ON A GENERALIZED KELLER-SEGEL SYSTEM IN ONE SPATIAL DIMENSION

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Abstract. In this paper we study a Keller-Segel system with diffusion given by fractional laplacians in one spatial dimension. We obtain several local and global well-posedness results. In presence of a logistic term, this model is known for exhibit a spatio-temporal chaotic behaviour where a number of peaks appears and mix together. We also study the dynamical properties of the system with the logistic term and we prove the existence of an attractor and provide a bound on the number of peaks that the solution may develop. These results generalize the known results where the diffusion is local.

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1. Introduction

The mathematical study of chemotaxis, i.e. the chemically prompted motion of cells, is a hot research topic with large related literature. In this context, the system

\begin{align*}
\partial_t u &= \mu \partial_x^2 u + \partial_x \cdot (u(-\partial_x v)) + ru(1 - u) \\
\tau \partial_t v &= \nu \partial_x^2 v - \lambda v + u.
\end{align*}

was proposed (see [40] and [35]). It is assumed that these cells \( u \) move towards increasing concentrations of some chemical substance \( v \). For instance, for the slime mold Dictyostelium Discoideum, the signal is produced by the cells, and, consequently, cell populations might form aggregates in
finite time. It also appears in certain bacterial populations, such as *Escherichia coli* and *Salmonella typhimurium*, and it results in their arrangement into a variety of spatial patterns. During embryogenesis, chemotaxis plays a role in angiogenesis, pigmentation patterning and neuronal development. Chemotaxis is also related to tumour growth. Specifically, in presence of the logistic term, this model is of particular importance because of its relationship with the three-component urokinase plasminogen invasion model (see [36]).

The system (1)-(2) with $\tau = r = 0$ also appears as a model of gravitational collapse. Indeed the system (1)-(2) is very similar in spirit to the Zel’dovich approximation [58] used in Cosmology to study the formation of large-scale structure in the primordial universe (see also [1]).

There is a huge literature on the mathematical study of (1)-(2) and its parabolic-elliptic simplification even in high dimensions. In particular, it is well-known that, in the parabolic-elliptic case, there is a threshold phenomena: there exists a constant $c_d$ depending on the dimension, $d$, such that if $\|u_0\|_{L^1} < c_d$ there exists a global in time classical solution but, on the other hand, if $\|u_0\|_{L^1} > c_d$ then there is a finite time blow-up (see [5, 6, 8, 9, 10, 11, 13, 14, 16, 17, 18, 22, 27, 28, 29, 38, 44, 46]).

The doubly parabolic case has been addressed by [7, 26, 43, 42, 48, 56, 54]. For instance, M. Winkler obtained blow up in finite time for initial data with radial symmetry and any prescribed initial data in a ball of $\mathbb{R}^d$, $d \geq 3$. Moreover, P. Biler, I. Guerra and G. Karch recently proved that for every finite Radon measure there exist $\tau_0$ and a global in time mild solution for (1)-(2) with $\tau > \tau_0$. In [26] it has been proved that if the initial data $(u_0, v_0)$ is small in $L^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ there exists a global solution. This result was recently generalized by X. Cao (see [23]).

Finally, the case with a nonlinear diffusion has been studied by several authors (see for instance [3, 4, 15, 20, 24]).

T. Hillen, M. Winkler, J. Tello and K. Painter have studied the case with a logistic growth (see [36, 49, 51, 50, 55, 54, 53] and references therein). In particular, in [54], the author prove that there exists a global in time solution for the doubly parabolic case with a sufficiently strong logistic parameter $r$. The same result holds for the parabolic-elliptic case (see [50]). Notice that if $r = 0$ the global existence has been addressed in [37]. Another remarkable feature of this model is its spatio-temporal chaotic behaviour. In particular, the numerical solutions reported in [49] for the system (1)-(2) develop a number of peaks that emerge and, eventually, mix with other peaks. These peaks are maxima for the functions $u, v$ such that they are very close to a region with slope bigger than one. This conduct materializes in the preliminary numerical study of system (3)-(4) with different values of $\alpha, \beta < 2$ (see Figure 1 for the case $\alpha = \beta = 1$). There are some mathematical results addressing this behaviour. For instance, in [48, 47] the existence of attractors is obtained for the case $\alpha = \beta = 2$ and spatial dimensions $d = 1, 2$. As noted in [54] the dynamical features of these models in high dimensions, in particular the existence of global attractors and bounded solutions, is an important topic.
Figure 1. Evolution in the case $\alpha = \beta = 1$.

The case with fractional powers of the laplacian instead of local derivatives (systems (3)-(4) and (5)-(6) and $\beta = 2$) has been addressed by several authors (see [1, 12, 19, 30, 33, 45, 57]). In particular, for the parabolic-elliptic case, in [1] the authors proved the existence of global in time solutions corresponding to small initial mass, while in [45] the authors proved finite time singularities by constructing a particular set of initial data showing this behaviour. In both papers and with different methods, the authors proved that any $L^1_t L^\infty_x$ bounded solution is global. In [1], the authors obtain a threshold phenomenon for the case $d = \alpha = 1, \beta = 2$ and global existence for every (maybe large) initial data if $\alpha > 1$ and $\beta = 2$. The doubly parabolic case with fractional operators has been addressed by P. Biler, G. Wu and X. Zheng. In particular, these authors proved local existence of solutions, global existence of solutions for initial data satisfying some smallness requirements and ill-posedness in a variety of Besov spaces.

This paper is devoted to the study of the following generalized, doubly parabolic ($\tau = 1$) Keller-Segel system with a logistic source

\begin{equation}
\partial_t u = -\mu \Lambda^\alpha u + \partial_x \cdot (u \Lambda^{\beta-1} H v) + ru(1-u) 
\end{equation}

and its parabolic-elliptic counterpart ($\tau = 0$)

\begin{equation}
\tau \partial_t v = -\nu \Lambda^\beta v - \lambda v + u,
\end{equation}

A similar model has been mentioned in [12, Section 5]. From this point onwards, we assume $\nu, \beta > 0, \alpha \geq 0$ and that the corresponding initial data is non-negative. Notice that in the case $\beta = \alpha = 2, \tau = 1$, we recover the usual doubly parabolic Keller-Segel system (1)-(2). In particular, the results in this paper applies to (1)-(2).

In this paper we prove local and global existence of solutions for (3)-(4) as well as a continuation criteria. In particular, we prove global existence for the hypoviscous case $\alpha = 1$ if the derivative of the initial data is small in the Wiener’s algebra. Notice that the constant in the smallness condition is a $O(1)$ number depending on the parameters present in the problem and $\|u_0\|_{L^1}$. As far as we know, this is the first time that a Wiener’s algebra bound is used in the Keller-Segel context. We prove global existence for
arbitrary initial data if $\alpha > 1$, $\beta \geq \alpha/2$. Furthermore, if $1 < \alpha \leq \beta \leq 2$, $r, \lambda > 0$, we obtain that the solutions remain bounded for every time. We also study the smoothing effect and the dynamical properties of the system. In particular we prove that the solution becomes analytic for every positive time and the existence of an attractor. These results can be roughly explained in Figure 2. The solution in a neighbourhood of this attractor develop a number of peaks that eventually merge themselves while other peaks emerge. Our last result is a bound on the number of peaks that the solution may develop. To the best of our knowledge, the global existence and the dynamical properties of the system are only known result when $\alpha = \beta = 2$. Moreover, the bound on the number of peaks seems new even in the local case.

For the parabolic-elliptic counterpart (5)-(6) we prove local and global existence of solutions. In particular, we characterize the conditions leading to global classical solutions depending on the value of $r$, $\alpha$ and $\|u_0\|_{L^1}$.

As we said before, these results generalize the known results for the case $\beta = 2$, $0 < \alpha \leq 2$.

Finally, let us mention that in the forthcoming paper [21], the authors perform a numerical study of the systems (3)-(4) and (5)-(6).

1.1. **Notation.** We write $H$ for the Hilbert transform and $\Lambda = \sqrt{-\Delta}$, i.e.

$$\widehat{Hu}(\xi) = -i\frac{\xi}{|\xi|} \hat{u}(\xi), \text{ and } \Lambda^s u(\xi) = |\xi|^s \hat{u}(\xi),$$

where $\hat{\cdot}$ denotes the usual Fourier transform. Notice that, in one dimension $\Lambda = \partial_x H$ and $\widehat{Hu}(0) = 0$. We write $H^s(\Omega)$ for the usual $L^2$-based Sobolev spaces with norm

$$\|f\|_{H^s} = \|f\|_{L^2}^2 + \|f\|_{H^s}^2, \|f\|_{H^s} = \|\Lambda^s f\|_{L^2}.$$

We define

$$\langle f \rangle = \frac{1}{|T|} \int_T f(x)dx, \quad T = [-\pi, \pi].$$
In what follows we will assume \( \tau = 1 \) for the doubly parabolic case and \( \tau = 0 \) for the parabolic-elliptic. We write \( T_{\max} \) for the maximum lifespan of the solution. For a given initial data \((u_0, v_0)\), we define

\[
N = \max\{\|u_0\|_{L^1(T)}, 2\pi\}.
\]

For a periodic function \( u \), we define the Wiener’s algebra based seminorms:

\[
|u|_s = \sum_{k \in \mathbb{Z}} |k|^s |\hat{u}(k)|.
\]

1.2. Plan of the paper. We provide the statements of our main results as well as some remarks in Section 2. In Section 3 we prove local existence of solutions and a continuation criteria. We study the global existence for the fully parabolic system in Section 4 and the global existence for the parabolic-elliptic case in Section 5. In Section 6 we study the smoothing properties of the systems (3)-(4) and (5)-(6). Finally, in Section 7 we study the dynamical properties of (3)-(4).

2. Statement of results

Let \( u_0(x), v_0(x) \geq 0 \) be the initial data for the system (3)-(4). Then we have the following definition of a solution:

**Definition 1.** Let \( 0 < T < \infty \) be a positive parameter. The couple \((u(t), v(t)) \in L^\infty([0, T], L^2(\Omega)) \times L^\infty([0, T], H^{3/2}(\Omega)) \) is a solution of (3)-(4) if

\[
\int_0^T \int_\Omega [\partial_t \phi - \mu \Lambda^\alpha \phi] u + \partial_x \phi (u \Lambda^{\beta-1} Hv) + \phi ru(1-u)dxdt - \int_\Omega \phi(x, 0) u_0 dx = 0,
\]

\[
\int_0^T \int_\Omega [\partial_t \varphi - \nu \Lambda^\beta \varphi - \lambda \varphi] v + \varphi ud\nu dxdt - \int_\Omega \varphi(x, 0) v_0 dx = 0,
\]

for every test functions \( \phi(x, t), \varphi(x, t) \in C^\infty_c([0, T] \times \Omega) \).

The parabolic-elliptic case only requires one initial data, \( u_0(x) \geq 0 \). Then, we have the following definition of solution for the system (5)-(6):

**Definition 2.** Let \( 0 < T < \infty \) be a positive parameter. The couple \((u(t), v(t)) \in L^\infty([0, T], L^2(\Omega)) \times L^\infty([0, T], H^{3/2}(\Omega)) \) is a solution of (5)-(6) if

\[
\int_0^T \int_\Omega [\partial_t \phi - \mu \Lambda^\alpha \phi] u + \partial_x \phi (u \Lambda^{\beta-1} Hv) + \phi ru(1-u)dxdt - \int_\Omega \phi(x, 0) u_0 dx = 0,
\]

\[
\int_0^T \int_\Omega [-\nu \Lambda^\beta \varphi - \lambda \varphi] v + \varphi ud\nu dxdt = 0,
\]

for every test function \( \phi(x, t), \varphi(x, t) \in C^\infty_c([0, T] \times \Omega) \).

**Definition 3.** If a solution \((u(t), v(t))\) verifies one of the previous definitions for every \( 0 < T < \infty \), this solution is called a global solution.

Now, for (3)-(4) and (5)-(6) with \( \lambda > 0 \), we obtain local in time existence of classical solution. In particular, we have the following result

**Theorem 1.** Given \( \Omega = \mathbb{R}, T, s \geq 3, \nu > 0 \) and \( 0 < \beta, \alpha \leq 2 \) then
Theorem 2. Assume that, for a finite time $T$ and initial data $(u_0, v_0) \in H^s(\Omega) \times H^{s+\beta/2}(\Omega)$, $s \geq 3$ and $\alpha, \beta, \nu, \mu > 0$, $r \geq 0$, the solution to (3)-(4) satisfies
\[
\int_0^T \| \Lambda^\beta v(s) \|_{L^\infty} + \| \partial_x u(s) \|_{L^\infty} \, ds < \infty,
\]
then, the solution can be continued up to time $T + \delta$ for a small enough $\delta > 0$. Moreover, if $\lambda > 0$, $\mu \geq \frac{1}{2\nu}$, $\alpha \geq \beta$ (if $\Omega = \mathbb{T}$) or $\lambda > 0$, $\alpha = \beta$ (if $\Omega = \mathbb{R}$), the previous condition can be replaced by
\[
\int_0^T \| u(s) \|_{L^{2\nu}}^\beta + \| \Lambda^\beta v(s) \|_{L^\infty} + \| u(s) \|_{L^\infty} \| \Lambda^\beta v(s) \|_{L^\infty} \, ds < \infty.
\]

Using this result we conclude that if $(u, v)$ is a solution showing finite time existence and being $T_{\text{max}}$ its maximum lifespan, we have
\[
\limsup_{t \to T_{\text{max}}} \| \Lambda^\beta v(t) \|_{L^\infty} + \| \partial_x u(t) \|_{L^\infty} = \infty.
\]
And, if $\lambda > 0$, $\mu \geq \frac{1}{2\nu}$, $\alpha \geq \beta$ (if $\Omega = \mathbb{T}$) or $\lambda > 0$, $\alpha = \beta$ (if $\Omega = \mathbb{R}$), the previous equation is replaced by
\[
\limsup_{t \to T_{\text{max}}} \| u(t) \|_{L^{2\nu}} + \| \Lambda^\beta v(t) \|_{L^\infty} = \infty.
\]

For the parabolic-elliptic case (5)-(6), we have

Corollary 1. Assume that, for a finite time $T$ and initial data $v_0 \in H^s(\Omega)$, $s \geq 3$ and $\alpha, \beta, \nu, \mu, \lambda > 0$, $r \geq 0$, the solution to (5)-(6) satisfies
\[
\int_0^T \| u(s) \|_{L^\infty} \, ds < \infty,
\]
then, the solution can be continued up to time $T + \delta$ for a small enough $\delta > 0$.

Let us emphasize that these results are general in the sense that there is no extra assumptions on the value of the parameters $\alpha, \beta, r, \lambda$ and they should be compared with the results in [54, Lemma 1.1] for the doubly parabolic case and in [1, 45] for the parabolic-elliptic case.

Using a Wiener’s algebra approach we obtain a global solution for small, periodic initial data.
Theorem 3. Let $\Omega = \mathbb{T}$, $(u_0, v_0) \in H^3 \times H^{3+1/2}$ be the domain and the initial data respectively, and assume $1 \leq \beta \leq 2 \leq 1 + \alpha$ and $r = 0, \mu > 1$ in the system (3)-(4). Then, if the initial data satisfy

$$|u_0|_1 + |v_0|_\beta \leq \min \{\mu - 1, \nu - \langle u_0 \rangle, \lambda/2\} \quad (\text{for } \lambda > 0),$$

or

$$|u_0|_1 + |v_0|_\beta \leq \min \{\mu - 1, (\nu - \langle u_0 \rangle)/3\} \quad (\text{for } \lambda = 0),$$

the solution is global and

$$|u(t)|_1 + |v(t)|_\beta \leq |u_0|_1 + |v_0|_\beta.$$

This result has the same flavour as [2, 43]. The case where $\alpha = 1$ is particularly interesting because for the case $\alpha > 1$ we prove below the existence of global solutions corresponding to arbitrary large initial data. Notice that the constant in the smallness condition depends explicitly on the parameters present in the problem and $\|u_0\|_{L^1}$.

