The $N = 2$ open string.

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Abstract

We show that the $N = 2$ open string describes a theory of self-dual Yang Mills (SDYM) in $(2, 2)$ dimensions. The coupling to the closed sector is described by SDYM in a Kähler background, with the Yang-Mills fields providing a source term to the self-duality equation in the gravity sector. The four-point S-matrix elements of the theory vanish, so the tree-level unitarity constraints leading to the Chan-Paton construction are relaxed. By considering more general group-theory ansätze the $N = 2$ string can be written for any gauge group, and not just the classical groups allowed for the bosonic and $N = 1$ strings. Such ad hoc group-theory factors can not be appended to the closed $N = 2$ string, explaining why the $\mathbb{Z}_n$ closed $N = 2$ strings are trivial extensions of the $\mathbb{Z}_1$ theory.

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1 Introduction.

$N = 2$ strings have had a short but rather convoluted history. The $N = 2$ open string was first studied by M. Ademollo et al. [1]. They found that the critical dimension of the theory was $D = 2$ and that the theory contained only massless scalars in the adjoint of the SU(2) Chan-Paton group. The on-shell three-point vertex vanished, but the theory had nontrivial local four-point vertices. It was argued that only $4N$-point vertices existed, and that the theory could be described by an effective sigma-model field theory.

Many years later, M.B. Green considered also closed and heterotic $N = 2$ strings in two dimensions [2]. These heterotic strings are classified by the 24-dimensional lattice of the bosonic sector, which could be, for instance, an $E_8^3$ lattice or a Moonshine module. While such heterotic strings have interactions similar to those of the open theory, the closed string was seen to have a trivial S-matrix. The effective theory of the heterotic string was again found to be a sigma model, but with corrections coming from diagrams with intermediate gravitons.

At about this time it was pointed out by A. D’Adda and F. Lizzi that, because of the U(1) symmetry structure of the $N = 2$ Virasoro algebra [3], the theory should be considered to exist in two complex, or four real, dimensions [4]. They therefore argued that the theory should have an $SO(2,2) \sim SL(2,\mathbb{R})_L \otimes SL(2,\mathbb{R})_R$ spacetime Lorentz group* and, since the four-point function vanished, argued that the open theory in four dimensions had a trivial S matrix.

Recently, H. Ooguri and C. Vafa re-examined the closed $N = 2$ string [5]. They took the theory to live in an intrinsically Kähler two-complex dimensional spacetime, with Lorentz group $U(1,1) \sim SL(2,\mathbb{R}) \otimes U(1)$. They found that the four-point S matrix elements vanish also for this case, but noted that the three-point function does not vanish on shell, so the theory has a nontrivial S matrix. They argued that all higher-point functions should vanish, and interpreted the theory as having a topological or integrable structure. The massless scalar of the theory was interpreted as the Kähler potential of the spacetime. It satisfies a quadratic equation of motion, the Plebanski equation [6], which is the condition that the spacetime have a self-dual Riemann tensor. Thus the closed $N = 2$ string can be thought of as a theory of self-dual gravity in $(2,2)$ dimensions. A $\mathbb{Z}_n$ generalization of this theory, containing $n$ particles with spins under the $U(1)$ part of the Lorentz group, was also studied in ref. [5].

*We shall always consider the Minkowski space theories.
Ooguri and Vafa also reconsidered the \( N = 2 \) heterotic string in the light of these new developments [7]. Since the left-handed \( N = 2 \) string has to be married to a bosonic or superstring with only one time direction, it turns out that the dependence of the theory on a timelike (and sometimes also a spatial) direction has to be fixed, resulting in a theory in a \( (2, 1) \) or a \( (1, 1) \) dimensional subspace. The equation of motion in the gauge sector of the three-dimensional theory [7] is essentially a dimensional reduction of Yang’s equation for self-dual Yang Mills (SDYM) in four dimensions [8] (which can be obtained from the five-dimensional action of Nair and Schiff [9]). Somewhat surprisingly, the resulting three-dimensional scalars of the theory are tachyonic! In addition, the theory also contains massless vector-like particles in the gravitational sector, whose couplings to the scalars are somewhat messy and poorly understood. Diagrams with intermediate vector particles induce an \( O(\alpha') \) modification to the equation of motion of the scalars, so that the gauge sector of the theory is not simply SDYM. The two-dimensional theory is also a dimensional reduction of Yang’s equation, which is equivalent to the equation of motion of the sigma model. Ref. [2] shows that the theory has \( O(\alpha') \) corrections modifying the SDYM (or sigma model) structure also in this case, in disagreement with ref. [7].

In this paper, we shall reconsider the \( N = 2 \) open string, now taken to be in \( (2, 2) \) dimensional spacetime. In section 2 we calculate several three and four point tree-level amplitudes of open and closed strings. In section 3 we consider the effective field theory of the open string, and in section 4 we give our conclusions.

2  \( N = 2 \) open-string amplitudes.

The tree-level action of open strings is found both from genus zero amplitudes of the closed sector and “genus 1\( \frac{1}{2} \)” amplitudes of the open and closed sectors. In order to find the genus zero amplitudes, to illustrate the issues involved in working in a \( (2, 2) \) space, and to establish notation, we shall, with apologies, first briefly review section 2 of ref. [5].

