Graph Inverse Semigroups and Leavitt Path Algebras

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Abstract

We study two classes of inverse semigroups built from directed graphs, namely graph inverse semigroups and a new class of semigroups that we refer to as Leavitt inverse semigroups. These semigroups are closely related to graph $C^*$-algebras and Leavitt path algebras. We provide a topological characterization of the universal groups of the local submonoids of these inverse semigroups. We study the relationship between the graph inverse semigroups of two graphs when there is a directed immersion between the graphs. We describe the structure of graphs that admit a directed cover or directed immersion into a circle and we provide structural information about graph inverse semigroups of finite graphs that admit a directed cover onto a bouquet of circles. We also find necessary and sufficient conditions for a homomorphic image of a graph inverse semigroup to be another graph inverse semigroup. We find a presentation for the Leavitt inverse semigroup of a graph in terms of generators and relations. We describe the structure of the Leavitt inverse semigroup and the Leavitt path algebra of a graph that admits a directed immersion into a circle. We show that two graphs that have isomorphic Leavitt inverse semigroups have isomorphic Leavitt path algebras and we classify graphs that have isomorphic Leavitt inverse semigroups. As a consequence, we show that Leavitt path algebras are $0$-retracts of certain matrix algebras.

1 Introduction

The notion of a Leavitt path algebra is an outgrowth of the seminal paper by W.G. Leavitt [23] providing a construction of what is now referred to as the Leavitt algebra $L_F(1, n)$ corresponding to a positive integer $n$ and a field $F$. The algebras $L_F(1, n)$ are the universal examples of algebras that do not have the invariant basis number property, namely if $R = L_F(1, n)$ then the free left $R$-modules $R$ and $R^n$ are isomorphic.

If $F$ is a field and $\Gamma$ is a directed graph, then we may form the Leavitt path algebra $L_F(\Gamma)$, whose elements correspond roughly to directed paths in the graph. The precise definition is given in Section 6 below. Leavitt path algebras for $F = \mathbb{C}$ are closely related to Cuntz-Krieger graph $C^*$-algebras in the sense of Kumjian, Pask and Raeburn [19]. The Leavitt algebra $L_F(1, n)$ is the Leavitt path algebra constructed from the graph $\Gamma = B_n$, the bouquet of $n$ circles (that is, the graph with one vertex and $n$ directed edges). The general study of Leavitt path algebras was initiated independently by Abrams and Aranda Pino [5] and by Ara, Moreno and Pardo [9] around 2004. It has deep connections to ring theory and the theory of graph $C^*$-algebras. We refer the reader to the survey article by Abrams [1] or the book by Abrams, Ara and Siles Molina [3] for much information about Leavitt path algebras.
A certain amount of structural information about Leavitt path algebras may be gleaned from the theory of inverse semigroups. We recall that an inverse semigroup is a semigroup \( S \) such that for every \( a \in S \) there exists a unique element \( a^{-1} \in S \) such that \( a = aa^{-1}a \) and \( a^{-1} = a^{-1}aa^{-1} \).

The book by Lawson [21] is a standard reference for the theory of inverse semigroups and their connections to other fields: any undefined notation and concepts about inverse semigroups that are used in this paper may be found in [21]. In particular, we shall make use (often without comment) of the elementary fact that idempotents of an inverse semigroup commute. We shall also make use of the natural partial order on an inverse semigroup \( S \) (defined by \( a \leq b \) if \( a = eb \) for some idempotent \( e \) of \( S \)).

Most of the inverse semigroups that arise in this paper (except groups!) have a zero, which we denote by 0 (or 0\(_S\) if we need to specify the inverse semigroup \( S \) under consideration) and all homomorphisms under consideration will map 0 to 0. Thus, unless stated otherwise, an inverse semigroup \( S \) has a zero, and a homomorphism \( f : S \to T \) between inverse semigroups \( S \) and \( T \) will be assumed to map 0\(_S\) onto 0\(_T\).

The most obvious way to associate an inverse semigroup to a Leavitt path algebra is to study the connection between Leavitt path algebras and graph inverse semigroups. The graph inverse semigroup \( I(\Gamma) \) associated with a directed graph \( \Gamma \) is defined in Section 2 below. Leavitt path algebras may be viewed as algebras constructed from the contracted semigroup algebra of a graph inverse semigroup by imposing some additional algebra relations known as the Cuntz-Krieger relations. It is known (see [25], Theorem 20) that if two graph inverse semigroups \( I(\Gamma) \) and \( I(\Delta) \) are isomorphic, then the corresponding graphs \( \Gamma \) and \( \Delta \) are isomorphic, and hence the Leavitt path algebras \( L_F(\Gamma) \) and \( L_F(\Delta) \) are isomorphic, but the converse is far from true.

In the present paper we study several structural properties of graph inverse semigroups. We also introduce another inverse semigroup \( LI(\Gamma) \) naturally associated with a directed graph \( \Gamma \). The inverse semigroup \( LI(\Gamma) \) is the multiplicative subsemigroup of the Leavitt path algebra \( L_F(\Gamma) \) generated by the vertices and edges (and “inverse edges”) of the graph \( \Gamma \) (these elements generate \( L_F(\Gamma) \) as an \( F \)-algebra). It is a quotient of the graph inverse semigroup \( I(\Gamma) \) and again has the property that \( L_F(\Gamma) \cong L_F(\Delta) \) if \( LI(\Gamma) \cong LI(\Delta) \). While the converse is certainly false in general, these inverse semigroups provide significantly more information about Leavitt path algebras than do graph inverse semigroups.

The definition and basic notation for graph inverse semigroups is introduced in Section 2 of this paper. In Section 3 we study the relationship between the graph inverse semigroups \( I(\tilde{\Gamma}) \) and \( I(\Gamma) \) when there is a directed cover or directed immersion \( f : \tilde{\Gamma} \to \Gamma \). In this case the map \( f \) induces homomorphisms between corresponding local submonoids of the graph inverse semigroups (Theorem 3.4). We provide a description of graphs that admit a directed cover or directed immersion into a circle (Theorem 3.1) and we prove a structural property of finite directed covers of a bouquet of circles (Theorem 3.6). Section 4 is concerned with groups naturally associated with graph inverse semigroups. We examine the universal group of a graph inverse semigroup and provide a topological description of the universal groups of its local submonoids (Theorems 4.5 and 4.9). In Section 5 we determine necessary and sufficient conditions for a quotient of a graph inverse semigroup \( I(\Gamma) \) to be another graph inverse semigroup (Theorem 5.3) and as a consequence we show that the quotient graph inverse semigroup is a retract of \( I(\Gamma) \) (Corollary 5.5).

Section 6 is concerned with Leavitt path algebras and Leavitt inverse semigroups. We define the Leavitt inverse semigroup \( LI(\Gamma) \) associated with a directed graph \( \Gamma \) and find a presentation for \( LI(\Gamma) \) as an inverse semigroup in terms of generators and relations (Theorem 6.3).
describe the structure of the Leavitt inverse semigroup and the Leavitt path algebra of a graph that admits a directed immersion into a circle (Theorems 6.6 and 6.7). We show that two graphs that have isomorphic Leavitt inverse semigroups have isomorphic Leavitt path algebras (Theorem 6.11). In the final section (Section 7) we classify graphs that have isomorphic Leavitt inverse semigroups (Theorem 7.12). As a consequence, we obtain structural results for Leavitt path algebras of a restricted class of graphs and we show that Leavitt path algebras are 0-retracts of matrix algebras of a restricted type (Theorem 7.20).

2 Graph inverse semigroups

All graphs under consideration in this paper will be directed graphs with either finitely many or countably infinitely many vertices and edges. We denote the set of vertices of a graph Γ by Γ₀ and the set of edges of Γ by Γ₁. If e ∈ Γ₁ then e is a directed edge from a vertex that we will denote by s(e) to a vertex that we will denote by r(e). In fact, s and r can be considered as mappings of Γ₁ into Γ₀, respectively called the source mapping and the range mapping for Γ. Thus for each vertex v ∈ Γ₀, s⁻¹(v) = {e ∈ Γ₁ : s(e) = v} and the out-degree of a vertex v is |s⁻¹(v)|, the number of directed edges with source v. (This is referred to as the index of v by some authors.) We allow for the possibility that s(e) = r(e) = v ∈ Γ₀, in which case e is a loop at v. A directed path in Γ is a finite sequence p = e₁e₂...eₙ of edges eᵢ ∈ Γ₁ with r(eᵢ) = s(eᵢ₊₁) for i = 1, ..., n − 1. We define s(p) = s(e₁) and r(p) = r(eₙ) and refer to p as a directed path from s(p) to r(p). We also consider a vertex v as being an empty (directed) path (i.e. a path with no edges) based at v and with s(v) = r(v) = v.

It is convenient to extend the notation so as to allow paths in which edges are read in either the positive or negative direction. To do this, we associate with each edge e an “inverse edge” e* (sometimes called a “ghost edge” by some authors) with s(e*) = r(e) and r(e*) = s(e). Also define (e*)* = e. We denote by (Γ₁)* the set {e* : e ∈ Γ₁} and assume that Γ₁ ∩ (Γ₁)* = ∅ and that the map e → e* is a bijection from Γ₁ to (Γ₁)*. With this convention, we can define a path in Γ as a sequence p = e₁e₂...eₙ with eᵢ ∈ Γ₁ ∪ (Γ₁)* and r(eᵢ) = s(eᵢ₊₁) for i = 1, ..., n − 1 and for each path p = e₁e₂...eₙ we define the inverse path to be p* = eₙ*...e₂*e₁*. As usual, s(p) = s(e₁) and r(p) = r(pₙ). The graph Γ is said to be connected if for all v, w ∈ Γ₀ there is at least one path p with s(p) = v and r(p) = w while Γ is said to be strongly connected if for all v, w ∈ Γ₀ there is at least one directed path p with s(p) = v and r(p) = w. A path p is a circuit at v if s(p) = r(p) = v. Thus, for example, a path of the form ee* where e is an edge with s(e) = v is a circuit at v. A path p = e₁e₂...eₙ is called reduced if eᵢ ≠ eᵢ₊₁ for each i. A reduced circuit is a circuit p = e₁e₂...eₙ that is a reduced path and such that e₁ ≠ eₙ*. A directed circuit is a directed path that is a circuit. A cycle is a directed circuit e₁e₂...eₙ such that s(eᵢ) ≠ s(eⱼ) if i, j ∈ {1, 2, ..., n} and i ≠ j. Two cycles C₁ and C₂ are said to be conjugate if C₁ = e₁e₂...eₙ and C₂ = eᵢeᵢ₊₁...eₙe₁...eᵢ₋₁ for some i. The graph Γ is acyclic if it has no non-trivial cycles. Γ is called a tree if it is connected and has no non-trivial reduced circuits. Equivalently (see for example Hatcher’s book [16]), Γ is a tree if it is connected and its fundamental group π₁(Γ) is trivial. Thus trees are connected acyclic graphs but connected acyclic graphs are not necessarily trees.

Graph inverse semigroups were first introduced by Ash and Hall [11] (for a restricted class of directed graphs) in connection with their study of the partially ordered set of J*-classes of a semigroup. Graph inverse semigroups generalize the polycyclic monoids introduced by Nivat and Perrot [24] and arise very naturally in the extensive theories of graph C*-algebras and
Leavitt path algebras. Graph inverse semigroups have been studied in their own right by several authors, for example Costa and Steinberg [12], Jones and Lawson [17], Lawson [22], Krieger [18], Mesyan and Mitchell [25] and Wang [29].

Define the graph inverse semigroup $I(\Gamma)$ of a directed graph $\Gamma$ as the semigroup generated by $\Gamma^0 \cup \Gamma^1 \cup (\Gamma^1)^*$ together with a zero $0$ subject to the relations

1. $s(e)e = er(e) = e$ for all $e \in \Gamma^0 \cup \Gamma^1 \cup (\Gamma^1)^*$;
2. $uv = 0$ if $u, v \in \Gamma^0$ and $u \neq v$;
3. $e^* f = 0$ if $e, f \in \Gamma^1$ and $e \neq f$;
4. $e^* e = r(e)$ if $e \in \Gamma^1$.

We emphasize that condition (1) of the definition above implies that $v^2 = v$ for all $v \in \Gamma^0$; that is, the vertices of $\Gamma$ are idempotents in $I(\Gamma)$. Condition (1) also implies that $e^* s(e) = r(e)e^* = e^*$ for all $e \in \Gamma^1$.

It is not difficult to see that $I(\Gamma)$ is in fact an inverse semigroup. A straightforward argument shows that every non-zero element of a graph inverse semigroup $I(\Gamma)$ may be uniquely written in the form $pq^*$ where $p$ and $q$ are directed (possibly empty) paths with $r(p) = r(q)$. We refer to this as the canonical form of a non-zero element of $I(\Gamma)$. The inverse of an element $pq^*$ is of course $qp^*$. If $pq^*$ and $rs^*$ are non-zero elements of $I(\Gamma)$, then the product $pq^* rs^*$ is non-zero if and only if either $q$ is a prefix of $r$ (i.e. $r = qt$ for some directed (possibly empty) path $t$, in which case $pq^* rs^* = p(qt)^*$), or else $r$ is a prefix of $q$ (i.e. $q = rt$ for some directed (possibly empty) path $t$, in which case $pq^* rs^* = (pt)^*$). The non-zero idempotents are of the form $pp^*$ for some (possibly empty) directed path $p$, and $pp^* \geq qq^*$ in the natural partial order on $I(\Gamma)$ if and only if $q = pt$ for some directed (possibly empty) path $t$. Thus the vertices of $\Gamma$ are the maximal idempotents in the natural partial order on $I(\Gamma)$.

If $\Gamma$ is the graph with one vertex and one edge, then $I(\Gamma)$ is the bicyclic monoid with a (removable) zero. If $\Gamma = B_n$ is the bouquet of $n > 1$ circles (i.e. the graph with one vertex and $n$ directed edges), then $I(\Gamma)$ is isomorphic to the polycyclic monoid $P_n$. This is the inverse monoid generated (as an inverse monoid) by a set $A$ with $|A| = n$ subject to the defining relations $a^{-1} a = 1$ and $a^{-1} b = 0$ for all $a, b \in A$ with $b \neq a$. (Here we regard $A$ as a set of labels for the directed edges of $B_n$). The monoid $P_n$ was introduced by Nivat and Perrot [26] as the syntactic monoid of the “correct parenthesis” language with $n$ sets of parentheses. It was rediscovered in the operator algebra literature where it is referred to as the Cuntz monoid, used in the construction of the Cuntz algebra $\mathcal{O}_n$ (see Paterson’s book [27] for details). The algebra constructed from the graph $B_n$ in the original paper by Leavitt [23] is what is now referred to as the Leavitt path algebra of this graph (see [3] for details).

3 Directed Covers and Directed Immersions

A morphism from the (directed) graph $\tilde{\Gamma}$ to the (directed) graph $\Gamma$ is a function $f : \tilde{\Gamma} \to \Gamma$ that takes vertices to vertices and edges to edges, and preserves incidence and orientation of edges; that is, $f(s(\tilde{e})) = s(f(\tilde{e}))$ and $f(r(\tilde{e})) = r(f(\tilde{e}))$ for all $\tilde{e} \in \tilde{\Gamma}^1$. (Here we abuse notation slightly by using the same symbol $f$ to denote the corresponding function that takes vertices to vertices and the function that takes edges to edges.) We extend the notation by defining $f(\tilde{e}^*) = f(\tilde{e})^*$ for all $\tilde{e} \in \tilde{\Gamma}^1$ and $f(\tilde{e}_1 \tilde{e}_2 \ldots \tilde{e}_n) = f(\tilde{e}_1)f(\tilde{e}_2)\ldots f(\tilde{e}_n)$ for each path $\tilde{p} = \tilde{e}_1 \tilde{e}_2 \ldots \tilde{e}_n$. In fact we will often use the notation $f(\tilde{p}) = p$ to denote the image of a path $\tilde{p} = \tilde{e}_1 \tilde{e}_2 \ldots \tilde{e}_n$ in $\tilde{\Gamma}$, where it is understood that $p$ is the path $p = e_1 e_2 \ldots e_n$ in $\Gamma$ and $e_i = f(\tilde{e}_i)$.
A morphism \( f : \bar{\Gamma} \to \Gamma \) between directed graphs induces maps \( f_\bar{v} : s^{-1}(\bar{v}) \to s^{-1}(f(\bar{v})) \) in the obvious way. We say that \( f \) is a directed cover if the induced maps \( f_\bar{v} \) are bijections for each \( \bar{v} \in \bar{\Gamma} \) and that \( f \) is a directed immersion if the induced maps \( f_\bar{v} \) are injections for each \( \bar{v} \in \bar{\Gamma} \).

This is closely related to the classical notion of covers and immersions of graphs in Stallings’ paper [28], the distinction being that Stallings defines \( f \) to be a cover if the induced maps \( f_\bar{v} : s^{-1}(\bar{v}) \cup r^{-1}(\bar{v}) \to s^{-1}(f(\bar{v})) \cup r^{-1}(f(\bar{v})) \) are bijections for each \( \bar{v} \in \bar{\Gamma} \) and \( f \) is an immersion if these induced maps \( f_\bar{v} \) are injections for each \( \bar{v} \in \bar{\Gamma} \).

There is a significant difference between directed immersions (or directed covers) of graphs and immersions (or covers) of graphs in the classical sense. Connected covers of a connected graph \( \Gamma \) are classified by conjugacy classes of subgroups of the fundamental group \( \pi_1(\Gamma) \) of the graph (see [16] or [28]). For example, connected covers of the circle \( B_{\{a\}} \) (the graph with one vertex and one directed edge) are classified by subgroups of \( \mathbb{Z} \). This yields the circuits \( C_n \) with \( n \) edges (the finite covers of \( B_{\{a\}} \)) and the universal cover \( B_{\{a\}} \) (the Cayley graph of \( \mathbb{Z} \) relative to the usual presentation \( \mathbb{Z} = \langle p(a : 0) \rangle \)). The only connected immersions into \( B_{\{a\}} \) are the connected covers and the connected subgraphs of the universal cover. However the description of directed covers of \( B_{\{a\}} \) and directed immersions into \( B_{\{a\}} \) is more complicated. Let \( L_\infty \) be the linear graph with vertices \( v_{-k}, k \geq 0 \) and an edge \( e_k \) from \( v_{-k} \) to \( v_{-k+1} \) for \( k > 0 \). For each integer \( n \geq 0 \) let \( L_n \) be the induced subgraph of \( L_\infty \) spanned by the vertices \( v_{-k}, 0 \leq k \leq n \).

**Theorem 3.1** A graph \( \Gamma \) admits a directed immersion into \( B_{\{a\}} \) if and only if the out-degree of every vertex of \( \Gamma \) is at most 1. If \( \Gamma \) is a connected graph, all of whose vertices have out-degree at most 1, then

(a) \( \Gamma \) has at most one sink. If \( \Gamma \) does have a sink \( v_0 \) and \( v \) is any other vertex in \( \Gamma \), then there is a unique directed path from \( v \) to \( v_0 \) and \( \Gamma \) is a tree. \( \Gamma \) has a sink if and only if it admits a directed cover onto the graph \( L_n \) where \( n \) is the maximum length of a directed path from some vertex of \( \Gamma \) to \( v_0 \) (and \( n = \infty \) if there are directed paths of arbitrary length ending at \( v_0 \)).

(b) If \( \Gamma \) is not a tree then \( \Gamma \) has a non-trivial cycle and any two non-trivial cycles are cyclic conjugates of each other. Furthermore, if \( v' \) is any vertex on one of these cycles \( C \) and \( v \) is any other vertex of \( \Gamma \), then there is a unique directed path from \( v \) to \( v' \) that does not include the cycle \( C \) as a subpath. In this case \( \Gamma \) is a directed cover of \( B_{\{a\}} \).

(c) \( \Gamma \) is a directed cover of \( B_{\{a\}} \) if and only if it is either a tree that has no sink or \( \pi_1(\Gamma) \cong \mathbb{Z} \), in which case \( \Gamma \) has a structure as described in part (b).

**Proof.** It is clear from the definition of a directed immersion that if there is a directed immersion from \( \Gamma \) into \( B_{\{a\}} \), then the out-degree of every vertex of \( \Gamma \) is at most 1 and that if \( \Gamma \) covers \( B_{\{a\}} \), then the out-degree of every vertex of \( \Gamma \) is 1. Conversely, if the out-degree of every vertex is at most 1 then the obvious map from \( \Gamma \) to \( B_{\{a\}} \) is a directed immersion, which is a directed cover if the out-degree of every vertex is 1.

(a) Observe first that if \( p = e_1e_2 \ldots e_n \) is a path in \( \Gamma \) such that \( e_1 \in (\Gamma^1)^* \), then we must have \( e_i \in (\Gamma^1)^* \) for all \( i \) since every vertex has out-degree at most 1. Suppose that \( v_0 \) and \( v_1 \) are sinks of \( \Gamma \). Since \( \Gamma \) is connected there is a path \( p = e_1e_2 \ldots e_k \) from \( v_0 \) to \( v_1 \). But since \( v_0 \) and \( v_1 \) are both sinks we must have \( e_1 \in (\Gamma^1)^* \) and \( e_k \in \Gamma^1 \). This violates the observation above unless \( v_0 = v_1 \), so \( \Gamma \) has a unique sink if it has one. Suppose that \( \Gamma \) does have a sink \( v_0 \) and that \( v \) is any vertex in \( \Gamma \) with \( v \neq v_0 \). There is a path \( p \) containing no circuits from \( v \) to \( v_0 \) that must be directed by the argument above. If \( p' \) is another directed path from \( v \) to \( v_0 \), then \( p' \) has no circuits since the out-degree of every vertex in \( p' \) other than \( v_0 \) is 1. We may write \( p = p_1p_2 \) and \( p' = p_1p_2' \) where the first edge of \( p_2 \) is different from the first edge of \( p_2' \). But this yields a
vertex \( r(p_1) \) of degree at least 2, a contradiction. So there is a unique directed path from \( v \) to 
\( v_0 \). If \( p = e_1 e_2 \ldots e_n \) is a reduced circuit such that \( s(e_i) \neq s(e_j) \) for \( i \neq j \), then either \( p \) or \( p^* \) is 
a cycle since the out-degree of every vertex in the circuit must be 1. But the graph \( \Gamma \) cannot 
contain any non-trivial cycle since \( v_0 \) is not a vertex of any such cycle and every vertex \( v \) in 
a cycle must have out-degree 1, which is impossible since there is a directed path from \( v \) to \( v_0 \). It 
follows that \( \Gamma \) is a tree.

Suppose that \( n \) is the maximum length of a directed path in \( \Gamma \) ending at \( v_0 \). For each vertex \( v \) 
of \( \Gamma \) let \( d(v) \) be the length of the directed path from \( v \) to \( v_0 \). If \( e \) is an edge of \( \Gamma \) with 
\( d(s(e)) = k \), then \( d(r(e)) = k - 1 \). The graph map that takes such an edge \( e \) to the edge \( e_k \) of \( L_n \) (and takes 
\( s(e) \) to \( v_{-k} \) and \( r(e) \) to \( v_{-k+1} \)) is a covering map, and every graph that admits a surjective cover 
onto \( L_n \) is of this form. The argument easily extends to the case when \( n = \infty \). A graph that 
admits a directed cover of \( L_4 \) is illustrated in Diagram 3.1.

![Diagram 3.1 A directed cover of \( L_4 \)](image)

(b) If \( \Gamma \) is not a tree then it must have at least one non-trivial reduced circuit, and hence 
\( \Gamma \) must have a nontrivial cycle since every vertex has out-degree 1 by part (a). Suppose that \( \Gamma \) 
has two distinct cycles \( C_1 \) and \( C_2 \) that are not just cyclic conjugates of each other. These cycles 
cannot be disjoint. To see this, note that if \( v_1 \) is a vertex in \( C_1 \) and \( v_2 \) is a vertex in \( C_2 \) then 
there is a path \( p = e_1 \ldots e_k \) (containing no cycles) from \( v_1 \) to \( v_2 \). Since all vertices in a cycle 
have out-degree 1, there must be indices \( i \) and \( j \) with \( 1 \leq i < j \leq k \) such that \( e_i \in (\Gamma^1)^* \) and 
\( e_j \in \Gamma^1 \). But this violates the observation in the proof of part (a). So the cycles \( C_1 \) and \( C_2 \) must 
have some vertex \( v \) in common. Then the cyclic conjugates of \( C_1 \) and \( C_2 \) starting at \( v \) must be 
equal or else there is some vertex \( w \) in \( C_1 \cap C_2 \) of out-degree at least 2, a contradiction. Hence 
\( C_1 \) and \( C_2 \) are cyclic conjugates of each other.

Suppose that \( v' \) is any vertex in a non-trivial cycle \( C \) and \( v \) is any other vertex. If \( v \) is on 
the cycle \( C \) then there is a directed path on \( C \) from \( v \) to \( v' \) that does not include the cycle \( C \) 
as a subpath. So suppose that \( v \) is not on \( C \). Then there is a path \( p = e_1 \ldots e_l \) from \( v \) to \( v' \). 
There is a largest integer \( i \) such that \( e_i \) is not an edge in \( C \). Since \( r(e_i) \in C \) has out-degree 1 
and there is an edge of \( \Gamma \) in \( C \) starting at \( r(e_i) \) we must have \( e_i \in \Gamma^1 \). It follows that all of the 
edges \( e_1, \ldots, e_i \) are in \( \Gamma^1 \). Also there is a unique (possibly empty) directed path \( p' \) in \( C \) from 
\( r(e_i) \) to \( v' \) that does not include \( C \) as a subpath. Hence the path \( e_1 \ldots e_i p' \) is a directed path 
from \( v \) to \( v' \) that does not include \( C \) as a subpath. The uniqueness of such a path follows easily 
by an argument similar to that used in part (a). It is clear that in this case \( \Gamma \) is a directed cover 
of \( B_{\{a\}} \) since every vertex has out-degree 1. A graph that admits a directed cover of \( B_{\{a\}} \) is 
illustrated in Diagram 3.2.
Diagram 3.2  A directed cover of $B_{\{a\}}$

(c) If $\Gamma$ is a tree with no sinks, then every vertex of $\Gamma$ has out-degree 1, so $\Gamma$ is a directed cover of $B_{\{a\}}$. If $\Gamma$ is not a tree then $\Gamma$ has the structure described in case (b), and hence it is a directed cover of $B_{\{a\}}$. Also, in this case, since $\Gamma$ has a unique cycle $C$ (up to cyclic conjugates), a spanning tree for $\Gamma$ contains every edge of $\Gamma$ except one edge in $C$, so $\pi_1(\Gamma) \cong \mathbb{Z}$. Conversely, if $\Gamma$ is a directed cover of $B_{\{a\}}$, then by part (a) it does not have a sink. So if $\Gamma$ is a tree, it is a tree with no sinks. If it is not a tree then it has the structure described in case (b), whence $\pi_1(\Gamma) \cong \mathbb{Z}$ by the argument above.

