Contextual Search for General Hypothesis Classes

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Abstract

We study a general version of the problem of online learning under binary feedback: there is a hidden function \( f : \mathcal{X} \to \mathcal{Y} \) in a certain hypothesis class \( \mathcal{H} \). A learner is given adversarially chosen inputs (contexts) \( x_t \in \mathcal{X} \) and is asked to submit a guess \( y_t \in \mathcal{Y} \) for the value \( f(x_t) \). Upon guessing the learner incurs a certain loss \( L(y_t, f(x_t)) \) and learns whether \( y_t \leq f(x_t) \) or \( y_t > f(x_t) \). The special case where \( \mathcal{H} \) is the class of linear functions over the unit ball has been studied in a series of papers. We both generalize and improve these results. We provide a \( O(d^2) \) regret bound where \( d \) is the covering dimension of the hypothesis class. The algorithms are based on a novel technique which we call Steiner potential since in the linear case it reduces to controlling the value of the Steiner polynomial of a convex region at various scales. We also show that this new technique provides optimal regret (up to log factors) in the linear case (i.e. the original contextual search problem), improving the previously known bound of \( O(d^4) \) to \( O(d \log d) \). Finally, we extend these results to a noisy feedback model, where each round our feedback is flipped with fixed probability \( p < 1/2 \).

1 Introduction

Puzzle  
We start with a puzzle: there is a hidden vector \( w \in [0,1]^d \), and for \( T \) periods you are given subsets \( S_t \subseteq [d] \) and asked to submit a guess \( y_t \) for \( y_t^* := \max_{i \in S_t} w_i \). After guessing you learn whether \( y_t \leq y_t^* \) or \( y_t > y_t^* \). You wish to minimize the total \( \ell_1 \)-loss of your guesses: \( \sum_{t=1}^T |y_t - y_t^*| \). When \( d = 1 \) this is essentially binary search and it is straightforward to get \( O(1) \) loss. Is it possible to get \( O(\text{poly}(d)) \) loss (independent of \( T \)) for this problem?

Framework for learning with binary feedback  
We develop a general framework for online learning problems under binary feedback. Binary feedback arises naturally in many settings such as pricing [15][8][11][13][14][16][20] (where the learner quotes a price and learns only if the customer purchased or not), personalized medicine [2] (where the learner chooses the dosage of a medicine and observes whether the patient was over-dosed or under-dosed), and one-bit compressed sensing [6][19] (where you only learn the sign of your measurement). In such situations the learner typically cannot even evaluate the actual loss from the feedback. Learning with binary feedback has also been used as a primitive for designing learning algorithm with an unknown fairness metric [7].

In our general setup, the learner is trying to learn a function \( f \) in a hypothesis class \( \mathcal{H} \) containing functions mapping from a context space \( \mathcal{X} \) to an outcome space \( \mathcal{Y} \). In each step a context \( x_t \in \mathcal{X} \) is chosen adversarially and the learner is asked to submit a guess \( y_t \) for the value of \( f(x_t) \) and incurs a loss \( L(f(x_t), y_t) \). The goal of the learner is to minimize the total loss \( \sum_{t=1}^T L(f(x_t), y_t) \).
If $\mathcal{X}$ is the unit ball $\mathbb{B} \subseteq \mathbb{R}^d$, $\mathcal{Y} = [-1, 1]$, $\mathcal{H}$ is the class of all linear functions $f_w(x) = w \cdot x$ for $w \in \mathbb{B}$, and the loss is $L(y, y') = |y - y'|$, this is the (linear case of the) contextual search problem. This problem has been extensively studied in recent years: Cohen et al [5] gave a $O(d^2 \log T)$ regret algorithm, which was later improved to $O(d \log T)$ by Lobel et al [16]. Paes Leme and Schneider [14] gave the first bound that is independent of time horizon of $O(d^4)$. These results for the linear case are driven by techniques from convex geometry which are not applicable to general functions. In the puzzle earlier in the introduction, for example, the set of vectors $w$ consistent with previous observations is typically not convex.

The natural lower bound on the achievable regret for the linear case is $\Omega(d)$ (since orthogonal contexts are equivalent to $d$ independent binary search problems). There are two natural approaches for the linear case of contextual search: keep track of all vectors $w \in \mathbb{R}^d$ consistent with the feedback obtained so far and either: (i) choose the guess that will cause the volume to be decreased by half; or (ii) choose the mid-point between the smallest and largest consistent guess. A lower bound in Section 8 of [16] shows that approach (i) has a regret of at least $\Omega(d^2)$ (but the best upper bound from [14] shows only that this has regret at most $2^{O(d \log d)}$). For approach (ii), a lower bound in [5] shows the regret can be at least $2^{\Omega(d)}$.

Our results We generalize previous results from linear functions to general hypothesis classes $\mathcal{H}$. By using the same techniques, we additionally obtain an algorithm with optimal total regret (up to log factors) for the linear case of contextual search. We also extend the results to noisy binary feedback. Our main results are as follows:

- If $d$ is the covering dimension of the hypothesis class $\mathcal{H}$ (see Definition 2.3) we provide an algorithm with regret $O(d^2)$ (Section 3.3).

- For linear contextual search (i.e. the case with linear functions over the unit ball) we provide an algorithm with nearly optimal regret $O(d \log d)$ (Section 3.5). The previous best known bound is $O(d^4)$ in [14].

- If the feedback is noisy (i.e. flipped with constant probability $p < 1/2$) we give an algorithm with regret $O(d \log T)$ for a hypothesis class with covering dimension $d$ (Section 4.1).

- For linear contextual search with noisy feedback, we give an algorithm with regret $O(\text{poly}(d))$ (Section 4.2).

- For the case of full-feedback (i.e. the algorithm learns $f(x_t)$) we give matching upper and lower bounds (up to constant factors) on the achievable regret. This is based on the notion of tree-dimension of a hypothesis class, which is a continuous analogue of Littlestone dimension (Section 5).

Our techniques To give an idea of the main techniques in the paper, let us start by discussing the specific case of linear contextual search. The previous best algorithm in this setting [14] (achieving $O(d^4)$ regret) used the approach of constructing a potential function based on the intrinsic volumes of the set $S$ of currently feasible hidden points. The intrinsic volumes in $\mathbb{R}^d$ (a concept from integral geometry [10]) are a set of $d$ “volume functions” $V_i(S)$ of convex sets, one for each dimension.

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1 We use the terms “regret” and “loss” interchangeably throughout this paper.
$1 \leq i \leq d$. These functions capture standard notions of volume (for example, $V_d(S)$ is the volume of $S$, and $V_{d-1}(S)$ is proportional to the “surface area” of $S$) and have many nice properties (invariant under isometries of the plane, monotone, etc.).

One way to define these intrinsic volumes is through Steiner’s formula, which expresses the volume of the Minkowski sum $\text{Vol}(S + tB)$ (where $B$ is the unit ball) as a polynomial of degree $d$ in $t$. The $d$ intrinsic volumes are exactly the coefficients of this polynomial (after appropriate normalization by the volume $\kappa_j$ of the $j$-dimensional unit ball):

$$\text{Vol}(S + tB) = \sum_{j=0}^{d} V_j(S)\kappa_{d-j}t^{d-j}.$$

The $O(d^4)$ algorithm in [14] works by showing that we can always guess so to decrease one of the intrinsic volumes $V_j(S)$ by some amount proportional to the loss we incur. In this paper, we instead show how to guess so to directly decrease $\text{Vol}(S + tB)$ by a constant factor — with some care in how we choose $t$ and our guess, this leads to an algorithm with $O(d\log d)$ total loss. This quantity $\text{Vol}(S + tB)$ can itself be thought of as a potential function; we call it (and the general technique) the Steiner potential.

Intriguingly, it is possible to generalize (in some sense) this geometric technique to arbitrary classes $H$ of hypotheses. Instead of keeping track of all functions in the hypothesis class that are consistent with the feedback so far, we keep track of an expanded set of functions that don’t violate the feedback up to a certain margin (in the linear case, this is exactly $S + tB$). This has the effect of regularizing the set of consistent hypotheses and allows for faster progress. Instead of volume, in the general case we control the size of an $\epsilon$-net of the set of these approximately valid hypotheses.

A second technique we use is adaptive scaling, which involves keeping track of multiple levels of discretization. For the linear case, this boils down to controlling the value of the Steiner polynomial at different values of $t$. More generally, at each step, we can estimate the maximum possible loss achievable in this round given the previous feedback. Based on this value, we will choose a scale, which will dictate the granularity of the $\epsilon$-net and the margin with which we prune inconsistent hypotheses. After picking the scale we show that it is possible to pick a (random) cut that will either: (i) reduce the number of valid hypotheses in the chosen granularity by half; or (ii) eliminate one valid hypothesis at a much coarser granularity. This will require a careful coupling between the discretizations at two different levels. This coupling between two levels is what allows us to overcome the fact that in the general case we can’t rely on techniques from convex geometry. See Section 3.2 for details.

One important feature of all our algorithms (not shared by previous algorithms) will be our use of randomness, in particular perturbed guesses. Every round, we compute the median $m_t$ of the set $f(x_t)$ where $f$ ranges over the set of approximately valid hypotheses. However, instead of guessing the median $m_t$ directly we guess a random value in $\{m_t - u, m_t + u\}$ (where the size of perturbation $u$ depends on our current scale). Our guarantee is that the potential function will decrease significantly for one of the cases (and thus in expectation).

We then consider the setting of binary feedback with noise. In this setting we move from keeping track of a set of approximately valid hypotheses to a pseudo-Bayesian approach, where we maintain a distribution $w$ over approximately valid hypotheses and update it as we receive feedback. By carefully bounding the weight of hypotheses within a ball of radius $1/T$, this results in an algorithm with loss $O(d\log T)$. 

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Ideally, it would be possible to combine this algorithm with the adaptive scaling approach of the noiseless setting, resulting in an $O(\text{poly}(d))$ regret algorithm for general hypothesis classes. While there are difficulties with this approach for general hypothesis classes, we show how to use this idea along with the geometric structure of linear hypothesis classes to achieve an $O(\text{poly}(d))$ regret algorithm for noisy linear contextual search. This is the first algorithm we are aware of in the noisy setting which gets any regret independent of $T$ for $d > 1$ and is one of the main technical contributions of this paper.

**Other related work** Our results in the full feedback case—parameterizing the optimal regret in terms of the tree dimension—can be seen as a generalization of similar results for Littlestone dimension [15] (indeed, in the case where $\mathcal{Y} = \{0, 1\}$, our notion of tree dimension reduces to Littlestone dimension). To the best of our knowledge, the results in Section 5 are the first results extending Littlestone dimension for realizable hypotheses to arbitrary classes of labels (but would not be surprised if this idea already exists in some form).

The idea of “adaptive discretization” or “zooming” is present in many online learning algorithms, especially those that operate in continuous action spaces or continuous context spaces [4, 12, 21]. However, these algorithms are usually designed for settings where 1. one cannot hope for better than $O(\sqrt{T})$ regret (let alone regret independent of $T$), 2. feedback is not binary but rather zeroth-order ([18] study a pricing setting where feedback is binary, but where the hypothesis class is large enough that one must incur $O(\text{poly}(T))$ regret). As such, the techniques used in these papers are markedly different than the techniques used in this work. In particular, we believe our technique of coupling together the potentials for different scales in the analysis of Theorem 5.3 is novel.

## 2 Framework for Online Learning with Binary Feedback

Consider a hypothesis class $\mathcal{H}$ consisting of functions mapping $\mathcal{X}$ to $\mathcal{Y}$. We refer to $\mathcal{X}$ as the context space and $\mathcal{Y}$ as the output space. We assume that the output space $\mathcal{Y}$ is a totally ordered set, i.e., for each $y_1, y_2 \in \mathcal{Y}$ with $y_1 \neq y_2$ we have either $y_1 < y_2$ or $y_2 < y_1$.

