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A New Dynamic Algorithm for Densest Subhypergraphs

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ABSTRACT
Computing a dense subgraph is a fundamental problem in graph mining, with a diverse set of applications ranging from electronic commerce to community detection in social networks. In many of these applications, the underlying context is better modelled as a weighted hypergraph that keeps evolving with time. This motivates the problem of maintaining the densest subhypergraph of a weighted hypergraph in a dynamic setting, where the input keeps changing via a sequence of updates (hyperedge insertions/deletions). Previously, the only known algorithm for this problem was due to Hu et al. [HWC17]. This algorithm worked only on unweighted hypergraphs, and had an approximation ratio of $(1 + \epsilon)r^2$ and an update time of $O(poly(r, \log n))$, where $r$ denotes the maximum rank of the input across all the updates. We obtain a new algorithm for this problem, which works even when the input hypergraph is weighted. Our algorithm has a significantly improved (near-optimal) approximation ratio of $(1 + \epsilon)$ that is independent of $r$, and a similar update time of $O(poly(r, \log n))$. It is the first $(1 + \epsilon)$-approximation algorithm even for the special case of weighted simple graphs. To complement our theoretical analysis, we perform experiments with our dynamic algorithm on large-scale, real-world data-sets. Our algorithm significantly outperforms the state of the art [HWC17] both in terms of accuracy and efficiency.

CCS CONCEPTS
• Theory of computation → Dynamic graph algorithms.

KEYWORDS
hypergraphs, densest subgraph, dynamic algorithms

1 INTRODUCTION
In the weighted densest subhypergraph (WDSH) problem, we are given a weighted hypergraph $G = (V, E, w)$ as input, where $w : E \rightarrow \mathbb{R}^+$ is a weight function. The density of any subset of vertices $U \subseteq V$ in $G$ is defined as $\rho_G(U) := \frac{\sum_{e \in E(U)} w_e)}{|U|}$, where $E(U)$ is the set of hyperedges induced by $U$ on $G$. The goal is to find a subset of vertices $U \subseteq V$ in $G$ with maximum density. We consider the dynamic WDSH problem, where the input hypergraph $G$ keeps changing via a sequence of updates. Each update either deletes a hyperedge from $G$, or inserts a new hyperedge and specifies its weight $w_e$. In this setting, the update time of an algorithm refers to the time it takes to handle an update in $G$.

Example 1: Real-time story identification. The wide popularity of social media yields overwhelming activity by millions of users at all times in the form of, say, tweets, status updates, or blog posts. These are important to be promptly identified. An efficient technique for these applications, the underlying context is better modelled as a weighted hypergraph rather than a standard graph. In order to appreciate the significance of these three features, consider two concrete real-world examples.

Example 2: Community detection. It is often more beneficial to model the underlying network as a hypergraph rather than a standard graph.
real-time story identification is focusing on certain "entities" associated with a story, e.g., famous personalities, places, organizations, products, etc. They consistently appear together in the numerous posts on the related story. In the example of the Israel-Palestine conflict, countless online posts have turned up about the several events, many of which feature the same small set of entities, e.g., Israel, Palestine, Hamas, Gaza, Sheikh Jarrah, and airstrike, or subsets thereof. This correlation can be leveraged: the set of all possible real-world entities (which can be billions) represented by nodes, with an edge connecting each pair if they appear together in a post, define a graph that changes dynamically over time; maintaining a dense subgraph of this network helps us to identify the group of most strongly-related entities (in the example above, this group might be {Hamas, Gaza, Sheikh Jarrah, airstrike}), and in turn, the trending story [AKSS12].

Note the significance of feature (I) here: the number of posts keeps growing rapidly, thus dynamically modifying the underlying graph. Further, a large number of posts gets deleted over time. This is often driven by the proliferation of fake news and its eventual removal upon detection. Also notice that feature (II) is crucial for this task. Every minute, millions of entities get mentioned in a small number of posts. The few entities in the story of interest, however, collectively appear in a massive number of posts. Therefore, to make them stand out, we can assign to the graph edges weights proportional to the number of posts they represent. Thus, the densest subgraph is induced by the union of the entities in the story. Finally, observe the importance of feature (III) in this context. For a post mentioning multiple entities, instead of adding an edge between each pair of them, we can simply include all of them in a single hyperedge. The standard graph formulation creates a clique among those nodes, which makes the density of the post proportional to the number of entities mentioned. This is inaccurate for several applications. In contrast, having a single hyperedge represent a post removes this bias. The task of real-time story identification thus reduces to precisely the dynamic WDSH problem.

Example 2: Trending Topics Identification. Consider the setting where we wish to identify a set of recently trendy topics in a website like Stack Overflow. We can model this scenario as a network where each node corresponds to a tag, and there is a hyperedge containing a set of nodes iff there is a post with the corresponding set of tags. The weight of a hyperedge represents the reach of a post, captured by, say, the number of responses it generates. The set of recently trendy topics will be given by the set of tags that form the densest subhypergraph in this network. The network is dynamic: posts are added very frequently and deletions are caused not only by their actual removal but also by our interest in only the ones that appeared (say) within the last few days.

Other applications of the WDSH problem include identifying a group of researchers with the most impact [HWC17] and analysing spectral properties of hypergraphs [CLTZ18].

Previous work. Starting with the work of Angel et al. [AKSS12], in recent years a sequence of papers have dealt with the densest subgraph problem in the dynamic setting. Epasto et al. [ELS15] considered a scenario where the input graph undergoes a sequence of adversarial edge insertions and random edge deletions, and designed a dynamic $(2 + \epsilon)$-approximation algorithm with $O(\text{polylog } n)$ update time. In the standard (adversarial) fully dynamic setting, Bhatcharya et al. [BHNT15] gave a $(4 + \epsilon)$-approximation algorithm with $O(\text{polylog } n)$ update time. This latter result was recently improved upon by Sawlani and Wang [SW20], who obtained a $(1 + \epsilon)$-approximation algorithm with $O(\text{polylog } n)$ update time. All these results, however, hold only on unweighted simple graphs (i.e., hypergraphs with rank 2). Our algorithm, in contrast, works for weighted rank-$r$ hypergraphs and is the first $(1 + \epsilon)$-approximation algorithm with $O(\text{polylog } n)$ update time even for the special case of edge-weighted simple graphs.

