Convergence and limits of linear representations of finite groups *

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Abstract
Motivated by the theory of graph limits, we introduce and study the convergence and limits of linear representations of finite groups over finite fields. The limit objects are infinite dimensional representations of free groups in continuous algebras. We show that under a certain integrality condition, the algebras above are skew fields. Our main result is the extension of Schramm’s characterization of hyperfiniteness to linear representations.

Keywords. linear representations, amenability, soficity, continuous rings, skew fields

Contents
1 Introduction 2
2 Definitions and results 4
3 Universal localizations 6
4 Continuous algebras 8
5 Sofic algebras 10
6 Amenable and non-amenable skew fields 12
7 Limits of linear representations 15
8 Linear tilings 16
9 Convergent sequences and sofic approximations 18
10 Amenable limit fields 23
11 Non-amenable limit fields 24

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1 Introduction

Unitary representations. Our primary motivation and model example is the view of infinite dimensional unitary representations into tracial von Neumann algebras as limits of finite dimensional unitary representations. By a finite dimensional unitary representation (of degree $r$), we mean a homomorphism $\kappa : F_r \to U(n)$ of the free group on $r$ generators into the unitary group $U(n)$. Note that such representations can be given by the $r$-tuple $\{\kappa(\gamma_i)\}_{i=1}^r$, where $\{\gamma_i\}_{i=1}^r$ are the standard generators of the free group $F_r$. We say that a sequence $\{\kappa_k : F_r \to U(n_k)\}_{k=1}^\infty$ of finite dimensional unitary representations are convergent if for any $w \in F_r$

$$\text{Tr}(w) = \lim_{k \to \infty} \text{Tr}_{n_k}(\kappa_k(w))$$

exists, where $\text{Tr}_{n_k}$ stands for the normalized trace function on the complex matrix algebra $\text{Mat}_{n_k \times n_k}(\mathbb{C})$. Note that from each sequence of representations one can choose a convergent subsequence. The limit of a convergent system $\{\kappa_k : F_r \to U(n_k)\}_{k=1}^\infty$ is defined as a representation of $F_r$ into the unitary group of a tracial von Neumann algebra via the GNS-construction, given below.

The function $\text{Tr}$ extends to the group algebra $\mathbb{C}F_r$ as a trace, that is a linear functional satisfying

$$\text{Tr}(ab) = \text{Tr}(ba).$$

Let $I \subset \mathbb{C}F_r$ be the set of elements $a$, for which $\text{Tr}(a^*a) = 0$. It is not hard to see that $I$ is an ideal of $\mathbb{C}F_r$ and the trace function $\text{Tr}$ vanishes on $I$. Let $A = \mathbb{C}F_r/I$, then

$$\langle [p], [q] \rangle = \text{Tr}[pq]$$

is well defined and gives rise to an inner product structure on the algebra $A$. Let $\mathcal{H}$ be the Hilbert completion of $A$. Then the left multiplication $L_{[p]} : A \to A$ defines a bounded linear representation of $A$ on $\mathcal{H}$. The weak closure of the image of $A$ is a tracial von Neumann algebra $\mathcal{N}$ equipped with a trace (the extension of $\text{Tr}$). Also, we have a natural homomorphism $S : F_r \to U(\mathcal{N})$ such that for any $w \in F_r$,

$$\text{Tr}(S(w)) = \lim_{k \to \infty} \text{Tr}_{n_k}(\kappa_k(w)).$$

Thus $S$ can be viewed as the limit object of the finite dimensional unitary representations $\{\kappa_k\}_{k=1}^\infty$. One can ask the following question. If $S : F_r \to U(\mathcal{N})$ is a representation of the free group into the unitaries of a tracial
von Neumann algebra, is it true that $S$ is the limit of finite dimensional unitary representations. This question is called the Connes Embedding Problem [23].

**Graph limits.** We are also strongly motivated by the emerging theory of graph limits. The original definition of graph convergence is due to Benjamini and Schramm [3] (see also the monography of Lovász [19]). Here we consider the limit of Schreier graphs (see e.g. [10]). Let $\lambda : \mathbb{F}_r \to S(n)$ be a homomorphism of the free group into the symmetric group of permutation on $n$ elements. These homomorphism are in bijective correspondence with Schreier graphs. The vertex set of the corresponding graph $G_\lambda$ is the set $[n] = \{1, 2, \ldots, n\}$. Two vertices $a$ and $b$ are connected with an edge labeled by the generator $\gamma_i$, if $\lambda(\gamma_i)(a) = b$. A sequence of permutation representations (or Schreier graphs) $\{\lambda_k : \mathbb{F}_r \to S(n_k)\}_{k=1}^\infty$ is called convergent if for any $m \geq 1$ and $m$-tuple $(w_1, w_2, \ldots, w_m) \subset \mathbb{F}_r$

$$\lim_{k \to \infty} \frac{\left| \text{Fix}_k(\lambda_k(w_1)) \cap \text{Fix}_k(\lambda_k(w_2)) \cap \cdots \cap \text{Fix}_k(\lambda_k(w_m)) \right|}{n_k}$$

exists, where $\text{Fix}_k(\lambda_k(w))$ is the fixed point set of the permutation $\lambda_k(w)$. Note that the original definition is somewhat different from the one above, nevertheless a simple inclusion-exclusion argument shows that the two definitions are equivalent. One can define various limit objects for such convergent sequences e.g. the invariant random subgroups (see [1]). A notion of limit, analogous to the unitary case, can be defined the following way. Let $(X, \mu)$ be the standard Borel probability measure space equipped with a countable measure preserving equivalence relation $E$ (see [16]). An $E$-transformation is a measure preserving bijection $T : X \to X$ such that for almost all $p \in X$, $p \equiv_E T(p)$. A zero transformation is an $E$-transformation $S$ such that $\mu(\text{Fix}(S)) = 1$. The full group of $E$, $[E]$ is the quotient of the group of $E$-transformations by the normal subgroup of zero transformations. Note that if $Q \in [E]$, then $\text{Fix}(Q)$ is well-defined up to a zero measure perturbation. We call a homomorphism $\lambda : \mathbb{F}_r \to [E]$ generating if for almost all $p \in X$: for any $q$ such that $p \equiv_E q$, there exists $w \in \mathbb{F}_r$ such that $\lambda(w)(p) = q$. A generating representation $\lambda : \mathbb{F}_r \to [E]$ is a limit of the convergence system of permutation representations $\{\lambda_k\}_{k=1}^\infty$ if for any $m \geq 1$ and $m$-tuple $(w_1, w_2, \ldots, w_m) \subset \mathbb{F}_r$

$$\lim_{k \to \infty} \frac{\left| \text{Fix}_k(\lambda_k(w_1)) \cap \text{Fix}_k(\lambda_k(w_2)) \cap \cdots \cap \text{Fix}_k(\lambda_k(w_m)) \right|}{n_k} = \mu(\text{Fix}(\lambda(w_1)) \cap \text{Fix}(\lambda(w_2)) \cap \cdots \cap \text{Fix}(\lambda(w_m))).$$
Again, for each convergent sequence \( \{\lambda_k\}_{k=1}^{\infty} \) one can find a limit representation into some full group. On the other hand, it is not known, whether any generating representation \( \lambda : \mathbb{F}_r \rightarrow [E] \) can be obtained as a limit.

**Schramm’s Theorem.** Let \( \lambda : \mathbb{F}_r \rightarrow S(n) \) be a permutation representation with corresponding Schreier graph \( G_\lambda([n], E_\lambda) \). We say that a convergence sequence of representations \( \{\lambda_k : \mathbb{F}_r \rightarrow S(n_k)\}_{k=1}^{\infty} \) is hyperfinite if for any \( \varepsilon > 0 \), there exists an integer \( K_\varepsilon > 0 \) such that for any \( k \geq 1 \), one can remove \( \varepsilon n_k \) edges from the graph \( G_{\lambda_k} \) in such a way, that all the components of the remaining graph have at most \( K_\varepsilon \) vertices. Schramm [22] proved that the hyperfiniteness of the sequence is equivalent to the amenability of its limit (see also [10]). A generating representation \( \lambda : \mathbb{F}_r \rightarrow [E] \) is amenable if \( E \) is a hyperfinite (amenable) equivalence relation [16].

## 2 Definitions and results

In the course of our paper we fix a finite field \( K \). Our goal is to study the convergence and limits of finite dimensional representations \( \theta : \mathbb{F}_r \rightarrow \text{Mat}_{n \times n}(K) \). Note that such representations are in one to one correspondence with linear representations \( \pi : \Gamma \rightarrow GL(n, K) \), where \( \Gamma \) is a finite group of \( r \) marked generators.

**Definition 2.1.** A sequence of finite dimensional representations \( \{\theta_k : \mathbb{F}_r \rightarrow \text{Mat}_{n_k \times n_k}(K)\}_{k=1}^{\infty} \) is convergent if for all \( n \geq 1 \) and matrix \( A \in \text{Mat}_{n \times n}(K\mathbb{F}_r) \),

\[
\lim_{k \to \infty} r_{n_k}^n(\theta_k(A))
\]

exists, where \( \theta_k(A) \) is the image of \( A \) in

\[
\text{Mat}_{n,n \times n \times n}(K) \cong \text{Mat}_{n \times n}(\text{Mat}_{n_k \times n_k}(K))
\]

and

\[
r_{n_k}^n(\theta_k(A)) = \frac{\text{Rank}(\theta_k(A))}{n_k}.
\]

Note that \( \theta_k \) naturally extends to the group algebra \( K\mathbb{F}_r \) and we denote the extension by \( \theta_k \), as well. We will make clear at the end of Section 3 why we consider matrices instead of single elements of the group algebra \( K\mathbb{F}_r \). Now we define the limit objects for convergent sequences. The objects we need, continuous algebras, were defined by John von Neumann in the thirties [21]. Let \( R \) be a separable, continuous \( K \)-algebra with rank function \( r_{k_R} \) (see Section 4 for a brief survey on continuous algebras).
Definition 2.2. Let \( \{\theta_k\}_{k=1}^{\infty} \) be finite dimensional representations as above. A representation \( \theta : \mathbb{F}_r \to R \), that is a homomorphism of the free group into the group of invertible elements of the continuous algebra \( R \) is a limit of \( \{\theta_k\}_{k=1}^{\infty} \), if for any \( n \geq 1 \) and \( A \in \operatorname{Mat}_{n \times n}(\mathbb{K}\mathbb{F}_r) \)

\[
\lim_{k \to \infty} \operatorname{rk}_{n_k}^n(\theta_k(A)) = \operatorname{rk}_R^n(\theta(A)),
\]

where \( \operatorname{rk}_R^n \) is the matrix rank on \( \operatorname{Mat}_{n \times n}(R) \).

