INDEPENDENT SETS IN CERTAIN CLASSES OF (ALMOST) REGULAR GRAPHS

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Abstract. We enumerate the independent sets of several classes of regular and almost regular graphs and compute the corresponding generating functions. We also note the relations between these graphs and other combinatorial objects and, in some cases, construct the corresponding bijections.

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1. Introduction

Let \( I(G) \) denote the number of independent sets of a graph \( G \). This number can be determined for some special classes of graphs (see [7] for a survey). For instance, \( I(G) \) was studied for grid graphs (see [1]), multipartite complete graphs, and path- and cyclic-schemes. Many of these numbers are given by certain combinations of Fibonacci numbers, some others by Lucas numbers.

In this paper, we study four classes of graphs. To define these classes, we recall that a line graph \( L(G) \) of a graph \( G \) is obtained by associating a vertex with each edge of \( G \) and connecting two vertices with an edge if and only if the corresponding edges of \( G \) are adjacent. Also, recall that a cycle graph \( C_\ell \) is a graph on \( \ell \) nodes containing a single cycle through all nodes.
We now give the definitions of our classes.

**Class 1:** Let $G_1^\ell = C_\ell$, the $\ell$-cycle graph. We obtain $G_2^\ell$ by superimposing the line graph $L(G_1^\ell)$ onto the graph $G_1^\ell$, that is splitting each edge of $G_1^\ell$ with the corresponding vertex of $L(G_1^\ell)$ and then adding the edges of $L(G_1^\ell)$. More generally, $G_n^\ell$ is obtained by superimposing $L^{n-1}(G_1^\ell)$ onto $G_{n-1}^\ell$. For example, in Figure 1, we have the graphs $G_4^3$ and $G_3^4$, respectively, if one ignores the dashed edges. Clearly, all but $\ell$ of the $\ell^n$ nodes of $G_n^\ell$ have degree 4, and we say that $G_n^\ell$ is an almost $4$-regular graph.

**Figure 1.** Examples of (almost) $4$-regular graphs under consideration.

**Class 2:** The graph $R_n^\ell$ is obtained from $G_n^\ell$ by duplicating the edges of $G_1^\ell$. For example, in Figure 1, the extra edges are the dashed edges, and by adding them we get the graphs $R_4^3$ and $R_3^4$ respectively. So, to get $R_n^\ell$ we add $\ell$ additional edges to $G_n^\ell$, and it is easy to see that $R_n^\ell$ is a $4$-regular graph.

**Class 3:** Let $K_1^\ell = K_\ell$, a complete graph on $\ell$ nodes. Put the $\ell$ nodes of $K_1^\ell$ on a circle and draw the remaining $(\ell - 1)! - \ell$ edges. Call the first $\ell$ edges external and the remaining edges, internal. Then construction of $K_n^\ell$ is similar to that of $G_n^\ell$, except that:

1. The basis of the construction is now $K_\ell$, rather than $C_\ell$.
2. On each iteration $i$, the graph superimposed onto $R_{i-1}^\ell$ is not $L^i(K_\ell)$ but rather the complete graph on the nodes of $L^i(C_\ell)$, the line graph of $C_\ell$ formed by the external edges of $K_1^\ell$.

In Figure 2 we show how to construct $K_3^3$ from $K_4^2$ (the dashed edges should be ignored). In that figure, the external edges of respective complete graphs are in bold. We also remark that the internal edges do not intersect each other. Moreover, Figure 2 suggests a convenient way of representing $K_4^3$, where each node of the graph lies only on internal edges incident with that node. We achieve that by putting the nodes of the line graphs under consideration off the centers of the corresponding edges. Indeed, if we were using the centers of the external edges, $K_4^3$ would look as in Figure 3, which is misleading since, for example, the node $a$ does not lie on the edge $bc$.

