On the Impact of Social Media Recommendations on Opinion Consensus

Vincenzo Auletta  Antonio Coppola  Diodato Ferraioli
Università degli Studi di Salerno
Email: {auletta,ancoppola,dferraioli}@unisa.it

Abstract

We consider a discrete opinion formation problem in a setting where agents are influenced by both information diffused by their social relations and from recommendations received directly from the social media manager. We study how the “strength” of the influence of the social media and the homophily ratio affect the probability of the agents of reaching a consensus and how these factors can determine the type of consensus reached.

In a simple 2-symmetric block model we prove that agents converge either to a consensus or to a persistent disagreement. In particular, we show that when the homophily ratio is large, the social media has a very low capacity of determining the outcome of the opinion dynamics. On the other hand, when the homophily ratio is low, the social media influence can have an important role on the dynamics, either by making harder to reach a consensus or inducing it on extreme opinions.

Finally, in order to extend our analysis to more general and realistic settings we give some experimental evidences that our results still hold on general networks.

1 Introduction

Over the last years, we witnessed a rapid rise of the role of online social networking platforms, such as Facebook or Twitter, in our life. As a consequence, individuals increasingly rely on these social platforms to get news and form their opinions. E.g., according to Pew Research Center survey in 2018 [Shearer and Matsa, 2018] 68% of American adults get news on social media, a significant rise from 49% of 2012. Moreover, it has been observed that social media may have a relevant effect in many real-world critical settings, such as in electoral campaigns [Androniciuc, 2016, Fujiwara et al., 2021]. For example, some studies showed that the social media may lead to extremism [Benigni et al., 2017] and polarization in individuals’ opinions [Allcott et al., 2020].

Hence, it urges to understand how the social media may affect the process of opinion formation of their users. To this aim, several models have been introduced to describe how the opinions of agents evolve under the effect of the social influence. The first such model, due to DeGroot [1974], states that each agent adopts an opinion that averages among the ones of individuals which she interacts with. One of the most relevant extensions of this model is, undoubtedly, the dynamics described by Friedkin and Johnsen [1990] (see also, the work of Bindel et al. [2015]), that limits the effects of social influence by holding agents close to their original ideology. These models assume that opinions may take values in a continuous space, and agents may express any value in this space. However, in several real settings, i.e., electoral contexts, the number of alternatives around which opinions should converge are limited. Moreover, even if opinions can take values that do not match any alternative, these cannot be expressed due to the limitedness of the options according to which opinions are expressed (e.g., polls, finite-precision ranks, etc.). For these reasons, continuous models turn out to be scarcely representative in some settings, and discrete versions of these models have been proposed in which agents’ opinions must belong to a discrete set Chierichetti et al. [2018], Ferraioli et al. [2016].
However, in several settings it is not sufficient to take into account only the social influence among agents’, but we have also to understand how the social media may influence the opinion formation process, and whether and how it is necessary to mitigate in some way the effects it provokes.

There has been recently an increasing interest on these questions. In particular, most of the recent literature in the social choice area focuses on the opportunity for the social media to manipulate the opinion formation process in order to support a target opinion. Different forms of manipulations have been studied, such as seeding, edge addition/deletion, and alteration of the order of changes (see Related Works section for more details).

In this work, we deviate from this approach, and we do not consider the social media as a manipulator. That is, the social media does not have a target that should be promoted, but it only acts as a platform for sharing information. However, social media’s goal is to maximize the activity of the agents on the platform and it implements policies about which, when, and to whom information are shared, in order to maximize engagement of users to their service. While the actual implementation of these policies is private, it is evident that users are more likely to be exposed to information closer to their own opinion [Bakshy et al. 2015, Levy 2021, Halberstam and Knight 2016] have proved that agents have larger probability of interacting (by viewing, liking, or re-sharing) with this kind of information, witnessing in this way their major engagement with the social media.

In this paper we want to answer the following question: how much a social media implementing these policies can influence the opinion formation process? This problem has been recently addressed by [Anunrojwong et al. 2020] in the context of continuous opinion formation processes. Their answer depends on the strength of the influence of the social media platform on individuals: if this is high, then agents’ opinions tend to extremes; if low, agents’ opinion tend to converge; in the middle, instead, some non-extreme disagreement can occur.

However, the continuous approach adopted by [Anunrojwong et al. 2020] does not fit with many real world critical contexts, such as in voting, in which we usually have a discrete and limited number of candidates around which opinions should converge. For this reason, in this work, we will depart from the work of [Anunrojwong et al. 2020], by focusing on the discrete opinion formation process, as defined by [Ferraioli et al. 2016].

Our Contribution In this work, we evaluate the impact of social media recommendations with respect to their influence on the ability of users to reach a consensus. Indeed, the likelihood that a consensus is reached has been widely adopted for comparing different opinion models, and for evaluating the impact that variations on the model may have on opinion formation [DeGroot 1974, Hegselmann and Krause 2002, Auletta et al. 2019a] (see Related Works for more details). Note also that consensus is a required goal in many practical settings: from the analysis of collective behaviour of flocks and swarms [Olfati-Saber 2006, Savkin 2004], to sensor fusion [Olfati-Saber and Shamma 2005], to formation control for multi-robot systems [Egerstedt and Hu 2001, Tanner et al. 2004, Lin et al. 2005].

In this work we first focus on a very simple class of networks, namely symmetric two-block model, already analyzed by [Anunrojwong et al. 2020], in which agents are separated in two components, and agents from the same component have the same initial opinion and receive the same influence from individuals inside and outside their component. Despite of the simplicity of this network, it highlights a very important difference with respect to the results given by [Anunrojwong et al. 2020]: namely, the impact of the social media not only depends on the strength of the social media influence, but also on the homophily ratio, that is how much individuals weight their similars with compared to others. This measure has been often showed to be a key attribute in opinion formation dynamics (see, e.g., [Dandekar et al. 2013]). Hence, our results show a better alignment with respect to the previous literature than the one given by [Anunrojwong et al. 2020].

Specifically, we will show that whenever the strength of the social media influence is large, consensus is essentially impossible to achieve whenever the initial opinions of the two groups are far from each other. Interestingly, for these initial opinions, consensus is also impossible to achieve when the homophily ratio is large, but the strength of the social media is very small. We also show how the chance of reaching a consensus changes with respect to how extreme are the initial opinions in the two groups. Finally, when initial opinions are instead close to each other, we show that consensus is always possible, but the likelihood of reaching a consensus increases when the homophily ratio is large or the strength of the social media is low.

We conjecture that these findings hold not only for the simple symmetric two-block model, but also for
more complex networks whenever initial opinions can be partitioned in two macro-blocks. As an evidence of this conjecture, we provide a massive set of experiments both on synthetic and on real networks: all our experiments show that the dynamics essentially follows the behaviour prescribed by results on the symmetric two-block model as the strength of the social media, the homophily ratio, and the value of initial opinions change.

Related Works Several extensions have been recently proposed to the seminal models by DeGroot and by Friedkin and Johnsen (and their discrete counterparts), by considering only limited interaction by agents [Potakis et al., 2016, 2018], or an evolving environment [Hegselmann and Krause, 2002] [Bhawalkar et al., 2013] [Bilò et al., 2018] [Ferraioli and Ventre, 2017] [Auletta et al., 2019a], or both repulsive and attractive interaction [Auletta et al., 2016] [Acar et al., 2017]. Despite their larger adherence with many real world aspects, however none of these variants has received the same level of interest as the models by DeGroot and by Friedkin and Johnsen. Moreover, the simplicity of the latter models allows a more clear analysis of the influence of social media, by untying it from the complexities of the former models.

Consensus in opinion formation has been object of intense research since the seminal work of DeGroot [1974]. Indeed, most works aim to evaluate opinion formation models based on their ability to reach a consensus [Hegselmann and Krause, 2002] [Auletta et al., 2019a]. Many other works try to characterize the parameters that enable a given dynamics to reach consensus [Feldman et al., 2014] [Mossel et al., 2014] [Auletta et al., 2020]. In this work we pursue both approaches: on one side, we investigate on how the social media recommendations may vary the probability that a consensus is reached; on the other side, we identify the settings, in terms of homophily ratio, strength of the social media influence, and initial agents’ opinions, where the probability of consensus is larger.

The study of the influence of a (non-manipulating) social media on the opinion formation process has been initiated by [Amnojwong et al., 2020], where, as described above, the focus is on continuous opinions, while we here consider discrete opinions.

Many works instead focus on manipulation of the opinion formation process in social networks, in particular in the framework of election manipulation. The first and most studied manipulation technique is seeding, that consists in selecting a set of sources of news from which to start a successful viral campaign in favour of a designed candidate or against her competitors [Kempe et al., 2003] [Wilder and Vorobeychik, 2018] [Corò et al., 2019] [Abouei Mehrizi et al., 2020] [Castiglioni et al., 2021] [Bredereck and Elkind, 2017]. Another kind of manipulation that received large interest consists in adding or deleting links [Sina et al., 2015] [Auletta et al., 2019b] [Bredereck and Elkind, 2017] [Castiglioni et al., 2021]; these may be implemented by social media by hiding the content of a “friend” or “neighbour” in the social network, or promoting the content of non-friends (e.g., as advertised content or through the mechanism of friend suggestion). A last kind of manipulation that recently received a lot of interest consists in guiding the dynamics by influencing the order in which agents are prompted to update their opinion (e.g., by delaying the visualization of a news) so that they will update only when there are enough friends to push them towards the desired candidate [Auletta et al., 2015] [2017a,b] [2020] [2021]. We note that, as described above, our work differs from all these works, since we are not considering a social media operating with the goal of promoting a specific candidate.

2 The Model

We consider \( n \geq 2 \) agents whose relationships are embedded into a social network modelled as an undirected weighted graph \( G = (V, E, w) \), where each vertex of the graph represents an agent. Each agent \( i \) keeps an opinion \( x_i^0 \in \Theta = \{-1, -1 + \delta, \ldots, -\delta, 0, \delta, \ldots, 1 - \delta, 1\} \) for some \( 0 < \delta \leq \frac{1}{2} \). We will sometimes denote \( \delta \) as the discretization factor of \( \Theta \). One may think about \( \Theta \) as the set of alternatives (e.g., candidates to an election) on which agents’ opinions need to converge: note that we are assuming that there is no way for an agent to express an opinion that does not corresponds to an alternative, as it is the case, whenever opinions are expressed, e.g., through polls. Observe that \( |\Theta| = 2\left\lceil \frac{1}{\delta} \right\rceil + 1 \). Let \( x = (x_1, \ldots, x_n) \) be a profile of opinions held by players, where \( x_i \) is the opinion kept of player \( i \).

The opinions of agents are influenced by their social relationships. Specifically, we assume that, for each edge \((i, j) \in E\), opinions of agents \( i \) and \( j \) are mutually influenced and the weight \( w_{ij} > 0 \) of the edge models the strength of this influence.

Moreover, we assume that the opinion of an agent can be also influenced by recommendations received
directly from the social media and not diffused through their own neighbours. We assume that the social media can present to the agents different recommendations, tailoring them on their interests. In particular, we assume that the social media has a discrete subset $\Omega$ of $[-1, 1]$, representing the available information, and it decides to present to an agent with opinion $x$ the information $s(x) \in \Omega$, where the function $s: \Theta \to \Omega$ models the recommendation procedure adopted by the media. Clearly, since the social media is interested in increasing the engagement of their users to the platform, it is interested in advertising to users information that best matches their profile. Thus, e.g., in an electoral setting, the social media will recommend right parties to right-oriented agents, left parties to left-oriented agents, and moderate party to remaining agents.

Thus, at each time step $t$ agents update their opinions depending on the opinions held by their social relations and the recommendations received by the social media. We denote by $x^t$ the profile of opinions held by agents at time $t$.

In this work, following the model introduced by Anunrojwong et al.

[2020], we will consider a specific choice for $\Omega$ and $s$: in particular, we assume $\Omega = \{-1, 0, 1\}$ (we will sometimes refer to the elements of $\Omega$ as “extreme left”, “extreme moderate”, and “extreme right” information or opinions), and assume $s$ being a symmetric threshold function such that $s(x) = -1$ if $x < -\lambda$, $s(x) = 1$ if $x > \lambda$, and $s(x) = 0$ otherwise, for some $0 < \lambda < 1$. While this choice is clearly simplifying the model, it still leads to interesting results about how these social media recommendations may affect the chance that agents may reach a consensus. Moreover, w.l.o.g., we will assume that $\lambda = 1/2$. This essentially means that the social media shows to each agent the information that is closest to her opinion (by breaking ties in favour of the “moderate” information). We remark that all our results about the impact of social media recommendations may be easily extended to arbitrary values of $\lambda$.

The combined influence of neighbours and social media recommendations may lead an agent to update her opinion. In this work, we follow the principles of the model presented by DeGroot

[1974] to represent how the opinion is updated. Specifically, since our focus is on a setting with discrete opinions, we will adapt to our model the discrete generalization of the DeGroot model defined by Ferraioli et al.