Note, that if \( \mathcal{N} \) is a tracial von Neumann algebra, then \( \mathcal{N} \) is equipped with a natural rank function and its completion is a continuous rank regular ring; the algebra of affiliated operators [21]. Hence the limit of finite dimensional unitary representations can also be viewed as a homomorphism into the group of invertible elements of a continuous algebra. Our first theorem is about the existence of limits.

**Theorem 1.** For any convergent sequence of finite dimensional representations \( \{\theta_k\}_{k=1}^{\infty} \), there exists a separable, continuous \( \mathbb{K} \)-algebra \( R \) and a representation \( \theta : \mathbb{F}_r \to R \) such that \( \theta \) is the limit of \( \{\theta_k\}_{k=1}^{\infty} \).

It turns out that under a certain integrality condition the limit is unique. We say that the convergence sequence \( \{\theta_k\}_{k=1}^{\infty} \) satisfies the Atiyah condition, if for any \( n \geq 1 \) and \( A \in \operatorname{Mat}_{n \times n}(\mathbb{K}\mathbb{F}_r) \)

\[
\lim_{k \to \infty} \operatorname{rk}_{n_k}^n(\theta_k(A)) \in \mathbb{Z}.
\]

The condition above is intimately related to Atiyah’s Conjecture on the integrality of the \( L^2 \)-betti numbers (see [17] and [18]).

**Theorem 2.** If the convergence sequence of linear representations \( \{\theta_k\}_{k=1}^{\infty} \) satisfies the Atiyah condition, then there exists a skew field \( D \) over the base field \( \mathbb{K} \) and a homomorphism \( \theta : \mathbb{F}_r \to D \) (that is a homomorphism into the multiplicative group of non-zero elements of \( D \)) such that \( \theta \) is the limit of \( \{\theta_k\}_{k=1}^{\infty} \) and \( \operatorname{Im}(\theta) \) generates \( D \). Moreover, if \( \theta' : \mathbb{F}_r \to D' \) is another generating limit homomorphism into a skew field \( D' \), then there exists a skew field isomorphism \( \pi : D \to D' \) such that \( \pi \circ \theta = \theta' \).

If the Atiyah condition is satisfied, we will be able to generalize Schramm’s Theorem cited in the Introduction. It is worth to note that hyperfinite sequences of graphs are basically the opposites of expander sequences. The notion of expander sequences for linear representations were introduced by Lubotzky and Zelmanov (see also [7]). We say that a sequence of linear representations \( \{\theta_k : \mathbb{F}_r \to \operatorname{Mat}_{n_k \times n_k}(\mathbb{K})\}_{k=1}^{\infty} \) is a...
**dimension expander** if there exists $\alpha > 0$ such that for all $k \geq 1$ and linear subspace $W \subset \mathcal{K}^{n_k}$ with $\dim_{\mathcal{K}}(W) \leq \frac{n_k}{2}$

$$\dim_{\mathcal{K}}(W + \sum_{i=1}^{r} \theta_k(\gamma_i)(W)) \geq (1 + \alpha) \dim_{\mathcal{K}}(W),$$

where $\{\gamma_i\}_{i=1}^{r}$ are the standard generators of $\mathbb{F}_r$. Note that

$$\sup_{W, \dim_{\mathcal{K}}(W) \leq \frac{n_k}{2}} \frac{\dim_{\mathcal{K}}(W + \sum_{i=1}^{r} \theta_k(\gamma_i)(W))}{\dim_{\mathcal{K}}(W)}$$

is the linear analogue of the Cheeger constant of a graph. It was observed in [7] that a random choice of $\{\cup_{i=1}^{r} \theta_k(\gamma_i)\}_{k=1}^{\infty}$ leads to a dimension expander with probability one, provided that $r$ is large enough. Later, Bourgain and Yehudayoff [4] constructed explicit families of dimension expanders. Using the linear graph theory vocabulary: subsets $\rightarrow$ linear subspaces, disjoint $\rightarrow$ independent, cardinality $\rightarrow$ dimension, we can define the hyperfiniteness for sequences of linear representations.

**Definition 2.3.** The linear representations

$\{\theta_k : \mathbb{F}_r \rightarrow \text{Mat}_{n_k \times n_k}(\mathcal{K})\}_{k=1}^{\infty}$ form a hyperfinite sequence if for any $\epsilon > 0$ there exists $K_{\epsilon} > 0$ such that for all $k \geq 1$, we have $\mathcal{K}$-linear subspaces $V_1^k, V_2^k, \ldots, V_t^k \subset \mathcal{K}^{n_k}$ such that

- For any $1 \leq j \leq t_k$, $\dim_{\mathcal{K}}(V_j^k) \leq K_{\epsilon}$.
- $\{V_j^k + \sum_{i=1}^{r} \theta_k(\gamma_i)V_j^k\}_{j=1}^{t_k}$ are independent subspaces such that

$$\dim_{\mathcal{K}}(V_j^k + \sum_{i=1}^{r} \theta_k(\gamma_i)V_j^k) < (1 + \epsilon) \dim_{\mathcal{K}}(V_j^k).$$

- $\sum_{j=1}^{t_k} \dim_{\mathcal{K}}(V_j^k) \geq (1 - \epsilon)n_k$.

Our main result is the generalization of Schramm’s Theorem for convergent sequences of linear representations.

**Theorem 3.** Let $\{\theta_k\}_{k=1}^{\infty}$ be a convergent sequence of linear representations satisfying the Atiyah condition. Let $\theta : \mathbb{F}_r \rightarrow D$ be the unique limit representation of $\{\theta_k\}_{k=1}^{\infty}$. Then $\{\theta_k\}_{k=1}^{\infty}$ is hyperfinite if and only if $D$ is an amenable skew field.

### 3 Universal localizations

In this section we recall the notion of universal localization from the book of Cohn [5]. Let $R$ be a unital ring, $\Sigma$ be a set of matrices over $R$. 

and \( f : R \to S \) be a unital ring homomorphism. Let \( \Sigma^f \) be the image of \( \Sigma \) under \( f \). If \( \Sigma \) is the set of matrices whose images under \( f \) are invertible, then \( R^f(S) \) denotes the set of entries in the inverses \( f(M)^{-1} \), for \( M \in \Sigma^f \). We call \( R^f(S) \) the rational closure of \( R \) in \( S \). According to Theorem 1.2 \([5]\) \( R^f(S) \) is a ring containing \( \text{Im}(f) \). An important tool for the understanding of the ring \( R^f(S) \) is the following variant of Cramer’s Rule.

**Lemma 3.1** (Proposition 1.3 \([5]\)). For any element \( p \in R^f(S) \), there exists \( n \geq 1 \) and \( Q \in \text{Mat}_{n \times n}(\text{Im}(f)) \), \( A \in \text{Mat}_{n \times n}(\text{Im}(f)) \) invertible in \( \text{Mat}_{n \times n}(S) \) and \( B \in \text{Mat}_{n \times n}(R^f(S)) \) in the form of \( B = \begin{pmatrix} I & u \\ 0 & 1 \end{pmatrix} \) such that

\[
Q = A \begin{pmatrix} I & 0 \\ 0 & p \end{pmatrix} B.
\]

Note that \( I \) denotes the unit matrix of size \( n - 1 \).

Recall that the division closure of \( \text{Im}(f) \) in \( S \) is the smallest ring in \( S \) containing \( \text{Im}(f) \) closed under taking inverses. The following lemma is given as an exercise in \([5]\).

**Lemma 3.2.** The division closure \( D(f) \) is a subring of \( R^f(S) \).

*Proof.* It is enough to prove that if \( p \in R^f(S) \) is invertible in \( S \), then \( p^{-1} \in R^f(S) \). By Lemma 3.1 we can write

\[
Q = A \begin{pmatrix} I & 0 \\ 0 & p \end{pmatrix} B.
\]

so \( \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = A^{-1}QB^{-1} \) that is \( \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} = BQ^{-1}A \). Since all the entries of \( B, Q^{-1} \) and \( A \) are, by definition, in the subring \( R^f(S) \), we have that \( p^{-1} \in R^f(S) \). \( \square \)

Now, let \( \Sigma \) be a set of square matrices over \( R \). The universal localization of \( R \) with respect to \( \Sigma \) is the unique ring \( R_\Sigma \) equipped with a homomorphism \( \lambda : R \to R_\Sigma \) such that if \( f : R \to S \) is an arbitrary homomorphism and the elements of \( \Sigma^f \) are all invertible matrices, then there exists a unique homomorphism \( \overline{f} : R_\Sigma \to S \) such that \( \overline{f} \circ \lambda = f \). Let \( D \) be a skew field and \( f : R \to D \) be a homomorphism. We call \( D \) epic if \( \text{Im}(f) \) generates \( D \).

**Proposition 3.1** (Theorem 2.2 \([5]\)). If \( D \) is epic and \( \Sigma_f \) is the set of matrices over \( R \) whose images in \( D \) are invertible, then the universal localization \( R_\Sigma_f \) is a local ring with residue-class field isomorphic to \( D \).
We have the following corollary.

**Corollary 3.1.** If \( f_1 : R \to D_1 \) and \( f_2 : R \to D_2 \) are two epic maps and for all \( n \geq 1 \) and for all matrices \( A \in \text{Mat}_{n \times n}(R) \)

\[
\text{rk}_{D_1}(f_1(A)) = \text{rk}_{D_2}(f_2(A)),
\]

then \( D_1 \cong D_2 \) and there is an isomorphism \( \iota : D_1 \to D_2 \) such that

\[
\iota \circ f_1 = f_2 \quad (1)
\]

**Proof.** By our condition, \( \Sigma f_1 = \Sigma f_2 = \Sigma \). Let \( \lambda' : R \to R_{\Sigma}/M \) be the natural map, where \( M \) is the unique maximal ideal of \( R_{\Sigma} \). Then there exist two skew field isomorphisms \( \pi_1 : R_{\Sigma}/M \to D_1 \) and \( \pi_2 : R_{\Sigma}/M \to D_2 \) such that \( \pi_1 \circ \lambda' = f_1 \) and \( \pi_2 \circ \lambda' = f_2 \). Then we can choose \( \iota = \pi_2 \circ \pi_1^{-1} \) to satisfy (1).

\[\square\]

4 **Continuous algebras**

In this section we recall the notion of a continuous algebra from the book of Goodearl [12] and present some important examples. A ring \( R \) is called von Neumann regular if for any \( a \in R \) there exists \( x \in R \) such that \( axa = a \). In other words, \( R \) is von Neumann regular if any finitely generated left ideal is generated by a single idempotent. A rank regular ring is a unital regular ring \( R \) equipped with a rank function \( \text{rk}_R \) satisfying the conditions below.

