A surge of interest related to the proposal of constructing the so-called Coherent Perfect Absorbers (CPA) [1] has been evident in recent years. The simplest CPA’s can be looked at as a scattering system (e.g. a cavity) with a small amount of loss which completely absorbs a monochromatic wave incident at a particular frequency [2]. Nonlinear effects may further help to make CPA’s a broad-band phenomenon [3] increasing prospects of its practical implementation, including optical filters and switches or logic gates for use in optical computers. Recently, a CPA in a rectangular cavity with randomly positioned scatterers and absorption due to a single antenna has been realised experimentally [4], paving a way to the construction of CPAs based on disordered cavities. In another recent experiment, a CPA has been realised with a two-port microwave graph system, both with and without time-reversal symmetry [5], and yet in another way with chaotic cavity with programmable meta-atom inclusions [6]. In those settings the relevant framework is that of chaotic wave scattering, and the CPA state corresponds to an eigenstate of the scattering matrix $S(\omega)$ with zero eigenvalue at a real frequency. This fact naturally motivates rising interest in a more general question of characterizing S-matrix complex zeros, as well as their manifestation in physical observables, which has not been systematically studied for wave chaotic systems with losses until very recently [7][12].

Although CPA’s are clearly a new exciting development, a somewhat related phenomena in lossy chaotic scattering have been discussed actually long ago. To this end one may invoke an early influential experiment by Doron, Smilansky and Frenkel who studied the frequency-dependent reflection coefficient $R(\omega) = |S(\omega)|^2$ of an electromagnetic wave sent via a single-mode waveguide to a cavity shaped in the form of a chaotic billiard. This line of experimental activity maintained its strength over several decades [14][18] providing important insights and stimulating theoretical research.

Returning to the setting of [13] one should recall that the scattering matrix of a single-channel system without gain or loss must be unimodular due to flux conservation: $S(\omega) = e^{i\delta(\omega)}$, where real $\delta(\omega)$ is known as the scattering phaseshift. The frequency derivative of the scattering phaseshift is an important scattering characteristics called Wigner time delay [19] and is well-described by the sum of Lorentzians of widths $\Gamma_n$ centered at positions $E_n$,

$$\tau_W(\omega) := \frac{d}{d\omega} \delta(\omega) = 2 \sum_{n=1}^{N} \frac{\Gamma_n}{(\omega - E_n)^2 + \Gamma_n^2}$$  \hspace{1cm} (1)

One of the main experimental observations made in [13] was that the reflection coefficient $R(\omega)$ showed considerable variations with frequency $\omega$, with many pronounced dips to low values $R(\omega) \lesssim 0.1$ at some frequencies, reminiscent of an ”imperfect” version of modern CPA. Such a behaviour is clearly incompatible with flux conservation and has been reasonably attributed to presence of uniform losses in resonator walls. Such losses had been then taken into account phenomenologically by adding a small imaginary increment to the real frequency: $\omega \rightarrow \omega + i\epsilon$ with $\epsilon > 0$. Assuming that absorption is weak, i.e. $\epsilon \ll \Delta$ , where $\Delta$ stands for the mean spacing between eigenfrequencies in the closed cavity in a given frequency range, one may expand in $\epsilon$ yielding a relation between $R(\omega)$ and the Wigner time-delay: $|S(\omega + i\epsilon)|^2 \approx e^{-\Delta \tau_W(\omega)}$. The paper thus was among the first promoting interest in statistics of Wigner time delays in wave-chaotic scattering, which after three decades still remains an active research topic, see e.g. [20][24] as well as [29][31] and references therein. Measuring the phaseshifts $\delta(\omega)$ independently allowed to test experimentally the relation between $R(\omega)$ and the Wigner time-delay, and overall good agreement has been reported in [13], with discrepancies close to the deepest minima attributed to inaccuracies in numerical differentiation.
Our goal in this paper is to have a closer look at \( R(\omega) \) in the above setting and to demonstrate that its behaviour at the deepest CPA-like dips bears important physical information which seems to have been not discussed before when addressing the shape of CPA minima, as e.g. in [6]. The proper framework for such an analysis is the so called effective Hamiltonian formalism for wave-chaotic scattering [35-41]. In such a formalism the parameters \( E_n \) and \( \Gamma_n \) featuring in \([\text{Eq.}(1)]\) describe the positions \( z_n = E_n - i\Gamma_n, \) \( n = 1, \ldots, N \) of \( S^- \) matrix poles in the lower half of the complex frequency plane, which in turn are considered to be complex eigenvalues of an \( N \times N \) non-selfadjoint matrix \( \mathcal{H}_{\text{eff}} = \mathbf{H}_N - i\omega \otimes \mathbf{w}^\dagger \) known as the effective Hamiltonian. In this setting the eigenfrequencies \( \omega_n \) of the closed resonator cavity are associated with the real eigenvalues of the self-adjoint part \( \mathbf{H}_N = \mathbf{H}_N^\dagger, \) with corresponding eigenmodes forming an orthonormal basis. Those eigenfrequencies are converted to the complex \( S^- \) matrix poles due to coupling of the cavity to continuum via a single open channel supporting waves coming from (and escaping to) infinity. The coupling between the channel and the cavity is then characterised by the vector \( \mathbf{w} = (w_1, \ldots, w_N) \) of coupling amplitudes. The single-channel scattering matrix \( S(\omega + i\epsilon) \) of a system with spatially-uniform losses \( \epsilon > 0 \) in this approach can be represented as

\[
S(\omega + i\epsilon) = \prod_{n=1}^{N} \frac{\omega + i\epsilon - z_n^*}{\omega + i\epsilon - z_n},
\]

where \( z_n^* \) stands for the complex conjugate of \( z_n \). Such relation immediately implies that in the absence of losses \( S(\omega) = e^{i\theta(\omega)} \) as dictated by flux conservation, and Eq.\([1]\) then easily follows. In presence of a uniform absorption \( \epsilon > 0 \) unimodularity is lost, and it is completely clear that the deepest dips in \( |S(\omega + i\epsilon)|^2 \) happen when the condition \( \omega + i\epsilon = E_n + i\Gamma_n \) is approximately satisfied.

