$\mathcal{N} = 5, 6$ Superconformal Chern-Simons Theories and M2-branes on Orbifolds

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Abstract: We explore further our recent generalization of the $\mathcal{N} = 4$ superconformal Chern-Simons theories of Gaiotto and Witten. We find and construct explicitly theories of enhanced $\mathcal{N} = 5$ or 6 supersymmetry, especially $\mathcal{N} = 5$, $Sp(2M) \times O(N)$ and $\mathcal{N} = 6$, $Sp(2M) \times O(2)$ theories. The $U(M) \times U(N)$ theory coincides with the $\mathcal{N} = 6$ theory of Aharony, Bergman, Jafferis and Maldacena (ABJM). We argue that the $\mathcal{N} = 5$ theory with $Sp(2N) \times O(2N)$ gauge group can be understood as an orientifolding of the ABJM model with $U(2N) \times U(2N)$ gauge group. We briefly discuss the Type IIB brane construction of the $\mathcal{N} = 5$ theory and the geometry of the M-theory orbifold.
1. Introduction and Concluding Remarks

Superconformal Chern-Simons theories have become a subject of intensive research recently. Schwarz [1] suggested that Chern-Simons theories without Yang-Mills kinetic term may be used to describe the $\mathcal{N} = 8$ superconformal M2-brane world-volume theory. The idea was crystalized by Bagger and Lambert [2–4] and Gustavsson [5, 6] (BLG) who proposed an $\mathcal{N} = 8$ Chern-Simons theory based on 3-algebra, and gave an explicit example with $SO(4) = SU(2) \times SU(2)$ gauge group. The $SO(4)$ BLG theory can be rewritten as an ordinary gauge theory [7, 8]. It is conjectured to be a specific theory of some M2 brane configuration [9, 10].

More recently, Aharony, Bergman, Jafferis and Maldacena (ABJM) [11] have found $\mathcal{N} = 6$ superconformal Chern-Simons theories of $U(M) \times U(N)$ gauge group, and have argued that the theories with $U(N) \times U(N)$ gauge group and Chern-Simons level $k$ is the holographic dual of the M-theory geometry background arising from $N \text{ M2 branes}$ on the
orbifold $\mathbb{C}^4/\mathbb{Z}_k$. Especially for $k = 1, 2$ cases, it has been argued that the supersymmetry is enhanced to $\mathcal{N} = 8$.

On the other hand, with somewhat different motivation, Gaiotto and Witten [12] have constructed a class of $\mathcal{N} = 4$ superconformal Chern-Simons theories coupled to hyper-multiplets. The gauge group and matter content are severely restricted and determined with the help of general classification of Lie super-algebra. In particular, the theory can come with $U(M) \times U(N)$ or $Sp(2M) \times O(N)$ gauge group and a single bi-fundamental hyper-multiplet. In a subsequent work [13], we have constructed more general $\mathcal{N} = 4$ linear quiver-type theories where bi-fundamental hyper and twisted multiplets alternate between connected nodes. The special case of $SU(2) \times SU(2)$ gauge group with both hyper and twisted hyper-multiplets becomes identical to the BLG theory with enhanced $\mathcal{N} = 8$ supersymmetry. It was also suggested in [13] that the embedding of the Chern-Simons theories into IIB string theory [12] can be T-dualized to make contact with M2-branes on orbifolds.

In this work, we show by an explicit construction that the $\mathcal{N} = 4$ theories with two gauge groups and both hyper and twisted hyper-multiplets in the same gauge representation always have an enhanced $\mathcal{N} = 5$ or more supersymmetry. Especially those with $U(M) \times U(N)$ gauge group coincides with the $\mathcal{N} = 6$ ABJM theory. We also find new $\mathcal{N} = 5$ theories of $Sp(2M) \times O(N)$ ($N > 2$) and $\mathcal{N} = 6$ theories of $Sp(2M) \times O(2)$. We argue that the $Sp(2N) \times O(2N)$ theory can be obtained by a simple orientifolding of $\mathcal{N} = 6$ $U(2N) \times U(2N)$ ABJM theory and can be regarded as the holographic dual of the M-theory geometry for M2 branes exploring the orbifold $\mathbb{C}^4/D_k$ where $2k - 2$ is the level of the Chern-Simons coefficient.

The $\mathcal{N} = 3$ theories can come with arbitrary gauge group and matter hyper-multiplets, and are not subject to any quantum correction to the Chern-Simons level. The superconformal theory of $\mathcal{N} = 2, 3$ theories has been studied extensively by Gaiotto and Yin [14]. ABJM have shown that the $\mathcal{N} = 3$ theory with $U(M) \times U(N)$ gauge group and a ‘pair’ of bi-fundamental matter field have an enhanced $\mathcal{N} = 6$ supersymmetry, by arguing that the theory has a $SU(2) \times SU(2)$ global symmetry which does not commute with the $SU(2)$ $R$-symmetry. The ABJM theory falls into our category of theories with enhanced supersymmetry.

We begin with a brief summary of $\mathcal{N} = 4$ theories with both types of hyper-multiplets in Sec. 2, and then show by an explicit construction that whenever the hyper-multiplets belong to the same gauge representation, there is an enhancement of the supersymmetry to at least $\mathcal{N} = 5$. Then we work out the Lagrangian for the theories of $Sp(2M) \times O(N)$ gauge group in detail.

In Sec. 3, we study a further enhancement of $\mathcal{N} = 5$ to $\mathcal{N} = 6$ supersymmetry, which occurs whenever the matter representation can be decomposed into a complex representation ($R$) and its complex conjugate representation ($\bar{R}$), or the matter representation is purely real. The $U(M) \times U(N)$ gauge theory becomes the ABJM model, and the $Sp(2M) \times O(2)$ gauge theory provides a new example of $\mathcal{N} = 6$ superconformal field theories.
We could pursue our analysis further and construct general $\mathcal{N} = 7, 8$ theories as well. From the $\mathcal{N} = 6$ point of view, enhancement of supersymmetry to $\mathcal{N} \geq 7$ requires that the representation $R$ be real; the $SO(4)$ BLG theory is such an example. Hopefully, there could be other cases where the matter representation is real besides the BLG case.

Given the IIB brane configuration of the ABJM model, it is easy to relate our new $\mathcal{N} = 5$ $OSp(M|N)$ theories to an orientifold of the ABJM model. We elaborate on this point in Sec. 4. Taking the T-duality to M-theory as in the ABJ M model, we obtain a $D_k$ orbifold $\mathbb{C}^4$. Orbifolds of $\mathbb{C}^4$ preserving $\mathcal{N} = 5$ supersymmetry were discussed earlier in [15] and very recently in [16].

In Appendices, we provide the detailed computations for the $\mathcal{N} = 5, 6$ cases and express the mass deformation of the $\mathcal{N} = 4$ Lagrangian [13] in $\mathcal{N} = 5, 6$ language. It shows that all of the $\mathcal{N} = 5, 6$ supersymmetries are preserved, while the $SO(5)$ or $SO(6)$ $R$-symmetries are partially broken to $SO(4)$ or $SO(4) \times U(1)$, respectively. This is somewhat expected from the mass deformation of the BLG theory [17, 18].

We close this introduction with some directions for further study. The 3-algebra structure played a crucial role in making the BLG model compatible with $\mathcal{N} = 8$ supersymmetry, while the ABJM model at $k = 1$ is argued to have $\mathcal{N} = 8$ without using the 3-algebra structure. It would be desirable to understand the relation between the two approaches. A recent paper [19] has taken a step in this direction (see also [20–26]). More work is required to establish the AdS/CFT correspondence of the Chern-Simons theories. The relevant topics include extension to $\mathcal{N} = 4$ orbifolds [27, 28], superconformal indices [29], Penrose limit [30], integrability [31], non-supersymmetric generalization [32], and partition function [16]. Finally, it remains to derive, from first principles, much richer family of $\mathcal{N} \geq 2$ superconformal theories with known M-theory geometry [27, 33–37].

2. $\mathcal{N} = 5$ Superconformal Theories

2.1 Review of $\mathcal{N} = 4$

We review the construction of general $\mathcal{N} = 4$ superconformal Chern-Simons-matter theories [12, 13]. We start with an $Sp(2n)$ group and let $A, B$ indices run over a $2n$-dimensional representation. We denote the anti-symmetric invariant tensor of $Sp(2n)$ by $\omega_{AB}$ and choose all the generators $t^A_B$ to be anti-Hermitian $(2n \times 2n)$ matrices such that $t^A_B = \omega^{AC}t^C_B$ are symmetric matrices. We consider a Chern-Simons gauge theory whose gauge group is a subgroup of $Sp(2n)$ and denote anti-Hermitian generators of the gauge group as $(t^m)^A_B$ which satisfy

$$[t^m, t^n] = f^{mn}_{\quad p}t^p,$$

The gauge field is denoted by $(A_m)_\mu$, and the adjoint indices are raised or lowered by an invariant quadratic form $k^{mn}$ or its inverse $k_{mn}$ of the gauge group.

We couple the gauge theory with hyper and twisted hyper-multiplet matter fields $(q^A_\alpha, \tilde{q}^A_\dot{\alpha}, \psi^A_\alpha, \tilde{\psi}^A_\dot{\alpha})$ satisfying the reality conditions

$$\tilde{q}^A_\alpha = (q^A_\alpha)^\dagger = \epsilon^{\alpha\beta} \omega_{AB} q^B_\beta, \quad \tilde{\psi}^A_\alpha = (\psi^A_\alpha)^\dagger = \epsilon^{\dot{\alpha}\dot{\beta}} \omega_{AB} \psi^B_\dot{\beta},$$

(2.1)
and similar conditions for $\tilde{q}^A_\alpha$ and $\tilde{\psi}^A_\alpha$. We use $(\alpha, \beta; \dot{\alpha}, \dot{\beta})$ doublet indices for the $SU(2)_L \times SU(2)_R$ $R$-symmetry group. We also have the inverse tensors, $\omega^{AB}, \epsilon_{\alpha \beta}, \epsilon_{\dot{\alpha} \dot{\beta}}$ such that, say, $\omega_{AC}\omega^{CB} = \delta^B_A$, and $\epsilon^{\alpha \gamma \epsilon_{\gamma \beta} = \delta^\gamma_{\beta}$.

Both types of hyper-multiplets share the same gauge symmetry, so the structure constants $f_{\mu \rho}^\nu$ and the quadratic form $k_{\mu \nu}$ are identical. But, they can take different representations in general $\mathcal{N} = 4$ theories. For $\mathcal{N} > 4$ supersymmetry, however, the two types of hyper-multiplets should be combined together into a bigger multiplet, so they have to take the same representation.

In the construction of Ref. [12,13], $\mathcal{N} = 1$ super-field formulation was used and conditions on the super-potentials for enhancement to $\mathcal{N} = 4$ was examined. It was found that there is essentially one constraint equation (called “fundamental identity” in [12]),

$$k_{\mu \nu}t_{\lambda(B)}^\mu t_{\lambda(CD)}^\nu = 0,$$

where the expression is summed over the cyclic permutation of indices $B, C, D$. When this condition is satisfied, all $\mathcal{N} = 1$ super-potentials are uniquely determined, and we end up with an $\mathcal{N} = 4$ theory.

