LIPSCHITZ CONTINUITY OF THE WASSERSTEIN PROJECTIONS IN THE CONVEX ORDER ON THE LINE

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Abstract. Wasserstein projections in the convex order were first considered in the framework of weak optimal transport, and found application in various problems such as concentration inequalities and martingale optimal transport. In dimension one, it is well-known that the set of probability measures with a given mean is a lattice w.r.t. the convex order. Our main result is that, contrary to the minimum and maximum in the convex order, the Wasserstein projections are Lipschitz continuity w.r.t. the Wasserstein distance in dimension one. Moreover, we provide examples that show sharpness of the obtained bounds for the Wasserstein distance with index 1.

Keywords: Optimal transport, Weak optimal transport, Projection, Convex order.

1. Introduction and main result

Motivated by the restoration of the convex order between discrete approximations of two probability measures on \( \mathbb{R}^d \), Alfonsi, Corbetta and one of the authors [2] introduced for \( p \in [1, +\infty) \) the Wasserstein projections in the convex order:

\[
I_p(\mu, \nu) := \arg \min \left\{ W_p(\mu, \eta): \eta \leq_c \nu \right\},
\]
\[
J_p(\mu, \nu) := \arg \min \left\{ W_p(\eta, \nu): \mu \leq_c \eta \right\}.
\]

Here \( W_p \) denotes the celebrated Wasserstein distance on the set \( \mathcal{P}_p(\mathbb{R}^d) \) of Borel probability measures on \( \mathbb{R}^d \) with finite \( p \)-th moment:

\[
W_p(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(dx, dy), \quad \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d),
\]

where we write \( \Pi(\mu, \nu) \) for the subset of couplings \( \pi \in \mathcal{P}_p(\mathbb{R}^d \times \mathbb{R}^d) \) with first marginal \( \mu \) and second marginal \( \nu \). Moreover, the inequality \( \mu \leq_c \nu \) means that \( \mu \) is smaller than \( \nu \) in the convex order, i.e., for all convex \( f: \mathbb{R}^d \to \mathbb{R} \):

\[
\int_{\mathbb{R}^d} f(x) \mu(dx) \leq \int_{\mathbb{R}^d} f(x) \nu(dy).
\]

When \( d = 1 \), the projections (1.1) and (1.2) enjoy additional properties according to [2]: the probability measures \( I(\mu, \nu) \) resp. \( J(\mu, \nu) \) defined in (1.10) below are (when \( p > 1 \) unique) optimizers of (1.1) resp. (1.2) for all \( p \geq 1 \). Our main result is the following Lipschitz continuity property of \( I \) and \( J \).

Theorem 1.1 (Lipschitz continuity). When \( d = 1 \) and \( p \in [1, \infty) \), the Wasserstein projections \( I, J \) are Lipschitz continuous. For \( \mu, \nu, \mu', \nu' \in \mathcal{P}_p(\mathbb{R}) \), we have

\[
W_p(I(\mu, \nu), I(\mu', \nu')) \leq 2W_p(\mu, \mu') + W_p(\nu, \nu'),
\]
\[
W_p(J(\mu, \nu), J(\mu', \nu')) \leq W_p(\mu, \mu') + 2W_p(\nu, \nu').
\]

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Theorem 1.1 generalises [2, Proposition 3.1 and Proposition 4.3] when \( d = 1 \). These propositions state that for any dimension \( d \in \mathbb{N} \) and probabilities \( \mu, \nu, \mu', \nu' \in \mathcal{P}_p(\mathbb{R}^d) \) with \( \mu \leq \nu \), so that \( I_p(\mu, \nu) = \mu \) and \( I_p(\mu, \nu) = \nu \), we have
\[
\begin{align*}
&W_p(I_p(\mu, \nu), I_p(\mu', \nu')) \leq 2W_p(\mu, \nu') + W_p(\nu', \nu), \\
&W_p(I_p(\mu, \nu), J_p(\mu', \nu')) \leq W_p(\mu, \mu') + 2W_p(\nu, \nu').
\end{align*}
\]
Hence, by Theorem 1.1 it is possible (for \( d = 1 \)) to drop the convex ordering constraint \( \mu \leq \nu \). The extension of Theorem 1.1 to dimensions \( d > 1 \) is to the authors’ understanding an interesting open question.

1.1. Discussion on Wasserstein projections and related problems. Gozlan, Roberto, Samson, and Tetal [14] introduced a generalization of optimal transport, the weak optimal transport, in order to study measure concentration inequalities. The following barycentric weak optimal transport problem received in recent years special attention, see for example [14, 13, 12, 1, 2, 4, 5]: for \( \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d) \), consider
\[
\mathcal{W}_p^p(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d} |x - \int_{\mathbb{R}^d} y \pi_x(dy)|^p \mu(dx),
\]
where we write \((\pi_x)_{x \in \mathbb{R}^d}\) for a disintegration kernel of \( \pi \) w.r.t. its \( \mu \)-marginal: \( \pi(dx, dy) = \mu(dx)\pi_x(dy) \). This barycentric weak optimal transport problem has an intrinsic connection with the problem of finding Wasserstein projection. Indeed, we have that the values of \( \mathcal{W}_p(\mu, \nu) \) and \( \mathcal{W}_p(\mu, I_p(\mu, \nu)) \) coincide, see [2, 4]. Moreover, if \( \pi^* \) is an optimizer of (1.7) then the image of the first marginal \( \mu \) under the map \( x \mapsto \int_{\mathbb{R}^d} \pi^*_x(y) dy \) is a minimizer of (1.1) and coincides with \( I_p(\mu, \nu) \) when \( p > 1 \). Therefore, when \( \mu, \nu \in \mathcal{P}_p(\mathbb{R}^d) \) are finitely supported, (1.7) can be used to compute the Wasserstein projection. In particular, \( I_2(\mu, \nu) \) can be computed by solving a quadratic optimization problem with linear constraints. We refer to [14, 3] for dual formulations of weak optimal transport problems with additional martingale constraints, to [4] for the existence of optimal couplings and necessary and sufficient optimality conditions, to [7] for continuity of their value function in terms of the marginal distributions \( \mu \) and \( \nu \), and to [6] for applications of such problems. We point out the connection of Wasserstein projections to Cafarelli’s contraction theorem that was discovered in [11]. Note that dual formulations of the minimization problems defining \( I_p(\mu, \nu) \) and \( J_p(\mu, \nu) \) have recently been studied by Kim and Ruan [17].

