Routing in Unit Disk Graphs without Dynamic Headers

Wolfgang Mulzer* and Max Willert

Institut für Informatik, Freie Universität Berlin, Germany
{mulzer,willerma}@inf.fu-berlin.de

Abstract
Let $V \subset \mathbb{R}^2$ be a set of $n$ sites in the plane. The unit disk graph $DG(V)$ of $V$ is the graph with vertex set $V$ in which two sites $v$ and $w$ are adjacent if and only if their Euclidean distance is at most 1.

We develop a compact routing scheme $R$ for $DG(V)$. The routing scheme $R$ preprocesses $DG(V)$ by assigning a label $l(v)$ to every site $v$ in $V$. After that, for any two sites $s$ and $t$, the scheme $R$ must be able to route a packet from $s$ to $t$ as follows: given the label of a current vertex $r$ (initially, $r = s$) and the label of the target vertex $t$, the scheme determines a neighbor $r'$ of $r$. Then, the packet is forwarded to $r'$, and the process continues until the packet reaches its desired target $t$. The resulting path between the source $s$ and the target $t$ is called the routing path of $s$ and $t$. The stretch of the routing scheme is the maximum ratio of the total Euclidean length of the routing path and of the shortest path in $DG(V)$, between any two sites $s, t \in V$.

We show that for any given $\varepsilon > 0$, we can construct a routing scheme for $DG(V)$ with diameter $D$ that achieves stretch $1 + \varepsilon$ and label size $O(\log D \log^3 n / \log \log n)$ (the constant in the $O$-Notation depends on $\varepsilon$). In the past, several routing schemes for unit disk graphs have been proposed. Our scheme is the first one to achieve poly-logarithmic label size and arbitrarily small stretch without storing any additional information in the packet.

1 Introduction

The routing problem is a well-known problem in distributed graph algorithms [13,17]. We are given a graph $G$ and want to preprocess it by assigning labels to each node of $G$ such that the following task can be solved: a data packet is located at a source node and has to be routed to a target node. A routing scheme should have several properties. First, routing must be local: a node can only use the label of the target node as well as its own local information to compute a neighbor to which the packet is sent next. Second, the routing should be efficient: the ratio of the routed path and the shortest path — the stretch factor — should be close to 1. Finally, the routing scheme should be compact: the size of the labels (in bits) must be small.

In the literature, one can find many different techniques and models for routing. A common tool is the use of routing tables. A routing table is a sequence of bits stored in a node. Typically, routing tables contain more information about the topology of the graph and are different from labels. In this article, we do not use routing tables, but store all the information in the labels. Moreover, many routing schemes use additional headers. The header contains mutable information and is stored in the data packet. Thus, the header moves with the data packet through the graph. The usage of an additional header makes it possible to implement recursive routing strategies or to remember information from past positions of the packet.

Furthermore, the literature distinguishes two types of input models. In the fixed-port model, the given graph already has a complete list of ports for each node $v$, i.e., a fixed numbering of the neighbors of $v$ used to identify the next hop of the packet. In particular, it is not possible to renumber the ports. In

*Supported by ERC StG 757609.
contrast, the designer-port model allows us to assign arbitrary port numbers during the preprocessing, see [10,11,22]. Below, we will briefly discuss the advantages and disadvantages of these two models.

A trivial solution to solve the routing problem is to store the complete shortest path tree in every label. Then it is easy to route the data packets along a shortest path. However, such a routing scheme is not compact. Moreover, Peleg and Upfal [17] proved that in general graphs, any routing scheme that achieves a constant stretch factor must store a polynomial number of bits for each node.

Nevertheless, there is a rich collection of routing schemes for general graphs [13,5,6,9,18,19]. For example, the scheme by Roditty and Tov [19] uses labels of size $mnO(1/\sqrt{\log n})$ and routes a packet from $s$ to $t$ on a path of length $O(k\Delta + m^{1/k})$, where $\Delta$ is the shortest path distance between $s$ and $t$, $k > 2$ is any fixed integer, $n$ is the number of nodes, and $m$ is the number of edges. Their routing scheme needs headers of poly-logarithmic size.

The lower bound result by Peleg and Upfal [17] shows that it is hopeless to find efficient routing schemes for general graphs that are compact as well, meaning that at most a poly-logarithmic number of bits in the labels/tables are necessary. Thus, it is natural to investigate special interesting graph classes.

Our graph class of interest comes from the study of mobile and wireless networks. These networks are usually modeled as unit disk graphs [7]. Nodes in this network are points in the plane and two nodes are connected if their distance is at most one. This is equivalent to a disk intersection graph in which all disks have diameter one. For unit disk graphs there are known routing schemes. The first routing scheme is by Kaplan et al. [14] and uses the fixed-port model. They present a routing scheme with stretch $1 + \varepsilon$ and routing table size $O(\log^2 n \log^2 D)$, where $D$ is the diameter of the given unit disk graph. Their routing is recursive and needs an additional header of size $O(\log n \log D)$. The second routing scheme is due to Yan, Xiang, and Dragan [24]. They present a routing scheme with label size $O(\log^3 n)$ and show that a data packet routes along a path of length at most $5\Delta + 13$, where $\Delta$ is the length of the optimal path. The designer-port model is used.

Here, we present the first compact routing scheme that is headerless and achieves stretch $1 + \varepsilon$. We obtain label size $O(\log D \log^3 n / \log \log n)$.\footnote{The constant in the $O$-Notation depends on $\varepsilon$.} We use the fixed-port model. In the conclusion, we will discuss how our scheme compares to the other schemes.

2 Preliminaries

We explain our graph theoretic notation and discuss how the routing scheme can access the input graph. Then, we provide a precise definition of our notion of a routing scheme and give some background on unit disk graphs.

We are given a simple and undirected graph $G = (V, E)$ with $n$ vertices. The edges are weighted by a non-negative weight function $w: E \to \mathbb{R}_+$. We write $d_G(s, t)$ for the (weighted) shortest path distance between the vertices $s, t \in V$ and we omit the subscript $G$ if it follows from the context. Throughout the whole article we assume that the graph is connected.

