DESIGNS FROM MAXIMAL SUBGROUPS AND CONJUGACY CLASSES OF REE GROUPS

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(Communicated by Leo Storme)

Abstract. In this paper, using a method of construction of 1-designs which are not necessarily symmetric, introduced by Key and Moori in [5], we determine a number of 1-designs with interesting parameters from the maximal subgroups and the conjugacy classes of the small Ree groups $2G_2(q)$. The designs we obtain are invariant under the action of the groups $2G_2(q)$.

1. Introduction

The aim of the present paper is to construct designs from the maximal subgroups and the conjugacy classes of the family of small Ree group $2G_2(q)$, where $q$ is an odd power of 3. The method that we use is one of the two methods introduced by Key and Moori in [4, 5]. The first method called Method 1 in some papers concerns the construction of self-dual symmetric 1-designs from primitive permutation representations of finite simple groups. This method has been applied to several sporadic simple groups ([4, 6, 9, 10, 11, 12, 13, 15]) and to some other families of simple groups (see [14, 16, 17]). The second method introduced in [5] and henceforth called Method 2, outlines the construction of 1-designs which are not necessarily symmetric. The construction in this method uses a maximal subgroup $M$ of a finite simple group $G$ and a conjugacy class in $G$ of some element $x \in M$. In their recent paper [14], the authors constructed designs from Ree groups using Method 1. The
A key-Moori Method 2, which we restate in Section 3 (Lemma 3.1), we construct a family of groups discovered by Ree in the 60s [18]. He showed that these groups are simple except the first one $2G(3)$, which is isomorphic to $PSL_2(8):3$. In [22], Wilson presented a simplified construction of the Ree groups, as the automorphism of a 7-dimensional vector space over the field of $q$ elements. Let $G = 2G_2(q)$ be a small Ree group (we always assume that $q \geq 27$ to avoid the non-simple case). The order of $G$ is $q^3(q^3 + 1)(q^2 - 1)$ and $G$ acts doubly transitive on a set $\Omega$ of size $q^3 + 1$. Moreover, every non-trivial element of $G$ that fixes more than two points in $\Omega$ is an involution. A Sylow 3-subgroup $P$ of $G$ is a TI-subgroup, i.e. for $g \notin N_G(P)$, we have $P \cap P^g = \{1_G\}$. The group $P$ is a 3-group of order $q^3$, where $q$ is a fixed prime. If $P$ is a 3-group of nilpotence class 3 with $|Z(P)| = q$ and $|P'| = q^2$. Both $P'$ and $P/P'$ are elementary abelian 3-groups. Moreover, all elements of order 3 lie in $P'$. All non-identity elements of $Z(P)$ are conjugate in $G$ and

$$P' \setminus Z(P) = b^G \cup (b^{-1})^G,$$

for $b \in P' \setminus Z(P)$. Also, we have $P \setminus P'$ is a union of three conjugacy classes of $G$. For more detailed information on Ree groups, we refer the reader to [14] and [20]. Throughout the rest of this paper, we fix $q$ and assume that $G = 2G_2(q)$. Our notation for groups is mainly from ATLAS [2]. The following theorem which can be found in [21] gives the classification of the maximal subgroups of $G$.

**Theorem 2.1.** The maximal subgroups of $G$, up to conjugacy, are

1. $q^{1+1+1};C_{q-1}$;
2. $C_{q-\sqrt{q+1}}:C_6$;
3. $C_{q+\sqrt{q+1}}:C_6$;
4. $2 \times PSL_2(q)$;
5. $(2^2 \times D_{2+q})C_3$;
6. $2G_2(q_0)$, where $q_0 = \sqrt[3]{q}$.

**Proof.** See [21, Theorem 4.2].

**Notation.** We use the following notation throughout the rest of the paper. Let $t_1 = (q-1)/2$, $t_2 = (q+1)/4$, $t_3 = q - \sqrt{3q} + 1$, $t_4 = q + \sqrt{3q} + 1$, $t_5 = q^3$. Also set $O(q) = \{t_1, t_2, t_3, t_4, t_5\}$. For $1 \leq i \leq 5$, we denote by $B_i$ the set of all subgroups
of $G$ of order $t_1$. From [14], we collect the following results which will be useful for our discussions.

