Fractional Boundary Value Problems and elastic sticky Brownian motions, II: The bounded domain

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Abstract

Sticky diffusion processes on bounded domains spend finite time (and finite mean time) on the lower-dimensional space given by the boundary. Once the process hits the boundary, then it starts again after a random amount of time. While on the boundary it can stay or move according to dynamics that are different from those in the interior. Such processes may be characterized by a time-derivative appearing in the boundary condition for the governing problem. We use time changes obtained by right-inverses of suitable processes in order to describe fractional sticky conditions and the associated boundary behaviours. We obtain that fractional boundary value problems (involving fractional dynamic boundary conditions) lead to sticky diffusions spending an infinite mean time (and finite time) on a lower-dimensional boundary. Such a behaviour can be associated with a trap effect in the macroscopic point of view.

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1 Introduction

We show that a fractional dynamic boundary condition for the heat equation can be associated with a suitable time-change for a Brownian motion in a bounded domain. The associated fractional boundary value problem leads to the characterization of the time and the mean time the process spends on the boundary. The behaviour on the boundary can be static or diffusive (sometimes termed penetrating) introducing respectively holding or excursion times. Here, we consider in details only holding times.

In the literature, the author found many papers dealing with fractional boundary value problems but the meaning is completely different. Indeed, they deal with fractional Cauchy problems (FCP) or non local PDEs with local boundary conditions. As far as we know, this problem is new and there are no works dealing with it. The novelty is also given by the probabilistic representation of the solution for which we have a clear reading in terms of Brownian motion and its special behaviour near the boundary. We refer to the FCP as a fractional initial value problem (FIVP). The problem we are interested in will be presented as a fractional boundary value problem (FBVP).

1.1 Presentation of the results

For an open bounded set \( \Omega \subset \mathbb{R}^d \) with smooth boundary \( \partial \Omega \), we can equip the compact Euclidean space \( \overline{\Omega} = \Omega \cup \partial \Omega \) with a finite Borel measure, say \( m \), given by the sum of the \( d \)-dimensional Lebesgue measure \( dx \) on \( \Omega \) and the \((d-1)\)-dimensional Hausdorff measure on \( \partial \Omega \). Our aim is to study the heat equation with non-local dynamic condition

\[
\eta D_t^\alpha u = -\sigma \partial_n u - cu \quad \text{on} \quad \partial \Omega
\]

where \( D_t^\alpha \) is a non-local operator in time. As we will discuss below, for \( \alpha = 1 \) the previous condition takes the form

\[
\eta \frac{\partial u}{\partial t} = -\sigma \partial_n u - cu \quad \text{on} \quad \partial \Omega
\]

which corresponds to the Wentzell-Robin boundary condition

\[
\eta \Delta u = -\sigma \partial_n u - cu \quad \text{on} \quad \partial \Omega.
\]

Thus, in case \( \alpha = 1 \) we obtain a well-known problem, the Laplacian generates a positive contraction semigroup ([2]). If \( \partial \Omega \) is \( C^\infty \) then we may consult [46] whereas, if \( \partial \Omega \) is Lipschitz, then we may refer to [45]. From the probability view point, the Wentzell boundary condition leads to a sticky Brownian motion and the Robin boundary condition leads to an elastic Brownian motion.

We deal here with the case \( \alpha \in (0,1) \) and provide a description of the associated stochastic process. For this process we focus on the stochastic dynamic near the boundary, then we mainly restrict our analysis to a bounded domain \( \Omega \in \mathbb{R}^d \) meaning an open, connected and non-empty set with \( C^\infty \) boundary \( \partial \Omega \).

Let \( \eta, \sigma, c \) be positive constants. We introduce briefly the processes we deal with further on. As usual, we denote by \( E_x \) the mean operator under the measure \( P_x \) where \( x \) is a starting point:

i) \( X^+ = \{ X^+_t \}_{t \geq 0} \) is a reflecting Brownian motion on \( \overline{\Omega} \) with boundary local time \( \gamma^+ = \{ \gamma^+_t \}_{t \geq 0} \). The process \( X^+ \) has generator \( (G^+, D(G^+)) \) where \( G^+ = \Delta \) is the Neumann Laplacian with

\[
D(G^+) = \{ \varphi, \Delta \varphi \in C(\overline{\Omega}), \varphi \in H^1(\Omega) : \partial_n \varphi = 0 \};
\]
ii) $X^\dagger = \{X^\dagger_t\}_{t \geq 0}$ is a Brownian motion on $\Omega$ killed upon reaching the boundary $\partial \Omega$ for which

$$\mathbb{E}_x[f(X^\dagger_t)] = \mathbb{E}_x[f(X^\dagger_t), t < \tau_{\partial \Omega}], \quad t > 0, \ x \in \Omega$$

where $\tau_{\partial \Omega} = \inf\{t \geq 0 : X^\dagger_t \in \partial \Omega\}$. The process $X^\dagger$ has generator $(G^\dagger, D(G^\dagger))$ where $G^\dagger = \Delta$ is the Dirichlet Laplacian and

$$D(G^\dagger) = \{\varphi, \Delta \varphi \in C(\overline{\Omega}) : \varphi|_{\partial \Omega} = 0\};$$

iii) $X = \{X_t\}_{t \geq 0}$ is an elastic sticky Brownian motion on $\overline{\Omega}$ with boundary local time $\gamma = \{\gamma_t\}_{t \geq 0}$. The process is termed elastic sticky according with the Wentzell-Robin boundary condition. We consider the measure

$$m(dx) = 1_{\Omega} \ dx + (\eta/\sigma) 1_{\partial \Omega} m_{\sigma}(dx)$$

(1)
defined as the sum of the $d$-dimensional Lebesgue measure supported on the interior and the $(d - 1)$-dimensional Hausdorff measure supported on the boundary. In particular, $m_{\sigma}$ is the $(d - 1)$-dimensional Hausdorff measure on $\partial \Omega$. The sticky process $X$ with elastic kill has generator $(G, D(G))$ with $G = \Delta$ and

$$D(G) = \{\varphi, \Delta \varphi \in C(\overline{\Omega}) : \varphi|_{\partial \Omega} = -\sigma \partial_n \varphi - c \varphi|_{\partial \Omega}, \ \eta, \sigma, c > 0\}$$

where $\partial_n u$ is the outer normal derivative with respect to $m$ introduced in (1). We recall that $u|_{\partial \Omega}$ is the trace function (continuous operator) from $H^1(\Omega)$ into $L^2(\partial \Omega, m_{\sigma})$, such that

$$u \mapsto T u = u|_{\partial \Omega}$$

for $u \in H^1(\Omega) \cap C(\overline{\Omega})$. The semigroup generated by $(G, D(G))$ is a compact, positive $C_0$-semigroup on $C(\overline{\Omega})$. As $\sigma \to \infty$ (or equivalently $\eta = c = 0$), we have the process $X^+$ with generator $(G^+, D(G^+))$ whereas, as $c \to \infty$ (or equivalently $\eta = \sigma = 0$) we have the process $X^\dagger$ with generator $(G^\dagger, D(G^\dagger))$. The case $\eta \to \infty$ leads to the (pure) sticky condition;

iv) $H = \{H_t\}_{t \geq 0}$ is a stable subordinator with $\mathbb{E}_0[\exp(-\lambda H_t)] = \exp(-t \lambda^\alpha)$, $\alpha \in (0, 1)$;

v) $L = \{L_t\}_{t \geq 0}$ is the inverse $L_t = \inf\{s \geq 0 : H_s > t\}$ to $H$.

Our main result is concerned with the processes:

vi) $X^L = \{X^L_t\}_{t \geq 0}$ is the time-changed process $X^L_t := X \circ L_t$. The process $X^L$ can be associated with a FIVP;

vii) $\tilde{X} = \{\tilde{X}_t\}_{t \geq 0}$ with boundary local time $\tilde{\gamma} = \{\tilde{\gamma}_t\}_{t \geq 0}$ where $\tilde{X}_t = X \circ \tilde{L}_t$ and $\tilde{L} = \{\tilde{L}_t\}_{t \geq 0}$ is a special time change defined below in Section 2.6 (see also Section 2.7);

viii) $\bar{X} = \{\bar{X}_t\}_{t \geq 0}$ with boundary local time $\bar{\gamma} = \{\bar{\gamma}_t\}_{t \geq 0}$ where $\bar{X}_t = X_{\text{el}} \circ \bar{V}_t^{-1}$ and $\bar{V} = \{\bar{V}_t\}_{t \geq 0}$ is defined below in formula (2) together with its inverse $\bar{V}^{-1}$ (see also Section 2.8). The elastic Brownian motion $X_{\text{el}}$ will be written in terms of $(X^+, \gamma^+)$. The time change obtained via $\bar{V}^{-1}$ introduces a fractional dynamic condition. The process $\bar{X}$ can be associated with a FBVP.

With no abuse of notation we refer to $\tau_{\partial \Omega}$ for the first hitting time of the boundary $\partial \Omega$ of the processes previously introduced if no confusion arises.
Recall that $\eta, \sigma, c$ are positive constants. Let us consider the dynamic boundary value problems

\[
\begin{align*}
\frac{\partial v}{\partial t}(t, x) &= Av(t, x), \quad t > 0, \ x \in \Omega, \\
v(0, x) &= f(x), \quad x \in \Omega, \\
v(t, x) &= Tv(t, x), \quad t > 0, \ x \in \partial \Omega, \\
\eta \frac{\partial v}{\partial t}(t, x) &= Bv(t, x), \quad t > 0, \ x \in \partial \Omega, \\
v(0, x) &= Tv(0, x) = f(x), \quad x \in \partial \Omega,
\end{align*}
\]

(2)

\[
\begin{align*}
D_\alpha^\eta w(t, x) &= Aw(t, x), \quad t > 0, \ x \in \Omega, \\
w(0, x) &= f(x), \quad x \in \Omega, \\
w(t, x) &= Tw(t, x), \quad t > 0, \ x \in \partial \Omega, \\
\eta D_\alpha^\eta w(t, x) &= Bw(t, x), \quad t > 0, \ x \in \partial \Omega, \\
w(0, x) &= f(x), \quad x \in \partial \Omega,
\end{align*}
\]

(3)

and

\[
\begin{align*}
\frac{\partial u}{\partial t}(t, x) &= Au(t, x), \quad t > 0, \ x \in \Omega, \\
u(0, x) &= f(x), \quad x \in \Omega, \\
u(t, x) &= Tu(t, x), \quad t > 0, \ x \in \partial \Omega, \\
\eta D_\alpha^\eta u(t, x) &= Bu(t, x), \quad t > 0, \ x \in \partial \Omega, \\
u(0, x) &= f(x), \quad x \in \partial \Omega,
\end{align*}
\]

(4)

for $f \in C(\bar{\Omega})$ and $D(B) \subset D(A)$, where

\[
D_\alpha^\eta u(t, x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u}{\partial s}(s, x)(t-s)^{-\alpha} ds, \quad \alpha \in (0, 1)
\]

is the Caputo-Dzherbashian fractional derivative. We recall that, for $z > 0$, $\Gamma(z) = \int_0^\infty s^{z-1} e^{-s} ds$ is the absolutely convergent Euler integral. As usual, we refer to $\Gamma(\cdot)$ as the gamma function. Focus on the generator $(G, D(G))$. Then, for the problem (2) with $Av = \Delta v$ and $Bv = -\sigma \partial_n v - cv$, we have that (consult for example [28])

\[
\eta \frac{\partial v}{\partial t} = \eta T v = \eta T \Delta v = -\sigma \partial_n v - cv \quad \text{on} \ \partial \Omega
\]

(5)

The solution to the problem (2) can be therefore written as

\[
v(t, x) = E_x[f(X_t)], \quad t \geq 0, \ x \in \bar{\Omega}.
\]
With $A$ and $B$ as defined above, under $\Delta w \in C(\overline{\Omega})$, the problem (3) can be written as

$$\begin{cases}
D^\alpha_t w(t, x) = Gw(t, x), & t > 0, \ x \in \overline{\Omega}, \\
w(0, x) = f(x), & x \in \overline{\Omega}, \ f \in D(G),
\end{cases}$$

(6)

and this actually brings our FBVP in the context of FIVPs. Since $(G, D(G))$ generates the elastic sticky Brownian motion $X$, we have the probabilistic representation

$$w(t, x) = E_x[f(X \circ L_t)], \ t \geq 0, \ x \in \overline{\Omega}$$

for the solution $w$ to (3). We observe that the problem associated with $X \circ L_t$ can be formulated in terms of the operator matrix on $C(\overline{\Omega})$ given by

$$A = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}, \ D(A) = \left\{ \begin{pmatrix} w \\ Tw \end{pmatrix} \in D(A) \times C(\partial \Omega) \right\}$$

(7)

for which we have

$$D^\alpha_t \begin{pmatrix} w \\ Tw \end{pmatrix} = A \begin{pmatrix} w \\ Tw \end{pmatrix}.$$

This formulation does not hold for the problem (4).

By following the same arguments as in (5), we write (4) as

$$\begin{cases}
\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x), & t > 0, \ x \in \Omega, \\
\eta D^\alpha_t Tu(t, x) = -\sigma \partial_n u(t, x) - cu(t, x), & t > 0, \ x \in \partial \Omega, \\
u(0, x) = f(x), & x \in \overline{\Omega}.
\end{cases}$$

(8)

We relate the problem (8) with $\bar{X}$. In particular, we show that the solution to (8) can be written as

$$u(t, x) = E_x[f(X_t)] = E_x[f(X^+ \circ \bar{V}_t^{-1}) M \circ \bar{V}_t^{-1}]$$

where $\bar{V}_t^{-1} := \inf\{s : \bar{V}_s > t\}$ is the inverse to the process

$$\bar{V}_t = t + H \circ (\eta/\sigma)_{\gamma_t}^\gamma$$

(9)

and $M_t = e^{-(c/\sigma)_{\gamma_t}^\gamma}$ is the multiplicative functional associated to the elastic condition. This representation introduces a time change written in terms of a subordinator $H$ independent from $X^+$ and provides a clear meaning for the non-local condition appearing in (8). Since the process $H$ may have jumps, then the inverse to $\bar{V}_t$ slows down the process according with the plateaus associated with that jumps. Since $H$ is independent from $X^+$, then the delay on the boundary for $\bar{X}$ can be regarded as a trapping effect to be charged to exogenous causes, for example to some property of the boundary. Thus, we say that $\bar{X}$ on a regular domain can be considered to describe an elastic Brownian motion $X^e$ on domains with irregular boundary. We clarify this point in the last section of the work (open problem).
1.2 State of the art in a nutshell

Non-local initial value problems (NLIVP). Problems involving boundary conditions may differ radically depending on the fractional operators we are dealing with. For fractional (or non-local) operators in space we have an exterior value problem except in case we use the spectral representation of the fractional operator introduced by the Phillips’ representation. Concerning the fractional Laplacian \((-\Delta)^\alpha\), such a representation coincides in \(\mathbb{R}^d\) with the usual representation given in terms of the Cauchy principal value whereas, in a bounded domain the Phillips’ formula gives a spectral representation which is, actually, a local representation and therefore, this introduces a local boundary value problem. It is well-known that the probabilistic representation of the solution is obtained through time change. Indeed, the solution to the Cauchy problem \(\partial u/\partial t = -(\Delta)^\alpha u,\ u_0 = f\) is obtained via subordination, that is we consider the base process (the Brownian motion) time-changed by a stable subordinator. This leads to subordinate semigroups. An interesting discussion on the different role played by subordination and killing has been given in [53].

Concerning fractional operators in time we consider the Caputo-Dzherbashian fractional derivative \(D_t^\alpha\) introduced in introduced in [17, 18, 19] by the first author and separately in a series of works starting from [26, 27] by the second author. Due to the simple name, very often only the first author is considered. The fractional Cauchy problem \(D_t^\alpha u = \Delta u,\ u_0 = f\) has been investigated by many researchers from both analytical and probabilistic point of view (see for example [16, 85, 59, 48, 49] and the recent works [23, 44, 11, 17]). Here, we have local boundary conditions. It is well-known that the probabilistic solution is obtained through time change as well as some problems involving fractional operators in space. The time change is now given by the inverse to an \(\alpha\)-stable subordinator. This introduces an interesting connection between operators in time and space which has inspired many researchers in the last years.