It is important to note that for a single-channel wave-chaotic system the set of parameters \( \Gamma_n \) (known as the resonance widths) is generically random. Note that the value of the uniform loss \( \epsilon \) can be reliably extracted from scattering data, as discussed, e.g. in [42]. By changing the frequency \( \omega \) one always can ensure satisfying \( \omega = E_n \) for a given \( n \), but if choosing uniform absorption rate \( \epsilon \) is not fully under experimental control the condition \( \epsilon = \Gamma_n \) can be satisfied only purely accidentally. If such coincidence however does happen, one comes exactly to the CPA situation of vanishing reflection, see Fig.\([1]\). In what follows we assume that the reflection dip is located at the frequency \( \omega = E_n \) but may be not perfect, though deep enough, i.e the value at the minimum \( 0 \leq R_{\text{min}} < 1 \). This is ensured by imposing the relations \( \omega = E_n + \delta\omega \) and \( \Gamma_n = \epsilon + \delta\epsilon \) assuming the range of frequencies around the dip \( \delta\omega \ll \Delta \) and the resonance mismatch parameter \( |\delta_n| \ll \epsilon \). Under these conditions using Eq.\([2]\) we immediately see that the shape of the reflection dip centered at the frequency \( \omega = E_n \) should be well described by the following profile:

\[
R(\omega) = K_n \frac{(\omega - E_n)^2 + \delta_n^2}{(\omega - E_n)^2 + 4\epsilon^2}, \quad K_n = \prod_{k \neq n} \left| z_k - z_n^* \right|^2 / \left| z_k^* - z_n \right|^2.
\]

Such a profile is then uniquely characterized by its depth \( R_{\text{min}} \) and the curvature at the minimum \( C := \frac{d^2}{d\omega^2} R(\omega) |_{\omega=E_n} \) which are given by

\[
R_{\text{min}} = K_n \frac{\delta_n^2}{4\epsilon^2}, \quad C = \frac{K_n}{2\epsilon^2}.
\]

The fluctuations of the dip shapes over different reflection minima are thus controlled by the statistics of resonance widths \( \Gamma_n \) and that of the factors \( K_n \). Whereas the statistical information about \( \Gamma_n \) in a few-channel system is available theoretically [43-46] and is in a good agreement with experiments [17], the factor \( K_n \) looks less familiar, and its meaning is not immediately obvious.

To this end we observe below that such a factor can be actually given a direct interpretation, by relating it to nonorthogonal eigenfunctions of the effective Hamiltonian \( \mathcal{H}_{\text{eff}} \). The latter Hamiltonian is not only nonselfadjoint, but also non-normal, i.e. does not commute with its adjoint, hence is characterized by the pair of left \( \mathbf{I}_n \) and right \( \mathbf{r}_n \) eigenvectors corresponding to the same complex eigenvalue \( z_n \), or equivalently \( \mathcal{H}_{\text{eff}} \mathbf{r}_n = z_n \mathbf{r}_n \) and \( \mathcal{H}_{\text{eff}}^\dagger \mathbf{l}_n = z_n^\dagger \mathbf{l}_n \). One can always choose the sets of right- and left eigenvectors to be bi-orthogonal, with scalar products satisfying \( \mathbf{l}_n^\dagger \mathbf{r}_m = \delta_{nm} \). The overlap non-orthogonality matrix is then defined as \( O_{mn} = \)}
perturbation of the self-adjoint matrix $H$.

In particular, self-overlaps are controlling sensitivity of the eigenvalue condition numbers. Various aspects of non-normal random matrices is a topic of active research in the mathematical literature the eigenvalue condition numbers.

To this end, the phenomenon of wave-chaotic scattering provides quite a unique framework for addressing eigenfunction non-orthogonality and its manifestations not only theoretically [44] but also experimentally [72]. In the chaotic scattering context self-overlaps are known as Petermann factors, and being responsible for the excess noise in lasing resonators have been addressed in an early paper [65]. Whereas the mean value for self-overlaps for wave-chaotic scattering in systems with broken time-reversal invariance has been calculated nonperturbatively using random Matrix Theory (RMT) techniques in [65], any other statistical characteristics of self-overlaps were only so far addressed in the framework of second-order perturbation theory in the limit of weak channel coupling: $\gamma := |\mathbf{w}|^2 \ll 1$.

In this paper we develop a method of non-perturbative evaluation of the probability density for the variable $t_n = O_{nn} - 1$ valid for the single-channel chaotic scattering at any coupling $\gamma > 0$. For generic non-normal matrices the relations between overlaps $O_{nn}$ and complex eigenvalues $z_n$ are very involved and hard to utilize. However in the special case of Hamiltonian belonging to the Gaussian Unitary ensemble of RMT. In the physically relevant limit $N \gg 1$ the corresponding eigenvalue density is then given by $\rho = \frac{1}{4\pi} \sqrt{4 - X^2}$, hence the mean eigenvalue spacing $\Delta = 1/(N\rho) \sim N^{-1}$. The coupling amplitudes $w_{i,i} = 1/(N\rho)$ are normalized in such a way that $\gamma = 1 + |w_{i,i}|^2$ stays finite as $N \to \infty$.

With this choice as $N \gg 1$ statistics of all scattering characteristics varying at frequency scales of the order of $\Delta$ around a spectral point $|X| < 2$ can be shown to be universal and depending only on the “renormalized” coupling strength $g = \frac{1}{\pi\rho^2} \left( \gamma + \frac{1}{2} \right) \geq 1$ but not on the particular choice of $w_{i,i}$, see e.g. [39, 70]. E.g. we can choose $\mathbf{w} = \sqrt{\gamma} \mathbf{w}$, with components $w_{ij}$ independent, mean zero complex Gaussian, with covariance $\langle \bar{w}_i w_j \rangle = N^{-1} \delta_{ij}$, where the angular brackets always denote averaging over relevant distributions. Equivalently, one may choose $\mathbf{w}$ to be any fixed of unit length, in particular $\mathbf{w} = \mathbf{e}$ with $\mathbf{e}^2 = (1, 0, \ldots, 0)$. Under these assumptions typically $\Gamma_n \sim \Delta$, and we introduce $y_n = \pi \Gamma_n/\Delta$. This is the main parameter controlling both non-Hermiticity and non-normality of the effective Hamiltonian $H_{eff}$. As long as $g \gg 1$ the departure from Hermiticity/normality is small and to a large extent can be studied perturbatively, whereas in the so-called perfect coupling case $g = 1$ those effects are the strongest.

