New vacuum solutions of conformal Weyl gravity

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Abstract

The Bach equation, i.e., the vacuum field equation following from the Lagrangian $L = C_{ijkl}C^{ijkl}$, will be completely solved for the case that the metric is conformally related to the cartesian product of two 2-spaces; this covers the spherically and the plane symmetric space-times as special subcases.

Contrary to other approaches, we make a covariant 2+2-decomposition of the field equation, and so we are able to apply results from 2-dimensional gravity. Finally, some cosmological solutions will be presented and discussed.

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1 Introduction

We consider the Lagrangian

\[ L = C_{ijkl} C^{ijkl} \]  

(1)

where \( C_{ijkl} \) is the conformally invariant Weyl tensor. The variational derivative of \( L \sqrt{-g} \) (where \( g \) is the determinant of the metric \( g_{ij} \)) gives rise to the Bach tensor \[ B_{ij} = 2 C_{ijkl}^{k} + C_{ijkl}^{l} R_{lk} \] (2)

The purpose of the present paper is to characterize several solutions of the Bach equation \( B_{ij} = 0 \), and thereby we cover the spherically and the plane symmetric metrics. In other words: We look for vacuum solutions of conformal Weyl gravity [2].

2 Another form of the field equation

Subtracting the divergence \[ \delta \]

\[ L_{GB} = R_{ijkl} R^{ijkl} - 4 R_{ij} R^{ij} + R^2 \] (3)

from the Lagrangian \( L \) eq. (1) we get

\[ \tilde{L} = 2 R_{ij} R^{ij} - \frac{2}{3} R^2 \]

3 more exactly:

\[ B^{ij} = \frac{1}{\sqrt{-g}} \frac{\delta L \sqrt{-g}}{\delta g_{ij}} \]

4 which represents the Gauss-Bonnet term in 4 dimensions
The variation of $\tilde{L}\sqrt{-g}$ with respect to the metric gives, of course, a vacuum equation identical to eq. (2), but now in a form [3], where neither the Weyl tensor not the full Riemann tensor explicitly appear.

$$B_{ij} = B_{ij}^{(1)} + B_{ij}^{(2)}$$

where

$$B_{ij}^{(1)} = -\Box R_{ij} + 2 R_{i;jk}^k - \frac{2}{3} R_{;ij} + \frac{1}{6} g_{ij} \Box R$$

and

$$B_{ij}^{(2)} = \frac{2}{3} R_{;ij} - 2 R_{ik} R_{jk}^k - \frac{1}{6} R^2 g_{ij} + \frac{1}{2} g_{ij} R_{kl} R^{kl}$$

This form of the field equation is also given in [4]; the two details where the equation given in [4] differs from ours are explained as follows:

1. Instead of $+2 R_{i;jk}^k$ they write $+R_{i;jk}^k + R_{j;ik}^k$. However, the tensor $R_{i;jk}^k$ is already symmetric in $ij$ due to the Bianchi identity.

2. In our eq. (4) the authors of ref. [4] write $-$ instead of $+$. But this is only due to the different sign conventions. Our conventions are defined by $R_{ij} = R_{ikj}$ and the condition that the Euclidean sphere has $R > 0$.

### 3 The trivial solutions

If $R_{ij} = \lambda g_{ij}$ for any constant $\lambda$, then by eqs. (4-6) we see that $B_{ij} = 0$ is identically fulfilled. In other words: Every Einstein space solves the Bach equation. From eqs. (1,2) it becomes clear that the Bach equation is conformally invariant. So we get: If we apply a conformal transformation to a solution, then the resulting space-time solves the Bach equation, too. Combining both properties it proves useful to define:

A solution of the Bach equation is called trivial if it is conformally related.

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5i.e. vacuum solutions of the Einstein field equation with arbitrary value of $\Lambda$. 

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to an Einstein space. In [6], the spherically symmetric solutions of the Bach equation have been analyzed, and the result is: Every spherically symmetric solution of the Bach equation is almost everywhere trivial. The restriction “almost everywhere” refers to possibly existing hypersurfaces where the necessary conformal factor becomes singular. Further details about the Bach tensor can be found in [7].