- \( 0 \leq \text{rk}_R(a) \leq 1 \), for any \( a \in R \).
- \( \text{rk}_R(a) = 0 \) if and only if \( a = 0 \).
- \( \text{rk}_R(1) = 1 \).
- \( \text{rk}_R(a + b) \leq \text{rk}_R(a) + \text{rk}_R(b) \).
- \( \text{rk}_R(ab) \leq \min(\text{rk}_R(a), \text{rk}_R(b)) \)
- \( \text{rk}_R(e + f) = \text{rk}_R(e) + \text{rk}_R(f) \) if \( e \) and \( f \) are orthogonal idempotents.

Note, that in a rank regular ring an element is invertible if and only if it has rank one. Also, a rank regular ring is a metric space with respect to the distance function

\[
d_R(a, b) := \text{rk}_R(a - b).
\]

If a rank regular ring \( R \) is complete with respect to the distance function, then \( R \) is called a **continuous ring**. We are particularly interested in
continuous \( K \)-algebras. The simplest examples are skew fields over \( K \) and matrix rings over such skew fields. For these continuous algebras the rank function may take only finitely many values. An other important example is due to John von Neumann. Let us consider the diagonal maps

\[
d_n : \text{Mat}_{2^n \times 2^n}(K) \to \text{Mat}_{2^{n+1} \times 2^{n+1}}(K).
\]

The maps preserve the normalized rank functions, hence the direct limit

\[
\varprojlim \text{Mat}_{2^n \times 2^n}(K)
\]

is a rank regular ring. Its metric completion \( \overline{R} \) is a continuous \( K \)-algebra, with a rank function \( \text{rk}_{\overline{R}} \) taking all values in between zero and one.

Note that if \( R \) is a rank regular ring, the metric completion of \( R \) is always a continuous ring [13]. Also, if \( R \) is a rank regular ring, then for any \( n \geq 1 \) the matrix ring \( \text{Mat}_{n \times n}(R) \) can be equipped with a unique matrix rank function \( \text{rk}_R^n \) such that

\[
\text{rk}_R^n(\text{Id}) = n \quad \text{and} \quad \text{rk}_R^n(A) = \text{rk}_R(A) + \text{rk}_R^{n-k}(B) \quad (2)
\]

if \( A \in \text{Mat}_{k \times k}(R), B \in \text{Mat}_{(n-k) \times (n-k)}(R) \) [14]. Finally, let us recall the notion of the ultraprodut of finite dimensional matrix algebras. This construction will be crucial in our paper. Let \( M = \{ \text{Mat}_{n_k \times n_k}(K) \}_{k=1}^\infty \) be a sequence of matrix algebras over our base field \( K \) equipped with the normalized rank functions \( \{ \text{rk}_{n_k} \}_{k=1}^\infty \) such that \( n_k \to \infty \). Let \( \omega \) be an ultrafilter on the natural numbers and let \( \lim_{\omega} \) be the associated ultralimit. The ultraproduct \( M_M \) of the algebras \( \{ \text{Mat}_{n_k \times n_k}(K) \}_{k=1}^\infty \) can be defined the following way. Consider the elements

\[
\{a_k\}_{k=1}^\infty \in \prod_{k=1}^\infty \text{Mat}_{n_k \times n_k}(K),
\]

for which \( \lim_{\omega} \text{rk}_{n_k}(a_k) = 0 \). It is easy to see that these elements form an ideal \( I_M \). The ultraproduct is defined as

\[
M_M := \prod_{k=1}^\infty \text{Mat}_{n_k \times n_k}(K)/I_M.
\]

The \( k \)-algebra \( M_M \) is a simple continuous algebra equipped with a rank function [14]

\[
\text{rk}_M(\{a_k\}_{k=1}^\infty) = \lim_{\omega} \text{rk}_{n_k}(a_k),
\]

where \( \{a_k\}_{k=1}^\infty \) denotes the class of \( \{a_k\}_{k=1}^\infty \in \prod_{k=1}^\infty \text{Mat}_{n_k \times n_k}(K) \) in \( M_M \).
5 Sofic algebras

Let $R$ be a countable $\mathcal{K}$-algebra over our finite base field $\mathcal{K}$, with $\mathcal{K}$-linear basis $\{1 = r_1, r_2, r_3, \ldots\}$. Following Arzhantseva and Paunescu [2], we call $R$ (linearly) sofic if there exists a function $j : R \backslash 0 \to \mathbb{R}^+$ and a sequence of positive numbers $s_i \to 0$ such that for any $i \geq 1$ there exists $n_i \geq 1$ and a $\mathcal{K}$-linear map $\phi_i : R \to \text{Mat}_{n_i \times n_i}(\mathcal{K})$ satisfying the conditions below:

- $\phi_i(1) = \text{Id}$
- $\text{rk}_{n_i}(\phi_i(a)) \geq j(a)$ if $0 \neq a \in \text{Span}\{r_1, r_2, \ldots, r_i\}$
- $\text{rk}_{n_i}(\phi_i(ab) - \phi_i(a)\phi_i(b)) < s_i$ if $a, b \in \text{Span}\{r_1, r_2, \ldots, r_i\}$.

Such a system is called a sofic representation of $R$. Clearly, the soficity of an algebra does not depend on the particular choice of the basis $\{r_1, r_2, \ldots\}$. Using the maps above, we can define a map $\phi : R \to M_M$, by

$$\phi(s) := \left[\{\phi_i(s)\}_{i=1}^\infty\right],$$

where $M = \{\text{Mat}_{n_i \times n_i}(\mathcal{K})\}_{i=1}^\infty$. By our assumptions, $\phi$ is a unital embedding. Conversely, we have the following proposition.

**Proposition 5.1.** Let $R$ be a countable algebra over our base field $\mathcal{K}$. If $R$ can be embedded into an ultraproduct $M_M$, then $R$ is sofic.

**Proof.** Let $\{1 = r_1, r_2, \ldots\}$ be a basis for $R$ and let $j(a) := \frac{1}{3}\text{rk}_M(\phi(a))$. It is enough to prove that for any $\epsilon > 0$ and $i \geq 1$, there exists an integer $n \geq 1$ and a linear, unit preserving map $\sigma : R \to \text{Mat}_{n \times n}(\mathcal{K})$ such that

- $\text{rk}_{n}(\sigma(a)) \geq j(a)$ if $0 \neq a \in \text{Span}\{r_1, r_2, \ldots, r_i\}$.
- $\text{rk}_{n}(\sigma(ab) - \sigma(a)\sigma(b)) < \epsilon$ if $a, b \in \text{Span}\{r_1, r_2, \ldots, r_i\}$.

Let $\phi : R \to M_M$ be the embedding, where $M = \{\text{Mat}_{n_j \times n_j}(\mathcal{K})\}_{j=1}^\infty$. Let

$$0 < \delta = \inf_{a \neq 0, a \in \text{Span}\{r_1, r_2, \ldots, r_i\}} \frac{1}{3}\text{rk}_M(\phi(a)).$$

Note that we used the finiteness of our base field in the definition of $\delta$. Let us define the unit preserving not necessarily linear maps $\psi_j : R \to \text{Mat}_{n_j \times n_j}(\mathcal{K})$ by

$$\phi(a) = \left[\{\psi_j(a)\}_{j=1}^\infty\right].$$
Let $F$ be the linear subspace of $R$ spanned by the elements \( \{ r_k r_l \}_{1 \leq k, l \leq i} \). Choose a basis \( \{ r_1, r_2, \ldots, r_i, \cup_{z=1}^{t} r_{m_z} r_{n_z} \} \) for $F$. For $1 \leq k, l \leq i$ let

\[
r_k r_l = \sum_{p=1}^{i} \alpha^{k,l}_p r_p + \sum_{z=1}^{t} \beta^{k,l}_z r_{m_z} r_{n_z}.
\]

By the properties of the ultraproduct, we have some $j \geq 1$ such that for any $1 \leq k, l \leq i$

\[
\text{rk}_{n_j} (\psi_j(r_k r_l) - \psi_j(r_k) \psi_j(r_l)) < \frac{\varepsilon}{2i^2} \tag{3}
\]

\[
\text{rk}_{n_j} \left( \psi_j(r_k r_l) - \left( \sum_{p=1}^{i} \alpha^{k,l}_p \psi_j(r_p) + \sum_{z=1}^{t} \beta^{k,l}_z \psi_j(r_{m_z} r_{n_z}) \right) \right) < \frac{\varepsilon}{2i^2} \tag{4}
\]

Also,

\[
\text{rk}_{n_j} \left( \psi_j(a) - \sum_{p=1}^{i} \lambda_p \psi_j(r_p) \right) < \frac{\delta}{2} \tag{5}
\]

\[
\text{rk}_{n_j} (\psi_j(a)) > \text{rk}_M(\phi(a)) - \frac{\delta}{2} \tag{6}
\]

whenever $a = \sum_{p=1}^{i} \lambda_p r_p$. Now we define the linear map $\sigma$ on $F$ by extending $\psi_j$ from the basis. That is,

\[
\sigma(r_p) := \psi_j(r_p)
\]

\[
\sigma(r_{m_z} r_{n_z}) := \psi_j(r_{m_z} r_{n_z}).
\]

Then we extend $\sigma$ linearly onto the whole algebra $R$ in an arbitrary fashion. Now let $a = \sum_{p=1}^{i} \lambda_p r_p$, $b = \sum_{q=1}^{i} \mu_q r_q$. Then

\[
\sigma(ab) = \sum_{p=1}^{i} \sum_{q=1}^{i} \lambda_p \mu_q \sigma(r_p r_q).
\]

Observe that for any $1 \leq p, q \leq i$,

\[
\text{rk}_{n_j} (\sigma(r_p r_q) - \sigma(r_p) \sigma(r_q)) \leq \text{rk}_{n_j} (\sigma(r_p r_q) - \psi_j(r_p r_q)) + \text{rk}_{n_j} (\psi_j(r_p r_q) - \psi_j(r_p) \psi_j(r_q)).
\]

Hence, by (3) and (4)

\[
\text{rk}_{n_j} (\sigma(r_p r_q) - \sigma(r_p) \sigma(r_q)) \leq \frac{\varepsilon}{i^2}.
\]
Thus we have that

\[
\text{rk}_{n_j} (\sigma(ab) - \sigma(a)\sigma(b)) = \\
= \text{rk}_{n_j} \left( \sum_{p=1}^{i} \sum_{q=1}^{i} \lambda_p \mu_q (\sigma(r_p r_q) - \sigma(r_p)\sigma(r_q)) \right) \leq \varepsilon .
\]

Also, by (5) and (6), \( \text{rk}_{n_j}(\sigma(a)) > 1/2\text{rk}_M(\phi(a)) \), whenever \( a \in \text{Span}\{r_1, r_2, \ldots, r_i\} \). Hence, our proposition follows.

