ON EXTENSIONS OF THE ALON-TARSI LATIN SQUARE CONJECTURE

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Abstract. Expressions involving the product of the permanent with the \((n-1)\)th power of the determinant of a matrix of indeterminates, and of \((0,1)\)-matrices, are shown to be related to an extension to odd dimensions of the Alon-Tarsi Latin Square Conjecture. These yield an alternative proof of a theorem of Drisko, stating that the extended conjecture holds for odd primes. An identity involving an alternating sum of permanents of \((0,1)\)-matrices is obtained.

1. Introduction

A Latin square of order \(n\) is an \(n \times n\) array of numbers in \([n] := \{1, \ldots, n\}\) so that each number appears exactly once in each row and each column. Let \(L_n\) be the number of Latin squares of order \(n\). Let \(\text{Sym}(n)\) be the symmetric group of permutations of \([n]\). For a permutation \(\pi \in \text{Sym}(n)\) we denote its sign by \(\epsilon(\pi)\). Viewing the rows and columns of a Latin square \(L\) as elements of \(\text{Sym}(n)\), the row-sign (column-sign) of \(L\) is defined to be the product of the signs of the rows (columns) of \(L\). The sign of \(L\), denoted \(\epsilon(L)\), is the product of the row-sign and the column-sign of \(L\). The parity of a Latin square is even (resp. odd) if its sign is 1 (resp. -1). The row parities and column parities of a Latin square are defined analogously. We denote by \(L_n^{\text{EVEN}}\) (\(L_n^{\text{ODD}}\)) the number of even (odd) Latin squares of order \(n\). The Alon-Tarsi Latin Square Conjecture [1] asserts that for even \(n\), \(L_n^{\text{EVEN}} - L_n^{\text{ODD}} \neq 0\). Values of \(L_n^{\text{EVEN}} - L_n^{\text{ODD}}\) for small \(n\) can be found in [10]. Drisko [2] proved the conjecture for \(n = p + 1\), where \(p\) is an odd prime, and Glynn [4] proved it for \(n = p - 1\). Since for odd \(n\) \(L_n^{\text{EVEN}} = L_n^{\text{ODD}}\) some extensions of this conjecture, that hold for odd \(n\), were proposed, as will be described shortly.

A Latin square is called normalized if its first row is the identity permutation, and unipotent if all the elements of its main diagonal are equal. Let \(U_n^{\text{E}}\) and \(U_n^{\text{O}}\) be the numbers of normalized unipotent even and odd Latin squares, respectively. Zappa [11] defined the Alon-Tarsi constant \(AT(n) := U_n^{\text{E}} - U_n^{\text{O}}\) and introduced the following extension of the Alon-Tarsi conjecture:

**Conjecture 1.1.** For all \(n\), \(AT(n) \neq 0\)

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A Latin square is called reduced if its first row and first column are the identity permutation. Let \( R^E_n \) and \( R^O_n \) denote the numbers of even and odd reduced Latin squares of order \( n \), respectively. Another possible extension of the Alon-Tarsi conjecture is the following (see [6] and [10]):

**Conjecture 1.2.** For all \( n, R^E_n - R^O_n \neq 0 \)

If \( n \) is even then these two conjectures are equivalent to the Alon-Tarsi conjecture. However, for odd \( n \) it is not clear whether the two conjectures are equivalent, despite the existence of a bijection between reduced Latin squares and normalized unipotent Latin squares of order \( n \) (see [11]). Drisko [3] proved Conjecture 1.1 in the case that \( n \) is an odd prime. Conjecture 1.2 is known to be true for small values of \( n \) (see [10]).

A Latin square \( L \) of order \( n \) determines \( n \) permutation matrices \( P_s, s \in [n] \), defined by \((P_s)_{ij} = 1\) if and only if \( L_{ij} = s\). Let \( S_n \) be the collection of all \( n \times n \) permutation matrices. For \( P \in S_n \) let \( \alpha_p \) be the corresponding permutation in \( \text{Sym}(n) \). The symbol-sign \( \epsilon_{\text{sym}}(L) \) is the product of the \( \epsilon(\alpha_p) \), \( s = 1, \ldots, n \). A Latin square \( L \) is symbol-even if \( \epsilon_{\text{sym}}(L) = 1 \) and symbol-odd if \( \epsilon_{\text{sym}}(L) = -1 \).

