SUPERSYMMETRIC GRAND UNIFIED THEORIES AND YUKAWA UNIFICATION

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INTRODUCTION

In this paper, I intend to motivate supersymmetric grand unified theories (SUSY GUTs), briefly explain an extension of the standard model based on them and present a calculation performed using certain properties of some SUSY GUTs to constrain the available parameter space.

Why GUTs?

Much work has been done on the running of the gauge couplings in the standard model, as prescribed by the renormalisation group. Amazingly, when the couplings $\alpha_1$, $\alpha_2$ and $\alpha_3$ were run up to fantastically high energies $\sim 10^{14}$ GeV, they seemed to be converging to one value. This is a feature naturally explained by many GUTs such as SU(5)$^{2,3}$ and reflects the fact that the strong, weak and electromagnetic forces seen today are different parts of the same grand unified force. It was realised that GUTs could also provide relations between the masses of the observed fermions, the structure and hierarchy of which are as yet unexplained. Despite these attractive features, several problems arose which detracted from the idea.

Unfortunately, the three couplings do not quite converge by $\sim 7\sigma$, and many GUTs, notably SU(5), predict proton decay much faster than the lower experimental bounds. Also incredible fine tuning is required for the so-called 'hierarchy problem'. This stems from the fact that $M_W$ changes through radiative corrections of order the new physics scale (Fig.1), say the Planck mass $\sim 10^{19}$ GeV, if there is no new physics at smaller energies. $M_W$ is therefore unstable to the corrections and vast cancellations in the couplings are required to motivate the correct phenomenology.
Figure 1: One loop corrections to $m_h^2$. The first diagram gives a $\sim M_{Pl}^2$ contribution.

Why SUSY?

Supersymmetry is an extra symmetry relating fermions and bosons, and so provides some explanation of how particles of differing spin should be related to one another. In an unbroken supersymmetric theory, each fermion has a degenerate bosonic partner. Of course, these so called superpartners are not observed, so that if supersymmetry was ever the correct theory, it must have been broken. However, with the introduction of superpartners at some rough energy scale $M_{SUSY}$, the renormalisation group running of the gauge couplings changes. The coupling constants are now seen to meet at a scale $\sim 10^{16}$ GeV, as reflected by the correct $\sin^2 \theta_w$ prediction. $M_W$ becomes stabilised because supersymmetry induces cancellations between the bosonic and fermionic loop corrections to the mass. The quadratic divergences induced by the loop corrections now add to zero and one is left with merely logarithmic divergences.

THE MINIMAL SUPERSYMMETRIC STANDARD MODEL (MSSM)

The MSSM is a minimal extension of the standard model into supersymmetry. In the model, every particle of the standard model has a superpartner associated with it that transforms identically under the standard model gauge group but have spin different by $\frac{1}{2}$. So for example, each quark has a scalar "squark" superpartner, the gluons have "gluinos" etc. At first sight however, the model has a $U(1)_Y$ gauge anomaly. This originates from the diagram with three B gauge bosons connected to an internal loop through which any fermions may run (cf Fig.2) and the counter term to it would destroy gauge invariance. The diagram is proportional to $\sum_i (Y_i/2)^3$ where $i$ runs over all active fermions. Through the hypercharge assignments, this cancels in the standard model but in the MSSM the superpartner of the Higgs called the Higgsino with $Y = 1$ may run around the loop. To cancel this effect, a second Higgs $H_2$ must be introduced which transforms in the same way to $H_1$ except for having $Y = -1$. 

Figure 2: One loop triple hypercharge boson anomaly.
The new Higgs must also develop a vev \( v_2 \) to give masses to up quarks and the two vevs are related by

\[
\tan \beta = \frac{v_2}{v_1}
\]  

where \( v_1^2 + v_2^2 = v^2 \) and \( v = 246 \) GeV, the measured vev of the standard model.

In chiral superfield form, the superpotential looks like

\[
W_{MSSM} = UQH_2u^c + DQH_1d^c + ELH_1e^c + \mu H_1H_2
\]

where U, D and E are the up, down and charged lepton Yukawa matrices respectively and all gauge and family indices have been suppressed.

One possible problem with this superpotential is the dimensionful parameter \( \mu \). \( \mu \) needs to be \( \sim M_Z \) to give the right electroweak symmetry breaking behaviour whereas one would expect it to be of order of the new physics scale \( M_{GUT} \). One solution to this problem is described in the Next to Minimal Supersymmetric Standard Model (NMSSM).

