HOPF AUTOMORPHISMS AND TWISTED EXTENSIONS

SUSAN MONTGOMERY, MARIA D. VEGA, AND SARAH WITHERSPOON

Abstract. We give some applications of a Hopf algebra constructed from a group acting on another Hopf algebra as Hopf automorphisms, namely Molnar’s smash coproduct Hopf algebra. We find connections between the exponent and Frobenius-Schur indicators of a smash coproduct and the twisted exponents and twisted Frobenius-Schur indicators of the original Hopf algebra A. We study the category of modules of the smash coproduct.

1. Introduction

Molnar [Ml] defined smash coproducts of Hopf algebras, putting them on equal footing with the better-known smash products by viewing both as generalizations of semidirect products of groups. Recently smash coproducts have made an appearance as examples of new phenomena in representation theory [BW, DE]. In this paper we propose several applications of smash coproducts. In particular, the smash coproduct construction will allow us to “untwist” some invariants defined via the action of a Hopf algebra automorphism, such as the twisted exponents and the twisted Frobenius-Schur indicators.

We note that considering Hopf automorphisms is a timely topic, since there has been recent progress in determining the automorphism groups of some Hopf algebras [AD, Ke, R3, SV, Y]. There has also been much recent work on indicators; their importance lies in the fact that they are invariants of the category of representations of the Hopf algebra, and may be defined for more abstract categories [NSc]. Moreover the notion of twisted indicators can be extended to pivotal categories [SV3].

We start by defining the smash coproduct A ⨁ kg, for any Hopf algebra A with an action of a finite group G by Hopf automorphisms, in the next section. In Section 3 we recall the notions of exponent and twisted exponent [SV2] of a Hopf algebra, and find connections between the exponent of A ⨁ kg and twisted exponents of A itself. In Section 4 we assume the Hopf algebra A is semisimple. We recall definitions of Frobenius-Schur indicators [KSZ] and twisted Frobenius-Schur indicators [SV] for simple modules over the Hopf algebra, and give relationships between the indicators of the smash coproduct A ⨁ kg and twisted indicators of A itself.

In Section 5 we do not assume the Hopf algebra is semisimple. We introduce the twisted Frobenius-Schur indicators of the regular representation of such a Hopf algebra, simultaneously generalizing indicators for not necessarily semisimple Hopf algebras [KMN] and twisted indicators for semisimple Hopf algebras [SV]. Again we find a connection with the Frobenius-Schur indicator of a smash coproduct. We compute an example for which the Hopf algebra
A is of dimension 8 in Section 6. Finally in Section 7 we study the structure of categories of modules of $A \rtimes k^G$, showing that they are equivalent to semidirect product tensor categories $\mathcal{C} \rtimes G$, where $\mathcal{C}$ is a category of $A$-modules.

Throughout, $k$ will be an algebraically closed field of characteristic 0.

2. The smash coproduct

Our Hopf algebra was defined by Molnar [Ml, Theorem 2.14], who called it the smash coproduct, although our definition seems different at first glance. See also [R2, p. 357].

Let $A$ be a Hopf algebra over a field $k$ and let a finite group $G$ act as Hopf algebra automorphisms of $A$. Let $k^G$ be the algebra of set functions from $G$ to $k$ under pointwise multiplication; that is, if $\{p_x \mid x \in G\}$ denotes the basis of $k^G$ dual to $G$, then $p_x p_y = \delta_{x,y} p_x$ for all $x, y \in G$. Recall that $k^G$ is a Hopf algebra with comultiplication given by $\Delta(p_x) = \sum_{y \in G} p_y \otimes p_y^{-1} x$, counit $\varepsilon(p_x) = \delta_{1,x}$ and antipode $S(p_x) = p_{x^{-1}}$ for all $x \in G$.

Then we may form the smash coproduct Hopf algebra

$$K = A \rtimes k^G$$

with algebra structure the usual tensor product of algebras. Denote by $a \rtimes p_x$ the element $a \otimes p_x$ in $K$, for each $a \in A$ and $x \in G$. Comultiplication is given by

$$\Delta(a \rtimes p_x) = \sum_{y \in G} (a_1 \rtimes p_y) \otimes ((y^{-1} \cdot a_2) \rtimes p_y^{-1} x)$$

for all $x \in G$, $a \in A$. The counit and antipode are determined by

$$\varepsilon(a \rtimes p_x) = \delta_{1,x} \varepsilon(a) 1 \quad \text{and} \quad S(a \rtimes p_x) = (x^{-1} \cdot S(a)) \rtimes p_{x^{-1}}.$$

If $\Lambda_A$ is an integral for $A$, then $\Lambda_K = \Lambda_A \rtimes p_1$ is an integral for $K$.

Note that Molnar defines the smash coproduct for the right coaction of any commutative Hopf algebra $H$. We show that our construction is actually his smash coproduct with $H = k^G$, by dualizing our $G$-action to a $k^G$-coaction.

Lemma 2.1. (1) $K$ as above is isomorphic to the smash coproduct as in [Ml, Theorem 2.14], and thus is a Hopf algebra.

(2) If $A$ is finite-dimensional, then $K^* \cong A^* \# kG$, the smash product Hopf algebra as in [Ml, Theorem 2.13].

Proof. (1) Given the left action of $G$ on $A$, we define $\rho : A \to A \otimes k^G$ by $a \mapsto \sum_{x \in G} (x \cdot a) \otimes p_x$. Then $\rho$ is a right comodule map, using the fact that the $G$-action on $A$ satisfies $x \cdot (y \cdot a) = (xy \cdot a)$ and $1 \cdot a = a$ for all $x, y \in G$ and $a \in A$.