6 Amenable and non-amenable skew fields

In this section we recall the notion of amenability for skew fields from [10]. Let \( D \) be a countable dimensional skew field over a commutative base field \( K \). We say that \( D \) is amenable if for any \( \varepsilon > 0 \) and finite dimensional \( K \)-subspace \( E \subset D \), there exists a finite dimensional \( K \)-subspace \( V \subset D \) such that

\[
\dim_K EV < (1 + \varepsilon) \dim_K V .
\]

All commutative fields and skew fields of finite Gelfand-Kirillov dimension are amenable. Also, if for a torsion-free amenable group \( G \), the group algebra \( KG \) is a domain, then its classical field of fraction is an amenable skew field.

**Proposition 6.1.** All amenable skew fields are sofic.

**Proof.** It is enough to prove that for any finite dimensional subspace \( E \subset D \) and \( \varepsilon > 0 \), there exists \( n \geq 1 \) and a \( K \)-linear map \( \tau : D \to \text{Mat}_{n \times n}(K) \) such that

- \( \tau(1) = \text{Id} \)
- \( \text{rk}_n(\tau(d)) > 1 - \varepsilon \) if \( 0 \neq d \in F \)
- \( \text{rk}_n(\tau(fg) - \tau(f)\tau(g)) < \varepsilon \) for any pair \( f, g \in F \).

First we need a technical lemma.

**Lemma 6.1.** If \( D \) is an amenable skew field, then for any \( \delta > 0 \) and finite dimensional subspace \( E \subset D \), there exists a pair of linear subspaces \( V_1 \subset V \subset D \) such that \( EV_1 \subset V \) and

\[
\dim_K(V_1) \geq (1 - \delta) \dim_K(V) .
\]
Proof. Let \( \{ s_i \}_{i=1}^m \) be a \( K \)-basis for \( E \). Let \( V \subset D \) be a finite dimensional subspace such that
\[
\dim_K(EV) \leq \dim_K(V)(1 + \frac{\delta}{2m}).
\]
Observe that for any \( 1 \leq i \leq m \)
\[
\dim_K(V \cap s_i^{-1}V) = \dim_K(s_iV \cap V) \geq (1 - \frac{\delta}{2m}) \dim_K(V).
\]
Hence,
\[
\dim_K(\cap_{i=1}^m(V \cap s_i^{-1}V)) \geq (1 - \delta) \dim_K(V).
\]
That is, if \( V_1 = \cap_{i=1}^m(V \cap s_i^{-1}V) \) then \( EV_1 \subset V \) and \( \dim_K(V_1) \geq (1 - \delta) \dim_K(V) \).

Now let \( H \subset D \) be the linear subspace spanned by \( F \cdot F \). Let \( V \subset D \) be an \( n \)-dimensional linear subspace such that for some linear subspace \( V_1 \subset V \), \( HV_1 \subset V \) and \( \dim_K(V_1) \geq (1 - \epsilon) \dim_K(V) \) hold. Let \( W \subset D \) be a linear subspace complementing \( V \) and let \( P : D \to V \) be the \( K \)-linear projection onto \( V \) such that \( P|_V = \text{Id} \) and \( P|_W = 0 \).

Now we define \( \tau(d) \) as \( P \circ M_d \), where \( M_d \) is the left-multiplication by \( d \), Clearly, \( \tau : D \to \text{End}_K(V) \cong \text{Mat}_{n \times n}(K) \) is a \( K \)-linear map satisfying \( \tau(1) = \text{Id} \). If \( d \in F \), then \( \text{Ker}(\tau(d)) \cap V_1 = 0 \), that is \( \text{rk}_n(\tau(d)) \geq 1 - \epsilon \).

Also, if \( f, g \in F \), then \( \tau(fg) = \tau(f)\tau(g) \) restricted on \( V_1 \). Therefore, \( \text{rk}_n(\tau(fg) - \tau(f)\tau(g)) < \epsilon \).

By Theorem 1. of [9], if \( K \subset E \) is a sub-skew field of \( D \) and \( E \) is non-amenable, then \( D \) is non-amenable as well. If \( K = \mathbb{C} \) then the free skew field on \( r \) generators over \( K \) is a non-amenable skew field. We conjecture that this is the case for all base fields. Recall that the free skew field is the universal localization of \( K\mathbb{F}_r \) with respect to the set of full matrices \([5]\) (see also \([6]\) for various other definitions of the free skew field).

**Proposition 6.2.** The free skew field on \( r \) generators over our finite base field \( K \) is sofic.

Proof. We will be following the paper of Eizenbud and Lichtman \([8]\). Let us consider the free group \( \mathbb{F}_r \) and a series of normal subgroups
\[
\mathbb{F}_r = N_1 \supset N_2 \supset \ldots, \cap_{k=1}^\infty N_k = \{1\}.
\]
where \( \mathbb{F}_r / N_k \) are torsion-free nilpotent groups. Thus, we have linear homomorphisms \( \pi_k : \mathcal{K} \mathbb{F}_r \to \mathcal{K}(\mathbb{F}_r / N_k) \). Since the algebra \( \mathcal{K}(\mathbb{F}_r / N_k) \) is an Ore domain, it embeds to the classical field of fractions \( M_k \). That is, we have composition maps \( \sigma_k : \mathcal{K} \mathbb{F}_r \to M_k \). Let \( \omega \) be an ultrafilter on the naturals. Consider the set-theoretical ultraproduct of the skew fields \( M_k, \prod_{\omega} M_k \). Note this construction is slightly different from the algebraic ultraproduct construction of Section 4. Here, we consider the ideal \( I \) of sequences \( \{a_k\}_{k=1}^\infty \in \prod_{k=1}^\infty M_k \) for which

\[ \{ k \in \mathbb{N} \mid a_k = 0 \} \in \omega. \]

Then \( \prod_{\omega} M_k = \prod_{k=1}^\infty M_k / I \). Note that \( \prod_{\omega} M_k \) is also a skew field. Let \( \sigma : \mathcal{K} \mathbb{F}_r \to \prod_{\omega} M_k \) be the ultraproduct embedding. That is \( \sigma(f) = [\{\sigma_1(f), \sigma_2(f), \ldots\}] \), where \([x]\) denotes the class of \( x \in \prod_{k=1}^\infty M_k \) in \( \prod_{\omega} M_k \). By Theorem 6.1 of [8], the division closure of \( \sigma(\mathcal{K} \mathbb{F}_r) \) in \( \prod_{\omega} M_k \) is isomorphic to the free skew field \( D_r \). Thus we have a homomorphism \( \hat{\sigma} : D_r \to \prod_{\omega} M_k \) extending \( \sigma \).

**Lemma 6.2.** Let \( 1 \in E \subset D_r \) be a finite dimensional \( \mathcal{K} \)-linear subspace. Then, there exists \( k \geq 1 \) and a \( \mathcal{K} \)-linear unit preserving map \( \rho : D_r \to M_k \) such that \( \rho(ab) = \rho(a)\rho(b) \) for any pair \( a, b \in E \).

**Proof.** Our proof is somewhat similar to the one of Proposition 5.1. For \( d \in D_r \), let \( \hat{\sigma}(d) = [\{\sigma_1(d), \sigma_2(d), \ldots\}] \), where \( \sigma_k(d) \in M_k \). Let \( \{1 = r_1, r_2, \ldots, r_s\} \) be a basis for \( E \) and let \( \bigcup_{p=1}^s r_p \cup \bigcup_{z=1}^t r_{m_z} r_{n_z} \) be a basis for the subspace \( F \) spanned by the elements \( \{r_i r_j\}_{1 \leq i, j \leq n} \). For \( 1 \leq i, j \leq n \), let

\[
    r_i r_j = \sum_{p=1}^s \alpha_{i,j}^p r_p + \sum_{z=1}^t \beta_{i,j}^z r_{m_z} r_{n_z}.
\]

Since \( \hat{\sigma} \) is a homomorphism, there exists \( k \geq 1 \) such that

- \( \sigma_k(r_i r_j) = \sigma_k(r_i) \sigma_k(r_j) \) for any \( 1 \leq i, j \leq s \)
- \( \sigma_k(r_i r_j) = \sum_{p=1}^s \alpha_{i,j}^p \sigma_k(r_p) + \sum_{z=1}^t \beta_{i,j}^z \sigma_k(r_{m_z} r_{n_z}) \).

Hence, if we extend \( \sigma_k \) from the basis \( \bigcup_{p=1}^s r_p \cup \bigcup_{z=1}^t r_{m_z} r_{n_z} \) to a linear map \( \rho : D_r \to M_k \), then for any pair \( 1 \leq i, j \leq s, \sigma_k(r_i r_j) = \rho(r_i r_j) \). Observe that if \( a, b \in E \) then \( \rho(ab) = \rho(a)\rho(b) \). Indeed, if \( a = \sum_{i=1}^s \lambda_i r_i, b = \sum_{j=1}^s \mu_j r_j \), then

\[
    \rho(ab) = \rho\left( \sum_{i=1}^s \lambda_i r_i \cdot \sum_{j=1}^s \mu_j r_j \right) = \sum_{i=1}^s \sum_{j=1}^s \lambda_i \mu_j \sigma_k(r_i r_j) = \sum_{i=1}^s \sum_{j=1}^s \lambda_i \mu_j \sigma_k(r_i r_j).
\]
This finishes the proof of Lemma 6.2.

Observe that $M_k$ is an amenable skew field. Indeed, it is the classical field of fractions of the amenable group algebra $K(F_r/N_k)$. Such skew fields are amenable by Proposition 2.1 [9]. Therefore by Lemma 6.2 and Proposition 6.1 our Proposition follows.

