Asymptotic behavior of a quasilinear Keller–Segel system with signal-suppressed motility

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Abstract
This paper is concerned with the density-suppressed motility model:

\[
\begin{align*}
    u_t &= \Delta \left( \frac{u^m}{v^\alpha} \right) + \beta u f(w), \\
    v_t &= D \Delta v - v + u, \\
    w_t &= \Delta w - u f(w)
\end{align*}
\]

in a smoothly bounded convex domain \( \Omega \subset \mathbb{R}^2 \), where \( m > 1, \alpha > 0, \beta > 0 \) and \( D > 0 \) are parameters, the response function \( f \) satisfies \( f \in C^1([0, \infty)), f(0) = 0, f(w) > 0 \) in \( (0, \infty) \). This system describes the density-suppressed motility of E. coli cells in the process of spatio-temporal pattern formation via so-called self-trapping mechanisms. Based on the duality argument, it is shown that for suitable large \( D \) the problem admits at least one global weak solution \((u, v, w)\) which will asymptotically converge to the spatially uniform equilibrium \((u_0 + \beta w_0, u_0 + \beta w_0, 0)\) with \( \bar{u}_0 = \frac{1}{|\Omega|} \int_{\Omega} u(x, 0) \, dx \) and \( \bar{w}_0 = \frac{1}{|\Omega|} \int_{\Omega} w(x, 0) \, dx \) in \( L^\infty(\Omega) \).

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1 Introduction

Chemotaxis, a kind of oriented motion of cells and organisms in response to certain chemicals in the environment, plays an outstanding role in the life of many cells and microorganisms, such as the transport of embryonic cells to developing tissues and immune cells to infection sites ([13,25]). The celebrated mathematical model describing chemotactic migration processes at population level is the Keller–Segel system of the form

\[
\begin{align*}
    u_t &= \nabla \cdot (\gamma(u, v) \nabla u - u \phi(u, v) \nabla v), \quad x \in \Omega, t > 0, \\
    v_t &= d \Delta v - v + u, \quad x \in \Omega, t > 0
\end{align*}
\]  

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in a bounded domain \( \Omega \subset \mathbb{R}^n \) where \( u = u(x, t) \) denotes the population density and \( v = v(x, t) \) is the concentration of chemical substance secreted by the population itself ([17]). The prominent feature of (1.1) is the ability of the constitutive ingredient cross-diffusion thereof to describe the collective behavior of cell populations mediated by a chemoattractant. Indeed, a rich literature has revealed that the Neumann initial-boundary value problem for the classical Keller–Segel system

\[
\begin{align*}
  u_t &= \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\
  v_t &= d \Delta v - v + u, & x \in \Omega, t > 0
\end{align*}
\]  

possesses solutions blowing up in finite time with respect to the spatial \( L^\infty \) norm of \( u \) in two- and even higher-dimensional frameworks under some condition on the mass and the moment of the initial data ([9,10,35], see also the surveys [2]). Apart from that, when \( \phi \) and \( \gamma \) in (1.1) are only smooth positive functions of \( u \) on \([0, \infty)\), a considerable literature underlines the crucial role of asymptotic behavior of the ratio \( \frac{\gamma(u)}{\phi(u)} \) at large values of \( u \) with regard to the occurrence of singularity phenomena (see recent progress in [12,33,34]).

As a simplification of (1.1), the Keller–Segel system with density dependent motility

\[
\begin{align*}
  u_t &= \nabla \cdot (\gamma(v) \nabla u - u \phi(v) \nabla v), & x \in \Omega, t > 0, \\
  v_t &= d \Delta v - v + u, & x \in \Omega, t > 0
\end{align*}
\]  

was proposed to describe the aggregation phase of Dictyostelium discoideum (Dd) cells in response to the chemical signal cyclic adenosine monophosphate (cAMP) secreted by Dd cells in [18]. Here the signal-dependent diffusivity \( \gamma(v) \) and chemotactic sensitivity function \( \phi(v) \) are linked through

\[\phi(v) = (\alpha - 1)\gamma'(v),\]

where \( \alpha \geq 0 \) denotes the ratio of effective body length (i.e. distance between the signal-receptors) to the walk length (see [19] for details). Notice that when \( \alpha = 0 \), there is only one receptor in a cell and hence chemotaxis is driven by the indirect effect of chemicals in the absence of the chemical gradient sensing. In this case, (1.3) reads as

\[
\begin{align*}
  u_t &= \Delta (\gamma(v) u), & x \in \Omega, t > 0, \\
  v_t &= d \Delta v - v + u, & x \in \Omega, t > 0
\end{align*}
\]  

where the considered diffusion process of the population is essentially Brownian, and the assumption \( \gamma'(v) < 0 \) accounts for the repressive effect of the chemical concentration on the population motility ([5]). In the context of acyl-homoserine lactone (AHL) density-dependent motility, the extended model of (1.4)

\[
\begin{align*}
  u_t &= \Delta (u \gamma(v)) + \beta \frac{uw^2}{w^2 + \lambda}, & x \in \Omega, t > 0, \\
  v_t &= D \Delta v + u - v, & x \in \Omega, t > 0, \\
  w_t &= \Delta w - \frac{uw^2}{w^2 + \lambda}, & x \in \Omega, t > 0
\end{align*}
\]  

was proposed in [20] to advocate that spatio-temporal pattern of Eshhricheia coli cells can be induced via so-called “self-trapping” mechanisms, that is at low AHL level, the bacteria undergo run-and-tumble random motion, while at high AHL levels, the bacteria tumble incessantly and become immotile at the macroscale.

In comparison with plenty of results on the Keller–Segel system where the diffusion depends on the density of cells, the respective knowledge seems to be much less complete.
when the cell dispersal explicitly depends on the chemical concentration via the motility function \( \gamma(v) \), which is due to considerable challenges of the analysis caused by the degeneracy of \( \gamma(v) \) as \( v \to \infty \) from the mathematical point of view. Indeed, to the best of our knowledge, Yoon and Kim ([39]) showed that in the case of \( \gamma(v) = \frac{c_0}{v^k} \) for small \( c_0 \), problem (1.4) admits a global classical solutions in any dimensions. The smallness condition on \( c_0 \) is removed lately in [1] for the parabolic-elliptic version of (1.4) with \( 0 < k < \frac{n}{(n - 2)^2} \).

Furthermore, for the full parabolic system (1.4) in the three-dimensional setting, Tao and Winkler ([29]) showed the existence of certain global weak solutions, which become eventually smooth and bounded for suitably small initial data \( u_0 \) under the assumption

\[
(H) \quad \gamma(v) \in C^3([0, \infty)), \quad \text{and there exist} \quad \gamma_1, \gamma_2, \eta > 0 \quad \text{such that} \quad 0 < \gamma_1 \leq \gamma(v) \leq \gamma_2, \quad |\gamma'(v)| < \eta \quad \text{for all} \quad v \geq 0.
\]

It should be remarked that based on the comparison method, Fujie and Jiang ([8]) obtained the uniform-in-time boundedness to (1.4) in two-dimensional setting for the more general motility function \( \gamma \), and in the three-dimensional case under a stronger growth condition on \( 1/\gamma \) respectively. In addition, they investigated the asymptotic behavior to the parabolic-elliptic analogue of (1.4) under the assumption \( \max_{0 \leq v < +\infty} |\gamma'(v)|^2 < +\infty \) or \( \gamma(v) = v^{-k} \) with \( 0 < k < \frac{n}{(n - 2)^2} \) in [7,14].

On the considered time scales of cell migration, e.g. metastatic cells moving in semi-solid medium, often it is relevant to take into account the growth of the population. A prototypical choice to accomplish this is the addition of logistic growth terms of the medium, often it is relevant to take into account the growth of the population. A prototypical choice to accomplish this is the addition of logistic growth terms in the higher dimensions has been proved for large \( \mu > 0 \) ([32]), while for small \( \mu \), the respective model can generate pattern formation (see [24]). The reader is referred to [21,22] for the other studies on the related variants involving super-quadratic degradation terms.

In the context of the diffusion of cells in a porous medium (see the discussions in [3,31]), Winkler ([38]) considered the cross--diffusion system

\[
\begin{aligned}
u_t &= \Delta (\gamma(v) u^n), \quad x \in \Omega, \quad t > 0, \\
\gamma v_t &= \Delta v - v + u, \quad x \in \Omega, \quad t > 0
\end{aligned}
\]

in smoothly bounded convex domains \( \Omega \subset \mathbb{R}^n \), where \( m > 1, \gamma \) generalizes the prototype \( \gamma(v) = a + b(v + d)^{-\alpha} \) with \( a \geq 0, b > 0, d \geq 0 \) and \( \alpha \geq 0 \), and proved the boundedness of global weak solutions to the associated initial-boundary value problem under some constriction on \( m \) and \( \alpha \), which particularly indicates that increasing \( m \) in the cell equation goes along with a certain regularizing effect despite both the diffusion and the cross-diffusion mechanisms implicitly contained in (1.7) are simultaneously enhanced.
In a recent paper [16], Jin et al. considered the three-component system
\begin{equation}
\begin{aligned}
  u_t &= \Delta (\gamma(v) u) + \beta u f(w) - \theta u, \quad x \in \Omega, \ t > 0, \\
  v_t &= D \Delta v + u - v, \quad x \in \Omega, \ t > 0, \\
  w_t &= \Delta w - u f(w), \quad x \in \Omega, \ t > 0
\end{aligned}
\end{equation}
\tag{1.8}
in a bounded domain \( \Omega \subset \mathbb{R}^2 \), where \( \beta, D > 0 \) and \( \theta \geq 0 \), the random motility function \( \gamma(v) \) satisfies (H) and functional response function \( f(w) \) fulfills the assumption
\[ f(w) \in C^1([0, \infty)), \quad f(0) = 0, \ f(w) > 0 \text{ in } (0, \infty) \text{ and } f'(w) > 0 \text{ on } [0, \infty). \]  \tag{1.9}

Based on the method of energy estimates and Moser iteration, they showed the uniform boundedness to initial–boundary value problem of (1.8), inter alia the asymptotic behavior thereof when parameter \( D \) is suitably large. Note that the authors of [23] showed the existence of global classical solutions to system (1.8) without the restriction (H) on \( \gamma(v) \). In synopsis of the above results, one natural problem seems to consist in determining to which extent nonlinear diffusion of porous medium type may influence the solution behavior in chemotaxis systems involving density-suppressed motility. Accordingly, the purpose of the present work is to address this question in the context of the particular choice \( \gamma(v) = v^{-\alpha} \) with \( \alpha > 0 \) instead of assumption (H) in (1.8). Specifically, we consider the asymptotic behavior to the initial–boundary value problem
\begin{equation}
\begin{aligned}
  u_t &= \Delta \left( \frac{u^m}{v^{\alpha}} \right) + \beta u f(w), \quad x \in \Omega, \ t > 0, \\
  v_t &= D \Delta v + u - v, \quad x \in \Omega, \ t > 0, \\
  w_t &= \Delta w - u f(w), \quad x \in \Omega, \ t > 0
\end{aligned}
\end{equation}
\tag{1.10}
along with the initial conditions
\[ u(x, 0) = u_0, \ v(x, 0) = v_0, \ w(x, 0) = w_0, \ x \in \Omega \]  \tag{1.11}
and under the boundary conditions
\[ \frac{\partial u}{\partial v} = \frac{\partial v}{\partial v} = \frac{\partial w}{\partial v} = 0 \text{ on } \partial \Omega \]  \tag{1.12}
in a bounded convex domain \( \Omega \subset \mathbb{R}^2 \) with smooth boundary \( \partial \Omega \).

In what follows, for simplicity we shall drop the differential element in the integrals without confusion, namely abbreviating \( \int_\Omega f(x)dx \) as \( \int_\Omega f \) and \( \int_0^t \int_\partial \Gamma f(x, \tau)dxd\tau \) as \( \int_0^t \int_\partial \Gamma f(. \tau)d\tau \). As an important step towards comprehensive understanding of the effect of nonlinear diffusion on the density-suppressed motility model. Our main result asserts that the weak solutions to the density-suppressed motility system (1.10) may approach the relevant homogeneous steady state in the large time limit if \( D \) is suitably large, which is stated as follows.