[2016]: at each step $t \geq 1$, agent $i$ will choose the opinion $x$ that minimizes $c_i(x, x^{t-1}) = b(x - s(x^{t-1}))^2 + \sum_{(i,j) \in E} w_{ij}(x - x_{j}^{t-1})^2$, where $b > 0$ is the weight of the influence of the social media on agents, and $x^{t-1} = (x_1^{t-1}, \ldots, x_n^{t-1})$ is the opinion profile at the previous time step. We notice that this setting can be equivalently described as a game: agents are the players, opinions are their strategies, and the function $c_i$ is the cost function of player $i$. According to this game-theoretic viewpoint, the opinion update consists essentially of selecting the best-response strategy, i.e. the one that minimizes the cost of the player given the strategies currently selected by other players and the social media.

We say that an opinion profile $x^t = (x_1^t, \ldots, x_n^t)$ is a consensus (on opinion $\bar{x}$) if $x_i^t = \bar{x}$ for every $i$. Moreover, we say that an opinion profile $x^t = (x_1^t, \ldots, x_n^t)$ is stable if it is a Nash equilibrium of the corresponding game, i.e. $x_i^t$ minimizes $c_i(x, x^t)$ for every agent $i$. It is easy to see that a consensus on an extreme opinion, say, e.g., $1$, is always a stable profile. Hence, in this opinion game, a Nash Equilibrium always exists.

Although a stable profile always exists, for given $G$ and $b$, there may be multiple stable opinion profiles, and which one is reached depends on the way in which agents update their opinions. In the literature, the DeGroot model has been associated to different update rules: The most popular rules are: i) synchronous rule, where at each time step $t$ all the agents update their options; ii) asynchronous rule, where at each time step $t$, a single agent, arbitrarily chosen, is allowed to update his/her opinion.

In the next section we will focus on the synchronous case. We will analyze the dynamics with the asynchronous update rule later in section 5.

## 3 Synchronous Updates

In this section we will analyze the dynamics when updates are synchronous. As we will see, the synchronism of the updates allows representing the opinions’ dynamics in a simple and tractable way. Through the analysis of the dynamics with synchronous updates, we obtain interesting findings about the effect of social media recommendations on opinion consensus. We will see in section 5 that these findings extend even to the asynchronous case.
3.1 Symmetric Two-Block Model

We will start our study by focusing on a simple setting: an \((a_{in}, a_{out})\)-symmetric two-block model. This is defined as follows: given an undirected graph \(G = (V, E)\) and a value \(b \geq 0\), we partition the set \(V\) of agents in two subsets, \(L\) and \(R\) such that, for each agent \(u \in P\) with \(P \in \{L, R\}\), we set \(x_u^0 = x_p^0\). Moreover, we set weights \(w_{ij}\) for each edge \((i, j) \in E\), such that for each agent \(i \in P\), we have that \(\sum_{j \in P} w_{ij} = a_{in}\) and \(\sum_{j \in P} w_{ij} = a_{out}\), where \(P = \{L, R\}\) and \(a_{in}, a_{out} > 0\). Roughly speaking, in a symmetric two-block model we assume that agents hold only two opinions and we can partition them in two symmetric communities depending on their opinions. Moreover, the cumulative influence is limited on communities with a large homophily ratio. However, when agents become more prone to extremize the opinions of the two groups, by leading them to diverge.

Moreover, we will show that if either the homophily ratio \(h\) or the media influence \(b\) are very large, consensus is only possible on extreme opinions, namely \(-1, 0\) and \(1\), even for small values of \(h\), and even for non-diverging initial opinion profiles.

In conclusion, our results show that in the two-block model the effect of the social media influence is limited on communities with a large homophily ratio. However, when agents become more prone to heterogeneous influence, then the social media may play an important role, by making consensus either harder to reach, or reachable only on extreme opinions.

3.1.1 Preliminary Results

In the following, we will give some useful characterizations of feasible opinion profiles, i.e., profiles that can be reached during the evolution of the dynamics, and stable profiles.

**Lemma 1.** Given an \((a_{in}, a_{out})\)-symmetric two-block model \(G = (L \cup R, E, w)\) and a social media influence \(b\), an opinion profile \((x_1, \ldots, x_n)\) is feasible only if \(x_i = x_L\) for every \(i \in L\), \(x_j = x_R\) for every \(j \in R\).

**Proof.** From the definition of cost, the opinion of agent \(i\) at time \(t + 1\) is equal to

\[
x_i^{t+1} = \arg \min_{y \in \Theta} \left\{ b(y - s(x_i^t))^2 + \sum_{j \in \Theta} w_{i,j}(y - x_j^t)^2 \right\}.
\]

Therefore, in an \((a_{in}, a_{out})\)-symmetric two-block model, for each agent \(i \in P\) the opinion at step 1 is:

\[
x_i^1 = \arg \min_{y \in \Theta} \left\{ b(y - s(x_p^0))^2 + a_{in}(y - x_p^0)^2 + a_{out}(y - x_p^0)^2 \right\}.
\]

Consequently, at step 1 \(x_i^1 = x_L^1\) for every \(i \in L\), and \(x_j^1 = x_R^1\) for every \(j \in R\).

The lemma follows by iteratively applying the same argument for all the following steps. \(\square\)

Next, we provide a characterization of best responses. This is our key lemma.
Lemma 2. Given an \((a_{in}, a_{out})\)-symmetric two-block model \(G = (L \cup R, E, w)\), a social media influence \(b\), and a feasible opinion profile \(x\), \(x^*\) is a best-response for agent \(i \in P\), for \(P \in \{L, R\}\), in the profile \(x\) only if
\[
x^* - \frac{\delta}{2} \leq \frac{b s(x_P) + a_{out} x_P + a_{in} x_P}{b + a_{out} + a_{in}} \leq x^* + \frac{\delta}{2},
\]
where \(\mathcal{P} = \{L, R\} \setminus P\), \(x_L = x_i\) for some \(i \in L\), and \(x_R = x_j\) for some \(j \in R\).

Proof. By Lemma 1 all the agents in the same community \(P\) have the same best response in \(x\), and, from the definition of cost, it is
\[
x^* = \arg\min_{y \in \Theta} \left\{ b(y - s(x_P))^2 + a_{in}(y - x_P)^2 + a_{out}(y - x_P^*)^2 \right\}
= \arg\min_{y \in \Theta} \left\{ (a_{out} + a_{in} + b)y^2 - 2y(b s(x_P) + a_{out} x_P + a_{in} x_P) + b s(x_P)^2 + a_{out} x_P^2 + a_{in} x_P^2 \right\}.
\]
Since, by definition, \(a_{out} + a_{in} + b > 0\), then the argument of \(\arg\min\) describes a parabola with convexity upwards. If \(\Theta\) was a continuous interval, the minimum value would be achieved by setting \(y = \frac{b s(x_P) + a_{out} x_P + a_{in} x_P}{b + a_{out} + a_{in}} \in [-1, 1]\), where the membership in this interval follows since \(|s(x_P)| \leq 1\), \(|x_P| \leq 1\), and \(|x_P^*| \leq 1\). However, since \(\Theta\) is discrete, then we have that
\[
x^* = \arg\min_{y \in \Theta} \left\{ \left( y - \frac{b s(x_P) + a_{out} x_P + a_{in} x_P}{b + a_{out} + a_{in}} \right)^2 \right\}.
\]
Since \(\delta\) is the distance among two consecutive elements in \(\Theta\), then there is an \(y \in \Theta\) such that
\[
\left| y - \frac{b s(x_P) + a_{out} x_P + a_{in} x_P}{b + a_{out} + a_{in}} \right| \leq \frac{\delta}{2},
\]
from which the lemma follows.

Finally, next lemma provides a characterization of the stable opinion profiles. To this aim, we define the relative amount of media influence as \(\tilde{b} = \frac{b}{a_{out}}\).

Lemma 3. Given an \((a_{in}, a_{out})\)-symmetric two-block model \(G = (L \cup R, E, w)\), and a social media influence \(b\), a feasible opinion profile \(x\) is stable only if the following conditions are satisfied:
\[
\begin{align*}
\hat{b} (s(x_L) - x_L + \frac{\delta}{2}) & \leq (x_L - x_R) + \frac{\delta}{2} (h + 1); \\
\hat{b} (s(x_L) - x_L + \frac{\delta}{2}) & \geq (x_L - x_R) - \frac{\delta}{2} (h + 1); \\
\hat{b} (s(x_R) - x_R - \frac{\delta}{2}) & \leq (x_R - x_L) + \frac{\delta}{2} (h + 1); \\
\hat{b} (s(x_R) - x_R + \frac{\delta}{2}) & \geq (x_R - x_L) - \frac{\delta}{2} (h + 1).
\end{align*}
\]

Proof. By Lemma 1 an opinion profile is feasible only if every agent \(i \in P\) with \(P \in \{L, R\}\) has the same opinion \(x_P\). Moreover, this opinion profile is stable if for every agent \(i \in P\) its best response with respect to \(x\) is still \(x_P\). By Lemma 2 we then have that if \(x\) is stable then it must hold that:
\[
\begin{align*}
\frac{b s(x_L) + a_{out} x_R + a_{in} x_L}{b + a_{out} + a_{in}} & \leq x_L + \frac{\delta}{2}; \\
\frac{b s(x_R) + a_{out} x_L + a_{in} x_R}{b + a_{out} + a_{in}} & \leq x_R + \frac{\delta}{2}.
\end{align*}
\]
By simple algebraic manipulations, we can rewrite above conditions as follows:
\[
\begin{align*}
\hat{b} (s(x_L) - x_L + \frac{\delta}{2}) & \leq (x_L - x_R) + \frac{\delta}{2} \left( \frac{a_{in}}{a_{out}} + 1 \right); \\
\hat{b} (s(x_L) - x_L + \frac{\delta}{2}) & \geq (x_L - x_R) - \frac{\delta}{2} \left( \frac{a_{in}}{a_{out}} + 1 \right); \\
\hat{b} (s(x_R) - x_R - \frac{\delta}{2}) & \leq (x_R - x_L) + \frac{\delta}{2} \left( \frac{a_{in}}{a_{out}} + 1 \right); \\
\hat{b} (s(x_R) - x_R + \frac{\delta}{2}) & \geq (x_R - x_L) - \frac{\delta}{2} \left( \frac{a_{in}}{a_{out}} + 1 \right).
\end{align*}
\]
The lemma follows by observing that \(\hat{b} = \frac{b}{a_{out}}\) and \(\frac{a_{in}}{a_{out}} = \gamma\). \(\square\)
Next lemmas prove some properties of the best-response opinion in presence of a social media influence that will be useful in characterizing the type of stable profile achieved by our opinion dynamics. Specifically, Lemma 4 provides conditions for an agent with an opinion close to the extremes, \(-1\) and \(1\), to change her idea and adopt an opinion that is far from these extremes. Lemma 5 and Lemma 7 provide instead conditions for an agent to adopt an opinion of opposite sign with respect to her actual opinion. All these lemmas are proved in a very similar way, hence, for the seek of readability, we will only show proofs of Lemma 4 and Lemma 5. We refer interested reader to Appendix A for the remaining proofs. In what follows we will assume that \(\lambda \in \Theta\). This is without loss of generality: indeed, all results below still hold by simply replacing \(\lambda\) with the largest \(\lambda' \in \Theta\) smaller than \(\lambda\), since for every \(x \in \Theta\) it must be the case that \(x > \lambda\) (resp. \(x < -\lambda\)) if and only if \(x > \lambda'\) (resp. \(x < -\lambda'\)).

**Lemma 4.** Given an \((a_{in}, a_{out})\)-symmetric two-block model \(G = (L \cup R, E, w)\) and a social media influence \(b\), if \(x_i^t > \lambda\), then the opinion of the agent \(i\) at next step is \(x_i^{t+1} \leq \lambda\) only if \(\bar{b} \leq \tau_1(h)\), where

\[
\tau_1(h) = \frac{2 + 2 \lambda - 2 \delta}{2 - 2 \lambda - \delta}. 
\]

**Proof.** We consider only the case that \(x_i^t > \lambda\). The case for \(x_i^t < -\lambda\) is symmetric and hence omitted.

