SCALING PROPERTIES OF THE THUE–MORSE MEASURE

MICHAEL BAAKE, PHILIPP GOHLKE, MARC KESSEBÖHMER, AND TANJA SCHINDLER

Abstract. The classic Thue–Morse measure is a paradigmatic example of a purely singular continuous probability measure on the unit interval. Since it has a representation as an infinite Riesz product, many aspects of this measure have been studied in the past, including various scaling properties and a partly heuristic multifractal analysis. Some of the difficulties emerge from the appearance of an unbounded potential in the thermodynamic formalism. It is the purpose of this article to review and prove some of the observations that were previously established via numerical or scaling arguments.

1. Introduction

As is well known, see [3, Sec. 10.1] and references therein, the Thue–Morse diffraction measure for the balanced-weight case is given by the infinite Riesz product

$$\mu_{\text{TM}} = \prod_{\ell=0}^{\infty} \left(1 - \cos(2\pi 2^\ell k)\right),$$

where convergence is understood in the vague topology. As such, $\mu_{\text{TM}}$ is a translation-bounded, positive measure on $\mathbb{R}$ that is purely singular continuous and 1-periodic. Clearly,

$$\mu_{\text{TM}} = \nu \ast \delta_\mathbb{Z},$$

with $\nu = \mu_{\text{TM}}|_{[0,1)}$ being a probability measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the latter represented by $[0,1)$ with addition modulo 1. More precisely, $\nu$ is the weak limit of probability measures $\nu_N$ with Radon–Nikodym densities

$$\frac{d\nu_N}{d\lambda}(k) = \prod_{\ell=0}^{N-1} \left(1 - \cos(2\pi 2^\ell k)\right),$$

relative to Lebesgue measure $\lambda$. In this setting, $\nu$ is a natural choice for the maximal spectral measure in the orthocomplement of the pure point sector of the Thue–Morse dynamical system [23]. Since $\nu$ is a continuous measure, it is often advantageous to simultaneously consider $\nu$ as a measure on $\mathbb{T}$ and on $[0,1]$, as we shall see later several times. It is the aim of this paper to obtain information about the local structure of the Thue–Morse measure from multifractal analysis and its relation to thermodynamic formalism. As there are many singular continuous measures with similar properties that are of interest in number theory, see [2] for a recent example, we hope that this approach will prove useful there as well, as it did in [17].

2010 Mathematics Subject Classification. 37D35, 37C45, 52C23.

Key words and phrases. Thue–Morse sequence, spectral measure, Riesz product, multifractal analysis.
Let \( g_\mathbb{E} \) denote the Euclidean metric on \( \mathbb{T} \), \( B_\mathbb{E}(x,r) \) the closed ball around \( x \in \mathbb{T} \) with Euclidean radius \( r \), and let \( B_2(x,r) \) denote the closed ball around \( x \in \mathbb{T} \) with radius \( r \) with respect to the shift space metric \( g_2 \), where \( g_2(x,y) := 2^{-k} \) when \( k \in \mathbb{N}_0 \) is the largest integer such that \( x_i = y_i \) for all \( i \leq k \). We note here that this metric is not well defined for the dyadic points \( x \in \mathbb{T} \); see our discussion in Section 2.1. However, as the dyadic points form only a countable subset of \( \mathbb{T} \), we do not consider those points any further. A discussion about the differences between the two metrics will also be given in Section 2.1.

One way to quantify how concentrated the measure \( \nu \) is at a given point \( x \in \mathbb{T} \) is to determine its local dimension, given by

\[
\dim_{\nu, \tau}(x) = \lim_{r \to 0} \frac{\log \nu(B_r(x,r))}{\log(r)} \quad \text{with} \quad \tau \in \{\mathbb{E}, 2\},
\]

provided that the limit exists. Such a limit then also applies to the local dimension of \( \mu_{TM} \) at any \( x + n \) with \( n \in \mathbb{Z} \). Due to their highly irregular structure, we cannot hope to pin down the level sets of \( \dim_{\nu, \tau} \) explicitly. However, the corresponding Hausdorff dimension with respect to the metric \( g_\tau \),

\[
f_\tau(\alpha) = \dim_{\mathbb{H}, \tau}\{x \in \mathbb{T} : \dim_{\nu, \tau}(x) = \alpha\},
\]

yields a properly behaved function of \( \alpha \). The analysis of \( f_\tau(\alpha) \) is one of the open questions considered in [24]. The problem to determine the local dimension at a given point \( x \in \mathbb{T} \) turns out to be intimately related to pointwise scaling properties of the approximants in Eq. (1).

More precisely, we consider

\[
\beta(x) := \lim_{n \to \infty} \frac{1}{n \log(2)} \log \prod_{\ell=0}^{n-1} (1 - \cos(2^{\ell+1} \pi x)),
\]

for all \( x \in \mathbb{T} \) for which the limit exists. Since \( \prod_{\ell=0}^{n-1} (1 - \cos(2^{\ell+1} \pi x)) \) is related to the diffraction measure resulting from the first \( 2^n \) letters of the Thue–Morse chain, we obtain some physical interpretation of the quantity \( \beta(x) \). By standard results, compare [4] and references therein, it is known that the scaling exponent \( \beta(x) \) exists and equals \(-1\) for Lebesgue-a.e. \( x \in \mathbb{T} \). For some particular examples of non-typical points, see [6, 4].

There is a natural way to interpret \( \beta \) in terms of the Birkhoff average of some function \( \psi : \mathbb{T} \to [-\infty, \log(2)] \),

\[
\psi(x) = \log(1 - \cos(2 \pi x)), \quad \beta(x) = \lim_{n \to \infty} \frac{\psi_n(x)}{n \log(2)},
\]

where \( \psi_n(x) = \sum_{\ell=0}^{n-1} \psi(2^{\ell} x) \). Considered as a Borel probability measure on the dynamical system \((\mathbb{T}, T)\) with \( T(x) = 2x \) (mod 1), \( \nu \) is an equilibrium measure for the thermodynamic potential \( \psi \) — in fact, it is a \( g \)-measure in the sense of Keane [14]; see the explanation in Section 9. In analogy to known results for Hölder continuous potentials [22, Cor. 1], we expect to find some simple relation between \( f_\tau(\alpha) \) and the Birkhoff spectrum

\[
b_\tau(\alpha) = \dim_{\mathbb{H}, \tau} \mathcal{B}(\alpha) \quad \text{for} \quad \tau \in \{\mathbb{E}, 2\}, \quad \text{with}
\]

\[
\mathcal{B}(\alpha) = \left\{ x \in \mathbb{T} : \lim_{n \to \infty} \frac{\psi_n(x)}{n} = \alpha \right\} = \left\{ x \in \mathbb{T} : \beta(x) = \frac{\alpha}{\log(2)} \right\},
\]
where \( \tau \) indicates the metric type. It is one of the strengths of the thermodynamic formalism to connect such locally defined functions to the Legendre transform of a globally defined quantity. An adequate choice for the latter in our situation is the topological pressure of the function \( t\psi, t \in \mathbb{R} \), defined by

\[
p(t) := \mathcal{P}(t\psi) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{J \in I_n} \sup_{x \in J} \exp(t\psi_n(x)),
\]

where, for each \( n \in \mathbb{N} \), \( I_n \) forms a partition of \([0, 1]\) into intervals of length \( 2^{-n} \).

With this at hand, we can state our main result as follows, where \( p^* \) denotes the Legendre transform of \( p \).

**Theorem 1.1.** The Birkhoff spectrum of the function \( \psi \) from (2) is given by

\[
b_{\tau}(\alpha) = b(\alpha) = \max \left\{ \frac{-p^*(\alpha)}{\log(2)}, 0 \right\},
\]

for \( \tau \in \{E, 2\} \). The function \( b \) is concave on \(( -\infty, \log(3/2) ] \), constantly equal to 1 for all \( \alpha \leq -\log(2) \), strictly less than 1 for \( \alpha > -\log(2) \), and equal to 0 for \( \alpha \geq \log(3/2) \); see Figure 1 for the graph of the spectrum. Moreover, the level sets \( \mathcal{B}(\alpha) \) are empty for \( \alpha > \log(3/2) \).

Finally, the dimension spectrum of the measure \( \nu \) is related to the Birkhoff spectrum by

\[
f_{\tau}(\alpha) = f(\alpha) = b(\log(2)(1 - \alpha)).
\]

The main technical difficulty in proving this result can be traced back to the fact that the potential \( \psi \) exhibits a singularity at \( 0 \in \mathbb{T} \). Since the authors are not aware of any general
framework covering this case, we choose to present not only the general line of arguments but also the individual steps in detail. One major part in proving this result is the calculation of the pressure function given in Eq. (4). This can be seen as a tractable example for a pressure function with an unbounded potential for which we establish an interesting and important approximation result in Section 5. We will present the crucial properties of \( p \) needed for the proof of our main theorem in Section 7; see [18] for related results on non-integrable locally constant potentials with singularities.\(^1\) Finite-size scaling arguments for the validity of Theorem 1.1, and a numerical approach that is equivalent to Figure 1, have been provided by [11]. In [9], a general approach for the multifractal analysis of measures generated by infinite products is given, which does not allow for singularities in the potential function, however. Finally, we will use classical results to show that \( \nu \) is an \textit{equilibrium} measure. In this context, we will also give a precise numerical value for the metric entropy of \( \nu \).

The paper is organised as follows. In Section 2, we introduce an alternative view to relate \((T, T)\) to a symbolic shift space and establish some properties of the potential \( \psi \). This enables us to prove the maximal scaling exponent \( \beta \) in Section 3. Some properties that are reminiscent of Gibbs measures are shown, in Section 4, to hold for \( \nu \). In Section 5, we show that the full pressure function emerges as a limit of pressure functions that are restricted to certain subshifts of finite type. This allows us to employ results on Hölder continuous potentials and thus to give a proof of our main theorem in Section 6. Further properties of the pressure function are established in Section 7, thus providing all necessary properties to prove the remaining part of Theorem 1.1 in Section 8, whereas Section 9 is devoted to embed the previous results into the realm of equilibrium measures and the variational principle.

2. Preliminaries and Notation

2.1. Norms and metrics. Consider the ergodic dynamical system \((T, T, \lambda)\), with mapping \( T: x \mapsto 2x \pmod{1} \) and \( \lambda \) the Lebesgue measure on \( T \). This enables us to regard the function \( \psi_n = \sum_{\ell=0}^{n-1} \psi \circ T^\ell \) as a Birkhoff sum with ergodic transformation \( T \). It is sometimes more convenient to consider the closed unit interval instead of \( T \), with or without identifying the endpoints — which will be clear from the context. When doing so, we extend \( T \) from \( T \) to \([0, 1]\) in the obvious way, with \( T(1) = 1 \) and \( T(1/2) = 1 \). This is then consistent with the identification of 1 and 0.

Information about the orbit of a point \( x \in T \) under the action of \( T \) is most easily obtained if we represent \( x \) as a binary sequence. Let \( \mathbb{X} = \{0, 1\}^\mathbb{N} \) be the one-sided binary shift space, and use \( x = (x_1, x_2, \ldots) \) to denote a sequence \( x \in \mathbb{X} \). By slight abuse of notation, we also use \( x \) to denote the corresponding number,

\[
x = \sum_{i=1}^{\infty} x_i 2^{-i},
\]

\(^1\)We would like to mention that Jörg Schmeling et al. are independently working on general results for locally constant potentials with singularities.
where the actual meaning will always be clear from the context. This way, \( x \in \mathbb{X} \) is always a number in \([0, 1]\), which we will often simply write as the binary string \( x_1 x_2 \cdots \), in line with \( 2 \cdot x = x_2 x_3 \cdots \) etc. Thus, the shift action \( \sigma \) on \( \mathbb{X} \) can be represented via the doubling map \( T \) on \([0, 1]\).

With the bit flip \( \hat{0} = 1 \) and \( \hat{1} = 0 \), which extends to \( x \) bitwise, one obtains \( \hat{x} = 1 - x \) for any \( x \in [0, 1] \), with the latter written in binary expansion. For many of our purposes, it does not matter that the dyadic numbers have two representations as a sequence. We also introduce notation for the set of finite binary strings, \( \Sigma^* = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n \), and for the periodic continuation of some \( \omega \in \Sigma^* \) as the infinite sequence \( \overline{\omega} = \omega \omega \cdots \). With \( q_1, \ldots, q_n \in \{0, 1\} \), we denote the corresponding cylinder for the string \( q_1 \cdots q_n \) as

\[ \langle q_1 \cdots q_n \rangle := \{ q_1 \cdots q_n x : x \in \mathbb{X} \}, \]

where we employ the standard convention for the concatenation of sequences. In particular, by our identification, we have \( \langle 0 \rangle = [0, \frac{1}{2}] \) and \( \langle 1 \rangle = [\frac{1}{2}, 1] \).