Let \( X = (X_{ij}) \) be the \( n \times n \) matrix of indeterminates. The following theorem is due to MacMahon [7]:

**Theorem 1.3.** \( L_n \) is the coefficient of \( \prod_{i=1}^n \prod_{j=1}^n X_{ij} \) in \( \text{per}(X)^n \),

where \( \text{per}(A) \) denotes the permanent of \( A \). Stones [9] showed that if we replace permanent by determinant in the expression in Theorem 1.3 an expression for the Alon-Tarsi conjecture is obtained, namely

**Theorem 1.4.** \( L_n^{\text{even}} - L_n^{\text{odd}} \) is the coefficient of \( (-1)^{n(n-1)/2} \prod_{i=1}^n \prod_{j=1}^n X_{ij} \) in \( \det(X)^n \).

The idea of taking the \( n^{th} \) power of the determinant was used by Stones [9] to obtain another expression for \( L_n^{\text{even}} - L_n^{\text{odd}} \).

**Theorem 1.5.** Let \( B_n \) be the set of \( n \times n \) \((0,1)\)-matrices. For \( A \in B_n \) let \( \sigma_0(A) \) be the number of zero elements in \( A \). Then

\[
(1.1) \quad L_n^{\text{even}} - L_n^{\text{odd}} = (-1)^n \sum_{A \in B_n} (-1)^{\sigma_0(A)} \det(A)^n
\]

It will be shown in the Section 2 that when \( n \) is odd, “hybrid” expressions involving one permanent and \( n-1 \) determinants yield analogous results for \( AT(n) \). In Section 3 an alternative proof of Drisko’s result that \( AT(p) \neq 0 \) for odd primes is shown. In Section 4 a formula linking Conjectures 1.1 and 1.2 is presented. Section 5 introduces a formula relating the permanents of all distinct regular \( p \times p \) adjacency matrices of bipartite graphs (up to renaming the vertices of one of the sides).

2. Formulae for \( AT(n) \)

For \( \alpha \in \text{Sym}(n) \) let \( L_n^{SE}(\alpha) \) (resp. \( L_n^{SO}(\alpha) \)) be the number of symbol-even (resp. symbol-odd) Latin squares with \( \alpha = \alpha_p \). Let \( L_n^{CE}(\alpha) \) (resp. \( L_n^{CO}(\alpha) \)) be the number of column-even (resp. column-odd) Latin squares with \( \alpha \) as the first column. Let \( L_n^{CE}(\alpha, \beta) \) (resp. \( L_n^{CO}(\alpha, \beta) \)) be the number of column-even (resp. column-odd) Latin squares with \( \alpha \) as the first row and \( \beta \) as the first column. We have:
Lemma 2.1. If \( n \) is odd then
\[
\sum_{\pi \in \text{Sym}(n)} \epsilon(\pi)(L^\text{SE}_n(\pi) - L^\text{SO}_n(\pi)) = (-1)^{\frac{n(n-1)}{2}} n!(n-1)! \text{AT}(n)
\]

