ON SOME ESTIMATES INVOLVING FOURIER COEFFICIENTS OF MAASS CUSP FORMS

QINGFENG SUN AND HUI WANG

Abstract. Let \( f \) be a Hecke-Maass cusp form for \( \text{SL}_2(\mathbb{Z}) \) with Laplace eigenvalue \( \lambda_f(\Delta) = 1/4 + \mu^2 \) and let \( \lambda_f(n) \) be its \( n \)-th normalized Fourier coefficient. It is proved that, uniformly in \( \alpha, \beta \in \mathbb{R} \),

\[
\sum_{n \leq X} \lambda_f(n)e(\alpha n^2 + \beta n) \ll X^{7/8+\varepsilon} \lambda_f(\Delta)^{1/2+\varepsilon},
\]

where the implied constant depends only on \( \varepsilon \). We also consider the summation function of \( \lambda_f(n) \) and under the Ramanujan conjecture we are able to prove

\[
\sum_{n \leq X} \lambda_f(n) \ll X^{1/3+\varepsilon} \lambda_f(\Delta)^{4/9+\varepsilon}
\]

with the implied constant depending only on \( \varepsilon \).

1. Introduction

The Fourier coefficients of automorphic forms contain many mysterious properties, especially its oscillation properties, which have been intensively studied by many number theorists. Let \( \lambda_F(n) \) be the normalized Fourier coefficients of an automorphic form \( F \) on \( \text{GL}_n \). Among the many criteria for evaluating the nature of oscillation, the summation function \( \sum_{n \leq X} \lambda_F(n) \) is certainly the most basic one, and the related exponential sum \( \sum_{n \leq X} \lambda_F(n)e(g(n)) \), where as usual, \( e(x) = e^{2\pi ix} \) and \( g(n) \) is a real-valued function, is another important target, with obvious application indications. In this paper, we are concerned with the oscillating behavior of the Fourier coefficients of cusp forms on the full modular group. More precisely, let \( f \) be a holomorphic Hecke cusp form of weight \( k \) or a Hecke-Maass cusp form of Laplace eigenvalue \( \lambda_f(\Delta) = 1/4 + \mu^2 \) (\( \mu > 0 \)) for \( \text{SL}_2(\mathbb{Z}) \) with normalized Fourier coefficients \( \lambda_f(n) \). Then for \( f \) holomorphic, one has the Fourier expansion

\[
f(z) = \sum_{n \geq 1} \lambda_f(n)n^{(k-1)/2}e(nz), \quad \text{Im}(z) > 0,
\]

while for \( f \) a Maass form, we can write its Fourier expansion as

\[
f(z) = \sqrt{y} \sum_{n \neq 0} \lambda_f(n)K_{i\mu}(2\pi|n|ye(nx),
\]

where \( K_{i\mu} \) is the modified Bessel function of the third kind. The famous Ramanujan–Petersson conjectures assert that \( \lambda_f(n) \ll n^{\varepsilon} \) for any \( \varepsilon > 0 \). This was proved by Deligne [2] for \( f \) holomorphic. For \( f \) a Maass cusp form, the best result is \( \lambda_f(n) \ll n^{7/64+\varepsilon} \) due to Kim and

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Sarnak [18]. By Rankin–Selberg theory (see [3, Proposition 19.6]), we have the following average result
\[ \sum_{n \leq X} |\lambda_f(n)|^2 \ll_{\varepsilon} X(X|\mu|)^{\varepsilon}. \] (1.1)

The Rankin-Selberg’s estimate in (1.1) shows that the Fourier coefficients $\lambda_f(n)$ behave like constants on average. However, as $n$ grows $\lambda_f(n)$ in fact vary greatly in sign, which can be seen in the estimate (see [4, 7])
\[ \sum_{n \leq X} \lambda_f(n)e(n\alpha) \ll_f X^{1/2 + \varepsilon} \lambda_f(\Delta)^{1/4 + \varepsilon}, \] (1.2)
where the implied constant depends only on $\varepsilon$. This is an improvement over the bound $X^{1/2 + \varepsilon} \lambda_f(\Delta)^{1/2 + \varepsilon}$ in [12, Section 8.3]. For the associated quadratic exponential sums, Pitt [24] first proved for any $\alpha, \beta \in \mathbb{R}$ and any $\varepsilon > 0$,
\[ \sum_{n \leq X} \lambda_f(n)e(\alpha n^2 + \beta n) \ll X^{15/16 + \varepsilon}, \] (1.3)
where the implied constant depends only on the form $f$ and $\varepsilon$. Later, Liu and Ren [19] improved the above bound to $X^{7/8 + \varepsilon}$.

It is interesting and useful to prove an estimate for the exponential sum in (1.3), which does not depend on the form $f$. So the first aim of our paper is to prove the following.

**Theorem 1.1.** Let $\lambda_f(n)$ be the normalized Fourier coefficients of a Hecke-Maass cusp form for $\text{SL}_2(\mathbb{Z})$ with Laplacian eigenvalue $\lambda_f(\Delta) = 1/4 + \mu^2$. For any $\alpha, \beta \in \mathbb{R}$ and any $\varepsilon > 0$, we have
\[ \sum_{n \leq X} \lambda_f(n)e(\alpha n^2 + \beta n) \ll X^{7/8 + \varepsilon} \lambda_f(\Delta)^{1/2 + \varepsilon}, \]
where the implied constant depends only on $\varepsilon$.

**Remark 1.** The methods of proving Theorem 1.1 can also be adopted for holomorphic forms. Let $f$ be a holomorphic cusp form of weight $k$ for $\text{SL}_2(\mathbb{Z})$ with normalized Fourier coefficients $\lambda_f(n)$. It would have, uniformly in $f$ and $\alpha, \beta \in \mathbb{R}$,
\[ \sum_{n \leq X} \lambda_f(n)e(\alpha n^2 + \beta n) \ll X^{7/8 + \varepsilon} k^{1 + \varepsilon}. \]

The proof for Theorem 1.1 is based on the ideas introduced by [24]. That is, taking two different approaches according to the Dirichlet approximation of $\alpha$. Let $Q \geq 1$ be any given number. By Dirichlet’s theorem, for any $\alpha \in \mathbb{R}$, there exists a reduced rational number $\ell/q$ with $1 \leq q \leq Q$ such that
\[ \left| \alpha - \frac{\ell}{q} \right| \leq \frac{1}{qQ}. \] (1.4)
For “larger” $q$, i.e., the oscillation of the exponential function is large, we separate the Fourier coefficients $\lambda_f(n)$ and the exponential function by the $\delta$-method as in Pitt [24] and deal the resulting sum using techniques in Liu and Ren [19]. However, with the demand that we shall make clear the dependence on $f$ in mind, we need to adopt a different form of the $\delta$-method, i.e., the $\delta$-method due to Duke, Friedlander and Iwaniec (see Section 2.5). For “smaller” $q$, we deal with it as in Pitt [24], except for applying the result of Godber in (1.2).

Another purpose of our paper is to consider the summation function of $\lambda_f(n)$, i.e.,

$$\sum_{n \leq X} \lambda_f(n).$$

For holomorphic cusp forms, this was first studied by Hecke [9] in 1927. Later, Walfisz [26] proved the following estimate, i.e.,

$$\sum_{n \leq X} \lambda_f(n) \ll_{f,\vartheta} X^{1/2+\vartheta},$$

where $\vartheta$ is exponent towards the Ramanujan-Petersson conjecture, i.e., $|\lambda_f(n)| \leq n^{\vartheta}$. Then by Deligne [2], one has the estimate $O_{f,\varepsilon}(X^{1/3+\varepsilon})$ for any $\varepsilon > 0$. Subsequently, Hafner and Ivić [8] removed the factor $X^\varepsilon$ of Deligne’s result and obtained the bound $O_f(X^{1/3})$. The current best record is $O_f(X^{1/3}(\log X)^{-0.1185})$ due to Wu [27]. For the case of Maass cusp forms, there are fewer results. Assuming the Ramanujan-Petersson conjecture, one also has the estimate (see for example [5])

$$\sum_{n \leq X} \lambda_f(n) \ll_{f,\varepsilon} X^{1/3+\varepsilon}.$$ (1.5)

Other interesting results without assuming the Ramanujan-Petersson conjecture but weaker than (1.5) can be found in Hafner and Ivić [8], Lü [20], Jiang and Lü [14] and some references therein.

Notice that all the above mentioned estimates depend on the form $f$. So the second purpose of this paper is to strengthen the estimate in (1.5) by making the dependence on the form $f$ explicit. Our result is the following.

**Theorem 1.2.** Let $\lambda_f(n)$ be the normalized Fourier coefficients of a Hecke-Maass cusp form for $\text{SL}_2(\mathbb{Z})$ with Laplacian eigenvalue $\lambda_f(\Delta) = 1/4 + \mu^2$. Under the Ramanujan conjecture, we have, for any $\varepsilon > 0$,

$$\sum_{n \leq X} \lambda_f(n) \ll X^{1/3+\varepsilon} \lambda_f(\Delta)^{4/9+\varepsilon},$$

where the implied constant depends only on $\varepsilon$.