The graph $K_n^\ell$ is almost $(\ell + 1)$-regular, since all but $\ell$ of its $\ell^n$ nodes have degree $\ell + 1$. Moreover, it follows from our definitions that $K_3^3 = G_3^3$. 
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Class 4: The graph $P^n_\ell$ is obtained from the graph $K^n_\ell$ by duplicating the external edges in the graph $K^1_\ell$. For example, in Figure 2, the extra edges are the dashed edges, and by adding them we construct the graph $P^3_4$ from $K^3_4$. So to get $P^n_\ell$ we add $\ell$ extra edges to $K^n_\ell$, and it is easy to see that $P^n_\ell$ is an $(\ell + 1)$-regular graph.

Figure 2. A way of constructing $K^3_4$ and $P^3_4$.

Let $g_\ell(n)$, $r_\ell(n)$, $k_\ell(n)$ and $p_\ell(n)$ denote the number of independent sets in the graphs $G^n_\ell$, $R^n_\ell$, $K^n_\ell$ and $P^n_\ell$ respectively. In our paper we study these numbers. In Section 3 we give an algorithm for calculating all these numbers. For the numbers $p_\ell(n)$, we provide an explicit generating function (see Theorem 3.4). However, in order to illustrate our approach to the problem, in Section 2 we consider $g_3(n)$ and find an explicit formula for it.

Our choice of the graphs to study was motivated by the so called de Bruijn graphs, which are defined as follows. A de Bruijn graph is a directed graph $\bar{G}_n = \bar{G}_n(V, E)$, where the set of vertices $V$ is the set of all the words of length $n$ in a finite alphabet $A$, and there is an arc from $v_i = (v_{i1}, \ldots, v_{in})$ to $v_j = (v_{j1}, \ldots, v_{jn})$ if

$$v_{i2} = v_{j1}, \ v_{i3} = v_{j2}, \ldots, v_{in} = v_{j(n-1)},$$

that is when the words $v_i$ and $v_j$ overlap by $(n - 1)$ letters.

The de Bruijn graphs were first introduced (for the alphabet $A = \{0, 1\}$) by de Bruijn in 1944 for enumerating the number of code cycles. However, these graphs proved to be a useful tool for various problems related to the subject of combinatorics on words (e.g. see [2, 3, 5]). It is known that the graph $\bar{G}_n$ can be defined recursively as $\bar{G}_n =$
The authors were interested in studying other graphs defined recursively using the operation of taking line graphs (with natural bases), which could give interesting applications. Also, with our choice of graphs \( G^n_\ell \) and \( K^n_\ell \), it is natural to complete them to regular graphs \( R^n_\ell \) and \( P^n_\ell \) and study these graphs. It turns out that there are combinatorial interpretations (relations to other combinatorial objects) for the number of independent sets for some of our graphs, and we mention these relations in Sections 2 and 3. Moreover, we construct a direct bijection describing such a relation for \( P^n_4 \) (see Proposition 3.5 and the discussion that follows).

2. The numbers \( g_3(n) \).

Let us first find an explicit formula for \( g_3(n) \).

It is clear that for any independent set of the graph \( G^n_3 \), we can label a node of \( G^n_3 \) 1 if this node is in the independent set, and label it 0 otherwise. Thus, our purpose is to count the number of triangles having either 0 or 1 in each node and such that no two adjacent nodes are both assigned 1s. We call such triangles legal.

In order to get a recursion for \( g_3(n) \), we introduce three auxiliary parameters \( a_n, b_n, \) and \( c_n \), which are the numbers of legal triangles that, up to rotation, have specific numbers in the nodes of the biggest triangle (see Figure 4). Since we consider only legal graphs, the 1s in the nodes of the biggest triangle induces 0s in certain nodes of a smaller triangle (this 0s are shown in Figure 4).

