LOCAL CYCLIC HOMOLOGY FOR NONARCHIMEDEAN BANACH ALGEBRAS

RALF MEYER AND DEVARSHI MUKHERJEE

ABSTRACT. Let $V$ be a complete discrete valuation ring with uniformiser $\pi$. We introduce an invariant of Banach $V$-algebras called local cyclic homology. This invariant is related to analytic cyclic homology for complete, bornologically torsionfree $V$-algebras. It is shown that local cyclic homology only depends on the reduction mod $\pi$ of a Banach $V$-algebra and that it is homotopy invariant, matricially stable, and excisive.

1. INTRODUCTION

This article is part of a programme to define analytic cyclic homology theories for bornological algebras over nonarchimedean fields, with the goal of defining well behaved homology theories for algebras over their residue fields, which may have finite characteristic (see [1,2,7,8]). In this article, we define a version of cyclic homology that gives good results for Banach algebras over nonarchimedean local fields such as $\mathbb{Q}_p$. Our prototype here is the local cyclic homology theory in the archimedean case, defined first by Puschnigg for inductive systems of “nice” Fréchet algebras over $\mathbb{C}$ (see [10]) and then simplified by the first author in the setting of complete bornological algebras over $\mathbb{C}$ (see [6]). The remarkable property of local cyclic homology for algebras over $\mathbb{C}$ is that it remains well behaved for $C^*$-algebras. For instance, it is invariant under homotopies that are merely continuous, and it is stable under the $C^*$-algebraic tensor product with the compact operators.

The definition of archimedean local cyclic homology has two key ingredients. The first one is analytic cyclic homology for complete bornological algebras over $\mathbb{C}$. The first author arrived at its definition by rewriting Connes’ entire cyclic cohomology in terms of bornologies and taking the most natural “predual” of that cohomology theory. The second key ingredient is to turn a Banach algebra into a bornological algebra using the bornology of precompact subsets. This allows to use locally defined linear maps, which map a continuous function to a nearby smooth function.

Before we can talk about the nonarchimedean version of this, we fix some notation. Let $V$ be a complete discrete valuation ring. Let $\pi$ be its uniformiser, $F$ its residue field, and $F$ its fraction field. We assume throughout that $F$ has characteristic zero. The nonarchimedean version of analytic cyclic homology has already been defined in [2]. It is a functor $\text{HA}_*$ from the category of complete, torsionfree bornological $V$-algebras to the category of $F$-vector spaces. We usually work with the enriched version $\mathbb{H}A_*$ of $\text{HA}_*$, which takes values in the derived category of the quasi-Abelian category of countable projective systems of inductive systems of Banach $F$-vector spaces; to define $\mathbb{H}A_*(A)$, we first take the homotopy limit of $\mathbb{H}A(A)$, then a colimit. This gives a chain complex of $F$-vector spaces, whose homology is $\text{HA}_*(A)$.

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In the archimedean case, the functors \( \mathbb{H}A \) and \( HA_\ast \) are only homotopy invariant for smooth homotopies, and they are not expected to behave well for C*-algebras. The nonarchimedean versions of \( \mathbb{H}A \) and \( HA_\ast \) are shown in [2] to be invariant under dagger homotopies, which take values in the completed tensor product with the algebra \( V[t] \). The latter algebra is the Monsky–Washnitzer algebra of the affine plane. It consists of those power series \( \sum c_n t^n \) with \( c_n \in V \) where the valuations of the coefficients \( c_n \) grow at least linearly. It is unclear whether \( \mathbb{H}A \) and \( HA_\ast \) behave well for larger algebras such as the \( \pi \)-adic completion \( V[t] \), which is defined by asking only for \( \lim c_n = 0 \).

In this article, we define a nonarchimedean analogue of the precompact bornology, namely, the compactoid bornology. A subset \( S \) of a bornological \( V \)-module \( M \) is called compactoid if there is a bounded \( V \)-submodule \( T \subseteq M \) such that, for every \( n \in \mathbb{N} \), there is a finite set \( F_n \subseteq T \) with \( S \subseteq VF_n + \pi^n \cdot T \). Let \( B \) be a Banach algebra over \( F \) with a submultiplicative norm. Then its unit ball \( D \subseteq B \) is an algebra over \( V \) with some nice extra properties, which make it a dagger algebra. For any dagger algebra \( D \), let \( D' \) be \( D \) with the compactoid bornology. We define the local cyclic homology \( \mathbb{H}L(D) \) as \( \mathbb{H}A(D') \). We are going to prove that this theory has good homological properties and gives good results for many interesting Banach \( \pi \)-algebras. Namely, local cyclic homology is invariant under continuous homotopy, tensoring with the \( \pi \)-adic completion of finite matrices, and satisfies excision for all extensions of dagger algebras. And we compute it for Banach algebra versions of Leavitt path algebras, Laurent polynomials in several variables, and the Tate algebras of curves over \( V \).

The hardest part of the work was already done in our previous paper [2], where we built an analytic cyclic homology theory for algebras over the residue field \( F \). A key result in [2] says that \( \mathbb{H}A(A) \) for an algebra \( A \) over \( F \) is naturally isomorphic to \( \mathbb{H}A(D) \) if \( D \) is a dagger algebra with \( D/\pi D \cong A \) and such that the quotient bornology on \( D/\pi D \) is the fine one; we briefly call \( D \) fine mod \( \pi \). Roughly speaking, any dagger algebra lifting of \( A \) that is also fine mod \( \pi \) may be used to compute \( \mathbb{H}A(A) \). The compactoid bornology is always fine mod \( \pi \), and we prove that a dagger algebra with the compactoid bornology remains a dagger algebra. Therefore, \( \mathbb{H}A(A) \cong \mathbb{H}L(D) \) for any dagger algebra \( D \) with \( D/\pi D \cong A \). Here \( D \) may be \( \pi \)-adically complete or, in other words, a Banach \( \pi \)-algebra.

We could also have defined the local cyclic homology for dagger algebras by \( \mathbb{H}L(D) := \mathbb{H}A(D/\pi D) \). This is equivalent to our definition using the precompact bornology. This definition looks very quick, and we use it implicitly to prove the formal properties of local cyclic homology and compute some examples, based on the results in [2] showing that analytic cyclic homology for algebras over \( F \) is polynomially homotopy invariant, stable with respect to the \( \pi \)-algebra of finite matrices, and excusive, and based on examples computed there.

While the definition of \( \mathbb{H}L(D) \) as \( \mathbb{H}A(D/\pi D) \) looks rather quick, it just shifts all the difficulties to the analytic cyclic homology for \( F \)-algebras. This is defined by lifting an \( F \)-algebra to a projective system of inductive systems of \( V \)-algebras and then taking an analytic cyclic homology complex for the latter object. Results in [2] allow to replace the lifting that is used to define \( \mathbb{H}A(D/\pi D) \) by a pro-dagger algebra. The choices of such liftings that the general theory in [2] provides are, however, still rather unwieldy. The main observation in this article is that we may choose the given Banach algebra \( D \) with its precompact bornology to compute \( \mathbb{H}A(D/\pi D) \).
Thus our definition of $\mathcal{H}_L(D)$ turns out to be more direct. In addition, it clarifies the similarity with local cyclic homology for bornological $\mathbb{C}$-algebras.

We may also prove the formal properties of $\mathcal{H}_L$ using that $\mathcal{H}_L(D_1) \cong \mathcal{H}_L(D_2)$ if $D_1 \subseteq D_2$ is a dagger subalgebra with $D_1/\pi D_1 = D_2/\pi D_2$. For instance, the homotopy invariance of $\mathcal{H}_L$ reduces to the homotopy invariance of analytic cyclic homology for dagger homotopies that is proven in [2]. This proof would be very close to the proof in [5] that local cyclic homology for bornological $\mathbb{C}$-algebras is homotopy invariant. In the end, we switched to proofs that reduce to $\mathbb{F}$-algebras because these are shorter than proofs that remain within the realm of dagger algebras.

