Acoustic radiation force and radiation torque beyond particles: Effects of non-spherical shape and Willis coupling

Shahrokh Sepehrirahnama,1,† Sebastian Oberst,† Yan Kei Chiang,2,‡ and David Powell2,‡

1Centre for Audio, Acoustics and Vibration, University of Technology Sydney, Sydney, Australia
2School of Engineering and Information Technology, University of New South Wales, Canberra, Australia

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Acoustophoresis deals with the manipulation of sub-wavelength scatterers in an incident acoustic field. The geometric details of manipulated particles are often neglected by replacing them with equivalent symmetric geometries such as spheres, spheroids, cylinders or disks. It has been demonstrated that geometric asymmetry, represented by Willis coupling terms, can strongly affect the scattering of a small object, hence neglecting these terms may miss important force contributions. In this work, we present a generalized formalism of acoustic radiation force and radiation torque based on the polarizability tensor, where Willis coupling terms are included to account for geometric asymmetry. Following Gorkov’s approach, the effects of geometric asymmetry are explicitly formulated as additional terms in the radiation force and torque expressions. By breaking the symmetry of a sphere along one axis using intrusion and protrusion, we characterize the changes in the force and torque in terms of partial components, associated with the direct and Willis Coupling coefficients of the polarizability tensor. We investigate in detail the cases of standing and travelling plane waves, showing how the equilibrium positions and angles are shifted by these additional terms. We show that while the contributions of asymmetry to the force are often negligible for small particles, these terms greatly affect the radiation torque. Our presented theory, providing a way of calculating radiation force and torque directly from polarizability coefficients, shows that in general it is essential to account for shape of objects undergoing acoustophoretic manipulation, and this may have important implications for applications such as the manipulation of biological cells.

Keywords: Acoustic bianisotropy, Primary Force, Primary Torque, Non-spherical Scattering

I. INTRODUCTION

Acoustic radiation forces play a key role in the field of acoustic particle manipulation, also known as acoustophoresis [1-8]. Acoustic sorting, separation, levitation and other similar applications have been developed to manipulate particulate phase in a fluid, e.g. to migrate biological cells to certain locations using incident plane waves, by inducing a radiation force field [3][7][16], as shown in Fig. 1. In the design of such applications, it is customary to treat the objects in the host fluid as spheres or other simple geometries and neglect the details of their shapes. In most cases, this leads to a design based solely on acoustic radiation force and neglecting the radiation torque. When a sub-wavelength object is treated as a particle, its dynamic equilibrium and force balance is independent of its shapes and its rotation is neglected. This assumption can be reasonable for small objects with approximately spherical shape; however, non-spherical objects lacking radial symmetry may be better approximated as rigid bodies than point-like particles.

Acoustic radiation stress consists of radiation pressure, which is related to radiated momentum through scattering fields, and Reynolds stress arising from surface oscillation. The primary radiation force acting on a scatterer corresponds to the incident-scattering portion of the time-averaged radiation stresses [2][17][25]. In the case of multiple objects, pairwise secondary radiation forces, also known as acoustic interaction forces, emerge from the scattering-scattering part of the radiation stresses [2][26]. The acoustic radiation torque is the resultant moment due to such stresses with respect to the centroid of the object [27][28]. Acoustic radiation force and torque contribute to the dynamic equilibrium state of the scatterers in combination with other forces such as hydrodynamic drag or gravitational force. The magnitudes of acoustic radiation force and torque are proportional to the incident energy density and their direction is determined from their normalized value, also referred to as the acoustic contrast factor [2][18][24], as shown in Fig. 1.

Current theories of acoustic radiation force and torque use either the radiation stresses at the surface of the object [17][18][29][31], or convert the surface integral of radiation...
stresses in the far-field to a volume integral using the Divergence theorem \[2, 19, 24\]. The advantage of the latter approach is that the object can be replaced by a set of acoustic multipole sources to evaluate the volume integral, capturing the essential geometric features and material properties of the object with the simplest possible model. Acoustic radiation force and torque have been developed analytically or modelled numerically for objects with spheroidal, cylindrical and disk shapes \[27, 31–38\]. However, these shapes exhibit a high degree of symmetry (at least axial symmetry), and are not fully representative of three-dimensional objects with arbitrary shape. Although the numerical approaches based on surface or volume integrals can be applied to objects of arbitrary shape, they do not directly show how the geometric asymmetry impacts on the force and torque. For acoustic scattering from small particles, it has been shown that asymmetry impacts on the force and torque. We respond to both the incident pressure and velocity fields. We make use of the far-field approach, also presented in the scattering field of a sphere is given as \[2, 19, 24\]

