Abstract

We consider a fourth order partial differential equation in $n$ dimensional space introduced by Abreu in the context of Kähler metrics on toric orbifolds. Similarity solutions depending only on the radial coordinate in $\mathbb{R}^n$ are determined in terms of a second order ordinary differential equation. A local asymptotic analysis of solutions in the neighbourhood of singular points is carried out. The integrability (or otherwise) of Abreu’s equation is discussed.

1 Introduction

In recent work Abreu has considered toric Kähler metrics on toric varieties or toric orbifolds of dimension $2n$. Following a construction due to Guillemin, each such variety or orbifold is completely determined by its moment polytope $\Delta$ in $\mathbb{R}^n$, and the scalar curvature $S$ of the Kähler metric is given by a formula

$$S = -\partial_j \partial_k g^{jk}$$  \hspace{1cm} (1.1)

where the matrix elements $g^{jk}$ are functions of $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, and $\partial_j$ denotes the partial derivative $\partial/\partial x_j$ (the summation convention is assumed). More precisely, in terms of a potential function $g(x) = g(x_1, \ldots, x_n)$ on $\mathbb{R}^n$, the metric on the interior of the polytope $\Delta$ is

$$ds^2 = g_{jk} dx_j dx_k,$$

where $g_{jk}$ are the elements of the Hessian matrix $G$, i.e.

$$G = (g_{jk}), \quad g_{jk} = \partial_j \partial_k g,$$

and in (1.1) the curvature is determined in terms of the inverse matrix

$$G^{-1} = (g^{jk}).$$
In \cite{4} it is shown that the condition for extremal toric metrics is that the curvature is an affine function of \( x \), in other words
\[ \partial_j S = \text{constant}, \quad j = 1, \ldots, n. \]  
(1.2)
Abreu constructs such metrics from potentials of the form
\[ g = \frac{1}{2} \sum_{l=1}^d \ell_l(x) \log \ell_l(x), \]  
(1.3)
where the \( \ell_l(x) \) are affine functions which determine the facets of the polytope by the equations \( \ell_l(x) = 0, \quad l = 1, \ldots, d \). For the simplest case of Guillemin’s construction \cite{3}, the polytope is a cuboid in \( \mathbb{R}^n \), so that \( d = 2n \), and the toric variety is just \((S^2)^x \), i.e. \( n \) copies of the 2-sphere obtained by attaching an \( n \)-torus \( T^n \) to each point in the interior of the cuboid.

The purpose of this note is to construct other types of solution to the equation (1.1) for the case of constant scalar curvature, namely
\[ \partial_j \partial_k g^{jk} = -\kappa, \quad \text{constant}. \]  
(1.4)
Henceforth we shall follow \cite{7} and refer to (1.4) as Abreu’s equation, and show that for \( n > 1 \) it admits \( O(n) \) invariant solutions of a different form compared with (1.3).

Rather than being geometric, our motivation for considering the partial differential equation (PDE) (1.4) comes from the theory of integrable systems. Taking derivatives of (1.3) we have
\[ g_{jk} = \frac{1}{2} \sum_{l=1}^d c_{l,j} c_{l,k} \ell_l(x), \]  
(1.5)
where the constants \( c_{l,j} \) are the coefficients of the affine functions \( \ell_l(x) \).

The expression (1.5) is reminiscent of the potential for vanishing rational solutions \cite{12, 13} of the integrable Kadomtsev-Petviashvili (KP) equation,
\[ \partial_x \left( 4 \partial_u \partial_{t_3} - 12u \partial_x - \partial^3 u \right) - 3 \partial^2 u \partial_{t_2}^2 = 0. \]  
(1.6)
which takes the form
\[ w = -\sum_{l=1}^d \frac{1}{x - x_l(t)}, \quad u = -2 \frac{\partial w}{\partial x}, \]  
(1.7)
where \( \ell \) denotes the dependence of the poles on the times \( t_2, t_3 \) (or an infinite sequence of such times in the full KP hierarchy). The dynamics of the \( d \) poles \( x_l(t) \) is governed by the integrable Calogero-Moser system for \( d \) particles.
The potential $w$ in (1.7) is a central object in the Sato formulation of KP theory [15], since the pseudo-differential Lax operator $L$ is constructed with a dressing operator $W$ such that

$$L = W \partial_x W^{-1}, \quad W = 1 + w \partial_x^{-1} + \ldots.$$  \hspace{1cm} (1.8)

