Supersymmetric Gauge Theories with Matters, Toric Geometries and Random Partitions

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Abstract

We derive the relation between the Hilbert space of certain geometries under the Bohr-Sommerfeld quantization and the perturbative prepotentials for the supersymmetric five-dimensional $SU(N)$ gauge theories with massive fundamental matters and with one massive adjoint matter.

The gauge theory with one adjoint matter shows interesting features. A five-dimensional generalization of Nekrasov’s partition function can be written as a correlation function of two-dimensional chiral bosons and as a partition function of a statistical model of partitions. From a ground state of the statistical model we reproduce the polyhedron which characterizes the Hilbert space.

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1 Introduction

Gauge/Gravity correspondences are fascinating properties suggested from the string theory. Recently, interesting correspondences were found [1] [2] [3] [4] which relate the five-dimensional $SU(N)$ gauge theory compactified on a circle with eight super-charges, a certain toric variety called a local $SU(N)$ geometry and a certain statistical model of partitions which is called a random plane partition model. In this paper we generalize the correspondence to the case of the gauge theory with matters.

The four-dimensional gauge theory with eight super-charges was solved by Seiberg and Witten [5] and its low energy effective action is determined by a holomorphic function which is called a prepotential. The prepotential consists of perturbative terms and non-perturbative terms. Nekrasov and Okounkov solved the theory without matters by using a localization technique and a random partition model [6]. They generalized the partition function to the case of the gauge theory with fundamental matters, with one adjoint matter and the five-dimensional gauge theory on a circle. The partition functions are called Nekrasov’s partition functions. Each of them is factorized to two parts, which are called the perturbative part and the instanton part. The perturbative (non-perturbative) prepotential is obtained from the perturbative (instanton) part of the partition function.

Seiberg et al. determined the perturbative prepotentials for the five-dimensional gauge theories with or without matters [7]. The perturbative prepotential for the $SU(N)$ gauge theory is as follows:

$$\mathcal{F}_{\text{pert}}(a_r; m_f) = \frac{1}{2g_{YM}^2} \sum_{r=1}^{N} a_r^2 + \frac{k_{\text{C.S.}}}{6} \sum_{r=1}^{N} a_r^3 + \frac{1}{12} \left( 2 \sum_{r>s} |a_r - a_s|^3 - \sum_{f} \sum_{w \in W_f} |w \cdot a + m_f|^3 \right)$$

where $a_r$ are vev. of the adjoint scalars, $g_{YM}$ is the gauge coupling constant, $f$ labels the matters, $m_f$ is the mass of the $f$-th matter, $W_f$ are the weights for the $f$-th matter, and $k_{\text{C.S.}}$ is the Chern-Simons coupling constant. For each theory, the cubic terms in the perturbative prepotential are derived from triple-intersection numbers of divisors of a certain Calabi-Yau manifold. However triple-intersection numbers of non-compact divisors are subtle. The author derived the cubic and quadratic terms in the perturbative prepotential for the theory without matters by calculating volume of a certain polyhedron [3].

If the geometry is a toric variety, it is described by a polyhedron $\mathcal{P}$ defined on a lattice $M^\vee$ [8]. Integer lattice points in $\mathcal{P} \cap M^\vee$ can be seen as physical states of the Hilbert space for the geometry under the Bohr-Sommerfeld quantization. The cardinality of $\mathcal{P} \cap M^\vee$ becomes the dimension of the Hilbert space. For the local $SU(N)$ geometry, which is non-compact and can be seen as an ALE space fibred over $\mathbb{CP}^1$, the cardinality is infinite and the dimension is naively infinite. Nevertheless it is regularized by using a complement of $\mathcal{P}$, which is a deviation from a singular limit. With a suitable identification between geometric parameters and gauge theoretic parameters, the perturbative prepotential for the five-dimensional gauge theory without matters emerges from the dimension.

$\mathcal{P}$ can be introduced by means of a Weil divisor $D$. When the toric variety is compact and the polyhedron is enough large for the lattice to be regarded as continuous, the self-intersection
numbers of $D$ and the volume of $\mathcal{P}$ are related as follows:

$$\frac{1}{d!} D^d = \text{Vol}(\mathcal{P}),$$

where $d$ is the dimension of the variety and $D^d$ denotes $d$-times self-intersection of $D$. However for non-compact toric varieties there are no equation like (1.2).

The instanton part of Nekrasov’s partition functions is obtained from the amplitude of the topological A-model strings on the local geometry by using topological vertexes [9] [10]. The topological vertex can be seen as a random plane partition model [12]. The perturbative and instanton part of Nekrasov’s partition function for the five-dimensional gauge theory without matters are derived from the random plane partition model [11]. The polyhedron for the local geometry is reproduced from a ground state of the random plane partition model [3].

In this paper, we generalize the triality, between gauge theories, geometries and statistical models. We reconstruct the perturbative prepotentials for the five-dimensional $SU(N)$ gauge theories with fundamental matters and with one adjoint matter from the dimension of the Hilbert spaces of certain geometries. We express a five-dimensional generalization of Nekrasov’s partition function for the $U(1)$ gauge theory with one adjoint matter as a correlation function of two-dimensional free chiral bosons and as a partition function of a certain statistical model of partitions. From this statistical model we can reproduce the polyhedron for the $SU(N)$ gauge theory with one adjoint matter.

This paper is organized as follows. In Section 2 we derive the relation between certain geometries and perturbative prepotentials for the five-dimensional $SU(N)$ gauge theories with fundamental matters in the fundamental representation. In Section 3 we derive the relation between a certain geometry and the five-dimensional gauge theory with one adjoint matter. In Section 4 we study several properties of Nekrasov’s partition function for the five-dimensional gauge theory with one adjoint matter. In particular, we show that it is expressed as a correlation function of two-dimensional free chiral bosons. In Section 5 we rewrite the correlation function as a partition function of a statistical model of partitions and reproduce the polyhedron for the geometry from a ground state of the statistical model. Relations between fermions and partitions are in appendix A.

2 Perturbative prepotentials with fundamental matters from polyhedrons

2.1 The case of $N_F = 0$

Our goal in this section is to describe the relation between the Hilbert space for a certain toric variety and the perturbative prepotential for the five-dimensional $SU(N)$ gauge theory with massive matters in the fundamental representation. We conveniently start with the case of no matters. In this case, the relevant geometry is called the local $SU(N)$ geometry as described in [3]. It is a non-compact Calabi-Yau manifold, a three-dimensional toric variety and seen as an ALE space fibred over $\mathbb{CP}^1$. The fibration is characterized by an integer $k_{c.s.} \in [1, N]$, which
is called framing. The geometry is quantized by the Bohr-Sommerfeld quantization. States in
the Hilbert space turn to be represented by integer lattice points in a certain polyhedron. The
dimension of the Hilbert space approximates to the volume of this polyhedron as the string
coupling constant $g_{st}$ goes to 0. By a certain identification between geometric parameters
and gauge theoretic parameters, the volume yields the perturbative prepotential for the five-
dimensional gauge theory with no matters.

Recall that in quantum mechanics the Bohr-Sommerfeld quantization is carried out by
imposing a condition on classical paths in a phase space. In the case of a particle moving in
d-dimensional space, the condition for the Bohr-Sommerfeld orbits $C$ is

$$\frac{1}{\hbar} \oint_C \sum_{i=1}^d p_i \, dq^i \in \mathbb{Z},$$

\hspace{3cm} (2.1)

where $q^i$ is the $i$-th component of the coordinate of the particle, $p_i$ is the $i$-th component of
its momentum and $\hbar$ is the Planck constant. There exist a symplectic two-form denoted by
$\sum_{i=1}^d dp_i \wedge dq^i$ on the phase space. For the quantization to be applicable to the sys-
tem, integrals of the symplectic two-form over compact two-cycles in the phase space must be in $\hbar \mathbb{Z}$.

Every $d$-dimensional toric variety is determined by a fan [8]. The fan consists of cones in
$M_{\mathbb{R}}$, where $M_{\mathbb{R}} = M \otimes \mathbb{R}$ and $M$ is a $d$-dimensional lattice. Each cone is generated by a finite
number of vectors in the lattice, which are called primitive vectors, and its apex is at the origin
of $M$. Let $e_i$ ($1 \leq i \leq 3$) be the generators of $M$. If the toric variety is a Calabi-Yau manifold,
the $e_3$ components of all the primitive vectors can be set to one. We denote such vectors by $v_{ij}$:

$$v_{ij} = ie_1 + je_2 + e_3, \quad i, j \in \mathbb{Z}.$$  \hspace{3cm} (2.2)

The fan for the local $SU(N)$ geometry consists of the following three-dimensional cones
besides their faces:

$$\mathbb{R}_{\geq 0} v_{0j} + \mathbb{R}_{\geq 0} v_{0j+1} + \mathbb{R}_{\geq 0} v_{10},$$

$$\mathbb{R}_{\geq 0} v_{0j} + \mathbb{R}_{\geq 0} v_{0j+1} + \mathbb{R}_{\geq 0} v_{-1N-k_{C.S.}},$$

\hspace{3cm} for all $0 \leq j \leq N - 1$,  \hspace{3cm} (2.3)

where $v_{ij}$ are as above. We denote the fan by $\Sigma_{\text{pure}}$ and the set of indices of the primitive
vectors by $S_{\text{pure}}$.

$$S_{\text{pure}} = \{(0, 0), \ldots, (0, N), (1, 0), (-1, N - k_{C.S.})\}.$$  \hspace{3cm} (2.4)

An example of $\Sigma_{\text{pure}}$ is shown in Figure 1.

