A spatio-temporalisation of $\text{ALC}(D)$ and its translation into alternating automata augmented with spatial constraints

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Abstract

The aim of this work is to provide a family of qualitative theories for spatial change in general, and for motion of spatial scenes in particular. To achieve this, we consider a spatio-temporalisation $\text{MTALC}(D_x)$ of the well-known $\text{ALC}(D)$ family of Description Logics (DLs) with a concrete domain: the $\text{MTALC}(D_x)$ concepts are interpreted over infinite $k$-ary $\Sigma$-trees, with the nodes standing for time points, and $\Sigma$ including, additionally to its uses in classical $k$-ary $\Sigma$-trees, the description of the snapshot of an $n$-object spatial scene of interest; the roles split into $m+n$ immediate-successor (accessibility) relations, which are serial, irreflexive and anti-symmetric, and of which $m$ are general, not necessarily functional, the other $n$ functional; the concrete domain $D_x$ is generated by an $\text{RCC8}$-like spatial Relation Algebra (RA), and is used to guide the change by imposing spatial constraints on objects of the "followed" spatial scene, eventually at different time points of the input trees. In order to capture the expressiveness of most modal temporal logics encountered in the literature, we introduce weakly cyclic Terminological Boxes (TBoxes) of $\text{MTALC}(D_x)$, whose axioms capture the decreasing property of modal temporal operators. We show the important result that satisfiability of an $\text{MTALC}(D_x)$ concept with respect to a weakly cyclic TBox can be reduced to the emptiness problem of a Büchi weak alternating automaton augmented with spatial constraints. In another work, complementary to this one, also submitted to this conference, we thoroughly investigate Büchi automata augmented with spatial constraints, and provide, in particular, a translation of an alternating into a nondeterministic, and an effective decision procedure for the emptiness problem of the latter.

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Introduction

The goal of the present work is to enhance the expressiveness of modal temporal logics with qualitative spatial constraints. What we get is a family of qualitative theories for spatial change in general, and for motion of spatial scenes in particular. The family consists of domain-specific spatio-temporal (henceforth s-t) languages, and is obtained by spatio-temporalising a well-known family of description logics (DLs) with a concrete domain, known as $\text{ALC}(D)$ ($\text{Baader and Hanschke 1991}$). $\text{ALC}(D)$ originated from a pure DL known as $\text{ALC}$ ($\text{Schmidt-Schauß and Smolka 1991}$), with $m \geq 0$ roles all of which are general, not necessarily functional relations, and which Schild ($\text{Schild 1991}$) has shown to be expressively equivalent to Halpern and Moses’ $\text{K}_{m}$ modal logic ($\text{Halpern and Moses 1985}$). $\text{ALC}(D)$ is obtained by adding to $\text{ALC}$ functional roles (better known as abstract features), a concrete domain $D$, and concrete features (which refer to objects of the concrete domain). The spatio-temporalisation of $\text{ALC}(D)$ is obtained, as the name suggests, by performing two specialisations at the same time: (1) temporalisation of the roles, so that they consist of $m+n$ immediate-successor (accessibility) relations $R_1, \ldots, R_m, f_1, \ldots, f_n$, of which the $R_i$’s are general, the $f_i$’s functional; and (2) spatialisation of the concrete domain $D$: the concrete domain is now $D_x$, and is generated by a spatial RA $\rho$, such as the Region-Connection Calculus $\text{RCC8}$ ($\text{Randell and Cui 1992}$).

The final spatio-temporalisation of $\text{ALC}(D)$ will be referred to as $\text{MTALC}(D_x)$ ($\text{MTALC}$ for Modal Temporal $\text{ALC}$). Constraint-based languages candidate for generating a concrete domain for a member of our family of spatio-temporal theories, are spatial RAs for which the atomic relations form a decidable subset —i.e., such that consistency of a CSP expressed as a conjunction of $p$-ary relations on $p$-tuples of objects, where $p$ is the arity of the RA relations, is decidable. These include, the Region-Connection Calculus $\text{RCC8}$ in ($\text{D.A Randell and Cui 1992}$) (see also ($\text{Egenhofer 1991}$)), the Cardinal Directions Algebra $\text{CDÂ}$ in ($\text{Frank 1992}$), and the rectangle algebra in ($\text{Balbiani, Condotta, and del Cerro 1998}$) (see also ($\text{Gütsch 1989}$) $\text{Mukerjee and Joe 1990}$), for the binary case; and the RA CYC$_t$ of 2D orientations in ($\text{Isli and Cohn 1998}$) ($\text{Isli and Cohn 2000}$) for the ternary case. As our illustrating spatial RA, we will be using the ternary RA in ($\text{Isli and Cohn 1998}$) ($\text{Isli and Cohn 2000}$).

It is known that, in the general case, satisfiability of an
$\text{ALC}(D)$ concept with respect to a cyclic Terminological Box (TBox) is undecidable (see, e.g., (Lutz 2001)). In order to capture the expressiveness of most modal temporal logics encountered in the literature, we introduce in this work weakly cyclic TBoses of $\text{MTALC}(D_x)$, whose axioms capture the increasing property of modal temporal operators. We show the important result that satisfiability of an $\text{MTALC}(D_x)$ concept with respect to a weakly cyclic TBox can be reduced to the emptiness problem of a B"uchi weak alternating automaton augmented with spatial constraints. In another work, complementary to this one, also submitted to this conference, we thoroughly investigate B"uchi automata augmented with spatial constraints, and provide, in particular, a translation of an alternating into a nondeterministic, and an effective decision procedure for the emptiness problem of the latter.

The $\text{MTALC}(D_x)$ description logics

Temporalisations of DLs are known in the literature (see, e.g., (Artale and Franconi 2000; Bettini 1997); as well as spatialisations of DLs (see, e.g., (Haarslev, Lutz, and Möller 1999)). The present work considers a spatio-temporalisation of the well-known family $\text{ALC}(D)$ of DLs with a concrete domain (Baader and Hanschke 1991). Specifically, we consider, at the same time, a temporalisation of the roles of the family and a spatialisation of its concrete domain.

Concrete domain

Definition 1 (concrete domain (Baader and Hanschke 1991))

A concrete domain $D$ consists of a pair $(\Delta_D, \Phi_D)$, where $\Delta_D$ is a set of (concrete) objects, and $\Phi_D$ is a set of predicates over the objects in $\Delta_D$. Each predicate $P \in \Phi_D$ is associated with an arity $n$ and we have $P \subseteq (\Delta_D)^n$.

Definition 2 (admissibility (Baader and Hanschke 1991))

A concrete domain $D$ is admissible if: (1) the set of its predicates is closed under negation and contains a predicate for $\Delta_D$; and (2) the satisfiability problem for finite conjunctions of predicates is decidable.

The concrete domains $D_x$, with $x$ spatial RA

Any spatial RA $x$ for which the atoms are Jointly Exhaustive and Pairwise Disjoint (henceforth JEPD), and such that the atomic relations form a decidable subclass, can be used to generate a concrete domain $D_x$ for members of the family $\text{MTALC}(D_x)$ of qualitative theories for spatial change. Such a concrete domain is used for representing knowledge on regions of a topological space $TS$; $\text{2DP}$ is the set of 2D points; $\text{2DO}$ is the set of 2D orientations; and $x$-at, as we have seen, is the set of $x$ atoms $-2^x$-at is thus the set of all $x$ relations.