Next we note that $A$ is a right comodule algebra under $\rho$ since the $G$-action is multiplicative, that is $(x \cdot a)(x \cdot b) = x \cdot (ab)$. Also $A$ is a right comodule coalgebra, as the $G$-action preserves the coalgebra structure of $A$, that is, $x \cdot (\sum a_1 \otimes a_2) = \sum (x \cdot a_1) \otimes (x \cdot a_2)$. Thus $A$ is a right $k^G$-comodule bialgebra.

Finally the antipode also dualizes to the antipode given by Molnar, and thus Molnar’s theorem [Ml, Theorem 2.14] applies.

(2) This is a special case of Molnar’s result [Ml, Theorem 5.4].
3. Hopf powers and exponents

In any Hopf algebra $H$, we denote the $n$th Hopf power of an element $x \in H$ by $x^{[n]} = \sum x_1 x_2 x_3 \ldots x_n$; that is, first apply $\Delta_H$ $n - 1$ times to $x$ and then multiply. Note that $x \mapsto x^{[n]}$ is a linear map.

For $H$ semisimple, recall that the exponent of $H$, $\exp(H)$, is the smallest positive integer $n$, if it exists, such that $x^{[n]} = \varepsilon(x)1$ for all $x \in H$. More generally, this definition makes sense whenever $S^2 = id$. We assume this property of $S$ unless stated otherwise.

Recently [SV2] introduced the twisted exponent in \cite{SV2}. Let $H$ has order $n$ of $H_n$.

**Proposition 3.2.** Suppose that the Hopf automorphism $H_n$ of the semisimple Hopf algebra $H$ is twisted by an automorphism of $H$ of finite order. Assume that $\tau \in Aut(H)$ and that $n$ is a multiple of the order of $\tau$. Define the $n$th $\tau$-twisted Hopf power of $x$ to be

$$x^{[n,\tau]} := \sum x_1(\tau \cdot x_2)(\tau^2 \cdot x_3)\ldots(\tau^{n-1} \cdot x_n).$$

**Definition 3.1.** $\exp_\tau(H)$ is the smallest positive integer $n$, if it exists, such that $n$ is a multiple of the order of $\tau$ and $x^{[n,\tau]} = \varepsilon(x)1$ for all $x \in H$.

Since $\tau$ is a Hopf automorphism, $\varepsilon(\tau \cdot x) = \varepsilon(x)$ for any $x \in H$, and thus $\varepsilon(x^{[n,\tau]}) = \varepsilon(x^{[n]}) = \varepsilon(x)$. If $H$ is not semisimple and $S^2 \neq id$ yet $S$ is still bijective, there is a more general definition of the twisted exponent in \cite{SV2}.

We will need the following proposition which is a special case of \cite[Proposition 3.4]{SV2}.

**Proposition 3.3.** Suppose that the Hopf automorphism $\tau$ of the semisimple Hopf algebra $H$ has order $r$, $\exp_\tau(H)$ is finite, and $m$ is a positive integer. Then $x^{[mr,\tau]} = \varepsilon(x)1$ for all $x \in H$ if and only if $\exp_\tau(H)$ divides $m$.

Next we give some formulas for our Hopf algebras $K = A \wr k^G$.

**Lemma 3.3.** Let $w = a \wr p_x \in A \wr k^G$, the smash coproduct as above. Then

$$(a \wr p_x)^{[n]} = \sum_{z \in G, z^n = x} a^{[n, z^{-1}]} \wr p_x.$$

In particular for $w = \Lambda_K = \Lambda_A \wr p_1$, replace $z$ by $z^{-1}$. Then

$$\Lambda_K^{[n]} = \sum_{z \in G, z^n = 1} \Lambda_A^{[n, z]} \wr p_z^{-1}.$$

**Proof.** A calculation shows that

$$(a \wr p_x)^{[n]} = \sum_{z \in G, z^n = x} a_1(z^{-1} \cdot a_2)(z^{-2} \cdot a_3)\ldots(z^{-(n-1)} \cdot a_n) \wr p_z,$$

which gives the first equation in the lemma. The second follows from the first. \qed

We now find a relation among the (twisted) exponents of $A$, $G$, and $K = A \wr k^G$.

**Theorem 3.4.** Assume that $S^2 = id$ in $A$. Then the exponent of $K$ is the least common multiple of $\exp(G)$ and $\exp_\tau(A)$ for all $z \in G$.

**Proof.** Let $n = \exp(K)$, so that

$$(a \wr p_x)^{[n]} = \varepsilon(a \wr p_x)1 = \varepsilon(a)\delta_{x,1}1 = \varepsilon(a)\delta_{x,1} \sum_z p_z.$$
Lemma 3.3, for all divides $n$ and $m$.

Question 3.7. We ask if Corollary 3.6 is true more generally. That is, if the order of $n$ divides $\exp(G)$ and $\exp_p(A)$ for all $z \in G$. By Proposition 3.2, $\exp(K)$ is a common multiple of $\exp(G)$ and $\exp_p(A)$ for all $z \in G$.