\[ \begin{align*}
&\text{Figure 4. Auxiliary parameters } a_n, b_n, \text{ and } c_n. \\
&\text{If we consider all the possibilities for the numbers of the biggest triangle, we have that } \\
g_3(n) &= g_3(n - 1) + 3a_n + 3b_n + c_n, \\
\text{where } g_3(n - 1) \text{ corresponds to all 0s, and we have multiple 3 two times because of possible rotations. Similarly, we get that } a_n = g_3(n - 2) + a_{n - 1}, \text{ and } b_n = c_n = g_3(n - 2). \text{ This leads to the recursion} \\
(1) \quad g_3(n) &= 2g_3(n - 1) + 6g_3(n - 2) - 4g_3(n - 3), \\
\text{which, under the same initial conditions, is equivalent to the recursion} \\
(2) \quad g_3(n) &= 4g_3(n - 1) - 2g_3(n - 2). 
\end{align*} \]
We define $g_3(0) = 1$, since we associate the graph $G_3^0$ with the empty graph, in which case there is only one independent set, the empty set. Thus,

$$g_3(n) = \frac{1}{2\sqrt{2}} \left((2 + \sqrt{2})^{n+1} - (2 - \sqrt{2})^{n+1}\right),$$

and the generating function for the numbers $g_3(n)$ is $1/(1 - 4x + 2x^2)$. The initial values for the numbers $g_3(n)$ are:

$$1, 4, 14, 48, 164, 560, 1912, 6528, 22288, 76096, \ldots.$$ 

Preceded by 0, the sequence $\{g_3(n)\}$ is the binomial transform of the Pell numbers $P_n = (1 + \sqrt{2})^n - (1 - \sqrt{2})^n$, (see [10, A007070]). These numbers can also be interpreted as maximum bets in a poker game (also see [10, A007070]), where the first player bets 1 dollar into a pot and the $i$th player bets the amount of the $(i - 1)$st player’s bet plus the resulting amount of money in the pot. Then the number of dollars $d_n$ in the pot after $n$ bets is given by

$$d_n = 2(d_{n-1} + (d_{n-1} - d_{n-2})) = 4d_{n-1} - 2d_{n-2}, \quad d_0 = 1, d_1 = 4,$$

which yields $d_n = g_3(n)$.

We remark that it would be interesting to obtain recurrence (2) directly from the graph $G_3^n$, rather than via recurrence (1). Unfortunately, we were unable to do this.

### 3. An algorithm for calculating $g_\ell(n)$, $r_\ell(n)$, $k_\ell(n)$ and $p_\ell(n)$

In this section we present an algorithm for calculating $g_\ell(n)$, $r_\ell(n)$, $k_\ell(n)$ and $p_\ell(n)$ by using the transfer matrix method (see [9, Theorem 4.7.2]).

#### 3.1. An algorithm for calculating $g_\ell(n)$

In this section, we use the transfer matrix method to obtain an information about the sequence of $g_\ell(n)$.

Similarly to Section 2 for any independent set of the graph $G_\ell^n$, consider a labeling of $G_\ell^n$, where the nodes of of the independent set are labeled 1 and the remaining nodes are labeled 0. For a given graph $G_\ell^n$, we define the $n$-th level of $G_\ell^n$ to be $G_\ell^n \setminus G_\ell^{n-1}$, which is isomorphic to $G_1^n$. Thus, we may think of an independent set of the graph $G_\ell^n$ as assembled from elements chosen on each level, making sure that when we add a new level, we create no conflict with the previous level.

The collection $L_\ell$ of possible level labelings is the set of all $(0, 1)$ $\ell$-vectors $v = (v_1, \ldots, v_\ell)$. It will be convenient to define $v_{\ell+1} := v_1$. Then $v = (v_1, \ldots, v_\ell)$ and $w = (w_1, \ldots, w_\ell)$ in $L_\ell$ are a possible consecutive pair of levels in an independent set of $G_\ell^n$ (with $w$ following $v$) if and only if

$$v_i = v_{i+1} = 0 \text{ or } w_i = 0, \quad \text{where } i = 1, 2, \ldots, \ell.$$
Thus, to obtain any independent set in the graph $G_{\ell}^n$, we begin with a vector of $L_{\ell}$, then keep adjoining each next vector $w \in L_{\ell}$ so that it satisfies (3) together with the previously chosen vector $v \in L_{\ell}$, until $n$ vectors have been selected.