With those preparations we can formulate our main result. Define, using Dirac $\delta(x)$ and $\delta^{(2)}(z) := \delta(Re z) \delta(Im z)$ the following object

$$P(t;z) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(O_{nn} - 1 - t)\delta^{(2)}(z - z_n) \right\rangle$$

interpreted as the (conditional) probability density of the non-orthogonality factor $t = O_{nn} - 1$ corresponding to eigenvalues in the vicinity of a point $z = X - iY, Y > 0$ in the complex plane. Then the limiting density $P_y^{(2)}(t) := \lim_{N \to \infty} \frac{1}{\pi \rho N} P(t;z = X - i\frac{y}{\pi \rho N})$ takes the following form for the present symmetry class:

$$P^{(2)}_y(t) = \frac{16}{t^3} e^{-2\gamma y} \text{L}_2 e^{-2\gamma y(1+\gamma)} I_0 \left( \frac{4y}{t} \right) \sqrt{(g^2 - 1)(1 + t)}$$

where $\text{L}_2(x)$ stands for the modified Bessel function and $\text{L}_2$ is a differential operator acting on functions $f(y)$ as

$$\text{L}_2 f(y) = \left\{ 1 + \left( \frac{\sinh 2y}{2y} \right)^2 + \frac{1}{2y} - \frac{1}{4y} \right\} \frac{d^2}{dy^2} y f(y).$$

The following remarks on the above formula are due. First, the most prominent feature in [63] is the heavy-tail behaviour $P^{(2)}_y(t) \sim t^{-3}$ for $t \gg 1$ rendering all moments $\langle O_{nn}^l \rangle, l \geq 2$ divergent. This tail behaviour is exactly the same as found earlier in other complex-valued non-normal random matrices [53, 58] and seems to be the most universal feature of random diagonal overlaps.

Second, the integrals $\int P^{(2)}_y(t) dt$ can be explicitly performed for $l = 0$ and $l = 1$, reproducing the known mean density of the resonance widths [43] and the mean diagonal overlap [63]. Finally, with suitable adjustments, the
method should be applicable to the rank-one family of subunitary deformations of the Haar-distributed unitary random CUE matrices. The corresponding model introduced in [77, 78] naturally appears in the context of time-periodic scattering. The statistics of non-orthogonality factors are then expected to be exactly the same as given by [9], with the perfect coupling case $g = 1$ corresponding in that context to the so-called truncated CUE [79].

Our starting point is to consider (negative) moments of the diagonal overlap defined for $z = X - iY$ and real $p$ as $\mathcal{O}_p = \left\langle \frac{1}{N} \sum_{n=1}^{N} \delta^{(z)}(z - z_n)O_{n}^{-p} \right\rangle$. Fixing the coupling parameter $\gamma > 0$ it is technically convenient to consider first the fixed coupling amplitudes $w = \sqrt{\gamma} \mathbf{e}$. We note that as $\sum_n Imz_n = -\gamma$ and $Imz_n \leq 0, \forall n$ it is enough to consider $0 \leq Y < \gamma$. Following the procedure explained for $p = -1$ in [68], we denote $\gamma_s = \gamma - Y > 0$, use the known joint probability density of all $z_n, n = 1, \ldots, N$ [44, 80, 81], and rely on GUE invariance. After defining $\tilde{z} = \sqrt{\gamma_s} \frac{1}{N} z$, considering $(N - 1) \times (N - 1)$ matrices $\tilde{H}_{eff} := H_{N-1} - i \sqrt{\gamma_s} \mathbf{e}_{N-1} \otimes \mathbf{e}_{N-1}^T$ and disregarding the explicit proportionality constants we arrive at the following representation

$$\mathcal{O}_p(z) \propto \frac{\gamma_s^{N-2}}{\gamma_s^{N-1}} e^{-\frac{\gamma}{2}(X^2 - Y^2 + \gamma^2) + \frac{X^2}{2} + \gamma^2 \mathcal{M}_p^{(\beta=2)}(z)}$$

(8)

with the average now going over the reduced GUE matrices $H_{N-1}$ of the size $(N - 1) \times (N - 1)$. The major task is to evaluate $\mathcal{M}_p^{(\beta=2)}(z)$ as $N \to \infty$ for a given $\gamma > 0$ and real $p$ assuming the scaling $Imz = Y \sim N^{-1}$. It is easy to check that in such a limit one may safely replace $\tilde{H}_{eff}$ with the original $H_{eff} = H_N - i \gamma \mathbf{e} \otimes \mathbf{e}$, without changing the result. When evaluating $\mathcal{M}_p^{(\beta=2)}(z)$ we also find it more technically convenient to consider the random complex gaussian channel vector $\tilde{w}$ rather than keeping it of unit length, and can show that both choices lead to the same result.

The crucial trick which allows to evaluate $\mathcal{M}_p^{(\beta=2)}(z)$ efficiently is to define a version of the matrix element of the GUE resolvent (Green’s function), also known in scattering theory as the $K$-matrix, via

$$G(z) := \tilde{w}^\dagger \frac{1}{z - H_N} \tilde{w} = u + iv, v > 0$$

(10)

and to use the identity $\det(z - H_{eff}) = \det(z - H_N)(1 + i\gamma G(z))$ to rewrite our main object of interest as

$$\mathcal{M}_p^{(\beta=2)}(z) = \left\langle \det(z - H_N) \frac{1 + i\gamma G(z)^{2(p+1)}}{1 + i\gamma G(z)^{2p}} \right\rangle$$

(11)

in terms of the function

$$\mathcal{P}_z^{(\beta=2)}(u, v) = (\langle \det(z - H_N) \rangle^2 \delta(u - \text{Re}(G(z))) \delta(v - \text{Im}(G(z))))$$