We end this article with an illuminating counterexample. A key open question in the study of analytic cyclic homology for dagger algebras is when the analytic and periodic cyclic homology of a nice dagger algebra agree. This is important because periodic cyclic homology is shown in [1] to specialise to rigid cohomology for Monsky–Washnitzer algebras, whereas analytic cyclic homology is shown in [8] to depend only on the reduction mod $\pi$. In the archimedean case, Khalkhali [5] proved that entire and periodic cyclic cohomology are isomorphic for Banach $\mathbb{C}$-algebras of finite projective dimension. This may lead to the hope that finite projective dimension may suffice to show that nonarchimedean analytic and periodic cyclic homology are isomorphic. We prove that this is not the case, by showing that the Tate algebra $\overline{V[[t]]}$ with the compactoid bornology is quasi-free – which means of projective dimension 1. But its periodic and analytic cyclic homology differ.

The paper is structured as follows. We begin by recalling some basic definitions and results about bornologies and dagger algebras in Section 2. We define the compactoid bornology and nuclearity in Section 3. Section 4 deals with inheritance properties of the compactoid bornology. We define local cyclic homology in Section 5. We prove that it has the expected homological properties in Section 6. We compute the local cyclic homology of some examples in Section 7.

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2. Preliminaries

Let $V$ be a complete discrete valuation ring. Let $\pi$ be its uniformiser, $\mathbb{F}$ its residue field, and $F$ its fraction field. We assume throughout that $F$ has characteristic zero.

As in [1,2,7,8], we use the framework of bornologies to do nonarchimedean analysis. A bornology on a set $X$ is a collection of its subsets, which are called bounded subsets, such that finite subsets are bounded and finite unions and subsets of bounded subsets remain bounded.

A bornological $V$-module is a $V$-module $M$ with a bornology such that every bounded subset is contained in a bounded $V$-submodule. We call a $V$-module map $f:M \rightarrow N$ bounded if it maps bounded subsets of $M$ to bounded subsets of $N$. A bornological $V$-algebra is a bornological $V$-module with a bounded multiplication map. A complete bornological $V$-module is a bornological $V$-module in which every bounded subset is contained in a bounded, $\pi$-adically complete $V$-submodule. Every bornological $V$-module $M$ has a completion $\overline{M}$ (see [1, Proposition 2.14]).

Example 2.1. The most basic example of a bornology on a $V$-module is the fine bornology, which consists of those subsets that are contained in a finitely generated $V$-submodule. Any fine bornological $V$-module is complete. By default, we equip modules over the residue field $\mathbb{F}$ with the fine bornology.
Definition 2.2 [7, Definition 4.1]). We call a bornological $V$-module $M$ (bornologically) torsionfree if multiplication by $\pi$ is a bornological embedding, that is, $M$ is algebraically torsionfree and $\pi^{-1} \cdot S := \{ x \in M : \pi x \in S \}$ is bounded for every bounded subset $S \subseteq M$. A $V$-module with the fine bornology is bornologically torsionfree if and only if it is torsionfree in the purely algebraic sense. For the rest of this article, we briefly write “torsionfree” instead of “bornologically torsionfree”.

Definition 2.3 [13]. We call a bornological $V$-algebra $D$ semidagger if, for every bounded subset $S \subseteq D$, the $V$-submodule $\sum_{i=0}^{\infty} \pi^{i} S^{i+1}$ is bounded in $D$. A complete, torsionfree, semidagger bornological $V$-algebra is called a dagger algebra.

Example 2.4. Any $F$-algebra with the fine bornology is semidagger and complete.

Example 2.5. Let $B$ be a Banach $F$-algebra. We assume the norm of $B$ to be submultiplicative. Let $D \subseteq B$ be the unit ball. Then $D \cdot D \subseteq D$, and $D$ becomes a $\pi$-adically complete, torsionfree $V$-algebra. Conversely, if such an algebra $D$ is given, then $D \to D \otimes F$ and there is a unique norm on $D \otimes F$ with unit ball $D$.

Let $D$ be the unit ball of a Banach $F$-algebra as above. Then we call $D$ with the bornology where all subsets are bounded a Banach $V$-algebra. This bornology makes $D$ a dagger algebra.

Definition 2.6. Any bornology on a $V$-algebra $D$ is contained in a smallest semidagger bornology, namely, the bornology generated by the $V$-submodules of the form $\sum_{i=0}^{\infty} \pi^{i} S^{i+1}$, where $S \subseteq D$ is bounded in the original bornology. This is called the linear growth bornology. We denote $D$ with the linear growth bornology by $D_{lg}$.

If $D$ is torsionfree, then the completion $D^{\dagger} := \overline{D_{lg}}$ is a dagger algebra (see [7, Proposition 3.8] or, in slightly different notation, [1, Lemma 3.1.12]).

Definition 2.7. A bornological $V$-module $M$ is called fine mod $\pi$ if the quotient bornology on $M/\pi M$ is the fine one. Equivalently, any bounded subset is contained in $F + \pi M$ for a finitely generated $V$-submodule $F \subseteq M$.

3. Compactoid bornologies

Let $A$ be an algebra over the residue field and let $D$ be a dagger algebra with $D/\pi D \cong A$. If, in addition, $D$ is fine mod $\pi$, then the main result of [8] gives a natural quasi-isomorphism $\mathbb{H}(D) \cong \mathbb{H}(A)$. It seems likely, however, that this fails when $D$ is a Banach $V$-algebra as in Example 2.5. We are going to introduce a nonarchimedean analogue of the precompact bornology on a topological $C$-vector space, which makes any dagger algebra fine mod $\pi$. This is the basis for our definition of local cyclic homology, exactly as in the archimedean case in [6].

Definition 3.1. Let $M$ be a bornological $V$-module. A subset $S \subseteq M$ is compactoid if it is contained in a bounded $V$-submodule $T \subseteq M$ such that, for every $n \in \mathbb{N}$, there is a finite set $F_{n} \subseteq T$ with $S \subseteq \bigvee F_{n} + \pi^{n} \cdot T$.

We call $M$ nuclear if any bounded subset of $M$ is already compactoid.

Remark 3.2. The analogous concept of a compactoid $V$-submodule of a locally convex $F$-vector space has already been studied by Peter Schneider (see [11, Section 12]).

Compactoid subsets are bounded, and the compactoid subsets in $M$ form another bornology. This bornology is always fine mod $\pi$. 
Example 3.3. Let $M$ be a Banach $V$-module as in Example 2.5. Then $S \subseteq M$ is compactoid if and only if for every $n \in \mathbb{N}$, the image of $S$ in $M/\pi^n M$ is finitely generated. This is the largest torsionfree bornology on $M$ that is fine mod $\pi$.

Example 3.4. For any $V$-module $M$, its fine bornology is nuclear.

The following proposition characterises the compactoid bornology in a different way, which is analogous to a very useful description of the precompact bornology on a Fréchet space over $\mathbb{C}$.

Proposition 3.5. Let $M$ be a torsionfree bornological $V$-module and let $S$ be a $V$-submodule. The following are equivalent:

(i) $S \subseteq M$ is compactoid;

(ii) there is a bounded $V$-submodule $T$ containing $S$ such that for each $n$, the $V$-submodule of $T/\pi^n T$ generated by the image of $S$ is finitely generated;

(iii) there are a bounded $V$-submodule $T \subseteq M$ and a null sequence $(t_n)$ in $T$ with

$$S = \left\{ s = \sum_{n=0}^{\infty} c_n t_n : (c_n) \in \ell^\infty(\mathbb{N}, V) \text{ and } s \text{ converges in } T \right\};$$

(iv) there are a bounded $V$-submodule $T \subseteq M$ with $S \subseteq T$ and a null sequence $(t_k)$ in $T$ such that $S \subseteq \sum_{k=0}^{\infty} V t_k + \pi^n T$ for each $n \in \mathbb{N}$.