Moreover, for the fully parabolic case with $1 < \alpha, \alpha/2 \leq \beta \leq 2$ we have

Theorem 4. Let $\Omega = \mathbb{T}$, $\mu, \nu > 0$, $2 \geq \alpha > 1$, $r, \lambda \geq 0$, $2 \geq \beta \geq \alpha/2$ and the initial data $(u_0, v_0) \in L^2 \times H^{3-\alpha/2}$ be given. Then there exists at least one global in time weak solution corresponding to $(u_0, v_0)$ satisfying

$$u(t) \in L^\infty([0,T], L^2(\mathbb{T})) \cap L^2([0,T], H^{\alpha/2}(\mathbb{T})), \forall T < \infty,$$

$$v(t) \in L^\infty([0,T], H^{3-\alpha/2}(\mathbb{T})) \cap L^2([0,T], H^{3\beta/2-\alpha/2}(\mathbb{T})), \forall T < \infty.$$

If, in addition, the initial data $(u_0, v_0) \in H^{k\alpha} \times H^{k\alpha+\beta/2}$, $k \in \mathbb{N}$, $k\alpha \geq 3$, then there exists a unique global in time solution corresponding to $(u_0, v_0)$

$$u(t) \in C([0,T], H^{k\alpha}(\mathbb{T})), \forall T < \infty,$$

$$v(t) \in C([0,T], H^{k\alpha+\beta/2}(\mathbb{T})), \forall T < \infty.$$

Furthermore, in the case $r, \lambda > 0, \alpha \leq \beta$, there exist positive numbers $T^*$, $S(\cdot)$ such that

$$\|u(t+1)\|_{H^{k\alpha/2}}^2 \leq S(\dot{H}^{k\alpha/2}), \forall t \geq T^*, 0 \leq k\alpha.$$

In particular,

$$\lim_{t \to \infty} \sup \|u(t)\|_{H^{\alpha/2}}^2 \leq S(H^{\alpha/2})$$

with

$$S(H^{\alpha/2}) \leq S(L^2) + S(\dot{H}^{\alpha/2}) = S(L^2) \left( 1 + \frac{2(1+e^{-r})}{\mu} e^{2t} \right),$$

where

$$S(L^2) = e^{\left[ 3N + \frac{C_{KP}(\alpha)^2 \lambda^3}{\mu} 2C_{FS}(\beta, \alpha, \lambda, \nu) + \frac{(C_{KP}(\alpha)C_{GN}(\alpha))^{\frac{\alpha+\beta}{\alpha}} 4\lambda^2}{\mu} (2NC_{FS}(\beta, \alpha, \lambda, \nu))^{\frac{\alpha+\beta}{\alpha+1}} \right].}$$
\begin{align*}
I = r + \frac{r^2C_E^2(\alpha)}{\mu}6N + \frac{N}{\nu} \left( \frac{3}{\nu} + 2C_F(\beta, \alpha, \lambda, \nu) \right) \\
\times \left( \frac{1}{2} + \frac{2C_E^2(\alpha) + (C_I(\alpha))^2}{\mu} \right) \left[ C_{KP}V(\alpha)(C^3_E(\alpha)C^j_E(\alpha) + C^j_E(\alpha)) \right]^2, \\
\end{align*}

where all the above constants are given later in (26)-(29).

Furthermore, there exists $C$, depending on the parameters present in the problem and on the initial data, such that

\[
\max_{0 \leq t < \infty} \{ \| u(t) \|^2_{H^{\alpha/2}} + \| v(t) \|^2_{H^{\beta+(k-1)\alpha/2}} \} \leq C,
\]

so, the solution is globally bounded.

**Remark 1.** We provide an estimate for these numbers $S(\cdot)$ along the proof of this Theorem. We also notice that the hypothesis $\beta \geq \alpha$ is not required to obtain the existence of $S(L^2)$.

Concerning the global existence for the parabolic-elliptic case (5)-(6) we have the following result

**Theorem 5.** Given $\Omega = \mathbb{R}, T, \mu, \alpha, r \geq 0, \lambda, \beta, \nu > 0$, and the initial data $u_0 \in H^3(\Omega)$. Assume that at least one of the following conditions holds:

- $r \geq \frac{1}{\nu}$,
- $2 > \alpha > 1$,
- $\alpha = 1$ and $N \leq \frac{\mu^2}{2\pi}$,

then, there is a global in time solution corresponding to $u_0$

\[ u(t) \in C([0, T], H^3(\Omega)), \forall T < \infty. \]

This result is in accordance with [54] and [1].

**Remark 2.** Notice that the previous result gives us a bound

\[ \| u(t) \|_{H^s} \leq \| u_0 \|_{H^s}e^{ct}. \]

This is enough to conclude the global existence, however, to the best of our knowledge, there is not a result addressing the optimality of such a bound.

We prove a result showing the smoothing effect, but first we need some basic notation and definitions:

Let’s define

\[(8) \quad 0 < \omega_0 = \min \left\{ \frac{\nu}{3}, \frac{\mu}{8} \right\}, \]

and consider $\omega$ a positive constant that will be fixed later. This constant may depend on the parameters present in the problem and on the initial data. We define the (time dependent) complex strip

\[ S_\omega = \{ x + i\xi, \ x \in \Omega, \ |\xi| \leq \omega t \}, \]
and
\[ C_1 = \max_{\xi \in \mathbb{R}^+} 4\omega \xi - \frac{\mu}{2} \xi^\alpha \]
(9)
\[ = 4\omega \left( \frac{8\omega}{\mu \alpha} \right)^{\frac{1}{\alpha-1}} - \frac{\mu}{2} \left( \frac{8\omega}{\mu \alpha} \right)^{\frac{\alpha}{\alpha-1}}, \]
\[ C_2 = \max_{\xi \in \mathbb{R}^+} 3\omega \xi - \nu \xi^\beta \]
(10)
\[ = 3\omega \left( \frac{3\omega}{\nu \beta} \right)^{\frac{1}{\beta-1}} - \nu \left( \frac{3\omega}{\nu \beta} \right)^{\frac{\beta}{\beta-1}}, \]
\[ C_3 = 2 \left[ \frac{\omega^2}{\alpha} \left( \frac{\omega^2}{(C_{SI}(\alpha))^2} \right)^{-1/(\alpha-1)} + 2\omega^{\frac{1}{\alpha-1}} \right], \]
(11)
\[ C_4 = 1.5 + \frac{2(C_{KPV}C_{SE}(1.1))^2}{\mu}, \]
(12)
\[ C_5 = 2 \left[ \frac{\mu}{4} + 1.25 + 2r + (17.5C_{SE}(1.1))^2 + \frac{9rC_{SE}(1.1)^2}{2} \right], \]
(13)
\[ C_6 = 1.5 + \frac{2(C_{KPV}C_{SE}(1.1))^2}{\mu}, \]
(14)
Finally, we denote
\[ K_1(\alpha, \beta, \mu, \nu, r, \lambda) = 1 + (\mu+2+2r+\nu+\lambda+1)C_{SE}(\alpha)+C_1+C_2+C_3+C_4, \]
(15)
\[ K_2(\alpha, \beta, \mu, \nu, r, \lambda) = 1 + (1+\mu+2+2r+\nu+\lambda+1)C_{SE}(1.1)+C_5+C_6, \]
(16)
\[ K(\alpha, \beta, \mu, \nu, r, \lambda) = \begin{cases} K_1 & \text{if } \alpha, \beta > 1, \\ K_2 & \text{if } \min\{\alpha, \beta\} = 1, \end{cases} \]
(17)
and
\[ \hat{T} = \frac{1 + \|u_0\|^2_{H^3(T)} + \|v_0\|^2_{H^4(T)}}{3K}. \]
(18)

Then, we have

**Theorem 6.** Given \( \Omega = \mathbb{T}, \mathbb{R}, 2 \geq \alpha, \beta \geq 1, \mu, \nu > 0, \lambda, r \geq 0 \) and the initial data \((u_0, v_0) \in H^3 \times H^4\), then the solution \((u(x, t), v(x, t))\) becomes analytic for every \(0 < t < \hat{T}\). Furthermore, \((u(x, t), v(x, t))\) becomes complex analytic in the growing in time, complex strip \(S_\omega\) with \(\omega \leq \omega_0\) and we have the bounds
\[ \|u(t)\|_{L^\infty(S_\omega)} \leq \sqrt{2}\|u_0\|_{L^\infty(\Omega)}, \|v(t)\|_{L^\infty(S_\omega)} \leq \sqrt{2}\|v_0\|_{L^\infty(\Omega)}. \]

and, as a consequence,

**Corollary 2.** If \(\alpha, \beta > 1,\) and \(\min\{\mu, \nu\} > 0, \lambda, r \geq 0,\) the solutions \((u(t), v(t)) \in H^3(\mathbb{T}) \times H^4(\mathbb{T})\) to the problem (3)-(4) are real analytic for every \(0 < t\).
Corollary 3. If $\alpha, \beta \geq 1$, and $\min\{\mu, \nu\} < 0$, the problem is ill-posed, i.e. there are solutions $(u(t), v(t))$ to the problem (3)-(4) such that

$$\|u_0\|_{H^3(\Omega)} + \|v_0\|_{H^4(\Omega)} < \epsilon$$

and

$$\limsup_{t \to -\delta^-} \|u(t)\|_{H^3(\Omega)} + \|v(t)\|_{H^4(\Omega)} = \infty,$$

for every $\epsilon > 0$ and small enough $\delta > 0$.

Remark 3. If, in addition, $\min\{\alpha, \beta\} > 1$, the restriction $\omega \leq \omega_0$ can be relaxed.

We can apply Theorem 6 to study some dynamical properties of the system (3)-(4). In particular

Theorem 7. Let $\Omega = \mathbb{T}$, $N \geq 3$, $N \in \mathbb{N}$, $2 \geq \alpha, \beta \geq 1$, $\mu, \nu > 0$, $\lambda, r \geq 0$ and the initial data $(u_0, v_0) \in H^3 \times H^4$ be given and write

$$W = \frac{\omega_0 \tilde{T}}{N},$$

where $\omega_0$ and $\tilde{T}$ are defined in (8) and (18) respectively. Then, for any $\epsilon > 0$, $0 < T/(N-1) < t < \tilde{T}$, $T = I^0 \cup R^u = I^0 \cup R^v$, where $I^0, I^v$ are the union of at most $\left[\frac{4\pi}{W\epsilon}\right]$ intervals open in $\mathbb{T}$, and

- $|\partial_x u(x)| \leq \epsilon$, for all $x \in I^u$;
- $\text{card}\{x \in R^u : \partial_x u(x) = 0\} \leq \frac{2}{\log 2} \frac{2^\nu}{W\epsilon} \log \left(\frac{\sqrt{2}(N-1)\|u_0\|_{L^\infty(\Omega)}}{W\epsilon}\right)$;
- $|\partial_x v(x)| \leq \epsilon$, for all $x \in I^v$;
- $\text{card}\{x \in R^v : \partial_x v(x) = 0\} \leq \frac{2}{\log 2} \frac{2^\nu}{W\epsilon} \log \left(\frac{\sqrt{2}(N-1)\|v_0\|_{L^\infty(\Omega)}}{W\epsilon}\right)$.

and

Theorem 8. Given $\Omega = \mathbb{T}$, $r, \lambda > 0$, $2 \geq \beta \geq \alpha \geq 8/7$, the system (3)-(4) has a maximal, connected, compact attractor in the space $H^{3\alpha}(\mathbb{T}) \times H^{3\alpha+\beta/2}$.

Notice that Theorem 7 gives us an estimate of the number of peaks appearing in the evolution (and reported in the numerical simulations). Indeed, we have the following corollary

Corollary 4. Given $\Omega = \mathbb{T}$, $r, \lambda > 0$, $2 \geq \beta \geq \alpha \geq 8/7$, and let $(u(t), v(t))$ be a solution in the attractor, then, the number of peaks for $u(t)$ can be bounded as

$$\text{card}\{\text{peaks for } u\} \leq \frac{12\pi K^1_1}{\log 2} \log \left(6\sqrt{2}K_1C_{SE}^2(\alpha)S(H^{\alpha/2})\right),$$

where $S(H^{\alpha/2})$ is defined in (7).

In [49], they perform a numerical study of the case $\alpha = \beta = 2$, $\mu = \nu$, $r = \lambda$ and different values of $\lambda$ and $\nu$. These authors take initial data satisfying

$$\|u_0\|_{L^\infty} = 1, \|v_0\|_{L^\infty} \leq 1.01.$$

We can use our previous results to give an analytical bound on the number of peaks that the solutions in [49] develop:
the lower order terms can be bounded as
\[ \text{card}\{\text{peaks for } u\} \leq \frac{12\pi K_1}{\log 2} \log \left( 6\sqrt{2}K_1 \right), \]
\[ \text{card}\{\text{peaks for } v\} \leq \frac{12\pi K_1}{\log 2} \log \left( 6\sqrt{2}K_1 1.01 \right), \]
where \( K_1 = K_1(2, 2, v, \nu, \lambda, \lambda) \) is defined in (15).

3. Local existence and continuation criteria

We prove now the local well-posedness result:

**Proof of Theorem 1.** We prove the case \( s = 3 \), being the other cases similar. We start with the fully parabolic problem (3)-(4). We compute

\[ \frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 \leq -\nu \|v\|_{H^\beta/2}^2 - \lambda \|v\|_{H^\beta/2}^2 + \|u\|_{L^2} \|v\|_{L^2}, \]

\[ \frac{1}{2} \frac{d}{dt} \|\Lambda^{\beta/2} \partial_x^2 v\|_{L^2}^2 = \int_\Omega \Lambda^{\beta/2} \partial_x^2 v \partial_x^2 \partial_t v = -\nu \|v\|_{H^{3+\beta}/2}^2 - \lambda \|v\|_{H^{3+\beta}/2}^2 + \|u\|_{H^3} \|v\|_{H^3}, \]

\[ \frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 = -\mu \|u\|_{H^{\alpha/2}}^2 - \int_\Omega \partial_x uuH \Lambda^{\beta-1} v dx + r \int_\Omega u^2 (1 - u) dx \]

\[ \leq \|u(t)\|_{L^2}^2 \|\Lambda^{\beta} v\|_{L^\infty} + r \|u\|_{L^2}^2, \]

and

\[ \frac{1}{2} \frac{d}{dt} \|\partial_x^3 u\|_{L^2}^2 = -\mu \|u\|_{H^{3+\alpha/2}}^2 + \int_\Omega \partial_x^3 uu \Lambda^{\beta-1} v dx \]

\[ + r \int_\Omega \partial_x^3 uu (1 - u) dx. \]

The higher order terms are

\[ J_1 = \int_\Omega \partial_x^3 uu \Lambda^{\beta-1} v dx \leq \|\Lambda^{\beta} v\|_{L^\infty} \|u\|_{H^3}^2, \]

\[ J_2 = \int_\Omega \partial_x^2 uu \Lambda^{\beta} \partial_x^2 v dx \leq \|\Lambda^{\beta} \partial_x^2 v\|_{L^2} \|u\|_{L^3} \|u\|_{L^{\infty}}, \]

\[ J_3 = r \int_\Omega (\partial_x^3 u)^2 (1 - 2u) dx \leq r \|\partial_x^3 u(t)\|_{L^2}^2. \]

Using the Gagliardo-Niremberg inequality

\[ \|\partial_x f\|_{L^4}^2 \leq 3 \|f\|_{L^\infty} \|\partial_x^2 f\|_{L^2}, \]

the lower order terms can be bounded as

\[ J_4 = \int_\Omega \partial_x^3 uu \partial_x^3 v dx \leq \|\Lambda^{\beta} v\|_{L^\infty} \|u\|_{H^3}^2, \]
\[
J_5 = \int_\Omega \partial_x^2 u \partial_t^2 u \Lambda^\beta \partial_x \nu dx \leq \| \Lambda^\beta \partial_x \nu \|_{L^4} \| u \|_{H^3} \| \partial_x^2 u \|_{L^4} \\
\leq C \| \Lambda^\beta \nu \|_{L^\infty}^{0.5} \| \Lambda^\beta \partial_x^2 v \|_{L^2}^{0.5} \| u \|_{H^3} \| \partial_x u \|_{L^\infty}^{0.5},
\]
\[
J_6 = \int_\Omega \partial_x^2 u \partial_x u \Lambda^\beta \partial_x v dx \leq \| \Lambda^\beta \partial_x^2 v \|_{L^2} \| u \|_{H^3} \| \partial_x u \|_{L^\infty},
\]
and
\[
J_7 = 6r \int_\Omega \partial_x^2 u \partial_x u \partial_t^2 u dx \leq 6r \| u \|_{H^3} \| \partial_x u \|_{L^4} \| \partial_x^2 u \|_{L^4} \leq C \| u \|_{H^3}^{0.5} \| \partial_x u \|_{L^\infty}^{0.5}.
\]

We define the energy
\[
E = \| u \|_{H^3}^2 + \| \nu \|_{H^{3+\beta/2}}^2.
\]
We have
\[
\frac{d}{dt} E \leq C (E + 1)^3 + \| \Lambda^\beta \partial_x^2 v \|_{L^2} \| u \|_{H^3} \| u \|_{L^\infty} \\
- \frac{\nu}{2} \| \nu \|_{H^{3+\beta}}^2 - \lambda \| u \|_{H^3}^2 + \frac{2}{\nu} \| \| u \|_{H^{3+\alpha}}^2 - \mu \| u \|_{H^{3+\alpha/2}}^2 \\
\leq C (\nu) (E + 1)^4 - \mu \| u \|_{H^{3+\alpha/2}}^2 - \frac{\nu}{4} \| \nu \|_{H^{3+\beta}}^2.
\]
In the grating the previous inequality we obtain the desired bound for the energy. Moreover, from this latter inequality we get that \((u, v) \in L^2_t H_x^{3+\alpha/2} \times L^2_t H_x^{3+\beta}.\) To prove the uniqueness we argue by contradiction.

