Linking Solutions for \( p \)-Laplace Equations with Nonlinear Boundary Conditions and Indefinite Weight

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Abstract We apply the linking method for cones in normed spaces to \( p \)-Laplace equations with various nonlinear boundary conditions. Some existence results are obtained.

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Key words: \( p \)-Laplace equation, nonlinear boundary condition, cohomological index, linking structure over cones.

1 Introduction and main results

In this paper, we consider the following problems:

Steklov boundary problem

\[
\begin{aligned}
\Delta_p u &= \varepsilon |u|^{p-2} u, & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} &= \lambda V(x)|u|^{p-2} u + h(x, u), & \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

No-flux boundary problem

\[
\begin{aligned}
-\Delta_p u + \varepsilon |u|^{p-2} u &= \lambda V(x)|u|^{p-2} u + h(x, u), & \text{in } \Omega, \\
u &= \text{constant}, & \text{on } \partial \Omega, \\
\int_{\partial \Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial n} dS_x &= 0,
\end{aligned}
\]

(1.2)

Neumann boundary problem

\[
\begin{aligned}
-\Delta_p u + \varepsilon |u|^{p-2} u &= \lambda V(x)|u|^{p-2} u + h(x, u), & \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0, & \text{on } \partial \Omega,
\end{aligned}
\]

(1.3)

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Robin boundary problem

\[
\begin{cases}
-\Delta_p u + \varepsilon |u|^{p-2} u = \lambda V(x)|u|^{p-2} u + h(x, u), & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} + \gamma(x)|u|^{p-2} u = 0, & \text{on } \partial \Omega.
\end{cases}
\] (1.4)

Here \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial \Omega\), \(dS_x\) is the surface element on \(\partial \Omega\), \(\frac{\partial u}{\partial n}\) is the outer normal derivative of \(u\) with respect to \(\partial \Omega\), \(\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)\) is the \(p\)-Laplacian operator with \(p > 1\), \(\varepsilon > 0\) is a constant and \(V(x) \in L^r(\partial \Omega)\) (in the case of (1.1) or \(L^r(\Omega)\) (in the case of (1.2)-(1.4)), where \(r = r(N, p)\) is defined by

\[
\begin{aligned}
r > (N - 1)/(p - 1), & \quad \text{if } 1 < p < N, \\
r > 1, & \quad \text{if } p = N, \\
r = 1, & \quad \text{if } p > N.
\end{aligned}
\] (1.5)

In problem (1.4), the function \(\gamma(x)\) satisfies \(\gamma(x) \in L^\infty(\partial \Omega)\) and \(\gamma(x) \geq 0\) for a.e. \(x \in \partial \Omega\).

In [14], the authors established and applied the linking method for cones in normed spaces to consider the following the problem

\[
\begin{cases}
-\Delta_p u + \varepsilon |u|^{p-2} u = \lambda V(x)|u|^{p-2} u + h(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega
\end{cases}
\]

for \(\lambda \in \mathbb{R}, V \in L^\infty(\Omega)\) and \(h\) satisfying \((h1')-(h4')\) below, they obtained the existence of a nontrivial solution. The main goal of this paper is to apply this method to study the problems (1.1)-(1.4).

For problem (1.1), we assume that \(h : \partial \Omega \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function (i.e., \(h(x, s)\) is continuous in \(s\) for a.e. \(x \in \partial \Omega\) and measurable in \(x\) for all \(s \in \mathbb{R}\)) satisfying the following conditions:

\((h1)\) if \(p < N, \forall \varepsilon > 0\), \(\exists a_\varepsilon \in L^r(\partial \Omega)\) such that \(|h(x, s)| \leq a_\varepsilon(x)|s|^{p-1} + \varepsilon|s|^{q-1}\), \(p^* = \frac{Np}{N-p}\),

\[
\frac{H(x, s)}{|s|^p} \to 0 \quad \text{and} \quad \lim_{|s| \to \infty} \frac{H(x, s)}{|s|^p} = +\infty,
\]

\((h2)\) for a.e. \(x \in \partial \Omega\), there hold \(\lim_{s \to 0} \frac{H(x, s)}{|s|^p} = 0\) and \(\lim_{|s| \to \infty} \frac{H(x, s)}{|s|^p} = +\infty\),

\((h3)\) there exist \(\mu > p, \gamma_0 \in L^1(\partial \Omega)\) and \(\gamma_1 \in L^r(\partial \Omega)\) such that

\[
\mu H(x, s) \leq \varepsilon h(x, s) + \gamma_0(x) + \gamma_1(x)|s|^p \quad \text{for a.e. } x \in \partial \Omega, \text{ and every } s \in \mathbb{R},
\]

\((h4)\) \(H(x, s) \geq 0\) for a.e. \(x \in \partial \Omega\) and every \(s \in \mathbb{R}\), where \(H(x, s) = \int_0^s h(x, t)dt\).

For problems (1.2)-(1.4), we assume \(h\) satisfies the same conditions \((h1)-(h4)\) with \(\partial \Omega\) replaced by \(\Omega\) and \(p^* = \frac{Np}{N-p}\).

The main results read as follow.
Theorem 1.1 Suppose the function \( h \) satisfies the conditions (h1)–(h4) and \( V \in L^r(\partial\Omega) \). Then for every \( \varepsilon > 0 \) and \( \lambda \in \mathbb{R} \), (1.1) has a nontrivial solution \( u \in W^{1,p}(\Omega) \).

Theorem 1.2 Suppose \( h : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function satisfying the conditions (h1)–(h4) with \( \partial\Omega \) replaced by \( \Omega \), \( p^* = \frac{Np}{N-p} \) and \( V \in L^r(\Omega) \). Then for every \( \varepsilon > 0 \) and \( \lambda \in \mathbb{R} \), the problems (1.2), (1.3) and (1.4) possess respectively a nontrivial solution \( u \in W^{1,p}(\Omega) \).

We note that when \( \lambda \neq 0 \), problems (1.2), (1.3) and (1.4) are respectively equivalent to the following problems:

No-flux boundary problem
\[
\begin{align*}
-\Delta_p u &= \lambda(V(x) - \frac{\varepsilon}{\lambda})|u|^{p-2}u + h(x, u), & \text{in } \Omega, \\
u &= \text{constant}, & \text{on } \partial\Omega, \\
\int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial n} \, dS_x &= 0,
\end{align*}
\]

Neumann boundary problem
\[
\begin{align*}
-\Delta_p u &= \lambda(V(x) - \frac{\varepsilon}{\lambda})|u|^{p-2}u + h(x, u), & \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0, & \text{on } \partial\Omega,
\end{align*}
\]

Robin boundary problem
\[
\begin{align*}
-\Delta_p u &= \lambda(V(x) - \frac{\varepsilon}{\lambda})|u|^{p-2}u + h(x, u), & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} + \gamma(x)|u|^{p-2}u &= 0, & \text{on } \partial\Omega.
\end{align*}
\]

Because \( V(x) - \frac{\varepsilon}{\lambda} \) is still in \( L^r(\Omega) \), the above three problems are exactly the following problems respectively:

No-flux boundary problem
\[
\begin{align*}
-\Delta_p u &= \lambda V(x)|u|^{p-2}u + h(x, u), & \text{in } \Omega, \\
u &= \text{constant}, & \text{on } \partial\Omega, \\
\int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial n} \, dS_x &= 0,
\end{align*}
\]

Neumann boundary problem
\[
\begin{align*}
-\Delta_p u &= \lambda V(x)|u|^{p-2}u + h(x, u), & \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0, & \text{on } \partial\Omega,
\end{align*}
\]

Robin boundary problem
\[
\begin{align*}
-\Delta_p u &= \lambda V(x)|u|^{p-2}u + h(x, u), & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} + \gamma(x)|u|^{p-2}u &= 0, & \text{on } \partial\Omega.
\end{align*}
\]
So Theorem 1.2 is equivalent to the following theorem (Note that the case \( \lambda = 0 \) is covered by the case \( \lambda \neq 0 \) with \( V \equiv 0 \)).

**Theorem 1.3** Suppose that the functions \( h \) and \( V \) satisfy the conditions as in Theorem 1.2. Then, for every \( \lambda \in \mathbb{R} \), the problems (1.6), (1.7) and (1.8) possess a nontrivial solution \( u \in W^{1,p}(\Omega) \), respectively.

In Theorems 1.1-1.3, if we replace (h3) by the following condition (h5) which was introduced in [22] for \( p = 2 \) and in [26] for general \( p \), the results are still true.

(h5) There exists a real number \( \theta \geq 1 \) such that

\[
\theta H(x, s) \geq H(x, ts), \quad \text{for a.e. } x \in \partial \Omega, \text{and every } s \in \mathbb{R}, t \in [0,1],
\]

where \( H(x, s) := h(x, s)s - pH(x, s) \).

That is to say we have the following three results.

**Theorem 1.1’** Suppose the function \( h \) satisfies the conditions (h1),(h2),(h4) and (h5), then for every \( \epsilon > 0 \) and \( \lambda \in \mathbb{R} \), (1.1) has a nontrivial solution \( u \in W^{1,p}(\Omega) \).

**Theorem 1.2’** Suppose \( h : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function satisfying the conditions (h1),(h2),(h4),(h5) with \( \partial \Omega \) replaced by \( \Omega \), \( p^* = \frac{Np}{N-p} \) and \( V \in L^r(\Omega) \). Then for every \( \epsilon > 0 \) and \( \lambda \in \mathbb{R} \), the problems (1.2), (1.3) and (1.4) possess respectively a nontrivial solution \( u \in W^{1,p}(\Omega) \).

**Theorem 1.3’** Suppose that the functions \( h \) and \( V \) satisfy the conditions as in Theorem 1.2’. Then, for every \( \lambda \in \mathbb{R} \), the problems (1.6), (1.7) and (1.8) possess a nontrivial solution \( u \in W^{1,p}(\Omega) \), respectively.

Let \( h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) be a continuous function satisfying the following conditions:

(h1’) if \( p < N \), \( \exists C > 0 \) and \( q \) satisfying \( p < q < p^* \), such that \( |h(x, s)| \leq C(1 + |s|^{q-1}) \), \( p^* = \frac{Np}{N-p} \); if \( p = N \), \( \exists C > 0 \) and \( q \) satisfying \( q > p \), such that \( |h(x, s)| \leq C(1 + |s|^{q-1}) \);

if \( p > N \), there is no restriction,

(h2’) \( \lim_{s \to 0} \frac{h(x,s)}{|s|^{p-1}} = 0 \) uniformly for \( x \in \overline{\Omega} \),

(h3’) there exist \( \mu > p \), \( R > 0 \) such that

\[
0 < \mu H(x, s) \leq sh(x, s), \text{ for } |s| \geq R,
\]

(h4’) \( sh(x, s) \geq 0 \), where \( H(x, s) = \int_0^s h(x, t)dt \).

