INEQUALITIES OF HERMITE-HADAMARD TYPE FOR HIGHER ORDER CONVEX FUNCTIONS, REVISITED

TOMASZ SZOSTOK

Institute of Mathematics, University of Silesia
Bankowa 14, 40-007 Katowice, Poland

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Abstract. In this paper we present a very short proof of inequalities of Hermite-Hadamard type obtained by M. Bessenyei and Zs. Páles. This proof is based on the recently developed method connected with use of stochastic orderings of random variables. In the second part of the paper we present a way to extend these known inequalities. Namely, we describe completely the possible inequalities of Hermite-Hadamard type for longer expression than it was the case in the results of Bessenyei and Páles.

1. Introduction. In the first part of the paper we will present a short proof of main results from [1]. Thus now we say a few words concerning these results. The starting point for Bessenyei and Páles was the well known Hermite-Hadamard inequality

\[
 f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2} 
\]  

(satisfied by every convex function \( f : [a, b] \rightarrow \mathbb{R} \)). In [1] inequality (1) was generalized for higher order convex functions. Therefore we recall now the definition of convex functions of higher orders. To this end we need the notion of divided differences of higher orders. These expressions are defined by the following recurrent formulas:

\[
 f[x_1] = f(x_1) 
\]

and

\[
 f[x_1, \ldots, x_n] = \frac{f[x_1, \ldots, x_{n-1}] - f[x_2, \ldots, x_n]}{x_n - x_1}. 
\]

Let \( I \subset \mathbb{R} \) be an interval. We say that function \( f \) is convex of order \( n \) if

\[
 f[x_1, \ldots, x_{n+2}] \geq 0, x_1, \ldots, x_{n+2} \in I.
\]

Note that 0–convexity means nondecreasingness and 1–convexity is equivalent to standard convexity.

In this paper \( \mathcal{G}_n[a, b], \mathcal{L}_n[a, b], \mathcal{R}_n[a, b], \mathcal{R}_n^*[a, b] \) stand, respectively, for the: \( n \)–point Gauss quadrature operator on the interval \( [a, b] \), \( n \)–point Lobatto quadrature operator on the interval \( [a, b] \), \( n \)–point Radau quadrature operator on the interval \( [a, b] \) containing the point \( a \) as a node and \( n \)–point Radau quadrature operator on...
the interval \([a, b]\) containing the point \(b\) as a node (for details see [12, 13, 14]). For example

\[
\mathcal{G}_2[a, b](f) = \frac{f\left(\frac{3-\sqrt{3}}{6}a + \frac{3+\sqrt{3}}{6}b\right) + f\left(\frac{3+\sqrt{3}}{6}a + \frac{3-\sqrt{3}}{6}b\right)}{2},
\]

\[
\mathcal{L}_3[a, b](f) = \frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)
\]

and

\[
\mathcal{R}_2^r[a, b](f) = \frac{1}{4}f(a) + \frac{3}{4}f\left(\frac{a+2b}{3}\right).
\]

In [1] it was proved that for all \(n\)-convex functions \(f\) with odd \(n\) we have

\[
\mathcal{G}_n[a, b](f) \leq \frac{1}{b-a} \int_a^bf(x)dx \leq \mathcal{L}_n[a, b](f)
\]

whereas for even \(n\) we have

\[
\mathcal{R}_n^r[a, b](f) \leq \frac{1}{b-a} \int_a^bf(x)dx \leq \mathcal{R}_n^r[a, b](f).
\]

In the first part of this paper we are going to present a short proof of these inequalities. To this end we will use an extension of Ohlin lemma obtained in [3]. However, before we do this we need the following definition.

**Definition 1.1.** We say that a function \(H : [a, b] \to \mathbb{R}\) changes sign at least \(s\) times if there exist \(a \leq x_0 < x_1 < \cdots < x_s \leq b\) such that \(H(x_i)H(x_{i+1}) < 0\) for \(i \in \{0, \ldots, s-1\}\). We say that \(H\) changes sign exactly \(s\) times if it changes sign at least \(s\) times but it does not change sign at least \((s+1)\) times. If \(F, G : [a, b] \to \mathbb{R}\), then we say that \(F\) and \(G\) cross (exactly) \(s\) times if \(H = F - G\) changes sign exactly \(s\) times.

Now we may formulate Theorem 4.3 from [3], mentioned above.