Let’s assume that there are two different solutions corresponding to the same initial data \((u_0, v_0) \in L^2 \times H^{3/2} .\) We write \((u_i, v_i), i = 1, 2\) for these solutions and define \(\tilde{u} = u_1 - u_2, \tilde{v} = v_1 - v_2.\) Then we have the bounds
\[
\frac{d}{dt} \| \tilde{u}(t) \|_{H^{3/2}}^2 + \nu \| \tilde{v}(t) \|_{H^3}^2 \leq c(\nu) \| \tilde{u}(t) \|_{L^2}^2,
\]
\[
\frac{d}{dt} \| \tilde{u}(t) \|_{L^2}^2 \leq 2 \left| \int_\Omega \partial_x \tilde{u}[\tilde{u} \Lambda^{\beta-1} H \nu v_1 + u_2 \Lambda^{\beta-1} H \tilde{v}] dx \right| \\
\leq \| \tilde{u}(t) \|_{L^2}^2 \| \Lambda^\beta v_1(t) \|_{L^\infty} + \| \tilde{u}(t) \|_{L^2}^2 \| \partial_x u_2(t) \|_{L^\infty} \| \Lambda^{\beta-1} \tilde{v}(t) \|_{L^2} \\
+ \| \tilde{u}(t) \|_{L^2} \| u_2(t) \|_{L^\infty} \| \Lambda^\beta \tilde{v}(t) \|_{L^2}.
\]

We use \(\beta - 1 \leq \beta/2\) and Young’s inequality to get
\[
\frac{d}{dt} (\| \tilde{u}(t) \|_{L^2}^2 + \| \tilde{v}(t) \|_{H^{3/2}}^2) \leq (\| \tilde{u}(t) \|_{L^2}^2 + \| \tilde{v}(t) \|_{H^{3/2}}^2) \\
\times \left[ c(\nu) + \| \Lambda^\beta v_1(t) \|_{L^\infty} + \frac{1}{2} \| \partial_x u_2(t) \|_{L^\infty} + c(\nu) \| u_2(t) \|_{L^\infty}^2 \right].
\]
Finally we conclude
\[
\| \tilde{u}(t) \|_{L^2}^2 + \| \tilde{v}(t) \|_{H^{3/2}}^2 \leq (\| \tilde{u}_0 \|_{L^2}^2 + \| \tilde{v}_0 \|_{H^{3/2}}^2) \\
\times e^{c(\nu) + \int_0^t \| \Lambda^\beta v_1(s) \|_{L^\infty} + 0.5 \| \partial_x u_2(s) \|_{L^\infty} + c(\nu) \| u_2(s) \|_{L^\infty}^2} dt.
\]
From this latter inequality we obtain the uniqueness. To conclude with the fully parabolic case, we have to prove that the sign of the solutions propagates, i.e. as the initial data is non-negative, the solution remains non-negative as well. We can prove this by contradiction. Let’s consider
the case $\Omega = \mathbb{T}$ first. Let $u(x, t)$ be a classical solution with a non-negative initial data and write $x_1 \in \mathbb{T}$ for the point such that $\min_x u(x, t) = u(x_1, t)$. Evaluating the equation (3) at the point of minimum and using the kernel expression for $\Lambda^\alpha$, we have

$$\frac{d}{dt}u(x_1, t) \geq u(x_1, t) \left[ \Lambda^\beta v(x_1, t) + r(1 - u(x_1, t)) \right], \ t \geq 0,$$

and solving this ode, we have

$$u(x_1, t) = u_0(x_1)e^{\int_0^t \Lambda^\beta v(x_1, s) + r(1 - u(x_1, s))ds}.$$

Thus, $u(t)$ has the same sign as $u_0(x) \geq 0$ and we conclude the claim. For the equation (4) we can proceed similarly and we get

$$v(x_1, t) = v_0(x_1)e^{-\lambda t} + e^{-\lambda t} \int_0^t u(x_1, s)ds \geq 0.$$

If the domain is $\Omega = \mathbb{R}$ and the function $u(x, t) \in H^3(\mathbb{R})$ then it tends to zero (from above) at infinity. In order the solution may become negative in some region there should exists a time $s < 0$ at infinity. In order the solution may become negative in some region there should exists a time $s < 0$ at infinity. In order the solution may become negative in some region there should exists a time $s < 0$ at infinity. In order the solution may become negative in some region there should exists a time $s < 0$ at infinity.

Due to the definition of $\Lambda^\alpha$ and solving this ode, we have

$$\nu \Lambda^\beta v(x_1) + \lambda v(x_1) \geq \min_x u(x, t) \geq 0.$$

We have

$$\Lambda^\beta v(x_1) = c(\alpha) \left( \sum_{k \in \mathbb{Z}, k \neq 0} \int_\mathbb{T} \frac{v(x_1) - v(x_1 - \eta)}{|\eta + 2k\pi|^{1+\alpha}} d\eta \right),$$

$$+ \text{P.V.} \int_\mathbb{T} \frac{v(x_1) - v(x_1 - \eta)}{|\eta|^{1+\alpha}} d\eta,$$

where

$$c(\beta) = \frac{\Gamma(1 + \beta) \cos((1 - \beta)\pi/2)}{\pi} \geq 0.$$

Due to the definition of $x_1$, we have $v(x_1) - v(x_1 - \eta) \leq 0$, and, consequently $\Lambda^\beta v(x_1) \leq 0$. Using this fact we get

$$\lambda v(x_1) \geq \nu \Lambda^\beta v(x_1) + \lambda v(x_1) \geq \min_x u(x, t) \geq 0.$$

In the same way, if we evaluate at $\tilde{x}_1$ such that $\max_x v(x, t) = v(\tilde{x}_1, t)$, we have $\Lambda^\beta v(\tilde{x}_1) \geq 0$ and we get the bound

$$\|v\|_{L^\infty} \leq v(\tilde{x}_1) + \frac{\nu \Lambda^\beta v(\tilde{x}_1)}{\lambda} \leq \frac{\|u(t)\|_{L^\infty}}{\lambda}.$$

Now we evaluate the equation (6) at $X_1$ such that $\max \Lambda^\beta v(x, t) = \Lambda^\beta v(X_1, t) \geq 0$. We have

$$\nu \Lambda^\beta v(X_1) \leq \nu \Lambda^\beta v(X_1) + \lambda v(X_1) \leq \|u(t)\|_{L^\infty}.$$
Defining \( \tilde{X}_t \) such that \( \min_x \Lambda^3 v(x, t) = \Lambda^3 v(\tilde{X}_t, t) \leq 0 \). We have

\[
-v \Lambda^3 v(\tilde{X}_t) = -u(\tilde{X}_t) + \lambda v(\tilde{X}_t) \leq \lambda \|v\|_{L^\infty} \leq \|u(t)\|_{L^\infty}.
\]

Collecting both estimates we conclude

\[
\|\Lambda^3 v(t)\|_{L^\infty(T)} \leq \|u(t)\|_{L^\infty(T)}. \tag{19}
\]

Taking \( k \) derivatives of the equation (6) and testing against \( \Lambda^3 \partial_x^k v \) we get

\[
\nu \|\Lambda^3 \partial_x^k v\|_{L^2}^2 + \lambda \|\Lambda^{3/2} \partial_x^k v\|_{L^2}^2 \leq \frac{1}{2\nu} \|\partial_x^k v\|_{L^2}^2 + \frac{\nu}{2} \|\Lambda^3 \partial_x^k v\|_{L^2}^2.
\]

With the previous bounds for \( v \) we can estimate the terms \( J_i \). Consequently, if we define the new energy

\[
E(t) = \|u(t)\|_{H^1}^2,
\]

we have the bound

\[
\frac{d}{dt} E(t) \leq C(\nu)(1 + E(t))^3 - \mu \|u(t)\|_{H^{3+\alpha/2}}^2.
\]

Using Gronwall, we conclude the existence of solution. Let’s explain briefly how to handle the case \( \Omega = \mathbb{R} \). We define \( v \) using Fourier transform. This function \( v \) is not entirely negative and \( v \) tends to zero at \( |x| \gg 1 \). If the function is negative in some region, then there exists \( x_t \) contained in a compact set such that \( v(x_t, t) = \min_x v(x, t) \). We evaluate the equation at that point and we get a contradiction. Consequently, the function \( v \) is non-negative. With the same reasoning, we get \( \|v(t)\|_{L^\infty} \leq \|u(t)\|_{L^\infty}/\lambda \). Using \( u \in H^3 \), we get that \( \Lambda^3 v \in H^3 \). Due to this fact, there exists \( X_t \) and \( \tilde{X}_t \) (defined as in the periodic case) contained in a compact. Evaluating at these points, we get the (19). With these remarks the proof follows. The uniqueness can be obtained as in the fully parabolic case.

Now we proceed with the proof of Theorem 2:

**Proof of Theorem 2. Step 1:** Let’s write

\[
\int_0^T \|\Lambda^3 v(s)\|_{L^\infty} + \|\partial_x u(s)\|_{L^\infty} ds = M < \infty.
\]

This bound implies

\[
\sup_{0 \leq t \leq T} \|u(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 e^{M+\nu T},
\]

and

\[
\sup_{0 \leq t \leq T} \|v(t)\|_{L^2}^2 \leq \left( \|u_0\|_{L^2}^2 e^M + \|v_0\|_{L^2}^2 \right) e^T.
\]

Moreover, we have

\[
\sup_{0 \leq t \leq T} \|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} e^{M+\nu T}, \quad \sup_{0 \leq t \leq T} \|v(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} e^{MT}
\]

thus, \( u, v \in L^\infty_T L^2_x \cap L^\infty_T L^\infty_x \). Notice that up to now we needed only the first part of \( M \). For the higher seminorm,

\[
y(t) = \|u(t)\|_{H^3}^2 + \|v(t)\|_{H^{3+\beta/2}}^2
\]
we have
\[ \frac{d}{dt} y(t) \leq c(M, \nu)(\|\Lambda^\beta v\|_{L^\infty} + \|\partial_x u\|_{L^\infty} + \|u_0\|_{L^\infty} e^{M+\nu T} y(t)), \]
and using Gronwall’s inequality we conclude the result.

**Step 2:** To simplify notation we write
\[ \int_0^T \|u(s)\|_{L^\infty} + \|\Lambda^\beta v(s)\|_{L^\infty} + \|u(s)\|_{L^\infty} \|\Lambda^\beta v(s)\|_{L^\infty} ds = \tilde{M}. \]

First we compute
\[ \frac{1}{2} \frac{d}{dt} \|\Lambda^\beta \partial_x^2 v\|_{L^2}^2 + \lambda \|\Lambda^\beta \partial_x^2 v\|_{L^2}^2 \leq -\nu \|\Lambda^{1.5} \partial_x^2 v\|_{L^2}^2 + \|\Lambda^{\beta/2} \partial_x^2 u\|_{L^2} \|\Lambda^{1.5} \partial_x^2 v\|_{L^2}, \]
thus,
\[ \frac{1}{2} \frac{d}{dt} \|\Lambda^\beta \partial_x^2 v\|_{L^2}^2 + \lambda \|\Lambda^\beta \partial_x^2 v\|_{L^2}^2 \leq \frac{1}{2\nu} \|\Lambda^{\beta/2} \partial_x^2 u\|_{L^2}^2. \]

Now we have
\[ \frac{1}{2} \frac{d}{dt} \|\partial_x^2 u\|_{L^2}^2 \leq -\mu \|\Lambda^{\alpha/2} \partial_x^2 u\|_{L^2}^2 + c \|\Lambda^\beta v\|_{L^\infty} \|\partial_x^2 u\|_{L^2}^2 \]
\[ + c \|\partial_x^2 u\|_{L^2} \left( \|u\|_{L^\infty}^{0.5} \|\partial_x^2 u\|_{L^2}^{0.5} \|\Lambda^\beta v\|_{L^\infty}^{0.5} \|\Lambda^{\beta/2} v\|_{L^2}^{0.5} \right) \]
\[ + \|u\|_{L^\infty} \|\Lambda^\beta \partial_x^2 v\|_{L^2} \|\partial_x^2 u\|_{L^2} \]
\[ + \nu \|\partial_x^2 u\|_{L^2}^2 + 2r \|\partial_x^2 u\|_{L^2} \|\partial_x u\|_{L^4}. \]

Using the previous bound and Young’s inequality, we get
\[ \frac{1}{2} \frac{d}{dt} \left( \|\partial_x^2 u\|_{L^2}^2 + \|\Lambda^\beta \partial_x^2 v\|_{L^2}^2 \right) \leq -\mu \|\Lambda^{\alpha/2} \partial_x^2 u\|_{L^2}^2 + \frac{1}{2\nu} \|\Lambda^{\beta/2} \partial_x^2 u\|_{L^2}^2 \]
\[ + c \|\Lambda^\beta v\|_{L^\infty} \|\partial_x^2 u\|_{L^2}^2 + \frac{1}{2\nu} \|\Lambda^\beta \partial_x^2 u\|_{L^2}^2 \]
\[ + c \|\partial_x^2 u\|_{L^2} \|u\|_{L^\infty} \|\partial_x^2 u\|_{L^2} \]
\[ + c \|\partial_x^2 u\|_{L^2} \|u\|_{L^\infty} \|\Lambda^\beta v\|_{L^\infty} \]
\[ + c \|\partial_x^2 u\|_{L^2} \|u\|_{L^\infty} + 1 \]

Thus, we obtain a bound for the $H^2$ seminorm. In the same way we obtain a bound for the $L^2$ norm. Since we have a bound for $H^2$, using Sobolev embedding, we get a bound for
\[ \int_0^T \|\partial_x u(s)\|_{L^\infty} ds \leq c \int_0^T \|u(s)\|_{H^2} ds \leq cT \|u_0\|_{H^2} \exp \left( c(\lambda) \tilde{M} \right). \]

**Proof of Corollary 1.** Using
\[ \|\Lambda^\beta \partial_x v\|_{L^\infty} \leq \frac{\|\partial_x u(t)\|_{L^\infty}}{\nu}, \]
we get
\[ \frac{d}{dt} \|\partial_x u(t)\|_{L^\infty} \leq c \|u(t)\|_{L^\infty} \|\partial_x u(t)\|_{L^\infty}. \]

Thus, if
\[ \int_0^T \|u(s)\|_{L^\infty} = M < \infty, \]

\[ \square \]
we get
\[
\int_0^T \|\Lambda^\beta v(s)\|_{L^\infty} + \|\partial_x u(s)\|_{L^\infty} \, ds \leq M + e^{cM} < \infty.
\]
\[\square\]

4. Global existence for the fully parabolic case

We start this section with two preliminary results concerning lower order norms. The behaviour is quite different depending on the value of \( r \). If \( r = 0 \), we have

**Lemma 1.** Let \((u_0, v_0)\) be two non-negative, smooth initial data for equation (3)-(4) with \( r = 0 \). Then, the solutions \((u, v)\) are non-negative functions. Moreover, if, in addition, the initial data \((u_0, v_0)\) are in \( L^1(\Omega) \), the solutions \((u, v)\) verify

- \( \|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \forall 0 \leq t \leq T_{\text{max}} \)
- \( \|v(t)\|_{L^1(\Omega)} = \frac{\|u_0\|_{L^1(\Omega)}}{\lambda} \left( \|v_0\|_{L^1(\Omega)} - \frac{\|u_0\|_{L^1(\Omega)}}{\lambda} \right) e^{-\lambda t} \forall 0 \leq t \leq T_{\text{max}} \) (if \( \lambda > 0 \)),

or

- \( \|v(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} t + \|v_0\|_{L^1(\Omega)} \forall 0 \leq t \leq T_{\text{max}} \) (if \( \lambda = 0 \)).