It was proved in [14] that (h1’)-(h4’) imply conditions (h1)-(h4)(with \( \partial \Omega \) replaced by \( \Omega \), \( p^* = \frac{Np}{N-p} \)). So we have the following direct consequence.
Corollary 1.4 Suppose $h : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying the conditions \((h^1')-(h^4')\) and $V \in L^\infty(\Omega)$. Then for every $\varepsilon > 0$ and $\lambda \in \mathbb{R}$, the problems (1.2), (1.3) and (1.4) possess respectively a nontrivial solution $u \in W^{1,p}(\Omega)$. Equivalently, for every $\lambda \in \mathbb{R}$, the problems (1.6), (1.7) and (1.8) possess respectively a nontrivial solution $u \in W^{1,p}(\Omega)$.

Similarly, we have

Corollary 1.5 Suppose $h : \partial \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying the conditions \((h^1')-(h^4')\) with $\Omega$ replaced by $\partial \Omega$, $p^* = \frac{Np - p}{N - p}$ and $V \in L^\infty(\partial \Omega)$. Then for every $\varepsilon > 0$ and $\lambda \in \mathbb{R}$, (1.1) has a nontrivial solution $u \in W^{1,p}(\Omega)$.

The problems (1.1)–(1.4), (1.6)–(1.8) arise in different areas, for example, the study of optimal constants for the Sobolev embedding theorems (c.f. [7, 10, 6, 36]), Non Newtonian fluids (c.f. [2, 1, 3, 15]) and differential geometry (c.f. [16]). Similar nonlinear boundary value problems has been extensively studied, one can refer to [4, 5, 6, 11, 12, 21, 27, 29, 30, 35, 38, 39, 42] for details. In [11], the authors considered the following problem

$$
\begin{cases}
\Delta_p u = |u|^{p-2}u, & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} = f(u), & \text{on } \partial \Omega.
\end{cases}
$$

They proved among other cases that when $f$ has the form $\lambda |u|^{q-2}u$ with subcritical growth, the above problem has infinitely many solutions. In [29], the authors considered the following problem

$$
\begin{cases}
\Delta_p u = |u|^{p-2}u + f(x, u), & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} = \lambda |u|^{p-2}u - h(x, u), & \text{on } \partial \Omega.
\end{cases}
$$

They obtained the existence of a solution when $f$ and $h$ satisfy some integral conditions of Landesmann-Laser type, and $\lambda$ equals to the first eigenvalue of the Steklov problem, i.e. the first (minimal) $\lambda$ such that the problem

$$
\begin{cases}
\Delta_p u = |u|^{p-2}u, & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} = \lambda |u|^{p-2}u, & \text{on } \partial \Omega,
\end{cases}
$$

has a nontrivial solution. In [42], the authors considered the following problem

$$
\begin{cases}
-\Delta_p u + \lambda(x)|u|^{p-2}u = f(x, u), & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} = \eta |u|^{p-2}u, & \text{on } \partial \Omega,
\end{cases}
$$
where $\lambda \in L^\infty(\Omega)$ and $\text{essinf}_{x \in \Omega} \lambda(x) > 0$. They proved that if $f$ is a superlinear and subcritical odd Carathéodory function, then the problem they considered has infinitely many solutions for $\eta$ less than some constant. In [38], the following problem was considered, the author proved that there exist a positive, a negative and a sign-changing solution when the parameter $\lambda$ is greater than the second eigenvalue of the Steklov problem (1.9), $f, g$ satisfying $\lim_{s \to 0} \frac{f(x,s)}{|s|^{p-2}s} = \lim_{s \to 0} \frac{g(x,s)}{|s|^{r-2}s} = 0$ and there exist $\delta_f > 0$ such that $f(x,s)|s|^{p-2}s \geq 0$ when $0 < |s| < \delta_f$. (It was proved in [24] that the first eigenvalue of the Steklov problem is isolated, so the second eigenvalue is the minimal eigenvalue greater than the first one). In [4], the authors considered the following problem

$$
\begin{cases}
-\Delta_p u + m(x)|u|^{p-2}u = \lambda a(x)|u|^{q-2}u, & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} = b(x)|u|^{r-2}u, & \text{on } \partial \Omega,
\end{cases}
$$

where $1 < q < p < r < p^*$, $\|m\|_\infty > 0$, $a(x) \in C(\overline{\Omega})$, $\|a\|_\infty = 1$ and $b(x) \in C(\partial \Omega)$, $\|b\|_\infty = 1$. They proved that for $0 < \lambda < \lambda^*(\lambda^* \text{ is a constant depends on } p, q, r \text{ and the best Sobolev constants of the embedding } W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ and } W_0^{1,p}(\Omega) \hookrightarrow L^r(\partial \Omega))$, the above problem has two solutions.

We note that all the problems listed above deal with the existence or multiplicity problems for definite weight (i.e. the weight does not change sign) or a restricted $\lambda$. However, Theorem 1.1-1.3, 1.1'-1.3' and corollary 1.4-1.5 are for indefinite weight and every $\lambda \in \mathbb{R}$.

For the no-flux problem (1.6), if we set $N = 1$ and $\Omega = (0, T)$, we get the following periodic problem for one-dimensional $p$-Laplace equation:

$$
\begin{cases}
-(|u'|^{p-2}u')' = \lambda V(x)|u|^{p-2}u + h(x,u), \\
u(0) = u(T), \\
u'(0) = u'(T).
\end{cases}
$$

The periodic solution of $p$-laplace equation has been considered in many papers, for example, [8, 9, 28]. To the author’s knowledge, when applied to this one-dimensional case, our results as stated in Theorem 1.3 and Corollary 1.4 are also new.

This paper is organized as follows. In section 2, we recall some notations, definitions and some useful lemmas. In section 3, we study the eigenvalue problems with Steklov, No-flux, Neumann, Robin boundary value conditions respectively. We prove the existence of
a divergent sequence of eigenvalues by critical point theory for even functionals on Finsler manifolds. In section 4, we prove Theorems 1.1, 1.2 and Theorems 1.1', 1.2'.

2 Notations, definitions and known results

Let \( X \) be a closed linear subspace of \( W^{1,p}(\Omega) \) such that \( W^{1,p}_0(\Omega) \subseteq X \subseteq W^{1,p}(\Omega) \) with the norm \( \| \cdot \| \) induced from the usual norm in \( W^{1,p}(\Omega) \). In this paper, we will also use an equivalent norm on \( X \) defined by
\[
\| u \|_p^p = \int_{\Omega} (|\nabla u|^p + \varepsilon |u|^p) \, dx
\]
for a positive number \( \varepsilon \). By Pettis’s theorem, \( X \) is reflexive.

2.1 Sobolev embedding theorem

In the following, we will use Sobolev embedding theorem and trace theorem frequently. So we list them as the following lemmas (see [23]).

**Lemma 2.1** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary, there hold
(i) If \( p < N \), then \( W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \) for \( 1 \leq q \leq \frac{Np}{N-p} \), moreover, \( W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \)
when \( 1 \leq q < \frac{Np}{N-p} \),
(ii) If \( p = N \), then \( W^{1,p}(\Omega) \hookrightarrow C^{1-\frac{N}{p}}(\overline{\Omega}) \) and \( W^{1,p}(\Omega) \hookrightarrow C^{1-\frac{N}{p}}(\overline{\Omega}) \) for \( 0 \leq \beta < 1 - \frac{N}{p} \), here and in the sequel, \( \hookrightarrow \) means continuous embedding map, and \( \hookrightarrow \hookrightarrow \) means compact embedding map.

**Lemma 2.2** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary, there hold
(i) If \( p < N \), then \( W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega) \) for \( 1 \leq q \leq \frac{Np}{N-p} \), \( W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega) \)
for \( 1 \leq q < \frac{Np}{N-p} \),
(ii) If \( p = N \), then \( W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega) \) for \( 1 \leq q < \infty \),
(iii) If \( p > N \), we have \( W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega) \) for \( q \geq 1 \).

2.2 Weak solution

We give the following definitions on weak solution (See, for example, [24], for details).

(i) Let \( u \in W^{1,p}(\Omega) \), we say it is a weak solution of \( (1.1) \) if it satisfies the equation
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \int_{\Omega} \varepsilon |u|^{p-2} uv \, dx = \lambda \int_{\partial \Omega} V(x)|u|^{p-2}uv \, dS + \int_{\partial \Omega} h(x,u)vdS,
\]
for any \( v \in W^{1,p}(\Omega) \),

(ii) Let \( u \in W^{1,p}_0(\Omega) \oplus \mathbb{R} \), we say it is a weak solution of (1.2) if it satisfies the equation

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \int_{\Omega} \varepsilon |u|^{p-2} u v \, dx = \lambda \int_{\Omega} V(x) |u|^{p-2} u v \, dx + \int_{\Omega} h(x,u) v \, dx
\]

for any \( v \in W^{1,p}_0(\Omega) \oplus \mathbb{R} \),

(iii) Let \( u \in W^{1,p}(\Omega) \), we say it is a weak solution of (1.3) if it satisfies the equation

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \int_{\Omega} \varepsilon |u|^{p-2} u v \, dx = \lambda \int_{\Omega} V(x) |u|^{p-2} u v \, dx + \int_{\Omega} h(x,u) v \, dx
\]

for any \( v \in W^{1,p}(\Omega) \),

(iv) Let \( u \in W^{1,p}(\Omega) \), we say it is a weak solution of the (1.4) if it satisfies the equation

\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \int_{\Omega} \varepsilon |u|^{p-2} u v \, dx + \int_{\partial \Omega} \gamma(x) |u|^{p-2} u v \, d\gamma \, S_x = \lambda \int_{\Omega} V(x) |u|^{p-2} u v \, dx + \int_{\Omega} h(x,u) v \, dx
\]

for any \( v \in W^{1,p}(\Omega) \).

2.3 Cohomological index

In this subsection, we recall the construction and some properties of the cohomological index of Fadell-Rabinowitz for a \( \mathbb{Z}_2 \)-set, see [17, 18, 34] for details. For simplicity, we only consider the usual \( \mathbb{Z}_2 \)-action on a linear space, i.e., \( \mathbb{Z}_2 = \{1, -1\} \) and the action is the usual multiplication. In this case, the \( \mathbb{Z}_2 \)-set \( A \) is a center symmetric set with \(-A = A\).

Let \( W \) be a normed linear space. We denote by \( \mathcal{S}(W) \) the set of all center symmetric subset of \( W \) not containing the origin in \( W \). For \( A \in \mathcal{S}(W) \), denote \( \tilde{A} = A/\mathbb{Z}_2 \). Let \( f : \tilde{A} \to \mathbb{R}P^\infty \) be the classifying map and \( f^* : H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\omega] \to H^*(\tilde{A}) \) the induced homomorphism of the cohomology rings. The cohomological index of \( A \), denoted by \( i(A) \), is defined by \( \sup\{k \geq 1 : f^*(\omega^{k-1}) \neq 0\} \). Here, we list some properties which will be useful for us in this paper. Let \( A, B \in \mathcal{S}(W) \)

(i) (monotonicity) if \( A \subseteq B \), then \( i(A) \leq i(B) \).

(ii) (invariance) if \( f : A \to B \) is an odd homeomorphism, then \( i(A) = i(B) \).