Note that both the Hausdorff dimension and the local dimension of a measure implicitly depend on the metric chosen. Following our discussion above, in addition to \( (\mathbb{X}, \varrho_\mathbb{X}) \) with the Euclidean distance \( \varrho_\mathbb{X} \), we also want to consider \( (\mathbb{X}, \varrho_2) \), with \( \varrho_2 \) as defined earlier. We note that, even with the above identification of points between \( \mathbb{X} \) and \([0, 1]\), this metric is not equivalent to \( \varrho_\mathbb{X} \). In particular, \( 0\mathbb{X} \) and \( 1\mathbb{X} \) have Euclidean distance 0 but distance 1 in the shift space. Nevertheless, \( \varrho_2 \) and \( \varrho_\mathbb{X} \) show some consistency in the sense that both metrics assign the same length \( 2^{-\ell} \) to cylinders (intervals) of the form \( \langle q_1 \cdots q_\ell \rangle \). This will be sufficient to conclude that Theorem 1.1 holds irrespective of whether \( \nu \) is considered as a measure on the metric space \( (\mathbb{X}, \varrho_2) \) or on the space \( ([0, 1], \varrho_\mathbb{X}) \).

2.2. Basic properties of the measure \( \nu \) and the potential \( \psi \). Let \( P_0 := 1 \) and set \( P_1 (x) := 1 - \cos(2\pi x) \). Now, for \( n \geq 0 \), define the trigonometric polynomial \( P_n \) on \( \mathbb{T} \) recursively by

\[ P_{n+1}(x) := P_n(2x)P_1(x). \tag{7} \]

Clearly, one has \( P_n(x) = \prod_{\ell=0}^{n-1} P_1(2^\ell x) \). A simple calculation gives the following result.

**Fact 2.1.** For each \( n \in \mathbb{N}_0 \), the trigonometric polynomial \( P_n \) is non-negative and defines a probability density, both on \( \mathbb{T} \) and on \([0, 1]\). \( \square \)

For our study of growth estimates and asymptotic properties, we write

\[ \psi_n(x) = \log(P_n(x)). \tag{8} \]

We abbreviate \( \psi = \psi_1 \), which has a unique maximum at \( x = \frac{1}{2} \) and singularities at \( x = 0 \) and \( x = 1 \) with value \(-\infty\). The central role of \( \psi \) as a thermodynamic potential will emerge when we discuss the variational principle in Section 9. The first two derivatives of \( \psi \) are

\[ \psi'(x) = \frac{2\pi \sin(2\pi x)}{1 - \cos(2\pi x)} \quad \text{and} \quad \psi''(x) = \frac{-4\pi^2}{1 - \cos(2\pi x)}, \tag{9} \]

where \( \psi''(x) < 0 \). This, together with Eq. (7), immediately implies the following result.
Fact 2.2. On $[0,1]$, the function $\psi$ is strictly concave and satisfies the symmetry relation $\psi(1-x) = \psi(x)$. Moreover, for all $n \in \mathbb{N}$, the functions $\psi_n$ from Eq. (8) satisfy the recursions

$$
\psi_{n+1}(x) = \psi_n(2x) + \psi(x) \quad \text{and} \quad \psi_{n+1}(x) = \psi_n(x) + \psi(2^n x)
$$

as well as the symmetry relations $\psi_n(1-x) = \psi_n(x)$. □

Figure 2. Illustration of the graphs of $\psi(x)$ (solid line), $\psi(2x)$ (dashed line) and $\psi(4x)$ (dotted line).

Observe that $\psi(2^m x)$ is a periodic function, with fundamental period $2^{-m}$, which means that $\psi(2^m x)$ consists of $2^m$ ‘humps’ of identical shape; see Figure 2 for an illustration. Moreover, if $m \geq 1$, it is symmetric under a reflection in $x = \frac{1}{2}$, with two humps on any cylinder of the form $\langle q_1 \cdots q_{m-1} \rangle$. A simple calculation then yields the following property, which is illustrated in Figure 3.

Fact 2.3. For any integer $n \geq 2$, the function $\psi_n$ is strictly concave on every cylinder of the form $\langle q_1 \cdots q_{n-1} \rangle$ with $q_i \in \{0,1\}$, and has singularities on the boundary points of them. In other words, $\psi_n$ consists of $2^{n-1}$ humps on $[0,1]$. □

3. Maximal scaling exponent

This section is devoted to the proof of the following proposition, which is crucial also for our main theorem.

Proposition 3.1. The maximal value of $\beta$ is given by

$$
\max_{x \in [0,1]} \limsup_{n \to \infty} \frac{\psi_n(x)}{n \log(2)} = \beta\left(\frac{1}{3}\right) = \beta\left(\frac{2}{3}\right) = \frac{\log(3/2)}{\log(2)} \approx 0.584963.
$$
To prove this statement, we need to identify all cylinders \( \langle q_1 \cdots q_n \rangle \) where the maximal value of \( \psi_n \) occurs. Since it will turn out that such a maximum never lies on one of the endpoints of such a cylinder, we can go one step further and consider the cylinders \( \langle q_1 \cdots q_n \rangle \) for \( \psi_n \). From Fact 2.2, we know that \( \psi_n(\hat{x}) = \psi_n(x) \) holds for all \( n \in \mathbb{N} \). It thus suffices to consider \( x \in [0, \frac{1}{2}] \).

To continue, we introduce \( \psi(n)(x) := \psi(2^n-1x) \) for \( n \in \mathbb{N} \), so that \( \psi(1) = \psi \) and 
\[
\psi_n(x) = \psi(n)(x) + \psi(n-1)(x) + \cdots + \psi(1)(x).
\]
Furthermore, we need to go into some detail on the prefixes of a given sequence. Here, if \( x = x_1x_2\cdots \) and \( m \in \mathbb{N} \), we use the shorthand \( x[m] := x_1x_2\cdots x_m \) for the prefix of \( x \) of length \( m \).

Let us also fix a notation for the alternating sequences in \( \mathbb{X} \), namely 
\[
y = \overline{01} \quad \text{and} \quad \hat{y} = \overline{10},
\]
where one implies the other.

**Lemma 3.2.** Let \( y \in \mathbb{X} \) be the sequence from (10). Then, the estimate 
\[
(\psi + \psi^{(2)})(y^{[2k]}0x) > (\psi + \psi^{(2)})(y^{[2k]}1\hat{x})
\]
holds for any \( k \in \mathbb{N} \) and all \( x \in \mathbb{X} \) with \( x \neq \overline{11} := 111\cdots \).

**Proof.** Using the identities \( \psi^{(2)}(y^{[2k]}0x) = \psi(y^{[2k-1]}0x) = \psi(y^{[2k-1]}1\hat{x}) \) in conjunction with \( \psi^{(2)}(y^{[2k]}1\hat{x}) = \psi(\hat{y}^{[2k-1]}1\hat{x}) = \psi(y^{[2k-1]}0x) \), we can rewrite Eq. (11) equivalently as 
\[
\psi(y^{[2k-1]}1\hat{x}) - \psi(y^{[2k-1]}0x) > \psi(y^{[2k]}1\hat{x}) - \psi(y^{[2k]}0x).
\]

![Figure 3. Illustration of \( \psi_3(x) \) (dotted line) and \( \psi_5(x) \) (solid line).](image-url)
Clearly, all arguments are in \([0, \frac{1}{2}]\). On this interval, \(\psi\) is strictly increasing, while \(\psi'\) is positive and decreasing. Note also that \(y^{2k-1}0x < y^{2k}0x\) and \(y^{2k-1}1x < y^{2k}1x\), and that we have the inequality
\[
0 < y^{2k}1x - y^{2k}0x = 2^{-2k}(1x - 0x) < 2^{-2k+1}(1x - 0x) = y^{2k-1}1x - y^{2k-1}0x.
\]
Thus, we get
\[
\psi(y^{2k-1}1x) - \psi(y^{2k-1}0x) = \int_{y^{2k-1}0x}^{y^{2k-1}1x} \psi'(u) \, du > \int_{y^{2k-1}1x - (y^{2k}1x - y^{2k}0x)}^{y^{2k}1x} \psi'(u) \, du
\geq \int_{y^{2k}0x}^{y^{2k}1x} \psi'(v) \, dv = \psi(y^{2k}1x) - \psi(y^{2k}0x),
\]
where the first inequality is due to a reduction of the integration region and the second follows via the substitution \(v = u + (y^{2k}1x - y^{2k-1}1x) \geq u\) so that \(\psi'(u) \geq \psi'(v)\).

This observation has the following consequence.

**Corollary 3.3.** Suppose that \(y\) is the alternating sequence from \((10)\). Then, for any \(n \in \mathbb{N}\), one has \(\psi_n(y^{[n+1]}x) > \psi_n(y^{[n]}y_n+1x)\), provided that \(x \neq \overline{a} := \overline{aaa} \ldots\) with \(a = y_{n+2}\).

**Proof.** We show the statement for odd and even \(n\) separately. First, assume \(n = 2m\) with \(m \in \mathbb{N}\). In this case, Lemma 3.2 implies
\[
\psi_{2m}(y^{[2m]}0x) = \left(\psi + \psi^{(2)} + \psi^{(3)} + \psi^{(4)} + \ldots + (\psi(2m-1) + \psi(2m))\right) (y^{[2m]}0x)
= (\psi + \psi^{(2)}) (y^{[2m]}0x) + (\psi + \psi^{(2)}) (y^{[2m-2]}0x) + \ldots + (\psi + \psi^{(2)}) (y^{[2]}0x)
> (\psi + \psi^{(2)}) (y^{[2m]}1x) + (\psi + \psi^{(2)}) (y^{[2m-2]}1x) + \ldots + (\psi + \psi^{(2)}) (y^{[2]}1x)
= \psi_{2m}(y^{[2m]}1x).
\]
From this result, we can also proceed to \(n = 2m + 1\) with \(m \in \mathbb{N}\),
\[
\psi_{2m+1}(y^{[2m+1]}1x) = \psi_{2m}(y^{[2m]}1x) + \psi(y^{[2m+1]}1x) \geq \psi_{2m}(y^{[2m]}0x) + \psi(y^{[2m+1]}0x)
> \psi_{2m}(y^{[2m]}1x) + \psi(y^{[2m+1]}0x) = \psi_{2m}(y^{[2m]}0x) + \psi(y^{[2m+1]}0x)
= \psi_{2m+1}(y^{[2m+1]}0x).
\]
For the first inequality, we have used that \(\psi\) is increasing on \([0, \frac{1}{2}]\), while the second inequality uses the corresponding identity for even indices. When \(n = 1\), the statement directly follows from the strictly increasing nature of \(\psi\) on \([0, \frac{1}{2}]\).

In order to narrow down the position of the maximum of \(\psi_n\), we would like to compare the values of \(\psi_n\) on cylinders of the form \(\langle q_1 \cdots q_n \rangle\), where we may choose \(q_1 = 0\) due to the symmetry of \(\psi_n\). To this end, we partition the interval \([0, \frac{1}{2}]\) into sets which contain exactly one element of each cylinder. We will show that it is possible to choose these sets in such a way that the maximum of \(\psi_n\) on each such set lies always in the cylinder \(\langle y_1 \cdots y_n \rangle\). The
sets can be constructed via an iterative reflection at the midpoints of appropriately chosen cylinders. Let us make this more precise as follows.

**Proposition 3.4.** Let \( y \) be the alternating sequence from (10) and denote the involution on \( X \) by \( I \), so \( I(x) = \overline{x} \), with \( I^0 = \text{id} \) as usual. Further, define

\[
A_n(x) := \left\{ q_1 \cdots q_n I^n(x) : q_1 = 0 \text{ and } q_2, \ldots, q_n \in \{0,1\} \right\},
\]

for \( n \in \mathbb{N} \) and \( x \in X \). Then, one has \( \left[ 0, \frac{1}{2} \right] = \bigcup_{x \in X} A_n(x) \).

Moreover, for all \( n \in \mathbb{N} \) and \( x \in X \), the maximum of \( \psi_n \) on \( A_n(x) \) is taken at \( y^{[n]} I^n(x) \). This maximum is strict as long as \( x \neq \overline{0} \), whereas it is given by \( -\infty \) for \( x = \overline{0} \).

**Proof.** The first claim is obvious, and the second can be shown by induction. For \( n = 1 \), one has \( A_1(x) = \{0x\} \), which is a singleton set, and the assertion is trivial. Suppose it is true up to \( n \). Now, observe that

\[
A_{n+1}(x) = A_n(0x) \cup A_n(1\overline{x}).
\]

By the induction assumption, \( \psi_n \) takes its maximum on \( A_n(0x) \) in the point

\[
y^{[n]} I^n(0x) = y^{[n]} y_n I^n(x) = y^{[n]} g_{n+1} I^n(x).
\]

Similarly, the position of the maximum on \( A_n(1\overline{x}) \) is given by \( y^{[n]} I^n(1\overline{x}) = y^{[n+1]} I^n(\overline{x}) \). Comparing these two points, we obtain that \( \psi_n \) takes its maximum on \( A_{n+1}(x) \) at the position \( y^{[n+1]} I^{n+1}(x) \), by an application of Corollary 3.3 (note that \( \psi_n \) is strictly larger than \( -\infty \) at that point). The proof is completed by the observation that \( \psi_{n+1} - \psi_n = \psi^{(n+1)} \) is constant on \( A_{n+1}(x) \), because

\[
\psi^{(n+1)}(q_1 \cdots q_{n+1} I^{n+1}(x)) = \psi(q_{n+1} I^{n+1}(x)) = \begin{cases} 
\psi(0x), & \text{for } q_{n+1} = 0, \\
\psi(1\overline{x}), & \text{for } q_{n+1} = 1,
\end{cases}
\]

where \( \psi(1\overline{x}) = \psi(0x) \) by the reflection symmetry of \( \psi \) on \([0,1]\). Obviously, \( \psi(0x) = -\infty \) if and only if \( x = \overline{0} \).

