A note on the Weingarten function

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Abstract

The aim of this note is to compare work of Formanek [6] on a certain construction of central polynomials with that of Collins [3] on integration on unitary groups.

These two quite disjoint topics share the construction of the same function on the symmetric group, which the second author calls Weingarten function.

By joining these two approaches we succeed in giving a simplified and very natural presentation of both Formanek and Collins’s Theory.

1 Schur Weyl duality

1.1 Basic results

We need to recall some basic facts on the representation Theory of the symmetric and the linear group.

Let $V$ be a vector space of finite dimension $d$ over a field $F$ which in this note can be taken as $\mathbb{Q}$ or $\mathbb{C}$. On the tensor power $V^\otimes k$ act both the symmetric group $S_k$ and the linear group $GL(V)$, Formula (1.1), furthermore if $F = \mathbb{C}$ and $V$ is equipped with a Hilbert space structure one has an induced Hilbert space structure on $V^\otimes k$. The unitary group $U(d) \subset GL(V)$ acts on $V^\otimes k$ by unitary matrices.

The first step of Schur Weyl duality is the fact that the two operator algebras $\Sigma_k(V), B_{k,d}$ generated respectively by $S_k$ and $GL(V)$ acting on $V^\otimes k$, are both semisimple and each the centralizer of the other.

In particular the algebra $\Sigma_k(V) \subset \text{End}(V^\otimes k) = \text{End}(V)^{\otimes k}$ equals the subalgebra $\Sigma_k(V) = (\text{End}(V)^{\otimes k})^{GL(V)}$ of invariants under the conjugation action of the group $GL(V) \rightarrow \text{End}(V)^{\otimes k}$, $g \mapsto g \otimes g \otimes \ldots \otimes g$.

From this, the double centralizer Theorem and work of Frobenius and Young one has that, under the action of these two commuting groups, $V^\otimes k$ decomposes into the direct sum

$$V^\otimes k = \bigoplus_{\lambda \vdash k, \, ht(\lambda) \leq d} M_\lambda \otimes S_\lambda(V)$$

(2)
over all partitions $\lambda$ of $k$ of height $\leq d$, (the height $ht(\lambda)$ denotes the number of elements, nonzero, of $\lambda$).

$M_\lambda$ is an irreducible representation of $S_k$ while $S_\lambda(V)$, called a Schur functor is an irreducible polynomial representation of $GL(V)$, which remains irreducible also when restricted to $U(d)$.

The character theory of the two groups can be deduced from the representations. We shall denote by $\chi_\lambda(\sigma)$ the character of the permutation $\sigma$ on $M_\lambda$. As for $S_\lambda(V)$ its character is expressed by a symmetric function $S_\lambda(x_1, \ldots, x_d)$ restriction to the first $d$ variables of a stable symmetric function called Schur function. Of this deep and beautiful Theory, see [13], [7], [8], [21], [16], we shall use only two remarkable formulas, the hook formula due to Frame, Robinson and Thrall [15], expressing the dimension of $\chi_\lambda(1)$ of $M_\lambda$ and the hook-content formula of Stanley, cf. [19, Corollary 7.21.4] expressing the dimension of $s_\lambda(d) := S_\lambda(1, \ldots, 1) = S_\lambda(1^d)$ of $S_\lambda(V)$.

We display partitions by Young diagrams, as in (1.3), by $\tilde{\lambda}$ we denote the dual partition obtained by exchanging rows and columns. The boxes, cf. (1.15), of the diagram are indexed by pairs $(i, j)$ of coordinates. Given then one of the boxes $u$ we define its hook number $h_u$ and its content $c_u$ as follows:

**Definition 1.2.** Let $\lambda$ be a partition of $n$ and let $u = (i, j) \in \lambda$ be a box in the corresponding Young diagram. The hook number $h_u = h(i, j)$ and the content $c_u$ are defined as follows:

$$h_u = h(i, j) = \lambda_i + \tilde{\lambda}_j - i - j + 1, \quad c_u = c(i, j) := j - i.$$  \hfill (3)

**Example 1.3.** Note that the box $u = (3, 4)$ defines a hook in the diagram $\lambda$, and $h_u$ equals the length (number of boxes) of this hook:

![Young diagram](https://example.com/diagram.jpg)

In this figure, we have $\lambda = (13, 11, 10, 8, 6^3)$, $ht(\lambda) = 7$ with $u = (3, 4)$. Then $\lambda = (7^6, 4^2, 3^2, 2, 1^2)$ and $h_u = \lambda_3 + \tilde{\lambda}_4 - 3 - 4 + 1 = 10 + 7 - 6 = 11$.

Here is another example: In the following diagram of shape $\lambda = (8, 3, 2, 1)$, each hook number $h_u$, respectively content $c_u$ is written inside its box in the diagram $\lambda$:
Theorem 1.4 (The hook and hook–content formulas). Let \( \lambda \vdash k \) be a partition of \( k \) and \( \chi_\lambda(1) \) and \( s_\lambda(d) \) the dimension of the corresponding irreducible representation \( M_\lambda \) of \( S_k \) and \( S_\lambda(V) \) of \( GL(V) \), \( \dim(V) = d \). Then

\[
s_\lambda(d) = \prod_{u \in \lambda} \frac{d + c_u}{h_u}, \quad \chi_\lambda(1) = \frac{k!}{\prod_{u \in \lambda} h_u}.
\] (4)

The remarkable Formula of Stanley exhibits \( s_\lambda(d) \) as a polynomial of degree \( k = |\lambda| \) in \( d \) with zeroes the integers \(-c_u\) and leading coefficient \( \prod_{u \in \lambda} h_u^{-1} \).

1.4.1 Matrix invariants

The dual of the algebra \( \text{End}(V)^{\otimes k} \) can be identified, in a \( GL(V) \) equivariant way, to \( \text{End}(V)^{\otimes m} \) by the pairing formula:

\[
\langle A_1 \otimes A_2 \cdots \otimes A_k \mid B_1 \otimes B_2 \cdots \otimes B_k \rangle := \text{tr}(A_1 \otimes A_2 \cdots \otimes A_k \circ B_1 \otimes B_2 \cdots \otimes B_k) = \prod_{i=1}^k \text{tr}(A_i B_i).
\]

Under this isomorphism the multilinear invariants of matrices are identified with the \( GL(V) \) invariants of \( \text{End}(V)^{\otimes m} \) which in turn are spanned by the elements of the symmetric group, hence by the elements of Formula (5). These are explicited by Formula (6) as in Kostant [11]

Proposition 1.5. The space \( T_d(k) \) of multilinear invariants of \( k \), \( d \times d \) matrices is identified with \( \text{End}_{GL(V)}(V^{\otimes k}) \) and it is linearly spanned by the functions:

\[
T_\sigma(X_1, X_2, \ldots, X_d) := \text{tr}(\sigma^{-1} \circ X_1 \otimes X_2 \otimes \cdots \otimes X_d), \quad \sigma \in S_k. \tag{5}
\]

If \( \sigma = (i_1 i_2 \ldots i_h) \ldots (j_1 j_2 \ldots j_e) (s_1 s_2 \ldots s_t) \) is the cycle decomposition of \( \sigma \) then we have that \( T_\sigma(X_1, X_2, \ldots, X_d) \) equals

\[
= \text{tr}(X_{i_1} X_{i_2} \cdots X_{i_h}) \ldots \text{tr}(X_{j_1} X_{j_2} \cdots X_{j_e}) \text{tr}(X_{s_1} X_{s_2} \cdots X_{s_t}). \tag{6}
\]

Proof. Since the identity of Formula (6) is multilinear it is enough to prove it on the decomposable tensors of \( \text{End}(V) = V \otimes V^* \) which are the endomorphisms of rank 1, \( u \otimes \varphi : v \mapsto \langle \varphi | v \rangle u \).

So given \( X_i := u_i \otimes \varphi_i \) and an element \( \sigma \in S_k \) in the symmetric group we have

\[
\sigma^{-1} \circ u_1 \otimes \varphi_1 \otimes u_2 \otimes \varphi_2 \otimes \cdots \otimes u_k \otimes \varphi_k(v_1 \otimes v_2 \otimes \cdots \otimes v_k)
\]
is also the

\[ (1) \prod_{i=1}^{k} \langle \varphi_i \mid v_i \rangle u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(k)} \]

\[ u_1 \otimes \varphi_1 \otimes u_2 \otimes \varphi_2 \otimes \cdots \otimes u_k \otimes v_m \circ \sigma^{-1}(v_1 \otimes v_2 \otimes \cdots \otimes v_k) \]

\[ = \prod_{i=1}^{m} \langle \varphi_i \mid v_{\sigma(i)} \rangle u_1 \otimes u_2 \otimes \cdots \otimes u_k = \prod_{i=1}^{k} \langle \varphi_{\sigma^{-1}(i)} \mid v_i \rangle u_1 \otimes u_2 \otimes \cdots \otimes u_k \]

\[ \implies \sigma^{-1}u_1 \otimes \varphi_1 \otimes u_2 \otimes \varphi_2 \otimes \cdots u_m \otimes \varphi_k = u_{\sigma(1)} \otimes \varphi_1 \otimes u_{\sigma(2)} \otimes \varphi_2 \otimes \cdots u_{\sigma(k)} \otimes \varphi_k \]

\[ \implies u_1 \otimes \varphi_1 \otimes u_2 \otimes \varphi_2 \otimes \cdots u_k \otimes \varphi_k \circ \sigma = u_1 \otimes \varphi_{\sigma(1)} \otimes u_2 \otimes \varphi_{\sigma(2)} \otimes \cdots u_k \otimes \varphi_{\sigma(k)}. \]

So we need to understand in matrix formulas the invariants

\[ tr(\sigma^{-1}u_1 \otimes \varphi_1 \otimes u_2 \otimes \varphi_2 \otimes \cdots u_k \otimes \varphi_k) = \prod_{i=1}^{k} \langle \varphi_i \mid u_{\sigma(i)} \rangle. \]

We need to use the rules

\[ u \otimes \varphi \circ v \otimes \psi = u \otimes \langle \varphi \mid v \rangle \psi, \quad tr(u \otimes \varphi) = \langle \varphi \mid u \rangle \]

from which the formula easily follows by induction. \qed

Remark 1.6. We can extend the Formula (5) to the group algebra

\[ t(\sum_{\tau \in S_d} a_{\tau} \tau)(X_1, \ldots, X_d) := \sum_{\tau \in S_d} a_{\tau} T_{\tau}(X_1, X_2, \ldots, X_d). \]

1.7 The symmetric group

The algebra of the symmetric group \( S_k \) decomposes into the direct sum

\[ F[S_k] = \oplus_{\lambda \vdash k} End(M_\lambda) \]

of the matrix algebras associated to the irreducible representations \( M_\lambda \) of partitions \( \lambda \vdash k \). Denote by \( \chi_\lambda \) the corresponding character of \( S_k \) and by \( e_\lambda \in End(M_\lambda) \subset F[S_k] \) the corresponding central unit. These elements form a basis of orthogonal idempotents of the center of \( F[S_k] \).

