ON WEYL QUANTIZATION FROM GEOMETRIC QUANTIZATION

P. de M. Rios & G.M. Tuynman

Abstract. In [We] a nice looking formula is conjectured for a deformed product of functions on a symplectic manifold in case it concerns a hermitian symmetric space of non-compact type. We derive such a formula for simply connected symmetric symplectic spaces using ideas from geometric quantization and prequantization of symplectic groupoids. We compute the result explicitly for the natural 2-dimensional symplectic manifolds $\mathbb{R}^2$, $\mathbb{H}^2$, and $\mathbb{S}^2$. For $\mathbb{R}^2$ we obtain the well known Moyal-Weyl product. The other cases show that the original idea in [We] should be interpreted with care. We conclude with comments on the status of our result.

Introduction

In [We] the author discusses the quantization by groupoids program as a means to obtain a deformed multiplication of the Poisson algebra $C^\infty(M)$ associated to a symplectic manifold $M$ in the form
\[(fg)(z) = \int_{M \times M} f(x)g(y)K(x, y, z) \, dx \, dy,\]

with a kernel $K_\hbar$, a function of a deformation parameter $\hbar$, of the form $K_\hbar(x, y, z) = \hbar^{-\dim M} \exp(iS(x, y, z)/\hbar)$, eventually multiplied by an “amplitude” $A(x, y, z)$. It is suggested that for hermitian symmetric spaces the function $S(x, y, z)$ should be the symplectic surface of a geodesic triangle for which the points $x$, $y$, and $z$ are the midpoints of the sides.

In this paper we will derive such a formula (formula (5) below) for simply connected symmetric symplectic spaces $M$ by means of geometric quantization of the symplectic groupoid $M \times M$ and its prequantization as described in [WX]. Our approach is inspired by the center-chord representation on euclidean spaces as described in [OdA]. We then apply this procedure to three simple 2-dimensional examples: the euclidean plane $\mathbb{R}^2$, the 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, and the hyperbolic plane $\mathbb{H}^2 \subset \mathbb{R}^3$. The first example, already worked out in [GBV] for $\mathbb{R}^{2n}$, gives us the well known Moyal-Weyl quantization of observables. In the hyperbolic plane we see that we have to interpret the amplitude function in a rather large sense: the phase function $S$ is defined only on “half” of $\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$, forcing the amplitude function to be zero outside this (open) domain, and it blows up at the boundary of this domain. In the example of the 2-sphere we encounter the additional problem that midpoints do not always determine a unique triangle.

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Preliminaries.
Let \((M, \omega)\) be a symplectic manifold and let \(\hbar \in \mathbb{R}^+\) be a parameter. Let \((Y, \theta)\) be a prequantization of \((M, \omega/\hbar)\), meaning that \(\pi : Y \to M\) is a principal \(S^1\)-bundle equipped with a connection form \(\theta\) whose curvature is \(\omega/\hbar\) (which implies that the group of periods of \(\omega\) is a discrete subgroup of \(\mathbb{R}\)). Using the identity representation of the circle \(S^1 \subset \mathbb{C}\) on \(\mathbb{C}\), we let \(L \to M\) be the associated complex line bundle over \(M\) with connection \(\nabla\) and compatible hermitian structure. It follows that we can identify \(Y\) with the subset of \(L\) of points of length 1 (with respect to the hermitian structure). We now assume that the curvature of \(\nabla\) also equals \(\omega/\hbar\), which implies that \(\omega/\hbar\) represents an integral cohomology class. This imposes a quantization condition on \(\hbar\) in case \(\omega\) is not exact.

Our purpose is to construct a map \(C^\infty(M) \times C^\infty(M) \to C^\infty(M)\) by means of geometric quantization of \(M \times M\) as a symplectic groupoid. Our strategy will be to use a polarization such that the polarized sections of geometric quantization can be identified with functions on \(M\) (usually these sections form the Hilbert space, but here we will interpret them as observables). Using a groupoid structure on the prequantization, we construct the looked-for product. To make this work, we will have to restrict our attention to symplectic spaces with a complete affine connection for which geodesic inversion with respect to a point is a symplectomorphism, and whose first homology group is zero. This brings us in the category of simply connected symmetric symplectic spaces, which includes all simply connected hermitian symmetric spaces. We will use extensively the results of [WX], as well as its notation, but we will restrict to the barest minimum of terminology. For more of that, the interested reader is referred to [WX] and the references therein.