7 Limits of linear representations

Let $\{\theta_k : F_r \to \text{Mat}_{n \times n}(K)\}_{k=1}^\infty$ be a convergent sequence of linear representations. Then we can consider the ultraproduct representation $\theta : F_r \to M_M$ and the extended algebra homomorphism (we denote it with the same letter) $\theta : KF_r \to M_M$. One should notice that $KF_r/Ker(\theta)$ is a sofic algebra. By definition, for any $n \geq 1$ and $A \in \text{Mat}_{n \times n}(KF_r)$

$$\lim_{k \to \infty} \text{rk}_{n_k}^n(\theta_k(A)) = \text{rk}_{M}^n(\theta(A)).$$

Note that $\text{Mat}_{n \times n}(M_M)$ is the algebraic ultraproduct of the matrix algebras $\{\text{Mat}_{n \times n}(\text{Mat}_{n \times n}(K))\}_{k=1}^\infty$ and the unique extended rank function [13] on $\text{Mat}_{n \times n}(M_M)$ is exactly the ultralimit of the matrix ranks $\text{rk}_{n_k}^n$. Now, let us prove Theorem 1. It is enough to see that there exists a von Neumann regular countable subalgebra $S \subset M_M$ containing $\text{Im}(\theta)$.

Indeed, the limit object sought after in the theorem is the metric closure of $S$ in $M_M$ (recall that the completion of a rank regular algebra is a continuous algebra [13]). If $T$ is a countable subset of $M_M$, then let $R(T)$ be the $K$-algebra generated by $T$. It is easy to see that $R(T)$ is still countable. Let $X : M_M \to M_M$ be a function such that for any $a \in M_M$,

$$aX(a)a = a.$$  

Let $R_1 = R(\text{Im}(\theta) \cup X(\text{Im}(\theta)))$ and inductively, let $R_{n+1} = R_n(\text{Im}(\theta) \cup X(\text{Im}(\theta)))$. Then the ring $S = \cup_{n=1}^\infty R_n$ is a countable von Neumann regular algebra containing $\text{Im}(\theta)$. This finishes the proof of Theorem 1.

Now let us suppose that the convergent sequence of representations $\{\theta_k\}_{k=1}^\infty$ satisfies the Atiyah condition, that is for any $n \geq 1$ and $A \in \text{Mat}_{n \times n}(KF_r)$

$$\lim_{k \to \infty} \text{rk}_{n_k}^n(\theta_k(A)) \in \mathbb{Z}.$$
Proposition 7.1. The division closure $D$ of $\text{Im}(\theta)$ in the algebra $\mathcal{M}_M$ is a skew field. That is, $\theta : \mathbb{F}_r \rightarrow D$ is the limit of the sequence $\{\theta_k\}_{k=1}^{\infty}$.

Proof. We use an idea of Linnell [17]. Let $p$ be an element of the rational closure of $\text{Im}(\theta)$ in $\mathcal{M}_M$. By Lemma 3.1 we have matrices $Q, A, B$ such that all the entries of $Q$ are from $\text{Im}(\theta)$ and

$$Q = A \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} B,$$

where $A$ and $B$ are invertible, and $\text{rk}_M(Q)$ is an integer. Since the matrix rank on $\text{Mat}_{n \times n}(\mathcal{M}_M)$ is the ultralimit of the matrix ranks $\text{rk}_{m_k}^n$ (or by [14])

$$n - 1 < \text{rk}_M^n \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \leq n,$$

by the integrality condition, $\text{rk}_M^n \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = n$. Thus, $p$ is invertible. By Lemma 3.2 the division closure of $\text{Im}(\theta)$ in $\mathcal{M}_M$ is contained by the rational closure. Therefore, each element of the division closure is invertible. Hence, the division closure $D$ is a skew field. \qed

Now suppose that for some skew field $D'$, $\theta' : \mathbb{F}_r \rightarrow D'$ is another limit for the sequence $\{\theta_k\}_{k=1}^{\infty}$. Then for any $n \geq 1$ and $A \in \text{Mat}_{n \times n}(\mathcal{K}\mathbb{F}_r)$

$$\text{rk}_{D'}^n(\theta(A)) = \text{rk}_{D}^n(\theta(A)).$$

Therefore, by Corollary 3.1 there exists an isomorphism $\pi : D \rightarrow D'$ such that $\pi \circ \theta = \theta'$. This finishes the proof of Theorem 2. \qed

Remark. Note that the proof of Proposition 7.1 shows that if $\theta' : \mathbb{F}_r \rightarrow S$ is a limit of the sequence $\{\theta_k\}_{k=1}^{\infty}$, where $S$ is a continuous algebra, then the division closure of $\text{Im}(\theta')$ in $S$ is still isomorphic to $D$.

8 Linear tilings

In this section we prove a key technical result of our paper. Let $D$ be a countable skew field over $\mathcal{K}$ with $\mathcal{K}$-basis $\{1 = r_1, r_2, \ldots \}$ and let
$\phi : D \to \text{Mat}_{n \times n}(K) \cong \text{End}_K(K^n)$ be a unit preserving linear map. We say that $x \in K^n$ is $i$-good with respect to $\phi$ if

$$x \in \text{Ker}(\phi(ab) - \phi(a)\phi(b)),$$

whenever $a, b \in \text{Span}\{r_1, r_2, \ldots, r_i\}$. The $i$-good elements form the $K$-subspace $G_n^{i,\phi} \subset K^n$. By definition, if $\{\phi_k : D \to \text{Mat}_{n \times n}(K)\}_{k=1}^{\infty}$ is a sofic representation, then for any fixed $i \geq 1$,

$$\lim_{k \to \infty} \frac{\dim_K G_n^{i,\phi}}{n_k} = 1.$$  

**Definition 8.1.** A unit preserving linear map $\phi : D \to \text{Mat}_{n \times n}(K)$ is an $i$-good map if

$$\dim_K G_n^{i,\phi} \geq 1 - \frac{1}{i}.$$ 

Let $\phi : D \to \text{Mat}_{n \times n}(K)$ and let $1 \in F \subset D$ and $H \subset K^n$ be finite dimensional linear subspaces. We call a subset $T \subset K^n$ a set of $(i, F, H)$-centers with respect to $\phi$ if

- For any $x \in T$, $\phi(F)(x) \subset H$ is a $\dim_K(F)$-dimensional $K$-subspace.
- If $x \in T$, then for any $0 \neq f \in F$, $\phi(f)(x)$ is $i$-good.
- The subspaces $\{\phi(F)(x)\}_{x \in T}$ are independent.

We say that $\phi$ has an $(F, H, i, \delta)$-tiling if there exists a set of $(i, F, H)$-centers $T$ for $\phi$ such that

$$|T| \dim_K(F) \geq (1 - \delta)n.$$  

**Theorem 4.** Let $D$ be a countable skew field over the base field $K$, with basis $\{1 = r_1, r_2, \ldots\}$. Then for any finite dimensional subspace $1 \in F \subset D$ and $\delta > 0$, we have a positive constant $N_{F,\delta}$ such that if $i, n \geq N_{F,\delta}$, $\phi : D \to \text{Mat}_{n \times n}(K)$ is an $i$-good unit preserving linear map and $\dim_K H \geq (1 - \frac{1}{i})n$, then $\phi$ has an $(F, H, i, \delta)$-tiling.

**Proof.** Let $\phi : D \to \text{Mat}_{n \times n}(K)$ be an $i$-good unit preserving linear map, the exact values of $i$ and $n$ will be given later. First note, that if $i$ is larger than some constant $N_F^1$, then $F^{-1}, F \subset \text{Span}\{r_1, r_2, \ldots, r_i\}$. Note that if

$$x \in \text{Ker}(\phi(f^{-1})\phi(f) - 1)$$

for any $0 \neq f \in F$, then $\dim_K \phi(F)(x) = \dim_K(F)$. Thus, if $i \geq N_F^1$, then $\dim_K \phi(F)(x) = \dim_K(F)$. Let

$$A_{F,i} = \{x \in K^n \mid \phi(f)(x) \in G_n^{i,\phi} \cap H\, \text{for any } f \in F\}.$$
It is easy to see that there exists some constant $N_{F,\delta}^2$ such that if $i \geq N_{F,\delta}^2$ then
\[ \dim_K(A_F;i) \geq (1 - \frac{\delta}{4})n. \] (7)
and for any $0 \neq f \in F$,
\[ \dim_K \ker \phi(f) = n - \dim_K \text{Im} \phi(f) \leq \frac{\delta}{3}n. \] (8)

Finally, let $N_{\delta}^3 > 0$ such that if $n > N_{\delta}^3$, then $|F| \leq |K|^\frac{\delta}{4}n$ and let $N_{F,\delta} = \max(N_1^2, N_{F,\delta}^2, N_{\delta}^3)$. For $0 \neq f \in F$ and $v \in K^n$, let $L(f,v)$ denote the set of points $y$ such that $\phi(f)(y) = v$. Then we have the estimate
\[ |L(f,v)| \leq |\ker \phi(f)| \leq |K|^\frac{\delta}{4}n. \] (9)

Let $T$ be a maximal set of $(i,F,H)$-centers for $\phi$. We need to prove that
\[ |T| \dim_K(F) \geq (1 - \delta)n. \]

Let $V$ be the span of the subspaces $\bigcup_{x \in T} \phi(F)(x)$. Then,
\[ |V| = |K|^{\dim_K(V)} = |K|^{|T| \dim_K(F)}. \]

Assume that $|V| < |K|^{(1-\delta)n}$. Now, suppose that $i,n \geq N_{F,\delta}$. Then
\[ |\bigcup_{v \in V} \bigcup_{f \in F} L(f,v)| \leq |V||F||K|^\frac{\delta}{4}n \leq |K|^{(1-\delta)n} \leq |A_{F,i}|. \]

Therefore, there exists $x \in A_{F,i}$ such that $\phi(F)(x) \cap V = 0$. Hence, $x \cup T$ is a set of $(i,F,H)$-centers for $\phi$, leading to a contradiction. \qed

9 Convergent sequences and sofic approximations

The goal of this section is to prove the following theorem.

**Theorem 5.** Let $D$ be a countable skew field over $K$ and $\phi : \mathbb{F}_r \to D$ be a generating homomorphism. Then $\phi$ is the limit of a convergent sequence of finite dimensional representations satisfying the Atiyah condition if and only if $D$ is sofic.

**Proposition 9.1.** Let $\phi : \mathbb{F}_r \to D$ as above, where $D$ is sofic. Then $\phi$ is the limit of a convergent sequence of finite dimensional representations.

