Vortex Loop Phase Transitions in Liquid Helium, Cosmic Strings, and High-T\textsubscript{c} Superconductors

Gary A. Williams
Department of Physics and Astronomy, University of California, Los Angeles, CA 90095
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The distribution of thermally excited vortex loops near a superfluid phase transition is calculated from a renormalized theory. The number density of loops with a given perimeter is found to change from exponential decay with increasing perimeter to algebraic decay as T\textsubscript{c} is approached, in agreement with recent simulations of both cosmic strings and high-T\textsubscript{c} superconductors. Predictions of the value of the exponent of the algebraic decay at T\textsubscript{c} and of critical behavior in the vortex density are confirmed by the simulations, giving strong support to the vortex-folding model proposed by Shenoy.

Here A\textsubscript{o} = 4\pi\textsuperscript{3}/3, K\textsubscript{c} is the "bare" superfluid density resulting from the spin waves (and is the initial value of K\textsubscript{c}), and U(a) is the renormalized energy of a ring, given by

$$U(a)/k_B T = \pi^2 \int_{a_o}^a K_c (\ln\left(\frac{a}{a_o}\right) + 1) \frac{da}{a_o} + \pi^2 K_o C$$  \hspace{1cm} (2)$$

where C is a nonuniversal constant characterizing the core energy. For helium C and a\textsubscript{o} are determined from two experimental inputs, T\textsubscript{c} = 2.172 K and the amplitude of the superfluid density \textsuperscript{3}, yielding C = 1.03 and a\textsubscript{o} = 2.5 \text{ Å}. The effective core size a\textsubscript{o} in Eq. (2) was suggested by Shenoy and co-workers \textsuperscript{4} to be a result of the random walk of the loop giving rise to radial fluctuations of order a\textsubscript{o} about the average diameter. This folding of the loop occurs because antiparallel vortex segments lower the energy. A simple polymer-type calculation \textsuperscript{11} using energy-entropy arguments yields a\textsubscript{o}/a = (K\textsubscript{o}/a\textsubscript{o})\textsuperscript{3}, where \theta = d/(d+2) = 0.6 in d = 3 dimensions has the same form as the well-known Flory exponent of the self-avoiding walk.

Eqs. (1) and (2) then constitute a coupled set of integral equations for the renormalized superfluid density, and can be solved recursively starting from the bare scale a\textsubscript{o} and iterating to distances greater than the correlation length \xi = a\textsubscript{o}/K\textsubscript{c}. In practice these are converted to a set of coupled differential equations similar to the Kosterlitz-Thouless recursion relations \textsuperscript{12} for the two-dimensional case, and are solved using a Runge-Kutta technique. As T is increased (K\textsubscript{o} decreased) the solution for \rho\textsubscript{s} falls to zero as (T\textsubscript{c} - T)\nu, with \nu = 0.6717 for \theta = 0.6. This can be better matched to the most precise experimental value \nu = 0.6705 by adjusting to \theta = 0.594, which is reasonable since it is known that the Flory-type arguments are not exact in three dimensions \textsuperscript{13}.

The arguments of Ref. 10 also yield a result for the average perimeter of a loop of diameter a,
\[ \frac{P}{a_o} = B \left( \frac{a}{a_o} \right)^{1/\delta} \quad (3) \]

where \( B \) is a constant and the exponent \( \delta = 2/(D+2) = 0.4 \). This form for the perimeter was at least partially verified in the computer simulations of Ref. 10, and in the XY model simulations of Epiney [14]. For \( P/a_o \) greater than about 20 the Epiney data for the average loop size versus average perimeter can be fit by Eq. (3) with \( B = 1.8 \), although the resolution is poor because of scatter in the data resulting from the relatively small lattice (16 \(^2\)) that was simulated.