**Notation.** Throughout the paper, $\varepsilon$ and $A$ are arbitrarily small and arbitrarily large positive numbers, respectively, which may be different at each occurrence. As usual, $e(x) = e^{2\pi ix}$ and the symbol $n \sim X$ means $X < n \leq 2X$.

2. **Preliminaries**

2.1. **Maass cusp forms for $\text{GL}_2$.** Let $f$ be a Hecke-Maass cusp form for $\text{SL}_2(\mathbb{Z})$ with Laplace eigenvalue $1/4 + \mu^2$, with the normalized Fourier coefficients $\lambda_f(n)$. For $\text{Re}(s) > 1$, the $L$-function
associated to $f$ is given by
\[ L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}, \]  
which satisfies the functional equation
\[ L(1-s, f) = (-1)^\eta \gamma(s)L(s, f), \]  
where $\eta = 0$ or 1 according as $f$ is even or odd, and
\[ \gamma(s) = \pi^{1-2s} \prod_{\pm} \Gamma \left( \frac{s + \eta \pm i\mu}{2} \right) \Gamma \left( \frac{1-s + \eta \pm i\mu}{2} \right)^{-1}. \]  

2.2. Summation formulas. We first recall the Poisson summation formula over an arithmetic progression.

**Lemma 2.1.** Let $\beta \in \mathbb{Z}$ and $c \in \mathbb{Z}_{\geq 1}$. For a Schwartz function $f : \mathbb{R} \to \mathbb{C}$, we have
\[ \sum_{n \equiv \beta \mod c} f(n) = \frac{1}{c} \sum_{n \in \mathbb{Z}} \hat{f} \left( \frac{n}{c} \right) e \left( \frac{n\beta}{c} \right), \]  
where $\hat{f}(y) = \int_{\mathbb{R}} f(x)e(-xy)dx$ is the Fourier transform of $f$.

**Proof.** See e.g. [13, Eq.(4.24)]. \qed

We have the following Voronoi formula for SL$_2(\mathbb{Z})$ (see [21, Eqs. (1.12), (1.15)]).

**Lemma 2.2.** Let $\varphi(x)$ be a smooth function compactly supported on $\mathbb{R}^+$. Let $a, \overline{a}, c \in \mathbb{Z}$ with $c \neq 0, (a, c) = 1$ and $a\overline{a} \equiv 1 \pmod{c}$. Then
\[ \sum_{m=1}^{\infty} \lambda_f(m) e \left( \frac{am}{c} \right) \varphi(m) = c \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \lambda_f(m) e \left( \frac{\pm \overline{a}m}{c} \right) \Psi_{\pm} (\frac{m}{c^2}), \]  
where for $\sigma > -1$,
\[ \Psi_{\pm} (x) = \frac{1}{4\pi i} \int_{(\sigma)} (\pi^2 x)^{-s} \rho_f^\pm(s) \hat{\varphi}(-s) ds, \]  
with
\[ \rho_f^\pm(s) = \prod_{\pm} \Gamma \left( \frac{1+s\pm i\mu}{2} \right) \Gamma \left( \frac{2-s\pm i\mu}{2} \right) \]  
Here $\hat{\varphi}(s) = \int_0^{\infty} \varphi(u) u^{s-1} du$ is the Mellin transform of $\varphi$.

2.3. Stirling’s formula. By Stirling asymptotic formula (see [23, Section 8.4, Eq. (4.03)]), for $|\arg s| \leq \pi - \varepsilon$, $|s| \gg 1$ and any $\varepsilon > 0$,
\[ \ln \Gamma(s) = \left( s - \frac{1}{2} \right) \ln s - s + \frac{1}{2} \ln(2\pi) + \sum_{j=1}^{K_1} \frac{B_{2j}}{2j(2j-1)s^{2j-1}} + O_K,\varepsilon \left( \frac{1}{|s|^{2K+1}} \right), \]  
where $B_j$ are Bernoulli numbers. Thus for $s = \sigma + it$, $\sigma$ fixed and $|t| \geq 2$,
\[ \Gamma(\sigma + it) = \sqrt{2\pi} (it)^{\sigma - 1/2} e^{-\pi|t|/2} \left( \frac{|t|}{e} \right)^{it} \left( 1 + \sum_{j=1}^{K_2} \frac{c_j}{t^j} + O_{\sigma,K_2,\varepsilon} \left( \frac{1}{|t|^{K+1}} \right) \right), \]  
\[ \sigma \geq 1, \quad K_1, K_2, \varepsilon > 0, \quad |t| \geq 2, \]  
where $K_1, K_2$ are positive integers.
for some \( H \).

Suppose that Lemma 2.3.

We need the following evaluation for exponential integrals which are Lemma 8.1 and Proposition 2.4.

\[
\delta \text{ inert if for each } \delta \text{ where the } \delta\text{-inert functions (see [17, Lemma 3.1]).}
\]

We will use a version of the circle method. Let

\[
C(j_1, \ldots, j_d) = \sup_{T \in \mathbb{F}} \sup_{(y_1, \ldots, y_d) \in \mathbb{R}^d_0} Y_T^{-j_1 \cdots - j_d} \left| y_1^{j_1} \cdots y_d^{j_d} w_T^{(j_1, \ldots, j_d)}(y_1, \ldots, y_d) \right| < \infty.
\]

**Lemma 2.3.** Suppose that \( w = w_T(y) \) is a family of \( Y \)-inert functions, with compact support on \([Z,2Z]\), so that \( w^{(j)}(y) \ll (Z/Y)^{-j} \). Also suppose that \( \varphi \) is smooth and satisfies \( \varphi^{(j)}(y) \ll H/Z^j \) for some \( H/Y^2 \gg R \gg 1 \) and all \( y \) in the support of \( w \).

1. If \( |\varphi'(y)| \gg H/Z \) for all \( y \) in the support of \( w \), then \( I \ll_A ZR^{-A} \) for \( A \) arbitrarily large.
2. If \( \varphi''(y) \gg H/Z^2 \) for all \( y \) in the support of \( w \), and there exists \( y_0 \in \mathbb{R} \) such that \( \varphi'(y_0) = 0 \) (note \( y_0 \) is necessarily unique), then

\[
I = \frac{e^{i\varphi(y_0)}}{\sqrt{\varphi''(y_0)}} F(y_0) + O_A(ZR^{-A}),
\]

where \( F(y_0) \) is an \( Y \)-inert function (depending on \( A \)) supported on \( y_0 \approx Z \).

We also need the second derivative test (see [11, Lemma 5.1.3]).

**Lemma 2.4.** Let \( \varphi(x) \) be real and twice differentiable on the open interval \([a,b]\) with \( \varphi''(x) \gg \lambda_0 > 0 \) on \([a,b]\). Let \( w(x) \) be real on \([a,b]\) and let \( V_0 \) be its total variation on \([a,b]\) plus the maximum modulus of \( w(x) \) on \([a,b]\). Then

\[
I \ll \frac{V_0}{\sqrt{\lambda_0}}.
\]

2.5. **The circle method.** Let \( \delta : \mathbb{Z} \to \{0,1\} \) be defined as \( \delta(0) = 1 \) and \( \delta(n) = 0 \) for \( n \neq 0 \). We will use a version of the \( \delta \)-method by Duke, Friedlander and Iwaniec (see [13, Chapter 20]) which states that for any \( n \in \mathbb{Z} \) and \( C \in \mathbb{R}^+ \), we have

\[
\delta(n) = \frac{1}{C} \sum_{1 \leq c \leq C} \frac{1}{c} \sum_{a \mod c} \varepsilon \left( \frac{na}{c} \right) \int_{\mathbb{R}} g(c, \zeta) e \left( \frac{n\zeta}{c} \right) d\zeta,
\]

where the \( \star \) on the sum indicates that the sum over \( a \) is restricted to \((a,c) = 1\). The function \( g \) has the following properties (see [13, (20.158), (20.159)]) and [10, Lemma 15])

\[
g(c, \zeta) \ll |\zeta|^{-A}, \quad g(c, \zeta) = 1 + O \left( \frac{C}{c} \left( \frac{c}{C} + |\zeta| \right)^A \right)
\]

(2.9)
for any $A > 1$ and
\[
\frac{\zeta^j}{\zeta} g(c, \zeta) \ll |\zeta|^{-j} \min \left( |\zeta|^{-1}, \frac{C}{c} \right) \log C, \quad j \geq 1.
\]

2.6. Some estimates. We quote the following results from Pitt (see [24, Section 2]), Tolev (see [25, Section 2]) and Liu and Ren (see [19, Lemma 3.3]) respectively.

