ON MEASURE SOLUTIONS OF THE BOLTZMANN EQUATION PART II: RATE OF CONVERGENCE TO EQUILIBRIUM

XUGUANG LU AND CLÉMENT MOUHOT

Abstract. The paper considers the convergence to equilibrium for measure solutions of the spatially homogeneous Boltzmann equation for hard potentials with angular cutoff. We prove the exponential sharp rate of strong convergence to equilibrium for conservative measure solutions having finite mass and energy. The proof is based on the regularizing property of the iterated collision operators, exponential moment production estimates, and some previous results on the exponential rate of strong convergence to equilibrium for square integrable initial data. We also obtain a lower bound of the convergence rate and deduce that no eternal solutions exist apart from the trivial stationary solutions given by the Maxwellian equilibrium. We finally use these convergence rates in order to deduce global-in-time strong stability of measure solutions.

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1. Introduction

The Boltzmann equation describes evolution of a dilute gas. Investigations of the spatially homogeneous Boltzmann equation have made a lot of progresses in the last decades
and it is hoped to provide useful clues for the understanding of the complete (spatially in-homogeneous) Boltzmann equation. The latter is more realistic and interesting to physics and mathematics but remains still largely out of reach mathematically and will most likely need long term preparations and efforts. For review and references of these areas, the reader may consult for instance [28, 13, 21].

The present paper is a follow-up to our previous work [21] on measure-valued solutions to the spatially homogeneous Boltzmann equation for hard potentials. In this second part, we prove that, under some angular cutoff assumptions (which include the hard sphere model), solutions with measure-valued initial data having finite mass and energy converge strongly to equilibrium in the exponential rate $e^{-\lambda t}$, where $\lambda > 0$ is the spectral gap of the corresponding linearized collision operator. This sharp exponential rate was first proved in [24] for initial data with bounded energy, and belonging to $L^1$ (for the hard sphere model) or to $L^1 \cap L^2$ (for all hard potentials with cutoff). The core idea underlying our improvement of this result to measure solutions is that instead of considering a one-step iteration of the collision integral which produces the $L^1 \cap L^2$ integrability for the hard sphere model (as first observed by Abrahamsson [1], elaborating upon an idea in [22]), we consider a multi-steps iteration which produces the $L^1 \cap L^\infty$ integrability for all hard potentials with angular cutoff. This, together with approximation by $L^1$ solutions through the Mehler transform, and the property of the exponential moment production, enables us to apply the results of [24] and obtain the same convergence rate $e^{-\lambda t}$ for measure solutions. We also obtain a lower bound of the convergence rate and establish the global in time strong stability estimate. As a consequence we prove that, for any hard potentials with cutoff, there are no eternal measure solutions with finite and non-zero temperature, apart from the Maxwellians.

1.1. The spatially homogeneous Boltzmann equation. The spatially homogeneous Boltzmann equation takes the form

$$\frac{\partial}{\partial t} f_t(v) = Q(f_t, f_t)(v), \quad (v, t) \in \mathbb{R}^N \times (0, \infty), \quad N \geq 2$$

with some given initial data $f_t(v)|_{t=0} = f_0(v) \geq 0$, where $Q$ is the collision integral defined by

$$Q(f, f)(v) = \int_{\mathbb{R}^N \times S^{N-1}} B(v - v_s, \sigma) \left( f(v') f(v'_s) - f(v) f(v_s) \right) d\sigma dv_s.$$  

In the latter expression, $v, v_s$ and $v', v'_s$ stand for velocities of two particles after and before their collision, and the microscopic conservation laws of an elastic collision

$$v' + v'_s = v + v_s, \quad |v'|^2 + |v'_s|^2 = |v|^2 + |v_s|^2.$$  

As in our previous work [1], the “measure-valued solutions” shall also be simply called “measure solution”.
induce the following relations:

$$v' = \frac{v + v_s}{2} + \frac{|v - v_s|}{2}/\sigma, \quad v_s' = \frac{v + v_s}{2} - \frac{|v - v_s|}{2}/\sigma$$

for some unit vector $\sigma \in \mathbb{S}^{N-1}$.

The collision kernel $B(z, \sigma)$ under consideration is assumed to have the following product form

$$B(z, \sigma) = |z|^{\gamma} b\left(\frac{z}{|z|} \cdot \sigma\right), \quad \gamma \in (0, 2],$$

where $b$ is a nonnegative Borel function on $[-1, 1]$. This corresponds to the so-called inverse power-law interaction potentials between particles. Throughout this paper we assume that the function $b$ satisfies Grad’s angular cutoff:

$$0 < A_0 := \int_{\mathbb{S}^{N-1}} b\left(\frac{z}{|z|} \cdot \sigma\right) d\sigma = |\mathbb{S}^{N-2}| \int_0^\pi b(\cos \theta) \sin^{N-2} \theta d\theta < +\infty$$

where $|\mathbb{S}^{N-2}|$ denotes the Lebesgue measure of the $(N-2)$-dimensional sphere $\mathbb{S}^{N-2}$. This enables us to split the collision integral as

$$Q(f, g) = Q^+(f, g) - Q^-(f, g)$$

with the two bilinear operators

$$Q^+(f, g)(v) = \iint_{\mathbb{R}^N \times \mathbb{S}^{N-1}} B(v - v_s, \sigma)f(v')g(v_s') d\sigma dv_s,$$

$$Q^-(f, g)(v) = A_0 f(v) \int_{\mathbb{R}^N} |v - v_s|^{\gamma}g(v_s) dv_s,$$

which are nonnegative when applied to nonnegative functions.

The bilinear operators $Q^\pm$ are bounded from $L^1_{s+\gamma}(\mathbb{R}^N) \times L^1_{s+\gamma}(\mathbb{R}^N)$ to $L^1_s(\mathbb{R}^N)$ for $s \geq 0$, where $L^1_s(\mathbb{R}^N)$ is a subspace of $L^1(\mathbb{R}^N) := L^1(\mathbb{R}^N)$ defined by

$$f \in L^1_s(\mathbb{R}^N) \iff \|f\|_{L^1_s} = \int_{\mathbb{R}^N} \langle v \rangle^s |f(v)| dv < +\infty.$$
1.2. The definition of the solutions. The equation (1.1) is usually solved as an integral equation as follows. Given any $0 \leq f_0 \in L^1_2(\mathbb{R}^N)$, we say that a nonnegative Lebesgue measurable function $v(t) \mapsto f_t(v)$ on $[0, \infty) \times \mathbb{R}^N$ is a mild solution to (1.1) if for every $t \geq 0$, $v \mapsto f_t(v)$ belongs to $L^1_2(\mathbb{R}^N)$, $\sup_{t \geq 0} \| f_t \|_{L^1_2} < \infty$, and there is a Lebesgue null set $Z_0$ (which is independent of $t$) such that

$$
\begin{align*}
\forall t &\in [0, \infty), \quad \forall v \in \mathbb{R}^N \setminus Z_0, \quad \int_0^t Q^\pm(f_\tau, f_\tau)(v) \, d\tau < \infty, \\
\forall t &\in [0, \infty), \quad \forall v \in \mathbb{R}^N \setminus Z_0, \quad f_t(v) = f_0(v) + \int_0^t Q(f_\tau, f_\tau)(v) \, d\tau.
\end{align*}
$$

The bilinear operators $f, g \mapsto Q^\pm(f, g)$ can now be extended to measures. For every $s \geq 0$, let $\mathcal{B}_s(\mathbb{R}^N)$ be the Banach space of real Borel measures on $\mathbb{R}^N$ with the norm

$$
\| F \|_s := \int_{\mathbb{R}^N} \langle v \rangle^s \, d|F|(v) < +\infty,
$$

where the positive Borel measure $|F|$ is the total variation of $F$. This norm $\| \cdot \|_s$ can also be defined by duality:

$$
\| F \|_s = \sup_{\varphi \in C_c(\mathbb{R}^N), \| \varphi \|_{L^\infty} \leq 1} \left| \int_{\mathbb{R}^N} \varphi(v) \langle v \rangle^s \, dF(v) \right|.
$$

The latter form is convenient when dealing with the difference of two positive measures. The norms $\| \cdot \|_s$ and $\| \cdot \|_{L^1_2}$ are related by

$$
\| F \|_s = \| f \|_{L^1_2} \quad \text{if} \quad dF(v) = f(v) \, dv.
$$

For any $F, G \in \mathcal{B}_\gamma(\mathbb{R}^N)$ (where $\gamma \in (0, 2]$ appears in (1.5)), we define the Borel measures $Q^\pm(F, G)$ and

$$
Q(F, G) = Q^+(F, G) - Q^-(F, G)
$$

through Riesz’s representation theorem by

$$
\int_{\mathbb{R}^N} \psi(v) \, dQ^+(F, G)(v) = \int_{\mathbb{R}^N \times \mathbb{R}^N} L_B[\psi](v, v_*) \, dF(v) \, dG(v_*),
$$

$$
\int_{\mathbb{R}^N} \psi(v) \, dQ^-(F, G)(v) = A_0 \int_{\mathbb{R}^N \times \mathbb{R}^N} |v - v_*|^{\gamma} \psi(v) \, dF(v) \, dG(v_*)
$$

for all bounded Borel functions $\psi$, where

$$
L_B[\psi](v, v_*) = |v - v_*|^{\gamma} \int_{N-1} b(\sigma) \psi(v') \, d\sigma, \quad n = \frac{v - v_*}{|v - v_*|}
$$

and in case $v = v_*$ we define $n$ to be a fixed unit vector $e_1$. It is easily shown (see Proposition 2.3 of [21]) that the extended bilinear operators $Q^\pm$ are also bounded from $\mathcal{B}_{s+\gamma}(\mathbb{R}^N) \times \mathcal{B}_{s+\gamma}(\mathbb{R}^N)$ to $\mathcal{B}_s(\mathbb{R}^N)$ for $s \geq 0$: if $F, G \in \mathcal{B}_{s+\gamma}(\mathbb{R}^N)$ then $Q^\pm(F, G) \in \mathcal{B}_s(\mathbb{R}^N)$ and

$$
\| Q^\pm(F, G) \|_s \leq 2^{(s+\gamma)/2} A_0 (\| F \|_{s+\gamma} \| G \|_0 + \| F \|_0 \| G \|_{s+\gamma}),
$$

where $A_0$ is a positive constant.
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\[ \|Q^\pm(F, F) - Q^\pm(G, G)\|_s \leq 2^{(s+\gamma)/2} A_0 (\|F + G\|_{s+\gamma} \|F - G\|_0 + \|F + G\|_0 \|F - G\|_{s+\gamma}). \]

Let us finally define the cone of positive distributions with \( s \) moments bounded:

\[ B^+_s(\mathbb{R}^N) := \left\{ F \in B_s(\mathbb{R}^N) \mid F \geq 0 \right\}. \]

We can now define the notion of solutions we shall use in this paper.

**Definition 1.1** (Measure strong solutions). Let \( B(z, \sigma) \) be given by (1.5) with some \( \gamma \in (0, 2] \), with the condition (1.6). Let \( \{F_t\}_{t \geq 0} \subset B^+_2(\mathbb{R}^N) \). We say that \( \{F_t\}_{t \geq 0} \), or simply \( F_t \), is a measure strong solution of equation (1.1) if it satisfies the following:

(i) \( \sup_{t \geq 0} \|F_t\|_2 < \infty \),

(ii) \( t \mapsto F_t \in C([0, \infty); B_2(\mathbb{R}^N)) \cap C^1([0, \infty); B_0(\mathbb{R}^N)) \) and

\[ \forall t \in [0, \infty), \quad \frac{d}{dt} F_t = Q(F_t, F_t). \]

Furthermore \( F_t \) is called a conservative solution if \( F_t \) conserves the mass, momentum and energy, i.e.

\[ \forall t \geq 0, \quad \int_{\mathbb{R}^N} \left( \frac{1}{v} \right) dF_t(v) = \int_{\mathbb{R}^N} \left( \frac{1}{v} \right) dF_0(v). \]

Observe that (1.18) and (1.19) imply the strong continuity of \( t \mapsto F_t \in C([0, \infty); B_2(\mathbb{R}^N)) \) and therefore the strong continuity of \( t \mapsto Q(F_t, F_t) \in C([0, \infty); B_0(\mathbb{R}^N)) \). Hence the differential equation (1.20) is equivalent to the integral equation

\[ \forall t \geq 0, \quad F_t = F_0 + \int_0^t Q(F_\tau, F_\tau) \, d\tau, \]

where the integral is taken in the sense of the Riemann integration or more generally in the sense of the Bochner integration. Recall also that here the derivative \( d\mu_t/dt \) and integral \( \int_a^b \nu_t \, dt \) are defined by

\[ \left( \frac{d}{dt} \mu_t \right)(E) = \frac{d}{dt} \mu_t(E), \quad \left( \int_a^b \nu_t \, dt \right)(E) = \int_a^b \nu_t(E) \, dt \]

for all Borel sets \( E \subset \mathbb{R}^N \).

1.3. **Recall of the main results of the first part.** The following results which will be used in the present paper are extracted from our previous paper [21].

**Theorem 1.2** ([21]). Let \( N \geq 2, \gamma \in (0, 2] \) and \( B(z, \sigma) \) as defined in (1.5). Then for any \( F_0 \in B^+_2(\mathbb{R}^N) \) with \( \|F_0\|_0 > 0 \), there exists a unique conservative measure strong solution \( F_t \) of equation (1.1) satisfying \( F_t|_{t=0} = F_0 \). Moreover this solution satisfies:
(a) $F_t$ satisfies the moment production estimate:

\[ \forall t > 0, \quad \forall s \geq 0, \quad \| F_t \|_s \leq K_s \left( 1 + \frac{1}{t} \right)^{(s-2)+} \]

where

\[ K_s := K_s(\| F_0 \|_0, \| F_0 \|_2) = \| F_0 \|_2 \left[ 2^{s+\gamma} \frac{\| F_0 \|_2}{\| F_0 \|_0} \left( 1 + \frac{1}{16\| F_0 \|_2 A_2 \gamma} \right) \right]^{(s-2)+} \]

\[ A_2 := |S^{N-2}| \int_0^\pi b(\cos \theta) \sin^N \theta \, d\theta. \]

(b) If $\gamma \in (0, 2)$ or if

\[ \gamma = 2 \quad \text{and} \quad \exists 1 < p < \infty \quad \text{s.t.} \quad \int_0^\pi [b(\cos \theta)]^p \sin^{N-2} \theta \, d\theta < \infty \]

then $F_t$ satisfies the exponential moment production estimate:

\[ \forall t > 0, \quad \int_{\mathbb{R}^N} e^{\alpha(t)(v)\gamma} \, dF_t(v) \leq 2\| F_0 \|_0 \]

where

\[ \alpha(t) = 2^{-s_0} \frac{\| F_0 \|_0}{\| F_0 \|_2} \left( 1 - e^{-\beta t} \right), \quad \beta = 16\| F_0 \|_2 A_2 \gamma > 0, \]

and $1 < s_0 < \infty$ depends only on the function $b$ and $\gamma$.

(c) Let $G_t$ be a conservative measure strong solutions of equation (1.1) on the time-interval $[\tau, \infty)$ with an initial datum $G_t|_{t=\tau} = G_\tau \in B_2^+(\mathbb{R}^N)$ for some $\tau \geq 0$. Then:

- If $\tau = 0$, then

\[ \forall t \geq 0, \quad \| F_t - G_t \|_2 \leq \Psi_{F_0}(\| F_0 - G_0 \|_2) e^{C(1+t)}, \]

where

\[ \Psi_{F_0}(r) = r + r^{1/3} + \int_{|v| > r^{-1/3}} |v|^2 \, dF_0(v), \quad r > 0, \quad \Psi_{F_0}(0) = 0, \]

and $C = R(\gamma, A_0, A_2, \| F_0 \|_0, \| F_0 \|_2)$ is an explicit positive continuous function on $(\mathbb{R}_+^+)^5$.

- If $\tau > 0$, then

\[ \forall t \in [\tau, \infty), \quad \| F_t - G_t \|_2 \leq \| F_\tau - G_\tau \|_2 e^{C_\tau(t-\tau)}, \]

where

\[ C_\tau := 4 (K_{2+\gamma} + \| F_0 \|_2) \left( 1 + \frac{1}{\tau} \right), \]

and $K_{2+\gamma}$ is defined by (1.23) with $s = 2 + \gamma$. 
(d) If $F_0$ is absolutely continuous with respect to the Lebesgue measure, i.e.

$$dF_0(v) = f_0(v) \, dv \quad \text{with} \quad 0 \leq f_0 \in L^1_2(\mathbb{R}^N),$$

then $F_t$ is also absolutely continuous with respect to the Lebesgue measure: $dF_t(v) = f_t(v) \, dv$ for all $t \geq 0$, and $f_t$ is the unique conservative mild solution of equation (1.1) with the initial datum $f_0$.

(e) If $F_0$ is not a single Dirac distribution, then there is a sequence $f_{k,t}$, $k \geq 1$, of conservative mild solutions of equation (1.1) with initial data $0 \leq f_{k,0} \in L^1_2(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} \left( \frac{1}{|v|^2} \right) f_{k,0}(v) \, dv = \int_{\mathbb{R}^N} \left( \frac{1}{|v|^2} \right) dF_0(v), \quad k = 1, 2, \ldots$$

such that

$$\forall \varphi \in C_b(\mathbb{R}^N), \quad \forall t \geq 0, \quad \lim_{k \to \infty} \int_{\mathbb{R}^N} \varphi(v) f_{k,t}(v) \, dv = \int_{\mathbb{R}^N} \varphi(v) \, dF_t(v).$$

Besides, the initial data can be chosen of the form $f_{k,0} = I_n[F_0]$, $k = 1, 2, 3, \ldots$ where $\{I_n[F_0]\}_{k=1}^\infty$ is a subsequence of the Mehler transforms $\{I_n[F_0]\}_{n=1}^\infty$ of $F_0$.

Remarks 1.3. (1) In the physical case, $N = 3$ and $0 < \gamma \leq 1$, the moment estimates (1.22) and (1.26) also hold for conservative weak measure solutions of equation (1.1) without angular cutoff (see [21]).

(2) The Mehler transform $I_n[F]$ of a measure $F \in \mathcal{B}^+_2(\mathbb{R}^N)$ will be given in Section 4 below. It satisfies the convenient property that

$$\lim_{n \to +\infty} \|I_n[F] - M\|_2 = \|F - M\|_2$$

and thus is a useful tool in order to reduce the study of properties of measure solutions to that of $L^1$ solutions.

1.4. Normalization. In most of the estimates in this paper, we shall try as much as possible to make explicit the dependence on the basic constants in the assumptions. But first let us study the reduction that can be obtained by scaling arguments.

Under the assumption (1.6), it is easily seen that $F_t$ is a measure solution of equation (1.1) with the angular function $b$ if and only if $t \mapsto F_{A_0^{-1}t}$ is a measure solution of equation (1.1) with the scaled angular function $A_0^{-1}b$. Therefore without loss of generality we can assume the normalization

$$A_0 = 1.$$  

(1.33)

Next given any $\rho > 0$, $u \in \mathbb{R}^N$ and $T > 0$, we define the bounded positive linear operator $\mathcal{N}_{\rho,u,T}$ on $\mathcal{B}_2(\mathbb{R}^N)$ as follows: for any $F \in \mathcal{B}_2(\mathbb{R}^N)$, there is a unique $\mathcal{N}_{\rho,u,T}(F) \in \mathcal{B}_2(\mathbb{R}^N)$
such that (thanks to Riesz’s representation theorem),
\[ \forall \psi \text{ Borel function s.t. } \sup_{v \in \mathbb{R}^N} |\psi(v)|(v)^{-2} < +\infty, \]
\[ \int_{\mathbb{R}^N} \psi(v) d\mathcal{N}_{\rho,u,T}(F)(v) = \frac{1}{\rho} \int_{\mathbb{R}^N} \psi \left( \frac{v - u}{\sqrt{T}} \right) dF(v). \]
We call \( \mathcal{N}_{\rho,u,T} \) the normalization operator associated with \( \rho, u, T \). The inverse \( \mathcal{N}_{\rho,u,T}^{-1} \) of \( \mathcal{N}_{\rho,u,T} \) is given by \( \mathcal{N}_{\rho,u,T}^{-1} = \mathcal{N}_{1/\rho,-u/\sqrt{T},1/T}^{-1} \), i.e.
\[ \int_{\mathbb{R}^N} \psi(v) d\mathcal{N}_{\rho,u,T}^{-1}(F)(v) = \rho \int_{\mathbb{R}^N} \psi \left( \sqrt{T} v + u \right) dF(v). \]
It is easily seen that for every \( F \in \mathcal{B}_2(\mathbb{R}^N) \)
\[ ||\mathcal{N}_{\rho,u,T}(F)||_0 = \frac{1}{\rho} ||F||_0, \]
\[ ||\mathcal{N}_{\rho,u,T}(F)||_2 \leq C_{\rho,u,T} ||F||_2, \]
\[ ||\mathcal{N}_{\rho,u,T}^{-1}(F)||_2 \leq C_{1/\rho,u/\sqrt{T},1/T} ||F||_2 \]
where
\[ C_{\rho,u,T} = \frac{1}{\rho} \max \left\{ 1 + \frac{|u|^2 + |u|}{T}, \frac{1 + |u|}{T} \right\}, \]
\[ C_{1/\rho,u/\sqrt{T},1/T} = \rho \max \left\{ 1 + |u|^2 + \sqrt{T}|u|; T + \sqrt{T}|u| \right\}. \]

We then introduce the subclass \( \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N) \) of \( \mathcal{B}_2^+(\mathbb{R}^N) \) by
\[ F \in \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N) \iff F \in \mathcal{B}_2^+(\mathbb{R}^N) \text{ and } \]
\[ \int_{\mathbb{R}^N} dF(v) = \rho, \quad \int_{\mathbb{R}^N} v dF(v) = u, \quad \frac{1}{N \rho} \int_{\mathbb{R}^N} |v - u|^2 dF(v) = T. \]
When restricting \( \mathcal{N}_{\rho,u,T} \) on \( \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N) \), it is easily seen that
\[ \mathcal{N}_{\rho,u,T} : \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N) \to \mathcal{B}_{1,0,1}^+(\mathbb{R}^N), \quad \mathcal{N}_{\rho,u,T}^{-1} : \mathcal{B}_{1,0,1}^+(\mathbb{R}^N) \to \mathcal{B}_{\rho,u,T}^+(\mathbb{R}^N). \]

Similarly we define \( L_{\rho,u,T}^1(\mathbb{R}^N) \) by
\[ f \in L_{\rho,u,T}^1(\mathbb{R}^N) \iff \]
\[ \left\{ 0 \leq f \in L_2^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} f(v) dv = \rho, \right\} \quad \int_{\mathbb{R}^N} v f(v) dv = u, \quad \frac{1}{N \rho} \int_{\mathbb{R}^N} |v - u|^2 f(v) dv = T. \]
In this case, the normalization operator \( \mathcal{N} = \mathcal{N}_{\rho,u,T} : L_{\rho,u,T}^1(\mathbb{R}^N) \to L_{1,0,1}^1(\mathbb{R}^N) \) writes
\[ \mathcal{N}(f)(v) = \frac{T^{N/2}}{\rho} f \left( \sqrt{T} v + u \right). \]
We denote the Maxwellian \( M \in L_{\rho,u,T}^1(\mathbb{R}^N) \) which is defined by
\[ M(v) := \frac{\rho}{(2\pi T)^{N/2}} \exp \left( -\frac{|v - u|^2}{2T} \right). \]
For notational convenience we do not distinguish between a Maxwellian distribution \( M \in \mathcal{B}_{p,u,T}^+(\mathbb{R}^N) \) and its density function \( M \in L^1_{p,u,T}(\mathbb{R}^N) \): we write without risk of confusion that
\[
dM(v) = M(v) \, dv.
\]

Due to the homogeneity of \( z \mapsto B(z, \sigma) \), we have by translation
\[
L_B \left[ \psi \left( \frac{\cdot - u}{\sqrt{T}} \right) \right] \langle v, v_s \rangle = T^{\gamma/2} L_B[\psi] \left( \frac{v - u, v_s - u}{\sqrt{T}} \right)
\]
and then by Fubini’s theorem we get (denoting simply \( N = N_{p,u,T} \) when no ambiguity is possible)
\[
\forall F \in \mathcal{B}^+_2(\mathbb{R}^N), \quad N \left( Q^\pm(F,F) \right) = \rho T^{\gamma/2} Q^\pm(N(F),N(F)) .
\]

Since \( N \) is linear and bounded, this implies that if \( F_t \) is a measure strong solution of equation (1.1) and \( c = \rho T^{\gamma/2} \), then
\[
\frac{d}{dt} N(F_{t,c}) = Q \left( N(F_{t,c}), N(F_{t,c}) \right).
\]
This, together with (1.34)-(1.36), leads to the following statement:

**Proposition 1.4** (Normalization). Let \( B(z, \sigma) \) be defined by (1.5) for some \( \gamma \in (0,2] \), with the condition (1.6). Let \( F_0 \in \mathcal{B}^+_{p,u,T}(\mathbb{R}^N) \) with \( \rho > 0, u \in \mathbb{R}^N \) and \( T > 0 \), and let \( F_t \) be the unique conservative measure strong solution of equation (1.1) with the initial datum \( F_0 \). Let \( M \in \mathcal{B}^+_{p,u,T}(\mathbb{R}^N) \) be the Maxwellian defined by (1.42), let \( N := N_{p,u,T} \) be the normalization operator, and let \( c = \rho T^{\gamma/2} \). Then:

(I) The application \( t \mapsto N(F_{t,c}) \) is the unique conservative measure strong solution of equation (1.1) with the initial datum \( N(F_0) \in \mathcal{B}^+_{1,0,1}(\mathbb{R}^N) \).

