Infrared behavior of the running coupling constant and bound states in QCD

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Abstract

The perturbative expression of the running strong coupling constant $\alpha_s(Q^2)$ has an unphysical singularity for $Q^2 = \Lambda_{QCD}^2$. Various modifications have been proposed for the infrared region. The effect of some of such proposals on the quark-antiquark spectrum is tested on a Bethe-Salpeter (second order) formalism which was successfully applied in previous papers to an overall evaluation of the spectrum in the light-light, light-heavy and heavy-heavy sectors (the only serious discrepancy with data being for the light pseudoscalar meson masses). In this paper only the $c\bar{c}$, $b\bar{b}$ and $q\bar{q}$ ($q = u$ or $d$) cases are considered and fine structure is neglected. It is found that in the $b\bar{b}$ and $c\bar{c}$ cases the results are little sensitive to the specific choice. In the light-light case the Dokshitzer et al. prescription is again essentially equivalent to the truncation prescription used in the previous calculation and it is consistent with the same $a$ priori fixing of the quark light masses on the typical current values $m_u = m_d = 10$ MeV (only the pion mass resulting completely out of scale of about 500 MeV). With the Shirkov-Solovtsov prescription, on the contrary, a reasonable agreement with the data is obtained only at the price of using a phenomenological momentum dependent effective mass for the quark. The use of such an effective mass should amount to a correction of the free quark propagator. It is remarkable that this has also the effect of bringing the pion mass in the correct range.

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I. INTRODUCTION

In perturbation theory the running coupling constant in QCD is usually written up to one loop as

\[ \alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln (Q^2/\Lambda^2)} \] (1)

or also up to two loops

\[ \alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln (Q^2/\Lambda^2)} \left[ 1 + \frac{2\beta_1 \ln (\ln (Q^2/\Lambda^2))}{\beta_0^2 \ln (Q^2/\Lambda^2)} \right], \] (2)

\(Q\) being the relevant energy scale, \(\beta_0 = 11 - \frac{2}{3}n_f\), \(\beta_1 = 51 - \frac{19}{3}n_f\) and \(n_f\) the number of flavors with masses smaller than \(Q\).

Such expressions have been largely tested in the large \(Q\) processes and are normally used to relate data obtained at different \(Q\) using the appropriate number of “active” flavors \(n_f\) and different values of \(\Lambda\) in the ranges between the various quark thresholds.

Both expressions become singular and completely inadequate as \(Q^2\) approaches \(\Lambda^2\). Therefore they must be somewhat modified in the infrared region.

Various proposals have been done in this direction. The most naive assumption consists in cutting the curve (1) at a certain maximum value \(\alpha_s(0) = \bar{\alpha}_s\) to be treated as a mere phenomenological parameter (truncation prescription). Alternatively, on the basis of general analyticity arguments, Shirkov and Solovtsov \cite{1} replace (1) with

\[ \alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln (Q^2/\Lambda^2)} \left( \frac{1}{\ln (Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} \right), \] (3)

This remains regular for \(Q^2 = \Lambda^2\) and has a finite \(\Lambda\) independent limit \(\alpha_s(0) = 4\pi/\beta_0\), for \(Q^2 \to 0\). Finally, inspired also by phenomenological reasons, Dokshitzer et al. \cite{2} write

\[ \alpha_s(Q^2) = \frac{\sin(\pi \mathcal{P})}{\pi \mathcal{P}} \alpha_s^0(Q^2), \] (4)

where \(\alpha_s^0(Q^2)\) is the perturbative running coupling constant as given by Eq. (1) and \(\mathcal{P} = d/d(\ln(Q^2/\Lambda^2))\) is a derivative acting on \(\alpha_s^0(Q^2)\). The various curves are reported in Fig. \[\text{I}\].

The above modified expressions have been applied to study various effects in which infrared behavior turns out to be important. Electron-positron annihilation into hadrons, \(\tau\)-lepton decay, lepton-hadron deep inelastic scattering, jet shapes, pion form-factors etc. are of this type.