**Lemma 2.5.** Let $c_1$ be the largest square-free factor of $c \in \mathbb{N}$ such that $c = c_1 c_2$, $(c_1, c_2) = 1$. Then for any $\varepsilon > 0$ and $\beta \in \mathbb{R}$, we have
\[
\sum_{n \sim X} S(m, n, c)(\alpha n^2 + \beta n) \ll (Xc)^{1/2+\varepsilon} + T^{1/2},
\]
where
\[
T = (m, c)^{1/2} \tau^2(c_1) c_1^{1/2} c_2 \sum_{c = c_3 c_4} c_4^{1/2} \sum_{u \equiv c_3 \mod{c_1}} \sum_{1 \leq h < X} \min \left\{ X, \left( \frac{2}{2\alpha h + \frac{u}{c_3}} \right)^{-1} \right\}.
\]

Here $\|x\|$ denote the distance from $x$ to the nearest integer.

**Lemma 2.6.** Let $M \geq 2$. Then for any $\varepsilon > 0$, there exists a smooth function $G(M, x)$, which is periodic with period one and satisfies
\[
\min(M, \|x\|^{-1}) \ll G(M, x).
\]
Moreover, $G(M, x)$ has a Fourier expansion
\[
G(M, x) = \sum_n b(n) e(nx)
\]
with coefficients satisfying $b(n) \ll \log M$ and
\[
\sum_{|n| > M^{1+\varepsilon}} |b(n)| \ll A, \varepsilon \ll M^{-A}
\]
for any constant $A > 0$.

**Lemma 2.7.** Let $f(n)$ denote the largest square-free factor of $n$ such that $(f(n), n/f(n)) = 1$. Then we have
\[
\sum_{n \leq X} f^{-1/4}(n) \ll X^{3/4}
\]
and
\[
\sum_{n \leq X} f^{-1/2}(n) \ll X^{1/2} \log X.
\]

We also need the following estimate (see Karatsuba [16, Chapter VI, §2, Lemma 5]).

**Lemma 2.8.** Let
\[
\alpha = \frac{\ell}{q} + \frac{\theta}{q^2}, \quad (\ell, q) = 1, \quad q \geq 1, \quad |\theta| \leq 1.
\]
Then for any $\beta \in \mathbb{R}$, $U > 0$ and $P \geq 1$, we have
\[
\sum_{x=1}^{P} \min \left\{ U, \|\alpha x + \beta\|^{-1} \right\} \leq 6 \left( \frac{P}{q} + 1 \right) (U + q \log q).
\]
3. Proof of Theorem 1.1

We assume \( \mu > X^\xi \), otherwise Theorem 1.1 follows from Liu and Ren [19]. By dyadic subdivision it suffices to prove the required estimate for the sum

\[
S(X, \alpha, \beta) = \sum_{n \sim X} \lambda_f(n) e \left( \alpha n^2 + \beta n \right).
\]

Let \( V(x) \in C^\infty_c(3/4, 9/4) \) be identically one on \([1, 2]\) with derivatives satisfying \( V^{(j)}(x) \ll 1 \) for any integer \( j \geq 0 \). Then we can write \( S(X, \alpha, \beta) \) as

\[
S(X, \alpha, \beta) = \sum_{n \sim X} e \left( \alpha n^2 + \beta n \right) \sum_{m=1}^{\infty} \lambda_f(m) V \left( \frac{m}{X} \right) \delta(n - m),
\]

(3.1)

where \( \delta(n) = \begin{cases} 
1 & \text{if } n = 0, \\
0 & \text{if } n \neq 0
\end{cases} \) is the Kronecker delta function.

Plugging the identity (2.8) for \( \delta(n) \) into (3.1) and exchanging the order of integration and summations, we get

\[
S(X, \alpha, \beta) = \frac{1}{C} \int_{\mathbb{R}} \sum_{1 \leq c \leq C} \frac{g(c, \zeta)}{c} \sum_{n \sim X} e \left( \alpha n^2 + \beta n + \frac{n \zeta}{cC} \right)
\]

\[
\times \sum_{a \mod c}^* e \left( \frac{na}{c} \right) \sum_{m=1}^{\infty} \lambda_f(m) e \left( -\frac{ma}{c} \right) V \left( \frac{m}{X} \right) e \left( -\frac{m \zeta}{cC} \right) d\zeta,
\]

where \( C > 1 \) is a parameter to be chosen later. Note that the contribution from \( |\zeta| \ll X^{-G} \) for \( G > 0 \) sufficiently large is negligible. Moreover, by the first property in (2.9), we can restrict \( \zeta \) in the range \( |\zeta| \ll X^\xi \) up to an negligible error. So we can insert a smooth partition of unity for the \( \zeta \)-integral and write \( S(X, \alpha, \beta) \) as

\[
S(X, \alpha, \beta) = \sum_{X^{-G} \ll \zeta \ll X^\xi} \frac{1}{C} \int_{\mathbb{R}} \varpi \left( \frac{\zeta}{X} \right) \sum_{1 \leq c \leq C} \frac{g(c, \zeta)}{c} \sum_{n \sim X} e \left( \alpha n^2 + \beta n + \frac{n \zeta}{cC} \right)
\]

\[
\times \sum_{a \mod c}^* e \left( \frac{na}{c} \right) \sum_{m=1}^{\infty} \lambda_f(m) e \left( -\frac{ma}{c} \right) V \left( \frac{m}{X} \right) e \left( -\frac{m \zeta}{cC} \right) d\zeta + O_A(X^{-A}),
\]

(3.2)

where \( \varpi(x) \in C^\infty_c(1, 2) \) satisfying \( \varpi^{(j)}(x) \ll 1 \) for any integer \( j \geq 0 \). Without loss of generality, we only consider the contribution from \( \zeta > 0 \) (the proof for \( \zeta < 0 \) is entirely similar). By abuse of notation, we still write the contribution from \( \zeta > 0 \) as \( S(X, \alpha, \beta) \).

Next we break the \( c \)-sum \( \sum_{1 \leq c \leq C} \) into dyadic segments \( c \sim C_0 \) with \( 1 \ll C_0 \ll C \) and write

\[
S(X, \alpha, \beta) = \sum_{X^{-G} \ll \zeta \ll X^\xi} \sum_{1 \ll C_0 \ll C} S(X, \alpha, \beta, C_0, \Xi) + O_A(X^{-A})
\]

(3.2)

with

\[
S(X, \alpha, \beta, C_0, \Xi) = \frac{1}{C} \int_{\mathbb{R}} \varpi \left( \frac{\zeta}{X} \right) \sum_{c \ll C_0} \frac{g(c, \zeta)}{c} \sum_{n \sim X} e \left( \alpha n^2 + \beta n + \frac{n \zeta}{cC} \right)
\]

\[
\times \sum_{a \mod c}^* e \left( \frac{na}{c} \right) \sum_{m=1}^{\infty} \lambda_f(m) e \left( -\frac{ma}{c} \right) V \left( \frac{m}{X} \right) e \left( -\frac{m \zeta}{cC} \right) d\zeta.
\]

(3.3)
We now proceed to estimate \( S(X, \alpha, \beta, C_0, \Xi) \) for \( 1 \ll C_0 \ll C \). Applying Lemma 2.2 with \( \varphi(x) = V(x/X) e(-x \zeta/(cC)) \) to transform the sum over \( m \) we get
\[
\sum_{m=1}^{\infty} \lambda_f(m) e \left( -\frac{ma}{c} \right) V \left( \frac{m}{X} \right) e \left( -\frac{mc}{C} \right) = c \sum_{\pm} \sum_{m=1}^{\infty} \lambda_f(m) \frac{m}{m} e \left( \mp \frac{\pi m}{c} \right) \Psi^\pm \left( \frac{m}{c^2}, c, \zeta \right),
\]
where by \((2.4), \)
\[
\Psi^\pm(x, c, \zeta) = \frac{1}{4\pi^2} \int_{\mathbb{R}} (\pi^2 x) x X)^{-\sigma - i\tau} \rho_f^\pm(\sigma + i\tau) V \left( \frac{\zeta X}{cC}, -\sigma - i\tau \right) d\tau
\]
with \( \rho_f^\pm(s) \) defined in \((2.5) \) and
\[
V^\dagger(r, s) = \int_0^\infty V(x) e(-r x)x^{s-1} dx.
\]
Plugging \((3.4) \) into \((3.3) \), we obtain
\[
S(X, \alpha, \beta, C_0, \Xi) = \frac{1}{C} \sum_{\pm} \sum_{c \sim C_0} g(c, \zeta) \sum_{m=1}^{\infty} \lambda_f(m) \frac{m}{m} \Psi^\pm \left( \frac{m}{c^2}, c, \zeta \right)
\times \sum_{n} e \left( \alpha n^2 + \beta n + \frac{n \zeta}{C} \right) S(n, \mp m; c) d\zeta,
\]
where \( S(n, m; c) \) is the classical Kloosterman sum.

