}(\rho_i \vec{u}^{(i)}) = 0 \quad \text{in} \; Q_T, \quad i = 1, 2. \tag{2}
\]

Here viscous stress tensors \(\sigma^{(i)}\) are defined by the equalities

\[
\sigma^{(i)}(\vec{u}^{(1)}, \vec{u}^{(2)}) = \sum_{j=1}^{2} \left[2\mu_{ij} D(\vec{u}^{(j)}) + \lambda_{ij} \text{div} \vec{u}^{(j)} \cdot I\right], \tag{3}
\]

where \(D(\vec{u}) = \frac{1}{2} \left(\nabla \vec{u} (\nabla \vec{u})^T\right)\), \(I\) is the identity tensor, constant viscosity coefficients \(\mu_{ij}\) and \(\lambda_{ij}\), \(i, j = 1, 2\), satisfy the conditions

\[
\mu_{11} > 0, \quad 4\mu_{11}\mu_{22} - (\mu_{12} + \mu_{21})^2 > 0, \quad \nu_{11} > 0, \quad \nu_{22} > 0, \quad 4\nu_{11}\nu_{22} - (\nu_{12} + \nu_{21})^2 > 0, \quad \nu_{ij} = \lambda_{ij} + 2\mu_{ij}, \quad i, j = 1, 2. \tag{4}
\]

It is assumed the following relationship between the pressures \(p_i\) and the density \(\rho_i\) for the \(i\)-th component of the mixture \(p_i = \rho_i^{\gamma_i}\), where \(\gamma_i > 1, i = 1, 2\), are adiabatic exponents. The summands \(\vec{J}^{(i)} = (-1)^{i+1} \cdot a(\vec{u}^{(2)} - \vec{u}^{(1)})\), \(a = \text{const} > 0, \ i = 1, 2\), characterize the intensity of the momentum exchange between the mixture components [2, 3]. Equations (1) and (2) are supplemented with initial conditions

\[
\rho_i \big|_{t=0} = \rho_i^0, \quad \rho_i \vec{u}^{(i)} \big|_{t=0} = \vec{q}^{(i)} \big|_{t=0} = \vec{q}^{(i)}_0 \quad \text{in} \; \Omega, \quad i = 1, 2. \tag{5}
\]
and with boundary conditions
\[ \tilde{u}^{(i)} = 0 \text{ on } (0, T) \times \partial \Omega, \; i = 1, 2. \] (6)

It means, that boundary of flow region is motionless and impermeable.

Global theorems of existence and results of stabilisation of solutions for the unsteady equations for multivelocity continuum similar to (1)-(4) are received now only in case of one-dimensional movement with flat waves when the solution depends only on one spatial variable [3-5]. The first results for Stoks-like mixture model in case of more than one spatial variable are received in [6-7].

In [8] research of so-called quasistationary mixture model of compressible fluids was conducted in the bounded region with special boundary conditions. In [9] the stationary solutions of the Dirichlet problem for the full equations (1)-(4) in case of three spatial variables was constructed, and in [10-13] existence of weak solutions for equations describing the three-dimensional steady motions of binary mixtures of heat-conductive compressible viscous fluids was proved.

The reader is referred to [10, 15] for standard notation of functional spaces and terminology.

2. Statement of the problem
Motion of continuum is characterised by potential energy
\[ E = E[\rho](t) = \sum_{i=1}^{2} \int_{\Omega} P_i(\rho_i(t, x)) dx, \; \rho = (\rho_1, \rho_2), \]
where nonnegative functions \( P_i \) are defined from the equations \( s P_i'(s) - P_i(s) = p_i(s) \) up to a nonessential linear function (in particular, if \( p_i = \rho_i^{\gamma_i}, \gamma_i > 1 \), then \( P_i(\rho_i) = (\gamma_i - 1)^{-1} \rho_i^{\gamma_i} \)) and by energy dissipation rate
\[ D = D[\tilde{u}](t) = \sum_{i=1}^{2} \int_{\Omega} |\nabla \tilde{u}^{(i)}|^2 dx, \; \tilde{u} = (\tilde{u}^{(1)}, \tilde{u}^{(2)}). \]

