Comment on “Schwinger’s model of angular momentum with GUP” by Verma Harshit et al.

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Introduction. – In [1], we studied the modification of the angular-momentum algebra dictated by a modification of the commutation relations between position and momentum. This led to interesting results concerning the angular-momentum spectrum, energy spectrum of the hydrogen atom, interactions between magnetic fields and angular momenta, and a description of multi-particle systems, including corrections to Clebsch-Gordan coefficients. Among the noteworthy results of that paper, we have that the commutation relations for angular-momentum operators now involve also functions of the linear-momentum operator. However, following the usual steps, we showed that the spectrum involves only expectation values of this function. Such corrections do not result from any approximation or assumption. On the other hand, when the radial Schrödinger equation for the hydrogen atom is studied, the assumption that the modification appears only via its expectation value is considered. This approximation is useful in that the equation resembles that of standard quantum mechanics, with corrections that have implications in the emission/absorption wavelengths. It is important to notice that this approximation is considered only in the study of the hydrogen atom and of the interaction between magnetic fields and angular momenta. The rest of [1] does not rely on this assumption.

That paper concludes with an analysis of the theory of angular momentum for systems of many particles. Specifically, we computed some of the Clebsch-Gordan coefficients. We showed that the Clebsch-Gordan coefficients are modified due to GUP. However, we also showed that the modification is associated with an ambiguity: coefficients for the same system but computed starting from higher or lower angular-momentum states are in general different. This is something to expect, because the spectrum of the angular-momentum operators, as well as the action of the ladder operators $J_-$ and $J_+$, is influenced by the linear momentum of the states to which they are applied. This ambiguity is a direct consequence of the modified position-momentum uncertainty relation considered in [1] and needs further study.

Comments on [2]. – Verma et al., in [2], have tried a possible resolution of the above ambiguity. Below, we comment on the main aspects of their paper.

Ladder operators for a simple harmonic oscillator (SHO). The authors define a couple of operators, $A$ and $A^\dagger$, using the usual definition of ladder operators in terms of the position and momentum operators. However, following the usual steps, we showed that the spectrum involves only expectation values of this function. Such corrections do not result from any approximation or assumption. On the other hand, when the radial Schrödinger equation for the hydrogen atom is studied, the assumption that the modification appears only via its expectation value is considered. This approximation is useful in that the equation resembles that of standard quantum mechanics, with corrections that have implications in the emission/absorption wavelengths. It is important to notice that this approximation is considered only in the study of the hydrogen atom and of the interaction between magnetic fields and angular momenta. The rest of [1] does not rely on this assumption.

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However, the rest of the results in [1] is not based on such substitution. In particular, the commutation relations between angular-momentum operators are computed implementing directly the modified commutation relation between position and momentum operators. Furthermore, the appearance of the expectation value $⟨C⟩$ in the angular-momentum spectrum was derived from rigorous standard procedures. Nonetheless, this does not motivate in general an indiscriminate substitution $C→⟨C⟩$.

The Jordan map and Schwinger’s model for the angular momentum. The Jordan map connects a set of matrices to a set of harmonic-oscillator operators [5]. In the specific case of the angular momentum, to each matrix representing an angular-momentum operator, with elements $J_{ij}$, the Jordan map associates an operator $J = \sum_{i,j} a_i^1 J_{ij} a_j$.

Therefore, in the standard theory of quantum mechanics, the representation of the angular-momentum operators as Pauli matrices, we find the following expressions [6]:

\[
J_+ = J_x + i J_y = h a_1^1 a_2, \tag{1}
\]
\[
J_- = J_x - i J_y = h a_1^1 a_2. \tag{2}
\]
\[
J^2 = \frac{\hbar^2}{4} (N_1 + N_2)(N_1 + N_2 + 2), \tag{3}
\]
\[
J_\pm = J_x \pm i J_y = h a_1^1 a_2. \tag{4}
\]

These expressions correspond to those used in [2]. A property of the Jordan map is that it preserves commutation relations. Therefore, we observe that the procedure used in [2] is tautological. In fact, starting from a representation of the standard (non-GUP) angular-momentum algebra, the authors use the Jordan map to find the commutation relations for the angular-momentum operators, which necessarily are the standard one.

As a final comment, it is worth noticing that their results concerning the commutation relations of the angular-momentum operators do not match the results in [1], obtained via direct computations.

Clebsch-Gordan coefficients for a system of two particles with GUP. The authors compute the coefficients starting from a maximal angular-momentum state and applying the operator $J_-$ on the top state $|L, L⟩$. They proceed by imposing normalization conditions.

It is important at this point to focus on a particular aspect. Consider the tower of states derived by repeated applications of $J_-$ on the top state $|L, L⟩$. Each of these states has to be normalized independently of the others. On the other hand, given a total angular-momentum state $|L, M⟩$, this describes any composition of two systems, with angular-momentum states $|l_1, m_1⟩$ and $|l_2, m_2⟩$, such that $|l_1 - l_2| \leq L \leq l_1 + l_2$ and $M = m_1 + m_2$. For each value $M$, in general several configurations will be possible such to obtain different values of $L$. Each choice of these compositions is described by a different Clebsch-Gordan coefficient.

Having this picture in mind, it is clear that the normalization of Clebsch-Gordan coefficients has to be performed considering the independent normalization of states $|L, M⟩$ with different values $M$ and the orthogonality among the states of equal $M$ but different $L$. However, in [2] the authors seem to normalize states belonging to different values of $M$ simultaneously. This results in constraints such as

\[
⟨C_1⟩ = ⟨C_2⟩ = ⟨C⟩, \quad \text{with} \quad C_i = 2α p_i - 4α^2 p_i^2, \tag{5}
\]

where $C_1$ and $C_2$ represent the modifications associated with the two constituent systems, while $C$ is the modification associated with the entire system. Such a relation is clearly unacceptable, since the constituents’ momenta are independent physical quantities. Furthermore, the only condition on the composite momentum is that it is the vector sum of the constituent momenta.

The results of [1], therefore, still stand, and the objections raised in [2] are invalid. Nevertheless, it is possible that a correct application of the Jordan map to the GUP-modified angular-momentum operators may be a useful tool in gaining insight into the problem in question and its possible resolution.

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