Admissibility of the concrete domains $D_x$ is an immediate consequence of (decidability and tractability of the subset $\{r\}_r \subseteq x$-at) of $x$ atomic relations, for each $x \in \{\text{RCC}8, \text{CDA}, \text{CYC}_1\}$. The reader is refereed to (Renz and Nebel 1999) for $x = \text{RCC}8$, to (Ligozat 1998) for $x = \text{CDA}$, and to (Ishi and Cohn 1998; Ishi and Cohn 2000) for $x = \text{CYC}_1$.

Theorem 1 Let $x \in \{\text{RCC}8, \text{CDA}, \text{CYC}_1\}$. The concrete domain $D_x$ is admissible.

Syntax of $\text{MTALC}(D_x)$ concepts

Definition 3 (MTALC($D_x$) concepts) Let $x$ be an RCC8-like $p$-ary spatial RA. Let $N_C$, $N_R$, and $N_{x^*}$ be mutu-ally disjoint and countably infinite sets of concept names, role names, and concrete features, respectively; and $N_{a^*}$ a countably infinite set of $N_R$ whose elements are abstract features. A (concrete) feature chain is any finite composition $f_1 \ldots f_n g$ of $n \geq 0$ abstract features $f_1, \ldots, f_n$ and one concrete feature $g$. The set of $\text{MTALC}(D_x)$ concepts is the smallest set such that:

1. $\top$ and $\bot$ are $\text{MTALC}(D_x)$ concepts
2. $\text{an MTALC}(D_x)$ concept name is an $\text{MTALC}(D_x)$ (atomic) concept
3. if $C$ and $D$ are $\text{MTALC}(D_x)$ concepts: $R$ is a role (in general, and an abstract feature in particular); $u_1, \ldots, u_p$ are feature chains; and $P$ is an $\text{MTALC}(D_x)$ predicate, then the following expressions are also $\text{MTALC}(D_x)$ concepts:

(a) $\lnot C, C \cap D, C \cup D, \exists R.C, \forall R.C$; and
(b) $\exists(u_1) \ldots (u_p).P$.

We denote by $\text{MTALC}$ the sublanguage of $\text{MTALC}(D_x)$ given by rules $[\text{I}]$ and $[\text{II}]$ in Definition 3, which is the temporal component of $\text{MTALC}(D_x)$. It is worth noting that $\text{MTALC}$ does not consist of a mere temporalisation of $\text{ALC}$ (Schmidt-Schauss and Smolka 1991). Indeed, $\text{ALC}$ contains only general roles, whereas $\text{MTALC}$ contains abstract features as well. A mere temporalisation of $\text{ALC}$ (i.e., $\text{MTALC}$ without abstract features) cannot capture the expressiveness of well-known modal temporal logics, including Propositional Linear Temporal Logic $\text{PLTL}$, the computation tree logic $\text{CTL}$, and the subsuming full branching modal temporal logic $\text{CTL}^*$ (Emerson 1990). Given two integers $p \geq 0$ and $q \geq 0$, the sublanguage of $\text{MTALC}(D_x)$ (resp. $\text{MTALC}$) whose concepts involve at most $p$ general roles, and $q$ abstract features will be referred to as $\text{MTALC}_{p,q}(D_x)$ (resp. $\text{MTALC}_{p,q}$). The particular case $(p, q) = (0, q)$ with $q \geq 0$ is discussed in Section 2, where we provide a translation of $\text{CTL}^*$ to $\text{MTALC}_{0,q}$.

Definition 4 (subconcept) The set $\text{Subc}(C)$ of subconcepts of an $\text{MTALC}(D_x)$ concept $C$ is defined inductively as follows:

1. $\text{Subc}(\top) = \{\top\}$, $\text{Subc}(\bot) = \{\bot\}$
1. Subc(A) = {A}, Subc(¬A) = {¬A}, for all atomic concepts A.
2. Subc(C ∩ D) = {C ∩ D} ∪ Subc(C) ∪ Subc(D),
3. Subc(C ∪ D) = {C ∪ D} ∪ Subc(C) ∪ Subc(D),
4. Subc(¬(C ∩ D)) = {¬(C ∩ D)} ∪ Subc(¬C) ∪ Subc(¬D),
5. Subc(¬(C ∪ D)) = {¬(C ∪ D)} ∪ Subc(¬C) ∪ Subc(¬D),
6. Subc(∃R.C) = {∃R.C} ∪ Subc(C),
7. Subc(∀R.C) = {∀R.C} ∪ Subc(C),
8. Subc(∃R.C) = subc(∃R.C),
9. Subc(∀R.C) = subc(∀R.C),
10. Subc(∃u1 . . . (up).P) = {∃(u1) . . . (up).P},
11. Subc(∀u1 . . . (up).P) = {∀(u1) . . . (up).P}.

We now define weakly cyclic TBoxes.

Weakly cyclic TBoxes

An (MTALC(Dc)) terminological axiom is an expression of the form A ⊑ C, A being a concept name and C a concept. A TBox is a finite set of axioms, with the condition that no concept name appears more than once as the left hand side of an axiom.

Let T be a TBox. T contains two kinds of concept names: concept names appearing as the left hand side of an axiom of T are defined concepts; the others are primitive concepts. A defined concept A “directly uses” a defined concept B iff B appears in the right hand side of the axiom defining A. If “uses” is the transitive closure of “directly uses” then T contains a cycle iff there is a defined concept A that “uses” itself. T is cyclic if it contains a cycle; it is acyclic otherwise. T is weakly cyclic if it satisfies the following two conditions:

1. Whenever A uses B and B uses A, we have B = A — the only possibility for a defined concept to get involved in a cycle is to appear in the right hand side of the axiom defining it.
2. All possible occurrences of a defined concept B in the right hand side of the axiom B ⊑ C defining B itself, are within the scope of exactly one quantifier (in other words, there is no free occurrence of B in C, and no occurrence of B in C is within the scope of more than one quantifier).

Definition 5 (depths of a defined concept) Let B be a defined concept, and C a concept. The set of depths of B in C, depths(B, C), is the set of all integers d such that B has an occurrence in C within the scope of d quantifiers.

depths(B, C) is defined inductively as follows:

1. If B has no occurrence in C, depths(B, C) = {0},
2. depths(B, B) = {0},
3. depths(B, ¬C) = depths(B, C),
4. depths(B, C ∩ D) = depths(B, C ∪ D) = depths(B, C) ∪ depths(B, D),
5. depths(B, ∃R.C) = depths(B, ∀R.C) = {d + 1 : d ∈ depths(B, C)}

Remark 1 In Definition 5

1. Item 5 includes the following particular case: depths(B, ∃(u1) . . . (up).P) = {0}; and
2. Item 5 includes the particular case: if depths(B, C) = {0} then depths(B, ∃R.C) = depths(B, ∀R.C) = {0}.

A weakly cyclic TBox can now be defined formally as follows:

Definition 6 (weakly cyclic TBox) A TBox T is weakly cyclic if and only if it satisfies what follows:

1. whenever two defined concepts A and B are such that A uses B and B uses A, we have B = A; and
2. all axioms B ⊑ C of T verify the following:

depths(B, C) = {0} or depths(B, C) = {1}.