(II) For all \( t \geq 0 \)
\[
\begin{align*}
\| F_t - M \|_0 & = \rho \| N(F_t) - N(M) \|_0, \\
\| F_t - M \|_2 & \leq C_{1/\rho,|u|/\sqrt{T},1/T} \| N(F_t) - N(M) \|_2, \\
\| N(F_t) - N(M) \|_2 & \leq C_{\rho,|u|,T} \| F_t - M \|_2
\end{align*}
\]
where \( C_{\rho,|u|,T} \) and \( C_{1/\rho,|u|/\sqrt{T},1/T} \) are given in (1.37)-(1.38).

1.5. **Linearized collision operator and spectral gap.** For any nonnegative Borel function \( w \) on \( \mathbb{R}^N \) we define the weighted Lebesgue space \( L^p(\mathbb{R}^N, w) \) with \( 1 \leq p < \infty \) by
\[
f \in L^p(\mathbb{R}^N, w) \iff \| f \|_{L^p(w)} := \left( \int_{\mathbb{R}^N} |f(v)|^p w(v) \, dv \right)^{1/p} < +\infty.
\]

Let \( B(z, \sigma) \) as defined in (1.5) with \( \gamma \in (0,2] \) with \( b \) satisfying the condition (1.6). Let \( M \) be the Maxwellian with mass \( \rho > 0 \), mean velocity \( u \) and temperature \( T > 0 \) defined in (1.42), and let
\[
L_M : L^2(\mathbb{R}^N, M^{-1}) \to L^2(\mathbb{R}^N, M^{-1})
\]
be the linearized collision operator associated with \( B(z,\sigma) \) and \( M(v) \), i.e.
\[
L_M(h)(v) = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} B(v-v_*,\sigma)M(v_*) \left( h' + h'_* - h - h_* \right) \, d\sigma \, dv_*.
\]
It is well-known that the spectrum \( \Sigma(L_M) \) of \( L_M \) is contained in \((-\infty, 0]\) and has a positive spectral gap \( S_{b,\gamma}(\rho,\mu,T) > 0 \), i.e.
\[
S_{b,\gamma}(\rho,\mu,T) := \inf \{ \lambda > 0 \mid -\lambda \in \Sigma(L_M) \} > 0.
\]
Moreover by simple calculations, one has the following scaling property on this spectral gap
\[
S_{b,\gamma}(\rho,u,T) = \rho T^{\gamma/2} S_{b,\gamma}(1,0,1).
\]
In the spatially homogeneous case, the study of the linearized collision operator goes back to Hilbert [17, 18] who computed the collisional invariant, the linearized operator and its kernel in the hard spheres case, and showed the boundedness and “complete continuity” of its non-local part. Carleman [6] then proved the existence of a spectral gap by using Weyl’s theorem and the compactness of the non-local part proved by Hilbert. Grad [14, 15] then extended these results to the case of hard potentials with cutoff. All these results are based on non-constructive arguments. The first constructive estimates in the hard spheres case were obtained only recently in [2] (see also [23] for more general interactions, and [25] for a review). Let us also mention the works [30, 3, 4] for the different setting of Maxwell molecules where the eigenbasis and eigenvalues can be explicitly computed by Fourier transform methods. Although these techniques do not apply here, the explicit formula computed are an important source of inspiration for dealing with more general physical models.

1.6. Main results. In order to use the results obtained in [24] (see also [29, 26]) for \( L^1 \) solutions, we shall need the following additional assumptions for some of our main results:
\[
\|b\|_{L^\infty} := \sup_{t \in [-1,1]} b(t) < \infty,
\]
\[
\inf_{t \in [-1,1]} b(t) > 0.
\]
Recall that for the hard sphere model, i.e. \( \gamma = 1, N = 3 \) and \( b \equiv \text{const.} > 0 \), the conditions \((1.44) - (1.45)\) are satisfied.

The first main result of this paper is concerned with the upper bound of the rate of convergence to equilibrium when the dimension \( N \) is greater or equal to 3.

**Theorem 1.5** (Sharp exponential relaxation rate). Suppose \( N \geq 3 \) and let \( B(z,\sigma) \) be given by \((1.5)\) for some \( \gamma \in (0,1] \) with the condition \((1.6)\) and the bounds \((1.44)\) and \((1.45)\). Let \( \rho > 0, u \in \mathbb{R}^N \) and \( T > 0 \), and let
\[
\lambda = S_{b,\gamma}(\rho,\mu,T) = S_{b,\gamma}(1,0,1)\rho T^{\gamma/2} > 0
\]
be the spectral gap for the linearized collision operator \( L_F \) associated with \( B(z, \sigma) \) and the Maxwellian \( M \in \mathcal{B}_{\rho,u,T}(\mathbb{R}^N) \). Then for any conservative measure strong solution \( F_t \) of the equation \((1.1)\) with \( F_0 \in \mathcal{B}_{\rho,u,T}(\mathbb{R}^N) \) we have:
\[
\forall t \geq 0, \quad \| F_t - M \|_2 \leq C \| F_0 - M \|_2^{1/2} e^{-\lambda t}
\]
where
\[
C := C_0 C_{1/\rho,|u|/\sqrt{T},1/T} [C_{\rho,|u|,T}]^{1/2}
\]
with \( C_{\rho,|u|,T} \) and \( C_{1/\rho,|u|/\sqrt{T},1/T} \) given in (1.35) and (1.36), and with some constant \( C_0 < \infty \) which only depends on \( N, b, \gamma \).

**Remark 1.6.** Applying Theorem 1.5 to the standard initial data and the Maxwellian \( F_0, M \in \mathcal{B}_{1,0,1}(\mathbb{R}^N) \), and using \( \| F_0 - M \|_2^{1/2} \leq (\| F_0 \|_2 + \| M \|_2)^{1/2} = (2(1 + N))^{1/2} \) we have
\[
\forall t \geq 0, \quad \| F_t - M \|_2 \leq C_0 e^{-\lambda t}, \quad \lambda = S_{b,\gamma}(1,0,1)
\]
where \( C_0 < \infty \) only depends on \( N, b, \gamma \). Then by normalization (using Proposition 1.4) and the relation \( S_{b,\gamma}(\rho, \mu, T) = S_{b,\gamma}(1,0,1) \rho T^{\gamma/2} \), we conclude that if \( F_0, M \in \mathcal{B}_{\rho,u,T}(\mathbb{R}^N) \), then
\[
\| F_t - M \|_2 \leq C_0 C_{1/\rho,|u|/\sqrt{T},1/T} e^{-\lambda t}, \quad t \geq 0; \quad \lambda = S_{b,\gamma}(\rho, u, T)
\]
for the same constant \( C_0 \). This estimate will be used in proving our next results Corollary 1.8 and Theorem 1.9.

The second main result is concerned with the lower bound of the rate of convergence to equilibrium.

**Theorem 1.7** (Lower bound on the relaxation rate). Suppose \( N \geq 2 \) and let \( B(z, \sigma) \) be given by (1.6) for some \( \gamma \in (0,2] \) with the condition (1.9). Let \( \rho > 0, u \in \mathbb{R}^N, T > 0 \) and let \( M \in \mathcal{B}_{\rho,u,T}(\mathbb{R}^N) \) be the Maxwellian.

Then for any conservative measure strong solution \( F_t \) of equation \((1.1)\) with initial data \( F_0 \in \mathcal{B}_{\rho,u,T}(\mathbb{R}^N) \) we have:

(i) If \( 0 < \gamma < 2 \), then
\[
\forall t \geq 0, \quad \| F_t - M \|_0 \geq (4\rho)^{1-\alpha} \| F_0 - M \|_0^2 \exp \left(-\beta t \frac{2}{\gamma^2} \right)
\]
where
\[
\alpha = \left( \frac{2}{\gamma} \right)^{\frac{1}{2-\gamma}} \quad \text{and} \quad \beta = \left( 1 - \frac{2}{\gamma} \right) \left( 2^6 (N + 1)^2 \rho T^{\gamma/2} \right)^{\frac{2}{\gamma^2}}.
\]

(ii) If \( \gamma = 2 \), then
\[
\forall t \geq 0, \quad \| F_t - M \|_0 \geq 4\rho \left( \frac{\| F_0 - M \|_0}{4\rho} \right)^{e^{\kappa t}}
\]
with \( \kappa = 2^6 (N + 1)^2 \rho T \).
Let us state an important corollary of Theorem 1.5 and Theorem 1.7, where part (iii) of the corollary gives a positive answer (for hard potentials) to the question of eternal solutions raised in [28, Chapter 1, subsection 2.9].

**Corollary 1.8.** We make the same assumptions on $N, \gamma$ and $B(z, \sigma)$ as in Theorem 1.5. Let $F_0 \in B^+_{\rho, u, T}(\mathbb{R}^N)$ with $\rho > 0$, $u \in \mathbb{R}^N$ and $T > 0$, and let $M \in B^+_{\rho, u, T}(\mathbb{R}^N)$ be the Maxwellian. We have:

(i) Let $F_t \in B^+_{\rho, u, T}(\mathbb{R}^N)$ be the unique conservative measure solution of equation (1.1) on $[0, \infty)$ with the initial datum $F_0$. If $F_0 \neq M$, then $F_t \neq M$ for all $t \geq 0$. In other words, $F_t$ cannot arrive at equilibrium state in finite time unless $F_0$ is an equilibrium.

(ii) Let $F_t \in B^+_{\rho, u, T}(\mathbb{R}^N)$ be a conservative backward measure strong solution of equation (1.1) on an interval $(-t_\infty, 0]$ for some $0 < t_\infty \leq +\infty$, i.e.

$$\frac{d}{dt}F_t = Q(F_t, F_t), \quad t \in (-t_\infty, 0].$$

Then if $F_0 \neq M$, then $(-t_\infty, 0]$ must be bounded, and if $F_0 = M$, then $F_t \equiv M$ on $(-t_\infty, 0]$. In particular we have

(iii) [Eternal measure solutions are stationary.] If a conservative measure strong solution $F_t$ of equation (1.1) in $B^+_{\rho, u, T}(\mathbb{R}^N)$ is eternal, i.e. defined on $t \in \mathbb{R}$, then it has to be stationary and $F_t = M$ for all times $t \in \mathbb{R}$.

The proof of this Corollary is short and we can present it here.

**Proof of Corollary 1.8.** Part (i) is follows simply from the lower bound in Theorem 1.7. Part (iii) follows from part (ii). In fact let $F_t$ be an eternal solution of equation (1.1) as defined in the part (iii) of the statement. Then $F_t$ is also a backward measure strong solution of equation (1.1) on the unbounded interval $(-\infty, 0]$. By part (ii) we conclude that $F_0 = M$ and thus $F_t \equiv M$ on $(-\infty, 0]$. Then by the uniqueness of forward solutions we conclude that $F_t = M$ for all $t \in \mathbb{R}$.

To prove part (ii), we use the existence and the uniqueness theorem of conservative measure strong solutions (see Theorem 1.2) to extend the backward solution $F_t$ to the whole interval $(-t_\infty, +\infty)$. Fix any $\tau \in (-t_\infty, 0)$. Then $t \mapsto F_{\tau+t}$ is a conservative measure strong solution of equation (1.1) on $[0, +\infty)$ with the initial datum $F_\tau$. By using the upper bound of the convergence rate in Theorem 1.5 (see also (1.46)), together with the conservation of mass and energy we have (with $\lambda = S_{b, \gamma}(\rho, u, T)$)

$$\|F_{\tau+t} - M\|_0 \leq Ce^{-\lambda t}, \quad \forall t \geq 0,$$

where $C > 0$ only depends on $N, b, \gamma, \rho, u, T$. Taking $t = -\tau$ gives

$$\|F_0 - M\|_0 \leq Ce^{\lambda \tau}.$$
Thus if $\|F_0 - M\|_0 > 0$, then

$$-\tau \leq \frac{1}{\lambda} \log \left( \frac{C}{\|F_0 - M\|_0} \right) < \infty.$$  

Letting $\tau \to -t_\infty$ leads to

$$t_\infty \leq \frac{1}{\lambda} \log \left( \frac{C}{\|F_0 - M\|_0} \right) < \infty.$$  

Next, applying Theorem 1.7, we have for all $t \geq 0$

$$\|F_{\tau+t} - M\|_0 \geq (4\rho)^{1-\alpha} \|F_\tau - M\|^\alpha_0 e^{-\beta t^{2\gamma}} \quad \text{(if } 0 < \gamma < 2)$$  

$$\|F_{\tau+t} - M\|_0 \geq 4\rho \left( \frac{\|F_\tau - M\|}{4\rho} \right)^\alpha e^{\kappa t} \quad \text{(if } \gamma = 2).$$  

Now suppose $\|F_0 - M\|_0 = 0$. Then taking $t = -\tau$ so that $\|F_{\tau+t} - M\|_0 = 0$ we obtain from (1.48), (1.49) that $\|F_\tau - M\|_0 = 0$. Since $\tau \in (-t_\infty, 0)$ is arbitrary, this shows that $F_t \equiv M$ on $(-t_\infty, 0]$ and concludes the proof. 

The third main result is concerned with the global-in-time stability of measure strong solutions.

**Theorem 1.9.** Let $N, \gamma$ and $B(z, \sigma)$ satisfy the same assumptions in Theorem 1.5. Let $\rho_0 > 0$, $u_0 \in \mathbb{R}^N$, $T_0 > 0$ and let $M \in B_{\rho_0, u_0, T}^+ (\mathbb{R}^N)$ be the Maxwellian.

Then for any conservative measure strong solutions $F_t, G_t$ of equation (1.1) with $F_0 \in B_{\rho_0, u_0, T}^+ (\mathbb{R}^N)$, there are explicitable constants $\eta \in (0, 1), C \in (0, \infty)$ only depending on $N, b, \gamma, \rho_0, u_0, T_0$, such that

$$\sup_{t \geq 0} \|F_t - G_t\|_2 \leq \bar{\Psi}_{F_0} (\|F_0 - G_0\|_2)$$  

where

$$\forall r \geq 0, \quad \bar{\Psi}_{F_0}(r) := C \left( r + [\Psi_{F_0}(r)]^\eta \right)$$  

with $\Psi_{F_0}(r)$ defined in (1.29).

1.7. Previous results and references. Apart from the paper [24] already mentionned concerning the sharp rate of relaxation for $L^1$ solutions in the case of hard spheres or hard potentials with cutoff, let us mention the many previous works that developed quantitative estimates on the rate of convergence [7, 8, 9, 10, 11, 11, 11, 11]. Let us also mention the recent work [16] obtaining sharp rates of relaxation for $L^1_t L^\infty_x$ solutions in the spatially inhomogeneous case in the torus.

1.8. Strategy and plan of the paper. The rest of the paper is organized as follows: In Section 2 we give an integral representation for the one-step iterated collision operator $(f, g, h) \mapsto Q^+(f, Q^+(g, h))$ and prove an $L^p$ gain of integrability for this operator. This is a generalization of Abrahamsson’s result [1] which is concerned with $\gamma = 1$ and $N = 3$. In order to obtain the required regularities of such iterated collision operators, the technical
difficulty is to deal with small values of $\gamma$. In that case one needs multi-step iteration of $Q^+$. In Section 3 we use iteratively the previous multi-step estimates on $Q^+$ to give a series of positive decompositions $f_t = f_t^n + h_t^n$ for $t \in [t_0, \infty)$ with $t_0 > 0$, for an $L^1$ mild solution $f_t$. In this decomposition the $f_t^n$ are bounded (in $L^\infty(\mathbb{R}^N)$) and regular (they belong at least to $H^1(\mathbb{R}^N)$ for instance) when $n$ is large enough; whereas $h_t^n$ decays in $L^1$ norm exponentially fast as $t \to \infty$. By approximation we then extend such positive decompositions to the measure strong solutions $F_t$. In Section 4 we first use the results of [24] and those obtained in Section 3 to prove Theorem 1.5 for then we use approximation by $L^\infty$ decompositions to the measure strong solutions. The proof of Theorem 1.7 is given in Section 5. In Section 6 we prove (2.1) belongs at least to $H^1_{\mathrm{f}}$ solution $S$ nonnegative measurable on $f$. Let $\sigma_\gamma$ s.t.

$$
\sigma_\gamma = 0, \quad \forall \gamma \in (0,2].
$$

where $\sigma_\gamma$ is given by

$$
\sigma_\gamma := \begin{cases} 
-\n & \text{if } t \leq -1 \\
\n + \sqrt{1 - t^2} \omega & \text{if } t \in (-1,1) \\
\n & \text{if } t \geq 1.
\end{cases}
$$

Lemma 2.1. Let $N \geq 3$ and let $B(z, \sigma)$ be given by (1.5) with some $\gamma \in (0,2]$ and with the angular function $b$ satisfying (1.44). Let $f \in L^1_{\omega}(\mathbb{R}^N)$ and $g, h \in L^1_{\omega}(\mathbb{R}^N)$. 

2. $L^p$-ESTIMATES OF THE ITERATED GAIN TERM

We introduce the weighted Lebesgue spaces $L^p_s(\mathbb{R}^N)$ for $1 \leq p \leq \infty$, $0 \leq s < \infty$ as:

\[
\begin{align*}
    f \in L^p_s(\mathbb{R}^N) \iff & \quad \|f\|_{L^p_s} = \left(\int_{\mathbb{R}^N} \langle v \rangle^{ps} |f(v)|^p \, dv \right)^{1/p} < +\infty, \quad 1 \leq p < +\infty \\
    f \in L^\infty_s(\mathbb{R}^N) \iff & \quad \|f\|_{L^\infty_s} = \sup_{v \in \mathbb{R}^N} \langle v \rangle^s |f(v)| < +\infty, \quad p = +\infty.
\end{align*}
\]

In the case $s = 0$, we denote $L^p_0(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ as usual.

We shall use the following formula of change of variables. For any $n \in S^{N-1}$ and $\psi$ nonnegative measurable on $S^{N-1}$:

\[
(2.1) \quad \int_{S^{N-1}} \psi(\sigma) \, d\sigma = \int_{-1}^1 (1 - t^2)^{(N-3)/2} \left( \int_{S^{N-2}(n)} \psi \left(tn + \sqrt{1 - t^2} \omega \right) \, d^1\omega \right) \, dt
\]

where $S^{N-2}(n) = \{\omega \in S^{N-1} | \omega \perp n\}$ and $d^1\omega$ denotes the sphere measure element of $S^{N-2}(n)$.

For convenience we rewrite (2.1) as follows:

\[
(2.2) \quad \int_{S^{N-1}} \psi(\sigma) \, d\sigma = \int_{\mathbb{R}} \zeta(t) \left( \int_{S^{N-2}(n)} \psi(\sigma_n(t, \omega)) \, d^1\omega \right) \, dt
\]

where

\[
(2.3) \quad \forall t \in \mathbb{R}, \quad \zeta(t) := (1 - t^2)^{\frac{N-3}{2}} 1_{(-1,1)}(t)
\]

\[
(2.4) \quad \sigma_n(t, \omega) := \begin{cases} 
-\n & \text{if } t \leq -1 \\
\n + \sqrt{1 - t^2} \omega & \text{if } t \in (-1,1) \\
\n & \text{if } t \geq 1.
\end{cases}
\]

The proof of Theorem 1.7 is given in Section 5. In Section 6 we prove Theorem 1.9 which is an application of Theorem 1.2 and Theorem 1.5.
Then \(Q^+(f, Q^+(g, h)) \in L^1(\mathbb{R}^N)\) with the estimate
\[
\|Q^+(f, Q^+(g, h))\|_{L^1} \leq A_0^2\|f\|_{L^1_{\gamma}}\|g\|_{L_{2\gamma}^1}\|h\|_{L_{2\gamma}^1}.
\]

Moreover we have the following representation: for almost every \(v \in \mathbb{R}^N\)
\[
Q^+(f, Q^+(g, h))(v) = \iiint_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N} K_B(v, v_*, w, w_*) f(v_*) g(w) h(w_*) \, dv_* dw dw_*
\]
where \(K_B : \mathbb{R}^{4N} \to [0, \infty)\) is defined by
\[
K_B(v, v_*, w, w_*) := \begin{cases} 
\frac{2^N}{|v - v_*||w - w_*|} \zeta \left( n \cdot \frac{2v - (w + w_*)}{|w - w_*|} \right) & 
\int_S N - 2(n) \frac{B(w - w_*, \sigma) B(w' - v_*, \sigma')}{|w' - v_*|^{N - 2}} q^{1/2} \omega \\
0 & |v - v_*||w - w_*| \neq 0
\end{cases}
\]
where the function \(\zeta\) is given by \((2.3)\), and
\[
n := \frac{v - v_*}{|v - v_*|}, \quad w' = \frac{w + w_*}{2} + \frac{|w - w_*|}{2} \sigma, \quad \sigma' = \frac{2v - v_* - w'}{|2v - v_* - w'|}
\]
with
\[
\sigma = \sigma(\omega) = \sigma_n(t, \omega) \text{ at } t = \frac{2v - (w + w_*)}{|w - w_*|}.
\]

**Remark 2.2.** Inserting the formula \((1.6)\) for \(B(z, \sigma)\) into \((2.7)\) gives the more detailed expression of \(K_B\):
\[
K_B(v, v_*, w, w_*) = \frac{2^N}{|w - w_*|^{1 - \gamma}|v - v_*|} \zeta \left( n \cdot \frac{2v - (w + w_*)}{|w - w_*|} \right) \int_{S N - 2(n)} \frac{b\left(\frac{w - w_*}{|w - w_*|} \cdot \sigma\right) b\left(\frac{w' - v_*}{|w' - v_*|} \cdot \sigma'\right)}{|w' - v_*|^{N - 2 - \gamma}} \, d\omega,
\]
for \(|w - w_*||v - v_*| \neq 0\). Also we note that
\[
K_B(v, v_*, w, w_*) > 0 \implies |v - v_*||w - w_*| \neq 0 \quad \text{and} \quad \left| n \cdot \frac{2v - (w + w_*)}{|w - w_*|} \right| < 1
\]
which implies, by using the formula for \(\sigma_n(t, \omega)\) in this case and the value of \(t\), that \((v - w') \cdot (v - v_*) = 0\) and therefore by Pythagoras’ theorem
\[
|w' - v_*| = |w' - v + v_*| = \sqrt{|v - v_*|^2 + |v - w'|^2},
\]
\[
|w' - v_*| \geq |v - v_*|.
\]

**Proof of Lemma 2.1.** We shall use the following formula of change of variables (see [28] Chapter 1, Sections 4.5-4.6): For every nonnegative measurable function \(\psi\) on \(\mathbb{R}^{4N}\), one
has
\begin{equation}
\prod_{R^N \times R^N \times R^N} B(v - v, \sigma) \psi(v', v, v_*) d\sigma \, dv = \prod_{R^N \times R^N \times R^N} B(v - v, \sigma) \psi(v, v_*, v') d\sigma \, dv.
\end{equation}

We can assume that \( f, g, h \) are all nonnegative. Applying (1.10), (2.14), and recalling definition of \( L_B[\varphi] \) (see (1.17)) we have, for any nonnegative measurable function \( \varphi \) on \( R^N \),
\begin{align*}
\left\{ \begin{array}{l}
\int_{R^N} Q^+(f, Q^+(g, h))(v) \varphi(v) \, dv = \int_{R^N} f(v) \left( \int_{R^N} Q^+(g, h)(w) L_B[\varphi](w, v_*) \, dw \right) \, dv_* , \\
\int_{R^N} Q^+(g, h)(w) L_B[\varphi](w, v_*) \, dw = \int_{R^N \times R^N} L_B[L_B[\varphi](\cdot, v_*)](w, w_*) g(w) h(w_*) \, dw \, dw_* ,
\end{array} \right.
\end{align*}
and so
\begin{equation}
\int_{R^N} Q^+(f, Q^+(g, h))(v) \varphi(v) \, dv = \prod_{R^N \times R^N \times R^N} L_B[L_B[\varphi](\cdot, v_*)](w, w_*) f(v_*) g(w) h(w_*) \, dv_* \, dw \, dw_* .
\end{equation}
Taking \( \varphi = 1 \) and using the inequalities
\begin{equation}
\left\{ \begin{array}{l}
|w - w_*| \leq \langle w \rangle \langle w_* \rangle , \\
|\gamma - v_*| \leq \frac{|w + w_*|}{2} + \frac{|w - w_*|}{2} + |v_*| \leq \langle w \rangle \langle w_* \rangle \langle v_* \rangle ,
\end{array} \right.
\end{equation}
we obtain
\begin{equation}
L_B[L_B[1](\cdot, v_*)](w, w_*) \leq A_0^2 \langle v_* \rangle \langle w \rangle \langle w_* \rangle \langle v_* \rangle \langle v_* \rangle \langle v_* \rangle ,
\end{equation}
and it follows from (2.15) that
\begin{equation}
0 \leq Q^+(f, Q^+(g, h)) \in L^1(R^N)
\end{equation}
and (2.15) holds true.