For general rank-$r$ hypergraphs, the only dynamic algorithm currently known in the literature was designed by Hu et al. [HWC17]: in the fully dynamic setting, their algorithm has an approximation ratio of $(1 + \epsilon)r^2$ and an amortized update time of $O(\text{poly}(r, \log n))$. In direct contrast, as summarized in Theorem 1.1, our approximation ratio is near-optimal (and independent of $r$), and our update time guarantee holds in the worst case. Furthermore, our algorithm works even when the hyperedges in the input hypergraph have large weights in $[1, \text{poly}(r, n)]$, whereas the algorithm in [HWC17] needs to assume that the input hypergraph is either unweighted or has very small weights in $[1, \text{poly}(r, \log n))]$.

Significance of our results. Given this background, let us now emphasize three aspects of our result as stated in Theorem 1.1.

First, the approximation ratio of our algorithm can be made arbitrarily close to 1, and in particular, it is independent of the rank $r$ of the input hypergraph. For example, if $r = 3$, then [HWC17] can only guarantee that in the worst case, the objective value of the solution maintained by their algorithm is at least $(100/r^2)\% \approx 11\%$ of the optimal objective value. In contrast, for any $r$, we can guarantee that the objective value of the solution maintained by our algorithm is always within $\approx 99\%$ of the optimal objective value. In fact, since $r$ can be, in theory, as large as $n$, the improvement over the approximation ratio is massive.

Second, the update time of our algorithm is $O(r^2 \cdot \text{polylog}(n, m))$. Note that any dynamic algorithm for this problem will necessarily have an update time of $\Omega(r)$, since it takes $\Theta(r)$ time to even specify an update. It is not surprising, therefore, that the update time of [HWC17] also had a polynomial dependency on $r$. Since $r$ is a small constant in most practical applications, our update time is essentially $O(\text{polylog}(n, m))$ in these settings.

Third, our dynamic algorithm works for weighted graphs, which, as noted above, are crucial for applications. Throughout the rest of this paper, we assume that the weight of every hyperedge is a positive integer. This is without loss of generality: if the weights are positive real numbers, then we can scale them appropriately and round them to integers without affecting the approximation factor (see full version [BBC21] for details). Finally, if the weights of the hyperedges are known to be integers in the range $[1, W]$, then a naive approach would be to make $W$ copies of every hyperedge when it gets inserted, and maintain a near-optimal solution in the resulting unweighted hypergraph. This, however, leads to an update time of $\Theta(W)$. This is prohibitive when $W$ is large. In contrast, our algorithm has polylogarithmic update time for any $W$.

Overview of Techniques. We obtain the result stated in Theorem 1.1 in two major steps. First, we use random weight scaling to reduce
the weighted version of the problem to the unweighted case, while incurring only a small polylogarithmic overhead in update time (Section 3). Next, to solve the unweighted version, we extend the techniques of [SW20] to handle any general hypergraph (Section 4). Our analysis shows that the approximation factor achieved is $1 + \epsilon$ for hypergraphs of any rank $r$ and in particular, does not grow with $r$. See Section 1.2 of the full version of our paper [BBCG21] for a detailed overview of our techniques.

Overview of Experimental Evaluations. We conduct extensive experiments to demonstrate the effectiveness of our algorithm in both fully dynamic and insertion-only settings with weighted and unweighted hypergraphs. We test our algorithm on several real-world temporal hypergraph datasets. For the unweighted case, in both the insertion-only and fully dynamic settings, our algorithm significantly outperforms the state of the art of [HWC17] both in terms of accuracy and efficiency. In comparison against an LP solver for computing the exact solution, our algorithm shows massive speed-up while incurring less than a few percentage points of relative error. See Section 5 of this paper (and Sections 1.3 and 5 of the full version [BBCG21]) for a detailed account of our experimental results.

2 PRELIMINARIES AND NOTATIONS

Let us fix the notations that we use throughout the paper. Our input weighted hypergraph is always a rank-$r$ hypergraph denoted by $G = (V, E, w)$, where $w : E \rightarrow \mathbb{N}$ is a weight function. We denote the number of vertices $|V|$ and hyperedges $|E|$ (or an upper bound on it) by $n$ and $m$ respectively. The maximum weight of a hyperedge in $G$ is given by $w_{\text{max}}(G) := \max_{e \in E} w_e$. The multiplicity of an edge in a multi-hypergraph is its number of copies in the hypergraph. For a subset of nodes $U \subseteq V$, denote its density in $G$ by $\rho_G(U) := (\sum_{e \in E[U]} w_e) / |U|$, where $E[U]$ is the set of hyperedges induced by $U$ on $G$. If the hypergraph is unweighted, then the density of $U$ is simply $\rho_G(U) = |E[U]| / |U|$. We denote the maximum density of $G$ by $\rho^*(G) := \max_{U \subseteq V} \rho_G(U)$. We drop the argument $G$ from each of the above when the hypergraph is clear from the context.

We use the shorthands WDSH and UDSH for weighted and unweighted multi-hypergraphs respectively. For the dynamic WDSH and UDSH problems, we get two types of queries: (a) max-density query, which asks the value of the maximum density over all subsets of nodes of the hypergraph, and (b) densest-subset query, which asks for a subset of nodes with the maximum density. We say an algorithm maintains an $\alpha$-approximation ($\alpha > 1$) to either of these problems if it answers every max-density query with a value that lies in $[\rho^*/\alpha, \rho^*]$ and every densest-subset query with a subset of nodes whose density lies in $[\rho^*/\alpha, \rho^*]$.

Given any weighted hypergraph $G$, we denote its unweighted multi-hypergraph version by $G^{\text{unw}}$, which is obtained by replacing each edge $e$ having weight $w_e$ by $w_e$ many unweighted copies of $e$. Note that $G$ and $G^{\text{unw}}$ are equivalent in terms of subset densities.

We say that a statement holds whp (with high probability) if it holds with probability at least $1 - 1/\text{poly}(n)$. We use the following version of the Chernoff bound.

**Fact 2.1.** (Chernoff bound) Let $X$ be a sum of mutually independent indicator random variables. Let $\mu$ and $\delta$ be real numbers such that $\mathbb{E}[X] \leq \mu$ and $0 \leq \delta \leq 1$. Then, $\Pr \left[ |X - \mu| \geq \delta \mu \right] \leq \exp \left( - \frac{\mu \delta^2}{3} \right)$.

3 REDUCTION TO UNWEIGHTED CASE

In this section, we show that we can use an algorithm for the dynamic UDSH problem to obtain one for the dynamic WDSH problem while incurring only a small increase in the update and query times.

