MAXIMAL EQUICONTINUOUS GENERIC FACTORS AND WEAK MODEL SETS

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Abstract. The orbit closures of regular model sets generated from a cut-and-project scheme given by a co-compact lattice \( \mathcal{L} \subseteq G \times H \) and compact and aperiodic window \( W \subseteq H \), have the maximal equicontinuous factor (MEF) \( (G \times H) / \mathcal{L} \), if the window is topologically regular. This picture breaks down, when the window has empty interior, in which case the MEF is trivial, although \( (G \times H) / \mathcal{L} \) continues to be the Kronecker factor for the Mirsky measure. As this happens for many interesting examples like the square-free numbers or the visible lattice points, a weaker concept of topological factors is needed, like that of generic factors [24]. For topological dynamical systems that possess a finite invariant measure with full support (E-systems) we prove the existence of a maximal equicontinuous generic factor (MEGF) and characterize it in terms of the regional proximal relation. This part of the paper profits strongly from previous work by McMahon [33] and Auslander [2]. In Sections 3 and 4 we determine the MEGF of orbit closures of weak model sets and use this result for an alternative proof (of a generalization) of the fact [34] that the centralizer of any B-free dynamical system of Erdős type is trivial.

1. Introduction. Let \( G \) and \( H \) be locally compact second countable groups. In many examples \( G = \mathbb{Z}^d \) or \( \mathbb{R}^d \), whereas \( H \) will often be a more general group. Each pair \( (\mathcal{L}, W) \) consisting of a cocompact lattice \( \mathcal{L} \subseteq G \times H \) and a compact subset \( W \) of \( H \), also called the window, defines a weak model set \( \Lambda(\mathcal{L}, W) \) as the set of all points \( x_G \in G \), for which there exists a point \( x_H \in W \) such that \( (x_G, x_H) \in \mathcal{L} \). There is an abundant literature on model sets, see e.g. the collection of references cited in [4]. Many of these sets exemplify aperiodic order, a concept which, so far, is mostly defined by a wealth of examples [5, 3]. The following seems to be a common feature of all of them: No \( g \in G \setminus \{0\} \) satisfies \( g + \Lambda(\mathcal{L}, W) = \Lambda(\mathcal{L}, W) \), but the orbit closure \( \{g + \Lambda(\mathcal{L}, W) : g \in G\} \) as a \( G \)-dynamical system has a nontrivial maximal equicontinuous factor (MEF) and/or a nontrivial Kronecker factor (KF) capturing the quasiperiodic aspects of the dynamics.

Many of the simpler examples are uniquely ergodic, so that one can talk unambiguously about their KF, and quite often this KF is just the MEF equipped with its Haar measure. But, more recently, dynamically richer examples, like the set of square-free numbers [11, 37], the set of visible lattice points [6] and their generalizations [17], have attracted much attention. They share the common feature that the orbit closure \( \{g + \Lambda(\mathcal{L}, W) : g \in G\} \) has a fixed point, so that the MEF must be trivial, whereas there are plenty of invariant measures that have non-trivial KFs.

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Equipped with a very natural invariant measure (called Mirsky measure in some cases) these systems are actually isomorphic to their KFs. The approach to the dynamics of weak model sets from [28] encompasses all these examples. It suggests that the notion of \textit{maximal equicontinuous generic factor (MEGF)} might be an appropriate unifying concept. Equicontinuous generic factors for abstract topological dynamical systems were studied in [24], see also [30, 14] for generic eigenfunctions.

In the next section we review some facts on equicontinuous generic factors and prove the existence of a unique MEGF for \(E\)-systems, i.e. for continuous actions of infinite (abstract) groups on compact metric spaces, which admit an invariant probability measure with full topological support (Theorem 2.5). The proof is inspired by work of McMahon [33] and Auslander [2]. If the acting group of such a system is abelian, we also show that the MEGF is trivial if and only if the system is topologically weakly mixing, generalizing a result of Huang and Ye [24] from the case \(G = \mathbb{Z}\) to general abelian groups \(G\).

In Section 3 we test the potential usefulness of the construction of MEGFs on the dynamics on weak model sets, where there are good reasons to expect that this construction leads to the canonical \(G\)-action on the group \((G \times H)/\mathcal{L}\), see e.g. [28, 17]. Indeed, we will see that when the window \(W\) is aperiodic and Haar regular, then \((G \times H)/\mathcal{L}, G)\) is both, the MEGF and the KF of the orbit closure of any weak model set that is generic for its Mirsky measure (Corollary 3). It should be noted, however, that there are other invariant measures of full support for the same system, which have a strictly larger KF, see Example 1.

In Section 4 we restrict to the dynamics of very particular weak model sets, namely to \(B\)-free dynamics, see e.g. [32, 17, 26]. If the set \(B \subseteq \mathbb{N}\) defining such a system is taut and contains an infinite pairwise co-prime subset (this is the case e.g. for the square free numbers), then centralizer of the system is trivial (Theorem 4.1).

2. Equicontinuous generic factors.

2.1. \textbf{Topologically transitive \(E\)-systems.} Let \((X, G)\) be a topological dynamical system with an abstract group \(G\) acting by homeomorphisms on a compact metrizable space \(X\). For this action we adopt the short hand notation \(x \mapsto gx\). By \(Gx\) we denote \(\{gx : g \in G\}\), and for \(A \subseteq X\) we write \(gA = \{gx : x \in A\}\). Let

\[X_t := \{x \in X : Gx = X\}\]

be set of \textit{transitive points}. It is a \(G\)-invariant \(G_\delta\)-set. The system \((X, G)\) is called \textit{minimal}, if \(X_t = X\).

\textbf{Remark 1.} The following are equivalent (see e.g. [9, Lem. 1 and Prop. 1]):

i) \((X, G)\) is \textit{topologically transitive}, i.e. for all non-empty open sets \(U, V \subseteq X\) there is \(g \in G\) such that \(gU \cap V \neq \emptyset\).

ii) \(X_t \neq \emptyset\)

iii) \(X_t\) is a residual subset of \(X\).

iv) Each closed \(G\)-invariant subset \(A\) of \(X\) is either nowhere dense in \(X\) or equal to \(X\).

\textbf{Remark 2.} As a topological space (endowed with the induced topology from the compact metric space \(X\)), \(X_t\) is second countable and in particular separable [36, Theorem 30.3]. The same is true for each \(G\)-invariant subset \(X_0 \subseteq X_t\) with the induced topology.
As usual, a Borel probability measure $\lambda$ on $X$ is called $G$-invariant, if $\lambda(gA) = \lambda(A)$ for each Borel subset $A$ of $X$ and each $g \in G$. Such a measure has full support, if $\lambda(U) > 0$ for each non-empty open set $U \subseteq X$.

**Definition 2.1.** A topologically transitive system $(X,G)$ is an $E$-system, if there exists a $G$-invariant probability measure $\lambda$ on $X$ with full support, see [20].

If $G$ acts on some topological space $Z$, we denote $N(U) := \{g \in G : gU \cap U \neq \emptyset\}$ for any open set $U \subseteq Z$. The action is syndetically nonwandering, if all such sets $N(U)$ are syndetic, i.e. if for each open set $U \subseteq Z$ there is a finite set $F \subseteq G$ such that

$$F^{-1}N(U) := \{f^{-1}\hat{g} : f \in F, \hat{g} \in N(U)\} = G. \quad (1)$$

**Lemma 2.2.** If $(X,G)$ is an $E$-system, then it is syndetically nonwandering.

**Proof.** When the acting group is $Z$, this is contained in the proof of Theorem 4.4 in [21]. Here we adapt the proof from [22, Lem. 2.3] to the case of general $G$: Let $\emptyset \neq U \subseteq X$ be open. The set $\hat{U} := \bigcup_{g \in G} gU \subseteq X$ is open and $G$-invariant. As $X$ is second countable, there is a countable subset $G' \subseteq G$ such that $\hat{U} = \bigcup_{g \in G'} gU$. Let $\lambda$ be the measure from Definition 2.1. As it has full support, we have $\lambda(U) > 0$. Hence there exists a finite subset $F \subseteq G'$ such that $\lambda \left( \bigcup_{f \in F} f^{-1}U \right) > \lambda(\hat{U}) - \lambda(U)$. As $gU \cup \bigcup_{f \in F} f^{-1}U \subseteq \hat{U}$ and $\lambda(gU) = \lambda(U)$ for each $g \in G$, this implies $\lambda \left( gU \cap \bigcup_{f \in F} f^{-1}U \right) > 0$ for each $g \in G$, in particular $G = F^{-1}N(U)$. \hfill $\square$

Here is how we will use the property of being syndetically nonwandering:

**Lemma 2.3.** Let $Z$ be a separable metric space on which $G$ acts isometrically and transitively. If the action is syndetically nonwandering, then $Z$ is totally bounded.

**Proof.** Suppose for a contradiction that $Z$ is not totally bounded. Then there is $\epsilon > 0$ such that $Z$ cannot be covered by finitely many $12\epsilon$-balls. We construct inductively an infinite sequence $z_1, z_2, \ldots$ of points in $Z$ such that the $6\epsilon$-balls $B_{6\epsilon}(z_i)$ are pairwise disjoint: Fix a transitive point $z_1 = z \in Z$. Suppose that $z_i$ are chosen for $i = 1, \ldots, k$. By choice of $\epsilon$, there is $z_{k+1} \notin \bigcup_{i=1}^k B_{12\epsilon}(z_i)$. Hence $B_{6\epsilon}(z_{k+1})$ is disjoint from all $B_{6\epsilon}(z_i)$ for $i = 1, \ldots, k$. As $z$ is a transitive point, there are $g_1, g_2, \cdots \in G$ such that $g_iz \in B_\epsilon(z_i)$ for all $i \geq 1$. Choose $F \subseteq G$ for $U = B_\epsilon(z)$ as in (1). Then, for each $g_i$, there exists $f_i \in F$ such that $f_ig_i \in N(B_\epsilon(z))$. As $G$ acts isometrically on $Z$, $d(f_ig_iz, z) < 2\epsilon$. As $F$ is finite, we find two different $i$ and $j$ with $f_i = f_j$. Hence $d(z_i, z_j) < d(g_iz, g_jz) + 2\epsilon = d(f_ig_iz, f_jg_jz) + 2\epsilon < 6\epsilon$, in contradiction to the disjointness of $B_{6\epsilon}(z_i)$ and $B_{6\epsilon}(z_j)$. \hfill $\square$

### 2.2. The main result on maximal equicontinuous generic factors

From now on we assume that $(X,G)$ is topologically transitive.

For topological dynamical systems with $G = \mathbb{Z}$ (even with $G = \mathbb{N}$), Huang and Ye [24] introduced the notion of an equicontinuous generic factor, that we adapt here to general $G$:

**Definition 2.4.** a) The system $(Y,G)$ is an equicontinuous generic factor of the system $(X,G)$, if $(Y,G)$ is an equicontinuous, transitive (and hence minimal) system, and if there is a continuous map $\pi : X \rightarrow Y$ equivariant under the action of $G$. (As $(Y,G)$ is minimal, the image $\pi(X)$ is dense in $Y$.) We also write $\pi : (X,G) \rightarrow (Y,G)$.  