For the sake of brevity we don’t write the proof. For \( r > 0 \) in the periodic case, the result reads (see also \([36]\))

**Lemma 2.** Let \((u_0, v_0)\) be two non-negative, smooth initial data for equation (3)-(4) with \( r > 0 \) and \( \Omega = \mathbb{T} \). Let’s define

\[
\mathcal{N} = \max\{\|u_0\|_{L^1(\mathbb{T})}, 2\pi\}.
\]

Then the solutions \((u, v)\) verify

- \( \|u(t)\|_{L^1(\mathbb{T})} \leq \mathcal{N}, \int_0^t \|u(s)\|_{L^2(\mathbb{T})}^2 \, ds \leq \mathcal{N} t + 2\mathcal{N}, \forall 0 \leq t \leq T_{\text{max}}, \)
- \( \|v(t)\|_{L^1(\mathbb{T})} \leq \max\{\|v_0\|_{L^1}, \mathcal{N}/\lambda\}, \forall 0 \leq t \leq T_{\text{max}}, \) (if \( \lambda > 0 \)),

or

- \( \|v(t)\|_{L^1(\mathbb{T})} \leq \|v_0\|_{L^1(\mathbb{T})} + \mathcal{N} t, \forall 0 \leq t \leq T_{\text{max}}, \) (if \( \lambda = 0 \)).

**Proof.** We take \( r = 1 \) without losing generality. The ODE for \( \|u(t)\|_{L^1} \) is

\[
\frac{d}{dt} \|u(t)\|_{L^1} = \|u(t)\|_{L^1} - \|u(t)\|_{L^2}^2.
\]

Recalling Jensen’s inequality

\[
\|u(t)\|_{L^1}^2 \leq 2\pi \|u(t)\|_{L^2}^2,
\]
we get
\[ \frac{d}{dt} \|u(t)\|_{L^1} = \|u(t)\|_{L^1} \left(1 - \frac{1}{2\pi} \|u(t)\|_{L^1}\right). \]

From this inequality we conclude the first part of the result. Given \( t > 0 \), we integrate in \((0, t)\) between 0 and \( t \) and we get
\[ \int_0^t \|u(s)\|_{L^2(\Omega)}^2 ds \leq \|u_0\|_{L^1} - \|u(t)\|_{L^1} + \sup_{0 \leq s \leq t} \|u(s)\|_{L^1} t \leq \mathcal{N} t + 2 \mathcal{N}, \]
and we conclude the second part. The bound for the \( L^1 \) norm of \( v \) is straightforward and we get
\[ \|v(t)\|_{L^1(\Omega)} \leq \frac{\mathcal{N}}{\lambda} + \left(\|v_0\|_{L^1(\Omega)} - \frac{\mathcal{N}}{\lambda}\right) e^{-\lambda t}, \forall t \geq 0, (\lambda > 0), \]
or
\[ \|v(t)\|_{L^1(\Omega)} \leq \mathcal{N} t + \|v_0\|_{L^1(\Omega)}, \forall t \geq 0, (\lambda = 0). \]

□

For the periodic case with \( 1 \leq \beta \leq 2 \leq 1 + \alpha \) and \( \mu > 1, r = 0 \), using a Wiener’s algebra approach we prove a global in time existence for small initial data:

**Proof of Theorem 3.** We denote \( \hat{f}(k) \) the \( k \)-th Fourier mode of the function \( f \). Then, as stated in Lemma 1 we have \( \hat{u}(0, t) = \langle u_0 \rangle \). We use a Wiener’s algebra approach and we study the evolution of
\[ \mathcal{E}(t) = |u(t)|_1 + |v(t)|_\beta. \]

The system (3)-(4) reads
\[
\frac{d}{dt} |\hat{u}(k)|k| = -\mu|k|^{1+\alpha}|\hat{u}(k)| + \frac{|k|\hat{\alpha}(k)}{|\hat{u}(k)|} \sum_j j\hat{u}(j) \frac{k-j}{|k-j|} \hat{v}(k-j) + \frac{|k|\hat{\alpha}(k)}{|\hat{u}(k)|} \sum_j \hat{u}(k-j) |j|^{\beta-1} \hat{v}(j),
\]
\[
\frac{d}{dt} |\hat{v}(k)|k|^\beta = -\nu|k|^{2\beta}|\hat{v}(k)| - \lambda |\hat{v}(k)||k|^\beta + \frac{\bar{\alpha}(k)}{|\bar{v}(k)|} |k|^\beta \hat{u}(k),
\]
so, using Tonelli’s Theorem,
\[
\frac{d}{dt} |u(t)|_1 \leq -\mu|u|_{1+\alpha} + 2|u|_1 |v|_\beta + |u|_2 |v|_{\beta-1} + |u|_0 |v|_{\beta+1},
\]
and
\[
\frac{d}{dt} \mathcal{E} \leq -\mu|u|_{1+\alpha} + 2|u|_1 |v|_\beta + |u|_2 |v|_{\beta-1} + |u|_0 |v|_{\beta+1} - \nu |v|_{2\beta} - \lambda |v|_{\beta} + |u|_{\beta}.
\]

Using Young’s inequality and the hypotheses in the theorem, we get
\[
\frac{d}{dt} \mathcal{E} \leq (|v|_{\beta} + 1 - \mu)|u|_2 + (|u|_1 + \langle u_0 \rangle - \nu)|v|_{2\beta} + (2|u|_1 - \lambda)|v|_{\beta}
\]
\[
\leq (\mathcal{E} + 1 - \mu)|u|_2 + (\mathcal{E} + \langle u_0 \rangle - \nu)|v|_{2\beta} + (2\mathcal{E} - \lambda)|v|_{\beta},
\]
thus, if
\[ \mathcal{E}(0) < \min\{\mu - 1, \nu - \langle u_0 \rangle, \lambda/2\}, \]
we obtain decay (consequently, a global bound) for $E(t)$. If $\lambda = 0$ and $E(0) < \min\{\mu - 1, (\nu - \langle u_0 \rangle)/3\}$, we arrive to the same conclusion for $E(t)$. Using Fourier series we get

$$\partial_x u = \sum_j ij \hat{u}(j)e^{ijx} \Rightarrow \|\partial_x u\|_{L^\infty} \leq |u|_1,$$

$$\Lambda^\beta v = \sum_j |j|^\beta \hat{v}(j)e^{ijx} \Rightarrow \|\Lambda^\beta v\|_{L^\infty} \leq |v|_\beta.$$

Using the continuation argument in the previous result we conclude. □

Now we proceed with the proof of the global existence of solutions for large data:

**Proof of Theorem 4.** As $\alpha > 1$, we can define $\frac{\alpha - 1}{2} = \delta > 0$ a fixed parameter. We take $0 < T < \infty$ a parameter and consider times $0 \leq t \leq T$. Notice that $T$ is fixed but arbitrary. The proof goes as follows: in the first three steps, we obtain the a priori estimates, in Step 4, we construct the solutions and in Step 5, we obtain the existence of an absorbing set and, as a consequence, the boundedness of the solutions.

**Step 1:** $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^{\alpha/2}$ and $v \in L_t^\infty H_x^{\beta-\alpha/2} \cap L_t^2 H_x^{3\beta/2-\alpha/2}$. We compute the evolution of the $L^2$ norm of $u$ in the case $r > 0$. If $r = 0$ the proof is analogous. We get

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 = -\mu \int_T \|\Lambda^{\alpha/2} u\|^2 dx + \frac{1}{2} \int_T u^2 \Lambda^\beta v dx + r \|u(t)\|_{L^2}^2 - r \|u(t)\|_{L^3}^3 \leq -\mu \|u(t)\|_{H^{\alpha/2}}^2 + \frac{1}{2} \|\Lambda^{\alpha/2} (u(t))\|^2_{L^2} \|\Lambda^{\beta-\alpha/2} v(t)\|_{L^2}$$

$$+ 2r \|u(t)\|_{L^2}^2 - r \|u(t)\|_{L^3}^3 \leq -\mu \|u(t)\|_{H^{\alpha/2}}^2 + \|u(t)\|_{L^2}^2 - r \|u(t)\|_{L^3}^3 + C_{KP}(\alpha) \|u\|_{L^\infty} \|u\|_{H^{\alpha/2}} \|\Lambda^{\beta-\alpha/2} v(t)\|_{L^2}$$

$$+ C_{KP}(\alpha) C_{GN}(\alpha) \|u(t) - \langle u(t)\rangle_{L^1} \|_{L^1}^{\delta/(1+\delta)} \|u(t)\|_{L^1}^{(2+\delta)/(1+\delta)} \times \|\Lambda^{\beta-\alpha/2} v(t)\|_{L^2},$$

where we have used Lemma 4 together with the following interpolation inequality

$$\|u\|_{L^\infty} - \langle u \rangle \leq \|u - \langle u \rangle\|_{L^\infty} \leq C_{GN}(\alpha) \|u - \langle u \rangle\|_{L^1}^{\delta/(1+\delta)} \|u\|_{H^{\alpha/2}}^{1/(1+\delta)}.$$

Using Young’s inequality and Lemmas 1 and 2, we obtain

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 \leq -\mu \|u(t)\|_{H^{\alpha/2}}^2 + r \|u(t)\|_{L^2}^2 - r \|u(t)\|_{L^3}^3 + \frac{\mu}{2} \|u(t)\|_{H^{\alpha/2}}^2$$

$$+ \frac{(C_{KP}(\alpha) C_{GN}(\alpha))^{\frac{2+\delta}{1+\delta}} (2\mathcal{N})^{\frac{2+\delta}{1+\delta}}} {\mu} \|\Lambda^{\beta-\alpha/2} v(t)\|_{L^2}^{\frac{2+\delta}{1+\delta}}$$

$$+ \frac{(C_{KP}(\alpha) \mathcal{N})^2}{\mu} \|\Lambda^{\beta-\alpha/2} v(t)\|_{L^2}^2.$$

(22)
Now, fix $t > 0$ and consider the equation for the $k$-th Fourier coefficient of $v$:
\[
\frac{d}{dt} \hat{v}(k, t) = -\nu |k|^\beta \hat{v}(k, t) - \lambda \hat{\varphi}(k, t) + \hat{u}(k, t).
\]
Solving this ODE, we get
\[
e^{(\nu |k|^\beta + \lambda)t} \hat{v}(k, t) = \hat{v}_0(k) + \int_0^t e^{(\nu |k|^\beta + \lambda)s} \hat{u}(k, s) ds.
\]
As $v_0 \in H^\gamma$ with $\gamma = \beta - \alpha/2 < \beta - 0.5$, using (23), we get
\[
|k|^{\beta - \alpha/2} e^{(\nu |k|^\beta + \lambda)t} |\hat{v}(k, t)| \leq |k|^{\beta - \alpha/2} |\hat{v}_0(k)|
\]
\[
+ \int_0^t |k|^{\beta - \alpha/2} e^{(\nu |k|^\beta + \lambda)s} |\hat{u}(k, s)| ds
\]
\[
\leq |k|^{\beta - \alpha/2} |\hat{v}_0(k)| + \frac{|k|^{\beta - \alpha/2}}{\nu |k|^\beta + \lambda} N e^{(\nu |k|^\beta + \lambda)t},
\]
we obtain
\[
\int_0^t \|\Lambda^{\beta - \alpha/2} v(s)\|^p_{L^2} ds \leq TC(\alpha, \beta, \lambda, \|u_0\|_{L^1(T)}, \|v_0\|_{H^{\beta - \alpha/2}(T)}, \nu, p).
\]
Consequently, using Lemma 2, we have
\[
\|u(t)\|^2_{L^2} + \frac{\mu}{2} \int_0^t \|u(s)\|^2_{H^{\alpha/2}} + r \|u(s)\|^3_{L^3} ds
\]
\[
\leq \|u_0\|^2_{L^2} + N t + 2N + tC(\alpha, \beta, \nu, \lambda, \|u_0\|_{L^1(T)}, \|v_0\|_{H^{\beta - \alpha/2}(T)}).
\]
Now, trivially, if $\lambda > 0$, we get
\[
\|v(t)\|^2_{L^2} + 2\nu \int_0^t \|v(s)\|^2_{H^{\beta/2}} ds \leq \|v_0\|^2_{L^2} + c(\lambda) \int_0^t \|u(s)\|^2_{L^2} ds.
\]
For the case $\lambda = 0$, the previous bound should be replaced by
\[
\|v(t)\|_{L^2} \leq \|v_0\|_{L^2} + \int_0^t \|u(s)\|_{L^2} ds,
\]
and
\[
2\nu \int_0^t \|v(s)\|^2_{H^{\beta/2}} ds \leq \int_0^t \|u(s)\|_{L^2} \|v(s)\|_{L^2} ds.
\]
Testing the equation for $v$ against $\Lambda^{2\beta - \alpha} v$ and using the self-adjointness we get
\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|^2_{H^{\beta - \alpha/2} + \lambda \|v(t)\|^2_{H^{\beta - \alpha/2} + \nu \|v(t)\|^2_{H^{3\beta/2 - \alpha/2}}} \leq \|u\|_{H^{\alpha/2}} \|v(t)\|_{H^{2\beta - \alpha - \alpha/2}}.
\]
As $\beta \leq 2 < 2\alpha$, we get $2\beta - \alpha - \alpha/2 \leq 1.5\beta - \alpha/2$ and we can use Young’s and Poincaré’s inequality to conclude this step.

