Weighted jump and variational inequalities for rough operators

Revised Version

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Abstract In this paper, we systematically study weighted jump and variational inequalities for rough operators. More precisely, we show some weighted jump and variational inequalities for the families $T := \{T_\varepsilon\}_{\varepsilon > 0}$ of truncated singular integrals and $M_\Omega := \{M_{\Omega,t}\}_{t > 0}$ of averaging operators with rough kernels, which are defined respectively by

\[ T_\varepsilon f(x) = \int_{|y| > \varepsilon} \frac{\Omega(y')}{|y|^n} f(x - y) dy \]

and

\[ M_{\Omega,t} f(x) = \frac{1}{t^n} \int_{|y| < t} \Omega(y') f(x - y) dy, \]

where the kernel $\Omega$ belongs to $L^q(S^{n-1})$ for $q > 1$.

1 Introduction

The jump and variational inequalities have been the subject of many recent articles in probability, ergodic theory and harmonic analysis. To present related results in a precise way, let us fix some notations. Given a family of complex numbers $a = \{a_t : t \in \mathbb{R}\}$ and $\rho \geq 1$, the $\rho$-variation norm of the family $a$ is defined by

\[ \|a\|_{V_\rho} = \sup \left( |a_{t_0}| + \sum_{k \geq 1} |a_{t_k} - a_{t_{k-1}}|^\rho \right) ^{\frac{1}{\rho}}, \]

where the supremum runs over all finite increasing sequences $\{t_k : k \geq 0\}$. It is trivial that

\[ \|a\|_{L^\infty(\mathbb{R})} := \sup_{t \in \mathbb{R}} |a_t| \leq \|a\|_{V_\rho} \text{ for } \rho \geq 1. \]

Via the definition (1.1) of the $\rho$-variation norm of a family of numbers, one may define the strong $\rho$-variation function $V_\rho(\mathcal{F})$ of a family $\mathcal{F}$ of functions. Given a family of Lebesgue measurable functions $\mathcal{F} = \{F_t : t \in \mathbb{R}\}$ defined on $\mathbb{R}^n$, for any fixed $x$ in $\mathbb{R}^n$, the value of the strong $\rho$-variation function $V_\rho(\mathcal{F})$ of the family $\mathcal{F}$ at $x$ is defined by

\[ V_\rho(\mathcal{F})(x) = \|\{F_t(x)\}_{t \in \mathbb{R}}\|_{V_\rho}, \quad \rho \geq 1. \]

Suppose $\mathcal{A} = \{A_t\}_{t > 0}$ is a family of operators on $L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$. The strong $\rho$-variation operator is simply defined as

\[ V_\rho(\mathcal{A}f)(x) = \|\{A_t(f)(x)\}_{t > 0}\|_{V_\rho}, \quad \forall f \in L^p(\mathbb{R}^n). \]

It is easy to observe from the definition of $\rho$-variation norm that for any $x$ if $V_\rho(\mathcal{A}f)(x) < \infty$, then $\{A_t(f)(x)\}_{t > 0}$ converges when $t \to 0$ or $t \to \infty$. In particular, if $V_\rho(\mathcal{A}f)$ belongs to some function spaces such as $L^p(\mathbb{R}^n)$ or $L^{p,\infty}(\mathbb{R}^n)$, then the sequence converges almost everywhere without any additional condition. This is why the mapping property of the strong $\rho$-variation operator is so interesting in ergodic theory and harmonic analysis. Also, by (1.2), for any $f \in L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we have

\[ A^*(f)(x) \leq V_\rho(\mathcal{A}f)(x) \text{ for } \rho \geq 1, \]
where $A^*$ is the maximal operator defined by
\[ A^*(f)(x) := \sup_{t>0} |A_t(f)(x)|. \]

Let $\mathcal{F} = \{F_t(x) : t \in \mathbb{R}_+ \}$ be a family of Lebesgue measurable functions defined on $\mathbb{R}^n$. For $\lambda > 0$, we introduce the $\lambda$-jump function $N_\lambda(\mathcal{F})$ of $\mathcal{F}$, its value at $x$ is the supremum over all $N$ such that there exist $s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_N < t_N$ with
\[ |F_{t_k}(x) - F_{s_k}(x)| > \lambda \]
for all $k = 1, \ldots, N$.

Using the fact that $\ell^{2,\infty}(\mathbb{N})$ (the weak $L^2$ space on $\mathbb{N}$) embeds into $\ell^\rho(\mathbb{N})$ for all $\rho > 2$, it is easy to check that the following pointwise domination holds
\[ \sup_{\lambda>0} \lambda \sqrt{N_\lambda(\mathcal{F})} \geq V_\rho(\mathcal{F}) \]
for all families of functions $\mathcal{F}$.

The first variational inequality was proved by Lépingle [26] for martingales (see [37] for a simple proof). Bourgain [2] is the first one using Lépingle’s result to obtain similar variational estimates for the ergodic averages, and then directly deduce pointwise convergence results without previous knowledge that the pointwise convergence holds for a dense subclass of functions, which are not available in some ergodic models. In particular, Bourgain’s work [2] has inaugurated a new research direction in ergodic theory and harmonic analysis. In their papers [18, 20, 19, 4, 5], Jones and his collaborators systematically studied jump and variational inequalities for ergodic averages and truncated singular integrals (mainly of homogeneous type). Since then many other publications came to enrich the literature on this subject (cf. e.g. [14, 25, 9, 21, 30, 36, 31, 32]).

For $\varepsilon > 0$, suppose $T_{\varepsilon}$ is the truncated singular integral operator defined by
\begin{equation}
T_{\varepsilon} f(x) = \int_{|y| > \varepsilon} \frac{\Omega(y')}{|y|^n} f(x-y) dy,
\end{equation}
where $\Omega \in L^1(S^{n-1})$ satisfies the cancelation condition
\begin{equation}
\int_{S^{n-1}} \Omega(y') d\sigma(y') = 0.
\end{equation}
Denote the family of operators $\{T_{\varepsilon}\}_{\varepsilon>0}$ by $\mathcal{T}$. For $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$, the Calderón-Zygmund singular integral operator $T$ with homogeneous kernel is defined by
\begin{equation}
T f(x) = \lim_{\varepsilon \to 0^+} T_{\varepsilon} f(x), \text{ a.e. } x \in \mathbb{R}^n.
\end{equation}
The fact that the limit exists almost everywhere is deduced from the density of $C_0^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ and the boundedness of the maximal singular integral $T^*$ which is given by $T^* f(x) = \sup_{\varepsilon>0} |T_{\varepsilon} f(x)|$. In fact, variational and jump inequalities for $\mathcal{T}$ can also be used to study the existence of the above limit and give extra information on the convergence.
The famous Hilbert transform $H$, which is defined by $H(f)(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy$, is an example of homogeneous singular integral operators $T_{\Omega}$ when the dimension $n = 1$. In 2000, Campbell et al [4] first considered the $L^p(\mathbb{R})$ $(1 < p < \infty)$ boundedness of the strong $\rho$-variation operator of the family of the truncated Hilbert transforms denoted by $\mathcal{H} := \{H_{\varepsilon}\}_{\varepsilon > 0}$. In 2002, Campbell et al [5] gave the $L^p(\mathbb{R}^n)$ boundedness of the strong $\rho$-variation operator of $\mathcal{T}$, the family of homogenous singular integrals with $\Omega \in L^r(\mathbb{S}^{n-1})$ and $n \geq 2$ for $\rho > 2$. In 2008, using the Fourier transform and the square function estimates given in [11], Jones, Seeger and Wright [21] developed a general method, which allows one to obtain some jump inequalities for families of the truncated singular integral operators $\mathcal{T}$ and of other integral operators arising from harmonic analysis.

**Theorem A.** ([21]) Suppose $\Omega$ satisfies (1.6) and $\Omega \in L^r(\mathbb{S}^{n-1})$ for $r > 1$. Then the $\lambda$-jump inequality

$$\sup_{\lambda > 0} \|\lambda \sqrt{N_\lambda(Tf)}\|_{L^p(\mathbb{R}^n)} \leq C_{p,n} \|f\|_{L^p(\mathbb{R}^n)} \quad (1 < p < \infty)$$

holds, where and in the sequel, the constant $C_{p,n} > 0$ depends only on $p$ and $n$. Whence, for $\rho > 2$ and $1 < p < \infty$, there exists a constant $C(p, \rho)$ such that

$$\|V_{\rho}(Tf)\|_{L^p(\mathbb{R}^n)} \leq C(p, \rho) \|f\|_{L^p(\mathbb{R}^n)}.$$

The purpose of this paper is to give the weighted jump and variational inequalities for singular integrals and averaging operators with rough kernels. In order to state our main results, let us first recall some definitions. We first recall the definition and some properties of $A_p$ weight on $\mathbb{R}^n$. Let $w$ be a non-negative locally integrable function defined on $\mathbb{R}^n$. We say $w \in A_1$ if there is a constant $C > 0$ such that $M(w)(x) \leq C w(x)$, where $M$ is the classical Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{r > 0} \frac{1}{r^n} \int_{|y| \leq r} |f(x-y)| dy.$$ 

Equivalently, $w \in A_1$ if and only if there is a constant $C > 0$ such that for any cube $Q$

$$\frac{1}{|Q|} \int_Q w(x) dx \leq C \inf_{x \in Q} w(x). \quad (1.8)$$

For $1 < p < \infty$, we say that $w \in A_p$ if there exists a constant $C > 0$ such that

$$\sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right)^{\frac{1}{p}} \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p'-1} \leq C. \quad (1.9)$$

The smallest constant appearing in (1.8) or (1.9) is denoted by $[w]_{A_p}$. $A_\infty = \bigcup_{p \geq 1} A_p$. It is well known that if $w \in A_\infty$, then there exist $\delta \in (0, 1]$ and $C > 0$ such that for any cube $Q$ and measurable subset $E \subset Q$

$$\frac{w(E)}{w(Q)} \leq C \left(\frac{|E|}{|Q|}\right)^\delta. \quad (1.10)$$

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The boundedness of many important operators in harmonic analysis on $L^p(w)$ for $w \in A_p$, $1 < p < \infty$, has been known for a long time. We refer the reader to [12, 13, 35, 33, 34, 10, 40, 15] for more details on this topic. Recently there has been renewed interest in weighted jump and variational inequalities. The weighted $\rho$-variational inequality $2 < \rho < \infty$ for singular integrals with Lipschitz kernels has been shown recently in [28, 29] (see also [22] for averaging operators). It arises naturally as an open problem that whether the $\rho$-variations for singular integrals with rough kernels are bounded on weighted $L^p$ spaces, even though the boundedness on unweighted $L^p$ spaces has been proved in [4, 5, 21, 8]. In the present paper, we give a positive solution to the problem and prove a weighted jump inequality which implies all $\rho$-variational inequalities for the singular integrals with rough kernels. We also show corresponding weighted jump inequalities for the averaging operators with rough kernels.

