Breakdown of scale invariance in the vicinity of the Tonks-Girardeau limit

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(Dated: June 2, 2014)

In this article, we consider the monopole excitations of the harmonically trapped Bose gas in the vicinity of the Tonks-Girardeau limit. Using Girardeau’s Fermi-Bose duality and subsequently an effective fermion-fermion odd-wave interaction, we obtain the dominant correction to the scale-invariance-protected value of the excitation frequency, for microscopically small excitation amplitudes. We produce a series of diffusion Monte Carlo results that confirm our analytic prediction for three particles. And less expectedly, our result stands in excellent agreement with the result of a hydrodynamic simulation (with the Lieb-Liniger equation of state as an input) of the microscopically large but macroscopically small excitations. We also show that the frequency we obtain coincides with the upper bound derived by Menotti and Stringari using sum rules. Surprisingly, however, we found that the usually successful hydrodynamic perturbation theory predicts a shift that is 9/4 higher than its ab initio numerical counterpart. We conjecture that the sharp boundary of the cloud in local density approximation—characterized by an infinite density gradient—renders the perturbation inapplicable. All our results also directly apply to the 3D p-wave-interacting waveguide-confined fermions.

PACS numbers: 67.85.De,02.30.Ik

I. INTRODUCTION

In scale-invariant systems, a state at one density can be expressed through a state at another, via trivial rescaling of space. Scale invariance is always associated with an inability of the interaction potential to introduce a distinct length scale. Several examples emerged recently in the physics of quantum gases. In three dimensions, the δ-interaction with infinite coupling strength, even when properly regularized, ensures the scale invariance of the unitary gases [1–3]. In 2D, the unregularized δ-potential, for any coupling constant, induces the scale invariance of two-dimensional Bose [6, 7] and spin-1/2 Fermi [8] gases at the classical field level, which is however broken by quantization [9–11]. Finally, in 1D, we have the Tonks-Girardeau (TG) gas [12–14]—a one-dimensional quantum Bose gas with an infinite strength δ-interaction—which is the subject of this article.

Scale invariance enables a robust frequency gauge: when a scale-invariant gas is placed in a symmetric harmonic trap of frequency ω and a monopole oscillation is induced, the signal shows neither damping nor amplitude-dependent frequency shifts—its frequency is fixed to 2ω, for all scale invariant systems and for all spatial dimensions [15]. As a consequence, monopole excitations in scale-invariant systems are very sensitive to changes in the equation of state [16], whether produced by a quantum anomaly [9–11], by an influence of the confining dimension [11], or just by a small shift in the coupling constant away from the scale-invariant point.

In this article, we study the effect of a small deviation from the TG point on the frequency of the monopole excitations of a one-dimensional harmonically trapped Bose gas [15–17], both for microscopically small and for microscopically large but macroscopically small excitation amplitudes. Our results also directly apply to the 3D p-wave-interacting waveguide-confined fermions [20], thanks to the mapping by Granger and Blume [21].

II. BOSONIC HAMILTONIAN OF INTEREST AND ITS EFFECTIVE FERMIONIC COUNTERPART

Our object of study is the system of N bosons of mass m with contact interaction in a 1D harmonic trap of frequency ω. In the second quantized form, the Hamiltonian reads

\[ \hat{H}_B = \int_{-\infty}^{+\infty} dx \left\{ \frac{\hbar^2}{2m} \left( \partial_x \hat{\Psi}_B^\dagger \right) \left( \partial_x \hat{\Psi}_B \right) + \frac{m \omega^2}{2} \hat{\Psi}_B^\dagger \hat{\Psi}_B + \frac{g_{1D}}{2} \hat{\Psi}_B^\dagger \hat{\Psi}_B^\dagger \hat{\Psi}_B \hat{\Psi}_B \right\} \]  

(1)

Here \( g_{1D} \) is the one-dimensional coupling constant, and \( \hat{\Psi}_B(x) \) is the bosonic quantum field.

In this article, we will be interested in the monopole excitations in the vicinity of the TG limit, \( g_{1D} \to \infty \).
This regime is both easy and difficult to work in. On one hand, right at the limit, the model maps to free fermions, via the Fermi-Bose map by Girardeau \cite{12}. On the other hand, away from the limit—even remaining finitely close to it—there exist conceptual difficulties in interpreting the resulting system as one governed by a Hamiltonian for free fermions plus a small correction \cite{22}.