3.1 Weight Scaling

Given a weighted hypergraph, we want to scale down the weights to make the max-weight small and simultaneously scale down the max-density by a known factor so that we can retrieve the original density value from the scaled one. Since we want to reduce the problem to the unweighted case, we work with the unweighted multi-hypergraph versions (see Section 2) of the weighted hypergraphs in question. Thus, the maximum edge-weight would correspond to the max-multiplicity of an edge in the unweighted version. Informally, given a weighted hypergraph $G$ on $n$ vertices, we want to obtain an unweighted multi-hypergraph $H$ such that (a) maximum multiplicity of an edge in $H$ is roughly $O(\log n)$ and (b) given $\rho^*(H)$, we can easily obtain an approximate value of $\rho^*(G)$. We achieve these in Lemmas 3.1 and 3.2 respectively.

Given any weighted hypergraph $G$, we define $G_q$ as the random hypergraph obtained by independently sampling each hyperedge of $G^{\text{unw}}$ with probability $q$.

For a parameter $\tilde{\rho}$, define $q(\tilde{\rho}) := \min \left\{ ce^{-2} \cdot \frac{\log n}{\tilde{\rho}}, 1 \right\}$ for some large constant $c$ and an error parameter $\epsilon > 0$.

Our desired multi-hypergraph $H$ will be given by $G_q(\tilde{\rho})$ for some appropriate $\tilde{\rho}$. The following lemma (proof in the full version [BBCG21]) shows that the max-multiplicity of $H$ is indeed small.

**Lemma 3.1.** For $\tilde{\rho} \geq w_{\text{max}}(G)/r$, set $H = G_q(\tilde{\rho})$. Then, maximum multiplicity of an edge in $H$ is $O(\text{er}^{-2} \log n)$ whp.

At the same time, we also need to ensure that we can retrieve the max-density and a densest subset of $G$ from that of $H$. The next lemma, which follows directly from Theorem 4 of [MPP+15], handles this.

**Lemma 3.2.** Given a weighted hypergraph $G = (V, E, w)$, let $H = G_q(\tilde{\rho})$ for a parameter $\tilde{\rho}$. Then, following hold simultaneously whp:

(i) $\forall U \subseteq V : \rho_{G}(U) \geq (1 + \epsilon)\tilde{\rho} \Rightarrow \rho_{H}(U) \geq ce^{-2} \log n$

(ii) $\forall U \subseteq V : \rho_{G}(U) < (1 - 2\epsilon)\tilde{\rho} \Rightarrow \rho_{H}(U) < (1 - \epsilon)ce^{-2} \log n$

It follows from the above lemma that $\rho^*(H) \approx ce^{-2} \log n$ if $\tilde{\rho}$ is very close to $\rho^*(G)$. We can now make parallel guesses $\tilde{\rho}$ for $\rho^*(G)$ and find the correct one by identifying the guess that gives the desired value of $\rho^*(H)$. We explain this in detail and prove it formally in the next section.

3.2 Fully Dynamic Algorithm for WDSH using UDSH

We handle the unweighted case UDSH and obtain the following theorem in Section 4.
Theorem 3.3. Given an unweighted rank-\(r\) (multi-)hypergraph \(H\) on \(n\) vertices and at most \(m\) edges with max-multiplicity at least \(w^*\), there exists a fully dynamic data structure \(\text{Udshp}\) that deterministically maintains a \((1+\epsilon)\) -approximation to the densest subhypergraph problem. The worst-case update time is \(O(\max((64r\epsilon^{-2} \log n)/w^*, 1) \cdot r e^{-4} \log^2 n \log m)\) per edge insertion or deletion. The worst-case query times for max-density and densest-subset queries are \(O(1)\) and \(O(\beta + \log n)\) respectively, where \(\beta\) is the output-size.

Here, we describe a way to use the above theorem as a subroutine to efficiently solve the dynamic WDSH problem. For the input weighted hypergraph \(G\), assume that we know the value of \(w_{\text{max}}(G)\) and an upper bound \(m\) on the number of hyperedges (across all updates) in advance.\(^1\) First, we observe the following.

Observation 3.4. In a rank-\(r\) weighted hypergraph \(G\) with at most \(m\) edges, we have \(w_{\text{max}}(G)/r \leq \rho^*(G) \leq mw_{\text{max}}(G)\).

Our algorithm for the dynamic WDSH problem is as follows.

Preprocessing. We keep guesses \(\bar{\rho}_i = (w_{\text{max}}(G)) \cdot (1 + \epsilon)^i\) for \(i = 0, 1, \ldots, \lceil \log_{1+\epsilon} \max_{i \in \{1, \ldots, n\}} \omega_{i, e}\rceil\). Note that by Observation 3.4, these are valid guesses for \(\rho^*(G)\). For each guess \(\bar{\rho}_i\) and each \(j \in \{1, \ldots, \omega_{i, e}\}\), we construct a data structure \(\text{Sample}(i, j)\) that, when queried, generates independent samples from the probability distribution Bin\((\lceil (1 + \epsilon)^i \rceil, q(\bar{\rho}_i))\). Each such data structure can be constructed in \(O(\max_{i})\) time so that each query is answered in \(O(1)\) time ([BP17], Theorem 1.2). Parallel to this, for each \(i\), we have a copy of the data structure for the UDSH problem, given by \(\text{Udshp}(i)\). The value of \(w^*\) that we set for \(\text{Udshp}(i)\) is \(w_{\text{max}}(G) \cdot q(\bar{\rho}_i)/2\).

Update processing. On insertion of the edge \(e\) with weight \(w_e\), for each guess \(\bar{\rho}_i\), query \(\text{Sample}(i, \lceil \log_{1+\epsilon} w_e \rceil)\) to get a number \(s\), and insert \(s\) copies of the unweighted edge \(e\) using the data structure \(\text{Udshp}(i)\). Similarly, on deletion of edge \(e\), for each \(i\), use \(\text{Udshp}(i)\) to delete all copies of the edge added during its insertion.

Query processing. Denote the value of maximum density returned by \(\text{Udshp}(i)\) as \(G^\prime\) \((q(\bar{\rho}_i))\), where \(G^\prime\) is the hypergraph obtained by rounding up each edge weight of \(G\) to the nearest power of \((1 + \epsilon)\). Thus, \(\rho^*(G^\prime) \leq \rho^*(G) \leq \rho^*(G^\prime) \leq (1 + \epsilon)\rho^*(G)\).