(13)

where now averaging includes one over $\tilde{w}$ as well. Such function represents a “deformed” version of the probability density for the Green’s function entry, Eq. [10]. Without deformation it enjoyed thorough attention, starting from [82], and methods to evaluate it within RMT context (and beyond) has been discussed e.g. in [83, 84]. Following one of those methods, we consider the Fourier-Laplace transform $\mathcal{F}(s, p) = \int du e^{-\frac{is}{i\gamma} \frac{1}{2} \int_0^\infty dv e^{-p v} \mathcal{P}_z(u, v)}$, and perform the average over $\tilde{w}$ first, before that over the GUE matrices $H_N$. Along those lines we notice that

$$is \text{Re}(G(z)) - p \text{Im}(G(z)) = \tilde{w}^\dagger i\gamma (X - H_N) - pY \tilde{w}$$

so that the average over $\tilde{w}$ amounts to performing a simple Gaussian integral. For simplicity of the presentation we set $X = 0$ below, the results for any $|X| < 2$ recovered by a simple rescaling, and scale $Y = iy/N$. Doing this we arrive after simple algebraic manipulations to the following identity

$$\mathcal{F}^{(\beta=2)}(s, p) = \left\langle \frac{[\det(iy / N - H_N)]^4}{\det(-i \frac{s}{2N} - H_N) \det(-i \frac{s}{2N} - H_N)} \right\rangle$$

(14)

where we defined $\tau_+ = s + \sqrt{s^2 + 4y(y + p)} > 0$ and $\tau_- = s - \sqrt{s^2 + 4y(y + p)} < 0$. The objects like those featuring in the right-hand side of [14] have a long history of studies, starting from [85], and then in increasing generality in [86, 87], culminating in [88]. Using the latter results an explicit expression for $\mathcal{F}(s, p)$ as $N \to \infty$ can be found as explained in [73]. Then on performing the Laplace-Fourier inversion of $\mathcal{F}(s, p)$ [75] one arrives at a quite compact result:

$$\mathcal{P}_z^{(\beta=2)}(u, v) \propto \mathbb{L}_2 \frac{1}{|z|^\beta} e^{-y \frac{2z + 2s + 1}{4}}$$

(15)

with the differential operator $\mathbb{L}_2$ defined in [7]. Now we can substitute [15] back to [12] and after introducing a new variable $x = \frac{u + iv + 1}{2}$ arrive to the formula for $M_p^{(\beta=2)}(y) = \lim_{N \to \infty} M_p^{(\beta=2)}(z = \frac{-y}{N})$ [75]:

$$M_p^{(\beta=2)}(y) \propto \mathbb{L}_2 \int_1^{\infty} da \frac{(a - 1)^{p+1}}{(a + 1)^p}$$

(16)

$$\times e^{-2gya} I_0(2y \sqrt{(a^2 - 1)(g^2 - 1)})$$
The above formula is well-defined for any $p > -2$. Evaluating straightforwardly the limit $N \to \infty$ in (8) while keeping $X = 0, Y = y/N$ gives us the moments of the self-overlap as $O_p \propto e^{-2gy} M_p(y)$ and the main result [6] then follows by a straightforward moment inversion after setting $a = 2t^{-1} + 1$.

The same method can be adapted *mutatis mutandis* for deriving a similar result for systems with preserved time-reversal symmetry, with the matrix $H_N$ belonging to the Gaussian Orthogonal Ensemble of RMT and the channel coupling vector being real-valued [39]. In that setting the central result needed for our goal is a correlation function involving half-integer powers of characteristic polynomials evaluated in [89]. Necessary steps of the derivation are presented in [76], with the final result being now

$$P_y^{(1)}(t) = \frac{1}{2} e^{-gy} L_1 e^{-gy(1+\frac{t}{2})} I_0 \left( \frac{2y}{t} \sqrt{(g^2 - 1)(1 + t)} \right)$$

(17)

where $L_1$ is the following differential operator

$$L_1 = 2 \sinh 2y - \left( \cosh 2y - \frac{\sinh 2y}{2y} \right) \left( \frac{3}{y^2} + 2 \frac{d}{dy} \right)$$

(18)

We see that the heavy-tail behaviour $P_y^{(1)}(t) \sim t^{-3}$ for $t \gg 1$ is also manifest in such a case. One can further integrate over $t$ to obtain the mean density of the scaled resonance widths $y = \pi \Gamma_n / \Delta$ in the form:

$$\rho^{(1)}(y) = \frac{1}{4\sqrt{2}} e^{-gy} L_1 \Phi(y)$$

(19)

where

$$\Phi(y) = \int_0^\infty da e^{-gay} \frac{(a - 1)}{\sqrt{a + 1}} I_0 \left( y \sqrt{(g^2 - 1)(a^2 - 1)} \right)$$

(20)

This expression looks considerably simpler than the 3-fold integral representation derived earlier for the same density in [15], though one can check numerically that they are fully equivalent [17].

In conclusion, we related the shape of deep, CPA-like dips in a single-channel reflection from lossy cavities with the diagonal non-orthogonality factor of eigenfunctions of the underlying effective Hamiltonian. Using RMT framework we derived, fully non-perturbatively, the explicit distribution of these factors for wave-chaotic scattering in systems with both broken and preserved time-reversal symmetry. The results imply that $O_{nn}$ are heavy-tail distributed, sharing this feature with other instances of non-orthogonality factors of non-Hermitian ensembles [85, 84, 88, 83] and supporting the claim of the universality of such a behaviour. Experimentally, statistics of $O_{nn}$ should be accessible within the framework of “harmonic inversion” method [47], or via accurate study of the shape of reflection dips.

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Finally recalling the definition of overlap non-orthogonality matrix as

\[ O_{mn} = (r^\dagger_m r_n)(l^\dagger_n l_m) \]

we see that the two are reciprocal.

Finally note that as residues of the scattering matrix at resonance poles can be extracted from the same harmonic inversion procedure used to get experimental access to widths \( \Gamma_n \), the relation (24) could provide a basis for experimentally studying the statistics of the non-orthogonality factors.