Multiplying the momentum balance equations (1) by \( \tilde{u}^{(i)} \) and integrating by parts, we arrive (using (6)) at the identity
\[ \frac{d}{dt} \left\{ \sum_{i=1}^{2} \left( \frac{1}{2} \int_{\Omega} \rho_i |\tilde{u}^{(i)}|^2 dx + \int_{\Omega} \frac{\rho_i^{\gamma_i}}{\gamma_i - 1} dx \right) \right\} + \sum_{i=1}^{2} \int_{\Omega} \sigma^{(i)} : \nabla \tilde{u}^{(i)} dx + a \int_{\Omega} |\tilde{u}^{(2)} - \tilde{u}^{(1)}|^2 dx = 0, \] (7)
which gives the mathematical relationship for the total energy balance. From identity (7) we obtain estimates of the energy dissipation rate \( D(t) \) and total energy \( E(t) \)
\[ E(t) = E[\rho, \tilde{u}](t) = K[\rho, \tilde{u}](t) + E[\rho](t) = \sum_{i=1}^{2} \int_{\Omega} \left[ \frac{1}{2} \rho_i |\tilde{u}^{(i)}|^2 + \frac{\rho_i^{\gamma_i}}{\gamma_i - 1} \right] dx. \] (8)

Note that for any continuously differentiable function \( a_i : R \rightarrow R \), every smooth solution of the continuity equations (2) satisfies the equalities
\[ \partial_t a_i(\rho_i) + div (a_i(\rho_i) \tilde{u}^{(i)}) + (\rho_i a_i'(\rho_i) - a_i(\rho_i)) div \tilde{u}^{(i)} = 0, \; i = 1, 2, \] (9)
which are called renormalized form of the equation (2). The functions \( \rho_i \), satisfying this system are called renormalized solutions of equations (2). Relations (7)-(9) determine the specific definition of a generalized solution of the initial-boundary value problem (1)-(6).

Definition A. A generalized solution to the initial-boundary value problem (1)-(6) are non-negative functions \( \rho_i \in L^\infty (0, T; L^{\gamma_i}(\Omega)), \; i = 1, 2 \), and vector fields \( \tilde{u}^{(i)} \in L^2 \left( 0, T; W^{1,2}_0(\Omega) \right), \; i = 1, 2 \), such that:
(A1) The total energy $\mathcal{E}^i(\rho, \vec{u}^i)(t)$ is locally integrable on an interval $(0, T)$, and the total energy equation (7) is fulfilled in the space of distributions $D'(0, T)$;

(A2) The momentum balance equations (1) are fulfilled in the space $D'((0, T) \times \Omega)$;

(A3) The continuity equations (2) are satisfied in the sense of renormalized solutions, i.e., for any differentiable functions $a_i$ such that, $a_i(z) \equiv 0$ for all sufficiently large $z \in R$, for example $z \geq M$ (constants $M$ are different for each of the functions $a_i$), equations (9) are fulfilled in $D'((0, T) \times \Omega)$. In addition, equations (2) hold true in the space of distributions $D'((0, T) \times R^3)$ if the functions $\rho_i, \vec{u}^i$ extended by zero over the whole space $R^3 \setminus \Omega$.

Remark 1. It follows from equations (1)-(2) that the generalized solution of problem (1)-(6) belongs to the class

$$\rho_i \in C([0, T]; L^\infty_{weak}(\Omega)), \quad \rho_i \vec{u}^i \in C([0, T]; L^\infty_{weak}(\Omega)), \quad i = 1, 2,$$

and, therefore, the initial conditions (5) make sense. This means that it is required that the initial data $\rho^0_i, \vec{q}^0_i, i = 1, 2$, satisfy the conditions:

$$\rho^0_i \in L^\infty(\Omega), \quad \rho^0_i \geq 0 \text{ a.e. in } \Omega, \quad \frac{\delta w_i^0}{\rho^0_i} \cdot 1_{\{\rho^0_i \leq 0\}} = 0 \text{ a.e. in } \Omega, \quad \frac{1}{\rho^0_i} |(\vec{q}^0_i)^2| \cdot 1_{\{\rho^0_i > 0\}} \in L^1(\Omega). \quad (10)$$

The main result of the article is given by the following theorem.

