THE FEICHTINGER CONJECTURE AND REPRODUCING KERNEL HILBERT SPACES

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Abstract. We prove two new equivalences of the Feichtinger conjecture that involve reproducing kernel Hilbert spaces. We prove that if for every Hilbert space, contractively contained in the Hardy space, each Bessel sequence of normalized kernel functions can be partitioned into finitely many Riesz basic sequences, then a general bounded Bessel sequence in an arbitrary Hilbert space can be partitioned into finitely many Riesz basic sequences. In addition, we examine some of these spaces and prove that for these spaces bounded Bessel sequences of normalized kernel functions are finite unions of Riesz basic sequences.

1. Introduction

We study the Feichtinger conjecture in the setting of reproducing kernel Hilbert spaces. The Feichtinger conjecture originated in harmonic analysis and currently is a topic of high interest as it has been shown to be equivalent to the celebrated Kadison-Singer Problem (KSP) [8]. The Feichtinger conjecture dates back to at least 2003 and appeared in print in [6]. There is a significant body of work on this conjecture [2, 3, 4, 5, 8, 9, 6, 7, 10].

There are several versions of the Feichtinger conjecture, all of which are equivalent to the Kadison-Singer problem, but we shall be interested in the version involving Bessel sequences.

Conjecture 1.1. Feichtinger Conjecture (FC). Every bounded Bessel sequence in a Hilbert space can be partitioned into finitely many Riesz basic sequences.

In this paper, we specialize this conjecture to the case where the underlying Hilbert space belongs to a special family of reproducing kernel Hilbert spaces on the unit disk, namely, the Hilbert spaces that are contractively contained in the Hardy space $H^2$ and require, in addition, that the bounded Bessel sequence consists of normalized kernel functions for a sequence of points in the disk. One of our results is that this special version of the Feichtinger conjecture is equivalent to the Feichtinger conjecture.

Our work is motivated by some work of Nikolski. In his lecture at the AIM workshop “The Kadison-Singer Problem” in 2006, Nikolski proved that any Bessel sequence consisting of normalized kernel functions in the Hardy space $H^2$ could be partitioned into finitely many Riesz basic sequences. Later in
2009, Baranov and Dyakonov \cite{BaranovDyakonov} proved the analogous result for two families of model subspaces of $H$. Thus, we were motivated to seek a converse. That is, to find a sufficiently large family of reproducing kernel Hilbert spaces, so that if one verify that each Bessel sequence of normalized kernel functions in those spaces could be partitioned into finitely many Riesz basic sequences, then that would guarantee the full FC.

In addition, we also prove that in order to verify the FC it is enough to test a specific family of sequences which is related to kernel functions in $H^2$. To state these equivalences formally we need the following basic notations and terminologies.

Given a set $J$ we let $\ell^2(J)$ denote the usual Hilbert space of square-summable functions on $J$ with canonical orthonormal basis $\{\varepsilon_i\}_{i \in J}$. We let $I_{\ell^2(J)}$ denote the identity operator on $\ell^2(J)$. When $J \subseteq \mathbb{N}$, we regard $\ell^2(J) \subseteq \ell^2(\mathbb{N}) = \ell^2$ and let $P_J$ denote the orthogonal projection of $\ell^2(\mathbb{N})$ onto $\ell^2(J)$.

A set of vectors $\{f_i\}_{i \in J}$ in a Hilbert space $\mathcal{H}$ is called a **frame** for $\mathcal{H}$ if there exist constants $A,B > 0$ such that

$$A\|x\|^2 \leq \sum_{i\in J} |\langle x, f_i \rangle|^2 \leq B\|x\|^2 \tag{1}$$

for every $x \in \mathcal{H}$. A countable collection $\{f_i\}_{i \in J}$ in a Hilbert space $\mathcal{H}$ is called a **frame sequence** if it is a frame for $\text{span}\{f_i : i \in J\}$. If only the right hand side inequality holds in Inequality (1), then $\{f_i\}_{i \in J}$ is called a Bessel set. A countable Bessel set is called a **Bessel sequence**. Thus, every frame sequence is a Bessel sequence. A Bessel sequence $\{f_i\}_{i \in J}$ is called **bounded**, if there exists a constant $\delta > 0$ such that $\|f_i\| \geq \delta$ for every $i \in J$. Note that a Bessel sequence is always bounded above.

Further, a set $\{f_i\}_{i \in J}$ in a Hilbert spaces $\mathcal{H}$ is called a **Riesz basis** for $\mathcal{H}$ if there exists an orthonormal basis $\{u_i\}_{i \in J}$ for $\mathcal{H}$ and an invertible operator $S \in B(\mathcal{H})$ such that $S(u_i) = f_i$ for every $i \in J$. It is easy to verify that a countable set $\{f_i\}_{i \in J}$ is a Riesz basis for $\mathcal{H}$ if and only if its linear span is dense in $\mathcal{H}$ and there exist constants $A,B > 0$ such that

$$A\sum_{i \in J} |\alpha_i|^2 \leq \| \sum_{i \in J} \alpha_i f_i \|^2 \leq B\sum_{i \in J} |\alpha_i|^2$$

for all square summable sets $\{\alpha_i\}_{i \in J}$. A countable set $\{f_i\}_{i \in J}$ is called a **Riesz basic sequence** if it is a Riesz basis for $\text{span}\{f_i : i \in J\}$. It is well-known that every Riesz basic sequence is a frame sequence.

Given a Bessel set $\{f_i\}_{i \in J}$ in a Hilbert space $\mathcal{H}$, the corresponding analysis operator, $F : \mathcal{H} \to \ell^2(J)$ defined by $F(x) = (\langle x, f_i \rangle)_{i \in J}$ is bounded. It is easy to check that $F^* : \ell^2(J) \to \mathcal{H}$ is given by $F^*(\varepsilon_i) = f_i$ for all $i \in J$, and $FF^* = (\langle f_j, f_i \rangle)$. The operators $F^*$ and $FF^*$ are called the **synthesis operator** and the **Grammian** of the set $\{f_i\}_{i \in J}$, respectively. Note that:

i. $\{f_i\}_{i \in J}$ is a Bessel sequence iff the map $F$ is bounded iff $FF^*$ is bounded;
ii. \( \{f_i\}_{i \in I} \) is a frame sequence iff \( F : \text{span}\{f_i : i \in I\} \rightarrow \ell^2(I) \) is bounded and bounded below;

iii. \( \{f_i\}_{i \in I} \) is a Riesz basic sequence iff \( F : \text{span}\{f_i : i \in I\} \rightarrow \ell^2(I) \) is invertible.

