On the derivative at $t = 1$ of the skew-growth functions for Artin monoids.

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Abstract

Let $G^+_M$ be the Artin monoid of finite type generated by the letters $a_i, i \in I$ with respect to a Coxeter matrix $M$ that is equipped with the degree map $\deg : G^+_M \to \mathbb{Z}_{\geq 0}$ defined by assigning to each equivalence class of words the length of the words, and let $N_{M, \deg}(t) := \sum_{J \subseteq I} (-1)^{|J|} t^{\deg(\Delta_J)}$ be the skew-growth function, where the summation index $J$ runs over all subsets of $I$ and $\Delta_J$ is the fundamental element in $G^+_M$ associated to the set $J$. In this article, we will calculate the derivative at $t = 1$ of the polynomial $N_{M, \deg}(t)$. As a result, we show that the polynomial $N_{M, \deg}(t)$ has a simple root at $t = 1$.

Keywords: Artin monoid, growth function, zeroes of polynomial

1. Introduction

Let $G^+_M$ be the Artin monoid of finite type (B-S §5) generated by the letters $a_i, i \in I$ with respect to a Coxeter matrix $M$ (B). Due to the homogeneity of the defining relations in $G^+_M$, we naturally define a map $\deg : G^+_M \to \mathbb{Z}_{\geq 0}$ defined by assigning to each equivalence class of words the length of the words. The spherical growth function for the monoid $G^+_M$ is defined as

$$P_{G^+_M, \deg}(t) := \sum_{u \in G^+_M} t^{\deg(u)}.$$ 

In [A-N][B][S1], they show that the inversion function $P_{G^+_M, \deg}(t)^{-1}$ is given by the following function, called the skew-growth function,

$$N_{M, \deg}(t) := \sum_{J \subseteq I} (-1)^{|J|} t^{\deg(\Delta_J)},$$

where the summation index $J$ runs over all subsets of $I$ and $\Delta_J$ is the fundamental element in $G^+_M$ associated to the set $J$ (B-S §5). That has been investigated by several authors ([A-N][B][B][D][I1][I2][S1][S2][S3][S4][X]). In [B-S §4], the authors show that the monoid $G^+_M$ satisfies the LCM condition (i.e. any two elements $\alpha$ and $\beta$ in it admit the left (resp. right) least common multiple). By using this property, for a subset $J \subseteq I$, they defined the fundamental element $\Delta_J$, as the right least common multiple of all the letters $a_i, i \in J$. In [S1 §4], it
is observed that the polynomial $N_{M, \deg}(t)$ has a simple root at $t = 1$. In this article, we will calculate the derivative at $t = 1$ of the polynomial $N_{M, \deg}(t)$. As a result, we show that the polynomial $N_{M, \deg}(t)$ has a simple root at $t = 1$.

Our main theorem is the following.

**Theorem 1.1.** For a Coxeter matrix $M$, the derivative at $t = 1$ of the polynomial $N_{M, \deg}(t)$ is given by the following list:

- $A_l \geq 1$: $N'_{M, \deg}(1) = (-1)^l l$,
- $E_8$: $N'_{M, \deg}(1) = 44$,
- $B_l \geq 2$: $N'_{M, \deg}(1) = (-1)^l l$,
- $F_4$: $N'_{M, \deg}(1) = 10$,
- $E_6$: $N'_{M, \deg}(1) = (-1)^l (l - 2)$,
- $H_3$: $N'_{M, \deg}(1) = -8$,
- $E_7$: $N'_{M, \deg}(1) = 7$,
- $H_4$: $N'_{M, \deg}(1) = 42$,
- $I_2(p \geq 5)$: $N'_{M, \deg}(1) = p - 2$.

The above statement can be verified by hand calculation for the types $E_6, E_7, E_8, F_4, H_3, H_4$ and $I_2(p \geq 5)$. In §3, we will prove Theorem 1.1 for the type $A_l$. By using the results in §3, we will prove Theorem 1.1 for the type $B_l$ and $D_l$ in §4, §5. As a corollary of Theorem 1.1, we obtain the following.