Definition 7 Let T be a weakly cyclic TBox.

1. An axiom B ⊑ C of T is cyclic if depths(B, C) = {1}; it is acyclic otherwise.
2. A defined concept B of T is cyclic if the axiom B ⊑ C defining it is cyclic; it is acyclic otherwise.
3. A cyclic axiom of T is said to be a necessity axiom if it is of either of the following forms:

(a) B ⊑ C ∩ ∀R.B where R is a role, either general or functional; and C a concept such that depths(B, C) = ∅,
(b) B ⊑ C ∩ ∀R.B where R is a role, either general or functional; and C a concept such that depths(B, C) = {0}.

4. A cyclic axiom of T is said to be an eventuality axiom if it is of either of the following forms:

(a) B ⊑ C ∩ ∃R.B where R is a role, either general or functional; and C a concept such that depths(B, C) = ∅,
(b) B ⊑ C ∩ ∃R.B where R is a role, either general or functional; and C a concept such that depths(B, C) = {0}.

5. A defined concept of T is a necessity defined concept if the axiom defining it is a necessity axiom.
6. A defined concept of T is an eventuality defined concept if the axiom defining it is an eventuality axiom.
7. The necessity defined concept B1 and the eventually defined concept B2 defined, respectively, by the axioms B1 = C ∩ ∀R.B1 and B2 = ¬C ∩ ∃R.B1 are each other’s duals.
8. The necessity defined concept B1 and the eventually defined concept B2 defined, respectively, by the axioms B1 = C ∩ (∃R.B1) and B2 = ¬C ∩ (∃R.B2) are each other’s duals.

From now on, we restrict ourselves, exclusively, to weakly cyclic TBoxes T such that

1. for all necessity or eventuality defined concepts B of T, T also has the defined concept consisting of the dual of B; and
2. all defined concepts B verify the following:

(a) B is acyclic,
(b) B is a necessity defined concept, or
(c) B is an eventuality defined concept.
In the rest of the paper, unless explicitly stated otherwise, we denote concepts reducing to concept names by the letters \( A \) and \( B \), possibly complex concepts by the letters \( C, D, E \), general roles by the letter \( R \), abstract features by the letter \( f \), concrete features by the letters \( g \) and \( h \), feature chains by the letter \( u \), (possibly complex) predicates by the letter \( P \).

**Example 1** Due to lack of space, an example supposed to come here is added as additional material, as a separate file including a brief background on the ternary spatial RA CYC, \( \text{Ish} \text{I, and Cohn 1998} \) \( \text{Ish} \text{I, and Cohn 2000} \) and an illustration of the use of MTALC\(_{0,1}(D_{CYC}) \) in robot navigation.

**Semantics of MTALC\(_{D_2}\)**

Let \( D_2 \) be an admissible spatial concrete domain generated by a \( p \)-ary spatial RA \( x \). MTALC\(_{D_2}\) concepts will be interpreted over \( k \)-ary \( \Sigma \)-trees.

**Definition 8 (k-ary \( \Sigma \)-tree)** Let \( \Sigma \) and \( K = \{d_1, \ldots, d_k\} \), \( k \geq 1 \), be two disjoint alphabets; \( \Sigma \) is a labelling alphabet and \( K \) an alphabet of directions. \( A \) (full) \( k \)-ary tree is an infinite tree whose nodes \( \alpha \in K^* \) have exactly \( k \) immediate successors each, \( od_1d_2 \ldots od_k \). \( A \)-tree is a tree whose nodes are labelled with elements of \( \Sigma \). \( A \) (full) \( k \)-ary \( \Sigma \)-tree is a \( k \)-ary tree \( t \) which is also a \( \Sigma \)-tree, which we consider as a mapping \( t : K^* \rightarrow \Sigma \) associating with each node \( \alpha \in K^* \) an element \( t(\alpha) \in \Sigma \). The empty word, \( \epsilon \), denotes the root of \( t \). Given a node \( \alpha \in K^* \) and a direction \( d \in K \), the concatenation of \( \alpha \) and \( d \), \( od \), denotes the \( d \)-successor of \( \alpha \). The level \( |\alpha| \) of a node \( \alpha \) is the length of \( \alpha \) as a word. We can thus think of the edges of \( t \) as being labelled with directions from \( K \), and of the nodes of each \( d \) as being labelled with letters from \( \Sigma \). A partial \( k \)-ary \( \Sigma \)-tree (over the set \( K \) of directions) is a \( \Sigma \)-tree with the property that a node may not have a \( d \)-successor for each direction \( d \); in other terms, a partial \( k \)-ary \( \Sigma \)-tree is a \( \Sigma \)-tree which is a prefix-closed \( \Sigma \)-tree.

MTALC\(_{D_2}\) is equipped with a Tarski-style possible worlds semantics. MTALC\(_{D_2}\) interpretations are spatio-temporal structures consisting of \( k \)-ary trees \( t \), representing \( k \)-immediate-successor branching time, together with an interpretation function associating with each primitive concept \( A \) the nodes of \( t \) at which \( A \) is true, and, additionally, associating with each concrete feature \( g \) and each node \( u \) of \( t \), the value at \( u \) (seen as a time instant) of the spatial concrete object referred to by \( g \).

**Definition 9 (interpretation)** Let \( x \) be an RCC8-like \( p \)-ary spatial RA and \( K = \{d_1, \ldots, d_k\} \) a set of \( k \) directions. An interpretation \( I \) of MTALC\(_{D_2}\) consists of a pair \( I = (t_I, I) \), where \( t_I \) is a \( k \)-ary tree and \( I \) is an interpretation function mapping each primitive concept \( A \) to a subset \( A_I \) of \( K^* \); each role \( R \) to a subset \( R_I \) of \( \{u, ud \in K^* \times K^* : d \in K\} \), so that \( R_I \) is functional if \( R \) is an abstract feature; and each concrete feature \( g \) to a total function \( g_I \) from \( K^* \) onto the set \( \Delta_{D_2} \) of (concrete) objects of the concrete domain \( D_2 \).

Given an MTALC\(_{D_2}\) interpretation \( I = (t_I, I) \), a feature chain \( u = f_1 \ldots f_m g \), and a node \( v_1 \), we denote by \( u^I(v_1) \) the value \( g_I^I(v_2) \), where \( v_2 \) is the \( f_1 \ldots f_m \)-successor of \( v_1 \); i.e., \( v_2 \) is so that there is a sequence \( v_1 = w_0, w_1, \ldots, w_n = v_2 \) verifying \( (w_i, w_{i+1}) \in f^I_1 \) for all \( i \in \{0, \ldots, n-1\} \) (in other words, \( v_2 = (f^I_1 \circ \ldots \circ f^I_m)(v_1) = f^I_n(\ldots( f^I_1(v_1)) \ldots) \).