**Remark 3.5.** In Proposition 3.4, we have singled out the points with \( x = \overline{0} \), because these are the positions where divergences of \( \psi_n \) occur. At first sight, it seems natural to also consider the special case \( x = \overline{1} \), which corresponds to the opposite boundary points of the cylinders of the form \( \langle q_1 \cdots q_n \rangle \). However, these points are just the midpoints of the cylinders of type \( \langle q_1 \cdots q_{n-1} \rangle \). Although the cardinality of the set \( A_n(x) \) gets reduced by a factor of 2 in that case, the argument we employed above remains unaltered.

Let us also mention that the statement in Proposition 3.4 can slightly be sharpened to hold also for all \( \psi_m \), with \( m > n \), which take the maximum on \( A_n(x) \) in the same point as \( \psi_n \). The value of the maximum is \( -\infty \) if and only if \( q_1 \cdots q_n I^n(x) \) is a singularity point for \( \psi_m \) for some (equivalently every) choice of \( q_1 \cdots q_n \). This follows from the fact that \( \psi^{(m)} \) is constant on \( A_n(x) \) for all \( m \geq n \).

**Corollary 3.6.** For any \( n \in \mathbb{N} \), the function \( \psi_n \) takes its maximal value on \([0, \frac{1}{2}]\) in the cylinder \( C_n = \langle y_1 \cdots y_n \rangle \), with \( y = \overline{01} \) as in (10), while the maximal value on \([\frac{1}{2}, 1]\) is taken
in \( I(C_n) \), and these two maximal values are equal. Moreover, in terms of cylinders that are coarser by one level, the two maxima of \( \psi_n \) lie in the interior of \( C_{n-1} \) and \( I(C_{n-1}) \).

**Proof.** Clearly, \( \psi_n \) has a maximum on the cylinder \( \langle y_1 \cdots y_n \rangle \), say in the point \( x^* \). Since the only singularity in this cylinder is at \( y_1^{[n-1]} y_n \), we know that \( \psi_n(x^*) \geq -\infty \).

Let \( z \in \langle q_1 \cdots q_n \rangle \) with \( q_1 = 0 \), so \( z \in [0, \frac{1}{2}] \). Then, by Proposition 3.4, \( z \in A_n(x) \) for some \( x \in X \), and we have

\[
\psi_n(x^*) \geq \psi_n(y_1^{[n]} I y_n(x)) \geq \psi_n(z).
\]

Unless \( z = x^* \), at least one of the inequalities is strict, which proves the first claim. The second is an obvious consequence of the symmetry under \( I \).

For the last statement, we know from Proposition 3.4 that the position of the maximum, \( x^* \), is not at the boundary point of \( C_n \) with the singularity. Consequently, it is either an interior point or the other boundary point, and hence an interior point of the coarser cylinder, \( C_{n-1} \). The mirror statement holds for \( I(C_n) \), and we are done. \( \square \)

**Lemma 3.7.** Let \( y \) be as in Eq. (10). Then, there exists a constant \( K > 0 \) such that the inequality \( \max_{x \in [0,1]} \psi_n(x) - \psi_n(y) \leq K \) holds for all \( n \in \mathbb{N} \).

**Proof.** Due to symmetry, it suffices to consider the maximum on the interval \([0, \frac{1}{2}]\). By Corollary 3.6, we obtain

\[
\max_{x \in [0,1]} (\psi_{n+1}(x) - \psi_{n+1}(y)) = \max_{x \in C_{n+1}} (\psi_{n+1}(x) - \psi_{n+1}(y)) \leq \max_{x \in C_{n+1}} (\psi_n(2x) - \psi_n(y)) + \max_{x \in C_{n+1}} (\psi(x) - \psi(y)).
\]

Note that \( \max_{x \in C_{n+1}} \psi_n(2x) = \max_{x \in I(C_n)} \psi_n(x) = \max_{x \in C_n} \psi_n(x) \) by the definition of the cylinders. Since \( y = \overline{01} \in C_n \) for all \( n \in \mathbb{N} \), we infer from the mean value theorem that

\[
\max_{x \in C_{n+1}} (\psi(x) - \psi(y)) \leq \frac{|\psi'(\xi)|}{2^n+1},
\]

with some \( \xi \in C_{n+1} \). For any \( n \in \mathbb{N} \), one has \( C_{n+1} \subset \left[ \frac{1}{4}, \frac{5}{4} \right] \), and since \( |\psi'| \) is decreasing on this interval, \( |\psi'(\xi)| \leq \psi' \left( \frac{1}{4} \right) = 2\pi \). Consequently,

\[
\max_{x \in [0,1]} (\psi_{n+1}(x) - \psi_{n+1}(y)) \leq \max_{x \in C_n} (\psi_n(x) - \psi_n(y)) + \frac{\pi}{2^n}.
\]

Recursively, we then obtain

\[
\max_{x \in [0,1]} (\psi_{n+1}(x) - \psi_{n+1}(y)) \leq \max_{x \in C_1} (\psi(x) - \psi(y)) + \sum_{k=1}^{n} \frac{\pi}{2^k} \leq \pi + \log \left( \frac{4}{3} \right),
\]

where we have used \( \psi(y) = \log \left( \frac{3}{2} \right) \) and \( \max_{x \in C_1} \psi(x) = \psi \left( \frac{1}{2} \right) = \log (2) \). \( \square \)

**Proof of Proposition 3.1.** Let \( n \geq 2 \). From Corollary 3.6, we know that the maximum of \( \psi_n \) is taken in the interior of the cylinders \( (0101 \cdots 01) \) and \( (1010 \cdots 10) \) when \( n \) is even, and in the cylinders \( (0101 \cdots 010) \) and \( (1010 \cdots 101) \) when \( n \) is odd. Since \( \frac{1}{3} = \overline{y} = y \) and \( \frac{2}{3} = \overline{y} = \hat{y} \), see Eq. (10), the claim on the location follows from Lemma 3.7, with the value given by a simple calculation as in [4]. \( \square \)
Remark 3.8. Both Corollary 3.1 and Lemma 3.7 (with an improved constant $K$) also follow from bounds on $\|P_n\|_\infty$ with $n \in \mathbb{N}$ that were established in [10]; see also [24, Thm. 1.1]. ♦

4. Gibbs-type properties

So far, we have shown that the level sets $B(\alpha)$ are indeed empty for $\alpha > \log\left(\frac{3}{2}\right)$. As a next step towards the proof of Theorem 1.1, we will establish a link between the local dimension of $\nu$ and the Birkhoff average of $\psi$ at certain points $x \in X$. Since our arguments will evolve along similar lines, let us first sketch how the corresponding relation arises for Gibbs measures.

For any H"older continuous potential $\phi$ on $(X, \varphi_2)$, there is a unique $T$-invariant Borel probability measure $\mu$ that satisfies

\[
\mu(\{x_1 \cdots x_n\}) \leq \exp(-Pn + \phi_n(x)) \leq \frac{1}{c_2},
\]

for any $x \in X$, $n \in \mathbb{N}$ and some constants $c_1, c_2 > 0$ and $P \in \mathbb{R}$. Here, $\phi_n = \sum_{\ell=0}^{n-1} \phi \circ \sigma^n$, in analogy to $\psi_n$, and $P$ turns out to be the topological pressure of $\phi$; compare Section 9. Following the terminology of [5], we call $\mu$ an invariant Gibbs measure for $\phi$. The property in (14) immediately allows to conclude that

\[
\lim_{n \to \infty} \frac{\phi_n(x)}{n} = P - \dim_\mu(x) \log(2)
\]

holds for all $x \in X$, see [22, Prop. 1], thus establishing a connection between the dimension spectrum and the Birkhoff spectrum at the same time.

Due to the singularity of $\psi$ at 0, the function $\psi_n$ has infinite variation on any cylinder of the form $\{x_1 \cdots x_n\}$, prohibiting the analogue of (14) for any measure on $X$. The aim of this section is to establish a slightly weaker but similar relation for $\nu$ that suffices to derive a relation that is analogous to (15), at least for points $x$ in certain subshifts of $X$.

Given $m \in \mathbb{N}$, restricting our space to the subshift of finite type (SFT)

\[
X_m := \{x \in X : x_\ell \cdots x_{\ell+m} \notin \{0^{m+1}, 1^{m+1}\}, \ell \in \mathbb{N}\}
\]

ensures that all $x \in X_m$ are bounded away from the singularity points of $\psi$ by at least $2^{-m}$.

We define the set of admissible words of length $n > m$ in $X_m$ as

\[
\Sigma^m_n := \{\omega \in \{0,1\}^n : \omega_\ell \cdots \omega_{\ell+m} \notin \{0^{m+1}, 1^{m+1}\}, \ell \in \{1,\ldots,n-m\}\},
\]

and use $\Sigma^n = \{0,1\}^n$ for the set of all binary words of length $n$. One can verify that the restriction of $\psi$ to $X_m$ is indeed H"older continuous, and estimate its modulus of continuity.

Lemma 4.1. For any $x \in [0,1]$, we have

\[
|\psi'(x)| \leq 2 \max\left\{\frac{1}{x}, \frac{1}{1-x}\right\}
\]

with respect to the Euclidean metric $\varrho_\mathbb{E}$ on $[0,1]$. Moreover, $\psi$ is H"older continuous on $X_m$ with respect to the metric $\varrho_2$ on the shift space $X$. 
Proof. Using (9), we obtain

\[ \lim_{x \to 0^+} x \psi'(x) = \lim_{x \to 0^+} \frac{2\pi x (1 + \cos(2\pi x))}{\sin(2\pi x)} = 2 \]

and

\[ \frac{d}{dx} (x \psi'(x)) = \frac{2\pi \sin(2\pi x) - 4\pi^2 x}{1 - \cos(2\pi x)}. \]

Since \( \sin(x) < x \) for \( x > 0 \) and \( 1 - \cos(2\pi x) > 0 \) on \( (0, 1) \), we see that the derivative of \( x \psi'(x) \) is negative, so \( x \psi'(x) \) is monotonically decreasing on \( (0, \frac{1}{2}) \). Combining this with Eq. (16) gives \( \psi'(x) \leq \frac{2}{x} \). The estimate \( |\psi'(x)| \leq \frac{2}{1-x} \) follows from the symmetry of \( \psi \).

To prove the second claim, we note that, for all \( m, n \in \mathbb{N} \) with \( n > m \), we have

\[ \sup_{\omega \in \Sigma_n} \sup_{x,y \in (\omega) \cap \mathbb{X}_m} |\psi(x) - \psi(y)| \leq \sup_{\omega \in \Sigma_n} \sup_{x \in (\omega) \cap \mathbb{X}_m} |\psi'(x)| |\omega| = \sup_{x \in \mathbb{X}_m} \frac{|\psi'(x)|}{2^n}. \]

By the concavity of \( \psi \), see Fact 2.2 and the first statement of the lemma, we have the estimate \( \sup_{x \in \mathbb{X}_m} |\psi'(x)| < \psi'(2^{-m-1}) \leq 2^{m+2} \). Consequently,

\[ \sup_{\omega \in \Sigma_n} \sup_{x,y \in (\omega) \cap \mathbb{X}_m} |\psi(x) - \psi(y)| \leq 2^{m+2-n} \]

and hence \(|\psi(x) - \psi(y)| \leq 2^n(2 - x_y) \). Thus, \( \psi \) is Lipschitz continuous with Lipschitz constant \( 2^{m+2} \) on \( \mathbb{X}_m \).

For \( x \in \mathbb{X}_n \), denote by \( \mathbb{C}_n(x) = \langle x_1 \cdots x_n \rangle \) the (unique) cylinder of length \( 2^{-n} \) that contains \( x \). We are concerned with the values of \( \psi_n \) on such cylinders. Recall that \( \psi_n(x) \) comprises \( 2^{n-1} \) humps of the same total width so that \( \psi_n \) is concave on \( \mathbb{C}_n(x) \), singular at one boundary point, and non-singular at the other; compare Figure 3. Taking the intersection of \( \mathbb{C}_n(x) \) with \( \mathbb{X}_m \) removes a neighbourhood around the singularity, and we find that the variation of \( \psi_n \) on such a set is bounded in a suitable way.

