QUASI-BIHAMILTONIAN SYSTEMS AND SEPARABILITY

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ABSTRACT. Two quasi–biHamiltonian systems with three and four degrees of freedom are presented. These systems are shown to be separable in terms of Nijenhuis coordinates. Moreover the most general Pfaffian quasi-biHamiltonian system with an arbitrary number of degrees of freedom is constructed (in terms of Nijenhuis coordinates) and its separability is proved.

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1. Preliminaries

As it is known, the biHamiltonian structure is a peculiar property of integrable systems, both finite and infinite dimensional [1, 2]. We recall some definitions. Let $M$ be a differentiable manifold, $TM$ and $T^*M$ its tangent and cotangent bundle and $P_0, P_1 : T^*M \rightarrow TM$ two compatible Poisson tensors on $M$ [1]: a vector field $X$ is said to be biHamiltonian w.r.t. $P_0$ and $P_1$ if there exist two smooth functions $H$ and $F$ such that

$$X = P_0 \, dH = P_1 \, dF,$$  \hspace{1cm} (1.1)

$d$ denoting the exterior derivative. Moreover, if $P_0$ is invertible, the tensor $N := P_1 P_0^{-1}$ is a Nijenhuis (or hereditary) tensor; in terms of the gradients of the Hamiltonian functions, the biHamiltonian property (1.1) entails that $N^*$ (the adjoint map of $N$) maps iteratively $dH$ into closed one–forms, so that $d(N^i dH) = 0$ ($i = 1, 2, \ldots$).

As a matter of fact, it is in general quite difficult to construct directly a biHamiltonian structure for a given integrable Hamiltonian vector field; so one can try to use some reduction procedure, starting from a few “universal” Poisson structures defined in an extended phase space. On the other hand, in the case of finite–dimensional systems arising as restricted or stationary flows from soliton equations [3, 4], the final result of the reduction procedure are some physically interesting dynamical systems (for example the Hénon–Heiles system) which, in their natural phase space, satisfy a weaker condition than the biHamiltonian one. So, the notion of quasi-biHamiltonian (QBH) system can be introduced [5, 6]; it was applied in [7] to dynamical systems with two degrees of freedom. One of the aims of this paper is just to give explicit examples of QBH systems with more than two degrees of freedom.

According to [7], a vector field $X$ is said to be a quasi–biHamiltonian (QBH) vector field w.r.t. two compatible Poisson tensors $P_0$ and $P_1$ if there are three smooth functions $H, F, \rho$ such that

$$X = P_0 \, dH = \frac{1}{\rho} P_1 \, dF,$$  \hspace{1cm} (1.2)

($\rho$ playing the role of an integrating factor). From this equation it follows that $F$ is an integral of motion for $X$, in involution with $H$, so that a QBH vector field with two degrees of freedom is Liouville-integrable. Of course, Eq.(1.2) can be studied for an arbitrary number $n$ of degrees of freedom, but the knowledge of $F$ and $H$ is no more sufficient to assure the integrability of $X$ for $n > 2$. In this case, the search for the integrability can be pursued using a sufficient criterion, which was recently introduced by one of the present authors (G.T.). Indeed, one can show that
Proposition 1.1. [4] Let $M$ be a $2n$ dimensional symplectic manifold equipped with an invertible Poisson tensor $P_0$, and $X$ be a Hamiltonian vector field with Hamiltonian $H$: $X = P_0 dH$. Let there exist a tensor $N : TM \rightarrow TM$ such that the tensor $P_1 : T^*M \rightarrow TM$ defined by $P_1 := NP_0$ is skew-symmetric. Denote by $X_i := N^{i-1} dH$ ($i = 1, 2, \ldots$) the vector fields and the one-forms obtained by the iterated action of $N$ and $N^*$.