**Step 2:** $u \in L^\infty_t H^{\alpha/2}_x \cap L^2_t H^{\beta}_x$ and $v \in L^\infty_t H^{\beta/2 + \alpha/2}_x \cap L^2_t H^{\beta + \alpha/2}_x$. Testing the equation for $v$ against $\Lambda^{\alpha + \beta} v$, we get
\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|^2_{H^{\beta/2 + \alpha/2} + \nu \|v(t)\|^2_{H^{\beta + \alpha/2}} \leq \|u(t)\|_{H^{\alpha/2}} \|v(t)\|_{H^{\beta + \alpha/2}}.
\]
The previous inequality implies
\[
\|v(t)\|_{H^{\beta/2+n/2}}^2 + \nu \int_0^t \|v(s)\|_{H^{\beta+n/2}}^2 ds \\
\leq \|v_0\|_{H^{\beta/2+n/2}}^2 + c(\nu) \int_0^t \|\Lambda^{\alpha/2} u(s)\|_{L^2}^2 \leq \|v_0\|_{H^{\beta/2+n/2}}^2 + C.
\]
with constant
\[
C = C(\alpha, \beta, \mu, \nu, \lambda, \|u_0\|_{L^1(T)}, \|u_0\|_{L^2(T)}, \|v_0\|_{H^{\beta-n/2}(T)}), T).
\]
We compute
\[
\frac{d}{dt} \|u(t)\|_{H^{\alpha/2}}^2 = -\mu \int_T |\Lambda^\alpha u|^2 dx + \int_T u \Lambda^2 v \Lambda^\alpha u dx + r \int_T |\Lambda^{\alpha/2} u|^2 dx \\
+ \int_T \partial_u \Lambda^{\beta-1} H v \Lambda^\alpha u dx - \int_T \Lambda^{\alpha/2}(u^2) \Lambda^\alpha u dx
\leq -\frac{3\mu}{4} \|u(t)\|_{H^\alpha}^2 + C(\mu) \|u(t)\|_{L^2}^2 \|\Lambda^\beta v(t)\|_{L^\infty}^2 \\
+ r \|u(t)\|_{H^{\alpha/2}}^2 + \|u\|_{H^1} \|\Lambda^{\beta-1} H v\|_{L^\infty} \|u\|_{H^\alpha} \\
+ c(r) \|\Lambda^{\alpha/2} u(t)\|_{L^\infty} \|u\|_{L^2} \|u(t)\|_{H^{\alpha/2}}.
\]
We use the interpolation inequalities
\[
\|u\|_{H^1} \leq C \|u\|_{H^\alpha}^{1/\alpha} \|u\|_{L^2}^{(\alpha-1)/\alpha}, \|\Lambda^{\alpha/2} u\|_{L^\infty} \leq c \|u\|_{H^\alpha},
\]
\[
\|\Lambda^\beta v(t)\|_{L^\infty}^2 \leq c \|v(t)\|_{H^{\beta+n/2}}^2, \|\Lambda^{\beta-1} H v\|_{L^\infty} \leq c \|v\|_{H^{\beta-1+n/2}} \leq c \|v\|_{H^{\beta/2+n/2}},
\]
and to get
\[
\|u(t)\|_{H^{\alpha/2}}^2 + \mu \int_0^t \|u(t)\|_{H^\alpha}^2 \leq \|u_0\|_{H^{\alpha/2}}^2 + C,
\]
where the constant depends on
\[
C = C(\alpha, \beta, \mu, \nu, \lambda, \|u_0\|_{L^1(T)}, \|u_0\|_{L^2(T)}, \|v_0\|_{H^{\beta-n/2}(T)}), \|v_0\|_{H^{\beta/2+n/2}(T)}, T).
\]
Notice that, in the case \(\alpha > 1.5\), we have
\[
\int_0^T \|\partial_u u(t)\|_{L^\infty} + \|\Lambda v(t)\|_{L^\infty} dt \leq C \int_0^T \|u(t)\|_{H^\alpha} + \|v(t)\|_{H^{\beta+n/2}} dt \leq C,
\]
so, in this case, we are done.

**Step 3:** \(u \in L^\infty_t H_{x,T}^{\beta/2} \cap L^2_t H_{x,T}^{3\alpha/2}\) and \(v \in L^\infty_t H_{x,T}^{\beta/2+\alpha} \cap L^2_t H_{x,T}^{\beta+\alpha}\) Testing the equation for \(v\) against \(\Lambda^{\beta+\alpha}\), we get
\[
\|v(t)\|_{H^{\beta/2+\alpha}}^2 + \nu \int_0^t \|v(s)\|_{H^{\beta+\alpha}}^2 ds \leq \|v_0\|_{H^{\beta/2+\alpha}}^2 + C.
\]
We compute
\[
\frac{d}{dt}\|u(t)\|_{H^\alpha}^2 = -\mu \int_T |\Lambda^{3\alpha/2}u|^2 dx + \int_T u\Lambda^\beta v\Lambda^{2\alpha} u dx + r \int_T |\Lambda^\alpha u|^2 dx \\
+ \int_T \partial_x u\Lambda^{\beta-1} H v\Lambda^{2\alpha} u dx - r \int_T \Lambda^\alpha (u^2)\Lambda^\alpha u dx
\]
\[
\leq -\mu\|u(t)\|_{H^{3\alpha/2}}^2 + \|\Lambda^{\alpha/2}(u\Lambda^\beta v)\|_{L^2}\|u(t)\|_{H^{3\alpha/2}}
+ r\|u(t)\|_{H^\alpha}^2 + \|\Lambda^{\alpha/2}(\partial_x u\Lambda^{\beta-1} H v)\|_{L^2}\|u(t)\|_{H^{3\alpha/2}}
+ c\|u(t)\|_{L^\infty}\|u(t)\|_{H^\alpha}^2
\]
\[
\leq -\frac{\mu}{2}\|u(t)\|_{H^{3\alpha/2}}^2 + c(\mu)\|u(t)\|_{H^\alpha}^2 \|\Lambda^\beta v\|_{L^\infty}^2
+ \|u(t)\|_{L^\infty}\|\Lambda^{\beta+\alpha/2} v\|_{L^2}^2 + r\|u(t)\|_{H^\alpha}^2
+ c(\mu)\|u(t)\|_{H^{3\alpha/2}} \|\Lambda^{\beta-1} H v\|_{L^\infty}^2 + \|\partial_x u\|_{L^2}^2 \|\Lambda^{\beta-1+\alpha/2} v\|_{L^\infty}^2
+ c\|u(t)\|_{L^\infty}\|u(t)\|_{H^\alpha}^2
.
\]

We use \( \beta - 1 + \alpha/2 \leq \beta/2 + \alpha/2 \) to get
\[
\|\Lambda^{\beta-1+\alpha/2} v(t)\|_{L^\infty}^2, \|\Lambda^{\beta-1} H v\|_{L^\infty}^2 \in L^\infty_t, \|\Lambda^{\beta+\alpha/2} v(t)\|_{L^2}^2, \|\Lambda^{\beta} v\|_{L^\infty}^2 \in L^1_t.
\]
To conclude the result we use interpolation. Then the bound \( u \in L^2_t H^{3\alpha/2}_x \) implies
\[
\int_0^T \|\partial_x u(s)\|_{L^\infty} ds \leq C(T),
\]
and we get the global existence of classical solutions.

**Step 4: Constructing solutions** If the initial data \((u_0, v_0) \in H^3 \times H^{3+\beta/2}\), we have local existence of solutions. Then, the previous bounds show, using a standard continuation argument, that the solution corresponding to this initial data is global.

Now let’s consider the case where the initial data is not that smooth, but merely \((u_0, v_0) \in L^2 \times H^{3-\alpha/2}\). After mollification, we have an initial data \((u_0^\epsilon, v_0^\epsilon)\) with the desired regularity. Applying the previous reasoning, we have a global smooth regularized solution \((u^\epsilon(t), v^\epsilon(t))\). Due to Step 1, these functions are uniformly bounded in
\[
u^\epsilon(t) \in L^\infty([0,T], L^2(T)) \cap L^2([0,T], H^{\alpha/2}(T)),
\]
\[
\nu^\epsilon(t) \in L^\infty([0,T], H^{3-\alpha/2}(T)) \cap L^2([0,T], H^{3\beta/2-\alpha/2}(T)).
\]
Testing \(\partial_t u^\epsilon, \partial_t v^\epsilon\) against \(\phi \in H^2\) and using the duality pairing, we get a uniform bound
\[
\partial_t u^\epsilon(t), \partial_t v^\epsilon(t) \in L^\infty([0,T], H^{-2}(T)).
\]
Applying Aubin-Lions’s Compactness Theorem (with \(H^{\alpha/2} \subset L^2 \subset H^{-2}\) for \(u^\epsilon\) and \(H^{3\beta/2-\alpha/2} \subset H^{3-\alpha/2} \subset H^{-2}\) for \(v^\epsilon\), we take a subsequence (denoted again by \(\epsilon\)) such that
\[
u^\epsilon(t) \to u(t) \text{ in } L^2_x L^2_t, u^\epsilon(t) \to u(t) \text{ in } L^2_t \tilde{H}^{\alpha/2}_x,
\]
\[
u^\epsilon(t) \to v(t) \text{ in } L^2_x H^{\beta-\alpha/2}_x, v^\epsilon(t) \to v(t) \text{ in } L^2_x \tilde{H}^{3\beta/2-\alpha/2}_x.
\]
Using the properties of the mollifier, we have
\[ u'(0) \to u_0 \text{ in } L^2, \quad v'(0) \to v_0 \text{ in } L^2. \]

With the previous strong convergence, we can pass to the limit in the weak formulations in Definition 1. We conclude this step.

**Step 5: Absorbing set** We write
\[ C_{FS}(\beta, \alpha, \lambda, \nu) = \sum_{k \in \mathbb{Z}} \left( \frac{|k|^{\beta-\alpha/2}}{\nu|k|^{\beta+\lambda}} \right)^2 \]

According to (24), for every \( t \geq 0 \), we have
\[ \|v(t)\|_{H^{\beta-\alpha/2}}^2 \leq \|v_0\|_{H^{\beta-\alpha/2}}^2 e^{-\lambda t} + NC_{FS}(\beta, \alpha, \lambda, \nu), \]
so,
\[ \int_t^{t+1} \|v(s)\|_{H^{\beta-\alpha/2}}^2 ds \leq \frac{\|v_0\|_{H^{\beta-\alpha/2}}^2}{\lambda} (1 - e^{-\lambda}) e^{-\lambda t} + NC_{FS}(\beta, \alpha, \lambda, \nu), \]
\[ \int_t^{t+1} \|v(s)\|_{H^{\beta-\alpha/2}}^2 ds \leq \left( \|v_0\|_{H^{\beta-\alpha/2}}^2 + NC_{FS}(\beta, \alpha, \lambda, \nu) \right)^{2+2^{2+2^{2+2}}} \]

Notice that if
\[ t \geq t_0 = \max \left\{ 0, \frac{1}{-\lambda} \log \left( \frac{NC_{FS}(\beta, \alpha, \lambda, \nu)}{\|v_0\|_{H^{\beta-\alpha/2}}^2 (1 - e^{-\lambda})} \right) \right\}, \]
we have an inequality that is independent of \( v_0 \):
\[ \|v(t)\|_{H^{\beta-\alpha/2}}^2 \leq \frac{2NC_{FS}(\beta, \alpha, \lambda, \nu)}{2}. \]

Then, from (22), we obtain
\[ \frac{d}{dt} \|u(t)\|_{L^2}^2 + \frac{\mu}{2} \|u(t)\|_{H^n/2}^2 \leq r \|u(t)\|_{L^2}^2 + \frac{(CKP(\alpha)N)^2}{\mu} \|v(t)\|_{H^{\beta-\alpha/2}}^2 + \frac{(CKP(\alpha)CGN(\alpha))^{2+2^{2+2^{2+2}}}}{\mu} 4N^2 \|v(t)\|_{H^{\beta-\alpha/2}}^{2+2^{2+2^{2+2}}}. \]

Due to Lemma 2, we obtain
\[ \int_t^{t+1} \|u(s)\|_{L^2}^2 \leq 3N. \]

Using Uniform Gronwall estimate (Lemma 5) and the previous inequality, we get
\[ \|u(t+1)\|_{L^2}^2 \leq S(L^2) \forall t \geq t_0, \]
where
\[ S(L^2) = e^r \left[ 3N + \frac{(CKP(\alpha))^2N^3}{\mu} 2C_{FS}(\beta, \alpha, \lambda, \nu) \right. \]
\[ \left. + \frac{(CKP(\alpha)CGN(\alpha))^{2+2^{2+2^{2+2}}}}{\mu} 4N^2 (2NC_{FS}(\beta, \alpha, \lambda, \nu))^{2+2^{2+2^{2+2}}} \right]. \]
Using the previous inequality we also obtain
\[
\int_t^{t+1} \|u(s)\|_{H^{\alpha/2}}^2 ds \leq \frac{2(S(L^2) + S(L^2)e^{-r})}{\mu}, \quad \forall t \geq t_0 + 1.
\]

Let’s consider \(\alpha < 2\) (the case \(\alpha = 2\) can be done straightforwardly). We look for a commutator-type structure in the nonlinearity:
\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^{\alpha/2}}^2 = -\mu T_{\alpha} u \partial_x u \lambda^\alpha u dx + \int_T^t |\lambda^\alpha u|^2 dx + \int_T u \lambda^\beta v \lambda^\alpha u dx
\]
\[
+ r \int_T^t |\lambda^{\alpha/2} u|^2 dx - r \int_T u^2 \lambda^\alpha u dx
\]
\[
+ \int_T^t \left[ \lambda^{\alpha/2}, \lambda^{\beta-1} H v \right] \partial_x u \lambda^{\alpha/2} u dx
\]
\[
+ \int_T \lambda^{\beta-1} H v \frac{\partial_x (\lambda^{\alpha/2} u)^2}{2} dx.
\]

We estimate
\[
I_1 = \int_T u \lambda^\beta v \lambda^\alpha u dx \leq \frac{2}{\mu} \|u\|_{L^\infty}^2 \|v\|_{H^\beta}^2 + \frac{\mu}{8} \|u\|_{H^{\alpha}}^2,
\]
\[
I_2 = r \int_T u^2 \lambda^\alpha u dx \leq \frac{2r^2}{\mu} \|u\|_{L^\infty}^2 \|u\|_{L^2}^2 + \frac{\mu}{8} \|u\|_{H^{\alpha}}^2,
\]
\[
I_3 = \int_T \lambda^{\beta-1} H v \frac{\partial_x (\lambda^{\alpha/2} u)^2}{2} dx \leq C_I(\alpha) \|v\|_{H^\beta} \|u\|_{H^{\alpha/2}} \|u\|_{H^{\alpha}},
\]
where we have
\[
\|f\|_{L^4}^2 \leq C_I(\alpha) \|f\|_{L^2} \|f\|_{H^{\alpha/2}}.
\]

The last term is
\[
I_4 = \int_T \left[ \lambda^{\alpha/2}, \lambda^{\beta-1} H v \right] \partial_x u \lambda^{\alpha/2} u dx
\]
\[
\leq \left\| \lambda^{\alpha/2}, \lambda^{\beta-1} H v \right\|_{L^2} \|\partial_x u\|_{L^2} \|u\|_{H^{\alpha/2}}.
\]