The NMSSM

The \( \mu \) term in Eq.2 is replaced by \( \lambda NH_1H_2 \) where N is a gauge singlet and therefore doesn’t affect the coupling constant unification. In certain supergravity models, N develops a vev naturally of order \( M_Z \) and so the \( \mu \) term is generated without having to put \( \mu \) in ”by hand.” The superpotential now has a discrete Peccei-Quinn symmetry which leads to phenomenologically unacceptable low energy axions and so a term \(-\frac{k}{3}N^3\) is added which breaks it.

GUTS WITH YUKAWA UNIFICATION

GUTs can quite naturally provide Yukawa unification relations between the quarks and/or leptons. For example in SU(5), the right handed down quarks and conjugated lepton doublet lie in a 5 representation. When a mass term \( \sim 5^i5_i \) is formed, the Yukawa relation

\[
\lambda_b(M_{GUT}) = \lambda_\tau(M_{GUT})
\]

applies. Also in SO(10), the whole of one family and a right handed neutrino is contained in one 16 representation, leading to triple Yukawa unification, where the top, bottom and charged lepton Yukawa couplings are equal at the GUT scale.

These relations can be used to constrain the parameter space of \( m_t \) and \( \tan \beta \), which has been done for the MSSM\(^5\). Our idea was to repeat this calculation for the NMSSM, to see how much the viable parameter space changes in the model.

THE CALCULATION

The basic idea is to choose some \( \tan \beta \) and \( m_t \) and run \( \lambda_b \) and \( \lambda_\tau \) up to \( M_{GUT} \approx 10^{16} \) GeV. Then, to some arbitrary accuracy, one can determine whether the GUT relation Eq.3 holds. If it does, then SU(5) and the other Yukawa unifying extensions of the standard model are possible on this point in parameter space. The procedure is iterated over all reasonable values of \( \tan \beta \) and \( m_t \). The calculation is presented in more detail in Ref.6.

\(^1\) \( \lambda \) and \( k \) are merely coupling constants
Figure 3: Viable Range of Parameter Space For $\alpha_S(M_Z) = 0.11$, $m_b = 4.25$ GeV. $\lambda$ and $k$ values are quoted at $m_t$.

**Starting Point $M_Z$.**

We use the definitions of the gauge couplings at $M_Z$: $\alpha_1^{-1}(M_Z) = 58.89$, $\alpha_2^{-1}(M_Z) = 29.75$ and $\alpha_3^{-1}(M_Z) = 0.11 \pm 0.01$. The first two gauge couplings are determined accurately enough for our purposes whereas the third needs to be used as a parameter, on account of its large uncertainty.

In order to convert masses of quarks to Yukawa couplings, we simply need to read them off the potential Eq.2 at some energy scale (taken here to be $m_t$):

$$\lambda_t(m_t) = \frac{\sqrt{2} m_t(m_t)}{v \sin \beta}$$

$$\lambda_b(m_t) = \frac{\sqrt{2} m_b(m_b)}{\eta_b v \cos \beta}$$

$$\lambda_\tau(m_t) = \frac{\sqrt{2} m_\tau(m_\tau)}{\eta_\tau v \cos \beta}.$$  

where

$$\eta_f = \frac{m_f(m_f)}{m_f(m_t)}.$$  

Note that whereas the $m_t$ referred to here is always the running one, it can be related to the physical mass by

$$m_t^{\text{phys}} = m_t(m_t) \left[ 1 + \frac{4}{3\pi} \alpha_3(m_t) + O\left(\alpha_3^2\right) \right].$$

To determine $\eta_b$ and $\eta_\tau$, the masses are run up from the on shell mass to $m_t$ using effective 3 loop QCD $\otimes$ 1 loop QED $7,8,9,10$. Note that these factors will depend of $m_b = 4.25 \pm 0.15$ GeV and $\alpha_3(M_Z)$. $m_t$ is assumed to be the rough energy scale when the whole supersymmetric spectrum kicks in. While being unrealistic, trials with $M_{\text{SUSY}} = 1$ TeV show only a few percent deviation from the predictions with $M_{\text{SUSY}} = m_t$. So, having determined the gauge and relevant Yukawa couplings at $m_t$,
we need RG equations to run them up to $M_{GUT}$ in the NMSSM. To derive these, we
used results from a general superpotential
11 to obtain