**Proposition 1.** Let $B_i$ be as above, and suppose that $B_i \in B_i$ are chosen arbitrarily for $1 \leq i \leq 5$. Then

(i) every element of $B_i$ is a Hall subgroup of $G$; in particular every two elements of $B_i$ for a fixed $j$ are conjugate in $G$;

(ii) $N_G(B_1) = B_1; 2 \cong D_2(q-1)$;

(iii) $N_G(B_2) = (Q_1 \times (B_2; 2)); 3$ and $N_G(Q_1) = N_G(B_2)$, where $Q_1 \cong 2^2$;

(iv) $N_G(B_3) = B_3; 6$;

(v) $N_G(B_4) = B_4; 6$;

(vi) if $i \neq 5$ then $B_i$ is cyclic; and if $\{1_G\} \neq S \leq B_j$ then $N_G(S) = N_G(B_j)$.

**Proof.** All parts follow from [8] and [20].

**Remark 1.** If $A \in B_i$ for $2 \leq i \leq 5$, then $N_G(A)$ is a maximal subgroup of $G$. If $B \in B_1$ then $N_G(B)$ lies in a maximal subgroup of the form $2 \times PSL_2(q)$.

**Lemma 2.2.** Assume that $B_i \in B_i$ for $1 \leq i \leq 5$ and $x \in G$ is non-trivial. Then the following statements hold.

(i) if $x \in B_1$ then $|x^G| = q^3(q^3 + 1)$;

(ii) if $x \in B_2$ then $|x^G| = q^3(q^2 - q + 1)(q - 1)$;

(iii) if $x \in B_3$ then $|x^G| = q^3(q + 1)(q + \sqrt{3q} + 1)(q - 1)$;

(iv) if $x \in B_4$ then $|x^G| = q^3(q + 1)(q - \sqrt{3q} + 1)(q - 1)$;

(v) if $x \in B_5$ then $|x^G| = q^3(q + 1)(q - 1)$;

(vi) if $o(x) = 2$ then $|x^G| = q^2(q^2 - q + 1)$;

(vii) if $o(x) = 6$ then $|x^G| = q^2(q^3 + 1)(q - 1)$;

(viii) if $o(x) = 7$ then $|x^G| = q^3(q^3 + 1)$;

(ix) if $o(x) = \frac{q^2 + 1}{2}$ then $|x^G| = q^3(q^2 - q + 1)(q - 1)$.

**Proof.** It follows by [14, Lemma 2.6] and the fact that $|x^G| = |G : C_G(x)|$ for all $x \in G$.

**Lemma 2.3.** Let $H$ and $K$ be distinct subgroups of $G$ of equal order $t \in \mathcal{O}(q)$. Then we have $H \cap K = \{1_G\}$.

**Proof.** See [14, Corollary 2.8].

**Lemma 2.4.** All subgroups of order $t \in \mathcal{O}(q)$ are conjugate in $G$.

**Proof.** It follows from part (a) of [19, Theorem 1.1].

**Remark 2.** Let $G$ be a group and $H$ be a subgroup of $G$. The subgroup $H$ is called TI subgroup if for every $g \in G$, $H \cap H^g = 1$ or $H \cap H^g = H$.

**Corollary 1.** All subgroups of order $t \in \mathcal{O}(q)$ are TI subgroups in $G$.

**Proof.** It follows from Lemma 2.3 and Lemma 2.4.

**Lemma 2.5.** Let $G$ be a group and $H$ be a subgroup of $G$. Then for each $x \in G$, $x^G \cap H$ is a union of conjugacy classes of $H$. 

**Proof.** It follows from Lemma 2.3 and Lemma 2.4.
Proof. The proof is straightforward.

Lemma 2.6. Let \( H \leq G \) be a subgroup of order 6. If \( x \in H \) is either of order 3 or 6 then \( x \) and \( x^{-1} \) are not conjugate in \( G \).

Proof. See [14, Lemma 2.11].

Lemma 2.7. If \( M \cong 2 \times \text{PSL}_2(q) \), then it has two conjugacy classes of elements of order 3 and 6.