In the quark-antiquark bound state problem the variable \(Q^2\) can be identified with the squared momentum transfer \(Q^2 = (k - k')^2\) and formally the use of a running coupling constant amounts to include higher order terms in the perturbative part of the potential or the Bethe-Salpeter kernel. In this case all values of \(Q^2\) are involved and an infrared regularization becomes essential. Furthermore \(\langle Q^2 \rangle\) ranges typically between \((1 \text{ GeV})^2\) and \((0.1 \text{ GeV})^2\) for different quark masses and internal excitations and values of \(Q^2\) smaller than \(\Lambda^2\) can be important. The specific infrared behavior is therefore expected to affect the spectrum and other properties of mesons.

The purpose of this paper is to test such kind of effects in a particular formalism we have developed and used in previous papers.
II. FORMALISM

In reference [3] we have obtained a good reproduction of the entire meson spectrum in terms of only four adjustable parameters, by solving numerically the eigenvalue equation for the squared mass operator

\[ M^2 = M_0^2 + U \] (5)

or the mass operator

\[ M = M_0 + V, \] (6)

where \( M_0 = w_1 + w_2 = \sqrt{m_1^2 + \vec{k}^2} + \sqrt{m_2^2 + \vec{k}^2} \) is the kinetic term and \( U \) and \( V \) are complicated momentum dependent potentials. Up to the first order in the running coupling constant \( \alpha_s(Q^2) \) and in terms of the string tension \( \sigma \), the “quadratic potential” \( U \) is given by

\[
\langle k|U|k' \rangle = \sqrt{\frac{(w_1 + w_2)(w'_1 + w'_2)}{w_1w_2w'_1w'_2}} \left\{ \frac{4}{3} \alpha_s(Q^2) \right\} - \frac{1}{Q^2} \left( q_{10}q_{20} + q^2 - \frac{(Q \cdot q)^2}{Q^2} \right) + \\
+ \frac{i}{2Q^2} \vec{k} \times \vec{k}' \cdot (\vec{\sigma}_1 + \vec{\sigma}_2) + \frac{1}{2Q^2} \left[ q_{20} (\vec{\alpha}_1 \cdot \vec{Q}) - q_{10} (\vec{\alpha}_2 \cdot \vec{Q}) \right] + \\
+ \frac{1}{6} \vec{\sigma}_1 \cdot \vec{\sigma}_2 + \frac{1}{4} \left( \frac{1}{3} \vec{\sigma}_1 \cdot \vec{\sigma}_2 - \frac{(Q \cdot \vec{\sigma}_1)(Q \cdot \vec{\sigma}_2)}{Q^2} \right) + \frac{1}{4Q^2} (\vec{\alpha}_1 \cdot \vec{Q})(\vec{\alpha}_2 \cdot \vec{Q}) + \\
+ \int \frac{d^3 \vec{r}}{(2\pi)^3} e^{iQ \cdot \vec{r}} j_{\text{inst}}(\vec{r}, \vec{q}_{10}, \vec{q}_{20}) \right\},
\] (7)

with

\[
j_{\text{inst}}(\vec{r}, \vec{q}_{10}, \vec{q}_{20}) = \frac{\sigma r}{q_{10} + q_{20}} \left[ \frac{q_{20}^2 \sqrt{q_{10}^2 - q_T^2}}{2} + q_{10}^2 \sqrt{q_{20}^2 - q_T^2} + \right. \\
+ \frac{q_{10}q_{20}}{|q_T|} \left( \arcsin \frac{|q_T|}{q_{10}} + \arcsin \frac{|q_T|}{q_{20}} \right) - \sigma \frac{q_{20}}{r \sqrt{q_{20}^2 - q_T^2}} (\vec{r} \times \vec{q} \cdot \vec{\sigma}_1 + \\
+ iq_{10}(\vec{r} \cdot \vec{\alpha}_1)) + \left. \frac{q_{10}}{\sqrt{q_{20}^2 - q_T^2}} (\vec{r} \times \vec{q} \cdot \vec{\sigma}_2 - iq_{20}(\vec{r} \cdot \vec{\alpha}_2)) \right]\]
\] (8)

and \( \sigma_j^k = \gamma_j^0 \gamma_j^k, \ \sigma_j^k = \frac{i}{2} \varepsilon^{k \ell m} [\gamma_j^\ell, \gamma_j^m], \ q_{j0} = (w_j + w'_j)/2, \ j = 1, 2. \ \vec{Q} = \vec{k} - \vec{k}', \ \vec{q} = (\vec{k} + \vec{k}')/2, \ q_T^k = (\delta^{jk} - r^j r^k)q_k, \ \vec{r} = \vec{r}/r. \)