1.2. Wasserstein projection in dimension one. Wasserstein projections in the special case \( d = 1 \), were further studied in [2] by means of quantile functions. The quantile function of a probability measure \( \mu \) on \( \mathbb{R} \) is the left-continuous pseudoinverse of its cumulative distribution function:
\[
\forall u \in (0, 1), \quad F^{-1}_\mu(u) := \inf\{ x \in \mathbb{R} : \mu((-\infty, x]) \geq u \}.
\]
It is well-known that the comonotonous coupling is an optimizer in (1.3):
\[
\forall \mu, \nu \in \mathcal{P}_p(\mathbb{R}), \quad \mathcal{W}_p(\mu, \nu) = \left( \int_0^1 |F^{-1}_\mu(u) - F^{-1}_\nu(u)|^p du \right)^{1/p} =: ||F^{-1}_\mu - F^{-1}_\nu||_p. \tag{1.8}
\]
Moreover, by [19, Theorem 3.A.5] the convex order can be characterised in terms of quantile functions: for \( \mu, \nu \in \mathcal{P}_1(\mathbb{R}) \) that share the same barycentre we have
\[
\mu \leq \nu \iff \forall u \in [0, 1], \quad \int_0^u F^{-1}_\mu(v) dv \geq \int_0^u F^{-1}_\nu(v) dv. \tag{1.9}
\]
As, according to [13, Theorem 1.5] and [2, Proposition 4.2], in one dimension, there exist optimizers of (1.1) and (1.2) not depending on the power \( p \geq 1 \), we drop...
the subscript. Moreover, by [2, Theorem 2.6 and Proposition 4.2], the respective quantile functions of such optimizers \( I(\mu, \nu) \) and \( J(\mu, \nu) \) are obtained for all \( u \in (0, 1) \) by

\[
F_{I(\mu, \nu)}^{-1}(u) = F_{\mu}^{-1}(u) - \partial_- \text{co}(G)(u) \quad \text{and} \quad F_{J(\mu, \nu)}^{-1}(u) = F_{\nu}^{-1}(u) + \partial_- \text{co}(G)(u),
\]

(1.10)

where \( G : v \mapsto \int_0^v (F_{\mu}^{-1} - F_{\nu}^{-1})(u) \, du \), \( \text{co} \) denotes the convex hull, and \( \partial_- \) the left-hand derivative. The optimizers \( I(\mu, \nu) \) and \( J(\mu, \nu) \) are not necessarily unique, see [2, Remark 2.3] for \( I_1 \), hence, the maps \( I \) and \( J \) provide continuous selections of optimizers for \( p = 1 \). A complete geometric characterization of \( I(\mu, \nu) \) and \( J(\mu, \nu) \) is given in [5]. Note that (1.10) implies

\[
W_p(I(\mu, \nu), \mu) = W_p(J(\mu, \nu), \nu) \quad \text{and} \quad W_p(I(\mu, \nu), \nu) = W_p(J(\mu, \nu), \mu),
\]

(1.11)

where, according to [2, Corollary 4.4], the first equality still holds for \( d \geq 2 \) when \( I \) and \( J \) are replaced by \( I_\mu \) and \( J_\mu \) respectively. The next examples show that the constants in (1.5) and (1.6) are sharp for \( p = 1 \).

**Example 1.2.** Let \( \mu \in \mathcal{P}_1(\mathbb{R}) \) and \( \nu \) be a Dirac measure. As \( \nu \) is the only measure dominated by itself in the convex order, \( I(\mu, \nu) = \nu \) and, as a consequence of (1.11), \( J(\mu, \nu) = \mu \). When \( \nu \) is also a dirac mass, we deduce for any \( \mu' \in \mathcal{P}_1(\mathbb{R}) \) that

\[
W_p(I(\mu, \nu), I(\mu, \nu')) = W_p(J(\mu, \nu), \nu) \quad \text{and} \quad W_p(J(\mu, \nu), J(\mu, \nu')) = W_p(\nu, \mu').
\]

Hence the factor \( 1 \) multiplying \( W_p(\nu, \nu') \) in the right-hand side of (1.5) and multiplying \( W_p(\mu, \mu') \) in the right-hand side of (1.6) is optimal. \( \diamond \)

**Example 1.3.** We fix \( \mu := \delta_0 \) and define, for \( \alpha \in (0, 1) \),

\[
\nu^\alpha := (1 - \alpha)\delta_{-a} + \alpha\delta_1.
\]

We have \( I(\nu^\alpha, \nu^\alpha) = \nu^\alpha \) and \( I(\mu, \nu^\alpha) = \delta_{\alpha(1-\alpha)(-a)} \), so that

\[
\begin{align*}
W_1(I(\nu^\alpha, \nu^\alpha), I(\mu, \nu^\alpha)) &= 2 \left( \alpha + \alpha^2(\alpha(1-\alpha) - 1) \right), \\
W_1(I(\mu, \nu^\alpha)) &= \alpha + \alpha^2(1-\alpha).
\end{align*}
\]

Then, an application of the de l’Hôpital rule yields \( \lim_{\alpha \to 0} \frac{W_1(I(\nu^\alpha, \nu^\alpha), I(\mu, \nu^\alpha))}{W_1(I(\mu, \nu^\alpha))} = 2 \). Hence, the factor 2 in (1.5) is optimal when \( p = 1 \). Since \( I(\delta_{\alpha(1-\alpha)(-a)}, \nu^\alpha) = \nu^\alpha \) and \( J(\delta_{\alpha(1-\alpha)(-a)}, \delta_0) = \delta_{\alpha(1-\alpha)(-a)} \), we find in the same way that the factor 2 is also optimal in (1.6) when \( p = 1 \). \( \diamond \)

**Example 1.4.** Let \( \mu, \mu', \nu, \nu' \) be the probability measures with quantile functions:

\[
F_\mu^{-1}(u) = u \mathbb{1}_{(0,1)}(u) + \frac{1 + u}{2} \mathbb{1}_{(\frac{1}{2},1)}(u), \quad F_\nu^{-1}(u) = \frac{u}{2},
\]

\[
F_{\mu'}^{-1}(u) = u \mathbb{1}_{(0,\frac{1}{2})}(u) + \frac{12 + 5u}{18} \mathbb{1}_{(\frac{1}{2},1)}(u), \quad F_{\nu'}^{-1}(u) = \frac{u}{2} \mathbb{1}_{(0,\frac{1}{2})}(u) + \frac{u}{2} \mathbb{1}_{(\frac{1}{2},1)}(u).
\]