We use the inequalities (see Lemma 4 for the commutator estimate)
\[
\|[\lambda^{\alpha/2}, \lambda^{\beta-1} H v] \partial_x u\|_{L^2} \leq C_{KPV}(\alpha) \left( \|\partial_x u\|_{L^2} \right)^{\frac{4-\alpha}{2}} \|\lambda^{\beta-1+\alpha/2} H v\|_{L^2(\alpha-1)}
\]
\[
+ \|u\|_{W^{\alpha/2, \infty}} \|v\|_{H^\beta},
\]
\[
\|f\|_{L^{2/(\alpha-1)}} \leq C_{SE}(\alpha) \|\lambda^{1-\alpha/2} f\|_{L^2}, \quad \|f\|_{L^\infty} \leq C_{SE}(\alpha) \|\lambda^{\alpha/2} f\|_{L^2},
\]
\[
\|f\|_{L^{2+(\alpha-1)/2}} \leq C_{SE}(\alpha) \|\lambda^{(\alpha-1)/2} f\|_{L^2}, \quad \|f\|_{W^{\alpha/2, \infty}} \leq C_{SE}(\alpha) \|f\|_{H^{\alpha}}
\]
to get
\[
I_4 \leq C_{KPV}(\alpha) \|u\|_{H^{\alpha/2}} \|u\|_{H^{\beta}} \|v\|_{H^{\beta}} \left( C_{SE}(\alpha) C_{SE}^1(\alpha) + C_{SE}^4(\alpha) \right),
\]
and, using Young’s inequality,
\[
I_4 \leq \|v\|_{H^{\beta}}^2 \|u\|_{H^{\alpha/2}}^2 \left[ C_{KPV}(\alpha) (C_{SE}(\alpha) C_{SE}^1(\alpha) + C_{SE}^4(\alpha)) \right]^2 \frac{\mu}{\mu} + \frac{\mu}{4} \|u\|_{H^{\alpha}}^2.
\]
Collecting every estimate, we get
\[ \frac{d}{dt} \|u\|^2_{H^{\alpha/2}} + \mu \|u\|^2_{H^{\alpha}} \leq 2 \|u\|^2_{H^{\alpha/2}g(t)}, \]
with
\[ g(t) = r + \frac{2C^2_{SE}(\alpha)\alpha}{\mu} \|u\|^2_{L^2} + \frac{1}{2} \|v\|^2_{H^\beta} + \frac{2C^2_{SE}(\alpha) + (C_I(\alpha))^2}{\mu} \|v\|^2_{H^\beta} \]
\[ + \left[ C_{KPV}(\alpha)(C^3_{SE}(\alpha)C^1_{SE}(\alpha) + C^4_{SE}(\alpha)) \right]^2 \|v\|^2_{H^\beta}. \]

Testing the equation for \( v \) against \( \Lambda^\beta \) and using \( \beta \geq \alpha \), we have
\[ \int^t_{t+1} \|v(s)\|^2_{H^\beta} ds \leq \frac{N}{\nu} \left( \frac{3}{\nu} + 2C_{FS}(\beta, \alpha, \lambda, \nu) \right) \|v(s)\|^2_{H^\beta} ds \leq \frac{N}{\nu} \left( \frac{3}{\nu} + 2C_{FS}(\beta, \alpha, \lambda, \nu) \right) \forall t \geq t_0, \]
so, if \( t \geq t_0 \),
\[ \int^t_{t+1} g(s) ds \leq r + \frac{r^2C^2_{SE}(\alpha)}{\mu} \frac{6N}{\nu} \left( \frac{3}{\nu} + 2C_{FS}(\beta, \alpha, \lambda, \nu) \right) \]
\[ \times \left( \frac{1}{2} + \frac{2C^2_{SE}(\alpha) + (C_I(\alpha))^2}{\mu} \right) \]
\[ \times \left[ C_{KPV}(\alpha)(C^3_{SE}(\alpha)C^1_{SE}(\alpha) + C^4_{SE}(\alpha)) \right]^2 \|v\|^2_{H^\beta}. \]

Using Lemma 5, we get
\[ \|u(t+1)\|^2_{H^{\alpha/2}} \leq S(\hat{H}^{\alpha/2}) \forall t \geq t_0 + 1, \]
\[ \int^t_{t+1} \|u(s)\|^2_{H^\alpha} ds \leq \frac{2}{\mu} S(\hat{H}^{\alpha/2}) \left( \frac{1}{2} + \int^t_{t+1} g(s) ds \right) \forall t \geq t_0 + 2, \]
with
\[ S(\hat{H}^{\alpha/2}) = \frac{2(S(L^2) + S(L^2)e^{-r})}{\mu} \right) e^2 \int^t_{t+1} g(s) ds. \]

Hence we have obtained the absorbing set in \( H^{k\alpha} \) with \( k = 1 \).

Due to the linear character of the equation for \( v \), we get
\[ \|v(t)\|^2_{H^{\alpha/2}} \leq \|v_0\|^2_{H^{\alpha/2}} e^{-\lambda t} + M(\hat{H}^{\alpha/2}) C_{FS}(\beta, \alpha, \lambda, \nu), \]
where, for a given space \( X \), we set
\[ M(X) = \sqrt{2\pi} \max \left\{ \max_{0 \leq s \leq T^*} \|u(s)\|_X, S(X) \right\}, \]
for \( T^* >> 1 \) that will be fixed later. We remark that \( \|\Lambda^{\alpha/2}u(t)\|_{L^1} \leq M(\hat{H}^{\alpha/2}). \)

Now we can continue in the same way using induction. Once we have the absorbing set for \( u \) in \( L^\infty([0, \infty], H^{k\alpha/2}(\mathbb{T})) \) \((k \geq 1)\) and the bound \( u \in L^2([t, t+1], H^{(k+1)\alpha/2}(\mathbb{T})) \), we can ensure that \( v \in L^\infty([0, \infty], H^{\beta+(k-1)\alpha/2}(\mathbb{T})) \).
and $v \in L^2([t, t+1], H^{\beta+\alpha/2}(\mathbb{T}))$. Now we test the equation for $u$ against $\Lambda^{(k+1)\alpha} u$:

$$
\frac{1}{2} \frac{d}{dt} \|u(t)\|_{H^{(k+1)\alpha}}^2 = -\mu \int_T |\Lambda^{(\frac{k}{2}+1)\alpha} u|^2 dx + r \int_T |\Lambda^{(k+1)\alpha/2} u|^2 dx \\
+ \int_T \Lambda^{\alpha/2} (u\Lambda^\beta v) \Lambda^{(k/2+1)\alpha} u dx \\
- r \int_T \Lambda^{\alpha/2} (u^2) \Lambda^{(k/2+1)\alpha} u dx \\
+ \int_T \frac{\Lambda^{(k+1)\alpha/2} \Lambda^\beta H v}{2} \partial_x u \Lambda^{(k+1)\alpha/2} u dx \\
+ \int_T \Lambda^{\beta-1} H v \partial_x (\Lambda^{(k+1)\alpha/2} u)^2 dx.
$$

To conclude the existence of $S \left(\frac{(k+1)\alpha}{2}\right)$ we use Lemma 4 and the same ideas. Finally notice that at each iteration we have to add one to the initial value $t_0$. Consequently, we can take $T^* = T^*(k)$ large enough to cover up to $H^{k\alpha}(\mathbb{T})$. For instance, to deal up to $H^3$, $T^* = t_0 + 10$ is enough. \qed

5. Global existence for the parabolic-elliptic case

Proof of Theorem 5. As we have already proved the existence of smooth solutions (locally in time), we will focus on the estimates allowing continuation arguments.

Part 1: Global existence result using the logistic term Notice that, due to Corollary 1, it is enough to obtain a bound for $u(t) \in L^1_1 L^\infty$. We define $x_t$ such that $\max_x u(x, t) = u(x_t, t)$. Due to Rademacher’s Theorem, this function is differentiable (in time) a.e.. Using the kernel representation for $\Lambda^\alpha u$, we get

$$
\frac{d}{dt} \|u(t)\|_{L^\infty} \leq u(x_t) \Lambda^\beta v(x_t) + ru(x_t) - ru(x_t)^2.
$$

Using the bound (19) and the hypothesis of the Theorem, we obtain

$$
\frac{d}{dt} \|u(t)\|_{L^\infty} \leq \frac{u(x_t)^2}{\nu} + ru(x_t) - ru(x_t)^2 \leq r \|u(t)\|_{L^\infty}.
$$

This bound implies

$$
\int_0^T \|u(t)\|_{L^\infty} dt \leq \frac{\|u_0\|_{L^\infty}}{r} e^{rT}.
$$

Part 2: Global existence result using $\alpha > 1$ Let’s assume first that $\Omega = \mathbb{T}$ and write $x_t$ for the point where $u(x_t)$ reaches its maximum. Using Lemma 6 we have that, if $\|u(t)\|_{L^\infty} \geq 2N/\pi$,

$$
-\mu \Lambda^\alpha u(x_t) \leq -\mu \frac{\Gamma(1+\alpha) \cos((1-\alpha)\pi/2)}{\pi} \frac{1}{2^\alpha} \frac{(u(x_t))^{1+\alpha}}{N^\alpha}.
$$

Using the previous argument, we have

$$
\frac{d}{dt} \|u(t)\|_{L^\infty} \leq -\mu \frac{\Gamma(1+\alpha) \cos((1-\alpha)\pi/2)}{\pi} \frac{1}{2^\alpha} \frac{(u(x_t))^{1+\alpha}}{N^\alpha} \frac{u(x_t) \Lambda^\beta v(x_t) + ru(x_t) - ru(x_t)^2}{\nu}.
$$
and we conclude the existence of $0 < M(\alpha, \mu, \|u_0\|_{L^1}) < \infty$ such that
$$\|u(t)\|_{L^\infty} \leq M(\alpha, \mu, \|u_0\|_{L^1}) \forall t.$$ Notice that if $\|u(t)\|_{L^\infty} \geq 2N/\pi$ does not hold we have a better bound of $\|u(t)\|_{L^\infty}$. In the case with $\Omega = \mathbb{R}$, the previous differential inequality is true even if $\|u(t)\|_{L^\infty} \geq 2N/\pi$ does not hold.

**Part 3: Threshold in the case $\alpha = 1$** In the case $\alpha = 1$, the previous inequality reads
$$\frac{d}{dt} \|u(t)\|_{L^\infty} \leq \left[1 - \frac{\mu}{2\pi N} - r\right] u(x_t)^2 + ru(x_t).$$

So, if $\mathcal{N} \leq \frac{\mu}{2\pi N}$ we conclude the result. □

**Remark 4.** Notice that this result holds in the case where the nonlinearity $u(1-u)$ is replaced with a (more general) nonlinearity $f(u)$ such that $f(y) \approx y(1-y)$ for $|y| >> 1$.

### 6. Smoothing effect

Here we prove our main result concerning the smoothing effect of the system (3)-(4):

**Proof of Theorem 6.** We consider $\Omega = \mathbb{T}$, but the proof is analogous for $\Omega = \mathbb{R}$. Let’s consider the Hardy-Sobolev norm
$$\|f\|^2_{H^\alpha(\mathbb{S}_\omega)} = \int_\Omega |f(x \pm i\omega t)|^2 dx + \int_\Omega |\partial_x f(x \pm i\omega t)|^2 dx.$$ Notice that, for $t > 0$, the finiteness of this norm implies the analyticity on the real line. We define $z = x \pm i\omega t$. In this complex strip the extended system is

\begin{align}
\partial_t u(z) &= -\mu \lambda^\alpha u(z) + \partial_z \cdot (u(z) \lambda^{\beta-1} Hv(z)) \\
&\quad + ru(z)(1 - u(z)), \\
\partial_t v(z) &= -\nu \lambda^\beta v(z) - \lambda v(z) + u(z).
\end{align}

The Hardy-Sobolev space for the complex extension of a real function is given by
$$H^\alpha(\mathbb{S}_\omega) = \{f(z, t), \ z \in \mathbb{S}_\omega \text{ s.t. } \|f\|_{H^\alpha(\mathbb{S}_\omega)} < \infty \text{ and } f(x, t) \in \mathbb{R}\}.$$ We are going to perform new energy estimates in this space for the appropriate value of $\omega$. Notice that, as the functions $u$ and $v$ are complex for complex arguments, the integration by parts is a delicate matter for some terms. Consequently there are several new terms appearing that are not present in the real case.

We deal first with the case $\alpha, \beta > 1$. At the end of the proof we explain how to cover the extreme case $\alpha = \beta = 1$.

Let’s start with the estimates for the equation (31). Using $\int f \bar{g} = \int \bar{f} g$, we have
$$\frac{d}{dt} \|v\|^2_{L^2(\mathbb{S}_\omega)} = 2\text{Re} \int_{\mathbb{T}} \bar{v}(z) (\partial_t v(z) \pm i\omega \partial_x v(z)) \ dx.$$ Using Plancherel’s Theorem, we have
$$\text{Re} \int_{\mathbb{T}} \bar{v}(z)(-\nu \lambda^\beta v(z) - \lambda v(z)) \ dx = -\nu \|v\|^2_{H^{\beta/2}(\mathbb{S}_\omega)} - \lambda \|v\|^2_{L^2(\mathbb{S}_\omega)} \leq 0.$$
Consequently, using (31),
\[
\frac{1}{2} \frac{d}{dt} \|v\|^2_{L^2(S_{\omega})} \leq \|v\|_{L^2(S_{\omega})} (\|\omega\|_{H^1(S_{\omega})} + \|u\|_{L^2(S_{\omega})}).
\]

Taking 4 derivatives of the equation (31) and testing against \( \partial^4_x v \), we get
\[
\frac{1}{2} \frac{d}{dt} \|v\|^2_{H^4(S_{\omega})} = \text{Re} \int_T \partial^4_x \bar{v}(z) (\partial_t \partial^4_x v(z) + i \omega \partial^5_x v(z)) \, dx \leq I_1 + I_2 + I_3,
\]
with
\[
I_1 = -\nu \text{Re} \int_T \partial^4_x \bar{v}(z) \partial^4_x \Lambda^\beta v(z) \, dx = -\nu \|v\|^2_{H^{4+\beta/2}(S_{\omega})},
\]
\[
I_2 = -\text{Re} \int_T \partial^4_x \bar{v}(z) \Lambda \partial^3_x u(z) \, dx
\]
\[
= -\text{Re} \int_T \Lambda^{0.5} \partial^4_x \bar{v}(z) \Lambda^{0.5} \partial^3_x u(z) \, dx
\]
\[
\leq 2 \|v\|_{H^{4+1/2}(S_{\omega})} \|u\|_{H^{3+1/2}(S_{\omega})},
\]
\[
I_3 = +\omega \text{Re} \int_T \partial^4_x \bar{v}(z) \Lambda \partial^3_x v(z) \, dx \leq 2\omega \|v\|^2_{H^{4.5}(S_{\omega})}.
\]

Using Young’s inequality and the interpolation inequality
\[
\|f\|_{H^{0.5}} \leq C_{SI}(\alpha) \|f\|_{L^2}^{(\alpha-1)/\alpha} \|f\|_{H^{\alpha/2}}^{1/\alpha},
\]
we get
\[
I_2 \leq \omega \|v\|^2_{H^{4+1/2}(S_{\omega})} + \left(\frac{C_{SI}(\alpha)^2}{\omega}\right) \|u\|^2_{H^{3}(S_{\omega})} \|u\|^2_{H^{3+\alpha/2}(S_{\omega})}
\]
\[
\leq \omega \|v\|^2_{H^{4+1/2}(S_{\omega})} + \left(\frac{C_{SI}(\alpha)^2}{\omega}\right) \left( \frac{\|u\|^2_{H^{3}(S_{\omega})}}{\alpha - \alpha/(\alpha-1)} + \epsilon^\alpha \frac{\|u\|^2_{H^{3+\alpha/2}(S_{\omega})}}{\alpha} \right),
\]
for every \( \epsilon > 0 \). Now we fix
\[
\epsilon = \left(\frac{\mu}{4(C_{SI}(\alpha))^2}\right)^{1/\alpha}
\]
to get
\[
I_2 \leq \omega \|v\|^2_{H^{4+1/2}(S_{\omega})} + \frac{\mu}{4} \|u\|^2_{H^{3+\alpha/2}(S_{\omega})}
\]
\[
+ \left(\frac{C_{SI}(\alpha)^2}{\omega}\right) \frac{\alpha - 1}{\alpha} \left( \frac{\mu \omega \alpha}{(C_{SI}(\alpha))^2} \right)^{-1/(\alpha-1)} \|u\|^2_{H^{3}(S_{\omega})}.
\]