Following [12], we introduce the “moment map” and “current” operators,

$$\mu_{\alpha \beta}^\mu \equiv t_{\lambda AB}^m q_\alpha^A q_\beta^B,$$

Using these notations, we can write down the Lagrangian of general $\mathcal{N} = 4$ superconformal Chern-Simons-matter theories in a fully covariant form [13],

$$\mathcal{L} = \frac{\epsilon^{\mu \nu \lambda}}{4\pi} \left(k_{\mu \nu}A_\lambda^\mu \partial_\nu A_\lambda^\alpha + \frac{1}{3} f_{\mu \rho \nu} A_\mu^\alpha A_\rho^\beta A_\nu^\gamma \right)$$

$$+ \frac{1}{2} \omega_{AB} \left(-\epsilon^{\alpha \beta} D_{\alpha} q_{\beta}^A D_{\beta} q_{\beta}^B + \epsilon^{\dot{\alpha} \dot{\beta}} \tilde{\psi}_{\dot{\alpha}}^A D_{\dot{\beta}} \tilde{\psi}_{\dot{\beta}}^B \right) + \frac{1}{2} \omega_{AB} \left(-\epsilon^{\dot{\alpha} \dot{\beta}} D_{\dot{\alpha}} \tilde{q}_{\dot{\beta}}^A D_{\dot{\beta}} \tilde{q}_{\dot{\beta}}^B + \epsilon^{\alpha \beta} \psi_{\alpha}^A D_{\beta} \psi_{\beta}^B \right)$$

$$- i \pi k_{\mu \alpha} \epsilon^{\dot{\alpha} \dot{\beta}} \epsilon^{\dot{\gamma} \dot{\delta}} j_{\dot{\alpha} \dot{\beta}} j_{\dot{\gamma} \dot{\delta}} + 4 \pi i k_{\mu \nu} \epsilon^{\alpha \gamma} \epsilon^{\beta \delta} j_{\alpha \beta} j_{\gamma \delta}$$

$$+ i \pi k_{\mu \nu} \left(\epsilon^{\dot{\alpha} \dot{\beta} \dot{\gamma} \dot{\delta}} \tilde{\mu}_{\dot{\alpha} \dot{\beta}} \tilde{\psi}_{\dot{\gamma}}^A \hat{t}_{\dot{\delta}} \tilde{\psi}_{\dot{\delta}}^B + \epsilon^{\alpha \beta} \epsilon^{\gamma \delta} \tilde{\mu}_{\alpha \beta} \tilde{\psi}_{\gamma}^A \hat{t}_{\delta} \tilde{\psi}_{\delta}^B \right)$$

$$- \frac{\pi^2}{6} f_{\mu \rho \nu} (\mu_{\mu})^\gamma_{\beta} (\mu_{\nu})^\gamma_{\alpha} (\mu_{\rho})^\gamma_{\delta}$$

$$+ \frac{\pi^2}{6} f_{\mu \rho \nu} (\mu_{\mu})^\gamma_{\beta} (\mu_{\nu})^\gamma_{\alpha} (\mu_{\rho})^\gamma_{\delta} + \frac{\pi^2}{6} f_{\mu \rho \nu} (\mu_{\mu})^\gamma_{\beta} (\mu_{\nu})^\gamma_{\alpha} (\mu_{\rho})^\gamma_{\delta}.$$

The supersymmetry transformation law reads,

$$\delta q_{\alpha}^A = +i \eta_{\alpha}^A \psi_{\alpha}^A, \quad \delta \tilde{q}_{\dot{\alpha}}^A = -i \eta_{\dot{\alpha}}^A \tilde{\psi}_{\dot{\alpha}}^A,$$

$$\delta A_{\mu}^\mu = 2 \pi i \eta_{\alpha}^A \gamma_{\mu} (j_{\alpha}^m - \bar{j}_{\alpha}^m),$$

$$\delta \psi_{\alpha}^A = + \left[ D_{\alpha} q_{\alpha}^A + \frac{2 \pi}{3} (t_{m})_{AB} (\mu_{\beta}^m)^{\beta} q_{\alpha}^B \right] \eta_{\alpha}^A - 2 \pi (t_{m})_{AB} (\mu_{\beta}^m)^{\beta} \eta_{\beta},$$

$$\delta \tilde{\psi}_{\dot{\alpha}}^A = - \left[ D_{\dot{\alpha}} \tilde{q}_{\dot{\alpha}}^A + \frac{2 \pi}{3} (\tilde{t}_{m})_{AB} (\mu_{\dot{\beta}}^m)^{\dot{\beta}} \tilde{q}_{\dot{\alpha}}^B \right] \eta_{\dot{\alpha}}^A + 2 \pi (\tilde{t}_{m})_{AB} (\mu_{\dot{\beta}}^m)^{\dot{\beta}} \eta_{\dot{\beta}}.$$

The spinor parameter $\eta_{\alpha \dot{\beta}}$ satisfies the reality condition

$$\eta_{\alpha}^{\beta} \eta_{\beta}^{\alpha} = \eta_{\alpha}^{\dot{\alpha}} \eta_{\dot{\alpha}}^{\dot{\beta}} = \epsilon^{\alpha \beta} \epsilon_{\dot{\alpha} \dot{\beta}} \eta_{\alpha}^{\dot{\alpha}} \eta_{\dot{\beta}}^{\dot{\beta}}.$$

(2.6)
In Ref. [12], it was noticed that the fundamental identity can be understood as the Jacobi identity for three fermionic generators of a Lie super-algebra,

\[ [M^m, M^n] = f^{mn}_p M^p, \quad [M^m, Q_A] = Q_B(t^m)_A^B, \quad \{Q_A, Q_B\} = t^m_{AB} M_m. \tag{2.7} \]

This turns out to be a rather strong constraint on the field content of the theory. Namely, the gauge group and matter should be such that the gauge symmetry algebra can be extended to a Lie super-algebra by adding fermionic generators associated to hyper-multiplets.

The notion of Lie super-algebra characterizing $\mathcal{N} = 4$ theories will be useful throughout the rest of the paper, as we investigate the conditions for enhanced supersymmetry ($\mathcal{N} > 4$).

### 2.2 General construction

A necessary condition for supersymmetry enhancement is that the two types of hyper-multiplets in the $\mathcal{N} = 4$ theory take the same representation of the gauge group. In this section, we will show that this is also sufficient for enhancement to $\mathcal{N} = 5$. In other words, for \emph{any} (extended) $\mathcal{N} = 4$ Gaiotto-Witten theory, if the two types of hyper-multiplets are in the same representation of the gauge group so that $t^m_{AB} = \tilde{t}^m_{AB}$, the supersymmetry is automatically enhanced to $\mathcal{N} = 5$.

The lift from $\mathcal{N} = 4$ to $\mathcal{N} = 5$ is an exercise of embedding the $R$-symmetry group $SO(4) = SU(2)_L \times SU(2)_R = Sp(2) \times Sp(2)$ into $Sp(4) = SO(5)$ in the standard way. We combine the $\mathcal{N} = 4$ hyper and twisted hyper-multiplets to form $\mathcal{N} = 5$ multiplets,

\[
\Phi^A_\alpha = \begin{pmatrix} q^A_\alpha \\ \bar{q}^A_\alpha \end{pmatrix}, \quad \Psi^A_\alpha = \begin{pmatrix} \bar{\psi}^A_\alpha \\ \psi^A_\dot{\alpha} \end{pmatrix}. \tag{2.8} \]

The reality conditions can be rewritten in the $\mathcal{N} = 5$ covariant way as

\[
\bar{\Phi}^A_\alpha = (\Phi^A_\alpha)^\dagger = C^\alpha_\beta \omega_{AB} \Phi^B_\beta, \quad \bar{\Psi}^A_\alpha = (\Psi^A_\alpha)^\dagger = C^{\alpha\beta} \omega_{AB} \Psi^B_\beta, \tag{2.9} \]

where the invariant tensor of $Sp(4)$,

\[
C^\alpha_\beta = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}, \tag{2.10} \]

can be understood as the charge conjugation matrix for the $SO(5)$ Clifford algebra in a suitably chosen basis. The “moment map” and the “current” operators also take the $\mathcal{N} = 5$ form,

\[
\mathcal{M}^m_{\alpha\beta} \equiv t^m_{AB} \Phi^A_\alpha \Phi^B_\beta, \quad \mathcal{J}^m_{\alpha\beta} \equiv t^m_{AB} \Phi^A_\alpha \Psi^B_\beta. \tag{2.11} \]

After a slightly lengthy algebra (see appendix [A]), we can uplift the $\mathcal{N} = 4$ Lagrangian (2.4) in the $\mathcal{N} = 5$ Lagrangian,

\[
\mathcal{L} = \frac{\epsilon^{\mu\nu\lambda}}{4\pi} \left( k_{mn} A^m_\mu \partial_\nu A^n_\lambda + \frac{1}{3} f_{mnp} A^m_\mu A^n_\nu A^p_\lambda \right) \\
+ \frac{1}{2} \omega_{AB} C^{\alpha\beta} \left( -D \Phi^A_\alpha D \Phi^B_\beta + i \Psi^A_\alpha D \Psi^B_\beta \right) - i \pi k_{mn} C^{\alpha\beta} C^{\gamma\delta} \left( \mathcal{J}^m_{\alpha\gamma} \mathcal{J}^n_{\beta\delta} - 2 \mathcal{J}^m_{\alpha\gamma} \mathcal{J}^m_{\beta\delta} \right) \\
+ \frac{2\pi^2}{15} f_{mnp} (\mathcal{M}^m)^\alpha_\beta (\mathcal{M}^n)^\beta_\gamma (\mathcal{M}^p)^\gamma_\alpha + \frac{3\pi^2}{5} (\mathcal{M}^m)^\alpha_\beta (\mathcal{M}^n)^\beta_\alpha, \tag{2.12} \]

\[ \]
and the supersymmetry transformation law,

\[
\delta \Phi_\alpha^A = i \eta^\alpha_\beta \Psi_\beta^A, \quad \delta A^m_\mu = 2\pi i \eta^{\alpha\beta} \gamma_\mu \epsilon^{m\beta}_\alpha, \\
\delta \Psi_\alpha^A = \left[ \partial_\alpha \Phi_\beta^A + \frac{2\pi}{3} (t_m)^B \Phi_\beta^B (M^m)^B_\gamma \right] \eta^{\gamma}_\alpha - \frac{4\pi}{3} (t_m)^A_B \Phi_\beta^B (M^m)^\gamma_\alpha \eta^{\beta}_\gamma.
\]

(2.13)

The parameter \( \eta_{\alpha\beta} \) satisfies

\[
\eta_{\alpha\beta} = -\eta_{\beta\alpha}, \quad (\eta^\dagger)^{\alpha\beta} = -\epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \eta_{\gamma\delta}, \quad \epsilon^{\alpha\beta} \eta_{\alpha\beta} = 0.
\]

(2.14)

### 2.3 OSp\((N|2M)\) example

**Symplectic embedding** Let us denote the generators of \( O(N) \) and \( Sp(2M) \) as \( M_{ab} \) and \( M_{\dot{a}\dot{b}} \), respectively. The invariant anti-symmetric tensor of \( Sp(2M) \) is denoted by \( \omega_{\dot{a}\dot{b}} \). We denote the bi-fundamental matter fields \( \Phi_\alpha^A, \Psi_\alpha^A \) by

\[
\Phi_\alpha^{\dot{a}}, \quad \Psi_\alpha^{\dot{a}}.
\]

(2.15)

We choose the symplectic invariant tensor \( \omega_{AB} \) as \( \omega_{\dot{a}\dot{b},\dot{a}\dot{b}} = \delta_{\dot{a}\dot{b}} \cdot \omega_{\dot{a}\dot{b}} \). The matter fields obey the reality condition of the form

\[
\tilde{\Phi}_\alpha^{\dot{a}} = (\Phi_\alpha^{\dot{a}})^\dagger = \delta_{\dot{a}\dot{b}} \omega_{\dot{a}\dot{b}} \epsilon^{\alpha\beta} \Phi_\beta^{\dot{b}},
\]

(2.16)

and similarly for the fermions. In the following the \( O(N) \) vector indices are raised or lowered sloppily while the \( Sp(2M) \) vector indices are raised or lowered by \( \omega_{\dot{a}\dot{b}} \) and \( \omega^{\dot{a}\dot{b}} \). Later we find it convenient to regard the matter fields as \( N \times 2M \) matrices and omit the indices.