**Definition 10 (satisfiability w.r.t. a TBox)** Let \( x \) be an RCC8-like \( p \)-ary spatial RA, \( K = \{d_1, \ldots, d_k\} \) a set of \( k \) directions, \( C \) an MTALC\(_{D_2}\) concept, \( T \) an MTALC\(_{D_2}\) weakly cyclic TBox, and \( I = (t_I, I) \) an MTALC\(_{D_2}\) interpretation. The satisfiability, by a node \( s \) of \( t_I \) w.r.t. \( T \), denoted \( I, s \models (C, T) \), is defined inductively as follows:

1. \( I, s \models (\top, T) \)
2. \( I, s \not\models (\perp, T) \)
3. For all primitive concepts \( A \):
   a. \( I, s \models (A, T) \) iff \( s \in A_I \)
   b. \( I, s \not\models (\neg A, T) \) iff \( s \not\in A_I \)
4. \( I, s \models (B, T) \) iff \( I, s \models (C, T), I, s \models (\neg B, T) \) iff \( I, s \models (\neg C, T), \) for all defined concepts \( B \) defined by the axiom \( B \models C \) of \( T \), such that \( B \) does not occur in \( C \), the right hand side of the axiom (in other words, such that \( \text{depths}(B, C) = 0 \)).
5. For all eventually defined concepts \( B \) defined by the axiom \( B \models C \lor \exists R.B, I, s \models (B, T) \) iff there exists \( s_0 = s, \ldots, s_i, \) with \( i \geq 0 \), such that:
   a. \( (s_j, s_{j+1}) \in R_I \), for all \( j \) such that \( 0 \leq j < i \); and
   b. \( I, s_i \models (C, T) \), for all \( j \) such that \( 0 \leq j < i \); and
   c. \( I, s_i \models (C, T) \)
6. For all eventually defined concepts \( B \) defined by the axiom \( B \models C \land \forall R.B, I, s \models (B, T) \) iff there exists \( s_0 = s, \ldots, s_i, \) with \( i \geq 0 \), such that:
   a. \( (s_j, s_{j+1}) \in R_I \), for all \( j \) such that \( 0 \leq j < i \);
   b. \( I, s_j \models (C, T) \), for all \( j \) such that \( 0 \leq j < i \); and
   c. \( I, s_i \models (C, T) \)
7. For all necessity defined concepts \( B \) defined by the axiom \( B \models C \land \forall R.B, I, s \models (B, T) \) iff
   a. \( I, s \models (C, T) \); and
   b. \( I, s' \models (B, T), \) for all \( s' \) such that \( (s, s') \in R_I \)
8. For all necessity defined concepts \( B \) defined by the axiom \( B \models C \lor \exists R.B, I, s \models (B, T) \) iff
   a. \( I, s \models (C, T) \); and
   b. \( I, s \models (C, T) \) or \( I, s' \models (B, T), \) for all \( s' \) such that \( (s, s') \in R_I \)
9. \( I, s \models (\neg (C \land D, T) \) iff \( I, s \models (C, T) \) and \( I, s \models (D, T) \)
10. \( I, s \models (C \lor D, T) \) iff \( I, s \models (C, T) \) or \( I, s \models (D, T) \)
11. \( I, s \models (\neg B_1, T) \) iff \( I, s \models (B_2, T), \) for all necessity or eventualy defined concepts \( B_1 \) whose dual is \( B_2 \)
12. \( I, s \models (\neg (C \land D), T) \) iff \( I, s \models (\neg C, T) \) or \( I, s \models (\neg D, T) \)
13. $I, s \models \langle -(C \sqcup D), T \rangle$ iff $I, s \models \langle -C, T \rangle$ and $I, s \models \langle -D, T \rangle$.

14. $I, s \models \langle \exists R.C, T \rangle$ iff $I, s' \models \langle C, T \rangle$, for some $s'$ such that $(s, s') \in R^I$.

15. $I, s \models \langle \forall R.C, T \rangle$ iff $I, s' \models \langle C, T \rangle$, for all $s'$ such that $(s, s') \in R^I$.

16. $I, s \models \langle -\exists R.C, T \rangle$ iff $I, s \models \langle \forall R, -C, T \rangle$.

17. $I, s \models \langle -\forall R.C, T \rangle$ iff $I, s \models \langle \exists R, -C, T \rangle$.

18. $I, s \models \langle \exists(u_1), \ldots, (u_p).P, T \rangle$ iff $p(u_1(s), \ldots, u_p(s))$.
$I, s \models \langle -\exists(u_1), \ldots, (u_p).P, T \rangle$ iff $\neg p(u_1(s), \ldots, u_p(s))$.

A concept $C$ is satisfiable w.r.t. a TBox $I$ iff $I, s \models \langle C, T \rangle$, for some MTALC($D_c$) interpretation $I$ and some state $s \in t_I$, in which case the pair $(I, s)$ is a model of $C$ w.r.t. $T$; $C$ is insatisfiable (has no models) w.r.t. $T$, otherwise. $C$ is valid w.r.t. $T$ iff the negation, $\neg C$, of $C$ is insatisfiable w.r.t. $T$.

The satisfiability of an MTALC($D_c$) concept w.r.t. a weakly cyclic TBox

Let $C$ be an MTALC($D_c$) concept and $T$ an MTALC($D_c$) weakly cyclic TBox. We define $T \oplus C$ as the TBox $T$ augmented with the axiom $B_{init} \models C$, $B_{init}$ being a fresh defined concept (not occurring in $T$):

$T \oplus C = T \cup \{B_{init} \models C\}$

In the sequel, we refer to $T \oplus C$ as the TBox $T$ augmented with $C$. The idea now is that, satisfiability of $C$ w.r.t. $T$ has (almost) been reduced to the emptiness problem of $T \oplus C$, seen as a weak alternating automaton on $k$-ary $\Sigma$-trees, for some labelling alphabet $\Sigma$ to be defined later, with the defined concepts as the states of the automaton, $B_i$ as the initial state of the automaton, the axioms as defining the transition function, with the accepting condition derived from those defined concepts that are not eventuality concepts, and with $k$ standing for the number of concepts of the form $\exists R.D$ in a certain closure, to be defined later, of $T \oplus C$.

The Disjunctive Normal Form

The notion of Disjunctive Normal Form (DNF) of a concept $C$ w.r.t. to a TBox $T$, $dfn1(C, T)$, is crucial for the rest of the paper. Such a form results, among other things, from the use of De Morgan’s Laws to decompose a concept so that, in the final form, the negation symbol outside the scope of a (existential or universal) quantifier occurs only in front of primitive concepts.

Given a (concrete) feature chain $u$, we define $Exists(u)$ as follows:

$Exists(u) = \begin{cases} \emptyset & \text{if } u \text{ reduces to a concrete feature,} \\ \{f_1, f_2, \ldots, (\exists f_m, T) \} & \text{otherwise (} u \text{ of the form } f_1 f_2 \cdots f_m) \end{cases}$

Definition 11 (first DNF) The first Disjunctive Normal Form ($dfn1$) of an MTALC($D_c$) concept $C$ w.r.t. an MTALC($D_c$) weakly cyclic TBox $T$, $dfn1(C, T)$, is defined inductively as follows:

1. For all primitive concepts $A$: $dfn1(A, T) = \{A\}$.
$dfn1(-A, T) = \{\neg A\}$.

2. For all acyclic defined concepts $B$: $dfn1(B, T) = dfn1(E, T), dfn1(-B, T) = dfn1(-E, T)$, where $E$ is the right hand side of the axiom $B \models E$ defining $B$.

3. For all eventuality defined concepts $B$ defined by the axiom $B \models C \sqcup \exists R.B$, $dfn1(B, T) = dfn1(C, T) \cup \{\exists R.B\}$.

