Twisted Fermat curves over totally real fields

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1. Introduction

Let \( p \) be a prime number, \( F \) a totally real field such that \( [F(\mu_p) : F] = 2 \) and \( [F : \mathbb{Q}] \) is odd. For \( \delta \in F^\times \), let \([\delta]\) denote its class in \( F^\times /F^\times p \). In this paper, we show

**Main Theorem.** There are infinitely many classes \([\delta]\) such that the twisted affine Fermat curves

\[
W_\delta : \quad X^p + Y^p = \delta
\]

have no \( F \)-rational points.

**Remark.** It is clear that if \([\delta] = [\delta']\), then \( W_\delta \) is isomorphic to \( W_{\delta'} \) over \( F \). For any \( \delta \in F^\times \), \( W_\delta /F \) has rational points locally everywhere.

To obtain this result, consider the smooth open affine curve:

\[
C_\delta : V^p = U(\delta - U),
\]

and the morphism:

\[
\psi_\delta : W_\delta \rightarrow C_\delta; \quad (x,y) \mapsto (x^p, xy).
\]

Let \( C_\delta \rightarrow J_\delta \) be the Jacobian embedding of \( C_\delta /F \) defined by the point \((0,0)\).

We will show that:

1. If \( L(1,J_\delta/F) \neq 0 \), then \( J_\delta(F) \) is a finite group (cf. Theorem 2.1. of §2).

   The proof is based on Zhang’s extension of the Gross-Zagier formula to totally real fields and on Kolyvagin’s technique of Euler systems. One might use techniques of congruence of modular forms to remove the restriction that the degree \([F : \mathbb{Q}]\) is odd.

2. There are infinitely many classes \([\delta]\) such that \( L(1,J_\delta/F) \neq 0 \) (cf. Theorem 3.1. of §3; see also 2.2.4.).

   The proof is based on the theory of double Dirichlet series. The condition that \([F(\mu_p) : F] = 2 \) is essential for the technique we use here.
Combining (1) and (2), one can see that the set
\[ \Pi := \left\{ [\delta] \in F^{\times}/F^{\times p} \mid J_{\delta}(F) \text{ is torsion} \right\} \]
is infinite.

1.1. Proof of the Main Theorem assuming (1) and (2). For any \(\delta \in F^{\times}\), consider the twisting isomorphism (defined over \(F(\sqrt[\cdot]{\delta})\)):
\[ \iota_{\delta} : C_{\delta} \rightarrow C_1; \quad (u, v) \mapsto (u/\delta, v/\sqrt{\delta^2}) . \]
Define \(\eta_{\delta} : J_{\delta} \rightarrow J_1\) to be the homomorphism associated to \(\iota_{\delta}\).

Let \(\Sigma_{\delta}\) denote the set \(\iota_{\delta}(C_{\delta}(F))\). It is easy to see that:
(i) \(\Sigma_{\delta} = \Sigma_{\delta'}\), if \([\delta] = [\delta']\),
(ii) \(\Sigma_{\delta} \cap \Sigma_{\delta'} = \{(0, 0), (1, 0)\}\), otherwise.

For any \(\delta \in F^{\times}\) with \([\delta] \in \Pi\), and \([\delta] \neq 1\), the diagram
\[ \begin{array}{ccc}
W_{\delta}(F) & \xrightarrow{\psi_{\delta}} & C_{\delta}(F) \\
& \downarrow \iota_{\delta} & \downarrow \eta_{\delta} \\
C_1(F(\sqrt[\cdot]{\delta})) & \hookrightarrow & J_1(F(\sqrt[\cdot]{\delta}))
\end{array} \]
commutes.

Since the set
\[ \bigcup_{\delta \in F^{\times}} J_1(F(\sqrt[\cdot]{\delta}))_{\text{tor}} \subset J_1(F) \]
is finite by the Northcott theorem, the set \(\bigcup_{[\delta] \in \Pi} \Sigma_{\delta}\) is finite. Thus, for all but finitely many \([\delta] \in \Pi \setminus \{[1]\}\), \(\Sigma_{\delta} = \{(0, 0), (1, 0)\}\), and therefore \(W_{\delta}\) has no \(F\)-rational points.

Remark. Our method is, in fact, effective: for any \([\delta] \in F^{\times}/F^{\times p}\), let
\[ \text{Supp}^{(p)}([\delta]) = \left\{ \text{prime of } F \mid p \nmid v_{\cdot}(\delta) \right\} . \]
Let \(L'\) be the Galois closure of \(F(\mu_p)\), and let \(S\) be the set of places of \(F\) above \(2D_{L'/Q}\), where \(D_{L'/Q}\) is the discriminant of \(L'/Q\). If \(\text{Supp}^{(p)}([\delta])\) is not contained in \(S\) and \(L(1, J_{\delta}) \neq 0\), then the twisted Fermat curve \(W_{\delta}\) has no \(F\)-rational points (see Proposition 2.2).

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2. Arithmetic methods

Fix $\delta \in F^\times \cap \mathcal{O}_F$ such that $(\delta, p) = 1$. Let $\zeta = \zeta_p$ be a primitive $p$-th root of unity. The abelian variety $J_\delta$ is absolutely simple, of dimension $g = \frac{p-1}{2}$, and has complex multiplication by $\mathbb{Z}[\zeta]$ over the field $F(\mu_p)$. In this section we show:

**Theorem 2.1.** If $L(1, J_\delta/F) \neq 0$, then $J_\delta(F)$ is finite.

**Notation.** In this section, for an abelian group $M$, set $\hat{M} = M \otimes \mathbb{Z} \prod_p \mathbb{Z}_p$, where $p$ runs over all primes. For any ring $R$, let $R^\times$ denote the group of invertible elements. For any ideal $a$ of $F$, denote the norm $N_{F/\mathbb{Q}}(a)$ by $N_a$. Let $A$ denote the adele ring of $F$, and $A_f$ its finite part. Sometimes, we shall not distinguish a finite place from its corresponding prime ideal.

2.1. *The Hilbert newform associated to $J_\delta$.* We first recall some facts about $L$-functions of twisted Fermat curves over arbitrary number fields (see [14], [32]). Let $F$ be any number field, $L = F(\mu_p)$, $L_0 = \mathbb{Q}(\mu_p)$, and $F_0 = L_0 \cap F$.

For any place $w$ of $L$, denote by $w_0$ and $v$ its restrictions to $\mathbb{Q}(\mu_p)$ and $F$, respectively. Let $\chi_{w_0}$ and $\chi_w$ be the $p$-th power residue symbols on $L_0^\times$ and $L^\times$, respectively, given by class field theory. Then $\chi_w = \chi_{w_0} \circ N_{L/\mathbb{Q}(\mu_p)}$. The Jacobi sum $j(\chi_w, \chi_w) = \sum_{a \neq 0, 1} \chi_w(a)\chi_w(1-a)$ is an integer in $L_0$ satisfying $j(\chi_w, \chi_w) = j(\chi_{w_0}, \chi_{w_0})^{i_w/w_0}$ and the Stickelberger relation:

$$ j(\chi_{w_0}, \chi_{w_0}) = \prod_{i=1}^n \sigma_i^{-1}(w_0) $$

as an ideal in $L_0$. Here, $i_w/w_0$ is the inertial degree for $w/w_0$, and $\sigma_i \in \text{Gal}(L_0/\mathbb{Q})$ is the image of $i$ under the isomorphism $(\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \text{Gal}(L_0/\mathbb{Q})$.

Since $\delta \in \mathcal{O}_F$ is coprime to $p$, $C_\delta$ has good reduction at $w$ for any $w \nmid p\delta$. We know that the zeta-function of the reduction $\hat{C}_\delta$ of $C_\delta$ at a place $v$ of $F$ is

$$ Z(\hat{C}_\delta, T) = \frac{P_v(T)}{(1-T)(1-N_v T)}, $$

with

$$ P_v(T) = \prod_{w|v} \prod_{\sigma} (1 - \chi_w(\delta^2)^\sigma j(\chi_w, \chi_w)^\sigma T^{f_v}), $$

where $f_v$ is the order of $N_v$ modulo $p$, and $\sigma$ runs over representatives in $\text{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$ of $\text{Gal}(F_0/\mathbb{Q})$. Then the number of points on $J_\delta$ (the reduction of $J_\delta$ at $v$) is $P_v(1)$. 

Now we give a bound on torsion points of $J_\delta(F)$. Let $F'$ be the Galois closure of $F/\mathbb{Q}$, and assume that $F \cap L_0 = F' \cap L_0$. This assumption is satisfied if $F$ is as in the main theorem, or $F$ is Galois over $\mathbb{Q}$. Let $L' = F'(\mu_p)$, and let $q \nmid 2D_{L'/\mathbb{Q}}$ be a prime. Let $\ell$ be a prime for which there exists a place $w'|\ell$ of $L'$ such that $\text{Frob}_{L_0/F_0}(w'|L_0)$ is a generator of $\text{Gal}(L_0/F_0)$, $\text{Frob}_{F'/F_0}(w'|F') = 1$ and $\text{Frob}_{\mathbb{Q}(\mu_p)/\mathbb{Q}(w'|\mathbb{Q}(\mu_p))} = 1$. Then, $\ell \equiv 1 \mod q$. Let $v$, $w$ and $w_0$ be the places of $F$, $L$ and $L_0$, respectively, below $w'$. Then, $v$ is inert in $L/F$ and $i_{w/w_0} = 1$. We have

$$P_v(1) = \prod_{\sigma}(1 - \chi_w(\delta^2)^\sigma j(\chi_w, \chi_w)^\sigma).$$

Since $v$ is inert in $L/F$ and $\delta \in F^\times$, we have $\chi_w(\delta^2) = 1$. Using the Stickelberger relation and the fact that $j(\chi_{w_0}, \chi_{w_0}) \equiv 1 \mod (1 - \zeta_p)^2$, one can show that $j(\chi_w, \chi_w) = -\ell^f$, for $f = \frac{p-1}{2[\mathbb{F}_q: \mathbb{Q}]}$. Then, $P_v(1) = (1 - \ell^f)^{[\mathbb{F}_q: \mathbb{Q}]} \equiv \ell^f \mod q$. Consequently, there are no $q$-torsion points in $J_\delta(F)$.

