Backstepping Transformation of Input Delay Nonlinear Systems

Delphine Bresch-Pietri and Miroslav Krstic

Abstract

We present here the details of a backstepping transformation aiming at reformulating the dynamics of a nonlinear systems subject to unknown long input delay in a form which is suitable for Lyapunov stability analysis. The control law underlying this transformation is predictor-based [2], [5], [6], as often considered for long delays. The proposed transformation follows recent results of the literature, based on the representation of the constant actuator delay as a transport Partial Differential Equation (PDE).

I. PROBLEM STATEMENT

Consider the following nonlinear plant

\[ \dot{X}(t) = f(X, U(t - D)) \]  

in which \( X \in \mathbb{R}^n \), \( f \) is a nonlinear function of class \( C^2 \) such that \( f(0,0) = 0 \), \( U \) is scalar and \( D \) is an unknown delay belonging to a known interval \([D, \bar{D}]\) (with \( \underline{D} > 0 \)).

Assumption 1: The plant \( \dot{X} = f(X, \Omega) \) is strongly forward complete.

Assumption 2: There exists a feedback law \( U(t) = \kappa(X(t)) \) such that the nominal delay-free plant is globally exponentially stable and such that \( \kappa \) is a class \( C^2 \) function, i.e. there exist (see resp. Theorem 4.14 and Theorem 2.207 in [3], [7]) \( \lambda > 0 \) and a class \( C^\infty \) radially unbounded...
positive definite function $V$ such that for $x \in \mathbb{R}^n$

$$
\frac{dV}{dX}(X)f(X, \kappa(X)) \leq -\lambda V(X)
$$

(2)

$$
|X|^2 \leq V(X) \leq c_1 |X|^2
$$

(3)

$$
\left| \frac{dV}{dX}(X) \right| \leq c_2 |X|
$$

(4)

for given $c_1, c_2 > 0$.

Assumption 1 guarantees that (1) does not escape in finite time and, in particular, before the input reaches the system at $t = D$. This is a reasonable assumption to enable stabilization. The difference from the standard notion of forward completeness comes from the fact that we assume that $f(0, 0) = 0$. Assumption 2 guarantees that the delay-free plant is (globally) exponentially stabilisable.

To analyze the closed-loop stability despite delay uncertainties, we use the systematic Lyapunov tools introduced in [4] and first reformulate plant (1) in the form

$$
\begin{align*}
    \dot{X}(t) &= f(X(t), u(0, t)) \\
    Du_t(x, t) &= u_s(x, t) \\
    u(1, t) &= U(t)
\end{align*}
$$

(5)

by introducing the following distributed input

$$
u(x, t) = U(t + D(x - 1)), \quad x \in [0, 1]\n$$

(6)

In details, the input delay is now represented as a coupling with a transport PDE driven by the input and with unknown convection speed $1/D$. We now propose to reformulate this plant thanks to a backstepping transformation of the (estimated) distributed input to obtain a dynamics compliant with Lyapunov analysis,

II. BACKSTEEPING TRANSFORMATION FOR UNMEASURED DISTRIBUTED INPUT

In this paper, we consider the actuator state $u(\cdot, t)$ to be unmeasured, as is typically the case in applications. To deal with this fact, we introduce a distributed input estimate

$$
\hat{u}(x, t) = U(t + \hat{D}(t)(x - 1)), \quad x \in [0, 1]
$$

(7)
Applying the certainty equivalence principle to the nominal dynamics (i.e. from the case of a known input delay), the control law is chosen as

$$U(t) = \kappa(\hat{p}(1,t))$$

(8)

in which the distributed predictor estimate is defined in terms of the actuator state estimate as

$$\hat{p}(x,t) = x(t + \hat{D}(t)x) = X(t) + \hat{D}(t) \int_0^x f(\hat{p}(y,t),\hat{u}(y,t)) \, dy$$

(9)

and the delay estimate \( \hat{D} \) is a time-differentiable function.