Proof. The equivalence between (i) and (ii) is trivial. It is easy to prove that (iii) implies (iv). It remains to prove that (ii) implies (iii) and that (iv) implies (ii).

Assume (iv) and let $T$ and $(t_k)$ be as in (iv). For fixed $n \in \mathbb{N}$, there is $k_0 \in \mathbb{N}$ with $t_k \in \pi^n T$ for $k > k_0$. Then $S \subseteq \sum_{k=k_0}^{\infty} V t_k + \pi^n T$. Thus (iv) implies (ii). Conversely, assume (ii) and let $S$ and $T$ be as in (ii). We construct a null sequence $(t_k)$ as in (iii) inductively, by choosing an increasing sequence $(k_n)_{n \in \mathbb{N}}$ and $t_{k_n+1}, \ldots, t_{k_{n+1}} \in S \cap \pi^n T$ for all $n$, such that if $s \in S$ is written as $s \equiv \sum_{i=0}^{k_n} c_i t_i$ mod $\pi^n T$, then there are $c_i$ for $i = k_n + 1, \ldots, k_{n+1}$ with $s \equiv \sum_{i=k_n+1}^{k_{n+1}} c_i t_i$ mod $\pi^{n+1} T$. Since $S \subseteq T$, we may start the induction with $k_0 = -1$. Let $\text{Im}_n(S)$ be the image of $S$ in $T/\pi^n T$. In the induction step from $n$ to $n+1$, we use that $\text{Im}_{n+1}(S)$ is finitely generated. Then $S \subseteq \left( \sum_{i=0}^{k_n} V t_i + \text{Im}_{n+1}(S) \right) \cap \pi^n T/\pi^{n+1} T$ because $V$ is Noetherian.

Let $t_{k_n+1}, \ldots, t_{k_{n+1}} \in \pi^n T$ be representatives of generators for this submodule. Let $s \in S$. Then there are coefficients $c_i$ for $0 \leq i \leq k_n$ with $s \equiv \sum_{i=0}^{k_n} c_i t_i$ mod $\pi^n T$. Then $s - \sum_{i=0}^{k_n} c_i t_i \in \left( \sum_{i=0}^{k_n} V t_i + \text{Im}_{n+1}(S) \right) \cap \pi^n T/\pi^{n+1} T$ may be written as $\sum_{i=k_n+1}^{k_{n+1}} c_i t_i$. Then $s \equiv \sum_{i=k_n+1}^{k_{n+1}} c_i t_i$ mod $\pi^{n+1} T$. The construction ensures that the infinite series $\sum_{i=0}^{k_n} c_i t_i$ converges $\pi$-adically in $T$ towards $s$. \qed

Definition 3.6. Let $M'$ denote $M$ with the compactoid bornology.

Proposition 3.7. Let $M$ be a torsionfree bornological $V$-module. Then the bornological $V$-module $M'$ is nuclear.

Proof. Let $S \subseteq M'$ be bounded. That is, $S$ is compactoid in the bornology of $M$. Proposition 3.5 provides a bounded $V$-submodule $T \subseteq M$ and a null sequence $(x_n)$ in $T$ such that

$$S = \left\{ s = \sum_{n=0}^{\infty} c_n x_n : (c_n) \in \ell^\infty(\mathbb{N}, V) \text{ and } s \text{ converges in } T \right\}. $$
Since \((x_n)\) is a null sequence in \(T\), there are \(a_n \in \mathbb{N}\) and \(y_n \in T\) with \(x_n = \pi^{a_n}y_n\), \(\lim a_n = \infty\) and \(\lim y_n = 0\) in \(T\). The subset

\[
S' = \left\{ s = \sum_{n=0}^{\infty} c_n y_n : (c_n) \in \ell^\infty(\mathbb{N}, V), s \text{ converges in } T \right\}
\]

is still compactoid in \(T\) by Proposition \(3.5\). We claim that \(S\) is compactoid as a subset of \(S'\). For the proof, we show that any \(\pi\)-adically convergent series \(s = \sum_{n=0}^{\infty} c_n x_n\) is \(\pi\)-adically convergent in \(S'\). That \(s\) converges in \(T\) means that for each \(e \geq 1\), there is an \(n_0\) such that \(s - \sum_{n=0}^{n_1} c_n x_n \in \pi^e T\) for all \(n_1 \geq n_0\). Therefore, there are \(\beta_{e,n_1} \in T\) with

\[
\pi^e \beta_{e,n_1} = s - \sum_{n=0}^{n_1} c_n x_n = \sum_{n=n_1+1}^{\infty} c_n x_n = \sum_{n=n_1+1}^{\infty} c_n \pi^{a_n} y_n.
\]

If \(n_1\) is big enough, then \(a_n \geq e\) for all \(n > n_1\). Since \(M\) is torsionfree, we get

\[
\beta_{e,n_1} = \sum_{n=n_1+1}^{\infty} c_n \pi^{a_n-e} y_n.
\]

This series converges in \(T\) because \(\beta_{f,n_2}\) exist for \(n_2 \gg n_1\). Then its limit \(\beta_{e,n_1}\) belongs to \(S'\). Therefore, the series \(\sum_{n=0}^{\infty} c_n x_n\) converges in \(S'\) as required. \(\square\)

4. Inheritance properties of the compactoid bornology

Recall that \(M'\) denotes \(M\) with the compactoid bornology. We are going to prove that \(M'\) inherits many properties from \(M\) and that the property of being nuclear is preserved by several constructions with bornologies. We will use many of these results in our study of local cyclic homology. Let \(M\) be a bornological \(V\)-module.

**Lemma 4.1.** If the bornological \(V\)-module \(M\) is complete, then so is \(M'\).

*Proof.* Let \(S\) be a compactoid subset of \(M\). Then there is a bounded \(V\)-submodule \(T \subseteq M\) such that for any \(m \in \mathbb{N}\), there is a finite subset \(F_m \subseteq T\) with \(S \subseteq VF_m + \pi^m T\). Since \(M\) is complete, we may choose \(T\) to be a bounded \(\pi\)-adically complete \(V\)-submodule. Let \(T' := \cap_{m \in \mathbb{N}} (VF_m + \pi^m T)\). Then \(S \subseteq T'\) and \(T'\) is \(\pi\)-adically complete because it is \(\pi\)-adically closed in \(T\). It is compactoid by construction. Thus \(M'\) is complete. \(\square\)

**Lemma 4.2.** If the bornological \(V\)-module \(M\) is torsionfree, then so is \(M'\).

*Proof.* Let \(S \subseteq M\) be a compactoid subset. Let \(T\) and \(F_n\) be as in the definition of being compactoid and let \(n \geq 1\). Since \(M\) is torsionfree, \(\pi^{-1} T := \{x \in M : \pi x \in T\}\) is bounded. By construction,

\[
\pi^{-1} S \subseteq \pi^{-1} VF_{n+1} + \pi^n T \subseteq \pi^{-1} VF_{n+1} + \pi^n \pi^{-1} T.
\]

Multiplication by \(\pi\) on \(M\) is injective because \(M\) is torsionfree. It identifies \(\pi^{-1} VF_{n+1}\) with a submodule of \(VF_{n+1}\). Since \(V\) is Noetherian, it follows that \(\pi^{-1} VF_{n+1}\) is again finitely generated. Therefore, \(\square\) witnesses that \(\pi^{-1} S\) is compactoid.