Theorem 2. Let $\Omega \subset R^3$ be a bounded domain with $C^{2+\nu}, \nu > 0$, boundary. Let the initial data $\rho^0_i, \vec{q}^0_i, i = 1, 2$, satisfy conditions (10), $\gamma_i > 3/2$ and total viscosity matrix $H = \{\nu_{ij}\}_{i,j=1}^2$ is triangular. Then for any arbitrary $T > 0$, there exists at least one generalized solution of the problem (1)-(6).

Let us characterize briefly the main steps of the proof of this theorem.

Along with problem (1)-(6) we consider its regularization

$$\partial_t (\rho_i \vec{u}^i) + div(\rho_i \vec{u}^i \otimes \vec{u}^i) + \nabla(\rho_i \vec{u}^i) + \delta \nabla(\rho_i \vec{u}^i) +$$

$$+ \varepsilon \nabla \vec{a}^i \cdot \nabla \rho_i \text{ in } Q_T = (0, T) \times \Omega, \quad i = 1, 2, \quad (11)$$

$$\partial_t (\rho_i) + div(\rho_i \vec{u}^i) = \varepsilon \Delta \rho_i \text{ in } Q_T, \quad i = 1, 2, \quad (12)$$

$$\rho_i|_{t=0} = \rho^0_i, \quad \rho_i \vec{u}^i|_{t=0} = \vec{q}^0_i \text{ in } \Omega, \quad i = 1, 2, \quad (13)$$

$$\vec{u}^i = 0 \text{ on } (0, T) \times \partial \Omega, \quad i = 1, 2, \quad (14)$$

$$\nabla \rho_i \cdot \vec{n} = 0 \text{ on } (0, T) \times \partial \Omega, \quad i = 1, 2, \quad (15)$$

where $\varepsilon > 0, \delta > 0$ are small parameters and values $\beta_i, i = 1, 2$ should be chosen sufficiently large. Equations (12) supplemented with Neumann boundary conditions (15).

A generalized solution of problem (1)-(6) will be obtained as the limit of a sequence of solutions to the regularized problem (11)-(15). Therefore, the first step in the proof of theorem 2 will be to construct solutions of the regularized problem by the method of finite-dimensional approximation. The next stage of the proof will be to justify the limit passages when the dissipation parameter $\varepsilon$ go to zero. The difficulty of this procedure is due to insufficiency of estimates for the densities $\rho_i, i = 1, 2$. Here we use the technique developed in [13-16] for the classical model of Navier-Stokes, based on the weak regularity properties of the effective viscous fluxes. In this work also introduced analogues of the effective viscous fluxes

$$\tau_i (\rho_i, \vec{u}^1, \vec{u}^2) = \rho_i \gamma_1^i + \delta \rho_i \beta_i - 2 \sum_{j=1}^3 (\lambda_{ij} + 2 \mu_{ij}) div \vec{u}^j$$

and the following properties are proved

$$\tau_i (\rho_i, \vec{u}^1, \vec{u}^2) \cdot \rho_k = \tau_i (\rho_i, \vec{u}^1, \vec{u}^2) \cdot \rho_{ik}, \quad i, k = 1, 2. \quad (16)$$
Here $\bar{b}(\rho)$ denotes the weak limit of a sequence $b(\rho_i)$. Properties (16) allow us to establish strong convergence of the densities and complete justification of the limit passages when $\varepsilon \to 0$. The final stage in the proof of theorem 2 consist in exclusion of the artificial pressures $\delta\rho^{\beta_i, i = 1, 2}$. To this end we apply the so-called truncation operators introduced in [17,18].

3. The existence of a strong solution of the auxiliary problem (11)-(15)

The following theorem gives the main result on the well-posedness of the regularized problem.

**Theorem 3.** Let the coefficients $\mu_{ij}$ and $\lambda_{ij}$ satisfy the conditions (4) and adiabatic exponents $\gamma_i > \frac{3}{2}, i = 1, 2$. Let the parameters $\varepsilon$, $\delta$, $\beta_i$ be chosen so that $\varepsilon > 0$, $\delta > 0$, $\beta_i \geq 15, i = 1, 2$. Suppose that $\Omega$ be a bounded domain in $\mathbb{R}^3$ with $C^{2,\theta}$ boundary, $\theta \in (0,1]$, and

$$0 < \rho_1 \leq \rho_0 \leq \bar{\rho} < \infty, \quad \rho_0^\beta \in W^{1,\infty}(\Omega), \quad \bar{q}_0(i) \in L^2(\Omega).$$