We check that \( F_{I(\mu, \nu)}^{-1}(u) = \frac{u}{2} \) and \( F_{I(\mu', \nu')}^{-1}(u) = \frac{u}{2} \mathbb{1}_{(0,\frac{1}{2})}(u) + \frac{3 + 5u}{18} \mathbb{1}_{(\frac{1}{2},1)}(u) \), whence,

\[
\begin{align*}
W_p(I(\mu, \nu), I(\mu', \nu')) &= W_p(\mu, \mu') + W_p(\nu, \nu')
\end{align*}
\]

with two positive summands. \( \diamond \)
1.3. **On the convex-order lattice in dimension one.** The restaturation of the convex order which was the original motivation for the introduction of the Wasserstein projections can also be achieved in dimension one using that $\mathcal{P}_1(\mathbb{R})$ is a complete lattice for the increasing and decreasing convex orders (see [16]). They both coincide with the convex order on the subset $\mathcal{P}_p^0(\mathbb{R})$ of $\mathcal{P}_p(\mathbb{R})$ consisting in probability measures with barycentre $x_0 \in \mathbb{R}$. On $\mathcal{P}_1(\mathbb{R})$, the minimum $\land_c$ and maximum $\lor_c$ can be expressed in terms of potential functions: the potential function of $\mu \in \mathcal{P}_1(\mathbb{R})$ is defined by

$$u_\mu(x) := \int_{\mathbb{R}} |x - y| \mu(dy).$$

For $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$, $\mu \land_c \nu$ and $\mu \lor_c \nu$ are uniquely determined by

$$u_{\mu \land_c \nu} = \min(u_\mu, u_\nu) \quad \text{and} \quad u_{\mu \lor_c \nu} = \max(u_\mu, u_\nu).$$

On the domain $\mathcal{P}_1(\mathbb{R}) \times \mathcal{P}_1(\mathbb{R})$,

$$(\mu, \nu) \mapsto \mu \land_c \nu \quad \text{and} \quad (\mu, \nu) \mapsto \mu \lor_c \nu,$$

are continuous mappings into $\mathcal{P}_1(\mathbb{R})$: [8, Lemma 4.1] provides continuity for $p = 1$ and [2, Lemma 4.3], which ensures uniform integrability, permits to deduce continuity for general $p \geq 1$. However, unlike $\Pi$ and $\mathcal{J}$, the minimum and maximum in the convex order are not Lipschitz continuous.

**Example 1.5.** Consider for $n \geq 3$ the measures in $\mathcal{P}_p(\mathbb{R})$:

$$\nu := \frac{1}{2n} \delta_0 + \frac{1}{n} \sum_{i=1}^{n-1} \delta_{\frac{i}{n}} + \frac{1}{2n} \delta_1,$$

$$\mu := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\frac{i}{n}},$$

$$\eta := \frac{3}{2n} \delta_0 + \frac{1}{n} \sum_{i=2}^{n-1} \delta_{\frac{i}{n}} + \frac{3}{2n} \delta_{\frac{n-1}{n}}.$$
Proof of Proposition 1.6. The map \( u \mapsto F_{I(u,v)}^{-1}(u) - F_{\mu}^{-1}(u) \) is non-increasing due to (1.10). Therefore, monotonicity of the integrand yields, for \( u \in (0,1) \) that
\[
\frac{1}{u} \int_0^u (F_{I(u,v)}^{-1}(v) - F_{\mu}^{-1}(v)) dv \geq \int_0^1 (F_{I(u,v)}^{-1}(v) - F_{\mu}^{-1}(v)) dv = \int_{\mathbb{R}} y(v) dv - \int_{\mathbb{R}} x(\mu) dx ,
\]
where the last equality comes from the inverse transform sampling and the fact that, as \( I(u,v) \leq c \), \( I(u,v) \) and \( v \) share the same barycentre. If \( \mu \) and \( v \) have the same barycenter, or put equivalently, \( \int_{\mathbb{R}} y(v) dv = \int_{\mathbb{R}} x(\mu) dx \), then \( \tilde{I}(\mu,v) \) shares this barycenter and we deduce by (1.9) that \( \tilde{I}(\mu,v) \leq \mu \), thus, \( \tilde{I}(\mu,v) \leq c \) \( \mu \wedge v \). Analogously, we have \( \mu \leq c \tilde{I}(\mu,v) \), and if \( \mu \) and \( v \) share the same barycentre, \( v \leq c \tilde{I}(\mu,v) \). Hence, \( \mu \forall c \leq \tilde{I}(\mu,v) \). □

2. Proof of Theorem 1.1

The proof of Theorem 1.1 relies on the next two results whose proofs are postponed.

Lemma 2.1. For \( p \geq 1 \), \( I \) and \( I \) are continuous maps on \( \mathcal{P}_p(\mathbb{R}) \times \mathcal{P}_p(\mathbb{R}) \) to \( \mathcal{P}_p(\mathbb{R}) \).

Proposition 2.2. Let \( f \) and \( g \) be real-valued càdlàg functions on \([0,1]\) with respective antiderivatives \( F \) and \( G \). We have, for \( p \geq 1 \),
\[
\|\partial_+ (\text{co}(F) - \text{co}(G))\|_p \leq \|f - g\|_p .
\]

Proof of Theorem 1.1. Let \( \mu, \mu', \nu, \nu' \in \mathcal{P}_p(\mathbb{R}) \). Assume for a moment that (1.5) and (1.6) hold for probability measures with bounded support. Since such measures are dense in \( \mathcal{P}_p(\mathbb{R}) \), there exist \( \mu_n, \mu_n', \nu_n, \nu_n' \in \mathcal{P}_p(\mathbb{R}) \), \( n \in \mathbb{N} \) with bounded support such that
\[
\lim_{n \to +\infty} W_p(\mu, \mu_n) + W_p(\mu', \mu_n') + W_p(\nu, \nu_n) + W_p(\nu', \nu_n') = 0 .
\]