**Theorem 1.1.** Let $T$ be given as in (1.5) with $\Omega \in L^q(S^{n-1})$, $q > 1$ satisfying (1.6). Then the following $\lambda$-jump inequality holds

\[(1.11) \quad \| \sup_{\lambda > 0} \lambda \sqrt{N_\lambda(Tf)} \|_{L^p(w)} \leq C_{p,w} \| f \|_{L^p(w)}, \]

if $w$ and $p$ satisfy one of the following conditions:

(i) If $q' \leq p < \infty$, $p \neq 1$ and $w \in A_{p'/q'}$,

(ii) If $1 < p \leq q$, $p \neq \infty$ and $w^{-\frac{1}{(p-1)}} \in A_{p'/q'}$.

Whence, for $\rho > 2$, there exists a constant $C(p, \rho)$ such that

\[(1.12) \quad \| V_{\rho}(Tf) \|_{L^p(w)} \leq C(p, \rho) \| f \|_{L^p(w)}, \]

if $w$ and $p$ satisfy one of the conditions (i) or (ii).

**Remark 1.2.** Theorem 1.1 covers Theorem A with $w \equiv 1$. Also, (1.12) is an improvement of the weighted $L^p$ boundedness for $T^*$ (see [10] and [40]), because of the pointwise estimate $T^*f(x) \leq V_{\rho}(Tf)(x)$. Restricted to the singular integrals of homogeneous type, this result significantly improves Corollary 1.4 in [29] where $\Omega$ is assumed to be in the Hölder class of order $\alpha$.

As in [21], the first step to prove Theorem 1.1 is that the desired estimate (1.11) is reduced to the estimate over short 2-variation and the estimate over dyadic $\lambda$-jump function through the following pointwise inequality (see for instance Lemma 1.3 in [21])

\[(1.13) \quad \lambda \sqrt{N_\lambda(Tf)}(x) \leq C \left[ S_2(Tf)(x) + \lambda \sqrt{N_{\lambda/3}([T_{2k}f]_{k \in \mathbb{Z}})}(x) \right], \]

where

\[ S_2(Tf)(x) = \left( \sum_{j \in \mathbb{Z}} |V_{2,j}(Tf)(x)|^2 \right)^{1/2}, \]

with

\[ V_{2,j}(Tf)(x) = \left( \sup_{t_1 < \cdots < t_N \in [j, j+1]} \sum_{i=1}^{N-1} |T_{t_{i+1}}f(x) - T_{t_i}f(x)|^2 \right)^{1/2}. \]
That is, we are reduced to prove
\[ \| \sup_{\lambda > 0} \lambda \sqrt{N_\lambda(T_f^2)} \|_{L^p(w)} \leq C_{p,w} \| f \|_{L^p(w)} \]  
and
\[ \| S_2(T_f) \|_{L^p(w)} \leq C_{p,w} \| f \|_{L^p(w)} \]  
for \( w \) and \( p \) satisfying the conditions (i) or (ii) in Theorem 1.1.

In order to show estimate (1.14), we prove some vector-valued weighted estimates such as (2.5) in Section 2 and use the generalized Rubio de Francia’s extrapolation theorem—Lemma 2.2 in Section 2—as well as Stein and Weiss’s interpolation theorem with change of measure. On the other hand, to establish estimate (1.15), we discover a new phenomenon, that is, the short 2-variation can be dominated by the vector-valued maximal function with rough kernel—inequality (3.5) in Section 3, which actually provides an alternate proof instead of the rotation argument used in [5] and [21]. Then we are reduced to prove vector-valued weighted estimates such as (3.6) and (3.14) in Section 3. The main idea behind the proof is Rubio de Francia’s extrapolation theorem, see for instance Remark 3.3 below.

The proof of Theorem 1.1 can be adapted to the situation of averaging operators with rough kernels \( M_\Omega = \{ M_{\Omega,t} \}_{t > 0} \), where \( M_{\Omega,t} \) is defined as
\[ M_{\Omega,t} f(x) = \frac{1}{t^n} \int_{|y|<t} \Omega(y') f(x - y) dy, \]
where \( \Omega \in L^1(S^{n-1}) \).

**Theorem 1.3.** Suppose the family \( M_\Omega = \{ M_{\Omega,t} \}_{t > 0} \) is defined in (1.16). Let \( \Omega \in L^{q}(S^{n-1}) \) for \( q > 1 \). Then
\[ \| \sup_{\lambda > 0} \lambda \sqrt{N_\lambda(M_{\Omega,t} f)} \|_{L^p(w)} \leq C_{p,w} \| f \|_{L^p(w)} \]
if \( w \) and \( p \) satisfy (i) or (ii) in Theorem 1.1. The similar inequality holds for the strong \( \rho \)-variation operator \( V_{\rho}(M_{\Omega,t} f) \) with \( \rho > 2 \).

**Remark 1.4.** When \( \Omega \equiv 1 \), the weighted jump and variational inequalities have been proved in [28], [29] and [22]. We will explain briefly the proof of Theorem 1.3 in Section 4.

## 2 Proof of Theorem 1.1 (I)

As we have stated in the previous section, to prove Theorem 1.1 it suffices to show (1.14) and (1.15). In this section, we give the proof of (1.14). Let us begin with one definition. For \( j \in \mathbb{Z} \), let \( \nu_j(x) = \frac{\chi_{|y|<2^j} x \{ 2^j \leq |x| < 2^{j+1} \}}{|y|^n} \), then
\[ \nu_j * f(x) = \int_{2^j \leq |y| < 2^{j+1}} \frac{\Omega(y)}{|y|^n} f(x - y) dy. \]
Obviously, for \( k \in \mathbb{Z} \),
\[
T_{2k} f(x) = \int_{|x-y| \geq 2^k} \frac{\Omega(x-y)}{|x-y|^m} f(y) \, dy = \sum_{j \geq k} \nu_j * f(x).
\]

Let \( \phi \in \mathcal{S}(\mathbb{R}^n) \) be a radial function such that \( \hat{\phi}(\xi) = 1 \) for \( |\xi| \leq 2 \) and \( \hat{\phi}(\xi) = 0 \) for \( |\xi| > 4 \). We have the following decomposition
\[
T_{2k} f = \phi_k * T f + \sum_{s \geq 0} (\delta_0 - \phi_k) * \nu_{k+s} * f - \phi_k * \sum_{s < 0} \nu_{k+s} * f
\]
\[
:= T^1_k f + T^2_k f - T^3_k f,
\]
where \( \phi_k \) satisfies \( \hat{\phi}_k(\xi) = \hat{\phi}(2^k \xi) \), \( \delta_0 \) is the Dirac measure at 0 and \( s \in \mathbb{N} \cup \{0\} \). Let \( \mathcal{T}^i f \) denote the family \( \{T^i_k f\}_{k \in \mathbb{Z}} \) for \( i = 1, 2, 3 \). Obviously, to show (1.14) it suffices to prove the following inequalities:
\[
(2.1) \quad \| \sup_{\lambda > 0} \lambda [N_\lambda(\mathcal{T}^i f)]^{1/2} \|_{L^p(w)} \leq C_{p,w} \| f \|_{L^p(w)}, \quad i = 1, 2, 3,
\]
for \( w \) and \( p \) satisfying the conditions (i) or (ii) in Theorem 1.1.

**Estimate of (2.1) for \( i = 1 \).** This estimate will follow easily from the weighted \( L^p \)-boundedness of \( T \) (see [10] or [40]) and the following Proposition 2.1 which is a simplified and weighted version of Theorem 1.1 in [21]. The proof of Proposition 2.1 will be postponed to the end of the section.

**Proposition 2.1.** Let \( \mathcal{U} \) be a family of operators given by \( \mathcal{U} f = \{\phi_k * f\}_k \). Then for \( 1 < p < \infty \) and \( w \in A_p \), we have
\[
\| \sup_{\lambda > 0} \lambda [N_\lambda(\mathcal{U} f)]^{1/2} \|_{L^p(w)} \leq C_{p,w} \| f \|_{L^p(w)}.
\]

Indeed,
\[
\| \sup_{\lambda > 0} \lambda [N_\lambda(\mathcal{T}^1 f)]^{1/2} \|_{L^p(w)} \leq \| \sup_{\lambda > 0} \lambda [N_\lambda(\{\phi_k * T f\})]^{1/2} \|_{L^p(w)}
\]
\[
\leq C_{p,w} \| T f \|_{L^p(w)} \leq C_{p,w} \| \Omega \|_{L^p(S^{n-1})} \| f \|_{L^p(w)},
\]
whenever \( w \) and \( p \) satisfy the conditions (i) or (ii) in Theorem 1.1.

**Estimate of (2.1) for \( i = 2 \).** By the Minkowski inequality, we get
\[
\sup_{\lambda > 0} \lambda [N_\lambda(\mathcal{T}^2 f(x))]^{1/2} \leq \sum_{s \geq 0} \left( \sum_{k \in \mathbb{Z}} \| (\delta_0 - \phi_k) * \nu_{k+s} * f(x) \|^2 \right)^{1/2}
\]
\[
:= \sum_{s \geq 0} G_s f(x).
\]

We are reduced to establish the estimate of \( \| G_s f \|_{L^p(w)} \) as sharp as possible so that we are able to sum up over \( s \in \mathbb{N} \cup \{0\} \). Let \( \psi \in C_0^\infty(\mathbb{R}^n) \) be a radial function such that \( 0 \leq \psi \leq 1 \),
supp$\psi \subset \{1/2 \leq |\xi| \leq 2\}$ and $\sum_{l \in \mathbb{Z}} \psi^2(2^{-l} \xi) = 1$ for $|\xi| \neq 0$. Define the multiplier $\Delta_l$ by $\hat{\Delta_l f}(\xi) = \psi(2^{-l} \xi) \hat{f}(\xi)$. Therefore,

$$G_s f(x) = \left( \sum_{k \in \mathbb{Z}} \left| \left( \delta_0 - \phi_k \right) * \nu_{s+k} \right| \sum_{l \in \mathbb{Z}} \Delta_{l-k}^2 f(x) \right)^{1/2} \leq \sum_{l \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} \left| \Delta_{l-k} \left( \delta_0 - \phi_k \right) * \nu_{s+k} \right| \Delta_{l-k} f(x) \right)^{1/2} := \sum_{l \in \mathbb{Z}} G_{s,l} f(x).$$

(2.3)