Prequantization of the groupoid.
The construction starts by giving the manifold \(M \times M\) the symplectic structure \((\omega, -\omega)\). More precisely, if \(\alpha\) and \(\beta\) denote the canonical projections \(M \times M \to M\) onto the first and second factor, then the symplectic form on \(M \times M\) is \(\alpha^*\omega - \beta^*\omega\). The manifold \(Y \times Y\) is in a natural way a principal \(S^1 \times S^1\)-bundle over \(M \times M\). Quotienting out the diagonal action of \(S^1\) on \(Y \times Y\), \(e^{i\phi} \cdot (y_1, y_2) = (e^{i\phi} \cdot y_1, e^{i\phi} \cdot y_2)\), we obtain a principal \(S^1\)-bundle \([Y] = Y \times Y/S^1 \to M \times M\). We will denote points in \([Y]\) as \([y_1, y_2]\) with \(y_i \in Y\). The induced \(S^1\)-action is taken to be \(e^{i\phi} \cdot [y_1, y_2] = [e^{i\phi} \cdot y_1, y_2] = [y_1, e^{-i\phi} \cdot y_2]\). Moreover, the 1-form \((X, -\theta)\) induces a connection form \([\theta] \equiv [\theta, -\theta]\) on \([Y]\), whose curvature is \((\omega/\hbar, -\omega/\hbar)\). We thus obtain a (particular) prequantization of \(M \times M\). Let \([L] \to M \times M\) be the associated complex line bundle with connection and compatible hermitian structure. And as before we identify \([Y]\) with the subset of \([L]\) of points of length 1. We define the diagonal section \(\epsilon' : M \to [Y]\) as \(\epsilon'(m) = [y, y]\) with \(y \in Y\) such that \(\pi(y) = m\). This section is horizontal for the connection \([\theta]\). It then follows from [WX, theorem 3.1, proposition 3.2] that there exists a unique groupoid structure on \([Y]\) with given properties. In our case this means that there exists a smooth map \(\circ\) with values in \([Y]\) and defined on pairs \([x, y_1], [y_2, z] \in [Y]\) such that \(\pi(y_1) = \pi(y_2)\). Using that \(S^1\) acts transitively on the fibres of \(Y \to M\), and the diagonal \(S^1\) action on \(Y \times Y\), this condition means that there exists a \(z' \in Y\) such that \([y_2, z'] = [y_1, z']\). With such a representation of the points, this “multiplication” \(\circ\) is given by

\[
[x, y] \circ [y, z] = [x, z].
\]
The polarization.

The next step in the geometric quantization procedure is the choice of a polarization on $M \times M$. However, for generic $M$ we know of no natural choices for a polarization which “mixes” both factors $M$. We thus try to find symplectic spaces for which we can define a rather natural mixing polarization. Here is the idea. For any complete affine connection $\nabla$ on $M$ we can define a smooth map $F : TM \to M \times M$ by

$$F(m, v) = (\exp_m(-v), \exp_m(v)),$$

where $\exp_m : T_mM \to M$ denotes the geodesic flow at time $t = 1$, starting at $m \in M$ and in the direction of the tangent vector $v \in T_mM$. Since $\exp_m$ is a diffeomorphism in a neighborhood of $0 \in T_mM$, $F$ is a diffeomorphism in a neighborhood of the zero section of $TM$. We define $U \subset TM$ as a maximal connected and symmetric (with respect to inversion in the fibres of the tangent bundle) open neighborhood of the zero section on which $F$ is a diffeomorphism. Associated to $U$ we define $V = F(U) \subset M \times M$. Note that if the (complete) affine connection $\nabla$ has no closed geodesics, then $U = TM$ and $V = M \times M$.

On $TM$ we have a natural foliation $\mathcal{F}_v$ whose leaves are just the fibres $T_mM$ of the tangent bundle. Our idea is that its image $\mathcal{P} = F_*\mathcal{F}_v$ should be a polarization for the restriction of the symplectic form $(\omega, -\omega)$ to $V$. An elementary computation shows that $\mathcal{P}$ is a polarization on $V$ if and only if for each $m \in M$ the map $\exp_m(v) \mapsto \exp_m(-v)$ is a symplectomorphism on $\exp_m(U \cap T_mM) \subset M$. We thus require that the symplectic manifold $M$ admits a complete affine connection for which geodesic inversion is a symplectomorphism.

We thus arrive in the category of symmetric symplectic spaces $[\text{Bi}] [\text{RO}]$, which includes the category of hermitian symmetric spaces because the connection associated to the natural (complete) metric on a hermitian symmetric space satisfies this condition. When this condition is satisfied, we obtain a (real) polarization $\mathcal{P}$ on $V \subset M \times M$. Moreover, as is obvious from the definition of $\mathcal{P}$ via $\mathcal{F}_v$, the space of leaves $V/\mathcal{P}$ is naturally isomorphic to $M$, seen either as the diagonal in $M \times M$ or as the zero section in $TM$.