4 The 2+2 decomposition of the Bach equation

In this section we perform a 2+2 decomposition of the metric, and then we apply results [8] from 2-dimensional gravity to solve the Bach equation.

4.1 The metric ansatz

For the metric

$$ds^2 = g_{ij} dx^i dx^j, \quad i, j = 0, \ldots, 3$$  \hspace{1cm} (7)

we make the following ansatz:

$$ds^2 = d\sigma^2 + d\tau^2$$  \hspace{1cm} (8)

where $d\sigma^2$ and $d\tau^2$ are both 2-dimensional. The metric

$$d\sigma^2 = g_{AB} dx^A dx^B, \quad A, B = 0, 1$$  \hspace{1cm} (9)

where $g_{AB}$ depends on the $x^A$ only, has curvature scalar $P$ and signature $(-+)$. The other 2-dimensional metric

$$d\tau^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \quad \alpha, \beta = 2, 3$$  \hspace{1cm} (10)

In [5], conditions have been found to decide whether a given space-time is conformally related to an Einstein space; however, as already mentioned there, these conditions are applicable only in such cases where a nonvanishing scalar constructed from the Weyl tensor exists.
where $g_{\alpha\beta}$ depends on the $x^\alpha$ only, has curvature scalar $Q$ and signature $(++)$.

For the 4-dimensional metric (7) we get signature $(-+++)$ and curvature scalar $R$ via

$$R(x^i) = P(x^A) + Q(x^\alpha). \quad (11)$$

### 4.2 The Einstein spaces of this type

The Einstein spaces of the type defined in section 4.1. are already known for a long time, see [9] for the history of these metrics. Here we deduce them for two reasons: First, we want to elucidate the method which we will apply to the Bach equation afterwards, and second, it is not yet general knowledge, that a spherically symmetric Einstein space (eq. (16) below) exists which cannot be written in Schwarzschild coordinates. The Einstein spaces can be found as extremals of the action

$$I = \int (R - 2\Lambda) \sqrt{-\det g_{ij}} d^4x \quad (12)$$

where $\Lambda$ has an arbitrary constant value. Extremality implies constancy of $R$, and because of eq. (11), we find both $P$ and $Q$ as constants. Let us assume that space-time is compact[7]. We denote the volumes of $d\sigma^2$ by $V_1$ and of $d\tau^2$ by $V_2$, i.e.

$$V_1 = \int \sqrt{-\det g_{AB}} d^2x^A, \quad V_2 = \int \sqrt{\det g_{\alpha\beta}} d^2x^\alpha \quad (13)$$

Due to the Gauss-Bonnet theorem we have two topological invariants which do not change by a smooth variation of the metric:

$$K_1 = \int P \sqrt{-\det g_{AB}} d^2x^A, \quad K_2 = \int Q \sqrt{\det g_{\alpha\beta}} d^2x^\alpha \quad (14)$$

Because of the constancy of $P$ and $Q$ we have $P = K_1/V_1$ and $Q = K_2/V_2$. We insert eqs. (11,13,14) into eq. (12) and get

$$I = K_1 V_2 + K_2 V_1 - 2\Lambda V_1 V_2 \quad (15)$$

[7]If not, we restrict to the corresponding local consideration.
Extremality of the action $I$ implies $\frac{\partial I}{\partial V_n} = 0$ for $n = 1, 2$:

$$0 = K_2 - 2\Lambda V_2, \quad 0 = K_1 - 2\Lambda V_1.$$ 

These equations imply $P = Q = 2\Lambda$ and $R = 4\Lambda$. For $\Lambda = 0$ we get only the flat Minkowski space-time. For $\Lambda > 0$ however, we get a nonflat spherically symmetric space-time

$$ds^2 = \Lambda^{-1} \left[ -t^{-2} dt^2 + t^2 dx^2 + d\Omega^2 \right]$$

(16)

(where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the metric of the standard 2-sphere) representing a cartesian product of two spaces of equal positive constant curvature which is non-singular and not asymptotically flat. Metric (16) represents a cosmological model of Kantowski-Sachs type and possesses a 6-dimensional isometry group including a time-like isometry, (i.e., the time-dependence of metric (16) is only due to the choice of the coordinates).