**Lemma 1:** The backstepping transformation of the distributed input estimate (7)\(^\dagger\)

$$\hat{w}(x,t) = \hat{u}(x,t) - \kappa(\hat{p}(x,t)),$$

(10)

in which the distributed predictor estimate is defined in (9), together with the control law (8), transforms plant (5) into

$$\dot{X}(t) = f(X(t), \kappa(X(t) + \hat{w}(0,t) + \hat{u}(0,t)))$$

(11)

$$\dot{\hat{D}}(t)\hat{w}_t = \hat{w}_x + \dot{\hat{D}}(t)q_1(x,t) - q_2(x,t)f(\hat{u}(t))$$

(12)

$$\hat{w}(1,t) = 0$$

(13)

$$\dot{D}\hat{u}_t = \hat{u}_x - \dot{\hat{D}}(t)p_1(x,t) - \dot{\hat{D}}(t)p_2(x,t)$$

(14)

$$\hat{u}(1,t) = 0$$

(15)

in which

$$\hat{u}(x,t) = u(x,t) - \hat{u}(x,t)$$

(16)

is the distributed input estimation error and

$$p_1(x,t) = \frac{D}{\hat{D}(t)} \left[ \hat{w}_x(x,t) + \dot{\hat{D}}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x,t)) f(\hat{p}(x,t),\hat{w}(x,t) + \kappa(\hat{p}(x,t))) \right]$$

(17)

$$p_2(x,t) = \frac{D}{\hat{D}(t)}(x - 1) \left[ \hat{w}_x(x,t) + \dot{\hat{D}}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x,t)) f(\hat{p}(x,t),\hat{w}(x,t) + \kappa(\hat{p}(x,t))) \right]$$

(18)

$$q_1(x,t) = (x - 1) \left[ \hat{w}_x(x,t) + \dot{\hat{D}}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x,t)) f(\hat{p}(x,t),\hat{w}(x,t) + \kappa(\hat{p}(x,t))) \right]$$

$$- \dot{\hat{D}}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x,t)) \int_0^x \Phi(x,y) \left[ f(\hat{p}(y,t),\hat{w}(y,t) + \kappa(\hat{p}(y,t))) \right] \, dy$$

$$\times (y - 1) \left[ \hat{w}_x(y,t) + \dot{\hat{D}}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(y,t)) f(\hat{p}(y,t),\hat{w}(y,t) + \kappa(\hat{p}(y,t))) \right] \, dy$$

(19)
\begin{align}
q_2(x,t) &= \mathcal{D}(t) \frac{d\kappa}{d\hat{\rho}}(\hat{\rho}(x,t)) \Phi(x,0) \\
f_\hat{u}(t) &= f(\hat{\rho}(0,t),u(0,t)) - f(\hat{\rho}(0,t),\hat{u}(0,t))
\end{align}

where \(\Phi\) is the transition matrix associated with the space-varying time-parametrized equation
\[
\frac{d\hat{x}}{dt}(x) = \mathcal{D}(t) \frac{df}{d\hat{\rho}}(\hat{\rho}(x,t),\hat{w}(x,t) + \kappa(\hat{\rho}(x,t)))r(x).
\]

**Proof:** First, Eq. (11) can be directly obtained from definitions (6), (10) and the one of \(\hat{u}\). Second, one can easily obtain from (7) that the estimate distributed input satisfies
\[
\mathcal{D}(t)\hat{u}_t(x,t) = \hat{u}_x(x,t) + \hat{D}(t)(x-1)\hat{u}_x(x,t)
\]

Matching this equation with \((\hat{s})\) gives (14) and (15), in which we have used (10) to express the functions \(p_1\) and \(p_2\) in terms of \(\hat{w}\) and \(\hat{w}_x\). Before studying the governing equation of the distributed input, we focus on the dynamics of the distributed predictor. The temporal and spatial derivative of \(\hat{\rho}(x,t)\) can be expressed as follows
\[
\hat{\rho}_t = f(\hat{\rho}(0,t),u(0,t)) + \hat{D}(t) \int_0^x f(\hat{\rho}(y,t),\hat{u}(y,t))dy
\]
\[
\hat{\rho}_x = \mathcal{D}(t)f(\hat{\rho}(0,t),\hat{u}(0,t)) + \hat{D}(t) \int_0^x \left[ \frac{\partial f}{\partial \hat{\rho}}(\hat{\rho}(y,t),\hat{u}(y,t))\hat{\rho}_t(y,t) + \frac{\partial f}{\partial \hat{u}}(\hat{\rho}(y,t),\hat{u}(y,t))\hat{u}_t(y,t) \right] dy
\]