Lemma 4.2. Let \( x \in \mathbb{X}_m \) for some \( m \in \mathbb{N} \). Then, there exists a constant \( K = K(m) > 0 \) such that, for all \( n \in \mathbb{N} \),

\[ \sup_{y \in \mathbb{C}_n(x) \cap \mathbb{X}_m} \exp(\psi_n(y)) \leq K \inf_{y \in \mathbb{C}_n(x) \cap \mathbb{X}_m} \exp(\psi_n(y)). \]

Proof. The Hölder continuity of \( \psi \) on \( \mathbb{X}_m \), see Lemma 4.1, implies that there exists a bounded distortion constant \( W > 0 \) such that

\[ \sup_{y \in \mathbb{C}_n(x) \cap \mathbb{X}_m} \psi_n(y) - \inf_{y \in \mathbb{C}_n(x) \cap \mathbb{X}_m} \psi_n(y) \leq W. \]

Setting \( K = \exp(W) \) gives the claim.
Lemma 4.3. If $x \in \mathbb{X}$, we have the following bound for the value of $\nu$ on $C_n(x)$,

$$\nu(C_n(x)) \leq 2^{-n} \sup_{y \in C_n(x)} \exp(\psi_n(y)).$$

Further, for $x \in \mathbb{X}_m$ and $m \in \mathbb{N}$, there exists a constant $K'$, independently of $n$, so that

$$\frac{1}{2^n} \inf_{\nu \in C_n(x) \cap \mathbb{X}_m} \exp(\psi_n(y)) \leq \nu(C_n(x)) \leq \frac{K'}{2^n} \sup_{\nu \in C_n(x) \cap \mathbb{X}_m} \exp(\psi_n(y)).$$

Proof. To establish (17), we note that, for $N > n$, one has

$$\nu_N(C_n(x)) = \int_{C_n(x)} P_N(\xi) \, d\xi \leq \sup_{\nu \in C_n(x)} P_n(y) \int_{C_n(x)} P_{N-n}(2^n \xi) \, d\xi$$

$$= \sup_{\nu \in C_n(x)} \exp(\psi_n(y)) 2^{-n} \int_0^1 P_{N-n}(\xi) \, d\xi = 2^{-n} \sup_{\nu \in C_n(x)} \exp(\psi_n(y)),$$

where we used the fact that $P_{N-n}$ is a probability density on $[0,1]$. Taking $N \to \infty$ in the above relation yields (17).

Let $C_n^0(x) = \langle x_1 \cdots x_n \rangle$ and $C_n^1(x) = \langle x_1 \cdots x_n 1 \rangle$ denote the left and right half of this interval, respectively. For $j \in \{0,1\}$, we find

$$\nu_N(C_n(x)) \geq \int_{C_n^j(x)} P_N(\xi) \, d\xi \geq \inf_{\nu \in C_n^j(x)} P_n(y) \int_{C_n(x)} P_{N-n}(2^n \xi) \, d\xi$$

$$= \inf_{\nu \in C_n^j(x)} \exp(\psi_n(y)) 2^{-n} \int_{j/2}^{(j+1)/2} P_{N-n}(\xi) \, d\xi = 2^{-n-1} \inf_{\nu \in C_n^j(x)} \exp(\psi_n(y)),$$

using that $P_{N-n}$ is symmetric under $x \mapsto 1-x$ in $[0,1]$. Again, performing $N \to \infty$ gives

$$\nu(C_n(x)) \geq 2^{-n-1} \max_{j \in \{0,1\}} \inf_{\nu \in C_n^j(x)} \exp(\psi_n(y)).$$

By assumption, we have $m \geq 1$. To continue, for $B \subset [0,1]$, define a $B$-truncated version of $\psi$ as $\psi^B := \psi 1_B$. With $D(m) := [2^{-m-1},1-2^{-m-1}]$, consider the function

$$\psi^{D(m)} := \psi 1_{D(m)} \geq \psi,$$

viewed as a function on $\mathbb{X}$. Clearly, $\psi^{D(m)}$ is Hölder continuous relative to the metric $\varrho_2$, and $\psi^{D(m)}(z) = \psi(z)$ as long as $z \neq \langle \omega \rangle$ for $\omega \in \{0 \cdots 0,1 \cdots 1\} \subset \{0,1\}^{m+1}$. In particular, this holds for $z \in \mathbb{X}_m$. We denote the bounded distortion constant of $\psi^{D(m)}$ by $W'$.

Now, choose $j = j(n) = \widehat{x}_n$. Since $x \in \mathbb{X}_m$, this implies

$$\inf_{\nu \in C_n^j(x) \cap \mathbb{X}_m} \psi_n(y) - \inf_{\nu \in C_n^j(x)} \psi_n(y) \leq \sum_{k=0}^{n-1} \sup_{\nu \in C_n^j(x) \cap \mathbb{X}_m} \psi(2^k y) - \inf_{\nu \in C_n^j(x)} \psi(2^k y) \leq \sum_{k=0}^{n-1} \sup_{\nu \in \langle x_{k+1} \cdots x_n \rangle \cap \mathbb{X}_m} \psi^{D(m)}(y) - \inf_{\nu \in \langle x_{k+1} \cdots x_n \rangle \cap \mathbb{X}_m} \psi^{D(m)}(y).$$

Assume $n \geq m$. Now, using Hölder continuity and setting $K' = e^{W'}$ gives

$$\inf_{\nu \in C_n^j(x) \cap \mathbb{X}_m} \psi_n(y) - \inf_{\nu \in C_n^j(x)} \psi_n(y) \leq W' = \log(K').$$
Since $\inf_{y \in C_n(x) \cap \mathbb{X}_m} \psi_n(y) \geq \inf_{y \in C_\mathcal{N}(x) \cap \mathbb{X}_m} \psi_n(y)$, we have established the lower bound.

The upper bound follows by a similar calculation. Using $\psi_n(y) \leq \psi_n^{D(m)}(y)$ for $y \in \mathbb{X}$, we find
\[
\sup_{y \in C_n(x)} \psi_n(y) - \sup_{y \in C_n(x) \cap \mathbb{X}_m} \psi_n(y) \leq \sup_{y \in C_n(x)} \psi_n^{D(m)}(y) - \sup_{y \in C_n(x) \cap \mathbb{X}_m} \psi_n^{D(m)}(y) < \log(K')
\]
as claimed. \qed

The following two results should be compared with Eq. (15). We first consider only cylinders as shrinking neighbourhoods of a point $x \in \mathbb{X}_m$ before we allow for more general balls $B(x, r)$ with $r > 0$, which may take different forms when built with respect to the Euclidean metric.

**Proposition 4.4.** Let $m \in \mathbb{N}$. For all $x \in \mathbb{X}_m$, one has
\[
\lim_{n \to \infty} \frac{1}{n} \log(\nu(C_n(x))) = \lim_{n \to \infty} \frac{1}{n} \log(2^n \exp(\psi_n(x))) = -\log(2) + \lim_{n \to \infty} \frac{\psi_n(x)}{n},
\]
provided that any of the limits exists.

**Proof.** By Lemmas 4.2 and 4.3, we have
\[
\frac{1}{KK'} \leq \frac{2^{-n}(K')^{-1} \inf_{y \in C_n(x) \cap \mathbb{X}_m} \exp(\psi_n(y))}{2^{-n} \sup_{y \in C_n(x) \cap \mathbb{X}_m} \exp(\psi_n(y))} \leq \frac{\nu(C_n(x))}{2^{-n} \exp(\psi_n(x))}
\]
(20)
\[
\leq \frac{2^{-n}K' \sup_{y \in C_n(x) \cap \mathbb{X}_m} \exp(\psi_n(y))}{2^{-n} \inf_{y \in C_n(x) \cap \mathbb{X}_m} \exp(\psi_n(y))} \leq KK'.
\]
If any of the limits exists, we obtain our assertion by (20). \qed

**Corollary 4.5.** Consider $\mathbb{X}_m$ as a metric subspace of either $(\mathbb{X}, \varrho_2)$ or $(\mathbb{T}, \varrho_\mathcal{E})$. Then, for any $\tau \in \{E, 2\}$ and $x \in \mathbb{X}_m$, one has
\[
\dim_{\nu, \tau}(x) = \lim_{r \to 0} \frac{\log(\nu(B_\tau(x, r)))}{\log(r)} = 1 - \frac{1}{\log(2)} \lim_{n \to \infty} \frac{\psi_n(x)}{n},
\]
provided that any of the limits exists.

**Proof.** Consider the metric space $(\mathbb{X}, \varrho_2)$ first. For $r < 1$, one has
\[
B_2(x, r) = \langle x_1 \cdots x_M \rangle = C_M(x),
\]
where $M = M(r) = \lfloor \log_{1/2}(r) \rfloor$ and the claim is immediate from Proposition 4.4 since, for any $(r_n)_{n \in \mathbb{N}}$ with $r_n > 0$ and $\lim_{n \to \infty} r_n = 0$, we have
\[
\lim_{n \to \infty} \frac{\log(\nu(B_2(x, r_n)))}{\log(r_n)} = \lim_{n \to \infty} \frac{\log(\nu(C_M(r_n)\rangle)}{-M(r_n)\log(2)} = 1 - \frac{1}{\log(2)} \lim_{n \to \infty} \frac{\psi_n(x)}{n}.
\]
Next, we consider $(\mathbb{T}, \varrho_\mathcal{E})$ and regard any ball $B_2(x, r)$ as a subset of $\mathbb{T}$ with obvious meaning. Since, for any $n \in \mathbb{N}$, $B_2(x, 2^{-n})$ has length $2^{-n}$ as an interval in Euclidean space, it is $B_2(x, 2^{-n}) \subseteq B_\mathcal{E}(x, 2^{-n})$. By the definition of $\mathbb{X}_m$, the Euclidean distance of $x$ from the
boundary points of the interval $B_2(x, 2^{-n}) = \langle x_1 \cdots x_n \rangle$ is at least $2^{-n-m-1}$ for any $n \in \mathbb{N}$. Thereby, for $n \geq m + 1$, we have
\[
\frac{\log(\nu(B_2(x, 2^{-n})))}{\log(2^{-n})} \leq \frac{\log(\nu(B_{E}(x, 2^{-n})))}{\log(2^{-n})} \leq \frac{\log(\nu(B_2(x, 2^{-n+m+1})))}{\log(2^{-n})},
\]
which gives the desired result as $n \to \infty$ for the sequence with $r_n = 2^{-n}$. For general sequences $(r_n)_{n \in \mathbb{N}}$ that tend to 0, the corresponding identity follows by interpolation. □

**Remark 4.6.** An immediate consequence of Corollary 4.5 and Proposition 3.1 is that, for $y = \overline{01}$ and $\tau \in \{E, 2\}$, one has
\[
\dim_{\nu, \tau}(y) = 2 - \frac{\log(3)}{\log(2)} \approx 0.415.
\]

On the other hand, Eq. (17) together with Corollary 3.1 gives $\dim_{\nu, \tau}(x) \geq 2 - \frac{\log(3)}{\log(2)}$ for all $x \in \mathbb{X}$ and $\tau \in \{E, 2\}$ by direct calculation. So, we actually get $\inf_{x \in \mathbb{X}} \dim_{\nu, \tau}(x) = 2 - \frac{\log(3)}{\log(2)}$, which verifies a conjecture from [24, Sec. 4.4.2].

5. **Restricted pressure function and the exhaustion principle**

Interpreting $\psi$ as a function on the symbolic space $\mathbb{X}$, the topological pressure from Eq. (4) for $t\psi$ can be rewritten as
\[
(21) \quad p(t) = \mathcal{P}(t\psi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in \Sigma^n} \sup_{x \in \langle \omega \rangle} \exp(t\psi_n(x)),
\]
where the limit exists by subadditivity. We shall see that the mapping $p: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defines a proper, convex function. We denote its Legendre transform by
\[
(22) \quad p^*(a) := \sup_{q \in \mathbb{R}} (qa - p(q)).
\]
Note that, since $p(t) = +\infty$ for $t < 0$, see Proposition 7.1 below, we are in the particular situation that $p^*(a) = \sup_{q \geq 0} (qa - p(q))$.

For a closed subshift $\mathbb{X}' \subseteq \mathbb{X}$ that is invariant under the left shift, let us define the restricted pressure by
\[
\mathcal{P}(t\psi|\mathbb{X}') := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in \Sigma^n} \sup_{x \in \langle \omega \rangle \cap \mathbb{X}'} \exp(t\psi_n(x)).
\]
For any $\omega$ with $\langle \omega \rangle \cap \mathbb{X}' = \emptyset$, we set $\sup_{x \in \langle \omega \rangle \cap \mathbb{X}'} \exp(t\psi_n(x)) = 0$. Clearly, $\mathcal{P}(t\psi) = \mathcal{P}(t\psi|\mathbb{X})$ gives back the pressure defined above, and $\mathcal{P}(t\psi|\mathbb{X}') \leq \mathcal{P}(t\psi)$ by definition. As in the previous section, we will be interested in the SFTs $\mathbb{X}_n$.