**SUPPLEMENTARY MATERIALS**

Non-orthogonality factors in the single-channel effective Hamiltonian

Recall the definitions of the effective Hamiltonian \( H_{e,f} = H_N - iw \otimes w^\dagger \) and its left-right eigenvectors and eigenvalues:

\[
H_{e,f} r_n = z_n r_n, \quad l^\dagger_n H_{e,f} = z_n l^\dagger_n \quad \text{and} \quad r^\dagger_n H_{e,f} = z_n^* r^\dagger_n \quad H_{e,f} l_n = z_n^* l_n.
\]

We choose the sets of right- and left eigenvectors to be bi-orthogonal, with scalar products satisfying \( l^\dagger_n r_m = \delta_{nm} \).

In these terms the single-channel scattering matrix at the complex plane \( z = \omega + i\varepsilon \) can be rewritten, after a simple algebra, in several equivalent forms

\[
S(z) = \prod_{n=1}^{N} \frac{z - z_n^*}{z - z_n} = \frac{\det \left( z - H_{e,f}^\dagger \right)}{\det (z - H_{e,f})}
\]

\[
= 1 - 2iw^\dagger \frac{1}{z - H_{e,f}} w = 1 - 2i \sum_{n=1}^{N} \left( w^\dagger r_n \right) \left( l^\dagger_n w \right)
\]

Comparing the residues at the poles \( z_n \) in (22) and (23) one finds the relation:

\[
-2i \left( w^\dagger r_n \right) \left( l^\dagger_n w \right) = (z_n - z_n^*) \prod_{k \neq n} \frac{z_k - z_n^*}{z_k - z_n}
\]

Conjugation gives also:

\[
2i \left( w^\dagger l_n \right) \left( r^\dagger_n w \right) = (z_n^* - z_n) \prod_{k \neq n} \frac{z_k^* - z_n}{z_k^* - z_n^*}
\]

Multiplying (24) and (25) and rearranging gives:

\[
\left[ 2i \left( l^\dagger_n w \right) \left( w^\dagger l_n \right) \right] \left[ -2i \left( r^\dagger_n w \right) \left( w^\dagger r_n \right) \right] = - (z_n - z_n^*)^2 \prod_{k \neq n} \frac{|z_k^* - z_n|}{|z_k - z_n|}
\]

On the other hand one may notice from the definition that \( 2iw \otimes w^\dagger = H_{e,f}^\dagger - H_{e,f} \), hence

\[
2i \left( l^\dagger_n w \right) \left( w^\dagger l_n \right) = l^\dagger_n \left[ 2iw \otimes w^\dagger \right] l_n = l^\dagger_n \left[ H_{e,f}^\dagger - H_{e,f} \right] l_n = (z_n^* - z_n) \left( l^\dagger_n l_n \right)
\]

and similarly:

\[
-2i \left( r^\dagger_n w \right) \left( w^\dagger r_n \right) = r^\dagger_n \left[ -2iw \otimes w^\dagger \right] r_n = r^\dagger_n \left[ H_{e,f} - H_{e,f}^\dagger \right] r_n = (z_n - z_n^*) \left( r^\dagger_n r_n \right)
\]

Substituting back to (25) shows that:

\[
\left( l^\dagger_n l_n \right) \left( r^\dagger_n r_n \right) = \prod_{k \neq n} \frac{|z_k^* - z_n|^2}{|z_k - z_n|^2}
\]

Finally recalling the definition of overlap non-orthogonality matrix as \( O_{mn} = (r^\dagger_m r_n)(l^\dagger_n l_m) \) we see that the right-hand side of (29) provides the expression for diagonal entries \( O_{nn} \), and comparing with the definition Eq.(3) of the factor \( K_n \) we see that the two are reciprocal.
Correlation function of characteristic polynomials for GUE matrices, eq.(14)

For convenience of the reader we collect the formulas from Theorem 1.3.2 in [1] below most useful for our purposes. Denoting the limiting mean eigenvalue density of $N \times N$ GUE matrices $H_N$ to be $\rho = \frac{1}{2\pi} \sqrt{4 - x^2}$ we define $D(\alpha) = \det \left( X + \frac{\alpha}{N\rho(X)} - H_N \right)$ and keep $\alpha$ fixed as $N \to \infty$. Then in such a limit

$$\left( \frac{D(\alpha_1) D(\alpha_2) D(\beta_1) D(\beta_2)}{D(\alpha_+ D(\beta_+)} \right) \propto e^{N \sum_{\alpha, \beta} \frac{1}{\rho(\alpha, \beta)} \left[ (\alpha_1 - \alpha_2)(\beta_1 - \beta_2) (\beta_2 - \beta_+)(\beta_2 - \beta_+) \right]} \det D$$ (30)

where $D$ is the following $3 \times 3$ matrix:

$$D = \begin{pmatrix}
S(\alpha_1, \beta_1) & S(\alpha_1, \beta_2) & S(\alpha_1, \alpha_+)
S(\alpha_2, \beta_1) & S(\alpha_2, \beta_2) & S(\alpha_2, \alpha_+)
S(\beta_+, \beta_1) & S(\beta_+, \beta_2) & S(\beta_+, \alpha_+)
\end{pmatrix}$$ (31)

where in our particular case:

$$\alpha_1^-, \beta_1^-, \beta_{1,2}^- = i\pi \rho(X) y_{1,2}, \quad \alpha_2^-, \beta_2^-, \beta_{1,2}^- = -i\pi \rho(X) y_{1,2}, \quad \alpha_+ = -i\frac{\pi \rho(X)}{2} \tau_-, \quad \beta_+ = -i\frac{\pi \rho(X)}{2} \tau_+$$

As $\text{Im} \alpha_+ > 0$, $\text{Im} \beta_+ < 0$ we have:

$$S(\alpha_1^-, \beta_1^-) = \frac{1}{\pi} \frac{\sin \left( \alpha_1^- - \beta_1^- \right)}{\alpha_1^- - \beta_1^-} = \frac{1}{\pi} \frac{\sin \left( 2\pi \rho(X) y_1 \right)}{2\pi \rho(X) y_1}$$ (32)

and similarly

$$S(\alpha_2^-, \beta_2^-) = \frac{1}{\pi} \frac{\sin \left( 2\pi \rho(X) y_2 \right)}{2\pi \rho(X) y_2}, \quad S(\alpha_1^-, \beta_1^-) = \frac{1}{\pi} \frac{\sin \left( 2\pi \rho(X) (y_1 + y_2) \right)}{2\pi \rho(X) (y_1 + y_2)}.$$ (33)

