Symmetry in music

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Abstract. Music and Physics are very close because of the symmetry that appears in music. A periodic wave is what music really is, and there is a field of Physics devoted to waves researching. The different musical scales are the base of all kind of music. This article tries to show how this musical scales are made, how the consonance is the base of many of them and how symmetric they are.

1. Introduction
At first sight connection between music and physics could seem to be strange, fields so different like art and science, but in the end, music is sound, sound is a wave and physics studies waves. Both subjects can research the same thing but from a different point of view.
This article, tries to show, how Physics can contribute to the study of Music, and how we can see some elements of this art from a scientific point of view.

2. Theoretical introduction of musical Acoustics
First of all, let’s talk about acoustics, just the items that will appear later.
The first item to talk about is sound. Sound is a travelling wave, a pressure oscillation transmitted through a solid, a liquid, or a gas, composed of frequencies within the range of hearing and of a level strong enough to be heard, or the sensation stimulated in organs of hearing by such vibrations. So to produce a sound, you need a sender, a transmitter and a receiver. In the experiment referred to in this article, it is used: as a sender, musical instruments (violin, bowed string; clarinet, semi-open tube; pipe organ, open tube, monochord); as a transmitter, the air; and as a receiver, the human ear has been replaced by a spectrum analyzer, with characteristics similar to the human ear which allows us to obtain numerical results and graphs of what the ear would receive.
Another item needed to understand this experiment is the Principle of Superposition. This Principle says that if two sounds are emitted, with the same amplitude but with different frequencies \( f_1 \) and \( f_2 \), a new sound is produced, its amplitude changes in time and whose frequency is the semi-sum of both frequencies.

\[
y_1 = A \cos(\sigma_1 t + \varphi) = A \cos(2\pi f_1 t + \varphi) \\
y_2 = A \cos(\sigma_2 t + \varphi) = A \cos(2\pi f_2 t + \varphi) \\
y_1 + y_2 = 2A \cos \left( \frac{2\pi f_1 - f_2}{2} t \right) \cos \left( \frac{2\pi f_1 + f_2}{2} t + \varphi \right)
\]  

\( f_1 \) and \( f_2 \) are the fundamental frequencies of the two waves, \( \sigma_1 \) and \( \sigma_2 \) their respective frequencies, \( \varphi \) the initial phase of each wave.

\[ y \] }
The *Fourier theorem* is necessary for this experiment. In our context, this theorem says: the sound from a musical instrument is usually not a pure sine wave, but a “fundamental frequency accompanied by a number of overtones of frequencies 2, 3, 4, etc., times the frequency of the fundamental. The combination of the fundamental and the overtones is a complicated periodic function with the period of the fundamental. Given the complicated function, we could ask how to write it as a sum of terms corresponding to the various harmonics. In general it might require all the harmonics, that is, an infinite series of terms. This is called Fourier series. Expanding a function into a Fourier series then amounts to breaking it down into its various harmonics. In fact, this process is sometimes called harmonic analysis [1], and it represented musically:

![Fourier Theorem](image1)

**Figure 1. Fourier Theorem**

In this graph one can see the fundamental frequency with the number 1 and its representation on a staff, and the musical representation of spectral analysis.

All of these items are theoretical knowledge needed in order to interpret the results but to give them a real sense it is necessary to define the *critical bandwidth* discovered by Békésy, it is the range of frequencies where the brain knows that the received sound is not just one but it cannot recognise two. When the Fourier Theorem is applied to these sounds, the spectrum shows how the peaks of frequency that the main cannot separate fuse in one, with a frequency that is the semi-sum of harmonics that fuse, and its intensity changes in time with a beat frequency similar to the difference in frequency between peaks. The hard sound for quick beats can be perceived in the critical bandwidth and the nice sound for slow beats.

![Critical Bandwidth](image2)

**Figure 2. Critical Bandwidth**

### 3. Theoretical introduction of musical Dissonance

The relationship between Music, Mathematics and Physics has been known since s.VI b.C. when Pythagoras and his pupils studied vibrant string. These studies set the beginning of the study of Harmony in Music.

The interest of the consonance and dissonance come from this time to nowadays. Throughout this time many people worked on it trying to set a definition. Some of the last theories about it are exposed in this article.

*John Tyndall*, for example, said in 1893 [2], that an interval would be more consonant if the ratios are more simple.

*Hermann von Helmholtz* [3] was the first person who introduced the sense of beat. *Georg von Békésy* focused his studies on acoustic hardness. The existence of the beats can create an unpleasant sensation, but not on all occasions. If the frequency of a beat is low, simply a tremolo is
appreciated, not dissonance. By contrast, two sounds whose frequencies differ enough to produce no audible pulsation, you can create, when they sound together, a feeling of roughness. Carl Stumpf said [4], in 1898, neither harmonics nor difference tones are essential to discriminate between consonant and dissonant intervals, whereas he rejected the frequency-ratio theory as more speculation. Stumpf called attention to the fact, investigated by him before [5] and confirmed by many others after him, that the degree of fusion of intervals depends on simple frequency ratio in the same order as consonance does. By fusion, he meant the tendency of two simultaneous tones to be perceived as a unity.[6] Plomp and Levelt add [7], that they assume that the total dissonance of such an interval is equal to the sum of the dissonances of each pair of adjacent partials. Kameoka and Kuriyagawa [8] based their studies about consonance and dissonance on the harmonic structure of the emitted sounds. They said that the consonance between two frequencies without harmonics only depends on the distance. And the intensity has an important role, because it can mask one of the sounds and have influence on the consonance. Ernst Terhardt thinks that musical dissonance has to be studied in two ways, one of them is sensorial dissonance, the concept of sounding well, on the other hand, the harmony or the influence of the culture.