If $f$ is a graph morphism from $\tilde{\Gamma}$ to $\Gamma$ with $f(\tilde{v}) = s(p)$ for some path $p$ in $\Gamma$ and some vertex $\tilde{v}$ in $\tilde{\Gamma}$, then we say that $p$ lifts to $\tilde{v}$ if there is a path $\tilde{p}$ in $\tilde{\Gamma}$ with $f(\tilde{p}) = p$ and $s(\tilde{p}) = \tilde{v}$. Note that directed paths must lift to directed paths if they lift, by the definition of a graph morphism. It is well-known and easy to prove that if $f : \tilde{\Gamma} \to \Gamma$ is a covering map between graphs, then every path in $\Gamma$ starting at a vertex $v$ lifts to a path at $\tilde{v}$ for every vertex $\tilde{v} \in f^{-1}(v)$. This is a very special case of the path lifting theorem in topology. See Hatcher’s book [16] for details. The following easy lemma is the analogous version of this for directed paths in directed graphs.

**Lemma 3.2 (Path lifting lemma for directed covers)** A graph morphism $f : \tilde{\Gamma} \to \Gamma$ is a directed cover if and only if, for every vertex $v \in \Gamma^0$ and every vertex $\tilde{v} \in f^{-1}(v)$, every directed path $p$ in $\Gamma$ with $s(p) = v$ lifts to a unique path $\tilde{p}$ with $s(\tilde{p}) = \tilde{v}$.

**Proof.** If $f$ is a directed cover and $e$ is an edge in $\Gamma$ with $s(e) = v$ then by the definition of a directed cover, there is a unique edge $\tilde{e}$ in $\tilde{\Gamma}$ with $f(\tilde{e}) = e$ and $s(\tilde{e}) = \tilde{v}$. This is the basis for an easy inductive proof that directed paths starting at $v$ lift uniquely to directed paths starting at $\tilde{v}$. The proof of the converse statement is equally straightforward.

The directed path lifting lemma above does not hold for directed immersions that are not directed covers in general, but it is easy to see that maximum initial segments of directed paths in $\Gamma$ lift uniquely to directed paths in $\tilde{\Gamma}$, as described in the following lemma, the proof of which is a simple adaptation of the proof of Lemma 3.2. The analogous observation for immersions between graphs may be found in [15].

**Lemma 3.3 (Path lifting lemma for directed immersions)** Let $f : \tilde{\Gamma} \to \Gamma$ be a directed immersion between graphs, let $v$ be a vertex of $f(\tilde{\Gamma})$ and let $p$ be a directed path in $\Gamma$ with $s(p) = v$. Then for every vertex $\tilde{v} \in f^{-1}(v)$ there is a unique (possibly empty) maximum initial segment $p_1$ of $p$ that lifts to a directed path at $\tilde{v}$. Furthermore, the lift of $p_1$ at $\tilde{v}$ is unique.

For each vertex $v$ of a graph $\Gamma$, let $vI(\Gamma)v$ be the local submonoid of $I(\Gamma)$ with identity $v$. Since $vpq^*v = 0$ if $pq^*$ is not a circuit at $v$ it follows that the non-zero elements of $vI(\Gamma)v$ are...
the circuits of the form $pq^*$ where $p$ and $q$ are directed (possibly empty) paths with $r(p) = r(q)$ and $s(p) = s(q) = v$. Clearly $vI(\Gamma)v$ is non-trivial (i.e. does not consist of just $v$ and 0) if and only if $v$ is not a sink in the graph $\Gamma$ since if $e$ is an edge of $\Gamma$ with $s(e) = v$, then $ee^* \in vI(\Gamma)v$ and $ee^* \neq v$.

Recall our convention that if $f : S \rightarrow T$ is a homomorphism between inverse semigroups then $f(0_S) = 0_T$. The homomorphism $f$ is called a 0-restricted homomorphism from $S$ to $T$ if in addition $f^{-1}(0_T) = \{0_S\}$. We call a function $f : \tilde{\Gamma} \rightarrow \Gamma$ a 0-morphism if $f(0_{\tilde{\Gamma}}) = 0_{\Gamma}$ and $f(st) = f(s)f(t)$ if $st \neq 0$ and we say that it is a 0-restricted morphism if in addition $f^{-1}(0_T) = \{0_S\}$. Note that a homomorphism from $S$ to $T$ is a 0-morphism, but not every 0-morphism is a homomorphism since we may have non-zero elements $s, t \in S$ with $st = 0$ but $f(s)f(t) \neq 0$. For example, let $S$ be the three-element semilattice $S = \{e_1, e_2, 0\}$ where $e_1$ and $e_2$ are idempotents with $e_1e_2 = 0$ and let $T$ be the two-element semilattice $T = \{1, 0\}$. The function $f : S \rightarrow T$ that takes $e_1$ and $e_2$ to 1 and 0 to 0 is a 0-morph that is not a homomorphism. In general, it is clear that a function $f : \tilde{\Gamma} \rightarrow \Gamma$ is a homomorphism if and only if it is a 0-morphism with the property that $f(s)f(t) = 0$ if $st = 0$.

A graph morphism $f : \tilde{\Gamma} \rightarrow \Gamma$ induces a natural function (which we denote by $f_*$) from $I(\tilde{\Gamma})$ to $I(\Gamma)$ in the obvious way. This induced function $f_*$ maps 0 to 0 and maps a nonzero element $\tilde{pq}^*$ of $I(\tilde{\Gamma})$ to $pq^*$ (where $f(\tilde{p}) = p$ and $f(\tilde{q}) = q$). By the definition of a graph morphism it is clear that $pq^*$ is a non-zero element of $I(\Gamma)$ if $\tilde{pq}^*$ is a non-zero element of $I(\tilde{\Gamma})$ since $r(p) = r(q)$ if $r(\tilde{p}) = r(\tilde{q})$. The induced function $f_*$ is well-defined by the uniqueness of canonical forms for elements of $I(\tilde{\Gamma})$ but it is not in general a homomorphism: in fact by Theorem 20 of [25] it is a homomorphism if and only if the graph morphism $f : \tilde{\Gamma} \rightarrow \Gamma$ is injective. However, we have the following fact.

**Theorem 3.4** Let $f : \tilde{\Gamma} \rightarrow \Gamma$ be a morphism of graphs with $f(\tilde{v}) = v$ for vertices $\tilde{v} \in \tilde{\Gamma}$ and $v \in \Gamma$. Then the following hold.

(a) $f_*$ is a 0-restricted morphism from $I(\tilde{\Gamma})$ to $I(\Gamma)$.
(b) $f_*$ induces a 0-restricted morphism of $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ into $vI(\Gamma)v$ for all vertices $\tilde{v}$ of $\tilde{\Gamma}$.
(c) $f$ is a directed immersion if and only if the 0-morphisms from $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ into $vI(\Gamma)v$ induced by $f_*$ are all injective homomorphisms (i.e. embeddings).
(d) $f$ is a directed cover if and only if the induced 0-morphisms from $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ to $vI(\Gamma)v$ are all full embeddings: that is, the image of $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ is a full inverse submonoid of $vI(\Gamma)v$ for all vertices $\tilde{v}$ of $\tilde{\Gamma}$.

**Proof.** (a) Suppose that $\tilde{pq}^*$ and $\tilde{p}'q'^*$ are non-zero elements of $I(\tilde{\Gamma})$ and denote their images under $f_*$ by $pq^*$ and $p'q'^*$ respectively. If $\tilde{pq}^*\tilde{p}'(\tilde{q}')^*$ is non-zero in $I(\tilde{\Gamma})$, then from the multiplication of canonical forms in graph inverse semigroups we either have $\tilde{q}$ is a prefix of $\tilde{p}'$ or $\tilde{p}'$ is a prefix of $\tilde{q}$. This easily implies that either $q$ is a prefix of $p'$ or $p'$ is a prefix of $q$. From this and the definition of the multiplication of canonical forms it is easy to see that the induced map $f_*$ is a 0-morphism. It is in fact a 0-restricted morphism since $f_*(\tilde{pq}^*) \neq 0$ if $\tilde{pq}^* \neq 0$.

(b) If $\tilde{pq}^*$ is a non-zero element of $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ then clearly $pq^*$ is a non-zero element of $vI(\Gamma)v$. It follows from part (a) that the restriction of $f_*$ to $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ is a 0-restricted morphism to $vI(\Gamma)v$.

(c) Now suppose that $f$ is a directed immersion from $\Gamma$ to $\Gamma$ and let $f(\tilde{v}) = v$. Let $\tilde{pq}^*$ and $\tilde{p}'q'^*$ be non-zero elements of $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ and denote their images under $f_*$ by $pq^*$ and $p'q'^*$ respectively. Suppose that $\tilde{pq}^*\tilde{p}'(\tilde{q}')^* = 0$ in $I(\tilde{\Gamma})$. Then $\tilde{q}$ is not a prefix of $\tilde{p}'$ and $\tilde{p}'$ is not a prefix of $\tilde{q}$. Hence we may write $\tilde{q} = \tilde{e}_1\ldots\tilde{e}_k\tilde{e}_{k+1}\ldots \tilde{e}_m$ and $\tilde{p}' = \tilde{e}_1\ldots\tilde{e}_k\tilde{e}'_{k+1}\ldots \tilde{e}'_m$ for some edges $\tilde{e}_i$ and $\tilde{e}'_j$ in $\tilde{\Gamma}$ with $s(\tilde{e}_1) = \tilde{v}$ and $\tilde{e}_{k+1} \neq \tilde{e}'_{k+1}$. (We allow for the possibility that the
common prefix $\tilde{e}_1\ldots\tilde{e}_k$ of $\tilde{q}$ and $\tilde{p}'$ might be empty.) It follows that $f_*(\tilde{q}) = e_1 \ldots e_k e_{k+1} \ldots e_n$ and $f_*(\tilde{p}') = e_1 \ldots e_k e'_{k+1} \ldots e'_n$ (where $f(\tilde{e}_i) = e_i$ and $f(\tilde{e}'_j) = e'_j$). Then since $f$ is a directed immersion and $\tilde{e}_{k+1} \neq \tilde{e}'_{k+1}$ it follows that $e_{k+1} \neq e'_{k+1}$, whence $q$ is not a prefix of $p'$ and $p'$ is not a prefix of $q$. Hence $pq^*p'q'^* = 0$. This implies that the restriction of $f_*$ to $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ is a 0-restricted homomorphism since it is a 0-restricted morphism by part (b).

Conversely, suppose that the restriction of $f_*$ to $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ is a homomorphism for all $\tilde{v}$. Suppose that there are two edges $\tilde{e}_1$ and $\tilde{e}_2$ in $\tilde{\Gamma}$ with $s(\tilde{e}_1) = s(\tilde{e}_2) = \tilde{v}$ and $f(\tilde{e}_1) = f(\tilde{e}_2) = e \in \Gamma^1$. Then $\tilde{e}_1\tilde{e}_1^*, \tilde{e}_2\tilde{e}_2^* \in \tilde{v}I(\tilde{\Gamma})\tilde{v}$ and $f_*(\tilde{e}_1\tilde{e}_1^*) = f_*(\tilde{e}_2\tilde{e}_2^*) = e e^* \in vI(\Gamma)v$. If $\tilde{e}_1 \neq \tilde{e}_2$ then $\tilde{e}_1\tilde{e}_2 = 0$ and so $\tilde{e}_1\tilde{e}_1^*\tilde{e}_2\tilde{e}_2^* = 0$. But $f_*(\tilde{e}_1\tilde{e}_1^*\tilde{e}_2\tilde{e}_2^*) = (ee^*)(ee^*) = ee^* \neq 0$. This violates the assumption that $f_*$ is a homomorphism, and so we must have $\tilde{e}_1 = \tilde{e}_2$. Hence $f$ is a directed immersion.

Now suppose that $f$ is a directed immersion and $f_*(\tilde{p}q^*) = f_*(\tilde{r}s^*) = pq^*$ for some non-zero elements $\tilde{p}q^*$ and $\tilde{r}s^*$ of $\tilde{v}I(\tilde{\Gamma})\tilde{v}$. Since $f$ maps directed edges to directed edges, $f(\tilde{p}) = p = f(\tilde{r})$. That is, $\tilde{p}$ and $\tilde{r}$ are lifts of $p$ at $\tilde{v}$. By the “uniqueness” part of Lemma 3.3 this forces $\tilde{p} = \tilde{r}$. Similarly $\tilde{q} = \tilde{s}$, so $\tilde{p}q^* = \tilde{r}s^*$. Hence $f_*$ is an injective map from $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ to $vI(\Gamma)v$.

(a) Suppose now that $f$ is a directed covering map from $\tilde{\Gamma}$ to $\Gamma$, let $\tilde{v}$ be a vertex in $\tilde{\Gamma}$ and $f(\tilde{v}) = v$. By part (c), $f_*$ is an injective map from $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ to $vI(\Gamma)v$. From the multiplication in $I(\Gamma)$ it follows that the non-zero idempotents of $I(\Gamma)$ are of the form $pp^*$ for some directed path $p$ starting at $v$. By Lemma 3.2 the path $p$ lifts to a unique path $\tilde{p}$ at $\tilde{v}$ and so $pp^*$ lifts to $\tilde{p}\tilde{p}^*$, an idempotent of $I(\tilde{\Gamma})$, so $f$ induces a full embedding of $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ into $vI(\Gamma)v$. Conversely, suppose that $f_*$ induces a full embedding of $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ into $vI(\Gamma)v$. Then if $p$ is a directed path in $\Gamma$ starting at $v$, the circuit $pp^*$ is an idempotent in $vI(\Gamma)v$, so it is the image under $f_*$ of some idempotent in $\tilde{v}I(\tilde{\Gamma})\tilde{v}$, which must be of the form $\tilde{p}\tilde{p}^*$ for some lift $\tilde{p}$ of $p$ at $\tilde{v}$. Hence all directed paths in $\Gamma$ starting at all vertices $v \in \Gamma^1$ lift to all preimages $\tilde{v} \in f^{-1}(v)$, whence $f$ is a directed covering map by Lemma 3.2.

We remark that the induced maps from $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ to $vI(\Gamma)v$ are in general not surjective since directed circuits in $\Gamma$ do not necessarily lift to directed circuits in $\tilde{\Gamma}$, even when $f$ is a cover. However powers of directed circuits do lift to directed circuits via finite directed covers of $\Gamma$.

Lemma 3.5 Let $f : \tilde{\Gamma} \to \Gamma$ be a directed cover of finite graphs, let $v$ be a vertex in $f(\tilde{\Gamma})$ and let $p$ be a directed circuit at $v$. Then there is a vertex $\tilde{v}' \in f^{-1}(v)$ and a positive integer $n$ such that $p^n$ lifts to a directed circuit at $\tilde{v}'$.

Proof. Let $\tilde{v}_0$ be any vertex in $f^{-1}(v)$. By the directed path lifting lemma (Lemma 3.2), $p$ lifts to a directed path $\tilde{p}_1$ from $\tilde{v}_0$ to some vertex $\tilde{v}_1$. Then $f(\tilde{v}_1) = v$ so again $p$ lifts to a directed path $\tilde{p}_2$ from $\tilde{v}_1$ to some vertex $\tilde{v}_2$. Continue like this to obtain a sequence of lifted paths $\tilde{p}_i$ from $\tilde{v}_{i-1}$ to $\tilde{v}_i$. By finiteness of $\Gamma$ we must have $\tilde{v}_i = \tilde{v}_{i+n}$ for some $n > 0$ and $i \geq 0$. Then $p^n$ lifts to the directed circuit $\tilde{p}_{i+1} \ldots \tilde{p}_{i+n}$ at $\tilde{v}' = \tilde{v}_i$.

Theorem 3.6 Let $f : \tilde{\Gamma} \to B_A$ be a finite directed cover of the bouquet of $|A|$ circles. Then for every vertex $\tilde{v}$ in $\tilde{\Gamma}$, $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ contains a submonoid isomorphic to the polycyclic monoid $PA$ if $|A| > 1$, and $\tilde{v}I(\tilde{\Gamma})\tilde{v}$ contains a submonoid isomorphic to the bicyclic monoid if $|A| = 1$.

Proof. Let $V_a$ denote the loop in $B_A$ labeled by $a \in A$. For each $a \in A$ let $V_a$ be the set of vertices $\tilde{w}$ in $\tilde{\Gamma}$ that lie on a non-trivial cycle $\tilde{e}_1 \ldots \tilde{e}_s$ such that $s(\tilde{e}_1) = \tilde{w}$ and $f(\tilde{e}_1) = e_a$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and let $V_m = \bigcap_{i=1, \ldots, n} V_{a_i}$ for $m \leq n$. We claim that $V_m \neq \emptyset$ and that if $\tilde{v}$ is any vertex in $\tilde{\Gamma}$ and $\tilde{e}$ is any edge with $s(\tilde{e}) = \tilde{v}$, then there is a directed path $\tilde{p} = \tilde{e}_1 \tilde{e}_2 \ldots \tilde{e}_l$ from $\tilde{v}$ to some vertex $\tilde{v}' \in V_m$. By the proof of Lemma 3.5, some power of the loop $e_a$, lifts to
a directed path \( \tilde{p}' \) starting at \( r(\tilde{e}') \) and ending at a vertex in \( V_1 \), so the directed path \( \tilde{e}' \tilde{p}' \) leads from \( \tilde{v} \) to a vertex in \( V_1 \), and hence the claim is true if \( m = 1 \). Assume inductively that it is true if \( m = k \). Let \( \tilde{v} \) be any vertex in \( \tilde{\Gamma} \), \( \tilde{e}' \) an edge starting at \( \tilde{v} \), and let \( \tilde{e}'_1 \) be the (unique) edge in \( \tilde{\Gamma} \) with \( s(\tilde{e}'_1) = r(\tilde{e}') \) and \( f(\tilde{e}'_1) = e_{a_{k+1}} \). By the induction assumption, \( \tilde{e}'_1 \) can be extended to some directed path \( \tilde{p}_0 \) from \( \tilde{v}_0 = r(\tilde{e}'_1) \) to a vertex \( \tilde{v}_1 \in V_k \). But then again by the induction hypothesis there is a directed path \( \tilde{p}_1 \) from \( \tilde{v}_1 \) to some vertex \( \tilde{v}_2 \in V_k \) whose first edge projects by \( f \) to \( e_{a_{k+1}} \). Continue in this fashion to obtain a sequence of directed paths \( \tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_i, \ldots \) whose first edge projects onto \( e_{a_{k+1}} \) and with \( \tilde{v}_i = s(\tilde{p}_i) \in V_k \) for all \( i \geq 1 \). By finiteness of \( \tilde{\Gamma} \) there must be a directed circuit \( \tilde{p}_i \tilde{p}_{i+1} \ldots \tilde{p}_j \) based at \( \tilde{v}_i \) for some \( i < j \). If the first edge of \( \tilde{p}_i \) is a loop at \( \tilde{v}_i \) (that projects onto \( e_{a_{k+1}} \)) then \( \tilde{v}_i \in V_{k+1} \). So assume this is not the case. Choosing \( i \) and \( j \) minimal, we may assume that this circuit \( \tilde{p}_i \tilde{p}_{i+1} \ldots \tilde{p}_j \) is a cycle. But then since the first edge of \( \tilde{p}_i \) projects onto \( e_{a_{k+1}} \) we see that in fact \( \tilde{v}_i \in V_{k+1} \). The claim then follows by induction on \( k \).

Thus for every vertex \( \tilde{v} \) in \( \tilde{\Gamma} \) there is a directed path \( \tilde{p} \) from \( \tilde{v} \) to some vertex \( \tilde{w} \in V_a \). Denote by \( \tilde{q}_a \) a cycle at \( \tilde{w} \) whose first edge projects onto \( e_a \). Then we see that the paths \( \tilde{r}_a = \tilde{p} \tilde{q}_a \tilde{p}^* \) are in \( \tilde{\nu} I(\tilde{\Gamma}) \tilde{v} \) for all \( a \in A \). From the relations in \( I(\tilde{\Gamma}) \) it easily follows that \( \tilde{r}_a^* \tilde{r}_a = \tilde{p} \tilde{p}^* \) and \( \tilde{r}_a \tilde{r}_b = 0 \) if \( a \neq b \), so the inverse subsemigroup of \( \tilde{\nu} I(\tilde{\Gamma}) \tilde{v} \) generated by the elements \( \tilde{r}_a \) (for \( a \in A \)) is a homomorphic image of the copy of the polycyclic monoid \( P_A \) with identity \( \tilde{p} \tilde{p}^* \) (provided \( |A| > 1 \)). Since the polycyclic monoid is congruence free (see [21]) it follows that this monoid is isomorphic to \( P_A \). A similar argument yields a copy of the bicyclic monoid if \( |A| = 1 \).

**Remarks**

(a) The conclusion of Theorem 3.6 is in general false if \( \tilde{\Gamma} \) is an infinite directed cover of \( B_A \). For example, if \( \tilde{\Gamma} \) is the universal cover of the circle \( B_{\{a\}} \) and \( \tilde{v} \) is any vertex of \( \Gamma \), then no power of the loop in \( B_{\{a\}} \) lifts to a circuit at \( \tilde{v} \) and \( I(\tilde{\Gamma}) \) does not contain a copy of the bicyclic monoid.

(b) The conclusion of Theorem 3.6 also fails if \( f \) is a directed immersion that is not a directed cover. For example, if \( \tilde{\Gamma} \) is the graph with two vertices \( \tilde{v}_1 \) and \( \tilde{v}_2 \) and one directed edge \( \tilde{e} \) from \( \tilde{v}_1 \) to \( \tilde{v}_2 \), then there is a directed immersion of \( \tilde{\Gamma} \) into the circle \( B_{\{a\}} \), but \( I(\tilde{\Gamma}) \) is finite and so does not contain a copy of the bicyclic monoid.

(c) It is not true in general that if \( \tilde{\Gamma} \) is a finite directed cover of \( \Gamma \), then \( I(\Gamma) \) embeds in \( I(\tilde{\Gamma}) \). For example, let \( \Gamma \) be the graph with two vertices \( v \) and \( w \) and two edges \( a \) and \( b \) from \( v \) to \( w \), and let \( \tilde{\Gamma} \) be graph with three vertices, \( v_1, w_1 \) and \( w_2 \) and two edges, namely \( a_1 \) from \( v_1 \) to \( w_1 \) and \( b_1 \) from \( v_1 \) to \( w_2 \). Then the map that sends \( v_1 \) to \( v \), \( w_1 \) to \( w \), \( a_1 \) to \( a \) and \( b_1 \) to \( b \) is a directed cover but \( I(\Gamma) \) does not embed in \( I(\tilde{\Gamma}) \). Thus Theorem 3.6 is specific to finite directed covers of a graph \( B_A \).

### 4 Universal groups

Recall that if \( S \) and \( T \) are inverse semigroups with 0, then a function \( \theta : S \to T \) is called a 0-morphism if \( \theta(0) = 0 \) and \( \theta(st) = \theta(s)\theta(t) \) if \( st \neq 0 \). We define the *universal group* \( U(S) \) of an inverse semigroup \( S \) with 0 to be the group generated by the set \( S^* = S \setminus \{0\} \) of non-zero elements of \( S \) subject to the relations \( s \cdot t = st \) if \( st \neq 0 \). Equivalently \( U(S) \) may be defined (up to isomorphism) by the following universal property. Namely, \( U(S) \) is the group with the property that there is a 0-morphism \( r_S : S \to U(S)^0 \) such that if \( \alpha : S \to H^0 \) is a 0-morphism from \( S \) to a group \( H \) with 0 adjoined, then there exists a unique 0-restricted homomorphism \( \beta : U(S)^0 \to H^0 \) such that \( \beta \circ r_S = \alpha \). We say that \( S \) is strongly \( E^*\)-unitary if the 0-morphism
\( \tau_S \) is \textit{idempotent-pure}, that is \( \tau_S^{-1}(1_{U(S)}) \) is the set of non-zero idempotents of \( S \). Lawson shows in [22] that graph inverse semigroups are strongly \( E^* \)-unitary.

A homomorphism \( \phi : S \to T \) between inverse semigroups is called \textit{idempotent-pure} if, for every idempotent \( a \) in \( T \), \( \phi^{-1}(a) \) is a semilattice (i.e. every preimage of an idempotent of \( T \) is an idempotent of \( S \)). An inverse monoid \( S \) is called \textit{factorizable} if for all \( a \in S \) there is an element \( b \) in the group of units of \( S \) such that \( a \leq b \).

**Proposition 4.1** Let \( S \) and \( T \) be inverse semigroups with zero and \( \phi \) a 0-restricted homomorphism from \( S \) to \( T \). Then

(a) \( \phi \) induces a homomorphism \( \phi_U \) from \( U(S) \) to \( U(T) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
S^* & \xrightarrow{\tau_S} & U(S) \\
\downarrow{\phi} & & \downarrow{\phi_U} \\
T^* & \xrightarrow{\tau_T} & U(T)
\end{array}
\]

(b) \( \phi_U \) is surjective if \( \phi \) is surjective;

(c) If \( \phi_U \) is injective and \( S \) is strongly \( E^* \)-unitary, then \( \phi \) is idempotent-pure;

(d) If \( S \) is factorizable and \( T \) is strongly \( E^* \)-unitary, then \( \phi_U \) is injective if \( \phi \) is idempotent-pure.

**Proof.** Since \( \phi \) is 0-restricted it follows that \( \tau_T \circ \phi \) is a 0-morphism from \( S \) to \( U(T) \). Hence by the universal property of \( U(S) \), there is a unique homomorphism \( \phi_U \) from \( U(S) \) to \( U(T) \) such that \( \phi_U(\tau_S(a)) = \tau_T(\phi(a)) \) for all \( a \in S^* \). Since \( \tau_S \) maps \( S^* \) onto the generators of \( U(S) \) and \( \tau_T \) maps \( T^* \) onto the generators of \( U(T) \), it follows that \( \phi_U \) is surjective if \( \phi \) is surjective.

