The components of the spin operator satisfy the commutation relation (hereafter we put $\hbar = 1$)

$$[S^\alpha, S^\beta] = i\epsilon^{\alpha\beta\gamma}S^\gamma$$

where $\alpha, \beta, \gamma = x, y, z$, and $\epsilon^{\alpha\beta\gamma}$ is the perfect anti-symmetric tensor with $\epsilon^{xyz} = 1$. This means that the spin has the definite handedness, i.e., right-handed nature, and it has been believed that this can never be challenged. Therefore, the handedness of the spin is so fundamental and important for the quantum mechanics, and it is striking if one can change it.

In solids, the magnetization $M$ is the sum of the spin of each electron, i.e., $M = \sum_r S_r$, which satisfies also the commutation relation (CR) $[M^\alpha, M^\beta] = i\epsilon^{\alpha\beta\gamma}M^\gamma$. In the ferromagnetic state, one can replace $M^2$ by the thermodynamic average $\langle M^2 \rangle = M^2 > 0$ on the r.h.s. of the CR, and obtains $[M^\alpha, M^\beta] = iM$ which is analogous to that between the position $x$ and the momentum $p$ of a particle. Namely, the correspondence $x \rightarrow M^x$, $p \rightarrow M^y$ works for the dynamics of the small amplitude fluctuation of the transverse magnetization, i.e., spin wave or magnon, in the ferromagnetic state. With the easy axis anisotropy energy $D$, the Hamiltonian is given by

$$H = D[(M^x)^2 + (M^y)^2]$$

which is equivalent to that of a harmonic oscillator. As can be easily seen from the analogy to the trajectory of $(x, p)$-point in the phase space, the direction of the rotation in $(M^x, M^y)$-space is clockwise.

The above consideration seems to be quite general and never be challenged. However, the influence of the relativistic spin-orbit interaction on the CR has not been studied carefully to the best of authors knowledge. For the central field of force, the spin-orbit interaction can be written as $H_{SO} = \lambda S \cdot L$ with $\lambda$ being the spin-orbit coupling energy and $L$ is the orbital angular momentum. Then, the total angular momentum $J = L + S$ is the conserved observable, which again satisfies the CR $[J^\alpha, J^\beta] = i\epsilon^{\alpha\beta\gamma}J^\gamma$. When $\lambda$ is larger than the crystal field as in the case of rare-earth ions, the magnetization $M$ is given by $M \sim \sum_r J_r$ and satisfy the same CR. Therefore, the above situation does not change.

In transition metal ions, on the other hand, the orbital angular momentum is quenched by the crystal field and band formation, and the spin angular momentum mostly contributes to the magnetization. Furthermore, in the metallic ferromagnet, the CR of the magnetization is determined by the quantum mechanical Berry phase of the many-body wavefunction and shows sometimes highly nontrivial behavior as shown below.

The repulsion between electrons is the origin of the magnetization, because it suppresses the double occupancy of electron on each orbital and hence tends to induce the spin moment. Typically it can be written as

$$H_U = -U \sum_r S_r \cdot S_r$$

where $S_r = \frac{1}{2} \sum_b \epsilon_{\alpha\beta\gamma}^b \sigma_{\alpha\beta}c_{b\beta,r}$ is the spin operator summed over the orbital $b$ on each site $r$. One can introduce the collective degrees of freedom as

$$H_\phi = U \sum_r (\phi_r \cdot \phi_r - 2\phi_r \cdot S_r)$$

Here after integrating over $\phi_r$, we obtain $H_U$. This $\phi_r$ represents the collective dynamics of magnetization, which is described by the effective action obtained by integrating over the electron degrees of freedom.