We first prove a rapid decay estimate of $\|G_{s,l} f\|_{L^2(\mathbb{R}^n)}$ for $l \in \mathbb{Z}$ and $s \in \mathbb{N} \cup \{0\}$. Since

$$\text{supp}(1 - \hat{\phi}_k) \nu_{k+s} \subset \{ \xi : |2^k \xi| > 2 \}$$

and $\Omega(x')$ satisfies [1,6], by a well-known Fourier transform estimate of Duandikoetxea and Rubio de Francia (see [11, p.551-552]), it is easy to show that for any fixed $q > 1$ and some $\gamma \in (0,1)$,

$$|1 - \hat{\phi}_k(\xi)| |\nu_{k+s}(\xi)| \leq C \|\Omega\|_{L^q(\mathbb{S}^{n-1})} 2^{-\gamma s} \min \{2^k |\xi|, |2^k \xi|^\gamma \}$$

and

$$|1 - \hat{\phi}_k(\xi)| |\nu_{k+s}(\xi)| |\psi(2^{k-1}\xi)| \leq C \|\Omega\|_{L^q(\mathbb{S}^{n-1})} 2^{-\gamma s} \min \{2^k, 2^{-\gamma l} \}.$$

Applying the above estimates and the Littlewood-Paley theory, we get

$$\|G_{s,l} f\|_{L^2} \leq C \|\Omega\|_{L^q(\mathbb{S}^{n-1})} |2^{-\gamma s} \min \{2^k, 2^{-\gamma l} \}| \left( \sum_{k \in \mathbb{Z}} \left| \Delta_{l-k} f \right| \right)^{1/2} \left( \sum_{k \in \mathbb{Z}} \left| \Delta_{l-k} f \right| \right)^{1/2} \|f\|_{L^2} \leq C \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \|f\|_{L^2}.$$

(2.4)

Now we give the weighted $L^p$ estimate of $G_{s,l} f$ for $l \in \mathbb{Z}$ and $s \in \mathbb{N} \cup \{0\}$. Define $T_{s,k} f = [(\delta_0 - \phi_k) * \nu_{k+s}] * f$. If we accept the following estimate for a moment

$$\left( \sum_{k \in \mathbb{Z}} |T_{s,k} f|^2 \right)^{1/2} \leq C_{p,w} \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \left( \sum_{k \in \mathbb{Z}} \left| \Delta_{l-k} f \right|^2 \right)^{1/2} \|f\|_{L^p(\mathbb{R}^n)},$$

whenever $w$ and $p$ satisfy the conditions (i) or (ii) in Theorem [23], then by (2.4) and the weighted Littlewood-Paley theory (see [2]), we get

$$\|G_{s,l} f\|_{L^p(\mathbb{R}^n)} = \left( \sum_{k \in \mathbb{Z}} \left| T_{s,k} \Delta_{l-k} f \right|^2 \right)^{1/2} \leq C_{p,w} \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \left( \sum_{k \in \mathbb{Z}} \left| \Delta_{l-k} f \right|^2 \right)^{1/2} \|f\|_{L^p(\mathbb{R}^n)}.$$

(2.5)

Repeating the same argument in [10] or [40] and using Stein and Weiss’s interpolation theorem with change of measure between (2.4) and (2.5), we get for some $\theta_0, \beta_0 \in (0,1)$

$$\|G_{s,l} f\|_{L^p(\mathbb{R}^n)} \leq C_{p,w} 2^{-\beta s} 2^{-\theta_0 |l|} \|\Omega\|_{L^q(\mathbb{S}^{n-1})} \|f\|_{L^p(\mathbb{R}^n)}.$$

(2.6)
Combining (2.2), (2.3) and (2.7), we obtain
\[ \| \sup_{\lambda > 0} \lambda [N_\lambda (T^2 f)(x)] \|_{L^p(w)}^{1/2} \leq \sum_{s \geq 0} \sum_{t \in \mathbb{Z}} \| G_{s,t} f \|_{L^p(w)}^{1/2} \]
\[ \leq C_{p,w} \sum_{s \geq 0} \sum_{t \in \mathbb{Z}} 2^{\beta_0 s + \theta_0 t} \| \Omega \|_{L^q(s^{n-1})} \| f \|_{L^p(w)} \]
\[ \leq C_{p,w} \| \Omega \|_{L^q(s^{n-1})} \| f \|_{L^p(w)}, \]
whenever \( w \) and \( p \) satisfy the conditions (i) or (ii) in Theorem 1.1.

Now we turn to the proof of (2.5). First of all, we have the following estimate
\[ (\sum_{k \in \mathbb{Z}} |T_{s,k} f_k|^2)^{1/2} \leq C_{p,w} \left( \sum_{k \in \mathbb{Z}} |\nu_{k+s} * f_k|^2 \right)^{1/2}, \]
which follows from
\[ (\sum_{k \in \mathbb{Z}} |\varphi_k * f_k|^2)^{1/2} \leq C_{p,w} \left( \sum_{k \in \mathbb{Z}} |f_k|^2 \right)^{1/2}, \]
for \( 1 < p < \infty \) and \( w \in A_p \) (see [1]). Secondly, we claim that for \( w \) and \( p \) satisfying the conditions (i) or (ii) in Theorem 1.1,
\[ (\sum_{k \in \mathbb{Z}} |\nu_{k+s} * f_k|^2)^{1/2} \leq C_{p,w} \| \Omega \|_{L^q(s^{n-1})} \left( \sum_{k \in \mathbb{Z}} |f_k|^2 \right)^{1/2}. \]
Then (2.5) is a consequence of (2.8) and (2.10).

Now we turn to the proof of claim (2.10). We only show that (2.10) holds when \( w \) and \( p \) satisfy the condition (i) in Theorem 1.1; the proof is similar for the condition (ii). Obviously,
\[ \left( \sum_{k \in \mathbb{Z}} |\nu_{k+s} * f_k|^2 \right)^{1/2} \leq \left( \sup_{k \in \mathbb{Z}} |\nu_{k+s} * f_k| \right) \| M_{\Omega} \|_{L^p(w)}, \]
where \( M_{\Omega} \) denotes the rough maximal operator defined by
\[ M_{\Omega} g(x) = \sup_{t > 0} \frac{1}{t^n} \int_{|y| < t} |\Omega(y') g(x - y)| dy. \]
By (10) (see also [27, p.106]), if \( w \) and \( p \) satisfy the conditions (i) or (ii) in Theorem 1.1 then
\[ \| M_{\Omega} g \|_{L^p(w)} \leq C_{p,w} \| \Omega \|_{L^q(s^{n-1})} \| g \|_{L^p(w)}. \]
Hence, from (2.11) and (2.12), if \( w \) and \( p \) satisfy the conditions (i) or (ii) in Theorem 1.1 then
\[ \left( \sum_{k \in \mathbb{Z}} |\nu_{k+s} * f_k|^2 \right)^{1/2} \leq C_{p,w} \| \Omega \|_{L^q(s^{n-1})} \sup_{k \in \mathbb{Z}} |f_k| \|_{L^p(w)}, \]
8
Using (2.13) under the condition (ii) and the duality, we see that if \( w \) and \( p \) satisfy the condition (i), then
\[
\left\| \sum_{k \in \mathbb{Z}} \nu_{k+s} \ast f_k \right\|_{L^p(w)} \leq C_{p, w} \left\| \Omega \right\|_{L^q(S^{n-1})} \sum_{k \in \mathbb{Z}} \left\| f_k \right\|_{L^p(w)}.
\]
Interpolating between (2.14) and (2.13) (under the condition (i)), we show that (2.10) holds if \( w \) and \( p \) satisfy the condition (i) in Theorem 1.1.

**Estimate of (2.1) for \( i = 3 \).** Similarly, we have the following pointwise estimate
\[
\sup_{\lambda > 0} \lambda |N_\lambda(\mathcal{F}^3 f)(x)|^{1/2} \leq \sum_{s < 0} \left( \sum_{k \in \mathbb{Z}} \left| \phi_k \ast \nu_{k+s} \ast f(x) \right|^2 \right)^{1/2}
\]
\[
:= \sum_{s < 0} H_s f(x).
\]
We are reduced to establish the estimate of \( \left\| H_s f \right\|_{L^p(w)} \) as sharp as possible so that we are able to sum up over all negative integers \( s \). By the Minkowski inequality, we get
\[
\left\| H_s f \right\|_{L^p(w)} = \left\| \left( \sum_{k \in \mathbb{Z}} \left| \sum_{l \in \mathbb{Z}} \phi_k \ast \nu_{k+s} \ast \Delta_{l-k} f \right|^2 \right)^{1/2} \right\|_{L^p(w)}
\]
\[
\leq \sum_{l \in \mathbb{Z}} \left\| \left( \sum_{k \in \mathbb{Z}} \left| \phi_k \ast \nu_{k+s} \ast \Delta_{l-k} f \right|^2 \right)^{1/2} \right\|_{L^p(w)}
\]
\[
:= \sum_{l \in \mathbb{Z}} \left\| H^l_s f \right\|_{L^p(w)},
\]
where the multipliers \( \{ \Delta_{l-k} \} \) were defined in Estimate of (2.1) for \( i = 2 \). We first prove a rapid decay estimate of \( \left\| H^l_s f \right\|_{L^2(\mathbb{R}^n)} \) for \( l \in \mathbb{Z} \) and \( s < 0 \). Since \( \text{supp} \hat{\phi}_k \subset \{ \xi : |2^k \xi| \leq 4 \} \), we have
\[
|\hat{\phi}_k(\xi)\nu_{k+s}(\xi)| \leq C \left\| \Omega \right\|_{L^q(S^{n-1})} 2^s \min\{2^k|\xi|, |2^k\xi|^{-\gamma}\}
\]
and
\[
|\hat{\phi}_k(\xi)\nu_{k+s}(\xi)\psi(2^{k-l}\xi)| \leq C \left\| \Omega \right\|_{L^q(S^{n-1})} 2^s \min\{2^l, 2^{-\gamma l}\}.
\]
Applying the above estimates and the Littlewood-Paley theory, we get
\[
\left\| H^l_s f \right\|_{L^2} \leq C \left\| \Omega \right\|_{L^q(S^{n-1})} 2^s \min\{2^l, 2^{-\gamma l}\} \left\| \left( \sum_{k \in \mathbb{Z}} |\Delta_{l-k} f|^2 \right)^{1/2} \right\|_{L^2}
\]
\[
\leq C \left\| \Omega \right\|_{L^q(S^{n-1})} 2^s \min\{2^l, 2^{-\gamma l}\} \left\| f \right\|_{L^2}.
\]
Now we give the weighted \( L^p \) norm of \( H^l_s f \) for \( l \in \mathbb{Z} \) and \( s < 0 \). By (2.14), (2.10) and the weighted Littlewood-Paley theory (see [23]), we get
\[
\left\| H^l_s f \right\|_{L^p(w)} \leq C_{p, w} \left\| \Omega \right\|_{L^q(S^{n-1})} \left\| \left( \sum_{k \in \mathbb{Z}} |\Delta_{l-k} f|^2 \right)^{1/2} \right\|_{L^p(w)}
\]
\[
\leq C_{p, w} \left\| \Omega \right\|_{L^q(S^{n-1})} \left\| f \right\|_{L^p(w)},
\]
(2.18)
whenever \( w \) and \( p \) satisfy the conditions (i) or (ii). Repeating the same argument in [10] or [40] and using Stein and Weiss’s interpolation theorem with change of measure between (2.17) and (2.18), we get for some \( \beta_1, \theta_1 \in (0, 1) \),
\[
\|H_{s}^{l}f\|_{L^p(w)} \leq C_{p,w}2^{\beta_1 s}2^{-\theta_1 |l|}||\Omega||_{L^p(S^{n-1})}\|f\|_{L^p(w)}.
\]

It follows from (2.15) and (2.16) that
\[
\|\sup_{\lambda > 0} \lambda |N_{\lambda}^{d}(\mathcal{D}^3 f)(x)|^{1/2}\|_{L^p} \leq \sum_{s < 0} \sum_{l \in \mathbb{Z}} \|H_{s}^{l}f\|_{L^p(w)} \leq C_{p,w} \sum_{s < 0} \sum_{l \in \mathbb{Z}} 2^{\beta_1 s}2^{-\theta_1 |l|}||\Omega||_{L^p(S^{n-1})}\|f\|_{L^p(w)} \leq C_{p,w} \|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(w)},
\]
whenever \( w \) and \( p \) satisfy the conditions (i) or (ii). We therefore finish the estimate of (2.1) in the case \( i = 3 \).