The above expression was obtained by reducing of a Bethe-Salpeter like equation which was obtained in reference [4] from first principle QCD under the only assumption that the logarithm of the Wilson loop correlator \( W \) could be written as the sum of its perturbative expression and an area term

\[
i \ln W = i (\ln W)_{\text{pert}} + \sigma S
\] (9)

(advantage was taken in the derivation of an appropriate Feynman-Schwentger representation of the quark propagator in an external field).
An expression for $\langle k|V|k'\rangle$ can be obtained by a direct comparison of Eq. (3) with Eq. (5). Neglecting terms in $V^2$ (what is consistent with the other approximations), this amounts simply to change the kinematical factor in front of the right-hand side of Eq. (7). Properly one should divide $\langle k|U|k'\rangle$ by $w_1 + w_2 + w'_1 + w'_2$, practically the simpler replacement $\sqrt{\frac{(w_1 + w_2)(w'_1 + w'_2)}{w_1w_2w'_1w'_2}} \to \frac{1}{2\sqrt{w_1w_2w'_1w'_2}}$ is essentially equivalent.

The interest of the more conventional Eq. (3) is that it makes more immediate a comparison with ordinary potential approaches and the consideration of the non-relativistic limit. In particular $V$ coincides with the Cornell potential $\langle k|V|k'\rangle = \langle k|(-\frac{3m^2}{r} + \sigma r)|k'\rangle$ in the static limit and with the potential obtained in [5] when the first relativistic corrections are included. In this paper, however, we shall refer only to Eq. (5), as more directly related to the original B-S equation.

The method used in [3] consists in solving first the eigenvalue equation for $M$ in the static limit of $\bar{\alpha}$ (or $\alpha$) and (4), with an appropriate redefinition of the adjustable parameters. In columns (b) and (c) we used the same values $n_f = 4$, $\Lambda = 0.2$ GeV frozen at $\bar{\alpha}_s = 0.35$; we have also taken $\sigma = 0.2$ GeV$^2$, $m_u = m_d = 10$ MeV, $m_c = 1.394$ GeV, $m_b = 4.763$ GeV. The quantities $\Lambda$ and $m_u = m_d$ were fixed a priori from high energy data; $\bar{\alpha}_s$, $\sigma$, $m_c$ and $m_b$ were adjusted on the ground $c\bar{c}$ and $b\bar{b}$ states, the $c\bar{c}$ hyperfine splitting and the Regge trajectory slope.

In this paper, to test the sensitivity of the results to the infrared behavior of $\alpha_s(Q^2)$, we have performed the same calculation in the $b\bar{b}$, $c\bar{c}$ and $q\bar{q}$ ($q = u$ or $d$) case using Eqs. (3) and (4), with an appropriate redefinition of the adjustable parameters.

III. RESULTS

In tables [3], [4], and [5] we give the $b\bar{b}$, $c\bar{c}$, and $q\bar{q}$ quarkonium masses respectively obtained for the different running coupling constant prescriptions. In column (a) we report the results obtained in ref. [3] for the truncated $\alpha_s(Q^2)$, in column (b) those obtained by means of Eq. (3) proposed by Shirkov-Solovtsov and in column (c) those obtained by means of the $\alpha_s(Q^2)$ of Eq. (4) proposed by Dokshitzer et al.

In columns (b) and (c) we used the same values $n_f = 4$, $\Lambda = 0.2$ GeV and $m_u = m_d = 10$ MeV as in [3], but we have slightly redefined the adjustable parameters taking $\sigma = 0.18$ GeV$^2$ in both cases and $m_c = 1.545$ GeV and $m_b = 4.898$ GeV for prescription (3) (column (b)), $m_c = 1.383$ GeV and $m_b = 4.7605$ GeV for prescription (4) (column (c)).

