WELL-POSEDNESS FOR A FAMILY OF PERTURBATIONS OF THE KdV EQUATION IN PERIODIC SOBOLEV SPACES OF NEGATIVE ORDER

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Abstract
We establish local well-posedness in Sobolev spaces $H^s(\mathbb{T})$, with $s \geq -1/2$, for the initial value problem issues of the equation

$$u_t + u_{xxx} + \eta Lu + uu_x = 0; \quad x \in \mathbb{T}, \ t \geq 0,$$

where $\eta > 0$, $(Lu)^\wedge(k) = -\Phi(k)\hat{u}(k)$, $k \in \mathbb{Z}$ and $\Phi \in \mathbb{R}$ is bounded above. Particular cases of this problem are the Korteweg-de Vries-Burgers equation for $\Phi(k) = -k^2$, the derivative Korteweg-de Vries-Kuramoto-Sivashinsky equation for $\Phi(k) = k^2 - k^4$, and the Ostrovsky-Stepanyans-Tsimring equation for $\Phi(k) = |k| - |k|^3$.

Keywords: Cauchy Problem, Local Well-Posedness, KdV equation.

1 Introduction
We consider the $\lambda$-periodic Cauchy problem for

$$\begin{cases}
u_t + u_{xxx} + \eta Lu + uu_x = 0, & x \in [0, \lambda], \ t \in [0, +\infty), \\
u(x, 0) = u_0(x),
\end{cases}$$

(1.1)

where $\eta > 0$ is a constant, the linear operator $L$ is defined via the Fourier transform by

$$(Lu)^\wedge(k) = -\Phi(k)\hat{u}(k), \quad \text{where} \quad k \in \mathbb{Z}/\lambda,$$

(1.2)

and the Fourier symbol $\Phi(k)$ is a real valued function which is bounded above; i.e., there is a constant $\alpha$ such that $\Phi(k) \leq \alpha$. We take $\alpha \geq 1$ without lost of generality.

Before stating the main result of this work we give some important examples that belong to the model considered in (1.1), where $u = u(x, t)$ is a real-valued function and $\eta > 0$ is a constant. The first example is the Korteweg-de Vries-Burgers equation

$$\begin{cases}
u_t + u_{xxx} - \eta u_{xx} + uu_x = 0, & t \geq 0, \\
u(x, 0) = u_0(x).
\end{cases}$$

(1.3)

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Molinet and Ribaud considered the initial value problem \((1.3)\) in \([17]\) and proved that it is globally well-posed for given data in \(H^s(\mathbb{R})\), \(s > -1\), and ill-posed in \(H^s(\mathbb{R})\) for \(s < -1\) in the sense that one cannot solve the Cauchy problem for \((1.3)\) by a Picard iterative method implemented on the integral formulation. They show that these results are also valid in the periodic setting. These results are surprising because the index \(s = -1\) is lower than the exponents \(s = -3/4\) and \(s = -1/2\) which are boundaries indexes that determine the Sobolev spaces where it is possible to obtain well-posedness results using a Picard iterative method implemented on the integral formulation for the KdV equation on \(\mathbb{R}\) and \(\mathbb{T}\), respectively. This was the first almost sharp result to a dispersive-dissipative equation using the Fourier restriction norm method or Bourgain method. It is not known what happen when \(s = -1\).

Other model that fits in the family \([11]\) is the derivative Korteweg-de Vries-Kuramoto Sivashinsky equation

\[
\begin{aligned}
  u_t + u_{xxx} + \eta(u_{xx} + u_{xxxx}) + uu_x &= 0, & t \geq 0, \\
  u(x, 0) &= u_0(x),
\end{aligned}
\tag{1.4}
\]

This equation arises as a model for long waves in a viscous fluid flowing down an inclined plane and also describes drift waves in a plasma (cf. \([9][21]\)). The equation \((1.4)\) is a particular case of Benney-Lin equation \([2][21]\), i.e.,

\[
\begin{aligned}
  u_t + u_{xxx} + \eta(u_{xx} + u_{xxxx}) + \beta u_{xxxxx} + uu_x &= 0, & x \in \mathbb{R}, & t \geq 0, \\
  u(x, 0) &= u_0(x),
\end{aligned}
\tag{1.5}
\]

when \(\beta = 0\). The initial value problem associated to \((1.4)\) was studied by Biagioni, Bona, Iorio and Scialom in \([3]\). They also determined the limiting behavior of solutions as the dissipation tends to zero. Biagioni and Linares proved global well-posedness for the initial value problem \((1.5)\) for initial data in \(L^2(\mathbb{R})\) in \([3]\). The Benney-Lin equation was studied by Chen and Li in \([8]\) using the Fourier restriction norm method or Bourgain method. Indeed, they proved that the initial value problem \((1.5)\) is globally well-posed in the Sobolev spaces \(H^s(\mathbb{R})\) for \(0 \geq s > -2\) and ill-posed in \(H^s(\mathbb{R})\) for \(s < -2\) in the sense that one cannot solve the Cauchy problem for \((1.5)\) by a Picard iterative method implemented on the integral formulation.

Another example of this type is the Ostrovsky-Stepanyams-Tsimring (OST) equation:

\[
\begin{aligned}
  u_t + u_{xxx} - \eta(\mathcal{H}u_x + \mathcal{H}u_{xx}) + uu_x &= 0, & t \geq 0, & p = 1, \\
  u(x, 0) &= u_0(x),
\end{aligned}
\tag{1.6}
\]

where \(\mathcal{H}\) denotes the Hilbert transform:

\[
\mathcal{H}f(x) = -\frac{1}{\pi} \text{pv.} \int f(y) \frac{dy}{x - y}
\tag{1.7}
\]

The equation \((1.6)\), with \(p = 1\), was derived by Ostrovsky-et al. in \([18]\) to describe the radiational instability of long waves in a stratified shear flow. The earlier well-posedness results for \((1.6)\), with \(p = 1\), can be found in \([11]\), for given data in \(H^s(\mathbb{R})\), local result when \(s > 1/2\) and global result for \(s \geq 1\). Carvajal and Scialom in \([7]\) considered the initial value problem \((1.6)\) in the real case and proved the local well-posedness results for given data in \(H^s(\mathbb{R})\), \(s \geq 0\) when \(p = 1, 2, 3\). They also obtained the global well-posedness results for data in \(L^2(\mathbb{R})\) with \(p = 1\). In \([11][12]\) Cui and Zhao obtained a low regularity result on the \((1.6)\) with \(p = 1\) by Fourier restriction norm method. Indeed, they proved that the initial value problem \((1.6)\) is locally well-posed in \(H^s(\mathbb{R})\) for \(s > -1\). Finally, Zhao in \([22]\) proved that \((1.6)\) is locally well-posed in \(H^s(\mathbb{R})\) for \(s > -5/4\).

The next Cauchy problem of a dissipative version of the KdV equation with rough initial data

\[
\begin{aligned}
  u_t + u_{xxx} + Lu + uu_x &= 0, & t \geq 0, \\
  u(x, 0) &= u_0(x),
\end{aligned}
\tag{1.8}
\]
where \( L = |\partial_x|^{2\gamma} \) is defined by a multiplier with symbol \(|k|^{2\gamma}\) and \( \gamma \geq 1 \), is other example that belongs to the class \((1.1)\). \((1.8)\) was studied by Han and Peng in \([14]\). They proved working in Bourgain type space the local and global well posedness results for Sobolev spaces \( H^s(\mathbb{R}) \) of negative order, and the order number is lower than the well known value \(-\frac{3}{4}\), i.e., \( s > -s_\gamma \), where \( s_\gamma \) denotes the boundary index and it is given by:

\[
\begin{cases}
\frac{3 - \gamma}{4 - 2\gamma}, & \text{if } 1 \leq \gamma \leq \frac{3}{2}, \\
\gamma, & \text{if } \gamma > \frac{3}{2}.
\end{cases}
\]

When \( \gamma = 1 \), this result agrees with that in \([17]\), and it improves the result obtained in \([16]\) in the case \( \gamma \geq 1 \).

Carvajal and Panthee proved in \([6]\) local well-posedness in Sobolev spaces \( H^s(\mathbb{R}) \) with \( s > -\frac{3}{4} \) to the initial value problem \((1.1)\) but only to the real case. In particular, they obtained that result when the symbol \( \Phi \) is given by

\[
\Phi(\xi) = \sum_{j=0}^{n} \sum_{l=0}^{2m} C_{l,j} \xi^l |\xi|^j; \quad C_{l,j} \in \mathbb{R}, \quad C_{2m,0} = -1.
\]

The examples above correspond to this case. They followed the theory developed by Bourgain \([5]\) and Kenig, Ponce and Vega \([15]\). They used the usual Bourgain’s space associated to the KdV equation instead of the Bourgain’s space associated to the linear part of the initial value problem \((1.28)\).