We refer to fractional operators only in case we have a clear identification of the fractional order \(\alpha \in (0, 1)\). The general class of non-local operators obviously includes the fractional ones. However, in case of non-local operators, we may not refer to any fractional order, we may only refer to the symbol \(\Phi\), for example we have \(D_t^\Phi\) and \(\Phi(-\Delta)\). Examples in case of time-operators can be found in [20, 40, 41, 55].

Non-local boundary value problems (NLBVP). In the present paper we are interested in local problems with non-local boundary conditions written in terms of fractional (time) operators. We consider sticky Brownian motions which are intimately related with dynamical boundary value problems. The sticky (or Feller-Wentzell) condition has been considered by Feller and Wentzell in the pioneering works [30, 58] and the probabilistic representation in terms of sample paths has been obtained by Itô and McKean [44, Section 10]. Further investigation has been carried out in the subsequent works, for example [1, 57] or [21, 33, 54] and many other recent papers. Feller and Bessel sticky diffusions have been considered in [52, 51]. In [36], the authors constructed diffusion processes on bounded domain with sticky reflection. On the boundary they considered both cases of static and diffusive behaviour. In [29] the sticky reflected distorted Brownian motion has been considered. For a discussion on the martingale problem and the time-change theory associated with generalized SDEs, see [35]. Many others can be listed. For the physical interpretation of the dynamic boundary condition we refer to [32]. From the probabilistic viewpoint, the (pure) sticky condition (given by \(\Delta u = 0\) on \(\partial\Omega\)) represents instantaneous absorption whereas, the (reflecting) sticky condition \((\eta\Delta u + \sigma\partial_n u = 0\) on \(\partial\Omega\)) represents slow reflection. Thus, a reflected sticky (or slowly reflected) Brownian motion spends some time on the boundary according to an inverse process written in terms of its boundary local time. This means that the process is forced to stay on the boundary with each visit (see Section 2.3 below).

Our result is concerned with the elastic sticky Brownian motion \((\eta\Delta u + \sigma\partial_n u + cu = 0\) on \(\partial\Omega\)) and the fractional dynamic boundary condition written in terms of the so called Caputo-
Dzherbashian derivative $D_t^\alpha$. We show that the associated process turns out to be delayed on the boundary so that the slow reflections slow down even more. This can be regarded as a simple case of trapping boundary. The boundary behaviour associated with general non-local operators (for example $D^\rho_t$ and $\Phi(-\Delta)$) have been considered in [22] for Brownian motions on the real line.

1.3 Organization of the work

In Section 2 we introduce some notations and preliminaries. Then, we state and discuss our main results in Section 3.1. We conclude with a discussion on the results together with their possible readings and extensions in Section 4.

2 Notations and preliminaries

2.1 Positive Continuous Additive Functionals

We recall some properties of this special class of functionals. Consider the set $A_+^+$ of PCAFs and denote by $\mu_A$ the Revuz measure of $A \in A_+^+$. We denote by $A_+^+(\Lambda)$ the set of PCAFs associated to Revuz measures compactly supported in $\Lambda \subseteq \Omega$. That is,

$$A_+^+(\Lambda) = \{ A \in A_+^+ : \text{supp}[\mu_A] = \Lambda \}, \quad \Lambda \subseteq \Omega.$$  

Let $Z = \{ Z_t \}_{t \geq 0}$ be a process on $\Omega$ with $P_x(Z_s \in dy) = p_Z(s, x, y)\mu(dy)$ and $\mu$ with support on $\Omega$. Denote by $E^\mu_x$ the integral $\int E_x \mu(dx)$ where, as usual, $E_x$ is the mean value under $P_x$ and $x = Z_0$ is a point in $\Omega$.

Let $B(\Lambda)$ be the set of Borel functions on $\Lambda \subseteq \Omega$. If $A_t \in A_+^+(\Lambda)$ is a PCAF of the process $Z$ on $\Omega$, we have that, for $f \in B(\Lambda),$

$$\lim_{t \downarrow 0} E^\mu_x \left[ \frac{1}{t} \int_0^t f(Z_s) dA_s \right] = \int_\Lambda f(x)\mu_A(dx)$$  \hfill (10)

and

$$E^\mu_x \left[ \int_0^t f(Z_s) dA_s \right] = \int_0^t \int_\Lambda f(y)p_Z(s, x, y)\mu_A(dy)ds, \quad x \in \Omega.$$  \hfill (11)

Further on we also use the fact that

$$\lim_{t \downarrow 0} E^\mu_x \left[ \frac{1}{t} \int_0^t f(Z_s) dA_s \right] = \lim_{\lambda \rightarrow -\infty} \lambda E^\mu_x \left[ \int_0^\infty e^{-\lambda t} f(Z_t) dA_t \right]$$  \hfill (12)

and

$$E_x \left[ \int_0^\infty e^{-\lambda t} f(Z_t) dA_t \right] = \lambda \int_0^\infty e^{-\lambda t} E_x \left[ \int_0^t f(Z_s) dA_s \right] dt.$$  \hfill (13)

In particular, we deal with Lebesgue-Stieltjes integrals as $\int_0^t dA_s$ and $\int_0^t dA_s^{-1}$ where $A_t$ will be the boundary local time or the interior occupation time of a process on $\Omega$ and $A_t^{-1}$ will be the inverse process

$$A_t^{-1} = \inf\{ s : A_s > t \}, \quad 0 \leq t < A_\zeta \quad \text{with} \quad A^{-1} = 0, \quad t \geq A_\zeta$$

where $\zeta = \inf\{ s : Z_s \in \partial \}$ is the lifetime of $Z$ and $\partial$ is a ”cemetery point”.

We recall the following lemma.
Lemma 1. ([9, Lemma 2.2, Chapter V]) Let $a(t)$ be a function from $[0, \infty)$ to $[0, \infty)$ which is non decreasing and right continuous and satisfies $a(0) = 0, a(\infty) = \lim_{t \to \infty} a(t)$. Define

$$a^{-1}(t) = \inf\{s : a(s) > t\}$$

for $0 \leq t < \infty$ where as usual we set $a^{-1}(t) = \infty$ if the set in braces is empty. The function $a^{-1}$ from $[0, \infty)$ to $[0, \infty)$ is called the inverse of $a$. It is right continuous and non decreasing. Define $a^{-1}(\infty) = \lim_{t \to \infty} a^{-1}(t)$. Then

$$a(t) = \inf\{s : a^{-1}(s) > t\}, \quad 0 \leq t < \infty$$

and if $f$ is non negative Borel measurable function on $[0, \infty]$ vanishing at $\infty$ one has

$$\int_{(0, \infty)} f(t) da(t) = \int_0^\infty f(a^{-1}(t)) dt.$$

We also observe that in case $a$ is continuous, then

$$a^{-1}(t) = \max\{s : a(s) = t\}, \quad 0 \leq t < a(\infty)$$

and $a^{-1}(t) = \infty$ for $t > a(\infty)$. The inverse $a^{-1}$ is strictly increasing on $[0, a(\infty))$ and $a(a^{-1}(t)) = t$ for $0 \leq t < a(\infty)$.

### 2.2 The process $X^+$

We recall that $X^+$ is the reflecting Brownian motion on $\Omega$ for which we respectively denote by

$$\gamma_t^+ = \int_0^t 1_{\partial\Omega}(X^+_s) ds, \quad t \geq 0 \quad \text{and} \quad \Gamma_t^+ = \int_0^t 1_{\Omega}(X^+_s) ds, \quad t \geq 0$$

the local time on the boundary $\partial\Omega$ and the occupation time of (the interior) $\Omega$. It is well known that, by using Ito’s formula, the local time can be obtained (in the $L^2$ sense $\forall t$ and a.s. uniformly on bounded interval of time) as

$$\gamma_t^+ = \lim_{\epsilon \downarrow 0} \int_0^t \frac{1_{\Lambda}(X^+_s)}{\epsilon} ds$$

where $\Lambda_\epsilon = \{x \in \overline{\Omega} : d(x, \partial\Omega) \leq \epsilon\}$.

Let us consider $\tilde{m}(dy) = 1_{\Omega}dy + 1_{\partial\Omega}m_\partial(dy)$. Let $f \in B(\overline{\Omega})$. We recall that $\gamma_t^+$ and $\Gamma_t^+$ are PCAFs for which

$$\lim_{t \downarrow 0} E_{\tilde{m}} \left[ \frac{1}{t} \int_0^t f(X^+_s) d\gamma_s^+ \right] = \int_{\partial\Omega} f(x) m_\partial(dx)$$

(14)

and

$$\lim_{t \downarrow 0} E_{\tilde{m}} \left[ \frac{1}{t} \int_0^t f(X^+_s) d\Gamma_s^+ \right] = \int_{\Omega} f(x) dx.$$

(15)

Now we quote some results on the path integral representation of the solution to the boundary value elliptic problem

$$\begin{cases}
\Delta g = c_1 g, & \text{in } \Omega, \\
\partial_n g + c_2 g = f, & \text{on } \partial \Omega,
\end{cases}$$

with $f \in B(\partial\Omega)$.

**Lemma 2.** ([56]) Let $c_1, c_2$ be two positive constants.
i) The gauge of\( (16) \)

\[ g(x) = E_x \left[ \int_0^\infty e^{-c_1 t - c_2 \gamma_1^+} d\gamma_1^+ \right] \]

is continuous on \( \Omega \) if there exists \( x_0 \in \Omega \) such that \( g(x_0) < \infty \).

ii) The unique (weak) solution to \( (16) \) has the probabilistic representation

\[ g(x) = E_x \left[ \int_0^\infty e^{-c_1 t - c_2 \gamma_1^+} f(X_t^+) d\gamma_1^+ \right], \quad f \in \mathcal{B}(\partial \Omega) \]

if, for some \( x_0 \in \Omega \), \( g(x_0) < \infty \).

iii) Given the Feynman-Kac semigroup on \( L^2(\Omega) \)

\[ T_t f(x) = E_x \left[ e^{-c_1 t - c_2 \gamma_1^+} f(X_t^+) \right] = \int_\Omega f(y) k(t, x, y) \tilde{m}(dy) \]

we have that

\[ g(x) = \int_{\partial \Omega} \left( \int_0^\infty k(t, x, y) dt \right) f(y) \tilde{m}(dy). \]

Moreover, \( k \) is a symmetric continuous kernel whose eigenfunctions are in \( C(\Omega) \) and \( T_t \) is a compact operator in \( C(\Omega) \). \( T_t \) is a Feller semigroup.

We notice that Lemma (2) deals with continuous (and bounded) functions on \( \Omega \) where \( \Omega \) is a bounded domain (open, connected and non-empty set) with \( C^3 \) boundary \( \partial \Omega \).

### 2.3 The process \( X \)

We have already introduced the process \( X \) with generator \( (G, D(G)) \), now we complete the presentation of \( X \) by recalling some further details. Our discussion is mainly based on the well-known book [9] and the pioneering work [44]. Let us consider the natural filtration \( \mathcal{F}_t = \sigma\{X_s, 0 \leq s < t\} \) and a good function \( f \) for which \( E[f(X_s)|\mathcal{F}_t] = E[f(X_s)|X_t], \ t \leq s, \) and \( E_x[f(X_{t+s})|\mathcal{F}_t] = E_{X_t}[f(X_s)], \ s, t > 0. \) We say that \( X \) is an elastic process meaning that

\[ E_x[f(X_t)] = E_x[f(X_t), t < \zeta] \]

is written in terms of the multiplicative functional \( M_t = 1_{t<\zeta} \) where the lifetime \( \zeta \) gives the elastic kill. In particular, there exists an independent exponential random variable (with parameter \( c/\eta \)) for which

\[ E[M_t|X_t] = e^{-(c/\eta)\gamma_1}. \] (17)

On the other hand, from the sticky condition, \( \{t : X_t \in \partial \Omega\} \) is a set of positive Lebesgue measure obtained from the holding times of \( X \) on \( \partial \Omega \). In particular, we may consider a sequence \( \{e_i\} \) of (identically distributed) independent exponential random variables (with parameter \( \sigma/\eta \)) for which

\[ P(e_0 > t, X_{e_0} \in dy) = e^{-(\sigma/\eta)t} P(X_{e_0} \in dy) \] and

\[ P(e_0 > t|X_{e_0}) = e^{-(\sigma/\eta)t}. \] (18)

For the Markov process \( X \) we introduce the semigroup

\[ S_t f(x) = E_x[f(X_t)]. \]
The semigroup $S_t$ generated by $(G, D(G))$ is a compact, positive $C_0$-semigroup on $C(\overline{\Omega})$ as shown in [2]. We remark that ([3]) the $C_0$-semigroup on $L^2(\Omega)$ has a continuous kernel $p$ such that

$$E_x[f(X_t)] = \int_{\Omega} f(y)p(t, x, y) m(dy)$$

with

$$f \in L^2(\Omega, m) = L^2(\Omega, dx) \oplus L^2(\partial \Omega, (\eta/\sigma)m_\partial(dx)).$$

In particular, the $C_0$-semigroup on $C(\Omega)$ can be given as

$$E_x[f(X_t)] = \int_{\Omega} f(y)p(t, x, y)dy + (\eta/\sigma)\int_{\partial \Omega} f(y)p(t, x, y)m_\partial(dy), \quad f \in C(\Omega)$$

from which we also write

$$S_t f = S_t f|_\Omega + S_t f|_{\partial \Omega}$$

where by $f|_\Omega$ we mean the restriction of $f$ on $\Omega$. In general, we use the notation $S_t f|_K$ meaning that $S_0 f|_K = f|_K$ and $f|_K(x) = f(x), \ x \in K \subseteq \overline{\Omega}$. We denote by

$$R_\lambda f(x) := E_x \left[ \int_0^\infty e^{-\lambda t} f(X_t) dt \right] \quad \lambda > 0, \quad x \in \overline{\Omega}$$

the associated $\lambda$-potential and write

$$P_x(X_t \in dy) = p(t, x, y)m(dy)$$

where $X_0 = x \in \overline{\Omega}$ and $m(dy)$ has been given in [1].

For the local time and the occupation time

$$\gamma_t = \int_0^t 1_{\partial \Omega}(X_s) ds, \quad t \geq 0 \quad \text{and} \quad \Gamma_t = \int_0^t 1_{\Omega}(X_s) ds, \quad t \geq 0$$

we observe that the inverses $\gamma_t^{-1} = \inf\{s \geq 0 : \gamma_s > t\}$ and $\Gamma_t^{-1} = \inf\{s \geq 0 : \Gamma_s = t\}$ are such that $\gamma \circ \gamma_t^{-1} = t$ and $\Gamma \circ \Gamma_t^{-1} = t$ almost surely. Let us introduce the jointly continuous local time $\gamma_t^{+}(z)$ of $X^+$ for $z \in \overline{\Omega}$ and $t > 0$. The occupation time formula says that

$$\int_{\Omega} 1_{\Omega}(z)\gamma_t^{+}(z)dz = \int_0^t 1_{\Omega}(X_s^{+})ds = t$$

and

$$(\eta/\sigma)\int_{\partial \Omega} \gamma_t^{+}(z)m_\partial(dz) = (\eta/\sigma)\int_0^t 1_{\partial \Omega}(X_s^{+})ds = (\eta/\sigma)\gamma_t^{+}.$$ 

Thus, by considering the measure ([1]), we get

$$\int_{\Omega} \gamma_t^{+}(z)m(dz) = t + (\eta/\sigma)\gamma_t^{+} =: V_t$$

which plays a role in the representation of $X$. Define the inverse

$$V_t^{-1} = \inf\{s : V_s > t\}.$$
The special case $\alpha = 1$ in our Theorem 3.3 below gives
\[ E_x[f(X_t)] = E_x[f(X^+ \circ V_t^{-1}) M \circ V_t^{-1}], \quad t > 0, \quad x \in \Omega \quad (19) \]
where $M_t = \exp \left(-\frac{c}{\eta}\gamma_t \right)$ is the multiplicative functional associated with the elastic condition introduced in [17]. The representation (19) for the sticky Brownian motion on the half line has been proved in [44, Section 10]. Under the representation (19) for $X$, the boundary local time $\gamma$ of $X$ can be given as the composition (from the same arguments as in [44], formula 18, Section 10)
\[ \gamma_t = \left(\frac{\eta}{\sigma}\right)^{\gamma^+ \circ V_t^{-1}}, \quad t > 0 \quad (20) \]

We proceed with the discussion on the process $X$ on $\Omega$ with generator $(G, D(G))$. Recall that slow reflections depend on the process $X$. Indeed, they are obtained via $\gamma$, the boundary local time of $X$.