From the commutation relation of \( OSp(N|2M) \) generators,

\[
[M_{ab}, M_{cd}] = \delta_{bc} M_{ad} - \delta_{bd} M_{ac} - \delta_{ac} M_{bd} + \delta_{ad} M_{bc}, \\
[M_{\dot{a}\dot{b}}, M_{\dot{c}\dot{d}}] = \omega_{\dot{a}\dot{b}} M_{\dot{c}\dot{d}} + \omega_{\dot{a}\dot{c}} M_{\dot{b}\dot{d}} + \omega_{\dot{a}\dot{d}} M_{\dot{c}\dot{b}} + \omega_{\dot{a}\dot{b}} M_{\dot{c}\dot{d}}, \\
[M_{ab}, Q_{\dot{c}\dot{c}}] = \delta_{bc} Q_{\dot{a}\dot{c}} - \delta_{ac} Q_{\dot{b}\dot{c}}, \\
[M_{\dot{a}\dot{b}}, Q_{\dot{c}\dot{c}}] = \omega_{\dot{a}\dot{c}} Q_{\dot{b}\dot{c}} + \omega_{\dot{b}\dot{c}} Q_{\dot{a}\dot{c}}, \\
\{Q_{\dot{a}\dot{a}}, Q_{\dot{b}\dot{b}}\} = \frac{k}{2\pi} (\omega_{\dot{a}\dot{b}} M_{ab} + \delta_{ab} M_{\dot{a}\dot{b}}),
\]

(2.17)

one can read off the representation matrices on matters,

\[
(t_{ab})_{\dot{c}i,\dot{d}d} = \omega_{\dot{c}i} (\delta_{\dot{a}\dot{d}} \delta_{\dot{b}i} - \delta_{\dot{a}\dot{i}} \delta_{\dot{b}d}) , \quad (t_{\dot{a}\dot{b}})_{\dot{c}i,\dot{d}d} = -\delta_{\dot{c}\dot{d}} (\omega_{\dot{a}\dot{d}} \omega_{\dot{b}\dot{i}} + \omega_{\dot{b}\dot{d}} \omega_{\dot{a}\dot{i}}),
\]

(2.18)

and the quadratic invariant tensor (Chern-Simons coupling)

\[
\epsilon^{ab,cd} = \frac{k}{8\pi} (\omega^{ab} \omega^{cd} - \omega^{ad} \omega^{bc}), \quad \epsilon^{\dot{a}\dot{b},\dot{c}\dot{d}} = -\frac{k}{8\pi} (\omega^{\dot{a}\dot{d}} \omega^{\dot{b}\dot{c}} + \omega^{\dot{b}\dot{d}} \omega^{\dot{a}\dot{c}}).
\]

(2.19)
Lagrangian  The kinetic terms for matters are given by

$$\mathcal{L}_{\text{kin}} = \frac{1}{2} \text{tr} \left( -D_\mu \Phi^\alpha D^\mu \Phi_\alpha + i \bar{\Psi}^\alpha \gamma^\mu D_\mu \Psi_\alpha \right).$$  \hspace{1cm} (2.20)

We normalize the gauge fields for each gauge group $O(N)$ and $Sp(2M)$ as

$$A_{O(N)} = \frac{1}{2} t_{ab} A^{ab} , \quad A_{Sp(2M)} = \frac{1}{2} t_{\dot{a} \dot{b}} \left( \omega^{\dot{b} \dot{c}} \dot{A}^\dot{c} \right).$$  \hspace{1cm} (2.21)

Then the Chern-Simons term becomes

$$\mathcal{L}_{\text{CS}} = \frac{\tau_{\mu \nu \rho}}{4k} \text{tr} \left( -A_\mu \partial_\nu A_\rho - \frac{2}{3} A_\mu A_\nu A_\rho + \dot{A}_\mu \partial_\nu \dot{A}_\rho + \frac{2}{3} \dot{A}_\mu \dot{A}_\nu \dot{A}_\rho \right).$$  \hspace{1cm} (2.22)

The Yukawa and bosonic potential terms are computed by substituting the following expressions into the currents and moment maps,

$$(J_{ab})_{\alpha \beta} = (\Phi_a \bar{\Psi}_\beta + \bar{\Psi}_a \Phi_\beta)_{ab} , \quad (J_{\dot{a} \dot{b}})_{\alpha \beta} = (\Phi_\alpha \bar{\Psi}_{\dot{b}} + \bar{\Psi}_{\dot{a}} \Phi_\beta)_{ab} ,$$

$$(M_{ab})_{\alpha \beta} = (\Phi_a \Phi_\beta + \Phi_\beta \Phi_a)_{ab} , \quad (M_{\dot{a} \dot{b}})_{\alpha \beta} = (\Phi_\alpha \Phi_{\dot{b}} + \Phi_{\dot{a}} \Phi_\beta)_{ab} ,$$

and so on.

For the computation of the interaction terms, it is useful to write the currents (2.23) and the moment maps (2.24) into the trace form,

$$(J_{ab})_{\alpha \beta} = -\text{tr} \left[ \bar{\Psi}_\beta \tau_{ab} \Phi_\alpha \right] , \quad (J_{\dot{a} \dot{b}})_{\alpha \beta} = \text{tr} \left[ \bar{\Psi}_\beta \Phi_\alpha \tau_{\dot{a} \dot{b}} \right] ,$$

$$(M_{ab})_{\alpha \beta} = -\text{tr} \left[ \bar{\Phi}_\beta \tau_{ab} \Phi_\alpha \right] , \quad (M_{\dot{a} \dot{b}})_{\alpha \beta} = \text{tr} \left[ \bar{\Phi}_\beta \Phi_\alpha \tau_{\dot{a} \dot{b}} \right] ,$$

Similarly, we also have

$$(M_{ab,cd})_{\alpha \beta} = +\text{tr} \left[ \bar{\Phi}_\beta \tau_{cd} \tau_{ab} \Phi_\alpha \right] ,$$

$$(M_{\dot{a} \dot{b},\dot{a} \dot{b}})_{\alpha \beta} = -\text{tr} \left[ \bar{\Phi}_\beta \tau_{ab} \Phi_\alpha \tau_{\dot{a} \dot{b}} \right] ,$$

$$(M_{\dot{a} \dot{b},\dot{c} \dot{d}})_{\alpha \beta} = +\text{tr} \left[ \bar{\Phi}_\beta \Phi_\alpha \tau_{\dot{a} \dot{b}} \tau_{\dot{c} \dot{d}} \right] ,$$

Here $\tau_{ab}$ and $\tau_{\dot{a} \dot{b}}$ are the matrices in the defining representation,

$$(\tau_{ab})_{cd} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} , \quad (\tau_{\dot{a} \dot{b}})_{\dot{c} \dot{d}} = -\omega_{\dot{a} \dot{c}} \delta_{\dot{b} \dot{d}} - \omega_{\dot{b} \dot{c}} \delta_{\dot{a} \dot{d}} .$$

Using the completeness relations, we can rewrite the product of traces into a single trace,

$$\frac{k_{ab,cd}}{\pi} \text{tr} \left[ X \tau_{ab} \right] \text{tr} \left[ Y \tau_{cd} \right] = -\frac{k}{\pi} \text{tr} \left[ X_+ Y_- \right] ,$$

$$\frac{k_{\dot{a} \dot{b},\dot{c} \dot{d}}}{\pi} \text{tr} \left[ X \tau_{\dot{a} \dot{b}} \right] \text{tr} \left[ Y \tau_{\dot{c} \dot{d}} \right] = +\frac{k}{\pi} \text{tr} \left[ X_+ Y_+ \right] ,$$

where $X_-$ and $X_+$ are the projections of $X$ satisfying $X^T = -X$ and $(X \omega)^T = +X \omega$.

Some straightforward computations give the Yukawa Lagrangian,

$$\mathcal{L}_{\text{Yukawa}} = -\frac{i k}{2} \text{tr} \left[ -\bar{\Psi}_\beta \Phi_a \Phi^\alpha \Psi^\beta + \bar{\Psi}_\beta \Phi^\alpha \Phi_\alpha \Psi^\beta + 2 \bar{\Psi}_\beta \Phi_\beta \Phi^\alpha \Psi^\beta - 2 \Psi^\beta \Phi_\alpha \Phi^\beta \Phi_\alpha \bar{\Psi} \right]$$

$$-i k \epsilon^{\alpha \beta \gamma \delta} \text{tr} \left[ \Phi_\alpha \bar{\Psi}_\beta \Phi_\gamma \Psi_\delta \right] ,$$

\hspace{1cm} (2.30)
and the bosonic potential,

\[-V = \frac{k^2}{6} \text{tr} \left\{ \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\alpha + \Phi_\alpha \Phi_\beta \Phi_\gamma - 6 \Phi_\gamma \Phi_\beta \Phi_\alpha \Phi_\beta \Phi_\gamma - 6 \Phi_\gamma \Phi_\beta \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\beta \Phi_\gamma \right\} \, (2.31)\]

In the computation of these we used the $Sp(4)$ identities

\[
\epsilon^{\alpha \beta \gamma} = C^{\alpha \beta} C^{\gamma \delta} + C^{\alpha \gamma} C^{\beta \delta} + C^{\alpha \delta} C^{\beta \gamma},
\]

and an equality which follows directly from (2.33),

\[
0 = \text{tr} \left\{ - \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\alpha - \Phi_\alpha \Phi_\alpha \Phi_\beta \Phi_\gamma - \Phi_\beta \Phi_\alpha \Phi_\gamma \Phi_\alpha + 3 \Phi_\gamma \Phi_\alpha \Phi_\gamma \Phi_\alpha \Phi_\beta \Phi_\alpha \Phi_\beta \right. \\
+ 3 \Phi_\alpha \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\alpha \Phi_\beta \Phi_\alpha \Phi_\beta \\
\left. + 3 \Phi_\beta \Phi_\alpha \Phi_\gamma \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\beta + 3 \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\alpha \Phi_\beta \Phi_\gamma \right\} \, (2.34)
\]

In summary, the full Lagrangian for the $O(N) \times Sp(2M)$ theory is

\[
\mathcal{L} = \frac{k^2}{4k} \text{tr} \left( -A_\mu \partial_\nu A_\rho - \frac{2}{3} A_\mu A_\nu A_\rho + \tilde{A}_\mu \partial_\nu \tilde{A}_\rho + \frac{2}{3} \tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho \right) \\
+ \frac{1}{2} \text{tr} \left( -D_\mu \tilde{\Phi}^\alpha D^\mu \Phi_\alpha + i \tilde{\Psi}^\alpha \tilde{\Psi} \Phi_\alpha \right) - i k \epsilon^{\alpha \beta \gamma \delta} \text{Tr} \left( \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\delta \right) \\
+ \frac{i k}{2} \text{tr} \left( -\tilde{\Psi}_\beta \Phi_\alpha \tilde{\Psi}^\beta + \tilde{\Psi}_\beta \Phi_\alpha \tilde{\Psi}^\beta + 2 \tilde{\Psi}_\alpha \tilde{\Psi}_\beta \tilde{\Psi}^\alpha \Psi^\beta - 2 \Psi^\beta \tilde{\Psi}^\alpha \Phi_\beta \Phi_\alpha \right) \\
+ \kappa^2 \left( 4 \Phi_\beta \tilde{\Phi}^\alpha \Phi_\beta \tilde{\Phi}^\gamma \Phi_\alpha \Phi_\beta \right. \\
\left. + 4 \Phi_\beta \tilde{\Phi}^\alpha \Phi_\beta \tilde{\Phi}^\gamma \Phi_\alpha \Phi_\beta - 6 \Phi_\gamma \Phi_\beta \tilde{\Phi}^\alpha \Phi_\beta \right) \right) \, (2.35)
\]

**Supersymmetry transformation rules** The $\mathcal{N} = 5$ supersymmetry transformation rule for $OSp(N|2M)$ model is given by

\[
\delta \Phi_\alpha = i \eta_\alpha^\beta \Psi_\beta, \\
\delta A_\mu = i k \eta^{\alpha \beta \gamma} \mu (\Phi_\alpha \Psi_\beta + \Psi_\beta \Phi_\alpha), \\
\delta \tilde{A}_\mu = \frac{i k}{2} \eta^{\alpha \beta \gamma} \mu (\tilde{\Phi}_\alpha \Psi_\beta + \tilde{\Psi}_\beta \Phi_\alpha), \\
\delta \Psi_\alpha = 2 \Phi_\gamma \eta_\gamma^\alpha + 2 \frac{k}{3} (\Phi_\gamma \tilde{\Phi}_\beta \Phi_\beta + \Phi_\beta \tilde{\Phi}_\gamma \Phi_\beta) \eta_\gamma - \frac{4 k}{3} (\Phi_\alpha \tilde{\Phi}_\beta \Phi_\beta \Phi_\alpha) \eta_\beta. \, (2.36)
\]

3. $\mathcal{N} = 6$ Superconformal Theories

In general, the Gaiotto-Witten construction we reviewed in section 2 assumes that the matter fields form a pseudo-real representation ($R$) of the gauge group; see the reality conditions (2.1). If $R$ can be decomposed into a complex representation ($\bar{R}$) and its complex-conjugate representation ($\bar{\bar{R}}$), then the $\mathcal{N} = 5$ supersymmetry is further enhanced to $\mathcal{N} = 6$. 