**Theorem 1.2.** Let \(X\) and \(Y\) be two random variables such that

\[
\mathbb{E}(X^j - Y^j) = 0, \ j = 1, 2, \ldots, s.
\]

If the distribution functions \(F_X, F_Y\) cross exactly \(s\) times and the last sign of \(F_X - F_Y\) is positive then

\[
\mathbb{E}f(X) \leq \mathbb{E}f(Y)
\]

for all \(s\)-convex functions \(f : \mathbb{R} \to \mathbb{R}\).

In our proof we will also need a result from [2]. In this theorem the following notation is used: if \(I \subset \mathbb{R}\) then \(\Delta(I)\) and \(D(I)\) are defined by the formulas:

\[
\Delta(I) := \{(x, y) \in I^2 : x \leq y\}
\]

and

\[
D(I) := \{(x, x) : x \in I\}.
\]

Let \(\mu\) be a non-zero bounded Borel measure on the interval \([0, 1]\), then its moments \(\mu_n\) are defined by the usual formulas,

\[
\mu_n := \int_0^1 t^n d\mu(t), \ n = 0, 1, 2, \ldots.
\]

The main results of [2] is given by the following theorem.
Theorem 1.3. Let $I \subset \mathbb{R}$ be an open interval let $\Omega \subset \Delta(I)$ be an open subset containing the diagonal $D(I)$ of $I \times I$ and let $\mu$ be a non-zero bounded Borel measure on $[0, 1]$. Assume that $n$ is the smallest non-negative integer such that $\mu_n \neq 0$. If $f : \Omega \rightarrow \mathbb{R}$ is a continuous function satisfying the integral inequality

$$\int_0^1 f(x + t(y - x))d\mu(t) \geq 0$$

then $\mu_n f$ is $(n - 1)$-convex.

2. A short proof of inequalities (2) and (3). The inequalities obtained in [1] are valid for higher order convex functions. As we can see from Theorem 1.2, if we are interested in inequalities satisfied by higher order convex functions then essential role is being played by the number of crossing points of the cumulative distribution functions involved. In the following lemma we obtain an upper estimation of the number of such points.

Lemma 2.1. Let $a, b \in \mathbb{R}$, $a < b$ be given numbers, let $n \in \mathbb{N}$, let $x_1, \ldots, x_n \in [a, b]$ be such that $x_1 < x_2 < \cdots < x_n$ and let $\alpha_1, \ldots, \alpha_n$ be positive numbers such that $\alpha_1 + \cdots + \alpha_n = 1$. Let further $F$ be the cumulative distribution function connected with the measure which is uniformly distributed on the interval $[a, b]$ and let $G$ be the cumulative distribution function of the measure

$$\alpha_1 \delta_{x_1} + \cdots + \alpha_n \delta_{x_n}.$$

Then the following assertions hold true:

(i) if $x_1 > a$ and $x_n < b$ then $F$ and $G$ have at most $2n - 1$ crossing points,

(ii) if $(x_1 = a$ and $x_n < b)$ or $(x_1 > a$ and $x_n = b)$ then $F$ and $G$ have at most $2n - 2$ crossing points,

(iii) if $x_1 = a$ and $x_n = b$ then $F$ and $G$ have at most $2n - 3$ crossing points.

Proof. Since the graph of $G$ consists of horizontal segments and $F$ is increasing, it is visible that $F$ and $G$ can cross at most once in each interval $(x_i, x_{i+1})$. Observe also that $F$ and $G$ can cross at each $x_i$ which lies in the open interval $(a, b)$. This finishes the proof.

In the next theorem we obtain exact number of crossing points for the quadratures considered in this paper.

Theorem 2.2. If $F$ and $G$ are defined so as in the above lemma and $\alpha_i, x_i$ are the weights and nodes (respectively) of $n$–point Gauss quadrature then $F$ and $G$ have exactly $2n - 1$ crossing points. Further if $\alpha_i, x_i$ are the weights and nodes (respectively) of $n$–point Radau quadrature (containing left or right endpoint) then $F$ and $G$ have exactly $2n - 2$ crossing points and, finally if $\alpha_i, x_i$ are the weights and nodes (respectively) of $n$–point Lobatto quadrature then $F$ and $G$ have exactly $2n - 3$ crossing points.

Proof. From Lemma 2.1 we have the upper estimation of the number of crossing points of $F$ and $G$. Thus, for the indirect proof, suppose that for some $n$ the function $G$ connected with the Gauss quadrature crosses $F$ $k$–times with some $k < 2n - 1$ points. Then from Theorem 1.2 we know that the inequality

$$\alpha_1 f(x_1) + \cdots + \alpha_n f(x_n) \leq \frac{1}{b - a} \int_a^b f(x)dx$$

(4)
is satisfied by all \(k\)-convex functions. But, on the other hand, the \(n\)-point Gauss quadrature rule is exact for polynomials of degree \(2n-1\) exactly. Using Theorem 1.3, we can see that every function satisfying (4) must be \((2n-1)\)-convex. This means that we have a contradiction, since not every \(k\)-convex function is \((2n-1)\)-convex.

