Energy characteristics of a radiating thin layer of bubbles in a liquid

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Abstract. The radiation of a thin bubble layer in an infinite liquid excited by a single wave of large amplitude is studied. It is shown that, after excitation, the layer emits a strong nonlinear wave in both directions. The radiation of the layer becomes close to sinusoidal with one line in the spectrum at large times. It is shown that in the nonlinear phase of the sound wave radiation, the time variation of the bubble energy in the layer is characterized by a stepwise dependence.

1. Introduction
The dynamics of the gas bubble is studied for over a century, beginning with the work of Rayleigh [1]. In his work, Rayleigh derived an equation describing the collapse of a spherical cavity in an incompressible fluid. The Rayleigh equation became the basis for studies of hydrodynamics and wave processes in a liquid with bubbles.

Interest in bubble liquids is due to the variety and importance of applications. The parameters of the reaction of the bubble to external action are due to the high compressibility of the gas in the bubble and the inertia of the added mass of the liquid, the viscosity of the liquid, acoustic radiation, and heat and mass transfer processes. In practice, processes in bubble media are largely determined by the dynamic response of a single bubble in its interaction with hydrodynamic and wave disturbances. An overview of these works can be found in [2].

An important direction is the study of the interaction of nonlinear waves with limited bubble regions and layers in a liquid. These studies are performed using the wave equation with the right-hand side of the Rayleigh equation [3, 4] and model [5]. An important influence on the acoustic properties of a liquid with bubbles is exerted by the interaction of bubbles. The equation for the pulsation of an individual bubble in a system of interacting bubbles was obtained in [6]. The self-consistent nonlinear wave set of equations allowing to study acoustic radiation of the localized areas in a pure liquid of a bubbles excited by a powerful sound wave was offered in [7, 8]. This system of equations allows the study of waves with an amplitude of up to 2 MPa.

The object of study of this work is the radiation of a thin bubble layer in an infinite liquid excited by a wave of high intensity. The goal is to determine the radiation characteristics and energy parameters of the bubble in the layer.

2. Physical formulation of the problem
A plane wave of large amplitude falls on the bubble layer. The thickness of the layer is substantially less than the length of the incident wave. The wave duration is chosen such that the bubbles make only one cycle of non-linear compression and expansion during the passage of the wave. After passing the wave, the bubbles perform volume pulsations and emit a sound wave. The solutions describing the
sound wave of radiation on both sides of the layer and the dynamics of the bubble located on the x axis are considered.

3. Nonlinear wave equation system

Investigations of the energy of a bubble were carried out in a one-dimensional formulation using a wave model [7, 8]. The wave system of equations in the one-dimensional case has the form

\[
\frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = -\frac{1}{c_0^2} \frac{\partial}{\partial t} \left(p \frac{\partial}{\partial t} \ln(1-\alpha)\right),
\]

\[
R_k \frac{d^2 R_k}{dt^2} + \frac{3}{2} \left(\frac{dR_k}{dt}\right)^2 + \frac{4\mu}{\rho_0 R_k} \frac{dR_k}{dt} + \frac{2\sigma}{\rho_0 R_k} = \frac{1}{\rho_0} \left[P_0 + \frac{2\sigma R_k}{R_k}\right]\left(\frac{R_k}{R_{k_0}}\right)^{3\gamma} - \frac{P_0}{\rho_0} - \frac{p(x_k, t)}{\rho_0},
\]

where volumetric gas content \(\alpha(x, t) = \sum_{k=1}^{N} V_k(t) / \Omega\) and \(V_k(t) = \frac{4}{3} \pi R_k^3(t)\).

Here \(k = 1, \ldots, N\), and \(N\) determines the total number of bubbles per unit volume \(\Omega\) with a coordinate \(x\). For a single-layer veil of bubbles, a unit volume is equal to the product of a unit area by the diameter of the bubble. The change in volumetric gas content is determined by the change in the number of bubbles in this volume. \(p(x, t)\) is pressure in the wave, \(P_0\) is initial pressure in the medium, \(c_0\) is the speed of sound in pure liquid, \(R_k\) is radius of the \(k\)-th bubble, \(\nu_k\) is volume of the \(k\)-th bubble, \(\rho\) is density of liquid, \(\sigma\) is surface tension of liquid, \(\mu\) is water viscosity, \(\gamma\) is isentropic exponent, \(t\) is time, \(x\) is space coordinate. Water under normal conditions was chosen as liquid for calculations.