We now claim that there exists a section $s_o : V \to [Y]$ which is horizontal in the direction of $\mathcal{P}$ and which coincides with $\varepsilon'$ on its domain of definition. The easiest way to construct this section is by pulling back all structures on $M \times M$ to $TM$ by means of the map $F$. More precisely, we define $\Omega$ as the closed 2-form $F^*(\omega, -\omega)$ on $TM$ and $(B, \Theta)$ as the principal $S^1$-bundle with connection over $TM$ obtained by pulling back the bundle $(|Y|, \theta)$. Obviously the curvature form of $\Theta$ is $\Omega/\hbar$. As argued above, $\Omega$ is identically zero on the fibres of $TM$, i.e., on the leaves of $\mathcal{F}_v$. The section $\varepsilon'$ of $[Y]$ gets transformed to a section $\sigma$ of $B$ above $M$ seen as the zero section of $TM$. Since the fibres of $TM$ are simply connected and since the curvature of $\Theta$ is identically zero on these fibres, we can extend the section $\sigma$ to a global section $TM \to B$ which is horizontal when restricted to a leaf of $\mathcal{F}_v$. Restricting this section to $U$ and then pushing it to $V$ by means of $F$ we obtain our section as claimed.

In order to get a better grip on this section, let $(m_1, m_2) \in V \subset M \times M$ be arbitrary. We then can define the curve $\gamma : [0, 1] \to V \subset M \times M$ by $\gamma(t) = F(m, tv)$, with $TM \supset U \ni (m, v) = F^{-1}(m_1, m_2)$. More or less by construction $s_o(m_1, m_2)$ is the end point of the horizontal lift of $\gamma$ starting at $\varepsilon'(m)$. But the two components $\gamma_1(t) = \exp_m(-tv)$ and $\gamma_2(t) = \exp_m(tv)$ of the curve $\gamma$ form together the geodesic from $m_1$ to $m_2$ with $m$ as midpoint. Choosing $\mu \in \pi^{-1}(m)$ arbitrary, we thus can define $\tilde{\gamma}_1(t)$ as the horizontal lift of $\gamma_1(t)$ in $Y$ starting at $\mu$. Together they form a horizontal lift in $Y$ above the geodesic between
m_1 and m_2. By definition of the connection form on [Y], the curve \( \tilde{\gamma}(t) = [\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)] \in [Y] \) is the horizontal lift of \( \gamma \) starting at \( \varepsilon'(m) = [\mu, \mu] \). It follows that \( s_o(m_1, m_2) = [x, y] \) in which \( x \) and \( y \) are the end points of a horizontal curve above the geodesic (unique in \( V \)) between \( m_1 \) and \( m_2 \).

We continue with the geometric quantization program and we look at the space of all sections of \([L]\) above \( V \) that are covariantly constant in the direction of \( \mathcal{P} \). We will call such sections \( \mathcal{P} \)-constant. Viewing \([Y]\) as a subset of \([L]\), the section \( s_o \) constructed above is \( \mathcal{P} \)-constant. Moreover, it is a smooth nowhere vanishing section. It follows that \( \mathcal{P} \)-constant sections \( s : V \rightarrow [L] \) are in 1-1 correspondence with functions \( f \) that are constant on the leaves of \( \mathcal{P} \), i.e., with functions on \( M = V/\mathcal{P} \). The identification is given by \( s = f \cdot s_o \).

A product of sections.

We now stop the geometric quantization program and we turn our attention to the groupoid structure on \([Y]\). We extend the groupoid multiplication \( \circ \) to \([L]\) by the following prescription. Any \( p \in [L] \) can be written in a unique way as \( p = \lambda[x, y] \) with \( \lambda \in [0, \infty) \) and \([x, y] \in [Y] \subset [L] \). Now, for \( p_i = \lambda_i[x_i, y_i] \) such that \( \pi(y_1) = \pi(x_2) \) we define

\[
p_1 \circ p_2 = \lambda_1 \lambda_2 [x_1, y_1] \circ [x_2, y_2].
\]

With this extended quasi-groupoid structure (quasi because now not every element has an inverse), we construct a product on sections of \([L]\). If \( s_1 \) and \( s_2 \) are two sections of \([L]\) (not necessarily above \( V \), not necessarily \( \mathcal{P} \)-constant), we define a new section \( s_1 \circ s_2 \) of \([L]\) by

\[
(s_1 \circ s_2)(m_1, m_3) = \int_M s_1(m_1, m_2) \circ s_2(m_2, m_3) \, dm_2.
\]

In this formula the measure \( dm_2 \) is the Liouville measure on \( M \) associated to the symplectic form \( \omega \). The integration makes sense because all groupoid products \( s_1(m_1, m_2) \circ s_2(m_2, m_3) \) lie in the same fibre of \([L] : \) the one above \( (m_1, m_3) \). Of course there is no guarantee that this integral converges, but we will not deal with these delicate analytical issues here.

We now, for the moment, restrict our attention to the case in which the metric \( g \) has no closed geodesics, i.e., the case in which \( F \) is a diffeomorphism from \( TM \) onto \( M \times M \).

