Poitou–Tate duality for totally positive Galois cohomology

Hassan Asensouyis, Jilali Assim, Zouhair Boughadi, and Youness Mazigh

Department of Mathematics, Faculty of Sciences, Moulay Ismail University of Meknès, Meknes, Morocco

ABSTRACT
In this paper, we establish a Poitou–Tate’s global duality for totally positive Galois cohomology. We illustrate this result in the case of the twisted module “à la Tate” $\mathbb{Z}_2(l)$, $l$ integer.

1. Introduction

Let $F$ be a number field and let $p$ be a rational prime. For a finite set $S$ of primes of $F$ containing the $p$-adic and the infinite primes, we denote by $S_f$ the set of finite primes in $S$ and by $G_S(F)$ the Galois group of the maximal algebraic extension $F_S$ of $F$ which is unramified outside $S$. Let $M$ be a finite discrete $\mathbb{Z}_p[[G_S(F)]]$-module.

For an odd prime $p$ and $n = 1, 2$, the global Poitou–Tate duality (e.g. [7, 5.1.6, p. 114]) states that there is a perfect pairing

$$\prod_{S_f}^n(M) \times \prod_{S_f}^{2-n}(M^*) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

where

$$\prod_{S_f}^n(M) := \ker(H^n(G_S(F), M) \longrightarrow \bigoplus_{v \in S_f} H^n(F_v, M))$$

and $(,)^*$ means the Kummer dual.

In the case $p = 2$, the non-triviality of the cohomology groups of the local absolute Galois group at real places leads to several complications. To control these contributions from infinite real places, several authors (e.g. [1, 2, 5, 8]) use a slight variant of Galois cohomology, the so-called totally positive Galois cohomology introduced by Kahn in [3] after ideas of Milne [6].

In this paper, we establish a global Poitou–Tate duality for totally positive Galois cohomology. For $n = 1, 2$, let $\prod_{S_f}^{n+}(M)$ be the kernels of the localization maps

$$\prod_{S_f}^{n+}(M) := \ker(H^n_+(G_S(F), M) \longrightarrow \bigoplus_{v \in S_f} H^n(F_v, M)),$$
where $H^i_+(\cdot, \cdot)$ denotes the $j$th totally positive Galois cohomology group (Section 3). The following theorem summarizes the main result of this paper (Theorem 3.10):

**Theorem.** Let $F$ be a number field and let $S_f$ be the set of finite primes in $S$. For a finite discrete $\mathbb{Z}_2[[G_3(F)]]$-module $M$

i. There is a perfect pairing, functorial in $M$,

$$
\bigwedge^2_{\mathcal{S}_Y}(M) \times \bigwedge^1_{\mathcal{S}_Y}(M^\vee) \longrightarrow \mathbb{Q}/\mathbb{Z}_2.
$$

ii. We have an exact sequence, functorial in $M$,

$$
0 \longrightarrow Z(M) \longrightarrow \bigwedge^1_{\mathcal{S}_Y}(M) \longrightarrow \bigwedge^2_{\mathcal{S}_Y}(M^\vee) \longrightarrow 0,
$$

where an explicit description of the kernel $Z(M)$ is given in Lemma 3.9.

The superscripts $(.)^*$ and $(.)^\vee$, respectively denote the Kummer dual and the Pontryagin dual.

Using results from [7, 11], the above theorem can be extended to $\mathbb{Z}_2[[G_3(F)]]$-modules of finite or co-finite type over $\mathbb{Z}_2$ (Theorem 3.13).

For an integer $i$, let $M = \mathbb{Z}_2(i)$ be the twisted module “à la Tate” of the ring of dyadic integers $\mathbb{Z}_2$. We have the following exact sequence

$$
0 \longrightarrow Z(\mathbb{Z}_2(i)) \longrightarrow \bigwedge^1_{\mathcal{S}_Y}(\mathbb{Z}_2(i)) \longrightarrow \bigwedge^2_{\mathcal{S}_Y}(\mathbb{Q}/\mathbb{Z}_2(1 - i)) \longrightarrow 0,
$$

with

$$
Z(\mathbb{Z}_2(i)) \cong \begin{cases} 
(\mathbb{Z}_2(i))^{r_{2} - 1} & \text{if } i = 0 \text{ and } r_1 = 0; \\
(\mathbb{Z}_2(i))^{r_{2}} \oplus (2\mathbb{Z}_2(i))^{r_{1} - 1} & \text{if } i = 0 \text{ and } r_1 \neq 0; \\
(\mathbb{Z}_2(i))^{r_{2}} \oplus (2\mathbb{Z}_2(i))^{r_{1}} & \text{if } i \neq 0 \text{ is even}; \\
(\mathbb{Z}_2(i))^{r_{2}} & \text{if } i \text{ is odd},
\end{cases}
$$

where $r_1$ and $r_2$ are the number of real and complex places of $F$, respectively.

As an application, for a number field $F$ with ring of $S$-integers $\mathcal{O}_{F,S}$, we realize the positive étale wild kernel [1, Definition 2.2] $WK^{\text{et},+}_{2i-2}\mathcal{O}_{F,S} := \bigwedge^2_{\mathcal{S}_Y}(\mathbb{Z}_2(i))$, for $i \geq 2$ an integer, as an Iwasawa module (Proposition 3.15). In particular, we get that the group $WK^{\text{et},+}_{2i-2}\mathcal{O}_{F,S}$ is independent of the set $S$ containing the infinite and dyadic places of $F$.

### 2. Some homological algebra

We fix an abelian category and work with the corresponding category of complexes. For a complex $X = (X^i, d_X^i)_{i \in \mathbb{Z}}$ and an integer $n \in \mathbb{Z}$, let $X[n]$ denote the complex given by the objects $(X[n])^{i} = X^{i+n}$ and the differentials $d^i_{X[n]} = (-1)^n d_X^{i+n}$. For a morphism of complexes $X \rightarrow Y$, the mapping cone corresponding to $u$ is the complex

$$
\text{Cone}(u) := Y \oplus X[1]
$$

with the differential

$$
d^i_{\text{Cone}(u)} = \begin{pmatrix} d^i_Y & u^{i+1} \\
0 & -d_X^{i+1} \end{pmatrix} : Y^i \oplus X^{i+1} \longrightarrow Y^{i+1} \oplus X^{i+2}.
$$

The distinguished triangle corresponding to $u$ is

$$
X \rightarrow Y \longrightarrow \text{Cone}(u) \xrightarrow{p(u)} X[1],
$$

(2)
where \( j(u) \) and \( p(u) \) are the canonical injection and projection respectively [7, §1.1.3, p. 30]. By definition, a distinguished triangle is isomorphic (in the derived category) to a distinguished triangle of the form (2), or equivalently to

\[
\text{Cone}(u) [-1] \xrightarrow{p(u)[-1]} X \xrightarrow{u} Y \xrightarrow{j(u)} \text{Cone}(u).
\]

Furthermore, a distinguished triangle

\[
X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]
\]

gives rise to a long exact sequence in cohomology

\[
\cdots \longrightarrow H^r(X) \longrightarrow H^r(Y) \longrightarrow H^r(Z) \longrightarrow H^{r+1}(X) \longrightarrow \cdots
\]

Recall also that for a given commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{g} \\
X_1 & \xrightarrow{v} & Y_1
\end{array}
\]

we have a morphism of mapping cones [10, §3.1, p. 66]

\[
(g,f[1]) : \text{Cone}(u) \longrightarrow \text{Cone}(v),
\]

where

\[
(g,f[1])^i = \begin{pmatrix} g^i & 0 \\ 0 & f^{i+1} \end{pmatrix} : Y^i \oplus X^{i+1} \longrightarrow Y^i \oplus X^{i+1}.
\]

The following proposition gives two interesting results about the mapping cone which will be useful in the sequel.

**Proposition 2.1.**
1. For any maps \( u : X \longrightarrow Y \) and \( v : Y \longrightarrow Z \) of complexes, there is a distinguished triangle

\[
\begin{array}{ccc}
\text{Cone}(u) & \xrightarrow{(v, \text{id}_X)} & \text{Cone}(v \circ u) \\
& \xrightarrow{(\text{id}_Y, u[1])} & \text{Cone}(v[1]) \\
& \xrightarrow{-j(u)[1]p(v)} & \text{Cone}(u)[1].
\end{array}
\]

2. An exact commutative diagram of complexes

\[
\begin{array}{ccc}
0 & \longrightarrow & X \xrightarrow{f} \ Y \xrightarrow{g} \ Z \xrightarrow{\gamma} \ 0 \\
\downarrow{\alpha} & & \downarrow{\beta} \\
0 & \longrightarrow & X_1 \xrightarrow{f_1} \ Y_1 \xrightarrow{g_1} \ Z_1 \xrightarrow{\gamma} \ 0
\end{array}
\]

gives rise to a distinguished triangle

\[
\begin{array}{ccc}
\text{Cone}(\alpha) & \longrightarrow & \text{Cone}(\beta) \\
& \longrightarrow & \text{Cone}(\gamma) \\
& \longrightarrow & \text{Cone}(\alpha)[1].
\end{array}
\]

**Proof.** See [6, chap. II, §0, Prop. 0.10].

**Proposition 2.2.** Let

\[
0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0
\]

be a short exact sequence of complexes. Then, the maps
are quasi-isomorphisms. Further, if (3) is a split exact sequence and \( r : Y \longrightarrow X \) (resp. \( s : Z \longrightarrow Y \)) is a retraction (resp. section), then the inverse of \( \ell \) up to homotopy is:

\[
h := (-w - r[1]) : \text{Cone}(v) \longrightarrow X[1],
\]

where \( w := r[1] d_Y s \).