For a finite group \( G \) let \( e \) be the central idempotent of an irreducible representation with character \( \chi \). One has the Formula:

\[ e = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g^{-1}) g. \]

As for the algebra \( \Sigma_k(V) \), it is isomorphic to \( F[S_k] \) if and only if \( d \geq k \). Otherwise it is a homomorphic image of \( F[S_k] \) with kernel the ideal generated by any antisymmetrizer in \( d+1 \) elements. This ideal is the direct sum of the \( End(M_\lambda) \) with \( ht(\lambda) > d \), where \( ht(\lambda) \), the height of \( \lambda \), cf. page 2 is also the length of its first column. So that

\[ \Sigma_k(V) = \oplus_{\lambda \vdash k, \; ht(\lambda) \leq d} End(M_\lambda) \]
1.8 The function $Wg(d, \mu)$

We start with a computation of a character.

Definition 1.9. Given a permutation $\rho \in S_k$ we denote by $c(\rho)$ the number of cycles into which it decomposes, and $\pi(\rho) \vdash k$ the partition of $k$ given by the lengths of these cycles. Notice that $c(\rho) = \text{ht}(\pi(\rho))$.

Given a partition $\mu \vdash k$ we denote by $
(\mu) := \{ \rho \mid \pi(\rho) = \mu \}$, $C_{\mu} := \sum_{\rho \in (\mu)} \rho = \sum_{\rho \in (\mu)} \rho$. \hspace{1cm} (12)

The sets $(\mu) := \{ \rho \mid \pi(\rho) = \mu \}$ are the conjugacy classes of $S_k$ and, thinking of $F[S_k]$ as functions from $S_k$ to $F$ we have that $C_{\mu}$ is the characteristic function of the corresponding conjugacy class. Of course the elements $C_{\mu}$ form a basis of the center of the group algebra $F[S_k]$.

Proposition 1.10. 1) For every pair of integers $k,d$ the function $P$ on $S_k$ given by $P: \rho \mapsto d c(\rho)$ is the character of the permutation action of $S_k$ on $V \otimes k$, $\dim_F(V) = d$.

2) The symmetric bilinear form on $F[S_k]$ given by $\langle \sigma \mid \tau \rangle := d c(\sigma \tau)$ has as kernel the ideal generated by the antisymmetrizer on $d + 1$ elements. In particular if $k \leq d$ it is non degenerate.

Proof. 1) If $e_1, \ldots, e_d$ is a given basis of $V$ we have the induced basis of $V \otimes k$, $e_{i_1} \otimes \ldots \otimes e_{i_k}$ which is permuted by the symmetric group. For a permutation representation the trace of an element $\sigma$ equals the number of the elements of the basis fixed by $\sigma$.

If $\sigma = (1, 2, \ldots, k)$ is one cycle then $e_{i_1} \otimes \ldots \otimes e_{i_k}$ is fixed by $\sigma$ if and only if $i_1 = i_2 = \ldots = i_k$ are equal, so equal to some $e_j$ so $tr(\sigma) = d$.

For a product of $a$ cycles of lengths $b_1, b_2, \ldots, b_a$ which up to conjugacy we may consider as

$$(1, 2, \ldots, b_1)(b_1 + 1, b_1 + 2, \ldots, b_1 + b_2)\ldots(k - b_{a_1}, \ldots, k)$$

we see that $e_{i_1} \otimes \ldots \otimes e_{i_k}$ is fixed by $\sigma$ if and only if it is of the form

$$e_{i_1}^{\otimes b_1} \otimes e_{i_2}^{\otimes b_2} \otimes \ldots \otimes e_{i_a}^{\otimes b_a},$$

giving $d^a$ choices for the indices $i_1, i_2, \ldots, i_a$.

2) In fact this is the trace form of the image $\Sigma_k(V)$ of $F[S_k]$ in the operators on $V \otimes m$, $\dim V = d$. Since $\Sigma_k(V)$ is semisimple its trace form is non degenerate. \hfill $\square$

Corollary 1.11.

$I)$ $P = \sum_{\lambda \vdash k, \text{ht}(\lambda) \leq d} s_\lambda(d) \chi_\lambda$, $II)$ $d c(\rho) = \sum_{\lambda \vdash k, \text{ht}(\lambda) \leq d} s_\lambda(d) \chi_\lambda(\rho)$. \hspace{1cm} (13)

Proof. This is immediate from Formula (2). \hfill $\square$
We thus have, with $ht(\mu)$ the number of parts of $\mu$ (cf. page 5), that

$$P := \sum_{\rho \in S_k} d^{c(\rho)} \rho = \sum_{\mu \vdash k} d^{ht(\mu)} C_\mu$$

(14)

is an element of the center of the algebra $\Sigma_k(V)$ which we can thus write

$$P = \sum_{\lambda \vdash k, \; ht(\lambda) \leq d} s_\lambda(d) \chi_\lambda = \sum_{\rho \in S_k} d^{c(\rho)} \rho = \sum_{\lambda \vdash k, \; ht(\lambda) \leq d} r_\lambda(d) e_\lambda$$

(15)

and we have:

**Proposition 1.12.**

$$r_\lambda(d) = \prod_{u \in \lambda}(d + c_u).$$

(16)

**Proof.** By Formula (10) and the orthogonality of characters we have:

I) $e_\lambda = \frac{\chi_\lambda(1)}{k!} \sum_{\sigma \in S_k} \chi_\lambda(\sigma)^{\sigma_\lambda}$, II) $\chi_\lambda(e_\mu) = \begin{cases} \chi_\lambda(1) & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$. (17)

One has thus, from Formulas (13) I) and (17) II) and denoting by $(\chi_\lambda, P)$ the usual scalar product of characters:

$$r_\lambda(d) = \frac{\chi_\lambda(\sum_{\rho \in S_k} d^{c(\rho)} \rho)}{\chi_\lambda(1)} = \frac{k!(\chi_\lambda, P)}{\chi_\lambda(1)} = \frac{k! s_\lambda(d)}{\chi_\lambda(1)} (4) \sum_{u \in \lambda}(d + c_u).$$

□

**Corollary 1.13.** The element $\sum_{\rho \in S_k} d^{c(\rho)} \rho$ is invertible in $\Sigma_k(V)$ with inverse

$$(\sum_{\rho \in S_k} d^{c(\rho)} \rho)^{-1} = \sum_{\lambda \vdash k, \; ht(\lambda) \leq d} \left(\prod_{u \in \lambda}(d + c_u)\right)^{-1} e_\lambda.$$

(18)

As we shall see in §2.1, it is interesting to study $(\sum_{\rho \in S_k} d^{c(\rho)} \rho)^{-1}$ where $k$ is fixed and $d$ is a parameter. We can thus use formula (18) for $d \geq k$ and following Collins [3] we write

$$(\sum_{\rho \in S_k} d^{c(\rho)} \rho)^{-1} = \sum_{\rho \in S_k} Wg(d, \rho) \rho := Wg(d, k).$$

(19)

Since $Wg(d, \rho)$ is a class function it depends only on the cycle partition $\mu = c(\rho)$ of $\rho$, so we may denote it by $Wg(d, \mu)$.

From definition (12) $C_\mu = \sum_{c(\rho) = \mu} \rho$ we can rewrite, $d \geq k$

$$C_\mu = \sum_{\rho \in S_k \mid c(\rho) = \mu} \rho, \quad Wg(d, k) = (\sum_{\rho \in S_k} d^{c(\rho)} \rho)^{-1} = \sum_{\mu \vdash k} Wg(d, \mu) C_\mu.$$
Substituting $e_\lambda$ in formula (18) with its expression of Formula (17)

$$e_\lambda = \frac{\chi(1)}{k!} \sum_{\sigma \in S_k} \chi_\lambda(\sigma)\sigma = \prod_{u \in \lambda} h_u^{-1} \sum_{\sigma \in S_k} \chi_\lambda(\sigma)\sigma$$

$$W_g(d, k) := \sum_{\rho \in S_k} W_g(d, \rho) = \sum_{\lambda \vdash k} \prod_{u \in \lambda} h_u(d + c_u) \sum_{\sigma} \chi_\lambda(\sigma)\sigma = \frac{\chi(1)^2}{k!^2 s_\lambda(d)} \chi_\lambda(\sigma)\sigma$$

(21)

**Theorem 1.14.**

$$W_g(d, \sigma) = \sum_{\lambda \vdash k} \prod_{u \in \lambda} h_u(d + c_u) \chi_\lambda(\sigma) = \sum_{\lambda \vdash k} \chi_\lambda(\sigma)\sigma$$

(22)

In particular $W_g(d, \sigma)$ is a rational function of $d$ with poles at the integers $-k + 1 \leq i \leq k - 1$ of order $p$ at $i$, $p(p + 1) \leq k$.

**Proof.** We only need to prove the last estimate. By symmetry we may assume that $i \geq 0$ then the $p^{th}$ entry of $i$ is placed at the lower right corner of a rectangle of height $p$ and length $i + p$ (cf. Figure at page 8). Hence if $\lambda \vdash k$, we have $i(p + i) \leq k$ and the claim. □

1.14.1 A more explicit formula

Formula (22), although explicit, is a sum with alternating signs so that it is not easy to estimate a given value or even to show that it is nonzero.

For $\sigma_0 = (1, 2, \ldots, k)$ a full cycle a better Formula is available. First Formula (23) by Formanek when $k = d$, and then by Collins Formula (24) in general.

When $k = d$ we write $W_g(d, \sigma) = a_\sigma$ and then:

$$d!^2 a_{\sigma_0} =\begin{cases} (-1)^{d+1} & \frac{d}{2d - 1} \neq 0 \end{cases}$$

(23)

Collins extends Formula (23) to the case $W_g(d, \sigma_0)$ getting:

$$W_g(d, \sigma_0) = (-1)^{k-1} c_{k-1} \prod_{-k+1 \leq j \leq k-1} (d - j)^{-1}$$

(24)

with $c_i := \frac{(2i)!}{(i+1)i!} = \frac{1}{1 + 1} \binom{2i}{i}$ the $i^{th}$ Catalan number. Which, since

$$c_{d-1} = \frac{(2d-2)!}{d!(d-1)!}, \prod_{-d+1 \leq j \leq d-1} (d - j) = (2d - 1)!$$

agrees, when $k = d$, with Formanek.

In order to prove Formula (24) we need the fact that $\chi_\lambda(\sigma_0) = 0$ except when $\lambda = (a, 1^{k-a})$ is a hook partition, with the first row of some length $a$, $1 \leq a \leq k$ and then the remaining $k - a$ rows of length 1.
This is an easy consequence of the Murnaghan–Nakayama formula, see [16].
In this case we have \( \chi_\lambda(\sigma_0) = (-1)^{k-a} \). We thus need to make explicit the integers \( s_\lambda(d), \chi_\lambda(1) \) for such a hook partition.