Similarly, for the case $q|2D_{L'/\mathbb{Q}}$, let $c_q \geq 1$ be the smallest positive integer such that there is a $\sigma \in \text{Gal}(L'(\mu_{q^c})/\mathbb{Q})$ for which $\sigma|_L$ is a generator of $\text{Gal}(L/F)$, $\sigma|_{F'} = 1$, and the restriction of $\sigma$ to $\text{Gal}(\mathbb{Q}(\mu_{q^c})/\mathbb{Q})$ has order greater than $f = \frac{p-1}{2[\mathbb{F}_q: \mathbb{Q}]}$. Then, $P_v(1) \equiv 0 \mod q^{c_q}[\mathbb{F}_q: \mathbb{Q}]$. Let $M$ be defined by $M := \prod_{q|2D_{L'/\mathbb{Q}}} q^{c_q}[\mathbb{F}_q: \mathbb{Q}]$. It follows that $J_\delta(F)_{\text{tor}} \subset J_\delta[M]$, the subgroup of $M$-torsion points of $J_\delta(F)$.

Let $F$ be a totally real field as in the main theorem. We have:

**Proposition 2.2.** Let $S$ be the set of places of $F$ above $2D_{L'/\mathbb{Q}}$. If $\text{Supp}^p([\delta])$ is not contained in $S$ and $L(1, J_\delta/F) \neq 0$, then the twisted Fermat curve $W_\delta$ has no $F$-rational points.

Let $F$ be as in the introduction. Then $F_0 = \mathbb{Q}(\mu_p)^+$ is the maximal totally real subfield of $L_0 = \mathbb{Q}(\mu_p)$. By the reciprocity law, one can see that $w \mapsto \chi_w(\delta^2)$ defines a Hecke character, which we denote by $\chi_{[\delta^2]}$. It depends only on the class of $\delta^2$ and has conductor above $\delta$. By Weil [32], the map $w \mapsto j(\chi_w, \chi_w)N_{L/\mathbb{Q}} w^{-\frac{1}{2}}$ also defines a Hecke character on $L$, denoted by $\psi$, which has conductor above $p$. Thus, we have a (unitary) Hecke character on $L$,

$$\chi_{[\delta^2]} \psi : \mathbb{A}_L^\times \rightarrow \mathbb{C}^\times,$$

which is not of the form $\phi \circ N_{L/F}$, for any Hecke character $\phi$ over $F$. Then, there exists a unique holomorphic Hilbert newform $f/F$ of pure weight 2 with trivial central character such that,

$$L_v(s, f/F) = \prod_{w|v} L_w(s - 1/2, \chi_{[\delta^2]} \psi),$$

for all places $v$ of $F$. Actually, the field over $\mathbb{Q}$ generated by the Hecke eigenvalues attached to $f$ is $F_0 = \mathbb{Q}(\mu_p)^+$, and for the CM abelian variety $J_\delta$, we
have
\[ L(s, J_\delta/F) = \prod_{\sigma \in \text{Gal}(L_0/Q)/\text{Gal}(L_0/F_0)} L(s - 1/2, \chi_{[\delta]}^\sigma \psi^\sigma) = \prod_{\sigma : F_0 \hookrightarrow \mathbb{C}} L(s, f^\sigma / F). \]

Note that \( L(s, J_\delta) \) only depends on the class \([ \delta ]\) of \( \delta \), and the above equality holds for any local factor.

2.2. A nonvanishing result. Let \( \pi \) be the automorphic representation associated to \( f \), and let \( N \) be its conductor. Let \( S_0 \) be any finite set of places of \( F \), including all infinite places and the places dividing \( N \). Choose a quadratic Hecke character \( \xi \) corresponding to a totally imaginary quadratic extension of \( F \), unramified at \( N \), where \( \xi(N) \cdot (-1)^g = -1 \) (since \( F \) is of odd degree, we have \((-1)^g = -1\)); i.e., the epsilon factor of \( L(s, \pi \otimes \xi) \) is \(-1\). Let \( D(\xi; S_0) \) denote the set of quadratic characters \( \chi \) of \( F \times \mathbb{Z} / \mathbb{A} \times F \), for which \( \chi_v = \xi_v \) for all \( v \in S_0 \). With the above notation and assumptions, by a theorem of Friedberg and Hoffstein [11], there exist infinitely many quadratic characters \( \chi \in D(\xi; S_0) \) such that \( L(s, \pi \otimes \chi) \) has a simple zero at the center \( s = 1/2 \).

Choose such a \( \chi \), and let \( K \) be the totally imaginary quadratic extension of \( F \) associated to it. The conductor of \( \chi \) is coprime to \( N \), and the \( L \)-function \( L(s, f/K) = L(s - 1/2, \pi)L(s - 1/2, \pi \otimes \chi) \) has a simple zero at \( s = 1 \). Let \( d \) denote the discriminant of \( K/F \).

2.3. Zhang’s formula.

2.3.1. The \((N, K)\)-type Shimura curves. Let \( \mathcal{O} \) be the subalgebra of \( \mathbb{C} \) over \( \mathbb{Z} \) generated by the eigenvalues of \( f \) under the Hecke operators. In our case, \( \mathcal{O} = \mathbb{Z}[\zeta + \zeta^{-1}] \) is the ring of integers of \( F_0 \). In [33] (see also [5], [6]), Zhang constructs a Shimura curve \( X \) of \((N, K)\)-type, and proves that there exists a unique abelian subvariety \( A \) of the Jacobian \( \text{Jac}(X) \) of dimension \([\mathcal{O} : \mathbb{Z}] = g\), such that
\[ L_v(s, A) = \prod_{\sigma : \mathcal{O} \hookrightarrow \mathbb{C}} L_v(s, f^\sigma / F), \]
for all places \( v \) of \( F \). By the construction of \( f \), it follows that \( L_v(s, A/F) = L_v(s, J_\delta/F) \) for all places \( v \) of \( F \). Therefore, by the isogeny conjecture proved by Faltings, \( A \) is isogenous to \( J_\delta \) over \( F \). In particular, the complex multiplication by \( \mathcal{O} \subset \mathbb{Q}(\mu_p)^+ \) on \( A \) is defined over \( F \).

Now, let us recall the constructions of \( X \) and \( A \).

The \( L \)-function of \( \pi \otimes \chi \) satisfies the functional equation
\[ L(1 - s, \pi \otimes \chi) = (-1)^{[\mathbb{Z}] N_F/Q(Nd)^{2s-1} L(s, \pi \otimes \chi)}, \]
where $\Sigma = \Sigma(N, K)$ is the following set of places of $F$:

$$\Sigma(N, K) = \left\{ v \mid v|\infty, \text{ or } \chi_v(N) = -1 \right\}.$$ 

Since the sign of the functional equation is $-1$, by our choice of $K$, the cardinality of $\Sigma$ is odd. Let $\tau$ be any real place of $F$. Then, we have:

1. Up to isomorphism, there exists a unique quaternion algebra $B$ such that $B$ is ramified at exactly the places in $\Sigma \setminus \{\tau\}$;

2. There exist embeddings $\rho : K \hookrightarrow B$ over $F$.

From now on, we fix an embedding $\rho : K \hookrightarrow B$ over $F$.

Let $G$ denote the algebraic group over $F$, which is an inner form of $\PGL_2$ with $G(F) \cong B^\times/F^\times$. The group $G(F_\tau) \cong \PGL_2(\mathbb{R})$ acts on $\mathcal{H}^\pm = \mathbb{C} \setminus \mathbb{R}$. Now, for any open compact subgroup $U$ of $G(A_f)$, we have an analytic space

$$S_U(\mathbb{C}) = G(F)_+ \setminus \mathcal{H}^+ \times G(\mathbb{A}_f)/U,$$

where $G(F)_+$ denotes the subgroup of elements in $G(F)$ with positive determinant via $\tau$.

Shimura has shown that $S_U(\mathbb{C})$ is the set of complex points of an algebraic curve $S_U$, which descends canonically to $F$ (as a subfield of $\mathbb{C}$ via $\tau$). The curve $S_U$ over $F$ is independent of the choice of $\tau$.

There exists an order $R_0$ of $B$ containing $\mathcal{O}_K$ with reduced discriminant $N$. One can choose $R_0$ as follows. Let $\mathcal{O}_B$ be a maximal order of $B$ containing $\mathcal{O}_K$, and let $N$ be an ideal of $\mathcal{O}_K$ such that

$$N_{K/F}N \cdot \text{disc}_{B/F} = N,$$

where $\text{disc}_{B/F}$ is the reduced discriminant of $\mathcal{O}_B$ over $\mathcal{O}_F$. Then, we take

$$R_0 = \mathcal{O}_K + N \cdot \mathcal{O}_B.$$

Take $U = \prod_v R_v^\times/\mathcal{O}_v^\times$. The corresponding Shimura curve $X := S_U$ is compact.

Let $\xi \in \text{Pic}(X) \otimes \mathbb{Q}$ be the unique class whose degree is 1 on each connected component and such that

$$T_m \xi = \deg(T_m)\xi,$$

for all integral ideals $m$ of $\mathcal{O}_F$ coprime to $Nd$. Here, the $T_m$ are the Hecke operators.