**Lemma 4.3.** On torsionfree bornological \(F\)-modules, the assignment \(M \mapsto M'\) is the right adjoint functor of the inclusion of the subcategory of nuclear, torsionfree bornological \(V\)-modules.
Proof. We assume \( N \) to be torsionfree to ensure that \( N' \) is nuclear by Proposition 3.7. Lemma 4.2 shows that \( N' \) is again torsionfree. Let \( M \) and \( N \) be torsionfree bornological \( V \)-modules. Assume that the bornology on \( M \) is nuclear. We claim that a \( V \)-linear map \( \varphi: M \to N \) is bounded if and only if it is bounded as a map to \( N' \). One implication is trivial because compactoid subsets are bounded. Let \( S \subseteq M \) be a bounded subset. By assumption, \( S \) is compactoid. That is, there is a bounded \( V \)-submodule \( T \subseteq M \) containing \( S \), such that for each \( n \), there is a finite subset \( F_n \subseteq T \) with \( S \subseteq V F_n + \pi^n T \). The inclusions \( \varphi(S) \subseteq V \varphi(F_n) + \pi^n \varphi(T) \) witness that \( \varphi(S) \subseteq N \) is compactoid. That is, \( \varphi \) is bounded as a map \( M \to N' \). \( \square \)

Lemma 4.4. If \( M \) is a nuclear bornological \( V \)-module, then so is \( \overline{M} \).

Proof. Let \( S \subseteq \overline{M} \) be a bounded subset. The inductive limit description of \( \overline{M} \) shows that \( S \) is contained in the image of the \( \pi \)-adic completion \( \widehat{U} \) of some bounded submodule \( U \subseteq M \). Since \( M \) is compactoid, there is a bounded submodule \( U' \subseteq M \) such that the image of \( U \) in \( U'/\pi^k U' \) is finitely generated for each \( k \in \mathbb{N} \). This is equal to the image of \( \widehat{U} \) in \( \widehat{U}/\pi^k \widehat{U} \cong U'/\pi^k U' \). This witnesses that \( S \) is compactoid.

We now investigate the behaviour of the compactoid bornology under tensor products and extensions.

Proposition 4.5. Let \( M \) and \( N \) be complete, torsionfree bornological \( V \)-modules. Then there is an isomorphism of complete bornological \( V \)-modules

\[
M' \otimes N' \cong (M \otimes N)'.
\]

If \( M \) and \( N \) are nuclear, then so is \( M \otimes N \).

Proof. Lemma 4.1 shows that \( M' \) and \( N' \) are complete. A submodule in \( M' \otimes N' \) is bounded if it is in the image of \( S \otimes T \), where \( S \) and \( T \) are \( \pi \)-adically complete, compactoid submodules of \( M \) and \( N \). By definition, this means that there are bounded, \( \pi \)-adically complete \( V \)-submodules \( S' \) and \( T' \) of \( M \) and \( N \) such that the images of \( S \) in \( S' \to S'/\pi^n S' \) and of \( T \) in \( T' \to T'/\pi^n T' \) are finitely generated for each \( n \). Then the image of \( S \otimes T \) in \( S' \otimes T'/\pi^n (S' \otimes T') \) is finitely generated for each \( n \). This says that \( S \otimes T \) is compactoid in \( M \otimes N \).

Conversely, let \( X \subseteq M \otimes N \) be a compactoid subset of \( M \otimes N \). Then there is a bounded \( \pi \)-adically complete submodule \( U \subseteq M \otimes N \), such that the image of \( X \) under the quotient map to \( U/\pi^n U \) is finitely generated for each \( n \). There are bounded \( \pi \)-adically complete submodules \( S' \subseteq M \) and \( T' \subseteq N \) such that \( U \) is contained in the image of \( S' \otimes T' \) in \( M \otimes N \). We have \( S' \otimes T'/\pi^n (S' \otimes T') \cong (S' \otimes T')/\pi^n (S' \otimes T') \), and the image of \( X \) remains finitely generated in \( (S' \otimes T')/\pi^n (S' \otimes T') \) for all \( n \in \mathbb{N} \). By Proposition 3.5, there is a null sequence \((x_k)_{k \in \mathbb{N}} \) in \( S' \otimes T' \) such that \( X \) is contained in the image of \( \sum_{k=0}^{\infty} V x_k + \pi^n (S' \otimes T') \) in \( M \otimes N \). Since \( \lim x_k = 0 \), there are \( l_k \to \infty \) with \( x_k \in \pi^{2l_k} S' \otimes T' \). Since \( S' \otimes T'/\pi^n (S' \otimes T') \cong (S' \otimes T')/\pi^n (S' \otimes T') \), the proof of Proposition 3.5 shows that we may arrange \( x_k \in \pi^{2l_k} S' \otimes T' \). Then \( x_k = \pi^{2l_k} \sum_{i=1}^{m_k} a_i \otimes b_i \) for some \( m_k \in \mathbb{N} \), \( a_i \in S' \) and \( b_i \in T' \) for all \( i \). Let \( S \) and \( T \) be the \( \pi \)-adically closed \( V \)-submodules in \( S' \) and \( T' \) generated by the bounded subsets \( \{ \pi^{l_k} a_i \} \) and \( \{ \pi^{l_k} b_i \} \), respectively. By Proposition 3.3, \( S \) and \( T \) are compactoid submodules of \( S' \) and \( T' \), and \( X \subseteq S \otimes T \). That is, \( X \) is bounded in \( M' \otimes N' \).

By definition, a bornological \( V \)-module \( M \) is nuclear if and only if the canonical map \( M' \to M \) is a bornological isomorphism. If \( M' = M \) and \( N' = N \), then \((M \otimes N)' = M' \otimes N' = M \otimes N \) as well. \( \square \)
Proposition 4.6. Let \( K \hookrightarrow L \hookrightarrow P \) be an extension of complete, torsionfree bornological \( V \)-modules and let \( P \) be algebraically torsionfree. Then \( K' \hookrightarrow L' \hookrightarrow P' \) is an extension of complete bornological \( V \)-modules as well. In addition, \( L \) is nuclear if and only if both \( K \) and \( P \) are nuclear.

Proof. The functoriality of the compactoid bornology (Lemma 4.3) shows that a subset \( S \subseteq K \) that is compactoid in \( K \) remains compactoid in \( L \). Conversely, we claim that if \( S \subseteq K \) is compactoid as a subset of \( L \), it is compactoid as a subset of \( K \). Let \( T \supseteq S \) be a bounded subset of \( L \) such that the image of \( S \) in \( T/\pi^n T \) is finitely generated for all \( n \in \mathbb{N} \). Since \( S \subseteq K \), it follows that \( S \subseteq T \cap K \). Since \( P \) is torsionfree, \( \pi^n x \in K \) for \( x \in L \) can only happen if \( x \in K \). This says that the canonical map \((T \cap K)/\pi^n(T \cap K) \to T/\pi^n T \) is injective. Therefore, the image of \( S \) in \((T \cap K)/\pi^n(T \cap K) \) is finitely generated for all \( n \in \mathbb{N} \). This finishes the proof of the claim that the inclusion \( K' \hookrightarrow L' \) is a bornological embedding.

The quotient map clearly remains bounded as a map \( q : L' \to P' \). It remains to prove that it is a bornological quotient map, that is, any compactoid subset of \( P \) is the image of a compactoid subset of \( L \).