Proof. Viewing a Latin squares as a set of \( n^2 \) triples \((i,j,k)\), such that \( L_{ij} = k \), and applying the mapping \( \tau : (i,j,k) \rightarrow (i,k,j) \), the \( k^{\text{th}} \) column of \( \tau(L) \) is the permutation \( \alpha_{\pi(k)} \) corresponding to the permutation matrix \( \pi(k) \) in \( L \). Thus \( L^\text{SE}_n(\alpha) = L^\text{CE}_n(\alpha) \) and \( L^\text{SO}_n(\alpha) = L^\text{CO}_n(\alpha) \). We have:
\[
\sum_{\pi \in \text{Sym}(n)} \epsilon(\pi)(L^\text{SE}_n(\pi) - L^\text{SO}_n(\pi)) = \sum_{\pi \in \text{Sym}(n)} \epsilon(\pi)(L^\text{CE}_n(\pi) - L^\text{CO}_n(\pi))
\]
By applying \( \pi^{-1} \) to the columns of each Latin squares with \( \pi \) as its first column we see that if \( n \) is odd then \( \epsilon(\pi)(L^\text{CE}_n(\pi) - L^\text{CO}_n(\pi)) = L^\text{CE}_n(\pi(id)) - L^\text{CO}_n(\pi(id)) \). Thus
\[
\sum_{\pi \in \text{Sym}(n)} \epsilon(\pi)(L^\text{SE}_n(\pi) - L^\text{SO}_n(\pi)) = n!(L^\text{CE}_n(\pi(id)) - L^\text{CO}_n(\pi(id))
\]
Since exchanging columns of a Latin square does not alter the column parity we have that for each \( \beta \in \text{Sym}(n) \) such that \( \beta(1) = 1 \), \( L^\text{CE}_n(\beta(id)) - L^\text{CO}_n(\beta(id)) = L^\text{CE}_n(\beta(id)) - L^\text{CO}_n(\beta(id)) \). Thus
\[
\sum_{\pi \in \text{Sym}(n)} \epsilon(\pi)(L^\text{SE}_n(\pi) - L^\text{SO}_n(\pi)) = n! \sum_{\beta \in \text{Sym}(n) \atop \beta(1)=1} L^\text{CE}_n(\beta(id)) - L^\text{CO}_n(\beta(id))
\]
\[
= n!(n-1)! L^\text{CE}_n(\pi(id)) - L^\text{CO}_n(\pi(id))
\]
We use the notation \( R_{n}^{(+,-)} \) for the number of reduced Latin squares with even row parity and odd column parity \( (R_{n}^{(+,+)}, R_{n}^{(-,+)} \) and \( R_{n}^{(-,-)} \) are defined accordingly). Since \( L^\text{CE}_n(\pi(id)) \) is the number of column-even reduced Latin squares, we have:
\[
L^\text{CE}_n(\pi(id)) - L^\text{CO}_n(\pi(id)) = R_{n}^{(+,+)} + R_{n}^{(-,+)} - R_{n}^{(+,-)} - R_{n}^{(-,-)}
\]
Since
\[
\text{AT}(n) = \begin{cases} R_{n}^{(+,+)} - R_{n}^{(-,-)}, & \text{if } n \equiv 0, 1 \pmod{4} \\ R_{n}^{(-,+)} - R_{n}^{(+,-)}, & \text{if } n \equiv 2, 3 \pmod{4}, \end{cases}
\]
by Section 5 in \cite{1}, the result follows. \( \square \)

We now have a result, analogous to Theorem 1.4 for \( \text{AT}(n) \):

Theorem 2.2. Let \( n \) be odd and let \( X = (X_{ij}) \) be the \( n \times n \) matrix of indeterminates. Then \( \text{AT}(n) \) is the coefficient of \((-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} \prod_{j=1}^{n} X_{ij} \) in \( \frac{1}{n!(n-1)!} \text{per}(X) \text{det}(X)^{n-1} \).

Proof. For \( P \in (S_n)^n \) let \( P = (P_1, P_2, \ldots, P_n) \) and for \( s = 1, \ldots, n \) let \( \alpha_s = \alpha_{P_s} \).
Expanding \( \text{per}(X) \) and \( \text{det}(X) \) we obtain
\[
(2.1) \quad \text{per}(X) \text{det}(X)^{n-1} = \sum_{\pi \in \text{Sym}(n)} \prod_{i} X_{i\pi(i)} \sum_{P \in (S_n)^n} \prod_{s=2}^{n} \epsilon(\alpha_s) \prod_{k=1}^{n} X_{k\alpha_s(k)}.
\]
Now, for each \( \pi \in \text{Sym}(n) \) the number of square-free terms in
\[
\prod_{i=1}^{n} X_{i\pi(i)} \prod_{\pi \in \text{Sym}(n)} \prod_{j=2}^{n} \epsilon(\alpha_j) \prod_{i=1}^{n} X_{i\alpha_j(i)}
\]
is equal to \( \epsilon(\pi)(L_{n}^{\text{SE}}(\pi) - L_{n}^{\text{SO}}(\pi)) \). Hence, by (2.1), the coefficient of \( \prod_{i=1}^{n} \prod_{j=1}^{n} X_{ij} \) in \( \text{per}(X) \text{det}(X) \) is
\[
\sum_{\pi \in \text{Sym}(n)} \epsilon(\pi)(L_{n}^{\text{SE}}(\pi) - L_{n}^{\text{SO}}(\pi)),
\]
and the result follows from Lemma 2.1.