Considering the small amplitude vibration of $\phi_r$ around the ordered moment, the quadratic effective action is enough as given by

$$S[\phi] = \sum_{n, \kappa} \phi_{\kappa, n}^\dagger (-i\nu_n) \Pi_\kappa^{\alpha\beta} (i\nu_n) \phi_\kappa^{\beta}(i\nu_n)$$

Left-handed Ferromagnet

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The dynamics of the total magnetization in metallic ferromagnet is studied theoretically taking into account the relativistic spin-orbit interaction. It is found that its quantum dynamics is seriously influenced by the band crossings near the Fermi energy, and sometimes the direction of the precession can be reversed from what expected from the commutation relation $[S^x, S^y] = i\hbar S^z$ ($\hbar = 2\pi\hbar$: Planck constant), i.e., the left-handed ferromagnet can be realized.

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with \( \nu_n \) being the Matsubara frequency, and \( \mathbf{k} \) the wave vector. The function \( \Pi_\mathbf{k}^{\alpha\beta}(i\nu_n) \) is given by

\[
\Pi_\mathbf{k}^{\alpha\beta}(i\nu_n) = U\delta^{\alpha\beta} - 2U^2\chi_{\mathbf{k}}^{\alpha\beta}(i\nu_n),
\]

\[
\chi_{\mathbf{k}}^{\alpha\beta}(i\nu_n) = \frac{1}{V} \sum_{m,m',k} \frac{f(\epsilon_{m+k}\mathbf{k}) - f(\epsilon_{m}\mathbf{k})}{\epsilon_{m+k} - \epsilon_{m} - i\nu_n} \times \langle mk|S^\alpha|m'k + \mathbf{k}\rangle\langle m'k + \mathbf{k}|S^\beta|m\rangle,
\]

(6)

where \( U \) is the Coulomb repulsion energy, and \( |mk\rangle \) is the Bloch wavefunction with crystal momentum \( \mathbf{k} \) and band index \( n \) in the ferromagnetic state. It should be noted here that the quantum dynamics of the collective coordinates \( \phi_r \) is influenced by the spin-orbit interaction, which is incorporated in the Bloch wavefunction \( |mk\rangle \).

Then the next question is how we can see the CR for which is incorporated in the Bloch wavefunction band index \( n \). It should be also noted that, near the resonance, the sign of \( \chi \) respectively. It is also expected that, near the resonance, the sign of \( \chi \) changes from negative to positive as the temperature is lowered. This means the effective sign reversal of the commutator \([M^x, M^y]\). This is analogous to the physics of negative-refraction metamaterials, where negative refractive index appears effectively and causes novel phenomena, while the microscopic refractive index is not negative nor does not violate causality [3]. The spin anisotropy energy \( D \) in Eq. (2) which is given by \([\chi_{\mathbf{k}}^{xx}(0) + \chi_{\mathbf{k}}^{yy}(0)]/2 - \chi_{\mathbf{k}}^{zz}(0)\) apart from a constant factor, on the other hand, remains almost constant and positive [4].

Now we demonstrate the nontrivial behavior of \( C^{xy} \) using an effective two-dimensional model for \( t_{2g} \) electrons given as [2],

\[
H = H_0 + H_U - \sum_r \mu_B \mathbf{H} \cdot (g_s S_r + g_I L_r),
\]

\[
H_0 = \sum_r \left[ \lambda_{SO} c_r^\dagger (l \cdot \sigma) c_r + \left\{ \sum_{\mu=\pm} (-t_0) c_r^{\dagger \mu} c_r + (\pm t_1)(c_{xx,r \pm y}^\dagger c_{xy,r} + c_{yy,r \pm x}^\dagger c_{xy,r}) + h.c. \right\} \right],
\]

(9)

where \( \mu_B = |e|/2m_e \) is the Bohr magneton, \( g_s = -2 \), \( g_I = -1 \), and \( c_r^{\dagger l} \) is the column (row) vector of the annihilation (creation) operators of the electron at site \( r \) with spin and orbital indices, and the summations with respect to these indices are abbreviated when they are not explicitly indicated. The spin and orbital angular momenta are represented by \( \mathbf{S} = \frac{1}{2} c_r^\dagger \sigma c_r \) and \( \mathbf{L} = c_r^\dagger c_r \), respectively. The matrix \( \mathbf{I} \) is originally \( I = 2 \) representation of \( SU(2) \), and is projected to \( t_{2g} \)-space in the above model. The transfer integrals are determined by the oxygen p-orbitals between the two transition metal ions; \( t_0 \) is nonzero even for the perfect perovskite structure, while \( t_1 \) becomes nonzero due to the shift of the oxygen atoms out of plane in the \( z \)-direction.