For \( \lambda = (a, 1^{k-a}) \), we get that the boxes are

\[
\begin{align*}
u &= (1, j), \ j = 1, \ldots, a, \ c_u = j - 1, \ h_u = \begin{cases} k & \text{if } j = 1 \\ a - j + 1 & \text{if } j \neq 1 \end{cases} \\
u &= (i + 1, 1), \ i = 1, \ldots, k - a, \ c_u = -i, \ h_u = k - a - i + 1. \end{align*}
\]

\[
\prod_u h_u = k \prod_{j=2}^{a} (a - j + 1) \prod_{i=1}^{k-a} (k - a - i + 1) = k(a - 1)! (k - a)!. 
\]

**Example 1.15.** \( a = 8, \ k = 11, \ (8, 1^3) \vdash 11 \) in coordinates

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 3 & 4 & 5 & 6 & 7 & & \\
3 & 4 & 5 & 6 & 7 & & & \\
4 & & & & & & & \\
\end{array}
\]

Hooks and content:

\[
\begin{array}{cccccccc}
11 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
3 & & & & & & & \\
2 & & & & & & & \\
1 & & & & & & & \\
\end{array}
, \quad \begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
-1 & & & & & & & \\
-2 & & & & & & & \\
-3 & & & & & & & \\
\end{array}
\]

Thus we finally have, substituting in Formula (22), that

\[
Wg(\sigma_0, d) = \sum_{a=1}^{k} (-1)^{k-a} \frac{1}{k(a - 1)! (k - a)!} \prod_{i=1-a}^{k-a} (d - i)^{-1} 
\]  \hspace{1cm} (25)

\[
= \sum_{a=1}^{k} (-1)^{k-a} \frac{\prod_{i=k-a+1}^{k-1} (d - i) \prod_{i=k-a+1}^{k} (d - i)}{k(a - 1)! (k - a)!} \prod_{-k+1 \leq j \leq k-1} (d - j)^{-1}. \hspace{1cm} (26)
\]

One needs to show that

\[
\sum_{a=1}^{k} (-1)^{a} \frac{\prod_{i=k-a}^{k-a+1} (d - i) \prod_{i=k-a+1}^{k} (d - i)}{k(a - 1)! (k - a)!} = \sum_{a=1}^{k} (-1)^{a} \frac{\prod_{i=k-a+1}^{k-1} i(d - i) \prod_{i=a}^{k-1} i(d + i)}{k!(k - 1)!} 
\]

\[
= P_k(d) := \frac{1}{k!} \sum_{b=0}^{k-1} (-1)^{b+1} \binom{k-1}{b} \prod_{i=k-b}^{k-1} (d - i) \prod_{i=k-b+1}^{k-1} (d + i) = (-1)^{k-1} c_{k-1}. \hspace{1cm} (27)
\]
By partial fraction decomposition we have that
\[
\prod_{i=1}^{k-a} (d-i)^{-1} = \sum_{i=1}^{k-a} \frac{b_j}{d-j},
\]
\[
b_0 = \prod_{i=1-a, \ i \neq 0}^{k-a} (-i)^{-1} = [(-1)^{k-a}(a-1)!/(k-a)!]^{-1}.
\]
Therefore the partial fraction decomposition of \(Wg(\sigma_0, d)\), from Formula (25), is
\[
\sum_{a=1}^{k} \frac{1}{k[(a-1)!((k-a)!)]^2} \frac{1}{d} + \sum_{-k+1 \leq j \leq k-1, \ j \neq 0} \frac{c_j}{d-j}.
\]
On the other hand the partial fraction decomposition of the product of Formula (26),
\[
\prod_{-k+1 \leq j \leq k-1} (d-j)^{-1} = \frac{(-1)^{k-1}}{(k-1)!^2} \frac{1}{d} + \sum_{-k+1 \leq j \leq k-1, \ j \neq 0} \frac{e_j}{d-j}.
\]
It follows that the polynomial \(P_k(d)\) of Formula (27) is a constant \(C\) with
\[
C \frac{(-1)^{k-1}}{(k-1)!^2} = \sum_{a=1}^{k} \frac{1}{k[(a-1)!((k-a)!)]^2} \implies C = (-1)^{k-1} \sum_{a=1}^{k} \frac{(k-1)!^2}{k[(a-1)!((k-a)!)]^2}.
\]
So finally we need to observe that
\[
\sum_{a=1}^{k} \frac{(k-1)!^2}{k[(a-1)!((k-a)!)]^2} = \frac{1}{k} \sum_{a=0}^{k-1} \left(\frac{k-1}{a}\right)^2 = \frac{1}{k} \left(\frac{2k-2}{k-1}\right) = C_{k-1}.
\]
In fact
\[
\sum_{a=0}^{n} \left(\frac{n}{a}\right)^2 = \binom{2n}{n}
\]
as one can see simply noticing that a subset of \(n\) elements in \(1, 2, \ldots, 2n\) distributes into \(a\) numbers \(\leq n\) and the remaining \(n-a\) which are \(> n\).

\[\square\]

1.15.1 A Theorem of Collins, [3] Theorem 2.2

For a partition \(\mu \vdash k\) we have defined, in Formula (12) \(C_\mu := \sum_{\sigma|\pi(\sigma) = \mu} \sigma\). Clearly we have for a sequence of partitions \(\mu_1, \mu_2, \ldots, \mu_i\)
\[
C_{\mu_1}C_{\mu_2} \cdots C_{\mu_i} = \sum_{\mu \vdash k} A[\mu; \mu_1, \mu_2, \ldots, \mu_i] C_\mu
\] (28)
where \(A[\mu; \mu_1, \mu_2, \ldots, \mu_i]\) counts the number of times that a product of \(i\) permutations \(\sigma_1, \sigma_2, \ldots, \sigma_i\) of types \(\mu_1, \mu_2, \ldots, \mu_i\) give a permutation \(\sigma\) of type \(\mu\). These numbers are classically called connection coefficients.
Remark 1.16. Notice that this number depends only on $\mu$ and not on $\sigma$.

Set:

$$ A[\mu, i, h] := \sum_{\mu_1, \mu_2, \ldots, \mu_i | \mu_j \neq 1^k \sum_j (k - ht(\mu_j)) = h} A[\mu; \mu_1, \mu_2, \ldots, \mu_i] $$

(29)

$$ A[\mu, h] := \sum_{i=1}^p (-1)^i A[\mu, i, h]. $$

Remark 1.17. For a permutation $\sigma \in S_k$ with $\pi(\sigma) = \mu$ we will write

$$ |\sigma| = |\mu| := k - ht(\mu). $$

(30)

This is the minimum number of transpositions with product $\sigma$.

We have $|\sigma\tau| \leq |\sigma| + |\tau|$, see Stanley [18] p.446 for a poset interpretation.

From Formula (22) we know that each $Wg(\sigma, d)$ is a rational function of $d$ with poles in $0, \pm 1, \pm 2, \ldots, \pm (k - 1)$ of order $< k$, so we can expand it in a power series in $d^{-1}$ converging for $d > k - 1$ as in Formula (31):

Theorem 1.18 ([3] Theorem 2.2). We have an expansion for $(\sum_{\rho \in S_k} d^{c(\rho)} \rho)^{-1}$ as power series in $d^{-1}$:

$$ P = \sum_{\rho \in S_k} d^{c(\rho)} \rho = d^k (1 + \sum_{\mu \vdash k|\mu| \neq 1^k} d^{k - ht(\mu)} C_\mu) = d^k (1 + \sum_{\mu \vdash k|\mu| \neq 1^k} d^{-|\mu|} C_\mu) $$

$$ P^{-1} = d^{-k} (1 + \sum_{i=1}^{\infty} (-1)^i \sum_{\mu \vdash k|\mu| \neq 1^k} d^{-i|\mu|} C_\mu) $$

$$ = d^{-k} (1 + \sum_{i=1}^{\infty} (-1)^i \sum_{\mu_1, \mu_2, \ldots, \mu_i | \mu_j \neq 1^k} d^{-\sum_{j=1}^i |\mu_j|} C_{\mu_1, \mu_2, \ldots, \mu_i}) $$

$$ = d^{-k} (1 + \sum_{i=1}^{\infty} (-1)^i \sum_{\mu_1, \mu_2, \ldots, \mu_i | \mu_j \neq 1^k} d^{-\sum_{j=1}^i |\mu_j|} A[\mu; \mu_1, \mu_2, \ldots, \mu_i] C_\mu) $$

$$ = d^{-k} (1 + \sum_{\mu \vdash k} \left( \sum_{h=1}^{\infty} d^{-h} A[\mu, h] \right) C_\mu) $$

since $\mu_1 + \mu_2 + \ldots + \mu_i = \mu$ implies $|\mu| \leq \sum_{j=1}^i |\mu_j|$. 

Proof. Recall that we denote by $|\mu| := k - ht(\mu)$, (30).
Remark 1.19. We want to see now, for $\mu \neq 1^k$ that the series $\sum_{h=|\mu|}^{\infty} d^{-h} A[\mu, h]$ starts with $h = |\mu|$, i.e. $A[\mu, |\mu|] \neq 0$. Thus we compute the top coefficient $A[\mu, |\mu|]$ which gives the asymptotic behaviour of $Wg(\sigma, d)$.

Let us denote by

$$C[\mu] := A[\mu, |\mu|] \implies \lim_{d \to \infty} d^{k+|\sigma|} Wg(\sigma, d) = C[\mu]. \quad (32)$$

From Formula (23) we have $C[(k)] = (-1)^{k-1} C_{k-1}$ (Catalan number) and a further and more difficult Theorem of Collins states

Theorem 1.20. [[3] Theorem 2.12 (ii)]

$$C[(k)] = (-1)^{k-1} C_{k-1}, \quad C[(a_1, a_2, \ldots, a_i)] = \prod_{j=1}^{i} C[(a_j)]. \quad (33)$$

Fixing $\sigma \in S_k$ with $\pi(\sigma) = \mu$ we have that $A[\mu; \mu_1, \mu_2, \ldots, \mu_i]$ is also the number of sequences of permutations $\sigma_j$, $\pi(\sigma_j) = \mu_j$ with $\sigma = \sigma_1 \sigma_2 \ldots \sigma_i$.

So we shall also use the notation, for $\pi(\sigma) = \mu$:

$$A[\sigma; \mu_1, \mu_2, \ldots, \mu_i] = A[\mu; \mu_1, \mu_2, \ldots, \mu_i], \quad C[\sigma] := A[\sigma, |\sigma|].$$

Thus

$$C[\mu] = A[\mu, |\mu|] = \sum_{i=1}^{\mu_1, \mu_2, \ldots, \mu_i \neq 1^k} \sum_{\sum_{j=1}^{i} |\mu_j| = |\mu|} (-1)^i A[\mu; \mu_1, \mu_2, \ldots, \mu_i] \quad (34)$$

We call a coefficient $A[\mu; \mu_1, \mu_2, \ldots, \mu_i]$ with $\mu_1, \mu_2, \ldots, \mu_i \neq 1^k$, and $\sum_{j=1}^{i} |\mu_j| = |\mu|$ a top coefficient.

1.20.1 **Top coefficients and a degeneration of $Q[S_k]$**

The study of $C[\mu]$ can be formulated in terms of a degeneration: $Q[\tilde{S}_k]$ of the multiplication in the group algebra whose elements now denote by $\tilde{\sigma}$.