Then there exist a couple $(\rho_{i, \varepsilon}, \bar{u}_{i, \varepsilon}^{(i)}), (\rho_{i, \varepsilon}, \bar{u}_{i, \varepsilon}^{(i)})$, $\bar{u}_{i, \varepsilon}^{(i)} = \bar{u}_{i, \varepsilon}^{(i)}, i = 1, 2$, with the following properties:

(i) $\rho_{i, \varepsilon} \in C^0(\bar{\Omega}, L^\beta_{\text{weak}}(\Omega)), \quad \rho_{i, \varepsilon} \in C^0(\bar{\Omega}, L^\beta(\Omega)) \cap L^\beta_{i+1} (\Omega_T), \quad 1 \leq p < \beta_i, \quad I = (0, T)$.

(ii) Equations (11) are fulfilled in the space of distributions $D'(\Omega_T)$.

(iii) Equations

$$\frac{d}{dt} \int_\Omega \rho_{i, \varepsilon} \eta dx - \int_\Omega \rho_{i, \varepsilon} \bar{u}_{i, \varepsilon}^{(i)} \nabla \eta dx + \varepsilon \int_\Omega \rho_{i, \varepsilon} \cdot \nabla \eta dx dt = 0, \quad \eta \in C^\infty(R^3), \quad i = 1, 2,$$

are fulfilled in $D'(I)$.

(iv) The functions $\rho_{i, \varepsilon}, \bar{u}_{i, \varepsilon}^{(i)}$ satisfy the energy inequalities

$$\frac{d}{dt} \mathcal{E}_\delta(\rho_{i, \varepsilon}, \bar{u}_{i, \varepsilon}^{(i)}) + c_0 \int_\Omega (|\nabla \bar{u}_{i, \varepsilon}^{(i)}|^2 + |\nabla \bar{u}_{i, \varepsilon}^{(i)}|^2) dx +$$

$$+ \varepsilon \int_\Omega \sum_{i=1}^2 (\gamma_i \rho_{i, \varepsilon}^\beta - \rho_{i, \varepsilon}^\beta) |\nabla \rho_{i, \varepsilon}|^2 dx + a \int_\Omega |\bar{u}_{i, \varepsilon}^{(i)}|^2 dx - |\bar{u}_{i, \varepsilon}^{(i)}|^2 dx \leq 0, \quad \text{in} \quad D'(I),$$

where $\mathcal{E}_\delta(\rho_{i, \varepsilon}, \bar{u}_{i, \varepsilon}^{(i)}) = \sum_{i=1}^2 \int_\Omega \left\{ \frac{1}{2} \rho_{i, \varepsilon} |\bar{u}_{i, \varepsilon}^{(i)}|^2 + \frac{\rho_{i, \varepsilon}^\gamma_i}{\gamma_i - 1} + \delta \rho_{i, \varepsilon}^\beta_i - 1 \right\} dx.$
(v) For any $\eta_i \in C_0^\infty(\Omega)$ and $\varphi^{(i)} \in C_0^\infty(\Omega)$ the following relations hold true

$$
\lim_{t \to +0} \int_\Omega \rho_{i,t} \eta_idx = \int_\Omega \rho_{i,0}^0 \eta_idx,
\lim_{t \to +0} \int_\Omega \rho_{i,\varepsilon} \varphi^{(i)}(t) dx = \int_\Omega \tilde{q}_0^{(i)} \cdot \varphi^{(i)} dx.
$$

(vi) If $\delta \in (0, 1)$, then we have the following estimates, uniform with respect to the parameter $\varepsilon$:

$$
\sum_{i=1}^2 \left( \|\tilde{u}_i^{(i)}\|_{L^2(I, W^{1,2}(\Omega))} + \|\rho_{i,t}\|_{L^\infty(I, L^{\gamma_i}(\Omega))} + \delta^{1/\beta_i} \|\rho_{i,\varepsilon}\|_{L^\infty(I, L^{l_i}(\Omega))} \right) \leq L
$$

$$
\sum_{i=1}^2 \left( \sqrt{\varepsilon} \|\nabla \rho_{i,\varepsilon}\|_{L^2(Q_T)} + \varepsilon \|\nabla \rho_{i,\varepsilon}\|_{L^{10\delta_i - 6}(Q_T)} \right) \leq L
$$