Notice that, in spite of the reduced number of adjustable parameters, the spectra of bottomonium and charmonium are not essentially modified by the new choice for $\alpha_s(Q^2)$, with perhaps the exceptions of the highest $c\bar{c}$ states that are lower in the Dokshitzer et al. case. This indicate little sensitivity of such spectra to the infrared behavior of $\alpha_s(Q^2)$.

Notice that the results are very little sensitive to the precise values of $m_u$ and $m_d$ if these are small.
The situation is completely different for the light-light spectrum of table III. While in front of the experimental and theoretical uncertainties columns (a) and (c) can be considered not really distinguishable, the values reported in column (b) are definitely systematically too low (in particular $\langle M^2 \rangle < 0$ for $\pi$-meson).

Notice however that in the above reported calculations we have used for $m_u$ and $m_d$ the current mass value of 10 MeV. This amounts to assume the difference between the current and the constituent masses to be essentially related to kinematical relativistic correction (cf. [3]) or, what is the same, that the free quark propagator is a good approximation for the complete one in the B-S equation. The inability of the formalism to reproduce a reasonable value for the $\pi$ mass and all experience gained by the chiral symmetry problematic (see in particular [6] and reference herein) suggests that this should not be the case for the light-light systems.

For this reason we have repeated the calculation for choice (3) with various constituent values for the light quark masses. Two sets of results are reported in columns (d) and (e) of table IV. Notice that for $m_u = m_d = 0.30$ GeV the situation is again very similar to those of column (a), table III. Notice also however that as $m_{u,d}$ increases the bound state masses uniformly increase and that for low value of $m_{u,d}$ the lowest bound state masses can be made to agree fairly well with the data, for high value the same occurs for higher states. This could suggest the use of a kind of running constituent mass.

In column (h) the results are reported for the squared effective mass

$$m_{\text{eff}}^2 = 0.11 k - 0.025 k^2 + 0.265 k^4$$

$k$ denoting the quark momentum in the center of mass frame. In Eq. (10) the coefficients are chosen in order to obtain $m_{\text{eff}} = 0.22, 0.28, 0.35$ GeV for $k^2 = 0.26, 0.41, 0.58$ GeV$^2$ approximately corresponding to the $\langle k^2 \rangle$ values for the 1S, 2D and 1G states respectively. As it can be seen the agreement with the data is much improved in this way and finally even a reasonable value for the $\pi$ mass is obtained.

Notice that the use of an effective running mass is in agreement with the general perspectives of combining Dyson-Schwinger equation with B-S equation. Obviously a fine tuning of the coefficients in Eq. (10) to further improve the results would be meaningless in the present context, due even to the approximation used (e.g. the triplet-singlet splitting has been essentially evaluated perturbally).

IV. CONCLUSIONS AND DISCUSSION

In conclusion the heavy quarkonium spectrum does not seem to be very sensitive to the specific infrared behavior of the running constant as it could be expected on general ground.

For the light quarkonium Dokshitzer et al. prescription (3) does not essentially change the results in comparison with the truncation assumption adopted in [3], in spite of the very different appearance of the corresponding curve in Fig. [1].

On the contrary the situation changes drastically for prescription (3).

Actually such prescription would be definitely ruled out if we insisted in using a current mass for the light quarks. If however we give to $m_{u,d}$ a constituent value the results are again similar to those obtained with the truncated $\alpha_s(Q^2)$ and can be strongly improved if we use the running effective mass (10).
To understand better the meaning of Eq. (11), notice that, in the context of the second order formalism developed in ref. [4], the free quark propagator $H_2^{(0)}(p)$ occurring in the BS equation is $i/(p^2 - m^2)$, where we have to set $p = (M_B/2 \pm k_0, \pm k)$ (the upper and lower signs referring to the quark and the antiquark respectively) in the C.M. frame. However if, consistently with the other approximations made, we neglect the spin dependent terms, the full quark propagator $H_2(p)$ can be written as $i/(p^2 - m^2 + \Gamma(p))$. Then, recalling that the instantaneous approximation consists in setting $k_0 = 0$ in the BS kernel, in the same order of ideas we can replace the slowly varying expression $\Gamma(p_0, |p|)$ by $\Gamma(M_B/2, |k|)$. Eventually we obtain the operator $M^2$ as given by Eq. (5) but with $m_1 = m_2$ replaced by