The similarity between (1.5) and the rational KP potential (1.7) raises the question of whether Abreu’s equation (1.4) might be integrable in some sense. Following the Ablowitz-Ramani-Segur conjecture [2] that all reductions of integrable PDEs should have the Painlevé property, our natural instinct is to seek similarity reductions of (1.4) and examine the structure of their singularities in the complex plane. To avoid begging the question, by an integrable PDE we mean one having a Lax pair (ensuring solvability by the inverse scattering transform [3]) and/or infinitely many symmetries [3].

2 Similarity solutions

For each $n$, Abreu’s equation (1.4) has $O(n)$ invariant solutions with the potential $g$ being a function of the radial distance only, that is

$$g = g(r), \quad r = |x|.$$  

In that case the Hessian matrix and its inverse are given by

$$G = \frac{f}{r} I + \frac{1}{r} \left( \frac{f}{r} \right)' xx^T, \quad G^{-1} = \frac{r}{f} I + \left( \frac{1}{r^2 f'} - \frac{1}{rf} \right) xx^T,$$

(2.1)

where

$$f = g',$$

and the prime denotes $d/dr$.

Substituting the form (2.1) of the inverse Hessian matrix into (1.4) yields the following third order ordinary differential equation (ODE) for $f$:

$$\left( n + r \frac{d}{dr} \right) \left( \frac{A'}{r} + r B' + (n + 1)B \right) = -\kappa; \quad A = \frac{r}{f}, \quad B = \frac{1}{r^2 f'} - \frac{1}{rf}.$$  \hspace{1cm} (2.2)

After an integration, this yields

$$\frac{A'}{r} + r B' + (n + 1)B = \lambda r^{-n} - \frac{\kappa}{n},$$

(2.3)

for constant $\lambda$. As a second order ODE for $f = f(r)$, the equation (2.3) takes the explicit form

$$f'' = \left( \frac{\kappa}{n} - \frac{\lambda}{r^{n-1}} - \frac{(n-1)}{f} \right) (f')^2 + \frac{(n-1)}{r} f'.$$

(2.4)
For the purposes of asymptotic analysis, (2.4) may be conveniently rewritten as
\[
\frac{d}{dr} \log \left( \frac{f^{n-1} f'}{r^{n-1}} \right) = \left( \frac{\kappa}{n} r - \lambda r^{1-n} \right) f'.
\]  
(2.5)

(By rescaling \( f \) we can always set \( \kappa = 1 \), but we choose to leave \( \kappa \) arbitrary.)

In the case \( n = 1 \) it is straightforward to integrate (2.4) to obtain the general solution
\[
f = \frac{1}{\rho} \log \left( \frac{\rho - \lambda + \kappa r}{\rho + \lambda - \kappa r} \right) + \alpha,
\]
so that the potential is
\[
g = \rho^{-1}(r_{-} - r_{-}) \log(r_{-} - r_{-}) + \rho^{-1}(r_{+} - r) \log(r_{+} - r) + \alpha \rho + \beta, \quad r_{\pm} = (\lambda \pm \rho)/\kappa,
\]
for arbitrary constants \( \rho > 0, \alpha, \beta \). Up to shifting by an affine function, the solution for \( n = 1 \) is of the form (1.3), with \( g(r) \) defined on a single interval \((r_{-}, r_{+})\) in \( \mathbb{R} \), which leads to a toric metric on \( S^2 \). Clearly for any \( n \) there is the trivial solution \( f = \text{constant} \), for which the inverse Hessian \( G^{-1} \) in (2.1) becomes infinite. For \( n > 1 \) we are unable to integrate (2.4) explicitly, and must resort to asymptotic analysis around singular points, before considering the solutions of the initial value problem.

**Asymptotics at \( r = 0 \):** There are several different types of behaviour near the origin.

- For any \( \lambda \), (2.4) admits the expansion
\[
f \sim f_0 + a r^n - \frac{\lambda a^2 n^2}{n+1} r^{n+1} + O(r^{n+2}),
\]  
(2.6)

with arbitrary constants \( f_0 \neq 0 \) and \( a \).

