AdS braneworld with Backreaction

Neven Bilić*1,3 and Gary B. Tupper†2,4

1Rudjer Bošković Institute, 10002 Zagreb, Croatia
2Centre for Theoretical and Mathematical Physics, Department of physics, University of Cape Town, Rondebosch 7701, South Africa
3Departamento de Física, Universidade Federal de Juiz de Fora, 36036-330, Juiz de Fora, MG, Brazil
4Associate Member, National Institute for Theoretical Physics

Abstract

We review the tachyon model derived from the dynamics of a 3-brane moving in the AdS5 bulk. The bulk geometry is based on the Randall–Sundrum II model extended to include the radion. The effective tachyon Lagrangian is modified due to the backreaction of the brane on the bulk geometry.

1 Introduction

Braneworld cosmology is based on the scenario in which matter is confined on a brane moving in the higher dimensional bulk with only gravity allowed to propagate in the bulk [1, 2, 3, 4]. It is usually assumed that extra dimensions are compact and if their size is large enough compared to the Planck scale, such a scenario may explain the large mass hierarchy between the electroweak scale and the fundamental scale of gravity. The Randall–Sundrum solution [3] to the hierarchy problem is a five dimensional universe containing two four dimensional branes with opposite brane tensions separated in the fifth dimension: the observer’s brane is placed on the negative tension brane and the separation proposed is such that the strength of gravity on observer’s brane is equal to the observed four-dimensional Newtonian gravity. At the same time it was realized that the Randall–Sundrum model, as well as any similar braneworld model, may have interesting cosmological implications [5]. In particular, owing to the presence of an extra dimension and the AdS5 bulk cosmological constant related to the brane tension, the usual Friedmann equations are modified [6] so the model can have

*bilic@irb.hr
†gary.tupper@uct.ac.za
predictions different from the standard cosmology and is therefore subject to cosmological tests [7].

In the second Randall–Sundrum model (RSII) [4] the negative tension brane is pushed off to infinity in the fifth dimension and the Planck mass scale is determined by the curvature of the five-dimensional space-time rather than the size of the fifth dimension. Hence, the model provides an alternative to compactification [4]. In RSII the bulk metric is $\text{AdS}_5/Z_2$

$$ds^2_{(5)} = e^{-2ky}g^{\mu\nu}dx^\mu dx^\nu - dy^2$$

with the observer brane at $y = 0$ and a negative tension brane at the AdS horizon at $y = \infty$. The fifth dimension can be integrated out to obtain a purely four-dimensional action with a well defined value for the Planck mass of the order $m^2_{Pl} \simeq (kK_{(5)})^{-1}$.

The new degree of freedom corresponding to the fluctuations of the interbrane distance along the extra dimension implies the existence of a massless scalar field: the radion which may cause a distortion of the simple AdS$_5$ geometry. Besides, the correct description must also include matter on observers brane which also distorts the naive bulk geometry [8, 9] (see also [10]).

Various technical and phenomenological aspects of the radion have been extensively discussed. Goldberger and Wise [11] proposed a bulk scalar field propagating in the background solution of the metric that generates a potential that can stabilize the radion. The minimum of the potential can be arranged to give the desired value of the separation distance $d_5$ between the branes without fine-tuning of parameters. The mass and the wave function of the radion is determined including the back reaction of the bulk stabilization field on the metric [12], giving a typical radion mass of the order of the weak scale between 0.100 and 1 TeV and the strength of its coupling to the SM fields of the order of 1 TeV. Quite recently, it has been speculated that the evidence for the ”Higgs boson” recently found at CERN may in fact be the evidence for the radion [13].

In this paper we investigate the dynamics of a moving 3-brane in an extended second Randal Sundrum (RSII) model which includes the back reaction due to the radion field. A 3-brane moving in AdS$_5$ background of the RSII model behaves effectively as a tachyon with the inverse quartic potential. The RSII model may be extended to include the back reaction due to the radion field. Then we show that the tachyon Lagrangian is modified by the interaction with the radion and, as a consequence, the effective equation of state obtained by averaging over large scales describes a warm dark matter (DM).

## 2 Gravity in the bulk

Unless stated otherwise, we work in units $c = \hbar = 1$ and keep the Newton constant $G$ explicit. It is convenient to choose a coordinate system such that $g_{(5)\mu5} = 0$ with metric

$$ds^2_{(5)} = g_{(5)MN}(X)dX^M dX^N = \Psi^2(x, y)g_{\mu\nu}(x)dx^\mu dx^\nu - \phi^2(x, y)dy^2,$$
which admits Einstein spaces of constant 4-curvature. Using (2) the bulk action may be expressed as \[ S_{\text{bulk}} = \frac{1}{K(5)} \int d^5x \sqrt{g(5)} \left[ -R(5) \frac{1}{2} - \Lambda(5) \right] \]

\[ = \frac{1}{K(5)} \int d^4x \sqrt{-g} \int dy \left[ -\frac{R}{2} \Psi^2 \varphi - 3g^{\mu\nu}(\Psi \varphi)_{,\mu} \Psi_{,\nu} + 6 \frac{\Psi^2(\partial_y \Psi)^2}{\varphi} - \Lambda(5) \Psi^4 \varphi \right] \] (3)

The consistency with Einstein’s equations outside the brane requires

\[ R_{(5)\mu5} = 0. \] (4)

This leads to

\[ \Psi = \exp \left( \int dy \varphi \frac{\partial_y W}{W} \right) \] (5)

where the function \( W = W(y) \) is a background warp that does not depend on \( x \). A choice of \( \varphi \) (gauge choice) is basically the choice of parametrization of the distance along the fifth dimension at fixed \( x \). It is convenient to impose the gauge condition

\[ \Psi^2 \varphi = W^2 \] (6)

so that the coefficient of \( R \) in (3) is entirely fixed by the background. With this gauge condition we find \[ \Psi(x, y) = \left[ W^2(y) + \phi(x) \right]^{1/2}, \quad \varphi(x, y) = \frac{W^2(y)}{W^2(y) + \phi(x)}. \] (7)

where \( \phi(x) \) is a function of \( x \). This yields

\[ S_{\text{bulk}} = \frac{1}{K(5)} \int d^4x \sqrt{-g} \int dy \left\{ -\frac{R}{2} W^2 + \frac{3}{4} \frac{W^2}{(W^2 + \phi)^2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right. \]

\[ + \left. [6(\partial_y W)^2 - \Lambda(5)] (W^2 + \phi) \right\} \] (8)