\[
\phi_s \approx -\frac{a_3^3}{3\rho_f} f_1 \partial_t \rho_i e^{ikr} - \frac{a_3^3}{2} \nabla \cdot \left( f_2 \mathbf{v} i e^{ikr} \right),
\]

\[
f_1 = 1 - \frac{\kappa_i}{\kappa_f}, \quad f_2 = \frac{2 \rho_s - 2 \rho_f}{2 \rho_s + \rho_f},
\]

where \(\rho_f\) and \(\mathbf{v} i\) denote the value of incident density and velocity fields, respectively, \(a_3\) denotes the sphere radius, \(r\) is the radial distance measured from the center, \(\rho_i\) and \(\kappa_i\) denote the density and compressibility of the scatterer, respectively, and \(k = \omega/c_f\) denotes the wave number. To generalize this approach to arbitrary small objects, including those exhibiting Willis coupling, we employ the method of multipole moments, up to dipole accuracy, and express the scattered pressure as \[40, 44\]

\[
p_s \approx -\omega^2 \Omega M G + \omega^2 \mathbf{v} \cdot \left[ \mathbf{D} \mathbf{G} \right], \quad G = G(kr) = \frac{e^{ikr}}{4\pi kr},
\]

\[
\phi_s = \frac{\rho_s}{j\rho_f^2} \omega.
\]

where \(G\) denote the Green’s function in 3D domain, and \(M\) and \(D\) denote the volumetric monopole and dipole moments, respectively. The term \(a_3^3\) from Eq. (4) is generalized to the volume of the object, denoted \(\Omega\). Furthermore, a characteristic size, denoted by \(a\), is required for non-spherical shapes to calculate the Rayleigh index of \(ka\). For this study, we propose \[6\]

\[
a = 3 \frac{\Omega}{\Gamma},
\]

where \(\Gamma\) is the outer surface area. This allows us to incorporate the volume and surface area as two measures of three-dimensional geometries in our numerical analysis. A factor of 3 is included to normalize \(a\) to the radius \(a_i\) for the case of a spherical object.

The scattering moments are expressed in terms of incident pressure \(p_i\) and velocity \(v_i\) fields through the polarization tensor \(\alpha\) as follows,

\[
\begin{bmatrix}
M
\end{bmatrix} = \frac{1}{\Omega} \begin{bmatrix}
\alpha_{pp} & \alpha_{pv}^T \\
\alpha_{vp} & \alpha_{vv}
\end{bmatrix} \begin{bmatrix}
p_i \\
v_i
\end{bmatrix}, \quad \alpha_{pp} = \left[ \begin{array}{c}
\alpha_{pp} \\
\alpha_{pv} \end{array} \right], \quad \alpha_{vv} = \left[ \begin{array}{c}
\alpha_{vv} \\
\alpha_{vy} \end{array} \right]
\]

\[
\alpha_{vp} = \left[ \begin{array}{c}
\alpha_{vp} \\
\alpha_{vp}
\end{array} \right], \quad \alpha_{vv} = \left[ \begin{array}{c}
\alpha_{xx} \alpha_{yy} \alpha_{zz} \\
\alpha_{xy} \alpha_{yz} \alpha_{zx}
\end{array} \right]
\]

where \(\alpha_{pp}\) and \(\alpha_{vp}\) denote the Willis coupling coefficients, \(\alpha_{vv}\) denotes direct-dipole polarization tensor, and superscript T denotes the transpose operator. The Cartesian form of these sub-tensors are given in Eq. (7). The entries of \(\alpha\) are often normalized as follows \[40\],

\[
\begin{bmatrix}
-\sqrt{3} M_jkD
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
\alpha_{pp} & \alpha_{pv}^T \\
\alpha_{vp} & \alpha_{vv}
\end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix}
p_i \\
v_i
\end{bmatrix}, \quad \alpha = \left[ \begin{array}{c}
-3\alpha_{pp} \frac{\sqrt{3}}{\rho_f^2} \alpha_{pv}^T \\
\frac{\sqrt{3}}{\rho_f^2} \alpha_{vv}
\end{array} \right].
\]

\[
\begin{bmatrix}
1
\end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix}
\alpha_{pp} & \alpha_{pv}^T \\
\alpha_{vp} & \alpha_{vv}
\end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix}
p_i \\
v_i
\end{bmatrix}, \quad \alpha = \left[ \begin{array}{c}
-3\alpha_{pp} \frac{\sqrt{3}}{\rho_f^2} \alpha_{pv}^T \\
\frac{\sqrt{3}}{\rho_f^2} \alpha_{vv}
\end{array} \right].
\]