Proof. It follows from the fact that \( \text{PSL}_2(q) \) has two conjugacy classes of elements of order 3.

Lemma 2.8. Assume that \( M \cong 2 \times \text{PSL}_2(q) \) then all subgroups of order \( q^2 + 1 \) are conjugate in \( M \).

Proof. According to [7, Theorem 2.1], all subgroups of order \( q^2 + 1 \) are conjugate in \( \text{PSL}_2(q) \). It is straightforward to generalize the result to \( M \cong 2 \times \text{PSL}_2(q) \).

3. Constructing designs using Method 2

In this section, we determine the parameters of all possible designs obtained by Method 2 from Ree groups. The following result is the method that we use to construct our non-symmetric 1-designs. Recall that \( \chi_M \) is the permutation character afforded by the action of \( G \) on the set of conjugates of \( M \) in \( G \). According to [16], we define \( A_M = \{|M \cap M^g| | g \in G \} \) for a maximal subgroup \( M \) of \( G \). In [14] we computed \( A_M \) for every maximal subgroup \( M \) of \( G \).

Lemma 3.1. (Method 2) Let \( S \) be a finite simple group, \( M \) a maximal subgroup of \( S \) and \( x \) a conjugacy class of elements of order \( n \) in \( S \) such that \( M \cap x^S \neq \emptyset \). Let \( B = \{(M \cap x^y)^g | y \in S \} \). Then we have \( 1 - (|x^S|, |M \cap x^S|, \chi_M(x)) \) design \( D \). The group \( S \) acts as an automorphism group on \( D \), primitive on blocks and transitive (not necessarily primitive) on points of \( D \).

Proof. See [9, Theorem 12].

If \( S = G \) is the Ree group, then we denote this design by \( D(x, M) \). The following lemma shows that if we obtain two of the three parameters of the design, then the other one is directly computed.

Lemma 3.2. [16, Lemma 4.2] Let \( D = (v, k, \lambda) \) be a design obtained by the construction Method 2. Then \( |G:M| = \lambda v/k \).

3.1. Maximal subgroups of the form \( q^{1+1+1} : C_{(q-1)} \). Let \( M \) be a maximal subgroup of \( G \) of the form \( q^{1+1+1} : C_{(q-1)} \). The following results give the third parameter \( \lambda = \chi_M(x) \) of the designs from \( M \) using Method 2. The first parameter \( v = |x^G| \) is given in Lemma 2.2 and the second parameter \( k = x^G \cap M \) can be computed using Lemma 3.2.

Lemma 3.3. Let \( M \) be a maximal subgroup of \( G \) of the form \( q^{1+1+1} : C_{(q-1)} \) and suppose that \( \chi_M \) is the permutation character afforded by the action of \( G \) on the set of conjugates of \( M \) in \( G \). Then we have

\[
\chi_M(x) = \begin{cases} 1, & \text{if } 3 \mid o(x) \\ q + 1, & \text{if } o(x) = 2 \\ 2, & \text{if } o(x) \mid q - 1 \text{ and } o(x) \neq 2 \end{cases}
\]
Proof. The action of $G$ on the set of conjugates of $M$ is doubly transitive of degree $r := q^3 + 1$. So by [3, Corollary 5.17], the permutation character of $G$ with respect to this action is $1 + \psi$, where $\psi$ is an irreducible character of $G$. Hence $\psi(1) = r - 1 = q^3$. Looking at the character table of $G$ [20], we can see that $G$ has a unique irreducible character of degree $q^3$, which we call $\psi$. The value of $\psi$ on conjugacy classes of $G$ are $-1, 0, 1$ and $q$. More precisely, if $x^G$ is a conjugacy class of $G$, then we have

$$\psi(x^G) = \begin{cases} -1, & \text{if } x^G \cap M = \emptyset \\ 0, & \text{if } 3 | o(x) \\ q, & \text{if } o(x) = 2 \\ 1, & \text{if } o(x) | q - 1 \text{ and } o(x) \neq 2 \end{cases}$$

The proof is now completed. \(\square\)

3.2. Maximal subgroups of the form $C_{q \pm \sqrt{3q} + 1}:C_6$ and $2G_2(q_0)$. Let $H$ be a subgroup of $G$. We say that $H$ controls $G$-fusion in itself if each pair of elements in $H$ which are conjugate in $G$ are also conjugate in $H$. Equivalently, if for $x \in H$ we have $x^G \cap H = x^H$.