Lemma 3. The following statements hold true:

i) $X$ may only have instantaneous reflections iff $\eta = 0$;

ii) $X$ may only have slow reflections iff $\eta > 0$.

Proof. For $\eta = 0$, $X$ is a strong Markov process on $\Omega$, that is the elastic Brownian motion $X^{el}$ introduced in Section 1.1, see point iii), with generator $G^{el} = \Delta$ and
\[ D(G^{el}) = \{ \varphi, \Delta \varphi \in C(\Omega), \varphi \in H^1(\Omega) : 0 = \sigma \partial_n \varphi + c \varphi |_{\partial \Omega} \}, \quad \sigma, c > 0. \]
The random time $V_t$ becomes $t$. The Robin boundary condition says that $X$ reflects instantaneously until the elastic kill. For $\eta > 0$, $X$ is a strong Markov process on $\Omega$. Due to the holding times, the process $X$ reflects slowly.

Since $P(e_0 > t) = \exp\left(-\frac{\sigma}{\eta}t\right)$, we conclude that, for all $\sigma < \infty$, if the process $X$ has only instantaneous reflections, then $\eta = 0$. Conversely, if the process has holding times, then $\eta > 0$.

We underline the fact that a strong Markov process must leave an holding point only by a jump. Indeed, a strong Markov process cannot stay in a point for a positive (Lebesgue) amount of time and then leave that point as a continuous motion. It must jump away or reflects (leave) instantaneously. Thus, we refer to $X$ as a strong Markov process on $\Omega$. Evidently $X$ is a Markov process on $\Omega$.

2.4 The random time $L$

The process $L$ is the inverse to the $\alpha$-stable subordinator $H$ defined by $L_t = \inf\{ s \geq 0 : H_s > t \}$ as already introduced in Section 1.1 and for which $P_0(L_t < s) = P_0(t < H_s), t, s > 0$. We assume that $H_0 = 0 = L_0$ and write accordingly $P_0$ for the associated probability measure. Denote by $l$ and $h$ the corresponding probability densities for which
\[ P_0(H_t \in ds) = h(t, s)ds, \quad P_0(L_t \in ds) = l(t, s)ds \]
and
\[ \int_0^\infty e^{-\xi s} h(t, s)ds = e^{-\xi t^\alpha}, \quad \int_0^\infty e^{-\lambda t} l(t, s)dt = \frac{\lambda^\alpha}{\lambda} e^{-s\lambda^\alpha}, \quad \xi, \lambda > 0. \quad (21) \]
We recall that
\[ \int_0^\infty e^{-\xi s} l(t, s)ds = E_{\alpha}(-\xi t^\alpha) \quad \text{with} \quad \int_0^\infty e^{-\lambda t} E_{\alpha}(-\xi t^\alpha)dt = \frac{\lambda^{\alpha-1}}{\lambda^\alpha + \xi}, \quad \lambda > 0 \quad (22) \]
where the Mittag-Leffler function $E_\alpha$ is analytic and such that
\[
|E_\alpha(-\mu t^\alpha)| \leq \frac{c_E}{1 + \mu t^\alpha}, \quad t \geq 0, \mu > 0 \quad \text{for a constant } c_E > 0
\] (see [8,42]). We underline that $E_\alpha \notin L^1(0,\infty)$ for $\alpha \in (0,1)$ and $E_\alpha \in L^2(0,\infty)$ for $\alpha \in (1/2,1)$. Moreover,
\[
D_t^\alpha E_\alpha(-\mu t^\alpha) = -\mu E_\alpha(-\mu t^\alpha), \quad t > 0, \mu > 0, \quad E_\alpha(0) = 1.
\]
The relation (24) can be easily verified from (22) and the fact that $D_t^\alpha$ is a convolution with
\[
\int_0^\infty e^{-\lambda t} D_t^\alpha E_\alpha dt = \left(\int_0^\infty e^{-\lambda t} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} dt\right) \left(\int_0^\infty e^{-\lambda_\alpha} dE_\alpha dt\right).
\]

### 2.5 The time-changed process $X^L$

We introduce the process $X^L := X \circ L_t$ as the process $X$ delayed by $L$. In particular, the random time $L$ may have plateau determining plateaux for the path of $X \circ L$. The process $L$ has been extensively investigated in relation with the FCPs on bounded and unbounded domains (see for example [19,20]). For the Markov process $X$ on $\Omega$ with generator $(G,D(G))$, we define the time-changed process $X^L = \{X^L_t\}_{t \geq 0}$ and recall that
\[
E_x\left[\int_0^\infty e^{-\lambda t} f(X \circ L_t) dt\right] = \int_\Omega f(y) \int_0^\infty e^{-\lambda t} \int_0^\infty P_x(X_s \in dy) P(L_t \in ds) dt
\]
\[
= \int_\Omega f(y) \int_0^\infty \frac{\lambda^\alpha}{\lambda} e^{-\lambda^\alpha s} P_x(X_s \in dy) ds
\]
\[
= \frac{\lambda^\alpha}{\lambda} E_x\left[\int_0^\infty e^{-\lambda^\alpha t} f(X_t) dt\right]
\]
is the $\lambda$-potential for the non-Markov process $X^L$ driven by the FCP
\[
D_t^\alpha w = Gw, \quad w_0 = f \in D(G).
\]
Thus, $X^L$ can be considered to solve (3) or (6) as we already discussed above. In particular,
\[
w(t,x) = E_x[f(X^L_t)] = \int_0^\infty S_s f(x) l(t,s) ds, \quad t \geq 0, x \in \Omega
\]
where $S_t$ is the $C_0$-semigroup on $C(\Omega)$ generated by $(G,D(G))$. References on the FIVP (26) have been given in Section 1.2. We also refer to [14] for a general discussion and a short review. As discussed in [13], the process $X^L$ can be considered as a delayed process with infinite occupation measure for every set $\Lambda \subset \Omega$. As we can see from (25), $\forall x \in \overline{\Omega}$,
\[
\lim_{\lambda \to 0} E_x\left[\int_0^\infty e^{-\lambda \alpha 1_\Lambda(X^L_t)} dt\right] = E_x\left[\int_0^\infty 1_\Lambda(X^L_t) dt\right] = \infty \quad \forall \Lambda \subset \Omega
\]
We refer to [25] for a discussion on the occupation measures of $X$ on $[0,\infty)$.

### 2.6 The random time $\hat{L}$

We discuss on the process $\hat{L}$ to be considered in the time-changed process $\hat{X} := X \circ \hat{L}$. The main difficulty in the characterization of this process relies on the fact that $\hat{L}$ depends on the base process $X$, that is the process serving as the base process for the time change leading to $\hat{X}$. The random time $\hat{L} = \hat{L}(X)$ of $X$ can be roughly defined as
\[
\hat{L}_t(X) = \begin{cases} 
  t \geq 0, & \text{if the base process } X \in \Omega, \\
  L_t, t \geq 0, & \text{if the base process } X \in \partial \Omega.
\end{cases}
\]
Formula (27) gives a clear picture. However, it does not allow a clear construction of \( \hat{X} \) via time change.

**Definition 1.** Let \( \hat{A} \) be a PCAF for the process \( X \circ \hat{L} \) on \( \Omega \). The random time \( \hat{L} = \hat{L}(X) \) is the time change such that, for \( f \in B(\Omega) \):

i) \( \hat{A} \in A^+_t(\partial \Omega) \) implies that

\[
\int_0^t f(X \circ \hat{L}_s) d\hat{A}_s = \int_0^t f(X \circ L_s) d\hat{A}_s, \tag{28}
\]

ii) \( \hat{A} \in A^+_t(\Omega) \) implies that

\[
\int_0^t f(X \circ \hat{L}_s) d\hat{A}_s = \int_0^t f(X_s) d\hat{A}_s. \tag{29}
\]

We proceed with the following further characterization of the process \( \hat{L} \). We introduce the function \( \alpha : \Omega \to (0, 1] \) defined as

\[
\alpha(y) = \begin{cases} 
1, & y \in \Omega, \\
\alpha \in (0, 1), & y \in \partial \Omega.
\end{cases}
\]

Denote by \( \hat{\lambda} \) the density of \( \hat{L} \) and write

\[
P_x(\hat{L}_t(y) \in ds, X_s \in dy) = \hat{\lambda}(t, s, y) p(s, x, y) ds m(dy)
\]

with

\[
\int_\Omega \hat{\lambda}(t, s, y) m(dy) = \delta(t-s) m(\Omega) \quad \text{and} \quad \int_{\partial \Omega} \hat{\lambda}(t, s, y) m(dy) = l(t, s) m(\partial \Omega)
\]

where \( \delta \) is the Dirac delta function (recall that \( L_t \to t \) almost surely as \( \alpha \uparrow 1 \)) and

\[
\int_0^\infty \hat{\lambda}(t, s, y) ds = 1, \quad \forall y \in \Omega, t > 0.
\]

In particular, for \( \Lambda \in \overline{\Omega} \),

\[
P_x(X \circ \hat{L}_t \in \Lambda) = \int_\Lambda \int_0^\infty P_x(\hat{L}_t(y) \in ds, X_s \in dy)
\]

\[
= \int_\Lambda p(t, x, y) dy + (\eta/\sigma) \int_{\Lambda \cap \partial \Omega} \left( \int_0^\infty p(s, x, y) l(t, s) ds \right) m(\partial \Omega).
\]

From (21),

\[
\int_0^\infty e^{-\lambda} P_x(\hat{L}_t(y) \in ds, X_s \in dy) dt = \frac{\lambda^{\alpha(y)}}{\lambda} e^{-\lambda^{\alpha(y)}} p(s, x, y) m(dy), \quad \lambda > 0, x, y \in \Omega. \tag{30}
\]

The time change \( \hat{L} \) can be also regarded as the multi-parameter process \( \{\hat{L}_t(y), (t, y) \in (0, \infty) \times \overline{\Omega}\} \) by means of which we obtain the switching process \( X \circ \hat{L}_t \). The law \( \hat{l} \) of \( L \), for a given \( y \in \overline{\Omega} \), solves the problem to find \( \hat{l} \in C^{1,1}(W^{1,\infty}(0, \infty), (0, \infty)) \) such that (consult for example [22, 24]),

\[
D_t^{\alpha(y)} \hat{l} = -\frac{\partial \hat{l}}{\partial x}, \quad \hat{l}(0, x) = f(x)
\]

for a good function \( f \), that is

\[
\hat{l}(t, x) = \int_0^t f(x-s) \hat{l}(t, s, y) ds = E_0 \left[ f(x-\hat{L}_t(y)) 1_{(\hat{L}_t(y) < x)} \right].
\]

The case \( \alpha(y) = 1 \) (for which we have the ordinary derivative) must be managed in view of the fact that \( \hat{l} = \delta \) is the Dirac delta function. In case \( \alpha(y) \in (0, 1) \), we observe that \( l = \hat{l} \) is a smooth function in both time and space variables.
2.7 The process $\hat{X}$

We introduce $\hat{X}$ in place of $X^L$ because of its Markov behaviour on the interior $\Omega$. The process $\hat{X}$ can be regarded as a switching process, that is a process changing its behaviour in a given region $\Lambda$ of the domain $\Omega$. In general, we may consider $\alpha(y)$ as the order for which

$$\alpha(\Lambda) = \alpha \quad \text{and} \quad \alpha(\Lambda^c) = 1, \quad \text{for some } \Lambda \subset \Omega.$$ 

Here we continue our discussion in case $\Lambda = \partial \Omega$. That is, the process $\hat{X}$ changes its behaviour on the boundary.

For the process $\hat{X} := X \circ \hat{L}$, we respectively consider the local time and the occupation time

$$\hat{\gamma}_t = \int_0^t 1_{\partial \Omega}(\hat{X}_s)ds, \quad t \geq 0 \quad \text{and} \quad \hat{\Gamma}_t = \int_0^t 1_{\Omega}(\hat{X}_s)ds, \quad t \geq 0.$$ 

With Definition 1 at hand, we are able to introduce the following facts:

- A first consequence is given by the identities

$$\int_0^t f(X \circ \hat{L}_s)d\hat{\gamma}_s = \int_0^t f(X \circ L_s)d\hat{\gamma}_s, \quad \int_0^t f(X \circ \hat{L}_s)d\hat{\Gamma}_s = \int_0^t f(X_s)d\Gamma_s$$

where we also underline the relation between occupation times on $\Omega$, that is $\forall t, \hat{\Gamma}_t = \Gamma_t$ for any given path of $X$.

- Let us consider $\gamma_t^{-1} = \inf\{s \geq 0 : \gamma_s > t\}$ and $\Gamma_t^{-1} = \inf\{s \geq 0 : \Gamma_s = t\}$, that is the right-inverses of the boundary local time $\gamma$ and the occupation time $\Gamma$ for $X$ on $\Omega$. The compositions $X \circ \gamma_t^{-1}$ and $X \circ \Gamma_t^{-1}$ respectively consider the parts of $X$ on $\partial \Omega$ and $\Omega$. For the time change $\hat{L}$ of $X$ we therefore have a random time $\hat{L} = \hat{L}(X)$ for which, for $t > 0$,

$$(X \circ \gamma_t^{-1}) \circ \hat{L}_t = (X \circ \gamma_t^{-1}) \circ L_t \quad \text{and} \quad (X \circ \Gamma_t^{-1}) \circ \hat{L}_t = X \circ \Gamma_t^{-1}. \quad (31)$$

The local time $\gamma$ of $X$ is a non-decreasing PCAF, thus $\gamma_t^{-1}$ may have jumps gluing together the paths of $X$ on $\partial \Omega$. This means that $X \circ \gamma_t^{-1}$ on $\partial \Omega$ describes the motion of $X$ on $\partial \Omega$, we provide below a further discussion on this process. On the other hand, since $\Gamma_t^{-1}$ may have jumps as well, $X \circ \Gamma_t^{-1}$ gives the path of the reflecting Brownian motion $X^+$ on $\Omega$ only if $X$ has no excursion on $\partial \Omega$ (there is no diffusion on the boundary, the boundary process is a pure jump process).