Graph Access Model. Let $\Sigma = \{0, 1\}$, and $[m] = \{0, 1, \ldots, m\}$, for $m \in \mathbb{N}$. We explain how the routing scheme may access the input graph $G = (V, E)$. Every vertex $v \in V$ has an identifier $v_d \in \Sigma^+$ of length $|v_d| = \lfloor \log n \rfloor$. We use the fixed-port model [10,11,22]. In this model the port numbers are assigned arbitrarily. The neighbors of a vertex $v \in V$ are accessed through ports. More precisely, there is a partial function node: $V \times [n - 1] \to V$, that assigns to every vertex $v \in V$ and to every port number $p \in [n - 1]$ the neighbor $w = \text{node}(v, p)$ that can be reached through the port $p$ at vertex $v$. For simplicity,
we set \( \text{node}(v, 0) = v \), for all \( v \in V \). In our algorithms, we use broadcast functions \( \beta_v : \Sigma^+ \to [n] \), for every vertex \( v \in V \). It is defined as follows:

\[
\beta_v(w_{id}) = \begin{cases} 
p, & \text{if node}(v, p) = w, \text{ and} \\
n, & \text{otherwise.}
\end{cases}
\]

The broadcast functions can be implemented with the node functions as follows: ask all neighbors of a node \( v \) whether they have the identifier \( w_{id} \). If there is one, then this neighbor will answer on the corresponding port \( p \). Otherwise, we output \( n \).

Other authors also use the designer-port model \([10, 11, 22, 24]\). In this model, the routing scheme can determine the assignment of port numbers to the incident edges of each vertex \( v \in V \) during the preprocessing phase. This additional power in the model can lead to more efficient routing schemes \([10, 11, 22, 24]\). However, a routing scheme that uses the designer-port cannot easily be used as a building block for more complicated routing schemes, since additional lookup tables become necessary in order to store the assignments of the port numbers.

**Routing Schemes.** Let \( G \) be a graph class. A routing scheme \( R \) for \( G \) consists of a family of labeling functions \( \ell_G : V(G) \to \Sigma^+ \), for each \( G \in G \). The labeling function \( \ell_G \) assigns a bit string \( \ell_G(v) \) to every node \( v \) of \( G \). The label \( \ell_G(v) \) serves as the address of the node \( v \) in \( G \). In contrast to the identifier of a node, the label usually contains the identifier, but some more information about the topology of the graph \( G \). While the identifier is given as fixed input, the label is chosen by the routing scheme during the preprocessing. As before, we omit the index \( G \) if the context is clear. Furthermore, \( R \) has a routing function \( \sigma : \Sigma^+ \times \Sigma^+ \times \mathbb{N}^{2^r} \to \mathbb{N} \). The routing function \( \sigma \) describes the behavior of the routing scheme, as follows: assume a data packet is located at a vertex \( s \in V \) and must be routed to a destination \( t \in V \). Then, \( \sigma(\ell(s), \ell(t), \beta_s) \) has to compute a port \( p \) so that the next hop of the data packet is from \( s \) to \( \text{node}(s, p) \). Now, let \( v_0 = s \) and \( v_{i+1} = \text{node}(v_i, \sigma(\ell(v_i), \ell(t), \beta_{v_i})) \), for \( i \geq 0 \). The sequence \((v_i)_{i \in \mathbb{N}}\) is called routing sequence. The routing scheme \( R \) is correct, for \( G \in G \), if and only if for all distinct \( s, t \in V(G) \), there is a number \( m(s, t) \in \mathbb{N} \) such that \( v_j = t \), for all \( j \geq m(s, t) \), for all \( j = 0, \ldots, m(s, t) - 1 \). If \( R \) is correct for \( G = (V, E) \), then \( \delta_G(s, t) = \sum_{i=1}^{m(s, t)} w(v_i-1, v_i) \) is called the routing length between \( s \) and \( t \) (in \( G \)). The stretch of the routing scheme is the largest ratio \( \delta_G(s, t)/d_G(s, t) \) over all distinct vertices \( s, t \in V \). The goal is to achieve a routing scheme that minimizes the stretch factor as well as the number of bits stored in the labels. Many routing schemes use additional headers during the routing. These headers as well as the target labels are stored in the data packet. In contrast to the target label, the header might change while the packet is routed through the graph. This gives additional power and makes it possible to develop recursive routing schemes. However, we will not make use of this technology.

**Unit Disk Graphs.** Our graph class of interest are the unit disk graphs. Let \( V \subset \mathbb{R}^2 \) be a set of \( n \) points in the Euclidean plane. The unit disk graph \( DG(V) \) of \( V \) has vertex set \( V \) and an edge between two vertices \( v, w \in V \) if and only if the Euclidean distance \( |vw| \) is at most 1, see Figure 1. The weight of the edge \( vw \) is \( |vw| \). Throughout, we will assume that \( DG(V) \) is connected, and we will use \( D \) to denote the diameter \( \max_{u, v \in V} d(u, v) \) of \( DG(V) \). Clearly, we have \( D \leq n - 1 \).

3 Building Blocks

In this section, we describe the building blocks for our routing scheme. For this, we review some simple routing schemes from the literature, and we show how to obtain a new routing scheme for unit disk graphs that achieves an additive stretch. This later scheme is based on the data structure of Chan and Skrepetos.

3.1 Simple Routing Schemes

The first routing scheme is for trees. There are many different such schemes, based on similar ideas. We would like to point out that some of these routing schemes can achieve label size \( O(\log n) \), see \([10, 22]\).
There is a routing scheme with label size two sets with label size two.

Lemma 3.3. The idea of the routing scheme is illustrated in Figure 2. First, we use Lemma 3.2 to find

Proof. are adjacent, since

If not, we use the bit

Using

β

v

and

ℓ

T

rooted at

Z

factor

1 + 64

Lemma 3.2. Let

DG(

ε>

T

be an n-vertex tree with arbitrary edge weights. There is a routing scheme for

T

with label size

O(\log^2 n / \log \log n)

whose routing function

σ_{tree}

sends a data packet along a shortest path, for any pair of vertices.

The second routing scheme is efficient for unit disk graphs with small diameter. The idea of the scheme was first described by Kaplan et al. [14]. They use the following lemma, which is based on a method by Gao and Zhang [12].

Lemma 3.2. Let

ε > 0

and

DG(V)

be an n-vertex unit disk graph with diameter

D. We can compute two sets

R ⊆ Z ⊆ V

with the following properties:

(i) |R| ∈ \(O(D\epsilon^{-2})\) and |Z| ∈ \(O(D\epsilon^{-3})\);

(ii) for every vertex

v ∈ V,

there is a cluster vertex

v′ \in R

with

d(v, v′) ≤ \epsilon;

and

(iii) for every

s, t \in R,

we have

dZ(s, t) ≤ (1 + 12\epsilon)d(s, t) + 12\epsilon,

where

dZ(s, t)

denotes the shortest path distance between

s

and

t

in

DG(Z).