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- 8 -
3.1 General construction

The construction is an exercise of embedding the $R$-symmetry group $SO(5) = Sp(4)$ into $SU(4) = SO(6)$. The $\mathcal{N} = 5$ fields are decomposed into $\mathcal{N} = 6$ fields as

$$
(\Phi^A_{\alpha})_{\mathcal{N}=5} = \begin{pmatrix}
\Phi^A_{\alpha} \\
C_{\alpha\beta} \Phi^\beta_A
\end{pmatrix},
(\Psi^A_{\alpha})_{\mathcal{N}=5} = \begin{pmatrix}
C_{\alpha\beta} \Psi^{\beta A} \\
-\bar{\Psi}_{\alpha A}
\end{pmatrix}.
$$

(3.1)

With the symplectic invariant tensor,

$$
(\omega_{AB})_{\mathcal{N}=5} = \begin{pmatrix}
0 & \delta_{AB} \\
-\delta_{BA} & 0
\end{pmatrix},
$$

(3.2)

the reality conditions reduce to

$$
(\Phi^A_{\alpha})^\dagger = \bar{\Phi}^\alpha_A, 
(\Psi^{A\alpha})^\dagger = \bar{\Psi}_{A\alpha},
$$

(3.3)

which is consistent with the following assignments we need for the lift to $\mathcal{N} = 6$.

| Gauge | $\Phi^A_{\alpha}$ | $\bar{\Phi}^\alpha_A$ | $\Psi^{A\alpha}$ | $\bar{\Psi}_{A\alpha}$ |
|-------|------------------|-----------------|-----------------|------------------|
| $SO(6)_R$ | $\mathbf{4}$ | $\mathbf{4}$ | $\mathbf{4}$ | $\mathbf{4}$ |

(3.4)

The gauge generators are written in a block diagonal form as

$$
(t^A_B)_{\mathcal{N}=5} = \begin{pmatrix}
t^A_B & 0 \\
0 & -t^B_A
\end{pmatrix}, 
(t_{AB})_{\mathcal{N}=5} = -\begin{pmatrix}
0 & t^B_A \\
t^A_B & 0
\end{pmatrix},
$$

(3.5)

and the fundamental identity in the $\mathcal{N} = 6$ notation reads

$$
(t^m_B)^c_D + (t^m_D)^c_B = 0.
$$

(3.6)

The “moment map” and “current” operators have the decomposition,

$$
(M^m)_{\alpha\beta} = -(M^m)_{\beta\alpha} - C^{\alpha\gamma}C_{\beta\delta}(M^m)^{\gamma\delta},
$$

(3.7)

$$
(M^{mn})_{\alpha\beta} = -(M^{mn})_{\beta\alpha} + C^{\alpha\gamma}C_{\beta\delta}(M^{mn})^{\gamma\delta},
$$

(3.8)

$$
(J^m)_{\alpha\beta} = (J^m)_{\beta\alpha} - C_{\alpha\gamma}C_{\beta\delta}(J^m)^{\gamma\delta},
$$

(3.9)

where we introduced the $\mathcal{N} = 6$ covariant quantities

$$
(M^m)_{\alpha\beta} = \bar{\Phi}^\alpha_A(t^m)^A_B \bar{\Phi}^B, 
(M^{mn})_{\alpha\beta} = \bar{\Phi}^\alpha_A(t^m)^A_B \bar{\Psi}_{B}^B,
$$

(3.10)

$$
(J^m)_{\alpha\beta} = \bar{\Phi}^\alpha_A(t^m)^A_B \bar{\Psi}_{A\beta}, 
(\bar{J}^m)_{\alpha\beta} = \bar{\Phi}^\alpha_A(t^m)^A_B \bar{\Psi}_{A\beta}.
$$

(3.11)

To rewrite the $\mathcal{N} = 5$ Lagrangian of the previous section in an $\mathcal{N} = 6$ covariant form, we have to make sure that all references to $C_{\alpha\beta}$ disappear. Using the $Sp(4)$ identity \cite{2.32}

$$
C^{\alpha\beta}C^{\gamma\delta} + C^{\alpha\gamma}C^{\beta\delta} + C^{\alpha\delta}C^{\beta\gamma} = \epsilon^{\alpha\beta\gamma\delta},
$$

\footnote{To avoid introducing new set of indices in every page, we are recycling not only the $\alpha, \beta$ indices, but also the $A, B$ indices. They run from 1 to $2n$ in $\mathcal{N} = 5$ formulas, but 1 to $n$ in $\mathcal{N} = 6$ formulas. Hopefully, the context would make it clear which notation is being used.}
we can remove all $C_{\alpha\beta}$ at the expense of introducing $\epsilon_{\alpha\beta\gamma\delta}$ which survives the lift. After a slightly lengthy algebra (see appendix [3]), we obtain the $N = 6$ lift of the $N = 5$ Lagrangian,

$$
\mathcal{L} = \frac{\epsilon_{\mu\nu\lambda}}{4\pi} \left( k_{mn} A_{\mu}^m \partial_{\nu} A_{\lambda}^n + \frac{1}{3} f_{mpn} A_{\mu}^m A_{\nu}^n A_{\lambda}^p \right) - D\Phi_{\alpha}^A D\Phi_{\alpha}^A + i \bar{\Psi}_{\alpha A} \bar{D} \Psi^{A\alpha} 
$$

$$
+ i \pi \left[ 2(J_m)^{\alpha\beta} (J^m)^{\beta\alpha} - 4(J_m)^{\alpha\beta} (J^m)_\gamma^{\beta\alpha} + \epsilon_{\alpha\beta\gamma\delta} (J_m)^{\alpha\beta} (J^m)_{\gamma\delta} + \epsilon_{\alpha\beta\gamma\delta} (J_m)^{\alpha\beta} (J^m)^{\gamma\delta} \right] 
$$

$$
- \frac{4\pi^2}{3} f_{mpn} (M^m)^{\alpha\beta} (M^n)^{\alpha\gamma} (M^p)^{\beta\gamma} + 4\pi^2 (M^{mn})^{\alpha\beta} (M_m)^{\beta\gamma} (M_n)^{\gamma\alpha}.
$$

(3.12)

and the supersymmetry transformation law,

$$
\delta \Phi_{\alpha}^A = -i \eta_{\alpha\beta} \Psi_{\beta}^A, \quad \delta A_{\mu}^m = 2\pi i \left( \eta^{\alpha\beta} \gamma_{\mu} (J^m)^{\beta\alpha} + \eta_{\alpha\beta} \gamma_{\mu} (J^m)^{\alpha\beta} \right),
$$

$$
\delta \Psi^{A\alpha} = \bar{D} \Phi_{\alpha}^A - \frac{2\pi}{3} (t_m)^A_B \Phi_{\gamma}^B (M^m)^{\beta\gamma} \eta^{\gamma\alpha} + \frac{4\pi}{3} (t_m)^A_B \Phi_{\beta}^B (M^m)^{\alpha\gamma} \eta^{\gamma\beta}
$$

$$
- \frac{2\pi}{3} \epsilon_{\alpha\beta\gamma\delta} (t_m)^A_B \Phi_{\gamma}^B (M^m)^{\rho\gamma} \eta_{\delta\rho}.
$$

(3.13)

The parameter $\eta_{\alpha\beta}$ satisfies

$$
\eta_{\alpha\beta} = -\eta_{\beta\alpha}, \quad (\eta^*)^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \eta_{\gamma\delta}.
$$

(3.14)

### 3.2 $U(M|N)$ example

**Symplectic embedding** Let us denote the $U(M)$ and $U(N)$ generators as $M_{a\dot{a}}$ and $M_{\dot{a}a}$, respectively. Here the indices without underlines indicate fundamental representation while those with underlines indicate anti-fundamental representation. For the present model, the complex matter fields $\Phi_{\alpha}^A$ and $\Psi^{A\alpha}$ are described as

$$
\Phi_{\alpha}^A = (\Phi_{\alpha})^{a\dot{a}}, \quad \Psi^{A\alpha} = (\Psi^{\alpha})^{a\dot{a}},
$$

(3.15)

and their complex conjugate fields as

$$
\bar{\Phi}_{\dot{\alpha}}^A = (\bar{\Phi}_{\dot{\alpha}})^{\dot{a}a}, \quad \bar{\Psi}_{\dot{\alpha}A} = (\bar{\Psi}_{\dot{\alpha}})^{\dot{a}a}.
$$

(3.16)

Hereafter we omit the indices $a, \dot{a}, a, \dot{a}$ and regard $\Phi_{\alpha}, \Psi^{\alpha}$ as $M \times N$ matrices. We choose the symplectic invariant tensor as

$$
\omega_{a\dot{a}b\dot{b}} = -\omega_{b\dot{b}a\dot{a}} = \delta_{ab} \delta_{\dot{a}\dot{b}}.
$$

(3.17)

From the commutation relation of the Lie super-algebra $U(M|N)$

$$
[M_{a\dot{b}}, Q_{c\dot{c}}] = +\delta_{a\dot{b}} Q_{c\dot{c}}, \quad [M_{a\dot{b}}, Q_{c\dot{c}}] = -\delta_{a\dot{b}} Q_{c\dot{c}}, \quad [M_{a\dot{b}}, M_{c\dot{d}}] = \delta_{a\dot{b}} M_{c\dot{d}} - \delta_{a\dot{d}} M_{c\dot{b}},
$$

$$
[M_{a\dot{b}}, Q_{c\dot{c}}] = -\delta_{a\dot{b}} Q_{c\dot{c}}, \quad [M_{a\dot{b}}, Q_{c\dot{c}}] = +\delta_{a\dot{b}} Q_{c\dot{c}}, \quad [M_{a\dot{b}}, M_{c\dot{d}}] = \delta_{a\dot{b}} M_{c\dot{d}} - \delta_{a\dot{d}} M_{c\dot{b}},
$$

$$
\{Q_{a\dot{c}}, Q_{b\dot{d}}\} = \frac{k}{2\pi} \left( \delta_{a\dot{c}} M_{b\dot{d}} + \delta_{a\dot{d}} M_{b\dot{c}} \right),
$$

(3.18)
one reads off the representation matrix of gauge group on matters

\[ (t_{ab})_{c\dot{c},d\dot{d}} = -\delta_{\dot{c}\dot{d}}\delta_{a\dot{d}}\delta_{b\dot{c}}, \quad (t_{ab})_{c\dot{c},d\dot{d}} = \delta_{\dot{c}\dot{d}}\delta_{a\dot{d}}\delta_{b\dot{c}}, \]  