Following the previous section, we write

$$\mathbf{P}_x(\hat{X}_t \in dy) = \mathbf{P}_x(X \circ L_t \in dy) = \int_0^\infty \mathbf{P}_x(X_s \in dy, L_t(y) \in ds)$$

and, for $f \in C(\Omega)$,

$$\mathbf{E}_x[f(\hat{X}_t)] = \int_\Omega f(y)p(t, x, y)m(dy) \quad (32)$$

$$= \int_\Omega f(y)p(t, x, y)dy + \eta/\sigma \int_{\partial \Omega} f(y) \int_0^\infty l(t, s)p(s, x, y)ds m_\partial(dy)$$

$$= S_t f|\Omega(x) + \int_0^\infty S_s f|_{\partial \Omega}(x) l(t, s)ds, \quad x \in \Omega$$

where $l(t, s)ds = \mathbf{P}_0(L_t \in ds)$ and $S_t$ is the semigroup generated by $(G, D(G))$. Further on we also write

$$w(t, x) = \mathbf{E}_x[f(\hat{X}_t)], \quad w = w_f|\Omega + w_f|_{\partial \Omega}. \quad (33)$$
Notice that, both \( l \) and \( p \) are continuous functions. For \( \alpha = 1 \), since \( L_t = t \) almost surely, then
\[
E_x[f(\bar{X}_t)] = \int_\Omega f(y)p(t, x, y)dy + (\eta/\sigma) \int_{\partial\Omega} f(y)p(t, x, y)m_\partial(dy)
\]
for \( t > 0, x \in \overline{\Omega} \) and \( f \in C(\overline{\Omega}) \). Indeed, \( \bar{X} \) equals in law \( X \) for \( \alpha = 1 \). Let us consider, for \( f \in C(\overline{\Omega}) \),
\[
f = f_1 + f_2 \quad \text{where} \quad f_1 = f1_\Omega \quad \text{and} \quad f_2 = f1_{\partial\Omega}.
\]
We notice that, from (32),
\[
\forall \lambda \in \mathbb{R}, \text{for} \lambda > 0, \quad \bar{T} = \inf_{\tau} \tau \geq 0 \quad \text{for} \quad \bar{X} \in \overline{\Omega}
\]
Indeed, \( \tau \) is the lifetime of \( \bar{X} \) on \( \overline{\Omega} \) written in terms of the local time \( \bar{\tau} \).
\[
\text{We also notice that, from (30),} \quad \forall x \in \overline{\Omega}, \quad \lim_{\lambda \to 0} E_x \left[ \int_0^\infty e^{-\lambda t}1_\Lambda(X_t)dt \right] = E_x \left[ \int_0^\infty 1_\Lambda(X_t)dt \right] < \infty \quad \text{only if} \quad \Lambda \cap \partial\Omega = \emptyset
\]
whereas,
\[
\lim_{\lambda \to 0} E_x \left[ \int_0^\infty e^{-\lambda t}1_\Lambda(X_t)dt \right] = \infty \quad \text{if} \quad \Lambda \cap \partial\Omega \neq \emptyset.
\]
Indeed,
\[
E_x \left[ \int_0^\infty 1_{\Lambda \cap \partial\Omega}(X_t^L)dt \right] = \infty.
\]

2.8 The process \( \bar{X} \)
We show that \( \forall t > 0 \), there exists a continuous kernel \( \bar{p}(t, \cdot, \cdot) : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R} \) such that
\[
E_x[f(\bar{X}_t)] = \int_\Omega f(y)\bar{p}(t, x, y)m(dy), \quad f \in C(\overline{\Omega})
\]
and write
\[
E_x[f(\bar{X}_t)] = E_x[f(\bar{X}_t), t < \bar{\tau}]
\]
where \( \bar{\tau} \) is the lifetime of \( \bar{X} \) on \( \overline{\Omega} \) written in terms of the local time \( \bar{\tau} \) according with (17). As in (20) for the process \( X \), we have
\[
\bar{\tau}_t = (\eta/\sigma)\gamma^+ \circ \bar{\gamma}^{-1}
\]
for the process \( \bar{X} \). We introduce the \( \lambda \)-potential
\[
\bar{R}_\lambda f(x) := E_x \left[ \int_0^\infty e^{-\lambda t}f(\bar{X}_t)dt \right], \quad \lambda > 0, \quad x \in \overline{\Omega}
\]
and \( \bar{\gamma} \) from which we can write
\[
\bar{R}_\lambda f(x) = E_x \left[ \int_0^\infty e^{-\lambda t}f(\bar{X}_t^+dt) \right] + E_x \left[ e^{-\lambda \bar{\tau}} E_{\bar{X}\in \overline{\Omega}} \left[ \int_0^\infty e^{-\lambda t}f(\bar{X}_t)dt \right] \right]
\]
\[
= R_\lambda^d f(x) + \bar{R}_\lambda f(x).
\]
Notice that
\[
\bar{R}_\lambda^d f(x) = \int_0^\infty e^{-\lambda t} \int_{\overline{\Omega}} \bar{R}_\lambda f(y)P_x(X_{t+\bar{\tau}} \in dy, \bar{\tau} \in dt)
\]
and \( T\bar{R}_\lambda f = T\bar{R}_\lambda^d f \). We close this section by observing that
\[
\Delta \bar{R}_\lambda f = \Delta(R_\lambda^d f + \bar{R}_\lambda f), \quad \text{and} \quad \lambda \bar{R}_\lambda f - f = \lambda(R_\lambda^d f + \bar{R}_\lambda^d f) - f, \quad \text{in} \ \overline{\Omega}.
\]
In particular,
\[
\lambda \bar{R}_\lambda f - f = (\lambda R_\lambda^d f - f) + \lambda \bar{R}_\lambda^d f \quad \text{in} \ \Omega \quad \text{and} \quad \lambda \bar{R}_\lambda f = f = (\lambda R_\lambda^d f - f) \quad \text{on} \ \partial\Omega.
\]
2.9 Discussion on $X$ and $\hat{X}$

We study the process $\hat{X}$ defined as $\hat{X} = X \circ \hat{L}$ in comparison with the (base) process $X$. Observe that $X$ and therefore $\hat{X}$ are both defined on $\Omega = \Omega \cup \partial\Omega$. The process $\hat{X}$ is obtained via time change and therefore, it represents a motion along the path of $X$ with the new clock $\hat{L}$. Concerning the path of the process $X$ with $X_0 = x \in \Omega$ we can consider the sets

$$\bigcup_{i \geq 0} [\tau_i, \tau_i + e_i), \text{ time for which } X \in \partial\Omega$$

and

$$\bigcup_{i \geq 0} [\tau_i + e_i, \tau_{i+1}), \text{ time for which } X \in \Omega$$

where

$$\tau_i := \inf\{t > \tau_{i-1} : X_t \notin \Omega\}, \quad i \geq 1$$

is the first return of $X$ on $\partial\Omega$ with the first hitting time $\tau_0 = \inf\{t \geq 0 : X_t \notin \Omega\}$ and $e_i$ are independent holding times, that is the sequence $\{e_i\}_i$ gives the time the process $X$ spends on $\partial\Omega$. The sticky condition implies that (see (18))

$$\forall x \in \partial\Omega, \quad P_x(e_i > t | X_{e_i}) = e^{-\sigma/\eta}t, \quad t > 0.$$

From the paths of $X$ we characterize the paths of $\hat{X}$ by considering the sets

$$\bigcup_{i \geq 0} [\hat{\tau}_i, \hat{\tau}_i + \hat{e}_i), \text{ time for which } \hat{X} \in \partial\Omega$$

and

$$\bigcup_{i \geq 0} [\hat{\tau}_i + \hat{e}_i, \hat{\tau}_{i+1}), \text{ time for which } \hat{X} \in \Omega$$

where

$$\hat{\tau}_i := \inf\{t > \hat{\tau}_{i-1} : \hat{X}_t \notin \Omega\} = \tau_i + \sum_{k \leq i} (\hat{e}_k - e_k), \quad i \geq 1 \text{ and } \hat{\tau}_0 = \tau_0.$$

The total amount of time the processes $X$ and $\hat{X}$ spend on $\partial\Omega$ can be respectively given (up to the associated kill) by

$$\sum_{i \geq 0} e_i \text{ and } \sum_{i \geq 0} \hat{e}_i$$

whereas, for the total amount of time the process $\hat{X}$ (and also $X$) spends on $\Omega$ we have

$$\tau_0 + \sum_{i \geq 0} \left(\tau_{i+1} - (\tau_i + e_i)\right).$$

Standard arguments say that

$$\gamma_t = \sum_{i} e_i, \quad N_t = \max\{i : \tau_i < t\}. \quad (39)$$
The reader may consult the book [7, Section 5, page 121] for a discussion on holding and irregular points. Analogous arguments lead to

\[
\hat{\gamma}_t = \sum_{i} \hat{\gamma}_t \hat{N}_i = \max\{i : \hat{\tau}_i < t\} = \max\{i : \tau_i + \sum_{k<i} (\hat{e}_k - e_k) < t\}. \tag{40}
\]

We observe that \(P(e_0 > t|N_i \leq 0) = 1\). In general, \(P(e_i > t|N_i > i) = P(e_i > t)\) with \(P(N_i > i) = P(\tau_i < t)\). Moreover, we have that

\[
P(\hat{N}_t > i) = P\left(\tau_i + \sum_{k<i} (\hat{e}_k - e_k) < t\right) \tag{41}
\]

from which we see a completely different structure. Notice that as \(\alpha \to 1\), \(L_t \to t\) (and \(H_t \to t\)) almost surely, then \(\hat{\tau}_i \to \tau_i\) almost surely. In this case we obviously have that \(\hat{X} = X\) on \(\Omega\).

We continue our description. The process \(\hat{X}\) must approach continuously the boundary. Indeed, \(X\) has continuous paths on \([0, \infty)\) and the process \(X\) must move on the given path of \(X\) according with the new times

\[
\hat{L}_t = t - (\hat{\tau}_i - \tau_i) - (\hat{e}_i - e_i), \quad \hat{\tau}_i + \hat{e}_i \leq t < \hat{\tau}_{i+1}, \quad i \geq 0 \quad \text{for which} \quad X \circ \hat{L}_t \in \Omega
\]

and

\[
\hat{L}_t = \tau_i + L_{t-\hat{\tau}_i}, \quad \hat{\tau}_i \leq t < \hat{\tau}_i + \hat{e}_i, \quad i \geq 0 \quad \text{for which} \quad X \circ \hat{L}_t \in \partial\Omega.
\]

The time-changed process runs continuously (indeed \(L\) is continuous) on the continuous path of \(X\). Moreover, it can be regarded as a Markov process on \(\Omega\) but the boundary diffusion is not Markov. Assume \(\hat{X}_0 = x \in \Omega\), then

\[
\hat{X}_t = \begin{cases} 
X \circ (t - (\hat{\tau}_i - \tau_i) - (\hat{e}_i - e_i)), & t \in \bigcup_{i \geq 0} (\hat{\tau}_i + \hat{e}_i, \hat{\tau}_{i+1}), \\
X \circ (\tau_i + L_{t-\hat{\tau}_i}), & t \in \bigcup_{i \geq 0} (\hat{\tau}_i, \hat{\tau}_i + \hat{e}_i) 
\end{cases}, \quad t \geq 0.
\]

Observe that, for \(\hat{X}_0 = x \in \Omega\) (which coincides with \(X_0 = x \in \Omega\))

\[
E_x \left[ \int_0^\infty e^{-\lambda t} f(\hat{X}_t) dt \right] \tag{42}
\]

\[
= E_x \left[ \int_{\hat{\tau}_0}^{\hat{\tau}_0+H_{\hat{e}_0}} e^{-\lambda t} f(\hat{X}_t) dt \right] + E_x \left[ e^{-\lambda \hat{\tau}_0} E_{\hat{X}_{\hat{\tau}_0}} \left[ \int_0^\infty e^{-\lambda t} f(\hat{X}_t) dt \right] \right], \quad \lambda > 0.
\]

In particular, \(42\) can be written by taking into account the fact that

\[
E_x \left[ \int_{\hat{\tau}_0}^{\hat{\tau}_0+H_{\hat{e}_0}} e^{-\lambda t} f(\hat{X}_t) dt \right] = 0.
\]

\[
E_x \left[ \sum_{i \geq 0} \int_{\hat{\tau}_i}^{\hat{\tau}_i+H_{\hat{e}_i}} e^{-\lambda t} f(\hat{X}_t) dt \right] + \sum_{i \geq 0} \int_{\hat{\tau}_i+H_{\hat{e}_i}}^{\hat{\tau}_{i+1}} e^{-\lambda t} f(\hat{X}_t) dt, \quad \lambda > 0. \tag{43}
\]

With \(\hat{X} = X\) on \(\Omega\) and \(X_0 = x \in \Omega\) (otherwise, assume \(\tau_0 = 0\), we have that

\[
E_x \left[ \int_0^{\hat{\tau}_0} e^{-\lambda t} f(\hat{X}_t) dt + \sum_{i \geq 0} \int_{\hat{\tau}_i+H_{\hat{e}_i}}^{\hat{\tau}_{i+1}} e^{-\lambda t} f(\hat{X}_t) dt \right] = E_x \left[ \int_0^\infty e^{-\lambda t} f(\hat{X}_t) d\hat{\tau}_t \right], \quad \lambda > 0
\]

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and
\[
E_x \left[ \sum_{t \geq 0} \int_{\tau_t}^{\tau_{t+H\circ e_0}} e^{-\lambda t} f(\hat{X}_t) dt \right] = E_x \left[ \int_0^\infty e^{-\lambda t} f(\hat{X}_t) d\gamma_t \right].
\]

Notice that \( \hat{X} \circ \hat{\Gamma}_t^{-1} \) and \( \hat{X} \circ \hat{\gamma}_t^{-1} \), because of the cutting jumps of \( \hat{\Gamma}^{-1} \) and \( \hat{\gamma}^{-1} \), give rise to trace processes.

**Lemma 4.** \( \hat{e}_0 \) equals in law \( H \circ e_0 \). In particular, \( \hat{e}_0 \) is a Mittag-Leffler random variable.

**Proof.** Let us consider the stopped process
\[
\hat{X}_t = \begin{cases} 
X_t, & 0 \leq t < \hat{\tau}_0 \\
X \circ (\tau_0 + L_t - \tau_0), & \hat{\tau}_0 \leq t < \hat{\tau}_0 + \hat{e}_0
\end{cases}
\]
with
\[
\varsigma(x) = E_x \left[ \int_0^{\hat{\tau}_0 + \hat{e}_0} e^{-\lambda t} f(\hat{X}_t) dt \right], \quad x \in \Omega.
\]

First we prove that, for \( \lambda > 0 \), \( f \in C(\Omega) \),
\[
T\varsigma(x) = \frac{\lambda^\alpha}{\lambda} E_x \left[ \int_0^{\hat{e}_0} e^{-\lambda t} f(X_t) dt \right], \quad x \in \partial\Omega.
\]

The process \( \hat{X}_t, 0 \leq t < \hat{\tau}_0 + \hat{e}_0 \) can be associated with a (time) non-local equation. The reader should have in mind the \( \lambda \)-potentials (21) and (25).

We are able to write \( \varsigma(x) = \varsigma_1(x) + \varsigma_2(x) \) with \( T\varsigma(x) = T\varsigma_2(x) \). In particular,
\[
\varsigma_1(x) = E_x \left[ \int_0^{\hat{\tau}_0} e^{-\lambda t} f(\hat{X}_t) dt \right] = E_x \left[ \int_0^{\hat{\tau}_0} e^{-\lambda t} f(X_t) dt \right]
\]
and, with \( \hat{\tau}_0 = \tau_0 \) by definition and \( X \) strong Markov on \( \Omega \),
\[
\varsigma_2(x) = E_x \left[ \int_{\tau_0}^{\tau_0 + H\circ e_0} e^{-\lambda t} f(\hat{X}_t) dt \right] = E_x \left[ \int_{\tau_0}^{\tau_0 + H\circ e_0} e^{-\lambda t} f(X_t) dt \right]
\]
\[
= E_x \left[ \int_{\tau_0}^{\tau_0 + H\circ e_0} e^{-\lambda t} f(X_{\tau_0} + L_t) dt \right] = E_x \left[ \int_{\tau_0}^{\tau_0 + H\circ e_0} e^{-\lambda t} f(X_{\tau_0} + L_t) dt \right]
\]
\[
= E_x \left[ \int_{\tau_0}^{\tau_0 + H\circ e_0} \frac{1}{\lambda} f(X_{\tau_0} + L_t) dt \right] = E_x \left[ \int_{\tau_0}^{\tau_0 + H\circ e_0} \frac{1}{\lambda} f(X_{\tau_0} + L_t) dt \right]
\]
\[
= E_x \left[ \int_{\tau_0}^{\tau_0 + H\circ e_0} \frac{1}{\lambda} f(X_{\tau_0} + L_t) dt \right] = E_x \left[ \int_{\tau_0}^{\tau_0 + H\circ e_0} \frac{1}{\lambda} f(X_{\tau_0} + L_t) dt \right]
\]
\[
= E_x \left[ \int_{\tau_0}^{\tau_0 + H\circ e_0} \frac{1}{\lambda} f(X_{\tau_0} + L_t) dt \right], \quad \lambda > 0, \quad x \in \Omega.
\]

The first claim immediately follows.
By definition of holding time, $X_t = x$ for $0 \leq t < e_0$ and

$$T_t(x) = \frac{\lambda^\alpha}{\lambda} \mathbb{E}_x \left[ \int_0^{e_0} e^{-\lambda^\alpha t} f(X_t) dt \right] = 1 - \mathbb{E}[e^{-\lambda x}] \int_0^{e_0} e^{-\lambda^\alpha t} f(x) dt = \frac{\lambda^\alpha}{\lambda} \frac{1}{\lambda^\alpha + (\sigma/\eta)} f(x), \quad x \in \partial \Omega.$$  

where, from (22),

$$T_t(x) = E_\alpha(- (\sigma/\eta) t^\alpha) T f(x).$$

We therefore obtain, $\forall x \in \partial \Omega$,

$$P_x(\hat{e}_i > t | X_{e_i}) = P_x(e_i > L_t | X_{e_i}) = E[\hat{e}_i] = E_\alpha(- (\sigma/\eta) t^\alpha)$$

which is the result.