4. For all necessity defined concepts $B$ defined by the axiom $B \models C \cap \forall R.B$, $dfn1(B, T) = \{\{dfn1(C, T) \cap \{\exists R.B\}\}\}$.

5. For all necessity defined concepts $B$ defined by the axiom $B \models C \cap (C \cap \forall R.B)$, $dfn1(B, T) = \{\{dfn1(C, T) \cap \{\exists R.B\}\}\}$.

6. For all necessity defined concepts $B$ defined by the axiom $B \models C \cap \forall R.B$, $dfn1(B, T) = \{\{dfn1(C, T) \cap \{\exists R.B\}\}\}$.

7. For all necessity defined concepts $B$ defined by the axiom $B \models C \cap \forall R.B$, $dfn1(B, T) = \{\{dfn1(C, T) \cap \{\exists R.B\}\}\}$.

8. For all necessity or eventuality defined concepts $B_i$ whose dual is the defined concept $B_2$, $dfn1(\neg B_1, T) = dfn1(B_2, T)$.

9. $dfn1(C \cap D, T) = \{\{dfn1(C, T) \cap \{\exists R.B\}\}\}$.

10. $dfn1(C \cup D, T) = dfn1(C, T) \cup dfn1(D, T)$.

11. $dfn1(\exists R.C, T) = \{\{\exists R.C\}\}$.

12. $dfn1(\forall R.C, T) = \{\{\forall R.C\}\}$.

13. $dfn1(\exists(u_1), \ldots, (u_p).P, T) = \{\{\exists(u_1), \ldots, (u_p).P\} \cup Exists(u_1), \ldots, (u_p)\}$.

14. $dfn1(\forall(u_1), \ldots, (u_p).P, T) = \{\{\forall(u_1), \ldots, (u_p).P\} \cup Exists(u_1), \ldots, (u_p)\}$.

15. $dfn1(\neg(C \cap D), T) = dfn1(-C, T) \cup dfn1(-D, T)$.

16. $dfn1(\neg(C \cup D), T) = \{\{dfn1(\neg C, T) \cap \neg dfn1(-D, T)\}\}$.

17. $dfn1(\neg\forall R.C, T) = \{\{\forall R.C\}\}$.

18. $dfn1(\neg\exists(u_1), \ldots, (u_p).P, T) = \{\{\exists(u_1), \ldots, (u_p).P\} \cup Exists(u_1), \ldots, (u_p)\}$.

where $\Pi$ is defined as follows:

1. $\Pi(\{S\}, \{T\}) = \{\emptyset \text{ if } A, \neg A \models S \cup T \text{ for some primitive concept } A,\
\{S \cup T\} \text{ otherwise}$.\n
2. $\Pi(\{S_1, \ldots, S_n\}, \{T_1, \ldots, T_m\}) = \bigcup_{i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}} \{\{S_i\}, \{T_j\}\}$.

Note that the $dfn1$ function checks satisfiability at the propositional level, in the sense that, given a concept $C$, $dfn1(C, T)$ is either empty, or is such that for all $S \models dfn1(C, T)$, $S$ does not contain both $A$ and $\neg A$, $A$ being a primitive concept. Furthermore, given a set $S \models dfn1(C, T)$, all elements of $S$ are concepts of either of the following forms: $A$ or $\neg A$, where $A$ is a primitive concept; $\forall R.D$; or $\exists(u_1), \ldots, (u_p).P$.

Definition 12 (the $pc\exists$-partition) Let $C$ be an MTALC($D_c$) concept, $T$ an MTALC($D_c$) TBox, $S \models dfn1(C, T)$ and $N_{df}^*$ the language of all finite words over the alphabet $N_{df}$. The $pc\exists$-partition of $S$, $pc\exists(S)$, is defined as $pc\exists(S) = \{S_{prop}, S_{exp}, S_{\exists}\}$, where:

$S_{prop} = \{A : A \models S \text{ and } A \text{ a \textbf{primitive} concept}\}$

$S_{exp} = \{\exists(u_1), \ldots, (u_p).P : \exists(u_1), \ldots, (u_p).P \models S\}$
and $S_3$ is computed as follows:

1. Initialise $S_3$ to the empty set: $S_3 = \emptyset$

2. For all $\exists R.C$ in $S$ with R general role: $S_3 = S_3 \cup \{\exists R.(C \cap C_1 \cap \cdots \cap C_k) : \{C_1, \ldots, C_k\} = \{D : \forall R.D \in S\}\}$

3. For all abstract features $f$ such that $S$ contains elements of the form $\exists f.C$: $S_3 = S_3 \cup \{\exists f.(C_1 \cap \cdots \cap C_k \cap D_1 \cap \cdots \cap D_l) : \{C_1, \ldots, C_k\} = \{E : \exists f.E \in S\} \text{ and } \{D_1, \ldots, D_l\} = \{E : \forall f.E \in S\}\}$

The second $dnf$ of a concept $C$ w.r.t. a TBox $T$, $dnf2(C, T)$, is now introduced. This consists of the $dnf1$ of $C$ w.r.t. $T$, $dnf1(C, T)$, as given by Definition 11 in which each element $S$ is replaced with $S' = S_{prop} \cup S_{exp} \cup S_3$. Formally:

**Definition 13 (second DNF)** Let $x \in \{RCC8, CDA, CYC\}$, $C$ be an MTALC($D_x$) concept, and $T$ an MTALC($D_x$) weakly cyclic TBox. The second Disjunctive Normal Form (dnf2) of $C$ w.r.t. $T$, $dnf2(C, T)$, is defined as $dnf2(C, T) = \{S' : S \in dnf1(C, T)\}$.

Given an MTALC($D_x$) concept $C$ and an MTALC($D_x$) TBox $T$, we can now use the second DNF, $dnf2$, to define the closure $(T \circ C)^*$ of $T \circ C$, the TBox $T$ augmented with $C$.

**Definition 14 (closure of $T \circ C$)** Let $C$ be an MTALC($D_x$) concept and $T$ an MTALC($D_x$) weakly cyclic TBox. The closure $(T \circ C)^*$ of $T \circ C$ is defined by the procedure of Figure 1 which also outputs a partial order $PO$ on the defined concepts of $(T \circ C)^*$.

**Remark 2** The axioms of $(T \circ C)^*$ are of the form $B = \{S_1, \ldots, S_m\}$; for all $S \in S_1, \ldots, S_m$, all elements of $S$ are of either of the following forms:

1. $A$ or $\neg A$, where $A$ is a primitive concept;
2. $\exists R.(B_1 \cap \cdots \cap B_k)$, $R$ being a general role or an abstract feature, and $B_j$ a defined concept, for all $j \in \{1, \ldots, k\}$;
3. $\exists (u_1) \cdots (u_p).P$.