**Proof.** Consider the short exact sequences

\[
0 \longrightarrow \text{Cone}(\text{id}_X) \xrightarrow{(u, \text{id}_X[1])} \text{Cone}(u) \xrightarrow{q} Z \longrightarrow 0
\]

and

\[
0 \longrightarrow X[1] \xrightarrow{\ell} \text{Cone}(v) \xrightarrow{(\text{id}_Z, v[1])} \text{Cone}(\text{id}_Z) \longrightarrow 0.
\]

Since \( \text{Cone}(\text{id}_A) \) is acyclic for any complex \( A \), the long exact sequences of cohomology imply that \( q \) and \( \ell \) are quasi-isomorphisms.

Suppose now that the exact sequence (3) splits, and denote by \( r \) a retraction and \( s \) a section. Recall that for all integer \( i \), we have

\[
r' u^i = \text{id}_X^i, \quad r' s^i = 0, \quad v' s^i = \text{id}_Z^i \quad \text{and} \quad u' r^i + s^i v^i = \text{id}_Y^i.
\]

Using these equalities, we can see that the map

\[
(-w - r[1]) : \text{Cone}(v) \longrightarrow X[1]
\]

is a homomorphism of complexes, and for all integer \( i \) it satisfies

\[
\text{id}_{\text{Cone}(v)}^i - \ell^i h^i = d_{\text{Cone}(v)}^{i-1} \begin{pmatrix} 0 & 0 \\ s^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ s^{i+1} & 0 \end{pmatrix} d_{\text{Cone}(v)}^i \quad \text{and} \quad h^i \ell^i = \text{id}_{X[1]}^i.
\]

Now, given a commutative cubic diagram of complexes:

![Commutative cubic diagram](image)

we have the following proposition.

**Proposition 2.3.** The above commutative cubic diagram gives the following commutative diagram
where the two first rows and columns are distinguished triangles.

**Proof.** Obviously, the commutativity is deduced from the commutative cubic diagram. Further, Proposition 2.1, 1. shows that the two first rows and columns are distinguished triangles. □

3. Poitou–Tate duality for totally positive Galois cohomology

For a field $K$, let $K^{sep}$ denote a fixed separable closure of $K$ and $G_K = \text{Gal}(K^{sep}/K)$. Let $F$ be a number field and let $S$ be a finite set of primes of $F$ containing the set $S_2$ of dyadic primes and the set $S_\infty$ of archimedean primes. For a place $\nu$ of $F$, we denote by $F_\nu$ the completion of $F$ at $\nu$. Notice that for any **infinite** place $\nu$ of $F$, a fixed extension $F^{sep} \hookrightarrow F_\nu^{sep}$ of the embedding $F \hookrightarrow F_\nu$ defines a continuous homomorphism $\rho_\nu : G_{F_\nu} \rightarrow G_S(F)$, where $G_S(F)$ is the Galois group $\text{Gal}(F_S/F)$ of the maximal algebraic extension $F_S$ of $F$ which is unramified outside $S$. For a finite discrete $\mathbb{Z}_2[[G_S(F)]]$-module $M$, we write $M_+$ for the cokernel of the map

$$M \rightarrow \bigoplus_{\nu | \infty} \text{Ind}_{G_{F_\nu}}^{G_S(F)} M,$$

where $\text{Ind}_{G_{F_\nu}}^{G_S(F)} M$ denotes the induced module

$$\text{Ind}_{G_{F_\nu}}^{G_S(F)} M := \left\{ f : G_S(F) \longrightarrow M \mid f \text{ continuous}, \quad f(\rho_\nu(\sigma)g) = \rho_\nu(\sigma)f(g) \text{ for all } \sigma \in G_{F_\nu}, g \in G_S(F) \right\}.$$

**Remark 3.1.** If we choose another embedding $F^{sep} \hookrightarrow F_\nu^{sep}$, we can in fact see that $\rho_\nu = \rho'_\nu$ since $G_\nu$ is at most of order two. Hence $\text{Ind}_{G_{F_\nu}}^{G_S(F)} M$ is independent of the choice of the embedding.

Consider the exact sequence

$$0 \longrightarrow M \longrightarrow \bigoplus_{\nu | \infty} \text{Ind}_{G_{F_\nu}}^{G_S(F)} M \longrightarrow M_+ \longrightarrow 0. \quad (4)$$

Following [3, §5] and [2, Section 2], we define the $n$th totally positive Galois cohomology group $H_+^n(G_S(F), M)$ of $M$ by

$$H_+^n(G_S(F), M) := H^{n-1}(G_S(F), M_+), \ n \in \mathbb{Z}. \quad (5)$$
Remark 3.2. It is well known that the cohomology groups $H^i(G_S(F), M)$ and $H^i(F_v, M)$ are finite for every finite discrete module $M$. Thus, taking the cohomology of the exact sequence (4) and using Shapiro’s lemma, we can see that the groups $H^i_S(G_S(F), M)$ are also finite.

Let us give an equivalent definition of the totally positive Galois cohomology in terms of the mapping cone.

Notice that a discrete $\mathbb{Z}_2[[G_S(F)]]$-module is an union of its finite $\mathbb{Z}_2[[G_S(F)]]$-submodules. In particular, it is ind-admissible in the sense of [7, Definition 3.3.4]. It is well known that $\text{Ind}_{G_v}^{G_S(F)} M$ is a discrete module when $M$ is, hence $\text{Ind}_{G_v}^{G_S(F)} M$ is an ind-admissible $\mathbb{Z}_2[[G_S(F)]]$-module. According to [7, Proposition 3.4.2, p. 83], the exact sequence (4) gives rise to the exact sequence of complexes of $\mathbb{Z}_2$-modules

$$0 \rightarrow C_{\text{cont}}^*(G_S(F), M) \rightarrow \bigoplus_{v \mid \infty} C_{\text{cont}}^*(G_S(F), \text{Ind}_{G_v}^{G_S(F)} M) \rightarrow C_{\text{cont}}^*(G_S(F), M_+) \rightarrow 0, \quad (6)$$

where $C_{\text{cont}}^*(G_S(F), \cdot)$ denotes the complex of continuous cochains [7, Definition 3.4.1.1].

On the one hand, Proposition 2.2 shows that the complex $\text{Cone}(\cdot)$ is quasi-isomorphic to the complex $C_{\text{cont}}^*(G_S(F), M_+)$. On the other hand, for any infinite prime $v$, the compatible maps

$$G_{F_v} \hookrightarrow G_S(F) \text{ and } \text{Ind}_{G_v}^{G_S(F)} M \rightarrow M$$

induce a morphism of complexes, [7, §3.4.1.6],

$$\text{Sh}_v : C_{\text{cont}}^*(G_S(F), \text{Ind}_{G_v}^{G_S(F)} M) \rightarrow C_{\text{cont}}^*(G_{F_v}, M)$$

which is a quasi-isomorphism by Shapiro’s lemma. It follows that

$$\text{Sh}_\infty := \bigoplus_{v \mid \infty} \text{Sh}_v : \bigoplus_{v \mid \infty} C_{\text{cont}}^*(G_S(F), \text{Ind}_{G_v}^{G_S(F)} M) \rightarrow \bigoplus_{v \mid \infty} C_{\text{cont}}^*(G_{F_v}, M)$$

is a quasi-isomorphism and that $\text{Cone}(\text{Sh}_\infty)$ is acyclic.

Further, by Proposition 2.1, 1. there is a distinguished triangle

$$\text{Cone}(\text{Sh}_\infty)[−1] \rightarrow \text{Cone}(\cdot) \rightarrow \text{Cone}(\text{Sh}_\infty \circ (\cdot)) \rightarrow \text{Cone}(\text{Sh}_\infty).$$

Writing the long exact sequence of cohomology, we obtain that $\text{Cone}(\text{Sh}_\infty \circ (\cdot))$ is quasi-isomorphic to $\text{Cone}(\cdot)$. We deduce that the complexes $\text{Cone}(\text{Sh}_\infty \circ (\cdot))$ and $C_{\text{cont}}^*(G_S(F), M_+)$ are quasi-isomorphic. Then, for all $n \geq 0$

$$H^n(C_{\text{cont}}^*(G_S(F), M_+)) = H^n(\text{Cone}(\text{Sh}_\infty \circ (\cdot))).$$

Using (5), we get the following description of the totally positive Galois cohomology in terms of the mapping cone.

Lemma 3.3. For any integer $n \geq 0$, we have

$$H^n_+(G_S(F), M) = H^{n−1}(\text{Cone}(\text{Sh}_\infty \circ (\cdot))) \cong H^n(\text{Cone}(\text{Sh}_\infty \circ (\cdot)))[−1].$$

The aim now is to prove an analogue of the Poitou–Tate exact sequence for the prime $p = 2$ involving the totally positive Galois cohomology groups $H^i_+(\cdot, i = 1, 2$. The proof uses the methods of [7].

