Warped products and black holes

Soon-Tae Hong*
Department of Science Education, Ewha Womans University, Seoul 120-750 Korea
(Dated: August 1, 2003)

We apply the warped product spacetime scheme to the Banados-Teitelboim-Zanelli black holes and the Reissner-Nordström-anti-de Sitter black hole to investigate their interior solutions in terms of warped products. It is shown that there exist no discontinuities of the Ricci and Einstein curvatures across event horizons of these black holes.

I. INTRODUCTION

A warped product spacetime was introduced by Bishop and O’Neill long ago [1]. The warped product scheme was later applied to general relativity [2] and semi-Riemannian geometry [3]. Recently, this warped products were extended to multiply warped products with non-smooth metric [4], and the Banados-Teitelboim-Zanelli (BTZ) black hole [5] and Reissner-Nordström-anti-de Sitter (RN-AdS) black hole [6].

On the other hand, the concept of the warped products was used in higher dimensional theories. For instance, the warped products were exploited in the Randall-Sundrum model in five-dimensions [7, 8] and in the Kaluza-Klein supergravity theory in seven-dimensions [9].

In this paper we will briefly analyze the multiply warped product manifold associated with the (2+1) BTZ black holes and (3+1) RN-AdS metric to investigate the physical properties inside the event horizons. In Section 2 we will introduce the concepts of multiply warped product manifolds. We will then apply this warped product scheme to the (2+1) BTZ black holes in Section 3, and to the (3+1) RN-AdS black hole in Section 4 so that we can explicitly obtain the Ricci and Einstein curvatures inside the event horizons of these black holes.

II. WARPED PRODUCT SPACETIME

Mathematically, a multiply warped product manifold is defined as \((M = B \times F_1 \times \ldots \times F_n, g)\) consisting of the Riemannian base manifold \((B, g_B)\) and fibers \((F_i, g_i) (i = 1, \ldots, n)\) associated with the Lorentzian metric,

\[ g = \pi_B^*g_B + \sum_{i=1}^{n}(f_i \circ \pi_B)^2\pi_i^*g_i, \tag{2.1} \]

where \(\pi_B, \pi_i\) are natural projections of \(B \times F_1 \times \ldots \times F_n\) onto \(B\) and \(F_i\), respectively, and \(f_i\) are positive warping functions.

For a specific case of \((B = R, g_B = -d\mu^2)\), the Lorentzian metric is of the form

\[ g = -d\mu^2 + \sum_{i=1}^{n}f_i^2g_i. \tag{2.2} \]

For instance, the Randall-Sundrum model metric can be regarded as the warped product manifold with the metric

\[ g = -N^2(t, y)dt^2 + A^2(t, y)dx^2 + dy^2. \tag{2.3} \]

Next, we consider the warped products with single discontinuity. Let \(M = M_0 \times_{f_i} F_1 \times \cdots \times_{f_n} F_n\) be multiply warped products with Riemannian curvature tensor \(R\). If \(X, Y \in V(M_0), U_i, V_i \in V(F_i) (i = 1, 2, \ldots, n), d_i = \text{dim} F_i, f_i \in C^0(S)\) at a single point \(p \in M_0\) and \(S = \{p\} \times_{f_i} F_1 \times \cdots \times_{f_n} F_n\), we then obtain the Ricci components of the form

\[ \text{Ric}(X, Y) = -\sum_{i=1}^{n}d_iX^iY^i \frac{f_i'' + \delta(p)(f_i'^+-f_i'^-)}{f_i}. \]

*Electronic address: soonhong@ewha.ac.kr
\[
\text{Ric}(X, U_i) = 0,
\]
\[
\text{Ric}(U_i, V_i) = F_i \text{Ric}(U_i, V_i) + \langle U_i, V_i \rangle \frac{f''_i + \delta(p)(f'_i^+ - f'_i^-)}{f_i} + \langle U_i, V_i \rangle \delta(p) \left[ (d_i - 1) \frac{f'_i^+ - f'_i^-}{f_i^2} + \sum_{j \neq i} d_j \frac{f'_i^+ - f'_j^-}{f_i f_j} \right],
\]
\[
\text{Ric}(U_i, U_j) = 0 \quad \text{for} \quad i \neq j,
\]
where \( X = X^1 \partial/\partial t \) and \( Y = Y^1 \partial/\partial t \). Here note that we have discontinuity contributions associated with \( \delta(p) \) at the point \( p \) since we assume \( f_i \) are \( C^0 \)-functions.