Now let $m$ be any common multiple of $\exp(G)$ and $\exp_p(A)$ for all $z \in G$. By Lemma 3.3 and Proposition 3.2,

\[
(a \circ p_x)^{[n]} = \sum_{z \in G, \ z^n = x} a^{[m,z^{-1}]} \circ p_x = \delta_{1,x} \sum_{z \in G} a^{[m,z^{-1}]} \circ p_x = \delta_{1,x} \varepsilon(a) \sum_{z \in G} p_x = \varepsilon(a \circ p_x)^1_K.
\]

Again by Proposition 3.2, $\exp(K)$ divides $m$.

We will use the following lemma in calculations.

**Lemma 3.5.** Let $H$ be a Hopf algebra and let $\tau$ be a Hopf automorphism of $H$ whose order divides $n$. Then $S(x^{[n,\tau]}) = \tau^{-1} \cdot (S(x)^{[n,\tau^{-1}]})$ for all $x \in H$.

**Proof.** Since $S$ is an anti-algebra and anti-coalgebra map and $\tau^n = 1$ by hypothesis,

\[
S(x^{[n,\tau]}) = S \left( \sum_x x_1 (\tau \cdot x_2) (\tau^2 \cdot x_3) \cdots (\tau^{n-1} \cdot x_n) \right)
\]

\[
= \sum_x \left( \tau^{n-1} \cdot S(x_n) (\tau^{n-2} \cdot S(x_{n-1})) \cdots (\tau^2 \cdot S(x_3)) (\tau \cdot S(x_2)) S(x_1) \right)
\]

\[
= \sum_x \left( \tau^{-1} \cdot S(x_n) (\tau^{-2} \cdot S(x_{n-1})) \cdots (\tau^{-1(n-2)} \cdot S(x_3)) (\tau^{-1(n-1)} \cdot S(x_2)) S(x_1) \right)
\]

\[
= \tau^{-1} \cdot \left( \sum_x S(x_n) (\tau^{-1} \cdot S(x_{n-1})) \cdots (\tau^{-1(n-3)} \cdot S(x_3)) (\tau^{-1(n-2)} \cdot S(x_2)) (\tau^{-1(n-1)} \cdot S(x_1)) \right)
\]

\[
= \tau^{-1} \cdot (S(x)^{[n,\tau^{-1}]})
\]

**Corollary 3.6.** Let $H$ be a Hopf algebra for which $S^2 = \text{id}$ and let $\tau$ be a Hopf automorphism of $H$. Then $\exp_{\tau^{-1}}(H) = \exp_{\tau}(H)$.

**Proof.** It is clear from Lemma 3.5 that $x^{[n,\tau]} = \varepsilon(x)1 \iff S(x)^{[n,\tau^{-1}]} = \varepsilon(x)1$ since $\tau$ and $S$ are bijective. Thus the two twisted exponents are the same.

**Question 3.7.** We ask if Corollary 3.6 is true more generally. That is, if the order of $\tau$ is $n$ and $m$ is relatively prime to $n$, then is $\exp_{\tau^m}(H) = \exp_{\tau}(H)$?
4. Modules and Frobenius-Schur indicators

In this section, we assume \( A \) is a semisimple Hopf algebra, and thus we may assume that \( \Lambda_A \) is a normalized integral, that is, \( \varepsilon(\Lambda_A) = 1 \). Then the integral \( \Lambda_K = \Lambda_A \natural K \) is a normalized integral of \( K = A \natural k^G \).

For any (left) \( K \)-module \( M \), we may write

\[
M = \bigoplus_{x \in G} M_x
\]

where \( M_x = p_x \cdot M \) is a \( K \)-submodule of \( M \) for each \( x \in G \). Note that each \( M_x \) is also an \( A \)-module, by restricting the action to \( A \).

Let \( \nu^K_m \) denote the \( m \)th Frobenius-Schur indicator for \( K \)-modules as in [KSZ], and let \( \nu^A_{m,x} \) denote the \( m \)th twisted Frobenius-Schur indicator for \( A \)-modules, twisted by \( x \), as in [SV]. That is, if \( V \) is a \( K \)-module with character (or trace function) \( \chi_V \), then

\[
\nu^K_m(V) = \chi_V(\Lambda^K_m).
\]

If \( W \) is an \( A \)-module with character \( \chi_W \) and \( x \) is an automorphism of \( A \) whose order divides \( m \), then

\[
\nu^A_{m,x}(W) = \chi_W(\Lambda^A_{m,x}).
\]

See [SV] for general results on twisted indicators and for computations of \( \nu^A_{m,x} \) when \( A = H_8 \), the smallest semisimple noncommutative, noncocommutative Hopf algebra.

Our next theorem gives a relationship between the Frobenius-Schur indicators of \( K \) and the twisted Frobenius-Schur indicators of \( A \).

**Theorem 4.1.** For every \( K \)-module \( M \),

\[
\nu^K_m(M) = \sum_{x \in G, \ x^m = 1} \nu^A_{m,x^{-1}}(M_x).
\]

**Proof.** Write \( M = \bigoplus_{x \in G} M_x \) as before. Then \( \nu^K_m(M) = \sum_{x \in G} \nu^K_m(M_x) \), and we will now compute \( \nu^K_m(M_x) \) for an element \( x \) of \( G \), writing \( \Lambda = \Lambda_A \) for ease of notation: By Lemma 3.3

\[
\nu^K_m(M_x) = \chi_{M_x}(\Lambda^K_m) = \chi_{M_x} \left( \sum_{z \in G, \ z^m = 1} \Lambda^{m,z} \natural p_{z^{-1}} \right) = \delta_{x^m,1} \chi_{M_x}(\Lambda^{m,x^{-1}}) = \delta_{x^m,1} \nu^A_{m,x^{-1}}(M_x).
\]

Summing over all elements of \( G \), we obtain the stated formula. \( \square \)

As a consequence, for example, if \( x \) is an element of \( G \) of order \( n \) and \( M \) is a \( K \)-module for which \( M = M_x \) (i.e. \( M_y = 0 \) for all \( y \neq x \)), then \( \nu^K_m(M) = 0 \) for all \( m < n \).