Henceforth, given a Bessel sequence \( \{f_i\}_{i \in I} \) in a Hilbert space \( \mathcal{H} \), \( F \) is reserved for the analysis operator from \( \mathcal{H} \) to \( \ell^2(I) \), as defined above.

We now wish to recall some basic terminology from reproducing kernel Hilbert space theory.

Recall that a reproducing kernel Hilbert space (RKHS) on a set \( X \) is a Hilbert space \( \mathcal{H} \) of functions on \( X \) such that evaluation at any point in \( X \) is a bounded linear functional. The function \( K : X \times X \rightarrow \mathbb{C} \) given by \( K(x, y) = k_y(x) \) where \( f(y) = \langle f, k_y \rangle \) for all \( f \in \mathcal{H} \) is called the reproducing kernel for \( \mathcal{H} \). The function \( k_y, y \in X \), is called the kernel function for the point \( y \). We set \( \tilde{k}_y = \frac{k_y}{\|k_y\|} = \frac{k_y}{\sqrt{K(y,y)}} \) whenever \( k_y \neq 0 \) and call it the normalized kernel function for the point \( y \).

Let \( \mathbb{D} \) denote the open unit disk in the complex plane, and let \( H^2 \) denote the familiar Hardy space on \( \mathbb{D} \). Recall that it is a reproducing kernel Hilbert space on \( \mathbb{D} \) with reproducing kernel \( K(z, w) = \frac{1}{1-z\overline{w}}, \; z, w \in \mathbb{D} \), which is called the Szegő kernel.

A Hilbert space \( \mathcal{H} \) is said to be contractively contained in \( H^2 \) if it is a vector subspace of \( H^2 \) and the inclusion of \( \mathcal{H} \) into \( H^2 \) is a contraction, that is, \( \|h\|_{H^2} \leq \|h\|_\mathcal{H} \) for every \( h \) in \( \mathcal{H} \). Lastly, recall that every Hilbert space that is contractively contained in \( H^2 \) is a RKHS. Given a Hilbert space \( \mathcal{H} \), that is contractively contained in \( H^2 \), \( k^\mathcal{H}_z \) will denote the kernel function for \( z \in \mathbb{D} \) and the corresponding normalized kernel function will be denoted by \( \tilde{k}^\mathcal{H}_z \).

Henceforth, \( k_z \) and \( \tilde{k}_z \) shall be reserved to denote the kernel function and the corresponding normalized kernel function in \( H^2 \) for the point \( z \in \mathbb{D} \), respectively.

**Theorem 1.2.** The following are equivalent:

(i) every bounded Bessel sequence in a Hilbert space can be partitioned into finitely many Riesz basic sequences (FC),

(ii) every bounded Bessel sequence of the form \( \{P\tilde{k}_{z_i}\}_{i \in \mathbb{N}}, \) where \( P \in B(H^2) \) is a positive operator and \( \{z_i\}_{i \in \mathbb{N}} \subseteq \mathbb{D} \) is a sequence, can be partitioned into finitely many Riesz basic sequences,

(iii) every bounded Bessel sequence of the form \( \{\tilde{k}^\mathcal{H}_{z_i}\}_{i \in \mathbb{N}}, \) where \( \mathcal{H} \) is contractively contained in \( H^2 \) and \( \{z_i\}_{i \in \mathbb{N}} \subseteq \mathbb{D} \) is a sequence, can be partitioned into finitely many Riesz basic sequences.

In fact we can also assume a much more restrictive condition on the sequence \( \{z_i\}_{i \in \mathbb{N}} \) which is that \( \{\tilde{k}_{z_i}\}_{i \in \mathbb{N}} \) is a Riesz basic sequence in \( H^2 \) or, equivalently, that \( \{z_i\}_{i \in \mathbb{N}} \) satisfy Carleson’s condition (C). This concept will be defined in the next section.
This leads us to formulate the following:

**Conjecture 1.3. Feichtinger Conjecture for Kernel Functions (FCKF).**

*Every Bessel sequence of normalized kernel functions in every reproducing kernel Hilbert space can be partitioned into finitely many Riesz basic sequences.*

From this point forward, we will say that a particular reproducing kernel Hilbert space $\mathcal{H}$ satisfies the FCKF if every Bessel sequence of normalized kernel functions in $\mathcal{H}$ can be partitioned into finitely many Riesz basic sequences.

Thus, the content of our theorem is that not only are the FC and the FCKF equivalent, but that the FC is equivalent to the FCKF holding for the family of Hilbert spaces contractively contained in $H^2$.

2. **History**

We shall now give a brief history and motivation of our problem. The study of Bessel sequences of normalized kernel functions was initiated by Shapiro and Shields in 1961 [17]. They analyzed these sequences purely in the context of interpolation problems in the corresponding RKHS. In this course, they proved a beautiful result about interpolating sequences in $H^2$ which in late 60’s was reformulated by Nikolski and Pavlov [13, 14] as follows:

**Theorem 2.1.** A sequence \( \{ \tilde{k}_z \}_i \in \mathbb{N} \) of normalized kernel functions in $H^2$ is a Riesz basic sequence iff there exists a constant $\delta > 0$ such that

\[
(C) \quad \prod_{i \neq j} \left| \frac{z^*_i z_j}{z^*_i z^*_j} \right| \geq \delta, \quad j = 1, 2, \ldots
\]

In the late 70’s, independent of the work of Nikolski and Pavlov, McKenna was also studying kernel functions. In [11] McKenna proved some partial converses to Shapiro and Shields results [17] and thereby brought some more insight to the area. In particular, he proved the following interesting result:

**Theorem 2.2.** Let \( \{ \tilde{k}_z \}_i \in \mathbb{N} \) be a Bessel sequence of normalized kernel functions in $H^2$. Then \( \{ z_i \}_i \in \mathbb{N} \) can be partitioned into finitely many subsequences each of which satisfies the condition (C).

Nikolski gave a completely different proof of the above theorem which he included in [12]. The FC motivated Nikolski to combine the above two results as follows:

**Theorem 2.3.** Every Bessel sequence \( \{ \tilde{k}_z \}_i \in \mathbb{N} \) of normalized kernel functions in $H^2$ can be partitioned into finitely many Riesz basic sequences.

Thus, Theorem 2.3 shows that $H^2$ satisfies the FCKF. This introduced methods from reproducing kernel Hilbert space theory to the FC.
3. Preliminary Results

We begin by recording a few elementary observations that we shall use later. Recall that a sequence \( \{f_i\}_{i \in J} \), in a Hilbert space \( \mathcal{H} \) is a Bessel sequence iff \( FF^* = (\langle f_j, f_i \rangle) \in B(\ell^2(J)) \), that is iff its Grammian is bounded. The following give characterization of other properties that we shall need in terms of Grammians.