**Proof.** Let $\{\psi_k : D \to \text{Mat}_{n_k \times n_k}(K)\}_{k=1}^{\infty}$ be a sofic approximation sequence for the skew field $D$. 18
Lemma 9.1. For any $B \in \text{Mat}_{n \times n}(D)$
\[
\lim_{k \to \infty} \text{rk}_{n_k}^n (\psi_k(B)) = \text{rk}_D^n(B),
\]
where $\text{rk}_{n_k}^n$ is the matrix rank on $\text{Mat}_{n \times n}(\text{Mat}_{n_k \times n_k}(K))$.

Proof. By taking a subsequence, we may suppose that
\[
\lim_{k \to \infty} \text{rk}_{n_k}^n (\psi_k(B))
\]
does not necessarily define a linear representation of $\mathbb{F}_r$. However, we have the following lemma.

Lemma 9.3. Let $\{\hat{\phi}_k\}_{k=1}^\infty$ be the maps as above. Let $\{\phi_k : \mathbb{F}_r \to \text{Mat}_{n \times n}(K)\}_{k=1}^\infty$ be linear representations such that for any generator $\gamma_i$ of the free group
\[
\lim_{k \to \infty} \text{rk}_{n_k}^n (\phi_k(\gamma_i) - \hat{\phi}_k(\gamma_i)) = 0.
\]

Then for any $n \geq 1$ and $A \in \text{Mat}_{n \times n}(\mathbb{K}\mathbb{F}_r)$
\[
\lim_{k \to \infty} \text{rk}_{n_k}^n (\phi_k(A) - \hat{\phi}_k(A)) = 0.
\]

Proof. Clearly, it is enough to show that for any $a \in \mathbb{K}\mathbb{F}_r$
\[
\lim_{k \to \infty} \text{rk}_{n_k}^n (\phi_k(a) - \hat{\phi}_k(a)) = 0
\]

First we prove (10) in a special case.

Lemma 9.4. Let $\{\hat{\phi}_k\}_{k=1}^\infty$ be the maps as above. Let $\{\phi_k : \mathbb{F}_r \to \text{Mat}_{n \times n}(K)\}_{k=1}^\infty$ be linear representations such that for any generator $\gamma_i$ of the free group
\[
\lim_{k \to \infty} \text{rk}_{n_k}^n (\phi_k(\gamma_i^{-1}) - \hat{\phi}_k(\gamma_i^{-1})) = 0.
\]
Proof. By soficity,

$$\lim_{k \to \infty} \text{rk}_{n_k}(\hat{\phi}_k(\gamma_i^{-1})\phi_k(\gamma_i) - 1) = 0.$$ 

Hence by our assumption,

$$\lim_{k \to \infty} \text{rk}_{n_k}(\hat{\phi}_k(\gamma_i^{-1})\phi_k(\gamma_i) - 1) = 0.$$ 

Thus

$$\lim_{k \to \infty} \text{rk}_{n_k} \left( (\hat{\phi}_k(\gamma_i^{-1}) - \phi_k(\gamma_i^{-1}))\phi_k(\gamma_i) \right) = 0.$$ 

Since $\phi_k(\gamma_i)$ is an invertible element for all $k$, the lemma follows. 

Now suppose that for some $w_1, w_2 \in F_r$

$$\lim_{k \to \infty} \text{rk}_{n_k}(\phi_k(w_1) - \hat{\phi}_k(w_1)) = 0 \quad \text{and} \quad \lim_{k \to \infty} \text{rk}_{n_k}(\phi_k(w_2) - \hat{\phi}_k(w_2)) = 0.$$ 

By soficity,

$$\lim_{k \to \infty} \text{rk}_{n_k}(\phi_k(w_1w_2) - \hat{\phi}_k(w_1)\hat{\phi}_k(w_2)) = 0.$$ 

Since

$$\phi_k(a)\phi_k(b) - \hat{\phi}_k(a)\hat{\phi}_k(b) = (\phi_k(a) - \hat{\phi}_k(a))\phi_k(b) - \hat{\phi}_k(a)(\hat{\phi}_k(b) - \phi_k(b))$$

we have that

$$\lim_{k \to \infty} \text{rk}_{n_k}(\phi_k(w_1w_2) - \hat{\phi}_k(w_1w_2)) = 0.$$ 

Therefore by induction, for any $w \in F_r$

$$\lim_{k \to \infty} \text{rk}_{n_k}(\phi_k(w) - \hat{\phi}_k(w)) = 0.$$ 

Now (10) follows easily. 

We finish the proof of Proposition [9.1]. Observe that

$$\lim_{k \to \infty} \text{rk}_{n_k}\hat{\phi}_k(\gamma_i) = 1$$

for all the generators, hence we have invertible elements $a^k_i \in \text{Mat}_{n_k \times n_k}(K)$ such that

$$\lim_{k \to \infty} \text{rk}_{n_k}(\hat{\phi}_k(\gamma_i) - a^k_i) = 0.$$ 

Now, let us define $\phi_k : F_r \to \text{Mat}_{n_k \times n_k}(K)$ by setting $\phi_k(\gamma_i) = a^k_i$. Then

the proposition immediately follows from Lemma [9.2].
Proposition 9.2. Let \( \{ \theta_k : \mathbb{F}_r \to \text{Mat}_{n_k \times n_k}(\mathcal{K}) \}_{k=1}^{\infty} \) be a convergent sequence of linear representations satisfying the Atiyah condition. Suppose that for some skew field \( D \) and generating map \( \theta : \mathbb{F}_r \to D \), \( \theta \) is the limit of \( \{ \theta_k \}_{k=1}^{\infty} \). Then \( D \) is sofic.

We will prove a stronger version of Proposition 9.2 that will be used in the next section. We call \( a \in k \mathbb{F}_r \) an element of length at most \( l \), if all the non-vanishing terms of \( a = \sum k_iw_i \) have length (as reduced words) at most \( l \). A matrix \( A \in \text{Mat}_{s \times s}(\mathcal{K}\mathbb{F}_r) \) is of length at most \( l \), if all the entries of \( A \) have length at most \( l \). Now, let \( \theta : \mathbb{F}_r \to D \) be a generating homomorphism, where \( D \) is a skew field and \( \{1 = r_1, r_2, \ldots \} \) is a \( \mathcal{K} \)-basis for \( D \). Let \( \rho : \mathbb{F}_r \to \text{Mat}_{s \times s}(\mathcal{K}) \) be a linear representation. We say that a linear map \( \phi : D \to \text{Mat}_{s \times s}(\mathcal{K}) \) is an \((m, \delta)\)-approximate extension of \( \rho \) if

- \( \phi \) is an \( m \)-good map (see Definition 8.1).
- For any element \( a \in \mathcal{K}\mathbb{F}_r \) of length at most \( m \)
  \[ \text{rk}_s(\phi(\theta(a)) - \rho(a)) < \delta. \]

Proposition 9.3. Let \( \theta : \mathbb{F}_r \to D \) be a linear representation into a countable skew field \( D \) such that \( \text{Im}(\theta) \) generates \( D \). Let \( \{1 = r_1, r_2, \ldots \} \) be a \( \mathcal{K} \)-basis of \( D \). Then for any \( m \geq 1 \) and \( \delta > 0 \) there exists a constant \( l_{m,\delta} \) such that if for a homomorphism \( \rho : \mathbb{F}_r \to \text{Mat}_{s \times s}(\mathcal{K}) \),

\[ |\text{rk}_s^n(\rho(A)) - \text{rk}_D^n(\theta(A))| < \frac{1}{l_{m,\delta}} \]  

(11)

whenever \( A \in \text{Mat}_{n \times n}(\mathcal{K}\mathbb{F}_r), n \leq l_{m,\delta} \) is a matrix of length at most \( l_{m,\delta} \), then there exists a \( \mathcal{K} \)-linear unit preserving map \( \phi : D \to \text{Mat}_{s \times s}(\mathcal{K}) \) that is an \((m, \delta)\)-approximate extension of \( \rho \).

Proof. Suppose that the Proposition does not hold for some pair \( m, \delta \). Then there exists a sequence \( \{ \theta_k : \mathbb{F}_r \to \text{Mat}_{n_k \times n_k}(\mathcal{K}) \}_{k=1}^{\infty} \) of linear representations converging to \( \theta \) such that none of the \( \theta_k \)'s have \((m, \delta)\)-approximate extension onto \( D \). Consider that ultraproduct map \( \hat{\theta} : \mathbb{F}_r \to \mathcal{M} \). By Proposition 7.1 and Theorem 2, we can extend \( \hat{\theta} \) onto an embedding \( \phi : D \to \mathcal{M} \) (that is \( \phi \circ \theta = \hat{\theta} \)). From now on, we follow the proof and the notation of Proposition 5.1. For \( d \in D \), let

\[ \phi(d) = [\{ \psi_j(d) \}_{j=1}^{\infty}], \]

where \( \{ \psi_j : D \to \text{Mat}_{n_j \times n_j}(\mathcal{K}) \}_{j=1}^{\infty} \) are not necessarily linear unit preserving maps. For a fixed \( i \geq 1 \), let \( \varepsilon = 1/i \). As observed in the proof of
Proposition 5.1, we have \( j = k_i \geq 1 \) such that equations (3), (4), (5) and (6) are satisfied. Observe that for any \( a \in \mathbb{K}^\mathbb{F}_r \),

\[
\lim_{\omega} (\psi_j(\theta(a)) - \theta_j(a)) = 0.
\]

Therefore, we have \( j = k_i \geq 1 \) such that all the equations above are satisfied, and furthermore

\[
\text{rk}_{n_k_i}(\psi_{k_i}(\theta(a)) - \theta_{k_i}(a)) \leq \frac{1}{i},
\]

for all \( a \in \mathbb{K}^\mathbb{F}_r \) of length at most \( i \). Therefore, mimicking the proof of Proposition 5.1 we can obtain a linear map \( \sigma_i : D \rightarrow \text{Mat}_{n_k_i \times n_k_i}(\mathbb{K}) \) such that

- \( \text{rk}_{n_k_i}(\sigma_i(a)) \geq j(a) \) if \( 0 \neq a \in \text{Span}\{r_1, r_2, \ldots, r_i\} \).
- \( \text{rk}_{n_k_i}(\sigma_i(ab) - \sigma_i(a)\sigma_i(b)) < \frac{1}{i} \) if \( a, b \in \text{Span}\{r_1, r_2, \ldots, r_i\} \).
- \( \text{rk}_{n_k_i}(\sigma_i(\theta(a)) - \theta_{k_i}(a)) \geq \frac{2}{i} \)

for all \( a \in \mathbb{K}^\mathbb{F}_r \) of length at most \( i \).