The proofs concerning Radau and Lobatto quadratures are completely analogous and are omitted.

**Corollary 1.** Let \(n\) be a positive integer. If \(n\) is odd then inequalities (2) are satisfied for every \(n\)-convex function \(f\), if \(n\) is even then inequalities (3) are satisfied for every \(n\)-convex function \(f\).

**Proof.** Let us consider the \(n\)-point Gauss quadrature. This quadrature coincides with the integral for polynomials of degree at most \(2n-1\). This means that first \(2n-1\) moments of the measure connected with this quadrature are equal to the respective moments of the measure equally distributed in the interval \([a,b]\). On the other hand observe that in Theorem 2.2 we proved that the number of crossing points of the cumulative distribution functions connected with this quadrature with the distribution function connected with the usual integral is also equal to \(2n-1\). Thus it is enough to use Theorem 1.2 to obtain the first of inequalities from (2). The remaining cases are completely analogous.

3. **A general inequality satisfied by all \(2\)-convex functions.** In this part of the paper we show a way to extend the results already presented. Namely, in papers [10] and [11] the inequalities of the form

\[ \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \]

or

\[ \frac{1}{b-a} \int_a^b f(x) \, dx \leq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n) \]

valid for convex functions were studied. Now we will use the results generalizing Theorem 1.2 to get inequalities of this type for higher order convex functions. To present our main theorem we need some definitions and notations.

Let \(X,Y\) be two random variables with the cumulative distribution functions \(F,G\). Denote \(F^{[0]}(t) := F(t)\) and \(F^{[k]}(t) := \int_{-\infty}^t F^{[k-1]}(s) \, ds\). Using this notation, we formulate the following theorem from [9](see also [8]).

**Theorem 3.1.** Let \(X,Y\) be two random variables with the cumulative distribution functions \(F,G\), respectively. Then inequality

\[ \mathbb{E} f(X) \geq \mathbb{E} f(Y) \]

holds for all \(m\)-convex functions \(f : \mathbb{R} \to \mathbb{R}\) if and only if the following two conditions are satisfied:

\[ \mathbb{E}(X^k) = \mathbb{E}(Y^k), k = 1, \ldots, m \] (5)

and

\[ (-1)^m (G^{[m]}(t) - F^{[m]}(t)) \geq 0, \text{ for all } t \in \mathbb{R}. \] (6)

We will also need the following simple observation
Remark 1. Let \( X, Y \) be two random variables concentrated on the interval \([a, b]\) with the cumulative distribution functions \( F, G \), respectively. If we have \( EX = EY \) and \( EX^2 = EY^2 \) then
\[
\int_a^b F(x)dx = \int_a^b G(x)dx
\]
and
\[
\int_a^b F^{[1]}(x)dx = \int_a^b G^{[1]}(x)dx.
\]
Indeed, integrating by parts and using the equality of expectations, we get
\[
\int_a^b F(x)dx = F(b)b - F(a)a - \int_a^b xdF(x) = G(b)b - G(a)a - \int_a^b xdG(x) = \int_a^b G(x)dx.
\]
To get (8) we again integrate by parts, arriving at
\[
\int_a^b x^2dF(x) = F(b)b^2 - F(a)a^2 - \int_a^b x^2dF(x) = F(b)b^2 - F(a)a^2 - 2 \int_a^b xF(x)dx.
\]
Further, we have
\[
\int_a^b xF(x)dx = \int_a^b xF^{[1]}(x) = F^{[1]}(b)b - F^{[1]}(a)a - \int_a^b F^{[1]}(x)dx,
\]
and hence we may rewrite (10) in the form
\[
\int_a^b x^2dF(x) = F(b)b^2 - F(a)a^2 - 2F^{[1]}(b)b + 2F^{[1]}(a)a - 2 \int_a^b F^{[1]}(x)dx.
\]
Performing the same calculations for \( \int_a^b x^2dG(x) \) and using the equalities: \( F(a) = G(a), F(b) = G(b), F^{[1]}(a) = G^{[1]}(a), F^{[1]}(b) = G^{[1]}(b) \) and the assumed equality of second moments
\[
\int_a^b x^2dF(x) = \int_a^b x^2dG(x),
\]
we get (8).