We have by Lemma 2.1 that \( (\tilde{I}(\mu_n,v_n))_{n \in \mathbb{N}} \) and \( (\tilde{I}(\mu_n',v_n'))_{n \in \mathbb{N}} \) converge to \( \tilde{I}(\mu,v) \) and \( \tilde{I}(\mu',v') \) resp. in \( \mathcal{P}_p(\mathbb{R}) \). Therefore,
\[
W_p(\tilde{I}(\mu,v), \tilde{I}(\mu',v')) = \lim_{n \to +\infty} W_p(\tilde{I}(\mu_n,v_n), \tilde{I}(\mu_n',v_n'))
\leq \lim_{n \to +\infty} 2W_p(\mu_n, \mu_n') + W_p(\nu_n, \nu_n')
= 2W_p(\mu, \mu') + W_p(\nu, \nu').
\]
Hence, we may assume that \( \mu, \nu, \mu', \nu' \) have bounded supports. This implies that the associated quantile functions are bounded on \((0,1)\) and since they are non-decreasing and have at most countably many jumps, coincide \( \lambda \)-a.s. with càdlàg functions on \([0,1]\), where \( \lambda \) denotes the Lebesgue measure on \([0,1]\). Therefore,
\[
G : v \mapsto \int_0^v (F^{-1}_\mu - F^{-1}_\nu)(u) \, du \quad \text{and} \quad G' : v \mapsto \int_0^v (F^{-1}_{\mu'} - F^{-1}_{\nu'})(u) \, du,
\]
are the antiderivatives of càdlàg functions on \([0,1]\). By Proposition 2.2 and (1.8),
\[
\|\partial_+ (\text{co}(G) - \text{co}(G'))\|_p \leq \|\partial_+ (G - G')\|_p = \|F^{-1}_\mu - F^{-1}_\nu - F^{-1}_{\mu'} + F^{-1}_{\nu'}\|_p
\leq \|F^{-1}_\mu - F^{-1}_{\mu'}\|_p + \|F^{-1}_\nu - F^{-1}_{\nu'}\|_p = W_p(\mu, \mu') + W_p(\nu, \nu').
\]
By (1.10), we have for \( \lambda \)-almost every \( u \in (0,1) \) that
\[
F_{\tilde{I}(\mu,v)}^{-1}(u) = F_{\mu}^{-1}(u) - \partial_+ \text{co}(G)(u) \quad \text{and} \quad F_{\tilde{I}(\mu',v')}(u) = F_{\mu'}^{-1}(u) - \partial_+ \text{co}(G')(u),
F_{\tilde{I}(\mu,v)}^{-1}(u) = F_{\nu}^{-1}(u) + \partial_+ \text{co}(G)(u) \quad \text{and} \quad F_{\tilde{I}(\mu',v')}(u) = F_{\nu'}^{-1}(u) + \partial_+ \text{co}(G')(u).
\]
Therefore, using (1.8), we obtain
\[
\mathcal{W}_p(I(\mu, \nu), I(\mu', \nu')) = \left\| F_{I(\mu, \nu)}^{-1} - F_{I(\mu', \nu')}^{-1} \right\|_p \leq \left\| F_{\mu}^{-1} - F_{\mu'}^{-1} \right\|_p + \left\| \partial_+ (\co(G) - \co(G')) \right\|_p \\
\leq \left\| F_{\mu}^{-1} - F_{\mu'}^{-1} \right\|_p + \left\| \partial_+ (\co(G) - \co(G')) \right\|_p \leq 2\mathcal{W}_p(\mu, \mu') + \mathcal{W}_p(\nu, \nu').
\]

In the same way,
\[
\mathcal{W}_p(I(\mu, \nu), I(\mu', \nu')) = \left\| F_{\nu}^{-1} - F_{\nu'}^{-1} \right\|_p + \left\| \partial_+ (\co(G) - \co(G')) \right\|_p \leq \mathcal{W}_p(\mu, \mu') + 2\mathcal{W}_p(\nu, \nu').
\]

**Proof of Lemma 2.1.** In the following we will only show continuity of \(I\) and remark that continuity of \(I\) follows mutatis mutandis (and can be even shown with a simpler line of argument, since \(I(\mu, \nu) \leq \nu\)).

Let \((\mu_n)_{n \in \mathbb{N}}, (\nu_n)_{n \in \mathbb{N}}\) be sequences that converge to \(\mu, \nu\) resp. in \(\mathcal{P}_p(\mathbb{R})\).

**Step 1.** We show that \((I(\mu_n, \nu_n))_{n \in \mathbb{N}}\) is a precompact subset of \(\mathcal{P}_p(\mathbb{R})\). As a consequence of the de la Vallée-Poussin theorem, see for example [10, Theorem 4.5.9 and proof], there exists a continuous, increasing and strictly convex function \(\theta: \mathbb{R}_+ \rightarrow \mathbb{R}_+\) such that \(\lim_{r \rightarrow \infty} \frac{1}{\theta(r)} = 0\) and
\[
\sup_{n \in \mathbb{N}} \int \theta(|x|^p) \mu_n(dx) < \infty, \quad \sup_{n \in \mathbb{N}} \int \theta(|y|^p) \nu_n(dy) < \infty.
\]

Note that, when \(p > 1\), \(\theta \circ (|\cdot|^p/2p)\) is also strictly convex as the composition of a convex function, increasing function with a strictly convex function. On the other hand, when \(p = 1\), \(\theta = \cos\) or the inequality \(|x + (1 - \alpha)| \leq \alpha|x| + (1 - \alpha)|y|\) is strict, so that, since \(\theta\) is increasing and strictly convex, \(\theta \circ (|\cdot|/2)\) is again strictly convex. We conclude that in any case \(\theta \circ (|\cdot|^p/2p)\) is strictly convex.

Consider the transport problem \(\mathcal{W}_\theta\) given by
\[
\mathcal{W}_\theta(\eta, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int \theta\left(\frac{|x|^p}{2p}\right) \pi(dx, dy),
\]
for \(\eta, \nu \in \mathcal{P}_p(\mathbb{R})\), and observe that
\[
\mathcal{W}_\theta(\eta, \nu) \leq \int \theta\left(\frac{|x|^p}{2p}\right) \eta \otimes \nu(dx, dy) \leq \frac{1}{2} \left( \int \theta(|x|^p) \eta(dx) + \int \theta(|y|^p) \nu(dy) \right).
\]

We have by [5, Theorem 1.4] that
\[
\mathcal{W}_\theta(\mu, \nu) := \inf_{\eta \in \Pi(\mu, \nu)} \mathcal{W}_\theta(\eta, \nu) = \mathcal{W}_\theta(I(\mu, \nu), \nu) \leq \mathcal{W}_\theta(\mu, \nu),
\]
from which we deduce that \((\mathcal{W}_\theta(\mu_n, \nu_n))_{n \in \mathbb{N}}\) is a bounded sequence. For \(n \in \mathbb{N}\), let \(\pi_n\) be an optimizer of \(\mathcal{W}_\theta(I(\mu_n, \nu_n), \nu_n)\) in \(\Pi(I(\mu_n, \nu_n), \nu_n)\). We then find by monotonicity and convexity of \(\theta\)
\[
\int \theta\left(\frac{|x|^p}{2p}\right) I(\mu_n, \nu_n)(dx) \leq \frac{1}{2} \left( \int \theta\left(\frac{|x|^p}{2p}\right) \pi_n(dx, dy) \right) + \int \theta\left(\frac{|y|^p}{2p}\right) \nu_n(dy) \leq \frac{1}{2} \left( \mathcal{W}_\theta(\mu_n, \nu_n) + \int \theta\left(\frac{|y|^p}{2p}\right) \nu_n(dy) \right).
\]