Comparing (2.15) with (2.16), it appears that in order to prove the integral representation (2.16) we only need to prove that the following identity
\begin{equation}
\forall 0 \leq \varphi \in C_c(R^N), \quad L_B[L_B[\varphi](\cdot, v_*)](w, w_*) = \int_{R^N} K_B(v, v_*, w, w_*) \varphi(v) \, dv
\end{equation}
holds for all \( w, w_*, v_* \in R^N \) satisfying
\begin{equation}
0 \neq \frac{w + w_*}{2} - v_* \neq \frac{|w - w_*|}{2} \neq 0.
\end{equation}
Observe that
\begin{equation*}
\left\{ \begin{array}{l}
L_B[L_B[\varphi](\cdot, v_*)](w, w_*) = L_B[L_B[\varphi(v_*)](\cdot, 0)](w - v_*, w_* - v_*), \\
K_B(v, v_*, w, w_*) = K_B(v - v_*, 0, w - v_*, w_* - v_*).
\end{array} \right.
\end{equation*}
By replacing respectively $\varphi(v_+ \cdot), w - v_+ \cdot w_+ - v_+ \cdot w, w_+ - v_+ \cdot w$ with $\varphi(\cdot), w_+ \cdot w, w_+ - v_+ \cdot w$, we can assume without loss of generality that $v_+ = 0$. That is, in order to prove (2.17), we only need to prove
\begin{equation}
(2.19) \quad \forall 0 \leq \varphi \in C_c(\mathbb{R}^N), \quad L_B [L_B[\varphi](\cdot, 0)] (w, w_*) = \int_{\mathbb{R}^N} K_B(v, 0, w, w_*) \varphi(v) \, dv.
\end{equation}

To do this we first assume that $b \in C([-1, 1])$ so that the use of the Dirac distribution is fully justified. We compute
\begin{equation}
(2.20) \quad L_B [L_B[\varphi](\cdot, 0)] (w, w_*) = \int_{S^{N-1}} B(w - w_*, \sigma) \int_{S^{N-1}} B(w', \omega) \varphi \left( \frac{w'}{2} + \frac{|w'|}{2} \omega \right) \, d\omega \, d\sigma,
\end{equation}
and, by using (2.8) and (2.18) with $v_+ = 0$, we have $w' \neq 0$ for all $\sigma \in S^{N-1}$. Let $\delta = \delta(x)$ be the one-dimensional Dirac distribution. Applying the integral representation
\begin{equation}
\forall \psi \in C((0, \infty)), \quad \rho > 0, \quad \psi(\rho) = \frac{2}{\rho^{N-2}} \int_0^\infty r^{N-1} \psi(r) \delta(\rho^2 - r^2) \, dr
\end{equation}
to the function
\[ \psi(\rho) := \varphi \left( \frac{w'}{2} + \rho \omega \right) \]
and then taking $\rho = |w'|/2$ and changing variable $r \omega = z$ we have
\[ \int_{S^{N-1}} B(w', \omega) \varphi \left( \frac{w'}{2} + \frac{|w'|}{2} \omega \right) \, d\omega = 2 \left| \frac{w'}{2} \right|^{-(N-2)} \int_{\mathbb{R}^N} B \left( \frac{w'}{2}, \frac{z}{|z|} \right) \varphi \left( \frac{w'}{2} + \frac{z}{2} \right) \delta \left( \frac{|w'|^2}{4} - |z|^2 \right) \, dz. \]
We then use the change of variable $z = v - w'/2$ and Fubini’s theorem:
\begin{equation}
L_B [L_B[\varphi](\cdot, 0)] (w, w_*) = \int_{\mathbb{R}^N} \varphi(v) \left( \int_{S^{N-1}} 2 \left| \frac{w'}{2} \right|^{-(N-2)} B(w - w_*, \sigma) B \left( \frac{w'}{2}, \frac{v - w'/2}{|v - w'/2|} \right) \delta \left( \frac{|w'|^2}{4} - |v - w'/2|^2 \right) \, d\sigma \right) \, dv.
\end{equation}

We now assume that $v \neq 0$ and $n = v/|v|$ satisfy
\[ \frac{2v - (w + w_*)}{|w - w_*|} \neq 1, \quad \frac{v - w + w_*}{4} \neq \frac{|w - w_*|}{4}. \]
We deduce that $|v - w'/2| > 0$ for all $\sigma \in S^{N-1}$ and we compute using \(2.2\)–\(2.3\)–\(2.4\) with $n = v/|v|$ that

$$
\int_{S^{N-1}} 2^{-\frac{(N-2)}{2}} B(w - w_*, \sigma) B\left( w', \frac{v - w'/2}{|v - w'|/2} \right) \delta\left( \frac{|w'|^2}{4} - \frac{|v - w'|^2}{4} \right) \, d\sigma
$$

$$
= \int_{\mathbb{R}} \zeta(t) \int_{S^{N-2}(n)} 2^{-\frac{(N-2)}{2}} B(w - w_*, \sigma) B\left( w', \frac{v - w'/2}{|v - w'|/2} \right) \left| \sigma = \sigma_n(t, \omega) \right| \, d\sigma \, dt
$$

$$
\times \delta\left( |v| \frac{|w - w_*|}{2} t - v \cdot \left( \frac{v - w + w_*}{2} \right) \right) \, dt
$$

$$
= \frac{2^N}{|v||w - w_*|^2} \zeta\left( \frac{v \cdot (2v - (w + w_*))}{|v||w - w_*|} \right) \int_{S^{N-2}(n)} |w'|^{-\frac{(N-2)}{2}} B(w - w_*, \sigma) B\left( w', \frac{2v - w'}{|2v - w'|} \right) \, d\omega \, dv.
$$

where $\sigma$ in the last line is given by \(2.9\). Thus we obtain

\[ L_B[L_B[\varphi](\cdot, 0)](w, w_*) = \frac{2^N}{|w - w_*|^2} \int_{\mathbb{R}^N} \frac{\varphi(v)}{|v|} \zeta\left( \frac{n \cdot \frac{2v - (w + w_*)}{|w - w_*|}}{w - w_*} \right) \int_{S^{N-2}(n)} |w'|^{-\frac{(N-2)}{2}} B(w - w_*, \sigma) B\left( w', \frac{2v - w'}{|2v - w'|} \right) \, d\omega \, dv. \]

This proves \(2.19\).

Finally we use standard approximation arguments in order to prove that \(2.19\) still holds without the continuity assumption on the function $b$. We skip these classical calculations.

\[ \square \]

**Lemma 2.3.** Let $N \geq 3$, $B(z, \sigma)$ as defined in \(1.5\) with the angular function $b$ satisfying \(1.14\). Let $1 \leq p, q \leq \infty$ satisfy $1/p + 1/q = 1$.

Then, in the case where we have

\[ 0 < \gamma < N - 2, \quad \frac{N - 1}{N - 1 - \gamma} \leq p < \frac{N}{N - 1 - \gamma}, \]

the following estimate holds

\[ \left( \int_{\mathbb{R}^N} [K_B(v, v_*, w, w_*)]^p \, dv \right)^{1/p} \leq C_p \|b\|_{L^\infty}^2 |w - w_*|^{2\gamma - N/q}. \]

Second, in the case where we have

\[ \gamma \geq N - 2, \quad 1 \leq p < N, \]

then the following estimate holds

\[ \left( \int_{\mathbb{R}^N} [K_B(v, v_*, w, w_*)]^p \, dv \right)^{1/p} \leq C_p \|b\|_{L^\infty}^2 \|v_*\|^{2\gamma - N/q} \langle w \rangle^{2\gamma - N/q} \langle w_* \rangle^{2\gamma - N/q}. \]

The constants $C_p$ only depend on $N, \gamma, p$. 
Proof. By replacing the function $b$ with $b/\|b\|_{L^\infty}$ we can assume for notation convenience that $\|b\|_{L^\infty} = 1$. Fix $w, w_*, v_* \in \mathbb{R}^N$. To prove the lemma we may assume that $w \neq w_*$. Recall that $N \geq 3$ implies $\zeta(t) \leq 1_{(-1,1)}(t)$. Then from (2.10)-(2.13) we have

\begin{equation}
K_B(v, v_*, w, w_*) \leq \frac{2^N|S^{N-2}|}{|w-w_*|^{1-\gamma}} \frac{1}{|v-v_*|^{N-1-\gamma}} 1_{(-1,1)} \left( n \cdot \frac{2v - (w + w_*)}{|w - w_*|} \right)
\end{equation}

for $0 < \gamma < N - 2$, whereas for $\gamma \geq N - 2$ we have

\begin{equation}
K_B(v, v_*, w, w_*) \leq \frac{2^N|S^{N-2}| |v_*|^\gamma 2^{N-(1-\gamma)} |w|^{2\gamma + 1 - N} |w_*|^{2\gamma + 1 - N}}{|v-v_*|^{1-\gamma}} 1_{(-1,1)} \left( n \cdot \frac{2v - (w + w_*)}{|w - w_*|} \right)
\end{equation}

where we used $N - 2 \geq 1$ and the inequalities in (2.16).

Let us define

\begin{equation}
J_\beta(w, w_*) = \int_{\mathbb{R}^N} \frac{1}{|v-v_*|^\beta} 1_{(-1,1)} \left( n \cdot \frac{2v - (w + w_*)}{|w - w_*|} \right) \, dv.
\end{equation}

We need to prove that

\begin{equation}
J_\beta(w, w_*) \leq |S^{N-1}| |v_*|^\gamma 2^{N-(1-\gamma)} |w - w_*| \quad \text{when} \quad 0 < \beta < N - 1,
\end{equation}

\begin{equation}
J_\beta(w, w_*) \leq |S^{N-1}| (N - \beta) |w - w_*|^N \quad \text{when} \quad N - 1 \leq \beta < N.
\end{equation}

To do this we use the change of variable

\begin{equation}
v = v_* + \frac{|w - w_*|}{2} r \sigma, \quad dv = \frac{|w - w_*|}{2} r^{N-1} \, dr \, d\sigma
\end{equation}

to compute

\begin{equation}
J_\beta(w, w_*) = \left| \frac{w - w_*}{2} \right|^{N-\beta} \int_{\mathbb{S}^{N-1}} I(u \cdot \sigma) \, d\sigma
\end{equation}

where

\begin{equation}
I(u \cdot \sigma) = \int_0^\infty r^{N-1-\beta} 1_{\{|r + u \cdot \sigma| < 1\}} \, dr, \quad u = \frac{2v_* - (w + w_*)}{|w - w_*|}.
\end{equation}

Let us now estimate $I(u \cdot \sigma)$ uniformly in $\sigma$. If $u \cdot \sigma \geq 1$, then $I(u \cdot \sigma) = 0$. Suppose $u \cdot \sigma < 1$. If $0 < \beta < N - 1$, then

\begin{equation}
I(u \cdot \sigma) \leq 2(1 + |u|)^{N-1-\beta} \leq 2 \left( \frac{|w - w_*|}{2} + \frac{|w + w_*|}{2} + |v_*| \right)^{N-1-\beta}
\end{equation}

which together with (2.30) and the third inequality in (2.16) gives (2.28). If $N - 1 \leq \beta < N$, then $0 < N - \beta \leq 1$ so that

\begin{equation}
I(u \cdot \sigma) \leq \frac{2^{N-\beta}}{N - \beta}
\end{equation}

which implies (2.29).
Now suppose \((2.22)\) is satisfied. Then using \((2.26)\) and \((2.28)\) with 

\[
N - 1 \leq \beta = (N - 1 - \gamma)p < N
\]

we obtain \((2.23)\):

\[
\left(\int_{\mathbb{R}^N} [K_B(v, v_s, w, w_s)]^p \, dv\right)^{1/p} \leq C_p |w - w_s|^{\gamma-1} \left(\int_{\mathbb{R}^N} (J_{(N-1)\gamma}(w, w_s))^{1/p} \right)
\]

\[
= C_p |w - w_s|^{\gamma-1} w_s^{\frac{N-p(N-1)}{p}} = C_p |w - w_s|^{2\gamma-N/q}
\]

Next suppose \((2.23)\) is satisfied. If \(N - 1 \leq p < N\), then by \((2.27)-(2.29)\) with \(\beta = p\) and using \(|w - w_s| \leq \langle w \rangle \langle w_s \rangle\) we have

\[
\left(\int_{\mathbb{R}^N} [K_B(v, v_s, w, w_s)]^p \, dv\right)^{1/p} \leq C_p \langle v_s \rangle^{\gamma+2-N} \langle w \rangle^{2\gamma+1-N} \langle w_s \rangle^{2\gamma+1-N} (J_p(w, w_s))^{1/p}
\]

\[
\leq C_p \langle v_s \rangle^{\gamma+2-N} \langle w \rangle^{2\gamma-N/q} \langle w_s \rangle^{2\gamma-N/q}
\]

Similarly if \(1 \leq p < N - 1\), then using \((2.27)-(2.28)\) with \(\beta = p\) we have

\[
\left(\int_{\mathbb{R}^N} [K_B(v, v_s, w, w_s)]^p \, dv\right)^{1/p} \leq C_p \langle v_s \rangle^{\gamma-(N-1)/q} \langle w \rangle^{2\gamma-N/q} \langle w_s \rangle^{2\gamma-N/q}
\]

Since \(\gamma \geq N - 2 \geq 1\) and \(1 \leq p < N\) imply

\[
\max \left\{ \gamma + 2 - N, \gamma - \frac{N-1}{q} \right\} \leq 2\gamma - N/q
\]

it follows that

\[
\max \left\{ \langle v_s \rangle^{\gamma+2-N}, \langle v_s \rangle^{\gamma-N-1} \right\} \leq \langle v_s \rangle^{2\gamma-N/q}
\]

This concludes the proof of \((2.25)\). \qed

**Lemma 2.4.** Let \(K(v, v_s)\) be a measurable function on \(\mathbb{R}^N \times \mathbb{R}^N\) and let

\[
\forall v \in \mathbb{R}^N, \quad T(f)(v) := \int_{\mathbb{R}^N} K(v, v_s) f(v_s) \, dv_s.
\]

Assume that \(1 \leq r < \infty, 0 \leq s < \infty\) and that there is \(0 < A < \infty\) such that

\[
(2.31) \quad \left(\int_{\mathbb{R}^N} |K(v, v_s)|^r \, dv_s\right)^{\frac{1}{r}} \leq A \langle v_s \rangle^s \quad \text{a.e. } v_s \in \mathbb{R}^N.
\]

Then

\[
(2.32) \quad \forall f \in L^r_s(\mathbb{R}^N), \quad \|T(f)\|_{L^r} \leq A \|f\|_{L^1}.
\]

Furthermore let \(1 \leq p, q \leq \infty\) satisfy

\[
\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1
\]

and assume that there is \(0 < B < \infty\) such that

\[
(2.33) \quad \left(\int_{\mathbb{R}^N} (|K(v, v_s)|^s |v_s|^{-s})^r \, dv_s\right)^{\frac{1}{r}} \leq B \quad \text{a.e. } v \in \mathbb{R}^N.
\]
Then
\[ (2.34) \quad \forall f \in L^2_s(\mathbb{R}^N), \quad \| T(f) \|_{L^p} \leq A^{\frac{\alpha}{p}} B^{1-\frac{\alpha}{p}} \| f \|_{L^2}. \]

Proof. Let us define
\[ \forall v \in \mathbb{R}^N, \quad T_s(f)(v) := \int_{\mathbb{R}^N} K(v, v_*) (v_*)^{-s} f(v_*) \, dv_*. \]
By Minkowski’s inequality and (2.31) we have
\[ (2.35) \quad \forall f \in L^1(\mathbb{R}^N), \quad \| T_s(f) \|_{L^r} \leq A \| f \|_{L^1}. \]
Also by (2.33) we have
\[ (2.36) \quad \forall f \in L^{r'}(\mathbb{R}^N), \quad \| T_s(f) \|_{L^{r'}} \leq B \| f \|_{L^{r'}} \]
where \( 1 \leq r' \leq \infty \) satisfies \( 1/r + 1/r' = 1 \).

Let \( \theta = 1 - r/p \). By assumption on \( p, q, r \) we have \( 0 \leq \theta \leq 1 \) and
\[ \frac{1}{p} = \frac{1 - \theta}{r} + \frac{\theta}{\infty}, \quad \frac{1}{q} = \frac{1 - \theta}{1} + \frac{\theta}{r'}. \]
So by (2.35), (2.36) and Riesz-Thorin’s interpolation theorem (see e.g. [27, Chapter 5]) we have
\[ (2.37) \quad \forall f \in L^q(\mathbb{R}^N), \quad \| T_s(f) \|_{L^p} \leq A^{1 - \theta} B^\theta \| f \|_{L^q}. \]
Now if we set \( (f)_s(v_*) = \langle v_* \rangle^s f(v_*) \), then
\[ T(f) = T_s((f)_s), \quad \| (f)_s \|_{L^1} = \| f \|_{L^1}, \quad \| (f)_s \|_{L^q} = \| f \|_{L^q} \]
and thus (2.32)-(2.34) follow from (2.35)-(2.37). \( \square \)

In order to highlight structures of inequalities, we adopt the following notational convention: Functions \( f, g, h \) appeared below are arbitrary members in the classes indicated. Whenever the notation (for instance) \( \| f \|_{L^q} \) appears, it always means that \( f \in L^q_s(\mathbb{R}^N) \) with the norm \( \| f \|_{L^q} \); and if \( \| f \|_{L^q}, \| f \|_{L^q} \) appear simultaneously, it means that \( f \in L^q_s(\mathbb{R}^N) \cap L^q_s(\mathbb{R}^N) \).

Lemma 2.5. Let \( 0 < \alpha < N, 1 \leq \alpha q < N, 1 < p \leq \infty \) with \( 1/p + 1/q = 1 \). Then
\[ \sup_{v_* \in \mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(v_*)|}{|v - v_*|^\alpha} \, dv_* \leq 2 \left( \frac{|S^{N-1}|}{N - \alpha q} \right)^{\frac{\alpha}{q}} \| f \|_{L^1}^{1 - \frac{\alpha q}{N}} \| f \|_{L^p}^{\frac{\alpha q}{N}}. \]

Proof. This follows from Hölder’s inequality and a minimizing argument. \( \square \)

Lemma 2.6. Suppose \( N \geq 3, \gamma > 0, B(z, \sigma) \) as defined in (1.5) with the condition (1.44). For any \( w, w_* \in \mathbb{R}^N \) with \( w \neq w_* \), let
\[ (2.38) \quad \forall v \in \mathbb{R}^N, \quad T_{w,w_*}(f)(v) := \int_{\mathbb{R}^N} K_B(v, v_*, w, w_*) f(v_*) \, dv_* \]
for nonnegative measurable or certain integrable functions \( f \) as indicated below.
(i) Suppose $0 < \gamma < N - 2$. Let $p_1 = (N - 1)/(N - 1 - \gamma)$. Then
\begin{equation}
\|T_{w,w_1}(f)\|_{L^{p_1}} \leq C_{p_1} \|b\|_{L^\infty} \|f\|_{L^1} \langle w \rangle^\gamma \langle w_* \rangle^\gamma.
\end{equation}

Also if $1 < p, q < \infty$ satisfy
\[ \frac{1}{p} = \frac{1}{q} + \frac{1}{p_1} - 1 \]
then
\begin{equation}
\|T_{w,w_1}(f)\|_{L^p} \leq C_p \|b\|_{L^\infty} \|f\|_{L^1} \langle w \rangle \langle w_* \rangle^\frac{1-p_1}{p_1} \langle w - w_* \rangle^{1-\gamma - \frac{1}{p}}.
\end{equation}

And if $1 < p \leq \infty$, $1 \leq q < N/(N - 1 - \gamma)$ satisfy $1/p + 1/q = 1$, then
\begin{equation}
\|T_{w,w_1}(f)\|_{L^\infty} \leq C_p \|b\|_{L^\infty} \|f\|_{L^1} \langle w \rangle \langle w_* \rangle^{N-1-\gamma} \langle w - w_* \rangle^{1/q} \|f\|_{L^{N-1-\gamma}}.
\end{equation}

(ii) Suppose $\gamma \geq N - 2$ and $1 < p < N$, $1/p + 1/q = 1$. Then
\begin{equation}
\|T_{w,w_1}(f)\|_{L^p} \leq C_p \|b\|_{L^\infty} \|f\|_{L^1} \langle w \rangle^{2\gamma - N/q} \langle w_* \rangle^{2\gamma - N/q}.
\end{equation}

Furthermore if $N/(N-1) < p < N$, then
\begin{equation}
\|T_{w,w_1}(f)\|_{L^\infty} \leq C_p \|b\|_{L^\infty} \|f\|_{L^1} \langle w \rangle^{N-1-\gamma} \langle w_* \rangle^{2\gamma - N - (N-1)/\gamma}.
\end{equation}

The constants $C_p < \infty$ only depend on $N, \gamma, p$.

**Proof.** As in the proof of Lemma 2.3 we can assume that $\|b\|_{L^\infty} = 1$. 

**Case (i).** Suppose $0 < \gamma < N - 2$. By Lemma 2.3 we have
\begin{equation}
\left( \int_{\mathbb{R}^N} [K_B(v,v_*,w,w_*)]^{p_1} dv \right)^{1/p_1} \leq C_{p_1} |w - w_*|^{2\gamma - N/q_1}
\end{equation}
where $q_1 = (p_1)/(p_1 - 1) = (N-1)/\gamma$. Since
\[ 0 < 2\gamma - N/q_1 < \gamma \quad \text{and} \quad |w - w_*| \leq \langle w \rangle \langle w_* \rangle,
\]
[2.39] follows from [2.32] with $r = p_1, s = 0$. Next recalling [2.13] and the second inequality in [2.16] we see that that
\[ K_B(v,v_*,w,w_*) > 0 \implies \sqrt{1 + |v-v_*|^2} \leq \sqrt{2} \langle w \rangle \langle w_* \rangle \langle v_* \rangle
\]
so that
\begin{equation}
K_B(v,v_*,w,w_*)^{-1} \leq \sqrt{2} \langle w \rangle \langle w_* \rangle \left( \frac{K_B(v,v_*,w,w_*)}{\sqrt{1 + |v-v_*|^2}} \right).\end{equation}

This together with [2.26] and $(N - 1 - \gamma)p_1 = N - 1$ gives
\begin{equation}
\left( \int_{\mathbb{R}^N} (K_B(v,v_*,w,w_*)^{-1})^{p_1} dv_* \right)^{1/p_1} \leq C_{p_1} \langle w \rangle \langle w_* \rangle \left( \int_{\mathbb{R}^N} \frac{dv_*}{|w - w_*|^{1-\gamma}} \right)^{1/p_1} = C_{p_1} \langle w \rangle \langle w_* \rangle \langle w - w_* \rangle^{1-\gamma}.
\end{equation}
Note that the above integral is finite since \( p_1 > 1 \). If we set

\[
A_{w,w_*} := C_{p_1} |w - w_*|^{2\gamma - N/q_1}, \quad B_{w,w_*} := C_{p_1} \frac{\langle w \rangle \langle w_* \rangle}{|w - w_*|^{1 - \gamma}}
\]

then we see from (2.44) and (2.46) that Lemma 2.4 can be used for \( T_{w,w_*}(f) \) with \( r = p_1 \) and \( s = 1 \), and thus for all \( f \in L^q_1(\mathbb{R}^N) \)

\[
\|T_{w,w_*}(f)\|_{L^p} \leq \left( A_{w,w_*} \right)^{\frac{p_1}{p}} \left( B_{w,w_*} \right)^{\frac{1 - p_1}{p}} \|f\|_{L^q_1} = C_p \frac{\langle w \rangle^{1 - \frac{p_1}{p}} \langle w_* \rangle^{\frac{1 - p_1}{p}}}{|w - w_*|^{1 - \gamma \frac{1 - p_1}{p}}} \|f\|_{L^q_1}
\]

where we have computed (using the definitions of \( p_1, q_1 \))

\[
\frac{p_1}{p} \left( 2\gamma - \frac{N}{q_1} \right) - \left( 1 - \frac{p_1}{p} \right) (1 - \gamma) = \frac{1}{p} + \gamma - 1.
\]

This proves (2.40). To prove (2.41) we use (2.26) to get

\[
|T_{w,w_*}(f)(v)| \leq 2^N |\mathbb{S}^{N-2}| \int_{\mathbb{R}^N} \frac{|f(v_*)| \, dv_*}{|w - w_*|^{1 - \gamma}} \int_{\mathbb{R}^N} \frac{|f(v_*)| \, dv_*}{|v - v_*|^{N - 1 - \gamma}}
\]

Since \( 1 \leq (N - 1 - \gamma)q < N \), it follows from Lemma 2.5 that

\[
\|T_{w,w_*}(f)\|_{L^\infty} \leq \frac{C_p}{|w - w_*|^{1 - \gamma}} \|f\|_{L^1}^{1 - \frac{N - 1 - \gamma}{N - q}} \|f\|_{L^{N/q}_1}^{\frac{N - 1 - \gamma}{N - q}}.
\]

Case (ii). Suppose \( \gamma \geq N - 2 \). In this case we recall the inequality (2.25). Then applying Lemma 2.4 to \( T_{w,w_*}(f) \) with \( r = p, s = 2\gamma - N/q \) gives (2.42). Finally suppose \( N/(N - 1) < p < N \). Recalling (2.38) and using (2.27) we have

\[
|T_{w,w_*}(f)(v)| \leq C_{N,\gamma} \langle w \rangle^{2\gamma + 1 - N} \langle w_* \rangle^{2\gamma + 1 - N} \int_{\mathbb{R}^N} \frac{\langle v_* \rangle^{\gamma + 2 - N}}{|v - v_*|} |f(v_*)| \, dv_*
\]

Since \( q = p/(p - 1) < N \), it follows from Lemma 2.5 that

\[
\int_{\mathbb{R}^N} \frac{\langle v_* \rangle^{\gamma + 2 - N}}{|v - v_*|} |f(v_*)| \, dv_* \leq C_p \|f\|_{L^{1/(N - 2) - N}} \|f\|_{L^{N/(N - 2) - N}}^{\frac{N}{N - 2}}.
\]

This proves (2.43). \( \square \)

Let \( f, g, h \) be nonnegative measurable functions on \( \mathbb{R}^N \). Define for any \( s \geq 0 \)

\[
(f)_s(v) := \langle v \rangle^s f(v).
\]

Then applying the inequality \( \langle v \rangle \leq \langle v' \rangle \langle v'_* \rangle \) we have

\[
\left\{ \begin{aligned}
(Q^+ (f,g))_s &\leq Q^+ ((f)_s, (g)_s), \\
(Q^+ (f,Q^+(g,h)))_s &\leq Q^+ ((f)_s, Q^+(g)_s, (h)_s),
\end{aligned} \right.
\]

and so on and so forth. Consequently we have for all \( s \geq 0 \) and \( 1 \leq p \leq \infty \):

(2.47) \[
\|Q^+ (f,Q^+(g,h))\|_{L^p_s} \leq \|Q^+ ((f)_s, Q^+(g)_s, (h)_s)\|_{L^p}
\]

provided that the RHS makes sense.
Now we are going to prove the $L^p$ and $L^\infty$ boundedness of the iterated operator $Q^+(f, Q^+(g, h))$. Let

\begin{equation}
N_\gamma = \begin{cases} \left\lfloor \frac{N-1}{\gamma} \right\rfloor & \text{if } 0 < \gamma < N - 2 \\
1 & \text{if } \gamma \geq N - 2 \end{cases}
\end{equation}

where $[x]$ denotes the largest integer not exceeding $x$.

