GABOR FRAMES: CHARACTERIZATIONS AND COARSE STRUCTURE

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Abstract. This chapter offers a systematic and streamlined exposition of the most important characterizations of Gabor frames over a lattice.

1. Introduction

Given a point \( z = (x, \xi) \in \mathbb{R}^d \) in time-frequency space (phase space), we define the corresponding time-frequency shift \( \pi(z) \) acting on a function \( f \in L^2(\mathbb{R}^d) \) by

\[
\pi(z)f(t) = e^{2\pi i \xi \cdot t} f(t - x).
\]

Gabor analysis deals with the spanning properties of sets of time-frequency shifts. Specifically, for a window function \( g \in L^2(\mathbb{R}^d) \) and a discrete set \( \Lambda \subseteq \mathbb{R}^{2d} \), which we will always assume to be a lattice, we would like to understand when the set

\[
\mathcal{G}(g, \Lambda) = \{ \pi(\lambda)g : \lambda \in \Lambda \}
\]

is a frame. This means that there exist positive constants \( A, B > 0 \) such that

\[
A \| f \|_{L^2}^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B \| f \|_{L^2}^2 \quad \forall f \in L^2(\mathbb{R}^d).
\]

For historical reasons a frame with this structure is called a Gabor frame, or sometimes a Weyl-Heisenberg frame.

The motivation for studying sets of time-frequency shifts is in the foundations of quantum mechanics by J. von Neumann [35] and in information theory by D. Gabor [17]. Since 1980 the investigation of Gabor frames has stimulated the interest of many mathematicians in harmonic, complex, and numerical analysis and engineers in signal processing and wireless communications.

Whereas (1) expresses a strong form of completeness (with stability built in the definition), a complementary concept is the linear independence of time-frequency shifts. Specifically, we ask for constants \( A, B > 0 \) such that

\[
A \| c \|_{\ell^2}^2 \leq \left\| \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g \right\|_{L^2}^2 \leq B \| c \|_{\ell^2}^2 \quad \forall c \in \ell^2(\Lambda),
\]

and in this case \( \mathcal{G}(g, \Lambda) \) is called a Riesz sequence in \( L^2(\mathbb{R}^d) \). Riesz sequences are important in wireless communications: a data set \( (c_\lambda)_{\lambda \in \Lambda} \) is transformed into an analog signal \( f = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g \) and then transmitted. The task at the receiver is to decode the data \( (c_\lambda) \). In this context (2) expresses the fact that the coefficients \( c_\lambda \) are uniquely determined by \( f \) and that their recovery is feasible in a robust way.
In this chapter we restrict our attention to sets of time-frequency shifts over a lattice \( \Lambda \), i.e., \( \Lambda = A \mathbb{Z}^{2d} \) for an invertible, real-valued \( 2d \times 2d \) matrix \( A \). The lattice structure implies the translation invariance \( \pi(\lambda)G(g,\Lambda) = G(g,\Lambda) \) (up to phase factors) and is at the basis of a beautiful and deep structure theory and many characterizations of \( 1 \) and \( 2 \).

After three decades we have a clear understanding of the structures governing Gabor systems. Our goal is to collect the most important characterizations of Gabor frames and offer a systematic exposition of these structures. In the center of these characterizations is the duality theorem for Gabor frames. To our knowledge all other characterizations within the \( L^2 \)-theory follow directly from this fundamental duality. In particular, the celebrated characterizations of Janssen and Ron-Shen are consequences of the duality theorem, and the characterization of Zeevi and Zibulski for rational lattices also becomes a corollary.

Even with this impressive list of different criteria at our disposal, it remains very difficult to determine whether a given window function and lattice generate a Gabor frame. Ultimately, each criterion (within the \( L^2 \)-theory) is formulated by means of the invertibility of some operator, and proving invertibility is always difficult. This fact explains perhaps why there are so many general results about Gabor frames, but so few explicit results about concrete Gabor frames.

Yet, there are some success stories due to Lyubarski [34], Seip [38], Janssen [30, 31], and some recent progress for totally positive windows [23, 24]. All these results have applied some of the characterizations presented here, or even invented some new ones. On the other hand, most questions about concrete Gabor systems remain unanswered, and so far every explicit conjecture about Gabor frames (with one exception) has been disproved by counter-examples.

To document some of the many white spots on the map of Gabor frames, let us mention two specific examples. (i) Let \( g_1(t) = (1 - |t|)_+ \) be the hat function (or \( B \)-spline of order 1). It is known that for all \( \alpha > 0 \) the Gabor system \( \mathcal{G}(g_1, \alpha \mathbb{Z} \times 2\mathbb{Z}) \) is not a Gabor frame. But it is not known whether \( \mathcal{G}(g_1, 0.33 \mathbb{Z} \times 2.001 \mathbb{Z}) \) is a frame. (ii) Let \( h_1(t) = te^{-\pi t^2} \) be the first Hermite function. It is known that \( \mathcal{G}(h_1, \alpha \mathbb{Z} \times \beta \mathbb{Z}) \) is not a Gabor frame whenever \( \alpha \beta = 2/3 \) [33]. But it is not known whether \( \mathcal{G}(h_1, \mathbb{Z} \times 0.666666 \mathbb{Z}) \) is a frame. In both cases, there is numerical evidence that these Gabor systems are frames, but so far there is no proof despite an abundance of precise criteria to check.

The novelty of our approach is the streamlined sequence of proofs, so that most of the structure theory of Gabor frames fits into a single, short chapter. In view of dozens of efforts on every aspect of Gabor analysis, we hope that this survey will be useful and inspire work on concrete open questions. The only prerequisite is the thorough mastery of the Poisson summation formula and some basic facts about frames and Riesz sequences.

The chapter is organized as follows: Section 2 covers the main objects of Gabor analysis. Section 3 is devoted to the interplay between the short-time Fourier transform, the Poisson summation formula, and commutativity of time-frequency shifts. The central Section 4 offers a complete proof of the duality theorem for
Gabor frames. Section 5 sketches the main theorems about the coarse structure of Gabor frames. A list of criteria that are tailored to rectangular frames is discussed and proved in Section 6. In Section 7 we derive the criterium of Zeevi and Zibulski for rational lattices, and Section 8 presents a number of (technically more advanced) criteria some of which have recently become useful. Except for the last section, we fully prove all statements.

2. The Objects of Gabor Analysis

Let \( g \in L^2(\mathbb{R}^d) \) be a non-zero window function and \( \Lambda \subseteq \mathbb{R}^{2d} \) a lattice. The set \( G(g, \Lambda) \) is called a Gabor frame if there exist positive constants \( A, B > 0 \) such that

\[
A \| f \|^2_{L^2} \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B \| f \|^2_{L^2}, \quad \forall f \in L^2(\mathbb{R}^d).
\]  

The frame inequality (3) can be recast by means of functional analytic properties of certain operators associated to a Gabor system \( G(g, \Lambda) \). We will use the frame operator \( S = S_{g, \Lambda} \) defined by

\[
S_{g, \Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g.
\]

Then \( G(g, \Lambda) \) is a frame if and only if \( S_{g, \Lambda} \) is bounded and invertible on \( L^2(\mathbb{R}^d) \). The extremal spectral values \( A, B \) are called the frame bounds. If they can be chosen to be equal \( A = B \), then the frame operator is a multiple of the identity, and \( G(g, \Lambda) \) is called a tight frame.

We will also use the Gramian operator \( G = G_{g, \Lambda} \) defined by its entries

\[
G_{\lambda \mu} = \langle \pi(\mu)g, \pi(\lambda)g \rangle.
\]

In this notation, \( G(g, \Lambda) \) is a Riesz sequence if and only if \( G_{g, \Lambda} \) is bounded and invertible on \( \ell^2(\Lambda) \).

If the upper inequality in (3) is satisfied, then the frame operator is well-defined and bounded on \( L^2(\mathbb{R}^d) \) and the Gramian operator is bounded on \( \ell^2(\Lambda) \). In this case, we call \( G(g, \Lambda) \) a Bessel sequence.

The underlying object of this definition is the short-time Fourier transform of \( f \) with respect to the window function \( g \in L^2(\mathbb{R}^d) \), which is defined by

\[
V_g f(z) = V_g f(x, \xi) = \int_{\mathbb{R}^d} f(t) \bar{g}(t - x) e^{-2\pi i \xi t} \, dt.
\]

We will need the following properties of the short-time Fourier transform.

Lemma 2.1 (Covariance property). Let \( f, g \in L^2(\mathbb{R}^d) \) and \( w, z \in \mathbb{R}^{2d} \). Then

\[
(4) \quad V_g(\pi(w)f)(z) = e^{-2\pi i (z - w) \cdot w_1} V_g f(z - w) \quad \text{and}
\]

\[
(5) \quad V_{\pi(w)g}(\pi(w)f)(z) = e^{2\pi i z I_{2d}^T w} V_g f(z),
\]

where \( I_d = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \) denotes the standard symplectic matrix and \( I_d \) is the \( d \)-dimensional identity matrix.

Proposition 2.2 (Orthogonality relations). Let \( f, g, h, \gamma \in L^2(\mathbb{R}^d) \).
(i) Then $V_g f, V_\gamma h \in L^2(\mathbb{R}^{2d})$ and

$$\langle V_g f, V_\gamma h \rangle_{L^2(\mathbb{R}^{2d})} = \langle f, h \rangle_{L^2(\mathbb{R}^d)}\overline{\langle g, \gamma \rangle}_{L^2(\mathbb{R}^d)}.$$ 

In particular, if $\|h\|_{L^2} = 1$, then $V_h$ is an isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$.

(ii) Furthermore, for all $z \in \mathbb{R}^{2d}$,

$$
(V_g f \cdot V_\gamma h) \hat{\gamma}(z) = (V_g f \cdot V_0 f)(Iz).
$$

**Proof.** (i) The orthogonality relations (6) are a well established fact from representation theory. For a direct proof using only Plancherel’s theorem we refer to the textbooks [8, 18].

(ii) By the Cauchy-Schwarz inequality the product $V_g f \cdot V_\gamma h$ is in $L^1(\mathbb{R}^{2d})$, therefore the Fourier transform is defined pointwise, and we obtain

$$
(V_g f \cdot V_\gamma h) \hat{\gamma}(z) = \int_{\mathbb{R}^{2d}} V_g f(w) \cdot V_\gamma h(w) e^{-2\pi i w \cdot z} dw
$$

$$
= \int_{\mathbb{R}^{2d}} V_{\pi(Iz)g}(\pi(Iz)f)(w) \cdot V_\gamma h(w) dw
$$

$$
= \langle \gamma, \pi(Iz)g \rangle \langle h, \pi(Iz)f \rangle,
$$

where we first used $I^2 = -I_{2d}$ and the covariance property (5), then the orthogonality relations (6) to separate the integral into two inner products. \(\square\)

### 3. Commutation Rules and the Poisson Summation Formula in Gabor Analysis

In this section we exploit the invariance properties of a Gabor system for the structural interplay between the short-time Fourier transform and time-frequency lattices.

