Connections between Quasi-periodicity and Modulation in Pulsating Stars

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Abstract. The observations of the photometric space telescopes CoRoT and Kepler show numerous Blazhko RR Lyrae stars which have non-repetitive modulation cycles. The phenomenon can be explained by multi-periodic, stochastic or chaotic processes. From a mathematical point of view, almost periodic functions describe all of them. We assumed band-limited almost periodic functions either for the light curves of the main pulsation or for the modulation functions. The resulting light curves can generally be described analytically and it can also be examined by numerical simulations. This presentation is a part of our systematic study on the modulation of pulsating stars (Benkő et al. 2009, 2011, 2012).

1. Definitions

When we speak about quasi-periodic signals in physics we generally think about signals which can be described mathematically by almost periodic functions and not quasi-periodic ones. Before defining them let we remind the well-known definition of periodic function. \( x(t) = x(t + P) \), or \( |x(t) - x(t + P)| = 0 \) for all \( t \).

\[
x(t) = a_0 + \sum_{n=1}^{\infty} a_n \sin(2\pi n f_0 t + \varphi_n),
\]

where \( f_0, a_n, \varphi_n \) are constants.

The definition of almost periodic function is quite intuitive. \( x(t) \) real function is a band-limited almost periodic function with the period \( P \) if

\[
x(t) \approx x(t + P), \quad \text{or} \quad |x(t) - x(t + P)| < \varepsilon \quad \text{for all} \ t,
\]

where

\[
0 < \varepsilon \ll \|x\| = \sqrt{x^2} = \sqrt{\lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} x^2(t)dt}.
\]

\(^1\)There are numerous non-equivalent definitions for almost periodic functions in the literature (see Bredikhin [2001]). We use now this simple definition.
Its Fourier representation is

\[ x(t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \sin \left[ 2\pi \int_{0}^{t} f_n(\tau) \, d\tau + \varphi_n(0) \right]. \tag{4} \]

Taking into account Eq. (4) the instantaneous frequency is

\[ f_n(t) = n f_0(t) + \frac{1}{2\pi} \varphi'_n(t), \]

where \( f_0(t), a_n(t), \varphi_n(t) \) are changing in time, but slowly. An important consequence of this expression is that the “harmonics” are not exactly harmonic: \( f_n(t) \neq n f_0(t) \! \).)

For the numerical tests we need a base light curve. Since this light curve should be strictly periodic we use the form Eq. (2) for producing it. In practice, we chose Fourier parameters of a typical RR Lyrae star’s light curve and an artificial light curve prepared from them as it is shown in Fig. 1.

2. Quasi-periodic Pulsation

First we made quasi-periodic signals using the light curve in Fig. 1 with the assumptions: \( f_0(t_{i+1}) = f_0(t_i) + k^i e_i; a_n(t_{i+1}) = a_n(t_i) + k^i e_i \) and \( \varphi_n(t_{i+1}) = \varphi_n(t_i) + k^i e_i \), where \( e_i \) is the standard white noise process and \( k \)-s are constants. The indices \( i = 1, 2, \ldots \), mean the consecutive sampling points. In other words, we supposed the time variability to be an auto-regressive (AR) random process. This approximation tests stochastic signals (see Fig. 2). The simulated light curves show correlated amplitude and frequency
connections, but the time scale of the variations support only the longest period Blazhko cycles. The corresponding Fourier spectra show harmonics surrounded by side peaks (even multiplets), but generally no peaks can be detected in the low frequency regime where the Blazhko frequency $f_m$ can be found for observed stars. If we change the zero point strongly some peaks appear in this low frequency range, but we parallelly get a strange random walk in the average brightness that we never observed in real stars.

If the quasi-periodic variation is caused by (multi)periodic variation of which has a characteristic period(s) $f_m^j \ll f_0$ for all $j$, we are facing a general amplitude and frequency modulated signal. Here the modulation functions depend on the harmonics as it was introduced by Szeidl et al. (2012). It is easy to verify: since $a_n(t)$, $f_0(t)$ and $\varphi_n(t)$ functions are periodic, they can be represented by Fourier series in a form of (2). Substituting these forms into expression (4) we get the equations of Szeidl et al. (2012). If the modulation function always depends on the harmonics of the Blazhko stars as stated by Szeidl et al. (2012), our formalism suggests that even the simplest Blazhko stars have a quasi-periodic (at least a multi-periodic) nature. This consequence, however, needs a careful check and validation.

3. Simultaneous Pulsation and Modulation

As we have seen in the previous section the multi-periodicity is in close connection with the modulation. For simplicity, we investigated the modulated signals with global (harmonic independent) modulation functions as they were used in Benkő et al. (2011).
We constructed light curves using the strictly periodic unmodulated light curve in Fig. 1. It is changed with random modulation, namely, the modulation functions were assumed as $f^m(t_{i+1}) = f^m(t_i) + k^t e_i$ where $e_i$ and $k^t$ are the white noise process and a constant term, respectively. As an alternative we tested cases where the modulation functions were strictly periodic and the pulsation varied randomly as it is described in Sec. 2.

Since the average pulsation and modulation are described exactly the consecutive runs demonstrate the possible light curve and Fourier spectra deviations. The simulated light curves have very similar characteristics to the observed ones (see left panels in Fig. 3). They show changes of the envelope curves, changes in the Fourier amplitudes of the modulation frequency and the side peaks; sometimes additional peaks appear (right panels in Fig. 3). All of these features are in the same magnitude as the observed effects.

4. Conclusions

We checked two options and their sub-cases: almost periodic pulsation alone and simultaneous pulsation and modulation where (at least) one of them was almost periodic. Random pulsation yields light curves with amplitude and frequency variation but they
show significant differences compared to the observed light curves. We showed that the strictly periodic sub-case results in a general modulation description.

Simultaneous modulation and pulsation where one of them has quasi-periodic behaviour yields completely analogous light curves and Fourier spectra with the observed data. The key question is whether the random perturbation is caused by a stochastic or a chaotic process. The best method for determining which one is the so-called “phase space reconstruction” which will be the next step in our future analysis.

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