Lemma 3.4. Let $G$ be a simple group with a maximal subgroup $M$ and assume that $M$ controls $G$-fusion in itself. Then the designs constructed by Method 2 are $1 - (|x^G|, |x^M|, |\mathcal{C}_G(x) : \mathcal{C}_M(x)|)$ designs, where $x$ is an element of $M$.

Proof. See [16, Proposition 3.4]. \(\square\)

By Lemma 3.4, if $M$ is a maximal subgroup of $G$ that controls $G$-fusion in itself then the parameters of designs using Method 2 can be easily computed. Our aim in this section is to prove that the maximal subgroups of the form $C_{q \pm \sqrt{3q} + 1}:C_6$ and $2G_2(q_0)$ satisfy this property.

Definition 3.5. Let $H \leq G$ and $k$ be a positive integer. We define $cn_H^G(k) := |\{x^G | x \in H, o(x) = k\}|$. Also we write $cn_H(k) := cn_H^G(k)$. It is easy to see that $cn_H^G(k) \leq cn_H(k)$ and if the equality holds then for every $x \in H$ with $o(x) = k$ we have $x^G \cap H = x^H$.

Lemma 3.6. Let $M$ be a maximal subgroup of $G$ of the form $C_{q \pm \sqrt{3q} + 1}:C_6$. Then $M$ controls $G$-fusion in itself.

Proof. Let $x \in M$ be a non-trivial element. Since $cn_M(2) = 1$, then for every involution $x \in M$ we have $x^G \cap M = x^M$. Now assume that $o(x) = 3$ or 6. Clearly, $M$ is a solvable group. So a subgroup of order 6 is a Hall subgroup. Hence all subgroups of order 6 in $M$ are conjugate in $M$. We conclude that there is a Hall subgroup $H$ of order 6 such that $x \in H$. By Lemma 2.6, $x$ is not conjugate to its inverse. This implies that $cn_M^G(t) = cn_M(t) = 2$, for $t = 3$ or 6.

Finally assume that $o(x)|q \pm \sqrt{3q} + 1$ and $y \in M \cap x^G$. So there exists an element $g \in G$ such that $g = x^q$. Hence $y \in H \cap H^g$, where $H \leq M$ and $|H| = q \pm \sqrt{3q} + 1$. Since the subgroups of order $q \pm \sqrt{3q} + 1$ are TI subgroups, we must have $H = H^g$. That is, $g$ lies in $N_G(H) = M$ and the result follows. \(\square\)

Lemma 3.7. Let $M$ be a maximal subgroup of $G$ of form $2G_2(q_0)$. Then $M$ controls $G$-fusion in itself.
Proof. Assume that \( x \in M \) is a non-trivial element of order \( t \). If \( t \in \{2, 3, 6, 9\} \) then \( cn_G^M(t) = cn_M(t) \). So assume that \( t \notin \{2, 3, 6, 9\} \) and \( x \) is conjugate to \( y \) in \( G \). We claim that \( x \) and \( y \) are conjugate in \( M \). First suppose that \( t \) divides \( q_0 - 1 \). Let \( x \in H_1 \) and \( y \in H_2 \), where \( H_1 \) and \( H_2 \) are subgroups of order \( q_0 - 1 \) in \( M \). By Lemma 2.4, \( H_1 \) and \( H_2 \) are conjugate in \( M \). Hence \( y^m \in H_1 \) for some \( m \in M \). Now let \( H \) be a subgroup of order \( q - 1 \) in \( G \), containing \( H_1 \). By Lemma 2.2 we have \( |x^G| = q^3(q^3 + 1) \). Also by Proposition 1, \( H \) contains exactly \( |G : N_G(H)| = q^3(q^3 + 1)/2 \) conjugates in \( G \). Therefore \( x \) has exactly two conjugates in \( H \). On the other hand, \( x \) has exactly two conjugates in \( N_M(H_1) \). Hence \( x \) and \( y^m \) are conjugate in \( M \) and the result follows. By a similar argument, we can prove the lemma for the other cases.