The learning protocol is as follows: an adversary chooses some $f_0 \in H$ and in each round they choose some $x_t \in \mathcal{X}$. The learner makes a prediction $y_t \in \mathcal{Y}$ and incurs loss $L(f_0(x_t), y_t)$ for some loss function $L : \mathcal{Y} \times \mathcal{Y} \to [0, 1]$. Upon making a prediction, the learner receives feedback on whether $y_t \leq f_0(x_t)$ or $y_t \geq f_0(x_t)$ (the feedback is arbitrary in case of equality). It will be convenient to represent the feedback as a variable $\sigma_t \in \{-1, +1\}$ such that $\sigma_t = 1$ if $y_t > f_0(x_t)$ and $\sigma_t = -1$ otherwise.

We make the following assumptions about the loss function throughout the paper:

- **Reflexive**: $L(y, y) = 0$ for all $y \in \mathcal{Y}$.
- **Symmetry**: $L(y_1, y_2) = L(y_2, y_1)$ for all $y_1, y_2 \in \mathcal{Y}$.
- **Triangle inequality**: $L(y_1, y_2) \leq L(y_1, y') + L(y', y_2)$ for all $y_1, y_2, y' \in \mathcal{Y}$.
- **Order consistency**: If $y_1 < y_2 < y_3$ then $\max\{L(y_1, y_2), L(y_2, y_3)\} \leq L(y_1, y_3)$.
- **Continuity**: If $0 < \ell < L(y_1, y_2)$ then there are $y', y'' \in \mathcal{Y}$ such that $\ell = L(y_1, y') = L(y'', y_2)$.

If $\mathcal{Y} = \mathbb{R}$, then for any continuous increasing function $\phi : \mathbb{R} \to [0, 1]$ and parameter $\alpha \leq 1$, the loss $L(y_1, y_2) = |\phi(y_1) - \phi(y_2)|^\alpha$ satisfies the desired properties.
2.1 Covering Dimension

Our loss bounds will be in terms of the covering dimension of the hypothesis class \( \mathcal{H} \). We start by defining a metric \( d_\infty(\cdot, \cdot) \) on \( \mathcal{H} \) induced by the loss function:

**Definition 2.1.** For two hypotheses \( f_1, f_2 \in \mathcal{H} \), let \( d_\infty(f_1, f_2) = \sup_{x \in \mathcal{X}} (L(f_1(x), f_2(x))) \)

We can now introduce the notions of \( \epsilon \)-net and covering dimension.

**Definition 2.2 (\( \epsilon \)-net).** For an \( \epsilon \), we say that a set \( S \subseteq \mathcal{H} \) is an \( \epsilon \)-net of \( \mathcal{H} \) under the \( d_\infty \) metric if for every \( h \in \mathcal{H} \), there is \( h' \in S \) with \( d_\infty(h, h') \leq \epsilon \). Let \( N_\epsilon(\mathcal{H}) \) be an \( \epsilon \)-net of \( \mathcal{H} \) of minimum cardinality.

**Definition 2.3 (Covering Dimension).** Define the covering dimension \( \mathcal{H} \) as

\[
Cdim(\mathcal{H}) = \sup_{0 < \epsilon \leq \frac{1}{2}} \frac{\log |N_\epsilon(\mathcal{H})|}{\log \frac{1}{\epsilon}}
\]

Note that this definition of \( Cdim \) differs from Hausdorff dimension in that we care not just about the limit \( \epsilon \to 0 \), but the largest value for any \( \epsilon \in [0, 1/2] \); importantly, this guarantees us that, for any \( \epsilon \in (0, 1) \),

\[
N_\epsilon(\mathcal{H}) \leq \max(\epsilon^{-1}, 2)^{Cdim(\mathcal{H})}.
\]

Note that we specify \( \epsilon \leq \frac{1}{2} \) to avoid issues when \( \epsilon \) is close to 1 - any other fixed constant upper bound \( p \) only changes this dimension by a constant factor of at most \( 1/\log(1/p) \).

We give a few quick examples to give intuition about covering dimension.

**Example 1.** The space of functions \( f : [d] \to \{0, 1\} \) with loss function \( L(y_1, y_2) = 1_{y_1 \neq y_2} \) has covering dimension \( d \).

**Example 2 (Linear contextual Search).** Let \( \mathcal{B} \) be the unit-ball in \( \mathbb{R}^d \), i.e. \( \mathcal{B} = \{x \in \mathbb{R}^d ; \|x\|_2 \leq 1\} \). For each \( v \in \mathcal{B} \), let \( f_v : \mathcal{B} \to \mathbb{R} \) be defined by the dot product \( f_v(x) = v \cdot x \). The linear contextual search problem is defined as the learning problem for class \( \mathcal{H} = \{f_v; v \in \mathcal{B}\} \) with loss function \( L(y_1, y_2) = |y_1 - y_2| \). This class has covering dimension \( O(d) \).

To see that the covering dimension is \( O(d) \), first note that \( d_\infty(f_v, f_u) = \|u - v\|_2 \). It suffices to show that for any \( 0 < \epsilon \leq \frac{1}{2} \), there is an \( \epsilon \)-net of the sphere of size \( (1/\epsilon)^{O(d)} \). To do this, we can greedily place points in the unit ball such that no two are within \( \epsilon \) of each other. If we draw an \( \frac{\epsilon}{2} \)-radius ball around each point, these balls must be disjoint and contained in a ball centered at the origin of radius \( 1 + \frac{\epsilon}{2} \). Thus, the maximum number of points we will place is

\[
\left(1 + \frac{\epsilon}{2}\right)^d = \left(1 + \frac{2}{\epsilon}\right)^d,
\]

giving us an \( \epsilon \)-net of the same size.

**Example 3 (Sparse Contextual Search).** The sparse version of the contextual search problem is given by class \( \mathcal{H} = \{f_v; v \in \mathcal{B}, \|v\|_0 \leq s\} \) where \( \|v\|_0 := |\{i; v_i \neq 0\}| \). The loss function is still \( L(y_1, y_2) = |y_1 - y_2| \). The covering dimension of this class is \( O(s \log d) \) (where we treat \( s \) as a
constant and \(d\) as tending to \(\infty\). To see that the covering dimension is \(O(s \log d)\), note that for any \(\epsilon\) we can use the result in the previous example to obtain an \(\epsilon\)-net of size
\[
\left(\frac{d}{s}\right) \left(1 + \frac{2}{\epsilon}\right)^s \leq s^d \left(1 + \frac{2}{\epsilon}\right)^s.
\]

**Example 4** (Unit demand). In the unit demand version of contextual search the set \(X = \{0, 1\}^d\) and the hypothesis class consists of functions \(f_w(x) = \max_{i \in [d]} w_i x_i\) for \(w \in [0, 1]^d\). The covering dimension of this class is \(O(d)\). This example corresponds to the puzzle in the introduction and corresponds to the economic situation where a seller wants to price a bundle of goods (represented by the context) but the buyer has an unit-demand valuation, i.e., only cares about the highest-valued item in the bundle.

To see that the covering dimension is \(O(d)\), note that the set \(S = \{\epsilon x | x \in \{0, 1, \ldots, \lfloor 1/\epsilon \rfloor\}^d\}\) forms an \(\epsilon\)-net and \(|S| \sim \left(\frac{1}{\epsilon}\right)^d\).

### 3 Loss Bounds from Noiseless Binary Feedback

For simplicity, in the following theorems we will assume our hypotheses map \(X \rightarrow \mathbb{R}\) and that loss is \(\ell_1\), i.e., \(L(y, y') = |y - y'|\). We will then remark on how to generalize our proof to other loss functions.

#### 3.1 \(O(d \log(T))\) regret via Single-scale Steiner Potential

We will obtain an algorithm which incurs expected loss \(O(d^2)\) for \(d = \dim(H)\). It is instructive to start with a simpler algorithm with regret \(O(d \log T)\) which illustrates the idea of the Steiner potential: instead of keeping track of the hypotheses that are consistent with the feedback so far, we will keep an inflated version of that set.

This algorithm starts with a \(T^{-1}\)-net of the hypothesis class and keeps a set of candidate hypothesis that are approximately consistent with observations seen so far. For each \(x_t\), it queries a random point around the median, halving the set of hypotheses with at least half probability.

**Algorithm 1** SINGLE-SCALE STEINER POTENTIAL

\[
\text{Initialize } H_1 = N_{1/T}(H) \\
\text{for } t \text{ in } 1, 2, \ldots, T \text{ do} \\
\quad \text{Adversary picks } x_t \\
\quad \text{Let } m_t \text{ be the median of } \{f(x_t); f \in H_t\} \\
\quad \text{Choose } y_t = m_t - 2/T \text{ or } m_t + 2/T, \text{ each with probability half} \\
\quad \text{Update } H_{t+1} = \{f \in H_t; \sigma_i(y_t - f(x_t)) \geq -1/T\}
\]

The analysis is based on the following lemma:

**Lemma 3.1.** If \(|m_t - f_0(x_t)| > 2/T\), then \(|H_t| \leq \frac{1}{2}|H_{t-1}|\) with probability at least \(1/2\).

**Proof.** Assume \(f_0(x_t) > m_t + 2/T\) (the other case is analogous), then with probability half, the algorithm guesses \(y_t = m_t + 2/T\) and gets the feedback that \(f_0(x_t) \geq y_t\), which eliminates all the hypotheses \(f\) such that \(f(x_t) < m_t + 1/T\). These hypotheses constitute at least half of \(H_t\). \(\square\)
Corollary 3.2. The Single-scale Steiner Potential obtains loss $O(d \log T)$ in expectation.

Proof. The total loss from periods where $|m_t - f_0(x_t)| \leq 2/T$ is at most $O(1)$. For the remaining periods, the size of $|H_t|$ is halved with at least $\frac{1}{2}$ probability. Note that for such a $t$,

$$
\mathbb{E}\left[ \frac{1}{|H_{t+1}|} \right] \geq \frac{3}{2} \mathbb{E}\left[ \frac{1}{|H_t|} \right]
$$

Next, $|H_t| \geq 1$ for all $t$ since there is some element in $H_1$ that is $\frac{1}{d}$ close to $f_0$ which is never eliminated. However, $\frac{1}{|H_t|} \geq \frac{1}{Td}$ and $\frac{1}{|H_t|} \leq 1$ for all $t$. Thus for any integer $c$, the probability that there are $c$ periods with loss greater than $\frac{1}{d}$ is at most $\left(\frac{1}{T}\right)^c$. Thus, the expected number of periods with loss larger than $2/T$ is at most

$$
10d \log T + \sum_{i=[10d \log T]}^{\infty} \frac{T^d}{\left(\frac{2}{T}\right)^i} = O(d \log T).
$$

\[\blacksquare\]

3.2 Strategy to achieve constant (in $T$) loss

In this subsection we provide some intuition on how to improve from $O(d \log T)$ to $O(d^2)$. In Single-scale Steiner Potential a loss larger than $1/T$ causes $H_t$ to half in size, but whenever it halves in size the only bound we can get for the loss is 1. To improve this bound, we need to guarantee that a loss of 1 can’t occur very often. Our strategy for doing that involves keeping multiple levels of discretization. Given $y_t$ and the feedback $\sigma_t \in \{-1, +1\}$ we will keep for each $u > 0$:

$$
H_1^u = N_u(H) \quad \text{and} \quad H_{t+1}^u = \{f \in H_t^u; \sigma_t(y_t - f(x_t)) \geq -u\}
$$

In other words, we keep an $u$-discretization of hypotheses along with all the hypotheses that are consistent with the feedback so far with an $u$-margin. The $u$-margin is important to guarantee that any hypothesis that is $u$-close to $f_0$ will never be eliminated. We will also refer to $H_0^0$ as the set of hypotheses consistent with observations so far without any discretization or margin.