2.1  The closed string and its amplitudes.

The superspace-vertex operator for emitting a closed-string scalar of momentum \( k \) is* 

\[
V_c = \frac{K}{\pi} e^{i(k \cdot \bar{X} + \bar{k} \cdot X)} ,
\]

*Here the “dots” indicate contractions over the two complex coordinates.
where $X$ is an $N = 2$ chiral superfield. Thus, the vertex at $\theta = \bar{\theta} = 0$ is simply the momentum insertion
\begin{equation}
V^0_c|_{\theta=\bar{\theta}=0} = \frac{\kappa}{\pi} e^{i(k \cdot x + \bar{k} \cdot \bar{x})},
\end{equation}
while the vertex integrated over the fermionic coordinates is
\begin{equation}
V^{int}_c = \frac{\kappa}{\pi} (i k \cdot \partial \bar{x} - i \bar{k} \cdot \partial x - k \cdot \bar{\psi}_R \bar{k} \cdot \psi_R) (i k \cdot \bar{\partial} \bar{x} - i \bar{k} \cdot \bar{\partial} x - k \cdot \bar{\psi}_L \bar{k} \cdot \psi_L) e^{i(k \cdot x + \bar{k} \cdot \bar{x})}.
\end{equation}

To obtain the three-point function, one needs to fix three bosonic coordinates and two $\theta$'s (and $\bar{\theta}$'s). Thus one calculates the correlation function of one $V^{int}_c$ and two $V^0_c$'s, at fixed positions. The result is\(^\dagger\)
\begin{equation}
A_{ccc} = \kappa c_{12}^2,
\end{equation}
where
\begin{equation}
c_{12} \equiv (k_1 \cdot \bar{k}_2 - \bar{k}_1 \cdot k_2)
\end{equation}
is the extra invariant product of the momenta (other than the dot product) that exists when the Lorentz group is reduced to $SL(2,\mathbb{R}) \otimes U(1)$. Note that $c_{ij}$ is antisymmetric with respect to its two indices, and is additive in the sense that $c_{i,j} + c_{i,k} = c_{i,j+k}$. Using momentum conservation, one sees that $A_{ccc}$ is totally symmetric, as it should be. The four-point function is given by the correlation function of two $V^{int}_c$'s and two $V^0_c$'s, with the position of one of the vertices integrated over the sphere. The result is
\begin{equation}
A_{cccc} = \frac{\kappa^2}{\pi} F^2 \frac{\Gamma(1 - s/2) \Gamma(1 - t/2) \Gamma(1 - u/2)}{\Gamma(s/2) \Gamma(t/2) \Gamma(u/2)}.
\end{equation}
The crucial point of ref. [5] is that the kinematic factor
\begin{equation}
F \equiv 1 - \frac{c_{12}c_{34}}{su} - \frac{c_{23}c_{41}}{tu}
\end{equation}
vanishes on shell in the scattering of massless particles in $(2,2)$ dimensions, so the four-point function of the theory vanishes identically. The local three-point function and vanishing four-point function can then be obtained from the action [5]
\begin{equation}
\mathcal{L}_c = \int d^4 x \left( \frac{1}{2} \partial^i \phi \partial_i \phi + \frac{2\kappa}{3} \phi \partial \bar{\phi} \wedge \partial \bar{\phi} \right).
\end{equation}
The equation of motion of this action is the Plebanski equation [6]—the equation for self-dual gravity written in terms of the Kähler potential in the $(2,2)$ space. In ref. [5], it is argued that this cubic action is exact.\(^\dagger\)
\(^\dagger\)We differ from ref. [5] by a factor of $\pi$. See, for example, ref. [10].
Note that if the theory is restricted to two real dimensions, the invariant $c_{12}$ vanishes and the theory has no three-point vertex. The kinematic factor $F$ is now 1, but the four-point function of eq. (6) still vanishes, since $stu = 0$ in the scattering of massless particles in two dimensions.

2.2 Open string amplitudes and group theory.

As argued in ref. [1] in two real dimensions, and in ref. [4] in two complex dimensions, the only particle in the open sector of the $N=2$ theory is a massless scalar (which will become a multiplet of scalars after the addition of group-theory factors to the theory). The superspace vertex for emitting the scalar is

$$V_o = g e^{i(k \cdot \bar{X} + \bar{k} \cdot X)} ,$$

where the vertex now lives on the boundary of the super-Riemann surface: $z = \bar{z} \equiv \sigma$, $\theta = \bar{\theta} \equiv \theta$. The boundary conditions of the fields are $\partial x = \bar{\partial} x|_{z=\bar{z}}$ and $\psi_R = \psi_L \equiv \psi|_{z=\bar{z}}$. Integrating out the fermionic coordinates, one obtains

$$V_o^{int} = \int d^2 \theta V_o = \frac{g}{2} (ik \cdot \partial_\sigma \bar{x} - i\bar{k} \cdot \partial_\sigma x - 4k \cdot \bar{\psi} \bar{k} \cdot \psi) e^{i(k \cdot \bar{x} + \bar{k} \cdot x)} .$$