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**References:**

1. [21], [22, Lem. 2.3], [23], [24]
b) An equicontinuous generic factor $\pi : (X, G) \to (Z, G)$ is maximal, if the following holds:

If $\pi_Y : (X, G) \to (Y, G)$ is another equicontinuous generic factor, then there is a factor map $\pi' : (Z, G) \to (Y, G)$ such that $\pi_Y = \pi' \circ \pi$.

The following theorem is the main result of this paper:

**Theorem 2.5.** Each $E$-system $(X, G)$ has a unique maximal equicontinuous generic factor (MEGF).

A slight extension of this theorem, which also provides more details of the construction of the factor map, is stated and proved below in Subsection 2.6.

**Remark 3.** An equicontinuous generic factor map $\pi$ is in particular a continuous map from a dense subset $X_t$ of $X$ to $Y$. If $X$ and $Y$ are both compact, such a map can always be extended to a measurable map from $X$ to $Y$, continuous at each point of $X_t$. Indeed, denote by $\Pi := \{(x, \pi(x)) : x \in X_t\}$ the closure of the graph of $\pi$ in $X \times Y$. The multivalued map $\phi : x \mapsto \Pi_x$, that associates to each point $x \in X$ the (compact) $x$-section of $\Pi$, is upper semi-continuous and hence measurable [10, Corollary III.3] so that there is a measurable selector $\tilde{\pi} : X \to Y$ such that $\pi(x) \in \Pi_x$ for each $x \in X$ [10, Theorem III.6]. As $\Pi_x = \{\pi(x)\}$ at all $x \in X_t$ ($\pi$ is continuous at all these points!), $\tilde{\pi}$ extends $\pi$, and as the graph of $\tilde{\pi}$ is contained in $\Pi$, $\tilde{\pi}$ is continuous at all $x \in X_t$.

Note, however, that in general $\tilde{\pi}$ cannot be chosen to be equivariant under the actions of $G$, because fixed points are always mapped to fixed points under equivariant maps.

2.3. An application to weak scattering and weak mixing.

**Definition 2.6.** a) The system $(X, G)$ is weakly scattering, if $(X \times Z, G)$ (the product action) is topologically transitive for each minimal equicontinuous system $(Z, G)$.

b) The system $(X, G)$ is topologically weakly mixing, if the system $(X \times X, G)$ (the product action) is topologically transitive.

For the case $G = Z$ (and also $G = N$), Huang and Ye proved that a system $(X, G)$ has a trivial MEGF (i.e. that the MEGF is a singleton) if and only if it is weakly scattering [24, Thm. 3.8], and that if $(X, G)$ is an $E$-system, this happens if and only if the system is topologically weakly mixing [24, Prop. 4.14]. Here we prove for general acting groups $G$:

**Theorem 2.7.** Assume that $(X, G)$ is topologically transitive.

a) If $(X, G)$ is weakly scattering, then the maximal equicontinuous generic factor of $(X, G)$ is trivial.

b) If $(X, G)$ is an $E$-system and has a trivial maximal equicontinuous generic factor, then $(X, G)$ is topologically weakly mixing.

For the proof see Subsection 2.7. As topological weak mixing implies weak scattering when the group is abelian, we have the following corollary, generalizing [24, Prop. 4.14(2)]:

1For $G = Z$ this is well known [18, 8]. Otherwise one can follow Furstenberg’s proof [18, Prop. II.11] for $G = Z$, which carries over to general $G$ (with simple modifications), except that it uses the following property: for all finite subsets $F \subseteq G$ and all non-empty open sets $U, V \subseteq X$ there exists $g \in G$ such that $gU \cap fF \neq \emptyset$ ($f \in F$). I do not know whether this is true for general $E$-systems, but in the abelian case the proof of [18, Prop. II.11] can easily be adapted, see also [9, Thm. 2].
Corollary 1. Assume that $G$ is abelian and that $(X,G)$ is an $E$-system. Then $(X,G)$ is weakly scattering if and only if it is topologically weakly mixing if and only if its maximal equicontinuous generic factor is trivial.

2.4. The regional proximal and the equicontinuous structure relation.

The equicontinuous structure relation. During all of this note, $X_0$ denotes a dense, $G$-invariant subset of $X_t$. We endow $X_t$ and $X_0$ with the induced topology inherited from $X$, and we denote by $O, O_t$ and $O_0$ these topologies on $X, X_t$ and $X_0$, respectively. The corresponding product topologies on $X^2, X_t^2$ and $X_0^2$ are denoted by $O^2, O_t^2$ and $O_0^2$, respectively. We recall the following elementary observation:

Remark 4. For $A \subseteq X$ and $B \subseteq X^2$ we have \(^2\)

$$A \cap X_0^{O_0} = A \cap X_0^O \cap X_0 \quad \text{and} \quad B \cap X_0^{O_0^2} = B \cap X_0^{O^2} \cap X_0^2.$$  \hspace{1cm} (2)

For $U \in O$ and $V \in O^2$ this implies

$$U \cap X_0^{O_0} = U^O \cap X_0 \quad \text{and} \quad V \cap X_0^{O_0^2} = V^{O^2} \cap X_0^2.$$  \hspace{1cm} (3)

If $X_0 = X_t$, eqn. (2) yields $A \cap X_t^{O_t} = A \cap X_t^O \cap X_t$, and this identity applied to $A = X_0$ gives

$$X_0^{O_0} = X_0 \cap X_t^{O_t} = X_0 \cap X_t^O \cap X_t = X_0^O \cap X_t.$$

For a homomorphism (i.e. a continuous $G$-equivariant map) $\pi : X_0 \to Y$, where $Y$ is any compact metric group on which $G$ acts minimally by translation (so that $\pi(X_0)$ is dense in $Y$ \(^3\)), denote

$$S_0^\pi := \{(x,x') \in X_0^2 : \pi(x) = \pi(x')\},$$

and let

$$S_0^{eq} := \bigcap_{\pi : X_0 \to Y} S_0^\pi.$$  \hspace{1cm} (4)

(The intersection is taken over all such spaces $Y$ and all homomorphisms $\pi : X_0 \to Y$.) Following [2], we call $S_0^{eq}$ the equicontinuous structure relation on $X_0$. It is obviously an $O_0$-closed, $G$-invariant equivalence relation on $X_0$.

Remark 5. Recall that each topologically transitive, equicontinuous $G$-action on a compact metric space $Y$ induces a group structure on $Y$, such that $Y$ (with its original topology) becomes a topological group, on which $G$ acts by translation. \(^4\)

In particular, using an equivalent metric on $Y$ if necessary, we can always assume that the action of $G$ on $Y$ is isometric. \(^5\)

\(^2\)Here is the proof for $A \subseteq X$. The one for $B \subseteq X^2$ is nearly identical:

$$x \in A \cap X_0^{O_0} \Leftrightarrow x \in X_0 \land \forall V \in O : x \in V \cap X_0 \Rightarrow (A \cap X_0) \cap (V \cap X_0) \neq \emptyset$$

$$\Leftrightarrow x \in X_0 \land \forall V \in O : x \in V \Rightarrow (A \cap X_0) \cap V \neq \emptyset$$

$$\Leftrightarrow x \in X_0 \land x \in A \cap X_0^O.$$  

\(^3\)Observe that, vice versa, denseness of $\pi(X_0)$ in $Y$ implies minimality of the action of $G$ on $Y$.

\(^4\)This fact goes back to work of Ellis. For details see Auslander’s book [2, Thm. 3.6 and Lem. 2.3]. (Our setting fits into his one if we endow the group $G$ with the discrete topology.) A comprehensive presentation of the corresponding circle of ideas (in the setting of abelian groups) can be found in [1, Sec. 3.1].

\(^5\)Denote by $d$ the given metric on $Y$. Then $d'(y_1,y_2) := \sup \{ d(gy_1, gy_2) : g \in G \}$ has the desired properties.
The regional proximal relation. Denote
\[ \Delta_0 := \{(x,x) : x \in X_0\} , \]
\[ U_0 := \{U_0 := U \cap X_0^2 : U \in \mathcal{O}^2 \text{ G-invariant, } \Delta_0 \subset U\} , \]
and let
\[ Q_0 := \bigcap_{U_0 \in U_0} \overline{U_0} \cap X_0^2 , \]
and denote by
\[ S_0^* \] the smallest \( \mathcal{O}_0^2 \)-closed, G-invariant equivalence relation
on \( X_0^2 \) containing \( Q_0 \).

Because of (3) in Remark 4, it does not matter how the topological hull operation
on \( U_0 \subseteq X_0^2 \) is interpreted.

**Lemma 2.8.** \( Q_0 \) is the regional proximal relation of the non-compact dynamical
system \((X_0,G)\), i.e.
\[ Q_0 = \{(x,y) \in X_0^2 : \text{for all neighbourhoods } A \in \mathcal{O} \text{ of } x, B \in \mathcal{O} \text{ of } y \text{ and } \]
\[ V \in \mathcal{O}^2 \text{ of } \Delta_0 \text{ there exist } x' \in A \cap X_0, y' \in B \cap X_0 \text{ and } \]
\[ g \in G \text{ s.t. } (gx',gy') \in V \} . \]

**Proof.** Suppose first that \((x,y) \in Q_0 \subseteq X_0^2\) and that \(A, B\) and \(V\) are
neighbourhoods as in (5). Define \( U := \bigcup_{g \in G} gV \). Then \( U \in \mathcal{O}^2 \) is a G-invariant
neighbourhood of \( \Delta_0 \), i.e. \( U_0 \in U_0 \). Hence \((x,y) \in \overline{U_0} \cap X_0^2\), and there exists \((x',y') \in U_0 \cap (A \times B)\).

So there is \( g \in G \) such that \((x',y') \in gV \cap (A \times B) \cap (X_0 \times X_0)\), which means that
\( x' \in A \cap X_0, y' \in B \cap X_0 \) and \((g^{-1}x',g^{-1}y') \in V\).

Conversely, suppose that \((x,y)\) belongs to the set on the r.h.s. of (5), and consider
any \( U_0 = U \cap X_0^2 \in U_0 \) with a G-invariant neighbourhood \( U \in \mathcal{O}^2 \) of \( \Delta_0 \). Given
any neighbourhood \( O \in \mathcal{O}^2 \) of \((x,y)\), fix neighbourhoods \( A \in \mathcal{O} \) of \( x \) and \( B \in \mathcal{O} \) of
\( y \) such that \( A \times B \subseteq O \). Then there are \( x' \in A \cap X_0, y' \in B \times X_0 \) and \( g \in G \) such that
\((gx',gy') \in U \), i.e. \((x',y') \in g^{-1}U \cap (X_0 \times X_0) = U_0 \). Hence \((x,y) \in \overline{U_0} \cap X_0^2\),
and as this holds for each \( U_0 \in U_0 \), we have \((x,y) \in Q_0\). \( \square \)

**Remark 6.** The first part of the proof goes through without any changes, if \( V \in \mathcal{O}^2 \)
just has non-empty intersection with \( \Delta_0 \), because also in this case the set \( U \in \mathcal{O}^2 \)
is a G-invariant neighbourhood of \( \Delta_0 \); for \((z,z) \in \Delta_0 = \Delta \cap (X_0 \times X_0)\) there is \( \tilde{g} \in G \) such that \((\tilde{g}z,\tilde{g}z) \in V\), so that \((z,z) \in U\). In this modified form the lemma
shows - when applied to \( X_0 = X_t \) - that the regional proximal relation \( Q_t \) is the
relation \( Q_m(\varphi) \) of McMahon [33, p.226], when the setting described at that place
of his paper is specialized to the situation treated here, i.e. when McMahon’s \( Y \) is
the trivial one-point system. In that case his sets \( X_m \) and \( R_m(\varphi) \) coincide with
our \( X_t \) and \( X_t \times X_t \), respectively, and, as already noted above, the definition of
his \( Q_m(\varphi) \) coincides with that of our \( Q_t \) so that his \( S_m(\varphi) \) is our \( S_t^* \). For the special
case of minimal dynamics, i.e. for the case \( X_t = X \), this setting is reproduced in
Auslander’s book [2].