We also need the closure of a concept $C$ w.r.t. a TBox $T$, $cl(C, T)$, which is defined as the union of the right-hand sides of the axioms in $(T \circ C)^*$. Formally:

**Definition 15 (closure of a concept w.r.t. a TBox)** The closure of an MTALC($D_x$) concept $C$ w.r.t. an MTALC($D_x$) TBox $T$, $cl(C, T)$, is defined as follows:

$$cl(C, T) = \bigcup_{B \in E \text{ axiom of } (T \circ C)^*} E$$

**Definition 16** Let $C$ be an MTALC($D_x$) concept and $T$ an MTALC($D_x$) TBox. We denote by:

1. $cFeatures(S)$, where $S \in cl(C, T)$, the set of concrete features of $S$:
   - in other words, $cFeatures(S)$ is the set of concrete features $g$ for which there exists a feature chain $u$ prefixed by $g$, such that $S$ contains a predicate concept $\exists (u_1) \cdots (u_p).P$, with $u \in \{u_1, \ldots, u_p\}$.

Figure 1: The closure $(T \circ C)^*$ of a weakly cyclic TBox $T$ augmented with a concept $C$, $T \circ C$; and a partial order $PO$ on the defined concepts of $(T \circ C)^*$.
2. \( \text{cFeatures}(C, T) = \bigcup_{S \in \text{cl}(C, T)} \text{cFeatures}(S) \), the set of concrete features of \( C \) w.r.t. \( T \);
3. \( \text{ncf}(C, T) = |\text{cFeatures}(C, T)| \), the number of concrete features of \( C \) w.r.t. \( T \);
4. \( \text{afFeatures}(C, T) = \{ f \in N_{aF} : \text{for some concept } E \text{ there exists } S \in \text{cl}(C, T) \text{ s.t. } \exists f.E \in S \} \), the set of abstract features of \( C \) w.r.t. \( T \);
5. \( \text{naf}(C, T) = |\text{afFeatures}(C, T)| \), the number of abstract features of \( C \) w.r.t. \( T \);
6. \( \text{pConcepts}(C, T) = \{ A : \exists S \in \text{cl}(C, T) \text{ s.t. } \{ A, \neg A \} \cap S_{\text{prop}} \neq \emptyset \} \), the set of primitive concepts of \( C \) w.r.t. \( T \);
7. \( \text{dConcepts}(C, T) = \) the set of defined concepts in \( (T \oplus C)^* \);
8. \( \text{rConcepts}(C, T) \), the set of relational existential (sub)concepts of \( C \) w.r.t. \( T \), is the set of all \( \exists R.D \) such that \( R \) is a general role and there exists an axiom \( B \models E \) in \( (T \oplus C)^* \) and \( S \in E \) so that \( \exists R.D \in S \);
9. \( \text{bf}(C, T) = \text{naf}(C, T) \), the functional branching factor of \( C \) w.r.t. \( T \);
10. \( \text{rbf}(C, T) = |\text{rConcepts}(C, T)| \), the relational branching factor of \( C \) w.r.t. \( T \);
11. \( \text{fbf}(C, T) = \text{bf}(C, T) + \text{rbf}(C, T) \), the branching factor of \( C \) w.r.t. \( T \).

We suppose that the relational existential concepts in \( \text{rConcepts}(C, T) \) are ordered, and refer to the \( i \)-th element of \( \text{rConcepts}(C, T) \), \( i = 1 \ldots \text{rbf}(C, T) \), as \( \text{rec}_i(C, T) \). Similarly, we suppose that the abstract features in \( \text{afFeatures}(C, T) \) are ordered, and refer to the \( i \)-th element of \( \text{afFeatures}(C, T) \), \( i = 1 \ldots \text{rbf}(C, T) \), as \( \text{af}_i(C, T) \). Together, they constitute the directions of the weak alternating automaton to be associated with the satisfiability of \( \text{MTALC}(D) \) w.r.t. \( T \).

**Definition 17 (branching tuple)** Let \( C \) be an \( \text{MTALC}(D) \) concept and \( T \) an \( \text{MTALC}(D) \) weakly cyclic TBox. The branching tuple of \( C \) w.r.t. \( T \) is given by the ordered \( \text{bf}(C, T) \)-tuple \( \text{bt}(C, T) = (\text{rec}_1(C, T), \ldots, \text{rec}_{\text{rbf}(C, T)}(C, T), \text{af}_1(C, T), \ldots, \text{af}_{\text{rbf}(C, T)}(C, T)) \) of the \( \text{rbf}(C, T) \) relational existential concepts in \( \text{rConcepts}(C, T) \) and the \( \text{fbf}(C, T) \) abstract features in \( \text{afFeatures}(C, T) \).

Given an \( \text{MTALC}(D) \) concept \( C \) and an \( \text{MTALC}(D) \) weakly cyclic TBox \( T \), we will be interested in \( k \)-ary \( \Sigma \)-trees (see Definition 3), \( t \), verifying the following:
1. \( k = \text{bf}(C, T) \); and
2. \( \Sigma = \text{pConcepts}(C, T) \times \Theta(\text{cFeatures}(C, T), \Delta_{D_C}) \), where \( \Theta(\text{cFeatures}(C, T), \Delta_{D_C}) \) is the set of total functions \( \theta : \text{cFeatures}(C, T) \rightarrow \Delta_{D_C} \) associating with each concrete feature \( g \) in \( \text{cFeatures}(C, T) \) a concrete value \( \theta(g) \) from the spatial concept \( \Delta_{D_C} \). This kind of trees will be seen as representing a class of interpretations of the satisfiability of \( C \) w.r.t. \( T \); the label \((X, \theta)\) of a node \( \alpha \in \{1, \ldots, \text{bf}(C, T)\}^* \) with \( X \subseteq \text{pConcepts}(C, T) \) and \( \theta \in \Theta(\text{cFeatures}(C, T), \Delta_{D_C}) \), is to be interpreted as follows:
1. \( X \) records the information on the primitive concepts that are true at \( \alpha \), in all interpretations of the class; and
2. \( \theta : \text{cFeatures}(C, T) \rightarrow \Delta_{D_C} \) records the values, at the abstract object represented by node \( \alpha \), of the concrete features \( g_1, \ldots, g_{\text{nbf}(C, T)} \) in \( \text{cFeatures}(C, T) \).

The crucial question is when we can say that an interpretation of the class is a model of \( C \) w.r.t. \( T \). To answer the question, we consider (weak) alternating automata on \( k \)-ary \( \Sigma \)-trees, with \( k = \text{bf}(C, T) \) and \( \Sigma = \text{pConcepts}(C, T) \times \Theta(\text{cFeatures}(C, T), \Delta_{D_C}) \). We then show how to associate such an automaton with the satisfiability of an \( \text{MTALC}(D) \) concept \( C \) w.r.t. a weakly cyclic TBox \( T \) in such a way that the models of \( C \) w.r.t. \( T \) coincide with the \( k \)-ary \( \Sigma \)-trees accepted by the automaton. The background on alternating automata has been adapted from [Muller, Saoudi, and Schupp 1992].
of constraints of the form \( P(u_1, \ldots, u_p) \) with \( P \) being an \( x \)
relation, \( u_1, \ldots, u_p \), \( K \)-\( N_{e,F} \)-chains (i.e., \( u_i \in \{1, \ldots, p\} \)),
is of the form \( g \) or \( d_i, \ldots, d_n, g \), \( n \geq 1 \) and \( n \) finite, the \( d_i \)'s
being directions in \( K \), and \( g \) a concrete feature).

