Universal scaling of the velocity field in crack front propagation

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The propagation of a crack front in disordered materials is jerky and characterized by bursts of activity, called avalanches. These phenomena are the manifestation of an out-of-equilibrium phase transition originated by the disorder. As a result avalanches display universal scalings which are however difficult to characterize in experiments at finite drive. Here we show that the correlation functions of the velocity field along the front allow to extract the critical exponents of the transition and to identify the universality class of the system. We employ these correlations to characterize the universal behavior of the transition in simulations and in an experiment of crack propagation. This analysis is robust, efficient and can be extended to all systems displaying avalanche dynamics.

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The presence of disorder is often at the origin of physical behaviors that are not observed in pure systems. In particular, under a slow drive, a disordered system does not respond smoothly but is characterized by quick and large rearrangements called avalanches followed by long quiescent periods. The earthquakes in tectonic dynamics [1–3], the plastic rearrangements in amorphous materials [4, 5] or the Barkhausen noise in soft magnets [6–8] are examples of such avalanches. If the drive is very slow, avalanches are triggered one by one: the system is driven to a first instability and then evolves freely until it stops. In this quasi-static limit one can measure the size and the duration of each avalanche. Their statistics are scale-free on many decades, revealing a critical behavior independent of many microscopic details.

This behavior is well established for earthquakes and Barkhausen noise where events are well separated in time. However in most experimental systems, the driving velocity is finite, so that a subsequent avalanche is often triggered before the previous one stops. One of the standard propositions to define avalanches is to threshold the global velocity signal. However this kind of analysis raises important issues: If the threshold is too large, a single avalanche may be interpreted as a series of seemingly distinct events, while if it is too small subsequent avalanches can be merged into a single event [9–12]. Hence disentangling avalanches becomes nearly impossible and accurately measuring critical exponents is then particularly challenging. Yet the local velocity signal displays two features which are a clear manifestation of the presence of avalanches (see in Fig. 1 the speed signal of two neighboring points extracted from a fracture experiment) : (i) it is very intermittent in time, as it is close to zero for long periods and much larger than the mean velocity for short ones, and (ii) it displays strong correlations in space, as neighboring points often move together. Then an alternative method to define avalanches is to threshold the local velocity signal in order to establish the state (quiescent or active) of each point in the system [13]. The issue is then to decide whether two active regions separated in space and/or time do belong to the same avalanche or not. The latter problem is particularly severe when studying the propagation of cracks [14–17] and wetting fronts [18–20] in disordered materials. In these systems the interactions are proven to be long-ranged [21, 22] and quasi-static avalanches are spatially disconnected objects [23]. Hence reconstructing avalanches from the resulting map of activity clusters remains very difficult, as for large systems there are active points at any time.

In this Letter we develop an alternative strategy. We show that the study of the space and time correlations of the local velocity field, a quantity which is experimentally accessible, allows to capture the universal features of the dynamics without any arbitrariness nor any tunable parameter, even at finite driving speed. We propose and characterize the scaling forms of these functions and show how they relate with the critical exponents of

![FIG. 1: (Color online) Left: Sketch of the experimental setup of planar crack propagation. Profile view (top left): a PMMA plate is detached from a thick PDMS substrat at fixed velocity. Top view (bottom left): the crack front (red line) separates the broken region from the unbroken one. Defects are dots of diameter $d_0 = 100 \mu$m. Right: Local velocity of two points which are $3d_0$ apart along the front. The grey line corresponds to the average speed $v_d$. Each signal is intermittent and the two signals display clear correlations.](image-url)
the avalanche dynamics and with the range of the interactions in the system. Our predictions are tested on numerical simulations and experimental data of crack propagation. For concreteness we focus our discussion on this system, however, as will be discussed in the conclusion, our main results hold for all systems displaying avalanche dynamics but driven at finite velocity.

In Fig. 1 we show a sketch of the crack front where $u(x,t)$ is the front position at point $x$ and time $t$. Its equation of motion in adimensional units writes:

$$\frac{\partial_t u(x,t)}{\mu} = f + \eta(x,u(x,t)) + \frac{1}{\pi} \int \frac{u(x',t) - u(x,t)}{|x' - x|^{1+\alpha}} dx'. \quad (1)$$

The mobility $\mu$ has the dimension of a velocity. In an ideal elastic material it coincides with the Rayleigh velocity $c_R$ but it is in general much smaller (see Supplemental Material (SupMat)). The first term $f$ is the driving force, $\eta(x,u)$ is the normalized toughness fluctuations and the last term accounts for the elasticity along the interface. In general this interaction is long-range with $0 < \alpha \leq 2$ ($\alpha = 2$ corresponds to short-range elasticity). In particular for the crack [24] and the wetting fronts [22] it was shown that $\alpha = 1$.