3.3. Maximal subgroups of \( G \) of form \( 2 \times PSL_2(q) \) and \( (2^2 \times D_{2q+1}) : 3 \). In this section, we deal with the remaining maximal subgroups of \( G \). We start with the following result which is true for any group \( G \).

**Lemma 3.8.** Let \( M \) be a maximal subgroup of \( G \) and \( x \in G \). Then
\[
\chi_M(x) = |\{M^g | g \in G, x \in M^g\}|.
\]

Proof. We have \( \chi_M(x) = |Fix(x)| \), where \( Fix(x) \) is the number of the fixed point elements of \( x \). So we can write
\[
Fix(x) = \{M^g | g \in G, M^g = M^xg\}
\]
\[
= \{M^g | g \in G, x \in N_G(M^g)\}
\]
\[
= \{M^g | g \in G, x \in M^g\}.
\]

**Lemma 3.9.** Let \( M \) be a maximal subgroup of \( G \) and \( x \in M \) be an involution. Then we have
\[
(i) \text{ if } M \cong 2 \times PSL_2(q), \text{ then } |x^G \cap M| = q(q - 1) + 1,
\]
\[
(ii) \text{ if } M \cong (2^2 \times D_{2q+1}) : 3 \text{ then } |x^G \cap M| = q + 4.
\]

Proof. In both cases, \( G \) has only one class of involutions. Hence \( |x^G \cap M| \) equals the number of involutions in \( M \). So the result follows by \cite[Lemma 2.10]{14}.

**Lemma 3.10.** Let \( M \) be a maximal subgroup of \( G \) of the form \( 2 \times PSL_2(q) \). Then for a non-trivial \( x \in M \)
\[
(i) \text{ if } o(x) = 2 \text{ then } |x^G \cap M| = q(q - 1) + 1,
\]
\[
(ii) \text{ if } o(x) = 3 \text{ then } |x^G \cap M| = \frac{q^2 - 1}{2},
\]
\[
(iii) \text{ if } o(x)|\frac{q+1}{2} \text{ then } |x^G \cap M| = 3q(q - 1),
\]
\[
(iv) \text{ if } o(x)|\frac{q+1}{2} \text{ then } |x^G \cap M| = 3q(q - 1),
\]
\[
(v) \text{ if } x \text{ is none of the above then } \chi_M(x) = 1.
\]

Proof. By \cite[Proposition 3.9]{14}, \( A_M = \{1, 2, 4, q, 2(q + 1), |M|\} \). Let \( x \in G \) be non-trivial. If \( o(x) \) does not divide an element of \( A_M \setminus |M| \), then by Lemma 3.8 we have \( \chi_M(x) = 1 \). So we can assume that \( o(x) \in \{2, 3, \frac{q+1}{2}, t\} \) where \( t|\frac{q+1}{2} \).

**Case 1:** If \( o(x) = 2 \) then \( |x^G \cap M| = q(q - 1) + 1 \) by Lemma 3.9.
Case 2: Now assume that \( o(x) = 3 \) and \( x = y^{9} \) for some \( y \in M \) and \( g \in G \). By Lemma 2.7, \( M \) has two conjugacy classes of elements of order 3. On the other hand \( x \) is not conjugate to \( x^{-1} \) in \( G \) from Lemma 2.6. Hence \( cn_{M}^{G}(3) = cn_{M}(3) = 2 \) and we conclude that \( x^{G} \cap M = x^{M} \). But \( |x^{M}| = 2^{3-1} \) and the result follows.