Further

$$S(\beta_+, \beta_{1,2}^-) = \frac{1}{i\pi \rho(X)} \left( y_{1,2} - \frac{\tau_+}{2} \right), \quad S(\alpha_{1,2}^-, \alpha_+) = \frac{1}{i\pi \rho(X)} \left( y_{1,2} + \frac{\tau_+}{2} \right)$$ (34)

and finally

$$S(\beta_+, \alpha_+) = -2\pi i \frac{e^{\alpha_+ - \beta_+}}{\alpha_+ - \beta_+} = 2\pi \frac{e^{\frac{\pi \rho(X)}{2}}}{}$$ (35)

Eventually we need to consider the limit $y_1 \to y_2 = y$ in (30) which after straightforward manipulations produces (after setting $X = 0$, hence $\pi \rho(x) = 1$) the following expression for the Laplace-Fourier transform defined in Eq.(14) of the main paper:

$$F^{(\beta=2)}(s, p) = A(s, p) + B(s, p) + C(s, p)$$ (36)

where

$$A(s, p) = 2 \frac{e^{\frac{1}{2} \rho(X)}}{2 \left( \tau_+ \right)} \left( y + \frac{\tau_+}{2} \right) \left( y - \frac{\tau_+}{2} \right) \left( \frac{\sinh^2 2y}{4y^2} - 1 \right),$$ (37)

$$B(s, p) = \frac{1}{2} e^{\frac{1}{2} \rho(X)} \left( y + \frac{\tau_+}{2} \right) \left( y - \frac{\tau_+}{2} \right) \frac{1}{4y^2} \left( e^{4y} - 1 - 4y - 8y^2 \right),$$ (38)

and

$$C(s, p) = \frac{1}{2} e^{\frac{1}{2} \rho(X)} \left( y + \frac{\tau_+}{2} \right) \left( y - \frac{\tau_+}{2} \right) \frac{1}{4y^2} \left( 1 + 4y - e^{4y} \right) + \frac{1}{2} e^{\frac{1}{2} \rho(X)} \frac{1}{2y} \left( e^{4y} - 1 \right).$$ (39)
Inverting the Laplace-Fourier transform

Substituting the definitions

\[ \tau_+ = s + \sqrt{s^2 + 4y(y + p)} \quad \text{and} \quad \tau_- = s - \sqrt{s^2 + 4y(y + p)} \]

(40)

to the expression for the Fourier-Laplace transform \( F^{(\beta=2)}(s, p) \) presented in Eqs. (37)-(39) above, we notice that it can be represented as

\[
A(s, p) = 2 e^{-\sqrt{s^2 + 4y(y + p)}} \left(2y + p - \sqrt{s^2 + 4y(y + p)}\right)^2 \frac{1}{4} \left(\sinh^2 \frac{2y}{y} - 1\right),
\]

(41)

\[
B(s, p) = \frac{1}{2} e^{-\sqrt{s^2 + 4y(y + p)}} \left(2y + p - \sqrt{s^2 + 4y(y + p)}\right) \frac{1}{4y^2} \left(e^{4y} \sinh^{2} \frac{2y}{y} - 1\right),
\]

(42)

and

\[
C(s, p) = \frac{1}{2} e^{-\sqrt{s^2 + 4y(y + p)}} \left\{ \left(2y - \sqrt{s^2 + 4y(y + p)}\right) \frac{1}{4y^2} \left(1 + 4y - e^{4y}\right) + \frac{1}{2y} \left(e^{4y} - 1\right) \right\}.
\]

(43)

To represent above as an explicit Laplace-Fourier transform we define for \( p > 0 \), any real \( s \) and \( \alpha \) the functions:

\[
\phi_\alpha(s, p; y) = \int_0^\infty dv e^{-pv} \int_{-\infty}^\infty \frac{du}{\sqrt{2\pi}} e^{i su} \frac{1}{\sqrt{\alpha + 2}} e^{-\frac{y u^2}{\alpha + 2} + \frac{1}{\alpha + 2}} = \frac{1}{2\sqrt{\alpha + 2}} e^{-\frac{y}{2\alpha}} \sqrt{\alpha + 2} K_{\alpha + 2} \left(\sqrt{s^2 + 4y(y + p)}\right)
\]

(44)

where we performed first the Gaussian integral over \( u \) and then used the formula

\[
\int_0^\infty dv \nu^{\nu - 1} e^{-\frac{\beta}{\alpha} - \alpha v} = 2 \left(\frac{\beta}{\alpha}\right)^{\nu/2} K_{\nu}(2\sqrt{\alpha}), \quad \beta > 0, \alpha > 0
\]

removing \( K_{\nu}(x) = K_{-\nu}(x) \).

In particular:

\[
\phi_1(s, p; y) = \sqrt{\frac{\pi}{2}} e^{-\frac{1}{2} \sqrt{s^2 + 4y(y + p)}} \left(1 + \sqrt{s^2 + 4y(y + p)}\right),
\]

(45)

and using this we can easily check that

\[
A(s, p) = \frac{1}{2} \left(\sinh^2 \frac{2y}{y} - 1\right) \left(\frac{1}{2} \frac{d^2}{dy^2} + 2 \frac{d}{dy} + 2\right) \sqrt{\frac{2}{\pi}} y^2 \phi_1(s, p; y),
\]

(46)

Further, after rewriting

\[
B(s, p) + C(s, p) = \frac{1}{2} e^{-\sqrt{s^2 + 4y(y + p)}} \left(2y + p\right) \frac{1}{4y^2} \left(e^{4y} - 1 - 4y - 8y^2\right) + e^{-\sqrt{s^2 + 4y(y + p)}} \left(1 + \sqrt{s^2 + 4y(y + p)}\right),
\]

(47)

it turns out to be possible to represent it in the form

\[
B(s, p) + C(s, p) = \left(2 - \frac{1}{8y^2} \left(e^{4y} - 1 - 4y - 8y^2\right) \frac{d}{dy}\right) \sqrt{\frac{2}{\pi}} y^2 \phi_1(s, p; y),
\]

(48)

Finally, one may check that adding (46) and (48) produces, after simple algebra, eq.(15) of the main text.