At the end of this section, let us give the proof of Proposition 2.1.

**Proof of Proposition 2.1** We first introduce some notations. For \( j \in \mathbb{Z} \) and \( \beta = (m_1, \cdots, m_n) \in \mathbb{Z}^n \), we denote the dyadic cube \( \prod_{k=1}^{n} (m_k2^j, (m_k + 1)2^j] \) in \( \mathbb{R}^n \) by \( Q_{\beta}^j \), and the set of all dyadic cubes with sidelength \( 2^j \) by \( D_j \). The conditional expectation of a locally integrable \( f \) with respect to \( D_j \) is given by
\[
E_j f(x) = \sum_{Q \in D_j} \frac{1}{|Q|} \int_{Q} f(y) dy \cdot \chi_Q(x)
\]
for all \( j \in \mathbb{Z} \). We also define the dyadic martingale difference operator \( D_j \) as \( D_j f(x) = E_j f(x) - E_{j-1} f(x) \). Thus for \( f \in L^p(\mathbb{R}^n) \), by the Lebesgue differential theorem we see that
\[
(2.19) \quad f(x) = - \sum_{j} D_j f(x) \quad \text{a. e.} \ x \in \mathbb{R}^n.
\]

We need to use a known extrapolation result:

**Lemma 2.2.** ([1 Corollary 1.2]) Let \( T \) be a sublinear operator such that \( T : L^1(w) \to L^{1,\infty}(w) \) for all \( w \in A_1 \). Then \( \|Tf\|_{L^p(w)} \leq C\|f\|_{L^p(w)} \) for \( 1 < p < \infty \) and \( w \in A_p \).

Note that \( N_\lambda \) is subadditive, then
\[
N_\lambda(\mathcal{D} f) \leq N_{\lambda/2}(\mathcal{D} f) + N_{\lambda/2}(\mathcal{E} f),
\]
where
\[
\mathcal{D} f = \{ \phi_k * f - E_k f \}_k \quad \text{and} \quad \mathcal{E} f = \{ E_k f \}_k.
\]
It has been proved in [22] p.8 that
\[
\|\sup_{\lambda > 0} \lambda \sqrt{N_\lambda(\mathcal{E} f)}\|_{L^p(w)} \leq C_p \|f\|_{L^p(w)}, \ 1 < p < \infty, \ w \in A_p.
\]
On the other hand, we observe that
\[
\sup_{\lambda > 0} \lambda \sqrt{N_\lambda(Df)} \leq \left( \sum_{k \in \mathbb{Z}} |\phi_k * f - \mathbb{E}_k f|^2 \right)^{1/2} := \mathcal{G}f.
\]

By Lemma 2.2, we just need to prove

\[
(2.20) \quad \sup_{\alpha > 0} \alpha w(\{x : \mathcal{G}f(x) > \alpha\}) \leq C\|f\|_{L^1(w)}, \quad w \in A_1.
\]

For any fixed \(\alpha > 0\), we perform the Calderón-Zygmund decomposition of \(f\) at height \(\alpha\) using dyadic cubes, then there exists \(\Lambda \subseteq \mathbb{Z} \times \mathbb{Z}^n\) such that the collection of dyadic cubes \(\{Q^j_{\beta}\}_{(j, \beta) \in \Lambda}\) are disjoint and the following hold:

(i) \(|\bigcup_{(j, \beta) \in \Lambda} Q^j_{\beta}| \leq \frac{1}{\alpha}\|f\|_{L^1(\mathbb{R}^n)}|;
(ii) |f(x)| \leq \alpha, \text{ if } x \not\in \bigcup_{(j, \beta) \in \Lambda} Q^j_{\beta};
(iii) \alpha < \frac{1}{|Q^j_{\beta}|} \int_{Q^j_{\beta}} |f(x)| dx \leq 2^n \alpha \text{ for each } (j, \beta) \in \Lambda.

We set
\[
g(x) = \begin{cases} f(x), & \text{if } x \not\in \bigcup_{(j, \beta) \in \Lambda} Q^j_{\beta}, \\ \frac{1}{|Q^j_{\beta}|} \int_{Q^j_{\beta}} f(y) dy, & \text{if } x \in Q^j_{\beta}, (j, \beta) \in \Lambda \end{cases}
\]
and
\[
b(x) = \sum_{(j, \beta) \in \Lambda} [f(x) - \mathbb{E}_j f(x)] \chi_{Q^j_{\beta}}(x) := \sum_{(j, \beta) \in \Lambda} b_{j, \beta}(x).
\]
Clearly, \(f = g + b\), \(|g|\|_{L^\infty(\mathbb{R}^n)} \leq 2\alpha\), \(|g|\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}\) and \(|b|\|_{L^1(\mathbb{R}^n)} \leq 2\|f\|_{L^1(\mathbb{R}^n)}\).

We accept the following fact for a moment

\[
(2.21) \quad \|\mathcal{G}f\|_{L^2(w)} \leq C\|f\|_{L^2(w)} , \quad w \in A_1,
\]
which will be proved later. By (2.21), the definition of \(g\) and (1.8), we have
\[
w(\{x : \mathcal{G}g(x) > \alpha\}) \leq \frac{C}{\alpha^2} \|\mathcal{G}g\|_{L^2(w)}^2 \leq \frac{C}{\alpha^2} \int_{\mathbb{R}^n} |g(x)|^2 w(x) dx \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |g(x)| w(x) dx
\]
\[
\leq \frac{C}{\alpha} \int_{(\bigcup Q^j_{\beta})^c} |f(x)| w(x) dx + \frac{C}{\alpha} \sum_{(j, \beta) \in \Lambda} \int_{Q^j_{\beta}} |f(y)| \frac{w(Q^j_{\beta})}{|Q^j_{\beta}|} dy
\]
\[
\leq \frac{C}{\alpha} \int_{(\bigcup Q^j_{\beta})^c} |f(x)| w(x) dx + \frac{C}{\alpha} \sum_{(j, \beta) \in \Lambda} \int_{Q^j_{\beta}} |f(y)| w(y) dy
\]
\[
\leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(x)| w(x) dx.
\]
Let $Q^j_{\beta}$ be the cube with center of $Q^j_{\beta}$ and 16 times sidelength of $Q^j_{\beta}$. Observe that $w \in A_1$ has the doubling property,

\[
w(\bigcup_{(j,\beta) \in \Lambda} \tilde{Q}^j_{\beta}) \leq C \sum_{(j,\beta) \in \Lambda} w(Q^j_{\beta}) \leq C \sum_{(j,\beta) \in \Lambda} \frac{w(Q^j_{\beta})}{|Q^j_{\beta}|} |Q^j_{\beta}|
\]

\[
\leq C \sum_{(j,\beta) \in \Lambda} \inf_{x \in Q^j_{\beta}} w(x) \frac{1}{\alpha} \int_{Q^j_{\beta}} |f(y)|dy
\]

\[
\leq C \sum_{(j,\beta) \in \Lambda} \frac{1}{\alpha} \int_{Q^j_{\beta}} |f(y)|w(y)dy
\]

\[
\leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(y)|w(y)dy.
\]

Notice that $E_k b_{j,\beta}(x) = 0$ for $x \not\in Q^j_{\beta}$ (see [21, p.6726]), then

\[
\alpha w(\{(\bigcup_{Q^j_{\beta}} \setminus Q^j_{\beta})^c : \mathfrak{G} b(x) > \alpha\}) \leq \int (\bigcup_{Q^j_{\beta}} \setminus Q^j_{\beta}) \sum_{k \in \mathbb{Z}} |\phi_k * b(x) - E_k b(x)|w(x)dx
\]

\[
\leq \sum_{(j,\beta) \in \Lambda} \sum_{k \in \mathbb{Z}} \int (Q^j_{\beta})^c |\phi_k * b_{j,\beta}(x)|w(x)dx.
\]

Recall that $\phi$ is a Schwartz function, so

\[
|\phi_k(x)| \leq \frac{C2^{-kn}}{(1 + |2^{-k}x|)^{n+1}} \quad \text{and} \quad |\partial_x \phi_k(x)| \leq \frac{C2^{-k(n+1)}}{(1 + |2^{-k}x|)^{n+2}}.
\]

For $k \leq j$, by [21], we obtain

\[
\int_{(Q^j_{\beta})^c} |\phi_k * b_{j,\beta}(x)|w(x)dx \leq \int_{Q^j_{\beta}} |b_{j,\beta}(y)| \int_{(Q^j_{\beta})^c} |\phi_k(x-y)|w(x)dx dy
\]

\[
\leq \int_{Q^j_{\beta}} |b_{j,\beta}(y)| \int_{|x-y| \geq 2^j} 2^{n-k} |2^{-k}(x-y)|^{n+1} w(x)dx dy
\]

\[
\leq C2^{k-j} \int_{Q^j_{\beta}} |b_{j,\beta}(y)| M(w)(y) dy
\]

\[
\leq C2^{k-j} \left[ \int_{Q^j_{\beta}} |f(y)|w(y)dy + \int_{Q^j_{\beta}} |f(z)| \frac{w(Q^j_{\beta})}{|Q^j_{\beta}|} dz \right]
\]

\[
\leq C2^{k-j} \int_{Q^j_{\beta}} |f(y)|w(y)dy.
\]
For $k \geq j$, we use the fact $\int_{Q^j_\beta} b_{j,\beta}(y)dy = 0$. Let $z^j_\beta$ be the center of $Q^j_\beta$, then
\[
\int_{(Q^j_\beta)^c} |\phi_k * b_{j,\beta}(x)||w(x)|dx \leq \int_{Q^j_\beta} |b_{j,\beta}(y)| \int_{(Q^j_\beta)^c} |\phi_k(x) - \phi_k(x - z^j_\beta)|w(x)dx dy \\
\leq 2^{j-k} \int_{Q^j_\beta} |b_{j,\beta}(y)| \int_{\mathbb{R}^n} \frac{2^{-kn}}{(1 + 2^{-k}|x - y|)^{n+2}} w(x)dx dy \\
\leq C \cdot 2^{j-k} \int_{Q^j_\beta} |f(y)|w(y)dy \\
\leq C \cdot 2^{j-k} \int_{Q^j_\beta} |f(y)|w(y)dy.
\]