1.1 Notation and Main Result

We recall the theory developed by T. Tao in \([20]\). We define the Fourier transform of a function \( f \) defined on \([0,\lambda]\) by

\[
\hat{f}(k) = \int_0^\lambda e^{-2\pi i k x} f(x) \, dx
\]

and we have the Fourier inversion formula

\[
f(x) = \int e^{2\pi i k x} \hat{f}(k) (dk)_\lambda
\]

where \((dk)_\lambda\) is the normalized counting measure on \( \mathbb{Z}/\lambda \) given by

\[
\int a(k) (dk)_\lambda = \frac{1}{\lambda} \sum_{k \in \mathbb{Z}/\lambda} a(k).
\]

The usual properties of the Fourier transform hold:

\[
\|f\|_{L^2([0,\lambda])} = \|\hat{f}\|_{L^2((dk)_\lambda)} \quad \text{(Plancherel)},
\]

\[
\int_0^\lambda f(x) g(x) \, dx = \int \hat{f}(k) \hat{g}(k) (dk)_\lambda \quad \text{(Parseval)},
\]

\[
\int \hat{f}(k) \hat{g}(k) (dk)_\lambda = \int \hat{f}(k-k_1) g(k_1) (dk_1)_\lambda \quad \text{(Convolution)},
\]

and so on. If we apply \( \partial_x^m \), \( m \in \mathbb{N} \), to \((1.10)\), we obtain

\[
\partial_x^m f(x) = \int e^{2\pi i k x} (2\pi i k)^m \hat{f}(k) (dk)_\lambda.
\]
This, together with (1.12), motivates us to define the Sobolev space $H^s([0,\lambda])$ with the norm
\[
\|f\|_{H^s([0,\lambda])} = \left\| \langle k \rangle^s \hat{f}(k) \right\|_{L^2((dk)_{\lambda})}.
\] (1.16)

We will often denote this space by $H^s_{\lambda}$ for simplicity. For a function $v = v(x,t)$ which is $\lambda$-periodic with respect to the $x$ variable and with the time variable $t \in \mathbb{R}$, we define the space-time Fourier transform $\hat{v} = \hat{v}(k,\tau)$ for $k \in \mathbb{Z}/\lambda$ and $\tau \in \mathbb{R}$ by
\[
\hat{v}(k,\tau) = \int_\mathbb{R} \int_0^\lambda e^{-2\pi i k x} e^{-2\pi i \tau t} v(x,t) \,dx \,dt.
\] (1.17)

This transform is inverted by
\[
v(x,t) = \int_\mathbb{R} \int_0^\lambda e^{2\pi i k x} e^{2\pi i \tau t} \hat{v}(k,\tau) \,(dk)_{\lambda} \,d\tau.
\] (1.18)

Similarly, $\hat{v}(k,\tau)$ and $\hat{v}(x,\tau)$ will denote the Fourier transform of $v(x,t)$ respect to the variables $x$ and $t$, respectively. $C$ will be denote a positive constant which may be different even in a single chain of inequalities. If $X,Y$ are Banach spaces, $\mathcal{B}(X;Y)$ is the space of the linear continue operators of $X$ in $Y$ with the norm $\|T\|_{\mathcal{B}(X;Y)} = \sup_{\|x\|_X = 1} \|Tx\|_Y$. If $X = Y$ we will write $\mathcal{B}(X)$ inside of $\mathcal{B}(X;X)$. The solution to the linear KdV equation:
\[
\begin{cases}
  u_t + u_{xxx} = 0, & x \in [0,\lambda], \quad t \in \mathbb{R}, \\
  u(x,0) = u_0(x),
\end{cases}
\] (1.19)
is given by
\[
u(x,t) = U_\lambda(t) u_0(x) = \int \int_\mathbb{R} e^{2\pi i k x} e^{-2\pi i k^3 t} \hat{u}_0(k) \,(dk)_{\lambda},
\] (1.20)

which may be rewritten as a space-time inverse Fourier transform,
\[
U_\lambda(t) u_0(x) = \int_\mathbb{R} \int_0^\lambda e^{2\pi i \tau t} e^{2\pi i k x} \delta(\tau - 4\pi^2 k^3) \hat{u}_0(k) \,(dk)_{\lambda} \,d\tau,
\] (1.21)

where $\delta(k)$ represents a 1-dimensional Dirac mass at $\kappa = 0$. This shows that $U_\lambda(\cdot) u_0$ has its space-time Fourier transform supported precisely on the cubic $\tau = 4\pi^2 k^3$ in $\mathbb{Z}/\lambda \times \mathbb{R}$. So, we recall the known Bourgain’s space associated to the KdV equation. For $s,b \in \mathbb{R}$, we define the $\mathcal{Y}_{s,b}([0,\lambda] \times \mathbb{R})$ spaces for $\lambda$-periodic KdV via the norm
\[
\|u\|_{\mathcal{Y}_{s,b}([0,\lambda] \times \mathbb{R})} \equiv \left\| \langle \tau - 4\pi^2 k^3 \rangle^b \langle k \rangle^s \hat{u}(k,\tau) \right\|_{L^2((dk)_{\lambda})L^2_x} = \left\| \langle \tau \rangle^b \langle k \rangle^s (U_\lambda(-t)u)(k,\tau) \right\|_{L^2((dk)_{\lambda})L^2_x} = \left( \int \int_\mathbb{R} \langle \tau \rangle^{2b} \langle k \rangle^{2s} \left| (U_\lambda(-t)u)(k,\tau) \right|^2 \,d\tau \,(dk)_{\lambda} \right)^{1/2}.
\] (1.22)

**Remark 1.1.** The spatial mean $\int_x u(x,t) \,dx$ is conserved during the evolution of the KdV equation. We may assume that the initial data $\phi$ satisfies a mean-zero assumption $\int_x \phi(x) \,dx$ since otherwise we can replace the dependent variable $u$ by $v = u - \int_x \phi$ at the expense of a harmless linear first order term. This observation was used by Bourgain in [3].

Since the $\lambda$-periodic initial value problem for KdV is equivalent to the integral equation
\[
u(t) = U_\lambda(t) \phi - \frac{1}{2} \int_0^t U_\lambda(t-t') \partial_x (u^2(t')) \,dt',
\] (1.23)

the study of periodic KdV in [5] and [15] was based in solve (1.23) using the contraction principle in the Bourgain’s spaces $\mathcal{Y}_{s,1/2}$ which was possible in virtue of the optimal bilinear estimate for $\partial_x u^2$, from Kenig, Ponce and Vega in the periodic case.
The Fourier inversion formula (1.10) allows us to write the solution of (1.29): zero solution. So, we wish to solve the linear homogeneous

\[ \lambda \] the companion space \( Z \)

Tao, in [10], introduced the slightly smaller space \( \Psi \) defined via the norm

Note that, if \( u \in Y^s \), then \( u \in L^\infty_t H^s_x \). Thus, they solve the integral equation (1.23) based around the iteration in the space \( Y^s \). They obtained the bilinear estimate for \( \partial_x u^2 \):

**Proposition 1.2.** If \( u \) and \( v \) are \( \lambda \)-periodic functions of \( x \), also depending upon \( t \) having zero \( x \)-mean for all \( t \), then

\[
\| \Psi(t) \partial_x (uv) \|_{L^1_x \cap L^2_t} \lesssim \lambda^0 \| u \|_{Z_{-1/2} \cap L^2_t} \| v \|_{Z_{-1/2} \cap L^2_t},
\]

where \( \Psi \in C^0_0(\mathbb{R}) \) is a cut-off function such that \( 0 \leq \Psi(t) \leq 1 \) and is supported on \([-2,2]\) with \( \Psi = 1 \) on \([-1,1]\).

**Remark 1.2.** Note that (1.20) implies \( \| \Psi(t) \partial_x (uv) \|_{Z_{-1/2} \cap L^2_t} \lesssim \lambda^0 \| u \|_{Y^{-1/2} \cap L^2_t} \| v \|_{Y^{-1/2} \cap L^2_t} \).

So, Colliander, Keel, Staffilani, Takaoka and Tao in [10] reproved that the initial value problem for KdV on \( \mathbb{T} \) is locally well-posed for \( s \geq -1/2 \). Our interest here is to obtain well-posedness results for the \( \lambda \)-periodic initial value problem (1.1) with given data \( u_0 \) in the Sobolev space \( H^s \) of negative order:

**Theorem 1.1** (Main Result). The initial value problem (1.1) with \( \eta > 0 \) and \( L \) given by (1.2) is locally well-posed for any data \( u_0 \in H^s(\mathbb{T}) \), for \( s \geq -1/2 \).

To prove this theorem we use Bourgain’s type space. So, we should be able to write (1.1) for all \( t \in \mathbb{R} \). For this, we define

\[
\eta(t) \equiv \eta \text{ sgn}(t) = \begin{cases} 
\eta, & \text{if } t \geq 0, \\
-\eta, & \text{if } t < 0,
\end{cases}
\]

and write (1.1) in the form

\[
\begin{cases} 
u + u_{xxx} + \eta(t) L u + u u_x = 0, & x \in [0, \lambda], \ t \in \mathbb{R}, \\
u(x,0) = u_0(x).
\end{cases}
\]

We first want to build a representation formula for the solution of the linearization of (1.1) about the zero solution. So, we wish to solve the linear homogeneous \( \lambda \)-periodic initial value problem

\[
\begin{cases} w + w_{xxx} + \eta(t) L w = 0, & x \in [0, \lambda], \ t \in \mathbb{R}, \\
w(x,0) = w_0(x).
\end{cases}
\]

The Fourier inversion formula (1.10) allows us to write the solution of (1.29):}

\[
w(x,t) = V_\lambda(t) w_0(x) = \int e^{2\pi i kx} e^{-(2\pi i k)^3 t + \eta \Phi(k)|t|} \hat{w}_0(k)(dk) \lambda.
\]