4. Experiment
The objective of the experiment is to prove experimentally the results that Plomp and Levelt obtained in 1965 [6] statistically and numerically about the consonance and dissonance, and the application of these results to the construction of musical scales (western and eastern).
This experiment tries to show the dissonance as something objective. All the theories until now show it as something subjective, something that depends on culture, taste or statistics and numerical results, but this experiment tries to give a value of dissonance. Obviously, the concept of dissonance is different for everyone, but with this experiment a scale can be made, with values, and it can be known from what value something dissonant is listened. Some intervals produced by different instruments have been recorded and then, a Fourier analysis with different precision has been done to obtain a very precise measurement or numerical results, very similar to the ear reception capacity. A spectrum has been used with a bandwidth as the human ear. The same experiment has been done with the violin and the clarinet. With the first one, a violinist played two sounds at the same time. The violinist tried to find a specific tune, the tune that allows an integer ratio between both sounds. The same procedure was done with the clarinet, but with two clarinettists, one of the sounds stays constant and the other rises from the unison to the octave chromatically. Analyzing the results with two different types of spectrum, one of them with a very thin bandwidth, to obtain very exact results, and the other one with a bandwidth similar to the human ear, in order to obtain numerical results about the human ear reception capacity. With the thin bandwidth, we obtained the exact frequencies of each peak. In the analysis with wide bandwidth, a spectrum was done every millisecond during the second second that was chosen, making a movie with them. The second second, because in the first one there would be imprecise tunes. In that movie we can see how the intensity of some peaks of frequency (the peaks which are very closed in the spectrum with thin bandwidth) change in time, the beats. Representing the results on a graph, intensity to time:

![Figure 3. Variation of intensity vs. time for minor second](image-url)
In this graph it can be seen how intensity changes in time for most of the harmonics, that is to say the beats.

Numerical results for this interval are shown in the next table:

| i | \(f_1\) | \(f'_1\) | \(f_2\) | \(f'_2\) | \(f_2-f_1\) | \(f'_2-f'_1\) | \(f_b\) | Dissonance |
|---|---|---|---|---|---|---|---|---|
| 1 | 300 | 294 | 313 | 34 | 19 | 303.5 | 17.79 | 0.96 |
| 2 | 600 | 589 | 623 | 34 | 36 | 606.0 | 34.72 | 0.99 |
| 3 | 900 | 880 | 936 | 56 | 46 | 908.0 | 55.17 | 0.87 |
| 4 | 1250 | 1173 | 1247 | 74 | 74 | 1210.0 | 72.80 | 0.83 |
| 5 | 1500 | 1467 | 1559 | 92 | 92 | 1513.0 | 87.79 | 0.79 |
| 6 | 1850 | 1761 | 1872 | 111 | 111 | 1816.5 | 114.58 | 0.72 |

This table shows how the beat frequency is very close to the difference of original frequencies and how the semisum of these frequencies is very similar to the fundamental frequency of each harmonic.

The value of the dissonance was calculated taking into account the percentage of bandwidth that corresponds to the beat frequency and applying the calculation of Plomp and Levelt. The total value for the dissonance of this interval is the addition of the dissonance of all of harmonics for this interval.

If the beat of frequency is outside the bandwidth, this beat doesn’t contribute to the dissonance. These results correspond to the minor second from Violin. The same procedure was followed with the rest of intervals and with both instruments. The experiment started with the unison, one of the sounds stays in the same frequency and the other goes up chromatically to the octave. When the results are represented in a similar graph to Plomp and Levelt’s, we see the following:

The red points correspond to the results of the experiment. A continuous graph couldn’t be done, like Plomp and Levelt, since all the rank of frequencies were not swept, they only go up chromatically. They only work with harmonics until 6, so they can only obtain a relationship from frequency in natural numbers up to 6 (1:1, 5:6, 4:5, 3:4, 2:3, 3:5, 1:2) that is why frequency relationship for numbers greater than 6 (15: 16, 8:9, 5:7, 5:8, 4:7, 8:15) don’t appear, their sounds are artificial (for that reason they can control the number of overtones), whereas the results of the experiment are natural. That is the reason why there are more overtones.

It can also be observed that the frequency ratios that agree with that are in the graph. In the experiment there are more consonant relationship than for Plomp and Levelt.

