Analytic Eigensystems for Isotropic Membrane Energies

Julian Panetta

August 8, 2019

This document follows the approach of [2] to derive the Hessian eigenvalues and eigenmatrices for isotropic membrane energy densities $\psi(F)$, where $F$ is a $3 \times 2$ deformation gradient. We assume that the energy is expressed in terms of the following generalizations for $3 \times 2$ matrices of the $2 \times 2$ tensor invariants:

$$I_{1}^{3\times2} := \sigma_1 + \sigma_2$$
$$I_{2}^{3\times2} := F \cdot F = \sigma_1^2 + \sigma_2^2$$
$$I_{3}^{3\times2} := \sigma_1 \sigma_2.$$

In these definitions, $\sigma_1$ and $\sigma_2$ are the singular values of $F$ obtained from the singular value decomposition:

$$F = U \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} \Sigma V^T \quad U \in O(3), V \in O(2).$$

We note that the third column of $U$ is the deformed surface normal $\hat{n}$.

1 Differentiating the SVD

We will need formulas for how $U$, $\Sigma$, and $V$ change as $F$ is perturbed with “velocity” $\dot{F}$, which we find by differentiating both sides of the SVD:

$$\dot{F} = \dot{U} \Sigma V^T + U \dot{\Sigma} V^T + U \Sigma \dot{V}^T \quad \implies \quad U^T \dot{F} V = U^T \dot{U} \Sigma + \dot{\Sigma} + \Sigma \dot{V}^T V. \quad (1)$$

Differentiating the relationships $U^T U = \text{Id}_{3\times3}$ and $V^T V = \text{Id}_{2\times2}$ reveals that $U^T \dot{U}$ and $\dot{V}^T V$ are skew symmetric and can be written as the infinitesimal rotations:

$$U^T \dot{U} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \quad \dot{V}^T V = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}.$$

Plugging these into (1), we obtain a formula for the infinitesimal rotations and singular value perturbations induced by $\dot{F}$:

$$U^T \dot{F} V = \begin{bmatrix} \dot{\sigma}_1 & -\sigma_2 \omega_z + \sigma_1 \alpha \\ \sigma_1 \omega_z + \sigma_2 \alpha & \sigma_2 \\ -\sigma_1 \omega_y & \sigma_2 \omega_x \end{bmatrix}. \quad (2)$$

Geometrically, $\omega_z$ indicates a rotation of the surface element about the current normal $\hat{n}$, while $\omega_x$ and $\omega_y$ are rotations around the principal stretch axes. When $\omega_x = \omega_y = 0$, the deformed surface element simply rotates in-plane around $\hat{n}$ (and $\hat{n}$ does not change). However, nonzero $\omega_x$ and $\omega_y$ indicate that $\dot{F}$ induces a rotation of $\hat{n}$.

\footnote{The $I_2$ invariant used here is from [2]; the other standard definition of principal invariant $I_2 = \frac{1}{2} (\text{tr}(A)^2 - \|A\|^2_F)$ actually coincides with $I_3$ in the 2D case.}
1.1 Example Perturbations

According to (2), a perturbation of the form
\[
\dot{F} = U \begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix} V^T
\]
leaves \( \hat{n} \) unchanged as it stretches/rotates the surface element in-plane. Specifically, we have \( \sigma_1 = a, \sigma_2 = d \) and the following system for \( \omega_z \) and \( \alpha \):
\[
\begin{align*}
\sigma_2 \omega_z + \sigma_1 \alpha &= -b \\
\sigma_1 \omega_z + \sigma_2 \alpha &= c
\end{align*}
\] (3)

On the other hand, perturbation
\[
\dot{F} = U \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ e & f \end{bmatrix} V^T
\]
rotates the surface element’s normal by angular velocities \( \omega_x = f/\sigma_2, \omega_y = -e/\sigma_1 \) without any in-plane stretch/rotation.