(1.30)
Observe that, defining $\tilde{U}_\lambda(t)$ by
\[
(\tilde{U}_\lambda(t)u_0)(k) = e^{\eta|t|\Phi(k)}\hat{u}_0(k), \quad k \in \mathbb{Z}/\lambda,
\]
the semigroup $V_\lambda(t)$ can be written as $V_\lambda(t) = U_\lambda(t)\tilde{U}_\lambda(t)$ where $U_\lambda(t)$ is the unitary group of the KdV \cite{[12]}. We next find a representation for the solution of the linear inhomogeneous $\lambda$-periodic initial value problem
\[
\begin{cases}
v_t + v_{xxx} + \eta(t)Lv = F, & x \in [0, \lambda], \quad t \in \mathbb{R}, \\
v(x,0) = 0,
\end{cases}
\tag{1.31}
\]
with $F = F(x, t)$ a given time-dependent $\lambda$-periodic (in $x$) function. By Duhamel’s principle,
\[
v(x, t) = \int_0^t V_\lambda(t-t') F(x, t') \, dt'.
\tag{1.32}
\]
We apply (1.30), rewrite $\hat{F}(k, t')$ using the Fourier inversion formula in the time variable and rearrange integrations to find
\[
v(x, t) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\pi i k x} e^{2\pi i (4\pi^2 k^3) t + \eta(\tau) \Phi(\tau) t} \int_0^t e^{2\pi i (\tau - 4\pi^2 k^3) - \eta(t) \Phi(k) t'} \, dt' \hat{F}(k, \tau) \, (dk) \lambda \, d\tau.
\tag{1.33}
\]
Performing the $t'$-integration, we find
\[
v(x, t) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\pi i k x} e^{2\pi i (4\pi^2 k^3) t + \eta(\tau) \Phi(\tau) t} \left(\frac{1}{2\pi i (\tau - 4\pi^2 k^3) - \eta(t) \Phi(k)}\right) \hat{F}(k, \tau) \, (dk) \lambda \, d\tau.
\tag{1.34}
\]
Then, the $\lambda$-periodic initial value problem for (1.28) is equivalent to the integral equation
\[
u(t) = V_\lambda(t)u_0 - \frac{1}{2} \int_0^t V_\lambda(t-t') \partial_x (u^2(t')) \, dt'.
\tag{1.35}
\]
The integral equation (1.35) can be solved using the contraction principle in the space $Y_{s,1/2}$ following the ideas of Carvajal and Panthee in \cite{[5]}. The main difficulty to resolve it of this way is the periodic bilinear estimate for $\partial_x u^2$, given in the Proposition 1.1 because it’s very restrictive compared with the bilinear estimate (see Theorem 1.1 in \cite{[13]}) of the real case in which $b \in (1/2, 1)$. So, we shall obtain a refined estimate of the forcing term of the integral equation associated to (1.28) which will permits us to use the Proposition 1.1 to solve (1.35). This refinement is made in the Proposition 3.1 but we prove our main result via the contraction principle in the space $Y^s$ and the bilinear estimate (1.26) from Colliander, Keel, Staffilani, Takaoka and Tao.

The layout of this paper is as follows. In Section 2 we present some basic results. In Section 3 we give the boundedness results for linear operators involving the spaces $Y^s$, $Z^s$ and $Y_{s,1/2}$. The proof of the main Theorem 1.1 will be given in Section 4.

2 Preliminary Results

Lemma 2.1. Let $a \leq 0$, $\psi \in C_0^\infty$ with support in $[-2, 2]$ and $\psi_T(t) = \psi(t/T)$. Then,
\[
\left\|\psi_T(t) \int_0^t e^{a|t-x|} g(x) \, dx \right\|_{L^2} \leq \frac{C(1 + T)}{1 + |a|} \|g\|_{L^2}.
\tag{2.1}
\]
If $g(0) = 0$,
\[
\left\|\psi_T(t) \frac{d}{dt} \int_0^t e^{a|t-x|} g(x) \, dx \right\|_{L^2} \leq \frac{C(1 + T)}{1 + |a|} \left\|\frac{dg}{dt}\right\|_{L^2},
\tag{2.2}
\]
Proposition 2.1. Let operation on the spaces have

We are going to argue by duality to obtain (2.1). We take

Proof. Let us define

Then,

Remark 2.1. We consider a cut-off function \( \Psi \in C^\infty(\mathbb{R}) \), such that \( 0 \leq \Psi(t) \leq 1 \),

Let us define \( \Psi_T(t) = \Psi(\frac{t}{T}) \) and \( \widetilde{\Psi}_T(t) = \text{sgn}(t) \Psi_T(t) \). Note that multiplication by \( \Psi(t) \) is a bounded operation on the spaces \( Y_s^a, Z^a \) and \( J_{a,b} \).

The next result will allow us to prove the Lemma 3.5 and to reduce the proof of (3.55).
We obtain (2.6) when \( b = 0 \) as consequence of (2.7), (2.1) and
\[
\left\| \alpha_1 \text{sgn}(t) \Psi_T(t) \int_0^t e^{a|t-x|} g(x) \, dx \right\|_{L^2} \leq \frac{\left| \alpha_1 \right| C (1 + T)}{1 + |\alpha_1| + |\alpha_2|} \| \Psi_{2T} g \|_{L^2}.
\]
Now, we are going to obtain (2.6) when \( b = 1 \). We know from (2.7) that
\[
\left\| \Psi_T(t) \int_0^t e^{a|t-x|} f(x) \, dx \right\|_{H^1} \leq \| \Psi_T(t) g(t) \|_{H^1} + |\alpha_1| \left\| \text{sgn}(t) \Psi_T(t) \int_0^t e^{a|t-x|} g(x) \, dx \right\|_{H^1}.
\]
(2.8)
Since \( \| \Psi_T g \|_{H^1} \leq C \| \Psi_{2T} g \|_{H^1} \), by virtue of (2.3) it is sufficient to estimate
\[
\left\| \frac{d}{dt} \left( \Psi_T(t) \int_0^t e^{a|t-x|} g(x) \, dx \right) \right\|_{L^2}
\]
which is bounded by
\[
\left\| \frac{d}{dt} \left( \Psi_T(t) \int_0^t e^{a|t-x|} g(x) \, dx \right) \right\|_{L^2} + \left\| \Psi_T(t) \frac{d}{dt} \left( \int_0^t e^{a|t-x|} g(x) \, dx \right) \right\|_{L^2}.
\]
(2.9)
For the first term above we can apply (2.1) and
\[
\left\| \frac{d}{dt} \left( \int_0^T e^{a|t-x|} (\Psi_{2T} g)(x) \, dx \right) \right\|_{L^2} \leq \frac{C(T + 1)}{1 + |a|} \| \Psi_{2T} g \|_{L^2} \leq \frac{C(T + 1)T}{1 + |a|} \left\| \frac{d}{dt} (\Psi_{2T} g) \right\|_{L^2},
\]
where, in the last inequality, it was used that
\[
\| \Psi_{2T} g \|^2_{L^2} = \int_{-4T}^{4T} \| \Psi_{2T} g(t) \|^2 \, dt \leq CT \| \Psi_{2T} g \|^2_{L^\infty} \leq CT \| \Psi_{2T} g \|_{L^2} \left\| \frac{d}{dt} (\Psi_{2T} g) \right\|_{L^2}.
\]
For the second term from (2.9) we used (2.2) with \( \Psi_{2T} g \) instead of \( g \) because \( g = \Psi_{2T} g \) on \([-T, T]\). This implies (2.6) when \( b = 1 \). The result (2.6) is obtained interpolating the cases \( b = 0 \) and \( b = 1 \).

The following Lemma plays a central role estimating the free term of the integral equation (1.3). This Lemma allows us to work in the usual \( Y_{s,1/2} \) space associated to the KdV equation.

**Lemma 2.2.** Let \( 0 < T \lesssim 1 \) and \( a \leq \alpha \). Then we have
\[
\left\| \Psi_T(\cdot) \right\|_{H^b} \leq C (T^{1/2} + T^{1/2-b}) \quad \forall b \geq 0,
\]
(2.10)
\[
\left\| \Psi_T(\cdot)e^{a|\cdot|} \right\|_{H^{1/2}} \leq C e^{2a},
\]
(2.11)
\[
\left\| \Psi_T(\cdot)e^{a|\cdot|} \right\|_{L^1} \leq C e^{2a},
\]
(2.12)
\[
\left\| (t|\Psi_T(t)e^{a|t|})^\wedge(\tau) \right\| \leq \frac{C T^2}{1 + (\tau^2 + a^2/T^2)},
\]
(2.13)
where \( C = C_\psi = \max \left\{ \| \phi \|_{L^\infty}, \left\| \frac{d^2 \phi}{dt^2} \right\|_{L^\infty}, \left\| \frac{d^3 \phi}{dt^3} \right\|_{L^\infty} \right\} \) is a constant depending on \( \psi \).