- For \( \lambda \neq 0 \) there is an alternative expansion
\[
f \sim f_0 + \frac{1}{\lambda(n-1)} r^{n-1} + O(r^{n+1})
\]  
(2.7)

with \( f_0 \neq 0 \), for \( n \neq 2 \). When \( n = 2 \) we have instead
\[
f \sim f_0 + \lambda^{-1} r - \frac{1}{4\lambda^2 f_0} r^2 + O(r^3).
\]

- For \( \lambda \neq 0 \) and \( n \neq 2 \) there is another local expansion in (2.4) that is regular at \( r = 0 \), with leading order behaviour
\[
f \sim -\frac{n(n-2)}{\lambda(n-1)} r^{n-1} + O(r^{n+1}).
\]  
(2.8)

for \( n \neq 2 \).
• In the case $\lambda = 0$, there is an exact solution that is singular at the origin, namely

$$f = \frac{2n^2}{\kappa} r^{-1}.$$

**Asymptotics at infinity:** The equation (2.4) admits two types of asymptotic expansion at infinity:

• In the first, $f$ is asymptotic to a nonzero constant, i.e.

$$f \sim f_{\infty} + \frac{n(n+1)}{\kappa} r^{-1} + \ldots,$$

for $f_{\infty} \neq 0$.

• In the second case, $f$ tends to zero:

$$f \sim \frac{2n^2}{\kappa} r^{-1}, \quad r \to \infty. \quad (2.10)$$

**Movable singularities at $r = r_0 \neq 0$:** The ODE (2.4) is outside the Painlevé class of second order equations whose general solution has no movable singularities other than poles, as described in [10], since its solutions admit algebraic branching at arbitrary points $r_0 \neq 0$ in the complex $r$ plane. The leading order behaviour in the neighbourhood of such a branch point is

$$f \sim c_0 (r - r_0)^{1/n}, \quad (2.11)$$

where the constant $c_0$ is arbitrary. Since (2.4) is second order, we expect that (2.11) should be the first term in an expansion in powers of $(r - r_0)^{1/n}$ providing a local representation of the general solution, since it contains the two arbitrary constants $r_0$ and $c_0$. If it has only algebraic branching around movable singularities, an ODE can possess the weak Painlevé property as defined in [18], and there are many examples of integrable ordinary and partial differential equations which have this property (see for instance [1, 4, 8]). However, the ODE (2.4) also admits movable logarithmic branch points in its solutions, with

$$f \sim \left( \frac{\lambda}{r_0^{\alpha-1}} - \frac{\kappa r_0}{n} \right)^{-1} \log(r - r_0) \quad (2.12)$$

in the neighbourhood of $r = r_0$. Logarithmic branching such as (2.12) is taken as a strong indicator of non-integrability in differential equations [2].

We would expect that generically the solutions of the ODE should have infinitely many branch points in the complex plane. If a solution has a branch point on the real axis, then for even $n$ it cannot be real-valued for all real $r$. In that case, for $\lambda \neq 0$ it would make sense to consider a real-valued
branch defined on the interval \([0, r^*]\), where \(r^* > 0\) is the position of the first positive real branch point. Such a solution might have the behaviour (2.3) or tend to a nonzero constant at the origin, and as \(r \to r^*\) from below would look like

\[
f \sim c(r^* - r)^{1/n}
\]

or

\[
f \sim \hat{c} \log(r^* - r)
\]

for real constants \(c, \hat{c}\). In that case the Hessian matrix \(G\) would be entirely determined by the function \(f\) and its first derivative, giving a metric on the ball of radius \(r^*\) in \(\mathbb{R}^n\), with a singularity at \(r = r^*\) where \(f'\) blows up. Exactly how to extend this to a metric on a 2\(n\)-dimensional (symplectic) manifold is not clear to us.