Nevertheless, an effective fermionic Hamiltonian,
\[
\hat{H}_{\text{F, eff.}} = \int_{-\infty}^{\infty} dx \left\{ \frac{\hbar^2}{2m} (\partial_x \hat{\Psi}_F^\dagger) (\partial_x \hat{\Psi}_F) + \frac{m\omega^2}{2} x^2 \hat{\Psi}_F^\dagger \hat{\Psi}_F - \frac{2\hbar^4}{m^2 g_{1D}} (\partial_x \hat{\Psi}_F^\dagger) \hat{\Psi}_F^\dagger (\partial_x \hat{\Psi}_F) \right\},
\]
(2)
\[23\] can be proven \cite{24, 25} to produce the correct eigen spectrum if used as the kernel of a variational energy functional. One can further show that in this case, the first order of the perturbation theory—with the quartic term in \cite{22} as a perturbation—produces the correct $1/g_{1D}$ correction to the eigenenergies. Here, $\hat{\Psi}_F(x)$ is the fermionic quantum field.

III. FREQUENCY OF THE MONOPOLE EXCITATION OF A MICROSCOPICALLY SMALL AMPLITUDE

The fermionic field in Eq. (2) can be expanded onto a series over the eigenstates of the harmonic trap: $\hat{\Psi}_F(x) = \sum_n b_n \phi_n(x)$, $\phi_n(x) = [1/(2^n n! \sqrt{\pi\ell})]^{1/2} e^{-x^2/(2\ell^2)} H_n(x/\ell)$ where $H_n(\xi)$ is the $n$-th Hermite polynomial, and $\ell \equiv \sqrt{\hbar/(m\omega)}$. The operator $b_n$ is the fermionic annihilation operator that removes one particle from the $n$-th eigenstate. The operators $b_n$ obey the standard fermionic commutation relations and the Hamiltonian in Fock space is of the form
\[
\hat{H}_{\text{F, eff.}} = \frac{\hbar^4}{2m^2 g_{1D}} \sum_{n<0} \sum_{k<0} t_{mk} b_n^\dagger b_m b_k^\dagger b_k + \mathcal{O}(1/g_{1D}^3)
\]
(3)
where $t_{mk} = 2 \int_{-\infty}^{\infty} dx (\phi_n^\dagger \phi_k - \phi_k \phi_n)$.

The ground state of whole system is
\[
|\Psi_0\rangle = \left( \prod_{n=0}^{N-1} b_n^\dagger \right) |\text{vac}\rangle
\]
(4)
where $|\text{vac}\rangle$ stands for the vacuum with no particle at all. The energy correction of ground state is analyzed in Appendix A, where we will show our result recovers the formula in Ref. \cite{32}.