Our strategy will be to choose in each round some discretization level $u$ based on the maximum possible loss achievable in this round. We will divide the space of losses in exponentially sized buckets and define $i$ to be the index of the bucket where the maximum loss falls:

$$
\text{MAXLOSS}_t := \max_{f \in H_t^u} f(x_t) - \min_{f \in H_t^u} f(x_t) \in \left[10 \cdot 2^{-(i+1)}, 10 \cdot 2^{-i}\right].
$$

Now we choose $u = u_i$ based on $i$, compute the median $m_i$ of $\{f(x_t); f \in H_t^u\}$ and guess either $y_t = m_t - 2u_i$ or $m_t + 2u_i$ with half probability each. Now one of two things can happen:

- If the loss is larger than $2u_i$, the set $H_t^{u_i}$ will decrease by a factor of 2 with half probability. This should happen at most $\log |N_{u_i}|$ times in expectation, generating loss $10 \cdot 2^{-i} \log |N_{u_i}| = O(2^{-i} \cdot d \log (1/u_i)).$

- If the loss is smaller than $2u_i$ we will show that the set $H_t^{2^{-i}}$ will decrease by at least 1 element in expectation (Lemma 3.5), so we get loss $u_i \cdot |N_{2^{-i}}| = O(u_i 2^{di})$.
This leads to a total loss of:

\[ O \left( \sum_i 2^{-i} \cdot d \log(1/u_i) + u_i 2^{d_i} \right) = O(d^2) \quad \text{for} \quad u_i = \frac{1}{3^d(i+1)} \]

### 3.3 Analysis of the $O(d^2)$ algorithm

**Theorem 3.3.** Let $d = \text{Cdim}(H)$. The Multi-scale Steiner Potential algorithm incurs expected total loss $O(d^2)$ in the binary feedback model.

**Algorithm 2** Multi-scale Steiner Potential

Let $u_i = 3^{-d(i+1)}$ and $v_i = 2^{-i}$ for all $i$

for $t$ in $1, 2, \ldots, T$

Adversary picks $x_t$

Let $i$ be the largest index such that $\max_{f \in H_0^t} f(x_t) - \min_{f \in H_0^t} f(x_t) \leq 10v_i$

Let $m_t$ be the median of the set $\{ f(x_t) | f \in H_0^t \}$

Query either $m_t + 2u_i$ or $m_t - 2u_i$ each with half probability

Note if there does not exist an index $i$ such that $\max_{f \in S_t} f(x_t) - \min_{f \in S_t} f(s_t) \leq 10v_i$, then we must actually have $\max_{f \in S_t} f(x_t) = \min_{f \in S_t} f(x_t) = f_0(x_t)$. In this case we know the value of $f_0(x_t)$ for sure so we simply query this value and incur 0 loss.

**Lemma 3.4.** If $i$ is the index chosen in the $t$-th step and $|m_t - f_0(x_t)| > 2u_i$ then $|H_0^t| \leq \frac{1}{2} |H_0^{t+1}|$ with probability at least $1/2$.

**Proof.** The same as the proof of Lemma 3.1 replacing $1/T$ by $u_i$. ■

The new ingredient is a “potential” argument when the loss is small:

**Lemma 3.5.** If $i$ is the index chosen in the $t$-th step and $|m_t - f_0(x_t)| \leq 2u_i$, then with probability at least $1/2$, $|H_0^{t+1}| \leq |H_0^t| - 1$.

**Proof.** By the choice of the index $i$, $\max_{f \in H_0^t} f(x_t) - \min_{f \in H_0^t} f(x_t) > 10v_i$, so there must exist $f \in H_0^t$ such that $|f(x_t) - m_t| \geq 5v_{i+1}$. Let’s assume that $f(x_t) \geq m_t + 5v_{i+1}$ (the other case is analogous). The algorithm will query $m_t + 2u_i$ with half probability and learn that $f_0(x_t) \leq m_t + 2u_i$ (by the assumption that $|m_t - f_0(x_t)| \leq 2u_i$). Such a query must eliminate some hypothesis $f' \in H_0^{t+1}$ since there must be some $f' \in H_0^{t+1}$ with $d_\infty(f, f') \leq v_{i+1}$, so this hypothesis must satisfy $f'(x_t) \geq m_t + 4v_{i+1}$ and hence will be ruled out by the information from querying $m_t + 2u_i$. ■

We can now proceed to prove Theorem 3.3.

**Proof of Theorem 3.3.** Let $A_i$ be the number of times that index $i$ is chosen by the algorithm and $|m_t - f_0(x_t)| > 2u_i$. Let $B_i$ be the number of times that index $i$ is chosen and $|m_t - f_0(x_t)| \leq 2u_i$. Combining the previous two claims (Lemmas 3.4 and 3.5), we have that

\[
\mathbb{E}[A_i] \leq 2 \log_2 N_{u_i}(H) \\
\mathbb{E}[B_i] \leq 2N_{v_{i+1}}(H)
\]

8
For each query with index \(i\), the loss is at most \(10v_i\). Also for queries with \(|mt - f_0(x_t)| \leq 2u_t\), the loss is at most \(2u_t\). Thus the total loss is at most \(\sum_i (10v_iA_i + 2u_iB_i)\). It remains to note that

\[
E \left[ \sum_i (10v_iA_i + 2u_iB_i) \right] \leq \left( \sum_{i=1}^{\infty} 20v_i \log_2 N_{u_t}(H) + 4u_t N_{u_{t+1}}(H) \right) = O(d^2).
\]

\[\blacksquare\]

**Remark.** To adjust our proof to deal with any loss functions satisfying the assumptions outlined in Section 2, we make the following adjustments. We replace \(\max_{f \in S_t} f(x_t) - \min_{f \in S_t} f(x_t)\) with \(L(\max_{f \in S_t} f(x_t), \min_{f \in S_t} f(x_t))\). Also, we replace \(mt + 2u_t\) with any \(y \in Y\) such that \(mt < y\) and \(L(mt, y) = 2u_t\) and similar for \(mt - 2u_t\) (note this y exists by the continuity of the loss function).

### 3.4 Connection to the Steiner Polynomial

We reinterpret this algorithm in the special case of linear contextual search (Example 2). We will exploit the geometric structure of this problem to improve the bound to \(O(d \log(d))\), which is optimal up to log factors. Reasoning about the continuous set of hypotheses will also lead to an algorithm with \(O(poly(d, T))\) running time, as opposed to keeping track of \(\epsilon\)-nets of \(T^{O(d)}\) size. The other advantage will be conceptual: we will be able to contrast our approach to the approach in [14] based on Intrinsic Volumes.

**Intrinsic Volumes and the Steiner Polynomial**

Intrinsic volumes can be defined via the Steiner polynomial [10]. Given a convex set \(S \subseteq \mathbb{R}^d\), Steiner showed that the volume of the Minkowski sum \(\text{Vol}(S + tB)\) is a polynomial of degree \(d\) in \(t\) (where \(B\) is the unit ball). The intrinsic volumes \(V_j\) of \(S\) correspond to the coefficients of this polynomial after normalization by volume \(\kappa_j\) of the \(j\)-dimensional ball:

\[
\text{Vol}(S + tB) = \sum_{j=0}^{d} V_j(S) \kappa_{d-j} t^{d-j}
\]

Both our algorithm and the intrinsic volumes based algorithm keep track of the set \(S_t\) of consistent hypotheses

\[
S_t := \{v \in B; \sigma_{\tau}(y_{\tau} - f_v(x_{\tau})) \geq 0 \text{ for all } \tau < t\}
\]

The intrinsic volumes based approach keeps track of \(V_j(S_t)\) and relates the loss incurred in round \(t\) to the decrease of \(V_j(S_j)^{1/j}\) for some index \(j \in \{1, \ldots, d\}\). Instead of keeping track of each of the coefficients we control the value of the Steiner polynomial itself.

**Obtaining \(O(d \log T)\) in poly-time for linear contextual search**

To make this concrete, we describe how the algorithm in Section 3.1 looks like in the continuous case.

\[\footnote{Intrinsic volumes have equivalent definitions as the volume of random projections and as the basis of the set of continuous and rigid valuations (Hadwiger’s Theorem). We refer to the text by Klain and Rota [10] for a comprehensive discussion.}\]
Algorithm 3 Single-scale Steiner Potential for Linear Contextual Search

Initialize $S_1 = B$

for $t$ in $1, 2, \ldots, T$

Adversary picks $x_t$

Let $m_t$ such that \( \text{Vol}(\{v \in S_t + \frac{1}{T}B; v \cdot x_t \geq m_t\}) = \frac{1}{2} \text{Vol}(S_t + \frac{1}{T}B) \)

Choose $y_t = m_t - 1/T$ or $m_t + 1/T$, each with probability half

Update $S_{t+1} = \{v \in S_t; \sigma_t(y_t - v \cdot x_t) \geq 0\}$

The analysis of this algorithm mirrors the one in Section 3.1 but controlling $\text{Vol}(S_t + (1/T)B)$ instead of $|H_t|$. Let $v_0 \in B$ be the true point, i.e., $f_0 = f_{v_0}$. First we argue that if $|m_t - v_0 \cdot x_t| > 1/T$ then $\text{Vol}(S_{t+1}) \leq \frac{1}{2} \text{Vol}(S_t)$ with at least probability $1/2$. This can be seen in the following picture:

Figure 1

The dotted black line depicts the $\{v; v \cdot x_t = m_t\}$ line and the surrounding blue dotted lines correspond to the two possible choices of $y_t$. If if $|m_t - v_0 \cdot x_t| > 1/T$, the true point $v_0$ is outside of the band defined by the blue lines, so if the blue line closest to $v_0$ is selected, then we will have:

$$\{v \in S_t; \sigma_t(y_t - v \cdot x_t) \geq 0\} + \frac{1}{T}B \subseteq \{v \in S_t + \frac{1}{T}B; \sigma_t(m_t - v \cdot x_t) \geq 0\}$$

and hence:

$$\text{Vol}(S_{t+1} + \frac{1}{T}B) \leq \frac{1}{2} \text{Vol}(S_t + \frac{1}{T}B)$$

Since the potential $\text{Vol}(S_t + \frac{1}{T}B) \geq \text{Vol}(\frac{1}{T}B) = \text{Vol}(B)/T^d$, it can’t decrease by half more than $\Omega(d \log T)$ times. Hence, in expectation, $|m_t - v_0 \cdot x_t| > 1/T$ occurs for at most $\Omega(d \log T)$ periods, leading to the desired regret bound.

Poly-time implementation We note that $m_t$ can be approximated using binary search as long as we can compute the volume of $(S_t \cap H) + \frac{1}{T}B$ for any half-space $H$. It is enough to notice that we only need a constant approximation of the volume in the previous proof and in order to approximate the volume we only need access to a separation oracle \cite{3, 9, 17}. Since $S_t$ is a ball intersected with at most $t$ halfspaces, it is trivial to obtain a separation oracle for it. To obtain a separation oracle for $S_t + \frac{1}{T}B$ it is enough to solve the problem of computing the distance from a query point to the convex set $S_t$ which is itself a convex problem (technically, this requires $S_t$ to not be too small, but we can guarantee this by introducing a small amount of noise in the selection of $y_t$; full details are provided in Appendix A).
3.5 \( O(d \log(d)) \) regret for linear contextual search

We can push the technique in Section 3.4 further to get optimal regret up to a log factor for linear contextual search. The main idea is to get a better version of Lemma 3.5 when the set of hypotheses has nice enough geometry. As in the previous section, let \( S_t \) will be the set of hypotheses consistent with the feedback seen so far. Consider the following algorithm:

**Algorithm 4 Multiscale Steiner Potential for Linear Contextual Search**

Initialize \( S_1 = B \) and let \( v_i = 2^{-i} \) and \( u_i = v_i/(16d) \) for all \( i \)

for \( t \) in \( 1, 2, \ldots, T \) do

  Adversary picks \( x_t \)

  Let \( i \) be the largest index such that \( \max_{v \in S_t} v \cdot x_t - \min_{v \in S_t} v \cdot x_t \leq v_i \)

  Let \( m_t \) such that \( \text{Vol}(\{v \in S_t + u_i B; v \cdot x_t \geq m_t\}) = \frac{1}{2} \text{Vol}(S_t + u_i B) \)

  Query either \( m_t + u_i \) or \( m_t - u_i \) each with half probability

The first lemma is exactly like in the previous subsection (see Figure 1):

**Lemma 3.6.** If \( |m_t - v_0 \cdot x_t| > u_i \) then \( \text{Vol}(S_{t+1} + u_i B) \leq \frac{1}{2} \text{Vol}(S_t + u_i B) \) with probability at least \( 1/2 \).