**Definition 20 (Büchi alternating automaton)** Let \( k \geq 1 \) be an integer and \( K = \{d_1, \ldots, d_k\} \) a set of directions. An alternating automaton on \( k \)-ary \( \Sigma(x, N_p, N_{e,F}) \)-trees is a tuple \( A = (L(\operatorname{Lit}(N_p) \cup \operatorname{constr}(x, K, N_{e,F}) \cup K \times Q), \Sigma(x, N_p, N_{e,F}), \delta, q_0, F) \), where \( Q \) is a finite set of states; \( \Sigma(x, N_p, N_{e,F}) \) is the input alphabet (labelling the
nodes of the input trees); \( \delta : Q \to L(\operatorname{Lit}(N_p) \cup \operatorname{constr}(x, K, N_{e,F}) \cup K \times Q) \) is the transition function; \( q_0 \in Q \) is the initial state; and \( F \) is the set of accepting states. \( A \) is said to be a weak alternating automaton if there exists a partial order \( \preceq \) on \( Q \), so that the transition function \( \delta \) has the property that, given two states \( q, q' \in Q \), if \( q' \) occurs in \( \delta(q) \) then \( q \preceq q' \).

Let \( A \) be an alternating automaton on \( k \)-ary \( \Sigma(x, N_p, N_{e,F}) \)-trees, as defined in **Definition 20** and \( t \) a \( k \)-ary \( \Sigma(x, N_p, N_{e,F}) \)-tree. A run, \( r(A, t) \), of \( A \) on \( t \) is a partial \( k \)-ary \( \Sigma(2^{H_{\leq \infty}} \times N_p, x, K, N_{e,F}) \)
tree defined inductively as follows. For all directions \( d \in K \), and for all nodes \( u \in K^* \) of \( r(A, t) \), \( u \) has at most one outgoing edge labelled with \( d \), and leading to the \( d \)-successor \( ud \) of \( u \). The label \( (Y, L_u, X_u) \) of the root belongs to \( 2^{H_{\leq \infty}} \times c(\operatorname{Lit}(N_p)) \times 2^{\operatorname{constr}(x, K, N_{e,F})} \) —in other words, \( Y_u = \{q_0\} \). If \( u \) is a node of \( r(A, t) \) of level \( n \geq 0 \), with label \( (Y_u, L_u, X_u) \), then calculate \( e = \bigwedge_{i \in Y_u} \operatorname{dist}(h, \delta(L(h))) \), where \( \delta \) is a function associating with each pair \((h_1, e_1)\) of \( H_{\leq \infty} \times L(\operatorname{Lit}(N_p) \cup \operatorname{constr}(x, K, N_{e,F}) \cup K \times Q) \) an element of \( L(\operatorname{Lit}(N_p) \cup \operatorname{constr}(x, K, N_{e,F}) \cup H_{\leq \infty}) \) defined inductively in the following way:

\[
\operatorname{dist}(h_1, e_1) = \begin{cases} e_1 & \text{if } e_1 \in \operatorname{Lit}(N_p) \cup \operatorname{constr}(x, K, N_{e,F}), \\
q&q \in K \times Q, \\
\operatorname{dist}(h_1, e_2) \lor \operatorname{dist}(h_1, e_3) & \text{if } e_1 = e_2 \lor e_3, \\
\operatorname{dist}(h_1, e_2) \land \operatorname{dist}(h_1, e_3) & \text{if } e_1 = e_2 \land e_3. 
\end{cases}
\]

Write \( e \) in \( n \)-dnf as \( e = \bigvee_{i \in Y_u} (L_i \land X_i) \), where the \( L_i \)'s are conjunctions of literals from \( \operatorname{Lit}(N_p) \), the \( X_i \)'s are conjunctions of constraints from \( \operatorname{constr}(x, K, N_{e,F}) \), and the \( Y_i \)'s are conjunctions of \( n+1 \)-histories. Then there exists \( i = 1 \ldots r \) such that

1. \( L_u = \{t \in \operatorname{Lit}(N_p) : \ell \text{ occurs in } L_i\}; \\
2. \( X_u = \{x \in \operatorname{constr}(x, K, N_{e,F}) : x \text{ occurs in } X_i\}; \\
3. \text{for all } d \in K, \text{ such that the set } Y = \{h_dq \in H_{n+1} : (h \in H_{n+1}) \text{ and } (q \in Q) \} \text{ is nonempty, and only for those } d, \text{ has a } d \text{-successor, ud, whose label } (Y_{ud}, X_{ud}, L_{ud}) \text{ is such that } Y_{ud} = Y; \text{ and} \\
4. \text{the label } t(u) = (P_u, \theta_u) \in 2^{N_{e,F}} \times \Theta(N_{e,F}, \Delta_D) \text{ of the node } u \text{ of the input tree } t \text{ verifies the following, where, given a node } v \text{ in } t, \text{ the notation } \theta_v \text{ consists of the function } \theta_v : N_{e,F} \to \Delta_D \text{ which is the second argument of } t(v):}
\bullet \text{for all } A \in N_p : \text{ if } A \in L_u \text{ then } A \in P_u; \text{ and if } \neg A \in L_u \text{ then } A \notin P_u \text{ (the elements } A \text{ of } N_p \text{ such that, neither } A \text{ nor } \neg A \text{ occur in } L_u \text{ may or may not occur in } P_u); \\
\bullet \text{for all } P(d_1, \ldots, d_{n-1}, g_1, \ldots, d_p, \ldots, g_p) \text{ appearing in } X_u, \\
\text{if } P(\theta_{ud_1}, \ldots, \theta_{ud_{n-1}}, g_1, \ldots, \theta_{ud_p}, \ldots, \theta_{ud_p}) \text{ holds.} \\
\text{In other words, the values of the concrete features } g_i, \\
i \in \{1, \ldots, p\}, \text{ at the } d_i \text{-successors of } u \text{ in } t \text{ are related by the } x \text{ relation } P."

A partial \( k \)-ary \( \Sigma(2^{H_{\leq \infty}} \times N_p, x, K, N_{e,F}) \)-tree \( A \) is a run of \( A \) if there exists a \( k \)-ary \( \Sigma(x, N_p, N_{e,F}) \)-tree \( t \) such that \( \sigma \) is a run of \( A \) on \( t \).

**Definition 22 (CSP of a run)** Let \( A \) be an alternating automaton on \( k \)-ary \( \Sigma(x, N_p, N_{e,F}) \)-trees, as defined in **Definition 20** and \( \sigma \) a run of \( A 

1. For all nodes \( v \) of \( \sigma \), of label \( \sigma(v) = (Y_v, L_v, X_v) \in 2^{H_{\leq \infty}} \times c(\operatorname{Lit}(N_p)) \times 2^{\operatorname{constr}(x, K, N_{e,F})} \), the argument \( X_v \) gives rise to the CSP of \( \sigma \) at \( v \), \( \operatorname{CSP}_v(\sigma) \), whose set of variables, \( V_v(\sigma) \), and set of constraints, \( C_v(\sigma) \), are defined as follows:

(a) Initially, \( V_v(\sigma) = \emptyset \) and \( C_v(\sigma) = \emptyset \)
(b) for all $K^*N_F$-chains $d_1 \ldots d_n g$ appearing in $X_v$, create, and add to $V_v(\sigma)$, a variable $(v_{d_1} \ldots d_n, g)$

(c) for all $P(d_1, \ldots d_{n_1}, g, \ldots, d_{m}, g_p)$ in $X_v$, add the constraint

$$P((v_{d_1} \ldots d_{n_1}, g), \ldots, (v_{d_m} \ldots d_{m}, g_p)) \text{ to } C_v(\sigma)$$

2. The CSP of $\sigma$, $CSP(\sigma)$, is the CSP whose set of variables, $V(\sigma)$, and set of constraints, $C(\sigma)$, are defined as $V(\sigma) = \bigcup_v V_v(\sigma)$ and $C(\sigma) = \bigcup_v C_v(\sigma)$.