Recall that, by Lemma 1, \(x_i^t = x_i^P\), where \(P\) is the block \(i\) belongs to, and every \(j \notin P\) has \(x_j^t = x_j^P\). Since \(x_i^t > \lambda\), then \(s(x_i^P) = 1\). Thus, by Lemma 2, \(x_i^{t+1} \leq \lambda\) only if \(\frac{b + a_{out}}{b + a_{out} + a_{in}} x_i^P \leq \lambda + \frac{\delta}{2}\). By dividing both sides by \(a_{out}\) and recalling that \(\frac{b}{a_{out}} = \bar{b}\) and \(\frac{a_{in}}{a_{out}} = h\), we have that \(x_i^{t+1} \leq \lambda\) only if

\[
\bar{b} \leq \frac{-x_i^P + \lambda + \frac{\delta}{2} + h(-x_i^P + \lambda + \frac{\delta}{2})}{1 - \frac{\delta}{2} - \lambda}. 
\]

It is immediate to check that the r.h.s. of (2) is maximized by taking \(x_i^P = \lambda + \delta\), and \(x_i^P = -1\). By substituting these values in (2), we achieve that \(x_i^{t+1} \leq \lambda\) only if \(\bar{b} \leq \frac{2 + 2 \lambda - 2 \delta}{2 - 2 \lambda - \delta}\), as desired. \(\square\)

**Lemma 5.** Given an \((a_{in}, a_{out})\)-symmetric two-block model \(G = (L \cup R, E, w)\) and a social media influence \(b\), if \(x_i^t \in [-\lambda, 0]\) (resp. \(x_i^t \in [0, \lambda]\)), then \(x_i^{t+1} > 0\) (resp. \(x_i^{t+1} < 0\)) only if \(\bar{b} \leq \tau_2(h)\), where

\[
\tau_2(h) = \frac{2 + 2 \lambda - 2 \delta}{2 - 2 \lambda - \delta}. 
\]

**Proof.** We consider only the case that \(x_i^t \in [-\lambda, 0]\). The case for \(x_i^t \in [0, \lambda]\) is symmetric and hence omitted.

Recall that, by Lemma 1, \(x_i^t = x_i^P\), where \(P\) is the block at which \(i\) belongs to, and every \(j \notin P\) has \(x_j^t = x_j^P\). Since \(x_i^t \in [-\lambda, 0]\), then \(s(x_i^P) = 0\). Thus, by Lemma 2, \(x_i^{t+1} \geq 0\) only if \(\frac{a_{out} x_i^P + a_{in} x_i^P}{b + a_{out} + a_{in}} \leq \frac{\delta}{2}\). By dividing both sides by \(a_{out}\) and recalling that \(\frac{a_{in}}{a_{out}} = h\), we have that \(x_i^{t+1} \geq 0\) only if

\[
\bar{b} \leq \frac{2}{\delta} \left( x_i^P - \frac{\delta}{2} + h(x_i^P - \frac{\delta}{2}) \right). 
\]

It is immediate to check that the r.h.s. of (3) is maximized by taking \(x_i^P = 0\), and \(x_i^P = 1\). By substituting these values in (3), we achieve that \(x_i^{t+1} \geq 0\) only if \(\bar{b} \leq (\frac{2}{\delta} - 1) - h\), as desired. \(\square\)

**Lemma 6.** Given an \((a_{in}, a_{out})\)-symmetric two-block model \(G = (L \cup R, E, w)\) and a social media influence \(b\), if \(x_i^t < -\lambda\) (resp. \(x_i^t > \lambda\)), then \(x_i^{t+1} > 0\) (resp. \(x_i^{t+1} < 0\)) only if \(\bar{b} \leq \tau_3(h)\), where

\[
\tau_3(h) = \frac{2 + 2 \lambda - 2 \delta}{2 - 2 \lambda - \delta}. 
\]

**Lemma 7.** Given an \((a_{in}, a_{out})\)-symmetric two-block model \(G = (L \cup R, E, w)\) and a social media influence \(b\), if \(x_i^t \in (0, \lambda]\) (resp. \(x_i^t \in [-\lambda, 0]\)), then \(x_i^{t+1} \leq 0\) only if \(\bar{b} \geq \tau_4(h)\), where \(\tau_4(h) = h - (\frac{\delta}{2} + 1)\).

**Lemma 8.** Given an \((a_{in}, a_{out})\)-symmetric two-block model \(G = (L \cup R, E, w)\) and a social media influence \(b\), if \(|x_i^t| > \lambda\), then \(x_i^{t+1} \leq 0\) only if \(\bar{b} \leq \tau_5(h)\), where \(\tau_5(h) = \frac{2 + 4 - (2 \lambda + 2 \delta)}{2 - 2 \lambda - \delta}\).

We next show that some interesting relationships exist among these thresholds.
Given an \( \tau \) quantity that will play a fundamental role in our characterization: on the initial opinions held by the players in the two blocks. To this aim let us define the following on the homophily ratio and the social media influence. We will distinguish different cases, depending in this subsection, we will study which type of consensus can be achieved in a two-block model, depending

### Lemma 9

The following relationships hold:

\[
\begin{align*}
\tau_1(h) &> \tau_4(h) \quad \text{only if } h < \frac{2}{\delta} + \frac{1}{1 - \lambda}; \quad (4) \\
\tau_2(h) &< \tau_4(h) \quad \text{if } h > \frac{2}{\delta}; \quad (5) \\
\tau_3(h) &< \tau_4(h) \quad \text{for every } h; \quad (6) \\
\tau_5(h) &< 0 \quad \text{if } \tau_4(h) > 0. \quad (7)
\end{align*}
\]

**Proof.** (4) This relationship immediately follows by observing that

\[
\tau_1(h) - \tau_4(h) = \frac{2 + 2\lambda + \delta - \delta h}{2 - 2\lambda - \delta} - h + \left(\frac{2}{\delta} + 1\right) = \frac{2}{2 - 2\lambda - \delta} \left[\frac{2\delta(1 - \lambda) + 1}{2}\right].
\]

(5) This relationship immediately follows by observing that

\[
\tau_2(h) - \tau_4(h) = \left(\frac{2}{\delta} - 1\right) - h - h + \left(\frac{2}{\delta} + 1\right) = 2 \left(\frac{2}{\delta} - h\right).
\]

(6) This relationship immediately follows by observing that

\[
\tau_3(h) - \tau_5(h) = \frac{2 - \delta - (2\lambda + 3\delta)h}{2 + \delta} = \frac{2 + \delta - (2\lambda + \delta)h}{2 - \delta} = \frac{4\delta}{4 - \delta^2} \left[(\lambda + \delta - 1)h - 1\right] < 0,
\]

where the last inequality follows because \( \lambda + \delta < 1 \).

(7) If \( \tau_4(h) > 0 \), then \( h > \frac{2}{\delta} + 1 \). Then \( \tau_5(h) = \frac{2 + \delta - (2\lambda + \delta)h}{2 - \delta} < -\frac{2\lambda(2\delta + 1)}{\delta^2} < 0 \).

#### 3.1.2 Consensus Characterization

In this subsection, we will study which type of consensus can be achieved in a two-block model, depending on the homophily ratio and the social media influence. We will distinguish different cases, depending on the initial opinions held by the players in the two blocks. To this aim let us define the following quantities that will play a fundamental role in our characterization: \( \tau_1(h) = \frac{2 + 2\lambda + \delta - \delta h}{2 - 2\lambda - \delta} \), \( \tau_2(h) = \left(\frac{2}{\delta} - 1\right) - h \), \( \tau_3(h) = \frac{2 - \delta - (2\lambda + 3\delta)h}{2 + \delta} \), \( \tau_4(h) = h - \left(\frac{2}{\delta} + 1\right) \), and \( \tau_5(h) = \frac{2 + \delta - (2\lambda + \delta)h}{2 - \delta} \).

#### Divergent and Extreme Initial Opinions

We start by considering the case where the starting opinions of the two blocks diverge (i.e., one is positive and the other is negative) and are both far away from 0.

**Theorem 1.** Given an \((a_{in}, a_{out})\)-symmetric two-block model \( G = (L \cup R, E, w) \) and a social media influence \( b \), if \( |x^L_0| > \lambda \) and \( |x^R_0| > \lambda \) and \( x^L_0 \cdot x^R_0 < 0 \), then

\[
\begin{align*}
\begin{cases}
\text{if } \hat{b} > \tau_1(h), & \text{no consensus can be stable;} \\
\text{if } \max\{0, \tau_2(h), \tau_3(h), \tau_4(h)\} < \hat{b} \leq \tau_1(h), & \text{only consensus on } 0 \text{ can be stable;} \\
\text{if } \max\{0, \tau_4(h)\} < \hat{b} \leq \max\{\tau_2(h), \tau_3(h)\} & \text{non-extreme consensus can be stable;} \\
\text{if } 0 < \hat{b} \leq \max\{0, \tau_4(h)\}, & \text{no consensus can be stable.}
\end{cases}
\end{align*}
\]

**Proof.** Suppose first that \( \hat{b} > \tau_1(h) \). Then, by Lemma 4, no agent \( i \in L \) can take an opinion \( x^L_i \) such that \( |x^L_i| \leq \lambda \), and no agent \( j \in R \) can take an opinion \( x^R_j \) such that \( |x^R_j| \leq \lambda \). Hence, after the first time step \( |x^L_i| > \lambda \) and \( |x^R_j| > \lambda \) and \( x^L_i \cdot x^R_j < 0 \). Then, we can iteratively apply the same argument to conclude that the opinions of the two blocks never converge to a consensus profile.

Suppose now that \( \max\{0, \tau_2(h), \tau_3(h), \tau_4(h)\} < \hat{b} \leq \tau_1(h) \). W.l.o.g., suppose that \( x^L_0 \leq 0 \) and \( x^R_0 \geq 0 \). Since \( \hat{b} > \max\{\tau_2(h), \tau_3(h)\} \), then, by Lemma 5 and Lemma 6, it follows that no agent \( i \in L \) can take
an opinion $x^i_L > 0$, and no agent $j \in R$ can take an opinion $x^j_R < 0$. Hence, after the first time step $x^i_L \leq 0$ and $x^j_R \geq 0$. Then, we can iteratively apply the same argument above to conclude that the unique opinion on which the two blocks can converge is 0.

Finally, suppose that $0 < b \leq \max \{0, \tau_4(h)\}$. Note that this interval is non-empty only if $\tau_4(h) > 0$, and thus, according to (7), $\tau_3(h) < 0$. W.l.o.g., suppose that $x^i_L < 0$ and $x^j_R > 0$. Then, by Lemma 7 and Lemma 8, it follows that no agent $i \in L$ can take an opinion $x^i_L \geq 0$, and no agent $j \in R$ can take an opinion $x^j_R \leq 0$. Hence, after the first time step $x^i_L < 0$ and $x^j_R > 0$. Then, we can iteratively apply the same argument above to conclude that the two blocks never converge to a consensus.

Remark 1. We observe that it is impossible that the interval corresponding to last two cases of (8) are both non-empty. Indeed, if the last interval is non-empty, then $\tau_3(h) > 0$ and thus $h > \frac{2}{3} + 1$. It is not hard to check that this implies that $\max \{\tau_2(h), \tau_3(h)\} < \tau_3(h) = \max \{0, \tau_4(h)\}$. Hence for $b \leq \max \{0, \tau_2(h), \tau_3(h), \tau_4(h)\}$, either no consensus can be stable, or it is possible to achieve consensus also on non-extreme opinions.

Roughly speaking, Theorem 2 shows that if we have initial opinions that are divergent and far away from 0, consensus is impossible to achieve for high values of the media influence $b$, while it can be achieved on non-extremal opinions only for small values of $b$ and under opportune conditions. Remark 1 also shows that the outcome also depends on the value of $h$. Specifically, for small values of $h$, it appears that the chance of having a consensus decreases as $b$ increases, since we go from a range in which non-extreme consensus can be stable to a range in which only consensus on 0 is stable, and finally to a range in which consensus is impossible. Instead, for large $h$ the behaviour is less “monotone”: indeed, we go from no consensus to possible consensus (on opinion 0) and again to no consensus. Moreover, from Theorem 2 it’s easy to prove that consensus is impossible when $h$ is large, regardless of the strength of the social media influence. This confirms the fundamental role played by the homophily ratio.

**Divergent Initial Opinion: Only One is Extreme** Consider now the case of divergent initial opinions, but we assume that only one of them is far from 0. It will turn out that, as above, consensus is impossible whenever either the social media influence or the homophily ratio is large, and it is possible on non-extremal opinions only for small values of $b$ and under opportune conditions on $h$. The proof of Theorem 2 is very similar to the proof of Theorem 1; hence, for the sake of readability, we postpone its proof to the Appendix 2.

**Theorem 2.** Given an $(a_{in}, a_{out})$-symmetric two-block model $G = (L \cup R, E, w)$ and a social media influence $b$, if $|x^i_L| > \lambda$ or $|x^j_R| > \lambda$ and $x^i_L \cdot x^j_R < 0$, then

$$\begin{cases} \text{if } b > \tau^*(h), & \text{no consensus can be stable;} \\ \text{if } \max \{0, \tau_2(h), \tau_3(h), \tau_4(h)\} < b \leq \tau^*(h), & \text{only consensus on 0 can be stable;} \\ \text{if } \max \{0, \tau_4(h)\} < b \leq \max \{\tau_2(h), \tau_3(h)\}, & \text{non-extreme consensus can be stable;} \\ \text{if } 0 < b \leq \max \{0, \tau_4(h)\}, & \text{no consensus can be stable}, \end{cases}$$

where $\tau^*(h) = \max \{\tau_1(h), \tau_2(h)\}$.

The following corollary highlights that for large values of the homophily ratio convergence to consensus is impossible, regardless of the strength of the social media influence.