\[
\begin{bmatrix}
1
\end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix}
\alpha_{pp} & \alpha_{pv}^T \\
\alpha_{vp} & \alpha_{vv}
\end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix}
p_i \\
v_i
\end{bmatrix}, \quad \alpha = \left[ \begin{array}{c}
-3\alpha_{pp} \frac{\sqrt{3}}{\rho_f^2} \alpha_{pv}^T \\
\frac{\sqrt{3}}{\rho_f^2} \alpha_{vv}
\end{array} \right].
\]

where \(\Gamma\) is the outer surface area. This allows us to incorporate the volume and surface area as two measures of three-dimensional geometries in our numerical analysis. A factor of 3 is included to normalize \(a\) to the radius \(a_i\) for the case of a spherical object.

The scattering moments are expressed in terms of incident pressure \(p_i\) and velocity \(v_i\) fields through the polarization tensor \(\alpha\) as follows,
By employing the reciprocity principle for Green’s function, it has been proven that [40]
\[ \vec{\alpha} = \alpha^T = \left( \begin{array}{cc} \alpha_{pp} & -\alpha_{pr}^T \\ -\alpha_{rp} & \alpha_{vv} \end{array} \right). \] (9)

This yields the relation between the Willis coupling coefficients \( \alpha_{rr} = -\alpha_{pr} \), which are later required for the formulation of the acoustic radiation force and the acoustic radiation torque. By substituting Eq. (7) into Eq. (5) and comparing the results with Eq. (4), the monopole and dipole moments for a spherical scatterer become [2] [24]
\[ \begin{bmatrix} M \\ D \end{bmatrix}_{\text{sphere}} = \frac{1}{\Omega} \left( \begin{array}{cc} \Omega \rho_f k_f f_1 & 0 \\ 0 & -\frac{3\Omega}{2\rho_f} \rho_f f_2 I \end{array} \right) \begin{bmatrix} p_i \\ v_i \end{bmatrix}, \] (10)
where \( 0 \) and \( I \) denote the zero column vector and the identity matrix of size three, respectively. The polarizability tensor can be expressed as sum of two tensors, as follow,
\[ \alpha = \alpha_{\text{sym}} + \alpha_{\text{asym}} = \begin{bmatrix} \alpha_{pp} & \alpha_{pr}^T \\ 0 & \alpha_{vv} \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_{rp} \end{bmatrix}, \] (11)
where \( \alpha_{\text{sym}} \) denotes the tensor that only includes the direct polarizability coefficients, and \( \alpha_{\text{asym}} \) denotes the tensor of polarization arising from pure Willis coupling effect. We will utilize this decomposition later to characterize the role of direct and Willis coupling coefficients in determining the acoustic radiation force and radiation torque. Subscripts \( \text{sym} \) and \( \text{asym} \) refers to direct and Willis coupling polarizability hereinafter.

Considering the Green’s function identity \( \left( \nabla^2 + k^2 \right) G = -\delta(r) \), one can write,
\[ \left( \nabla^2 - \frac{1}{c_f^2} \partial_t \right) \phi_i \approx -\frac{j\omega}{\rho_f} \Omega M \delta + \frac{j\omega}{\rho_f} \nabla \cdot [\Omega \mathbf{D} \delta] \] (12)
where \( \delta \) denotes the Dirac delta impulse, and \( r \) denotes the distance from the center of the smallest sphere enclosing the scatterer. We will use Eq. (12), which gives the approximation of the scattered field by a set of monopole and dipole sources, to derive the acoustic radiation force and torque.