#### 3.1. Poisson Summation Formula.

If $\Lambda$ is a lattice, then the function $\Phi(z) = \sum_{\lambda \in \Lambda} |\langle f, \pi(z + \lambda)g \rangle|^2$ satisfies $\Phi(z + \nu) = \Phi(z)$ for $\nu \in \Lambda$ and thus is periodic with respect to $\Lambda$. It is therefore natural to study the Fourier series of $\Phi$. The mathematical tool is the Poisson summation formula, and this is in fact the mathematical core of all existing characterizations of Gabor frames over a lattice (though the terminology is often a bit different, e.g., fiberization technique in [37]).

We formulate the Poisson summation formula explicitly for an arbitrary lattice $\Lambda = AZ^{2d}$ where $A$ denotes an invertible, real-valued $2d \times 2d$ matrix. We write $\Lambda^\perp = (A^T)^{-1}Z^{2d}$ for the dual lattice and $\Lambda^0 = I\Lambda^\perp$ for the adjoint lattice with $I = (\begin{smallmatrix} 0 & I_d \\ -I_d & 0 \end{smallmatrix})$.

The volume of the lattice is $\text{vol} \Lambda = |\det(A)|$, and the reciprocal value $D(\Lambda) = \text{vol}(\Lambda)^{-1}$ is the density or redundancy of $\Lambda$.

We first formulate a sufficiently general version of the Poisson summation formula [39].

**Lemma 3.1.** Assume that $\Lambda = AZ^{2d}$ and $F \in L^1(\mathbb{R}^{2d})$. Then the periodization $\Phi(x) = \sum_{\lambda \in \Lambda} F(x - \lambda)$ is in $L^1(\mathbb{R}^{2d}/\Lambda)$.

(i) The Fourier coefficients of $\Phi$ are given by $\hat{\Phi}(\nu) = \hat{F}(\nu)$ for all $\nu \in \Lambda^\perp$. 
(ii) Poisson summation formula – general version: Let \( K_n \) be a summability kernel\(^1\), then

\[
\sum_{\lambda \in \Lambda} F(z + \lambda) = \text{vol}(\Lambda)^{-1} \lim_{n \to \infty} \sum_{\nu \in \Lambda^\perp} K_n(\nu) \hat{F}(\nu) e^{2\pi i \nu \cdot z}.
\]

with convergence in \( L^1(\mathbb{R}^d/\Lambda) \).

(iii) If \( (\hat{F}(\nu))_{\nu \in \Lambda^\perp} \in \ell^1(\Lambda^\perp) \), then the Fourier series converges absolutely and \( \Phi \) coincides almost everywhere with a continuous function.

By applying the Poisson summation formula to the function \( V_g f \cdot \overline{\gamma h} \) and a lattice \( \Lambda \), we obtain an important identity for the analysis of Gabor frames. This technique is so ubiquitous in Gabor analysis, that Janssen \cite{29} and later Feichtinger and Luef \cite{16} called it the “Fundamental Identity of Gabor Analysis”.

**Theorem 3.2.** Let \( f, g, h, \gamma \in L^2(\mathbb{R}^d) \), and \( \Lambda = \mathbb{A} \mathbb{Z}^d \) be a lattice.

(i) Then

\[
\sum_{\lambda \in \Lambda} V_g f(z + \lambda) \overline{V_\gamma h(z + \lambda)} = \text{vol}(\Lambda)^{-1} \lim_{n \to \infty} \sum_{\mu \in \Lambda^\circ} K_n(-\mathcal{I} \mu) V_g \gamma(\mu) \overline{V_f h(\mu)} e^{2\pi i \mu \cdot z}
\]

with convergence in \( L^1(\mathbb{R}^d/\Lambda) \).

(ii) Assume in addition that both \( \mathcal{G}(g, \Lambda) \) and \( \mathcal{G}(\gamma, \Lambda) \) are Bessel sequences and that \( \sum_{\mu \in \Lambda^\circ} |V_g \gamma(\mu)| < \infty \). Then

\[
\sum_{\lambda \in \Lambda} V_g f(z + \lambda) \overline{V_\gamma h(z + \lambda)} = \text{vol}(\Lambda)^{-1} \sum_{\mu \in \Lambda^\circ} V_g \gamma(\mu) \overline{V_f h(\mu)} e^{2\pi i \mu \cdot z} \quad \forall z \in \mathbb{R}^d.
\]

**Proof.** (i) We apply the Poisson summation formula to the product \( V_g f \cdot \overline{\gamma h} \) and the lattice \( \Lambda \) and obtain

\[
\sum_{\lambda \in \Lambda} V_g f(z + \lambda) \overline{V_\gamma h(z + \lambda)} = \text{vol}(\Lambda)^{-1} \lim_{n \to \infty} \sum_{\nu \in \Lambda^\perp} K_n(\nu) (V_g f \cdot \overline{\gamma h})^\circ(\nu) e^{2\pi i \nu \cdot z}
\]

\[
= \text{vol}(\Lambda)^{-1} \lim_{n \to \infty} \sum_{\nu \in \Lambda^\perp} K_n(\nu) (V_g \gamma \cdot \overline{\gamma h})(\mathcal{I} \nu) e^{2\pi i \nu \cdot z}
\]

\[
= \text{vol}(\Lambda)^{-1} \lim_{n \to \infty} \sum_{\mu \in \Lambda^\circ} K_n(-\mathcal{I} \mu) V_g \gamma(\mu) \overline{V_f h(\mu)} e^{2\pi i \mu \cdot z},
\]

where we used Proposition 2.2 to rewrite the Fourier transform in the first line.

(ii) If \( \sum_{\mu \in \Lambda^\circ} |V_g \gamma(\mu)| < \infty \), then the right-hand side of (9) converges absolutely to a continuous function, and we do not need the summability kernel. Next we rewrite the left-hand side with the help of identity (4) as

\[
\Phi(z) = \sum_{\lambda \in \Lambda} V_g(\pi(-z)f)(\lambda) \overline{V_\gamma(\pi(-z)h)(\lambda)}.
\]

\(^1\)It suffices to take the Fourier coefficients of the multivariate Fejer kernel \( \hat{F}_n(k) = \prod_{j=1}^d (1 - |k_j|/\pi + 1) \) and set \( K_n(\nu) = \hat{F}_n(\mathcal{A}^T \nu) = \hat{F}_n(k) \) for \( \nu = (\mathcal{A}^T)^{-1} k \in \Lambda^\perp \).
where, as so often, the phase factors cancel. Since \( G(g, \Lambda) \) is a Bessel sequence with Bessel bound \( B_g \), we know that
\[
\|V_g(\pi(-z)f - f)|_\Lambda\|_{L^2} \leq B_g^{1/2}\|\pi(-z)f - f\|_{L^2}.
\]
This means that the map \( z \mapsto V_g(\pi(-z)f)|_\Lambda \) is continuous from \( \mathbb{R} \) to \( \ell^2(\Lambda) \) for all \( f \in L^2(\mathbb{R}^d) \). Likewise, the map \( z \mapsto V_g(\pi(-z)h)|_\Lambda \) is continuous for all \( h \in L^2(\mathbb{R}^d) \).

This observation implies that the left-hand side is also a continuous function. Thus both sides of (9) are continuous and coincide almost everywhere, therefore (9) must hold everywhere. \( \square \)

3.2. **Commutation Rules.** In the fundamental identity (8) the adjoint lattice \( \Lambda^\circ \) appears as a consequence of the Poisson summation formula. We now present a more structural property of the adjoint lattice.

**Lemma 3.3.** Let \( \Lambda \subseteq \mathbb{R}^{2d} \) be a lattice. Then its adjoint lattice is characterized by the property
\[
\Lambda^\circ = \{ \mu \in \mathbb{R}^{2d} : \pi(\lambda)\pi(\mu) = \pi(\mu)\pi(\lambda) \quad \forall \lambda \in \Lambda \}.
\]

**Proof.** Let \( z \in \mathbb{R}^{2d} \) and \( \lambda = Ak \in \Lambda \) for some \( k \in \mathbb{Z}^{2d} \). A straight forward computation yields \( \pi(z)\pi(\lambda) = e^{2\pi i(\Lambda_1 \cdot z_2 - \Lambda_2 \cdot z_1)} \pi(\lambda)\pi(z) \). Consequently, the time-frequency shifts commute if and only if
\[
1 = e^{2\pi i(\lambda_1 \cdot z_2 - \lambda_2 \cdot z_1)} = e^{2\pi i\lambda \cdot Iz} = e^{2\pi iAk \cdot Iz}.
\]
This holds for all \( k \in \mathbb{Z}^{2d} \) if and only if \( Ak \cdot Iz = k \cdot A^Tz \in \mathbb{Z} \) for all \( k \in \mathbb{Z}^{2d} \), which is precisely the case when \( A^Tz \in \mathbb{Z}^{2d} \), or equivalently when \( z \in T^{-1}(A^T)^{-1}\mathbb{Z}^{2d} = \Lambda^\circ \).

This interpretation of the adjoint lattice is crucial for an important technical point.

**Lemma 3.4** (Bessel duality). Let \( g \in L^2(\mathbb{R}^d) \) and \( \Lambda \subseteq \mathbb{R}^{2d} \) be a lattice. Then \( G(g, \Lambda) \) is a Bessel sequence if and only if \( G(g, \Lambda^\circ) \) is a Bessel sequence.

**Proof.** The proof is inspired by [10].

Fix \( h \in \mathcal{S}(\mathbb{R}^d) \) with \( \|h\|_{L^2} = 1 \). Then \( G(h, M) \) is a Bessel sequence for every lattice \( M \subseteq \mathbb{R}^{2d} \). Next let \( Q = A[0, 1)^{2d} \) be a fundamental domain of \( \Lambda = AZ^{2d} \), i.e., \( \mathbb{R}^{2d} = \bigcup_{\lambda \in \Lambda} \lambda + Q \) as a disjoint union. As a consequence we may write \( \int_{\mathbb{R}^{2d}} f(z) \, dz = \int_Q \sum_{\lambda \in \Lambda} f(z - \lambda) \, dz \) for \( f \in L^1(\mathbb{R}^{2d}) \).