Lemma 3.3. Let

DG(V)

be an n-vertex unit disk graph with diameter

D. Furthermore, let

0 < \epsilon ≤ 1.

There is a routing scheme with label size

O(\epsilon^{-3}D\log n)

whose routing function

σ_{diam}

achieves stretch factor

1 + 64\epsilon.

Proof. The idea of the routing scheme is illustrated in Figure 2. First, we use Lemma 3.2 to find

R

and

Z. Next, let

z \in R.

We use Dijkstra’s algorithm to compute a shortest path tree

T_z

of the vertices

Z

rooted at

z. We store the concatenation of

z_{id}

and the pairs

(u_{id}, v_{id})

in the label

ℓ(z),

for all edges

uv

of

T_z. Next, let

v \in V \setminus R.

We pick an arbitrary cluster vertex

v′ \in R

with

d(v, v′) ≤ \epsilon.

We store

v_{id}

and

ℓ(v′)

in the label of

v. Finally, for every

v \in V

we store a bit

b(v)

in

ℓ(v)

that is true if and only if

v \in R.

Since |Z| ∈ \(O(D\epsilon^{-3})\) by Lemma 3.2, we immediately get

|ℓ(v)| \in \(O(\epsilon^{-3}D\log n))\).

The routing function

σ_{diam}

now works as follows: we are given

ℓ(s), \ell(t)

and the broadcast-function

β_s.

Using

β_s,

we check whether

s

and

t

are adjacent. If so, we use the port

β_s(ℓ_{id})

to route the data packet. If not, we use the bit

b(v)

to check whether

s

is a cluster vertex in

R.

If

s

is not a cluster vertex,

s

and

s′

are adjacent, since

\epsilon ≤ 1.

We extract

s′_{id}

from

ℓ(s′)

and route the data packet via the port

β_s(s′_{id})

to

s′.

\footnote{In fact, there is a lower bound that shows that label size

O(\log n)

cannot be achieved in the fixed-port model [11].}
The last two inequalities hold because $|\sigma_d| = d(s,t)$. Otherwise, let $d(s,t) > 1$ and let $s'$ and $t'$ be their clusters ($s = s'$ and $t = t'$ is possible). Observe that $d(s',t') = d_Z(s',t')$. Hence, we can use Lemma 3.2 to derive
\[
\delta(s,t) = |ss'| + \delta(s',t') + |t't| \leq \varepsilon + d_Z(s',t') + \varepsilon \leq (1 + 12\varepsilon)d(s',t') + 14\varepsilon \\
\leq (1 + 12\varepsilon)(d(s,t) + 2\varepsilon) + 14\varepsilon = d(s,t) + 12\varepsilon d(s,t) + 16\varepsilon + 24\varepsilon^2 \\
\leq (1 + 28\varepsilon + 24\varepsilon^2)d(s,t) \leq (1 + 52\varepsilon)d(s,t).
\]
The last two inequalities hold because $d(s,t) > 1$ and $\varepsilon \leq 1$. Hence, $\delta(s,t) \leq (1 + 2^6 \cdot \varepsilon)d(s,t)$.

### 3.2 The Distance Oracle of Chan and Skrepetos

Our routing scheme is based on the recent approximate distance oracle for unit disk graphs by Chan and Skrepetos [5]: we are given a set $V \subset \mathbb{R}^2$ of $n$ points in the plane and a parameter $\varepsilon \geq D^{-1}$, where $D$ is the diameter of $DG(V)$. Chan and Skrepetos show how to compute in $O((1/\varepsilon)^4 n \log n)$ time a data structure of size $O((1/\varepsilon)n \log n)$ that can answer approximate distance queries in $DG(V)$ in $O((1/\varepsilon) \log n)$ time: given two vertices $s, t \in V$, compute a number $\theta \in \mathbb{R}$ with $d(s,t) \leq \theta \leq d(s,t) + O(\varepsilon D)$. The main tool for this data structure is a suitable hierarchical decomposition of $DG(V)$. More precisely, Chan and Skrepetos show that given $V$, one can compute in $O(n \log n + (1/\varepsilon) n)$ time a decomposition tree $T$ for $DG(V)$ with the following properties:

- Every node $\mu$ of $T$ is assigned two sets: $\text{port}(\mu) \subseteq V(\mu) \subseteq V$. The subgraph of $DG(V)$ induced by $V(\mu)$ is connected and the vertices in $\text{port}(\mu)$ are called portals.
- If $\mu$ is the root, then $V(\mu) = V$.
- If $\mu$ is an inner node with $k$ children $\sigma_1, \ldots, \sigma_k$, the sets $\text{port}(\mu), V(\sigma_1), \ldots, V(\sigma_k)$ are pairwise disjoint, and we have $V(\sigma_i) \subseteq V(\mu)$, for $1 \leq i \leq k$.
- If $\mu$ is a leaf, then $V(\mu) = \text{port}(\mu)$.

\[\text{for notational ease, we restrict the explicit constants in our stretch bounds to powers of two.}\]

\[\text{The reader familiar with the work of Chan and Skrepetos may notice that we have slightly extended the notion of portals: while Chan and Skrepetos define portals only for inner nodes, we also define portals for the leaves. This does not change the essence of the decomposition, but makes the presentation more unified.}\]
The height of $\mathcal{T}$ is in $O(\log n)$, and for every node $\mu$ of $\mathcal{T}$, we have $|\text{port}(\mu)| \in O(1/\varepsilon)$.