We define a matrix $G = G_{\ell}$, the transfer matrix of the problem, as follows. $G$ is a $2^\ell \times 2^\ell$ matrix of 0s and 1s whose rows and columns are indexed by vectors of $L_{\ell}$. The entry of $G$ in position $(v, w)$ is 1 if the ordered pair of vectors $(v, w)$ satisfies (3), and is 0 otherwise. $G$ depends only on $\ell$, not on $n$. Hence, the number of independent sets of $G_{\ell}^n$, $g_{\ell}(n)$, is the first entry of the vector $G^n \cdot (u_1, \ldots, u_{2^\ell})^T$, where $u_i = 1$ if there are the $i$th vector $v$ in the collection $L_{\ell}$ has no two consecutive 1s, even after wrapping, (i.e. $u_i + u_{i+1} \leq 1$ for all $i = 1, 2, \ldots, \ell$), and $u_i = 0$ otherwise. Hence,

$$g_{\ell}(n) = (1, 0, \ldots, 0) \cdot G^n \cdot (u_1, \ldots, u_{2^\ell})^T.$$  

For instance, when $\ell = 3$, the possible level vectors are

$$(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1),$$

except for the last level, where we only have

$$(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0).$$

If we index the rows and the columns of the transfer matrix $G$ in this order, then we get

$$G = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$  

The vector $(u_1, \ldots, u_{2^\ell})^T$ in this case is $(1, 1, 1, 0, 1, 0, 0, 0)$. If we now find the first entry of the vector $(I - xG)^{-1} \cdot (1, 1, 1, 0, 1, 0, 0, 0)^T$, where $I$ is the unit matrix, then we get that the generating function for $g_3(n)$ is given by $1/(1 - 4x + 2x^3)$. We obtain the results for larger $\ell$ similarly.

**Theorem 3.1.** The generating functions for the numbers $g_4(n)$, $g_5(n)$ and $g_6(n)$ are given, respectively, by

$$\frac{1 + 4x - x^2 - 2x^3}{1 - 3x - 14x^2 + 15x^3 + 7x^4},$$  

$$\frac{1 - 5x - 30x^2 + 69x^3 + 31x^4 - 22x^5}{(1 + x)(1 + 5x - 8x^2)},$$  

$$\frac{1 + 10x - 12x^2 - 50x^3 + 10x^4 + 20x^5 - 12x^6}{1 - 8x - 66x^2 + 280x^3 + 178x^4 - 532x^5 - 84x^6 + 108x^7}.$$  

We remark that the algorithm for finding the generating function for $g_{\ell}(n)$ has been implemented in Maple, and yielded explicit results for $\ell \leq 6$. 


3.2. **An algorithm for calculating** $r_\ell(n)$. In this section we use the transfer matrix method to obtain an information about the numbers $r_\ell(n)$. This case is similar to that of $g_\ell(n)$ with some small differences.

We partition $R_\ell^n$ into levels just as in the case of $G_\ell^n$, so the $n$-th level of $R_\ell^n$ is $R_\ell^n \setminus R_\ell^{n-1}$.

The collection of possible levels $\mathcal{L}_\ell$ is the set of all $\ell$-vectors $v$ of 0s and 1s such that there no consecutive 1s in $v$, that is, $v_i + v_{i+1} \neq 2$ (where we define $v_{\ell+1} := v_1$). Clearly, the set $\mathcal{L}_\ell$ contains exactly $L_\ell$ vectors where $L_\ell$ is the $\ell$th Lucas number. For instance, $L_3$ contains the vectors $(0,0,0), (0,0,1), (0,1,0), (1,0,0)$.

The condition that vectors $v$ and $w$ in $\mathcal{L}_\ell$ are a possible consecutive pair of levels in an independent set of $R_\ell^n$ is given by (3) just as for $G_\ell^n$. To obtain any independent set in the graph $R_\ell^n$, we begin with a vector of $\mathcal{L}_\ell$, then keep adjoining each next vector $w \in \mathcal{L}_\ell$ so that it satisfies (3) together with the previously chosen vector $v \in \mathcal{L}_\ell$, until $n$ vectors have been selected.