**Theorem 1.1** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded convex domain with smooth boundary, and suppose that \( m > 1, \alpha > 0, \beta > 0 \) and \( f \) satisfies (1.9). Assume that initial data \( (u_0, v_0, w_0) \in (W^{1, \infty}(\Omega))^3 \) with \( u_0 \geq 0, v_0 \geq 0 \) and \( v_0 > 0 \) in \( \Omega \). Then problem (1.10)–(1.12) admits at least one global weak solution \((u, v, w)\) in the sense of Definition 2.1 below. Moreover, there exists constant \( D_0 > 0 \) such that if \( D > D_0 \),
\[ \lim_{t \to \infty} \| u(\cdot, t) - u_* \|_{L^\infty(\Omega)} + \| v(\cdot, t) - v_* \|_{L^\infty(\Omega)} + \| w(\cdot, t) \|_{L^\infty(\Omega)} = 0 \]  \tag{1.13}
with \( u_* = \frac{1}{|\Omega|} \int_\Omega u_0 + \frac{\beta}{|\Omega|} \int_\Omega w_0 \).
Asymptotic behavior of a quasilinear Keller–Segel system with…

As the first step to prove the above claim, in Sect. 2 we give the definition of a global weak solution to problem (1.10)–(1.12) and recall that problem (1.10)–(1.12) with \( m > 1 \) and \( \alpha > 0 \) possesses a globally defined weak solution in two-dimensional setting by the approximation procedure (2.5) used in [38]. With respect to the convergence properties asserted in (1.13), our analysis is essentially different from that of [16]. In fact, thanks to \( \gamma_1 \leq \gamma(v) \leq \gamma_2 \) for all \( v \geq 0 \) in (H), authors of [16] derived the estimate of \( \|u(\cdot, t)\|_{L^2(\Omega)} \), which is the start point of a priori estimate of \( \|u(\cdot, t)\|_{L^\infty(\Omega)} \). In particular, the assumption \( \gamma_1 \leq \gamma(v) \) plays an essential role in constructing energy function \( F(u, v) := \|u(\cdot, t) - u_*\|_{L^2(\Omega)} + \|v(\cdot, t) - u_*\|_{L^2(\Omega)} \), which leads to the convergence of \((u, v)\) if \( D \) is suitable large (see the proofs of Lemma 4.8 and Lemma 4.10 in [16] for the details). Whereas our asymptotic analysis consists at its core in an analysis of the functional

\[
\int_\Omega u^2 + \eta \int_\Omega |\nabla v|^2
\]

for solutions of certain regularized versions of (1.10), provided that in dependence on the model parameter \( D \) the positive constant \( \eta \) is suitably chosen when \( D \) is suitable large. This yields the finiteness of \( \int_0^\infty \int_\Omega |\nabla u|^{\frac{m+1}{2}} |v|^{2} \) and \( \int_0^\infty \int_\Omega |\nabla v|^2 \) (see Lemma 5.3), and then entails that as a consequence of these integral inequalities, all our solutions asymptotically become homogeneous in space and hence satisfy (1.13) (Lemmas 5.4–5.7).

**Remark 1.1**

1. Note that as an apparently inherent drawback, assumption (H) in [16] excludes \( \gamma(v) \) decay functions such as \( v^{-\alpha} \). Indeed, despite \( v \) is bounded below by \( \delta \) with the help of Lemma 2.3 and thereby the upper bound for \( \gamma(v) \) can be removed, an lower bound for \( \gamma(v) \) in (H) is essentially required therein.

2. Due to the results on existence of global solutions in [38], the asymptotic behavior of solutions herein seems to be achieved for the higher-dimensional version of (1.10) at the cost of additional constraint on \( m \) and \( \alpha \).

### 2 Preliminaries

Throughout this paper, we shall pursue weak solutions to problem (1.10)–(1.12) specified as follows.

**Definition 2.1** Let \( m > 1, \alpha > 0, \beta > 0 \) and \( f \) satisfies (1.9). Then a triple \((u, v, w)\) of nonnegative functions

\[
\begin{align*}
  u &\in L^1_{loc}(\Omega \times [0, \infty)) \\
v &\in L^1_{loc}([0, \infty); W^{1,1}(\Omega)) \\
w &\in L^1_{loc}([0, \infty); W^{1,1}(\Omega))
\end{align*}
\]

will be called a global weak solution of problem (1.10)–(1.12) if

\[
u^m / v^\alpha \in L^1_{loc}(\Omega \times [0, \infty))
\]

and

\[
- \int_0^\infty \int_\Omega u \phi_t - \int_\Omega u_0 \phi(\cdot, 0) = \int_0^\infty \int_\Omega \frac{u^m}{v^\alpha} \Delta \phi + \beta \int_0^\infty \int_\Omega uf(w) \phi
\]
for all \( \varphi \in C_0^\infty(\Omega \times [0, \infty)) \) such that \( \frac{\partial \varphi}{\partial t} |_{\partial \Omega} = 0 \) and

\[
- \int_0^\infty \int_\Omega \varphi_t - \int_\Omega \varphi_v(\cdot, 0) = -D \int_0^\infty \int_\Omega \nabla v \cdot \nabla \varphi - \int_0^\infty \int_\Omega \varphi + \int_0^\infty \int_\Omega w \varphi
\]

(2.3)

for all \( \varphi \in C_0^\infty(\Omega \times [0, \infty)) \) as well as

\[
\int_0^\infty \int_\Omega \varphi_t - \int_\Omega \varphi_v(\cdot, 0) = -\int_0^\infty \int_\Omega \nabla w \cdot \nabla \varphi - \int_0^\infty \int_\Omega u \varphi
\]

(2.4)

for all \( \varphi \in C_0^\infty(\Omega \times [0, \infty)) \).

For \( \varepsilon \in (0, 1) \), we denote by \((u_\varepsilon, v_\varepsilon, w_\varepsilon)\) the solution of the regularized problem

\[
\begin{aligned}
u_{ef} &= \varepsilon \Delta (u_\varepsilon + 1)^M + \Delta \left( u_\varepsilon (u_\varepsilon + \varepsilon)^{m-1} v_\varepsilon^\alpha \right) + \beta u_\varepsilon f(w_\varepsilon), \quad x \in \Omega, \ t > 0, \\
u_{ef} &= D \Delta v_\varepsilon + u_\varepsilon - v_\varepsilon, \quad x \in \Omega, \ t > 0, \\
w_{ef} &= \Delta w_\varepsilon - u_\varepsilon f(w_\varepsilon), \quad x \in \Omega, \ t > 0, \\
\frac{\partial u_\varepsilon}{\partial v} &= \frac{\partial v_\varepsilon}{\partial v} = \frac{\partial w_\varepsilon}{\partial v} = 0, \quad x \in \partial \Omega, \ t > 0, \\
u_\varepsilon(x, 0) &= u_0, \ v_\varepsilon(x, 0) = 0, \ w_\varepsilon(x, 0) = w_0,
\end{aligned}
\]

(2.5)

with \( M > m \). Note that due to the a priori boundedness of \( w_\varepsilon \), the global smooth solvability of (2.5) can be derived by the argument in Lemma 2.4 of [38] with evident minor adaptations, and we may refrain from giving the details for brevity here. As for the global weak solutions of (1.10)–(1.12), we can state as follows.

**Lemma 2.1** Let \( m > 1, \alpha > 0, \beta > 0 \) and \( f \) satisfies (1.9). Then there exist \((\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)\) as well as nonnegative functions

\[
\begin{aligned}
u_j \not\to 0 \text{ as } j \to \infty \text{ and as } \varepsilon_j \not\to 0, \text{ we have}
\end{aligned}
\]

(2.7)

\[
\begin{aligned}
u_j \to \nu \text{ a.e. in } \Omega \times (0, \infty), \\
u_j \to \nu \text{ in } \bigcap_{p \geq 1} L^p_{loc}(\Omega \times [0, \infty)),
\end{aligned}
\]

(2.8)

\[
\begin{aligned}
u_j \to \nu \text{ in } C^{0}_{loc}(\Omega \times [0, \infty)), \\
u_j \to \nu \text{ in } L_{loc}^2(\Omega \times [0, \infty)),
\end{aligned}
\]

(2.9)

\[
\begin{aligned}
u_j \to \nu \text{ in } L_{loc}^2(\Omega \times [0, \infty)), \\
\nabla v_j \to \nabla \nu \text{ in } L_{loc}^2(\Omega \times [0, \infty)), \\
\nabla w_j \to \nabla w \text{ in } L_{loc}^2(\Omega \times [0, \infty)).
\end{aligned}
\]

(2.10)

(2.11)

(2.12)

Moreover, \( \nu > 0 \) in \( \Omega \times (0, \infty) \) and \((\nu, \nu, \nu, \nu)\) forms a global weak solution of (1.10)–(1.12) in the sense of Definition 2.1.

**Proof** The existence of global weak solutions of (1.10)–(1.12) can be verified on the basis of straightforward extraction procedures as in [38]. Indeed, due to the a priori boundedness of \( w_\varepsilon \), one can derive some necessary a priori estimation for \((u_\varepsilon, v_\varepsilon, w_\varepsilon)\) such as \( \int_{t}^{t+1} \int_{\Omega} u_{\varepsilon}^p \) with all \( p < m + 1, (u_\varepsilon, v_\varepsilon, w_\varepsilon) \) in \( (W^{1,q}(\Omega))^2 \) with some \( q > 2 \) and \( u_\varepsilon \in L^\infty(\Omega) \), and finally apply an Aubin-Lions lemma to obtain a weak solution of (1.10)–(1.12) with the additional information (2.7)–(2.12) (we refer the reader to the proof of Lemma 7.1 in [38] for detail). □
The following basic properties of the spatial $L^1$ norms of $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ as well as the $L^\infty$ norm of $w_\varepsilon$ are easily verified.

**Lemma 2.2** Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be the classical solution of (2.5) in $\Omega \times (0, \infty)$. Then we have

\[
\|u_\varepsilon(\cdot, t)\|_{L^1(\Omega)} + \beta \|w_\varepsilon(\cdot, t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} + \beta \|w_0\|_{L^1(\Omega)},
\]

(2.13)

\[
\|u_\varepsilon(\cdot, t)\|_{L^1(\Omega)} \geq \|u_0\|_{L^1(\Omega)},
\]

(2.14)

\[
\int_\Omega v_\varepsilon(\cdot, t) \leq \int_\Omega v_0 + \int_\Omega u_0 + \beta \int_\Omega w_0
\]

(2.15)

as well as

\[
\|w_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \text{ is nonincreasing in } [0, \infty).
\]

(2.16)

**Proof** Multiplying $w_\varepsilon$-equation by $\beta$ and adding the result to $u_\varepsilon$-equation in (2.5), we get

\[
\beta \frac{d}{dt} \int_\Omega w_\varepsilon + \frac{d}{dt} \int_\Omega u_\varepsilon = 0,
\]

(2.17)

which immediately yields (2.13). An integration of the first equation in (2.5) gives us

\[
\frac{d}{dt} \int_\Omega u_\varepsilon = \int_\Omega u_\varepsilon f(w_\varepsilon) \geq 0
\]

(2.18)

which readily entails (2.14). Upon the integration of the second equation in (2.5), we can see that

\[
\frac{d}{dt} \int_\Omega v_\varepsilon + \int_\Omega v_\varepsilon \leq \int_\Omega u_\varepsilon
\]

which, along with (2.13) leads to (2.15). Due to the fact that $f$ and $w_\varepsilon$ are nonnegative, the claim in (2.16) results upon an application of the maximum principle to $w_\varepsilon$-equation in (2.5). \(\square\)

Let us first derive a positive uniform-in-time lower bound for $v_\varepsilon$ which will alleviate the difficulties caused by the singularity of signal-dependent motility function $v^{-a}$ near zero. Despite the quantitative lower estimate for solutions of the Neumann problem was established in the related literature ([11,38]), we present a proof of our results with some necessary details to make the lower bound accessible to the sequel analysis.

**Lemma 2.3** For all $D \geq 1$ and $\varepsilon \in (0, 1)$, there exists $\delta > 0$ such that

\[
v_\varepsilon(x, t) > \delta \text{ for all } x \in \Omega \text{ and } t > 0.
\]

(2.19)

**Proof** According to the pointwise lower bound estimate for the Neumann heat semigroup $(e^{t\Delta})_{t \geq 0}$ on the convex domain $\Omega$, one can find $c_1(\Omega) > 0$ such that

\[
e^{t\Delta} \varphi \geq c_1(\Omega) \int_\Omega \varphi \text{ for all } t \geq 1 \text{ and each nonnegative } \varphi \in C^0(\overline{\Omega})
\]

(e.g. [6,11]).