The three-point open-string amplitude is again found from the correlation function of one $V_o^{int}$ and two $V_o^0$’s at fixed positions. As usual [10], the result is basically the square root of the closed-string amplitude of eq. (4):

$$A_{ooo} = g c_{12} .$$

This amplitude is totally antisymmetric with respect to the three scalars, so one needs to insert group-theory factors to prevent the vanishing of the amplitude. We shall not restrict ourselves to the standard Chan-Paton factors [11], but shall rather take the most general possible ansatz consistent with principles to be given later. We thus add an index $a$ to the open-string scalars, taking their kinetic term to be diagonal with a group factor of $\delta^{ab}$. The most general three-point function is now

$$A_{ooo} = -ig c_{12} f^{abc} ,$$

with $f^{abc}$ an as yet unspecified totally antisymmetric tensor. (In the Chan-Paton ansatz, one would have $f^{abc} = \text{Tr} (\Lambda^a [\Lambda^b \Lambda^c])$.) Unsurprisingly, the vertex of eq. (12) is the same

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*The strange factors of 2 are the result of using closed-string conventions.*
as that of the heterotic string, where the \( f^{abc} \)'s are the structure constants of the SDYM gauge group \([7]\). However, unlike the heterotic case, the theory is defined in the full \((2,2)\) dimensional space, and does not need to be dimensionally reduced.

Temporarily putting aside the issue of the group theory, we now proceed to the four-point function. This amplitude depends on the cyclic ordering of the vertices on the boundary, which we choose to place at 0, \( x \), 1 and \( \infty \), with \( x \) integrated between 0 and 1. Two of the vertices are of the type \( V^{\text{int}}_0 \), and the other two are \( V^{\text{0}}_0 \)'s. The result is

\[
A_{oooo} = g^2 \left[ \frac{t(t + 1/2)}{(1-x)^2} + \frac{c_{12}c_{34}}{z} + \frac{c_{23}c_{41}}{1-x} \right] x^{-2s}(1-x)^{-2t} \\
= \frac{g^2}{4} F \frac{\Gamma(1-2s) \Gamma(1-2t)}{\Gamma(2u)},
\]

with \( F \) the vanishing kinematic factor of eq. (7). Thus the four-point amplitude is zero, and the theory is “topological” in the sense of Ooguri and Vafa. Note that if the theory is reduced to two real dimensions the three-point function vanishes, but the four-point function becomes

\[
A_{oooo} \xrightarrow{D \to 2} \frac{g^2}{2} u,
\]

in agreement with the result of Ademollo \textit{et al.} \([1]\).

Since the four-point function vanishes, the usual factorization constraints \([13]\) leading to the Chan-Paton \textit{ansatz} \([11]\) no longer apply. However, one does find a constraint on the \( f^{abc} \)'s by demanding consistency between the vanishing four-point amplitude of the string theory and the same amplitude calculated from the effective field theory. The three-point amplitude of eq. (12) can be obtained from the lagrangian \([7]\)

\[
\mathcal{L}_{3o} = \int d^4x \left( \frac{1}{2} \partial^i \varphi^a \partial_i \varphi^a - i \frac{g}{3} f^{abc} \varphi^a \partial^i \varphi^b \partial_i \varphi^c \right).
\]

The field-theory four-point function is now calculated by sewing together the vertices (12) from this action, and adding some local four-point vertex \( V_{4o} \). This gives

\[
A_{oooo}^{\text{FT}} = -g^2 \left\{ A f^{axb} f^{xcd} + B f^{bcx} f^{xda} + C f^{cax} f^{xbd} \right\} - V_{4o},
\]

where we have defined

\[
A = \frac{c_{12}c_{34}}{s} \quad B = \frac{c_{23}c_{41}}{t} \quad C = \frac{c_{31}c_{24}}{u}
\]

\(^\dagger\)This amplitude was essentially obtained in ref. \([12]\), but there it was not realized that it vanishes on shell, giving a very different interpretation of the result.
for the poles of the four-point amplitude in the three channels. The kinematic relation
\[ F = 0 \]
and its permutations imply that
\[ B = u - A \quad \text{and} \quad C = t + A . \]  
(18)

These identities allow us to reduce the amplitude to
\[ A_{0000}^{FT} = -g^2 \left\{ A \left( f^{abx} f^{xcd} - f^{acx} f^{zbd} - f^{bcx} f^{zda} \right) + uf^{bcx} f^{xda} + tf^{cax} f^{zbd} \right\} - V_{4o} . \]  
(19)

For the amplitude to be zero, in agreement with eq. (13), the factor multiplying \( A \) must
vanish, since \( V_{4o} \) is a local vertex. This factor is simply the Jacobi identity. Thus the \( f^{abc} \)'s are the structure constants of any semisimple group times a product of an arbitrary
number of \( U(1) \) factors, as might have been expected. The resulting vertex \( V_{4o} \) is the
same as that of the heterotic string [7] (excluding the gravitational sector), and gives a
correction to the lagrangian of eq. (15). The modified lagrangian will be further discussed
in the next section.