The integral \( \Psi^\pm(x, c, \zeta) \) has the following properties.

**Lemma 3.1.** Let \( r = \zeta X/cC \) and \( \zeta \ll \Xi \).

1. Suppose \( X\Xi/(cC) \gg X^\varepsilon \). Then for \( \mu^{1-\varepsilon} \ll r \ll \mu^{1+\varepsilon} \), \( \Psi^\pm(x, c, \zeta) = \Psi_1 + \Psi_2 \), where \( \Psi_1 \) is negligibly small unless \( x X \ll \mu^{1+\varepsilon} \), in which case
   \[
   \Psi_1 \ll (rXx)^{1/2},
   \]
   and \( \Psi_2 \) is negligibly small unless \( x X \ll r \mu^{1+\varepsilon} \), in which case
   \[
   \Psi_2 \ll (xX)^{1/2}.
   \]

2. Suppose \( X\Xi/(cC) \gg X^\varepsilon \). Then for \( r \ll \mu^{1-\varepsilon} \) or \( r \gg \mu^{1+\varepsilon} \), \( \Psi^\pm(x, c, \zeta) \) is negligibly small unless \( x = \max\{r^2, \mu^2\}/X \), in which case
   \[
   \Psi^\pm(x, c, \zeta) \ll (xX)^{1/2}.
   \]

3. If \( X\Xi/(cC) \ll X^\varepsilon \), then \( \Psi^\pm(x, c, \zeta) \) is negligibly small unless \( x X \ll \mu^{2+\varepsilon} \), in which case
   \[
   \Psi^\pm(x, c, \zeta) \ll (xX)^{1/2+\varepsilon}.
   \]

**Proof.** For the case \( r \approx X\Xi/(cC) \gg X^\varepsilon \), we apply the stationary phase to the integral \( V^\dagger(r, -\sigma - i\tau) \). Write
\[
V^\dagger(r, -\sigma - i\tau) = \int_0^\infty V(u) u^{-\sigma-1} \exp (i g(u)) du,
\]
where \( g(u) = -2\pi ru - \tau \log u \). Note that
\[
\begin{align*}
\frac{d}{du} g(u) &= -2\pi r - \tau/u, \\
\frac{d}{du} g^{(j)}(u) &= \tau(-1)^j(j-1)!u^{-j} \approx |\tau|, \quad j = 2, 3, \ldots.
\end{align*}
\]
By repeated integration by parts one shows that \( V^\dagger (r, -\sigma - i\tau) \) is negligibly small unless \( |\tau| \asymp r \) and \( \tau < 0 \) (note that \( r > 0 \) here). The stationary point is \( u_0 = -\tau/(2\pi) \). Applying Lemma 2.3 (2) with \( Y = Z = 1 \) and \( H = R = \tau \asymp X^\varepsilon \), we have

\[
V^\dagger (r, -\sigma - i\tau) = \tau^{-1/2} V_\sigma^\dagger \left( \frac{-\tau}{2\pi} \right) e \left( -\frac{\tau}{2\pi} \log \frac{-\tau}{2\pi e r} \right) + O_A (X^{-A}), \tag{3.7}
\]

where \( V_\sigma^\dagger (x) \) is an inert function (depending on \( A \) and \( \sigma \)) supported on \( x \asymp 1 \). Plugging (3.7) into (3.5), we obtain

\[
\Psi^\pm (x, c, \zeta) = \frac{1}{4\pi^2} \int_{-\infty}^{0} (\pi^2 x X)^{-\sigma-i\tau} \rho^\dagger_f (\sigma + i\tau) \times \tau^{-1/2} V_\sigma^\dagger \left( \frac{-\tau}{2\pi} \right) e \left( -\frac{\tau}{2\pi} \log \frac{-\tau}{2\pi e r} \right) d\tau + O_A (X^{-A}),
\]

where \( r = \zeta X/(cC) > 0 \). Making a change of variable \( \tau \to -r\tau \),

\[
\Psi^\pm (x, c, \zeta) = \frac{\sqrt{-1}}{4\pi^2} \int_{0}^{\infty} (\pi^2 x X)^{-\sigma+i\tau} \rho^\dagger_f (\sigma - i\tau) \times \tau^{-1/2} V_\sigma^\dagger \left( \frac{\tau}{2\pi} \right) e \left( \frac{\tau}{2\pi} \log \frac{\tau}{2\pi e r} \right) d\tau + O_A (X^{-A}), \tag{3.8}
\]

where by (2.5),

\[
\rho^\dagger_f (\sigma - ir\tau) = \prod \pm \Gamma \left( \frac{1+\sigma-i(r\tau+\mu)}{2} \right) \pm \prod \pm \Gamma \left( \frac{2+\sigma-i(r\tau+\mu)}{2} \right). \tag{3.9}
\]

(1) For \( \mu^{1-\varepsilon} \ll r \ll \mu^{1+\varepsilon} \), we divide the range of \( \tau \) into two pieces:

\[
(0, \infty) = \{ \tau \parallel r\tau - \mu \parallel \ll \mu^\varepsilon \} \cup \{ \tau \parallel r\tau - \mu \parallel > \mu^\varepsilon \} := I_1 + I_2
\]

and correspondingly denote by the integral over \( I_j \) by \( \Psi_j, j = 1, 2 \). Then by (2.7),

\[
\Psi_1 \ll r^{1/2} (xX)^{-\sigma} \int_{I_1} \left| \tau \parallel r\tau - \mu \parallel \right|^{\sigma+1/2} \left| V_\sigma^\dagger \left( \frac{\tau}{2\pi} \right) \right| d\tau \ll r^{1/2} \mu^{1/2+\varepsilon} (xX/\mu^{1+\varepsilon})^{-\sigma}.
\]

By taking \( \sigma \) sufficiently large, one sees that \( \Psi_1 \) is negligibly small unless \( xX \ll \mu^{1+\varepsilon} \), in which case by taking \( \sigma = -1/2 \) we have the estimate

\[
\Psi_1 \ll (xX)^{1/2}. \tag{3.10}
\]

For \( \tau \in I_2 \), by (3.9) and Stirling’s approximation in (2.6), we have

\[
\rho^\dagger_f (\sigma - ir\tau) = \left( \prod \pm \left( \frac{|r\tau + \mu|}{2e} \right)^{-i(r\tau+\mu)} \right) \left| r\tau + \mu \right|^{\sigma+1/2} \times (h_{\sigma,1}(r\tau + \mu)h_{\sigma,1}(r\tau - \mu) \pm h_{\sigma,2}(r\tau + \mu)h_{\sigma,2}(r\tau - \mu)) + O_{\sigma,A} (X^{-A}), \tag{3.11}
\]
where $h_{\sigma,j}(x)$, $j = 1, 2$, satisfy $h_{\sigma,j}(x) \ll_{\sigma,j,A} 1$ and $x^\ell h_{\sigma,j}(x) \ll_{\sigma,j,\ell,A} x^{-1}$ for any integer $\ell \geq 1$. Then by (3.8) and (3.11),

\[
\Psi_2 = \frac{\sqrt{-r}}{4\pi^2} \int_0^\infty (\pi^2 x X)^{-\sigma + ir\tau} \left( \prod_{\pm} \left( \frac{|r\tau \pm \mu|}{2e} \right)^{-i(r\tau \pm \mu)} |r\tau \pm \mu|^{\sigma+1/2} \right) \\
\times (h_{\sigma,1}(r\tau + \mu)h_{\sigma,1}(r\tau - \mu) \pm h_{\sigma,2}(r\tau + \mu)h_{\sigma,2}(r\tau - \mu)) \\
\times \tau^{-1/2} V_{\sigma}^2 \left( \frac{\tau}{2\pi} \right) e \left( \frac{\tau}{2\pi} \log \frac{\tau}{2\pi} \right) d\tau + \Psi_3,
\]

(3.12)

where

\[
\Psi_3 \ll r^{1/2}(x X)^{-\sigma} \int_{|\tau| = 1} (|r\tau - \mu||r\tau + \mu|)^{\sigma + 1/2} |V_{\sigma}^2 \left( \frac{\tau}{2\pi} \right)| d\tau \ll r^{1/2} \mu^{1/2+\epsilon} (x X/\mu^{1+\epsilon})^{-\sigma},
\]

which can be negligibly small unless $x X \ll \mu^{1+\epsilon}$, in which case by taking $\sigma = -1/2$ we have

\[
\Psi_3 \ll (r X)^{1/2}.
\]

(3.13)