The calculations were carried out for the values of the environment parameters \(P_0 = 0.1\) MPa, \(R_0 = 2.5 \times 10^{-4}\) m, \(\gamma = 1.4\), \(\rho_0 = 10^3\) kg/m\(^3\), \(c_0 = 1500\) m/s. The width of the layer is \(h = 5.0 \times 10^{-4}\) m. The amplitude of the wave \(P_b = 0.5\) MPa, duration \(\tau = 25 \times 10^{-6}\) s. Volumetric gas content \(\alpha = 10^{-3}\). The influence of the interaction of bubbles in a layer on the energy parameters of a bubble has little effect when the volume gas content is \(\alpha = 10^{-3}\).

The system of equations (1) - (4) is reduced to a dimensionless form using the relations

\[
p = p / P_0, \quad \bar{R}_k = R_k / R_0, \quad \bar{t} = t / \tau, \quad \bar{x} = x / (\tau c_0), \quad \bar{\tau} = R_0 \sqrt{\rho_0} / (3\gamma(P_0 + P_b)).
\]

The value \(\tau\) is of the same order as the Rayleigh time, which is equal to the collapse time of the empty cavity in a constant pressure field \(P_0 + P_b\).

The presence of bubbles in a liquid is described by the right-hand side of equation (1). Pulsations of the bubbles, which are described by equations (2) change the right side of equation (1), which forms a sound wave of radiation. Verification of the system of equations (1) - (2) is given in [7].

4. The energy of the volume pulsations of a single bubble

The energy equation for nonlinear pulsations of a single bubble is obtained from the equations of the self-consistent model (1) - (2) in the same way as the energy equation is derived for forced oscillations of a linear oscillator. In the self-consistent model (1) - (2), there is an inverse relationship between each bubble and wave. The power of the source of external force is always limited. The energy ratio takes the form

\[
\frac{d}{dt} (T + U) = -16\pi\mu R R_t - 4\pi R^3 p_b R.
\]

Thus, the total energy of the bubble will change under the influence of external pressure, determined by the superposition of the wave field and the reaction field of the bubbles and dissipative losses due to friction.

The principal difference of the obtained expression from the energy relation (3) is that the driving force changes due to changes in the parameters of the bubble at each time point. This is the essence of the self-consistent model (1) - (2).

Expressions for kinetic and potential energies have the form
\[ T = 2\pi\rho_o R^3 \dot{R}^2, \]  
\[ U = -\frac{\left( P_o + \frac{2\sigma}{R_o} \right)}{(1 - \gamma)} \cdot \Omega_o \left( \frac{R}{R_o} \right)^{3(1-\gamma)} - 1 + P_o \Omega_o \left( \frac{R}{R_o} \right)^3 - 1 + \sigma \cdot (S - S_0), \]  
where \( S \) and \( S_0 \) the current and equilibrium values of the surface areas of the bubble. A graph of potential energy versus \( R/R_0 \) is shown in Fig. 1. The minimum value of the potential energy is zero and achieved when \( R/R_0 = 1 \).

Figure 1. The dependence of the potential energy of the bubble on the ratio \( R/R_0 \).

The asymmetry of the graph of the potential energy of a single bubble is due to the nonlinearity of the pulsations. Most of the time, the bubble is in an expanded state with nonlinear pulsations. To characterize the bubble, it is convenient to introduce the concept of a local time scale (current scale). Compression can be described as a phase with a fast local time scale and an expansion phase can be described with a slower local time scale. The ratio of local time scales can reach the order and more. For experiments [12], and when the bubble radius changes from \( 0.5 R_0 \) to \( 1.1 R_0 \), the ratio of local time scales reaches.