We also have an analogue of Theorem 1.5 for \( \text{AT}(n) \):

**Theorem 2.3.** Let \( B_n \) be the set of \( n \times n \) (0,1)-matrices. For \( A \in B_n \) let \( \sigma_0(A) \) be the number of zero elements in \( A \). If \( n \) is odd then
\[
\text{AT}(n) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{n!(n-1)!} \sum_{A \in B_n} (-1)^{\sigma_0(A)} \text{per}(A) \text{det}(A)^{n-1}
\]

**Proof.** Most of the proof is similar to Stones’ proof of Theorem 1.5. By (2.1),
\[
\sum_{A \in B_n} (-1)^{\sigma_0(A)} \text{per}(A) \text{det}(A)^{n-1} = \sum_{(A,P) \in B_n \times (S_n)^n} Z(A,P)
\]
where
\[
Z(A,P) = (-1)^{\sigma_0(A)} \prod_{i=1}^{n} A_{i\alpha_1(i)} \prod_{s=2}^{n} \epsilon(\alpha_s) \prod_{k=1}^{n} A_{k\alpha_s(k)}.
\]
If for \( (A,P) \) there exists \( i,j \in [n] \) such that \( (P_s)_{ij} = 0 \) for all \( s = 1, \ldots, n \), then let \( A^c \) be the matrix formed by toggling \( A_{ij} \) in the lexicographically first such coordinate \( ij \). Thus \( Z(A,P) = -Z(A^c,P) \) and these two terms cancel in the sum in (2.3). So, on the right hand side of (2.3) we are left only with with \( \sum_{P \in S^*} \epsilon(P) \), where \( S^* = \{(P_1, \ldots, P_n) : \sum_{s=1}^{n} sP_s \text{ is a Latin square}\} \) and \( A \) is the all-1 matrix.

Now,
\[
\sum_{\pi \in \text{Sym}(n)} \prod_{s=2}^{n} \epsilon(\alpha_s) = \sum_{\pi \in \text{Sym}(n)} \epsilon(\pi) \prod_{\pi_{P_1} = \pi \alpha_s} \prod_{s=1}^{n} \epsilon(\alpha_s)
\]
\[
= \sum_{\pi \in \text{Sym}(n)} \epsilon(\pi) \sum_{P \in S^*} \sum_{\pi_{P_1} = \pi} \epsilon_{\text{sym}} \left( \sum_{s=1}^{n} sP_s \right)
\]
\[
= \sum_{\pi \in \text{Sym}(n)} \epsilon(\pi)(L_{n}^{\text{SE}}(\pi) - L_{n}^{\text{SO}}(\pi)),
\]
and the result follows from Lemma 2.1. \( \square \)
3. An alternative proof of Drisko’s theorem

The main result of this section (Corollary 3.6) was first proved by Drisko [3]. An alternative proof, based on the results of Section 2, is presented here. I am indebted to an anonymous reviewer for suggesting this proof.

In this section the rows and columns of an \( n \times n \) matrix will be indexed by the numbers \( 0, 1, \ldots, n - 1 \).

**Definition 3.1.** Let \( A \) be an \( n \times n \) matrix and let \( B \) be a subset of cells of \( A \). Let \( k \) be an integer. The \( k \)-left shift of \( B \) is the set of cells \( \{ b_{i,(j-k) \mod n} : b_{i,j} \in B \} \). The \( k \)-down shift of \( B \) is the set of cells \( \{ b_{(i+k) \mod n,j} : b_{i,j} \in B \} \).