Derivation of eq.(16) of the main text.

According to Eqs.(12) and (15) of the main text we need to deal with the following integral:

\[
F_p^{(2)}(y) = \int_{-\infty}^\infty du \int_0^\infty dv \left[(1 - \gamma v)^2 + \gamma^2 u^2\right]^{p+1} \frac{1}{\nu^3} e^{-\frac{u^2 + y^2}{\nu^2}}
\]

(49)
which after introducing the combination \( x = \frac{1}{2} (v + \frac{1}{v}) + \frac{\gamma^2}{2v} > 1 \) as the integration variable instead of \( u^2 \) allows to rewrite the above as

\[
F_p(y) \propto \int_1^\infty dx e^{-2x} \int_{v_1}^{v_2} \frac{dv}{v} \sqrt{(v - v_1)(v_2 - v)} \left[ \frac{1 - \gamma^2}{2v} - \gamma + \gamma^2 x \right]^{p+1} \left[ \frac{1 - \gamma^2}{2v} + \gamma - \gamma^2 x \right]^{p-1} \tag{50}
\]

where \( v_{1,2} = x \mp \sqrt{x^2 - 1} \), with \( v_1 v_2 = 1 \). Changing \( v \to 1/v \) and then substituting \( v = \frac{1}{2} (v_1 + v_2) + \frac{1}{2} (v_2 - v_1) \cos \theta \), where \( \theta \in [0, \pi] \) brings the right-hand side to

\[
F_p(y) \propto \int_1^\infty dx e^{-2x} \int_0^\pi \frac{\left[ x(1 + \gamma^2) - 2\gamma + (1 - \gamma^2)\sqrt{x^2 - 1} \cos \theta \right]^{p+1}}{\left[ x(1 + \gamma^2) + 2\gamma + (1 - \gamma^2)\sqrt{x^2 - 1} \cos \theta \right]^{p-1}} d\theta \tag{51}
\]

After introducing \( g = \frac{1}{2} (\gamma + \frac{1}{\gamma}) \) and using that \( \frac{1}{2} \left( \frac{1}{\gamma} - \gamma \right) = \pm \sqrt{\gamma^2 - 1} \) with sign dependent on whether \( \gamma < 1 \) or \( \gamma > 1 \) (this sign is immaterial for the value of the integral and can be omitted) the above is brought to the following form:

\[
F_p(y) \propto \gamma \int_1^\infty dx e^{-2x} \int_0^\pi \frac{g x - 1 + \cos \theta \sqrt{(g^2 - 1)(x^2 - 1)}}{g x + 1 + \cos \theta \sqrt{(g^2 - 1)(x^2 - 1)}} d\theta \tag{52}
\]

Further introducing \( a = g x + \cos \theta \sqrt{(g^2 - 1)(x^2 - 1)} > 1 \) as a new variable one can transform (52) to

\[
F_p(y) \propto \gamma \int_1^\infty da \frac{(a - 1)^{p+1}}{(a + 1)^p} \int_{x_1(a)}^{x_2(a)} e^{-2y x} dx / (x - x_1(a))(x_2(a) - x) \tag{53}
\]

where \( x_{1,2}(a) = g a \mp \sqrt{(a^2 - 1)(g^2 - 1)} \). Substituting \( x = \frac{1}{2} (x_1(a) + x_2(a)) + \frac{1}{2} (x_2(a) - x_1(a)) \cos \phi \), where \( \phi \in [0, \pi] \) brings the right-hand side to

\[
F_p(y) \propto \gamma \int_1^\infty da \frac{(a - 1)^{p+1}}{(a + 1)^p} e^{-2y a \sqrt{(a^2 - 1)(g^2 - 1)}} \cos \phi d\phi \tag{54}
\]

which is equivalent to (16) of the main text.

Numerical simulations provide a reasonably good support for the final result, Eq.(6) of the main text, already for relatively small matrices.

**FIG. 2.** Eq.6 compared with the empirical density of overlaps \( O_{nn} \) for \( n = 10^6 \) realizations of \( 32 \times 32 \) GUE matrices, with the choice \( y = 0.05 \) and \( \gamma = 0.5 \).
Our starting point is the well-known joint probability density (JPD) of eigenvalues for $N \times N$ non-selfadjoint matrix $\mathcal{H}_{eff} = H_N - i\gamma e \otimes e^T$, with $H_N$ taken from the Gaussian Orthogonal Ensemble[2,3]:

$$\mathcal{P}_N(z_1, \ldots, z_N) \propto \frac{e^{-N(\frac{1}{2}(\gamma^2 + \sum_{j=1}^{N} (Rz_j^2)))}}{\sqrt{\prod_{j,k=1}^{N} |z_j - z_k|^2 \delta(\sum_{j=1}^{N} z_j + \gamma)}}$$

Using it we follow essentially the same procedure of reduction to averaging over matrices of size $(N - 1) \times (N - 1)$ as in the GUE case (cf. also [4]) and come to the following relation:

$$\mathcal{O}_p^{(\beta=1)}(z) \propto \frac{1}{(\gamma Y)^{1/2}} e^{-\frac{N}{2}(X^2 - Y^2 + \gamma^2) + \frac{N-1}{2} \gamma^2 \mathcal{M}_p^{(\beta=1)}(z)}$$

where

$$\mathcal{M}_p^{(\beta=1)}(z) = \left\langle \left| \det(z - H_{eff}) \right| \right| \frac{1 + i\gamma G(z)}{1 + i\gamma G(z^*)} \right|^{2(p+1)} \rangem_{p}^{(\beta=1)}(z)$$

which, assuming for simplicity $X = 0$ and the scaling $Y = y/N$, asymptotically in the limit $N \to \infty$ is equivalent to

$$\mathcal{O}_p^{(\beta=1)}(z) \propto \lim_{N \to \infty} \mathcal{M}_p^{(\beta=1)}(z)$$

where one replaces $\mathcal{H}_{eff}$ with the original $\mathcal{H}_{eff} = H_N - i\gamma \hat{w} \otimes \hat{w}^\dagger$ without changing the result and may consider the random real gaussian channel vector $\hat{w}$ rather than keeping it of unit length. The same steps as in the GUE case then give:

$$\mathcal{M}_p^{(\beta=1)}(z) = \left\langle \left| \det(z - H_N) \right| \right| \frac{1 + i\gamma G(z)}{1 + i\gamma G(z^*)} \right|^{2(p+1)}$$