**Proposition 3.1.** Let \( \{f_i\}_{i \in J} \subseteq \mathcal{H} \). Then \( \{f_i\}_{i \in J} \) is a Riesz basis for \( \mathcal{H} \) iff it is a Bessel sequence with closed linear span equal to \( \mathcal{H} \), and there exists a constant \( c > 0 \) such that \( FF^* \geq cI_{\ell^2(J)} \).

**Proposition 3.2.** A sequence \( \{f_i\}_{i \in J} \) in a Hilbert space \( \mathcal{H} \) is a Riesz basic sequence iff it is a Bessel sequence and there exists a constant \( c > 0 \) such that \( FF^* \geq cI_{\ell^2(J)} \).

Thus, we get the following reformulation of the FC.

**Proposition 3.3.** A Bessel sequence \( \{f_i\}_{i \in J} \) can be partitioned into \( n \) Riesz basic sequences iff there exist a partition \( A_1, \ldots, A_n \) of \( J \) and constants \( c_1, \ldots, c_n > 0 \) such that \( P_{A_i} FF^* P_{A_i} \geq c_i P_{A_i} \) for all \( 1 \leq i \leq n \).

From now on, whenever a sequence in a Hilbert space can be partitioned into finitely many Riesz basic sequences we will say that it satisfies the FC.

**Proposition 3.4.** Let \( \{f_i\}_{i \in J} \subseteq \mathcal{H} \) and \( \{g_i\}_{i \in J} \subseteq \mathcal{K} \), \( J \subseteq \mathbb{N} \) be two sequences such that \( (\langle f_j, f_i \rangle) = D(\langle g_j, g_i \rangle)D^* \), where \( D \) is an invertible, diagonal operator in \( B(\ell^2(J)) \). Then:

(i) \( \{f_i\}_{i \in J} \) is a Bessel sequence iff \( \{g_i\}_{i \in J} \) is a Bessel sequence,

(ii) \( \{f_i\}_{i \in J} \) is a frame sequence iff \( \{g_i\}_{i \in J} \) is a frame sequence,

(iii) \( \{f_i\}_{i \in J} \) satisfies the FC iff \( \{g_i\}_{i \in J} \) satisfies the FC.

**Proof.** Note that (i) and (iii) are immediate consequences of the Grammian characterizations of these properties. To prove (ii) note that a Bessel sequence \( \{f_i\}_{i \in J} \) is a frame sequence iff the Grammian is bounded below on the orthogonal complement of its kernel. \qed

While the following result is not necessary for the development of any of our further results, it does serve to explain why we have chosen to study the Bessel sequence version of the FC in this context instead of the frame version. Given a bounded operator \( T \in B(\mathcal{H}, \mathcal{K}) \) we shall denote the kernel of \( T \) by \( \text{Ker}(T) \).

**Theorem 3.5.** A sequence \( \{k_{z_i}\}_{i \in J} \) of normalized kernel functions in \( H^2 \) is a frame sequence iff it is a Riesz basis basic sequence. Moreover, in this case there is no other kernel function in the closed linear span of \( \{k_{z_i}\}_{i \in J} \).

**Proof.** Let \( \mathcal{H} \) be the closed linear span of \( \{k_{z_i}\}_{i \in J} \) in \( H^2 \). If \( \{k_{z_i}\}_{i \in J} \) is a Riesz basis for \( \mathcal{H} \), then clearly it is a frame for \( \mathcal{H} \). To prove the converse, suppose
\(\{k_{z_i}\}_{i \in J}\) is a frame for \(\mathcal{H}\). Then the analysis operator \(F : \mathcal{H} \to \ell^2(J)\), given by \(F(x) = (x, k_{z_i})\) is bounded and \(F^*\) is onto. Hence, to prove \(\{k_{z_i}\}_{i \in J}\) is a Riesz basis, it is enough to prove that \(F^*\) is one-to-one.

To this end, let \(\{\lambda_i\}_{i \in J} \subseteq \text{Ker}(F^*)\). Then \(\sum_{i \in J} \lambda_i k_{z_i} = 0\), which implies that \(\langle f, \sum_{i \in J} \lambda_i \tilde{k}_{z_i} \rangle = 0\) for all \(f \in \mathcal{H}^2\), which further implies that \(\sum_{i \in J} \lambda_i \|k_{z_i}\|^2 = 0\) for all \(f \in \mathcal{H}^2\).

Note that, \(\{\tilde{k}_{z_i}\}_{i \in J}\) is a frame and so is a Bessel sequence. Therefore, by Theorem 2.2, \(\{z_i\}_{i \in J}\) is a finite union of sets satisfying (C) and hence satisfies the Blaschke condition. Let \(f_j\) denote the Blaschke product with zeroes at \(z_i : i \neq j\). Then each \(f_j\) is in \(\mathcal{H}^2\) and so \(\sum_{i \in J} \lambda_i \|f_j(z_i)\|^2 = 0\) for all \(j \in J\). This forces, \(\lambda_j = 0\) for all \(j \in J\). Thus, \(\text{Ker}(F^*) = 0\). So, \(F^*\) is an invertible operator and hence \(\{k_{z_i}\}_{i \in J}\) is a Riesz basis for \(\mathcal{H}\).

The moreover part follows by observing that since \(\{z_i\}_{i \in J}\) is a Blaschke sequence, the corresponding Blaschke product \(B \in \mathcal{H}^2\) is orthogonal to the span of the kernel functions and \(B(w) \neq 0\) at any point not in the set. \(\square\)

### 4. Proof Of The Main Theorem

The following theorem is a stepping-stone to our main result.

**Theorem 4.1.** Fix a sequence \(\{z_i\}_{i \in \mathbb{N}}\) in \(\mathbb{D}\) so that \(\{\tilde{k}_{z_i}\}_{i \in \mathbb{N}}\) is a frame sequence in \(\mathcal{H}^2\). Let \(Q \in B(\ell^2)\) be a positive operator such that there exists a constant \(\delta > 0\) with \(\langle Qe_i, e_i \rangle \geq \delta\) for each \(i\). Then there exists a positive operator \(P \in B(\mathcal{H}^2)\) such that

\[
Q = \left(\langle \tilde{P}k_{z_i}, P\tilde{k}_{z_i} \rangle\right)
\]

with \(\|P\tilde{k}_{z_i}\|^2 \geq \delta\) for all \(i\).