In order to keep a close connection with the Randall-Sundrum models, we take

\[ W = e^{-ky} \text{ on } 0 \leq y \leq l. \] (9)

The bulk metric is then given by

\[ ds_{(5)}^2 = (e^{-2ky} + \phi) g^{\mu\nu} dx^\mu dx^\nu - \left( \frac{e^{-2ky}}{e^{-2ky} + \phi} \right)^2 dy^2 \] (10)

and the integration over \( y \) yields \footnote{Because the fifth dimension is \( S^1/Z \) the \( y \)-integrals \( \int_0^l dy \) are doubled.}
where we identified the four-dimensional Newton constant
\[
\frac{1}{8\pi G} = \frac{2}{K(5)} \int_0^l dy W^2 = \frac{1 - e^{-2kl}}{kK(5)}.
\] (12)

Then the function \(\omega\) is expressed as
\[
\omega(\phi) = 16\pi G \int_0^l dy \frac{W^2}{(W^2 + \phi)^2} = \frac{1}{(1 + \phi)(e^{-2kl} + \phi)},
\] (13)

and we use the abbreviation
\[
\tilde{k} = k - \frac{\Lambda(5)}{6k}.
\] (14)

The field \(\phi(x)\) dubbed “radion” parameterizes the interbrane distance at fixed \(x^\mu\)
\[
d_5 = \int_0^l dy \phi = \int_0^l dy \frac{W^2}{W^2 + \phi} = \frac{1}{2k} \ln \frac{1 + \phi}{e^{-2kl} + \phi},
\] (15)

so that the distance to the AdS horizon \(\lim_{l \to \infty} d_5\) remains finite. As in the RSII model, the metric (10) will be a solution to Einstein’s equations provided
\[
k^2 = -\frac{\Lambda(5)}{6},
\] (16)

where \(\Lambda(5)\) on the right-hand side is negative for AdS\(_5\).

The bulk action (11) may be further simplified. First, as we shall shortly see, the last term in curly brackets in (11) is canceled by the brane action if the RSII fine tuning is imposed. Second, the radion kinetic term may be brought to the standard form by introducing the canonically normalized radion \(\Phi\) via the transformation [8]
\[
\phi = (1 + e^{-2kl}) \sinh^2 \left( \sqrt{\frac{4\pi G}{3}} \Phi \right) + e^{-kl} \sinh \left( \sqrt{\frac{16\pi G}{3}} \Phi \right).
\] (17)

Then, the bulk action takes a simple form
\[
S_{\text{bulk}} = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} + \frac{1}{2} g^{\mu\nu} \Phi_{,\mu} \Phi_{,\nu} \right)
\] (18)

3 Brane action

Consider a 3-brane moving in the 4+1 bulk spacetime with metric (10). The points on the brane are parameterized by \(X^\mu(x^\mu)\), and \(g^{\text{ind}}_{\mu\nu} = g_{(5)MN} X^M X^N\) is the induced metric. Taking the Gaussian normal parameterization
\[
X^M(x^\mu) = (x^\mu, y(x^\mu))
\] (19)

we have
\[
g^{\text{ind}}_{\mu\nu} = \left( \frac{e^{-2ky} + \phi}{e^{-2ky} + \phi} \right)^2 \left[ \frac{(e^{-2ky} + \phi)^3}{(e^{-2ky})^2} g_{\mu\nu} - y_{,\mu} y_{,\nu} \right]
\] (20)
The brane action is then given by

$$S_{\text{brane}} = -\sigma \int d^4x \sqrt{-\det g_{\mu\nu}^{\text{ind}}} = -\sigma \int d^4x \sqrt{-g} (e^{-2ky} + \phi)^2 \left(1 - \frac{(e^{-2ky} + \phi)^2 g_{\mu\nu} y_{\mu}y_{\nu}}{(e^{-2ky} + \phi)^3 g_{\mu\nu}}\right)^{1/2}$$

(21)

From this we find the contribution of observer’s brane at \(y = 0\) and the negative tension brane at \(y = l\) as

$$S_{\text{brane}|y=0} + S_{\text{brane}|y=l} = -\sigma_0 \int d^4x \sqrt{-g} (1 + \phi)^2 - \sigma_l \int d^4x \sqrt{-g} (e^{-2kl} + \phi)^2$$

(22)

With the RSII fine tuning

$$\sigma_0 = -\sigma = \frac{3\tilde{k}}{K(5)} = \frac{6k}{K(5)}$$

(23)

the brane contributions cancel the last term on the right-hand side of (11).

Hence, the appearance of a massless mode - the radion - causes two effects. First, according to (20), matter on observers brane sees the (induced) metric

$$\tilde{g}_{\mu\nu} = g_{\mu\nu}^{\text{ind}|y=0} = (1 + \phi)g_{\mu\nu}$$

(24)

and second, the physical distance to the AdS\(_5\) horizon at coordinate infinity

$$d_5 = \frac{1}{2k} \ln \frac{1 + \phi}{\phi},$$

(25)

is no longer infinite if \(\phi \neq 0\). The physical size of the 5-th dimension is of the order \(1/k \sim l_{P_1}\) although its coordinate size is infinite.