By the time rescaling $\tilde{t} = Dt$, we can see that $\tilde{v}(x, \tilde{t}) := v_\varepsilon(x, \frac{\tilde{t}}{D})$ satisfies

\[
\frac{\partial \tilde{v}}{\partial \tilde{t}} = \Delta \tilde{v} - D^{-1} \tilde{v} + D^{-1} u_\varepsilon(x, D^{-1} \tilde{t}).
\]

(2.20)
Now applying the variation-of-constant formula to (2.20), we have

$$\tilde{v}(\cdot, \tilde{t}) = e^{(\Delta-D^{-1})} v_0(\cdot) + D^{-1} \int_0^{\tilde{t}} e^{(\tilde{t}-s)(\Delta-D^{-1})} u_\varepsilon(\cdot, D^{-1} s) ds \quad t > 0,$$  \hspace{1cm} (2.21)

where by the comparison principle, we can see

$$e^{(\Delta-D^{-1})} v_0(\cdot) \geq e^{-2} \inf_{x \in \Omega} v_0(x)$$

for all \( x \in \Omega, \tilde{t} \leq 2D \).

and

$$D^{-1} \int_0^{\tilde{t}} e^{(\tilde{t}-s)(\Delta-D^{-1})} u_\varepsilon(\cdot, D^{-1} s) ds \geq D^{-1} \int_0^{\tilde{t}} e^{(\tilde{t}-s)(\Delta-D^{-1})} u_\varepsilon(\cdot, D^{-1} s) ds$$

$$\geq c_1(\Omega) D^{-1} \left( \int_0^{\tilde{t}} e^{(\tilde{t}-s)(\Delta-D^{-1})} ds \right) \inf_{s \in (0, \infty)} \int_{\Omega} u_\varepsilon(\cdot, s)$$

$$\geq c_1(\Omega) (e^{-D^{-1}} - e^{-D^{-1} \tilde{t}}) \int_{\Omega} u_0$$

$$\geq \frac{c_1(\Omega)}{2e} \int_{\Omega} u_0$$

for all \( x \in \Omega \) and \( \tilde{t} \geq 2D \),

due to \( D \geq 1 \). Therefore inserting above inequalities into (2.21) readily establish (2.19) with

$$\delta = \min\{ \frac{c_1(\Omega)}{2e} \int_{\Omega} u_0, e^{-2} \inf_{x \in \Omega} v_0(x) \}. \quad \Box$$

Through a straightforward semigroup argument, we formulate a favorable dependence of \( \| v_\varepsilon(\cdot, t) \|_{L^p(\Omega)} \) with respect to parameter \( D \).

**Lemma 2.4** For \( p > 1 \), there exists \( C(p) > 0 \) such that

$$\| v_\varepsilon(\cdot, t) \|_{L^p(\Omega)} \leq C(p) \left( 1 + D^{\frac{1}{p} - 1} \right) \quad \text{for all } t > 0.$$  \hspace{1cm} (2.22)

**Proof** Applying a Duhamel’s formula to the equation

$$\frac{\partial \tilde{v}}{\partial t} = \Delta \tilde{v} - D^{-1} \tilde{v} + D^{-1} u_\varepsilon(x, D^{-1} t)$$

satisfied by \( \tilde{v}(x, \tilde{t}) := v_\varepsilon(x, \frac{\tilde{t}}{D}) \) and employing well-known smoothing properties of the Neumann heat semigroup \( (e^{t \Delta})_{t \geq 0} \) on \( \Omega \) (see Lemma 3 of [27] or Lemma 1.3 of [36] for...
example), we can find $c_\rho > 0$ such that for any $\tilde{t} > 0$

$$\| \tilde{v}_\epsilon (\cdot, \tilde{t}) \|_{L^p(\Omega)}$$

$$= \left\| e^{-D^{-1}\tilde{t}}e^{i\Delta v_0(\cdot)} + D^{-1}\int_0^{\tilde{t}} e^{i(\tilde{t}-s)(\Delta-D^{-1})}u_\epsilon(\cdot, D^{-1}s)ds \right\|_{L^p(\Omega)}$$

$$\leq e^{-D^{-1}\tilde{t}}\| v_0 \|_{L^p(\Omega)} + \frac{c_\rho}{D}\int_0^{\tilde{t}} e^{-D^{-1}(\tilde{t}-s)}(1 + (\tilde{t} - s)^{-\frac{1}{p}})\| u_\epsilon(\cdot, D^{-1}s) \|_{L^1(\Omega)}ds$$

$$\leq \| v_0 \|_{L^p(\Omega)} + \frac{c_\rho}{D}(\| u_0 \|_{L^1(\Omega)} + \beta\| w_0 \|_{L^1(\Omega)})\int_0^{\tilde{t}} e^{-D^{-1}(\tilde{t}-s)}(1 + (\tilde{t} - s)^{-\frac{1}{p}})ds$$

$$= \| v_0 \|_{L^p(\Omega)} + \frac{c_\rho}{D}(\| u_0 \|_{L^1(\Omega)} + \beta\| w_0 \|_{L^1(\Omega)})\int_0^{\tilde{t}} e^{-D^{-1}\sigma}(1 + \sigma^{-\frac{1}{p}})d\sigma$$

$$\leq \| v_0 \|_{L^p(\Omega)} + (1 + D^{\frac{1}{p}-1})c_\rho(\| u_0 \|_{L^1(\Omega)} + \beta\| w_0 \|_{L^1(\Omega)})\left(1 + \int_0^{\infty} e^{-\sigma}\sigma^{-1}\frac{1}{p}d\sigma\right),$$

which ends up (2.22) with \( C(p) = \| v_0 \|_{L^p(\Omega)} + 2c_\rho(\| u_0 \|_{L^1(\Omega)} + \beta\| w_0 \|_{L^1(\Omega)})(1 + \int_0^{\infty} e^{-\sigma}\sigma^{-1}\frac{1}{p}d\sigma). \)

\[\square\]

### 3 Space-time \( L^1 \)-estimates for \( u_\epsilon^{m+1}v^{-\alpha}_\epsilon \)

In this section, taking advantage of the special structure of the diffusive processes in (2.5) (also (1.8)), the classical duality arguments (cf. [4,29]) is used to obtain the fundamental regularity information for a bootstrap argument. To this end, we denote by \( A \) the self-adjoint realization of \(-\Delta + 1\) under homogeneous Neumann boundary condition in \( L^2(\Omega) \) with its domain given by \( D(A) = \left\{ \varphi \in W^{2,2}(\Omega) \big| \frac{\partial \varphi}{\partial n} = 0 \right\} \) and \( A \) is self-adjoint and possesses a family \((A^\beta)_{\beta \in \mathbb{R}}\) of corresponding densely defined self-adjoint fractional powers.

**Lemma 3.1** Assume that \( m > 1 \) and \( D \geq 1 \), then for \( t > 0 \)

$$\frac{d}{dt}\int_\Omega |A^{-\frac{1}{2}}(u_\epsilon + 1)|^2 + \int_\Omega u_\epsilon^{m+1}v^{-\alpha}_\epsilon \leq C \int_\Omega |A^{-1}(u_\epsilon + 1)|^{m+1} + C$$

(3.1)

with constant \( C > 0 \) independent of \( D \).

**Proof** Due to \( \partial_t(u_\epsilon + 1) = u_\epsilon f \), the first equation in (2.5) can be written as

$$\frac{d}{dt}A^{-1}(u_\epsilon + 1) + \varepsilon(u_\epsilon + 1)^M + u_\epsilon(u_\epsilon + \varepsilon)^{m-1}v^{-\alpha}_\epsilon$$

$$= A^{-1}\left\{ \varepsilon(u_\epsilon + 1)^M + u_\epsilon(u_\epsilon + \varepsilon)^{m-1}v^{-\alpha}_\epsilon + \beta u_\epsilon f(w_\epsilon) \right\}. \quad (3.2)$$
Testing (3.2) by \( u_\varepsilon + 1 \), one has

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |A^{-\frac{1}{2}}(u_\varepsilon + 1)|^2 + \varepsilon \int_\Omega (u_\varepsilon + 1)^{M+1} + \int_\Omega u_\varepsilon (u_\varepsilon + \varepsilon)^{m-1}(u_\varepsilon + 1)v_\varepsilon^{-\alpha} = \varepsilon \int_\Omega (u_\varepsilon + 1)^{M} A^{-1}(u_\varepsilon + 1) + \int_\Omega u_\varepsilon (u_\varepsilon + \varepsilon)^{m-1}v_\varepsilon^{-\alpha}A^{-1}(u_\varepsilon + 1) + \beta \int_\Omega u_\varepsilon f(w_\varepsilon) A^{-1}(u_\varepsilon + 1). \tag{3.3}
\]

Thanks to \( W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega) \) in two-dimensional setting and the standard elliptic regularity in \( L^2(\Omega) \), one can find \( c_1 > 0 \) and \( c_2 > 0 \) such that

\[
\|\varphi\|_{L^{M+1}(\Omega)} \leq c_1 \|\varphi\|_{W^{2,2}(\Omega)}^{M+1} \leq c_2 \|A\varphi\|_{L^2(\Omega)}^{M+1} \tag{3.4}
\]

for all \( \varphi \in W^{2,2}(\Omega) \) such that \( \frac{\partial \varphi}{\partial \nu} \vert_{\partial \Omega} = 0 \). Hence by the Young inequality, we can see that

\[
\varepsilon \int_\Omega (u_\varepsilon + 1)^{M} A^{-1}(u_\varepsilon + 1) \leq \varepsilon \int_\Omega (u_\varepsilon + 1)^{M+1} + \varepsilon \int_\Omega |A^{-1}(u_\varepsilon + 1)|^{M+1} \leq \frac{\varepsilon}{2} \|u_\varepsilon + 1\|_{L^{M+1}(\Omega)}^{M+1} + \frac{\varepsilon c_1}{2} \|A^{-1}(u_\varepsilon + 1)\|_{W^{2,2}(\Omega)}^{M+1} = \frac{\varepsilon}{2} \int_\Omega (u_\varepsilon + 1)^{M+1} + \frac{\varepsilon c_1 c_2}{2} \|u_\varepsilon + 1\|_{L^{2}(\Omega)}^{M+1}, \tag{3.5}
\]

which along with the Young inequality implies that for any \( \varepsilon_1 > 0 \), there exits \( c(\varepsilon_1) > 0 \) such that \( \|\varphi\|_{L^2(\Omega)} \leq \varepsilon_1 \|\varphi\|_{L^{M+1}(\Omega)} + c(\varepsilon_1) \|\varphi\|_{L^1(\Omega)} \) due to \( M > 1 \), entails that

\[
\varepsilon \int_\Omega (u_\varepsilon + 1)^{M} A^{-1}(u_\varepsilon + 1) \leq \frac{3\varepsilon}{4} \int_\Omega (u_\varepsilon + 1)^{M+1} + c_3 \|u_\varepsilon + 1\|_{L^1(\Omega)}^{M+1}. \tag{3.6}
\]

Furthermore, since \( \|w_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)} \), we apply Lemma 2.3 and Young’s inequality to obtain that for \( t > 0 \),

\[
\int_\Omega u_\varepsilon(u_\varepsilon + \varepsilon)^{m-1}v_\varepsilon^{-\alpha}A^{-1}(u_\varepsilon + 1) \leq \frac{1}{4} \int_\Omega \left\{ u_\varepsilon(u_\varepsilon + \varepsilon)^{m-1}v_\varepsilon^{-\alpha} + c_4 \int_\Omega |A^{-1}(u_\varepsilon + 1)|^{m+1}v_\varepsilon^{-\alpha} \right\} \tag{3.7}
\]

and

\[
\beta \int_\Omega u_\varepsilon f(w_\varepsilon) A^{-1}(u_\varepsilon + 1) \leq \frac{1}{4} \int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha} + c_5 \int_\Omega v_\varepsilon^{\alpha} |A^{-1}(u_\varepsilon + 1)|^{m+1} \leq \frac{1}{4} \int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha} + \int_\Omega |A^{-1}(u_\varepsilon + 1)|^{m+1} + c_6 \int_\Omega v_\varepsilon^{\alpha}. \tag{3.8}
\]

Noticing that \( u_\varepsilon + 1 \geq \max\{u_\varepsilon, \varepsilon\} \), we have

\[
\int_\Omega u_\varepsilon(u_\varepsilon + \varepsilon)^{m-1}(u_\varepsilon + 1)v_\varepsilon^{-\alpha} \geq \frac{1}{4} \int_\Omega u_\varepsilon^{m+1}(u_\varepsilon + \varepsilon)^{m-1}v_\varepsilon^{-\alpha} \frac{3}{4} \int_\Omega u_\varepsilon^{m+1}v_\varepsilon^{-\alpha}. \]
and hence insert \((3.8)\) and \((3.7)\) into \((3.3)\) to get
\[
\frac{d}{dt} \int_{\Omega} |A^{-\frac{1}{2}}(u_{\varepsilon} + 1)|^2 + \int_{\Omega} u_{\varepsilon}^{m+1} v_{\varepsilon}^{-\alpha} \\
\leq 2(c_4 \delta^{-\alpha} + 1) \int_{\Omega} |A^{-1}(u_{\varepsilon} + 1)|^{m+1} + 2c_6 \int_{\Omega} v_{\varepsilon}^{\frac{\alpha}{m-1}},
\]
which along with Lemma 2.4 and \(D \geq 1\) readily arrive at \((3.1)\).