The competition between elasticity and disorder is at the origin of a second order dynamical phase transition called depinning [25, 26]. The force $f$ is the control parameter and the velocity $v$ is the order parameter which vanishes at a critical force $f_c$. In analogy with equilibrium phase transitions, two independent exponents can be defined: the exponent $\beta$ associated to the order parameter, $v \sim (f - f_c)^\beta$ and the roughness exponent $\zeta$ associated to the fluctuations of the front position, $(u(x,t) - u(0,t))^2 \sim x^{2\zeta}$; the brackets (...) denote the average over different realizations of the disorder.

Below $f_c$ the velocity is zero, but a local perturbation can induce an extended reorganization of the front, the avalanche, up to a scale $\xi \sim |f - f_c|^{-\nu}$, the divergent correlation length of the transition. Symmetries and dimensional analysis allow to link all the exponents of the avalanche statistics (size, duration, ...) to $\beta$ and $\zeta$. In particular, the statistical tilt symmetry ensures the scaling relation $\nu = 1/(\alpha - \zeta)$.

In the moving phase, the dynamical regime we are interested in, it is customary to work with a fixed driving velocity $v_d$ instead of a fixed force $f$. In practice, this is achieved by replacing $f$ with a parabolic potential of curvature $m^2$ moving at velocity $v_d$: $f \rightarrow m^2(v_d t - u(x,t))$ (see SupMat for details). When $v_d$ is small, the local velocity field $v(x,t) = \partial_t u(x,t)$ along the front displays intermittency and correlations (see Fig. 1). Instead of trying to identify avalanches we focus on this quantity and its correlation functions:

$$C_v(x) := \langle v(0,t) v(x,t) \rangle^c = v_d^2 f \left( \frac{x}{\xi_v} \right), \quad (2)$$

$$G_v(\tau) := \langle v(x,t) v(x,t + \tau) \rangle^c = v_d^2 g \left( \frac{\tau}{\tau^*} \right). \quad (3)$$

The proposed scaling forms rely on the existence of two scales: $\xi_v \sim v_d^{-\nu/\beta}$ and $\tau^* \sim v_d^{-\nu_2/\beta}$. The first one is the correlation length at finite velocity and arises naturally from the combination of the scalings of the velocity $v \sim (f - f_c)^\beta$ and of the correlation length $\xi \sim (f - f_c)^{-\nu}$. The time scale $\tau^*$ is linked to $\xi_v$ through the dynamical exponent $z$ [27] : $\tau^* \sim \xi_v^z$. Note that these assumptions are reasonable provided that $m^2$ is small enough, otherwise the parabolic potential confines the interface at length scales $\sim m^{-2/\alpha}$.

**Asymptotic forms** We derive the asymptotic forms of $f(y)$ and $g(y)$ via a scaling analysis based on the existence of a unique correlation length (and a unique correlation time) when $v_d$ is small. Below this length (and time), one expects to find the critical behavior while above it, the $f \rightarrow \infty$ behavior (equivalent to $v_d \rightarrow \infty$) should be recovered. For a slow drive, $v_d \rightarrow 0$, the local velocity is intermittent : $v(0,t)$ is large (with a value of order $v^{\text{max}}$ independent of $v_d$) with probability $\propto v_d$ and almost zero otherwise. The main contribution to $C_v(x)$ comes from the realizations for which both $v(0,t)$ and $v(x,t)$ are of order $v^{\text{max}}$. In the critical regime, one expects from dimensional analysis that if $v(0,t)$ is of order $v^{\text{max}}$, then $v(x,t)$ is also of order $v^{\text{max}}$ with a probability that decays as $x^{-\beta/\nu}$. This gives $C_v(x) \sim v_d x^{-\beta/\nu} \sim v_d^2 (x/\xi_v)^{-\beta/\nu}$. For temporal correlations, a similar reasoning yields $G_v(\tau) \sim v_d^2 \delta(\tau - \tau^{\ast}) \sim v_d^2 (\tau/\tau^{\ast})^{-\beta/\nu}$.

Concerning the large scale behavior, it is convenient to rewrite equation (1) in the comoving frame : $u(x,t) \rightarrow v_d t + u(x,t)$. Only two terms are affected : the parabolic drive becomes $-m^2 u(x,t)$ and the disorder becomes $\eta(x,v_d t + u(x,t))$. For simplicity, we take the disorder to be a white noise:

$$\langle \eta(0,0,0) \rangle \eta(0,v_d t + u(x,t)) \rightarrow \delta(x) \delta(v_d t + \Delta u),$$