In that case \( \mathcal{P} \)-constant sections of \([L]\) are globally defined sections. For two \( \mathcal{P} \)-constant sections \( s_i = f_i \cdot s_o, \; i = 1, 2 \) with \( f_i \in C^\infty(M) \) we thus get the formula

\[
(s_1 \circ s_2)(m_1, m_3) = \int_M f_1(m_12)f_2(m_23)s_o(m_1, m_2) \circ s_o(m_2, m_3) \, dm_2,
\]

in which \( m_{ijk} \) denotes the midpoint of the geodesic between \( m_j \) and \( m_k \). Since \( s_o \) is nowhere vanishing, there must be a constant \( \lambda \) such that \( s_o(m_1, m_2) \circ s_o(m_2, m_3) = \lambda s_o(m_1, m_3) \). In order to determine this constant we argue as follows. We choose \( x_1, x_2, x_3, \) and \( x'_3 \) such that \( s_o(m_1, m_2) = [x_1, x_2], \ s_o(m_2, m_3) = [x_2, x_3], \) and \( s_o(m_1, m_3) = [x_1, x'_3] \). Note that we may take the same \( x_1 \) and \( x_2 \) because of the equivalence relation defining the points in \([Y]\). It follows from formula (1) that \( s_o(m_1, m_2) \circ s_o(m_2, m_3) \) equals \([x_1, x_3]\). But we know that \( x_1 \) and \( x_2 \) are the endpoints of a horizontal lift above the geodesic between \( m_1 \) and \( m_2 \), and
similarly for the pairs $x_2, x_3$ and $x_1, x'_3$. We thus have a geodesic triangle $m_3m_2m_1$ and a horizontal lift starting at $x_3$ above $m_3$, passing through $x_2$ and $x_1$ and coming to $x'_3$, again above $m_3$. It follows that $x'_3 = \lambda x_3$ with $\lambda \in S^1$ the holonomy of the geodesic triangle $m_3m_2m_1$. In particular we have $[x_1, x_3] = \lambda [x_1, x'_3]$. Now if $\Delta(m_3m_2m_1)$ is any 2-chain whose boundary is the geodesic triangle $m_3m_2m_1$, then $\lambda = \exp(i\int_{\Delta(m_3m_2m_1)} \omega/\hbar)$. The result does not depend upon the choice for $\Delta$ because the curvature form $\omega/\hbar$ represents an integral cohomology class. We are thus led to introduce the phase function $\tilde{S}(m_3, m_2, m_1) = \int_{\Delta(m_3m_2m_1)} \omega$ representing the symplectic area of the surface $\Delta(m_3m_2m_1)$ whose boundary is the geodesic triangle with corners at $m_3$, $m_2$, and $m_1$. Actually $\tilde{S}$ is in general multiple valued because there is (in dimensions higher than 2) no unique such 2-chain $\Delta$, but this indeterminacy disappears when taking the exponential. On the other hand, in order to be sure that such a 2-chain exists for all geodesic triangles, we further restrict our attention to spaces $M$ without homology in dimension 1. This excludes for instance the 2-torus, but all simply connected hermitian symmetric spaces satisfy this condition, and thus in particular the hermitian symmetric spaces of compact and non-compact type.

Thus, substituting these results in formula (2) we obtain $(f_1 \circ s_o \circ f_2 \cdot s_o)(m_1, m_3) = g(m_1, m_3) \cdot s_o(m_1, m_3)$, where $g$ is given by

$$g(m_1, m_3) = \int_M f_1(m_{12}) f_2(m_{23}) e^{i\tilde{S}(m_3, m_2, m_1)/\hbar} \, dm_2 .$$

If we forget the trivializing section $s_o$, we thus have associated to two functions $f_1$, $f_2$ on $M$ a new function $g$ on $M \times M$. In general, the product $s_1 \circ s_2$ of two $P$-constant sections will not be $P$-constant. In terms of the function $g$ this means that, in general, the function $g : M \times M \to \mathbb{C}$ is not constant on the leaves of $P$, i.e., of the form $g(m_1, m_3) = \hat{g}(m_13)$ for some function $\hat{g} : M \to \mathbb{C}$ with $m_{13}$ the midpoint of the geodesic between $m_1$ and $m_3$.

**A new product of functions on $M$.**

In order to get a $P$-constant section, or, in other words, in order to associate to two functions $f_1$ and $f_2$ on $M$ a new function $f_1 \ast f_2$ on $M$ (not on $M \times M$), we integrate (average) over the leaves of $P$. This is done most easily in terms of the fibres of $TM$ and we get

$$f_1 \ast f_2(m) = \int_{T_m M} dv \, g(m_1, m_3) = \int_{T_m M} dv \, g(F(m, v))$$

$$= \int_{T_m M} dv \int_M dm_2 \, f_1(m_{12}) f_2(m_{23}) e^{i\tilde{S}(m_3, m_2, m_1)/\hbar} ,$$

with $(m_1, m_3) = F(m, v)$ and $m_{jk}$ the midpoint on the geodesic between $m_j$ and $m_k$.