2.3 Closed string group theory?

Since we have taken an agnostic approach to inserting group theory factors into string
diagrams, we can ask if it is also possible to add such factors in the closed sector. Thus
consider adding an index \( a \) to the closed-string scalar, and take the three-point function
on the sphere to be
\[ A_{ccc} = \kappa c_{12}^2 d^{abc} , \]  
(20)

where, in this case, \( d^{abc} \) is a totally symmetric tensor. The field-theory amplitudes are
calculated using the vertex derived from the action of (8), in which \( c_{12}^2 \) is replaced by [5]
\[ \tilde{c}_{12}^2 \sim c_{12}^2 - s^2 . \]  
(21)

(This difference is a field redefinition that affects only local terms.) The four-point field
theory amplitude is given by
\[ A_{cccc}^{FT} = \kappa^2 \left\{ \frac{c_{12}^2 c_{34}^2}{s} d^{abx} d^{xcd} + \frac{c_{23}^2 c_{41}^2}{t} d^{cbx} d^{xda} + \frac{c_{31}^2 c_{24}^2}{u} d^{cax} d^{zbd} \right\} - V_{4c} . \]  
(22)

Using the identities of eq. (18), this can be rewritten as
\[ A_{cccc}^{FT} = \kappa^2 \left\{ s \left( A^2 + s^2 - c_{12}^2 - c_{34}^2 \right) d^{abx} d^{xcd} + t \left( B^2 + t^2 - c_{23}^2 - c_{41}^2 \right) d^{cbx} d^{xda} 
+u \left( C^2 + u^2 - c_{31}^2 - c_{24}^2 \right) d^{cax} d^{zbd} \right\} - V_{4c} , \]  
(23)
In order for this amplitude to be zero, in accordance with the string amplitude of eq. (6), the factors multiplying \( sA^2 \) and \( tB^2 \) must separately vanish. This means that the \( d^{abc} \) factors are essentially trivial: Define the matrices

\[
(D^b)_{ax} \equiv d^{abx}. \tag{24}
\]

The vanishing of the factor multiplying \( sA^2 \) tells us that this set of matrices is mutually commuting. They can therefore be simultaneously diagonalized by an orthogonal matrix \( O \), giving

\[
(D^{'b})_{a'd'c'} = (O^T D^b O)_{a'd'c'} = \delta_{a'd'} m_{a'c'}^b \quad \text{(no sum)}.
\tag{25}
\]

Performing this orthogonal rotation on the fields of the theory, the modified \( d' \)'s become

\[
d^{a'b'c'} = \delta_{a'd'} m_{b'}^{c'} \quad \text{(no sum)}, \tag{26}
\]

where \( m_{a'}^{b'} = O_{b'}^a m_a^{b} \). Since \( d^{a'b'c'} \) is totally symmetric, this equation means that it must be completely “diagonal”:

\[
d^{a'b'c'} = \delta_{a'd',b',c'} m(a'). \tag{27}
\]

Thus the theory is simply a sum of noninteracting copies of the theory without the group factors. Returning to eq. (22), one can now see that \( V_{4c} = 0 \). For the rest of this paper, we shall not consider the rather uninteresting case of a sum of separate theories, and shall return to the case of a single closed-string scalar, without group factors in the closed sector.

Note that the \( \mathbb{Z}_n \) theory of Ooguri and Vafa [5] is an example of a closed string with such group-theory factors. The theory contains \( n \) fields \( \phi^l \), which are considered to have “spin” \( l/n \) with respect to the \( U(1) \) part of the Kähler Lorentz group \( U(1,1) \). However, while the little group of a massless scalar with a full \( SO(2,2) \) Lorentz group would be \( \mathbb{R}^2 \), this group is completely broken in the Kähler case. The notion of the spin of a particle therefore no longer exists, and the theory becomes a theory of \( n \) “scalars”. Since its three-point vertex is of the form \( c_{12}^2 \) times constant factors, the theory must be the sum of \( n \) noninteracting theories, by our arguments above. This was indeed found to be the case in ref. [5], after taking appropriate linear combinations of the fields.

2.4 Mixed open/closed amplitudes.

Thus far, we have collected all the pure open and closed three and four-point amplitudes. Since the effective action of the open theory also contains terms mixing the gauge and gravitational sectors, we shall also need to consider mixed amplitudes. We shall thus
calculate the remaining three-point amplitudes, and another four-point amplitude, to lead us to the full equations of motion of the theory.

Mixed open/closed amplitudes are relatively unfamiliar, so we shall first review a few basic facts that we shall need: The Green functions on the upper-half plane \((\text{Im} z \geq 0)\) are:

\[
\begin{align*}
\langle x^i(z) \bar{x}_j(w) \rangle &= -\delta^i_j \left( \log |z - w| + \log |z - \bar{w}| \right) \\
\langle \psi^i_R(z) \bar{\psi}^j_R(w) \rangle &= \delta^i_j \frac{1}{z - w} \\
\langle \psi^i_L(z) \bar{\psi}^j_L(w) \rangle &= \delta^i_j \frac{1}{z - \bar{w}} \\
\langle \bar{\psi}^i_R(z) \bar{\psi}^j_L(w) \rangle &= \delta^i_j \frac{1}{z - \bar{w}} \\
\langle \bar{\psi}^i_R(z) \psi^j_L(w) \rangle &= \delta^i_j \frac{1}{z - w} .
\end{align*}
\] (28)