(3.19)

and the quadratic invariant tensor

\[ k^{ab,cd} = -k \frac{1}{2\pi} \delta^{ad}\delta^{bc}, \quad k^{\dot{a}\dot{b},\dot{c}\dot{d}} = +k \frac{1}{2\pi} \delta^{\dot{a}\dot{d}}\delta^{\dot{b}\dot{c}}. \]  

(3.20)

**Lagrangian** Once we normalize gauge fields for each gauge group \( U(M) \) and \( U(N) \) as

\[ A_{U(M)} = t_{ab} A_{a\dot{b}}, \quad \tilde{A}_{U(N)} = t_{\dot{a}\dot{b}} \tilde{A}_{\dot{a}\dot{b}}, \]  

(3.21)

it is easy to write down the Chern-Simons term and the matter kinetic term in the matrix form. To express the remaining interactions in terms of matrix fields \( \Phi \) and \( \Psi \), it is useful to write down the Chern-Simons term and the matter kinetic term in the matrix form.

\[ (J_{ab})_{\alpha\beta} = \text{tr} [\Phi_{a\dot{b}} \Psi_{\alpha\beta}], \quad (J_{\dot{a}\dot{b}})_{\alpha\beta} = -\text{tr} [\Phi_{a\dot{b}} \Psi_{\alpha\beta}], \]

\( (J_{\dot{a}b})_{\alpha\beta} = \text{tr} [\Psi_{\dot{a}b} \Phi_{\alpha\beta}], \quad (J_{ab})_{\alpha\beta} = -\text{tr} [\Psi_{\dot{a}b} \Phi_{\alpha\beta}], \]

\( (M_{ab})_{\alpha\beta} = \text{tr} [\Phi_{a\dot{b}} \tilde{\Phi}_{\alpha\beta}], \quad (M_{\dot{a}\dot{b}})_{\alpha\beta} = -\text{tr} [\Phi_{a\dot{b}} \tilde{\Phi}_{\alpha\beta}], \)

\( (M_{\dot{a}b})_{\alpha\beta} = \text{tr} [\Psi_{\dot{a}b} \tilde{\Phi}_{\alpha\beta}], \quad (M_{ab})_{\alpha\beta} = -\text{tr} [\Psi_{\dot{a}b} \tilde{\Phi}_{\alpha\beta}], \)  

(3.22)

The products of traces can be simplified using completeness relations

\[ k^{a,b}_{c,d} \text{tr}[X_{\tau_{a\dot{b}}} \text{tr}[Y_{\tau_{cd}}] = -k \frac{1}{2\pi} \text{tr}[XY], \quad \tilde{k}^{a\dot{b},b\dot{d}} \text{tr}[X_{\tau_{a\dot{b}}} \text{tr}[Y_{\tau_{cd}}] = \frac{k}{2\pi} \text{tr}[XY]. \]  

(3.24)

To summarize, the full \( \mathcal{N} = 6 \) supersymmetric Lagrangian for the \( U(M) \times U(N) \) theory is

\[ \mathcal{L} = -\frac{\epsilon_{\mu\nu\rho}}{2k} \text{tr} \left( A_{\mu} \partial_{\nu} A_{\rho} + \frac{2}{3} A_{\mu} A_{\nu} A_{\rho} - \tilde{A}_{\mu} \partial_{\nu} A_{\rho} - \frac{2}{3} A_{\mu} \tilde{A}_{\nu} \tilde{A}_{\rho} \right) \]

\[ -\text{tr} \left( D_{\mu} \Phi^\alpha D^\mu \Phi_{\alpha} - i \bar{\Psi}_{\alpha} \gamma^\mu D_{\mu} \Psi_{\alpha} \right) - ik \epsilon_{\alpha\beta\gamma\delta} \text{tr} \left( \Phi_{\alpha} \bar{\Psi}_{\beta} \Phi_{\gamma} \Psi_{\delta} + i k \epsilon_{\alpha\beta\gamma\delta} \text{tr} \left( \Phi_{\alpha} \bar{\Psi}_{\beta} \Psi_{\gamma} \Phi_{\delta} \right) \right) \]

\[ -ik \epsilon_{\alpha\beta\gamma\delta} \text{tr} \left( \Phi_{\alpha} \bar{\Psi}_{\beta} \Phi_{\gamma} \Psi_{\delta} + i k \epsilon_{\alpha\beta\gamma\delta} \text{tr} \left( \Phi_{\alpha} \bar{\Psi}_{\beta} \Psi_{\gamma} \Phi_{\delta} \right) \right) \]

\[ + \frac{1}{3} k^2 \text{tr} \left( \Phi_{\alpha} \bar{\Phi}_{\alpha} \Phi_{\beta} \bar{\Phi}_{\alpha} \Phi_{\gamma} \bar{\Phi}_{\alpha} \Phi_{\delta} + \Phi_{\alpha} \bar{\Phi}_{\alpha} \Phi_{\beta} \bar{\Phi}_{\alpha} \Psi_{\gamma} \Phi_{\delta} \right) + \frac{4}{3} k^2 \text{tr} \left( \Phi_{\alpha} \bar{\Phi}_{\alpha} \Phi_{\beta} \bar{\Phi}_{\alpha} \Phi_{\gamma} \Phi_{\delta} \right) \]

\[ -2k^2 \text{tr} \left( \Phi_{\alpha} \bar{\Phi}_{\alpha} \Phi_{\beta} \bar{\Phi}_{\alpha} \Psi_{\gamma} \Phi_{\delta} \right). \]  

(3.25)

The bosonic part is precisely that of the ABJM model [11] and the Yukawa term agrees with that obtained in Ref. [27]. Note that the \( \mathcal{N} = 6 \) Lagrangian above looks almost identical to the \( \mathcal{N} = 5 \) Lagrangian (2.35) of the \( \text{OSp}(N|2M) \) model, except for the reality condition (2.16) for the latter. It follows that the moduli space of vacua of the \( \text{OSp}(N|2M) \) theory should be that of the \( U(N|2M) \) theory modded out by the reality condition. We will come back to this point in section 4.
Supersymmetry transformation rules For scalar and gauge fields, one find
\[
\delta \Phi_\alpha = -i\eta_{\alpha\beta} \Psi^\beta, \quad \delta A_\mu = i\eta^{\alpha\beta}\gamma_\mu \Phi_\alpha \Psi^\beta + \eta_{\alpha\beta}\gamma_\mu \Psi^\beta \Phi_\alpha, \\
\delta \tilde{A}_\mu = i\eta^{\alpha\beta}\gamma_\mu \bar{\Psi}_\alpha \bar{\Phi}^\beta + \eta_{\alpha\beta}\gamma_\mu \bar{\Phi}^\beta \bar{\Psi}_\alpha .
\] (3.26)

The supersymmetry transformation rule for fermions now becomes
\[
\delta \Psi^\alpha = \left[ \gamma^\mu D_\mu \Phi_\gamma - \frac{2k}{3} (\Phi_\beta \bar{\Phi}^\beta \Phi_\gamma) \right] \eta^\gamma \alpha + \frac{4k}{3} (\bar{\Phi}^\alpha \Phi_\gamma) \eta^\gamma \beta - \frac{2k}{3} \epsilon^{\alpha\beta\gamma\delta} (\Phi_\beta \bar{\Phi}^\delta \Phi_\gamma) \eta_{\delta \rho} ,
\] (3.27)
in agreement with a recent independent work [38].

3.3 OSp(2|2M) example

We now describe new \( N = 6 \) superconformal Chern-Simons theories for the super-algebra \( OSp(2|2M) \). The commutation relations were already discussed in section 2.3.

Symplectic embedding \( U(1) = SO(2) \) and \( Sp(2M) \) generators are denoted by \( M_{+-} \) and \( M_{ab} \), respectively. Matter fields \( \Phi^A \) and \( \Psi^{A\alpha} \) are denoted by
\[
\Phi^A = (\Phi_a)^{+a}, \quad \Psi^{A\alpha} = (\Psi^\alpha)^{+a},
\] (3.28)
and their complex conjugate fields by
\[
\bar{\Phi}^\alpha = (\bar{\Phi}^\alpha)^{+a}, \quad \bar{\Psi}_{A\alpha} = (\bar{\Psi}_{a\alpha})^{+a} .
\] (3.29)

For clarity, we hereafter suppress the \( U(1) \) and symplectic indices of the matter fields. We choose the symplectic invariant tensor to be
\[
\omega_{+a,-b} = \omega_{-a,+b} = \omega_{ab} ,
\] (3.30)
and the representation of gauge group on matter fields to be
\[
(t_{+-})_{-a,+b} = -(t_{+-})_{+a,-b} = \omega_{ab} ,
\] (3.31)
\[
(t_{ab})_{\pm c,\mp d} = -\left( \omega_{ac}\omega_{bd} + \omega_{ad}\omega_{bc} \right) .
\] (3.32)

The canonical expression used in section 3.1 can be obtained by
\[
(t_m)^A_B : \quad (t_{+-})^{+a}_{+b} = \omega^{+a,-c}(t_{+-})^{-c,+b} = \delta^a_b , \\
(t_{ab})^{+c}_{+d} = \omega^{+c,-f}(t_{ab})^{-f,+d} = \delta^c_a \omega_{bd} + \delta^c_b \omega_{ad} .
\] (3.33)

The quadratic invariant tensor for \( OSp(2\!\!|2M) \) reads
\[
k^{+-,+-} = -\frac{k}{2\pi}, \quad k^{ab,cd} = -\frac{k}{8\pi} \left( \omega^{ac}\omega_{bd} + \omega^{ad}\omega_{bc} \right) .
\] (3.34)

Before closing this paragraph, let us present some useful quantities to be used below,
\[
(t_{+-})_{\pm a,\mp b} = \omega_{ab} , \\
(t_{ab})_{\pm f,\mp g} = -\left( \omega_{af}\omega_{bc}\omega_{dg} + \omega_{bf}\omega_{ac}\omega_{dg} + \omega_{af}\omega_{bd}\omega_{cg} + \omega_{bf}\omega_{ad}\omega_{cg} \right) , \\
(t_{+-})_{\pm c,\mp d} = \pm \left( \omega_{ac}\omega_{bd} + \omega_{ad}\omega_{bc} \right) .
\] (3.35)
Lagrangian  In our convention, we obtain the kinetic terms for matter fields

\[ \mathcal{L}_{\text{kin}} = -D^\mu \Phi_\alpha D_\mu \Phi^\alpha + i \bar{\Psi}^\alpha \gamma^\mu D_\mu \Psi_\alpha, \]  

(3.36)

together with the Chern-Simons term

\[ \mathcal{L}_{\text{CS}} = \frac{\epsilon^{\mu
\nu\rho}}{2k} A_\mu \partial_\nu A_\rho + \frac{\epsilon^{\mu
\nu\rho}}{4k} \text{tr} \left( \tilde{A}_\mu \partial_\nu \tilde{A}_\rho + \frac{2}{3} \tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho \right). \]  

(3.37)

Here we normalized the gauge fields as

\[ A_{U(1)} = t_+ - A, \quad \tilde{A}_{Sp(2M)} = \frac{1}{2} t_{ab} \tilde{A}^{ab}, \]  

(3.38)

As usual, explicit expressions of the Yukawa interactions and scalar potentials can be easily computed once we substitute the moment map and current operators for the present model,

\[ (M_{+ -})^{a}_\beta = (+ \Phi^a \Phi_\beta), \quad (M_{ab})^{a}_\beta = (+ \Phi^a \Phi_\beta)_{ab} + (\Phi^a \Phi_\beta)_{ba}, \]  

(3.39)

and

\[ (J_{+ -})_{\alpha\beta} = (+ \Phi^\alpha \Phi_\beta), \quad (J_{ab})_{\alpha\beta} = (+ \Phi^\alpha \Phi_\beta)_{ab} + (\Phi^\alpha \Phi_\beta)_{ba}, \]  

(3.40)

As for the symplectic summation convention, we take \( \Phi_1 \Phi_2 \equiv \Phi_1^a \Phi_2^a \).