**Remark 2.1.** Recall that $e_i, \forall i$, is a sequence of i.i.d. exponential random variables. Thus, $\hat{e}_i = H \circ e_i, \forall i$ is a sequence of i.i.d. holding times. Indeed, $H$ is independent from $X$. As a first consequence we have that

$$P(\hat{e}_i > t | X_{e_i}) = P(e_i > L_t | X_{e_i}) = E[\hat{e}_i] = E_\alpha(- (\sigma/\eta) t^\alpha)$$

(as announced in the previous proof) where the last equality comes from (22). Thus, by recalling that $E_\alpha \notin L^1(0, \infty)$, we deduce that the process $\hat{X}$ spends an infinite mean amount of (holding) time on $\partial \Omega$ with each visit. Moreover, since $\sum_i H \circ e_i$ equals in law $H \circ \sum_i e_i$, formula (41) becomes

$$P(\hat{N}_i > i) = P(\tau_i + H \circ T_{e_i} - T_{e_i} < t) \quad \text{with} \quad T_{e_i} = \sum_{k<i} e_k.$$  

Now we state the following result. Notice that a slow reflection of $X$ may have an independent delay.

**Lemma 5.** The following statements hold true:

i) $\hat{X}$ may only have instantaneous reflections iff $\eta = 0$;

ii) $\hat{X}$ may only have delayed (slow) reflections iff $\eta > 0$.

**Proof.** We follow the same arguments as in the proof of Lemma 3. Recall that

$$P(\hat{e}_0 > t) = E_\alpha(- (\sigma/\eta) t^\alpha)$$

where the Mittag-Leffler function $E_\alpha$ has been introduced in Section 2.4. Thus, we may have instantaneous reflection only if $\eta = 0$ whereas, we may have Mittag-Leffler holding times if $\eta > 0$.

On the other hand, for $\eta = 0$, we have that

$$E_x[f(\hat{X}_t)] = \int_\Omega f(y)p(t, x, y) dy$$

where $p(t, x, y)$ is now the continuous kernel for the elastic Brownian motion $X^{el}$. In case $\eta > 0$, the process $\hat{X}$ is defined as a time-changed elastic sticky Brownian motion $X$ for which the new clock $L$ introduces a delay effect ([15]). Since $X$ slowly reflects on the boundary, then $\hat{X} = X \circ L$ turns out to be delayed.

\[19\]
3 The fractional boundary value problem

3.1 Main results

The Caputo-Dzherbashian derivative of a function \( \varrho \) turns out to be well-defined if, for \( \varrho' = \partial \varrho/\partial s \), we have \( \varrho'(s,\cdot)(t-s)^{-\alpha} \in L^1(0,t) \ \forall t \). In a bounded interval, \( D_t^\alpha \varrho \) is well defined for \( \varrho(\cdot, x) \in AC(a, b) \ \forall x \in \Omega, (a, b) \subset (0, \infty) \) where the set \( AC(a, b) \) of the absolutely continuous functions on \((a, b)\) is the natural set of functions to be considered for the convolution operator \( D_t^\alpha \). We recall that \( AC(a, b) = \{ \varphi \in C((a, b) : \varphi' \in L^1(a, b) \} \) and observe that \( AC(a, b) \) coincides with \( W^{1,1}(a, b) \). On the half line we may consider \( W^{1,1}(0, \infty) \) which embeds into \( L^\infty(0, \infty) \). In particular, \( D_t^\alpha \varrho \) is well defined for \( \varrho(\cdot, x) \in W^{1,\infty}(0, \infty), \forall x \in \Omega \) and this ensures existence of the Laplace transform

\[
\int_0^\infty e^{-\lambda t} D_t^\alpha \varrho(t, x) \, dt = \left( \int_0^\infty e^{-\lambda t} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \, dt \right) \left( \int_0^\infty e^{-\lambda t} \frac{\partial \varrho}{\partial t}(t, x) \, dt \right) = \frac{\lambda^\alpha}{\lambda} (\varrho(\lambda, x) - \varrho(0, x)), \quad \lambda > 0. \tag{45}
\]

Formula (45) is obtained from the fact that \( D_t^\alpha \) is a convolution operator. Notice that we are not asking for \( \varrho(\cdot, x) \in L^1(0, \infty) \), for some \( x \in \Omega \) in the spaces below.

We introduce the spaces

\[
\hat{D}_L = \left\{ \varphi : (0, \infty) \times \Omega \to \mathbb{R} \text{ with } \varphi = T \varphi \text{ such that } \frac{\partial \varphi}{\partial t}, D_t^\alpha \varphi \in C((0, \infty) \times \partial \Omega) \text{ and } \left| \frac{\partial \varphi}{\partial t}(t, x) \right| \leq \varrho(x)t^{\alpha-1}, \ \varrho \in L^\infty(\partial \Omega), \ t > 0 \right\}
\]

and

\[
\check{D}_L = \left\{ \varphi : (0, \infty) \times \Omega \to \mathbb{R} \text{ with } \varphi = T \varphi \text{ such that } \frac{\partial \varphi}{\partial t} \in C((0, \infty) \times \partial \Omega) \text{ and } \frac{\partial \varphi}{\partial t}(s, x)(t-s)^{-\alpha} \in L^1(0, t), \ \forall x \in \partial \Omega, \ t > s > 0 \right\}.
\]

Concerning the problems \([6]\) and \([5]\), for \( f \in C(\Omega) \), we address the problem to find a solution in \( C((0, \infty), \Omega) \cap \hat{D}_L \) associated with \( X \) and a solution in \( C((0, \infty), \Omega) \cap \check{D}_L \) associated with \( \check{X} \). Then, we discuss probabilistic representations. We observe that \( u \in \check{D}_L \) ensures existence of \( D_t^\alpha u \) on the boundary \( \partial \Omega \). On the other hand, for \( u \in \hat{D}_L \) we get

\[
\left| D_t^\alpha T^u \right| \leq \| \varrho \|_{L^\infty(\partial \Omega)} \frac{1}{\Gamma(1-\alpha)} \int_0^t s^{\alpha-1}(t-s)^{-\alpha} \, ds = \Gamma(\alpha) \| \varrho \|_{L^\infty(\partial \Omega)}.
\]

The process \( X \) with generator \((G, D(G))\) is driven by \([8]\) in case \( \alpha = 1 \). For \( \alpha \in (0, 1) \), we do not have a semigroup on \( C(\Omega) \) and the associated process in not a Markov process on \( \Omega \).

We first obtain the following preliminary results.

**Lemma 6.** For \( f \in \mathcal{B}(\Omega) \), \( \lambda > 0 \), \( x \in \Omega \):

i) \[
E_x \left[ \int_0^\infty e^{-\lambda t} f(\tilde{X}_t) \, d\tilde{\gamma}_t \right] = \frac{\lambda^\alpha}{\lambda} E_x \left[ \int_0^\infty e^{-\lambda^\alpha t} f(\check{X}_t) \, d\gamma_t \right]
\]

ii)
\[
E_x \left[ \int_0^\infty e^{-\lambda t} f(\hat{X}_t) d\hat{\Gamma}_t \right] = E_x \left[ \int_0^\infty e^{-\lambda t} f(X_t) d\Gamma_t \right]
\]

**Proof.** For a PCAF \( \hat{A} \) of \( \hat{X} \) with \( \text{supp}[\mu_\hat{A}] = \Lambda \subset \Omega \), according with (11) and (13), we obtain

\[
E_x \left[ \int_0^t f(\hat{X}_s) d\hat{A}_s \right] = \int_0^t \int_\Lambda f(y) \hat{p}(s, x, y) \mu_\Lambda(dy) ds \tag{46}
\]
and

\[
E_x \left[ \int_0^\infty e^{-\lambda t} f(\hat{X}_t) d\hat{A}_t \right] = \int_\Lambda f(y) \hat{p}(\lambda, x, y) \mu_\Lambda(dy), \quad \lambda > 0 \tag{47}
\]

where, from (30),

\[
\hat{p}(\lambda, x, y) := \int_0^\infty e^{-\lambda t} \hat{p}(t, x, y) dt = \int_0^\infty \frac{\lambda^\alpha(y)}{\lambda} e^{-s\lambda^\alpha(y)} p(s, x, y) ds =: \frac{\lambda^\alpha(y)}{\lambda} p(\lambda^\alpha(y), x, y).
\]

Recall that \( p(t, x, y) \) is the transition kernel of \( X \) on \( \Omega \). Then,

\[
E_x \left[ \int_0^\infty e^{-\lambda t} f(\hat{X}_t) d\hat{A}_t \right] = \begin{cases} 
\frac{\lambda^\alpha}{\lambda} \int_\Lambda f(y) p(\lambda^\alpha, x, y) \mu_\Lambda(dy), & \Lambda \subseteq \partial \Omega, \\
\int_\Lambda f(y) \hat{p}(\lambda, x, y) \mu_\Lambda(dy), & \Lambda \subseteq \Omega.
\end{cases} \tag{48}
\]

From (48) and (47) we get the claim. \( \square \)

**Theorem 3.1.** Let us consider (33) and define \( \tilde{w} = \tilde{w}(\lambda, x) \) as

\[
\tilde{w}(\lambda, x) = \int_0^\infty e^{-\lambda t} w(t, x) dt, \quad \lambda > 0, \quad x \in \Omega.
\]

Then:

i) For a continuous function \( f \) compactly supported on \( \Omega \), \( \tilde{w} \) has the representation

\[
\tilde{w}(\lambda, x) = E_x \left[ \int_0^\infty e^{-\lambda t} f(\hat{X}_t) d\hat{\Gamma}_t \right]
\]

and satisfies

\[
\begin{cases} 
\lambda \tilde{w} - f = \Delta \tilde{w}, & \Omega, \\
\eta \lambda \tilde{w} = -\sigma \partial_n \tilde{w} - c \tilde{w}, & \partial \Omega.
\end{cases} \tag{49}
\]

ii) For a continuous function \( f \) compactly supported on \( \partial \Omega \), \( \tilde{w} \) has the representation

\[
\tilde{w}(\lambda, x) = E_x \left[ \int_0^\infty e^{-\lambda t} f(\hat{X}_t) d\hat{\gamma}_t \right]
\]

and satisfies

\[
\begin{cases} 
\lambda^\alpha \tilde{w} = \Delta \tilde{w}, & \Omega, \\
\eta \frac{\lambda^\alpha}{\lambda} (\lambda \tilde{w} - f) = -\sigma \partial_n \tilde{w} - c \tilde{w}, & \partial \Omega.
\end{cases} \tag{50}
\]
Proof. From (33) write \( w = w_1 + w_2 \) and
\[
\tilde{w}_i(\lambda, x) = \int_0^\infty e^{-\lambda t}w_i(t, x)dt, \quad \lambda > 0, \quad x \in \overline{\Omega}, \quad i = 1, 2.
\]
It follows that \( w = w_i \) with \( i \in \{1, 2\} \) depending on \( \text{supp}[f] \). From (32) and ii) of Lemma 6 we write
\[
\tilde{w}_1(\lambda, x) = \int_{\Omega} f(y)p(\lambda, x, y)dy = \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t}f(\hat{X}_t)dt \right], \quad t > 0, \quad x \in \overline{\Omega}
\]
which solves
\[
\begin{aligned}
\lambda \tilde{w}_1 - f &= \Delta \tilde{w}_1, \quad \Omega, \\
\eta \lambda \tilde{w}_1 - f &= -\sigma \partial_n \tilde{w}_1 - c\tilde{w}_1, \quad \partial \Omega.
\end{aligned}
\]
It suffices to observe that \( p(\lambda, \cdot, \cdot) \in D(G) \). Since \( f(\partial \Omega) = 0 \) we get the claim.

From (32) and i) of Lemma 6 and the definition of \( w \) we get
\[
\tilde{w}_2(\lambda, x) = \eta(\sigma) \int_{\Omega} f(y) \int_0^\infty e^{-\lambda t}l(t, z)dt \int_0^\infty p(z, x, y)dz m_\partial(dy) = (\eta(\sigma) \lambda^\alpha \int_{\partial \Omega} f(y)p(\lambda, x, y) m_\partial(dy)
\]
\[
= \frac{\lambda^\alpha}{\lambda} \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda^\alpha t}f(X_t)dt \right], \quad t > 0, \quad x \in \overline{\Omega}
\]
\[
= \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t}f(\hat{X}_t)dt \right]
\]
We recognize (25) in (51). In particular, \( p(\lambda^\alpha, \cdot, \cdot) \in D(G) \) solves a FCP with initial datum \( f \in C(\partial \Omega) \). Thus, \( \tilde{w}_2 \) can be associated with the problem
\[
\begin{aligned}
\frac{\lambda^\alpha}{\lambda}(\lambda \tilde{w}_2 - f) &= \Delta \tilde{w}_2, \quad \Omega, \\
\eta \frac{\lambda^\alpha}{\lambda}(\lambda \tilde{w}_2 - f) &= -\sigma \partial_n \tilde{w}_2 - c\tilde{w}_2, \quad \partial \Omega
\end{aligned}
\]
in which \( f(\Omega) = 0 \). Thus, we get the claim. Indeed, from (19) and (20) we write
\[
\mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t}f(X_t)dt \right] = (\eta(\sigma) \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t-\lambda(\eta(\sigma) \gamma t)+(c/\sigma)\gamma t}f(X_t^+)d\gamma \right]
\]
and from Lemma 6 we get
\[
\mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t}f(\hat{X}_t)dt \right] = (\eta(\sigma) \frac{\lambda^\alpha}{\lambda} \mathbb{E}_x \left[ \int_0^\infty e^{-c_1 t-c_2 \gamma t}f(X_t^+)d\gamma \right] = \tilde{w}_2(\lambda, x) \tag{52}
\]
where \( c_1 = \lambda^\alpha \) and \( c_2 = \lambda^\alpha(\eta(\sigma) + c/\sigma) \). According with Lemma 2 \( \tilde{w}_2 \) solves (16) with boundary condition
\[
\partial_n \tilde{w}_2(\lambda, x) + c_2 \tilde{w}_2(\lambda, x) = (\eta(\sigma) \frac{\lambda^\alpha}{\lambda} f(x) \quad x \in \partial \Omega, \quad \lambda > 0,
\]
that is
\[
\partial_n \tilde{w}_2(\lambda, x) + (c/\sigma)\tilde{w}_2(\lambda, x) + (\eta(\sigma) \left( \lambda^\alpha \tilde{w}_2(\lambda, x) - \frac{\lambda^\alpha}{\lambda} f(x) \right) = 0, \quad x \in \partial \Omega, \quad \lambda > 0. \tag{53}
\]
Thus, \( \tilde{w}_2(\lambda, x) \) solves (50).
Now we focus on 
\[ w_{f|\Omega}(t, x) = \int_{\partial\Omega} \hat{p}(t, x, y) f(y) m(dy), \quad t \geq 0, \ x \in \Omega \]
and study its compact representation on \( L^2(\Omega, m) \). Recall that 
\[ \tilde{w}_{f|\Omega}(\lambda, x) = E_x \left[ \int_0^{\infty} e^{-\lambda t} \hat{X}_t \, d\gamma_t \right], \quad \lambda > 0, \ x \in \Omega. \]

**Theorem 3.2.** The following hold true:

i) \( w_{f|\Omega}(t, x) \) is continuous and bounded on \([0, \infty) \times \Omega\);

ii) \( w_{f|\Omega}(t, x) \) has a compact representation \( \forall t \geq 0 \) on \( L^2(\Omega) \) under absolute and uniform convergence;

iii) \( w_{f|\Omega}(t, x) = w_{f|\Omega} \in C((0, \infty); \Omega) \cap \hat{D}_{L} \) is the unique solution to
\[ D^t_{\alpha} w = Gw \text{ in } \Omega \quad \text{with} \quad w_0 = f_{|\partial\Omega}, \quad f \in D(G); \quad (54) \]

iv) \( w_{f|\Omega}(t, x) = E_x[f_{|\partial\Omega}(\hat{X}_t)], \ t \geq 0, \ x \in \Omega. \)

**Proof.** Notice that \( \forall \lambda \geq 0, \ \tilde{w}_{f|\Omega}(\lambda, \cdot) = \tilde{w}_2(\lambda, \cdot) \) which is continuous (and bounded) on \( \overline{\Omega} \) as a consequence of the previous theorem and the point i) of Lemma 2. From Lemma 2 we also conclude that \( e^{-\lambda t}\hat{p}(t, x, y) \) is continuous on \( \overline{\Omega} \times \overline{\Omega} \) for every \( t \geq 0 \). Thus, \( w_{f|\Omega} \) is continuous on \([0, \infty) \times \Omega\). This proves i).