### III. BTZ BLACK HOLES IN WARPED PRODUCTS

In this section, we apply the warped product scheme to the BTZ metric to investigate the inner structure of the black hole. We start with the static BTZ three-metric of the form
\[
ds^2 = N^2 dt^2 - N'^2 dr^2 + r^2 d\phi^2.
\]  
Here the lapse function for interior solution is given by
\[
N^2 = m - \frac{r^2}{l^2},
\]
which can be rewritten in terms of the event horizon \( r_H = m^{1/2}l \) in the region \( r < r_H \) as follows
\[
N^2 = \frac{(r_H + r)(r_H - r)}{l^2}.
\]
As you see, the lapse function is positive in the region \( r < r_H \) and thus this lapse function is well-defined inside the event horizon. So, the BTZ three-metric has a signature \((+,-,+,+)^\text{.}

Next we define a new coordinate \( \mu \) as
\[
d\mu^2 = N'^2 dr^2.
\]
From this definition, \( \mu \) can be given by a integration
\[
\mu = \int_0^r dx \frac{1}{[(r_H + x)(r_H - x)]^{1/2}}.
\]
whose analytic solution is simply written as follows
\[
\mu = l \sin^{-1} \left( \frac{r}{r_H} \right) = F(r).
\]
The function \( F(r) \) then satisfies boundary conditions
\[
\lim_{r \to r_H} F(r) = \frac{l \pi}{2}, \quad \lim_{r \to 0} F(r) = 0,
\]
and \( dr/d\mu > 0 \) implies \( F^{-1} \) is well-defined function.

Using the coordinate \( \mu \), we rewrite the BTZ metric in terms of the warped products
\[
ds^2 = -d\mu^2 + f_1(\mu)^2 dt^2 + f_2^2(\mu)d\phi^2,
\]
where \( f_1 \) and \( f_2 \) are warping functions given in terms of \( \mu \) as follows
\[
f_1(\mu) = \left( m - \frac{F^{-2}(\mu)}{l^2} \right)^{1/2}, \quad f_2(\mu) = F^{-1}(\mu).
\]
After some algebra using the warped product metric in (3.8), we can obtain the nonvanishing Ricci curvature components

\[
R_{\mu\mu} = -\frac{f_1''}{f_1} - \frac{f_2''}{f_2},
R_{tt} = \frac{f_1f_1'f_2'}{f_2} + f_1f_1'',
R_{\phi\phi} = \frac{f_1'f_2'f_1}{f_1} + f_2f_2''.
\]

(3.10)

Exploiting the explicit expressions for \( f_1 \) and \( f_2 \) in (3.9), we can obtain the identities for \( f_1, f_1' \) and \( f_1'' \) in terms of \( f_1, f_2 \) and their derivatives as below

\[
f_1 = f_2', 
\frac{f_1'}{f_2} = -\frac{f_2}{f_2'^2}, 
\frac{f_1''}{f_2} = \frac{f_1'f_2'}{f_2}.
\]

(3.11)

so that inside event horizons we can obtain the Ricci curvatures of the simple form

\[
R_{\mu\mu} = -\frac{2f_1'}{f_2},
R_{tt} = \frac{2f_1f_1'}{f_2},
R_{\phi\phi} = 2f_2f_1'.
\]

(3.12)

Similarly, we can also evaluate the Einstein scalar curvature inside the event horizon as follows

\[
R = -\frac{6}{f_2^2}.
\]

(3.13)

Next, we investigate the relations between the inner solutions above and the well-known exterior solutions. As you see, outside event horizon \( r_H \), the BTZ three-metric is given by

\[
ds^2 = -(-m + \frac{r^2}{l^2})^2 dt^2 + (-m + \frac{r^2}{l^2})^{-2} dr^2 + r^2 d\phi^2.
\]

(3.14)

Using this exterior metric, we can obtain the Ricci and Einstein curvatures explicitly in terms of the warping functions \( f_i \) defined in (3.9),

\[
R_{rr} = -\frac{2f_1'}{f_1'f_2},
R_{tt} = \frac{2f_1f_1'}{f_2},
R_{\phi\phi} = 2f_2f_1',
R = -\frac{6}{f_2^2}.
\]