Now assume that \( G(g, \Lambda) \) is a Bessel sequence. Let \( c = (c_\mu)_{\mu \in \Lambda^\circ} \in \ell^2(\Lambda^\circ) \) be a finite sequence and \( f = \sum_{\mu \in \Lambda^\circ} c_\mu \pi(\mu)g \). Since \( V_h : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d}) \) is an isometry by Proposition 2.2, we obtain
\[
\|f\|_{L^2(\mathbb{R}^{2d})}^2 = \|V_h f\|_{L^2(\mathbb{R}^{2d})}^2 = \int_Q \sum_{\lambda \in \Lambda^\circ} \left( \sum_{\mu \in \Lambda^\circ} c_\mu \pi(\mu)g, \pi(-\lambda + z)h \right)^2 \, dz = \int_Q I(z) \, dz.
\]

We now reorganize the sum over \( \mu \). First we use \( \pi(-\lambda + z) = \gamma_{z,\lambda} \pi(\lambda)^* \pi(z) \) for some phase factor \( |\gamma_{z,\lambda}| = 1 \). Then we use the commutativity \( \pi(\lambda)\pi(\mu) = \pi(\mu)\pi(\lambda) \)
for all $\lambda, \mu \in \Lambda^\circ$ (Lemma 3.3). This is the heart of the proof, and the reader should convince herself that the proof does not work without this property. We obtain

$$\langle \sum_{\mu \in \Lambda^\circ} c_\mu \pi(\mu) g, \pi(-\lambda + z) h \rangle = \bar{\gamma}_{\lambda, z} \langle \pi(\lambda) g, \sum_{\mu \in \Lambda^\circ} \bar{e}_\mu \pi(\mu)^* \pi(z) h \rangle = \bar{\gamma}_{\lambda, z} \langle \pi(\lambda) g, \pi(z) \sum_{\mu \in \Lambda^\circ} \bar{c}_- \gamma_{\mu, z} \pi(\mu) h \rangle.$$ 

Since both Gabor families $G(g, \Lambda)$ and $G(h, \Lambda^\circ)$ are Bessel sequences by assumption (with constants $B_g$ and $B_h$), we obtain a pointwise estimate for the integrand $I(z)$ in (10):

$$I(z) = \left| \sum_{\lambda \in \Lambda} \bar{\gamma}_{\lambda, z} \langle \pi(z) \sum_{\mu \in \Lambda^\circ} \bar{c}_- \gamma_{\mu, z} \pi(\mu) h, \pi(\lambda) g \rangle \right|^2 \leq B_g \left\| \pi(z) \sum_{\mu \in \Lambda^\circ} \bar{c}_- \gamma_{\mu, z} \pi(\mu) h \right\|_{L^2(\mathbb{R}^d)}^2 \leq B_g B_h \left\| \bar{c} \right\|_{\ell^2}^2,$$

Integration over $z$ now yields

$$\left\| \sum_{\mu \in \Lambda^\circ} c_\mu \pi(\mu) g \right\|_{L^2}^2 = \int_Q I(z) \, dz \leq B_g B_h \left\| \bar{c} \right\|_{\ell^2}^2,$$

and thus $G(g, \Lambda^\circ)$ is a Bessel sequence.

Since $\Lambda = (\Lambda^\circ)^\circ$, the proof of the converse is the same.

4. Duality Theory

The duality theory relates the spanning properties of a Gabor family $G(g, \Lambda)$ on a lattice $\Lambda$ to the spanning properties of $G(g, \Lambda^\circ)$ over the adjoint lattice. The following duality theorem is the central result of the theory of Gabor frames. We will see that most structural results about Gabor frames can be derived easily from it.

**Theorem 4.1** (Duality theorem). Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then the following are equivalent:

(i) $G(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$.

(ii) $G(g, \Lambda^\circ)$ is a Bessel sequence and there exists a dual window $\gamma \in L^2(\mathbb{R}^d)$ such that $G(\gamma, \Lambda^\circ)$ is a Bessel sequence satisfying

$$\langle \gamma, \pi(\mu) g \rangle = \text{vol}(\Lambda) \delta_{\mu, 0} \quad \forall \mu \in \Lambda^\circ.$$

(iii) $G(g, \Lambda^\circ)$ is a Riesz sequence for $L^2(\mathbb{R}^d)$.

We follow the proof sketch given in the survey article [20].
Proof. (i) \( \Rightarrow \) (ii): Since \( G(g, \Lambda) \) is a Gabor frame, there exists a dual window \( \gamma \) in \( L^2(\mathbb{R}^d) \) such that \( G(\gamma, \Lambda) \) is a frame and the reconstruction formula

\[
    f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)\gamma
\]

holds for all \( f \in L^2(\mathbb{R}^d) \) with unconditional \( L^2 \)-convergence. We apply the reconstruction formula to \( \pi(z)^*f \) and take the inner product with \( \pi(z)^*h \) for \( z \in \mathbb{R}^{2d} \) and \( h \in L^2(\mathbb{R}^d) \). Then we have

\[
    \langle f, h \rangle = \langle \pi(z)^*f, \pi(z)^*h \rangle = \sum_{\lambda \in \Lambda} \langle \pi(z)^*f, \pi(\lambda)g \rangle \langle \pi(\lambda)\gamma, \pi(z)^*h \rangle
\]

\[
    = \sum_{\lambda \in \Lambda} V_g f(z + \lambda) V_\gamma h(z + \lambda) = \Phi(z)
\]

for all \( f, h \in L^2(\mathbb{R}^d) \) and all \( z \in \mathbb{R}^{2d} \). This means that the \( \Lambda \)-periodic function \( \Phi \) on the right-hand side is constant.

By Proposition 2.2 (ii) the Fourier coefficients of the right-hand side are given by

\[
    \hat{\Phi}(\nu) = (V_g f \cdot V_\gamma h)^\gamma(\nu) = V_\gamma(\mu) V_f h(\mu),
\]

where \( \nu \in \Lambda^\perp \) and \( \mu = T\nu \in \Lambda^\circ \). Since these are the Fourier coefficients of a constant function, they must satisfy

\[
    \text{vol}(\Lambda)^{-1} V_\gamma(\mu) V_f h(\mu) = \langle f, h \rangle \delta_{\mu,0} \quad \forall \mu \in \Lambda^\circ.
\]

As this identity holds for all \( f, h \in L^2(\mathbb{R}^d) \), we obtain the biorthogonality relation

\[
    \text{vol}(\Lambda)^{-1} \langle \gamma, \pi(\mu)g \rangle = \delta_{\mu,0} \quad \forall \mu \in \Lambda^\circ.
\]

By assumption both \( G(g, \Lambda) \) and \( G(\gamma, \Lambda) \) are frames and thus Bessel sequences, therefore Lemma 3.4 implies that both \( G(g, \Lambda^\circ) \) and \( G(\gamma, \Lambda^\circ) \) are Bessel sequences.

(ii) \( \Rightarrow \) (i): We use the biorthogonality and read the fundamental identity (9) backwards:

\[
    \text{vol}(\Lambda)^{-1} \sum_{\mu \in \Lambda^\circ} V_\gamma(\mu) V_f h(\mu) e^{2\pi i \mu Tz} = \sum_{\lambda \in \Lambda} V_g f(z + \lambda) V_\gamma h(z + \lambda).
\]

Since both \( G(g, \Lambda^\circ) \) and \( G(\gamma, \Lambda^\circ) \) are Bessel sequences, Lemma 3.4 implies that the Gabor systems \( G(g, \Lambda) \) and \( G(\gamma, \Lambda) \) are also Bessel sequences. Furthermore \( \sum_{\mu \in \Lambda^\circ} |V_\gamma(\mu)| < \infty \) by the biorthogonality relation (11), hence all assumptions of Theorem 3.2 are satisfied and guarantee that (9) holds pointwise. For \( z = 0 \) and \( f = h \) we thus obtain

\[
    \|f\|^2_{L^2} = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \langle \pi(\lambda)\gamma, f \rangle.
\]

Since both sets \( G(g, \Lambda) \) and \( G(\gamma, \Lambda) \) are Bessel sequences with Bessel bounds \( B_g \) and \( B_\gamma \) respectively, the frame inequality for \( G(g, \Lambda) \) is obtained as follows:

\[
    \|f\|^4_{L^2} \leq \left( \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \right) \left( \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)\gamma \rangle|^2 \right)
\]
Hence, the family\( \mathcal{G}(g, \Lambda^0) \) and \( \mathcal{G}(\gamma, \Lambda^0) \) satisfy the biorthogonality condition (11), thus
\[
(\gamma, \Lambda^0) \ni \chi \mapsto \left\langle \chi, \nu \right\rangle = e^{-2\pi i (\mu - \nu) \cdot \nu} = e^{-2\pi i \nu_1} \quad \forall \nu, \mu \in \Lambda^0.
\]
We remark that the duality theory also holds verbatim for general locally compact Abelian groups admitting a lattice.

(ii) \( \Rightarrow \) (iii): By assumption, the Bessel sequences \( \mathcal{G}(g, \Lambda^0) \) and \( \mathcal{G}(\gamma, \Lambda^0) \) satisfy the biorthogonality relations (11) and is a Riesz sequence.

(iii) \( \Rightarrow \) (ii): By assumption \( \mathcal{G}(g, \Lambda^0) \) is a Riesz sequence, i.e., a Riesz basis for its closed linear span, which we denote by \( \mathcal{K} := \text{span}\{\mathcal{G}(g, \Lambda^0)\} \). By the general properties of Riesz bases [5], there exists a Bessel sequence \( \{e_\nu : \nu \in \Lambda^0\} \) in \( \mathcal{K} \) such that
\[
\langle e_\nu, \pi(\mu) g \rangle = \delta_{\nu, \mu} \quad \forall \nu, \mu \in \Lambda^0.
\]
On the other hand, since \( \mathcal{K} \) is invariant with respect to \( \pi(\nu) \) for all \( \nu \in \Lambda^0 \), we have that \( \pi(\nu) e_0 \) is also in \( \mathcal{K} \) and satisfies the biorthogonality
\[
\langle \pi(\nu) e_0, \pi(\mu) g \rangle = e^{-2\pi i (\mu - \nu) \cdot \nu} \langle e_0, \pi(\mu - \nu) g \rangle = \delta_{0, \mu - \nu}.
\]
This implies that \( e_\nu - \pi(\nu) e_0 \in \mathcal{K} \cap \mathcal{K}^\perp = \{0\} \).