**Corollary 1.** Given an $(a_{in}, a_{out})$-symmetric two-block model $G = (L \cup R, E, w)$ and a social media influence $b$, if $|x^i_L| > \lambda$ or $|x^j_R| > \lambda$ and $x^i_L \cdot x^j_R < 0$, then no consensus opinion profile can be stable if $h \geq \frac{2}{3} + \frac{1}{3\sqrt{x}},$ regardless of the value of the social media influence $b$. Moreover, consensus can be stable on non-extremal opinions only if $h < \max \left\{ \frac{2}{3} - 1, \frac{2-\delta}{2\sqrt{x}+33} \right\}$.

**Proof.** If $h \geq \frac{2}{3} + \frac{1}{3\sqrt{x}}$, we have that $\max \{\tau_2(h), \tau_4(h)\} \leq 0 \leq \max \{0, \tau_4(h)\}$, and, from (11), $\tau_1(h) \leq \tau_4(h) \leq \max \{0, \tau_2(h), \tau_3(h), \tau_4(h)\}$. Hence, there cannot be any value of $b$ for which consensus can be stable.

Suppose instead that $h \geq \max \left\{ \frac{2}{3} - 1, \frac{2-\delta}{2\sqrt{x}+33} \right\}$. Then, $\max \{\tau_2(h), \tau_4(h)\} \leq 0 \leq \max \{0, \tau_4(h)\}$, and thus the interval allowing for consensus on non-extremal opinions is empty. □
Divergent Initial Opinion: Both are Moderate

Consider now the case that initial opinions of the two blocks are still divergent but both close to opinion 0. Clearly, in this case, large values of the social influence would push these opinions to 0, thus leading to a consensus on this opinion. However, we show that this is the only possible consensus in this setting when $b$ is large. Proof of Theorem 3 in a very similar to proof of Theorem 1; consequently, for the seek of readability, we postpone this proof to Appendix B.

Theorem 3. Given an $(a_{in}, a_{out})$-symmetric two-block model $G = (L \cup R, E, w)$ and a social media influence $b$, if $|x_L^0| \leq \lambda$ and $|x_R^0| \leq \lambda$ and $x_L^0 \cdot x_R^0 < 0$, then

$$
\begin{cases}
\text{if } \hat{b} > \max \{\tau_2(h), \tau_3(h)\}, & \text{only consensus on 0 can be stable;} \\
\text{if } \max \{0, \tau_4(h)\} < \hat{b} \leq \max \{\tau_2(h), \tau_3(h)\}, & \text{non-extreme consensus can be stable;} \\
\text{if } 0 < \hat{b} \leq \max \{0, \tau_4(h)\}, & \text{no consensus can be stable.}
\end{cases}
$$

Convergent Initial Opinions

We conclude this section by considering the case that initial opinions do not diverge. We observe that, in this case, a large influence of the social media (with respect to homophily ratio) may lead only to consensus on extreme opinions, namely $-1, 0, 1$.

Theorem 4. Given an $(a_{in}, a_{out})$-symmetric two-block model $G = (L \cup R, E, w)$ and a social media influence $b$, if $x_L^0 \cdot x_R^0 \geq 0$, then consensus on opinions different from $-1, 0, 1$ can be stable only if $\hat{b} \leq h + 1$.

Proof. Suppose that consensus is achieved on $x \in \{\delta, \ldots, 1 - \delta\}$ (the case in which consensus is achieved on a non-extreme negative opinion is symmetric and hence omitted).

Suppose first that $x \leq \lambda$. By Lemma 3, consensus on $x$ is a stable opinion profile only if both the following conditions hold:

$$
\begin{aligned}
\hat{b} &\geq -\frac{\hat{b} \delta (1 + h)}{x + \delta} \\
\hat{b} &\leq \frac{\hat{b} \delta (1 + h)}{x - \delta}
\end{aligned}
$$

It is immediate to check that the first condition always holds, since $\hat{b}, h, \delta, x \geq 0$. As for the second condition, its r.h.s. is maximized when $x = \delta$. Hence, a non-extreme consensus can be stable only if $\hat{b} \leq \frac{\hat{b} \delta (1 + h)}{\delta - \delta} = h + 1$, as desired.

Suppose now that $x > \lambda$. By Lemma 3, consensus on $x$ is a stable opinion profile only if both the following conditions hold:

$$
\begin{aligned}
\hat{b} &\leq \frac{\hat{b} \delta (1 + h)}{1 - x - \delta} \\
\hat{b} &\geq -\frac{\hat{b} \delta (1 + h)}{1 + x - \delta}
\end{aligned}
$$

As above, the second condition is always satisfied. As for the first one, its r.h.s. can be maximized taking $x = 1 - \delta$. Hence, a non-extreme consensus can be stable only if $\hat{b} \leq \frac{\hat{b} \delta (1 + h)}{1 - (1 - \delta)} = h + 1$, as desired. \hfill \Box

Next corollary highlights a fundamental difference between the discrete and the continuous setting. Indeed, stable profiles that are non-extreme consensus are feasible in the discrete case while it is known that they are not feasible in the continuous case.

Corollary 2. Given an $(a_{in}, a_{out})$-symmetric two-block model $G = (L \cup R, E, w)$ and a social media influence $b$, if $\delta \to 0$, then consensus on opinions different from $-1, 0, 1$ can not be stable.

Proof. Suppose that consensus is achieved on $x \in \{\delta, \ldots, 1 - \delta\}$ (the case in which consensus is achieved on a non-extreme negative opinion is symmetric and hence omitted).

From the proof of Theorem 5 we know that, if $x \in \{\delta, \ldots, \lambda\}$, then consensus on $x$ can be stable only if $\hat{b} \leq \frac{\hat{b} \delta (1 + h)}{x - \delta}$. When $\delta \to 0$, the condition became $\hat{b} \leq 0$ that is impossible by definition.

If $x \in \{1 + \delta, \ldots, 1 - \delta\}$, from the proof of Theorem 5, we know that consensus can be stable only if $\hat{b} \leq \frac{\hat{b} \delta (1 + h)}{1 - x - \delta}$. When $\delta \to 0$, the condition became $\hat{b} \leq 0$ that is impossible by definition. \hfill \Box
3.2 Convergence

Previous sections characterized when and which consensus can be achieved depending on the homophily and the social media influence in opinion dynamics with synchronous update rules.

However, we should highlight that, although, as suggested above, a stable state always exists, synchronous updates can make the opinion dynamics unable to converge to this state from specific initial opinions profiles. In the following we provide a simple example.

**Proposition 1.** An opinion dynamics with synchronous updates may not converge to a stable state.

*Proof.* Let consider a two-player social network. We note the two player as $l$ and $r$. Let $w_{l,r} = 10$ the weight of the edge, $\Theta = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ the opinions’ set and $b = 1$ the strength of social media influence. The initial opinions of the agents are $x_l^0 = -1, x_r^0 = 1$.

At step 1, the agent $l$ choose the opinion $x \in \Theta$ that minimizes $c_l(x^0) = 1(x + 1)^2 + 10(x - 1)^2$. It’s easy to see why $x_l^1 = \arg\min_x c_l(x^0) = 1$. In the same way, the opinion of agent $r$ at the step 1 is $x_r^1 = \arg\min_x c_r(x^0) = -1$. Consequently, at step 2 we have $x_l^2 = 1, x_r^2 = -1$. Therefore, the dynamics continues indefinitely since $x_l^{t-1} \neq x_l^t$ and $x_r^{t} \neq x_r^{t-1}$ for all $t \geq 0$.

However whenever the opinion dynamics with synchronous updates converges, the stable state at which it converges can be computed, since it deterministically depends only on the initial opinion profile $x^0 = (x_l^0, x_r^0)$.

In the section 4 we will study the opinion dynamics with asynchronous updates. We will show a specular behaviour: indeed, it will turn out that in this case the dynamics always converges to a stable state, but more than one stable states may exist. Interestingly, in section 4.2 we will also show that the findings about when and which consensus are reached by the dynamics obtained through the analysis of the synchronous case even extend to the asynchronous case.

4 General Networks

In previous section we presented some results related to the two-block model of a social network. We conjecture that our results hold in more general settings under the hypothesis that it is possible to distinguish in the network two well separated sets of similar agents. In this section we present some experimental evidences to support our conjecture. In particular, we run our experiments on stochastic two-block model graphs, random graphs and on real graphs. In the latter two cases we use algorithmic techniques to separate nodes in two components and then we define weights of the edges in order to define the influence coming on an agent from her own component and from the other component.

Observe that in a symmetric two-block model network, for each agent $i$, $\sum_{j \in P} w_{ij}$, that is the influence that she receives from the other agents in the same component, is equal to a constant $a_{in}$. At the same time, for each agent $i$, $\sum_{j \in R} w_{ij}$, that is the influence that she receives from the other agents in the different component, is equal to a constant $a_{out}$. Moreover, agents in the same component have the same initial opinion. In our first experiment we extend this model by relaxing some of these assumptions. In particular, the set of vertices is $V = L \cup R$, where $|L| = |R| = N$; each edge between two agents in the same component exists with probability $p_{in}$, while each edge between agents in different components exists with probability $p_{out}$. However, all edges have the same weight. Thus, two agents in the same component may have different neighbours, even if they receive the same expected influences. Indeed, the expected influence received by her component is equal to $(N - 1)p_{in}$ and the expected influence that an agent receive from her opposite component is equal to $Np_{out}$. Furthermore, agents in the same component can have different initial opinions.

We set $N = 50$ and simulate our opinion dynamics with different values of $p_{in}$, $p_{out}$, $\delta$, and $b$. For each setting we run $n_p = 1000$ simulations. For each simulation, given the two blocks, say $L$ and $R$, we assume that for each agent in $L$, the initial opinion $x_l^0$ is drawn at random in the interval $[l_L, h_L]$, and for each agent in $R$ the choice is drawn at random in the interval $[l_R, h_R]$, where $h_L$ and $l_R$ are set respectively to $-\xi$ and $\xi$, where $\xi$ is drawn uniformly at random in the interval $[0, \lambda + \delta]$, $l_L$ is drawn at random in the interval $[-1, h_L]$, and $h_R$ is drawn at random in the interval $[l_R, 1]$. Let $m$ be the
number of runs in which the dynamics converges to consensus, we measure the consensus probability as
\[ p_c = \frac{m}{np} \] and the 95% confidence interval as \[ p_c \pm 2\sqrt{p_c(1-p_c)/np} \]

Next we consider networks generated using three well-known network formation models: the Random
Graphs model [Gilbert, 1959], the Watts-Strogatz model [Watts and Strogatz, 1998] and the Hyperbolic
Random Graph model [Krioukov et al., 2010]. We remark that the Random Graph model is generally
used to generate random networks. The other two graph models are known to generate networks enjoying
properties usually more similar to real social networks. In particular, the Watts-Strogatz model is known
to generate small-world networks (i.e., network with small diameter and a large clustering index). In
Krioukov et al., 2010 it has been showed that the Hyperbolic Random Graph model, for a special choice
of parameters, generate networks that are small-worlds with a degree distribution that is a power law, a
characteristic that can be find in several real-life social networks.

In the Random Graphs model, for each pair of vertices \( u \) and \( v \) the edge \((u,v)\) is created with
probability \( p \). Notice that, in general, the random graph \( G = (V,E) \) generated in this way cannot be
separated in well-defined components of the same size. However, we can partition the set of vertices in
two components, \( L \) and \( R \), by running the well-know algorithm of Kernighan and Lin [1970], that returns
the partition generated by the sparsest cut. We then assign weight \( w_{in} \) to edges among nodes in the
same component, and \( w_{out} \) to all the remaining edges. Note that, as in the two block model, here each node
receives a different social influence from nodes within the same component and nodes of the opposite
component. However, the homophily ratio as \( h = \frac{w_{in}}{w_{out}} \), where \( w_{in} = \frac{1}{|V|} \sum_{a \in V} a_{in}(v) \) and \( w_{out} = \frac{1}{|V|} \sum_{a \in V} a_{out}(v) \).

Watts-Strogatz networks are generated by positioning nodes in a metric (usually Euclidean) space and
linking nodes through two classes of links: two nodes whose distance is below a given threshold \( r \) are
linked through so-called strong ties; each node has \( k \) additional links, termed weak ties, connecting to
randomly selected endpoints. Hyperbolic Random Graph networks are generated in a similar way but the
metric space is restricted to be hyperbolic with negative curvature. For both these two classes of networks we partition nodes in two communities and set the weights of the links as described for
the Random Graphs model. For each of these three classes of graphs we set the number of agents to be 100
and we run our opinion dynamics for different values of \( w_{in}, w_{out}, \delta, \) and \( b \). For each setting, we run 1000
simulations and we compute the initial opinions, the consensus probability and the confidence interval
as stated before.