Therefore, the left-hand side is uniformly bounded in \(n \in \mathbb{N}\). Recall that \(\frac{r}{\theta(r)}\) vanishes for \(r \rightarrow \infty\) and so does \(\sup_{s \geq r, t \geq r} \frac{r}{\theta(r)}\). Since
\[
\int \mathbb{1}_{(r, \infty)}\left(\frac{|x|^p}{2p}\right) I(\mu_n, \nu_n)(dx) \leq \sup_{s \geq r, t \geq r} \frac{r}{\theta(r)} \int I(\mu_n, \nu_n)(dx),
\]
with the integrals on the right-hand side uniformly bounded in \(n \in \mathbb{N}\), we deduce that
\[
\lim_{r \rightarrow \infty} \sup_{n \in \mathbb{N}} \int \mathbb{1}_{(r, \infty)}(|x|^p) I(\mu_n, \nu_n)(dx) = 0.
\]
In particular, by Markov's inequality, we get that the sequence is tight and by [21, Definition 6.8] precompact in \( P_p(\mathbb{R}) \).

Step 2. Precompactness allows us to pass to a subsequence convergent in \( P_p(\mathbb{R}) \) with limit \( \gamma \). Consider the continuous, increasing, and strictly convex function \( \hat{\theta}(x) := \sqrt{x^2 + 1} \) on \( \mathbb{R}_+ \) with \( \hat{\theta}(x) \leq x^p + 1 \). By stability, that is [5, Theorem 1.5], we obtain

\[
\mathcal{V}_\theta(\mu, \nu) = \lim_{n \to \infty} \mathcal{V}_\theta(\mu_n, \nu_n) = \lim_{n \to \infty} \mathcal{W}_\theta(\mathcal{F}(\mu_n, \nu_n), \nu_n) = \mathcal{W}_\theta(\gamma, \nu).
\]

Since \( \mathcal{W}_p \)-convergence preserves convex ordering, we get that \( \mu \leq_c \gamma \) and by uniqueness of the optimizer of \( \mathcal{V}_\theta(\mu, \nu) \), \( \gamma = \mathcal{F}(\mu, \nu) \). Hence, \( (\mathcal{F}(\mu_n, \nu_n))_{n \in \mathbb{N}} \) converges to \( \mathcal{F}(\mu, \nu) \) in \( P_p(\mathbb{R}) \).

The proof of Proposition 2.2 relies on the next three lemmas. The proof of the first one is postponed after the one of the proposition.

**Lemma 2.3.** Let \( 0 \leq a < b \) and \( F, G : [0, b] \to \mathbb{R} \) be continuous on \([0, b]\), convex on \([0, a]\) and affine on \([a, b]\). Then for any increasing convex function \( \theta : \mathbb{R} \to \mathbb{R} \) we have

\[
\int_0^b \theta(\partial_+(\co(F) - \co(G))) (u) \, du \leq \int_0^b \theta(\partial_+(F - G)) (u) \, du
\]

where \( \co \) denotes the convex hull and \( \partial_+ \) the right-hand derivative.

**Lemma 2.4.** Let \( 0 \leq a < b < \infty \), \( F : [0, b] \to \mathbb{R} \), and

\[
H(x) := \begin{cases} 
\co(F|_{[0,a]})(x) & \text{if } x \in [0, a) \\
F(x) & \text{if } x \in [a, b].
\end{cases}
\]

Then \( \co(H) = \co(F) \).

**Proof of Lemma 2.4.** The function \( \co(H) \) is convex and satisfies \( \co(H) \leq H \leq F \).

By definition of the convex hull, we deduce that \( \co(H) \leq \co(F) \). Conversely, \( \co(F) \) is convex and bounded from above by \( F \), so that the restriction \( \co(F)|_{[0,a]} \) is convex and bounded from above by \( F|_{[0,a]} \). Hence \( \co(F)|_{[0,a]} \leq \co(F|_{[0,a]}) \) and \( \co(F) \leq H \).

By definition of the convex hull, \( \co(F) \leq \co(H) \), which concludes the proof.

**Lemma 2.5.** Let \( f : [0, 1] \to \mathbb{R} \) be a càdlàg function. Then there exists for every \( \varepsilon > 0 \) a piecewise constant, càdlàg function \( g : [0, 1] \to \mathbb{R} \) with at most finitely many jumps such that \( \sup_{x \in [0, 1]} |f(x) - g(x)| < \varepsilon \).

**Proof of Lemma 2.5.** This is follows from [9, Section 12, Lemma 1] and the discussion below.

**Proof of Proposition 2.2.** For the moment we assume that the assertion of the proposition holds true for antiderivatives of piecewise constant, càdlàg functions. Since \( f \) and \( g \) are càdlàg, there exist by Lemma 2.5 for every \( n \in \mathbb{N}^* \) piecewise constant, càdlàg functions \( f_n, g_n : [0, 1] \to \mathbb{R} \) with finitely many discontinuities such that

\[
\sup_{x \in [0, 1]} (|f(x) - f_n(x)| + |g(x) - g_n(x)|) < 1/n.
\]

Therefore, \((f_n)_{n \in \mathbb{N}^*}\) and \((g_n)_{n \in \mathbb{N}^*}\) converge in \( L^p(\lambda) \) to \( f \) and \( g \), respectively. Let, for \( n \in \mathbb{N}^*, u \in [0, 1] \),

\[
F_n(u) := F(0) + \int_0^u f_n(v) \, dv \quad \text{and} \quad G_n(u) := G(0) + \int_0^u g_n(v) \, dv.
\]
We have \( \|F - F_n\|_\infty < \frac{1}{n}, \|G - G_n\|_\infty < \frac{1}{n} \). Since \( \text{co}(F) - \|F - F_n\|_\infty \) (resp. \( \text{co}(F_n) - \|F - F_n\|_\infty \)) is a convex function bounded from above by \( F - \|F - F_n\|_\infty \leq F_n \) (resp. \( F_n \)), \( \|\text{co}(F) - \text{co}(F_n)\|_\infty \leq \|F - F_n\|_\infty < \frac{1}{n} \) and, in the same way, \( \|\text{co}(G) - \text{co}(G_n)\|_\infty < \frac{1}{n} \).