**Theorem 2.7.** Suppose $N \geq 3$, $\gamma > 0$ and let $B(z, \sigma)$ defined by (1.35) with the condition (1.44). Given any $s \geq 0$ we have:

(i) Suppose $0 < \gamma < N - 2$. Let $N_\gamma$ be defined in (2.48) and let

\[ p_n = \frac{N - 1}{N - 1 - n\gamma} \in (1, \infty], \quad n = 1, 2, \ldots, N_\gamma. \]

Then

(a) For all $n = 1, 2, \ldots, N_\gamma$, we have

\begin{equation}
\|Q^+(f, Q^+(g, h))\|_{L^{p_n}_1} \leq C_p \|b\|_{L^\infty}^2 \|f\|_{L^1_{s+\gamma}} \|g\|_{L^1_{s+\gamma}} \|h\|_{L^1_{s+\gamma}}.
\end{equation}

(b) If $1 \leq n \leq N_\gamma - 1$, then

\begin{equation}
\|Q^+(f, Q^+(g, h))\|_{L^{p_{n+1}}_1} \leq C_{p_{n+1}} \|b\|_{L^\infty}^2 \|f\|_{L^1_{s+\gamma}} \|g\|_{L^1_{s+\gamma}} \|h\|_{L^1_{s+\gamma}} \|h\|_{L^{p_{n+1}}_{s+\gamma}}^{1-\alpha_n} \|h\|_{L^{p_{n+1}}_{s+\gamma}}^{\alpha_n},
\end{equation}

where

\[ \gamma_1 = \max\{\gamma, 1\}, \quad \eta_n = \frac{1}{N} \left(1 - \frac{N - 2}{n}\right), \quad n \geq 1. \]

(c) Finally if $n = N_\gamma$, then

\begin{equation}
\|Q^+(f, Q^+(g, h))\|_{L^\infty} \leq C_\infty \|b\|_{L^\infty}^2 \|f\|_{L^1_{s+\gamma}}^{1-\alpha_1} \|f\|_{L^1_{s+\gamma_1}}^{\alpha_1} \|g\|_{L^1_{s+\gamma_1}} \|h\|_{L^1_{s+\gamma_1}} \|h\|_{L^\infty_{s+\gamma_1}}^{1-\alpha_2} \|h\|_{L^\infty_{s+\gamma_1}}^{\alpha_2},
\end{equation}

where $\gamma_* = (\gamma - 1)_+$ and

\[ 0 < \alpha_1 := \left(\frac{N - 1}{\gamma N_\gamma}\right) \left(\frac{N - 1 - \gamma}{N}\right) < 1, \quad 0 \leq \alpha_2 := \left(\frac{N - 1}{\gamma N_\gamma}\right) \left(\frac{(1 - \gamma)_+}{N}\right) < 1. \]

(ii) Suppose $\gamma \geq N - 2$. Let $1 < p < N$ and $q = p/(p - 1)$. Then

\begin{equation}
\|Q^+(f, Q^+(g, h))\|_{L^p_1} \leq C_p \|b\|_{L^\infty}^2 \|f\|_{L^1_{s+2\gamma - N/q}} \|g\|_{L^1_{s+2\gamma - N/q}} \|h\|_{L^1_{s+2\gamma - N/q}}.
\end{equation}

Furthermore if $N/(N - 1) < p < N$, then

\begin{equation}
\|Q^+(f, Q^+(g, h))\|_{L^\infty} \leq C_p \|b\|_{L^\infty}^2 \|f\|_{L^1_{s+\gamma+2-N}}^{1-\frac{N}{p}} \|f\|_{L^p_{s+\gamma+2-N}}^{\frac{N}{p}} \|g\|_{L^1_{s+\gamma+2-N}} \|h\|_{L^1_{s+\gamma+2-N}} \|h\|_{L^1_{s+\gamma+2-N}}.
\end{equation}

The constants $C_p < \infty$ only depend on $N, \gamma, p$.

**Remark 2.8.** Observe that in the case $\gamma \geq N - 2$, the iterated operator maps (forgetting about the weights) $L^1 \times L^1 \times L^1$ to $L^p$ for $1 < p < N$. This can recovered heuristically from Lions’ theorem (see also [26]): $Q^+$ maps $L^1 \times H^s$ to $H^{s+(N-1)/2}$. Then $L^1$ is contained in $H^{-N/2-0}$ and applying twice Lions’ theorem one gets that the iterated operator maps...
\[ L^1 \times L^1 \times L^1 \text{ to } H^{2(N-1)/2-N/2-0} = H^{N/2-1-0} \]. And the Sobolev embedding for the space \( H^{N/2-1} \) is precisely \( L^N \).

**Proof.** The proof is a direct application of the inequalities obtained in Lemma \( \text{(2.5)} \) and Lemma \( \text{(2.6)} \). We can assume that \( f, g, h \) are all nonnegative. And because of \( \text{(2.47)} \), we need only to prove the theorem for \( s = 0 \). By the integral representation of \( Q^+(f, Q^+(g, h)) \) and the definition of \( T_{w,w} \), we have

\[ Q^+(f, Q^+(g, h))(v) = \int_{\mathbb{R}^N \times \mathbb{R}^N} T_{w,w}(f)(v)g(w)h(w)dw \, dw, \]

and therefore

\[ \|Q^+(f, Q^+(g, h))\|_{L^p} \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} \|T_{w,w}(f)\|_{L^p} g(w)h(w)dw \, dw \]

for all \( 1 \leq p \leq \infty \).

**Case (i).** Suppose \( 0 < \gamma < N - 2 \). By \( \text{(2.39)} \) we have

\[ \|T_{w,w}(f)\|_{L^p_1} \leq C_{p_1} \|b\|_{L^\infty}^2 \|f\|_{L^1} \gamma \langle w \rangle^{\gamma} \langle w^*_n \rangle^{\gamma} \]

and so by \( \text{(2.56)} \)

\[ \|Q^+(f, Q^+(g, h))\|_{L^p_1} \leq C_{p_1} \|b\|_{L^\infty}^2 \|f\|_{L^1} \|g\|_{L^1} \|h\|_{L^1}. \]

This proves \( \text{(2.49)} \) (with \( s = 0 \)).

- Suppose \( 1 \leq n \leq N_\gamma - 1 \). By definition of \( p_n \) we have

\[ \frac{1}{p_{n+1}} = \frac{1}{p_n} + \frac{1}{p_1} - 1, \quad n = 1, 2, \ldots, N_\gamma - 1. \]

By \( \text{(2.30)} \) and \( \text{(2.56)} \) we have

\[ \begin{align*}
\|T_{w,w}(f)\|_{L^{p_{n+1}}} &\leq C_{p_{n+1}} \|b\|_{L^\infty}^2 \frac{\langle w \rangle^{1-p_1} \langle w^*_n \rangle^{1-p_1}}{|w - w^*_n|^{1-p_1}} \|f\|_{L^{p_n}}, \\
\|Q^+(f, Q^+(g, h))\|_{L^{p_{n+1}}} &\leq C_{p_{n+1}} \|b\|_{L^\infty}^2 \|f\|_{L^{p_n}} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\langle w \rangle^{1-p_1} \langle w^*_n \rangle^{1-p_1}}{|w - w^*_n|^{1-p_1}} g(w)h(w)dw \, dw.
\end{align*} \]

Let \( q_n \geq 1 \) be defined by

\[ \frac{1}{q_n} + \frac{1}{p_n} = 1, \quad \text{i.e.} \quad q_n = \frac{N - 1}{n \gamma}. \]

We have

\[ 1 - \gamma - \frac{1}{p_{n+1}} = \frac{n + 2 - N}{N - 1} \gamma, \quad q_n \left(1 - \gamma - \frac{1}{p_{n+1}}\right) = 1 - \frac{N - 2}{n}. \]
If \( n \leq N - 2 \), then \( 1 - \gamma - 1/p_{n+1} \leq 0 \) and using \( |w - w_s| \leq \langle w \rangle \langle w_s \rangle \) we have
\[
\langle w \rangle^{1-\frac{q}{p_{n+1}}} \langle w_s \rangle^{1-\frac{q}{p_{n+1}}} \leq \langle w \rangle^\gamma \langle w_s \rangle^\gamma
\]
and so by (2.57) we have
\[
\|Q^+(f, Q^+(g, h))\|_{L^{p_{n+1}}} \leq C_{p_{n+1}} \|b\|_{L^\infty}^2 \|f\|_{L^1} \|g\|_{L^1} \|h\|_{L^1}.
\]
If \( n > N - 2 \), then \( 0 < q_n(1 - \gamma - 1/p_{n+1}) < 1 \) so that applying Lemma 2.5 (with \( q = q_n, \alpha = 1 - \gamma - 1/p_{n+1} \)) and recalling the definition of \( \theta_n \) we have
\[
\int_{\mathbb{R}^N} \frac{\langle w \rangle^{1-\frac{q}{p_{n+1}}} \langle w_s \rangle^{1-\frac{q}{p_{n+1}}} g(w)h(w_s) \, dw \, dw_s \leq C_{p_{n+1}} \|g\|_{L^1} (\|h\|_{L^1})^{1-\theta_n} (\|h\|_{L^{p_{n+1}}}^{\theta_n})
\]
and thus (2.56) (with \( s = 0 \)) follows from (2.57) and (2.58).

- Now let \( n = N_\gamma \). Let us recall that
\[
q_{N_\gamma} = \frac{N - 1}{N_\gamma - 1} > \frac{N - 1}{\gamma} - 1 > 0,
\]
hence
\[
q_{N_\gamma}(N - 1 - \gamma) < N - 1.
\]
Using (2.41) and (2.56) (for the \( L^\infty \) norm) we have
\[
\|T_{w, w_s}(f)\|_{L^\infty} \leq \frac{C_{\infty} \|b\|_{L^\infty}^{2} \|f\|_{L^1}^{\frac{N-1-\gamma}{N}} q_{N_\gamma} \|f\|_{L^{p_{N_\gamma}}}^{\frac{N-1-\gamma}{N} q_{N_\gamma}},
\]
\[
\|Q^+(f, Q^+(g, h))\|_{L^\infty} \leq C_{\infty} \|b\|_{L^\infty}^{2} \|f\|_{L^1}^{\frac{N-1-\gamma}{N} q_{N_\gamma}} (\|f\|_{L^{p_{N_\gamma}}}^{\frac{N-1-\gamma}{N} q_{N_\gamma}}) \int_{\mathbb{R}^N} \frac{g(w)h(w_s)}{|w - w_s|^{1-\gamma}} \, dw \, dw_s.
\]
If \( 0 < \gamma < 1 \), then from (2.59) and \( N \geq 3 \) we have \( 0 < q_{N_\gamma}(1 - \gamma) < N - 1 \) so that using Lemma 2.4 gives
\[
\int_{\mathbb{R}^N} \frac{g(w)h(w_s)}{|w - w_s|^{1-\gamma}} \, dw \, dw_s \leq C_{\gamma} \|g\|_{L^1} \|h\|_{L^1}^{\frac{1-\gamma}{N} q_{N_\gamma}} \|h\|_{L^{p_{N_\gamma}}}^{\frac{1-\gamma}{N} q_{N_\gamma}}.
\]
If \( 1 \leq \gamma < N - 2 \), then \( |w - w_s|^{\gamma-1} \leq \langle w \rangle^{\gamma-1} \langle w_s \rangle^{\gamma-1} \) and so
\[
\int_{\mathbb{R}^N} \frac{g(w)h(w_s)}{|w - w_s|^{1-\gamma}} \, dw \, dw_s \leq \|g\|_{L^{1-\gamma}} \|h\|_{L^{1-\gamma}}.
\]
Thus (2.52) (with \( s = 0 \)) follows from (2.60), (2.61) and (2.62).

**Case (ii).** Suppose \( \gamma \geq N - 2 \). If \( 1 < p < N \), then (2.51) (with \( s = 0 \)) follows from (2.56) and (2.42). If \( N/(N-1) < p < N \), then (2.55) (with \( s = 0 \)) follows from (2.56) (for the \( L^\infty \) norm) and (2.43). \( \square \)
3. Iteration and Decomposition of Solutions

We begin by the study of the process of iteration the collision operator and decomposition of solutions though the following lemma. Roughly speaking the strategy of the decomposition is the following. We use the Duhamel formula to decompose the flow associated with the equation into two parts, one of which is more regular than the initial datum, while the amplitude of the other decreases exponentially fast with time, and we repeat this process in order to increase the smoothness, starting each time a new flow having the smoother part of the previous solution as initial datum. Each time we start a new flow, we depart from the true solution, since the initial datum is not the real solution, and this error by a Lipschitz stability estimate. Finally the times of the decomposition are chosen in such a way that the time-decay of the non-smooth parts dominates this Lipschitz stability error.

Lemma 3.1. Let $B(z, \sigma)$ be defined in (1.5) with the condition (1.33) and for some $\gamma \in (0,2]$. Let $f_t \in L^1_2(\mathbb{R}^N)$ be a mild solution of equation (1.1). Let us define

$$E^t_s(v) := \exp \left( - \int_s^t \int_{\mathbb{R}^N} |v - v_\ast| \gamma f_\tau(v_\ast) \, dv_\ast \, d\tau \right).$$

Given any $t_0 \geq 0$ we also define for all $t \geq t_0$

$$f^0_t(v) = f_t(v), \quad h^0_t(v) = 0,$$

$$f^n_t(v) = \int_{t_0}^t E^t_{t_1}(v) \int_{t_0}^{t_1} Q^+ (f^0_{t_2}, f^{n-1}_{t_2}, f^{n-1}_{t_2}) E^t_{t_2} (v) \, dt_2 \, dt_1,$$

$$h^1_t(v) = f_{t_0}(v) E^t_{t_0}(v) + \int_{t_0}^t Q^+ (f_{t_1}, f_{t_0} E^t_{t_0}) (v) E^t_{t_1}(v) \, dt_1,$$

$$h^n_t(v) = h^1_t(v) + \int_{t_0}^t E^t_{t_1}(v) \int_{t_0}^{t_1} Q^+ (f_{t_1}, Q^+ (f_{t_2}, h^{n-1}_{t_2}, f^{n-1}_{t_2}) E^t_{t_2}) (v) \, dt_2 \, dt_1$$

$$+ \int_{t_0}^t E^t_{t_1}(v) \int_{t_0}^{t_1} Q^+ (f_{t_1}, Q^+ (f^{n-1}_{t_2}, h^{n-1}_{t_2}, E^t_{t_2}) (v) \, dt_2 \, dt_1$$

$$+ \int_{t_0}^t E^t_{t_1}(v) \int_{t_0}^{t_1} Q^+ (h^{n-1}_{t_1}, Q^+ (f^{n-1}_{t_2}, f^{n-1}_{t_2}) E^t_{t_2}) (v) \, dt_2 \, dt_1$$

for $n = 1, 2, 3, \ldots$

Then $f^n_t \geq 0$, $h^n_t \geq 0$ and there is a null set $Z \subset \mathbb{R}^N$ which is independent of $t$ and $n$ such that for all $v \in \mathbb{R}^N \setminus Z$

$$\forall t \in [t_0, \infty), \quad n = 1, 2, 3, \ldots, \quad f_t(v) = f^n_t(v) + h^n_t(v).$$

Proof. In the following we denote by $Z_0, Z_1, Z_2, \cdots \subset \mathbb{R}^N$ some null sets (i.e. $\text{meas}(Z_n) = 0$) which are independent of the time variable $t$. The decomposition (3.6) is based on the
Duhamel representation formula for the solution $f_t$: for all $v \in \mathbb{R}^N \setminus Z_0$

\begin{equation}
\forall t \geq t_0, \quad f_t(v) = f_{t_0}(v) + \int_{t_0}^{t} Q^+(f_{t_1}, f_{t_1})E_{t_1}^t(v) \, dt.
\end{equation}

Applying (3.7) to $f_t$ at time $t = t_1$ and inserting it into the second argument of $Q^+(f_{t_1}, f_{t_1})$ we obtain for all $t \geq t_0$ and all $v \in \mathbb{R}^N \setminus Z_1$

\begin{equation}
\forall t \geq t_0, \quad f_t(v) = h_1^t(v) + \int_{t_0}^{t} Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2})E_{t_2}^t_1) \, dt_2 \, dt_1.
\end{equation}

That is, we have the decomposition

\[ \forall t \geq t_0, \quad f_t(v) = h_1^t(v) + f_t^1(v). \]

Suppose for some $n \geq 1$, the decomposition $f_t(v) = h_n^t(v) + f_t^n(v)$ holds for all $t \geq t_0$ and all $v \in \mathbb{R}^N \setminus Z_n$. Let us insert $f_{t_2} = h_n^t + f_n^t$ and $f_{t_1} = h_t^n + f_t^n$ into $Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2})E_{t_2}^t_1)$ in the following way:

\[ \begin{cases} 
Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2})E_{t_2}^t_1) = Q^+(f_{t_1}, Q^+(f_{t_2}, h_n^t)E_{t_2}^t_1) + Q^+(f_{t_1}, Q^+(f_{t_2}, f_n^t)E_{t_2}^t_1), \\
Q^+(f_{t_1}, Q^+(f_{t_2}, h_n^t)E_{t_2}^t_1) = Q^+(h_t^n, Q^+(f_{t_2}, h_n^t)E_{t_2}^t_1) + Q^+(f_{t_1}, Q^+(f_{t_2}, f_n^t)E_{t_2}^t_1), \\
Q^+(f_{t_1}, Q^+(f_{t_2}, f_n^t)E_{t_2}^t_1) = Q^+(h_t^n, Q^+(f_{t_2}, f_n^t)E_{t_2}^t_1) + Q^+(f_{t_1}, Q^+(f_{t_2}, f_n^t)E_{t_2}^t_1). 
\end{cases} \]

Then

\[ Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2})E_{t_2}^t_1) = Q^+(f_{t_1}, Q^+(f_{t_2}, h_n^t)E_{t_2}^t_1) + Q^+(f_{t_1}, Q^+(f_{t_2}, f_n^t)E_{t_2}^t_1) + Q^+(h_t^n, Q^+(f_{t_2}, h_n^t)E_{t_2}^t_1) + Q^+(h_t^n, Q^+(f_{t_2}, f_n^t)E_{t_2}^t_1) + Q^+(f_{t_1}, Q^+(f_{t_2}, f_n^t)E_{t_2}^t_1). \]

Inserting this into (3.8) yields

\[ \forall t \geq t_0, \quad f_t(v) = h_{n+1}^t(v) + f_{n+1}^t(v). \]

This proves the lemma by induction with $Z := \bigcup_{n=1}^{\infty} Z_n$. \hfill \Box

Note that the above iterations make sense because by induction $f_0^t = f_t$ and then all $f_n^t$ are nonnegative. Note also that if $t_0 > 0$ then, by moment production estimate (Theorem 1.2), we have

\[ \forall t \geq t_0, \quad f_t \in \bigcap_{s \geq 0} L_s^1(\mathbb{R}^N). \]

This enables us to use moment estimates for $Q^+(f, Q^+(g, h))$

\begin{equation}
\forall s \geq 0, \quad \|Q^+(f, Q^+(g, h))\|_{L_s^1} \leq \|f\|_{L_{s+2}^2} \|g\|_{L_{s+2}^2} \|h\|_{L_{s+2}^2}.
\end{equation}

Before we can show the regularity property of $f_n^t$ and the exponential decay (in norm) of $h_n^t$ we need further preparation.

Let us recall the following definition of the Sobolev space $H^s(\mathbb{R}^N)$ ($s > 0$): this is subset of functions $f \in L^2(\mathbb{R}^N)$ so that

\[ f \in H^s(\mathbb{R}^N) \iff \|f\|_{H^s} = \|\widehat{f}\|_{L_s^2} = \left( \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2} < +\infty. \]
where $\hat{f}(\xi)$ is the Fourier transform of $f$:

$$
\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^N} f(v) e^{-i\xi \cdot v} \, dv.
$$

As usual we denote the homogeneous seminorm as

$$
\| f \|_{H^s} = \| \hat{f} \|_{L^2} = \left( \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}
$$

and the norm and seminorm are related by

$$
(3.10) \quad \| g \|_{H^s} \leq \| g \|_{H^s} \leq (2\pi)^{N/2} 2^{s/2} \| g \|_{L^2} + 2^{s/2} \| g \|_{H^s}.
$$

It is easily proved (see [20, pp. 416-417]) that if the angular function $b$ satisfies

$$
(3.11) \quad \| b \|_{L^2} := \left| S^{N-2} \right| \int_0^{\pi} b(\cos \theta)^2 \sin^{N-2} \theta \, d\theta < +\infty
$$

then $Q^+: L^2_{N+\gamma}(\mathbb{R}^N) \times L^2_{N+\gamma}(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ is bounded with

$$
(3.12) \quad \| Q^+(f, g) \|_{L^2} \leq C \| b \|_{L^2} \| f \|_{L^2_{N+\gamma}} \| g \|_{L^2_{N+\gamma}}
$$

where $C < \infty$ only depends on $N$ and $\gamma$. This together with (3.10), (3.12) and the estimate of $\| Q^+(f, g) \|_{H^s}$ obtained in [5, 20] for $s = (N - 1)/2$ leads to the following lemma.

**Lemma 3.2** ([5, 20]). Suppose $N \geq 2, \gamma \geq 0$. Let $B(z, \sigma)$ as defined in (1.5) with the condition (3.11). Then $Q^+: L^2_{N+\gamma}(\mathbb{R}^N) \times L^2_{N+\gamma}(\mathbb{R}^N) \rightarrow H^{\frac{N-1}{2}}(\mathbb{R}^N)$ is bounded with the estimate

$$
\| Q^+(f, g) \|_{H^{\frac{N-1}{2}}} \leq C \| b \|_{L^2} \| f \|_{L^2_{N+\gamma}} \| g \|_{L^2_{N+\gamma}}
$$

where $C < \infty$ only depends on $N, \gamma$.

The following lemma will be useful to prove the $H^1$-regularity of $f^n_t$ in the decomposition $f_t = f^n_t + h^n$. 

**Lemma 3.3.** Let $B(z, \sigma)$ as defined in (1.5) with the condition $x \in (1.33)$ and (1.44), for some $0 < \gamma \leq 2$. Let $F_t \in B^+_{1,0,1}(\mathbb{R}^N)$ be a conservative measure strong solution of equation (1.1). Then for any $t_0 > 0$ we have

$$
(3.13) \quad \forall t \geq t_0, \forall v \in \mathbb{R}^N, \quad \int_{\mathbb{R}^N} |v - v_*|^\gamma \, dF_t(v_*) \geq a(v)^\gamma \geq a
$$

where

$$
(3.14) \quad a := \left[ K_4 \left( 1 + \max\{1, 1/t_0\} \right) \right]^{-2/\gamma}
$$

and $K_4 = K_4(1,1 + N)$ is the constant in (1.23). In particular if $t_0 \geq 1$, then $a$ is independent of $t_0$.

Moreover for any $t_0 \leq t_1 \leq t < \infty$, let

$$
E^n_{t_1}(v) := \exp \left( - \int_{t_1}^t \int_{\mathbb{R}^N} |v - v_*|^\gamma \, dF_t(v_*) \, d\tau \right).
$$
Then for any \( f \in L^\infty(\mathbb{R}^N) \cap L^4_2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \) we have \( fE^t_{1i} \in H^1(\mathbb{R}^N) \) and

\[
\| f E^t_{1i} \|_{H^1(\mathbb{R}^N)} \leq C \left[ \| f \|_{L^\infty(\mathbb{R}^N)} + \| f \|_{L^4_2(\mathbb{R}^N)} + \| f \|_{H^1(\mathbb{R}^N)} \right] e^{-\alpha(t-t_1)}(1+t-t_1)
\]

where \( C \) only depends on \( N, \gamma \).

**Proof.** Let

\[
L_s(F_t)(v) := \int_{\mathbb{R}^N} |v - v_*|^s \, dF_t(v_*).
\]

By conservation of mass, momentum and energy, the inequality (3.13) is obvious when \( \gamma = 2 \). Suppose \( 0 < \gamma < 2 \). In this case we use the moment production estimate (1.22) with \( s = 4 \) to get

\[
L_4(F_t)(v) \leq \langle v \rangle^4 \| F_t \|_4 \leq \langle v \rangle^4 \mathcal{K}_4 \left( 1 + \frac{1}{t_0} \right)^2.
\]

Then using Hölder inequality with

\[
\frac{1}{p} = \frac{2}{4 - \gamma}, \quad \frac{1}{q} = \frac{2 - \gamma}{4 - \gamma},
\]

we have

\[
\langle v \rangle^2 < |v|^2 + N = L_2(F_t)(v) \leq L_\gamma(F_t)(v)^{\frac{2}{1-\gamma}} L_4(F_t)(v)^{\frac{2-\gamma}{2-\gamma}}
\]

\[
\leq L_\gamma(F_t)(v)^{\frac{2}{1-\gamma}} \langle v \rangle^{\frac{4(2-\gamma)}{4-\gamma}} \left[ \mathcal{K}_4 \left( 1 + \frac{1}{t_0} \right)^2 \right]^{\frac{2-\gamma}{2-\gamma}}.
\]

This gives

\[
\forall t \geq t_0, \quad \langle v \rangle^\gamma \leq L_\gamma(F_t)(v) \left[ \mathcal{K}_4 \left( 1 + \frac{1}{t_0} \right)^2 \right]^{\frac{2-\gamma}{2-\gamma}} \leq \frac{1}{a} L_\gamma(F_t)(v)
\]

and (3.13) follows.