2 Gradients of the Invariants

We can now use the formulas for \( \dot{\sigma}_1 \) and \( \dot{\sigma}_2 \) to differentiate the invariants:
\[
\frac{\partial I_3^{3 \times 2}}{\partial F} : \dot{F} = \hat{\Sigma} : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = (U^T \dot{F} V) : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \dot{F} : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} V^T = \frac{\partial I_1^{3 \times 2}}{\partial F} = U \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} V^T,
\]
\[
\frac{\partial I_3^{1 \times 2}}{\partial F} : \dot{F} = \hat{\Sigma} : \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \\ 0 & 0 \end{bmatrix} = (U^T \dot{F} V) : \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \\ 0 & 0 \end{bmatrix} = \dot{F} : \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \\ 0 & 0 \end{bmatrix} V^T = \frac{\partial I_3^{3 \times 2}}{\partial F} = U \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_1 \end{bmatrix} V^T,
\]
\[
\frac{\partial I_1^{3 \times 2}}{\partial F} : \dot{F} = 2F : \dot{F} = \frac{\partial I_3^{3 \times 2}}{\partial F} = 2F.
\]

3 Hessians of the Invariants

We evaluate the Hessian applied to an arbitrary perturbation \( \dot{F} \). First, the easy invariant:
\[
\frac{\partial^2 I_3^{3 \times 2}}{\partial F^2} : \dot{F} = 2\dot{F},
\]
which means \( \frac{\partial^2 I_3^{3 \times 2}}{\partial F^2} \) is a multiple of the fourth order identity tensor. Any orthogonal basis can be chosen as a set of eigenmatrices, and their corresponding eigenvalues are all 2.

Next, we consider \( I_3^{1 \times 2} \):
\[
U^T \left( \frac{\partial^2 I_3^{3 \times 2}}{\partial F^2} : \dot{F} \right) V = U^T \dot{U} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{V}^T V = \begin{bmatrix} 0 & -(\omega_z + \alpha) \\ \omega_z + \alpha & 0 \\ -\omega_y & \omega_x \end{bmatrix}.
\]

We plug in \( \dot{F} = U \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} V^T \) and note that summing the equations in (3) yields \( \omega_z + \alpha = \frac{c-b}{\sigma_1+\sigma_2} \). Thus:
\[
\frac{\partial^2 I_3^{3 \times 2}}{\partial F^2} : \dot{F} = U \begin{bmatrix} 0 & \frac{c-b}{\sigma_1+\sigma_2} \\ \frac{b-c}{\sigma_1+\sigma_2} & 0 \\ \frac{d}{\sigma_1} & \frac{e}{\sigma_2} \end{bmatrix} V^T.
\]
We further deduce the three eigenmatrices with nonzero eigenvalues:

\[
\frac{1}{\sqrt{2}} U \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} V^T, \quad \frac{1}{\sqrt{2}} U \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} V^T, \quad \frac{1}{\sqrt{2}} U \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} V^T \quad (\lambda = 0).
\]

We further deduce the three eigenmatrices with nonzero eigenvalues:

\[
\frac{1}{\sqrt{2}} U \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} V^T, \quad U \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} V^T, \quad U \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} V^T.
\]

\[\lambda = \frac{1}{\sqrt{3} + \sqrt{2}}, \quad \lambda = \frac{1}{\sqrt{3}}, \quad \lambda = \frac{1}{\sqrt{2}}\]

Finally, we consider \(I_3^{3 \times 2}\):

\[
U^T \left( \frac{\partial^2 I_3^{3 \times 2}}{\partial F^2} : \dot{F} \right) V = U^T \dot{U} \begin{bmatrix} \sigma_2 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \delta_2 & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{V}^T V = \begin{bmatrix} \dot{\sigma}_2 & -\sigma_1 \omega_z + \sigma_2 \alpha \\ \sigma_2 \omega_z + \sigma_1 \alpha & \delta_1 \\ -\sigma_2 \omega_y & \sigma_1 \omega_x \end{bmatrix}.
\]