**Proof.** It’s clear that
\[
\left\| \Psi_T \right\|^2_{L^2} = \int_\mathbb{R} \left| \Psi \left( \frac{t}{T} \right) \right|^2 \, dt = \int_\mathbb{R} T |\Psi(t)|^2 \, dt = T \left\| \Psi \right\|_{L^2}^2.
\]
(2.14)
By the definition of the space \( H^b \), we have
\[
\left\| \Psi_T \right\|_{H^b} \leq C \left\| \Psi_T \right\|_{L^2} + C \left\| D_T^b \Psi_T \right\|_{L^2} = CT^{1/2} \left\| \Psi \right\|_{L^2} + CT^{1/2-b} \left\| D_T^b \Psi \right\|_{L^2}.
\]
(2.15)
where we have used the fact
\[
\|D_t^b \Psi_T\|_{L^2}^2 = \int_\mathbb{R} |\tau|^{2b} |\tilde{\Psi}(T \tau)|^2 \, d\tau = T^{1-2b} \|D_t^b \Psi\|_{L^2}^2.
\]
Since \(\|\Psi\|_{L^2}\) and \(\|D_t^b \Psi\|_{L^2}\) are bounded by a constant because of the form of the function \(\Psi\), then from (2.15) we obtain (2.10). We call \(h(t) = \Psi(t) e^{a|t| T}\), and so \(h_T(t) = \Psi_T(t) e^{a|t|}\), to get like in (2.16):
\[
\|\Psi_T(\cdot)e^{a|\cdot|}\|_{H_t^{1/2}} = \|h_T\|_{H_t^{1/2}} \leq C T^{1/2} \|h\|_{L^2} + C \|D_t^{1/2} h\|_{L^2}.
\]
We know that
\[
\|h\|_{L^2}^2 = \int_{-2T}^{2T} |\Psi(t)|^2 e^{2a|t| T} \, dt \leq 4T e^{4aT^2} \|\Psi\|_{L^\infty}^2.
\]
To bounded the term \(\|D_t^{1/2} h\|_{L^2}\) we are going to explore \(\hat{h}(\tau)\) integrating by parts two times,
\[
\hat{h}(\tau) = \int_0^{+\infty} \Psi(t) e^{at T} e^{-i\tau t} \, dt + \int_{-\infty}^0 \Psi(t) e^{-at T} e^{-i\tau t} \, dt \quad (2.18)
\]
\[
= -\frac{1}{aT - i\tau} \left(1 + \int_0^{+\infty} \frac{d\Psi}{dt}(t) e^{i(t(aT - i\tau))} \, dt\right) - \frac{1}{aT + i\tau} \left(1 - \int_{-\infty}^0 \frac{d\Psi}{dt}(t) e^{-i(t(aT + i\tau))} \, dt\right)
\]
\[
= -\frac{2aT}{(aT)^2 + \tau^2} + \frac{1}{(aT - i\tau)^2} \int_0^{+\infty} \frac{d^2\Psi}{dt^2}(t) e^{i(aT - i\tau)} \, dt + \frac{1}{(aT + i\tau)^2} \int_{-\infty}^0 \frac{d^2\Psi}{dt^2}(t) e^{-i(aT + i\tau)} \, dt.
\]
From this we have that
\[
|\hat{h}(\tau)| \leq \frac{2|a| T}{(aT)^2 + \tau^2} + \frac{2(2T) e^{2aT^2} \|\frac{d^2\Psi}{dt^2}\|_{L^\infty}}{(aT)^2 + \tau^2}, \quad (2.19)
\]
and, from (2.18)
\[
|\hat{h}(\tau)| \leq 4T e^{2aT^2} \|\Psi\|_{L^\infty} \leq 4 e^{2a} \|\Psi\|_{L^\infty} = C_1 e^{2a}.
\]
Hence, with \(C_0 e^{2a} = 4 e^{2a} \|\frac{d^2\Psi}{dt^2}\|_{L^\infty} \geq 4T e^{2aT^2} \|\frac{d^2\Psi}{dt^2}\|_{L^\infty}\), from (2.19) and (2.20), we obtain that
\[
|\hat{h}(\tau)| \leq \frac{2|a| T + C e^{2a}}{1 + (aT)^2 + \tau^2},
\]
where \(C = C_0 + C_1\). Multiplying by \(|\tau|^{1/2}\) in (2.21), taking square and integrating on \(\mathbb{R}\), we have that
\[
\|D_t^{1/2} h\|_{L^2}^2 = \|\tau|^{1/2} \hat{h}(\tau)\|_{L^2}^2 \leq 4a^2T^2 \int_\mathbb{R} \frac{|\tau|}{(1 + a^2T^2 + \tau^2)^2} \, d\tau + Ce^{4a} \int_\mathbb{R} \frac{|\tau|}{(1 + a^2T^2 + \tau^2)^2} \, d\tau
\]
\[
\leq 4a^2T^2 \int_\mathbb{R} \frac{|\tau|}{(a^2T^2 + \tau^2)^2} \, d\tau + Ce^{4a} \int_\mathbb{R} \frac{|\tau|}{(1 + \tau^2)^2} \, d\tau
\]
\[
\leq 4 + Ce^{4a} \leq C e^{4a},
\]
where in the second inequality we used \(\tau = |a| T x\). From (2.10), (2.17), (2.22) and since \(T \leq 1\), we conclude (2.11). Integrating on \(\mathbb{R}\) the next inequality which is consequence of (2.21)
\[
|\hat{h}(\tau)| \leq \frac{|a| T}{(aT)^2 + \tau^2} + \frac{Ce^{2a}}{1 + \tau^2},
\]
we have proved (2.12). The proof of (2.13) is equal to that of (2.6) in the Lemma 2.3 in [6].
3 Linear Estimates

Here we study the linear operator $Ψ V_λ$ as well as the linear operator $M_λ$ defined as

$$M_λ : f \mapsto Ψ(t) \int_0^t V_λ(t - t') f(t') \, dt'.$$  \hfill (3.1)

3.1 Linear Estimates for the Free Term in $Y^s$

The next proposition gives a bounded to the free term of the integral equation (1.35).

**Lemma 3.1.**

$$\|Ψ(t) V_λ(t) φ\|_{Y^s} \lesssim \|φ\|_{H^s}. \hfill (3.2)$$

**Proof.** We denote $Θ_k(t) = Ψ(t) e^{η Φ(k) |t|}$. Then

$$(Ψ(t) V_λ(t) φ)^\wedge(k, τ) = Θ_k(t) * (e^{-2πi k t})^\wedge τ • φ(k) = \hat{Θ}_k(τ - 4π^2 k^3) • φ(k). \hfill (3.3)$$

So, $\|Ψ(t) V_λ(t) φ\|_{Y^s}$

$$\leq \left\| (τ - 4π^2 k^3)^{1/2} (k)^s • Θ_k(τ - 4π^2 k^3) • φ(k) \right\|_{L^2((dk) \times L^2(τ))}^2 + \left\| (k)^s • Θ_k(τ - 4π^2 k^3) • φ(k) \right\|_{L^2((dk) \times L^1(τ))}^2 \hfill (3.4)$$

3.1 with (2.11) and (2.12) imply (3.2). \hfill ∎

3.2 Linear Estimates for the Forcing Term in $Y^s$

**Lemma 3.2.**

$$\left\| Ψ(t) \int_0^t V_λ(t - t') F(t') \, dt' \right\|_{Y^s} \lesssim \|F\|_{Z^s}. \hfill (3.5)$$

**Proof.** By applying a smooth cutoff function, we may assume that $F$ is supported on $T \times [-3, 3]$. Let $a(t) = sgn(t)b(t)$, where $b$ is a smooth bump function supported on $[-10, 10]$ which equals 1 on $[-5, 5]$. The identity

$$χ_{[0, t]}(t') = \frac{1}{2} (a(t') + a(t - t')),$$

valid for $t \in [-2, 2]$ and $t' \in [-3, 3]$, allows us to rewrite

$$Ψ(t) \int_0^t V_λ(t - t') F(t') \, dt' = Ψ(t) \int_χ_{[0, t]}(t') V_λ(t - t') F(t') \, dt'$$

$$= \frac{1}{2} Ψ(t) V_λ(t) \int_{R} a(t') V_λ(-t') F(t') \, dt' + \frac{1}{2} Ψ(t) \int_{R} a(t - t') V_λ(t - t') F(t') \, dt'. \hfill (3.6)$$

We consider the contribution of each one of the addend of (3.6). We denote $\tilde{a}_k(t') = a(t') e^{η Φ(k) |t'|}$ and we use (5.2) to obtain

$$\left\| Ψ(t) V_λ(t) \int_{R} a(t') V_λ(-t') F(t') \, dt' \right\|_{Y^s} \leq \int_{R} a(t') V_λ(-t') F(t') \, dt' \|_{H^s}$$

$$= \left\| (k)^s \int_{R} e^{-2πi (4π^2 k^3) t'} \tilde{a}_k(t') F(k, t') \, dt' \right\|_{L^2((dk) \times L^2(τ))} \hfill (3.7)$$

$$= \left\| (k)^s \int_{R} \tilde{a}_k(t')(4π^2 k^3 - τ) F(k, τ) \, dτ \right\|_{L^2((dk) \times L^2(τ))}.$$
As in the proof of (2.24), integrating by part twice we obtain
\[ | \hat{a}_k(\tau) | \leq \frac{C(\eta, \alpha)(1 + |\tau|)}{1 + \eta^2 k^2 + \tau^2} \leq \frac{C(\eta, \alpha)}{1 + \tau} \]

We have from (3.7) that
\[ \left\| \Psi(t) \int_{\mathbb{R}} a(t') V_{\lambda}(t - t') F(t') \, dt' \right\|_{Y^s} \leq C(\eta, \alpha) \left\| \int_{\mathbb{R}} \langle k \rangle^s \hat{F}(k, \tau) \, dk \right\|_{L^2((dk)_\lambda)} \]  \tag{3.8}

The contribution of the second term in (3.6) is calculated using that multiplication by \( \Psi(t) \) is a bounded operation on the space \( Y^s \) (See Remark 2.1) and note that the space-time Fourier transform of \( \int_{\mathbb{R}} a(t - t') V_{\lambda}(t - t') F(t') \, dt' \) is given by
\[ \left( \int_{\mathbb{R}} a(t - t') V_{\lambda}(t - t') F(t') \, dt' \right) (k, \tau) = \left( \int_{\mathbb{R}} a(t - t') e^{-i2\pi k(t - t')} \hat{F}(k, t') \, dt' \right) (k, \tau) \]
\[ = \left( \hat{a}_k(\cdot) e^{-2\pi i(\tau - 4\pi k^3)} \star \hat{F}(k, \cdot) (t) \right) (\tau) = \hat{a}_k(\tau - 4\pi k^3) \hat{F}(k, \tau). \]  \tag{3.9}

From the definitions (1.23), (1.27) and from the estimate for \( \hat{a} \) used above we have:
\[ \left\| \int_{\mathbb{R}} a(t - t') V_{\lambda}(t - t') F(t') \, dt' \right\|_{Y^s} \]
\[ = \left\| \langle \tau - 4\pi^2 k^3 \rangle^{1/2} \langle k \rangle^s \hat{a}_k(\tau - 4\pi^2 k^3) \hat{F}(k, \tau) \right\|_{L^2((dk)_\lambda) L^2(\mathbb{R})} + \left\| \langle k \rangle^s \hat{a}_k(\tau - 4\pi^2 k^3) \hat{F}(k, \tau) \right\|_{L^2((dk)_\lambda) L^2(\mathbb{R})} \]
\[ \leq C(\eta, \alpha) \left\| \langle \tau - 4\pi^2 k^3 \rangle^{-1/2} \langle k \rangle^s \hat{F}(k, \tau) \right\|_{L^2((dk)_\lambda) L^2(\mathbb{R})} + C(\eta, \alpha) \left\| \langle k \rangle^s \hat{F}(k, \tau) \right\|_{L^2((dk)_\lambda) L^2(\mathbb{R})} \]  \tag{3.10}

(3.8) and (3.10) give (3.5).