We apply the Bohr-Sommerfeld quantization to the local $SU(N)$ geometry by identifying
the variety with the phase space, the Kähler two-form with the symplectic two-form and the
string coupling $g_{st}$ with the Planck constant. There are $N$ pieces of $\mathbb{C}P^1$ in the variety: One
is the base $\mathbb{C}P^1$ and the others are in the fiber. The Kähler parameters for the base $\mathbb{C}P^1$ and
$\mathbb{C}P^1$ in the fiber are denoted by $t_B$ and $t_r$ respectively, where $r$ runs from 1 to $N - 1$. These
parameters are quantized for the consistent quantization.

$$T_r = t_r / g_{st}, \quad T_B = t_B / g_{st} \in \mathbb{Z}_{\geq 0}.$$  \hspace{3cm} (2.5)
Figure 1: Σ_{pure} in the case of \( N = 3 \) and \( k_{c.s.} = N \).

We further require the \( SU(N) \) condition \[1\] on the parameters. Let us write \( T_r \) by using \( N \) integers \( p_r \) as \( T_r = p_{N-r+1} - p_{N-r} \). We will impose the condition \( \sum_{r=1}^{N} p_r = 0 \).

The Hilbert space of the quantization can be read from a polyhedron. Let \( M \) the three-dimensional lattice dual to \( M \). The dual pairing is denoted by \( \langle , \rangle \). Let \( e_i^* \) \((1 \leq i \leq 3)\) be the generators of \( M^\vee \), which satisfy \( \langle e_i^*, e_j \rangle = \delta_{ij} \). The polyhedron \( P_{pure} \) is defined as follows:

\[
P_{pure} = \left\{ m \in M^\vee_{\mathbb{R}} \mid \langle m, v_{ij} \rangle \geq -d_{ij}, \forall (i,j) \in S_{pure} \right\}, \tag{2.6}
\]

where \( d_{ij} \) are certain integers and \( M^\vee_{\mathbb{R}} = M^\vee \otimes \mathbb{R} \). The integers appear as the coefficients of a Weil divisor \( D = \sum_{(i,j) \in S_{pure}} d_{ij}D_{ij} \), where each \( D_{ij} \) is an irreducible divisor corresponding to the edge \( \mathbb{R}_{\geq 0}^i v_{ij} \). The polyhedron can be regarded as an image of the moment map of real 3d torus actions. A compact edge of the polyhedron corresponds to \( \mathbb{C}P^1 \) in the geometry and its length corresponds to the Kähler parameter. Therefore \( d_{ij} \) are chosen such that the following combinations give the Kähler parameters.

\[
\begin{align*}
T_r &:= d_{0r-1} - 2d_{0r} + d_{0r+1} \geq 0, \\
T_B &:= -2d_{00} + d_{10} + d_{-1N-k_{c.s.}} + \frac{N - k_{c.s.}}{N} (d_{00} - d_{0N}) \geq 0. \tag{2.7}
\end{align*}
\]

It turns out that the base vectors of the Hilbert space \( H_{pure} \) are labelled by integer lattice points in \( P_{pure} \cap M^\vee \) and the dimension of \( H_{pure} \) is the cardinality of \( P_{pure} \cap M^\vee \). The dimension is infinite since the cardinality is infinite. Nevertheless, we can regularize it \[3\] by considering a deviation from the singular limit where \( T_r \to 0 \) with keeping \( T_B \) fixed. Let \( \Theta_{pure} \) be the polyhedron \( P_{pure} \) which appears at the limit. It is surrounded by four planes: \( K_{00}, K_{0N}, K_{10} \) and \( K_{-1N-k_{c.s.}} \), where each \( K_{ij} \) is a plane and orthogonal to \( v_{ij} \).

\[
K_{ij} = \left\{ m \in M^\vee_{\mathbb{R}} \mid \langle m, v_{ij} \rangle = -d_{ij} \right\}. \tag{2.8}
\]

Let \( P_{pure}^c \) be the complement of \( P_{pure} \) in \( \Theta_{pure} \).

\[
P_{pure}^c = \text{Cl}(\Theta_{pure} \setminus P_{pure}), \tag{2.9}
\]

where Cl means to take the closure. An example of \( P_{pure}^c \) is shown in Figure 2. We define the dimension of \( H_{pure} \) as the cardinality of \( P_{pure}^c \cap M^\vee \):

\[
g_{st} \cdot \dim H_{pure} = -g_{st} \text{Card}(P_{pure}^c \cap M^\vee), \tag{2.10}
\]
Figure 2: The shaded solid in the figure is $\mathcal{P}_{\text{pure}}^c$ in the case of $N = 3$ and $k_{\text{C.S.}} = N$.

where the sign is a convention.

As $g_{\text{st}} \to 0$ with keeping $t_r$ and $t_B$ fixed, the cardinality becomes the volume of $\mathcal{P}_{\text{pure}}^c$, which is denoted by $\text{Vol}(\mathcal{P}_{\text{pure}}^c)$. It is calculated in [3] as follows:

$$\text{Vol}(\mathcal{P}_{\text{pure}}^c) = \sum_{n=1}^{N-1} \frac{1}{3n(n+1)} \left( \sum_{r=1}^{n} rT_r \right)^3 + T_B \sum_{n=1}^{N-1} \frac{1}{2n(n+1)} \left( \sum_{r=1}^{n} rT_r \right)^2$$

$$+ \frac{N - k_{\text{C.S.}}}{6} \sum_{n=1}^{N} \left( \frac{1}{N} \sum_{r=1}^{n-1} (N - r)T_r - \sum_{r=1}^{k-1} T_r \right)^3. \tag{2.11}$$

Let us make a relation between the geometric parameters and the gauge theoretic parameters as follows:

$$g_{\text{st}} = \beta \hbar, \quad \tilde{p}_r = \frac{a_r}{\hbar}, \quad g_{\text{st}}T_B = -2N \ln(\beta \Lambda), \tag{2.12}$$

where $\beta$ is the circumference of the circle in the fifth-direction, $\hbar$ is a parameter, $\tilde{p}_r = p_r + \xi_r$, $\xi_r = \frac{1}{N} \left( r - \frac{N+1}{2} \right)$, $a_r$ are vev. of the adjoint scalar and $\Lambda$ is the scale parameter for the underlying four-dimensional theory. With this identification, the perturbative prepotential for the five-dimensional $SU(N)$ gauge theory with no matters emerges from $\text{dim} \mathcal{H}_{\text{pure}}$ as $\hbar \to 0$.

$$- g_{\text{st}} \text{Vol}(\mathcal{P}_{\text{pure}}^c) \xrightarrow{\hbar \to 0} - \frac{1}{\hbar^2} \mathcal{F}_{\text{pure}}^{\text{pert}}(a_r; \beta, \ln(\beta \Lambda), k_{\text{C.S.}}) + \mathcal{O}(\hbar^{-1}), \tag{2.13}$$

where

$$\mathcal{F}_{\text{pure}}^{\text{pert}}(a_r; \beta, \ln(\beta \Lambda), k_{\text{C.S.}}) = \frac{\beta}{6} \sum_{r<s} a_{rs}^3 - \frac{\beta k_{\text{C.S.}}}{6} \sum_{r=1}^{N} a_r^3 - \ln(\beta \Lambda) \sum_{r>s} a_{rs}^2. \tag{2.14}$$
2.2 The case of $1 \leq N_F \leq 2N$

We describe the relation between the Hilbert space for a certain non-compact toric variety under
the Bohr-Sommerfeld quantization and the perturbative prepotential for the five-dimensional
$SU(N)$ gauge theory with $N_F$ ($1 \leq N_F \leq 2N$) massive fundamental matters. There are more
$N_F$ one-dimensional cones in the fan for the variety than the number of one-dimensional cones in
$\Sigma_{pure}$. As in the previous section, the geometry is quantized and states in the Hilbert space turn
to be labelled by integer lattice points in a certain polyhedron. We can obtain the perturbative
prepotential for the five-dimensional gauge theory with $N_F$ massive fundamental matters from
the volume of the polyhedron.

We introduce the following integers $N_L, N_R, \{l_f\}$ and $\{r_f\}$:

$$N_L \in [1, N], \quad N_R \in [0, N] \quad N_L + N_R = N_F \quad (2.15)$$

The following three-dimensional cones besides their faces provide a fan for the geometry.

$$R \geq 0 v_{0j} + R \geq 0 v_{0j+1} + R \geq 0 v_{1f} \quad \text{for } l_f \leq j \leq l_{f+1} - 1, \quad 0 \leq f \leq N_L,$$
$$R \geq 0 v_{0l_f} + R \geq 0 v_{1f-1} + R \geq 0 v_{1f} \quad \text{for } 1 \leq f \leq N_L,$$
$$R \geq 0 v_{0j} + R \geq 0 v_{0j+1} + R \geq 0 v_{-1f} \quad \text{for } r_f \leq j \leq r_{f+1} - 1, \quad 0 \leq f \leq N_R,$$
$$R \geq 0 v_{0r_f} + R \geq 0 v_{-1f-1} + R \geq 0 v_{-1f} \quad \text{for } 1 \leq f \leq N_R. \quad (2.17)$$

We denote the fan by $\Sigma_{N_F}$ and the set of indices of the primitive vectors by $S_{N_F}$.

$$S_{N_F} = \{(0, 0), \cdots, (0, N), (1, 0), \cdots, (1, N_L), (-1, 0), \cdots, (-1, N_R)\}. \quad (2.18)$$

An example of $\Sigma_{N_F}$ is shown in Figure 3. The other choices of $l_f, r_f$ in (2.16) yield geometries
which are related with each other by flop transitions.

Figure 3: $\Sigma_{N_F}$ in the case of $N = 3, N_L = 3, (l_1, l_2, l_3) = (1, 1, 2), N_R = 1$ and $r_1 = 1$.