If there exist $(n-1)$ independent functions $H_i$ ($i = 2, \ldots, n$) and $(n(n+1)/2 - 1)$ functions $\rho_{ij}$ ($i = 2, \ldots, n; 1 \leq j \leq i$) with $\rho_{i1} = 1$ and $\rho_{ii} \neq 0$ ($i = 2, \ldots, n$), such that the 1-forms $\alpha_i$ can be written as $\alpha_i = \sum_{j=1}^{i} \rho_{ij} dH_j$ ($i = 1, 2, \ldots, n$), then:

1. the vector fields $X_i$ satisfy the recursion relations $X_{i+1} = P_0 \alpha_{i+1} = P_1 \alpha_i$ ($i = 1, \ldots, n-1)$;
2. the functions $H_i$ are in involution with respect to the Poisson bracket defined by $P_0$ and they are constants of motion for each field $X_k$ ($k = 1, \ldots, n$);
3. the Hamiltonian system corresponding to the vector field $X$ is Liouville-integrable.

Moreover, if $P_1$ is a Poisson tensor, then also $X_2$ is an integrable Hamiltonian vector field and the functions $H_i$ are in involution also with respect to the Poisson bracket defined by $P_1$.

This result is applied in the next section of this paper, where we consider two Hénon–Heiles type systems with three and four degrees of freedom.

To fix the notations, on any open set of a $2n$ dimensional symplectic manifold $M$, let $(q = (q_1, \ldots, q_n); p = (p_1, \ldots, p_n))$ be a set of canonical coordinates and $P_0$ the Poisson tensor $P_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ ($I$ denoting the $n \times n$ identity matrix). Let $P_1$ be a compatible Poisson tensor w.r.t. $P_0$, such that the Nijenhuis tensor $N := P_1 P_0^{-1}$ is maximal, i.e., it has $n$ distinct eigenvalues $\Lambda = (\lambda_1, \ldots, \lambda_n)$. As it is known [8], in a neighborhood of a regular point, where the eigenvalues $\Lambda$ are independent, one can construct a canonical transformation $(q, p) \mapsto (\Lambda; \mu)$ ($(\Lambda; \mu)$ referred to as Nijenhuis coordinates) such that $P_1$ and $N$ take the Darboux form

$$P_1 = \begin{bmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{bmatrix}, \quad N = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \quad (\Lambda := diag(\lambda_1, \ldots, \lambda_n)).$$

A QBH vector field is said to be Pfaffian [7] if the integrating factor $\rho$ in Eq.(1.2) is the product of the eigenvalues of $N$, i.e.,

$$\rho = \prod_{i=1}^{n} \lambda_i.$$  \hspace{1cm} (1.4)

Working in this setting,

• in Sect.2 we present two Hénon–Heiles type systems with three and four degrees of freedom, which are Pfaffian QBH systems; passing to a set of Nijenhuis
coordinates, we show that the Hamilton-Jacobi equations for these systems are separable;

- in Sect.3 we obtain the general solution of Eq.(1.2) for a Pfaffian QBH vector field with an arbitrary number of degrees of freedom; the Hamiltonian $H$ and the function $F$ contain $n$ arbitrary smooth functions $f_i$, each one of them depending on a single pair $(\lambda_i; \mu_i)$ of Nijenhuis coordinates. Finally, we prove that the Hamiltonian $H$ is separable.

2. Two Hénon–Heiles type systems with three and four degrees of freedom

In this section we present two separable QBH systems with three and four degrees of freedom; they belong to a family of integrable flows obtained in [4] as stationary flows of the Korteweg–de Vries hierarchy [9]. This family contains the classical Hénon–Heiles system as its second member, so the higher members can be considered as multi-dimensional extensions of Hénon–Heiles.

The third member of this family, which is a stationary reduction of the seventh order KdV flow, is defined in a six dimensional phase space (with coordinates $q = (q_1, q_2, q_3)$, $p = (p_1, p_2, p_3)$) by the Hamiltonian vector field $X = P_0 dH$, with Hamiltonian function

$$H = \frac{1}{2} (2p_1p_2 + p_3^2) - \frac{5}{8} q_1^4 + \frac{5}{2} q_1^2 q_2 + \frac{q_1 q_3^2}{2} - \frac{q_2^2}{2}. \tag{2.1}$$