In our next result, we show that a twisted Frobenius-Schur indicator may always be realized as a Frobenius-Schur indicator for a smash coproduct. Let \( \tau \) be any Hopf automorphism of \( A \) of finite order \( n \), and let \( G = \langle \tau \rangle \) be the cyclic subgroup of the automorphism group generated by \( \tau \). Set \( K = A \natural k^G \).

**Theorem 4.2.** For any \( A \)-module \( N \), extend \( N \) to be a \( K \)-module by letting \( M_{\tau^{-1}} = N \) and \( M_x = 0 \) for all \( x \in G, \ x \neq \tau^{-1} \). Then for every positive integer multiple \( m \) of \( n \),

\[
\nu^A_{m,\tau}(N) = \nu^K_m(M).
\]
Thus every value of a twisted indicator for \( A \) is the value of an ordinary indicator for a smash coproduct over \( A \).

**Proof.** By Theorem 4.1,

\[
\nu^K_m(M) = \sum_{x \in G, x^m = 1} \nu^A_{m,x^{-1}}(M_x) = \nu^A_{m,\tau}(M_{\tau^{-1}}) = \nu^A_{m,\tau}(N).
\]

\[\square\]

**Example 4.3.** We illustrate the theorem using a non-trivial automorphism of \( A = H_8 \), the Kac-Palyutkin algebra of dimension 8 which is neither commutative nor cocommutative. The Hopf automorphism group was found in \([SV]\), Section 4.2. Let \( A \) be generated by \( x, y, z \) with the usual relations \( x^2 = y^2 = 1, z^2 = \frac{1}{2}(1 + x + y - xy), xy = yx, xz = zy \) and \( yz = zx \), where \( x, y \) are group-like and \( \Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z) \).

Let \( \tau = \tau_4 \) be the automorphism of \( A \) of order 2 that interchanges \( x \) and \( y \) and sends \( z \) to \( \frac{1}{2}(z + xz + yz + xyz) \), and let \( \chi \) be the character of the unique two-dimensional simple module \( N \) of \( A \). Then from \([SV]\), \( \nu^A_{2,\tau}(N) = -1 \).

Letting \( G = \langle \tau \rangle \) and \( K = A \hat{\otimes} k^G \), \( N \) becomes a \( K \)-module \( M \) by setting \( M_\tau = N \) and \( M_1 = 0 \). Then \( \nu^K_2(M) = -1 \).

5. **Frobenius-Schur indicators for non-semisimple Hopf algebras**

Let \( A \) be a finite-dimensional Hopf algebra that is not necessarily semisimple and for which \( S^2 \) is not necessarily the identity map. When \( A \) is not semisimple, there does not exist a normalized integral, and so we cannot use the definition of indicator from the previous section. Instead we extend the work in \([KMN]\) and define twisted Frobenius-Schur indicators for \( A \) itself and obtain connections to Frobenius-Schur indicators of smash coproducts. Fix \( \tau \), a Hopf automorphism of \( A \) whose order divides the positive integer \( m \). We define a variant of the \( m \)th twisted Hopf power map of \( A \) to be \( P_{m-1,\tau} : A \to A \), given by

\[
P_{m-1,\tau}(a) = \sum_a (\tau^{m-1} \cdot a_1)(\tau^{m-2} \cdot a_2) \cdots (\tau^2 \cdot a_{m-2})(\tau \cdot a_{m-1})
\]

for all \( a \in A \). We will use this map to define twisted Frobenius-Schur indicators, and then we will show how it relates to the twisted Hopf power maps defined in Section 3 by giving equivalent definitions of twisted Frobenius-Schur indicators in Theorem 5.1 and Corollary 5.2.

The \( m \)th twisted Frobenius-Schur indicator of \( A \) is

\[
\nu_{m,\tau}(A) := \text{Tr}(S \circ P_{m-1,\tau}),
\]

the trace of the map \( S \circ P_{m-1,\tau} \) from \( A \) to \( A \), where \( S \) is the antipode of \( A \).

We choose this definition as it specializes to the definition of the Frobenius-Schur indicator of the regular representation \( A \) for an arbitrary finite-dimensional Hopf algebra in \([KMN]\) when \( \tau \) is the identity, and also to the definition of twisted Frobenius-Schur indicators in the semisimple case given in \([SV]\, \text{Theorem 5.1}\). The indicator of the regular representation has also been considered in \([SH]\).

The following theorem generalizes part of \([KMN]\, \text{Theorem 2.2}\).