B. Acoustic Radiation Force in the Rayleigh Limit

Using Gorkov’s far-field approach [2] [19] [24] [45], as shown in Supplementary Notes I, the acoustic radiation force acting on a sub-wavelength particle can be expressed as follows,
\[ \mathbf{F} = -\rho_f \int_{\Omega_x} \left\langle v_i \left[ \nabla^2 - \frac{1}{c_f^2} \partial_t \right] \phi_i \right\rangle d\Omega, \] (13)
where \( \Omega_x \) denotes the unbounded volume of the fluid domain, \( \left\langle \cdot \right\rangle \) denotes the time-averaging operator over one wave period. For any two harmonically varying fields \( \mathcal{F} \) and \( \mathcal{G} \), the time-averaged product is \( \left\langle \mathcal{F} \mathcal{G} \right\rangle = \Re \left[ \mathcal{F} \mathcal{G}^* \right] \) with \( \Re \) denoting the real part of a complex quantity and \( * \) denoting the complex conjugation operator. By substituting Eq. (12) into (13) and making use of the properties of the Dirac delta impulse [2] [24], the force expression changes to
\[ \mathbf{F} = \left\langle j\omega\Omega \mathbf{D} \cdot \nabla v_i \right\rangle_{r=0} + \left\langle j\omega\Omega \mathbf{E} \cdot \nabla v_i \right\rangle_{r=0}. \] (14)
Substituting Eq. (7) into (14), the radiation force expression expands in terms of incident fields to
\[ \mathbf{F} = \left\langle j\omega\alpha_{pp} p_i v_i \right\rangle_{r=0} + \left\langle j\omega\alpha_{pr} v_i \nabla v_i \right\rangle_{r=0} + \left\langle j\omega\alpha_{vp} v_i \nabla v_i \right\rangle_{r=0}. \] (15)
By using \( \nabla p_i = j\omega p_i v_i \), \( \left\langle j\omega \mathcal{F} \mathcal{G} \right\rangle = -\left\langle (\partial_t \mathcal{F}) \mathcal{G} \right\rangle = -\left\langle \mathcal{F} (\partial_t \mathcal{G}) \right\rangle \) and rearranging the terms, the force expression reads
\[ \mathbf{F} = -\left\langle \frac{\alpha_{pp}}{2\rho_f} p_i \nabla p_i \right\rangle_{r=0} - \left\langle \frac{1}{\rho_f} \alpha_{pr} \nabla \cdot [p_i \nabla v_i] \right\rangle_{r=0} + \left\langle j\omega \alpha_{vp} v_i \nabla v_i \right\rangle_{r=0}. \] (16)

Considering from (8) and Eq. (9), the relation between Willis coupling coefficients \( \alpha_{pr} = j\rho_f \alpha_{pp} \), the force expression simplifies further to
\[ \mathbf{F} = -\left\langle \frac{\alpha_{pp}}{2\rho_f} \nabla \left[ p_i^2 \right] \right\rangle_{r=0} + \left\langle j\omega \alpha_{vp} v_i \nabla v_i \right\rangle_{r=0} + \left\langle \frac{1}{\rho_f} \alpha_{pr} \cdot [p_i \nabla v_i - v_i \nabla p_i] \right\rangle_{r=0}. \] (17)
This general expression of the radiation force holds true for an object of arbitrary shape and of sub-wavelength size.

1. Acoustic radiation force of a spherical object

Due to the symmetry, the Willis coupling terms \( \alpha_{pv} = \alpha_{vp} \), \( \alpha_{pp} \), \( \alpha_{vp} \) become zero. Furthermore, \( \alpha_{pv} = \alpha_{vp} = \alpha_{vp} = \alpha_{pp} \), hence, the second term on the right-hand side of Eq. (17) changes to
\[ \left\langle j\omega \alpha_{vp} v_i \cdot \nabla v_i \right\rangle_{r=0} = \left\langle j\omega \alpha_{vp} \nabla \left[ v_i^2 \right] \right\rangle_{r=0}, \] (18)
where \( v_i \) is the magnitude of the velocity vector and \( v_i^2 = v_i \cdot v_i \). Finally, Eq. (17) becomes
\[ \mathbf{F} = -\left\langle \frac{\alpha_{pp}}{2\rho_f} \nabla \left[ p_i^2 \right] \right\rangle_{r=0} + \left\langle j\omega \alpha_{vp} \nabla \left[ v_i^2 \right] \right\rangle_{r=0}. \] (19)
Substituting Eq. (10) into Eq. (19), one could derive the Green’s function potential, which is the basis of radiation force potential theory [2] [19] [24] [45]. \( \mathbf{F} = -\nabla G \) with
\[ G = \frac{4\pi a^3}{3} \left\langle \int f_1 \frac{k_f}{2} f_2 \frac{3}{4} \rho_f v_i^2 \right\rangle. \] (20)
2. Nonspherical scatterer with rotational and mirror symmetry

Next, we look at non-spherical geometries which still hold axial or mirror symmetries, such as prolate (elongated) and oblate (flattened) spheroids. Again, due to mirror symmetry, the Willis Coupling coefficients become zero. The three terms on the diagonal of $\alpha_{ij}$ are no longer all equal and Eq. (18) is invalid for an incident wave with an arbitrary 3D wave-front. However, for incident plane waves normal to the object’s planes of symmetry, Eq. (18) can be employed to derive the force potential since the force only acts in the direction of incidence. And, for nonspherical scatterers with rotational and mirror symmetry, the force expression becomes

$$\mathbf{F} = -\left\langle \frac{\alpha_{pp}}{2\rho_f} \nabla |p_f|^2 \right\rangle_{r=0} + \left\langle j\omega \alpha_{i_0} \nabla \phi \right\rangle_{r=0}. \quad (21)$$

The expression in Eq. (21) shows that the Gorkov force potential theory is no longer applicable owing to $\alpha_{ij} \nabla [v_i \cdot v_j]$, despite the rotational or mirror symmetry of the scatterer.