**Corollary 1.2.** The polynomial $N_{M, \deg}(t)$ has a simple root at $t = 1$.

2. Preliminary results

Let $l$ be a positive integer and let $I = \{1, 2, \ldots, l\}$. The Coxeter matrix $M = (m(\alpha, \beta))_{\alpha, \beta \in I}$ of the type $X_l \in \{A_l, B_l, D_l\}$ is given by the following list. For the type $A_l$, we give

$$m(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 3 & \text{if } |\alpha - \beta| = 1 \\ 2 & \text{if } |\alpha - \beta| > 1 \end{cases}$$

For the type $B_l$, we give

$$m(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 3 & \text{if } |\alpha - \beta| = 1 \text{ and } \alpha + \beta > 3 \\ 2 & \text{if } |\alpha - \beta| > 1 \\ 4 & \text{if } \alpha + \beta = 3 \end{cases}$$

For the type $D_l$, we give

$$m(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 2 & \text{if } |\alpha - \beta| > 1 \text{ and } \alpha + \beta \neq 2l - 2 \\ 2 & \text{if } \alpha + \beta = 2l - 1 \\ 3 & \text{if } |\alpha - \beta| = 1 \text{ and } \alpha + \beta < 2l - 1 \\ 3 & \text{if } \alpha + \beta = 2l - 2 \text{ and } \alpha \neq \beta \end{cases}$$

\[\text{For the type } B_l, \text{ we adopt for convenience the different definition of the Coxeter matrix from that in [B].}\]
We simply write the polynomial $N_{M, \deg}(t)$ by $N_{X_1}(t)$. Namely, we put

$$N_{X_1}(t) := \sum_{J \subset I} (-1)^{\#J} t^{\deg(\Delta_{X_1,J})},$$

where $\Delta_{X_1,J}$ is the fundamental element in the Artin monoid $G^+_M$ associated to the set $J$. Moreover, for a non-negative integer $j \in \{0, \ldots, l\}$ we put

$$N_{X_1,j}(t) := \sum_{J \subset I, \#J = j} t^{\deg(\Delta_{X_1,J})}, \quad C_{X_1,j} := \frac{dN_{X_1,j}(t)}{dt} \bigg|_{t=1}.$$

Therefore, we have the following equations:

$$N'_{X_1}(1) = \sum_{j=1}^{l} (-1)^j C_{X_1,j}, \quad C_{X_1,j} = \sum_{J \subset I, \#J = j} \deg(\Delta_{X_1,J}).$$

To a Coxeter matrix $M = (m(\alpha, \beta))_{\alpha, \beta \in I}$ of the type $X_l$, we attach a Coxeter graph $\Gamma_{X_l}$, whose vertices are indexed by the set $I$ and two vertices $\alpha$ and $\beta$ are connected by an edge iff $m(\alpha, \beta) \geq 3$. For a subset $J \subset I$, we associate a full subgraph $\Gamma_{X_l}(J)$, whose vertices are indexed by the set $J$. The edge is labeled by $m(\alpha, \beta)$ (omitted if $m(\alpha, \beta) = 3$). We note that $\Gamma_{X_l}(I)$ corresponds to the graph $\Gamma_{X_l}$. For a subgraph $\Gamma_{X_l}(J)$ of $\Gamma_{X_l}$, we write the number of connected components of $\Gamma_{X_l}(J)$ by $k_{X_l}(J)$. Let $\Gamma_{X_l}(J)$ be a full subgraph of $\Gamma_{X_l}$ with $k$-connected components $\Gamma_{X_l}(J_1), \Gamma_{X_l}(J_2), \ldots, \Gamma_{X_l}(J_k)$. Then, we write

$$\Gamma_{X_l}(J) = \bigsqcup_{i=1}^k \Gamma_{X_l}(J_i).$$

We recall a fact from [B-S].

**Proposition 2.1.** For a subset $J \subset I$, we suppose that the full subgraph $\Gamma_{X_l}(J)$ has a decomposition $\Gamma_{X_l}(J) = \bigsqcup_{i=1}^k \Gamma_{X_l}(J_i)$. Then:

1. For $1 \leq i < j \leq k$, $\Delta_{X_l,J_i}$ and $\Delta_{X_l,J_j}$ commute with each other.
2. Then, the fundamental element $\Delta_{X_l,J}$ can be written as a product of the fundamental elements $\Delta_{X_l,J_1}, \ldots, \Delta_{X_l,J_k}$:

$$\Delta_{X_l,J} \equiv \Delta_{X_l,J_1} \cdots \Delta_{X_l,J_k}.$$

Since the map $\deg$ is an additive map, we can compute

$$\deg(\Delta_{X_l,J}) = \sum_{i=1}^k \deg(\Delta_{X_l,J_i}). \quad (2.1)$$

**3. Proof of the type $A_l$**

Let $l$ be a positive integer and let $I = \{1, 2, \ldots, l\}$. In this section, we will prove Theorem 1.1 for the type $A_l$. First, we have a remark on $k_{A_l}(J)$.  