(iii) (continuity) if \( C \) is a closed symmetric subset of \( A \), then there exists a closed symmetric neighborhood \( N \) of \( C \) in \( A \), such that \( i(N) = i(C) \), hence the interior of \( N \) is also a neighborhood of \( C \) in \( A \) and \( i(\text{int}N) = i(C) \).

(iv) (neighborhood of zero) if \( U \) is bounded closed symmetric neighborhood of the origin in \( W \), then \( i(\partial U) = \dim W \).

For more properties about the cohomological index, we refer to [31].
2.4 Some useful lemmas

In this subsection, we recall some known results which will be useful in section 3 and section 4. The first one is a linking theorem for cones in normed spaces which is the theoretical tool of this paper. It is contained in Corollary 2.9, Theorem 2.8, Proposition 2.4 and Theorem 2.2 of [14]. Here, we write it as one lemma.

**Lemma 2.3** ([14]) Let \( X \) be a real normed space and let \( C_-, C_+ \) be two symmetric cones in \( X \) such that \( C_+ \) is closed in \( X \), \( C_- \cap C_+ = \{0\} \) and

\[
i(C_- \setminus \{0\}) = i(X \setminus C_+) = m < \infty.
\]

Define the following four sets by

\[
D_- = \{u \in C_- : \|u\| \leq r_-\},
\]

\[
S_+ = \{u \in C_+ : \|u\| = r_+\},
\]

\[
Q = \{u + te : u \in C_-, t \geq 0, \|u + te\| \leq r_-, e \in X \setminus C_-, t \geq 0, \|u + te\| = r_-, e \in X \setminus C_+\},
\]

\[
H = \{u + te : u \in C_-, t \geq 0, \|u + te\| = r_-, e \in X \setminus C_-\}.
\]

Then \((Q, D_- \cup H)\) links \( S_+ \) cohomologically in dimension \( m+1 \) over \( \mathbb{Z}_2 \). Moreover, suppose \( f \in C^1(X, \mathbb{R}) \) satisfying the \((PS)\) condition, and

\[
\sup_{x \in D_- \cup H} f(x) < \inf_{x \in S^+} f(x), \sup_{x \in Q} f(x) < \infty.
\]

Then \( f \) has a critical value \( c \geq \inf_{x \in S^+} f(x) \).

**Remark:** Recently, in [13], the author extended it to more general case (the functional space is completely regular topological space or metric space). If the functional space \( X \) is a real Banach space, according to the proof of Theorem 6.10 in [13], the Cerami condition is sufficient for the compactness of the set of critical points at a fixed level and the first deformation lemma to hold (see [34]). So this critical point theorem still hold under the Cerami condition.

The results in section 3 is based on the following theorem.

**Lemma 2.4** (Proposition 3.52 in [37]) Suppose \( M \) is a \( C^1 \) Finsler manifold with free \( \mathbb{Z}_2 \)-action, \( \Phi \in C^1(M, \mathbb{R}) \) and \( \Phi \) is even (i.e. \( \mathbb{Z}_2 \)-invariant). Set

\[
\mathcal{F}_k = \{M : M \text{ is } \mathbb{Z}_2 \text{-invariant and } i(M) \geq k\} \text{ and } c_k = \inf_{M \in \mathcal{F}_k} \sup_{u \in M} \Phi(u).
\]

Then the following two statements are true:

(i) If \( -\infty < c_k = \cdots = c_{k+m-1} = c < +\infty \) and \( \Phi \) satisfies \((PS)_c\), then we have \( i(K^c) \geq m \).
Moreover, if \(-\infty < c_k \leq \cdots \leq c_{k+m-1} < +\infty\) and the functional \(\Phi\) satisfies (PS)\(_c\) for \(c = c_k, \cdots, c_{k+m-1}\), then all \(c_k, \cdots, c_{k+m-1}\) are critical values and \(\Phi\) has at least \(m\) distinct pairs of critical points.

(ii) If \(-\infty < c_k < +\infty\) for all sufficiently large \(k\) and \(\Phi\) satisfies (PS), then \(c_k \nearrow +\infty\).

In the proof of the main results, we will also use the following technical lemma.

**Lemma 2.5** (Lemma 4.2 in [14]) Let \(E\) be a measurable subset of \(\mathbb{R}^n\), let \(1 \leq \alpha < \infty\), \(1 \leq \beta < \infty\) and \(h : E \times \mathbb{R} \rightarrow \mathbb{R}\) be Carathéodory function. Assume that, for every \(\epsilon > 0\), there exists \(a_\epsilon \in L^\beta(E)\) such that \(|h(x,s)| \leq a_\epsilon(x) + \epsilon|s|^{\frac{\beta}{\alpha}}\) for a.e. \(x \in E\) and every \(s \in \mathbb{R}\).

Then, if \((u_k)\) is a bounded sequence in \(L^\alpha(E)\) and convergent to \(u\) a.e. in \(E\), we have that \((h(x,u_k))\) is convergent to \(h(x,u)\) strongly in \(L^\beta(E)\).

**Remark 2.6** If the condition \(|h(x,s)| \leq a_\epsilon(x) + \epsilon|s|^{\frac{\beta}{\alpha}}\) only holds for a.e. \(x \in E\) and \(|s| \leq S\) (S is a positive constant), the conclusion also holds if \(\|u_n\|_\infty \leq S, \|u\|_\infty \leq S\).

### 3 Existence of a divergent sequence of eigenvalues

In this section, we assume that \(\text{meas}\{x \in \Omega : V(x) > 0\} > 0\) if \(V\) is defined on \(\Omega\), \(\text{meas}\{x \in \partial\Omega : V(x) > 0\} > 0\) if \(V\) is defined on \(\partial\Omega\). We consider the following eigenvalue problems

**Steklov problem**

\[
\begin{cases}
\Delta_p u = \varepsilon |u|^{p-2} u, & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} = \lambda V(x)|u|^{p-2} u, & \text{on } \partial\Omega,
\end{cases}
\]

**No-flux problem**

\[
\begin{cases}
-\Delta_p u + \varepsilon |u|^{p-2} u = \lambda V(x)|u|^{p-2} u, & \text{in } \Omega, \\
u = \text{constant}, & \text{on } \partial\Omega, \\
\int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial n} dS_x = 0,
\end{cases}
\]

**Neumann problem**

\[
\begin{cases}
-\Delta_p u + \varepsilon |u|^{p-2} u = \lambda V(x)|u|^{p-2} u, & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega,
\end{cases}
\]

**Robin problem**

\[
\begin{cases}
-\Delta_p u + \varepsilon |u|^{p-2} u = \lambda V(x)|u|^{p-2} u, & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial n} + \gamma(x)|u|^{p-2} u = 0, & \text{on } \partial\Omega.
\end{cases}
\]
Lemma 3.1
For any $\varepsilon > 0$ in the cases $P(\Omega)_{\varepsilon}$, $N(\Omega)_{\varepsilon}$, $R(\Omega)_{\varepsilon}$, the author proved that these four problems has a divergent sequence of eigenvalues respectively by Ljusternik-Schnirelman principle. By critical point theory for functionals on Finsler manifolds, we also get a divergent sequence of eigenvalues respectively.

3.1 A general eigenvalue problem
Let $a \in L^r(\Omega)$, $b \in L^r(\partial \Omega)$, $\beta \in L^\infty(\partial \Omega)$ and $\beta(x) \geq 0$ for a.e. $x \in \partial \Omega$. We suppose that $a$, $b$ satisfy the following assumption:

(A): If $\text{meas}\{x \in \Omega : a(x) > 0\} = 0$, then $a \equiv 0$, $\text{meas}\{x \in \partial \Omega : b(x) > 0\} > 0$ and $X = W^{1,p}(\Omega)$.

Define on $X$ the functional

$$F(u) = \frac{1}{p} \int_\Omega a(x)|u(x)|^p dx + \frac{1}{p} \int_{\partial \Omega} b(s)|u(x)|^p dS_x,$$

and

$$G_\varepsilon(u) = \frac{1}{p} \int_\Omega (|\nabla u|^p + \varepsilon|u|^p) dx + \frac{1}{p} \int_{\partial \Omega} \beta(s)|u(x)|^p dS_x.$$ 

We want to solve the problem

$$G_\varepsilon'(u) = \lambda F'(u). \tag{3.10}$$

Clearly, we have

$$F \in C^1, \quad \langle F'(u), v \rangle = \int_\Omega a|u|^{p-2}uv dx + \int_{\partial \Omega} b|u|^{p-2}v dS_x,$$

and

$$G_\varepsilon \in C^1, \quad \langle G_\varepsilon'(u), v \rangle = \int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + \varepsilon|u|^{p-2}uv) dx + \int_{\partial \Omega} \beta|u|^{p-2}v dS_x.$$ 

First, we consider the case $\varepsilon > 0$.

Lemma 3.1 For any $u, v \in X$, we have

$$\langle G_\varepsilon'(u) - G_\varepsilon'(v), u - v \rangle \geq (\|u\|_{\varepsilon}^{p-1} - \|v\|_{\varepsilon}^{p-1})(\|u\|_{\varepsilon} - \|v\|_{\varepsilon}).$$

Proof : Its proof is the same as Lemma 2.3 in [24]. For reader’s convenience we give it here.