Let \( S \subseteq P \) be compactoid. Since \( P \) is complete, Proposition 4.6 implies that there is a bounded, \( \pi \)-adically complete \( V \)-submodule \( T \subseteq P \) containing \( S \) and a null sequence \((x_n)\) in \( T \) such that \( S \subseteq \{s = \sum_{n \in \mathbb{N}} c_n x_n : (c_n) \in \ell^\infty(\mathbb{N}, V)\} \); these sequences converge automatically because \( \lim x_n = 0 \) and \( T \) is complete. Since \( L \) is complete, there is a bounded, \( \pi \)-adically complete \( V \)-submodule \( \hat{W} \subseteq L \) that lifts \( T \). We may lift \((x_n)\) to a null sequence \((\hat{x}_n)\) in \( \hat{W} \). Then \( \hat{S} = \{\sum_{n=0}^\infty c_n \hat{x}_n : (c_n) \in \ell^\infty(\mathbb{N}, V)\} \) is a compactoid submodule of \( \hat{L} \) that lifts \( S \).

By definition, a bornological \( V \)-module \( M \) is nuclear if and only if the canonical map \( M' \to M \) is a bornological isomorphism. The category of bornological \( V \)-modules is quasi-Abelian. This implies a version of the Five Lemma. For our two extensions \( K' \hookrightarrow L' \hookrightarrow P' \) and \( K \hookrightarrow L \hookrightarrow P \), it says that the map \( L' \to L \) is a bornological isomorphism if the maps \( K' \to K \) and \( P' \to P \) are bornological isomorphisms. Conversely, it is easy to see that subspaces and quotients inherit nuclearity. \( \square \)

Lemma 4.7. Let \( D \) be a bornological algebra and let \( D' \) be \( D \) with the compactoid bornology. This is again a bornological algebra. If \( D \) is semidagger, then so is \( D' \).

Proof. It is easy to see that the product of compactoid subsets is again compactoid, so that \( D' \) is a bornological algebra. Now assume \( D \) is semidagger. We are going to prove that \( D' \) is semidagger as well. Let \( S \) be a compactoid \( V \)-submodule of \( D \). We must prove that \( S_1 := \sum_{n=0}^\infty \pi^n S^{n+1} \) is compactoid as well. Since \( S \) is compactoid, there are a bounded subset \( T \) and a null sequence \((x_n)_{n \in \mathbb{N}}\) in \( T \) such that \( S \subseteq \sum_{n \in \mathbb{N}} V x_n + \pi^n T \) for each \( h \in \mathbb{N} \). Since \( D \) is semidagger, the subset \( T_{1/2} := \sum_{i=0}^\infty \pi^i/2^{i+1} T_{i+1} \supseteq T \) is still bounded (see Lemma 3.1.10). For \( n \in \mathbb{N} \), \( i = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1} \), let \( x_i := x_{i_0} \cdots x_{i_n} \in T^n \). Then

\[
S_1 = \sum_{n=0}^\infty \pi^n S^{n+1} \subseteq \sum_{n=0}^\infty \left( \sum_{i \in \mathbb{N}^{n+1}} V \pi^n x_i + \pi^k \pi^n T^{n+1} \right) \subseteq \left( \sum_{n=0}^\infty \sum_{i \in \mathbb{N}^{n+1}} V \pi^n x_i \right) + \pi^k T_{1/2}.
\]

We may arrange the countable set of elements \((\pi^n x_i)\) for \( n \in \mathbb{N} \), \( i \in \mathbb{N}^{n+1} \) into a sequence that converges to zero in \( T_{1/2} \). Therefore, the inclusion above witnesses that \( S_1 \) is compactoid. \( \square \)

Lemma 4.8. If \( D \) is a dagger algebra, then so is \( D' \).
Proof. This follows from Lemmas 4.1, 4.2 and 4.7.

Proposition 4.9. Let \( R \) be a nuclear bornological \( V \)-algebra. Then \( R_{lg} \) and \( R^! = \overline{R_{lg}} \) are nuclear.

Proof. By Lemma 4.4, we only need to prove that \( R_{lg} \) is nuclear. Let \( S \) be a subset of linear growth in \( R \). Then there is a bounded submodule \( T \subseteq R \) with \( S \subseteq T \) := \( \sum_{j=0}^{\infty} \pi^j T^{j+1} \). Since \( R \) is compactoid, Proposition 3.5 gives a bounded submodule \( \tilde{T} \) of \( R \) and a null sequence \( (x_\nu) \in \tilde{T} \) such that \( \tilde{T} \subseteq \sum_{n\in \mathbb{N}} V x_\nu + \pi^k \tilde{T} \) for each \( k \geq 1 \). Now we consider the countable set of all products \( \pi^j x_{i_1} \cdots x_{i_{1+k}} \) for \( j, i_1, \ldots, i_{j+1} \in \mathbb{N} \). For each \( k \), there are only finitely many such terms that do not belong to \( \pi^k \tilde{T} \). Therefore, we may order these products into a sequence \( (y_m)_{m\in \mathbb{N}} \) that converges to 0 in the \( \pi \)-adic topology on \( \tilde{T} \). Now it follows from the construction that \( T_1 \subseteq \sum_{m \in \mathbb{N}} V y_m + \pi^k \tilde{T} \) for each \( k \in \mathbb{N} \). Thus \( T_1 \) is compactoid. Hence so is \( S \).

Corollary 4.10. Let \( R \) be a torsionfree bornological algebra with the fine bornology. Then \( R^! = \overline{R_{lg}} \) is nuclear.

Proof. The fine bornology on \( R \) is nuclear by Example 3.4. Then apply Proposition 4.9.

Let \( R \) be any torsionfree bornological \( V \)-algebra. There is a canonical bounded algebra homomorphism \( R \to R^! \). Since the compactoid bornology is functorial, it induces a bounded algebra homomorphism \( R' \to (R^!)' \). The target is a dagger algebra by Lemma 4.8. So this homomorphism extends uniquely to a bounded homomorphism

\[(R^!)' \to (R^!)'.\]

In general, the homomorphism \((R^!)' \to (R^!)'\) need not be invertible. The following proposition describes the two situations where we know this:

Proposition 4.11. Let \( R \) be a torsionfree bornological algebra. If \( R \) is nuclear or dagger, then the canonical map \((R^!)' \to (R^!)'\) is an isomorphism.

Proof. Suppose first that \( R \) is nuclear. Then \( R \cong R' \), so that \((R^!)' = R^!\). The canonical map \((R^!)' \to R^!\) generates the required inverse map \((R^!)' \to (R^!)'). Next assume that \( R \) is a dagger algebra. Then \( R \cong R^!\) and so \((R^!)' \cong R'\). The canonical map \( R' \to (R^!)'\) provides the required inverse map \((R^!)' \to (R^!)').

5. Definition of local cyclic homology

Our definition of local cyclic homology is based on the analytic cyclic homology theories introduced in [2,8]. We do not recall how they are defined. Suffice it to say the following. Let \( \text{Ind}(\text{Ban}_F) \) be the quasi-Abelian category of countable projective systems of inductive systems of Banach \( F \)-vector spaces. There are functors

\[\mathbb{H}: \{\text{complete, torsionfree bornological } V\text{-algebras}\} \to \text{Der}\left(\text{Ind}(\text{Ban}_F)\right),\]

\[\mathbb{H}: \{F\text{-algebras}\} \to \text{Der}\left(\text{Ind}(\text{Ban}_F)\right).\]

Their definitions are based on the Cuntz–Quillen approach to cyclic homology. The first functor is invariant under dagger homotopies and Morita equivalences and satisfies excision for extensions with a bounded \( V \)-linear section. The second functor is invariant under polynomial homotopies, stable for matrices and satisfies excision.
Then there is a canonical quasi-isomorphism

\[ R \to \frac{R}{\pi R} \]

bounded. There are natural quasi-isomorphisms

\[ \varrho : D_n \to A \]

for all extensions of \( F \)-algebras. To be more precise, the first functor above is the composite of the functor defined in \([2]\) with the dissection functor from the category of complete bornological \( F \)-vector spaces to the category of inductive systems of Banach \( F \)-vector spaces. This composite is already used in \([8]\). The main result in \([8]\) is the following:

**Theorem 5.1** ([8] Theorem 5.9). Let \( \mathcal{A} \) be a \( \mathbb{F} \)-algebra. Let \( D = (D_n)_{n \in \mathbb{N}} \) be a projective system of dagger algebras that is fine mod \( \pi \). Let \( \varrho : D \to \mathcal{A} \) be a pro-homomorphism, represented by a coherent family of surjective homomorphisms \( \varrho_n : D_n \to A \) for \( n \in \mathbb{N} \). Assume that \( \ker \varrho = (\ker \varrho_n)_{n \in \mathbb{N}} \) is analytically nilpotent. Then there is a canonical quasi-isomorphism \( \mathbb{H} \mathcal{A}(D) \cong \mathbb{H} \mathcal{A}(D) \).