**Inclusions between the various relations.**

**Lemma 2.9.** \( Q_0 \subseteq S_0^{eq} \) and hence also \( S_0^* \subseteq S_0^{eq} \).

**Proof.** Let \( \pi : X_0 \rightarrow Y \) be as in (4), and recall that w.l.o.g. we can assume that the
action of \( G \) on \( Y \) is isometric. Let \((x,y) \in Q_0\). Suppose for a contradiction that
\[ \pi(x) \neq \pi(y). \] Let \( \delta := d(\pi(x), \pi(y)) > 0. \) There are neighbourhoods \( A \in \mathcal{O} \) of \( x \) and \( B \in \mathcal{O} \) of \( y \) such that

\[ d(\pi(gx), \pi(gy)) = d(g(\pi(x)), g(\pi(y))) = d(\pi(x'), \pi(y')) > \delta/2 \]

for all \( x' \in A \cap X_0, y' \in B \cap X_0 \) and \( g \in G. \)

Let \( M := \{(y, y') : y, y' \in Y, \; d(y, y') < \delta/2\}. \) The set \( M \) is an open neighbourhood of the diagonal in \( Y \times Y, \) and as the metric is translation invariant, the set \( M \) is \( G \)-invariant. Furthermore, \((\pi(gx'), \pi(gy')) \notin M \) for all \( x' \in A \cap X_0, y' \in B \cap X_0 \) and \( g \in G. \) Let \( \tilde{V} := \bigcup_{g \in G} g((\pi \times \pi)^{-1}M). \) Then \( \tilde{V} \subseteq X_0^2 \) and \((A \times B) \cap \tilde{V} = \emptyset. \)

As \( \pi : X_0 \to Y \) is continuous, \( \tilde{V} \in \mathcal{O}_0^2 \) is a \( G \)-invariant neighbourhood of \( \Delta_0. \) Hence \( \tilde{V} = V \cap X_0^2 \) for some \( V \in \mathcal{O}^2. \) Let \( U := \bigcup_{g \in G} gV. \) This set is clearly \( \mathcal{O}^2 \)-open and \( G \)-invariant, and it contains \( \Delta_0. \) Note also that

\[ U_0 = \left( \bigcup_{g \in G} gV \right) \cap X_0^2 = \bigcup_{g \in G} g(V \cap X_0^2) = \bigcup_{g \in G} g\tilde{V} = \tilde{V} \].

Hence \( U_0 \in \mathcal{U}_0 \) and \((A \times B) \cap U_0 = (A \times B) \cap \tilde{V} = \emptyset. \) Therefore \((x, y) \notin U_0, \) which contradicts \((x, y) \in Q_0. \)

2.5. The role of invariant measures with full topological support. We follow McMahon [33] and Auslander [2] in order to study the relation between \( Q_0 \) and \( S^*_0. \) Although some parts of their proofs carry over directly, we prefer to give full details here.

General assumptions and notations

- \( \mathcal{N} \) denotes the family of all closed \( G \)-invariant subsets of \( X^2. \)
- For any \( N \in \mathcal{N} \) and \( x \in X \) denote by \( N_x := \{y \in X : (x, y) \in N\} \) the \( x \)-section of \( N. \)
- We fix a \( G \)-invariant Borel probability measure \( \lambda \) on \( X. \) As \( X \) is compact metrizable, \( \lambda \) is regular. \(^6\)

Lemma 2.10 ([33, 2]). Let \( N \in \mathcal{N}. \) Then

a) \( N_{gx} = gN_x \) and \( \lambda(N_{gx}) = \lambda(N_x) \) for all \( x \in X \) and \( g \in G. \)

b) The map \( x \mapsto \lambda(N_x) \) (\( x \in X \)) is upper semicontinuous.

c) \( \lambda(N_x) \leq \lambda(N_{gx}) \) for all \( x \in X \) and \( x' \in \overline{Gx}. \)

d) \( \lambda(N_x) \leq \lambda(N_{gx}) \) for all \( x \in X_t \) and \( x' \in X. \)

\( \lambda(N_x) = \lambda(N_{gx}) \) for all \( x, x' \in X_t. \)

f) The map \( x \mapsto \lambda(N_x) \) (\( x \in X \)) is continuous at each \( x \in X_t. \)

g) \( \lambda(N_{gx} \Delta N_{gx'}) = \lambda(N_x \Delta N_{gx'}) \) for all \( x, x' \in X \) and \( g \in G. \)

h) The map \( (x, x') \mapsto \lambda(N_x \Delta N_{gx'}) \) (\( x, x' \in X \)) is continuous at each \((x, y) \in X_t \times X_t. \)

Proof. a) \( N_{gx} = \{y \in X : (gx, y) \in N\} = \{y \in X : (x, g^{-1}y) \in N\} = \{gy' \in X : (x, y') \in N\} = gN_x. \)

b) Let \( \epsilon > 0. \) There is an open neighbourhood of \( V \supseteq N_x \) such that \( \lambda(V) < \lambda(N_x) + \epsilon. \) By compactness of \( N, \) there is a neighbourhood \( U \subseteq X \) of \( x \) such that \( N_{gx} \subseteq V \) for all \( x' \in U. \) Hence \( \lambda(N_{gx'}) \leq \lambda(V) < \lambda(N_x) + \epsilon \) for all \( x' \in U. \)

c) This follows from a) and b).

d) This is a special case of c).

\(^6\)The existence of invariant probability measures is discussed in [2, Ch. 12]. Such a measure always exists if the group \( G \) is solvable, in particular if it is abelian.
e) This follows from d).

f) Let $\lambda_{\min} := \inf_{x' \in X} \lambda(N_{x'})$, let $x \in X_I$ and $\epsilon > 0$. In view of b) and d), there is a neighbourhood $U$ of $x$ such that $\lambda(N_x) = \lambda_{\min} \leq \lambda(N_{x'}) < \lambda(N_x) + \epsilon$ for each $x' \in U$.

g) This follows from a):
$$\lambda(N_{gx} \triangle N_{g'x'}) = \lambda(gN_x \triangle gN_{x'}) = \lambda(g(N_x \triangle N_{x'})) = \lambda(N_x \triangle N_{x'})$$

h) This follows from f).

Following [2] we define pseudometrics $d_N$ on $X_I$: \(^7\) given $N \in \mathcal{N}$ let
$$d_N(x,x') := \lambda(N_x \triangle N_{x'}) .$$
Their restrictions to $X_0^2$ yield pseudo-metrics on $X_0$. For $X_0 \subseteq X_I$ let \(^8\)
$$K_0(N) = \{(x,x') \in X_0^2: d_N(x,x') = 0\} .$$
If $X_0 = X_I$, we denote this set by $K_I(N)$. Observe that
$$K_0(N) = K_I(N) \cap X_0^2 .$$

**Remark 7.** By Lemma 2.10, $d_N$ is $G$-invariant and continuous, so that $K_0(N)$ is a $G$-invariant $\mathcal{O}_0$-closed equivalence relation on $X_0$. Let $Z_N^* := X_0/K_0$ and define $d_N^*[x],[y]) := d_N(x,y)$ for $x,y \in X_0$. Then $(Z_N^*,d_N^*)$ is a metric space, and the canonical projection $\pi_N : X_0 \to Z_N^*$ is continuous. As $K_0(N)$ is $G$-invariant, $G$ acts in a canonical way on $Z_N^*$, and this action is isometric. Hence it extends isometrically to the completion of $Z_N^*$, which we denote by $X_N$. As $Z_N^*$ is the continuous image of a separable space, it is separable, and so is its completion $(X_N,d_N)$. Finally, as $Z_N^*$ is the continuous image of a subset $X_0$ of the set of transitive points, also the action of $G$ on $Z_N^*$ is topologically transitive, and as that action is equicontinuous, the action of $G$ on $X_N$ is in fact minimal. In order to conclude that $S^*_0 \subseteq S^*_0 = K_0(N)$, we would need to know that $(X_N,d_N)$ is compact. As this space is complete by construction, all that remains to be proved is that it is totally bounded, equivalently that $Z_N^*$ is totally bounded. This is achieved in the following lemma. \(^9\)

**Lemma 2.11.** Assume that $\lambda$ has full topological support in $X$. Then the space $Z_N^*$ constructed in Remark 7 is totally bounded.

**Proof.** In view of Lemma 2.3, it suffices to show that the action of $G$ on $Z_N^*$ is syndetically nonwandering. So let $U$ be any open subset of $Z_N^* = \pi_N(X_0)$. Then $\pi_N^{-1}(U)$ is an open subset of $X_0$, i.e. there exists $V \in \mathcal{O}$ such that $\pi_N^{-1}U = V \cap X_0$. As $X_0$ is a $G$-invariant set, we have for each $g \in G$
$$\pi_N^{-1}(gU \cap U) = g(\pi_N^{-1}U) \cap \pi_N^{-1}U = g(V) \cap V \cap X_0 .$$
In particular, $gU \cap U \neq \emptyset$ if and only if $gV \cap V \neq \emptyset$, so that $N(U) = N(V)$. Hence it suffices to show that $N(V)$ is syndetic, which follows from Lemma 2.2. \(\square\)

Now we can finish the discussion from Remark 7 with the following lemma:

---

\(^7\)In [2] this is written down for minimal $(X,G)$.

\(^8\)See also [33, Lemma 1.2].

\(^9\)The construction of a nontrivial equicontinuous generic factor in the proof of Lemma 4.7 in [24] bears a certain resemblance to the procedure from [2], which motivated the present proof. It differs, though, from our proof in the way how compactness of the generic factor is secured.
Lemma 2.12. Assume that \( \lambda \) has full topological support in \( X \). Then \( S_0^eq \subseteq \bigcap_{N \in \mathcal{N}} K_0(N) \).

Proof. It suffices to show that \( S_0^eq \subseteq S_0^eq = K_0(N) \) for each \( N \in \mathcal{N} \). In view of Remark 7 (and Remark 5) this follows from Lemma 2.11.