**Proof.** Let \(\mathcal{H}\) be the closed linear span of \(\{\tilde{k}_{z_i}\}_{i \in \mathbb{N}}\) in \(\mathcal{H}^2\). Then, by Theorem 3.5, \(\{k_i\}_{i \in \mathbb{N}}\) is a Riesz basis for \(\mathcal{H}\), since \(\{k_{z_i}\}_{i \in \mathbb{N}}\) is a frame for \(\mathcal{H}\). So, the analysis operator \(F : \mathcal{H} \to \ell^2\), given by \(F(x) = (x, k_{z_i})\) is invertible with \(F^*(e_i) = k_{z_i}\) for each \(i\). Set \(R = F^{-1}Q(F^{-1})^*\). Then \(R : \mathcal{H} \to \mathcal{H}\) is a positive, bounded operator. We now extend \(R\) to \(\mathcal{H}^2\) by defining it be 0 on \(\mathcal{H}^\perp\). We claim that \(P = R^{1/2}\) satisfies the required conditions. To prove the claim, we fix \(i, j \in \mathbb{N}\), and consider

\[
\langle \tilde{P}k_{z_i}, P\tilde{k}_{z_i} \rangle = \langle R^{1/2}k_{z_j}, R^{1/2}k_{z_i} \rangle = \langle Rk_{z_j}, k_{z_i} \rangle = \langle Q(F^{-1})^*k_{z_j}, (F^{-1})^*k_{z_i} \rangle = \langle Qe_j, e_i \rangle
\]

Hence, \(Q = \left(\langle \tilde{P}k_{z_j}, P\tilde{k}_{z_i} \rangle\right)\). Also, as obtained above \(\|P\tilde{k}_{z_i}\|^2 = \langle \tilde{P}k_{z_i}, P\tilde{k}_{z_i} \rangle = \langle Qe_i, e_i \rangle \geq \delta\) for all \(i\). This completes the proof. \(\square\)
Remark 4.2. Note that in Theorem 4.1, we have a great deal of freedom in the choice of the frame sequence \( \{ \tilde{k}_{zi} \}_{i \in \mathbb{N}} \) and hence on the Hilbert space \( \mathcal{H} = \text{span}\{ \tilde{k}_{zi} : i \in \mathbb{N} \} \), and also on the behavior of \( P \) on \( \mathcal{H} \).

We are now ready to give the proof of our main theorem. But, before proving the theorem we first recall a characterization of the Hilbert spaces contractively contained in the Hardy space \( H^2 \) found in the work of Sarason [16], which is very crucial for our proof as it reveals their connection with positive contractions on \( H^2 \).

Let \( \mathcal{H} \) be a Hilbert space that is contractively contained in the Hardy space \( H^2 \) with norm \( \| \cdot \|_{\mathcal{H}} \). Let \( T : \mathcal{H} \to H^2 \) be the inclusion map, then \( T \) and \( T^* : H^2 \to \mathcal{H} \) are both contractions. Thus, \( P = TT^* \) is a bounded, positive contraction in \( B(H^2) \). This gives rise to another Hilbert space, the range space \( \mathcal{R}(P^{1/2}) \), which one obtains by equipping the range of \( P^{1/2} \) with the norm, \( \| y \|_P = \| x \|_{H^2} \), where \( x \) is the unique vector in the orthogonal complement of the kernel of \( P^{1/2} \) such that \( y = P^{1/2} x \). One has that \( \mathcal{H} = \mathcal{R}(P^{1/2}) \) as sets and the two norms coincide. Thus, if a Hilbert space \( \mathcal{H} \) is contractively contained in \( H^2 \), then there exists a positive contraction \( P \in B(H^2) \) such that \( \mathcal{H} \) is the range space, \( \mathcal{R}(P^{1/2}) \).

On the other hand, given a positive contraction \( P \in B(H^2) \) the range space \( \mathcal{R}(P^{1/2}) \), as defined above, is always contractively contained in \( H^2 \).

Henceforth, given a positive contraction \( P \in B(H^2) \), we shall denote the Hilbert space \( \mathcal{R}(P^{1/2}) \) by \( \mathcal{H}(P) \) and the kernel function in it for a point \( w \in \mathbb{D} \) by \( k^P_w \). For the normalized kernel function \( \tilde{k}^P_w \), we shall use \( \tilde{k}^P_w \).

Lastly, we note that \( k^P_w = Pk_w \) for \( w \in \mathbb{D} \) and \( ||Px||_P = ||P^{1/2}x|| \) for all \( x \in H^2 \).

Proof of Theorem 1.2 (i) implies (iii) is trivially true. We now prove (iii) implies (ii). Let \( P \in B(H^2) \) be a positive operator and let \( \{ z_i \}_{i \in \mathbb{N}} \) be a sequence in \( \mathbb{D} \) such that \( \{ P^2 k_{zi} \}_{i \in \mathbb{N}} \) is a bounded Bessel sequence in \( H^2 \) with \( ||P^2 k_{zi}|| \geq \delta > 0 \) for all \( i \). Then \( T = P^2/||P^2|| \) is a positive contraction in \( B(H^2) \) and thus \( \mathcal{H}(T) = \mathcal{R}(T^{1/2}) \) is a Hilbert space that is contractively contained in \( H^2 \). Further, note that for fixed \( i, j \),

\[
\langle \tilde{k}^T_{z_j}, \tilde{k}^T_{z_i} \rangle_T = \frac{\langle Tk_{z_j}, Tk_{z_i} \rangle}{\|Tk_{z_j}\|_T \|Tk_{z_i}\|_T} = \frac{\langle T^{1/2}k_{z_j}, T^{1/2}k_{z_i} \rangle}{\|T^{1/2}k_{z_j}\| \|T^{1/2}k_{z_i}\|} = \frac{\langle Pk_{z_j}, Pk_{z_i} \rangle}{\|Pk_{z_j}\| \|Pk_{z_i}\|} = \frac{||k_{z_j}|| \langle P\tilde{k}_{z_j}, P\tilde{k}_{z_i} \rangle}{||Pk_{z_j}|| \|P\tilde{k}_{z_i}||}.
\]
Hence,
\[
\left( \langle \tilde{k}^T_{z_i}, \tilde{k}^T_{z_i} \rangle_T \right) = D \left( \langle P \tilde{k}_{z_i}, P \tilde{k}_{z_i} \rangle \right) D^*,
\]
where \( D \in B(\ell^2) \) is an invertible, diagonal operator with \( i^{th} \) diagonal entry \( D_{ii} = \|k_{z_i}\|/\|P k_{z_i}\| \), since \( P \in B(H^2) \) and \( \|P k_{z_i}\| \geq \delta \|k_{z_i}\| \) for all \( i \). Then (i) of Proposition 3.4 implies that \( \{\tilde{k}^T_{z_i}\}_{i \in \mathbb{N}} \) is a Bessel sequence in \( H(T) \), since \( \{P \tilde{k}_{z_i}\}_{i \in \mathbb{N}} \) is a Bessel sequence in \( H^2 \). Thus, by assuming (iii), we have that it satisfies the FC. Hence, by using (iii) of Proposition 3.4, this completes the proof of (iii) implies (ii).