For each prime $v \in S$, fix an embedding $F_v^{\text{sep}} \hookrightarrow F_v^{\text{sep}}$ extending $F \hookrightarrow F_v$. This defines a continuous homomorphism $G_{F_v} \hookrightarrow G_F \rightarrow G_S(F)$, hence a “restriction” map [7, p. 113]

$$\text{res}_v : C_{\text{cont}}^*(G_S(F), M) \rightarrow C_{\text{cont}}^*(G_{F_v}, M).$$
Let \( res_S := \bigoplus_{v \in S} res_v \) and \( \pi_\infty \) be the projection map
\[
\bigoplus_{v \in S} C^*_{cont}(G_{F_v}, M) \xrightarrow{\pi_\infty} \bigoplus_{v \in \infty} C^*_{cont}(G_{F_v}, M).
\]

For simplicity we will use the following additional notations:
\[
C^*_S := \bigoplus_{v \in S} C^*_{cont}(G_{F_v}, M);
C^*_S := \bigoplus_{v \in S} C^*_{cont}(G_{F_v}, M);
C^*_{Sf} := \bigoplus_{v \in S} C^*_{cont}(G_{F_v}, M).
\]

By Proposition 2.1, 1. the composite morphism of complexes
\[
C^*(M) \xrightarrow{\text{res}_S} C^*_S \xrightarrow{\pi_\infty} C^*,
\]
induces a distinguished triangle
\[
\text{Cone}(\text{res}_S) \xrightarrow{\pi_\infty, \text{id}_{\text{res}_S}[1]} \text{Cone}(\pi_\infty \circ \text{res}_S) \xrightarrow{\text{id}_{\pi_\infty}, \text{res}_S[1]} \text{Cone}(\pi_\infty) \xrightarrow{-j(\text{res}_S)[1]p(\pi_\infty)} \text{Cone}(\text{res}_S)[1].
\]

Let us compare the objects of this distinguished triangle with the local and global continuous complexes. Remark that for an infinite place \( v \), the morphism \( res_v \) is the composite
\[
C^*(M) \longrightarrow C^*(\text{Ind}_{G_v}(F)) \xrightarrow{\text{Sh}_v} C^*_{cont}(G_{F_v}, M).
\]

Then the composite
\[
C^*(M) \xrightarrow{\tau} \bigoplus_{v \in \infty} C^*(\text{Ind}_{G_{F_v}}(F)) \xrightarrow{\text{Sh}_{\infty}} C^*
\]
is exactly the morphism \( \pi_\infty \circ \text{res}_S \). Let \( \text{Sh}^{-1} \) be the inverse of \( \text{Sh} \) in the derived category, then there exists an homotopy \( b \) between the maps \( \text{Sh}^{-1} \circ \pi_\infty \circ \text{res}_S \) and \( \tau \).

We have the following diagram
\[
\begin{array}{ccc}
C^*(M) & \xrightarrow{\pi_\infty \circ \text{res}_S} & C^* \\
| & | & | \\
C^*(M) & \xrightarrow{\tau} & \bigoplus_{v \in \infty} C^*(\text{Ind}_{G_{F_v}}(F)) \\
| & | & | \\
C^*(M) & \xrightarrow{\tau} & \bigoplus_{v \in \infty} C^*(\text{Ind}_{G_{F_v}}(F))
\end{array}
\]
\[
\begin{array}{ccc}
C^*(M) & \xrightarrow{j(\pi_\infty \circ \text{res}_S)} & \text{Cone}(\pi_\infty \circ \text{res}_S) \xrightarrow{-p(\pi_\infty \circ \text{res}_S)} C^*(M)[1] \\
| & | & | \\
\text{Cone}(\text{res}_S) & \xrightarrow{\pi_\infty, \text{id}_{\text{res}_S}[1]} & \text{Cone}(\pi_\infty \circ \text{res}_S) \xrightarrow{\text{id}_{\pi_\infty}, \text{res}_S[1]} \text{Cone}(\pi_\infty) \\
| & | & | \\
\text{Cone}(\text{res}_S) & \xrightarrow{\pi_\infty, \text{id}_{\text{res}_S}[1]} & \text{Cone}(\pi_\infty \circ \text{res}_S) \xrightarrow{\text{id}_{\pi_\infty}, \text{res}_S[1]} \text{Cone}(\pi_\infty)
\end{array}
\]

where \( q \) is the quasi-isomorphism of Proposition 2.2 given by the exact sequence (6) and \( \delta = -p(\tau) \circ q^{-1} \) with \( q^{-1} \) is the inverse of \( q \) in the derived category.

In the above diagram, the lines are distinguished triangles, the top left and the bottom right squares are commutative up to homotopy and the other squares are commutative.

Taking the cohomology in the two upper distinguished triangles of the diagram (7) and using the five lemma, we can see that
\[
\left( \begin{array}{cc}
\text{Sh}^{-1} & b \\
0 & \text{id}
\end{array} \right) : \text{Cone}(\pi_\infty \circ \text{res}_S) \longrightarrow \text{Cone}(\tau)
\]
is a quasi-isomorphism. Hence, we obtain a quasi-isomorphism
\[
\tilde{q} : \text{Cone}(\pi_\infty \circ \text{res}_S) \longrightarrow C^*(M_+),
\]
and a distinguished triangle
We define \( \text{res}_+ \) as the composite map

\[
\text{res}_+ : C^*(M_+) \xrightarrow{\delta} C^*(M)[1] \xrightarrow{\text{res}_f[1]} C^*_S[1].
\]

Now, we consider the following split exact sequence

\[
0 \longrightarrow C^*_S \xrightarrow{\text{id}} C^* \xrightarrow{\text{res} \cdot \pi_\infty} C^* \longrightarrow 0,
\]

where \( \text{id} \) is the canonical injection. Notice that a retraction and a section of this exact sequence are, respectively, the canonical projection \( \pi_f \) and injection \( \pi_\infty \). By Proposition 2.2, we have a quasi-isomorphism

\[
h : \text{Cone}(\pi_\infty) \longrightarrow C^*_S[1].
\]

**Proposition 3.4.** We have an isomorphism of distinguished triangles up to homotopy

\[
\begin{array}{ccc}
\text{Cone}(\text{res}_S) & \xrightarrow{\pi_\infty \circ \text{res}_S[1]} & \text{Cone}(\pi_\infty) \\
\xrightarrow{\pi_\infty \circ \text{id}_{C^*(M)[1]}} & \xrightarrow{\text{id}_{C^* \circ \text{res}_S[1]}} & \xrightarrow{h} \text{Cone}(\text{res}_S)[1] \\
\xrightarrow{\pi_\infty \circ \text{id}_{C^*(M)[1]}} & \xrightarrow{\text{id}_{C^* \circ \text{res}_S[1]}} & \xrightarrow{h} \text{Cone}(\text{res}_S)[1]
\end{array}
\]

**Proof.** Recall that the quasi-isomorphism \( h \) is given by \(-w - \pi_f[1]\), where \( w = \pi_f[1]d_{C^*_S}i_\infty \) (see Proposition 2.2). Notice that \( i_\infty \) is a homomorphism of complexes, so

\[
w = \pi_f[1]d_{C^*_S}i_\infty = \pi_f[1]i_\infty[1]d_{C^*_S} = 0.
\]

Hence,

\[
h(\pi_\infty \circ \text{res}_S[1]) = 0 \quad \pi_f[1] \quad 0 \\
\quad \pi_\infty[\pi_\infty \circ \text{res}_S] \\
\quad 0 \quad \text{res}_S[1].
\]

Using the diagram (7), we have a commutative diagram

\[
\begin{array}{ccc}
C^*(M) & \xrightarrow{\pi_\infty \circ \text{res}_S} & C^*_S \\
\xrightarrow{\text{res}_+ \circ \tilde{q}} & & \xrightarrow{\delta} C^*(M)[1] \\
\end{array}
\]

In particular, we have \( \delta \circ \tilde{q} = -p(\pi_\infty \circ \text{res}_S) \). Thus, we get

\[
\begin{align*}
\text{res}_+ \circ \tilde{q} &= \text{res}_S[1] \circ \delta \circ \tilde{q} \\
&= -\text{res}_S[1] \circ p(\pi_\infty \circ \text{res}_S) \\
&= (0 - \text{res}_S[1]).
\end{align*}
\]
Observe that the quasi-isomorphism \( \ell : X[1] \longrightarrow \text{Cone}(v) \) (see Proposition 2.2) satisfies
\[ -j(\text{res}_S)[1]p(\pi_\infty)\ell = j(\text{res}_S)[1]j. \]

Since \( \ell \) and \( h \) are inverse up to homotopy, the right side square is commutative up to homotopy.

In the sequel, we adopt the following usual notations:

\( (.)^* \): the Kummer dual \( \text{Hom}(., \mu_{2^\infty}) \), where \( \mu_{2^\infty} \) is the group of all roots of unity of 2-power order;
\( (.)^\vee \): the Pontryagin dual \( \text{Hom}(., Q_2/\mathbb{Z}_2) \).

As a consequence of Proposition 3.4, we get the following analogue of the Poitou–Tate exact sequence for totally positive Galois cohomology.

**Proposition 3.5.** Let \( M \) be a finite discrete \( \mathbb{Z}_2[[G_S(F)]] \)-module. Then there is a long exact sequence

\[
0 \rightarrow \bigoplus_{v \in S_F} H^0(F_v, M) \rightarrow H^0(\text{Cone}(\text{res}_S)) \rightarrow H^1_+(G_S(F), M) \rightarrow \bigoplus_{v \in S_F} H^1(F_v, M)
\]

\[
\xrightarrow{\text{Hodge pairing}} H^1(G_S(F), M^\vee) \rightarrow H^2_+(G_S(F), M) \rightarrow \bigoplus_{v \in S_F} H^2(F_v, M) \rightarrow H^0(G_S(F), M^\vee) \rightarrow 0.
\]

**Proof.** It is well known that for \( n \geq 3 \), the cohomology group \( H^n(G_S(F), M) \) is isomorphic to \( \bigoplus_v \text{real } H^n(F_v, M) \). Hence, from the exact sequence (4) it follows that, for all \( n \geq 3 \)

\[ H^n_+(G_S(F), M) = 0. \]

Also, remark that \( H^n_+(G_S(F), M) \) is trivial by definition. Thus, to obtain the desired exact sequence it suffices to take the cohomology of the diagram of Proposition 3.4 and calculate \( H^n(\text{Cone}(\text{res}_S)) \) for \( n = 1, 2 \). Let us consider the map

\[ \widehat{\text{res}}_S : C^*(M) \longrightarrow C^*_S \bigoplus_{v | \infty} \check{C}^*_\text{cont}(G_F, M), \]

where \( \check{C}^*_\text{cont}(G_F, M) \) is the complete Tate cochain complex, \( \widehat{\text{res}}_S = \bigoplus_{v \in S_F} \text{res}_v \bigoplus_{v | \infty} \text{res}_v \) and for \( v | \infty \) the map \( \widehat{\text{res}}_v \) is the composite map

\[ \widehat{\text{res}}_v : C^*(M) \xrightarrow{\text{res}_v} C^*_\text{cont}(G_F, M) \xrightarrow{\tau_v} \check{C}^*_\text{cont}(G_F, M), \]

the map \( \tau_v \) being the canonical injection. Recall that the continuous cohomology group with compact support \( \hat{H}^n_{c, \text{cont}}(G_S(F), M) \) is defined by

\[ \hat{H}^n_{c, \text{cont}}(G_S(F), M) := H^n(\text{Cone}(\widehat{\text{res}}_S)[-1]) \]

cf. [7, ch. 5, p. 132]. Then, for all \( n \geq 1 \),

\[ H^n(\text{Cone}(\text{res}_S)) = H^n(\text{Cone}(\widehat{\text{res}}_S)) = \hat{H}^{n+1}_{c, \text{cont}}(G_S(F), M). \]

In particular, \( H^n(\text{Cone}(\text{res}_S)) \cong H^{2-n}(G_S(F), M^\vee) \) as a consequence of [7, Proposition 5.7.4].