Define a new (still associative) multiplication by

$$Q[\tilde{S}_k] := \oplus_{\sigma \in S_k} Q\tilde{\sigma}, \quad \tilde{\sigma}_1 \tilde{\sigma}_2 := \begin{cases} \tilde{\sigma}_1 \tilde{\sigma}_2 & \text{if } |\sigma_1 \sigma_2| = |\sigma_1| + |\sigma_2| \\ 0 & \text{otherwise} \end{cases}. \quad (35)$$

Contrary to the semisimple algebra $Q[S_k]$ the algebra $Q[\tilde{S}_k]$ is a graded algebra, with $Q[\tilde{S}_k]_h = \oplus_{\sigma \in S_k | |\sigma| = h} Q\tilde{\sigma}$ and has

$$I := \oplus_{\sigma \in S_k | \sigma \neq 1} Q\tilde{\sigma} = \oplus_{h=1}^{k-1} Q[\tilde{S}_k]_h$$

---

1 I have made a considerable effort trying to understand, and hence verify, the proof of this Theorem in [3], to no avail. I hope somebody has verified it. At any rate the Theorem is true as I will show presently with a simple natural proof.
as a nilpotent ideal, \( I^k = 0 \), its nilpotent radical. Observe that

\[
|\sigma_1\sigma_2| = |\sigma_1| + |\sigma_2| \iff c(\sigma_1\sigma_2) = c(\sigma_1) + c(\sigma_2) - k
\]

so if \( c(\sigma_1) + c(\sigma_2) \leq k \) we know a priori that the product \( \tilde{\sigma}_1\tilde{\sigma}_2 = 0 \).

In this algebra the multiplication of two elements \( \tilde{C}_{\mu_1}, \tilde{C}_{\mu_2} \) associated to conjugacy classes as in (12) involves only the top coefficients and is:

\[
\tilde{C}_{\mu_1}\tilde{C}_{\mu_2} = \sum_{|\mu|=|\mu_1|+|\mu_2|} A[\mu; \mu_1, \mu_2]\tilde{C}_{\mu}. \tag{36}
\]

We then have

\[
(\sum_{\rho \in S_k} d^{(\rho)}\tilde{\rho})^{-1} = d^{-k}(1 + \sum_{\mu+k|\mu \neq 1^k} d^{-|\mu|}\tilde{C}_{\mu})^{-1} = d^{-k}(1 + \sum_{\mu+k} d^{-|\mu|}C[\mu]\tilde{C}_{\mu})
\]

\[
= d^{-k}(1 + \sum_{h=1}^{k-1} d^{-h}(\sum_{\mu+k|\mu = h} C[\mu]\tilde{C}_{\mu})). \tag{37}
\]

Notice that if \( h = k - 1 \) the only partition \( \mu \) with \( |\mu| = k - 1 \) is \( \mu = (k) \) the partition of the full cycle.

Hence in Formula (37) the lowest term is \( d^{-2k+1}C[(k)]\tilde{C}_{(k)} \).

An example, which the reader can skip, the connection coefficients for \( S_4 \), in box the top ones (write the elements \( C_{\mu} \) with lowercase):

\[
\begin{array}{cccc}
\text{c}_{1,1,2} & \text{c}_{1,3} & \text{c}_{2,2} & \text{c}_4 \\
\text{c}_{1,1,2} & 6c_{1,1,1,1} + 3c_{1,3} + 2c_{2,2} & 4c_{1,1,2} + 4c_4 & c_{1,1,2} + 2c_4 & 3c_{1,3} + 4c_{2,2} \\
\text{c}_{1,3} & 4c_{1,1,2} + 4c_4 & 8c_{1,1,1,1} + 4c_{1,3} + 8c_{2,2} & 3c_{1,3} & 4c_{1,1,2} + 4c_4 \\
\text{c}_{2,2} & c_{1,1,2} + 2c_4 & 3c_{1,3} & 3c_{1,1,1,1} + 2c_{2,2} & 2c_{1,1,2} + c_4 \\
\text{c}_4 & 3c_{1,3} + 4c_{2,2} & 4c_{1,1,2} + 4c_4 & 2c_{1,1,2} + c_4 & 6c_{1,1,1,1} + 3c_{1,3} + 2c_{2,2}
\end{array}
\]

Setting \( a = c_{1,1,2}, \ b = c_{1,3}, \ c = c_{2,2}, \ d = c_4 \) compute (37)

\[
a^2 = 3b + 2c, \ ab = 4d, \ ac = 2d
\]

\[
P = 1 + T, \ T = x^{-1}a + x^{-2}(b + c) + x^{-3}d, \ (1 + T)^{-1} = 1 - T + T^2 - T^3
\]

\[
T^2 = x^{-2}a^2 + 2x^{-3}a(b+c) = x^{-2}(3b+2c)+x^{-3}12d, \ T^3 = x^{-3}a(3b+2c) = x^{-3}(12+4)d = x^{-3}16d
\]

\[
-T + T^2 - T^3 = -x^{-1}a - x^{-2}(b+c) - x^{-3}d + x^{-2}(3b+2c) + x^{-3}12d - x^{-3}16d
\]

The conjugacy classes and their cardinality in \( S_5 \):

\[
(1, c_{1,1,1,1,1} \ 10, c_{1,1,1,2} \ 20, c_{1,1,3} \ 15, c_{1,2,2} \ 30, c_{1,4} \ 20, c_{2,3} \ 24, c_5)
\]
Here is a table of the top connection coefficients for $S_5$. The numbers to the right are the degrees $|\mu|$:

|     | $c_{1,1,1,2}$ | $c_{1,1,3}$ | $c_{1,2,2}$ | $c_{1,4}$ | $c_{2,3}$ | $c_5$ |
|-----|---------------|--------------|--------------|-----------|----------|-------|
| $c_{1,1,1,2}$ | $3c_{1,1,3} + 2c_{1,2,2}$ | $2c_{1,4} + 3c_{2,3}$ | $5c_5$ | $5c_5$ | $0$ | $0$ |
| $c_{1,1,3}$ | $4c_{1,4} + c_{2,3}$ | $5c_5$ | $0$ | $0$ | $0$ |
| $c_{1,2,2}$ | $2c_{1,4} + 3c_{2,3}$ | $5c_5$ | $0$ | $0$ | $0$ |
| $c_{1,4}$ | $5c_5$ | $0$ | $0$ | $0$ | $0$ |
| $c_{2,3}$ | $5c_5$ | $0$ | $0$ | $0$ | $0$ |
| $c_5$ | $0$ | $0$ | $0$ | $0$ | $0$ |

$a = c_{1,1,1,2}, 1  b = c_{1,1,3}, 2  c = c_{1,2,2}, 2  d = c_{1,4}, 3  e = c_{2,3}, 3  f = c_5, 4$

$a^2 = 3b+2c, \ ab = 4d+e, \ ac = 2d+3e, \ ad = 5f, \ ae = 5f, \ b^2 = 5f, \ bc = 5f, \ c^2 = 5f,$

$$1 + T, \ T = x^{-1}a + x^{-2}(b + c) + x^{-3}(d + e) + x^{-4}f$$

$$T^2 = x^{-2}a^2 + x^{-4}(b + c)^2 + 2x^{-3}a(b + c) + 2x^{-4}a(d + e)$$

$$= x^{-2}(3b + 2c) + 2x^{-3}(6d + 4e) + 40x^{-4}f$$

$$T^3 = x^{-3}a(3b + 2c) + 2x^{-4}a(6d + 4e) + x^{-4}(b + c)(3b + 2c)$$

$$= x^{-3}(12d + 3e + 4d + 6e) + x^{-4}(100 + 15 + 10 + 15 + 10)f$$

$$= x^{-3}(16d + 9e) + x^{-4}150f$$

$$T^4 = x^{-4}a(16d + 9e) = x^{-4}(16 \cdot 5 + 45)f = x^{-4}125f$$

$$125 - 150 + 40 - 1 = 14$$

$c_i=$Catalan(i): 1, 2, 5, 14, 42,...  Catalan(4)=14!

$$-T + T^2 - T^3 + T^4 =$$

$$-(x^{-1}a + x^{-2}(b+c) + x^{-3}(d+e)) + x^{-2}(3b+2c) + 2x^{-3}(6d+4e) - x^{-3}(16d+9e) + 14f$$

$$= -x^{-1}a - x^{-2}(b+c) - x^{-3}(d+e) + x^{-2}(3b+2c) + 2x^{-3}(6d+4e) - x^{-3}(16d+9e)$$

$$= -x^{-1}a + x^{-2}(3b + 2c - b - c) + x^{-3}(12d + 8e - 16d - 9e - d - e)$$

$$= -x^{-1}a + x^{-2}(2b + c) + x^{-3}(-5d - 2e) + 14f.$$
1.20.2 Young subgroups

Let $\Pi := \{A_1, A_2, \ldots, A_j\}$, $|A_i| = a_i$ be a decomposition of the set $[1, 2, \ldots, k]$:

\[
i.e. \quad A_1 \cup A_2 \cup \ldots \cup A_j = [1, 2, \ldots, k], \quad A_i \cap A_j = \emptyset, \quad \forall i \neq j.
\]

**Definition 1.21.**

1. The subgroup of $S_k$ fixing this decomposition is the product $\prod_{i=1}^{j} S_{A_i} = \prod_{i=1}^{j} S_{a_i}$ of the symmetric groups $S_{a_i}$. It is usually called a *Young subgroup* and will be denoted by $Y_\Pi$.

2. Given two decompositions of $[1, 2, \ldots, k]$, $\Pi_1 := \{A_1, A_2, \ldots, A_j\}$, and $\Pi_2 := \{B_1, B_2, \ldots, B_h\}$ we say that $\Pi_1 \leq \Pi_2$ if each set $A_i$ is contained in one of the sets $B_d$. This is equivalent to the condition $Y_{\Pi_1} \subset Y_{\Pi_2}$.

3. In particular, if $\sigma \in S_k$ we denote by $\Pi_\sigma$ the decomposition of $[1, 2, \ldots, k]$ induced by its cycles and denote $Y_\sigma := Y_{\Pi_\sigma}$.

**Remark 1.22.** Observe that $\tau \in Y_\Pi$ if and only if $\Pi_\tau \leq \Pi$. The conjugacy classes of $Y_\Pi$ are the products of the conjugacy classes in the blocks $A_i$.

Then we have for the group algebra and $\tau = (\tau_1, \tau_2, \ldots, \tau_j) \in Y_\Pi$:

\[
\mathbb{Q}[Y_\Pi] = \bigotimes_{i=1}^{j} \mathbb{Q}[S_{a_i}] \subset \mathbb{Q}[S_k], \quad (\tau_1, \tau_2, \ldots, \tau_j) = \tau_1 \otimes \tau_2 \otimes \ldots \otimes \tau_j.
\]  (38)

We denote by $c_\tau$ the sum of the elements of the conjugacy class of $\tau$ in $Y_\Pi$ in order to distinguish it from $C_\tau$ the sum over the conjugacy class in $S_k$.