By direct computations, we have

$$\langle G_\varepsilon'(u) - G_\varepsilon'(v), u - v \rangle = \int_\Omega \left([|\nabla u|^p + |\nabla v|^p - |\nabla u|^{p-2}\nabla u \cdot \nabla v - |\nabla v|^{p-2}\nabla v \cdot \nabla u] dx + \varepsilon \int_\Omega (|u|^p + |v|^p - |u|^{p-2}uv - |v|^{p-2}vu) dx + \int_{\partial \Omega} \beta(|u|^p + |v|^p - |u|^{p-2}uv - |v|^{p-2}vu) dS_x.\right.$$
It follows from the proof of Lemma 2.3 in [24] that
\[
\int_{\partial \Omega} \beta(|u|^p + |v|^p - |u|^{p-2}uv - |v|^{p-2}vu)\,dS_x \geq 0.
\]
Hence
\[
\langle G'_\varepsilon(u) - G'_\varepsilon(v), u - v \rangle \geq \int_\Omega \left( |\nabla u|^p \cdot \nabla v + \varepsilon |u|^{p-2}uv \right) \, dx
\]
\[
+ \varepsilon \int_\Omega \left( |u|^p + |v|^p - |u|^{p-2}uv - |v|^{p-2}vu \right) \, dx
\]
\[
= \|u\|_\varepsilon^p + \|v\|_\varepsilon^p - \int_\Omega \left( |\nabla u|^{p-2} \nabla v \cdot \nabla v + \varepsilon |u|^{p-2}uv \right) \, dx
\]
\[
- \int_\Omega \left( |\nabla v|^{p-2} \nabla u \cdot \nabla v + \varepsilon |v|^{p-2}vu \right) \, dx.
\]
Applying Hölder inequality, we have
\[
\int_\Omega \left( |\nabla u|^{p-2} \nabla u \cdot \nabla v + \varepsilon |u|^{p-2}uv \right) \, dx
\]
\[
\leq \left( \int_\Omega |\nabla u|^p \, dx \right)^{\frac{p-1}{p}} \left( \int_\Omega |\nabla v|^p \, dx \right)^{\frac{1}{p}} + \left( \int_\Omega \varepsilon |u|^p \, dx \right)^{\frac{p-1}{p}} \left( \int_\Omega \varepsilon |v|^p \, dx \right)^{\frac{1}{p}}.
\]
Similar to the proof of Lemma 2.3 in [24], we use the following inequality
\[
(a + b)^\alpha(c + d)^{1-\alpha} \geq a^\alpha c^{1-\alpha} + b^\alpha d^{1-\alpha}
\]
which holds for any $\alpha \in (0, 1)$ and for any $a > 0$, $b > 0$, $c > 0$, $d > 0$. Set
\[
a = \int_\Omega |\nabla u|^p \, dx, \quad b = \int_\Omega \varepsilon |u|^p \, dx, \quad c = \int_\Omega |\nabla v|^p \, dx, \quad d = \int_\Omega \varepsilon |v|^p \, dx, \quad \alpha = \frac{p-1}{p},
\]
we can deduce that
\[
\int_\Omega \left( |\nabla u|^{p-2} \nabla u \cdot \nabla v + \varepsilon |u|^{p-2}uv \right) \, dx \leq \|u\|_\varepsilon^{p-1} \|v\|_\varepsilon.
\]
Similarly, we can obtain
\[
\int_\Omega \left( |\nabla v|^{p-2} \nabla v \cdot \nabla u + \varepsilon |v|^{p-2}vu \right) \, dx \leq \|v\|_\varepsilon^{p-1} \|u\|_\varepsilon.
\]
Therefore, we have
\[
\langle G'_\varepsilon(u) - G'_\varepsilon(v), u - v \rangle \geq \|u\|_\varepsilon^p + \|v\|_\varepsilon^p - \|u\|_\varepsilon^{p-1} \|v\|_\varepsilon - \|v\|_\varepsilon^{p-1} \|u\|_\varepsilon
\]
\[
= ((\|u\|_\varepsilon^{p-1} - \|v\|_\varepsilon^{p-1})(\|u\|_\varepsilon - \|v\|_\varepsilon)
\]
\[
\geq 0.
\]
Lemma 3.2 If \( u_n \to u \), \( \langle G'_\varepsilon(u_n), u_n - u \rangle \to 0 \), then \( u_n \to u \) in \( X \).

Proof : By Sobolev’s compact embedding theorem we have \( u_n \to u \) in \( L^p(\Omega) \). Since \( X \) is a reflexive Banach space, weak convergence and norm convergence imply strong convergence (see the proof of Proposition 2.4 in [24]). So we only need to show that \( \|u_n\|_{\varepsilon} \to \|u\|_{\varepsilon} \).

Notice that

\[
\lim_{n \to \infty} (G'_\varepsilon(u_n) - G'_\varepsilon(u), u_n - u) = \lim_{n \to \infty} (\langle G'_\varepsilon(u_n), u_n - u \rangle - \langle G'_\varepsilon(u), u_n - u \rangle) = 0.
\]

By the Lemma 3.1 we have

\[
\langle G'_\varepsilon(u_n) - G'_\varepsilon(u), u_n - u \rangle \geq (\|u\|_{\varepsilon}^{p-1} - \|u\|_{\varepsilon}^{p-1})\|u_n\|_{\varepsilon} - \|u\|_{\varepsilon}) \geq 0.
\]

Hence \( \|u_n\|_{\varepsilon} \to \|u\|_{\varepsilon} \) as \( n \to \infty \) and the assertion follows.

Lemma 3.3 \( F' \) is weak-to-strong continuous, i.e. \( u_n \to u \) in \( X \) implies \( F'(u_n) \to F'(u) \).

Proof : Let \( u_n \to u \) in \( X \). We have to show that \( F'(u_n) \to F'(u) \) in \( X^* \). The proof is similar to the proof of Proposition 2.2 in [24].

If \( 1 < p < N \). For any \( v \in X \), by Hölder inequality, Sobolev embedding theorem and the identity

\[
\frac{p}{N} + \frac{p-1}{N-p} + \frac{1}{N-p} = 1, \quad \frac{p-1}{N-1} + \frac{p-1}{N-p} + \frac{1}{N-p} = 1,
\]

we have that

\[
\begin{align*}
|\langle F'(u_n) - F'(u), v \rangle| &
\leq |\int_{\Omega} a(|u_n|^{p-2}u_n - |u|^{p-2}u)vdx| + \left| \int_{\partial\Omega} b(|u_n|^{p-2}u_n - |u|^{p-2}u)v\nu| \right|
\leq C_1|a||L^r(\Omega)||u_n|^{p-2}u_n - |u|^{p-2}u||L^{\frac{p}{p-1}}(\Omega)||v||L^{\frac{Np}{N-p}}(\Omega)
+ C_2|b||L^r(\partial\Omega)||u_n|^{p-2}u_n - |u|^{p-2}u||L^{\frac{\gamma}{p-1}}(\partial\Omega)||v||L^{\frac{Np}{N-p}}(\partial\Omega)
\leq C_1|a||L^r(\Omega)||u_n|^{p-2}u_n - |u|^{p-2}u||L^{\frac{p}{p-1}}(\Omega)||v|
+ C_2|b||L^r(\partial\Omega)||u_n|^{p-2}u_n - |u|^{p-2}u||L^{\frac{\gamma}{p-1}}(\partial\Omega)||v|.
\end{align*}
\]

Here \( \beta \) and \( \gamma \) satisfy \( \max\{p-1,1\} < \beta < \frac{Np}{N-p} \), and \( \max\{p-1,1\} < \gamma < \frac{Np}{N-p} \).

To prove the conclusion, we only need to show that \( |u_n|^{p-2}u_n \to |u|^{p-2}u \) in \( L^{\frac{\beta}{p-1}}(\Omega) \) and \( |u_n|^{p-2}u_n \to |u|^{p-2}u \) in \( L^{\frac{\gamma}{p-1}}(\partial\Omega) \). To see this, let \( w_n = |u_n|^{p-2}u_n \) and \( w = |u|^{p-2}u \). Since \( u_n \to u \) in \( W^{1,p}(\Omega) \), \( u_n \to u \) in \( L^{\beta}(\Omega) \), it follows that \( w_n(x) \to w(x) \), a.e. in \( \Omega \)
and \( \int_{\Omega} |w_n|^{\frac{\beta}{p-t}} \, dx \to \int_{\Omega} |w|^{\frac{\beta}{p-t}} \, dx \), by Proposition 2.4 in [19], we conclude that \( w_n \to w \) in \( L^{\frac{\beta}{p-t}}(\Omega) \). The proof of \( u_n \to u \) in \( L^{\frac{\beta}{p-t}}(\partial \Omega) \) is similar.

If \( p > N \). For any \( v \in X \), we have

\[
|\langle F'(u_n) - F'(u), v \rangle| \leq |\int_{\Omega} a(|u_n|^{p-2}u_n - |u|^{p-2}u)vdx| + |\int_{\partial \Omega} b(|u_n|^{p-2}u_n - |u|^{p-2}u)v\, d\Sigma_x| \\
\leq \|a\|_{L^1(\Omega)} \|u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^\infty(\Omega)} \|v\|_{L^\infty(\Omega)} \\
+ \|b\|_{L^1(\partial \Omega)} \|u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^\infty(\partial \Omega)} \|v\|_{L^\infty(\partial \Omega)} \\
\leq C_1 \|a\|_{L^1(\Omega)} \|u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^\infty(\Omega)} \|v\| \\
+ C_2 \|b\|_{L^1(\partial \Omega)} \|u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^\infty(\partial \Omega)} \|v\|.
\]

By the Sobolev embedding theorem, we have that \( u_n, u \in C(\Omega) \) and \( u_n \to u \) uniformly, so \( \|u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^\infty(\Omega)} \to 0 \) and \( \|u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^\infty(\partial \Omega)} \to 0 \). So the conclusion follows in this case.

If \( p = N \). For any \( v \in X \), by Hölder inequality and the Sobolev embedding theorem it follows that

\[
|\langle F'(u_n) - F'(u), v \rangle| \\
\leq |\int_{\Omega} a(|u_n|^{p-2}u_n - |u|^{p-2}u)vdx| + |\int_{\partial \Omega} b(|u_n|^{p-2}u_n - |u|^{p-2}u)v\, d\Sigma_x| \\
\leq \|a\|_{L^r(\Omega)} \|u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^{\frac{p}{p-t}}(\Omega)} \|v\|_{L^s(\Omega)} \\
+ \|b\|_{L^r(\partial \Omega)} \|u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^{\frac{p}{p-t}}(\partial \Omega)} \|v\|_{L^s(\partial \Omega)} \\
\leq C_1 \|a\|_{L^r(\Omega)} \|u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^{\frac{p}{p-t}}(\Omega)} \|v\| \\
+ C_2 \|b\|_{L^r(\partial \Omega)} \|u_n|^{p-2}u_n - |u|^{p-2}u\|_{L^{\frac{p}{p-t}}(\partial \Omega)} \|v\|.
\]

Here \( \beta \) and \( \gamma \) satisfy \( \beta > \max\{p - 1, 1\} \), \( \gamma > \max\{p - 1, 1\} \), and \( s, t > 1 \) are real number such that

\[
\frac{1}{r} + \frac{p-1}{\beta} + \frac{1}{s} = 1, \frac{1}{r} + \frac{p-1}{\gamma} + \frac{1}{t} = 1.
\]

To prove the conclusion, we only need to show that \( |u_n|^{p-2}u_n \to |u|^{p-2}u \) in \( L^{\frac{\beta}{p-t}}(\Omega) \) and \( |u_n|^{p-2}u_n \to |u|^{p-2}u \) in \( L^{\frac{\beta}{p-t}}(\partial \Omega) \). The proof is similar to the case \( p < N \).

**Lemma 3.4** If \( u_n \to u \), then \( F(u_n) \to F(u) \).

**Proof:** By the definition of \( F \), there holds

\[
p|F(u_n) - F(u)| = |\langle F'(u_n), u_n \rangle - \langle F'(u), u \rangle|.
\]
Lemma 3.5 \[ \text{If } u \in M \text{ satisfies } G_{\varepsilon}(u) = \lambda \text{ and } G_{\varepsilon}'(u) = 0, \text{ then } (\lambda, u) \text{ is a solution to } (3.10). \]

Proof: By Proposition 3.54 in [34], the norm of \( \tilde{G}_{\varepsilon}'(u) \in T^*_u M \) is given by \( \| \tilde{G}_{\varepsilon}'(u) \|_{u}^* = \min_{\mu \in \mathbb{R}} \| G'(u) - \mu F'(u) \|^{\ast} \) (here the norm \( \| \cdot \|_{u}^{\ast} \) is the norm in the fibre \( T^*_u M \), and \( \| \cdot \|^{\ast} \) is the operator norm). Hence there exist \( \mu \in \mathbb{R} \) such that \( G_{\varepsilon}'(u) - \mu F'(u) = 0 \), that is \( (\mu, u) \) is a solution of (3.10) and \( \lambda = \tilde{G}_{\varepsilon}(u) = \mu \).