3.3 Linear Estimates for the Forcing Term in \( Y_{s,1/2} \)

Proposition 3.1. Let \( T \in \left( \frac{\sqrt{2}}{\sqrt{3} - 1}, \frac{1}{3} \right], s \in \mathbb{R} \) and \( \beta > 8 \). Then,
\[ \left\| \Psi(t) \int_{\mathbb{R}} V_{\lambda}(t - t') F(t') \, dt' \right\|_{Y_{s,1/2}} \leq C \eta \alpha (\beta + (\eta \alpha)^2) e^{\eta T^1/2} \| F \|_{Y_{-1/2}}. \]  \tag{3.11}

where \( C \) is a constant.

Remark 3.1. The Proposition 3.1 together with the inequality (2.11), which implies a linear estimate for the free term in \( Y_{s,1/2} \), that is, \( \| \Psi(t) V_{\lambda}(t) \phi \|_{Y_{s,1/2}} \leq C e^{\eta T^1} \| \phi \|_{H^s} \), and the bilinear estimate from Kenig, Ponce and Vega given in the Proposition 1.1 guarantee the local well-posedness result to the \( \lambda \)-periodic initial value problem (1.1) in the Sobolev spaces \( H^s(\mathbb{T}) \) to \( s > -1/2 \) at least for small initial data.

To prove this Proposition 3.1 we need the next Lemmas:

Lemma 3.3 (Schur’s Lemma). Let \( f \) be in \( S(\mathbb{R}) \) and \( L \) the integral operator, given by
\[ (Lf)(x) = \int_{\mathbb{R}} N(x, y) f(y) \, dy \]
where the kernel $N$ is such that
\[
\sup_x \int_{\mathbb{R}} |N(x,y)| \, dy \leq 1, \quad \text{and} \quad \sup_y \int_{\mathbb{R}} |N(x,y)| \, dx \leq 1.
\]

Then, $\|L\|_{L^2 \rightarrow L^2} \leq 1$.

Proof. See the section 2.4.1, page 284 of [19].

Lemma 3.4. Let $\frac{\sqrt{\beta}}{\sqrt{\alpha}} \leq T \leq 1$, $\beta \geq 2$, $\alpha \geq 1$ and $|a| \leq \alpha$. Then
\[
\|\Psi_T(\cdot) I_a(\cdot)\|_{H^{1/2}} \leq C \alpha (\beta + \alpha^2) e^{2\alpha} T^{1/2} \|f\|_{H^{-1/2}},
\]
where $I_a(t) := \int_0^t e^{a|t-t'|} f(t') \, dt'$.

Proof. Rewrite $I_a(t)$ as in the proof of the Lemma 2.4 in [6]. By Fourier inverse transform, we have
\[
I_a(t) = \int_{\mathbb{R}} \hat{f}(\tau)e^{iet} \frac{1}{\tau} + \text{sgn}(t)a \, d\tau.
\]
Since, $\frac{1}{\text{sgn}(t)a - it} = \text{sgn}(t)p_a(\tau) + iq_a(\tau)$ where $p_a(\tau) = \frac{a}{\alpha^2 + \tau^2}$ and $q_a(\tau) = \frac{1}{\alpha^2 + \tau^2}$ then, replacing $\tau$ by $t'$, we obtain
\[
I_a(t) = \text{sgn}(t) \int_{\mathbb{R}} p_a(t')[e^{a|t'|} - e^{it'}] \hat{f}(t') \, dt' + i \int_{\mathbb{R}} q_a(t')[e^{a|t'|} - e^{it'}] \hat{f}(t') \, dt' := I_{a,1}(t) + I_{a,2}(t).
\]

Estimate for $I_{a,1}$. We write
\[
I_{a,1}(t) = \text{sgn}(t) \int_{|t'| > 1/T} p_a(t')[e^{a|t'|} - e^{it'}] \hat{f}(t') \, dt' + \text{sgn}(t) \int_{|t'| \leq 1/T} p_a(t')[e^{a|t'|} - e^{it'}] \hat{f}(t') \, dt'
\]
\[
:= I_{a,1}^>(t) + I_{a,1}^<(t)
\]

Case 1: $|t'| > 1/T$. In this case $|t'| \geq (t')$.
\[
\Psi_T(t) I_{a,1}^>(t) = \Psi_T(t) \text{sgn}(t) \int_{|t'| > 1/T} p_a(t')[e^{a|t'|} - e^{it'}] \hat{f}(t') \, dt' = ah_T(t),
\]
where $h_T(t) = h(t/T)$ and
\[
h(t) = \Psi(t) \text{sgn}(t) \int_{|t'| > 1/T} \frac{\hat{f}(t')}{a^2 + (t')^2} \sum_{t} e^{it't'} dt',
\]
\[
\hat{h}(t)(\tau) = \int_{|t'| > 1/T} \frac{\hat{f}(t')}{a^2 + (t')^2} K(a, T, \tau, t') \, dt'
\]
with
\[
K(a, T, \tau, t') = \int_{|t'| > 1/T} \text{sgn}(t) \Psi(t)[e^{aT|t'|} - e^{iTt'}] e^{-i\tau t} dt.
\]

Integrating by parts, we have
\[
K(a, T, \tau, t') = \int_{\mathbb{R}} \text{sgn}(t)\Psi(t)[e^{aT|t'|} - e^{-i\tau t'}] dt - \int_{\mathbb{R}} \text{sgn}(t)\Psi(t)[e^{iTt'} e^{-i\tau t} dt
\]
\[
= -\frac{2i}{\tau} \int_{\mathbb{R}} \text{sgn}(t) \left( d\Psi(t) + aT \text{sgn}(t) \Psi(t) \right) e^{aT|t'|} e^{-i\tau t} dt + \frac{2i}{\tau} \int_{\mathbb{R}} \text{sgn}(t) \left( d\Psi(t) + iTt' \Psi(t) \right) e^{iT(t'-\tau)t} dt
\]
\[
= K_1(a, T, \tau) + K_2(T, \tau, t'),
\]

where

\[ K_1(a, T, \tau) = -\frac{i}{T} \int_{\mathbb{R}} sgn(t) \left( \frac{d}{dt} \Psi(t) + aT sgn(t) \Psi(t) \right) e^{aT \tilde{t}} e^{-iTt} dt \]

\[ = \frac{1}{T^2} \int_{\mathbb{R}} sgn(t) \left( \frac{d^2}{dt^2} \Psi(t) + 2aT sgn(t) \frac{d}{dt} \Psi(t) + (aT)^2 \Psi(t) \right) e^{aT \tilde{t}} e^{-iTt} dt, \tag{3.20} \]

\[ |K_1(a, T, \tau)| \leq \frac{1}{|\tau|^2} \int_{-2}^{2} \left( \frac{d^2}{dt^2} \Psi(t) \right) + 2|a|T \left| \frac{d}{dt} \Psi(t) \right| + (|a|T)^2 \left| \Psi(t) \right| e^{aT \tilde{t}} dt \leq \frac{C (1 + \alpha T)^2 e^{2a}}{|\tau|^2} , \tag{3.21} \]

and

\[ K_2(T, \tau, t') = \frac{i}{T} \int_{\mathbb{R}} sgn(t) \left( \frac{d}{dt} \Psi(t) + iTt' \Psi(t) \right) e^{iT \tilde{t}} e^{-iTt} dt \tag{3.22} \]

\[ = \frac{2iTt'}{T^2} - \frac{1}{T^2} \int_{\mathbb{R}} sgn(t) \left( \frac{d^2}{dt^2} \Psi(t) + 2iTt' \frac{d}{dt} \Psi(t) + (iTt')^2 \Psi(t) \right) e^{iT \tilde{t}} e^{-iTt} dt \tag{3.23} \]

\[ = \frac{1}{T(T \tilde{t} - \tau)} \int_{\mathbb{R}} sgn(t) \frac{d^2}{dt^2} \Psi(t) e^{iT \tilde{t} - \tau} dt + \frac{iTt'}{T(T \tilde{t} - \tau)} \int_{\mathbb{R}} sgn(t) \frac{d}{dt} \Psi(t) e^{iT \tilde{t} - \tau} dt - \frac{2iTt'}{T(T \tilde{t} - \tau)}. \tag{3.24} \]

Thus, from (3.22):

\[ |K_2(T, \tau, t')| \leq C \frac{|t'|}{|\tau|}, \tag{3.25} \]

from (3.23):

\[ |K_2(T, \tau, t')| \leq C \frac{|t'|^2}{|\tau|^2}, \tag{3.26} \]

and from (3.24):

\[ |K_2(T, \tau, t')| \leq \frac{1}{T(T \tilde{t} - \tau)} \int_{-2}^{2} \frac{d^2}{dt^2} \Psi(t) |dt| + \frac{T|t'|}{T(T \tilde{t} - \tau)} \int_{-2}^{2} \frac{d}{dt} \Psi(t) |dt| + \frac{2T|t'|}{T(T \tilde{t} - \tau)} \]

\[ \leq \frac{CT|t'|}{|T(T \tilde{t} - \tau)|}. \tag{3.27} \]