where $\Delta u = u(x,t) - u(0,0)$. Let us discuss the behavior of this noise correlation as a function of the length scale. From dimensional analysis $\tau \sim x^\zeta$ while $\Delta u \sim x^\xi$. Hence $v_d \tau/\Delta u \sim \xi_v^{1/\nu} x^{\zeta - \xi} \sim (x/\xi_v)^{\beta/\nu}$. So we see that at large length scale, when $x > \xi_v$, $\Delta u$ is subdominant compared to $v_d \tau$ (the same occurs when one focus on the time scale instead of the length scale). We conclude that the behavior at scales larger than $\xi_v$, or times larger than $\tau^{\ast}$ is captured by the Langevin equation:

$$\partial_t u(x,t) = \eta(x,v_d t) + \frac{1}{\mu} \int \frac{u(x',t) - u(x,t)}{|x' - x|^{1+\alpha}} dx', \quad (4)$$

where for simplicity we took $m^2 = 0$. This equation is linear in $u$ and can be easily solved in Fourier space. Plugging its solution into the correlation function, we obtain (see SupMat for details) :

$$C_v(x \gg \xi_v) \sim \begin{cases} 1/s^{1+\alpha} e^{-s/\xi_v} & \text{for } \alpha < 2, \\ 1/s^{1+\alpha} e^{-s/\xi_v} & \text{for SR}, \end{cases} \quad (5)$$
takes values ranging from 0 to $f$ with

\[ \text{ular automaton version of the variant of equation (1)} \]

ior of $G$

elastic interactions. Interestingly, the long time behav-

relation function provides exactly the range $1 + \alpha$

Note that at large distance, the decay of the spatial cor-

price, we assume periodic boundary conditions along

x. The local velocity is defined as :

\[ v(x, t) = \theta(F(x, t) + \eta(x, u(x, t))) \]  

FIG. 2: (Color online) Spatial correlations in the cellular au-

aton for driving velocities $v_d = 0.002, 0.01, 0.05, 0.1$ and

0.3 (from upper left to lower right). **Main panel** : A perfect collapse is observed using the scaling form (2). The asymptotic behaviors (7) is verified. In particular going at large distances the decay $y^{-2}$ of the elastic interaction is recovered while at small distances the critical behavior $\beta/\nu \simeq 0.385$ is captured. From the crossover the length $\xi_v$ is estimated to be $\xi_v \simeq 0.07 v_d^{-\nu/\beta}$. **Inset** : Non rescaled correlation function $C_v(x)$. System size : $L = 4096$, mass: $m^2 = 10^{-3}$

\[ G_v(\tau \gg t^*) \sim -1/\tau^{1+\frac{1}{\nu}} \quad \text{for} \; \alpha \leq 2. \tag{6} \]

Note that at large distance, the decay of the spatial corre-

tion function provides exactly the range $1 + \alpha$ of the elas-

tic interactions. Interestingly, the long time behavior of $G_v(\tau)$ displays anticorrelations with an $\alpha$ dependent power law decay. This is consistent with the anti-

correlation between the sizes of dynamical avalanches, pre-

dicted and numerically measured in [28]. Collecting all these informations, we can write the full scaling forms (here for $\alpha = 1$, i.e. for crack and wetting fronts) :

\[ f(y) \sim \begin{cases} y^{-\frac{\beta}{\nu}} & \text{if} \; y \ll 1, \\ y^{-2} & \text{if} \; y \gg 1, \end{cases} \tag{7} \]

\[ g(y) \sim \begin{cases} y^{-\frac{\Delta}{\nu}} & \text{if} \; y \ll 1, \\ -y^{-2} & \text{if} \; y \gg 1. \end{cases} \tag{8} \]

**Simulation and experiment** We implemented a cell-

ar automaton version of the variant of equation (1) with $f$ replaced by $m^2 (v_d t - u(x, t))$ and $\alpha = 1$. The three variables $u, x$ and $t$ are integer, in particular $x$ takes values ranging from 0 to $L - 1$, $L$ being the length of the front. Furthermore, in order to compute the elas-

tic force, we assume periodic boundary conditions along $x$. The local velocity is defined as :

\[ v(x, t) = \theta(F(x, t) + \eta(x, u(x, t))) \]  

FIG. 3: (Color online) Spatial correlations in the experiment for driving velocities $v_1 = 132 \pm 3 \text{ nm.s}^{-1}$ and $v_2 = 31 \pm 1 \text{ nm.s}^{-1}$. The large distance decay $x^{-2}$ of the elastic interactions is observed. No rescaling of the $x$-axis was performed.

\[ F(x, t) = m^2 (v_d t - u(x, t)) + \sum_{x'} \frac{u(x', t) - u(x, t)}{|x' - x|^2}, \]

$\theta$ being the Heaviside function. Here the disorder pinning

force $\eta$ should be negative. In practice, we take identical

and independent variables whose distribution is the neg-

ative part of the normal law. At each time step all the points feeling a positive total force jump one step forward while the other points - which feel a negative force - stay pinned. Then the time is incremented, $t \to t + 1$, and the forces are recomputed : new pinning forces are drawn for the jumping points, the elastic force is updated by using a Fast Fourier Transform (FFT) algorithm and the driving force is incremented by $m^2 v_d$. For the numerical implementation, we started from a flat configuration, turned on the dynamics and waited until reaching the steady state before computing the two correlation functions.