The UHP has three real bosonic Möbius transformations, and \(2N\) real fermionic ones (with \(N\) the number of supersymmetries). Since closed-string vertices are parameterized by a complex point in superspace, even three-point functions with closed vertices involve some integrations over the positions of the vertices. If one gauge fixes the fermionic transformations by demanding that some \(\theta\)'s be put to zero, the jacobian of the super-Möbius group splits into a product of bosonic and fermionic jacobians. Denoting positions of open vertices by \(a, b, c\ldots\), and those of closed vertices by \(z = x + iy, \ldots\), one finds that the bosonic jacobians for various gauge fixings are (in an obvious notation):

\[
\begin{align*}
J_{abc} &= |(a - b)(b - c)(c - a)| \\
J_{az} &= y|a - z|^2 \\
J_{aby} &= y|(a - b)(2x - b - a)| \\
J_{zy'} &= 2yy'|x - x'| ,
\end{align*}
\] (29)

while the jacobians of the fermionic transformations are:

\[
\begin{align*}
J_{\theta_a, \theta_b} &= 1/|a - b|^N \\
J_{\theta_z} &= 1/y^N .
\end{align*}
\] (30)

We can now proceed with the calculation of the mixed three and four-point amplitudes, taking them in order of increasing number of closed vertices. We first calculate the amplitude of two open strings with one closed one. The simplest way to do the calculation is to fix \(\theta_a, \theta_b, a\) and \(z\), and to integrate \(b\) over the real line, although of course the final
result is independent of this choice. One obtains

$$A_{oooc} = \frac{\kappa}{\pi} \delta^{ab} c_{12} \int_{-\infty}^{\infty} db \frac{y}{(x - b)^2 + y^2} = \kappa \delta^{ab} c_{12}^2.$$ \hfill (31)

Since this amplitude gives the coupling of the gravitational sector to the kinetic term of the open scalars, we have chosen to append the group-theory factor $\delta^{ab}$, as in the kinetic term. This vertex is the same as the gravitational self-interaction of eq. (4), showing a “universality” in the couplings of the various fields to gravity. The meaning of the interaction will be further considered in the next section.

One can also find the amplitude of one closed string with three open ones, needed for the gravitational coupling of the cubic Yang-Mills vertex. A simple gauge-fixing for this amplitude is to fix the positions of the three open vertices to be at 0, 1 and $\infty$, and to fix the $\theta$'s of the two vertices at 0 and $\infty$. One then needs to integrate the closed vertex over the UHP. Since the integrand is symmetric under $z \rightarrow \bar{z}$, one can transform this integral to an integral over the entire plane, and use the integration results of ref. [10]. The result is

$$A_{oooc} = \frac{i}{2} \kappa g f^{abc} F \frac{\Gamma(s) \Gamma(t) \Gamma(u)}{\Gamma(-s) \Gamma(-t) \Gamma(-u)} (c_{12} + c_{23}) \Gamma(-s) \Gamma(-t) \Gamma(-u),$$ \hfill (32)

which vanishes on shell because of the presence of the $F$ factor of eq. (7).

We now turn to amplitudes with two closed strings. Since the open string carries a group index, one would expect that any amplitude with only one open string and an arbitrary number of closed ones should vanish. That this is indeed the case follows from the twist symmetry of the theory: The UHP is invariant under the diffeomorphism $z \rightarrow -\bar{z}$. Since open string vertices are odd under the diffeomorphism, while closed vertices are even, one has the relation $\langle V_1^o V_2^o \ldots V_n^o \rangle = (-)^n \langle V_1^o V_2^{n-1} \ldots V_1^o \rangle$. This means that any amplitude with only one open string vanishes. This conclusion is borne out for $V_{oooc}$ by explicitly examining the integrand of the amplitude, an exercise which we shall spare the reader.

The above argument shows that we do not need to consider the four-point amplitude with one open and three closed strings. The only remaining four-point amplitude contains two open and two closed strings. Since its calculation involves carrying out a complicated three-dimensional integral, and since it should presumably again vanish by the presence of an $F$ factor, we shall not perform the calculation.

A more interesting amplitude, and the last one we shall consider, is the scattering of three closed strings at genus $\frac{1}{2}$. Unlike the previous amplitudes, these diagrams can be
thought of as giving a quantum correction to the three-point amplitude on the sphere, as
given in eq. (4). (We remind the reader that the notion of a tree-level amplitude in the
open string is not that well defined, and that one has the relation \( \kappa \sim \hbar g^2 \) between the
closed and open-string couplings.) The amplitude on the UHP is calculated by fixing \( z \)
and \( \theta_z \) of one vertex, and \( y' \) of a second vertex, but still involves the integration over \( x' \)
and the position of the third vertex. Examining the integrand, one sees that the answer
has the form

\[
A'_{ccc} \propto \frac{\kappa^3}{g^2} c_{12}^4 ,
\]

with some finite coefficient\(^\ast\). Since this amplitude contains no open-string vertices, it
has a different structure to all the previous amplitudes, and needs to be allocated some
(unknown) group-theory factor. In addition, there is a similar graph on the projective
plane \( \mathbb{R}P_2 \), which should be combined with the UHP graph. (In the Chan-Paton
scheme the resulting group-theory factor would be \( N - 2 \) for the group \( SO(N) \), etc.) We shall
consider this counterterm with an arbitrary coefficient in the following, although the most
reasonable choice would be to take it to vanish.