Analogously we get for the case $\Lambda < 0$ the solution

$$ds^2 = |\Lambda|^{-1} \left[ -x^{-2} dt^2 + x^2 dx^2 + d\bar{\Omega}^2 \right]$$

(17)

(where $d\bar{\Omega}^2 = d\theta^2 + \sinh^2 \theta d\phi^2$ is the metric of the standard plane of constant negative curvature) representing a cosmological model of Bianchi type III.

4.3 Curvature for this type of metrics

For metric (7), the non-vanishing components $R_{ijkl}$ of the Riemann tensor are

$$R_{ABCD} = \frac{P}{2} (g_{AC}g_{BD} - g_{BC}g_{AD})$$

In the usual deduction we get from the action (12) the Einstein equation

$$R_{ij} - \frac{R}{2} g_{ij} = -\Lambda g_{ij},$$

i.e., $R_{ij} = \Lambda g_{ij}$ and $R = 4\Lambda$, whose spherically symmetric solution are almost everywhere given in Schwarzschild coordinates as

$$ds^2 = - \left(1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2 \right) dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda}{3} r^2 \right)^{-1} dr^2 + r^2 d\Omega^2$$
and
\[ R_{\alpha\beta\gamma\delta} = \frac{Q}{2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\beta\gamma}g_{\alpha\delta}) \] (19)

For the Ricci tensor we get analogously \( R_{\alpha A} = 0 \), and
\[ R_{AB} = \frac{P}{2}g_{AB}, \quad R_{\alpha\beta} = \frac{Q}{2}g_{\alpha\beta} \] (20)

The Weyl tensor reads\[^9\]
\[ C_{\alpha\beta\gamma\delta} = \frac{R}{6}(g_{\alpha\gamma}g_{\beta\delta} - g_{\beta\gamma}g_{\alpha\delta}) \] (21)
\[ C_{ABCD} = \frac{R}{6}(g_{AC}g_{BD} - g_{BC}g_{AD}) \] (22)

As a byproduct we get: conformal flatness of metric (7) implies the vanishing of its curvature scalar \( R \). However, in contrast to the Riemann tensor, the Weyl tensor possesses also non-vanishing mixed components:
\[ C_{\alpha A\beta B} = -\frac{R}{12}g_{\alpha\beta}g_{AB} \] (23)

Summing up eqs. (21-23) we get
\[ C_{ijkl}C^{ijkl} = \frac{1}{3}R^2 \]

### 4.4 Solving the Bach equation – constant \( P \) and \( Q \)

As first part we make the analogous calculation as in section 4.2. Inserting eqs.(11,18-20) into eq. (3) we get \( L_{GB} = 2PQ \), i.e., the 4-dimensional topological invariant is the double product of the corresponding 2-dimensional ones:
\[ \int L_{GB}\sqrt{-\det g_{ij}}\,d^4x^i = 2K_1K_2 \] (24)

Further we get \( R^2 = (P + Q)^2 \) and
\[ R_{ij} \, R^{ij} = \frac{1}{2}(P^2 + Q^2) \] (25)

\[^9\text{The definition is}\]
\[ C_{ijkl} = R_{ijkl} - \frac{1}{2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{jk}g_{il} - R_{il}g_{jk}) + \frac{R}{6}(g_{ik}g_{jl} - g_{jk}g_{il}) \]
thus, up to a divergence, eq. (1) now reads

$$\hat{L} = \frac{1}{3} (P^2 + Q^2)$$

(26)

and with the notation from eq. (13) and

$$L_1 = \int P^2 \sqrt{-\det g_{AB}} d^2x^A, \quad L_2 = \int Q^2 \sqrt{\det g_{\alpha\beta}} d^2x^\alpha$$

(27)

$$\hat{I} \equiv \int \hat{L} \sqrt{-\det g_{ij}} d^4x^i = \frac{1}{3} (L_1 V_2 + L_2 V_1)$$

(28)

In the set of spaces with constant $P$ and $Q$ we get for $n = 1, 2$: $L_n = K_n^2/V_n$, i.e.

$$\hat{I} = \frac{1}{3} \left( K_1^2 V_2/V_1 + K_2^2 V_1/V_2 \right)$$

(29)

Consequently,

$$\frac{\partial \hat{I}}{\partial V_n} = 0$$

implies $0 = K_1^2/V_1 - K_2^2 V_1/V_2^2$, i.e. $P^2 = Q^2$. Thus, besides the Einstein spaces of this type we additionally get spaces with $P = -Q \neq 0$. These are just the cartesian products of two 2-spaces of constant non-vanishing curvature with the additional condition that they have $R = 0$, i.e., that they are conformally flat, which can be shown by eqs. (21-23).