It remains to be decided what measure $dv$ to take on $T_m M$, but there exists a rather canonical way to obtain one. Using that $F$ is a global diffeomorphism (we are still in that case), $F^*(\omega, -\omega)$ is a symplectic form on $TM$, and thus we have its Liouville volume form $d\mu_{TM}(m, v)$ on $TM$. On the other hand, the zero section of $TM$ is diffeomorphic to the symplectic manifold $(M, \omega)$, and thus on the zero section of $TM$ we have its Liouville volume form $d\mu_M(m)$. It follows that there exists a unique volume form $dv_m(v) \equiv dv$ on each fibre $T_m M$ such that $d\mu_M(m) \cdot dv_m(v) = d\mu_{TM}(m, v)$.
Formula (4) presents our deformed product of functions on \( M \). In order to write it in a nicer way, we look at the map \( \Psi : (v, m_2) \mapsto (m_{12}, m_{23}) \) from \( T_m M \times M \) to \( M \times M \). We conjecture that this map is injective; it certainly need not be surjective as can be seen in the case of the hyperbolic plane. If we denote by \( dm_{12} \) the Liouville measure on the first factor of \( M \times M \) and by \( dm_{23} \) the Liouville measure on the second factor, then there exists a positive function \( A_m \) on \( W_m = \Psi(T_m M \times M) \subset M \times M \) such that \( \Psi^*(A_m \, dm_{12} \, dm_{23}) = dv \, dm_2 \). Associated to \( W_m \) we define the set \( W \subset M^3 \) as \( W = \{(m, m_{12}, m_{23}) \in M^3 \mid (m_{12}, m_{23}) \in W_m \} \). We then can interpret the family of functions \( A_m \) as a single function \( A : W \to [0, \infty) \) by \( A(m, m_{12}, m_{23}) = A_m(m_{12}, m_{23}) \). Still under the assumption that \( \Psi : T_m M \times M \to W_m \) is bijective, we define the function \( S \) on \( W \) by \( S(m, m_{12}, m_{23}) = \tilde{S}(m_3, m_2, m_1) \), where the points \( m_i \) are defined by the equations \( (v, m_2) = \Psi^{-1}(m_{12}, m_{23}) \) and \( F(m, v) = (m_1, m_3) \). The function \( S \) can be described as the symplectic area of a surface \( \Delta \) whose boundary is the geodesic triangle whose midpoints of its three sides are \((m, m_{12}, m_{23})\). With these preparations, and denoting \( m' \equiv m_{12} \), \( m'' \equiv m_{23} \), formula (4) can be written as

\[
(f_1 \ast f_2)(m) = \iint_{W_m} f_1(m') f_2(m'') \, e^{iS(m,m',m'')} / \hbar \, A(m, m', m'') \, dm' \, dm''.
\]

Except for the restriction of the integration to \( W_m \) instead of \( M \times M \), this is exactly of the form for a deformed product as conjectured in [We].

The general case.

We have derived formula (5) under the assumption that \( F \) is a global diffeomorphism from \( TM \) to \( M \times M \). If this is not the case, we were led to introduce the subsets \( U \subset TM \) and \( V = F(U) \subset M \times M \), and the section \( s_0 \) defined only above \( V \). It follows that the integration procedure which led us to formula (3) can only be performed for those values of \( m_2 \) such that \((m_1, m_2)\) and \((m_2, m_3)\) both lie in \( V \). The next step of “averaging” over the leaves of \( \mathcal{P} \) should also be done with care. These leaves are only defined in \( V \) (elsewhere \( \mathcal{P} \) is not defined), which means in terms of \( TM \) that we have to integrate, not over the whole tangent space \( T_mM \), but only over the part in \( U \), i.e., over \( T_mM \cap U \). On the other hand, the argument which led to the measure \( dv \) remains valid: the pull-back by \( F \) of the Liouville measure on \( V \) to \( U \) gives us a measure on \( U \). The zero section still carries its natural Liouville measure, and thus there exists a natural measure \( dv_m \) on \( T_mM \cap U \) such that it completes the natural Liouville measure on the zero section to the pull back of the Liouville measure on \( V \). We conclude that formula (4) still defines a deformed product of functions, provided we restrict integration to the appropriate subset of \( T_mM \times M \).