The formula for the total energy of a nonlinear single bubble in an incompressible fluid allows us to reduce a number of problems related to bubble pulsations and wave processes in gas-liquid media to problems of the classical Lagrange dynamics. The full Lagrange function can be written as follows

\[ L = 2\pi\rho_o R^3 \dot{R}^2 + \left[ P_o + \frac{2\sigma}{R_o} \right] \cdot \Omega_o \left( \frac{R}{R_o} \right)^{3(1-\gamma)} - 1 - P_o \Omega_o \left( \frac{R}{R_o} \right)^3 - 1 - \sigma \cdot (S - S_0). \]  

For small pulsations, when \( \epsilon=\delta(R/R_0) \ll 1 \), the expressions for the kinetic and potential energies can be expanded in a degree of smallness \( \epsilon \) near the equilibrium value of the bubble radius \( R_0 \). Then the law of conservation of energy coincides in appearance with the expression for a single particle with full energy \( E \), when the particle makes movements with one degree of freedom in a potential field \( U \).

\[ \frac{m\dot{R}^2}{2} + U(\delta R) = E \]
Here \( m = 4 \pi \rho_0 R_0^3 \) the added mass of the liquid is a bubble, and the potential energy is estimated using the expression

\[
U(\delta R) \approx (6\pi \gamma P_b R_0 + 4\pi \sigma(3\gamma - 1)) \cdot \delta R^2
\]  

Expression (6) is useful for comparing a dynamic bubble system with other dynamic systems in the description of which the Lagrange formalism is used.

5. Results and discussion

The layer is excited by a falling wave of pressure of finite duration and energy. A wave is a symmetric pulse. A pressure equal to zero outside the interval of the pulse duration. The wave moves towards the layer from left to right along the X axis. Solutions are given as a time dependence at two points located in front of the layer and behind the layer with respect to the incident wave.

Fig. 2 shows two typical time dependences of the wave field near the bubble layer. Figure 2a shows the incident wave and the radiation wave of a layer that propagates in the opposite direction to the incident wave. The first impulse is the incident wave. The second wave is the radiation layer. Figure 2b shows the wave passing through the layer and the radiation of the layer. The initial amplitude of the wave is \( P_b = 5 \cdot 10^4 \) Pa, the energy in the incident wave is \( E_b = 1.18 \times 10^{-2} \) J.

![Figure 2](image)

**Figure 2.** a) Incident impulse and radiation of the layer in the opposite direction.

b) Impulse that passed through the bubble layer and radiation of the layer.

The Fig. 3 shows the combined graphs of the change in total energy inside the bubble and the graph of the change in the radius of the bubble with time. The total energy of a bubble of \( E \) is

![Figure 3](image)

**Figure 3.** Change in bubble energy and bubble pulsations.
normalized on energy of a bubble at rest $E_0 = 2 \pi \rho R_0^5/\tau^2$, the change in radius $\delta R$ is dimensionless by the value of the initial radius $R_0$. On the curve of the dependence of the bubble total energy on time there are clearly distinguishable levels of constant energy values. The behavior of the total energy over time is characterized by a stepwise dependence. In this case, the energy state of the bubble changes in a quasidiscrete manner by passing from one high energy level to the next low one. The figure shows that changes in energy levels are consistent with pulsations and occur when the direction of movement of the bubble surface changes.

**Conclusions**

Researches showed that power characteristics of a bubble which makes pulsations in a layer and radiates a sound wave in liquid are characterized by step dependence of change of energy on time. This effect is most expressed in a nonlinear phase of a pulsation of a bubble. In a nonlinear phase of pulsations the bubble quickly loses energy because of radiation.

In the self-consistent model (1) - (2), the total energy of the bubble in the layer changes under the influence of the pressure field, which is a superposition of the wave field and the reaction field of the bubbles. The main energy losses are associated with radiation by a layer of a nonlinear sound wave and to a lesser extent due to the viscosity of the liquid.

This is a fundamental difference between the identity of energy (3) and the identity of the energy of linear pulsations of the bubble, which is obtained under the assumption that the external driving force does not change. In the latter case, there is no feedback, and the response of the generator does not appear in the characteristics of this external force.

**References**

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