The experimental data presented here correspond to planar crack propagation. A 5mm thick plate of PMMA is detached from a thick PDMS substrate using the beam cantilever geometry depicted in Fig. 1 [29]. To introduce disorder, we print obstacles of diameter $d_0 = 100 \mu$m with a density of 20 % on a commercial transparency that is then bonded to the PMMA plate before bringing it in contact with the PDMS. Crack front pinning results from the strong adhesion of the ink dots with PDMS. Images of $1800 \times 1800$ pixels are taken normal to the mean fracture plane every second. As the system is fully transparent, the crack front appears as the interface between the clear and the dark region observed on the image. The pixel size is 35 $\mu$m, so the observed front length is 63 mm.

We tested two different velocity regimes : $v_1 = 132 \pm 3 \text{ nm.s}^{-1}$ and $v_2 = 31 \pm 1 \text{ nm.s}^{-1}$. The local crack
speed is computed using the methodology proposed in Refs. [13, 30] based on the waiting time matrix: the number of frames during which the front stays inside each pixel provides the waiting time in this pixel, from which the local speed is inferred. This procedure significantly reduces the noise level in comparison to the velocity signal obtained from the difference between two successive crack front positions.

Both experiments and the cellular automaton are expected to belong to the universality class of a one-dimensional interface with $\alpha = 1$. The depinning exponents of this class have been computed numerically: $\zeta = 0.388 \pm 0.002$ [31], $\nu = 1/(1 - \zeta) = 1.634 \pm 0.005$, $\beta = 0.625 \pm 0.005$, $\gamma = 0.770 \pm 0.005$ [32] in agreement with renormalization group calculations [33]. The spatial correlations of the local velocity are shown on Figs. 2 and 3. The results of the simulation perfectly collapse on the scaling form (2) showing that a unique correlation length $\xi$ controls the dynamics. The asymptotic form proposed in (7) is verified, in particular the decay in $1/\xi^2$ is the fingerprint of the long-range nature of the elasticity.

In our experimental conditions, we are not able to detect the short distance behavior, as it occurs at distances comparable with the characteristic size $d_0$ of the disorder, but we confirm the large distance decay. This proves that the elastic kernel of the crack front is long-range in this experiment, confirming the interpretation provided by Chopin et al. [29] who used the same experimental system to investigate the depinning dynamics of interfaces from a single obstacle. Interestingly, varying the crack speed $v_d$ does not change the distance to the critical depinning point. This rather counter-intuitive behavior results from the velocity dependence of the material toughness. As shown in [29], this implies that the characteristic mobility $\mu$ involved in the equation (1) scales with the mean crack speed $v_d$ so that the distance $v_d/\mu$ to the critical point remains constant (see also SupMat).

We now turn to the temporal correlation function. The results are shown on Figs. 4 and 5 where we plot the non-connected correlation function which corresponds to $G_v(\tau) + v_d^2$ and normalize it by dividing by $v_d^2$. The parts of the curves below 1 correspond to anticorrelation. Again numerical simulations show a perfect collapse on the scaling form (3) with a unique $t^*$ and the asymptotic form of equation (8) is verified: the exponent $\beta/(\nu z) \approx 0.50$ at small scale is recovered and the anticorrelation displays a power law decay $1/\tau^2$ (see inset in Fig. 4). A similar behavior with a crossover from a power law decay to anticorrelation is observed in the experiment. However, curves corresponding to different crack speeds are collapsed using $v_d$ instead of $t^* = v_d^{\beta/(\nu z)}$. This is also explained by the relation $\mu \sim v_d$, specific to our material. Finally, note the anticorrelation at large time scale. This is the first time that anticorrelation is predicted and observed in depinning systems at finite drive.

Discussion Our findings open new perspectives for the experimental study of disordered elastic interfaces. As the correlations of the local velocity display universal features of the depinning even when the driving speed is finite, the critical behavior can be investigated far from the critical point. This provides a robust and efficient method to identify the universality class of the transition.
and to test the relevance of specific depinning models.