Now we will come to the 2$^{nd}$ excitations. The unperturbed manifold of energy $E_0^{(0)} + 2\hbar\omega$ is of twofold degeneracy. The set of unperturbed eigenstates is
\[
\{ |\Psi_{2n}\rangle = \hat{b}_{N+1}^\dagger \hat{b}_{N+1}^\dagger |\Psi_0\rangle, |\Psi_{2n+1}\rangle = \hat{b}_{N-2}^\dagger \hat{b}_{N-2}^\dagger |\Psi_0\rangle \}
\]
Based on the perturbation theory, corrections to the energies are represented by the spectrum of the $2 \times 2$ matrix of the perturbation term in the space spanned by the members of the manifold
\[
\hat{\nu} = \frac{\hbar^4}{m^2 g_{1D}} \left( \frac{I_N^{(2\alpha)} - \Omega_N}{\Omega_N} - \frac{1}{2} \right)
\]
(5)
where $\Omega_N = \Omega_{N-2}^{N-1} + I_N^{(2\alpha)}$, $I_N^{(2\alpha)} = \sum_{m=1}^{N+1} \sum_{m=1}^{N-1} \omega_{nm}$, and $I_N^{(2\alpha)} = \sum_{m=1}^{N+1} \sum_{n=0}^{N-1} \omega_{nm}$. Hence the transition frequencies for microscopically small amplitude read
\[
\hbar\omega_{2\pm,0} = 2\hbar\omega + \frac{\hbar^4}{2m^2 g_{1D}} \left[ I_N^{(2\alpha)} + I_N^{(2\beta)} - 2I_N^{(0)} + \sqrt{\left(I_N^{(2\alpha)} - I_N^{(2\beta)}\right)^2 + 4\Omega_N^2} \right] + \mathcal{O}(1/g_{1D}^2)
\]
(6)
where $I_N^{(0)} \equiv \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \omega_{nm}$ and $\omega_{nm}$ takes the form
\[
\omega_{nm} = \sqrt{\frac{2(m-n)^2}{\pi^3} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} \frac{\Gamma(m+1)}{\Gamma(m+1)}}
\times _3 F_2 \left[ \frac{3}{2} - n, -m; \frac{3}{2} - n, -m+1 \right]
\]
(7)
After a lengthy but straightforward calculation (see the Appendices), we obtain the analytical form of $1/g_{1D}$ corrections to the relevant transition frequencies:
\[
\hbar\omega_{2+,0} = 2\hbar\Omega_D = \left( 2 + \mathcal{O}(\frac{1}{\gamma_0(N)}) \right) \hbar\omega
\]
(8)
\[
\hbar\omega_{2-,0} = 2\hbar\Omega_M = \left( 2 - \frac{\sqrt{N}\Gamma(N-\frac{5}{2})}{\sqrt{\pi}} \frac{\Gamma(N+\frac{1}{2})}{\Gamma(N+2)} \right)
\times _3 F_2 \left[ \frac{3}{2} - 1, -N, -N; \frac{3}{2} - 1, -N; \frac{3}{2} - N, \frac{3}{2} - N+1 \right] \frac{1}{\gamma_0(N)} + \mathcal{O}(\frac{1}{\gamma_0(N)}) \right) \hbar\omega,
\]
(9)
where $\hbar\omega_{2+,0} = E_{2+} - E_0$, are the transition frequencies, $E_0$ is the ground state energy, and $E_{2+}$ are the energies of the states that, in the strict TG limit, form a two-fold degenerate manifold, $2\hbar\omega$ above the ground state. The effective Lieb-Liniger parameter $\gamma_0(N) \equiv (mg_{1D})/(m\hbar^2)$ \cite{20} uses the TG (i.e. $g_{1D} \rightarrow \infty$) density in the center of the trap, $m\hbar^2 \equiv (\sqrt{2/\pi})\sqrt{m\hbar^2/\hbar}$ instead of the true density. Here, $_3 F_2 [a_1, a_2, a_3; b_1, b_2; z]$ is the generalized hypergeometric function of order $(3, 2)$.

The interpretation of the $2\pm$ eigenstates can be inferred from the corresponding transition frequencies.

The first one (2+) is the second state of an infinite $\hbar\omega$-spaced “dipole” ladder: coherent wave packets formed out of the members of the ladder represent finite amplitude dipole excitations; their frequency $\Omega_D$ is equal to
the frequency of the trap exactly, interactions notwithstanding \[37\]. The first state of the ladder is analyzed in Appendix B. The zeroth state is the ground state.

The second eigenstate \((2^-)\) in the \(E_0^{(0)} + 2\hbar\omega\) manifold, is the first (ground state being the zeroth) step in the “monopole” ladder, that corresponds to the breathing excitations of frequency \(\Omega_M\). In the noninteracting case, the ladder (exactly \(2\hbar\omega\)-spaced) can be obtained by a recurring application of the creation operator \(\hat{L}_+\) of an appropriate \(SO(2,1)\) group to the ground state \[15\]. The excitation dynamics consists of a periodic scaling transformation of frequency \(2\omega\). The existence of this structure is a direct consequence of the scale invariance of the TG gas and its free-fermionic counterpart in the harmonic potential.

A deviation from the TG limit (and the corresponding fermion-fermion interactions \(2\)) breaks the scale invariance weakly. The goal of this article is to assess the impact that this effect has on the excitations of a microscopic amplitude and compare it to the corresponding predictions for the microscopically large but macroscopically small excitations.

As far as the microscopic amplitude excitations are concerned, our program is already fulfilled. Indeed, a linear combination of the ground state and a small admixture of the state \(2^-\) is already a small amplitude monopole excitation. Its frequency is given by the formula \(9\) that constitutes the central result of this article.

IV. COMPARISON TO THE OTHER FEW-BODY RESULTS

We verified that for two atoms \((N = 2)\), the formula \(9\) for the frequency of the small amplitude monopole excitations coincides with the known exact results \[25\].

In the three-body case \((N = 3)\) we perform a Diffusion Monte Carlo simulation of the imaginary time evolution and extract the \(\omega_{2^-,0}\) transition frequency from the inverse Laplace transform components of the imaginary-time dynamic structure factor.