### 3.1 Dynamical brane as a tachyon

Consider an additional 3-brane moving in the bulk with metric (10). In this case, the fifth coordinate is treated as a dynamical scalar field \(y(x)\). Changing \(y(x)\) to a new field

$$\theta(x) = e^{ky(x)}/k$$

(26)

from (21) we obtain [14]

$$S_{\text{brane}} = -\int d^4x \sqrt{-g} \frac{\sigma}{k^4\theta^4} (1 + k^2\theta^2\phi)^2 \sqrt{1 - \frac{g^{\mu\nu} \theta_{\mu} \theta_{\nu}}{(1 + k^2\theta^2\phi)^3}}$$

(27)

When \(\phi = 0\) we have the pure undistorted AdS\(_5\) and

$$S_{\text{brane}}^{(0)} = -\int d^4x \sqrt{-g} \frac{\sigma}{k^4\theta^4} \sqrt{1 - g^{\mu\nu} \theta_{\mu} \theta_{\nu}}$$

(28)

This action describes a tachyon with inverse quartic potential. A related model is discussed by Silverstein and Tong [15] where a D3-brane action is given by

$$S_{\text{D3}} = \int d^4x \sqrt{-g} \frac{\sigma}{k^4\theta^4} \left[1 - \sqrt{1 - g^{\mu\nu} \theta_{\mu} \theta_{\nu}}\right]$$

(29)
In this case, the pressure \( p = \mathcal{L} \) is positive definite so there is no dark energy resulting (at low “velocity” there is no force on the D-brane). Although in our case \( p < 0 \), the steep potential drives a dark matter attractor \([16]\) so \( p \to 0^- \) very quickly and this “tachyon dust” clusters efficiently on caustics \([17]\). One can get inflation or DE by adding a potential term \( V(\phi) \) to \([29]\) but that is somewhat ad-hoc. Reversing the brane charge (D3-brane) in \([29]\) gives \( p < 0 \) but the steepness of the potential remains an obstruction. What makes the tachyon intriguing is that even if \( k^{-1} \sim l_{Pl} \) as the AdS horizon is approached \( e^{2k_{y}} \) may be so large that \( \theta \) is \( \mathcal{O}(H^{-1}) \) without any fine tuning or dimensionful parameters.

On the other hand, if \( \phi \) is not strictly zero within \( L \), the tachyon can drive a transition from \( k^2 \theta^2 \phi \ll 1 \) regime to \( k^2 \theta^2 \phi \gg 1 \). In the latter regime the brane action \([27]\) takes the form

\[
S_{\text{brane}} \simeq -\int d^4x \sqrt{-g} \sigma \phi^2 \sqrt{1 - g_{\mu\nu} \bar{\theta}_{,\mu} \bar{\theta}_{,\nu}} \tag{30}
\]

where \( \bar{\theta}_{,\mu} = \theta_{,\mu} / (k^3 \theta^3 \phi^{3/2}) \). One sees an obvious similarity to the Chaplygin gas \([18, 19, 20, 21]\): The Hubble drag drives the brane velocity towards vanishing such that \( \sigma \phi^2 \) serves as a variable tension, or potential for \( \theta \) through an implicit dependence of \( \phi \) on \( \theta \). The latter is similar to ”quartessence” \([22]\), the model for DE/DM unification. Although not a single field model, this two component model has a potential to give both DE and DM out of a single geometric structure.

### 3.2 Pressureless matter on the \( y = 0 \) brane

If matter is placed on the \( y = 0 \) brane, its action is

\[
S_{\text{matt}} = \int d^4x \sqrt{-\tilde{g}} \mathcal{L}_{\text{matt}} \tag{31}
\]

where \( \tilde{g} \) is the determinant of the metric \([24]\) induced on the \( y = 0 \) brane. Pressureless matter can be modeled using a complex scalar field. Consider a Lagrangian of the type

\[
\mathcal{L}_{\text{matt}} = g^{\mu\nu} \Psi^*_{,\mu} \Psi_{,\nu} - V(m^2 |\Psi|^2) \tag{32}
\]

for a complex scalar field

\[
\Psi = \frac{\varphi}{\sqrt{2m}} \exp(-im\chi) \tag{33}
\]

where \( m \) is the mass appearing in the potential \( V \). In the Thomas-Fermi approximation \([19]\) the Lagrangian \([32]\) becomes

\[
\mathcal{L}_{\text{mattTF}} = \frac{\varphi^2}{2} g^{\mu\nu} \chi_{,\mu} \chi_{,\nu} - V(\varphi^2/2). \tag{34}
\]

with the equations of motion for the fields \( \varphi \) and \( \chi \)

\[
g^{\mu\nu} \chi_{,\mu} \chi_{,\nu} = V'(\varphi^2/2), \tag{35}
\]

\[
(\sqrt{-g} \varphi^2 g^{\mu\nu} \chi_{,\mu})_{,\nu} = 0, \tag{36}
\]
where $V'(x) = dV/dx$. Assuming $V' > 0$, the field $\chi$ may be treated as a velocity potential for the fluid 4-velocity

$$u^\mu = g^{\mu\nu} \chi_{,\nu}/\sqrt{V'},$$

(37)

As a consequence, the stress-energy tensor $T^{\mu\nu}$ constructed from the Lagrangian (32) takes the perfect fluid form, with the parametric equation of state

$$\rho = \frac{\varphi^2}{2} V' + V, \quad p = \frac{\varphi^2}{2} V' - V.$$

(38)

Now we assume $p = 0$. In this case we obtain an equation

$$\frac{\varphi^2}{2} V' = V.$$

(39)

with solution

$$V = \frac{1}{2} m^2 \varphi^2.$$

(40)

Defining a new field $\alpha = m^2 \varphi^2$, redefining $\chi \rightarrow m \chi$, and replacing $g^{\mu\nu} \rightarrow \tilde{g}^{\mu\nu} = g^{\mu\nu}(1 + \phi)^{-1}$ we finally obtain the Lagrangian for pressureless matter as

$$\mathcal{L}_{\text{matt}} = \alpha \left[ (1 + \phi)^{-1} g^{\mu\nu} \chi_{,\mu} \chi_{,\nu} - 1 \right]$$

(41)

and the matter action as

$$S_{\text{matt}} = \int d^4x \sqrt{-g} \alpha \left[ (1 + \phi) g^{\mu\nu} \chi_{,\mu} \chi_{,\nu} - (1 + \phi)^2 \right].$$

(42)

The field $\alpha$ is not dynamical and, as we shall shortly see, will be eliminated from the field equations.