$$
\sum_{i=1}^2 \left( \|\rho_{i,\varepsilon} \tilde{u}_i^{(i)}\|_{L^\infty(I, L^{2\beta_i + \gamma_i}(\Omega))} + \|\rho_{i,\varepsilon} \tilde{u}_i^{(i)}\|_{L^2(I, L^{6\beta_i + \gamma_i}(\Omega))} \right) \leq L
$$

$$
\sum_{i=1}^2 \left( \|\rho_{i,\varepsilon} \tilde{u}_i^{(i)}\|_{L^{10\delta_i - 6}(Q_T)} + \varepsilon \|\nabla \rho_{i,\varepsilon} \cdot \nabla \tilde{u}_i^{(i)}\|_{L^{5\beta_i + 3}(Q_T)} \right) \leq L
$$

Here $L$ — positive constant independent of $\varepsilon$ and $\delta$.

4. The study of equations with artificial pressure

In this section we present the results of limit passages when the dissipation parameter $\varepsilon$ go to zero.

**Theorem 4** Let the coefficients $\mu_{ij}$, $\lambda_{ij}$ and the parameters $\gamma_i$, $\delta$, $\beta_i$, $i = 1, 2$ meet all requirements of theorem 2 and theorem 3. Suppose that $\Omega$ be a bounded domain in $R^3$ with boundary of class $C^{2,\theta}$, $\theta \in (0, 1)$, and

$$
\rho_i^0 \in L^{\beta_i}(\Omega), \quad \rho_i^0 \geq 0 \ a.e. \ in \ \Omega,
$$

$$
\tilde{q}_0^{(i)} \in L^{2\beta_i}(\Omega), \quad \tilde{q}_0^{(i)} \cdot 1_{\{\rho_i^0 = 0\}} = 0 \ a.e. \ in \ \Omega, \quad \frac{\tilde{q}_0^{(i)}}{\rho_i^0} \cdot 1_{\{\rho_i^0 > 0\}} \in L^1(\Omega).
$$

Then there exists generalized solution $(\rho_{i,\delta}, \tilde{u}_{\delta}^{(i)})$, $i = 1, 2$, of problem

$$
\partial_t (\rho_i \tilde{u}_i^{(i)}) + \text{div}(\rho_i \tilde{u}_i^{(i)} \otimes \tilde{u}_i^{(i)}) + \nabla \rho_i^{\gamma_i} + \delta \nabla \rho_i^{\beta_i} = \text{div} \sigma^{(i)} + \tilde{f}^{(i)} \quad \text{in} \ Q_T,
$$

$$
\partial_t \rho_i + \text{div}(\rho_i \tilde{u}_i^{(i)}) = 0 \quad \text{in} \ Q_T,
$$

$$
\tilde{u}_i^{(i)} = 0 \ on \ \partial \Omega \times (0, T), \quad \rho_i|_{t=0} = \rho_i^0, \quad \tilde{q}_i^{(i)}|_{t=0} = \rho_i \tilde{u}_i^{(i)}|_{t=0} = \tilde{q}_0^{(i)}, \quad i = 1, 2,
$$

with the properties:

$$
\rho_{i,\delta} \in L^{\beta_i + 1}(Q_T), \quad \rho_{i,\delta} \in C^0(\bar{I}, L^{\beta_i}(\Omega)) \cap C^0(\bar{I}, L^p(\Omega)), \quad 1 \leq p < \beta_i,
$$

$$
\rho_{i,\delta} \geq 0 \ in \ Q_T, \quad \rho_{i,\delta} = 0 \ in \ (R^3 \setminus \Omega) \times I,
$$

$$
\tilde{u}_{\delta}^{(i)} \in L^2(I, W^{1,2}_0(\Omega)), \quad \tilde{q}_{\delta}^{(i)} = 0 \ in \ (R^3 \setminus \Omega) \times I, \quad \rho_{i,\delta} \tilde{u}_{\delta}^{(i)} \in L^2(I, L^{6\beta_i \gamma_i}(\Omega)) \cap C^0(\bar{I}, L^{2\beta_i}(\Omega)),
$$

$$
\rho_{i,\delta} \tilde{q}_{\delta}^{(i)} \in L^\infty(I, L^1(R^3)) \cap L^2(I, L^{6\beta_i}(\Omega)) \cap L^1(I, L^{6\beta_i}(\Omega)).
$$

(ii) Equations (17) are fulfilled in $D'(Q_T)$.
(iii) Equations (18) are fulfilled in the sense \( \partial_t \rho_{i,\delta} + \text{div}(\rho_{i,\delta} \vec{u}_{\delta}^{(i)}) = 0 \) in \( D'(R^3 \times \Omega) \).