Lemma 3.6 \( \tilde{G}_{\varepsilon} \) satisfies the \((PS)\) condition, i.e. if \( (u_n) \) is a sequence on \( M \) such that \( \tilde{G}_{\varepsilon}(u_n) \to c \), and \( \tilde{G}_{\varepsilon}'(u_n) \to 0 \), then up to a subsequence \( u_n \to u \in M \) in \( X \).

Proof: First, from the definition of \( G_{\varepsilon} \), we can deduce that \( (u_n) \) is bounded. Since \( X \) is reflexive, up to a subsequence, \( u_n \) converges weakly to some \( u \in X \).

From \( \tilde{G}_{\varepsilon}'(u_n) \to 0 \), we have \( G_{\varepsilon}'(u_n) - \mu_n F'(u_n) \to 0 \) for a sequence of real numbers \( (\mu_n) \). Then applying this formula to \( u_n \), we get \( \mu_n \to c \). By Lemma 3.3 \( G_{\varepsilon}'(u_n) \to cF'(u) \).

Hence \( \langle G_{\varepsilon}'(u_n), u_n - u \rangle \to 0 \). By Lemma 3.2 we get \( u_n \to u \).

Let \( F \) denote the class of symmetric subsets of \( M \), let \( F_n = \{ M \in F : i(M) \geq n \} \) and \( \lambda_{n, \varepsilon} = \inf_{M \in F_n} \sup_{u \in M} \tilde{G}_{\varepsilon}(u) \). Since \( F_n \supset F_{n+1} \), \( \lambda_{n, \varepsilon} \leq \lambda_{n+1, \varepsilon} \).

Lemma 3.7 There exists a compact set in \( F_n \).

Proof: If \( \text{meas}\{ x \in \Omega : a(x) > 0 \} = 0 \), then by assumption \( (A) \), \( a \equiv 0 \), \( \text{meas}\{ x \in \partial \Omega : b(x) > 0 \} > 0 \) and \( X = W^{1,p}(\Omega) \). We follow the idea in the proof of Theorem 3.2 in [20]. In this case, we can infer that \( \forall n \in \mathbb{N}^* \), there exist \( n \) open balls \( (B_i)_{1 \leq i \leq n} \) in \( \partial \Omega \) such that \( B_i \cap B_j = \emptyset \) if \( i \neq j \) and \( \text{meas}(\{ x \in \partial \Omega : b(x) > 0 \} \cap B_i) > 0 \). Approximating the characteristic function \( \chi_{\{x \in \partial \Omega : b(x) > 0\}} \) by \( C^\infty(\partial \Omega) \) functions in \( L^{\frac{np}{n-1}}(\partial \Omega) \), we can infer that there exists a sequence \( (u_i)_{1 \leq i \leq n} \subseteq C^\infty(\partial \Omega) \) such that \( \int_{\partial \Omega} b(s)|u_i|^p ds > 0 \).
for all $i = 1, \ldots, n$ and $\text{supp} u_i \cap \text{supp} u_j = \emptyset$ when $i \neq j$. From trace theorem, we can find a sequence $(w_i)_{1 \leq i \leq n} \in X$ such that $\Gamma(w_i) = u_i$, here $\Gamma$ is the trace map. So $F(w_i) = \frac{1}{p} \int_{\partial \Omega} b(s) |u_i|^p ds > 0$. Normalizing $w_i$, we assume that $F(w_i) = 1$. Denote $W_n$ the space generated by $(w_i)_{1 \leq i \leq n}$. \( \forall w \in W_n \), we have $w = \sum_{i=1}^{n} \alpha_i w_i$ and $F(w) = \sum_{i=1}^{n} |\alpha_i|^p$. So $w \to \left( F(w) \right)^{\frac{1}{p}}$ defines a norm on $W_n$. Since $W_n$ is finite-dimensional, this norm is equivalent to $\| \cdot \|_\varepsilon$. So $\{ w \in W_n : F(w) = 1 \} \subseteq M$ is compact with respect to the norm $\| \cdot \|_\varepsilon$ and by (i4) in section 2.3, $i(\{ w \in W_n : F(w) = 1 \}) = n$. So $\{ w \in W_n : F(w) = 1 \} \in \mathcal{F}_n$.

If $\text{meas}\{ x \in \Omega : a(x) > 0 \} > 0$, the proof is similar, see also the proof of Theorem 3.2 in [20].

Hence, $\lambda_{k,\varepsilon}$ is finite. Finally, from Lemma 2.4 and Lemma 3.6, we have $\lambda_{n,\varepsilon}$ is a divergent sequence of critical values of $\tilde{G}_\varepsilon$. So by Lemma 3.5 we get a divergent sequence of eigenvalues for problem (3.10).

**Lemma 3.8** There holds

$$\lambda_{n,\varepsilon} = \inf_{K \in \mathcal{F}_n} \sup_{u \in K} G_\varepsilon(u),$$

where $\mathcal{F}_n^c = \{ K \in \mathcal{F}_n : K \text{ is compact} \}$.

**Proof:** Indeed, the same reason as the proof of Proposition 3.1 in [14], we have that for every symmetric, open subset $A$ of $\mathcal{M}$, $i(A) = \sup\{ i(K) : K \text{ is compact and symmetric with } K \subseteq A \}$. This combines (i3) in section 2.3, can deduce the assertion easily.

Next, we consider the case $\varepsilon = 0$.

Put $G(u) = G_0(u) = \frac{1}{p} \int |\nabla u|^p dx + \frac{1}{p} \int_{\partial \Omega} \beta(s) |u(s)|^p ds$ and $\lambda_n = \inf_{K \in \mathcal{F}_n^c} \sup_{u \in K} G(u)$. To solve the eigenvalue problem $G'(u) = \lambda F'(u)$, we follow the method in [20].

**Lemma 3.9** We have the following two statements:

(i) $\lim_{\varepsilon \to 0^+} \lambda_{n,\varepsilon} = \lambda_n$,

(ii) $\lambda_n \to +\infty$ as $n \to +\infty$.

**Proof:** (i) Let $\varepsilon > 0$, from the definition, we have $\lambda_{n,\varepsilon} \geq \lambda_n$. \( \forall \delta > 0 \), there exist $K = K(\delta) \in \mathcal{F}_n^c$ such that $\lambda_n \leq \sup_{u \in K} G(u) < \lambda_n + \delta$. Set $\gamma = \sup_{u \in K} \| u \|_p ^p$, then there holds

$$\lambda_n \leq \lambda_{n,\varepsilon} \leq \sup_{u \in K} G(u) + \frac{\varepsilon \gamma}{p}.$$  

When $\varepsilon$ is sufficiently small, we obtain $\sup_{u \in K} G(u) + \frac{\varepsilon \gamma}{p} \leq \lambda_n + \delta$. Thus $\lambda_n \leq \lambda_{n,\varepsilon} \leq \lambda_n + \delta$ for all $\varepsilon$ small enough. From this we get the desired result.
(ii) Fix $a(x) \in L^r(\Omega)$, $b(x) \in L^r(\partial\Omega)$, since $\lambda_n$, $F$ and $\mathcal{F}_c^n$ depends on $a$ and $b$, we write $\lambda_n = \lambda_n(a, b)$ and $F(u) = F(a, b)(u)$, $\mathcal{F}_c^n = \mathcal{F}_c^n(a, b)$. Let $\tau > 0$ be small, define

$$\tilde{a}(x) = \begin{cases} a(x), & \text{if } a(x) \geq \tau, \\ \tau, & \text{if } a(x) < \tau, \end{cases}$$

and

$$\tilde{b}(x) = \begin{cases} b(x), & \text{if } b(x) \geq \tau, \\ \tau, & \text{if } b(x) < \tau. \end{cases}$$

Then $\tilde{a}, \tilde{b}$ still satisfy the assumption (A), hence we have $\lambda_{n, \varepsilon}(\tilde{a}, \tilde{b}) \leq \lambda_n(\tilde{a}, \tilde{b}) + \frac{\varepsilon}{\mu^2}$, Since $(\lambda_{n, \varepsilon}(\tilde{a}, \tilde{b}))_n \not\to \infty$, $\lim_{n \to \infty} \lambda_n(\tilde{a}, \tilde{b}) = +\infty$.

We claim that $\lambda_n(a, b) \geq \lambda_n(\tilde{a}, \tilde{b})$, so we get $\lim_{n \to \infty} \lambda_n = \lim_{n \to \infty} \lambda_n(a, b) = +\infty$.

Suppose that $K$ is a compact symmetric set such that $\bar{i}(K) \geq n$ and $F(\tilde{a}, \tilde{b})(u) = 1$, $\forall u \in K$. Then the map $\Psi : K \to \Psi(K)$ defined by $u \mapsto \frac{u}{F(a, b)(u)}$, is an odd homeomorphism, and $F(a, b)(w) = 1$, $\forall w \in \Psi(K)$. Since $\tilde{a} \geq a$ and $\tilde{b} \geq b$, we have $F(a, b)(u) \leq 1$, $\forall u \in K$. So $\sup_{u \in \Psi(K)} G(u) \geq \sup_{w \in K} G(w)$ and $i(\Psi(K)) = i(K) \geq n$. Therefore we have

$$\sup_{u \in \Psi(K)} G(u) \geq \lambda_n(\tilde{a}, \tilde{b}).$$

But any set in $\mathcal{F}_c^n(a, b)$ can be write as the image of a set in $\mathcal{F}_c^n(\tilde{a}, \tilde{b})$ under the map $\Psi$, so we get $\lambda_n(a, b) \geq \lambda_n(\tilde{a}, \tilde{b})$.

**Lemma 3.10** $(\lambda_n)_n$ is sequence of eigenvalues associated to the problem $G'(u) = \lambda F'(u)$.

**Proof**: Fix $n \in \mathbb{N}^*$, let $\varepsilon = \frac{1}{k}$, $k \in \mathbb{N}^*$. From the above discussion there exists a sequence $(u_k)_{k \in \mathbb{N}^*}$ of eigenfunctions associated to $(\lambda_n, \frac{1}{k})_k$ satisfying $G(u_k) + \|u_k\|_p^p = 1$. Hence $(u_k)_k$ is bounded in $X$, thus, up to a subsequence, $(u_k)_k$ converges weakly in $X$ to some $u \in X$. Since $u_k$ satisfies $G'(u_k) + \frac{1}{k} |u_k|^{p-2}u_k = \lambda_n, \frac{1}{k} F'(u_k)$, from Lemma 3.3, we have $|u_k|^{p-2}u_k \to |u|^{p-2}u$, $F'(u_k) \to F'(u)$ in $X$ as $k \to \infty$, so $G'(u_k) \to \lambda_n F'(u)$ as $k \to \infty$, $G'(u_k) + |u_k|^{p-2}u_k \to \lambda_n F'(u) + |u|^{p-2}u$ in $X$ as $k \to \infty$. Thus there holds

$$\langle G'(u_k) + |u_k|^{p-2}u_k, u_k - u \rangle \to 0.$$  

By Lemma 3.2 with $\varepsilon = 1$, we have $u_k \to u$, so $G'(u) = \lambda_n F'(u)$ and $(\lambda_n, u)$ is a solution of the problem $G'(u) = \lambda F'(u)$.
3.2 Existence results

The following theorems is direct consequence of subsection 3.1.