$$m_{\text{eff}}^2(|k|) = m^2 - \Gamma(M_B/2, |k|).$$

Eq. (11) corresponds to a parametrization of right hand side of Eq. (11). Obviously in principle $\Gamma(p)$ should be obtained by actually solving the DS equation for $H_2(p)$ (ref. [3b,4b,4c]), but this is a complicated task that we reserve to a forthcoming paper.

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FIG. 1. Running coupling constant $\alpha_s(Q)$ on logarithmic scale. Truncation prescription (full line), Shirkov-Solovtsov prescription (dashed line), Dokshitzer et al. prescription (dot-dashed line).
### TABLE I

$\bar{b}\bar{b}$. $n_f = 4$, $\Lambda = 0.2$ GeV. (a) $m_b = 4.763$ GeV, $\sigma = 0.2$ GeV$^2$, $\alpha_s(0) = 0.35$, Ref. [3]. (b) $m_b = 4.898$ GeV, $\sigma = 0.18$ GeV$^2$, Shirkov-Solovtsov $\alpha_s(Q^2)$. (c) $m_b = 4.7605$ GeV, $\sigma = 0.18$ GeV$^2$, Dokshitzer et al. $\alpha_s(Q^2)$.

| States  | exp.  | (a) | (b) | (c) |
|---------|-------|-----|-----|-----|
| $1^1S_0$ |       | 9.374 | 9.374 | 9.375 |
| $1^3S_1$ | $\Upsilon(1S)$ | 9.46037 ± 0.00021 | 9.460 | 9.460 | 9.460 |
| $2^1S_0$ |       | 9.975 | 9.988 | 9.983 |
| $2^3S_1$ | $\Upsilon(2S)$ | 10.02330 ± 0.00031 | 10.010 | 10.023 | 10.017 |
| $3^1S_0$ |       | 10.322 | 10.342 | 10.328 |
| $3^3S_1$ | $\Upsilon(3S)$ | 10.3553 ± 0.0005 | 10.348 | 10.368 | 10.352 |
| $4^1S_0$ |       | 10.598 | 10.618 | 10.391 |
| $4^3S_1$ | $\Upsilon(4S)$ | 10.5800 ± 0.0035 | 10.620 | 10.639 | 10.612 |
| $5^1S_0$ |       | 10.837 | 10.854 | 10.816 |
| $5^3S_1$ | $\Upsilon(10860)$ | 10.865 ± 0.008 | 10.857 | 10.872 | 10.834 |
| $6^1S_0$ |       | 11.060 | 11.070 | 11.026 |
| $6^3S_1$ | $\Upsilon(11020)$ | 11.019 ± 0.008 | 11.079 | 11.089 | 11.044 |
| $1^1P_1$ |       |       |       | 9.908 | 9.918 | 9.914 |
| $1^3P_2$ | $\chi_{b2}(1P)$ | 9.9132 ± 0.0006 |       |       |       |
| $1^3P_1$ | $\chi_{b1}(1P)$ | 9.8919 ± 0.0007 |       |       |       |
| $1^3P_0$ | $\chi_{b0}(1P)$ | 9.8598 ± 0.0013 | 9.900 | 9.908 | 9.920 | 9.917 |
| $2^1P_1$ |       | 10.260 | 10.279 | 10.269 |
| $2^3P_2$ | $\chi_{b2}(2P)$ | 10.2685 ± 0.0004 |       |       |       |
| $2^3P_1$ | $\chi_{b1}(2P)$ | 10.2552 ± 0.0005 |       |       |       |