**Definition 3.2.** An \( n \times n \) matrix \( A \) will be said to be \( k \)-left row shifted, for \( 0 < k < n \), if for all \( i = 1, \ldots, n - 1 \), the \( i^{th} \) row of \( A \) is equal to the \( k \)-left shift of the \((i - 1)^{st}\) row, and the \( 0^{th} \) row is equal to the \( k \)-left shift of the \((n - 1)^{st}\) row.

**Remark 3.3.** If \( p \) is an odd prime and \( A \) is a \( p \times p \) \( k \)-left row shifted matrix, then the set of cells of \( A \) is the disjoint union of \( p \) diagonals, where the elements of each diagonal are all equal. These diagonals will be referred to as the principal diagonals of \( A \).

**Lemma 3.4.** Let \( p \) be an odd prime. Let \( A \) be a \( p \times p \) \( k \)-left row shifted \((0,1)\)-matrix. Let \( b \) be the first row of \( A \) and let \( |b| \) be the number of 1’s in \( b \). Then

(i) \( \text{per}(A) \equiv |b| \pmod{p} \)

(ii) \( \text{det}(A) \equiv \pm|b| \pmod{p} \)

**Proof.** Part (i) can be easily obtained from Ryser’s permanent formula ([8], see also [http://mathworld.wolfram.com/RyserFormula.html](http://mathworld.wolfram.com/RyserFormula.html)). However, a different approach, that will also apply to Part (ii), is used here. We define a mapping \( s \) on the set of diagonals of \( A \) as follows: For a diagonal \( d \) in \( A \), \( s(d) \) is obtained by taking the \( k \)-left shift of \( d \) and then taking the 1-down shift of the result. Note that the fixed points of \( s \) are exactly the principal diagonals defined in Remark 3.3. The mapping \( s \) is a bijection and, since \( A \) is \( k \)-left row shifted, \( s(d) \) contain the same set of values as \( d \). In particular, if \( d \) consists only of 1’s, so does \( s(d) \). Also note that \( s^{p}(d) = d \) for all \( d \) and thus, since \( p \) is prime, each orbit under \( s \) is of size one or \( p \). As mentioned above, the orbits of size one are those containing the principal diagonal. Thus, \( \text{per}(A) \pmod{p} \) is equal to the number of principal diagonals consisting only of 1’s, and since there are \( |b| \) such diagonal Part (i) follows.

For Part (ii), it remains to show that all principal diagonals correspond to permutations of the same parity and that \( s \) preserves the parity of the permutation corresponding to the diagonal acted upon. Let \( d_{1} \) and \( d_{2} \) be two diagonals, such that \( d_{1} \) is the \( k \)-left shift of \( d_{2} \). This means that if \( \pi_{1} \) and \( \pi_{2} \) are the corresponding permutations, then \( \pi_{2} = \nu^{k} \circ \pi_{1} \) (application from right to left), where \( \nu = (12 \ldots p) \), which is an even permutation, since \( p \) is odd. If \( d_{1} \) and \( d_{2} \) are principal diagonals then \( d_{1} \) is the \( k \)-left shift of \( d_{2} \) for some \( k \). Thus, all fixed diagonals correspond to permutations of the same parity. If \( d_{1} \) is the \( k \)-down shift of \( d_{2} \), then the corresponding permutations satisfy \( \pi_{1} = \pi_{2} \circ \nu^{k} \). Since \( s \) consists of a left shift and a down shift, \( s \) preserves the parity. This proves (ii). □
Theorem 3.5. Let $p$ be an odd prime. Let $B_p$ be the set of $p \times p$ $(0,1)$-matrices. Then
\[
\frac{1}{p} \sum_{A \in B_p} (-1)^{\sigma_0(A)} \text{per}(A) \det(A)^{p-1} \equiv -1 \pmod{p}.
\]