Fix a closed invariant subshift $\mathbb{X}' \subseteq \mathbb{X}$ and, for every $n \in \mathbb{N}$, consider
\[
a_n := \log \sum_{\omega \in \Sigma^n} \sup_{x \in \langle \omega \rangle \cap \mathbb{X}'} \exp(t\psi_n(x)).
\]
Now, the sequence \((a_n)_{n \in \mathbb{N}}\) is finite and subadditive, which follows by
\[
a_{n+k} = \log \sum_{\omega \in \Sigma^{n+k}} \sup_{x \in (\omega) \cap X'} \exp(t\psi_{n+k}(x)) \\
\leq \log \left( \sum_{\omega \in \Sigma^n} \sup_{x \in (\omega) \cap X'} \exp(t\psi_n(x)) \sum_{\omega \in \Sigma^k} \sup_{x \in (\omega) \cap X'} \exp(t\psi_k(x)) \right) = a_n + a_k.
\]

This guarantees that the limit in the definition of \(\mathcal{P}(t\psi|X')\) always exists and is given by the infimum, so we can use
\[
\mathcal{P}(t\psi|X') := \inf_{n \in \mathbb{N}} \frac{1}{n} \log \sum_{\omega \in \Sigma^n} \sup_{x \in (\omega) \cap X'} \exp(t\psi_n(x)). 
\]

**Proposition 5.1.** For \(m \in \mathbb{N}\), consider the function \(p_m : \mathbb{R} \rightarrow \mathbb{R}\) defined by \(t \mapsto \mathcal{P}(t\psi|X_m)\). Then, for each \(t \in \mathbb{R}\), one has
\[
\mathcal{P}(t\psi) = \lim_{m \to \infty} p_m(t).
\]

**Proof.** Since \(\mathcal{P}(t\psi|X_m)\) is monotonically increasing in \(m\), the limiting expression in Proposition 5.1 exists. From the fact that \(\mathcal{P}(t\psi|X_m) \leq \mathcal{P}(t\psi)\) for every \(m \in \mathbb{N}\), we obtain
\[
\mathcal{P}(t\psi) \geq \limsup_{m \to \infty} \mathcal{P}(t\psi|X_m).
\]

Next, we prove \(\limsup_{m \to \infty} \mathcal{P}(t\psi) - \mathcal{P}(t\psi|X_m) \leq 0\) in several steps. Our approach is to find, for each word \(\omega \in \Sigma^n \setminus \Sigma^n_m\), a corresponding word \(\omega' \in \Sigma^n_m\), and compare \(\sup_{x \in (\omega)} t\psi_n(x)\) with \(\sup_{x \in (\omega') \cap X_m} t\psi_n(x)\). This will be done in Lemma 5.3. Furthermore, for a given \(\omega' \in \Sigma^n_m\), we will estimate the number of words \(\omega \in \Sigma^n \setminus \Sigma^n_m\) which will be compared with \(\omega'\). This will be done in Lemma 5.2.

To construct such an \(\omega'\) for \(\omega = \omega_1 \cdots \omega_n\), we use the following algorithm. Start at \(\omega_1\). Look at the first letter where 0 or 1 has appeared \((m+1)\) times in a row. Say this happens at \(\omega_j\). Then, build the new word \(\tilde{\omega} := \omega_1 \cdots \omega_{j-1} \omega_j \cdots \omega_n\). Repeat the algorithm with \(\tilde{\omega}\) and keep repeating until the final word \(\omega'\) lies in \(\Sigma^n_m\). We denote the map given by this algorithm by \(h\), so \(h(\omega) = \omega'\).

**Lemma 5.2.** For \(\omega' \in \Sigma^n_m\), one has \(\operatorname{card} \{\omega \in \Sigma^n \setminus \Sigma^n_m : h(\omega) = \omega'\} < 2^{|n/m|}\).

**Proof.** Each \(\omega' \in \Sigma^n_m\) contains at most \(\lfloor n/m \rfloor\) single-letter subwords of length \(m\). The following algorithm gives a possibility to find pre-images of \(\omega'\). Let \(\gamma(1), \ldots, \gamma(i)\) be the integers such that \(\omega'_{\gamma(j)-m} = \ldots = \omega'_{\gamma(j)-1}\) for some \(1 \leq j \leq i\), with \(i\) the total number of such sequences. We denote by \(v_{i,1}\) the identity and set \(v_{j,2}(\omega') := \omega'_{\gamma(j)-i} \cdots \omega'_{\gamma(j)-1} \cdots \omega'_{\gamma(j)-1} \cdots \omega'\). Then, we have that \(v_{i,k_1} \circ \cdots \circ v_{i,k_t}(\omega')\) with \(k_\ell \in \{1, 2\}\) are all pre-images of \(h\). Since there are \(2^i\) possibilities to choose \(\{k_1, \ldots, k_t\}\), there are \(2^i\) pre-images (with all but one in \(\Sigma^n \setminus \Sigma^n_m\)).

**Lemma 5.3.** For any pair \(\omega \in \Sigma^n \setminus \Sigma^n_m\) and \(\omega' = h(\omega) \in \Sigma^n_m\), we have
\[
\sup_{x \in (\omega') \cap X_m} \psi_n(x) + 4 \lfloor n/m \rfloor + 2^{m+2} \geq \sup_{y \in (\omega)} \psi_n(y).
\]
Proof. Clearly, we have

\[ \sup_{y \in (\omega^\ell)} \psi_n(y) - \sup_{x \in (\omega^\ell) \cap \mathcal{X}_m} \psi_n(x) \leq \sum_{\ell=0}^{n-1} \sup_{y \in (2^\ell \omega^\ell) \cap \mathcal{X}_m} (\psi(y) - \psi(z)), \]

where for \( \omega = \omega_1 \cdots \omega_n \) we have used the notation \( 2^\ell \omega = \omega_{1+\ell} \cdots \omega_n \), for \( 0 \leq \ell \leq n - 1 \).

In the next steps, we aim to estimate

\[ \sup_{y \in (2^\ell \omega)} \psi(y) - \psi(z), \]

separately for each \( 0 \leq \ell \leq n - 1 \). Let \( \gamma(j) \) be the position of the first digit which gets inverted \( j \) times by the above algorithm.

With \( i \) defined as in the proof of Lemma 5.2, it is \( i \leq \lfloor n/m \rfloor \). For \( \ell \in \{0, \ldots, \gamma(1) - 2 - m\} \), we have

\[ \sup_{y \in (2^\ell \omega)} \psi(y) - \psi(z) \leq \sup_{y,z \in (\omega_{\ell+1} \cdots \omega_{\gamma(1)-1})} (\psi(y) - \psi(z)) \]

\[ = \sup_{y \in (\omega_{\ell+1} \cdots \omega_{\gamma(1)-1})} \psi(y) - \inf_{z \in (\omega_{\ell+1} \cdots \omega_{\gamma(1)-1})} \psi(z). \]

Here, we have \( (\omega_{\ell+1} \cdots \omega_{\gamma(1)-1}) \subset \Sigma_2^{(1)} \) and, by the choice of \( \ell \), we also have the inequality \( \gamma(1) - \ell - 1 > m \). This implies for all \( x \in (\omega_{\ell+1} \cdots \omega_{\gamma(1)-1}) \) with \( \ell \in \{0, \ldots, \gamma(1) - 2 - m\} \) that \( x \in [2^{-m-1}, 1 - 2^{-m-1}] =: J_m \), which follows directly by considering the dual representation of the unit interval.

For the next estimate, we use the elementary formula

\[ |\psi(x + h) - \psi(x)| \leq h \max_{u \in [x,x+h]} |\psi'(u)|. \]

For \( y, z \) as above (i.e., the points where the supremum and the infimum are attained), we have \( |y - z| \leq 2^{-\gamma(1) + \ell + 1} \). Since \( y, z \in J_m \), we see via Lemma 4.1 that \( |\psi'(y)| \leq 2^{m+2} \). Combining these observations gives

\[ \sup_{y \in (2^\ell \omega)} \psi(y) - \psi(z) \leq 2^{m+\ell+3-\gamma(1)}. \]

Recall that \( \psi(x) = \psi(\tilde{x}) \) holds for all \( x \in \mathbb{X} \). In analogy to above, for any choice of \( \ell \in \{\gamma(j) - 1, \ldots, \gamma(j+1) - 2 - m\} \) with \( j \in \{1, \ldots, i - 1\} \), we then have

\[ \sup_{y \in (2^\ell \omega)} \psi(y) - \psi(z) \leq \sup_{y,z \in (\omega_{\ell+1} \cdots \omega_{\gamma(j+1)-1})} (\psi(y) - \psi(z)) \leq 2^{m+\ell+3-\gamma(j+1)}. \]

When \( \ell \in \{\gamma(j) - 1 - m, \ldots, \gamma(j) - 2\} \) with \( j \in \{1, \ldots, i\} \), the word \( 2^\ell \omega \) starts with a consecutive sequence of a single letter that is one digit longer than the corresponding sequence in \( 2^\ell \omega' \). Note that the consecutive letter in \( 2^\ell \omega \) and in \( 2^\ell \omega' \) does not have to be the same but
this issue can be handled by the fact that \( \psi(x) = \psi(\hat{x}) \). By the monotonicity on \([0, \frac{1}{2}]\) and on \([\frac{1}{2}, 1]\), we obtain
\[
\sup_{y \in (2^\ell \omega)} \left( \psi(y) - \psi(z) \right) \leq 0.
\]

Finally, we consider \( \ell \in \{ \gamma(i) - 1, \ldots, n - 1 \} \) with \( \gamma(i) \) as above. This gives
\[
\sup_{y \in (2^\ell \omega)} \left( \psi(y) - \psi(z) \right) = \sup_{y \in (2^\ell \omega')} \inf_{z \in (2^\ell \omega \cap \mathcal{X}_m)} \psi(z).
\]

We note that each of the words \( \omega_{\ell+1} \cdots \omega_n \) with \( \ell \in \{ \gamma(i) - 1, \ldots, n - 1 \} \) starts with at most \( m \) times the same digit. If \( \ell < n - m \), the word \( \omega_{\ell+1} \cdots \omega_n \) is longer than \( m \) digits, which implies that the supremum of \( \psi \) on the cylinder \( \langle \omega_{\ell+1} \cdots \omega_n \rangle \) is attained in \( J_m \).

If \( \ell \geq n - m \), the supremum of \( \psi \) on the cylinder \( \langle \omega_{\ell+1} \cdots \omega_n \rangle \) is attained at the word starting with \( \omega_{\ell+1} \cdots \omega_n \) followed by an infinite sequence of the letter \( \widehat{\omega}_{\ell+1} \). This follows from the fact that \( \psi \) is monotonically increasing on the interval \([0, \frac{1}{2}]\) and monotonically decreasing on \([\frac{1}{2}, 1]\). This implies in particular that the supremum is also attained on \( J_m \) in this case. By the restriction \( z \in \mathcal{X}_m \) for the infimum, it follows directly that \( z \in J_m \). Thus, for \( \ell \in \{ \gamma(i) - 1, \ldots, n - 1 \} \), an analogous argument as above gives
\[
\sup_{y \in (2^\ell \omega)} \left( \psi(y) - \psi(z) \right) \leq 2^{m+\ell+2-n}.
\]

As usual, for \( \gamma' < \gamma \), we define the sum \( \sum_{\ell=\gamma}^{\gamma'} \) over any quantity to be zero. Putting things together, with the integer \( i \) from above, we get
\[
\sum_{\ell=0}^{n-1} \sup_{y \in (2^\ell \omega)} \left( \psi(y) - \psi(z) \right)
\]

\[
\leq \sum_{\ell=0}^{\gamma(1)-m-2} 2^{m+\ell+3-\gamma(1)} + \sum_{j=1}^{i-1} \sum_{\ell=\gamma(j)-1}^{\gamma(j+1)-2-m} 2^{m+\ell+1-\gamma(j+1)} + \sum_{\ell=\gamma(i)-1}^{n-1} 2^{m+\ell+2-n}
\]

\[
\leq \sum_{\ell'=0}^{\infty} 2^{-\ell'+1} + \left( \sum_{j=1}^{i-1} \sum_{\ell'=0}^{\infty} 2^{-\ell'+1} \right) + \sum_{\ell'=0}^{\infty} 2^{-\ell'+m+1} \leq 4i + 2^{m+2} \leq 4\lfloor n/m \rfloor + 2^{m+2}.
\]

For the last estimate, we used the fact that \( i \leq \lfloor n/m \rfloor \), as in the proof of Lemma 5.2. \( \Box \)

If we apply Lemmas 5.2 and 5.3 for \( t > 0 \), we obtain
\[
\sum_{\omega \in \Sigma_n} \sup_{y \in (\omega)} \exp(t \psi_n(x)) \leq \sum_{\omega' \in \Sigma_n} 2^{\lfloor n/m \rfloor} \exp(t(4 \lfloor n/m \rfloor + 2^{m+2})) \sup_{x \in (\omega') \cap \mathcal{X}_m} \exp(t \psi_n(x)).
\]
This implies
\[
\mathcal{P}(t\psi) \leq \lim_{n \to \infty} \frac{1}{n} \log \left( 2^{n/m} \exp(t(2^{m+2} + 4[n/m])) \sum_{\omega' \in \Sigma^n} \sup_{x \in (\omega') \cap X_m} \exp(t\psi_n(x)) \right)
\]
\[= \lim_{n \to \infty} \frac{|n/m| \log(2) + t(2^{m+2} + 4[n/m])}{n} \sup_{x \in (\omega') \cap X_m} \exp(t\psi_n(x))
\]
\[= \frac{\log(2) + 4t}{m} + \mathcal{P}(t\psi|X_m).
\]
Consequently, \( \limsup_{m \to \infty} \mathcal{P}(t\psi) - \mathcal{P}(t\psi|X_m) \leq 0. \)