Combining the above estimates, we obtain
\[
aw(|\bigcup_{(j,\beta) \in A} Q^j_\beta : \mathcal{G}b(x) > \alpha|) \leq C \sum_{(j,\beta) \in A} \sum_{k \in \mathbb{Z}} 2^{-|j-k|} \int_{Q^j_\beta} |f(y)|w(y)dy \leq C\|f\|_{L^1(w)}.
\]

Now we turn to the proof of (2.21), which can be showed in a similar way as Lemma 3.2 in [21]. Estimate (2.21) is a consequence of the following fact: there exists a $\theta > 0$ such that
\[
(2.22) \quad \|\phi_{k+j} \ast \mathcal{D}_j f - \mathcal{E}_{k+j} \mathcal{D}_j f\|_{L^2(w)} \leq 2^{-\theta|k|}\|\mathcal{D}_j f\|_{L^2(w)}.
\]

Indeed, using (2.19), (2.22) and the Minkowski inequality,
\[
\|\mathcal{G} f\|_{L^2(w)} \leq \left( \sum_k \left( \sum_j \|\phi_k \ast \mathcal{D}_j f(x) - \mathcal{E}_k \mathcal{D}_j f\|_{L^2(w)}^2 \right)^{1/2} \right)^{1/2} \leq \left( \sum_k \left( \sum_j 2^{-\theta|k-j|}\|\mathcal{D}_j f\|_{L^2(w)}^2 \right)^{1/2} \right)^{1/2} \leq C_\theta \left( \sum_j \|\mathcal{D}_j f\|_{L^2(w)}^2 \right)^{1/2} \leq C_\theta \|f\|_{L^2(w)},
\]

the last inequality follows from the fact that the dyadic martingale square function is bounded on $L^2(w)$, see for instance [33 Theorem 3.6] or [24].

Finally we prove (2.22). When $k \geq 0$, $\mathcal{E}_{k+j} \mathcal{D}_j f = 0$. In [21 p.6722], Jones et al have proved that $|\phi_{k+j} \ast \mathcal{D}_j f| \leq C 2^{-k} M(\mathcal{D}_j f)$. The weighted $L^p$-boundedness of the Hardy-Littlewood maximal function implies
\[
(2.23) \quad \|\phi_{k+j} \ast \mathcal{D}_j f - \mathcal{E}_{k+j} \mathcal{D}_j f\|_{L^2(w)} \leq 2^{-k}\|\mathcal{D}_j f\|_{L^2(w)}.
\]

When $k < 0$, $\mathcal{E}_{k+j} \mathcal{D}_j f = \mathcal{D}_j f$. Thus
\[
\phi_{k+j} \ast \mathcal{D}_j f(x) - \mathcal{E}_{k+j} \mathcal{D}_j f(x) = \int_{|y| \leq 2^{k+j}} \phi_{k+j}(y) [\mathcal{D}_j f(x - y) - \mathcal{D}_j f(x)]dy \\
+ \sum_{d \geq 1} \int_{E_{k+j,d}} \phi_{k+j}(y) [\mathcal{D}_j f(x - y) - \mathcal{D}_j f(x)]dy \\
:= I_0(x) + \sum_{d \geq 1} I_d(x),
\]

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where \( E_{k,j,d} = \{ y : 2^{k+j+d-1} < |y| \leq 2^{k+j+d+1} \} \). For a fixed integer \( k \leq -8 \), we estimate \( \| I_d \|_{L^2(w)} \) for the cases \( d > |k|/2 \) and \( 0 \leq d \leq |k|/2 \), respectively.

For the case \( d > |k|/2 \), \( y \in E_{k,j,d} \) and \( N \geq n + 1 \), we have \( |\phi_{k+j}(y)| \leq C2^{-Nd}2^{-(k+j)} \), since \( \phi \) is a Schwartz function. Using the Minkowski inequality,

\[
\| I_d \|_{L^2(w)} \leq C2^{-Nd-n(k+j)} \int_{\mathbb{R}^n} \left( \int_{E_{k,j,d}} |D_j f(x-y) - D_j f(x)|^2 w(x)dx \right)^{1/2} dy 
\leq C2^{-Nd-n(k+j)} \int_{\mathbb{R}^n} \left[ \int_{E_{k,j,d}} (|D_j f(x-y)|^2 + |D_j f(x)|^2) w(x)dx \right]^{1/2} dy 
\leq C2^{-Nd-n(k+j)} \int_{\mathbb{R}^n} \left[ \int_{E_{k,j,d}} |D_j f(x-y)|^2 w(x)dx \right]^{1/2} dy + \frac{C}{2^d} \| D_j f \|_{L^2(w)}.
\]

Let \( Q \) be the dyadic cube containing \( Q \) with sidelength \( 2l(Q) \). We write

\[
D_j f = \sum_{Q \in \mathcal{D}_{-1}} a_Q \chi_Q, \quad \text{where} \quad a_Q = \frac{1}{|Q|} \int_Q f(z)dz - \frac{1}{|Q|} \int_Q f(z)dz.
\]

Note that \( Q^k_\beta \cap Q^k_\alpha = \emptyset \) when \( \beta \neq \alpha \). By a trivial calculation,

\[
\| D_j f \|_{L^2(w)} = \left( \sum_{Q \in \mathcal{D}_{-1}} a_Q^2 w(Q) \right)^{1/2}.
\]

The Cauchy-Schwarz inequality and the properties of \( A_1 \) weight imply

\[
\int_{E_{k,j,d}} \left[ \int_{\mathbb{R}^n} |D_j f(x-y)|^2 w(x)dx \right]^{1/2} dy 
\leq C \int_{E_{k,j,d}} \left( \sum_{Q \in \mathcal{D}_{-1}} a_Q^2 \int_{\mathbb{R}^n} \chi_Q(x-y)w(x)dx \right)^{1/2} dy 
\leq C|E_{k,j,d}|^{1/2} \left( \sum_{Q \in \mathcal{D}_{-1}} a_Q^2 \int_{E_{k,j,d}} w(x+y)dydx \right)^{1/2} 
\leq C|E_{k,j,d}|^{1/2} 2^{(k+j+d+1)n/2} \left( \sum_{Q \in \mathcal{D}_{-1}} a_Q^2 \int_Q M(w)(x)dx \right)^{1/2} 
\leq C|E_{k,j,d}| \left( \sum_{Q \in \mathcal{D}_{-1}} a_Q^2 w(Q) \right)^{1/2} 
\leq C|E_{k,j,d}| \left\| D_j f \right\|_{L^2(w)}.
\]

Then, we conclude that

\[
(2.24) \quad \| I_d \|_{L^2(w)} \leq C(2^{-Nd-n(k+j)}|E_{k,j,d}| + 2^{-d}) \| D_j f \|_{L^2(w)} \leq C2^{-d} \| D_j f \|_{L^2(w)}.
\]

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For each $0 \leq d \leq |k|/2$ and $Q^{j-1}_\beta$, we have $\mathcal{D}_j f(x - y) = \mathcal{D}_j f(x)$ for $y \in E_{k,j,d}$ and $x \in Q^{j-1}_\beta$ such that $\text{dist}(x, (Q^{j-1}_\beta)^c) \geq 2^{j+d+k+1}$. By (1.14), there is a $\delta > 0$ such that

$$
\|I_d\|_{L^2(w)}^2 \leq C \sum_{\beta} \sup_{x \in Q^{j-1}_\beta} |I_d(x)|^2 w(\{x \in Q^{j-1}_\beta : \text{dist}(x, (Q^{j-1}_\beta)^c) \leq 2^{j+d+k+1}\})
$$

$$
\leq C \sum_{\beta} \frac{|\{x \in Q^{j-1}_\beta : \text{dist}(x, (Q^{j-1}_\beta)^c) \leq 2^{j+d+k+1}\}|^\delta}{|Q^{-1}_{\beta}|^\delta} w(Q^{j-1}_\beta) \sup_{x \in Q^{j-1}_\beta} |I_d(x)|^2
$$

$$
\leq C2^{-\delta|k|/2} \sum_{\beta} w(Q^{j-1}_\beta) \sup_{x \in Q^{j-1}_\beta} |I_d(x)|^2.
$$

For $x \in Q^{j-1}_\beta$, $|I_d(x)| \leq C \sup_{y \in B(z^{j-1}_\beta, 2^{j+1})} |\mathcal{D}_j f(y)|$. Note that there are at most $9^n Q^{j-1}_\alpha$’s jointing with $B(z^{j-1}_\beta, 2^{j+1})$. Therefore there exists a multi-index $\alpha_\beta$ such that

$$
\sup_{x \in B(z^{j-1}_\beta, 2^{j+1})} \|\mathcal{D}_j f(x)\| = |a_{Q^{j-1}_\alpha}| = |\mathcal{D}_j f(x)| \chi_{Q^{j-1}_\alpha}(x).
$$

Thus we obtain

$$
\|I_d\|_{L^2(w)}^2 \leq C2^{-\delta|k|/2} \sum_{\beta} w(Q^{j-1}_\beta) |a_{Q^{j-1}_\alpha}|^2
$$

$$
\leq C2^{-\delta|k|/2} \frac{w(Q^{j-1}_\beta)}{w(B(z^{j-1}_\beta, 2^{j+1}))} \int_{B(z^{j-1}_\beta, 2^{j+1})} |\mathcal{D}_j f(x)|^2 \chi_{Q^{j-1}_\alpha}(x) w(x) dx
$$

$$
\leq C2^{-\delta|k|/2} \|\mathcal{D}_j f\|_{L^2(w)}^2.
$$

(2.25)

For $k \leq -8$, by estimates (2.24) and (2.25), we have

$$
|\phi_{k+j} \ast \mathcal{D}_j f - \mathcal{E}_{k+j} \mathcal{D}_j f|_{L^2(w)} \leq \| \sum_{0 \leq d \leq |k|/2} I_d \|_{L^2(w)} + \| \sum_{d>|k|/2} I_d \|_{L^2(w)}
$$

$$
\leq C|k|2^{-\delta|k|/4} \|\mathcal{D}_j f\|_{L^2(w)} + C \sum_{d>|k|/2} \frac{1}{2^d} \|\mathcal{D}_j f\|_{L^2(w)}
$$

(2.26)

for some $\theta > 0$. Finally (2.23) and (2.26) together imply the desired estimate (2.22) and we finish the proof of Proposition 2.1.