For simplicity, we write \(G^\prime\) \((q(\bar{\rho}_i))\) as \(G^\prime\). Note that the value of \(w^*\) provided to each \(\text{Udshp}(i)\) satisfies the condition in Theorem 3.3 w.h.p (by the Chernoff bound (Fact 2.1) since the expected value of max-multiplicity of \(G^\prime\) is \(w_{\text{max}}(G) \cdot q(\bar{\rho}_i)\). By Theorem 3.3, \(\text{Udshp}(i)\) returns value \(\bar{\rho}_{i^*}\) such that

\[
(1 + \epsilon)\rho^*(G^\prime) \leq \bar{\rho}_{i^*} \leq \rho^*(G^\prime).
\]

By the definition of \(i^*\), we have \(\bar{\rho}_{i^*} \geq (1 + \epsilon)ce^{-2} \log n\). This means \(\rho^*(G^\prime) \geq (1 + \epsilon)ce^{-2} \log n\). Then, by Lemma 3.2 (ii), we get

\[
\rho^*(G^\prime) \geq (1 + 2\epsilon)\bar{\rho}_{i^*} \geq (1 + 2\epsilon)\rho^*(G).
\]

Thus, from eqs. (1) and (2), we get

\[
\rho^*(G) \geq \frac{1 - 2\epsilon}{1 + \epsilon} \bar{\rho}_{i^*} \geq \frac{1 - 2\epsilon}{1 + \epsilon} \rho^*(G).
\]

Again, let \(U^*\) be the densest subset returned by \(\text{Udshp}(i^*)\). By Lemma 3.2 (ii), we see that

\[
\rho_{G^\prime}^\prime(U^*) \geq (1 - 2\epsilon)\bar{\rho}_{i^*} \geq \frac{1 - 2\epsilon}{1 + \epsilon} \rho^*(G).
\]

Therefore, by the definition of \(G^\prime\), we have

\[
\rho^*(G) \geq \rho_{G^\prime}^\prime(U^*) \geq \frac{\rho_{G^\prime}^\prime(U^*)}{1 + \epsilon} \geq \frac{1 - 2\epsilon}{1 + \epsilon} \rho^*(G).
\]

Given any \(0 < \delta < 1\), we set \(\epsilon = \Theta(\delta)\) small enough so that \(\frac{1 - 2\epsilon}{1 + \epsilon} \geq \frac{1}{1 + \delta}\). Therefore, by eqs. (3) and (4), the value and the subset that we return on the max-density and densest-subset queries respectively are \((1 + \delta)\)-approximations to \(\rho^*(G)\).

Runtime. As noted before, we feed \(G^\prime\) to \(\text{Udshp}(i)\). Fix an \(i\). Let \(o_i\) be the max-multiplicity of an edge in \(G^\prime\). When a hyperedge of \(G^\prime\) is inserted/deleted, we insert/delete at most \(o_i\) unweighted copies of that edge to \(\text{Udshp}(i)\). Therefore, by Theorem 3.3, the worst case update time for \(\text{Udshp}(i)\) is \(O(o_i \cdot \max((64r\epsilon^{-2} \log n)/w^*, 1) \cdot re^{-4} \log^2 n \log m)\). Using the Chernoff bound (Fact 2.1), we have \(o_i \leq 2w_{\text{max}}(G) \cdot q(\bar{\rho}_i) = 4w^*\) w.h.p. Also, since \(\bar{\rho}_i \geq w_{\text{max}}(G)/r\) for each \(i\), we can apply Lemma 3.1 to get that \(o_i \leq O(re^{-2} \log n)\). Hence, the expression simplifies to \(O(r^2e^{-6} \log n \cdot re^{-4} \log^2 n \log m)\) \(= O(r^2e^{-6} \log^2 n \log m)\). Finally, accounting for all the \(O(\log_{1+\epsilon} r^m)\) values of \(i\), the total update time is \(O(r^2e^{-6} \log^2 n \log^2 m)\) (recall that \(\delta = \Theta(\epsilon)\)). The max-density query for WDSH is answered by binary-searching on the \(O(\log^2 m)\) copies of \(\text{Udshp}\), which gives a query time of \(O(\log \delta + \log m)\) by Theorem 3.3. Note that the densest-subset query is made only on the relevant copy \(i^*\) after we find it, and hence, by Theorem 3.3, it takes \(O(\beta + \log n)\) time, where \(\beta\) is solution-size. Therefore, we obtain the following theorem that captures our main result.

Theorem 3.5. (Formal version of Theorem 1.1) Given a weighted rank-\(r\) hypergraph on \(n\) vertices and at most \(m\) edges, for any \(0 < \delta < 1\), there exists a randomized fully dynamic algorithm that maintains a \((1 + \delta)\)-approximation to the densest subhypergraph problem. The worst-case update time is \(O(r^2e^{-6} \log^2 n \log^2 m)\) per hyperedge insertion or deletion. The worst-case query times for max-density and densest-subset queries are \(O(\log \delta + \log m)\) and \(O(\beta + \log n)\) respectively, where \(\beta\) is the output-size. The preprocessing time is \(O(w_{\text{max}}^2 \log^2 m \log w_{\text{max}})\), where \(w_{\text{max}}\) is the max-weight of a hyperedge.

Now all it remains is to solve the unweighted case and prove Theorem 3.3. We do this in Section 4.

\(^1\)These assumptions can be removed with very small increase in update time while preserving the approximation ratio (details in the full version [BBCG21]).

\(^2\)Bin\((n, p)\) is the Binomial distribution with parameters \(n\) and \(p\).
4 FULLY DYNAMIC ALGORITHM FOR UDSH

Here, due to limited space, we give a sketch of our algorithm and analysis for the dynamic UDSH problem and provide the complete details in the full version [BBCG21].

Our Algorithm and Analysis. We extend the techniques of [SW20] for the densest subgraph problem and take the primal-dual approach to solve the UDSH problem. Recall that the input is an unweighted multi-hypergraph $H = (V, E)$ and we want to find the approximate max-density as well as an approximately densest subset of $H$. As is standard, we associate a variable $x_v \in \{0, 1\}$ with each vertex $v$ and $y_e \in \{0, 1\}$ with each hyperedge $e$ such that $x_v = 1$ and $y_e = 1$ respectively denote that we include $v$ and $e$ in the solution subset. Relaxing the variables, the primal LP for UDSH (Primal$(H)$) is given below. Following notations similar to [SW20], for each vertex $u$ and edge $e$, let $f_e(u)$ and $D$ be the dual variables corresponding to constraints (5) and (6) respectively. Then, the dual program Dual$(H)$ is as follows.