We apply the Bohr-Sommerfeld quantization to the geometry. The base vectors of the
Hilbert space $\mathcal{H}_{N_F}$ of the quantization is labelled by points in the intersection of $M^\vee$ and the
following polyhedron $\mathcal{P}_{N_F}$:

$$\mathcal{P}_{N_F} = \{m \in M^\vee_\mathbb{R} | \langle m, v_{ij} \rangle \geq -d_{ij}, \forall (i, j) \in S_{N_F} \}. \quad (2.19)$$
where \( d_{ij} \) are certain integers. The integers appear as the coefficients of a Weil divisor \( D = \sum_{(i,j) \in S_{NF}} d_{ij} D_{ij} \), where each \( D_{ij} \) is an irreducible divisor corresponding to the edge \( \mathbb{R}_{ \geq 0 } u_{ij} \). Their linear combinations are identified with the Kähler parameters by using the moment map. It turns out that the Kähler parameters are determined by \( T_B, T_r \), which are given in (2.7), and the following \( N_F \) parameters \( T_{mf} \):

\[
T_{mf} = \begin{cases} 
\frac{1}{N} (d_0 - d_0 N) - d_1 f_{-1} + d_1 f & \text{for } 1 \leq f \leq N_L \\
\frac{1}{N} (d_0 - d_0 N) - d_{-1} f_{-N_L - 1} + d_{-1} f_{-N_L} & \text{for } N_L + 1 \leq f \leq N_F.
\end{cases} (2.20)
\]

We require

\[
T_B, T_r, T_{mf} \geq 0, \quad \forall r \in [1, N - 1], \quad \forall f \in [1, N_F]. (2.21)
\]

We further require the \( SU(N) \) condition \([\dagger]\) on \( T_r \) and the following conditions\(^2\):

\[
\begin{align*}
T_{1f} & \geq \frac{1}{N} \sum_{r=1}^{N} (N-r) T_r - \sum_{r=1}^{f-1} T_r + T_{mf} \geq 0 & \text{for } 1 \leq f \leq N_L, \\
T_{rf} & \geq \frac{1}{N} \sum_{r=1}^{N} (N-r) T_r - \sum_{r=1}^{f-1} T_r + T_{m_{N_L} + f} \geq 0 & \text{for } 1 \leq f \leq N_R. (2.22)
\end{align*}
\]

\( T_{mf} \) appear in \( \mathcal{M}_R' \) as distances between intersection points of \( K_{ij} \), where \( K_{ij} \) are defined in (2.8).

\[
T_{mf} = \begin{cases} 
(K_{00} \cap K_{0N} \cap K_{10}) e_1^* - (K_{0N} \cap K_{1f_{-1}} \cap K_{1f}) e_2^* & \text{for } 1 \leq f \leq N_L, \\
(K_{00} \cap K_{0N} \cap K_{-10}) e_2^* - (K_{0N} \cap K_{-1f_{-N_L - 1}} \cap K_{-1f_{-N_L}}) e_2^* & \text{for } N_L + 1 \leq f \leq N_F. (2.23)
\end{cases}
\]

where the subscript means to take the \( e_i^* \) component of the points.

Let \( \Theta_{NF} \) be the polyhedron \( \mathcal{P}_{NF} \) which appears at the limit where \( T_r \to 0 \) with keeping \( T_B \) and \( T_{mf} \) fixed. \( \Theta_{NF} \) is surrounded by \( N + N_F + 3 \) planes: \( K_{00}, K_{0N}, K_{10}, \cdots, K_{1N_L}, K_{-10}, \cdots, K_{-1N_R} \). Each apex of \( \Theta_{NF} \) is on \( \mathcal{M}' \) due to the \( SU(N) \) condition. Let \( \mathcal{P}_{NF}^C \) be the complement of \( \mathcal{P}_{NF} \) in \( \Theta_{NF} \).

\[
\mathcal{P}_{NF}^C = \text{Cl}(\Theta_{NF} \setminus \mathcal{P}_{NF}), (2.24)
\]

where \( \text{Cl} \) means to take the closure. An example of \( \mathcal{P}_{NF}^C \) is shown in Figure 4. We regularize the dimension of the Hilbert space by using \( \mathcal{P}_{NF}^C \).

\[
g_{st} \cdot \dim \mathcal{H}_{NF} = -g_{st} \text{Card}(\mathcal{P}_{NF}^C \cap \mathcal{M}'), (2.25)
\]

As \( g_{st} \to 0 \) with keeping \( g_{st} T_B, g_{st} T_r \) and \( g_{st} T_{mf} \) fixed, \( \dim \mathcal{H}_{NF} \) approximates to the volume of \( \mathcal{P}_{NF}^C \) times minus one. As a solid, \( \mathcal{P}_{NF}^C \) is obtained from \( \mathcal{P}_{pure}^C \) in the case of \( k_{C.S.} = N \) by removing two solid \( P_L \) and \( P_R \).

\[
\mathcal{P}_{NF}^C = \mathcal{P}_{pure}^C|_{k_{C.S.}=N} \setminus (P_L \cup P_R) (2.26)
\]

An example of \( P_L \) and \( P_R \) is shown in Figure 5 and the relation of \( \mathcal{P}_{pure}^C, \mathcal{P}_{NF}^C, P_L \) and \( P_R \) is shown in Figure 6. We see below that the matter terms in the perturbative prepotential emerge from the volume of \( P_L \) and \( P_R \).
Figure 4: The shaded solid in the figure is $P_{N_F}^c$ in the case of $N = 3$, $N_L = 3$, $(l_1, l_2, l_3) = (1, 1, 2)$, $N_R = 1$ and $r_1 = 1$.

Figure 5: $P_L$ (left) and $P_R$ (right) in the case of $N = 3$, $N_L = 3$, $(l_1, l_2, l_3) = (1, 1, 2)$, $N_R = 1$ and $r_1 = 1$.

We calculate the volume of $P_L$ and $P_R$. $P_L$ is sliced into triangular pyramids by $K_{1_n}$ ($1 \leq n \leq N_L - 1$) and $K_{0_m}$ ($l_1 + 1 \leq m \leq N - 1$):

$$P_L = \bigcup_{j=1}^{N_L} \bigcup_{i=l_1}^{N-1} P_{i,j-1},$$

where each $P_{i,j}$ is a triangular pyramid surrounded by four planes $K_{0_i}$, $K_{0_{i+1}}$, $K_{1_j}$ and $K_{1_{j+1}}$. The pyramids are classified into two types as in Figure 4 according as $T = -d_{0_i} + d_{0_{i+1}} + d_{1_j} - d_{1_{j+1}}$ takes a positive or negative value. The pyramids in $P_L$ belong to type (a) in Figure 4.

By this condition, the line bundle $\mathcal{O}(D)$ is ample.
The relation of $P_{\text{pure}}$, $P_{N_F}$, $P_L$ and $P_R$ in the case of $N = 3$, $N_L = 3$, $(l_1, l_2, l_3) = (1, 1, 2)$, $N_R = 1$ and $r_1 = 1$.

$\begin{align*}
\text{Figure 6: } & \text{The relation of } P_{\text{pure}}, P_{N_F}, P_L \text{ and } P_R \text{ in the case of } N = 3, N_L = 3, (l_1, l_2, l_3) = (1, 1, 2), N_R = 1 \text{ and } r_1 = 1.
\end{align*}$

$p_{ij}$ in the case of (a) $-d_{0i} + d_{0i+1} + d_{1j} - d_{1j+1} \geq 0$, and (b) $-d_{0i} + d_{0i+1} + d_{1j} - d_{1j+1} \leq 0$.

$\begin{align*}
\text{Figure 7: } & p_{ij} \text{ in the case of (a) } -d_{0i} + d_{0i+1} + d_{1j} - d_{1j+1} \geq 0, \text{ and (b) } -d_{0i} + d_{0i+1} + d_{1j} - d_{1j+1} \leq 0.
\end{align*}$

The volume of such pyramid $p_{ij}$ is $\frac{1}{6}T^3$. Then the volume of $P_L$ becomes

\[
\text{Vol}(P_L) = \sum_{j=1}^{N_L} \sum_{i=l_j}^{N-1} \frac{1}{6} \left( \sum_{r=1}^{i} T_r - \frac{1}{N} \sum_{r=1}^{N-1} (N - r)T_r - T_{m_j} \right)^3, 
\]

where we have used the identities

\[
-d_{0i} + d_{0i+1} + d_{1j} - d_{1j+1} = \sum_{r=1}^{i} T_r - \frac{1}{N} \sum_{r=1}^{N-1} (N - r)T_r - T_{m_j}
\]

for $l_j \leq i \leq N - 1$, $1 \leq j \leq N_L$.

Similarly, $P_R$ is sliced into triangular pyramids and its volume becomes

\[
\text{Vol}(P_R) = \sum_{j=1}^{N_R} \sum_{i=r_j}^{N-1} \frac{1}{6} \left( \sum_{r=1}^{i} T_r - \frac{1}{N} \sum_{r=1}^{N-1} (N - r)T_r - T_{m_{NL+j}} \right)^3.
\]

Let us make a relation between the geometric parameters and the gauge theoretic parameters
as follows:

\[ g_{st} = \beta \hbar, \quad \bar{p}_r = \frac{a_r}{\hbar}, \quad g_{st} T_B = \frac{\beta}{g_{YM}^2} - \frac{\beta}{2} \sum_{f=1}^{N_F} m_f, \quad T_{m_f} = \frac{m_f}{\hbar}, \]

\[
\frac{\beta}{g_{YM}^2} = \begin{cases} 
(2N - N_F) \ln(\beta \Lambda) & \text{if } 1 \leq N_F < 2N \\
\text{const.} & \text{if } N_F = 2N 
\end{cases}
\]

(2.31)

where \( m_f \) are the mass of the fundamental matters and the meanings of the other parameters are same as (2.12). With this identification we have

\[
-g_{st} \text{Card}(P^c_{N_F} \cap M^\vee) \xrightarrow{\hbar \to 0} -g_{st} \text{Vol}(P^c_{N_F}) \\
= -g_{st} \text{Vol}(P^c_{\text{pure}|k_{C.S.}=N}) + g_{st} \text{Vol}(P_L) + g_{st} \text{Vol}(P_R) \\
= -\frac{1}{\hbar^2} F^\text{pert}_{N_F} (a_r; \beta, \frac{\beta}{g_{YM}^2}, N, m_f) - \frac{N \beta}{12 \hbar^2} \sum_{f=1}^{N_F} m_f^3 + O(\hbar^{-1})
\]