3 Proof of Theorem [1,1] (II)

In this section we will finish the proof of (1.15). For $t \in [1, 2]$, we define $\nu_{0,t}$ as

$$
\nu_{0,t}(x) = \frac{\Omega(|x|)}{|x|^n} \chi_{\{t \leq |x| \leq 2\}}(x)
$$
and \( \nu_{j,t}(x) = 2^{-jn}\nu_{0,t}(2^{-j}x) \) for \( j \in \mathbb{Z} \). Observe that \( V_{2,j}(\mathcal{T}f)(x) \) is just the strong 2-variation function of the family \( \{\nu_{j,t} * f(x)\}_{t \in [1,2]} \), the Minkowski inequality implies

\[
S_2(\mathcal{T}f)(x) = \left( \sum_{j \in \mathbb{Z}} |V_{2,j}(\mathcal{T}f)(x)|^2 \right)^\frac{1}{2} = \left( \sum_{j \in \mathbb{Z}} \|\{\nu_{j,t} * f(x)\}_{t \in [1,2]}\|_{V_2}^2 \right)^\frac{1}{2} \\
\leq \sum_{k \in \mathbb{Z}} \left( \sum_{j \in \mathbb{Z}} \|\nu_{j,t} * (\Delta_{k-j}^2 f)(x)\|_{t \in [1,2]}^2 \right)^\frac{1}{2} \\
:= \sum_{k \in \mathbb{Z}} S_{2,k}(\mathcal{T}f)(x). \tag{3.1}
\]

By (3.1), to get (1.15) it remains to show that if \( p \) and \( w \) satisfy the conditions (i) or (ii) in Theorem 1.1 then there exists an \( \varepsilon > 0 \) such that for all \( k \in \mathbb{Z} \)

\[
\|S_{2,k}(\mathcal{T}f)\|_{L^p(w)} \leq C_{p,w} 2^{-\varepsilon|k|} \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^p(w)}. \tag{3.2}
\]

Note that the authors of [8] proved that for given \( 1 < p < \infty \), there exists a constant \( \delta \in (0,1) \) such that for \( k \in \mathbb{Z} \),

\[
\|S_{2,k}(\mathcal{T}f)\|_{L^p} \leq C_p 2^{-\delta|k|} \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^p}. \tag{3.3}
\]

Thus, if we can prove that when \( p \) and \( w \) satisfy the conditions (i) or (ii) in Theorem 1.1 for all \( k \in \mathbb{Z} \)

\[
\|S_{2,k}(\mathcal{T}f)\|_{L^p(w)} \leq C_{p,w} \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^p(w)}, \tag{3.4}
\]

then by applying Stein and Weiss’s interpolation theorem with change of measure between (3.3) and (3.4), there is a constant \( 0 < \delta' < 1 \) such that

\[
\|S_{2,k}(\mathcal{T}f)\|_{L^p(w)} \leq C_{p,w} 2^{-\delta'|k|} \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^p(w)}. \]

In other words, we get (3.2) and complete the proof of (1.15). Therefore, we reduce the proof of (1.15) to showing (3.4). The estimate of (3.4) is dealt with differently according to whether \( q \geq 2 \) or \( q < 2 \).

### 3.1 The estimate of (3.4) for \( q \geq 2 \)
We have the following observation:

\[
S_{2,k}(Tf)(x) = \left( \sum_{j \in \mathbb{Z}} \sup_{t_1 < \cdots < t_N, [t_i, t_{i+1}] \subset [1, 2]} \sum_{l=1}^{N-1} \left| \nu_{j,t_l} \ast \Delta_{k-j}^2 f(x) - \nu_{j,t_{l+1}} \ast \Delta_{k-j}^2 f(x) \right|^2 \right)^{1/2}
\]

\[
= \left( \sum_{j \in \mathbb{Z}} \sup_{t_1 < \cdots < t_N, [t_i, t_{i+1}] \subset [1, 2]} \sum_{l=1}^{N-1} \left| \int_{2^{l}t_{l+1} \leq |y| \leq 2^{l+1} t_{l+1}} \frac{\Omega(y)}{|y|^n} \Delta_{k-j}^2 f(x - y) dy \right|^2 \right)^{1/2}
\]

\[
= \left( \sum_{j \in \mathbb{Z}} \sup_{t_1 < \cdots < t_N, [t_i, t_{i+1}] \subset [1, 2]} \sum_{l=1}^{N-1} \left| \int_{2^{l}t_{l+1} \leq |y| \leq 2^{l+1} t_{l+1}} \frac{\Omega(y)}{|y|^n} \Delta_{k-j}^2 f(x - y) dy \right|^2 \right)^{1/2}
\]

\[
\leq \left( \sum_{j \in \mathbb{Z}} \sup_{t_1 < \cdots < t_N, [t_i, t_{i+1}] \subset [1, 2]} \left( \int_{2^{l}t_{l+1} \leq |y| \leq 2^{l+1} t_{l+1}} \frac{\Omega(y)}{|y|^n} \left| \Delta_{k-j}^2 f(x - y) \right| dy \right)^2 \right)^{1/2}
\]

\[
\leq \left( \sum_{j \in \mathbb{Z}} \left( \frac{1}{2^n} \int_{|y| < 2^{l+1}} \frac{\Omega(y)}{|y|^n} \left| \Delta_{k-j}^2 f(x - y) \right| dy \right)^2 \right)^{1/2}
\]

\[
\leq C \left( \sum_{j \in \mathbb{Z}} \left| M_{\Omega}(\Delta_{k-j}^2 f)(x) \right|^2 \right)^{1/2},
\]

where \(M_{\Omega}\) is the rough maximal operator defined in Section 2. Therefore, we get

\[
(3.5) \quad S_{2,k}(Tf)(x) \leq C \left( \sum_{j \in \mathbb{Z}} \left| M_{\Omega}(\Delta_{k-j}^2 f)(x) \right|^2 \right)^{1/2}.
\]

To continue the proof, we need the following proposition which will be proved later.
Proposition 3.1. For \( q \geq 2 \), if \( p \) and \( w \) satisfy the conditions (i) or (ii) in Theorem 1.1 then

\[
(3.6) \quad \left\| \left( \sum_{j \in \mathbb{Z}} |M_{\Omega} f_j|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C_{p,w}(1 + \|\Omega\|_{L^q(S^{n-1})}) \left\| \left( \sum_{j \in \mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{L^p(w)}.
\]

Using (3.5), (3.6) and the weighted Littlewood-Paley theory (see for instance [23]), we get for \( q \geq 2 \),

\[
\|S_{2,k}(T f)\|_{L^p(w)} \leq C_{p,w}\|\Omega\|_{L^q(S^{n-1})} \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_{k-j} f_j|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C_{p,w}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p(w)},
\]

whenever \( w \) and \( p \) satisfy the conditions (i) or (ii).

Now we prove Proposition 3.1. Write

\[
|\Omega(x')| = |\Omega(x)| - \frac{1}{\omega_{n-1}} \int_{S^{n-1}} |\Omega(y')|d\sigma(y') + \frac{1}{\omega_{n-1}} \int_{S^{n-1}} |\Omega(y')|d\sigma(y')
\]

\[
:= \Omega_0(x') + C(\Omega, n),
\]

where \( \omega_{n-1} \) denotes the area of \( S^{n-1} \). It is easy to check that

\[
(3.7) \quad M_{\Omega} f(x) \leq Cg_{\Omega_0}(|f|)(x) + CM f(x),
\]

where the operator \( g_{\Omega_0} \) is defined by

\[
g_{\Omega_0}(f)(x) = \left( \sum_k |T_{k,\Omega_0} f(x)|^2 \right)^{1/2} \quad \text{with} \quad T_{k,\Omega_0} f(x) = \int_{2^k < |y| < 2^{k+1}} \frac{\Omega_0(y')}{|y|^n} f(x - y)dy
\]

and \( M \) denotes the Hardy-Littlewood maximal operator. By the properties of \( A_p \) weights (see [13]), if \( w \) and \( p \) satisfy the conditions (i) or (ii) in Theorem 1.1 then \( w \in A_p \). Thus, by (3.7) and using the weighted norm inequality of the operator \( M \) for vector-valued functions (see II, Theorem 3.1)), to get (3.6) it suffices to prove that if \( q \geq 2 \) and \( w, p \) satisfy the conditions (i) or (ii) in Theorem 1.1 then

\[
(3.8) \quad \left\| \left( \sum_{j \in \mathbb{Z}} |g_{\Omega_0}(f_j)|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C_{p,w}\|\Omega_0\|_{L^q(S^{n-1})} \left\| \left( \sum_{j \in \mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{L^p(w)}.
\]

Consider the linear operators \( T_{\varepsilon,\Omega_0} f = \sum_k \varepsilon_k T_{k,\Omega_0} f \), where \( \varepsilon = \{\varepsilon_k\} \) is a sequence with \( \varepsilon_k = \pm 1 \). By the usual argument with Rademacher function [39], to show (3.8) it needs only to prove that if \( q \geq 2 \) and \( w, p \) satisfy the conditions (i) or (ii) in Theorem 1.1 then

\[
(3.9) \quad \left\| \left( \sum_{j \in \mathbb{Z}} |T_{\varepsilon,\Omega_0} f_j|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C_{p,w}\|\Omega_0\|_{L^q(S^{n-1})} \left\| \left( \sum_{j \in \mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{L^p(w)}
\]

where \( C_{p,w} \) is independent of \( \varepsilon \). We need the following lemma, which is only an application of [16] Lemma 9.5.4.
Lemma 3.2. (a) Let \( q' \leq 2 < p < \infty \) and \( w \in A_{p/q'} \). Then there exists a constant \( C_1 = C_1(n,p, [w]_{A_{p/q'}}) \) such that for every nonnegative function \( g \) in \( L^{(p/2)'}(w) \), there is a function \( G(g) \) such that

\begin{itemize}
  \item[(i)] \( g \leq G(g) \)
  \item[(ii)] \( \|G(g)\|_{L^{(p/2)'}(w)} \leq 2\|g\|_{L^{(p/2)'}(w)} \)
  \item[(iii)] \( [G(g)]_{A_{2/q'}} \leq C_1 \).
\end{itemize}

Moreover, both constants \( C_1(n,p,B) \) and \( C_2(n,p,B) \) increase as \( B \) increases.

By the duality, we only need to prove (3.9) when \( w \) and \( p \) satisfy the condition (i) in Theorem 1.1. We first prove (3.9) for \( q' < p < \infty \).

Case 1. \( q' < p = 2 \). Note that \( T_{\epsilon, \Omega_0} \) is weighted \( L^2 \) bounded for \( A_{2/q'} \) weight uniformly in \( \epsilon \) (see for instance [10] or [40]), so (3.9) is an obvious consequence of the fact above since \( w \in A_{2/q'} \) in this case.