We define a matrix the transfer matrix of the problem $R = R_\ell$ in the same way as $G = G_\ell$ in subsection 3.1. Then $R$ is an $L_\ell \times L_\ell$ matrix, and the number of independent sets of $R_\ell^n$, $r_\ell(n)$, is the first entry of of the vector $R_\ell^n \cdot 1$ where $1 = (1,1,\ldots,1)$. Hence,

$$r_\ell(n) = (1,0,\ldots,0) \cdot R_\ell^n \cdot 1.$$  

For instance, when $\ell = 3$, the possible level vectors in an independent set are

$$(0,0,0), (0,0,1), (0,1,0), (1,0,0).$$

If we indexed the rows and columns in this order, then the transfer matrix is

$$R = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{bmatrix}.$$  

If we find the first entry of the vector $(I - xG)^{-1} \cdot 1$, we get that the generating function for $r_3(n)$ is given by $\frac{1+4x-4x^2}{1-3x-4x^2+4x^3}$. The initial values for the numbers $r_3(n)$ are

$$1, 4, 10, 28, 76, 208, 568, 1552, 4240, \ldots$$

This sequence appears as A026150 in [10].

Similarly to the case $\ell = 3$, we obtain the following results for $\ell = 4, 5, 6$.

**Theorem 3.2.** The generating functions for the numbers $r_4(n)$, $r_5(n)$ and $r_6(n)$ are given, respectively, by

$$\frac{1+4x-4x^2}{1-3x-4x^2+4x^3},$$

$$\frac{1+7x-6x^2}{1-4x-8x^2+6x^3},$$

$$\frac{1+12x-24x^2+8x^4}{(1-8x+4x^2+4x^3)(1+2x-2x^2)}.$$
We remark that the algorithm for finding the generating function for \( r_\ell(n) \) has been implemented in Maple, and yielded explicit results for \( \ell \leq 6 \).

### 3.3. An algorithm for calculating \( k_\ell(n) \)

In this section we use the transfer matrix method yet again to obtain information about the sequences \( k_\ell(n) \). This case is also similar to that of \( g_\ell(n) \), so we will only sketch it briefly.

We partition \( K^n_\ell \) into levels just as in the case of \( G^n_\ell \), so the \( n \)-th level of \( K^n_\ell \) is \( K^n_\ell \setminus K^{n-1}_\ell \). The collection of possible levels \( L_\ell \) is the set of all \( \ell \)-vectors \( v \) of 0s and 1s. Vectors \( v \) and \( w \) in \( L_\ell \) are a possible consecutive pair of levels in an independent set of \( K^n_\ell \) if they satisfy \( \text{(3)} \).

We define the transfer matrix of the problem, \( K = K_\ell \), in the same way as \( G_\ell \). Then \( K \) is a \( 2\ell \times 2\ell \) matrix, and the number of independent sets of \( K^n_\ell \), \( k_\ell(n) \), is the first entry of \( K^n \cdot (u_1, \ldots, u_{2\ell})^T \) where \( u_i = 1 \) if the \( i \)-th vector in the collection \( L_\ell \) contains at most one nonzero entry. Hence,

\[ k_\ell(n) = (1, 0, \ldots, 0) \cdot K^n \cdot (u_1, \ldots, u_{2\ell})^T. \]

For instance, when \( \ell = 3 \), the possible level vectors in an independent set are

\[ (0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1). \]

If we index the rows and columns in this order, then the transfer matrix is

\[
K = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

If we find the first entry of the vector \((I - xK)^{-1} \cdot (u_1, \ldots, u_{2\ell})^T\), we get that the generating function for \( k_3(n) \) is given by \( \frac{1}{1 - 4x + 2x^2} \). In particular, we get that \( k_3(n) = g_3(n) \) which is can also be seen directly from the definitions. Similarly, we have the following result.