In the remaining case, we will also show that the volume decreases by a constant factor. This will make strong use of the geometry in linear contextual search:

**Lemma 3.7.** If \( |m_t - v_0 \cdot x_t| \leq u_i \) then \( \text{Vol}(S_{t+1} + u_i B) \leq \frac{3}{4} \text{Vol}(S_t + u_i B) \) with probability at least \( 1/2 \).

**Proof.** In the left part of Figure 2 we depict the setting of the lemma as follows: the yellow region is \( S_t + u_i B \), the dashed black line in the middle corresponds to the median and the blue dashed lines to the \( u_i \) perturbations (note that those “lines” are actually hyperplanes, but we refer to them as lines for simplicity). The dashed red lines correspond to a \( 2u_i \) perturbation. We start by noting that if the algorithm queries the blue line on the left, all the \( (S_t + u_i B) \) volume left of the leftmost red line is removed (see middle part of Figure 2). Similarly, if the algorithm queries the blue line on the right, all the volume right of the rightmost red line is removed. All the volume outside of the red lines is removed with at least \( 1/2 \) probability. Now, we are only left to argue that at least a constant fraction of the volume is outside the dashed red lines.

To make this argument, let \( C \) be the largest volume of a section of \( S_t + u_i B \) in the direction \( x_t \) inside the band between the dashed red lines. The volume of \( S_t + u_i B \) inside the band is at most \( 4u_i \cdot C \). Now, the total volume of \( S_t + u_i B \) can be bound by taking the two cones formed by taking the convex hull of the section of largest volume inside the band and the extreme points \( q_1 \) and \( q_2 \), which are at least \( v_{i+1} \) apart in the \( x_t \) direction. See the red region in the right part of Figure 2.

The volume of the two cones is at least:

\[
\frac{v_{i+1} C}{d} = \frac{C 2^{-i}}{2d} = \frac{C \cdot 16u_i d}{2d} = 8u_i \cdot C
\]

which is at least twice the volume of the band between the dotted red lines. Hence one of the sides has at least \( 1/4 \) of the total volume. \( \square \)
Theorem 3.8. The expected regret of the Multiscale Steiner Potential for Linear Contextual Search is at most $O(d \log d)$.

Proof. Everytime we choose index $i$, the volume of $\text{Vol}(S_t + u_i B)$ decreases by a constant factor with half probability. Since $\text{Vol}(S_t + u_i B) \geq \text{Vol}(u_i B) = u_i^d \text{Vol}(B)$ this can’t happen more than $O(d \log(1/u_i))$ times in expectation, so the total regret is at most:

$$O \left( \sum_i v_i d \log(u_i) \right) = O \left( \sum_i d 2^{-i} \log(d) \right) = O(d \log d)$$

Remark. Is the Steiner potential necessary? One natural algorithm for this problem is to query $y_t$ such that $\text{Vol}(\{ v \in S_t; v \cdot x_t \geq y_t \}) = \frac{1}{2} \text{Vol}(S_t)$ (i.e. guess the median without inflating the set). This algorithm has regret at least $\Omega(d^2)$ by the example in Section 8 of [16]. Inflating the set by taking the Minkowski sum with a ball seems to be the appropriate regularization that allows us to overcome the $d^2$ lower bound.

Another natural algorithm is to guess $y_t = \frac{1}{2} (\min_{v \in S_t} v \cdot x_t + \max_{v \in S_t} v \cdot x_t)$. This algorithm was shown to have regret at least $2^{\Omega(d)}$ in [5].

4 Noisy Feedback

We now consider the binary feedback model with noise, where each round the feedback is (independently) flipped with probability $p < 1/2$. We will no longer be able to eliminate a hypothesis based on the feedback since it is always possible that the feedback was flipped, instead we will keep a weight function expressing the likelihood of each hypothesis given observations.

We first give a $O(d \log T)$ algorithm for general hypothesis classes and then we show how to get $\text{poly}(d)$ for linear contextual search exploiting the geometrical structure of the hypothesis class of linear functions.
4.1 \(O(d \log T)\) algorithm

The usual approach in Bayesian inference is to start with a uniform prior over the set of hypotheses and given each observation, compute the posterior. It is important to emphasize that the true hypothesis \(f_0\) in our model is still chosen adversarially. The Bayesian inference only serves to provide the intuition.

The algorithm will be as follows: as before we will start with a discretized version of the hypothesis class \(N_{T-2}(H)\) which we will call \(N\) for short in this section. We will keep a weight function \(w_t: N \rightarrow \mathbb{R}\) which roughly expresses the likelihood that a hypothesis is close to the true hypothesis. Our guess will be a perturbed version of the weighted median \(m_t\) of the set \(\{f(x_t); x_t \in N\}\). Formally, the weighted median \(m_t\) is a number that satisfies:

\[
\sum_{f \in N; f(x_t) \geq m_t} w_t(f) \geq \frac{1}{2} \quad \text{and} \quad \sum_{f \in N; f(x_t) \leq m_t} w_t(f) \geq \frac{1}{2}
\]

After receiving the feedback, we will update the weights in the following way (we will choose \(y_t\) at random such that \(y_t = f(x_t)\) occurs with zero probability):

\[
\hat{w}_{t+1}(f) = \begin{cases} 
p' \cdot w_t(f) & \text{if } \sigma_t(y_t - f(x_t)) < 0 \\
(1 - p') \cdot w_t(f) & \text{if } \sigma_t(y_t - f(x_t)) > 0
\end{cases}
\]

for some parameter \(p'\) and re-normalizing afterwards:

\[
w_{t+1}(f) = \frac{\hat{w}_{t+1}(f)}{\sum_{f' \in N} \hat{w}_{t+1}(f')}
\]

In standard Bayesian inference, we would normally use \(p = p'\). For this algorithm, we will choose any parameter \(p'\) with \(p < p' < 1/2\). The actual choice of parameter will only affect the constants. Also note that unlike Bayesian inference we don’t choose the guess with largest likelihood but a perturbed version of the median.

**Algorithm 5** Single-Scale Steiner Potential with Noise

Initialize \(w_1(f) = 1/|N|\) for all \(f \in N := N_{T-2}(H)\).

for \(t\) in 1, 2, \ldots, \(T\) do

Adversary picks \(x_t\)

Let \(m_t\) be the weighted median \(\{f(x_t); f \in N\}\) with respect to weights \(w_t\).

Choose \(y_t \in [m_T - \frac{1}{T}, m_t + \frac{1}{T}]\) uniformly at random

Choose \(\hat{w}_{t+1}(f) = p' \cdot w_t(f)\) if \(\sigma_t(y_t - f(x_t)) < 0\) and \(\hat{w}_{t+1}(f) = (1 - p') \cdot w_t(f)\) otherwise

Normalize the weights: \(w_{t+1}(f) = \hat{w}_{t+1}(f)/\sum_{f' \in N} \hat{w}_{t+1}(f')\)

---

**Theorem 4.1.** In the noisy feedback model, the above algorithm incurs expected total loss \(O(d \log T)\).

We will denote the true hypothesis by \(f_0\) as usual. Since \(f_0\) might not belong to the discretized set \(N\), we will control the weight that is placed on the closest hypothesis. Let \(f_1\) be a hypothesis in \(N\) with \(d_\infty(f_1, f_0) \leq T^{-2}\).
Lemma 4.2. If \( f_1(x_t), f_0(x_t) \) are on the same side of \( y_t \) and \( W_t^- \) is the total weight mass on the other side of \( y_t \), then we have the following inequality where the expectation is taken over the randomness in the feedback

\[
\mathbb{E} \left[ \frac{1}{w_{t+1}(f_1)} \right] = \frac{1 - cW_t^-}{w_t(f_1)}
\]

for some constant \( 0 < c < 1 \) given by

\[
c = (p' - p) \left[ \frac{1 - p'}{p'} - \frac{p}{1 - p} \right].
\]

Proof. Assume wlog that \( f_1(x_t), f_0(x_t) \geq y_t \) and let \( W_t^- = \sum_{f \in N: f(x_t) < y_t} w_t(f) \) be the weight on hypotheses in the opposite direction and \( W_t^+ = 1 - W_t^- \) the remaining weight. Since \( y_t \) is chosen from a continuous distribution, the event that some \( f \) has \( f(x_t) = y_t \) occurs with zero probability. With probability \( 1 - p \) we have

\[
w_{t+1}(f_1) = \frac{(1 - p') \cdot w_t(f_1)}{(1 - p') \cdot W_t^+ + p' \cdot W_t^-}
\]

with the remaining probability \( p \) we have:

\[
w_{t+1}(f_1) = \frac{p' \cdot w_t(f_1)}{p' \cdot W_t^+ + (1 - p') \cdot W_t^-}
\]

Averaging them we have:

\[
\mathbb{E} \left[ \frac{1}{w_{t+1}(f_1)} \right] = (1 - p) \cdot \frac{(1 - p') \cdot W_t^+ + p' \cdot W_t^-}{(1 - p') \cdot w_t(f_1)} + p \cdot \frac{p' \cdot W_t^+ + (1 - p') \cdot W_t^-}{p' \cdot w_t(f_1)}
\]

\[
= \frac{1}{w_t(f_1)} \left[ W_t^+ + \left( \frac{p'}{1 - p} (1 - p) + \frac{1 - p'}{p} p \right) W_t^- \right] = \frac{1 - cW_t^-}{w_t(f_1)}
\]

since \( 0 < \frac{p'}{1 - p} (1 - p) + \frac{1 - p'}{p} p = 1 + (p' - p) \left[ \frac{p'}{1 - p} - \frac{1 - p'}{p} \right] = 1 - c. \)

Lemma 4.3. In expectation over both the randomness in the choice of \( y_t \) and the randomness in the feedback, we have:

\[
\mathbb{E} \left[ \frac{1}{w_{t+1}(f_1)} \right] \leq \left( 1 + \frac{c'}{T} \right) \cdot \frac{1}{w_t(f_1)} \quad \text{for} \quad c' = \frac{1 - p'}{2p'}
\]

Proof. With at least \( 1 - \frac{1}{2T} \) probability, \( f_1(x_t) \) and \( f_0(x_t) \) are on the same side of \( y_t \) since

\[
|f_1(x_t) - f_0(x_t)| < 1/T^2
\]

and the magnitude of the perturbation is \( 1/T \). In this case, by the previous lemma, we have:

\[\mathbb{E} \left[ 1/w_{t+1}(f_1) \right] = 1/w_t(f_1).\] With the remaining probability \( f_1(x_t) \) and \( f_0(x_t) \) are on opposite sides of \( y_t \) and we can use the trivial bound in the equation below.

\[
\frac{1}{w_{t+1}(f_1)} \leq \frac{1 - p'}{p'} \cdot \frac{1}{w_t(f)}.
\]

Combining, we get the desired inequality. \( \blacksquare \)
Lemma 4.4. If $|f_0(x_t) - m_t| > \frac{2}{T}$ then for the constant $c$ in Lemma 4.2, then in expectation over both the randomness in the choice of $y_t$ and the randomness in the feedback, we have:

$$
\mathbb{E} \left[ \frac{1}{w_{t+1}(f_1)} \right] \leq \frac{1 - c/4}{w_t(f)}
$$

Proof. Assume without loss of generality that $f_0(x_t) > m_t + \frac{2}{T}$. Since the magnitude of the perturbation is $1/T$, $f_1(x_t)$ will be on the same side of our guess as $f_0(x_t)$ and hence we can apply Lemma 4.2. With probability at least $1/2$ we have $y_t > m_t$ and hence $W_t^- \geq 1/2$. With the remaining probability we use the trivial bound $W_t^- \geq 0$. Combining those we get the bound in the statement. 

The previous lemmas imply that $w_t(f)$ grows by a constant factor (in expectation) whenever the median is far from the true point. We conclude the proof by showing that this can’t happen too often since weights are bounded.