It is interesting that, in the corresponding amplitude on the torus, one finds a similar
result [14]:

\[
A''_{ccc} \sim \kappa^3 c_{12}^6 ,
\]

but with a coefficient that diverges in the infrared!

### 2.5 The partition function

In addition to scattering amplitudes, one would also like to find the partition function
of the theory. In an open theory this involves calculating the path integral on all the
genus zero graphs: the torus, the Klein bottle, the Möbius band and the annulus. The
torus amplitude has been calculated in ref. [15], and was found, in their normalization,
to be the volume form on moduli space:

\[
Z_{\text{torus}} = \frac{1}{4\pi} \int_M \frac{d\tau d\bar{\tau}}{\tau_2^2} \left( = \frac{1}{12} \right) .
\]

Note that, because of the extra ghosts of the \( U(1) \) symmetry, the partition function
appears as the partition function of a scalar in 2 real dimensions. In particular, this
means that if one compactifies two of the dimensions, as was done in ref. [5] using the
techniques of ref. [16], one again finds an infrared divergence.

\(^\ast\)Which Mathematica thinks is \(-2/3\).
Instead of performing the path-integral for our case, we shall obtain the partition function from the free energy \[17\], knowing the spectrum of the open and closed sectors of the theory (considered to live in two dimensions). Recall that, while the torus amplitude is defined over the keyhole region of moduli space, the moduli space of the other surfaces is simply the real line, and their partition functions are integrated over the same range as their free energies (see, for example, ref. \[18\]). Defining the proper time \(t\), the total partition function is given by

\[
Z = \frac{1}{2} Z_{\text{torus}} + \frac{1}{4\pi} \int_0^\infty \frac{dt}{t^2} \left( \frac{1}{2} + \frac{1}{2} c_{\text{annulus}} + \frac{1}{2} c_{\text{M"obius}} \right),
\]

where the coefficients \(c\) are group-theory factors. The factors of \(1/2\) are due to the nonorientability of the theory, and should be dropped if nonorientable graphs are discarded. We have put \(c_{\text{Klein}} = 1\), since we want the sum of the contributions of the torus and Klein bottle to be that of the one scalar of the closed sector. In the Chan-Paton scheme one would have \(c_{\text{annulus}} = N^2\), and \(c_{\text{M"obius}} = \pm N\) for the groups \(SO(N)\) and \(USp(N)\), resulting in a correct description of the spectrum of the open sector. If one chooses to implement a more general \emph{ansatz} for the group \(G\), one needs to enforce the condition

\[
c_{\text{annulus}} + c_{\text{M"obius}} = 2 \dim G,
\]

which is rather unnatural.

Note also that the partition function of eq. (36) shows yet another infrared divergence as \(t \to 0\). In the \(N = 0\) and \(N = 1\) theories, such divergences can be isolated by performing modular-like transformations rotating the diagrams by \(90^\circ\). We thus define \(q\) for the three graphs by \[18\]:

\[
q_{\text{Klein}} \equiv e^{-\pi/t} \\
q_{\text{annulus}} \equiv e^{-4\pi/t} \\
q_{\text{M"obius}} \equiv -e^{-\pi/t}.
\]

The partition function then becomes

\[
Z = \frac{1}{2} Z_{\text{torus}} + \frac{1}{16\pi} \left[ \int_0^1 \frac{dq}{q} \left( 2 + \frac{1}{2} c_{\text{annulus}} \right) + \int_{-1}^0 \frac{dq}{q} \cdot 2c_{\text{M"obius}} \right].
\]

Thus, in the Chan-Paton case, the divergence at \(q = 0\) can be regulated by a principle-part prescription for the (very uninteresting) group \(SO(2)\). This is the analogue of the groups \(SO(32)\) for the superstring \[19\] and \(SO(8192)\) for the bosonic string \[18\]. However, since there are no known anomalies for the bosonic and \(N = 2\) strings, the argument for these special groups is not very compelling in these cases, especially since the theories retain similar divergences, even when these special groups are chosen.
3 Space-time interpretation of the open string.

In the previous section, we began constructing actions for the purely open and purely closed sectors. Here we shall continue this process, using the three-point amplitudes we have found and the vanishing of the four-point amplitudes to determine the effective field theory string up terms quartic in the fields. This will be sufficient to allow us to deduce the complete equations of motion of the theory.

The actions of the open and closed sectors to cubic order are given in eqs. (8) and (15), using the open and closed three-point amplitudes of eqs. (4) and (12). Similarly, the three-point vertex of one “graviton” and two “gluons”, given in eq. (31), can be obtained from the interaction

$$\mathcal{L}_{oooc} = \int d^4x \left( 2\kappa \phi \partial \bar{\phi}^a \wedge \partial \bar{\phi}^a \right).$$

(40)

This gives the complete action to cubic level, excluding the quantum “counterterms” of eqs. (33) and (34) which we shall discuss later.