Collecting all the estimates for \( v \) we obtain
\[
\frac{1}{2} \frac{d}{dt} \|v\|^2_{H^4(S_{\omega})} \leq -\nu \|v\|^2_{H^{4+\beta/2}(S_{\omega})} + 3\omega \|v\|^2_{H^{4+1/2}(S_{\omega})} + \frac{\mu}{4} \|u\|^2_{H^{3+\alpha/2}(S_{\omega})}
\]
\[
+ \left(\frac{C_{SI}(\alpha)^2}{\omega}\right) \frac{\alpha - 1}{\alpha} \left( \frac{\mu \omega \alpha}{(C_{SI}(\alpha))^2} \right)^{-1/(\alpha-1)} \|u\|^2_{H^{3}(S_{\omega})}
\]
\[
+ \left( \omega + \frac{1}{2} \right) \|v\|^2_{H^4(S_{\omega})} + \frac{\|u\|^2_{H^{3}(S_{\omega})}}{2}.
\]
Now we proceed with the equation for $u$. The lower order term can be bounded easily

$$
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2(S_w)}^2 = \text{Re} \int_T \bar{u}(z) \left( \partial_t u(z) + i\omega \partial_x u(z) \right) dx
$$

$$\leq \|u\|_{L^2(S_w)} \left( \omega \|u\|_{H^1(S_w)} + \|u\|_{L^\infty(S_w)} \|v\|_{H^3(S_w)} \right)
$$

$$+ r \|u\|_{L^2(S_w)} + r \|u\|_{L^2(S_w)} \|u\|_{L^\infty(S_w)}
$$

$$+ \|u\|_{H^1(S_w)} \|\Lambda^{\beta-1} H v\|_{L^\infty(S_w)}).
$$

The higher order seminorm contributes with

$$
\frac{1}{2} \frac{d}{dt} \|u\|_{H^3(S_w)}^2 = \text{Re} \int_T \partial_x^2 \bar{u}(z) \left( \partial_t \partial_x^3 u(z) \pm i\omega \partial_x^4 u(z) \right) dx = I_4 + I_5 + I_6 + I_7,
$$

with

$$I_4 = -\mu \text{Re} \int_T \partial_x^3 \bar{u}(z) \partial_x^3 \Lambda^\alpha u(z) dx = -\mu \|u\|_{H^{3+\alpha/2(S_w)}}^2,$$

$$I_5 = +\omega \text{Re} \int_T \partial_x^4 \bar{u}(z) \Lambda H \partial_x^3 u(z) dx \leq 2\omega \|u\|_{H^{3+\alpha/2(S_w)}}^2,$$

$$I_6 = r \text{Re} \int_T \partial_x^3 \bar{u}(z) \partial_x^4 \left[ u(z) (1 - u(z)) \right] dx
$$

$$\leq r \|u\|_{H^3(S_w)}^2 \left( 1 + 2 \|u\|_{L^\infty(S_w)} + 6 \|\partial_x u\|_{L^\infty(S_w)} \right),$$

and

$$I_7 = \text{Re} \int_T \partial_x^3 \bar{u}(z) \partial_x^4 \left[ u(z) \Lambda^{\beta-1} H v(z) \right] dx = K_1 + K_2 + K_3 + K_4 + K_5,$$

with

$$K_1 = \text{Re} \int_T \partial_x^2 \bar{u}(z) u(z) \partial_x^3 \Lambda^\beta v(z) dx \leq \|u\|_{H^3(S_w)} \|u\|_{L^\infty(S_w)} \|v\|_{H^{3+\delta}(S_w)},$$

$$K_2 = 4\text{Re} \int_T \partial_x^3 \bar{u}(z) \partial_x^2 u(z) \Lambda^\beta v(z) dx \leq 4 \|u\|_{H^3(S_w)} \|\Lambda^\beta v\|_{L^\infty(S_w)},$$

$$K_3 = 6\text{Re} \int_T \partial_x^4 \bar{u}(z) \partial_x^2 u(z) \partial_x \Lambda^\beta v(z) dx \leq 6 \|u\|_{H^3(S_w)} \|\Lambda^\beta \partial_x v\|_{L^\infty(S_w)},$$

$$K_4 = 4\text{Re} \int_T \partial_x^5 \bar{u}(z) \partial_x u(z) \partial_x^2 \Lambda^\beta v(z) dx \leq 4 \|\partial_x u\|_{L^\infty(S_w)} \|u\|_{H^3(S_w)} \|v\|_{H^{2+\delta}(S_w)},$$

$$K_5 = \text{Re} \int_T \partial_x^3 \bar{u}(z) \partial_x^4 u(z) \Lambda^{\beta-1} H v(z) dx.$$
Using $\Lambda H = -\partial_x$ and the self-adjointness of $\Lambda^*$, we find a commutator

$$L_2 \leq \left\| \left[ \Lambda^{0.5}, \text{Im}\Lambda^\beta H \right] \right\|_{L^2(S_\omega)} \left\| \partial_x \text{Re} u \right\|_{L^2(S_\omega)} \left| H^{3.5}(S_\omega) \right| \left. \left| u \right|_{H^{3.5}(S_\omega)} \right|^2 + \left| \text{Im}\Lambda^\beta H \right|_{L^\infty(S_\omega)} \left| u \right|_{H^{3.5}(S_\omega)}^2 \right).$$

We use Lemma 4 to get the commutator estimate

$$\left\| \left[ \Lambda^{0.5}, \text{Im}\Lambda^\beta H \right] \right\|_{L^2(S_\omega)} \leq C_{KP}^2 \left| \text{Im}\Lambda \right|_{L^\infty(S_\omega)} \left| u \right|_{H^{3.5}(S_\omega)}^2 .$$

Putting all the estimates together and using (28) and $3 + \beta \leq 4 + \beta/2$, we get

$$\frac{d}{dt} \left| u \right|_{H^3(S_\omega)}^2 \leq 2 \left| u \right|_{H^4(S_\omega)}^2 \left( \omega + 17.5 C_{SE}^2(\alpha) \left| v \right|_{H^4(S_\omega)} + 2r \right)
+ 2 C_{SE}^2(\alpha) \left| v \right|_{H^{4+\beta/2}(S_\omega)} + \left| u \right|_{H^3(S_\omega)}^3 9r C_{SE}^2(\alpha)
- 2\mu \left| u \right|_{H^{3+\alpha/2}(S_\omega)}^2 + 4\omega \left| u \right|_{H^{3.5}(S_\omega)}^2
+ 2 \left| \text{Im}\Lambda^\beta H \right|_{L^\infty(S_\omega)} \left| u \right|_{H^{3.5}(S_\omega)}^2
+ 2 \left| u \right|_{H^{3.5}(S_\omega)} \left| u \right|_{H^1(S_\omega)} C_{KP}^2 \left| \text{Im}\Lambda \right|_{L^\infty(S_\omega)} \left| v \right|_{H^4(S_\omega)}^2 .$$

Notice that, using Poincaré inequality

$$\left| \text{Im}\Lambda^\beta H \right|_{L^\infty(S_\omega)} \leq \sqrt{2} \left| \text{Im} v \right|_{H^\beta} \leq \sqrt{2} \left| \text{Im} v \right|_{H^2} .$$

We define the energy

$$E(t) = 1 + \left| u \right|_{H^3(S_\omega)}^2 + \left| v \right|_{H^4(S_\omega)}^2 + \frac{1}{\frac{\mu}{4} - \sqrt{2} \left| \text{Im} v \right|_{H^2}^2} \left| u \right|_{L^\infty(T)} - \left| u(z) \right|^2 \left| L^\infty \right| + \frac{1}{\frac{\mu}{4} - \sqrt{2} \left| \text{Im} v \right|_{H^2}^2} \left| v \right|_{L^\infty(T)} - \left| v(z) \right|^2 \left| L^\infty \right| .$$

We have

$$\frac{d}{dt} \left| u \right|_{H^2} \frac{1}{\frac{\mu}{4} - \sqrt{2} \left| \text{Im} v \right|_{H^2}^2} \leq \sqrt{2} \left( E(t) \right)^{\frac{\mu}{4}} - \sqrt{2} \left| \text{Im} v \right|_{H^2} \leq \left. \left| E(t) \right| \right| \frac{1}{\frac{\mu}{4} - \sqrt{2} \left| \text{Im} v \right|_{H^2}^2} \left| u \right|_{L^\infty} \left| \partial_t u \right|_{L^\infty} .$$

Integrating this ode, taking the absolute value and using the limit definition for the derivative, we get (see [25, 32] for further details)

$$\frac{d}{dt} \left| \frac{1}{\frac{\mu}{4} - \sqrt{2} \left| \text{Im} v \right|_{H^2}^2} \right|_{L^\infty} \leq \left( \mu + 2r \right) C_{SE}^2(\alpha) \left| E(t) \right|^4 .$$

In the same way

$$\frac{d}{dt} \left| \frac{1}{\frac{\mu}{4} - \sqrt{2} \left| \text{Im} v \right|_{H^2}^2} \right|_{L^\infty} \leq \left( \nu + \lambda + 1 \right) C_{SE}^2(\alpha) \left| E(t) \right|^3 .$$
Using Plancherel, we have that
\[
\frac{d}{dt} E(t) \leq -2\nu \|v\|_{H^{3+\beta/2}(S_\omega)}^2 + 6\omega \|v\|_{H^{3+1/2}(S_\omega)}^2 + 2\left(\frac{\mu}{4} - \mu\right)\|u\|_{H^{3+\alpha/2}(S_\omega)}^2
\]
\[\begin{aligned}
+ & 2\left[\left(\frac{(C_{SI}(\alpha))^2}{C_{SI}(\alpha)}\right)^{\alpha - 1} - 1 \alpha \left(\frac{\omega^2 \alpha}{C_{SI}(\alpha)}\right)^{-1/(\alpha - 1)} + 2\omega \\
+ & 1.25 + 2r + (17.5C_{SE}^2(\alpha))^2 + \frac{(9rC_{SE}^2(\alpha))^2}{2}\right] E(t) \\
+ & (E(t))^2 \left(1.5 + \frac{2(2C_{PV}C_{SE}^2(\alpha))^2}{\mu}\right) \\
+ & (1 + (\mu + 2 + 2r + \nu + \lambda + 1)C_{SE}^2(\alpha))(E(t))^4 \\
+ & \left(\frac{\mu}{2} + (\sqrt{2})^3 \|\text{Im } v\|_{H^2} + 4\omega\right) \|u\|_{H^{3+\gamma}(S_\omega)}^2.
\end{aligned}\]
Now observe that, as long as \(E(t) < \infty\), we have \(\frac{d}{dt} E(t) \leq 0\), and, using Poincaré inequality if needed, we get
\[
\frac{d}{dt} E(t) \leq -2\nu \|v\|_{H^{3+\beta/2}(S_\omega)}^2 + 6\omega \|v\|_{H^{3+1/2}(S_\omega)}^2 - \frac{\mu}{2} \|u\|_{H^{3+\alpha/2}(S_\omega)}^2
\]
\[\begin{aligned}
+ & 2\left[\left(\frac{(C_{SI}(\alpha))^2}{C_{SI}(\alpha)}\right)^{\alpha - 1} - 1 \alpha \left(\frac{\omega^2 \alpha}{C_{SI}(\alpha)}\right)^{-1/(\alpha - 1)} + 2\omega \\
+ & 1.25 + 2r + (17.5C_{SE}^2(\alpha))^2 + \frac{(9rC_{SE}^2(\alpha))^2}{2}\right] E(t) \\
+ & (E(t))^2 \left(1.5 + \frac{2(2C_{PV}C_{SE}^2(\alpha))^2}{\mu}\right) \\
+ & (1 + (\mu + 2 + 2r + \nu + \lambda + 1)C_{SE}^2(\alpha))(E(t))^4 \\
+ & 4\omega \|u\|_{H^{3+\gamma}(S_\omega)}^2.
\end{aligned}\]
Using Plancherel, we have that \(\alpha, \beta > 1\),
\[
4\omega \|u\|_{H^{3+\gamma}(S_\omega)}^2 - \frac{\mu}{2} \|u\|_{H^{3+\alpha/2}(S_\omega)}^2 \leq C_1 \|u\|_{H^3(S_\omega)},
\]
\[
3\omega \|v\|_{H^{3+\gamma}(S_\omega)}^2 - \nu \|v\|_{H^{3+\beta/2}(S_\omega)}^2 \leq C_2 \|v\|_{H^3(S_\omega)}^2,
\]
with \(C_1, C_2\) given by (9), (10). Consequently, we can choose any positive value for \(\omega > 0\) and we get the inequality
\[
\frac{d}{dt} E(t) \leq K_1(E(t))^4,
\]
with \(K_1, C_i\) defined in (9), (10) and (15). Solving the ode, we obtain
\[
E(t) \leq \frac{1}{\sqrt{1 + \|u_0\|_{H^3(T)}^2 + \|v_0\|_{H^4(T)}^2} - 3tK_1},
\]
and, using (15), we conclude that \((u(t), v(t))\) are analytic functions at least for time
\[
\tilde{T} = \frac{1}{3K_1(1 + \|u_0\|_{H^3(T)}^2 + \|v_0\|_{H^4(T)}^2)}.
Notice that in the extreme cases \( \min\{\alpha, \beta\} = 1 \), we can take
\[
0 < \omega \leq \omega_0,
\]
(with \( \omega_0 \) defined in (8)) to obtain the inequality
\[
\frac{d}{dt} E(t) \leq K_2(E(t))^4,
\]
with \( K_2 \) given by (16). From this inequality we get
\[
E(t) \leq \frac{1}{\sqrt[4]{1 + \|u_0\|^2_{H^4(T)} + \|v_0\|^2_{H^4(T)}} - 3tK_2}
\]
and we conclude that the solution \((u(t), v(t))\) is analytic for time \( t < \tilde{T} \) with
\[
\tilde{T} = \frac{1}{3K_2(1 + \|u_0\|^2_{H^4(T)} + \|v_0\|^2_{H^4(T)})}.
\]

\[\square\]

**Remark 5.** A similar Theorem holds for the system (5)-(31). We refer the reader to [1] for details on how to adapt the proof.

The proof of Corollary 2 is obtained by a standard continuation argument. First notice that the solution \((u(t), v(t)) \in H^4(T) \times H^4(T)\) globally and it is unique. In particular, at \( t = \tilde{T} \), we can restart the evolution with initial data \((u_0^0, v_0^0) = (u(\tilde{T}), v(\tilde{T}))\). The initial data may not be analytic, but there exists a \( \delta \) small enough so \((u_1(t), v_1(t)) = (u(\tilde{T} + t), v(\tilde{T} + t))\) is analytic for \( 0 < t < \delta \). As we can find such a positive \( \delta \) for every initial data, we conclude. In other words, if we can not find such a positive \( \delta \) is because \((u_0^0, v_0^0)\) is not in \( H^4(T) \times H^4(T)\), and this is a contradiction. For the proof of Corollary 3 we refer to [1, 25, 32].