In particular, if \( D \) is a dagger algebra and fine mod \( \pi \), then \( \mathbb{H} \mathcal{A}(D) \cong \mathbb{H} \mathcal{A}(D/\pi D) \).

This result will be the basis for our study of local cyclic homology. Here it is crucial to restrict to dagger algebras that are fine mod \( \pi \). We do not know how to compute \( \mathbb{H} \mathcal{A}(D) \) when \( D \) is a Banach \( V \)-algebra as in Example 2.5.

**Definition 5.2.** Let \( D \) be a dagger algebra and let \( D' \) be the algebra \( D \) with the compactoid bornology. The local cyclic homology complex of \( D \) is the chain complex \( \mathbb{H} \mathbb{L}(D) := \mathbb{H} \mathcal{A}(D') \), viewed as an object in the derived category of \( \text{Ind}(\text{Ban}_F) \).

**Lemma 5.3.** \( \mathbb{H} \mathbb{L} \) is a functor on the category of dagger algebras.

**Proof.** The compactoid bornology is functorial by Lemma 4.3 and so is \( \mathbb{H} \mathcal{A} \). \( \square \)

**Theorem 5.4.** Let \( D_1 \) and \( D_2 \) be dagger algebras. If \( D_1/\pi D_1 \cong D_2/\pi D_2 \), then \( \mathbb{H} \mathbb{L}(D_1) \cong \mathbb{H} \mathbb{L}(D_2) \).

**Proof.** By Lemma 4.8 the algebras \( D'_1 \) and \( D'_2 \) are again dagger algebras. They are fine mod \( \pi \) by construction of the compactoid bornology, and \( A := D'_1/\pi D'_1 \cong D'_2/\pi D'_2 \). Theorem 5.1 implies

\[ \mathbb{H} \mathbb{L}(D_1) \cong \mathbb{H} \mathcal{A}(D'_1) \cong \mathbb{H} \mathcal{A}(A) \cong \mathbb{H} \mathcal{A}(D'_2) \cong \mathbb{H} \mathbb{L}(D_2) \].

Theorem 5.4 explains our choice of the compactoid bornology in the definition of local cyclic homology. Its crucial features are that it is fine mod \( \pi \) and that it still gives a dagger algebra. This ensures that \( \mathbb{H} \mathbb{L}(D) \) for a dagger algebra \( D \) depends only on \( D/\pi D \). There are, however, other bornologies that may work just as well. For instance, we could give a dagger algebra the linear growth bornology generated by the fine bornology. Some work would, however, be required to check whether this bornology is complete. Another advantage of the compactoid bornology is that it does not use the multiplication and that it is analogous to the precompact bornology in the archimedean case, which is used in that setting to define local cyclic homology (see \([6]\)).

**Corollary 5.5.** Let \( R \) be a torsionfree \( V \)-algebra and let \( \hat{R} \) be its \( \pi \)-adic completion. Give \( R \) the fine bornology and equip \( \hat{R} \) with the bornology where all subsets are bounded. There are natural quasi-isomorphisms

\[ \mathbb{H} \mathcal{A}(R^1) \cong \mathbb{H} \mathbb{L}(R^1) \cong \mathbb{H} \mathbb{L}(\hat{R}) \].

**Proof.** The algebra \( \hat{R} \) is a Banach \( V \)-algebra as in Example 2.5 and \( R^1 \) is a dagger algebra by construction. It is nuclear by Corollary 4.10. So \( \mathbb{H} \mathcal{A}(R^1) = \mathbb{H} \mathbb{L}(R^1) \). We compute

\[ \hat{R}/\pi \hat{R} \cong R/\pi R = R_{\text{lg}}/\pi R_{\text{lg}} \cong \hat{R}_{\text{lg}}/\pi \hat{R}_{\text{lg}} = R^1/\pi R^1 \].
the isomorphism $R_{1g}/\pi R_{1g} \cong \overline{R}_{1g}/\pi R_{1g}$ follows because $R_{1g}/\pi R_{1g}$ as an $F$-algebra is bornologically complete as a $V$-algebra (compare [2, Proposition 2.3.3]). Now Theorem 5.4 implies $\mathbb{HL}(R') \cong \mathbb{HL}(\overline{R})$.

6. Homotopy invariance, excision and matricial stability

We are going to prove that local cyclic homology satisfies some nice formal properties, namely, homotopy invariance, matricial stability, excision, and an analogue of Bass’ Fundamental Theorem. These will be used in Section 7 to compute the local cyclic homology of certain Banach $V$-algebras.

**Theorem 6.1 (Homotopy invariance).** Let $D$ be a dagger algebra. Give $\overline{V[t]}$ the bornology where all subsets are bounded. There is a quasi-isomorphism

$$\mathbb{HL}(D) \cong \mathbb{HL}(D \circledast \overline{V[t]}).$$

**Proof.** The bornological algebras $D'$ and $(D \circledast \overline{V[t]})'$ are both dagger by Proposition 4.3 and fine mod $\pi$. Then Theorem 5.1 and [8, Theorem 8.1] imply

$$\mathbb{HL}(D) \cong \mathbb{HA}(D/\pi D) \cong \mathbb{HA}(D/\pi D \circledast F[t]) \cong \mathbb{HA}((D \circledast \overline{V[t]})').$$

An elementary homotopy between two bounded homomorphisms $f, g: D_1 \Rightarrow D_2$ is a bounded homomorphism $F: D_1 \to D_2 \circledast \overline{V[t]}$ with $ev_0 \circ F = f$ and $ev_1 \circ F = g$. Let homotopy be the equivalence relation generated by elementary homotopy. If $f, g: D_1 \Rightarrow D_2$ are homotopic, then $\mathbb{HL}(f) = \mathbb{HL}(g)$ by Theorem 6.1.

**Corollary 6.2.** For any Banach $V$-algebra $D$, there is a natural quasi-isomorphism $\mathbb{HL}(D) \cong \mathbb{HL}(\overline{D[t]}).$ Here $\overline{D[t]}$ is the Banach $V$-algebra of all polynomials $\sum_{n=0}^{\infty} d_n t^n$ with $d_n \in D$ for all $n \in \mathbb{N}$ and $\lim n = 0$ $\pi$-adically in $D$.