The quartic terms in the action are found by calculating four-point field-theory amplitudes, and demanding that they vanish. In the discussion following eq. (22) (without closed group factors), we saw that $V_{4c} = 0$, showing that the closed action does not receive quartic corrections [5]. In the open sector, on the other hand, we saw that one needs a nonvanishing vertex $V_{4o}$ to cancel the result of the graphs built from two $A_{ooo}$'s. $V_{4o}$ is determined from eq. (19), and can be derived from the interaction

$$\mathcal{L}_{4o} = \int d^4x \left( -\frac{g^2}{6} f^{abc} f^{xbe} \partial_i \phi^a \phi^b \bar{\partial}_i \phi^c \phi^d \right),$$

(41)
in agreement with the result of the pure Yang-Mills sector of the heterotic string [7]. Note that, unlike the case of the heterotic string, one does not receive additional contribution from graphs with an intermediate graviton. In an open theory such graphs have the topology of an annulus, and should be considered to be true quantum corrections to the theory.

There is also a mixed quartic term in the action, which is found by examining the four-point function with one closed and three open scalars. In the field theory, the new vertex $V_{oooc}$ must cancel the graphs made by sewing together the vertex $A_{ooo}$ (from eq. (12)) with the vertex $\bar{A}_{oooc} = \kappa \delta^{ab} \bar{c}_{12}^2$ derived from the interaction of eq. (40). Calling the
closed scalar particle 1, and summing over the three channels, one obtains
\[
A_{\text{ooc}} = -i \kappa g f^{abc} \left\{ \left( \frac{c_{12}}{s} - s \right) c_{34} - \left( \frac{c_{13}}{u} - u \right) c_{24} + \left( \frac{c_{14}}{t} - t \right) c_{23} \right\} - V_{\text{ooc}}
\]
\[= -i \kappa g f^{abc} \left\{ A c_{12} - s c_{34} + C c_{13} + u c_{24} - B c_{14} - t c_{23} \right\} - V_{\text{ooc}} \]  \tag{42}
\[= -2i \kappa g f^{abc} \left\{ t c_{13} - u c_{14} \right\} - V_{\text{ooc}}. \]

The resulting vertex can be derived from the interaction
\[
\mathcal{L}_{\text{ooc}} = \int d^4x \left( -\frac{4}{3} i g \kappa f^{abc} \partial \bar{\partial} \phi \wedge \varphi^a \partial \varphi^b \bar{\partial} \varphi^c \right). \tag{43}
\]

Finally, we can consider the amplitude with two closed and two open scalars. This can be obtained in two ways: by sewing together two \(\tilde{A}_{\text{ooc}}\)’s with an intermediate gluon, or by sewing an \(A_{\text{ccc}}\) to an \(\tilde{A}_{\text{ooc}}\) with an intermediate graviton. Since these vertices are both essentially \(c^2\), the result (including relative symmetry factors) is proportional to the amplitude of four closed strings, which we know to vanish. There is thus no term coupling two gravitons to the open lagrangian. Redefining \(\phi\) by a factor of \(4\kappa\), and \(\varphi\) by a factor of \(2g\), and taking \(\varphi\) to be an antihermitian matrix in the Lie algebra, the total lagrangian to this order can be written as
\[
\mathcal{L} = \frac{1}{16\kappa^2} \int d^4x \left( \frac{1}{2} \partial^i \phi \bar{\partial}_i \phi + \frac{1}{6} \phi \bar{\partial} \partial \phi \wedge \bar{\partial} \partial \phi \right) + \frac{1}{4g^2} \int d^4x \left( -\frac{1}{2} \text{Tr} \left( \partial^i \varphi \bar{\partial}_i \varphi \right) - \frac{i}{3!} \text{Tr} \left( \bar{\partial}_i \varphi \left[ \partial^i \varphi, \varphi \right] \right) + \frac{1}{4!} \text{Tr} \left( \bar{\partial}_i \varphi \left[ \partial^i \varphi, \varphi \right], \varphi \right) + \cdots \right. \\
\left. + \frac{1}{2} \partial \bar{\partial} \phi \wedge \text{Tr} \left( \varphi \partial \bar{\partial} \phi \right) + \frac{i}{3!} \partial \bar{\partial} \phi \wedge \text{Tr} \left( \varphi \left[ \partial \varphi, \bar{\partial} \varphi \right] \right) + \cdots \right). \tag{44}
\]

This lagrangian appears to be rather arbitrary, but its resulting equations of motion can be given a simple geometrical interpretation if one considers the \((2, 2)\) dimensional space to be intrinsically Kähler [5]. One can then define the Kähler potential
\[
K = \eta_{ij} x^i \bar{x}^j + \phi, \tag{45}
\]
giving rise to the spacetime metric \(g_{ij} = \partial_i \bar{\partial}_j K = \eta_{ij} + \partial_i \bar{\partial}_j \phi\). Consider first the purely closed terms in \(\mathcal{L}\). With this interpretation, the equation of motion of \(\phi\) becomes the Plebanski equation [6]
\[
\det g_{ij} = -1. \tag{46}
\]
This equation can also be written in terms of forms as
\[
\partial \bar{\partial} K \wedge \bar{\partial} \partial K = 2\omega \wedge \bar{\omega}, \tag{47}
\]
where $\omega$ is the holomorphic $(2,0)$ form $dx^1 \wedge dx^2$ on the space. As discussed in ref. [5], the Plebanski equation leads to the vanishing of the Ricci tensor of the $(2,2)$ space, and is equivalent to the statement that the space have a self-dual Riemann tensor. The closed theory is thus a theory of self-dual gravity.