3. Acoustic radiation force for objects of arbitrary shape

A lack of symmetry in the shape of scatterer with respect to the incident field results in Willis coupling. Since generic shapes can always be found in real-life, it is important to investigate the role of shape complexity and asymmetry. For instance red and white blood cells and worm-like bacteria in bio-acoustophoretic applications or bianisotropic metamaterials in acoustic/photon beam forming, wave manipulation and holography show natural or engineered asymmetries, which are yet to be investigated in the context of acoustic radiation force and radiation torque. Equation (17) applies to this general case as long as the characteristic length of the object is within the Rayleigh limit. Finally, it is evident from Eq. (17) that Gorkov potential for acoustic radiation force is inapplicable to the general case of objects with arbitrary shapes; hence, the force is required to be calculated directly from Eq. (17).

C. Acoustic Radiation Torque

Acoustic radiation torque $\mathbf{T}$ is obtained from the radiation stresses using the far-field approach, as shown in Supplementary Notes I [21][27][28][56], as follows,

$$\mathbf{T} = -\rho_f \int_{\Omega_m} \mathbf{x} \times \left\langle \mathbf{v_i} [\nabla^2 - \frac{1}{c_f^2} \partial_{tt}] \phi \right\rangle d\Omega. \quad (22)$$

where $\mathbf{x}$ denote the position vector. Substituting Eq. (12) into Eq. (22) and using $\mathbf{x} \times \mathbf{n} = 0$, the torque expression becomes

$$\mathbf{T} = \left\langle j\omega \mathbf{p} \mathbf{D} \times \mathbf{v_i} \right\rangle_{r=0}. \quad (23)$$

This expression is the same as Eq. (11) in reference [47], in which the radiation force and the radiation torque were derived from their canonical momentum and spin densities. Finally, using Eq. (7), we obtain the radiation torque for an arbitrarily shaped scatterer, as follows,

$$\mathbf{T} = \left\langle j\omega \rho \alpha_{ip} \mathbf{p} \times \mathbf{v_i} \right\rangle_{r=0} + \left\langle j\omega \mathbf{a}_{ip} \mathbf{v_i} \times \mathbf{v_i} \right\rangle_{r=0}. \quad (24)$$

This formulation of radiation torque not only gives the term corresponding to the spin density, which is proportional to $\langle \mathbf{v_i} \times \mathbf{v_i} \rangle$ [47], but also shows the role of Willis coupling effects distinctively. Supplementary Notes II outlines the details of how to calculate the polarizability tensor from the scattering of incident standing waves in three dimensions using Boundary Element Method.

III. RESULTS

A. Case of Standing Plane Wave

The incident pressure, velocity and their derivative fields are expressed as

$$p_i = P_e \cos(kz)e^{-j\omega t}, \quad \mathbf{v_i} = \frac{jP_e}{\rho_f} \sin(kz)e^{-j\omega t} \mathbf{e}_z,$$

$$\nabla \mathbf{p}_i = -P_e k \sin(kz)e^{-j\omega t} \mathbf{e}_z, \quad \nabla \mathbf{v}_i = \left( \frac{jP_e}{\rho_f} \right) k \cos(kz)e^{-j\omega t} \mathbf{e}_z, \quad (25)$$

Without loss of generality, the propagation direction is assumed to be along the z-axis, denoted by $\mathbf{e}_z$. Substituting Eq. (25) into Eq. (17) and time averaging, the force and torque acting on an arbitrarily shaped object become

$$\mathbf{F} \cdot \mathbf{e}_z = F_{sym} + F_{asym},$$

$$F_{sym} = \frac{kE_i}{\rho_f} \left( \Re \left[ \alpha_{pp} \right] - k \frac{c_f}{\kappa_f} \Im \left[ \alpha_{vr} \mathbf{e}_z \cdot \mathbf{e}_z \right] \right) \sin(2kz),$$