Now we may proceed with the generalization of the results from \([1]\). We will do only one step concerning the \(2\)–convex functions. After this, the procedure which is needed for further extension of the results will be clear. As it is known from \([2]\) and from inequality (3), the inequality
\[
\frac{1}{4}f(a) + \frac{3}{4}f\left(\frac{a + 2b}{3}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx
\]
is satisfied by all \(2\)–convex functions.

To generalize this result we will deal with inequalities of the form
\[
\alpha_1f(x_1) + \alpha_2f(x_2) + \alpha_3f(x_3) \leq \frac{1}{b - a} \int_a^b f(x)dx.
\]

Remark 2. It is enough to study the problem of finding the numbers \( \alpha_i, x_i \) such that (13) is satisfied by all \(2\)–convex functions on the interval \([0, 1]\). The obtained values may easily be re-scaled to a given interval \([a, b]\).

To study this problem we have to evaluate the cumulative distribution functions connected with the expressions occurring in (13).
Remark 3. Let the function $G : [0, 1] \to [0, 1]$ be given by

$$G(t) := \begin{cases} 
0 & t \in [0, x_1) \\
\alpha_1 & t \in [x_1, x_2) \\
\alpha_1 + \alpha_2 & t \in [x_2, x_3) \\
1 & t \in [x_3, 1] 
\end{cases} \quad (14)$$

and let $F(t) = t, t \in [0, 1]$. Then $F$ is the distribution function of the measure which is uniformly distributed on the interval $[0, 1]$ and $G$ is the cumulative distribution function of the measure

$$\alpha_1 \delta_{x_1} + \alpha_2 \delta_{x_2} + (1 - \alpha_1 - \alpha_2) \delta_{x_3}.$$

We begin with a result which gives necessary conditions to obtain an inequality valid for all 2–convex functions.

Lemma 3.2. Let numbers $\alpha_1, \alpha_2, \alpha_3, x_1, x_2, x_3 \in [0, 1]$, satisfying $x_1 < x_2 < x_3$ be such that inequality

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) + \alpha_3 f(x_3) \leq \int_0^1 f(x)dx \quad (15)$$

is satisfied by all 2–convex functions $f : [0, 1] \to \mathbb{R}$. Then the following conditions are satisfied:

$$x_1 = 0,$$

if $\alpha_3 > 0$ then $x_3 < 1$,

$$\alpha_1 + \alpha_2 + \alpha_3 = 1, \quad (16)$$

$$\alpha_2 = \frac{\frac{1}{2} x_3 - \frac{1}{3}}{x_2(x_3 - x_2)}, \quad (17)$$

and

$$\alpha_3 = \frac{\frac{1}{2} x_2 - \frac{1}{3}}{x_3(x_2 - x_3)}. \quad (18)$$

Proof. Let $F$ and $G$ be the functions defined in Remark 3. From the condition (5) we know that for functions $x \mapsto 1, x \mapsto x$ and $x \mapsto x^2$, (15) must be satisfied with equality instead of inequality. Thus we have

$$\alpha_1 + \alpha_2 + \alpha_3 = \int_0^1 1dx = 1.$$

Now we show that $x_1 = 0$. Assume, for the indirect proof, that $x_1 > 0$. Then $G^{[1]}(t) = 0, t \in [0, x_1]$. This means that $G^{[2]}$ is also equal to zero on the interval $[0, x_1]$. This yields a contradiction with (6), since $F^{[2]}(t) = \frac{t^2}{\theta}$ is positive on the whole interval $[0, 1]$.

Now we will evaluate the numbers $\alpha_2, \alpha_3$. Since (15) is satisfied for all 2–convex functions we have the equality (5), thus

$$\alpha_2 x_2 + \alpha_3 x_3 = \int_0^1 xdx = \frac{1}{2} \quad (19)$$

and

$$\alpha_2 x_2^2 + \alpha_3 x_3^2 = \int_0^1 x^2dx = \frac{1}{3} \quad (20)$$
Multiplying both sides of (19) by \( x_3 \), we get
\[
\alpha_2 x_2 x_3 + \alpha_3 x_3^2 = \frac{1}{2} x_3.
\]
This, equation together with (20), yields
\[
\alpha_2 x_2 (x_3 - x_2) = \frac{1}{2} x_3 - \frac{1}{3}
\]
which gives us (17). Similarly we get (18).