6. Proof of the first part of Theorem 1.1

If we restrict our attention to any of the SFTs \(X_m\), there is a natural correspondence between the local dimension of \(\nu\) and the Birkhoff average of \(\psi\) due to Corollary 4.5. Moreover, standard multifractal formalism for H"older continuous potentials, compare [12], provides us with the relation
\[
\dim_{H,\tau} \left\{ x \in X_m : \lim_{n \to \infty} \frac{\psi_n(x)}{n} = \alpha \right\} = \max \left\{ \frac{-p_m^*(\alpha)}{\log(2)}, 0 \right\},
\]
which holds for both \(\tau \in \{E,2\}\) because the metric spaces with the two different metrics restricted to \(X_m\) are bi-Lipschitz equivalent. To show Theorem 1.1, we need to make sure that these relations extend properly to the full shift \(X\). Intuitively, \(X_m\) captures the relevant information about \(\nu\) (in the limit of large \(m\)) because it is designed in such a way that, given \(\omega \in \Sigma^n\), \(\nu(\langle \omega \rangle)\) is small if \(\langle \omega \rangle \cap X_m = \emptyset\). In parts, this is made more precise by Proposition 5.1 showing that the restricted pressure function indeed converges to the full pressure function. Some additional work is necessary to ensure that the information which we obtain from \(X_m\) with \(m \in \mathbb{N}\) is consistent with the Hausdorff dimension of the full level sets.

In what follows, we will prove the two identities from Theorem 1.1. We postpone the proof of all the properties of the spectrum to Section 8, as we shall need further properties of the pressure function \(p\). The latter will be derived in Section 7.

Proof of the first part of Theorem 1.1. Let us begin with proving the upper bounds in (5) and (6) by giving an upper bound for the Hausdorff dimension of a sufficiently large superset.

For arbitrary \(n \in \mathbb{N}\), we introduce a neighbour relation on the set of words \(\Sigma^n\) as follows. First, \(\omega\) is called a neighbour of \(\omega'\) if the intersection of \(\langle \omega \rangle\) and \(\langle \omega' \rangle\) as subsets of \([0,1]\) consists of precisely one point, that is, if the corresponding intervals are adjacent in \([0,1]\). Then, for each \(n \geq 2\) and \(x \in X\), we can choose a neighbour \(\omega_{x}^{n-1}\) of \(C_{n-1}(x)\) such that \(B_E(x,2^{-n}) \subset C_{n-1}(x) \cup \langle \omega_{x}^{n-1} \rangle\). Next, let
\[
\bar{\psi}_n(x) := \log \left( \sup_{z \in C_{n-1}(x)} \exp(\psi_{n-1}(z)) + \sup_{z' \in \langle \omega_{x}^{n-1} \rangle} \exp(\psi_{n-1}(z')) \right),
\]
and, for \(n \in \mathbb{N}\) and \(\alpha \in \mathbb{R}\), consider the set
\[
\mathcal{G}(n, \alpha) := \{ x \in T : \bar{\psi}_n(x) - n\alpha > 0 \}.\]
For \( \alpha \in \mathbb{R} \) and \( \varepsilon > 0 \), we define \( \Gamma_n := \{ \omega \in \Sigma^n : \mathcal{G}(n, \alpha - \varepsilon) \cap \langle \omega \rangle \neq \emptyset \} \). Then, for fixed \( q \geq 0 \) and any \( s > \frac{q(\alpha - \varepsilon) - \mathcal{P}(q\psi)}{\log(2)} \), we obtain

\[
\sum_{n>m} \sum_{\omega \in \Gamma_n} |\langle \omega \rangle|^s = \sum_{n>m} \sum_{\omega \in \Gamma_n} e^{-sn \log(2)} \leq \sum_{n>m} \sum_{\omega \in \Gamma_n} e^{-sn \log(2)} \sup_{x \in \langle \omega \rangle} (q\psi_n(x) - nq(\alpha - \varepsilon)) \\
\leq \sum_{n>m} \sum_{\omega \in \sum_n} \sup_{x \in \langle \omega \rangle} (-sn \log(2) + q\psi_n(x) - nq(\alpha - \varepsilon)) \\
\leq \sum_{n>m} \exp\left(-n\left(s \log(2) + q(\alpha - \varepsilon) - \frac{1}{n} \log \sum_{\omega \in \sum_{n-1}} \sup_{x \in \langle \omega \rangle} e^{q\psi_{n-1}(x)}\right)\right) \\
m \to \infty \to 0,
\]

where we have used the definition of \(-\mathcal{P}(q\psi)\) and the fact that any \( \omega \in \Sigma_n \) can be a neighbour of at most two distinct words in \( \Gamma_{n-1} \). Consequently, the \( s \)-dimensional Hausdorff measure (with respect to both \( g_E \) and \( g_2 \)) equals zero. Since this holds for an arbitrary \( s > \frac{q(\alpha - \varepsilon) - \mathcal{P}(q\psi)}{\log(2)} \) with \( q \geq 0 \), we may conclude for the Hausdorff dimension of the \( \limsup \)-set that

\[
\dim_{H,s} \limsup_{n \to \infty} \mathcal{G}(n, \alpha - \varepsilon) \leq \inf_{q \geq 0} \frac{q(\alpha - \varepsilon) - \mathcal{P}(q\psi)}{\log(2)} = \frac{p^\tau(\alpha - \varepsilon)}{\log(2)}.
\]

Let us proceed to the final step. Making use of the immediate inequality \( \tilde{\psi}_n \geq \psi_{n-1} \), and combining the inclusions \( B_2(x, 2^{-n}) \subseteq B_E(x, 2^{-n}) \subseteq C_{n-1}(x) \cup \langle \omega_n \rangle \) with the inequality \( \nu(C_n(x)) \leq 2^{-n} \sup_{y \in C_n(x)} \exp(\psi_n(y)) \) from Lemma 4.3, we can deduce that, for each \( \varepsilon > 0 \), each of the sets

\[
\left\{ x \in \mathbb{T} : \limsup_{n \to \infty} \frac{\psi_n(x)}{n} \geq \alpha \right\} \quad \text{and} \quad \left\{ x \in \mathbb{T} : \liminf_{n \to \infty} \frac{\log(\nu(B_\tau(x, 2^{-n})))}{\log(2)} \leq 1 - \frac{\alpha}{\log(2)} \right\},
\]

for \( \tau \in \{2, E\} \), is a subset of \( \limsup_{n \to \infty} \mathcal{G}(n, \alpha - \varepsilon) \). Hence, by the continuity of the Legendre transform, we obtain via \( \varepsilon \to 0 \) the desired upper bounds in (5) and (6).

For the lower bound, fix \( m \in \mathbb{N} \). Then, for \( \alpha \leq \log\left(\frac{2}{2}\right) \) and \( \tau \in \{E, 2\} \), we have

\[
b_\tau(\alpha) \geq \dim_{H,\tau}\left\{ x \in \mathbb{X}_m : \lim_{n \to \infty} \frac{\psi_n(x)}{n} = \alpha \right\} = \max\left\{ \frac{-p^\tau(\alpha)}{\log(2)}, 0 \right\} \quad \text{max} \to \infty \to \max\left\{ \frac{-p^\tau(\alpha)}{\log(2)}, 0 \right\},
\]

where the equality is due to (24) and the convergence part follows from \( \lim_{m \to \infty} p_m(t) = p(t) \), see Proposition 5.1, in conjunction with [12, Thm 1.8]. For the remaining case, we use Corollary 4.5 to deduce that, for any \( \tau \in \{E, 2\} \) and \( m \in \mathbb{N} \), one has

\[
\mathcal{F}_\tau(\alpha) \supseteq \left\{ x \in \mathbb{X}_m : \dim_{\nu,\tau}(x) = \alpha \right\} = \left\{ x \in \mathbb{X}_m : \lim_{n \to \infty} \frac{\psi_n(x)}{n} = (1 - \alpha)\log(2) \right\},
\]

where \( \mathcal{F}_\tau(\alpha) = \left\{ x \in \mathbb{X} : \dim_{\nu,\tau}(x) = \alpha \right\} \). This yields the lower bound in (6). \( \square \)
7. Further properties of the pressure function

Before we can continue with the proof of Theorem 1.1, we need to establish some further properties of the pressure function \( p \), which might also be of independent interest.

**Proposition 7.1.** The function \( p : \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \) with \( p(t) = \mathcal{P}(t\psi) \) is well defined and convex. The sequence of real-analytic convex functions \( p_m \) from Proposition 5.1 converges pointwise to \( p \); see Figure 4 for an illustration of the graph of \( p \).

Further, the domain \( \{ t \in \mathbb{R} : p(t) < \infty \} \) of \( p \) is equal to \( [0, +\infty) \), and one has

\[
(25) \quad p(0) = \log(2), \\
(26) \quad p(1) = \log(2), \\
(27) \quad \lim_{t \to \infty} p(t) - \log(3/2)t = 0, \\
(28) \quad p'(0+) = -\log(2).
\]

**Proof.** The pointwise convergence follows from Proposition 5.1, while convexity and analyticity of the functions \( p_m \) follow from the general thermodynamic formalism for Hölder continuous functions; compare [7]. Convexity of \( p \) follows from the fact that the pointwise limit of convex functions is again convex.

Next, \( p(t) = \infty \) for \( t < 0 \) follows from

\[
\frac{1}{n} \log \sum_{\omega \in \Sigma_n} \sup_{x \in \langle \omega \rangle} \exp(t\psi_n(x)) \geq \frac{1}{n} \log \sup_{x \in \langle \langle \rangle \rangle} \exp(t\psi_n(x)) = \infty.
\]

The finiteness of \( p \) for \( t \in [0, +\infty) \) follows from the obvious estimate \( p(t) \leq \log(2)(1+t) \) with equality for \( t = 0 \), the latter giving (25).

---

**Figure 4.** The graph of the pressure function \( p \) (solid line) with the two asymptotes \( x \mapsto \log(3/2)x \) and \( x \mapsto (1-x)\log(2) \) (dashed lines). The dotted lines are added to illustrate \( p(0) = p(1) = \log(2) \).
To establish (26), we follow the approach of [5, p. 19], where we first prove \( p(1) \geq \log(2) \).

For \( x \in \mathbb{X} \), Lemma 4.3 implies

\[
\frac{\nu(C_n(x))}{\exp(-n \log(2) + \sup_{y \in C_n(x)} \psi_n(y))} \leq 1.
\]

Summing the measure over all possible cylinder sets of length \( n \) gives 1 and thus

\[
\exp(-n \log(2) + \log \sum_{\omega \in \Sigma^n} \sup_{x \in \langle \omega \rangle} \exp(\psi_n(x))) \geq 1,
\]

and we get

\[
\frac{1}{n} \log \sum_{\omega \in \Sigma^n} \sup_{x \in \langle \omega \rangle} \exp(\psi_n(x)) \geq \log(2).
\]

Letting \( n \to \infty \) then gives the first inequality.

Next, we prove \( p(1) \leq \log(2) \). Let \( x \in \mathbb{X}_m \) and \( y \in C_n(x) \cap \mathbb{X}_m \). Then, a combination of Lemmas 4.2 and 4.3 shows that there exists an \( R > 0 \) such that

\[
\frac{\nu(C_n(x))}{\exp(-n \log(2) + \sup_{y \in C_n(x) \cap \mathbb{X}_m} \psi_n(y))} \geq R.
\]

Summing the measure of all possible cylinder sets in \( \mathbb{X}_m \) of length \( n \) gives a number \( \leq 1 \).

Consequently,

\[
R \exp(-n \log(2) + \log \sum_{\omega \in \Sigma^n} \sup_{x \in \langle \omega \rangle \cap \mathbb{X}_m} \exp(\psi_n(x))) \leq 1,
\]

which implies

\[
\frac{1}{n} \log \sum_{\omega \in \Sigma^n} \sup_{x \in \langle \omega \rangle \cap \mathbb{X}_m} \exp(\psi_n(x)) \leq \log(2) - \frac{\log(R)}{n}.
\]

Letting \( n \to \infty \) gives \( P(\psi|\mathbb{X}_m) \leq \log(2) \). Proposition 5.1 now gives the second inequality.

Let us next establish (27). Observe first that, by an application of Proposition 3.1, we have

\[
p(t) \geq \lim_{n \to \infty} \frac{1}{n} \log \exp(t \psi_n(01)) = \log(3/2) t
\]

for all \( t > 0 \). Hence, it remains to show

\[
\lim_{t \to \infty} p(t) - \log(3/2) t \leq 0.
\]

We will prove this inequality in a series of lemmas, following some ideas developed in [17].