We have:

\[
\tau = (\tau_1, \tau_2, \ldots, \tau_j) \in Y_\Pi, \quad c_\tau \overset{(12)}{=} C_{\tau_1} \otimes C_{\tau_2} \otimes \ldots \otimes C_{\tau_j}.
\]  (39)

The first remark is:

**Remark 1.23.** If $\tau = (\tau_1, \tau_2, \ldots, \tau_j) \in Y_\Pi$ then

\[
c(\tau) = c(\tau_1) + c(\tau_2) + \cdots + c(\tau_j),
\]

\[
\implies |\tau| = \sum_i a_i - c(\tau) = \sum_i (a_i - c(\tau_i)) = |\tau_1| + |\tau_2| + \cdots + |\tau_j|.
\]  (40)

As a consequence if $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_j), \tau = (\tau_1, \tau_2, \ldots, \tau_j) \in Y_\Pi$ we have

\[
|\gamma \tau| = |\gamma| + |\tau| \iff |\gamma_i \tau_i| = |\gamma_i| + |\tau_i|, \quad \forall i.
\]  (41)

If we then consider the associated discrete algebras, From Formulas (41) and (38) we deduce an analogous of Formula (38) for the discrete algebras:

\[
\mathbb{Q}[\tilde{Y}_\Pi] = \bigotimes_{i=1}^{j} \mathbb{Q}[\tilde{S}_{a_i}] \subset \mathbb{Q}[\tilde{S}_k], \quad \tau = (\tau_1, \tau_2, \ldots, \tau_j), \quad \tilde{\tau} = \tilde{\tau}_1 \otimes \tilde{\tau}_2 \otimes \cdots \otimes \tilde{\tau}_j.
\]  (42)

Formula (40) tells us that $\mathbb{Q}[\tilde{Y}_\Pi] = \bigotimes_{i=1}^{j} \mathbb{Q}[\tilde{S}_{a_i}]$ as graded tensor product and the inclusion in $\mathbb{Q}[\tilde{S}_k]$ preserves the degrees.
1.23.1 A proof of Theorem 1.20

In particular let $\sigma \in S_k$ and $\sigma = c_1 c_2 \ldots c_j$ its cycle decomposition.

Let $A_i$ be the support of the cycle $c_i$ of $\sigma$ and $a_i$ its cardinality, so that $\Pi_\sigma = \{A_1, \ldots, A_j\}$ and $Y_\sigma = Y_{\Pi_\sigma}$. We have $\sigma \in Y_\sigma$ and its conjugacy class in $Y_\sigma$ is the product of the conjugacy classes of the cycles $(a_i) \subset S_{a_i}$. (12)

We denote, as before, by $c_\sigma$ the sum of the elements of this conjugacy class.

We have now a very simple but crucial fact;

**Proposition 1.24.**

1. Let $(i, i_1, \ldots, i_a), (j, j_1, \ldots, j_b)$ be two disjoint cycles and take the transposition $(i, j)$ then:

\[
(i, i_1, \ldots, i_a)(j, j_1, \ldots, j_b)(i, j) = (i, j_1, \ldots, j_b, i_1, \ldots, i_a, j) \tag{43}
\]

\[
(i, j)(i, i_1, \ldots, i_a)(j, j_1, \ldots, j_b) = (j, j_1, \ldots, j_b, i_1, \ldots, i_a, i) \tag{44}
\]

2. Let $\sigma \in S_k$ and $\tau = (i, j)$ a transposition. Then $|\sigma\tau| = |\tau\sigma| = |\sigma| \pm 1$ and $|\sigma\tau| = |\tau\sigma| = |\sigma| - 1$ if and only if the two indices $i, j$ both belong to one of the sets of the partition of $\sigma$, i.e. $\tau = (i, j) \in Y_\sigma$.

**Proof.** 1) is clear and 2) follows immediately from 1).

Notice that $\Pi_{\sigma\tau} < \Pi_\sigma$ and is obtained from $\Pi_\sigma$ by replacing the support of the cycle in which $i, j$ appear with two subsets support of the 2 cycles in which this splits. Similarly for $\Pi_{\tau\sigma}$.

From this we deduce the essential result of this section:

**Corollary 1.25.** Let $\sigma \in S_k$. Consider a decomposition $\sigma = \sigma_1\sigma_2 \ldots \sigma_h$, $\sigma_i \in S_k$, $\sigma_i \neq 1, \forall i$ with $|\sigma| = |\sigma_1| + |\sigma_2| + \ldots + |\sigma_h|$. Then for all $i$ we have $\sigma_i \in Y_{\Pi_\sigma} = Y_\sigma$ (Definition 1.21).

**Proof.** By induction on $h$, if $h = 1$ there is nothing to prove.

If $\sigma_1 = (i, j)$ is a transposition then the theorem follows by induction on $\sigma = \sigma_1 \sigma_2 \ldots \sigma_h$, and Proposition 1.24.

If $|\sigma_1| > 1$ we split $\sigma_1 = \tau\sigma_1$ with $|\tau\sigma_1| = |\sigma_1| - 1$ and we are reduced to the previous case.

We are now ready to prove the Theorem of Collins, Formula (33).

Let $\sigma \in S_k$ and $\sigma = c_1 c_2 \ldots c_j$ its cycle decomposition. Let $A_i$ be the support of the cycle $c_i$ and $a_i$ its cardinality, so that $\Pi_\sigma = \{A_1, \ldots, A_j\}$.

By the previous Corollary 1.25 and Remark 1.16 the contribution to $\sigma$ in the terms of Formula (29) are all in the subgroup $Y_\sigma$ so that finally

\[
C[\sigma] = C[\tilde{\sigma}] \quad \text{with } C[\tilde{\sigma}] \text{ computed in } \mathbb{Q}[\tilde{Y}_\sigma].
\]

In order to compute $C[\tilde{\sigma}]$ we observe that the term $d^{-k-|\tilde{\sigma}|}C[\tilde{\sigma}]c_{\tilde{\sigma}} = d^{-k-|\sigma|}C[\tilde{\sigma}]c_{\tilde{\sigma}}$ is the lowest term in $d^{-1}$ in

\[
\left( \sum_{\rho \in Y_\sigma} d^c(\rho) \tilde{\rho}^{-1} \right)^{-1} = \bigotimes_{i=1}^{j} \left( \sum_{\rho \in S_{a_i}} d^c(\rho) \tilde{\rho}^{-1} \right)^{-1}. \tag{45}
\]
From Formula (37) applied to the various full cycles \( c_i \in S_{a_i} \), we have that the lowest term in \( (\sum_{\rho \in S_{a_i}} d^{(\rho)} \rho)^{-1} \) is \( d^{-2a_i+1} C[(a_i)] C(a_1) \) so that we have finally that the lowest term in Formula (45) is

\[
d^{-k-|\sigma|} C[\bar{\sigma}] C_{\bar{\sigma}}^{(39)} = \prod_{i=1}^{j} d^{-2a_i+1} C[(a_i)] C(a_1) \otimes \ldots \otimes C(a_j),
\]

\[
\Rightarrow C[\sigma] = C[\bar{\sigma}] = \prod_{i=1}^{j} C[(a_i)]^{(23)} = \prod_{i=1}^{j} (-1)^{a_i-1} c_{a_i-1}. \tag{46}
\]

We have proved, Formula (23) that \((-1)^{a_i-1} C[(a_i)]\) is the Catalan number \( c_{a_i-1} \) and the proof of Theorem 1.20 is complete. \( \square \)

### 1.25.1 A table

The case \( k = d \) is of special interest, see §2.7. We write \( Wg(d, \mu) = a_\mu \) so that \( \sum_{\mu=d} Wg(d, \mu) c_\mu = \sum_{\mu} a_\mu c_\mu \) in Formula (20).

A computation using Mathematica gives \( d \leq 8 \) the list \( d!^2 \sum_{\mu=d} a_\mu c_\mu \):

\[
\begin{align*}
&\frac{2}{3} c_2^2 + \frac{4}{3} c_1,1 \\
&\frac{9}{10} c_1,2 + \frac{3}{5} c_3 + \frac{21}{10} c_1^3 \\
&\frac{48}{35} c_1^2,2 - \frac{4}{7} c_4 + \frac{22}{35} c_2^2 + \frac{29}{35} c_1,3 + \frac{134}{35} c_1^4.
\end{align*}
\]

\[
\begin{align*}
&-\frac{299}{126} c_1^3,2 - \frac{101}{126} c_1,4 - \frac{37}{63} c_2,3 + \frac{5}{9} c_5 + \frac{115}{126} c_1,2^2 + \frac{1}{63} c_2,3 + \frac{145}{18} c_1^5 \\
&-\frac{2538}{539} c_1^2,4 - \frac{668}{539} c_1^2,3 - \frac{459}{539} c_1,2,3 - \frac{338}{539} c_2^2 - \frac{6}{11} c_6 + \frac{300}{539} c_3,3 \\
&+ \frac{922}{1617} c_2,4 + \frac{26}{33} c_1,5 + \frac{2396}{1617} c_1^2,2^2 + \frac{1180}{539} c_1,3,3 + \frac{10508}{539} c_1^6.
\end{align*}
\]

\[
\begin{align*}
&-\frac{12319}{1144} c_3^2,4 - \frac{7369}{3432} c_1^3,4 - \frac{196}{143} c_1^2,2,3 - \frac{1087}{1144} c_1^2,2,3 - \frac{223}{286} c_1,2^3 - \frac{1061}{1716} c_2,5 - \frac{156}{18} c_3,4 + \frac{7}{13} c_7 \\
&+ \frac{1015}{1716} c_2,3^2 + \frac{1379}{1716} c_1,3^2 + \frac{259}{312} c_2,3^2 + \frac{9401}{3432} c_1^3,2^2 + \frac{7385}{1716} c_1^4,3 + \frac{184894}{3432} c_1^7.
\end{align*}
\]

\[
\begin{align*}
&3245092 + \frac{546368}{19305} c_1^8 + \frac{14434}{19305} c_1^7,2 + \frac{12828}{19305} c_1^5,3 + \frac{112828}{19305} c_1^4,2,2 - \frac{16336}{3861} c_1^4,4 - \frac{4384}{1755} c_1^3,2,3 + \frac{41332}{19305} c_1^3,5 \\
&- \frac{10432}{6435} c_1^2,2,3 + \frac{8608}{6435} c_1^2,2,4 + \frac{24718}{19305} c_1^2,3^2 - \frac{2624}{2145} c_1^2,6 + \frac{17122}{19305} c_2,1,2,3 - \frac{1216}{1485} c_1^2,5 - \frac{1384}{1755} c_1,3,4
\end{align*}
\]
the reader will notice certain peculiar properties of these sequences.

Conjecture There is a sign change at the cycle, in the sense that for $n$ odd $Wg((n))$ is the smallest of the positive values and for $n$ even $Wg((n))$ is the biggest of the negative values. The negatives are strictly increasing in the lexicographic order the positives are strictly decreasing in the lexicographic order.

I verified this up to $d = 14$.

These deserve further investigation, maybe the factorization of Jucys:

$$\sum_{\rho \in S_k} \rho = \prod_{i=1}^{k} (d + J_i), \quad J_i = (1, i) + (2, i) + \ldots + (i-1, i)$$

see [10] and [14] can be used.

1.26 The algebra $(\wedge M_d)^G$

Preliminary to the next step we need to recall the theory of antisymmetric conjugation invariant functions on $M_d$. This is a classical theory over a field of characteristic 0 which one may take as $\mathbb{Q}$.