$$F_{asym} = \frac{kE_i}{\rho_f} \frac{c_f}{\kappa_f} \Im \left[ 2 \alpha_{pr} \mathbf{e}_z \cdot \mathbf{e}_z \right] \cos(2kz),$$

$$\mathbf{T} = \mathbf{T}_{sym} + \mathbf{T}_{asym}, \quad (26)$$

$$\mathbf{T}_{sym} = -\frac{kE_i}{\rho_f} \frac{c_f}{\kappa_f} \Im \left[ 2 \alpha_{vr} \mathbf{e}_z \right] \times \mathbf{e}_z \sin^2(kz),$$

$$\mathbf{T}_{asym} = \frac{kE_i}{\rho_f} \left( \frac{1}{\kappa_f} \Re \left[ \alpha_{ip} \right] \times \mathbf{e}_z \right) \sin(2kz),$$

where $E_i = \frac{p_e^2}{4\rho_f c_f^2}$ denotes the acoustic energy density of the incident wave, and $\Im$ denotes the imaginary part of a complex variable. Subscripts sym and asym, previously defined in Eq. (11), refer to the contributions from direct and Willis coupling coefficients, respectively. $F_{sym}$ and $\mathbf{T}_{sym}$ denote partial force and torque terms that arise from the Willis coupling representation of shape complexity. The spatial dependence of partial force $F_{sym}$ is $\cos(2kz)$, which gives the stable zero-force location with negative force gradient at $\lambda/8$. Compared to $\mathbf{F}_{sym}$, which is classically referred to as acoustic radiation force, with $\sin(2kz)$ leading to the prediction of acoustic traps at pressure/velocity nodes under plane standing waves for sub-wavelength spherical and spheroidal particles.
the location of zero net force is shifted by up to $\lambda/8$ along the wave direction. The effects of geometrical complexity on the primary radiation force is the largest at pressure and velocity nodal planes, where $\sin(2kz) = 0$. Furthermore, the actual location of stable zero-force for acoustic traps in a plane standing wave, considering the contribution of Willis coupling, are shifted from the nodal locations as a result of the additional force induced by the Willis Coupling effects. However, this shift depends on how large the Willis coupling effect is. This result implies that it is possible to obtain anomalous force and torque fields by engineering the Willis coupling coefficients through shape manipulation. Finally, it is noted that changing the object symmetry also changes the $\alpha_{pp}$ and $\alpha_{vv}$, since a portion of the scattered energy goes to the Willis coupling effect.

**B. Case of Travelling Plane Wave**

For a travelling wave in the $z$-direction, the incident pressure, velocity and their derivative fields are expressed as

$$p_i = P_0 e^{ikz} e^{-i\omega t}, \quad v_i = \frac{P_0}{\rho_f c_f} e^{ikz} e^{-i\omega t} e_z,$$

$$\nabla p_i = -jkP_0 e^{ikz} e^{-i\omega t} e_z, \quad \nabla v_i = \frac{jP_0 k}{\rho_f c_f} e^{ikz} e^{-i\omega t} e_z e_z.$$ (27)

Substituting Eq. (27) into (17), the force and torque under a plane travelling wave becomes

$$F = \frac{2E_i}{\rho_f} \left( -3 \left[ \alpha_{pp} \right] e_z + kc_f \mathcal{R} \left[ \alpha_{vv} e_z \right] \right),$$

$$T = -\frac{2kE_i}{\rho_f} \left( \frac{1}{k} \mathcal{R} \left[ \alpha_{pp} \right] \times e_z + \frac{c_f}{3} \left[ \alpha_{vv} e_z \right] \times e_z \right).$$ (28)

This expression shows that the direct contribution of the Willis Coupling terms $\alpha_{pp}$ and $\alpha_{vv}$ to the force under traveling wave is zero. However, the force generally has also transverse components in $x$- and $y$- directions due to the $\alpha_{vv} e_z$ term, despite the incident wave’s one-dimensional propagation line.

**C. Numerical results**

The shapes of objects are constructed by adding a tail-like attachment (protrusion) to or by creating a hole (intrusion) in a sphere, to engineer a geometrical anti-symmetry in one direction. The circular tail/hole as shown in Fig. 2 are considered to generate a non-spherical and axisymmetric shape. The apparent symmetry is further reduced by considering the cross-section of the attachment being a square but having the same edge length as the diameter of the circular one; however, they both exhibit zero Willis coupling in the normal to the length directions due to the mirror symmetry. The later design was used to investigate the shape effects over a given range of size within the Rayleigh limit, which in practice is reasonably $ka < 0.3$.