That is we have a sofic approximation of \( D \), \( \{\sigma_i : D \rightarrow \text{Mat}_{n_k_i \times n_k_i}(\mathbb{K})\}_{i=1}^{\infty} \) such that for any \( a \in \mathbb{K}^\mathbb{F}_r \)

\[
\lim_{i \rightarrow \infty} \text{rk}_{n_k_i}(\sigma_i(\theta(a)) - \theta_{k_i}(a)) = 0
\]

in contradiction with our assumption. \( \square \)

**Remark.** One should note that for a domain \( R \) (provided it is not an Ore domain) it is possible to have many non-isomorphic skew fields with epic embeddings \( \phi : R \rightarrow D \). In fact, according to our knowledge, there is no finitely generated skew field \( D \) countable dimensional over \( \mathbb{K} \) for which epic embeddings \( \phi : \mathbb{K}^\mathbb{F}_r \rightarrow D \) known not to exist. In [15], infinitely many examples of different epic embeddings of \( \theta : \mathbb{K}^\mathbb{F}_r \rightarrow \mathbb{Q} \) are given, where the skew fields \( \mathbb{Q} \) are the quotient fields of certain amenable domains. Since all these skew fields \( \mathbb{Q} \) are amenable, they are sofic, hence by our result above these \( \theta \)'s are limits of finite dimensional representations. We cannot make the difference between these embeddings using only the ranks of group algebra elements. This observation shows why the use of matrices in Definition 2.1 is crucial.
10 Amenable limit fields

The goal of this section is to prove the first part of Theorem 3.

**Proposition 10.1.** Let \( \{ \theta_k : \mathbb{F}_r \to \text{Mat}_{n_k \times n_k}(K) \}_{k=1}^\infty \) be a convergent sequence of representations satisfying the Atiyah condition. Let \( \theta : \mathbb{F}_r \to D \) be a limit representation of \( \{ \theta_k \}_{k=1}^\infty \), where \( D \) is an amenable skew field and \( \text{Im}(\theta) \) generates \( D \). Then \( \{ \theta_k \}_{k=1}^\infty \) is a hyperfinite sequence.

**Proof.** Let \( \{ 1 = r_1, r_2, \ldots \} \) be a \( K \)-basis for \( D \). By Proposition 9.3, we have a sofic approximation sequence \( \{ \phi_k : D \to \text{Mat}_{n_k \times n_k}(K) \}_{k=1}^\infty \) such that for any \( a \in K \)

\[
\lim_{k \to \infty} \text{rk}_{n_k}(\phi_k(\theta(a)) - \theta_k(a)) = 0.
\]

Let \( \epsilon > 0 \) and choose \( \delta > 0 \) in such a way that \((1 - \delta)^2 > 1 - \epsilon\). By Lemma 6.1 we have finite dimensional \( K \)-subspaces \( F_1 \subset F \subset D \) such that \( \theta(\gamma_s)F_1 \subset F \) holds for any generator \( \gamma_s \) and

\[
\dim_K(F_1) > (1 - \delta) \dim_K(F).
\]

Now let \( N_{F,\delta} > 0 \) be the constant in Theorem 4. Let \( i \geq N_{F,\delta} \) be an integer such that

\[
\cup_{s=1}^r \theta(\gamma_s) \cup F \subset \text{Span} \{ r_1, r_2, \ldots, r_i \}.
\]

By definition, there exists \( q \geq 1 \) such that if \( k \geq q \), then

- \( \phi_k \) is \( i \)-good.
- \( \dim_K H_k > (1 - \frac{1}{i})n_k \), where \( H_k = \cap_{s=1}^r \{ x \in K^{n_k} \mid \phi_k(\theta(\gamma_s))(x) = \theta_k(\gamma_s)(x) \} \).

By Theorem 4 if \( k > q \), then \( \phi_k \) has an \( (F, H_k, i, \delta) \)-tiling. Let \( T_k \) be the set of centers of the tiling above. For \( x \in T_k \), let

\[
V_x = \{ \phi_k(F_1)(x) \}.
\]

By our assumptions,

- For any \( x \in T_k \), \( \dim_K(V_x + \sum_{s=1}^r \theta_k(\gamma_s)V_x) \leq \dim_K(F) \).
- \( \sum_{x \in T_k} \dim_K(V_x) \geq (1 - \delta)^2 n_k \).
- The subspaces \( \{ W_x = V_x + \sum_{s=1}^r \theta_k(\gamma_s)V_x \}_{x \in T_k} \) are independent.

Hence, \( \{ \theta_k \}_{k=1}^\infty \) is a hyperfinite sequence. Indeed, \( K_\varepsilon \) can be chosen as \( \{ \max_{1 \leq j \leq q} n_j, \dim_K(F) \} \).

\[23\]
11 Non-amenable limit fields

The goal of this section is to finish the proof of Theorem 3 by proving the following proposition.

**Proposition 11.1.** Let \( \{\theta_k : \mathbb{F}_r \to \text{Mat}_{n_k \times n_k}(\mathcal{K})\}_{k=1}^{\infty} \) be a convergent sequence of finite dimensional representations satisfying the Atiyah condition. Suppose that the generating map \( \theta : \mathbb{F}_r \to D \) is a limit of \( \{\theta_k\}_{k=1}^{\infty} \), where \( D \) is a non-amenable skew field. Then \( \{\theta_k\}_{k=1}^{\infty} \) is not hyperfinite.

The proof of the proposition will be given in several steps. Let \( \theta : \mathbb{F}_r \to D \) be a generating map, where \( D \) is a \( \mathcal{K} \)-skew field with basis \( \{1 = r_1, r_2, \ldots\} \). Let \( \rho_k : D \to \text{Mat}_{n_k \times n_k}(\mathcal{K}) \cong \text{End}_{\mathcal{K}}(\mathcal{K}^{n_k}) \) be a sequence of unit preserving linear maps such that some non-trivial \( \mathcal{K} \)-subspaces \( L_k \subset \mathcal{K}^{n_k} \) are fixed with uniform bound

\[
\dim_{\mathcal{K}} L_k \leq C \in \mathbb{N} \quad \text{for any } k \geq 1.
\]

**Proposition 11.2.** Suppose that \( D, \theta, \{\rho_k\}_{k=1}^{\infty} \) are as above, \( \delta > 0 \), and for any \( k \geq 1 \)

- \( \dim_{\mathcal{K}}(L_k + \sum_{i=1}^{r} \rho_k(\theta(\gamma_i))L) \leq (1 + \delta) \dim_{\mathcal{K}} L_k \).
- \( \rho_k(ab)(x) = \rho_k(a)\rho_k(b)(x) \) if \( x \in L_k \) and \( a, b \in \text{Span} \{r_1, r_2, \ldots, r_k\} \).

Then there exists some integer \( m \geq 1 \) and a finite dimensional \( \mathcal{K} \)-linear subspace \( L \subset D^m \) such that

\[
\dim_{\mathcal{K}}(L + \sum_{i=1}^{r} \theta(\gamma_i)L) \leq (1 + \delta) \dim_{\mathcal{K}}(L).
\]

**Proof.** Again, let \( \omega \) be an ultrafilter on the natural numbers. If \( \{V_k\}_{k=1}^{\infty} \) are finite dimensional \( \mathcal{K} \)-linear vectorspaces, then their ultraproduct \( V = \prod_{\omega} V_k \) is defined the following way. Let \( Z \subset \prod_{k=1}^{\infty} V_k \) be the subspace of sequences \( \{x_k\}_{k=1}^{\infty} \) such that

\[
\{k \mid x_k = 0\} \in \omega.
\]

Then \( V = \prod_{\omega} V_k := \prod_{k=1}^{\infty} V_k/Z \). Observe, that if \( \{W_k \subset V_k\}_{k=1}^{\infty} \) is a sequence of subspaces, then \( \prod_{\omega} W_k \subset \prod_{\omega} V_k \). Also, if \( \{\xi_k : D \to \text{End}_{\mathcal{K}}(V_k)\}_{k=1}^{\infty} \) is a sequence of linear maps, the ultraproduct map \( \zeta : D \to \text{End}_{\mathcal{K}}(V) \) is defined as \( \zeta(d)(x) = [\{\xi_k(x_k)\}_{k=1}^{\infty}] \), where \( x = [\{x_k\}_{k=1}^{\infty}] \in V \).

**Lemma 11.1.** \( L = \prod_{\omega} L_k \) is a non-trivial finite dimensional subspace of \( K = \prod_{\omega} \mathcal{K}^{n_k} \). Also, \( \dim_{\mathcal{K}}(L) = t \), where \( \{k \mid \dim_{\mathcal{K}}(L_k) = t\} \in \omega \).
Proof. Let \( \{a_k^1, a_k^2, \ldots, a_k^C\} \) be a \( K \)-generator system for \( L_k \). Let \( x = \{x_k^1\}_{k=1}^\infty \in L \). Then by finiteness, there exist elements \( \{\lambda_i \in K\}_{i=1}^C \) such that
\[
\{k \mid \sum_{i=1}^C \lambda_i a_k^i = x_k\} \in \omega.
\]
Therefore, \( x = \sum_{i=1}^C \lambda_i a_i \), where \( a_i = \{a_k^i\}_{k=1}^\infty \in L \).

Lemma 11.2. The ultraproduct of the finite dimensional spaces \( \{L_k + \sum_{i=1}^r \rho_k(\theta(\gamma_i))(L_k)\} \) is \( L + \sum_{i=1}^r \rho(\theta(\gamma_i))(L) \).