Then the proof of (3.15) is based on the following a priori estimates (a full justification requires standard approximation arguments that we skip here):

**Case 1:** \( 0 < \gamma < 1 \). In this case we have \( 0 < 2(1-\gamma) < 2 < N \) so that

\[
\left\{ \begin{array}{l}
\int_{\mathbb{R}^N} \frac{1}{|v - v_*|^{2(1-\gamma)}} \, dv \leq C \| f \|_{L^\infty}^2 + \| f \|_{L^2}^2, \\
\sum_{j=1}^N \| f \partial_{v_j} E^t_{1i}(v) \|_{L^2}^2 \leq C \left( \| f \|_{L^\infty}^2 + \| f \|_{L^2}^2 \right) e^{-2\alpha(t-t_1)}(t-t_1)^2.
\end{array} \right.
\]

**Case 2:** \( \gamma \geq 1 \). Since \( \gamma \leq 2 \), this implies \( |v - v_*|^{2(\gamma-1)} \leq \langle v \rangle^2 \langle v_* \rangle^2 \). Then recalling \( \| F_t \|_2 = 1 + N \) and \( f \in L^\infty(\mathbb{R}^N) \) we have

\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} |f(v)|^2 |v - v_*|^{2(\gamma-1)} \, dv \right) \, dF_t(v_*)(v) \leq (1 + N) \| f \|_{L^\infty} \int_{\mathbb{R}^N} |f(v)| \langle v \rangle^2 \, dv.
\]
which shows that
\[
\sum_{j=1}^{N} \left\| f \partial_{v_j} E_{t_1}^t \right\|_{L^2}^2 \leq C \left\| f \right\|_{L^\infty} \left\| f \right\|_{L^1} e^{-2a(t-t_1)(t-t_1)}.
\]

Combing the two cases and using
\[
\left\| f \right\|_{L^\infty} \left\| f \right\|_{L^2} \leq \frac{1}{2} \left\| f \right\|_{L^2}^2 + \frac{1}{2} \left\| f \right\|_{L^2}^2
\]
we obtain
\[
\left\| f E_{t_1}^t \right\|_{H^1(\mathbb{R}^N)}^2 \leq C \left( \left\| f \right\|_{L^\infty(\mathbb{R}^N)}^2 + \left\| f \right\|_{L^1(\mathbb{R}^N)}^2 \right) e^{-2a(t-t_1)(1 + t-t_1)}.
\]

**Theorem 3.4.** Suppose \( N \geq 3 \) and let \( B(z, \sigma) \) be defined by (1.33) with the conditions (1.33) and (1.44). Let \( f_t \in L^1_{1,0,1}(\mathbb{R}^N) \) be a conservative mild solution of equation (1.1).

Then for any \( t_0 > 0 \), the positive decomposition \( f_t = f_t^n + h_t^n \) given in (3.1)-(3.5) on \([t_0, \infty)\) satisfy the following estimates for all \( s \geq 0 \):

\[
\sup_{n \geq N_\gamma+1, t \geq t_0} \left\| f_t^n \right\|_{L^\infty} \leq C_{t_0,s} \tag{3.16}
\]
\[
\sup_{n \geq N_\gamma+2, t \geq t_0} \left\| f_t^n \right\|_{H^1} \leq C_{t_0} \tag{3.17}
\]
\[
\forall t \geq t_0, \forall n \geq 1, \quad \left\| h_t^n \right\|_{L^1} \leq C_{t_0,s,n} e^{-\frac{a}{2}(t-t_0)} \tag{3.18}
\]
\[
\forall t_1, t_2 \geq t_0, \quad \sup_{n \geq 1} \left\| f_{t_1}^n - f_{t_2}^n \right\|_{L^1} \quad \text{and} \quad \sup_{n \geq 1} \left\| h_{t_1}^n - h_{t_2}^n \right\|_{L^1} \leq C_{t_0,s} |t_1 - t_2| \tag{3.19}
\]

Here \( N_\gamma \) is defined by (2.48), \( a = a_{t_0} > 0 \), \( s \geq 0 \) is given in (3.14), and \( C_{t_0}, C_{t_0,s}, C_{t_0,s,n} \) are finite constants depending only on \( N \), the function \( b, \gamma \), \( \max\{1, 1/t_0\} \), \( s \), as well as \( n \) in the case of \( C_{t_0,s,n} \). In particular if \( t_0 \geq 1 \), all these constants are independent of \( t_0 \).

**Proof.** Let
\[
\forall s \geq 0, \quad \left\| f \right\|_s = \sup_{t \geq t_0} \left\| f_t \right\|_{L^1}.
\]

By using Theorem 1.2 (1.14) and the fact that
\[
\left\| f_0 \right\|_{L^1} = 1, \quad \left\| f_0 \right\|_{L^2} = 1 + N
\]
we have with \( \mathcal{K}_s = \mathcal{K}_s(1, 1 + N) \) that
\[
\forall s \geq 0, \quad \left\| f \right\|_s \leq \mathcal{K}_s \left( 1 + \max\left\{1, \frac{1}{t_0}\right\} \right)^{(s-2)/\gamma}.
\]

We first prove (3.16) and (3.18). To do this it suffices to prove the following estimates (3.20)-(3.21):

- For all \( s \geq 0 \) and all \( t \geq t_0 \)

\[
\forall n \geq 1, \quad \left\| h_t^n \right\|_{L^1} \leq \left\| f \right\|_{s+(2n-1)\gamma}^{2n-1} (1 + t + t_0)^{2n-1} e^{-a(t-t_0)} \tag{3.20}
\]
• If $0 < \gamma < N - 2$, we then define

$$p_n := \frac{N - 1}{N - 1 - n\gamma}, \quad n = 1, 2, 3, \ldots, N_\gamma,$$

and then for all $s \geq 0$

(3.21) \quad \max_{1 \leq n \leq N_\gamma} \sup_{t \geq 0} \| f_t^n \|_{L^p} \leq C_{a, p} \| f \|_{s + s_1}^{\beta_*},

(3.22) \quad \sup_{n \geq N_\gamma + 1} \sup_{t \geq 0} \| f_t^n \|_{L^\infty} \leq C_{a, p} \| f \|_{s + s_1}^{\beta_*}.

where $s_1 = N_\gamma + \gamma$ and $0 < \beta_*, \beta_* < \infty$ depend only on $N$ and $\gamma$.

• If $\gamma \geq N - 2$ and $1 < p < N$, then for all $s \geq 0$,

(3.23) \quad \sup_{t \geq 0} \| f_t^1 \|_{L^p} \leq C_{a, p} \| f \|_{s + 2\gamma - N/q}^{3},

(3.24) \quad \sup_{n \geq t_2 \geq 0} \| f_t^n \|_{L^\infty} \leq C_{a, p} \| f \|_{s + s_1}^{3 + 4/N},

where $s_2 = 3\gamma + 2 - 3N/2$.

In the following we denote by $C, C_*, C_{*, *}$ the finite positive constants (larger than 1) that only depend on $N, A_2, \| b \|_{L^\infty}, \gamma,$ and the arguments “*,” “*”; they may have different values in different places. By definition of $E_t^p$ and Lemma 3.3 we have

(3.25) \quad \forall t_0 \leq t_2 \leq t_1 \leq t,

$$\begin{cases} E_{t_1}^t \leq e^{-a(t-t_1)}, \\ E_{t_2}^{t_1} \leq e^{-a(t-t_2)}. \end{cases}$$

By (3.25) and $0 \leq f_t^n \leq f_t$ we have

(3.26) \quad h_t^n(v) \leq h_t^1(v) + 2 \int_{t_0}^{t} \int_{t_0}^{t_1} e^{-a(t-t_2)} Q^+ \left( f_t^n, Q^+(f_{t_2}, h_{t_2}^{n-1}) \right)(v) \, dt_2 \, dt_1 \\
+ \int_{t_0}^{t} \int_{t_0}^{t_1} e^{-a(t-t_2)} Q^+ \left( h_{t_1}^{n-1}, Q^+(f_{t_2}, f_t^n) \right)(v) \, dt_2 \, dt_1,

(3.27) \quad f_t^n(v) \leq \int_{t_0}^{t} \int_{t_0}^{t_1} e^{-a(t-t_2)} Q^+ \left( f_{t_1}^{n-1}, Q^+(f_{t_2}^{n-1}, f_{t_2}^{n-1}) \right)(v) \, dt_2 \, dt_1.

By definition of $h_t^1$ in (3.4) and using (3.25) and $\| f_t \|_{L^1} \geq \| f_t \|_{L^1} = 1$ we have

$$\| h_t^1 \|_{L^1} \leq \| f \|_{s + \gamma}^2 (1 + t - t_0) e^{-a(t-t_0)}.$$

Suppose (3.20) holds for some $n \geq 1$. Using (3.25) for $h_t^{n+1}$ we have

$$\| h_t^{n+1} \|_{L^1} \leq e^{-a(t-t_0)} \| f \|_{s + \gamma}^2 (1 + t - t_0) + 2 \| f \|_{s + 2\gamma}^2 \int_{t_0}^{t} \int_{t_0}^{t_1} \| h_{t_2}^{n} \|_{L^1 + 2\gamma} e^{-a(t-t_2)} \, dt_2 \, dt_1 \\
+ \| f \|_{s + 2\gamma}^2 \int_{t_0}^{t} \int_{t_0}^{t_1} e^{-a(t-t_2)} \, dt_2 \, dt_1 \\
= \| f \|_{s + 2\gamma}^2 e^{-a(t-t_0)} \left( 1 + t - t_0 + \int_{t_0}^{t} \int_{t_0}^{t_1} e^a(t_2-t_0) \left( \| h_{t_2}^{n} \|_{L^1 + 2\gamma} + \| h_{t_1}^{n} \|_{L^1 + \gamma} \right) \, dt_2 \, dt_1 \right).$$
and by inductive hypothesis on $h^n$ we have for all $t_0 \leq t_2 \leq t_1$,
\begin{align*}
2 \|h^n_{t_2}\|_{L^1_{s+2\gamma}} + \|h^n_{t_1}\|_{L^1_{s+\gamma}} & \leq 2 (\|f\|_{s+2\gamma + (2n-1)\gamma})^{2n} (1 + t_2 - t_0)^{2n-1} e^{-a(t_2-t_0)} \\
& + \|f\|_{s+\gamma + (2n-1)\gamma})^{2n} (1 + t_1 - t_0)^{2n-1} e^{-a(t_1-t_0)} \\
& \leq 3 \|f\|_{s+2(n+1)\gamma}^{2n} (1 + t_1 - t_0)^{2n-1} e^{-a(t_2-t_0)}.
\end{align*}

So
\[ \|h^{n+1}\|_{L^1_{s}} \leq \|f\|_{s+(2n+1)\gamma}^{2(n+1)} e^{-a(t-t_0)} \left( 1 + t - t_0 + 3 \int_{t_0}^{t} (1 + t_1 - t_0)^{2n-1} (t_1 - t_0) \, dt_1 \right). \]

It is easily checked that
\[ \forall t \geq t_0, \quad 1 + t - t_0 + 3 \int_{t_0}^{t} (1 + t_1 - t_0)^{2n-1} (t_1 - t_0) \, dt_1 \leq (1 + t - t_0)^{2n+1}. \]

Thus
\[ \|h^{n+1}\|_{L^1_{s}} \leq \|f\|_{s+(2n+1)\gamma}^{2(n+1)} (1 + t - t_0)^{2n+1} e^{-a(t-t_0)}. \]

This proves (3.20).

Now we are going to prove (3.21)–(3.24). First of all by (3.27) and the inequality
\[ \int_{t_0}^{t} \int_{t_0}^{t_1} e^{-a(t-t_2)} \, dt_2 \, dt_1 \leq \frac{1}{a^2} \]
we have
\[ (3.28) \quad \sup_{t_2 \geq t_0} \|f^n_t\|_{L^p_s} \leq \frac{1}{a^2} \left( \sup_{t_1 \geq t_2 \geq t_0} \|Q^+(f^n_{t_1}, Q^+(f^n_{t_2}, f^n_{t_2}))\|_{L^p_s} \right) \]
for all $s \geq 0, 1 \leq p \leq \infty$, provided that the RHS makes sense.

**Case 1:** $0 < \gamma < N - 2$. We first prove that
\[ (3.29) \quad \forall s \geq 0, \quad \sup_{t \geq t_0} \|f^n_t\|_{L^p_s} \leq C_{a,n} \|f\|_{s+n-1+\gamma_1}^{\beta_n} \quad n = 1, 2, \ldots, N_{\gamma} \]
where $\gamma_1 = \max\{\gamma, 1\}$ and
\[ \beta_n := 2(N + 1) \left( 1 + \frac{1}{N} \right)^{n-1} + 1 - 2N. \]

By part (i) of Theorem 2.24 we have
\[ \forall t_1 \geq t_2 \geq t_0, \quad \|Q^+(f_{t_1}, Q^+(f_{t_2}, f_{t_2}))\|_{L^p_{s+\gamma}} \leq C_1 \|f_{t_1}\|_{L^1_s} \|f_{t_2}\|_{L^1_{s+\gamma}}^2 \leq C_1 \|f\|_{s+\gamma}^3. \]

Using (3.28) with $p = p_1$ and $n = 1$ (recalling $f^{(0)}_{t_1} = f_{t_1}$) gives
\[ \forall s \geq 0, \quad \sup_{t \geq t_0} \|f^n_t\|_{L^p_s} \leq C_{a,1} \|f\|_{s+\gamma}^3. \]

Since $\beta_1 = 3$, this proves that the inequality in (3.20) holds for $n = 1$. 
Suppose the inequality in (3.29) holds for some $1 \leq n \leq N_\gamma - 1$. Then we compute using $0 \leq f_t^n \leq f_t$ and part (I) of Theorem 2.7 that, for all $s \geq 0$,

$$
(3.30) \quad \left\| Q^+ \left(f_{t_1}^{n+1}, Q^+ (f_{t_2}^n, f_{t_2}^n) \right) \right\|_{L^p_{s+1}} \leq C_n \left\| f_t^n \right\|_{L^p_{s+1}} \left\| f_{t_1}^{n+1} \right\|_{L^p_{s+1}} \left\| f_{t_2}^n \right\|_{L^p_{s+1}} \left\| f_{t_2}^n \right\|_{L^p_{s+1}}^{1-\theta_n} \left\| f_{t_2}^n \right\|_{L^p_{s+1}}^{\theta_n} \leq C_n \left\| f_t \right\|_{L^p_{s+1}}^{2-\theta_n} (\sup_{t \geq t_0} \left\| f_t^n \right\|_{L^p_{s+1}}) \right\|_{L^p_{s+1}}^{1+\theta_n}.
$$

By inductive hypothesis on $f_t^n$ we compute

$$
(3.31) \quad \left\| f \right\|_{L^p_{s+\gamma_1}}^{2-\theta_n} (\sup_{t \geq t_0} \left\| f_t^n \right\|_{L^p_{s+1}}) \right\|_{L^p_{s+\gamma_1}}^{1+\theta_n} \leq C_{a,n} \left\| f \right\|_{L^p_{s+\gamma_1}}^{2-\theta_n + \beta_n (1+\theta_n)}.
$$

Also by definition of $\theta_n$ and $\beta_n$ it is easily checked that $2 - \theta_n + \beta_n (1 + \theta_n) < \beta_{n+1}$. It then follows from (3.28), (3.30) and (3.31) that

$$
\forall s \geq 0, \sup_{t \geq t_0} \left\| f_t^{n+1} \right\|_{L^p_{s+1}} \leq C_{a,n+1} \left\| f \right\|_{L^p_{s+\gamma_1}}^{\beta_{n+1}}.
$$

This proves that the inequality in (3.29) holds for all $n = 1, 2, \ldots, N_\gamma$. From (3.29) and $N_\gamma - 1 + \gamma_1 < N_\gamma + \gamma = s_1$, we obtain (3.21) with $\beta_s = \beta_{N_\gamma}$.

Next let us prove (3.22). By Theorem 2.7 (see (2.52), (2.53)) and using (3.21) with $n = N_\gamma$ we have

$$
\left\| Q^+ (f_{t_1}^{N_\gamma}, Q^+ (f_{t_2}^{N_\gamma}, f_{t_2}^{N_\gamma})) \right\|_{L^\infty} \leq C \left\| f_{t_1}^{N_\gamma} \right\|_{L^\infty}^{1-\alpha_1} \left\| f_{t_1}^{N_\gamma} \right\|_{L^\infty}^{\alpha_1} \left\| f_{t_2}^{N_\gamma} \right\|_{L^\infty}^{1-\alpha_2} \left\| f_{t_2}^{N_\gamma} \right\|_{L^\infty}^{\alpha_2} \left\| f_{t_2}^{N_\gamma} \right\|_{L^\infty} \leq C \left( \left\| f \right\|_{L^{s+1}} \right)^{3 + (\beta_{N_\gamma} - 1)(\alpha_1 + \alpha_2)}.
$$

This together with (3.28) gives

$$
(3.32) \quad \sup_{t \geq t_0} \left\| f_t^{N_\gamma+1} \right\|_{L^\infty} \leq C_a \left\| f \right\|_{L^{s+1}}^\eta, \quad \eta := 3 + (\beta_{N_\gamma} - 1)(\alpha_1 + \alpha_2).
$$

Using (3.28) with $p = \infty$, Theorem 2.7 and

$$
\left\| f_t^{N_\gamma+k} \right\|_{L^p_{s+\gamma}} \leq \left\| f \right\|_{L^s}^{1/pN_{\gamma}} \left\| f_t^{N_\gamma+k} \right\|_{L^\infty}^{1/qN_{\gamma}}
$$
It follows from (3.32) and for all $s \geq 0$,

$$
(3.33) \quad \sup_{t \geq t_0} \left\| f_t^{N_s+k+1} \right\|_{L^\infty_s} \leq \frac{1}{a_1} \sup_{t_1 \geq t_2 \geq t_0} \left\| Q^+ \left( f_{t_1}^{N_s+k}, Q^+ \left( f_{t_2}^{N_s+k}, f_{t_2}^{N_s+k} \right) \right) \right\|_{L^\infty_s} \\
\leq C_a \sup_{t_1 \geq t_2 \geq t_0} \left\{ \left\| f_{t_1}^{N_s+k} \right\|_{L^1_s}^{1-\alpha_1} \left\| f_{t_1}^{N_s+k} \right\|_{L^{P_N}\gamma_s}^{\alpha_1} \times \left\| f_{t_2}^{N_s+k} \right\|_{L^1_s}^{1-\alpha_2} \left\| f_{t_2}^{N_s+k} \right\|_{L^{P_N}\gamma_s}^{\alpha_2} \right\} \\
\leq C_a \| f \|_{s+\gamma}^{3-(\alpha_1+\alpha_2)/qN_\gamma} \left( \sup_{t \geq t_0} \left\| f_t^{N_s+k} \right\|_{L^\infty_s} \right)^{1+(\alpha_1+\alpha_2)/qN_\gamma} \\
\leq C_a \| f \|_{s+\gamma}^{3-\delta} \left( \sup_{t \geq t_0} \left\| f_t^{N_s+k} \right\|_{L^\infty_s} \right)^{\delta} < \infty, \quad k = 1, 2, 3, \ldots
$$

where

$$\delta := \frac{\alpha_1 + \alpha_2}{qN_\gamma} = \frac{N - 1 - \gamma + (1-\gamma)+}{N} \quad (< 1).$$

Now fix any $s \geq 0$ and let us define

$$A = C_a \| f \|_{s+\gamma}^{3-\delta} \quad \text{and} \quad Y_k = \sup_{t \geq t_0} \left\| f_t^{N_s+k} \right\|_{L^\infty_s}.$$

Then, from (3.33),

$$Y_{k+1} \leq AY_k^\delta, \quad k = 1, 2, \ldots$$

which gives

$$Y_{k+1} \leq A^{1+\delta+\ldots+\delta k} Y_1^\delta = A^{1+\delta k} Y_1^\delta \leq A^{1+\delta k} Y_1, \quad k = 1, 2, \ldots$$

It follows from (3.32) and $\gamma < s_1$ that

$$\sup_{t \geq t_0} \left\| f_t^{N_s+k+1} \right\|_{L^\infty_s} = Y_{k+1} \leq \left( C_a \| f \|_{s+\gamma}^{3-\delta} \right)^{1+\delta} C_a \| f \|_{s+s_1}^{1+(1-\delta)} \leq C_a \| f \|_{s+s_1}^{3+(1-\delta)}$$

for all $k = 1, 2, 3, \ldots$ This gives (3.22) with $\beta^* = (3-\delta)/(1-\delta) + \eta$.

Case 2: $\gamma \geq N - 2$. By Theorem 2.4, we have for any $1 < p < N$ and $s \geq 0$

$$\left\| Q^+ \left( f_{t_1}, Q^+ (f_{t_2}, f_{t_2}) \right) \right\|_{L^p_s} \leq C_p \left\| f_{t_1} \right\|_{L^1_{s+\gamma-N/q}} \left\| f_{t_2} \right\|_{L^1_{s+2\gamma-N/q}} \leq C_p \left\| f \right\|_{s+2\gamma-N/q}^3.$$

This together with (3.28), with $n = 1$ implies that

$$\forall \ s \geq 0, \quad \sup_{t \geq t_0} \left\| f_t^1 \right\|_{L^p_s} \leq C_{a,p} \left\| f \right\|_{s+2\gamma-N/q}^3.$$

This proves (3.29). In particular for $p = 2$ we have

$$\forall \ s \geq 0, \quad \sup_{t \geq t_0} \left\| f_t^1 \right\|_{L^2_s} \leq C_a \left\| f \right\|_{s+2\gamma-N/2}^3.$$
Then using (3.28) with $p = \infty$, Theorem 2.7 with
\[ p = q = 2 \in \left( \frac{N}{N-1}, N \right), \]
and induction on $n$ starting from $n = 1$ with the $L_s^2$-boundedness (3.34) we have, for all $s \geq 0$,
\begin{align*}
(3.35) \quad \sup_{t \geq t_0} \left\| f_t^{n+1} \right\|_{L_s^\infty} & \leq \frac{1}{a^s} \sup_{t \geq t_1 \geq t_0} \left\| Q^+ (f_{t_1}^n, Q^+ (f_{t_2}^n, f_{t_2}^n)) \right\|_{L_s^\infty} \\
& \leq C_a \sup_{t \geq t_1 \geq t_0} \left( \left\| f_{t_1}^n \right\|^{1-2/N}_{L_s^{1+\gamma+2-N}} \left\| f_t^n \right\|^{2/N}_{L_s^{1+\gamma+2-N}} \left\| f_{t_2}^n \right\|^{2}_{L_s^{1+\gamma+1-N}} \right) \\
& \leq C_a \left\| f \right\|_{s+2\gamma+1-N}^{3-2/N} \left( \sup_{t \geq t_0} \left\| f_t^n \right\|_{L_s^{1+\gamma+2-N}} \right)^{2/N} < +\infty, \quad n = 1, 2, 3, \ldots
\end{align*}
Taking $n = 1$ and using (3.34) and $2\gamma + 1 - N < 3\gamma + 2 - 3N/2 =: s_2$ we have
\begin{align*}
(3.36) \quad \sup_{t \geq t_0} \left\| f_t^2 \right\|_{L_s^\infty} & \leq C_a \left\| f \right\|_{s+s_2}^{3+4/N}.
\end{align*}

Further, using
\[ \forall n \geq 2, \quad \left\| f_t^n \right\|_{L_s^{1+\gamma+2-N}} \leq \left\| f_t^{1/2} \right\|_{s+\gamma+4-2N} \left\| f_t^n \right\|_{L_s^\infty}^{1/2} \]
and $2\gamma + 4 - 2N \leq 2\gamma + 1 - N \leq s_2$ (because $\gamma \geq N - 2 \geq 1$) we get from (3.35) that
\[ \sup_{t \geq t_0} \left\| f_t^n \right\|_{L_s^\infty} \leq C_a \left\| f \right\|_{s+s_2}^{3-1/N} \left( \sup_{t \geq t_0} \left\| f_t^n \right\|_{L_s^\infty} \right)^{1/N}, \quad n = 2, 3, \ldots
\]
By iteration we have, as shown above with $\delta = 1/N$, and using (3.36) that
\begin{align*}
\sup_{t \geq t_0} \left\| f_t^{n+1} \right\|_{L_s^\infty} & \leq \left( C_a \left\| f \right\|_{s+s_2}^{3-1/N} \right)^{1-\delta^{n-1}} \left( \sup_{t \geq t_0} \left\| f_t^2 \right\|_{L_s^\infty} \right)^{\delta^n} \\
& \leq C_a \left( \left\| f \right\|_{s+s_2} \right)^{(3-1/N)\frac{1-\delta^{n-1}}{1-\delta} + (3+4/N)\delta^n} \leq C_a \left( \left\| f \right\|_{s+s_2} \right)^{3+4/N}, \quad n = 2, 3, \ldots
\end{align*}
This proves (3.24).