2.3.2. Gross-Zagier-Zhang formula. Now, we define the basic class in $\text{Jac}(X)(K) \otimes \mathbb{Q}$, where $\text{Jac}(X)$ is the connected component of $\text{Pic}(X)$, from the CM-points on the curve $X$. The CM points corresponding to $K$ on $X$ form a set:

$$\mathcal{C} : G(F)_+ \setminus G(F)_+ \cdot h_0 \times G(\mathbb{A}_f)/U \cong T(F) \setminus G(\mathbb{A}_f)/U; \quad [(h_0, g)] \leftrightarrow [g],$$

where $h_0 \in \mathcal{H}^+$ is the unique fixed point of the torus $T(F) = K^\times/F^\times$. 
For a CM point \( z = [g] \in \mathcal{C} \), represented by \( g \in G(\mathbb{A}_f) \), let

\[
\Phi_g : K \longrightarrow \hat{B}, \quad t \mapsto g^{-1} \rho(t) g.
\]

Then, \( \text{End}(z) := \Phi_g^{-1}(\hat{R}_0) \) is an order of \( K \), say \( \mathcal{O}_n = \mathcal{O}_F + n\mathcal{O}_K \), for a (unique) ideal \( n \) of \( F \). The ideal \( n \), called the conductor of \( z \), is independent of the choice of the representative \( g \). By Shimura’s theory, every CM point of conductor \( n \) is defined over the abelian extension \( H'_n \) of \( K \) corresponding to \( K^\times \setminus \hat{K}^\times /\hat{F}^\times \hat{\mathcal{O}}^\times_n \) via class field theory.

Let \( P_1 \) be a CM point in \( X \) of conductor 1, which is defined over \( H'_1 \), the abelian extension of \( K \) corresponding to \( K^\times \setminus \hat{K}^\times /\hat{F}^\times \hat{\mathcal{O}}^\times_1 \). The divisor \( P = \text{Gal}(H'_1/K) \cdot P_1 \) together with the Hodge class defines a class

\[
x := [P - \deg(P) \xi] \in \text{Jac}(X)(K) \otimes \mathbb{Q},
\]

where \( \deg P \) is the multi-degree of \( P \) on the geometric components. Let \( x_f \) be the \( f \)-typical component of \( x \). In [34], Zhang generalized the Gross-Zagier formula to the totally real field case, by proving that

\[
L'(1, f/K) = \frac{2^{g+1}}{\sqrt{N(d)}} \cdot \|f\|^2 \cdot \|x_f\|^2,
\]

where \( \|f\|^2 \) is computed on the invariant measure on

\[
PGL_2(F) \setminus \mathcal{H}^g \times PGL_2(\mathbb{A}_f)/U_0(N)
\]

induced by \( dx dy/y^2 \) on \( \mathcal{H}^g \), and where

\[
U_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\hat{\mathcal{O}}_F) \middle| c \in \hat{N} \right\} \subset \text{GL}_2(\hat{F}),
\]

and \( \|x_f\|^2 \) is the Neron-Tate pairing of \( x_f \) with itself.

2.3.3. The equivalence of nonvanishing of \( L \)-factors. For any \( \sigma : F \hookrightarrow \mathbb{C} \), it is known by a result of Shimura that \( L(1, f/F) \neq 0 \) is equivalent to \( L(1, f^\sigma/F) \neq 0 \). One can also show this using Zhang’s formula above. To see this, assume \( L(1, f/F) \neq 0 \). Then, \( \|x_f\| \neq 0 \), and therefore, \( \|x_f\| \neq 0 \). It follows that \( L'(1, f^\sigma/K) \neq 0 \). Since \( L(1, f/F) \neq 0 \), the \( L \)-function \( L(s, f^\sigma/F) \) has a positive sign in its functional equation. Thus, \( L(1, f^\sigma/F) \neq 0 \). In fact, to obtain our main theorem, we do not need this equivalence, but we may see that Theorem 3.1 is equivalent to statement (2) in the introduction.

2.4. The Euler system of CM points. We now assume that \( L(1, \chi_{[\delta^2]}\psi) \neq 0 \), or equivalently, \( L(1, f/F) \neq 0 \). Then by the equivalence of nonvanishing of \( L(1, f^\sigma) \) for all embeddings \( \sigma : F \hookrightarrow \mathbb{C} \), we have that \( L(1, J_\delta/F) \neq 0 \). By Zhang’s formula, we also know that \( \|x_f\| \neq 0 \).
Let \( \mathcal{N} \) be the set of square-free integral ideals of \( F \) whose prime divisors are inert in \( K \) and coprime to \( N \). For any \( n \in \mathcal{N} \), define

\[
H_n = \prod_{\ell | n} H'_{\ell} \subset H'_n, \quad H_1 = H'_1.
\]

Let \( u_n \) denote the cardinality of \( (\hat{O}_n^x \cap K^x \hat{F}^x) / \hat{O}_F^x \). Then, \( H_\ell / H_1 \) is a cyclic extension of degree \( t(\ell) = \frac{N(\ell) + 1}{u_1 / u_\ell} \).

For each \( n \in \mathcal{N} \), let \( P_n \) be a CM point of order \( n \) such that \( P_n \) is contained in \( T_\ell P_m \) if \( n = m\ell \in \mathcal{N} \) and \( \ell \) is a prime ideal of \( F \). Let \( y_n = \text{Tr}_{H_\ell / H_1} \pi(P_n) \in A(H_n) \), where \( \pi \) is a morphism from \( X \) to \( \text{Jac}(X) \) defined by a multiple of the Hodge class.

The points \( \{y_n\}_{n \in \mathcal{N}} \) form an Euler system (see [29, Prop. 7.5], or [33, Lemma 7.2.2]) so that, for any \( n = m\ell \in \mathcal{N} \) with \( \ell \) a prime ideal of \( F \),

(1) \( u_n^{-1} \sum_{\sigma \in \text{Gal}(H_n / H_m)} y_{\sigma}^m = u_m^{-1} a_{\ell} y_m; \)

(2) For any prime ideal \( \lambda_m \) of \( H_m \) above \( \ell \), and for \( \lambda_n \) the unique prime above \( \lambda_m \),

\[
\text{Frob}_{\lambda_m} y_m \equiv y_n \mod \lambda_n;
\]

(3) The class \( x_f \) is equal to \( y_K := \text{tr}_{H_1 / K} y_1 \) in \( (A(K) \otimes \mathbb{Q}) / \mathbb{Q}^x \).

Theorem 2.1 follows with the nontrivial Euler system by Kolyvagin’s standard argument (see [21], [23], [13], and [33, Th. A]).

3. Analytic methods

Let \( r = 4 \) or an odd prime, and let \( L = F(\zeta_r) \), with \( [L : F] = 2 \). Let \( \psi \) be a unitary Hecke character of \( L \). In this section, we show:

**Theorem 3.1.** There are infinitely many classes \( \delta \in F^x / F^{xr} \) such that \( L \left( \frac{1}{2}, \chi | \delta, | \psi \right) \) does not vanish.

Let \( \rho \) be a unitary Hecke character of \( F \). The purpose of this section is to construct a perfect double Dirichlet series \( Z(s, w; \psi; \rho) \) similar to an Asai-Flicker-Patterson type Rankin-Selberg convolution, which possesses meromorphic continuation to \( \mathbb{C}^2 \) and functional equations. Then, Theorem 3.1 will follow from the analytic properties of \( Z(s, w; \psi; \rho) \) (when \( r = 4 \), see [7]). To do this, it is necessary to recall the Fisher-Friedberg symbol in [9].

3.1. The \( r \)-th power residue symbol. Let \( S' \) be a finite set of non-archimedean places of \( L \) containing all places dividing \( r \), and such that the ring of \( S' \)-integers \( \mathcal{O}_{L}^{S'} \) has class number one. We shall also assume that \( S' \) is closed under conjugation and that \( \psi \) and \( \rho \) are both unramified outside \( S' \).
Let $S_\infty$ denote the set of all archimedean places of $L$, and set $S = S' \cup S_\infty$. Let $I_L(S)$ (resp. $I_L(S)$) denote the group of fractional ideals (resp. the set of all integral ideals) of $O_L$ coprime to $S'$. In [9], Fisher and Friedberg have shown that the $r$-th order symbol $\chi_n$ can be extended to $I_L(S)$ i.e., $\chi_n(m)$ is defined for $m, n \in I_L(S)$. Let us recall their construction.

For a non-archimedean place $v \in S'$, let $\mathfrak{P}_v$ denote the corresponding ideal of $L$. Define $\mathfrak{c} = \prod_{v \in S'} \mathfrak{P}_v^{r_v}$ with $r_v = 1$ if $\text{ord}_v(r) = 0$, and $r_v$ sufficiently large such that, for $a \in L_v$, $\text{ord}_v(a - 1) \geq r_v$ implies that $a \in (L_v^\times)^r$. Let $P_L(c)$ be the subgroup of principal ideals $(\alpha)$ with $\alpha \equiv 1 \mod \mathfrak{c}$, and let $H_c = I_L(S)/P_L(c)$ be the ray class group modulo $c$. Set $R_c = H_c \otimes \mathbb{Z}/r\mathbb{Z}$, and write the finite group $R_c$ as a direct product of cyclic groups. Choose a generator for each, and let $\mathfrak{c}_0$ be a set of ideals of $O_L$, prime to $S$, which represent these generators. For each $\mathfrak{c}_0 \in \mathfrak{c}_0$, choose $m_{\mathfrak{c}_0} \in L^\times$ such that $m_{\mathfrak{c}_0}O_L^{S'} = m_{\mathfrak{c}_0}O_L^{S'}$. Let $\mathfrak{c}$ be a full set of representatives for $R_c$ of the form $\prod_{\mathfrak{c}_0 \in \mathfrak{c}_0} \mathfrak{c}_0^{\lambda_{\mathfrak{c}_0}}$. Note that $rO_L^{S'} = mO_L^{S'}$ for all $c \in \mathfrak{c}$. Without loss, we suppose that $O_L^{S'} \in \mathfrak{c}$ and $mO_L^{S'} = 1$.