The equation of motion in the purely open sector also has a simple meaning in (flat) Kähler space [7]—it is Yang’s equation for SDYM [8]. This can be seen by defining:

\[
A \equiv e^{-i\varphi/2} \partial e^{i\varphi/2} \Rightarrow F \wedge \bar{\omega} = 0
\]

\[
\bar{A} \equiv e^{i\varphi/2} \bar{\partial} e^{-i\varphi/2} \Rightarrow F \wedge \omega = 0.
\] (48)

The equation of motion of $\varphi$, to this order and ignoring $\phi$, is

\[
\bar{\partial} \left( e^{-i\varphi} \partial e^{i\varphi} \right) = 0 \Rightarrow F \wedge k_0 = 0,
\] (49)

where $k_0$ is the flat Kähler form. Eqs. (48) and (49) are equivalent to the self-duality of the Yang-Mills field strength in the Kähler space.

We can now easily find the interpretation of the equations of motion of the full lagrangian $\mathcal{L}$ of eq. (44). The equation of motion of $\varphi$ is the obvious generalization of eq. (49)*:

\[
g_{ij} \bar{\partial}^j \left( e^{-i\varphi} \partial e^{i\varphi} \right) = 0 \Rightarrow F \wedge k = 0,
\] (50)

where $k$ is now the full Kähler form of the Kähler potential of eq. (45). With the definitions of eq. (48), this is the statement of the self-duality of the Yang-Mills field strength on the curved Kähler space. On the other hand, the Ricci-flatness condition of the closed theory gets modified. The full equation of motion of $\phi$ is:

\[
\partial \bar{\partial} K \wedge \partial \bar{\partial} \bar{K} = 2\omega \wedge \bar{\omega} - \frac{4\kappa^2}{g^2} \text{Tr} \left( F \wedge F \right),
\] (51)

or

\[
\det g_{ij} = -1 - \frac{2\kappa^2}{g^2} \text{Tr} \left( F_{ij} F^{ij} \right).
\] (52)

One thus sees that, as in Einstein gravity, the matter sector provides a source term to the gravitational field equations. If one was to include the gravitational counterterms of eqs. (33) and (34), coming from the closed scattering amplitudes on the higher-genus diagrams, the right-hand side of eq. (51) would also get corrections of the form $R \wedge R$, and terms with higher derivatives.

Since all these equations have a clear meaning, it is reasonable to assume that they are, in fact, the equations of the string theory to all orders in the fields.

*In this equation the indices are raised with the $\epsilon$ symbol.
4 Conclusions

As in the closed and heterotic $N = 2$ theories, the $N = 2$ open string theory in $(2, 2)$ dimensional spacetime has the peculiar property that its four-point scattering amplitudes vanish identically. All these theories can be written as “scalar” field theories in a two-complex dimensional spacetime, although the heterotic string has to be dimensionally reduced. The closed theory has the interpretation of being a theory of self-dual gravity \[5\]. The heterotic string is related to a theory of SDYM, but not in a straightforward manner \[7\]. First, the theory must be reduced to a $(2, 1)$ dimensional spacetime, in which the particles become tachyonic, and second, the Yang-Mills self-duality equation receives corrections from diagrams with internal gravitons. By contrast, we have seen that the open theory is simply described by SDYM in the Kähler background of the closed sector. The gravitational equation of motion is however modified by terms of the form $\text{Tr} \ (F \wedge F)$, so the resulting spacetime is no longer self-dual.

Because of the “topological” nature of the $N = 2$ theories, with all amplitudes being local, the usual constraints on string theories coming from unitarity and factorization are greatly weakened (assuming that we understand these issues in spacetimes with two time-like directions). In particular, the open theory can be defined for any gauge group, and one does not have to use the standard Chan-Paton ansatz. At this stage, there is little to constrain more general ansätze, although their ad hoc nature may be considered rather unpleasant. The further calculation of loop amplitudes in all $N = 2$ strings may be important for clarifying their nature, and, in particular, knowing loop amplitudes of $N = 2$ open strings may further constrain the allowed SDYM groups. Infrared divergences in the partition function suggests that in the standard Chan-Paton ansatz, the trivial group $SO(2)$ may be favoured, analogously to the group $SO(32)$ of the superstring, but this conclusion is very weak.

In general, our understanding of $N = 2$ strings is still in its infancy. The theories appear to be related to self-duality equations in $(2, 2)$ dimensional spaces, but the spacetimes have to be taken as being basically Kähler, and the expected Lorentz invariance is absent. Amplitudes in the theories are local, or vanish identically, but this is understood only from explicit calculation. Calculations of loop amplitudes also indicate peculiar problems. One-loop results show basic infra-red divergences, and the string theory results appear to agree with those of the effective field theories only if the theories are considered as being in two, and not in four dimensions. Clearly, there should be deep results to be discovered here.
Note added.

While this paper was in preparation, we received two papers from W. Siegel [20]. By considering the Ramond sector, one of the continuously degenerate sectors of the theory, he argued that the theory should actually be described by an $N = 4$ supersymmetric SDYM theory. He also argued that the $N = 2$ string is the same as the “$N = 4$” $SU(2)$ string of ref. [1], and that the theory should then be able to be written in a fully $SO(2,2)$ Lorentz-invariant form.

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