**Theorem 3.3.** The generating functions for the numbers \( k_4(n) \), \( k_5(n) \) and \( k_6(n) \) are given, respectively, by

\[
\frac{1 + 2x + 3x^2}{1 - 3x - 14x^2 + 15x^3 + 7x^4}, \\
\frac{1 + x + 12x^2 - 8x^3}{1 - 5x - 30x^2 + 69x^3 + 31x^4 - 22x^5}, \\
\frac{1 - x + 38x^2 - 72x^3 - 8x^4 + 30x^5}{1 - 8x - 66x^2 + 280x^3 + 178x^4 - 532x^5 - 84x^6 + 108x^7}.
\]

We remark that the algorithm for finding the generating function for \( k_\ell(n) \) has been implemented in Maple, and yielded explicit results for \( \ell \leq 6 \).
3.4. An algorithm for calculating $p_\ell(n)$. In this section we use the transfer matrix method to obtain information about the sequences $p_\ell(n)$.

In this section we use the transfer matrix method once more to obtain information about the sequences $k_\ell(n)$. This case is also similar to that of $g_\ell(n)$, so we will only sketch it briefly.

We partition $P^n_\ell$ into levels the same way as $G^n_\ell$, so the $n$-th level of $P^n_\ell$ is $P^n_\ell \setminus P^{n-1}_\ell$. The collection of possible levels $L_\ell$ is the set of all $\ell$-vectors $v$ of 0s and 1s. Vectors $v$ and $w$ in $L_\ell$ are a possible consecutive pair of levels in an independent set of $P^n_\ell$ if they satisfy (3).

The collection of possible levels $L_\ell$ is the set of all $\ell$-vectors $v = (v_1, \ldots, v_\ell)$ of 0s and 1s such that $v_1 + \cdots + v_\ell \leq 1$. Clearly, the set $L_\ell$ contains exactly $\ell + 1$ vectors which are $(0, \ldots, 0)$ and $(0, \ldots, 0, 1, 0, \ldots, 0)$. The condition that vectors $v$ and $w$ in $L_\ell$ are a possible consecutive pair of levels in an independent set of $P^n_\ell$ is given by (3).

We define the transfer matrix of the problem, $P = P_\ell$, in the same way as $G_\ell$, $R_\ell$ and $K_\ell$. We define a matrix $P = P_\ell$, the transfer matrix of the problem, as follows. $P$ is an $(\ell + 1) \times (\ell + 1)$ matrix of 0s and 1s whose rows and columns indexed by vectors of $L_\ell$.

Therefore, it is easy to see that

$$P = \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 & 0
\end{bmatrix}_{(\ell + 1) \times (\ell + 1)}.$$

Theorem 3.4. The generating function for $p_\ell(n)$ is given by

$$\sum_{n \geq 0} p_\ell(n)x^n = \frac{1 + 2x}{1 - (\ell - 1)x - 2x^2}.$$

Proof. We want to find the first entry of the vector $(I - xP)^{-1} \cdot 1$ which means we must find the first row, say $(e_1, \ldots, e_{\ell+1})$, of the matrix $(I - xP)^{-1}$. By solving the system of equations

$$(I - xP)^{-1} \cdot (e_1, \ldots, e_{\ell+1})^T = (1, 0, \ldots, 0)^T,$$

we get that

$$e_1 = \frac{1 - (\ell - 2)x}{1 - (\ell - 1)x - 2x^2} \quad \text{and} \quad e_j = \frac{x}{1 - (\ell - 1)x - 2x^2} \quad \text{for } j \geq 2.$$

Hence, the first entry of the vector $(I - xP)^{-1} \cdot 1$ is given by

$$\sum_{i=1}^{\ell+1} e_i = \frac{1 + 2x}{1 - (\ell - 1)x - 2x^2}.$$
For instance, when $\ell = 3$, the generating function for $p_3(n)$ is given by $\frac{1+2x}{2-x-2x^2}$. One can see, in particular, that $p_3(n) = r_3(n)$, which follows directly from the definitions.