To state the final (and most important) property of $\mathcal{T}$, we first need to introduce some additional notation. The properties of $\mathcal{T}$ so far imply that the portal sets of two different nodes in $\mathcal{T}$ are disjoint. For every portal $p$, we let $\mu(p)$ be the unique node in $\mathcal{T}$ with $p \in \text{port}(\mu(p))$. Moreover, let $\mu$ be a node of $\mathcal{T}$ and $s, t \in V(\mu)$. We denote by $d_\mu(s, t)$ the shortest path distance between $s$ and $t$ in the subgraph of $\text{DG}(V)$ induced by $V(\mu)$. Now, the decomposition tree of Chan and Skrepetos has the property that for every pair of vertices $s, t \in V$, if we set

$$\theta(s, t) = \min_{p, t \in V(\mu(p))} d_{\mu(p)}(s, p) + d_{\mu(p)}(p, t)$$

then

$$\theta(s, t) \leq d(s, t) + O(\varepsilon D). \quad (1)$$

3.3 A Routing Scheme with Additive Stretch

In the last section we presented a routing scheme that is efficient for unit disk graphs with low diameter. In this section we present a routing scheme that is efficient for unit disk graphs with large diameter. Let $\text{DG}(V)$ be an $n$-vertex unit disk graph with diameter $D$, and let $\varepsilon > D^{-1}$. First, we set $c = n \cdot (\varepsilon D)^{-1}$ and define $x_c = \lfloor x \cdot c \rfloor$, for each $x \in \mathbb{R}_+$. Next, we compute the decomposition tree $\mathcal{T}$, as explained in Section 3.2.

First, we describe the labels of the routing scheme. Let $v \in V$, and let $p$ be a portal with $v \in V(\mu(p))$. We compute the shortest path tree $T_p$ of $\text{DG}(V(\mu(p)))$ rooted at $p$ and enumerate its vertices in postorder. The postorder number of $v$ in $T_p$ is denoted by $r_p(v)$. Next, the subtree of $T_p$ rooted at $v$ is called $T_p(v)$ and we use $l_p(v)$ to denote the smallest postorder number in $T_p(v)$. Thus, since we enumerated the vertices in postorder, a vertex $w \in V(\mu(p))$ is in the subtree $T_p(v)$ if and only if $r_p(w) \in [l_p(v), r_p(v)]$. Finally, we apply the tree routing from Lemma 3.1 to $T_p$ and denote by $\ell_p(v)$ the corresponding label of $v$. We store $(p_{\text{id}}, d_{\mu(p)}(v, c), l_p(v), r_p(v), \ell_p(v))$ in $\ell(v)$ and get the following lemma.

**Lemma 3.4.** For every vertex $v \in V$, we have $|\ell(v)| \in O \left( \frac{\log^3 n}{\varepsilon \log \log n} \right)$.

**Proof.** Since $\mathcal{T}$ has height $O(\log n)$, we know that $v$ is in $O(\log n)$ different sets $V(\mu)$. Moreover, for every node $\mu$, there are at most $O(1/\varepsilon)$ portals. Thus, the label of $v$ contains $O(\varepsilon^{-1} \log n)$ different entries. The value $d_{\mu(p)}(v, c)$ is a natural number, and since $c \leq n$, we have

$$d_{\mu(p)}(v, c) = \lfloor d_{\mu(p)}(v, c) \cdot c \rfloor \leq n^2.$$ 

Thus, we need $O(\log n)$ bits for the number $d_{\mu(p)}(v, c)$. Moreover, the identifier $p_{\text{id}}$ as well as the postorder numbers stored in one entry only need $O(\log n)$ bits. Finally, we apply Lemma 3.1 to conclude that one entry of the routing label has size $O(\log^2 n / \log \log n)$. The claim follows. \qed

Next, we describe the routing function. We are given the labels $\ell(s)$ and $\ell(t)$ for the current vertex $s$ and the target vertex $t$. First, we identify all portals $p$ with $s, t \in V(\mu(p))$. We can do this by identifying all vertices $p$ such that the entry $(p_{\text{id}}, d_{\mu(p)}(v, c), l_p(s), r_p(s), \ell_p(s))$ is in $\ell(s)$ and the entry $(p_{\text{id}}, d_{\mu(p)}(t, c), l_p(t), r_p(t), \ell_p(t))$ is in $\ell(t)$. Next, let $\theta(s, t; p) = d_{\mu(p)}(t, p) + d_{\mu(p)}(p, s)$, if $t$ is not in the subtree $T_p(s)$, and $\theta(s, t; p) = d_{\mu(p)}(t, p) - d_{\mu(p)}(p, s)$, otherwise; see Figure 3 for an illustration of the two cases. Let $p_{\text{opt}}$ be the portal that minimizes $\theta(s, t; p)$ among all portals $p$. Then, it is easy to see, that $\theta(s, t; p_{\text{opt}}) \leq \theta(s, t)$ (recall from Section 3.2 that $\theta(\cdot, \cdot)$ denotes the result of the distance oracle by Chan and Skrepetos). Hence, $\theta(s, t; p_{\text{opt}})$ is a good approximation for the distance between $s$ and $t$. However, the routing function cannot compute the optimal portal $p_{\text{opt}}$, since we do not have direct access to the real value $d_{\mu(p)}(s, p_{\text{opt}})$. Instead, we use the values $d_{\mu(p)}(\cdot, p)$, to compute a near-optimal portal. We define $\theta_c(s, t; p) = d_{\mu(p)}(t, p_c) + d_{\mu(p)}(p, s_c)$, if $t$ is not in the subtree $T_p(s)$, and $\theta_c(s, t; p) = d_{\mu(p)}(t, p_c) - d_{\mu(p)}(p, s_c)$, otherwise. Let $p_0$ be the portal that lexicographically minimizes
(θ_c(s, t; p), p_0), among all portals p. We call p_0 the s-t-portal and set θ_c(s, t) = θ_c(s, t; p_0). Observe that the s-t-portal can be computed by using only the labels of s and t. The routing function now uses the labels ℓ_{p_0}(s) and ℓ_{p_0}(t) as well as the broadcast function β_s to compute the next vertex in T_{p_0} and forwards the data packet to this vertex.

Finally, we have to show that the routing scheme is correct and routes along a short (not necessarily shortest) path. For this, we first show that the routing process terminates.

**Lemma 3.5.** Let s be the current vertex, t the target vertex, and suppose that the routing scheme sends the packet from s to v. Moreover, let p_0 be the s-t-portal. Then, p_0 is a possible candidate for the v-t-portal, and we have θ_c(s, t; p_0) ≥ θ_c(v, t; p_0) + |sv|_c.