The analysis of the local speed correlations has already been performed in previous simulations and experiments. But the link with the critical exponents was missing. In the simulations of Ref. [34] of an interface with short-range elasticity, the correlation function $C_v(x)$ was used to extract the scale $\xi$, and the exponential cutoff was observed but the small scale exponent $\beta/\nu$ was not predicted. In the fracture experiments of Tallakstad et al. [30], the correlation functions of the local velocity were found to scale as $C_v(x) \sim x^{-\tau_x}$ and $G_v(\tau) \sim \tau^{-\tau_t}$ with exponents $\tau_x = 0.53 \pm 0.12$ and $\tau_t = 0.43$ a bit away from the depinning predictions $\beta/\nu \simeq 0.38$ and $\beta/(\nu z) \simeq 0.50$. However exponential cutoffs at large distances and time were used for the fit and the anticorrelation in time was not observed. Note that standard log-log plot routines discard negative values and one must use alternative plots to see the anticorrelation. It would be interesting to test how far the behavior predicted in this study could capture the Tallakstad et al.’s experiments, as their systems allow the exploration of the crack behavior closer to the critical point than the one used in this study. Finally we note that Gjerden et al. [35] computed the same correlation functions in simulations of a fiber bundle model that mimics the presence of damages in front of the crack. They used parameter values such that their model should fall into the depinning universality class with long-range elasticity [35] and measured $\tau_x = \tau_t = 0.43$ with cutoffs faster than exponential.

Finally it is important to remark that the scaling forms (2) and (3) are very general and valid for all out-of-equilibrium transitions with avalanche dynamics. The asymptotic forms (7) and (8) are also very general, as beyond $\xi_v$ the spatial correlations decay as $1/x^{d+\alpha}$ for a long-range model ($d$ being the spatial dimension) and exponentially fast for short-range elasticity. It would certainly be insightful to probe this behavior in various problems, including those where the nature of the elastic interactions still needs to be deciphered or in the context of the yielding transition where avalanches of plastic events are observed [4].

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Supplemental Material for *Universal scaling of the velocity field in crack front propagation*

We give the principal details of some of the calculations described in the main text of the Letter.

A) Equation of motion for crack propagation in disordered materials

In our experiment a crack propagates at constant velocity $v_d$. The derivation of the equation of motion for the front in presence of impurities can be found in [29]. Here for the sake of completeness we recall the main steps of the derivation and provide an explanation for the surprising observation that the data with $v_d = 132 \pm 3 \text{ nm.s}^{-1}$ and $v_d = 31 \pm 1 \text{ nm.s}^{-1}$ seem to display the same distance from the critical depinning point.

Crack propagation in homogeneous elastic material

It is convenient to start with the homogeneous elastic material. Here the front is perfectly flat and is characterized by its position $u$ and speed $\dot{u}$ (see Fig. 6). Following Griffith’s idea [36], the evolution in time of the crack is determined by the energy balance (per unit surface) between the energy released when the material is fractured and the energy needed to create new fracture surfaces:

$$G^{\text{dyn}}(u, \dot{u}) = G_c.$$  \hspace{1cm} (10)

$G_c$ is the fracture energy, which is constant for homogeneous elastic materials. $G^{\text{dyn}}$ is the energy release rate which accounts for the release of potential elastic energy minus the kinetic term. It displays a simple velocity dependence [37]:

$$G^{\text{dyn}}(u, \dot{u}) = \left(1 - \frac{\dot{u}}{c_R}\right) G^{\text{el}}(u),$$  \hspace{1cm} (11)

where $c_R$ is the Rayleigh wave velocity and $G^{\text{el}}(u)$ is the elastic energy release rate.

In the experimental setup sketched in Fig. 6 we impose a displacement $\Delta$ at the end of the upper plate. The elastic energy associated with the deformation of the plate is a function of the imposed displacement $\Delta$ and of the crack length $u$. In particular if one describes the plate as an Euler-Bernoulli cantilever beam of unit width and height $h$, the elastic energy writes [37]:

$$E^{\text{el}}(u, \Delta) = \frac{E h^3 \Delta^2}{8 u^3},$$  \hspace{1cm} (12)

with $E$ the Young modulus. The elastic energy release rate is then

$$G^{\text{el}}(u, \Delta) = - \frac{dE^{\text{el}}}{du}(u, \Delta) = \frac{3 E h^3 \Delta^2}{8 u^4}.$$  \hspace{1cm} (13)

FIG. 6: Sketch of a crack of length $u$ for an homogeneous elastic material of unit width (along the $x$ direction). The upper plate has height $h$ and a vertical displacement $\Delta$ is imposed at its end. This plate can be described as an Euler-Bernoulli cantilever beam.
The experiment starts by imposing an initial displacement $\Delta_0$ which opens the crack up to a length $u_0$ such that $G^\text{el}(u_0, \Delta_0) = G_c$. Then the displacement is increased as $\Delta(t) = \Delta_0 + v_0 t$ and the crack moves from $u_0$ to $u_0 + u(t)$. Keeping $v_0 t \ll \Delta_0$, $u(t) \ll u_0$ one can write the first order expansion of the elastic energy release rate:

$$G^\text{el}(u_0 + u(t), \Delta_0 + v_0 t) = G_c + \dot{G}_0 t + G_0 u(t), \quad (14)$$