Case 2. \( q' \leq 2 < p < \infty \). Note that

\[
\left\| \left( \sum_{j \in \mathbb{Z}} |T_{\epsilon, \Omega_0}f_j|^2 \right)^{1/2} \right\|_{L^p(w)} = \sup_{\|g\|_{L^{(p/2)'}(w)} \leq 1} \left| \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} |T_{\epsilon, \Omega_0}f_j|(x)^2 g(x)w(x) \, dx \right|^{1/2}.
\]

Since \( w \in A_{p/q'} \) and \( g \in L^{(p/2)'}(w) \), by Lemma 3.2 (a), there exists a function \( G(|g|) \) such that

\[
\left\| \left( \sum_{j \in \mathbb{Z}} |T_{\epsilon, \Omega_0}f_j|^2 \right)^{1/2} \right\|_{L^p(w)} \leq \sup_{\|g\|_{L^{(p/2)'}(w)} \leq 1} \left| \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |T_{\epsilon, \Omega_0}f_j|(x)^2 G(|g|)(x)w(x) \, dx \right|^{1/2}.
\]

Note that \( G(|g|)w \in A_{2/q'} \), hence by the weighted uniform \( L^2 \) boundedness of \( T_{\epsilon, \Omega_0} \), Hölder’s inequality and Lemma 3.2 (a), we get

\[
\left\| \left( \sum_{j \in \mathbb{Z}} |T_{\epsilon, \Omega_0}f_j|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C\|\Omega_0\|_{L^q(S^{n-1})} \left( \left\| \sum_{j \in \mathbb{Z}} |f_j|^2 \right\|_{L^{p/2}(w)} \right)^{1/2} \left\| G(|g|) \right\|_{L^{(p/2)'}(w)}^{1/2}
\]

\[
\leq C\|\Omega_0\|_{L^q(S^{n-1})} \left( \left\| \sum_{j \in \mathbb{Z}} |f_j|^2 \right\|_{L^{p/2}(w)} \right)^{1/2} \left\| G(|g|) \right\|_{L^{(p/2)'}(w)}^{1/2}
\]

\[
\leq C\|\Omega_0\|_{L^q(S^{n-1})} \left( \left\| \sum_{j \in \mathbb{Z}} |f_j|^2 \right\|_{L^{p/2}(w)} \right)^{1/2} \left\| G(|g|) \right\|_{L^{(p/2)'}(w)}^{1/2}
\]

\[
\leq C\|\Omega_0\|_{L^q(S^{n-1})} \left( \left\| \sum_{j \in \mathbb{Z}} |f_j|^2 \right\|_{L^{p/2}(w)} \right)^{1/2} \left\| G(|g|) \right\|_{L^{(p/2)'}(w)}^{1/2}
\]

\[
\leq C\|\Omega_0\|_{L^q(S^{n-1})} \left( \left\| \sum_{j \in \mathbb{Z}} |f_j|^2 \right\|_{L^{p/2}(w)} \right)^{1/2} \left\| G(|g|) \right\|_{L^{(p/2)'}(w)}^{1/2}
\]

(3.10)
Case 3. $q' < p < 2$. Since $w \in A_{p/q'}$ and $(\sum_{j \in \mathbb{Z}} |f_j|^2)^{1/2} \in L^p(w)$, thus $h = (\sum_{j \in \mathbb{Z}} |f_j|^2)^{\frac{2/p}{2}} \in L^{p/p}(w)$. Let $H(h)$ be the function in Lemma 3.2 (b). Then applying Hölder’s inequality, we have
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |T_{\epsilon, \Omega_0} f_j|^2 \right)^{1/2} \right\|_{L^p(w)} = \left\| \sum_{j \in \mathbb{Z}} |T_{\epsilon, \Omega_0} f_j|^2 h^{-1/2} H(h) \right\|_{L^{2/p}(w)}^{1/2} 
\leq \left\| \sum_{j \in \mathbb{Z}} |T_{\epsilon, \Omega_0} f_j|^2 h^{-1/2} \right\|_{L^1(w)}^{1/2} \left\| H(h) \right\|_{L^{2/p}(w)}^{1/2}.
\]
Since $H(h)^{-1} w \in A_{2/q'}$, using again the weighted $L^2$ uniform estimate of $T_{\epsilon, \Omega_0}$ and Lemma 3.2 (b), we have
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |T_{\epsilon, \Omega_0} f_j|^2 \right)^{1/2} \right\|_{L^p(w)} 
\leq C \left\| \Omega_0 \right\|_{L^q(S^{n-1})} \left( \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |f_j(x)|^2 H(h(x))^{-1} w(x) \, dx \right)^{1/2} \left\| \left( \sum_{i \in \mathbb{Z}} |f_i|^2 \right)^{1-\frac{2}{p}} \right\|_{L^{2/p}(w)}^{1/2} 
\leq C \left\| \Omega_0 \right\|_{L^q(S^{n-1})} \left( \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} |f_j|^2 \left( \sum_{i \in \mathbb{Z}} |f_i|^2 \right)^{1-\frac{2}{p}} w(x) \, dx \right)^{1/2} \left\| \left( \sum_{i \in \mathbb{Z}} |f_i|^2 \right)^{1/2} \right\|_{L^p(w)}^{1-\frac{2}{p}}.
\]
Combining (3.10) and (3.11), we show (3.9) for $w \in A_{p/q'}$ and $q' < p < \infty$.

Finally, the estimate of (3.9) for the endpoint case $p = q'$, $p \neq 1$ can follow by interpolating between (3.9) for some $p_0 > q'$ and the unweighted estimate for some $p_1 < q'$ (see for instance [10]) by using the properties of $A_p$ weight.

Remark 3.3. The main idea behind the proof of estimate (3.9) is Rubio de Francia’s extrapolation theorem [38, Theorem 3]. In particular, the result in the case $q > 2$ and $p > q'$ is just a consequence of Rubio de Francia’s extrapolation theorem by the fact that $T_{\epsilon, \Omega_0}$ is bounded on $L^2(w)$ with $w \in A_{2/q'}$ (see [10] or [40]). However, other cases $q = 2$ or $p = q'$ cannot be directly deduced from Theorem 3 in [38]. Thus we prefer having given a detailed proof of estimate (3.9) above.

3.2 The estimate of (3.1) for $q < 2$

For $q < 2$, we first give the estimate of (3.1) for $p$ and $w$ satisfy the condition (ii) in Theorem 1.1 that is, $q < 2$, $1 < p \leq q$ and $w^{-\frac{1}{p-1}} \in A_{p/q'}$. We claim that (3.6) holds also for $q < 2$, $p$ and $w$ satisfy the condition (ii) in Theorem 1.1 In fact, by the weighted $L^p$ boundedness of $M$ (see [10] or [27]), we have
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |M_{\epsilon} f_j|^p \right)^{1/p} \right\|_{L^p(w)} \leq C_{p,w} \left\| \Omega \right\|_{L^q(S^{n-1})} \left\| \left( \sum_{j \in \mathbb{Z}} |f_j|^p \right)^{1/p} \right\|_{L^p(w)}
\]
and
\[ \left\| \sup_{j \in \mathbb{Z}} |M_{\Omega} f_j| \right\|_{L^p(w)} \leq C_{p,w} \| \Omega \|_{L^q(S^{n-1})} \left\| \sup_{j \in \mathbb{Z}} |f_j| \right\|_{L^p(w)} \]
for \(1 < p \leq q\) and \(w^{-\frac{1}{(p-1)}} \in A_{p'/q'}\). Interpolating between the above two estimates, we get
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |M_{\Omega} f_j|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C_{p,w} \| \Omega \|_{L^q(S^{n-1})} \left\| \left( \sum_{j \in \mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{L^p(w)}
\]
for \(q < 2, 1 < p \leq q\) and \(w^{-\frac{1}{(p-1)}} \in A_{p'/q'}\). Thus, by (3.5), (3.6) and the weighted Littlewood-Paley theory (see [23]), we have
\[
\| S_{2,k}(Tf) \|_{L^p(w)} \leq \left\| \left( \sum_{j \in \mathbb{Z}} |M_{\Omega} (\Delta^2_{k-j} f)|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C_{p,w} \| \Omega \|_{L^q(S^{n-1})} \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta^2_{k-j} f|^2 \right)^{1/2} \right\|_{L^p(w)}
\]
whenever \(q < 2, 1 < p \leq q\) and \(w^{-\frac{1}{(p-1)}} \in A_{p'/q'}\).

Secondly, for \(q < 2\), we estimate (3.4) for \(p\) and \(w\) satisfying the condition (i) in Theorem 1.1 that is, \(q < 2, q' \leq p < \infty\) and \(w \in A_{p'/q'}\). Using the idea of proving Lemma B.1 in [22], we may show that for \(x \in \mathbb{R}^n\),
\[
\| a_{\ell} \|_{\tilde{Y}_2} \leq 8 \| a_{\ell} \|_{X} \left\| \frac{d}{dt} a_{\ell} \right\|_{X}^{1/2},
\]
where \(X = L^2([1, 2], \frac{dt}{t})\) and \(a_{\ell} = T_{j,\ell} \Delta^2_{k-j} f(x)\) with \(T_{j,\ell} h(x) = \nu_{j,\ell} * h(x)\). (3.13) means that
\[
|S_{2,k}(Tf)(x)|^2 \leq 8 \sum_{j \in \mathbb{Z}} \left( \int_{1}^{2} |T_{j,\ell} \Delta^2_{k-j} f|^2 \frac{dt}{l} \right)^{1/2} \left( \int_{1}^{2} \left\| \frac{d}{dt} T_{j,\ell} \Delta^2_{k-j} f \right\|^2 \frac{dt}{l} \right)^{1/2}.
\]
By the Cauchy-Schwarz inequality, we have for any \(1 < p \leq \infty\),
\[
\left\| S_{2,k}(Tf) \right\|_{L^p(w)}^2 \\
\leq \left\| \left( \sum_{j \in \mathbb{Z}} \int_{1}^{2} |T_{j,\ell} \Delta^2_{k-j} f|^2 \frac{dt}{l} \right)^{1/2} \right\|_{L^p(w)} \left\| \left( \sum_{j \in \mathbb{Z}} \int_{1}^{2} \left\| \frac{d}{dt} T_{j,\ell} \Delta^2_{k-j} f \right\|^2 \frac{dt}{l} \right)^{1/2} \right\|_{L^p(w)}
\]:= \| I_{1,k} f \|_{L^p(w)} \cdot \| I_{2,k} f \|_{L^p(w)}.
\]
We now estimate \(\| I_{1,k} f \|_{L^p(w)}\) and \(\| I_{2,k} f \|_{L^p(w)}\), respectively. For \(\| I_{1,k} f \|_{L^p(w)}\), since \(p \geq q' > 2\), by the Minkowski inequality, we get
\[
\| I_{1,k} f \|_{L^p(w)} \leq \left( \int_{1}^{2} \left\| \left( \sum_{j \in \mathbb{Z}} |T_{j,\ell} \Delta^2_{k-j} f|^2 \right)^{1/2} \right\|_{L^p(w)}^2 \frac{dt}{l} \right)^{1/2}.
\]
By [10], for $q' \leq p < \infty$, $p \neq 1$ and $w \in A_{p/q'}$

$$\left\| \sup_{j \in \mathbb{Z}, t \in [1, 2]} |T_{j,t} f| \right\|_{L^p(w)} \leq C \left\| M_{\Omega} f \right\|_{L^p(w)} \leq C_{p,w} \left\| \Omega \right\|_{L^q(S^{n-1})} \left\| f \right\|_{L^p(w)}.$$ 