Finally, we considered two samples of real social networks that are freely available in the SNAP library
Leskovec and Krevl, 2014. The first one, ego-Facebook, is a sample of 4039 nodes and 88234 edges
retrieved from Facebook network McAuley and Leskovec, 2012. The second one, feather-lastfm-social
consists in a less dense network of 7624 nodes and 27806 edges Rozemberczki and Sarkar, 2020. In
order to run multiple simulations on these networks we do not use a deterministic partitioning algorithm
to retrieve communities, but for each simulation an agent is assigned to cluster \( L \) with a probability \( p_L \)
drawn uniformly at random in \([0.4,0.6]\), and to cluster \( R \) otherwise. We will show below that, despite
this random choice of the partitions, we still are able to achieve results that are similar to previous more
regular networks. Weights, homophily ratio, the influence of the social media, initial opinions, consensus probability and the confidence interval are then computed as described above (but mediated over only
500 simulations, due to the larger size of these networks).

We observe that numerical oscillations can make impossible to reach consensus even if opinions of agents
are very close to each other. For this reason, we consider a relaxed definition of consensus. In
particular, following the analysis of Anunrojwong et al., 2020, we will focus on the average opinion
\( \pi_P = \frac{\sum_{i \in P} x_i}{|P|} \) for each partition \( P \), and on its projection \( \bar{\xi}_P \) on \( \Theta \), being the opinion in \( \Theta \) closest to \( \pi_P \). Then, a stable opinion profile \( x = (x_1, x_2, \ldots, x_n) \) is a consensus if \( \bar{\xi}_L = \bar{\xi}_R \). We performed
extensive simulations to determine if the relaxed definition of consensus can make our results inaccurate.
Specifically, we compute the distributions of opinions when a relaxed consensus is reached. We observe
that when a relaxed consensus is reached, the mean of the opinions is near to the relaxed consensus
value and the variance is lower than \( 0.5 \cdot \delta \). Remember that the opinion of each agent belongs to the
set \( \Theta = \{-1, -1 + \delta, \ldots, 0, \ldots, 1 - \delta, 1\} \). In Figure 1 we show the value of the relaxed consensus and
the distribution of opinions at relaxed consensus on the ego-Facebook social network, as \( \delta \) increases.
In particular, we can observe that the mean is near to the relaxed consensus value and the variance is tight w.r.t the value of $\delta$. The results showed are confirmed by repeating the same simulations 30 times. We obtained very similar results for different values of $\delta$ and in most of the models considered in our analysis: Two-Block Model, Random graph, Watts-Strogatz graphs, Hyperbolic Random graphs and feather-lastfm-social. We omit these other results for the seek of readability.

Figure 1: We show the value of relaxed consensus, the opinion’s mean and the opinion’s variance at the relaxed consensus for ego-facebook social network when $\delta = 0.5$ and $b$ is set to have $\bar{b} \in \{0.5, 1, 1.5, \ldots, 3\}$. Similar results are obtained for different value of $\delta$ and different network models.

In the light of these results, in the following we consider the relaxed consensus since it is numerically stabler than consensus and preserves the accuracy of the results. We will refer to the relaxed consensus simply as consensus.

Note that the stochastic networks strongly depend on the particular choice of the input parameters of the randomized generative algorithm. In particular, the Random Graph model strongly depends on the probability ($p$) of creating an edge between each pair of nodes of the graph, while the Watts-Strogatz model depends on the particular choices of $r$ and $k$. We observed that our results are essentially independent from these particular choices of the parameters. Indeed, we are interested on the trend of the consensus probability and not on the absolute values. We observed that the former does not change by varying the model’s parameters and it reflects the findings discussed in section 3.1. For example, in Figure 2, we show the trend of the consensus probability for the Random Graph Model for different values of $p$. In particular, we show how the trend of the consensus probability changes as the social media influence (Figure 2a), the initial opinion of agents (Figure 2b), and the homophily ratio (Figure 2c) changes. In Figure 2c, we observe that there are not negligible numerical differences between $p = 0.6$ or $p = 0.8$ and $p = 0.2$, and they increase as the average initial opinion of agents in a component increases. Specifically we have a difference larger than 0.2 when $\bar{b} > 0.8$. However, as we specified above, we are interested on the consensus probability. We can observe that, while we have a change of scale, the consensus probability follows the same monotone trend for all the density values $p$ of the Random Graph.

For the Watts Strogatz Model we observe almost the same results. In particular, we computed how the trend of the consensus probability varies as the influence of social media, the opinion of agents and the homophily ratio changes for different combination of parameters. We omit these results for the seek of readability since they are very similar to the ones showed in Figure 2.

We shown that the trend of the consensus probability does not depends on the particular choice of the parameters of the generative algorithms. In the light of this, we decided to do not assume any specific value for these parameters. Specifically, for Random Graph networks, at each simulation we draw the value of $p$ uniformly at random in the interval $[0.3, 0.7]$. Similarly, for Watts-Strogatz networks, at each simulation we draw $r$ uniformly at random in the interval $[5, 40]$ and $k$ uniformly at random in the interval $[4, 30]$. 

13
Figure 2: We show the trend of the consensus probability for the Random Graph model for $p \in \{0.2, 0.6, 0.8\}$. In Figure 2a we show the trend of the consensus probability when $\delta = 0.125, w_{\text{in}} = w_{\text{out}} = 1$, and $b$ is set to have $\tilde{b} \in \{0.5, 1.5, \ldots, 14.5\}$.
In Figure 2b we show how the consensus probability changes as the average opinion of agents in a component varies when $\delta = 0.25, w_{\text{in}} = 1, w_{\text{out}}$ is drawn at each simulation uniformly at random in $[0.3, 4]$ and $\tilde{b} = 1.2\tau_1(h)$.
In Figure 2c we show the consensus probability trend when $\delta = 0.125, \tilde{b} = 0.5, w_{\text{out}} = 1$ and $w_{\text{in}} = \{0.5, 1, 2, 4\}$. Notice that, $h$ is the expected value of the homophily ratio over all runs involving the same value of $w_{\text{in}}$.

On the other hand, for the Hyperbolic Random model we adopted a different approach. We fixed some of the parameters of the generative algorithm to make the networks similar to real social networks [Krioukov et al., 2010]. Specifically, the Hyperbolic Random Model takes three parameters: the exponent of the power-law distribution $\gamma$, the temperature $T$ and the target average degree of each node $K$. In order to have a coefficient of clustering significantly higher than 0 and an efficient decentralized search, properties frequently observed in real social networks, we must set $T < 1$ and $\gamma < 3$. Specifically, we set $T = 0.6$ and $\gamma = 2.5$. Moreover, we observed that the trend of consensus probability does not significantly change by varying the value of $K$, we omit these results for the seek of readability since they are very similar to the one showed in Figure 2. Hence, as we did for the other network formation models, at each simulation we draw $K$ uniformly at random in the interval $[0.07 \cdot |V|, 0.75 \cdot |V|]$.

Our Experimental Results Our experiments highlight that the consensus probability essentially depends only on the relative social media influence $\tilde{b}$ and on the homophily ratio $h$, and not on the absolute values of the media influence $b$, the inter-cluster influence $a_{\text{out}}$, and the intra-cluster influence $a_{\text{in}}$. Indeed, Figure 3 shows that the probability of reaching a consensus is essentially the same when $\tilde{b}$ and $h$ are unchanged, even if we change the values of $b, a_{\text{out}}, a_{\text{in}}$.

The analysis of the symmetric two-block model also highlights that the probability of consensus
the homophily ratio is equal to models. The confidence intervals are shown as error bars. Similar results have been obtained for the other network ratio is always 2, and $b \in \{25p_{\text{out}}, 25p_{\text{out}} + 50p_{\text{out}}, \ldots, 525p_{\text{out}}\}$ (so $\tilde{b} \in \{0.5, 0.5 + 1, \ldots, 10.5\}$). In Figure 3b, we show the consensus probability for $\tilde{b} \in \{0.5, 2.5, \ldots, 14.5\}$, and for each of these values, we evaluate this probability on two different settings. In the first, $w_{\text{in}} = w_{\text{out}} = 1$ (from which the homophily ratio is $h_1$) and $b$ such that $ba_{\text{out}} = \tilde{b}$; in the second, with different blocks, we set $w_{\text{in}} = 1$, $w_{\text{out}}$ is such that the homophily ratio is is equal to $h_1$, and $\tilde{b}$ is as in the first setting. The confidence intervals are shown as error bars. Similar results have been obtained for the other network models.

usually decreases when either $\tilde{b}$ or $h$ increases. This behaviour is confirmed in all our experiments, even for the more complex networks. Specifically, Figure 4 shows how the probability of consensus changes as $\tilde{b}$ increases for different values of $h$. It is immediate to see that, except for low values of $\tilde{b}$, the probability of consensus effectively decreases with $\tilde{b}$. Moreover, our results show that, for each value of $\tilde{b}$, the probability of consensus usually appears to be lower when $h$ is large (notice that, due to the fact that for very large $\tilde{b}$ the probability of consensus is very small, in this range the results showed in Figure 4 are highly affected by statistical noise, as it is also highlighted by the fact that the 95% confidence interval are much larger in this range). An apparently strange behaviour occurs for low values of $\tilde{b}$. Indeed, in this range we have that the consensus probability increases. However, this behaviour is still in line with the theoretical results achieved for the symmetric two-block model. Indeed, as observed above, for large values of $h$, the probability of consensus is expected to have this non-monotone behaviour: it first increases (by going from no consensus to possible consensus on 0), and then decreases (by going from possible consensus on 0 to no consensus again).

Results in Section 3.1 show that convergence to consensus is affected by the initial opinions of agents: indeed conditions for non-consensus in case both initial opinions are larger than $\lambda$ in absolute value are stricter than in the case of a single initial opinion far from 0, and the latter are much more stricter in the case of both initial opinion are close to zero. This behaviour still holds even in more complex graph structures. Specifically, Figure 5 shows that the consensus probability decreases as the average opinion of the agents in a component goes to 1. Interestingly, the figure highlights that a sharp change of probability occurs exactly around $\lambda = 0.5$, by confirming our findings.

Actually, we also run experiments in which we impose initial opinions to be larger or smaller than $\lambda$. Again, we observe that the behaviour on complex networks is very close to the one described for the simple symmetric two-block model. Specifically, we first considered the case in which initial opinions are restricted to be larger than $\lambda$. Figure 6 shows how the probability of consensus changes as $\tilde{b}$ increase in this setting. We observe that the figure shows that the decrement of the probability of consensus occurs as soon as the value of $\tilde{b}$ is around $\tau_1(h)$. Hence, not only the general behaviour that emerges from the symmetric two-block model extends to more general networks, but we can also say that the given thresholds turn out to be quite precise in describing the behaviour also in more general networks.

Similar observations hold when we consider that agents are allowed to take opinions smaller than $\lambda$ in absolute values (see Figure 7). Here, there are three possible phases: when $\tilde{b}$ is small, we have high probability of consensus; for intermediate values, the probability is smaller, but still far away from zero; finally, for large $\tilde{b}$, the probability of consensus get close to zero. Interestingly, the phases changes occurs,
In Figure 4a we show how the consensus probability changes when $\delta = 0.25$, $p_{\text{out}} = 0.99$, $p_{\text{in}}$ is set in order to have the desired $h$, and $b$ is set to have $\tilde{b} \in \{0.3, 0.3 + \frac{1}{25p_{\text{out}}}, \ldots, 14.88\}$.

In Figure 4b we show how the consensus probability changes when $\delta = 0.25$, $w_{\text{out}} = 1$, $w_{\text{in}} \in \{1, 4, 8\}$, and $b$ is set to have $\tilde{b} \in \{1, 2, 3, \ldots, 10\}$). Note that, in this setting, $h$ is the expected value of the homophily ratio over all runs involving the same value for $w_{\text{in}}$.

The confidence intervals are shown as error bars. Similar results have been obtained for the other network models.

as indicated by results above, around $\tau_1(h)$ and $\tau_2(h)$.

Finally, we consider the case in which initial opinions do not diverge. Specifically, Figure 8 shows that, except for real datasets, it is possible to distinguish two phases: for low values of $\tilde{b}$, there is an high probability of consensus, whereas for larger values this probability decreases (but it does not go to zero). Moreover, Figure 8 confirms that the smaller is the homophily ratio the smaller $\tilde{b}$ need to be to make the probability of consensus large by allowing consensus on an opinion different from $-1, 0, 1$, as observed in Theorem 4. Interestingly, for real datasets this behaviour is not confirmed, since the probability of consensus is close to 1 regardless the values of $\tilde{b}$ and $h$. We conjecture that this different behaviour depends on the large density of these networks with respect to the remaining ones. However, this will requires a more careful analysis that would focus on the link between the impact of social media recommendations and the structural and topological properties of the network. However, differently from what happens for the case of divergent initial opinions, we here highlight a difference between experimental and theoretical results. The latter ones show that the thresholds among the two phases should depend on the homophily ratio $h$. However, this dependence does not appear in experiments. We leave open the problem of investigating about the reasons behind this discrepancy.

5 Asynchronous Updates

In this section we will focus on opinion dynamics with asynchronous updates, where at each time step, a single agent, arbitrarily chosen, is allowed to update his/her opinion.

We will show that in this setting, the game admits a generalized potential function [Monderer and Shapley, 1996]. Consequently, unlike the synchronous case, the dynamics always converges to a stable state.