**Theorem 6.3 (Matrix stability).** Let $D$ be a dagger algebra. For a set $\Lambda$, let $M_\Lambda(V)$ be the $V$-algebra of finitely supported matrices indexed by $\Lambda \times \Lambda$. Let $\lambda \in \Lambda$. The canonical map $i_\lambda: D \to D \circledast M_\Lambda(V)$, $x \mapsto e_{\lambda,\lambda} \otimes x$, induces a quasi-isomorphism

$$\mathbb{HL}(D) \cong \mathbb{HL}(D \circledast M_\Lambda(V)).$$

**Proof.** Let $D/\pi D = A$. We compute

$$\mathbb{HL}(D) \cong \mathbb{HA}(A) \cong \mathbb{HA}(M_\Lambda(A)) \cong \mathbb{HA}((D \circledast M_\Lambda(V))') = \mathbb{HL}(D \circledast M_\Lambda(V));$$

here the second quasi-isomorphism uses [8, Proposition 8.2], and the third quasi-isomorphism uses Theorem 5.1 and $D \circledast M_\Lambda(V)/\pi \cdot (D \circledast M_\Lambda(V)) \cong M_\Lambda(A)$. □

**Corollary 6.4.** Let $D$ be a Banach $V$-algebra. There is quasi-isomorphism $\mathbb{HL}(D) \cong \mathbb{HL}(M_\Lambda(D))$. Here $M_\Lambda(D)$ is the Banach $V$-algebra of all matrices $(d_{\lambda,\mu})_{\lambda,\mu \in \Lambda}$ with $d_{\lambda,\mu} \in D$ for all $\lambda, \mu \in \Lambda$ and such that, for each $k \in \mathbb{N}$, there are only finitely many $\lambda, \mu \in \Lambda$ with $d_{\lambda,\mu} \not\in \pi^k D$.

Next, we prove that local cyclic homology satisfies excision for any extension of dagger algebras; no bounded linear section is needed.

**Theorem 6.5 (Excision Theorem).** An extension of dagger algebras $K \overset{i}{\to} E \overset{\pi}{\to} Q$ induces an exact triangle

$$\mathbb{HL}(K) \overset{i}{\to} \mathbb{HL}(E) \overset{\pi}{\to} \mathbb{HL}(Q) \overset{\delta}{\to} \mathbb{HL}(K)[-1]$$

in the derived category of the quasi-abelian category $\text{Ind}(\text{Ban}_F)$. 

Proof. The given extension induces an extension of \(\mathbb{F}\)-algebras
\[ K \otimes_V \mathbb{F} \to E \otimes_V \mathbb{F} \to Q \otimes_V \mathbb{F} \]
because \(Q\) is torsionfree. This induces a natural exact triangle
\[
\mathbb{H}A(K \otimes_V \mathbb{F}) \to \mathbb{H}A(E \otimes_V \mathbb{F}) \to \mathbb{H}A(Q \otimes_V \mathbb{F}) \to \mathbb{H}A(K \otimes_V \mathbb{F})[-1]
\]
in the derived category by \[8\,\text{Theorem 8.3}\]. By Theorem 5.4 this exact triangle is isomorphic to an exact triangle as in the statement of the theorem. \(\square\)

An exact triangle in a derived category implies a long exact sequence in homology (compare \[2\,\text{Theorem 5.1}]\).

**Theorem 6.6** (Bass Fundamental Theorem). Let \(D\) be a dagger algebra. Then \(\mathbb{H}L(D \otimes \mathbb{V}[t, t^{-1}]) \cong \mathbb{H}L(D) \oplus \mathbb{H}L(D)[1]\); here \([1]\) means a degree shift.

**Proof.** Let \(A = D/\pi D\). Then we compute
\[
\mathbb{H}L(D \otimes \mathbb{V}[t, t^{-1}]) \cong \mathbb{H}A((D \otimes \mathbb{V}[t, t^{-1}])^\dagger) \cong \mathbb{H}A(A \oplus \mathbb{F}[t, t^{-1}])
\]
\[
\cong \mathbb{H}A(A) \oplus \mathbb{H}A(A)[1] \cong \mathbb{H}A(D') \oplus \mathbb{H}A(D')[1] \cong \mathbb{H}L(D) \oplus \mathbb{H}L(D)[1],
\]
where \(\mathbb{H}A(A \oplus \mathbb{F}[t, t^{-1}]) \cong \mathbb{H}A(A) \oplus \mathbb{H}A(A)[1]\) follows from \[8\,\text{Corollary 8.5}]\). \(\square\)

## 7. Some computations of local cyclic homology

In this section, we compute local cyclic homology for some Banach \(V\)-algebras.

**Example 7.1.** Let \(E\) be a directed graph. Let \(L(F, E)\) and \(C(F, E)\) be its Leavitt and Cohn path algebras over \(F\), respectively. Their lifts \(L(V, E)\) and \(C(V, E)\) are torsionfree \(V\)-algebras, which we equip with the fine bornology. Their \(\pi\)-adic completions \(L(V, E)\) and \(C(V, E)\) are Banach \(V\)-algebras. The analytic cyclic homology of the dagger completions \(L(V, E)^\dagger\) and \(C(V, E)^\dagger\) is computed in \[2\,\text{Theorem 8.1}]\). With the compactoid bornology, these dagger completions are still dagger algebras by Lemma 4.8 and still fine mod \(\pi\). Corollary 4.10 shows directly that \(L(V, E)^\dagger\) is nuclear.

Theorem 5.4 implies that the local cyclic homology of \(L(V, E)\) and \(C(V, E)\) is naturally isomorphic to the analytic cyclic homology computed in \[2\], namely,
\[
\mathbb{H}L(L(V, E)) \cong \mathbb{H}A(L(F, E)) \cong \text{coker}(N_E) \oplus \ker(N_E)[1],
\]
\[
\mathbb{H}L(C(V, E)) \cong \mathbb{H}A(C(F, E)) \cong F(\mathbb{L}^\pi).
\]

**Example 7.2.** Consider the \(V\)-algebra of Laurent polynomials in \(n\) variables \(L_n(V) = V[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]\). We may write this as \(L_{n-1}(V) \otimes V[t_n, t_n^{-1}]\). Taking \(\pi\)-adic completions gives
\[
\overline{L_n(V)} \cong L_{n-1}(V) \otimes V[t, t^{-1}].
\]
Then Theorem 6.6 implies
\[
\mathbb{H}L(\overline{L_n(V)}) \cong \mathbb{H}L(L_{n-1}(V)) \oplus \mathbb{H}L(L_{n-1}(V))[1] \cong \mathbb{H}L(L_{n-1}(V)) \otimes (F \oplus F[1]).
\]
Iterating this and using \(\mathbb{H}L(L_0(F)) = \mathbb{H}A(V) = F\), we identify \(\mathbb{H}L(L_n(V))\) with the \(F\)-vector space \(\Lambda^*(F^n)\), the exterior algebra, with the usual \(\mathbb{Z}/2\)-grading and
but the \( \pi \mathcal{R} \) that in addition, there is a canonical chain map

\[
\text{Theorem 7.4.}
\]

This is a Banach varieties of higher dimension. It is unclear, however, how to generalise these results\
variety over \( \mathcal{C} \). Elkik \cite{4} shows that there is a smooth \( \mathcal{V} \)-algebra \( \mathcal{X} \) be a smooth affine variety over \( \mathcal{F} \).

\[
\text{Example 7.3.}
\]

Let \( \mathcal{X} \) be a smooth affine variety over \( \mathcal{F} \) and let \( \mathcal{A} = \mathcal{O}(\mathcal{X}) \) be its coordinate ring. Elkik \cite{4} shows that there is a smooth \( \mathcal{V} \)-algebra \( \mathcal{X} \) is an isomorphism of \( \mathcal{V} \)-algebras.

\[
\text{Proposition 7.5.}
\]

by projective \( \mathcal{A} \) for \( \mathcal{A} \) is quasi-isomorphisms; here \( \mathcal{H} \mathcal{P} \) denotes the chain complex that computes periodic cyclic homology (see \cite{1} for its definition in the current setting).

\[
\text{Theorem 7.4.}
\]

In the situation above, we also know that \( \mathcal{H} \mathcal{L}(\mathcal{R}) = \mathcal{H} \mathcal{A}(\mathcal{R}) \). If \( \mathcal{X} \) has dimension 1, then \( \mathcal{H} \mathcal{A}_1(\mathcal{X}) \) is the rigid cohomology of \( \mathcal{X} \) with coefficients in \( \mathcal{F} \) by \cite{8} Corollary 5.6).