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**Proposition 3.1.** For a subset $J \subset I$, we put $j := \# J$. Then:
(1) If the number $j$ satisfies an inequality $1 \leq j \leq \left[ \frac{l}{2} \right]$, then the number of connected components $k_{A_l}(J)$ can run over from $1$ to $j$.
(2) If the number $j$ satisfies an inequality $j > \left[ \frac{l}{2} \right]$, then the number of connected components $k_{A_l}(J)$ can run over from $1$ to $l - j + 1$.

We put $\beta_{l,j} := \min\{j, l - j + 1\}$. Then, the summary of Proposition 3.1 is that the number of connected components $k_{A_l}(J)$ can run over from $1$ to $\beta_{l,j}$. For two positive integers $j$ and $k$ with $j \leq l$ and $k \leq \beta_{l,j}$, we put

$$N_{A_l,j}^{(k)}(t) := \sum_{J \subset I, \# J = j, k_{A_l}(J) = k} t^{\deg(\Delta_{A_l,J})}, \quad C_{A_l,j}^{(k)} := \left. \frac{dN_{A_l,j}^{(k)}(t)}{dt} \right|_{t=1}. $$

Therefore, we have the following equation:

$$C_{A_l,j}^{(k)} = \sum_{J \subset I, \# J = j, k_{A_l}(J) = k} \deg(\Delta_{A_l,J}).$$

By definition, we have

$$C_{A_l,j} = \sum_{k=1}^{\beta_{l,j}} C_{A_l,j}^{(k)}. $$

We recall a fact from [B-S].

**Proposition 3.2.** For a subset $J \subset I$, we suppose that the full subgraph $\Gamma_{A_l}(J)$ is connected. Then, the degree $\deg(\Delta_{A_l,J})$ of the fundamental element is given by

$$\deg(\Delta_{A_l,J}) = \left( \frac{\# (J) + 1}{2} \right).$$

From the equation (2.1), we easily show the following formula.

**Proposition 3.3.** For a subset $J \subset I$, we suppose that the full subgraph $\Gamma_{A_l}(J)$ has a decomposition $\Gamma_{A_l}(J) = \bigsqcup_{i=1}^{k} \Gamma_{A_l}(J_i)$. Then, the degree of the fundamental element $\Delta_{A_l,J}$ can be written as

$$\deg(\Delta_{A_l,J}) = \sum_{i=1}^{k} \deg(\Delta_{A_l,J_i}) = \sum_{i=1}^{k} \left( \frac{\# (J_i) + 1}{2} \right).$$

**Proposition 3.4.** Let $j$ and $k$ be two positive integers with $j \leq l$ and $k \leq \beta_{l,j}$. For given positive integers $\tau_1, \ldots, \tau_k$ with $\sum_{i=1}^{k} \tau_i = j$, we define the set $S_{j,\tau_1,\ldots,\tau_k}$ by

$$\{ J \subset I \mid \# J = j, \Gamma_{A_l}(J) = \bigsqcup_{i=1}^{k} \Gamma_{A_l}(J_i) \text{ with } \min(J_1) < \cdots < \min(J_k) \text{ s.t. } \# J_i = \tau_i (i = 1, \ldots, k) \}$$

Then, we have the following equation

$$\# S_{j,\tau_1,\ldots,\tau_k} = \left( \frac{l - j + 1}{k} \right).$$
We remark that the result does not depend on the choice of positive integers \( \tau_1, \ldots, \tau_k \). Hence, the number \( \binom{l-j+1}{k} \) divides the number \( C_{A,i,j}^{(k)} \). Then, we define the number \( \tilde{C}_{A,i,j}^{(k)} \) by the equation

\[
C_{A,i,j}^{(k)} = \tilde{C}_{A,i,j}^{(k)} \cdot \binom{l-j+1}{k}.
\]

For two positive integers \( j, k \) with \( k \leq j \), we put

\[
T_{k,j} := \{ (\tau_1, \ldots, \tau_k) \in \mathbb{Z}^k_{>0} | \sum_{i=1}^{k} \tau_i = j \}.
\]