| $2^3P_0$ | $\chi_{b0}(2P)$ | 10.2321 ± 0.0006 | 10.260 | 10.260 | 10.280 | 10.271 |
| States               | exp. (MeV) | (a) (MeV) | (b) (MeV) | (c) (MeV) |
|----------------------|------------|-----------|-----------|-----------|
| $1^1S_0$             | $\eta_c(1S)$ | 2979.8 ± 2.1 | 2982 | 2977 | 2982 |
| $1^3S_1$             | $J/\psi(1S)$ | 3096.88 ± 0.04 | 3097 | 3097 | 3097 |
| $1\Delta S$         |            | 117       | 115       | 119       | 116     |
| $2^1S_0$             | $\eta_c(2S)$ | 3594 ± 5 | 3575 | 3606 | 3573 |
| $2^3S_1$             | $\psi(2S)$ | 3686.00 ± 0.09 | 3642 | 3670 | 3636 |
| $2\Delta S$         |            | 92        | 67        | 64        | 63       |
| $3^1S_0$             |            |           | 3974      | 4005      | 3950    |
| $3^3S_1$             | $\psi(4040)$ | 4040 ± 10 | 4025 | 4054 | 3998 |
| $4^1S_0$             |            |           | 4298      | 4323      | 4252    |
| $4^3S_1$             | $\psi(4415)$ | 4415 ± 6 | 4341 | 4364 | 4291 |
| $1^1P_1$             |            |           | 3529      | 3556      | 3528    |
| $1^3P_2$             | $\chi_{c2}(1P)$ | 3556.17 ± 0.13 | 3525 | 3530 | 3561 | 3531 |
| $1^3P_1$             | $\chi_{c1}(1P)$ | 3510.53 ± 0.12 | 3525 | 3561 | 3531 |
| $1^3P_0$             | $\chi_{c0}(1P)$ | 3415.1 ± 1.0 | 3525 | 3530 | 3561 | 3531 |
| $2^1P_1$             |            |           | 3925      | 3954      | 3904    |
| $2^3P$               |            |           | 3927      | 3958      | 3906    |
| $1^1D_2$             |            |           | 3813      | 3853      | 3811    |
| $1^3D_3$             |            |           |           |           |         |
| $1^3D_2$             | $\psi(3836)$ | 3836 ± 13 | 3813 | 3854 | 3811 |
| $1^3D_1$             | $\psi(3770)$ | 3769.9 ± 2.5 | 4149 | 4183 | 4121 |
| $2^1D_2$             |            |           | 4149      | 4184      | 4121    |
| $2^3D_3$             |            |           |           |           |         |
| $2^3D_2$             |            |           | 4149      | 4184      | 4121    |
| $2^3D_1$             | $\psi(4160)$ | 4159 ± 20 | 4149 | 4184 | 4121 | 4121 |