(2.32)

where

\[
F^\text{pert}_{N_F} (a_r; \beta, \frac{\beta}{g_{YM}^2}, k_{C.S.}, m_f) = \frac{\beta}{6} \sum_{r>s}^N a_{rs}^3 - \frac{\beta}{12} \sum_{r=1}^N (a_{rs} + m_f)^3 - \frac{\beta (2k_{C.S.} - N_F)}{12} \sum_{r=1}^N a_r^3 \\
+ \frac{\beta}{g_{YM}^2} \frac{1}{2} \sum_{r=1}^N a_r^2.
\]

(2.33)

### 3 Perturbative prepotentials with one adjoint matter from polyhedrons

In this section we derive the relation between the Hilbert space for a certain non-compact geometry with a periodic boundary condition and the perturbative prepotential for the five-dimensional \( SU(N) \) gauge theory with one massive adjoint matter. The non-compact geometry is obtained from a toric variety with certain conditions. As in the previous sections, the geometry is quantized and states in the Hilbert space of the quantization are labelled by integer lattice points in a certain polyhedron. The periodic boundary condition is archived by imposing a periodic boundary condition on the polyhedron. This allows us to consider only a fundamental region of it. The perturbative prepotential for the five-dimensional gauge theory with one massive adjoint matter can be obtained from the volume of the fundamental region.

First of all, forgetting the periodic boundary condition, we consider the toric variety. The fan for the variety consists of the following three-dimensional cones besides their faces:

\[
\mathbb{R}_{\geq 0} v_{i,j-1} + \mathbb{R}_{\geq 0} v_{i,j} + \mathbb{R}_{\geq 0} v_{i+1,j-1}, \quad \forall i \in \mathbb{Z}, \forall j \in [1, N].
\]

(3.1)

We denote the fan by \( \Sigma_{\text{adj}} \). An example of \( \Sigma_{\text{adj}} \) are shown in Figure 8.
We apply the Bohr-Sommerfeld quantization to the geometry. The Hilbert space $H_{adj}$ of the quantization can be read from the following polyhedron $P_{adj}$:

$$P_{adj} = \{ m \in \mathbb{M}^\vee | \langle m, v_{ij} \rangle \geq -d_{ij}, \forall i \in \mathbb{Z}, \forall j \in [0, N] \},$$

(3.2)

where $d_{ij}$ are certain integers. The integers appear as the coefficients of a Weil divisor $D = \sum_{i \in \mathbb{Z}, 0 \leq j \leq N} d_{ij}D_{ij}$, where each $D_{ij}$ is an irreducible divisor corresponding to the edge $\mathbb{R}_{\geq 0}v_{ij}$. Their linear combinations are identified with the Kähler parameters by using the moment map. It turns out that the Kähler parameters are determined by the following parameters:

$$T_{B_i} = d_{i-1}N - d_{i0} - d_{iN} + d_{i+10}, \quad \forall i \in \mathbb{Z}$$
$$T_{r_i} = d_{i-1} - 2d_{ir} + d_{ir+1}, \quad \forall r \in [1, N-1], \forall i \in \mathbb{Z}$$
$$T_{m_i} = \frac{1}{N}(d_{i0} - d_{iN} - d_{i+10} + d_{i+1N}) \quad \forall i \in \mathbb{Z}.$$  

(3.3)

We require the parameters are non-negative and independent of the subscript $i \in \mathbb{Z}$.

$$T_B = T_{B_i} \geq 0, \quad T_r = T_{r_i} \geq 0, \quad T_m = T_{m_i} \geq 0, \quad \forall r \in [1, N-1], \forall i \in \mathbb{Z}.$$  

(3.4)

We further require the following conditions$^3$:

$$T_{s+1} \geq \sum_{r=1}^{s}(T_{r} - T_{r+1}) - \frac{1}{N} \sum_{r=1}^{N-1}(N - r)(T_{r} - T_{r+1}) - T_{m_i} \geq 0, \quad \forall s \in [1, N-2],$$
$$\frac{1}{N} \sum_{r=1}^{N-1}(T_{r} - T_{r+1}) - T_{m_i} \geq 0.$$  

(3.5)

An example of $P_{adj}$ is shown in Figure 9.

By the condition (3.4), $P_{adj}$ can be seen as periodic with periodicity $T_B + NT_m$ along the $e_1^*$ direction. The fundamental region $\tilde{P}_{adj}$ can be taken as follows:

$$\tilde{P}_{adj} = \left\{ m \in \mathbb{M}^\vee \left| \begin{array}{c}
\langle m, v_{ij} \rangle \geq -d_{ij}, \forall i \in [0, 1], \forall j \in [0, N], \\
-T_B/2 - NT_m < \langle m, e_1 \rangle - (K_{00} \cap K_{10}) e_1^* \leq T_B/2
\end{array} \right. \right\}.$$  

(3.6)

$^3$By this condition, the line bundle $\mathcal{O}(D)$ is ample.
Integer lattice points in $\tilde{P}_{\text{adj}} \cap M^\vee$ label the base vectors of the Hilbert space $H_{\text{adj}}$ and the cardinality of $\tilde{P}_{\text{adj}} \cap M^\vee$ is the dimension of the Hilbert space. The dimension can be regularized by considering a deviation from the singular limit where $T_r \to 0$ with keeping $T_B$ and $T_m$ fixed. Let $\tilde{\Theta}_{\text{adj}}$ be the polyhedron $\tilde{P}_{\text{adj}}$ which appears at the limit. Each apex of $\tilde{\Theta}_{\text{adj}}$ is on $M^\vee$ due to the $SU(N)$ condition. Let $\tilde{P}^c_{\text{adj}}$ be the complement of $\tilde{P}_{\text{adj}}$ in $\tilde{\Theta}_{\text{adj}}$.

$$\tilde{P}^c_{\text{adj}} = \text{Cl}(\tilde{\Theta}_{\text{adj}} \setminus \tilde{P}_{\text{adj}}),$$ (3.7)

where Cl means to take the closure. The dimension of the Hilbert space is defined as the cardinality of $\tilde{P}^c_{\text{adj}} \cap M^\vee$.

$$g_{st} \cdot \dim H_{\text{adj}} = -g_{st} \text{Card}(\tilde{P}_{\text{adj}} \cap M^\vee).$$ (3.8)

As $g_{st} \to 0$, the dimension approximates to the volume of $\tilde{P}^c_{\text{adj}}$. As a solid, $\tilde{P}^c_{\text{adj}}$ is obtained by using certain two solids $P_0$ and $P_1$:

$$\tilde{P}^c_{\text{adj}} = (P_0 \cup P_1) \cap \tilde{\Theta}_{\text{adj}},$$ (3.9)

$$P_0 = \left\{ \sum_{i=1}^{3} m_i e_i^* \in M^\vee_R \begin{array}{|c|} \hline \sum_{i=1}^{3} m_i e_i^* \in \mathcal{P}^c_{\text{pure}}|_{k_{C.S.}=0}, \ m_1 - (K_{00} \cap K_{10}) e_1^* \leq T_B/2 \hline \end{array} \right\}$$ (3.10)

$$P_1 = \left\{ \sum_{i=1}^{3} m_i e_i^* \in M^\vee_R \begin{array}{|c|} \hline \begin{aligned} & (m_1 + T_B + N T_m) e_1^* + (m_2 + T_m) e_2^* \\ & + (m_1 + m_3 - (K_{00} \cap K_{10}) e_1^*) e_3^* \in \mathcal{P}^c_{\text{pure}}|_{k_{C.S.}=0}, \end{aligned} \end{array} \begin{array}{|c|} \hline m_1 - (K_{00} \cap K_{10}) e_1^* > -T_B/2 - NT_m \hline \end{array} \right\}. \quad (3.11)$$

The sum of the volume of $P_0$ and $P_1$ is equal to $\text{Vol}(\mathcal{P}^c_{\text{pure}}|_{k_{C.S.}=0})$.

$$\text{Vol}(P_0) + \text{Vol}(P_1) = \text{Vol}(\mathcal{P}^c_{\text{pure}}|_{k_{C.S.}=0}).$$ (3.12)
Then the volume of $\tilde{P}_{adj}^c$ is
\[
\text{Vol}(\tilde{P}_{adj}^c) = \text{Vol}(P^c_{\text{pure}|k_{C.S.}=0}) - \text{Vol}(P_{adj}),
\] (3.13)
where $P_{adj}$ is
\[
P_{adj} = (P_0 \cap P_1) \cup ((P_0 \cup P_1) \setminus ((P_0 \cup P_1) \cap \tilde{\Theta}_{adj})).
\] (3.14)
An example of $P_{adj}$ is shown in Figure 10.