After the normalization \( \gamma := \text{vol}(\Lambda)^{-1} e_0 \), the set \( \mathcal{G}(\gamma, \Lambda^0) = \{\text{vol}(\Lambda)^{-1} e_\nu : \nu \in \Lambda^0\} \) satisfies the biorthogonality relations (11) and is a Bessel sequence by the properties of Riesz bases.

The duality theory was foreshadowed by Rieffel’s abstract work on non-commutative tori [36]. The biorthogonality relations (11) were discovered by the engineers Wexler and Raz [41] and characterize all possible dual windows (see Corollary 4.4). Janssen [27, 28], Daubechies et al. [10] and Ron-Shen [37] made the results of Wexler and Raz rigorous and further expanded upon them which became the duality theory for separable lattices. The theory for general lattices is due to Feichtinger and Kozek [15]. Recently, Jakobsen and Lemvig [26] formulated density and duality theorems for Gabor frames along a closed subgroup of the time-frequency plane. We remark that the duality theory also holds verbatim for general locally compact Abelian groups admitting a lattice.

Remark 4.2 (Frame bounds and an alternative proof). By rewriting Janssen’s proof of the duality theory in [28] for general lattices, one can show that
\[
AI_{L^2} \leq S_{g, \Lambda} \leq BI_{L^2} \quad \iff \quad AI_{L^2} \leq \text{vol}(\Lambda)^{-1} G_{g, \Lambda^0} \leq BI_{L^2}.
\]
Hence, the family \( \mathcal{G}(g, \Lambda) \) is a frame with frame bounds \( A, B > 0 \) if and only if \( \mathcal{G}(g, \Lambda^0) \) is a Riesz sequence with bounds \( \text{vol}(\Lambda) A, \text{vol}(\Lambda) B > 0 \) respectively.

Definition 4.3. Let \( g \in L^2(\mathbb{R}^d) \) and \( \mathcal{G}(g, \Lambda) \) be a Bessel sequence. We call \( \gamma \in L^2(\mathbb{R}^d) \) a dual window for \( \mathcal{G}(g, \Lambda) \) if \( \mathcal{G}(\gamma, \Lambda) \) is a Bessel sequence and the reconstruction property
\[
f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) g = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) \gamma \rangle \pi(\lambda) g
\]
holds for all \( f \in L^2(\mathbb{R}^d) \).
The duality theorem now yields the following characterization of all dual windows.

**Corollary 4.4.** Suppose \( g, \gamma \in L^2(\mathbb{R}^d) \) and \( \Lambda \subseteq \mathbb{R}^{2d} \) such that \( G(g, \Lambda) \) and \( G(\gamma, \Lambda) \) are Bessel sequences. Then \( \gamma \) is a dual window for \( G(g, \Lambda) \) if and only if the Wexler-Raz biorthogonality relations (11) are satisfied.

**Proof.** This is simply equivalence (i) \( \Leftrightarrow \) (ii) of Theorem 4.1. \( \square \)

We conclude this section with a characterization of tight Gabor frames.

**Corollary 4.5.** A Gabor system \( G(g, \Lambda) \) is a tight frame if and only if \( G(g, \Lambda^o) \) is an orthogonal system. In this case the frame bound satisfies \( A = \text{vol}(\Lambda)^{-1} \|g\|_{L^2}^2 \).

**Proof.** If \( G(g, \Lambda) \) is a tight frame, then the frame operator is just a multiple of the identity, i.e., \( S = A I_{L^2} \). Hence the canonical dual window is of the form \( \gamma = S^{-1}g = \frac{1}{A}g \) and the biorthogonality relations (11) yield

\[
\langle g, \pi(\mu)g \rangle = A \langle \gamma, \pi(\mu)g \rangle = A \text{vol}(\Lambda) \delta_{\mu,0} \quad \forall \mu \in \Lambda^o.
\]

Therefore, \( G(g, \Lambda^o) \) is an orthogonal system and in particular \( A = \text{vol}(\Lambda)^{-1} \|g\|_{L^2}^2 \).

Conversely, let \( G(g, \Lambda^o) \) be an orthogonal system, i.e.,

\[
\langle g, \pi(\mu)g \rangle = \|g\|_{L^2}^2 \delta_{\mu,0} \quad \forall \mu \in \Lambda^o.
\]

Then by Theorem 3.2 with \( \gamma = g, h = f \) and \( z = 0 \), we obtain

\[
\text{vol}(\Lambda)^{-1} \|g\|_{L^2}^2 \|f\|_{L^2}^2 = \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2
\]

and thus \( G(g, \Lambda) \) is a tight frame. \( \square \)

5. The Coarse Structure of Gabor Frames

Many of the fundamental properties of Gabor frames can be derived with little effort from the duality theorem. In the following we deal with the density theorem, the Balian-Low theorem, and the existence of Gabor frames.

5.1. Density Theorem. To recover \( f \) from the inner products \( \langle f, \pi(\lambda)g \rangle \), we need enough information. The density theorem quantifies this statement. The density theorem has a long history and has been proved many times. We refer to Heil’s comprehensive survey [25]. Our point is that it follows immediately from the duality theory.

**Theorem 5.1** (Density theorem). Let \( g \in L^2(\mathbb{R}^d) \) and \( \Lambda \subseteq \mathbb{R}^{2d} \) be a lattice. Then the following holds:

(i) If \( G(g, \Lambda) \) is a frame for \( L^2(\mathbb{R}^d) \), then \( 0 < \text{vol}(\Lambda) \leq 1 \).

(ii) If \( G(g, \Lambda) \) is a Riesz sequence in \( L^2(\mathbb{R}^d) \), then \( \text{vol}(\Lambda) \geq 1 \).

(iii) \( G(g, \Lambda) \) is a Riesz basis for \( L^2(\mathbb{R}^d) \) if and only if it is a frame and \( \text{vol}(\Lambda) = 1 \).
Proof. (i) Let $\gamma = S^{-1}g$ be the canonical dual window of $G(g, \Lambda)$. Then $g$ possesses the following two distinguished representations with respect to the frame $G(g, \Lambda)$:

$$g = 1 \cdot g = \sum_{\lambda \in \Lambda} \langle g, \pi(\lambda)\gamma \rangle \pi(\lambda)g.$$ 

By the general properties of the dual frame [11], the latter expansion has the coefficients with the minimum $\ell^2$-norm, therefore

$$\sum_{\lambda \in \Lambda} |\langle g, \pi(\lambda)\gamma \rangle|^2 \leq 1 + \sum_{\lambda \neq 0} 0 = 1.$$

Consequently, with the biorthogonality (11) (in fact, we only need the condition for $\mu = 0$) we obtain

$$(13) \quad \text{vol}(\Lambda)^2 = \langle g, \gamma \rangle^2 \leq \sum_{\lambda \in \Lambda} |\langle g, \pi(\lambda)\gamma \rangle|^2 \leq 1,$$

which is the density theorem.

(ii) The volume of the adjoint lattice is $\text{vol}(\Lambda^\circ) = \text{vol}(\Lambda)^{-1}$. Therefore, the claim is equivalent to (i) by Theorem 4.1.

(iii) A Riesz basis is a Riesz sequence that is complete in the Hilbert space, and therefore is also a frame. Consequently, both (i) and (ii) apply and thus $\text{vol}(\Lambda) = 1$.

Conversely, if $G(g, \Lambda)$ is a frame with $\text{vol}(\Lambda) = 1$, then we have equality in (13) and thus $\langle g, \pi(\lambda)g \rangle = \delta_{\lambda,0}$ for $\lambda \in \Lambda$. Since the Gabor system $G(\gamma, \Lambda)$ is a Bessel sequence and biorthogonal to $G(g, \Lambda)$, we deduce that $G(g, \Lambda)$ is a Riesz sequence, and by the assumed completeness it is a Riesz basis for $L^2(\mathbb{R}^d)$. \hfill \Box

The above proof of the density theorem is due to Janssen [27].

5.2. Existence of Gabor Frames for sufficiently dense lattices. In the early treatments of Gabor frames one finds many qualitative statements that assert the existence of Gabor frames. Typically they claim that for a window function $g \in L^2(\mathbb{R}^d)$ with “sufficient” decay and smoothness and for a “sufficiently dense” lattice $\Lambda$ the Gabor system $G(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$. For a sample of results we refer to [8, 12, 40]. In this section we derive such a qualitative result as a consequence of the duality theorem.

We will measure decay and smoothness by means of time-frequency concentration as follows: we say that $g$ belongs to the modulation space $M_{s_\infty}(\mathbb{R}^d)$ if

$$|V_0 g(z)| \leq C(1 + |z|)^{-s} \quad \forall z \in \mathbb{R}^{2d}.$$

This is not the standard definition of the modulation space, but it is the most convenient definition for our purpose. A systematic exposition of modulation spaces is contained in [18].

To quantify the density of a lattice $\Lambda = A\mathbb{Z}^{2d}$, we set simply

$$\|\Lambda\| = \|A\|_{op},$$

with the understanding that this definition is highly ambiguous and depends more on the choice of a basis $A$ for the lattice than on the lattice itself.
Theorem 5.2. Assume that $g \in M^\infty_\psi(R^d)$ for some $s > 2d$. Then there exists a $\delta_0$ depending on $g$ such that for every lattice $\Lambda = AZ^{2d}$ with $\|A\|_{op} < \delta_0$ the Gabor system $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(R^d)$.

In other words, there exists a sufficiently small neighborhood $V$ of the zero matrix such that $\mathcal{G}(g, AZ^{2d})$ is a frame for every $A \in V$.

Proof. Invariably, qualitative existence theorems in Gabor analysis (and more generally in sampling theory) use the fact that an operator that is close enough to the identity operator is invertible. For this proof, we use the duality theorem and show that on the adjoint lattice, which is sufficiently sparse, the Gramian matrix is diagonally dominant and therefore invertible.

Without loss of generality we assume that $\|g\|_{L^2} = 1$, then the Gramian can be written as $G = I + R$, where $G_{\mu,\nu} = \langle \pi(\nu)g, \pi(\mu)g \rangle$ and $R$ is the off-diagonal part of $G$.