**Remark 3.6.** Proposition 3.5 is a slight generalization of [5, Proposition 2.6]. However, a certain argument concerning the continuous cohomology with compact support in [5, §2.2, (4), p. 6] turns out to be incorrect. In [5, §2.2, (4), p. 6] the author claimed that \( H^n(G_S(F), M_S) = \)
\[ \hat{H}_{n, \text{cont}}(G_S(F), M), \] where \( M \) is the cokernel of the canonical map

\[ M \longrightarrow \bigoplus_{v \in S} \text{Ind}^G_{G_v} M. \]

This is not always true but the results of [5] remain unchanged using Proposition 3.5.

Let us recall the local duality theorem (e.g. [6, Corollary 2.3, p. 34]). Let \( M \) be a finite discrete \( \mathbb{Z}_2[[G_S(F)]] \)-module. For \( n = 0, 1, 2 \) and for every place \( v \) of \( F \), the cup products

\[ H^n(F_v, M) \times \hat{H}^{2-n}(F_v, M^*) \longrightarrow H^2(F_v, \mu_{2^n}) \cong \mathbb{Q}_2/\mathbb{Z}_2, \]

if \( v \) is finite

\[ H^n(F_v, M) \times \hat{H}^{2-n}(F_v, M^*) \longrightarrow H^2(F_v, \mu_{2^n}), \]

if \( v \) is infinite

are perfect pairings, where \( H^n(F_v, M) \) are the Tate cohomology groups.

For \( n = 1, 2 \), we define the groups \( \text{III}^n_{S_f}(M) \) and \( \text{III}^{n,+}_{S_f}(M) \) to be the kernels of the localization maps

\[ \text{III}^n_{S_f}(M) := \ker(H^n(G_S(F), M) \longrightarrow \bigoplus_{v \in S} H^n(F_v, M)), \]

\[ \text{III}^{n,+}_{S_f}(M) := \ker(H^n(G_S(F), M) \xrightarrow{\text{res}^n} \bigoplus_{v \in S} H^n(F_v, M)), \]

where \( \text{res}^n \) is induced from (8).

We are interested in the study of the analogue of the pairing (1) for the groups \( \text{III}^{n,+}_{S_f}(M) \) with \( n = 1, 2 \).

Let’s start with the case \( n = 2 \). We have the following proposition:

**Proposition 3.7.** Let \( M \) be a finite discrete \( \mathbb{Z}_2[[G_S(F)]] \)-module. There is a perfect pairing, functorial in \( M \),

\[ \text{III}^2_{S_f}(M) \times \text{III}^1_{S_f}(M^*) \longrightarrow \mathbb{Q}_2/\mathbb{Z}_2. \]

**Proof.** Taking cohomology of the diagram in Proposition 3.4, the map

\[ H^2(G_S(F), M) \longrightarrow \bigoplus_{v \in S} H^2(F_v, M) \]

in Proposition 3.5 is identified with the map \( \text{res}^2 : H^2(G_S(F), M) \longrightarrow \bigoplus_{v \in S} H^2(F_v, M) \). Thus, by the definition of \( \text{III}^{2,+}_{S_f}(M) \), the exact sequence of Proposition 3.5 gives the following one:

\[ \bigoplus_{v \in S} H^1(F_v, M) \longrightarrow H^1(G_S(F), M^*)^\vee \longrightarrow \text{III}^{2,+}_{S_f}(M) \longrightarrow 0. \]

Dualizing this exact sequence and using the local duality (9), we get

\[ \text{III}^{2,+}_{S_f}(M)^\vee \cong \text{III}^1_{S_f}(M^*). \]

The case \( n = 1 \) is more complicated, it requires more than a simple manipulation of the long exact sequence of Proposition 3.5. Let’s start by recalling the morphisms already defined in this section.
In addition, we consider the projection map $\tilde{\pi}_\infty : C_S^\bullet \to \bigoplus_{v|\infty} \hat{C}_{cont}^\bullet (G_F, M)$. We have the following proposition:

**Proposition 3.8.** Let $M$ be a finite discrete $\mathbb{Z}_2[[G_S(F)]]$-module. We have an exact sequence, functorial in $M$,

$$0 \to Z(M) \to \mathbb{I}^1_{\mathcal{H}}(M) \to \mathbb{I}^2_{\mathcal{H}}(M^*)^\vee \to 0,$$

where

$$Z(M) := \ker(H^0(\text{Cone}(\pi_\infty \circ \text{res}_S))) \to H^0(\text{Cone}(\tilde{\pi}_\infty \circ \tilde{\text{res}}_S))).$$

**Proof.** Consider the commutative exact diagram

$$0 \to C_S^\bullet _f \to C_{S_f}^\bullet \oplus C_{C}^\bullet \xrightarrow{\pi_\infty} C_{C}^\bullet \to 0$$

and

$$0 \to C_S^\bullet _f \to C_{S_f}^\bullet \oplus \hat{C}_{\infty}^\bullet \xrightarrow{\hat{\pi}_\infty} \hat{C}_{\infty}^\bullet \to 0.$$

where $\hat{C}_{\infty}^\bullet := \bigoplus_{v|\infty} \hat{C}_{cont}^\bullet (G_F, M)$ and $\tau_\infty := \bigoplus_{v|\infty} \tau_v$.

Since Cone(id) is acyclic, we obtain, by Propositions 2.1 and 2.2, that

$$\text{Cone}((\text{id}, \tau_\infty)) \sim \text{Cone}(\tau_\infty) \quad \text{and} \quad \text{Cone}(\pi_\infty) \sim \text{Cone}(\tilde{\pi}_\infty),$$

where $X \sim Y$ means that the complexes $X$ and $Y$ are quasi-isomorphic. Using Proposition 2.3, the commutative cubic diagram

\[ \begin{array}{ccc}
C^\bullet (M) & \xrightarrow{\text{res}_S} & C_{S_f}^\bullet \oplus C_{C}^\bullet \\
\downarrow{\text{id}} & & \downarrow{\text{id}, \tau_\infty} \\
C^\bullet (M) & \xrightarrow{\tilde{\text{res}}_S} & C_{S_f}^\bullet \oplus \hat{C}_{\infty}^\bullet \\
\downarrow{\text{id}} & & \downarrow{\hat{\pi}_\infty} \\
C^\bullet (M) & \xrightarrow{\pi_\infty \circ \text{res}_S} & C_{\infty}^\bullet \\
\downarrow{\text{id}} & & \downarrow{\tau_\infty} \\
C^\bullet (M) & \xrightarrow{\pi_\infty \circ \tilde{\text{res}}_S} & \hat{C}_{\infty}^\bullet \\
\end{array} \]
gives rise to the commutative diagram

\[
\begin{array}{c}
\text{Cone}(\text{res}_S) \longrightarrow \text{Cone}(\widetilde{\text{res}}_S) \longrightarrow \text{Cone}((\text{id}, \tau_\infty)) \longrightarrow \text{Cone}(\text{res}_S)[1] \\
\text{Cone}(\pi_\infty \circ \text{res}_S) \longrightarrow \text{Cone}(\widetilde{\pi}_\infty \circ \widetilde{\text{res}}_S) \longrightarrow \text{Cone}(\tau_\infty) \longrightarrow \text{Cone}(\pi_\infty \circ \text{res}_S)[1] \\
\text{Cone}(\pi_\infty) \longrightarrow \text{Cone}(\widetilde{\pi}_\infty) \\
\text{Cone}(\text{res}_S)[1] \longrightarrow \text{Cone}(\widetilde{\text{res}}_S)[1].
\end{array}
\]

Taking the cohomology, we obtain for all \( n \in \mathbb{Z} \) an exact commutative diagram

\[
\begin{array}{c}
H^n(\text{Cone}(\text{id}, \tau_\infty)) \longrightarrow H^n(\text{Cone}(\tau_\infty)) \\
H^n(\text{Cone}(\pi_\infty)) \longrightarrow H^{n+1}(\text{Cone}(\text{res}_S)) \longrightarrow H^{n+1}(\text{Cone}(\pi_\infty \circ \text{res}_S)) \longrightarrow H^{n+1}(\text{Cone}(\pi_\infty)) \\
H^n(\text{Cone}(\widetilde{\pi}_\infty)) \longrightarrow H^{n+1}(\text{Cone}(\widetilde{\text{res}}_S)) \longrightarrow H^{n+1}(\text{Cone}(\pi_\infty \circ \widetilde{\text{res}}_S)) \longrightarrow H^{n+1}(\text{Cone}(\widetilde{\pi}_\infty)) \\
H^{n+1}(\text{Cone}((\text{id}, \tau_\infty))) \longrightarrow H^{n+1}(\text{Cone}(\tau_\infty)).
\end{array}
\]