4 Backreaction

The total action as seen on observer’s brane is

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{matt}},$$

(43)

where $S_{\text{bulk}}$, $S_{\text{brane}}$, and $S_{\text{matt}}$ are defined in (18), (27), and (42), respectively. For the moment we ignore the pressureless matter on observer’s brane and let $l \rightarrow \infty$. Then, the relation (17) between $\phi$ and $\Phi$ becomes

$$\phi = \sinh^2 \left( \sqrt{4\pi G/3} \Phi \right),$$

(44)

and the Newton constant defined in (12) is simply related to the bulk gravitational constant as

$$8\pi G = kK_{(5)}.$$ 

(45)

From now on we work in units $8\pi G = 1$. It is convenient to replace $\theta$ with a new field

$$\Theta(x) = 3e^{-2k\theta(x)} = \frac{3}{k^2 \theta(x)^2}$$

(46)
and introduce new constants
\[ \lambda = \sigma / (6k^2), \quad \ell = \sqrt{6} / k. \] (47)

Then the combined radion and brane Lagrangian becomes
\[ \mathcal{L} = \frac{1}{2} X - \frac{\lambda}{\ell^2} \psi^2 \sqrt{1 - \ell^2 Y / \psi^3} \] (48)

where we have used the abbreviations
\[ X = g^{\mu\nu} \Phi_{,\mu} \Phi_{,\nu}, \quad Y = g^{\mu\nu} \Theta_{,\mu} \Theta_{,\nu}, \] (49)

and
\[ \psi = 2 \Theta + 6 \sinh^2 \left( \sqrt{\frac{1}{6}} \Phi \right), \] (50)

The energy-momentum tensor corresponding to the above Lagrangian
\[ T_{\mu\nu} = 2 \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} - \mathcal{L} g_{\mu\nu} = \Phi_{,\mu} \Phi_{,\nu} + \frac{\lambda}{\ell^2 \psi} \sqrt{1 - \ell^2 Y / \psi^3} \Theta_{,\mu} \Theta_{,\nu} - \mathcal{L} g_{\mu\nu}, \] (51)

may be expressed as a sum of two components
\[ T_{\mu\nu} = T_{1\mu\nu} + T_{2\mu\nu} \] (52)

each representing a perfect fluid with
\[ T_{i\mu\nu} = (p_i + \rho_i) u_{i\mu} u_{i\nu} - p_i g_{\mu\nu}, \quad i = 1, 2. \] (53)

The corresponding velocities, pressures and densities are given by
\[ u_{1\mu} = \frac{\Phi_{,\mu}}{\sqrt{X}}, \quad u_{2\mu} = \frac{\Theta_{,\mu}}{\sqrt{Y}}, \] (54)
\[ p_1 = \frac{1}{2} X, \quad p_2 = -\frac{\lambda \psi^2}{\ell^2} \sqrt{1 - \ell^2 Y / \psi^3}, \] (55)
\[ \rho_1 = \frac{1}{2} X, \quad \rho_2 = \frac{\lambda \psi^2}{\ell^2} \frac{1}{\sqrt{1 - \ell^2 Y / \psi^3}}, \] (56)

### 4.1 Conjugate fields

\( \mathcal{L} \) and \( T_{\mu\nu} \) may be expressed in terms of the conjugate fields (or conjugate “momenta”) \( \pi^\mu_\Phi \) and \( \pi^\mu_\Theta \) defined as
\[ \pi^\mu_\Phi = \frac{\partial \mathcal{L}}{\partial \Phi_{,\mu}} = g^{\mu\nu} \Phi_{,\nu}, \] (57)
\[ \pi^\mu_\Theta = \frac{\partial \mathcal{L}}{\partial \Theta_{,\mu}} = \frac{\lambda}{\psi} g^{\mu\nu} \Theta_{,\nu} \sqrt{1 - \ell^2 Y / \psi^3}, \] (58)
For timelike $\Phi, \mu$ and $\Theta, \mu$ we may also define the norms

$$\pi_\Phi = \sqrt{g_{\mu\nu} \pi^\mu_{\Phi} \pi^\nu_{\Phi}}, \quad \pi_{\Theta} = \sqrt{g_{\mu\nu} \pi^\mu_{\Theta} \pi^\nu_{\Theta}}.$$ \hspace{1cm} (59)

Using these equations one finds a useful expression

$$1 - \ell^2 \frac{Y}{\psi^3} = \frac{1}{1 + \ell^2 \pi^2_{\Theta}/(\lambda^2 \psi)}.$$ \hspace{1cm} (60)

Using (57)-(60) we obtain

$$L = \frac{1}{2} \pi^2_{\Phi} - \frac{\lambda \psi^2}{\ell^2} \sqrt{1 + \ell^2 \pi^2_{\Theta}/(\lambda^2 \psi)},$$ \hspace{1cm} (61)

$$T_{\mu\nu} = \pi_{\Phi,\mu} \pi_{\Phi,\nu} + \frac{\ell^2 \psi}{\lambda} \pi_{\Theta,\mu} \pi_{\Theta,\nu} - g_{\mu\nu} \mathcal{L}.$$ \hspace{1cm} (62)

and

$$p_1 = \frac{1}{2} \pi^2_{\Phi}; \quad p_2 = -\frac{\lambda \psi^2}{\ell^2} \frac{1}{\sqrt{1 + \ell^2 \pi^2_{\Theta}/(\lambda^2 \psi)}},$$ \hspace{1cm} (63)

$$\rho_1 = \frac{1}{2} \pi^2_{\Phi}; \quad \rho_2 = \frac{\lambda \psi^2}{\ell^2} \sqrt{1 + \ell^2 \pi^2_{\Theta}/(\lambda^2 \psi)}.$$ \hspace{1cm} (64)

The same expression for $T_{\mu\nu}$ is obtained by making use of the canonical definition

$$\mathcal{T}_{\mu\nu}^\text{can} = \sum_{\varphi, \pi} \varphi_{,\mu} \pi_{,\nu} - \mathcal{L} g_{\mu\nu}.$$ \hspace{1cm} (65)

### 4.2 Hamilton’s equations

The Hamiltonian may be identified with the total energy density

$$\mathcal{H} = T^\mu_{\mu} + 3 \mathcal{L} = \rho_1 + \rho_2,$$ \hspace{1cm} (66)

which yields

$$\mathcal{H} = \frac{1}{2} \pi^2_{\Phi} + \frac{\lambda \psi^2}{\ell^2} \sqrt{1 + \ell^2 \pi^2_{\Theta}/(\lambda^2 \psi)}.$$ \hspace{1cm} (67)