We now make the following observations about $R$:
(i) If $\|A\|_{op} = \delta$ and $\mu = T(A^T)^{-1}k \in \Lambda^\circ$, then $|k| = |A^T T^{-1} T(A^T)^{-1}k| \leq \|A\|_{op} |\mu| = \delta |\mu|$ and therefore

$$(1 + |\mu|)^{-s} \leq (1 + \delta^{-1}|k|)^{-s}.$$ 

(ii) By applying a simplified version of Schur’s test to the self-adjoint operator $R$, the operator norm of $R$ can estimated by

$$\|R\|_{op} \leq \sup_{\mu \in \Lambda^\circ} \sum_{\nu \neq \mu} |\langle \pi(\nu)g, \pi(\mu)g \rangle|$$

$$= \sup_{\mu \in \Lambda^\circ} \sum_{\nu \neq \mu} |\langle g, \pi(\mu - \nu)g \rangle|$$

$$\leq \sum_{\mu \neq 0} (1 + |\mu|)^{-s}$$

$$\leq \sum_{k \in \mathbb{Z}^d, k \neq 0} (1 + \delta^{-1}|k|)^{-s} := \varphi(\delta).$$

(iii) Since $s > 2d$, $\varphi(\delta)$ is finite for all $\delta > 0$, and $\varphi$ is a continuous, increasing function that satisfies

$$\lim_{\delta \to 0^+} \varphi(\delta) = 0.$$ 

Consequently, there is a $\delta_0$ such that $\varphi(\delta_0) = 1$. For $\delta < \delta_0$ we then obtain that

$$\|R\|_{op} \leq \varphi(\delta) < 1,$$

therefore $G$ is invertible on $\ell^2(\Lambda^\circ)$. This means that $\mathcal{G}(g, \Lambda^\circ)$ is a Riesz sequence, and by duality $\mathcal{G}(g, \Lambda)$ is a frame, whenever the matrix $A$ defining $\Lambda = AZ^{2d}$ satisfies $\|A\|_{op} < \delta_0$. \hfill \Box

The above proof highlights the role of the duality theorem in the qualitative existence proof. By emphasizing some technicalities about modulation spaces, one
may prove a slightly more general version of the existence theorem. We say that \( g \) belongs to the modulation space \( M^1(\mathbb{R}^d) \) if

\[
\int_{\mathbb{R}^{2d}} |\langle g, \pi(z)g \rangle| \, dz < \infty.
\]

The proof of Theorem 5.2 can be extended to yield the following result.

**Theorem 5.3** ([12,13]). Assume that \( g \in M^1(\mathbb{R}^d) \). Then there exists a \( \delta_0 \) depending on \( g \) such that for every lattice \( \Lambda = AZ^{2d} \) with \( \|A\|_{\text{op}} < \delta_0 \) the Gabor system \( G(g, \Lambda) \) is a frame for \( L^2(\mathbb{R}^d) \).

These existence results are complemented by an important theorem of Bekka [3]: For every lattice \( \Lambda \) with \( \text{vol}(\Lambda) \leq 1 \), there exists a window \( g \in L^2(\mathbb{R}^d) \) such that \( G(g, \Lambda) \) is a frame.

### 5.3. Balian-Low Theorem

The Balian-Low theorem (BLT) states that for a window with a mild decay in time-frequency the necessary density condition must be strict. In the standard formulation, the window a Gabor frame \( G(g, \Lambda) \) at the critical density \( \text{vol}(\Lambda) = 1 \) lacks time-frequency localization. We refer to the surveys [4,7] for a detailed discussion of the Balian-Low phenomenon in dimension 1. For higher dimensions and arbitrary lattices, the BLT follows from an important deformation result of Feichtinger and Kaiblinger [14] with useful subsequent improvements in [1,22].

**Theorem 5.4.** Assume that \( g \in M^\infty_v(\mathbb{R}^d) \) for some \( s > 2d \) and that \( G(g, \Lambda) \) is a frame for \( L^2(\mathbb{R}^d) \). Then there exists an \( \varepsilon_0 > 0 \) such that \( G(g, (1 + \tau)\Lambda) \) is a frame for every \( \tau \) with \( |\tau| < \varepsilon_0 \).

**Proof.** We only give the proof idea and indicate where the duality theorem enters. Let \( \tilde{\Lambda} = (1 + \tau)\Lambda \), then its adjoint lattice is \( \tilde{\Lambda}^\circ = (1 + \tau)^{-1}\Lambda^\circ \). If \( G(g, \Lambda) \) is a frame, then the Gramian operator \( G = G_{g,\Lambda^\circ} \) is invertible on \( \ell^2(\Lambda^\circ) \). Set \( \rho = (1 + \tau)^{-1} \) and we consider the cross-Gramian operator \( \tilde{G}^\rho \) with entries

\[
\tilde{G}^\rho_{\mu\nu} = \langle \pi(\rho\nu)g, \pi(\mu)g \rangle, \quad \mu, \nu \in \Lambda^\circ.
\]

We argue that

\[
\lim_{\rho \to 1} \|\tilde{G}^\rho - G\|_{\text{op}} = 0.
\]

This implies that for \( |\rho - 1| < \varepsilon_0 \) for some \( \varepsilon_0 \) the cross-Gramian operator \( \tilde{G}^\rho \) is invertible on \( \ell^2(\Lambda^\circ) \). Now a perturbation result for Riesz bases that goes back to Paley-Wiener (see, e.g., [5]) implies that \( G(g, (1 + \tau)^{-1}\Lambda) \) is a Riesz sequence, and by the duality theorem \( G(g, (1 + \tau)\Lambda) \) is a frame.

The proof of (15) is similar to the proof of Theorem 5.2. We apply Schur’s test to estimate the operator norm of \( \tilde{G}^\rho - G \). Given \( \delta > 0 \), we may choose \( R > 0 \) such that

\[
\sum_{\mu:|\mu - \nu| > R} |\tilde{G}^\rho_{\mu\nu} - G_{\mu\nu}| < \delta/2 \quad \text{for all } \nu \in \Lambda^\circ \text{ and } 1/2 < \rho < 2,
\]
and likewise \( \sum_{|\mu - v| > R} |\tilde{G}^\rho_{\mu v} - G_{\mu v}| < \delta/2 \) for all \( \mu \in \Lambda^o \). As in (14), this is possible because \( g \in M^\infty_{v_s}(\mathbb{R}^d) \) guarantees the off-diagonal decay of \( G \) and \( \tilde{G}^\rho \).

Next, we choose \( \epsilon_0 > 0 \) such that for \( |\rho - 1| < \epsilon_0 \)

\[
\sum_{\mu:|\mu - v| \leq R} |\tilde{G}^\rho_{\mu v} - G_{\mu v}| = \sum_{\mu:|\mu - v| \leq R} |\langle \pi(\rho \nu) g - \pi(\nu) g, \pi(\mu) g \rangle| < \delta/2
\]

and \( \sum_{|\mu - v| \leq R} |\tilde{G}^\rho_{\mu v} - G_{\mu v}| < \delta/2 \). Combining both estimates yields \( \|\tilde{G}^\rho - G\|_{op} < \delta \).

Again, the optimal assumption on \( g \) in Theorem 5.4 is that it belongs to \( M^1(\mathbb{R}^d) \).

**Corollary 5.5.** Assume that \( \text{vol}(\Lambda) = 1 \) and \( \mathcal{G}(g, \Lambda) \) is a frame for \( L^2(\mathbb{R}^d) \). Then \( g \notin M^\infty_{v_s}(\mathbb{R}^d) \) for all \( s > 2d \).

**Proof.** If \( g \in M^\infty_{v_s} \) and \( \mathcal{G}(g, \Lambda) \) were a frame for some lattice \( \Lambda \) with \( \text{vol}(\Lambda) = 1 \), then by Theorem 5.4 the Gabor system \( \mathcal{G}(g, (1 + \tau)\Lambda) \) would also be a frame for some \( \tau > 0 \). But \( \text{vol}((1 + \tau)\Lambda) = (1 + \tau)^{2d} \text{vol}(\Lambda) > 1 \), and this contradicts the density theorem. \qed

### 5.4. The Coarse Structure of Gabor Frames and Gabor Riesz Sequences

One of the principal questions of Gabor analysis is the question under which conditions on a window \( g \) and a lattice \( \Lambda \) the Gabor system \( \mathcal{G}(g, \Lambda) \) is a frame for \( L^2(\mathbb{R}^d) \) or a Riesz sequence in \( L^2(\mathbb{R}^d) \). To formalize this, we define the full frame set \( \mathcal{F}_{\text{full}}(g) \) of \( g \) to be the set of all lattices \( \Lambda \) such that \( \mathcal{G}(g, \Lambda) \) is a frame and the reduced frame set \( \mathcal{F}(g) \) to be the set of all rectangular lattices \( \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \) such that \( \mathcal{G}(g, \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d) \) is a frame. Formally,

\[
\mathcal{F}_{\text{full}}(g) = \{ \Lambda \subseteq \mathbb{R}^{2d} \text{ lattice : } \mathcal{G}(g, \Lambda) \text{ is a frame } \}
\]

\[
\mathcal{F}(g) = \{ (\alpha, \beta) \subseteq \mathbb{R}^2_+ : \mathcal{G}(g, \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d) \text{ is a frame } \}.
\]

We summarize the results of the previous sections in the main result about the coarse structure of Gabor frames, i.e., results that hold for arbitrary Gabor systems over a lattice.

**Theorem 5.6.** If \( g \in M^\infty_{v_s}(\mathbb{R}^d) \) for some \( s > 2d \) or in \( M^1(\mathbb{R}^d) \), then \( \mathcal{F}_{\text{full}}(g) \) is an open subset of \( \{ \Lambda : \text{vol}(\Lambda) < 1 \} \) and contains a neighborhood of \( 0 \).

Likewise, \( \mathcal{F}(g) \) is open in \( \{ (\alpha, \beta) \in \mathbb{R}^2_+ : \alpha \beta < 1 \} \) and contains a neighborhood of \( (0, 0) \) in \( \mathbb{R}^2_+ \).

Theorem 5.6 should not be underestimated. It compresses the efforts of dozens of articles into a single statement. It contains the existence of Gabor frames, the density theorem, and the Balian-Low theorem. For each result there are now several different proofs (with subtle differences in the hypotheses) and many ramifications. What is perhaps new is the close connection of the coarse structure of Gabor frames to the duality theory.
6. The Criterion of Janssen, Ron and Shen for Rectangular Lattices

In this and the following section, we consider rectangular lattices of the form \( \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \) for \( \alpha, \beta > 0 \). Observe that the adjoint of such a lattice is

\[
\Lambda^\circ = I \left( \begin{array}{cc}
\frac{1}{\alpha} I_d & 0 \\
0 & \frac{1}{\beta} I_d \\
\end{array} \right) \mathbb{Z}^{2d} = \frac{1}{\beta} \mathbb{Z}^d \times \frac{1}{\alpha} \mathbb{Z}^d,
\]

and is again a rectangular lattice.

For convenience, we denote \( \mathcal{G}(g, \alpha, \beta) := \mathcal{G}(g, \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d) \).