**Proof.** First, let μ = μ(p_0). Since sv is an edge of the shortest path tree T_{p_0}, it follows that v ∈ V(μ(p_0)). This gives the first part of the claim. For the second part, we distinguish two cases:

**Case 1:** t ∈ T_{p_0}(s). In this case, we have t ∈ T_{p_0}(v), and thus θ_c(v, t; p_0) = d_μ(t, p_0)_c - d_μ(p_0, v)_c. Moreover, we have
\[ d_μ(p_0, v)_c = |d_μ(p_0, v) · c| = |d_μ(p_0, s) · c + |sv| · c| ≥ |d_μ(p_0, s) · c| + |sv| · c = |d_μ(p_0, s) · c| + |sv|_c, \]

since s is on the path in T_{p_0} from p_0 to v. Hence,
\[ θ_c(s, t; p_0) = d_μ(t, p_0)_c - d_μ(p_0, s)_c ≥ d_μ(t, p_0)_c - d_μ(p_0, v)_c + |sv|_c = θ_c(v, t; p_0) + |sv|_c, \]
as desired.

**Case 2:** t ∉ T_{p_0}(s). Similarly to the first case, we have d_μ(p_0, s)_c ≥ d_μ(p_0, v)_c + |sv|_c and θ_c(v, t; p_0) ≤ d_μ(t, p_0)_c + d_μ(p_0, v)_c. Thus, we get
\[ θ_c(s, t; p_0) = d_μ(t, p_0)_c + d_μ(p_0, s)_c ≥ d_μ(t, p_0)_c + d_μ(p_0, v)_c + |sv|_c ≥ θ_c(v, t; p_0) + |sv|_c, \]
and the claim follows.

**Corollary 3.6.** Let s, t, and v be as in Lemma 3.5. Then, θ_c(s, t) ≥ θ_c(v, t) + |sv|_c.

**Proof.** Let p_0 be the s-t-portal. From Lemma 3.5, we get
\[ θ_c(s, t) = θ_c(s, t; p_0) ≥ θ_c(v, t; p_0) + |sv|_c ≥ θ_c(v, t) + |sv|_c. \]
The claim follows.
Lemma 3.7. Let $s$, $t$ and $v$ be as in Lemma 3.5. Let $p$ be the $s$-$t$-portal and $q$ be the $v$-$t$-portal. Then, if $\theta_c(s,t) = \theta_c(v,t)$, it follows that $p_{id} \geq q_{id}$.

Proof. From Lemma 3.5 we have

$$\theta_c(v,t; q) = \theta_c(v,t) = \theta_c(s,t) = \theta_c(s,t;p) \geq \theta_c(v,t;p) + |sv|_c \geq \theta_c(v,t;p) \geq \theta(v,t;q).$$

Hence, $\theta_c(v,t;p) = \theta_c(v,t;q)$. Furthermore, by construction, we have $(\theta_c(v,t;p), p_{id}) \geq (\theta_c(v,t;q), q_{id})$. Thus, the claim follows.

Lemma 3.8. The routing scheme is correct.

Proof. First, we show that the data packet eventually arrives at the target vertex $v$. Let $T_v$ be the progress towards $v$ that leads from $s$ towards $t$, and $T_q$ is an edge of $T_p$ that leads from $s$ to $v$. Now, since the triples $(\theta_c(s,t), p_{id}, h_p(s,t))$ lie in $\mathbb{N}^3$ and since $(0,0,0)$ is a global minimum, it follows that the data packet eventually arrives at the target vertex $v$.

Lemma 3.9. For any two vertices $s$ and $t$, we have $\delta(s,t) \leq d(s,t) + O(\epsilon D)$.

Proof. First, we show that $\theta_c(s,t) \leq c \cdot \theta(s,t) + 1$: let $p_0$ be the $s$-$t$-portal, and let $p_{opt}$ be the portal minimizing $\theta(s,t; \cdot)$ among all portals. Let $\mu = \mu_{opt}$. We obtain

$\theta_c(s,t) = \theta_c(s,t; p_0) \leq \theta_c(s,t; p_{opt}) = \lfloor c \cdot d_p(t, p_{opt}) \rfloor \leq \lfloor c \cdot d_p(t, p_{opt}) + 1 \rfloor \leq c \cdot \theta(s,t) + 1$

where the $\pm$-operator is used to cover the two possible cases in the definition of $\theta_c$, and because $|a| + |b| \leq |a + b|$ and $|a| - |b| \leq |a - b| + 1$, for all $a, b \geq 0$. By Lemma 3.8 we know that the routing terminates. Let $\pi : s = w_0, \ldots, w_m = t$ be the routing path. From Corollary 3.6 we get $\lfloor \frac{c}{c} \rfloor \leq \theta_c(w_i, t) - \theta_c(w_{i+1}, t)$, and thus

$$\delta(s,t) = \sum_{i=0}^{m-1} |w_i w_{i+1}| \leq \sum_{i=0}^{m-1} \frac{|w_i w_{i+1}|}{c} = \frac{m}{c} + \frac{1}{c} \sum_{i=0}^{m-1} |w_i w_{i+1}| = \frac{m}{c} + \frac{\theta_c(s,t)}{c}$$

Now, using Equation (1) from Section 3.2, the choice of $c = n \cdot (\epsilon D)^{-1}$, and the fact that $m \leq n - 1$, we get

$$\frac{m}{c} + \theta(s,t) \leq \frac{n}{n \cdot (\epsilon D)^{-1}} + d(s,t) + O(\epsilon D) = d(s,t) + O(\epsilon D),$$

as claimed.

We can now conclude with our first theorem.

Theorem 3.10. Let $DG(V)$ be an $n$-vertex unit disk graph with diameter $D$. Furthermore, let $\epsilon > D^{-1}$. There is a routing scheme with label size $O(e^{-1} \log^3 n / \log \log n)$ whose routing function $\sigma_{add}$ routes any data packet on a path with additive stretch $O(\epsilon D)$. 

8
4 A Routing Scheme with Stretch $1 + \varepsilon$

Let DG(V) be an $n$-vertex unit disk graph with diameter $D$, and let $\varepsilon > 0$. Furthermore, without loss of generality, we can assume that $\varepsilon \leq 1$. For our routing scheme, we need the following two ingredients from the literature.

**Planar spanners.** Let $c \geq 1$. A $c$-spanner for DG(V) is a subgraph $H$ of DG(V) with vertex set $V$ such that for any $s, t \in V$, we have $d_H(s, t) \leq c \cdot d(s, t)$. The following lemma shows the existence of good planar spanners for unit disk graphs and was proven by Li, Calinescu, and Wan [16].

**Lemma 4.1.** For any $n$-vertex unit disk graph DG(V), there exists a planar 4-spanner $H \subseteq$ DG(V). The spanner $H$ can be found in $O(n \log n)$ time.