Finally, we provide the proof of Theorem 7:

**Proof of Theorem 7.** Using Theorem 6, for \( \tilde{T}/(N - 1) < t < \tilde{T}, N \geq 3 \), and \( \omega = \omega_0 \) defined in (8), we have that the solutions become analytic in a strip with width at least
\[
W = \frac{\omega}{N} \frac{1 + \|u_0\|^2_{H^3(T)} + \|v_0\|^2_{H^4(T)}}{\mathcal{K}},
\]
and \( \mathcal{K} \) given by (17). We have
\[
\omega \tilde{T} \left( \frac{1}{N - 1} - \frac{1}{N} \right) \leq \omega t - W
\]
and using Cauchy’s formula and Hadamard’s three lines theorem,
\[
\|\partial_z u\|_{L^\infty(\{|z| \leq W\})} \leq \frac{N(N - 1)\|u\|_{L^\infty(\{|z| \leq \omega t\})}}{\omega \tilde{T} W} \leq \frac{\sqrt{2}(N - 1)\|u_0\|_{L^\infty(T)}}{W},
\]
\[
\|\partial_z v\|_{L^\infty(\{|z| \leq W\})} \leq \frac{N(N - 1)\|v\|_{L^\infty(\{|z| \leq \omega t\})}}{\omega \tilde{T} W} \leq \frac{\sqrt{2}(N - 1)\|v_0\|_{L^\infty(T)}}{W}.
\]
Using Lemma 7, we have that for any \( \epsilon > 0, 0 < \tilde{T}/(N - 1) < t < \tilde{T}, \)
\[
T = I^u_\epsilon \cup R^w_\epsilon = I^v_\epsilon \cup R^w_\epsilon,
\]
where \( I^u_\epsilon, I^v_\epsilon \) are the union of at most \( \frac{1}{W} \) intervals open in \( T \), and
Lemma 3. Given a compact semiflow in $H$

Proof. As in Theorem 4 we have

$$\Lambda^{3\alpha}u \in L^\infty([0, T], L^2) \cap L^2([0, T], H^{\alpha/2}),$$

$$\Lambda^{3\alpha + \beta/2}v \in L^\infty([0, T], L^2) \cap L^2([0, T], H^{\beta/2}).$$

We have to prove that

$$\partial_t \Lambda^{3\alpha}u \in L^2([0, T], H^{-\alpha/2}), \partial_t \Lambda^{3\alpha + \beta/2}v \in L^2([0, T], H^{-\beta/2}).$$
By duality and Kato-Ponce inequality, we have

\[
\|\partial_t u\|_{H^{-\alpha/2}} = \sup_{\|\phi\|_{H^{\alpha/2}} \leq 1} \left| \int_T \partial_t \Lambda^{3\alpha} u \phi dx \right|
\]

\[
\leq \mu \|\Lambda^{3\alpha} u\|_{L^2} + \|\Lambda^{2.5\alpha} (u \Lambda^\beta v)\|_{L^2} + \|\Lambda^{2.5\alpha} (\partial_x u \Lambda^\beta H v)\|_{L^2} + r \|\Lambda^{2.5\alpha} u\|_{L^2} + r \|\Lambda^{2.5\alpha} (v^2)\|_{L^2}
\]

\[
\leq C \left( \|\Lambda^{3.5\alpha} u\|_{L^2} + \|\Lambda^{2.5\alpha} u\|_{L^2} \Lambda^\beta v\|_{L^\infty} + \|\Lambda^{\beta+2.5\alpha} v\|_{L^2} \|u\|_{L^\infty} + \|\Lambda^{2.5\alpha} \partial_x u\|_{L^2} \|\Lambda^{\beta-1} H v\|_{L^\infty} + \|\partial_x u\|_{L^\infty} \|\Lambda^{\beta+2.5\alpha} - v\|_{L^2} + \|\Lambda^{2.5\alpha} u\|_{L^2} + \|u\|_{L^\infty} \|\Lambda^{2.5\alpha} u\|_{L^2} \right)
\]

\[
\leq C \left( \|u\|_{H^{1.5\alpha}} + \|\|H^{\alpha}\|v\|_{H^{\alpha+\beta/2}} + \|\|H^{\beta+3\alpha}\|u\|_{H^{3\alpha}} + \|\|H^{3\alpha}\|v\|_{H^{\beta+3\alpha}} + \|\|H^{3\alpha}\|u\|_{H^{\beta+3\alpha}} \right),
\]

and we conclude the desired bound for \(\partial_t \Lambda^{3\alpha} u\). We proceed in the same way for \(\partial_t \Lambda^{3\alpha+\beta/2} v\). These inequalities implies the continuity as a map from \([0, T]\) to \(H^{3\alpha} \times H^{\beta+2+3\alpha}\). To get the full norm we use

\(u \in L^\infty([0, T], L^2) \cap L^2([0, T], H^{\alpha/2}), v \in L^\infty([0, T], L^2) \cap L^2([0, T], H^{\beta/2}).\)

and repeat the argument for

\(\partial_t u \in L^2([0, T], H^{-\alpha/2}), \partial_t v \in L^2([0, T], H^{-\beta/2}).\)

This implies

\(S(\cdot)(u_0, v_0) \in C([0, T], H^{3\alpha} \times H^{\beta+2+3\alpha}).\)

The semigroup property follows from the uniqueness of the classical solution. Fixed \(s_0\), the continuity of

\(S(s_0)(\cdot, \cdot) : H^{3\alpha} \times H^{\beta+2+3\alpha} \rightarrow H^{3\alpha} \times H^{\beta+2+3\alpha},\)

can be obtained with the energy estimates. Finally, we use Theorem 6 to get that \(S(t)(u_0, v_0) \in H^{3.5\alpha} \times H^{\beta+3\alpha}\) if \(t \geq \delta\), for every initial data and \(\delta > 0\). As in Theorem 4, we obtain the existence of \(T^*\) and a constant \(C\) such that

\(\max_{t \leq T^*} \{\|u(t)\|_{H^{3.5\alpha}} + \|v(t)\|_{H^{\beta+3\alpha}} \} \leq C.\)

Using the compactness of the embeddings \(H^\epsilon \hookrightarrow L^2\), we conclude the result.

\(\square\)

**Remark 6.** The restriction \(\alpha \leq \beta\) is to get the existence of the absorbing sets applying Theorem 4. The restriction \(8/7 \leq \alpha\) is to get

\(3\alpha + \beta/2 \geq 3.5\alpha \geq 4,\)

to invoke Theorem 6.

**Proof of Theorem 8.** We can use the previous Lemma together with Theorem 4 and Theorem 1.1 in [52] to conclude Theorem 8.

\(\square\)

**Proof of Corollary 4.** Notice that in the case \(\min\{\alpha, \beta\} > 1\), we are free to choose

\(\omega = \frac{N}{1 + \|u_0\|_{H^3(T)}^2 + \|v_0\|_{H^4(T)}^2},\)
and we can improve the statement in Theorem 7. Applying Lemma 7, we have that for any \( \epsilon > 0 \), \( T = I_{\epsilon}^u \cup R_{\epsilon}^u = I_{\epsilon}^c \cup R_{\epsilon}^c \), with \( I_{\epsilon}^u, I_{\epsilon}^c \) are the union of at most \( 12\pi K_1 \) intervals open in \( T \), and

\[
\begin{align*}
& \cdot |\partial_x u(x)| \leq \epsilon, \text{ for all } x \in I_{\epsilon}^u, \\
& \cdot \text{card}\{x \in R_{\epsilon}^u : \partial_x u(x) = 0\} \leq \frac{12\pi K_1 \log^2}{\epsilon}, \\
& \cdot |\partial_x v(x)| \leq \epsilon, \text{ for all } x \in I_{\epsilon}^c, \\
& \cdot \text{card}\{x \in R_{\epsilon}^c : \partial_x v(x) = 0\} \leq \frac{12\pi K_1 \log^2}{\epsilon}.
\end{align*}
\]

We are interested in the points of maximum such that they are close to regions with slope bigger than one (the, so-called, peaks). Consequently, we take \( \epsilon = 1 \) and \( N = 3 \). Finally, notice that, in the attractor, we have

\[
\|u(t)\|_{L^\infty} \leq C_{SE}(\alpha) S(H^{\alpha/2})
\]

to conclude the result. \( \square \)

**Proof of Corollary 5.** The proof follows the same ideas as before. \( \square \)

### Appendix A. Auxiliary Lemmas

We state the Kato-Ponce inequality and the Kenig-Ponce-Vega commutator estimate for \( [\Lambda^s, F]G = \Lambda^s(FG) - F\Lambda^s G \) and where \( \Lambda = \sqrt{-\Delta} \) (see [31, 39, 41]).

**Lemma 4.** Let \( F, G \) be two smooth functions and \( \Omega \in \mathbb{R}^d, \mathbb{T}^d \). Then we have the following inequalities:

\[
\|[\Lambda^s, F]G\|_{L^p} \leq C(s, p, p_1) (\|F\|_{W^{s, p_1}} \|G\|_{L^{p_2}} + \|G\|_{W^{s-1, p_3}} \|\nabla F\|_{L^{p_4}}),
\]

with

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad p, p_1, p_3 \in (1, \infty), \quad p_2, p_4 \in [0, \infty], \quad s > 0.
\]

and

\[
\|\Lambda^s(FG)\|_{L^p} \leq \frac{C_{KP}(s, p, p_1)}{2} (\|\Lambda^s F\|_{L^{p_1}} \|G\|_{L^{p_2}} + \|\Lambda^s G\|_{L^{p_3}} \|F\|_{L^{p_4}}),
\]

with

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, \quad 1 < p < \infty, 1 < p_1 \leq \infty, \quad s > \max\{0, d/p - d\}.
\]

**Remark 7.** In particular, we are using the notation

\[
C_{KPV}(\alpha) = C \left( \alpha, 2, 2 + \frac{2\alpha - 2}{2 - \alpha}, \frac{2}{\alpha - 1}, \infty, 2 \right).
\]

We require the following uniform Gronwall lemma (see [52]).

**Lemma 5.** Suppose that \( g, h, y \) are non-negative, locally integrable functions on \((0, \infty)\) and \( dy/dt \) is locally integrable. If there are positive constants \( a_1, a_2, a_3, b \) such that

\[
\frac{dy}{dt} \leq gy + h, \quad \int_t^{t+b} g(s)ds \leq a_1, \quad \int_t^{t+b} h(s)ds \leq a_2, \quad \int_t^{t+b} y(s)ds \leq a_3
\]

then

\[
\int_t^{t+b} y(s)ds \leq \max \left\{ \frac{a_1}{g(t)}, \frac{a_2}{g(t+b)} \right\}.
\]
for \( t \geq 0 \), then
\[
y(t + b) \leq \left( \frac{a_3}{b} + a_2 \right) e^{a_1}.
\]

We also give some estimates for the fractional laplacian

**Lemma 6.** Let \( h \in C^2(T) \) be a positive function and write \( h(x^*) = \max_x h(x) = \|h\|_{L^\infty} \).

- if \( h \) verifies the bounds
  \[
  \|h\|_{L^\infty}/2 \geq \langle h \rangle, \quad \|h\|_{L^p(T)} \leq \gamma_p, \quad \text{with } \gamma_p \leq \frac{\sqrt{\pi} \|h\|_{L^\infty}}{2^{1-p}},
  \]
  then
  \[
  \Lambda^\alpha h(x^*) \geq \frac{\Gamma(1 + \alpha) \cos((1 - \alpha)\pi/2)}{\pi} \frac{1}{2^{1+\alpha}} \frac{\|h\|_{L^\infty}^{1+\alpha}}{\gamma_p^\alpha}.
  \]

- \[
  \|e^{-\Lambda^\alpha t} h\|_{L^\infty} - \langle h \rangle \leq \left( \|h\|_{L^\infty} - \langle h \rangle \right) \exp \left( -\frac{2\Gamma(1 + \alpha) \cos((1 - \alpha)\pi/2)}{\pi^{1+\alpha}} t \right).
  \]

**Proof.** **Step 1; Pointwise estimate:** We take \( r > 0 \) a fixed constant (that will be defined later) and define
\[
\mathcal{U}_1 = \{ \eta \in B(0, r) \text{ s.t. } h(x^*) - h(x^* - \eta) > h(x^*)/2 \},
\]
and \( \mathcal{U}_2 = B(0, r) - \mathcal{U}_1 \). Notice that if the function is sharp enough, i.e., if \( h(x^*)/2 \geq \langle h \rangle, \mathcal{U}_1 \neq \emptyset \). We have
\[
\gamma_p^\alpha \geq \|h\|_{L^p}^p = \int_T |h(x^* - \eta)|^p d\eta \geq \int_{\mathcal{U}_2} |h(x^* - \eta)|^p d\eta \geq \frac{|h(x^*)|^p}{2^p} |\mathcal{U}_2|,
\]
so,
\[
(32) \quad - \left( \frac{2\gamma_p}{|h(x^*)|} \right)^p \leq -|\mathcal{U}_2|.
\]
Writing
\[
c(\alpha) = \frac{\Gamma(1 + \alpha) \cos((1 - \alpha)\pi/2)}{\pi},
\]
we have
\[
\Lambda^\alpha h(x) = c(\alpha) \left( \sum_{k \in \mathbb{Z}, k \neq 0} \int_{\mathbb{T}} \frac{h(x) - h(x - \eta)}{|\eta + 2k\pi|^{1+\alpha}} d\eta \right. + \text{P.V.} \int_{\mathbb{T}} \frac{h(x) - h(x - \eta)}{|\eta|^{1+\alpha}} d\eta \right).
\]
Thus
\[
\Lambda^\alpha h(x^*) = c(\alpha) \sum_k P.V. \int_{\mathbb{T}} \frac{h(x^*) - h(x^* - \eta)}{|\eta + k2\pi|^{1+\alpha}} d\eta
\]
\[
\geq c(\alpha) P.V. \int_{\mathcal{U}_1} \frac{h(x^*) - h(x^* - \eta)}{|\eta|^{1+\alpha}} d\eta
\]
\[
\geq c(\alpha) \frac{h(x^*)}{2r^{1+\alpha}} |\mathcal{U}_1|
\]
\[
\geq c(\alpha) \frac{h(x^*)}{2r^{1+\alpha}} (2r - |\mathcal{U}_2|)
\]
\[
\geq c(\alpha) \frac{h(x^*)}{2r^{1+\alpha}} \left(2r - \left(\frac{2\gamma_p}{h(x^*)}\right)^p\right).
\]

We take
\[
r = \left(\frac{2\gamma_p}{h(x^*)}\right)^p,
\]
thus
\[
\Lambda^\alpha h(x^*) \geq c(\alpha) \frac{h(x^*)}{2r^{1+\alpha}} \left(\frac{2\gamma_p}{h(x^*)}\right)^p \geq c(\alpha) \frac{h(x^*)^{1+\alpha}}{2r^{1+\alpha}} \gamma_p^p.
\]

Finally notice that due to the boundedness of the domain we have to impose the restriction
\[
r = \left(\frac{2\gamma_p}{h(x^*)}\right)^p \leq \pi \Rightarrow \gamma_p \leq \frac{\sqrt{\pi} \|h\|_{L^\infty}}{2}.
\]

**Step 2; Estimates for the propagator:** Now, using |\eta|^{1+\alpha} \leq \pi^{1+\alpha} for \eta \in \mathbb{T} and denoting the mean as \langle h \rangle,
\[
\Lambda^\alpha h(x^*) \geq \frac{2\Gamma(1 + \alpha) \cos((1 - \alpha)\pi/2)}{\pi^{1+\alpha}} (h(x^*) - \langle h \rangle)
\]
and
\[
\|e^{-\Lambda^\alpha t} h\|_{L^\infty} - \langle h \rangle \leq (\|h\|_{L^\infty} - \langle h \rangle) \exp \left(-\frac{2\Gamma(1 + \alpha) \cos((1 - \alpha)\pi/2)}{\pi^{1+\alpha}} t\right).
\]

□

The last Lemma studies the number of critical points of an analytic function (see [34])

**Lemma 7.** Let w > 0, and let u be analytic in the neighbourhood of \{z : |\Im z| \leq w\} and 2\pi-periodic in the x-direction. Then, for any \epsilon > 0, \mathbb{T} = I_\epsilon \cup R_\epsilon, where I_\epsilon is an union of at most \left\lfloor \frac{4\pi}{\epsilon} \right\rfloor intervals open in \mathbb{T}, and

- |\partial_x u(x)| \leq \epsilon, for all x \in I_\epsilon,
- card\{x \in R_\epsilon : \partial_x u(x) = 0\} \leq \frac{2\log 2}{\log w} \log \left(\frac{\max_{|z| \leq w} |\partial_x u(z)|}{\epsilon}\right).

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