**Definition 6.1.** Let \( g \in L^2(\mathbb{R}^d) \) and \( \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \) with \( \alpha, \beta > 0 \). The pre-Gramian matrix \( P(x) \) is defined by

\[
P(x)_{j,k} = \overline{g(x + \alpha j - \frac{k}{\beta})} \quad \forall j, k \in \mathbb{Z}^d,
\]

and the Ron-Shen matrix \( R(x) := P(x)^* P(x) \) has the entries

\[
R(x)_{k,l} = \sum_{j \in \mathbb{Z}^d} g(x + \alpha j - \frac{k}{\beta}) \overline{g(x + \alpha j - \frac{l}{\beta})} \quad \forall k, l \in \mathbb{Z}^d.
\]

**Theorem 6.2.** Let \( g \in L^2(\mathbb{R}^d) \) and \( \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \) with \( \alpha, \beta > 0 \) be a rectangular lattice. Then the following are equivalent:

(i) \( \mathcal{G}(g, \alpha, \beta) \) is a frame for \( L^2(\mathbb{R}^d) \).

(ii) \( \mathcal{G}(g, \alpha, \beta) \) is a Bessel sequence and there exists a dual window \( \gamma \in L^2(\mathbb{R}^d) \) such that \( \mathcal{G}(\gamma, \alpha, \beta) \) is a Bessel sequence satisfying

\[
\sum_{j \in \mathbb{Z}^d} \gamma(x + \alpha j) \overline{g(x + \alpha j - \frac{k}{\beta})} = \beta^d \delta_{k,0} \quad \forall k \in \mathbb{Z}^d \text{ and a.e. } x \in \mathbb{R}^d.
\]

(iii) \( \mathcal{G}(g, \frac{1}{\beta}, \frac{1}{\alpha}) \) is a Riesz sequence for \( L^2(\mathbb{R}^d) \).

(iv) There exist positive constants \( A, B > 0 \) such that for all \( c \in \ell^2(\mathbb{Z}^d) \) and almost all \( x \in \mathbb{R}^d \)

\[
A \|c\|_{\ell^2}^2 \leq \sum_{j \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} |c_k \overline{g(x + \alpha j - \frac{k}{\beta})}|^2 \right) \leq B \|c\|_{\ell^2}^2.
\]

(v) There exist positive constants \( A, B > 0 \) such that the spectrum of almost every Ron-Shen matrix is contained in the interval \([A, B]\). This means

\[
\sigma(R(x)) \subseteq [A, B] \quad \text{for a.e. } x \in \mathbb{R}^d.
\]

(vi) The set of pre-Gramians \( \{P(x)\} \) is uniformly bounded on \( \ell^2(\mathbb{Z}^d) \) and has a set of uniformly bounded left-inverses. This means that there exist \( \Gamma(x) : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d) \) such that

\[
\Gamma(x) P(x) = I_{\ell^2(\mathbb{Z}^d)} \quad \text{for a.e. } x \in \mathbb{R}^d,
\]

\[
\|\Gamma(x)\| \leq C \quad \text{for a.e. } x \in \mathbb{R}^d.
\]
Proof. The equivalence of (i) and (iii) is Theorem 4.1. The equivalence of conditions (iv), (v) and (vi) is mainly of linguistic nature, the mathematical content is in the equivalence (iii) \(\Leftrightarrow\) (iv).

(iv) \(\Leftrightarrow\) (v): For all sequences \(c \in \ell^2(\mathbb{Z}^d)\), we have
\[
\sum_{j \in \mathbb{Z}^d} \left| \sum_{k \in \mathbb{Z}^d} c_k g(x + \alpha j - \frac{k}{b}) \right|^2 = \langle P(x)c, P(x)c \rangle = \langle R(x)c, c \rangle.
\]
Hence, inequality (17) becomes
\[
A \|c\|_{\ell^2}^2 \leq \langle R(x)c, c \rangle \leq B \|c\|_{\ell^2}^2 \quad \forall c \in \ell^2(\mathbb{Z}^d),
\]
for almost all \(x \in \mathbb{R}^d\), which is equivalent to \(\sigma(R(x)) \subseteq [A, B] \) for almost all \(x \in \mathbb{R}^d\).

(iv) \(\Rightarrow\) (iii): Let \(c \in \ell^2(\mathbb{Z}^{2d})\) be a finite sequence and \(f = \sum_{k,l \in \mathbb{Z}^d} c_{k,l} M_{\frac{\alpha}{b}} T_k g\). For fixed \(k\) the sum over \(l\) is a trigonometric polynomial
\[
p_k(x) := \sum_{l \in \mathbb{Z}^d} c_{k,l} e^{2\pi i \frac{1}{\alpha} \cdot x}
\]
with period \(\alpha\) in each coordinate, and its \(L^2\)-norm over a period \(Q_{\alpha} := [0, \alpha]^d\) given by
\[
\int_{Q_{\alpha}} |p_k(x)|^2 \, dx = \alpha^d \sum_{l \in \mathbb{Z}^d} |c_{k,l}|^2.
\]
To calculate the \(L^2\)-norm of \(f\) we use the periodization trick and obtain
\[
\|f\|_{L^2}^2 = \left\| \sum_{k \in \mathbb{Z}^d} p_k \cdot T_{\frac{\alpha}{b}} g \right\|_{L^2}^2
\]
\[
= \int_{\mathbb{R}^d} \left| \sum_{k \in \mathbb{Z}^d} p_k(x) g(x - \frac{k}{b}) \right|^2 \, dx
\]
\[
= \int_{Q_{\alpha}} \sum_{j \in \mathbb{Z}^d} \left| \sum_{k \in \mathbb{Z}^d} p_k(x) g(x + \alpha j - \frac{k}{b}) \right|^2 \, dx.
\]
Next, for every \(x \in Q_{\alpha}\) we apply assumption (17) to the integrand and obtain
\[
\|f\|_{L^2}^2 \geq \int_{Q_{\alpha}} A \sum_{k \in \mathbb{Z}^d} |p_k(x)|^2 \, dx
\]
\[
= \alpha^d A \sum_{k,l \in \mathbb{Z}^d} |c_{k,l}|^2 = \alpha^d A \|c\|_{\ell^2}^2.
\]
for all finite sequences \(c \in \ell^2(\mathbb{Z}^{2d})\). The upper bound follows analogously, and thus \(G(g, \frac{1}{b}, \frac{1}{\alpha})\) is a Riesz sequence.

(iii) \(\Rightarrow\) (iv): By assumption,
\[
A \|c\|_{\ell^2}^2 \leq \left\| \sum_{k,l \in \mathbb{Z}^d} c_{k,l} M_{\frac{\alpha}{b}} T_k g \right\|_{L^2}^2 \leq B \|c\|_{\ell^2}^2
\]
for all \(c \in \ell^2(\mathbb{Z}^{2d})\). We apply this fact to sequences \(c\) of the form \(c_{k,l} := a_k b_l\) for \(a, b \in \ell^2(\mathbb{Z}^d)\). Then \(\|c\|_{\ell^2(\mathbb{Z}^{2d})}^2 = \|a\|_{\ell^2(\mathbb{Z}^d)}^2 \|b\|_{\ell^2(\mathbb{Z}^d)}^2\).
Every $p \in L^2(Q_\alpha)$ can be written as Fourier series $p(x) = \sum_{l \in \mathbb{Z}^d} b_l e^{2\pi i \frac{\ell}{\beta}}$ with coefficients $b \in \ell^2(\mathbb{Z}^d)$. Hence, we obtain for arbitrary $a \in \ell^2(\mathbb{Z}^d)$ and $p \in L^2(Q_\alpha)$

$$\frac{A}{\alpha^d} \|a\|_{\ell^2(\mathbb{Z}^d)}^2 \int_{Q_\alpha} |p(x)|^2 \, dx = A \|a\|_{\ell^2(\mathbb{Z}^d)}^2 \|b\|_{\ell^2(\mathbb{Z}^d)}^2 = A \|c\|_{\ell^2(\mathbb{Z}^d)}^2$$

$$\leq \left\| \sum_{k,l \in \mathbb{Z}^d} a_kb_l M_\frac{1}{\alpha} T_{\frac{\ell}{\beta}} g \right\|_{L^2}^2$$

$$= \int_{\mathbb{R}^d} |p(x)|^2 \left| \sum_{k \in \mathbb{Z}^d} a_k g(x - \frac{k}{\beta}) \right|^2 \, dx$$

$$= \int_{Q_\alpha} \sum_{j \in \mathbb{Z}^d} |p(x + \alpha j)|^2 \left| \sum_{k \in \mathbb{Z}^d} a_k g(x + \alpha j - \frac{k}{\beta}) \right|^2 \, dx$$

$$= \int_{Q_\alpha} |p(x)|^2 \sum_{j \in \mathbb{Z}^d} \left| \sum_{k \in \mathbb{Z}^d} a_k g(x + \alpha j - \frac{k}{\beta}) \right|^2 \, dx.$$

(18)

Since $L^2(Q_\alpha)$ contains all characteristic functions of measurable subsets in $Q_\alpha$, (18) implies

$$\frac{A}{\alpha^d} \|a\|_{\ell^2(\mathbb{Z}^d)}^2 \leq \sum_{j \in \mathbb{Z}^d} \left| \sum_{k \in \mathbb{Z}^d} a_k g(x + \alpha j - \frac{k}{\beta}) \right|^2$$

for all $a \in \ell^2(\mathbb{Z}^d)$. The upper bound follows analogously.

(v) $\Rightarrow$ (vi): Suppose that the spectrum of almost all $R(x)$ is contained in the interval $[A, B]$ for some positive constants $A, B > 0$. Then the set of pre-Gramians is uniformly bounded by $B^{1/2}$ since $R(x) = P(x)^*P(x)$.

As $R(x)$ is invertible, we may define the pseudo-inverse $\Gamma(x) := R(x)^{-1}P(x)^*$. Then

$$\Gamma(x)P(x) = I_{\ell^2(\mathbb{Z}^d)}$$

and

$$\|\Gamma(x)\| \leq \|R(x)^{-1}\| \|P(x)\| \leq A^{-1}B^{1/2}.$$

(vi) $\Rightarrow$ (v): By assumption, every $P(x)$ possesses a left inverse $\Gamma(x)$ with control of the operator norm. This implies

$$\|c\|_{\ell^2}^2 = \|\Gamma(x)P(x)c\|_{\ell^2}^2 \leq \|\Gamma(x)\|^2 \|P(x)c\|_{\ell^2}^2$$

$$\leq C^2 \langle R(x)c, c \rangle \leq C^2 \|P(x)c\|^2 \|c\|_{\ell^2}^2 \leq C^2D^2 \|c\|_{\ell^2}^2.$$

for all $c \in \ell^2(\mathbb{Z}^d)$ and almost all $x \in \mathbb{R}^d$. This is (v).