Proof of Theorem 4.1. The regret bound follows directly from the fact that the probability of having $|f_0(x_t) - m_t| > 2/T$ for more than $\Omega(d \log T)$ periods is at most $O(1/T)$. Our strategy for proving this is to define a random process $Y_t$ that is a super-martingale, i.e. $\mathbb{E}[Y_{t+1}] \leq Y_t$ and argue that if $|f_0(x_t) - m_t| > 2/T$ happens too often, then $Y_T$ will be much larger than $Y_1$. This happens with small probability by Markov’s inequality.

We will first define an auxiliary sequence of real numbers $s_t \geq 0$ for $t \in \{1 \ldots T\}$ as follows. Let $s_1 = 1/|N| \geq T^{-2d}$ and let:

$$
 s_{t+1} = \begin{cases} 
 (1 - c/4)^{-1} \cdot s_t, & \text{if } |f_0(x_t) - m_t| > 2/T \\
 (1 + c'/T)^{-1} \cdot s_t, & \text{otherwise}
 \end{cases}
$$

Now define the following stochastic process:

$$
 Y_t = \frac{s_t}{w_t(f_1)}
$$

It is simple to see that $Y_1 = 1$. The previous lemmas imply that $Y_t$ is a super-martingale, i.e. $\mathbb{E}[Y_{t+1}] \leq Y_t$ and hence $\mathbb{E}[Y_T] \leq 1$. Now, in the case that $|f_0(x_t) - m_t| > 2/T$ for more than $\Omega(d \log T)$ periods, we have

$$
 s_T \geq T^{-2d} \cdot (1 - c/4)^{-\Omega(d \log T)} \cdot (1 + c'/T)^{-T} \geq T
$$

and hence:

$$
 Y_T = \frac{s_T}{w_T(f_1)} \geq s_T \geq \Omega(T)
$$

but since $\mathbb{E}[Y_T] \leq 1$, this can happen with at most $O(1/T)$ probability by Markov’s inequality. 

Remark. We’ve assumed here that $p$ is a constant bounded away from $1/2$. How does the regret of this algorithm depend on $p$ as $p$ approaches $1/2$? If $p = \frac{1}{2} - \delta$, then we can set $p' = \frac{1}{2} - \delta'$ where $\delta' = \delta/2$. This leads to $c = O(\delta^2)$ – adapting the proof of Theorem 4.1 then shows we can have at most $O\left(\frac{d \log T}{\delta^2}\right)$ inaccurate rounds, for a total of at most $O\left(\frac{d \log T}{\delta^2}\right)$ regret.
4.2 Noisy Linear Contextual Search

In this section we study the specific problem of linear contextual search in the noisy feedback model. We show that here we can achieve total loss $O(\text{poly}(d))$ independent of $T$. Throughout this section we will use $q_0 \in \mathcal{B}$ to denote the true point, i.e. $f_0(x) = q_0 \cdot x$.

Our approach (Algorithm 6) builds off the Bayesian inference approach in the previous section (Algorithm 5) by combining it with the multi-scale discretization ideas in Section 3.3. At a high (and slightly inaccurate) level, Algorithm 6 works as follows. Throughout the algorithm, we maintain a distribution $w$ over the unit ball $\mathcal{B}(0,1)$ where $w(q)$ represents the likelihood that $q$ is our true point $q_0$. Each round $t$, we are provided a direction $x_t$ by the adversary. We begin by measuring the “width” of our distribution in the direction $x_t$ – i.e., the length of the smallest interval in this direction which contains almost all of the mass of our distribution $w$. Then (similarly as in Algorithm 5), we will guess a perturbed version of median of $w$ in the direction $x_t$, where the size of the perturbation depends on the width. Finally, we multiplicatively update the distribution $w$, penalizing points on the wrong side of our guess (again, similarly as in Algorithm 5).

In the actual algorithm, we maintain a separate distribution $w_i$ for each possible scale $\gamma_i$ for the width (in particular, we are in scale $w_i$ if almost all of the mass of $w_{i-1}$ is concentrated in a small strip in direction $x_t$). This aids analysis in letting us guarantee we operate in each scale for at most a bounded number of rounds, which lets us bound the total loss of this algorithm.
Algorithm 6 Noisy Linear Contextual Search

Initialization:
Let $\eta = 1/(2d^{10})$.

for each integer $i > 0$ do

Let $\beta_i = \frac{1}{2^{i+1}}$ and $\gamma_i = \frac{1}{2^i}$.
Construct a distribution $w_i : B(0, 1) \rightarrow \mathbb{R}$. Let $w_{i,t}$ denote the weight function $w_i$ at round $t$.
Initialize $w_{i,0}(q) = 1/\text{Vol}(B(0, 1))$ for all $q \in B(0, 1)$.
Initialize $C_i = 0$. ($C_i$ will store the number of times we have been in scale $i$).

Algorithm:
for $t$ in $1, 2, \ldots, T$ do

Adversary picks $x_t$.
Let $i_t$ be the largest index $i$ such that at least one of the following is true:
- There exists $a, b \in \mathbb{R}$ such that $|a - b| \leq 10\gamma_i$ and $\int_{a \leq x_t \cdot q \leq b} w_{i,t}(q) dq \geq 1 - \gamma_i d^4$.
- $C_{i_t} + 1 > 100 \left( \frac{d^4(i-1)}{\gamma_i d^4} + d^{25}(i-1) \right)$.

Let $y$ be the weighted median of $w_{i_t,t}$ in the direction $x_t$ i.e. $y$ satisfies
\[
\int_{x_t \cdot q < y} w_{i_t,t}(q) dq = \int_{x_t \cdot q > y} w_{i_t,t}(q) dq = \frac{1}{2}.
\]
Query $\hat{y} = y + \delta$ where $\delta$ is chosen uniformly at random from $[-2\beta_t, 2\beta_t]$.
Update weights:
for $i$ in $\{i_t, i_t + 1\}$ do

For each $q \in B(0, 1)$, if $q$ violates the feedback then $w_{i,t+1}(p) = (1 - \eta) w_{i,t}(q)$.
Normalize $w_i$ so that $\int_{B(0,1)} w_{i,t+1}(p) dp = 1$.

$C_{i_t} \leftarrow C_{i_t} + 1$.

Theorem 4.5. Algorithm $\$ incurs $O(\text{poly}(d))$ expected total loss for the problem of noisy linear contextual search.

The proof of Theorem 4.5 is structured roughly as follows. Let $L_i$ be the (expected) loss sustained at scale $i$. We wish to show that $\sum_i L_i = \text{poly}(d)$. To bound $L_i$, we’ll start by roughly following the analysis in Theorem 4.1. Specifically, we’ll look at the total weight (according to $w_i$) of a tiny ball surrounding the true point $q_0$. Let the weight of this ball at time $t$ be $W_{i,t}$. We’ll again show that $1/W_{i,t}$ when suitably scaled is a super-martingale: it decreases in expectation by a large amount whenever our guess is far from accurate and cannot increase very much in expectation even if our guess is close to accurate. Since $1/W_{i,t}$ cannot decrease below 1, this lets us upper bound the number of rounds where we are far from accurate.

Now, even when we are far from accurate, we know that since we are in scale $i$, almost all of the mass of $w_i$ is concentrated on some thin strip in direction $x_t$. If the true point $q_0$ is located in or near this strip, this lets us bound the loss each round when we are far from accurate (since the median will lie in this strip). So it suffices to show that if a weight function $w_i$ concentrates on some thin strip, then with high probability, the true point $q_0$ lies close to this strip.

To prove this, we again look at the weight of a small ball $B_\alpha$ with radius $\alpha$ around $q_0$ (see left
side of Figure 3. If we know that the weight on some strip is at least some threshold \( \tau \), then if \( w_{i,t}(\mathcal{B}_\alpha) + \tau > 1 \), we know that the ball and strip intersect, and therefore \( q_0 \) is at most distance \( \alpha \) away from this strip. It thus suffices to show that \( w_{i,t}(\mathcal{B}_\alpha) > 1 - \tau \) with high probability.

Now, we will choose \( \alpha \) and \( \tau \) large enough so that this inequality is satisfied at time \( t = 0 \). We therefore only need to show that this is still true with high probability for all times \( t \). Intuitively, this should be true – the amount of weight on a ball centered at the true point \( q_0 \) should only increase as time goes on and we get more feedback (the feedback is noisy, so we might occasionally decrease the weight of this ball, but overall the increases should drown out the decreases). Proving this formally, however, is technically challenging and where we need to use the Euclidean geometry specific to linear contextual search.

To show this, we use the following lemma. Choose two points \( q_1 \) and \( q_2 \) on a line through \( q_0 \) so that \( q_1 \) lies between \( q_0 \) and \( q_2 \). We claim that with high probability (for all times \( t \)), \( w_{i,t}(q_1) \geq \kappa \cdot w_{i,t}(q_2) \) for some constant \( \kappa \). To show this, observe that there is no half space which contains both \( q_0 \) and \( q_2 \) but not \( q_1 \). This means that the only way \( w_{i,t}(q_2) \) can increase relative to \( w_{i,t}(q_1) \) is if a guess separates \( q_2 \) from \( q_1 \) and if the feedback on this guess is noisy (right side of Figure 3). This occurs with probability \( p < 1/2 \) and is unlikelier than the alternative (which increases the weight of \( q_1 \) relative to \( q_2 \)). We can thus bound the ratio of \( w_{i,t}(q_1)/w_{i,t}(q_2) \) from below with high probability over all rounds.

If we could union bound over all points in \( \mathcal{B}_\alpha \) we would be done (this inequality allows us to relate the weight of all the points outside \( \mathcal{B}_\alpha \) to the weight of points inside \( \mathcal{B}_\alpha \)). Unfortunately there are infinitely many points inside \( \mathcal{B}_\alpha \) so we cannot apply a naive union bound. Luckily, we can show that nearby points are very likely to have similar weights: the only way the relative weight of two nearby points \( q \) and \( q' \) changes is if we guess a hyperplane separating \( q \) and \( q' \) – and since we add a perturbation to our guess every round, we can bound the probability of this happening. This allows us to repeat the previous geometric argument with \( \epsilon \)-nets instead of single points, which completes the proof.

**Notation.** Below, we will use \( q_0 \) to denote the hidden point. We let \( \mathcal{B}(0,1) \) denote the unit ball and in general \( \mathcal{B}(q,r) \) to denote the ball of radius \( r \) centered at \( q \). We will use \( w_{i,t} \) to denote the
weight function \(w_i\) at round \(t\). For a set \(S \subseteq B(0,1)\), we use the notation
\[
w_{i,t}(S) = \int_S w_{i,t}(q) dq.
\]

Let \(\alpha_i = \frac{1}{2^{10^4 x_i^2}}\). Define the set \(S_{\alpha_i}\) to be the \(\alpha_i\)-net consisting of all points in the unit ball whose coordinates are integer multiples of \(\alpha_i\). Note that \(|S_{\alpha_i}| \leq \left(\frac{2d}{\alpha_i}\right)^d\). For all \(i\), let \(\Gamma_i = B(q_0, \alpha_i) \cap B(0,1)\) be the ball of radius \(\alpha_i\) centered at \(q_0\). For simplicity, throughout this proof we will assume that the feedback noise is fixed at \(p = 1/3\) (it is straightforward to adapt this proof for any other \(p < 1/2\); doing so only affects the constant factor of the loss bound).

**Step 1: Understanding \(1/w_{i,t}\)**

As in the analysis of Algorithm 5, we begin by understanding how the reciprocal of our weight function \(1/w_{i,t}(p)\) evolves over time. This will allow us to construct various helpful super-martingales (for example, allowing us to bound the number of rounds we spend in each scale).

The following claim relates how \(1/w_{i,t}(q_1)\) changes when \(q_1\) and \(q_0\) are on the same side of the hyperplane \(x_t \cdot q = \hat{y}\) (i.e. is more likely to be consistent with feedback).