Lemma 2.13. Assume that \( \lambda \) has full topological support in \( X \). Then
\[
\bigcap_{N \in \mathcal{N}} K_0(N) \subseteq Q_0
\]

Proof. We follow the arguments in the proof of [2, Theorem 8] for the minimal case: Let \( (x,y) \in \bigcap_{N \in \mathcal{N}} K_0(N) \subseteq X_0^2 \), let \( V \in O^2 \) be a neighbourhood of \( \Delta_0 \), and let \( A \in \mathcal{O} \) be a neighbourhood of \( x \). Without loss we can assume that \( A \times A \subseteq V \), because \( (x, x) \in \Delta_0 \). Define
\[
N := \bigcup_{g \in G} ((\{y\} \times A)^2).
\]

Obviously, \( N \) is \( O^2 \)-closed and \( G \)-invariant, i.e. \( N \in \mathcal{N} \), and \( \{y\} \times A \subseteq N \) (consider \( g = e \)), so that \( A \subseteq N_y \). As \( (x, y) \in K_0(N) \), it follows that \( \lambda(A \setminus N_x) \leq \lambda(N_y \setminus N_x) \leq d_N(x, y) = 0 \). As \( A \) is open and \( N_x \) is closed, this implies \( A \subseteq N_x \) and hence \( (x, x) \in \{x\} \times A \subseteq N \). Therefore, there are \( g \in G \) and \( x' \in A \) such that \( (gy, gx') \in A \times A \). Hence \( (gx', gy) \in A \times A \subseteq V \). As \( X_0 \) is dense in \( X \) by assumption and as \( A \in \mathcal{O} \), one can choose \( x'' \in A \cap X_0 \). As \( y \in X_0 \), this proves that \( (x, y) \in Q_0 \). (Observe that this proves a bit more, namely that in (5) of Lemma 2.8 the point \( y' \) could instead be chosen to be equal to \( y \). Interchanging the roles of \( x \) and \( y \), one could instead choose \( x' = x \), see [2, Ch. 9, Cor. 9].)

\[ \blacksquare \]

Theorem 2.14. Suppose there exists a \( G \)-invariant Borel probability measure \( \lambda \) on \( X \) with full topological support. Then
\[
Q_0 = S_0^2 = S_0^eq = \bigcap_{N \in \mathcal{N}} K_0(N) = \bigcap_{N \in \mathcal{N}} K_\lambda(N) \cap X_0^2,
\]
where the equivalence relations \( K_0(N) \) are determined as above using \( \lambda \). In particular, \( S_0^eq = S_\lambda^eq \cap X_0^2 \). (Observe that these identities hold for each such measure \( \lambda \).)

Proof. \( Q_0 \subseteq S_0^2 \) by definition, \( S_0^2 \subseteq S_0^eq \) by Lemma 2.9, \( S_0^eq \subseteq \bigcap_{N \in \mathcal{N}} K_0(N) = \bigcap_{N \in \mathcal{N}} K_\lambda(N) \cap X_0^2 \) by Lemma 2.12, and \( \bigcap_{N \in \mathcal{N}} K_0(N) \subseteq Q_0 \) by Lemma 2.13. \[ \blacksquare \]

2.6. Existence of maximal equicontinuous generic factors. The natural question that arises now is whether there actually exists a maximal equicontinuous factor of \( (X_\ell, G) \), or, in the terminology of [24], a maximal equicontinuous generic factor of \( (X, G) \). We precede the proof of this fact with a more technical lemma.

Lemma 2.15. For each measure \( \lambda \) as in Theorem 2.14, there exists an at most countable family \( \mathcal{N}_c \subseteq \mathcal{N} \) such that \( \bigcap_{N \in \mathcal{N}_c} K_0(N) = \bigcap_{N \in \mathcal{N}_c} K_0(N) \) for all invariant dense sets \( X_0 \subseteq X_\ell \).

Proof. As \( X \) is second countable, there is a countable base \( O_1, O_2, \ldots \) for the topology of \( X^2 \). Hence, for each \( N \in \mathcal{N} \), there is an index set \( J_N \subseteq \mathbb{N} \) such that \( X_0^2 \setminus K_\lambda(N) = \bigcup_{j \in J_N} O_j \cap X_0^2 \). Let \( J = \bigcup_{N \in \mathcal{N}} J_N \). With each \( j \in J \) we can
associate a set $N_j \in \mathcal{N}$ such that $j \in J_{N_j}$, i.e. such that $O_j \cap X_i^2 \subseteq X_i^2 \setminus K_i(N_j)$. Let $\mathcal{N}_c = \{N_j : j \in J\}$. Then

$$X_i^2 \setminus \left( \bigcap_{N \in \mathcal{N}_c} K_i(N) \right) \subseteq X_i^2 \setminus \left( \bigcap_{N \in \mathcal{N}} K_i(N) \right) = \bigcup_{j \in J} X_i^2 \setminus K_i(N_j) = \bigcup_{N \in \mathcal{N}_c} O_j \cap X_i^2 \subseteq \bigcup_{j \in J} X_i^2 \setminus K_i(N_j) = \bigcup_{N \in \mathcal{N}_c} O_j \cap X_i^2 \subseteq \bigcup_{N \in \mathcal{N}_c} X_i^2 \setminus K_i(N).$$

Hence we have equalities everywhere in this chain of inclusions. As $K_i(N) \subseteq X_i^2$ for all $N \in \mathcal{N}$, this implies $\cap_{N \in \mathcal{N}_c} K_i(N) = \cap_{N \in \mathcal{N}} K_i(N)$ and hence

$$\bigcap_{N \in \mathcal{N}} K_i(N) = X_i^0 \cap \bigcap_{N \in \mathcal{N}_c} K_i(N) = X_i^0 \cap \bigcap_{N \in \mathcal{N}_c} K_i(N) = \bigcap_{N \in \mathcal{N}_c} K_i(N).$$

The following theorem is the announced more detailed version of Theorem 2.5. The additional information is contained in its part b).

**Theorem 2.16.** Suppose that $(X, G)$ is an E-system, i.e. there exists a G-invariant Borel probability measure $\lambda$ on $X$ with full topological support.

a) $(X, G)$ has a maximal equicontinuous generic factor $\pi : (X, G) \xrightarrow{\pi c} (Z, G)$, where $(Z, G)$ is a compact, metrizable, equicontinuous system, unique up to isomorphism, and one can choose $(Z, G)$ as a minimal group translation.

b) If $X_0$ is a $G$-invariant subset of $X_1$, if $\pi_Y : X_0 \to Y$ is another homomorphism to a minimal equicontinuous compact system $(Y, G)$, then there is a factor map $\pi' : (Z, G) \to (Y, G)$ such that $\pi_Y = \pi' \circ \pi|_{X_0}$. In particular, $\pi_Y$ extends continuously to $X_1$.  

*Proof.* a) Enumerate the countable set $\mathcal{N}_c$ from the previous lemma as $\mathcal{N}_c = \{N_n : n \in \mathbb{N}\}$. Define $D : X_i \times X_i \to \mathbb{R}$ as $D(x, y) = \sum_{n \in \mathbb{N}} 2^{-n} d_{N_i}(x, y)$, where $d_{N_i}$ is the G-invariant continuous pseudo-metric on $X_i$ associated with $N_i$. Then also $D$ is a G-invariant continuous pseudo-metric on $X_i$ and

$$X_i^D := \{(x, y) \in X_i^2 : D(x, y) = 0\} = \bigcap_{n \in \mathbb{N}} K_i(N_n) = \bigcap_{N \in \mathcal{N}} K_i(N)$$

is an $\mathcal{O}_2$-closed G-invariant equivalence relation on $X_i$. Let $Z^* := X_i/X_i^D$ and define $d_Z([x], [y]) := D(x, y)$ for $x, y \in X_i$. Then $(Z^*, d_Z)$ is a metric space, the canonical projection $\pi : X_i \to Z^*$ is continuous, and just as in Remark 7 one shows that its completion $(Z, d_Z)$ is compact and that $G$ acts in a canonical way isometrically on $(Z, d_Z)$. As $X_i^D = S^c_i$ by Theorem 2.14, and as $d_Z([x], [y]) = D(x, y)$ for $x, y \in X_i$, the system $(Z, G)$ is a MEGF for $(X, G)$.

We turn to the proof of the uniqueness (up to conjugacy) of the MEGF: Suppose that $\pi_Y : (X, G) \xrightarrow{\pi c} (Y, G)$ is a further MEGF of $(X, G)$. Then there are factor maps $\pi' : Z \to Y$ and $\pi'' : Y \to Z$ such that $\pi_Y = \pi' \circ \pi$ and $\pi = \pi'' \circ \pi_Y$. Hence $\pi = \pi'' \circ \pi' \circ \pi$, which implies that $(\pi'' \circ \pi')|_{\pi(X)} = \text{id}_{\pi(X)}$. As $\pi(X_i)$ is dense in $Z$ and as $\pi'' \circ \pi' : Z \to Z$ is continuous, this implies that $\pi'' \circ \pi' = \text{id}_Z$. In particular, $\pi'$ is 1-1, i.e. the factor map $\pi' : Z \to Y$ is a homeomorphism.
b) We have to prove that \((Z,G)\) is maximal among all compact metrizable equicontinuous generic factors of \((X,G)\) (in the strengthened sense of part b) of the theorem). So suppose there are a \(G\)-invariant subset \(X_0 \subseteq X_t\) and a homomorphism \(\pi_Y : X_0 \to Y\) to a compact metrizable equicontinuous minimal system \((Y,G)\).

Then \(X_0^D \cap X_0^2 = \bigcap_{N \in \mathcal{N}} K_0(N) = S^{eq}_0 \subseteq S^{eq}_0\) by Theorem 2.14. It follows that \(\pi_Y\) factorizes over \(\pi_{|X_0} : X_0 \to Z\). \(\square\)

2.7. Weak mixing and maximal equicontinuous generic factors.

**Proof of Theorem 2.7.** a) We follow the proof of Theorem 2.7 in [24]. Denote by \(\pi : X_t \to Z\) the MEGF of \((X,G)\). As \((X,G)\) is weakly scattering, there is \((x_0,z_0) \in X_t \times Z\) such that \(G(x_0,z_0) = X \times Z\). Define \(\phi : X_t \times Z \to Z\), \(\phi(x,z) = z^{-1}\pi(x)\).

Then \(\phi\) is continuous, and \(\phi(gx_0,gz_0) = (gz_0)^{-1}\pi(gx_0) = z_0^{-1}g^{-1}\pi(x_0) = \phi(x_0,z_0)\) for all \(g \in G\). Hence \(\phi(x,z) = \phi(x_0,z_0)\) for all \((x,z) \in X_t \times Z\). In particular, \(\pi(x) = \phi(x,e) = \phi(x_0,z_0)\) for all \(x \in X_t\), so that \(Z = \pi(X_t) = \{\phi(x_0,z_0)\}\) is a singleton.

b) Let \((X,G)\) be an \(E\)-system, and denote by \(\lambda\) any \(G\)-invariant Borel probability measure on \(X\) with full topological support. Assume that the MEGF of \((X,G)\) is trivial. That means that the equicontinuous structure relation \(S^{eq}_t\) defined in (4) is maximal, i.e. \(S^{eq}_t = X_t^2\). Hence \(K_t(N) = X_t^2\) for all \(N \in \mathcal{N}\) by Lemma 2.12, where, as before, \(\mathcal{N}\) denotes the family of all closed \(G\)-invariant subsets of \(X^2\). Therefore \(\lambda(N_x \triangle N_{x'}) = 0\) for all \(N \in \mathcal{N}\) and all \(x, x' \in X_t\).