**Theorem 5.1.** Let \( A \) be a left integral of \( A \) and \( \lambda \) a right integral of \( A^* \) for which \( \lambda(A) = 1 \). Then

\[
\nu_{m,\tau}(A) = \lambda(S(A)^{[m,\tau]}).
\]
Proof. By [R, Theorem 1],

\[
\text{Tr}(S \circ P_{m-1, \tau}) = \sum \lambda(S(\Lambda_2)S \circ P_{m-1, \tau}(\Lambda_1)) \\
= \sum \lambda(S(\Lambda_m)S((\tau^{m-1} \cdot \Lambda_1)(\tau^{m-2} \cdot \Lambda_2) \cdots (\tau \cdot \Lambda_{m-1}))) \\
= \sum \lambda(S(\Lambda_m)(\tau \cdot (\tau^{m-1} \cdot S(\Lambda_{m-1}))) \cdots (\tau^{m-1} \cdot S(\Lambda_1))) \\
= \sum \lambda(S(\Lambda_1)(\tau \cdot (\tau^{m-1} \cdot S(\Lambda_2)) \cdots (\tau^{m-1} \cdot S(\Lambda_m))) = \lambda(S(\Lambda)^{[m, \tau]}).
\]

A similar proof to that of [KMN, Corollary 2.6] yields the following result that will be useful for computations.

**Corollary 5.2.** Let \( \Lambda_r \) be a right integral of \( A \) and \( \lambda_r \) be a right integral of \( A^\ast \) for which \( \lambda_r(\Lambda_r) = 1 \). Then

\[
\nu_{m, \tau}(A) = \lambda_r(\Lambda_r^{[m, \tau]}).
\]

Similarly let \( \Lambda_l \) be a left integral of \( A \) and \( \lambda_l \) be a left integral of \( A^\ast \) for which \( \lambda_l(\Lambda_l) = 1 \). Then

\[
\nu_{m, \tau}(A) = \lambda_l(\tau^{-1} \cdot \Lambda_l^{[m, \tau^{-1}]}).
\]

Proof. The first statement follows immediately from Theorem 5.1 and the fact that if \( \Lambda_l \) is a left integral, then \( \Lambda_r := S(\Lambda_l) \) is a right integral, and the value of \( \lambda_r \) on each is the same.

For the second statement, if \( \lambda_r \) is a right integral, let \( \lambda_l := \lambda_r \circ S \), a left integral of \( A^\ast \). Then again by Theorem 5.1 and also Lemma 3.5

\[
\lambda_l(\tau^{-1} \cdot \Lambda_l^{[m, \tau^{-1}]}) = \lambda_r(S(\tau^{-1} \cdot \Lambda_l^{[m, \tau^{-1}]})) \\
= \lambda_r((\tau^{-1} \cdot (S(\Lambda_l^{[m, \tau^{-1}]})))) \\
= \lambda_r(S(\Lambda_l)^{[m, \tau^{-1}]}) = \lambda_r(\Lambda_r^{[m, \tau]}).
\]

Now let \( G \) be a group of Hopf algebra automorphisms of \( A \), as in Section 2. The next result is a connection between twisted indicators of \( A \) and indicators of the smash coproduct \( K = A \bowtie kG \).

**Theorem 5.3.** \( \nu_m(K) = \sum_{g \in G, \ g^m = 1} \nu_{m, g}(A) \).
Proof. Note that $\Lambda_K = \Lambda \otimes p_1$ and $\lambda_K^* = \lambda \otimes (\sum_{z \in G} z)$ (since e.g. $\varepsilon(z \cdot a) = \varepsilon(a)$). By [KMN, Theorem 2.2] and our Lemmas 3.3 and 3.5,

$$\nu_m(K) = \lambda_K^*(S_K(\Lambda_K^m))$$

$$= \left( \lambda \otimes \left( \sum_{z \in G} \right) \right) \left( S_K \left( \sum_{g \in G, g^m = 1} \Lambda_K^{m,g} \otimes p_{g^{-1}} \right) \right)$$

$$= \sum_{g \in G, g^m = 1} \lambda(g \cdot S(\Lambda_1(g \cdot \Lambda_2) \cdots (g^{m-1} \cdot \Lambda_m))$$

$$= \sum_{g \in G, g^m = 1} \lambda(S(\Lambda)^{m,g^{-1}})$$

$$= \sum_{g \in G, g^m = 1} \nu_{m,g^{-1}}(A) = \sum_{g \in G, g^m = 1} \nu_{m,g}(A).$$

□

In the next section, we compute an example, a non-semisimple Hopf algebra of dimension 8 and its Hopf automorphism group.

6. A NON-SEMISIMPLE EXAMPLE

Let $A$ be the Hopf algebra defined as

$$A = k\langle g, x, y \mid gx = -xg, \; gy = -yg, \; xy = -yx, \; g^2 = 1, \; x^2 = y^2 = 0 \rangle$$

with coalgebra structure given by:

$$\Delta(g) = g \otimes g, \; \varepsilon(g) = 1, \; S(g) = g,$$

$$\Delta(x) = x \otimes g + 1 \otimes x, \; \varepsilon(x) = 0, \; S(x) = gx,$$

$$\Delta(y) = y \otimes g + 1 \otimes y, \; \varepsilon(y) = 0, \; S(y) = gy.$$  

The element $\Lambda = xy + xyg$ is both a right and left integral for $A$, and $\lambda = (xy)^*$ is both a right and left integral for $A^*$ such that $\lambda(\Lambda) = 1$.

Lemma 6.1. Let $V$ be the $k$-span of $x$ and $y$. Then $\text{Aut}(A) \cong GL_2(V)$.

Proof. This is close to the examples considered in [AD], as $A$ is pointed and generated by its group-like and skew-primitive elements. However we provide an elementary proof for completeness.

The coradical of $A$ is given by $A_0 = k\langle g \rangle$. Any automorphism $\tau$ of $A$ stabilizes $A_0$ and so fixes $g$. The next term of the coradical filtration is

$$A_1 = A_0 \oplus V \oplus gV,$$

since $V$ is the set of $(g, 1)$-primitives and $gV$ is the set of $(1, g)$-primitives. Consequently $V$ and $gV$ are each stable under the action of $\tau$. But the $\tau$-action on $V$ determines the $\tau$-action on $gV$, and also on $A = A_1 \oplus W$, where $W$ is the span of $xy$ and $gxy$.