From the Proposition 3.3, we have

\[
\tilde{C}_{A,i,j}^{(k)} = \sum_{(\tau_1, \ldots, \tau_k) \in T_{k,j}} \sum_{i=1}^{k} \binom{\tau_i + 1}{2}.
\]

Lemma 3.5. Let \( j \) and \( k \) be two positive integers with \( j \leq l \) and \( k \leq \beta_{l,j} \). Then, the following equation \( E_{j,k} \) holds.

\[
\tilde{C}_{A,i,j}^{(k)} = k \binom{j+1}{k+1}.
\]

Proof. We will show the general equation \( E_{j,k} \) by double induction. First, for \( k = 1 \), the subgraph \( \Gamma_{A_1}(J) \) is connected. Hence, we easily compute \( \deg(\Delta_{A_1,J}) = \binom{j+1}{2} \). Therefore, we say the equation \( E_{j,1} \) is true. Next, for induction hypothesis, we assume

(A) \( E_{j,k} \) is true for \( j = 1, \ldots, r \) and arbitrary \( k \), and

(B) \( E_{r+1,k} \) is true for \( 1 \leq k \leq s - 1 \).

We will show the equation \( E_{r+1,k} \). For a positive integer \( i \in \{1, \ldots, r-s+2\} \), we consider the set \( \{(\tau_1, \ldots, \tau_s) \in T_{s,j} | \tau_1 = i \} \). Then, we easily count the number \( \#\{(\tau_1, \ldots, \tau_s) \in T_{s,j} | \tau_1 = i \} = \binom{s-1}{r-i} \). By applying the induction hypothesis (A) and (B) to \( (\tau_2, \ldots, \tau_s) \), we have

\[
\tilde{C}_{A,i,j}^{(s)} = \binom{r+1}{2} \binom{r-i}{s-2} + \tilde{C}_{A,i,r+1-i}^{(s-1)}
\]

\[
= \binom{r+1}{2} \binom{r-i}{s-2} + (s-1) \sum_{i=1}^{r-s+2} \binom{r+2-i}{s}
\]

\[
= \binom{r+2}{s+1} + (s-1) \binom{r+2}{s} = s \binom{r+2}{s+1}.
\]

This completes the proof. \( \square \)
Theorem 3.6. For positive integers \( l, j \) with \( l \geq j \), the following equation holds:

\[
C_{A_{l+1},j+1} - C_{A_l,j} = (j + 1) \binom{l + 1}{j + 1}.
\]

Proof. First, we remark the following.

Proposition 3.7. Theorem 3.6 implies Theorem 1.1 for the type \( A_l \).

Proof. It is easy to show \( N'_{A_1}(1) = -1 \). Hence, it suffices to show that

\[
N'_{A_{l+1}}(1) + N'_{A_l}(1) = 0
\]

for any positive integer \( l \).

\[
N'_{A_{l+1}}(1) + N'_{A_l}(1)
\]

\[
= -C_{A_{l+1},1} + \sum_{j=1}^{l} (C_{A_{l+1},j+1} - C_{A_l,j})(-1)^{j+1}
\]

\[
= -C_{A_{l+1},1} + \sum_{j=1}^{l} (-1)^{j+1}(j + 1) \binom{l + 1}{j + 1}
\]

\[
= \sum_{j=0}^{l} (-1)^{j+1}(j + 1) \binom{l + 1}{j + 1}
\]

\[
= -(l + 1) \sum_{j=0}^{l} (-1)^{j} \binom{l}{j}
\]

\[
= -\frac{d}{dx} (1 + x)^{l+1} \bigg|_{x=-1} = 0.
\]

To prove Theorem 3.6, we prepare a lemma.

Lemma 3.8. (1) For positive integers \( l, j \) with \( j + 1 \leq \left\lceil \frac{l+1}{2} \right\rceil \), the following equation holds:

\[
\sum_{k=1}^{j+1} \binom{j}{k-1} \binom{l - j + 1}{k} = \binom{l + 1}{j + 1}.
\]

(2) For positive integers \( l, j \) with \( j + 1 > \left\lceil \frac{l+1}{2} \right\rceil \), the following equation holds.

\[
\sum_{k=1}^{l-j+1} \binom{j}{k-1} \binom{l - j + 1}{k} = \binom{l + 1}{j + 1}.
\]

Proof. (1) We rewrite the equation as follows

\[
\sum_{k=1}^{j+1} \binom{j}{k-1} \binom{l - j + 1}{l-j+1-k} = \binom{l + 1}{l-j}.
\]

This follows from \( (1 + x)^{l+1} = (1 + x)^l(1 + x)^{l-j+1} \).