In order to prove that \((X,G)\) is weakly mixing we must show that each \(N \in \mathcal{N}\) is either nowhere dense in \(X^2\) or equal to \(X^2\), see Remark 1. So assume that \(N \in \mathcal{N}\) is not nowhere dense, i.e. that \(\text{int}(N) \neq \emptyset\). Then there are open sets \(U,V \subseteq X\) such that \(U \times V \subseteq N\). Fix any \(x_0 \in X_t \cap U\). Then \(V \subseteq N_{x_0}\), and \(\lambda(N_x \triangle N_{x_0}) = 0\) for all \(x \in X_t\). As \(\lambda\) has full topological support, this implies \(V \subseteq N_x\) for all \(x \in X_t\), i.e. \(X_t \times V \subseteq N\). Let \(W := \bigcup_{g \in G} gV\). Then \(W\) is open and \(G\)-invariant, and \(W\) is dense in \(X\), because \((X,G)\) is topologically transitive. It follows that

\[
X^2 = X_t \times W = \bigcup_{g \in G} g(X_t \times V) \subseteq \bigcup_{g \in G} gN = N = N.
\]

\(\square\)

3. Maximal equicontinuous generic factors and weak model sets. The dynamics of weak model sets are an excellent testing ground for the relevance of MEGFs. We start by summarizing some essential notations and results from [28].

3.1. Some recollections on weak model sets.

**Assumptions and notations.**

1. \(G\) and \(H\) are locally compact second countable abelian groups with Haar measures \(m_G\) and \(m_H\). Then the product group \(G \times H\) is locally compact second countable abelian as well, and we choose \(m_{G \times H} = m_G \times m_H\) as Haar measure on \(G \times H\).

2. \(\mathcal{L} \subseteq G \times H\) is a cocompact lattice, i.e., a discrete subgroup whose quotient space \((G \times H)/\mathcal{L}\) is compact. Thus \(\hat{X} := (G \times H)/\mathcal{L}\) is a compact second countable abelian group. Denote by \(\pi_G : G \times H \to G\) and \(\pi_H : G \times H \to H\) the canonical projections. We assume that \(\pi_G|\mathcal{L}\) is 1-1 and that \(\pi_H|\mathcal{L}\) is dense in \(H\).

3. \(G\) acts on \(G \times H\) by translation: \(gx := (g,0) + x\).
(4) Elements of $G \times H$ are denoted as $x = (x_G, x_H)$, elements of $\hat{X}$ as $\hat{x}$ or as $x + L = (x_G, x_H) + L$, when a representative $x$ of $\hat{x}$ is to be stressed. We normalise the Haar measure $m_\chi$ on $X$ such that $m_\chi(\hat{X}) = 1$. Thus $m_\chi$ is a probability measure.

(5) The window $W$ is a compact subset of $H$. We assume that $m_H(W) > 0$.

**Consequences of the assumptions.**

(1) Being locally compact second countable abelian groups, $G$, $H$ and $G \times H$ are metrizable with a translation invariant metric with respect to which they are complete metric spaces. In particular they have the Baire property. As such groups are $\sigma$-compact, $m_G$, $m_H$ and $m_{G \times H}$ are $\sigma$-finite.

(2) As $G \times H$ is $\sigma$-compact, the lattice $L \subseteq G \times H$ is at most countable. Note that $G \times H$ can be partitioned by shifted copies of a relatively compact fundamental domain $D$. Denote $\text{dens}(L) = 1 / m_{G \times H}(D)$. As $m_\chi$ is a probability measure, we have $m_\chi(A) = \text{dens}(L) \cdot m_{G \times H}(D \cap (\pi^x)^{-1}(A))$ for any measurable $A \subseteq \hat{X}$, where $\pi^x : G \times H \to \hat{X}$ denotes the quotient map. As a factor map between topological groups, $\pi^x$ is open.

(3) $L$ acts on $(H, m_H)$ by $h \mapsto \ell_H + h$ metrically transitively, i.e., for every measurable $A \subseteq H$ such that $m_H(A) > 0$ there exist at most countably many $\ell^i \in L$ such that $m_H(\bigcup_i (\ell^i + A)) = 0$, see [31, Ch. 16, Ex. 1].

(4) The action $\hat{x} \mapsto (g, 0) + \hat{x}$ of $G$ on $\hat{X}$ is minimal and uniquely ergodic.

(5) Denote by $M$ and $M^G$ the spaces of all locally finite measures on $G \times H$ and $G$, respectively. They are endowed with the topology of vague convergence. As $G$ and $G \times H$ are complete metric spaces, this is a Polish topology, see [25, Thm. A.2.3].

**The objects of interest.** The pair $(L, W)$ assigns to each point $\hat{x} \in \hat{X}$ a discrete point set in $G \times H$. Such point sets $P$ are identified with the measures $\sum_{y \in P} \delta_y \in M$. More precisely:

(1) For $\hat{x} = x + L \in \hat{X}$ define

$$\nu_w(\hat{x}) := \sum_{y \in (x + L) \cap (G \times W)} \delta_y.$$  

It is important to understand $\nu_w$ as a map from $\hat{X}$ to $M$. The canonical projection $\pi^G : G \times H \to G$ projects measures $\nu \in M$ to measures $\pi^G_\nu$ on $G$ defined by $\pi^G_\nu(A) := \nu((\pi^G)^{-1}(A))$. We abbreviate

$$\nu^G_w := \pi^G_\nu \circ \nu_w : \hat{X} \to M^G.$$

The set of continuity points of $\nu_w$ and $\nu^G_w$ is a dense $G_\delta$-subset of $\hat{X}$.

(2) Denote by

- $M_w$ the vague closure of $\nu_w(\hat{X})$ in $M$,  
- $M^G_w$ the vague closure of $\nu^G_w(\hat{X})$ in $M^G$.

The group $G$ acts continuously by translations on all these spaces: $(g\nu)(A) := \nu(g^{-1}A)$. As $\nu_w(\hat{x})(g^{-1}A) = (g\nu_w(\hat{x}))(A) = \nu_w(g\hat{x})(A)$, it is obvious that all $\nu_w(\hat{x})$ are uniformly translation bounded, and it follows from [7, Thm. 2] that both spaces are compact.

(3) $Q_M := m_\chi \circ \nu_w^{-1}$ and $Q_{M^G} := m_\chi \circ (\nu^G_w)^{-1}$ are the Mirsky measures on $M_w$ and $M^G_w$, respectively. Note that $Q_{M^G} = Q_M \circ (\pi^G_\nu)^{-1}$. 


Observe that $\nu_G^G(\hat{X}) \subseteq \mathcal{M}_W^G$ is the space of weak model sets that is of primary interest.

### 3.2. The MEGF of the Mirsky measure

The following facts are taken from [28] and [29]:

#### (1) Facts about the MEF [28, Theorem 1]:

- a) If $\text{int}(W) \neq \emptyset$, then $(\hat{X}, G)$ is the MEF of $(\mathcal{M}_W, G)$.
- b) If $\text{int}(W) = \emptyset$, then the MEFs of $(\mathcal{M}_W, G)$ and $(\mathcal{M}_W^G, G)$ are trivial.

#### (2) Facts about the KF [28, Theorems 2] and [29, Theorem B1]:

- a) $(\hat{X}, m_{\hat{X}}, G)$ is the KF of $(\mathcal{M}_W, Q_{\mathcal{M}_W}, G)$. Even more, both systems are isomorphic.
- b) If the window $W$ is Haar aperiodic, then the same is true for the system $(\mathcal{M}_W^G, Q_{\mathcal{M}_W^G}, G)$.

Here $W$ is Haar aperiodic, if $m_H((h + W)\Delta W) = 0$ implies $h = 0$.

Denote by $X \subseteq \mathcal{M}_W$ the topological support of $Q_{\mathcal{M}_W}$ and by $X^G \subseteq \mathcal{M}_W^G$ that of $Q_{\mathcal{M}_W^G}$. Although both sets may be strictly contained in their ambient spaces, they capture the most important aspects of the dynamics. For example, $m_{\hat{X}}$-a.a. configurations $\nu_W(\hat{x})$ belong to these spaces [28, Theorem 5d].

**Corollary 2.** Statements (1) and (2) above remain true for the subsystems $(X, G)$ and $(X^G, G)$.

**Proof.** For (2) this is trivial, because $Q_{\mathcal{M}}(X) = Q_{\mathcal{M}G}(X^G) = 1$. For (1) observe first, that $(\mathcal{M}_W, G)$ has a unique minimal subsystem [28, Lemma 6.3], so this system is contained in $X$. In case a) it is an almost automorphic extension of $(\hat{X}, G)$ [28, Theorem 1a], so that $(\hat{X}, G)$ is its MEF. But then $(\hat{X}, G)$ is also the MEF of $(X, G)$. In case b), the minimal system is a fixed point, so that the MEF of any subsystem containing this fixed point is trivial. \( \square \)

The facts listed as (1) above, and also Corollary 2, provide no useful information on the MEF of the systems $(\mathcal{M}_W^G, Q_{\mathcal{M}_W^G}, G)$, respectively. This changes completely when the MEGF of $(X^G, G)$ is considered. (Recall that $X^G$ denotes the support of the Mirsky measure on $\mathcal{M}_W^G$.)

We denote by $W_{\text{reg}}$ the topological support of $m_H|W$. This is the smallest closed subset of $W$ which has full Haar measure inside $W$. We say that the window $W$ is **Haar regular**, if $W_{\text{reg}} = W$ [29, Def. 3.10]. A Haar regular window is Haar aperiodic if and only it is aperiodic, i.e. if $W + h = W$ implies $h = 0$ [29, Sec. 3.2].

**Theorem 3.1.** Suppose that the window $W$ is Haar regular.

a) $(\hat{X}, G)$ is the MEF of $(X, G)$.

b) If the window $W$ is aperiodic, then the same is true for the system $(X^G, G)$.

In combination with Corollary 2 this implies:

**Corollary 3.** Suppose that the window $W$ is Haar regular and aperiodic. Recall that, by definition, $X^G$ is the orbit closure of any weak model set which is generic for the Mirsky measure $Q_{\mathcal{M}_W^G}$. Then

- $(\hat{X}, G)$ is the MEGF of $(X^G, G)$, and
- $(\hat{X}, m_{\hat{X}}, G)$ is the Kronecker factor of $(X^G, Q_{\mathcal{M}_W^G}, G)$.