The Hamiltonian $\mathcal{H}$ (defined in (67) as a function of $\pi^\mu_{\Phi}, \pi^\mu_{\Theta}, \Phi, \text{and } \Theta$) is related to $\mathcal{L}$ (defined in (48) as a function of $\Phi, \mu, \Theta, \mu, \Phi, \Theta$) through the Legendre transformation

$$\mathcal{H}(\pi^\mu, \varphi) = \sum_{\{\pi, \varphi\}} \pi^\mu \varphi_{,\mu} - \mathcal{L}(\varphi_{,\mu}, \varphi),$$ \hspace{1cm} (68)

where

$$\varphi_{,\mu} = \frac{\partial \mathcal{H}}{\partial \pi^\mu},$$ \hspace{1cm} (69)

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \varphi_{,\mu}}.$$ \hspace{1cm} (70)
In (68)-(70) \( \pi \) stands for \( \pi_\Phi \) or \( \pi_\Theta \), and \( \varphi \) stands for \( \Phi \) or \( \Theta \). The first pair of Hamilton’s equations is obtained by multiplying (69) by \( u_1^\mu \) and \( u_2^\mu \) for \( \Phi \) and \( \Theta \) fields, respectively.

From (67) we derive

\[
\dot{\pi}_1 \equiv \dot{\Phi} = \frac{\partial H}{\partial \pi_\Phi}, \quad \dot{\pi}_2 \equiv \dot{\Theta} = \frac{\partial H}{\partial \pi_\Theta}.
\]  

(71, 72)

The remaining two Hamilton’s equations are obtained by applying the covariant divergence to (70) and using the Euler-Lagrange equations

\[
\frac{\partial \mathcal{L}}{\partial \varphi} = \left( \frac{\partial \mathcal{L}}{\partial \varphi, \mu} \right)_{;\mu}.
\]  

(73)

Then, with the help of (54) we find

\[
\dot{\pi}_1 + 3H_1\pi_1 = -\frac{\partial H}{\partial \Phi},
\]  

\[
\dot{\pi}_2 + 3H_2\pi_2 = -\frac{\partial H}{\partial \Theta},
\]  

(74, 75)

The quantities \( H_i, i = 1, 2 \), are related to the expansions of \( u_i \)

\[
3H_i = u_i^{\mu \mu}.
\]  

(76)

The set of equations (71), (72), (74), and (75) are solved assuming spatially flat FRW spacetime, in which case

\[
H_1 = H_2 = H
\]  

(77)

where \( H \) is the Hubble expansion rate.

For a more complete description we add to the total Lagrangian the contribution of pressureless matter on the observer’s brane (41). In this case, there is an additional contribution to the Hamiltonian

\[
\mathcal{H}_\chi = \frac{\pi_\chi^2}{2\alpha(1 + \phi)} + \frac{\alpha}{2}(1 + \phi)^2.
\]  

(78)

where

\[
\pi_\chi = \sqrt{g_{\mu\nu}\pi_\chi^\mu\pi_\chi^\nu}.
\]  

(79)

and \( \pi_\chi^\mu \) is the conjugate momentum of the field \( \chi \). The non-dynamical field \( \alpha \) can be eliminated by the Hamilton’s equation

\[
\frac{\partial \mathcal{H}}{\partial \alpha} = 0,
\]  

(80)

which follows from the Euler-Lagrange equation \( \partial \mathcal{L}/\partial \alpha = 0 \) and (68). Then we find the total Hamiltonian

\[
\mathcal{H} = \frac{1}{2}\pi_\phi^2 + \frac{\lambda\psi^2}{\ell^2} \sqrt{1 + \ell^2\pi_\Theta^2/(\lambda^2\psi)} + \pi_\chi \sqrt{1 + \phi},
\]  

(81)
and we have two additional Hamilton’s equations
\[\dot{\chi} = \frac{\partial H}{\partial \pi_\chi},\]  
\[\dot{\pi}_\chi + 3H\pi_\chi = 0.\]  
(82) (83)

Finally, from (71)–(75) and (82)–(83) using (81) we obtain the following set of equations
\[\dot{\Phi} = \pi_\Phi,\]  
\[\dot{\Theta} = \psi \lambda \pi_\Theta \sqrt{1 + \ell^2 \pi^2_\Theta / (\lambda^2 \psi)} - \frac{3}{\ell^2 \lambda} \left( \frac{4 \lambda^2 \psi + 3 \ell^2 \pi^2_\Theta}{\sqrt{1 + \ell^2 \pi^2_\Theta / (\lambda^2 \psi)}} \right) \phi' - \pi_\chi \sqrt{\frac{1}{6} \phi} \]  
\[\dot{\pi}_\Theta = -3H\pi_\Theta - \frac{1}{\ell^2 \lambda} \left( \frac{4 \lambda^2 \psi + 3 \ell^2 \pi^2_\Theta}{\sqrt{1 + \ell^2 \pi^2_\Theta / (\lambda^2 \psi)}} \right) \]  
\[\dot{\pi}_\chi = -3H\pi_\chi,\]  
(84) (85) (86) (87) (88) (89)

together with the Friedmann equation for the scale \(a(t)\)
\[\frac{\dot{a}}{a} = H = \sqrt{\frac{1}{3} \mathcal{H}}\]  
(90)
where \(\phi\) is defined in (44) and
\[\phi' = \sqrt{\frac{1}{6} \sinh \left( \sqrt{\frac{2}{3}} \Phi \right)}\]  
(91)
Equation (89) is easily solved for \(a\)
\[\pi_\chi = \frac{\pi_\chi_0}{a^3},\]  
(92)
where \(\pi_\chi_0\) is a constant which could be fixed by physics. For example, we may require that the fraction of dust (which represents baryons) today is about 0.05\(\rho_{cr}\). More precisely, at \(t = t_0\) when \(a(t_0) = 1\) we require
\[\rho_\chi(t_0) \equiv \pi_\chi_0 \sqrt{1 + \phi(t_0)} = 0.05 \frac{3}{8 \pi G} H_0^2.\]  
(93)
5 Numerical results

To exhibit the main features we neglect the dust on observer’s brane and solve our equations assuming spatially flat FRW spacetime with line element
\[ ds^2 = dt^2 + a(t)^2(dr^2 + r^2dΩ^2) \] (94)

We evolve the radion-tachyon system in time measured in units of \( \ell \) and we take \( \lambda\ell^2 = 1/3 \). Equations (84)-(88) are integrated starting from \( t = 0 \) with the initial conditions \( \Theta = 1.01, \phi = 0.1, \pi_\Phi = \pi_\Theta = 0.00001 \). The results of integration are depicted in Figs. 1, 2, and 3. As one would anticipate from (84) with (87), the field \( \Phi \) undergoes damped oscillations with the amplitude decreasing as \( 1/t \) (Fig. 1). In the asymptotic region one finds an approximate solution
\[ \Phi = \frac{A}{t} \cos \frac{2t}{\ell} \] (95)

where \( A \) is the amplitude of the asymptotic oscillations. Comparing (95) with the exact solution for \( \Phi \) depicted in Fig. 1 we find \( A = 0.1518 \).