Suppose that \( \phi_U \) is injective and \( S \) is strongly \( E^* \)-unitary. If \( \phi(a) \) is an idempotent of \( T \) then \( \tau_T(\phi(a)) = \phi_U(\tau_S(a)) \) is the identity of \( U(T) \). Hence \( a \) is an idempotent of \( S \), and so \( \phi \) is idempotent-pure.

Now suppose that \( S \) is factorizable and \( T \) is strongly \( E^* \)-unitary and that \( \phi \) is idempotent-pure. Note that if \( S \) is factorizable, then for every element \( a \in S^* \), there exists an element \( a' \) in the group of units of \( S \) such that \( a = ea' \) for some idempotent \( e \). Hence, \( \tau_S(a) = \tau_S(a') \). It follows that, if \( a, b \in S^* \), then \( \tau_S(a)\tau_S(b) = \tau_S(a'\tau_S(b')) = \tau_S(a'b') \). So every element of \( U(S) \) is of the form \( \tau_S(a) \) for some element \( a \) in the group of units of \( S \). If \( \phi_U(\tau_S(a)) = \phi(\tau_T(a)) \) is the identity of \( U(T) \) then since \( \phi \) and \( \tau_T \) are both idempotent-pure it follows that \( a \) is the identity of \( S \), and so \( \tau_S(a) \) is the identity of \( U(S) \), whence \( \phi_U \) is injective.

**Remarks.** (1) The converse of part (b) of Proposition 4.1 is false in general. For example, let \( S \) be the two element semilattice \( S = \{e, 0\} \) and let \( T \) be the three element semilattice \( T = \{e, f, 0\} \) with \( ef = 0 \). Then \( U(S) \cong U(T) \) is the trivial group but the obvious embedding of \( S \) into \( T \) is a homomorphism that is not surjective.

(2) The converse of part (c) of Proposition 4.1 is also false in general. For example, let \( S = SIM(a, b) \), the symmetric inverse monoid on two letters, and let \( T = SIM(a, b, c) \), the symmetric inverse monoid on three letters. The identity map on \( S \) extends in the obvious way to an idempotent-pure homomorphism \( \phi : S \to T \). By Example 2.1 in [22], \( S \) is strongly \( E^* \)-unitary with maximal group image \( U(S) \cong \mathbb{Z}_2 \), the cyclic group of order 2, while \( U(T) \) is the trivial group. The homomorphism \( \phi_U \) is not injective.
The following fact is implicit in Lawson’s paper [22]. We provide a proof for completeness.

**Theorem 4.2** For any graph \( \Gamma \), the universal group \( U(I(\Gamma)) \) is isomorphic to \( FG(\Gamma^1) \), the free group on \( \Gamma^1 \).

**Proof.** First recall that the non-zero elements of \( \Gamma \) consists of all elements of form \( pq^* \) where \( p, q \) are directed paths satisfying \( r(p) = r(q) \). For each edge \( e \in \Gamma^1 \) define \( \tau(e) = e, \) regarded as a generator for \( FG(\Gamma^1) \) and define \( \tau(e^*) = e^{-1} \in FG(\Gamma^1) \). By the uniqueness of the canonical form for non-zero elements of \( I(\Gamma) \), this extends in the obvious way to a well-defined function \( \tau : I(\Gamma) \to FG(\Gamma^1)^0 \) with \( \tau(0) = 0 \) and \( \tau(pq^*) = \text{red}(pq^{-1}) \), the reduced form of \( pq^{-1} \), if \( r(p) = r(q) \). For any \( p_1q_1^*p_2q_2^* \in I(\Gamma)^* \), \( (p_1q_1^*)(p_2q_2^*) \in I(\Gamma)^* \) if and only if either \( q_1 \) is a prefix of \( p_2 \) or \( p_2 \) is a prefix of \( q_1 \). In either case, it is routine to see that \( \tau((p_1q_1^*)(p_2q_2^*)) = \tau(p_1q_1^*)\tau(p_2q_2^*) \).

That is to say, \( \tau \) is a 0-morphism. Now for any group \( H \) and any 0-morphism \( \alpha : I(\Gamma) \to H^0 \), we easily see that \( \alpha(e^*) = \alpha(e)^{−1} \) for every \( e \in \Gamma^1 \). Since \( FG(\Gamma^1) \) is freely generated by the elements \( e \in \Gamma^1 \), the map \( e \mapsto \alpha(e) \) for \( e \in \Gamma^1 \) extends to a unique homomorphism \( \beta : FG(\Gamma^1) \to H \) and clearly \( \alpha = \beta \circ \tau \). Hence \( FG(\Gamma^1) \cong U(I(\Gamma)) \). □

**Corollary 4.3** If \( \Delta \) is a subgraph of \( \Gamma \) then \( U(I(\Delta)) \) is a free factor of \( U(I(\Gamma)) \).

**Proof.** Clearly the set of edges of \( \Delta \) is a subset of the set of edges of \( \Gamma \), so the result is immediate from Theorem 4.2. □

We turn to a description of the universal groups of the local submonoids in \( I(\Gamma) \). The non-zero idempotents of \( I(\Gamma) \) are of the form \( pp^* \) where \( p \) is a directed path in \( \Gamma \). We denote by \( U(I(\Gamma),pp^*) \) the universal group of the local submonoid \( pp^*I(\Gamma)pp^* \). In particular, when \( p \) is the trivial path at the vertex \( v \), \( U(I(\Gamma),v) \) denotes the universal group of the local submonoid \( vI(\Gamma)v \).

Recall that the non-zero elements of the local submonoid \( vI(\Gamma)v \) are of the form \( pq^* \) where \( p \) and \( q \) are directed (or empty) paths with \( s(p) = s(q) = v \) and \( r(p) = r(q) \).

Let \( V_v = \{ w \in \Gamma^0 : \text{there is a (possibly empty) directed path } p \text{ in } \Gamma \text{ from } v \text{ to } w \} \) and let \( \Gamma_v \) denote the subgraph of \( \Gamma \) induced by the vertices in \( V_v \). A subtree \( T \) of \( \Gamma_v \) is called a directed tree at \( v \) if \( T \) contains the vertex \( v \) and every geodesic path in \( T \) from \( v \) to some other vertex \( w \) in \( T \) is directed. \( T \) is called a directed spanning tree of \( \Gamma_v \) at \( v \) if \( T \) is a directed tree at \( v \) that contains all of the vertices of \( \Gamma_v \).

**Lemma 4.4** Let \( v \) be a vertex of a graph \( \Gamma \) and let \( T \) be a directed subtree of \( \Gamma_v \) containing the vertex \( v \). Then \( T \) extends to a directed spanning tree \( T_v \) of \( \Gamma_v \).

**Proof.** The proof of this is a straightforward application of Zorn’s Lemma. Let \( \mathcal{T} \) be the set of all subtrees \( T' \) of \( \Gamma_v \) such that \( T' \) contains the tree \( T \) and \( T' \) is directed at \( v \). Then \( \mathcal{T} \) is a partially ordered set with respect to inclusion (i.e. \( T_1 \leq T_2 \) if and only if \( T_1 \) is a subtree of \( T_2 \)).

It is easy to see that the union of a chain of trees in \( \mathcal{T} \) is another tree in \( \mathcal{T} \), so by Zorn’s Lemma \( \mathcal{T} \) has a maximal element \( T_v \). If \( T_v \) is not a spanning tree of \( \Gamma_v \), then there is some vertex \( w \) of \( \Gamma_v \) that is not in \( T_v \). Since there is a directed path in \( \Gamma_v \) from \( v \) to \( w \), there is some directed path \( p \) that starts at a vertex \( w' \) in \( T_v \) and ends at \( w \) and has no edge or vertex other than \( w' \) in \( T_v \). Then \( T_v \cup \{ p \} \) is a tree strictly containing \( T_v \) as a subtree. If \( p' \) is the geodesic path in \( T_v \) from \( v \) to \( w' \), then \( p'p \) is the geodesic path in \( T_v \cup \{ p \} \) from \( v \) to \( w \) and \( p'p \) is directed, so \( T_v \cup \{ p \} \in \mathcal{T} \). This contradicts the maximality of \( T_v \), so \( T_v \) is a directed spanning tree at \( v \). □

**Theorem 4.5** If \( v \) is a vertex of a graph \( \Gamma \) then \( U(\Gamma,v) \cong \pi_1(\Gamma_v,v) \).
Proof. It follows from Lemma 4.4 (with $T = \{v\}$) that $\Gamma_v$ has a directed spanning tree $T_v$ at $v$. Denote the geodesic path in $T_v$ from $v$ to a vertex $w$ in $\Gamma_v$ by $p_w$. Thus each path $p_w$ is a directed path from $v$ to $w$. The group $\pi_1(\Gamma_v,v)$ is generated by the homotopy classes $[c(e)]$ of circuits of the form $c(e) = p_{s(e)}e p_{r(e)}^*$ for each edge $e$ of $\Gamma_v$ that is not in $T_v$ (see [28] or [16] for basic information about homotopy of graphs). We claim that the set $S_v$ consisting of these circuits, together with $\{0\}$ and the circuits of the form $pp^*$, for $p$ a directed path in $T_v$, starting at $v$, generates the local submonoid $vI(\Gamma)v$ as an inverse submonoid of $I(\Gamma)$.

To see this, suppose first that $w$ is a vertex in $\Gamma_v$, $q' = e'_1e'_2 \ldots e'_m$ is the geodesic path in $T_v$ from $v$ to $w$ and $p = e_1e_2 \ldots e_n$ is any other directed path from $v$ to $w$. We prove by induction on $n$ that $pq^*$ can be expressed as a product of elements in $S_v$ and their inverses. The result is clearly true if $p = q'$ or if $n = 0$ so assume $p \neq q'$ and $n \geq 1$. If $n = 1$ then $e_1 \neq q'$ and so $e_1$ is not an edge in $T_v$. So in this case $pq^* = c(e_1) \in S_v$. So assume that $n > 1$ and that the result is true for all directed paths $p$ of length less than $n$ from $v$ to some vertex $w$ in $\Gamma_v$. Since $p \neq q'$ we may write $p = e_1e_2 \ldots e_j e'_1e'_2 \ldots e'_m$ for some $j \leq n$ and $e_j \neq e_{j-1}'$. (We allow for the case that $e'_1 \ldots e'_m$ is empty.) Let $p_1$ be the geodesic in $T_v$ from $v$ to $s(e_j)$. By the induction assumption, the circuit $e_1e_2 \ldots e_{j-1}p_1^*$ can be written as a product of elements of $S_v$ and their inverses. Also $p_1 e_j(e'_1 \ldots e_{j-1}^*) = c(e_j)$, so $e_1 \ldots e_{j-1}e'_j e'_1 \ldots e_{j-1}^*$ is a product of elements of $S_v$ and their inverses. But then $pq^* = (e_1 \ldots e_j e'_1 \ldots e_{j-1}^*)(q'q^*)$ is a product of elements of $S_v$ and their inverses. Now let $w$ be any vertex in $\Gamma_v$ and $p$, $q$ any directed paths from $v$ to $w$ in $\Gamma_v$. Let $q'$ be the geodesic in $T_v$ from $v$ to $w$. Then by the argument above, the circuits $pq^*$ and $qp^*$ can be written as products of elements in $S_v$ and their inverses. It follows that the circuit $pq^* = (pq^*)(q'q^*)$ is in the inverse submonoid of $I(\Gamma)$ generated by $S_v$. So $vI(\Gamma)v$ is generated as an inverse monoid by the elements of $S_v$.

We now claim that every non-zero element of $vI(\Gamma)v$ can be written uniquely in the form $c(e_1) \ldots c(e_k) pp^*c(e_{k+1}) \ldots c(e_{k+s})^*$ for some edges $e_i$ in $\Gamma_v$ that are not in $T_v$ and some directed path $p$ in $T_v$ starting at $v$ such that both $p_{r(e_k)}$ and $p_{r(e_{k+1})}$ are prefixes of $p$. (We allow for the possibility that either $k$ or $s$ (or both) might be zero). To prove this, we first make three observations.

Observation 1. If $p$ is a directed path in $T_v$ starting at $v$, $e$ is an edge of $\Gamma_v$ that is not in $T_v$ and $pp^*c(e) \neq 0$, then $pp^*c(e) = c(e)$. To see this, note that if $pp^*c(e) = pp^*p_{s(e)}e p_{r(e)}^* \neq 0$, then $p$ is a prefix of $p_{s(e)}e$ since $e$ is not an edge in $T_v$. Hence $p$ is a prefix of $p_{s(e)}$ (again since $e$ is not an edge in $T_v$). Observation 1 follows easily from this.

Observation 2. If $e$ and $f$ are edges of $\Gamma_v$ that are not in $T_v$ and $c(e)c(f) \neq 0$, then $c(e)c(f) = p_{r(e)}p_{r(f)}^*$. To see this, note that $c(e)c(f) \neq 0$ then $p_{r(e)}e p_{s(f)}p_{r(f)}^* \neq 0$. Since neither $e$ nor $f$ is an edge in $T_v$ it follows that $p_{s(e)}e = p_{s(f)}f$ and so $e = f$. This implies that $c(e)c(f) = c(e)c(e)$, which easily yields Observation 2.

Observation 3. The set of elements $pp^*$ where $p$ is a directed path in $T_v$ starting at $v$ is a submonoid of $vI(\Gamma)v$. This follows easily since if $p_1 p_2 p_2^* \neq 0$ then either $p_1$ is a prefix of $p_2$ or $p_2$ is a prefix of $p_1$.

It follows from these three observations and the fact that $vI(\Gamma)v$ is generated as an inverse monoid by $S_v$ that every element of $vI(\Gamma)v$ can be written as a product of the form $c(e_1) \ldots c(e_k) pp^*c(e_{k+1}) \ldots c(e_{k+s})^*$ for some edges $e_i$ in $\Gamma_v$ that are not in $T_v$ and some directed path $p$ in $T_v$ starting at $v$. If $p$ is a prefix of $p_{r(e_k)}$ then $p_{r(e_k)} pp^* = p_{r(e_k)} p_{r(e_k)}^* p_{r(e_k)} p_{r(e_k)}^* p_{r(e_k)}^*$ and similarly if $p$ is a prefix of $p_{r(e_{k+1})}$ then $pp^* p_{r(e_{k+1})} = p_{r(e_{k+1})} p_{r(e_{k+1})} p_{r(e_{k+1})}$. So we may assume without loss of generality that both $p_{r(e_k)}$ and $p_{r(e_{k+1})}$ are prefixes of $p$.

Note that if $c(e_1)c(e_2) \neq 0$, then either $p_{s(e_2)}e_2$ is a prefix of $p_{r(e_1)}$ or $p_{r(e_1)}$ is a prefix
of $p_{s(e_2)}e_2$. Since $e_2$ is not an edge in $T_v$, we must have $p_{r(e_1)}$ is a prefix of $p_{s(e_2)}e_2$. Applying this to all products $c(e_1)c(e_{i+1})$ we see that if $c(e_1)c(e_2)\ldots c(e_k) \neq 0$ then we must have $c(e_1)c(e_2)\ldots c(e_k) = p_{s(e_1)}e_1 p_{r(e_1)} e_2 p_{r(e_2)} e_3 \ldots e_k p_{r(e_k)}$ for some directed paths $p_{s(e_1)}$ in $T_v$ from $r(e_1)$ to $s(e_{i+1})$. A similar argument applies to the non-zero product $c(e_{k+1})^* \ldots c(e_{k+s})^*$. Hence we have

$$c(e_1) \ldots c(e_k) p p^* c(e_{k+1})^* \ldots c(e_{k+s})^* = p_{s(e_1)} e_1 p_{r(e_1)} e_2 p_{r(e_2)} e_3 \ldots e_k p_{r(e_k)} p_{r(e_{k+1})} e_{k+1} p_{r(e_{k+1})} e_{k+2} \ldots e_{k+s} p_{s(e_{k+s})}$$

where the $p_{s(e_1)}$ are paths in $T_v$. The uniqueness of canonical forms in $I(\Gamma)$ and the fact that each $e_i$ is not in $T_v$ implies that if

$$c(e_1) \ldots c(e_k) p p^* c(e_{k+1})^* \ldots c(e_{k+s})^* = c(e'_1) \ldots c(e'_m) p p'^* c(e'_{m+1})^* \ldots c(e'_{m+n})^*$$

then $k = m, s = n, p = p'$ and $e_i = e'_i$ for all $i$.

For $e$ an edge of $\Gamma_v$ not in $T_v$ define $\theta(c(e)) = [c(e)] \in \pi_1(\Gamma_v, v)$ and also define $\theta(pp^*) = 1$ (the identity of $\pi_1(\Gamma_v, v)$) for $p$ a geodesic path in $T_v$ from $v$ to some vertex $w = r(p)$. By the uniqueness of the expression for non-zero elements of $vI(\Gamma)\nu$ established above, it follows that $\theta$ extends to a well-defined function (again denoted by $\theta$) from $vI(\Gamma)\nu$ to $\pi_1(\Gamma_v, v)^0$. A routine argument, using Observations 1, 2 and 3 above, shows that $\theta$ defines a 0-morphism from $vI(\Gamma)\nu$ to $\pi_1(\Gamma_v, v)^0$. If $\alpha$ is any other 0-morphism from $vI(\Gamma)\nu$ to $G^0$, for some group $G$ with 0, then since $\pi_1(\Gamma_v, v)$ is freely generated by the $\theta(c(e))$’s we see, as in the proof of Theorem 4.2, that there is a unique homomorphism $\beta : \pi_1(\Gamma_v, v) \to G$ that satisfies $\alpha = \beta \circ \tau$. Hence $\mathcal{U}(\Gamma, v) \cong \pi_1(\Gamma_v, v)$. ■

**Corollary 4.6** If $v$ is a vertex in a graph $\Gamma$ then $\mathcal{U}(\Gamma, v)$ is a free group. If $u$ and $v$ are vertices in the same strongly connected component of $\Gamma$, then $\mathcal{U}(\Gamma, v) \cong \mathcal{U}(\Gamma, u)$. In particular, if $\Gamma$ is a strongly connected graph, then $\mathcal{U}(\Gamma, v) \cong \pi_1(\Gamma, v)$ is a free group with rank independent of the choice of $v$.

**Proof.** Clearly $\mathcal{U}(\Gamma, v)$ is a free group since it is the fundamental group of a graph by Theorem 4.5. If $u$ and $v$ are in the strongly connected component of $\Gamma$ then there is a directed path $p$ from $u$ to $v$ and a directed path $q$ from $v$ to $u$. If $w$ is any vertex in $\Gamma_v$, there is a directed path $p'$ from $v$ to $w$ and hence there is a directed path $pp'$ from $u$ to $w$, whence $w$ is a vertex in $\Gamma_u$. Hence $\Gamma_v$ is a subgraph of $\Gamma_u$. Similarly $\Gamma_u$ is a subgraph of $\Gamma_v$, so $\Gamma_u = \Gamma_v$. Hence by Theorem 4.5 $\mathcal{U}(\Gamma, v) \cong \pi_1(\Gamma_v, v) \cong \pi_1(\Gamma_u, u) \cong \mathcal{U}(\Gamma, u)$. The result about strongly connected graphs follows immediately since if $\Gamma$ is strongly connected then $\Gamma_v = \Gamma$. ■

**Corollary 4.7** If $\Delta$ is a subgraph of a graph $\Gamma$ and $v$ is a vertex of $\Delta$, then $\mathcal{U}(\Delta, v)$ is a free factor of $\mathcal{U}(\Gamma, v)$.

**Proof.** If there is a directed path in $\Delta$ from $v$ to some vertex $w$ in $\Delta$, then the same path lies in $\Gamma$, so $\Delta_v$ is a subgraph of $\Gamma_v$. Let $T_v$ be a directed spanning tree for $\Delta_v$ at $v$. By Lemma 4.2 $T_v$ can be extended to a directed spanning tree $T_v'$ for $\Gamma_v$ at $v$. Notice that if $e$ is an edge of $\Delta_v$ that is not in $T_v'$, then it is not in $T_v'$ either. Hence the set of free generators for $\mathcal{U}(\Delta, v)$ obtained from $T_v$ is contained in the set of free generators for $\mathcal{U}(\Gamma, v)$ obtained from $T_v'$. It follows from Theorem 4.5 that $\mathcal{U}(\Delta, v)$ is a free factor of $\mathcal{U}(\Gamma, v)$. ■
Remark. We remark that in general if $\Delta$ is a subgraph of the graph $\Gamma$, then there may be vertices of $\Delta$ that are in $\Gamma_v$ but not in $\Delta_v$ since there may be directed paths in $\Gamma$ from $v$ to a vertex in $\Delta$ but no such directed path in $\Delta$. Also, while the proof of Corollary 4.7 shows that every directed spanning tree of $\Delta_v$ at $v$ may be extended to a directed spanning tree of $\Gamma_v$ at $v$, it is not necessarily true that every directed spanning tree of $\Gamma_v$ restricts to a directed spanning tree of $\Delta_v$. This is because in general a geodesic path from $v$ to some other vertex $w$ in $\Delta$ in a directed spanning tree for $\Gamma_v$ may pass through vertices and edges of $\Gamma \setminus \Delta$.

Lemma 4.8 If $a$ and $b$ are $D$-related idempotents in an inverse semigroup $S$ with 0, then $\mathcal{U}(aSa) \cong \mathcal{U}(bSb)$.

Proof. There exists an element $x \in S$ such that $xx^{-1} = a$ and $x^{-1}x = b$. So $u$ is a non-zero element of $aSa$ if and only if $x^{-1}ux$ is a non-zero element of $bSb$. It follows that the map defined by $u \mapsto x^{-1}ux$ induces an isomorphism from $\mathcal{U}(aSa)$ onto $\mathcal{U}(bSb)$.

Theorem 4.9 Let $p$ be a directed path from a vertex $v$ to a vertex $w$ in a graph $\Gamma$. Then

(a) $\mathcal{U}(\Gamma, pp^*) \cong \mathcal{U}(\Gamma, w)$.

(b) $\mathcal{U}(\Gamma, w)$ is isomorphic to a free factor of $\mathcal{U}(\Gamma, v)$.

Proof. (a) Note that $pp^* D \mathcal{I}(p) = w$ in $I(\Gamma)$ since $p^*p = r(p)$, so the result of part (a) follows immediately from Lemma 4.8.

(b) If $v$ and $w$ are in the same strongly connected component then the result follows from Corollary 4.6. So we may assume that there is a directed path from $v$ to $w$ but no directed path from $w$ to $v$. Let $p = e_1e_2 \ldots e_n$ be a directed path from $v$ to $w$ and suppose that $k$ is the largest index such that $s(e_k) \notin \Gamma_w$. That is, there is a directed path from $w$ to $r(e_k)$ but no directed path from $w$ to $s(e_i)$ for $i = 1, \ldots, k$. Clearly every vertex $s(e_i)$ for $i = k + 1, \ldots, n$ is in the same strongly connected component as $w$, so $\Gamma_w = \Gamma_{s(e_i)}$ for all of these vertices. By Lemma 4.4 we may choose a directed spanning tree $T_{r(e_k)}$ for $\Gamma_w$ at $r(e_k) = s(e_{k+1})$.

If in the directed path $e_1 e_2 \ldots e_k$ from $v$ to $r(e_k)$ we have $s(e_i) = s(e_j)$ for some $i \neq j$, then we may omit the subpath $e_i \ldots e_{j-1}$ to obtain a shorter directed path $e_1 \ldots e_{i-1}e_j \ldots e_k$ from $v$ to $r(e_k)$. By omitting all such circuits in the path $e_1 \ldots e_k$ we obtain a directed geodesic path $p'$ from $v$ to $r(e_k)$ consisting of some of the vertices and edges of the path $e_1 \ldots e_k$. Let $T'$ be the subgraph of $\Gamma$ consisting of all of the vertices and edges of $\Gamma$ contained in the paths $p'/q$, for $q$ a geodesic path in $T_{r(e_k)}$ starting at $r(e_k)$. Since no vertex in $p'$ other than $r(e_k)$ lies in $\Gamma_w = \Gamma_{r(e_k)}$, it follows that $T'$ is a tree with the property that every geodesic path in $T'$ from $v$ to some vertex in $T'$ is directed. Clearly, $T'$ contains all of the vertices in $\Gamma_w = \Gamma_{r(e_k)}$. By Lemma 4.4 we may extend $T'$ to a directed spanning tree $T_v$ for $\Gamma_v$ at $v$. If $e$ is an edge in $\Gamma_w$ that is not in $T_{r(e_k)}$, then $e$ is not in $T_v$ either, so the free generators for $\mathcal{U}(\Gamma, r(e_k))$ obtained from the spanning tree $T_{r(e_k)}$ are among the free generators for $\mathcal{U}(\Gamma, v)$ obtained from the spanning tree $T_v$. It follows that $\mathcal{U}(\Gamma, r(e_k))$ is a free factor of $\mathcal{U}(\Gamma, v)$. The result then follows since $\mathcal{U}(\Gamma, r(e_k)) \cong \mathcal{U}(\Gamma, w)$ by Corollary 4.6.

5 Quotients which are also graph inverse semigroups

Recall that if $J$ is an ideal of an inverse semigroup $S$, then $S/J$ denotes the Rees quotient of $S$ by the corresponding Rees congruence $\rho J$, where $a \rho J b$ if $a = b$ or $a, b \in J$. Rees quotients of graph inverse semigroups are again graph inverse semigroups as described in the following theorem [25, Theorem 7].
Theorem 5.1 Let $J$ be an ideal of $I(\Gamma)$. Then $I(\Gamma)/J \cong I(\Delta)$, where $\Delta^0 = \Gamma^0 \setminus (J \cap \Gamma^0)$, $\Delta^1 = \{e \in \Gamma^1 : r(e) \not\in J\}$, and the source mapping and range mapping of $\Delta$ are restrictions of those for $\Gamma$.