Primal$(H)$:
\[
\begin{align*}
\text{max} & \quad \sum_{e \in E} y_e \\
\text{s.t} & \quad y_e \leq x_u \quad \forall u \in e \quad \forall e \in E \\
\end{align*}
\]
\[
\sum_{v \in V} x_v \leq 1
\]
\[
\sum_{v \in V} x_v y_e \geq 0 \quad \forall u \in v \quad e \in E
\]

Dual$(H)$:
\[
\begin{align*}
\text{min} & \quad D \\
\text{s.t} & \quad \sum_{u \in e} f_e(u) \geq 1 \quad \forall e \in E \quad (8) \\
\sum_{u \in e} f_e(u) & \leq D \quad \forall v \in V \quad (9) \\
f_e(u) & \geq 0 \quad \forall u \in v \quad \forall e \in E \quad (10)
\end{align*}
\]

Think of $f_e(u)$ as a “load” that edge $e$ puts on node $u$. We can thus interpret Dual$(H)$ as a load balancing problem: each hyperedge needs to distribute a unit load among its vertices such that the maximum total load on a vertex due to all its incident edges is minimized. For each $v \in V$, define $\ell(v) := \sum_{e \ni v} f_e(v)$. Note that if for some feasible solution, some edge assigns $f_e(u) > 0$ to some $v \in e$ and $\ell(v) > \ell(u)$ for some $u \in e \setminus \{v\}$, then we can “transfer” some positive load from $f_e(u)$ to $f_e(u)$ while maintaining constraint (8) and without exceeding the objective value. Therefore, we can always find an optimal solution to Dual$(H)$ satisfying the following “local” property.

\[
\forall e \in E : f_e(v) > 0 \Rightarrow \ell(v) \leq \ell(u) \quad \forall u \in e \setminus \{v\}
\]

We can verify that property (11) is also sufficient to get a global optimal solution to Dual$(H)$ (see full version [BBCG21]). Next, we show in Theorem 4.1 (proof deferred to Appendix A) that “approximately” maintaining property (11) (see const. (14)) gives us a near-optimal solution to Dual$(H)$, i.e., an approximate value of $\rho^*(H)$. In this regard, we define a system of equations Dual$(H, \eta)$ as follows.

\[
\begin{align*}
\ell(v) & = \sum_{e \ni v} f_e(v) \quad \forall v \in V \quad (12) \\
\sum_{u \in e} f_e(u) & = 1 \quad \forall e \in E \quad (13) \\
\ell(v) & \leq \ell(u) + \eta \quad \forall u \in v \setminus \{v\}, \forall e \in E : f_e(u) > 0 \quad (14) \\
f_e(u) & \geq 0 \quad \forall u \in e \forall e \in E \quad (15)
\end{align*}
\]

Theorem 4.1. Given a feasible solution $(\hat{f}, \hat{D})$ to Dual$(H, \eta)$, we have $\rho^*(1 - \epsilon) \leq \hat{D}(1 - \epsilon) < \rho^*$, where $\hat{D} = \max_u \ell(u)$ and $\epsilon = \sqrt{\frac{8n \log n}{D}}$.

By Theorem 4.1, we see that if we can find $\hat{D}$, i.e., a feasible solution to Dual$(H, \eta)$, then we can get a $(1 + \epsilon)$-approximation to $\rho^*$, where $\epsilon = \sqrt{\frac{32 \log n}{\hat{D}}}$. This means that given $\epsilon$, we initially need to set $\eta = \frac{\epsilon^2 \hat{D}}{32 \log n}$. But we do not know the value of $\hat{D}$ initially, and in fact, that’s what we are looking for. However, we shall initially have an estimate $\hat{D}$ of $\hat{D}$ such that $\hat{D} \leq \hat{D} \leq 2\hat{D}$. Note that if we have $\eta \geq 1$, then we can maintain constraint (14) with some positive slack while having integer loads on the vertices. This means that we are allowed to simply assign the unit load of an edge $e$ entirely on some vertex $u \in e$. Assume that we know a lower bound $w^*$ on the max-multiplicity of a hyperedge in the graph. If $w^* \geq 64\epsilon^2 - 2\log n$, then it already implies that $\eta \geq 1$ since $\hat{D} \geq \rho^* \geq 64\epsilon^2 - 2\log n$ and hence, $\hat{D} \geq \hat{D}/2 \geq 32\epsilon^2 - 2\log n$. Otherwise, we duplicate each hyperedge $(64\epsilon^2 - 2\log n)/w^*$ times (hence, this factor appears in the update time of Theorem 3.3), so that we are ensured that $\rho^* \geq 64\epsilon^2 - 2\log n$, implying $\eta \geq 1$ as before. Once we have $\eta \geq 1$ and are allowed to assign the entire load of an edge on a single node in it, our problem reduces to the following hypergraph “orientation” problem.

Problem (Hypergraph Orientation). Given an unweighted multi-hypergraph $H = (V, E)$ and a parameter $\eta \geq 1$, for each edge $e \in E$, assign a vertex $v \in e$ as its head $h(e)$, such that

\[
\forall e \in E : h(e) = v \Rightarrow d_{in}(v) \leq d_{in}(u) + \eta \forall u \in e \setminus \{v\}
\]

where $d_{in}(v) := |\{e \in E : h(e) = v\}|$.

Given a parameter $\hat{D}$, we construct a data structure $\text{HOP}(\hat{D})$ that maintains the “oriented” hypergraph satisfying (16) with $\eta = \frac{\epsilon^2 \hat{D}}{32 \log n}$ and in turn, solves the UDSH problem. We describe it in detail in Data Structure 1. The following lemmas (see full version for proofs) give the correctness and runtime guarantees of the data structure.

Lemma 4.2. After each insertion/deletion, the data structure $\text{HOP}(\hat{D})$ maintains constraint (16) with $\eta = \frac{\epsilon^2 \hat{D}}{32 \log n}$.

Lemma 4.3. If $\hat{D} \leq \hat{D}$, then the operations querysubset and querydensity of $\text{HOP}(\hat{D})$ return a $(1 + \epsilon)$-approximation to the densest-subset and max-density queries respectively.