As a consequence, the original tachyon field \( \theta \) exhibits oscillations about a linear function which corresponds to a tachyon solution without radion (Fig. 2). Similar oscillations are seen in the momentum field
\[ \pi_\theta = -\ell\pi_\Theta(2\Theta)^{-3/2} \] (96)
conjugate to \( \theta \) (Fig. 3). To exhibit the oscillating behavior more clearly we have plotted \( \pi_\theta \) multiplied by \( a^3 \).

After the transient period the equation of state \( w = p/\rho \) becomes positive and oscillatory (Fig. 4). In the asymptotic regime \( t \to \infty \) we find an approximate expression
\[ w \simeq \frac{\dot{\phi}^2}{\phi^2 + 2\psi^{3/2} |\pi_\Theta|/\ell} \] (97)
which yields

\[ w \simeq \frac{3}{2} A^2 \sin^2 \frac{2t}{\ell} \]  

(98)
to leading order in the amplitude \( A \). Since the oscillations in \( w \) are rapid on cosmological timescales, it is most useful to time average co-moving quantities. The effective equation of state is then

\[ \langle p \rangle = \langle w \rangle \langle \rho \rangle , \]  

(99)
where \( \langle x \rangle \) denotes the time average of the quantity \( x \). By averaging (98) over long timescales we find

\[ \langle w \rangle \simeq \frac{3}{4} A^2 = 0.017 \]  

(100)
This estimate hints at the analysis of Avelino et al.\(^{23}\) who have recently shown that cosmological data favor a dark matter equation of state \( w_{\text{DM}} \approx 0.01 \) rather then a pressureless, or cold dark matter equation of state.

The nature of dark matter (DM) is still an open question. In spite of the large-scale successes of cold DM there is still some unresolved issues such as overproduction of small scale structure and halos with a central cusp.\(^{24}\) These problems are somewhat alleviated by warm DM and in particular by sterile neutrino warm DM.\(^{25, 26}\) However, a recent analysis\(^ {28}\) shows that a realistic warm DM scenario with \( m_{\text{DM}} \simeq 4 \text{ keV} \) in agreement with recent constraints from Lyman-\( \alpha \) forest\(^ {27}\) is not able to alleviate the small scale crisis of cold DM structure formation.

It is easy to demonstrate that the equation of state (100) may be associated with warm DM. We assume that our equation of state corresponds to that of DM thermal relics of mass \( m_{\text{DM}} \) at the time of radiation-matter equality \( t_{\text{eq}} \). Furthermore, assuming that DM particles constitute a non-relativistic gas at \( t \sim t_{\text{eq}} \), the corresponding equation of state is, to a good
approximation, given by

$$w_{\text{DM}} = \frac{T}{m_{\text{DM}}},$$  \hspace{1cm} (101)

where $T$ is the temperature of the gas. Taking $T = T_{\text{eq}} = 7.4$ eV at $t = t_{\text{eq}}$ and identifying $w_{\text{DM}}|_{\text{eq}} = < w > = 0.017$, we obtain $m_{\text{DM}} \simeq 430$ eV. These DM particles become non relativistic at the time when $T = T_{\text{NR}} \simeq m_{\text{DM}}$ corresponding to the cosmological scale

$$a_{\text{NR}} \simeq \frac{T_{\text{eq}}}{T_{\text{NR}}} = < w > a_{\text{eq}}$$  \hspace{1cm} (102)

We next show that the horizon mass at the time when the equivalent DM particles just become non-relativistic is of the order typically of a small galaxy. The horizon mass before equality evolves as [29]

$$M_{\text{H}} \simeq M_{\text{eq}} \left( \frac{a}{a_{\text{eq}}} \right)^{3},$$  \hspace{1cm} (103)

where $M_{\text{eq}} \simeq 2 \times 10^{15}M_{\odot}$ for a spatially flat universe. Thus, at $a = a_{\text{NR}}$ we obtain $M_{\text{H}} \simeq 10^{10}M_{\odot}$, the mass scale typical of a small galaxy and therefore the DM may be qualified as warm.

We have restricted attention to a homogeneous isotropic evolution for simplicity. A warm DM model in general has a non-vanishing sound speed and hence may face the problem of the well-known Jeans instability. The perturbations of the scale smaller than the sonic horizon will be prevented from growing. In our case, one cannot interpret $\sqrt{\langle w \rangle}$ as the adiabatic speed of perturbations. Note also that the quantity $\ddot{p}/\dot{\rho}$ cannot be identified with the speed of sound squared $c_s^2$ because $\ddot{p}/\dot{\rho}$ is, in our case, not positive semi-definite owing to interactions. The non-interacting radion is stiff matter, with unit speed of sound, whereas
The non-interacting tachyon asymptotically has vanishing speed of sound. As we have shown in appendix A the sound speed squared for the composite is the sum of the components weighted by their fraction of $\rho + p$:

$$c_s^2 = c_{s1}^2 \frac{X \mathcal{L}_X}{X \mathcal{L}_X + Y \mathcal{L}_Y} + c_{s2}^2 \frac{Y \mathcal{L}_Y}{X \mathcal{L}_X + Y \mathcal{L}_Y}$$  \hspace{1cm} (104)$$

where $c_{s1}$ and $c_{s2}$ are defined as

$$c_{s1}^2 = \frac{\mathcal{L}_X}{\mathcal{L}_X + 2X \mathcal{L}_{XX}}; \quad c_{s2}^2 = \frac{\mathcal{L}_Y}{\mathcal{L}_Y + 2Y \mathcal{L}_{YY}}. \hspace{1cm} (105)$$

The expression (104) agrees with the speed of sound for a multicomponent fluid defined in [32].