Recall that a congruence $\rho$ on an inverse semigroup $S$ with 0 is called 0-restricted if $0\rho = \{0\}$. Notice that if $\rho$ is an arbitrary congruence on a graph inverse semigroup $I(\Gamma)$ and $J = 0\rho$, then $J$ is an ideal of $I(\Gamma)$ and $\rho$ induces in the obvious way a 0-restricted congruence on the Rees quotient $I(\Gamma)/J \cong I(\Delta)$ where $\Delta$ is the graph constructed in Theorem 5.1. Thus the discussion of general congruences (other than Rees congruences) on graph inverse semigroups may be reduced to that of 0-restricted congruences on graph inverse semigroups.

For any $v \in \Gamma^0$ with out-degree 1 we denote the unique edge in $s^{-1}(v)$ by $e_v$. Let $W$ be a set of vertices with out-degree 1, let $\mathbb{Z}^+$ be the set of all positive integers and let $C(W)$ be the set of all cycles whose vertices lie in $W$. Since all vertices in $W$ have out-degree one, any two cycles in $C(W)$ are either disjoint or cyclic conjugates of each other. A cycle function $f : C(W) \to \mathbb{Z}^+ \cup \{\infty\}$ is a function that is invariant under cyclic conjugation. A congruence pair $(W,f)$ of $\Gamma$ consists of a subset $W$ of vertices of out-degree 1 and a cycle function $f$.

Let $\rho$ be a 0-restricted congruence on $I(\Gamma)$ and $W = \{v \in \Gamma^0 : ee^* \rho v = s(e)\}$. Then all vertices of $W$ have out-degree 1. For $c \in C(W)$, let $f(c)$ be the smallest positive integer $m$ such that $c^m \rho s(c)$. If no power of $c$ is equivalent to $s(c)$, then we define $f(c) = \infty$. Then $T(\rho) = (W,f)$ is a congruence pair. Conversely, let $(W,f)$ be a congruence pair of $\Gamma$ and let $\phi(W,f)$ denote the congruence generated by the relation $R$ consisting of all pairs $(e_v e^*_v, v)$ for $v \in W$ and $(c^{f(c)}, s(c))$ for $c \in C(W)$ with $f(c) \in \mathbb{Z}^+$. Then the following theorem is proved in [29, Theorem 1.3].

Theorem 5.2 The mapping $T$ from the set of all 0-restricted congruences on $I(\Gamma)$ to the set of all congruence pairs of $\Gamma$ and the mapping $\phi$ from the set of all congruence pairs of $\Gamma$ to the set of all 0-restricted congruences on $I(\Gamma)$ are inverses. In particular, there exists a one-to-one correspondence between 0-restricted congruences on $I(\Gamma)$ and congruence pairs of $\Gamma$.

Theorem 5.2 enables us to describe all 0-restricted congruences on a graph inverse semigroup for which the quotient is another graph inverse semigroup.

Theorem 5.3 Let $\rho$ be a 0-restricted congruence on $I(\Gamma)$ determined by the congruence pair $(W,f)$. Then $I(\Gamma)/\rho$ is isomorphic to a graph inverse semigroup if and only if

1. $W \subseteq \{v \in \Gamma^0 : v$ has out-degree 1, $e_v$ is a loop at $v\}$; and

2. for any $v \in W$, $f(e_v) = 1$.

Proof. Sufficiency. Suppose that conditions (1) and (2) are satisfied. We proceed to prove that $I(\Gamma)/\rho$ is isomorphic to the graph inverse semigroup $I(\Delta)$, where $\Delta$ is the graph with $\Delta^0 = \Gamma^0$, $\Delta^1 = \Gamma^1 \setminus \{e_v : v \in W\}$, and the source mapping for $\Delta$ is the restriction of the source mapping for $\Gamma$ and the range mapping for $\Delta$ is the restriction of the range mapping for $\Gamma$. By conditions (1), (2) and Theorem 5.2 $\rho$ is generated by all pairs $(e_v e^*_v, v)$ and $(e_v, v)$ where $v \in W$. However, in the inverse semigroup $I(\Gamma)$, the relation $(e_v, v) \in \rho$ implies the relation $(e_v e^*_v, v) \in \rho$. Hence $\rho$ is generated by all pairs $(e_v, v)$ where $v \in W$. Let $\phi$ be the function that maps a loop $e_v$ of $\Gamma$ at a vertex $v$ in $W$ to the vertex $v$ and fixes all other vertices and edges of $\Gamma$. Then $\phi$ is a function that maps the generators of $I(\Gamma)$ to the generators of $I(\Delta)$. This function $\phi$ extends
to a homomorphism which we again denote by \( \phi \) from \( I(\Gamma) \) onto \( I(\Delta) \). To see this, note that if \( pq^* \) is a non-zero element of \( I(\Gamma) \) then \( \phi(pq^*) = pq^* \) if neither \( p \) nor \( q \) contains an edge \( e_v \) that is a loop at some vertex \( v \in W \). If \( pq^* \) does contain such a loop \( e_v \) then we must have \( p = e_1e_2\ldots e_ne_v^k \) and \( q = f_1f_2\ldots f_me_v^t \) for some \( k, t \geq 0 \) (with at least one of \( k \) or \( t \) greater than \( 0 \)). Then we see that \( \phi(pq^*) \) is obtained from \( pq^* \) by removing the path \( e_v^k(e_v^*)^t \) at the vertex \( r(p) = r(q) = v \). Then from the definition of the multiplication of canonical forms in \( I(\Gamma) \) it is easy to see that \( \phi \) is a homomorphism from \( I(\Gamma) \) onto \( I(\Delta) \). But then since \( \rho \) is generated by the pairs \((e_v,v)\) where \( e_v \) is a loop at some vertex \( v \in W \), it follows that the kernel of \( \phi \) is \( \rho \) and so the inverse semigroups \( I(\Gamma)/\rho \) and \( I(\Delta) \) are isomorphic.

**Necessity.** We may assume that \( W \) is nonempty or else the congruence \( \rho \) determined by the pair \((W,f)\) is the identity congruence. Suppose that \( I(\Gamma)/\rho \cong I(\Delta) \) for some graph \( \Delta \).

Recall first that the idempotents of a graph inverse semigroup \( I(\Gamma) \) are of the form \( pp^* \) for some directed (possibly empty) path \( p \) and that the maximal idempotents in the partial order correspond to the vertices of \( \Gamma \) by [25, Lemma 15(3)]. Recall also that in any homomorphism between inverse semigroups, idempotents lift, and so the idempotents of the inverse semigroup \( I(\Gamma)/\rho \) are of the form \((pp^*)\rho \) for some directed (possibly empty) path \( p \) in \( \Gamma \). Suppose that \((pp^*)\rho \geq \nu p \) for some vertex \( \nu \) and directed path \( p \) in \( \Gamma \). Then \((pp^*)\rho = (\nu pp^*)\rho = \nu p \). Since \( \rho \) is 0-restricted, \( \nu p \neq 0 \rho \) and so we must have \( s(p) = \nu \), in which case it follows that \((pp^*)\rho = \nu p \).

Hence \( \nu p \) is maximal in the partial order in the graph inverse semigroup \( I(\Delta) \cong I(\Gamma)/\rho \), and so we may view \( \nu p \) as a vertex of \( \Delta \) for each vertex \( \nu \) of \( \Gamma \).

Now suppose that condition (1) fails. Then there exists a vertex \( \nu \) in \( W \) such that \( s(e_v) = \nu \) and \( r(e_v) = v \). We have \( e_v^*e_v = r(e_v) \) and \( (e_v,e_v^*, v) \in \rho \), and so \( \nu p \mathcal{D} r(e_v)p \) in \( I(\Gamma)/\rho \). But since \( \rho \) is 0-restricted, we cannot have \( \nu p = r(e_v)p \) since \( r(e_v) = 0 \) in \( I(\Gamma) \). Hence \( \nu p \) and \( r(e_v)p \) are distinct vertices of \( \Delta \) that are \( \mathcal{D} \)-related in \( I(\Delta) \cong I(\Gamma)/\rho \). This, together with condition (1) of the theorem and Theorem 5.2, contradicts [25, Corollary 2], and so condition (1) must hold. Then for any vertex \( v \in W \) the edge \( e_v \) is a loop at \( v \). This implies that \( (e_v,e_v^*, v) \rho = (e_v^*e_v)\rho = \nu p \), and so \( e_v\rho \) is in the \( \mathcal{H} \)-class of the idempotent \( \nu p \) in the graph inverse semigroup \( I(\Delta) \). But it is routine to see that if \( pq^*qp^* = qp^*pq^* \) in a graph inverse semigroup, then \( p = q \) and so \( pq^* = pp^* \), and so graph inverse semigroups are combinatorial. From this it follows that \( e_v\rho = \nu p \), that is \( f(e_v) = 1 \). Hence condition (2) must also hold. 

**Corollary 5.4** Let \( \rho \) be a congruence on \( I(\Gamma) \) such that \( I(\Gamma)/\rho \) is isomorphic to a graph inverse semigroup \( I(\Delta') \) and let \( J = 0\rho \). Then \( \Delta' \) is a subgraph of \( \Gamma \) with set \( \Gamma^0 \setminus (\Gamma^0 \cap J) \) of vertices. If \( v \) is a vertex of \( \Delta' \), then the universal group \( \mathcal{U}(\Delta',v) \) is a free factor of \( \mathcal{U}(\Gamma,v) \).

**Proof.** \( J \) is an ideal of \( I(\Gamma) \) and \( I(\Gamma)/J \) is a graph inverse semigroup \( I(\Delta) \) as described in Theorem 5.1. Since \( \Delta \) is obtained from \( \Gamma \) by omitting some of the vertices and edges of \( \Gamma \), we see that \( \Delta \) is a subgraph of \( \Gamma \) with \( \Delta^0 = \Gamma^0 \setminus (\Gamma^0 \cap J) \). Furthermore, if \( pq^* \) and \( p'q'^* \) are non-zero elements of \( I(\Gamma) \) that are \( \rho \)-related, either \( pq^* , p'q'^* \in J \) or \( (pq^*) , (p'q'^*) \in \rho' \) where \( \rho' \) is the 0-restricted congruence on \( I(\Delta) \) that is the restriction of \( \rho \) to \( I(\Delta) \). The quotient \( I(\Delta)/\rho' \) is the graph inverse semigroup \( I(\Delta') \) isomorphic to \( I(\Gamma)/\rho \) as described in Theorem 5.3. By the proof of Theorem 5.3, the graph \( \Delta' \) is obtained from \( \Delta \) by removing the loops at some of the vertices of \( \Delta \), so \( \Delta' \) is a subgraph of \( \Delta \) with the same set of vertices as \( \Delta \). Hence \( \Delta' \) is a subgraph of \( \Gamma \) with set \( \Gamma^0 \setminus (\Gamma^0 \cap J) \) of vertices. The description of the universal groups then follows from Corollary 4.7.

**Corollary 5.5** Let \( \rho \) be a congruence on \( I(\Gamma) \) such that \( I(\Gamma)/\rho \) is isomorphic to a graph inverse semigroup \( I(\Delta') \). Then \( I(\Delta') \) is a retract of \( I(\Gamma) \) and \( \mathcal{U}(I(\Delta')) \) is a free factor of \( \mathcal{U}(I(\Gamma)) \).
Proof. From the proof of Corollary 5.4 we see that \( \Delta' \) is a subgraph of \( \Gamma \) so \( I(\Delta') \) is an inverse subsemigroup of \( I(\Gamma) \). Again using the notation of Theorem 5.3 and Corollary 5.4 let \( \phi \) be the map from \( I(\Gamma) \) to \( I(\Delta') \) defined by \( \phi(pq^*) = 0 \) if \( r(p) \in J \) and \( \phi(pe_v^k(e_v^*)^lq^*) = pq^* \) if \( r(p) \notin J \), \( v \in W \), \( e_v \) is a loop at \( v \) and \( f(e_v) = 1 \). It is routine to check that \( \phi \) is a semigroup homomorphism from \( I(\Gamma) \) onto \( I(\Delta') \). Clearly the restriction of \( \phi \) to the inverse subsemigroup \( I(\Delta') \) of \( I(\Gamma) \) is the identity map, so \( \phi \) is a retraction map and \( I(\Delta') \) is a retract of \( I(\Gamma) \). The fact that \( U(I(\Delta')) \) is a free factor of \( U(I(\Gamma)) \) follows from Corollary 5.3. 

6 Leavitt path algebras and Leavitt inverse semigroups

Let \( F \) be a field and let \( \Gamma \) be a row finite graph; that is \( |s^{-1}(v)| \) is finite for every vertex \( v \) in \( \Gamma \). Recall (see [3]) that the Leavitt path algebra \( L_F(\Gamma) \) is the \( F \)-algebra generated by the set \( \Gamma^0 \cup \Gamma^1 \cup (\Gamma^1)^* \) subject to the relations (1)–(4) defining the graph inverse semigroup \( I(\Gamma) \) and the additional “Cuntz-Krieger” relations

\[
(5) \ v = \sum_{e \in s^{-1}(v)} ee^* \quad \text{for all } v \in \Gamma^0 \text{ such that } v \text{ is not a sink.}
\]

The following fact is immediate from the definition of a Leavitt path algebra.

Lemma 6.1. The Leavitt path algebra \( L_F(\Gamma) \) corresponding to a graph \( \Gamma \) and a field \( F \) is isomorphic to the graph \( F_0I(\Gamma)/(v - \sum_{e \in s^{-1}(v)} ee^*) \) where the sum is taken over all vertices that are not sinks and where \( F_0I(\Gamma) \) is the contracted semigroup algebra of \( I(\Gamma) \).

We denote by \( LI(\Gamma) \) the multiplicative subsemigroup of \( L_F(\Gamma) \) generated by \( \Gamma^0 \cup \Gamma^1 \cup (\Gamma^1)^* \). (Of course \( LI(\Gamma) \) is a proper subset of \( L_F(\Gamma) \) since the addition and scalar multiplication operations are not used in constructing elements of this subsemigroup.) We will see that \( LI(\Gamma) \) is in fact an inverse semigroup, which we refer to as the Leavitt inverse semigroup of the graph \( \Gamma \). We will give a presentation for this semigroup (as an inverse semigroup) by generators and relations.

To see this, we make use of a natural basis for \( L_F(\Gamma) \) as an \( F \)-vector space, as described in a paper by Alahmadi, Alsulami, Jain and Zelmanov [10]. For each vertex \( v \) which is not a sink, choose an edge \( \gamma(v) \) such that \( s(\gamma(v)) = v \) and refer to this as a special edge. Then the following theorem was proved in [10].

Theorem 6.2. The following set of elements form a basis for \( L_F(\Gamma) \): (i) \( v \), where \( v \in \Gamma^0 \); (ii) \( p, p^* \), where \( p \) is a directed path in \( \Gamma \); (iii) \( pq^* \) where \( p = e_1\ldots e_n, q = f_1\ldots f_m, e_i, f_j \in \Gamma^1, r(e_n) = r(f_m), \) and either \( e_n \neq f_m \) or \( e_n = f_m \) but this edge \( e_n = f_m \) is not special.

We refer to the basis constructed in Theorem 6.2 as the natural basis for \( L_F(\Gamma) \).

Let \( L(\Gamma) \) be the semigroup generated by the set \( \Gamma^0 \cup \Gamma^1 \cup (\Gamma^1)^* \) subject to the relations (1)–(4) used to define the graph inverse semigroup \( I(\Gamma) \) and the additional relations:

\[
(v) \ e_v e_v^* = v \quad \text{for each vertex } v \in \Gamma^0 \text{ of out-degree 1.}
\]

Clearly \( L(\Gamma) \) is an inverse semigroup since it is a homomorphic image of \( I(\Gamma) \).

Theorem 6.3. For each graph \( \Gamma \), \( LI(\Gamma) \cong L(\Gamma) \). In particular, \( LI(\Gamma) \) is an inverse semigroup. Every element of \( LI(\Gamma) \) is uniquely expressible in one of the forms...
Thus inductively, $i$ is in the natural basis. (Note that $p_q$ in the same canonical forms. $p_q$ is used to prove uniqueness of canonical forms of non-zero elements in $L_I(\Gamma)$.) Since elements of $L_I(\Gamma)$ are isomorphic since they have the same generators and their elements can be expressed either in form (a) or in form (b) are equal in the same edge $e_v$ (where $v = s(e_v)$ is a vertex of out-degree 1), it follows from (v) that $p_q = p_l q_l'$ where $p_1 = e_1...e_n$ and $q_1 = f_1...f_m$.

Thus by induction we see that all elements of $L_I(\Gamma)$ are expressible in the form (a) or (b) in the statement of the theorem. The elements of the form (a) are in the natural basis for $L_F(\Gamma)$ and the elements of the form (b) are also in the natural basis for $L_F(\Gamma)$ provided $e$ is not a special edge. If $e$ is a special edge, then the relations (5) imply that $p'e e' q'^* = p' q'^* - \sum_i q_i g_i q_i' q'^*$ in $L_F(\Gamma)$, where the sum is taken over all edges $g_i$ such that $s(g_i) = r'(p')$ and $g_i \neq e$. Again by applying the relations (v), we see that $p' q'^*$ is equal to an element in $L_I(\Gamma)$ of form (a) or (b). Thus inductively,

$$p'e e' q'^* = p_0 q_0^* - \sum_i p_1 g_1 q_1^* q_1' q'^* - \sum_i p_s g_i q_i^* q_s$$

where the sum $\sum_i$ is taken over all edges $g_i$ such that $s(g_i) = r(p)$ and $g_i \neq \gamma(r(p))$, $p = p'$, $q = q', p_0 q_0^*, p_1 q_1^*, \ldots, p_s q_s^*$ have strictly ascending lengths and $p_0 q_0^*$ is of form (a) or (b) which is in the natural basis. (Note that $p'e e' q'^*$ is essentially determined by the last sum in the above formula.) Since elements of $L_F(\Gamma)$ can be expressed uniquely as linear combinations of the elements in the natural basis, it follows that two elements $p_q$ and $r s^*$ of $L_I(\Gamma)$ that are written either in form (a) or in form (b) are equal in $L_F(\Gamma)$ (and hence in $L_I(\Gamma)$) if and only if $p = r$ and $q = s$. But since $L(\Gamma)$ satisfies the relations (1)-(4) and (v) it follows that every non-zero element of $L(\Gamma)$ may also be expressed in one of the forms (a) or (b). The same argument that is used to prove uniqueness of canonical forms of non-zero elements in $I(\Gamma)$ shows that two such elements $p_q$, $r s^*$ of $L(\Gamma)$ are equal in $L(\Gamma)$ if and only if $p = r$ and $q = s$. Hence $LI(\Gamma)$ and $L(\Gamma)$ are isomorphic since they have the same generators and their elements can be expressed in the same canonical forms.

We remark that an alternative proof of the fact that $LI(\Gamma) \cong L(\Gamma)$ using Gröbner-Shirshov bases has been provided by Fan and Wang [14].

**Corollary 6.4** For each graph $\Gamma$ and each vertex $v$ in $\Gamma$ the universal groups $U(I(\Gamma))$ and $U(LI(\Gamma))$ are isomorphic and the universal group of the local submonoid $vI(\Gamma)v$ is isomorphic to the universal group of the local submonoid $vLI(\Gamma)v$.

**Proof.** Imposing the additional relations $e_v e_v^* = v$ on the generators for $I(\Gamma)$ does not change the universal group since the relation $e_v e_v^{-1} = 1$ holds in any group. 

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1Based on an earlier version of this paper, David Milan asked whether the kernel of the natural homomorphism from a graph inverse semigroup $I(\Gamma)$ onto the corresponding Leavitt inverse semigroup $LI(\Gamma)$ coincides with the congruence $\leftrightarrow$ introduced by Lenz [21]. This is in fact the case. We present a proof of this in the addendum to this paper.
We say that the directed path \( p = e_1e_2\ldots e_n \) in a graph \( \Gamma \) has \textit{exits} if at least one of the vertices \( s(e_i) \) has out-degree greater than 1 (and in this case we say that \( p \) has an exit at \( s(e_i) \)). In particular, an edge \( e \in \Gamma^1 \) has exits if and only if \( s(e) \) has out-degree greater than 1. We say that the directed path \( p = e_1e_2\ldots e_n \) has \textit{no exits} (or that \( p \) is an \textit{NE path}) if every vertex \( s(e_i), \, i = 1, \ldots, n \) has out-degree 1. We also define the empty path at any vertex \( v \) to be an NE path.

**Corollary 6.5** For each graph \( \Gamma \) the non-zero idempotents of \( LI(\Gamma) \) are the elements of the form \( pp^* \) where \( p \) is a directed path in \( \Gamma \). Furthermore, \( pp^* = qq^* \) in \( LI(\Gamma) \) if and only if either \( p = qp_1 \) for some NE path \( p_1 \) or \( q = pq_1 \) for some NE path \( q_1 \). In particular, \( pp^* = v \) in \( LI(\Gamma) \) for some \( v \in \Gamma^0 \) if and only if \( v = s(p) \) and \( p \) is an NE path.

**Proof.** It is clear from the relations defining a Leavitt inverse semigroup that every non-zero element of \( LI(\Gamma) \) is of the form \( pq^* \) where \( p \) and \( q \) are directed paths with \( r(p) = r(q) \). It is also routine to see that \( pq^* \) is a non-zero idempotent of \( LI(\Gamma) \) if and only if \( p = q \), and that \( pp^* = pp_1p^*_p \) if \( p_1 \) is an NE path with \( s(p_1) = r(p) \). Suppose conversely that \( pp^* = qq^* \) for some directed paths \( p \) and \( q \). Then \( pp^* = s(p)pp^* = s(p)qq^* ≠ 0 \) so \( s(p) = s(q) \). If \( p \) is not a prefix of \( q \) and \( q \) is not a prefix of \( p \) then there exist edges \( e_1, e_2 \) and paths \( s.p', q' \) with \( p = se_1p', \, q = se_2q' \) and \( e_1 ≠ e_2 \). From \( se_1p^*p_1^*s^* = se_2q^*q_1^*s^* \) we see, on premultiplying by \( s^* \) and postmultiplying by \( s \) that \( e_1p^*p^*_1 = e_2q^*q^*_1 \). Hence \( e_2e_1p^*p^*_1 = e_2e_2q^*q^*_1 = q^*q^*_1 ≠ 0 \). But since \( e_1 ≠ e_2 \), we see that \( e_1e_2 = 0 \), a contradiction. Hence we must have either \( q \) is a prefix of \( p \) or \( p \) is prefix of \( q \). In the first case we have \( p = qp_1 \) for some directed path \( p_1 \). Then from \( qq^* = qp_1^*q^* \) we see as above that \( p_1p_1^* = q^*q = r(q) = s(p_1) \). Then by an argument very similar to the argument above, we see that \( p_1 \) is an NE path. Similarly, if \( p \) is a prefix of \( q \) then \( q = pq_1 \) for some NE path \( q_1 \).

Recall that a graph \( \Gamma \) admits a directed immersion into a circle \( B_{(a)} \) if and only if all of its vertices have out-degree at most 1: the structure of such graphs is described in Theorem [3.1]. We now provide a straightforward classification of the Leavitt inverse semigroups and Leavitt path algebras of such graphs. We may assume that such a graph is connected, since the Leavitt inverse semigroup of a graph is the 0-direct union of the Leavitt inverse semigroups of the connected components of the graph.

We recall (see [21]) that for each non-empty set \( A \) and each group \( G \), the \textit{Brandt semigroup} \( B_A(G) \) is the semigroup \( B_A(G) = \{(a, g, b) : a, b \in A, \, g \in G \} \cup \{0\} \) with multiplication

\[(a, g, b)(c, h, d) = (a, gh, d) \text{ if } b = c \text{ and } 0 \text{ otherwise.}\]

**Theorem 6.6** Let \( \Gamma \) be a connected graph that immerses into a circle.

\begin{enumerate}
\item[(a)] If \( \Gamma \) is a tree then \( LI(\Gamma) \cong B_{\Gamma^0}(1) \), the combinatorial \( |\Gamma^0| \times |\Gamma^0| \) Brandt semigroup;
\item[(b)] If \( \Gamma \) is not a tree then \( LI(\Gamma) \cong B_{\Gamma^0}(\mathbb{Z}) \), the \( |\Gamma^0| \times |\Gamma^0| \) Brandt semigroup with maximal subgroups isomorphic to \( \mathbb{Z} \).
\end{enumerate}

**Proof.** If \( e \) is an edge of \( \Gamma \) then from the relations defining \( LI(\Gamma) \) and the fact that \( s(e) \) has out-degree 1 we see that \( ee^* = s(e) \) and \( e^*e = r(e) \) so \( s(e) \) and \( r(e) \) are \( D \)-related in \( LI(\Gamma) \). Also, if \( p \) is a directed path starting at a vertex \( v \), then by induction on the length of \( p \) we easily see that \( pp^* = v \) in \( LI(\Gamma) \). These facts, together with the fact that \( \Gamma \) is connected, imply that \( LI(\Gamma) \) is a 0-bisimple inverse semigroup whose idempotents may be identified with the vertices of \( \Gamma \). Since \( v_1v_2 = 0 \) if \( v_1 \neq v_2 \in \Gamma^0 \), this implies that \( LI(\Gamma) \) is a homomorphic image of a Brandt
semigroup with \(|Γ^0|\) rows (\(R\)-classes) and \(|Γ^0|\) columns (\(L\)-classes). By Theorem 6.3, we see that every element of \(LI(Γ)\) may be expressed uniquely in the form \(pq^*\) where \(p\) and \(q\) are (possibly empty) directed paths with \(r(p) = r(q)\) and the last edge in the path \(p\) is different from the last edge in \(q\). Hence distinct vertices of \(Γ\) remain distinct as elements of \(LI(Γ)\) and so \(LI(Γ)\) is a Brandt semigroup with \(|Γ^0|\) rows and columns. The corresponding maximal subgroups are trivial if \(Γ\) is a tree and isomorphic to a homomorphic image of \(Z\) otherwise, by Theorem 3.1. But by the canonical form for elements of \(LI(Γ)\) described in Theorem 6.3, no two distinct powers of a circuit in \(Γ\) are equal in \(LI(Γ)\), so the maximal subgroups of \(LI(Γ)\) are isomorphic to \(Z\) if \(Γ\) is not a tree.