In particular, for \( n = -1 \) we have a commutative diagram

\[
\begin{array}{c}
H^{-1}(\text{Cone}(\tau_\infty)) \\
H^0(\text{Cone}(\pi_\infty \circ \text{res}_S)) \longrightarrow H^0(\text{Cone}(\pi_\infty)) \\
H^0(\text{Cone}(\widetilde{\pi}_\infty \circ \widetilde{\text{res}}_S)) \longrightarrow H^0(\text{Cone}(\widetilde{\pi}_\infty)) \\
H^0(\text{Cone}(\tau_\infty)) = 0,
\end{array}
\]

where the triviality of \( H^0(\text{Cone}(\tau_\infty)) \) follows from the surjectivity of \( H^0(F_v, M) \to \hat{H}^0(F_v, M) \) and the equality \( H^1(F_v, M) = \hat{H}^1(F_v, M) \), for all infinite place \( v \). Also we have an exact sequence

\[
H^{-1}(\text{Cone}(\pi_\infty)) \longrightarrow H^0(\text{Cone}(\widetilde{\text{res}}_S)) \longrightarrow \text{III}^{1+}_{S_y}(M) \longrightarrow 0,
\]

where

\[
\text{III}^{1+}_{S_y}(M) := \ker(H^0(\text{Cone}(\widetilde{\pi}_\infty \circ \widetilde{\text{res}}_S)) \longrightarrow H^0(\text{Cone}(\widetilde{\pi}_\infty))).
\]
From Proposition 3.4, we deduce that:

$$\text{III}_{S_1}^{1,+}(M) = \ker(H^0(\text{Cone}(\pi_\infty \circ \text{res}_S))) \rightarrow H^0(\text{Cone}(\pi_\infty)).$$

So, the diagram (11) leads to the following exact sequence

$$0 \rightarrow Z(M) \rightarrow \text{III}_{S_1}^{1,+}(M) \rightarrow \text{III}_{S_1}^{1,+}(M) \rightarrow 0,$$

where $Z(M) := \ker(H^0(\text{Cone}(\pi_\infty \circ \text{res}_S))) \rightarrow H^0(\text{Cone}(\pi_\infty \circ \text{res}_S))).$

Recall that, for all $n$, $H^n(\text{Cone}(\text{res}_S)) = \hat{F}_{e, \text{cont}}^n(G_S(F), M)$. Then, by [7, Proposition 5.7.4] we have

$$H^0(\text{Cone}(\text{res}_S)) \cong H^2(G_S(F), M^\vee).$$

According to the quasi-isomorphisms

$$\text{Cone}(\pi_\infty) \sim C_{S_1}^*[1] \text{ and } \text{Cone}(\pi_\infty) \sim \text{Cone}(\pi_\infty)$$

(see Proposition 3.4 and (10), respectively), we have $H^{-1}(\text{Cone}(\pi_\infty)) \cong \oplus_{v \in S_1} H^0(F_v, M)$. Therefore, the exact sequence (12) and the local duality (9) give the following isomorphism

$$\text{II}_{S_1}^{1,+}(M) \cong \text{II}_{S_1}^2(M^\vee).$$

(14)

Finally, by the exact sequence (13) and the isomorphism (14), we deduce the desired result. □

In the next lemma, we give an explicit description of the module

$$Z(M) := \ker(H^0(\text{Cone}(\pi_\infty \circ \text{res}_S))) \rightarrow H^0(\text{Cone}(\pi_\infty \circ \text{res}_S))).$$

First, recall that for every infinite place $v$ of $F$ a fixed extension $F^{sep} \hookrightarrow F_v^{sep}$ of the embedding $F \hookrightarrow F_v$ defines a continuous homomorphism

$$G_{F_v} \hookrightarrow G_S(F),$$

hence a restriction map $\text{res}_v = (\text{res}_v^x)$

$$C^0(M) = M \xrightarrow{d^0_{C(M)}} C^1(M) \rightarrow \cdots$$

$$C^0_{\text{cont}}(G_{F_v}, M) = M \xrightarrow{d^0_{\text{cont}}(G_{F_v}, M)} C^1_{\text{cont}}(G_{F_v}, M) \rightarrow \cdots$$

Second, observe that

$$\text{res}_v^0(x) = x \quad \text{for all } x \in \ker d^0_{C(M)} = M^{G_{v}(F)}.$$

Moreover, if $f : C^*(M) \rightarrow C^*_\text{cont}(G_{F_v}, M)$ and $\text{res}_v$ are homotopic, then

$$f^0(x) = \text{res}_v^0(x) = x \quad \text{for all } x \in \ker d^0_{C(M)}.$$

Indeed, since $f$ and $\text{res}_v$ are homotopic, there exists a morphism

$$h^1 : C^1(M) \rightarrow C^0_{\text{cont}}(G_{F_v}, M) \text{ such that } f^0 - \text{res}_v^0 = h^1 d^0_{C(M)}.$$

Thus, $f^0(x) = \text{res}_v^0(x)$ for all $x \in \ker d^0_{C(M)}.$
Now, since \( \hat{\pi}_\infty \circ \hat{\text{res}}_S = \tau_\infty \circ \pi_\infty \circ \text{res}_S \) and \((\tau_\infty)^i = \text{id}\) for all integers \(i \geq 0\), we get

\[
\begin{pmatrix}
  d^0_{\text{Cone}(\hat{\pi}_\infty \circ \hat{\text{res}}_S)} &=& d^0_{\text{Cone}(\pi_\infty \circ \text{res}_S)} \\
  d^{-1}_{\text{Cone}(\hat{\pi}_\infty \circ \hat{\text{res}}_S)} &=& \left( \begin{array}{cc}
  \oplus_{v|\infty} N_v & (\pi_\infty \circ \text{res}_S)^0 \\
  0 & -d^0_{\text{Cone}(\hat{\pi}_\infty \circ \hat{\text{res}}_S)}
\end{array} \right)
\end{pmatrix}
\]

where \( N_v = \sum_{\sigma \in G_v} \sigma \). Further, remark that \((\pi_\infty \circ \text{res}_S)^0 = \oplus_{v|\infty} \text{res}_v^0\).

Recall that

\[
\hat{C}_\infty := \oplus_{v|\infty} M \quad \text{and} \quad \hat{Z}(M) := \ker(H^0(\text{Cone}(\pi_\infty \circ \text{res}_S)) \longrightarrow H^0(\text{Cone}(\hat{\pi}_\infty \circ \hat{\text{res}}_S))).
\]

**Lemma 3.9.** Let \( M \) be a finite discrete \( Z_2[[G_S(F)]] \)-module. We have the following isomorphism of \( Z_2[[G_S(F)]] \)-modules

\[
Z(M) \cong (\oplus_{v|\infty} N_v)(\hat{C}^{-1}_\infty)/I_\infty,
\]

where \( I_\infty := (\oplus_{v|\infty} \text{res}_v^0)(\ker(d^0_{\text{Cone}(\hat{\pi}_\infty \circ \hat{\text{res}}_S)})) \cap (\oplus_{v|\infty} N_v)(\hat{C}^{-1}_\infty)\).

**Proof.** The following exact commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Im} \, d^{-1}_{\text{Cone}(\pi_\infty \circ \text{res}_S)} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \ker d^0_{\text{Cone}(\pi_\infty \circ \text{res}_S)}
\end{array}
\]

shows that

\[
Z(M) \cong \text{Im} \, d^{-1}_{\text{Cone}(\pi_\infty \circ \text{res}_S)}/\text{Im} \, d^{-1}_{\text{Cone}(\pi_\infty \circ \text{res}_S)};
\]

(15)

Since \( C^{-1}_\infty = 0 \), we can identify \((\text{Cone}(\pi_\infty \circ \text{res}_S))^{-1} = C^0(M)\). Consider the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \ker d^{-1}_{\text{Cone}(\pi_\infty \circ \text{res}_S)} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{Im} \, d^{-1}_{\text{Cone}(\pi_\infty \circ \text{res}_S)}
\end{array}
\]

(16)

The kernel \( \ker d^{-1}_{\text{Cone}(\pi_\infty \circ \text{res}_S)} \) is trivial. Indeed, let \( y \in C^0(M) \) such that \( d^{-1}_{\text{Cone}(\pi_\infty \circ \text{res}_S)}(0, y) = 0 \). This means that

\[
\begin{cases}
  \text{res}_v^0(y) = 0, \quad \text{for all } v|\infty \\
y \in \ker \hat{d}^0_{\text{Cone}(\hat{\pi}_\infty \circ \hat{\text{res}}_S)}
\end{cases}
\]

Since \( y = \text{res}_v^0(y) \), we get \( y = 0 \).
Therefore, the commutative diagram (16) induces the following exact sequence
\[
0 \longrightarrow \ker d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1} \overset{s}{\longrightarrow} \hat{C}_\infty^{-1} \longrightarrow \text{Im } d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1}/\text{Im } d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1} \longrightarrow 0,
\]
where \(s\) is the composite map
\[
\ker d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1} \hookrightarrow \hat{C}_\infty^{-1} \oplus C^0(M) \rightarrow \hat{C}_\infty^{-1}.
\]

Let \(\varphi\) to be the composite map
\[
\ker d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1} \overset{s}{\longrightarrow} \hat{C}_\infty^{-1} \oplus C^0(M) \longrightarrow \hat{C}_\infty^{-1}.
\]

We claim that \(\text{Im } (\varphi) = I_\infty\). Indeed, let \((x_\nu)_{\nu|\infty}\) be an element of \(\text{Im } (\varphi)\). Hence, there exists \(((a_\nu)_{\nu|\infty}, y)\) in \(\ker d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1}\) such that
\[
(x_\nu)_{\nu|\infty} = \varphi(((a_\nu)_{\nu|\infty}, y)) = \left( \oplus_{\nu|\infty} N_\nu \right)((a_\nu)_{\nu|\infty}).
\]

Since \(((a_\nu)_{\nu|\infty}, y)\) is an element of \(\ker d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1}\), we get
\[
\left\{
\begin{array}{l}
(\oplus_{\nu|\infty} N_\nu)((a_\nu)_{\nu|\infty}) + (\pi_{\infty} \circ \text{res}_S)^0(y) = 0 \\
-d_{\text{Cone}(M)}^0(y) = 0.
\end{array}
\right.
\]

Therefore, \((x_\nu)_{\nu|\infty}\) belongs to \((\oplus_{\nu|\infty} \text{res}_S^0)(\ker d_{\text{Cone}(M)}^0)\). This proves that \(\text{Im } (\varphi) \subseteq I_\infty\).