By [15, Theorem 6.2.7], we have \( \mathcal{A}\)-almost sure convergence of \((\partial_+ \text{co}(F_n))_{n \in \mathbb{N}^*}\) and \((\partial_+ \text{co}(G_n))_{n \in \mathbb{N}^*}\) to \( \partial_+ \text{co}(F) \) and \( \partial_+ \text{co}(G) \), respectively. Again, as \( f \) and \( g \) are càdlàg, we have \( \max(\|f\|_\infty, \|g\|_\infty) + 1 =: K < \infty \), and \( \max(\|f_n\|_\infty, \|g_n\|_\infty) \leq K \) for all \( n \in \mathbb{N}^* \), which yields by definition of the convex hull that \( \text{co}(F_n)(u) \geq F_n(0) - K u \) and \( \text{co}(F_n)(u) \geq F_n(1) - K(1 - u) \). Hence,

\[
\frac{\text{co}(F_n)(u) - \text{co}(F_n)(0)}{u} \geq -K, \quad \frac{\text{co}(F_n)(1) - \text{co}(F_n)(u)}{1 - u} \leq K,
\]

and by monotonicity of the one-sided derivatives (and the same reasoning for \( G_n \)) we obtain that \( \max(\|\partial_+ \text{co}(F_n)\|_\infty, \|\partial_+ \text{co}(G_n)\|_\infty) \leq K \). Then dominated convergence yields that \((\partial_+ \text{co}(F_n))_{n \in \mathbb{N}^*}\) and \((\partial_+ \text{co}(G_n))_{n \in \mathbb{N}^*}\) converge in \( L^1(\mathcal{A}) \) to \( \partial_+ \text{co}(F) \) and \( \partial_+ \text{co}(G) \), respectively. Finally, by applying (2.1) and the triangle inequality we get the desired inequality:

\[
\|\partial_+ (\text{co}(F) - \text{co}(G))\|_p = \lim_{n \to \infty} \|\partial_+ (\text{co}(F_n) - \text{co}(G_n))\|_p \\
\leq \lim_{n \to \infty} \|f_n - g_n\|_p = \|f - g\|_p.
\]

It remains to show (2.1) for piecewise constant, càdlàg functions \( f \) and \( g \) with finitely many jumps. To this end, let \((a_k)_{0 \leq k \leq n}\) be a partition of \([0, 1]\) adapted to \( f \) and \( g \), i.e., \( 0 = a_0 < \ldots < a_n = 1 \) and for all \( k \in \{0, \ldots, n - 1\} \), \( f|_{[a_k, a_{k+1}]} \) and \( g|_{[a_k, a_{k+1}]} \) are constant. For \( k \in \{0, \ldots, n\} \), we consider

\[
F^k : x \mapsto \begin{cases} 
\text{co}(F|_{[0, a_k]})(x) & \text{if } x \in [0, a_k), \\
F(x) & \text{else}; 
\end{cases}
\]

\[
G^k : x \mapsto \begin{cases} 
\text{co}(G|_{[0, a_k]})(x) & \text{if } x \in [0, a_k), \\
G(x) & \text{else}, 
\end{cases}
\]

and write \( f^k = \partial_+ F^k \) and \( g^k = \partial_+ G^k \).

Note that \( F^0 = F^1 = F, \ G^0 = G^1 = G \) and \( F^n = \text{co}(F), \ G^n = \text{co}(G) \). By induction we shall show that, for \( k \in \{0, \ldots, n - 1\} \),

\[
\|f^{k+1} - g^{k+1}\|_p \leq \|f^k - g^k\|_p. \tag{2.5}
\]

As the initial case is trivial, we assume that (2.5) holds for \( 0 \leq k \leq n - 2 \). First, observe

\[
\|f^{k+1} - g^{k+1}\|_p = \|f^{k+1} - g^{k+1}|_{[0, a_{k+1}]}\|_p + \|f^{k+1} - g^{k+1}|_{[a_{k+1}, 1]}\|_p \\
\geq \|\partial_+ (\text{co}(F|_{[0, a_{k+1}]})) - \text{co}(G|_{[0, a_{k+1}]}))\|_p + \|f^{k+1} - g^{k+1}|_{[a_{k+1}, 1]}\|_p.
\]

Applying Lemma 2.4 with \( a = a_k, b = a_{k+1} \) yields \( \text{co}(F|_{[0, a_{k+1}]})) = \text{co}(F|_{[0, a_{k+1}]})) \) and \( \text{co}(G|_{[0, a_{k+1}]})) = \text{co}(G|_{[0, a_{k+1}]})) \), so that,

\[
\|f^{k+1} - g^{k+1}\|_p = \|\partial_+ (\text{co}(F|_{[0, a_{k+1}]})) - \text{co}(G|_{[0, a_{k+1}]}))\|_p + \|f^{k+1} - g^{k+1}|_{[a_{k+1}, 1]}\|_p.
\]

Since \( f \) and \( g \) are absolutely bounded by some constant \( C > 0 \), we have \( \text{co}(F|_{[0, a_k]})(u) \geq F(a_k) - C(a_k - u), \text{co}(G|_{[0, a_{k+1}]})(u) \geq G(a_{k+1}) - C(a_{k+1} - u), \) thus, \( \lim_{u \searrow a_k} \text{co}(F|_{[0, a_k]})(u) = \)
Let \( a = a_k \) and \( \lim_{x \rightarrow a_k} \) \( \partial_s \) \( G(\lambda_{(a_{k-1})})(u) = G(a_k) \). In particular, \( F^k \) and \( G^k \) are continuous. We can apply Lemma 2.3 with \( a = a_k \), \( b = a_{k+1} \) to obtain
\[
\| \partial_s (\co(F^k)(\theta_{(a_{k+1})})) - \co(G^k(\theta_{(a_{k+1})})) \|_p \leq \| (F^k - G^k)(\theta_{(a_{k+1})}) \|_p,
\]
from which we deduce (2.5). In particular, we have shown the assertion:
\[
\| \partial_s (\co(F) - \co(G)) \|_p = \| f^n - g^n \|_p = \| f - g \|_p. \quad \square
\]

The proof of Lemma 2.3 relies on the next two lemmas.