**Corollary 6.7** Let \(Γ\) be a connected graph that immerses into a circle and let \(F\) be a field. Then
(a) If \(Γ\) is a tree then \(L_F(Γ) \cong M_{|Γ^0|}(F)\), the algebra of \(|Γ^0| × |Γ^0|\) matrices with entries in \(F\) and only finitely many non-zero entries in each row and column.
(b) If \(Γ\) is not a tree, then \(L_F(Γ) \cong M_{|Γ^0|}(F[x, x^{-1}])\) where \(F[x, x^{-1}]\) is the algebra of Laurent polynomials over \(F\) (i.e. the semigroup algebra \(FZ\)).

**Proof.** By Lemma 6.1 and the fact that all vertices have out-degree at most 1 we have \(L_F(Γ) \cong F_0 I(Γ)/(ee^* - s(e) : e \in Γ^1)\) where \(F_0 I(Γ)\) is the contracted semigroup algebra of \(I(Γ)\). But since the relation \(ee^* = s(e)\) holds in \(LI(Γ)\) for all \(e \in Γ^1\), this implies that \(L_F(Γ) \cong F_0 LI(Γ)\), the contracted semigroup algebra of the Leavitt inverse semigroup \(LI(Γ)\). The result then follows from Theorem 6.3.

**Remark** We remark that the characterization given in Corollary 6.7(b) of Leavitt path algebras of a graph \(Γ\) that admits a directed cover of the circle is a special case of the characterization of Leavitt path algebras of a class of graphs given in Proposition 3.5 of [6]. This is because by Theorem 3.1 there is a one-one correspondence between the vertices of \(Γ\) and the directed paths that end in a specified vertex of the unique cycle \(C\) in \(Γ\) and do not include \(C\) as a subpath.

**Theorem 6.8** Let \(Γ\) and \(Δ\) be connected graphs that immerse into a circle and let \(F\) be a field. Then the following are equivalent.
(a) \(LI(Γ) \cong LI(Δ)\);
(b) \(L_F(Γ) \cong L_F(Δ)\);
(c) \(|Γ^0| = |Δ^0|\) and either \(Γ\) and \(Δ\) are both trees or \(π_1(Γ) \cong π_1(Δ) \cong Z\).

**Proof.** The equivalence of (a) and (c) follows immediately from Theorems 3.1 and 6.6 since two Brandt semigroups are isomorphic if and only if they have isomorphic maximal subgroups and the same number of rows.

Suppose that \(LI(Γ) \cong LI(Δ)\). The non-zero elements of \(LI(Γ)\) are precisely the non-zero elements in a natural basis for \(L_F(Γ)\) so an isomorphism between \(LI(Γ)\) and \(LI(Δ)\) is a bijection between the natural bases of \(L_F(Γ)\) and \(L_F(Δ)\) that also preserves multiplication of basis elements in the algebras, so it induces an isomorphism between \(L_F(Γ)\) and \(L_F(Δ)\). Hence (a) implies (b).

Conversely suppose that \(L_F(Γ) \cong L_F(Δ)\). If \(Γ\) is a tree then in particular \(Γ\) is acyclic, so from Theorem 1 of [7] it follows that \(L_F(Γ)\) is von-Neumann regular. Hence \(L_F(Δ)\) is von-Neumann regular, from which it follows, again by Theorem 1 of [7], that \(Δ\) is acyclic and hence since the out-degree of every vertex of \(Γ\) is at most 1, \(Δ\) is a tree. Thus \(Γ\) is a tree if and only if \(Δ\) is a tree. If \(Γ\) and \(Δ\) are both trees, then by Corollary 6.7(a) \(L_F(Γ) \cong M_{|Γ^0|}(F)\) and \(L_F(Δ) \cong M_{|Δ^0|}(F)\). So if \(L_F(Γ) \cong L_F(Δ)\) it follows that \(|Γ^0| = |Δ^0|\) in this case. If \(Γ\) and \(Δ\) are not trees, then
by Corollary 6.7(b), \( L_F(\Gamma) \cong M_{|\Gamma^0|}(F[x, x^{-1}]) \) and \( L_F(\Delta) \cong M_{|\Delta^0|}(F[x, x^{-1}]) \). It is well-known that if \( R \) and \( S \) are commutative rings then \( M_n(R) \cong M_m(S) \) if and only if \( R \cong S \) and \( m = n \). Hence if \( L_F(\Gamma) \cong L_F(\Delta) \) and \( \Gamma \) is not a tree then it again follows that \( |\Gamma^0| = |\Delta^0| \). Hence (b) implies (c).

We will prove that the implication (a) implies (b) of Theorem 6.8 holds for arbitrary connected graphs; that is, if \( LI(\Gamma) \cong LI(\Delta) \) then \( L_F(\Gamma) \cong L_F(\Delta) \) (Theorem 6.11 below). However the converse is false in general as the following example shows.

**Example** Given the following two graphs,

\[
\Gamma_1: \bullet \xrightarrow{\cdots} \bullet \xrightarrow{\cdots} \bullet \quad \Gamma_2: \bullet \xrightarrow{\cdots} \bullet
\]

we see from [2, Example 2.2] that \( L_F(\Gamma_1) \cong L_F(\Gamma_2) \). However, \( LI(\Gamma_1) \) is not isomorphic to \( LI(\Gamma_2) \). This is because, according to Theorem 6.3, \( LI(\Gamma_1) \cong I(\Gamma_1) \) since every vertex in \( \Gamma_1 \) has out-degree 2 whereas \( LI(\Gamma_2) \) is not a graph inverse semigroup by Theorem 5.3. Alternatively we can use Theorem 6.12 below to see that these Leavitt inverse semigroups are not isomorphic.

**Lemma 6.9** For each graph \( \Gamma \) we have the following:

(a) \( \Gamma^0 \) is the set of maximal idempotents in \( LI(\Gamma) \).

(b) \( \{pee^{p^*} : p \) is an NE path, \( e \in \Gamma^1 \) and the out degree of \( s(e) \) is at least 2 \} is the set of maximal idempotents of \( LI(\Gamma) \setminus \Gamma^0 \).

**Proof.** (a) By Corollary 6.5, the non-zero idempotents of \( LI(\Gamma) \) are of the form \( pp^* \) for some (possibly empty) directed path \( p \) in \( \Gamma \). Now suppose that \( pp^* \geq v \) for some idempotent \( pp^* \) in \( LI(\Gamma) \) and some \( v \in \Gamma^0 \). Then \( pp^*v = vpp^* = v \in LI(\Gamma) \). This forces \( v = s(p) \), and \( vpp^* = pp^* \), so \( v = pp^* \). Hence \( v \) is a maximal idempotent in \( LI(\Gamma) \).

(b) If \( qq^* \geq pee^{p^*} \) where \( q \neq pe \), then we see from \( (qq^*)(pee^{p^*}) = pee^{p^*} \) that \( q \) is a prefix of \( p \). If \( p \) is an NE path then \( q \) is also an NE path so we get \( qq^* = s(q) = s(p) \in \Gamma^0 \). Furthermore, it is clear that any idempotent \( p_1p_1^* \) for which \( p_1 \) is not an NE path is less than or equal to some \( pee^{p^*} \) where \( p \) is an NE path, \( e \in \Gamma^1 \) and the out-degree of \( s(e) \) is at least 2.

**Lemma 6.10** Let \( \phi \) be an isomorphism between the Leavitt inverse semigroups \( LI(\Gamma) \) and \( LI(\Delta) \) for some graphs \( \Gamma \) and \( \Delta \). Then

(a) \( \phi \) preserves vertices; that is, \( \phi(v) \in \Delta^0 \) for each \( v \in \Gamma^0 \);

(b) for any nonzero \( pq^* \in LI(\Gamma) \), if \( \phi(pq^*) = p_1q_1^* \) and \( q \) is an NE path, then \( q_1 \) is an NE path, \( \phi(s(p)) = s(p_1) \), \( \phi(pp^*) = p_1p_1^* \) and \( \phi(s(q)) = q_1q_1^* = s(q_1) \);

(c) for any nonzero \( pq^* \in LI(\Gamma) \), if \( \phi(pq^*) = p_1q_1^* \) and \( p,q \) are NE paths, then \( p_1,q_1 \) are NE paths, \( \phi(s(p)) = p_1p_1^* = s(p_1) \) and \( \phi(s(q)) = q_1q_1^* = s(q_1) \);

(d) for any \( e \in \Gamma^1 \), if \( s(e) \) has out-degree at least 2, then there exist NE paths \( p_1,p_2,p_3 \) and an edge \( \hat{e} \) for which \( s(\hat{e}) \) has out-degree at least 2 such that \( \phi(e) = p_1\hat{e}p_2p_3^* \) and there exist NE paths \( q_1,q_2,q_3 \) such that \( \phi^{-1}(\hat{e}) = q_1eq_2q_3^* \);

(e) for any \( v \in \Gamma^0 \), if \( s^{-1}(v) = \{e_1,\ldots,e_n\} \) with \( n \geq 2 \), then there exist NE paths \( p_i,q_i \), and distinct edges \( \tilde{e}_i,i = 1,\ldots,n \) such that \( \phi(e_i) = p_\tilde{e}_ip_iq_i^*,i = 1,\ldots,n \) and \( s^{-1}(r(p)) = \{\tilde{e}_1,\ldots,\tilde{e}_n\} \).

**Proof.** (a) This follows from Lemma 6.9(a) since \( \phi \) must map maximal idempotents in \( LI(\Gamma) \) to maximal idempotents in \( LI(\Delta) \).
(b) If $\phi(pq^*) = p_1q_1^*$ and $q$ is NE, then $\phi(pp^*) = \phi(pr(q)p^*) = \phi(pq^*qp^*) = p_1q_1^*q_1p_1^* = p_1r(q_1)p_1^* = p_1p_1^*$ and $\phi(s(q)) = \phi(qq^*) = \phi(qr(p)q^*) = q_1p_1q_1^* = q_1r(p_1)q_1^* = q_1q_1^*$. Since $\phi(s(q)) \in \Delta^0$ by Lemma 6.9(a), this implies that $q_1q_1^* = v$ in $LI(\Delta)$ for some $v \in \Delta^0$. This forces $v = s(q_1)$ and $q_1$ is an NE path by Corollary 6.5. Also, $p_1q_1^* = \phi(pq^*) = \phi(s(p)pq^*) = \phi(s(p))\phi(pq^*) = \phi(s(p))p_1q_1^* \neq 0$ so we must have $\phi(s(p)) = s(p_1)$ since $\phi(s(p)) \in \Delta^0$ by part (a) of this lemma.

(c) Note that $\phi(pq^*) = p_1q_1^*$ implies $\phi(qp^*) = q_1p_1^*$. This part follows directly from part (b).

(d) Let $e$ be an edge with $s(e)$ of out-degree at least 2 and suppose that $\phi(e) = pq^*$. By part (b) we see that $q$ is an NE path. Also by Lemma 6.9(b), $ee^*$ is a maximal idempotent in $LI(\Gamma) \setminus \Gamma^0$ so $pp^* = pq^*qp^*$ is a maximal idempotent in $LI(\Delta) \setminus \Delta^0$. Then from Lemma 6.9(b) we see that there exists an NE path $p_1$ and an edge $\tilde{e}$ for which $s(\tilde{e})$ has out-degree at least 2 such that $pp^* = p_1\tilde{e}\tilde{e}^*p_1^*$ in $LI(\Delta)$. By Lemma 6.9(b) we have $p$ is not a prefix of $p_1$ since $p_1$ is an NE path and so, again Lemma 6.9(b), $p = p_1\tilde{e}p_2$ where $p_2$ is an NE path in $\Delta$. Moreover, we have $e = \phi^{-1}(p_1)\phi^{-1}(\tilde{e})\phi^{-1}(pq^*)$. From part (c) of this lemma, we observe that $\phi^{-1}(\tilde{e}) = (-\phi^{-1}(p_1))\phi^{-1}(pq^*)$. This forces the existence of NE paths $q_1, q_2, q_3$ such that $\phi^{-1}(\tilde{e}) = q_1q_2e\tilde{q}^3_2$.

(e) Take $v \in \Gamma^0$ such that $s^{-1}(v) = \{e_1, \ldots, e_n\}$ with $n \geq 2$. According to part (d) of the lemma, we have $\phi(e_i) = p_{i,1}e_i p_{i,2}p_{i,3}^*$ for some NE paths $p_{i,j}$ for all $i$ and since each $p_{i,1}$ is an NE path, we see that all $p_{i,1}$ are the same path. Since $p_{i,1}e_i e_i^* p_{i,1}^*$ is the image of $LI(\Delta)$ of $e_i e_i^*$ under $\phi$ and the $e_i$ are distinct, it follows that the $\tilde{e}_i$ are distinct, for $i = 1, \ldots, n$. Hence the out-degree of $r(p_{i,1}) = s(\tilde{e}_i)$ is at least $n$, which is the out-degree of $v = s(e_i)$. Similarly, from the second statement of part (d), we see that the out-degree of $s(e_i)$ is less than or equal to the out-degree of $s(\tilde{e}_i)$. It follows that $s^{-1}(r(p)) = \{\tilde{e}_1, \ldots, \tilde{e}_n\}$.

We are now in a position to prove the following theorem.

**Theorem 6.11** Let $\Gamma$ and $\Delta$ be connected graphs and let $F$ be a field. If $LI(\Gamma) \cong LI(\Delta)$, then $L_F(\Gamma) \cong L_F(\Delta)$.

**Proof.** By the definition of a Leavitt path algebra we observe that $L_F(\Gamma)$ is isomorphic to the quotient of the contracted semigroup algebra $F_0LI(\Gamma)$ of $LI(\Gamma)$ by the ideal $I_1$ generated by elements of the form $\sum_{e_i \in s^{-1}(v)} ee^* - v$ for $v \in \Gamma^0$ with the out-degree of $v$ at least 2. $L_F(\Delta)$ is isomorphic to the contracted semigroup algebra $F_0LI(\Delta)$ of $LI(\Delta)$ by the ideal $I_2$ generated by elements of the form $\sum_{d \in s^{-1}(u)} dd^* - u$ for $u \in \Gamma^0$ with the out-degree of $u$ at least 2. Suppose that $\phi$ is an isomorphism from $LI(\Gamma)$ onto $LI(\Delta)$. Then $\phi$ induces an algebra isomorphism, say $\eta$, from $F_0LI(\Gamma)$ onto $F_0LI(\Delta)$. Now for any $v \in \Gamma^0$ with out-degree greater than 1 and any $e_i \in s^{-1}(v)$ we see from Lemma 6.10(d), (e) that there exist NE paths $p, p_i, q_i$ and edges $\tilde{e}_i \in s^{-1}(r(p))$ such that $\phi(e_i) = p\tilde{e}_i p_1^* q_i^*$, $\phi(v) = s(p)$ and $|s^{-1}(v)| = |s^{-1}(r(p))|$. Distinct $e_i$ correspond to distinct $\tilde{e}_i$. Thus,

$$\eta(\Sigma_{e_i \in s^{-1}(v)} e_i e_i^* - v) = \Sigma_{e_i \in s^{-1}(v)} \phi(e_i)(\phi(e_i))^* - s(p)$$

$$= \Sigma_{\tilde{e}_i \in s^{-1}(u)} p\tilde{e}_i p_1^* p^* - pp^*$$

$$= p(\Sigma_{\tilde{e}_i \in s^{-1}(u)} \tilde{e}_i p_1^* - u)p^* \in I_2$$

which means that $\eta(I_1) \subseteq I_2$. Similarly, one can obtain that $\eta^{-1}(I_2) \subseteq I_1$. So we have $\eta(I_1) = I_2$ and $\eta^{-1}(I_2) = I_1$. It follows that $L_F(\Gamma) \cong L_F(\Delta)$ as required.

We remark that Ruy Exel outlined an alternative proof (also suggested by Benjamin Steinberg) of Theorem 6.11 using his notion of tight representations of inverse semigroups.
7 Some structural properties of Leavitt inverse semigroups

In this section we determine some structural properties of Leavitt inverse semigroups culminating in a description of necessary and sufficient conditions for two graphs to have isomorphic Leavitt inverse semigroups (Theorem 7.12) and some applications of that theorem to the structure of Leavitt path algebras for some classes of graphs. We will need some preliminary concepts and lemmas in order to formulate and prove this result and some other structural properties.

Let \( \Gamma \) be an arbitrary (directed) graph. Define a relation \( \sim \) on \( \Gamma^0 \) by \( v_1 \sim v_2 \) if there exist (possibly empty) NE paths \( p \) and \( q \) such that \( s(p) = v_1, s(q) = v_2 \) and \( r(p) = r(q) \). Note that this implies that \( v_1 \sim r(p) = r(q) \) for \( i = 1, 2 \) even if the out-degree of \( r(p) \) is at least 2 since the empty path at \( r(p) \) is an NE path.

**Lemma 7.1** The relation \( \sim \) is an equivalence relation on \( \Gamma^0 \).

**Proof.** The relation \( \sim \) is reflexive since we regard the empty path at any vertex \( v \in \Gamma^0 \) to be an NE path. Clearly \( \sim \) is symmetric. If \( v_1 \sim v_2 \) and \( v_2 \sim v_3 \) then there are NE paths \( p_1, q_1, p_2, q_2 \) such that \( s(p_1) = v_1, s(q_1) = v_2, r(p_1) = r(q_1), s(p_2) = v_2, s(q_2) = v_3 \) and \( r(p_2) = r(q_2) \). If \( q_1 \) is the empty path then \( r(p_1) = v_2 \) and so \( p_1p_2 \) is an NE path with \( s(p_1p_2) = v_1 \) and \( r(p_1p_2) = r(q_2) \), so in this case \( v_1 \sim v_3 \). Similarly \( v_1 \sim v_3 \) if \( p_2 \) is the empty path. If neither \( q_1 \) nor \( p_2 \) is the empty path then since all vertices in an NE path (except the range vertex) have out-degree 1 it follows that either \( q_1 \) is a prefix of \( p_2 \) or \( p_2 \) is a prefix of \( q_1 \). In the first case, there is an NE path \( t \) such that \( s(t) = r(q_1), r(t) = r(q_2) \) and \( p_2 = q_1t \), so \( p_1t \) is an NE path with \( s(p_1t) = v_1 \) and \( r(p_1t) = r(q_2) \), and so \( v_1 \sim v_3 \). Similarly \( v_1 \sim v_3 \) if \( p_2 \) is a prefix of \( q_1 \). Hence \( \sim \) is transitive. \( \blacksquare \)

**Corollary 7.2** If \( e \in \Gamma^1 \) is an edge that has exits then \( s(e) \sim r(e) \) if and only if \( e \) lies in a cycle which has exits only at \( s(e) \).

**Proof.** If \( s(e) \sim r(e) \), then there exist NE paths \( p, q \) such that \( r(e) = s(p), s(e) = s(q) \) and \( r(p) = r(q) \). This forces that \( q \) is trivial since \( q \) has no exit and \( s(e) \) has out-degree greater than 1. So the path \( ep \) is a cycle which has exits only at \( s(e) \). The converse part is clear. \( \blacksquare \)

The equivalence relation \( \sim \) enables a description of the Green relations on \( LI(\Gamma) \).

**Theorem 7.3** Let \( \Gamma \) be a graph and \( pq^* \), \( xy^* \) elements of \( LI(\Gamma) \) in canonical form as described in Theorem 6.3. Then the Green relations on \( LI(\Gamma) \) are described as follows.

- (a) \( pq^* \mathcal{R} xy^* \) iff \( pp^* = xx^* \).
- (b) \( pq^* \mathcal{L} xy^* \) iff \( qq^* = yy^* \).
- (c) \( pq^* \mathcal{D} r(p) \).
- (d) \( pq^* \mathcal{D} xy^* \) iff \( r(p) \sim r(x) \).
- (e) \( pq^* \mathcal{J} xy^* \) iff there exist vertices \( u, v \in \Gamma^0 \) with \( r(p) \sim u \) and \( r(x) \sim v \) such that \( u \) and \( v \) are in the same strongly connected component of \( \Gamma \).
- (f) If \( pq^* \mathcal{H} xy^* \) and \( pq^* \neq xy^* \) in \( LI(\Gamma) \) then either there is a (possibly empty) NE path \( p' \) from \( r(p) \) to \( r(x) \) and a non-trivial NE cycle \( C \) based at \( r(x) \) or there is a (possibly empty) NE path \( p' \) from \( r(x) \) to \( r(p) \) and a non-trivial NE cycle \( C \) based at \( r(p) \). In the former case \( xy^* = pp' C^n p'^* q^* \) for some non-zero integer \( n \) (where \( C^{-n} \) is interpreted as \( (C^*)^n \) for \( n > 0 \)); in the latter case \( pq^* = xp' C^n p'^* y^* \) for some non-zero integer \( n \).
- (g) The maximal subgroup of \( LI(\Gamma) \) containing the idempotent \( pp^* \) is either trivial or is isomorphic to the group \( (\mathbb{Z},+) \) of integers: it is non-trivial if and only if there is a path of the
form $p'C$ where $s(p') = r(p), p'$ is a (possibly trivial) NE path and $C$ is a non-trivial NE cycle in $\Gamma$ based at $r(p')$.

**Proof.** Note that $pq^* R x y^*$ iff $pq^* q^p = xy^* y x^*$. The result of part (a) follows since $pq^* q^p = pr(q) p^r = pr(p) p^r = pp^r$ and similarly $xy^* y x^* = xx^*$. The proof of part (b) is similar. For part (c), note that $pq^* R p'$ by part (a). But $p^r p = r(p) = r(p)^p r(p)$, so $p^r \mathcal{L} r(p)$. Hence $pq^* \mathcal{D} r(p)$. Now suppose that $r(p) \sim r(x)$. Then there exist NE paths $p_1, p_2$ with $s(p_1) = r(p), s(p_2) = r(x)$ and $r(p_1) = r(p_2)$. Since $p_1 p_2^* = r(p)$ and $p_1^* p_2 = r(p_1)$ it follows that $r(p) \mathcal{D} r(p_1)$. Similarly $r(x) \mathcal{D} r(p_2) = r(p_1)$, so $r(p) \mathcal{D} r(x)$ and hence by part (c) of this theorem, $pq^* xy^*$. Conversely, suppose that $pq^* \mathcal{D} xy^*$, so $r(p) \mathcal{D} r(x)$, again by part (c). Then there exists $p_1 q_1$ in canonical form such that $r(p) \mathcal{R} p_1 q_1 \mathcal{L} r(x)$. This implies that $r(p) = p_1 p_1^*$ and $r(x) = q_1 q_1^*$ by parts (a) and (b) of this theorem. Then by Corollary 6.5, $p_1$ and $q_1$ are NE paths, so $r(p) \sim r(x)$. This proves part (d).

Now suppose that there are vertices $u, v$ satisfying the conditions in part (e). By Corollary 2 of [25] we know that $u \mathcal{J} v$ in $I(\Gamma)$, so $u \mathcal{J} v$ in $LI(\Gamma)$. But also by part (d), $r(p) \mathcal{D} u$ and $v \mathcal{D} r(x)$ so $r(p) \mathcal{J} r(x)$ in $LI(\Gamma)$, whence $pq^* \mathcal{J} xy^*$ by part (c). Suppose conversely that $pq^* \mathcal{J} xy^*$. Then $r(p) \mathcal{J} r(x)$ by part (c). So there exist $p_1 q_1, p_2 q_2$ in canonical form such that $r(p) = p_1 q_1 r(x) p_2 q_2$. This forces $s(p_1) = s(q_2) = r(p)$ and $s(q_1) = s(p_2) = r(x)$. Also, either $p_2$ is a prefix of $q_1$ or $q_1$ is a prefix of $p_2$. Suppose that $q_1$ is a prefix of $p_2$. So there exists a (possibly empty) directed path $t_1$ with $p_2 = q_1 t_1$. Also since $q_2$ is a directed path from $r(p)$ to $r(q_2) = r(t_1)$ there exist (possibly empty) directed paths $t_2, t_3, t_4$ such that $p_1 = t_2 t_3$ and $q_2 = t_2 t_4$. Then $r(p) = p_1 q_1 r(x) p_2 q_2 = t_2 t_3 q_1 t_4 t_1 t_2 = p_3 q_3^*$ where $p_3 = t_2 t_3 t_4$ and $q_3 = t_2 t_4$. This forces $p_3 = q_3$ to be an NE path by Corollary 6.5 and hence $t_4 = t_3 t_4$ and also $r(p) \sim u' = r(p_3)$ and $p' = p_2$ is a directed path from $r(x)$ to $u'$. A similar argument applies in the case where $p_2$ is a prefix of $q_1$. Similarly, there is some vertex $v'$ with $r(x) \sim v'$ and a directed path $p''$ from $r(p)$ to $v'$. Thus in all cases we have some vertices $u', v'$ with $u' \sim r(p), v' \sim r(x)$ and directed paths $p'$ from $r(x)$ to $u'$ and $p''$ from $r(p)$ to $v'$.

Since $r(p) \sim u'$, there are NE paths $h_1, h_2$ with $s(h_1) = u', s(h_2) = r(p)$ and $r(h_1) = r(h_2) \sim r(p)$. Since $h_2$ is an NE path starting at $r(p)$ it must be a prefix of $p''$, so there exists a directed path $h_3$ such that $p'' = h_2 h_3$. Denote the vertex $r(h_1) = r(h_2) = s(h_3)$ by $u$. Similarly, there are directed paths $h_4, h_5, h_6$ such that $h_4, h_5, h_6$ are NE, $s(h_4) = v', s(h_5) = r(x), r(h_4) = r(h_5) = s(h_6)$ and $p' = h_5 h_6$. Denote the vertex $s(h_6) = r(h_4) = r(h_5)$ by $v$. Then $u \sim r(p), v \sim r(x), h_3 h_4$ is a directed path from $u$ to $v$ and $h_6 h_1$ is a directed path from $v$ to $u$. Thus $u$ and $v$ are in the same strongly connected component of $\Gamma$. This proves part (e).

Suppose that $pq^* R x y^*$ and $pq^* \neq x y^*$. Then by parts (a) and (b), $pp^* = xx^*$ and $qq^* = yy^*$ and also either $x \neq p$ or $y \neq q$. Assume that $x \neq p$. (The case $y \neq q$ is similar.) By Corollary 6.5 there is a non-empty NE path $t$ such that either $p = xt$ or $x = pt$. Assume that $x = pt$ since the other case is dual. Since $t$ is an NE path we must have $t = p'C^k$ for some NE path $p'$ containing no cycles, some NE cycle $C$, and some integer $k \geq 0$. Since $x \neq p$ we cannot have $p'$ and $C$ both trivial: also, if $p'$ is trivial then $k > 0$.