Although at first sight it would be possible to hope that since the greater number of overtones used, respecting to Plomp and Levelt, would contribute to a greater dissonance. There are a series of factors that
influence it, for example, the intensity. For that reason, the intensity of the overtones that Plomp and Levelt handle can produce greater dissonances than the experiment (still being less harmonic). Also it is necessary to consider that the point of greater dissonance obtained in the experiment has been done in accordance with the point of greater dissonance obtained by Plomp and Levelt. Relative values are used at any moment, the sounds are more or less consonant compared to others, then to be able to establish a comparison, the most dissonant value and the minus have been taken as a reference, comparing the rest based on that scale, i.e., if the most dissonant value for the experiment were more dissonant than the one of Plomp and Levelt, the differences of the other tips (between our results and those of Plomp and Levelt) would be different.

If the dots are connected, a graph that is very similar to the one of Plomp and Levelt can be obtained.

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The same was done with the clarinet and the pipe organ. The difference with these instruments is that they are tempered instruments, so the ratio between the frequencies is not of natural numbers.

The next graph shows the results of these three instruments.

The red points correspond to the violin, blue squares to the clarinet, and green rhombus to the pipe organ. The plot shows how the most consonant dots are the same for the three instruments and how for the ratio 5:7 the triton it can be seen that it is a consonant interval for no tempered instrument (when the ratio is simple) and very dissonant for the organ, with tempered tuning.

Most of the ratios haven’t got the same consonance or dissonance, for many reasons. If the intensity of one of the overtones is different to another, the frequency ratio would be different. For the clarinet there are only the pair overtones, so there are less than for the rest of the instruments, and this, influences in the consonance and dissonance.

5. Scales
These results try to prove how eastern and western scales are based on the sense of consonance.
5.1. Western scales

It is well-known that western music is based on the superposition of fifth and third. Some theoreticians will use only one of the intervals using therefore the second and third overtones, and others use both, that’s the reason why they would use overtones 2, 3 and 5 (remembering that the fifth corresponds with the ratio $\frac{3}{2}$, Mayor third $\frac{5}{4}$, minor third $\frac{5}{3}$ and the octave $\frac{2}{1}$).

Now some of the western scales will be shown, in first place the harmonic structure in which they are constructed and secondly the ordered frequencies ascending indicating the intervals, that in this case formed between consecutive frequencies. a-Figures show the way to make the scales, how all of them are made in the same way, and b-Figures show the periodicity or the symmetry in the scales, or in many of them.

5.1.1. Pythagoras

5.1.2. Ramos de Pareja

5.1.3. Salinas

5.1.4. Zarlino
All these scales are constructed, as it can be seen, from fifth $\frac{3}{2}$, Mayor third $\frac{5}{4}$ and minor third $\frac{5}{3}$, i.e. using overtones 2, 3 and 5, first of the series, therefore, simple ratios, which implies little dissonant intervals. Once the frequencies have been placed in ascending sequence, intervals are formed between each consecutive sound, but they are always powers of these same overtones. It can be said that western music is constructed with consonant intervals, where consonant means sound with few beats in the critical bandwidth [9].

In the case of Eastern music it will also be verified that the scales are made with consonant intervals from the first overtones of the series. The values for the frequency ratios have been taken from the works of Gruber [10], Sanchez González [11] and Barkechli [12].

### 5.2. Indian scales

#### 5.2.1. 14 Tones

As it can be seen how this 14 tone scale is made only with the fifth, as Pythagoras scale. But these scales show that, once the frequencies are placed in ascending sequence, the intervals between two consecutive sounds are always the same, and make system of symmetry.

#### 5.2.2. 23 Tones

This is the most common scale for Indian people. When they play their music, they never reach this frequency exactly, because when Indian people talk about a sound, they never talk about a specific sound, they just talk about an area around that sound, but the center of this area is this frequency.

Between two consecutive sounds, in this scale there are three different intervals, the first of them, $a = \frac{256}{243}$ was used by Pythagoras, and the others, $b = \frac{81}{80}$ and $c = \frac{25}{24}$ are used by Salinas. This fact shows how eastern and western music are linked.

The next figure shows the name that Indian musicians give to these frequencies.
5.2.3. 26 Tones

Figure 14.a  26-tone scale

5.2.4. 30 Tones

Figure 15.a  30-tone scale

Figure 15.b  30-tone scale
5.2.5. 32 Tones

All Indian scales are made using the same intervals as western scales, i.e. using the harmonics 2, 3 and 5, the most consonant intervals.

5.3. Arabian scales

These scales are made overlapping tetrachords. For this music, the experiment obtained very similar schemes to the ones for Indian music.
This music used two more overtones to make their scales, the 7 and the 11, but remember that the interval $\frac{7}{5}$ is consonant [9].

6. Conclusions
This experiment tries to demonstrate experimentally the Helmholtz beats theory, seeing them, and obtaining similar results to Plomp and Levelt’s, although they obtain statistical and numerical results and the experiment just obtains experimental results.

It demonstrates as well than triton is consonant if it is played with exact tuning $\frac{7}{5}$ and how eastern and western scales are made using consonant intervals.

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