In fact starting from initial data specified near the origin at \(r = \epsilon > 0\), the ODE (2.4) is regular. It is not possible to take initial data at the origin since \(r = 0\) is a singular point of (2.4). Clearly the right hand side of (2.4) is also singular at \(f = 0\). It is easy to see that given \(f'(\epsilon) \neq 0\), the solution cannot have stationary points for \(r > \epsilon\), since integrating (2.4) with initial data \(f(\epsilon) \neq 0, f'(\epsilon) = 0\) gives only the constant solution \(f(r) = f(\epsilon)\). It is then trivial to prove the following:

**Lemma:** Suppose that initial data is specified for the ODE (2.4) at a point \(r = \epsilon > 0\), with \(f(\epsilon), f'(\epsilon)\) both nonzero. Then the solution \(f(r)\) for \(r > \epsilon\) remains positive (negative) for \(f(\epsilon) > 0\) \((f(\epsilon) < 0)\), and for \(f'(\epsilon) > 0\) or \(f'(\epsilon) < 0\) it is monotone (increasing or decreasing, respectively), as long as it exists.

We wish to consider solutions of the ODE which are free of branch points on the whole positive real axis, or on a finite interval \([0, r^*)\) (with branching at \(r = r^*\)). In that case the components of the Hessian matrix \(G\) determine a metric on \(\mathbb{R}^n\), or on the ball of radius \(r^*\). The eigenvalues of \(G\) in (2.1) are \(f/r\) (repeated \(n - 1\) times) and \(f'\). Thus if we also require a Riemannian metric given by a positive definite Hessian, then we should consider only solutions of (2.4) defined with both initial data positive, i.e. \(f(\epsilon) > 0, f'(\epsilon) > 0\). The signs of \(f\) and \(f'\) remain constant and the Lemma also holds when the solution is analytically continued to \(r < \epsilon\), provided it exists. However, a priori we have no guarantee that when the solution is continued back towards \(r = 0\) (solving the initial value problem in reverse) it will not reach a singularity (branch point) at some point \(r = \epsilon'\) with \(0 < \epsilon' < \epsilon\). Hence we are led to consider a nonlinear connection problem for (2.4), to determine what asymptotic behaviours at \(r = 0\) are compatible with a solution defined on \([0, r^*)\) with specified branching at \(r = r^*\), or a solution on the whole positive real axis with specified asymptotics at infinity.

For completeness we will also consider the cases when either \(f, f'\) or both are negative, and henceforth we assume \(\kappa > 0, \lambda \neq 0\). Taking the four
different combinations of initial data in turn, we obtain the following result:

**Theorem 1:** Suppose non-zero initial data is specified for the ODE (2.4) at \( r = \epsilon > 0 \). There are four possibilities:

(i) \( f(\epsilon) > 0, f'(\epsilon) > 0 \). The solution reaches a logarithmic branch point with asymptotic behaviour \( (2.14) \) (with \( \hat{c} < 0 \)) at some finite point \( r = r^* > \epsilon \).

(ii) \( f(\epsilon) < 0, f'(\epsilon) > 0 \). The solution reaches an algebraic branch point with asymptotic behaviour \( (2.13) \) (with \( c < 0 \)) at some finite point \( r = r^* > \epsilon \).

(iii) \( f(\epsilon) > 0, f'(\epsilon) < 0 \). Either the solution reaches an algebraic branch point \( (2.13) \) (with \( c > 0 \)) at some finite point \( r = r^* > \epsilon \), or \( f \) is defined for all \( r \geq \epsilon \) and asymptotes to a non-negative constant with the behaviour \( (2.9) \) or \( (2.10) \) as \( r \to \infty \).

(iv) \( f(\epsilon) < 0, f'(\epsilon) < 0 \). Either the solution reaches a logarithmic branch point \( (2.14) \) (with \( \hat{c} > 0 \)) at some finite point \( r = r^* > \epsilon \), or \( f \) is defined for all \( r \geq \epsilon \) and asymptotes to a negative constant with the behaviour \( (2.9) \) as \( r \to \infty \).

**Proof.** (i) The solution cannot be defined for all \( r \geq \epsilon \), since the only possible asymptotics \( (2.3) \) or \( (2.10) \) at infinity necessitate \( f' < 0 \) for sufficiently large \( r \) (with our assumption of positive \( \kappa \)), which contradicts the Lemma. An algebraic branch point \( (2.13) \) would mean \( f \to 0 \) as \( r \to r^* \), contradicting the fact that \( f \) is monotone increasing by the Lemma. Hence the logarithmic branching \( (2.14) \) at some finite \( r = r^* \) is the only possibility.

(ii) Again by the Lemma, \( f \) must remain negative and monotone increasing. This rules out the asymptotic behaviours \( (2.3) \) or \( (2.10) \) at infinity for which \( f' < 0 \), and also a logarithmic branch point which would require \( f \to -\infty \).