**Sparse covers.** Let $H = (V, E)$ be a weighted planar graph, and let $r \in \mathbb{N}$. A sparse $r$-cover for $H$ is a collection of connected subgraphs $H_1, H_2, \ldots$ of $H$ with the following properties:

(i) for each vertex $v \in V$, there is at least one subgraph $H_i$ that contains all the vertices $w \in V$ with $d_H(v, w) \leq r$;

(ii) each vertex $v \in V$ is contained in $O(1)$ subgraphs $H_i$; and

(iii) diam($H_i$) $\leq 2^k \cdot r$, for every subgraph $H_i$, where diam($H_i$) is the diameter of $H_i$.

The following lemma establishes the existence of sparse covers for planar graphs and has been proven by Kawarabayashi, Sommer, and Thorup [15].

**Lemma 4.2.** For any weighted planar graph $H$ with $n$ vertices and for any $r \in \mathbb{N}$, we can compute a sparse $r$-cover for $H$ in $O(n \log n)$ time.

**The Routing Scheme.** Now we have all ingredients for our final routing scheme. In the preprocessing phase, we compute a planar 4-spanner $H$ of DG(V), as in Lemma 4.1. Then, we have diam($H$) $\leq 4D$. Next, for each $k \in I = \{[\log \frac{n}{8}], \ldots, [\log (4D)]\}$, we use Lemma 4.2 to construct a sparse $2^k$-cover ($H_1^k, H_2^k, \ldots$) of $H$. Let $G_i^k$ be the induced unit disk graph on the vertex set of $H_i^k$. Let $k_0 = [\log \frac{n}{8}]$, for each $G_i^k$, we apply the preprocessing mechanism of the low diameter routing scheme from Lemma 3.3. For each $k \in I \setminus \{k_0\}$, we apply to each $G_i^k$ the preprocessing step of the routing scheme with additive stretch from Theorem 3.10. We use $\ell_{k, i}$ to denote the resulting labeling for the graph $G_i^k$, for $k \in I$.

Now, we describe how to obtain the labels for our routing scheme. Let $v$ be a vertex of DG(V) and let $k \in I$. Since $H_1^k, H_2^k, \ldots$ is a sparse $2^k$-cover, there exists an index $i(v, k)$ such that $H_i^k(v, k)$ contains all vertices $w \in V$ with $d_H(v, w) \leq 2^k$. Now, for each $v \in V$, the label $\ell(v)$ is the concatenation of the tuples $(k, i, b(i, k, v), \ell_{k, i}(v))$, for each $k \in I$ and each $i$ with $v \in V(G_i^k)$. Here $b(i, k, v)$ is a Boolean value that is true if and only if $i = i(v, k)$. The following lemma bounds the maximum label size.

**Lemma 4.3.** For every vertex $v \in V$, we have $|\ell(v)| \in O\left(\frac{\log D \log^3 n}{\varepsilon} + \frac{\log(n)}{\varepsilon^4}\right)$.

**Proof.** Since there are $O(\log D)$ different values for $k$, and since for each $k$, the vertex $v$ appears in $O(1)$ subgraphs $G_i^k$, we have that $v$ lies in $O(\log D)$ different subgraphs $G_i^k$. For the subgraphs $G_i^k$, the label $\ell_{k, i}(v)$ comes from the low diameter routing scheme. Since diam($G_i^{k_0}$) $\in O(1/\varepsilon)$, Lemma 3.3 implies that $\ell_{k_0, i}(v)$ needs $O(\log(n)/\varepsilon^4)$ bits. Since $v$ lies in $O(1)$ subgraphs $G_i^k$, we can conclude that the corresponding tuples in $\ell(v)$ require $O(\log(n)/\varepsilon^4)$ bits in total. For the remaining $O(\log D)$ subgraphs, we derive the label $\ell_{k, i}(v)$ from the additive stretch routing scheme from Theorem 3.10. Hence, the corresponding tuples take $O(\varepsilon^{-1} \log D \log^3 n)$ bits in total. The claim follows. 

\[^{3}\text{Li, Calinescu, and Wan actually proved that there is a planar 2.42-spanner}\[16]. \text{Since we do not care about the exact constant, we use a power of 2 to simplify later calculations.}\]

\[^{4}\text{Actually, it is possible to prove an upper bound of 48r on the diameters of the subgraphs}\[15], \text{but we again prefer a power of two in order to simplify subsequent calculations.}\]
Figure 4: It is $2^{k-3} \leq \text{diam}(G^k_{i(t,k)}) \leq 2^{k+5}$. We use the additive stretch routing scheme to route within $G^k_{i(t,k)}$ until we find a vertex $v$ that is in $G^k_{i(t,k')}$ for $k' < k$. This process continues until we find a vertex that is in $G^{k_0}_{i(t,k_0)}$; here we use the low diameter routing scheme until we reach $t$.

We next describe the routing function $\sigma$, see Figure 4. Suppose we are given the labels $\ell(s)$ and $\ell(t)$ of the current vertex $s$ and the target $t$, together with the broadcast function $\beta_s$. The routing function works as follows: we find the smallest number $k = k(s,t) \in \mathcal{I}$ such that there is an index $i$ for which the tuple $(k, i, \text{true}, \ast)$ is in $\ell(t)$ and the tuple $(k, i, \text{true}, \ast)$ is in $\ell(s)$.

We can now derive the following observation:

**Observation 4.4.** Let $s, t$ be vertices of $G^k_s$ with $k = k(s, t)$. Then we have $d(s, t) \leq 2^{k+6}$. Moreover, if $k > k_0$ we have $d(s, t) \geq 2^{k-3}$.

**Proof.** By property (iii) of a sparse cover we get $d(s, t) \leq \text{diam}(G^k_s) \leq \text{diam}(H^k_s) \leq 2^{k+6}$. This proves the first inequality.

Next, let $k > k_0$. The minimality of $k$ and property (i) of a sparse cover show that $d_H(s, t) \geq 2^{k-1}$. Finally, since $H$ is a 4-spanner of $G$ we derive $d(s, t) \geq 2^{k-3}$ and the claim follows. □

Once we have $k$ and $i$, we can route in $G^k_s$ using the labels $\ell_{k,i}(s)$ and $\ell_{k,i}(t)$ as well as the broadcast function $\beta_s$. If $k = k_0$, we use $\sigma_{\text{diam}}$; otherwise, we use $\sigma_{\text{add}}$ to compute the correct port. It remains to show the correctness and to analyze the stretch factor. We start with the correctness. Its proof is quite similar to the correctness proof of $\sigma_{\text{add}}$.