Now we will show that \( x_3 < 1 \). Again for the indirect proof we suppose that
\( x_3 = 1 \). In such case the graph of the function \( G[1] \) in the interval \( (x_2, 1] \) is a line
with the slope \( \alpha_1 + \alpha_2 < 1 \). However, \( (F[1])'(1) = F(1) = 1 \), this means that
\( G[1](s) > F[1](s) \) on some interval \( (s_0, 1] \). Consequently, function \( \varphi \) given by
\[
\varphi(s) := \int_0^s G[1](t) - F[1](t)dt.
\]
is increasing in the interval \( (s_0, 1] \). However from Remark 1 we know that \( \varphi(1) = 0 \)
which means that \( \varphi(s) < 0, s \in [s_0, 1) \), which is a contradiction with (6).

**Remark 4.** Let \( x_2, x_3 \in (0, 1), x_2 < x_3 \) be given. Observe that from formulas (17)
and (18) it follows that \( x_2 < \frac{2}{3} < x_3 \) (if we want the numbers \( \alpha_2, \alpha_3 \) to be positive).
Further, from (16) we know that
\[
\alpha_1 = 1 - (\alpha_1 + \alpha_2) = \frac{1}{2} \left( x_2 + x_3 \right) - \frac{1}{3} \frac{x_2 x_3}{x_2 x_3}.
\]
Thus we must take \( x_2, x_3 \) such that
\[
\frac{x_2 + x_3}{2} > x_2 x_3 + \frac{1}{3},
\]
(22)
to keep \( \alpha_1 \) positive. Note that to get an inequality satisfied by \( 2 \)–convex functions
we must have \( \alpha_1 > 0 \), since for \( \alpha_1 \leq 0 \) we have
\[
\int_0^{x_2} G[1](t)dt < \int_0^{x_2} F[1](t)dt.
\]

Particularly interesting is the case:
\[
x_2 = \frac{3 - \sqrt{3}}{6}, x_3 = \frac{3 + \sqrt{3}}{6}
\]
where we have \( \alpha_1 = 0, \alpha_2 = \alpha_3 = \frac{1}{2} \). As it was mentioned above, in this case
we do not obtain an inequality for \( 2 \)–convex functions but we get an inequality
for \( 3 \)–convex functions, since the quadrature operator associated with such nodes
is exact for \( 1, x, x^2, x^3 \) i.e. first three moments coincide. It is interesting that
dealing with inequalities for \( 2 \)–convex we encounter a known inequality (see (2))
for \( 3 \)–convex functions. Of course such \( x_2, x_3 \) must be eliminated at some point
and indeed, they are eliminated by the inequality (22).

In view of the above lemma, the problem may be formulated as follows: let
\( x_2 \in \left( 0, \frac{2}{3} \right), x_3 \in \left( \frac{2}{3}, 1 \right) \) which satisfy (22) be given, let \( \alpha_1, \alpha_2, \alpha_3 \) be of the form
obtained in Lemma 3.2. Check if all \( 2 \)–convex functions satisfy
\[
\alpha_1 f(0) + \alpha_2 f(x_2) + (1 - \alpha_1 - \alpha_2) f(x_3) \leq \int_0^1 f(x)dx.
\]
(23)
We will give the solution to this problem but first we need some preliminary notations and observations.
Remark 5. It follows from Theorem 3.1 that inequality (23) is satisfied for all \( 2 \)-convex \( f \) if and only if
\[
G^{[2]}(t) = \int_0^t G^{[1]}(s) ds \geq \int_0^t F^{[1]}(s) ds = F^{[2]}(t)
\]
for all \( t \in [0, 1] \). This condition however is hard to check, therefore, in the main result of this section we will give a condition that is easier verifiable. Namely, it will be shown that it is enough to check if the inequality (24) is satisfied in one (appropriately chosen) point.

Lemma 3.3. Let \( x_2 \in (0, \frac{2}{3}) \), \( x_3 \in (\frac{2}{3}, 1) \) satisfy (22), let \( \alpha_2, \alpha_3 \) be given by (17) and (18), and let \( \alpha_1 := 1 - \alpha_2 - \alpha_3 \). Let further \( G \) be defined by
\[
G(t) := \begin{cases} 
0 & t = 0 \\
\alpha_1 & t \in (0, x_2) \\
\alpha_1 + \alpha_2 & t \in [x_2, x_3) \\
1 & t \in [x_3, 1] 
\end{cases}
\]
and let \( F(t) = t \), \( t \in [0, 1] \). If \( 2\alpha_1 \geq x_2 \) then functions \( G^{[1]} \) and \( F^{[1]} \) have exactly one crossing point. If \( 2\alpha_1 < x_2 \) then functions \( G^{[1]} \) and \( F^{[1]} \) can have either one or three crossing points.