In summary, the full Lagrangian is \( \mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{potential}} \), where

\[ \mathcal{L}_{\text{kin}} = -D^\mu \Phi_\alpha D_\mu \Phi^\alpha + i \bar{\Psi}^\alpha \gamma^\mu D_\mu \Psi_\alpha, \]

\[ \mathcal{L}_{\text{CS}} = \frac{\epsilon^{\mu\nu\rho}}{2k} A_\mu \partial_\nu A_\rho + \frac{\epsilon^{\mu\nu\rho}}{4k} \text{tr} \left( \tilde{A}_\mu \partial_\nu \tilde{A}_\rho + \frac{2}{3} \tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho \right), \]

\[ \mathcal{L}_{\text{Yukawa}} = -ik \left( \Phi^\alpha \bar{\Psi}^\beta \cdot \bar{\Psi}^\gamma \Phi_\delta - \Phi^\alpha \Phi_\beta \cdot \bar{\Psi}^\gamma \Phi_\delta - \Phi^\alpha \Phi_\delta \cdot \bar{\Psi}^\beta \Phi_\gamma \right) \]

\[ + 2ik \left( \Phi^\alpha \bar{\Psi}^\beta \cdot \bar{\Psi}^\gamma \Phi_\delta - \Phi^\alpha \Phi_\beta \cdot \bar{\Psi}^\gamma \Phi_\delta - \Phi^\alpha \Phi_\delta \cdot \bar{\Psi}^\beta \Phi_\gamma \right) \]

\[ - ik \epsilon^{\alpha\beta\gamma\delta} \left( \Phi^\alpha \bar{\Psi}^\beta \cdot \Phi_\gamma \bar{\Psi}_\delta - \frac{1}{2} \Phi^\alpha \omega \Phi_\gamma \cdot \bar{\Psi}_\delta \Phi_\beta \right) \]

\[ - ik \epsilon^{\alpha\beta\gamma\delta} \left( \bar{\Psi}^\beta \Phi^\alpha \cdot \bar{\Psi}^\gamma \Phi_\delta - \frac{1}{2} \bar{\Psi}^\beta \omega \Phi_\delta \cdot \Phi^\alpha \Phi_\gamma \right), \]

\[ \mathcal{L}_{\text{potential}} = -3k^2 \left( \Phi^\alpha \omega \Phi_\beta \cdot \Phi_\gamma \bar{\Phi}_\delta \cdot \Phi_\gamma \Phi_\delta \right) + \frac{5k^2}{3} \left( \Phi^\alpha \Phi_\beta \cdot \Phi_\gamma \bar{\Phi}_\delta \cdot \Phi_\gamma \Phi_\delta \right) \]

\[ - 2k^2 \left( \Phi^\alpha \Phi_\gamma \cdot \Phi_\beta \Phi^\gamma \cdot \bar{\Phi}_\delta \Phi_\delta \right) + \frac{k^2}{3} \left( \Phi^\alpha \Phi_\beta \cdot \Phi_\gamma \Phi_\delta \cdot \bar{\Phi}_\delta \Phi_\gamma \right). \]  

(3.41)

Here we used the notations

\[ \Phi_1 \Phi_2 = \Phi_1^a \omega_{ab} \Phi_2^b, \quad \bar{\Phi}_1 \Phi_2 = \bar{\Phi}_1^a \omega_{ab} \bar{\Phi}_2^b. \]  

(3.42)
Supersymmetry transformation rules  The $\mathcal{N} = 6$ supersymmetry transformation rules for $OSp(2|\mathcal{M})$ model are given by

$$
\delta \Phi_\alpha = -\eta_\alpha \beta \Psi_\beta, \quad \delta A_\mu = -ik(\eta_\alpha \beta \gamma_\mu \Phi_\alpha \bar{\Phi}_\beta + \eta_\alpha \beta \gamma_\mu \Psi_\beta \bar{\Phi}_\alpha),
$$

$$
\delta A^{ab}_\mu = -ik(\eta_\alpha \beta \gamma_\mu \Phi_\alpha \bar{\Phi}_\beta + \eta_\alpha \beta \gamma_\mu \Psi_\beta \bar{\Phi}_\alpha)^{(ab)}.
$$

(3.43)

and

$$
\delta \Psi^\alpha = \left(\gamma^\mu D_\mu \Phi_\gamma - \frac{2k}{3} \Phi_\beta \bar{\Phi}_\beta \cdot \Phi_\gamma + \frac{k}{3} \omega \bar{\Phi}_\beta \cdot \Phi_\beta \omega \Phi_\gamma \right) \eta^{\gamma \alpha} + \left(\frac{4k}{3} \Phi_\beta \bar{\Phi}_\beta \cdot \Phi_\gamma - \frac{2k}{3} \omega \bar{\Phi}_\beta \cdot \Phi_\beta \omega \Phi_\gamma \right) \eta^{\gamma \beta} + \epsilon^{\alpha \beta \gamma \delta} \left(\frac{2k}{3} \Phi_\beta \bar{\Phi}_\beta \cdot \Phi_\gamma - \frac{k}{3} \omega \bar{\Phi}_\beta \cdot \Phi_\beta \omega \Phi_\gamma \right) \eta_{\delta \beta}.
$$

(3.44)

4. IIB Orientifold and M2-branes on Orbifold

In [11] it was argued that the $\mathcal{N} = 6$ $U(N) \times U(N)$ ABJM model with CS coupling $k$ is the world-volume theory of $N$ M2-branes in orbifold $\mathbb{C}^4/\mathbb{Z}_k$. \footnote{To avoid confusion, note that the letter “$k$” used in earlier sections (2.3, 3.2, 3.3) of this paper is inversely related to the integer-quantized Chern-Simons level $k$ of this section.} Here we argue that our $\mathcal{N} = 5$ theory with the gauge group $SO(2N) \times Sp(2N)$ and the CS coupling $2k$ is the world-volume theory of $N$ M2-branes in orbifold $\mathbb{C}^4/\hat{D}_{k+2}$, where $\hat{D}_k$ is the binary dihedral group.

Our arguments closely follow that of [11]. We take the orientifold of a Type IIB brane configuration realizing the ABJM model, and consider its M-theory dual. We show how the orientifold breaks supersymmetry down to $\mathcal{N} = 5$ from the viewpoint of M-theory geometry as well as the world-volume field theory.

Brane construction of ABJM theory  The ABJM model with gauge group $U(N) \times U(N)$ can be embedded in IIB superstring theory in flat spacetime with compact $x_6$ direction. Consider $N$ D3-branes(0126) intersecting with an NS5-brane(012345) and an (1, $k$) 5-brane(01234’5’) at different points on the $S^1(x_6)$. The directions 3’, 4’, 5’ are given by rotating 3, 4, 5 by the same angle $\theta$ in the planes 37, 48 and 59 respectively. The D3-brane world-volume theory is a $\mathcal{N} = 3$ $U(N) \times U(N)$ Yang-Mills Chern-Simons theory which flows to the ABJM model in the IR limit.

T-duality along the $x_6$ direction followed by an M-theory lift gives a theory of $N$ M2-branes. The transverse space is a fibration of $T^2(\bar{x}_6, x_{10})$ over $\mathbb{R}^6(x_3, x_4, x_5, x_3’, x_4’, x_5’)$. The 5-branes turn into Taub-NUT type geometry after the duality chain. The M-theory geometry is given by a $\mathbb{Z}_k$ orbifold of the product of two Taub-NUTs, where the $\mathbb{Z}_k$ is the simultaneous translation along the two $S^1$ fibers by $1/k$-period. Thus the ABJM model describes $N$ M2-branes at the orbifold $\mathbb{C}^4/\mathbb{Z}_k$.

The four bi-fundamental scalars in ABJM model, which we denote by $(\phi_1, \phi_2, \phi_3, \phi_4)$ here, are identified with complex coordinates of the orbifold. As an example, for $N = 1$ the
scalars ($\phi_i$) are complex numbers. The $U(1) \times U(1)$ gauge fields removes one dimension of the moduli space through gauge equivalence and adds one back through the dual photon. The net effect is the $\mathbb{Z}_k$ orbifolding.

$$\alpha : \phi_i \rightarrow e^{2\pi i/k} \phi_i. \quad (4.1)$$

The $SU(4)$ $R$-symmetry of ABJM model has a geometric interpretation as the subgroup of transverse $SO(8)$ rotations which commutes with the $\mathbb{Z}_k$ orbifolding.

**Introduction of orientifold** Back in Type IIB setup, introducing the O3-plane on top of 2$N$ D3-branes gives an $SO(2N) \times Sp(2N)$ gauge theory [39]. Following the chain of duality to M-theory, one finds that the orientifold turns into an orbifold ($\beta$) that flips all the coordinates $(3, 4, 5; 3', 4', 5'; 6, 10)$. Since the directions 6 and 10 make the phase directions of the fields $\phi_i$ and $\beta$ reverses them, $\beta$ should act anti-holomorphically on these fields. Also, if one requires that the origin is the only fixed point under $\beta$, the action cannot be involutive.

Let us recall the simpler system of O6$^-$-plane and $k$ D6-branes that uplifts to the M-theory on $\hat{D}_{k+2}$ orbifold. The generators $\alpha, \beta$ of the orbifold group $\hat{D}_{k+2}$ correspond to the $1/2k$-period shift along the M-theory circle and the orientifold, respectively. They satisfy

$$\alpha^{2k} = 1, \quad \beta^2 = \alpha^k, \quad \beta \alpha \beta^{-1} = \alpha^{-1}. \quad (4.2)$$

So $\beta$ squares to the half-period shift along the M-theory circle. If the same rule applies to our case, then $\beta^2$ should flip the sign of all the fields $\phi_i$. From anti-holomorphicity and $\beta^2 = -1$, the action of $\beta$ on fields should be of this form

$$\beta : (\phi_1, \phi_2, \phi_3, \phi_4) \rightarrow (\phi_2^*, -\phi_1^*, \phi_4^*, -\phi_3^*), \quad (4.3)$$

up to a linear redefinition of fields. This is equivalent to the reality condition of matter fields in $\mathcal{N} = 5$ supersymmetric theory of section 2.2 involving the matrix $C_{\alpha\beta}$. This new orbifold element breaks the transverse rotation symmetry further to $Sp(4) \simeq SO(5)$, in consistency with the supersymmetry of the world-volume theory.

**Orientifolding the field theory** As noted in Ref. [11], the ABJM model written in $d = 3, \mathcal{N} = 2$ super-field notation closely resembles the conifold theory [40] in $d = 4, \mathcal{N} = 1$ notation. The reason is that both theories have dual descriptions in terms of D-branes winding around a circle and intersecting with two 5-branes at different points on the circle. Also, the two 5-branes are tilted relative to each other in both theories, albeit in somewhat different ways.