Denote the first term in (3.12) by $\Psi_2^0$. Then

\[
\Psi_2^0 \ll r^{1/2}(x X)^{-\sigma} \int_{|\tau| = 1} (|r\tau - \mu||r\tau + \mu|)^{\sigma + 1/2} d\tau \ll r^{1/2} \mu^{1/2+\epsilon} \left( \frac{x X}{r \mu^{1+\epsilon}} \right)^{-\sigma},
\]

which can be negligibly small unless $x X \ll r \mu^{1+\epsilon}$, in which case by taking $\sigma = -1/2$,

\[
\Psi_2^0 = (-r)^{1/2}(x X)^{1/2} \int_0^\infty G(\tau) \exp(i\eta(\tau)) d\tau,
\]

where, temporarily,

\[
G(\tau) = \frac{1}{4\pi \sqrt{r}} V_{\sigma}^2 \left( \frac{\tau}{2\pi} \right) (h_{\sigma,1}(r\tau + \mu)h_{\sigma,1}(r\tau - \mu) \pm h_{\sigma,2}(r\tau + \mu)h_{\sigma,2}(r\tau - \mu))
\]

with $\sigma = -1/2$, and

\[
\eta(\tau) = r\tau \log \frac{\pi x X}{2e} - (r\tau + \mu) \log \frac{|r\tau + \mu|}{2e} - (r\tau - \mu) \log \frac{|r\tau - \mu|}{2e} + r\tau \log \tau.
\]

Note that

\[
\eta'(\tau) = -r \log \frac{|r\tau - \mu||r\tau + \mu|}{2\pi x X \tau},
\]

\[
\eta''(\tau) = -r \left( \frac{1}{\tau - \mu/r} + \frac{1}{\tau + \mu/r} - \frac{1}{\tau} \right)
\]

and

\[
\int_{|r\tau - \mu| > \sqrt{r}} \left| \frac{dG(\tau)}{d\tau} \right| d\tau \ll \max_{\tau = 1} \left\{ \frac{r}{|r\tau - \mu|^2}, \frac{r}{|r\tau + \mu|^2} \right\} \ll 1.
\]

Moreover, for $|r\tau - \mu| > \sqrt{r}$ and $\mu^{1-\epsilon} \ll \mu^{1+\epsilon}$,

\[
\eta''(\tau) \approx r \max_{\tau = 1} |\tau - \mu/r|^{-1}.
\]
Then by Lemma 2.4,

\[
(-r)^{1/2}(xX)^{1/2} \int_{|r\tau - \mu| > \sqrt{r}} G(\tau) \exp(i\eta(\tau)) \, d\tau \\
\ll (xX)^{1/2} \min_{|r\tau - \mu| > \sqrt{r}, r = 1} |\tau - \mu/r|^{1/2} \\
\ll (xX)^{1/2}.
\]

Trivially, we have

\[
(-r)^{1/2}(xX)^{1/2} \int_{|r\tau - \mu| \leq \sqrt{r}} G(\tau) \exp(i\eta(\tau)) \, d\tau \ll (xX)^{1/2}.
\]

Assembling the above results, we conclude that

\[
\Psi_0^0 \ll (xX)^{1/2}. \tag{3.14}
\]

Then the first statement follows from (3.10) and (3.12)–(3.14).

(2) For \( r \ll \mu^{1-\varepsilon} \), we take \( \sigma = -1/2 \) in (3.8) to get

\[
\Psi^\pm(x, c, \zeta) = \frac{\sqrt{-r}}{4\pi^2} \int_0^\infty (x^2 X)^{1/2} \rho_f^\pm \left( \frac{1}{2} - ir\tau \right) \\
\times \tau^{-1/2} V^\pm(\frac{\tau}{2\pi}) e \left( \frac{r\tau}{2\pi} \log \frac{\tau}{2\pi e} \right) \, d\tau + O_A(X^{-A}), \tag{3.15}
\]

where \( V^\pm(x) = V^\pm_{-1/2}(x) \) and by (3.9),

\[
\rho_f^\pm \left( \frac{1}{2} - ir\tau \right) = \prod_{\pm} \Gamma \left( \frac{1/2 - i(r\tau + \mu)}{2} \right) \pm \prod_{\pm} \Gamma \left( \frac{1/2 + i(r\tau + \mu)}{2} \right).
\]

Since \( r \ll \mu^{1-\varepsilon} \), using Stirling’s approximation in (2.6), we derive

\[
\rho_f^\pm \left( \frac{1}{2} - ir\tau \right) = \left( \frac{\mu - r\tau}{2e} \right)^{-i(r\tau - \mu)} \left( \frac{r\tau + \mu}{2e} \right)^{-i(r\tau + \mu)} \\
\times \left( h_1(r\tau - \mu) h_1(r\tau + \mu) + h_2(r\tau - \mu) h_2(r\tau + \mu) \right) + O_A(X^{-A}), \tag{3.16}
\]

where \( h_j(x), j = 1, 2, \) satisfy \( h_j(x) \ll_j 1 \) and \( x^\ell h_j^{(\ell)}(x) \ll_{j, \ell, A} x^{-1} \) for any integer \( \ell \geq 1 \). Plugging (3.16) into (3.15), one has

\[
\Psi^\pm(x, c, \zeta) = \frac{(-rxX)^{1/2}}{4\pi} \int_0^\infty V^\pm_0(\tau) \exp(i\varrho_0(\tau)) \, d\tau + O_A(X^{-A}), \tag{3.17}
\]

where

\[
V^\pm_0(\tau) = \frac{1}{\sqrt{\tau}} V^\pm \left( \frac{\tau}{2\pi} \right) \left( h_1(r\tau - \mu) h_1(r\tau + \mu) + h_2(r\tau - \mu) h_2(r\tau + \mu) \right)
\]

satisfying \( d^\ell V^\pm_0(\tau)/d\tau^\ell \ll_\ell 1 \) for any integer \( \ell \geq 0 \), and

\[
\varrho_0(\tau) = r\tau \log \frac{\pi x X}{2e} - (r\tau - \mu) \log \frac{\mu - r\tau}{2e} - (r\tau + \mu) \log \frac{r\tau + \mu}{2e} + r\tau \log \tau.
\]
We compute
\[
\theta'_{0}(\tau) = -r \log \frac{\mu^2 - r^2 \tau^2}{2 \pi x X \tau},
\]
\[
\theta''_{0}(\tau) = -r \left( \frac{1}{\tau - \mu/r} + \frac{1}{\tau + \mu/r} - \frac{1}{\tau} \right) \approx r, \quad j = 2, 3, \ldots.
\]
By repeated integration by parts one shows that \( \Psi^{\pm}(x, c, \zeta) \) is negligibly small unless \( x X \approx \mu^2 \).

By the second derivative test in Lemma 2.4, we have
\[
\Psi^{\pm}(x, c, \zeta) \ll (x X)^{1/2}.
\]

For \( r \gg \mu^{1+\epsilon} \), the proof is similar as that for the case \( r \ll \mu^{1-\epsilon} \) and we will be brief. In this case, the formula (3.17) still holds. Thus repeated integration by parts shows that \( \Psi^{\pm}(x, c, \zeta) \) is negligibly small unless \( x X \approx r^2 \). Note that the total variation of \( V^+_{0}(\tau) \) is bounded by 1 and the second derivative of the phase function is of size \( r \). By the second derivative test in Lemma 2.4, we have
\[
\Psi^{\pm}(x, c, \zeta) \ll (x X)^{1/2}.
\]
This proves the second statement of the lemma.

(3) For \( X \Xi/(c C) \ll X^{\epsilon} \), we have \( r \approx X \Xi/(c C) \ll X^{\epsilon} \). By repeated integration by parts, one has (see [22, Lemma 5])
\[
V^{\uparrow}(r, \sigma + i \tau) \ll_{\sigma} \min \left\{ 1, \left( \frac{1 + |r|}{|\tau|} \right)^{j} \right\}.
\]
Thus
\[
V^{\uparrow} \left( \frac{X}{c C}, -\sigma - i \tau \right) \ll \left( \frac{X^{\epsilon}}{|\tau|} \right)^{j},
\]
which implies that the contribution from \( |\tau| \geq X^{\epsilon} \) can be arbitrarily small by taking \( j \) sufficiently large. Using (3.5) and the trivial estimate \( V^{\uparrow}(r, \sigma + i \tau) \ll 1 \), we have
\[
\Psi^{\pm}(x, c, \zeta) = \frac{1}{4 \pi^2} \int_{|\tau| \leq X^{\epsilon}} (\pi^2 x X)^{-\sigma - i \tau} \rho_{j}^{\pm}(\sigma + i \tau) V^{\uparrow} \left( \frac{X}{c C}, -\sigma - i \tau \right) d\tau + O_{A}(X^{-A})
\]
\[
\ll (x X)^{-\sigma} \int_{|\tau| \leq X^{\epsilon}} (|\tau + \mu||\tau - \mu|)^{\sigma + 1/2} d\tau
\]
\[
\ll \mu^{1+\epsilon} \left( x X/\mu^{2} \right)^{-\sigma},
\]
which implies that the contribution from \( x X \gg \mu^{2+\epsilon} \) is negligible. For \( x X \ll \mu^{2+\epsilon} \), we shift the line of integration in (3.5) to \( \sigma = -1/2 \) to get
\[
\Psi^{\pm}(x, c, \zeta) \ll (x X)^{1/2+\epsilon}.
\]
This finishes the proof of the lemma. \( \square \)