(2) In the same way, we obtain the result. 

\[
\square
\]
We consider two cases.

Case 1:  \( j + 1 \leq \left\lceil \frac{l+1}{2} \right\rceil \).

\[
C_{A_{l+1},j+1} - C_{A_{l},j} = \sum_{k=1}^{j+1} C_{A_{l+1},j+1}^{(k)} - \sum_{k=1}^{j} C_{A_{l},j}^{(k)}
\]

\[
= \sum_{k=1}^{j+1} k \binom{j+2}{k+1} \binom{l-j+1}{k} - \sum_{k=1}^{j} k \binom{j+1}{k+1} \binom{l-j+1}{k}
\]

\[
= (j+1) \binom{l-j+1}{j+1} + \sum_{k=1}^{j} k \binom{j+1}{k} \binom{l-j+1}{k}
\]

\[
= \sum_{k=1}^{j+1} k \binom{j+1}{k} \binom{l-j+1}{k}
\]

\[
= (j+1) \sum_{k=1}^{j+1} \binom{j}{k-1} \binom{l-j+1}{k}.
\]

Thanks to the Lemma 3.8 (1), we have

\[
C_{A_{l+1},j+1} - C_{A_{l},j} = (j+1) \binom{l+1}{j+1}.
\]

Case 2:  \( j + 1 > \left\lceil \frac{l+1}{2} \right\rceil \).

\[
C_{A_{l+1},j+1} - C_{A_{l},j} = \sum_{k=1}^{l-j+1} C_{A_{l+1},j+1}^{(k)} - \sum_{k=1}^{l-j+1} C_{A_{l},j}^{(k)}
\]

\[
= \sum_{k=1}^{l-j+1} k \binom{j+2}{k+1} \binom{l-j+1}{k} - \sum_{k=1}^{l-j+1} k \binom{j+1}{k+1} \binom{l-j+1}{k}
\]

\[
= \sum_{k=1}^{l-j+1} k \binom{j+1}{k} \binom{l-j+1}{k}
\]

\[
= (j+1) \sum_{k=1}^{l-j+1} \binom{j}{k-1} \binom{l-j+1}{k}.
\]

Thanks to the Lemma 3.8 (2), we have

\[
C_{A_{l+1},j+1} - C_{A_{l},j} = (j+1) \binom{l+1}{j+1}.
\]

This completes the proof of Theorem 3.6.
4. Proof of the type $B_l$

Let $l$ be a positive integer in $\mathbb{Z}_{\geq 2}$ and let $I = \{1, 2, \ldots, l\}$. In this section, we will prove Theorem 1.1 for the type $B_l$. We recall a fact from [B-S].

**Proposition 4.1.** For a subset $J \subset I$, we suppose that the full subgraph $\Gamma_{B_l}(J)$ is connected. Then, the degree $\deg(\Delta_{B_l}, J)$ of the fundamental element is given by

$$\deg(\Delta_{B_l}, J) = \begin{cases} \#(J)^2 & \text{if } J \supset \{1, 2\} \\ \frac{(\#(J)+1)}{2} & \text{if } J \not\supset \{1, 2\} \end{cases}$$

From the equation (2.1), if the full subgraph $\Gamma_{B_l}(J)$ for a subset $J \subset I$ has a decomposition $\Gamma_{B_l}(J) = \bigsqcup_{i=1}^{k} \Gamma_{B_l}(J_i)$ with $\min(J_1) < \cdots < \min(J_k)$, then we can compute the degree $\deg(\Delta_{B_l}, J)$ of the fundamental element. In the case of $J_1 \not\supset \{1, 2\}$, we compute

$$\deg(\Delta_{B_l}, J) = \sum_{i=1}^{k} \deg(\Delta_{B_l}, J_i) = \sum_{i=1}^{k} \left( \frac{\#(J_i)+1}{2} \right).$$