**Remark 8.** If the window is not aperiodic, the MEGF of $(X^G, G)$ is the $G$-action on a factor group of $\hat{X}$: the periods of $W$ must be factored out as in the corresponding theorems from [29].
We precede the proof of the theorem with some observations and notations from [28, 29]: The system $(\mathcal{M}_W^G, G)$ can be extended to the graph system $(\mathcal{G}_W^G, G)$, where $\mathcal{G}_W^G \subseteq \hat{X} \times \mathcal{M}^G$ is the closure of the graph of $\nu_W^G : \hat{X} \to \mathcal{M}^G$. This system is a topological joining of $(\hat{X}, G)$ and $(\mathcal{M}_W^G, G)$. The natural projections $\pi_x^G : \mathcal{G}_W^G \to \hat{X}$ and $\tilde{\pi}_G : \mathcal{G}_W^G \to \mathcal{M}_W^G$ are continuous. Finally, there is a homeomorphism $\Phi : \mathcal{G}_W^G \setminus (\hat{X} \times \{0\}) \to \mathcal{M}_W \setminus \{0\}$ that commutes with the respective actions of $G$. The natural projection $\pi_x^G : \mathcal{M}_W \to \mathcal{M}_W^G$ satisfies $\tilde{\pi}_G = \pi_x^G \circ \Phi$. 10 The composition $\hat{\pi} := \pi_x^G \circ \Phi^{-1} : \mathcal{M}_W \setminus \{0\} \to \hat{X}$ is the (continuous) map defined in [28, Def. 5.5]. In view of [28, Lemma 4.4], it satisfies $\nu \leq \nu_{\hat{\pi}}(\hat{\pi} \nu)$ for all $\nu \in \mathcal{M}_W \setminus \{0\}$.

Next, we recall one more concept from [29, Sec. 3.2 and 4]: Denote $\overline{\mathcal{M}}_W := \{\nu \in \mathcal{M} : \nu \leq \nu_{\hat{\pi}}(\hat{\pi}) \text{ for some } \hat{x} \in \hat{X}\}$. Then $\mathcal{M}_W = \nu_{\hat{\pi}}(\hat{X}) \subseteq \overline{\mathcal{M}}_W$, because $\nu_{\hat{\pi}}$ is upper semi-continuous, and the projection $\pi_x^G$ extends naturally from $\mathcal{M}_W$ to $\overline{\mathcal{M}}_W$. For each $\nu \in \overline{\mathcal{M}}_W$, $\pi_x^G$ is also denoted by $\pi_x^G$. $\pi_x^G$ and recall from (6) that $\mathcal{M}_W \subseteq \overline{\mathcal{M}}_W \setminus \{0\}$ so that $\nu \leq \nu_{\hat{\pi}}(\hat{\pi} \nu)$. Hence, given any tempered van Hove sequence $(\nu_n, n \in \mathbb{N})$ of subsets of $\mathcal{M}_W$ which is equipped with the topology generated by the Hausdorff distance.

Denote by $X^\circ$ the set of topologically transitive points of $\mathcal{M}^G$.

**Lemma 3.2.** If $\pi_x^G \nu \in X^\circ$ for some $\nu \in \overline{\mathcal{M}}_W$, then $W_{\text{reg}} \subseteq S_{\nu}(\nu)$.

**Proof.** Denote $W' = S_{\nu}(\nu)$. Suppose first that $m_H(W') = m_H(W)$. Then $m_H(W' \cap W_{\text{reg}}) = m_H(W)$ so that $W_{\text{reg}} \subseteq W_{\text{reg}} \cap W' \subseteq S_{\nu}(\nu)$.

Suppose now that $m_H(W') > m_H(W) > 0$. We will derive a contradiction: Let $\nu \in \overline{\mathcal{M}}_W \setminus \{0\}$ and recall from (6) that $\nu \leq \nu_{\hat{\pi}}(\hat{\pi} \nu)$. Then, for each $x \in G \times X$,

$$\nu\{x\} = 1 \Rightarrow \nu_{\hat{\pi}}(\hat{\pi} \nu)\{x\} = 1$$

just by the definition of $W'$, and so $\nu \leq \nu_{\hat{\pi}}(\hat{\pi} \nu)$. Hence, given any tempered van Hove sequence $(A_n, n \in \mathbb{N})$ of subsets of $G$, the upper density of any $S_{\nu} \nu$ w.r.t. $(A_n)$ is bounded in the following way [28, Thm. 3] 11:

$$\forall \epsilon > 0 \exists n_0 \in \mathbb{N} \forall n \geq n_0 \forall g \in G : \frac{(S_{\nu} \nu)(A_n \times H)}{m_G(A_n)} \leq \frac{(S_{\nu}(\nu_{\hat{\pi}})(A_n \times H))}{m_G(A_n)} = \frac{\nu_{\hat{\pi}}(T_g(\nu))(A_n \times H)}{m_G(A_n)} \leq \text{dens}(\mathcal{L}) \cdot m_H(W') + \epsilon.$$

This applies in particular to $\epsilon := \frac{\text{dens}(\mathcal{L})}{2} (m_H(W) - m_H(W')) > 0$ and yields

$$\frac{(S_g \pi_x^G \nu)(A_n)}{m_G(A_n)} = \frac{(\pi_x^G(S_g \nu))(A_n \times H)}{m_G(A_n)} = \frac{m_G(A_n)}{m_G(A_n)} \leq \text{dens}(\mathcal{L}) \cdot m_H(W') + \epsilon = \text{dens}(\mathcal{L}) \cdot m_H(W) - \epsilon$$

for all $g \in G$. Hence all points in the orbit closure of $\pi_x^G \nu$ have Banach density at most $\text{dens}(\mathcal{L}) \cdot m_H(W') - \epsilon$. As $X^\circ = \text{supp}(Q_{\mathcal{M}^G})$ and as $Q_{\mathcal{M}^G}$-a.a. $\nu^G \in X^\circ$ have

10 See [28, Lem. 5.3] and the discussion around [28, Lem. 2.1]. In the notation from that paper, $\Phi = \pi_x^G \circ (\pi_x^G)^{-1}$, and $\pi_x^G$ is also denoted by $\pi_x^G$.

11 This is based on a previous result by Moody [35].
Lemma 3.3. Suppose that $W$ is Haar regular and aperiodic.

a) If $\pi^0_\nu \in X^G_t$ for some $\nu \in \mathcal{M}_W$, then $(\pi^0_\nu)^{-1}\{\nu\} = \{\nu\}$.

b) $(\pi^0_\nu)^{-1}X^G_t: X^G_t \to \mathcal{M}_W \setminus \{0\}$ and $\hat{\pi}^G: X^G_t \to \hat{X}$, $\hat{\pi}^G := \hat{\pi} \circ (\pi^0_\nu)^{-1}X^G_t$ are well defined maps. Both are continuous (when $X^G_t \subseteq X^G$ is equipped with the subspace topology).

c) If $\nu^G \in X^G_t$, then $\nu^G \leq \nu^G(\hat{\pi}^G \nu^G)$.

d) If $\nu^G \in X^G_t$ and $\nu^G \leq \nu^G(\hat{\pi}^G \nu^G)$ for some $\hat{x} \in \hat{X}$, then $\hat{\pi}^G \nu^G = \hat{x}$.

Proof. a) This follows from [29, Lemma 4.6], because Haar regularity of $W$ and Lemma 3.2 imply in view of (7) that $S_{\hat{\nu}}(\nu) = W$.

b) As each $\nu^G \in X^G_t$ is the image of some $\nu \in \mathcal{M}_W \setminus \{0\}$ under $\pi^0_\nu$, the well definedness follows from part a). For the continuity of $(\pi^0_\nu)^{-1}X^G_t$ it suffices to observe that the preimage of any closed set $A \subseteq \mathcal{M}_W$ under this map is closed in $X^G_t$:

$$\{\nu^G \in X^G_t : (\pi^0_\nu)^{-1}(\nu^G) \in A\} = X^G_t \cap \pi^0_\nu \{\pi^0_\nu(\nu) = \nu\}.$$

and as $A$ is a closed subset of the compact metric space $\mathcal{M}_W$ and $\pi^0_\nu: \mathcal{M}_W \to \mathcal{M}_W$ is continuous, the set $\pi^0_\nu \{\pi^0_\nu(\nu) = \nu\}$ is closed.

c) Let $\nu^G \in X^G_t$ and denote $\nu := (\pi^0_\nu)^{-1} \nu^G$. As $\nu \in \mathcal{M}_W \setminus \{0\}$, we have $\nu \leq \nu^G(\hat{\pi}^G \nu^G)$. Hence $\nu^G = \pi^0_\nu^G \nu^G \leq \nu^G(\nu^G(\hat{\pi}^G \nu^G)) = \nu^G(\hat{\pi}^G \nu^G)$.

d) Let $\hat{x} \in \hat{X}$ and $\nu^G \in X^G_t$ with $\nu^G \leq \nu^G(\hat{x}) = \pi^0_\nu^G(\nu^G(\hat{x}))$, and denote $\nu := (\pi^0_\nu)^{-1} \nu^G$. Define $\hat{\nu} \in \mathcal{M}_W \setminus \{0\}$ by $\hat{\nu}\{g,h\} = 1$, if $\nu^G(\hat{x})\{g,h\} = 1$ and $\nu^G(g) = 1$, otherwise $\hat{\nu}\{g,h\} = 0$. Then $\pi^0_\nu^G \hat{\nu} = \nu^G$ and $\hat{\nu} \leq \nu^G(\hat{x})$. As $\pi^0_\nu^G \nu^G = \nu^G = \pi^0_\nu^G \nu^G$, Lemma 4.4 from [29] guarantees the existence of some $d \in H$ such that $\nu(A) = \nu(A - (0,d))$ for each Borel subset $A$ of $G \times H$, and consequently $S_{\hat{\nu}}(\nu) = S_d(\hat{\nu}) + d$. But $S_d(\nu) = S_{\hat{\nu}}(\nu) = W$ as shown in the proof of part a), so that $W = W + d$, and the aperiodicity of $W$ implies that $d = 0$. Hence $\hat{\nu} = \nu$, so that $\nu \leq \nu^G(\hat{x})$. On the other hand, $\hat{\pi}^G$ is the unique point in $\hat{X}$ for which $\nu \leq \nu^G(\hat{\pi}^G \nu^G)$ [28, Lem. 5.4], so that $\hat{x} = \hat{\pi}^G \nu^G$, where $\hat{\pi}^G = \hat{\pi}((\pi^0_\nu)^{-1} \nu^G) = \hat{\pi}^G \nu^G$.

Proof of Theorem 3.1. $(X,G)$ and $(X^G,G)$ are E-systems in the sense of Definition 2.1, where the respective Minkwski measures play the role of the measure $\lambda$. Hence Theorem 2.16 applies, and both systems have compact abelian groups as MEGFs, call them $(Z,G)$ and $(Z^G,G)$ with factor maps $\pi^G$ and $\pi^G$, respectively.

a) $\hat{\pi} = \pi^G \circ \Phi^{-1}: (X,G) \to (\hat{X},G)$ is a generic factor map. By Theorem 2.16b, there is a factor map $\pi': Z \to \hat{X}$ such that $\hat{\pi} = \pi' \circ \pi^G$ on $X_t$. Let $\hat{x}_0 := \pi'(0)$. Then $\pi' = \alpha + \hat{x}_0$ for some group homomorphism $\alpha : Z \to \hat{X}$.12 This shows that, for each $\hat{x} \in \hat{X}$, the set $(\pi')^{-1}\{\hat{x}\}$ is a coset of $\ker(\alpha)$, and in order to prove that $\pi' : (Z,G) \to (\hat{X},G)$ is a homeomorphism it suffices to observe that $\text{card}((\pi')^{-1}\{\hat{x}\}) = 1$ for each continuity point $\hat{x}$ of $\nu^G$: for such a point, $\nu^G(\hat{x}) = (\pi')^{-1}\{\hat{x}\} = (\pi^G)^{-1}(\pi')^{-1}\{\hat{x}\}$.