In the inequalities above \( C = C_{\Psi} = \max\{\|\Psi\|_{L^\infty}, \|\frac{d}{dt} \Psi\|_{L^\infty}, \|\frac{d^2}{dt^2} \Psi\|_{L^\infty}\}. \) From (3.17), (3.19), (3.21) and considering \( |\tau| > 1/2, \) which implies \( \langle \tilde{\tau} \rangle \leq |\tau|, \) we have

\[ |\tilde{h}(\tilde{t})| \leq \int_{|t'| > 1/T} \frac{|\tilde{f}(t')|}{a^2 + (t')^2} |K_1(a, T, \tau)| dt' + \int_{|t'| > 1/T} \frac{|\tilde{f}(t')|}{a^2 + (t')^2} |K_2(T, \tau, t')| dt' \]

\[ \leq \int_{|t'| > 1/T} \frac{|\tilde{f}(t')|}{a^2 + (t')^2} C \frac{(1 + \alpha T)^2 e^{2a}}{|\tau|^2} dt' + \int_{1/T < |t'| \leq |\tau|} \frac{|\tilde{f}(t')|}{a^2 + (t')^2} |K_2(T, \tau, t')| dt' + \]

\[ + \int_{|t'| \geq |\tau|} \frac{|\tilde{f}(t')|}{a^2 + (t')^2} |K_2(T, \tau, t')| dt' \]

\[ = J_1 + J_2 + J_3. \tag{3.28} \]
We obtained \( J_2 \) and \( J_3 \) splitting the set \( \{ |t'| > 1/T \} \) in \( \{ 1/T < |t'| < \beta |\tau|T \} \neq \emptyset \) (because \( T \geq \sqrt{2}/\sqrt{\beta} \)) and \( \{ |t'| \geq \beta |\tau|T \} \) where \( \beta \geq 2 \). We estimate \( J_1 \) so,

\[
J_1 \leq C \frac{(1 + \alpha T)^2 e^{2a}}{\gamma^2} \int_{|t'| > 1/T} \hat{f}(t') \frac{\langle t' \rangle^{1/2}}{(t')^2} dt' \\
\leq C \frac{(1 + \alpha T)^2 e^{2a}}{\gamma^2} \| f \|_{H^{-1/2}} \left( \int_{|t'| > 1/T} \frac{1}{(t')^2} dt' \right)^{1/2} \\
\leq C \frac{(1 + \alpha T)^2 e^{2a} T}{\gamma^2} \| f \|_{H^{-1/2}}.
\]

(3.29)

From (3.27) we obtain

\[
|K_2(T, \tau, t')| \leq \frac{C T|t'|}{|\tau(Tt' - \tau)|} \leq 2^{1-\gamma} \frac{C T|t'|}{|\tau||\tau^0|T^{1-\gamma} |t'|^{1-\gamma}} \quad \text{always that } |t'| \geq 2|\gamma| T
\]

(3.30)

because \( |Tt' - \tau| \geq |Tt'| - |\tau| \geq |\tau|, |Tt' - \tau| \geq |Tt'| - |\tau| \geq |Tt'|/2 \) and so, for \( 0 \leq \gamma \leq 1 \)

\[
|Tt' - \tau| \geq |\gamma| \frac{|Tt'|^{1-\gamma}}{2^{1-\gamma}}.
\]

Note that \( 1/T < 2|\gamma| T \leq \beta |\tau| T \). So, to estimate the integral \( J_3 \), since \( T \geq \frac{\sqrt{2}}{\sqrt{\beta}} \), then \( \frac{2|\gamma| T}{T} \leq \beta |\tau| T \leq |t'| \), hence

\[
J_3 \leq C \int_{|t'| \geq \beta |\tau| T} \left| \hat{f}(t') \right| \frac{\langle t' \rangle^{1/2} T |t'|}{|\tau| |\tau^0| T^{1-\gamma} |t'|^{3-\gamma}} dt' \\
\leq \frac{C T^2}{|\gamma|^{1+\gamma}} \| f \|_{H^{-1/2}} \left( \int_{|t'| > 1/T} \frac{1}{|t'|^{3-\gamma}} dt' \right)^{1/2} \\
\leq \frac{C T}{|\gamma|^{1+\gamma}} \| f \|_{H^{-1/2}}. \tag{3.31}
\]

To estimate \( J_2 \) we are going to use the Schur’s lemma 3.3

\[
J_2 \leq \int_{1/T < |t'| \leq 2|\gamma| |T|} \left| \hat{f}(t') \right| \frac{\langle K_2(T, \tau, t') \rangle}{|t'|^{2}} dt' \\
\leq \frac{1}{|\gamma|^{1/2}} \int_{1/T < |t'| \leq 2|\gamma| |T|} \left| \hat{f}(t') \right| \frac{|\gamma|^{1/2} |K_2(T, \tau, t')|}{|t'|^{3/2}} dt', \tag{3.32}
\]

from \( \langle \gamma \rangle \leq |\gamma| \) and from (3.28) we have

\[
|\hat{h}(\gamma)| \leq C \left( \frac{(1 + \alpha T)^2 e^{2a} T}{(\gamma)^2} + \frac{T}{(\gamma)^{1+\gamma}} \right) \| f \|_{H^{-1/2}} \\
+ \frac{C}{(\gamma)^{1/2}} \int_{1/T < |t'| \leq 2|\gamma| |T|} \left| \hat{f}(t') \right| \frac{|\gamma|^{1/2} |K_2(T, \tau, t')|}{|t'|^{3/2}} dt'. \tag{3.33}
\]

Now, we consider the integral operator \( (L_T g)(\gamma) = \int_{R} N_T(\gamma, t') g(t') dt' \) where \( g(t') = \frac{\left(\hat{f}(t')\right)}{|t'|^{1/2}} \) and \( N_T(\gamma, t') = \frac{|\gamma|^{1/2}}{|t'|^{3/2}} |K_2(T, \tau, t')| \chi_{[ \frac{1}{4} < |t'| \leq 2|\gamma| |T|]} \). So, we obtain from (3.33) that

\[
|\hat{h}(\gamma)| \leq C \left( \frac{(1 + \alpha T)^2 e^{2a} T}{(\gamma)^2} + \frac{T}{(\gamma)^{1+\gamma}} \right) \| f \|_{H^{-1/2}} + \frac{C}{(\gamma)^{1/2}} L_T g(\gamma), \tag{3.34}
\]

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We multiply (3.34) by $\langle \tau \rangle^{1/2}$, take the $L^2$ norm and obtain

$$
\| \langle \tau \rangle^{1/2} \hat{h}(t)(\tau) \|_{L^2} \leq C \left( \sqrt{\frac{\beta}{2} + \alpha} \right)^2 T^3 e^{2a} \| f \|_{H^{-1/2}} \left\| \frac{1}{\langle \tau \rangle^{1/2}} \right\|_{L^2} + C T \| f \|_{H^{-1/2}} \left\| \frac{1}{\langle \tau \rangle^{1/2}} \right\|_{L^2} + C \| L_T g(\tau) \|_{L^2}.
$$

(3.35)

It is sufficient to prove that the operator $L_T$ is bounded in $L^2$. We need to prove that

$$
\sup_{\tau} \int_{\mathbb{R}} |N_T(\tau, t')| \, dt' \leq C(T) \quad \text{and} \quad \sup_{t'} \int_{\mathbb{R}} |N_T(\tau, t')| \, d\tau \leq C(T)
$$

to apply the Schur’s Lemma. We proceed using (3.25)

$$
\sup_{\tau} \int_{\mathbb{R}} |N_T(\tau, t')| \, dt' \leq C \sup_{\tau} \int_{1/T < |t'| \leq |\tau|} \frac{|\tau|^{1/2}}{|\tau|^{3/2}} \, dt' \leq C \sup_{\tau} \int_{0}^{\beta |\tau| T} \frac{|\tau|^{-1/2}}{t'^{1/2}} \, dt' \\
\leq C \sup_{\tau} |\tau|^{-1/2} T^{1/2} (\beta |\tau|)^{1/2} = C \beta^{1/2} T^{1/2},
$$

(3.36)

and using (3.20)

$$
\sup_{t'} \int_{\mathbb{R}} |N_T(\tau, t')| \, d\tau \leq C \sup_{t'} \int_{|\tau| > 1/T} \frac{|\tau|^{1/2}}{|\tau|^{3/2}} \, d\tau \leq C \sup_{t'} \int_{|t'| > 1/T} \frac{|t'|^{1/2}}{|\beta T|} \, d\tau \\
\leq C \sup_{t'} \frac{|t'|^{1/2}}{|t'|^{3/2}} \left| \frac{t'}{\beta T} \right|^{-1/2} = C \beta^{1/2} T^{1/2}.
$$

(3.37)

Hence, $\| L_T g \|_{L^2} \leq C \beta^{1/2} T^{1/2}$. Then, from (3.35)

$$
\| \langle \tau \rangle^{1/2} \hat{h}(t)(\tau) \|_{L^2} \leq C \left[ (\sqrt{\frac{\beta}{2} + \alpha})^2 T^3 e^{2a} + T \right] \| f \|_{H^{-1/2}} + C \beta^{1/2} T^{1/2} \| g \|_{L^2} \\
\| h \|_{H^{1/2}} \leq C \left[ (\sqrt{\frac{\beta}{2} + \alpha})^2 T^3 e^{2a} + T + \beta T^{1/2} \right] \| f \|_{H^{-1/2}} \\
\leq C (\beta + \alpha^2) e^{2a} T^{1/2} \| f \|_{H^{-1/2}}
$$

(3.38)

when $|\tau| > 1/2$. If $|\tau| \leq 1/2$ we use that $|K(a, T, \tau, t')| \leq C (e^{2a} + 1)$ which is consequence of (3.18). So,

$$
\hat{h}(t)(\tau) \leq C (e^{2a} + 1) \int_{|t'| > 1/T} \frac{\hat{f}(t')}{|\alpha^2 + t'|^2} \, dt' = C (e^{2a} + 1) \int_{|t'| > 1/T} \frac{\hat{f}(t')}{|t'|^2} \frac{|t'|^{1/2}}{|\beta T|} \, dt' \\
\leq C (e^{2a} + 1) \| f \|_{H^{-1/2}} \left( \int_{|t'| > 1/T} \frac{dt'}{|t'|^3} \right)^{1/2} = C (e^{2a} + 1) T \| f \|_{H^{-1/2}},
$$

and,

$$
\int_{|\tau| \leq 1/2} \langle \tau \rangle |\hat{h}(t)(\tau)|^2 \, d\tau \leq C (e^{2a} + 1) T \| f \|_{H^{-1/2}} \left( \int_{|\tau| \leq 1/2} \langle \tau \rangle \, d\tau \right)^{1/2} = C (e^{2a} + 1) T \| f \|_{H^{-1/2}}.
$$

(3.39)

Thus, adding (3.35) and (3.39), we have that

$$
\| h \|_{H^{1/2}} \leq C \left[ (\beta + \alpha^2) e^{2a} T^{1/2} + (e^{2a} + 1) T \right] \| f \|_{H^{-1/2}} \\
\leq C \left[ (\beta + \alpha^2) e^{2a} + 1 \right] T^{1/2} \| f \|_{H^{-1/2}}.
$$

(3.40)
Finally, from (3.15) and (3.40),
\[ \| \Psi_T I_{a,1}^+ \|_{H^{1/2}} = \| a h_T \|_{H^{1/2}} \leq C |a| (T^{1/2} + 1) \| h \|_{H^{1/2}} \]
\[ \leq C \alpha (T^{1/2} + 1) [(\beta + \alpha^2) e^{2a} + 1] T^{1/2} \| f \|_{H^{-1/2}} \]
\[ \leq C \alpha [(\beta + \alpha^2) e^{2a} + 1] T^{1/2} \| f \|_{H^{-1/2}}, \] (3.41)
and (3.12) is proved in this case.