Conversely it is easy to check that any invertible linear action on $V$ preserves all of the relations of $A$, and thus gives an automorphism. □
For an automorphism \( \tau \) of order 2 or 3, we are able to compute some values of the indicators, using Corollary 5.2. We identify \( \tau \) with a matrix
\[
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\]
where \( a, b, c, d \in k \), such that \( \text{Det}(\tau) = ad - bc \neq 0 \).

**Proposition 6.2.** Case (1). If \( \tau^2 = 1 \) and \( m \) is even, then \( \nu_{m, \tau}(A) = \frac{m^2}{2} (1 + \text{Det}(\tau)) \). Consequently,
\[
\nu_{m, \tau}(A) = \begin{cases} m^2, & \text{if } \text{Det}(\tau) = 1 \\ 0, & \text{if } \text{Det}(\tau) = -1 \end{cases}.
\]

Case (2). If \( \tau^3 = 1 \), then \( \nu_{3, \tau}(A) = (\text{Tr}(\tau) + \text{Det}(\tau))^2 + (\text{Tr}(\tau) + 1) (1 - \text{Det}(\tau)) \). Consequently,
\[
\nu_{3, \tau}(A) = \begin{cases} 9, & \text{if } \tau = \text{id} \\ 0, & \text{if } \tau \neq \text{id} \end{cases}.
\]

**Proof.** We verify the formulas by using the first part of Corollary 5.2.

Case (1): Recall that \( \Lambda = xy + xyg \) and \( \lambda = (xy)^* \) are right integrals. We must find \( \lambda(\Lambda^{[m, \tau]}) \). First we will show that \( \lambda((xy)^{[m, \tau]}) = \frac{m^2}{4} (1 + \text{Det}(\tau)) \), and then we will argue that \( \lambda((xyg)^{[m, \tau]}) = \lambda((xy)^{[m, \tau]}) \). In order to find \( (xy)^{[m, \tau]} \), first note that
\[
\begin{align*}
\Delta^{m-1}(x) &= x \otimes g^{\otimes m-1} + 1 \otimes x \otimes g^{\otimes m-2} + \ldots + 1^{\otimes i-1} \otimes x \otimes g^{\otimes m-1} + \ldots + 1^{\otimes m-1} \otimes x, \\
\Delta^{m-1}(y) &= y \otimes g^{\otimes m-1} + 1 \otimes y \otimes g^{\otimes m-2} + \ldots + 1^{\otimes i-1} \otimes y \otimes g^{\otimes m-1} + \ldots + 1^{\otimes m-1} \otimes y,
\end{align*}
\]
each sum consisting of \( m \) terms. Set
\[
x_1 = x \otimes g^{\otimes m-1}, \ldots, x_i = 1^{\otimes i-1} \otimes x \otimes g^{\otimes m-1}, \ldots, x_m = 1^{\otimes m-1} \otimes x,
\]
the index indicating the position of \( x \) in the tensor product, and similarly define \( y_1, y_2, \ldots, y_m \).

Letting \( \mu \) denote the multiplication map, by definition we have
\[
(xy)^{[m, \tau]} = \mu((1 \otimes \tau)^{\otimes 2} \left( \sum_{i,j=1}^{m} x_i y_j \right)).
\]
Since \( \tau \cdot g = g \) and \( \lambda = (xy)^* \), in computing \( \lambda((xy)^{[m, \tau]}) \), the only terms in the above expansion of \( (xy)^{[m, \tau]} \) yielding a nonzero value of \( \lambda \) are those with an even number of factors of \( g \). These are precisely the terms \( x_i y_j \) for which \( i, j \) have the same parity, of which there are \( \frac{m^2}{2} \) terms. If \( i, j \) are both odd (of which there are \( \frac{m^2}{4} \) pairs), then in \( (xy)^{[m, \tau]} \), the \( (i, j) \) term is simply \( xy \) by the following observations: (1) \( \tau \) is applied only to factors of \( g \) or 1, which are fixed by \( \tau \), (2) if \( i \leq j \), there are an even number of factors of \( g \) between \( x \) and \( y \) after applying \( \mu \), and (3) if \( i > j \), there are an odd number of factors of \( g \) between \( x \) and \( y \) after applying \( \mu \) (since \( x_i \) is to the left of \( y_j \)), so moving factors of \( g \) to the right, past \( x \), results in a factor of \( (-1) \), and then applying the relation \( xy = -yx \) results in another factor of \( (-1) \), so that the end result is a term \( xy \). If \( i, j \) are both even (of which there are \( \frac{m^2}{4} \) pairs), then in \( (xy)^{[m, \tau]} \), the \( (i, j) \) term is \( \tau \cdot xy = \text{Det}(\tau) xy \), by similar reasoning. Therefore
\[
\lambda((xy)^{[m, \tau]}) = \lambda \left( \frac{m^2}{4} xy + \frac{m^2}{4} \text{Det}(\tau) xy \right) = \frac{m^2}{4} (1 + \text{Det}(\tau)).
\]
Finally, in order to compute $\lambda((xyg)^{[m, \tau]})$, note that we need only include an extra factor of $g^{\otimes m}$ on the right:

$$(xyg)^{[m, \tau]} = \mu((1 \otimes \tau)^{\otimes 2}) \left( \sum_{i,j=1}^{m} x_i y_j \right) (g^{\otimes m}).$$

Since $m$ is even, the number of new factors of $g$ to be included, in comparison to our previous calculation, is even, and so a similar analysis applies. One checks that the extra factors of $g$ do not affect the result, and so

$$\lambda((xyg)^{[m, \tau]}) = \lambda((xy)^{[m, \tau]}) = \frac{m^2}{4} (1 + \det(\tau)).$$

Consequently, $\nu_{m, \tau}(A) = \lambda(\Lambda^{[m, \tau]}) = \frac{m^2}{2} (1 + \det(\tau))$.