We will see that in the asynchronous case we may also establish useful bounds on the convergence of the dynamics.

Finally, we will focus on consensus, by providing experimental evidence that the results described in the previous section still hold in the asynchronous case.

5.1 Convergence

We start by showing that the proposed opinion game is a generalized ordinal potential game and thus the opinion dynamics with asynchronous updates always converges to a stable profile [Monderer and Shapley, 1996].
Figure 5: In Figure 5a we show how the consensus probability changes when \( \delta = 0.25, p_{\text{out}} = 0.2, p_{\text{in}} = 0.4 \), and \( b \) is set to have \( b = 1.2 \cdot \tau_1(h) \). In Figure 5b we show how the consensus probability changes when \( \delta = 0.25, w_{\text{in}} = 1, w_{\text{out}} \) is drawn at each simulation uniformly at random in \([0.3, 4]\), and \( b \) is set to have \( b = 1.2 \cdot \tau_1(h) \).

Note that in this experiment the initial opinions are still drawn uniformly at random in intervals \([l_L, h_L]\) and \([l_R, h_R]\), but these interval are fixed (they are chosen to be the same interval but with opposite sign) to have that the average opinions of agents in each component has absolute value \( x_0 \), with \( x \in \{0, 0.6, 0.8, 1, 1.5, 2, 3, 4\} \).

The confidence intervals are shown as error bars. Similar results have been obtained for the other network models.

[Shapley 1996.]

**Theorem 5.** For every \( G = (V, E, w) \) and every \( b \geq 0 \), the opinion dynamics with asynchronous updates always converges to a stable opinion profile \( x \).

**Proof.** The theorem follows by showing that the function \( \Phi(x) = \sum_{i \in V} b_i (x_i - s(x_i))^2 + P(x) \) is a generalized ordinal potential function for the game described above, where \( x = (x_1, \ldots, x_n) \) is an opinion profile and \( P(x) = \sum_{(i,j) \in E} w_{i,j} (x_i - x_j)^2 \). Let \( P_i(x) = \sum_{(i,j) \in E} w_{i,j} (x_i - x_j)^2 \), then the cost of agent \( i \) given the opinion profile \( x \) is \( c_i(x) = b(x_i - s(x_i))^2 + P_i(x) \).

We call an edge \( (e_1, e_2) \) a discarding edge if its endpoints have different opinions, i.e., \( x_{e_1} \neq x_{e_2} \). Since the graph \( G \) is undirected, then \( \sum_{i \in V} P_i(x) \) is twice the sum of weights of all the discarding edges with respect to the opinion profile \( x \). Hence, \( P(x) = \frac{1}{2} \sum_{i \in V} P_i(x) \).

Denote by \( (y_i, x_{-i}) \) the opinion profile obtained from \( x \) when player \( i \) switches from opinion \( x_i \) to opinion \( y_i \). Then

\[
c_i(x) - c_i(y_i, x_{-i}) = b_i \left[ (x_i - s(x_i))^2 - (y_i - s(x_i))^2 \right] + P_i(x) - P_i(y_i, x_{-i}).
\]

From the definition of function \( s \) and the choice of \( \lambda = \frac{1}{2} \), it follows that \( (y_i - s(x_i))^2 \geq (y_i - s(y_i))^2 \).

Hence:

\[
c_i(x) - c_i(y_i, x_{-i}) \leq b_i \left[ (x_i - s(x_i))^2 - (y_i - s(y_i))^2 \right] + P_i(x) - P_i(y_i, x_{-i}). \tag{9}
\]

The difference in the function \( \Phi \) between the same pair of profiles is:

\[
\Phi(x) - \Phi(y_i, x_{-i}) = \sum_{k \in V} b_k(x_k - s(x_k))^2 + P(x) - \sum_{j \in V, j \neq i} b_j(x_j - s(x_j))^2
\]

\[
- b_i(y_i - s(y_i))^2 + P(y_i, x_{-i})
\]

\[
= b_i \left[ (x_i - s(x_i))^2 - (y_i - s(y_i))^2 \right] + P(x) - P(y_i, x_{-i})
\]

Let \( D_i(x) = \sum_{j \in V, j \neq i} w_{j,i} (x_j - x_i)^2 \) be the sum of the weights of discarding edges not incident on \( i \) in the opinion profile \( x \). It is immediate to see that \( D_i(x) \) is not affected by the deviation of player \( i \), and
In Figure 6a we show how the consensus probability changes when 
and \( \lambda \sim \tilde{b} \). The confidence intervals are shown as error bars. Similar experiments have been run also on the remaining 
network models with very similar results.

In Figure 6b we show how the consensus probability changes when \( \delta = 0.25 \), \( w_{\text{in}} = w_{\text{out}} = 1 \), and \( b \) is set to have 
\( b \in \{0.5, 1.5, \ldots, 10.5\} \).

In order to have large divergent initial opinion, these are drawn uniformly at random in intervals \([-1, -\lambda - \delta] \) and \([\lambda + \delta, 1]\).

The confidence intervals are shown as error bars. Similar experiments have been run also on the remaining 
network models with very similar results.

thus \( D_i(x) = D_i(y_i, x_{-i}) \). Thus, we can write \( P(x) \) as:

\[
P(x) = \sum_{(j,k) \in E, j \neq i} w_{j,k}(x_j - x_k)^2 + \sum_{j \in \{i,j\} \in E} w_{i,j}(x_i - x_j)^2 = D_i(x) + P_i(x)
\]

Similarly, we have \( P(y_i, x_{-i}) = D_i(y_i, x_{-i}) + P_i(y_i, x_{-i}) = D_i(x) + P(x, y_i, x_{-i}) \). Consequently we have 
\( P(x) - P(y_i, x_{-i}) = P_i(x) - P_i(y_i, x_{-i}) \), and thus

\[
\Phi(x) - \Phi(x_{-i}, y_i) = b_i [(x_i - s(x_i))^2 - (y_i - s(y_i))^2] + P_i(x) - P_i(x_{-i}, y_i).
\]

(10)

By (9) and (10) it follows that if \( c_i(x) - c_i(y_i, x_{-i}) > 0 \), then \( \Phi(x) - \Phi(y_i, x_{-i}) > 0 \), and thus \( \Phi \) is a 
generalized ordinal potential function, as desired.

Observe that from an initial opinion profile \( x^0 = \{x^0_0, \ldots, x^0_n\} \), the opinion dynamics with asyn-
chronous updates may converge to different stable states, depending on the order in which agents are 
chosen for updating their opinions. For example, let \( G \) be a three-player social network, let \( V = \{0, 1, 2\}, \)
\( E = \{(0, 1), (1, 2), (0, 2)\}, x^0 = \{-1, 0.5, 0\}, \Theta = \{-1, -0.5, 0, 0.5, 1\}, b = 1 \), and \( \lambda = 0.5 \). Recall that the 
opinion of player \( i \) at step \( t \) is

\[
x^t_i = \arg \min_x \{b(x - s(x^{t-1}_{-i}))^2 + \sum_{j \in \{i,j\} \in E} w_{i,j}(x - x^{t-1}_j)^2\}.
\]

Thus, if players’ selection order is \{Player 0, Player 2, Player 1, Player 0, Player 1, Player 2\}, then the dynamics reach the stable opinion profile \( x^{eq} = \{-0.5, -0.5, -0.5\} \); if, instead, the selection order 
is \{Player 2, Player 1, Player 0, Player 0, Player 1, Player 2\}, then the stable opinion profile is \( x^{eq} = \{-1, -1, -1\} \).

We are also able to bound the time that the opinion dynamics takes to converge to a stable state. 
Specifically, we observe that convergence time can in general be exponential in the number of agents, 
as stated by the next theorem.

**Theorem 6.** There is a social network \( G = (V, E) \) with \( |V| = n \) and an opinion set \( \Theta \) such that the 
corresponding opinion dynamics with asynchronous updates 
takes an number of steps to converge to a 
stable state that is exponential in \( n \).
Nevertheless, through the analysis of the generalized potential function defined in Theorem 5, we can determine polynomial upper bounds on the number of steps needed to converge to a stable state, whenever the weights of edges and the social media influence have bounded precision $k$, i.e., they can be represented with at most $k$ digits after the decimal point.

**Theorem 7.** Given a social network $G = (V, E)$ with $|V| = n$ and an opinion set $\Theta$ with discretization factor $\delta$, if both the strength $b$ of social media influence and the weights $(w_e)_{e \in E}$ of the edges have bounded precision $k$, then the opinion dynamics with asynchronous updates converges to a Nash Equilibrium in $O(10^k b^n \frac{\delta^2}{2} + 10^k 4 w_{\text{max}} \frac{\delta^2}{2})$, where $w_{\text{max}} = \max_{e \in E} w_e$.

The proofs of Theorem 3 and Theorem 7 resemble the ones used for proving similar results by Ferraioli et al. [2016]. Anyway, we include them in Appendix C for sake of completeness.

### 5.2 Impact on Consensus

We will now focus on the behavior of the opinion dynamics with asynchronous updates with respect to convergence to consensus. We will give experimental evidence that, even in this case, the behavior of the dynamics resembles the one observed in the experiments shown in section 4 and hence it reflects the theoretical findings obtained through the analysis of the synchronous case (section 3). Specifically, we performed simulations in the very same settings described in section 4. Therefore, we run experiments on stochastic two-block model graphs, random graphs and real graphs. As observed in the synchronous case, we observe that the trend of the consensus probability does not change by varying the model’s parameters. We omit these results for the sake of readability, since they are very similar to the ones showed in section 4. Moreover, we compute weights, homophily ratio, social media influence, consensus probability and confidence interval exactly as discussed in section 4. Unlike the synchronous case, we have to define how, at each time step, we determine the agent that can update her opinion. Specifically, let $\mathcal{A}_t$ be the set of agents at step $t$ for which the best-response is to change their current opinion, the update policy is: at each round $t$ we sample an agent from $\mathcal{A}_t$ uniformly at random.

From Figure 9 we observe that the probability of reaching a consensus is the same when $\tilde{b}$ and $h$ are unchanged, even if we change the value of $b, a_{\text{out}}, a_{\text{in}}$, exactly as in the case of synchronous case.

The results showed in Figure 10 highlight that the findings obtained in the section 3 on the dependence between the relative amount of social media influence $\tilde{b}$, the homophily ratio $h$, and the consensus probability hold also for the asynchronous case. As discussed in section 4, we observe that convergence to consensus is affected by the initial opinions of agents. Specifically, we observe that the consensus probability decreases as the average opinion of agents in a component goes to 1, with a sharp change of probability around $\lambda = 0.5$. We run experiments in which we impose initial opinions to be larger or smaller than $\lambda$. We observe that the decrement of the consensus probability occurs as soon as the value
In Figure 8a we show the results in the same setting as Figure 4a. In Figure 8b we show how the consensus probability changes when \( \delta = 0.125 \), \( w_{\text{out}} = 1 \), \( w_{\text{in}} \in \{0.5, 1, 2, 4\} \) and \( b \) is set to have \( \tilde{b} \in \{0.5, 1.5, \ldots, 10.5\} \). As Figure 4, \( h \) is the expected value of the homophily ratio over all runs involving the same value of \( w_{\text{in}} \). In Figure 8c we show the results in the same setting as Figure 4b. Note that in this experiment the initial opinions are not constrained to diverge; indeed, they are still drawn uniformly at random in intervals \([l_L, h_L]\) and \([l_R, h_R]\), but both intervals are equal to \([-1, 1]\). The confidence intervals are shown as error bars. Similar experiments have been run also on the remaining network models with very similar results.

Moreover, we consider the case in which initial opinions do not diverge. The results confirm that the smaller is the homophily ratio the smaller \( \tilde{b} \) need to be to make the probability of consensus large by allowing consensus on an opinion different from \(-1, 0, 1\), as proved in Theorem 4. As observed in section 4 for real dataset this behaviour is not confirmed. Further, the results highlight a difference between experimental and theoretical results. Specifically, the latter ones show that the threshold should depend on the homophily ratio \( h \), but this dependence does not appear in experiments. We leave open the problem of investigating about the reasons behind these discrepancies. The results obtained are very similar to the ones shown in Figure 8, hence we omit them.

6 Conclusions

In this work we analyzed the impact of social media recommendations on opinion formation processes, when opinions may assume only discrete values, as is the case of several electoral settings. We focused mainly on how and how much the social media may influence the likelihood that agents reach a consensus.
Figure 9: Figures show the trend of consensus probability in the same setting as Figure 3 except that the update of opinions is asynchronous. The confidence intervals are shown as error bars. Similar results have been obtained for the other network models.

Figure 10: Figures show the trend of consensus probability in the same setting as Figure 4 except that the update of opinions is asynchronous. The confidence intervals are shown as error bars. Similar results have been obtained for the other network models.

Clearly, it would be interesting also to deepen our analysis by evaluating how the social media can influence, not only the probability of consensus, but also the kind of equilibria that can be reached by the opinion formation process.