Now let us prove the $H^1$-regularity (3.17) of $f_t^n$ for $n \geq N\gamma + 2$. For notation convenience we denote
\[ Q_{t_1,t_2}^{n-1}(v) := Q^+ (f_{t_1}^{n-1}, Q^+ (f_{t_2}^{n-1}, f_{t_2}^{n-1}) E_{t_1}^t) (v). \]
The iteration formula (3.3) is then written
\begin{align*}
(3.37) \quad \forall t \geq t_0, \quad f_t^n(v) = \int_{t_0}^t E_{t_1}^t(v) \int_{t_0}^{t_1} Q_{t_1,t_2}^{n-1}(v) \, dt_2 \, dt_1.
\end{align*}
Applying Theorem 2.7 and the $L_s^\infty$ estimate in (3.16) we have
\[ \left\| Q_{t_1,t_2}^{n-1} \right\|_{L_\infty} \leq e^{-a(t_1-t_2)} \left\| Q^+ (f_{t_1}^{n-1}, Q^+ (f_{t_2}^{n-1}, f_{t_2}^{n-1})) \right\|_{L_\infty} \leq C_{t_0} e^{-a(t_1-t_2)}. \]
Also by $f_t^{n-1} \leq f_t$ we have
\[ \left\| Q_{t_1,t_2}^{n-1} \right\|_{L_2^s} \leq e^{-a(t_1-t_2)} \left\| Q^+ (f_{t_1}^{n-1}, Q^+ (f_{t_2}^{n-1}, f_{t_2}^{n-1})) \right\|_{L_2^s} \leq C_{t_0} e^{-a(t_1-t_2)}. \]
And by Lemma 3.22 and the $L^\infty_s$ estimate in (3.16) we have
\[
\|Q^{n-1}_{t_1, t_2}\|_{H^1} \leq \|Q^{n-1}_{t_1, t_2}\|_{H^{N+1}} \leq C \|Q^+(f^{n-1}_{t_2}, f^{n-1}_{t_2})\|_{L^2_{N+\gamma}}^2 \leq C\|f^{n-1}_{t_1}\|_{L^2_{N+\gamma}}^2 e^{-a(t_1-t_2)} \leq C_0 e^{-a(t_1-t_2)}.
\]
Thus we conclude from Lemma 3.33 that $Q^{n-1}_{t_1, t_2} E^{t_1}_t \in H^1(\mathbb{R}^N)$ and
\[
(3.38) \quad \|Q^{n-1}_{t_1, t_2} E^{t_1}_t\|_{H^1} \leq C_0 e^{-a(t_1-t_2)} e^{-a(t_1-t_1)}(1 + t - t_1) = C_0 e^{-a(t-t_2)}(1 + t - t_1).
\]
Using Minkowski’s inequality to (3.37) we then conclude from (3.38) and the above estimates that $f^n_t \in H^1(\mathbb{R}^N)$ and
\[
\|f^n_t\|_{H^1} \leq \int_{t_0}^t \int_{t_0}^{t_1} \|Q^{n-1}_{t_1, t_2} E^{t_1}_t\|_{H^1} dt_2 dt_1 \leq C_0 \int_{t_0}^t \int_{t_0}^{t_1} e^{-a(t-t_2)}(1 + t - t_1) dt_2 dt_1 \leq C_0.
\]
This proves (3.17).

Finally let us prove (3.19). To do this we rewrite $f^n_t$ as follows (recall definition of $f^n_t$ in (3.3))
\[
f^n_t(v) = E^t_{t_0}(v) \int_{t_0}^t E^{t_1}_t(v) \int_{t_0}^{t_1} Q^+ \left( f^{n-1}_{t_1}, Q^+ \left( f^{n-1}_{t_2}, f^{n-1}_{t_2} \right) E^{t_1}_t \right) (v) dt_2 dt_1
\]
and recall that
\[
E^t_s(v) = \exp \left( -\int_s^t L_\gamma(f_\tau(v)) d\tau \right), \quad L_\gamma(f_\tau)(v) := \int_{\mathbb{R}^N} |v - v_\star|^\gamma f_\tau(v_\star) dv_\star.
\]
Then it is easily seen that the function $t \mapsto f^n_t(v)$ is absolutely continuous on every bounded subinterval of $[t_0, \infty)$ and
\[
\frac{\partial}{\partial t} f^n_t(v) = \int_{t_0}^t Q^+ \left( f^{n-1}_{t_1}, Q^+ \left( f^{n-1}_{t_2}, f^{n-1}_{t_2} \right) E^{t_1}_t \right) (v) dt_2 - L_\gamma(f_t)(v) f^n_t(v), \quad \text{a.e. } t \geq t_0.
\]
Since
\[
\|Q^+ \left( f^{n-1}_{t_1}, Q^+ \left( f^{n-1}_{t_2}, f^{n-1}_{t_2} \right) E^{t_1}_t \right)\|_{L^1} \leq \|Q^+ (f_t, Q^+ (f_{t_2}, f_{t_2}))\|_{L^1} e^{-a(t-t_2)} \leq \|f_t\|_{L^1_{N+\gamma}}^2 \|f_{t_2}\|_{L^1_{N+\gamma}}^2 e^{-a(t-t_2)} \leq C_{t_0,s} e^{-a(t-t_2)}
\]
and
\[
\|L_\gamma(f_t) f^n_t\|_{L^1} \leq \|f_t\|_{L^1_{N+\gamma}} \|f^n_t\|_{L^1_{N+\gamma}} \leq C_{t_0,s}
\]
it follows that
\[
\left\| \frac{\partial}{\partial t} f^n_t(v) \right\|_{L^1} \leq C_{t_0,s} \quad \text{a.e. } t \geq t_0.
\]
Thus, by the absolute continuity of $t \mapsto f^n_t(v)$, we deduce that
\[
\forall t_1, t_2 \geq t_0, \quad \|f^n_{t_1} - f^n_{t_2}\|_{L^1} \leq C_{t_0,s} |t_1 - t_2|.
\]
On the other hand, from
\[
\forall t \geq t_0, \quad f_t(v) = f_{t_0}(v) + \int_{t_0}^t \left[ Q^+(f_\tau, f_\tau)(v) - L_\gamma(f_\tau)(v) f_\tau(v) \right] d\tau
\]
we also have $\|f_{t_1} - f_{t_2}\|_{L^1_s} \leq C_{t_0,s}|t_1 - t_2|$ for all $t_1, t_2 \geq t_0$. Thus the function $t \mapsto h^n_t = f_t - f^n_t$ satisfies the same estimate. This proves (3.19) and completes the proof of the theorem.

**Corollary 3.5.** Suppose $N \geq 3$ and $B(z, \sigma)$ as defined in (1.3) with the conditions (1.33), (1.44) and some $\gamma \in (0, 2]$. Let $F_t \in B^+_{1,0,1}(\mathbb{R}^N)$ be a conservative measure strong solution of equation (1.1). Then for any $t_0 > 0$, $F_t$ can be decomposed as

$$\forall t \geq t_0, \quad dF_t(v) = f_t(v) \, dv + d\mu_t(v),$$

with

$$0 \leq f_t \in \bigcap_{s \geq 0} L^\infty_s \cap H^1(\mathbb{R}^N), \quad \mu_t \in \bigcap_{s \geq 0} B^+_s(\mathbb{R}^N),$$

satisfying for all $s \geq 0$

$$\sup_{t \geq t_0} \|f_t\|_{L^\infty_s} \leq C_{t_0,s}, \quad \sup_{t \geq t_0} \|f_t\|_{H^1} \leq C_{t_0},$$

$$\forall t \geq t_0, \quad \|\mu_t\|_s \leq C_{t_0,s} e^{-\frac{\gamma}{2}(t-t_0)},$$

$$\forall t_1, t_2 \in [t_0, \infty), \quad \|f_{t_1} - f_{t_2}\|_{L^1_s} \leq C_{t_0,s}|t_1 - t_2|,$$

where $a = a_{t_0} > 0$ is given in (3.14) and $C_{t_0}, C_{t_0,s}$ are finite constant depending only on $N$, $t_0$, $\gamma$, $b$, $\max\{1, 1/t_0\}$ and $s$.

**Proof.** By Theorem 1.2 there is a sequence $\{f_{k,t}\}_{k=1}^\infty \subset L^1_{1,0,1}(\mathbb{R}^N)$ of conservative mild solutions of equation (1.1) such that

$$\forall \varphi \in C_c(\mathbb{R}^N), \quad \forall t \geq 0, \quad \lim_{k \to \infty} \int_{\mathbb{R}^N} \varphi(v) f_{k,t}(v) \, dv = \int_{\mathbb{R}^N} \varphi(v) \, dF_t(v).$$

Let $n_\gamma = N_\gamma + 2$ with $N_\gamma$ defined in (2.48) and consider the positive decompositions of $f_{k,t}$

$$\forall k = 1, 2, 3, \ldots \forall t \geq t_0, \quad f_{k,t}(v) = f^{n_\gamma}_{k,t}(v) + h^{n_\gamma}_{k,t}(v),$$

given by (3.1), (3.6). By Theorem 3.3 we have for all $s \geq 0$

$$\sup_{k \geq 1, t \geq t_0} \left\|f^{n_\gamma}_{k,t}\right\|_{L^\infty_s} \leq C_{t_0,s}, \quad \sup_{k \geq 1, t \geq t_0} \left\|f^{n_\gamma}_{k,t}\right\|_{H^1} \leq C_{t_0},$$

$$\forall t \geq t_0, \quad \sup_{k \geq 1} \left\|h^{n_\gamma}_{k,t}\right\|_{L^1_s} \leq C_{t_0,s} e^{-\frac{\gamma}{2}(t-t_0)},$$

$$\forall t_1, t_2 \geq t_0, \quad \sup_{k \geq 1} \left\|f^{n_\gamma}_{k,t_1} - f^{n_\gamma}_{k,t_2}\right\|_{L^1_s} \leq C_{t_0,s}|t_1 - t_2|.$$
Since \( h_{k_j,t}^{n_{v_i}} = f_{k_j,t} - f_{k_j,t}^{n_{v_i}} \), it follows from (3.47) and the weak convergence (3.33) that for every \( t \geq t_0 \), \( h_{k_j,t}^{n_{v_i}} \) converges weakly to some \( \mu_t \in B_0^+ (\mathbb{R}^N) \) as \( j \to \infty \), i.e.

\[
\forall \varphi \in C_c (\mathbb{R}^N), \forall t \geq t_0, \lim_{j \to \infty} \int_{\mathbb{R}^N} \varphi (v) h_{k_j,t}^{n_{v_i}} (v) \, dv = \int_{\mathbb{R}^N} \varphi (v) \, d\mu_t (v).
\]

This leads to the decomposition (3.39). The inequalities (3.40), (3.41), (3.42) follow easily from (3.44), (3.45), (3.46), (3.47), (3.48) and the equivalent version (1.13) of measure norm \( \| \cdot \|_s \).

\[ \square \]

4. Rate of Convergence to Equilibrium

This section is devoted to the proof of Theorem 1.5. We first recall the results in [24] on the exponential rate of convergence to equilibrium for \( L^1 \) mild solutions.

**Theorem 4.1** (Cf. Theorem 1.2 of [24]). Suppose \( N \geq 3 \) and \( B(z, \sigma) \) as defined in (1.5) for some \( \gamma \in (0, 1] \) with the conditions (1.33), (1.44), and (1.45). Let \( \lambda = S_{b, \gamma} (1, 0, 1) > 0 \) be the spectral gap of the linear operator \( L_M \) in (1.43) associated with \( B(z, \sigma) \) and the Maxwellian \( M (v) = (2\pi)^{-N/2} e^{-|v|^2/2} \) in \( L^1_{1,0,1} (\mathbb{R}^N) \).

Let \( f_0 \in L^1_{1,0,1} (\mathbb{R}^N) \cap L^2 (\mathbb{R}^N) \) and let \( f_t \in L^2 (\mathbb{R}^N) \) be the unique conservative solution of equation (1.1) with the initial datum \( f_0 \). Then there is a constant \( 0 < C < \infty \), which depends explicitly only on \( N \), the function \( b, \gamma \) and on \( \| f_0 \|_{L^2} \), such that

\[
\forall t \geq 0, \quad \| f_t - M \|_{L^1} \leq C e^{-\lambda t}.
\]

In the important case of hard sphere model (i.e. \( N = 3, \gamma = 1 \) and \( b = \text{const.} \)), the assumption “\( f_0 \in L^1 \cap L^2 \)” can be relaxed to “\( f_0 \in L^1 \)” and the same result holds with the constant \( C \) depending only and explicitly on \( N \), the function \( b \) and \( \gamma \).

**Lemma 4.2** (Cf. Lemma 4.6 of [24]). Suppose \( N \geq 3 \) and \( B(z, \sigma) \) as defined in (1.5) for some \( \gamma \in (0, 1] \) with the conditions (1.33), (1.44), and (1.45). Let \( \lambda = S_{b, \gamma} (1, 0, 1) > 0 \) be the spectral gap of the linear operator \( L_M \) in (1.43) associated with \( B(z, \sigma) \) and the Maxwellian \( M (v) = (2\pi)^{-N/2} e^{-|v|^2/2} \) in \( L^1_{1,0,1} (\mathbb{R}^N) \). Let \( \alpha > 0, m (v) = e^{-\alpha |v|^2} \).

Then there are some explicitable finite constants \( \varepsilon > 0, C > 0 \) depending only and explicitly on \( N \), the function \( b, \gamma \) and \( \alpha \), such that if \( f_t \), with the initial datum \( f_0 \in L^1_{1,0,1} (\mathbb{R}^N) \cap L^1 (\mathbb{R}^N, m^{-2}) \) is a conservative solution to equation (1.1) satisfying

\[
\forall t \in [0, \infty), \quad \| f_t - M \|_{L^1 (m^{-2})} \leq \varepsilon
\]

then

\[
\forall t \in [0, \infty), \quad \| f_t - M \|_{L^1 (m^{-1})} \leq C \| f_0 - M \|_{L^1 (m^{-1})} e^{-\lambda t}.
\]
Remarks 4.3. (1) The original version of Theorem 4.1 and Lemma 4.2 in [24] were proved for the class \( L_{\pi N/2,0,1/2}^1(\mathbb{R}^N) \), i.e. for the Maxwellian \( M(v) = M_{\pi N/2,0,1/2}(v) = e^{-|v|^2} \). According to Proposition 4.4 (normalization), these are equivalent to the present version. In fact let \( g_t \in L_{\pi N/2,0,1/2}^1(\mathbb{R}^N) \), \( f_t \in L_{1,0,1}(\mathbb{R}^N) \) have the relation
\[
 f_t(v) = (2\pi)^{-N/2} g_t(v/\sqrt{2}), \quad \text{i.e.} \quad g_t(v) = (2\pi)^{N/2} f_{ct}(\sqrt{2}v), \quad t \geq 0
\]
where \( c = \pi N/2 - \gamma/2 \). Then \( f_t \) is a conservative solution of equation (1.1) if and only if \( g_t \) is a conservative solution of equation (1.1). And we have
\[
 \forall t \geq 0, \quad \|f_t - M_{1,0,1}\|_{L^1} = \pi^{-N/2} \|g_{t/\sqrt{2}} - M_{\pi N/2,0,1/2}\|_{L^1},
\]
and
\[
 S_{b,\gamma}(\pi N/2, 0, 1/2) = S_{b,\gamma}(1, 0, 1) \pi^{N/2} \gamma/2.
\]
(2) In order to prove the exponential rate of convergence to equilibrium, it was introduced in [24] the modified linearized collision operator (introduced first in [24])
\[
 \mathcal{L}_m(h) = m^{-1} M L M (m M^{-1} h), \quad m(v) = e^{-a|v|^s}
\]
with \( M(v) = e^{-|v|^2} \), \( a > 0 \) and \( 0 < s < 2 \). It is proved in [24] that \( \mathcal{L}_m \) and \( L_M \) has the same spectrum, but \( \mathcal{L}_m \) is available to connect the exponential moment estimates of solutions. The proof of the original version of Theorem 4.1 in [24] used additional technical assumptions: the angular function \( b \) is convex and non-decreasing in \((-1, 1)\), and the constant \( s \) in \( m(v) = e^{-a|v|^s} \) satisfies \( 0 < s < \gamma/2 \). These assumptions were only used to prove the exponential moment estimate of the form (1.20) (see Lemma 4.7 of [24]). According to Theorem 4.2 in Section 1, these additional assumptions on the function \( b \) can be removed and the restriction \( 0 < s < \gamma/2 \) can be relaxed to \( 0 < s \leq \gamma \). In particular one can choose \( s = \gamma \).

(3) Let \( f_t \) be a conservative mild solution of equation (1.1) on \([\tau, \infty)\) with the initial datum \( f_\tau \). Applying Theorem 4.1 and Lemma 4.2 to the solution \( t \to f_{t+\tau} \) on \([0, \infty)\), the exponential terms for the decay estimates of \( \|f_t - M\|_{L^1} \) on the time interval \([\tau, \infty)\) is given by \( e^{-\lambda(t-\tau)} \).

To prove Theorem 1.5 we need further preparation.

Lemma 4.4. Let \( 0 \leq f \in L_{k+l}^1(\mathbb{R}^N) \cap L_L^2(\mathbb{R}^N) \cap H^s(\mathbb{R}^N) \) with \( k \geq 0, \ l > 0, \ s > 0 \). Let
\[
 \mathcal{N}(f) = \mathcal{N}_{\rho,u,T}(f) \in L_{1,0,1}^1(\mathbb{R}^N)
\]
be the normalization of \( f \) defined in (1.40), (1.41), and suppose that \( |\rho - 1| + |u| + |T - 1| \leq 1/2 \). Then
\[
 \|f - \mathcal{N}(f)\|_{L_k^1} \leq C_{N,k,l,s}(f) \left( |\rho - 1| + |u| + |T - 1| \right)^{d/(1+2s)}
\]
where

\[ C_{N,k,l,s}(f) := C_{N,k,l} \max \left\{ \| f \|_{L^1_{k+l}}, \| f \|_{L^2_{k+l}}, \| f \|_{H^s} \right\} \]

and \( C_{N,k,l} < +\infty \) only depends on \( N,k,l \).

**Proof.** Recall that \( \mathcal{N}(f) = \rho^{-1} T^{N/2} f(\sqrt{T} v + u) \). Let \( \mathcal{N}_1(f) = T^{N/2} f(\sqrt{T} v + u) \). Then

\[ \| f - \mathcal{N}(f) \|_{L^1_k} \leq \| f - \mathcal{N}_1(f) \|_{L^1_k} + 2|\rho - 1| \| \mathcal{N}_1(f) \|_{L^1_k} \]

where we used \( |1 - \rho^{-1}| \leq 2|\rho - 1| \) because \( 1/2 \leq \rho \leq 3/2 \). We need to prove that

\[ \| f - \mathcal{N}_1(f) \|_{L^1_k} \leq C_N \| f - \mathcal{N}_1(f) \|_{L^1_{k+l}} \left( \| \hat{\mathcal{N}_1(f)} \|_{L^2_k} + \| f \|_{L^1_{k+l}} \right) \]

Let \( h = f - \mathcal{N}_1(f) \), \( R \in (0, \infty) \). We have

\[ \| h \|_{L^1_k} = \int_{(v) < R} (v)^k |h(v)| \, dv + \int_{(v) \geq R} (v)^k |h(v)| \, dv \leq C_N \| h \|_{L^2_k} R^{2k+N} + \| h \|_{L^1_{k+l}} \frac{1}{R^l} \]

Minimizing the RHS with respect to \( R \in (0, \infty) \) leads to

\[ \| h \|_{L^1_k} \leq 2C_N^{2l} \| h \|_{L^1_{k+l}}^{2k+N+2l} \| h \|_{L^2_k}^{2l} \| h \|_{L^1_{k+l}}^{2l} \]

which gives (4.2) by Plancherel’s theorem \( \| h \|_{L^2_k} = (2\pi)^{-N/2} \| \hat{h} \|_{L^2_k} \).

Since \( 1/2 \leq T \leq 3/2 \) and \( |u| \leq 1/2 \) imply

\[ 1 + \left| \frac{v - u}{\sqrt{T}} \right|^2 \leq 4(1 + |v|^2), \]

it follows that

\[ \| \mathcal{N}_1(f) \|_{L^1_k} = \int_{\mathbb{R}^N} \left( 1 + \left| \frac{v - u}{\sqrt{T}} \right|^2 \right)^{(k+l)/2} f(v) \, dv \leq 2^{k+l} \| f \|_{L^1_{k+l}} \]

and thus

\[ \| f - \mathcal{N}(f) \|_{L^1_k} \leq C_{N,k,l} \| f \|_{L^1_{k+l}}^{k+N+2l} \left( \| \hat{\mathcal{N}_1(f)} \|_{L^2_k} + 2^{k+1} |\rho - 1| \| f \|_{L^1_k} \right) \]

Next we compute

\[ \hat{\mathcal{N}_1(f)}(\xi) = e^{iT^{-1/2} \xi \cdot u} \hat{f}(T^{-1/2} \xi), \]

\[ \left| 1 - e^{iT^{-1/2} \xi \cdot u} \right| \leq 2 \left( 1 + |T^{-1/2} \xi|^2 \right)^{s/2} \max \{ |u|, |u|^s \}, \]

\[ \left| \hat{f}(\xi) - \hat{\mathcal{N}_1(f)}(\xi) \right| \leq \left| \hat{f}(\xi) - \hat{f}(T^{-1/2} \xi) \right| + 2 \left| \hat{f}(T^{-1/2} \xi) \right| \left( 1 + |T^{-1/2} \xi|^2 \right)^{s/2} \max \{ |u|, |u|^s \}, \]

hence

\[ \| \hat{f} - \hat{\mathcal{N}_1(f)} \|_{L^2_k} \leq \left\| \hat{f} - \hat{f}(T^{-1/2}) \right\|_{L^2_k} + 2^{1+N/4} \| f \|_{H^s} \max \{ |u|, |u|^s \}. \]
Write $\xi = (\xi_1, \xi_2, \ldots, \xi_N), v = (v_1, v_2, \ldots, v_N)$, and $f_j(v) = v_j f(v), j = 1, 2, \ldots, N$. Then

$$\hat{f}(\xi) - \hat{f} \left( T^{-1/2} \xi \right) = -i \int_{1/\sqrt{T}}^{1} \sum_{j=1}^{N} \hat{f}_j(t\xi) \xi_j \, dt.$$ 

By Cauchy-Schwarz inequality and

$$1/2 \leq T \leq 3/2 \implies \left| \frac{1}{\sqrt{T}} - 1 \right| \leq |T - 1|$$

we have

$$\left| \hat{f}(\xi) - \hat{f} \left( T^{-1/2} \xi \right) \right| \leq |T - 1|^{1/2} \left( \int_{1/\sqrt{T}}^{1} \sum_{j=1}^{N} |\hat{f}_j(t\xi)|^2 \, dt \right)^{1/2}$$

where $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $1/p + 1/q = 1$ with $p = (1+s)/s$ and $q = 1 + s$. Then

$$\left| \hat{f}(\xi) - \hat{f} \left( T^{-1/2} \xi \right) \right|^2 = \left| \hat{f}(\xi) - \hat{f} \left( T^{-1/2} \xi \right) \right|^{2/p} \left| \hat{f}(\xi) - \hat{f} \left( T^{-1/2} \xi \right) \right|^{2/q} \leq |T - 1|^{1/p} \left( \int_{1/\sqrt{T}}^{1} \sum_{j=1}^{N} |\hat{f}_j(t\xi)|^2 \, dt \right)^{1/p} |\xi|^{2/p} \left| \hat{f}(\xi) - \hat{f} \left( T^{-1/2} \xi \right) \right|^{2/q}.$$ 

It follows from Hölder’s inequality and $q/p = s$ that

$$\left\| \hat{f} - \hat{f} \left( T^{-1/2} \cdot \right) \right\|_{L^2}^2 \leq |T - 1| \left( \int_{1/\sqrt{T}}^{1} \sum_{j=1}^{N} \left( \int_{\mathbb{R}^N} |\hat{f}_j(t\xi)|^2 \, d\xi \right) \, dt \right)^{1/p} \left\| \hat{f} - \hat{f} \left( T^{-1/2} \cdot \right) \right\|_{L^2}^{2/q}$$

where $L^2_a$ denotes the weighted $L^2$ space with the homogeneous weight $|\xi|^s$. Since, by Plancherel’s theorem,

$$\int_{1/\sqrt{T}}^{1} \left\{ \sum_{j=1}^{N} \int_{\mathbb{R}^N} |\hat{f}_j(t\xi)|^2 \, d\xi \right\} \, dt = (2\pi)^N \left( \int_{1/\sqrt{T}}^{1} t^{-N} \, dt \right) \left( \int_{\mathbb{R}^N} |v|^2 f(v)^2 \, dv \right) \leq C_N \|f\|_{L^2}^2$$

and

$$\left\| \hat{f} - \hat{f} \left( T^{-1/2} \cdot \right) \right\|_{L^2} \leq \|\hat{f}\|_{L^2} + \left\| \hat{f} \left( T^{-1/2} \cdot \right) \right\|_{L^2} \leq \left( 1 + 2^{(N+2s)/4} \right) \|f\|_{H^s},$$

it follows that

$$\left\| \hat{f} - \hat{f} \left( T^{-1/2} \cdot \right) \right\|_{L^2} \leq C_N \|f\|_{L^2}^{1/p} \|f\|_{H^s}^{1/q} |T - 1|^{1/(2p)}.$$ 

Thus we get from (4.4), (4.3), and $1/p = s/(1+s)$ that

$$\left\| \hat{f} - \hat{N}_1(f) \right\|_{L^2} \leq C_N \max \left\{ \|f\|_{L^2}, \|f\|_{H^s} \right\} (|u| + |T - 1|)^{2/(1+s)}.$$
and so (4.4) follows from (4.3) and (4.1).

Let us define the *Mehler transform* and prove the key property on it.

**Definition 4.5.** Let \( \rho > 0, u \in \mathbb{R}^N \) and \( T > 0 \). The *Mehler Transform* \( I_n[F] \) of \( F \in \mathcal{B}^+_{\rho,u,T}(\mathbb{R}^N) \) is given by

\[
I_n[F](v) = e^{Nn} \int_{\mathbb{R}^N} M_{1,0,T} \left( e^u \left( v - u - \sqrt{1 - e^{-2n} (v_\ast - u)} \right) \right) dF(v_\ast), \quad n > 0,
\]

where \( M_{1,0,T}(v) = (2\pi T)^{-N/2} \exp(-|v|^2/(2T)) \).

**Lemma 4.6.** Let \( F \in \mathcal{B}^+_{\rho,u,T}(\mathbb{R}^N) \) and let \( M \in \mathcal{B}^+_{\rho,u,T}(\mathbb{R}^N) \) be the Maxwellian distribution. Then \( I_n[F] \in L^1_{\rho,u,T}(\mathbb{R}^N) \) and for any \( 0 \leq s \leq 2 \)

\[
(4.7) \quad \lim_{n \to \infty} \|I_n[F] - M\|_{L^s} = \|F - M\|_s.
\]

**Proof.** It is well-known that \( I_n[F] \in L^1_{\rho,u,T}(\mathbb{R}^N) \) for all \( n > 0 \) and

\[
(4.8) \quad \lim_{n \to +\infty} \int_{\mathbb{R}^N} \varphi(v) I_n[F](v) \, dv = \int_{\mathbb{R}^N} \varphi(v) \, dF(v)
\]

for all \( \varphi \in C(\mathbb{R}^N) \) satisfying \( \sup_{v \in \mathbb{R}^N} |\varphi(v)|/|v|^{-2} < \infty \) (see also (4.10) below).