First of all, we can show that the vector field $X$ is Liouville-integrable. Indeed, if one introduces the functions

$$H_1 = H,$$

$$H_2 = \frac{p_1^2}{2} + p_1 p_2 q_1 + p_3^2 q_1 - p_2^2 q_2 - p_2 p_3 q_3 - \frac{q_1^5}{2} - \frac{q_1^2 q_3^2}{4} + \frac{q_2 q_3^2}{2} + 2 q_1 q_2^2, \tag{2.2}$$

$$H_3 = \frac{p_3^2 q_1^2}{2} + p_3 q_2 - p_1 p_3 q_3 - p_2 p_3 q_1 q_3 + \frac{p_2^2 q_3^2}{2} + \frac{q_1 q_3^2}{4} - q_1 q_2 q_3 - \frac{q_3^4}{8},$$

$X$ satisfies the assumptions of Prop.1.1; the tensor $P_1$ is given by

$$P_1 = \begin{bmatrix} 0 & A \\ -A^T & B \end{bmatrix}, \quad A = \begin{bmatrix} q_1 & -1 & 0 \\ 2 q_2 & q_1 & q_3 \\ q_3 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -p_2 & -p_3 \\ p_2 & 0 & 0 \\ p_3 & 0 & 0 \end{bmatrix}, \tag{2.3}$$

and the functions $\rho_{ij}$ are: $\rho_{11} = \rho_{22} = \rho_{33} = 1$, $\rho_{21} = \rho_{32} = -2 q_1$, $\rho_{31} = (3 q_1^2 - 2 q_2)$.

Furthermore one easily verifies that $P_1$ is a Poisson tensor, compatible with $P_0$ (so that $N = P_1 P_0^{-1}$ is a Nijenhuis tensor). One can show that $X$ is a QBH vector field; in fact Eq.(1.2) is verified with $\rho$ and $F$ given by $\rho = q_3^2$ and $F = H_3$. 
At last, let us show the separability of this system in terms of Nijenhuis coordinates. In this case the construction of a canonical map \( \Phi : (\lambda; \mu) \mapsto (q; p) \) between a set of Nijenhuis coordinates \((\lambda; \mu)\) and the coordinates \((q; p)\) is quite simple. We observe that the matrix \( A \) in Eq.(2.3) depends only on the coordinates \( q \), so also the eigenvalues \( \lambda \) depend only on \( q \): \( q_k = f_k(\lambda) \). Then we introduce the generating function \( S = \sum_{k=1}^3 p_k f_k(\lambda) \) and we get

\[
q_1 = -\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3) ,
q_2 = -\frac{1}{8}(\lambda_1 + \lambda_2 + \lambda_3)^2 + \frac{1}{2}(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) ,
q_3 = (\lambda_1 \lambda_2 \lambda_3)^{1/2} ,
\]

\[
p_1 = \frac{\lambda_1 \mu_1}{\lambda_{12} \lambda_{13}} (-\lambda_1 + \lambda_2 + \lambda_3) + \frac{\lambda_2 \mu_2}{\lambda_{21} \lambda_{23}} (\lambda_1 - \lambda_2 + \lambda_3) + \frac{\lambda_3 \mu_3}{\lambda_{31} \lambda_{32}} (\lambda_1 + \lambda_2 - \lambda_3) ,
p_2 = -2\left( \frac{\lambda_1 \mu_1}{\lambda_{12} \lambda_{13}} + \frac{\lambda_2 \mu_2}{\lambda_{21} \lambda_{23}} + \frac{\lambda_3 \mu_3}{\lambda_{31} \lambda_{32}} \right) ,
p_3 = 2(\lambda_1 \lambda_2 \lambda_3)^{1/2} \left( \frac{\mu_1}{\lambda_{12} \lambda_{13}} + \frac{\mu_2}{\lambda_{21} \lambda_{23}} + \frac{\mu_3}{\lambda_{31} \lambda_{32}} \right) ,
\]

where we put, for brevity, \( \lambda_{ij} := \lambda_i - \lambda_j \). Since \( \rho = q_3^2 = \lambda_1 \lambda_2 \lambda_3 \), we are faced with a Pfaffian system. Written in the above mentioned Nijenhuis coordinates, the Hamiltonian function \( H \) given by Eq.(2.1) takes the form

\[
H = \frac{\lambda_1 (16 \mu_1^2 - \lambda_1^5)}{8 \lambda_{12} \lambda_{13}} + \frac{\lambda_2 (16 \mu_2^2 - \lambda_2^5)}{8 \lambda_{21} \lambda_{23}} + \frac{\lambda_3 (16 \mu_3^2 - \lambda_3^5)}{8 \lambda_{31} \lambda_{32}} . \tag{2.5}
\]