To see the conclusion of Case (1), note that since $\tau^2 = 1$, the determinant of $\tau$ is either 1 or $-1$.

Case (2): A similar analysis applies. Note that $\lambda((xy)^{[3, \tau]}) = \mu((1 \otimes \tau \otimes \tau^2)(\sum_{i,j=1}^{3} x_i y_j)$ and that $\tau^2(x) = (a^2 + bc)x + b(a + d)y$, $\tau^2(y) = c(a + d)x + (d^2 + bc)y$. In evaluating $\lambda((xy)^{[3, \tau]})$, we again need only consider $(i,j)$ terms for which $i, j$ have the same parity. By contrast, in evaluating $\lambda((xyg)^{[3, \tau]})$, we need only consider $(i,j)$ terms for which $i, j$ have different parity. Thus we find

$$\lambda((xy)^{[3, \tau]}) = \lambda \left( x + x(\tau^2 \cdot y) + yg(\tau^2 \cdot x) + (\tau^2 \cdot x) + \tau \cdot xy \right)$$

$$= 1 + (d^2 + bc) + (a^2 + bc) + (a^2 + bc)(d^2 + bc) - bc(a + d)^2 + (ad - bc),$$

$$\lambda((xyg)^{[3, \tau]}) = \lambda \left( xg^2(\tau \cdot y)g^4 + yg(\tau \cdot x)g^5 + g(\tau \cdot x)g^2(\tau^2 \cdot y)g + g(\tau \cdot y)g^2(\tau \cdot x)g^2 \right)$$

$$= d + a + a(d^2 + bc) - bc(a + d) - bc(a + d) + d(a^2 + bc).$$

Adding these together, we have

$$\lambda(\Lambda^{[3, \tau]}) = 1 + a + d + a^2 + ad + d^2 + a^2 d + ad^2 + a^2 d^2 - abc - 2abcd - bc + d + b^2 c^2$$

$$= (\text{Tr}(\tau) + \det(\tau))^2 + (1 + \text{Tr}(\tau))(1 - \det(\tau)).$$

To see the conclusion in Case (2), one can check the possible Jordan forms of the matrix for $\tau$. \hfill \Box

7. Tensor products and category of modules

The following theorem generalizes [BW, Theorem 2.1] from the case that $A$ is a group algebra, to the case that $A$ is a Hopf algebra. Let $K = A \otimes k^G$ as before, and recall that for $M$ a $K$-module and $x \in G$, $M_x$ denotes $p_x \cdot M$, a $K$-submodule of $M$, and $M = \bigoplus_{x \in G} M_x$. If $y \in G$, define $y M_x$ to be $M_x$ as a vector space, with $A$-module structure given by $a \cdot y M_x = (y^{-1} \cdot a) \cdot M_x$ for all $a \in A$, $x \in M$.

**Theorem 7.1.** Let $M, N$ be $K$-modules. Then

(i) $(M \otimes N)_x \cong \bigoplus_{y, z \in G} M_y \otimes \ y M_z$, and

(ii) $(M^*)_x = \ x^*(M_{x^{-1}})^*$. 

10
Proof. The proof is a straightforward generalization of that of [BW, Theorem 2.1]. We include details for completeness. We will prove the statement for modules of the form $M = M_y$, $N = N_z$. Let $\phi : M_y \otimes N_z \to M_y \otimes y N_z$, where the target module is a $K$-module on which $p_{yz}$ acts as the identity and $p_w$ acts as $0$ for $w \neq yz$, be defined by $\phi(m \otimes n) = m \otimes n$ for all $m \in M_y$, $n \in N_z$. We check that $\phi$ is a $K$-module homomorphism: Let $x \in G$, $a \in A$. Apply $\Delta$ to $x \otimes p_x$ to obtain
$$\phi((a \otimes p_x)(m \otimes n)) = \sum \delta_{x,yz} \phi(a_1 m \otimes (y^{-1} \cdot a_2)n).$$
On the other hand,
$$(a \otimes p_x)\phi(m \otimes n) = \sum \delta_{x,yz} a_1 m \otimes (y^{-1} \cdot a_2)n.$$ 
As $\phi$ is a bijection by its definition, it is an isomorphism of $K$-modules.