where $\dot{G}_0 = v_0 \delta G^\text{el}(u_0, \Delta_0)$ and $G'_0 = \partial_u G^\text{el}(u_0, \Delta_0)$. When $\dot{u} \ll c_R$, combining equation (14) with equations (10) and (11) to first order yields the following equation of motion:

$$\frac{1}{\mu} \ddot{u} = k (v_d t - u), \quad (15)$$

with $\mu = c_R$, $v_d = -\frac{\dot{G}_0}{G_c} = \frac{u_0}{2\Delta_0} v_0$ and $k = -\frac{G'_0}{G_c} = \frac{4}{u_0}$. Thus by varying $v_0$ one can control the steady velocity $v_d$ of the crack propagation.

**Crack propagation in disordered elastic material**

When the material is heterogeneous the fracture energy displays local fluctuations around its mean value $G_c$:

$$G_c(x, u) = G_c + \delta G_c(x, u). \quad (16)$$

As a consequence the crack front $u(x, t)$ becomes rough. This non trivial shape introduces a correction in the elastic energy release rate, which was computed to first order in perturbation by Rice [21]:

$$G^\text{el}(x, u(x, t), \Delta) = G^\text{el}(\bar{u}(t), \Delta) \left(1 + \frac{1}{\pi} \int \frac{u(x', t) - u(x, t)}{|x' - x|^2} dx'\right), \quad (17)$$

where $\bar{u}(t) = L^{-1} \int u(x, t) dx$ is the average front position. The balance between the energy release and the fracture energy still holds but must now be written at the local level:

$$G^\text{dyn}(x, u(x, t), \dot{u}(x, t)) = \left(1 - \frac{\dot{\bar{u}}(x, t)}{c_R}\right) G^\text{el}(x, u(x, t), \Delta) = G_c(x, u(x, t)). \quad (18)$$

In presence of impurities the first order expansion of $G^\text{el}$ becomes:

$$G^\text{el}(x, u(x, t), t) = G_c + \frac{G_c}{\pi} \int \frac{u(x', t) - u(x, t)}{|x' - x|^2} dx' + \dot{G}_0 t + G'_0 \bar{u}(t). \quad (19)$$

By combining together equations (19), (18), (16) and (11) one obtains:

$$\frac{1}{\mu} \ddot{u} = k (v_d t - \bar{u}(t)) + \frac{1}{\pi} \int \frac{u(x', t) - u(x, t)}{|x' - x|^2} dx' - \frac{\delta G_c}{G_c}, \quad (20)$$

where $\mu = c_R$. This equation is equivalent to equation (1) of the main text: $k$ plays the role of $m^2$, $-\frac{\delta G_c(x, u)}{G_c}$ is the disorder and the elasticity is long-range. However in equation (1) $\bar{u}(t)$ has been replaced by $u(x, t)$. Note that $k = \frac{4}{u_0}$ hence if $u_0$ is large enough the length $k^{-1}$ is much larger than $\xi_v$.

**Crack propagation law in visco-elastic materials**

The PDMS used in our experiments is not perfectly elastic but displays a visco-elastic behaviour. This impacts the fracture energy $G_c$ that shows a rather strong dependence with crack speed [29, 38]:

$$G_c = G_c(\dot{u}) \simeq \left(1 + \frac{\dot{u}}{u_c}\right)^\gamma. \quad (21)$$
In particular for our experiment, we have $v_c \ll v_d$ and $\gamma \simeq 3$ [29], so that $G_c(u) \sim \hat{u}^\gamma$ in the range of crack speeds investigated. Hence the expansion for the fracture energy in presence of impurities should be modified as follow:

$$\frac{G_c(x, u, \hat{u})}{G_c(v_d)} = 1 + \delta G_c(x, u, v_d) \frac{\gamma}{v_d} (\hat{u} - v_d).$$

(22)

The last term in (22) modifies the equation of motion (20) as follow:

$$\left(1 + \frac{\gamma}{v_d} \right) v = k (v_d t - \pi(t)) - \frac{\delta G_c}{G_c} + \frac{1}{\pi} \int \frac{u(x') - u(x)}{|x' - x|^2} dx'. $$

(23)

where the constant $\gamma$ has been absorbed in the loading. Note that equations (20) and (23) have the same form but the mobility has been renormalized. In the experiment the driving velocity satisfies $v_c \ll v_d \ll v_R$ so that $\mu \simeq \frac{v_c}{v_d}$. Functional Renormalization Group calculations have shown that the dynamical correlation length $\xi_v$ depends not only on the driving velocity $v_d$ but also on the mobility [39]:

$$\xi_v = L_c \left( \frac{\mu f_c}{v_d} \right)^{\frac{\nu}{2}}$$

(24)

where $L_c$ is the Larkin length and $f_c$ the critical force. For our experimental conditions the ratio $\frac{\nu}{v_d}$ does not depend on $v_d$. Hence tuning $v_d$ does not change $\xi_v$ and we cannot come closer to the depinning critical point.