**Claim 4.6.** Consider a round \(t\). Say the adversary picks direction \(x_t\) and the algorithm queries \(\hat{y}\).
Let \(q_1\) be a point such that \(q_1\) and \(q_0\) are on the same side of the hyperplane \(x_t \cdot q = \hat{y}\). Let
\[
X = \int_{\text{sign}(x_t \cdot q - \hat{y}) \neq \text{sign}(x_t \cdot q_0 - \hat{y})} w_{i,t}(q) dq
\]
Then for \(\eta \leq 1/4\) we have the following after applying weight updates
\[
\mathbb{E}\left[\frac{1}{w_{i,t+1}(q_1)}\right] \leq \left(1 - \frac{1}{10^4} \eta X\right) \frac{1}{w_{i,t}(q_1)}
\]
where the expectation is over the randomness in the feedback. In particular, we always have
\[
\mathbb{E}\left[\frac{1}{w_{i,t+1}(q_1)}\right] \leq \frac{1}{w_{i,t}(q_1)}.
\]

**Proof.** Recall that points that violate feedback have their weight multiplied by \((1 - \eta)\) (and then the distribution is renormalized). With probability \(1 - p = 2/3\) (when the feedback is not flipped), we thus have that
\[
w_{i,t+1}(q_1) = \frac{w_{i,t}(q_1)}{(1 - X) + (1 - \eta) X}.
\]
Likewise, with probability \(p = 1/3\) (when the feedback is flipped), we have that
\[
w_{i,t+1}(q_1) = \frac{(1 - \eta) w_{i,t}(q_1)}{(1 - \eta)(1 - X) + X}.
\]
Taking expectations over the feedback, we therefore have that

\[
\mathbb{E}
\left[
\frac{1}{w_{i,t+1}(q_1)}
\right]
= \left(\frac{2}{3} \cdot (1 - \eta X) + \frac{1}{3} \cdot \left(1 + \frac{\eta}{1 - \eta} \cdot X\right)\right) \frac{1}{w_{i,t}(q_1)}
= \left(1 - \eta X \left(\frac{2}{3} - \frac{1}{3(1 - \eta)}\right)\right) \frac{1}{w_{i,t}(q_1)}
\leq \left(1 - \frac{1}{10} \eta X\right) \frac{1}{w_{i,t}(q_1)}
\]

Points very close to \(q_0\) are likely to be on the same side of the hyperplane as \(q_0\), allowing us to apply Claim 4.6.

**Claim 4.7.** Let \(q_1 \in B(0, 1)\) such that \(\|q_1 - q_0\| \leq \alpha_i\). Then, in expectation both over the randomness in the feedback and the algorithm,

\[
\mathbb{E}
\left[
\frac{1}{w_{i,t+1}(q_1)}
\right]
\leq \frac{1}{1 - \frac{\alpha_i}{\beta_i}} \frac{1}{w_{i,t}(q_1)}
\]

regardless of the direction \(x_t\) that the adversary chooses.

**Proof.** Since \(\|q_1 - q_0\| \leq \alpha_i\), and since \(\hat{y}\) is chosen by adding a uniform \(\beta_i\) random variable to \(y\), the probability that \(q_1\) and \(q_0\) are on opposite sides of the plane \(x_t \cdot q = \hat{y}\) is at most \(\alpha_i \beta_i\). Combining this with Claim 4.6 gives us the desired result. \(\blacksquare\)

We now use Claims 4.7 and 4.6 to understand how \(1/w_{i,t+1}(\Gamma_{it+1})\) changes over time (generalizing from single points to small balls). This first claim bounds the decrease in \(1/w_{i,t+1,t+1}(\Gamma_{it+1})\) when our guess is close to accurate.

**Claim 4.8.** Assume \(|y - x_t \cdot q_0| \leq \beta_i\). Then, in expectation both over the randomness in feedback and in our algorithm,

\[
\mathbb{E}
\left[
\frac{1}{w_{i,t+1,t+1}(\Gamma_{it+1})}
\right]
\leq \left(1 - \gamma_{it}^{10d}\right) \frac{1}{w_{i,t+1,t+1}(\Gamma_{it+1})}
\]

**Proof.** Recall that \(y\) is the median of \(w_{i,t}\) in direction \(x_t\) as computed by our algorithm. Now, define the two quantities

\[
X^-=\int_{x_t \cdot q \leq y - 2\beta_{it}} w_{i,t+1,t}(q)\,dq \quad \text{and} \quad X^+=\int_{x_t \cdot q \geq y + 2\beta_{it}} w_{i,t+1,t}(q)\,dq.
\]

These quantities represent the mass of \(w_{i,t+1}\) above and below the strip of width \(2\beta_{it}\) around the median. Note that by the maximality of \(i_t\), either \(X^- \geq \gamma_{it+1}^{4d}/2\) or \(X^- \geq \gamma_{it}^{4d}/2\); if not, then there exists a strip of width \(2\beta_{it}\) \(\leq 10\gamma_{it+1}\) containing at least \(1 - \gamma_{it+1}^{4d}\). Without loss of generality, assume \(X^- \geq \gamma_{it+1}^{4d}/2\).

Now, recall that \(\hat{y}\) is chosen uniformly in the interval \([y - 2\beta_{it}, y + 2\beta_{it}].\) We will divide the expectation in the theorem statement into three cases, based on where \(\hat{y}\) lies.

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Claim 4.9. Assume $y - x_t \cdot q_0 > \beta_t$. Then, in expectation both over the randomness in feedback and in our algorithm,

$$
\mathbb{E} \left[ \frac{1}{w_{t+1,t+1}(\Gamma_{it+1})} \right] \leq \left( 1 - \frac{1}{d^{21}} \right) \frac{1}{w_{t+1,t}(\Gamma_{it})}
$$

\[\blacksquare\]
Proof. We essentially repeat the logic from the proof of Claim 4.8 with the change that we can more strongly lower bound $X^-$. Without loss of generality, assume that $y < x_t \cdot q_0 - \beta_i$. Define

$$X^- = \int_{x_t \cdot q \leq \hat{y}} w_{i,t}(q) dq.$$  

Note that since $y$ is the weighted median of $w_{i,t}$ in the direction $x_t$, $X^- \geq 1/2$.

Now, we again have three cases. To begin, with probability $\frac{1}{4} - \frac{\alpha_i}{\beta_i}$, $\hat{y}$ lies in the interval $[y, y + \beta_i - \alpha_i]$. Since $y + \beta_i < x_t \cdot q_0$, the hyperplane $x_t \cdot q = \hat{y}$ does not intersect $\Gamma_i$, and therefore we can apply Claim 4.6 to show that

$$\mathbb{E}\left[ \frac{1}{w_{i,t+1,\Gamma_i}} \right] \leq (1 - 0.1X^-) \frac{1}{w_{i,t+1,\Gamma_i}} \leq (1 - 0.05\eta) \frac{1}{w_{i,t+1,\Gamma_i}}.$$  

Likewise, the probability that the hyperplane $x_t \cdot q = \hat{y}$ intersects $\Gamma_i$ is at most $\frac{\alpha_i}{\beta_i}$ (in which case we can pessimistically bound the decrease in weight as in the proof of Claim 4.8), and with the remaining 3/4 probability the hyperplane $x_t \cdot q = \hat{y}$ does not intersect $\Gamma_i$, and we can apply the weaker variant of Claim 4.6. Combining these observations, we get that

$$\mathbb{E}\left[ \frac{1}{w_{i,t,\Gamma_i}} \right] \leq (1 - \frac{1}{d^{21}}) \frac{1}{w_{i,t,\Gamma_i}}.$$  

\[ \square \]

**Step 2: Bounding the number of rounds**

In this step we will bound the total number of rounds in each scale $i$. Specifically, our algorithm ensures that we move onto the next scale once either the weight concentrates on a strip or once $C_i$ grows large enough - we will show with high probability that this is always due to the weight concentrating on a small strip.

Let $A_i$ be the number of rounds $t$ such that $i_t = i$ and $|y - x_t \cdot q_0| \leq \beta_i$ (i.e. the number of rounds where we are “accurate”). Let $B_i$ be the number of rounds $t$ such that $i_t = i$ and $|y - x_t \cdot q_0| > \beta_i$ (i.e. the number of rounds where we are “inaccurate”). Note that $A_i + B_i = C_i$. Also, recall that our algorithm ensures that $C_i \leq 100(d_{ini}^{d_i} + d^{25}i) + 1$ for all rounds $t$.

We first show that with high probability, $B_i$ will be no larger than $O(\text{poly}(d)i)$.

**Claim 4.10.** For any constant $c > 0$, with probability at least $1 - 2^{-d^4ic}$ we have throughout all rounds that

$$B_i \leq 100d^{25}i(1 + c)$$

**Proof.** We will construct a sequence $Z_t$ so that $\frac{Z_t}{w_{i,t}(\Gamma_i)}$ is a super-martingale. Consider the sequence $Z_t$ defined as follows.
• \( Z_1 = \left( \frac{\alpha_i}{2} \right)^4 \).

• If \( i_t \not\in \{ i, i-1 \} \) then \( Z_{t+1} = Z_t \).

• If \( i_t = i \) and \( |y - x_t \cdot q| \leq \beta_i \) or \( i_t = i-1 \) then \( Z_{t+1} = \left( 1 - \frac{\alpha_i}{\beta_i} \right) Z_t \).

• If \( i_t = i \) and \( |y - x_t \cdot q| > \beta_i \) then \( Z_{t+1} = \left( 1 + \frac{1}{d^2} \right) Z_t \).

Consider the ratio \( Y_t = \frac{Z_t}{w_i(t)} \). Note that Claim 4.7 and Claim 4.9 imply that 
\[
\mathbb{E}[Y_{t+1} | Y_t] \leq Y_t,
\]
so \( Y_t \) is a super-martingale.

Now, note that \( Y_1 \leq 1 \) since 
\[
w_{i,1}(\Gamma_i) = \frac{\text{Vol}(B(q_0, \alpha_i) \cap B(0,1))}{\text{Vol}(B(0,1))} \geq \frac{1}{2^d} \cdot \frac{\text{Vol}(B(q_0, \alpha_i))}{\text{Vol}(B(0,1))} = \frac{\alpha_i^d}{2^d},
\]
(Here we have used the fact that \( \text{Vol}(B(q_0, \alpha_i) \cap B(0,1)) \) must contain a ball of radius \( \alpha_i/2 \).) Since \( Y_t \) is a non-negative super-martingale, by Doob’s martingale inequality it holds that for any constant \( M \)
\[
\Pr[\exists t | Y_t \geq M] \leq \frac{1}{M}.
\]
However note that if there exists a round where \( B_i \geq 100d^{25}i(1+c) \), then for \( t \) sufficiently large
\[
Z_t \geq Z_1 \left( 1 + \frac{1}{d^2} \right)^{B_i} \left( 1 - \frac{\alpha_i}{\beta_i} \right)^{C_{i+C_i-1}}
\geq \frac{\alpha_i^d}{2^d} \left( 1 + \frac{1}{d^2} \right)^{100d^{25}i(1+c)} \left( 1 - \frac{\alpha_i}{\beta_i} \right)^{1000 \left( \frac{d^4i}{100} + d^{25}i \right)}
\geq \frac{\alpha_i^d}{2^d} \cdot 2^{d^3i(1+c)}
\geq 2^{d^3ic}.
\]
(Here in the last inequality we have used the fact that \( 2^{d^3i} \geq (2/\alpha_i)^d \). Since \( w_{i,t}(\Gamma_i) \leq 1 \) for all \( t \), this implies that \( Y_t \geq 2^{d^3ic} \), which immediately implies the desired claim. ■

Recall that in our algorithm, we check the following two conditions for determining the scale \( i \) we use for the current query:

• There exists \( a, b \in \mathbb{R} \) such that \( |a - b| \leq 10\gamma_i \) and \( \int_{a \leq x_t \cdot q \leq b} w_i(q) dq \geq 1 - \gamma_i^{3d} \).

• \( C_{i-1} > 100 \left( \frac{d^4(i-1)}{i(i+1)} + d^{25}(i - 1) \right) \).

We now show that with high probability, only the first condition is ever relevant.
Claim 4.11. For an index $i$, the probability that we ever have
\[ C_i \geq 100 \left( \frac{d^4_i}{\gamma_1^{10d}} + d^{25}i \right) \]
is at most $2^{-d^4i}$.