**Lemma 2.6.** Let \( \theta : \mathbb{R} \rightarrow \mathbb{R} \) be convex, \( x, z \in \mathbb{R} \) with \( x < z \), and \( y, \tilde{y} \in [x, z] \). Then
\[
\theta(y) - \theta(x) \leq \sum_{z < \tilde{y}} \left( \theta(z) - \theta(y) \right). \quad (2.6)
\]

**Proof of Lemma 2.6.** Since \( \theta \) is convex, we have
\[
\theta(y) - \theta(x) \leq \frac{1}{z - y} \left( \theta(z) - \theta(x) \right) \quad \text{and} \quad \frac{1}{z - x} \left( \theta(z) - \theta(x) \right) \leq \theta(z) - \theta(y).
\]
Combining these two inequalities yields (2.6). \( \square \)

**Lemma 2.7.** Let \( \mu \in \mathcal{P}_1(\mathbb{R}) \) and \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a measurable map such that \( \mu \) is equal to the image \( \int_a \lambda \) of the Lebesgue measure \( \lambda \) on \( (0, 1) \) by \( f \). Then
\[
\forall x \in [0, 1], \quad \int_x^1 f(u) du \leq \int_x^1 F^{-1}(u) du.
\]

**Proof of Lemma 2.7.** Let \( v \in [0, 1] \), \( c : (x, y) \mapsto 1_{[x \leq c \leq y]} \) and \( U \) be a uniform random variable on \( (0, 1) \). Observe that for all \( x, x', y, y' \in \mathbb{R} \) with \( x \leq x' \) and \( y \leq y' \) we have
\[
c(x', y') - c(x, y') - c(x', y) + c(x, y) \leq 0.
\]
Then by [18, Theorem 3.1.2] we have
\[
\int_0^1 f(u) du = \mathbb{E}[c(1 - U, f(U))] \leq \mathbb{E}[c(1 - U, F^{-1}(U))] = \int_0^1 F^{-1}(u) du. \quad \square
\]

**Proof of Lemma 2.3.** Let \( f, g : [0, b) \rightarrow \mathbb{R} \) be given by the right-hand derivative of \( F \) and \( G \) resp., that is \( f := \partial_s F, \ g := \partial_s G \). Our first goal is to find an explicit representation of the convex hulls of \( F \) and \( G \). To this end, consider the infimum
\[
c := \inf \left\{ x \in [0, b) \mid f(x) \geq \frac{F(b)-F(a)}{b-a} \right\}, \quad (2.7)
\]
which is well-defined and not greater than \( a \) as \( f(a) = \frac{F(b)-F(a)}{b-a} \). Moreover, the infimum in (2.7) is attained by continuity of \( F \) and right-continuity of \( f \). When \( c > 0 \), we get by (2.7) and monotonicity of \( f \) that \( \sup_{x \in (0, c]} f(x) = F(\phi) = \lim_{x \rightarrow \phi} f(x) \leq \lim_{x \rightarrow \phi} \frac{F(b)-F(a)}{b-a} = \frac{F(b)-F(a)}{b-a} \). Under the convention \( \sup_{x \in [0, c]} f(x) = -\infty \) when \( c = 0 \), we therefore find
\[
\sup_{x \in [0, c]} f(x) \leq \frac{F(b)-F(a)}{b-a} =: \phi \leq f(c). \quad (2.8)
\]
For \( x \in [c, a] \), using (2.8) and the fact that \( f \) is non-decreasing on \( [0, a] \), we obtain
\[
F(b) - F(x) = F(b) - F(c) + F(c) - F(x) = (b - c)\phi - \int_c^x f(u) du \leq (b - c)\phi - (c - x)\phi = (b - x)\phi \leq (b - x)f(x). \quad (2.9)
\]
We claim that
\[
\co(F)(x) = F(\min(\phi, x)) + \max(\phi, x - \phi), \quad x \in [0, b]. \quad (2.10)
\]
Denote the right-hand side of (2.10) by \( F \). Note that \( F \) is convex since the right-hand derivative \( \partial_s F(x) = 1_{[0,c]}f(x) + 1_{[c,b)}\phi \) is non-decreasing by (2.8). We calculate
\[
F(x) = F(c) + F(x) - F(b) + F(b) - F(c) \geq F(c) - (b - x)\phi + (b - c)\phi = F(x),
\]
for \( x \in [c, a] \), which yields that \( \tilde{F} \leq F \) on \([0, a] \cup \{b\} \). Since both functions are affine on \([a, b]\) we conclude that \( \tilde{F} \leq F \) and therefore, by definition of the convex hull, \( \tilde{F} \leq \text{co}(F) \).

In order to show (2.10), it remains to verify that \( \text{co}(F) \leq \tilde{F} \). By convexity of \( \text{co}(F) \) and the inequality \( \text{co}(F) \leq F \), we have, for \( x \in [c, b] \),

\[
\text{co}(F)(x) = \frac{\tilde{F}(c)}{b-c} F(b) + \frac{\tilde{F}(a)}{b-c} F(a) = F(c) + (x-c)\phi = \tilde{F}(x),
\]

and \( \text{co}(F)(x) \leq F(x) = \tilde{F}(x) \) for \( x \in [0, c] \), thus, \( \text{co}(F) \leq \tilde{F} \).

Reasoning the same way for \( G \), we deduce that \( d \) defined analogously to (2.7)

\[
d := \inf \left\{ x \in [0, b] : g(x) \geq \frac{G(b) - G(x)}{b-x} \right\}
\]

is not greater than \( a \) and has the properties

\[
\begin{align*}
\sup_{x \in [0,d]} g(x) & \leq \frac{G(b) - G(d)}{b-d} =: \gamma \leq g(d), \\
\text{co}(G)(x) & = G(\min(x, d)) + \max(x - d, 0) \gamma.
\end{align*}
\]

After this preparatory work we proceed to show the assertion, that is (2.4). Without loss of generality, we assume that \( c \leq d \). Note that, by (2.10) and (2.8), \( \partial_+ \text{co}(F)(x) = \min(f(\min(x, c)), \phi) \) and, by (2.12) and (2.11), \( \partial_+ \text{co}(G)(x) = \min(g(\min(x, d)), \gamma) \). Therefore, the left-hand side of (2.4) coincides with

\[
\int_c^d \theta(\|f(u) - g(u)\|) \, du + \int_c^d \theta(\|\phi - g(u)\|) \, du + \int_d^b \theta(\|\phi - \gamma\|) \, du.
\]

We have to show

\[
\int_c^d \theta(\|\phi - g(u)\|) + \int_c^d \theta(\|\phi - \gamma\|) \, du \leq \int_c^d \theta(\|f(u) - g(u)\|) \, du.
\]

**Case I:** Suppose that \( \phi \geq \gamma \). By (2.8), the monotonicity of \( f \) on \([0, a] \) and (2.11),

\[
g(u) \leq \gamma \leq \phi \leq f(u) \quad u \in [c, d].
\]