Therefore, using the governing equation of the distributed input estimate given in (22),
\[
\mathcal{D}(t)\hat{\rho}_t(x,t) - \hat{\rho}_x(x,t) = \mathcal{D}(t)\left[ f(\hat{\rho}(0,t),u(0,t)) - f(\hat{\rho}(0,t),\hat{u}(0,t)) \right] + \hat{D}(t)\hat{D}(t) \int_0^x f(\hat{\rho}(y,t),\hat{u}(y,t))dy
\]
\[
+ \hat{D}(t) \int_0^x \left[ \frac{\partial f}{\partial \hat{\rho}}(\hat{\rho}(y,t),\hat{u}(y,t))\hat{D}(t)\hat{\rho}_t(y,t) - \hat{\rho}_x(y,t) \right] dy
\]
\[
+ \hat{D}(t)\hat{D}(t) \int_0^x \left[ \frac{\partial f}{\partial \hat{u}}(\hat{\rho}(y,t),\hat{u}(y,t))(y-1)\hat{u}_x(y,t) \right] dy
\]

Consider a given \(t \geq 0\) and denote \(r(x) = \hat{D}(t)\hat{\rho}_t(x,t) - \hat{\rho}_x(x,t)\). Taking a spatial derivative of the latter equality, one can obtain the following equation in \(x\), parametrized in \(t\),
\[
\begin{cases}
\frac{dr}{dx}(x) = \hat{D}(t) \frac{\partial f}{\partial \hat{\rho}}(\hat{\rho}(x,t),\hat{u}(x,t))r(x) + \hat{D}(t)\hat{D}(t) \left[ f(\hat{\rho}(x,t),\hat{u}(x,t)) + \frac{\partial f}{\partial \hat{u}}(\hat{\rho}(x,t),\hat{u}(x,t))(x-1)\hat{u}_x(x,t) \right] \\
r(0) = \hat{D}(t) \left[ f(\hat{\rho}(0,t),u(0,t)) - f(\hat{\rho}(0,t),\hat{u}(0,t)) \right]
\end{cases}
\]

\[\text{(27)}\]
Defining the transition matrix $\Phi$ associated to the corresponding homogeneous equation, one can solve this equation and obtain

$$\dot{D}(t)\hat{p}_i = \dot{\hat{p}}_i + \Phi(x,0,t)\dot{D}(t)[f(\hat{p}(0,t),u(0,t)) - f(\hat{p}(0,t),\hat{u}(0,t))]$$

$$+ \dot{D}(t)\dot{D}(t) \int_0^x \Phi(x,y,t) \left[ f(\hat{p}(y,t),\hat{u}(y,t)) + \frac{\partial f}{\partial \hat{u}}(\hat{p}(y,t),\hat{u}(y,t))(y-1)\hat{u}_y(y,t) \right] dy \quad (28)$$

Now, matching the time- and space-derivatives of the backstepping transformation (10)

$$\dot{w}_i(x,t) = \dot{\hat{u}}_i(x,t) - \frac{d\kappa}{d\hat{p}}(\hat{p}(x,t))\dot{\hat{p}}_i(x,t)$$

$$\dot{w}_x(x,t) = \dot{\hat{u}}_x(x,t) - \frac{d\kappa}{d\hat{p}}(\hat{p}(x,t))\dot{\hat{p}}_x(x,t)$$

with the governing equations (22) and (28), one can obtain (12) and use the backstepping transformation (10) to express the functions $q_1$ and $q_2$ in terms of $\dot{w}$ and its spatial-derivative.