Let $m, n \in I_L(S)$ be coprime. Write $m = (m)\mathfrak{c}g$ with $\mathfrak{c} \in \mathfrak{c}$, $m \in L^\times$, $m \equiv 1 \mod \mathfrak{c}$ and $g \in I_L(S)$, $(g, n) = 1$. Then the $r$-th power residue symbol $(\frac{mm}{n})_r$ is defined. If $m = (m')\mathfrak{c}'g'^r$ is another such decomposition, then $\mathfrak{c}' = \mathfrak{c}$ and $(\frac{mm}{n})_r = (\frac{mm'}{n})_r$.

In view of this, the $r$-th power residue symbol $(\frac{m}{n})_r$ is defined to be $(\frac{mm}{n})_r$, and the character $\chi_m$ is defined by $\chi_m(n) = \chi_m(n)^r$. This extension of the $r$-th power residue symbol depends on the above choices. Let $S_m$ denote the support of the conductor of $\chi_m$. It can be easily checked that if $m = m'a^r$, then $\chi_m(n) = \chi_m(n)$ whenever both are defined. This allows one to extend $\chi_m$ to a character of all ideals of $I_L(S \cup S_m)$.

The extended symbol possesses a reciprocity law: if $m, n \in I_L(S)$ are coprime, then $\alpha(m, n) = \chi_m(n)\chi_n(m)^{-1}$ depends only on the images of $m, n$ in $R_c$.

In our situation, we also need the following lemma:

**Lemma 3.2.** The natural morphism

$$I_F(S)/P_F(c) \longrightarrow I_L(S)/P_L(c)$$

has kernel of order a power of 2.

**Proof.** If $[n]$ is in the kernel, i.e., $n = (\alpha)$ in $I_L(S)$ is a principal ideal with $\alpha \equiv 1 \mod c$, then $\alpha/\overline{\alpha}$ is a root of unity with $\alpha/\overline{\alpha} \equiv 1 \mod c$. Now let $W$ be the set of roots of unity in $L$ which are $\equiv 1 \mod c$. Let $W_0$ be the subset of $W$ of elements of the form $u/\overline{u}$ for some unit $u$ in $O_L$ and $u \equiv 1 \mod c$. It is clear that $W_0 \supset W^2$. Then, the map

$$\text{Ker} \left( I_F(S)/P_F(c) \rightarrow I_L(S)/P_L(c) \right) \longrightarrow W/W_0; \quad n \mapsto \alpha/\overline{\alpha}$$
is obviously injective; i.e., the order of the kernel of the natural map in this lemma is a power of 2.

Since \( r \) is odd, using the lemma, we may choose a suitable set \( \mathfrak{c}_0 \) of representatives since the beginning such that if \( m \in I_F(S) \), then the decomposition \( m = (m) \mathfrak{c}_0^r \) is such that \( m \in F^x \), \( \mathfrak{c}_0 \in I_F(S) \).

Using the symbol \( \chi_n \), we shall construct a perfect double Dirichlet series \( Z(s, w; \psi; \rho) \) (i.e., possessing meromorphic continuation to \( \mathbb{C}^2 \)) of type:

\[
(3.1) \quad Z(s, w; \psi; \rho) = Z_S(s, w; \psi; \rho) = * \sum_{n \in I_F(S)} L_S(s, \psi \chi_n) \rho(n) N_{F/\mathbb{Q}}(n)^{-w},
\]

where the sum is over the set of all integral ideals of \( \mathcal{O}_F \) coprime to \( S' \), for \( n \in I_F(S) \) square-free, the function \( L_S(s, \psi \chi_n) \) is precisely the Hecke \( L \)-function attached to \( \psi \chi_n \) with the Euler factors at all places in \( S \) removed, and where \( * \) is a certain normalizing factor. For an arbitrary \( n \in I_F(S) \), write \( n = n_1n_2^r \) with \( n_1 \) \( r \)-th power free. If \( L_S(s, \psi \chi_n) \) denotes the Hecke \( L \)-series associated to \( \psi \chi_n \) with the Euler factors at all places in \( S \) removed, then \( L_S(s, \psi \chi_n) \) is defined as \( L_S(s, \psi \chi_n) \) multiplied by a Dirichlet polynomial whose complexity grows with the divisibility of \( n \) by powers (see (3.10), (3.12) and (3.13) for precise definitions).

Based on the analytic properties of \( Z(s, w; \psi; \rho) \), we show the following result which is stronger than Theorem 3.1.

**Theorem 3.3.** 1) There exist infinitely many \( r \)-th power free ideals \( n_1 \) in \( I_F(S) \) with trivial image in \( R_r \) for which the special value \( L_S(\frac{s}{2}, \chi_n \psi) \) does not vanish.

2) Let \( \kappa_r \) denote the number of characters of \( R_r \) whose restrictions to \( F \) are also characters of the ideal class group of \( F \), and let \( \kappa \) be the residue of the Dedekind zeta function \( \zeta_F(s) \) at \( s = 1 \). Then for \( x \to \infty \),

\[
(3.2) \quad \sum_{\substack{N_{F/\mathbb{Q}}(n) < x \\ n \in I_F(S) \\ n = (n) \\ [n] = 1}} L_S \left( \frac{1}{2}, \chi_n \psi \right) \sim \frac{\kappa \cdot \kappa_r}{h_F \cdot |R_r|} \frac{L_S(1, \psi) L_S(\frac{r}{2}, \psi^r)}{L_S(\frac{r}{2} + 1, \psi^r)} \prod_{v \in S'} (1 - q_v^{-1}) \cdot x,
\]

where \([n]\) denotes the image of the ideal \( n \) in \( R_r \).

**Remarks.** i) By the above definition of the extended \( r \)-th power residue symbol, it is easy to see that the first part of this theorem is equivalent to Theorem 3.1.

ii) In fact, by a well-known result of Waldspurger [30], it will follow that \( L_S(\frac{1}{2}, \chi_n \psi) \geq 0 \), for \( n \in I_F(S) \), \( n = (n) \) and trivial image in \( R_r \). We will see this in the course of the proof of Theorem 3.3.
iii) Following [8], by a simple sieving process, one can prove the more familiar variant of the above asymptotic formula where the sum is restricted to square-free principal ideals.

3.2. The series $Z_{aux}(s, w; \psi; \rho)$ and metaplectic Eisenstein series. To obtain the correct definition of $Z(s, w; \psi; \rho)$, let $G_0(n, m)$, for $m, n \in \mathcal{I}_L(S)$, be given by

$$G_0(n, m) = \prod_{\text{ord}_v(n) = k, \text{ord}_v(m) = l} G_0(p_v^k, p_v^l),$$

where, for $k, l \geq 0$,

$$G_0(p_v^k, p_v^l) = \begin{cases} 
1 & \text{if } l = 0, \\
q_v^k & \text{if } k + 1 = l; \ l \not\equiv 0 \pmod{r}, \\
q_v^{k-1}q_v^{l-1}(q_v - 1) & \text{if } k + 1 = l; \ l > 0; \ l \equiv 0 \pmod{r}, \\
0 & \text{otherwise}.
\end{cases}$$

Here $q_v$ denotes the absolute value of the norm of $v$. Also, let $G(\chi_{m_1}^r)$ (where $m_1$ denotes the $r$-th power free part of $m$ and $\chi_{a}^r(b) := \chi_{b}(a))$ be the normalized Gauss sum appearing in the functional equation of the (primitive) Hecke $L$-function associated to $\chi_m^r$. If $n^*$ denotes the part of $n$ coprime to $m_1$, then set

$$G(n, m) := \overline{\chi_{m_1}(n^*)} G(\chi_{m_1}^r) G_0(n, m).$$

Now, let $\psi$ be as above. For $n \in \mathcal{I}_L(S)$ and $\text{Re}(s) > 1$, let $\Psi_S(s, n, \psi)$ be the absolutely convergent Dirichlet series defined by

$$\Psi_S(s, n, \psi) = L_S(rs - \frac{r}{2} + 1, \psi^r) \sum_{m \in \mathcal{I}_L(S)} \frac{\psi(m)G(n, m)}{N_{L/Q}(m)^s}.$$

This series can be realized as a Fourier coefficient of a metaplectic Eisenstein series on the $r$-fold cover of $GL(2)$ (see [18] and [24]). It follows as in Selberg [28], or alternatively, from Langlands’ general theory of Eisenstein series [25] that $\Psi_S(s, n, \psi)$ has meromorphic continuation to $\mathbb{C}$ with only one possible (simple) pole at $s = \frac{1}{2} + \frac{1}{r}$. Moreover, this function is bounded when $|\text{Im}(s)|$ is large in vertical strips, and satisfies a functional equation as $s \to 1 - s$ (see Kazhdan-Patterson [18, Cor. II.2.4]).