Proof. Again, we will construct a sequence $Z_t$ so that $\frac{Z_t}{w_{i+1,t}(\Gamma_{i+1})}$ is a super-martingale. Consider the sequence $Z_t$ defined as follows.

- $Z_1 = \frac{d}{2^d}$
- If $i_t \notin \{i, i + 1\}$ then $Z_{t+1} = Z_t$.
- If $i_t = i$ and $|y - x_t \cdot q_0| > \beta_i$ or $i_t = i + 1$ then $Z_{t+1} = \left(1 - \frac{\alpha_{i+1}}{\beta_{i+1}}\right) Z_t$
- If $i_t = i$ and $|y - x_t \cdot q_0| \leq \beta_i$ then $Z_{t+1} = \left(1 + \gamma_i^{10d}\right) Z_t$

Consider the ratio $Y_t = \frac{Z_t}{w_{i+1,t}(\Gamma_{i+1})}$. Similarly as in the proof of Claim 4.10, $Y_1 \leq 1$. Note that Claim 4.7 and Claim 4.8 imply that $E[Y_{t+1}] \leq Y_t$ so $Y_t$ is a super-martingale.

Now assume that $C_i \geq 100 \left( \frac{d^4_i}{\gamma_1^{10d}} + d^{25}i \right)$. By the constraints of our algorithm, we are guaranteed that
\[ C_{i+1} \leq 100 \left( \frac{d^4(i + 1)}{\gamma_1^{10d}} + d^{25}(i + 1) \right) + 1 \leq \frac{1000d^{25}i}{\gamma_1^{10d}} \]
Also by Claim 4.10 with probability at least $1 - \frac{1}{2^{10d^4i}}$, $B_i \leq 1100d^{25}i$ over all rounds $t$. This implies that eventually
\[ A_i = C_i - B_i \geq 99 \left( \frac{d^4_i}{\gamma_1^{10d}} \right) \]
Thus, for sufficiently large $t$, we have that
\[ Z_t \geq \frac{d^4_{i+1}}{2^d} \left( 1 + \gamma_i^{10d} \right)^{99} \left( \frac{d^4_i}{\gamma_1^{10d}} \right) \left(1 - \frac{\alpha_{i+1}}{\beta_{i+1}}\right)^{C_{i+1} + B_i} \geq 2^d i \]
However note $Y_1 \leq 1$ and $Y_t$ is a supermartingale. Also, $w_{i+1,t}(\Gamma_{i+1}) \leq 1$ for all rounds $t$. Thus, by Doob’s martingale inequality, the probability we ever have $C_i \geq 100 \left( \frac{d^4_i}{\gamma_1^{10d}} + d^{25}i \right)$ is at most
\[ \frac{1}{2^{10d^4i}} + \frac{1}{2^{2d^4i}} \leq \frac{1}{2^{d^4i}} \]
(the first term is from the probability that at some point $B_i \geq 1100d^{25}i$).
Step 3: Proving $w_i$ concentrates near $q_0$

We now aim to show that with high probability, if the weight function $w_i$ is concentrated on a thin strip, this strip must be close to the true point $q_0$ (this is necessary to bound the total regret we incur each round in scale $i$). To do this, we will argue that we can “round” points to the $\alpha_i$-net $S_{\alpha_i}$ without significantly affecting their weight. We will then rely on the geometric observation mentioned earlier: that for points $q_1, q_2 \in S_{\alpha_i}$, for some $i$ such that $q_0, q_1, q_2$ are nearly collinear, we can relate the weights $w_{i,t}(q_1)$ and $w_{i,t}(q_2)$. We begin by relating the weights of collinear points.

Claim 4.12. Fix an index $i$. If $q_0, q_1, \text{ and } q_2$ are collinear in that order, then with probability at least $1 - 2^{-d_{10}}$, we have that for all rounds $t$

$$w_{i,t}(q_1) \geq \gamma_i^d w_{i,t}(q_2)$$

Proof. Consider a time step $t$ where $i_t = i$ or $i_t = i - 1$. We say a point is on the “good” side of the hyperplane $x_t \cdot q = \hat{y}$ if it is on the same side as $q_0$. Otherwise we say the point is on the “bad” side. Note for $q_1, q_2$ satisfying the conditions of the claim, one of the following statements must be true:

- **Case 1**: $q_1, q_2$ are on the same side of the hyperplane $x_t \cdot q = \hat{y}$.
- **Case 2**: $q_1$ is on the good side of the hyperplane and $q_2$ is on the bad side of the hyperplane.

We will now consider the quantity $R_t = \left( \frac{w_{i,t}(q_2)}{w_{i,t}(q_1)} \right)^{d^o}$. Note that in Case 1, then $R_{t+1} = R_t$ (the relative weights remain unchanged if both $q_1$ and $q_2$ are on the same side of the hyperplane). In Case 2,

$$E[R_{t+1}] = E \left[ \left( \frac{w_{i,t+1}(q_2)}{w_{i,t+1}(q_1)} \right)^{d^o} \right] = \left( \frac{2}{3} (1 - \eta)^{d^o} + \frac{1}{3} (1 - \eta)^{d^o} \right) \left( \frac{w_{i,t}(q_2)}{w_{i,t}(q_1)} \right)^{d^o} \leq \left( \frac{w_{i,t}(q_2)}{w_{i,t}(q_1)} \right)^{d^o}$$

Here the last inequality follows from the fact that $2x/3 + 1/(3x) \leq 1$ for all $x \in [1/2, 1]$, and $\eta = \Theta(d^{-10})$ so $(1 - \eta)^{d^o} = 1 - o(1) \in [1/2, 1]$. Note that this implies that $R_t$ is a non-negative super-martingale (with $R_1 = 1$).

Now, if $w_{i,t}(q_1) < \gamma_i^d w_{i,t}(q_2)$, this would mean that

$$R_t = \left( \frac{w_{i,t}(q_2)}{w_{i,t}(q_1)} \right)^{d^o} > \gamma_i^{-d_{10}}.$$  

By Doob’s martingale inequality, the probability that this ever happens is at most $\gamma_i^{d_{10}} \leq 2^{-d_{10}}$, as desired.

We next relate the weights of nearby points $q_1$ and $q_2$. If $q_1$ and $q_2$ are close together, then it is unlikely they are ever separated by a hyperplane, and their weights should be similar. The following claim captures this intuition.

Claim 4.13. Fix an index $i$. If $q_1, q_2$ satisfy $\|q_1 - q_2\| \leq 2\beta_i^{10}$, then with probability at least $1 - 2^{-d_{10}}$ we have that for all rounds $t$

$$w_{i,t}(q_1) \geq \gamma_i w_{i,t}(q_2)$$
Proof. We will bound the number of rounds $t$ such that $i_t = i$ or $i_t = i - 1$ and the hyperplane $x_t \cdot q = \hat{y}$ intersects the segment connecting $q_1$ and $q_2$. We will show that with high probability, this quantity is at most $d^{\beta_0}i$. Note that since $w_{i,t}(q_1)/w_{i,t}(q_2)$ is unchanged when $q_1$ and $q_2$ both lie on the same side of the hyperplane, and decreases by at most a factor of $(1 - \eta)$ when they lie on different sides, this will show that with high probability

$$w_{i,t}(q_1) \geq (1 - \eta)^{d^i}w_{i,t}(q_2) \geq (1 - 2d^{-10})^{d^i}w_{i,t}(q_2) \geq 2^{-i}w_{i,t}(q_2) = \gamma_i w_{i,t}(q_2),$$

as desired.

Now, for a fixed round $t$, note that the probability that the hyperplane $x_t \cdot q = \hat{y}$ intersects the segment connecting $q_1$ and $q_2$ is at most $\frac{\|q_1 - q_2\|}{2\beta_i} \leq \beta_i^{\beta_0}$. There are at most

$$C_i + C_{i-1} \leq \frac{1000d^i}{\sqrt[10]{i}d^{9i}}$$

indices $t$ for which $i_t = i$ or $i_t = i - 1$. The probability that at least $d^{\beta_0}i$ of the hyperplanes $x_t \cdot q = \hat{y}$ intersect the segment connecting $q_2$ and $q_1$ is at most

$$\left(\frac{1000d^i}{\sqrt[10]{i}d^{9i}}\right)^{d^i} \leq \left(\frac{1000d^i\beta_0^9}{\gamma_i^{10d}}\right)^{d^i} \leq 2^{-d^{\beta_0}i},$$

which implies our desired result.

Finally, we apply Claims 4.12 and 4.13 to bound the relative weights for all approximately collinear pairs of points in $S_a$.

Claim 4.14. Fix an index $i$. With probability at least $1 - 2^{-d^i}$ the following claim holds: for all $q_1, q_2 \in S_{\alpha_i}$ such that

- The angle between the vectors $q_1 - q_0$ and $q_2 - q_0$ is at most $\beta_i^{10}$
- $\|q_1 - q_0\| \leq \|q_2 - q_0\|$ we have for all rounds $t$

$$w_{i,t}(p_1) \geq \gamma_i^{2d}w_{i,t}(p_2)$$

Proof. Fix a pair of points $q_1, q_2 \in S_{\alpha_i}$ satisfying the conditions in the statement. Let $q_\perp$ be the foot of the perpendicular from $q_1$ to the segment connecting $q_2$ and $q_0$. Note that

$$\|q_\perp - q_1\| \leq 2\beta_i^{10}$$

Therefore, by the conditions of Claim 4.13 with probability at least $1 - 2^{-d^{\beta_0}i}$, for all rounds $t$,

$$w_{i,t}(q_1) \geq \gamma_i w_{i,t}(q_\perp).$$

Note that $q_0, q_\perp,$ and $q_2$ are collinear. Since $\|q_\perp - q_0\| \leq \|q_1 - q_0\| \leq \|q_2 - q_0\|$, $q_\perp$ lies between $q_0$ and $q_2$ on this line. By Claim 4.12 this means that with probability at least $1 - 2^{-d^{\beta_0}i}$, for all rounds $t$,

$$w_{i,t}(q_\perp) \geq \gamma_i^d w_{i,t}(q_2).$$
Combining these two claims, we know that with probability at least \(1 - 2^{-d^{10}i + 1}\), for all rounds \(t\),

\[w_{i,t}(q_1) \geq \gamma_i^{d+1} w_{i,t}(q_2).\]

This is for a specific pair of points in \(S_{\alpha_i}\). Union bounding over all \(|S_{\alpha_i}|^2\) pairs of points, we have that the theorem statement fails with probability at most

\[|S_{\alpha_i}|^2 2^{-d^{10}i + 1} = \left(\frac{2d}{\alpha_i}\right)^{2d} 2^{-d^{10}i + 1} \leq 2^{-d^i}.\]

Finally, we prove that if \(w_i\) concentrates on a strip, \(q_0\) is within \(\gamma_i\) of this strip. To show this, it suffices to show that the weight of \(B(q_0, \gamma_i)\) is large enough that it must intersect a sufficiently concentrated strip. We do this by using Claim 4.14 to relate the weight of points of \(S_{\alpha_i}\) inside and outside \(B(q_0, \gamma_i)\).

**Claim 4.15.** Fix an index \(i\). With probability at least \(1 - 2^{-d^i}\), the following statement holds for all \(t\):

- If there exist \(a, b\) such that \(|a - b| \leq 10\gamma_i\) and

\[
\int_{a \leq x_t \cdot q \leq b} w_{i,t}(q) dq \geq 1 - \gamma_i^{4d}
\]

then \(a - \gamma_i \leq x_t \cdot q_0 \leq b + \gamma_i\).