The distribution of the density of loops with a given perimeter \( P \) can be obtained from the theory outlined above, which is of interest because these distributions have now been measured in the cosmic-string [3] and high-\( T_c \) [11] simulations. The probability per unit volume for finding a loop of mean diameter between \( a \) and \( a+da \) is

\[ \frac{\pi}{2 a_o^3} \left( \frac{a}{a_o} \right)^2 \exp \left( \frac{-U(a)}{k_BT} - \frac{\partial (1/K_r)}{\partial a} \right) \quad (4) \]

Equating this to the probability \( D(P) \) \( dP/a_o \) of finding the corresponding loop of perimeter between \( P \) and \( P+\delta P \) gives the probability distribution

\[ D(P) = \frac{\pi}{2 a_o^3} \frac{\delta}{B} \left( \frac{a}{a_o} \right)^{\frac{3\delta+1}{\delta}} \exp \left( \frac{-U(a)}{k_BT} \right) \quad (5) \]

For temperatures well below \( T_c \), \( U(a) \sim a \ln a \), and hence \( D(P) \) decreases exponentially with \( P^{\delta} \). Near \( T_c \), however, the behavior is quite different. By differentiating Eq. (1) with respect to \( a \) and substituting for the exponential term in Eq. (5) gives

\[ D(P) = \frac{\pi}{2 a_o^3} \frac{\delta}{B} \left( \frac{a}{a_o} \right)^{-\frac{3\delta+1}{\delta}} \left[ \frac{\partial (1/K_r)}{\partial a} \right] \quad (6) \]

Precisely at \( T_c \), \( K_r \) from Eq. (1) satisfies the condition \( K_r a/a_o = D_o = 0.39 \), at least for values of \( a \) greater than about 5\( a_o \), and where \( D_o \) is a universal constant [3][3]. This condition is the three-dimensional equivalent of the universal "jump" of the superfluid density in two dimensions [1]. Inserting this result into Eq. (6) and employing Eq. (3) yields the prediction that at \( T_c \) the loop distribution will cross over from exponential to algebraic decay with \( P \),

\[ D(P) = \frac{\pi \delta B^{\frac{3\delta}{2}}}{2 a_o^3 A_o D_o} \left( \frac{P}{a_o} \right)^{-\gamma} \quad (7) \]

where the exponent \( \gamma = 3\delta + 1 \). For the "Flory" value \( \delta = 0.4 \) this would predict \( \gamma = 2.2 \). This form for \( D(P) \) signals the onset of loops of infinite size, since they no longer have a vanishing probability. It should be noted that the algebraic decay is a consequence of the strong renormalization at \( T_c \), where the screening of large loops by smaller ones causes the variation of \( U(a) \) to change from \( \sim a \ln a \) to \( \ln a \) at \( T_c \), causing the change from Eq. (5) to Eq. (7).

The crossover from exponential to algebraic decay is a central feature observed in the recent cosmic-string [3] and high-\( T_c \) [11] simulations using lattices larger than \( 96^3 \), and which was also seen with more limited resolution in the earlier XY model simulations of Epiney [14]. The results of Antunes and Bettencourt [2] yield \( \gamma = 2.23 \pm 0.04 \) at \( T_c \), which from the analysis above gives \( \delta = 0.41 \pm 0.01 \), in very good agreement with Shenoy’s Flory-scaling prediction. Fits to the high-\( T_c \) simulations of Nguyen and Sudbo [4] (in the zero-field, isotropic limit of their Villain model) give \( \gamma = 2.4 \) [10], leading to a higher value \( \delta = 0.48 \). However, the results of Ref. 2 show that \( \gamma \) increases rapidly above \( T_c \) to the value of 2.5 found by Vachaspati and Vilenkin [17], and hence an accurate determination requires bracketing temperatures very close to \( T_c \). It is interesting that the result \( \gamma = 3\delta + 1 \) apparently remains valid even above \( T_c \), since the Vachaspati-Vilenkin value of \( \gamma = 2.5 \) is based on the "Brownian" value \( \delta = 0.5 \).

The magnitude of the loop distribution \( D(P) \) at \( T_c \) as calculated from either Eq. (5) or (7) appears to be about a factor of 3 smaller than found in the simulations. Comparing the continuum calculation to the lattice results is made difficult by uncertainties in matching at the scale of the lattice spacing \( a_l \); in computing the magnitude of Eq. (7) for the comparison the geometric value \( a_o = \sqrt{2} a_l \) was employed, but it is not entirely clear that this is the correct choice.