Proof. Observe that, in the case \( 2\alpha_1 \geq x_2 \) we have \( \alpha_1 x_2 \geq \frac{x_2^2}{2} \) which means that
\[
G^{[1]}(x_2) \geq F^{[1]}(x_2),
\]
this case is presented on Figure 1.

Moreover the graph of the function \( F^{[1]} \) in the interval \([0, x_2]\) is a segment and \( G^{[1]} \) is convex. Therefore the graph of \( G^{[1]} \) lies above the graph of \( F^{[1]} \) in the whole interval \([0, x_2]\). Further the graph of \( G^{[1]} \) on the interval \([x_3, 1]\) is a line with the slope equal to 1, and \( (F^{[1]})'(1) = 1 \) which means that the graph of \( G^{[1]} \) lies below the graph of \( F^{[1]} \) in the interval \([x_3, 1]\). Therefore, there must be an odd number of crossing points of the functions \( F^{[1]} \) and \( G^{[1]} \) in the interval \((x_2, x_3)\). However the graph of \( G^{[1]} \) in this interval is a segment and the graph of \( F^{[1]} \) is a parabola. Consequently, they cannot cross three times, i.e. they cross only once.

![Figure 1](image-url)
Now, assume that $2\alpha_1 < x_2$. This means that there is one crossing point of $F^{[1]}$ and $G^{[1]}$ in the interval $[0, x_2]$, see Figure 2. Further, $G^{[1]}(x_2) < F^{[1]}(x_2)$ and (similarly as before) $G^{[1]}(x_3) < F^{[1]}(x_3)$. Thus there is an even number of crossing points of the functions $F^{[1]}$ and $G^{[1]}$ in the interval $(x_2, x_3)$. This even number may be either 0 or 2. If there are no crossing points in $(x_2, x_3)$ then we have one crossing point of the functions $F^{[1]}$ and $G^{[1]}$ in $[0, 1]$ and if there are two crossing points in $(x_2, x_3)$, then there are three such points in $[0, 1]$

![Figure 2](image)

**Figure 2.** The graphs of functions $F^{[1]}$ and $G^{[1]}$ (with two crossing points in the interval $(x_2, x_3)$) in the case $2\alpha_1 < x_2$

To finish the proof consider the case $2\alpha_1 = x_2$. Then we have $G^{[1]}(x_2) = \frac{x_2^2}{4} = F^{[1]}(x_2)$ and (as always) $F^{[1]}(x_3) < G^{[1]}(x_3)$. This means that there is one point $t_0$ in the interval $[x_2, x_3)$ such that $F^{[1]}(t_0) = G^{[1]}(t_0)$ and we have

$$F^{[1]}(t) \leq G^{[1]}(t), t \in [0, t_0] \text{ and } F^{[1]}(t) \geq G^{[1]}(t), t \in [t_0, 1]$$

This means that, in this case, we again have one crossing point.

**Theorem 3.4.** Let $x_2 \in (0, \frac{2}{3})$, $x_3 \in (\frac{2}{3}, 1)$ satisfy (22), let $\alpha_2, \alpha_3$ be given by (17) and (18), and let $\alpha_1 := 1 - \alpha_2 - \alpha_3$. Let further $G$ be defined by (25) and let $F(t) = t, t \in [0, 1]$.

Inequality (23) is satisfied by all 2–convex functions if and only if one of the following conditions is true:

(i) $2\alpha_1 \geq x_2$

(ii) $2\alpha_1 < x_2$ and $F^{[1]}(t) \geq G^{[1]}(t)$ for all $t \in (x_2, x_3)$

(iii) $2\alpha_1 < x_2, t_0, t_1, t_2$ are crossing points of the functions $F^{[1]}$ and $G^{[1]}$ and

$$\int_{t_0}^{t_1} G^{[1]}(t) - F^{[1]}(t)dt \geq 0.$$  \hspace{1cm} (26)

**Proof.** First assume that every 2–convex function $f$ satisfies (23), that $2\alpha_1 < x_2$ and that there are three crossing points of functions $F^{[1]}, G^{[1]}$. It is obvious that (26) is satisfied, since from Theorem 3.1 we know that

$$\int_{0}^{s} G^{[1]}(t) - F^{[1]}(t)dt \geq 0$$

for every $s \in [0, 1]$.
Now we want to show that the conditions (i),(ii),(iii) imply (23). To this end we define a function \( \varphi : [0, 1] \to \mathbb{R} \) by the formula (21).