Finally, we prove (ii) implies (i). Let \( \{f_i\}_{i \in \mathbb{N}} \) be a bounded Bessel sequence in a Hilbert space \( H \) with \( \|f_i\| \geq \delta > 0 \) for each \( i \). Then, \( FF^* \) is a bounded, positive operator in \( B(\ell^2) \), where \( F : H \to \ell^2 \) is the analysis operator associated with \( \{f_i\}_{i \in \mathbb{N}} \). Also, \( \langle FF^*(e_i), e_i \rangle = \|f_i\|^2 \geq \delta^2 \) for all \( i \). Thus, by Theorem 4.1, there exists a positive operator \( P \in B(H^2) \) with \( \|P k_{z_i}\| \geq \delta \) for each \( i \), such that
\[
FF^* = \left( \langle f_j, f_i \rangle \right) = \left( \langle P \tilde{k}_{z_j}, P \tilde{k}_{z_i} \rangle \right),
\]
where \( \{\tilde{k}_{z_i}\}_{i \in \mathbb{N}} \) is a frame sequence and hence is a Riesz basic sequence in \( H^2 \). Since \( FF^* \in B(\ell^2) \), \( \{P \tilde{k}_{z_i}\}_{i \in \mathbb{N}} \) is a Bessel sequence. Also, \( \{P \tilde{k}_{z_i}\}_{i \in \mathbb{N}} \) is bounded. Thus by assuming (ii), we have that it satisfies the FC. Hence, by (iii) of Proposition 3.4, \( \{f_i\}_{i \in \mathbb{N}} \) also satisfies the FC. This completes the proof of (ii) implies (i) and of the theorem.

Finally, note that the remark after the theorem follows immediately from the fact that in Theorem 4.1 we choose \( \{z_i\}_{i \in \mathbb{N}} \) so that \( \{k_{z_i}\}_{i \in \mathbb{N}} \) is a Riesz basic sequence.

5. Analysis Of New Equivalences

We can easily verify that statement (ii) of Theorem 1.2 can be reduced to the case of positive operators which are contractions. Thus, Theorem 1.2 motivates the study of sequences \( \{P \tilde{k}_{z_i}\}_{i \in \mathbb{N}}, \{k^P_{z_i}\}_{i \in \mathbb{N}} \), where \( P \in B(H^2) \) is a positive contraction and \( \{\tilde{k}_{z_i}\}_{i \in \mathbb{N}}, \{k^P_{z_i}\}_{i \in \mathbb{N}} \) are sequences of normalized kernel functions in \( H^2 \) and \( \mathcal{H}(P) \), respectively. By considering positive operators and kernel functions we have much more structure to exploit and thereby we can expect some interesting and fruitful research in this direction. The theorem suggests that it might not be easy to make any general statement about the whole family of these sequences. Instead we focus on some particular families of positive operators and investigate the FC for the corresponding sequences. In this direction we have the following results.

It is elementary to see that the FC holds for \( \mathcal{H}(P) \) when \( P \) is a positive, invertible operator in \( B(H^2) \). Note in this case \( \mathcal{H}(P) = H^2 \) and the two norms are equivalent. Thus, \( \{P \tilde{k}_{z_i}\}_{i \in \mathbb{N}} \) is a Bessel (frame or Riesz basic)
sequence iff \( \{ \tilde{k}^P_{zi} \}_{i \in \mathbb{N}} \) is Bessel (frame or Riesz basic) sequence iff \( \{ k_{zi} \}_{i \in \mathbb{N}} \) is a Bessel (frame or Riesz basic) sequence. Thus, every Bessel sequence of normalized kernel functions in \( \mathcal{H}(P) \) satisfies the FC, since the FCKF holds for the Hardy space and hence the space \( \mathcal{H}(P) \) satisfies the FCKF.

We now focus on some orthogonal projections. If \( P \) is an orthogonal projection, then \( \mathcal{H}(P) \) is just the range of \( P \) and the norm is just the usual Hardy space norm. In [4], Baranov and Dyakonov have considered the FC for what are often called de Branges spaces, that is spaces of the form \( H^2 \ominus \phi H^2 \) with some conditions on \( \phi \) and proved the following two theorems.

**Theorem 5.1** (Branov-Dyakonov). Let \( \phi \) be an inner function. If \( \{ z_i \}_{i \in \mathbb{N}} \) is a sequence in \( \mathbb{D} \) such that \( \text{sup}_i |\phi(z_i)| < 1 \) for all \( i \), then the corresponding sequence of normalized kernel functions in \( H^2 \ominus \phi H^2 \) satisfies the FC.

The second theorem of Baranov and Dyakonov uses one-component inner functions. An inner function \( \phi \) is said to be an **one-component** inner function if the set \( \{ z : |\phi(z)| < \epsilon \} \) is connected for some \( \epsilon \in (0, 1) \).

**Theorem 5.2** (Baranov-Dyakonov). Assume that \( \phi \) is a one-component inner function. Then every Bessel sequence of normalized kernel functions in \( H^2 \ominus \phi H^2 \) satisfies the FC.

Note that given an inner function \( \phi \), the model space \( H^2 \ominus \phi H^2 \) is the de Branges space \( \mathcal{H}(P) \), where \( P \in B(H^2) \) is the orthogonal projection onto \( H^2 \ominus \phi H^2 \). Hence, the above theorems of Baranov and Dyakonov analyze the class of de Branges spaces \( \mathcal{H}(P) \) for FCKF, where \( P \) belongs to the family of projections onto \( H^2 \ominus \phi H^2 \) and \( \phi \) is an inner function with properties, as stated in Theorem 5.1 and 5.2. In particular, their second theorem proves that when \( \phi \) is an one-component inner function and \( P \) is the orthogonal projection onto \( H^2 \ominus \phi H^2 \), then the de Branges space \( \mathcal{H}(P) \) satisfies FCKF.

The result for the complementary projection is elementary.

**Theorem 5.3.** Let \( \phi \) be an inner function and let \( P_\phi \) be the orthogonal projection onto \( \phi H^2 \). Then the space \( \mathcal{H}(P_\phi) \) satisfies the FCKF.