Proof. Define the group $G = \langle \nu \rangle \times \langle \nu \rangle$, where $\nu = (12\cdots p)$. The group $G$ acts on $B_p$ by permuting the rows and columns, so that for each element of $G$, its first component permutes the order of the rows and the second component permutes the order of the columns. By The Orbit-Stabilizer Theorem, an orbit has size $|G| = p^2$ unless each of its elements has a non-trivial stabilizer in $G$. If $g = (\nu^i, \nu^j)$ is a stabilizer of $A \in B_p$, so is any of its powers, including $(\nu, \nu^k)$ for some $k$, since $p$ is prime. Thus, an orbit has size smaller than $p^2$ if and only if for each matrix $A$ in that orbit there exists some $0 < k < p$ for which $(\nu, \nu^k)A = A$. Let
\[
D = \{ A \in B_p | (\nu, \nu^k)A = A \text{ for some } 0 < k < p \}.
\]
The action of $G$ preserves $\sigma_0$ and, since $\nu$ is an even permutation, it also preserves the permanent and the determinant. We have
\[
\frac{1}{p} \sum_{A \in B_p} (-1)^{\sigma_0(A)} \text{per}(A) \det(A)^{p-1} = \frac{1}{p} \sum_{A \in D} (-1)^{\sigma_0(A)} \text{per}(A) \det(A)^{p-1} \pmod{p}.
\]
Hence, it suffices to prove (3.1) with “$B_p$” replaced by “$D$”.

Suppose $(\nu, \nu^k)A = A$. Then, after applying $\nu^k$ to the $i$th row the $(i + 1)$st is obtained, for $i = 0, \ldots, p - 2$ and applying $\nu^k$ to the $(p - 1)$st row yields the 0th row. This implies that $A$ is a $(p - k)$-left row shifted matrix. Thus, $A$ is uniquely determined by its first row $b$ and the number $k$. We denote this by $A = A(b, k)$.

Now, suppose $A = A(b, k)$ is not the all-1 matrix and let $a = |b|$. Since $p$ is odd, $\sigma_0(A) \equiv a \pmod{2}$. Then, by Lemma 3.4 and Fermat’s Little Theorem, $(-1)^{\sigma_0(A)} \text{per}(A) \det(A)^{p-1} \equiv (-1)^a a \pmod{p}$. For a fixed $a \in \{1, \ldots, p - 1\}$, the number of distinct matrices $A(b, k)$ with $|b| = a$ is $\binom{p}{a}(p - 1)$. Therefore,
\[
\frac{1}{p} \sum_{A \in D} (-1)^{\sigma_0(A)} \text{det}(A)^{p-1} \equiv \frac{1}{p} \sum_{a=1}^{p-1} \binom{p}{a} (p - 1)(-1)^a a \pmod{p},
\]
where the cases that $a \in \{0, p\}$ have been discarded since they correspond to the all-0 and all-1 matrices, which have zero determinant. The result now follows from the binomial identity
\[
\sum_{a=0}^{p} \binom{p}{a} (-1)^a a = 0
\]
(see http://en.wikipedia.org/wiki/Binomial_coefficient).

The following result was first proved by Drisko [3].

Corollary 3.6. If $p$ is an odd prime, then
\[
AT(p) \equiv (-1)^{\frac{p - 1}{2}} \pmod{p}.
\]

Proof. When $n = p$ is an odd prime we can rearrange (1.1) to obtain
\[
AT(p) = (-1)^{\frac{p - 1}{2}} \times \frac{(-1)}{(n - 1)!} \times \frac{1}{p} \sum_{A \in B_p} (-1)^{\sigma_0(A)} \text{per}(A) \det(A)^{p-1}
\equiv (-1)^{\frac{p - 1}{2}} \times (-1) \times (-1) \pmod{p},
\]
by Theorem 3.3. The result follows. □