After performing the Fourier-Laplace transform of the above function and averaging over random Gaussian real vectors $\hat{w}$ we reduce the problem to evaluating of the following function

$$\mathcal{F}_p^{(\beta=1)}(s, p) = \left\langle \frac{|\det(i\gamma N - H_N)|^2}{\det(-i\tau_{\pm} N/2N - H_N) \det(-i\tau_{\mp} N/2N - H_N)} \right|$$

where in the present case $\frac{1}{2} \tau_+ = s + \sqrt{s^2 + y(y + 2p)} > 0$ and $\frac{1}{2} \tau_- = s - \sqrt{s^2 + 4y(y + p)} < 0$ and the averaging is to be performed over GOE matrices $H_N$ as $N \to \infty$. Such a correlation function has been evaluated in [5], with the result being given by

$$\mathcal{F}_p^{(\beta=1)}(s, p) \propto \frac{\sinh 2y}{2y} \left[ \frac{\tau_+ - \tau_-}{2} \right] K_1 \left( \frac{\tau_+ - \tau_-}{4} \right) + \beta(p, q),$$

where

$$\beta(p, q) = \frac{2}{(2y)^3} \left[ 2y \cosh 2y - \sinh 2y \right] \left( y^2 - \frac{1}{4} \tau_+ \tau_- \right) K_0 \left( \frac{\tau_+ - \tau_-}{4} \right)$$
Substituting here the values of $\tau_{\pm}$ one then arrives at

$$F(\beta=1)(s,p) \propto 2 \frac{\sinh 2y}{2y} \sqrt{s^2 + y(y + 2p)} K_1 \left( \sqrt{s^2 + y(y + 2p)} \right) + \beta(p,q),$$

(63)

with

$$\beta(p,q) = \frac{2}{(2y)^3} [2y \cosh 2y - \sinh 2y] (y + p) K_0 \left( \sqrt{s^2 + y(y + 2p)} \right).$$

The Fourier-Laplace inversion is performed using the functions defined in (44), namely

$$\phi_{-\frac{1}{2}}(s,p; y/2) = \frac{2}{\sqrt{y}} K_0(\sqrt{s^2 + y^2 + 2py}), \quad \phi_{\frac{1}{2}}(s,p; y/2) = \frac{2}{y^{3/2}} \sqrt{s^2 + y^2 + 2py} K_1(\sqrt{s^2 + y^2 + 2py}),$$

(64)

which also can be used to prove the following identity:

$$2p K_0(\sqrt{s^2 + y^2 + 2py}) = -\sqrt{y} \int_0^\infty e^{-pv} \int_{-\infty}^{\infty} e^{isu} e^{-\frac{u^2 + v^2 + 1}{v^2}} \left[ \frac{3}{2v} + \frac{y}{2} \left( 2 - \frac{u^2 + v^2 + 1}{v^2} \right) \right] \frac{du}{\sqrt{2\pi}}.$$

These identities when applied to the inversion of (62) yield

$$P_y^{(\beta=1)}(u,v) \propto \left( \cosh 2y - \frac{\sinh 2y}{2y} \right) \left( \frac{3}{2v} + \frac{u^2 + v^2 + 1}{v^2} \right) + 2 \sinh 2y \sqrt{\frac{y}{4v^2}} e^{-\frac{u^2 + v^2 + 1}{v^2}}$$

(65)

and the final expression (17) for $P_{y}^{(\beta=1)}(t)$ is straightforwardly obtained following the same manipulations as described for the GUE-based case.

Numerical simulations provide a reasonably good support for the final result, Eq.(18) of the main text, already for relatively small matrices.

![Graph](image)

**FIG. 3.** Eq.18 compared with the empirical density of overlaps $O_{nn}$ for $n = 10^6$ realizations of $32 \times 32$ matrices, with $y = 0.05$ and $\gamma = 0.5$.

**Comparison of the formula Eq.(19) for the mean resonance density with the form in [6]**

We start with recalling Eq.(4) from [6]:

$$\rho^{(\beta=1)}(y) = \frac{1}{4\pi} \frac{d^2}{dy^2} \int_{-1}^{1} (1 - \lambda^2) e^{2\lambda y} (g - \lambda) F(\lambda, y) d\lambda$$

(66)
where

\[ F(\lambda, y) = \int_0^\infty dp_1 \int_1^g dp_2 \frac{e^{-yp_1}}{(\lambda - p_1)^2 \sqrt{(p_1^2 - 1)(p_1 - g)}} \frac{e^{-yp_2}(p_1 - p_2)}{(\lambda - p_2)^2 \sqrt{(p_2^2 - 1)(g - p_2)}} \]

Below we consider only the perfect coupling case \( g \to 1 \). For this first change the variable: \( p_2 = 1 + (g - 1)t, t \in [0, 1] \), implying \( \frac{dp_2}{\sqrt{(p_2 - 1)(g - p_2)}} = \frac{dt}{\sqrt{(1-t)}} \). This allows to evaluate the integral over \( t \) in the limit \( g \to 1 \), the result being equal to \( \frac{\pi}{\sqrt{2}} e^{-y} \frac{(p_1 - 1)(\lambda - 1)}{2} \), yielding the formula:

\[ \rho_{g=1}^{(\beta=1)}(y) = \frac{1}{4\sqrt{2}} \int_1^\infty dp \int_1^1 e^{-yp_1} e^{2\lambda y} \frac{(1 + \lambda)(2\lambda - p - 1)^2}{(\lambda - p)^2} \]

which should be compared to the \( g = 1 \) limit of Eq.(19)-(20) in the paper. Although we were not yet able to show analytically the equivalence of the two formulas, the numerical comparison is flawless:

![Equation Graph](image)

**FIG. 4.** Eq.(19) of the main text compared with Eq.(67)

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