First, let $U$ be a vector space, a polynomial $g(x_1, \ldots, x_m)$ in $m$ variables $x_i \in U$ is antisymmetric or alternating in the variables $X := \{x_1, \ldots, x_m\}$ if for all permutations $\sigma \in S_m$ we have

$$g(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) = \epsilon_\sigma g(x_1, \ldots, x_m), \quad \epsilon_\sigma \text{ the sign of } \sigma.$$ 

A simple way of forming an antisymmetric polynomial from a given one $g(x_1, \ldots, x_m)$ is the process of alternation\(^2\)

$$\text{Alt}_X g(x_1, \ldots, x_m) := \sum_{\sigma \in S_m} \epsilon_\sigma g(x_{\sigma(1)}, \ldots, x_{\sigma(m)}). \quad (47)$$

Recall that the exterior algebra $\wedge U^*$, with $U$ a vector space, can be thought of as the space of multilinear alternating functions on $U$. Then exterior multiplication as functions is given by the Formula:

$$f(x_1, \ldots, x_h) \in \bigwedge^h U^*; \quad g(x_1, \ldots, x_k) \in \bigwedge^k U^*;$$

$$f \wedge g(x_1, \ldots, x_{h+k}) = \frac{1}{h!k!} \sum_{\sigma \in S_{h+k}} f(x_{\sigma(1)}, \ldots, x_{\sigma(h)})g(x_{\sigma(h+1)}, \ldots, x_{\sigma(h+k)})$$

$$= \frac{1}{h!k!} \text{Alt}_{x_1, \ldots, x_{h+k}} f(x_1, \ldots, x_h)g(x_{h+1}, \ldots, x_{h+k}) \in \bigwedge^{h+k} U^*. \quad (48)$$

It is well known that:

\(^2\)we avoid on purpose multiplying by $1/m!$
Proposition 1.27. A multilinear and antisymmetric polynomial \( g(x_1, \ldots, x_m) \) in \( m \) variables \( x_i \in \mathbb{C}^m \) is a multiple, \( a \det(x_1, \ldots, x_{d^2}) \), of the determinant.

In fact if the polynomial has integer coefficients \( a \in \mathbb{Z} \).

For a multilinear and antisymmetric polynomial map \( g(x_1, \ldots, x_m) \in U \) to a vector space, each coordinate has the same property so

\[
g(x_1, \ldots, x_m) = \det(x_1, \ldots, x_m)a, \quad a \in U.
\]

We apply this to \( U = M_d \). Let us identify \( M_d = \mathbb{C}^{d^2} \) using the canonical basis of elementary matrices \( e_{i,j} \) ordered lexicographically e.g.:

\[
d = 2, \quad e_{1,1}, \ e_{1,2}, \ e_{2,1}, \ e_{2,2}.
\]

Given \( d^2 \) matrices \( Y_1, \ldots, Y_{d^2} \in M_d \) we may consider them as elements of \( \mathbb{C}^{d^2} \) and then form the determinant \( \det(Y_1, \ldots, Y_{d^2}) \).

By Proposition 1.27 the 1 dimensional space \( \wedge^{d^2} M_d^* \) has as generator the determinant \( \det(Y_1, \ldots, Y_{d^2}) \) which, since the conjugation action by \( G := GL(d, \mathbb{Q}) \) on \( M_d \) is by transformations of determinant 1, is thus an invariant under the action by \( G \).

The theory of \( G \) invariant antisymmetric multilinear \( G \) invariant functions on \( M_d \) is well known and related to the cohomology of \( G \).

The antisymmetric multilinear \( G \) invariant functions on \( M_d \) form the algebra \( (\wedge M_d^*)^G \). This is a subalgebra of the exterior algebra \( \wedge M_d^* \) and can be identified to the cohomology of the unitary group. As all such cohomology algebras it is a Hopf algebra and by Hopf’s Theorem it is the exterior algebra generated by the primitive elements.

The primitive elements of \( (\wedge M_d^*)^G \) are, see [11]:

\[
T_{2i-1} = T_{2i-1}(Y_1, \ldots, Y_{2i-1}) := tr(S_{2i-1}(Y_1, \ldots, Y_{2i-1})) \quad (50)
\]

\[
S_{2i-1}(Y_1, \ldots, Y_{2i-1}) = \sum_{\sigma \in S_{2i-1}} \epsilon_\sigma Y_{\sigma(1)} \cdots Y_{\sigma(2i-1)}
\]

with \( i = 1, \ldots, d \). In particular, since these elements generate an exterior algebra we have:

Remark 1.28. A product of elements \( T_i \) is non zero if and only if the \( T_i \) involved are all distinct, and then it depends on the order only up to a sign.

The \( 2^n \) different products form a basis of \( (\wedge M_d^*)^G \). The non zero product of all these elements \( T_{2i-1}(Y_1, \ldots, Y_{2i-1}) \) is in dimension \( d^2 \). We denote

\[
T_d(Y_1, Y_2, \ldots, Y_{d^2}) = T_1 \wedge T_3 \wedge T_5 \wedge \cdots \wedge T_{2d-1}. \quad (51)
\]

Proposition 1.29. A multilinear antisymmetric function of \( Y_1, \ldots, Y_{d^2} \) is a multiple of \( T_1 \wedge T_3 \wedge T_5 \wedge \cdots \wedge T_{2d-1} \).

Remark 1.30. The function \( \det(Y_1, \ldots, Y_{d^2}) \) is an invariant of matrices so it must have an expression as in Formula (6). In fact up to a computable integer constant [6] this equals the exterior product of Formula (51).
The constant of the change of basis when we take as basis the matrix units can be computed up to a sign, see \[6\]:

\[
T_d(Y) = C_d \det(Y_1, \ldots, Y_d), \quad C_d := \pm \frac{1!3!5! \cdots (2d-1)!}{1!2! \cdots (d-1)!}.
\] (52)

2 Comparing Formanek, [6] and Collins [3]

Rather than following the historical route we shall first discuss the paper of Collins, since this will allow us to introduce some notations useful for the discussion of Formanek’s results.

2.1 The work of Collins

In the paper [3], Collins introduces the Weingarten function in the following context. He is interested in computing integrals of the form

\[
\int_{U(d)} \prod_{\ell=1}^{k_1} u_{j_\ell,h_\ell} \prod_{m=1}^{k_2} \bar{u}_{i_m,p_m} du
\]

where \( U(d) \) is the unitary group of \( d \times d \) matrices and the elements \( u_{i,j} \) the entries of a matrix \( X \in U(d) \) while \( \bar{u}_{j,i} \) the entries of \( X^{-1} = U^* = \bar{U}^t \).

Here \( du \) is the normalized Haar measure. If one translates by a scalar matrix \( \alpha, \|\alpha\| = 1 \) then the integrand is multiplied by \( \alpha^{k_1} \bar{\alpha}^{k_2} \), on the other hand Haar measure is invariant under multiplication so that this integral vanishes unless we have \( k_1 = k_2 \). In this case the computation will be algebraic based on the following considerations.

Let us first make some general remarks. A finite dimensional representation \( R \) of a compact group \( G \) (with the dual denoted by \( R^* \)), decomposes into the direct sum of irreducible representations. In particular if \( R^G \) denotes the subspace of \( G \) invariant vectors there is a canonical \( G \)-equivariant projection \( E : R \to R^G \). The projection \( E \) can be written as integral

\[
E(v) := \int_G g \cdot v \, dg, \quad dg \text{ normalized Haar measure.} \quad (53)
\]

In turn the integral \( E(v) = \int_G g \cdot v \, dg \) is defined in dual coordinates by

\[
\langle \varphi \mid E(v) \rangle = \langle \varphi \mid \int_G g \cdot v \, dg \rangle := \int_G \langle \varphi \mid g \cdot v \rangle \, dg, \quad \forall \varphi \in R^*. \quad (54)
\]

The functions, of \( g \in G, \langle \varphi \mid g \cdot v \rangle, \varphi \in R^*, v \in R \) are called representative functions; therefore an explicit formula for \( E \) is equivalent to the knowledge of integration of representative functions. In fact usually the integral is computed by some algebraic method of computation of \( E \).

In the case of \( V = \mathbb{C}^d \) with natural basis \( e_i \) and dual basis \( e^i \).
We take \( R = \text{End}(V) \) with the conjugation action of \( GL(V) \) or of its compact subgroup \( U(d) \) of unitary \( d \times d \) matrices:

\[
Xe_{h,p}X^{-1} = \sum_{i,j} u_{i,h} \bar{u}_{j,p} e_{i,j}, \quad X = \sum_{i,j} u_{i,j} e_{i,j} \in U(d), \quad X^{-1} = \sum_{i,j} \bar{u}_{j,i} e_{i,j}.
\]

A basis of representative functions for \( R = \text{End}(V) \) is

\[
\text{tr}(e_{i,j} X e_{h,p} X^{-1}) = \text{tr}(e_{i,j} \sum_{a,b} u_{a,h} \bar{u}_{b,p} e_{a,b}) = u_{j,h} \bar{u}_{i,p}, \quad i, j, h, p = 1, \ldots, d. \quad (55)
\]

Since a duality between \( \text{End}(V)^{\otimes k} \) and itself is the non degenerate pairing:

\[
\langle A \mid B \rangle := \text{tr}(A \cdot B)
\]

a basis of representative functions of \( \text{End}(V)^{\otimes k} \) is formed by the products

\[
\text{tr}(e_{i_{1,j_{1}} \otimes e_{i_{2,j_{2}}} \ldots \otimes e_{i_{k,j_{k}}} \cdot X e_{h_{1,p_{1}}} X^{-1} \otimes X e_{h_{2,p_{2}}} X^{-1} \ldots \otimes X e_{h_{k,p_{k}}} X^{-1}} = \text{tr} \left( e_{i_{l,j_{l}}} \cdot X e_{h_{l,p_{l}}} X^{-1} \right) = \prod_{\ell=1}^{k} \text{tr}(e_{i_{l,j_{l}}} \cdot X e_{h_{l,p_{l}}} X^{-1}) = \prod_{\ell=1}^{k} u_{j_{\ell},h_{\ell}} \bar{u}_{i_{\ell},p_{\ell}}, \quad (56)
\]

where in order to have compact notations we write

\[
i_{\vec{l}} := (i_{1}, i_{2}, \ldots, i_{k}), \quad e_{i_{\vec{l}}} = e_{i_{1,j_{1}}} \otimes e_{i_{2,j_{2}}} \ldots \otimes e_{i_{k,j_{k}}} \quad (57)
\]

Collins writes the explicit Formula (64) for

\[
\int_{U(d)} \prod_{\ell=1}^{k} u_{j_{\ell},h_{\ell}} \bar{u}_{i_{\ell},p_{\ell}} \, du = \int_{U(d)} \text{tr} \left( e_{i_{\vec{l}}} \cdot X e_{h_{\vec{u}}} X^{-1} \right) \, dX = \text{tr} \left( e_{i_{\vec{l}}} \cdot E(e_{h_{\vec{u}}}) \right) \quad (58)
\]

In order to do this, it is enough to have an explicit formula for the equivariant projection \( E \) of \( \text{End}(V)^{\otimes k} \) to the \( GL(V) \) (or \( U(d) \)) invariants \( \Sigma_{k}(V) \), the algebra generated by the permutation operators \( \sigma \in S_{k} \) acting on \( V^{\otimes k} \).