(ii) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (vi): Condition (ii) can be understood as an explicit version of (vi). Alternatively, it is a slight reformulation of the biorthogonality condition (11), once again with the Poisson summation formula:

$$\sum_{j \in \mathbb{Z}^d} \gamma(x + \alpha j)\bar{g}(x + \alpha j - \frac{k}{\beta}) = \frac{1}{\alpha^d} \sum_{j \in \mathbb{Z}^d} \langle \gamma, M_\frac{1}{\alpha} T_{\frac{k}{\beta}} g \rangle e^{2\pi i \frac{\ell}{\beta} \cdot x} = \beta^d \delta_{k,0}.$$

□
The formulation (16) of the biorthogonality is due to Janssen [29]. Conditions (iv) and (v) were discovered by Ron and Shen [37]. The criterion (vi) is from [24].

The results of Ron and Shen are more general and hold for separable lattices of the form \( P\mathbb{Z}^d \times Q\mathbb{Z}^d \) with invertible, real-valued \( d \times d \) matrices \( P, Q \). In this setting, Theorem 6.2 holds with the appropriate modifications (just replace the scalar-multiplication with \( \alpha, \beta, 1/\alpha, 1/\beta \) by the matrix-vector multiplication with \( P, Q, P^{-1}, Q^{-1} \) and use appropriate fundamental domains).

Condition (iv) has been the master tool of Janssen in his work on exponential windows [30] and “Zak transforms with few zeros” [31]. The construction of a dual window was used by Janssen [27] to give a signal-analytic proof of the Theorem of Lyubarski and Seip. Recently, the biorthogonality condition for the dual window was used successfully in the analysis of totally positive windows of finite type [24]. Christensen et al. [6] have used (16) to compute explicit formulas for dual windows.

Condition (iv) also lends itself to proving qualitative sufficient conditions. By imposing the diagonal dominance of \( R(x) \), one can derive some conditions on \( g \) to guarantee that \( \mathcal{G}(g, \alpha, \beta) \) is a frame. The easiest case is \( R(x) \) being a family of diagonal matrices. In this way one obtains the “painless non-orthogonal expansions” of Daubechies, Grossman, and Meyer [9]. This fundamental result precedes the era of wavelets and Gabor analysis, and yields all Gabor frames that are used for real applications, e.g., in signal analysis or speech processing.

**Theorem 6.3** (Painless non-orthogonal expansions). Suppose \( g \in L^\infty(\mathbb{R}^d) \) with \( \text{supp} \ g \subseteq [0, L]^d \). If \( \alpha \leq L \) and \( \beta \leq \frac{1}{L} \), then \( \mathcal{G}(g, \alpha, \beta) \) is a frame if and only if

\[
0 < \text{ess inf}_{x \in \mathbb{R}^d} \sum_{k \in \mathbb{Z}^d} |g(x - \alpha k)|^2.
\]

**Proof.** By assumption, we have \( \frac{1}{\beta} \geq L \). If \( \frac{1}{\beta} > L \), then the supports of \( T_{\frac{k}{\beta}} g \) and \( T_{\frac{l}{\beta}} g \) are disjoint for \( k \neq l \); if \( \frac{1}{\beta} = L \), then the supports of \( T_{\frac{k}{\beta}} g \) and \( T_{\frac{l}{\beta}} g \) overlap on a set of measure zero and we may modify \( g \) so that \( T_{\frac{k}{\beta}} g \cdot T_{\frac{l}{\beta}} g = 0 \) everywhere for \( k \neq l \). Consequently,

\[
R(x)_{k,l} = \sum_{j \in \mathbb{Z}^d} g\left(x + j\alpha - \frac{k}{\beta}\right) \overline{g}\left(x + j\alpha - \frac{l}{\beta}\right)
\]

\[
= \sum_{j \in \mathbb{Z}^d} |g\left(x + j\alpha - \frac{k}{\beta}\right)|^2 \delta_{k,l},
\]

and thus \( R(x) \) is a diagonal matrix for almost all \( x \). Clearly, a diagonal matrix is bounded and invertible if and only if its diagonal is bounded above and away from zero, therefore the assertion of Theorem 6.3 follows immediately. \( \square \)

Theorem 6.2 can also be reformulated in terms of frames for \( L^2(\mathbb{T}^d) \). For this we recall that the Zak transform with respect to the parameter \( \alpha > 0 \) is defined by

\[
Z_\alpha f(x, \xi) := \sum_{k \in \mathbb{Z}^d} f(x - \alpha k) e^{2\pi i \alpha k \cdot \xi}.
\]
Most characterizations of a Gabor frame over a rectangular lattice can be formulated by means of the Zak transform. Here is the general version attached to Theorem 6.2.

**Theorem 6.4.** Let \( g \in L^2(\mathbb{R}^d) \) and \( \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \) with \( \alpha, \beta > 0 \). Then the following are equivalent:

(i) \( \mathcal{G}(g, \alpha, \beta) \) is a frame for \( L^2(\mathbb{R}^d) \).

(ii) \( \{ Z_{\frac{1}{\beta}} g(x + \alpha j, \beta \cdot) : j \in \mathbb{Z}^d \} \) is a frame for \( L^2(\mathbb{T}^d) \) for almost all \( x \in \mathbb{R}^d \) with frame bounds independent of \( x \).

**Proof.** By Theorem 6.2, \( \mathcal{G}(g, \alpha, \beta) \) is a Gabor frame if and only if there exist positive constants \( A, B > 0 \) such that

\[
A \| c \|_{\ell^2}^2 \leq \sum_{j \in \mathbb{Z}^d} \left| \sum_{k \in \mathbb{Z}^d} c_k \mathcal{G}(x + \alpha j - \frac{k}{\beta}) \right|^2 \leq B \| c \|_{\ell^2}^2
\]

for all \( c \in \ell^2(\mathbb{Z}^d) \) and almost all \( x \in \mathbb{R}^d \).

Using Parseval’s identity for Fourier series, we interpret the inner sum over \( k \) as an inner product of periodic \( L^2 \)-functions. The Fourier series of \( c(\xi) = \sum_{k \in \mathbb{Z}^d} c_k e^{2\pi ik \cdot \xi} \), and the Fourier series of the sequence \( (g(x + \alpha j - k/\beta))_{k \in \mathbb{Z}^d} \) (for fixed \( x \)) is precisely the Zak transform

\[
Z_{\frac{1}{\beta}} g(x + \alpha j, \beta \cdot) = \sum_{k \in \mathbb{Z}^d} g(x + \alpha j - \frac{k}{\beta}) e^{2\pi ik \cdot \xi}.
\]

Consequently,

\[
\sum_{k \in \mathbb{Z}^d} c_k \mathcal{G}(x + \alpha j - \frac{k}{\beta}) = \int_{\mathbb{T}^d} \hat{c}(\xi) Z_{\frac{1}{\beta}} g(x + \alpha j, \beta \cdot) d\xi
\]

and (19) just says that the set \( \{ Z_{\frac{1}{\beta}} g(x + \alpha j, \beta \cdot) : j \in \mathbb{Z}^d \} \) is a frame for \( L^2(\mathbb{T}^d) \) for almost all \( x \in \mathbb{R}^d \). Furthermore, the frame bounds can be chosen to be \( A \) and \( B \) independent of \( x \).

\[\square\]

7. **Zak Transform Criteria for Rational Lattices — The Criteria of Zeevi and Zibulski**

All criteria formulated so far are expressed by the invertibility of an infinite matrix or of an operator on an infinite dimensional space. For rectangular lattices \( \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \) with \( \alpha \beta \in \mathbb{Q} \) one may further reduce the effort and study the invertibility of a family of finite-dimensional matrices.

Assume that \( \alpha \beta = p/q \leq 1 \) for \( p, q \in \mathbb{N} \). In order to simplify the labeling of vectors and matrices, we define \( E_q := \{0, 1, \ldots, q-1\}^d \) and \( E_p := \{0, 1, \ldots, p-1\}^d \). We then write \( j \in \mathbb{Z}^d \) uniquely as \( j = ql + r \) for \( l \in \mathbb{Z}^d \) and \( r \in E_q \). Using the quasi-periodicity of the Zak transform, we obtain

\[
Z_{\frac{1}{\beta}} g(x + \alpha j, \beta \cdot) = Z_{\frac{1}{\beta}} g(x + \frac{p}{q\beta}(ql + r), \beta \cdot) = e^{2\pi ip l \cdot \xi} Z_{\frac{1}{\beta}} g(x + \frac{p}{q\beta} r, \beta \cdot).
\]
Thus for rational values of $\alpha \beta$, we obtain a function system which factors into certain complex exponentials and some functions. The frame property of such a system is characterized in the following lemma.

**Lemma 7.1.** Let $\{h_r : r \in F\} \subseteq L^2(\mathbb{T}^d)$ be a finite set and $p \in \mathbb{N}$ such that $\text{card } F \geq p^d$. Furthermore, let $A(\xi)$ be the matrix with entries $A(\xi)_{r,s} = \overline{h_r(\xi + \alpha_p)}$ for $r \in F, s \in E_p$. Then the following are equivalent:

(i) The set $\{e^{2\pi i p} h_r(\xi) : l \in \mathbb{Z}^d, r \in F\}$ is a frame for $L^2(\mathbb{T}^d)$.

(ii) There exist $A, B > 0$ such that the singular values of $A(\xi)$ are contained in $[A^{1/2}, B^{1/2}]$ for almost all $\xi \in \mathbb{T}^d$.

(iii) There exist $A, B > 0$ such that $\sigma(A^*(\xi)A(\xi)) \subseteq [A, B]$ for almost all $\xi \in \mathbb{T}^d$.

The condition $\text{card } F \geq p^d$ is essential, otherwise the matrix $A(\xi)$ cannot be injective and $A^*(\xi)A(\xi)$ cannot be invertible.