(3.15)

Here note that the Einstein scalar curvature \( R \) is identical to that of the interior case and the Ricci components \( R_{tt} \) and \( R_{\phi\phi} \) are also the same as those of interior case. Moreover, from the definition of the coordinate \( \mu \) in (3.4), we can find the following identity

\[
R_{\mu\mu} = f_2^2 R_{rr}
\]

(3.16)

which is also attainable by comparing \( R_{\mu\mu} \) and \( R_{rr} \) components in (3.12) and (3.15). We can thus conclude that all the Ricci components and Einstein scalar curvature have identical forms both in exterior and interior of event horizon \( r_H \) without discontinuities.
Next, we consider the charged BTZ black hole whose lapse function for interior solution is given by

\[ N^2 = m - \frac{r^2}{l^2} + 2Q^2 \ln r, \quad (3.17) \]

where we have an additional term proportional to \( Q^2 \) where \( Q \) is the charge of the BTZ black hole. Similar to the static BTZ case, we can find the Ricci and Einstein curvatures inside event horizon \( r_H \) as follows

\[ R_{\mu\mu} = -\frac{2f_1}{f_2} + \frac{2Q^2}{f_2}, \]
\[ R_{tt} = \frac{2f_1 f_1'}{f_2} - \frac{2Q^2 f_1^2}{f_2}, \]
\[ R_{\phi\phi} = 2f_2 f_1', \]
\[ R = -\frac{6}{l^2} + \frac{2Q^2}{f_2}. \quad (3.18) \]

Note that we have charge contributions in the Ricci components \( R_{\mu\mu} \) and \( R_{tt} \) and also in the Einstein curvature \( R \).

### IV. RN-ADS BLACK HOLE IN WARPED PRODUCTS

Next, in this section we consider the RN-AdS metric in the warped product scheme to study the inner structure of the black hole. The (3+1) RN-AdS four-metric is given by

\[ ds^2 = N^2 dt^2 - N^{-2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.1) \]

whose lapse function in region between event horizons \( r_- \) and \( r_+ \) is described as

\[ N^2 = -1 + \frac{2m}{r} - \frac{Q^2}{r^2} - \frac{r^2}{l^2}, \quad (4.2) \]

with the charge \( Q \) and the cosmological constant \( 1/l^2 \). Again note that the four-metric has a positive signature in time direction as mentioned in the BTZ case. If we define the event horizons \( r_- \) and \( r_+ \) by solving the equation

\[ 0 = -1 + \frac{2m}{r} - \frac{Q^2}{r^2} - \frac{r^2}{l^2}, \quad (4.3) \]

we can then rewrite \( N^2, Q^2 \) and \( l^2 \) in terms of \( r_- \) and \( r_+ \) as follows

\[ N^2 = \frac{(r_+ - r)(r - r_-)}{r^2 l^2} \left( r^2 + (r_+ + r_-) r + \frac{Q^2 l^2}{r_+ r_-} \right), \]
\[ Q^2 = \frac{r_+ r_- [2m(r_+^2 + r_-^2 + r^2) - r_+ r_- (r_+ + r_-)]}{(r_+ + r_-)(r_+^2 + r_-^2)}, \]
\[ l^2 = \frac{(r_+ + r_-)(r_+^2 + r_-^2)}{2m - r_+ - r_-}. \quad (4.4) \]

As in the BTZ black hole case, we define a new coordinate \( \mu \) as

\[ d\mu^2 = N^{-2} dr^2, \quad (4.5) \]

which is integrated to yield the following somewhat complicated expression

\[ \mu = \int_{r_-}^{r} \frac{dx}{((r_+ - x)(x - r_-)(x + (r_+ + r_-) x + Q^2 l^2 / r_+ r_-))^{1/2}} = F(r). \quad (4.6) \]

Here note that \( dr/d\mu > 0 \) implies \( F^{-1} \) is well-defined function.