It is easy to show that the Hamilton-Jacobi equation \( H(\lambda, \frac{\partial W}{\partial \lambda}) = h \) is separable and has the complete integral \( W = \sum_{i=1}^3 W_i(\lambda_i; c_0, c_1, c_2) \), with \( W_1, W_2 \) and \( W_3 \) solutions of the following equations

\[
\frac{dW_i}{d\lambda_i} = \left( \frac{1}{16 \lambda_i} (\lambda_i^6 + c_2 \lambda_i^2 + c_1 \lambda_i + c_0) \right)^{1/2} c_2 = 8h , \quad (i = 1, 2, 3) . \tag{2.6}
\]

Our second example is a Hénon–Heiles system with four degrees of freedom. It can be constructed as a stationary reduction of the ninth order KdV flow [10]. Its phase space is eight dimensional, and the Hamiltonian is

\[
H = \frac{1}{2} (p_1^2 + 2p_1 p_3 + p_3^2) + \frac{3}{4} q_1^5 - \frac{5}{2} q_1^3 q_2 + 2q_1 q_2^2 + \frac{5}{2} q_1^2 q_3 + \frac{q_1 q_4^2}{2} - q_2 q_3 . \tag{2.7}
\]
Also in this case, the vector field $X = P_0 dH$ is Liouville-integrable. Indeed, let us consider the functions

\[ H_1 = H, \]
\[ H_2 = p_1 p_2 + p_2^2 q_1 + p_1 p_3 q_1 + p_1^2 q_1 - p_2 p_3 q_2 - p_3^2 q_3 - p_3 p_4 q_4 \]
\[ + \frac{5}{8} q_1^6 - \frac{5}{4} q_1^4 q_2 - q_1^2 q_2 - \frac{q_1^2 q_3^2}{4} + q_2^3 + \frac{q_2 q_3^2}{2} + 3 q_1 q_2 q_3 - \frac{1}{2} q_3^4, \]
\[ H_3 = \frac{1}{2} p_2^2 q_1 + \frac{1}{2} p_4 q_1^2 + \frac{1}{2} p_3^2 q_2 + p_2 p_3 q_1 q_2 + \frac{1}{2} p_3^2 q_4 - p_3 p_4 q_1 q_4 \]
\[ - 2 p_2 p_3 q_3 + p_4 q_2 + p_1 p_3 q_2 + p_1 p_2 q_1 - p_2 p_4 q_4 + \frac{1}{2} p_1^2 \]
\[ + \frac{5}{4} q_1^4 q_2 - 3 q_1^3 q_2 + \frac{1}{2} q_1 q_4^2 + \frac{5}{4} q_1^4 q_3 + q_1^3 q_2 - q_1 q_2 q_3 - \frac{1}{2} q_1 q_2 q_4^2 \]
\[ + \frac{1}{2} q_4 q_3^2 + q_2 q_3^2 + 2 q_1 q_3^2, \]
\[ H_4 = -p_2 p_4 q_1 q_4 - p_3 p_4 q_2 q_4 + p_2 p_3 q_1^2 + p_4^2 q_1 q_2 + p_2^2 q_3 - p_1 p_4 q_4 \]
\[ - \frac{5}{8} q_1^4 q_2^2 \]
\[ + \frac{3}{2} q_1^2 q_2 q_4^2 - \frac{1}{2} q_2 q_4^2 - q_1 q_3 q_4^2 - \frac{1}{8} q_4^4, \]

and the tensor $P_1 = \begin{bmatrix} 0 & A \\ -A^T & B \end{bmatrix}$, with the matrices $A$ and $B$ given by

\[ A = \begin{bmatrix} q_1 & -1 & 0 & 0 \\ q_2 & 0 & -1 & 0 \\ 2q_3 & q_2 & q_1 & q_4 \\ q_4 & 0 & 0 & 0 \end{bmatrix}, \]
\[ B = \begin{bmatrix} 0 & -p_2 & -p_3 & -p_4 \\ p_2 & 0 & 0 & 0 \\ p_3 & 0 & 0 & 0 \\ p_4 & 0 & 0 & 0 \end{bmatrix}. \]  \hspace{1cm} (2.9)