In the case $\ell = 4$ the initial values of the numbers $p_4(n)$ are

$$1, 5, 17, 61, 217, 773, 2753, 9805, 34921, 124373, \ldots$$

The same sequence turns out to appear in [6] (see [10, A007483]). Thus, the following proposition is true:

**Proposition 3.5.** The number of independent sets in the graph $P_4^n$ is equal to the number of (possibly empty) subsequences of the sequence $\{1, 2, \ldots, 2n + 1\}$ in which each odd member has an even neighbor.

Here, the neighbors of an integer $m$ are $m-1$ and $m+1$. In the case $n = 1$, the sequences appearing in the proposition are $\epsilon, 2, 23, 12, 123$, where $\epsilon$ is the empty sequence. In this case, we can find a direct bijection between the objects in Proposition 3.5 as described below.

We start by labeling the vertices of the (innermost) level $n$ of $P_4^n$ clockwise by 2, 23, 12, 123. The level $n-1$ is labeled starting from the vertex immediately to the left of vertex labeled 2 as follows: 4, 45, [3]4, [3]45 (the meaning of brackets will be discussed below). More generally, for $i < n$, given a level $n - i + 1$ labeled clockwise with $2i, 2i(2i + 1), [2i - 1]2i, [2i - 1][2i(2i + 1)]$, we label the (next outer) level $n - i$ clockwise from the inside out with $2i + 2, (2i + 2)(2i + 3), [2i + 1][2i + 2], [2i + 1][2i + 2][2i + 2][2i + 3]$ starting from the vertex immediately to the left of vertex labeled $2i$. (See Figure 5 for the case $n = 3$.) Each independent set has at most one vertex on each level. Now, given any independent set in $P_4^n$, we can write the labels of its nodes in increasing order and delete any integer $[2i - 1]$ in brackets if the sequence also contains $2i - 2$ or $2i - 1$ without brackets. Erasing all brackets now, if any, we obtain a sequence from Proposition 3.5.

**Example 3.6.** The independent sets $\{45, [5]67\}, \{4, [5]67\}, \{12, [5]6\}, \{[3]4, 67\}$ correspond to the sequences 4567, 467, 1256, 3467, respectively.
For convenience, we will write down the set of nonadjacent labels at level \( n - i \) for each label at level \( n - i + 1 \).

\[
\begin{align*}
(n \text{ no label}) \epsilon & \mapsto (2i + 2), (2i + 2)(2i + 3), [2i + 1](2i + 2), [2i + 1](2i + 2)(2i + 3) \\
2i & \mapsto [2i + 1](2i + 2), [2i + 1](2i + 2)(2i + 3) \\
2i(2i + 1) & \mapsto (2i + 2), [2i + 1](2i + 2)(2i + 3) \\
[2i - 1]2i & \mapsto (2i + 2), (2i + 2)(2i + 3) \\
[2i - 1]2i(2i + 1) & \mapsto (2i + 2)(2i + 3), [2i + 1](2i + 2)
\end{align*}
\]

Now it is not difficult to construct an independent set given a sequence of Proposition 3.5. We partition the sequence of integers from 1 to \( 2n + 1 \) as follows:

\[
123 \mid 45 \mid 67 \ldots \mid 2n(2n + 1),
\]

then choose the vertices of the independent set in the order of increasing labels using the rules (4). Notice that the label of the vertex at level \( n - i + 1 \) must contain \( 2i \).

**Example 3.7.**

\[
\begin{align*}
4567 & \mapsto 123\underline{45}67 \mapsto (\epsilon, 45, [5]67) \mapsto \{45, [5]67\} \\
467 & \mapsto 123\underline{45}67 \mapsto (\epsilon, 4, [5]67) \mapsto \{45, [5]67\} \\
1256 & \mapsto 12\underline{34}567 \mapsto (12, \epsilon, [5]6) \mapsto \{12, [5]6\}
\end{align*}
\]

It can be shown that the two maps described above are inverses of each other based on the recursive structure of sequences under consideration.

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