His idea is to consider first the map

\[
\Phi : \text{End}(V)^{\otimes k} \rightarrow \Sigma_{k}(V), \quad \Phi(A) := \sum_{\sigma} \text{tr}(A \circ \sigma^{-1}) \sigma. \quad (59)
\]

This map is a \( GL(V) \) equivariant map to \( \Sigma_{k}(V) \), but it is not a projection. In fact restricted to \( \Sigma_{k}(V) \), we have

\[
\Phi : \Sigma_{k}(V) \rightarrow \Sigma_{k}(V), \quad \Phi(\tau) := \sum_{\sigma \in S_{k}} \text{tr}(\tau \circ \sigma^{-1}) \sigma.
\]

Setting \( \sigma = \gamma \tau, \quad \tau \sigma^{-1} = \gamma^{-1} \) we have:

\[
\Phi(\tau) = \sum_{\gamma \in S_{k}} \text{tr}(\gamma^{-1}) \gamma \tau = \Phi(1) \tau = \tau \Phi(1) = \tau \sum_{\gamma \in S_{k}} \text{tr}(\gamma^{-1}) \gamma. \quad (60)
\]
We have seen, in Corollary 1.13, that
\[ \Phi(1) = \sum_{\gamma \in S_k} tr(\gamma^{-1}) \gamma = \sum_{\gamma \in S_k} d^{c(\gamma)} \gamma \]
is a central invertible element of \( \Sigma_k(V) \). So the equivariant projection \( E \) is \( \Phi \) composed with multiplication by the inverse \( W_g(d, k) \) of the element \( \Phi(1) = \sum_{\gamma \in S_k} tr(\gamma^{-1}) \gamma \) given by Formula (22) or (18).

\[ E = (\sum_{\gamma \in S_k} tr(\gamma^{-1}) \gamma)^{-1} \circ \Phi = \Phi(1)^{-1} \circ \Phi = W_g(d, k) \circ \Phi. \]  \hspace{1cm} (61)

Of course
\[ \Phi(e_i) = \sum_{\sigma} tr(e_i \circ \sigma)^{-1} \sigma \]

\[ \implies E(e_i) = \sum_{\gamma, \sigma \in S_k} W_g(d, \gamma) \gamma \sum_{\sigma} tr(e_i \circ \sigma^{-1}) \sigma \]

and Formula (58) becomes
\[ tr(e_i \circ \sum_{\gamma \in S_k} W_g(d, \gamma) \gamma \sum_{\sigma} tr(e_i \circ \sigma^{-1}) \sigma) \]
\[ = \sum_{\gamma, \sigma \in S_k} tr(e_i \circ \gamma) tr(e_i \circ \sigma^{-1}) W_g(d, \gamma \sigma^{-1}) \]  \hspace{1cm} (62)
\[ E(e_i) = \sum_{\gamma, \sigma \in S_k} \delta_j^i \delta_{\tau(h)}(\gamma) \delta_{\tau(p)}(\sigma) W_g(d, \gamma \sigma^{-1}). \]  \hspace{1cm} (63)

From Formulas (7) and (8) since \( e_i, j = e_i \otimes e_j \) we have
\[ tr(e_1, i_1 \otimes e_2, j_2 \otimes e_k, j_k \circ \gamma) = \prod_{h} \langle e_{i_1(h)} | e^{j_k} \rangle = \prod_{h} \delta^{j_k}_{i_1(h)} \]
\[ = \sum_{\gamma, \sigma \in S_k} \delta^{i_1}_{\tau(h)}(\gamma) \delta^{p(h)}_{\tau(h)} W_g(d, \gamma \sigma^{-1}) \]
\[ = \sum_{\gamma, \sigma \in S_k} \delta^{i_1}_{\tau(h)}(\gamma) \delta^{p(h)}_{\tau(h)} W_g(d, \gamma \sigma^{-1}). \]  \hspace{1cm} (64)

**Remark 2.2.** In particular for \( i_\ell = h_\ell = p_\ell = \ell \) and \( j_\ell = \tau(\ell), \ 1 \leq \ell \leq k \), Formula (64) gives \( W_g(d, \tau) \).

Collins then goes several steps ahead since he is interested in the asymptotic behaviour of this function as \( d \to \infty \) and proves an asymptotic expression for any \( \sigma \) in term of its cycle decomposition, Theorem 1.20. \hfill \Box

**2.3 Tensor polynomials**

In work in progress with Felix Huber, [9], we consider the problem of understanding tensor valued polynomials of \( k, d \times d \) matrices.

That is maps from \( n \) tuples of \( d \times d \) matrices \( x_1, \ldots, x_n \in End(V) \) to tensor space \( End(V)^{\otimes k} \) of the form
\[ G(x_1, \ldots, x_n) = \sum_\iota \alpha_\iota m_{1,i_1} \otimes m_{2,i_2} \otimes \ldots \otimes m_{k,i_k}, \ \alpha_\iota \in \mathbb{C} \quad m_{j,i} \quad \text{monomials in the } x_i. \]

A particularly interesting case is when the polynomial is multilinear and alternating in \( n = d^2 \) matrix variables.

In this case, by Proposition 1.27 we have
Theorem 2.4. 1. 
\[ G(x_1, \ldots, x_d^2) = \text{det}(x_1, \ldots, x_d) \bar{J}_G. \]

2. Moreover we have the explicit formula 
\[ G(e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}, \ldots, e_{d,d}) = \bar{J}_G. \]

3. The element \( \bar{J}_G \in M_d^{\otimes k} \) is \( GL(k) \) invariant and so \( \bar{J}_G \in \Sigma_k(V) \) is a linear combinations of the elements of the symmetric group \( S_n \subset M_d^{\otimes k} \) given by the permutations.

For theoretical reasons instead of computing \( \bar{J}_G \) it is better to compute its multiple, as in Formula (52): 
\[ G(x_1, \ldots, x_d^2) = T_d(X) J_G, \quad \bar{J}_G = C_d J_G. \]

Using Formula (59) we may first compute 
\[ \Phi(G(x_1, \ldots, x_d^2)) = \sum_{\sigma \in S_k} \text{tr}(\sigma^{-1} \circ G(x_1, \ldots, x_d^2)) = T_d(X) \Phi(J_G). \]

Consider the special case 
\[ G_d(Y_1, \ldots, Y_d^2) := \text{Alt}_Y(m_1(Y) \otimes \cdots \otimes m_d(Y)), \quad m_i(Y) = Y_{(i-1)^2+1} \ldots Y_{i^2}. \]

**Lemma 2.5.** 
\[ \text{Alt}_Y \text{tr}(\sigma^{-1} \circ m_1(Y) \otimes \cdots \otimes m_d(Y)) = \begin{cases} T_d(Y) & \text{if } \sigma = 1 \\ 0 & \text{otherwise} \end{cases} \]

**Proof.** 
\[ \text{tr}(\sigma^{-1} \circ m_1(Y) \otimes \cdots \otimes m_d(Y)) = \prod_{i=1}^{j} \text{tr}(N_i) \]

with \( N_i \) the product of the monomials \( m_j \) for \( j \) in the \( i^{th} \) cycle of \( \sigma \), cf. Formula (6). The previous invariant gives by alternation the invariant 
\[ \text{Alt}_Y \prod_{i=1}^{j} \text{tr}(N_i) = T_{a_1} \wedge T_{a_2} \wedge \cdots \wedge T_{a_j}, \quad a_i = \text{degree of } N_i \]
in degree \( d^2 \). If \( \sigma \neq 1 \) we have \( j < d \) hence the product is 0, since the only invariant alternating in this degree is \( T_1 \wedge T_3 \wedge T_5 \wedge \cdots \wedge T_{2d-1} \).

On the other hand if \( \sigma = 1 \) we have \( N_i = m_i \) and the claim follows. \( \square \)

**Proposition 2.6.** We have 
\[ G_d(Y_1, \ldots, Y_d^2) := \text{Alt}_Y(m_1(Y) \otimes \cdots \otimes m_d(Y)) = T_d(Y) W_g(d,d). \]

**Proof.** The previous Lemma in fact implies that \( \Phi(G_d(Y_1, \ldots, Y_d^2)) = T_d(Y) 1_d \)

therefore \( \Phi(J_{G_d}) = (60) \Phi(1) J_{G_d} = 1 \) so that \( J_{G_d} = (\Phi(1))^{-1} = W_g(d,d). \) \( \square \)
2.7 The construction of Formanek

Let us now discuss a theorem of Formanek relative to a conjecture of Regev, see [6] or [1], that a certain explicit central polynomial $F(X,Y)$ in $d^2$, $d \times d$ matrix variables $X = \{X_1, \ldots, X_{d^2}\}$ and another $d^2$, $d \times d$ matrix variables $Y = \{Y_1, \ldots, Y_{d^2}\}$ is non zero. This polynomial plays an important role in the theory of polynomial identities, see [1].

The definition of $F(X,Y)$ is this, decompose $d^2 = 1 + 3 + 5 + \ldots + (2d-1)$ and accordingly decompose the $d^2$ variables $X$ and the $d^2$ variables $Y$ in the two lists. Construct the monomials $m_i(X), i = 1, \ldots, d$ and similarly $m_i(Y)$ as product in the given order of the given $2i-1$ variables $X_i$ of the $i^{th}$ list as for instance

$$m_1(X) = X_1, m_2(X) = X_2X_3X_4, m_3(X) = X_5X_6X_7X_8X_9, \ldots$$

$$m_i(X) = X_{(i-1)^2+1} \ldots X_{i^2}, \quad m_i(Y) = Y_{(i-1)^2+1} \ldots Y_{i^2}.$$ 

We finally define

$$F(X,Y) := \text{Alt}_X \text{Alt}_Y (m_1(X)m_1(Y)m_2(X)m_2(Y) \ldots m_d(X)m_d(Y)), \quad (69)$$

where $\text{Alt}_X$ (resp. $\text{Alt}_Y$) is the operator of alternation, Formula (47), in the variables $X$ (resp. $Y$). By Theorem 2.4 it takes scalar values, a multiple of $T_d(X)T_d(Y)$, but it could be identically 0.

Theorem 2.8.

$$F(X,Y) = (-1)^{d-1} \frac{1}{(dl)^2(2d-1)} T_d(X)T_d(Y)Id_d \quad (70)$$

$$= (-1)^{d-1} \frac{C_d^2}{(dl)^2(2d-1)} \Delta(X) \Delta(Y)Id_d; \quad \Delta(X) = \det(X_1, \ldots, X_{d^2}).$$

Notice that by Formula (52) the coefficient is an integer (as predicted).

Thus $F(X,Y)$ is a central polynomial. In fact it has also the property of being in the conductor of the ring of polynomials in generic matrices inside the trace ring. In other words by multiplying $F(X,Y)$ by any invariant we still can write this as a non commutative polynomial. This follows by polarizing in $z$ the identity, cf. [1] Proposition 10.4.9 page 286.

$$\det(z)^d F(X,Y) = F(zX,Y) = F(X,zY) = F(Xz,Y) = F(X,Yz).$$

Let us follow Formanek’s proof. First, since $F(x,y)$ is a central polynomial Formula (70) is equivalent to:

$$\text{tr}(F(X,Y)) = (-1)^{d-1} \frac{d}{(dl)^2(2d-1)} T_d(X)T_d(Y). \quad (71)$$
Now we have, with $\sigma_0 = (1, 2, \ldots, d)$ the cycle:

$$tr(F(X,Y)) = tr(\sigma_0^{-1} \circ \text{Alt}_X \text{Alt}_Y (m_1(X)m_1(Y) \otimes m_2(X)m_2(Y) \otimes \ldots \otimes m_d(X)m_d(Y)), \tag{72}$$

$$= tr(\sigma_0^{-1} \circ \text{Alt}_X (m_1(X) \otimes m_2(X) \otimes \ldots \otimes m_d(X) \cdot Wg(d,d)) \mathcal{T}_d(Y).$$

Denote $Wg(d,d) = \sum_{\tau \in S_d} a_{\tau} \tau$, we have

$$tr(\sigma_0^{-1} \circ \text{Alt}_X (m_1(X) \otimes m_2(X) \otimes \ldots \otimes m_d(X) \cdot Wg(d,d))$$

$$= \sum_{\tau} a_{\tau} tr(\sigma_0^{-1} \circ \text{Alt}_X (m_1(X) \otimes m_2(X) \otimes \ldots \otimes m_d(X)))$$

which, by Lemma 2.5 equals $a_{\sigma_0} \mathcal{T}_d(X)$. Therefore the main Formula (70) follows from Formula (23).