Now we return to the evaluation of \( S(X, \alpha, \beta, C_{0}, \Xi) \) in (3.6). Applying Lemma 2.5, we have
\[
S(X, \alpha, \beta, C_{0}, \Xi) \ll \sup_{\zeta = \Xi} \frac{X^{\epsilon}}{C} \sum_{c \sim C_{0}} \sum_{m} \sum_{m} \frac{\lambda_{f}(m)}{m} \left| \Psi^{\pm} \left( \frac{m}{c}, c, \zeta \right) \right| \left( (X e)^{1/2+\epsilon} + T^{1/2} \right),
\]
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where $\mathbf{T}$ is given by (2.10). By Lemma 3.1, we obtain

$$
S(X, \alpha, \beta, C_0, \Xi) \ll \frac{1}{C} \sum_{c \ll C_0} \sum_{m \ll \epsilon c^2}^{2} \frac{|\lambda_f(m)|}{m} \left( \frac{mX^2}{c^2} \right)^{1/2} \left( (Xc)^{1/2+\epsilon} + \mathbf{T}^{1/2} \right),
$$

where $1_A = 1$ is $A$ is true and equals 0 otherwise,

$$
S_1 = \frac{X^\epsilon}{C} \sum_{c \ll C_0} \sum_{m \ll \epsilon c^2} \frac{|\lambda_f(m)|}{m} \left( \frac{mX^2}{c^2} \right)^{1/2} \left( (Xc)^{1/2+\epsilon} + \mathbf{T}^{1/2} \right),
$$

$$
S_2 = \frac{X^\epsilon}{C} \sum_{c \ll C_0} \sum_{m \ll \epsilon c^2 \xi/C} \frac{|\lambda_f(m)|}{m} \left( \frac{mX^2}{c^2} \right)^{1/2} \left( (Xc)^{1/2+\epsilon} + \mathbf{T}^{1/2} \right),
$$

$$
S_3 = \frac{X^\epsilon}{C} \sum_{c \ll C_0} \sum_{m \ll \epsilon c^2 \mu^2/X} \frac{|\lambda_f(m)|}{m} \left( \frac{mX^2}{c^2} \right)^{1/2} \left( (Xc)^{1/2+\epsilon} + \mathbf{T}^{1/2} \right),
$$

$$
S_4 = \frac{X^\epsilon}{C} \sum_{c \ll C_0} \sum_{m \ll X/2\xi/c^2} \frac{|\lambda_f(m)|}{m} \left( \frac{mX^2}{c^2} \right)^{1/2} \left( (Xc)^{1/2+\epsilon} + \mathbf{T}^{1/2} \right)
$$

and

$$
S_5 = \frac{X^\epsilon}{C} \sum_{c \ll C_0} \sum_{m \ll \epsilon c^2 \mu^2/X} \frac{|\lambda_f(m)|}{m} \left( \frac{mX^2}{c^2} \right)^{1/2+\epsilon} \left( (Xc)^{1/2+\epsilon} + \mathbf{T}^{1/2} \right).
$$

Obviously, $S_3$ can be dominated by $S_5$. We use the strategy of Liu and Ren [19] to deal with $S_1$, and $S_2$–$S_5$ can be estimated similarly. Firstly, by (2.10) and Lemma 2.6 we have,

$$
\mathbf{T} \ll (m, c)^{1/2} c_1^{1/2+\epsilon} c_2 \sum_{c = c_4 c_3} c_4^{1/2} \sum_{1 \leq h < X \mod c_3} G \left( X, 2ah + \frac{u}{c_3} \right)
$$

$$
= (m, c)^{1/2} c_1 c_2 \sum_{c = c_4 c_3} c_4^{1/2} \sum_{1 \leq h < X \mod c_3} \left\{ b(0) + \sum_{|n| \geq 1} b(n)e \left( 2nah + \frac{nu}{c_3} \right) \right\}
$$

$$
\ll |T_0| + |T_1| + |T_2| + X^{-A},
$$

say, where $c_1, c_2$ are defined as in Lemma 2.5. Trivially, we have

$$
T_0 = (m, c)^{1/2} c_1^{-1/2+\epsilon} \sum_{c = c_4 c_3} c_4^{1/2} \sum_{1 \leq h < X \mod c_3} b(0)
$$

$$
\ll X(m, c)^{1/2} c_1^{1/2+\epsilon} c_2^{-1/2}.
$$

Moreover,

$$
T_1 = (m, c)^{1/2} c_1^{-1/2+\epsilon} \sum_{c = c_4 c_3} c_4^{1/2} \sum_{1 \leq h < X \mod c_3} \sum_{1 \leq n < X^{1+\epsilon}} b(n) e \left( 2nah + \frac{nu}{c_3} \right)
$$

$$
\ll (\log X) (m, c)^{1/2} c_1^{-1/2+\epsilon} \sum_{c = c_4 c_3} c_4^{-1/2} \sum_{1 \leq h < X^{1+\epsilon} / c_3} \min \left\{ X, \|2c_3 k \alpha\|^{-1} \right\},
$$

(3.19)
which is based on the orthogonality of additive characters and the elementary estimate \( \sum_{h \leq X} e(\xi h) \ll \min(X, \| \xi \|^{-1}) \). Finally,

\[
T_2 = (m, c)^{1/2} cc_1^{-1/2 + \varepsilon} \sum_{c = c_3c_4} c_4^{1/2} \sum_{1 \leq h < X \mod c_3} \sum_{1 \leq n \leq X^{1+\varepsilon}} b(-n)e\left(-2n\alpha h - \frac{n\mu}{c_3}\right),
\]

which can be estimated similarly as \( T_1 \). Hence by Cauchy–Schwartz inequality, the Rankin–Selberg estimate in (1.1) and the above estimates, we have

\[
S_1 \ll \frac{X^{1+\varepsilon} \Xi^{1/2}}{C^{3/2}} \sum_{c \sim C_0} \frac{1}{c^{3/2}} \sum_{m \leq c \mu^{1+\varepsilon}/X} \frac{\left| \lambda_f(m) \right|}{m^{1/2}} \left( Xc \right)^{1/2+\varepsilon} + \frac{X^{1+\varepsilon} \Xi^{1/2}}{C^{3/2}} \sum_{c \sim C_0} \frac{1}{c^{3/2}} \sum_{m \leq c \mu^{1+\varepsilon}/X} \frac{\left| \lambda_f(m) \right|}{m^{1/2}} T_1^{1/2}
\]

\[
\ll \frac{X^{1+\varepsilon} \mu^{1/2+\varepsilon}}{C^{3/2}} C_0^{1+\varepsilon} + \sum_{i = 0}^2 S_{1i},
\]

recalling \( \Xi \ll X^\varepsilon \), where

\[
S_{1i} = \frac{X^{1+\varepsilon}}{C^{3/2}} \sum_{c \sim C_0} \frac{1}{c^{3/2}} \sum_{m \leq c \mu^{1+\varepsilon}/X} \frac{\left| \lambda_f(m) \right|}{m^{1/2}} T_i^{1/2}.
\]

By (3.19), Cauchy–Schwartz inequality, the Rankin–Selberg estimate in (1.1) and Lemma 2.7, we have

\[
S_{10} \ll \frac{X^{3/2+\varepsilon}}{C^{3/2}} \sum_{c \sim C_0} c^{-1/2+\varepsilon} c_1^{-1/4} \sum_{m \leq c \mu^{1+\varepsilon}/X} \frac{\left| \lambda_f(m) \right|}{m^{1/2}} (m, c)^{1/4}
\]

\[
\ll \frac{X^{3/2+\varepsilon}}{C^{3/2}} \sum_{c \sim C_0} c^{-1/2+\varepsilon} c_1^{-1/4} \left( \sum_{m \leq c \mu^{1+\varepsilon}/X} \left| \lambda_f(m) \right|^2 \right)^{1/2} \left( \sum_{m \leq c \mu^{1+\varepsilon}/X} \frac{(m, c)^{1/2}}{m} \right)^{1/2}
\]

\[
\ll \frac{X^{1+\varepsilon} \mu^{1/2+\varepsilon}}{C^{3/2}} C_0^{5/4+\varepsilon}.
\]  