In the general case the map \( \Psi \) need not be injective, not even on the relevant subset \((T_mM \cap U) \times M\) as described above, as can be seen in the example of the 2-sphere. However, inspired by the example of the 2-sphere, we conjecture that there still exists a positive function \( A_m \) on \( W_m = \Psi((T_mM \cap U) \times M) \) such that \( \Psi^*(A_m \, dm_{12} \, dm_{23}) = dv \, dm_2 \). We also conjecture that \( \Psi \) is injective outside a closed subset of measure zero in \((T_mM \cap U) \times M\). This means that we can copy the arguments leading to formula (5), and that this formula is valid also in the general case, but with the new subset \( W_m \).
Three examples

The Euclidean plane \( \mathbb{R}^2 \).

Let \( M = \mathbb{R}^2 \) be the Euclidean plane with the symplectic form \( \omega = dp \wedge dq = d(p dq) \). The (unique) prequantization is the bundle \( Y = M \times S^1 \) with connection form \( \hbar \theta = p dq + d\varphi \). The map \( F \), a global diffeomorphism, is given as \( F(p, q; v_p, v_q) = (p - v_p, q - v_q; p + v_p, q + v_q) \). A horizontal lift of the curve \( (p + tv_p, q + tv_q) \) is given by \( (q + tv_p, p + tv_p, \exp(i/\hbar (pt + \frac{1}{2} t^2 v_p) v_q)) \). An elementary calculation then gives for the section \( s_o \) the expression

\[
s_o(p_1, q_1; p_2, q_2) = [(p_1, q_1; 1), (p_2, q_2; \exp(i/\hbar(p_1 + p_2)(q_2 - q_1)))] ,
\]

where we used the equivalence relation on \([ , ]\) to put the first phase equal to 1. From there it follows immediately from formula (1) that the phase factor \( \lambda \) in \( s_o(m_1, m_2) \odot s_o(m_2, m_3) = \lambda s_o(m_1, m_3) \) is given by

\[
\lambda = \exp(i/2\hbar((p_1 + p_2)(q_1 - q_2) + (p_2 + p_3)(q_2 - q_3) + (p_3 + p_1)(q_3 - q_1))) .
\]

A trivial calculation shows that this is indeed \( \exp(i\tilde{S}(p_3, q_3; p_2, q_2; p_1, q_1)/\hbar) \) with \( \tilde{S} \) the symplectic area (oriented with respect to the volume form \( dp \wedge dq \)) of the triangle with corners at \((p_3, q_3), (p_2, q_2), \) and \((p_1, q_1)\).

In this example, the change of coordinates \((v, m_2) \mapsto (m_{12}, m_{23})\) is a linear bijection with Jacobian \( \frac{1}{4} \), which implies that the amplitude function \( A \) is constant \( \frac{1}{4} \). Moreover, in the Euclidean plane, the area \( \tilde{S}(p_3, q_3; p_2, q_2; p_1, q_1) \) is four times the area of the triangle determined by its midpoints, i.e., \( S(p, q; p_{12}, q_{12}; p_{23}, q_{23}) = 4\tilde{S}(p, q; p_{12}, q_{12}; p_{23}, q_{23}) \). Up to a scale factor this result is the usual formula one gives for Moyal-Weyl quantization of the Euclidean plane ([OdA]).

The hyperbolic plane \( \mathbb{H}^2 \).

Our next example is the hyperbolic plane \( \mathbb{H}^2 \) which we interpret as one sheet of the 2-sheeted hyperboloid in \( \mathbb{R}^3 \) determined by the equations \( z^2 - x^2 - y^2 = 1 \) and \( z > 0 \). We introduce the Lorentzian metric \( \langle | \rangle \) by the formula

\[
\langle (x, y, z) | (x', y', z') \rangle = zz' - xx' - yy' .
\]

This metric induces a surface element, which we take as symplectic form. An elementary but tedious calculation shows that the oriented hyperbolic area of a triangle determined by its three corners \( a, b, c \in \mathbb{H}^2 \subset \mathbb{R}^3 \) is given by the formula

\[
\tilde{S}(a, b, c) = 2 \text{Arg} \left( 1 + \langle a | b \rangle + \langle b | c \rangle + \langle c | a \rangle + i \det(abc) \right) ,
\]

where \( \text{Arg} \) denotes the argument of a complex number; it lies in the interval \(( -\pi, \pi )\). This formula is derived in [Ma] and [Ur] in the context of relativistic addition of velocities.