As a solid, $P_{adj}$ is $P_L$, which appear in section 2 in the case of $N_L = N - 1$ and $l_f = f$ for $1 \leq f \leq N - 1$. Hence $P_{adj}$ consists of the triangular pyramids $P_{i,j}$.

\[
P_{adj} = \bigcup_{j=0}^{N-2} \bigcup_{i=j+1}^{N-1} P_{i,j}.
\] (3.15)
By using equalities $-d_0 + d_{0,i+1} + d_{1,j} - d_{1,j+1} = \sum_{r=j+1}^{i} T_r - T_m$ for $0 \leq j < i \leq N - 1$, we obtain
\[
\text{Vol}(P_{adj}) = \frac{1}{6} \sum_{j=0}^{N-2} \sum_{i=j+1}^{N-1} \left( \sum_{r=j+1}^{i} T_r - T_m \right)^3.
\] (3.16)

Let us make an identification between the geometric parameters and the gauge theory parameters as follows:
\[
g_{st} = \beta \hbar, \quad T_r = p_{N-r+1} - p_{N-r}, \quad \tilde{p}_r = \frac{a_r}{\hbar}, \quad T_m = \frac{m}{\hbar},
\]
g_{st}T_B = -\ln q - N/\beta m, \quad q = \exp(2\pi i\tau), \quad \tau = \frac{4\pi i}{g_{YM}},
\] (3.17)
where \( m \) is the mass for the adjoint matter, \( 1/g^2_M \) is the gauge coupling constant and the meanings of the other parameters are same as (2.12). With this identification, we can obtain the perturbative prepotential for the five-dimensional gauge theory with one massive adjoint matter in the case of \( a_{r+1} - a_r > m \) from the volume of \( \tilde{P}_{adj}^c \).

\[
- g_{st} \text{Vol}(\tilde{P}_{adj}^c) = - g_{st} \text{Vol}(P_{pure|k_{C.S.}=0}^c) + g_{st} \text{Vol}(P_{adj}^c)
\]

\[
h \to 0 \quad \frac{1}{h^2} F_{adj}^{pert}(a_r; \beta, - \ln q, 0, m) - \frac{\beta N(N-1)m^3}{12h^2} + O(h^{-1}),
\]

where

\[
F_{adj}^{pert}(a_r; \beta, - \ln q, k_{C.S.}, m) = \frac{\beta}{6} \sum_{r>s}^N a^3_{rs} - \frac{\beta}{12} \sum_{r>s}^N (a_{rs} - m)^3 - \frac{\beta}{12} \sum_{r>s}^N \frac{n}{a_{rs} + m} - \frac{\beta k_{C.S.}}{6} \sum_{r=1}^N a^3_r - \frac{\ln q}{2N} \sum_{r>s}^N a^2_{rs}.
\]

\[ \tag{3.20} \]

4 Several features of the \( \mathcal{N} = 1^* \) gauge theory

The five-dimensional gauge theories with one massive adjoint matter, named the \( \mathcal{N} = 1^* \) gauge theories, have interesting features. We find that a five-dimensional generalization of Nekrasov’s partition function for \( U(1) \) gauge theory can be written as a correlation function of two-dimensional free chiral bosons. This property of Nekrasov’s partition function for the four-dimensional gauge theory is described in [6]. We investigate the partition function for the five-dimensional gauge theory by taking several limits of the parameters. By embedding \( N \) partitions into a single partition, we can also obtain a five-dimensional generalization of Nekrasov’s partition function for the \( SU(N) \) gauge theory from that for \( U(1) \) gauge theory.

4.1 A 5D generalization of Nekrasov’s partition function and 2D CFT

We describe here the relation between a five-dimensional generalization of Nekrasov’s partition function and a correlation function of two-dimensional free chiral bosons. Let

\[
\mu = \frac{m}{\hbar} \in \mathbb{Z}_{\geq 0}, \quad t = e^{-\beta h},
\]

where \( m \) is the mass of the adjoint matter. The following equation will be a five-dimensional generalization of Nekrasov’s partition function for \( U(1) \) gauge theory.

\[
Z_{N=1^*}(q, t, \mu) = \sum_{\lambda} q^{||\lambda||} \prod_{i \neq j} \frac{[\lambda_i - i - \lambda_j + j]_{t^{1/2}}}{[\lambda_i - i + j]_{t^{1/2}}} \frac{[\mu + j - i]_{t^{1/2}}}{[\mu + \lambda_i - i - \lambda_j + j]_{t^{1/2}}},
\]

\[ \tag{4.2} \]

where \( \lambda = (\lambda_1, \lambda_2, \cdots) \) is a partition, which is a non-increasing sequence of non-negative integers, and \( |\lambda| = \sum_{i=1}^{\infty} \lambda_i \). In the above equation, we have used

\[
[n]_{t^{1/2}} = \frac{t^{n/2} - t^{-n/2}}{t^{1/2} - t^{-1/2}},
\]

\[ \tag{4.3} \]
which is called a “$t^{1/2}$-integer”. This partition function is symmetric under $t \leftrightarrow t^{-1}$.

$$Z_{N=1^*}(q, 1/t, \mu) = Z_{N=1^*}(q, t, \mu) \quad (4.4)$$

A two-dimensional chiral boson $\varphi$ is introduced as follows:

$$\varphi(z) = -iJ_0 \ln z + i \sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{1}{n \, z^n}, \quad [J_k, J_{k'}] = k \delta_{k+k',0}. \quad (4.5)$$

We will see below that $Z_{N=1^*}(q, t, \mu)$ is expressed by using $\varphi(z)$ as follows:

$$Z_{N=1^*}(q, t, \mu) = \text{Tr} \left( q^{L_0} \cdot \prod_{n=1}^{\mu} \exp(-i \varphi(t^{-n+\mu+1 \over 2})) : \right), \quad (4.6)$$

where $L_0$ is the zero-mode of the Virasoro algebra and $::$ is the conformal normal ordering. It is convenient for the later use to rewrite (4.2) as follows:

$$Z_{N=1^*}(q,t,\mu) = \sum_{\lambda} q^{\lambda | \lambda} \left( \prod_{i=1}^{l(\lambda)} \prod_{n=1}^{\lambda_i} \frac{[n-i+l(\lambda)+\mu]_{l/2} [n-i+l(\lambda)-\mu]_{l/2}}{[n-i+l(\lambda)]_{l/2}^2} \right)$$

$$\times \left( \prod_{n=1}^{l(\lambda)} \frac{[n-l(\lambda)+\mu]_n [n-l(\lambda)-\mu]_n}{[n-l(\lambda)]_n^{2n} / l^{n/2}} \right)$$

$$\times [\mu]_{l/2} \times \det_{1 \leq i, j \leq l(\lambda)} \frac{[-\lambda_i + i + \lambda_j - j]_{l/2}}{[\mu - \lambda_i + i + \lambda_j - j]_{l/2}}, \quad (4.7)$$

where $l(\lambda)$ is the length of $\lambda$.

Let us see that (4.2) is actually written as (4.4). RHS of (4.6) can be computed by using two-dimensional free fermions $\psi$ and $\psi^*$, which are the fermionization of $\varphi$. The trace in (4.6) becomes the summation over the free fermion states, which are related to partitions as described in Section A.

Let $\Psi_\pm(x)$ and $\Psi^*_\pm(x)$ be the following dressed free fermions:

$$\Psi_\pm(x) = \left( \prod_{n=1}^{\mu} \Gamma_\pm(t^{-n+\mu+1 \over 2}) \right) \psi(x) \left( \prod_{n=1}^{\mu} \Gamma^{-1}_\pm(t^{-n+\mu+1 \over 2}) \right) \quad (4.8)$$

$$= \left( \prod_{n=1}^{\mu} (1 - t^{\pm(-n+\mu+1 \over 2})x^{\pm1})^{-1} \right) \psi(x), \quad (4.9)$$

$$\Psi^*_\pm(x) = \left( \prod_{n=1}^{\mu} \Gamma_\pm(t^{-n+\mu+1 \over 2}) \right) \psi^*(x) \left( \prod_{n=1}^{\mu} \Gamma^{-1}_\pm(t^{-n+\mu+1 \over 2}) \right) \quad (4.10)$$

$$= \left( \prod_{n=1}^{\mu} (1 - t^{\pm(-n+\mu+1 \over 2})x^{\pm1}) \right) \psi^*(x), \quad (4.11)$$
where
\[
\Gamma_{\pm}(z) = \exp(\sum_{k=1}^{\infty} \frac{1}{k} z^{\pm k} J_{\pm k}),
\]
(4.12)
and we have used the following relations in (4.9) and (4.11):
\[
\Gamma_{\pm}(z) \psi(x) \Gamma_{\pm}^{-1}(z) = (1 - z^{\pm 1} x^{\pm 1})^{-1} \psi(x),
\]
(4.13)
\[
\Gamma_{\pm}(z) \psi^*(x) \Gamma_{\pm}^{-1}(z) = (1 - z^{\pm 1} x^{\pm 1}) \psi^*(x).
\]
(4.14)
The functions \( \Gamma_{\pm}(z) \) are introduced in [11] to study Schur process, in particular, the random plane partition model. The mode expansions of \( \Psi_{\pm}(z) \) and \( \Psi_{\pm}^*(z) \) are
\[
\Psi_{\pm}(z) = \sum_{r \in \mathbb{Z}+1/2} \Psi_{r,\pm} z^{-r-1/2}, \quad \Psi_{\pm}^*(z) = \sum_{r \in \mathbb{Z}+1/2} \Psi_{r,\pm}^* z^{-r-1/2}.
\]
(4.15)
By noting (A.7), we find
\[
\text{Tr} \left( q^{L_0} : \prod_{n=1}^{\mu} \exp(-i\phi(t^{-n+\mu+1}/2)) : \right)
= \sum_{\lambda} q^{\lambda|\lambda} \det_{1 \leq i,j \leq l(\lambda)} G_{\lambda_i-1,\lambda_j-1}.
\]
(4.16)
where
\[
G_{ij}^{\lambda} = \langle \phi; -l | \Psi_{i+1/2,+} \Psi_{-j-1/2,+} | \phi; -l \rangle.
\]
(4.17)
The matrix element \( G_{ij}^{\lambda} \) has the following expression.
\[
G_{ij}^{\lambda} = \text{Coeff}_{x^{-1} y^{1/2}} \left( \prod_{n=1}^{t} \frac{1 - t^{n-\mu+1}/2 x^{-1}}{1 - t^{n+\mu+1}/2 y} \right) \left( \frac{x}{y} \right)^{t} \frac{1}{x-y}.
\]
(4.18)
For the case of \( G_{\lambda_i-1,\lambda_j-1}^{\lambda} \), which we want to evaluate, it is sufficient to expand the function in (4.19) by \( x^{-1} \) and \( y \) because \( \lambda_i - i + l \) is positive for all \( 1 \leq i \leq l(\lambda) \). The function in (4.19)
satisfies the following differential equation.

