A diffusion and osmosis system with bounded nonlinearity

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Abstract. A diffusion and osmosis system of ordinary differential equation is considered. The system may represent migration of finite particle among finite places. Such system in the linear form yields a solution in the form of matrix exponential. In this paper, a bounded nonlinear forcing to the system is considered. Although it is bounded, the nonlinear affected so much to the solution. Numerical simulation of some two-dimensional examples are presented.

Understanding the diffusion and osmosis system will be important for studying migration people among towns, cities or countries.

1. Diffusion and osmosis in applications

Diffusion and osmosis are physical processes, and terms mostly appear in physics, chemistry or biology. According to oxford dictionary [1], diffusion is ‘the spreading of something more widely’. In physics, it is ‘the intermingling of substances by the natural movement of their particles’. Basically, diffusion means the spread of particles from higher concentration to the lower one. On the other hand, osmosis is ‘a process by which molecules of a solvent tend to pass through a semi-permeable membrane from a less concentrated solution into a more concentrated one’. In short, it is the motion of particles from a lower to the higher concentration.

Diffusion has been studied theoretically, and for applications in many areas. Mathematically, diffusion equation is already a well-known, including porous medium equation (PME). PME has Barenblatt’s solution, which is related to fundamental solution of the heat equation. Due to the nonlinearity, PME introduces an interaction of Barenblatt’s solutions [2, 3]. PME has also in the interest of many researchers, among others [4, 5, 6]. On the other hand, in some simplest cases provide inverse problems for the heat equation, e.g. [7, 8]. On the application, inverse problem, for a more general and complex case, has been observed in the process of wood drying, [9, 10]. Diffusion has been applied to study the drying of food or food products [11, 12, 13], and also the drying and cracks of soil, clays or other media [14, 15, 16, 17, 18, 19].

Explanation on osmosis for understanding the physical phenomenon can be found in [20, 21]. To understand osmosis experimentally and by simulation was presented in [22]. Raghunathan and Aluru [23] studied osmosi of water through a semi-permeable membrane with an uncharged, a positively and a negatively charged nano-pore. This kind of study was intended to understand osmosis and possibility of design of advanced nano-porous membranes for various applications. Osmosis was also used to study nano-composites for application in bone repair [24].
All the literatures above, the process of diffusion and osmosis are considered in continuous media. Hence, the mathematical model comes to a partial differential equation. This paper focuses on a discrete media, where the model is in a system of ordinary differential equations.

2. A brief of ordinary differential equations, diffusion and osmosis system

This section will be started with the theory of ordinary differential equation [25, 26], then to introduce a diffusion and osmosis system. It will be started by the existence and uniqueness of the solution of a system of first order differential equations. Theorem 1, theorem 2 and theorem 3 are presented without proofs. The proofs, however, can be found in [25].

**Theorem 1 (Peano’s theorem)**

Let \( a > 0, \ b > 0, \ M > 0, \ x: \mathbb{R} \to \mathbb{R}^n \), and \( \Omega = \{ (x, t) \in \mathbb{R}^{n+1} : |x - x_0| < a, |t - t_0| < b \} \). Let \( f: \Omega \to \mathbb{R}^n \) be a continuous function that satisfies

\[
|f(x, t)| < M, \text{ for } (x, t) \in \Omega.
\]

Then, \( \frac{dx(t)}{dt} = f(x(t), t); \ x(t_0) = x_0; \) has a \( C^1 \) solution for \( |t - t_0| \leq T \) if \( T \leq b \) and \( T \leq a/M \).

**Theorem 2**

In addition to the hypothesis in Theorem 1, assume that

\[
(x - y, f(x, t) - f(y, t)) \leq C|x - y|^2, \text{ for } (x, t), (y, t) \in \Omega.
\]

Then (2) has a unique solution for \( t_0 \leq t \leq t_0 + b \). The solution with initial data \( x_0 \) is \( x(x_0, t) \) and it satisfies

\[
|x(x_0, t) - x(x_1, t)| \leq e^{C(t-t_0)}|x_0 - x_1|.
\]

The proof of Therem 1 and theorem 2 can be found in [25].