Then similarly to the proof of (2.5) and applying the weighted Littlewood-Paley theory with $w \in A_p$ [23], we get

$$\left\| I_{1,k} f \right\|_{L^p(w)} \leq C_{p,w} \left\| \Omega \right\|_{L^q(S^{n-1})} \left(\int_{1}^{2} \left\| \sum_{j \in \mathbb{Z}} |\Delta_{k-j}^2 f| \right\|_{L^{p/w}(w)}^{1/2} \frac{dt}{t} \right)^{1/2} \leq C_{p,w} \left\| \Omega \right\|_{L^q(S^{n-1})} \left\| f \right\|_{L^p(w)}.$$ 

Now we consider $\left\| I_{2,k} f \right\|_{L^p(w)}$. For any given Schwartz function $h$, by the spherical coordinate transformation, a trivial calculation shows

$$\frac{d}{dt} T_{j,t} h(x) = \frac{d}{dt} \left[ T_{j,t} * h(x) \right] = \frac{d}{dt} \left[ \int_{2t \leq |y| \leq 2t+1} \frac{\Omega(y')}{|y|^n} h(x-y) dy \right] = \frac{d}{dt} \left[ \int_{S^{n-1}} \Omega(y') \int_{2t}^{2t+1} \frac{1}{r} h(x-ry') drd\sigma(y') \right] = -\frac{1}{t} \int_{S^{n-1}} \Omega(y') h(x-2^j ty') d\sigma(y').$$

For $t \in [1, 2]$ and $\{h_j\} \in L^2(\ell^2)(\mathbb{R}^n)$, we define

$$T^{*}_{j,t} h_j(x) = \int_{S^{n-1}} |\Omega(y')| |h_j(x-2^j ty')| d\sigma(y').$$

It is easy to verify that

$$\left| \frac{d}{dt} T_{j,t} h_j(x) \right| \leq C T^{*}_{j,t} h_j(x).$$

Next, we claim that for $q < 2$, $q' \leq p < \infty$ and $w \in A_{p/q'}$,

$$\left\| \left( \int_{1}^{2} \sum_{j \in \mathbb{Z}} \left| T^{*}_{j,t} h_j \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(w)} \leq C_{p,w} \left\| \Omega \right\|_{L^q(S^{n-1})} \left\| \sum_{j \in \mathbb{Z}} |h_j|^2 \right\|_{L^p(w)}^{1/2}.$$ 

In fact, for $t \in [1, 2]$,

$$\left| T^{*}_{j,t} h_j(x) \right|^2 \leq \left\| \Omega \right\|_{L^q(S^{n-1})} \int_{S^{n-1}} \left| \Omega^{2-q}(y') \right| h_j(x-2^j ty')^2 d\sigma(y').$$

Then for $q < 2$ and $q' \leq p < \infty$, there exists a function $g \in L^1(\mathbb{R}^n)$ with $\left\| g \right\|_{L^q(\mathbb{R}^n)} = 1$, such
that
\[
\left\| \left( \int_1^2 \sum_{j \in \mathbb{Z}} |T^*_j h_j|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(w)}^2
= \left| \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} |T^*_j h_j(x)|^2 \frac{dt}{t} g(x) w(x) \, dx \right|
\leq C \|\Omega\|_{L^q(S^{n-1})}^q \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} \int_1^2 \int_{S^{n-1}} |\Omega^{2-q}(y')| |h_j(x - 2^j t y')|^2 \frac{dt}{t} |g(x)| w(x) \, dx
\]
\leq C \|\Omega\|_{L^q(S^{n-1})}^q \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} \int_{2^j < |y| \leq 2^{j+1}} \frac{|\Omega^{2-q}(y')|}{|y|^n} |h_j(x - y)|^2 \frac{dt}{t} |g(x)| w(x) \, dx
\leq C \|\Omega\|_{L^q(S^{n-1})}^q M_{\Omega^{2-q}}(gw)(x) \sum_{j \in \mathbb{Z}} |h_j(x)|^2 \, dx
\leq C \|\Omega\|_{L^q(S^{n-1})}^q M_{\Omega^{2-q}}(gw) \|L^p(w^{1/(1-p/2)}) \sum_{j \in \mathbb{Z}} |h_j|^2 \|_{L^p(w)}
\leq C_{p,w} \|\Omega\|_{L^q(S^{n-1})}^q \left( \sum_{j \in \mathbb{Z}} |h_j|^2 \right)^{1/2} \left\| \left( \sum_{j \in \mathbb{Z}} |h_j|^2 \right)^{1/2} \right\|^2_{L^p(w)},
\]
where in the above inequality we have used
\[
\left( \int |M_{\Omega^{2-q}}(uw)(x)|^{(p/2)'} w(x)^{1/(1-p/2)} \, dx \right)^{1/(p/2)'} \leq C_{p,w} \|\Omega\|_{L^q(S^{n-1})}^{2-q}
\]
for $\Omega^{2-q} \in L^{\frac{p}{2q}}$ and $w \in A_{p/q'}$ (see [10] or [27]).

Therefore, by (3.14) and the weighted Littlewood-Paley theory with $w \in A_p$ (see [23]), we have that for $q < 2$ and $q' \leq p < \infty$ and $w \in A_{p/q'}$,
\[
\|I_{2,k}f\|_{L^p(w)} \leq C \left\| \left( \int_1^2 \sum_{j \in \mathbb{Z}} |T^*_j \Delta_{k-j} f|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(w)}
\leq C_{p,w} \|\Omega\|_{L^q(S^{n-1})} \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_{k-j} f|^2 \right)^{1/2} \right\|_{L^p(w)}
\leq C_{p,w} \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^p(w)}.
\]

Combined with the estimate of $\|I_{1,k}f\|_{L^p(w)}$, we get for $k \in \mathbb{Z}$,
\[
\|S_{2,k}(Tf)\|_{L^p(w)} \leq C_{p,w} \|\Omega\|_{L^q(S^{n-1})} \|f\|_{L^p(w)},
\]
whenever $q < 2$, $q' \leq p < \infty$, and $w \in A_{p/q'}$. 

4 Proof of Theorem 1.3

We write
\[ \Omega(x') = \left[ \Omega(x') - \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \Omega(y')d\sigma(y') \right] + \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \Omega(y')d\sigma(y') := \Omega_0(x') + C(\Omega, n), \]
where \( \omega_{n-1} \) denotes the area of \( S^{n-1} \). Thus,
\[ M_{\Omega,t}f(x) = \frac{1}{t^n} \int_{|y|<t} \Omega_0(y')f(x-y)dy + C(\Omega, n) \frac{1}{t^n} \int_{|y|<t} f(x-y)dy \]
\[ := M_{\Omega_0,t}f + C(\Omega, n)M_tf, \]
where \( \Omega_0 \) satisfies the cancelation condition (1.6). Denote the operator family \( \{M_{\Omega_0,t}\}_{t>0} \) by \( \mathcal{M}_{\Omega_0} \) and \( \{M_t\}_{t>0} \) by \( \mathcal{M} \). It has been proved in [22] that
\[ \| \sup_{\lambda>0} \lambda \sqrt{N_\lambda(\mathcal{M}f)} \|_{L^p(w)} \leq C_{p,w} \| \Omega_0 \|_{L^1(S^{n-1})} \| f \|_{L^p(w)}. \]

To prove Theorem 1.3 it suffices to show
\[ \| \sup_{\lambda>0} \lambda \sqrt{N_\lambda(\mathcal{M}_{\Omega_0}f)} \|_{L^p(w)} \leq C_{p,w} \| \Omega_0 \|_{L^1(S^{n-1})} \| f \|_{L^p(w)}. \]
Thus by (1.13), the proof of (1.1) is reduced to prove
\[ \| \sup_{\lambda>0} \lambda \sqrt{N_\lambda(\{M_{\Omega_0,2^k}f\}_{k \in \mathbb{Z}})} \|_{L^p(w)} \leq C_{p,w} \| \Omega_0 \|_{L^1(S^{n-1})} \| f \|_{L^p(w)} \]
and
\[ \| S_2(\mathcal{M}_{\Omega_0}f) \|_{L^p(w)} \leq C_{p,w} \| \Omega_0 \|_{L^1(S^{n-1})} \| f \|_{L^p(w)}. \]

For (1.2), we define \( \sigma_k(y) = 2^{-kn} \Omega_0(y')\chi_{\{|y'|<2^k\}}(y) \) for \( k \in \mathbb{Z} \). For \( q, w \) and \( p \) satisfying the conditions (i) or (ii), we have
\[ \| \sup_{\lambda>0} \lambda \sqrt{N_\lambda(\{M_{\Omega_0,2^k}f\}_{k \in \mathbb{Z}})} \|_{L^p(w)} \leq C \| \left( \sum_{k \in \mathbb{Z}} |f \ast \sigma_k|^2 \right)^{1/2} \|_{L^p(w)} \leq C_{p,w} \| \Omega_0 \|_{L^1(S^{n-1})} \| f \|_{L^p(w)}, \]
where the second inequality is a known result in [10].

For (1.3), we define \( \mu_{j,t}(x) = (2^jt)^{-n} \Omega_0(x')\chi_{\{|x'|<2jt\}}(x) \) for \( j \in \mathbb{Z} \) and \( t \in [1,2] \). Observe that \( V_{2,j}(\mathcal{M}_{\Omega_0}f)(x) \) is just the strong 2-variation function of the family \( \{\mu_{j,t} \ast f(x)\}_{t \in [1,2]} \); hence
\[ S_2(\mathcal{M}_{\Omega_0}f)(x) = \left( \sum_{j \in \mathbb{Z}} |V_{2,j}(\mathcal{M}_{\Omega_0}f)(x)|^2 \right)^{1/2}. \]

Similar to the proof of (1.13) in Theorem 1.1 we get
\[ \| S_2(\mathcal{M}_{\Omega_0}f) \|_{L^p(w)} \leq C_{p,w} \| \Omega_0 \|_{L^1(S^{n-1})} \| f \|_{L^p(w)}, \]
whenever \( q, w \) and \( p \) satisfy the conditions (i) or (ii).

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