**Proof.** Let \( \{ k^P_{zi} \}_{i \in \mathbb{N}}, \{ z_i \}_{i \in \mathbb{N}} \subseteq \mathbb{D} \) be a Bessel sequence in \( \mathcal{H}(P_\phi) \). To prove that this sequence satisfies the FC, we first observe that \( P_\phi = T_\phi T_\phi^* \), where \( T_\phi \) is the Toeplitz operator with symbol \( \phi \) and \( T_\phi^* k_{zi} = \bar{\phi(z_i)} k_{zi} \). Also, \( \mathcal{H}(P_\phi) \) coincides with the range of \( P_\phi \) and the two norms are equal, since \( P_\phi \) is an orthogonal projection. To simplify notation, we set \( P = P_\phi \). Then,

\[
\langle \tilde{k}^P_{zi}, \tilde{k}^P_{zi} \rangle_P = \left\langle \frac{P k_{zi}}{\|P k_{zi}\|}, \frac{P k_{zi}}{\|P k_{zi}\|} \right\rangle = \frac{\bar{\phi(z_i)}}{\|\phi(z_i)\|} \langle \tilde{k}^P_{zi}, \tilde{k}^P_{zi} \rangle \frac{\phi(z_i)}{\|\phi(z_i)\|}
\]

Hence,

\[
\left( \left( \tilde{k}^P_{zi}, \tilde{k}^P_{zi} \right)_P \right) = D \left( \langle \tilde{k}^P_{zi}, \tilde{k}^P_{zi} \rangle \right) D^*,
\]
where $D \in B(\ell^2)$ is an invertible, diagonal operator with $i^{th}$ diagonal entry $\frac{\phi(z_i)}{|\phi(z_i)|}$. Finally, using Proposition 3.3 and Theorem 2.3 we conclude that \{k_{z_i}^p\}_{i \in \mathbb{N}} satisfies the FC and hence $\mathcal{H}(P_\phi)$ satisfies the FCKF. 

By taking a closer look at the proof of (iii) implies (ii) in Theorem 1.2 we notice that in order to prove that for an orthogonal projection $P$ a bounded Bessel sequence $\{P_\phi k_{z_i}\}_{i \in \mathbb{N}}$ satisfies the FC, all we need is that the corresponding sequence $\{k_{z_i}^p\}_{i \in \mathbb{N}}$ in $\mathcal{H}(P)$, for the same $P$, satisfies the FC. As an immediate consequence we get the following result.

**Theorem 5.4.** Let $\phi$ be an inner function and let $P_\phi$ be the orthogonal projection onto $\phi H^2$. If $\{z_i\}_{i \in \mathbb{N}}$ is a sequence in $\mathbb{D}$ such that $\{P_\phi k_{z_i}\}_{i \in \mathbb{N}}$ is a bounded Bessel sequence in $H^2$, then $\{P_\phi k_{z_i}\}_{i \in \mathbb{N}}$ satisfies the FC.

These last two results allow us to draw some conclusions about spaces that contain the image of an inner function, that look analogous to the theorems of Baranov and Dyakanov and require bounds away from 0 instead of away from 1. First we need a preliminary result.

**Proposition 5.5.** Let $\phi$ be an inner function and let $P_\phi$ be the orthogonal projection onto $\phi H^2$. Then for every sequence $\{z_i\}_{i \in \mathbb{N}} \subseteq \mathbb{D}$ such that there exists a $\delta > 0$ with $|\phi(z_i)| \geq \delta$ for all $i$, the following hold true:

(i) for each $i$, $\delta \|k_{z_i}\| \leq \|P_\phi k_{z_i}\| \leq \|k_{z_i}\|$.

(ii) $\{P_\phi k_{z_i}\}_{i \in \mathbb{N}}$ is a Bessel sequence iff $\{k_{z_i}\}_{i \in \mathbb{N}}$ is a Bessel sequence, 

(iii) $\{P_\phi k_{z_i}\}_{i \in \mathbb{N}}$ is a frame sequence iff it is a Riesz basic sequence,

(iv) $\{P_\phi k_{z_i}\}_{i \in \mathbb{N}}$ is a Riesz basic sequence iff $\{k_{z_i}\}_{i \in \mathbb{N}}$ is a Riesz basic sequence.

**Proof.**

Let $\{z_i\}_{i \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ and let $\delta > 0$ be a constant such that $|\phi(z_i)| \geq \delta$ for all $i$.

As noted earlier, $P_\phi = T_\phi T_\phi^*$, where $T_\phi$ is the Toeplitz operator with symbol $\phi$, and $T_\phi^* k_{z_i} = \phi(z_i) k_{z_i}$. Thus,

\begin{equation}
\langle P_\phi k_{z_j}, P_\phi k_{z_i} \rangle = \langle T_\phi^* k_{z_j}, T_\phi^* k_{z_i} \rangle = \phi(z_j) \overline{\phi(z_i)} \langle k_{z_j}, k_{z_i} \rangle
\end{equation}

Hence,

\begin{equation}
\langle P_\phi k_{z_j}, P_\phi k_{z_i} \rangle = D \left( \langle k_{z_j}, k_{z_i} \rangle \right) D^*,
\end{equation}

where $D \in B(\ell^2)$ is an invertible, diagonal operator with $\phi(z_i)$ as the $i^{th}$ diagonal entry, since $\delta \leq |\phi(z_i)| \leq 1$ for all $i$.

Clearly, (i) follows from Equation (2) and (ii), (iii) and (iv) follows from Equation (3), using Proposition 3.4 and Theorem 3.5.

We can generalize Proposition 5.5 and Theorem 5.4 as follows. Given an operator $T \in B(H^2)$ we shall denote the range of $T$ by $\text{Ran}(T)$. 

Proposition 5.6. Let $P \in B(H^2)$ be an orthogonal projection. Given a sequence $\{z_i\}_{i \in \mathbb{N}}$ in $\mathbb{D}$, if there exists an inner function $\phi$ such that $|\phi(z_i)| \geq \delta$ for all $i$ and $\phi H^2 \subseteq \text{Ran}(P)$, then:

(i) for each $i$, $\delta||k_{z_i}|| \leq ||P_k z_i|| \leq ||k_{z_i}||$,
(ii) $\{P_k z_i\}_{i \in \mathbb{N}}$ is a Bessel sequence iff $\{k_{z_i}\}_{i \in \mathbb{N}}$ is a Bessel sequence,
(iii) $\{P_k z_i\}_{i \in \mathbb{N}}$ is a frame sequence iff it is a Riesz basic sequence,
(iv) $\{P_k z_i\}_{i \in \mathbb{N}}$ is a Riesz basic sequence iff $\{k_{z_i}\}_{i \in \mathbb{N}}$ is a Riesz basic sequence.