We will prove that since $M = M_y$, its dual satisfies $M^* = (M^*)_{y^{-1}}$, and that the corresponding underlying $A$-module structure on the vector space $(M^*)_{y^{-1}}$ is isomorphic to $y^{-1}(M_y)^*$. To see this, first let $x \in G$, $f \in M^*$, and $m \in M$. Then
$$(1 \otimes p_x)(f)(m) = f((1 \otimes p_{x^{-1}})m) = \delta_{x^{-1},y} f(m).$$
It follows that $(M_y)^* = M^* = (M^*)_{y^{-1}}$, as claimed. The $A$-module structure on $(M^*)_{y^{-1}}$ may be determined by considering the action on $M^*$ of all elements of $K$ of the form $a \otimes p_{y^{-1}}$ where $a \in A$. Let $f \in M^*$ and $m \in M$. Then
$$(a \otimes p_{y^{-1}})(f)(m) = f(S(a \otimes p_{y^{-1}})m) = f((y \cdot S(a))m).$$
Considering the restriction of $M^* = (M^*)_{y^{-1}}$ to an $A$-module in this way, we see that the action of $a$ on the vector space $(M_y)^*$ is that of $a$ on the $A$-module $y^{-1}(M_y)^*$:
$$(a \cdot y^{-1} f)(m) = ((y \cdot a)f)(m) = f(S(y \cdot a)m) = f((y \cdot S(a))m).$$
Therefore the $A$-module structure on the vector space $(M^*)_{y^{-1}}$ is that of the $A$-module $y^{-1}(M_y)^*$. \hfill $\square$

Remark 7.2. As a consequence of the theorem, the category of $K$-modules is equivalent to the semidirect product tensor category $\mathcal{C} \rtimes G$ where $\mathcal{C}$ is the category of $A$-modules. By definition, $\mathcal{C} \rtimes G$ is the category $\oplus_{g \in G} \mathcal{C}$, with objects $\oplus_{g \in G}(M_g, g)$ where each $M_g$ is an object of $\mathcal{C}$, and tensor product $(M, g) \otimes (N, h) = (M \otimes g N, gh)$. See [T], where the notation $\mathcal{C}[G]$ is used instead for this semidirect product category. For other occurrences of $\mathcal{C} \rtimes G$ in the literature, see, for example, [GNa-Ni].

References

[AD] N. Andruskiewitsch and F. Dumas, On the automorphisms of $U_q^+(\mathfrak{g})$, In: Quantum groups, IRMA Lectures on Mathematical and Theoretical Physics, vol. 12, pp. 107–133. European Mathematical Society, Zurich (2008).

[BW] D. Benson and S. Witherspoon, Examples of support varieties for Hopf algebras with non-commutative tensor products, Archiv der Mathematik 102 (2014), no. 6, 513–520.

[DE] S. Danz and K. Erdmann, Crossed products as twisted category algebras, Algebr. Represent. Theor. DOI 10.1007/S10468-014-9493-8.

[EG] P. Etingof and S. Gelaki, On the exponent of finite-dimensional Hopf algebras. Math. Res. Lett., 6(2):131–140, 1999.

[GNa-Ni] S. Gelaki, D. Naidu, and D. Nikshych, Centers of graded fusion categories, Algebra Number Theory 3 (2009), no. 8, 959–990.
[KMN] Y. Kashina, S. Montgomery, and S. Ng, On the trace of the antipode and higher indicators, Israel J. Math. 188 (2012), 57–89.

[KSZ] Y. Kashina, Y. Sommerhuser, and Y. Zhu, On Higher Frobenius-Schur Indicators, Mem. Amer. Math. Soc. 181 (2006), no. 855, viii+65 pp.

[Ke] M. Keilberg, Automorphisms of the doubles of purely non-abelian finite groups, Algebras and Representation Theory, to appear; arXiv:1311.0573.

[LM] V. Linchenko and S. Montgomery, A Frobenius-Schur theorem for Hopf algebras, Algebr. Represent. Theory 3 (2000), 347–355.

[M] R. Molnar, Semi-direct products of Hopf algebras, J. Algebra 47 (1977), 29–51.

[MN] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Lectures Vol. 82, Amer. Math. Soc., Providence, 1997.

[NSc] S.-H. Ng and P. Schauenburg, Higher Frobenius-Schur indicators for pivotal categories, Hopf Algebras and Generalizations, AMS Contemp. Math. 441, AMS, Providence, RI, 2007, 63–90.

[Ni] D. Nikshych, Non-group-theoretical semisimple Hopf algebras from group actions on fusion categories, Selecta Math. 14 (2008), no. 1, 145–161.

[R] D. E. Radford, The group of automorphisms of a semisimple Hopf algebra over a field of characteristic 0 is finite, Amer. J. Math. 112 (1990), 331–357.

[R2] D. E. Radford, Hopf Algebras, World Scientific Publishing, 2012.

[R3] D. E. Radford, On automorphisms of biproducts, arXiv:1503.00381.

[SV] D. Sage and M. Vega, Twisted Frobenius-Schur indicators for Hopf algebras, J. Algebra 354 (2012), 136–147.

[SV2] D. Sage and M. Vega, Twisted exponents and twisted Frobenius-Schur indicators for Hopf algebras, Communications in Algebra, to appear; arXiv:1402.5201.

[SV3] D. Sage and M. Vega, Twisted Frobenius-Schur indicators for pivotal categories, in preparation.

[Sh] K. Shimizu, Some computations of Frobenius-Schur indicators of the regular representations of Hopf algebras, Algebr. Represent. Theory 15 (2012), 325–357.

[T] D. Tambara, Invariants and semi-direct products for finite group actions on tensor categories, J. Math. Soc. Japan 53 (2001), no. 2, 429–456.

[Y] M. Yakimov, Rigidity of quantum tori and the Andruskiewitsch-Dumas conjecture, Selecta Math (N.S.) 20 (2014), no. 2, 421–464.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA
E-mail address: smontgom@usc.edu

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NC
E-mail address: mvedega@ncsu.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX
E-mail address: sjw@math.tamu.edu