**Case 2:** \( |t'| \leq 1/T \). We proceed like in [6], so \( \widetilde{\Psi}_T(t) = \text{sgn}(t)\Psi_T(t) \) and
\[ (\Psi_T(t) I_{a,1}^- (t))^\wedge(\tau) = \int_{|t'| \leq 1/T} p_a(t') \hat{f}(t') \{ (\widetilde{\Psi}_T(t)e^{|a|t})^\wedge(\tau) - (\widetilde{\Psi}_T(t)e^{|a|t})^\wedge(\tau - t') \} \text{d}t' + \]
\[ + \int_{|t'| \leq 1/T} p_a(t') \hat{f}(t')(\widetilde{\Psi}_T(t)[e^{|a|t} - 1])^\wedge(\tau - t') \text{d}t' \]
\[ := I_{a,11}(\tau) + I_{a,12}(\tau). \] (3.42)
We can estimate the integral \( I_{a,11} \) with the ideas used to prove the Lemma 2.1 in [13].
\[ I_{a,11} = \int_{|t'| \leq 1/T} p_a(t') \hat{f}(t') \int_{\tau - t'}^{\tau} \frac{d}{du} (\widetilde{\Psi}_T(t)e^{|a|t})^\wedge(u) \text{d}u \text{d}t' \]
\[ = \int_{|t'| \leq 1/T} \frac{a t'}{a^2 + (t')^2} \hat{f}(t') \int_{0}^{1} \frac{d}{d\lambda} (\widetilde{\Psi}_T(t)e^{|a|t})^\wedge(\tau - \lambda t') \text{d}\lambda \text{d}t'. \] (3.43)
We multiply (3.43) by \( (\tau)^{1/2} \leq C((|t'|^{1/2} + |\tau - \lambda t'|^{1/2}) \), take the \( L^2 \) norm and obtain
\[ \| I_{a,11} \|_{H^{1/2}} \leq C \int_{|t'| \leq 1/T} \frac{|a| |t'| (t')^{1/2} \hat{f}(t')}{a^2 + (t')^2} \text{d}t' \left\| \frac{d}{dt} (\widetilde{\Psi}_T(t)e^{|a|t})^\wedge(\tau) \right\|_{L^2} + \]
\[ + C \int_{|t'| \leq 1/T} \frac{|a| |t'| (t')^{1/2} \hat{f}(t')}{a^2 + (t')^2} \text{d}t' \left\| |\tau|^{1/2} \frac{d}{dt} (\widetilde{\Psi}_T(t)e^{|a|t})^\wedge(\tau) \right\|_{L^2} \]
\[ \leq C \int_{|t'| \leq 1/T} \frac{\hat{f}(t')}{|a| |t'| (t')^{1/2}} \frac{\text{d}t'}{a^2 + (t')^2} \text{d}t' \left\| |\tau|^{1/2} (|t| \Psi_T(t)e^{|a|t})^\wedge(\tau) \right\|_{L^2} + \]
\[ + C \int_{|t'| \leq 1/T} \frac{\hat{f}(t')}{|a| |t'| (t')^{1/2}} \frac{\text{d}t'}{a^2 + (t')^2} \text{d}t' \left\| |\tau|^{1/2} (|t| \Psi_T(t)e^{|a|t})^\wedge(\tau) \right\|_{L^2} \]
\[ \leq C T^{3/2} \| f \|_{H^{-1/2}} \left( \int_{|t'| \leq 1/T} \frac{|a|^{2} |t'|^2 (t')^2}{a^2 + (t')^2} \text{d}t' \right)^{1/2} + \]
\[ + C T \| f \|_{H^{-1/2}} \left( \int_{|t'| \leq 1/T} \frac{|a|^{2} |t'|^2 (t')^2}{a^2 + (t')^2} \text{d}t' \right)^{1/2} \] (3.44)
\[ \leq C \alpha T \| f \|_{H^{-1/2}} + C \alpha T^{1/2} \| f \|_{H^{-1/2}} \] (3.45)
\[ \leq C \alpha T^{1/2} \| f \|_{H^{-1/2}}. \] (3.46)
We obtained (3.44) thanks to Cauchy-Schwartz’s inequality and (2.48). (3.35) is consequence from
\[ |a|^2 |t'|^4 \leq \alpha^2 |t'|^4 \leq \alpha^2 [a^2 + (t')^2]^2 \]
and this implies that the root square of the integrals in (3.44) are bounded by \( \sqrt{2} \alpha T^{-1/2} \).

The estimate of the integral \( I_{a,12} \) in [6] is not adequate but a small modification is sufficient to obtain a good result. From Case 2 in the proof of the Lemma 2.4 in [6] we know that
\[ |(\widetilde{\Psi}_T(t)[e^{|a|t} - 1])^\wedge(\tau - t')| \leq C T^2 \frac{|a|}{(1 + |\tau|T)^2}. \] (3.47)
So, using Cauchy-Schwartz’s inequality

\[ |I_{a,12}(\tau)| \leq C \int_{|\tau'| \leq 1/T} \frac{|a|^2}{a^2 + (\tau')^2} |\tilde{f}(\tau')| \frac{T^2}{(1 + |\tau|T)^2} d\tau' \]

\[ = \frac{CT^2}{(1 + |\tau|T)^2} \int_{|\tau'| \leq 1/T} \frac{|\tilde{f}(\tau')| |a|^2(\tau')^{-1/2}}{a^2 + (\tau')^2} d\tau' \]

\[ \leq \frac{CT^2}{(1 + |\tau|T)^2} \left( \int_{|\tau'| \leq 1/T} \frac{|a|^4(\tau')}{a^2 + (\tau')^2} d\tau' \right)^{1/2} \|f\|_{H^{-1/2}}. \quad (3.48) \]

Taking square, multiplying by \( \langle \tau \rangle \) and integrating on \( \mathbb{R} \) in (3.48) we obtain

\[ \int_{\mathbb{R}} \langle \tau \rangle |I_{a,12}(\tau)|^2 d\tau \leq CT^4 \left\{ \int_{\mathbb{R}} \frac{(1 + |\tau|)}{(1 + |\tau|T)^2} d\tau \right\} \left\{ \int_{|\tau'| \leq 1/T} \frac{|a|^4(1 + |\tau'|)}{|a| + |\tau'|^2} d\tau' \right\} \|f\|_{H^{-1/2}}^2 \]

\[ \leq C (\alpha + 1) T \|f\|_{H^{-1/2}}^2 \quad (3.49) \]

because

\[ \int_{\mathbb{R}} \frac{(1 + |\tau|)}{(1 + |\tau|T)^4} d\tau \leq C \left( \frac{1}{T} + \frac{1}{T^2} \right) \]

and

\[ \int_{|\tau'| \leq 1/T} \frac{|a|^4(1 + |\tau'|)}{|a| + |\tau'|^2} d\tau' \leq \frac{1 + |a|}{T} \leq \frac{\alpha + 1}{T}. \]

Hence, from (3.49),

\[ \|I_{a,12}\|_{H^{1/2}} \leq C \sqrt{\alpha + 1} T^{1/2} \|f\|_{H^{-1/2}}. \quad (3.50) \]

We conclude from (3.42), (3.46) and (3.50) that

\[ \|\Psi_T I_{a,1}\|_{H^{1/2}} \leq C (\alpha + \sqrt{\alpha + 1}) T^{1/2} \|f\|_{H^{-1/2}}. \quad (3.51) \]

Note that, from (3.41) and (3.51), we have actually proved that

\[ \|\Psi_T I_{a,1}\|_{H^{1/2}} \leq C \left\{ \alpha (\beta + \alpha^2) e^{a} + 1 \right\} T^{1/2} \|f\|_{H^{-1/2}}, \quad (3.52) \]

which gives (3.12).

**Estimate for** \( I_{a,2} \). The estimate for \( I_{a,2} \) is similar to that of \( I_{a,1} \), exchanging \( p_a \) by \( q_a \) and \( \overline{\Psi_T} \) by \( \Psi_T \). So, we omit its calculation.

**Corollary 3.1.** Let \( \frac{\sqrt{\beta}}{\sqrt{\alpha}} \leq T \leq 1, \beta \geq 2, \alpha \geq 1 \) and \(-\alpha \leq a \leq 0\).

\[ \|\Psi_T I_a\|_{H^{1/2}} \leq C \alpha (\beta + \alpha^2) T^{1/2} \|f\|_{H^{-1/2}}. \quad (3.53) \]

**Proof.** (3.53) is a direct consequence of (3.52). \( \square \)

**Lemma 3.5.** Let \( \frac{\sqrt{\beta}}{\sqrt{\alpha}} \leq T \leq \frac{1}{2} \) and \( \beta \geq 8 \). Then,

\[ \|\Psi_T (\cdot) I_a (\cdot)\|_{H^{1/2}} \leq C \alpha (\beta + \alpha^2) T^{1/2} \|f\|_{H^{-1/2}}, \quad \text{if} \quad a < -\alpha, \quad (3.54) \]

and,

\[ \|\Psi_T (\cdot) I_a (\cdot)\|_{H^{1/2}} \leq C \alpha (\beta + \alpha^2) e^{\alpha} T^{1/2} \|f\|_{H^{-1/2}}, \quad \text{if} \quad a \leq a. \quad (3.55) \]
Proof. (2.6) and (3.53) give (3.54), and (3.55) is consequence of (3.54) and (3.12).