The total length per unit volume \( \rho_o \) of the vortex loops can be found by multiplying \( D(P) \) by \( P \) and integrating. At \( T_c \) this can be found explicitly using Eq. (7),

\[ \rho_o(T_c) = \frac{\pi \delta B}{2 a_o^3 A_o D_o (3\delta - 1)} \quad (8) \]

The quantity \( \rho_o(T_c) a_o^2 \) was postulated in Ref. 1 to be universal, with a value of 0.6 in lattice units (0.2 per plaquette, with three plaquettes per unit volume). This means that the coefficient \( B \) in Eq. (8) characterizing the relationship between the perimeter and the average loop diameter must be universal, since all of the other parameters are. As with \( D(P) \) above, the magnitude of Eq. (8) must be multiplied by a factor of about 3 to match with the lattice results. For liquid helium, the vortex density [18] at \( T_c \) is then about \( 1 / a_o^2 \sim 1 \times 10^{15} /\text{cm}^2 \), which is many orders of magnitude higher than previous estimates [15] which did not use a renormalized theory.
FIG. 1. Normalized vortex density as a function of $T_c / T$, for several different lattice sizes.

A further prediction of the loop theory is the existence of critical behavior in the vortex density at $T_c$ \cite{18}, which has now been seen in the cosmic-string simulations \cite{1}. This is shown in Fig. 1, which plots the normalized density versus $T_c / T$, calculated by integrating Eq. (5) very near the transition. The behavior above $T_c$ (dashed lines) is only conjectured, as the theory is not valid there. To suppress the exponential variation arising from thermal excitation of the smallest loops, the core energy constant was taken to be the relatively large value $C = 4/3$, corresponding to the Villain model. The density just below $T_c$ is found to decrease from its value at $T_c$ as $-(T_c-T)^\delta$. This exponent was not measured in the cosmic-string simulations, probably because of the relatively low core energy in that model, which would make it difficult to separate the algebraic behavior from the exponential. However, the exponent of the density in the region just above $T_c$ was measured in Ref. 1 to be $0.39 \pm 0.01$, which is quite consistent with the value of $\delta \approx 0.41$ determined above from the same type of simulation.

It is well known from scaling and renormalization-group studies that many critical exponents are identical above and below the transition, and it is likely that this exponent constitutes a further measurement of $\delta$. Since $\delta$ is less than one, $T_c$ marks an inflection point in the density, where the curvature changes sign.

Also shown in Fig. 1 is the effect of a finite-size lattice on the density near $T_c$. For this the recursion relations are stopped at a finite scale that is a fraction $\beta = 0.75$ smaller than the lattice size, where $\beta$ was found in Ref. 8 by comparing to the helium simulations of Ref. 15. The effect of finite size is to smear out the critical behavior, and this explains why it was not apparent in the early simulations \cite{20}, which only used a maximum $10^3$ lattice size.

The coherence length in the loop theory can be identified with the diameter of the largest loop that is thermally excited \cite{3}. Since this is divergent at $T_c$, in a system of finite (but macroscopic) size the transition can be identified with the point where a loop just touches two opposing boundaries of the system. This is the percolation threshold for establishment of the infinite vortex cluster, as first proposed by Onsager 50 years ago, and verified in simulations \cite{21}. At higher temperature even larger loops can be excited; these can take the form of a single vortex line passing from one side of the system to the other, with the topological return path of the loop being around the outside of the system. These are known as "string" excitations in cosmology \cite{1,2,17}. The left-over strings from the rapid cooling through the phase transitions in the early universe are thought to be the source of matter \cite{14,22}. A laboratory example of this may be the observation of a few remnant vortex lines \cite{23} in a finite-sized sample of liquid $^4$He which has not been rotated or stirred, but simply cooled through the $\lambda$-transition.