Let \( 0 \leq s \leq 2 \). Applying the dual version (1.14) of the norm \( \| \cdot \|_s \) and the convergence (4.8) we have

\[
(4.9) \quad \|F - M\|_s \leq \lim inf_{n \to \infty} \|I_n[F] - M\|_{L^s}.
\]

On the other hand, for any Borel function \( \psi \) on \( \mathbb{R}^N \) satisfying \( \sup_{v \in \mathbb{R}^N} |\psi(v)|/|v|^{-2} < \infty \), we have by Fubini theorem and change of variable that

\[
(4.10) \quad \int_{\mathbb{R}^N} \psi(v) I_n[F](v) \, dv = \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \psi \left( e^{-n} z + u + \sqrt{1 - e^{-2n} (v_\ast - u)} \right) M_{1,0,T}(z) \, dz \right) dF(v_\ast).
\]

Now take

\[
\psi_n(v) = \langle v \rangle^s \text{sign} (I_n[F](v) - M(v)).
\]

Then

\[
\|I_n[F] - M\|_s = \int_{\mathbb{R}^N} \psi_n(v) I_n[F](v) \, dv - \int_{\mathbb{R}^N} \psi_n(v) M(v) \, dv
\]

\[
= \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \psi_n \left( e^{-n} z + u + \sqrt{1 - e^{-2n} (v_\ast - u)} \right) M_{1,0,T}(z) \, dz \right) d(F - M)(v_\ast)
\]

\[
+ \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \psi_n \left( e^{-n} z + u + \sqrt{1 - e^{-2n} (v_\ast - u)} \right) M_{1,0,T}(z) \, dz \right) M(v_\ast) \, dv_\ast - \int_{\mathbb{R}^N} \psi_n(v) M(v) \, dv
\]

\[:= I_{n,1} + I_{n,2}.
\]

Let \( h(v_\ast) = \text{sign}(F - M) \) be the sign function of \( F - M \), i.e.

\[
d|F - M|(v_\ast) = h(v_\ast) \, d(F - M)(v_\ast) \quad \text{and} \quad h(v_\ast)^2 \equiv 1.
\]
Then \(d(F - M)(v_*) = h(v_*)d|F - M|(v_*)\) so that
\[
I_{n,1} \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left\langle e^{-n}z + u + \sqrt{1 - e^{-2n}}(v_* - u) \right\rangle^s M_{1,0,T}(z) \, dz \, d|F - M|(v_*).
\]
Since
\[
\forall (z, v_*) \in \mathbb{R}^N \times \mathbb{R}^N, \lim_{n \to \infty} \left\langle e^{-n}z + u + \sqrt{1 - e^{-2n}}(v_* - u) \right\rangle^s = \langle v_* \rangle^s
\]
and
\[
\left\langle e^{-n}z + u + \sqrt{1 - e^{-2n}}(v_* - u) \right\rangle^s \leq 3^s \langle u \rangle^s \langle z \rangle^s \langle v_* \rangle^s
\]
so that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 3^s \langle u \rangle^s \langle z \rangle^s \langle v_* \rangle^s M_{1,0,T}(z) \, dz \, d|F - M|(v_*) = 3^s \langle u \rangle^s \|M_{1,0,T}\|_s \|F - M\|_s < +\infty,
\]
it follows from dominated convergence that
\[
(4.11) \quad \limsup_{n \to \infty} I_{n,1} \leq \|M_{1,0,T}\|_{L^1} \|F - M\|_s = \|F - M\|_s.
\]
Next we prove that \(\limsup_{n \to \infty} I_{n,2} \leq 0\). We compute by changing variable that
\[
\int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \psi_n \left( e^{-n}z + u + \sqrt{1 - e^{-2n}}(v_* - u) \right) M_{1,0,T}(z) \, dz \right) M(v_*) \, dv_*
\]
\[
= \frac{1}{(1 - e^{-2n})^{N/2}} \int_{\mathbb{R}^N} M_{1,0,T}(z) \, dz \int_{\mathbb{R}^N} \psi_n(v) M \left( \frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) \, dv.
\]
So we get
\[
I_{n,2}
\]
\[
= \frac{1}{(1 - e^{-2n})^{N/2}} \int_{\mathbb{R}^N} M_{1,0,T}(z) \, dz \int_{\mathbb{R}^N} \psi_n(v) \left[ M \left( \frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) - M(v) \right] \, dv
\]
\[
+ \left( \frac{1}{(1 - e^{-2n})^{N/2}} - 1 \right) \int_{\mathbb{R}^N} \psi_n(v) M(v) \, dv
\]
\[
\leq \frac{1}{(1 - e^{-2n})^{N/2}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \langle v \rangle^s \left[ M \left( \frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) - M(v) \right] M_{1,0,T}(z) \, dz \, dv
\]
\[
+ \left( \frac{1}{(1 - e^{-2n})^{N/2}} - 1 \right) \int_{\mathbb{R}^N} \langle v \rangle^s M(v) \, dv
\]
and finally (since the last last term above clearly converges to zero)
\[
(4.12) \quad \limsup_{n \to \infty} I_{n,2} \leq \limsup_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \langle v \rangle^s \left[ M \left( \frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) - M(v) \right] M_{1,0,T}(z) \, dz \, dv.
\]
It is obvious that the integrand in the RHS of (4.12) tends to zero as \( n \to \infty \). To find a dominated function for the integrand, we recall that

\[
M \left( \frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) = \frac{\rho}{(2\pi T)^{N/2}} \exp \left( -\frac{1}{2T} \left| \frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} - u \right|^2 \right).
\]

Elementary calculation shows that

\[
\frac{1}{2T} \left| \frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} - u \right|^2 \geq \frac{|v - u|^2}{4T} - \frac{|z|^2}{4T}.
\]

This gives

\[
M \left( \frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) M_{1,0,T}(z) \leq \frac{\sqrt{\rho}}{(2\pi T)^{N/2}} \sqrt{M(v)} \sqrt{M_{1,0,T}(z)}
\]

and thus

\[
(v)^s M \left( \frac{v - e^{-n}z - (1 - \sqrt{1 - e^{-2n}})u}{\sqrt{1 - e^{-2n}}} \right) - M(v) \leq \left( \frac{\sqrt{\rho}}{(2\pi T)^{N/2}} \right)^s \sqrt{M(v)} \sqrt{M_{1,0,T}(z)} + (v)^s M(v) M_{1,0,T}(z).
\]

By dominated convergence we then conclude that the limit in the RHS of (4.12) is zero. Therefore \( \lim \sup_{n \to \infty} I_{n,2} \leq 0 \) and

\[
\lim_{n \to \infty} \sup I_{n,F} - M \|_s \leq \lim_{n \to \infty} \sup I_{n,1} + \lim_{n \to \infty} \sup I_{n,2} \leq \| F - M \|_s,
\]

which together with (4.11) proves (4.7). \( \square \)

**Proof of Theorem 1.5** We first prove that the theorem holds true for all \( L^1 \) mild solutions in \( L^1_{1,0,1}(\mathbb{R}^N) \). We will then use approximation and normalization to extend it to general measure solutions.

**Step 1.** Let \( \lambda = S_{b,\gamma}(1, 0, 1) > 0 \) be the spectral gap of the linearized operator \( L_M \) associated with the kernel \( B(z, \sigma) \) and the Maxwellian \( M(v) = (2\pi)^{-N/2} e^{-|v|^2/2} \) in \( L^1_{1,0,1}(\mathbb{R}^N) \). Let \( f_0 \in L^1_{1,0,1}(\mathbb{R}^N) \) and let \( f_t \) be the unique conservative mild \( L^1 \) solution of equation (1.1) with the initial datum \( f_0 \). We shall prove that

\[
(4.13) \quad \forall t \geq 0, \quad \| f_t - M \|_{L^1_2} \leq C_0 \| f_0 - M \|_{L^2_2}^{1/2} e^{-\lambda t}
\]

where the constant \( 0 < C_0 < \infty \) depends only on \( N \), the function \( b \) and \( \gamma \).

To do this we use Theorem 3.24 to consider the positive decomposition:

\[
\forall t \geq 1, \quad f_t = g_t + h_t,
\]
where \(g_t = f_t^h \geq 0\) and \(h_t = h_t^v \geq 0\) are given in (3.1)-(3.5) with \(n = N, \gamma + 2\) and \(t_0 = 1\). In the following we denote by \(c_i > 0, C_i > 0 (i = 1, 2, \ldots)\) some finite constants that depend only and explicitly on \(N\), the function \(b\) and \(\gamma\). By Theorem 3.4 with \(t_0 = 1\) we have

\[
\forall t \geq 1, \quad \| f_t - g_t \|_{L^1} \leq \| h_t \|_{L^1} \leq C_1 e^{-\frac{4}{\gamma} t},
\]

\[
\sup_{t \geq 1} \left\{ \| g_t \|_{L^{N+4}}, \| g_t \|_{L^2}, \| g_t \|_{H^1} \right\} \leq C_2.
\]

We can assume that \(C_1 \geq 1\). Let \(\tau_0 = (2/a) \log(8C_1) (> 1)\) and let

\[
\begin{cases}
\rho_t = \int_{\mathbb{R}^N} g_t(v) \, dv, & u_t = \frac{1}{\rho_t} \int_{\mathbb{R}^N} v g_t(v) \, dv, \\
T_t = \frac{1}{N \rho_t} \int_{\mathbb{R}^N} |v - u_t|^2 g_t(v) \, dv,
\end{cases}
\]

\[
\mathcal{N}(g_t)(v) = \frac{T_t^{N/2}}{\rho_t} g_t(\sqrt{T_t} v + u_t), \quad t \geq \tau_0.
\]

Using the relation

\[
T_t = \frac{1}{N \rho_t} \int_{\mathbb{R}^N} |v|^2 g_t(v) \, dv - \frac{|u_t|^2}{N}
\]

we compute

\[
\forall t \geq \tau_0, \quad |\rho_t - 1| + |u_t| + |T_t - 1| \leq 4 \| f_t - g_t \|_{L^1} \leq 4C_1 e^{-\frac{4}{\gamma} t} \leq \frac{1}{2}.
\]

So by (4.15) and applying Lemma 4.3 (with \(k = 2, l = N + 2, s = 1\)) we have

\[
\forall t \geq \tau_0, \quad \| g_t - \mathcal{N}(g_t) \|_{L^1} \leq C_3 \left( 4C_1 e^{-\frac{4}{\gamma} t} \right)^{1/6} = C_4 e^{-c_1 t}.
\]

This together with (4.14) gives

\[
\forall t \geq \tau_0, \quad \| f_t - \mathcal{N}(g_t) \|_{L^1} \leq \| f_t - g_t \|_{L^1} + \| g_t - \mathcal{N}(g_t) \|_{L^1} \leq C_5 e^{-c_1 t}.
\]

Also by (4.16), \(\sup_{t \geq 1} \| g_t \|_{L^2} \leq C_2, \tau_0 > 1\) and \(\mathcal{N}(g_t) \in L^1_{1,0,1}(\mathbb{R}^N)\) we have

\[
C_6 \leq \inf_{t \geq \tau_0} \| \mathcal{N}(g_t) \|_{L^2}, \quad \sup_{t \geq \tau_0} \| \mathcal{N}(g_t) \|_{L^2} \leq C_7.
\]

The second inequality follows from elementary calculations and the bounds \(1/2 \leq \rho_t\) and \(T_t \leq 3/2\). To prove the first one, we consider some \(R > 0\) and write

\[
1 = \int_{|v| < R} N(g_t)(v) \, dv + \int_{|v| \geq R} N(g_t)(v) \, dv \leq |B^N|^{1/2} R^{N/2} \| \mathcal{N}(g_t) \|_{L^2} + R^{-2} N,
\]

where \(|B^N|\) is the volume of the unite ball \(B^N\). If we now fix \(R = \sqrt{2N}\), then

\[
\frac{1}{2} |B^N|^{-1/2} (2N)^{-N/4} \leq \| \mathcal{N}(g_t) \|_{L^2}
\]

for all \(t \geq \tau_0\) so that the first inequality in (4.18) holds for \(C_6 = (1/2)|B^N|^{-1/2}(2N)^{-N/4}\).

To prove (4.13) we use the following technique of “moving solutions” as used in [1] and [24]. For any \(\tau \geq \tau_0\), let \((t, v) \mapsto f_t^{(\tau)}(v)\) be the unique conservative solution on \([\tau, \infty) \times \mathbb{R}^N\) with the initial datum at time \(t = \tau\):

\[
f_t^{(\tau)}|_{t=\tau} = f^{(\tau)} = \mathcal{N}(g_{\tau}).
\]
On the one hand, by Theorem 4.11 we have
\[ \forall t \geq \tau, \quad \| f_t^{(\tau)} - M \|_{L^1} \leq C_{f_t^{(\tau)}} e^{-\lambda(t-\tau)} \]
where the coefficient \( 0 < C_{f_t^{(\tau)}} < \infty \) depends only and explicitly on \( N \), the function \( b, \gamma \) and \( \| f_t^{(\tau)} \|_{L^2} \). Since (4.18) implies \( C_6 \leq \| f_t^{(\tau)} \|_{L^2} \leq C_7 \), it follows that the constant \( C_{f_t^{(\tau)}} \) is uniformly bounded by some constant \( C_8 > 0 \). Thus
\[ \forall t \geq \tau, \quad \| f_t^{(\tau)} - M \|_{L^1} \leq C_8 e^{-\lambda(t-\tau)}. \]  
(4.19)

On the other hand using the stability estimate (1.30) we have
\[ \forall t \geq \tau, \quad \| f_t - f_t^{(\tau)} \|_{L^1} \leq \| f_t^{(\tau)} - f_t^{(\tau)} \|_{L^1} e^{2(t-\tau)}. \]  
(4.20)

Since (4.17) and \( \tau \geq \tau_0 \) imply
\[ \| f_t - f_t^{(\tau)} \|_{L^2} = \| f_t - N(g) \|_{L^2} \leq C_5 e^{-c_1 \tau}, \]
it follows from (4.19) and (4.20) that
\[ \forall t \geq \tau, \quad \| f_t - M \|_{L^1} \leq \| f_t - f_t^{(\tau)} \|_{L^1} + \| f_t^{(\tau)} - M \|_{L^1} \leq C_5 e^{-c_1 \tau} + C_8 e^{-\lambda(t-\tau)}. \]  
(4.21)

Now for any
\[ t \geq t_1 := \frac{c_1 + c_2 + \lambda}{c_2 + \lambda} \tau_0, \quad \text{we choose} \quad \tau = \frac{c_2 + \lambda}{c_1 + c_2 + \lambda} t. \]

Then \( t > \tau \geq \tau_0 \) and
\[ -c_1 \tau + c_2 (t - \tau) = -\lambda(t - \tau) = -\frac{\lambda c_1}{c_1 + c_2 + \lambda} t := -c_3 t, \]
and so (4.21) becomes
\[ \forall t \geq t_1, \quad \| f_t - M \|_{L^1} \leq (C_5 + C_8) e^{-c_3 t}. \]  
(4.22)

Now let
\[ m(v) := \exp \left( -\frac{\alpha(1)}{4} |v|^2 \right) \]
where \( \alpha(t) > 0 \) is given in Theorem 4.12 for the initial datum \( F_0 \in B_{1,0,1}^{+}(\mathbb{R}^N) \) defined by
\[ dF_0(v) = f_0(v) \, dv. \]
Then
\[ \sup_{t \geq 1} \| f_t \|_{L^1(m^{-1})} \leq 2, \quad \| M \|_{L^1(m^{-1})} \leq C_9. \]  
(4.23)

Therefore, using Cauchy-Schwarz’ inequality and (4.22), we get
\[ \forall t \geq t_1, \quad \| f_t - M \|_{L^1(m^{-2})} \leq \| f_t - f_t^{(\tau)} \|_{L^1(m^{-1})}^{1/2} \| f_t^{(\tau)} - M \|_{L^1}^{1/2} \leq C_{10} e^{-c_4 t}. \]  
(4.24)

Let \( \varepsilon > 0 \) be the constant in Theorem 4.1 corresponding to \( m(v) \), and let us choose
\[ t_2 = \max \left\{ t_1, \frac{1}{c_4} \log \left( \frac{C_{10}}{\varepsilon} \right) \right\}. \]

Then by (4.24) we have
\[ \forall t \geq t_2, \quad \| f_t - M \|_{L^1(m^{-2})} \leq C_{10} e^{-c_4 t} \leq \varepsilon. \]
It follows from Lemma 4.2 (see Remark 4.3 (3)) that
\begin{equation}
\forall t \geq t_2, \quad \|f_t - M\|_{L^1(m^{-1})} \leq C_{11} \|f_{t_2} - M\|_{L^1(m^{-1})} e^{-\lambda(t-t_2)}.
\end{equation}

Next, applying the elementary inequality
\[ 1 + |v|^2 \leq C e^{\delta |v|^2}, \quad \eta := \frac{\gamma}{2}, \quad \delta := \frac{\alpha(1)}{4}, \]
for some constant \( C > 0 \), we have
\begin{equation}
\|f_t - M\|_{L^2} \leq C_{12} \|f_t - M\|_{L^1(m^{-1})}.
\end{equation}

On the other hand, using the bound in (4.23), we have
\begin{equation}
\|f_{t_2} - M\|_{L^1(m^{-1})} \leq \|f_{t_2} - M\|^{1/2}_{L^1(m^{-2})} \|f_{t_2} - M\|^{1/2}_{L^2} \leq C_{13} \|f_{t_2} - M\|^{1/2}.
\end{equation}

It follows from (4.26), (4.25) and (4.27) that
\begin{equation}
\forall t \geq t_2, \quad \|f_t - M\|_{L^2} \leq C_{14} \|f_{t_2} - M\|^{1/2} e^{-\lambda t}.
\end{equation}

It remains to estimate \( \|f_t - M\|_{L^1} \) in terms of \( \|f_0 - M\|_{L^1} \) for \( t \in [0, t_2] \). To do this we use the estimate in [21, p. 3359] for the measure \( H_t := F_t - G_t \) where \( F_t, G_t \) are measure solutions of the equation (1.1). Here we define more precisely \( F_t, G_t \) to be
\[ \text{d}F_t(v) = M(v) \text{d}v \quad \text{and} \quad \text{d}G_t(v) = f_t(v) \text{d}v. \]

Then \( \|H_t\|_2 = \|M - f_t\|_{L^2}, \|f_t\|_{L^2} = \|M\|_{L^2} \), and thus (recalling \( A_0 = 1 \))
\[ \forall t \in (r, +\infty), \quad \|M - f_t\|_{L^2} \leq 2 \|(M - f_r)_+\|_{L^2} + 4 \left( \|M\|_{L^2}^1 \gamma + \|M\|_{L^1} \right) \int_r^t \|M - f_s\|_{L^2} \text{d}s. \]

Since \( t \mapsto f_t(v) \geq 0 \) is continuous on \([0, \infty)\) for a.e. \( v \in \mathbb{R}^N \), it follows from dominated convergence that
\[ 2 \|(M - f_r)_+\|_{L^2} \xrightarrow{r \to 0^+} 2 \|(M - f_0)_+\|_{L^2} = \|f_0 - M\|_{L^2}, \]
where the last equality follows from
\[ |f_0 - M| = f_0 - M + 2(M - f_0)_+ \quad \text{and} \quad \|f_0\|_{L^2} = \|M\|_{L^2}. \]

Thus letting \( r \to 0^+ \) gives
\[ \forall t \in (0, +\infty), \quad \|f_t - M\|_{L^2} \leq \|f_0 - M\|_{L^2} + c_5 \int_0^t \|f_s - M\|_{L^2} \text{d}s. \]

Since \( \|f_s - M\|_{L^1} \leq \|f_s - M\|_{L^2} \), it follows from Gronwall’s lemma that
\begin{equation}
\forall t \geq 0, \quad \|f_t - M\|_{L^2} \leq \|f_0 - M\|_{L^2} e^{c_5 t}.
\end{equation}

Inserting this estimate with \( t = t_2 \) into the right hand side of (4.28) gives
\begin{equation}
\forall t \geq t_2, \quad \|f_t - M\|_{L^2} \leq C_{15} \|f_0 - M\|^{1/2} e^{-\lambda t}.
\end{equation}
Also because of (1.29) and \( \|f_0 - M\|_{L^2} \leq 2(1 + N) \), we have

\begin{equation}
\forall t \in [0, t_2], \quad \|f_t - M\|_{L^2} \leq C_{16} \|f_0 - M\|_{L^2}^{1/2} e^{-\lambda t}.
\end{equation}

Combining (4.28), (4.31) we obtain (4.13) with \( C_0 = \max\{C_{15}, C_{16}\} \).

**Step 2.** Let us now prove that the theorem holds true for measure solutions in \( B^+_1(\mathbb{R}^N) \).

Given any \( F_0 \in B^+_1(\mathbb{R}^N) \), let \( F_t \) be the unique conservative measure strong solution of equation (1.1) with the initial datum \( F_0 \). By part (e) of Theorem 1.2 and Lemma 4.6, there is a sequence \( f_{k,t} \in L^1_{1,0,1}(\mathbb{R}^N) \) of solutions with the initial data \( f_{k,0} = I_\rho[F_0] \in L^1_{1,0,1}(\mathbb{R}^N) \) such that

\begin{equation}
\forall \varphi \in C_b(\mathbb{R}^N), \quad \forall t \geq 0, \quad \lim_{k \to \infty} \int_{\mathbb{R}^N} \varphi(v) f_{k,t}(v) \, dv = \int_{\mathbb{R}^N} \varphi(v) \, dF_t(v),
\end{equation}

\begin{equation}
\lim_{k \to \infty} \|f_{k,0} - M\|_{L^2} = \|F_0 - M\|_{L^2}.
\end{equation}

Using the formulation (1.13) of the norm \( \|\cdot\|_2 \), we conclude from (4.32) that

\begin{equation}
\forall t \geq 0, \quad \|F_t - M\|_2 \leq \lim_{k \to \infty} \|f_{k,t} - M\|_{L^2}.
\end{equation}

Applying (1.13) to \( f_{k,t} \), we have

\begin{equation}
\forall t \geq 0, \quad k = 1, 2, 3, \ldots \quad \|f_{k,t} - M\|_{L^2} \leq C_0 \|f_{k,0} - M\|_{L^2}^{1/2} e^{-\lambda t}.
\end{equation}

Combining (4.34), (4.35) and (4.33) we obtain

\begin{equation}
\forall t \geq 0, \quad \|F_t - M\|_2 \leq C_0 \|F_0 - M\|_2^{1/2} e^{-\lambda t}.
\end{equation}

**Step 3.** Finally, we show that for any \( \rho > 0, u \in \mathbb{R}^N \) and \( T > 0 \), the theorem holds true for measure solutions in \( B^+_\rho,u,T(\mathbb{R}^N) \). Let \( F_0 \in B^+_\rho,u,T(\mathbb{R}^N) \) and let \( F_t \) be the unique conservative measure strong solution with the initial datum \( F_0 \). Let \( M_{\rho,u,T} \in B^+_\rho,u,T(\mathbb{R}^N) \) be the Maxwellian and let \( \mathcal{N} = \mathcal{N}_{\rho,u,T} \) be the normalization operator. By Proposition 1.4 the flow \( t \mapsto \mathcal{N}(F_{t/c}) \) is the unique conservative measure strong solution of equation (1.1) with the initial datum \( \mathcal{N}(F_0) \in B^+_1(\mathbb{R}^N) \). Here \( c = \rho T^{\gamma/2} \). Since \( \mathcal{N}(M_{\rho,u,T}) \in B^+_1(\mathbb{R}^N) \) is the standard Maxwellian, it follows from the above result (4.36) that (writing \( \mathcal{N}(F_t) = \mathcal{N}(F_{ct/c}) \))

\begin{equation}
\forall t \geq 0, \quad \|\mathcal{N}(F_t) - \mathcal{N}(M_{\rho,u,T})\|_2 \leq C_0 \|\mathcal{N}(F_0) - \mathcal{N}(M_{\rho,u,T})\|_2^{1/2} e^{-\lambda ct}.
\end{equation}

Then, applying Proposition 1.4, we have

\begin{equation}
\forall t \geq 0, \quad \|F_t - M_{\rho,u,T}\|_2 \leq C_{1/\rho,u/\sqrt{T},1/T} \|\mathcal{N}(F_t) - \mathcal{N}(M_{\rho,u,T})\|_2
\end{equation}

\begin{equation}
\leq C_0 C_{1/\rho,u/\sqrt{T},1/T} \|\mathcal{N}(F_0) - \mathcal{N}(M_{\rho,u,T})\|_2^{1/2} e^{-\lambda ct}
\end{equation}

\begin{equation}
\leq C_0 C_{1/\rho,u/\sqrt{T},1/T} [C_{\rho,u,T}]^{1/2} \|F_0 - M_{\rho,u,T}\|_2^{1/2} e^{-\lambda ct}.
\end{equation}
Since $\lambda c = S_{h,\gamma}(1,0,1)\rho T^{\gamma/2} = S_{h,\gamma}(\rho, u, T)$ is the spectral gap of the linearized operator $L_{M,\rho,u,T}$, this completes the proof of Theorem 1.3.

\[\Box\]

5. Lower Bound of Convergence Rate

In this section we prove Theorem 1.7. We recall that we assume here that $N \geq 2, 0 < \gamma \leq 2$, and that the function $b$ satisfies only (1.33).