By use of the notation of sect. 3 we can say that they represent trivial solutions of the Bach equation.

### 4.5 Solving the Bach equation – variable $P$ or $Q$

From eq. (25) we see: $R_{ij} R^{ij} - \frac{1}{2} R^2$ represents a divergence. Thus, it seems tempting to use also this divergence to show that our Lagangian gives just the same field equation than $L = R^2$ would give. But this argument does not work from the following reason: The statement, that $R_{ij} R^{ij} - \frac{1}{2} R^2$ represents a divergence, is valid only within the class of metrics considered here, so the

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\(^{10}\)In the 2-parameter class of Lagrangians $L_{\alpha,\beta} = \alpha R_{ij} R^{ij} + \beta R^2$, the case $\alpha + 3\beta = 0$ leads to Weyl gravity, cf. sect. 2; and the Eddington case is defined by $\alpha + 2\beta = 0$, this case we meet here, cf. [10] for details.
vacuum field equation need not be the same for \( L \) eq. (1) and the Lagrangian \( R^2 \): The variation has to be made in comparison with all possible metrics. Therefore, we have now to use the full equation (2) or (4-6). For our purposes it turned out that eq. (2) is easier to handle. We write

\[ \Box R = \Box P + \Box Q \]  

(30)

with a context-dependent meaning of the symbol \( \Box \), cf. eqs. (7-11). E.g.: In \( \Box P \), \( \Box \) denotes the D’Alembertian within \( d\sigma^2 \). Analogously, we use only one symbol “;” for the covariant derivative. After a lengthy but straightforward calculation we get the \( AB \)-part of the Bach equation:

\[ B_{AB} \equiv \frac{1}{3} P_{;AB} + g_{AB} \left( \frac{1}{6} \Box Q - \frac{1}{3} \Box P + \frac{1}{12} Q^2 - \frac{1}{12} P^2 \right) = 0 \]  

(31)

From the trace of eq. (31) we see that \( \Box P + \frac{1}{2} P^2 \) and \( \Box Q + \frac{1}{2} Q^2 \) are both constant because they are equal but “live” in different spaces:

\[ \Box P + \frac{1}{2} P^2 = C, \quad \Box Q + \frac{1}{2} Q^2 = C = \text{const.} \]  

(32)

The fact that the trace-free part of the tensor \( P_{;AB} \) vanishes is equivalent to the requirement that \( \xi^A = \varepsilon^{AB} P_{;B} \) represents a Killing vector.\footnote{We use \( \varepsilon^{AB} \), the covariantly constant antisymmetric Levi-Civita pseudotensor in \( d\sigma^2 \).}

Now we have to use the \( \alpha\beta \)-part of the field equation. However, we need not really deduce it, because there is a duality \( A \leftrightarrow \alpha \). Thus, the only additional requirement is that \( \eta^\alpha = \varepsilon^{\alpha\beta} Q_{;\beta} \) represents a Killing vector, too.

Let us summarize this section: 1. There exists a double-Birkhoff theorem as follows: If a solution of the Bach equation is the cartesian product of two 2-spaces, then 2 independent Killing vectors exist. They are orthogonal to each other, and each of them is hypersurface orthogonal. If either \( P \) or \( Q \) is constant, then the number of Killing vectors equals 4.