Let’s prove the other inclusion. Let \((x_\nu)_{\nu|\infty} \in I_\infty\), so there exist \(y\) in \(\ker d_{\text{Cone}(M)}^0\) and \((a_\nu)_{\nu|\infty}\) in \(\hat{C}_\infty^{-1}\) such that for all infinite place \(\nu\), \(x_\nu = \text{res}_S^0(y) = N_\nu(a_\nu)\). Hence \(((a_\nu)_{\nu|\infty}, -y)\) belongs to \(\ker d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1}\). Therefore,
\[
\varphi(((a_\nu)_{\nu|\infty}, -y)) = (\oplus_{\nu|\infty} N_\nu)((a_\nu)_{\nu|\infty}) = (x_\nu)_{\nu|\infty}.
\]

This proves \(I_\infty \subseteq \text{Im } (\varphi)\).

By the definition of \(\varphi\), we then have a commutative diagram
\[
\begin{array}{c}
0 \rightarrow \ker d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1} \overset{s}{\rightarrow} \hat{C}_\infty^{-1} \rightarrow \text{Im } d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1}/\text{Im } d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1} \rightarrow 0 \\
0 \rightarrow I_\infty \rightarrow \hat{C}_0^0 \rightarrow \hat{C}_\infty^0/I_\infty \rightarrow 0.
\end{array}
\]

Observe that a pair \(((a_\nu)_{\nu|\infty}, y)\) in \(\ker d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1}\) belongs to \(\ker \varphi\) exactly when \((a_\nu)_{\nu|\infty}\) is an element of \(\ker (\oplus_{\nu|\infty} N_\nu)\) and \(y = 0\). Then \(s\) induces an isomorphism between \(\ker \varphi\) and \(\ker (\oplus_{\nu|\infty} N_\nu)\). By the snake lemma, we obtain
\[
\text{Im } d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1}/\text{Im } d_{\text{Cone}(\pi_{\infty} \circ \text{res}_S)}^{-1} \cong \ker (\hat{C}_\infty^0/I_\infty \longrightarrow \hat{C}_\infty^0/(\oplus_{\nu|\infty} N_\nu)(\hat{C}_\infty^{-1}))
\]

\[
= (\oplus_{\nu|\infty} N_\nu)(\hat{C}_\infty^{-1})/I_\infty.
\]

Therefore, the isomorphism (15) gives
\[
\text{Z}(M) \cong (\oplus_{\nu|\infty} N_\nu)(\hat{C}_\infty^{-1})/I_\infty.
\]

Summarizing, we get the main result of this paper namely, the global Poitou–Tate duality for totally positive Galois cohomology. \(\square\)
**Theorem 3.10.** Let $M$ be a finite discrete $\mathbb{Z}_2[[G_S(F)]]$-module.

i. There is a perfect pairing, functorial in $M$,

$$\mathbf{II}^+_S(M) \times \mathbf{II}^+_S(M') \longrightarrow \mathbb{Q}/\mathbb{Z}_2.$$

ii. We have an exact sequence, functorial in $M$,

$$0 \longrightarrow (\bigoplus_{v \mid \infty} N_v)(\mathcal{C}_\infty^{-1})/I_\infty \longrightarrow \mathbf{II}^+_S(M) \longrightarrow \mathbf{II}^+_S(M')^\vee \longrightarrow 0,$$

where $I_\infty = (\bigoplus_{v \mid \infty} \text{res}^0_{v})(\ker^0_{C(M)}) \cap (\bigoplus_{v \mid \infty} N_v)(\mathcal{C}_\infty^{-1}).$

The aim now is to extend the above theorem to a $\mathbb{Z}_2[[G_S(F)]]$-module $M$ of finite or co-finite type over $\mathbb{Z}_2$. To do this, we need the following two lemmas.

As in the finite case, we set

$$Z(M) := \ker(H^0(\text{Cone}(\pi_\infty \circ \text{res}_S))) \longrightarrow H^0(\text{Cone}(\widehat{\pi}_\infty \circ \widehat{\text{res}}_S)), $$

where

$$\pi_\infty \circ \text{res}_S : C^\bullet(M) \longrightarrow C^\bullet(\mathcal{M}) := \bigoplus_{v \mid \infty} C^\bullet_{\text{cont}}(G_{F_v}, M)$$

and

$$\widehat{\pi}_\infty \circ \widehat{\text{res}}_S : C^\bullet(M) \longrightarrow C^\bullet(\mathcal{M}) := \bigoplus_{v \mid \infty} \widehat{C}^\bullet_{\text{cont}}(G_{F_v}, M).$$

Let $T$ (resp. $A$) be a $\mathbb{Z}_2[[G_S(F)]]$-modules of finite (resp. co-finite) type over $\mathbb{Z}_2$. Hence

$$T = \lim_{\to} T/2^n T \quad \text{and} \quad A = \lim_{\to} A[2^n],$$

where $T/2^n T$ and $A[2^n]$ are the co-kernel and the kernel of the multiplication by $2^n$ on $T$ and $A$, respectively.

**Lemma 3.11.** We have the following isomorphisms

$$\lim_{\to} Z(T/2^n T) \simeq Z(T) \quad \text{and} \quad \lim_{\to} Z(A[2^n]) \simeq Z(A).$$

**Proof.** Consider the commutative diagram

$$C^\bullet(T/2^n+1 T) \xrightarrow{(\pi_\infty \circ \text{res}_S)_{n+1}} C^\bullet(\mathcal{T}/2^n+1 T)$$

$$\quad \downarrow \quad \downarrow$$

$$C^\bullet(T/2^n T) \xrightarrow{(\pi_\infty \circ \text{res}_S)_n} C^\bullet(\mathcal{T}/2^n T),$$

(17)

where the vertical maps are the canonical maps. Then, using [7, Lemma 4.1.2], we obtain the following commutative diagram

$$C^\bullet(T) \xrightarrow{\pi_\infty \circ \text{res}_S} C^\bullet(\mathcal{T})$$

$$\downarrow t \quad \downarrow t$$

$$\lim C^\bullet(T/2^n T) \xrightarrow{((\pi_\infty \circ \text{res}_S)_n)} \lim C^\bullet(\mathcal{T}/2^n T).$$
Thus, we have an isomorphism
\[ \text{Cone}(\pi_\infty \circ \text{res}_S) \simeq \text{Cone}((\pi_\infty \circ \text{res}_S)_n). \]

Since \( \lim \) is a covariant additive functor, we get
\[ \text{Cone}((\pi_\infty \circ \text{res}_S)_n) \simeq \lim \text{Cone}((\pi_\infty \circ \text{res}_S)_n). \]

Hence
\[ \text{Cone}(\pi_\infty \circ \text{res}_S) \simeq \lim \text{Cone}((\pi_\infty \circ \text{res}_S)_n). \]

The commutative diagram (17) ensures that the maps
\[ \text{Cone}((\pi_\infty \circ \text{res}_S)_{n+1}) \longrightarrow \text{Cone}((\pi_\infty \circ \text{res}_S)_n) \]
are onto, so the complex \( (\text{Cone}((\pi_\infty \circ \text{res}_S)_n)) \) satisfies the Mittag–Leffler condition (in the sense of [11, Definition 3.5.6]). Then, we have an exact sequence [11, p. 84]
\[
0 \rightarrow \lim^1 H^{i-1}(\text{Cone}((\pi_\infty \circ \text{res}_S)_n)) \rightarrow H^i(\lim \text{Cone}((\pi_\infty \circ \text{res}_S)_n)) \rightarrow \lim H^i(\text{Cone}((\pi_\infty \circ \text{res}_S)_n)) \rightarrow 0.
\]

Since \( H^i(\text{Cone}((\pi_\infty \circ \text{res}_S)_n)) \) are finite (as a consequence of the finiteness of \( H^i(C^\bullet(T/2^nT)) \) and \( H^i(C^\bullet(T/2^nT)) \)), the group \( \lim^1 H^{i-1}(\text{Cone}((\pi_\infty \circ \text{res}_S)_n)) \) is trivial. Then, we have an isomorphism
\[
H^i(\lim \text{Cone}((\pi_\infty \circ \text{res}_S)_n)) \simeq \lim H^i(\text{Cone}((\pi_\infty \circ \text{res}_S)_n)).
\]

It follows that
\[
H^i(\text{Cone}(\pi_\infty \circ \text{res}_S)) \simeq H^i(\lim \text{Cone}((\pi_\infty \circ \text{res}_S)_n)) \simeq \lim H^i(\text{Cone}((\pi_\infty \circ \text{res}_S)_n)).
\]

Proceeding in the same way with the complexes \( \text{Cone}((\text{res}_\infty \circ \text{res}_S)_n) \) instead of \( \text{Cone}((\pi_\infty \circ \text{res}_S)_n) \), we obtain
\[
H^i(\text{Cone}(\text{res}_\infty \circ \text{res}_S)) \simeq \lim H^i(\text{Cone}((\text{res}_\infty \circ \text{res}_S)_n)).
\]

Hence, by taking the inverse limit in the exact sequence (see (11) for the surjectivity)
\[
0 \longrightarrow Z(T/2^nT) \longrightarrow H^0(\text{Cone}((\pi_\infty \circ \text{res}_S)_n)) \longrightarrow H^0(\text{Cone}((\text{res}_\infty \circ \text{res}_S)_n)) \longrightarrow 0,
\]
we obtain a commutative diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & \lim Z(T/2^nT) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \lim H^0(\text{Cone}((\pi_\infty \circ \text{res}_S)_n)) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^0(\text{Cone}(\pi_\infty \circ \text{res}_S)) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^0(\text{Cone}(\text{res}_\infty \circ \text{res}_S)) \\
\end{array}
\]
\[
\longrightarrow 0.
\]

Thus, we get the first isomorphism \( \lim Z(T/2^nT) \simeq Z(T). \)

The second isomorphism \( \lim Z(A[2^n]) \simeq Z(A) \) is a consequence of the following isomorphisms
\[ H^i(\text{Cone}(\pi_\infty \circ \text{res}_S)) \cong \lim_{\to} H^i(\text{Cone}(\pi_\infty \circ \text{res}_S)_n) \text{ and} \]
\[ H^i(\text{Cone}(\widehat{\pi}_\infty \circ \text{res}_S)) \cong \lim_{\to} H^i(\text{Cone}(\widehat{\pi}_\infty \circ \text{res}_S)_n) \]
which are ensured by the exactness of the functor \( \lim \).