The next steps are to express the area of a hyperbolic triangle as a function of its midpoints and to determine the change of coordinates \((v, m_2) \mapsto (m_{12}, m_{23}) \). A straightforward calculation shows that if \( a, b, c \in \mathbb{H}^2 \subset \mathbb{R}^3 \) are the corners of a hyperbolic triangle, and if
\[ \alpha, \beta, \gamma \in H^2 \subset \mathbb{R}^3 \] denote the midpoints of the three sides, then the area of the triangle (see \cite{Tu}, \cite{RO}) is given by the simple formula

\[ S(\alpha, \beta, \gamma) = 2 \text{Arg} \left( \sqrt{1 - \det(\alpha\beta\gamma)^2} + i \det(\alpha\beta\gamma) \right) = 2 \arcsin(\det(\alpha\beta\gamma)) . \]  

(7)

The same analysis shows that the map \((a, b, c) \mapsto (\alpha, \beta, \gamma)\) is injective onto the triples \((\alpha, \beta, \gamma)\) satisfying \(\det(\alpha\beta\gamma)^2 < 1\), justifying the formula for \(S\). It follows immediately that the subsets \(W_\alpha\) are given as

\[ W_\alpha = \{ (\beta, \gamma) \in H^2 \times H^2 \mid \det(\alpha\beta\gamma)^2 < 1 \} . \]  

(8)

A final computation shows that the amplitude function \(A\) is given by

\[ A(\alpha, \beta, \gamma) = 16 \langle \alpha | \beta \rangle_L \cdot \langle \beta | \gamma \rangle_L \cdot \langle \gamma | \alpha \rangle_L \cdot \left( 1 - \det(\alpha\beta\gamma)^2 \right)^{-5/2} . \]  

(9)

The fact that this amplitude function diverges on the boundary of \(W_\alpha\) shows that we correctly restricted integration to this subset and that it is optimal.

**The sphere \(S^2\).**

In the last example we consider the compact hermitian symmetric space \(S^2\) seen as the unit sphere in \(\mathbb{R}^3\), i.e., determined by the equation \(z^2 + x^2 + y^2 = 1\). We equip \(\mathbb{R}^3\) with the Euclidean metric \(\langle | \rangle_E\) given by

\[ \langle (x, y, z) | (x', y', z') \rangle_E = z z' + x x' + y y' . \]

As for the hyperbolic plane, we take the induced surface element as symplectic form. And again, an elementary but tedious calculation shows that the oriented spherical area of a triangle determined by its three corners \(a, b, c \in S^2 \subset \mathbb{R}^3\) is given by the formula

\[ \tilde{S}(a, b, c) = 2 \text{Arg} \left( 1 + \langle a | b \rangle_E + \langle b | c \rangle_E + \langle c | a \rangle_E + i \det(abc) \right) , \]  

(10)

i.e., by exactly the same formula as in the hyperbolic case, except that we use the Euclidean metric instead of the Lorentzian one. However, this formula needs more explanation than its hyperbolic counter part, because on \(S^2\) there are several triangles with the same three corners. The area given by formula (10) is the area of the triangle whose three corners are \(a, b,\) and \(c\) and whose three sides all have length less than \(\pi\).

Elementary geometry shows that the subset \(U \subset TS^2\) is given by those tangent vectors that have length less than \(\pi/2\). In fact, if \(v \in T_mS^2\) has length \(\pi/2\), the two points \(\exp_m(-v)\) and \(\exp_m(v)\) are antipodal, and thus there is a circle of pairs \((m, v)\) having these antipodal points as image under \(F\). It follows that the image \(V = F(U)\) is the set of pairs \((m_1, m_2)\) such that \(m_1 \neq -m_2\). And indeed for any two non-antipodal points there is a unique geodesic with length less than \(\pi\) joining them. The integration over \(m_2\) in formula (3) has to be done over all those \(m_2\) such that the two pairs \((m_1, m_2)\) and \((m_2, m_3)\) belong to \(V\). Since in the definition of \(V\) we only exclude antipodal points, this means that we have to leave out a set of measure zero in the integration over \(m_2\). In other words, we can maintain
formula (3) as it stands. The factor $e^{i\tilde{S}(m_3,m_2,m_1)/\hbar}$ in the integration over $m_2$ in (3) is defined except on a set of measure zero (when $m_2$ is antipodal to either $m_1$ or $m_3$).

The integration over $v \in T_mM$ should not be done over the whole of $T_mM$ but only over $T_mM \cap U$, i.e., over tangent vectors of length less than $\pi/2$. That this indeed corresponds exactly to integrating over the leaves of $P$ can also be seen as follows. Two (pairs of) points in $V \subset S^2 \times S^2$ lie on the same leaf of $P$ if and only if they have the same midpoint on the geodesic segment joining them. Since we avoid antipodal pairs, there exists a unique geodesic segment of length less than $\pi$ joining $(m_1,m_2)$, on which the midpoint is given by the normalized average $(m_1 + m_2)/(1 + m_1 \cdot m_2)\in S^2$. We thus find at the same time that the space of leaves is characterized by $S^2$, the space of midpoints, and that the distance of such a midpoint to one of its endpoints is less than $\pi/2$, justifying the restriction to integrate only over tangent vectors of length less than $\pi/2$.