**Definition 1**

Let \( A(t) \) is a matrix in the form

\[
A(t) = \begin{bmatrix}
a_{1,1}(t) & \cdots & a_{1,n}(t) \\
\vdots & \ddots & \vdots \\
a_{n,1}(t) & \cdots & a_{n,n}(t)
\end{bmatrix},
\]

and \( b(t) \) be a vector

\[
b(t) = \begin{bmatrix}
b_1(t) \\
\vdots \\
b_n(t)
\end{bmatrix}
\]

A linear system of first order differential equations is

\[
\frac{dx(t)}{dt} = A(t)x(t) + b(t).
\]

Observe that all elements of matrix \( A(t) \) and vector \( b(t) \) are function of \( t \). Some theory are based on the assumption that \( A \) and \( b \) are constant, which is a homogeneous system.

**Theorem 3**

Let \( A(t) \) be a continuous function for \( |t - t_0| < b \) with values in matrix (5). Then, \( \frac{dx(t)}{dt} = A(t)x(t); \ x(t_0) = x_0; \) has a unique solution \( x(t) = F(t)x_0 \) where \( F(t) \) satisfies

\[
\frac{dF(t)}{dt} = A(t)F(t); \ F(t) = I;
\]

\( I \) is an identity matrix.
In the following definition, it will be introduced concepts of a diffusion system, an osmosis system, and a diffusion and osmosis system including with bounded nonlinearity.

**Definition 2**
Consider a homogeneous system

\[ \frac{dx(t)}{dt} = A(t)x(t), \]  

(10)

where \( A \) is given by (5). Equation (10) is called a diffusion system if \( a_{ii}(t) < 0 \) for \( i = 1, \cdots, n \). And, it called an osmosis system if \( a_{ii}(t) > 0 \) for \( i = 1, \cdots, n \). Equation (10) is called a diffusion and osmosis system if \( a_{ii}(t) < 0 \) for \( i \in J \subset \{1, \cdots, n\}, J \neq \emptyset \) and \( J \neq \{1, \cdots, n\} \), and \( a_{ii}(t) > 0 \) for \( i \in \{1, \cdots, n\}/J \).

This definition is based on the [1] for diffusion and [20, 21] for osmosis. Moreover, the definition is adjusted for system of first order differential equations.

Below are some examples of \( A \).

**Diffusion system**

\[
A = \begin{bmatrix}
-0.5 & 1 \\
0 & -0.2
\end{bmatrix}, \quad A = \begin{bmatrix}
-0.5 + 0.2 \sin t & 1 \\
0 & -0.2 - 0.1 \cos t
\end{bmatrix}
\]

Write \( x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \) and consider system (10) for these diffusion cases. Observe that for these cases \( x_i(t) \) decreases proportionally with respect to the value of \( x_i(t) \) itself for \( i = 1, 2 \).

**Osmosis system**

\[
A = \begin{bmatrix}
0.3 & 0 \\
-0.1 & 0.8
\end{bmatrix}, \quad A = \begin{bmatrix}
0.3 + 0.1 \cos t & 0 \\
-0.1 & 0.8 - 0.2 \sin t
\end{bmatrix}
\]

Similarly, consider system (10) for these osmosis cases. Observe that \( x_i(t) \) proportionally increases with respect to the value of \( x_i(t) \) itself for \( i = 1, 2 \).

**Diffusion and osmosis system**

\[
A = \begin{bmatrix}
-0.5 & 0.7 \\
-0.3 & 0.2
\end{bmatrix}, \quad A = \begin{bmatrix}
-0.5 + 0.2 \sin t & 0.7 \\
-0.3 & 0.2 + 0.1 \cos t
\end{bmatrix}
\]

Consider system (10) for these diffusion and osmosis cases. In these cases \( x_1(t) \) proportionally decreases with its value, but \( x_2(t) \) proportionally increases with respect to the value of \( x_2(t) \).