Here \((\pi_\infty \circ \text{res}_S)_n\) and \((\widehat{\pi}_\infty \circ \text{res}_S)_n\) are the maps
\[
\begin{align*}
(\pi_\infty \circ \text{res}_S)_n : & \quad C^*(A[2^n]) \longrightarrow C^*_\infty(A[2^{n+1}]) \quad \text{and} \\
(\widehat{\pi}_\infty \circ \text{res}_S)_n : & \quad C^*(A[2^n]) \longrightarrow \widehat{C}^*_\infty(A[2^{n+1}])
\end{align*}
\]

Lemma 3.12. For \(i = 1, 2\), we have the following isomorphisms

- \( \text{III}^+_S(T) \cong \lim_{\to} \text{III}^+_S(T/2^nT) \) and \( \text{III}^+_S(T) \cong \lim_{\to} \text{III}^+_S(T/2^nT) \);
- \( \text{III}^+_S(A) \cong \lim_{\to} \text{III}^+_S(A[2^n]) \) and \( \text{III}^+_S(A) \cong \lim_{\to} \text{III}^+_S(A[2^n]) \).

Proof. Let \(i \geq 0\) be an integer. Remark that the groups \(G_S(F)\) and \(G_{F_i}\) satisfy the finiteness condition (F) in [7, §4.2, p. 96]. Using [7, Lemma 4.2.2], we get the following isomorphisms
\[
\begin{align*}
H^i(G_S(F), T) & \cong \lim_{\to} H^i(G_S(F), T/2^nT) \\
H^i(F_n, T) & \cong \lim_{\to} H^i(F_n, T/2^nT).
\end{align*}
\]
By Remark 3.2, the groups \(H^i_+(G_S(F), T/2^nT)\) are finite. Thus the group
\[
\lim_{\to} H^i_+(G_S(F), T/2^nT)
\]
is trivial. According to [7, Corollary 4.1.3], we have an isomorphism
\[
H^i_+(G_S(F), T) \cong \lim_{\to} H^i_+(G_S(F), T/2^nT).
\]
Taking the inverse limit in the exact sequences
\[
\begin{align*}
0 & \longrightarrow \text{III}^+_S(T/2^nT) \longrightarrow H^i(G_S(F), T/2^nT) \longrightarrow \bigoplus_{n \in \mathbb{Z}} H^i(F_n, T/2^nT) \quad \text{and} \\
0 & \longrightarrow \text{III}^+_S(T/2^nT) \longrightarrow \text{HI}^+_1(G_S(F), T/2^nT) \longrightarrow \bigoplus_{n \in \mathbb{Z}} H^i(F_n, T/2^nT),
\end{align*}
\]
we obtain the isomorphisms
\[
\text{III}^+_S(T) \cong \lim_{\to} \text{III}^+_S(T/2^nT) \quad \text{and} \quad \text{III}^+_S(T) \cong \lim_{\to} \text{III}^+_S(T/2^nT).
\]
For the module \(A = \lim_{\to} A[2^n]\), the isomorphisms are ensured by the fact that the functor \(\lim\) is exact.

Theorem 3.13. Let \(M\) be a \(\mathbb{Z}_2[[G_S(F)]]\)-module of finite or co-finite type over \(\mathbb{Z}_2\).

i. There is a perfect pairing, functorial in \(M\),
\[
\text{III}^+_S(M) \times \text{III}^+_S(M^*) \longrightarrow \mathbb{Q}_2/\mathbb{Z}_2.
\]
ii. We have an exact sequence, functorial in $M$,
\[ 0 \rightarrow (\oplus_{v|\infty} N_v)(\hat{C}_\infty^{-1})/I_\infty \rightarrow \text{III}_{S_f}^{1+}(M) \rightarrow \text{III}_{S_f}^{2}(M^*)^\vee \rightarrow 0, \]
where $I_\infty = (\oplus_{v|\infty} \text{res}_v^0)(\text{ker}_d^0_{G(M)}) \cap (\oplus_{v|\infty} N_v)(\hat{C}_\infty^{-1})$.

Proof. Let $M$ be as in the theorem. Throughout the proof we denote by $M_n$ the module $M/2^n M$ (resp. $M[2^n]$) when $M$ is a $\mathbb{Z}_2[G_5(F)]$-module of finite (resp. co-finite) type over $\mathbb{Z}_2$.

i. Using Theorem 3.10, we have two commutative diagrams
\[
\begin{array}{ccc}
\text{III}_{S_f}^{2+}(M_{n+1}) & \times & \text{III}_{S_f}^{1}((M_{n+1})^*) \\
\downarrow & & \uparrow \\
\text{III}_{S_f}^{2+}(M_n) & \times & \text{III}_{S_f}^{1}((M_n)^*) \\
\end{array}
\]
if $M$ is of finite type over $\mathbb{Z}_2$, and
\[
\begin{array}{ccc}
\text{III}_{S_f}^{2+}(M_{n+1}) & \times & \text{III}_{S_f}^{1}((M_{n+1})^*) \\
\downarrow & & \uparrow \\
\text{III}_{S_f}^{2+}(M_n) & \times & \text{III}_{S_f}^{1}((M_n)^*) \\
\end{array}
\]
if $M$ is of co-finite type over $\mathbb{Z}_2$.

Taking the inverse or direct limit (according to $M$ is of finite or co-finite type over $\mathbb{Z}_2$) and using Lemma 3.12, we obtain a perfect pairing
\[ \text{III}_{S_f}^{2+}(M) \times \text{III}_{S_f}^{1}(M^*) \rightarrow \mathbb{Q}_2/\mathbb{Z}_2. \]

ii. Consider the exact sequence
\[ 0 \rightarrow Z(M_n) \rightarrow \text{III}_{S_f}^{1+}(M_n) \rightarrow \text{III}_{S_f}^{2}(M^*)^\vee \rightarrow 0. \]

Taking the inverse or direct limit (according to $M$ is of finite or co-finite type over $\mathbb{Z}_2$) of the above exact sequence, Lemmas 3.11 and 3.12 give rise to the following exact sequence
\[ 0 \rightarrow Z(M) \rightarrow \text{III}_{S_f}^{1+}(M) \rightarrow \text{III}_{S_f}^{2}(M^*)^\vee \rightarrow 0. \]

Proceeding exactly as in the proof of Lemma 3.9, we get that
\[ Z(M) \simeq (\oplus_{v|\infty} N_v)(\hat{C}_\infty^{-1})/((\oplus_{v|\infty} \text{res}_v^0)(\text{ker}_d^0_{G(M)}) \cap (\oplus_{v|\infty} N_v)(\hat{C}_\infty^{-1})). \]

Therefore, we find the exact sequence in ii. of the theorem. □

Let us compute the module $Z(M)$ for the $\mathbb{Z}_2[[G_{F,S}]]$-module $M = \mathbb{Z}_2(i)$, the twist “à la Tate” of the ring of dyadic integers. By Theorem 3.13, we have
\[ Z(M) \cong (\oplus_{v|\infty} N_v)(\hat{C}_\infty^{-1})/I_\infty. \]

First of all, recall that $\hat{C}_\infty^{-1} = \oplus_{v|\infty} M$ and $I_\infty = (\oplus_{v|\infty} \text{res}_v^0)(\text{ker}_d^0_{G(M)}) \cap (\oplus_{v|\infty} N_v)(\hat{C}_\infty^{-1})$.

Remark that for any infinite place $v$, the group $G_{F,S}$ is either trivial or equals to $\{1, \sigma\}$ according to $v$ is complex or real respectively, where $\sigma$ is the complex conjugation. Hence if $v$ is complex, $N_v$ coincides with the identity of $\mathbb{Z}_2(i)$. If $v$ is real, $N_v$ is the multiplication by $1 + \sigma$. Therefore,
Summarizing, we obtain that \( \mathcal{O}_{\mathbb{V}_\infty}^{-1}(\widehat{\mathcal{O}}_\infty) = \left\{ \begin{array}{ll} (Z_2(i))^{r_2} \oplus (2Z_2(i))^{r_1} & \text{if } i \text{ is even;} \\
(Z_2(i))^{r_2} & \text{if } i \text{ is odd,} \end{array} \right\} \)

where \( r_1 \) and \( r_2 \) are the number of real and complex places of \( F \), respectively.