Comparing (11)-(15) to plant (5), one can see that the main advantage of this new representation is that the boundary conditions (13) and (15) are now equal to zero, consistently with the choice of the control law (8), as opposed to the one stated in (5). This is particularly for stability analysis.

To provide a total description of the system dynamics, we also need the governing equation of spatial derivatives of the distributed variables, which are given in the following lemma.

**Lemma 2:** The spatial derivatives of the distributed input estimation error (16) and of the backstepping transformation (10) satisfy

$$\begin{cases}
D\hat{u}_{xt} = \hat{u}_{xx} - \dot{D}(t)p_3(x,t) - \dot{\hat{D}}(t)p_4(x,t) \\
\hat{u}_x(1,t) = \dot{\hat{D}}(t)p_1(1,t)
\end{cases} \quad (29)$$

$$\begin{cases}
\dot{D}(t)\hat{w}_x = \dot{\hat{w}}_{xx} + \dot{\hat{D}}(t)q_3(x,t) - q_4(x,t)f_{\hat{u}}(t) \\
\hat{w}_x(1,t) = -\dot{\hat{D}}(t)q_1(1,t) + q_2(1,t)f_{\hat{u}}(t)
\end{cases} \quad (30)$$

$$\begin{cases}
\dot{D}(t)\hat{w}_{xx} = \dot{\hat{w}}_{xxx} + \dot{\hat{D}}(t)q_5(x,t) - q_6(x,t)f_{\hat{u}}(t) \\
\hat{w}_{xx}(1,t) = -\dot{\hat{D}}(t)q_3(1,t) + q_4(1,t)f_{\hat{u}}(t) + \dot{\hat{D}}(t)q_7(t)
\end{cases} \quad (31)$$

in which $p_3, p_4, q_3, q_4, q_5, q_6$ and $q_7$ are given in Appendix.
Proof: Taking a spatial derivative of (14), one can obtain the governing equation in (29) and, from the boundary condition (15), that \( \tilde{u}_t(1,t) = 0 \) which gives, replacing in (14), the boundary condition in (29). The exact same arguments applied to (12)-(13) governing the backstepping transformation give system (30).

Taking a spatial derivative of the first equation in (30) give the one in (31). Finally, using the first equation in (30) for \( x = 1 \), one can obtain

\[
\hat{w}_{xx}(1,t) = -\dot{D}(t)q_3(x,t) + q_4(1,t)f_\hat{u}(t) + \dot{D}(t)\hat{w}_{xt}(1,t)
\]

in which \( \hat{w}_{xt}(1,t) = q_7(t) \) can be reformulated by taking a time derivative of the boundary condition in (30). Finally, the functions \( p_3, p_4, q_3, q_4, q_5, q_6 \) and \( q_7 \) given in Appendix can be expressed in terms of \( \hat{w}(\cdot,t) \) and its spatial derivative by using the backstepping transformation (10) and its spatial derivative versions.

\[ \]

III. Conclusion

In this paper, we have presented a backstepping transformation aiming at reformulating the dynamics of a nonlinear systems subject to unknown long input delay in a form which is suitable for Lyapunov stability analysis. This transformation will be particularly useful in future works to perform a Lyapunov analysis of closed-loop stability to delay uncertainties.

Appendix

A. Expression of the functions involved in Lemma 2

\[ p_3(x,t) = p_{1,x}(x,t) \]

\[ = \frac{D}{D(t)} \left[ \hat{w}_{xx} + \dot{D}(t) \frac{d\kappa}{d\hat{p}} (\hat{p}(x,t)) \frac{d}{dx} \left[ f(\hat{p}(x,t), \hat{w}(x,t) + \kappa(\hat{p}(x,t))) \right] + \dot{D}(t)^2 f(\hat{p}(x,t), \hat{w}(x,t) + \kappa(\hat{p}(x,t))) T \frac{d^2\kappa}{d\hat{p}^2} (\hat{p}(x,t)) f(\hat{p}(x,t), \hat{w}(x,t) + \kappa(\hat{p}(x,t))) \right] \]