Moreover, in the case of $J_1 \supset \{1, 2\}$, we compute

$$\deg(\Delta_{B_l}, J) = \sum_{i=1}^{k} \deg(\Delta_{B_l}, J_i) = \#(J_1)^2 + \sum_{i=2}^{k} \left( \frac{\#(J_i) + 1}{2} \right).$$

**Theorem 4.2.** The following equation holds:

$$N'_{B_l}(1) - N'_{A_l}(1) = (-1)^l(l-1).$$

**Proof.** We compute the difference between $N'_{B_l}(1)$ and $N'_{A_l}(1)$. From the equations (4.1) and (4.2), we only have to count the case when the set $J_1$ contains the index set $\{1, 2\}$. For a positive integer $u \in \{2, \ldots, l-2\}$ and $X_l \in \{A_l, B_l\}$, we put

$$S_{X_l, u} := \{ J \subset I \mid \Gamma_{X_l}(J) = \bigsqcup_{i=1}^{k} \Gamma_{X_l}(J_i) \text{ with } \min(J_1) < \cdots < \min(J_k) \text{ s.t. } J_1 = \{1, \ldots, u\} \}$$

For each $u \in \{2, \ldots, l-2\}$, the difference on $S_{X_l, u}$ is the following

$$\sum_{J \in S_{B_l, u}} (-1)^{\#J} \deg(\Delta_{B_l}, J) - \sum_{J \in S_{A_l, u}} (-1)^{\#J} \deg(\Delta_{A_l}, J)$$

$$= \sum_{J \in S_{B_l, u}} (-1)^{\#J} \left\{ \deg(\Delta_{B_l}, J) - \deg(\Delta_{A_l}, J) \right\}$$

$$= \sum_{J \in S_{B_l, u}} (-1)^{\#J} \left\{ \frac{u^2}{2} - \left( \frac{u + 1}{2} \right) \right\}$$

$$= \sum_{J \in S_{B_l, u}} (-1)^{\#J} \left\{ \frac{u^2}{2} - \left( \frac{u + 1}{2} \right) \right\}$$

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Hence, we only have to count the cases $J_1 = \{1, \ldots, l-1\}, \{1, \ldots, l\}$

\[ N'_{B_l}(1) - N'_{A_l}(1) = (-1)^{l-1} \left\{ (l-1)^2 - \left( \frac{l}{2} \right) \right\} + (-1)^l \left\{ l^2 - \left( \frac{l+1}{2} \right) \right\} \]

\[ = (-1)^l (l-1). \]

This completes the proof. \(\square\)

5. Proof of the type $D_l$

Let $l$ be a positive integer in $\mathbb{Z}_{\geq 4}$ and let $I = \{1, 2, \ldots, l\}$. In this section, we will prove Theorem 1.1 for the type $D_l$. We recall a fact from [B-S].

Proposition 5.1. For a subset $J \subset I$, we suppose that the full subgraph $\Gamma_{D_l}(J)$ is connected. Then, the degree $\deg(\Delta_{D_l},J)$ of the fundamental element is given by

\[
\deg(\Delta_{D_l},J) = \begin{cases} 
\#(J)+1 & \text{if } J \not\supset \{l-1, l\} \\
\#(J)(\#(J)-1) & \text{if } J \supset \{l-3, l-2, l-1, l\} \\
6 & \text{if } J = \{l-2, l-1, l\}
\end{cases}
\]

From the equation (2.1), if the full subgraph $\Gamma_{D_l}(J)$ for a subset $J \subset I$ has a decomposition $\Gamma_{D_l}(J) = \Gamma_{D_l}(J_1) \sqcup \cdots \sqcup \Gamma_{D_l}(J_k)$ with $\min(J_1) < \cdots < \min(J_k)$, then we can compute the degree $\deg(\Delta_{D_l},J)$ of the fundamental element.

Theorem 5.2. The following equality holds:

\[ N'_{D_l}(1) = (-1)^l (l-2). \]

Proof. We will show the statement by induction on $l$. First, for $l = 4$, we easily compute $N'_{D_4}(1) = \sum_{J \subset I} (-1)^{\#J} \deg(\Delta_{D_4},J) = 12$. Next, by applying the induction hypothesis, we will compute the difference $N'_{D_{l-1}}(1) - N'_{D_{l-1}}(1)$.