(iii) and (iv) are proved similarly.

Having considered the standard initial value problem for (2.4) at \( r = \epsilon \), we can now solve it in reverse, and consider continuing the solutions of types (i)-(iv) backwards towards the origin. (This is equivalent to solving the ODE for \( h(r) = f(-r) \) with initial data at \( r = -\epsilon \).) The solution to the nonlinear connection problem may be summarized thus:

**Theorem 2:** Solving the initial value problem for the ODE (2.4) in the reverse direction, with the four combinations of initial data as in Theorem 1, leads to the following possibilities for the solution \( f(r) \) with \( r < \epsilon \):

(i) Either the solution reaches an algebraic branch point at \( r = r_0 \) with \( (2.11) \) \( (c_0 > 0) \) for \( 0 < r_0 < \epsilon \), or it continues back to the origin with one of the asymptotic behaviours \( (2.7) \) (with \( f_0 > 0, a > 0 \)), \( (2.4) \) (with \( f_0 > 0 \), for \( \lambda > 0 \) only) or \( (2.8) \) (for \( \lambda < 0, n \neq 2 \) only).

(ii) Either the solution reaches a logarithmic branch point at \( r = r_0 \) with \( (2.14) \) for \( 0 < r_0 < \epsilon \), or it continues back to the origin with one of the asymptotic behaviours \( (2.7) \) (with \( f_0 < 0, a > 0 \)), or \( (2.4) \) (with \( f_0 < 0 \), for \( \lambda > 0 \) only).
(iii) Either the solution reaches a logarithmic branch point at \( r = r_0 \) with \( (2.12) \) for \( 0 < r_0 < \epsilon \), or it continues back to the origin with one of the asymptotic behaviours \( (2.6) \) (with \( f_0 > 0, a < 0 \)), or \( (2.7) \) (with \( f_0 < 0 \), for \( \lambda < 0 \) only).

(iv) Either the solution reaches an algebraic branch point at \( r = r_0 \) with \( (2.11) \) (c_0 < 0) for \( 0 < r_0 < \epsilon \), or it continues back to the origin with one of the asymptotic behaviours \( (2.6) \) (with \( f_0 > 0, a < 0 \)), \( (2.7) \) (with \( f_0 < 0 \), for \( \lambda < 0 \) only) or \( (2.8) \) (for \( \lambda > 0 \), \( n \neq 2 \) only).

**Proof.** The proof is straightforward, by a simple enumeration of the various possibilities \( (2.6), (2.7) \) or \( (2.8) \) at \( r = 0 \) allowed by the Lemma, as well as the different branching behaviours that can occur.

3 Conclusions

For any dimension \( n > 1 \), new solutions of Abreu’s equation \((1.4)\) have been found by making a similarity reduction to \( O(n) \) invariant solutions. The similarity solutions are determined in terms of a single function \( f(r) \) depending only on the radial coordinate \( r \), which satisfies a second order ODE \((2.4)\) including an arbitrary parameter \( \lambda \). Due to a singularity in the ODE at the origin, the initial value problem is not defined at \( r = 0 \). Nevertheless we have shown that for some suitable subset of the possible initial data specified at \( r = \epsilon > 0 \) the solution may have a continuation in both the forward \((r > \epsilon)\) and backward \((r < \epsilon)\) directions to define a solution either on the whole positive real axis, or on an interval \([0, r^*]\) with a branch point at \( r = r^* \). With the restriction to Riemannian metrics defined by a positive definite Hessian matrix \( G \) in \((2.1)\), only the latter case is relevant (corresponding to case (i) in Theorems 1 and 2 above). However, whether such solutions could have further geometric significance, by extension to a suitable metric on a symplectic manifold of dimension \( 2n \), is uncertain.

Performing numerical integrations of \((2.4)\) with \( n = 3 \) and \( \kappa = \lambda = 1 \), by different choices of initial data we have obtained solutions displaying some of the asymptotic behaviours included in cases (i)-(iv) of the Theorems. A more detailed understanding of such solutions would necessitate estimates on the initial data to ensure that branch points do not appear as \( r \to 0 \). Determining conditions for the existence of points of inflection would be an essential step in such an understanding. It would also be desirable to have some bounds on \( r^* \), the position of the branch point, in terms of \( f(\epsilon) \) and \( f'(\epsilon) \), but this is beyond the scope of the present work.