**Lemma 7.2.** Let \( y \) be as in Eq. (10). Then, there exists an \( \varepsilon > 0 \) such that

\[
(\psi + \psi^{(2)})(y) \geq (\psi + \psi^{(2)})(011z) + \varepsilon
\]

holds for all \( z \in \mathbb{X} \). Furthermore, for all \( k \in \mathbb{N} \) and \( z \in \mathbb{X} \), we have

\[
(\psi + \psi^{(2)})(y) \geq (\psi + \psi^{(2)})(y^{[2k]}1z).
\]

**Proof.** The proof can be done similarly to that of Lemma 3.2. An argument analogous to the one leading to (12) yields \( \psi^{(2)}(y) = \psi(y) \) and \( \psi^{(2)}(y^{[2k]}1z) = \psi(y^{[2k-1]}0z) \). Consequently, the two estimates claimed are equivalent to

\[
\psi(y^{[2k-1]}0z) + \psi(y^{[2k]}1z) \leq 2\psi(y) - W,
\]
where \( W = \varepsilon \) if \( k = 1 \) and \( W = 0 \) otherwise.

Observe that we have
\[
\psi(y^{[2k-1]}0\overline{z}) + \psi(y^{[2k]}1z) = \psi(y^{[2k-1]}0\overline{z}) + \psi(y^{[2k-1]}11z) \\
\leq \psi(y^{[2k-1]}0\overline{U}) + \psi(y^{[2k-1]}11\overline{U}) = \psi(y^{[2k]}\overline{U}) + \psi(y^{[2k]}1\overline{U}) \\
= 2\left(\frac{1}{3}\psi(y^{[2k]}\overline{U}) + \frac{2}{3}\psi(y^{[2k]}1\overline{U})\right) - W_k \leq 2\psi(y) - W_k,
\]
where the constant \( W_k \) is given by
\[
W_k = \frac{1}{3}\left(\psi(y^{[2k]}1\overline{U}) - \psi(y^{[2k]}\overline{U})\right) > 0.
\]

The first inequality now follows from the facts that the cylinder \( \langle y^{[2k-1]}0 \rangle \) is coarser than \( \langle y^{[2k-1]}11 \rangle \) and that \( \psi' \) is positive and decreasing on \([0, \frac{1}{2}]\). The last step is then a consequence of the concavity of \( \psi \) together with the observation that \( y = \frac{1}{3}y^{[2k]}\overline{0} + \frac{2}{3}y^{[2k]}1\overline{0} \).

**Corollary 7.3.** Let \( y \) be the alternating sequence from Eq. (10) and let \( \varepsilon > 0 \) be the constant from Lemma 7.2. Then, for all \( n \in \mathbb{N} \) and \( z \in \mathbb{X} \), one has
\[
\psi_n(y) \geq \psi_n(y^{[n]}\overline{y}_{n+1}z) + \varepsilon.
\]

**Proof.** The proof is similar to that of Corollary 3.3. To prove our claim, we employ Lemma 7.2 for \( n = 1 \). Analogously as in (13), we obtain
\[
\psi_{2m}(y) = \left( (\psi + \psi^{(2)}) + (\psi^{(3)} + \psi^{(4)}) + \ldots + (\psi^{(2m-1)} + \psi^{(2m)}) \right)(y) = m(\psi + \psi^{(2)})(y) \\
\geq (\psi + \psi^{(2)})(y^{[2m]}1z) + (\psi + \psi^{(2)})(y^{[2m-2]}1z) + \ldots + (\psi + \psi^{(2)})(y^{[2]}1z) + \varepsilon \\
= \psi_{2m}(y^{[2m]}1z) + \varepsilon.
\]
From this result, we can also proceed to \( n = 2m + 1 \) with \( m \in \mathbb{N} \),
\[
\psi_{2m+1}(y) = \psi_{2m}(\overline{y}) + \psi(y) = \psi_{2m}(y) + \psi(y) \geq \psi_{2m}(y^{[2m]}1\overline{z}) + \varepsilon + \psi(y^{[2m+1]}0\overline{z}) \\
= \psi_{2m}(\overline{y^{[2m]}0\overline{z}}) + \varepsilon + \psi(y^{[2m+1]}0\overline{z}) = \psi_{2m+1}(y^{[2m+1]}0\overline{z}) + \varepsilon,
\]
which completes the argument. \( \square \)

Next, we split the set \( \Sigma^n \) into the sets \( D_{\ell,n} \), where
\[
D_{\ell,n} := \left\{ a^{r_1}I(a^{r_2}) \cdots I^{r_{\ell-2}}(a^{r_{\ell-1}})I^{r_{\ell-1}}(a^{r_{\ell}}) : a \in \{0,1\}, r_k \in \mathbb{N}, \sum_{k=1}^{\ell} r_k = n \right\}
\]
with \( I \) defined as in Proposition 3.4. Put differently, we consider words of length \( n \) that consist of \( \ell \) blocks of consecutive 0s and 1s. With the help of Corollary 7.3, we can now prove the following result.

**Lemma 7.4.** There exist constants \( K, \delta > 0 \) such that
\[
\max_{\omega \in D_{\ell,n}} \sup_{x \in (\omega)} \psi_n(x) \leq K + n \log(3/2) - \delta(n - \ell)
\]
holds uniformly for all \( n, \ell \in \mathbb{N} \) with \( \ell \leq n \).
Proof. Let \( \omega \in D_{\ell,n} \), so \( \text{card}\{i \leq n : \omega_{i-1} = \omega_i\} = n - \ell \). We set \( s(0) := s(0, \omega) := 0 \), together with \( s(k + 1) := s(k + 1, \omega) := \min \{i > s(k) : \omega_i = \omega_{i+1}\} \) for \( k < n - \ell \) and \( s(n - \ell + 1) := n \). Then, we have

\[
\sup_{x \in (\omega)} \psi_n(x) \leq \sum_{k=1}^{n-\ell+1} \sup_{x \in \sigma^{s(k-1)}(\omega)} \psi_{s(k) - s(k-1)}(x).
\]

By the definition of \( s \) and the symmetry of \( \psi \), we have

\[
\sup_{x \in \sigma^{s(k-1)}(\omega)} \psi_{s(k) - s(k-1)}(x) \leq \sup_{x \in X} \psi_{s(k) - s(k-1)}(y^{[s(k) - s(k-1)]} \hat{\gamma}_{s(k) - s(k-1) + 1} x)
\]

for any \( k \leq n - \ell \). Next, when \( s(k) - s(k - 1) = 1 \), we get

\[
\sup_{x \in \sigma^{s(k-1)}(\omega)} \psi_{s(k) - s(k-1)}(x) \leq \sup_{x \in X} \psi(00x) = 0.
\]

If \( s(k) - s(k - 1) > 1 \), Corollary 7.3 implies

\[
\sup_{x \in \sigma^{s(k-1)}(\omega)} \psi_{s(k) - s(k-1)}(x) \leq \psi_{s(k) - s(k-1)}(y) - \varepsilon = (s(k) - s(k - 1)) \log(3/2) - \varepsilon.
\]

Finally, Corollary 3.6 and Lemma 3.7, with the constant \( K > 0 \) defined there, give

\[
\sup_{x \in \sigma^{s(n-\ell)}(\omega)} \psi_{n-s(n-\ell)}(x) = \sup_{x \in X} \psi_{n-s(n-\ell)}(y^{[n-s(n-\ell)]} x) = \sup_{x \in X} \psi_{n-s(n-\ell)}(x)
\]

\[
\leq \sup_{x \in X} \psi_{n-s(n-\ell)}(01) + K = (n - s(n - \ell)) \log(3/2) + K.
\]

Combining these five estimates and setting \( \delta = \min \{\varepsilon, \log(3/2)\} \) yields

\[
\sup_{x \in (\omega)} \psi_n(x) \leq \sum_{k=1}^{n-\ell} \left( (s(k) - s(k - 1)) \log(3/2) - \delta \right) + (n - s(n - \ell)) \log(3/2) + K
\]

\[
= n \log(3/2) - (n - \ell) \delta + K,
\]

and thus the statement of the lemma. \( \square \)

For the final step, we set \( \gamma := \log(3/2) \). Using Lemma 7.4 together with \( \text{card}(D_{\ell,n}) = \binom{n-1}{\ell-1} \) to justify the first inequality gives

\[
\frac{1}{n} \log \sum_{\omega \in \Sigma^n} \sup_{x \in (\omega)} \exp(t \psi_n(x)) - t \gamma = \frac{1}{n} \log \sum_{\ell=1}^{n} \sum_{\omega \in D_{\ell,n}} \sup_{x \in (\omega)} \exp(t \psi_n(x)) - t \gamma
\]

\[
\leq \frac{tK}{n} + \frac{1}{n} \log \sum_{\ell=1}^{n} \left( \frac{n-1}{\ell-1} \right) \exp(-(n - \ell) \delta t)
\]

\[
= \frac{tK}{n} + \frac{1}{n} \log(1 + \exp(-\varepsilon t))^{n-1} = \frac{tK}{n} + \frac{n-1}{n} \log(1 + \exp(-\delta t)).
\]

Letting first \( n \to \infty \) and then \( t \to \infty \) gives the desired result.

Let us finally prove (28). To see that \( p'(0) \geq -\log(2) \), we note that, by Theorem 1.1, the inequality \( p'(0) < -\log(2) \) would imply that \( b(-\log(2)) < 1 \), which contradicts the fact that \( B(-\log(2)) \) as defined in Eq. (3) has full Lebesgue measure.
Hence, it remains to show that \( p'(0) \leq -\log(2) \). Recall the definition of \( \psi^D(\ell) \) from (19) and set \( \psi^D(\ell)(x) := \sum_{k=1}^n \psi^D(\ell)(2^{k-1}x) \), where \( W_\ell \) denotes the bounded distortion constant for \( \psi^D(\ell) \). For \( \omega \in \Sigma^n \), we define \( \psi_n,\omega := \sup_{x \in (\omega)} \psi_n(x) \) and \( \psi_n,\omega := \inf_{x \in (\omega)} \psi_n(x) \), and analogously \( \psi^D_n,\omega \) and \( \psi^D_n,\omega \). Eq. (23) and the properties of the Hölder mean then imply that, for all \( n, \ell \in \mathbb{N} \),

\[
\begin{align*}
p'(0+) & \leq \inf_{t \geq 0} \frac{n^{-1} \sum_{\omega \in \Sigma^n} \exp(t \psi_n,\omega) - \log(2)}{t} \\
& = \inf_{t \geq 0} \frac{1}{n} \log \left( 2^{-n} \sum_{\omega \in \Sigma^n} \exp(\psi_n,\omega)^t \right)^{1/t} = \frac{1}{n} \log \left( \prod_{\omega \in \Sigma^n} \exp(\psi_n,\omega) \right)^{2^{-n}} \\
& = \frac{1}{n^2} \sum_{\omega \in \Sigma^n} \psi_n,\omega \leq \frac{1}{n^2} \sum_{\omega \in \Sigma^n} \psi^D_n,\omega \leq \frac{1}{n^2} \sum_{\omega \in \Sigma^n} (\psi^D_n,\omega + W_\ell) \\
& \leq \frac{1}{n} \int \psi^D_n d\lambda + \frac{W_\ell}{n} = \int \psi^D d\lambda + \frac{W_\ell}{n}.
\end{align*}
\]

Now, letting first \( n \) tend to infinity and then \( \ell \) gives the desired inequality. \( \square \)

**Remark 7.5.** The graph of the pressure function in Figure 4 was generated using approximants of the form

\[
p_n(t) = \frac{1}{n - 2} \log \sum_{j=1}^{2^{n-2}} \left( \frac{1}{2} \exp \left( \psi_n((2j-1)2^{-n}) \right) \right)^t,
\]

which can be shown to converge to \( p(t) \) by an explicit calculation. Thus, we have reduced the number of evaluations on cylinders by a factor of 4 using symmetry, and we have avoided singular points. The modification from \( n \) to \( n - 2 \) in the denominator and the additional factor \( \frac{1}{2} \) in front of the exponential ensure the normalisations \( p_n(0) = \log(2) \) and \( p_n(1) = \log(2) \) for all \( n \), respectively. It turns out that these approximants exhibit a considerably faster convergence than approximants without the above normalisations. Since the computational cost increases exponentially in \( n \), this is practically relevant. Note that this approach is equivalent to the numerics presented in [11]. \( \diamond \)

8. THE REMAINING PARTS OF THE PROOF OF THEOREM 1.1

Proposition 7.1 enables us to prove the remaining claims of Theorem 1.1. We are left to show that the Birkhoff spectrum \( b \) satisfies the following properties:

(B1) The function \( b \) is concave on \( (-\infty, \log(3/2)] \) and vanishes in the right boundary point of this interval.

(B2) The level sets \( \mathcal{B}(\alpha) \) are empty for \( \alpha > \log(3/2) \).