Then $X$ verifies the assumptions of Prop.1.1 with the following choices for the functions $\rho_{ij}$: $\rho_{11} = \rho_{22} = \rho_{33} = \rho_{44} = 1$, $\rho_{21} = \rho_{32} = \rho_{43} = -2q_1$, $\rho_{31} = \rho_{42} = (3q_1^2 - 2q_2)$, $\rho_{41} = (-4q_1^3 + 6q_1 q_2 - 2q_3)$.

Moreover, $P_1$ is a Poisson tensor, compatible with $P_0$ (so that $N = P_1 P_0^{-1}$ is a Nijenhuis tensor). The Hamiltonian vector field $X$ is a QBH vector field since it satisfies the equation $X = P_1 dF/\rho$, with $\rho = -q_3^2$, $F = -H_4$.

At last, let us consider the map between the coordinates $(q, p)$ and the Nijenhuis coordinates $(\lambda, \mu)$. Since also in this case the matrix $A$ in Eq.(2.9) depends only on $q$, we proceed as in the previous example. The result is
\[ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -2q_1, \]
\[ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = q_1^2 + 2q_2, \]
\[ \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_2\lambda_3\lambda_4 = -2(q_1q_2 + q_3), \]
\[ \lambda_1\lambda_2\lambda_3\lambda_4 = -q_4^2, \]
\[ \mu_1 = -\frac{p_1}{2} - \frac{p_4}{2} \frac{\lambda_2\lambda_3\lambda_4}{(-\lambda_1\lambda_2\lambda_3\lambda_4)^{1/2}} + \frac{p_2}{4}(-\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \]
\[ + \frac{p_3}{16}(-3\lambda_1^2 + 2\lambda_1\lambda_2 + \lambda_2^2 + 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 + \lambda_3^2 + 2\lambda_1\lambda_4 - 2\lambda_2\lambda_4 - 2\lambda_3\lambda_4 + \lambda_4^2), \]
\[ \mu_2 = -\frac{p_1}{2} - \frac{p_4}{2} \frac{\lambda_1\lambda_3\lambda_4}{(-\lambda_1\lambda_2\lambda_3\lambda_4)^{1/2}} + \frac{p_2}{4}(\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4) \]
\[ + \frac{p_3}{16}(\lambda_1^2 + 2\lambda_1\lambda_2 - 3\lambda_2^2 - 2\lambda_1\lambda_3 + 2\lambda_2\lambda_3 + \lambda_3^2 - 2\lambda_1\lambda_4 + 2\lambda_2\lambda_4 - 2\lambda_3\lambda_4 + \lambda_4^2), \]
\[ \mu_3 = -\frac{p_1}{2} - \frac{p_4}{2} \frac{\lambda_1\lambda_2\lambda_4}{(-\lambda_1\lambda_2\lambda_3\lambda_4)^{1/2}} + \frac{p_2}{4}(\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4) \]
\[ + \frac{p_3}{16}(\lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 + 2\lambda_1\lambda_3 + 2\lambda_2\lambda_3 - 3\lambda_3^2 - 2\lambda_1\lambda_4 - 2\lambda_2\lambda_4 + 2\lambda_3\lambda_4 + \lambda_4^2), \]
\[ \mu_4 = -\frac{p_1}{2} - \frac{p_4}{2} \frac{\lambda_1\lambda_2\lambda_3}{(-\lambda_1\lambda_2\lambda_3\lambda_4)^{1/2}} + \frac{p_2}{4}(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4) \]
\[ + \frac{p_3}{16}(\lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 + \lambda_3^2 + 2\lambda_1\lambda_4 + 2\lambda_2\lambda_4 + 2\lambda_3\lambda_4 - 3\lambda_4^2). \]