Lemma 4.4. If $\hat{D} \leq 2\hat{D}$, then the operations insert and delete of $\text{HOP}(\hat{D})$ take $O(\epsilon^{-3} \log^2 n)$ and $O(\epsilon^{-2} \log n)$ time respectively. The operation querydensity takes $O(1)$ time and querysubset takes $O(\beta + \log n)$ time, where $\beta$ is the solution-size.

Completing the Algorithm. The above lemmas prove Theorem 3.3 as long as we have an estimate $\hat{D}$ such that $\hat{D} \leq \hat{D} \leq 2\hat{D}$.
For this, we keep parallel data structures $\text{HOP}(\bar{D})$ for $O(\log m)$ guesses of $\bar{D}$ in powers of 2. Then, we show that we can maintain an “active” copy of $\text{HOP}$ corresponding to the correct guess, from which the solution is extracted. Thus, we incur only an $O(\log m)$ overhead on the total update time for an edge insertion/deletion. This part is very similar to Algorithm 3 of [SW20] and we discuss this in detail in the full version [BCG21] and formally prove Theorem 3.3.

5 EXPERIMENTS

In this section, we present extensive experimental evaluations of our algorithms. We consider weighted and unweighted hypergraphs in both insertion-only and fully dynamic settings, leading to a total of four combinations. However, due to space limitations, we discuss only the fully dynamic setting here and defer the incremental setting to Appendix B. We call our algorithms Udshp and Wdshp for the unweighted and weighted settings respectively and we compare their accuracy and efficiency to that of the baseline algorithms. Furthermore, we study the trade-off between accuracy and efficiency for Udshp and Wdshp.

Table 1: Description of our dataset with the key parameters, the drugs and a hyperedge corresponds to a combination of drugs abuse related events leading to an emergency hospital visit across the nation health surveillance system that records drug records at an interval of 3 months.

In both insertion only and dynamic settings, we report the maximum density at an interval of 3 months. In the fully dynamic case, we maintain a sliding window of 3 years, and we report the subset with the largest density among these choices. We implement all algorithms in C++ and all experiments are run on a workstation with 256 GB memory and Intel Xeon(R) 2.20 GHz processor running Ubuntu 20 operating system.

Baseline Algorithms. We consider two main baselines algorithms. (1) The first one is an exact algorithm, denoted as Exact, that computes the exact value of the densest subhypergraph at every reporting interval of the dataset. We use google OR-Tools to implement an LP based solver for the densest subhypergraph [HWC17, PF]. (2) Second one is the dynamic algorithm for maintaining densest subhypergraph by Hu et al. [HWC17]; we call it HWC. It takes $\epsilon_H$ as an input accuracy parameter and produces a $(1+\epsilon_H)\rho$ and $(1+\epsilon_H)^2\rho$-approximate densest subhypergraph in the insertion only and fully dynamic model respectively. For the weighted hypergraphs we modify the HWC implementation – each edge with weight $w_e$ is processed by creating $w_e$ many copies of that edge.

Parameter Settings. Both HWC and our algorithms Udshp and Wdshp take an accuracy parameter $\epsilon$ as an input. However, it is important to note that the accuracy parameter $\epsilon$ for both the algorithms are not directly comparable. Udshp or Wdshp guarantees to maintain a $(1+\epsilon)$-approximate solution in both insertion only and fully dynamic settings, whereas HWC maintains $(1+\epsilon_H)\rho$ and $(1+\epsilon_H)^2\rho$ approximate solutions for insertion and fully dynamic settings respectively. Thus, for a fair comparison between the algorithms, we run Udshp (or Wdshp) and HWC with different values of $\epsilon$ such that their accuracy is comparable. We use $\epsilon_H$ to denote the parameter for HWC to make this distinction clear. For various settings and datasets, we use different parameters and specify them in the corresponding plots. We emphasize here that the motivation behind the choices of the parameters is to compare the update time of Udshp and Wdshp to that of HWC while ensuring that Udshp and Wdshp has better accuracy than that of HWC. We restrict our focus to the small approximation error regime.

Accuracy and Efficiency Metrics. To measure the accuracy of Udshp, Wdshp, and HWC, we use relative error percentage with respect to Exact. It is defined as $\frac{|p(\text{Alg},t)−p(\text{Exact},t)|}{p(\text{Exact},t)} \times 100\%$, where $p(X, t)$ is the density estimate by algorithm $X$ at time interval $t$. We also compute the average relative error of an algorithm by taking the average of the relative errors over all the reporting intervals. For measuring efficiency we compare the average wall-clock time taken over the operations during each reporting interval and also overall
average time (taken over all reporting intervals) as an efficiency comparison metric.

5.1 Fully Dynamic Case
In this section, we consider the fully dynamic setting where the hyperedges can be both inserted and deleted. We perform experiments for hypergraphs with unweighted as well as weighted edges.
For both the cases, we first compare the accuracy and the efficiency of our algorithm against the baselines. And then we analyze the accuracy vs efficiency trade-off of Udshp and Wdshp.

**UNWEIGHTED HYPERGRAPHS:** We first discuss our findings for the unweighted case.

**Accuracy and Efficiency Comparison.** In Figure 1, we compare the accuracy and efficiency of Udshp against the baselines for the unweighted hypergraphs. In the top row, we compare the accuracy of Udshp and HWC in terms of relative error percentage with respect to Exact. In the bottom row, we plot the average time taken per operation by Exact, Udshp, and HWC during each reporting interval. For each dataset, the parameters are identical for the top row and bottom row plots. We reiterate that the input parameters for Udshp and HWC are chosen to compare Udshp and HWC in the low relative error regime. We highlight our main findings below.

We observe that for smaller hypergraphs (DAWN, tag-math-sx), Udshp and HWC achieve impressive accuracy, however Udshp is consistently more than 10x faster than HWC. In fact, HWC is several times slower compared to Exact. On the other hand, Udshp is 3x-5x times faster compared to Exact. As the sizes of the hypergraphs increase, Exact gets much slower compared to Udshp and HWC as LP solvers are known to have scaling issues. For larger datasets, Udshp maintains a clear edge in terms of accuracy over HWC even when their update times are almost identical or better for Udshp, as demonstrated by the last three columns. To quantify the gain further, in Figure 3a, we compare the performance of Udshp against HWC and Exact in terms of average relative error and average update time, where the average is taken over all the reporting intervals. We make several interesting observations.