An $n$-branch of a run $\sigma = r(A, t)$ is a path of length (number of edges) $n$ beginning at the root of $\sigma$. A branch is an infinite path. If $u$ is the terminal node of an $n$-branch $\beta$, then the argument $Y_u$ of the label $(Y_u, L_u, X_u)$ of $u$ is a set of $n$-histories. Following (Muller, Saoudi, and Schupp 1992), we say that each $n$-history in $Y_u$ lies along $\beta$. An $n$-history $h$ lies along $\sigma$ if there exists an $n$-branch $\beta$ such that $h$ lies along $\beta$. An (infinite) history is a sequence $q_0d_1,q_1, \ldots, d_nq_n \in \{q_0\}^\omega$. Given such a history, $h = q_0d_1,q_1, \ldots, d_nq_n \in \{q_0\}^\omega$:

1. $h$ lies along a branch $\beta$ if, for every $n \geq 1$, the prefix of $h$ consisting of the $n$-history $q_0d_1,q_1, \ldots, d_nq_n$ lies along the $n$-branch $\beta_n$ consisting of the first $n$ edges of $\beta$;
2. $h$ lies along $\sigma$ if there exists a branch $\beta$ of $\sigma$ such that $h$ lies along $\beta$;
3. $Q$-proj$(h)$ (the $Q$-projection of $h$) is the infinite word $q_0g_1, \ldots, g_n \in Q^\omega$ such that, for all $n \geq 1$, the $n$-length prefix $q_0g_1, \ldots, g_n$ is the $Q$-projection of $h_n$, the $n$-history which is the $2n$-th prefix of $h$.

We denote by $Inf(h)$ the set of states appearing infinitely often in $Q$-proj$(h)$

The acceptance condition is now defined as follows. A history $h$ is accepting if $Inf(h) \cap F \neq \emptyset$. A branch $\beta$ of $r(A, t)$ is accepting if every history lying along $\beta$ is accepting.

The condition for a run $\sigma$ to be accepting splits into two subconditions. The first subcondition is the standard one, and is related to (the histories lying along) the branches of $\sigma$, all of which should be accepting. The second subcondition is new: the CSP of $\sigma$, $CSP(\sigma)$, should be consistent. A accepts a $k$-ary $\Sigma(x, N_F, N_{c,F})$-tree $t$ if there exists an accepting run of $A$ on $t$. The language $L(A)$ accepted by $A$ is the set of all $k$-ary $\Sigma(x, N_F, N_{c,F})$-trees accepted by $A$.

**Associating a weak alternating automaton with the satisfiability of a concept w.r.t. a weakly cyclic TBox**

Summarising the previous steps, especially the work of the procedure of Figure 1, we get the following corollary.

**Corollary 1** Let $x$ be a spatial RA of arity $p$, $C$ an $MTALC(D_x)$ concept, $T$ an $MTALC(D_x)\oplus T$ weakly cyclic TBox, $T \oplus C$ the TBox $T$ augmented with $C$, and $B_\oplus$ the initial defined concept of $T \oplus C$. $C$ is satisfiable w.r.t. $T$ iff the language $L(A_C, T)$ accepted by weak alternating automaton $A_C, T = (L(Lit(N_F)) \cup \text{constr}(x, K, N_{c,F}) \cup K \times Q), \Sigma(x, N_F, N_{c,F}), \delta, q_0, F)$ on $k$-ary $\Sigma(x, N_F, N_{c,F})$-trees is nonempty. The parameters of the automaton are as follows:

1. $N_F = p\text{Concepts}(C, T), N_{c,F} = c\text{Features}(C, T), Q = d\text{Concepts}(C, T), q_0 = B_1$
2. $K$ is the set of relational existential concepts and abstract features appearing as arguments in the branching tuple of $C$ w.r.t. $T$: $K = \{d_1, \ldots, d_n : (d_1, \ldots, d_n) = \text{bt}(C, T)\}$(Definition 12)
3. $\delta(B)$ is obtained from the axiom $B = E$ in $(T \oplus C)^* \text{defining } B$, as follows. $E$ is of the form $\{s_1, \ldots, s_n\}$, with $S = S_{prop} \cup S_{exp} \cup S_{\exists}$, for all $S \in \{s_1, \ldots, s_n\}$.

(a) We transform $E$ into $E' = \{\mu(s_1), \ldots, \mu(S)\}$, with $\mu(S), S \in \{s_1, \ldots, s_n\}$, computed as follows:

1. Let $Set_1 = \{\exists R.D, B_1 \land \cdots \land \exists R.D, B_1\} : R \text{ general role and } \exists R.D \in S_3 \land D = B_1 \sqcap \cdots \sqcap B_1 \cup \{f, B_1\} \land \cdots \land \{f, B_1\} : f \text{ abstract feature and } \exists f.D \in S_3 \land D = B_1 \sqcap \cdots \sqcap B_1$

2. Let $Set_{exp} = \{P(u_1, \ldots, u_p) : u_1, \ldots, u_p \in K^*N_F \text{ and } \exists(u_1) \ldots \exists(u_p), P \in S_{exp}\}$

3. Let $\mu(S) = S_{prop} \cup Set_{exp} \cup Set_1$

(b) We now have $\delta(B) = \bigcup_{S \in E} X$.

4. The set $F$ of accepting states is the set of defined concepts in $d\text{Concepts}(C, T)$ that are not equivalently defined concepts

5. Finally, the partial order $\geq$ on the states in $Q$ is as computed by the procedure of of Figure 1

**Conclusion and future work**

We have investigated a spatio-temporalisation $MTALC(D_x)$ of the ALC($D$) family of description logics with a concrete domain (Baader and Hanschke 1991), obtained by temporalising the roles, so that they consist of $m + n$ immediate-successor (accessibility) relations, the first $m$ being general, the other $n$ functional; and spatialising the concrete domain, which is generated by an RCC8-like qualitative spatial language (DA Randell and Cui 1992; Egenhofer 1991). We have shown the important result that satisfiability of an $MTALC(D_x)$ concept with respect to a weakly cyclic TBox can be reduced to the emptiness problem of a Büchi weak alternating automaton augmented with spatial constraints.

In another work, complementary to this one, also submitted to this conference, we thoroughly investigate Büchi automata augmented with spatial constraints, and provide, in particular, a translation of an alternating into a nondeterministic, and a nondeterministic doubly depth-first polynomial space algorithm for the emptiness problem of the latter. Together, the two works provide an effective solution to the satisfiability problem of an $MTALC(D_x)$ concept with respect to a weakly cyclic TBox.

A future work worth mentioning whether one can keep the same spatio-temporalisation and define a form of TBox cyclicity stronger than the one considered in this work, and
expressive enough to subsume the semantics of the well-known mu-calculus.

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