**Proof.** For $f \in L^2(\mathbb{T}^d)$ and $\xi \in Q_{1/p} = [0, \frac{1}{p}]^d$ we write the vector $y(\xi) = (f(\xi + \alpha_p))_{s \in E_p}$. Then the inner product of $f$ with the frame functions $h_r(\xi)e^{2\pi i p l \xi}$ can be written as

$$\langle f, h_r e^{2\pi i p l \xi} \rangle = \int_{[0, \frac{1}{p}]^d} \sum_{s \in E_p} f(\xi + \alpha_p)\overline{h_r(\xi + \alpha_p)}e^{-2\pi i p l \xi} \ d\xi$$

$$= \int_{[0, \frac{1}{p}]^d} (A(\xi)y(\xi))_r e^{-2\pi i p l \xi} \ d\xi.$$

Since $\{p^{d/2} e^{2\pi i p l \xi} : l \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2(Q_{1/p})$, we now obtain

$$\sum_{l \in \mathbb{Z}^d} \sum_{r \in F} |\langle f, h_r e^{2\pi i p l \xi} \rangle|^2 = \sum_{r \in F} \sum_{l \in \mathbb{Z}^d} \left| \int_{Q_{1/p}} (A(\xi)y(\xi))_r e^{-2\pi i p l \xi} \ d\xi \right|^2$$

$$= \frac{1}{p^{2d}} \sum_{r \in F} \int_{Q_{1/p}} |(A(\xi)y(\xi))_r|^2 \ d\xi$$

$$= \frac{1}{p^d} \int_{Q_{1/p}} |A(\xi)y(\xi)|^2 \ d\xi.$$

If the singular values of $A$ are all in an interval $[A^{1/2}, B^{1/2}]$, then $|A(\xi)y(\xi)|^2 = \langle A(\xi)^*A(\xi)y(\xi), y(\xi) \rangle \geq A |y(\xi)|^2$. Therefore

$$\sum_{l \in \mathbb{Z}^d} \sum_{r \in F} |\langle f, h_r e^{2\pi i p l \xi} \rangle|^2 = \frac{1}{p^{d/2}} \int_{Q_{1/p}} |A(\xi)y(\xi)|^2 \ d\xi$$

$$\geq \frac{A}{p^d} \int_{Q_{1/p}} |y(\xi)|^2 \ d\xi$$

$$= \frac{A}{p^d} \int_{Q_{1/p}} \sum_{s \in E_p} |f(\xi + \alpha_p)|^2 \ d\xi = \frac{A}{p^d} \|f\|_{L^2(\mathbb{T}^d)}^2.$$

(20)
Similarly, for the upper frame inequality. Thus the set \( \{ h_r(\xi)e^{2\pi i p_l \xi} : l \in \mathbb{Z}^d, r \in F \} \) is a frame for \( L^2(\mathbb{T}^d) \).

Conversely, assume that \( \{ h_r(\xi)e^{2\pi i p_l \xi} : l \in \mathbb{Z}^d, r \in F \} \) is a frame for \( L^2(\mathbb{T}^d) \). Then (20) says that for all \( f \in L^2(\mathbb{T}^d) \) with associated vector-valued function \( y(\xi) = (f(\xi + \frac{r}{p}))_{s \in E_p} \) we must have

\[
\frac{A}{p^d} \int_{Q_{1/p}} |y(\xi)|^2 \, d\xi \leq \frac{1}{p^d} \int_{Q_{1/p}} |A(\xi)y(\xi)|^2 \, d\xi \leq \frac{B}{p^d} \int_{Q_{1/p}} |y(\xi)|^2 \, d\xi .
\]

Now we diagonalize \( A(\xi)^*A(\xi) \). Since \( A^*A \) is a measurable matrix-valued function on \( \mathbb{T}^d \), its diagonalization can be chosen to be measurable (see Azoff [2]). This means that there exist two measurable matrix-valued functions \( U, D \) such that \( U(\xi) \) is a unitary matrix, \( D(\xi) \) is of diagonal form and \( A(\xi)^*A(\xi) = U(\xi)^*D(\xi)U(\xi) \) for all \( \xi \in \mathbb{T}^d \).

Hence (21) is equivalent to

\[
A \int_{Q_{1/p}} |\tilde{y}(\xi)|^2 \, d\xi \leq \int_{Q_{1/p}} \langle D(\xi)\tilde{y}(\xi), \tilde{y}(\xi) \rangle \, d\xi \leq B \int_{Q_{1/p}} |\tilde{y}(\xi)|^2 \, d\xi
\]

for all vector-valued functions \( \tilde{y}(\xi) = U(\xi)y(\xi) \) with components in \( L^2(Q_{1/p}) \).

Clearly, inequality (22) can only hold if \( \sigma(D(\xi)) = \sigma(A(\xi)^*A(\xi)) \subseteq [A, B] \) for almost all \( \xi \in Q_{1/p} \).

We now apply this lemma to the set \( \{ e^{2\pi i p_l \xi}Z_{\frac{1}{p}}(x + \frac{p}{q^3}r, \beta \xi) : l \in \mathbb{Z}^d, r \in E_q \} \) and obtain the characterization of Zeevi and Zibulski for rational rectangular lattices [42, 43].

**Theorem 7.2.** Let \( g \in L^2(\mathbb{R}^d) \) and \( \alpha \beta = p/q \in \mathbb{Q} \) with \( p/q \leq 1 \). For \( x, \xi \in \mathbb{R}^d \) let \( Q(x, \xi) \) be the matrix with entries

\[
Q(x, \xi)_{r,s} = Z_{\frac{1}{p}}g(x + \frac{p}{q}r, \beta \xi + \frac{\beta s}{p}) \quad \forall r \in E_q, s \in E_p .
\]

The Gabor family \( \mathcal{G}(g, \alpha, \beta) \) is a frame for \( L^2(\mathbb{R}^d) \) if and only if the singular values of \( Q(x, \xi) \) are contained in an interval \([A^{1/2}, B^{1/2}] \subseteq (0, \infty) \) for almost all \( x, \xi \in \mathbb{R}^d \).

**Proof.** By Theorem 6.4, \( \mathcal{G}(g, \alpha, \beta) \) is a frame if and only if \( \{ Z_{\frac{1}{p}}g(x + \alpha j, \beta \xi) : j \in \mathbb{Z}^d \} = \{ e^{2\pi i p_l \xi}Z_{\frac{1}{p}}g(x + \alpha r, \beta \xi) : l \in \mathbb{Z}^d, r \in E_q \} \) is a frame for \( L^2(\mathbb{T}^d) \). Now, the claim follows from Lemma 7.1 with the functions \( h_r(\xi) = Z_{\frac{1}{p}}g(x + \alpha r, \beta \xi) \).}

The Zak transform has been used frequently to derive theoretical properties of Gabor frames. The Zeevi-Zibulski matrices in particular are very useful for computational issues, and several important counter-examples have been discovered first through numerical tests before being proved rigorously [32, 33]. On the other hand, it seems to be very difficult to apply directly and decide rigorously whether a concrete Gabor system is a frame or not.
8. Further Characterizations

So far we have discussed characterization of Gabor frames that work for arbitrary windows in $L^2(\mathbb{R}^d)$. On a technical level, we have not used more than the Poisson summation formula. Under mild additional conditions that are standard in time-frequency analysis, one can prove further characterizations for Gabor frames. These, however, require additional and more advanced mathematical tools, such as spectral invariance, a non-commutative version of Wiener’s lemma or Beurling’s method of weak limit. For this reason, we state these characterizations without proofs.

8.1. The Wiener amalgam space and irrational lattices. This condition refines Theorem 6.2 for irrational lattices. As the appropriate class of window we use the Wiener amalgam space $W_0 = W(C, \ell^1)$. It consists of all continuous functions $g$ for which the norm

$$\|g\|_W = \sum_{k \in \mathbb{Z}^d} \sup_{x \in Q_1} |g(x + k)|$$

is finite.

**Theorem 8.1** ([21]). Assume that $g \in W_0$ and $\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$ with $\alpha \beta \notin \mathbb{Q}$. Then $G(g, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d)$ if and only if there exists some $x_0 \in Q_\alpha$ such that $R(x_0)$ is invertible on $\ell^2(\mathbb{Z}^d)$.

Thus for irrational lattices it suffices to check the invertibility of a single Ron-Shen matrix $R(x)$ instead of all matrices. Although this condition looks useful, it has not yet found any applications.

8.2. Janssen’s criterium without inequalities.

**Theorem 8.2** ([23]). Assume that $g \in W_0$ and $\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$. Then $G(g, \alpha, \beta)$ is a frame if and only if the pre-Gramian $P(x)$ is one-to-one on $\ell^\infty(\mathbb{Z}^d)$ for all $x \in \mathbb{R}^d$.

Put differently, to show that $G(g, \alpha, \beta)$ is a frame, one has to show that

$$\sum_{k \in \mathbb{Z}^d} c_k g(x + \alpha j - \frac{k}{\beta}) = 0 \implies c \equiv 0,$$

with the added subtlety that $c$ is only a bounded sequence, but not necessarily in $\ell^2(\mathbb{Z}^d)$, as is the case in Theorem 6.2.

In general it is easier to verify the injectivity of an operator than to prove its invertibility, therefore Theorem 8.2 is a strong result. It has been applied successfully for the study of totally positive windows of Gaussian type in [23] and carries potential for further applications.

8.3. Gabor frames without inequalities. This group of conditions holds for arbitrary lattices and windows in $M^1(\mathbb{R}^d)$. As is well-known, the modulation space $M^1(\mathbb{R}^d)$ is a natural condition in many problems in time-frequency analysis, because it is invariant under the Fourier transform and many other transformations. By
choosing a suitable norm, $M^1(\mathbb{R}^d)$ becomes a Banach space and its dual space $M^\infty(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ that satisfy

$$\sup_{z \in \mathbb{R}^{2d}} |V_{\varphi} f(z)| < \infty$$

for some (or equivalently, for all) Schwartz functions $\varphi$.

**Theorem 8.3** ([19]). Let $g \in M^1(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ be a lattice. Then the following are equivalent:

(i) $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R}^d)$, i.e., $S_{g, \Lambda}$ is invertible on $L^2(\mathbb{R}^d)$.

(ii) The frame operator $S_{g, \Lambda}$ is one-to-one on $M^\infty(\mathbb{R}^d)$.

(iii) The analysis operator $C_{g, \Lambda} : f \mapsto (\langle f, \pi(\lambda) g \rangle)_{\lambda \in \Lambda}$ is one-to-one from $M^\infty(\mathbb{R}^d)$ to $\ell^\infty(\Lambda)$.

(iv) The synthesis operator $D_{g, \Lambda^o} : c \mapsto \sum_{\lambda \in \Lambda^o} c_\lambda \pi(\lambda) g$ is one-to-one from $\ell^\infty(\Lambda^o)$ to $M^\infty(\mathbb{R}^d)$.

(v) The Gramian operator $G_{g, \Lambda^o}$ is one-to-one on $\ell^\infty(\Lambda^o)$.

Conceptually, it seems easier to verify that an operator is one-to-one, therefore one may hope that these conditions will become useful when research on Gabor frames will move from rectangular lattices towards arbitrary ones.

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