\[ p_4(x,t) = \hat{p}_{2,x}(x,t) \]

\[ = \frac{D}{D(t)} \left[ \hat{w}_x + \dot{D}(t) \frac{d\kappa}{d\hat{p}} (\hat{p}(x,t)) f(\hat{p}(x,t), \hat{w}(x,t) + \kappa(\hat{p}(x,t))) + (x-1) \left[ \hat{w}_{xx}(x,t) \right] \right. \]

\[ \left. + \dot{D}(t)^2 f(\hat{p}(x,t), \hat{w}(x,t) + \kappa(\hat{p}(x,t))) T \frac{d^2\kappa}{d\hat{p}^2} (\hat{p}(x,t)) f(\hat{p}(x,t), \hat{w}(x,t) + \kappa(\hat{p}(x,t))) \right. \]

\[ \left. + \dot{D}(t) \frac{d\kappa}{d\hat{p}} (\hat{p}(x,t)) \frac{d}{dx} \left[ f(\hat{p}(x,t), \hat{w}(x,t) + \kappa(\hat{p}(x,t))) \right] \right] \]
\[ q_3(x, t) = q_{1,x}(x, t) \]
\[ = \hat{w}_x(x, t) + \hat{D}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) + (x - 1) \left[ \hat{w}_{xx}(x, t) \right. \]
\[ + \hat{D}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) \frac{d}{dx} \left[ f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right] \]
\[ + \hat{D}(t)^2 f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) T d^2 \kappa \frac{d^2}{d\hat{p}^2}(\hat{p}(x, t)) f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \]
\[ - \hat{D}(t)^2 \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) \left[ f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) + \frac{\partial f}{\partial \hat{u}}(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right] \]
\[ \times (x - 1) \left[ \hat{w}_x(x, t) + \hat{D}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right] \right] \] (35)

\[ q_4(x, t) = q_{2,x}(x, t) \]
\[ = \hat{D}(t)^2 \left[ f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) T d^2 \kappa \frac{d^2}{d\hat{p}^2}(\hat{p}(x, t)) f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right. \]
\[ + \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) \frac{\partial f}{\partial \hat{u}}(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \left[ \Phi(x, 0, t) \right. \]
\[ + (x - 1) \left[ \hat{w}_{xx}(x, t) + \frac{d}{dx} \left[ \hat{D}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) \frac{d}{dx} \left[ f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right] \right. \]
\[ + \hat{D}(t)^2 f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) T d^2 \kappa \frac{d^2}{d\hat{p}^2}(\hat{p}(x, t)) f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \]
\[ - \hat{D}(t) \frac{d}{dx} \left( \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) \left[ f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) + \frac{\partial f}{\partial \hat{u}}(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right] \hat{w}_x(x, t) + \hat{D}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right) \right] \right] \] (36)

\[ q_5(x, t) = q_{3,x}(x, t) \]
\[ = 2 \left[ \hat{w}_{xxx}(x, t) + \hat{D}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) \frac{d}{dx} \left[ f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right] \right. \]
\[ + \hat{D}(t)^2 f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) T d^2 \kappa \frac{d^2}{d\hat{p}^2}(\hat{p}(x, t)) f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \]
\[ + (x - 1) \left[ \hat{w}_{xxx}(x, t) + \frac{d}{dx} \left[ \hat{D}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) \frac{d}{dx} \left[ f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right] \right. \]
\[ + \hat{D}(t)^2 f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) T d^2 \kappa \frac{d^2}{d\hat{p}^2}(\hat{p}(x, t)) f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \]
\[ - \hat{D}(t) \frac{d}{dx} \left( \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) \left[ f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) + \frac{\partial f}{\partial \hat{u}}(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right] \hat{w}_x(x, t) + \hat{D}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right) \right] \times (x - 1) \left[ \hat{w}_x(x, t) + \hat{D}(t) \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right] \right] \] (37)