The various Painlevé tests are good heuristic tools for analysing ordinary and partial differential equations, and have been used to isolate new integrable systems (see Chapter 7 of [3] for a review). These tests are based on the Painlevé property, that the general solution of an equation should have only poles as movable singularities. However, as originally empha-
sized by the authors of [2], the Painlevé property is extremely sensitive to changes of variables. For example, transformations of hodograph type can change equations with movable algebraic branch points into equations that have only poles [6, 9]. The ODE (2.4) has not only algebraic branching but also movable logarithmic branch points, and so certainly fails the standard Painlevé test or its weak extension [13]. This is a strong indication that Abreu’s equation (1.4) is not integrable.

It has been observed [17] that when an equation is written in potential form it can have a single (or finitely many) logarithmic terms in a local expansion around singular points, and still be integrable. However, the asymptotic expansion for (2.4) with (2.12) as the first term seems to require infinitely many logarithms. In the case of the KP equation (1.6), the potential $w$ in (1.8) is given by the logarithmic derivative of the tau-function:

$$w = -\frac{\partial}{\partial x} \log \tau.$$  

The tau-function for the rational solutions (1.7) takes the form

$$\tau = \prod_{l=1}^{d} (x - x_l(t)),$$  

and this is a polynomial both in $x$ and the times $t$. For the more general algebro-geometric solutions of KP (1.1), the tau-function is a theta-function of an arbitrary Riemann surface. The rational and soliton solutions arise as degenerate limits of the theta-functions. It would be interesting to see if Abreu’s equation would admit quasiperiodic generalizations of (1.5), by dropping the extremality condition (1.2) on the curvature, allowing $S$ in (1.1) to be a more general (say, periodic) function.

Other methods of testing for integrability might provide useful information about Abreu’s equation. Since (1.4) is not of evolution type, a promising method would be the symmetry approach of Shabat et al [13], which has recently been extended [14] in order to deal with non-evolutionary equations. It would also be interesting to apply the methods of [16] to look for other sorts of group invariant solutions of Abreu’s equation (1.4).

References

[1] S. Abenda and Y. Fedorov, Acta Appl. Math. 60 (2000) 137-178.
[2] M.J. Ablowitz, A. Ramani and H. Segur, Lett. Nuovo Cim. 23 (1978) 333-338.
[3] M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press (1991).
[4] M. Abreu, Internat. J. Math. 9 (1998) 641-651.
[5] M. Abreu, J. Diff. Geom. 58 (2001) 151-187.
[6] P.A. Clarkson, A.S. Fokas and M.J. Ablowitz, SIAM J. Appl. Math. 49 (1989) 1188-1209.

[7] S.K. Donaldson, *Convex analysis and toric manifolds*, lecture at LMS Mary Cartwright meeting (2002).

[8] V. Guillemin, J. Diff. Geom. 40 (1994) 285-309.

[9] A.N.W. Hone, Phys. Lett. A 249 (1998) 46-54.

[10] E.L. Ince, *Ordinary Differential Equations* (1926). Reprint: New York: Dover Publications (1956).

[11] I.M. Krichever, Russian Math. Surveys 32 (1977) 185-213.

[12] I.M. Krichever, Funct. Anal. Appl. 12 (1978) 59-61.

[13] A.V. Mikhailov, A.B. Shabat and R.I. Yamilov, Russian Math. Surveys 42 (1987) 1-63.

[14] A.V. Mikhailov and V.S. Novikov, J. Phys. A 35 (2002) 4775-4790.

[15] Y. Ohta, J. Satsuma, D. Takahashi and T. Tokihiro, Prog. Theor. Phys. Suppl. 94 (1988) 210-241.

[16] P.J. Olver, *Applications of Lie Groups to Differential Equations*, 2nd Edition, Springer-Verlag (1993).

[17] A. Pickering, J. Math. Phys. 37 (1996) 1894-1927.

[18] A. Ramani, B. Dorizzi and B. Grammaticos, Phys. Rev. Lett. 49 (1982) 1538-1541.

[19] T. Shiota, J. Math. Phys. 35 (1995) 5844-5849.