Case 3: Now, suppose that \( o(x) = t \). As \( |x^{M}| = q(q-1) \), we need to prove that \( |x^{G} \cap M| = 3|x^{M}| \). Let \( y \in M \) such that \( y = x^{q} \). So we can find \( K_{1}, K_{2} \leq M \) of order \( \frac{q+1}{2} \), such that \( x \in K_{1} \) and \( y \in K_{2} \). By Lemma 2.8, \( K_{1} \) and \( K_{2} \) are conjugate in \( M \). Hence there is an element \( m \in M \) such that \( K_{2} = K_{1}^{m} \). Clearly, \( y = x^{q} \in K_{2} \cap K_{1}^{m} = K_{1}^{m} \cap K_{1}^{q} \). By Corollary 1 all subgroups of order \( \frac{q+1}{2} \) in \( G \) are TI subgroups, we have \( K_{1}^{m} = K_{1}^{q} \). Hence \( gm^{-1} \in NG(K_{1}) \). By the structure of \( M \), it is easy to check that \( |N_{M}(K_{1})| = 2(q+1) \). According to Proposition 1, \( N_{G}(K_{1}) = L:D \), where \( |L| = 2(q+1) \) and \( D \cong C_{3} \). Since \( L \) is a normal Hall subgroup of the solvable group \( N_{G}(K_{1}) \) and \( |L| = |N_{M}(K_{1})| \), we have \( L = N_{M}(K_{1}) \). Now let \( D = \langle d \rangle \). We claim that \( x^{G} \cap M = x_{0}^{M} \cup x_{1}^{M} \cup x_{2}^{M} \), where \( x_{i} = x^{id} \) and \( i \in \{0,1,2\} \). Clearly, \( x_{0}^{M} \cup x_{1}^{M} \cup x_{2}^{M} \subseteq x^{G} \cap M \). Let \( x^{q} = y \in M \). Arguing similarly as before we find an element \( m \in M \) such that \( gm^{-1} \in N_{G}(K_{1}) = L:D \). Hence there is \( l \in L \leq M \) such that \( gm^{-1} = dl \) and \( i \in \{0,1,2\} \). So \( x^{q} = x^{id} \in x^{M} \). Since the order of centralizer of all elements of order \( t \) are equal to \( q+1 \), we have \( |x^{G} \cap M| = 3|x^{M}| = 3q(q-1) \). Therefore, \( \chi_{M}(x) = 3 \).

Case 4: Finally, let \( o(x) = \frac{q+1}{2} \). Then we can easily find elements \( j_{1}, x_{1} \in M \) such that \( x = j_{1}x_{1} \), \( o(j_{1}) = 2 \) and \( o(x_{1}) = \frac{q+1}{4} \). Since all involutions in \( G \) are conjugate, the result follows by an argument similar to that used for \( x_{1} \) in Case 3.

Lemma 3.11. Let \( M \) be a maximal subgroup of the form \((2^{2} \times D_{\frac{q+1}{2}}):3\). Then for \( x \in M \nabla

\begin{itemize}
  \item[(i)] if \( o(x) = 2 \) then \( |x^{G} \cap M| = q + 4 \),
  \item[(ii)] if \( o(x) = 3 \) or 6 then \( x^{G} \cap M = x^{M} \),
  \item[(iii)] if \( o(x) \) is not as above then \( \chi_{M}(x) = 1 \).
\end{itemize}

Proof. By [14, Proposition 3.9], \( \mathcal{A}_{M} = \{1, 2, 3, 4, 6, 8, |M|\} \). Let \( x \in G \) be non-trivial. If \( o(x) \) does not divide an element of \( \mathcal{A}_{M} \setminus \{|M|\} \), then by Lemma 3.8 we have \( \chi_{M}(x) = 1 \). So we can assume that \( o(x) \in \{2, 3, 6\} \). If \( o(x) = 2 \), then by Lemma 3.9, we have \( |x^{G} \cap M| = q + 4 \). Now assume that \( o(x) = t \) where \( t = 3 \) or 6. Let \( y = x^{q} \) for some \( y \in M \) and \( g \in G \). By the structure of \( M \), \( \mathcal{C}_{M}(x) \cong C_{6} \). Hence by Lemma 2.6, \( x \) is not conjugate to \( x^{-1} \) in \( G \). Since the order of a Sylow 3-subgroup of \( M \) is 3, we have \( cn_{M}(t) \leq 2 \). On the other hand, \( 2 = cn_{M}^{G}(t) \leq cn_{M}(t) \). So \( cn_{M}^{G}(t) = cn_{M}(t) = 2 \) and we conclude that \( x^{G} \cap M = x^{M} \). \( \square \)

3.4. Main Theorem. By using results in Section 2 and Subsections 3.1, 3.2 and 3.3, we are able to state and prove our main result in Theorem 3.12. This subsection ends with Table 1, which gives the parameters of the constructed designs.