Now we observe that, \( \forall t \geq 0 \), we have the following representation on \( L^2(\Omega, m) \),
\[ w_{f|\Omega}(t, x) = \int_0^{\infty} l(t, s) S_{s\Omega}(f_{|\partial\Omega}|\Omega) \, ds, \quad t \geq 0, \ x \in \Omega \]
where \( l \) has been introduced in Section 2.4. Indeed, the semigroup \( S_{t\Omega} \) on \( C(\overline{\Omega}) \) coincides with a symmetric Markov semigroup on \( L^2(\Omega) \), see [2, Theorem 2.3]. Since the semigroup is compact ([2, Corollary 2.7]), there exists \( \{\mu_k, \psi_k\}_k \) such that (32) has a compact representation.

Let us introduce \( (f, g) = (f, g)_{\Omega} + (f, g)_{\partial\Omega} \) w.r.t. \( m(dy) = 1_{\Omega} dy + (\eta/\sigma) 1_{\partial\Omega} m_\partial(dy) \) and write
\[ S_{t\Omega} f = \sum_k e^{-\tau_k t}(f, \psi_k) \psi_k \quad (55) \]
from which we get
\[ w_{f|\Omega}(t, x) = \sum_k e^{-\mu_k t}(f_{|\partial\Omega}, \psi_k)(x) \quad (56) \]
and
\[ w_{f|\Omega}(t, x) = \sum_k E_{\alpha}(\mu_k t^\alpha)(f_{|\partial\Omega}, \psi_k)(x) \quad (57) \]
on \( L^2(\Omega, m) \) where the last equality follows from [22]. The monotonicity property given in [2, Theorem 3.5] says that
\[ S_t^1 \leq S_t \leq S_t^+ \quad (58) \]
where $S^+_t$ is the (Dirichlet) semigroup for the killed Brownian motion and $S^+_t$ is the semigroup for the reflected sticky Brownian motion (corresponding to the limits of $c \in (0, \infty)$ already mentioned in Section 1.1). Let us introduce $\{\mu^0_k, \psi^0_k\}$ corresponding to $S^+_t$ (that is, as $c \to 0$). From [56, Theorem 3.1] we know that

$$0 = \mu^0_0 < \mu^0_1 \leq \mu^0_2 \leq \cdots \leq \mu^0_k$$

and $\mu^0_k \to \infty$ as $k \to \infty$. With [56] and [57] at hand, by considering [58] and the Parseval identity, we have

$$\|w_{f|\Omega}\|_{L^2(\Omega)} \leq \exp\left(-\mu^0_k t\right) \|f\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}$$

and

$$\|w_{f|\partial\Omega}\|_{L^2(\partial\Omega)} \leq E_\alpha(-\mu^0_k t^\alpha) \|f\|_{L^2(\partial\Omega)} \leq \|f\|_{L^2(\partial\Omega)}.$$

The representation of [56] can be treated by known arguments, we refer to [5]. Then, we focus on [57] for which we recall that $E_\alpha(-z)$ defined in Section 2.4 is completely monotone (a non-increasing function of $z \geq 0$). We also notice that $f \in D(G)$ and $Tf$ is a continuous operator from $H^1(\Omega)$ into $L^2(\partial\Omega)$ for which $\exists C_H > 0$ such that

$$\|f|\partial\Omega\|_{L^2(\partial\Omega)} \leq C_H \|f\|_{H^1(\Omega)}. \tag{59}$$

We exploit the fact that the semigroup $S_t$ maps $L^2(\Omega)$ into $L^\infty(\Omega)$ ([2]).

Let $\mu^* = \min_k \{\mu_k\}$. Observe that

$$\left| \sum_k E_\alpha(-\mu_k t^\alpha) (f|\partial\Omega, \psi_k) \psi_k(x) \right| \leq \sum_k \left| E_\alpha(-\mu_k t^\alpha) \right| \|f|\partial\Omega, \psi_k\|_{L^\infty(\Omega)} \|\psi_k\|_{L^\infty(\Omega)} \leq \frac{c_E}{1 + \mu^* t^\alpha} \sum_k \|f|\partial\Omega, \psi_k\|_{L^\infty(\Omega)} \|\psi_k\|_{L^\infty(\Omega)},$$

where the last step has been obtained from (23). This shows that (57) converges in $L^\infty(\Omega)$ as well as (55). In particular, $\exists \beta$ such that $\|f|\partial\Omega, \psi_k\|_{L^\infty(\Omega)} \leq C k^\beta$ for some $C > 0$ and for which

$$\sum_k \|f|\partial\Omega, \psi_k\|_{L^\infty(\Omega)} \|\psi_k\|_{L^\infty(\Omega)} < \infty.$$ 

We conclude that (57) converges absolutely and uniformly. Now we observe that

$$\int_\Pi \left| \sum_k E_\alpha(-\mu_k t^\alpha) (f|\partial\Omega, \psi_k) \psi_k(x) \right|^2 m(dx) \leq \int_\Pi \left( \left| E_\alpha(-\mu^* t^\alpha) \right|^2 \sum_k \|f|\partial\Omega, \psi_k\|_{L^2(\Omega)}^2 \right) m(dx) \leq \left( \frac{c_E}{1 + \mu^* t^\alpha} \right)^2 \sum_k \|f|\partial\Omega, \psi_k\|_{L^2(\Omega)}^2 \|\psi_k\|_{L^2(\Omega)} \leq (c_E^2) \sum_k \|f|\partial\Omega, \psi_k\|^2$$

where (by the Parseval identity and (59))

$$\sum_k \|f|\partial\Omega, \psi_k\|^2 \leq C_H \|f\|_{H^1(\Omega)}^2.$$
Then, we conclude that \( \text{(57)} \) converges in \( L^2(\Omega) \) uniformly in \( t \geq 0 \). Thus we proved \( ii) \).

The fact that \( \text{(57)} \) solves \( \text{(54)} \) comes from \( \text{(24)} \) and the fact that \( \{\psi_k\}_k \in D(G) \). Thus, the identity

\[
(f|_{\partial \Omega}, \psi_k) \psi_k(x) D_t^\alpha E_\alpha(-\mu_k t^\alpha) = E_\alpha(-\mu_k t^\alpha)(f|_{\partial \Omega}, \psi_k) \Delta \psi_k(x)
\]

holds \( \forall k \). The continuity w.r.t. the initial datum can be obtained by considering that \( E_\alpha(z) \to 1 \) as \( z \to 0 \) \( \forall \alpha \in (0, 1] \) and

\[
\left\| \sum_k E_\alpha(-\mu_k t^\alpha)(f|_{\partial \Omega}, \psi_k) \psi_k(x) - f|_{\partial \Omega} \right\|^2_{L^2(\Omega)} \leq |E_\alpha(-\mu t^\alpha) - 1|^2 \|f|_{\partial \Omega}\|^2_{L^2(\partial \Omega)}.
\]

We move on the analysis about the space \( \hat{D}_L \). According with \( \text{(33)} \) and \( \text{(34)} \), we have that

\[
\frac{\partial}{\partial t} \sum_k E_\alpha(-\mu_k t^\alpha)(f|_{\partial \Omega}, \psi_k) \psi_k \leq Ct^{\alpha-1} \left| \sum_k (f|_{\partial \Omega}, \psi_k) \mu_k \psi_k \right| = Ct^{\alpha-1} |Gf|_{\partial \Omega}|
\]

where \( G = \Delta \) generates the semigroup \( S_t \). Since \( f, \Delta f \in C(\overline{\Omega}) \) we obtain that \( \text{(57)} \) is in \( \hat{D}_L \).

Uniqueness follows from energy method. Consider two solutions \( \varpi_1, \varpi_2 \) with initial datum \( f \). Then, \( \varpi_1, \varpi_2 \) have the representation \( \text{(57)} \) and \( T\varpi_1 = T\varpi_2 \). The function \( \varpi^* := \varpi_1 - \varpi_2 \) solves \( \text{(54)} \) with zero initial datum. Moreover, \( \varpi^* \in H^1_0(\Omega) \) and

\[
D^\alpha_t \|\varpi^*(t, \cdot)\|^2_{L^2(\Omega)} = \int_{\Omega} 2\varpi^* G \varpi^* m(dx) = \int_{\Omega} 2\varpi^* G \varpi^* dx = \int_{\Omega} |\nabla \varpi^*|^2 dx \leq 0.
\]

Since \( D^\alpha_t \) is a convolution operator involving a positive kernel, then for the energy \( E(t) = \|\varpi^*(t, \cdot)\|^2_{L^2(\Omega)} \) we have \( \frac{d}{dt} E(t) \leq 0 \) and therefore \( E(t) \) is a decreasing function with \( E(t) \leq E(0) = 0 \). We conclude that \( \varpi^* \) is m.a.e. zero and \( T\varpi^* = 0 \) for all \( t \geq 0 \) and almost all \( x \in \partial \Omega \).

Since \( \varpi^* \) is continuous, then \( \varpi_1 = \varpi_2 \) on \( [0, \infty) \times \overline{\Omega} \). This proves \( iii) \).

The point \( iv) \) follows from \( \text{(32)} \).

This concludes the proof. \( \square \)

We provide some comment based on the previous results and conclude the discussion about \( \hat{X} \). Recall formulas \( \text{(34)} \) and \( \text{(33)} \). We can write

\[
\hat{R}_\lambda f(x) = E_x \left[ \int_0^\infty e^{-\lambda t} f(\hat{\Gamma}_t + \hat{\gamma}_t) d(\hat{\Gamma}_t + \hat{\gamma}_t) \right], \quad \lambda > 0, \quad x \in \overline{\Omega}, \quad f \in C(\overline{\Omega})
\]

and \( w = w|_{\Omega} + w|_{\partial \Omega} \) for which we have that \( w|_{\Omega} : [0, \infty) \times \overline{\Omega} \to \mathbb{R} \) solves

\[
\begin{cases}
\frac{\partial}{\partial t} w = \Delta w, \quad (0, \infty) \times \Omega \\
\eta \frac{\partial}{\partial t} w = -\sigma \partial_n w - cw, \quad (0, \infty) \times \partial \Omega, \\
w(0, x) = f_1(x), \quad x \in \overline{\Omega}
\end{cases}
\]
whereas \( w_{f|\Omega} : [0, \infty) \times \overline{\Omega} \to \mathbb{R} \) solves

\[
\begin{cases}
D_t^\alpha w = \Delta w, & (0, \infty) \times \Omega, \\
\eta T D_t^\alpha w = -\sigma \partial_n w - cw, & (0, \infty) \times \partial\Omega, \\
w(0, x) = f_2(x), & x \in \overline{\Omega}.
\end{cases}
\]  
(61)

By following the same arguments as in [5], the conditions above take respectively the forms

\[
\eta \frac{\partial}{\partial t} T w_{f|\Omega} = -\sigma \partial_n w_{f|\Omega} - cw_{f|\Omega} \quad \text{on } \partial\Omega
\]

and

\[
\eta D_t^\alpha T w_{f|\Omega} = -\sigma \partial_n w_{f|\Omega} - cw_{f|\Omega} \quad \text{on } \partial\Omega.
\]

We recall that \((G, D(G))\) generates a semigroup on \(L^2(\overline{\Omega}, m)\) and the functions \(w_{f|\Omega}, w_{f|\Omega}\) are well-defined. The initial data \(f_1, f_2\) are not in \(D(G)\) but we still have a classical solution of the Cauchy problem \((60)\) equivalent to \((2)\) and the fractional Cauchy problem \((61)\) equivalent to \((3)\). Indeed, \(D(G) \subset L^2(\overline{\Omega})\) and \(S_t\) is associated with a maximal monotone (it a contraction of \(C_0\)-semigroup) and self-adjoint \((36)\) operator. Thus, \(f_1, f_2 \in L^2(\overline{\Omega})\) ensures existence and uniqueness of both solutions ([13, Theorem VII.7]). Moreover, in both cases we have a compact representation written in terms of the semigroup \(S_t\) of \(X\).

We state now our main result on the probabilistic representation of \(u\) with compact representation on \(L^2(\overline{\Omega}, m)\).

**Theorem 3.3.** For the solution \(u \in C((0, \infty), \overline{\Omega}) \cap \bar{D}_L\) to the FBVP \((8)\) with \(f \in C(\overline{\Omega})\), the following representation holds true

\[
u(t, x) = \mathbb{E}_x[f(\tilde{X}_t)], \quad t > 0, \ x \in \overline{\Omega}
\]

where the process \(\tilde{X} = \{\tilde{X}_t\}_{t \geq 0}\) can be obtained via time change of an elastic Brownian motion. In particular,

\[
\mathbb{E}_x[f(\tilde{X}_t)] = \mathbb{E}_x[f(X^+ \circ \tilde{V}_t^{-1}) \exp \left(-\frac{c}{\sigma} \gamma^+ \circ \tilde{V}_t^{-1}\right)], \quad t > 0, \ x \in \overline{\Omega}
\]

(62)

where \(\tilde{V}_t^{-1}\) is the inverse to \(\tilde{V}_t = t + H \circ (\eta/\sigma)\gamma^+_t\).

**Proof.** Our time-change via \(\tilde{V}_t\) is obtained by considering the elastic Brownian motion and the couple \((X^+, \gamma^+)\).