Proof. All the elements of \( L + \sum_{i=1}^r \rho(\theta(\gamma_i))(L) \) can be written in the form of
\[
x_0 + \sum_{i=1}^r \rho(\theta(\gamma_i))(x_i),
\]
where \( \{x_0, x_1, x_2, \ldots, x_r\} \subset V \). Hence
\[
\prod_{\omega} \{L_k + \sum_{i=1}^r \rho_k(\theta(\gamma_i))(L_k)\} \supset L + \sum_{i=1}^r \rho(\theta(\gamma_i))(L).
\]
On the other hand, all the elements of \( \prod_{\omega} \{L_k + \sum_{i=1}^r \rho_k(\theta(\gamma_i))(L_k)\} \) can be written as
\[
[\{x_0^k + \sum_{i=1}^r \rho_k(\theta(\gamma_i))x_i^k\}_{k=1}^\infty] = [\{x_0^k\}_{k=1}^\infty] + \sum_{i=1}^r \rho(\theta(\gamma_i))\{x_i^k\}_{k=1}^\infty].
\]
Therefore
\[
\prod_{\omega} \{L_k + \sum_{i=1}^r \rho_k(\theta(\gamma_i))(L_k)\} \subset L + \sum_{i=1}^r \rho(\theta(\gamma_i))(L).
\]
By our conditions, if \( k \) is large enough, then
\[
\begin{align*}
\rho_k(b)\rho_k(c)(x) &= \rho_k(bc)(x) \\
\rho_k(a)\rho_k(bc)(x) &= \rho_k(abc)(x) \\
\rho_k(ab)\rho_k(c)(x) &= \rho_k(abc)(x)
\end{align*}
\]
whenever \( x \in L_k \) and \( a, b, c \in D \). Hence for the ultraproduct map \( \rho \),
\[
\rho(ab)\rho(c)(x) = \rho(a)\rho(b)\rho(c)(x),
\]
whenever \( x \in L \).
Now, we finish the proof of Proposition 11.2. Define the $K$-vectorspace $T$, by

$$T := \rho(D)(L) \subset K.$$ 

Then by (12), we have an embedding

$$\psi : D \to \text{End}_K(T)$$

defined by $\psi(d)(z) = \sum_{i=1}^t \rho(dd_i)l_i$, where $\{l_1, l_2, \ldots, l_t\}$ is a $K$-basis of $L$ and $z = \sum_{i=1}^t \rho(d_i)l_i$. Hence, $T$ is a left $D$-vectorspace, with generating system $\{l_1, l_2, \ldots, l_t\}$. Also, by Lemma 11.1 and Lemma 11.2 $L$ is finite dimensional and for $L \subset T \cong D^m$,

$$\dim_K(L + \sum_{i=1}^r \theta(\gamma_i)L) \leq (1 + \delta) \dim_K(L).$$

This finishes the proof of Proposition 11.2.

Recall [9] (Proposition 3.1), that if $D$ is a countable non-amenable skew field over $K$, then there exist elements $d_1, d_2, \ldots d_l$ and $\varepsilon > 0$ such that for any $m \geq 1$ and finite dimensional $K$-subspace $W \subset D^m$,

$$\frac{\dim_K(W + \sum_{i=1}^t d_iW)}{\dim_K W} > 1 + \varepsilon. \quad (13)$$

Now let $\theta : \mathbb{F}_r \to D$ be a generating map.

**Lemma 11.3.** Let $\theta : \mathbb{F}_r \to D$ be a generating map, where $D$ is non-amenable. Then there exists $\delta > 0$ such that for any $m \geq 1$ and finite dimensional $K$-subspace $W \subset D^m$,

$$\frac{\dim_K(W + \sum_{i=1}^r \theta(\gamma_i)W)}{\dim_K W} > 1 + \delta,$$

where $\{\gamma_i\}_{i=1}^r$ is the standard generator system for $\mathbb{F}_r$.

**Proof.** Suppose that such $\delta > 0$ does not exist. Then, we have a sequence of finite dimensional $K$-subspaces $\{W_j \subset D^m\}_{j=1}^\infty$ such that

$$\lim_{j \to \infty} \frac{\dim_K(W_j + \sum_{i=1}^r \theta(\gamma_i)W_j)}{\dim_K W_j} = 1.$$ 

Let $S \subset D$ be the set of elements $s$ in $D$ such that

$$\lim_{j \to \infty} \frac{\dim_K(W_j + sW_j)}{\dim_K W_j} = 1.$$
Lemma 11.4. S is the division closure of $\text{Im} (KF_r)$, that is $S = D$.

Proof. Clearly, if $a, b \in S$, then $a + b \in S$ and $ab \in S$. We need to show that if $a \in S$ then $a^{-1} \in S$. Let $a \in S$. Since $a(a^{-1}W_j + W_j) = (W_j + aW_j)$, we get that

$$\lim_{j \to \infty} \frac{\dim_K(W_j + a^{-1}W_j)}{\dim_K W_j} = 1,$$

therefore $a^{-1} \in S$. \qed

Thus by (13), Lemma 11.3 follows. \qed

Now we finish the proof of Proposition 11.1. Let

$\{\theta_k : F_r \to \text{Mat}_{n_k \times n_k}(K)\}_{k=1}^\infty$ be a convergent, hyperfinite sequence of finite dimensional representations satisfying the Atiyah condition. Let $\theta : F_r \to D$ be the limit map, where $D$ is a non-amenable skew field and $\theta$ is a generating map. Let $\delta > 0$ be the constant in Lemma 11.3. Let $V_1^k, V_2^k, \ldots, V_t^k \subset K^{n_k}$ be $K$-linear subspaces such that

- For any $1 \leq j \leq t_k$, $\dim_K(V_j^k) \leq K_\delta$.
- $\{V_j^k + \sum_{i=1}^r \theta_k(\gamma_i)V_j^k\}_{j=1}^{t_k}$ are independent subspaces such that
  $$\dim_K(V_j^k + \sum_{i=1}^r \theta_k(\gamma_i)V_j^k) < (1 + \delta) \dim_K(V_j^k).$$
- $\sum_{j=1}^{t_k} \dim_K(V_j^k) \geq (1 - \delta)n_k$.

We say that the above subspaces $V_m^k$ and $V_n^l$ are equivalent if there exists a linear isomorphism

$$\zeta : (V_m^k + \sum_{i=1}^r \theta_k(\gamma_i)V_m^k) \to (V_n^l + \sum_{i=1}^r \theta_k(\gamma_i)V_n^l)$$

such that $\zeta(V_m^k) = V_n^l$ and

$$\zeta(\theta_k(\gamma_i))(x) = \theta_j(\gamma_i)(\zeta(x))$$

for any generator $\gamma_i$ and $x \in V_m^k$. By the finiteness of the base field and the uniform dimension bound, there are only finitely many equivalence classes. Hence, by taking a subsequence we can assume that there exists a constant $\tau > 0$ such that for each $k \geq 1$ there are elements $V_1^k, V_2^k, \ldots, V_t^k$ of the above subspaces with the following properties:
• All the $V^k_j$’s are equivalent.

• For each $k \geq 1$, $\sum_{j=1}^{s_k} \dim_{\mathcal{K}}(V^k_j) \geq \tau n_k$.

For $1 \leq j \leq s_k$, let $\zeta^k_j : (V^k_1 + \sum_{i=1}^r \theta_k(\gamma_i)V^k_j) \rightarrow (V^k_j + \sum_{i=1}^r \theta_k(\gamma_i)V^k_j)$ be the isomorphism showing the equivalence. If $\Delta = \{\lambda_1, \lambda_2, \ldots, \lambda_{s_k}\} \subset \mathcal{K}^{s_k}$ is a $s_k$-tuple of elements of $\mathcal{K}$, then $\zeta^k : \Delta \rightarrow \zeta^k$ defines an isomorphism from $(V^k_1 + \sum_{i=1}^r \theta_k(\gamma_i)V^k_j)$ to $(W^k_1 + \sum_{i=1}^r \theta_k(\gamma_i)V^k_j)$, where

• $W^k_\Delta \subset \mathcal{K}^{n_k}$ is a $\mathcal{K}$-subspace of dimension $\dim_{\mathcal{K}}(V^k_1)$

• $\zeta_\Delta(\theta_k(\gamma_i)(x)) = \theta_k(\gamma_i)\zeta_\Delta(x)$, for all $x \in V^k_1$.

**Lemma 11.5.** Let $\{H_k \subset \mathcal{K}^{n_k}\}_{k=1}^\infty$ be a sequence of subspaces such that $\lim_{k \to \infty} \frac{\dim_{\mathcal{K}}(H_k)}{s_k} = 1$. Then if $k$ is large enough, there exists $\Delta$ such that $W^k_\Delta \subset H_k$ (Note that we cannot assume that for large enough $k$, $V^k_j \subset H_k$ for some $j \geq 1$).

**Proof.** Let $y^k_1, y^k_2, \ldots, y^k_t, t \leq K_\varepsilon$ be a $\mathcal{K}$-basis for $V^k_1$. We define the linear map $\iota^k_j : \mathcal{K}^{s_k} \rightarrow \mathcal{K}^{n_k}$ by

$$\iota^k_j(\Delta) = \zeta^k(\delta^k).$$

Since $\{V^k_1, V^k_2, \ldots, V^k_{s_k}\}$ are independent subspaces, $\iota^k_j$ is always an embedding. Let

$$M^k_j = \{\Delta \in \mathcal{K}^{s_k} \mid \iota^k_j(\Delta) \in H_k\}.$$

By our assumption, for any $j \geq 1$,

$$\lim_{k \to \infty} \frac{\dim_{\mathcal{K}}(M^k_j)}{s_k} = 1.$$

Hence, if $k$ is large enough, then $\bigcap_{j=1}^t M^k_j \neq \emptyset$. Therefore, there exists $\Delta \in \mathcal{K}^{s_k}$ such that $\zeta_\Delta(V^k_1) \subset H_k$. \hfill $\Box$

By Proposition 9.3, there exists a sequence of maps $\rho_k : D \rightarrow \text{Mat}_{n_k \times n_k}(\mathcal{K})$ and subspaces $\{H_k \subset \mathcal{K}^{n_k}\}_{k=1}^\infty$ such that

• For any $j \geq 1$, there exists $k_j$ such that $\rho_k(ab)(x) = \rho_k(a)\rho_k(b)(x)$, if $k \geq k_j$, $a, b \in \text{Span}\{r_1, r_2, \ldots, r_j\}$ and $x \in H_k$.

• $\rho_k(\theta(\gamma_i))(x) = \theta_k(\gamma_i)(x)$, if $x \in H_k$ and $\gamma_i$ is a generator of $\mathbb{F}_r$.

• $\lim_{k \to \infty} \frac{\dim_{\mathcal{K}}(H_k)}{n_\varepsilon} = 1$. 

28
Hence, the sequence of maps \( \{ \rho_k \}_{k=1}^{\infty} \) and subspaces \( L_k = W_k \) satisfy the condition of Proposition 11.2. Therefore, there exists \( m \geq 1 \) and a finite \( K \)-dimensional subspace \( L \subset D^m \) such that
\[
\frac{\dim_K(L + \sum_{i=1}^r \theta(\gamma_i)L)}{\dim_K L} \leq 1 + \delta ,
\]
in contradiction with the statement of Lemma 11.3. This finishes the proof of Proposition 11.1.

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