By solving this system with respect to \((q; p)\) one can recover the canonical map \(\Phi : (\lambda; \mu) \mapsto (q; p)\) which allows one to write the Hamiltonian function \(H\) given by Eq. (2.7) in terms of Nijenhuis coordinates; it reads

\[ H = \frac{\lambda_1(16\mu_1^2 - \lambda_1^2)}{8\lambda_{12}\lambda_{13}\lambda_{14}} + \frac{\lambda_2(16\mu_2^2 - \lambda_2^2)}{8\lambda_{21}\lambda_{23}\lambda_{24}} + \frac{\lambda_3(16\mu_3^2 - \lambda_3^2)}{8\lambda_{31}\lambda_{32}\lambda_{34}} + \frac{\lambda_4(16\mu_4^2 - \lambda_4^2)}{8\lambda_{41}\lambda_{42}\lambda_{43}}. \]

Let us remark that also in this case the system is Pfaffian, since \(\rho = -q_4^2 = \lambda_1\lambda_2\lambda_3\lambda_4\). At last, one proves that the Hamilton–Jacobi equation \(H(\lambda; \frac{\partial W}{\partial \lambda}) = h\) is separable and has the complete integral \(W = \sum_{i=1}^{4} W_i(\lambda; c_0, c_1, c_2, c_3)\), with \(W_1, W_2, W_3\) and \(W_4\) solutions of the following equations

\[ \frac{dW_i}{d\lambda_i} = \left( \frac{1}{16\lambda_i} \left( \lambda_i^8 + c_3\lambda_i^3 + c_2\lambda_i^2 + c_1\lambda_i + c_0 \right) \right)^{1/2} \quad c_3 = 8h \quad (i = 1, 2, 3, 4). \]

3. **Quasi-biHamiltonian systems with \(n\) degrees of freedom**

Let us consider a 2\(n\) dimensional symplectic manifold \(M\), a Poisson tensor \(P_1\) compatible with \(P_0\), and let us assume to have introduced a set of Nijenhuis coordinates
(\(\lambda; \mu\)), so that \(P_1\) takes the Darboux form (1.3). We search for the general solution of the QBH equation (1.2) in the Pfaffian case (i.e., with \(\rho\) defined by Eq.(1.4)).

**Proposition 3.1.** In the Pfaffian case, the general solution of the equation \(P_0 dH = P_1 dF/\rho\) is given by

\[
H = \sum_{i=1}^{n} \frac{1}{\Delta_i} f_i(\lambda_i; \mu_i) , \quad F = \sum_{i=1}^{n} \frac{\rho_i}{\Delta_i} f_i(\lambda_i; \mu_i) ,
\]

where \((\lambda; \mu)\) are Nijenhuis coordinates, \(\Delta_i := \Pi_{j\neq i} \lambda_{ij} (\lambda_{ij} := \lambda_i - \lambda_j)\), \(\rho_i := \rho/\lambda_i\) and the \(n\) functions \(f_i(\lambda_i; \mu_i)\) (each one of them depending on one pair of coordinates) are arbitrary smooth functions.

**Proof.** Eq.(1.2) corresponds to the two sets of equations

\[
\frac{\partial H}{\partial \mu_i} = \frac{\lambda_i}{\rho} \frac{\partial F}{\partial \mu_i} \quad (i = 1, 2, \ldots, n) ,
\]

\[
\frac{\partial H}{\partial \lambda_i} = \frac{\lambda_i}{\rho} \frac{\partial F}{\partial \lambda_i} \quad (i = 1, 2, \ldots, n) .
\]

The general solution of the first set is

\[
H = \frac{1}{\rho} \sum_{i=1}^{n} \lambda_i G_i(\lambda; \mu_i) + K(\lambda) , \quad F = \sum_{i=1}^{n} G_i(\lambda; \mu_i) ,
\]

where the functions \(G_i = G_i(\lambda; \mu_i)\) and \(K = K(\lambda)\) are arbitrary. Indeed, the solution of the first equation (3.2), for \(i = 1\), is \(H = \frac{1}{\rho} \sum_{j=1}^{n} \lambda_j F(\lambda; \mu) + \phi_1(\lambda; \mu_2, \ldots, \mu_n)\), with \(\phi_1\) arbitrary; on account of this result, the equation (3.2) for \(i = 2\) has the solution