Theorem 3.12. Let \( M_{i}, (1 \leq i \leq 6) \) be a maximal subgroup of \( G \) of the form (i) as in Theorem 2.1 and let \( x \in M \) be a non-trivial element. Then the parameters of all non-trivial \( 1 \)-designs \( D(x, M_{i}) = (v, k, \lambda) \) are as given in Table 1.
Proof. By Lemma 3.1, $\mathcal{D}(x, M_i) = (|x^G|, |M \cap x^G|, \chi_M(x))$. The first parameter $|x^G|$ is given in Lemma 2.2. By the results in this section, either $|M_i \cap x^G|$ or $\chi_M(x)$ are known, and the other can be directly computed using Lemma 3.2. Note that in the last two rows of Table 1, (1) and (2) indicate that only one of these designs can be constructed, according to whether $q^k \parallel q^\pm$ or $q - 1$. The proof of the theorem is now complete.

| Max $t = o(x)$ | $s = |x^G|$ | $k = |M \cap x^G|$ | $\lambda = \chi_M(x)$ |
|---------------|-------------|-----------------|-----------------|
| $M_1$ | $t = 2$ | $q^2(q^2 - q + 1)$ | $q^2$ | $q - 1$ |
| $M_1$ | $t = 3$ | $(q^2 + 1)(q - 1)$ | $q - 1$ | 1 |
| $M_1$ | $t = 3$ | $q(q^2 + 1)(q - 1)$ | $q(q - 1)$ | 1 |
| $M_1$ | $t = 9$ | $q^2(q^2 + 1)(q - 1)$ | $q^2(q - 1)$ | 1 |
| $M_1$ | $t = 6$ | $q^2(q^2 + 1)(q - 1)$ | $q^2(q - 1)$ | 1 |
| $M_1$ | $t(q - 1), t \neq 2$ | $q^2(q^2 + 1)$ | $2q^2$ | 2 |
| $M_2, M_3$ | $t = 2$ | $q^2(q^2 - q + 1)$ | $q^2$ | $2(q^2 - 1)$ |
| $M_2, M_3$ | $t = 3$ | $(q^2 + 1)(q - 1)$ | $q^2$ | $q^2$ |
| $M_2, M_3$ | $t = 6$ | $q^2(q^2 + 1)(q - 1)$ | $q^2$ | 4 |
| $M_4$ | $t|q^2$, $t \neq 2$ | $q^2(q^2 - q + 1)$ | $q^2(q^2 - q + 1)(q - 1)$ | 3 |
| $M_4$ | $t = 3$ | $q^2(q^2 + 1)(q - 1)$ | $q^2(q^2 - q + 1)$ | 3 |
| $M_5$ | $t = 6$ | $q^2(q^2 + 1)(q - 1)$ | $q^2(q^2 - q + 1)$ | 3 |
| $M_5$ | $t|q^2$, $t \neq 2$ | $q^2(q^2 - q + 1)(q - 1)$ | $3q^2(q^2 - q + 1)$ | 3 |
| $M_6$ | $t = 2$ | $q^2(q^2 - q + 1)$ | $q^2(q^2 - q + 1)$ | 6 |
| $M_6$ | $t = 3$ | $(q^2 + 1)(q - 1)$ | $(q^2 + 1)(q - 1)$ | 9 |
| $M_6$ | $t = 6$ | $q^2(q^2 + 1)(q - 1)$ | $q^2(q^2 + 1)(q - 1)$ | 9 |
| $M_6$ | $t = 9$ | $q^2(q^2 + 1)(q - 1)$ | $q^2(q^2 + 1)(q - 1)$ | 9 |
| $M_6$ | $t(q - 1), t \neq 2$ | $q^2(q^2 + 1)$ | $q^2(q^2 + 1)$ | 9 |
| $M_6$ | $t|q^2$, $t \neq 2$ | $q^2(q^2 - q + 1)(q - 1)$ | $q^2(q^2 - q + 1)$ | 9 |
| $M_6$ | $t|q^2$, $t \neq 2$ | $q^2(q^2 - q + 1)$ | $q^2(q^2 - q + 1)$ | 9 |
| $M_6$ | $t|q^2$, $t \neq 2$ | $q^2(q^2 - q + 1)$ | $q^2(q^2 - q + 1)$ | 9 |

Acknowledgments

The authors wish to thank the referee for comments and suggestions.