12As $G$ acts on $Z$ and on $\hat{X}$ by translation, there are group monomorphisms $\eta : G \to Z$ and $\hat{\eta} : G \to \hat{X}$ such that $g\eta = z + \eta(g)$ and $g\hat{x} = \hat{x} + \hat{\eta}(g)$ for all $z \in Z, \hat{x} \in \hat{X}$ and $g \in G$. Consider $\alpha := \pi' - \hat{x}_0$. For $g,g' \in G$ we have

$$\alpha(g\eta + \eta(g')) = \pi'(g\eta + \eta(g')) - \hat{x}_0 = \pi'(0) + \hat{\eta}(g + g') - \hat{x}_0 = \hat{\eta}(g) + \hat{\eta}(g').$$

In other words: $\alpha|_{\eta(G)} : \eta(G) \to \hat{\eta}(G)$ is a group homomorphism, and as $\alpha : Z \to \hat{X}$ is continuous and $\eta(G)$ is dense in $Z$, this shows that $\alpha : Z \to \hat{X}$ is a group homomorphism.
Hence \( \hat{\pi} : (X_t, G) \to (\hat{X}, G) \) is a maximal generic factor.

b) \( \hat{\pi}^G = \hat{\pi} \circ (\pi^G)^{-1} : X^G \to \hat{X} \) is a continuous map by Lemma 3.3b. As \( \hat{\pi} \) and \( \pi^G \) commute with the respective actions of \( G \), the map \( \hat{\pi}^G : (X^G, G) \to (\hat{X}, G) \) is indeed an equicontinuous generic factor.

Next, \( \pi^G \circ \pi^G : (X_t, G) \to (Z^G, G) \) is an equicontinuous generic factor. As \( \hat{\pi} : (X_t, G) \to (\hat{X}, G) \) is the MEGF of \( (X, G) \) by part a) of the present theorem, Theorem 2.16b guarantees the existence of a factor map \( \pi' : (\hat{X}, G) \to (Z^G, G) \) such that \( \pi^G \circ \pi^G = \pi' \circ \hat{\pi} \) on \( X_t \). On the other hand, Theorem 2.16b applies as well to the equicontinuous generic factor \( \hat{\pi} \circ (\pi^G)^{-1} : (X^G, G) \to (\hat{X}, G) \), i.e. there is a factor map \( \pi'' : (Z^G, G) \to (\hat{X}, G) \) such that \( \hat{\pi} \circ (\pi^G)^{-1} = \pi'' \circ \pi^G \) on \( X^G \). Hence
\[
\hat{\pi} = \pi'' \circ \pi^G = \pi'' \circ \pi' \circ \hat{\pi} \text{ on } X_t,
\]
so that \( \pi'' \circ \pi' = \text{id}_\hat{X} \) on the dense subset \( \hat{\pi}(X_t) \) of \( \hat{X} \). As \( \pi'' \circ \pi' \) is continuous, this shows that \( \pi'' \circ \pi' = \text{id}_\hat{X} \), in particular \( \pi' \) is 1-1. Being a factor map from \( Z^G \) onto \( \hat{X} \), \( \pi' \) thus is a homeomorphism, so that \((\hat{X}, G)\) is (isomorphic to) the MEGF \((Z^G, G)\) of \((X^G, G)\).

4. Applications to \( B \)-free dynamics. \( B \)-free dynamical systems form a (very) special subclass of dynamical systems generated by weak model sets. We discuss the MEGF of these systems and use it to prove that the centralizer of \( B \)-free systems of (generalized) Erdős type is trivial.

4.1. A recollection of facts on \( B \)-free systems. For any given set \( B \subseteq \mathbb{N} \) one can define its set of multiples \( M_B = \bigcup_{b \in B} b\mathbb{Z} \) and the set of \( B \)-free numbers \( F_B = \mathbb{Z} \setminus M_B \). Investigations into the structure of \( M_B \) or, equivalently, of \( F_B \) have a long history [see 17 for references], and dynamical systems theory provides some useful tools for such studies. One way to see this is to interpret such systems as weak model sets, where \( G = \mathbb{Z}, H \) is a closed subgroup of \( \hat{H} := \prod_{b \in B} \mathbb{Z}/b\mathbb{Z} \), namely the closure of the canonically embedded integers: \( H = \Delta(\mathbb{Z}) \) where \( \Delta(n) = (n, n, n, \ldots) \in \hat{H} \).

Finally, \( L = \{(n, \Delta(n)) : N \in \mathbb{Z}\} \) and \( W = \{h \in H : h_b \neq 0 \forall b \in B\} \). It is easy to see that in this case \( \hat{X} \) is isomorphic to \( H \), so that instead of \( \nu^G_W : \hat{X} \to M^G \) one simply looks at \( \nu^G_W : H \to M^G, \) where \( \nu^G_W(h) = \sum_{n \in \mathbb{Z}} \delta_n \cdot 1_W(h + \Delta(n)) (h \in H) \). Then \( \nu^G_W(0) = \sum_{n \in \mathbb{Z}} 1_W(\Delta(n)) = \sum_{n \in F_B} \delta_n \).

In the literature on \( B \)-free dynamics (e.g. [32, 17, 26]) these point measures on \( \mathbb{Z} \) are represented as \( 0,1 \)-sequences. This means that the map \( \nu^G_W \) is replaced by a map \( \varphi : H \to \{0,1\}^\mathbb{Z} \) defined by \( (\varphi(h))_n = 1_W(T^nh) = 1_W(h + \Delta(n)) \), where \( \Delta : \mathbb{Z} \to H \) is the natural embedding of \( \mathbb{Z} \) into \( H \) and, correspondingly, the set \( M^G \) is replaced by the set \( X_\varphi := \varphi(H) \). The systems \((M^G_W, \mathbb{Z})\) and \((X_\varphi, \mathbb{Z})\), with the respective left shifts as \( \mathbb{Z} \)-actions are obviously isomorphic dynamical systems. The Mirkovsky measure \( Q_{M^G} \) is denoted by \( \nu_\eta \), because it is (quasi-)generic for the point \( \eta := \varphi(\Delta(0)) \in X_\varphi \).

If the set \( B \subseteq \mathbb{Z} \) is taut (a basic regularity property whose definition is recalled in the next subsection), then the window is always Haar aperiodic [26]. Hence \((H, \mathbb{Z})\) is the MEGF of \((X_\varphi, \mathbb{Z})\) by Theorem 3.1, where \( \mathbb{Z} \) acts on \( H \) by \( h \mapsto h + \Delta(n) \). The system \((H, m_H, \mathbb{Z})\) is also the Kronecker factor of \((X_\varphi, \nu_\eta, \mathbb{Z})\), and there are many other invariant measures with the same KF. That this need not be the case for all ergodic invariant probability measures on \((X_\varphi, \mathbb{Z})\) is demonstrated in the following example.

**Example 1.** Consider an Erdős set \( B \) as studied in [32], where the elements of \( B \) are pairwise co-prime and \( \sum_{b \in B} 1/b < \infty \). Even for this rather special class of
systems (which includes the square-free numbers) one can find ergodic invariant measures with full topological support, for which the KF is not supported by the MEGF. The following construction of such examples uses a result from [32] whose proof relies on [19, Theorem 2].

Let $\kappa$ be an ergodic shift-invariant probability measure on $\{0,1\}^Z$, and denote by $\nu_\varphi \ast \kappa := M_\varphi(\nu_\varphi \otimes \kappa)$ the “convolution” of $\nu_\varphi$ and $\kappa$, where $M : ((0,1)^Z)^2 \to (0,1)^Z$ is the coordinate-wise multiplication [32, Section 2]. Corollary 3.15 together with Remark 3.16 of that reference shows that the system $(X_\varphi, \nu_\eta, Z)$ is isomorphic to the direct product of the systems $(X_\varphi, \nu_\eta, Z)$ and $(\{0,1\}^Z, \kappa, Z)$, whenever the latter system is (isomorphic to) an irrational rotation. In order to make sure that $\nu_\varphi \ast \kappa$ has the same topological support as $\nu_\varphi$ itself, it suffices to produce a 0-1-coding of an irrational rotation for which each block of 1’s (of arbitrary length) has positive probability.

To this end fix any irrational number $\alpha \in (0,1]$ and a sequence $(J_n)_{n \geq 0}$ of open subintervals of $\mathbb{R}/Z$ with length $|J_n| = (2(2n+1)2^n)^{-1}$. Define

$$E := \bigcup_{n=1}^{\infty} \bigcup_{k=-n}^{n}(J_n + k\alpha).$$

Denote the Lebesgue measure on $\mathbb{R}/Z$ by $\lambda$. Then $0 < \lambda(E) \leq 1/2$. The coding map $\varphi_E : \mathbb{R}/Z \to \{0,1\}^Z$, $x \mapsto (1_E(x+k\alpha))_{k \in Z}$, is $\lambda$-almost surely 1-1, and the probability, that this coding produces only 1’s at positions $-n, \ldots, n$ is at least $|J_n| > 0$.

Formally this construction can be written as a kind of model set, where the internal space $H$ is replaced by $H \times (\mathbb{R}/Z)$ and where the set $W \times E$ is taken as a window. Observe however, that $E$ is an open dense subset of $\mathbb{R}/Z$, so that the closure of this window would be $W \times (\mathbb{R}/Z)$, which by itself is a window that reproduces precisely the original system. Hence this construction is far from any weak model set.

Remark 9. Example 2 in [32, Section 2.2.2] shows that many $\mathcal{B}$-free systems support invariant ergodic measures $P$ with the following two properties:

i) The KF of $P$ is bigger than the KF of the Mirsky measure and is hence not supported by the MEGF of the system.

ii) $P$ is obtained as the Mirsky measure of a compact sub-window of the original window.

These measures do not have full topological support, however.

4.2. Generalized Erdös-type $\mathcal{B}$-free systems have trivial centralizers. Recall that $X_\varphi = \varphi(H)$, $\eta = \varphi(\Delta(0))$, and denote $X_\eta := \varphi(\Delta(Z)) = \{S^n\eta : n \in Z\}$, where $S : X_\varphi \to X_\varphi$ denotes the left shift. In the sequel we denote the natural $\mathbb{Z}$-action on $H$ by $T : H \to H$, $h \mapsto h + \Delta(1)$. We noted already that

(P1) $S \circ \varphi = \varphi \circ T$, but in general $\varphi : H \to X_\varphi$ is not continuous.