For the Lagrangian (48) we have $\mathcal{L}_X = 1/2$, $\mathcal{L}_{XX} = 0$, and

$$\mathcal{L}_Y = \frac{\lambda}{2\psi} \frac{1}{\sqrt{1 - \ell^2 Y/\psi^3}} = \frac{\lambda}{2\psi} \sqrt{1 + \ell^2 \pi_\Theta^2 / (\lambda^2 \psi)}, \hspace{1cm} (106)$$

$$Y \mathcal{L}_{YY} = \frac{\lambda}{4\psi^4} \frac{\ell^2 Y}{(1 - \ell^2 Y/\psi^3)^{3/2}} = \frac{\ell^2 \pi_\Theta^2}{4\lambda \psi^2} \sqrt{1 + \ell^2 \pi_\Theta^2 / (\lambda^2 \psi)}. \hspace{1cm} (107)$$

Using this we obtain

$$c_s^2 = 1 - \frac{\ell^2 \pi_\Theta^4}{(\pi_\Theta^2 + \pi_\Theta^2 \sqrt{\lambda^2 / \psi^2 + \ell^2 \pi_\Theta^2 / \psi^3}(\ell^2 \pi_\Theta^2 + \lambda^2 \psi)} \hspace{1cm} (108)$$

Due to the rapid oscillations, it is more appropriate to define the effective speed of sound as the ratio of the co-moving acoustic to the co-moving particle horizon radii:

$$c_{\text{eff}} = \frac{\int dt c_s / a}{\int dt / a}. \hspace{1cm} (109)$$
In Fig. 5 we plot the effective speed of sound defined in (109) together with the approximate asymptotic value

\[ c_{\text{eff}}|_{\text{app}} \simeq \sqrt{3} A. \]  

(110)

Note that \( c_{\text{eff}}|_{\text{app}} \) is twice as large as the “average” speed of sound that one would naively expect from the equation of state (100).

Figure 5: Effective speed of sound. The horizontal red line represents the approximate asymptote given by (110).

6 Summary and Conclusions

We have presented a derivation of the effective tachyon Lagrangian in an AdS_5 geometry distorted by the radion back-reaction. The usual tachyon inverse quartic potential is modified due to the interaction with the radion. The field equation of the resulting tachyon-radion system is solved assuming homogeneous and isotropic evolution. The back-reaction causes the tachyon-radion system to behave as “warm” tachyon matter with a linear barotropic equation of state.

In addition, we have studied the sound speed in our model. We have derived the adiabatic speed of sound for a general model with two dynamical fields and Lagrangian that depends on the two kinetic terms and on the composite field in the form of a general function of the two fields. We have shown that the effective sound speed in our model approaches asymptotically a constant value of the order of 0.25.

The ultimate question regards the clustering properties of the model. A fluid with a nonzero sound speed has a characteristic scale below which the pressure effectively opposes gravity. At the linear level one expects a suppression of small-scale structure formation: initially growing modes undergo damped oscillations once they enter the co-moving acoustic
horizon. Perturbation theory is not the whole story – it would be worth studying the nonlinear effects, e.g., using the Press-Schechter formalism as in the pure tachyon model of [30].

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A Adiabatic speed of sound

The derivation of the so called adiabatic speed of sound follows the procedure described in Appendix A of [30] generalized to two dynamical fields Φ and Θ. Consider a Lagrangian of the form

\[ \mathcal{L} = \mathcal{L}(X, Y, \psi), \]  

(A.1)

where \( \psi = \psi(\Phi, \Theta) \) is an arbitrary function of \( \Phi \) and \( \Theta \). and

\[ X = g^{\mu\nu}\Phi_{\mu,\nu}, \quad Y = g^{\mu\nu}\Theta_{\mu,\nu}, \]  

(A.2)

For simplicity, we assume that the functional dependence of \( \mathcal{L} \) on \( X \) and \( Y \) is such that

\[ \mathcal{L}_{XY} = 0. \]  

(A.3)

The pressure \( p \) and the density \( \rho \) are functions of \( \psi \), \( X \) and \( Y \) through \( \mathcal{L} \)

\[ p = \mathcal{L}, \]  

(A.4)

\[ \rho = 2X\mathcal{L}_X + 2Y\mathcal{L}_Y - \mathcal{L}. \]  

(A.5)

The standard definition of the adiabatic speed of sound is

\[ c_s^2 = \frac{\partial p}{\partial \rho}_{s/n}, \]  

(A.6)

where the differentiation is taken at constant \( s/n \), i.e. for an isentropic process. Here \( s = S/V \) is the entropy density and \( n = N/V \) the particle number density associated with the particle number \( N \). We use the terminology and notation of Landau and Lifshitz [31] (see also [32]). Our Lagrangian is a function of three variables, namely \( X \), \( Y \) and \( \psi \). Hence

\[ c_s^2 = \left. \frac{\delta p}{\delta \rho} \right|_{s/n} = \left. \frac{\partial p/\partial X}\delta X + (\partial p/\partial Y)\delta Y + (\partial p/\partial \psi)\delta \psi}{\partial p/\partial X}\delta X + (\partial p/\partial Y)\delta Y + (\partial p/\partial \psi)\delta \psi} \right|_{s/n}, \]  

(A.7)
where the differentials $\delta X$, $\delta Y$ and $\delta \psi$ are subject to the constraint $\delta(s/n) = 0$. Next we show that this constraint implies $\delta \psi = 0$.