**B) Thermal approximation for the large scale behavior of the correlation functions**

For the purpose of computing the tails of the correlation functions it is convenient to rewrite equation (1) of the main text in the comoving frame : $u(x, t) \rightarrow v_d t + u(x, t)$. The disorder term then becomes $\eta(x, v_d t + u(x, t))$. To describes large scales $x > \xi_v$ or $t > t^*$, a reasonable approximation is to use an effective model where the disorder is replaced by white noise. Its correlations then read :

$$\langle \eta(0, u(0,0)) \eta(x, v_d \tau + u(x, \tau)) \rangle = \delta(x) \delta(v_d \tau + \Delta u(x, \tau))$$

(25)

where $\Delta u(x, \tau) := u(x, \tau) - u(0,0)$. As argued in the main text, at large length scale, $x > \xi_v$ or large time scale, $\tau > t^*$, $\Delta u(x, \tau)$ is subdominant compared to $v_d \tau$. The disorder can thus be replaced on these scales by an effective thermal noise $\eta(x, v_d t)$ and the equation of motion becomes a Langevin equation :

$$\partial_t u(x, t) = -m^2 u(x, t) + \eta(x, v_d t) + \int \frac{u(x', t) - u(x, t)}{|x' - x|^{1+\alpha}} dx', $$

(26)

$$\partial_t u(q, t) = -m^2 u(q, t) + \eta(q, v_d t) - |q|^\alpha u(q, t) = -b(q) u(q, t) + \eta(q, v_d t). $$

(27)

where $b(q) = m^2 + |q|^\alpha$. The Fourier transformation from equation (26) to (27) holds for $0 < \alpha < 2$. Equation (27) with $\alpha = 2$ corresponds to the short-range (SR) elasticity. It is a first order linear differential equation and its solution reads :

$$u(q, t) = \int_{-\infty}^{t} e^{-b(q)(t-t')} \eta(q, v_d t') dt'. $$

(28)

The general two-point correlation function, which we denote $C(x, \tau)$ can be written as a Fourier transform :

$$C(x, \tau) := \langle \partial_t u(y, t) \partial_t u(y + x, t + \tau) \rangle = \int \frac{dq_1 dq_2}{(2\pi)^2} e^{iq_1 y} e^{iq_2 (x+y)} \langle \partial_t u(q_1, t) \partial_t u(q_2, t + \tau) \rangle.$$

(29)

We can compute the correlation term in Fourier space by plugging in $\partial_t u(q, t) = \eta(q, v_d t) - b(q) \int_{-\infty}^{t} e^{-b(q)(t-t')} \eta(q, v_d t') dt'$ with Itô convention and using the noise correlation $\langle \eta(q_1, v_d t_1) \eta(q_2, v_d t_2) \rangle = 2\pi \delta(q_1 + q_2) \delta(t_1 - t_2) / v_d$:

$$\langle \partial_t u(q_1, t) \partial_t u(q_2, t + \tau) \rangle = \frac{2\pi \delta(q_1 + q_2)}{v_d} \left( \delta(\tau) - b(q_2) e^{-b(q_2)\tau} + b(q_1) b(q_2) e^{-b(q_2)\tau} \right).$$

(30)
Performing one integral over $q$, equation (29) now reads:

$$C(x, \tau) = \frac{1}{v_d} \delta(x) \delta(\tau) - \frac{1}{2v_d} \int \frac{dq}{2\pi} e^{iqx} e^{-b(q)\tau}.$$  \hspace{1cm} (31)\

The first term is local and originates from the delta function approximation to the noise, and represents the correlations at shorter scales, not described accurately by the present effective model. We now focus on the second term which describes the large scale tail.

Let us indicate the result for $\alpha = 1$, our case of most interest. Let us set the mass to zero. One finds, at large scales $x > \xi$ or $\tau > t^*_v$,

$$C(x, \tau) \simeq \frac{1}{2v_d} \partial_x \int \frac{dq}{2\pi} e^{iqx} e^{-|q|\tau} = \frac{1}{2\pi v_d} \partial_x \frac{\tau}{x^2 + \tau^2} = \frac{1}{2\pi v_d} \frac{x^2 - \tau^2}{(x^2 + \tau^2)^2}.$$  \hspace{1cm} (32)\

Consider now the spatial correlations, setting $\tau = 0$. For $\alpha = 1$ we find the decay

$$C_v(x) \simeq \frac{1}{2\pi v_d} \frac{1}{x^2}.$$  \hspace{1cm} (33)\

This result extends to general $0 < \alpha \leq 2$ as follows:

$$C_v(x) \simeq \frac{\Gamma(1 + \alpha) \sin \left(\frac{\pi\alpha}{2}\right)}{2\pi v_d} \frac{1}{x^{1+\alpha}} \quad \text{for} \quad \alpha \leq 2.$$  \hspace{1cm} (34)\