\[
\text{Theorem 7.4.}
\]

In addition, there is a canonical chain map \( \mathcal{H} \mathcal{A}(\mathcal{R}) \rightarrow \mathcal{H} \mathcal{P}(\mathcal{R} \otimes \mathcal{F}) \).

\[
\text{Proof.}
\]

By Lemma \cite{4,8}, both \( \mathcal{B} \) and \( (\mathcal{R})' \) are dagger algebras. Since they are also fine mod \( \pi \), Theorem \cite{5} gives a quasi-isomorphism \( \mathcal{H} \mathcal{L}(\mathcal{R}) \cong \mathcal{H} \mathcal{L}(\mathcal{B}) \). The periodic cyclic homology complex \( \mathcal{H} \mathcal{P}(\mathcal{R} \otimes \mathcal{F}) \) for bornological \( \mathcal{F} \)-algebras like \( \mathcal{R} \otimes \mathcal{F} \) is defined in \cite{1}. There is a canonical chain map \( \mathcal{H} \mathcal{A}(\mathcal{R}) \rightarrow \mathcal{H} \mathcal{P}(\mathcal{R} \otimes \mathcal{F}) \) because the chain complex \( \mathcal{H} \mathcal{A}(\mathcal{R}) \) is defined as a subcomplex of \( \mathcal{H} \mathcal{P}(\mathcal{R} \otimes \mathcal{F}) \).

\[
\text{Example 7.3.}
\]

Let \( \mathcal{B} := \mathcal{V}[t] \) be the coordinate ring of the affine plane. Then \( \mathcal{D}' \), the \( \pi \)-adic completion of \( \mathcal{D} \) equipped with the compactoid bornology, is quasi-free, but \( \mathcal{H} \mathcal{A}_1(\mathcal{D}') \neq \mathcal{H} \mathcal{P}_1(\mathcal{D}' \otimes \mathcal{F}) \).

\[
\text{Proposition 7.5.}
\]

Let \( \mathcal{D} := \mathcal{V}[t] \) be the coordinate ring of the affine plane. Then \( \mathcal{D}' \), the \( \pi \)-adic completion of \( \mathcal{D} \) equipped with the compactoid bornology, is quasi-free, but \( \mathcal{H} \mathcal{A}_1(\mathcal{D}') \neq \mathcal{H} \mathcal{P}_1(\mathcal{D}' \otimes \mathcal{F}) \).

\[
\text{Proof.}
\]

Define \( \Omega^1(\mathcal{D}) \) as in \cite{3} and let \( \Omega^1(\mathcal{D}') = \ker (\mathcal{D} \otimes \mathcal{D}' \rightarrow \mathcal{D}') \). It is easy to see that the map

\[
\mathcal{D} \otimes \mathcal{D} \rightarrow \Omega^1(\mathcal{D}), \quad f_1 \otimes f_2 \mapsto f_1 \cdot dt \cdot f_2
\]

is an isomorphism of \( \mathcal{D} \)-bimodules. Since \( \mathcal{D} \) is torsionfree, the \( \pi \)-adic completions form an extension \( \Omega^1(\mathcal{D}) \rightarrow \mathcal{D} \otimes \mathcal{D} \rightarrow \mathcal{D} \). Therefore,

\[
\Omega^1(\mathcal{D}) \cong \Omega^1(\mathcal{D}) \cong \mathcal{D} \otimes \mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}.
\]
Then $\widetilde{\Omega}(\mathcal{D})' \to \mathcal{D} \otimes \mathcal{D}' \to \mathcal{D}'$ is an extension by Proposition 4.6. Proposition 4.5 shows that $\mathcal{D} \otimes \mathcal{D}' \cong \mathcal{D}' \otimes \mathcal{D}'$. This implies an isomorphism

$$\Omega^{1}(\mathcal{D}') \cong \mathcal{D}' \otimes \mathcal{D}'$$

of $\mathcal{D}'$-bimodules. So $\Omega^{1}(\mathcal{D}')$ is a projective $\mathcal{D}'$-bimodule. Then $\mathcal{D}'$ is quasi-free.

Now let $\mathcal{D} := \mathcal{D}' \otimes F$. This is again quasi-free. Since $F$ has characteristic zero, $\text{HP}(\mathcal{D}) \cong \text{X}(\mathcal{D})$. The homology of this chain complex is the algebraic de Rham cohomology of $\mathcal{D}$. It is well known that this differs from the Monsky–Washnitzer or rigid cohomology of the plane (see [9]). At the same time,

$$\text{HA}(\mathcal{D}') = \text{HHL}(\mathcal{D}) \cong \text{HHL}(\mathcal{D}') = \text{HA}(\mathcal{D'}) \cong \text{HA}(F[t])$$

by Theorem 5.4. This agrees with the rigid cohomology of the affine plane. As a result, $\text{HA}_*(\mathcal{D}') \neq \text{HP}_*(\mathcal{D}' \otimes F)$. □

References

[1] Guillermo Cortiñas, Joachim Cuntz, Ralf Meyer, and Georg Tamme, *Nonarchimedean bornologies, cyclic homology and rigid cohomology*, Doc. Math. 23 (2018), 1197–1245, doi: 10.25537/dm.2018v23.1197-1245 MR 3874948

[2] Guillermo Cortiñas, Ralf Meyer, and Devarshi Mukherjee, *Non-Archimedean analytic cyclic homology*, Doc. Math. 25 (2020), 1353–1419, doi: 10.25537/dm.2020v25.1353-1419 MR 4164727

[3] Joachim Cuntz and Daniel Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. 8 (1995), no. 2, 251–289, doi: 10.1090/S0894-0347-1995-1303029-0 MR 1303029

[4] Renée Elkik, *Solutions d’équations à coefficients dans un anneau hensélien*, Ann. Sci. École Norm. Sup. (4) 6 (1973), 553–603 (1974), available at http://www.numdam.org/item?id=ASENS_1973_4_6_4_553_0 MR 0345966

[5] Masoud Khalkhali, *Algebraic connections, universal bimodules and entire cyclic cohomology*, Comm. Math. Phys. 161 (1994), no. 3, 433–446, available at https://projecteuclid.org/euclid.cmp/1104270005 MR 1269386

[6] Ralf Meyer, *Local and analytic cyclic homology*, EMS Tracts in Mathematics, vol. 3, European Mathematical Society (EMS), Zürich, 2007. doi: 10.4171/039 MR 2337277

[7] Ralf Meyer and Devashri Mukherjee, *Dagger completions and bornological torsion-freeness*, Q. J. Math. 70 (2019), no. 3, 1135–1156, doi: 10.1093/qmath/haz012 MR 4009486

[8] _____, *Analytic cyclic homology in positive characteristic* (2021), eprint. arXiv: 2109.01470

[9] Paul Monsky and G. Washnitzer, *Formal cohomology. I*, Ann. of Math. (2) 88 (1968), 181–217, doi: 10.2307/1970571 MR 0248141

[10] Michael Puschnigg, *Diffeotopy functors of ind-algebras and local cyclic cohomology*, Doc. Math. 8 (2003), 143–245, available at http://www.math.uni-bielefeld.de/documenta/vol-08/09.html MR 2029166

[11] Peter Schneider, *Nonarchimedean functional analysis*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. doi: 10.1007/978-3-662-04728-6 MR 1869547

Email address: rmeyer2@uni-goettingen.de

Email address: devarshi.mukherjee@mathematik.uni-goettingen.de

Mathematisches Institut, Universität Göttingen, Bunsenstrasse 3–5, 37073 Göttingen, Germany