**Definition 3**
Consider a non homogeneous system

\[ \frac{dx(t)}{dt} = A(t)x(t) + f(x), \]  

(11)

where \( f(x) \) is the nonlinear part of the system. Equation (11) is called a diffusion and osmosis system with bounded nonlinearity if the linear part

\[ \frac{dx(t)}{dt} = Ax(t) \]

is a diffusion and osmosis system, and \( f(x) \) is bounded, \( \|f(x)\| = M < \infty \).

3. Numerical simulation
In this section, numerical simulation of several diffusion and osmosis system is discussed. For graphical representation, only two-dimensional systems are presented. Consider a linear diffusion system
\[
\begin{align*}
\begin{bmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{bmatrix} &= 
\begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix} \begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix},
\end{align*}
\]  
(12)

and a linear osmosis system

\[
\begin{align*}
\begin{bmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{bmatrix} &= 
\begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix}.
\end{align*}
\]  
(13)

Phase portraits of (12) and (13) are presented in Fig 1(a) and Fig 1(b), respectively. Both systems have the same equilibrium line, i.e. \( y = x \). The equilibrium line of (12) is stable manifold. The solutions of the system starting from any initial points approach the line as time tends to infinity. On the other hand, the solutions of system (13) with initial points outside the equilibrium line leave away the line as time goes to infinity.

Previous figure is about a linear system, now consider a diffusion and osmosis system with bounded nonlinearity in the form

\[
\begin{align*}
\begin{bmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{bmatrix} &= 
\begin{bmatrix}
\alpha & -\alpha \\
\alpha & -\alpha
\end{bmatrix} \begin{bmatrix}
x(t) \\
y(t)
\end{bmatrix} + \begin{bmatrix}
\dot{f}_1(x, y) \\
\dot{f}_2(x, y)
\end{bmatrix},
\end{align*}
\]  
(14)

Figure 2 shows the simulation of system (14) for various values of \( \alpha \), and various sinusoidal function \( f_1 \) and \( f_2 \). The simulation was done in scilab, the parameter \( \alpha \) and the nonlinear functions are written in the caption of each figure. Observe that both figures indicate the existence of asymptotic curves which need to investigate further.

\begin{align*}
(a) & \quad \alpha = -0.2; \quad f_1(x, y) = \cos(xy); \quad f_2(x, y) = \sin(xy) \\
(b) & \quad \alpha = -0.2; \quad f_1(x, y) = \sin(xy); \quad f_2(x, y) = \cos(xy)
\end{align*}

Figure 2: Evolution of the state variables eqn (14): (a) Eqn (12); (b) Eqn (13).
For further simulation, a diffusion and/or osmosis system with bounded nonlinearity has also been considered. The system is in the form

\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{pmatrix} = \begin{bmatrix}
a & -a \\
a & a
\end{bmatrix}
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} + \begin{pmatrix}
f_1(x,y) \\
f_2(x,y)
\end{pmatrix}.
\] (15)

The simulation results are presented in Figure 3 for various values of \(a\), and various sinusoidal function \(f_1\) and \(f_2\). There are tendencies that the system has asymptotic manifolds.

Figure 3: Evolution of the state variables in (15).

Figure 4 shows the evolution of system (10) for a matrix \(A\) where its elements are functions of \(t\). In this case

\[
\begin{pmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{pmatrix} = \begin{bmatrix}
-0.2 + 0.1 \sin t & 0.2 \\
0.2 & 0.2 - 0.1 \sin t
\end{bmatrix}
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} + \begin{pmatrix}
\sin(xy) \\
\cos(xy)
\end{pmatrix}.
\] (16)
4. Conclusion and further research
A diffusion and osmosis system of first order ordinary differential equations has been discussed. It may represent migration of finite particle among finite places. For linear system, the solution is well-known in the form of matrix exponential. A bounded nonlinear contribution to the system has been considered. While underlying theory is left from the discussion, simulations of some examples have been presented. The bounded nonlinearity affects much to the solution that indicated in the phase portrait. Understanding the diffusion and osmosis system will be important for studying migration people among towns, cities or countries. The triggers of the migration may be economic factors, technology, culture, etc. Those factors may contribute linearly, or they interact to form nonlinear interactions.

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