Our fluid is a two component system with $p = p_1 + p_2$, $\rho = \rho_1 + \rho_2$. As in [30] we start from the standard thermodynamic relation for each component $i = 1, 2$

$$\delta(p_i V) = T\delta S_i - p_i \delta V. \quad (A.8)$$

Assuming that there exist a conserved particle number $N_i$ for each fluid, the volume may be expressed in terms of particle number densities $n_i = N_i/V$ so

$$\delta V = -V \frac{\delta n_i}{n_i}. \quad (A.9)$$

Equations (A.8) may then be written in the form

$$\delta h_i = T\delta \left( \frac{s_i}{n_i} \right) + \frac{1}{n_i} \delta p_i, \quad (A.10)$$

where

$$h_i = \frac{p_i + \rho_i}{n_i} \quad (A.11)$$

is the enthalpy per particle. In the case of two conserved particle numbers, for an isentropic process we must have $\delta(s/n_i) = 0$ for both $i = 1$ and $i = 2$, where $s = s_1 + s_2$. As a consequence

$$n_1 \delta \left( \frac{s_1}{n_1} \right) + n_2 \delta \left( \frac{s_2}{n_2} \right) = -s_2 \frac{n_1}{n_2} \delta \left( \frac{n_2}{n_1} \right) = 0, \quad (A.12)$$

where $\delta(n_2/n_1)$ vanishes because of (A.9). Using this, from (A.10) it follows

$$\delta p|_{s/n} = n_1 \delta h_1 + n_2 \delta h_2, \quad (A.13)$$

where $p = p_1 + p_2$.

Furthermore, for an isentropic relativistic flow one can define the velocity potentials $\phi_i$ such that

$$h_i u_{i\mu} = \phi_{i\mu}. \quad (A.14)$$

Comparing this with (54) and identifying

$$\phi_1 \equiv \Phi, \quad \phi_2 \equiv \Theta, \quad (A.15)$$

we find

$$h_1 = \sqrt{X}; \quad h_2 = \sqrt{Y}. \quad (A.16)$$

Using this in (A.13) we obtain

$$\delta p|_{s/n} = \frac{n_1}{2\sqrt{X}} \delta X + \frac{n_1}{2\sqrt{Y}} \delta Y. \quad (A.17)$$

Comparing this equation with the general expression for the total differential of $p$

$$\delta p = \frac{\partial p}{\partial X} \delta X + \frac{\partial p}{\partial Y} \delta Y + \frac{\partial p}{\partial \psi} \delta \psi \quad (A.18)$$
we conclude that an isentropic process implies
\[ \delta \psi = 0 \] (A.19)
or \( \psi = \text{const.} \). Furthermore, from (A.17) it follows
\[ n_1 = 2 \sqrt{X} \mathcal{L}_X, \] (A.20)
\[ n_2 = 2 \sqrt{Y} \mathcal{L}_Y. \] (A.21)
These two expressions are derived assuming an isentropic process, i.e., keeping \( \psi = \text{const.} \).

If we had \( \psi = \text{const} \) in (48), i.e., if \( \mathcal{L} = \mathcal{L}(X, Y) \), equations (A.20) and (A.21) would follow from the field equation for \( \Phi \) and \( \Theta \)
\[ (\mathcal{L}_X g^\mu{}_{\nu} \Phi,_{\nu} \psi) = 0; \quad (\mathcal{L}_Y g^\mu{}_{\nu} \Theta,_{\nu} \psi) = 0, \] (A.22)
which would imply conservation of two currents
\[ j_{1\mu} = 2 \mathcal{L}_X \Phi,_{\mu} = n_1 u_{\mu}; \quad j_{2\mu} = 2 \mathcal{L}_Y \Theta,_{\mu} = n_2 u_{\mu}. \] (A.23)
The particle number densities \( n_1 \) and \( n_2 \) in these expressions coincide with (A.20) and (A.21). However, in a more general case \( \mathcal{L} = \mathcal{L}(X, Y, \psi) \), the field equations for \( \Phi \) and \( \Theta \) do not imply conservation of the two currents in (A.23). Nevertheless, equations (A.20) are still valid expressions for conserved number densities when the condition \( \delta (s/n) = 0 \) is imposed.

The adiabatic speed of sound is now given by
\[ c_s^2 = \left. \frac{\delta p}{\delta \rho} \right|_\psi = \frac{\mathcal{L}_X (\delta X/\delta Y) + \mathcal{L}_Y}{(\mathcal{L}_X + 2X \mathcal{L}_{XX})(\delta X/\delta Y) + \mathcal{L}_Y + 2Y \mathcal{L}_{YY}} \] (A.24)
The ratio \( (\delta X/\delta Y) \) may be expressed in terms of \( X, Y, \) and the derivatives of \( \mathcal{L} \) using the condition
\[ \frac{\delta (\sqrt{X} \mathcal{L}_X)}{\sqrt{X} \mathcal{L}_X} = \frac{\delta (\sqrt{Y} \mathcal{L}_Y)}{\sqrt{Y} \mathcal{L}_Y} \] (A.25)
which follows from (A.9), (A.20), and (A.21). We find
\[ \frac{\delta X}{\delta Y} = \frac{X \mathcal{L}_X (\mathcal{L}_Y + 2Y \mathcal{L}_{YY})}{Y \mathcal{L}_Y (\mathcal{L}_X + 2X \mathcal{L}_{XX})} \] (A.26)
Using this we obtain
\[ c_s^2 = \frac{X \mathcal{L}_X^2 (\mathcal{L}_Y + 2Y \mathcal{L}_{YY}) + Y \mathcal{L}_Y^2 (\mathcal{L}_X + 2X \mathcal{L}_{XX})}{(X \mathcal{L}_X + Y \mathcal{L}_Y)(\mathcal{L}_X + 2X \mathcal{L}_{XX})(\mathcal{L}_Y + 2Y \mathcal{L}_{YY})} \] (A.27)
This expression may be written in the form
\[ c_s^2 = c_{s1}^2 \frac{\rho_1 + p_1}{\rho + p} + c_{s2}^2 \frac{\rho_2 + p_2}{\rho + p} \] (A.28)
where \( c_{s1} \) and \( c_{s2} \) are defined as
\[ c_{s1}^2 = \frac{\mathcal{L}_X}{\mathcal{L}_X + 2X \mathcal{L}_{XX}}; \quad c_{s2}^2 = \frac{\mathcal{L}_Y}{\mathcal{L}_Y + 2Y \mathcal{L}_{YY}}. \] (A.29)
\[ \rho_1 + p_1 = 2X \mathcal{L}_X; \quad \rho_2 + p_2 = 2Y \mathcal{L}_Y. \] (A.30)
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