Here we argue that the orientifold action (4.3) can be obtained by the standard open string analysis. The bi-fundamental matter fields of ABJM model arise from the open string connecting a pair of D-branes separated by a 5-brane. The orientifold projection on those fields should be independent of the relative angle of the two 5-branes.
We start with the conifold theory described as $U(2N) \times U(2N)$ gauge theory with two bi-fundamental fields $A_i (2N, 2\bar{N})$ and $B_i (2\bar{N}, 2N)$ and the super-potential

$$W = \text{Tr}(A_1 B_2 A_2 B_1 - A_1 B_1 A_2 B_2). \quad (4.4)$$

One encounters the same super-potential when writing the ABJM model in $d = 3, \mathcal{N} = 2$ chiral super-fields $A_i, B_i$. Their lowest components are combined into an $SU(4)$ multiplet,

$$\Phi = \begin{pmatrix} A_i \\ B^i \end{pmatrix}, \quad \bar{\Phi} = \begin{pmatrix} (A^\dagger)^i \\ B^i \end{pmatrix}. \quad (4.5)$$

The orientifold of the conifold model relevant for our discussion was discussed in section 4.3.2 of [41]. There it was shown that the orientifold acts on the matter fields as the $\mathbb{Z}_2$ identification

$$A_1 = B_2^T J \equiv A, \quad A_2 = -B_1^T J \equiv B. \quad (4.6)$$

where $J$ is a matrix form of the anti-symmetric invariant tensor of $Sp(2N)$ satisfying $J^2 = -1$. The reason why we should impose the condition $J^2 = -1$ instead of $J^2 = 1$ on the Chan-Paton factors is explained at [42]. The resulting theory is an $SO(2N) \times Sp(2N)$ gauge theory with two bi-fundamental fields $A, B$ and the super-potential

$$W = \text{Tr}(AB^T BA^T - BB^T AA^T). \quad (4.7)$$

One can see that (4.6) is nothing but the reality condition on the field $\Phi$ of section 2.2 up to a trivial change of basis.

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Appendix

A. Details of Computation

A.1 The \( \mathcal{N} = 5 \) Case

**Yukawa term** We start from the Yukawa terms of \( \mathcal{N} = 4 \) Lagrangian (2.4). By applying the fundamental identity for \( t^m \) to the last two terms we get,

\[
\mathcal{L}_Y = i\pi \epsilon^{\alpha\beta} \gamma^\delta \left\{ -(q_a t^m \psi_\gamma)(q_\beta t_m \psi_\delta) - (\tilde{q}_a t^m \tilde{\psi}_\gamma)(\tilde{q}_\beta t_m \tilde{\psi}_\delta) + 4(q_a t^m \psi_\gamma)(\tilde{q}_\delta t_m \tilde{\psi}_\beta) \\
- (q_a t^m \tilde{\psi}_\beta)(q_\gamma t_m \tilde{\psi}_\delta) - (\tilde{q}_a t^m \psi_\beta)(q_\gamma t_m \psi_\delta) \\
- (q_a t^m \tilde{\psi}_\delta)(q_\gamma t_m \tilde{\psi}_\beta) - (\tilde{q}_a t^m \psi_\delta)(q_\gamma t_m \psi_\beta) \right\}.
\]

Here we dropped dots on indices as there is no confusion. Now we replace the two terms in the second line of the RHS with similar terms with \( \beta, \gamma \) exchanged. This effect can be cancelled by doubling the two terms in the third line once the fundamental identities for \( \epsilon^{\alpha\beta} \) are used, and we get the following

\[
\mathcal{L}_Y = i\pi \epsilon^{\alpha\beta} \gamma^\delta \left\{ -(q_a t^m \psi_\gamma)(q_\beta t_m \psi_\delta) - (\tilde{q}_a t^m \tilde{\psi}_\gamma)(\tilde{q}_\beta t_m \tilde{\psi}_\delta) \\
- (q_a t^m \tilde{\psi}_\gamma)(q_\beta t_m \psi_\delta) - (\tilde{q}_a t^m \psi_\gamma)(q_\beta t_m \tilde{\psi}_\delta) \\
- 2(q_a t^m \tilde{\psi}_\delta)(q_\gamma t_m \tilde{\psi}_\beta) - 2(\tilde{q}_a t^m \psi_\delta)(\tilde{q}_\gamma t_m \psi_\beta) + 4(q_a t^m \psi_\gamma)(\tilde{q}_\delta t_m \tilde{\psi}_\beta) \right\} \\
= -i\pi k_{mn} C^{\alpha\beta} C^{\gamma\delta} \left\{ \mathcal{J}^m_{\alpha\gamma} \mathcal{J}^n_{\beta\delta} - 2\mathcal{J}^m_{\alpha\gamma} \mathcal{J}^n_{\gamma\delta} \right\}.
\]

(A.1)

**Potential** We begin by collecting some useful formulae. We start from

\[
0 = (\mu^{mn})^\alpha_\beta (\mu_m)^\beta_\gamma (\mu_n)^\gamma_\alpha + (\mu^{mn})^\alpha_\beta (\mu_m)^\alpha_\gamma (\mu_n)^\gamma_\alpha + (\mu^{mn})_\gamma^\beta (\mu_m)^\beta_\alpha (\mu_n)^\gamma_\alpha \\
= 2(\mu^{mn})^\alpha_\beta (\mu_m)^\beta_\gamma (\mu_n)^\gamma_\alpha + (\mu^{mn})_\gamma^\beta (\mu_m)^\beta_\alpha (\mu_n)^\gamma_\alpha \\
= 2 f_{\mu mn}(\mu^p)^\alpha_\beta (\mu_m)^\beta_\gamma (\mu_n)^\gamma_\alpha + 2(\mu^{mn})_\gamma^\beta (\mu_m)^\beta_\alpha (\mu_n)^\gamma_\alpha.
\]

(A.2)

Similar equalities hold if some \( q \) are replaced by \( \tilde{q} \), which we express by putting dots to the indices. Putting dots to \( \beta \) we get

\[
0 = 2(\mu^{mn})^\alpha_\beta (\mu_m)^\beta_\gamma (\mu_n)^\gamma_\alpha + (\mu^{mn})_\gamma^\beta (\mu_m)^\beta_\alpha (\mu_n)^\gamma_\alpha.
\]

(A.3)

Putting dots to \( \alpha \) we get

\[
0 = 2(\mu^{mn})^\alpha_\beta (\mu_m)^\beta_\gamma (\mu_n)^\gamma_\alpha + 2(\mu^{mn})_\gamma^\beta (\mu_m)^\beta_\alpha (\mu_n)^\gamma_\alpha + 2 f_{\mu mn}(\mu^p)^\alpha_\beta (\mu_m)^\beta_\gamma (\mu_n)^\gamma_\alpha + f_{\mu mn}(\mu^p)^\alpha_\beta (\mu_m)^\beta_\gamma (\mu_n)^\gamma_\alpha.
\]

(A.4)

Here the first term was rewritten using \( (\mu^{mn})^\alpha_\beta = -(\mu^{mn})^\alpha_\beta \) and (A.3), and the third term was decomposed into symmetric and antisymmetric parts in \( mn \). Now consider

\[
I \equiv 2 f_{\mu mn}(\mathcal{M}^m)^\alpha_\beta (\mathcal{M}^n)^\beta_\gamma (\mathcal{M}^p)^\gamma_\alpha + 9(\mathcal{M}^{mn})^\alpha_\alpha (\mathcal{M}_m)^\beta_\beta (\mathcal{M}_n)^\gamma_\gamma.
\]

(A.5)
Expanding this into $\mu$’s and using \((A.4)\) and \((A.2)\) we find
\[
I = -\frac{5}{2}f_{mnp}(\mu^m)^{\alpha}_\beta(\mu^n)^{\beta}_\gamma(\mu^p)^\gamma_\alpha - \frac{5}{2}f_{mnp}(\mu^m)^{\dot{\alpha}}_\dot{\beta}(\mu^n)^{\dot{\beta}}_\dot{\gamma}(\mu^p)^\dot{\gamma}_\dot{\alpha} + 15(\mu^{mn})^{\dot{\alpha}}_\dot{\beta}(\mu_m)^{\beta}_\gamma(\mu_n)^\gamma_\alpha + 15(\mu^{mn})^{\alpha}_\beta(\mu_m)^{\dot{\beta}}_\dot{\gamma}(\mu_n)^\dot{\gamma}_\dot{\alpha}.
\]
(A.6)
Hence the potential term in \((2.4)\) is $L_{\text{pot}} = -V = \pi^2I/15$.

**Supersymmetry transformation** Let $\eta_{\dot{\alpha}\dot{\alpha}}$ be the parameter of $\mathcal{N} = 4$ supersymmetry. We define $\eta_{\dot{\alpha}\dot{\alpha}} = -\eta_{\dot{\alpha}\dot{\alpha}}$, and introduce the $4 \times 4$ matrix valued spinor
\[
\hat{\eta}_{\dot{\alpha}}^\beta = \left( \begin{array}{cc} 0 & \eta_{\dot{\alpha}}^\beta \\ \eta_{\dot{\alpha}}^\beta & 0 \end{array} \right).
\]
(A.7)
Rewriting the $\mathcal{N} = 4$ transformation law \((2.5)\) in terms of $Sp(4)$ multiplets $\Phi^A_{\dot{\alpha}}, \Psi^A_{\alpha}$ and $\hat{\eta}_{\dot{\alpha}}^\beta$, we easily obtain \((2.13)\).

**Sp(4) R-symmetry** The R-symmetry acts on matter fields as
\[
\Phi^A_{\dot{\alpha}} = U_{\alpha}^\beta \Phi^A_{\beta}, \quad \Psi^A_{\alpha} = U_{\alpha}^\beta \Psi^A_{\beta}.
\]
(A.8)
$U$ is unitary and satisfies $U^* = CUC^{-1}, U^TCU = C$. They are equivalently $SO(5)$ spinors with charge conjugation matrix $C$. The $\mathcal{N} = 4$ supersymmetry parameter $\hat{\eta}$ satisfies,
\[
C\hat{\eta}C^{-1} = \hat{\eta}^T, \quad (\hat{\eta}^*) = C\hat{\eta}C^{-1}, \quad \text{Tr}[\hat{\eta}] = 0, \quad \Gamma_5\hat{\eta}\Gamma_5 = -\hat{\eta},
\]
where $(\Gamma_5)_{\alpha}^\beta = \text{diag}(+1, +1, -1, -1)$. The $Sp(4)$ R-invariance removes the last condition and uplifts the supersymmetry to $\mathcal{N} = 5$.

**A.2 The $\mathcal{N} = 6$ Case**

**Yukawa term** Substituting \((3.9)\) in the Yukawa terms of the $\mathcal{N} = 5$ Lagrangian \((2.12)\) and expanding, we get the $\mathcal{N} = 6$ Yukawa term $L_Y = i\pi I_Y$.
\[
I_Y = 2(J^n)_{\alpha\beta}(\bar{J}_n)^{\alpha\beta} - 4(J^n)_{\alpha\beta}(\bar{J}_n)^{\beta\alpha} + (J^n)_{\alpha\beta}(\bar{J}_n)^{\gamma\delta}(2C^{\alpha\delta}C^{\beta\gamma} - C^{\alpha\gamma}C^{\beta\delta}) + (J^n)_{\alpha\beta}(\bar{J}_n)^{\gamma\delta}(2C^{\alpha\delta}C^{\beta\gamma} - C^{\alpha\gamma}C^{\beta\delta}).
\]
(A.10)
Using the fundamental identity \((3.6)\) and the $Sp(4)$ identity \((2.32)\) we get the Yukawa terms in \((3.12)\).