By means of suitable interpolation arguments, one can appropriately estimate the integrals
\[
\int_{\Omega} |A^{-1}(u_{\varepsilon} + 1)|^{m+1} \quad \text{and} \quad \int_{\Omega} |A^{-\frac{1}{2}}(u_{\varepsilon} + 1)|^2
\]
in terms of \(\int_{\Omega} u_{\varepsilon}^{m+1} v_{\varepsilon}^{-\alpha}\) and thereby derive estimate of the form
\[
\int_{t}^{t+1} \int_{\Omega} u_{\varepsilon}^{m+1} v_{\varepsilon}^{-\alpha} \leq C
\]
with \(C > 0\) independent of \(D\), which can be stated as follows

**Lemma 3.2** Let \(m > 1\) and \(D \geq 1\). Then there exists \(C > 0\) such that for all \(D \geq 1\) as well as \(\varepsilon \in (0, 1)\)
\[
\int_{t}^{t+1} \int_{\Omega} u_{\varepsilon}^{m+1} v_{\varepsilon}^{-\alpha} \leq C \quad \text{for all} \quad t > 0.
\]

**Proof** By the standard elliptic regularity in \(L^2(\Omega)\), we have
\[
\int_{\Omega} |A^{-1}(u_{\varepsilon} + 1)|^{m+1} \leq c_1 \|u_{\varepsilon} + 1\|^{m+1}_{L^1(\Omega)}.
\]
Noticing that for the given \(p \in (2, m+1)\) (for example \(p := \frac{m+3}{2}\)), an application of Young’s inequality implies that for any \(\eta > 0\), there exists \(c_1(\eta) > 0\) such that
\[
c_1 \|u_{\varepsilon} + 1\|^{m+1}_{L^1(\Omega)} \leq \eta \|u_{\varepsilon} + 1\|^{m+1}_{L^p(\Omega)} + c_1(\eta) \|u_{\varepsilon} + 1\|^{m+1}_{L^1(\Omega)}.
\]
On the other hand, by the Hölder inequality, we can see that
\[
\int_{\Omega} |A^{-\frac{1}{2}}(u_{\varepsilon} + 1)|^2 = \int_{\Omega} (u_{\varepsilon}^{m+1} v_{\varepsilon}^{-\alpha})^{\frac{p}{m+1}} v_{\varepsilon}^{\frac{pa}{m+1}} \\
\leq \left( \int_{\Omega} u_{\varepsilon}^{m+1} v_{\varepsilon}^{-\alpha} \right)^{\frac{p}{m+1}} \left( \int_{\Omega} v_{\varepsilon}^{\frac{pa}{m+1-p}} \right)^{\frac{m+1-p}{m+1}}.
\]
Hence combining above estimates with Lemma 2.4, we arrive at
\[
\int_{\Omega} |A^{-1}(u_{\varepsilon} + 1)|^{m+1} \leq \eta \|u_{\varepsilon} + 1\|^{m+1}_{L^p(\Omega)} + c_1(\eta) \|u_{\varepsilon} + 1\|^{m+1}_{L^1(\Omega)} \\
\leq \eta \|u_{\varepsilon} + 1\|^{m+1}_{L^p(\Omega)} + c_2(\eta) \\
\leq \eta \left( \int_{\Omega} u_{\varepsilon}^{m+1} v_{\varepsilon}^{-\alpha} \right)^{\frac{p}{m+1-p}} + c_2(\eta) \\
\leq \eta c_3(\alpha, m) \left( \int_{\Omega} u_{\varepsilon}^{m+1} v_{\varepsilon}^{-\alpha} \right) + c_2(\eta) \quad \text{for all} \quad t > 0.
\]
On the other hand, by self-adjointness of $A^{-\frac{1}{2}}$ and Hölder’s inequality, we get
\[
\int_{\Omega} |A^{-\frac{1}{2}}(u_\epsilon + 1)|^2 = \int_{\Omega} (u_\epsilon + 1) A^{-1}(u_\epsilon + 1)
\leq \|u_\epsilon + 1\|_{L^2(\Omega)} \|A^{-1}(u_\epsilon + 1)\|_{L^2(\Omega)}
\leq c_4 \|u_\epsilon + 1\|_{L^2(\Omega)}^2
\leq c_5 \|u_\epsilon\|_{L^2(\Omega)}^2 + c_5
\leq c_6 \|u_\epsilon\|_{L^2(\Omega)}^{m+1} + c_6 \quad \text{for all } t > 0.
\] (3.11)

So in this position, proceeding in the same way as above we also have
\[
\int_{\Omega} |A^{-\frac{1}{2}}(u_\epsilon + 1)|^2 \leq c_6 \eta \|u_\epsilon + 1\|_{L^2(\Omega)}^{m+1} + c_1(\eta) c_6 \|u_\epsilon + 1\|_{L^2(\Omega)}^{m+1}
\leq \eta c_3(\alpha, m) \left( \int_{\Omega} u_\epsilon^{m+1} v_\epsilon^{-\alpha} \right) + c_7(\eta) \quad \text{for all } t > 0. \tag{3.12}
\]

Therefore inserting (3.10) and (3.12) into (3.1) and taking $\eta$ sufficiently small, we have
\[
\frac{d}{dt} \int_{\Omega} |A^{-\frac{1}{2}}(u_\epsilon + 1)|^2 + c_8 \int_{\Omega} |A^{-\frac{1}{2}}(u_\epsilon + 1)|^2 + c_8 \int_{\Omega} u_\epsilon^{m+1} v_\epsilon^{-\alpha} \leq c_9 \quad \text{for all } t > 0
\] (3.13)

with some $c_8 > 0, c_9 > 0$ for all $D \geq 1$. Furthermore, by Lemma 3.4 of [28], we immediately obtain (3.9). \hfill \Box

As the direct consequence of Lemma 3.2 and Lemma 2.4, we have

**Lemma 3.3** Let $m > 1, D \geq 1$, then for $p \in (\max\{2, \frac{m+1}{\alpha+1}\}, m+1)$ one can find a constant $C(p) > 0$ such that
\[
\int_{t}^{t+1} \int_{\Omega} u_\epsilon^p(\cdot, s) ds \leq C(p) \quad \text{for all } t > 0 \text{ and } D \geq 1. \tag{3.14}
\]

**Proof** For $p \in (2, m+1)$, we utilize Young’s inequality to estimate
\[
\int_{t}^{t+1} \int_{\Omega} u_\epsilon^p = \int_{t}^{t+1} \int_{\Omega} (u_\epsilon^{m+1} v_\epsilon^{-\alpha})^{\frac{p}{m+1}} v_\epsilon^{\frac{p\alpha}{m+1}} \frac{\partial u_\epsilon}{\partial t}
\leq \int_{t}^{t+1} \int_{\Omega} u_\epsilon^{m+1} v_\epsilon^{-\alpha} + \int_{t}^{t+1} \int_{\Omega} v_\epsilon^{\frac{p\alpha}{m+1}} \quad \text{for all } t > 0,
\]
which leads to (3.14) with the help of Lemma 2.4. \hfill \Box

### 4 Boundedness of solutions ($u_\epsilon, v_\epsilon, w_\epsilon$)

On the basis of the quite well established arguments from parabolic regularity theory, we can turn the space–time integrability properties of $u_\epsilon^p$ into the integrability properties of $\nabla v_\epsilon$ as well as $\nabla w_\epsilon$.

**Lemma 4.1** Let $m > 1, \alpha > 0$ and suppose that $D \geq 1$. Then for $q \in (2, \frac{2(m+1)}{3-m})$, there exists constant $C > 0$ such that for all $D \geq 1$ and $\epsilon \in (0, 1)$
\[
\|v_\epsilon(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C. \tag{4.1}
\]
as well as
\[ \| w_\varepsilon(\cdot, t) \|_{W^{1,q}(\Omega)} \leq C \] (4.2)
for all \( t > 0 \).

**Proof** From the continuity of function \( h(x) = \frac{2x}{(4-x)^+} \) for \( x \in [2, 4) \), it follows that for given \( q > 2 \) suitably close to the number \( \frac{2(m+1)}{(3-m)^+} \), one can choose \( p \in (2, m+1) \) in an appropriately small neighborhood of \( m+1 \) such that
\[ \frac{p}{p-1} \cdot \left( \frac{1}{2} + \frac{1}{p} - \frac{1}{q} \right) < 1. \] (4.3)

From the smoothing properties of Neumann heat semigroup \( (e^{t\Delta})_{t \geq 0} \), it follow that there exist \( c_i > 0 \) \((i = 1, 2)\) such that
\[ \| e^{\Delta \varphi} \|_{W^{1,q}(\Omega)} \leq c_1 \| \varphi \|_{L^1(\Omega)} \] for \( \varphi \in C^0(\overline{\Omega}) \) (4.4)
as well as
\[ \| e^{t\Delta} \varphi \|_{W^{1,q}(\Omega)} \leq c_2 t^{-\frac{1}{2} - \frac{1}{p} \left( \frac{1}{p} - \frac{1}{q} \right)} \| \varphi \|_{L^p(\Omega)} \] for all \( t \in (0, 1) \) and \( \varphi \in C^0(\overline{\Omega}) \). (4.5)

Therefore by the Duhamel representation to the second equation of (2.20), we obtain
\[ \| \tilde{v}_\varepsilon(\cdot, t) \|_{W^{1,q}(\Omega)} = \| e^{(t-(t-1))_+ (\Delta - D^{-1})} \tilde{v}_\varepsilon(\cdot, (t-1)_+) \]
\[ + \frac{1}{D} \int_{(t-1)_+}^t e^{(t-s)(\Delta - D^{-1})} u_\varepsilon(\cdot, \frac{s}{D}) ds \|_{W^{1,q}(\Omega)} \]
\[ \leq \| e^{(t-(t-1))_+ \Delta} \tilde{v}_\varepsilon(\cdot, (t-1)_+) \|_{W^{1,q}(\Omega)} \]
\[ + \frac{1}{D} \int_{(t-1)_+}^t \| e^{(t-s)\Delta} u_\varepsilon(\cdot, \frac{s}{D}) \|_{W^{1,q}(\Omega)} ds. \] (4.6)

Due to (4.4) and (4.5), we have
\[ \| e^{(t-(t-1)_+)\Delta} \tilde{v}_\varepsilon(\cdot, (t-1)_+) \|_{W^{1,q}(\Omega)} = \| e^{\Delta} \tilde{v}_\varepsilon(\cdot, t-1) \|_{W^{1,q}(\Omega)} \]
\[ \leq c_1 \| \tilde{v}_\varepsilon(\cdot, t-1) \|_{L^1(\Omega)} \] for \( t > 1 \), (4.7)
while for \( t \leq 1 \),
\[ \| e^{(t-(t-1)_+)\Delta} \tilde{v}_\varepsilon(\cdot, (t-1)_+) \|_{W^{1,q}(\Omega)} = \| e^{\Delta} v_0(\cdot) \|_{W^{1,q}(\Omega)} \]
\[ \leq c_1 \| v_0(\cdot) \|_{W^{1,\infty}(\Omega)} \].
On the other hand, we can see that for \( t > 0 \)

\[
\int_{(t-1)_+}^t \|e^{-(t-s)\Delta} u_\varepsilon(\cdot, D^{-1}s)\|_{W^{1,q}(\Omega)} ds \\
\leq c_2 \int_{(t-1)_+}^t (t-s)^{-\frac{1}{2} - \left(\frac{1}{p} - \frac{1}{q}\right)} \|u_\varepsilon(\cdot, D^{-1}s)\|_{L^p(\Omega)} ds \\
\leq c_2 \left\{ \int_{(t-1)_+}^t (t-s)^{-\frac{p}{p-1} \left(\frac{1}{2} + \frac{1}{p} - \frac{1}{q}\right)} ds \right\}^{\frac{p-1}{p}} \left\{ \int_{(t-1)_+}^t \|u_\varepsilon(\cdot, D^{-1}s)\|_{L^p(\Omega)} ds \right\}^{\frac{1}{p}} \\
\leq c_2 \left( \int_0^1 \sigma^{-\frac{p}{p-1} \left(\frac{1}{2} + \frac{1}{p} - \frac{1}{q}\right)} d\sigma \right)^{\frac{p-1}{p}} \left\{ \int_{(D^{-1}-D)}^{D^{-1}} \|u_\varepsilon(\cdot, s)\|_{L^p(\Omega)} ds \right\}^{\frac{1}{p}} \\
\leq c_3 D^{\frac{1}{p}},
\]

where due to \( D \geq 1 \) and the application of Lemma 3.3, we have

\[
\int_{(D^{-1}-D)}^{D^{-1}} \|u_\varepsilon(\cdot, s)\|_{L^p(\Omega)} ds \leq c_4
\]

and the finiteness of \( \int_0^1 \sigma^{-\frac{p}{p-1} \left(\frac{1}{2} + \frac{1}{p} - \frac{1}{q}\right)} d\sigma \) due to (4.3). Hence combining (4.6) with (4.7) and (4.8) gives

\[
\|v_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} \leq c_2 \|\tilde{v}_\varepsilon(\cdot, t-1)\|_{L^1(\Omega)} + c_3 D^{\frac{1}{p}-1} + c_1 \|v_0(\cdot)\|_{W^{1,\infty}(\Omega)} \\
\leq c_2 \left( \int_\Omega u_0 + \beta \int_\Omega w_0 \right) + c_3 + c_1 \|v_0(\cdot)\|_{W^{1,\infty}(\Omega)}
\]

for all \( t > 0 \) and thus completes the proof of (4.1).