**Lemma 4.5.** The routing scheme is correct.

**Proof.** Let $s$ be the current vertex, $t$ the desired target vertex, and suppose that the routing scheme sends the packet to vertex $v$ from $s$. Moreover, let $k = k(s, t)$ and $i = i(s, t)$ be two indices that were used by the routing function to determine $v$. Since the routing step from $s$ to $v$ takes place in the graph $G^k_s$, we know that $k$ is a potential candidate for $k(v, t)$. Thus, $k(v, t) \leq k$. If $k(v, t) < k$, we have made progress. However, if $k(v, t) = k$, it must be that $i(s, t) = i(v, t)$, since we defined $\ell(t)$ such that for each $k$, there is exactly one $i$ with $b(i, k, t) = \text{true}$. This means that if $k$ does not change, the routing continues in the subgraph $G^k_s$. We already proved in Lemma 3.3 and Lemma 3.8 that the underlying routing scheme for this task is correct. Hence, after a finite number of steps, we either reach $t$, or we decrease the value $k$. Since there is only a finite number of values for $k$, correctness follows. □

The next lemma bounds the additive stretch as a function of $k$.

**Lemma 4.6.** There is a constant $c > 0$ with the following property: let $s$ and $t$ be two vertices and let $k = k(s, t)$. Then, we have $d(s, t) \leq d(s, t) + c \cdot 2^k$.

---

7The $\ast$ is a placeholder for an arbitrary value. Note that $\ell(s)$ and $\ell(t)$ each contain at most one tuple that starts with $k, i$.
Proof. We use induction on \( k \geq k_0 \). First, suppose that \( k = k_0 = \lfloor \log(8/\varepsilon) \rfloor \) and let \( s, t \) be two vertices with \( k(s, t) = k_0 \). Let \( G^k_i \) be the graph that is used to determine the next vertex after \( s \). Since \( k \) can only decrease during the routing, and since \( k_0 \) is the minimum possible value of \( k \), we route within \( G^k_i \), using the low diameter routing scheme, until we reach \( t \). Moreover, by Lemma 3.3 and Observation 4.4 and for \( c \geq 2^{12} \) we get
\[
\delta(s, t) \leq (1 + \varepsilon \cdot 2^k)d(s, t) \leq d(s, t) + \varepsilon \cdot 2^{k+12} \leq d(s, t) + c\varepsilon \cdot 2^k.
\]
Next, assume that \( k > k_0 \). Let \( s, t \) be two vertices with \( k(s, t) = k \), and assume that for every vertex \( w \) with \( k(w, t) < k \), we have \( \delta(w, t) \leq d(w, t) + c\varepsilon \cdot 2^k(w, t) \). Let \( G^k_i \) be the graph in which our scheme chooses to route the data packet from \( s \) to the next node. Let \( v \) be the first node on the routing path from \( s \) to \( t \) for which \( k(v, t) < k \), see Figure 4. Moreover, let \( \delta'(\cdot, \cdot) \) measure the length of the routing path within the subgraph \( G^k_i \), using the additive stretch routing scheme. Next, by the definition of \( k_0 \) and since \( k > k_0 \) we get \( \text{diam}(G^k_i) \geq d(s, t) \geq 2^k-3 \geq 1/\varepsilon \) from Observation 4.4. Furthermore, we know that \( d(v, t) \leq \delta'(v, t) \), since \( t \) is a vertex in \( G^k_i \). Finally, we use the inductive hypothesis as well as Theorem 3.10 to derive
\[
\delta(s, t) = \delta'(s, v) + \delta(v, t) \leq \delta'(s, v) + d(v, t) + c\varepsilon \cdot 2^{k(v, t)} \leq \delta'(s, v) + \delta'(v, t) + c\varepsilon \cdot 2^{k-1} = \delta'(s, t) + c\varepsilon \cdot 2^{k-1} \leq d(s, t) + \varepsilon d(s, t) + c\varepsilon \cdot 2^k,
\]
for \( c \geq c_02^6 \), where \( c_0 \) is the constant from the O-Notation of the stretch in Theorem 3.10. Hence, the claim follows.

Finally, we can put everything together to obtain our main theorem.

**Theorem 4.7.** Let \( \text{DG}(V) \) be an \( n \)-vertex unit disk graph and \( D \) its diameter. Furthermore, let \( \varepsilon > 0 \). There is a routing scheme with \( O\left(\log D \log^3 n / \log \log n\right) \) label size whose routing function \( \sigma \) achieves the stretch factor \( 1 + \varepsilon \).

Proof. It remains to show the stretch factor. Here, it suffices to show that the stretch factor is \( 1 + O(\varepsilon) \). Let \( s \) and \( t \) be two vertices and \( k \equiv k(s, t) \). If \( k = k_0 \) the stretch factor immediately follows from Lemma 3.3. Thus, assume \( k \neq k_0 \). On the one hand we know from Observation 4.4 that \( 2^k-3 \leq d(s, t) \), and on the other hand we know from Lemma 4.6 that \( \delta(s, t) \leq d(s, t) + c\varepsilon \cdot 2^k \). Plugging everything together we get
\[
\delta(s, t) \leq d(s, t) + \varepsilon d(s, t) + c2^3\varepsilon \cdot d(s, t) = (1 + c^2\varepsilon) d(s, t).
\]
This gives the desired stretch factor and the theorem follows.

5 Conclusion

We presented the first efficient, compact, and headerless routing scheme for unit disk graphs. It achieves near-optimal stretch \( 1 + \varepsilon \) and uses \( O(\log D \log^3 n / \log \log n) \) bits in the label. It would be interesting to see if this result can be extended to disk graphs in general. If the radii of the disks are unbounded, the decomposition of Chan and Skrepetos cannot be applied immediately. However, the case of bounded radii is still interesting, and even there, it is not clear how the method by Chan and Skrepetos generalizes.