\[ q_6(x, t) = q_{4,x}(x, t) \]
\[ = \hat{D}(t)^2 \frac{d}{dx} \left[ f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) T d^2 \kappa \frac{d^2}{d\hat{p}^2}(\hat{p}(x, t)) f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right. \]
\[ + \frac{d\kappa}{d\hat{p}}(\hat{p}(x, t)) \frac{\partial f}{\partial \hat{u}}(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \left[ \Phi(x, 0, t) \right. \]
\[ + \hat{D}(t)^3 \left[ f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) T d^2 \kappa \frac{d^2}{d\hat{p}^2}(\hat{p}(x, t)) f(\hat{p}(x, t), \hat{w}(x, t) + \kappa(\hat{p}(x, t))) \right. \]

May 24, 2013

DRAFT
\[
q_7(t) = -\dot{D}(t)q_1(1,t) - \dot{D}(t)q_{1,t}(1,t) + \dot{D}(t) \frac{\partial \kappa}{\partial \hat{p}}(\hat{p}(x,t)) \Phi(x,0,t) f_{\tilde{u}}(t)
\]

\begin{align*}
+ \dot{D}(t) \hat{p}_1(x,t) T \frac{\partial^2 \kappa}{\partial \hat{p}^2}(\hat{p}(x,t)) \Phi(x,0,t) f_{\tilde{u}}(t) \\
+ q_2(1,t) f_{dp}(t) f(\hat{p}(0,t),\hat{u}(0,t) + \hat{w}(0,t) + \kappa(\hat{p}(0,t))) - q_2(1,t) f_{du}(t) \frac{\bar{u}(0,t) + \bar{u}_s(0,t)}{D} \\
+ q_2(1,t) \frac{\partial f}{\partial \hat{u}}(\hat{p}(0,t),\hat{w}(0,t) + \kappa(\hat{p}(0,t))) \frac{\bar{u}_s(0,t) - \bar{D}(t)p_1(0,t) - \dot{D}(t)p_2(0,t)}{D}
\end{align*}

and in which we have used

\[
f_{dp}(t) = \frac{\partial f}{\partial \hat{p}}(\hat{p}(0,t),u(0,t)) - \frac{\partial f}{\partial \hat{p}}(\hat{p}(0,t),\hat{u}(0,t))
\]

\[
f_{du}(t) = \frac{\partial f}{\partial \hat{u}}(\hat{p}(0,t),u(0,t)) - \frac{\partial f}{\partial \hat{u}}(\hat{p}(0,t),\hat{u}(0,t))
\]

\[
q_{1,t}(1,t) = -\dot{D}(t) \frac{d \kappa}{d \hat{p}}(\hat{p}(1,t)) \int_0^1 \Phi(1,y,t) \left[ f(\hat{p}(y,t),\hat{w}(y,t) + \kappa(\hat{p}(y,t))) \\
+ \frac{\partial f}{\partial \hat{u}}(\hat{p}(y,t),\hat{w}(y,t) + \kappa(\hat{p}(y,t))) \right] dy - \dot{D}(t) \hat{p}_1(1,t) T \frac{d^2 \kappa}{d \hat{p}^2}(\hat{p}(1,t)) \int_0^1 \Phi(1,y,t) \\
\times f(\hat{p}(y,t),\hat{w}(y,t) + \kappa(\hat{p}(y,t))) \left[ \hat{w}_x(y,t) + \dot{D}(t) \frac{d \kappa}{d \hat{p}}(\hat{p}(y,t)) \\
+ \hat{D}(t) \frac{d \kappa}{d \hat{p}}(\hat{p}(y,t)) f(\hat{p}(y,t),\hat{w}(y,t) + \kappa(\hat{p}(y,t))) \right] dy
\]