Again plugging in \(\dot{F} = U \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} V^T\) and using the formulas from Section 11, we find:

\[
\frac{\partial^2 I_3^{3 \times 2}}{\partial F^2} = U \begin{bmatrix} \frac{d}{\sigma_1} & -c \\ -b & \frac{a}{\sigma_1} \\ \frac{2}{\sigma_1} & \frac{2}{\sigma_2} f \end{bmatrix} V^T
\]

We deduce the following eigenmatrices and eigenvalues:

\[
\frac{1}{\sqrt{2}} U \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} V^T, \quad \frac{1}{\sqrt{2}} U \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} V^T, \quad \frac{1}{\sqrt{2}} U \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} V^T, \quad \frac{1}{\sqrt{2}} U \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} V^T.
\]

\[\lambda = 1, \quad \lambda = -1, \quad \lambda = \frac{1}{\sqrt{3} + \sqrt{2}}, \quad \lambda = \frac{1}{\sqrt{2}}\]

We note that for all invariants, four of the six Hessian eigenmatrices are simply padded versions of the 2D eigenmatrices from [2], while the last two are new and concern the rotation of the surface element’s normal.

### 4 Example: Incompressible neo-Hookean Sheet

We consider the membrane energy of a thin sheet of incompressible neo-Hookean material [1]:

\[
\psi_{\text{IncNeo}}(F_{3\text{D}}) = \frac{\mu}{2} \left( \text{tr}(F_{3\text{D}}^T F_{3\text{D}}) - 3 \right) = \frac{\mu}{2} \left( I_{3\text{D}}^3 - 3 \right)
\]

When the sheet experiences an in-plane deformation gradient \(F \in \mathbb{R}^{3 \times 2}\), it stretches or compresses in the normal direction to maintain \(J = 1\). We can solve for the normal stretch as \(\frac{1}{I_{3\text{D}}^3}\) and express \(\psi_{\text{IncNeo}}\) directly in terms of \(F\)’s invariants:

\[
\psi_{\text{Sheet}}(F) = \frac{\mu}{2} \left( I_{3\text{D}}^3 + \left( \frac{1}{I_{3\text{D}}^3} \right)^2 - 3 \right).
\]
The Hessian of this energy density is:

\[
\frac{\partial^2 \psi_{\text{sheet}}}{\partial F^2} = \frac{\mu}{2} \left[ \frac{\partial^2 I_3^{3 \times 2}}{\partial F^2} + 6 \left( \frac{1}{I_3^{3 \times 2}} \right)^4 \frac{\partial I_3^{3 \times 2}}{\partial F} \otimes \frac{\partial I_3^{3 \times 2}}{\partial F} - 2 \left( \frac{1}{I_3^{3 \times 2}} \right)^3 \frac{\partial^2 I_3^{3 \times 2}}{\partial F^2} \right]
\]

\[
= \mu \left[ I_d + 3 \left( \frac{1}{I_3^{3 \times 2}} \right)^4 \left( \begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & \sigma_1 \end{array} \right) V^T \right] \otimes \left( \begin{array}{c|c} \sigma_2 & 0 \\ \hline 0 & \sigma_1 \end{array} \right) V^T - \left( \frac{1}{I_3^{3 \times 2}} \right)^3 \frac{\partial^2 I_3^{3 \times 2}}{\partial F^2} .
\]

We note that \( \frac{\partial I_3^{3 \times 2}}{\partial F} \) is orthogonal to all but two of the eigenmatrices of \( \frac{\partial^2 I_3^{3 \times 2}}{\partial F^2} \) (and eigenmatrices for the fourth order identity tensor \( I_d \) can be chosen arbitrarily), so we immediately get the following four eigenpairs:

\[
\begin{align*}
\lambda &= \mu \pm \frac{\mu}{\sqrt{2}} \left( \frac{1}{I_3^{3 \times 2}} \right)^3, \\
\lambda &= \mu + \frac{\mu}{\sqrt{2}} \frac{1}{I_3^{3 \times 2}}.
\end{align*}
\]