Next due to \( \|w_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)} \), an application of Duhamel representation to the third equation in (2.5) yields

\[
\|w_\varepsilon(\cdot, t)\|_{W^{1,q}(\Omega)} \leq \|e^{\Delta} w_\varepsilon(\cdot, (t - 1)_)\|_{W^{1,q}(\Omega)} \\
+ f(\|w_0\|_{L^\infty(\Omega)}) \int_{(t-1)_+}^t \|e^{(t-s)\Delta} u_\varepsilon(\cdot, s)\|_{W^{1,q}(\Omega)} ds,
\]

and thereby (4.2) can be actually derived as above. \( \square \)

The following lemma will be used in the derivation of regularity features about spatial and temporal derivatives of \( u_\varepsilon \).
Lemma 4.2  Let $p > 0$ and $\varphi \in C^\infty(\bar{\Omega})$, then

$$
\frac{1}{p} \int_0^t \frac{d}{dt} (u_\varepsilon + \varepsilon)^p \cdot \varphi + (p - 1) M \int_{\Omega} (u_\varepsilon + \varepsilon)^p - (u_\varepsilon + \varepsilon)^p - (u_\varepsilon + \varepsilon)^{p-2} (u_\varepsilon + 1) M - 1 |\nabla u_\varepsilon|^2 \varphi \\
= (1 - p) \int_{\Omega} (mu_\varepsilon + \varepsilon)(u_\varepsilon + \varepsilon)^{p-1} v_\varepsilon ^{-\alpha - 1} \nabla u_\varepsilon \cdot \nabla \varphi \\
+ \alpha (p - 1) \int_{\Omega} (u_\varepsilon + \varepsilon)^{p-1} v_\varepsilon ^{-\alpha - 1} \nabla u_\varepsilon \cdot \nabla \varphi \\
+ (1 - p) \int_{\Omega} (mu_\varepsilon + \varepsilon)(u_\varepsilon + \varepsilon)^{p-1} v_\varepsilon ^{-\alpha - 1} \nabla u_\varepsilon \cdot \nabla \varphi \\
- M \int_{\Omega} (u_\varepsilon + \varepsilon)^{p-1} (u_\varepsilon + 1) M - 1 |\nabla u_\varepsilon|^2 \varphi \\
+ \alpha \int_{\Omega} u_\varepsilon (u_\varepsilon + \varepsilon)^{p-2} v_\varepsilon ^{-\alpha - 1} \nabla v_\varepsilon \cdot \nabla \varphi + \beta \int_{\Omega} u_\varepsilon f(w_\varepsilon)(u_\varepsilon + \varepsilon)^{p-1} \varphi
$$

(4.9)

for all $t > 0$ and $\varepsilon \in (0, 1)$.

**Proof**  This can be verified by the straightforward computation.  \(\square\)

Thanks to the boundedness of $\|\nabla v_\varepsilon(\cdot, t)\|_{L^q(\Omega)}$ with some $q > 2$ in Lemma 4.1, we can achieve the following $D$-independent $L^p$-estimate of $u_\varepsilon$ with finite $p$.

Lemma 4.3  Let $m > 1$. Then for all $D \geq 1$ and any $p > 1$, there exists a constant $C(p) > 0$ such that

$$
\|u_\varepsilon(\cdot, t)\|_{L^p(\Omega)} \leq C(p)
$$

(4.10)

for all $t > 0$ and $\varepsilon \in (0, 1)$.

**Proof**  According to Lemma 2.3 and Lemma 2.4, one can find $c_i > 0 (i = 1, 2)$ independent of $D \geq 1$ fulfilling

$$
v_\varepsilon ^{-\alpha}(x, t) \geq c_1, \quad v_\varepsilon ^{-\alpha - 2}(x, t) \leq c_2 \text{ in } \Omega \times (0, \infty)
$$

(4.11)

for all $\varepsilon \in (0, 1)$.

Letting $\varphi \equiv 1$ in (4.9) and by Young’s inequality, we have

$$
\frac{d}{dt} \int_{\Omega} (u_\varepsilon + \varepsilon)^p + p(p - 1) \int_{\Omega} (mu_\varepsilon + \varepsilon)(u_\varepsilon + \varepsilon)^{m+p-4} v_\varepsilon ^{-\alpha} |\nabla u_\varepsilon|^2 + \int_{\Omega} (u_\varepsilon + \varepsilon)^p \\
\leq \alpha p(p - 1) \int_{\Omega} u_\varepsilon (u_\varepsilon + \varepsilon)^{m+p-3} v_\varepsilon ^{-\alpha - 1} \nabla u_\varepsilon \cdot \nabla \varphi \\
+ \beta p \int_{\Omega} u_\varepsilon f(w_\varepsilon)(u_\varepsilon + \varepsilon)^{p-1} + \int_{\Omega} (u_\varepsilon + \varepsilon)^p \\
\leq \frac{p(p-1)}{2} \int_{\Omega} (mu_\varepsilon + \varepsilon)(u_\varepsilon + \varepsilon)^{m+p-4} v_\varepsilon ^{-\alpha} |\nabla u_\varepsilon|^2 \\
+ \frac{\alpha^2 p(p-1)}{2} \int_{\Omega} (u_\varepsilon + \varepsilon)^{m+p-3} v_\varepsilon ^{-\alpha - 2} |\nabla v_\varepsilon|^2 \\
+ \beta pf(\|w_0\|_{L^\infty(\Omega)}) \int_{\Omega} u_\varepsilon (u_\varepsilon + \varepsilon)^{p-1} + \int_{\Omega} (u_\varepsilon + \varepsilon)^p
$$
Furthermore, recalling (4.11), we can find $c_3 > 0$ and $c_4 > 0$ independent of $p$ such that
\[
\frac{d}{dt} \int_{\Omega} (u_\varepsilon + \varepsilon)^p + c_1 \int_{\Omega} |\nabla (u_\varepsilon + \varepsilon)|^{\frac{m+p-1}{2}} |x|^2 + \int_{\Omega} (u_\varepsilon + \varepsilon)^p \\
\leq c_4 p^2 \int_{\Omega} (u_\varepsilon + \varepsilon)^{m+p-1} |\nabla v_\varepsilon|^2 + c_4 p \int_{\Omega} (u_\varepsilon + \varepsilon)^p.
\]
(4.12)

According to (4.1), $\|\nabla v_\varepsilon\|_{L^q(\Omega)}^2 \leq c_5$ for any fixed $q \in (2, \frac{2(m+1)}{3-m_+})$, and hence the Hölder inequality yields
\[
c_4 p^2 \int_{\Omega} (u_\varepsilon + \varepsilon)^{m+p-1} |\nabla v_\varepsilon|^2 \\
\leq c_4 p^2 \left\{ \int_{\Omega} (u_\varepsilon + \varepsilon)^{\frac{m+p-1+q}{q-2}} \right\}^{\frac{q-2}{q}} \|\nabla v_\varepsilon\|_{L^q(\Omega)}^2 \\
\leq c_4 c_5 p^2 \| (u_\varepsilon + \varepsilon)^{\frac{m+p-1}{2}} \|^2_{L^{\frac{2q}{2-q}}(\Omega)} \\
\leq \frac{c_3}{4} \int_{\Omega} |\nabla (u_\varepsilon + \varepsilon)|^{\frac{m+p-1}{2}} |x|^2 + c_6(p),
\]
where we have used an Ehrling-type inequality due to $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2q}{2-q}}(\Omega)$ in two-dimensional setting and (2.13).

On the other hand, since
\[
c_4 p \int_{\Omega} (u_\varepsilon + \varepsilon)^p \leq \eta \int_{\Omega} (u_\varepsilon + \varepsilon)^{m+p-1} + \frac{(c_4 p)^{\frac{m-1+p}{m-1}} |\Omega|}{\eta^{\frac{m-1+p}{m-1}}} \\
\leq \eta \| (u_\varepsilon + \varepsilon)^{\frac{m+p-1}{2}} \|^2_{L^2(\Omega)} + \frac{(c_4 p)^{\frac{m-1+p}{m-1}} |\Omega|}{\eta^{\frac{m-1+p}{m-1}}}
\]
for any $\eta > 0$, we also have
\[
c_4 p \int_{\Omega} (u_\varepsilon + \varepsilon)^p \leq \frac{c_3}{4} \int_{\Omega} |\nabla (u_\varepsilon + \varepsilon)|^{\frac{m+p-1}{2}} |x|^2 + c_7(p)
\]
(4.14)
with some $c_7(p) > 0$. Now inserting (4.14) and (4.13) into (4.12), we infer that for all $t > 0$
\[
\frac{d}{dt} \int_{\Omega} (u_\varepsilon + \varepsilon)^p + \int_{\Omega} (u_\varepsilon + \varepsilon)^p \leq c_8(p)
\]
(4.15)
with $c_8(p) > 0$ independent of $D \geq 1$, which along with a standard comparison argument implies that
\[
\int_{\Omega} u_\varepsilon^p(\cdot, t) \leq \max\{c_8(p), \|u_0\|^p_{L^p(\Omega)} + 1\}
\]
(4.16)
for all $t \geq 0$ and thus yields the claimed conclusion. \hfill \Box

With the $L^p$-estimate of $u_\varepsilon$ at hand, the standard Moser-type iteration can be immediately applied in our approaches to obtain further regularity concerning $L^\infty$-norm of $u_\varepsilon$ (see Lemma A.1 of [30] for example) and so we refrain from giving the details here.

Lemma 4.4 Assume that $m > 1, \alpha > 0$ and $D \geq 1$, then there exists $C > 0$ such that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \geq 0.
\]
(4.17)
Remark 4.1 It should be mentioned that when \( m > 1, \alpha > 0 \) and \( D > 0 \), one can obtain the boundedness of \( L^\infty \)-norm of \( u_\varepsilon \) for all \( t > 0 \) by the above argument (also see [38] for reference). However, the explicit dependence of \( \| u_\varepsilon (\cdot, t) \|_{L^p(\Omega)} \) on \( D \) is required to investigate the large time behavior of solutions in the sequel. Hence \( D \geq 1 \) is imposed specially for the convenience of our discussion below.

At the end of this section, based on the above results we derive a regularity property for \( v \) which goes beyond those in Lemma 4.1.

Lemma 4.5 Let \( m > 1, \alpha > 0 \). Then there exists \( C > 0 \) such that for all \( D \geq 1 \) and \( \varepsilon \in (0, 1) \)

\[
\| \nabla v_\varepsilon (\cdot, t) \|_{L^\infty(\Omega)} \leq C \tag{4.18}
\]

as well as

\[
\| \nabla w_\varepsilon (\cdot, t) \|_{L^\infty(\Omega)} \leq C \tag{4.19}
\]

for all \( t > D \).

Proof Due to \( \| \nabla e^{\Delta (\tilde{t} - 1)} \|_{L^\infty(\Omega)} \leq c_1 \| \tilde{v}(\cdot, \tilde{t} - 1) \|_{L^1(\Omega)} \), as the proof of Lemma 4.1, we use Duhamel formula of (2.20) in the following way

\[
\| \nabla \tilde{v}(\cdot, \tilde{t}) \|_{L^\infty(\Omega)} = \left\| \nabla e^{\Delta (\tilde{t} - 1)} \tilde{v}(\cdot, \tilde{t} - 1) + D^{-1} \int_{\tilde{t} - 1}^{\tilde{t}} \nabla e^{(t-s)(\Delta - D^{-1})} u(\cdot, D^{-1}s) ds \right\|_{L^\infty(\Omega)} 
\]

\[
\leq \| \nabla e^{\Delta (\tilde{t} - 1)} \|_{L^\infty(\Omega)} + D^{-1} \int_{\tilde{t} - 1}^{\tilde{t}} \| \nabla e^{(t-s)\Delta} u(\cdot, D^{-1}s) \|_{L^\infty(\Omega)} ds
\]

\[
\leq c_1 \| \tilde{v}(\cdot, \tilde{t} - 1) \|_{L^1(\Omega)} 
\]

\[
+ c_2 D^{-1} \int_{\tilde{t} - 1}^{\tilde{t}} (1 + (\tilde{t} - s)^{-\frac{3}{4}}) ds \max_{\tilde{t} - 1 \leq s \leq \tilde{t}} \| u(\cdot, D^{-1}s) \|_{L^4(\Omega)}. 
\]

for all \( \tilde{t} > 1 \), which along with (4.17) readily leads to (4.18). It is obvious that (4.19) can be proved similarly. \( \square \)

5 Asymptotic behavior

5.1 Weak decay information

The standard parabolic regularity property becomes applicable to improve the regularity of \( u, v \) and \( w \) as follows.