Proof. Let $P_\phi$ denote the orthogonal projection onto $\phi H^2$. Then $P_\phi \leq P$ and thus, $||P_\phi k_{z_i}|| \leq ||P_k z_i|| \leq ||k_{z_i}||$. This proves (i), since $\delta||k_{z_i}|| \leq ||P_\phi k_{z_i}||$.

To prove (ii), we first note that for any $x \in H^2$,

$$\langle x, P_\phi \bar{k}_{z_i} \rangle = \langle P_\phi x, P \bar{k}_{z_i} \rangle$$

Thus, if $\{P \bar{k}_{z_i}\}_{i \in \mathbb{N}}$ is a Bessel sequence, then $\{P_\phi \bar{k}_{z_i}\}_{i \in \mathbb{N}}$ is a Bessel sequence and hence, $\{k_{z_i}\}$ is a Bessel sequence, using Proposition 5.5. The other implication follows trivially from the fact that $P \in B(H^2)$.

We shall now prove (iii) and (iv). Note that if $\{k_{z_i}\}_{i \in \mathbb{N}}$ is a Bessel sequence, then

$$\left(\langle P_\phi \bar{k}_{z_i}, P_\phi \bar{k}_{z_i} \rangle \right) \leq \left(\langle P \bar{k}_{z_i}, P \bar{k}_{z_i} \rangle \right) \leq \left(\langle \bar{k}_{z_i}, \bar{k}_{z_i} \rangle \right)$$

To prove (iii), we first assume that $\{P_k z_i\}_{i \in \mathbb{N}}$ is a frame sequence. Then it is a Bessel sequence and thus $\{\bar{k}_{z_i}\}_{i \in \mathbb{N}}$ is also a Bessel sequence. So, Equation (4) holds and we get $F_\phi F_\phi^* \leq FPF_P^* \leq FF^*$, where $F_\phi$, $F_P$ and $F$ are the analysis operators corresponding to the sequences $\{P_\phi \bar{k}_{z_i}\}_{i \in \mathbb{N}}$, $\{P \bar{k}_{z_i}\}_{i \in \mathbb{N}}$ and $\{k_{z_i}\}_{i \in \mathbb{N}}$, respectively. We claim that $\text{Ker}(F_P^*) = \{0\}$. To accomplish the claim, we first note that $\text{Ker}(F_P^*) \subseteq \text{Ker}(F_\phi^*)$, using Equation (4). Further, using the same ideas as used in the proof of Theorem 3.5, we conclude that $\text{Ker}(F^*) = \{0\}$. This implies that $\text{Ker}(F_\phi^*) = \{0\}$, using Equation (3). Thus it follows that $\text{Ker}(F_P^*) = \{0\}$.

Lastly, $F_P^*$ is onto as well, since $\{P \bar{k}_{z_i}\}_{i \in \mathbb{N}}$ is a frame sequence. Therefore $F_P^*$ and hence $F_P$ is an invertible operator, which implies that the sequence $\{P \bar{k}_{z_i}\}_{i \in \mathbb{N}}$ is a Riesz basic sequence. The other implication in (iii) is trivially true.

Finally, (iv) follows from Equation (4), using (iv) of Proposition 5.5 together with Proposition 3.2.\qed

Theorem 5.7. Let $P \in B(H^2)$ be an orthogonal projection. Let $\{z_i\}_{i \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that there exists an inner function $\phi$ with $|\phi(z_i)| \geq \delta$ for all $i$ and $\phi H^2 \subseteq \text{Ran}(P)$. If $\{P_k z_i\}_{i \in \mathbb{N}}$ is a Bessel sequence, then it satisfies the FC.

Proof. The proof follows immediately from Proposition 5.6.\qed
To illustrate the above theorem we would like to mention a few examples where the hypotheses of Proposition 5.6 are satisfied, and so the conclusions of Proposition 5.6 and Theorem 5.7 apply.

Example 5.8. Let \( P \in B(H^2) \) be the orthogonal projection onto the closed linear span of \( \{ z^j : j \neq j_1, \ldots, j_n \} \), \( j_1 < \cdots < j_n \), and let \( \{ z_i \}_{i \in \mathbb{N}} \) be a sequence in \( \mathbb{D} \) such that there exists a constant \( \delta > 0 \) with \( |z_i| \geq \delta \) for all \( i \). Then, \( \phi(z) = z^{j_n+1} \) is an inner function, \( \phi H^2 \subseteq \text{Ran}(P) \) and \( |\phi(z_i)| \geq \delta^{j_n+1} \) for all \( i \). Hence \( P \) and \( \{ z_i \}_{i \in \mathbb{N}} \) satisfy the conditions of Proposition 5.6.

Example 5.9. Given an inner function \( \phi \), \( \{ \mathbb{C} + \phi H^2 \} \) denotes the span of \( \mathbb{C} \) and \( \phi H^2 \) in \( H^2 \). These spaces were first introduced in [15]. Let \( \{ z_i \}_{i \in \mathbb{N}} \) be a sequence in \( \mathbb{D} \) and let \( \phi \) be an inner function such that \( |\phi(z_i)| \geq \delta > 0 \) for all \( i \). Then the orthogonal projection \( P \) onto \( \{ \mathbb{C} + H^2_{\phi} \} \) and \( \{ z_i \}_{i \in \mathbb{N}} \) satisfy the conditions of Proposition 5.6.

Example 5.10. Let \( P \in B(H^2) \) be an orthogonal projection such that the kernel of \( P \) is spanned by \( n \) inner functions \( \phi_1, \ldots, \phi_n \). Then \( \phi = z\phi_1 \cdots \phi_n \) is an inner function and \( \phi H^2 \subseteq \text{Ran}(P) \). Now, if \( \{ z_i \}_{i \in \mathbb{N}} \) is a sequence in \( \mathbb{D} \) such that there exists a constant \( \delta > 0 \) with \( |z_i| \geq \delta \), \( |\phi_k(z_i)| \geq \delta \) for all \( k \) and for all \( i \), then \( |\phi(z_i)| \geq \delta^{j_n+1} \) for all \( i \). Thus \( P \) and \( \{ z_i \}_{i \in \mathbb{N}} \) satisfy the conditions of Proposition 5.6.