Finally, let us compare our routing scheme to the known schemes. The model of the routing scheme of Kaplan et al. [14] is very close to ours. The routing scheme can be implemented using the fixed-port model. Moreover, they also use some kind of broadcasting function, since they claim that neighborhood can be checked locally. The scheme was generalized to non-unit disk graphs with constant bounded radii [23]. Nevertheless, in unit disk graphs, we achieve the same stretch factor and still have additional information of poly-logarithmic size. The main advantage of our routing scheme is that we do not use any additional headers. Therefore, whenever a data packet arrives at a node, it is not necessary to know what happened before or where the packet came from. In the routing scheme of Kaplan et al., it happens that a data packet visits a node more than once.

\(^8\)The constant in the O-Notation depends on \( \varepsilon \).
The routing scheme of Yan et al. [24] uses headers as well, but they are only computed in the first step and do not change again. The idea of their routing scheme is similar to ours: the graph is covered by $O(\log n)$ different trees. When the routing starts, the labels of the source and the target are used to determine the identity of a tree and an $O(\log n)$-bit label of the target within this tree. Finally, they completely forget the original labels and route within this tree until they reach $t$. Their stretch is bounded by a constant. Our routing scheme can also be turned into this model, but we have $O(\log D \log n)$ different trees that cover the unit disk graph and the label of a vertex in one of the trees has size $O(\log^2 n / \log \log n)$. Nevertheless, we achieve the near optimal stretch $1 + \varepsilon$. Moreover, Yan et al. use the designer-port model and thus, they can route within a tree using labels of size $O(\log n)$. But since nodes are contained in more than one tree, there have to be lookup-tables for the port assignments. Their routing scheme can easily be turned into the fixed-port model: the stretch would not change and the label size would increase to $O(\log^3 n / \log \log n)$. In conclusion, our routing scheme needs an $O(\log D)$-factor more in the label size but achieves near-optimal stretch $1 + \varepsilon$ and the underlying routing model is specified more clearly.

References

[1] Ittai Abraham and Cyril Gavoille. On approximate distance labels and routing schemes with affine stretch. In Proc. 25th Int. Symp. Dist. Comp. (DISC), pages 404–415, 2011.

[2] Ittai Abraham, Cyril Gavoille, Andrew V. Goldberg, and Dahlia Malkhi. Routing in networks with low doubling dimension. In 26th IEEE International Conference on Distributed Computing Systems (ICDCS), page 75, 2006.

[3] Baruch Awerbuch, Amotz Bar-Noy, Nathan Linial, and David Peleg. Improved routing strategies with succinct tables. J. Algorithms, 11(3):307–341, 1990.

[4] Bahareh Banyassady, Man-Kwun Chiu, Matias Korman, Wolfgang Mulzer, André van Renssen, Marcel Roeloffzen, Paul Seiferth, Yannik Stein, Birgit Vogtenhuber, and Max Willert. Routing in polygonal domains. In Proc. Annu. Internat. Sympos. Algorithms Comput. (ISAAC), pages 10:1–10:13, 2017.

[5] Timothy M. Chan and Dimitrios Skrepetos. Approximate shortest paths and distance oracles in weighted unit-disk graphs. J. of Computational Geometry, 10(2):3–20, 2019.

[6] Shiri Chechik. Compact routing schemes with improved stretch. In Proc. ACM Symp. Princ. Dist. Comp. (PODC), pages 33–41, 2013.

[7] Brent N Clark, Charles J Colbourn, and David S Johnson. Unit disk graphs. Discrete mathematics, 86(1-3):165–177, 1990.

[8] Lenore J Cowen. Compact routing with minimum stretch. J. Algorithms, 38(1):170–183, 2001.

[9] Tamar Eilam, Cyril Gavoille, and David Peleg. Compact routing schemes with low stretch factor. J. Algorithms, 46(2):97–114, 2003.

[10] Pierre Fraigniaud and Cyril Gavoille. Routing in trees. In Proc. 28th Internat. Colloq. Automata Lang. Program. (ICALP), pages 757–772, 2001.

[11] Pierre Fraigniaud and Cyril Gavoille. A space lower bound for routing in trees. In Annual Symposium on Theoretical Aspects of Computer Science, pages 65–75. Springer, 2002.

[12] Jie Gao and Li Zhang. Well-separated pair decomposition for the unit-disk graph metric and its applications. SIAM J. Comput., 35(1):151–169, 2005.

[13] Silvia Giordano and Ivan Stojmenovic. Position based routing algorithms for ad hoc networks: A taxonomy. In Ad hoc wireless networking, pages 103–136. Springer-Verlag, 2004.
[14] Haim Kaplan, Wolfgang Mulzer, Liam Roditty, and Paul Seiferth. Routing in unit disk graphs. *Algorithmica*, 80(3):830–848, 2018.

[15] Ken-ichi Kawarabayashi, Christian Sommer, and Mikkel Thorup. More compact oracles for approximate distances in undirected planar graphs. In *Proceedings of the twenty-fourth annual ACM-SIAM symposium on Discrete algorithms*, pages 550–563. Society for Industrial and Applied Mathematics, 2013.

[16] Xiang-Yang Li, Gruia Calinescu, and Peng-Jun Wan. Distributed construction of a planar spanner and routing for ad hoc wireless networks. In *Proceedings. Twenty-First Annual Joint Conference of the IEEE Computer and Communications Societies*, volume 3, pages 1268–1277. IEEE, 2002.

[17] David Peleg and Eli Upfal. A trade-off between space and efficiency for routing tables. *J. ACM*, 36(3):510–530, 1989.

[18] Liam Roditty and Roei Tov. New routing techniques and their applications. In *Proc. ACM Symp. Princ. Dist. Comp. (PODC)*, pages 23–32, 2015.

[19] Liam Roditty and Roei Tov. Close to linear space routing schemes. *Distributed Computing*, 29(1):65–74, 2016.

[20] Nicola Santoro and Ramez Khatib. Labelling and implicit routing in networks. *The Computer Journal*, 28(1):5–8, 1985.

[21] Mikkel Thorup. Compact oracles for reachability and approximate distances in planar digraphs. *J. ACM*, 51(6):993–1024, 2004.

[22] Mikkel Thorup and Uri Zwick. Compact routing schemes. In *Proc. 13th ACM Symp. Par. Algo. Arch. (SPAA)*, pages 1–10, 2001.

[23] Max Willert. Routing schemes for disk graphs and polygons. Master’s thesis, Freie Universität Berlin, 2016.

[24] Chenyu Yan, Yang Xiang, and Feodor F Dragan. Compact and low delay routing labeling scheme for unit disk graphs. *Comput. Geom. Theory Appl.*, 45(7):305–325, 2012.