(iv) The initial conditions in (19) understood in the sense of the parameter \( \delta \).

\[
\lim_{t \to 0^+} \int_\Omega \rho_{i,\delta}(t) \eta \, dx = \int_\Omega \rho_{i,0}^0 \eta \, dx, \quad \eta \in C_0^\infty(\Omega), \quad \lim_{t \to 0^+} \int_\Omega \rho_{i,\delta} \vec{u}_{\delta}^{(i)}(t) \vec{v} \, dx = \int_\Omega \vec{q}_{0}^{(i)} \vec{v} \, dx, \quad \vec{v} \in C_0^\infty(\Omega).
\]

(v) For \( \delta \in (0, 1) \) we have the following uniform estimates

\[
\left\{ \| \vec{u}_{\delta}^{(i)} \|_{L^2(I,W_0^{1,2}(\Omega))}, \| \rho_{i,\delta} \|_{L^\infty(I,L^{\gamma_i}(\Omega))}, \delta \frac{\beta_i}{\gamma_i} \| \rho_{i,\delta} \|_{L^\infty(I,L^{\gamma_i}(\Omega))}, \| \rho_{i,\delta} \vec{u}_{\delta}^{(i)} \|_{L^\infty(I,L^{1}(\Omega))} \right\} \leq L, \quad (20)
\]

\[
\| \rho_{i,\delta} \|_{L^{s_i}(Q_T)} \leq L, \quad s_i = \gamma_i + \theta_i, \quad \theta_i = \frac{2}{3} \gamma_i - 1,
\]

\[
\delta \frac{\beta_i}{\gamma_i} \| \rho_{i,\delta} \|_{L^{s_i}+\theta_i(Q_T)} + \| \rho_{i,\delta} \vec{u}_{\delta}^{(i)} \|_{L^\infty(I,L^{\gamma_i}(\Omega))} + \| \rho_{i,\delta} \vec{u}_{\delta}^{(i)} \|_{L^2(I,L^{\frac{2\gamma_i}{\gamma_i-1}}(\Omega))} \leq L, \quad (21)
\]

\[
\| \rho_{i,\delta} \vec{u}_{\delta}^{(i)} \|_{L^1(I,L^{\gamma_i}(\Omega))} + \| \rho_{i,\delta} \|_{L^2(I,L^{\frac{2\gamma_i}{\gamma_i-1}}(\Omega))} \leq L.
\]

Where the constant \( L \) depends on the problem date, namely on

\[
\mathcal{E}_{1,0} = \sum_{i=1}^{2} \int \left\{ \frac{1}{2} \rho_{i,0}^0 \| \vec{u}_{\delta}^{(i)} \|^2 + \frac{(\rho_{i,0}^0)^{\gamma_i}}{\gamma_i - 1} + \frac{(\rho_{i,0}^0)^{\beta_i}}{\beta_i - 1} \right\} \, dx.
\]

5. The limit passage when \( \delta \to 0 \).

In this section as a result of limit passage when \( \delta \to 0 \) we exclude artificial pressures and completes the proof of theorem 2.

Together with the inequalities (20)-(21) there are other estimates of the densities independent of the parameter \( \delta \).

**Lemma 5.** Let \( \rho_{i,\delta}, \, \vec{u}_{\delta}^{(i)} \) is solution of problem (17)-(19) constructed in theorem 4. Then we have the following uniform with respect to \( \delta \in (0, 1) \) estimates

\[
\| \rho_{i,\delta} \|_{L^{s_i}(Q_T)} \leq L(\mathcal{E}_{1,0}), \quad s_i = \gamma_i + \theta_i, \quad \theta_i = \frac{2}{3} \gamma_i - 1, \quad (\frac{3}{2} < \gamma_i < 6), \quad \theta_i = \frac{1}{2} \gamma_i, \quad (\gamma_i \geq 6), \quad (22)
\]

with constant \( L \) independent of the \( \delta \).