Proof of Proposition 4.2. From the definition of the $\mathcal{Y}_{s,\lambda}$ norm, we have

$$\left\| \Psi(t) \int_0^t V_{\lambda}(t-t') F(t') \, dt' \right\|_{\mathcal{Y}_{s,\lambda}}^2$$

$$= \sum_{k \neq 0} \langle k \rangle^{2s} \int_{-\infty}^{\infty} (1 + |\tau|) \left| e^{-ik^3 \tau} \Psi(t) \int_0^t e^{ik^3 (t-t') + \eta |t-t'|} \hat{F}(k, t') \, dt' \right|^2 \, d\tau$$

$$= \left\| \langle k \rangle^s \Psi(t) \int_0^t e^{\eta (t-t') \Phi(k)} \left[ e^{-ik^3 t'} \hat{F}(k, t') \right] \, dt' \right\|_{H_{s,\lambda}^{1/2}}^2$$

$$\leq \left\| \langle k \rangle^s C \eta \alpha (\beta + (\eta \alpha)^2) e^{2\eta \alpha \eta} T^{1/2} \left\| e^{-ik^3 t} \hat{F}(k, t) \right\|_{H_{s,\lambda}^{-1/2}}^2 \right\|_{l_k^2}^2$$

$$= C^2 (\eta \alpha)^2 (\beta + (\eta \alpha)^2) e^{4\eta \alpha T} \left\| \langle k \rangle^s (\tau - k^3)^{-1/2} \hat{F}(k, \tau) \right\|_{l_k^2 l_t^2}^2.$$  

In the inequality (3.56) we apply the Lemma 3.3, 3.57 implies 3.11.

4 Local Well-posedness in $H^s(T)$

Consider the $\lambda$-periodic initial value problem 1.1 with periodic initial data $u_0 \in H^s_{s,\lambda}$, $s \geq -1/2$. We show first that, for arbitrary $\lambda$, this problem is well-posed on a time interval of size $\sim 1$ provided $\|u_0\|_{H_{s,\lambda}^{-1/2}}$ is sufficiently small. Then we show by a rescaling argument that (1.1) is locally well-posed for arbitrary initial data $u_0 \in H^s_{s,\lambda}$. As mentioned before in Remark 1.1, we restrict our attention to initial data having zero $x$-mean.

Proof of the Theorem 1.1. Fix $u_0 \in H^s_{s,\lambda}$, $s \geq -1/2$ and for $w \in Z^{-1/2}$ define

$$(Aw)(t) = \Psi(t) V_{\lambda}(t) u_0 - \Psi(t) \int_0^t V_{\lambda}(t-t') \left( \Psi(t') w(t') \right) \, dt'.$$  

(4.1)

The bilinear estimate 1.26 shows that $u \in Y^{1/2}$ implies $\Psi(t) \partial_x (u^2) \in Z^{-1/2}$ so the (nonlinear) operator

$$\Gamma(u) = A \left( \frac{1}{2} \partial_x (u^2) \right)$$

is defined on $Y^{1/2}$. Observe that $\Gamma(u) = u$ is equivalent, at least for $t \in [-1, 1]$, to (1.35), which is equivalent to (1.1).

Claim 1. $\Gamma : (\text{bounded subsets of } Y^{1/2}) \longrightarrow (\text{bounded subsets of } Y^{1/2})$.

Since

$$\Gamma(u) = \Psi(t) V_{\lambda}(t) u_0 - \Psi(t) \int_0^t V_{\lambda}(t-t') \left( \frac{\Psi(t')}{2} \partial_x u^2(t') \right) \, dt',$$

we estimate using (3.2), (3.4) and the bilinear estimate 1.26:

$$\|\Gamma(u)\|_{Y^{-1/2}} \leq \|\Psi(t) V_{\lambda}(t) u_0\|_{Y^{-1/2}} + \left\| \Psi(t) \int_0^t V_{\lambda}(t-t') \left( \frac{\Psi(t')}{2} \partial_x u^2(t') \right) \, dt' \right\|_{Y^{-1/2}}$$

$$\leq C_1 \|u_0\|_{H_{s,\lambda}^s} + C_2 \|\Psi(t) \partial_x u^2\|_{Z^{-1/2}}$$

$$\leq C_1 \|u_0\|_{H_{s,\lambda}^s} + C_2 C_3 \lambda^{l+} \|u\|_{Y^{-1/2}}^2.$$  

(4.2)
and the claim is proved.

Now, we consider the ball

$$\mathcal{B} = \{ w \in Y^{-1/2} : \| w \|_{Y^{-1/2}} \leq C_4 \| u_0 \|_{H_\lambda^{-1/2}} \}. $$

**Claim 2.** $\Gamma$ is a contraction on $\mathcal{B}$ if $\| u_0 \|_{H_\lambda^{-1/2}}$ is sufficiently small.

We wish to prove that for some $\theta \in (0, 1)$,

$$\| \Gamma(u) - \Gamma(v) \|_{Y^{-1/2}} \leq \theta \| u - v \|_{Y^{-1/2}}$$

for all $u, v \in \mathcal{B}$. Since $u^2 - v^2 = (u + v)(u - v)$, we can see that

$$\| \Gamma(u) - \Gamma(v) \|_{Y^{-1/2}} \leq \left\| -\Psi(t) \int_0^t V_\lambda(t - t') \frac{\Phi(t')}{2} \partial_x (u^2 - v^2)(t') dt' \right\|_{Y^{-1/2}}$$

$$\leq C_2 \| \Psi(t) \partial_x (u + v)(u - v) \|_{Y^{-1/2}}$$

$$\leq C_2 C_3 \lambda^{0^+} (\| u \|_{Y^{-1/2}} + \| u \|_{Y^{-1/2}}) \| u - v \|_{Y^{-1/2}}$$

$$\leq 2 C_2 C_3 \lambda^{0^+} \| u_0 \|_{H_\lambda^{-1/2}} \| u - v \|_{Y^{-1/2}}.$$  \hspace{1cm} (4.3)

(4.3) holds because $u, v \in \mathcal{B}$. Hence, for fixed $\lambda$, if we take $\| u_0 \|_{H_\lambda^{-1/2}}$ so small such that

$$2 C_2 C_3 \lambda^{0^+} \| u_0 \|_{H_\lambda^{-1/2}} \ll 1$$

the contraction estimate is verified.

The preceding discussion establishes well-posedness of (1.1) on a $O(1)$-sized time interval for any initial data satisfying (4.4). To prove that our result holds for every given data $u_0$ in $H_\lambda$ and not only for small data as in (4.4), let us perform the following scale change

$$v(x, t) = \frac{1}{\sigma^2} u \left( \frac{x}{\sigma}, \frac{t}{\sigma^3} \right)$$

(4.5)

where $\sigma \geq \alpha$. So, $v$ is periodic with respect to the $x$ variable with period $\sigma \lambda$, hence for $k \in \mathbb{Z}/\sigma \lambda$

$$\hat{v}(k) = \frac{1}{\sigma^2} \int_0^{\sigma \lambda} e^{-2\pi ikx} u(x/\sigma) \, dx = \frac{1}{\sigma} \hat{u}(k\sigma),$$

(4.6)

and $v$ satisfies the equation

$$\sigma^3 v_t(x, t) + \sigma^3 v_{xxx}(x, t) + \sigma^3 v_{xx}(x, t) + \eta S v(x, t) = 0,$$

where the operator $S$ is defined by $(Sv)^\wedge(k) := -\Phi(\sigma k) \hat{v}(k)$ and so, $S\hat{v}(x, t) = \frac{1}{\sigma^3} Lu \left( \frac{x}{\sigma}, \frac{t}{\sigma^3} \right)$. Hence, $v_t + v_{xxx} + vv_x + \eta \bar{S} v = 0$. Considering $\bar{S} = \frac{1}{\sigma^3} S$, $v$ satisfies

$$\begin{cases}
  v_t + v_{xxx} + vv_x + \eta \bar{S} v = 0 & x \in [0, \sigma \lambda], \ t \in (0, +\infty) \\
  v(x, 0) = v_0(x) = \frac{1}{\sigma^2} u_0 \left( \frac{x}{\sigma} \right),
\end{cases}$$

(4.7)

where

$$(\bar{S}v)^\wedge(k) = \frac{1}{\sigma^3} (Sv)^\wedge(k) = -\frac{1}{\sigma^3} \Phi(\sigma k) \hat{v}(k) \quad \text{and} \quad \frac{1}{\sigma^3} \Phi(\sigma k) \leq \frac{\alpha}{\sigma^3} \leq 1.$$
Finally, consider (1.1) with $\lambda = \lambda_0$ fixed and $u_0 \in H^{s}_{\lambda_0}$, $s \geq -1/2$. This problem is well-posed on a small time interval $[0, \delta]$ if and only if the $\sigma$-rescaled problem (4.7) is well-posed on $[0, \sigma^3 \delta]$. A calculation shows that

$$\|v_0\|_{H^{-1/2}_{\sigma \lambda_0}} \leq \frac{1}{\sigma} \|u_0\|_{H^{-1/2}_{\lambda_0}}.$$

Observe that

$$(\sigma \lambda_0)^{0+} \|v_0\|_{H^{-1/2}_{\lambda_0}} \leq \frac{(\sigma \lambda_0)^{0+}}{\sigma} \|u_0\|_{H^{-1/2}_{\lambda_0}} \ll 1,$$

provided $\sigma = (\lambda_0, \|u_0\|_{H^{-1/2}_{\lambda_0}})$ is taken to be sufficiently large. This verifies (4.4) for the problem (4.7), proving well-posedness of (4.7) on the time interval, say $[0, 1]$. Hence, (1.1) is locally well-posed for $t \in [0, \sigma^{-3}]$.

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