(3.21)
Similarly, by (3.20), Cauchy–Schwartz inequality, the Rankin–Selberg estimate in (1.1) and Lemmas 2.7–2.8, we have

\[
S_{11} \ll \frac{X^{1+\varepsilon}}{C^{3/2}} \sum_{c \sim C_0} c^{-1/2} c_1^{-1/2+\varepsilon} \sum_{m \ll c_1^{1/2+\varepsilon} X} \frac{|\lambda_f(m)|}{m^{1/2}} \left( \sum_{c_3 \ll C_0} c_4^{-1/2} \sum_{1 \leq k \ll X^{1+\varepsilon}} \min \left\{ X, \|2c_3 k\alpha\|^{-1} \right\} \right)^{1/2}
\]

\[
\ll \frac{X^{1/2+\varepsilon} \mu^{1/2+\varepsilon}}{C^{3/2}} \left( \sum_{c \sim C_0} c_1^{-1/2} c_1^{-1/2+\varepsilon} \sum_{k' \ll X^{1+\varepsilon}} \tau(k') \min \left\{ X, \|2\alpha k'\|^{-1} \right\} \right)^{1/2}
\]

\[
\ll \frac{X^{1/2+\varepsilon} \mu^{1/2+\varepsilon}}{C^{3/2}} \left( C_0^{1/2+\varepsilon} \left( \frac{X^{1+\varepsilon}}{q} + 1 \right) (X + q \log q) \right)^{1/2}
\]

\[
\ll \frac{X^{1+\varepsilon} \mu^{1/2+\varepsilon}}{C^{3/2}} \left( \frac{X}{q} + \frac{q}{X} \right)^{1/2},
\]  

(3.22)

and the estimate for \( S_{12} \) is entirely similar as that for \( S_{11} \). Therefore, by (3.21) and (3.22), we have

\[
S_1 \ll \frac{X^{1+\varepsilon} \mu^{1/2+\varepsilon}}{C^{3/2}} C_0^{1+\varepsilon} \left( \frac{X}{q} + \frac{q}{X} \right)^{1/2}.
\]  

(3.23)

Similarly, we get

\[
S_2 \ll \frac{X^{3/2+\varepsilon} \mu^{1/2+\varepsilon}}{C^{3/2}} C_0^{1+\varepsilon} \left( \frac{X}{q} + \frac{q}{X} \right)^{1/2},
\]  

(3.24)

\[
S_4 \ll \frac{X^{3/2+\varepsilon} \mu^{1/2+\varepsilon}}{C^{3/2}} C_0^{1+\varepsilon} \left( \frac{X}{q} + \frac{q}{X} \right)^{1/2},
\]  

(3.25)

and

\[
S_3, S_5 \ll \frac{X^{1/2+\varepsilon} \mu^{1+\varepsilon}}{C} C_0^{3/4+\varepsilon} \left( \frac{X}{q} + \frac{q}{X} \right)^{1/2}.
\]  

(3.26)

Plugging (3.23)–(3.26) into (3.18) and then inserting the resulting upper bounds into (3.2), we have

\[
S(X, \alpha, \beta) \ll \frac{X^{9/4+\varepsilon} \mu^{1+\varepsilon}}{C^{11/4}} + \frac{X^{2+\varepsilon} \mu^{1+\varepsilon}}{C^{5/2}} \left( \frac{X}{q} + \frac{q}{X} \right)^{1/2}
\] 

\[
+ X^{1/2+\varepsilon} \mu^{1+\varepsilon} C^{3/4+\varepsilon} + X^{1/2+\varepsilon} \mu^{1+\varepsilon} C^{1/2+\varepsilon} \left( \frac{X}{q} + \frac{q}{X} \right)^{1/2}.
\]

We take \( C = X^{1/2} \) to balance the contribution and obtain

\[
S(X, \alpha, \beta) \ll X^{7/8+\varepsilon} \mu^{1+\varepsilon} + X^{3/4+\varepsilon} \mu^{1+\varepsilon} \left( \frac{X}{q} + \frac{q}{X} \right)^{1/2}.
\]  

(3.27)
We will apply (3.27) for larger $q$. For smaller $q$, we follow closely Pitt [24] and combine the results of Godber [6] to give an estimate for the aimed exponential sums in Theorem 1.1, which does not depend on $f$.

**Lemma 3.2.** Let $\lambda_f(n)$ be the normalized Fourier coefficients of a Maass cusp form for $\text{SL}_2(\mathbb{Z})$ with Laplacian eigenvalue $\lambda_f(\Delta) = 1/4 + \mu^2$. Let $\alpha$ satisfy (1.4). Then for any $\varepsilon > 0$, we have

$$\sum_{n \leq X} \lambda_f(n)e\left(an^2 + \beta n\right) \ll X^{1/2+\varepsilon}q^{1/2}\lambda_f(\Delta)^{1/4+\varepsilon} + X^{3/2+\varepsilon}Q^{-1/2}\lambda_f(\Delta)^{1/4+\varepsilon},$$

where the implied constant depends only on $\varepsilon$.

**Proof.** By Godber [6, Theorem 1.2], for any $\gamma \in \mathbb{R}$,

$$\sum_{n \leq X} \lambda_f(n)e(\gamma n) \ll X^{1/2+\varepsilon}\lambda_f(\Delta)^{1/4+\varepsilon},$$

(3.28)

where the implied constant depends only on $\varepsilon$. Then Lemma 3.2 holds by following closely [24, Section 5] and replacing the estimate $\sum_{n \leq X} \lambda_f(n)e(\gamma n) \ll X^{1/2}\log X$ by (3.28). □

**Completion of the proof of Theorem 1.1.** Take $Q = X^{5/4}$. If the Dirichlet approximant to $\alpha$ has $q \ll X^{3/4}\lambda_f(\Delta)^{1/2}$, we apply Lemma 3.2, whereas if $X^{3/4}\lambda_f(\Delta)^{1/2} < q \ll Q$ then we apply (3.27). In either case we obtain the estimate $O(X^{7/8+\varepsilon}\lambda_f(\Delta)^{1/2+\varepsilon})$ as in the statement of Theorem 1.1.

4. **Proof of Theorem 1.2**

By dyadic subdivision, we only need to estimate the sum

$$\sum_{n \sim X} \lambda_f(n).$$

Let $h$ be a nonnegative smooth function supported in $[1, 2]$ such that $h(x) = 1$ for $x \in [1+\eta, 2-\eta]$ ($\eta > 0$) and $h^{(j)}(x) \ll_j \eta^{-j}$ for any integer $j \geq 0$. Assume the Ramanujan–Petersson conjecture $\lambda_f(n) \ll n^\varepsilon$. Then we have

$$\sum_{n \sim X} \lambda_f(n) = \sum_{n \geq 1} \lambda_f(n)h\left(\frac{n}{X}\right) + O(\eta X^{1+\varepsilon}).$$

By Mellin inversion, one has

$$h(x) = \frac{1}{2\pi i} \int_{(2)} \tilde{h}(s)x^{-s}ds,$$

where $\tilde{h}(s) = \int_0^\infty h(x)x^{s-1}dx$. It follows that

$$\sum_{n \sim X} \lambda_f(n) = \frac{1}{2\pi i} \int_{(2)} \tilde{h}(s)L(s, f)X^sds + O(\eta X^{1+\varepsilon}).$$

(4.1)

By shifting the line of integration in (4.1) to $\text{Re}(s) = -\varepsilon$, we get

$$\sum_{n \sim X} \lambda_f(n) = \frac{X^{-\varepsilon}}{2\pi} \int_{\mathbb{R}} X^it\tilde{h}(-\varepsilon + it)L(-\varepsilon + it, f)dt + O(\eta X^{1+\varepsilon}).$$

(4.2)
Repeated integration by parts shows that for any integer $j \geq 1$,
\[
\hat{h}(s) = \frac{(-1)^j}{s(s+1) \cdots (s+j-1)} \int_0^\infty h^{(j)}(x)x^{s+j-1}dx \ll \eta(|s|)^{-j}. \tag{4.3}
\]
Thus for $T \geq 1$ and $j > 2$,
\[
\int_{|t| \geq T} X^{|it|} \hat{h}(-\varepsilon + it)L(-\varepsilon + it, f)dt 
\ll \int_{|t| \geq T} \eta(|t|)^{-j} \lambda_f(\Delta)^{1/2+\varepsilon} |t|^{1+\varepsilon}dt 
\ll \lambda_f(\Delta)^{1/2+\varepsilon} \eta T^2(\eta T)^{-j},
\]
where we have used the convexity bound (see [13, (5.20)])
\[
L(\sigma + it, f) \ll_{\sigma, \varepsilon} ((1 + |t| + \mu)(1 + |t| - \mu))^{(1-\sigma)/2+\varepsilon}.
\]
Consequently, by taking $j$ sufficiently large, the contribution from $T \geq (\lambda_f(\Delta)X)^\varepsilon \eta^{-1}$ is negligible. Moreover, by taking $j = 1$ in (4.3), for $T_1 = X^{1/3} \lambda_f(\Delta)^{-\theta}$, where $\theta$ is a parameter to be optimized later,
\[
\int_{|t| \leq T_1} X^{|it|} \hat{h}(-\varepsilon + it)L(-\varepsilon + it, f)dt 
\ll \int_{|t| \leq T_1} (1 + |t|)^{-1} \lambda_f(\Delta)^{1/2+\varepsilon}(1 + |t|)^{1+\varepsilon}dt 
\ll \lambda_f(\Delta)^{1/2-\theta+\varepsilon} X^{1/3}.
\]
Therefore, by inserting a dyadic smooth partition of unit to the $t$-integral in (4.2), one has
\[
\sum_{n \sim X} \lambda_f(n) = \frac{X^{-\varepsilon}}{2\pi} \sum_{\substack{T_1 < T < \lambda_f(\Delta)X^{\varepsilon - 1} \\
T \text{dyadic}}} \mathcal{I}(T) + O \left( \eta X^{1+\varepsilon} + \lambda_f(\Delta)^{1/2-\theta+\varepsilon} X^{1/3} \right), \tag{4.4}
\]
where
\[
\mathcal{I}(T) := \int_{\mathbb{R}} X^{|it|} \hat{h}(-\varepsilon + it)L(-\varepsilon + it, f)\omega \left( \frac{|t|}{T} \right) dt \tag{4.5}
\]
with $\omega(x) \in C^\infty_c(1,2)$ satisfying $\omega^{(j)}(x) \ll 1$ for any integer $j \geq 0$.