Then, by applying Lemma 2.6 (where in the notation of this lemma \( x = \phi - \gamma \), \( y = f(u) - \gamma \), \( z = f(u) - g(u) \)), we find

\[
\theta(f(u) - g(u)) - \theta(\phi - g(u)) \geq \theta(f(u) - \gamma) - \theta(\phi - \gamma), \quad u \in [c, d]
\]

so that

\[
\int_c^d \theta(\|f(u) - g(u)\|) - \theta(\|\phi - g(u)\|) \, du \geq \int_c^d \theta(f(u) - \gamma) - \theta(\phi - \gamma) \, du.
\]

Denoting by \( \bar{\phi} \) the fraction \( \frac{F(b) - F(d)}{b-d} \), we have, since \( \theta \) is increasing and convex,

\[
\int_c^d \theta(\|f(u) - g(u)\|) \, du \geq \theta \left( \int_c^d \frac{f(u) - g(u)}{b-d} \, du \right) = \theta(\|\bar{\phi} - \gamma\|).
\]

Summing the two last inequalities, then using the convexity of \( \theta \), we deduce that

\[
\begin{align*}
\int_c^b \theta(\|f(u) - g(u)\|) \, du - \int_c^d \theta(\|\phi - g(u)\|) \, du - (b-d)\theta(\phi - \gamma) & \\
\geq & \int_c^d \theta(f(u) - \gamma) - \theta(\phi - \gamma) \, du + \int_c^b \theta(\bar{\phi} - \gamma) - \theta(\phi - \gamma) \, du \\
\geq & (b-c) \left( \theta \left( \frac{1}{b-c} \left( \int_c^d f(u) - \gamma \, du + \int_c^b \bar{\phi} - \gamma \, du \right) \right) - \theta(\phi - \gamma) \right),
\end{align*}
\]

since \( \theta \) is non-decreasing \( \frac{1}{b-c} \left( \int_c^d f(u) - \gamma \, du + \int_c^b \bar{\phi} - \gamma \, du \right) = \phi - \gamma \), we find that (2.14) is non-negative, from which we derive (2.13).
Case 2: Suppose that $\phi < \gamma$ and let $e := \inf\{u \in [c, d] \mid g(u) \geq \phi\}$ where by convention the infimum over the empty set is defined as $d$. By (2.8) and the monotonicity of $f$ on $[0, a]$, we have $g(u) \leq \phi \leq f(u)$ for $u \in [c, e)$, thus,

$$\int_{c}^{e} \theta(\phi - g(u))\,du \leq \int_{c}^{e} \theta(f(u) - g(u))\,du. \tag{2.15}$$

On the other hand, by (2.10),

$$\forall x \in [c, b], \ F(x) \geq \co(F)(x) = F(b) + (x - b)\phi \tag{2.16}$$

so that $\tilde{\phi} = \frac{F(b) - F(d)}{b - d} \leq \phi$. As $\theta$ is increasing and convex, we obtain

$$\int_{b}^{e} \theta(\|f(u) - g(u)\|)\,du \geq \theta(\gamma - \tilde{\phi}). \tag{2.17}$$

As consequence of (2.15) and (2.17), the following inequality suffices to get (2.13):

$$\int_{c}^{d} \theta(g_1(u) - \phi)\,du + (b - d)\theta(\gamma - \phi) \leq \int_{c}^{d} \theta(\|f(u) - g(u)\|)\,du + (b - d)\theta(\gamma - \tilde{\phi}). \tag{2.18}$$

Showing (2.18) is equivalent to proving that the respective images $\mu$ and $\nu$ of the Lebesgue measure $\lambda$ on $(0, 1)$ by the non-decreasing maps

$$T^1(u) := \begin{cases} g(e + (b - e)u) - \phi & u < \frac{\lambda - e}{b - e}, \\ \gamma - \phi & \text{else}, \end{cases}$$

$$T^2(u) := \begin{cases} |f(e + (b - e)u) - g(e + (b - e)u)| & u < \frac{\lambda - e}{b - e}, \\ \gamma - \tilde{\phi} & \text{else}, \end{cases}$$

are in the increasing convex order ($\leq_{icx}$). By [19, Theorem 4.A.3], this is equivalent to

$$\int_{0}^{1} F^{-1}_\mu(u)\,du \leq \int_{0}^{1} F^{-1}_\nu(u)\,du, \quad v \in [0, 1]. \tag{2.19}$$

Since $T^1$ is non-decreasing, we have by [2, Lemma A.3] that $T^1(u) = F^{-1}_\nu(u)$ for $\lambda$-almost every $u \in (0, 1)$. This observation combined with Lemma 2.7 leads to

$$\int_{0}^{1} F^{-1}_\mu(u)\,du = \int_{0}^{1} T^1(u)\,du$$

and (2.19) is implied by

$$\int_{0}^{1} T^1(u)\,du \leq \int_{0}^{1} T^2(u)\,du, \quad v \in [0, 1]. \tag{2.20}$$

Recall that $\tilde{\phi} \leq \phi$, so that this inequality holds for $v \in \left[\frac{b - e}{b - d}, 1\right]$. Next, abbreviate $\frac{d - e}{b - d} =: w$, let $v \in [0, w)$, and write $\hat{e} := e + (b - e)v$. We have by the triangle inequality that

$$\int_{0}^{1} T^1(u)\,du \leq \int_{0}^{w} f(e + (b - e)u) + T^2(u) - \phi\,du + \int_{0}^{1} \gamma - \phi\,du
= \frac{F(d) - F(\hat{e}) - (d - \hat{e})\phi + (b - d)(\gamma - \phi)}{b - e} + \int_{0}^{w} T^2(u)\,du.$$

Remember that $(b - d)\tilde{\phi} = F(b) - F(d)$ and (2.16) applies to $x = \hat{e}$ since $\hat{e} \geq e \geq c$. Thus,

$$F(d) - F(\hat{e}) = F(b) - F(\hat{e}) - (b - d)\tilde{\phi} \leq (b - \hat{e})\phi - (b - d)\tilde{\phi}.$$

We obtain

$$\int_{0}^{1} T^1(u)\,du \leq \frac{(b - d)(\phi - \gamma - \tilde{\phi})}{b - e} + \int_{0}^{w} T^2(u)\,du
\leq \frac{b - d}{b - e}(\gamma - \phi) + \int_{0}^{w} T^2(u)\,du = \int_{0}^{1} T^2(u)\,du. \quad \Box$$
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