For a positive integer $u \in \{1, \ldots, l-2\}$, we put

\[ S_{D_l,u} := \left\{ J \subset I \mid \Gamma_{D_l}(J) = \bigcup_{i=1}^{k} \Gamma_{D_l}(J_i) \text{ with } \min(J_1) < \cdots < \min(J_k) \text{ s.t. } J_1 = \{1, \ldots, u\} \right\} \]

\[ I_u := I \setminus \{1, \ldots, u, u+1\}. \]

For each $u \in \{1, \ldots, l-5\}$, the difference on $S_{D_l,u}$ is the following

\[ \sum_{J \in S_{D_l,u}} (-1)^{\#J} \deg(\Delta_{D_l},J) \]
\[
= (-1)^u \sum_{K \subseteq I_u} (-1)^{|K|} \left\{ \deg(\Delta_{D_i, J_1}) + \deg(\Delta_{D_i, K}) \right\} \\
= (-1)^u \deg(\Delta_{D_i, J_1}) \sum_{K \subseteq I_u} (-1)^{|K|} + (-1)^u \sum_{K \subseteq I_u} (-1)^{|K|} \deg(\Delta_{D_i, K}) \\
= (-1)^u \sum_{K \subseteq I_u} (-1)^{|K|} \deg(\Delta_{D_i, K}) \\
= (-1)^u N'_{D_{l-1}} (1).
\]

From the induction hypothesis, this is equal to \((-1)^{l-1} (l - u - 3)\). For \(u = l - 4\), the difference on \(S_{D_l, l-4}\) is computed in a similar manner

\[
\sum_{J \in S_{D_l, l-4}} (-1)^{|J|} \deg(\Delta_{D_l, J}) \\
= (-1)^{l-4} N'_{A_l} (1).
\]

For \(u = l - 3\), we easily compute

\[
\sum_{J \in S_{D_l, l-3}} (-1)^{|J|} \deg(\Delta_{D_l, J}) = 0.
\]

Therefore, we can compute the difference

\[
N'_{D_l} (1) - N'_{D_{l-1}} (1) \\
= \sum_{u=1}^{l-3} \sum_{J \in S_{D_l, u}} (-1)^{|J|} \deg(\Delta_{D_l, J}) + (-1)^{l-2} \binom{l-1}{2} \\
= (-1)^{l-1} \left\{ (l - 4) + (l - 3) + \cdots + 1 \right\} + (-1)^{l-2} \binom{l-1}{2} \\
= (-1)^l (2l - 5).
\]

From the induction hypothesis, we have

\[
N'_{D_l} (1) = (-1)^l (l - 2).
\]

\hspace{1cm} \square

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References

[A-N] M. Albenque and P. Nadeau: Growth function for a class of monoids, 21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009), 25-38.

[B] N. Bourbaki: Groups et algèbres de Lie, Chapitres 4, 5 et 6. Éléments de Mathématique XXXIV. Paris, Hermann, 1968.

[Bro] A. Bronfman: Growth functions of a class of monoids, preprint, 2001.

[B-S] E. Brieskorn and K. Saito: Artin-Gruppen und Coxeter-Gruppen, Invent. Math. 17 (1972) 245-271, English translation by C. Coleman, R. Corran, J. Crisp, D. Easdown, R. Howlett, D. Jackson and A. Ram at the University of Sydney, 1996.

[D] P. Deligne: Les immeubles des groupes de tresses généralisée, Invent. Math. 17 (1972) 273-302.

[I1] T. Ishibe: On the monoid in the fundamental group of type $B_{ii}$, Hiroshima Math. J. 42, no.1, (2012), 99-114.

[I2] T. Ishibe: The skew growth functions for the monoid of type $B_{ii}$ and others, Preprint.

[S1] K. Saito: Growth functions associated with Artin monoids of finite type, Proc. Japan Acad. Ser. A Math. Sci. 84 (2008), no.10, 179-183.

[S2] K. Saito: Growth functions for Artin monoids, Proc. Japan Acad. Ser. A Math. Sci. 85 (2009), no.7, 84-88.

[S3] K. Saito: Limit elements in the Configuration Algebra for a Cancellative Monoid, Publ. Res. Inst. Math. Sci. 46 (2010), no.1, 37-113.

[S4] K. Saito: Growth partition functions for cancellative infinite monoids, preprint RIMS-1705 (2010).

[X] P. Xu: Growth of the positive braid semigroups, J. Pure Appl. Algebra 80 (1992), no.2, 197-215.