\[
- \dot{D}(t) \frac{d \kappa}{d \hat{p}}(\hat{p}(1,t)) \int_0^1 \Phi(1,y,t) \left[ \frac{\partial f}{\partial \hat{p}}(\hat{p}(y,t),\hat{w}(y,t) + \kappa(\hat{p}(y,t))) \hat{p}_1(y,t) \\
+ \frac{\partial f}{\partial \hat{u}}(\hat{p}(y,t),\hat{w}(y,t) + \kappa(\hat{p}(y,t))) \hat{u}_y(y,t) + \left[ \frac{\partial^2 f}{\partial \hat{u} \partial \hat{p}}(\hat{p}(y,t),\hat{w}(y,t) + \kappa(\hat{p}(y,t))) \hat{p}_1(y,t) \\
+ \frac{\partial^2 f}{\partial \hat{u}^2}(\hat{p}(y,t),\hat{w}(y,t) + \kappa(\hat{p}(y,t))) \hat{u}_y(y,t) \right] (y-1) \hat{w}_x(y,t) + \hat{D}(t) \frac{d \kappa}{d \hat{p}}(\hat{p}(y,t)) \\
\times f(\hat{p}(y,t),\hat{w}(y,t) + \kappa(\hat{p}(y,t))) \right] dy
\]

(38)
with
\[
\dot{p}(x,t) = \frac{1}{D(t)} \left[ \dot{p}(x,t) + \Phi(x,0,t)\dot{D}(t)f_u(t) + \dot{D}(t)\int_0^x \Phi(x,y,t) \left[ f(\dot{p}(y,t),\dot{u}(y,t)) + \frac{\partial f}{\partial \dot{u}}(\dot{p}(y,t),\dot{u}(y,t))(y-1) \right] \right] dy \]
\[
\dot{u}(x,t) = \frac{1 + \dot{D}(t)(x-1)}{D(t)} \left[ \dot{w}(x,t) + \dot{D}(t)\frac{d\kappa}{d\dot{p}}(\dot{p}(x,t))f(\dot{p}(x,t),\dot{w}(x,t) + \kappa(\dot{p}(x,t))) \right] + \dot{D}(t)(x-1) \times \left[ \dot{w}(x,t) + \frac{d\kappa}{d\dot{p}}(\dot{p}(x,t)) \frac{d}{dx} \left[ f(\dot{p}(x,t),\dot{w}(x,t) + \kappa(\dot{p}(x,t))) \right] + \dot{D}(t)^2 f(\dot{p}(x,t),\dot{w}(x,t) + \kappa(\dot{p}(x,t))) \right] \]
\[
\dot{u}_\alpha(x,t) = \frac{1}{D(t)} \left[ (1 + \dot{D}(t)) \left[ \dot{w}(x,t) + \dot{D}(t)\frac{d\kappa}{d\dot{p}}(\dot{p}(x,t))f(\dot{p}(x,t),\dot{w}(x,t) + \kappa(\dot{p}(x,t))) \right] + \dot{D}(t)(x-1) \times \left[ \dot{w}(x,t) + \frac{d\kappa}{d\dot{p}}(\dot{p}(x,t)) \frac{d}{dx} \left[ f(\dot{p}(x,t),\dot{w}(x,t) + \kappa(\dot{p}(x,t))) \right] + \dot{D}(t)^2 f(\dot{p}(x,t),\dot{w}(x,t) + \kappa(\dot{p}(x,t))) \right] \right] \]

REFERENCES

[1] D. Angeli and E. D. Sontag. Forward completeness, unboundedness observability, and their lyapunov characterizations. Systems & Control Letters, 38(4):209–217, 1999.
[2] Z. Artstein. Linear systems with delayed controls: a reduction. IEEE Transactions on Automatic Control, 27(4):869–879, 1982.
[3] H. Khalil. Nonlinear Systems. 3rd Edition, Prentice Hall, 2002.
[4] M. Krstic. Delay Compensation for Nonlinear, Adaptive, and PDE Systems. Birkhauser, 2009.
[5] W. Kwon and A. Pearson. Feedback stabilization of linear systems with delayed control. IEEE Transactions on Automatic Control, 25(2):266–269, 1980.
[6] A. Manitius and A. Olbrot. Finite spectrum assignment problem for systems with delays. IEEE Transactions on Automatic Control, 24(4):541–552, 1979.
[7] L Praly. Fonctions de lyapunov, stabilité et stabilisation. Classe notes, 2008.