In Fig. 1 we compare our (non-perturbative) numerical three-body results with the perturbative prediction \[9\]. In Fig. 2 the dominant corrections to the monopole frequency for \(N = 2\) and \(N = 3\), extracted from the non-perturbative data, are also compared to formula \(9\).

V. LARGE-\(N\) ASYMPTOTICS AND A COMPARISON WITH THE SUM-RULE PREDICTIONS

The frequency of the monopole excitations of a microscopically small amplitude can be bounded from above using the sum rules \[18\] (see Fig. 3). The order \(1/\gamma_0\) correction to this bound can also be computed analyti-

\[h\Omega_M \overset{N \gg 1}{=} \left( 2 - \frac{64}{15\pi} \frac{1}{\gamma_0(N)} + \mathcal{O}\left(\frac{1}{\gamma_0(N)}\right) \right) \hbar\omega \] 

We conjecture that the upper bound \(10\) actually equals the exact prediction \(9\) in the limit of large \(N\). To test this conjecture, we multiplied the \(\frac{1}{\gamma_0(N)}\)-term in the the bound \(10\) by \(AN^\sigma\), with \(A\) and \(\sigma\) being free parameters to be used to fit the \(\frac{1}{\gamma_0(N)}\) term in the series \[9\]. Indeed we found the values that support our conjecture, namely \(A = 1.000\) and \(\sigma = 0.0003\).
VI. COMPARISON TO THE FREQUENCIES OF THE EXCITATIONS OF A MICROSCOPICALLY LARGE BUT MACROSCOPICALLY SMALL AMPLITUDE

The monopole frequencies obtained above correspond to excitations of microscopically small amplitude: there the many-body energy of the excited atomic cloud is only a few one-body harmonic quanta above the ground state energy. A priori it is not obvious if the microscopic predictions will remain valid for microscopically large but macroscopically small excitations, whose spatial amplitude is smaller than but comparable to the size of the cloud.

To compare the two frequencies, we investigate the time dynamics using the hydrodynamic equations (see e.g. Eqs. (1) and (2) of Ref. [30]). We use the well-known thermodynamic limit for the dependence of the zero-temperature chemical potential $\mu(n)$ on the one-dimensional particle density $n$, for a uniform one-dimensional $\delta$-interacting Bose gas; this equation of state was obtained by Lieb and Liniger, using Bethe Ansatz [20]. We propagate the hydrodynamic equations numerically. To excite the monopole mode, we quench the trapping frequency. Fig. 3 shows a good agreement with the large-N asymptotics for the frequency of the microscopically small excitations [10].

In order to obtain an analytic expression for the frequency shift, we apply the perturbation theory developed by Pitaevskii and Stringari in Ref. [20] (with more technical details worked out in Ref. [17] for the purpose of computing an analytic expression for the dominant beyond-mean-field correction to the monopole frequency of a BEC. Here we use the TG equation of state (EoS), $\mu_0(n) = (\pi^2 h^2 / 2m)n^2$ as the unperturbed EoS, and the first-order (in $\gamma(n)^{-1}$) correction to the EoS, $\Delta \mu(n) = (8\pi^2 h^2 / 3m)n^2 \gamma(n)^{-1}$, as a perturbation. The function $\gamma(n) \equiv (\mu_0 - \mu_0 (1D)) / (\hbar n^2)$ is the so-called Lieb-Liniger parameter [20]. While most of the outlined steps of the study in Ref. [17, 20] are universally applicable to any EoS, the boundary conditions for the density mode functions $\delta n(z)$ at the edge of the atomic cloud $|z| = R_{TF}$ are typically dictated by the specific physical properties of the system at hand. (Here $R_{TF}$ is the Thomas-Fermi radius.) In the TG case, with or without further beyond-TG corrections to the EoS, those are given by

$$\delta n(z) = A(R_{TF} - |r|)^{-1/2} + B + O((R_{TF} - |r|)^{1/2})$$

$$B = 0.$$  

(11)

Indeed, following the analysis developed in Ref. [17], one can show (i) that the first two terms in (11) correspond to the near-edge asymptotic of the two linearly independent solutions of the mode equation, and (ii) that when rewritten in Lagrange form [31], the solutions that violate condition (11) lead to the appearance of crossing particle trajectories, incompatible with hydrodynamics. To our surprise, we found that the macroscopic perturbation theory leads to a frequency shift that is 9/4 times greater in magnitude than its microscopic counterpart Eq. (10) (see Fig. 3). This is definitely an artifact of the perturbative treatment of the macroscopic theory rather than of the macroscopic theory per se. Indeed, our macroscopic nonperturbative numerical results are consistent with the microscopic theory. We attribute the failure of the perturbation theory to the divergence of the spatial derivative of the steady-state density at the edge of the cloud: in a monopole excitation this will lead to an infinite time derivative of the density itself, possibly invalidating the perturbation theory.