We consider a standing plane wave and assume that objects are sound hard and immovable, allowing us to focus only on the effects of scatterer’s exterior shape. An incident pressure field with 10 mm wavelength in air with $c_f = 343,140 \text{ m/s}$ and $\rho_f = 1.204 \text{ kg/m}^3$ is considered. The acoustic radiation force and torque results are normalized, as follows,

$$Q = \frac{F}{E_i k \Omega}, \quad Z = \frac{T}{E_i \Omega},$$ (29)

where $Q$ and $Z$ are dimensionless vector quantities, as shown in Fig. 1. For one dimensional problems such as a sphere in plane waves, these quantities reduce to a scalar value, which has been referred to as force contrast factor an torque contrast factor, respectively, and used to determine the direction of the force and torque.

To study the effect of asymmetry in the axial direction $z$, the objects are perturbed from sphere by adding a tail or hole with square cross section. The non-zero polarizability coefficients are shown in Fig. 3 which indicate that the real parts of Willis coupling coefficients are negligible for the small object sizes considered here. For shapes with $z$-symmetry, i.e. the reference sphere and the case with a hole all the way through, both
real and imaginary parts of the Willis coupling coefficients become zero, marked by the red circles on the horizontal axis in Fig. 3. The polarizability coefficients are larger for the cases with the tail, added material volume (+), than those with the hole, subtracted material volume (−). Considering Eq. (26) which shows the relation between polarizability coefficients and the acoustic radiation force and torque, it is expected that addition of a tail produces larger magnitude of the torque than the cases with a hole. Finally, a comparison between the cases with square and circular cross-sections is given in Supplemental Notes III, showing that the polarizability coefficients are smaller for the square cross section, for any given tail length or depth hole in the studied range of 0 to 2a_{s}.

To investigate the radiation force and torque further, normalized values Q_{x}, Q_{z} and Z_{y} are shown in Fig. 4 for the object oriented at $\pi/4$ with respect to the incidence direction, in the y-plane. This means that the radiation force has the same components in the x and z directions, and the torque has only one component in the y direction. The spatial dependence of the force and torque components are according to Eq. (26). The normalized values of the partial force $F_{sym}$, due to direct-polarization coefficients, in Fig. 4a show that the magnitude change is negligible compared to the reference sphere. However, it was observed that the cases with a tail experiences almost similar forces as the reference sphere. For those with a hole, the force increases with increasing hole depth, except for the case of through-hole, which shows a sudden force reduction. In contrast, the Willis-coupling force $F_{asym}$ varies more significantly as the tail length or the depth hole increases. The only exception is the case of through hole, $l_{h}/a_{s} = 2$, which gives zero Willis-coupling force due to z-symmetry. Nonetheless, the Willis-coupling force $F_{asym}$ is at least two orders of magnitude smaller than the direct-polarization force $F_{sym}$; hence, it could be neglected for estimating the radiation force for practical applications.

The results of torque contrast factor $Z_{y}$ in Fig. 4b-d shows that radiation torque is more sensitive to the deviation from spherical shape. The changes of $T_{sym}$ are larger for the case with the tail, experiencing a negative torque that aligns the tail with the direction of the incident wave vector. Those with a hole are subjected to a positive torque that tends to align the hole in the normal to the wave vector direction. The case with a through hole is an exception as it experiences a large negative torque, similar to the cases with a tail. These results imply that the partial torque $T_{asym}$, due to direct-polarization effect, tends to bring the object to the orientation with the smallest cross-section normal to the incident wave direction. Similar relation between object orientation and the radiation torque was observed for prolate and oblate spheroids 33.

However, as shown in Fig. 4d these non-spherical objects are also subjected to the Willis-coupling torque $T_{asym}$, as expressed in Eq. (26). Unlike the large difference between partial radiation forces, this partial torque is just smaller than the $T_{sym}$ by less than an order of magnitude. It was found
that $T_{\text{asym}}$ opposes and reinforces the $T_{\text{sym}}$ before and after the pressure node at $z/\lambda = 0.25$, respectively. Therefore, the orientation of the objects under the action of the radiation torque needs to be determined by accounting for both components. These observations are particularly of interest as it captures not only the contribution of Willis coupling, but also the significance of calculating radiation torques in capturing the essence of shape complexity, even for sub-wavelength objects with $ka \ll 1$.