Case 1: $r(p') = r(p)$. Then $C$ is a cycle based at $r(p) = r(q)$ and $x = p'C^k$ for some $k > 0$. Since $yy^* = qq^*$, Corollary 6.5 implies that there is an NE path $p''$ such that either $y = qp''$ or $q = yy''$. Since $r(q) = r(x) = r(y)$ and $C$ is an NE cycle, this forces $p'' = C^m$ for some $m \geq 0$. If $y = qC^m$ then since the last edge in $C$ is an NE edge, the fact that $xy^*$ is in canonical form forces $m = 0$. So in this case $xy^* = p'C^k q^*$. If $q = yC^m$ then $xy^* = pC^k C^{-m} q^* = pC^{k-m} q^*$ since $CC^* = C^* C = s(C)$ in $LI(\Gamma)$. Thus in Case 1, $xy^* = pC^m q^*$ for some non-zero integer $n$.

Case 2: $r(p') \neq r(p)$. Then $r(x) = r(y) = r(p') \neq r(p) = r(q)$. As in Case 1, there is an NE
path \( p'' \) such that either \( y = qp'' \) or \( q = yp'' \). If \( q = yp'' \) then the path \( p'p'' \) is a non-trivial NE circuit based at \( r(p) \) so there is some non-trivial cycle \( C_1 \) based at \( r(p) \) and \( p'p'' = C^n \) for some \( n > 0 \). Since \( p'' \) is an NE path from \( r(y) = r(x) \) to \( r(p) = r(q) \) in this case we have \( p''p'^* = r(y) \) and so \( y = yr(y) = yp''p'^* = qp''p'^* \). Then \( xy'' = pp''p'^*q = pCq^lq^* \). If \( y = qp'' \) then \( p' p^k \) and \( p'' \) are non-trivial NE paths starting at \( r(p) = r(q) \) and ending at \( r(p') = r(p'') = r(x) = r(y) \). So \( p'' = p' C^l \) for some integer \( l \geq 0 \). The fact that \( xy'' \) is in canonical form forces that one and only one of \( k, l \) is nonzero. Thus, we have \( xy'' = pp'C^n p^* q^* \) for some non-zero integer \( n \). This completes the proof of part (f).

Suppose that \( xy'' \not\sim pp^* \) and \( xy'' \not= pp^* \). From part (f) we have an NE path \( p' \) and a non-trivial NE cycle \( C \) either with \( s(p') = r(p), s(C) = r(p') = r(x) = r(y) \) and \( xy'' = pp'C^n p^* p^* \) or with \( s(p') = r(x) = r(y), s(C) = r(p') = r(p) \) and \( pp^* = xp'C^n p^* y^* \) for some non-zero \( n \). This latter condition is impossible by Corollary 6.5 and the uniqueness of canonical forms, so we must have the former condition. Without loss of generality, we may suppose that \( p' \) does not contain an edge in \( C \). Otherwise we may assume that \( p' = p_1 p_2 \) where \( p_1 \) does not contain an edge in \( C \) and all edges in \( p_2 \) are contained in \( C \), and also that \( C = p_3 p_2 \). Thus, \( xy'' = pp_1 p_2 (p_3 p_2)^n p_2 p_1 p^* = pp_1 p_2 p_3 p_1 p^* \) where \( p_1 \) does not contain an edge in the cycle \( p_2 p_3 \) which is a conjugate of \( C \). Moreover, if \( pp'C^n p^* p^* = pp''(C')^n p^* p^* \), then since \( p', p'' \) and \( C' \) are NE, \( C' \) must be a conjugate of \( C \). Noticing that any edge contained in \( C' \) is also contained in \( C \), we see that \( p' = p'' \) and \( C^n = C'^n \) so that \( C'^{m-n} = r(C) \). By Corollary 6.5 this implies \( m = n \). Since \( (pp'C^n p^* p^*)pp'C^n p^* \) is isomorphic to the group \( (Z, +) \). Thus part (g) is verified.

\[ \square \]

**Remark** We remark that there is a significant difference between the Green relations on \( I(\Gamma) \) and the Green relations on \( LI(\Gamma) \). The Green relations on \( I(\Gamma) \) are given in [25], Corollary 2. In particular, \( I(\Gamma) \) is combinatorial for every graph \( \Gamma \), but by Theorem 7.3(f) \( LI(\Gamma) \) is combinatorial if and only if \( \Gamma \) has no non-trivial NE cycles. In particular, if \( \Gamma \) is acyclic (which is equivalent to the multiplicative semigroup of \( LI(\Gamma) \) being von-Neumann regular by [7]), then \( LI(\Gamma) \) is combinatorial. The converse is false in general of course since \( \Gamma \) may have non-trivial cycles but no non-trivial NE cycles.

Denote the \( \sim \)-equivalence class of a vertex \( v \in \Gamma^0 \) by \([v]\). Note that \([v] = \{v\}\) if and only if \( v \) does not have out-degree 1 and \( s(e) \) does not have out-degree 1 for every edge \( e \in r^{-1}(v) \). (In particular, \([v] = \{v\}\) if \( v \) is a source whose out-degree is not 1.) We denote by \( \Gamma_{[v]} \) the subgraph of \( \Gamma \) induced by the set of vertices in \([v]\). That is, \( \Gamma_{[v]}^0 = [v] \) and \( \Gamma_{[v]}^1 = \{e \in \Gamma^1 : s(e), r(e) \in [v]\} \).

**Lemma 7.4** Let \( v \) be a vertex in a graph \( \Gamma \).

(a) If there are at least two non-conjugate cycles \( C_1 \) and \( C_2 \) in \( \Gamma_{[v]} \) then \( C_1 \cap C_2 \) contains a vertex of out-degree greater than 1.

(b) \( \Gamma_{[v]} \) contains at most one vertex \( w \) of out-degree equal to 1. This vertex \( w \) is contained in every cycle \( C \) for which \( \Gamma \cap C_{[v]} \) (if there are any such cycles).

(c) If \( \Gamma_{[v]} \) contains an NE cycle, then every vertex in \([v]\) has out-degree 1. In particular, there is only one conjugacy class of cycles in \( \Gamma_{[v]} \).

**Proof.** (a) Suppose that \( \Gamma_{[v]} \) contains distinct cycles \( C_1 \) and \( C_2 \) that are not cyclic conjugates of each other. Let \( v_1 \) be a vertex in \( C_1 \) \( \setminus \) \( C_2 \) and \( v_2 \) a vertex in \( C_2 \) \( \setminus \) \( C_1 \). Then \( v_1 \sim v_2 \) so there are NE paths \( p \) and \( q \) with \( s(p) = v_1, s(q) = v_2 \) and \( r(p) = r(q) \in (C_1 \cap C_2)^0 \). If the out-degree of \( r(p) \) is greater than 1 we are done. If not, then \( r(p) \) has out-degree 1 and the edge \( e_1 = e_{r(p)} \) lies in \( C_1 \cap C_2 \). But then either \( r(e_1) \) has out-degree at least 2 or \( r(e_1) \) has out-degree 1, and in
the latter case the edge \( e_2 \) starting at \( r(e_1) \) lies on \( C_1 \cap C_2 \). Continuing in this fashion we see that there is some vertex \( w \) in \( C_1^0 \cap C_2^0 \) with out-degree at least 2.

(b) If \( v_1 \) and \( v_2 \) are distinct vertices in \([v]\) then \( v_1 \sim v_2 \). By the definition of the equivalence relation \( \sim \) this forces either \( v_1 = v_2 \) or else at least one of the vertices \( v_i \) has out-degree 1. So there is at most one vertex in \([v]\) of out-degree not equal to 1. Suppose that there is such a vertex in \([v]\) and denote it by \( w \). If \( C \) is a cycle in \( \Gamma_{[v]} \) and \( v_1 \) is a vertex in \( C \), then \( v_1 \sim w \) so there are NE paths \( p \) and \( q \) with \( s(p) = v_1, s(q) = w \) and \( r(p) = r(q) \). Since the out-degree of \( w \) is not 1, \( q \) must be the empty path at \( w \) and \( w = r(p) \). But then since every vertex in \( p \) except \( w \) has out-degree 1 this forces \( w \) to be a vertex of the cycle \( C \). If \( C_1 \) and \( C_2 \) are distinct non-conjugate cycles in \( \Gamma_{[v]} \) then by the proof above we see that \( w \) is in both cycles.

(c) Suppose that \( \Gamma_{[v]} \) contains an NE cycle \( C \). If \([v]\) contains a vertex of out-degree not equal to 1 then this vertex must lie on \( C \) by part (b), but this is a contradiction since \( C \) is an NE cycle. The fact that there is only one conjugacy class of cycles in \( \Gamma_{[v]} \) follows from part (a).

**Lemma 7.5** Let \( v \) be a vertex of the graph \( \Gamma \). Then

(a) If every vertex of \([v]\) has out-degree 1 then \( \Gamma_{[v]} \) is a directed cover of \( B_{(a)} \), in which case it is either an infinite tree or has structure determined by Theorem 3.1(b);

(b) If \([v]\) has a sink in \( \Gamma \), then \( \Gamma_{[v]} \) is an immersion over \( B_{(a)} \) whose structure is determined by Theorem 3.1(a);

(c) If \([v]\) has a vertex \( w \) of out-degree greater than 1 in \( \Gamma \) then either \( \Gamma_{[v]} \) contains no cycles, in which case \( w \) is a sink of the graph \( \Gamma_{[v]} \) and \( \Gamma_{[v]} \) is an immersion over \( B_{(a)} \) whose structure is determined by Theorem 3.1(a), or else \( \Gamma_{[v]} \) has at least one cycle and \( w \in C^0 \) for every cycle \( C \) in \( \Gamma_{[v]} \). In the latter case, if \( v' \) is any vertex of \([v]\) with \( v' \neq w \), then there is a unique directed path from \( v' \) to \( w \) that does not include a cycle in \( \Gamma_{[v]} \) as a subpath.

**Proof.** Part (a) is immediate from Theorem 3.1. If \([v]\) contains a sink of \( \Gamma \) then this vertex is also a sink of \( \Gamma_{[v]} \) so the result of part (b) is also immediate from Theorem 3.1. Suppose that \([v]\) has a vertex \( w \) of out-degree greater than 1 in \( \Gamma \). There is a unique such vertex \( w \) by Lemma 7.4(b). If \( \Gamma_{[v]} \) contains no cycles, then \( w \) is a sink in the graph \( \Gamma_{[v]} \), so \( \Gamma_{[v]} \) has the structure described in Theorem 3.1(a). If \( \Gamma_{[v]} \) has at least one cycle then \( w \in C^0 \) for every cycle \( C \) in \( \Gamma_{[v]} \) by Lemma 7.4(b). If \( v' \neq w \) is a vertex of \([v]\) then there is a directed path \( p \) from \( v' \) to \( w \) since \( v' \sim w \) and \( w \) has out-degree greater than 1. Since \( w \) is in every cycle in \( \Gamma_{[v]} \), we may assume that the path \( p \) does not contain any cycle in \( \Gamma_{[v]} \) as a subpath. The uniqueness of such a directed path \( p \) follows by an argument very similar to the argument used in the proof of Theorem 3.1.

**Lemma 7.6** If \( \phi \) is an isomorphism from \( LI(\Gamma) \) onto \( LI(\Delta) \), then the following statements hold.

(a) \( \phi \) induces a bijection from \( \Gamma^0 \) onto \( \Delta^0 \).

(b) For all vertices \( v_1, v_2 \) of \( \Gamma \), \( v_1 \sim v_2 \) if and only if \( \phi(v_1) \sim \phi(v_2) \).

(c) \( \phi \) induces a bijection of the equivalence class \([v]\) in \( \Gamma^0 \) onto the equivalence class \([\phi(v)]\) in \( \Delta^0 \) for all \( v \in \Gamma^0 \).

(d) For each integer \( n \geq 1 \), \([v]\) contains a vertex of out-degree \( n \) if and only if \([\phi(v)]\) contains a vertex of out-degree \( n \).

(e) If \( C \) is a cycle in \( \Gamma_{[v]} \), then \( \phi(C) \) is uniquely expressible in the form \( \phi(C) = pC'p^* \) or \( \phi(C) = pC'p^* \) in \( LI(\Delta) \) for some cycle \( C' \) and some NE path \( p \) contained in \( \Delta_{\phi(v)} \), and moreover \( \phi^{-1}(C') = p_1C_1p_1^* \) or \( \phi^{-1}(C') = p_1C_1p_1^* \) for some cyclic conjugate \( C_1 \) of \( C \) and some NE path \( p_1 \) contained in \( \Gamma_{[v]} \).
(f) $\phi$ induces a bijection between the set of distinct conjugacy classes of cycles in $\Gamma_{[v]}$ and the set of distinct conjugacy classes of cycles in $\Delta_{[\phi(v)]}$.

Proof. (a) By Lemma 6.10(a), $\phi$ maps vertices of $\Gamma$ to vertices of $\Delta$. The restriction of $\phi$ to $\Gamma^0$ is clearly injective since $\phi$ is injective. But by Lemma 6.10(a), the map $\phi^{-1}$ maps vertices of $\Delta$ to vertices of $\Gamma$, so the restriction of $\phi$ to $\Gamma^0$ is a bijection onto $\Delta^0$.

(b) This follows immediately from Theorem 7.3(d).

(c) By parts (a) and (b) of this lemma, $\phi$ induces an injection of the equivalence class $[v]$ into the equivalence class $[\phi(v)]$ for all $v \in \Gamma^0$. This map is surjective since $\phi^{-1}$ induces an injection of the equivalence class $[\phi(v)]$ into the equivalence class $[v]$.

(d) Suppose that $[v]$ contains a vertex $v$ of out-degree $n > 1$. Then, in the notation of Lemma 6.10(e), the vertex $s(p_{i+1})$ has out-degree $n$ and since $p_{i+1}$ is an NE path, $s(p_{i+1})$ is a prefix of $s(e)$. If all vertices of $[v]$ have out-degree 1 then by what we just proved, applied to $\phi^{-1}$, all vertices of $[\phi(v)]$ have out-degree 1.

(e) Suppose that $[v]$ contains a cycle $C = e_1 e_2 ... e_n$. Let $\phi(e_i) = p_i q_i$ where $p_i, q_i$ are directed paths in $\Delta$ with $r(p_i) = r(q_i)$. If $\phi(e_1)$ has out-degree 1 then by Lemma 6.10(c), $p_i, q_i$ are NE paths so all of their vertices are related via the equivalence relation $\sim$ on $\Delta^0$. By Lemma 7.4(b) $C$ contains at most one vertex (say $e_k$) whose out-degree is greater than 1. Then by Lemma 6.10(d), $\phi(e_k) = p_k q_k$ where $p_k, q_k$ are NE paths and the out-degree of $s(e_k)$ is at least 2. Then all vertices in $p_k$ are $\sim$-related to $r(p_k) = s(e_k)$ and all vertices in $p_k q_k$ are $\sim$-related to $r(e_k)$. Then since $s(q_1) = s(p_2), ..., s(q_{k-1}) = s(p_k), s(q_k) = s(p_{k+1}), ..., s(q_n) = s(p_1)$, we see that all vertices in $p_1 q_1 p_2 q_2 ... p_k$ are $\sim$-related to $s(p_1)$ and all vertices in $p_k q_k p_{k+1} q_{k+1} ... q_n$ are $\sim$-related to $s(q_n) = s(p_1)$. Thus all vertices in $\phi(C)$ are in the $\sim$-class $[\phi(v)]$ of $\Delta^0$, that is $\phi(C)$ is a path in $\Delta_{[\phi(v)]}$.

Since $C$ represents a non-zero element of $L\Gamma$, we have $\phi(C) = pq$ in $L\Delta$, where $p$ and $q$ are directed paths in $\Delta$ with $r(p) = s(q) = \phi(s(e_1))$ and $r(p) = r(q)$. Furthermore, all vertices in $pq$ are in $[\phi(v)]$ by the argument in the previous paragraph since these vertices are among the vertices in the union of the paths $p_i q_i$, $i = 1, ..., n$. Since $C$ is a cycle, $C^2$ is also a non-zero element of $L\Gamma$ and so $pq^* pq^*$ is a non-zero element in $L\Delta$. Hence either $p$ is a prefix of $q$ or $q$ is the prefix of $p$. Note from Lemma 6.10(b) and the multiplication in $L\Gamma$ that $q$ is NE. If the out-degree of $s(e_k)$ is at least 2, then by Lemma 6.10(d) $p$ contains the unique edge $e_k$ which has exits so that $q$ must be a prefix of $p$. That is $p = qp_1$ for some directed circuit $p_1$ which contains $e_k$. Now $p_1$ must be a cycle since only $e_k$ has exits and $e_k$ appears in $p_1$ only once. If $C$ is NE and $q$ is a prefix of $p$ which means that $p = qp_2$ for some directed circuit $p_2$ in $\Delta$, then by Lemma 6.10(c) $p_2$ is NE. So there must exist some NE cycle $C'$ such that $p_2 = (C')^k$ for some positive integer $k$. A similar discussion shows that $\phi^{-1}(qC'q^*) = C'$ for some positive integer $l$. These force that $k = l = 1$. If $p$ is a prefix of $q$, a similar argument shows that $\phi(C) = p(C^* p)^*$ for some NE cycle $C'$. The uniqueness of such expression follows from the canonical forms of elements in $L\Delta$.

Moreover, if $\phi(C) = pC'p^*$ for some NE cycle $C'$ and NE path $p$, then we get $C = \phi^{-1}(p) \phi^{-1}(C') \phi^{-1}(p^*)$ We see from Lemma 6.10(c) that there exist NE paths $p_1, q_1$ in $\Gamma$ such that $C = (q_1 p_1) \phi^{-1}(C') (p_1 q_1)$. That is, $\phi^{-1}(C') = p_1 q_1 C q_1 p_1^*$. It follows that $q_1 C q_1$ is a cyclic conjugate of $C$ since $q_1$ is NE. A similar argument applies if $\phi(C) = pC^* p^*$ for some NE cycle $C'$ and NE path $p$.

(f) By Lemma 7.5(a), if $\Gamma_{[v]}$ is not a tree and all vertices of $\Gamma_{[v]}$ have out-degree 1 then $\Gamma_{[v]}$ contains a unique cycle $C_v$ (up to cyclic conjugates). By part (c) of this lemma and Lemma 6.10(c) all vertices of $\Delta_{[\phi(v)]}$ have out degree 1 and $\Delta_{[\phi(v)]}$ has a unique cycle $C'_{[\phi(v)]}$ (up to cyclic
conjugates). If \( \Gamma_{[v]} \) has \( n \) distinct cycles \( C_1, \ldots, C_n \) (up to cyclic conjugates) for some \( n > 1 \), then by Lemma 7.3(b) there exists some unique vertex \( w \) (in \( [v] \)) whose out-degree is at least 2 in \( \Gamma \) such that \( w \) lies in all these cycles. Moreover, these cycles correspond to the edges in \( s^{-1}(w) \cap \Gamma_{[v]} \). Then by part (b) of this lemma and Lemma 6.10(e), \( \Delta_{[\phi(v)]} \) has \( n \) distinct cycles (up to cyclic conjugates).

For each vertex \( v \) of a graph \( \Gamma \) let \( T_{[v]} \) be a (directed) spanning tree of the subgraph \( \Gamma_{[v]} \). From the structure of the graphs \( \Gamma_{[v]} \) described in Lemma 7.5 it is clear that \( T_{[v]} = \Gamma_{[v]} \) if \([v]\) does not contain any cycle, while on the other hand if \( \Gamma_{[v]} \) does contain a cycle then \( T_{[v]} \) contains all edges of \( \Gamma_{[v]} \) except some particular edge \( e_C \) of \( C \) for each cycle \( C \) in \( \Gamma_{[v]} \). Note that these \( e_C \)'s can not be chosen arbitrarily for cycles with exits. For instance, consider the graph \( \Gamma \) represented in Diagram 7.1. The subgraph \( \Gamma_{[v]} \) has two conjugacy classes of cycles, namely the conjugacy classes of the cycles \( C_1 \) consisting of the edges \( e_1, e_2, e_3, e_5 \), and \( C_2 \) consisting of the edges \( e_1, e_2, e_4, e_6 \); we can not respectively choose \( e_1 \) as \( e_{C_1} \) and \( e_2 \) as \( e_{C_2} \) because the remaining subgraph is not a tree. On the other hand, \( \Gamma_{[u]} \) has one conjugacy class of cycles, namely the conjugacy class of the cycle \( C_3 \) consisting of the edges \( e_7 \) and \( e_8 \), and we can choose either \( e_7 \) or \( e_8 \) as \( e_{C_3} \).

Since cycles in distinct equivalence classes are clearly disjoint, the choice of edges \( e_C \) for cycles in distinct equivalence classes are disjoint and the spanning trees \( T_{[v]} \) (for \( v \in \Gamma^0 \)) are uniquely determined by the choice of these edges \( e_C \) for each cycle \( C \).

We call a set \( \{T_{[v]} : v \in \Gamma^0\} \) of spanning trees for the induced graphs \( \Gamma_{[v]} \) a set of NE spanning trees if every edge in each tree is an NE edge in \( \Gamma \).

**Lemma 7.7** For any \( v \in \Gamma^0 \) and any cycle \( C \) in \( \Gamma_{[v]} \), one obtains a set of NE spanning trees by choosing any edge in \( C \) as \( e_C \) if \( C \) is an NE cycle and choosing the edge with exits as \( e_C \) if \( C \) has exits. Every set of NE spanning trees is obtained this way.

**Proof.** If every vertex of \( C \) has out-degree 1, then by Lemma 7.4 we know that \( \Gamma_{[v]} \) has only one cycle. It follows from Theorem 3.1(a) that the subgraph \( \Gamma_{[v]} \setminus \{e_C\} \) is an NE tree (which is a spanning tree of \( \Gamma_{[v]} \)) since \( s(e_C) \) is a sink and every other vertex has out-degree 1. If \( C \) has exits, then we see from Lemma 7.4(b) that \( C \) contains only one vertex \( v_0 \) with out-degree at least 2. In this case, \( \Gamma_{[v]} \) may contain more than one cycle. Again it follows from Theorem 3.1(a) and Lemma 7.5(c) that the subgraph \( \Gamma_{[v]} \setminus s^{-1}(v_0) \) is an NE tree (which is a spanning tree of \( \Gamma_{[v]} \)) since \( v_0 \) is a sink and every other vertex has out-degree 1. In this way, we get a set of NE spanning trees. By the definition of a set of NE spanning trees, no set of NE spanning trees can contain an edge \( e \in s^{-1}(v_0) \) since all of these edges have exits in \( \Gamma \). □

Diagram 7.1 Two \( \sim \)-classes containing cycle(s)

Since cycles in distinct equivalence classes are clearly disjoint, the choice of edges \( e_C \) for cycles in distinct equivalence classes are disjoint and the spanning trees \( T_{[v]} \) (for \( v \in \Gamma^0 \)) are uniquely determined by the choice of these edges \( e_C \) for each cycle \( C \).

We call a set \( \{T_{[v]} : v \in \Gamma^0\} \) of spanning trees for the induced graphs \( \Gamma_{[v]} \) a set of NE spanning trees if every edge in each tree is an NE edge in \( \Gamma \).

**Lemma 7.7** For any \( v \in \Gamma^0 \) and any cycle \( C \) in \( \Gamma_{[v]} \), one obtains a set of NE spanning trees by choosing any edge in \( C \) as \( e_C \) if \( C \) is an NE cycle and choosing the edge with exits as \( e_C \) if \( C \) has exits. Every set of NE spanning trees is obtained this way.

**Proof.** If every vertex of \( C \) has out-degree 1, then by Lemma 7.4 we know that \( \Gamma_{[v]} \) has only one cycle. It follows from Theorem 3.1(a) that the subgraph \( \Gamma_{[v]} \setminus \{e_C\} \) is an NE tree (which is a spanning tree of \( \Gamma_{[v]} \)) since \( s(e_C) \) is a sink and every other vertex has out-degree 1. If \( C \) has exits, then we see from Lemma 7.4(b) that \( C \) contains only one vertex \( v_0 \) with out-degree at least 2. In this case, \( \Gamma_{[v]} \) may contain more than one cycle. Again it follows from Theorem 3.1(a) and Lemma 7.5(c) that the subgraph \( \Gamma_{[v]} \setminus s^{-1}(v_0) \) is an NE tree (which is a spanning tree of \( \Gamma_{[v]} \)) since \( v_0 \) is a sink and every other vertex has out-degree 1. In this way, we get a set of NE spanning trees. By the definition of a set of NE spanning trees, no set of NE spanning trees can contain an edge \( e \in s^{-1}(v_0) \) since all of these edges have exits in \( \Gamma \). □
Form a new graph $\bar{\Gamma}$ by contracting each spanning tree $T_{[v]}$, $(v \in \Gamma^0)$ to a point. More precisely, we may describe the graph $\bar{\Gamma}$ in the following way: $\bar{\Gamma}^0 = \{ [v] : v \in \Gamma^0 \}$ and $\bar{\Gamma}^1$ is a set in one-one correspondence with $\{ e \in \Gamma^1 \setminus \bigcup_{v \in \Gamma^0} T^1_{[v]} \}$. We denote the image of $e$ under this correspondence by $\bar{e}$. The source and range functions are defined for $\bar{e} \in \bar{\Gamma}^1$ by $s(\bar{e}) = [s(e)]$ and $r(\bar{e}) = [r(e)]$. Thus the edge $e_C$ of a cycle $C$ in $\Gamma_{[v]}$ gives rise to a loop $\bar{e}_C$ at $[v]$ in $\bar{\Gamma}$. There is a natural function $\chi : \Gamma \to \bar{\Gamma}$ defined by $\chi_{\Gamma}(v) = [v]$ for all $v \in \Gamma^0$, $\chi_{\Gamma}(e) = \bar{e}$ for $e \in \Gamma^1 \setminus \bigcup_{v \in \Gamma^0} T^1_{[v]}$, and $\chi_{\Gamma}(e) = [v]$ if $e \in T^1_{[v]}$ for some $v \in \Gamma^0$. The map $\chi_{\Gamma}$ is not a graph morphism since it maps some edges to vertices.