Because \( \frac{\partial I_3^{3 \times 2}}{\partial F} \) is generally not orthogonal to either of the remaining two eigenmatrices of \( \frac{\partial^2 I_3^{3 \times 2}}{\partial F^2} \) (whose eigenvalues are distinct) we must diagonalize the projection of \( \frac{\partial^2 \psi_{\text{sheet}}}{\partial F^2} \) onto their span to obtain the final two eigenpairs. We obtain simpler expressions using the basis \( D_1 := U \left[ \begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array} \right] V^T \) and \( D_2 := U \left[ \begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array} \right] V^T \) for this subspace, which results in the reduced Hessian:

\[
\begin{bmatrix}
D_1 : \frac{\partial^2 \psi_{\text{sheet}}}{\partial F^2} ; D_1 & D_1 : \frac{\partial \psi_{\text{sheet}}}{\partial F} ; D_2 \\
D_2 : \frac{\partial \psi_{\text{sheet}}}{\partial F} ; D_1 & D_2 : \frac{\partial^2 \psi_{\text{sheet}}}{\partial F^2} ; D_2
\end{bmatrix} = \mu \left[ \begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right] + \frac{\mu}{(I_3^{3 \times 2})^4} \left[ \begin{array}{c|c} 3 \sigma_2^2 & 2 I_3^{3 \times 2} \\ \hline 2 I_3^{3 \times 2} & 3 \sigma_1^2 \end{array} \right].
\]

The eigendecomposition of this \( 2 \times 2 \) matrix can be expressed by introducing quantities \( \beta := 3(\sigma_2^2 - \sigma_1^2) \) and \( \gamma := \sqrt{16 \left( \frac{I_3^{3 \times 2}}{I_3^{3 \times 2}} \right)^2 + \beta^2} \):

\[
\begin{align*}
v_1 &= \left[ \begin{array}{c}
\beta - \gamma \\
4 I_3^{3 \times 2}
\end{array} \right], & \lambda_1 &= \mu + \frac{\mu}{2} \frac{3 I_3^{3 \times 2}}{(I_3^{3 \times 2})^2} + \gamma, \\
v_2 &= \left[ \begin{array}{c}
\beta + \gamma \\
4 I_3^{3 \times 2}
\end{array} \right], & \lambda_2 &= \mu + \frac{\mu}{2} \frac{3 I_3^{3 \times 2}}{(I_3^{3 \times 2})^2} - \gamma,
\end{align*}
\]

making the final two eigenpairs of \( \frac{\partial^2 \psi_{\text{sheet}}}{\partial F^2} \):

\[
\begin{align*}
U \left[ \begin{array}{c|c}
\beta - \gamma & 0 \\ \hline 0 & 4 I_3^{3 \times 2}
\end{array} \right] , & \lambda = \mu + \frac{\mu}{2} \frac{3 I_3^{3 \times 2} + \gamma}{(I_3^{3 \times 2})^2}, \\
U \left[ \begin{array}{c|c}
\beta + \gamma & 0 \\ \hline 0 & 4 I_3^{3 \times 2}
\end{array} \right] , & \lambda = \mu + \frac{\mu}{2} \frac{3 I_3^{3 \times 2} - \gamma}{(I_3^{3 \times 2})^2}.
\end{align*}
\]

Note that these eigenmatrices do not have unit norm and should be normalized.

References

[1] Javier Bonet and Richard D Wood. *Nonlinear continuum mechanics for finite element analysis*. Cambridge university press, 1997.

[2] Breannan Smith, Fernando De Goes, and Theodore Kim. Analytic eigensystems for isotropic distortion energies. *ACM Trans. Graph.*, 38(1):3:1–3:15, February 2019.