In the same plot, we also present the sum-rule bound [18]. At weak fermion-fermion interactions, it reproduces well the perturbative prediction Eq. (10).

VII. CONCLUSION AND OUTLOOK

In this article, we obtained an analytic expression, Eq. (9) for the leading behavior of the deviation of the frequency of the microscopically small monopole excitations of a strongly-interacting one-dimensional Bose gas from the value predicted by the scale invariance in the TG limit.

We further compare this prediction with (a) the known non-perturbative analytic expressions for two atoms [28] and to (b) the Diffusion Monte Carlo predictions for three atoms. For large numbers of atoms, the prediction in Eq. (9) stands in excellent agreement with (c) the sum-rule bound [10, 18, 20]. It was not a priori obvious to us if our formula will also apply to microscopically large but macroscopically small excitations: they correspond to a large number of atoms (still covered by the
formula (1), and have a macroscopic magnitude (that is formally beyond the scope of Eq. (9)). We found that (d) the numerically propagated hydrodynamic equations produce the same leading-order frequency correction as the large-$N$ limit of Eq. (9). Finally, we find that (e) the hydrodynamic perturbation theory, which was so successful in predicting the beyond-mean-field corrections to the monopole frequency in both the three-dimensional and the two-dimensional Bose gases, fails to predict the analogous beyond-TG correction in our case: the hydrodynamic perturbative prediction turns out to be approximately 9/4 higher than the ab initio numerical value it was designed to approximate. We conjecture that the sharp boundary of the TG cloud, characterized by an infinite density gradient, renders the perturbation theory inapplicable.

Experimentally, the monopole excitation frequency of the Lieb-Liniger gas has been already studied, in Ref. [19]. In the range of parameters our article is devoted to, the beyond-scale-invariance shifts are too small to be reliably compared with the experimental data. However, we plan to extend our study of the frequency of microscopically large but macroscopically small monopole excitations to the whole range of the interaction strengths. One can already observe that in the intermediate range, the experimental frequencies depart from the sum-rule prediction to ground state energy ETH,,|\rangle. The formula for the energy is $E_0^{(1)} = \frac{\hbar \omega}{\sqrt{\pi N}} \gamma_0(N) \sum_{m=1}^{N-1} \sum_{n=0}^{m-1} \frac{(m-n)^2 \Gamma(m-\frac{1}{2})}{\Gamma(m+1)}$ \left( \frac{1}{\Gamma(n+1)} \right)^{3F_2} \left[ \frac{3}{2} - n, \frac{3}{2} - m, 1 \right]

and $3F_2[a_1, a_2, a_3; b_1, b_2, z]$ is the order (3, 2) generalized hypergeometric function. This result, as well as its derivation, is essentially identical to the formula obtained in Ref. [30].

**Appendix A**

There is only one eigenstate, $|\Psi_1\rangle = |\bar{\Phi}_{N}\rangle_{0}$, in the first excited state manifold, and thus the correction to the energy is $E_1^{(1)} = \langle \Psi_1 | \hat{V} | \Psi_1 \rangle$. The formula for the transition frequency, $\hbar \omega_{1,0} \equiv E_1 - E_0$, assumes a compact form, and it reads

$$h \omega_{1,0} \equiv \hbar \Omega_D = \hbar \omega + \mathcal{O}(\frac{1}{\gamma_0(N)^2})$$

**Appendix B**

The ground state of the unperturbed system is $|\Psi_0\rangle = \left( \prod_{n=0}^{N-1} \hat{b}_n^\dagger \right) |\text{vac}\rangle$ where $|\text{vac}\rangle$ is the vacuum with no particles at all. The first order perturbation theory correction to ground state energy $E_0^{(1)}$ is given by

$$E_0^{(1)} = \frac{\hbar \omega}{\sqrt{\pi N}} \gamma_0(N) \sum_{m=1}^{N-1} \sum_{n=0}^{m-1} \frac{(m-n)^2 \Gamma(m-\frac{1}{2})}{\Gamma(m+1)} \left( \frac{1}{\Gamma(n+1)} \right)^{3F_2} \left[ \frac{3}{2} - n, \frac{3}{2} - m, 1 \right]$$
We interpret the state $|\Psi_1\rangle$ as the first state of an infinite $\hbar\omega$-spaced “dipole” ladder: coherent wave packets formed out of the members of the ladder represent finite-amplitude dipole excitations (i.e. oscillations of the center of mass); their frequency $\Omega_0$ is equal to the frequency of the trap exactly, interactions notwithstanding [37]. The zeroth state of the ladder is the ground state.