For $T > |\mu|^{1+\varepsilon}$, we apply the functional equation (2.2) for $L(-\varepsilon + it, f)$, introduce the series (2.1) and then integrate termwise to obtain
\[
\mathcal{I}(T) \ll \max_{x \in [1,2]} \left| \int_{\mathbb{R}} \frac{(Xx)^{|it|}}{\varepsilon - it} \gamma(1 + \varepsilon - it)L(1 + \varepsilon - it, f)\omega \left( \frac{|t|}{T} \right) dt \right| 
\ll \max_{x \in [1,2]} \left| \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1+\varepsilon}} \int_{\mathbb{R}} \frac{(Xxn)^{|it|}}{\varepsilon - it} \gamma(1 + \varepsilon - it)\omega \left( \frac{|t|}{T} \right) dt \right|,
\]
where \( \gamma(s) \) is defined in (2.3). By Stirling’s formula in (2.6), for \(|t| \approx T > |\mu|^{1+\varepsilon}\),

\[
\gamma(\sigma + it) = \pi^{1-2\sigma-2it} \prod_{\pm} \frac{\Gamma\left(\frac{\sigma+\eta+\iota(t+\mu)}{2}\right)}{\Gamma\left(\frac{1-\sigma-\eta-\iota(t+\mu)}{2}\right)} = \pi^{1-2\sigma-2it} \prod_{\pm} \left( t \pm \mu \right)^{\sigma-\frac{1}{2}} g_\sigma(t + \mu) \exp \left( i \sum_{\pm} (t \pm \mu) \log \frac{|t \pm \mu|}{2e} \right),
\]

(4.6)

where \( g_\sigma(t) \) satisfies \( t^f g_\sigma(t) \ll_{\varepsilon, \sigma} 1 \). It follows that

\[
\gamma(\sigma + it) \ll (t^2 + \mu^2)^{\sigma-\frac{1}{2}} \ll t^{2\sigma-1}.
\]

(4.7)

Now we use (4.7) together with \((\varepsilon - it)^{-1} = it^{-1} + O(t^{-2})\) to derive

\[
\mathcal{I}(T) \ll \sup_{x \in [1,2]} \left| \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1+\varepsilon}} \mathfrak{J}(Xxn, T) \right| + T^\varepsilon,
\]

(4.8)

where

\[
\mathfrak{J}(P, T) = \int_{\mathbb{R}} P^{it} \gamma(1 + \varepsilon - it) \omega \left( \frac{|t|}{T} \right) \frac{dt}{t}.
\]

Making a change of variable \( t \to T t \), one has

\[
\mathfrak{J}(P, T) = \int_{\mathbb{R}} P^{iTt} \gamma(1 + \varepsilon - iTt) \omega \left( |t| \right) \frac{dt}{t}.
\]

By (4.6) we have

\[
\mathfrak{J}(P, T) = \int_{\mathbb{R}} u(t) e(\varsigma(t)) dt,
\]

where \( u(t) = \pi^{-1-2\varepsilon} t^{-1} (t^2 T^2 - \mu^2)^{1/2+\varepsilon} \omega(|t|) \prod_{\pm} g_{1+\varepsilon}(-tT \pm \mu) \) and

\[
\varsigma(t) = \frac{T}{2\pi} \log(\pi^2 P) - \frac{1}{2\pi} \sum_{\pm} (tT \pm \mu) \log \frac{|tT \pm \mu|}{2e}.
\]

We can apply the stationary phase analysis to \( \mathfrak{J}(P, T) \). Without loss of generality, we consider the case \( t > 0 \), since the case \( t < 0 \) can be treated similarly. Note that

\[
\varsigma'(t) = -\frac{T}{2\pi} \log \frac{t^2 T^2 - \mu^2}{4\pi^2 P}
\]

and

\[
\varsigma''(t) = -\frac{T}{2\pi} \frac{2t T^2}{t^2 T^2 - \mu^2} + T.
\]

We also have \( u^{(j_1)}(t) \ll j_1 T^{1+\varepsilon} \) and \( \varsigma^{(j_2)}(t) \ll j_2 T \) for any integer \( j_1 \geq 0 \) and \( j_2 \geq 2 \). Then integration by parts shows that the integral is negligibly small unless \( T \approx \sqrt{P} \). The stationary point which is the solution to the equation \( \varsigma'(t) = 0 \) is \( t_0 = T^{-1}(\mu^2 + 4\pi^2 P)^{1/2} \). By Lemma 2.3, we have

\[
\mathfrak{J}(P, T) = \int_{\mathbb{R}} u(t) e(\varsigma(t)) dt = T^{1/2+2\varepsilon} F_\varsigma(t_0) e(\varsigma(t_0)) + O_A\left( T^{-A} \right),
\]

(4.9)
where $F_{\lambda}(x)$ is an 1-inert function (depending on $A$) supported on $x \approx 1$. Plugging (4.9) into (4.8) and estimating the resulting sum over $n$ trivially, we obtain

\[ \mathcal{I}(T) \ll T^{1/2+\varepsilon}. \]  

(4.10)

If $T_1 > |\mu|^{1+\varepsilon}$, then $T \gg T_1 > |\mu|^{1+\varepsilon}$. If $T_1 \leq |\mu|^{1+\varepsilon}$, i.e., $X^{1/3} \lambda_f(\Delta)^{-\theta} \leq |\mu|^{1+\varepsilon}$, then

\[ X \leq \lambda_f(\Delta)^{3(1/2+\theta)}. \]  

(4.11)

In this case, for $T \leq |\mu|^{1+\varepsilon}$, we move the line of integration in (4.5) to $\text{Re}(s) = 1/2$ and apply (4.3) with $j = 1$ to get

\[
\mathcal{I}(T) = \int_{\mathbb{R}} X^{1/2+\varepsilon + \varepsilon} \hat{h} \left( \frac{1}{2} + it \right) L \left( \frac{1}{2} + it, f \right) \omega \left( \frac{1}{T} \frac{1 + it + \varepsilon}{T} \right) dt
\]

\[
= -X^{1/2+\varepsilon} \int_{1}^{2} h'(x)x^{1/2} \int_{1/2}^{1} \frac{(xX)^it}{1 + it} L \left( \frac{1}{2} + it, f \right) \omega \left( \frac{1}{T} \frac{1 + it + \varepsilon}{T} \right) dt dx.
\]

Applying the subconvexity bound due to Jutila and Motohashi [15],

\[ L \left( \frac{1}{2} + it, f \right) \ll (|t| + \mu)^{1/3+\varepsilon}, \]

we obtain

\[ \mathcal{I}(T) \ll X^{1/2+\varepsilon} \int_{|t| = T} \left. \frac{1}{t} \right| \left( |t| + \mu \right)^{1/3+\varepsilon} dt \ll X^{1/2+\varepsilon} \lambda_f(\Delta)^{1/6+\varepsilon}, \]

which by (4.11) is bounded by $X^{1/3} \lambda_f(\Delta)^{\theta/2+5/12+\varepsilon}$. This estimate combined with (4.10) when plugged into (4.4) implies

\[
\sum_{n \sim X} \lambda_f(n) \ll (\lambda_f(\Delta)X)^{\varepsilon} \eta^{-1/2} + \eta X^{1+\varepsilon} + \lambda_f(\Delta)^{1/2-\theta+\varepsilon} X^{1/3} + \lambda_f(\Delta)^{\theta/2+5/12+\varepsilon} X^{1/3}.
\]

Take $\eta = X^{-2/3}$ and $\theta = \frac{1}{18}$. We conclude that

\[ \sum_{n \sim X} \lambda_f(n) \ll X^{1/3+\varepsilon} \lambda_f(\Delta)^{1/9+\varepsilon}. \]

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