TABLE II. $c\bar{c}$. $n_f = 4$, $\Lambda = 0.2$ GeV. (a) $m_c = 1.394$ GeV, $\sigma = 0.2$ GeV$^2$, $\alpha_s(0) = 0.35$, Ref. [3]. (b) $m_c = 1.545$ GeV, $\sigma = 0.18$ GeV$^2$, Shirkov-Solovtsov $\alpha_s(Q^2)$. (c) $m_c = 1.383$ GeV, $\sigma = 0.18$ GeV$^2$, Dokshitzer et al. $\alpha_s(Q^2)$. 

TABLE III. \(q\bar{q}. m_{u,d} = 0.01\) GeV, \(n_f = 4\). (a) \(\alpha_s(0) = 0.35\), Ref. [3], \(\sigma = 0.2\) GeV\(^2\), \(\Lambda = 0.2\) GeV. (b) Shirkov-Solovtsov \(\alpha_s(Q^2), \sigma = 0.18\) GeV\(^2\), \(\Lambda = 0.2\) GeV. (c) Dokshitzer et al. \(\alpha_s(Q^2), \sigma = 0.18\) GeV\(^2\), \(\Lambda = 0.2\) GeV.

| States | exp. (MeV) | (a) (MeV) | (b) (MeV) | (c) (MeV) |
|--------|------------|-----------|-----------|-----------|
| \(1^1S_0\) | \(\pi^0\) \(134.9764 \pm 0.0006\) | 479 | - | 575 |
| \(1^1S_0\) | \(\pi^\pm\) \(139.56995 \pm 0.00035\) | 768.5 \(\pm 0.6\) | 846 | 423 | 904 |
| \(1\Delta SS\) | \(\rho(770)\) | 630 | 367 | - | 329 |
| \(2^1S_0\) | \(\pi(1300)\) | 1300 \(\pm 100\) | 1326 | 952 | 1338 |
| \(2^2S_1\) | \(\rho(1450)\) | 1465 \(\pm 25\) | 1461 | 1128 | 1459 |
| \(2\Delta SS\) | | 165 | 135 | 176 | 121 |
| \(3^1S_0\) | \(\pi(1800)\) | 1795 \(\pm 10\) | 1815 | 1485 | 1793 |
| \(3^3S_1\) | \(\rho(2150)\) | 2149 \(\pm 17\) | 1916 | 1600 | 1889 |
| \(3\Delta SS\) | | 354 | 101 | 115 | 96 |
| \(1^1P_1\) | \(b_1(1235)\) | 1231 \(\pm 10\) | | | |
| \(1^1P_2\) | \(a_2(1320)\) | 1318.1 \(\pm 0.7\) | | | |
| \(1^3P_1\) | \(a_1(1260)\) | 1230 \(\pm 40\) | 1303 | 1045 | 1365 |
| \(1^3P_0\) | \(a_0(1450)\) | 1450 \(\pm 40\) | | | |
| \(1^1D_2\) | \(\pi_2(1670)\) | 1670 \(\pm 20\) | | | |
| \(1^3D_2\) | \(\rho_3(1690)\) | 1691.1 \(\pm 5\) | 1701 | 1444 | 1715 |
| \(1^3D_2\) | | 1700 \(\pm 20\) | | | |
| \(1^1F_3\) | | | | | |
| \(1^3F_4\) | \(a_4(2040)\) | 2037 \(\pm 26\) | 1990 | 1743 | 1985 |
| \(1^3F_3\) | \(X(2000)\) | | | | |
| \(1^3F_2\) | | | | | |
| \(1^1G_4\) | | | | | |
| \(1^3G_5\) | \(\rho_5(2350)\) | 2330 \(\pm 35\) | 2238 | 1994 | 2214 |
| \(1^3G_4\) | | | | | |
| \(1^3G_3\) | \(\rho_3(2250)\) | | | | |
| \(1^1H_5\) | | | | | |
| \(1^3H_6\) | \(a_6(2450)\) | 2450 \(\pm 130\) | 2460 | 2215 | 2416 |
| \(1^3H_5\) | | | | | |
| \(1^3H_4\) | | | | | |
| States  | (MeV) | exp.  | (d)  | (e)  | (f)  |
|---------|-------|-------|------|------|------|
| $^1S_0$ | $\begin{cases} \pi^0 \\ \pi^\pm \end{cases}$ | $\begin{cases} 134.9764 \pm 0.0006 \\ 139.56995 \pm 0.00035 \end{cases}$ | 26   | 473  | 124  |
| $^3S_1$ | $\rho(770)$ | $768.5 \pm 0.6$ | 725  | 868  | 737  |
| $^3S_1$ | $\rho(1450)$ | $1465 \pm 25$ | 1344 | 1468 | 1508 |
| $^3P_0$ | $\pi(1300)$ | $1300 \pm 100$ | 1190 | 1326 | 1401 |
| $^3P_0$ | $\pi(1800)$ | $1795 \pm 10$ | 1688 | 1806 | 1993 |
| $^3P_0$ | $\rho(2150)$ | $2149 \pm 17$ | 1788 | 1900 | 2063 |
| $^3S_0$ | $\pi(1300)$ | $1300 \pm 40$ | 1230 | 1364 | 1319 |
| $^3S_0$ | $\pi(1800)$ | $1700 \pm 20$ | 1603 | 1715 | 1741 |
| $^3F_4$ | $a_4(2040)$ | $2037 \pm 26$ | 1881 | 1979 | 2043 |
| $^3F_4$ | $X(2000)$ | $2330 \pm 35$ | 2118 | 2209 | 2319 |
| $^3F_4$ | $\rho_5(2350)$ | $2450 \pm 130$ | 2329 | 2415 | 2569 |
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