For $\text{Re}(s), \text{Re}(w) > 1$, let $Z_{aux}(s, w; \psi; \rho)$ be the auxiliary double Dirichlet series defined by

$$Z_{aux}(s, w; \psi; \rho) = \sum_{n \in \mathcal{I}_F(S)} \frac{\Psi_S(s, n, \psi)\rho(n)}{N_{F/Q}(n)^w}.$$
Let $\tilde{\rho}$ be the Hecke character of $L$ given by $\tilde{\rho} = \rho \circ N_{L/F}$. As we shall shortly see, $Z_{\text{aux}}(s, w; \psi; \bar{\rho})$ is the type of object that constitutes a building block in the process of constructing the perfect double Dirichlet series $Z(s, w; \psi; \rho)$. Set

$$
\Gamma_{\text{aux}}^*(s, \psi \tilde{\rho}) = \prod_{v \in S_{\infty}} \prod_{j=1}^{r-1} L_v \left( s - \frac{1}{2} + \frac{j}{r}, \psi_v \tilde{\rho}_v \right),
$$

and let

$$
\tilde{Z}_{\text{aux}}(s, w; \psi; \bar{\rho}) := \Gamma_{\text{aux}}^*(s, \psi \tilde{\rho}) \cdot Z_{\text{aux}}(s, w; \psi; \bar{\rho}).
$$

Let $R_1$ be the tube region in $\mathbb{C}^2$ whose base $B_1$ is the convex region in $\mathbb{R}^2$ which lies strictly above the polygonal contour determined by $(0, 2)$, $(1, 1)$, and the rays $y = -2x + 2$ for $x \leq 0$ and $y = 1$ for $x \geq 1$. As a simple consequence of the analytic properties of $\Psi_S(s, n, \psi) \ (n \in I_L(S))$, we have the following:

**Proposition 3.4.** The double Dirichlet series $Z_{\text{aux}}(s, w; \psi; \bar{\rho})$ is holomorphic in $R_1$, unless $\psi^r \bar{\rho}^r = 1$ when it has only one simple pole at $s = \frac{1}{2} + \frac{1}{r}$.

Furthermore, $\tilde{Z}_{\text{aux}}(s, w; \psi; \bar{\rho})$ satisfies the functional equation

$$
\tilde{Z}_{\text{aux}}(s, w; \psi; \bar{\rho}) = \prod_{v \in S'} \left( 1 - (\psi \bar{\rho})^{-r}(\pi_v) q_v^{-\frac{r}{2}} \right),
$$

where each $A_{\eta, \tau}^{(\psi \bar{\rho})}(s)$ is a polynomial in the variables $q_v^\pm, q_v^{-\frac{r}{2}} \ (v \in S')$, and the sum is over a finite set of idèle class characters $\eta$ and $\tau$, unramified outside $S$ and with orders dividing $r$.

**3.3. The double Dirichlet series $\tilde{Z}(s, w; \psi; \rho)$.** It turns out that the function $Z_{\text{aux}}(s, w; \psi; \bar{\rho})$ possesses another functional equation. To describe it, we introduce a new double Dirichlet series $\tilde{Z}(s, w; \psi; \rho)$ defined for $\text{Re}(s), \text{Re}(w) > 1$ by

$$
\tilde{Z}(s, w; \psi; \rho) = L_S(rs + rw + 1 - r, \psi^r \bar{\rho}^r) \sum_{m \in I_L(S)} \frac{\psi(m) L_{W}(w, \chi_m^* \rho)}{N_{L/Q}(m)^s} \prod_{v \in S'} \frac{\psi_m^*(h_v)}{N_{F/Q}(h_v)^{2s-1}} \prod_{\text{ord}_v(h_0) > 0} \left[ (\chi_m^*(\pi_v) q_v^{-w} - q_v^{-1}) \right] \prod_{\text{ord}_{(h_v)} > 0} \left[ (\chi_m^*(\pi_v) q_v^{-w-1} + 1 - 2q_v^{-1}) \right] \prod_{\text{ord}_{(h_v)} > 0} \left[ 1 - (\chi_m^*(\pi_v) q_v^{-w-1}) \right].
$$
In the above formula, an ideal \( m \in \mathcal{I}_L(S) \) is called \textit{imaginary}, if it has no divisor in \( \mathcal{I}_F(S) \), other than \( \mathcal{O}_F \). The function \( L_S(w, \chi_m^w \rho) \) represents the \( L \)-series defined over \( F \) (not necessarily primitive) associated to \( \chi_m^w \rho \) with the Euler factors corresponding to places removed in \( S \). Also, all the products are over places of \( F \), \( \pi_v \) is the local parameter of \( F_v \) (\( F_v \) denoting the completion of \( F \) at \( v \)), and \( q_v \) is the absolute value of the norm in \( F \) of \( v \).

Let \( \mathcal{R}_2 \) denote the tube region in \( \mathbb{C}^2 \) whose base \( \mathcal{B}_2 \) is the convex region in \( \mathbb{R}^2 \) which lies strictly above the polygonal contour determined by \( (1,1), \left( \frac{3}{2}, 0 \right) \) and the rays \( y = -x + \frac{3}{2} \) for \( y \leq 0 \) and \( x = 1 \) for \( y \geq 1 \). Recall that \( L_S(w, \chi_m^w \rho) \) differs from a primitive \( L \)-series by only finitely many Euler factors (i.e., the factors corresponding to places in \( S \) and to places \( v \) for which \( \text{ord}_v(N_{L/F}(m)) \equiv 0 \) (mod \( r \))). Applying the functional equation of \( L_S(w, \chi_m^w \rho) \) and some standard estimates, one can easily show that the function \( \tilde{Z}(s,w;\psi;\rho) \) is holomorphic in \( \mathcal{R}_2 \), unless \( \rho = 1 \) where it has only one simple pole at \( w = 1 \). The following proposition gives the functional equation connecting the double Dirichlet series \( Z_{\text{aux}}(s,w;\psi;\rho) \) and \( \tilde{Z}(s,w;\psi;\rho) \).

**Proposition 3.5.** The function \( \tilde{Z}(s,w;\psi;\rho) \) is holomorphic in \( \mathcal{R}_2 \), unless \( \rho \) is the trivial character when it has a simple pole at \( w = 1 \). Furthermore, for \( \text{Re}(s), \text{Re}(w) > 1 \), there exist the functional equations

\[
(3.8) \quad \prod_{v \in S_\infty} L_v \left( 1 - w, \rho_v \right) \cdot \prod_{v \in S'} \left( 1 - \rho^{-r}(\pi_v) q_v^{-r w} \right) \cdot \tilde{Z}(s + w - \frac{1}{r}, 1 - w; \psi; \rho) \\
= \prod_{v \in S_\infty} L_v \left( w, \rho_v^{-1} \right) \cdot \sum_\tau B^{(\rho)}_\tau(w) Z_{\text{aux}}(s,w;\psi \tilde{\rho}, \tilde{\rho}),
\]

and

\[
(3.9) \quad \prod_{v \in S_\infty} L_v \left( w, \rho_v^{-1} \right) \cdot \prod_{v \in S'} \left( 1 - \rho^r(\pi_v) q_v^{r w - r} \right) \cdot Z_{\text{aux}}(s,w;\psi \tilde{\rho}, \tilde{\rho}) \\
= \prod_{v \in S_\infty} L_v \left( 1 - w, \rho_v \right) \cdot \sum_\tau C^{(\rho)}_\tau(1 - w) \tilde{Z}(s + w - \frac{1}{r}, 1 - w; \psi \tau; \rho),
\]

where, as before, \( B^{(\rho)}_\tau(w) \), \( C^{(\rho)}_\tau(w) \) are polynomials in the variables \( q_v^w, q_v^{-w} \) \( (v \in S') \). The above products are over the places of \( k \) corresponding to those in \( S \), and the sums are over a finite set of idèle class characters \( \tau \), unramified outside \( S \) and orders dividing \( r \).

The proof of this proposition will be given in the next section.

Let \( \alpha \) and \( \beta \) be the involutions on \( \mathbb{C}^2 \) given by

\[
\alpha: (s,w) \rightarrow (1 - s, 2s + w - 1) \quad \text{and} \quad \beta: (s,w) \rightarrow (s + w - \frac{1}{2}, 1 - w).
\]

It can be easily checked that these involutions generate the dihedral group \( D_8 \) of order 8. It follows directly from Propositions 3.2 and 3.3 that both
\( \bar{Z}(s + w - \frac{i}{2}, 1 - w; \psi; \rho) \) and \( Z_{\text{aux}}(s, w; \psi \hat{\rho}, \hat{\rho}) \) can be continued to \( \mathcal{R}_1 \cup \mathcal{R}_2 \). Clearly, this applies to \( Z_{\text{aux}}(s, w; \psi, \rho) \) (replace \( \psi \) by \( \psi^{-1} \rho \) and \( \rho \) by \( \hat{\rho} \)). It follows from the functional equation (3.6) that \( Z_{\text{aux}}(s, w; \psi \hat{\rho}, \hat{\rho}) \) can be continued to \( \mathcal{R}_1 \cup \mathcal{R}_2 \cup \alpha(\mathcal{R}_2) \), and hence, by (3.8), the function \( \bar{Z}(s + w - \frac{i}{2}, 1 - w; \psi; \rho) \) continues to this region. The double Dirichlet series \( Z_{\text{aux}}(s, w; \psi \hat{\rho}, \hat{\rho}) \) may have only one simple pole in \( \mathcal{R}_2 \), namely \( w = 1 \), and this pole occurs only if \( \rho \) is the trivial character. This fact follows easily by inspection of the proof of Proposition 3.3 (see \( \S 3.1 \)). Then from the functional equation (3.6), one can see that \( Z_{\text{aux}}(s, w; \psi \hat{\rho}, \hat{\rho}) \) may have a pole only at \( w = 2 - 2s \in \alpha(\mathcal{R}_2) \), provided \( \psi^r|_{\mathcal{O}_E} \cdot \rho^r \) is trivial. The last fact also applies to \( \bar{Z}(s + w - \frac{i}{2}, 1 - w; \psi, \rho) \), by the functional equation \( \beta \) in (3.8).