Lemma 5.1 Let \( (u, v, w) \) be the nonnegative global solution of (1.10)–(1.12) obtained in Lemma 2.1. Then there exist \( \kappa \in (0, 1) \) and \( C > 0 \) such that for all \( t > D \)

\[
\| u \|_{C^{\kappa, \frac{\kappa}{2}}(\Omega \times [t, t+1])} \leq C \tag{5.1}
\]

as well as

\[
\| v \|_{C^{2+\kappa, 1+\frac{\kappa}{2}}(\Omega \times [t, t+1])} + \| w \|_{C^{2+\kappa, 1+\frac{\kappa}{2}}(\Omega \times [t, t+1])} \leq C. \tag{5.2}
\]
Proof We rewrite the first equation of (2.5) in the form

\[ u_{\varepsilon t} = \nabla \cdot a(x, t, u_{\varepsilon}, \nabla u_{\varepsilon}) + b(x, t, u_{\varepsilon}, \nabla u_{\varepsilon}) \]

where

\[ a(x, t, u_{\varepsilon}, \nabla u_{\varepsilon}) = (\varepsilon M(u_{\varepsilon} + 1)^{M-1} + m u_{\varepsilon}^{m-1} v_{\varepsilon}^{-\alpha}) \nabla u_{\varepsilon} - \alpha u_{\varepsilon}^{m} v_{\varepsilon}^{-\alpha - 1} \nabla v_{\varepsilon} \]

and

\[ b(x, t, u_{\varepsilon}, \nabla u_{\varepsilon}) = \beta u_{\varepsilon} f(w_{\varepsilon}). \]

According to Lemmas 2.3, 2.4, 4.4 and 4.5, there exist two constants \( c_1 > 0 \) and \( c_2 > 0 \) independent of \( D \geq 1 \) satisfying

\[ c_1 \leq v_{\varepsilon}^{-\alpha}(x, t) \leq c_2 \text{ in } \Omega \times (D, \infty) \tag{5.3} \]

and

\[ \| v_{\varepsilon}^{-\alpha - 1} \cdot \|_{L^\infty(\Omega)} + \| \nabla v_{\varepsilon}(\cdot, t) \|_{L^\infty(\Omega)} + \| u_{\varepsilon}(\cdot, t) \|_{L^\infty(\Omega)} + \| w_{\varepsilon}(\cdot, t) \|_{L^\infty(\Omega)} \leq c_2 \text{ for } t \geq D. \]

This guarantees that for all \( (x, t) \in \Omega \times (D, \infty) \)

\[ a(x, t, u_{\varepsilon}, \nabla u_{\varepsilon}) \cdot \nabla u_{\varepsilon} \geq \frac{c_1 m}{2} u_{\varepsilon}^{m-1} |\nabla u_{\varepsilon}|^2 - c_3, \]

\[ |a(x, t, u_{\varepsilon}, \nabla u_{\varepsilon})| \leq mc_4 u_{\varepsilon}^{m-1} |\nabla u_{\varepsilon}| + c_4 |u_{\varepsilon}|^{m-1} \]

and

\[ |b(x, t, u_{\varepsilon}, \nabla u_{\varepsilon})| \leq c_5 \]

with some constants \( c_i > 0 \) \((i = 3, 4, 5)\) for all \( t > D \) and \( \varepsilon \in (0, 1) \). Therefore as an application of the known result on Hölder regularity in scalar parabolic Eq. ([26]), there exist \( k_1 \in (0, 1) \) and \( C > 0 \) such that for all \( t > D \) and \( \varepsilon \in (0, 1) \),

\[ \| u_{\varepsilon} \|_{C^{k_1, 3/4}(\Omega \times [t, t+1])} \leq C, \]

which along with (2.7) readily entails (5.1) with \( \kappa = k_1 \). Similarly one can also conclude that there exist \( k_2 \in (0, 1) \) and \( C > 0 \) such that

\[ \| u \|_{C^{k_2, 3/2}(\Omega \times [t, t+1])} + \| w \|_{C^{k_2, 3/2}(\Omega \times [t, t+1])} \leq C \text{ for all } t > D. \tag{5.4} \]

Moreover, since \( f \in C^1[0, \infty) \), we have

\[ \| uf(w) \|_{C^{k_3, 3/2}(\Omega \times [t, t+1])} \leq C \text{ for all } t > D \]

with \( k_3 = \min\{k_1, k_2\} \). Thereupon (5.2) with \( \kappa = k_3 \) follows from the parabolic regularity estimates ([19, Chapter IV, Theorem 5.3]).

The first step towards establishing the stabilization result in Theorem 1.1 consists in the following observation.

**Lemma 5.2** Assume that \( m > 1 \) and \( D \geq 1 \), we have

\[ \int_0^\infty \int_\Omega uf(w) < \infty \tag{5.5} \]
and
\[ \int_0^\infty \int_\Omega |\nabla w|^2 < \infty. \] (5.6)

**Proof** An integration of the third equation in (2.19) yields
\[ \int_\Omega w_\varepsilon(\cdot, t) + \int_0^t \int_\Omega u_\varepsilon f (w_\varepsilon) = \int_\Omega w_0 \quad \text{for all} \ t > 0. \]

Since \( w_\varepsilon \geq 0 \), this entails
\[ \int_0^\infty \int_\Omega u_\varepsilon f (w_\varepsilon) \leq \int_\Omega w_0 \] (5.7)
which implies (5.5) on an application of Fatou’s lemma, because \( u_\varepsilon f (w_\varepsilon) \to u f (w) \) a.e. in \( \Omega \times (0, \infty) \).

We test the same equation by \( w_\varepsilon \) to see that
\[ \frac{1}{2} \int_\Omega w_\varepsilon^2 (\cdot, t) + \int_0^t \int_\Omega |\nabla w_\varepsilon|^2 = \frac{1}{2} \int_\Omega w_0^2 - \int_0^t \int_\Omega u_\varepsilon f (w_\varepsilon) w_\varepsilon \leq \frac{1}{2} \int_\Omega w_0^2 \]
and thereby verifies (5.6) via (2.12). \( \square \)

The above decay information of \( w_\varepsilon \) seems to be weak for the derivation of the large-time behavior of \( u_\varepsilon \) and \( v_\varepsilon \). Indeed, under additional constraint on \( D \), we obtain the decay information concerning the gradient of \( u_\varepsilon \) and \( v_\varepsilon \) which makes our latter analysis possible.

**Lemma 5.3** Let \( m > 1 \) and \( \alpha > 0 \). There exists \( D_0 \geq 1 \) such that whenever \( D > D_0 \), the solution of \( (1.10)-(1.12) \) constructed in Lemma 2.1 satisfies
\[ \int_3^\infty \int_\Omega |\nabla u |^{\frac{m+1}{m-1}} < \infty \] (5.8)
as well as
\[ \int_3^\infty \int_\Omega |\nabla v |^2 < \infty \] (5.9)

**Proof** Testing the first equation of (2.5) by \( (u_\varepsilon + \varepsilon) \) and applying Young’s inequality, we obtain that
\[ \frac{1}{2} \frac{d}{dt} \int_\Omega (u_\varepsilon + \varepsilon)^2 + \int_\Omega (u_\varepsilon + \varepsilon)^m v_e^{-\alpha} |\nabla u_e|^2 \]
\[ \leq \alpha \int_\Omega (u_\varepsilon + \varepsilon)^m v_e^{-\alpha-1} u_e \cdot \nabla v_e + \beta \int_\Omega u_\varepsilon (u_\varepsilon + \varepsilon) f (w_\varepsilon) \]
\[ \leq \frac{1}{2} \int_\Omega (u_\varepsilon + \varepsilon)^m v_e^{-\alpha} |\nabla u_e|^2 + \frac{\alpha^2}{2} \int_\Omega v_e^{-\alpha-2} (u_\varepsilon + \varepsilon)^m v_e |\nabla v_e|^2 \]
\[ + \beta \int_\Omega (u_\varepsilon + \varepsilon) u_\varepsilon f (w_\varepsilon), \]
and hence
\[ \frac{d}{dt} \int_\Omega (u_\varepsilon + \varepsilon)^2 + \int_\Omega (u_\varepsilon + \varepsilon)^m v_e^{-\alpha} |\nabla u_e|^2 \]
\[ \leq \alpha^2 \int_\Omega v_e^{-\alpha-2} (u_\varepsilon + \varepsilon)^m v_e |\nabla v_e|^2 + 2 \beta \int_\Omega (u_\varepsilon + \varepsilon) u_\varepsilon f (w_\varepsilon). \] (5.10)
On the other hand, let \( \mu_\varepsilon(t) = \left( \frac{1}{|\Omega|} \int_{\Omega} u_\varepsilon^{m+1} (\cdot, t) \right)^{\frac{2}{m+1}} \), then testing the second equation of (2.5) by \(-\Delta v_\varepsilon\) shows

\[
\frac{d}{dt} \int_\Omega |\nabla v_\varepsilon|^2 + 2D \int_\Omega (\Delta v_\varepsilon)^2 + 2 \int_\Omega |\nabla v_\varepsilon|^2 = 2 \int_\Omega (u_\varepsilon(\cdot, t) - \mu_\varepsilon(t)) \Delta v_\varepsilon \\
\leq \frac{1}{D} \int_\Omega |u_\varepsilon(\cdot, t) - \mu_\varepsilon(t)|^2 + D \int_\Omega (\Delta v_\varepsilon)^2,
\]

and thus

\[
\frac{d}{dt} \int_\Omega |\nabla v_\varepsilon|^2 + D \int_\Omega (\Delta v_\varepsilon)^2 + 2 \int_\Omega |\nabla v_\varepsilon|^2 \leq \frac{1}{D} \int_\Omega |u_\varepsilon(\cdot, t) - \mu_\varepsilon(t)|^2.
\] (5.11)

Hence combining (5.10) and (5.11), we have

\[
\frac{d}{dt} \left( \int_\Omega (u_\varepsilon + \varepsilon)^2 + \eta \int_\Omega |\nabla v_\varepsilon|^2 \right) + \eta D \int_\Omega |\Delta v_\varepsilon|^2 + 2 \eta \int_\Omega |\nabla v_\varepsilon|^2 \\
+ \int_\Omega v_\varepsilon^{-\alpha}(u_\varepsilon + \varepsilon)^{m-1}|\nabla u_\varepsilon|^2 \\
\leq \frac{\eta}{D} \int_\Omega |u_\varepsilon(\cdot, t) - \mu_\varepsilon(t)|^2 + \alpha^2 \int_\Omega v_\varepsilon^{-\alpha-2}(u_\varepsilon + \varepsilon)^{m+1}|\nabla v_\varepsilon|^2 \\
+ 2 \int_\Omega (u_\varepsilon + \varepsilon)u_\varepsilon f(w_\varepsilon). 
\] (5.12)

for parameter \( \eta > 0 \) which will be determined later.

In view of Lemma 2.3 and Lemma 2.4, there exist \( c_i > 0 (i = 1, 2) \) independent of \( D \geq 1 \) satisfying

\[
v_\varepsilon^{-\alpha}(x, t) \geq c_1, \quad v_\varepsilon^{-\alpha-2}(x, t) \leq c_2 \text{ in } \Omega \times (2, \infty)
\] (5.13)

for all \( \varepsilon \in (0, 1) \). Therefore from (5.12), it follows that

\[
\frac{d}{dt} \left( \int_\Omega (u_\varepsilon + \varepsilon)^2 + \eta \int_\Omega |\nabla v_\varepsilon|^2 \right) + \eta D \int_\Omega |\Delta v_\varepsilon|^2 \\
+ 2 \eta \int_\Omega |\nabla v_\varepsilon|^2 + c_1 \int_\Omega (u_\varepsilon + \varepsilon)^{m-1}|\nabla u_\varepsilon|^2 \\
\leq \frac{\eta}{D} \int_\Omega |u_\varepsilon(\cdot, t) - \mu_\varepsilon(t)|^2 + \alpha^2 c_2 \int_\Omega (u_\varepsilon + \varepsilon)^{m+1}|\nabla v_\varepsilon|^2 \\
+ 2 \int_\Omega (u_\varepsilon + \varepsilon)u_\varepsilon f(w_\varepsilon). 
\] (5.14)

According to Lemma 4.3 with \( p = 2(m + 1) \), we have

\[
\left( \int_\Omega (u_\varepsilon + \varepsilon)^{2(m+1)} \right)^{\frac{1}{2}} \leq c_3,
\]
and then use the Gagliardo–Nirenberg inequality and Hölder inequality to arrive at
\[
\int_\Omega (u_\varepsilon + \varepsilon)^{m+1} |\nabla v_\varepsilon|^2 \\
\leq \left( \int_\Omega (u_\varepsilon + \varepsilon)^{2(m+1)} \right)^{\frac{1}{2}} \left( \int_\Omega |\nabla v_\varepsilon|^4 \right)^{\frac{1}{2}} \\
\leq c_4 \left( \int_\Omega (u_\varepsilon + \varepsilon)^{2(m+1)} \right)^{\frac{1}{2}} (\|\Delta v_\varepsilon\|^2 + \|\nabla v_\varepsilon\|^2_{L^2(\Omega)}) \\
\leq c_3 c_4 (\|\Delta v_\varepsilon\|^2 + \|\nabla v_\varepsilon\|^2_{L^2(\Omega)}).
\]

Therefore inserting (5.15) into (5.14) yields
\[
\begin{align*}
\frac{d}{dt} \left( \int_\Omega (u_\varepsilon + \varepsilon)^2 + \eta \int_\Omega |\nabla v_\varepsilon|^2 \right) + \eta D \int_\Omega |\Delta v_\varepsilon|^2 \\
+ 2\eta \int_\Omega |\nabla v_\varepsilon|^2 + \frac{4c_1}{(m+1)^2} \int_\Omega |\nabla (u_\varepsilon + \varepsilon)^{\frac{m+1}{2}}|^2 \\
\leq \frac{\eta}{D} \int_\Omega |u_\varepsilon(\cdot, t) - \mu_\varepsilon(t)|^2 + \alpha^2 c_2 c_3 c_4 (\|\Delta v_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla v_\varepsilon\|^2_{L^2(\Omega)}) \\
+ 2 \int_\Omega (u_\varepsilon + \varepsilon) u_\varepsilon f (w_\varepsilon).
\end{align*}
\]