Then, from the definition of \( \varphi \), it follows that \( \varphi(0) = 0 \). Further Remark 1 gives us \( \varphi(1) = 0 \).

Now, if (i) is satisfied then from Lemma 3.3 we know that there is only one crossing point of the functions \( F^{[1]} \) and \( G^{[1]} \), let us call this point \( t_0 \). Therefore the function \( \varphi_{[0,t_0]} \) is increasing and \( \varphi_{[t_0,1]} \) is decreasing. It is visible that \( \varphi \) must be nonnegative.

Similarly, in the case (ii), we have only one crossing point of \( F^{[1]} \) and \( G^{[1]} \), (lying in the interval \((0, x_2)\). This means that again \( \varphi \) is nonnegative.

If, on the other hand, (iii) is satisfied, then from (26) we know that \( \varphi(t_1) \geq 0 \). Recall that \( \varphi \) satisfies \( \varphi(0) = \varphi(1) = 0 \), functions \( \varphi_{[0,t_0]}, \varphi_{[t_1,t_2]} \) are increasing and \( \varphi_{[t_0,t_1]}, \varphi_{[t_2,1]} \) are decreasing. In view of the following properties it is clear that \( \varphi \) is nonnegative on the interval \([0, 1] \). \( \square \)

**Example 1.** Take \( x_2 = \frac{1}{7} \) and \( x_3 = \frac{7}{17} \), then from Lemma 3.2 we get \( \alpha_2 = \frac{3}{22}, \alpha_3 = \frac{50}{77} \) and \( \alpha_1 = \frac{3}{14} \). We can see that \( 2\alpha_1 > x_2 \) which means that the condition (i) of Theorem 3.4 is satisfied and, therefore,

\[
\frac{3}{14} f(x) + \frac{3}{22} \left( \frac{2}{3} x + \frac{1}{3} y \right) + \frac{50}{77} f \left( \frac{3}{10} x + \frac{7}{10} y \right) \leq \frac{1}{y - x} \int_x^y f(t) dt
\]

is satisfied for all 2–convex functions \( f : [x, y] \to \mathbb{R} \).

**Example 2.** Take \( x_2 = \frac{5}{17} \) and \( x_3 = \frac{3}{4} \), then from Lemma 3.2 we get \( \alpha_2 = \frac{3}{15}, \alpha_3 = \frac{1}{2} \) and \( \alpha_1 = \frac{1}{5} \). We can see that \( 2\alpha_1 < x_2 \) i.e. we must study the functions \( G^{[1]} \) and \( F^{[1]} \) in the interval \((x_2, x_3)\). We have

\[
G^{[1]}(t) = \alpha_1 x_2 + (\alpha_1 + \alpha_2)(t - x_2) = \frac{1}{12} + \frac{1}{2} \left( t - \frac{5}{12} \right) = \frac{1}{2} t - \frac{1}{8},
\]

for all \( t \in \left[ \frac{5}{17}, \frac{3}{4} \right] \) and

\[
F^{[1]}(t) = \frac{t^2}{2}.
\]

This means that the only point where \( G^{[1]}(t) = F^{[1]}(t) \) is \( t_1 = \frac{1}{2} \). The graph of the function \( G^{[1]}(t) \) in the interval \((x_2, x_3)\) is a segment and \( F^{[1]} \) is a convex function. In consequence,

\[
G^{[1]}(t) \leq F^{[1]}(t), t \in [x_2, 1].
\]

Thus we can see that the condition (ii) of Theorem 3.4 is satisfied and, similarly as in the previous remark, the inequality

\[
\frac{1}{5} f(x) + \frac{3}{10} \left( \frac{7}{12} x + \frac{5}{12} y \right) + \frac{1}{2} f \left( \frac{1}{4} x + \frac{3}{4} y \right) \leq \frac{1}{y - x} \int_x^y f(t) dt
\]

is satisfied for all 2–convex functions \( f : [x, y] \to \mathbb{R} \).