We observe that

\[
\mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} f(\tilde{X}_t) dt \right] = \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t - (c/\sigma)\gamma^+_t \circ \tilde{V}_t^{-1}} f(X^+ \circ \tilde{V}_t^{-1}) dt \right]
\]

\[
= \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda \tilde{V}_t - (c/\sigma)\gamma^+_t} f(X^+_t) d\tilde{V}_t \right]
\]

\[
= \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t - \lambda H_0(\eta/\sigma)\gamma^+_t - (c/\sigma)\gamma^+_t} f(X^+_t) d(t + H \circ (\eta/\sigma)\gamma^+_t) \right]
\]

\[
= I_1(x) + I_2(x)
\]

where

\[
I_1(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t - \lambda H_0(\eta/\sigma)\gamma^+_t - (c/\sigma)\gamma^+_t} f(X^+_t) dt \right]
\]

\[
I_2(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} f(X^+_t) dt \right]
\]

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and $I_2$ can be obtained as

$$
\mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t - \lambda \alpha(\eta/\sigma)\gamma_t^+ - (c/\sigma)\gamma_t^+} f(X_t^+) dt \right]
$$

for the multiplicative functional $M_t^\alpha = \exp(-\lambda H \circ (\eta/\sigma)\gamma_t^+)$ such that, from the independence of $H$,

$$
\mathbb{E}_x[M_t^\alpha|X_t^+] = \exp(-\lambda^\alpha(\eta/\sigma)\gamma_t^+).
$$

In particular,

$$
I_2(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t - \lambda \alpha(\eta/\sigma)\gamma_t^+ - (c/\sigma)\gamma_t^+} f(X_t^+) dt \right]
$$

as a new clock for the elastic Brownian motion written in terms of $(X^+, \gamma^+)$ and write

$$
\bar{R}_\lambda f(x) = I_1(x) + I_2(x), \quad \lambda > 0
$$

as follows

$$
\bar{R}_\lambda f(x) = \mathbb{E}_x \left[ \int_0^\infty f(X_t^+) \exp \left( -\lambda T_{\lambda,t} - (c/\sigma) \gamma_t^+ \right) dT_{\lambda,t} \right]
$$

$$
= \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} f(X_t^+) \exp \left( -(c/\sigma) \gamma_t^+ \circ T_{\lambda,t}^{-1} \right) dt \right], \quad \lambda > 0.
$$

As we can immediately check,

$$
T_{\lambda,t} = \int_{\Pi} \gamma_{t,z}^+ m_\lambda(dz) \quad \text{with} \quad m_\lambda(dz) = dz + \frac{\lambda^\alpha}{\lambda}(\eta/\sigma)m_\theta(dz)
$$

where the local time $\gamma_{t,z}^+$ has been introduced in Section 2.3.

We also have that $\bar{R}_\lambda f$ belongs to the domain of the generator of an elastic sticky Brownian motion obtained by considering $T_{\lambda,t}$ with $t > 0, \lambda > 0$. This implies that

$$
\Delta \bar{R}_\lambda f = \lambda \bar{R}_\lambda f - f, \quad \text{in } \Omega
$$

with

$$
\bar{R}_\lambda f, \Delta \bar{R}_\lambda f \in C(\overline{\Omega})
$$
and (the sticky boundary condition introduced by $T_{\lambda,t}$), for $\eta, \sigma, c, \lambda > 0$,

$$\frac{\lambda^\alpha}{\chi} (\eta/\sigma) \Delta \bar{R}_\lambda f = -\partial_n \bar{R}_\lambda f - (c/\sigma) \bar{R}_\lambda f, \quad \text{on } \partial \Omega. \quad (68)$$

From (67) and (68) we can write

$$\frac{\lambda^\alpha}{\chi} (\eta/\sigma) \Delta \bar{R}_\lambda f = \frac{\lambda^\alpha}{\chi} (\eta/\sigma) (\lambda \bar{R}_\lambda f - f), \quad \text{on } \partial \Omega$$

and therefore

$$(\eta/\sigma) \frac{\lambda^\alpha}{\chi} (\lambda \bar{R}_\lambda f - f) = -\partial_n \bar{R}_\lambda f - (c/\sigma) \bar{R}_\lambda f, \quad \text{on } \partial \Omega. \quad (69)$$

Recall (45). For $u \in D_L$, $D_L^\alpha Tu$ is well-defined, and from (69) we conclude that (69) leads to

$$(\eta/\sigma) D^\alpha T u(t, x) = -\partial_n u(t, x) - (c/\sigma) u(t, x), \quad t > 0, \quad x \in \partial \Omega \quad (70)$$

as claimed. Condition (70) with $Tu_0 = Tu$ given by $Tf$ on $\partial \Omega$ gives a unique solution for the heat equation. In particular, the couple $(u, Tu)$ is the unique solution (by uniqueness of the Laplace inverse under continuity for $u$ on $\overline{\Omega}$) of (8). From (65) with given positive constants $\eta, \sigma, c, \lambda$, we have the resolvent for a Feller-Wentzell semigroup. The semigroup is obtained via probabilistic techniques by replacing $V_t$ with (63) in Section 2.3. Then, uniqueness follows.

Focus on (66). We observe that

$$m_\lambda(dz) = \int_0^\infty e^{-\lambda t} (\delta_0(t)dz + \Pi(t) (\eta/\sigma) m_\theta(dz)) dt, \quad \lambda > 0$$

where $\Pi(t) := \Pi(t, \infty) = \int_t^\infty y^{-\alpha} dy$, $t > 0$ is the tail of the Lévy measure of $H$. The subordinator $H$ has no role as $X^+ \notin \partial \Omega$. This suggests that the positive continuous functional

$$\bar{A}_t = (\eta/\sigma) \int_0^t \Pi(t-s) d\gamma^+_s$$

can be considered in order to represent $\bar{X}$ via time change. This discussion will be given in a separate work.

Focus on the path integral

$$E_x \left[ \int_0^\infty e^{-\lambda t} f(\bar{X}_t) d(\bar{V}_t - t) \right] = \frac{\lambda^\alpha}{\chi} E_x \left[ \int_0^\infty e^{-\lambda t} f(\bar{X}_t) d\gamma^+_t \right]$$

$$= (\eta/\sigma) \frac{\lambda^\alpha}{\chi} E_x \left[ \int_0^\infty e^{-\lambda t} \gamma^+_t f(\bar{X}_t) d\gamma^+_t \right]$$

which has been previously denoted by $I_2$ with $\bar{V}_t - t = H \circ (\eta/\sigma) \gamma^+_t$. We discuss below in Lemma 3.5 the process $\bar{V}_t - t$ in terms of holding times. Moreover, with

$$c_1 = \lambda, \quad c_2 = \lambda^\alpha (\eta/\sigma) + (c/\sigma)$$

we can apply Lemma 2 where $(\eta/\sigma) \lambda^\alpha / \lambda f$ must be considered in place of $f$.

**Theorem 3.4.** \(\forall t > 0,\) there exists a continuous kernel $\bar{p}(t, \cdot, \cdot) : \overline{\Omega} \times \overline{\Omega} \rightarrow \mathbb{R}$ such that

$$E_x[f(\bar{X}_t)] = \int_{\overline{\Omega}} f(y) \bar{p}(t, x, y) m(dy), \quad f \in L^2(\overline{\Omega}, m).$$

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Proof. For \( \eta, \sigma, c, \lambda > 0 \), let us consider \( G_\lambda = \Delta \) with
\[
D(G_\lambda) = \left\{ \varphi, \Delta \varphi \in C(\bar{\Omega}), \varphi \in H^1(\Omega) : \eta \frac{\lambda^\alpha}{\lambda}(\Delta \varphi)|_{\partial \Omega} = -\sigma \partial_n \varphi - c \varphi|_{\partial \Omega} \right\}.
\]
Then \( (G_\lambda, D(G_\lambda)) \) generates an elastic sticky Brownian motion, say \( X_\lambda = \{X_{\lambda,t}\}_{t \geq 0} \) which is a strong Markov process on \( \Omega \) with \( C_0 \)-seminigroup on \( C(\bar{\Omega}) \), say \( S_{\lambda,t} \), with probabilistic representation
\[
S_{\lambda,t}f(x) = \mathbb{E}_x[f(X_{\lambda,t})] = \mathbb{E}_x\left[f(X^+ \circ \bar{V}_{\lambda,t}^{-1}) \exp\left(-\frac{c}{\sigma} \gamma^+ \circ \bar{V}_{\lambda,t}^{-1}\right)\right]
\]
obtained via time change by considering \( \bar{V}_{\lambda,t} = T_{\lambda,t} \) defined as in the previous proof. As in [65] we write
\[
\tilde{R}_\lambda f(x) := \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} f(X_t) dt \right] = \mathbb{E}_x \left[ \int_0^\infty e^{-\lambda t} f(X_{\lambda,t}) dt \right] \quad (71)
\]
For the \( C_0 \)-seminigroup \( S_{\lambda,t} \) on \( C(\bar{\Omega}) \) we know that \( S_{\lambda,t} L^2(\Omega) \subset C(\bar{\Omega}) \) and \( S_{\lambda,t} : L^2(\Omega) \to C(\bar{\Omega}) \) is compact. In particular, there exists a continuous kernel \( \varphi_\lambda \) on \( \bar{\Omega} \) such that
\[
\int_0^\infty e^{-\lambda t} S_{\lambda,t}f(x) dt = \int_{\Omega} f(y) \varphi_\lambda(x, y) m(y). \quad (72)
\]
From (71) and (72), we get that \( \tilde{R}_\lambda, \varphi_\lambda \in D(G_\lambda) \) and
\[
\tilde{R}_\lambda f(x) = \int_{\Omega} f(y) \varphi_\lambda(x, y) m(y)
\]
is continuous and bounded. That is, \( \varphi_\lambda \) is a continuous kernel on \( \bar{\Omega} \) for \( \tilde{R}_\lambda \). The claim follows. \( \Box \)

We conclude this section with the following result.

**Theorem 3.5.** Let \( \{\hat{e}_i\}_i \) be the sequence of holding times for \( \hat{X} \). Then \( \{\hat{e}_i\}_i \) are independent and identically distributed. Moreover, \( \hat{e}_0 \) has the same law of \( \hat{e}_0 \).

**Proof.** The process \( \hat{X} \) is obtained via time change from the process \( X \) for which, \( \forall \ i, \)
\[
\mathbb{P}_x(\hat{e}_i > t, X_{e_i} \in \Lambda) = \mathbb{P}_x(\hat{e}_i > t) \mathbb{P}_x(X_{e_i} \in \Lambda) = e^{-(\sigma/\eta)t} \mathbb{P}_x(X_{e_0} \in \Lambda), \quad x \in \partial \Omega, \ \Lambda \subset \Omega.
\]
Since \( \hat{e}_0 = H \circ e_0 \) as shown in Lemma [4] and \( H \) is independent from \( X \), then \( \{\hat{e}_i = H \circ e_i\}_i \) is a sequence of independent and identically distributed random variables. Indeed, by exploiting the Markov property of \( H \) and the independence of the increments, we get that
\[
(H_{e_0} - H_0) = H_{e_0} \perp (H_{e_1+e_0} - H_{e_0}) = H_{e_1}
\]
and therefore, \( H \circ e_i, \ i \in \mathbb{N} \) are independent random variables. The process \( \hat{X} \) moves on the path of \( X \) and therefore, by construction \( \mathbb{P}(\hat{X}_{e_0} = X_{e_0}) = 1 \) and
\[
\mathbb{P}_x(\hat{e}_0 > t, \hat{X}_{\hat{e}_0} \in \Lambda) = \mathbb{P}_x(e_0 > L_t, X_{e_0} \in \Lambda) = \mathbb{P}_x(e_0 > L_t, X_{e_0} \in \Lambda).
\]
Indeed, \( e_0 \) is an holding time for \( X \). We get (recall that \( H \perp X \) implies \( L \perp X \))
\[
\mathbb{P}_x(\hat{e}_0 > t, \hat{X}_{\hat{e}_0} \in \Lambda) = E_x(\exp(\frac{c}{\sigma} t_0) \mathbb{P}_x(X_{e_0} \in \Lambda)
\]
where the Mittag-Leffler has been introduced in [22]. Thus, \( \hat{e}_i \) are independent Mittag-Leffler random variables.
The process $\bar{X}$ moves on the path of $X^+$ according with the new clock $\bar{V}^{-1}$ which governs the time the process $\bar{X}$ waits (until a continuous reflection) on the boundary with each visit, that is the holding times for $\bar{X}$. Notice that $X^+$ may have only instantaneous reflections up to the elastic kill. Focus on $V$ and $\bar{V}$. By definition, $V_i$ and $\bar{V}_i$ equal $t$ in case $\eta = 0$, there are no holding times and we have instantaneous reflection as the path of $X^+$ entails. For $\eta > 0$, the extra-time $(\eta/\sigma)\gamma_i^+$ the process $X^+$ spends on the boundary is considered in the construction of $X$. Similarly, $H \circ (\eta/\sigma)\gamma_i^+$ gives the extra-time the process $X^+$ spends on the boundary in order to have $\bar{X}$. Since $H$ is independent from $X^+$ we may argue that $\{\bar{e}_i\}_i$ is a sequence of independent and identically distributed holding times. In particular, the special case $H_t = t$ exactly gives $\bar{e}_i = e_i$ for every $i$ where $e_i$ denotes the holding time for $\alpha = 1$, that is the case of $X$. Assume the process starts from the boundary, for $x \in \partial \Omega$, 

\[
P_x(\bar{e}_0 > t|\bar{X}_{\bar{e}_0} \in \Lambda), \quad \Lambda \in \Omega
\]

can be written by considering that, for $\alpha = 1$,

\[
e^{-\langle \sigma/\eta \rangle t} = P_x(e_0 > t) = P_x(e_0 > t|X_{e_0} \in \Lambda), \quad \Lambda \subset \Omega
\]

can be treated as the case $\alpha \in (0,1)$ from the independence of $H$. The variable $e_0$ is an exponential r.v. with parameter $(\sigma/\eta)$ and $\{e_i, i = 0, 1, \ldots\}$ are i.i.d., this implies that for every $\alpha \in (0,1)$, $\{\bar{e}_i\}_i$ are i.i.d. as well. By construction we have that $P(\bar{e}_i \geq e_i) = 1 \forall i$ and $\bar{e}_0$ can be obtained from $e_0$ by observing that $\bar{V}_i = t + H \circ (\eta/\sigma)\gamma_i^+$. Moreover, $\bar{V}_i$ may have jumps only from the independent process $H$. In particular, we consider $\bar{V}_i - t$ and follow here the same arguments as in the proof of Theorem 4.3 in [25]. Thus, $\bar{e}_0 = H \circ e_0$ does not depend on $(\bar{X}_{\bar{e}_0} \in \Lambda)$ for every $\Lambda \subset \Omega$ and

\[
P(\bar{e}_0 > t|\bar{X}_{\bar{e}_0} \in \Lambda) = P(H \circ e_0 > t) = P(e_0 > L_t) = E[\exp(-\langle \sigma/\eta \rangle t)] = E_\alpha(-\langle \sigma/\eta \rangle t^\alpha), \quad (73)
\]

that is, $\bar{e}_0$ is a Mittag-Leffler r.v. as well as $\bar{e}_i \forall i$.

We state the following result on the slow reflections of $X$ which are independently delayed.

**Lemma 7.** The following statements hold true:

i) $\bar{X}$ may only have instantaneous reflections iff $\eta = 0$;

ii) $\bar{X}$ may only have delayed (slow) reflections iff $\eta > 0$.

**Proof.** First we follow the first part of the proof of Lemma 5 by considering the holding time $\bar{e}_0$ with

\[
P(\bar{e}_0 > t) = E_\alpha(-\langle \sigma/\eta \rangle t^\alpha)
\]

and the associated reflection. Thus, instantaneous reflection implies $\eta = 0$ whereas we may have delayed slow reflection only as $\eta > 0$.