Remark 5.11. If \( \phi \) is a finite Blaschke product and \( \{ z_i \}_{i \in \mathbb{N}} \subseteq \mathbb{D} \) converges to 1, then the condition \( |\phi(z_i)| \geq \delta \), follows automatically for all, but finitely many \( z_i \)'s. Because, the zeroes of \( \phi \) lies in the set \( \{ z : |z| < r \} \) for some \( r > 0 \), and \( |z_i| \) converges to 1. Hence, when \( \phi \) is a finite Blaschke and \( \{ |z_i| \} \) converges to 1, then the bounded below assumption on \( \phi \) in Proposition 5.6 and Theorem 5.7 is redundant.

Remark 5.12. For the case, when \( P \) is an orthogonal projection, the Hilbert space \( H(P) \) coincides with \( \text{Ran}(P) \). Further, in this case, if there exists a constant \( \delta > 0 \) such that \( \delta \| k_{z_i} \| \leq \| PK_{z_i} \| \leq \| k_{z_i} \| \) for all \( i \), then \( \{ PK_{z_i} \}_{i \in \mathbb{N}} \) is a Bessel (frame or Riesz basic) sequence iff \( \{ k_{z_i} \}_{i \in \mathbb{N}} \) is Bessel (frame or Riesz basic) sequence. Hence, if \( P \) is an orthogonal projection and \( \{ z_i \}_{i \in \mathbb{N}} \) is a sequence in \( \mathbb{D} \) such that there exists an inner function \( \phi \) satisfying the condition of Theorem 5.7, then whenever \( \{ k_{z_i} \}_{i \in \mathbb{N}} \) is a Bessel sequence in \( H(P) \), it satisfies the FC.

Our last results focus on the weighted Hardy spaces on the unit disk. We shall briefly define these spaces here, for more details we refer to [18].

Let \( \{ \beta_n \} \) be a sequence of positive numbers with \( R = \lim \inf(\beta_n)^{1/n} > 0 \). Then the set \( \{ \sum_n a_n z^n : \sum_n \beta_n^2 |a_n|^2 < \infty \} \) is a reproducing kernel Hilbert space on the disk of radius \( R \) with norm \( \| \sum_n a_n z^n \|_\beta = \sum_n \beta_n^2 |a_n|^2 \) and reproducing kernel \( K_\beta(z,w) = \sum_n \frac{a_n z^n}{\beta_n^2} \). This Hilbert space is called a weighted Hardy space and is denoted by \( H^2(\beta) \). If we let \( P \in B(H^2) \) be a positive, diagonal contraction with \( n \)th diagonal entry \( p_n \). Then the space
\( \mathcal{H}(P) \) coincides with the weighted Hardy space \( H^2(\beta) \), where \( \beta_n = \frac{1}{\sqrt{p_n}} \) for every \( n \) and the functions in \( H^2(\beta) \) are restricted to the unit disk \( \mathbb{D} \).

**Proposition 5.13.** Let \( P \in B(H^2) \) be a positive operator and \( D \in B(H^2) \) be a positive, diagonal operator such that \( \alpha D \leq P \leq \beta D \) for some \( \alpha, \beta > 0 \). Then:

(i) \( \{P^{1/2}k_z\}_{i \in \mathbb{N}} \) is a Bessel sequence iff \( \{D^{1/2}k_z\}_{i \in \mathbb{N}} \) is a Bessel sequence,

(ii) \( \{P^{1/2}k_z\}_{i \in \mathbb{N}} \) is a frame sequence iff \( \{D^{1/2}k_z\}_{i \in \mathbb{N}} \) is a frame sequence,

(iii) \( \{P^{1/2}k_z\}_{i \in \mathbb{N}} \) is a Riesz basic sequence iff \( \{D^{1/2}k_z\}_{i \in \mathbb{N}} \) is a Riesz basic sequence.

**Proof.** As a direct consequence of the given inequalities we get

\[
\alpha \|P^{1/2}x\|^2 \leq \|D^{1/2}x\|^2 \leq \beta \|D^{1/2}x\|^2
\]

for every \( x \in H^2 \). Using this we can easily verify that (i), (ii) and (iii) holds true. \( \square \)

**Theorem 5.14.** Let \( P \) be a positive operator and \( D \in B(H^2) \) be a positive, diagonal operator such that \( \alpha D \leq P \leq \beta D \) for some \( \alpha, \beta > 0 \). Then \( \{P^{1/2}k_z\}_{i \in \mathbb{N}} \) satisfies the FC iff \( \{D^{1/2}k_z\}_{i \in \mathbb{N}} \) satisfies the FC.

**Proof.** This follows immediately from Proposition 5.13. \( \square \)

We conclude with a brief summary. As mentioned earlier, in the case of an orthogonal projection \( P \), in order to prove that a bounded Bessel sequence \( \{P^*k_z\}_{i \in \mathbb{N}} \) satisfies the FC, it is enough to have that the corresponding sequence \( \{k_z^P\}_{i \in \mathbb{N}} \) in \( \mathcal{H}(P) \) satisfies the FC. Hence, when \( P \) is an orthogonal projection we shall only mention the results about sequences of normalized kernel functions in \( \mathcal{H}(P) \).

- \( H^2 \) satisfies the FCKF (Nikolski, 2006). Note that \( H^2 = \mathcal{H}(P) \) with \( P = I \), the identity operator.
- Given an inner function \( \phi \), every Bessel sequence \( \{k_z^P\} \) in \( \mathcal{H}(P) \) such that \( \sup_z |\phi(z)| < 1 \) satisfies the FCKF, where \( P \) is the orthogonal projection onto \( H^2 \ominus \phi H^2 \) (Baranov and Dyakonov, 2009).
- Given a one-component inner function \( \phi \), the de Branges \( \mathcal{H}(P) \) satisfies the FCKF, where \( P \) is the orthogonal projection onto \( H^2 \ominus \phi H^2 \) (Baranov and Dyakonov, 2009).
- Given a positive, invertible operator \( P \in B(H^2) \), every Bessel sequence \( \{P^*k_z\} \) satisfies the FC.
- Given a positive, invertible operator \( P \in B(H^2) \), the space \( \mathcal{H}(P) \) satisfies the FCKF.
- Given an inner function \( \phi \), the space \( \mathcal{H}(P_\phi) \) satisfies the FCKF, where \( P_\phi \) is the orthogonal projection onto \( \phi H^2 \) (Theorem 5.13).
Given an orthogonal projection $P$, if $\{\tilde{k}^P_{zi}\}$ is a Bessel sequence in $\mathcal{H}(P)$ such that there exists an inner function $\phi$ with $\inf_i|\phi(z_i)| > 0$ and $\phi H^2$ contained in the range of $P$, then $\{\tilde{k}^P_{zi}\}_{i \in \mathbb{N}}$ satisfies the FC (Remark 5.12).

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