**Example 3.** Take \( x_2 = \frac{1}{4} \) and \( x_3 = \frac{3}{4} \), then from Lemma 3.2 we get \( \alpha_2 = \frac{1}{4}, \alpha_3 = \frac{5}{9} \) and \( \alpha_1 = \frac{1}{5} \). We can see that \( 2\alpha_1 < x_2 \) therefore we will find the crossing points of \( F^{[1]} \) and \( G^{[1]} \) in the interval \((x_2, x_3)\). We have

\[
G^{[1]}(t) = \alpha_1 x_2 + (\alpha_1 + \alpha_2)(t - x_2) = \frac{1}{36} + \frac{4}{9} \left( t - \frac{1}{4} \right) = \frac{4}{9} t - \frac{1}{12},
\]
for all $t \in \left[\frac{5}{2}, \frac{3}{4}\right]$ and

$$F[1](t) = \frac{t^2}{2}.$$  

This means that there are two crossing points of the functions $F[1]$ and $G[1]$ in the interval $(x_2, x_3)$: $t_1 = \frac{4}{9} - \frac{\sqrt{10}}{18}$ and $t_2 = \frac{4}{9} + \frac{\sqrt{10}}{18}$. Further

$$G[2](t_1) = \int_{x_2}^{x_1} \frac{4}{9}s \, ds + \int_{x_2}^{t_1} \frac{4}{12} \, ds = \frac{1}{96} - \frac{37\sqrt{10} - 80}{5832}$$

and

$$F[2](t_1) = \frac{t_1^3}{6} = \frac{376 - 101\sqrt{10}}{17496}.$$  

We can see that $F[2](t_1) < G[2](t_1)$ which means that (26) is satisfied and inequality

$$\frac{1}{9}f(x) + \frac{1}{3}f\left(\frac{3}{4}x + \frac{3}{4}y\right) + \frac{5}{9}f\left(\frac{1}{4}x + \frac{3}{4}y\right) \leq \frac{1}{y - x} \int_x^y f(t) \, dt$$

is satisfied for all 2-convex functions $f : [x, y] \to \mathbb{R}$.

We can see that checking if an inequality of the form (13) is satisfied is now very easy (in the most complicated case it is enough to compare two numbers) and it is possible to write a computer program that verifies the inequalities of this type. We end the paper with two remarks concerning possible generalizations of the above result and the remark concerning the results in the converse direction.

**Remark 6.** If we want to study an analogue problem for more points than three then we will have more crossing points than three and therefore we will have to check that the integral from (24) is positive in more points than one. The situation will be then similar to that of Lemma 2 from [6].

**Remark 7.** To study an analogue problem for $k$-convex functions with $k > 2$ we have to calculate the functions $F[k-1]$ and $G[k-1]$ and repeat the reasonings presented here for these functions.

**Remark 8.** Using Theorem 1.3, it is easy to see that all inequalities obtained in this paper in fact characterize 2-convex functions.

**REFERENCES**

[1] M. Bessenyei and Zs. Páles, Higher-order generalizations of Hadamard’s inequality, *Publicationes Math. (Debrecen)*, **61** (2002), 623–643.

[2] M. Bessenyei and Zs. Páles, Characterization of higher order monotonicity via integral inequalities, *P. Roy. Soc. Edinb. A*, **140** (2010), 723–736.

[3] M. Denuit, C. Lefèvre and M. Shaked, The $s$-convex orders among real random variables, with applications, *Math. Inequal. Appl.*, **1** (1998), 585–613.

[4] C. P. Niculescu and L. E. Persson, *Convex Functions and Their Applications*, Springer, New York, 2006.

[5] J. Ohlin, On a class of measures of dispersion with application to optimal reinsurance, *Astin Bull.*, **5** (1969), 249–266.

[6] A. Olbryś and T. Szostok, Inequalities of the Hermite-Hadamard type involving numerical differentiation formulas, *Results Math.*, **67** (2015), 403–416.

[7] T. Rajba, On The Ohlin lemma for Hermite-Hadamard-Fejer type inequalities, *Math. Inequal. Appl.*, **17**, (2014), 557–571.

[8] T. Rajba, On a generalization of a theorem of Levin and Steckin and inequalities of the Hermite-Hadamard type, *Math. Inequal. Appl.*, **20**, (2017), 363–375.

[9] M. Shaked and J. G. Shanthikumar, *Stochastic Orders*, Springer, New York, NY, 2007.
[10] T. Szostok, Ohlin’s lemma and some inequalities of the Hermite-Hadamard type, *Aequationes Math.*, 89 (2015), 915–926.

[11] T. Szostok, Levin-Stechkin theorem and inequalities of the Hermite-Hadamard type, arXiv:1411.7708.

[12] E. W. Weisstein, Legendre-Gauss quadrature, *MathWorld*, 2015.

[13] E. W. Weisstein, Lobatto quadrature, *MathWorld*.

[14] E. W. Weisstein, Radau quadrature, *MathWorld*.

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E-mail address: tomasz.szostok@us.edu.pl