The key role of thermally excited vorticity in the present model of the phase transition allows a rather different viewpoint of the Kibble-Zurek mechanism \cite{19,22} for defect formation in rapidly quenched transitions as discussed above. This is of interest because of the possibility of carrying out such experiments in liquid helium \cite{19,22,24,25}. In this view the vortices "created" in a quench of superfluid $^4$He through $T_c$ are simply a perturbation on the equilibrium vortex density. This perturbation is a consequence of the dynamics of large loops, which become slow in their response to external fields, and which are actually the source of the critical slowing down of the equilibrium transition \cite{1}. In a rapid quench the large loops in the vicinity of $T_c$ cannot keep up and fall out of equilibrium, forming an excess density that survives to lower temperatures, finally decaying to the equilibrium line density only after the quench is stopped.

It may be possible to model this process analytically using the Fokker-Planck equation for the loop distribution function formulated in the dynamic theory of Ref. 9, with the temperature being a ramp function in time. It should be noted that the exponent $\delta$ from above plays an important role in the vortex dynamics, since the frictional drag force on a loop is proportional to its total perimeter. It was found in Ref. 9 that the dynamic exponent characterizing critical slowing down is given by $x = z \nu$, where $z = (1-\delta)/\delta$. For the "Flory" value $\delta = 2/(d+2)$ this gives the scaling result $z = d/2 = 3/2$; for $\delta = 0.41$ as found above, $z = 1.44$, a few percent smaller. The possibility of deviations from the scaling result has been suggested previously in perturbative dynamic theories \cite{20}.

Vortex creation has been observed in superfluid $^3$He \cite{24}, where the quench is induced by absorbed neutrons depositing their energy in a small region of the liquid, which heats it above $T_c$, and which is then rapidly cooled back down by the surrounding cold liquid \cite{25}. Although $^3$He is a p-wave BCS superfluid, vortex loops will still be associated with the phase transition as above, but due to the Ginzburg criterion \cite{27} they will only be appreciable
in a very narrow temperature range near \( T_c \), since the zero-temperature coherence length of \(^3\)He is of order 500 Å, compared with \( a_0 = 2.5 \) Å in \(^4\)He. The theory \[25\] of the quenched \(^3\)He involves the exponents \( \delta \) and \( \gamma \) at \( T_c \); the use of the Brownian values \[17\] \( \gamma = 2.5 \) and \( \delta = 0.5 \) needs to be reexamined in light of the present results.

The vortex loop model also allows insights into the high-\( T_c \) superconducting transition in zero field. If \( T_c \) is the point where the loops of infinite size act to bring all supercurrents to a halt, then it is not to be identified as the point where thermal de-excitation of Cooper pairs is complete. The continued existence of pairs above \( T_c \) has been suggested in experiments, commonly known as the pseudogap phenomenon \[28\]. The vortex loops constitute a concrete physical picture of the “phase fluctuations” postulated to give rise to this effect in Ref. 28. The Cooper pairs above \( T_c \) will not be the same as those below, since they will no longer be part of a macroscopic BCS-type condensate, which is destroyed by the vortices. Presumably the pairs would be more localized excitations, on the scale of the 10-15 Å zero-temperature coherence length.

The vortices also offer a simple explanation \[29\] of the magnetic flux noise in YBCO samples that is observed to increase rapidly by many orders of magnitude over a temperature range of order 5 K near \( T_c \). When the loops being thermally excited terminate on the sample surface, they will induce dipolar current patterns on the surface, and this will generate magnetic flux that can be sensed by a detection loop above the surface. As \( T_c \) is approached from below both the number and size of the loops will increase, increasing the flux noise through the detector. This effect can also be observed in a low-\( T_c \) superconductor, since the same vortex-loop transition occurs also in that case, but with the considerable difference that the large zero-temperature coherence length (several thousand Å) causes the temperature range where the vortices are appreciable to be very much closer to \( T_c \). The experiments \[21\] showed that indeed a very sharp flux-noise peak could be observed in a lead sample only within about 2 mK of \( T_c \), and that this was only an upper limit to the width due to the resolution of the thermometry and the additional broadening that would be caused by a distribution of \( T_c \)'s across the sample.

In summary, a vortex-loop theory is able to provide physical insight into recent models of cosmic strings and high-\( T_c \) superconductors. The theory relates the critical exponents measured in the simulations to the random-walk nature of the loops, and the good agreement between the predicted and measured exponents provides strong support for the Flory-scaling ansatz of Shenoy \[7\].

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