It remains to express the phase function $\tilde{S}$ in terms of midpoints and to compute the amplitude function $A$. Contrary to the hyperbolic case, there always exists a geodesic triangle with given midpoints $\alpha, \beta, \gamma \in S^2$. More precisely, if $a,b,c \in S^2 \subset R^3$ are the corners of a spherical triangle, and if $\alpha, \beta, \gamma \in S^2 \subset R^3$ denote the midpoints of the three sides, then the oriented area $S$ of the triangle is given as (see [Tu], [RO])

$$S(\alpha, \beta, \gamma) = 2 \text{Arg} \left( \eta \sqrt{1 - \det(\alpha\beta\gamma)^2 + i \det(\alpha\beta\gamma)} \right), \quad (11)$$

where $\eta$ is a sign: the same as the majority of signs among the three scalar products $\langle \alpha|\beta \rangle_E$, $\langle \beta|\gamma \rangle_E$, and $\langle \gamma|\alpha \rangle_E$ (provided they are all non zero). We see that it is (up to the factor $\eta$) the same formula as in the hyperbolic case. Unlike the hyperbolic case, we do not have a restriction on the midpoints, a fact which is corroborated by the fact that for points on the unit sphere, the determinant $\det(\alpha\beta\gamma)^2$ is always less than or equal to 1.

However, the calculations leading to the formula for $S$ show that, if all three sides of a triangle have length less than $\pi$, then all three scalar products $\langle \alpha|\beta \rangle_E$, $\langle \beta|\gamma \rangle_E$, and $\langle \gamma|\alpha \rangle_E$ have the same sign, where the sign should be interpreted as a function on $R$ defined as being $+1$ for positive values, $-1$ for negative values, and 0 for zero. Thus, the set $W_\alpha$ is

$$W_\alpha = \{ (\beta, \gamma) \in S^2 \times S^2 \mid \text{sign}\langle \beta|\gamma \rangle_E = \text{sign}\langle \alpha|\beta \rangle_E = \text{sign}\langle \gamma|\alpha \rangle_E \} . \quad (12)$$

Moreover, the calculations also show that if all three inner products are zero, then there is an infinity of triangles having the given points as midpoints (roughly a set parametrized by a point on $S^2$). But this set has measure zero in $W_\alpha$ and hence can be neglected in the integration. Note that even though the triangle itself is not uniquely determined by its midpoints, its area is.

Since $W_\alpha$ is only half of $S^2 \times S^2$ (with respect to the natural measure), we have to take the restriction of the integration to $W_\alpha$ in formula (5) seriously. If the three inner products do not all have the same sign, there still exists a triangle $abc$ (unique if no inner product is zero) but one of its sides will be longer than or equal to $\pi$.

Computing finally the amplitude function $A$ we find

$$A(\alpha, \beta, \gamma) = 16 \left| \langle \alpha|\beta \rangle_E \cdot \langle \beta|\gamma \rangle_E \cdot \langle \gamma|\alpha \rangle_E \right| \cdot \left( 1 - \det(\alpha\beta\gamma)^2 \right)^{-5/2} . \quad (13)$$
Conclusions

Formula (5) partly defines a deformed product of functions on a symplectic manifold whose form was conjectured in [We], in the spirit of the central (Weyl) representation of quantum observables [OdA] and strict quantization [Ri]. We have derived this formula using basic ideas from geometric quantization and groupoids, for symplectic spaces without homology in dimension 1 and which admit a complete affine connection for which geodesic inversion is a symplectomorphism. It should be noted that the final result does not depend upon the choice of the prequantization bundle $Y$ for the symplectic manifold $M$.

We emphasize the pure and simple geometrical character of the result (and its derivation). Accordingly, its algebraic and analytical properties need to be clarified. Also, it should be compared with other approaches (e.g. [Bi][BM][Ka][Urt], see also [Qi]). On the other hand, a striking property of formula (5) is its asymptotic behaviour, specially when the functions are oscillatory: $f_1 \propto \exp(ig_1/h)$, $f_2 \propto \exp(ig_2/h)$. In this case, stationary phase evaluation of formula (5) often yields $f_1 \ast f_2 \propto \exp\{i(g_1 \triangle g_2)/h\}$, where $(g_1 \triangle g_2)(m) = Stat_{(m',m'')}\{g_1(m') + g_2(m'') + S(m,m',m'')\}$ is the composition of central generating functions (of canonical relations) $g_1$ and $g_2$ [RO]. This is a rather promising feature to be used in semiclassical analysis.

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PdMR: Laboratório Nacional de Computação Científica; Av. Getúlio Vargas 333; Petrópolis, RJ 25651-070; Brazil

Current address: Department of Mathematics; University of California at Berkeley; Berkeley, CA 94720-3840; USA

E-mail address: prios@math.berkeley.edu

GMT: CNRS UMR 8524 AGAT & UFR de Mathématiques; Université de Lille I; F-59655 Villeneuve d’Ascq Cedex; France

E-mail address: Gijs.Tuynman@univ-lille1.fr