1. Udshp is 3x-5x faster than Exact for small hypergraphs; the gain is massive (10x-15x) for larger graphs. (2) Compared against HWC, the avg. update time for Udshp can be 10x-12x smaller (DAWN and tag-math-sx) while maintaining almost the same average relative error of less than 1%. (3) At the other end of the spectrum, for almost the same average update time, Udshp can offer 55%-90% improvement in accuracy over HWC (Coauth-MAG and DBLP-A11). (4) HWC performs worse than Exact for smaller datasets, being slower by 3x-5x factors (DAWN and tag-math-sx).

Accuracy vs Efficiency trade off for Udshp. In Figure 4a we plot average update time, and average and max relative error for Udshp for different values of $\epsilon$. The max relative error is the maximum of the relative error over all the reporting intervals. As expected, when $\epsilon$ decreases, the update time increases and the average and maximum relative error incurred by Udshp decreases.

We observe that for the hypergraphs with high density values ($\Omega(\log n)$), e.g., DAWN, tag-math-sx, tag-stack-overflow, the average and maximum relative errors are quite low ($< 2 - 5\%$). Thus, we recommend using Udshp with larger values of $\epsilon$ (like $\epsilon = 1$) for them. Note that reduction in update time is quite dramatic ($\approx 8x$) when increasing $\epsilon$ from 0.5 to 1.0 for these graphs. For the hypergraphs with low density values ($o(\log n)$) the relative errors can go well above 30%-40% for larger values of $\epsilon$. Thus, we recommend using Udshp with smaller values of $\epsilon$ (like $\epsilon = 0.3$) for more accurate solutions, as for hypergraphs like Coauth-MAG, reducing $\epsilon$ from 1.0 to 0.5 reduces the average relative error from 70% to 30% (albeit at the cost of a 3-fold increase in average update time).

**WEIGHTED HYPERGRAPHS:** For the weighted case in Figure 2, we consider similar settings as in Figure 1. In the top row, we compare the relative error percentage of Wdshp and HWC, and the bottom row, shows the average update times of Wdshp, HWC, and Exact with same parameters (for each hypergraph). For a detailed discussion on the accuracy and efficiency comparison and tradeoffs, please refer to the full version [BBCG21].

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We try to get a lower bound on $\rho$ where the last implication is by (14). Hence, we get the following.

For Observation A.1.

Each hyperedge $e$ maintains a list of vertices that it contains and has a pointer $h(e)$ to the head vertex.

Algorithm 1 ROTATE($e, v$)
1: $z \leftarrow h(e)$
2: remove $e$ from In($z$)
3: remove $e$ from Out($u$) for each $u \in e \setminus \{z\}$
4: $h(e) \leftarrow v$; add $e$ to In($v$)
5: add $e$ to Out($u$) for each $u \in e \setminus \{v\}$

Algorithm 2 TIGHTENEDGE($e$)
1: for $v \in \{\text{next} 4d_{in}(e)/\eta \text{ edges in In}(e)\}$ do
2: $u \leftarrow \arg \min_{v \in e} d_{in}(v)$
3: if $d_{in}(u) \leq d_{in}(e) - \eta/2$ then
4: return $e$
5: return null

Algorithm 3 INSERT(e)
1: $v \leftarrow \arg \min_{v \in e} d_{in}(v)$
2: $h(e) \leftarrow v$
3: add $e$ to In($v$)
4: add $e$ to Out($u$) for each $u \in e \setminus \{v\}$
5: while tightheadge($e$) $\neq$ null do
6: $f \leftarrow \text{tightheadge}(e)$
7: $v \leftarrow \arg \min_{v \in f} d_{in}(v)$; rotate($f, v$)
8: increment($e$)

Algorithm 4 INCREMENT($v$)
1: $d_{in}(e) \leftarrow d_{in}(e) + 1$
2: Update $d_{in}(v)$ in Indegrees
3: for $v \in \{\text{next} 4d_{in}(e)/\eta \text{ edges in In}(v)\}$ do
4: for $u \in v$ do
5: $d_{in}(e) \leftarrow d_{in}(e)$

Algorithm 5 TIGHTOUTEDGE($e$)
1: $v \leftarrow \text{Out}(e)$ max
2: if $d_{out}(u) \geq d_{in}(e) + \eta/2$ then
3: return $v$
4: return null

Algorithm 6 DELETE(e)
1: $v \leftarrow h(e)$
2: remove $e$ from In($v$)
3: remove $e$ from Out($u$) for each $u \in e \setminus \{v\}$
4: while tightheadge($e$) $\neq$ null do
5: $f \leftarrow \text{tightheadge}(e)$; $z \leftarrow h(f)$
6: rotate($f, o$); $v \leftarrow z$
7: decrement($e$)

Algorithm 7 DECREMENT($v$)
1: $d_{in}(e) \leftarrow d_{in}(e) - 1$
2: Update $d_{in}(v)$ in Indegrees
3: for $v \in \{\text{next} 4d_{in}(e)/\eta \text{ edges in In}(v)\}$ do
4: for $u \in v$ do
5: $d_{in}(e) \leftarrow d_{in}(e)$

Algorithm 8 DENSESTSUBSET($\gamma$)
1: $\hat{D} \leftarrow \text{Indegrees}\max$; $A \leftarrow \{v : d_{in}(v) \geq \hat{D}\}$
2: $B \leftarrow \{v : d_{in}(v) \geq \hat{D} - \eta\}$
3: while $|B|/|A| \geq 1 + \gamma$ do
4: $\hat{D} \leftarrow \hat{D} - \eta$; $A \leftarrow B$
5: $B \leftarrow \{v : d_{in}(v) \geq \hat{D} - \eta\}$
6: return $B$

Algorithm 9 QUERYSUBSET()
1: $\hat{D} \leftarrow \text{Indegrees}\max$; $\gamma \leftarrow \sqrt{2\eta \log n/\hat{D}}$
2: return densestsubset($\gamma$)

Algorithm 10 QUERYDENSITY()
1: return $(\text{Indegrees}\max) \cdot (1 - \frac{1}{\gamma})$

A  MISSING PROOF

Proof of Theorem 4.1. Since $(\hat{f}, \hat{D})$ is a feasible solution to $\text{Dual}(H, \eta)$, we see that $(\hat{f}, \hat{D})$ is a feasible solution to $\text{Dual}(H)$. Since $\rho^*$ is an optimal solution to $\text{Dual}(H)$, we have $\hat{D} \geq \rho^*$ and the left inequality follows.