In Ref. [2], the anomalous Hall effect (AHE) of the model Eq. (2) has been studied. There, the sign of the transverse conductivity \( \sigma^{xy} \) represents the direction of the rotational motion of electrons. An essential observation is that \( \sigma^{xy} \) represents the topological nature, Berry phase, of the Bloch wavefunction, and is mostly determined by the band crossings acting as magnetic monopoles in momentum space. The expression of \( C^{xy} \) in Eq. (8) is related to that of \( \sigma^{xy} \) under the substitution \( S \leftrightarrow J_e \), where \( J_e \) the electric current [2]. Therefore, we also expect the sign change of \( C^{xy} \) for magnon, which is actually the case as shown below.

Starting from the ferromagnetic ordered state \( \langle \phi_r \rangle \) is studied. The denominator \( \epsilon_{m+k} - \epsilon_{m} \) in Eq. (8) suggests that the (near) degeneracies of the two bands contribute dominantly to \( C^{xy} \) compared with \( \chi_{\mathbf{k}}^{xx}(0) \) corresponding to the anisotropy energy \( D \), which contains only \( \epsilon_{m+k} - \epsilon_{m} \) in the denominator.

Precisely speaking, the handedness of spin dynamics itself is governed by the signs of \( \Im \chi_{\mathbf{k}}^{\alpha\beta}(\omega) \) and \( \Re \chi_{\mathbf{k}}^{\alpha\beta}(\omega) \) at a spin-wave resonance \( \omega = \omega_{\text{reso}} \). However, concerning the lowest excitation at low temperature, they correspond to those of \( C^{xy} \) and \( D \) respectively. It should be also noted that, near the resonance, the sign of \( \Im \chi_{\mathbf{k}}^{\alpha\beta}(\omega) \) is given by the sign of \( \Re \chi_{\mathbf{k}}^{\alpha\beta}(\omega) - \chi_{\mathbf{k}}^{\alpha\beta}(\omega) \), which can be observed by the spin-resolved neutron scattering as discussed later.

Now we demonstrate the nontrivial behavior of \( C^{xy} \) using an effective two-dimensional model for \( t_{2g} \) electrons.
The quantity $C^{xy}$ is closely related to the topological nature of Bloch functions similarly to the transverse conductivity $\sigma^{xy}$ [2]. Actually, at $T = 0$, it is written as $C^{xy} = 3\langle \frac{\partial \Psi}{\partial k_x} \frac{\partial \Psi}{\partial k_y} \rangle$ where $\Psi$ is the many-body ground state wavefunction of electrons under the magnetic field $H = (h_x, h_y, h_z)$ applied to the spins [6]. The Berry curvature is known to be singularly enhanced near the band degeneracy acting as "a magnetic monopole". Here we introduce an effective Hamiltonian which describes the nearly degenerate bands of up and down spins. $H = \epsilon_k + b_k \cdot \sigma$, where $\epsilon_k$ and $b_k$ are arbitrary functions of $k$ and magnetization. The contribution to $C^{xy}$ from these bands is estimated by

$$C^{xy}_{\text{Dirac}} = \frac{1}{V} \sum_k \left[ f(\epsilon_{k-}) - f(\epsilon_{k+}) \right] \frac{b_k^z}{(2 |b_k|)^3}, \quad (10)$$

where $\epsilon_{k\pm} = \epsilon_k \pm |b_k|$. In the case where the Fermi level is close to this nearly degenerate point, the above contribution becomes large as the separation between the bands, $2|b_k|$, decreases. When the dominantly minority spin state is lower in energy than that of the majority spin state, i.e., $b_k^z > 0$, this enhanced contribution has the opposite sign compared to that expected from the original CR Eq. (11). This occurs only when both the minority and majority bands have finite density of states at the Fermi energy, and does not occur for the half-metallic ferromagnets.