\[
\begin{align*}
16\pi^2 \frac{\partial \lambda_t}{\partial t} &= \lambda_t \left[ 6\lambda_t^2 + \lambda_b^2 + \lambda^2 - \left( \frac{13}{15} g_1^2 + 3 g_2^2 + \frac{16}{3} g_3^2 \right) \right] \\
16\pi^2 \frac{\partial \lambda_b}{\partial t} &= \lambda_b \left[ 6\lambda_b^2 + \lambda_t^2 + \lambda^2 - \left( \frac{7}{15} g_1^2 + 3 g_2^2 + \frac{16}{3} g_3^2 \right) \right] \\
16\pi^2 \frac{\partial \lambda_r}{\partial t} &= \lambda_r \left[ \lambda_r^2 + 3\lambda_b^2 + \lambda^2 - \left( \frac{9}{5} g_1^2 + 3 g_2^2 \right) \right] \\
16\pi^2 \frac{\partial \lambda}{\partial t} &= \lambda \left[ 4\lambda^2 + 2k^2 + 3\lambda_t^2 + 3\lambda_b^2 + 3\lambda_r^2 - \left( \frac{3}{5} g_1^2 + 3 g_2^2 \right) \right] \\
16\pi^2 \frac{\partial k}{\partial t} &= 6k \left[ \lambda^2 + k^2 \right]
\end{align*}
\]  

(9)

in the limit that the lighter two families have negligible contributions (a very good
approximation).

The Yukawa couplings can now be run from $m_t$ to $10^{16}$ GeV using numerical tech-
niques. The parameters $\lambda$ and $k$ particular to the NMSSM are unconstrained at
$m_t$ so they are merely varied for different curves.

Our results are displayed in Fig. 3 as contours in the $\tan \beta - m_t$ plane consistent
with Eq.1. We take $\alpha_3(M_Z) = 0.11$, $m_b = 4.25$ GeV and the NMSSM parameters
$\lambda(m_t)$ and $k(m_t)$ as indicated. The MSSM contour is shown for comparison and is
indistinguishable from the NMSSM contour with $\lambda(m_t) = 0.1$ and $k(m_t) = 0.5$. In fact
our plot for the MSSM based on 1-loop RG equations is very similar to the 2-loop result
in ref.5. The deviation of the NMSSM contours from the MSSM contour depends most
sensitively on $\lambda(m_t)$ rather than $k(m_t)$. Two of the contours are shortened due to either
$\lambda$ or $k$ blowing up at the GUT scale. For $\lambda(m_t) = 0.5$, $k(m_t) = 0.5$, no points in the
$m_t - \tan \beta$ plane are consistent with Eq.1 Yukawa unification, while for $\lambda(m_t) = 0.1,$
$k(m_t) = 0.1 - 0.5$ the contours are virtually indistinguishable from the MSSM contour.
In general we find that for any of the current experimental limits on $\alpha_3$ and $m_b$, the
maximum value of $\lambda(m_t)$ or $k(m_t)$ is $\sim 0.7$ for a perturbative solution to Eq.1.

Fig.4 shows the effects of particle thresholds, which can modify Eq.3 to $\lambda_b = 0.9\lambda_r$.
Our treatment does not treat supersymmetric or heavy thresholds exactly and so some

Figure 4: Viable Range of Parameter Space For $\alpha_S(M_Z) = 0.12$ and experimental bounds of
$m_b = 4.1–4.4$ GeV. The left most lines are for $\lambda_b = 0.9\lambda_r$. 
sort of corrections like those shown are expected. The curves are at $\alpha_s(M_Z) = 0.12$ and $m_b = 4.1\text{–}4.4\text{ GeV}$ to illustrate that uncertainties in these quantities make a large difference to the parameter space. These uncertainties are much bigger than those associated with the NMSSM, and so the MSSM and NMSSM would be practically indistinguishable given the parameters $m_t$ and $\tan \beta$.

**Other Yukawa Parameters**

The next useful step is to notice that Eqs.9 are all of the form

$$16\pi^2 \frac{\partial \lambda_a}{\partial t} = \lambda_a \left[ \sum_i M_i^a \lambda_i^2 - \sum_{j=1}^3 c_j^a g_j^2 \right],$$  

(10)

where $M_i^a$ and $c_j^a$ are constants supplied by the relevant RG equation. When the $\beta$ function

$$\frac{dq_i}{dt} = \frac{b_i g_i^3}{16\pi^2}$$  

(11)

is inserted, and the RG equations are reparameterised in terms of the flow and not the trajectory of the solutions, we obtain

$$\frac{\lambda_a(M_{\text{SUSY}})}{\lambda_a(M_{\text{GUT}})} = \xi^a \exp\left(-\sum_i M_i^a I_i\right),$$  

(12)

where

$$\xi^a = \prod_{i=1}^3 \left( \frac{\alpha(M_{\text{GUT}})}{\alpha_i(M_{\text{SUSY}})} \right) \frac{c_i^a}{\lambda_i^2}$$  

(13)

contains all the information about the gauge couplings and

$$I_i = \frac{1}{16\pi^2} \int_{\ln(M_{\text{SUSY}})}^{\ln(M_{\text{GUT}})} \lambda_i^2 dt$$  

(14)

cconcerns the Yukawa couplings.