For $\eta = 0$, $\bar{V}_i = t$ and $\bar{X}$ is an elastic Brownian motion which implies instantaneous reflections. Focus now on $\eta > 0$. Recall that $S_{\lambda,t}$ defined in the previous proof, generated by $G_\lambda = \Delta$ with

\[
D(G_\lambda) = \left\{ \varphi, \Delta \varphi \in C(\bar{\Omega}), \varphi \in H^1(\Omega) : \eta \frac{\lambda^\alpha}{\lambda} (\Delta \varphi)_{|\partial \Omega} = -\sigma \partial_n \varphi - c \varphi_{|\partial \Omega} \right\}.
\]

is the semigroup of the elastic sticky Brownian motion with speed measure

\[
m_\lambda(dx) = dx + \frac{\lambda^\alpha}{\lambda} (\eta/\sigma) m_\sigma(dx).
\]

Thus, for $\alpha = 1$, the process may only have slow reflection (see Lemma 3) whereas, for $\alpha \in (0,1)$ the exponential holding times are delayed even more by $L$. Indeed, form Theorem 3.3 $\bar{e}_0$ equals in law $\bar{e}_i \forall i$. 

\[\square\]
Remark 3.1. (Occupation measures) For \( \Lambda', \Lambda \in \partial \Omega \), consider the occupation measure

\[
\mu_{\Lambda'(x, \Lambda')} = E_x \left[ \int_0^{\tau(\Lambda')} 1_{\Lambda'}(\bar{X}_s) ds \right], \quad x \in \Lambda, \quad \Lambda' \subset \Lambda
\]

where \( \tau(\Lambda) = \inf \{ t : \bar{X}_t \notin \Lambda \} \) and

\[
\mu_{\Lambda}(x, \Lambda) \leq E_x[\tau(\Lambda)] := E_x \left[ \int_0^{\infty} 1_{\Lambda}(\bar{X}_s) ds \right], \quad x \in \Lambda.
\]

Here we only underline the interesting role of the “trapping” boundary for \( X^+ \) under the time change \( \bar{V} \). Thus, we focus on the fact that

\[
E_x[\tau(\Lambda)] = \lim_{\lambda \to 0} \bar{R}_\lambda 1_{\Lambda}(x), \quad x \in \Lambda.
\]

First we observe that

\[
\bar{R}_\lambda 1_{\Lambda}(x) = \bar{R}_\lambda 1_{\Lambda \cap \Omega}(x) + \bar{R}_\lambda 1_{\Lambda \cap \partial \Omega}(x)
\]

where

\[
\bar{R}_\lambda 1_{\Lambda \cap \Omega}(x) = E_x \left[ \int_0^{\infty} e^{-\lambda t} 1 - (\eta/\sigma) \gamma^+_{\Lambda}(X^+_t) ds \right]
\]

and

\[
\bar{R}_\lambda 1_{\Lambda \cap \partial \Omega}(x) = (\eta/\sigma) \Lambda^\alpha \bar{g}_2(x)
\]

can be respectively obtained from \( I_1 \) and \( I_2 \) in \( \lambda \). We have that

\[
\bar{R}_\lambda 1_{\Lambda \cap \Omega}(x) = \bar{g}_1(x), \quad \bar{R}_\lambda 1_{\Lambda \cap \partial \Omega}(x) = (\eta/\sigma) \Lambda^\alpha \bar{g}_2(x)
\]

where \( \bar{g}_1, \bar{g}_2 \) are continuous and bounded (see for example Lemma 3.1 for \( \bar{g}_2 \)). As \( \lambda \to 0 \), we get that, for \( \alpha \in (0, 1], \forall x \in \Lambda,
\]

\[
\bar{R}_0 1_{\Lambda \cap \Omega}(x) = E_x[\tau(\Lambda \cap \Omega)] < \infty, \forall \alpha \in (0, 1]
\]

and

\[
\bar{R}_0 1_{\Lambda \cap \partial \Omega}(x) = E_x[\tau(\Lambda \cap \partial \Omega)] \begin{cases} < \infty, & \alpha = 1, \\ = \infty, & \alpha \in (0, 1). \end{cases}
\]

Thus, for \( \alpha \in (0, 1) \), only in case \( \Lambda \cap \partial \Omega = \emptyset \) we have \( E_x[\tau(\Lambda)] < \infty, \forall x \in \Lambda \). This suggests a trap effect on the boundary for the Brownian motion.

Remark 3.2. For the processes \( X \) or \( \bar{X} \) started from \( x \in \partial \Omega \) we may respectively consider, for \( \Lambda \in \Omega, \)

\[
P_x(e_0 > t|X_{e_0} \in \Lambda) = P(\text{no excursions of } X \text{ on } \Omega \text{ until time } e_0) = P(N_t = 0) \quad (74)
\]

and

\[
P_x(\bar{e}_0 > t|\bar{X}_{\bar{e}_0} \in \Lambda) = P(\text{no excursions of } \bar{X} \text{ on } \Omega \text{ until time } \bar{e}_0) = P(\bar{N}_t = 0) \quad (75)
\]

where \( N_t \) is a Poisson process and \( \bar{N}_t = N \circ L_t \) is the fractional Poisson process with \( N \) independent from \( L \). In particular,

\[
P(N_t = k) = e^{-(t/\sigma)} (t/\sigma)^k / k!, \quad k \in \mathbb{N}_0
\]
and
\[ P(\tilde{N}_t = k) = \int_0^\infty P(N_s = k) l(t, s) ds = P(N \circ L_t = k), \quad k \in \mathbb{N}_0. \]

We say that "no Poissonian events" coincides with "no excursions on the interior". With \[ \text{at hand}, \text{we have that} \]
\[ P(\tilde{N}_t = 0) = \int_0^\infty P(N_s = 0) l(t, s) ds = \int_0^\infty e^{-\eta/s} t l(t, s) ds = E_\alpha(-\eta/s) t^\alpha \]
gives the time the process spends on the boundary until the next excursion on the interior \( \Omega \).

### 3.2 On the boundary processes

The following discussion is obtained as a by-product of the previous results. We therefore proceed
without a detailed proof. Recall that we deal only with static behaviour on the boundary (that
is we have no diffusive part). The boundary trace processes we are interested in are expected to
be pure jump processes written as
\[ X \circ \gamma_t^{-1} \quad t \geq 0 \quad \text{and} \quad \tilde{X} \circ \tilde{\gamma}_t^{-1} \quad t \geq 0. \]

Let us consider now the following processes on the boundary. Assume \( \tau_0^+ = 0 \) and define
\[ \tau_i^+ = \inf\{ t > \tau_{i-1}^+ : X_t^+ \notin \Omega \}, \quad i \geq 1. \]

Along the path of \( X^+ \), we have that
\[ X_t^0 := X^+ \circ \int_0^t (\tau_N^+ - \tau_N^-) dN_s = X^+ \circ \sum_{k=1}^{N_t} (\tau_k^+ - \tau_{k-1}^+) = X^+ \circ \tau_N^+ \]
and
\[ \tilde{X}_t^0 := X^+ \circ \int_0^t (\tau_{\tilde{N}}^+ - \tau_{\tilde{N}}^-) d\tilde{N}_s = X^+ \circ \sum_{k=1}^{\tilde{N}_t} (\tau_k^+ - \tau_{k-1}^-) = X^+ \circ \tau_{\tilde{N}}^+ \]

The inter-times for the jumps of \( X_t^0 \) and \( \tilde{X}_t^0 \) are given by the associated holding times, that is
we respectively have exponential and Mittag-Leffler inter-times.

By following the same arguments for the process \( \tilde{X} \) (and by Theorem 3.5) we are lead to the
following conjecture:
\[ E_{x}[f(\tilde{X}_t^0)] = E_{x}[f(\tilde{X}_t^0)], \quad f \in C(\partial \Omega), \quad t \geq 0, \quad x \in \partial \Omega \]
which must be understood in some sense. This should allow an equivalence between boundary
traces of \( w \) and \( u \) given above. We postpone this discussion to a separate work.

The boundary trace process has generator given by the Dirichlet-Neumann operator which
can be defined as the normal derivative of the solution to a Dirichlet problem (we refer to
\[ \text{37} \] for a general discussion and the probabilistic framework). Indeed, it is well-known that
the Dirichlet-Neumann (or the Dirichlet-to-Neumann) operator, say \( A_{DN} \), maps \( U \in D(A_{DN}) \)
to \( \partial_u U \in L^2(\partial \Omega) \) where \( U \in H^1(\Omega) \) is the harmonic function with \( T U = U \). In general,
this operator is non-local and the associated motion is right-continuous with left limits. This
operator turns out to be associated with the reflected Brownian motion \( X^+ \) (and therefore with instantaneous reflections).

It is clear that \( X_t^0 \) is a Markov process. Assume that \( A_{DN}^* \) is the generator of the boundary
process \( X_t^0 \). Then, \( E_x[f(X_t^0)] \) solves the problem
\[ \frac{\partial \varphi}{\partial t} = A_{DN}^* \varphi, \quad \varphi_0 = f \in D(A_{DN}^*). \]
Since $N_t$ can be obtained via time change as $N_t \circ L_t$ where $L$ is an inverse to a stable subordinator, then standard arguments says that $E_x[f(\tilde{X}_t^\theta)]$ solves the problem

$$D_t^\alpha \tilde{\varphi} = A^*_DN \tilde{\varphi}, \quad \tilde{\varphi}_0 = f \in D(A^*_DN).$$

Thus, the problem is of the type discussed in Section 2.5. Moreover, we notice that also the boundary process of $\hat{X}$ can be associated with FCPs. Indeed, $\hat{X}$ has a representation along the path of $X^+$ and i.i.d. holding times given by the Mittag-Leffler r.v. as stated in Lemma 3.5.

4 Conclusions and open problems

4.1 Conclusions

The problem (8) in the 1-dimensional case has been recently considered in [10] in order to describe motions on star graphs. This result can be also considered in traffic models or motions on metric graphs. The motion slows down in a vertex (node) according with an independent holding time. Here we propose results on smooth domains which is not restrictive for our purposes. Extensions to Lipschitz or irregular domains will be considered. However, our feeling is that the present paper will serve as a reference for the probabilistic description of the model.

The different processes $\hat{X}$ and $\bar{X}$ may be considered as two processes leading to the same behaviour on the boundary only in terms of holding times. The sticky condition leading to (37) implies a finite mean holding time whereas, for $\alpha \in (0, 1)$, the fractional sticky analogue leading to (44) implies an infinite mean holding time. We recall that (44) becomes (37) as $\alpha \to 1$ and for $\alpha \in (0, 1]$, we have

$$P(\hat{e}_i > t|\hat{X}_{\hat{e}_i}) = P(\bar{e}_i > t|\bar{X}_{\bar{e}_i})$$

which converges to

$$\begin{cases} 0 & \text{as } \eta \to 0, \text{ instantaneous reflection with each visit, } t < \hat{\zeta} \\ 1 & \text{as } \sigma \to 0, \text{ instantaneous absorption at the first visit, } t < \hat{\zeta} \end{cases}$$

according with the lifetimes $\hat{\zeta}$ and $\bar{\zeta}$ with the elastic coefficient $c \geq 0$ for which $P(t < \hat{\zeta})$ and $P(t < \bar{\zeta})$ have the limit

$$\begin{cases} 1 & \text{as } c \to 0, \text{ there is no elastic kill, a.s. } \hat{\zeta} = \infty \text{ and } \bar{\zeta} = \infty, \\ 0 & \text{as } c \to \infty, \text{ there is no reflection, a.s. } \hat{\zeta} = \tau_{\partial\Omega} \text{ and } \bar{\zeta} = \tau_{\partial\Omega}. \end{cases}$$

The holding times $\{\hat{e}_i\}_i$ and $\{\bar{e}_i\}_i$ are independent and identically distributed Mittag-Leffler random variables as entailed in Lemma 3.5. This implies that $E[\hat{e}_i]$ and $E[\bar{e}_i]$ are finite only for $\alpha = 1$ (the case of $X$ with generator $(G, D(G))$). These holding times can be taken as general as we need by considering the operator $D_t^\Phi \rho$ in place of $D_t^\alpha \rho$ where, according with (45),

$$\int_0^\infty e^{-\lambda D_t^\Phi \rho(t, x)} dt = \frac{\Phi(\lambda)}{\lambda} \left(\lambda \rho(\lambda, x) - \rho(0, x)\right), \quad \lambda > 0$$

and $H$ is such that $E_0[\exp(-\lambda H_t)] = \exp(-t\Phi(\lambda))$. As discussed in [15], we may therefore include holding times for $X$ and $\bar{X}$ characterized by the limit of $\Phi(\lambda)/\lambda$ as $\lambda \to 0$.

Many phenomena can exhibit the same behavior of the processes $\hat{X}$ and $\bar{X}$ near the boundary, that is as an endogenous property of the motion. An example is given in [11] where the authors deal with a model of sticky expectations in which investors update their beliefs too slowly. Our results in this regards give a control for the slow update depending on $\alpha \in (0, 1)$. A further example is given by the bank interest rates (or the Vasicek model for instance) which are termed...
Figure 1: The Koch domain (pre-fractal, first step) on the left and the modified Koch domain (pre-fractal, second step) introduced in [12] on the right. In the second picture the passage between triangles is blocked by a wall with a small opening. The conjecture in OP2 says that $\alpha \in (0, 1)$ for $\bar{X}$ on a smooth domain can be associated with the size of the openings in the second ”bad” domain (modified Koch). The approximation $\Omega^s$ can not be given by the pre-fractals (first picture) which are Lipschitz domains. However, we may introduce a smooth approximation $\Omega^s$ for the modified Koch domain (second picture) in order to relate $\bar{X}$ on $\Omega^s$ with a reflecting Brownian motion on the modified Koch domain (which is trap).

Figure 2: Koch domain (pre-fractal) with inward curves and outward curves. The construction is given by considering a square with Koch curves (with random scale factors) in each side. The same description in Figure 1 can be given here.

sticky if they react slowly to changes in the corresponding market rates or in the policy rate (31).

Moreover, an intermediate phenomenon between microscopic and macroscopic scales is the synthetic of biological adhesion of colloidal particles to one another or to other surfaces, typically immersed in a fluid medium. Sticky diffusions may arise when modeling systems of mesoscale particles (those with diameters around 1 micrometer) which form the building blocks for many common materials (33).

4.2 Open problem

We discuss briefly the following open problem which could be, in our view, a possible interesting extension and application.

Let $\Omega$ be an irregular domain, that is a domain with (possibly random) irregular boundary (see for example [16]). We recall the definition of trap domain given in [12]. Let us consider the
reflected Brownian motion $Z = \{ Z_t \}_{t \geq 0}$ on $\overline{\Omega}$. The domain $\Omega$ is a trap domain for $Z$ if
\[
\sup_{x \in \Omega^*} E_x[\tau_{\partial B_\rho}] = \infty
\] (78)
where $\tau_{\partial B_\rho}$ is the lifetime of $Z$ on $\Omega^*$ with $\Omega^* = \overline{\Omega} \setminus B_\rho$ where $B_\rho$ is a ball (with radius $\rho > 0$) with Dirichlet boundary. The process $Z$ on $\Omega^*$ reflects on $\partial \Omega$ and dies on $\partial B_\rho$. Thus, if the process spends an infinite mean amount of time in $\Omega^*$, this is due to the irregular boundary $\partial \Omega$.

We can provide some examples of trap domains. Figure 1 shows Koch and modified Koch curves to be considered in the construction of the pre-fractal domains. From these, we get the fractal domains, that is the irregular domains we are interested in. Figures 2 give two examples of irregular domains which are non-trap for the Brownian motion. They are obtained in the simple case of Koch boundaries via iterations with Koch curves. Conversely, by considering the modified Koch curves in Figure 1 we obtain trap domains for the Brownian motion under suitable conditions for the small openings of the modified curves, see [12].

Let $\Omega^s$ be a smooth approximation of $\Omega$, for example $\text{dist}(\partial \Omega^s, \partial \Omega) < s$ and $\partial \Omega^s$ is smooth. Consider the process $\bar{X}$ on $\overline{\Omega^s}$ in comparison with $Z$ on $\overline{\Omega}$ where $\Omega$ is a trap domain for $Z$ (in the previous examples the openings in Figure 1 are such that (78) is satisfied). The process $Z$ spends some time on the boundary due to the irregular nature of the domain. The behaviour of $Z$ on $\overline{\Omega}$ can be associated with the behaviour of $\bar{X}$ on $\overline{\Omega^s}$ for $\eta \neq 0$.

Our conjecture is that a motion on an irregular domain, to be considered in the microscopic analysis (under geometric characterization of the medium), can be described by an anomalous motion on a regular domain, in a macroscopic analysis (let say, by looking from a distance).

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