Now we turn to experimental consequences of the sign change in $C^{xy}$. One of the most direct detection of left-handedness is by means of polarization analysis of magnetic neutron scattering. Energy and angle resolved polarization of neutrons $P'$ scattered by magnetic interaction is proportional to the space and time Fourier transform of the statistical average $[7]$

$$P' \propto g^2 \left[ \langle S_r^{(\perp)}(0) \{ P \cdot S_r^{(\perp)}(t) \} + \langle P \cdot S_r^{(\perp)}(0) \rangle S_r^{(\perp)}(t) \right] - P \langle S_r^{(\perp)}(0) \cdot S_r^{(\perp)}(t) \rangle - i \langle S_r^{(\perp)}(0) \times S_r^{(\perp)}(t) \rangle \right]. \quad (11)$$

Here $P$ is polarization of the incident beam, $S_r^{(\perp)}(t) = \hat{k} \times \{ S_r(t) \times \hat{k} \}$, $r$ and $r'$ are the spin coordinates, $g$ is the gyromagnetic ratio or the Lande splitting factor, and $\hat{k}$ is a unit vector along the momentum transfer $k$. The last term in Eq. (11), proportional to $\langle S_r^{(\perp)}(0) \times S_r^{(\perp)}(t) \rangle$, has two characteristic features distinguishing it from the all other ones. First, it does not depend on the polarization of the incident beam $P$. Second, this average distinguishes between left- and right-handed coordinates since the direction of the resultant vector $c$ of vector product $a \times b$ is related to the vectors $a$ and $b$ by the right-hand rule. Therefore, to test the violation of the clockwise direction of the rotation of spin in the ferromagnet, one has to study polarization of scattered neutron beam when the incident beam is unpolarized. In this case the resulting polarization is determined purely by the average $\langle S_r^{(\perp)}(0) \times S_r^{(\perp)}(t) \rangle$.

For usual ferromagnet the expression Eq.(11) can be considerably simplified. For unpolarized incident beam the resultant polarization of the scattered beam is proportional to $P' \propto 2kg^2(\hat{\kappa} \cdot \hat{\eta}) (S^{(\times)}(0)S^{(\times)}(t) - S^{(\parallel)}(0)S^{(\parallel)}(t))$. Here $\hat{\eta}$ is a unit vector along $z$-direction. Note, the sign of the polarization of the scattered beam depends just of the sign of the average $\langle S^{(\times)}(0)S^{(\times)}(t) - S^{(\parallel)}(0)S^{(\parallel)}(t) \rangle$. For the particular case of the Heisenberg antiferromagnet the polarization of the scattered neutrons with unpolarized scattered beam is $P'_\pm \propto \pm 2\tilde{\kappa}(\hat{\kappa} \cdot \hat{\eta})/[1 + (\hat{\kappa} \cdot \hat{\eta})^2]$, where upper sign refers to the process of magnon creation and the lower one to its annihilation. Hence, if the average $\langle S^{(\times)}(0)S^{(\times)}(t) - S^{(\parallel)}(0)S^{(\parallel)}(t) \rangle$ changes its sign due to the spin-orbit coupling, one can observe "wrong" signs of polarizations when magnons are created and annihilated. Note, the strongest polarization observed for the momentum transfer parallel or anti-parallel to the magnetization axis. Therefore, in terms of the spin-resolved neutron scattering, we can obtain the information about the following value $A_{\kappa}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \Im \langle S_{\kappa}^{(\times)}(0)S_{\kappa}^{(\times)}(t) \rangle \propto \Re \{ \chi_{\kappa}^{xy}(\omega) - \chi_{\kappa}^{xy}(\omega) \}$. To detect the direction of the precession, we need to know the sign of $\Re \{ \chi_{\kappa}^{xy}(\omega) - \chi_{\kappa}^{xy}(\omega) \}$ at the lowest resonance. In addition, there is another important aspect of the neutron scattering spectrum associated with the sign change of $C^{xy}$. Namely, as the $C^{xy}$ approaches zero, there splits off the bound state from the continuum in $A_{\kappa=0}(\omega)$, which has the left-handed direction of precession. This state becomes lower in energy, and eventually becomes the lowest-energy peak as the sign of $C^{xy}$ changes. Therefore, this multiple peak structure is an important signature observed near the transi-
tion between right- and left-handedness of the magnon modes as expected in the neutron scattering experiment.