For the rest of this section we assume that the set $\mathcal{B}$ is taut (see e.g. [23, 17]). This means that for each $b \in \mathcal{B}$ the logarithmic density $\delta(\mathcal{M}_B \setminus \{b\})$ is strictly smaller

\footnote{Since I could not locate this statement in the literature, I provide a sketch of a proof: It suffices to prove that $\bigcup_{k \in \mathbb{Z}}(R_\alpha \times R_\alpha)^{k(E \times E^c)} \subseteq (\mathbb{R}/Z)^2$ has full 1-dimensional Lebesgue measure on the line $L_\delta := \{(x, x+\delta) : x \in \mathbb{R}/Z\}$ for every irrational $\delta$. By ergodicity of $R_\alpha \times R_\alpha$ along $L_\delta$, it suffices to show that the measure is not zero. Therefore suppose for a contradiction that $(E \times E^c) \cap L_\delta$ has Lebesgue measure 0. Then $E \subseteq E - \delta$ up to measure 0, which contradicts $0 < \lambda(E) < 1$ in view of the irrationality of $\delta$.}
than \( \delta(M_B) \), where \( \delta(M_B) := \lim_{n \to \infty} \frac{1}{\log n} \sum_{k \leq n, k \in M_B} k^{-1} \) is known to exist by the Theorem of Davenport and Erdös [12, 13].

**General assumption:** \( \mathcal{B} \) is taut and contains an infinite pairwise co-prime subset.

Under this assumption \((X, \eta, S)\) is proximal [17] and

(A1) \( X_\eta = X_\varphi = \text{supp}(\nu_\eta) \), where \( \nu_\eta = m_H \circ \varphi^{-1} \) denotes the Mirsky measure. We call this set simply \( X \) and denote by \( X_t \) the topologically transitive points in \( X \). As \( m_H \), and hence also \( \nu_\eta \), is ergodic, we have \( m_H(\varphi^{-1}(X_t)) = 1 \).

(A2) \( X \) is hereditary, i.e. if \( x \in X \) and \( y \in \{0, 1\}^\mathbb{Z} \) are such that \( y \leq x \), then also \( y \in X \).

(A3) The window \( W \) is Haar regular and aperiodic.

Properties (A1) and (A2) are finally proved in [27], but the proof relies significantly on previous work in [17] and to some extent also on [26, Prop. 2.2]. Property (A3) is the combination of [26, Thm. A and Prop. 5.1].

As \((X, S)\) is proximal with unique fixed point \( 0^\mathbb{Z} \), the MEF of this system is trivial and useless for the centralizer problem. Here we will use the MEGF of \((X, S)\) instead to give a purely dynamical alternative proof of a slight generalization of a result of Mentzen [34] (who assumes that \( \mathcal{B} \) is of Erdös type).\(^{14}\) See also Remark 11 below.

**Theorem 4.1** (Trivial centralizer, see also [34]). Suppose that \( \mathcal{B} \) is taut and contains an infinite pairwise co-prime subset. If \( F : X \to X \) is a homeomorphism that commutes with \( S \), then \( F \circ S^k = \text{id}_X \) for some \( k \in \mathbb{Z} \).

**Remark 10.** When the set \( \mathcal{B} \) is of opposite type, in the sense that it contains no scaled copy of any infinite pairwise co-prime set, the investigations into the structure of the centralizer seem to be much more subtle. A first example of this type was treated by A. Dymek [15]: she proved that the set \( \mathcal{B} = \{2^n c_n : n \in \mathbb{N}\} \) with pairwise co-prime odd \( c_n \) gives rise to a trivial centralizer. Generalizations of these examples are studied in [16], where also sets \( \mathcal{B} \) producing non-trivial centralizers are described.

4.2.1. The role of the MEGF. Some arguments below are based on our Lemma 3.3. For the convenience of the reader we rewrite the three relevant statements of that lemma using the special notation for \( \mathcal{B} \)-free systems:

**Lemma 4.2.** Assume that \( \mathcal{B} \) is taut and denote by \( X_t \) the set of topologically transitive points of \( X \). There is a well defined map \( \pi : X_t \to H \) with the following properties:

a) \( \pi : X_t \to H \) is continuous.

b) If \( x \in X_t \), then \( x \leq \varphi(\pi x) \).

c) If \( x \in X_t \) and \( x \leq \varphi(h) \), then \( h = \pi x \).

**Corollary 4.** \( \pi : (X_t, S) \to (H, T) \) is the MEGF of \((X, S)\). In particular,

(P2) \( S(X_t) = X_t \) and \( \pi \circ S|_{X_t} = T \circ \pi \).

**Proof.** If \( x \in X_t \), then also \( Sx \in X_t \), and as \( x \leq \varphi(\pi x) \) by Lemma 4.2b, also \( Sx \leq S(\varphi(\pi x)) \) (P1) \( \varphi(T(\pi x)) \), so that \( \pi(Sx) = T(\pi x) \) by Lemma 4.2c. The continuity of \( \pi : X_t \to H \) is proved in Lemma 4.2a. Hence \( \pi \) is an equicontinuous generic factor map. In view of (A3) and Theorem 3.1b, it is the MEGF.

\(^{14}\)There are also some unpublished notes by Lemańczyk et al. providing a related proof but also using explicitly arithmetic properties.
Let $F : X \to X$ be a continuous and surjective map that commutes with $S$. Then

(P3) $F(X_t) = X_t$,

so that the map $\pi \circ F|_{X_t} : X_t \to H$ is a generic factor map. Hence, by Corollary 4 and Theorem 2.16b, there is a (continuous!) factor map $f : (H, T) \to (H, T)$ satisfying

(P4) $\pi \circ F|_{X_t} = f \circ \pi$.

Combining (P2) and (P4) we get $T(f(\pi x)) = T(\pi(Fx)) = \pi(S(Fx)) = \pi(F(Sx)) = f(\pi(Sx)) = f(T(\pi x))$ for all $x \in X_t$, and as both $f$ and $T$ are continuous, $\pi(X_t)$ is dense in $H$, we have

(P5) $T \circ f = f \circ T$.

Remark 11. The whole setting described so far applies to general weak model sets, except assumptions (A1)–(A3) which, in the context of $\mathcal{B}$-free dynamics, follow from $\mathcal{B}$ being taut and containing an infinite pairwise co-prime subset. But, of course, they are formulated in dynamical terms without any recourse to arithmetics.

**Proposition 1.** If $F : X \to X$ is a continuous and surjective map that commutes with $S$, then there is $x \in S$ such that $F(S^k x) \leq x$ for all $x \in X$.

**Proof.** As in the first lines of the proof of Lemma 2 in [34] it follows that $F(0^\omega) = 0^\omega$ and $F(0^{\infty}10^{\infty}) \neq 0^\omega$ (one just uses the heredity and proximality of $X$.) Denote by $\rho : \{0,1\}^{[0,\alpha]}$ the block map that defines $F$. Then $\rho(0) = 0$, because $F(0^\omega) = 0^\omega$, and as $F(0^{\infty}10^{\infty}) \neq 0^\omega$, there is $k \in \mathbb{Z}$ such that $F(S^k(0^{\infty}10^{\infty})) = S^k(0^{\infty}10^{\infty})[0] = F(0^{\infty}10^{\infty})[k] = 1$.

From now on we consider $F \circ S^k$ instead of $F$ and denote this new map again by $F$. The new $F$ also satisfies (P3)–(P5), when the map $f$ is replaced by $f \circ S^k$.

We denote the block map for the new $F$ by $\rho$ again so that, defining $u := 0^a10^a$, we have $\rho(u) = 1$. We will prove that $F(x) \leq x$ for all $x \in X$ for the new $F$.

Our first goal is to show that

$$\varphi(f(h))[0] \geq \varphi(h)[0] \quad \text{for all } h \in H_0 := \varphi^{-1}(X_t). \quad (8)$$

Let $h \in H_0$, so that $x := \varphi(h) \in X_t$. If $\varphi(h)[0] = 0$, there is nothing to prove. So we may assume that $\varphi(h)[0] = 1$. Define $y \in \{0,1\}^\omega$ by $y[a] = u$ and $y[n] = x[n]$ for $|n| > a$. Then $y[n] \leq x[n] = \varphi(h)[n]$ for all $n \neq 0$ and $y[0] = 1 = \varphi(h)[0]$, so that $y \leq \varphi(h)$ and $y \in X$ by heredity (A2). As $y[n] = x[n]$ for all $n \in \mathbb{Z}$ except at most finitely many and as $x \in X_t$, also $y \in X_t$. Hence Lemma 4.2c applies to $y$, so that $\pi y = h$. We see that $F(y) \in X_t$ by (P3), so that Lemma 4.2b can be applied to $F(y)$:

$$F(y) \leq \varphi(\pi(Fy)).$$

It follows that

$$\varphi(f(h))[0] = \varphi(f(\pi y))[0] \overset{(P3)}{=} \varphi(\pi(Fy))[0] \geq F(y)[0] = \rho(y[a]) = \rho(u) = 1 = \varphi(h)[0],$$

that is (8). Now, for arbitrary $n \in \mathbb{Z}$, $\varphi(T^nh) = S^n(\varphi(h)) \in S^n(X_t) = X_t$, so that (8) applies to the point $T^nh$ as well, whence

$$\varphi(f(h))[n] = S^n(\varphi(f(h)))[0] \overset{(P3)}{=} \varphi(T^n(f(h)))[0] \overset{(P5)}{=} \varphi(f(T^n h))[0] \overset{(8)}{=} \varphi(T^n h)[0] \overset{(P1)}{=} S^n(\varphi(h))[0] = \varphi(h)[n].$$

\[15\] Indeed, they are both translations.
Hence
\[ \varphi(f(h)) \geq \varphi(h) \quad \text{for all } h \in H_0. \]
As \( m_H(H_0) = 1 \) in view of (A1), this shows that \( \varphi \circ f = \varphi \) \( m_H \)-a.s. So we have for \( m_H \)-almost all \( h \in H \)
\[ F(\varphi(h)) \overset{\text{Lemma 4.2b}}{\leq} \varphi(\pi(F(\varphi(h)))) \overset{(\text{P}4)}{=} \varphi(f(\varphi(h)))) \overset{\text{Lemma 4.2c}}{=} \varphi(f(h)) = \varphi(h), \]
equivalently, \( F(x) \leq x \) for \( \nu_\eta \)-a.a. \( x \in X \). As \( \text{supp}(\nu_\eta) = X \) by (A1) and as \( F \) is continuous, this shows that \( F(x) \leq x \) for all \( x \in X \).

\textbf{Proof of Theorem 4.1.} If \( F: X \to X \) is even a homeomorphism, then Proposition 1 applies to \( F^{-1} \) as well. Hence there are \( k, \ell \in \mathbb{Z} \) such that \( F(S^k x) \leq x \) and
\[ F^{-1}(S^{-\ell}x) \leq x \quad \text{for all } x \in X. \]
Thus, for all \( x \in X \),
\[ x = (F^{-1} \circ S^{-\ell}) \circ (F(S^k x)) \leq F(S^\ell x) = F(S^k(S^\ell-k)x) \leq S^\ell-k x. \]
\text{(9)}

Applied to the point \( x = 0^\infty 10^\infty \in X \) this shows that \( \ell = k \), and we conclude from \text{(9)} that \( x = F(S^k x) \) for all \( x \in X \).

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