2. The cartesian product of two 2-spaces is a solution of the Bach equation if and only if there exists a constant \( C \) such that both 2-spaces solve the
fourth-order field equation following from the 2-dimensional Lagrangian
\[ L = \frac{1}{2} (2) R^2 + C. \]

3. The solutions for \( L = \frac{1}{2} (2) R^2 + C \) are all known in closed form \([8, eq.(14)]\), so we are now able to list all these solutions of the Bach equation which possess exactly 2 Killing vectors:

\[
- (a + Cr - r^3/6) dt^2 + \frac{dr^2}{a + Cr - r^3/6} + (b + C\psi - \psi^3/6) d\phi^2 + \frac{d\psi^2}{b + C\psi - \psi^3/6}
\]

with 3 constants \(a, b, C\). Each of the two factor spaces gives one integration constant, but from eq. (32) it follows, that the third constant reflects that fact that the Bach equation is scale-invariant.

4. The number of Killing vectors of a solution of the Bach equation for the metrics discussed here equals 2, 4, 6, or 10. The solutions with 6 Killing vectors and the flat space-time solution with 10 Killing vectors have already been listed in sct. 4.4, the solutions with 2 Killing vectors are given by eq. (33) above; thus, it remains to find the solutions possessing 4 Killing vectors. This takes place if from \( P \) and \( Q \) one is constant, and the other one not. For \( C < 0 \), all solutions have exactly 2 Killing vectors because neither \( P \) nor \( Q \) can be const., cf. eq. (32). We restrict to the case that \( Q \) is constant, and \( P \) not constant; the other case is quite analogously to deal. Depending on the

\[ \text{Here, } (2) R \text{ is the curvature scalar in that 2-dimensional space. In principle, this result could have been guessed already from eq. (26): In the variation of (26) with respect to } g_{AB}, \text{ the scalar } Q^2 \text{ plays the role of a constant and vice versa. However, by such a consideration one looses the information about the fact, that both equations (32) contain the same constant } C. \]

\[ \text{With the ansatz } d(2)s^2 = dw^2/A(w) \pm A(w)dy^2 \text{ one gets – besides the constant curvature solution } (2) R^2 = 2C \text{ – the general solution as } A(w) = C_1 + Cw - w^3/6 \text{ where } C_1 \text{ is a further constants. One should note that the cubic term in } A(w) \text{ is necessarily unequal zero to have a non-constant curvature scalar, and that therefore, a term } \approx w^2 \text{ can be made vanish by a suitable } w\text{-translation. However, the fact that the cubic term is just } -1/6 \text{ was fixed by a corresponding coordinate transformation, a multiplication of } w \text{ and } y \text{ by the same constant. This } "-1/6" \text{ was chosen such that the factor } C \text{ of the linear term is just the } C \text{ defined in eq. (32).} \]
sign of $Q$, we have 3 different subcases.

The spherically symmetric solutions we get for the case that $d\tau^2 = d\Omega^2$, i.e., $Q = C = 2$:

$$ds^2 = -(a + 2r - r^3/6)dt^2 + \frac{dr^2}{a + 2r - r^3/6} + d\Omega^2$$  \hspace{1cm} (34)

This is – up to conformal transformations – the general spherically symmetric solution of the Bach equation, however, this form of the solution is not very common. Therefore, we multiply metric (34) by a conformal factor $\rho^2(r)$ and use $\rho = c \pm 1/r$ as new radial coordinate. As a result one gets the known old result (see [6] also for the details of that transformation) that the spherically symmetric solutions of the Bach equation are conformally related to the Schwarzschild–de Sitter solution.

The plane-symmetric solutions we get for $d\tau^2 = dy^2 + dz^2$, i.e. $Q = C = 0$.

$$ds^2 = -(a - r^3/6)dt^2 + \frac{dr^2}{a - r^3/6} + dy^2 + dz^2$$  \hspace{1cm} (35)

5  Cosmological solutions

Here we give some examples of cosmological solutions of the type eq. (34) and (35). The interpretation of the more general solution (33) as cosmological model shall be postponed to later work.