3.4. The double Dirichlet series \( Z(s, w; \psi; \rho) \). To define the perfect double Dirichlet series \( Z(s, w; \psi; \rho) \), let \( L_S(s, \chi_n \psi) \), for \( n \in \mathcal{I}_F(S) \), be given by

\[
L_S(s, \chi_n \psi) := L_S(s, \chi_n \psi)P_n(s, \psi),
\]

where \( n_1 \) denotes the \( r \)-th power free part of \( n \), and \( P_n(s, \psi) \) is the Dirichlet polynomial defined by

\[
P_n(s, \psi) = \prod_{\nu, \text{ord}_\nu(n_1) > 0} \left( 1 + \psi(\pi_\nu) q_\nu^{1-2s} + \cdots + \psi(\pi_\nu)^{\text{ord}_\nu(n_1)-1} q_\nu^{\text{ord}_\nu(n_1)-1} q_\nu^{(1-2s)} \right)
\]

\[
\times \prod_{\nu, \text{ord}_\nu(n_1) = r \mu, \nu \in \text{inert in } L} \left( 1 - \psi(\pi_\nu) q_\nu^{-2s} \right) \left( 1 + \psi(\pi_\nu) q_\nu^{1-2s} + \cdots + \psi(\pi_\nu)^{r \mu-1} q_\nu^{(r \mu-1) (1-2s)} + \psi(\pi_\nu)^{r \mu} q_\nu^{r \mu (1-2s)} \right) \left( 1 + q_\nu^{-1} \right)
\]

\[
\times \prod_{\nu, \text{ord}_\nu(n_1) = r \omega, \nu = \nu' \nu'' \in L} \left( 1 - (\chi_n \psi)(\pi_\nu') q_\nu'^{-s} \right) \left( 1 - (\chi_n \psi)(\pi_\nu') q_\nu'^{-s} \right) \left( 1 + \psi(\pi_\nu) q_\nu^{1-2s} + \cdots + \psi(\pi_\nu)^{r \omega-1} q_\nu^{(r \omega-1) (1-2s)} + \psi(\pi_\nu)^{r \omega} q_\nu^{r \omega (1-2s)} \right) \left( 1 - q_\nu^{-1} \right). \]

Here the products are over places \( \nu \) of \( F \), and \( \pi_\nu \) denotes the local parameter of \( F_\nu \). It can be seen that these polynomials satisfy a functional equation as \( s \to 1 - s \), and that we have the estimate

\[
P_n(s, \psi) \ll \varepsilon \text{ N}_F/Q(n)^{\varepsilon} \quad (\varepsilon > 0, \text{ Re}(s) \geq \frac{1}{2}).
\]

Furthermore, if \( \psi(m) = \overline{\psi(m)} \), for \( m \in \mathcal{I}_L(S) \), then \( P_n(s, \psi) \geq 0 \), for \( s \in \mathbb{R} \). Later, we shall specialize \( \psi \) to be (essentially) a normalized Jacobi sum, which obviously satisfies this property.
For $\text{Re}(s), \text{Re}(w) > 1$, we define $Z(s, w; \psi; \rho)$ as

$$(3.12) \quad Z(s, w; \psi; \rho) = Z_S(s, w; \psi; \rho) = L_S(rs + rw + 1 - r, \psi^r \bar{\rho}^r) \sum_{n \in \mathcal{I}_F(S)} \frac{L_S(s, \chi_n \psi) \rho(n)}{N_{F/Q}(n)^w}.$$  

Applying the functional equation and the convexity bound of $L_S(s, \chi_n \psi)$ ($n \in \mathcal{I}_F(S)$), we see that $Z(s, w; \psi; \rho)$ is holomorphic in $\mathcal{R}_1$, if the character $\psi^r$ is nontrivial. Representing the normalizing factor $L_S(rs + rw + 1 - r, \psi^r \bar{\rho}^r)$ by its Dirichlet series, then after multiplying and reorganizing, we can write $Z(s, w; \psi; \rho)$ as

$$(3.13) \quad Z(s, w; \psi; \rho) = \sum_{n \in \mathcal{I}_F(S)} \frac{L_S(s, \chi_n \psi) Q_n(s, \psi) \rho(n)}{N_{F/Q}(n)^w},$$

where $Q_n(s, \psi)$, for $n \in \mathcal{I}_F(S)$, is a new set of Dirichlet polynomials which can be easily expressed in terms of $P_n(s, \psi)$.

Referring to the definition of $\tilde{Z}(s, w; \psi; \rho)$ given in (3.7), replace $L_S(w, \chi_m^* \rho)$ by its Dirichlet series, the sum being over $n$, say. For fixed $m \in \mathcal{I}_L(S)$ imaginary, and $n \in \mathcal{I}_F(S)$, collect the terms contributing to $(\chi_m^* \rho)(n) N_{F/Q}(n)^{-w}$. Switching the order of summation, we obtain:

**Proposition 3.6.** For $\text{Re}(s), \text{Re}(w) > 1$,

$$(3.14) \quad Z(s, w; \psi; \rho) = L_S(2s, \psi) \tilde{Z}(s, w; \psi; \rho),$$

where the $L$-function is defined over $F$.

Assuming both $\psi^r$ and $\psi^r \bar{\rho}^r$ to be nontrivial, we see from Proposition 3.4 that

$L_S(2s + 2w - 1, \psi) \tilde{Z}(s + w - \frac{1}{2}, 1 - w; \psi; \rho)$ continues to $\beta(\mathcal{R}_1)$, and hence, from the above discussion, it continues to $\mathcal{R}_1 \cup \beta(\mathcal{R}_1) \cup \mathcal{R}_2 \cup \alpha(\mathcal{R}_2)$. Note that the convex closure of this tube region is $\mathbb{C}^2$. As $\psi^r \bar{\rho}^r \neq 1$, and therefore, by Propositions 3.2 and 3.3, the function $\tilde{Z}(s + w - \frac{1}{2}, 1 - w; \psi; \rho)$ does not have a pole at $s = \frac{1}{2} + \frac{1}{r}$, one can easily check that the only possible poles of $L_S(2s + 2w - 1, \psi) \tilde{Z}(s + w - \frac{1}{2}, 1 - w; \psi; \rho)$ are the hyperplanes $w = 0$ and $w = 2 - 2s$. Clearly, both are simple poles, and they may occur only if $\rho$ and $\psi^r|\mathcal{O}_F \cdot \rho^r$ are both trivial.

Consequently, by the convexity theorem for holomorphic functions of several complex variables (see [16]) and by Proposition 3.4, we have the following:

**Theorem 3.7.** When $\psi^r$ and $\psi^r \bar{\rho}^r$ are nontrivial, the function

$$(w - 1)(2s + w - 2)Z(s, w; \psi; \rho)$$

has analytic continuation to $\mathbb{C}^2$, and for any fixed $s$, it is (as a function of the variable $w$) of order one.
The fact that, for any fixed $s$, the above function is of order one follows as in [8, Prop. 3.11].

By Proposition 3.4 and (3.7), one finds that, for $\text{Re}(s) > \frac{1}{2}$,

(3.15)

$$\lim_{w \to 1} (w - 1)^2 Z_1(\frac{1}{2}, w; \psi) = \lim_{(s, w) \to (\frac{1}{2},1)} (w - 1)(2s + w - 2)Z_1(s, w; \psi) = 0,$$

and, therefore, $Z_1(\frac{1}{2}, w; \psi; 1)$ has at most a simple pole at $w = 1$. To compute its residue, recall the functional equation satisfied by $L(s, \chi_n, \psi)$ with $\mathfrak{n}_1 \in \mathcal{I}_F(S)$ r-th power free (see [31, Ch. VII, §7]). Combining this with the functional
equation of the polynomial $Q_n(s, \psi) \ (n \in \mathcal{I}_F(S))$, we find that
\[\sum_l L_S(s, \chi_n, \psi)Q_n(s, \psi) = \varepsilon(s, \chi_n, \psi) \cdot L_S(1-s, \chi_n, \psi)Q_n(1-s, \psi)\]
\[\cdot \prod_{v \in S_w} \frac{L_v(1-s, \psi_v)}{L_v(s, \psi_v)} \cdot \prod_{v \in S'} \frac{L_v(1-s, (\chi_n, \psi)_v)}{L_v(s, (\chi_n, \psi)_v)}.
\]
A simple local computation shows that $\varepsilon(\frac{1}{2}, \chi_n, \psi) = \psi(n)\varepsilon(\frac{1}{2}, \psi)$. It immediately follows that $Z_1(s, w; \psi; 1)$ satisfies the functional equation
\[(3.17) \prod_{v \in S_w} L_v(s, \psi_v) \cdot \prod_{v \in S'} \left(1 - \psi_v(\pi_v) q_v^{-s-r}\right) \cdot Z_1(s, w; \psi; 1) = \prod_{v \in S_w} L_v(1-s, \psi_v) \cdot \sum_{\rho} D_\rho(s) \cdot Z_1(1-s, 2s + w - 1; \psi; \rho),\]
where $D_\rho(s)$ are polynomials in the variables $q_v, q_v^{-s}, v \in S'$, and the sum is over a finite set of idèle class characters $\rho$, unramified outside $S$ and orders dividing $r$. As $r$ is odd, and $\psi$, restricted to the group of principal ideals of $F$, is quadratic and nontrivial, it follows that $Z_1(s, w; \psi; 1)$ does not have a pole at $w = 2 - 2s$. Then (3.15) yields
\[(3.18) \operatorname{Res}_{w=1} Z_1\left(\frac{1}{2}, w; \psi; 1\right) = \frac{\kappa \cdot \kappa_\epsilon}{h_F \cdot |R_\epsilon|} L_S(1, \psi)L_S\left(\frac{r}{2}, \psi^r\right) \prod_{v \in F, v \in S'} (1-q_v^{-1}),\]
where $\kappa_\epsilon$ denotes the number of characters of $R_\epsilon$ whose restrictions to $F$ are also characters of the ideal class group of $F$.