References

[1] E. F. Assmus Jr. and J. D. Key, Designs and Their Codes, Cambridge Tracts in Mathematics, 103, Cambridge University Press, Cambridge, 1992.
[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups, Oxford University Press, Eynsham, 1985.
[3] I. M. Isaacs, Character Theory of Finite Groups, Dover Publications, Inc., New York, 1994.
[4] J. D. Key and J. Moori, Codes, designs and graphs from the Janko groups $J_1$ and $J_2$, J. Combin. Math. Combin. Comput., 40 (2002), 143–159.
[5] J. D. Key and J. Moori, Designs from maximal subgroups and conjugacy classes of finite simple groups, J. Combin. Math. Combin. Comput., 99 (2016), 41–60.
[6] J. D. Key, J. Moori and B. G. Rodrigues, On some designs and codes from primitive representations of some finite simple groups, *Combin. Math. Combin. Comput.*, 45 (2003), 3–19.

[7] O. H. King, The subgroup structure of finite classical groups in terms of geometric configurations, in *Surveys in Combinatorics*, London Math. Soc. Lecture Note Ser., 327, Cambridge Univ. Press, Cambridge, 2005, 26–56.

[8] V. M. Levchuk and Y. N. Nuzhin, The structure of Ree groups, *Algebra i Logika*, 24 (1985), 26–41.

[9] J. Moori, Finite groups, designs and codes, in *Information Security, Coding Theory and Related Combinatorics*, NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur., 29, IOS, Amsterdam, 2011, 202–230.

[10] J. Moori and B. G. Rodrigues, A self-orthogonal doubly even code invariant under *McL*:2, *J. Combin. Theory Ser. A*, 110 (2005), 53–69.

[11] J. Moori and B. G. Rodrigues, Some designs and codes invariant under the simple group *Co*₂, *J. Algebra*, 316 (2007), 649–661.

[12] J. Moori and B. G. Rodrigues, A self-orthogonal doubly-even code invariant under the *McL*, *Ars Comb.*, 91 (2009), 321–332.

[13] J. Moori and B. G. Rodrigues, On some designs and codes invariant under the Higman-Sims group, *Util. Math.*, 86 (2011), 225–239.

[14] J. Moori, B. G. Rodrigues, A. Saeidi and S. Zandi, Some symmetric designs invariant under the small Ree groups, *Comm. Algebra*, 47 (2019), 2131–2148.

[15] J. Moori and A. Saeidi, Some designs and codes invariant under the Tits group, *Adv. Math. Commun.*, 11 (2017), 77–82.

[16] J. Moori and A. Saeidi, Some design invariant under the Suzuki groups, *Util. Math.*, 109 (2018), 105–114.

[17] J. Moori and A. Saeidi, Constructing some design invariant under the *PSL*₂(q), *q* even, *Comm. Algebra*, 46 (2018), 160–166.

[18] R. Ree, A family of simple groups associated with the simple Lie algebra of type (G₂), *Amer. J. Math.*, 83 (1961), 432–462.

[19] D. O. Revin and E. P. Vdovin, On the number of classes of conjugate Hall subgroups in finite simple groups, *J. Algebra*, 324 (2010), 3614–3652.

[20] H. Ward, On Ree’s series of simple groups, *Trans. Amer. Math. Soc.*, 121 (1966), 62–89.

[21] R. A. Wilson, *The Finite Simple Groups*, Graduate Texts in Mathematics, 251, Springer-Verlag London, Ltd., London, 2009.

[22] R. A. Wilson, Another new approach to the small Ree groups, *Arch. Math. (Basel)*, 94 (2010), 501–510.

Received August 2018; revised August 2019.

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