It follows from estimates (20), (22) that there are subsequences, such that

\[
\vec{u}_{\delta}^{(i)} \to \vec{u}^{(i)} \text{ weakly in } L^2(I,W_0^{1,2}(\Omega)) \text{ and } L^2(I,W^{1,2}(R^3)), \quad \vec{u}^{(i)} = 0 \text{ in } (R^3\setminus \Omega) \times I.
\]

\[
\rho_{i,\delta} \to \rho_i \text{ weakly in } L^{\gamma_i+\theta_i}(R^3 \times I),
\]

\[
\theta_i = \frac{2}{3} \gamma_i - 1 \text{ (if } \frac{3}{2} < \gamma_i < 6 \text{) and } 0 < \theta_i < \frac{1}{2} \gamma_i \text{ (if } \gamma_i \geq 6 \text{),}
\]

\[
\rho_{i,\delta} \to \rho_i \text{ -weakly in } L^{\infty}(I,L^{\gamma_i}(R^3)), \quad \rho_i \geq 0 \text{ a.e. in } Q_T, \quad \rho_i = 0 \text{ in } (R^3\setminus \Omega) \times I.
\]

\[
\rho_{i,\delta}^{\gamma_i} \to \rho_i^{\gamma_i} \text{ weakly in } L^{\frac{\gamma_i+\theta_i}{\gamma_i}}(Q_T), \quad \rho_i^{\gamma_i} \geq 0 \text{ a.e. in } R^3 \times I, \quad \rho_i^{\gamma_i} = 0 \text{ in } (R^3\setminus \Omega) \times I.
\]

\[
\delta \rho_{i,\delta}^{\beta_i} \to 0 \text{ weakly in } L^{\frac{\beta_i+\delta_i}{\beta_i}}(Q_T).
\]

In addition, we note the following properties

1. If \( \rho_{i,\delta}, \, \vec{u}_{\delta}^{(i)}, \, i = 1, 2, \) is solution of problem (17)-(19) then \( \rho_{i,\delta} \) are renormalized solutions of continuity equations (18).

2. The functions \( \rho_{1}, \, \vec{u}_{1}^{(i)}, \, \rho_{1}^{\gamma_i} \) defined by the formulas (23) satisfy the continuity equations (18) in \( D'(I \times R^3) \), the balance momentum equations (17) in \( D'(Q_T) \) up to terms \( \nabla \rho_{1}^{\gamma_i} \) (these terms

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will be replaced by the term $\nabla \rho_i^\gamma$. The limit function $\rho_i, i = 1, 2$, are renormalized solutions of continuity equations, i.e. for any functions $b_i \in C^0[0, \infty) \cap C^1(0, \infty)$ satisfying the conditions $|b_i'(t)| \leq c \cdot t^{-\lambda_0}, t \in (0, 1], \lambda_0 < 1$, $|b_i'(t)| \leq c \cdot t^{\lambda_1}, t \geq 1$, $1 < \lambda_0 < \frac{2}{p} - 1, c > 0$, the equations $\partial_t b_i(\rho_i) + \nabla (b_i(\rho_i) \cdot \vec{u}(t)) + [\rho_i b_i'(\rho_i) - b_i(\rho_i)]\nabla \vec{u}(t) = 0$ are fulfilled. We have the estimates

$$\|T_k(\rho_{i,\delta}) - T_k(\rho_i)\|_{L^{\gamma_i+1}(Q_T)} \leq c, \ k \geq 1, \ T_k(s) = \begin{cases} s, \text{ if } s \in [0, k), \\ k \text{ if } s \in [k, \infty). \end{cases}$$

The above facts allow us to prove that under the requirements of theorems 2 and 3, there exist a subsequence $\rho_{i,\delta}$, such that $\rho_{i,\delta} \to \rho_i$ strongly in $L^p(Q_T), i = 1, 2, 1 \leq p < \gamma_i + \theta_i(\gamma_i)$

From this result we finally obtain the equalities $\overline{\rho_i^\gamma} = \overline{\rho_i^\gamma}, i = 1, 2$. This completes the proof of theorem 2.

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