Exploiting the coordinate \( \mu \) defined in (4.6), we now rewrite the RN-AdS black hole metric in terms of the warped products

\[ ds^2 = -d\mu^2 + f_1^2(\mu) dr^2 + f_2^2(\mu) (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.7) \]
where the warping functions \( f_1 \) and \( f_2 \) are given by
\[
\begin{align*}
f_1(\mu) &= \left(-1 + \frac{2m}{F^{-1}(\mu)} - \frac{Q^2}{F^{-2}(\mu)} - \frac{F^{-2}(\mu)}{l^2}\right)^{1/2}, \\
f_2(\mu) &= F^{-1}(\mu).
\end{align*}
\] (4.8)

Again, after some tedious algebra using the metric (4.7), we can obtain the Ricci curvature components in terms of the warping functions
\[
\begin{align*}
R_{\mu\mu} &= -\frac{f''_1}{f_1} \frac{2f''_2}{f_2}, \\
R_{tt} &= \frac{f'_1 f'_2}{f_1} + f_1 f''_1, \\
R_{\theta\theta} &= \frac{f'_1 f'_2}{f_1} + f_2 f''_2 + f_2^2 + 1, \\
R_{\phi\phi} &= \left(\frac{f'_1 f'_2}{f_1} + f_2 f''_2 + f_2^2 + 1\right) \sin^2 \theta.
\end{align*}
\] (4.9)

As in the case of the BTZ black hole, we can find the following identities for \( f_1, f'_1 \) and \( f''_1 \)
\[
\begin{align*}
f_1 &= f'_2, \\
f'_1 &= -\frac{m}{f_2} + \frac{Q^2}{f_2^3} - \frac{f_2}{l^2}, \\
f''_1 &= -\frac{2f_1 f'_1}{f_2} - \frac{Q^2 f_1}{f_2^3} - \frac{3f_1}{l^2},
\end{align*}
\] (4.10)

which can be used to evaluate the Ricci curvatures in the region between \( r_- \) and \( r_+ \)
\[
\begin{align*}
R_{\mu\mu} &= \frac{Q^2}{f_2^3} + \frac{3}{l^2}, \\
R_{tt} &= -\frac{Q^2 f_1^2}{f_2^3} - \frac{3f_1^2}{l^2}, \\
R_{\theta\theta} &= \frac{Q^2}{f_2^3} - \frac{3f_2^2}{l^2}, \\
R_{\phi\phi} &= \left(\frac{Q^2}{f_2^3} - \frac{3f_2^2}{l^2}\right) \sin^2 \theta,
\end{align*}
\] (4.11)

to yield the Einstein curvature
\[
R = -\frac{12}{l^2}.
\] (4.12)

On the other hand, outside event horizon \( r_+ \), we have the RN-AdS four-metric of the well-known form
\[
ds^2 = -(1 - \frac{2m}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}) dt^2 + (1 - \frac{2m}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2})^{-2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\] (4.13)

to yield the Ricci and Einstein curvatures outside event horizons in terms of the warping functions \( f_1 \) and \( f_2 \)
\[
\begin{align*}
R_{rr} &= \frac{Q^2}{f_1 f_2} + \frac{3}{f_1^2 l^2}, \\
R_{tt} &= -\frac{Q^2 f_1^2}{f_2} - \frac{3f_1^2}{l^2}, \\
R_{\theta\theta} &= \frac{Q^2}{f_2} - \frac{3f_2^2}{l^2}, \\
R_{\phi\phi} &= \left(\frac{Q^2}{f_2} - \frac{3f_2^2}{l^2}\right) \sin^2 \theta, \\
R &= -\frac{12}{l^2}.
\end{align*}
\] (4.14)
Note that the Einstein scalar curvature $R$ and the Ricci components $R_{tt}$, $R_{\theta\theta}$ and $R_{\phi\phi}$ are equal to those in the region between $r_-$ and $r_+$. Moreover, using the definition of the coordinate $\mu$ in (4.5), we can find the following identity

$$R_{\mu\mu} = f_1^2 R_{rr},$$

which is also attainable from the Ricci components $R_{\mu\mu}$ and $R_{rr}$ in (4.11) and (4.14). As in the BTZ case, all the Ricci components and Einstein scalar curvature thus have identical forms both in exterior and interior of event horizon $r_+$ without discontinuities.

V. CONCLUSIONS

In conclusion, we have studied the warped product spacetime to obtain the interior solutions of the BTZ black holes and the RN-AdS black hole in terms of warping functions. In both cases, there exist no discontinuities of the Ricci and Einstein curvatures across event horizons of these black holes.

Acknowledgments

The author would like to thank Itzhak Bars for initial discussions. This work is supported in part by the Korea Science and Engineering Foundation Grant R01-2000-00015.

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