By the elementary inequality:
\[
\frac{\xi^\mu - \delta^\mu}{\xi - \delta} \geq \delta^\mu - 1 \quad \text{for} \quad \mu \geq 1, \xi \geq 0, \delta \geq 0 \quad \text{and} \quad \xi \neq \delta,
\]
we have
\[
|u_\varepsilon^{\frac{m+1}{2}}(\cdot, t) - \mu_\varepsilon^{\frac{m+1}{2}}| \geq \mu_\varepsilon^{\frac{m+1}{2}} |u_\varepsilon(\cdot, t) - \mu_\varepsilon(t)|
\]
and thus
\[
\mu_\varepsilon^{m-1}(t) \int_\Omega |u_\varepsilon(\cdot, t) - \mu_\varepsilon(t)|^2 \leq \int_\Omega |u_\varepsilon^{\frac{m+1}{2}}(\cdot, t) - \mu_\varepsilon^{\frac{m+1}{2}}(t)|^2.
\]

Furthermore by the Hölder inequality and the nonincreasing property of \( t \mapsto \int_\Omega u_\varepsilon(\cdot, t) \),
\[
\mu_\varepsilon(t) \geq \frac{1}{|\Omega|} \int_\Omega u_\varepsilon(\cdot, t) \geq \frac{1}{|\Omega|} \int_\Omega u_0
\]
and thereby the Poincaré inequality entails that for some \( c_5 > 0 \)
\[
\frac{\mu_\varepsilon^{m-1}}{u_0^{m-1}} \int_\Omega |u_\varepsilon(\cdot, t) - \mu_\varepsilon(t)|^2 \\
\leq \int_\Omega |u_\varepsilon^{\frac{m+1}{2}}(\cdot, t) - \mu_\varepsilon^{\frac{m+1}{2}}(t)|^2 \\
\leq c_5 \int_\Omega |\nabla u_\varepsilon^{\frac{m+1}{2}}|^2 \\
\leq c_5 \int_\Omega |\nabla (u_\varepsilon + \varepsilon)^{\frac{m+1}{2}}|^2.
\]
Hence substituting (5.18) into (5.16) shows that

\[
\frac{d}{dt} \left( \int_{\Omega} (u_\varepsilon + \varepsilon)^2 + \eta \int_{\Omega} |\nabla v_\varepsilon|^2 \right) + \left( \frac{4c_1}{(m+1)^2} - \frac{\eta c_5}{Du_0^{m-1}} \right) \int_{\Omega} |\nabla (u_\varepsilon + \varepsilon)^{m+1}|^2 \\
\leq (\alpha^2 c_2 c_3 c_4 - \eta D) \| \Delta v_\varepsilon \|^2_{L^2(\Omega)} + (\alpha^2 c_2 c_3 c_4 - 2\eta) \| \nabla v_\varepsilon \|^2_{L^2(\Omega)} \\
+ 2 \int_{\Omega} (u_\varepsilon + \varepsilon) u_\varepsilon f (w_\varepsilon) \\
\leq (\alpha^2 c_2 c_3 c_4 - \eta) \| \Delta v_\varepsilon \|^2_{L^2(\Omega)} + (\alpha^2 c_2 c_3 c_4 - 2\eta) \| \nabla v_\varepsilon \|^2_{L^2(\Omega)} \\
+ 2 \| u_\varepsilon (\cdot, t) \|_{L^\infty(\Omega)} + 1 \int_{\Omega} u_\varepsilon f (w_\varepsilon)
\]

and hence completes the proof upon the choice of \( D_0 := \max \{ 1, \frac{\alpha^2 c_2 c_3 c_4 (m+1)^2}{3c_1 u_0^{m-1}} \} \). Indeed, for any \( D > D_0 \), it is possible to find \( \eta > 0 \) such that

\[
\frac{3c_1}{(m+1)^2} \geq \frac{\eta c_5}{Du_0^{m-1}}, \quad \alpha^2 c_2 c_3 c_4 \leq \eta
\]

and thereby

\[
\frac{d}{dt} \left( \int_{\Omega} |u_\varepsilon + \varepsilon|^2 + \eta \int_{\Omega} |\nabla v_\varepsilon|^2 \right) + \frac{c_1}{(m+1)^2} \int_{\Omega} |\nabla (u_\varepsilon + \varepsilon)^{m+1}|^2 \mathbf{1} + \eta \int_{\Omega} |\nabla v_\varepsilon|^2 \\
\leq 2(\| u_\varepsilon (\cdot, t) \|_{L^\infty(\Omega)} + 1) \int_{\Omega} u_\varepsilon f (w_\varepsilon).
\]

Therefore, in view of (5.7), (4.17) and (4.18) we see that for any \( t > 3 \),

\[
\int_{3}^{t} \int_{\Omega} |\nabla (u_\varepsilon + \varepsilon)^{m+1}|^2 + \int_{3}^{t} \int_{\Omega} |\nabla v_\varepsilon|^2 \leq c_6 + c_6 \int_{3}^{\infty} \int_{\Omega} u_\varepsilon f (w_\varepsilon) \leq c_6 + c_6 \int_{\Omega} w_0.
\]

with constant \( c_6 > 0 \) independent of \( \varepsilon \) and time \( t \), which implies that (5.8) and (5.9) is valid due to the lower semicontinuity of norms.

\( \Box \)

### 5.2 Decay of \( w \)

The integrability statement in Lemma 5.2 can be turned into the decay property of \( w \) with respect to the norm in \( L^\infty(\Omega) \), thanks to the fact that \( \| u(\cdot, t) \|_{L^1(\Omega)} \) is increasing with time, while \( \| w(\cdot, t) \|_{L^\infty(\Omega)} \) is nonincreasing.

**Lemma 5.4** The third component of the weak solution of (1.10)–(1.12) constructed in Lemma 2.1 fulfills

\[
\| w(\cdot, t) \|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to \infty.
\]  

(5.20)
Proof Writing \( u_0 := \frac{1}{|\Omega_1|} \int_{\Omega_1} u_0 \) and \( f(w) := \frac{1}{|\Omega|} \int_{\Omega} f(w) \), we use the Cauchy–Schwarz inequality and Poincaré inequality to see that for all \( t > 0 \)
\[
\int_{\Omega_1} u_0 \cdot \int_{\Omega_1} f(w) = \int_{\Omega} u_0 f(w) - \int_{\Omega} u_0 (f(w) - f(w)) \leq \int_{\Omega_1} u_0 f(w) + c_1 \| u \|_{L^\infty(\Omega)} \| f'(w) \|_{L^\infty(\Omega)} \left\{ \int_{\Omega} |\nabla w|^2 \right\}^{\frac{1}{2}}.
\]
Thanks to the boundedness of \( u \) and \( w \), we have
\[
\int_{\Omega_1} u_0^2 \cdot \left\{ \int_{\Omega_1} f(w) \right\}^2 \leq 2 \left\{ \int_{\Omega} u_0 f(w) \right\}^2 + c_2 \int_{\Omega} |\nabla w|^2 \]
\[
\leq c_3 \int_{\Omega} u_0 f(w) + c_2 \int_{\Omega} |\nabla w|^2.
\]
Hence from Lemma 5.2 it follows that
\[
\int_{1}^{\infty} \| f(w(\cdot, t)) \|_{L^1(\Omega)}^2 dt < \infty.
\]
which, along with the uniform Hölder estimate from Lemma 5.1, implies that
\[
f(w(\cdot, t)) \to 0 \quad \text{in } L^1(\Omega) \quad \text{as } t \to \infty
\]
and thereby we may extract a subsequence \((t_j)_{j \in \mathbb{N}} \subset \mathbb{N}\) such that as \( t_j \to \infty, f(w(\cdot, t_j)) \to 0 \) almost everywhere in \( \Omega \). Recalling function \( f \) is positive on \((0, \infty)\) and \( f(0) = 0 \), this necessarily requires that \( w(\cdot, t_j) \to 0 \) almost everywhere in \( \Omega \) as \( t_j \to \infty \). Furthermore, the dominated convergence theorem ensures that
\[
w(\cdot, t_j) \to 0 \quad \text{in } L^1(\Omega) \quad \text{as } t_j \to \infty.
\]
Now invoking the Gagliardo–Nirenberg inequality in two dimensional setting, we have
\[
\| w(\cdot, t_j) \|_{L^\infty(\Omega)} \leq c_4 \| \nabla w(\cdot, t_j) \|_{L^4(\Omega)}^\frac{4}{5} \| w(\cdot, t_j) \|_{L^1(\Omega)}^\frac{1}{5} + c_4 \| w(\cdot, t_j) \|_{L^1(\Omega)}
\]
and thus
\[
\| w(\cdot, t_j) \|_{L^\infty(\Omega)} \to 0 \quad \text{as } t_j \to \infty.
\]
Since \( t \mapsto \| w(\cdot, t) \|_{L^\infty(\Omega)} \) is noncreasing by Lemma 2.3, \((5.20)\) indeed results from \((5.23)\).
\[\Box\]

5.3 Convergence of \( u \)

In this subsection, we will show that \( u \) stabilizes toward the constant \( \bar{u} \) as \( t \to \infty \). Note that a first step in this direction is provided by the finiteness of \( \int_{\Omega} \int_{\Omega} |\nabla u|^2 \) in Lemma 5.3, which implies that \( \| \nabla u \|_{L^2(\Omega)} \) along a suitable sequence of numbers \( t_k \to \infty \). However, in order to make sure convergence along the entire net \( t \to \infty \), a certain decay property of \( u_t \) seems to be required.
Lemma 5.5 We have
\[ \int_{3}^{\infty} \|u_t(t)\|_{(W^{1,2}_0(\Omega))^*}^2 \, dt < \infty. \] (5.24)

Proof For any \( \varphi \in C_0^\infty(\Omega) \), multiplying the first equation in (2.5) by \( \varphi \) and integrating by parts over \( \Omega \) yields
\[
|\int_{\Omega} u_{xt} \, \varphi| = \left| \int_{\Omega} \varepsilon \nabla (u_{x} + 1)^M \cdot \nabla \varphi + \nabla (u_{x} + \varepsilon) \frac{m-1}{\varepsilon} v_x^\alpha \cdot \nabla \varphi + \beta u_x f(w_x) \varphi \right|
\leq \int_{\Omega} (M (u_{x} + 1)^{M-1} |\nabla u_x| + m v_x^{-\alpha} (u_x + \varepsilon)^{m-1} |\nabla u_x| + \alpha (u_x + 1)^{m} v_x^{-\alpha-1} |\nabla v_x|) |\nabla \varphi|
+ \beta \int_{\Omega} |u_x f(w_x)| \|\varphi\|_{L^\infty(\Omega)}
\leq c_1 \left( \left\{ \int_{\Omega} |\nabla (u_x + \varepsilon)^{\frac{m+1}{2}}|^2 \right\}^{\frac{1}{2}} + \left\{ \int_{\Omega} |\nabla v_x| \right\}^{\frac{1}{2}} \right) \|\varphi\|_{W^{1,2}(\Omega)}
+ \beta \int_{\Omega} u_x f(w_x) \|\varphi\|_{L^\infty(\Omega)}
\]
with \( c_1 > 0 \) independent of \( \varphi \) and \( \varepsilon \), where we have used the boundedness of \( u_x \) and \( v_x \).

As in the considered two-dimensional setting we have \( W^{1,2}(\Omega) \hookrightarrow L^\infty(\Omega) \), the above inequality implies that
\[
\|u_{xt}(\cdot, t)\|_{(W^{1,2}_0(\Omega))^*}
\leq c_1 \left( \left\{ \int_{\Omega} |\nabla (u_x + \varepsilon)^{\frac{m+1}{2}}|^2 \right\}^{\frac{1}{2}} + \left\{ \int_{\Omega} |\nabla v_x| \right\}^{\frac{1}{2}} \right) + \beta \int_{\Omega} u_x f(w_x)
\]
for all \( t > 3 \) and hence for all \( T > 4 \),
\[
\int_{3}^{T} \|u_{xt}(\cdot, t)\|_{(W^{1,2}_0(\Omega))^*}^2 \, dt \leq c_2 \left( \int_{3}^{T} \int_{\Omega} |\nabla (u_x + \varepsilon)^{\frac{m+1}{2}}|^2 \, dt + \int_{3}^{T} \int_{\Omega} |\nabla v_x|^2 \, dt + \int_{3}^{T} \int_{\Omega} u_x f(w_x) \, dt \right)
\]
which together with (5.19) leads to
\[
\int_{3}^{\infty} \|u_{xt}(\cdot, t)\|_{(W^{1,2}_0(\Omega))^*}^2 \, dt \leq c_3
\]
with \( c_3 > 0 \) independent of \( \varepsilon \). Hence (5.24) results from lower semi-continuity of the norm in the Hilbert space \( L^2((3, \infty); (W^{1,2}_0(\Omega))^*) \) with respect to weak convergence. \( \square \)

Thanks to above estimates, we adapt the argument in [37] to show that \( u \) actually stabilizes toward \( \bar{u}_0 + \beta \bar{w}_0 \) in the claimed sense beyond in the weak*-sense in \( L^\infty(\Omega) \).