**Lemma 7.8** In a contracted graph $\bar{\Gamma}$, if the out-degree of $[v]$ is one, then the only edge in $s^{-1}([v])$ is a loop. Hence, the equivalence relation $\sim$ on $\Gamma^0$ is trivial.

**Proof.** For any edge $\bar{e}$ in $\bar{\Gamma}$, either $e$ does not belong to any graph $\Gamma_{[v]}$ or $e$ belongs to exactly one such graph. In the former case the out-degree of $s(e)$ is at least 2 and $s(e)$ is not $\sim$ related to $r(e)$. In the latter case either $e$ lies in a cycle which has exits only at $s(e)$ or $e$ lies in an NE cycle. In the first two cases, the out-degree of $s(\bar{e})$ is also at least 2. In the third case, $\bar{e}$ is a loop and the out-degree of $s(\bar{e})$ is 1. The second statement of the lemma thus follows directly from the definition of the relation $\sim$.

We remark that the graph $\bar{\Gamma}$ and the contraction map $\chi_{\Gamma}$ depends on the choice of the spanning trees $T_{[v]}$, $v \in \Gamma^0$. However the contracted graphs corresponding to different choices of NE spanning trees are isomorphic.

**Lemma 7.9** For any $v \in \Gamma^0$, arbitrarily choose sets of NE spanning trees $T_v$ and $T'_v$ for $\Gamma_{[v]}$. Then the contracted graph $\bar{\Gamma}_1$ corresponding to the spanning trees $T_v$, $v \in \Gamma^0$, is isomorphic to the contracted graph $\bar{\Gamma}_2$ corresponding to the spanning trees $T'_v$, $v \in \Gamma^0$.

**Proof.** By Lemma 7.7 an NE spanning tree $T_{[v]}$ for $\Gamma_{[v]}$ is determined by the choice of an edge $e_C$ in $C$ in an NE cycle $C$ in $\Gamma_{[v]}$ (if such a cycle exists) since the choices of the edges $e_C$ for a cycle $C$ that has exits is fixed. Similarly another spanning tree $T'_v$ for $\Gamma_{[v]}$ is determined by the choice of another edge $e'_C$ in each NE cycle $C$ in $\Gamma_{[v]}$. Then the map defined by $[v] \mapsto [v]$, $\bar{e}_C \mapsto \bar{e}'_C$ for all cycles $C$ in $\Gamma_{[v]}$ (and all $v \in \Gamma^0$), and $\bar{e} \mapsto \bar{e}$ for all other edges of $\bar{\Gamma}_1$ induces a graph isomorphism of $\bar{\Gamma}_1$ onto $\bar{\Gamma}_2$.

**Lemma 7.10** The mapping $\chi_{\Gamma} : \Gamma \to \bar{\Gamma}$ naturally induces a 0-restricted morphism $\bar{\chi}_{\Gamma}$ from $LI(\Gamma)$ onto $LI(\bar{\Gamma})$.

**Proof.** This is a routine calculation. The map $\chi_{\Gamma}$ defines a map $\bar{\chi}_{\Gamma}$ from the generators of $LI(\Gamma)$ onto the generators of $LI(\bar{\Gamma})$ that is easily seen to extend to a 0-restricted morphism if we define $\bar{\chi}(0) = 0$.

We note that in general $\bar{\chi}_{\Gamma}$ is not a homomorphism since if $e_1$ and $e_2$ are distinct edges in one of the spanning trees $T_{[v]}$ in $\Gamma$, then $e_1 e_2 = 0$ in $LI(\Gamma)$ but $\bar{\chi}_{\Gamma}(e_1^*) \bar{\chi}_{\Gamma}(e_2^*) = [v][v] = [v]$ in $LI(\bar{\Gamma})$. Note also that by the definition of the edges in $\bar{\Gamma}$, $\bar{\chi}_{\Gamma}(e) = \bar{e}$ if $e \in \Gamma^1 \setminus \bigcup_{v \in \Gamma^0} T^1_{[v]}$, so $\bar{\chi}_{\Gamma}$ induces a bijection from the edges $e \in \Gamma^1 \setminus \bigcup_{v \in \Gamma^0} T^1_{[v]}$ onto the edges $\bar{e}$ in $\bar{\Gamma}$.

**Lemma 7.11** Let $\phi : LI(\Gamma) \to LI(\Delta)$ be a semigroup isomorphism. Choose two sets of NE spanning trees $\{ T_{[v]} : v \in \Gamma^0 \}$ and $\{ T'_{[u]} : u \in \Delta^0 \}$. Then there exists a graph isomorphism $\phi$ from $\bar{\Gamma}$ to $\bar{\Delta}$ such that $\chi_{\Delta}(\psi(v)) = \bar{\phi} \chi_{\Gamma}(v)$ where $v \in \Gamma^0$ and $\psi$ is the restriction of $\phi$ to $\Gamma^0$.
Moreover, we also observe from Lemma 6.10(d) that \( \phi \) according to Lemma 7.6(e). Thus, it follows from Lemmas 6.10(e), 7.4(c) and 7.6(b), (d), (f) that the claim above that \( s(\phi(e)) \sim s(\phi(e)) \) in \( \Delta \) if \( e \) is NE. To see this, note Lemma 7.7 and take a set of NE spanning trees for \( \Gamma \). Then the graph is commutative.

Proof. For any \( [v] \in \hat{\Gamma}^0 \), define \( \bar{\phi}([v]) = \chi_{\Delta}\phi(v) \). We see from Lemma 7.7(b) that \( \bar{\phi} \) is well-defined and is an injection from \( \hat{\Gamma}^0 \) to \( \hat{\Delta}^0 \). For any \( [u] \in \hat{\Delta}^0 \), again by Lemma 7.7(b), \( [\phi^{-1}(u)] \in \Gamma^0 \) does not depend on the choice of \( u \). Moreover, \( \phi([\phi^{-1}(u)]) = \chi_{\Delta}(\phi([\phi^{-1}(u)]) = \chi_{\Delta}\phi^{-1}(u) = [u] \). So \( \phi \) is a bijection from \( \Gamma^0 \) onto \( \Delta^0 \). Clearly, \( \chi_{\Delta}\psi(v) = \phi\chi_{\Gamma}(v) \) for any \( v \in \Gamma^0 \).

We now claim that there exists a bijection \( \varphi \) from \( \Gamma^1 \setminus \bigcup_{v \in \Gamma^0} T^1_{[v]} \) onto \( \Delta^1 \setminus \bigcup_{u \in \Delta^0} T^1_{[u]} \) such that for any \( e \in \Gamma^1 \setminus \bigcup_{v \in \Gamma^0} T^1_{[v]} \), \( \varphi(e) \) contains \( \varphi(e) \) if \( e \) has exits and \( s(\varphi(e)) \sim s(\varphi(e)) \) in \( \Delta \) if \( e \) is NE. To see this, note Lemma 7.7 and take a set of NE spanning trees for \( \Gamma \). Then the edges in \( \Gamma^1 \setminus \bigcup_{v \in \Gamma^0} T^1_{[v]} \) can be divided into three types: edges \( e \) where the out-degree of \( s(e) \) is at least 2 and \( s(e) \) is related to \( r(e) \); edges \( e_C \) in a cycle \( C \) which has exits only at \( s(e_C) \); and edges \( e_{C'} \) in an NE cycle \( C' \). Define \( \varphi \) as the following: for the first two types, \( \varphi(e) \) is as in Lemma 6.10(d); for the third type, \( \varphi(e_{C'}) \) is \( e_{C'} \), where \( C' \) is the cycle corresponding to \( C \) according to Lemma 7.6(e). Thus, it follows from Lemmas 6.10(e), 7.4(c) and 7.6(b), (d), (f) that \( \varphi \) preserves the types of elements from \( \Gamma^1 \setminus \bigcup_{v \in \Gamma^0} T^1_{[v]} \) to \( \Delta^1 \setminus \bigcup_{u \in \Delta^0} T^1_{[u]} \) and is a bijection. Moreover, we also observe from Lemma 6.10(d) that \( \varphi(e) \) contains \( \varphi(e) \) if \( e \) has exits, and from Lemma 6.10(e) that \( s(\varphi(e)) \sim s(\varphi(e)) \) in \( \Delta \) if \( e \) is NE. This proves the claim.

Take a bijection \( \varphi \) from \( \Gamma^1 \setminus \bigcup_{v \in \Gamma^0} T^1_{[v]} \) onto \( \Delta^1 \setminus \bigcup_{u \in \Delta^0} T^1_{[u]} \) constructed as above. For any \( \tilde{e} \in \tilde{\Gamma}^1 \), define \( \tilde{\varphi}(\tilde{e}) = \chi_{\Delta}\varphi(e) \). By the definition of \( \chi_{\Delta} \), \( \tilde{\varphi} \) restricts to a (well-defined) bijection from \( \tilde{\Gamma}^1 \) onto \( \hat{\Delta}^1 \). For any \( \tilde{e} \in \tilde{\Gamma}^1 \), we know that \( e \in \Gamma^1 \setminus \bigcup_{v \in \Gamma^0} T^1_{[v]} \). It follows from the proof of the claim above that \( \varphi(s(e)) \sim s(\varphi(e)) \) and \( \varphi(r(e)) \sim r(\varphi(e)) \) in \( \Delta \). Thus, we observe from the definitions of \( \chi_{\Gamma} \), \( \chi_{\Delta} \) and \( \tilde{\varphi} \) that \( s(\tilde{\varphi}(\tilde{e})) = [s(\varphi(e))] = [\varphi(s(e))] = \tilde{\varphi}(s(e)) \). Similarly, \( r(\tilde{\varphi}(\tilde{e})) = \tilde{\phi}(r(e)) \). Therefore, \( \tilde{\phi} \) is an isomorphism from \( \hat{\Gamma} \) to \( \hat{\Delta} \).

We have proved the direct part of the following theorem, which classifies graphs with isomorphic Leavitt inverse semigroups.

**Theorem 7.12** Let \( \Gamma \) and \( \Delta \) be graphs. Then \( LI(\Gamma) \cong LI(\Delta) \) if and only if there is a bijection \( \psi : \Gamma^0 \to \Delta^0 \), sets of NE spanning trees \( \{T_{[v]} : v \in \Gamma^0\} \) and \( \{T_{[u]} : u \in \Delta^0\} \), and a graph isomorphism \( \bar{\phi} : \Gamma \to \Delta \) such that, for all \( v \in \Gamma^0 \), \( \bar{\phi}(\chi_{\Gamma}(v)) = \chi_{\Delta}(\psi(v)) \); that is, the following diagram is commutative.

\[
\begin{array}{ccc}
\Gamma^0 & \overset{\psi}{\longrightarrow} & \Delta^0 \\
\chi_{\Gamma} \downarrow & & \downarrow \chi_{\Delta} \\
\hat{\Gamma}^0 & \overset{\bar{\phi}}{\longrightarrow} & \hat{\Delta}^0 \\
\end{array}
\]

We need some additional notation and lemmas to prove the converse part of Theorem 7.12.

**Lemma 7.13** Suppose that there is a bijection \( \psi : \Gamma^0 \to \Delta^0 \), sets of NE spanning trees \( \{T_{[v]} : v \in \Gamma^0\} \) and \( \{T_{[u]} : u \in \Delta^0\} \), and a graph isomorphism \( \bar{\phi} : \Gamma \to \Delta \) such that, for all \( v \in \Gamma^0 \), \( \bar{\phi}(\chi_{\Gamma}(v)) = \chi_{\Delta}(\psi(v)) \). Then

(a) for all \( u \in \Delta^0 \), \( \bar{\phi}^{-1}(\chi_{\Delta}(u)) = \chi_{\Gamma}(\psi^{-1}(u)) \); and
(b) \( \psi \) restricts to a bijection from \( [v] \) to \( [\psi(v)] \) for all \( v \in \Gamma^0 \).
Proof. The proof of part (a) follows in a routine fashion from the fact that $\phi$ and $\psi$ are bijections. If $v_1 \sim v_2$ in $\Gamma^0$, then $\phi\Gamma(v_1) = \phi\Gamma(v_2)$, and so $\chi\Delta(\psi(v_1)) = \phi(\phi\Gamma(v_1)) = \phi(\phi\Gamma(v_2)) = \chi\Delta(\psi(v_2))$, that is $\psi(v_1) \sim \psi(v_2)$ in $\Delta^0$. Similarly, from part (a) it follows that if $\psi(v_1) \sim \psi(v_2)$ then $v_1 = \psi^{-1}(\psi(v_1)) \sim \psi^{-1}(\psi(v_2)) = v_2$. Part (b) easily follows from this and the fact that $\psi$ is a bijection.

Fixing an NE spanning tree of $\Gamma[v]$, for any $v_1, v_2 \in [v]$, we observe that there exist directed NE paths $p, q$ in $T_{[v]}$ such that $r(p) = r(q), v_1 = s(p)$ and $v_2 = s(q)$. These paths $p, q$ are not necessarily unique. However, there is a unique shortest such directed NE path $p$, and a unique shortest such path $q$. We denote this choice of $pq^*$ by $p[v_1, v_2]$. Clearly $p[v_1, v_2]$ has no non-trivial circuits.

Lemma 7.14 (a) Let $v_1, v_2, v_3, v$ be vertices of a graph $\Gamma$ such that $v_1, v_2, v_3 \in [v]$. Then as elements of $LI(\Gamma)$, we have $p[v_1, v_2]p[v_2, v_3] = p[v_1, v_3], p[v_1, v_2]p[v_2, v_1] = v_1$ and $p[v_1, v_2]^* = p[v_2, v_1]^*$.

(b) If $v_1 \sim v_2 \in \Gamma^0$ and $v_2 = s(e)$ for some edge $e$ that is not in $T_{[v_2]}$, then $p[v_1, v_2]$ is a directed NE path from $v_1$ to $v_2$.

Proof. The proof of part (a) follows easily from the definitions. For part (b) there are two cases. Suppose that $p[v_1, v_2] = pq^*$ for NE paths $p, q$ in $T_{[v_2]}$. If $v_2$ has out-degree greater than 1 then clearly $q$ must be $v_2$ (the empty path at $v_2$), so $p[v_1, v_2] = p$ is a directed path from $v_1$ to $v_2$. If $v_2$ has out-degree 1 but $e$ is not in the spanning tree $T_{[v_2]}$, then again $q$ must be the empty path at $v_2$. This is because if $q$ is not empty, then the first edge of $q$ must be $e$, a contradiction since $q$ is in the spanning tree. Hence again $p[v_1, v_2] = p$ is a directed NE path from $v_1$ to $v_2$.

Now we define a mapping $\tilde{\chi}_\Gamma$ from $LI(\Gamma)$ to $LI(\bar{\Gamma})$. For any $v \in \Gamma^0$, fix a vertex $v_0$ in the $\sim$-class $[v]$. Let $\tilde{\chi}_\Gamma$ map the vertex $[v]$ in $\bar{\Gamma}$ to $v_0$ in $\Gamma$. For any directed path $\bar{p} = \bar{e}_1 \ldots \bar{e}_m$ in $\bar{\Gamma}$, define

$$\tilde{\chi}_\Gamma(\bar{p}) = p[\tilde{\chi}_\Gamma(s(\bar{e}_1)), e_1 p[r(e_1), s(e_2)] e_2 \ldots e_{m-1} p[r(e_{m-1}), s(e_m)] e_m p[r(e_m), \tilde{\chi}_\Gamma(r(e_m))]]$$

Notice that since each edge $e_i$ is not in a spanning tree, we have $p[\tilde{\chi}_\Gamma(s(\bar{e}_i)), s(\bar{e}_i)] = p_1$ for some NE path $p_1$ and $p[r(e_i), s(e_{i+1})] = p_{i+1}$ for some NE path $p_{i+1}$ by Lemma 7.14(b). So, as an element of $LI(\Gamma)$, $\tilde{\chi}_\Gamma(\bar{p})$ is a non-zero element of the form $p_1 e_1 p_2 e_2 \ldots p_m e_m p_{m+1} p_{m+2}^*, r(p_{m+1}) = r(p_{m+2})$ and the $p_i$ are NE paths in the spanning trees. In particular, $\tilde{\chi}_\Gamma(\bar{e}) = p_1 e_2 p_3^*$ where $p_1, p_2, p_3$ are in the spanning trees. For a nonzero element $\bar{pq}^*$ in $LI(\bar{\Gamma})$, define $\tilde{\chi}_\Gamma(\bar{pq}^*) = \tilde{\chi}_\Gamma(\bar{p})(\tilde{\chi}_\Gamma(\bar{q}))^*$. If $\bar{q} = f_1 \ldots f_n$ then $\tilde{\chi}_\Gamma(\bar{pq}^*) = p_1 e_1 \ldots p_m e_m p' q'^* f_n p_s^* \ldots f_1 p_1^*$ for some NE paths $p_i, p'_j, p'$ in the spanning trees with $r(p') = r(q')$.

We call a 0-morphism $f : S \to T$ from an inverse semigroup $S$ onto an inverse subsemigroup $T$ of $S$ a 0-retraction (and we call $T$ a 0-retract of $S$) if the restriction of $f$ to $T$ is the identity map on $T$.

Lemma 7.15 The mapping $\tilde{\chi}_\Gamma$ is a monomorphism from $LI(\Gamma)$ to $LI(\bar{\Gamma})$ such that $\tilde{\chi}_\Gamma \tilde{\chi}_\Gamma$ is the identity mapping of $LI(\bar{\Gamma})$. Hence $LI(\bar{\Gamma})$ is isomorphic to a 0-retract of $LI(\Gamma)$.

Proof. Since the map $\tilde{\chi}_\Gamma$ induces a bijection from $\{e \in \Gamma^1 \setminus \bigcup_{v \in \Gamma^0} T_{[v]}^1\}$ onto $\Gamma^1$ the map from $\bar{\Gamma}$ into $\Gamma^1$ defined by $\bar{e} \mapsto e$ is an injection. But also the path $p[v_1, v_2]$ is uniquely determined by the vertices $v_1, v_2 \in [v]$ and the choice of spanning tree $T_{[v]}$, so it follows that $\tilde{\chi}_\Gamma$ is an
injection from $LI(\Gamma)$ to $LI(\Delta)$. A routine argument, using the characterization of $\hat{\chi}(pq^*)$ in the paragraph above, shows that $\hat{\chi}$ is a homomorphism, so it is a monomorphism. The fact that $\hat{\chi}r\hat{\chi}$ is the identity mapping of $LI(\Gamma)$ follows immediately from the definitions. Thus $\hat{\chi}r\hat{\chi}$ is a surjective 0-morphism from $LI(\Gamma)$ onto the inverse subsemigroup $\hat{\chi}(LI(\Gamma))$ of $LI(\Gamma)$ and $\hat{\chi}(\hat{\chi}(x)) = \hat{\chi}(\chi(x)) = \chi(x)$ for all $x \in LI(\Gamma)$; that is, the restriction of $\hat{\chi}r\hat{\chi}$ to $\hat{\chi}(LI(\Gamma))$ is the identity mapping. Hence $\hat{\chi}(LI(\Gamma))$ is a 0-retract of $LI(\Gamma)$. The result follows since $\hat{\chi}$ is an isomorphism from $LI(\Gamma)$ onto $\hat{\chi}(LI(\Gamma))$. ■

Let $\psi : \Gamma^0 \rightarrow \Delta^0$ be a bijection, $\{T[v] : v \in \Gamma^0\}$ and $\{T[u] : u \in \Delta^0\}$ be sets of NE spanning trees and $\phi : \Gamma \rightarrow \Delta$ be a graph isomorphism such that, for all $v \in \Gamma^0$, $\phi(\chi(v)) = \chi(\psi(v))$. Then $\phi$ naturally induces an isomorphism $\hat{\phi}$ from $LI(\Gamma)$ onto $LI(\Delta)$ which maps directed paths to directed paths. From Lemmas 7.10 and 7.15 we get a 0-restricted morphism $\hat{\phi} = \hat{\chi}\phi\hat{\chi}$ from $LI(\Gamma)$ into $LI(\Delta)$.

Note that $\phi(q) = \hat{\chi}\phi\hat{\chi}(v) = \hat{\chi}\phi\hat{\chi}\chi(v) \sim \psi(v)$, and also that $s(\phi(pq^*)) = s(\phi(p))$ since $\phi$ is a 0-morphism and $s(\phi(p)) = \phi(s(p))$ since $\phi$ is a 0-morphism. Similarly, $r(\phi(pq^*)) = s(\phi(q)) = \phi(s(q))$. In view of these facts we may define, for any nonzero element $pq^* \in LI(\Gamma)$,

$$\phi(pq^*) = p[\psi(s(p)), \phi(s(p))] \phi(pq^*) p[\phi(s(q)), \psi(s(q))]$$

and $\phi(0) = 0$. Then $\phi$ is a well-defined function from $LI(\Gamma)$ to $LI(\Delta)$ and $\phi(pq^*)$ is nonzero for a nonzero element $pq^* \in LI(\Gamma)$. In particular, for any directed path $p$ in $\Gamma$ (which we may think of as $pr(p)^*$, where $r(p)$ is the empty path at the vertex $r(p)$), we have

$$\phi(p) = p[\psi(s(p)), \phi(s(p))] \phi(p) p[\phi(s(r(p))), \psi(s(r(p)))]$$

and for any vertex $v$ in $\Gamma$, we have $\phi(v) = \psi(v)$.

Lemma 7.16 below provides a proof of the converse part of Theorem 7.12.

**Lemma 7.16** The map $\phi$ is an isomorphism from $LI(\Gamma)$ onto $LI(\Delta)$.

**Proof.** Let $p_1q_1^*$ and $pq_2^*$ be arbitrary non-zero elements in $LI(\Gamma)$. Then we obtain from the definition of $\phi$ that

$$\phi(p_1q_1^*)\phi(pq_2^*) = p[\psi(s(p_1)), \phi(s(p_1))] \phi(p_1q_1^*) p[\phi(s(q_1)), \psi(s(q_1))] \bullet$$

$$p[\psi(s(p_2)), \phi(s(p_2))] \phi(pq_2^*) p[\phi(s(q_2)), \psi(s(q_2))].$$

(7.2)

If $(p_1q_1^*)(pq_2^*) \neq 0$, then $s(q_1) = s(p_2)$. By Lemma 7.14 (a) and the fact that $\phi$ is a 0-morphism, we obtain

$$\phi(p_1q_1^*)\phi(pq_2^*) = p[\psi(s(p_1)), \phi(s(p_1))] \phi(p_1q_1^*) \phi(s(p_1))(pq_2^*) p[\phi(s(q_2)), \psi(s(q_2))]$$

$$= \phi((p_1q_1^*)(pq_2^*).$$

(7.3)

Suppose that $(p_1q_1^*)(pq_2^*) = 0$. If $s(q_1) \neq s(p_2)$, then since $\psi$ is injective, we see that $\psi(s(q_1)) \neq \psi(s(p_2))$ which means by (7.2) that $\phi(p_1q_1^*)\phi(pq_2^*) = 0$. If $s(q_1) = s(p_2)$, then we know from (7.2) that

$$\phi(p_1q_1^*)\phi(pq_2^*) = p[\psi(s(p_1)), \phi(s(p_1))] \phi(p_1q_1^*) \phi(pq_2^*) p[\phi(s(q_2)), \psi(s(q_2))].$$

(7.4)
Since \((p_1q_1^*)(p_2q_2^*) = 0\), then neither of \(q_1, p_2\) is a prefix of the other which means that in both
\(q_1 \) and \(p_2\), there exists a vertex whose out-degree is at least 2. Since our chosen spanning trees
both \(\Gamma\) and \(\Delta\) are NE and \(\phi\) is an isomorphism, we see that in both \(\tilde{\phi}(q_1)\) and \(\tilde{\phi}(p_2)\), there
exists a vertex whose out-degree is at least 2. So we obtain from (7.4) that \(\phi(p_1q_1^*)\phi(p_2q_2^*) = 0\).
Thus \(\phi\) is a semigroup homomorphism.

To see that \(\phi\) is surjective, we only need prove that every edge in \(\Delta\) has a preimage under
\(\phi\) since \(LI(\Delta)\) is generated by \(\Delta^0\) and \(\Delta^1\) and we already established that \(\phi(v) = \psi(v)\) for each
vertex \(v\) in \(\Gamma\). Given an edge \(d\) in \(\Delta\), we take

\[
x = p[\psi^{-1}(s(d)), \tilde{\chi}_\Gamma \tilde{\phi}^{-1} \chi_\Delta(s(d))] \tilde{\chi}_\Gamma \tilde{\phi}^{-1} \chi_\Delta(d) p[\tilde{\chi}_\Gamma \tilde{\phi}^{-1} \chi_\Delta(r(d)), \psi^{-1}(r(d))]
\]

which is nonzero by a similar discussion as in the first paragraph of the proof. Note that each
\(p[i_1, i_2]\) involves only NE paths in spanning trees and that \(\tilde{\phi}^{-1}(\chi_\Delta(u)) = \chi_G(\psi^{-1}(u))\) for any
\(u \in \Delta^0\) by Lemma 7.13(a). Then we observe from (7.3), Lemma 7.15 and the related definitions
that

\[
\phi(x) = p[\psi(\psi^{-1}(s(d))), \tilde{\phi}(\psi^{-1}(s(d)))\tilde{\phi}(p[\psi^{-1}(s(d)), \tilde{\chi}_\Gamma \tilde{\phi}^{-1} \chi_\Delta(s(d))])\tilde{\phi}(\tilde{\chi}_\Gamma \tilde{\phi}^{-1} \chi_\Delta(d))] \cdot \\
\tilde{\phi}(p[\tilde{\chi}_\Gamma \tilde{\phi}^{-1} \chi_\Delta(r(d)), \psi^{-1}(r(d))]) \cdot p[\tilde{\phi}(\psi^{-1}(r(d))), \psi(\psi^{-1}(r(d)))]
\]

\[
= p[s(d), \tilde{\phi}(\psi^{-1}(s(d)))\tilde{\phi}(\tilde{\chi}_\Gamma \tilde{\phi}^{-1} \chi_\Delta(d))p[\tilde{\phi}(\psi^{-1}(r(d))), r(d)]
\]

\[
= p[s(d), \chi_\Delta \chi_\Delta(s(d))] \chi_\Delta \chi_\Delta(d) p[\chi_\Delta \chi_\Delta(r(d)), r(d)].
\]

(7.5)

If \(d\) is in a spanning tree, then we see from (7.3) that \(\tilde{\chi}_\Delta \chi_\Delta(s(d)) = \tilde{\chi}_\Delta \chi_\Delta(d) = \tilde{\chi}_\Delta \chi_\Delta(r(d))\) so
that \(\phi(x) = p[s(d), r(d)] = d\). If \(d\) is not in any spanning tree, then again we observe from (7.3)
that

\[
\phi(x) = p[s(d), \tilde{\chi}_\Delta \chi_\Delta(s(d))p[\tilde{\chi}_\Delta \chi_\Delta(s(d)), s(d)] d p[r(d), \tilde{\chi}_\Delta \chi_\Delta(r(d)) p[\tilde{\chi}_\Delta \chi_\Delta(r(d)), r(d)] = d.
\]

Therefore, \(d\) has a preimage under \(\phi\).