Proof of Theorem 1.7. In the following we denote $\| \cdot \| = \| \cdot \|_0$. We first prove the theorem assuming that $F_0, M \in B_{1,0,1}^+(\mathbb{R}^N)$. By

$$\forall t_1, t_2 \in [0, \infty), \quad F_{t_2} = F_{t_1} + \int_{t_1}^{t_2} Q(F_\tau, F_\tau) \, d\tau$$

and $Q(M, M) = 0$, we have

(5.1) $\forall t_1, t_2 \in [0, \infty), \quad \|F_{t_2} - M\| - \|F_{t_1} - M\| \leq \left| \int_{t_1}^{t_2} \|Q(F_\tau, F_\tau) - Q(M, M)\| \, d\tau \right|.$

Using the inequalities in (1.19), $0 < \gamma \leq 2$, and the conservation of mass and energy (which implies $\|F_t\|_\gamma \leq \|F_t\|_2 = 1 + N$, etc.) we have

(5.2) $\|Q(F_t, F_t) - Q(M, M)\| \leq 2^4(1 + N) \|F_t - M\|_\gamma$.

Since $t \mapsto \|F_t - M\|_\gamma$ is bounded, we have by Hölder’s inequality

(5.3) $\|F_t - M\|_\gamma \leq \|F_t - M\|^{1-\gamma/2}_2 \|F_t - M\|^{\gamma/2}_2$,

and it follows from (5.1)-(5.3) that $t \mapsto \|F_t - M\|$ is Lipschitz continuous and

(5.4) $\left| \frac{d}{dt} \|F_t - M\| \right| \leq 2^4(N + 1) \|F_t - M\|^{1-\gamma/2}_2 \|F_t - M\|^{\gamma/2}_2$ a.e. $t \in (0, \infty)$.

Next using $|F_t - M| = F_t - M + 2(M - F_t)_+$ and the conservation of mass and energy we have

(5.5) $\|F_t - M\| = 2 \|(M - F_t)_+\|, \quad \|F_t - M\|_2 = 2 \|(M - F_t)_+\|_2$.

Let $0 < \delta < 1$. Applying Jensen’s inequality to the convex function $x \mapsto \exp(\delta x/2)$ and the measure $(M - F_t)_+$ (with $t$ fixed) and assuming $\|(M - F_t)_+\| > 0$ we have

(5.6) $\frac{1}{\|(M - F_t)_+\|} \int_{\mathbb{R}^N} \exp(\delta \langle v \rangle^2/2) \, d(M - F_t)_+ \geq \exp(\frac{\delta}{2} \cdot \frac{\|(M - F_t)_+\|}{ \|(M - F_t)_+\|} )$.

On the other hand, we gave

$$d(M - F_t)_+(v) \leq dM(v) = \frac{1}{(2\pi)^{N/2}} \exp(-|v|^2/2) \, dv$$

and we compute

(5.7) $\int_{\mathbb{R}^N} \exp(\delta \langle v \rangle^2/2) \, d(M - F_t)_+(v) \leq \int_{\mathbb{R}^N} \exp(\delta \langle v \rangle^2/2) \, dM(v) = e^{\delta/2} \left( \frac{1}{1 - \delta} \right)^{N/2}$.
We choose now \( \delta = \frac{1}{N+1} \). Then
\[
e^{\delta/2} \left( \frac{1}{1-\delta} \right)^{N/2} = e^{\frac{1}{2(N+1)}} (1 + 1/N)^{N/2} < 2
\]
and thus, from (5.5)-(5.7), we have
\[
\exp \left( \frac{1}{2(N+1)} \cdot \frac{\|F_t - M\|_2}{\|F_t - M\|} \right) \leq \frac{4}{\|F_t - M\|}.
\]
i.e.
\[
\|F_t - M\|_2 \leq 2(N + 1) \|F_t - M\| \log \left( \frac{4}{\|F_t - M\|} \right).
\]
If we adopt the convention \( x \log(1/x) = 0 \) for \( x = 0 \), the inequality (5.8) also holds for
\[
\|F_t - M\| = 0.
\]
However, to avoid discussing the case \( \|F_t - M\| = 0 \) for some \( t \), we consider
\[
U_\varepsilon(t) := \|F_t - M\| + \varepsilon, \quad 0 < \varepsilon < 1.
\]
Then
\[
\|F_t - M\| \leq \frac{4}{U_\varepsilon(t)} \|F_t - M\| \log \left( \frac{1}{U_\varepsilon(t)} \right).
\]
Thus by (5.4) we deduce
\[
\left| \frac{d}{dt} U_\varepsilon(t) \right| \leq A U_\varepsilon(t) \left[ \log \left( \frac{1}{U_\varepsilon(t)} \right) \right]^{\gamma/2} \quad \text{a.e. } t \in (0, \infty)
\]
where \( A = 2^6(N + 1)^2 \).

Case 1: \( 0 < \gamma < 2 \). In this case we have, by (5.9),
\[
\frac{d}{dt} \left[ \log \left( \frac{1}{U_\varepsilon(t)} \right) \right]^{1-\gamma/2} = -(1 - \gamma/2) \left[ \log \left( \frac{1}{U_\varepsilon(t)} \right) \right]^{-\gamma/2} \frac{1}{U_\varepsilon(t)} \cdot \frac{d}{dt} U_\varepsilon(t) \leq (1 - \gamma/2)A
\]
for almost every \( t \in (0, \infty) \). Observe that the function
\[
t \mapsto \left[ \log \left( \frac{1}{U_\varepsilon(t)} \right) \right]^{1-\gamma/2}
\]
is absolutely continuous on every bounded interval of \([0, \infty)\). It follows that
\[
\forall t \geq 0, \quad \left[ \log \left( \frac{1}{U_\varepsilon(t)} \right) \right]^{1-\gamma/2} \leq \left[ \log \left( \frac{1}{U_\varepsilon(0)} \right) \right]^{1-\gamma/2} + (1 - \gamma/2)A t.
\]
Next, using the convexity inequality
\[
\forall x, y \geq 0, \quad \left( x + \frac{1}{2} y \right)^{1-\gamma/2} \leq \frac{2 x}{\gamma} \left( \frac{y}{\gamma} \right)^{1-\gamma/2} + \left( 1 - \frac{\gamma}{2} \right) y^{1-\gamma/2},
\]
we have
\[
\left\{ \log \left( \frac{1}{U_\varepsilon(0)} \right)^{1-\gamma/2} + \left(1 - \frac{\gamma}{2}\right) At \right\} \leq \alpha \log \left( \frac{1}{U_\varepsilon(0)} \right) + \beta_1 t^{2-\gamma}
\]
where
\[
\alpha = \left( \frac{2}{\gamma} \right)^{\frac{2}{2-\gamma}}, \quad \beta_1 = \left(1 - \frac{\gamma}{2}\right) A \left(2^6 (N+1)^2 \right)^{\frac{2}{2-\gamma}}.
\]
Thus, from (5.10), we obtain
\[
\forall t \geq 0, \quad U_\varepsilon(t) \geq U_\varepsilon(0)^{\alpha} \exp \left( -\beta_1 t^{2-\gamma} \right).
\]
Using the definition of \( U_\varepsilon(t) \) and letting \( \varepsilon \to 0^+ \), we get finally
\[
\forall t \geq 0, \quad \|F_t - M\| \geq \left( \frac{\|F_0 - M\|}{4} \right)^\alpha \exp \left( -\beta_1 t^{2-\gamma} \right).
\]
This concludes the proof of the “standard case”, for \( 0 < \gamma < 2 \).

Case 2: \( \gamma = 2 \). In this case we have by (5.9) with \( \gamma = 2 \) that
\[
\frac{d}{dt} \log \left( \log \left( \frac{1}{U_\varepsilon(t)} \right) \right) = -\left[ \log \left( \frac{1}{U_\varepsilon(t)} \right) \right]^{-1} \frac{1}{U_\varepsilon(t)} \cdot \frac{d}{dt} U_\varepsilon(t) \leq A, \quad \text{a.e.} \quad t \in (0, \infty).
\]
Since the function
\[
t \mapsto \log \left( \log \left( \frac{1}{U_\varepsilon(t)} \right) \right)
\]
is absolutely continuous on every bounded interval of \([0, \infty)\), it follows that for all \( t > 0 \)
\[
\log \left( \log \left( \frac{1}{U_\varepsilon(t)} \right) \right) \leq \log \left( \log \left( \frac{1}{U_\varepsilon(0)} \right) \right) + At, \quad \text{i.e.} \quad U_\varepsilon(t) \geq (U_\varepsilon(0))^{e^{At}}.
\]
Letting \( \varepsilon \to 0^+ \) leads to
\[
\forall t \geq 0, \quad \|F_t - M\| \geq \left( \frac{\|F_0 - M\|}{4} \right)^\alpha \exp \left( -\beta_1 t^{2-\gamma} \right).
\]
This prove the standard case for \( \gamma = 2 \).

General non-normalized setting. The general case can be reduced to the standard case by using normalization. Let \( F_0 \in B^+_{\rho,u,T}(\mathbb{R}^N) \), \( c = \rho T^{\gamma/2} \) and let \( M \in B^+_{\rho,u,T}(\mathbb{R}^N) \) be the Maxwellian. Then by Proposition 1.4, the normalization \( t \mapsto N(F_t/c) \in B^+_{1,0,1}(\mathbb{R}^N) \) is a conservative measure strong solution of equation (1.1) with the initial datum \( N(F_0) \).

Applying the above estimates and \( \|F_t - M\| = \rho \|N(F_t) - N(M)\| \) we obtain that if \( 0 < \gamma < 2 \) then
\[
\forall t \geq 0, \quad \|F_t - M\| = \rho \|N(F_{c^{-1}t}) - N(M)\| \\
\geq 4\rho \left( \frac{\|N(F_0) - N(M)\|}{4} \right)^\alpha \exp \left( -\beta_1 (ct)^{\frac{2}{2-\gamma}} \right) \\
= 4\rho \left( \frac{\|F_0 - M\|}{4\rho} \right)^\alpha \exp \left( -\beta t^{2-\gamma} \right).
\]
with
\[ \beta = \beta_1 e^{\frac{2}{\gamma}} = (1 - \gamma) \left( 2^6(1 + N)^2 \rho T^{\gamma/2} \right)^{\frac{2}{\gamma}}. \]

Similarly if \( \gamma = 2 \), then
\[ \| F_t - M \| \geq 4 \rho \left( \frac{\| N(F_0) - N(M) \|}{4} \right)^{\kappa_{Ac}} = 4 \rho \left( \frac{\| F_0 - M \|}{4 \rho} \right)^{\kappa_t} \]
with \( \kappa = A_c = 2^6(N + 1)\rho T \). This completes the proof. \( \square \)

6. Global in Time Stability Estimate

In the last section we prove the the global in time strong stability of the measure strong solutions of equation (1.1).

Proof of Theorem 1.9 Let \( F_t \) be a conservative measure strong solution of equation (1.1) with the initial datum \( F_0 \in B_{\rho_0, u_0}^+(\mathbb{R}^N) \), and let \( G_t \) be any conservative measure strong solution of equation (1.1) with the initial datum \( G_0 \). Let
\[ D_0 := \min \left\{ \rho_0^2, \left( \frac{4 \| F_0 \|_2 + 6 \left( \frac{\| F_0 \|_2}{\rho_0^2} \right)^2 \right)^{-1} \frac{T_0}{2} \right\}. \]
If \( \| F_0 - G_0 \|_2 \geq D_0 \), then by conservation of mass and energy we have for all \( t \geq 0 \),
\[ \| F_t - G_t \|_2 \leq \| F_0 \|_2 + \| G_0 \|_2 \leq 2 \| F_0 \|_2 + \| G_0 - F_0 \|_2 \leq \left( \frac{2 \| F_0 \|_2}{D_0} + 1 \right) \| G_0 - F_0 \|_2. \]

In the following we assume that \( \| F_0 - G_0 \|_2 < D_0 \). By the uniqueness theorem, we can also assume that \( \| F_0 - G_0 \|_2 > 0 \). Due to our choice of \( D_0 \), we have that \( G_0 \) is non-zero and is not a Dirac distribution. Therefore let \( \rho > 0, \, u \in \mathbb{R}^N, \, T > 0 \) be the mass, mean velocity and temperature corresponding to \( G_0 \) (i.e. \( G_0 \in B_{\rho, u, T}^+(\mathbb{R}^N) \)). Using the condition \( \| F_0 - G_0 \|_2 < D_0 \) and elementary estimates we have
\[
\begin{align*}
|\rho - \rho_0| &\leq \| G_0 - F_0 \|_2, \quad 0 < \frac{\rho_0}{2} \leq \rho \leq \frac{3 \rho_0}{2}, \\
|u - u_0| &\leq \frac{2 \| F_0 \|_2}{\rho_0^2} \| G_0 - F_0 \|_2, \\
|T - T_0| &\leq \left( \frac{4 \| F_0 \|_2}{N \rho_0} + \frac{6}{N} \left( \frac{\| F_0 \|_2}{\rho_0^2} \right)^2 \right) \| G_0 - F_0 \|_2, \\
0 &< \frac{T_0}{2} \leq T \leq \frac{3T_0}{2}.
\end{align*}
\]

Let \( M_{F_0}, M_{G_0} \) be the Maxwellians associated with \( F_0, G_0 \) respectively, i.e. \( M_{F_0} \in B_{\rho_0, u_0, T_0}^+(\mathbb{R}^N), \, M_{G_0} \in B_{\rho, u, T}^+(\mathbb{R}^N) \). In the following calculations the constants \( 0 < C_i < \infty \) (\( i = 1, 2, \ldots, 9 \)) only depend on \( N \), the function \( b, \gamma, \rho_0, u_0 \) and \( T_0 \), and we recall that
\[ \| F_0 \|_2 = \rho_0(1 + N T_0 + |u_0|^2). \]
We need to estimate \(\|M_{G_0} - M_{F_0}\|_2\). Let us define
\[
\mathcal{M}(\rho, u, T; v) = (2\pi)^{-N/2} \rho T^{-N/2} \exp \left( -\frac{|v - u|^2}{2T} \right)
\]
and let us compute
\[
\frac{\partial}{\partial \rho} \mathcal{M}(\rho, u, T; v) = \mathcal{M}(1, u, T; v),
\]
\[
\nabla_u \mathcal{M}(\rho, u, T; v) = \mathcal{M}(\rho, u, T; v) \left( \frac{v - u}{T} \right),
\]
\[
\frac{\partial}{\partial T} \mathcal{M}(\rho, u, T; v) = \left( -\frac{(N/2 + 1)}{T} + \frac{|v - u|^2}{2T^2} \right) \mathcal{M}(\rho, u, T; v).
\]
If we set
\[
\forall \theta \in [0, 1], \quad \rho(\theta) = \theta \rho + (1 - \theta) \rho_0, \quad u(\theta) = \theta u + (1 - \theta) u_0, \quad T(\theta) = \theta T + (1 - \theta) T_0,
\]
then
\[
|\mathcal{M}(\rho(\theta), u(\theta), T(\theta); v) - \mathcal{M}(\rho_0, u_0, T_0; v)| \leq |\rho - \rho_0| \int_0^1 \mathcal{M}(1, u(\theta), T(\theta); v) \, d\theta
\]
\[
+ |\rho - \rho_0| \int_0^1 \mathcal{M}(\rho(\theta), u(\theta), T(\theta); v) \left( \frac{|v - u(\theta)|}{T(\theta)} \right) \, d\theta
\]
\[
+ |T - T_0| \int_0^1 \mathcal{M}(\rho(\theta), u(\theta), T(\theta); v) \left( \frac{(N/2 + 1)}{T(\theta)} + \frac{|v - u(\theta)|^2}{2T(\theta)^2} \right) \, d\theta.
\]
We then deduce
\[
\|M_{G_0} - M_{F_0}\|_2 = \int_{\mathbb{R}^N} \langle v \rangle^2 |\mathcal{M}(\rho, u, T; v) - \mathcal{M}(\rho_0, u_0, T_0; v)| \, dv
\]
\[
\leq C_1 \left( |\rho - \rho_0| + |\rho - \rho_0| + |T - T_0| \right)
\]
and thus using the above estimates for \(\rho - \rho_0\), \(u - u_0\) and \(T - T_0\), we obtain
\[
(6.3) \quad \|M_{G_0} - M_{F_0}\|_2 \leq C_2 \|G_0 - F_0\|_2.
\]
Next from the above estimates we have
\[
\lambda_0 = S_{b, \gamma}(1, 0, 1) \rho_0 T_0^{\gamma/2}, \quad \lambda = S_{b, \gamma}(1, 0, 1) \rho T^{\gamma/2} \geq 2^{-1 - \gamma/2} \lambda_0.
\]
Then using the convergence estimate (1.46) and recalling that
\[
C_{1/\rho, |u|/\sqrt{T}, 1/T} = \rho \max \left\{ 1 + |u|^2 + \sqrt{T} |u|, \ T + \sqrt{T} |u| \right\},
\]
we have
\[
\left\{ \begin{array}{l}
\|F_t - M_{F_0}\|_2 \leq C_3 e^{-\lambda_0 t},
\|G_t - M_{G_0}\|_2 \leq C_0 C_{1/\rho, |u|/\sqrt{T}, 1/T} e^{-\lambda t} \leq C_4 \exp \left( -2^{-1 - \gamma/2} \lambda_0 t \right).
\end{array} \right.
\]
Thus
\[
\forall t \geq 0, \quad \|F_t - M_{F_0}\|_2 + \|G_t - M_{G_0}\|_2 \leq C_5 e^{-C_0 t},
\]
and it follows from (6.3) that

\[(6.4) \quad \forall t \geq 0, \quad \|F_t - G_t\|_2 \leq C_5 e^{-C_6 t} + C_2 \|F_0 - G_0\|_2.\]

On the other hand by the stability estimate (1.28) we have

\[(6.5) \quad \forall t \geq 0, \quad \|F_t - G_t\|_2 \leq \Psi_{F_0}(\|F_0 - G_0\|_2) e^{C_7 (1 + t)}.\]

The remaining of the proof is concerning with balancing properly (6.4) and (6.5).

**Case 1:** \(\Psi_{F_0}(\|F_0 - G_0\|_2) < 1\). Note that \(\|F_0 - G_0\|_2 > 0\) implies \(\Psi_{F_0}(\|F_0 - G_0\|_2) > 0\).

Let

\[t_0 = \log \left\{ \left( \frac{1}{\Psi_{F_0}(\|F_0 - G_0\|_2)} \right)^{\frac{1}{C_6 + C_7}} \right\} .\]

For every \(t \geq 0\), if \(t \leq t_0\), then, using (6.5),

\[\|F_t - G_t\|_2 \leq \Psi_{F_0}(\|F_0 - G_0\|_2) e^{C_7 (1 + t_0)} = e^{C_7 \left( \Psi_{F_0}(\|F_0 - G_0\|_2) \right)^{\frac{C_6}{C_6 + C_7}}}.\]

If \(t \geq t_0\), then, using (6.4),

\[\|F_t - G_t\|_2 \leq C_5 \left( \Psi_{F_0}(\|F_0 - G_0\|_2) \right)^{\frac{C_6}{C_6 + C_7}} + C_2 \|F_0 - G_0\|_2.\]

Thus

\[(6.6) \quad \forall t \geq 0, \quad \|F_t - G_t\|_2 \leq C_8 \left( \Psi_{F_0}(\|F_0 - G_0\|_2) \right)^{\frac{C_6}{C_6 + C_7}} + C_2 \|F_0 - G_0\|_2.\]

**Case 2:** \(\Psi_{F_0}(\|F_0 - G_0\|_2) \geq 1\). In this case we have, by conservation of mass and energy and \(\|F_0 - G_0\|_2 \leq \rho_0/2 \leq \|F_0\|_2/2\) that

\[(6.7) \quad \forall t \geq 0, \quad \|F_t - G_t\|_2 \leq \frac{5}{2} \|F_0\|_2 \leq \frac{5}{2} \|F_0\|_2 \left( \Psi_{F_0}(\|F_0 - G_0\|_2) \right)^{\frac{C_6}{C_6 + C_7}}.\]

Combining (6.6), (6.7), and (6.2), we obtain

\[\forall t \geq 0, \quad \|F_t - G_t\|_2 \leq C_9 \left( \Psi_{F_0}(\|F_0 - G_0\|_2) \right)^{\frac{C_6}{C_6 + C_7}} \|F_0 - G_0\|_2 + \left\| \Psi_{F_0}(\|F_0 - G_0\|_2) \right\|^{\frac{C_6}{C_6 + C_7}}.\]

This proves Theorem 1.9 \(\square\)

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References

[1] Abrahamsson, F. Strong $L^1$ convergence to equilibrium without entropy conditions for the Boltzmann equation. Comm. Partial Differential Equations 24, 7-8 (1999), 1501–1535.
[2] Baranger, C., and Mouhot, C. Explicit spectral gap estimates for the linearized Boltzmann and Landau operators with hard potentials. Rev. Mat. Iberoamericana 21, 3 (2005), 819–841.
[3] Bobylev, A. V. The method of the Fourier transform in the theory of the Boltzmann equation for Maxwell molecules. Dokl. Akad. Nauk SSSR 225, 6 (1975), 1041–1044.
[4] Bobylev, A. V. The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules. In Mathematical physics reviews, Vol. 7, vol. 7 of Soviet Sci. Rev. Sect. C Math. Phys. Rev. Harwood Academic Publ., Chur, 1988, pp. 111–233.
[5] Bouchut, F., and Desvillettes, L. A proof of the smoothing properties of the positive part of Boltzmann’s kernel. Rev. Mat. Iberoamericana 14, 1 (1998), 47–61.
[6] Carleman, T. Problèmes mathématiques dans la théorie cinétique des gaz. Publ. Sci. Inst. Mittag-Leffler. 2. Almqvist & Wiksells Boktryckeri Ab, Uppsala, 1957.
[7] Carlen, E. A., and Carvalho, M. C. Strict entropy production bounds and stability of the rate of convergence to equilibrium for the Boltzmann equation. J. Statist. Phys. 67, 3-4 (1992), 575–608.
[8] Carlen, E. A., and Carvalho, M. C. Entropy production estimates for Boltzmann equations with physically realistic collision kernels. J. Statist. Phys. 74, 3-4 (1994), 743–782.
[9] Carlen, E. A., and Carvalho, M. C. Entropy production estimates for Boltzmann equations with physically realistic collision kernels. J. Statist. Phys. 74, 3-4 (1994), 743–782.
[10] Carlen, E. A., Carvalho, M. C., and Gabetta, E. Central limit theorem for Maxwellian molecules and truncation of the Wild expansion. Comm. Pure Appl. Math. 53, 3 (2000), 370–397.
[11] Carlen, E. A., Carvalho, M. C., and Lu, X. On strong convergence to equilibrium for the Boltzmann equation with soft potentials. J. Stat. Phys. 135, 4 (2009), 681–736.
[12] Carlen, E. A., and Lu, X. Fast and slow convergence to equilibrium for Maxwellian molecules via Wild sums. J. Statist. Phys. 112, 1-2 (2003), 59–134.
[13] Desvillettes, L., Mouhot, C., and Villani, C. Celebrating Cercignani’s conjecture for the Boltzmann equation. Kinet. Relat. Models 4, 1 (2011), 277–294.
[14] Grad, H. Principles of the kinetic theory of gases. In Handbuch der Physik (herausgegeben von S. Flügge), Bd. 12, Thermodynamik der Gase. Springer-Verlag, Berlin, 1958, pp. 205–294.
[15] Grad, H. Asymptotic theory of the Boltzmann equation. II. In Rarefied Gas Dynamics (Proc. 3rd Internat. Sympos., Palais de l’UNESCO, Paris, 1962), Vol. I. Academic Press, New York, 1963, pp. 26–59.
[16] Gualdani, M., Mischler, S., and Mouhot, C. Factorization of non-symmetric operators and applications. Tech. rep., preprint, 2013.
[17] Hilbert, D. Begründung der kinetischen Gastheorie. Math. Ann. 72, 4 (1912), 562–577.
[18] Hilbert, D. Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. Chelsea Publishing Company, New York, N.Y., 1912 (1953).
[19] Lions, P.-L. Compactness in Boltzmann’s equation via Fourier integral operators and applications. Tech. rep., preprint, 2013.
[20] Lu, X. A direct method for the regularity of the gain term in the Boltzmann equation. J. Math. Anal. Appl. 228, 2 (1998), 409–435.
[21] Lu, X., and Mouhot, C. On measure solutions of the Boltzmann equation, part I: moment production and stability estimates. J. Differential Equations 252, 4 (2012), 3305–3363.
[22] Mischler, S., and Wennberg, B. On the spatially homogeneous Boltzmann equation. Ann. Inst. H. Poincaré Anal. Non Linéaire 16, 4 (1999), 467–501.
[23] Mouhot, C. Explicit coercivity estimates for the linearized Boltzmann and Landau operators. Comm. Partial Differential Equations 31, 7-9 (2006), 1321–1348.
[24] Mouhot, C. Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation with hard potentials. Comm. Math. Phys. 261, 3 (2006), 629–672.
[25] Mouhot, C. Quantitative linearized study of the Boltzmann collision operator and applications. Commun. Math. Sci., suppl. 1 (2007), 73–86.
[26] Mouhot, C., and Villani, C. Regularity theory for the spatially homogeneous Boltzmann equation with cut-off. Arch. Ration. Mech. Anal. 173, 2 (2004), 169–212.
[27] Stein, E. M., and Weiss, G. Introduction to Fourier analysis on Euclidean spaces. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.
[28] Villani, C. A review of mathematical topics in collisional kinetic theory. In Handbook of mathematical fluid dynamics, Vol. I. North-Holland, Amsterdam, 2002, pp. 71–305.

[29] Villani, C. Cercignani’s conjecture is sometimes true and always almost true. Comm. Math. Phys. 234, 3 (2003), 455–490.

[30] Wang Chang, C. S., Uhlenbeck, G. E., and de Boer, J. In Studies in Statistical Mechanics, Vol. V. North-Holland, Amsterdam, 1970.

XUGUANG LU

TSINGHUA UNIVERSITY
DEPARTMENT OF MATHEMATICAL SCIENCES
BEIJING 100084, P.R., CHINA

E-MAIL: xglu@math.tsinghua.edu.cn

Clément Mouhot

UNIVERSITY OF CAMBRIDGE
DPMMS, CENTRE FOR MATHEMATICAL SCIENCES
WILBERFORCE ROAD, CAMBRIDGE CB3 0WA, UK

E-MAIL: C.Mouhot@dpmms.cam.ac.uk