\[
(\mu + x\partial_x + y\partial_y)x \left( \prod_{i=1}^{\mu} \frac{1 - t^{i-\frac{\mu+1}{2}}x^{-1}}{1 - t^{i-\frac{\mu+1}{2}}y} \right) \frac{1}{x - y} = \left( \prod_{i=1}^{\mu} \frac{1 - t^{i-\frac{\mu+1}{2}}x^{-1}}{1 - t^{i-\frac{\mu+1}{2}}y} \right) \left( \sum_{i=1}^{\mu} \frac{1}{(1 - t^{i-\frac{\mu+1}{2}}x^{-1})(1 - t^{i-\frac{\mu+1}{2}}y)} \right) = \sum_{n,m=0}^{\infty} \sum_{k=1}^{\mu} (-1)^n e_{n,k} h_{m,k} x^{-n} y^m ,
\]

where

\[
e_{n,k} = e_n(z_1^{-1}, \ldots, z_{k-1}^{-1}, 0, z_{k+1}^{-1}, \ldots, z_{\mu}^{-1}),
\]

\[
h_{m,k} = h_m(z_1, \ldots, z_{\mu}, z_k).
\]

\(e_n\) is the \(n\)-th elementary symmetric function, \(h_m\) is the \(m\)-th complete symmetric function and \(z_i = t^{-i+\frac{\mu+1}{2}}\) for \(1 \leq i \leq \mu\). From the properties of symmetric functions, \(e_{n,k}\) and \(h_{m,k}\) satisfy the following equations.

\[
e_{n,k} = \sum_{r=0}^{n} (-t^{-k+\frac{\mu+1}{2}})^r e_{n-r}(z_1^{-1}, \ldots, z_{\mu}^{-1}), \quad h_{m,k} = \sum_{r=0}^{m} (t^{-k+\frac{\mu+1}{2}})^r h_{m-r}(z_1, \ldots, z_{\mu}).
\]

The following equations can be found [13].

\[
e_n(z_1^{-1}, \ldots, z_{\mu}^{-1}) = \left[ \begin{array}{c} \mu \\ n \end{array} \right]_{\frac{\mu}{1/2}} , \quad h_m(z_1, \ldots, z_{\mu}) = \left[ \begin{array}{c} \mu + m - 1 \\ m \end{array} \right]_{\frac{\mu+m}{1/2}},
\]

where

\[
\left[ \begin{array}{c} \mu \\ n \end{array} \right]_{\frac{m}{1/2}} = \frac{[\mu]_{\frac{m}{1/2}}!}{[n]_{\frac{m}{1/2}}! [\mu-n]_{\frac{m}{1/2}}!} ,
\]

\[
[n]_{\frac{m}{1/2}}! = [n]_{\frac{m}{1/2}} [n-1]_{\frac{m}{1/2}} \cdots [1]_{\frac{m}{1/2}}.
\]

By using (4.19), (4.20), (4.23) and (4.24), we obtain \(G_{ij}^l\).

\[
G_{ij}^l = (-1)^{i+l} \left[ \begin{array}{c} \mu - n + m - 1 \\ m + l \end{array} \right]_{\frac{m+l}{1/2}} \left[ \begin{array}{c} \mu + m + l \\ n + l \end{array} \right]_{\frac{n+l}{1/2}}.
\]

By using (4.7), (4.17) and (4.26), we arrive the equation (4.6).

### 4.2 Closed expression of the partition function

\(Z_{N=1^*}(q,t,\mu)\) can be written as a summation of products of the skew Schur functions. This allows us to express \(Z_{N=1^*}(q,t,\mu)\) in a closed form.
For infinite variables \( \{x_1, x_2, \cdots \} \) and \( \{y_1, y_2, \cdots \} \), there are the following equations.

\[
\langle \nu | \prod_{i=1}^{\infty} \Gamma_+ (x_i) | \lambda \rangle = s_{\lambda/\nu}(x_i),
\]
(4.27)

\[
\langle \lambda | \prod_{i=1}^{\infty} \Gamma_- (y_i) | \nu \rangle = s_{\lambda'/\nu'}(y_i),
\]
(4.28)

where \( s_{\lambda/\nu} \) is the skew Schur function, \( \lambda \) and \( \nu \) are partitions and \( \lambda' \) is the transpose of \( \lambda \). The following equation is an identity [13].

\[
\sum_{\nu, \lambda} q^{\mid \lambda \mid \nu} s_{\lambda/\nu}(x_j) s_{\lambda'/\nu'}(y_k) = \prod_{i=1}^{\infty} \left\{ (1 - q^i)^{-1} \prod_{j,k=1}^{\mu} (1 + q^i x_j y_k) \right\}.
\]
(4.29)

Then we find the partition function is written as follows:

\[
Z_{N=1^*}(q, t, \mu) = \sum_{\nu, \lambda} q^{\mid \lambda \mid} s_{\lambda/\nu}(t^{\mu+1}) s_{\lambda'/\nu'}(-t^{\mu+1})
\]

\[
= \prod_{i=1}^{\infty} \left\{ (1 - q^i)^{-1} \prod_{j,k=1}^{\mu} (1 - q^j t^{-j-k+\mu+1}) \right\}
\]

\[
= \left( q^{-\frac{1}{24}} \eta(\tau) \right)^{-\frac{1}{2}(\mu-2)(\mu-1)} \prod_{n=1}^{\mu-1} \left( \frac{\theta_{11}(n \ln t, \tau)}{iq^{1/8}(tn/2 - t^{-n/2})} \right)^{\mu-n},
\]
(4.30)

where \( \eta(\tau) \) is the Dedekind eta function and \( \theta_{11}(\nu, \tau) \) is the theta function:

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),
\]
(4.31)

\[
\theta_{11}(\nu, \tau) = -2 \exp(\pi i \tau/4) \sin(\pi \tau) \prod_{n=1}^{\infty} (1 - q^n)(1 - zq^n)(1 - z^{-1}q^n),
\]
(4.32)

where \( q = \exp(2\pi i \tau) \) and \( z = \exp(2\pi i \nu) \).

### 4.3 Several limits of the gauge theory

We examine certain limits of the gauge theory, in particular, the limit to the four-dimensional gauge theory and to the five-dimensional gauge theory without matters.

As \( \beta \to 0 \ (t \to 1) \), \( Z_{N=1^*}(q, t, \mu) \) becomes Nekrasov’s partition function for the four-dimensional \( \mathcal{N} = 2^* \) gauge theory which is described in [6].

\[
Z_{N=1^*}(q, t, \mu) \xrightarrow{\beta \to 0} \sum_{\lambda} \langle \lambda; 0 | q^{L_0} \exp\left(-\sum_{k=1}^{\infty} \frac{\mu}{k} J_{-k} \right) \exp\left(\sum_{k=1}^{\infty} \frac{\mu}{k} J_k \right) | \lambda; 0 \rangle.
\]
(4.33)
To obtain the five-dimensional gauge theory without matters, we decouple the adjoint matter by letting $\mu \to \infty$ with keeping $qt^\mu = Q$ fixed.

$$Z_{N=1^*}(q, t, \mu) = \sum_\lambda t^{\mu|\lambda|} \langle \lambda; 0 \rangle \left( \prod_{n=1}^\mu \Gamma_+ (t^{n-1/2}) \right) \left( \prod_{n=1}^\mu \Gamma_- (t^{-n+1/2}) \right) |\lambda; 0\rangle,$$

$$\mu \to \infty, qt^\mu = Q \quad \langle \phi; 0 \rangle \left( \prod_{n=-\infty}^{-1} \Gamma_+ (t^{-(n+1/2)}) \right) Q^L_0 \left( \prod_{n=0}^\infty \Gamma_- (t^{-(n+1/2)}) \right) |\phi; 0\rangle, \quad (4.34)$$

where we have used

$$t^{\mu|\lambda|} \xrightarrow{\mu \to \infty} \begin{cases} 1 & \lambda = \phi \\ 0 & \lambda \neq \phi. \end{cases} \quad (4.35)$$

By setting $Q = \beta \Lambda$, (4.34) becomes Nekrasov’s partition function for the five-dimensional gauge theory without matters [6].

### 4.4 $N$ partitions and $N=1^*$ $SU(N)$ gauge theories

By embedding $N$ partitions ($N$ component fermions) into a single partition (one component fermions), we can obtain the partition function for the $N=1^*$ $SU(N)$ gauge theory from $Z_{N=1^*}(q, t, \mu)$. With the embedding, $Z_{N=1^*}(q, t, \mu)$ is factorized into two parts: One is written as summation over $N$ partitions and the other is the remaining part. We obtain the instanton part of Nekrasov’s partition function from the first part and the perturbative prepotential from the second part. For the case of the gauge theory with no matters this property is shown in [1].

Let $\psi^{(r)}$ and $\psi^{(r)*}$ be $N$ component fermions labelled by $r \in [1, N]$. We embed the $N$ component fermions into one component fermions in the standard fashion [14];

$$\psi_s^{(r)} = \psi_{N(s-\xi_r)} \quad \psi_s^{(r)*} = \psi_{N(s+\xi_r)} \quad (4.36)$$

With this embedding, states for the $N$ component fermions are mapped to states for the one component fermions bijectively. By using the standard correspondence between fermion states and partitions, for each set of $N$ partitions $\lambda^{(r)}$ and their charges $p_r$, the corresponding single partition $\nu(\lambda^{(r)}, p_r)$ and its charge $P$ are obtained. The relations between them are such that:

$$\{ x_i (\nu(\lambda^{(r)}, p_r)) + P; i \geq 1 \} = \bigcup_{r=1}^N \{ N(x_i (\lambda^{(r)}) + \bar{p}_r); i \geq 1 \}, \quad (4.37)$$

where $x_i(\lambda) = \lambda_i - i + 1/2$. From (4.37), the following two equations are found.

$$P = \sum_{p_r} p_r, \quad (4.38)$$

$$|\nu(\lambda^{(r)}, p_r)| = N \sum_{r=1}^N |\lambda^{(r)}| + N \sum_{r=1}^N p_r^2 + \sum_{r=1}^N r p_r. \quad (4.39)$$
For a set of charges \( \{p_r\} \), there is a unique partition which depends only on the charges. We call it a ground partition \( \lambda_{GP} \): \( \lambda_{GP} = \nu(\phi^{(r)}, p_r) \).