It can be obtained, e.g. by introducing a regularization factor $e^{-|q|^\alpha}$ in the inverse Fourier transform of $|q|\alpha$ and taking the limit $\epsilon \to 0$ at the end, using:

$$\int \frac{dq}{2\pi} e^{iqx-|q|\alpha} \frac{\Gamma(1 + \alpha)}{\pi(\epsilon^2 + x^2)^{1+\alpha}} \cos \left((1 + \alpha) \arctan \left(\frac{x}{\tau}\right)\right).$$  \hspace{1cm} (35)\

For the short-range elasticity ($\alpha = 2$) the prefactor in (34) vanishes, and the large distance decay is no more a power law within this model, but is much faster. Although a quantitative treatment goes beyond the present effective model, one can obtain some qualitative idea by considering a model with a finite correlation length along $x$, e.g. replacing $\delta(x) \to e^{-\frac{|q|}{\ell}}$. The disorder correlator then becomes:

$$\langle \eta(q_1, v_d t_1) \eta(q_2, v_d t_2) \rangle = \frac{2\ell}{1 + (q_2\ell)^2} \frac{(2\pi)\delta(q_1 + q_2)}{v_d} \frac{\delta(t_1 - t_2)}{v_d}.$$  \hspace{1cm} (36)\

When computing the spatial correlation function in the limit $m \to 0$, we obtain, discarding all $\delta(\tau)$ and $\delta(x)$ terms (i.e. assuming $\ell$ is the largest length)

$$C_v(x) \simeq \frac{1}{v_d} \int \frac{dq}{2\pi} e^{iqx} \frac{(2\pi)\delta(q_1 + q_2)}{v_d} \frac{\delta(t_1 - t_2)}{v_d} \simeq \frac{1}{2v_d} \frac{1}{x^{1+\alpha}} e^{-x/\ell}.$$  \hspace{1cm} (37)\

which is the rationale for the exponential decay quoted in the main text (6). It is then reasonable to expect that $\ell$ will be of order $\xi_v$. Again, this is not at the present stage an accurate calculation which would require to account for more details about the renormalized disorder.

We now turn to the temporal correlations. Consider first $\alpha = 1$. We obtain, setting $x = 0$ and $\tau > 0$ in (32)

$$G_v(\tau) \simeq -\frac{1}{2\pi v_d} \frac{1}{\tau^2}. \quad \text{for} \quad \tau > t^*_v.$$  \hspace{1cm} (38)\

For general $\alpha$ we obtain

$$G_v(\tau) \simeq -\frac{1}{2\pi v_d} \int \frac{dq}{2\pi} b(q) e^{-b(q)\tau} \simeq -\frac{e^{-m^2\tau}}{2\pi v_d} \int_0^\infty dq(m^2 + |q|\alpha) e^{-|q|\alpha \tau} \simeq -\frac{e^{-m^2\tau}(1 + m^2\alpha\tau)}{2\pi v_d \alpha^2} \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\tau^{1+\frac{2}{\alpha}}}.$$  \hspace{1cm} (39)\

In the limit $m \to 0$ this reduces to:

$$G_v(\tau) \simeq -\frac{\Gamma\left(\frac{1}{\alpha}\right)}{2\pi v_d \alpha^2} \frac{1}{\tau^{1+\frac{2}{\alpha}}}.$$  \hspace{1cm} (40)
Several remarks are in order. First we note that, at variance with the spatial decay, the temporal decay remains a power law even for local elasticity, a property of standard diffusion itself. Second, the negative sign in front of the result is the mark of anticorrelations. Although the regime described here $\tau > t^* v$ is far from the intermittent one $\tau < t^* v$, this is in qualitative agreement with the anti-correlation of dynamical avalanches found in [28] (see in particular the Fig. 5 there). We can thus expect a robust region of negative temporal correlations in a broad region of time scales, as observed. Note that in (32) the spatio-temporal correlation changes sign along the line $x = \tau$ (in the present units where all elastic and dynamic coefficients have been set to unity), which would be nice to observe. Equations (34) and (40) correspond to equations (6) and (7) of the main text.

Note that the approach used in this section, based on replacing the quenched noise with a velocity dependent thermal noise, does not allow to recover the correct dependence on $v_d$ but only the large scale dependence on $\tau$ and $x$. Indeed, in the replacement (25) we have not tried to be accurate: one could refine the model by multiplying by a prefactor with the correct dimension, and appropriate dependence in velocity (which could in principle be predicted by the renormalization group [39, 40] which goes beyond this study).