**Theorem 3.11** (Existence of eigenvalue sequence for $S(\Omega)_\varepsilon$) Let $F$ and $G_\varepsilon$ be defined in section 3.1 with $a \equiv 0$, $b \equiv V$ and $\beta(x) \equiv 0$. Let $X$ be $W^{1,p}(\Omega)$, then there exist a nondecreasing sequence of nonnegative eigenvalues $\{\lambda_{n,\varepsilon}\}$ of (3.10) (when $\varepsilon = 0$, set $\lambda_{n,\varepsilon} = \lambda_n$), that is, the eigenvalues of $S(\Omega)_\varepsilon$, moreover, this sequence is divergent.

**Theorem 3.12** (Existence of eigenvalue sequence for $P(\Omega)_\varepsilon$) Let $F$ and $G_\varepsilon$ be defined in section 3.1 with $a \equiv V$, $b \equiv 0$ and $\beta \equiv 0$. Let $X$ be $W^{1,p}_0(\Omega) \oplus \mathbb{R}$, then there exist a nondecreasing sequence of nonnegative eigenvalues $\{\lambda_{n,\varepsilon}\}$ of (3.10) (when $\varepsilon = 0$, set $\lambda_{n,\varepsilon} = \lambda_n$), that is, the eigenvalues of $P(\Omega)_\varepsilon$, moreover, this sequences is divergent.

**Theorem 3.13** (Existence of eigenvalue sequence for $N(\Omega)_\varepsilon$) Let $F$ and $G_\varepsilon$ be defined in section 3.1 with $a \equiv V$, $b \equiv 0$ and $\beta \equiv 0$. Let $X$ be $W^{1,p}(\Omega)$, then there exist a nondecreasing sequence of nonnegative eigenvalues $\{\lambda_{n,\varepsilon}\}$ of (3.10) (when $\varepsilon = 0$, set $\lambda_{n,\varepsilon} = \lambda_n$), that is, the eigenvalues of $N(\Omega)_\varepsilon$, moreover, this sequence is divergent.

**Theorem 3.14** (Existence of eigenvalue sequence for $R(\Omega)_\varepsilon$). Let $F$ and $G_\varepsilon$ be defined in section 3.1 with $a \equiv V$, $b \equiv 0$ and $\beta(x) \equiv \gamma(x)$. Let $X$ be $W^{1,p}(\Omega)$, then there exist a nondecreasing sequence of nonnegative eigenvalues $\{\lambda_{n,\varepsilon}\}$ of (3.10) (when $\varepsilon = 0$, set $\lambda_{n,\varepsilon} = \lambda_n$), that is, the eigenvalues of $R(\Omega)_\varepsilon$, moreover, this sequence is divergent.

3.3 Index computation for cones

Similar to Theorem 3.2 in [14], we have:

**Theorem 3.15** If $\lambda_{m,\varepsilon} < \lambda_{m+1,\varepsilon}$ for some $m \in \mathbb{N}^*$, then

$$i(\{u \in X \setminus \{0\} : G_\varepsilon(u) \leq \lambda_{m,\varepsilon} F(u)\}) = i(\{u \in X : G_\varepsilon(u) < \lambda_{m+1,\varepsilon} F(u)\}) = m.$$ 

Proof: Suppose $\lambda_{m,\varepsilon} < \lambda_{m+1,\varepsilon}$. If we set $A = \{u \in \mathcal{M} : G_\varepsilon(u) \leq \lambda_{m,\varepsilon}\}$ and $B = \{u \in \mathcal{M} : G_\varepsilon(u) < \lambda_{m+1,\varepsilon}\}$, clearly, we have $i(A) \leq m$. Assume that $i(A) \leq m - 1$. By (i3) in section 2.3, there exists a symmetric neighborhood $W$ of $A$ in $\mathcal{M}$ satisfying $i(W) = i(A)$. Notice that such a $W$ is also a neighborhood of the critical set of $G_\varepsilon|_{\mathcal{M}}$ at level $\lambda_{m,\varepsilon}$, by the equivariant deformation theorem, there exists $\delta > 0$ and an odd continuous map $\iota : \{u \in \mathcal{M} : G_\varepsilon(u) \leq \lambda_{m,\varepsilon} + \delta\} \to \{u \in \mathcal{M} : G_\varepsilon(u) \leq \lambda_{m,\varepsilon} - \delta\} \cup W = W$. It follows from (i2) in section 2.3 that $i(u \in \mathcal{M} : G_\varepsilon(u) \leq \lambda_{m,\varepsilon} + \delta) \leq m - 1$. This contradicts the
definition of $\lambda_{m,\varepsilon}$ and the monotonicity of the cohomological index. By the invariance of the cohomological index under odd homeomorphism, we have

$$i(\{u \in X \setminus \{0\} : G_\varepsilon (u) \leq \lambda_{m,\varepsilon} F(u)\}) = m.$$  

By the monotonicity of the cohomological index, we have $i(B) \geq m$. Assume that $i(B) \geq m + 1$. From the proof of Lemma 3.8 there exists a symmetric, compact subset $K$ of $B$ with $i(K) \geq m + 1$. Since $\max\{G_\varepsilon (u) : u \in K\} < \lambda_{m+1,\varepsilon}$, this contradicts to Lemma 3.8. By the invariance of the cohomological index under odd homeomorphism, we have $i(\{u \in X : G_\varepsilon (u) < \lambda_{m+1,\varepsilon} F(u)\}) = m$. 

\section{Proof of the main theorem}

In this section, we assume that $\varepsilon > 0$.

\subsection{Proof of Theorem 1.1 and Theorem 1.1'}

We consider the $C^1$ functional $f_\varepsilon : X = W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$f_\varepsilon (u) = \frac{1}{p} \int_\Omega (|\nabla u|^p + \varepsilon |u|^p) dx - \frac{\lambda}{p} \int_{\partial \Omega} V |u|^p dS_x - \int_{\partial \Omega} H(x, u) dS_x.$$ 

It is clear that critical points of $f_\varepsilon$ are weak solutions of (1.1).

In this case,

$$G_\varepsilon (u) = \frac{1}{p} \int_\Omega (|\nabla u|^p + \varepsilon |u|^p) dx$$

$$F(u) = \frac{1}{p} \int_{\partial \Omega} V(x) |u|^p dS_x,$$

Hence

$$f_\varepsilon (u) = G_\varepsilon (u) - \lambda F(u) - \int_{\partial \Omega} H(x, u) dS_x.$$ 

We follow the line of [14].

\textbf{Lemma 4.1} By (h1) and (h2), we have, $\int_{\Omega} \frac{H(x, u)}{|u|^p} dS_x \to 0$ as $\|u\|_\varepsilon \to 0$.

\textbf{Proof: Case 1:} $p < N$. Set

$$H_0(x, s) = \begin{cases} \frac{H(x, s)}{|s|^p}, & \text{if } s \neq 0, \\ 0, & \text{if } s = 0, \end{cases}$$
Applying Sobolev embedding theorem again, the conclusion follows in this case. 

\[ |H_0(x,s)| \leq \frac{1}{p}a_\varepsilon(x) + \varepsilon \frac{s^{q-1}}{N-p} |s|^{\frac{p-1}{N-p}}. \]

By the continuous embedding of \( X \) into \( L^{\frac{N-p}{N-p-p}}(\partial\Omega) \) and Lemma 2.5 it follows that \( H_0(x,u) \) converges to 0 in \( L^{\frac{N-1}{N-p-p}}(\partial\Omega) \) as \( \|u\|_\varepsilon \rightarrow 0 \). Using Hölder inequality we have
\[
\int_{\partial\Omega} |H(s,u)|dS_x = \int_{\partial\Omega} |H_0(s,u)||u|^p dS_x \leq \left( \int_{\partial\Omega} |H_0(s,u)|^\frac{N-1}{N-p} dS_x \right)^\frac{N-p}{N-1} \left( \int_{\partial\Omega} |u|^{\frac{N-p}{N-p-p}} dS_x \right)^\frac{N-p-p}{N-p}.
\]

Applying Sobolev embedding theorem again, the conclusion follows in this case.

**Case 2:** \( p = N \). In this case, by making \( q \) large enough, we can also write \( |H_0(x,s)| \leq \frac{1}{p}a_S(x) + \frac{1}{q}|s|^{q-p} \) and \( q - p > 1 \), here \( H_0(x,s) \) is defined as **Case 1**. By the continuous embedding of \( X \) into \( L^{(q-p)}(\partial\Omega) \), it follows from Lemma 2.5 that \( H_0(x,u) \) converges to 0 in \( L^{(q-p)}(\partial\Omega) \) as \( \|u\|_\varepsilon \rightarrow 0 \). Using the Hölder inequality we have
\[
\int_{\partial\Omega} |H(s,u)|dS_x = \int_{\partial\Omega} |H_0(s,u)||u|^p dS_x \leq \left( \int_{\partial\Omega} |H_0(s,u)|^r dS_x \right)^\frac{1}{r} \left( \int_{\partial\Omega} |u|^{\frac{r}{r-p}} dS_x \right)^{\frac{r-p}{r}}.
\]

Applying Sobolev embedding theorem again, the conclusion follows in this case.

**Case 3:** \( p > N \). In this case, we can also write \( |H_0(x,s)| \leq \frac{1}{p}a_S(x) + \frac{1}{q}|s|^{q-p} \), for \( |s| \leq S \) and \( q - p > 1 \), here \( H_0(x,s) \) is defined as **Case 1**. By Sobolev embedding theorem, we can also assume that \( \|u\|_{C^0(\partial\Omega)} < S \) for some \( S > 0 \) when \( \|u\|_\varepsilon \) is small. Since \( X \) continuously embeds into \( L^{(q-p)}(\partial\Omega) \) and from Lemma 2.5, Remark 2.6 we can deduce that \( H_0(x,u) \) goes to 0 in \( L^1(\partial\Omega) \) as \( \|u\|_\varepsilon \rightarrow 0 \). Using the Hölder inequality we have
\[
\int_{\partial\Omega} |H(s,u)|dS_x = \int_{\partial\Omega} |H_0(s,u)||u|^p dS_x \leq \left( \int_{\partial\Omega} |H_0(s,u)|dS_x \right)||u||_{L^\infty(\partial\Omega)}^p.
\]

Applying Sobolev embedding theorem again, the conclusion follows in this case.