Summation over partitions in \((4.2)\) is expressed as summation over \( N \) partitions \( \lambda^{(r)} \) and their charges \( p_r \). The charges are restricted to satisfy \( \sum_{r=1}^{N} p_r = 0 \) owing to the charge conservation \((4.38)\). We factorize \( Z_{N=1^*}(q, t, \mu) \) into two parts: One is written as summation over \( N \) partitions and the other is the remaining part. We denote the first part by \( Z_{\text{inst}}^{*}(q, p_r, \mu) \) and the second part by \( Z_{\text{pert}}^{*}(q, p_r, \mu) \).

\[
Z_{N=1^*}(q, t, \mu) = \sum_{\{p_r\}} \sum_{\{\lambda^{(r)}\}} \prod_{(r,i) \neq (s,j)} \frac{[N(x_i(\lambda^{(r)}) + \tilde{p}_r - x_j(\lambda^{(s)}) - \tilde{p}_s)]_{t_1/2}}{[N(x_j(\phi^{(r)}) - x_i(\phi^{(s)}))]_{t_1/2}} \frac{[N(x_j(\phi^{(r)}) - x_i(\phi^{(s)}))]_{t_1/2}}{[N(\mu/N + x_j(\phi^{(r)}) - x_i(\phi^{(s)}))]_{t_1/2}} \times [N(\mu/N + x_i(\lambda^{(r)}) + \tilde{p}_r - x_j(\lambda^{(s)}) - \tilde{p}_s)]_{t_1/2} \]

\[
= \sum_{\{p_r\}} \sum_{\{\lambda^{(r)}\}} \prod_{(r,i) \neq (s,j)} \frac{[N(x_i(\lambda^{(r)}) + \tilde{p}_r - x_j(\lambda^{(s)}) - \tilde{p}_s)]_{t_1/2}}{[N(x_j(\phi^{(r)}) - x_i(\phi^{(s)}))]_{t_1/2}} \frac{[N(x_j(\phi^{(r)}) - x_i(\phi^{(s)}))]_{t_1/2}}{[N(\mu/N + x_j(\phi^{(r)}) - x_i(\phi^{(s)}))]_{t_1/2}} [N(\mu/N + x_i(\lambda^{(r)}) + \tilde{p}_r - x_j(\lambda^{(s)}) - \tilde{p}_s)]_{t_1/2} \]

\[
= \sum_{\{p_r\}} Z_{\text{inst}}^{*}(q, p_r, \mu) \cdot Z_{\text{pert}}^{*}(q, p_r, \mu), \tag{4.40}
\]

where we have used \((4.39)\) in the third line.

With the following identification, we can interpret \( Z_{\text{inst}}^{*}(q, p_r, \mu) \) and \( Z_{\text{pert}}^{*}(q, p_r, \mu) \) respectively as the instanton part and the perturbative part of a five-dimensional generalization of Nekrasov’s partition function for \( SU(N) \) gauge theory.

\[
\tilde{p}_r = a_r/h, \quad t = \exp(-\frac{\beta \hbar}{N}), \quad \mu = \frac{N m}{\hbar}, \quad \ln q = 2 \pi i \tau, \quad \tau = \frac{4 \pi i}{g_Y^2 M}. \tag{4.41}
\]

\[
Z_{\text{inst}}^{*}(q, p_r, \mu) = \sum_{\lambda^{(r)}} e^{2 \pi i \tau \sum_{r=1}^{N} |\lambda^{(r)}|} \prod_{(r,i) \neq (s,j)} \frac{\sinh(\frac{\beta \hbar}{2}(a_{rs}/h + \lambda_i^{(r)} - \lambda_j^{(s)} + j - i))}{\sinh(\frac{\beta \hbar}{2}(a_{rs}/h + j - i))} \frac{\sinh(\frac{\beta \hbar}{2}((m + a_{rs})/h + j - i))}{\sinh(\frac{\beta \hbar}{2}((m + a_{rs})/h + j - i))}. \tag{4.42}
\]

To see the emergence of the perturbative prepotential from \( Z_{\text{pert}}^{*}(q, p_r, \mu) \), we rewrite it as
follows:

\[
Z_{\text{pert}}(q, p_r, \mu) = q^{\lambda_{GP}} \prod_{i \neq j} \frac{[\lambda_{GP} - \lambda_{GP} + j - i t_{1/2}]}{[j - i t_{1/2}]} \frac{[\mu + j - i t_{1/2}]}{[\mu + \lambda_{GP} - \lambda_{GP} + j - i t_{1/2}]} \\
= q^{\lambda_{GP}} \prod_{(i,j) \in \lambda_{GP}} \frac{[h(i, j) + \mu t_{1/2}]}{[h(i, j)]^{t_{1/2}}} \frac{[h(i, j) - \mu t_{1/2}]}{[h(i, j)]^{t_{1/2}}} \tag{4.43}
\]

where \( h(i, j) \) is the hook length for the box \((i, j) \in \lambda_{GP}: h(i, j) = \lambda_{GP} - i + \lambda_{GP} - j \). The hook length are easily obtained by bicolouring each box in \( \lambda_{GP} \). We bicolour each box \((i, j) \in \lambda_{GP} \) by \((r, s)\)-colour \((r, s \in [1, N])\) when \( i \) and \( j \) satisfy \( x_i(\lambda_{GP}) \in \mathbb{N}Z + r - \frac{1}{2} \) and \( x_j(\lambda_{GP}^t) \in \mathbb{N}Z - s + \frac{1}{2} \). An example of the bicolouring for the \( SU(3) \) ground partition is shown in Figure 11. The collection of \((r, s)\)-coloured box forms a partition, which we call \((r, s)\)-coloured partition \( \lambda_{rs} \). The partition is \( \lambda_{rs} = (T_{rs}, T_{rs} - 1, \ldots, 1) \), where

\[
T_{rs} = \sum_{n=N-r+1}^{N-s} T_n, \tag{4.44}
\]

and \( T_n \) are defined as differences of the charges: \( T_n = p_{N-n+1} - p_{N-n} \). The box \((i, j) \in \lambda_{rs} \) corresponds to a box in \( \lambda_{GP} \) and its hook length is \( N(T_{rs} - i - j + 2) + r - s \). Hence there are

\[
\begin{pmatrix}
(3, 2) \\
(2, 1) \\
(3, 1)
\end{pmatrix}
\]

Figure 11: The example of bicolouring in the case of \( N = 3 \), \( T_1 = 2 \) and \( T_2 = 5 \). In this case the ground partition is \( \lambda_{GP} = (9, 7, 5, 5, 4, 4, 3, 3, 2, 2, 1, 1) \). The upper-right parenthesis in the figure means the map between the shading patterns of boxes and the \((r, s)\)-colours of boxes.

partition \( \lambda_{rs} \). The partition is \( \lambda_{rs} = (T_{rs}, T_{rs} - 1, \ldots, 1) \), where

\[
T_{rs} = \sum_{n=N-r+1}^{N-s} T_n, \tag{4.44}
\]

and \( T_n \) are defined as differences of the charges: \( T_n = p_{N-n+1} - p_{N-n} \). The box \((i, j) \in \lambda_{rs} \) corresponds to a box in \( \lambda_{GP} \) and its hook length is \( N(T_{rs} - i - j + 2) + r - s \). Hence there are
The $T_{rs} - n$ boxes whose hook length are $N n + r - s$. As $\hbar \to 0$, each part of $Z_{\text{pert}}(q, p_r, \mu)$ become

$$|\lambda_{GP}| \xrightarrow{h \to 0} \frac{1}{\hbar^2} \sum_{r \succ s} a_{rs}^2 + O(h^{-1}),$$

$$\sum_{(i,j) \in \lambda_{GP}} \ln[h(i, j) + \mu]_{t/2} \xrightarrow{h \to 0, \beta \to \infty} \frac{\beta}{\hbar^2} \sum_{r \succ s} \left( \frac{1}{12} (a_{rs} + m)^3 - \frac{1}{12} (3 a_{rs} m^2 + m^3) \right) + O(h^{-1}),$$

$$\sum_{(i,j) \in \lambda_{GP}} \ln[h(i, j) - \mu]_{t/2} \xrightarrow{h \to 0, \beta \to \infty} \frac{\beta}{\hbar^2} \sum_{r \succ s} \left( \frac{1}{12} (a_{rs} - m)^3 + \frac{1}{12} (3 a_{rs} m^2 - m^3) \right) + \frac{i}{\hbar^2} O(T_{rs}) + O(h^{-1}),$$

$$\sum_{(i,j) \in \lambda_{GP}} \ln[h(i, j)]_{t/2} \xrightarrow{h \to 0, \beta \to \infty} \frac{\beta}{\hbar^2} \sum_{r \succ s} \left( \frac{1}{12} a_{rs}^3 \right) + O(h^{-1}),$$

where we assumed $a_{r+1} - a_r > \mu$ and $\beta \gg 1$ to compare the dimension counting argument in Section 3. We can obtain the perturbative prepotential for the five-dimensional $\mathcal{N} = 1^*$ SU$(N)$ gauge theory from $Z_{\text{pert}}(q, p_r, \mu)$.

$$\ln Z_{\text{pert}}(q, p_r, \mu) \xrightarrow{h \to 0, \beta \to \infty} \frac{-1}{\hbar^2} F_{\text{adj}}(a_r; \beta, -2 N \pi i \tau, 0, m) - \frac{\beta N (N - 1)}{12 \hbar^2} m^3 + \frac{i}{\hbar^2} O(T_{rs}) + O(h^{-1}),$$

where $F_{\text{adj}}(a_r; \beta, -2 N \pi i \tau, 0, m)$ is given in (3.20).