\[ I_N^{(2a)} - I_N^{(2b)} = \sqrt{\frac{32}{\pi^3}} \frac{\Gamma\left(\frac{3}{2}ight) \Gamma\left(N - \frac{3}{2}\right)}{\Gamma(N+1)\Gamma(N-1)} \left\{ \begin{array}{c} 1 \end{array} \right\} - \left(\frac{N - \frac{1}{2}}{N - \frac{3}{2}}\right)^2 \frac{\Gamma\left(\frac{3}{2}, 1 - N, -N\right)}{(N+1)(N-1)} \times 3F_2 \left[ \begin{array}{c} \frac{3}{2}, 1 - N, -N \end{array} \right] \] 

\[ = \sqrt{\frac{72}{\pi^3}} \frac{\Gamma\left(N - \frac{3}{2}\right) \Gamma\left(N + \frac{1}{2}\right)}{\Gamma(N)\Gamma(N+2)} \times 3F_2 \left[ \begin{array}{c} \frac{3}{2}, 1 - N, -N \end{array} \right] \] 

which leads to

\[ I_N^{(2a)} + I_N^{(2b)} - 2I_N^{(0)} = -\sqrt{\frac{72}{\pi^3}} \frac{N\Gamma\left(N - \frac{3}{2}\right) \Gamma\left(N + \frac{1}{2}\right)}{\Gamma(N)\Gamma(N+2)} \times 3F_2 \left[ \begin{array}{c} \frac{3}{2}, 1 - N, -N \end{array} \right] \] 

where $m < n$, $\varphi_n$ is the $n$-th eigenfunction of harmonic trap and $H_n(x)$ is the $n$-th Hermite polynomial. Then we have

\[ R_{mn}(x) = \frac{1}{2} [\varphi_m'(x)\varphi_n(x) - \varphi_m(x)\varphi_n'(x)] \]

then it can be shown

\[ R_{mn}(x) = \frac{m - n}{\sqrt{\pi} 2^{m+n} n!} \sum_{k=0}^{m} 2^k k! \binom{m}{k} \binom{n}{k} \times e^{-x^2} H_{m+n-2k-1}(x) \]
\[
\int_{-\infty}^{+\infty} R_{N-2,N+1} R_{N-1,N} dx = \frac{3}{\pi(N-2)!N!\sqrt{N^2} - 1} \frac{1}{2^{N-1}} \sum_{k=0}^{N-2} \sum_{l=0}^{N-1} (-1)^{k+l+1} k!! \binom{N-2}{k} \binom{N+1}{k} \binom{N-1}{l} \binom{N}{l} \Gamma\left(2N-k-l-\frac{3}{2}\right) \tag{C7}
\]

Thus \( \Omega_N \) is given

\[
\Omega_N = -8 \int_{-\infty}^{+\infty} R_{N-2,N+1} R_{N-1,N} dx = \sqrt{\frac{18(N^2-1)}{\pi^3}} \frac{\Gamma\left(\frac{N}{2}\right) \Gamma\left(N+\frac{1}{2}\right)}{\Gamma(N)\Gamma(N+2)} {}_3F_2\left[\frac{3}{2}, 1-N, -N; \frac{7}{2} - N, \frac{1}{2} - N, 1 \right] \tag{C8}
\]

where the formula for integral \([39]\) was used. The monopole frequency (3) and (4) are immediately arrived from these results.

\[
\int_{-\infty}^{+\infty} e^{-2x^2} H_p(x) H_q(x) dx = (-1)^{(p+3q)/2} 2^{p+q-1}/2 \Gamma\left(\frac{p+q+1}{2}\right) \tag{C9}
\]

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[22] It can be shown for example that if the energy shift induced by a single \( \delta \)-function scatterer situated in between two walls is reinterpreted in terms of the interaction present in the fermionic Hamiltonian (2), the second order of the Taylor expansion in the powers of its prefactor, proportional to \( 1/g_{1D} \), turns out to be positive.
contradicting the non-positivity of the second order per-
turbation theory shift; see Problem 4.1.11 in [35].
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