The changes of polarizability response in terms of size were investigated for the case of a tail with square cross section and $l_t/\alpha_s = 2$. A scaling ratio from 1:1 to 1:10 was applied, indicating an increase of size by one order of magnitude, as shown by the $ka$ values in Fig. 5. It was found that both real and imaginary parts of the non-zero polarization coefficients increases with increasing size ratio. Real part of Willis-coupling coefficients $\alpha_{\text{pv}}$ increases by at least seven orders of magnitude, while the direct polarizability coefficients grow by three. For the imaginary parts in Fig. 5b, the growth of Willis-coupling is around four orders of magnitude while it is six
IV. DISCUSSION

In the presented formulation, the assumption of a sub-wavelength scatterer, \( ka < 1 \), led to the monopole-dipole approximation of the scattering field. For \( ka > 1 \), a more accurate approximation including the quadrupole and higher order multipole moments is required to obtain the analytical expressions of the acoustic radiation force and radiation torque. This could be achieved by incorporating the Willis coupling factors into the partial-wave expansion series and adjusting the scattering coefficients accordingly.

We also observed that the normalized force, also called contrast factor, shows far less variation across the presented range of \( ka \) values (Rayleigh index), when \( k\Omega \) is used instead of the choice of \( \pi a^2 \). The difference is because \( k\Omega \propto \pi a^2 \times ka \). In previous studies mainly focused on spheres or spheroids, the extra \( ka \) factor was always applied to the contrast factor, leading to a size dependence which bears no helpful information \([18, 22, 24, 49, 53]\). Therefore, it is concluded that the normalisation of radiation force and torque with the respect to volume, as expressed in Eq. (29), better indicates the dependence on the shape and size features when it comes to non-spherical objects.

The assumptions of sound-hard immovable scatterers were made to focus this study on the effects of shape and size. However, including the effects of material properties and wave refraction is straightforward, since these can be readily incorporated into the multipole moments of a scatterer. Our choice of numerical approach for solving scattering problem was Boundary Element Method, which was combined by the analytical multipole translation and rotation to obtain the multipole moments from the four reference cases of plane standing wave. The alternative approach is to use Finite Element Method, which is available in commercial software packages, and use the polarizability retrieval techniques, based on surface integral of scattering pressure with appropriate spherical harmonics as weight functions. Nevertheless, it is noted that the discretisation of the scatterer’s surface for either of these methods requires extra care to ensure a uniform distribution of element size and aspect ratio across the surface. Our results were obtained after a numerical convergence test to achieve the optimal element size. The BEM scattering results was verified by comparing against FEM simulation of pressure on a fictitious spherical surface at \( 4\lambda \) distance, with
The advantage of using the presented novel formulation is its ability to calculate acoustic radiation force and acoustic radiation torque straightaway from the polarizability tensor, as a measure of acoustic transfer function and being independent of the incident wave field. If non-planar acoustic waves such as Bessel or Gaussian beams are of interest, one could simply calculate the radiation force and torque from the incident field values at the centroid of the object. Moreover, the effect of the angle of incidence is easily included by rotation of the polarizability tensor. For sub-wavelength objects, our explicit force and torque expressions can be implemented into commercial software packages such as COMSOL for multiphysics simulations of acoustofluidic processes. Our results show for the first time that geometrical complexities have no effects on the radiation force that is induced by a plane traveling wave. However, in the case of two counter-propagating waves, which gives the plane standing wave scenario, the radiation force and radiation torque will be influenced by such shape effects. This is an indication of the non-linear nature of radiation force and radiation torque, for which the wave superposition principle becomes inapplicable.

FIG. 6: Variation of normalized radiation force and radiation torque in terms of size for the case of a sphere with a rectangular tail, oriented at $\pi/4$ in the $y$-plane. Solid lines indicate $F_{sym}$ and $T_{sym}$ and the other parts are shown by dashed lines.

V. CONCLUSION

The theory of acoustic radiation force and torque was revisited to incorporate the Willis coupling arising from the shape complexities of an arbitrary object in incident acoustic fields. The mathematical expression of the acoustic radiation force and radiation torque were provided in terms of polarizability coefficients, applicable for any choice of incident pressure field. As examples, we derived these expressions for plane standing and travelling waves to characterize the impact of Willis coupling and changes to the polarizability tensor as compared to the case of simple sphere. We found that radiation torque is more sensitive to the shape effects than radiation force. As the size factor $ka$ increases, the contribution of the Willis coupling effect becomes larger for the radiation torque, while it remains negligible to the radiation force. It is concluded that knowing the acoustic radiation torque becomes imperative when the geometry of the object is accounted for even within the Rayleigh limit. Furthermore, there will be locations in a standing wave with stable angular balance and zero radiation torque. The presented work outlined the significance of the acoustic radiation force and radiation torque in the analysis and design of acoustophoresis processes involving non-spherical objects.

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