To complete the proof, we define the double Dirichlet series $Z_0(s, w; \psi; \rho)$ by simply replacing in (3.16) the polynomial $Q_n(s, \psi)$ by $P_n(s, \psi)$ defined in (3.10). Note that
\[Z_0(s, w; \psi; \rho) = \frac{1}{h_F \cdot |R_\epsilon|} \sum_{\rho_1, \rho_2} L_S(rs + rw + 1 - r, \psi^{r' \rho_1 \rho_2'}) Z(s, w; \rho_1 \rho_2)\]
and therefore, $Z_0(s, w; \psi; \rho)$ may have additional poles at the zeros of the incomplete $L$-functions $L_S(rs + rw + 1 - r, \psi^{r' \rho_1 \rho_2'})$. It is well-known that these zeros occur in the region $\operatorname{Re}(s + w) < 1$. In particular, the function $Z_0\left(\frac{1}{2}, w; \psi; 1\right)$ is holomorphic for $\operatorname{Re}(w) > \frac{1}{2}$, except for $w = 1$, where it has a simple pole. Using (3.18), we can compute its residue as
\[(3.19) \operatorname{Res}_{w=1} Z_0\left(\frac{1}{2}, w; \psi; 1\right) = \frac{\kappa \cdot \kappa_\epsilon}{h_F \cdot |R_\epsilon|} \frac{L_S(1, \psi)L_S\left(\frac{r}{2}, \psi^r\right)}{L_S\left(\frac{r}{2} + 1, \psi^r\right)} \prod_{v \in F, v \in S'} (1-q_v^{-1}) > 0.
\]
This implies that $L_S\left(\frac{1}{2}, \chi_n, \psi\right) \neq 0$ for infinitely many $r$-th power free ideals $n_1$ in $\mathcal{I}_F(S)$ with trivial image in $R_\epsilon$, which is the first assertion of Theorem 3.3.
For the remaining part, one needs to apply a Tauberian theorem. To keep the argument as simple as possible, note first that, as \( \psi(\overline{m}) = \psi(m) \), for \( m \in I_L(S) \), we have \( \mathcal{P}_n(s, \psi) \geq 0 \), for \( s \in \mathbb{R} \). On the other hand, by the comment made right after Lemma 3.2, any \( r \)-th power free ideal \( n_1 \) in \( I_F(S) \) with trivial image in \( R_\epsilon \) can be decomposed as \( n_1 = (n_1)g^r \) with \( n_1 \in F^\times \), \( n_1 \equiv 1 \mod \epsilon \) and \( g \in I_F(S) \). By definition, the character \( \chi_{n_1} \) coincides with the classical \( r \)-th power residue symbol \( \chi_{n_1} \), given by class field theory. It follows that the incomplete \( L \)-series \( L_S(s, \chi_{n_1}) \) differs from the complete Hecke \( L \)-series associated to \( L(s, \chi_{n_1}, \psi) \) by only finitely many local factors. Recall that the latter is the \( L \)-series associated to a Hilbert modular form. As the set \( S' \) is closed under conjugation, it follows from a well-known result of Waldspurger [31] that \( L_S(\frac{1}{2}, \chi_{n_1}) \geq 0 \), for \( n \in I_F(S) \), \( n = (n) \) and trivial image in \( R_\epsilon \). Hence, the function \( Z_0(\frac{1}{2}, w; \psi; 1) \), for \( \Re(w) > 1 \), is given by a Dirichlet series with nonnegative coefficients. The second part of Theorem 3.3 now follows from the Wiener-Ikehara Tauberian theorem.

Remark. With some additional effort, one can exhibit an error term on the order of \( O(x^\theta) \) with \( \theta < 1 \) in the asymptotic formula (3.2). Also, the remark following Theorem 3.3 implies that the Hecke \( L \)-series \( L_S(\frac{1}{2}, \chi_n, \psi) \neq 0 \) for infinitely many square-free principal ideals \( (n) \) in \( I_F(S) \) with trivial image in \( R_\epsilon \). Any such ideal has a generator \( n \in F \) with \( n \equiv 1 \mod \epsilon \).

3.5. Proof of Proposition 3.3. Recall that for \( a \in I_L(S) \), we defined \( \chi_a^* \) by \( \chi_a^*(b) := \chi_b(a) (b \in I_L(S)) \). Note that every ideal \( m \) of \( \mathcal{O}_L \) can be uniquely decomposed as \( m = m' \mathfrak{h} \), where \( m' \) is an imaginary ideal of \( \mathcal{O}_L \), and \( \mathfrak{h} \) is a real ideal; that is, \( \mathfrak{h} \in \mathcal{O}_F \). For \( m \in I_L(S) \) imaginary and \( r \)-th power free, let \( \varepsilon(w, (\chi_m^* \rho)^{-1}) \) denote the epsilon-factor in the functional equation of \( L(w, (\chi_m^* \rho)^{-1}) \) (as a Hecke \( L \)-function of \( F \)). Also, for \( m \in I_L(S) \) imaginary and \( \mathfrak{h} \in I_F(S) \), coprime and \( r \)-th power free, let \( G(\chi_{m \mathfrak{h}}^*) \) be the normalized Gauss sum in the functional equation of the Hecke \( L \)-function (of the field \( L \)) associated to \( \chi_{m \mathfrak{h}}^* \), i.e., \( \varepsilon(\frac{1}{2}, \chi_{m \mathfrak{h}}^*) \). We set \( m_0 \) and \( \mathfrak{h}_0 \) to be the product of all distinct prime ideals dividing \( m \) and \( \mathfrak{h} \), respectively.

The following lemma is a consequence of a standard local computation. The details will be omitted.

Lemma 3.8. Let \( m \) and \( \mathfrak{h} \) be integral ideals as above. Assume that the images of \( m \mathfrak{h} \) and \( m \) in \( R_\epsilon \) are \( \epsilon \) and \( \epsilon' \), respectively. Then,

\[
G(\chi_{m \mathfrak{h}}^*) \varepsilon \left( \frac{1}{2}, (\chi_m^* \rho)^{-1} \right) = C_{\epsilon, \epsilon', \rho} \eta(\epsilon)^{-1} \eta(\mathfrak{h}_1) \hat{\rho}(m_0)^{-1} \chi_m^*(\mathfrak{h}_0) \chi_{m \mathfrak{h}}^*(m_0) \chi_{m \mathfrak{h}}^*(m_0)^{-1},
\]

where \( \hat{\rho} = \rho \circ N_{L/F} \), \( C_{\epsilon, \epsilon', \rho} \) is a constant depending on just \( \epsilon \), \( \epsilon' \) and \( \rho \), and \( \eta \) is a Hecke character unramified outside \( S \) and order dividing \( r \). Furthermore,
Recall that if \( \mathfrak{e}' \) is replaced by \( \mathfrak{e}''' \) with \( \mathfrak{e}'/\mathfrak{e}''' \) a real ideal, then both \( C_{\mathfrak{e}, \mathfrak{e}', \rho} \) and \( \eta \) do not change.

**Proof of Proposition 3.3.** Using (3.5), we have

\[
(3.20) \\
Z_{aux}(s, w; \psi \tilde{\rho}, \tilde{\rho})
= \sum_{n \in \mathcal{I}_F(S)} \frac{\Psi_S(s, n, \psi \tilde{\rho}) \overline{\rho(n)}}{N_{F/Q}(n)^w} \\
= L_S \left( rs - \frac{r}{2} + 1, \psi^r \tilde{\rho}^r \right) \sum_{m \in \mathcal{I}_L(S)} \frac{(\psi \tilde{\rho})(m) \overline{\rho(n)} G(m, m)}{N_{L/F}(m)^s N_{F/Q}(n)^w} \\
= L_S \left( rs - \frac{r}{2} + 1, \psi^r \tilde{\rho}^r \right) \sum_{m \in \mathcal{I}_L(S)} \frac{(\psi \tilde{\rho})(m) \overline{\rho(n)} \chi_{m}^*(n^*) G(m, m)}{N_{L/F}(m)^s N_{F/Q}(n)^w},
\]

where \( n^* \) denotes the part of \( n \) coprime to \( m \). In the last sum, replace \( m \) by \( m \mathfrak{h} \) with \( m \in \mathcal{I}_L(S) \) imaginary and \( \mathfrak{h} \) real. As we shall see, the only contribution to the sum comes from \( m \) and \( \mathfrak{h} \) for which their \( r \)-th power free parts \( m_1 \) and \( \mathfrak{h}_1 \) are coprime. Then, we have

\[
(3.21) \\
\sum_{m \in \mathcal{I}_L(S)} \frac{(\psi \tilde{\rho})(m) \overline{\rho(n)} \chi_{m}^*(n^*) G(m, m)}{N_{L/F}(m)^s N_{F/Q}(n)^w} = \sum_{m \in \mathcal{I}_L(S)} \frac{(\psi \tilde{\rho})(m)}{N_{L/F}(m)^s} \\
\cdot \sum_{\mathfrak{h} \in \mathcal{I}_L(S)} \frac{(\psi \tilde{\rho})(\mathfrak{h}) \overline{\rho(n)} \chi_{m_1 \mathfrak{h}_1}^*(n^*) G(m_1 \mathfrak{h}_1, \mathfrak{h}_1)}{N_{L/F}(\mathfrak{h})^s N_{F/Q}(n)^w}.
\]