Lemma 5.6 Let \( m > 1 \), \( \alpha > 0 \) and suppose that \( D \geq D_0 \) with \( D_0 \) as in Lemma 5.3. Then we have
\[ \|u(\cdot, t) - u_*\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty, \] (5.25)
where \( u_* = \frac{1}{|\Omega|} \int_{\Omega} u_0 + \frac{\beta}{|\Omega|} \int_{\Omega} w_0 \).
Proof According to Lemma 5.3 and Lemma 5.5, one can conclude that
\[ u(\cdot, t) \xrightarrow{w^*} u_* \quad \text{in} \quad L^\infty(\Omega) \quad \text{as} \quad t \to \infty \]  
(5.26)

In fact, if this conclusion does not hold, then one can find a sequence \((t_k)_{k \in \mathbb{N}} \subset (0, \infty)\) such that \(t_k \to \infty\) as \(k \to \infty\), and some \(\tilde{\psi} \in L^1(\Omega)\) such that
\[
\int_{\Omega} u(x, t_k) \tilde{\psi} dx - \int_{\Omega} u_* \tilde{\psi} dx \geq c_1 \quad \text{for all} \quad k \in \mathbb{N}
\]
with some \(c_1 > 0\). Furthermore, by the boundedness of \(u\) and the density of \(C_0^\infty(\Omega)\) in \(L^1(\Omega)\), we can choose \(\psi \in C_0^\infty(\Omega)\) closing \(\tilde{\psi}\) in \(L^1(\Omega)\) enough that
\[
\int_{\Omega} u(x, t_k) \psi dx - \int_{\Omega} u_* \psi dx \geq \frac{3c_1}{4} \quad \text{for all} \quad k \in \mathbb{N},
\]
and then
\[
\int_{t_k}^{t_k+1} \int_{\Omega} u(x, t) \psi dx dt - \int_{t_k}^{t_k+1} \int_{\Omega} u_* \psi dx dt \geq \frac{c_1}{2} \quad \text{for all sufficiently large} \quad k \in \mathbb{N},
\]
(5.27)

where we have used the fact that
\[
\left| \int_{t_k}^{t_k+1} \int_{\Omega} (u(x, t) - u(x, t_k)) \psi dx \right| \\
= \left| \int_{t_k}^{t_k+1} \int_{t_k}^{t} (u_t(\cdot, s), \psi(\cdot)) ds dt \right| \\
\leq \left( \int_{t_k}^{t_k+1} \int_{t_k}^{t} \|u_t(\cdot, s)\|_{(W_0^{1,2}(\Omega))^*} ds dt \cdot \|\psi\|_{W_0^{1,2}(\Omega)} \right)^{\frac{1}{2}} (t - t_k)^{\frac{1}{2}} ds dt \cdot \|\psi\|_{W_0^{1,2}(\Omega)} \\
\leq \left( \int_{t_k}^{t_k+1} \int_{t_k}^{t} \|u_t(\cdot, s)\|^2_{(W_0^{1,2}(\Omega))^*} ds dt \right)^{\frac{1}{2}} \cdot \|\psi\|_{W_0^{1,2}(\Omega)} \\
\leq \left( \int_{t_k}^{\infty} \|u_t(\cdot, s)\|^2_{(W_0^{1,2}(\Omega))^*} ds \right)^{\frac{1}{2}} \cdot \|\psi\|_{W_0^{1,2}(\Omega)} \\
\to 0 \quad \text{as} \quad k \to \infty,
\]
due to Lemma 5.5.

Let \(\mu(t) = \left( \frac{1}{|\Omega|} \int_{\Omega} u^{\frac{m+1}{2}}(\cdot, t) \right)^{\frac{2}{m+1}}\). Then as in (5.18), we have
\[
\overline{u}_0^{m-1} \int_{\Omega} |u(\cdot, t) - \mu(t)|^2 \leq \int_{\Omega} |u^{\frac{m+1}{2}}(\cdot, t) - \mu^{\frac{m+1}{2}}(t)|^2 \leq c_5 \int_{\Omega} |\nabla u^{\frac{m+1}{2}}(\cdot, t)|^2
\]
and thus
\[
\overline{u}_0^{m-1} \int_{t_k}^{t_k+1} \int_{\Omega} |u(\cdot, t) - \mu(t)|^2 \leq c_5 \int_{t_k}^{t_k+1} \int_{\Omega} |\nabla u^{\frac{m+1}{2}}(\cdot, t)|^2.
\]
(5.28)

We now introduce
\[ u_k(x, s) := u(x, t_k + s), \quad (x, s) \in \Omega \times (0, 1) \]
for $k \in \mathbb{N}$. Then (5.28) implies that
\[
\overline{u}_{0}^{-m-1} \int_{0}^{1} \int_{\Omega} |u_{k}(\cdot, s) - \mu_{k}(s)|^{2} ds \leq c_{5} \int_{t_{k}}^{t_{k}+1} \int_{\Omega} |\nabla \overline{u}_{0}^{-m+1}(\cdot, t)|^{2} 
\rightarrow 0 \text{ as } k \rightarrow \infty,
\]
due to (5.8) in Lemma 5.3. This means that
\[
u_{k}(x, s) = \mu_{k}(x, t_{k} + s), \ (x, s) \in \Omega \times (0, 1)
\]
for $k \in \mathbb{N}$. Then (5.28) implies that
\[
\overline{u}_{0}^{-m-1} \int_{0}^{1} \int_{\Omega} |u_{k}(\cdot, s) - \mu_{k}(s)|^{2} ds \leq c_{5} \int_{t_{k}}^{t_{k}+1} \int_{\Omega} |\nabla \overline{u}_{0}^{-m+1}(\cdot, t)|^{2} 
\rightarrow 0 \text{ as } k \rightarrow \infty,
\]
which in particular allows us to get
\[
\int_{0}^{1} \int_{\Omega} (u_{k}(\cdot, s) - \mu_{k}(s)) \psi ds \rightarrow 0 \text{ as } k \rightarrow \infty
\]
as well as
\[
\int_{0}^{1} \int_{\Omega} (u_{k}(\cdot, s) - \mu_{k}(s)) ds \rightarrow 0 \text{ as } k \rightarrow \infty.
\]
Moreover, by Lemma 5.4, we have
\[
\int_{t_{k}}^{t_{k}+1} \int_{\Omega} w(\cdot, t) dt \leq |\Omega| \|w(\cdot, t_{k})\|_{L_{\infty}(\Omega)} \rightarrow 0 \text{ as } k \rightarrow \infty
\]
and thereby
\[
|\Omega| \int_{0}^{1} \mu_{k}(s) ds = \int_{0}^{1} \int_{\Omega} u_{k}(\cdot, s) ds - \int_{0}^{1} \int_{\Omega} (u_{k}(\cdot, s) - \mu_{k}(s)) ds
\]
\[
= |\Omega| u_{*} - \beta \int_{t_{k}}^{t_{k}+1} \int_{\Omega} w(\cdot, t) dt - \int_{0}^{1} \int_{\Omega} (u_{k}(\cdot, s) - \mu_{k}(s)) ds
\]
\[
\rightarrow |\Omega| u_{*} \text{ as } k \rightarrow \infty
\]
due to (5.31) and (5.20).

Therefore from (5.27), (5.30) and (5.32), it follows that
\[
\frac{c_{1}}{2} \leq \int_{t_{k}}^{t_{k}+1} \int_{\Omega} u(\cdot, t) \psi dt - \int_{t_{k}}^{t_{k}+1} \int_{\Omega} u_{*} \psi dt
\]
\[
= \int_{0}^{1} \int_{\Omega} (u_{k}(\cdot, s) - \mu_{k}(s)) \psi ds + \int_{0}^{1} \int_{\Omega} \mu_{k}(s) \psi ds - u_{*} \int_{\Omega} \psi
\]
\[
= \int_{0}^{1} \int_{\Omega} (u_{k}(\cdot, s) - \mu_{k}(s)) \psi ds + \int_{0}^{1} \mu_{k}(s) ds \int_{\Omega} \psi - u_{*} \int_{\Omega} \psi
\]
\[
\rightarrow 0 \text{ as } k \rightarrow \infty,
\]
which is absurd and hence proves that actually (5.26) is valid.

Let us suppose on the contrary that (5.25) be false. Then without loss of generality there exist sequence $\{x_{k}\}_{k \in \mathbb{N}}$ and $\{t_{k}\}_{k \in \mathbb{N}} \in (0, \infty)$ with $t_{k} \rightarrow \infty$ as $k \rightarrow \infty$ such that for some
$c_1 > 0$

\[u(x_k, t_k) - u_* = \max_{x \in \Omega} |u(x, t_k) - u_*| \geq c_1 \text{ for all } k \in \mathbb{N} .\]

In view of the compactness of $\overline{\Omega}$, where passing to subsequences we can find $x_0 \in \overline{\Omega}$ such that $x_k \to x_0$ as $k \to \infty$. Furthermore, because $u$ is uniformly continuous in $\bigcup_{k \in \mathbb{N}}(\overline{\Omega} \times t_k)$, this entails that one can extract a further subsequence if necessary such that

\[u(x, t_k) - u_* \geq \frac{c_1}{2} \text{ for all } x \in B := B_{\delta}(x_0) \cap \Omega \text{ and } k \in \mathbb{N} .\]

for some $\delta > 0$. Noticing that if $x_0 \in \partial \Omega$, the smoothness of $\partial \Omega$ ensures the existence of $\hat{x}_0 \in \partial \Omega$ and a smaller $\hat{\delta} > 0$ such that $B_{\hat{\delta}}(\hat{x}_0) \subset B$. Now taking the nonnegative function $\psi \in C^\infty_0(B_{\hat{\delta}}(\hat{x}_0))$ such as a smooth truncated function in $B_{\hat{\delta}}(\hat{x}_0)$, we then have

\[\int_{\Omega} (u(x, t_k) - u_*) \psi \, dx = \int_{B_{\hat{\delta}}(\hat{x}_0)} (u(x, t_k) - u_*) \psi \, dx \geq \frac{c_1}{2} \int_{\Omega} \psi \, dx ,\]

which contradicts (5.26) and hence proves the lemma.

\[\Box \]

### 5.4 Stabilization of $v$

In what follows, based on the uniform Hölder bounds of $v$ and decay of $\nabla v$ implied by (5.2) and (5.9) respectively, we shall show the corresponding stabilization result for $v$ by a contradiction argument.

**Lemma 5.7** Let $m > 1$ and $(u, v, w)$ be the solution of (1.10)–(1.12) obtained in Lemma 2.1. Then we have

\[\|v(\cdot, t) - u_*\|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty. \quad (5.33)\]

**Proof** According to the uniform Hölder bounds of $v$ and decay of $\nabla v$ implied by (5.2) and (5.9), respectively, (5.33) may be derived by a contradiction argument. Indeed, assume that (5.33) was false, then we can find a sequence $(t_k)_{k \in \mathbb{N}}$ with $t_k \to \infty$ as $k \to \infty$, and constant $c_1 > 0$ such that

\[\|v(\cdot, t_k) - u_*\|_{L^\infty} \geq c_1 .\]

Furthermore the uniform Hölder continuity of $v$ in $\Omega \times [t, t + 1]$ warrants the existence of $(x_k)_{k \in \mathbb{N}}$ and $r > 0$ such that

\[|v(x, t) - u_*| > \frac{c_1}{2} \text{ for every } x \in B_r(x_k) \text{ and } t \in (t_k, t_k + \tau) \text{ and hence}\]

\[\int_{t_k}^{t_k + \tau} \int_{\Omega} |v(\cdot, t) - u_*|^2 > \frac{|\Omega| \tau c_1^2}{4}. \quad (5.34)\]

On the other hand, the Poincaré inequality indicates

\[\int_{t_k}^{t_k + \tau} \int_{\Omega} |v(\cdot, t) - u_*|^2 \leq C \int_{t_k}^{t_k + \tau} \int_{\Omega} |\nabla v|^2 + C \int_{t_k}^{t_k + \tau} \int_{\Omega} |v(\cdot, t) - u_*|^2. \quad (5.35)\]
Therefore (5.35) yields a contradiction to (5.34) thanks to
\[
\int_{t_k}^{t_k + \tau} \int_{\Omega} |\nabla u|^2 \to 0 \quad \text{as } t_k \to \infty.
\]

Now the convergence result in the flavor of Theorem 1.1 has actually been proved already.

**Proof of Theorem 1.1** The claimed assertion in Theorem 1.1 is the consequence of Lemma 5.4, 5.6 and 5.7.

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