Figure 2 shows the explicit calculation for the model in Eq. (9). We can clearly see the interchange of the lowest and the next excitations with \( \kappa = 0 \) around \( k_B T \sim 0.08 \). The lowest-energy excitation corresponds to the left-handed spin motion below \( k_B T \sim 0.08 \), e.g. at \( k_B T \sim 0.01 \) in Fig. 2(a), while the lowest one is right-handed at \( k_B T \sim 0.09 \) in Fig. 2(b).

There are several other experimental techniques which are sensitive to the direction of the spin precession in the effective magnetic field, i.e., precession \( \frac{dS}{dt} = g_{eff}\mu_B H \times S \) which direction in external or internal magnetic field of a ferromagnet is determined by effective gyromagnetic ratio \( g_{eff} \). The most traditional one is the electronic spin resonance (ESR) technique with rotating magnetic field \( \mu B \) and the most recent advanced one is the magneto-optical polarimetric analysis where the real space trajectory of the magnetization dynamics can be detected \([8, 10, 11]\). However, both techniques are sensitive to the sign of effective \( g \)-factor. Note, the change of the sign of CR, signaling on the left-handedness, is an exotic situation. To the contrast, negative value of the \( g \)-factor of effective spin Hamiltonian is a frequent situation when an ion is influenced by a crystal field from the neighboring ions of the lattice \([12]\). Hence, we conclude that the polarization analysis of the scattered neutrons is a rather unique possibility to trace the left-handedness since the polarization of the scattered beam Eq.(11) is determined just by the sign of the correlation function and does not depend on the sign of the effective \( g \)-factor.

In conclusion, we have theoretically shown that the left-handed magnet can be realized and does not contradict with the commutation relation for the spin operators. The conditions for this novel possibility is that (i) the orbital angular momentum is basically quenched but still the spin-orbit interaction affects the spin dynamics, (ii) both majority and minority spins have the finite density of states, and (iii) the band degeneracy occurs near the Fermi energy and gives the dominant contribution to the Berry phase. These conditions are satisfied in the metallic ferromagnets including the transition metal ions, and most preferably in two dimensional systems. For example, candidate substances are listed up as follows: the transition metal intermetallic compounds \( \text{Co}_2\text{Pd(Pt)}_4 \) \([13]\), \( \text{CoPt}_3 \) \([14]\), and the series of ferromagnetic ruthenates \( \text{Sr}_{1-x}\text{Ca}_x\text{RuO}_3 \) \([15]\). Even though the left-handedness is not realized, we expect the nontrivial temperature dependence of the magnon dynamics there. Detailed first-principles calculations and experimental search for this left-handed magnet is highly desired and left for future investigations.

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![FIG. 2: Temperature dependence of the off-diagonal correlation obtained by spin-polarized neutron scattering. The interchange of two spin-wave spectra occurs around \( k_B T \sim 0.08 \). The sign of the lowest-energy resonance represents the left(+) and right(−) handedness.](image-url)
temperature as discussed for Fig.2.

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