4. Linking Conjectures 1.1 and 1.2

Proposition 4.1. Let $n$ be odd and let $A_1, A_2, \ldots, A_n$ be $n \times n$ matrices over a field. Then

\begin{equation}
\sum_{\rho, \sigma \in \text{Sym}(n)^n} \epsilon(\sigma_1) \epsilon(\sigma) \epsilon(\rho) \prod_{i,j=1}^n (A_j)_{\sigma(j), \rho(i)} = (n-1)! (R_n^R - R_n^O) \text{per}(A_1) \prod_{j=2}^n \text{det}(A_j).
\end{equation}

Here $\rho_1$ and $\sigma_1$ are the first components in $\rho$ and $\sigma$ respectively. Combining Proposition 4.1 with Theorem 2.2 yields the following identity, linking $AT(n)$ and $R_n^R - R_n^O$.

Theorem 4.2. Let $X = (X_{ij})$ be an $n \times n$ matrix of indeterminates. Then $AT(n) \cdot (R_n^R - R_n^O)$ is the coefficient of $(-1)^{n(n-1)/2} \prod_{i=1}^n \prod_{j=1}^n X_{ij}$ in

\begin{equation}
\frac{1}{n!(n-1)!^2} \sum_{\rho, \sigma \in \text{Sym}(n)^n, \rho_1 = \text{id}} \epsilon(\sigma_1) \epsilon(\sigma) \epsilon(\rho) \prod_{i,j=1}^n X_{\sigma(j), \rho(i)}.
\end{equation}

Proof. This follows by taking $A_1 = A_2 = \cdots = A_n = X$ in (4.1) and applying Theorem 2.2. □

Thus, showing that the above coefficient is nonzero would prove both conjectures.

5. On the Permanent of Adjacency Matrices

The evaluation of the permanent of a matrix is a complex problem, even for adjacency matrices of bipartite graphs (0,1-matrices) (see [3]). Theorem 2.3 leads to an interesting identity involving the permanents of (0,1)-matrices:

Theorem 5.1. Let $p$ be an odd prime, let $B_p$ be the set of $p \times p$ (0,1)-matrices, and let $B_p^r = \{ A \in B_p : \text{det}(A) \not\equiv 0 \pmod{p} \}$. Let $B_p^r$ be a set of representatives in $B_p$ of the row permutation classes. Then

\[ \sum_{A \in B_p^r \cap B_p^r} (-1)^{\sigma_0(A)} \text{per}(A) \equiv -1 \pmod{p}. \]

Proof. Let $B_p^r$ be the subset of $B_p$ containing the regular matrices. From [11] we have:

\[ AT(p) = \frac{(-1)^{\frac{p^2+1}{2}}}{p!(p-1)!} \sum_{A \in B_p^r} (-1)^{\sigma_0(A)} \text{per}(A) \text{det}(A)^{p-1} \]

If $A'$ can be obtained from $A$ by permuting the rows, then per$(A') = \text{per}(A)$ and det$(A')^{p-1} = \text{det}(A)^{p-1}$ (since $p$ is even). Since the rows of each $A \in B_p^r$ are all distinct, each row permutation class in $B_p^r$ contains exactly $p!$ matrices. Let $B_p^r$ be a set of representatives of the row permutation classes in $B_p$. Then

\[ AT(p) = \frac{(-1)^{\frac{p^2+1}{2}}}{(p-1)!} \sum_{A \in B_p^r \cap B_p^r} (-1)^{\sigma_0(A)} \text{per}(A) \text{det}(A)^{p-1}. \]
By Fermat’s little theorem and Wilson’s theorem we have
\[ AT(p) \equiv (-1)^{\frac{p-1}{2}} \sum_{A \in B^1_p \cap B^*_p} (-1)^{\sigma_0(A)} \text{per}(A) \pmod{p}. \]

The result follows from Corollary 3.6. \(\square\)

Remark 5.2. If we view an \(n \times n\) \((0,1)\)-matrix \(A\) as the adjacency matrix of a bipartite graph \(G_A\), having two parts of identical size \(n\), then \(\text{per}(A)\) is the number of perfect matchings in \(G_A\). A set \(B^1_p\), as in Theorem 5.1, represents all possible such graphs, up to renaming the vertices of one of the parts.

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