5.1  Axially symmetric Bianchi type I Universe

In order that to obtain some cosmological solutions in conformal Weyl gravity we have to do the following. The 2-metric $d\sigma^2$ for metric (35) can be written as:

$$d\sigma^2 = -(a + br^3)dt^2 + \frac{dr^2}{a + br^3}.$$  \hspace{1cm} (36)

\footnote{With $Q = -2$, $C = 2$ and $d\tau^2 = d\Omega^2$ we get the analogous case for a plane of negative curvature. The formulas can be straightforwardly written down.}
(Because of $C = 0$ the factor $b$ need not be put to $-1/6$.) It is evident that for $a, b < 0$ we can rename $t \to x$, $r \to t$, $a \to -a$, $b \to -b$ then we have:

$$d\sigma^2 = -\frac{dt^2}{a + bt^3} + \left(a + bt^3\right) dx^2. \quad (37)$$

Let we introduce the polar coordinate system $y = \rho \cos \varphi$, $z = \rho \sin \varphi$ then the solution (35) is given by:

$$ds^2 = -\frac{dt^2}{a + bt^3} + \left(a + bt^3\right) dx^2 + d\rho^2 + \rho^2 d\varphi^2. \quad (38)$$

The metric (38) describes the axially symmetric Kasner-like Universe with expanding $x$-dimension and constant $(y, z)$–plane.

The calculations of the scalar invariants for this metric give us:

$$R = -6bt, \quad (39)$$

$$R_{ik} R^{ik} = 18bt^2, \quad (40)$$

$$R_{iklm} R^{iklm} = 36bt^2. \quad (41)$$

At the surface defined by $t = -(a/b)^{1/3}$ there is only a coordinate peculiarity similar to that one of the Schwarzschild horizon.

### 5.2 Spherically symmetric Universe

Analogously to the previous case we can exchange $t \to r$ and $r \to t$ in the solution (34) and then we get:

$$ds^2 = -\frac{dt^2}{a + 2t - t^3/6} + \left(a + 2t - t^3/6\right) dr^2 + d\Omega^2. \quad (42)$$

The scalar invariants are:

$$R = t, \quad (43)$$

$$R_{ik} R^{ik} = \frac{1}{2} t^2, \quad (44)$$

$$R_{iklm} R^{iklm} = t^2. \quad (45)$$

Also, at $t_0$ (where $a + 2t_0 - t_0^3/6 = 0$) there is not a real singularity and we have only the peculiarity $g_{rr} = 0$ and $g_{tt} = -\infty$. 

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6 Discussion

In many papers, motivations for considering conformal Weyl gravity, i.e., motivations for solving the Bach equation, are given.

The Bach tensor (sometimes also called: Schouten-Haantjes tensor) plays also a role in the following contexts:

1. The integrability of the null-surface formulation of General Relativity imposes a field equation on the local null surfaces which is equivalent to the vanishing of the Bach tensor, see [11].

2. For asymptotically flat space-times it holds: It is conformally related to a Ricci-flat space-time if and only if the Bach tensor vanishes, see [12].

3. The Mannheim-Kazanas approach [4], see also its analysis in [13], essentially uses the Bach tensor and tries to relate it to observable astrophysical effects.

4. The Bach equation accompanied by conformally invariant matter (electromagnetic field) has been discussed in [14]. There, already the relation to the $R^2$-gravity in 2 dimensions has been mentioned, and a Birkhoff theorem has been deduced. However, at that time, the solutions of $R^2$-gravity in 2 dimensions (deduced in [15]) were not known, so this relation was not so useful as it is now.

5. In several approaches to quantum gravity, e.g. by compactification of 11-dimensional supergravity [16], one gets $R^2$-terms including the Weyl-term in the effective action. In [17], a theorem relating solutions of a 4-order theory of gravity to General Relativity has been deduced. In both the papers [16, 17], the general need to include the Weyl term is mentioned, but the calculations have been restricted to the case where this term is absent. So, here remains a general interest in these calculations, too.

6. Quite recently, renewed interest in the Bach equation lead to several concrete calculations, see [18], [19] and the references cited there. In [18],
a new approach to the Newtonian limit of conformal gravity has been presented, and in [19], the Hamiltonian formulation and exact solutions of the Bianchi type I spacetime in conformal gravity have been deduced. The exact solutions given there are more general for the Bianchi type I case than our cosmological solution given in scct. 5.1. However, our general solution (33) possessing only 2 Killing vectors is more general than the solutions given in [19].

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