With this formulation, the running of the physically relevant Yukawa eigenvalues and mixing angles can be expressed in simple terms as shown below,

$$\begin{align*}
\left( \frac{\lambda_{u,c}}{\lambda_t} \right)_{M_{\text{SUSY}}} &= \left( \frac{\lambda_{u,c}}{\lambda_t} \right)_{M_{\text{GUT}}} e^{3I_b + I_t}, \\
\left( \frac{\lambda_{d,s}}{\lambda_b} \right)_{M_{\text{SUSY}}} &= \left( \frac{\lambda_{d,s}}{\lambda_b} \right)_{M_{\text{GUT}}} e^{3I_t + I_b}, \\
\left( \frac{\lambda_{e,\mu}}{\lambda_t} \right)_{M_{\text{SUSY}}} &= \left( \frac{\lambda_{e,\mu}}{\lambda_t} \right)_{M_{\text{GUT}}} e^{3I_t}, \\
\left| \frac{V_{cb}}{V_{cb}} \right|_{M_{\text{GUT}}} &= e^{I_b + I_t},
\end{align*}$$  

(15)

with identical scaling behaviour to $V_{cb}$ of $V_{ub}, V_{ts}, V_{td}$. To a consistent level of approximation $V_{us}, V_{ud}, V_{cs}, V_{cd}, V_{tb}, \lambda_u/\lambda_c, \lambda_d/\lambda_s$ and $\lambda_c/\lambda_\mu$ are RG invariant. The CP violating quantity $J$ scales as $V_{cb}^2$. Eqs. [13], [14] also apply to the NMSSM since the extra $\lambda$ and $k$ parameters cancel out of the RG equations in a similar way to the gauge contributions as can easily be seen from Eq.[9]. The only difference to these physically relevant quantities is therefore contained in $I_t, I_b$ and $I_t$. 
CONCLUSIONS

We have discussed the unification of the bottom quark and tau lepton Yukawa couplings within the framework of the NMSSM. By comparing the allowed regions of the $m_t$-$\tan \beta$ plane to those in the MSSM we find that over much of the parameter space the deviation between the predictions of the two models which is controlled by the parameter $\lambda$ is small, and always much less than the effect of current theoretical and experimental uncertainties in the bottom quark mass and the strong coupling constant. We have also discussed the scaling of the light fermion masses and mixing angles, and shown that to within current uncertainties, the results of recent quark texture analyses\textsuperscript{12} performed for the minimal model also apply to the next-to-minimal model. There are however two distinguishing features of the NMSSM. Firstly, the scaling of the charged lepton masses will be somewhat different, depending on $\lambda$ and $k$. Although this will not affect the quark texture analysis of RRR, it may affect the success of the GJ ansatze\textsuperscript{13,14} for example. Secondly, the larger $\tan \beta$ regions may not be accessible in the NMSSM for large values of $\lambda$ and $k$, so that full Yukawa unification may not be possible in this case.
REFERENCES

[1] V. Barger and R. J. N. Phillips, Preprint MAD/PH/752 (1993).
[2] M. Chanowitz, J. Ellis, and M. K. Gaillard, Nuclear Physics B128, 506 (1977).
[3] A. J. Buras, J. Ellis, M. K. Gaillard, and D. V. Nanopoulos, Nucl. Phys. B135, 66 (1978).
[4] H. E. Haber, Preprint SCIPP 92/33 (1993).
[5] V. Barger, M. S. Berger, and P. Ohmann, Phys. Rev. D47, 1093 (1993).
[6] B. C. Allanach and S. F. King Phys. Lett. B328, 360 (1994).
[7] S. G. Gorishny, A. L. Kataev, S. A. Larin, and L. R. Surgaladze, Mod. Phys. Lett. A5, 2703 (1990).
[8] O. V. Tarasov, A. A. Vladimirov, and A. Yu. Zharkov, Phys. Lett. B93, 429 (1980).
[9] S. G. Korishny, A. L. Kataev, S. A. Larin, and P. Lett., Phys. Lett. B135, 457 (1984).
[10] L. Hall, Nucl. Phys. B75 (1981).
[11] S. P. Martin and M. T. Vaughn, NUB-3081-93TH hep-ph 9311340 (1993).
[12] P. Ramond, R. Roberts, and G. Ross, RAL-93-010 UFIFT-93-06 (1993).
[13] H. Georgi and C. Jarlskog, Phys. Lett. B86, 297 (1979).
[14] S. Dimopoulos, L. Hall, and S. Raby, Phys. Rev. D45, 4192 (1992).