In this work we focused on a classical opinion formation model. However, we believe that it would be undoubtedly interesting to analyze whether our results extend to more complex (but more realistic) opinion formation models.

In our analysis, we restrict the opinion space of the social media $\Omega$ to only three values. To consider an higher cardinality of $\Omega$ would be clearly of interest, but we will expect that such an analysis will give results very similar in spirit to the one proved in our work (but with an explosion of possible cases). Similar considerations can be done about extending our mono-dimensional representation of opinions to higher dimensional representations.

Even if, our experimental results highlight a large adherence to the theoretical findings obtained for the symmetric two-block model, some small differences exist among the results for different network structures (mainly, in the case that initial opinions are convergent). It would be then interesting to understand whether and how these difference may be motivated through a detailed study of the relationship among the impact of the social influence and the structural and topological properties of the social network.
A Missing Proofs from Section 3.1.1

Proof of Lemma 6. We consider only the case that \( x^i_1 < -\lambda \). The case for \( x^i_1 > \lambda \) is symmetric and hence omitted.

Recall that, by Lemma 1, \( x^i_1 = x^P_i \), where \( P \) is the block at which \( i \) belongs, and every \( j \notin P \) has \( x^j_1 = x^T_j \). Since \( x^i_1 < -\lambda \), then \( s(x^P_i) = 1 \). Thus, by Lemma 2, \( x^{i+1}_1 \geq 0 \) only if \( \frac{\bar{b} + a_{\text{out}}}{\bar{a}_{\text{out}}} + \frac{\lambda_{\text{out}}}{\lambda} \geq \frac{\lambda}{2} \).

By dividing both sides by \( a_{\text{out}} \) and recalling that \( \frac{\bar{b}}{a_{\text{out}}} = \bar{b} \) and \( \frac{\lambda_{\text{out}}}{a_{\text{out}}} = \lambda \), we achieve that
\[
\bar{b} \leq \frac{x^P_i - \frac{\lambda}{2} + h(x^P_i) - \frac{\lambda}{2}}{\frac{\lambda}{2} + 1}.
\]

(11)

It is immediate to check that the r.h.s. of (11) is maximized by taking \( x^P_i = -\lambda - \delta \), and \( x^P_j = 1 \). By substituting these values in (11), we achieve that \( x^{i+1}_1 \) can be greater than 0 only if \( \bar{b} \leq \frac{2 - \delta + (2\lambda + \delta)\lambda}{2\lambda + 3\lambda} \), as desired.

Proof of Lemma 7. We consider only the case that \( x^i_1 \in [-\lambda, 0) \). The case for \( x^i_1 \in (0, \lambda] \) is symmetric and hence omitted.

Recall that, by Lemma 1, \( x^i_1 = x^P_i \), where \( P \) is the block at which \( i \) belongs, and every \( j \notin P \) has \( x^j_1 = x^T_j \). Since \( x^i_1 \in [-\lambda, 0) \), then \( s(x^P_i) = 0 \). Thus, by Lemma 3, \( x^{i+1}_1 \geq 0 \) only if \( \frac{\bar{b} + a_{\text{out}}}{\bar{a}_{\text{out}}} + \frac{\lambda_{\text{out}}}{\lambda} \geq -\frac{\lambda}{2} \).

By dividing both sides by \( a_{\text{out}} \) and recalling that \( \frac{\bar{b}}{a_{\text{out}}} = \bar{b} \) and \( \frac{\lambda_{\text{out}}}{a_{\text{out}}} = \lambda \), we have that \( x^{i+1}_1 \) can be at least 0 if
\[
\bar{b} \geq -\frac{2}{\delta}\left(1 + h(x^P_i) + \frac{\lambda}{2}\right).
\]

(12)

It is immediate to check that the r.h.s. of (12) is minimized by choosing \( x^P_i = -\delta \), and for \( x^P_j = 1 \). By substituting these values in (12), we achieve that \( x^{i+1}_1 \geq 0 \) only if \( \bar{b} \geq \lambda - (\frac{\lambda}{2} + 1) \), as desired.

Proof of Lemma 8. We consider only the case that \( x^i_1 < -\lambda \). The case for \( x^i_1 > \lambda \) is symmetric and hence omitted.

Recall that, by Lemma 1, \( x^i_1 = x^P_i \), where \( P \) is the block at which \( i \) belongs, and every \( j \notin P \) has \( x^j_1 = x^T_j \). Since \( x^i_1 < -\lambda \), then \( s(x^P_i) = 1 \). Thus, by Lemma 4, \( x^{i+1}_1 \geq 0 \) only if \( \frac{\bar{b} + a_{\text{out}}}{\bar{a}_{\text{out}}} + \frac{\lambda_{\text{out}}}{\lambda} \geq -\frac{\lambda}{2} \).

By dividing both sides by \( a_{\text{out}} \) and recalling that \( \frac{\bar{b}}{a_{\text{out}}} = \bar{b} \) and \( \frac{\lambda_{\text{out}}}{a_{\text{out}}} = \lambda \), we have that \( x^{i+1}_1 \) can be at least 0 if
\[
\bar{b} \leq \frac{x^P_i + \frac{\lambda}{2} + h(x^P_i) + \frac{\lambda}{2}}{1 - \frac{\lambda}{2}}.
\]

(13)

It is immediate to check that the r.h.s. of (13) is minimized by taking \( x^P_i = -\lambda - \delta \), and \( x^P_j = 1 \). By substituting these values in (13), we achieve that \( x^{i+1}_1 \) can be at least 0 only if \( \bar{b} \leq \frac{2 \delta - (2\lambda + \delta)\lambda}{2\lambda + 3\lambda} \), as desired.

B Missing Proofs from Section 3.1.2

Proof of Theorem 2. Suppose w.l.o.g. that \( x^L_1 < -\lambda \), and \( x^R_0 \geq 0 \). Suppose first that \( \bar{b} > \tau^*(h) \). Since \( \tau^*(h) \geq \tau_1(h) \), then, by Lemma 4, no agent \( i \in L \) can take an opinion \( x^L_i \geq -\lambda \). Moreover, since \( \tau^*(h) \geq \tau_2(h) \), then, by Lemma 5, no agent \( j \in R \) can take an opinion \( x^R_j < 0 \). Hence, after the first time step we still have \( x^L_1 < -\lambda \) and \( x^R_0 \geq 0 \). Then, we can iteratively apply the same argument above to conclude that the opinions of the two blocks never converge to a consensus profile.

As for the remaining cases, the argument is exactly the same as discussed in the proof of Theorem 1.

Proof of Theorem 3. Suppose that \( \bar{b} > \max\{\tau_2(h), \tau_3(h)\} \). W.l.o.g., suppose that \( x^L_0 \leq 0 \) and \( x^R_0 \geq 0 \). Then by Lemma 5 and Lemma 6, no agent \( i \in L \) can take an opinion \( x^L_i > 0 \) and no agent \( j \in R \) can take an opinion such that \( x^R_j < 0 \). Hence, after the first step \( x^L_1 \leq 0 \) and \( x^R_1 \geq 0 \). Then, we can iteratively apply the same argument to conclude that only consensus on 0 is possible.

For the case where \( 0 < \bar{b} \leq \max 0, \tau_4(h) \), the argument is exactly the same as discussed in the proof of Theorem 4.
C Missing Proofs from Section 5.1

C.1 Proof of Theorem 6

Proof of Theorem 6. In the following construction we assume δ = 0.5.

A 6-gadget G is a graph (V, E), with |V| = 6, V = {A, B, C, D, E, F}, with edges (A, B), (B, C) and (C, D) having weights ε, 2ε, 3ε respectively, and edges (D, E), (B, F) and (D, F) all weighting 4ε, for some ε > 0. The value of the social media’s influence is b = ε < ε.

Let A₀ a further player with edges (A₀, B) and (A₀, D) of weight 4ε. A₀ allows G to switch between vectors (0, 0, 0, 0, 0, 1/2) and (0, 1/2, 0, 1/2, 0, 1/2). If the players \{A, B, C, D, E, F\} have opinions (0, 0, 0, 0, 0, 1/2) and A₀ is set to 1, then we can have the following best-response sequence, that will be referred as switch-on cycle:

\[(0, 0, 0, 0, 0, 1/2) \rightarrow (0, 1/2, 0, 0, 0, 1/2) \rightarrow (0, 0, 0, 0, 0, 1/2) \rightarrow (1/2, 0, 0, 1/2, 0, 0, 1/2) \rightarrow (0, 1/2, 0, 1/2, 0, 1/2) \rightarrow (1/2, 0, 1/2, 0, 1/2) \rightarrow (0, 1/2, 0, 1/2, 0, 1/2) \rightarrow (1/2, 0, 0, 1/2, 0, 1/2) \rightarrow (0, 0, 0, 0, 0, 1/2)\]

On the other hand, if the players \{A, B, C, D, E, F\} have opinions (0, 1/2, 0, 1/2, 0, 1/2) and A₀ is set to 0, then we can have the following best-response sequence that will be referred as switch-off cycle:

\[(0, 1/2, 0, 1/2, 0, 1/2) \rightarrow (1/2, 1/2, 0, 1/2, 0, 1/2) \rightarrow (0, 0, 1/2, 0, 1/2, 0, 1/2) \rightarrow (0, 1/2, 0, 1/2, 0, 1/2) \rightarrow (1/2, 1/2, 1/2, 0, 0, 1/2) \rightarrow (1/2, 1/2, 1/2, 0, 0, 1/2) \rightarrow (1/2, 1/2, 1/2, 0, 0, 1/2) \rightarrow (1/2, 1/2, 1/2, 0, 0, 1/2) \rightarrow (0, 1/2, 0, 1/2, 0, 1/2) \rightarrow (0, 0, 0, 0, 0, 1/2)\]

During the switch-on cycle the opinion of A does not change, while it follows the sequence 0 → 1/2 → 0 → 1/2 → 0 in the switch-off cycle.

We now define an opinion game characterized by an exponentially large difference between the largest and the smallest edge’s weight. Consider an n 6-gadgets \{G_i\} with players \{A_i, B_i, C_i, D_i, E_i, F_i\}, with i = 1, ..., n. The edge’ weights are parametrized by \(\epsilon_i\), with \(\epsilon_i < \epsilon_{i-1}\), \(\forall i = 1, ..., n\). Let the social media’s influence \(b < \epsilon_n\). For each i we connect \(G_i\) with \(G_{i-1}\) by having \(A_{i-1}\) acting as a switch for \(G_i\).

We finally add the switch player \(A_0\) for \(G_1\). Consequently, the total number of players is \(6n + 1\). We set the edges’ weight to make the behaviour of \(A_i\) not influenced by the edges \((A_i, B_i+1)\) and \((A_i, D_i+1)\) so that the opinion of \(A_i\) always follows the opinion of \(B_i\). Basically, we want that, whatever is the opinion of \(B_i+1\) and \(D_i+1\), the best-response for player \(A_i\) always is the opinion of \(B_i\). Specifically, the needed condition is \(\epsilon_i > 8\epsilon_{i+1} + b\). Hence, it is sufficient to set \(\epsilon_i > 9\epsilon_{i+1}\), since \(\epsilon_{i+1} \geq \epsilon_n > b\). Therefore, the largest edge’s weight is 4\(\epsilon_1\) and the smallest one is \(\epsilon_n\) and their ratio is grater than \(4 \cdot 9^{n-1}\).

Consider now the following initial opinion profile: players \(B_1\) and \(D_1\) have opinion 1/2, \(F_i = 1/2, \forall i = 1, ..., n\), all the renaming players have opinion 0. Basically, \(G_1\) is the starting configuration of a switch-off cycle. Since \(A_i\), for \(i = 1, ..., n\), act as a switch, when her opinion switches from 0 to 1/2 (1/2 to 0), a switch-off-cycle (switch-on) is executed on \(G_{i+1}\). Note that the last two cases occur two times during the switch-off cycle of \(G_1\). An exponentially long opinion dynamics can start by switching-off \(G_1\). Hence, \(G_2\) goes through 2 switch-on cycles and 2 switch-off cycles, \(G_3\) goes through 4 switch-on cycles and 4 switch-off cycles. Consequently, \(G_n\) goes through \(2^{n-1}\) switch-on cycles and \(2^{n-1}\) switch-off cycles.

C.2 Proof of Theorem 7

In order to prove Theorem 7, let us first consider the simpler case in which both the strength of social media’s influence (b) and the weight of the edges are integer.

Proposition 2. Given a social network \(G = (V, E)\) with \(|V| = n\) and an opinion set \(\Theta\) with discretization factor \(\delta\), if both the strength \(b\) of social media influence and the weights \((w_e)_{e \in E}\) of the edges are integers, then the opinion dynamics with asynchronous updates converges to a Nash Equilibrium in \(O(4b \frac{n^2}{\delta^2} + 4w_{max} \frac{n^2}{\delta^2})\), where \(w_{max} = \max_{e \in E} w_e\).