Define $S_i : = \{v : \hat{f}(v) \geq \hat{D} - \eta i\}$ for $i \geq 0$. For some parameter $0 < \gamma < 1$, let $k$ be the maximal number such that $|S_i| \geq (1 + \gamma)|S_{i-1}|$ for all $i \in [k]$. Thus, $|S_{k+1}| < (1 + \gamma)|S_k|$. For an edge $e$ incident on $v \in S_k$, consider $u \in e \setminus \{v\}$. We have

$$u \notin S_{k+1} \Rightarrow \hat{f}(u) < \hat{D} - \eta(k+1) \leq \hat{f}(v) - \eta \Rightarrow \hat{f}(e) = 0$$

where the last implication is by (14). Hence, we get the following.

Observation A.1. For $v \in S_k$, we have $\sum_{e \ni v} \hat{f}(e) = \sum_{e \subset S_{k+1}} \hat{f}(e)$.

We try to get a lower bound on $\rho(S_{k+1})$. We see that

$$|S_{k+1}| \leq \sum_{v \in S_k} \hat{f}(v) = \sum_{e \ni v} \hat{f}(e) = \sum_{e \in S_{k+1}} \hat{f}(e)$$

The second equality follows by Obs. A.1 and the last one by (13).

Therefore, by definition of $k$, we get

$$\rho(S_{k+1}) = \frac{|E(S_{k+1})|}{|S_{k+1}|} \geq \frac{(\hat{D} - \eta k)|S_k|}{|S_{k+1}|} \geq \frac{\hat{D} - \eta k}{1 + \gamma} > (\hat{D} - \eta k)(1 - \gamma).$$

Again, since $|S_k| \geq (1 + \gamma^k)|S_0| \geq (1 + \gamma)^k$, we have $k \leq \log_{1+\gamma} |S_k| \leq \log_{1+\gamma} n \leq 2 \log n/\gamma$. Therefore, we have

$$\rho(S_{k+1}) > \frac{\hat{D} - 2n \log n}{\gamma} (1 - \gamma) = \hat{D} \left(1 - \frac{2n \log n}{\gamma \hat{D}}\right) (1 - \gamma).$$
We set $\gamma$ so as to maximize the RHS. Clearly, it is maximized when $\gamma = \frac{2\eta \log n}{D}$, and so, we set $\gamma := \sqrt{\frac{2\eta \log n}{D}}$. Hence, we get

$$
\rho^* \geq \rho(S_{k+1}) > \hat{D}(1-\gamma)^2 > \hat{D}(1-2\gamma) = \hat{D} \left( 1 - \sqrt{\frac{8\eta \log n}{D}} \right).
$$

\[ \square \]

**B  EXPERIMENTS: INSERT-ONLY CASE**

Here, we give an account of our experiments for the insert-only setting with unweighted hyperedges. We defer the discussion on the weighted incremental setting to the full version [BBCG21].

![Figure 6: Avg. Accuracy and Efficiency Comparison for Unweighted Incremental Setting](image)

**Figure 6:** Avg. Accuracy and Efficiency Comparison for Unweighted Incremental Setting: On the left, we plot avg. relative err. of \textit{Udshp} and \textit{HWC}, and on the right, we compare the avg. update time of \textit{Udshp}, \textit{HWC}, and \textit{Exact}. (Average is taken over the entire duration)

**Accuracy vs Efficiency Trade-offs.** As similar to that in the dynamic setting, in Figure 7, we analyze the change in the average update time and the average and max relative error for \textit{Udshp} for different values of $\epsilon \in \{1.0, 0.7, 0.5\}$. We observe that even if the update time is sensitive to change in $\epsilon$, the average and maximum relative error for all the high density ($\Omega(\log n)$) hypergraphs (DAWN, tag-ask-ubuntu, tag-math-sx, tag-stack-overflow) is low ($<10\%$). And thus we recommend, using \textit{Udshp} with high value of $\epsilon$ (like $\epsilon = 1$) for these hypergraphs. On the other hand for the low density ($o(\log n)$) hypergraphs (like Coauth-MAG), we recommend using \textit{Udshp} with low value of $\epsilon$ (like $\epsilon = 0.5$ or 0.3).

**Accuracy and Efficiency Comparison.** In Figure 5 top row we compare the accuracy of \textit{Udshp} and \textit{HWC} with respect to \textit{Exact}. And in the bottom row, we plot the average time taken per operation by \textit{Exact}, \textit{Udshp}, and \textit{HWC} during each reporting interval. To further quantify the gain of \textit{Udshp}, in Figure 6, we compare the performance of \textit{Udshp} against \textit{HWC} and \textit{Exact} in terms of average relative error and average update time. We highlight some of our main findings below.

1. Performance of \textit{HWC} fluctuates quite a lot over time as evident from the saw-tooth behaviour in the relative error and the update time curves for \textit{HWC} in Figure 5. Thus, even if the average case update time for \textit{HWC} is low, the worst-case update time could be very high. In contrast, \textit{Udshp} exhibits a much more stable behavior over time, making it more suitable for practical use. Note that this is consistent with the theoretical results for the respective algorithms since \textit{HWC} only guarantees small amortized update time while \textit{Udshp} guarantees small worst-case update time.

2. For the first four datasets, on average \textit{Udshp} has 70% better accuracy while being 2x-4x faster (on average) compared to \textit{HWC} (Figure 6). For the largest dataset Coauth-MAG, \textit{HWC} indeed has an edge over \textit{Udshp} in terms of average update time while both incurring comparable loss in accuracy (Figure 6). However, as we noted before, the saw-tooth behavior of \textit{HWC} implies a higher worst-case update time for \textit{HWC} compared to \textit{Udshp} (Figure 5).

3. \textit{Exact} performs extremely poorly in the incremental settings, as one would expect. The sizes of the hypergraphs are much larger compared to the dynamic settings, making \textit{Exact} extremely unsuitable for any practical purpose.

![Figure 7: Accuracy vs Efficiency Trade-off for Unweighted Incremental Hypergraphs (Udshp)](image)

**Figure 7:** Accuracy vs Efficiency Trade-off for Unweighted Incremental Hypergraphs (Udshp): We plot the avg. update time (left), avg. relative err. (middle), and max. relative err. (right) over the reporting intervals for different settings of $\epsilon$. 