\[
H = \frac{\lambda_1}{\rho} G_1(\lambda; \mu_1) + \frac{\lambda_2}{\rho} \psi_1(\lambda; \mu_2, \ldots, \mu_n) + \phi_2(\lambda; \mu_3, \ldots, \mu_n) ,
\]

\[
F = G_1(\lambda; \mu_1) + \psi_1(\lambda; \mu_2, \ldots, \mu_n) ,
\]

with \(\psi_1\) and \(\phi_2\) arbitrary. Iterating this procedure for \(i = 3, \ldots, n\) one easily obtains the solution (3.4). Let us insert this solution into Eq.s(3.3), putting in evidence the dependence on \(\mu\); we conclude that \(K(\lambda)\) has to be a constant function (which can be taken equal to zero with no loss of generality) and that Eq.s(3.3) can be written as

\[
\frac{\partial}{\partial \lambda_i} \left( \sum_{j=1}^{n} \lambda_{ij} G_j \right) = \frac{1}{\lambda_i} \left( \sum_{j=1}^{n} \lambda_{ij} G_j \right) \quad (i = 1, 2, \ldots, n) .
\]
By integrating these equations \((i = 1, 2, \ldots, n)\) and taking into account the dependence on \(\mu\), we easily obtain that

\[
G_i(\lambda; \mu) = \frac{\rho_i}{\Delta_i} f_i(\lambda_i; \mu_i) \quad (i = 1, 2, \ldots, n) ,
\]

where each \(f_i\) is an arbitrary function depending only on the pair of variables \((\lambda_i; \mu_i)\).

\(\Box\)

Of course, the vector field \(X = P_0 dH\) is a QBH vector field in \(2n\) dimensions.

At last, on account of the above result, we can also prove that the Hamiltonian \(H\) and the function \(F\) are separable:

**Proposition 3.2.** The Hamiltonian \(H\) and the function \(F\), written in terms of the Nijenhuis coordinates \((\lambda; \mu)\) in the form \((3.1)\), are separable for each \(n\)-ple of functions \(f_i(\lambda_i; \mu_i)\).

**Proof.** The Hamilton-Jacobi equation for \(H\) is separable iff \(H\) verifies the Levi-Civita conditions \(L_{ij}(H) = 0\) \((i, j = 1, \ldots, n; i \neq j)\) where \([11]\)

\[
L_{ij}(H) = \frac{\partial H}{\partial \lambda_i} \frac{\partial^2 H}{\partial \lambda_j \partial \mu_i \partial \mu_j} + \frac{\partial H}{\partial \mu_i} \frac{\partial^2 H}{\partial \mu_j \partial \lambda_i \partial \lambda_j} - \frac{\partial H}{\partial \lambda_i} \frac{\partial^2 H}{\partial \mu_i \partial \mu_j \partial \lambda_j} - \frac{\partial H}{\partial \mu_i} \frac{\partial^2 H}{\partial \lambda_i \partial \mu_j \partial \lambda_j} .
\]

In our case, it is \(\partial^2 H/\partial \mu_i \partial \mu_j = 0\) and

\[
\frac{\partial \Delta_j}{\partial \lambda_j} = \Delta_j \sum_{\alpha \neq j} \lambda^{-1}_{j\alpha} ; \quad \frac{\partial \Delta_j}{\partial \lambda_{j\beta}} = -\Delta_j \lambda^{-1}_{j\beta} \quad (\beta \neq j) .
\]

It may be useful to decompose \(L_{ij}(H)\) as \(L_{ij}(H) = M_{ij}(H) + N_{ij}(H)\), where \(M_{ij}(H)\) depends linearly on the functions \(f_i\), and \(N_{ij}(H)\) depends on the derivatives \(\partial f_i/\partial \lambda_i\) but not on \(f_i\). Using Eq.\((3.9)\) one directly verifies that \(M_{ij}(H) = 0\) and \(N_{ij}(H) = 0\). Similarly, one can show that the Levi-Civita conditions \((3.8)\) are fulfilled also by the function \(F\) given in \((3.1)\). \(\Box\)

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