Next, we separate the contribution of \( \mathfrak{h} \) in the inner sum. To do so, let \( m_1 \) denote the \( r \)-th power free part of an ideal \( m \in \mathcal{I}_L(S) \), and set \( m_0 \) to be the product of all distinct prime ideals dividing \( m_1 \), and

\[
m_2 := \prod_{\text{ord}_v(m) = r e_v} p_v^{e_v}.
\]

For fixed \( m \), \( n \) and \( \mathfrak{h} \) as above, let \( p_v \) be a prime ideal of \( L \) dividing \( \mathfrak{h}_0 \). Upon replacing this prime ideal by its conjugate, we can assume that \( \text{ord}_v(m) = 0 \). Recall that

\[
G_0(n, m) = \prod_{\text{ord}_v(n) = k} G_0(p_v^k, p_v^l),
\]

where \( k \) and \( l \) are integers.
where \( G_0(p^k, p') \) is given by (3.4). As \( \text{ord}_v(mh) = \text{ord}_v(h) \neq 0 \) (mod \( r \)) (this condition implying that \( \text{ord}_v(n) = \text{ord}_v(h) - 1 \), and \( n \in \mathcal{I}_F(S) \), we can decompose \( n = (h/h_0h_2)n' \) with \( n' \in \mathcal{I}_F(S) \) coprime to \( h_1 \). Also, we have

\[
\text{ord}_v(n) = \text{ord}_v(n) \geq \text{ord}_v(mh) - 1
= \text{ord}_v(m) + \text{ord}_v(h) - 1 = \text{ord}_v(m) + \text{ord}_v(n),
\]

which implies \( \text{ord}_v(m) = 0 \). It immediately follows that \( m \) and \( h_1 \) are coprime. Then, by (3.4), we can write

\[
(3.22) \quad G(x^*_m h_2) G_0(n, mh) = G(x^*_n n') G_0(n', mh_2)
= G(x^*_m h_1) N_{L/Q} \left( \frac{h}{h_0h_2} \right) \frac{1}{2} G_0(n', mh_2).
\]

Furthermore, we have

\[
G_0(n', mh_2) = \prod_{\text{ord}_v(n') = k_v} G_0(p^{k_v}, p^{\ell_v + re_v})
= \prod_{\ell_v \equiv 0 (r)} G_0(p_{v}^{k_v}, p_{v}^{l_v + re_v}) \cdot \prod_{\ell_v \equiv 0 (r)} G_0(p_{v}^{k_v}, p_{v}^{l_v + re_v})
= \prod_{\ell_v \equiv 0 (r)} G_0(p_{v}^{k_v}, p_{v}^{l_v + re_v}) \cdot \prod_{\ell_v \equiv 0 (r)} G_0(p_{v}^{k_v}, p_{v}^{l_v + re_v})
= N_{L/Q} \left( \frac{mh_2}{m_0} \right) \frac{1}{2} \prod_{\ell_v \equiv 0 (r)} G_0(p_{v}^{k_v}, p_{v}^{l_v + re_v}) \cdot \prod_{\ell_v \equiv 0 (r)} G_0(p_{v}^{k_v}, p_{v}^{l_v + re_v})
= N_{L/F} \left( \frac{n'}{n_0} \right) \prod_{\ell_v \equiv 0 (r)} G_0(p_{v}^{k_v}, p_{v}^{l_v + re_v}) \cdot \prod_{\ell_v \equiv 0 (r)} G_0(p_{v}^{k_v}, p_{v}^{l_v + re_v})
\]

One can decompose \( n' \) as

\[
n' = n_1 \cdot N_{L/F} \left( \frac{m}{m_0} \right) \cdot h_2
\]
with \( n_1 \) coprime to \( mh \). Here, if \( v \) is complex such that \( l_v = l_0 = 0 \), then one chooses either \( v \) or \( \bar{v} \), but not both. As \( n = (h/h_0 h_2)n' \), we also have

\[
n = n_1 \cdot N_{L/F}(m) \cdot h_0 \cdot \prod_{v - \text{complex}} N_{L/F}(p_v)^{\alpha_v - 1} \cdot \prod_{v - \text{real}} q_v^{\beta_v - 1}.
\]

Recall that \( n^* \) denotes the part of \( n \) coprime to \( mh_1 \). It follows that

\[
n^* = n_1 \cdot \left( \frac{m}{m_0 m_2} \right) \cdot N_{L/F}(m_2) \cdot h_2 \cdot \prod_{v - \text{complex}} N_{L/F}(p_v)^{\alpha_v - 1} \cdot \prod_{v - \text{real}} q_v^{\beta_v - 1}.
\]

Combining all these with (4.26), we obtain

\[
\sum_{m \in \mathcal{I}_L(S)} \frac{(\psi \rho)(m)}{N_{L/Q}(m)^s} \sum_{n \in \mathcal{I}_F(S)} \frac{(\psi \rho)(h)^\rho(n)}{N_{L/Q}(h)^s N_{F/Q}(n)^w} G(\chi_{m, h_1}, G_0(n, mh)) \left[ \prod_{v - \text{split in } L} \left( \frac{\rho(n_1) \chi_{m, h_1}^{w-1} G(\chi_{m, h_1})}{\prod_{\text{ord}_v(N_{L/F}(m_2)) > 0} (1 - q_v^{w-1})} \right) \right. \\
\quad \left. \cdot \prod_{\text{ord}_v(N_{L/F}(m_2)) > 0} \left[ (\chi_{m, 1} \rho)(\pi_v) q_v^{w-1} + (1 - q_v^{w-1}) \cdot \sum_{\alpha_v \geq 0} \left( (\chi_{m, 1} \rho)^{w}(\pi_v) q_v^{w} \right)^{\alpha_v} \right] \right]
\]

where \( \pi_h = \pi_0 \).
Note that the last sum represents an incomplete Hecke $L$-function. After evaluating the geometric series inside the last four products, the missing Euler factors corresponding to places of $F$ dividing $N_{L/F}(m_2)h_2$ can be incorporated. Also, multiply and divide by the Euler factors corresponding to places of $F$ dividing $h_0$, forcing in this way $L_S(w, (\chi_{m_1}^* \rho)^{-1})$ to appear.

Let $R_\epsilon^+$ be the subgroup of $R_\epsilon$ generated by the images (in $R_\epsilon$) of all real fractional ideals of $L$ coprime to $S'$. Let $\epsilon'$ be a fixed element of $R_\epsilon$ which is the image of an imaginary ideal $m \in I_L(S)$. Replacing $\psi$ by $\psi \tau_1 \tau_2$ with $\tau_1$ and $\tau_2$ characters of $R_\epsilon$ and $R_\epsilon/R_\epsilon^+$, respectively, and making a standard linear combination, one can restrict the first two sums over ideals $\tau$ see [31], that the incomplete Hecke fractional ideals of $S$ characters unramified outside $\psi$. satisfies the functional equation

$$\prod_{v \notin S} L_v(w, (\chi_{m_1}^* \rho_{v})^{-1}) = \prod_{v \in S} [1 - (\chi_{m_1}^* \rho_{v})^{-1}(\pi_v) q_v^{-w}]^{-1}$$

satisfies the functional equation

$$L_S(w, (\chi_{m_1}^* \rho)^{-1}) = \varepsilon(w, (\chi_{m_1}^* \rho)^{-1}) \cdot L_S(1 - w, \chi_{m_1}^*) \cdot \prod_{v \in S_{\infty}} \frac{L_v(1 - w, \rho_v)}{L_v(w, \rho_v^{-1})} \cdot \prod_{\pi \in S'} \frac{L_v(1 - w, (\chi_{m_1}^*)_{v})}{L_v(w, (\chi_{m_1}^*)_{v}^{-1})}.$$ Replace $\psi$ by $\psi \eta^{-1}$, and combine the above functional equation with Lemma 3.6. Here $\text{Re}(s)$ is taken sufficiently large to ensure convergence. Using the Fisher-Friedberg extension of the reciprocity law [9], one can see that

$$\chi_{m_1}(m) \chi_{h_1}(m) = C'_{\epsilon', \tilde{\epsilon}} \cdot \chi_{m_1}(h_1),$$

where $C'_{\epsilon', \tilde{\epsilon}}$ is a constant depending on just $\epsilon$ and the class $\tilde{\epsilon}$ in $R_\epsilon/R_\epsilon^+$. Also, note that

$$\prod_{\pi \in S'} \left(1 - \rho^{-r}(\pi_v) q_v^{-w} \right)^{-1} \cdot \frac{L_v(1 - w, (\chi_{m_1}^*)_{v})}{L_v(w, (\chi_{m_1}^*)_{v}^{-1})} \left(1 - \rho^{r}(\pi_v) q_v^{w-r} \right)$$

is the inverse of a polynomial in the variables $q_v^w$, $q_v^{-w}$ corresponding to places $v \in S'$ of the totally real field $F$. The characters involved in its coefficients are trivial on real ideals. Now, the functional equation (3.8) immediately follows, after we replace $\psi$ with $\psi \tau$, where $\tau$ ranges over a finite set of idèle class characters unramified outside $S$ and orders dividing $r$, and make a combination such that the above product over $v \in S'$ disappears.

Starting from the definition of

$$\prod_{\pi \in S'} \left(1 - \rho^{r}(\pi_v) q_v^{w-r} \right)^{-1} \cdot \tilde{Z}(s + w - \frac{r}{2}, 1 - w; \psi; \rho),$$

one can easily check (3.9) by reversing the above argument. \qed
TWISTED FERMAT CURVES OVER TOTALLY REAL FIELDS

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1376  ADRIAN DIACONU AND YE TIAN

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