### 3 Appendix

If $k > d$ of course there is still an expression as in Formula (19) but it is not unique.

It can be made unique by a choice of a basis of $\Sigma_k(V)$. This may be done as follows.

**Definition 3.1.** Let $0 < d$ be an integer and let $\sigma \in S_n$.

Then $\sigma$ is called $d$–bad if $\sigma$ has a descending subsequence of length $d$, namely, if there exists a sequence $1 \leq i_1 < i_2 < \cdots < i_d \leq n$ such that $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_d)$. Otherwise $\sigma$ is called $d$–good.

**Remark 3.2.** $\sigma$ is $d$–good if any descending sub-sequence of $\sigma$ of length $\leq d - 1$. If $\sigma$ is $d$–good then $\sigma$ is $d'$–good for any $d' \geq d$.

Every permutation is $1$–bad.

**Theorem 3.3.** If $\dim(V) = d$ the $d + 1$–good permutations form a basis of $\Sigma_k(V)$.

**Proof.** Let us first prove that the $d + 1$–good permutations span $\Sigma_{k,d}$.

So let $\sigma$ be $d + 1$–bad so that there exist $1 \leq i_1 < i_2 < \cdots < i_{d+1} \leq n$ such that $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_{d+1})$. If $A$ is the antisymmetrizer on the $d + 1$ elements $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_{d+1})$ we have that $A \sigma = 0$ in $\Sigma_k(V)$, that is, in $\Sigma_{k,d}(V)$, $\sigma$ is a linear combination of permutations obtained from the permutation $\sigma$ with some proper rearrangement of the indices $\sigma(i_1), \sigma(i_2), \ldots, \sigma(i_{d+1})$. These permutations are all lexicographically $< \sigma$. One applies the same algorithm to any of these permutations which is still $d + 1$–bad. This gives an explicit algorithm which stops when $\sigma$ is expressed as a linear combination of $d + 1$–good permutations (with integer coefficients so that the algorithm works in all characteristics).

In order to prove that the $d + 1$–good permutations form a basis, it is enough to show that their number equals the dimension of $\Sigma_{k,d}$. This is insured by a classical result of Schensted which we now recall. \qed
3.3.1 The RSK and d-good permutations

The RSK correspondence\(^3\), see [12], [19], is a combinatorially defined bijection \(\sigma \leftrightarrow (P_\lambda, Q_\lambda)\) between permutations \(\sigma \in S_n\) and pairs \(P_\lambda, Q_\lambda\) of standard tableaux of same shape \(\lambda\), where \(\lambda \vdash n\).

In fact more generally it associates to a word, in the free monoid, a pair of tableaux, one standard and the other semistandard filled with the letters of the word. This correspondence may be viewed as a combinatorial counterpart to the Schur–Weyl and Young theory.

The correspondence is based on a simple game of inserting a letter.

We have some letters piled up so that lower letters appear below higher letters and we want to insert a new letter \(x\). If \(x\) fits on top of the pile we place it there otherwise we go down the pile, until we find a first place where we can replace the existing letter with \(x\). We do this and expel that letter, first creating a new pile or, if we have a second pile of letters then we try to place that letter there and so on.

So let us pile inductively the word strange.

\[
e \mapsto e, \quad g \mapsto g, \quad n \mapsto n, \quad r \mapsto r, \quad t \mapsto t, \quad s \mapsto s
\]

Notice that, as we proceed, we can keep track of where we have placed the new letter, we do this by filling a corresponding tableau.

\[
\begin{array}{ccccccc}
6 & & & & & & s \\
5 & & & & & r & \\
3 & & & n & & & \\
2 & 7 & g & t & & & \\
1 & 4 & a & e & & & \\
\end{array}
\]

It is not hard to see that from the two tableaux one can decrypt the word we started from giving the bijective correspondence.

Assume now that \(\sigma \leftrightarrow (P_\lambda, Q_\lambda)\), where \(P_\lambda, Q_\lambda\) are standard tableaux, given by the RSK correspondence. By a classical theorem of Schensted [17], \(ht(\lambda)\) equals the length of a longest decreasing subsequence in the permutation \(\sigma\). Hence \(\sigma\) is \(d+1\)-good if and only if \(ht(\lambda) \leq d\).

Now \(M_\lambda\) has a basis indexed by standard tableaux of shape \(\lambda\), see [16]. Thus the algebra \(\Sigma_k(V)\) has a basis indexed by pairs of tableaux of shape \(\lambda\). \(ht(\lambda) \leq d\) and the claim follows.

Therefore one may define the Weingarten function for all \(k\) as a function on the \(d+1\)-good permutations in \(S_k\).

\(^3\)Robinson, Schensted, Knuth
3.3.2 Cayley’s Ω process

It may be interesting to compare this method of computing the integrals of Formula (61) with a very classical approach used by the 19th century invariant theorists.

Let me recall this for the modern readers. Recall first that, given a \( d \times d \) matrix \( X = (x_{i,j}) \), its adjugate is \( \wedge^{d-1}(X) = (y_{i,j}) \) with \( y_{i,j} \) the cofactor of \( x_{j,i} \) that is \((-1)^{i+j} \) times the determinant of the minor of \( X \) obtained by removing the \( j \) row and \( i \) column. Then the inverse of \( X \) equals \( \det(X)^{-1} \wedge^{d-1}(X) \).

It is then easy to see that, substituting to \( u_{i,j} \) the variables \( x_{i,j} \) and to \( \bar{u}_{i,j} \) the polynomial \( y_{i,j} \) one transforms a monomial \( M = \prod_{k=1}^{\infty} u_{j,\ell} \overline{u}_{i,\ell} \) into a polynomial \( \pi_d(M) \) in the variables \( x_{i,j} \) homogeneous of degree \( dk \), the invariants under \( U_d \) become powers \( \det(X)^{k} \). Denote by \( S^{kd}(x_{i,j}) \) the space of these polynomials which, under the action of \( GL(d) \times GL(d) \), decomposes by Cauchy formula, cf. Formula 6.18, page 178, of [1]. Then we have also an equivariant projection from these polynomials to the 1–dimensional space spanned by \( \det(X)^{k} \), it is given through the Cayley Ω process used by Hilbert in his famous work on invariant theory. The Ω process is the differential operator given by the determinant of the matrix of derivatives:

\[
X = (x_{i,j}), \quad Y = \left( \frac{\partial}{\partial x_{i,j}} \right), \quad \Omega := \det(Y). \tag{73}
\]

We have that \( \Omega^k \) is equivariant under the action by \( SL(n) \) so it maps to 0 all the irreducible representations different from the 1–dimensional space spanned by \( \det(X)^k \) while

\[
\Omega \det(X)^k = k(k+1) \ldots (k+d-1) \det(X)^{k-1}.
\]

Both statements follow from the Capelli identity, see [16] §4.1 and [2].

\[
\det(X)\Omega = \det(a_{i,j}), \quad a_{i,i} = \Delta_{i,i} + n - i, \quad a_{i,j} = \Delta_{i,j}, \quad i \neq j
\]

the polarizations \( \Delta_{i,j} = \sum_{h=1}^{d} x_{i,h} \frac{\partial}{\partial x_{h,j}} \).

If we denote by \( \underline{x}_i := (x_{i,1}, \ldots, x_{i,n}) \) we have the Taylor series for a function \( f(\underline{x}_1, \ldots, \underline{x}_n) \) of the vector coordinates \( \underline{x}_i \).

\[
f(\underline{x}_1, \ldots, \underline{x}_j + \lambda \underline{x}_i, \ldots, \underline{x}_n) = \sum_{k=0}^{\infty} \frac{(\lambda \Delta_{i,j})^k}{k!} f(\underline{x}_1, \ldots, \underline{x}_n).
\]

Thus

\[
\int_U M \, du = \frac{\Omega^k \pi_d(M)}{\prod_{i=1}^{k} (i(i+1)\ldots(i+d-1))}. \tag{74}
\]
We can use Remark 2.2 to give a possibly useful formula:

\[
W_g(d, \gamma) = \frac{\Omega^k \pi_d(M)}{\prod_{i=1}^{k} (i(i+1) \ldots (i+d-1))}, \quad M = \prod_{i=1}^{k} u_i i^i \bar{u}_{i, \gamma(i)}.
\]  

Let me discuss a bit some calculus with these operators.

**Lemma 3.4.** If \( i \neq j \) then \( \Delta_{ij} \) commutes with \( \Omega \) and with \( \det(X) \) while

\[
[\Delta_{ii}, \det(X)] = \det(X), \quad [\Delta_{ii}, \Omega] = -\Omega.
\]

**Proof.** The operator \( \Delta_{ij} \) commutes with all of the columns of \( \Omega \) except the \( i^{th} \) column \( \omega_i \) with entries \( \frac{\partial}{\partial x_{it}} \). Now \( [\Delta_{ij}, \frac{\partial}{\partial x_{it}}] = -\frac{\partial}{\partial x_{jt}} \), from which \( [\Delta_{ij}, \omega_i] = -\omega_j \). The result follows immediately. \( \square \)

Let us introduce a more general determinant, analogous to a characteristic polynomial. We denote it by \( C_m(\rho) = C(\rho) \) and define it as:

\[
\begin{pmatrix}
\Delta_{1,1} + m - 1 + \rho & \Delta_{1,2} & \ldots & \Delta_{1,m} \\
\Delta_{2,1} & \Delta_{2,2} + m - 2 + \rho & \ldots & \Delta_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta_{m-1,1} & \Delta_{m-1,2} & \ldots & \Delta_{m-1,m} \\
\Delta_{m,1} & \Delta_{m,2} & \ldots & \Delta_{m,m} + \rho
\end{pmatrix}.
\]

We have now a generalization of the Capelli identity:

**Proposition 3.5.**

\[
\Omega C(k) = C(k+1)\Omega, \quad \det(X)C(k) = C(k-1)\det(X)
\]

\[
\det(X)^k \Omega^k = C(-(k-1))C(-(k-2)) \ldots C(-1)C,
\]

\[
\Omega^k \det(X)^k = C(k)C(k-1) \ldots C(1).
\]

**Proof.** We may apply directly Formulas (76) and then proceed by induction. \( \square \)

Develop now \( C_m(\rho) \) as a polynomial in \( \rho \) obtaining an expression

\[
C_m(\rho) = \rho^m + \sum_{i=1}^{m} K_i \rho^{m-i}.
\]

Capelli proved, [2], that, as the elementary symmetric functions generate the algebra of symmetric functions so the elements \( K_i \) generate the center of the enveloping algebra of the Lie algebra of matrices.

In [16] Chapter 3, §5 it is also given the explicit formula, also due to Capelli, of the action of \( C_m(\rho) \) (as a scalar) on the irreducible representations which classically appear as primary covariants.
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