Proof. In the proof of Theorem 6, we proved that \(\Phi(x) = \sum_{i \in V} b(x_i - s(x_i))^2 + \sum_{(i,j) \in E} w_{i,j} (x_i - x_j)^2\) is a generalized ordinal potential function for the opinion dynamics game proposed. Trivially, we have \(\Phi(x) \leq 4bn + 4w_{max} n^2\), where \(n\) is the number of player and it is equal to \(|V|\) and \(w_{max} = \max_{e \in E} w_e\).

Moreover, in the proof of Theorem 5 we also proved that

\[\Phi(x) - \Phi(y, x_{-i}) = b((x_i - s(x_i))^2 - (y_i - s(y_i))^2) + \Delta P\]
where \( \Delta P = P(x) - P(y, x_{-i}) = P(x) - P(y, x_{-i}) \).

If a player \( i \) change its opinion at step \( t \), the following inequalities it is satisfied:

\[
c_i(x_i^t) - c_i(x_{-i}, y_i) = b(x_i^t - s(x_i^t))^2 - b(y_i - s(x_i^t))^2 + \Delta P > 0
\]

where \( y_i \) is the best-response of player \( i \) at step \( t \).

Let \( \Delta B = b(x_i^t - s(x_i^t))^2 - b(y_i - s(x_i^t))^2 \), \( (x_i^t - s(x_i^t)) = k \delta \) with \( k \in \mathbb{N} \), therefore, knowing that \( b \in \mathbb{N} \), it’s easy to see why \( b(x_i^t - s(x_i^t))^2 = k' \delta^2 \), with \( k' \in \mathbb{N} \). Trivially, we have the same only for \( b(y_i - s(x_i^t))^2 \), hence \( \Delta B = Kk \delta^2 \), with \( K \in \mathbb{Z} \). In a very similar way, knowing that \( w_e \in \mathbb{N} \), \( \forall e \in E \), we can show that \( \Delta P = K' \delta^2 \), with \( K' \in \mathbb{Z} \).

Consequently, \( \Phi(x^t) - \Phi(y, x_{-i}) \geq c_i(x^t) - c_i(x_{-i}, y_i) = K' \delta^2 \geq \delta^2 \).

Basically, If a player change it’s opinion, the generalized potential function decreases at least by \( \delta^2 \).

Therefore the number of steep needed .

We now extend the bound for the convergence of the opinion dynamics to games whose influences have bounded precision \( k \). To this aim, let us first define the concept of best-response equivalent [Ferraioli et al. 2016] [Dyer and Mohanaraj, 2011].

**Definition 1.** Two best-response games \( B, B' \) are best-response equivalent if they have the same set of players and strategies, and for any player and strategy profile, the best-response of that player is the same in \( B \) as in \( B' \).

Let \( B_{OP} \) the best-response game described in section 2. \( B_{OP}' \) is exactly defined as \( B_{OP} \) but for any player \( i \) the cost is \( c_i(\cdot) = 10^k c_i(\cdot) \), where \( c_i(\cdot) \) is the cost for the player \( i \) in \( B_{OP} \).

**Observation 1.** \( B_{OP} \) and \( B_{OP}' \) are best-response equivalent. \( \Phi(\cdot) = 10^k \Phi(\cdot) \) is the generalized ordinal potential function for \( B_{OP}' \), where \( \Phi(\cdot) \) is the generalized ordinal potential function for \( B_{OP} \).

Now we are ready for proving Theorem 7 through an opportune extension of Proposition 2.

**Proof of Theorem 7.** Let \( B_{OP} \) the best-response opinion game whose edges’ weights and social media’s influence have bounded precision \( k \). \( B_{OP} \) is exactly the same as \( B_{OP} \), but the social media’s influence and edges’ weights are \( 10^k \) times greater than \( B_{OP} \). \( b' = 10^k b, w'_e = 10^k w_e, \forall e \in E \). Trivially, the cost of any player \( i \) in \( B_{OP}' \) is \( 10^k \) times the cost of that player in \( B_{OP} \). \( c_i(\cdot) = 10^k c_i(\cdot) \). Therefore, from observation 1 we know that \( B_{OP} \) and \( B_{OP}' \) are best-response equivalent. The influences of opinion game \( B_{OP}' \) are integer, hence from Proposition 2 it converges in \( O(4b' n^2 + 4w'_e n^2) \). Consequently \( B_{OP} \) converges in \( O(10^k b n^2 + 10^k 4w_{max} n^2) \). \( \square \)

**References**

E. Shearer and K. E. Matsa. News use across social media platforms 2018. Available at [https://www.pewresearch.org](https://www.pewresearch.org) 2018.

A. I. Andronicu. Using Social Media In Political Campaigns. Evidence From Romania. *SEA - Practical Application of Science*, (10):51–57, 2016.

T. Fujiwara, K. Müller, and C. Schwarz. The effect of social media on elections: Evidence from the united states. Technical report, National Bureau of Economic Research, 2021.

M. C. Benigni, K. Joseph, and K. M. Carley. Online extremism and the communities that sustain it: Detecting the isis supporting community on twitter. *PLOS ONE*, 12:1–23, 2017.

H. Allcott, L. Braghieri, S. Eichmeyer, and M. Gentzkow. The welfare effects of social media. *American Economic Review*, 110(3):629–76, 2020.

M. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121, 1974.

N. Friedkin and E. Johnsen. Social influence and opinions. *The Journal of Mathematical Sociology*, 15 (3-4):193–206, 1990.
D. Bindel, J. M. Kleinberg, and S. Oren. How bad is forming your own opinion? *Games and Economic Behavior*, 92:248–265, 2015.

F. Chierichetti, J. Kleinberg, and S. Oren. On discrete preferences and coordination. *Journal of Computer and System Sciences*, 93:11 – 29, 2018.

D. Ferraioli, P. Goldberg, and C. Ventre. Decentralized dynamics for finite opinion games. *Theoretical Computer Science*, 648:96–115, 2016.

E. Bakshy, S. Messing, and L. A Adamic. Exposure to ideologically diverse news and opinion on facebook. *Science*, 348(6239):1130–1132, 2015.

R. Levy. Social media, news consumption, and polarization: Evidence from a field experiment. *American economic review*, 111(3):831–70, 2021.

Y. Halberstam and B. Knight. Homophily, group size, and the diffusion of political information in social networks: Evidence from twitter. *Journal of public economics*, 143:73–88, 2016.

J. Anunrojwong, O. Candogan, and N. Immorlica. Social learning under platform influence: Extreme consensus and persistent disagreement. *SSRN*, 2020.

R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence: Models, analysis and simulation. *Journal of Artificial Societies and Social Simulation*, 5:1–24, 2002.

V. Auletta, A. Fanelli, and D. Ferraioli. Consensus in opinion formation processes in fully evolving environments. In *AAAI*, pages 6022–6029, 2019a.

R. Olfati-Saber. Flocking for multi-agent dynamic systems: Algorithms and theory. *IEEE Transactions on automatic control*, 51(3):401–420, 2006.

A. V Savkin. Coordinated collective motion of groups of autonomous mobile robots: Analysis of vicsek’s model. *IEEE Transactions on Automatic Control*, 49(6):981–982, 2004.

R. Olfati-Saber and J. S Shamma. Consensus filters for sensor networks and distributed sensor fusion. In *CDC*, pages 6698–6703, 2005.

M. Egerstedt and X. Hu. Formation control with virtual leaders and reduced communications. *IEEE Transactions on Robotics and Automation*, 17(6):947–951, 2001.

H. G Tanner, G. J Pappas, and V. Kumar. Leader-to-formation stability. *IEEE Transactions on Robotics and Automation*, 20(3):443–455, 2004.

Z. Lin, B. Francis, and M. Maggiore. Necessary and sufficient graphical conditions for formation control of unicycles. *IEEE Transactions on Automatic Control*, 50(1):121–127, 2005.

P. Dandekar, A. Goel, and D. T Lee. Biased assimilation, homophily, and the dynamics of polarization. *Proceedings of the National Academy of Sciences*, 110(15):5791–5796, 2013.

D. Fotakis, D. Palyvos-Giannas, and S. Skoulakis. Opinion dynamics with local interactions. In *IJCAI*, pages 279–285, 2016.

D. Fotakis, V. Kandiros, V. Kontonis, and S. Skoulakis. Opinion dynamics with limited information. In *WINE*, pages 282–296, 2018.

K. Bhawalkar, S. Gollapudi, and K. Munagala. Coevolutionary opinion formation games. In *STOC*, pages 41–50, 2013.

V. Bilò, A. Fanelli, and L. Moscardelli. Opinion formation games with dynamic social influences. *Theoretical Computer Science*, 746:444–458, 2018.

D. Ferraioli and C. Ventre. Social pressure in opinion games. In *IJCAI*, pages 3661–3667, 2017.

25
V. Auletta, I. Caragiannis, D. Ferraioli, C. Galdi, and G. Persiano. Generalized discrete preference games. In *IJCAI*, pages 53–59, 2016.

E. Acar, G. Greco, and M. Manna. Group reasoning in social environments. In *AAMAS*, pages 1296–1304, 2017.

M. Feldman, N. Immorlica, B. Lucier, and S. M. Weinberg. Reaching consensus via non-bayesian asynchronous learning in social networks. In *APPROX/RANDOM*, pages 192–208, 2014.

E. Mossel, J. Neeman, and O. Tamuz. Majority dynamics and aggregation of information in social networks. *Autonomous Agents and Multi-Agent Systems*, 28(3):408–429, 2014.

V. Auletta, D. Ferraioli, and G. Greco. On the complexity of reasoning about opinion diffusion under majority dynamics. *Artificial Intelligence*, 284:103288, 2020.

D. Kempe, J. Kleinberg, and E. Tardos. Maximizing the spread of influence through a social network. In *KDD*, pages 137–146, 2003.

B. Wilder and Y. Vorobeychik. Controlling elections through social influence. In *AAMAS*, pages 265–273, 2018.

F. Corò, E. Cruciani, G. D’Angelo, and S. Ponziani. Exploiting social influence to control elections based on scoring rules. In *IJCAI*, pages 201–207, 2019.

M. Abouei Mehrizi, F. Corò, E. Cruciani, and G. D’Angelo. Election control through social influence with unknown preferences. In *COCOON*, pages 397–410, 2020.

M. Castiglioni, D. Ferraioli, N. Gatti, and G. Landriani. Election manipulation on social networks: Seeding, edge removal, edge addition. *Journal of Artificial Intelligence Research*, 71:1049–1090, 2021.

R. Bredereck and E. Elkind. Manipulating opinion diffusion in social networks. In *IJCAI*, pages 894–900, 2017.

S. Sina, N. Hazon, A. Hassidim, and S. Kraus. Adapting the social network to affect elections. In *AAMAS*, pages 705–713, 2015.

V. Auletta, D. Ferraioli, and V. Savarese. Manipulating an election in social networks through edge addition. In *AI*IA, pages 495–510, 2019b.

V. Auletta, I. Caragiannis, D. Ferraioli, C. Galdi, and G. Persiano. Minority becomes majority in social networks. In *WINE*, pages 74–88, 2015.

V. Auletta, I. Caragiannis, D. Ferraioli, C. Galdi, and G. Persiano. Information retention in heterogeneous majority dynamics. In *WINE*, pages 30–43, 2017a.

V. Auletta, I. Caragiannis, D. Ferraioli, C. Galdi, and G. Persiano. Robustness in discrete preference games. In *AAMAS*, pages 1314–1322, 2017b.

V. Auletta, D. Ferraioli, and G. Greco. Optimal majority dynamics for the diffusion of an opinion when multiple alternatives are available. *Theoretical Computer Science*, 869:156–180, 2021.

E. N Gilbert. Random graphs. *The Annals of Mathematical Statistics*, 30(4):1141–1144, 1959.

D. J Watts and S. H Strogatz. Collective dynamics of ‘small-world’networks. *Nature*, 393(6684):440–442, 1998.

D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, and M. Boguná. Hyperbolic geometry of complex networks. *Physical Review E*, 82(3):036106, 2010.

B. W Kernighan and S. Lin. An efficient heuristic procedure for partitioning graphs. *The Bell system technical journal*, 49(2):291–307, 1970.
J. Leskovec and A. Krevl. SNAP Datasets: Stanford large network dataset collection. http://snap.stanford.edu/data, June 2014.

J. J McAuley and J. Leskovec. Learning to discover social circles in ego networks. In NIPS, volume 2012, pages 548–56, 2012.

B. Rozemberczki and R. Sarkar. Characteristic functions on graphs: Birds of a feather, from statistical descriptors to parametric models. In CIKM, pages 1325–1334, 2020.

Dov Monderer and Lloyd S. Shapley. Potential games. Games and Economic Behavior, 14(1):124–143, 1996. ISSN 0899-8256. doi: https://doi.org/10.1006/game.1996.0044. URL https://www.sciencedirect.com/science/article/pii/S0899825696900445

Martin Dyer and Velumailum Mohanaraj. Pairwise-interaction games. In Luca Aceto, Monika Henzinger, and Jiří Sgall, editors, Automata, Languages and Programming, pages 159–170, Berlin, Heidelberg, 2011. Springer Berlin Heidelberg.