(B3) We have \( b(\alpha) = 1 \) for \( \alpha \leq -\log(2) \) and \( b(\alpha) < 1 \) for \( \alpha > -\log(2) \).

(B1): By Proposition 7.1 the pressure function \( p \) is convex and thus its Legendre transform \( p^* \) is convex on its domain of definition given by \( \{ x \in \mathbb{R} : \sup_{t \in \mathbb{R}} (xt - p(t)) < \infty \} \) which is
equal to $(-\infty, \log(3/2)]$. Further, as a consequence of (27), we have
\[
-p^*(\log(3/2)) = \inf_{t \in \mathbb{R}} (p(t) - \log(3/2)t) = 0,
\]
and it follows that $b(\log(3/2)) = 0$.

(B3): The fact that the level sets are empty for $\alpha > \log(3/2)$ follows from Lemma 3.1.

(B3): Clearly, $b(\alpha) \leq 1$ for all $\alpha \in \mathbb{R}$ by definition. On the other hand, for $\alpha \leq -\log(2)$, we have
\[
b(\alpha) \geq \frac{-p^*(\alpha)}{\log(2)} = \frac{\inf_{t \geq 0} (p(t) - \alpha t)}{\log(2)} \geq \frac{\inf_{t \geq 0} (p(0) - t \log(2) - \alpha t)}{\log(2)} = 1,
\]
where we have used the fact that $p$ is convex on $[0, \infty)$ together with Eqs. (25) and (28). This proves the first claim. The second is an immediate consequence of the definition for the Legendre transform in conjunction with Eqs. (25) and (28).

9. Variational principle and equilibrium measure

So far, we have employed various concepts from the thermodynamic formalism and shown explicitly some results that, in general, were known to hold only for a more restrictive class of potentials. In particular, the relationship between the Birkhoff spectrum and the scaling or $f(\alpha)$-spectrum is of the same form as the one holding for equilibrium measures of Hölder continuous potentials [22, Prop. 1]. It is the aim of this section to show that our results embed nicely into the known formalism in the sense that $\nu$ is indeed an equilibrium measure for the potential $\psi$. Let us expand a bit on this concept. To an upper semi-continuous function $\phi$ on the compact dynamical system $(X, T)$, we assign its variational pressure
\[
P_T(\phi) = \sup_{\mu \in \mathcal{M}_T} (h(\mu) + \mu(\phi)),
\]
where $\mathcal{M}_T$ denotes the set of $T$-invariant probability measures on $X$ and $h(\mu)$ is the metric entropy of $\mu$; compare [16]. A measure $\mu \in \mathcal{M}_T$ that maximises the above expression is called an equilibrium measure.

In fact, our measure $\nu$ satisfies the even stronger property of being a $g$-measure in the sense of Keane [14]. We call a non-negative function on $X$ a $g$-function if
\[
\sum_{x \in T^{-1}(y)} g(x) = 1
\]
holds for all $y \in X$. The associated operator $\varphi_g$ on the space of bounded measurable functions $B(X)$ is defined via
\[
(\varphi_g f)(y) = \sum_{x \in T^{-1}(y)} g(x)f(x)
\]
and, for any $\mu \in \mathcal{M}_T$, its dual operation is given by $(\varphi_g^*\mu)(f) = \mu(\varphi_g f)$. Any Borel probability measure $\mu$ with $\varphi_g^*\mu = \mu$ necessarily lies in $\mathcal{M}_T$ and is called a $g$-measure.

It is a matter of direct calculation to verify that the function $g(x) = \frac{1}{2}(1 - \cos(2\pi x))$ is indeed a $g$-function for the dynamical system $(X, \sigma)$, as well as for $(T, T)$. Similarly, we find that $\nu$ is a $g$-measure with this choice of $g$ by a straightforward computation.
The following characterisation of $g$-measures is classic.

**Fact 9.1** ([19, Thm. 1]). Let $g$ be a $g$-function and $\mu$ a probability measure on $(X, \sigma)$. Then, the following characterisations are equivalent.

1. One has $\varphi_g^* \mu = \mu$.
2. The measure $\mu$ satisfies $\mu \in \mathcal{M}_\sigma$ and is an equilibrium measure for the potential $\log(g)$, with $\mathcal{P}_\sigma(\log(g)) = h(\mu) + \mu(\log(g)) = 0$. \hfill $\Box$

Note that $\log(g) = \psi - \log(2)$. We therefore find

$$\mathcal{P}_\sigma(\psi) = \sup_{\mu \in \mathcal{M}_\sigma} \left( h(\mu) + \mu(\log(g)) + \log(2) \right) = \mathcal{P}_\sigma(\log(g)) + \log(2) = \log(2),$$

and a measure is an equilibrium measure for $\psi$ if and only if it is an equilibrium measure for $\log(g)$ which, in turn, is true if and only if it is a $g$-measure. In [15, Section 2], it was shown that the $g$-measure on the dynamical system $(\mathbb{T}, T)$ with $T: x \mapsto 2x \pmod{1}$, is unique and strongly mixing if $g = 0$ at a single position in $[0,1)$. For a detailed discussion on the existence of $g$-measures for Riesz products, we refer to [8].

**Corollary 9.2.** The Thue–Morse measure $\nu$ is a strongly mixing $g$-measure. In particular, it is the unique equilibrium measure for the potential $\psi$ and

$$\mathcal{P}_\sigma(\psi) = h(\nu) + \nu(\psi) = \log(2)$$

as the corresponding variational pressure. \hfill $\Box$

In fact, by using Eq. (26), it turns out that the notions of variational pressure and topological pressure of $\psi$ coincide. This is well known for Hölder continuous potentials [5, Thm. 1.22]. The relation $h(\nu) + \nu(\psi) = \log(2)$ was also established in [24], using slightly different techniques.

**Remark 9.3.** The uniqueness of $\nu$ as an equilibrium measure can be obtained by generalising the Ruelle–Perron–Frobenius theorem [25] to our potential $\psi$. This provides additional information about the operator $\varphi_g$, such as the fact that 1 is the (up to normalisation) unique eigenfunction to the eigenvalue 1 and the fact that the remainder of the spectrum is contained in a disk around zero of radius strictly smaller than 1. \hfill $\Diamond$

**Remark 9.4.** It is worth noticing that Corollary 9.2 permits us to give a closed form for the metric entropy of $\nu$ in terms of the autocorrelation coefficients; compare [27, Eqs. (19) and (20)]. There, the authors established the relation

$$\lim_{n \to \infty} \nu_n(\psi) = - \log(2) - 2 \sum_{j=1}^{\infty} \frac{\eta(j)}{j},$$

where $\eta(j) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} v_k v_{k+j}$ are the autocorrelation coefficients, with $(v_k)_{k \in \mathbb{N}}$ denoting the one-sided Thue–Morse sequence in $\{\pm 1\}$ that starts with 1; see [3] for background. Note that the relation

$$\nu(\psi) = \lim_{n \to \infty} \nu_n(\psi)$$
can be checked easily. This is due to the fact that the singularities in $\psi$ are relatively ‘weak’ and the measures $\nu_n$ assign rapidly decreasing values to boundary intervals of $[0, 1]$.

Due to the well-known renormalisation equations for $\eta$, compare [3, Sec. 10.1], the series in Eq. (29) can be numerically evaluated with high precision. Following the procedure presented in [27], we find via Corollary 9.2 that

$$h(\nu) = 2 \log(2) + 2 \sum_{j=1}^{\infty} \frac{\eta(j)}{j} \approx 0.50638399544731967430,$$

with a precision of 20 correct digits. Note that this value is related to the information dimension $D_1$, as calculated in [27] and [11], via $h(\nu) = \log(2)D_1$, so $D_1 \approx 0.730$. Our numerical value is a significant improvement over the lower bound for the entropy derived in [21]. It is precise enough to affirmatively answer the question from [24, Sec. 4.4.1] whether the (information) dimension of $\nu$ is strictly larger than its energy exponent $e(\nu) := 1 - \log_2(\kappa)$, with $\kappa = (1 + \sqrt{17})/4$, which gives $e(\nu) \approx 0.643$.

Acknowledgements

MK acknowledges support from the German Research Foundation (DFG), through grant KE 1440/3-1, and would like to thank the Mittag-Leffler institute for its kind hospitality during the research program Fractal Geometry and Dynamics, where valuable discussions with F. Ekström and J. Schmeling took place. This work was also supported by the Research Centre of Mathematical Modelling (RCM$^2$) of Bielefeld University (for TS), and by the DFG Collaborative Research Centre 1283 at Bielefeld (for MB and PG).

References

[1] J.-P. Allouche and J. Shallit, Automatic Sequences, Cambridge University Press, Cambridge (2003).
[2] M. Baake and M. Coons, A natural probability measure derived from Stern’s diatomic sequence, Acta Arithm. 183 (2018) 87–99; arXiv:1706.00187.
[3] M. Baake and U. Grimm, Aperiodic Order. Vol. 1: A Mathematical Invitation, Cambridge University Press, Cambridge (2013).
[4] M. Baake, U. Grimm and J. Nilsson, Scaling of the Thue–Morse diffraction measure, Acta Phys. Pol. A 126 (2014) 431–434; arXiv:1311.4371.
[5] R. Bowen, Gibbs Measures and the Ergodic Theory of Anosov Diffeomorphisms, Springer, Berlin (2008).
[6] Z. Cheng, R. Savit and R. Merlin, Structure and electronic properties of Thue–Morse lattices, Phys. Rev. B 37 (1982) 4375–4382.
[7] M. Denker and M. Kesseböhmer, Thermodynamic formalism, large deviation, and multifractals, in Stochastic Climate Models, P. Imkeller and J.-S. von Storch (eds.), Birkhäuser, Basel (2001), pp. 159–169.
[8] A.-H. Fan, On uniqueness of $G$-measures and $g$-measures, Studia Math. 119 (1996) 255–269.
[9] A.-H. Fan, Multifractal analysis of infinite products, J. Stat. Phys. 5-6 (1997) 1313–1336.
[10] A.O. Gelfond, Sur les nombres qui ont des propriétés additives et multiplicatives données, Acta Arithm. 13 (1968) 259–265.
[11] C. Godrèche and J.M. Luck, Multifractal analysis in reciprocal space and the nature of the Fourier transform of self-similar structures, J. Phys. A: Math. Gen. 23 (1990) 3769–3797.
[12] J. Jaerisch and M. Kesseböhmer, Regularity of multifractal spectra of conformal iterated function systems, Trans. Amer. Math. Soc. 363 (2011) 313–330; arXiv:0902.2473.
[13] S. Kakutani, Strictly ergodic symbolic dynamical systems, in Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, L.M. Le Cam, J. Neyman and E.L. Scott (eds.), University of California Press, Berkeley, CA (1972), pp. 319–326.

[14] M. Keane, Sur les mesures invariantes d’un recouvrement régulier, C.R. Acad. Sci. Paris 272 (1971) A585–A587.

[15] M. Keane, Strongly mixing g-measures, Invent. Math. 16 (1972) 309–324.

[16] G. Keller, Equilibrium States in Ergodic Theory, Cambridge University Press, Cambridge (1998).

[17] M. Kesseböhmer and B.O. Stratmann, A multifractal analysis for Stern–Brocot intervals, continued fractions and Diophantine growth rates, J. Reine Angew. Math. (Crelle) 605 (2007) 133–163; arXiv:math.NT/0509603.

[18] D.H. Kim, L. Liao, M. Rams and B. Wang, Multifractal analysis of the Birkhoff sums of Saint-Petersburg potential; preprint arXiv:1707.06059.

[19] F. Ledrappier, Principe variationnel et systèmes dynamiques symboliques, Z. Wahrscheinlichkeitsth. Verw. Gebiete 30 (1974) 185–202.

[20] W. Parry and M. Pollicott, Zeta functions and the periodic orbit structure of hyperbolic dynamics, Astérisque 187–188 (1999).

[21] L. Peng and T. Kamae, Spectral measure of the Thue–Morse sequence and the dynamical system and random walk related to it, Ergodic Th. & Dynam. Syst. 36 (2016) 1246–1259.

[22] Y. Pesin and H. Weiss, The multifractal analysis of Birkhoff averages and large deviations, in Global Analysis of Dynamical Systems, H.W. Broer, B. Krauskopf and G. Vegter (eds.), IoP Publishing, Bristol and Philadelphia (2001), pp. 419–431.

[23] M. Queffélec, Substitution Dynamical Systems – Spectral Analysis, 2nd ed., LNM 1294, Springer, Berlin (2010).

[24] M. Queffélec, Questions around the Thue–Morse sequence, Unif. Distrib. Th. 13 (2018) 1–25.

[25] D. Ruelle, Statistical mechanics of a one-dimensional lattice gas, Commun. Math. Phys. 9 (1968) 267–278.

[26] P. Walters, Ruelle’s operator theorem and g-measures, Trans. Amer. Math. Soc. 214 (1975) 375–387.

[27] M.A. Zaks, A.S. Pikovsky and J. Kurths, On the generalized dimensions for the Fourier spectrum of the Thue–Morse sequence, J. Phys. A: Math. Gen. 32 (1999) 1523–1530.