**Potential** The potential term is $L_{\text{pot}} = -V = \pi^2I/15$, where
\[
I = 2f_{mnp}(M^m)^{\alpha}_\beta(M^n)^{\beta}_\gamma(M^p)^\gamma_\alpha + 9(M^{mn})^{\gamma}_\gamma(M_m)^{\alpha}_\beta(M_n)^{\beta}_\alpha \equiv 2I_1 + 9I_2.
\]
(A.11)
Substituting \((3.7)\) and \((3.8)\) we obtain the intermediate results,
\[
2I_1 = -4f_{mnp}(M^m)^{\alpha}_\beta(M^n)^{\beta}_\gamma(M^p)^\gamma_\alpha + 12f_{mnp}(M^m)^{\alpha}_\beta(M^n)^{\beta}_\gamma C^{\gamma\rho}(M^p)^\rho_{\sigma} C_{\alpha},
\]
\[
9I_2 = -36(M^{mn})^{\gamma}_\gamma(M^m)^{\alpha}_\beta(M^n)^{\beta}_\alpha + 36(M^{mn})^{\gamma}_\gamma(M^m)^{\alpha}_\beta C^{\beta\rho}(M^n)^{\rho}_\sigma C_{\alpha}.
\]
(A.12)
In the right hand side of both equations, the first term is itself $SU(4)$ invariant. The remaining terms, which we denote as $12X_1$ and $36X_2$, should combine into an $SU(4)$ invariant. To see this, we introduce a new $SU(4)$ invariant term and decompose it using the identities (2.32) and (3.6),

$$Z \equiv \epsilon_{\alpha\beta\gamma}e^{\alpha\sigma\tau}(M_m^{mn})^\beta(M_n)\gamma_{\sigma}(M_n)^\delta = 4X_2 - X_3 + 2X_4 + 2X_5, \quad (A.13)$$

$$X_3 \equiv (M_m)^\alpha\beta(M_n)^\gamma_{\sigma}(M_n)^\delta \rho C_{\rho\alpha}, \quad (A.14)$$

$$X_4 \equiv (M_m)\gamma_{\alpha}(M_m)^\beta\gamma_{\beta}(M_n)^\sigma C_{\rho\sigma\alpha}, \quad (A.15)$$

$$X_5 \equiv (M_n)^\gamma_{\alpha}(M_m)^\alpha\beta C_{\beta\rho}(M_m)^\gamma_{\rho\alpha}. \quad (A.16)$$

Inserting $f^{mn}_{pt} = [t^m, t^n]$ into different $t^m$ factors in $X_1$, one can show

$$X_1 = X_2 + X_3 = -X_2 + X_4 = -X_2 + X_5, \quad (A.17)$$

from which one can easily find $12X_1 + 36X_2 = 4Z$ as expected. Note also that

$$Z = (M_m^{mn})^\beta(M_m)^\gamma_{\sigma}(M_n)^\delta \left\{ \delta_{\rho\sigma}^\gamma \delta_{\tau}^\delta \pm (5 \text{ other terms}) \right\} = 4\text{Tr}(M_m^{mn}M_nM_m) + 2\text{Tr}(M_m^{mn}M_mM_n), \quad (A.18)$$

where the trace is with respect to the $SU(4)$ indices and the fundamental identity was used. The potential term $\mathcal{L}_{pot} = -V = \pi^2 I/15$ finally becomes

$$I = -4f_{mpn}\text{Tr}(M_m^{mn}M^p) + 44\text{Tr}(M_{mn}M_mM^n) + 16\text{Tr}(M_m^{mn}M_mM_n)$$

$$= 40\text{Tr}(M_m^{mn}M_mM_n) + 20\text{Tr}(M_m^{mn}M_nM_m)$$

$$= 60\text{Tr}(M_m^{mn}M_mM_n) - 20f_{mpn}\text{Tr}(M_mM_nM_p),$$

$$\mathcal{L}_{pot} = 4\pi^2 \text{Tr}(M_m^{mn}M_mM_n) - \frac{4\pi^2}{3} f_{mpn}\text{Tr}(M_mM_nM_p). \quad (A.19)$$

**B. Mass Deformation**

In this section, we present a supersymmetry preserving mass deformation of the $\mathcal{N} = 5$ and $\mathcal{N} = 6$ theories. It was shown in [13] that the extended Gaiotto-Witten theories allow a mass deformation which preserves the whole $\mathcal{N} = 4$ supersymmetry and $SO(4)$ $R$-symmetry.

The mass deformation adds the following terms to the Lagrangian [13],

$$\mathcal{L}_{mass} = -\frac{\omega_{AB}}{2} \left( m^2 e^{\alpha\beta}q^A q^B + m^2 e^{\dot{\alpha}\dot{\beta}}\bar{q}^A \bar{q}^B + im e^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\alpha}} A^B \psi^B - im e^{\alpha\beta} \bar{\psi}_{\dot{\alpha}} A^B \bar{\psi}_{\dot{\beta}} \right)$$

$$- \frac{2\pi}{3} m k_{mn} \left\{ (\mu^m)_{\alpha\beta}(\mu^n)_{\dot{\alpha}\dot{\beta}} - (\mu^m)_{\dot{\alpha}\dot{\beta}}(\mu^n)_{\alpha\beta} \right\}, \quad (B.1)$$

and the supersymmetry transformation rules,

$$\delta_{mass}\psi^A_{\dot{\alpha}} = m q^A_{\dot{\alpha}} \eta_{\dot{\alpha}}^{\alpha}, \quad \delta_{mass}\bar{\psi}^A_{\alpha} = m \bar{q}^A_{\alpha} \eta_{\alpha}^{\dot{\alpha}}. \quad (B.2)$$

We will generalize this result to the $\mathcal{N} = 5$ and $\mathcal{N} = 6$ theories of this paper. We will find that supersymmetries are all preserved, but the $R$-symmetry gets partially broken.
\( \mathcal{N} = 5 \) mass deformation

Using the notations introduced in Section 2.2, the \( \mathcal{N} = 4 \) mass deformed Lagrangian (B.1) is rewritten as

\[
\mathcal{L}_\text{mass} = -\frac{\omega_{AB}}{2} \left( m^2 \Phi_A^C \phi_B^D - im \Psi_A^\alpha (C \Gamma_5)^\alpha_\beta \Psi_B^\beta \right) + \frac{2\pi m}{3} (\mathcal{M}_A^\beta) (\mathcal{M}_B^\alpha)^\gamma (\Gamma_5)^\alpha_\gamma ,
\]

(B.3)

where the matrix \( \Gamma_5 \) is defined by \( (\Gamma_5)^\alpha_\beta = \text{diag}(+1, +1, -1, -1) \). The mass deformation to the supersymmetry transformation rule is

\[
\delta_m \Psi_A^\alpha = m (\Gamma_5)^\alpha_\gamma \eta^\gamma \Phi^\beta_\beta .
\]

(B.4)

The explicit dependence on \( \Gamma_5 \) implies that the \( SO(5) \) \( R \)-symmetry is broken down to the \( SO(4) \) subgroup. Nevertheless, one can show that the mass deformed theory is invariant under the whole \( \mathcal{N} = 5 \) supersymmetry deformed by (B.4). This is consistent with previous results [17, 18] on mass deformation of the \( \mathcal{N} = 8 \) BLG model of \( SO(4) \) gauge group. For the BLG model, the mass terms did not break any supersymmetry despite the \( R \)-symmetry breaking.

Checking the supersymmetry  Let us sketch the proof of the full \( \mathcal{N} = 5 \) invariance.

We work order by order in \( m \). The \( \mathcal{O}(m^2) \) terms in \( \delta \mathcal{L} \) arise from \( \delta \) of the boson mass term and \( \delta_{\text{mass}} \) of the fermion mass term.

\[
\delta \mathcal{L}|_{m^2} = -\omega_{AB} m^2 \Phi_A^C \phi_B^D + im \omega_{AB} \Psi_A^\alpha (C \Gamma_5)^\alpha_\beta \delta_{\text{mass}} \Psi_B^\beta = -im \omega_{AB} m^2 \Phi_A^C \phi_B^D + im \omega_{AB} \Psi_A^\alpha (C \Gamma_5)^\alpha_\beta \delta_{\text{mass}} \Psi_B^\beta = 0 .
\]

(B.5)

The \( \mathcal{O}(m) \) terms fall into two types, one proportional to \( m \Psi \Phi \) and the other to \( m \eta \Phi \). The terms of the first type arise from \( \delta \) of the fermionic mass term and \( \delta_{\text{mass}} \) of the fermionic kinetic term, and are easily shown to cancel each other. The remaining \( \mathcal{O}(m) \) terms are given by

\[
\delta \mathcal{L}|_m = \frac{2\pi im}{3} \left( (\Gamma_5)^\alpha_\beta \eta^\gamma (\mathcal{M}_A^\beta) (\mathcal{J}_m)_{\gamma}^\delta - 2(\Gamma_5)^\alpha_\beta (\mathcal{M}_A^\rho) (\mathcal{J}_m)_{\gamma}^\delta \eta^\rho \right)
\]

\[
+ \frac{2\pi im}{3} \left( 2(\Gamma_5)^\alpha_\beta (\mathcal{M}_A^\rho) (\mathcal{J}_m)_{\gamma}^\delta \eta^\rho + 2(\Gamma_5)^\alpha_\beta (\mathcal{J}_m)_{\gamma}^\delta \eta^\rho \right) 
\]

\[
+ \frac{2\pi im}{3} \left( 3(\Gamma_5)^\alpha_\beta (\mathcal{M}_A^\rho) (\mathcal{J}_m)_{\gamma}^\delta \eta^\rho - 6(\Gamma_5)^\alpha_\beta (\mathcal{J}_m)_{\gamma}^\delta \eta^\rho \right) ,
\]

(B.6)

where the three lines in the right hand side correspond respectively to \( \delta \) of the fermion mass term, \( \delta \) of the quartic potential term and \( \delta_{\text{mass}} \) of the Yukawa interaction. It appears difficult to show that this vanishes, but since we know it vanishes when \( \eta^\rho \) is proportional to \( \Gamma_1, \cdots, \Gamma_4 \), we only need to check that it vanishes when \( \eta \sim \Gamma_5 \).
\( \mathcal{N} = 6 \) mass deformation

Following the lifting procedure in section 3, one can show that the mass deformed Lagrangian is given by

\[
L_{\text{mass}} = -m^2 \Phi_\alpha^A \Phi_\alpha^A + im \bar{\Psi}_{\alpha A}(C \Gamma_5 C)_{\alpha \beta} \Psi^{A \beta} + \frac{2\pi m}{3} \left( M^m \right)_{\beta}^\alpha \left( M_m \right)^{\gamma}_{\delta} \left\{ -2 \delta^\alpha_{\beta} (\Gamma_5)_{\gamma}^\delta + (\Gamma_5 C)_{\alpha \gamma} C^{\beta \delta} + C_{\alpha \gamma} (C \Gamma_5)^{\beta \delta} \right\}. \tag{B.7}
\]

We regard \( \Phi_\alpha^A \) as a \( SO(6) \) left-handed spinor and \( \Psi^{A \alpha} \) as a right-handed spinor. Introducing the six-dimensional Gamma matrices

\[
\hat{\Gamma}_I = \begin{pmatrix} 0 & (\bar{\rho}_I)_{\alpha \beta} \\ (\rho_I)_{\alpha \beta} & 0 \end{pmatrix}, \quad \rho_I = (C \Gamma_I, iC), \quad \bar{\rho}_I = (\Gamma_I C, -iC), \tag{B.8}
\]

one can write \( L_{\text{mass}} \) in the following form

\[
L_m = -m^2 \Phi_\alpha^A \Phi_\alpha^A - m \bar{\Psi}_{\alpha A}(\rho_{56})_{\alpha \beta} \Psi^{A \beta} + \frac{2\pi im}{3} \left( M^m \right)_{\beta}^\alpha \left( M_m \right)^{\gamma}_{\delta} \left\{ 2 \delta^\alpha_{\beta} (\bar{\rho}_{56})_{\gamma}^\delta - (\bar{\rho}_5)_{\alpha \gamma} (\rho_6)^{\beta \delta} + (\bar{\rho}_6)_{\alpha \gamma} (\rho_5)^{\beta \delta} \right\}. \tag{B.9}
\]

The deformation to the supersymmetry transformation rule is

\[
\delta_{\text{mass}} \Psi^{A \alpha} = m(C \Gamma_5 C)_{\alpha \beta} \eta^{\beta \gamma} \Phi_\gamma^A = im(\rho_{56})_{\alpha \beta} \eta^{\beta \gamma} \Phi_\gamma^A. \tag{B.10}
\]

The mass deformation breaks the \( SO(6) \) \( R \)-symmetry down to \( SO(4) \times SO(2) \). But this deformation preserves all \( \mathcal{N} = 6 \) supersymmetry, since the \( SO(2) \) relates the sixth supersymmetry with the fifth one which has been shown to be the symmetry. See [11, 43] for related discussions.

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