We have seen that nonzero elements map to nonzero ones by \(\phi\). Note the canonical forms of
Leavitt inverse semigroups in Section 5. For any nonzero (reduced) elements \(p_1q_1^*, p_2q_2^*\) in \(LI(\Gamma)\),
if \(p_1q_1^* \neq p_2q_2^*\), then we may suppose that \(p_1 \neq p_2\). If \(s(p_1) \neq s(p_2)\), then
\(s(\phi(p_1)) = \psi(s(p_1)) \neq \psi(s(p_2)) = s(\phi(p_2))\) which means that \(\phi(p_1q_1^*) \neq \phi(p_2q_2^*)\). If \(s(p_1) = s(p_2)\), then we assume that
\(p_1 = e_1 \ldots e_m, p_2 = e_1' \ldots e_m', e_1 = e_1', \ldots, e_{i-1} = e_{i-1}', e_i \neq e_i'\) for some \(i \in \{1, \ldots, m\}\).
Thus, \(s(e_i)\) has out-degree at least 2. Since the chosen spanning trees for \(\Gamma\) are NE, we
obtain that \(\tilde{\phi}(e_i) \neq \tilde{\phi}(e_i')\) which leads to

\[
\phi(p_1) = p[\psi(s(e_1)), \tilde{\phi}(s(e_1))\tilde{\phi}(e_1) \ldots \tilde{\phi}(e_m)p[\psi(r(e_m)), \tilde{\phi}(r(e_m))]
\]

\[
\neq p[\psi(s(e_1')), \tilde{\phi}(s(e_1'))\tilde{\phi}(e_1') \ldots \tilde{\phi}(e_m')p[\psi(r(e_m')), \tilde{\phi}(r(e_m'))] = \phi(p_2)
\]

because \(\tilde{\phi}(e_i)\) and \(\tilde{\phi}(e_i')\) contain distinct edges which have the same source by Lemma 6.10(d). So
in this case, we also have \(\phi(p_1q_1^*) \neq \phi(p_2q_2^*)\). We proved that \(\phi\) is injective, so it is a semigroup
isomorphism. This completes the proof of the lemma, and hence of Theorem 7.12. 

**Example.** We illustrate the use of Theorem 7.12 to construct an isomorphism \(\phi\) from the
Leavitt inverse semigroup \(LI(\Gamma)\) onto the Leavitt inverse semigroup \(LI(\Delta)\) for the two graphs \(\Gamma\)
and \(\Delta\) in Diagram 7.2. By Theorem 6.11 the Leavitt path algebras \(L_F(\Gamma)\) and \(L_F(\Delta)\) of these
graphs are also isomorphic.
\[ \Gamma^0 \text{ has two } \sim \text{-classes} \ [v_1] = \{v_1, v_2, v_3\} \text{ and } [v_4] = \{v_4, v_5, v_6\}; \ \Delta^0 \text{ also has two } \sim \text{-classes} \ [u_1] = \{u_1, u_2, u_3\} \text{ and } [u_4] = \{u_4, u_5, u_6\}. \] As we discussed, \( e_{C_1} \) and \( e_{C_2} \) respectively determine a spanning tree of \( \Gamma_{[v_1]} \) and \( \Gamma_{[u_1]} \); \( e'_{C_1} \) and \( e'_{C_2} \) respectively determine a spanning tree of \( \Delta_{[u_4]} \) and \( \Delta_{[u_4]} \). Since \( \Gamma \) and \( \Delta \) have isomorphic contracted graphs, for convenience, we denote them as \( \bar{\Gamma} = \bar{\Delta} \), where \( \bar{\Gamma} = \bar{\Delta} \), and \( \Delta \) have isomorphic contracted graphs, for convenience, we denote them as \( \Gamma = \Delta \), where \( w_1 = [v_1] = [u_1] \), \( w_2 = [v_4] = [u_4] \), \( f_1 = \bar{e}_{C_1} = \bar{e}_{C_1} \), \( f_2 = \bar{e}_5 = \bar{d}_5 \) and \( f_3 = \bar{e}_{C_2} = \bar{e}_{C_2} \). 

\[ \begin{array}{c}
\text{Diagram 7.2 Two graphs with isomorphic Leavitt inverse semigroups and their contracted graph(s)}
\end{array} \]

Let \( \bar{\phi} \) be the identity mapping of \( \bar{\Gamma} \) and \( \psi(v_i) = u_i \) for \( i = 1, 2, \ldots, 6 \). To construct a semigroup isomorphism \( \phi \) from \( LI(\bar{\Gamma}) \) to \( LI(\bar{\Delta}) \) according to Theorem 7.12, we only need to list the image of generators, in fact only edges, under \( \phi \). First, by (7.11) and Lemma 7.14(a), for any \( e \in \Gamma^1 \), if \( e \) is in a spanning tree, then we see from \( \chi_\Gamma(s(e)) = \chi_\Gamma(e) = \chi_\Gamma(r(e)) \) that \[ \phi(e) = p[\psi(s(e)), \chi_\Delta\chi_\Gamma(s(e))]p[\chi_\Delta\chi_\Gamma(r(e)), \psi(r(e))] = p[\psi(s(e)), \psi(r(e))], \] and if \( e \) is not in a spanning tree, then \( \phi(e) = p[\psi(s(e)), s(\chi_\Delta(\bar{e}))][\bar{\chi}_\Delta(\bar{e})]p[r(\chi_\Delta(\bar{e})), \psi(r(e))] \). Thus,

\[
\begin{align*}
\phi(e_1) &= p[u_1, u_2] = d_2d_1^*; \\
\phi(e_2) &= p[u_2, u_3] = d_1; \\
\phi(e_3) &= p[u_4, u_5] = d_3^*; \\
\phi(e_4) &= p[u_6, u_4] = d_3d_4; \\
\phi(e_5) &= p[u_3, u_1]d_5 p[u_5, u_4] = d_2d_5d_4; \\
\phi(e_{C_1}) &= p[u_3, u_1]e_{C_1} p[u_3, u_1] = d_2e_{C_1}^*d_2; \\
\phi(e_{C_2}) &= p[u_5, u_4]e_{C_2} p[u_4, u_6] = d_4e_{C_2}^*d_3d_4^*. 
\end{align*}
\]

We close the paper with several results that follow from Theorem 7.12.

**Corollary 7.17** Let \( \bar{\Gamma} \) and \( \bar{\Delta} \) be contracted graphs. Then \( LI(\bar{\Gamma}) \) is isomorphic to \( LI(\bar{\Delta}) \) if and only if \( \bar{\Gamma} \) is isomorphic to \( \bar{\Delta} \).
Proof. The direct part follows from Lemma 17.8 and the direct part of Theorem 7.12. The converse part is trivial. 

Corollary 7.18 If $\Gamma$ and $\Delta$ are graphs such that $\bar{\Gamma} \cong \bar{\Delta} \cong B_X$ (the bouquet of $|X|$ circles), then $LI(\Gamma) \cong LI(\Delta)$ if and only if $|\Gamma^0| = |\Delta^0|$.

Proof. This is immediate from Theorem 7.12 since there is only one equivalence class of vertices of $\Gamma$ (or $\Delta$) under the equivalence relation $\sim$.

We may view Corollary 7.18 as a generalization of Theorem 6.8 since a connected graph that immerses into a circle has only one $\sim$-class of vertices.

Remark We remark that the hypotheses of Corollary 7.18 do not classify graphs whose contracted graphs are isomorphic to a bouquet of circles and which have isomorphic Leavitt path algebras. It clearly follows from Corollary 7.18 and Theorem 6.11 that if $\Gamma$ and $\Delta$ are two graphs with $\bar{\Gamma} \cong \bar{\Delta} \cong B_X$ and $|\Gamma^0| = |\Delta^0|$ then $L_F(\Gamma) \cong L_F(\Delta)$. But the conditions $L_F(\Gamma) \cong L_F(\Delta)$ and $\bar{\Gamma} \cong \bar{\Delta} \cong B_X$ do not necessarily imply that $|\Gamma^0| = |\Delta^0|$. In fact the following result follows easily from Theorem 7.12 and some results in the paper by Abrams, Ánh and Pardo [4].

Corollary 7.19 Let $\Gamma$ be a finite graph with $\bar{\Gamma} \cong B_X$ where $|X| = n \geq 2$. Then

(a) $L_F(\Gamma) \cong M_{|\Gamma^0|}(L_F(B_X))$ (where $L_F(B_X)$ is the Leavitt algebra $L_F(1, n)$);
(b) If $n = 2$ then $L_F(\Gamma) \cong L_F(B_X) \cong L_F(1, 2)$;
(c) If $\Gamma$ and $\Delta$ are two graphs with $\bar{\Gamma} \cong \bar{\Delta} \cong B_X$ where $|X| = n > 2$, then $L_F(\Gamma) \cong L_F(\Delta)$ if $g.c.d(|\Gamma^0|, n - 1) = g.c.d(|\Delta^0|, n - 1)$.

Proof. By Theorem 7.12 $LI(\Gamma) \cong LI(B^k(B_X))$ where $B^k(B_X)$ denotes the graph obtained from $B_X$ by attaching a directed NE path of length $k = |\Gamma^0| - 1$ ending in the (unique) vertex in the graph $B_X$. Hence $L_F(\Gamma) \cong L_F(B^k(B_X))$ by Theorem 6.11. By Lemma 5.1 of [2], $L_F(B^k(B_X)) \cong M_k(L_F(1, n))$, the algebra of $k \times k$ matrices over the Leavitt algebra $L_F(1, n)$. The results of parts (b) and (c) follow immediately from Theorem 12 of [1] (Theorems 4.14 and 5.12 of [4]).

The ideas employed in the proof of Corollary 7.19 may be extended somewhat to obtain a result relating the structure of the Leavitt path algebra $L_F(\Gamma)$ to the structure of the algebra $L_F(\bar{\Gamma})$ of the contracted graph $\bar{\Gamma}$ for any graph $\Gamma$ with finite $\sim$-equivalence classes.

We define a function $f : L_F(\Delta) \to L_F(\Gamma)$ between Leavitt path algebras to be a 0-morphism if $f$ is a linear transformation between the underlying vector spaces that restricts to a 0-morphism $LI(\Delta) \to LI(\Gamma)$ between the corresponding Leavitt inverse semigroups. We call a function $f : L_F(\Delta) \to L_F(\Gamma)$ a 0-retraction if $L_F(\Gamma)$ is a subalgebra of $L_F(\Delta)$ and $f$ is a 0-morphism from $L_F(\Delta)$ onto $L_F(\Gamma)$ that restricts to the identity function on $L_F(\Gamma)$; equivalently, we say that $L_F(\Gamma)$ is a 0-retract of $L_F(\Delta)$. We extend the notation slightly by saying that an $F$-algebra $A_1$ is a 0-retract of an $F$-algebra $A_2$ if there are graphs $\Gamma$ and $\Delta$ such that $A_1 \cong L_F(\Gamma)$, $A_2 \cong L_F(\Delta)$ and $L_F(\Gamma)$ is a 0-retract of $L_F(\Delta)$. Recall from [3] that a subgraph $\Gamma'$ of a graph $\Delta$ is called a complete subgraph of $\Delta$ if $e \in \Gamma'$ for every edge $e \in \Delta$ such that $s(e) \in \Gamma^0$.

Theorem 7.20 Let $\Gamma$ be a graph with finite equivalence classes $[v]$ for each $v \in \Gamma^0$ and let $n$ be the maximum order of any equivalence class $[v]$ for $v \in \Gamma^0$. Then $L_F(\Gamma)$ is a 0-retract of $M_n(L_F(\Gamma))$. 

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Proof. By Theorem 7.12, $L_F(\Gamma) \cong L_F(\Gamma_1)$ where $\Gamma_1$ is obtained from $\bar{\Gamma}$ by attaching an NE path $p_{[v]}$ of length $|v| - 1$ ending at the vertex $[v]$ of $\bar{\Gamma}$ for each $[v] \in \bar{\Gamma}$. Let $\Delta$ be the graph obtained from $\bar{\Gamma}$ by attaching an NE path $q_{[v]}$ of length $n$ ending at $[v] \in \bar{\Gamma}$ for each vertex $[v] \in \bar{\Gamma}$. By Proposition 9.3 of a paper by Abrams and Tomforde [3], $L_F(\Delta) \cong M_n(L_F(\bar{\Gamma}))$.

By identifying $p_{[v]}$ with the suffix of $q_{[v]}$ of length $|v| - 1$, we may view $\Gamma_1$ as a subgraph of $\Delta$, in fact a complete subgraph in the sense of [3]. Hence by Lemma 1.6.6 of [3], $L_F(\Gamma_1)$ is a subalgebra of $L_F(\Delta)$. We show by induction on the number of edges in $\Delta \setminus \Gamma_1$ that in fact there is a 0-retraction of $L_F(\Delta)$ onto $L_F(\Gamma_1)$.

Suppose that $\Delta_1$ is obtained by attaching one NE edge $e$ to $\Gamma_1$ with $r(e) = s(p_{[v]})$ and $s(e) \notin \Gamma_1$ and such that $\Delta_1$ is a subgraph of $\Delta$. We claim that we may construct a well-defined map $f : L_F(\Delta_1) \to L_F(\Gamma_1)$ by contracting the edge $e$ and the vertex $s(e)$ to the vertex $r(e) = s(p_{[v]})$ in $\Gamma_1$. More precisely, we proceed as follows. Choose the special edges of $\Delta_1$ (in the sense of [10]) in such a way that all special edges of $\Delta_1$ other than $e$ are special edges of $\Gamma_1$. Note that since $s(e)$ is a source in $\Delta_1$ of out-degree 1, a directed path $p$ in $\Delta_1$ contains the edge $e$ if and only if $p = et$ for some (possibly empty) directed path $t$ in $\Gamma_1$ with $s(t) = r(e)$. From this and the choice of special edges in $\Delta_1$ it is easy to see that the corresponding natural basis of the algebra $L_F(\Delta_1)$ consists of the elements in the natural basis of $L_F(\Gamma_1)$ together with the elements $s(e), e, e^*$ and the non-zero elements $epq^*, pq^*e^*$ and $epq^*e^*$ where $pq^*$ is a non-empty natural basis element for the algebra $L_F(\bar{\Gamma})$. Define $e' = s(e) = r(e)$ and for each other directed path $p$ in $\Delta_1$, define $p'$ to be the directed path in $\Gamma_1$ obtained by deleting the first edge of $p$ if this edge is $e$ and $p' = p$ if the first edge of $p$ is not $e$. Then define $f$ on natural basis elements of $L_F(\Delta_1)$ by $f(pq^*) = p'q'^*$. This extends by linearity to a linear transformation (that we again denote by $f$) from $L_F(\Delta_1)$ to $L_F(\Gamma_1)$, and this linear transformation is surjective since every natural basis element of $L_F(\Gamma_1)$ arises as the image of a natural basis element of $L_F(\Delta_1)$. Furthermore, $f$ fixes every natural basis element of $L_F(\Gamma_1)$ so the restriction of $f$ to $L_F(\Gamma_1)$ is the identity map.

We prove by induction on $|p| + |q|$ that $f(pq^*) = p'q'^*$ for all non-zero elements of $LI(\Delta_1)$. We may assume that $pq^*$ is not a natural basis element of $LI(\Delta_1)$. It follows from Theorem 6.3 that $|p| + |q|$ is at least 2 since any element $pq^*$ with $|p| + |q| < 2$ is a natural basis element. If $|p| + |q| = 2$, then again by Theorem 6.3 $pq^*$ is $\gamma^*$ for some special edge $\gamma$ of $\Delta_1$. If $\gamma = e$, then $\gamma^* = s(e)$; if $\gamma \neq e$, then $\gamma^*$ is in $\Gamma_1$. In both cases, $f(\gamma^*) = \gamma'^*$. Hence the result holds if $|p| + |q| \leq 2$. This is a basis for the induction.

Now suppose that $|p| + |q| > 2$ and that $f(wz^*) = w'z'^*$ for all non-zero elements $wz^*$ of $LI(\Delta_1)$ such that $|w| + |z| < |p| + |q|$. We aim to show that $f(pq^*) = p'q'^*$. We may assume that the edge $e$ is the first edge of either $p$ or $q$ or both and that $pq^*$ is not a natural basis element of $L_F(\Delta_1)$, so $p'q'^*$ is not a natural basis element of $L_F(\Gamma_1)$. Then $p'q'^* = xe_1e_2^*y^*$ for some directed paths $x$ and $y$ in $\Gamma_1$, where $e_1$ is the special edge $e_1 = \gamma(s(e_1))$. If $s^{-1}(s(e_1)) = \{e_1, e_2, ... e_n\}$ then $e_1^* = s(e_1) - e_2e_3 - ... - e_ne_n$, so $p'q'^* = xy^* - xe_2e_3y^* - ... - xe_ne_ny^*$. Consider the case where $e$ is the first letter of $q$ and $s(e) = ep'$ and $q = q'$. We have $pq^* = ep'q'^* = ey^* - xe_2e_3y^* - ... - xe_ne_ny^*$. So $f(pq^*) = f(xy^*) - f(xe_2e_3y^*) - ... - f(xe_ne_ny^*)$. Since the terms $xe_2e_3^*$ are basis elements of $L_F(\Delta_1)$, we have $f(xe_2e_3^*y^*) = xe_2e_3y^*$ for $i = 2, ..., n$. By the induction hypothesis, $f(xe_2e_3^*y^*) = xy^*$ since $|ex| + |y| < |e| + |e|$. Hence $f(pq^*) = xy^* - xe_2e_3y^* - ... - xe_ne_ny^* = p'q'^*$. A similar argument shows that $f(pq^*) = p'q'^*$ if $e$ is the first letter of $p$ but not the first letter of $p$ or if $e$ is the first letter of both $p$ and $q$, as required.

Now suppose that $p_1q_1^*$ and $p_2q_2^*$ are non-zero elements of $L_F(\Delta_1)$ such that $p_1q_1^*p_2q_2^* \neq 0$. 37
Assume without loss of generality that \( p_2 \) is a prefix of \( q_1 \) (the other case is dual), so \( q_1 = p_2 t \) for some directed path \( t \) in \( \Delta_1 \) which leads to \( f(p_1 q_1^* p_2 q_2^*) = f(p_1 (q_2 t)*) = p_1^* (q_2 t)^* \). If \( p_2 \) starts with \( e \), then so does \( q_1 \) and hence \( f(p_1 q_1^* p_2 q_2^*) = p_1^* (q_2 t)^* = p_1^* t^* q_2^* = p_1^* p_2^* t^* q_2^* = p_1 q_1^* p_2 q_2^* = f(p_1 q_1^*) f(p_2 q_2^*) \). If \( p_2 \) does not start with \( e \) but \( q_1 \) does, then \( p_2, q_2 \) must be trivial and \( p_2 = q_2 = s(e) \). So we have \( f(p_1 q_1^* p_2 q_2^*) = p_1^* (q_2 t)^* = p_1^* q_1^* p_2 q_2^* = f(p_1 q_1^*) f(p_2 q_2^*) \). If \( q_1 \) does not start with \( e \), then neither does \( p_2 \). So we have \( f(p_1 q_1^* p_2 q_2^*) = p_1^* (q_2 t)^* = p_1^* t^* q_2^* = (p_1 q_1^* f(p_2 q_2^*) = f(p_1 q_1^*) f(p_2 q_2^*) \). In summary, \( f \) restricts to a 0-morphism from \( L_I(\Delta_1) \) to \( L_I(\Gamma_1) \) so that \( f \) is a 0-retraction of \( L_F(\Delta_1) \) onto \( L_F(\Gamma_1) \).

If \( \Delta_1 = \Delta \) we stop. If \( \Delta_1 \neq \Delta \), we may attach another \( \text{NE}- \)edge to \( \Delta_1 \) to obtain another subgraph \( \Delta_2 \) of \( \Delta \) and proceed as before, obtaining a 0-retraction \( h \) of \( L_F(\Delta_2) \) onto \( L_F(\Delta_1) \). Then \( h f \) is a 0-retraction of \( L_F(\Delta_2) \) onto \( L_F(\Gamma_1) \). Continue adding \( \text{NE} \) paths to intermediate graphs until we eventually reach \( \Delta \) and obtain a 0-retraction of \( L_F(\Delta) \) onto \( L_F(\Gamma_1) \). This completes the proof since \( M_n(L_F(\Gamma)) \cong L_F(\Delta) \) and \( L_F(\Gamma) \cong L_F(\Gamma) \).

### 8 Addendum: The kernel of the map from \( I(\Gamma) \) onto \( LI(\Gamma) \)

In this addendum, we show that the kernel of the map from \( I(\Gamma) \) onto \( LI(\Gamma) \) is the congruence \( \leftrightarrow \) introduced by Lenz \[24\], answering a question raised by Milan based on an earlier version of this paper. We first recall the definition of Lenz’s congruence \( \leftrightarrow \).

Let \( S \) be an inverse semigroup (with 0) and for each \( a \in S \) let \( a^\perp = \{ x \in S : x \leq a \) in the natural partial order on \( S \} \). Given \( a, b \in S \) we define \( a \rightarrow b \) if, whenever \( 0 < x \leq a \), \( a \downarrow \cap b^\perp \neq \{ 0 \} \), and define \( a \leftrightarrow b \) if \( a \rightarrow b \) and \( b \rightarrow a \). Then \( \leftrightarrow \) is a 0-restricted congruence on \( S \) \([20]\), Proposition 3.4). The congruence \( \leftrightarrow \) was introduced by Lenz \[24\] in connection with his construction of various topological groupoids associated with inverse semigroups: it has been studied by several authors, in particular by Lalonde, Milan and Scott \[20\] who used it to study the ideal structure of the tight \( C^* \)-algebra of an inverse semigroup.

**Theorem 8.1** For any graph \( \Gamma \), the congruence \( \leftrightarrow \) is the kernel of the natural homomorphism from \( I(\Gamma) \) onto \( LI(\Gamma) \): that is \( LI(\Gamma) \cong I(\Gamma)/\leftrightarrow \).

**Proof.** From the definition of the Leavitt inverse semigroup \( LI(\Gamma) \) it is clear that the kernel of the natural homomorphism from \( I(\Gamma) \) onto \( LI(\Gamma) \) is the congruence \( \rho \) generated by \( \{(ee^*, s(e)) : s(e) \) has out-degree 1 in \( \Gamma^0\} \). We aim to show that \( \rho = \leftrightarrow \).

Suppose first that \( e \) is an edge of \( \Gamma \) such that \( s(e) \) has out-degree 1. Then \( s(e) \rho ee^* \). Let \( 0 < x \leq s(e) \) in \( \Gamma(\Gamma) \). Then \( x = pp^* \) for some directed path \( p \) with \( s(p) = s(e) \). Either \( p = s(e) \) or \( p = eq \) for some directed path \( q \) with \( s(q) = r(e) \). Clearly \( s(e)^\perp \cap (ee^*)^\perp \neq \{ 0 \} \). Also, if \( p = eq \) then \( (pp^*)^\perp = \{ e \vartriangleleft^* e^* s(e) : s(t) = r(q) \} \) \( \cap (ee^*)^\perp \neq \{ 0 \} \), so \( s(e) \rightarrow ee^* \). By essentially the same argument, \( ee^* \rightarrow s(e) \) and so \( \rho \subseteq \leftrightarrow \).

Now suppose that \( p_1 q_1^*, p_2 q_2^* \) are non-zero elements of \( I(\Gamma) \) such that \( p_1 q_1^* \leftrightarrow p_2 q_2^* \). Then \( (p_1 q_1^*)^\perp \cap (p_2 q_2^*)^\perp \neq \{ 0 \} \), so there exist paths \( t_1, t_2 \) such that \( p_1 t_1 q_1^* q_1^* = p_2 t_2 q_2^* q_2^* \). So either \( p_1 \) is a prefix of \( p_2 \) or \( p_2 \) is a prefix of \( p_1 \). Suppose without loss of generality that \( p_2 = p_1 z \) for some path \( z = e_1 e_2 \ldots e_n \). We claim that \( z \) is an \( \text{NE} \) path. If not, there is some index \( i \) with \( 1 \leq i \leq n \) such that \( s(e_i) \) has out-degree at least 2, and so there is some edge \( f \) with \( f \neq e_i \) and \( s(f) = s(e_i) \). Let \( z_1 = e_1 \ldots e_{i-1} \) and \( z_2 = e_i \ldots e_n \). Then \( (p_1 z_1 f f^* z_1^* q_1^*)^\perp \cap (p_2 q_2^*)^\perp \neq \{ 0 \} \). Hence there exist paths \( t_3, t_4 \) such that \( p_1 z_1 f t_3 t_4 f^* z_1^* q_1^* = p_2 t_2 t_4^* q_2^* = p_1 z_1 z_2 t_4^* q_2^* \). This implies that \( z t_4 = f t_3 \). But this is impossible since the first edge of \( z_2 \) is \( e_i \neq f \). Hence \( z \) is an \( \text{NE} \) path. Now we have \( p_1 t_1 q_1^* = p_2 t_2 t_4^* q_2^* = p_1 z t_2 t_4^* q_2^* \), so \( t_1 = z t_2 \) and hence \( q_2 t_2 = q_1 t_1 = q_1 z t_2 \). This
implies that \( q_2 = q_1 z \). So we have \( p_2 = p_1 z \) and \( q_2 = q_1 z \) for some NE path \( z \). But this means that \( (p_1 q_1^*) \rho (p_2 q_2^*) \). Hence \( \leftrightarrow \subseteq \rho \). It follows that the congruences \( \leftrightarrow \) and \( \rho \) coincide.

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