**Lemma 4.2** If there exists \( b > 0 \) and \( (u_k) \) in \( X \) such that \( \|u_k\|_\varepsilon \rightarrow \infty \) and \( \int_{\Omega}(|\nabla u_k|^p + \varepsilon|u_k|^p)dx \leq b \int_{\partial\Omega} V(x)|u_k|^p dS_x \). Then from (h2) and (h4) we have \( \int_{\partial\Omega} H(x,u_k)dS_x \rightarrow +\infty \).

**Proof:** Set \( v_k = \frac{u_k}{\|u_k\|_\varepsilon} \), then, up to a subsequence, \( (v_k) \) converges to some \( v \) weakly in \( X \) and a.e.in \( \partial\Omega \). By Lemma 3.4 it follows that \( b \int_{\partial\Omega} V|v|^p ds \geq 1 \). So \( |v| \neq 0 \) on a set with positive measure. Thus from (h2) we have
\[
\lim_{k \rightarrow \infty} \frac{H(s,u_k(s))}{\|u_k\|_\varepsilon^p} = \lim_{k \rightarrow \infty} \frac{H(s,\|u_k\|_\varepsilon v_k(s))}{\|u_k\|_\varepsilon^p |v_k(s)|^p} = +\infty
\]
on a set with positive measure. By (h4) we can apply Fatou’s lemma to the sequence \( \left( \frac{H(s,u_k)}{\|u_k\|_\varepsilon} \right)_k \) and the assertion follows.
Lemma 4.3 Suppose (h1) is satisfied. The map $T : X \to X^*$ defined by $T(u)(v) = \int_{\partial \Omega} h(x, u)v \, dS_x$ is weak-to-strong continuous.

Proof: If $p < N$, we set $\alpha = \frac{\frac{Np}{N-p}}{\frac{Np}{N-p} - 1} = \frac{Np}{N-p}$. Let $(u_k)$ be a sequence weakly convergent to $u$ in $X$, then $(u_k)$ is bounded in $L^{\frac{Np}{N-p}(\partial \Omega)}$ and up to subsequence, converges to $u$ a.e.in $\partial \Omega$. By (h1) and Young’s inequality we have

$$|h(x, s)| \leq a_\varepsilon(x)|s|^{p-1} + \varepsilon|s|^{\frac{Np}{N-p}-1}$$

$$\leq \frac{\alpha(p-1)}{N-1}(\frac{\varepsilon}{\alpha})^{\frac{N-1}{p-1}} + \frac{p-1}{N-1}(\frac{Np}{N-p} - 1)}{\frac{Np}{N-p} - 1} |s|^{\frac{Np}{N-p}-1} + \varepsilon|s|^{\frac{Np}{N-p}-1}.$$

From Lemma 2.5, $(h(x, u_k))$ is convergent to $h(x, u)$ strongly in $L^\alpha(\partial \Omega)$, hence strongly in $X^*$.

If $p = N$, in this case, by making $q$ large enough, we may assume that for every $\varepsilon > 0$, there exists $a_\varepsilon \in L^\alpha(\partial \Omega)$ such that $|h(x, s)| \leq a_\varepsilon(x)|s|^{p-1} + \varepsilon|s|^{q-1}$ and $q-p > 1$. Let $(u_k)$ be a sequence weakly convergent to $u$ in $X$. Then $(u_k)$ is bounded in $L^{(q-p)}(\partial \Omega)$ and up to subsequence, converges to $u$ a.e.in $\partial \Omega$. By (h1) and Young’s inequality we have

$$|h(x, s)| \leq a_\varepsilon(x)|s|^{p-1} + \varepsilon|s|^{q-1} \leq \frac{(q-p)}{q-1}(\frac{a_\varepsilon(x)}{\varepsilon})^{\frac{1}{q-p}} + \frac{p-1}{q-1}\varepsilon^{\frac{1}{q-p}}|s|^{q-1} + \varepsilon|s|^{q-1}.$$ 

From Lemma 2.5, we have $(h(x, u_k))$ is convergent to $h(x, u)$ strongly in $L^{\frac{q-p}{q-1}}(\partial \Omega)$, hence strongly in $X^*$.

If $p > N$, let $(u_k)$ be a sequence weakly convergent to $u$ in $X$. Then by Sobolev embedding theorem, $(u_k)$ converges to $u$ uniformly in $\partial \Omega$. By (h1), we have

$$|h(x, u_k) - h(x, u)| \leq a_S(x)(|u_k|^{p-1} + |u|^{p-1}) - |h(x, u_k) - h(x, u)|,$$

for some $S > 0$. Applying Fatou’s lemma to the sequence $a_S(x)(|u_k|^{p-1} + |u|^{p-1}) - |h(x, u_k) - h(x, u)|$, we obtain

$$2 \int_{\partial \Omega} a_S(s)|u|^{p-1} \, dS_x \leq \liminf_{k \to \infty} \int_{\partial \Omega} [a_S(s)(|u_k|^{p-1} + |u|^{p-1}) - |h(s, u_k) - h(s, u)|] \, dS_x$$

$$\leq 2 \int_{\partial \Omega} a_S(s)|u|^{p-1} \, dS_x - \limsup_{k \to \infty} \int_{\partial \Omega} |h(s, u_k) - h(s, u)| \, dS_x.$$

So $\limsup_{k \to \infty} \int_{\partial \Omega} |h(s, u_k) - h(s, u)| \, dS_x \leq 0$, that is, $h(x, u_k)$ converges to $h(s, u)$ in $L^1(\partial \Omega)$. From Sobolev embedding theorem, we have that $h(x, u_k)$ converges to $h(s, u)$ in $X^*$.

Lemma 4.4 Suppose (h1)–(h4) hold. For every $\lambda \in \mathbb{R}$ and $c \in \mathbb{R}$, the functional $f_\varepsilon$ satisfies $(PS)_c$ condition.
Proof: Let \((u_k)_k\) be a sequence in \(X\) satisfying \(f'_\varepsilon(u_k) \to 0\) in \(X^*\) and \(f_\varepsilon(u_k) \to c\).

Claim: \((u_k)\) is bounded in \(X\). By contradiction, we assume that \(\|u_k\|_\varepsilon \to \infty\). From (h3) we have

\[
\mu f'_\varepsilon(u_k) - \langle f'_\varepsilon(u_k), u_k \rangle = \left(\frac{\mu}{p} - 1\right) \int_\Omega (|\nabla u_k|^p + \varepsilon |u_k|^p) dx - \left(\frac{\mu}{p} - 1\right) \int_{\partial\Omega} \lambda V |u_k|^p dS_x
\]

\[
+ \int_{\partial\Omega} (h(s, u_k) u_k - \mu H(s, u_k)) dS_x \geq \left(\frac{\mu}{p} - 1\right) \int_\Omega (|\nabla u_k|^p + \varepsilon |u_k|^p) dx
\]

\[
- \left(\frac{\mu}{p} - 1\right) \int_{\partial\Omega} \lambda V |u_k|^p dS_x - \int_{\partial\Omega} (\gamma_0 + \gamma_1 |u_k|^p) dS_x.
\]

Since

\[
\mu f'_\varepsilon(u_k) - \langle f'_\varepsilon(u_k), u_k \rangle + \int_{\partial\Omega} \gamma_0 dS_x \leq \frac{1}{2} \left(\frac{\mu}{p} - 1\right) \int_\Omega (|\nabla u_k|^p + \varepsilon |u_k|^p) dx
\]

for \(k\) large enough, there exists \(b > 0\) such that

\[
\int_\Omega (|\nabla u_k|^p + \varepsilon |u_k|^p) dx \leq \int_{\partial\Omega} (2\lambda V + b\gamma_1) |u_k|^p dS_x
\]

for \(k\) large enough. \((2\lambda V + b\gamma_1)\) is still in \(L^*(\partial\Omega)\), from Lemma 1.2 we can deduce that

\[
\lim_{k \to \infty} \frac{\int_{\partial\Omega} H(s, u_k) dS_x}{\|u_k\|_\varepsilon} = +\infty.
\]

Moreover, by Sobolev embedding theorem, we have \(\int_{\partial\Omega} V |u|^p dS_x \leq C \|V\|_r \|u\|_\varepsilon^p\). Therefore

\[
0 = \lim_{k \to \infty} \frac{f_\varepsilon(u_k)}{\|u_k\|_\varepsilon} = \frac{1}{p} - \lim_{k \to \infty} \frac{\lambda \int_{\partial\Omega} V |u_k|^p dS_x}{p \|u_k\|_\varepsilon^p} + \frac{\int_{\partial\Omega} H(s, u_k) dS_x}{\|u_k\|_\varepsilon^p} = -\infty.
\]

It is a contradiction, so \((u_k)\) is bounded in \(X\).

Actually, \(f'_\varepsilon(u_k) = G'_\varepsilon(u_k) - \lambda F'(u_k) - T(u_k)\), here \(T : X \to X^*\) is defined in Lemma 4.3. By Lemma 3.3 and 4.3 we have, up to subsequence, \(F'(u_k)\) and \(T(u_k)\) converge, so \(G'_\varepsilon(u_k)\) converges in \(X^*\). By Lemma 3.2 we can deduce that \(u_k\) has a convergent subsequence. So we have proved the \((PS)_c\) condition.

In order to prove the Theorem 1.1', we need the following result.

**Lemma 4.4'** Suppose (h1), (h2), (h4), (h5) hold. For every \(\lambda \in \mathbb{R}\), \(f_\varepsilon\) satisfies the Cerami condition.

Proof: Let \((u_k)\) be a sequence in \(X\) satisfying \((1 + \|u_k\|_\varepsilon) f'_\varepsilon(u_k) \to 0\) in \(X^*\) and \(f_\varepsilon(u_k) \to c\).

Claim: \((u_k)\) is bounded in \(X\). Otherwise, if \(\|u_k\|_\varepsilon \to \infty\), we consider \(w_k := \frac{u_k}{\|u_k\|_\varepsilon}\).

Then, up to subsequence, we get \(w_k \rightharpoonup w\) in \(X\) and \(w_k(x) \to w(x)\) a.e. \(x \in \partial\Omega\) as \(k \to \infty\). If \(w \neq 0\) in \(X\), since \(f'_\varepsilon(u_k) u_k \to 0\), that is to say

\[
\int_\Omega (|\nabla u_k|^p + \varepsilon |u_k|^p) dx - \lambda \int_{\partial\Omega} V(x)|u_k|^p dS_x - \int_{\partial\Omega} h(x, u_k) u_k dS_x \to 0,
\]

(4.11)
by Schwartz inequality and Sobolev embedding theorem, we have
\[
\frac{\int_{\partial \Omega} |V(x)| |u_k|^p \, dS_x}{\|u_k\|_\varepsilon^p} \leq C \|V\|_r,
\]
so by dividing the left hand side of (4.11) with \(\|u_k\|_\varepsilon^p\) there holds
\[
\left| \int_{\partial \Omega} \frac{h(x, u_k) u_k}{\|u_k\|_\varepsilon} \, dS_x \right| \leq C. \tag{4.12}
\]
On the other hand, by condition (h5), we have \(h(x, s)s \geq H(x, s)\), so by condition (h2),
\[
\lim_{|s| \to \infty} \frac{h(x, s)s}{|s|^p} = +\infty. \tag{4.13}
\]
By Fatou’s lemma, we have
\[
\int_{\partial \Omega} h(x, u_k) u_k \, dS_x = \int_{\{w_k \neq 0\}} |w_k|^p h(x, u_k) u_k |u_k|^p \, dS_x \to \infty,
\]
this contradicts to (4.12).