## 5 Statistical models of partitions and Polyhedrons

Nekrasov’s partition function for the five-dimensional $U(1)$ gauge theory with no matters is expressed as a partition function of a random plane partition model [14]. For the case of the partition function $Z_{\mathcal{N}=1^*}(q, t, \mu)$, we can rewrite it as a partition function of a statistical model of partitions. Each ground partition corresponds to a ground state of the model. We can reproduce the polyhedron $\widetilde{\mathcal{P}}_{\text{adj}}$ from the ground state. The reproduction of the polyhedron in the case of the gauge theory with no matter is described in [3].

For partitions $\lambda$ and $\rho$, the following equations hold.

$$\langle \lambda | \Gamma_+(1) | \rho \rangle = \begin{cases} 1 & \text{if } \lambda \prec \rho \\ 0 & \text{if } \lambda \npreceq \rho \end{cases} \quad (5.1)$$

$$\langle \lambda | \Gamma_-^{-1}(1) | \rho \rangle = \begin{cases} (-1)^{|\lambda| - |\rho|} & \text{if } \lambda^t \succ \rho^t \\ 0 & \text{if } \lambda^t \nprecirc \rho^t \end{cases} \quad (5.2)$$

where $\lambda \succ \rho$ means $\lambda_1 \geq \rho_1 \geq \lambda_2 \geq \rho_2 \geq \cdots$. By using (5.1) and (5.2), $Z_{\mathcal{N}=1^*}(q, t, \mu)$ is written as a partition function of a statistical model of partitions.

$$Z_{\mathcal{N}=1^*}(q, t, \mu) = \sum_{\lambda} \langle \lambda | 0 | t^{-\mu} L_0 (t^{L_0} \Gamma_+(1) \cdots t^{L_0} \Gamma_+(1)) (q t^{-\mu}) L_0 (t^{L_0} \Gamma_-^{-1}(1) \cdots t^{L_0} \Gamma_-^{-1}(1)) | \lambda ; 0 \rangle$$

$$= \sum_{\pi} \left( \prod_{n=-\mu+1}^{\mu-1} t^{|\pi(n)|} \right) (-q t^{-\mu})^{|\pi(0)|} (-t^{-\mu+1})^{|\pi(\mu)|} \quad (5.3)$$

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where $\pi$ is a sequence of partitions $\pi(n)$ ($-\mu \leq n \leq \mu$), which satisfy the following relations,

$$
\pi(-\mu) < \pi(-\mu + 1) < \cdots < \pi(0), \quad \pi(0)^t > \pi(1)^t > \cdots > \pi(\mu)^t, \quad \pi(-\mu) = \pi(\mu).
$$

(5.4)

$\pi$ can be seen as a transposed version of plane partitions. In the case of a plane partition $\nu$, it is written as a sequence of partitions $\nu(n)$ satisfying $\cdots < \nu(-2) < \nu(-1) < \nu(0) > \nu(1) > \nu(2) > \cdots$ and is seen as evolutions of $\nu(n)$ along $n$ from $n = -\infty$ to $n = +\infty$. In the case of $\pi$, it can be seen as a periodic evolution of partition $\pi(n)$ along $n$. However, by the effect of $(q^{-\mu})^{\nu(0)}$ and $t^{\mu|\pi(n)|}$, the evolution becomes more complex than the one for plane partitions. The effect of $(q^{-\mu})^{\nu(0)}$ is moving $\pi(0)$ ahead by $-\ln q/(\alpha \beta) - \mu$ steps and the effect of $t^{\mu|\pi(n)|}$ is moving $\pi(\mu)$ behind by $\mu$ steps. Then the period of the evolution becomes $-\ln q/(\beta \hbar)$. This is equal to the period of $\tilde{P}_{adj}$.

For the ground partition $\lambda_{GP}$, we can define the following set $P_{GP}$ of sequences of partitions satisfying (5.4).

$$
P_{GP} = \{ \pi|\pi(0) = \lambda_{GP} \}.
$$

(5.5)

There exists only one sequence of partitions, which denoted by $\pi_{GPP}$, whose number of boxes achieves the minimum in $P_{GP}$.

$$
\exists \pi_{GPP} \in P_{GP}, \text{ s.t. } |\pi_{GPP}| \leq |\pi|, \forall \pi \in P_{GP}
$$

(5.6)

$\pi_{GPP}$ is the ground state of the statistical model. The explicit form of $\pi_{GPP}$ is as follows:

$$
\pi_{GPP}(n) = \begin{cases} 
\max\{\lambda_{GP} - n, \lambda_{i,\mu}^\mu\} & \text{for } (n \geq 0) \\
\max\{\lambda_{GP} + n, \lambda_{i,\mu}^\mu\} & \text{for } (n < 0),
\end{cases}
$$

(5.7)

where $\lambda_{i,\mu}^\mu$ is the following partition:

$$
\lambda_{i,\mu}^\mu = \max\{\lambda_{GP} + n, \lambda_{GP} - \mu\}.
$$

(5.8)

To make a relation between $\pi_{GPP}$ and $\tilde{P}_{adj}^c$, we define the following sequence of partitions $\Upsilon(n)$, $(\ln q/2\beta \hbar - \mu/2 < n \leq -\ln q/2\beta \hbar - \mu/2)$:

$$
\Upsilon(n)_i = \begin{cases} 
\pi_{GPP}(0)_i & \text{if } \frac{\ln q}{2\beta \hbar} - \frac{\mu}{2} < n \leq -\mu \\
\max\{\pi_{GPP}(n)_i, \pi_{GPP}(\mu + n)_i\} & \text{if } -\mu < n \leq 0 \\
\pi_{GPP}(n)_i & \text{if } 0 < n \leq -\frac{\ln q}{2\beta \hbar} - \frac{\mu}{2}.
\end{cases}
$$

(5.9)

Let $M_N^\nu$ be a lattice generated by $e_1^*, 1/N e_2^*$ and $e_3^*$. We can reproduce all points $m + (K_{0N} \cap K_{10} \cap K_{1N}) \in \tilde{P}_{adj}^c \cap M_N^\nu$ from the partitions $\Upsilon(n)$ $(\ln q/2\beta \hbar - \mu/2 < n \leq -\ln q/2\beta \hbar - \mu/2)$ bijectively by the following map:

$$
m = \begin{cases} 
ne_1^* + \frac{1}{N}(-\mu + j - i + 1)e_2^* + (\mu - n + i - 1)e_3^* & \text{for } \ln q/2\beta \hbar - \mu/2 < n \leq -\mu \\
n1N e_2^* + (n + j - i + 1)e_2^* + (-n + i - 1)e_3^* & \text{for } -\mu < n \leq 0 \\
ne_1^* + \frac{1}{N}(j - i + 1)e_2^* + (i - 1)e_3^* & \text{for } 0 < n \leq -\ln q/2\beta \hbar - \mu/2,
\end{cases}
$$

where $(i, j) \in \Upsilon(n)$.
A Partitions and fermions

A partitions \( \lambda \) is a sequence of non-negative integers: \( \lambda = (\lambda_1, \lambda_2, \cdots) \). \( l(\lambda) \) is the number of non-zero integers in \( \lambda \) and called the length of \( \lambda \).

Let \( \psi(z) = \sum_{r \in \mathbb{Z} + 1/2} \psi_r z^{-r-1/2} \) and \( \psi^*(z) = \sum_{r \in \mathbb{Z} + 1/2} \psi^*_r z^{-r-1/2} \) be two-dimensional free fermions with anti-commutation relations \( \{ \psi_r, \psi^*_s \} = \delta_{r+s,0} \). The vacuum state \( |\phi\rangle \) and its dual state \( \langle \phi| \) are defined as follows:

\[
\psi_r |\phi\rangle = 0, \quad \psi^*_r |\phi\rangle = 0 \quad \text{for } r > 0.
\]
\[
\langle \phi| \psi_r = 0, \quad \langle \phi| \psi^*_r = 0 \quad \text{for } r < 0. \tag{A.1}
\]

\( J_k \) are modes of the \( U(1) \) current for the free fermions and written as follows:

\[
J_k = \sum_{r \in \mathbb{Z} + 1/2} : \psi_{k-r} \psi^*_r : . \tag{A.2}
\]

There is a bijective map between states of the fermions and partitions.

\[
|\lambda; p\rangle = (\psi_{-x_1(\lambda)-p} \cdots \psi_{-x_1(\lambda)}-p)(\psi_{x_1(\phi)+p} \cdots \psi_{x_1(\phi)+p})|\phi; p\rangle \tag{A.3}
\]
\[
\langle \lambda; p| = \langle \phi; p| \psi_{-x_1(\phi)-p} \cdots \psi_{-x_1(\phi)-p})(\psi_{x_1(\lambda)+p} \cdots \psi_{x_1(\lambda)+p}), \tag{A.4}
\]

where \( \lambda \) is a partition, \( p \) is its charge, \( l = l(\lambda) \) and

\[
|\phi; p\rangle = \begin{cases} 
\psi_{-p+1/2} \cdots \psi_{-1/2} |\phi; 0\rangle & \text{if } p \geq 0 \\
\psi^*_{p+1/2} \cdots \psi^*_{-1/2} |\phi; 0\rangle & \text{if } p < 0 
\end{cases} \tag{A.5}
\]

\( |\lambda; 0\rangle \) and \( \langle \lambda; 0| \) are written as follows:

\[
|\lambda; 0\rangle = (\psi_{-x_1(\lambda)} \cdots \psi_{-x_1(\lambda)}) |\phi; -l(\lambda)\rangle \tag{A.6}
\]
\[
\langle \lambda; 0| = \langle \phi; -l(\lambda)| (\psi^*_{x_1(\lambda)} \cdots \psi^*_{x_1(\lambda)}). \tag{A.7}
\]

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