Further, observe that the kernel \( \ker(d_{\alpha}^{\alpha}(M)) = M^{G_\alpha(F)} \) and \( Z_2(i)^{G_\alpha(F)} = 0 \) if \( i \neq 0 \). Thus, we get that \( I_\infty = 0 \) when \( i \neq 0 \). If \( i = 0 \) we have \( \ker(d_{\alpha}^{\alpha}(Z_2)) = Z_2 \), so we can show that \( I_\infty \) is isomorphic, as a submodule of \( (\oplus_{\mathbb{V}_\infty} \mathcal{O}_\infty)(\widehat{\mathcal{O}}_\infty)^{-1} \), to \( Z_2 \) or \( 2Z_2 \) according to \( r_1 = 0 \) or not. Summarizing, we obtain that

\[
Z(Z_2(i)) \cong \left\{ \begin{array}{ll} (Z_2(i))^{r_2} & \text{if } i = 0 \text{ and } r_1 = 0; \\
(Z_2(i))^{r_2} \oplus (2Z_2(i))^{r_1 - 1} & \text{if } i = 0 \text{ and } r_1 \neq 0; \\
(Z_2(i))^{r_2} \oplus (2Z_2(i))^{r_1} & \text{if } i \neq 0 \text{ is even;}
(Z_2(i))^{r_2} & \text{if } i \text{ is odd.} \end{array} \right\}
\]

As a consequence of Theorem 3.13 we then obtain the following:

**Corollary 3.14.**

i. We have an isomorphism

\[ \mathcal{I}_{\mathbb{S}_\mathbb{S}}^2(Z_2(i)) \cong \mathcal{I}_{\mathbb{S}_\mathbb{S}}^1(Q_2/Z_2(1 - i)) \]

ii. We have an exact sequence

\[ 0 \longrightarrow Z(Z_2(i)) \longrightarrow \mathcal{I}_{\mathbb{S}_\mathbb{S}}^1(Z_2(i)) \longrightarrow \mathcal{I}_{\mathbb{S}_\mathbb{S}}^2(Q_2/Z_2(1 - i)) \longrightarrow 0. \]

In particular, if \( i \) is an odd integer and \( F \) is a totally real number field, we have a perfect pairing

\[ \mathcal{I}_{\mathbb{S}_\mathbb{S}}^1(Z_2(i)) \times \mathcal{I}_{\mathbb{S}_\mathbb{S}}^2(Q_2/Z_2(1 - i)) \longrightarrow Q_2/Z_2. \]

As an application, we give a description of \( \mathcal{I}_{\mathbb{S}_\mathbb{S}}^2(Z_2(i)) \) in terms of an Iwasawa module.

Let \( \mathcal{O}_{F,S} \) be the ring of \( S \)-integers of a number field \( F \). For an integer \( i \geq 2 \), the positive étale wild kernel \([1, \text{Definition 2.2}]\) is the group

\[ WK_{2i-2}^{\text{et}} \mathcal{O}_{F,S} := \mathcal{I}_{\mathbb{S}_\mathbb{S}}^{2i}(Z_2(i)) = \ker(H_+^2(G_\mathbb{S}(F), Z_2(i))) \oplus \bigoplus_{v \in \mathbb{S}} H^2(F_v, Z_2(i)). \]

For \( i = 1 \), the group \( \mathcal{I}_{\mathbb{S}_\mathbb{S}}^{22}(Z_2(1)) \) is isomorphic to \( A_{F,F}^{\text{et}} \), the 2-part of the narrow \( S \)-class group (see also Remark 2.3 in [2]), i.e. the 2-part of the quotient of the narrow class group by the subgroup generated by the class of finite primes in \( S \). In particular it depends on the set \( S \).

Let \( F_\infty := \bigcup_{n \geq 0} F_n \) be the cyclotomic \( Z_2 \)-extension of \( F \) with Galois group \( \Gamma = \text{Gal}(F_\infty/F) \), and let \( X_\infty^+ \) be the Galois group of the maximal abelian 2-extension \( L_\infty^+ \) of \( F_\infty \) which is unramified at finite places and completely decomposed at all primes above 2. Observe that for all finite non 2-adic place \( v \), the maximal unramified pro-2-extension of \( F_v \) is \( F_v \cdot \infty \), the cyclotomic \( Z_2 \)-extension of \( F_v \). Hence, the extension \( L_\infty^+ / F_\infty \) is completely decomposed at all finite places. The following proposition is an analogue of Schneider’s description of the classical étale wild kernel [9].

**Proposition 3.15.** Let \( i \geq 2 \) be an integer. If either \( i \) is odd, or \( i \) is even and \( \sqrt{-1} \in F \), then

\[ WK_{2i-2}^{\text{et}} \mathcal{O}_{F,S} \cong X_\infty^+(i - 1)_\Gamma. \]

In particular, in both cases we recover that the group \( WK_{2i-2}^{\text{et}} \mathcal{O}_{F,S} \) is independent of the set \( S \) containing the infinite and dyadic places of \( F \).
Proof. Let \( i \geq 2 \) be an integer such that, either \( i \) is odd or \( i \) is even and \( \sqrt{-1} \in F \). In the sequel, we set \( j = 1 - i \). For all finite place \( v \) of \( F \), we denote by \( F_{v, \infty} \) the cyclotomic \( \mathbb{Z}_2 \)-extension of \( F_v \) and \( \Gamma_v = \text{Gal}(F_{v, \infty}/F_v) \) the decomposition group of \( v \) in \( F_{\infty}/F \). Notice that \( \Gamma \) and \( \Gamma_v \) are of cohomological dimension one. In particular, for all finite place \( v \), we have

\[
H^2(\Gamma, Q_2/\mathbb{Z}_2(j)) = H^2(\Gamma_v, Q_2/\mathbb{Z}_2(j)) = 0.
\]

Consider the following exact commutative diagram

\[
\begin{array}{cccccc}
H^1(\Gamma, Q_2/\mathbb{Z}_2(j)) & \to & H^1(\mathbb{G}_S(F), Q_2/\mathbb{Z}_2(j)) & \to & H^1(\mathbb{G}_S(F_\infty), Q_2/\mathbb{Z}_2(j))^\Gamma & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{v \in S_f} H^1(\Gamma_v, Q_2/\mathbb{Z}_2(j)) & \to & \bigoplus_{v \in S_f} H^1(F_v, Q_2/\mathbb{Z}_2(j)) & \to & \bigoplus_{v \in S_f} H^1(F_{v, \infty}, Q_2/\mathbb{Z}_2(j))^\Gamma & \to & 0.
\end{array}
\]

We have (see, e.g., [1, Lemma 3.1])

\[
H^1(\Gamma, Q_2/\mathbb{Z}_2(j)) = 0 \quad \text{and} \quad H^1(\Gamma_v, Q_2/\mathbb{Z}_2(j)) = 0 \quad \text{for all} \quad v \in S_f,
\]

and then

\[
\begin{align*}
\text{III}_{S_f}^1(Q_2/\mathbb{Z}_2(j)) &= \ker(H^1(\mathbb{G}_S(F_\infty), Q_2/\mathbb{Z}_2(j))^\Gamma \to \bigoplus_{v \in S_f} H^1(F_{v, \infty}, Q_2/\mathbb{Z}_2(j))^\Gamma) \\
&= (\ker(H^1(\mathbb{G}_S(F_\infty), Q_2/\mathbb{Z}_2(j)) \to \bigoplus_{v \in S_f} H^1(F_{v, \infty}, Q_2/\mathbb{Z}_2(j))))^\Gamma.
\end{align*}
\]

The hypotheses on the integer \( i \) ensure that the groups \( \mathbb{G}_S(F_\infty) \) and \( \mathbb{G}_{F_v, \infty} \) act trivially on \( Q_2/\mathbb{Z}_2(j) \). Thus,

\[
H^1(\mathbb{G}_S(F_\infty), Q_2/\mathbb{Z}_2(j)) = \text{Hom}(\mathbb{G}_S(F_\infty), Q_2/\mathbb{Z}_2(j))
\]

and

\[
H^1(F_{v, \infty}, Q_2/\mathbb{Z}_2(j)) = \text{Hom}(F_{v, \infty}, Q_2/\mathbb{Z}_2(j)).
\]

Hence,

\[
\text{III}_{S_f}^1(Q_2/\mathbb{Z}_2(j)) = \text{Hom}(X^+_\infty, Q_2/\mathbb{Z}_2)(-j)^\Gamma.
\]

Therefore, using i. of Theorem 3.13, we obtain the isomorphism

\[
WK^n_{2l-2}O_{F,S} \cong X^+_\infty (i - 1)_{\Gamma}.
\]

In particular, the group \( WK^n_{2l-2}O_{F,S} \) is independent of the set \( S \) containing the infinite and dyadic places of \( F \).

From now on, we adopt the following notation

\[
WK^n_{2l-2}O_{F,S} := WK^n_{2l-2}O_{F}.
\]

Let \( F_\infty = \bigcup_n F_n \) be the cyclotomic \( \mathbb{Z}_2 \)-extension of \( F \) and for \( n \geq 0 \), \( G_n = \text{Gal}(F_n/F) \). Since for all \( n \geq 0 \)

\[
(X^+_\infty (i - 1)_{\Gamma})_{G_n} \cong X^+_\infty (i - 1)_{\Gamma},
\]

the above description of the positive étale wild kernel shows immediately that the positive étale wild kernel satisfies Galois co-descent in the cyclotomic tower (see as well [1, Corollary 3.3]).

Corollary 3.16. Let \( i \geq 2 \) be an integer. If either \( i \) is odd, or \( i \) is even and \( \sqrt{-1} \in F \), then the positive étale wild kernel satisfies Galois co-descent in the cyclotomic \( \mathbb{Z}_2 \)-extension:
Remark 3.17. For a number field $F$, we know that $WK_{2i-2}^+ F_n \cong WK_{2i-2}^+ F$. Furthermore if $\sqrt{-1} \in F$, the above result has been proved in [4, Theorem 2.18].

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