If \(w = 0\) in \(X\), inspired by [22], we choose \(t_k \in [0, 1]\) such that \(f_{\varepsilon}(t_k u_k) := \max_{t \in [0, 1]} f_{\varepsilon}(tu_k)\).

For any \(\beta > 0\) and \(\tilde{w}_k := (2p\beta)^{1/p} w_k\), by Lemma 3.3 and Lemma 4.3 we have that
\[
f_{\varepsilon}(t_k u_k) \geq f_{\varepsilon}(\tilde{w}_k) = 2\beta - \frac{\lambda}{p} \int_{\partial \Omega} V(x) |\tilde{w}_k|^p \, dS_x - \int_{\partial \Omega} H(x, \tilde{w}_k) \, dS_x \geq \beta,
\]
when \(k\) is large enough, this implies that
\[
\lim_{k \to \infty} f_{\varepsilon}(t_k u_k) = \infty. \tag{4.14}
\]
Since \(f_{\varepsilon}(0) = 0\), \(f_{\varepsilon}(u_k) \to c\), we have \(t_k \in (0, 1)\). By the definition of \(t_k\),
\[
\langle f'_{\varepsilon}(t_k u_k), t_k u_k \rangle = 0. \tag{4.15}
\]
From (4.13), (4.14), we have
\[
f_{\varepsilon}(t_k u_k) - \frac{1}{p} \langle f'_{\varepsilon}(t_k u_k), t_k u_k \rangle = \int_{\partial \Omega} \left( \frac{1}{p} h(x, t_k u_k) t_k u_k - H(x, t_k u_k) \right) \, dS_x \to \infty.
\]
By (h3), there exists \(\theta \geq 1\) such that
\[
\int_{\partial \Omega} \left( \frac{1}{p} h(x, u_k) u_k - H(x, u_k) \right) \, dS_x \geq \frac{1}{\theta} \int_{\partial \Omega} \left( \frac{1}{p} h(x, t_k u_k) t_k u_k - H(x, t_k u_k) \right) \, dS_x \to \infty. \tag{4.16}
\]
On the other hand,
\[
\int_{\partial \Omega} \left( \frac{1}{p} h(x, u_k) u_k - H(x, u_k) \right) \, dS_x = f_{\varepsilon}(u_k) - \frac{1}{p} \langle f'_{\varepsilon}(u_k), u_k \rangle \to c. \tag{4.17}
\]
(4.15) and (4.16) are contradiction. Hence \(\{u_k\}\) is bounded in \(X\). So up to a subsequence, we can assume that \(u_k \to u\) for some \(X\).
The same reason as Lemma 4.3 we can prove that \{u_k\} have a convergent subsequence. So \( f_\varepsilon \) satisfies the Cerami condition.

\[ \frac{1}{p} \int_\Omega (|\nabla u|^p + \varepsilon |u|^p)dx - \frac{\lambda}{p} \int_\Omega V(x)|u|^pdx - \int_\Omega H(x,u)dx \]

**Proof of Theorem 1.1:** Replacing \((\lambda, V)\) with \((-\lambda, -V)\), we can assume that \( \lambda \geq 0 \).

**Case 1:** \( \text{meas}\{x \in \partial \Omega : V(x) > 0\} > 0 \) (by Theorem 3.11) \( S(\Omega)_\varepsilon \) has a divergent sequence \((\lambda_m, \varepsilon)_m \) of eigenvalues), \( \lambda \geq \lambda_1, \varepsilon \).

Since the sequence \((\lambda_m, \varepsilon)_m \) is divergent, there exist \( m \geq 1 \) such that \( \lambda_m, \varepsilon \leq \lambda < \lambda_{m+1, \varepsilon} \).

Define

\[
C_- = \{ u \in X : G_\varepsilon(u) \leq \lambda_m, \varepsilon F(u) \}, \\
C_+ = \{ u \in X : G_\varepsilon(u) \geq \lambda_{m+1, \varepsilon} F(u) \},
\]

we have that \( C_- , C_+ \) are two symmetric closed cones in \( X \) with \( C_- \cap C_+ = \{0\} \).

By Theorem 3.15 we have that \( i(C_- \setminus \{0\}) = i(X \setminus C_+) = m \).

Since \( \lambda < \lambda_{m+1, \varepsilon} \), by Lemma 4.4 there exist \( r_+ > 0 \) and \( \alpha > 0 \) such that \( f_\varepsilon(u) > \alpha \) for \( u \in C_+ \) and \( ||u||_\varepsilon = r_+ \). Since \( \lambda \geq \lambda_{m, \varepsilon} \), by (h4) we have \( f_\varepsilon(u) \leq 0 \) for every \( u \in C_- \).

Let \( e \in X \setminus C_- \), we define another norm on \( X \) by \( ||u||_V := (\int_{\partial \Omega} (|V| + 1)|u|^p dS_x)^{1/p} \). If \( u \in C_- \) and \( t > 0 \), then

\[
||u + te||_\varepsilon = t||\frac{u}{t} + e||_\varepsilon \leq t(||\frac{u}{t}||_\varepsilon + ||e||_\varepsilon) \leq t(C||\frac{u}{t}||_V + \frac{||e||_\varepsilon}{||e||_V} ||e||_V) \leq Ct(||\frac{u}{t}||_V + ||e||_V).
\]

Notice that \( C_- \) is also closed in \( X \) with respect to the norm \( || \cdot ||_V \), by Proposition 2.12 in [14], there exists \( \beta \geq 1 \) such that \( \frac{\beta}{2} ||V||_V + ||e||_V \leq \beta \frac{u}{t} + e \). Hence, \( ||u + te||_\varepsilon \leq b||u + te||_V \) for every \( u \in C_- , t \geq 0 \) and some \( b > 0 \). Thus from Lemma 4.2 we have that

\[
\int_{\Omega} H(s, u_k) dS_x \rightarrow +\infty \text{ for } ||u_k||_\varepsilon \rightarrow +\infty \text{ and } u_k \in C_- + R^+ e.
\]

So there exists \( r_+ > r_- \) such that \( f_\varepsilon(u) \leq 0 \) for \( u \in C_- + R^+ e \) and \( ||u||_\varepsilon \geq r_- \).

If we define \( D_- , S_+ , Q , H \) as Lemma 2.3, then \( f_\varepsilon \) is bounded on \( Q \), \( f_\varepsilon(u) \leq 0 \) for every \( u \in D_- \cup H \) and \( f_\varepsilon(u) \geq \alpha > 0 \) for every \( u \in S_+ \). With Lemma 4.4 it follows that \( f_\varepsilon \) has a critical value \( c \geq \alpha > 0 \). Hence \( u \) is a nontrivial weak solution of (1.1).

**Case 2:** \( \text{meas}\{x \in \partial \Omega : V(x) > 0\} > 0 \) , \( 0 \leq \lambda < \lambda_{1, \varepsilon} \) or \( \text{meas}\{x \in \partial \Omega : V(x) > 0\} = 0 \), \( \lambda \geq 0 \). We set \( C_- = \{0\} \), \( C_+ = X \) and the proof is similar.

**Proof of theorem 1.1':** The process is the same as the proof of Theorem 1.1. With the aid of the remark after Lemma 2.3, we use Lemma 4.4 instead of Lemma 4.3.

### 4.2 Proof of Theorem 1.2 and Theorem 1.2'

For problem (1.2), we consider the \( C^1 \) functional \( f_\varepsilon : X = W^{1,p}_0(\Omega) \oplus R \rightarrow R \) defined by

\[
f_\varepsilon(u) = \frac{1}{p} \int_\Omega (|\nabla u|^p + \varepsilon |u|^p)dx - \frac{\lambda}{p} \int_\Omega V(x)|u|^pdx - \int_\Omega H(x,u)dx
\]
For problem (1.3), we consider the $C^1$ functional $f_\varepsilon : X = W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$f_\varepsilon(u) = \frac{1}{p} \int_\Omega \left( |\nabla u|^p + \varepsilon |u|^p \right) dx - \frac{\lambda}{p} \int_\Omega |V|^p dx - \int_\Omega H(x,u) dx$$

$$= G_\varepsilon(u) - \lambda F(u) - \int_\Omega H(x,u) dx \quad (see \ Theorem 3.12).$$

For problem (1.4), we consider the $C^1$ functional $f_\varepsilon : X = W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$f_\varepsilon(u) = \frac{1}{p} \int_\Omega \left( |\nabla u|^p + \varepsilon |u|^p \right) dx + \frac{1}{p} \int_{\partial\Omega} \beta |u|^p dS_x - \frac{\lambda}{p} \int_\Omega |V|^p dx - \int_\Omega H(x,u) dx$$

$$= G_\varepsilon(u) - \lambda F(u) - \int_\Omega H(x,u) dx \quad (see \ Theorem 3.13).$$

It is clear that critical points of $f_\varepsilon$ are weak solutions of (1.2), (1.3), (1.4), respectively. The following lemmas are needed in the proofs of Theorem 1.2 and 1.2'. Their proofs are similar to the proofs of Lemma 4.1, 4.2, 4.3, 4.4, see also [14]. In the following Lemmas 4.5-4.8, we always assume that (h1)-(h4) with $\partial\Omega$ replaced by $\Omega$ and $p^* = \frac{Np}{N-p}$.

**Lemma 4.5** There holds $\int_\Omega \frac{H(x,u) dx}{\|u\|_\varepsilon^p} \to 0$ as $\|u\|_\varepsilon \to 0$.

**Lemma 4.6** If there exists $b > 0$ and $(u_k)$ in $X$ such that $\|u_k\|_\varepsilon \to \infty$ and $\int_\Omega (|\nabla u_k|^p + \varepsilon |u_k|^p) dx \leq b \int_\Omega |V|^p dx$. Then from we have $\int_\Omega \frac{H(x,u_k) dx}{\|u_k\|_\varepsilon^p} \to +\infty$.

**Lemma 4.7** The map $T : X \to X^*$ defined by $T(u)(v) = \int_\Omega h(x,u) v dx$ is weak-to-strong continuous.

**Lemma 4.8** For every $\lambda \in \mathbb{R}$ and $c \in \mathbb{R}$, the functional $f_\varepsilon$ satisfies $(PS)_c$.

Under the conditions of Theorem 1.2', we have the following result.

**Lemma 4.8'** For every $\lambda \in \mathbb{R}$, $f_\varepsilon$ satisfies the Cerami condition.

**Proof of Theorem 1.2:** The proof is similar to the proof of Theorem 1.1 by using Lemmas 4.5-4.8. We omit the details here.

**Proof of Theorem 1.2':** The proof is similar to the proof of Theorem 1.1 by using Lemmas 4.5-4.7 and 4.8'. We omit the details here.

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