**Proof.** First for each point \(q \in S_{\alpha_i}\), consider an axis-parallel box centered at that point with side length \(\frac{\alpha_i}{\gamma_i}\). Now consider all rounds \(t\) with \(i_t = i\) or \(i_t = i - 1\) and all planes of the form \(x_t \cdot q = \hat{y}\) for these rounds. We show that with high probability, all boxes intersect at most \(d^i\) of these planes. Using essentially the same argument as in Claim 4.13, we find that this probability is at least

\[1 - \left(\frac{2d}{\alpha_i}\right)^d \left(\frac{1000\alpha_i^{d^i}}{d^i}\right) \beta_i^{d^i} \geq 1 - \frac{1}{2d^{20}i}\]

In particular, with at least \(1 - \frac{1}{2d^{20}i}\) probability, the following two inequalities hold:

\[
\int_{q \in B(q_0, \gamma_i) \cap B(0, 1)} w_{i,t}(q) dq \geq (1 - \eta)^{d^i} \left(\frac{\alpha_i}{d}\right)^d \sum_{q \in B(q_0, \gamma_i - \alpha_i) \cap S_{\alpha_i}} w_{i,t}(q). \quad (1)
\]

\[
1 = \int_{q \in B(0, 1)} w_{i,t}(q) dq \leq \frac{1}{(1 - \eta)^{d^i}} \left(\frac{\alpha_i}{d}\right)^d \sum_{q \in S_{\alpha_i}} w_{i,t}(q). \quad (2)
\]

In both of these inequalities we are using the fact that if at most \(d^i\) planes intersect any box, then the weights of any two points in the same box are within a factor of \((1 - \eta)^{d^i}\).

Next, consider the ball \(B(q_0, \gamma_i - \alpha_i)\). Let \(T_i = \{B(q_0, \gamma_i - \alpha_i) \cap S_{\alpha_i}\}\). Consider the following two transformations: \(f : S_{\alpha_i} \to B(q_0, \gamma_i - \alpha_i)\), which sends a point \(q\) to
\[ f(q) = \left(1 - \frac{\gamma_i}{2}\right) q_0 + \frac{\gamma_i}{2} q \]

and the transformation \( g : \mathcal{B}(q_0, \gamma_i - \alpha_i) \rightarrow T_i \), where \( g(q) \) is the point obtained by rounding the coordinates of \( q \) to the nearest integer multiple of \( \frac{\alpha_i}{d} \) (note that \( g(q) \in T_i \)). If we consider the map \( q \rightarrow q' = g(f(q)) \) given by the above, the number of points \( q \in S_\alpha \) that map to a fixed point \( q' \in T_i \) is at most \( \left(\frac{10}{\gamma_i}\right)^d \).

To see this, note that \( g^{-1}(q) \) is an axis-parallel box with side-length \( \frac{\alpha_i}{d} \), and thus \( f^{-1}(g^{-1}(q)) \) contains all the points in \( S_\alpha \) contained within an axis aligned box with side-length \( \frac{2\alpha_i}{\gamma_i} \), which contains at least \( 2^d / \gamma_i + 1 \) points.

Now, note that \( q \) and \( q' = g(f(q)) \) satisfy the conditions of Claim 4.14. Thus, by Claim 4.14, with probability at least \( 1 - 1/2^d \), we have that

\[ \sum_{q \in \mathcal{B}(q_0, \gamma_i - \alpha_i) \cap S_\alpha} w_{i,t}(q) \geq \gamma_i 2^d \sum_{q \in S_\alpha} w_{i,t}(q) = \gamma_i 2^d \sum_{q \in S_\alpha} w_{i,t}(q) \]

Combining the above with equations (1) and (2), we conclude that with probability at least

\[ 1 - \frac{1}{2 \gamma_i} - \frac{1}{2 \gamma_i^d \gamma_i} \geq 1 - \frac{1}{2 \gamma_i^d \gamma_i} \]

we have

\[ \int_{q \in \mathcal{B}(q_0, \gamma_i)} w_{i,t}(q) dq \geq \left(1 - \frac{1}{2^d \gamma_i^d \gamma_i}\right)^{2^d \gamma_i} \gamma_i^3 \frac{1}{2^d \gamma_i^d} \geq \gamma_i^d \]

This implies that the ball \( \mathcal{B}(q_0, \gamma_i) \) must intersect the strip \( a \leq x_t \cdot q \leq b \). If this happens then the desired condition is clearly satisfied.

**Step 4: Completing the proof**

Finally, we can proceed to prove the main theorem.

**Proof of Theorem 4.5** First, for each \( i \), let \( L_i \) be the total loss at scale \( i \). We will bound \( \mathbb{E}[L_i] \).

By Claim 4.11 with probability at least \( 1 - \frac{1}{2^d \gamma_i^d \gamma_i} \), \( C_{i-1} \leq 100(\frac{\alpha_i(i-1)}{\gamma_i^d} + d^5(i-1)) \) for all rounds. If this is true, then the only time we query at level \( i \), there must be some strip given by \( a \leq x_t \cdot q \leq b \) of width at most \( 10\gamma_i \) that contains \( 1 - \gamma_i^d \) of the total weight of \( w_i \). Thus, by Claim 4.15 with at least \( 1 - \frac{1}{2^d \gamma_i^d \gamma_i} - \frac{1}{2^d \gamma_i^d \gamma_i} \) probability, all queries at level \( i \) incur loss at most
12\gamma_i + 2\beta_i \leq 14\gamma_i. Now, by using Claim 4.10, we can bound the expected total loss at level \(i\) as

\[
\mathbb{E}[L_i] \leq \left( \frac{1}{2d^4(i-1)} + \frac{1}{2d^4(i-1)} \right) C_i + 2\beta_i C_i + 14\gamma_i \left( 100d^{25}i + \sum_{j=0}^{\infty} 100d^{25}i \Pr[B_i \geq 100d^{25}i(1 + j)] \right)
\]

\[
\leq \left( \frac{1}{2d^4(i-1)} + \frac{1}{2d^4(i-1)} + 2\beta_i \right) \cdot \left( 100 \left( \frac{d^{4i}}{\gamma_i^{10d}} + d^{25}i \right) + 1 \right) + 1400d^{25}i\gamma_i \left( 1 + \sum_{j=0}^{\infty} \frac{1}{2d^{4ij}} \right)
\]

\[
\leq 4\beta_i \cdot \left( 200 \frac{d^{25}i}{\gamma_i^{10d}} \right) + 2800d^{25}i\gamma_i
\]

\[
= 4 \cdot 2^{-100di} \cdot (200d^{25}i2^{10di}) + 2800d^{25}i2^{-i}
\]

\[
= 800d^{25}i2^{-90di} + 2800d^{25}i2^{-i}.
\]

It follows that

\[
\sum_{i=1}^{\infty} \mathbb{E}[L_i] \leq \sum_{i=1}^{\infty} \left( 800d^{25}i2^{-90di} + 2800d^{25}i2^{-i} \right) = O(d^{25}) = O(\text{poly}(d)).
\]

\[\square\]

**Remark.** Naively, one can implement Algorithm 6 with time complexity \(T^{O(d)}\), via the observation that \(T\) hyperplanes divide \(B(0, 1)\) into at most \(O(Td)\) pieces, so we can simply compute this division and the weight of each distribution \(w_i\) (we care about at most \(T\) scales) on each component of this division.

It is an interesting open question if it is possible to implement Algorithm 6 (or otherwise achieve \(O(\text{poly}(d))\) regret) with time complexity \(\text{poly}(d, T)\). To do so, it would suffice to be able to efficiently sample from the distributions \(w_i\).

## 5 Tight loss bounds for full feedback

We also study the problem where the learner has full feedback, i.e., after the prediction \(y_t\) the feedback is the actual value of \(f_0(x_t)\). We show that the optimal regret can be completely characterized (up to constant factors) by a continuous analogue of the Littlestone dimension.

For this section we don’t require the assumption that \(\mathcal{Y}\) is ordered, only that the loss function forms a valid metric (i.e. is symmetric and satisfies the triangle inequality).

### 5.1 Tree Dimension

**Definition 5.1.** A \((\mathcal{X}, \mathcal{Y})\)-tree of cost \(c\) is a labeled binary tree with the following properties

- There is a root node and each interior node has two children

- Each interior node is labeled with a triple \((x, y_1, y_2)\) where \(x \in \mathcal{X}, y_1, y_2 \in \mathcal{Y}\)

- For each leaf, the sum of \(L(y_1, y_2)\) over all nodes on the path from the root to the leaf is at least \(c\).
Definition 5.2. We say a \((\mathcal{X}, \mathcal{Y})\)-tree \(T\) is \(\mathcal{H}\)-satisfiable if we can label each leaf with some \(f \in \mathcal{H}\) such that for each node \((x, y_1, y_2) \in T\), all leaves of the left subtree satisfy \(f(x) = y_1\) and all leaves of the right subtree satisfy \(f(x) = y_2\).

Definition 5.3 (Tree dimension). We define \(\tau(\mathcal{H})\), the tree dimension of \(\mathcal{H}\), to be the maximum cost of a \((\mathcal{X}, \mathcal{Y})\)-tree that is \(\mathcal{H}\)-satisfiable.

Remark. Note we can naturally extend the above definition to any subset \(\mathcal{H}' \subset \mathcal{H}\).

It is worth noting that covering dimension is “more restrictive” than tree dimension in the sense that bounded covering dimension implies bounded tree dimension.

Theorem 5.4. Let \(\mathcal{H}\) be a hypothesis class consisting of functions mapping \(\mathcal{X} \to \mathcal{Y}\) and let \(L\) be a loss function that defines a metric on \(\mathcal{Y}\). If \(Cdim(\mathcal{H})\) is finite then
\[
\tau(\mathcal{H}) \leq 6 \cdot Cdim(\mathcal{H})
\]

Before proving the above we prove a few preliminary lemmas.

Lemma 5.5. Let \(\mathcal{H}\) be a hypothesis class and \(L\) be a loss function that defines a metric on \(\mathcal{Y}\). Let \(T\) be an \(\mathcal{H}\)-satisfiable \((\mathcal{X}, \mathcal{Y})\)-tree where all leaves have depth \(d\) and such that for each internal node \((x, y_1, y_2)\), \(L(y_1, y_2) > 2^{-i}\). Then
\[
d \leq (i + 1) \cdot Cdim(\mathcal{H})
\]

Proof. Since the tree is \(\mathcal{H}\)-satisfiable, we can label the leaves with functions \(f \in \mathcal{H}\). Any two of these functions \(f_1, f_2\) must satisfy \(d_\infty(f_1, f_2) \geq 2^{-i}\) since there must be some internal node \((x, y_1, y_2)\) where \(f_1(x) = y_1\) and \(f_2(x) = y_2\). Therefore, there are \(2^d\) functions in \(\mathcal{H}\) such that any two have \(d_\infty\) distance bigger than \(2^{-i}\). This implies that
\[
|N_{2^{-i+1}}(\mathcal{H})| \geq 2^d
\]
Now by the definition of covering dimension, we conclude \(d \leq (i + 1) \cdot Cdim(\mathcal{H})\). \(\blacksquare\)

Given a rooted binary tree \(T\), we say a rooted binary tree \(T'\) is contained in \(T\) if all of the nodes of \(T'\) are nodes of \(T\) and the nodes of \(T'\) form a binary tree where each interior node has two children under the topology given by \(T\).

Lemma 5.6. Consider a rooted binary tree \(T\) (where all interior nodes have exactly two children) and say its nodes are colored with colors \(1, 2, \ldots, c\). We say the colored tree satisfies property \((x_1, \ldots, x_c)\) if for each \(i \in [c]\), it does not contain a monochromatic complete binary tree of color \(i\) and depth \(x_i\). If the coloring of \(T\) satisfies property \((x_1, \ldots, x_c)\), there exists a leaf such that on the path from the root to the leaf, there are at most \(x_i\) nodes of color \(i\) for all \(i \in [c]\).

Proof. We prove the lemma by induction on \(x_1 + \cdots + x_c\). The base cases are obvious. Now say the root of \(T\) is colored with color \(i\). Clearly we must have \(x_i > 0\). Then either the left or right subtree of the root must satisfy property \((x_1, \ldots, x_i - 1, \ldots, x_c)\). Using the inductive hypothesis, we get the desired. \(\blacksquare\)
Proof of Theorem 5.4. Assume for the sake of contradiction that \( \tau(H) > 6 \cdot \text{Cdim}(H) \). Consider a \((\mathcal{X}, \mathcal{Y})\)-tree \( T \) that is \( H \)-satisfiable and has cost larger than \( 6 \cdot \text{Cdim}(H) \). Note we can assume that there are no nodes in \( T \) where \( L(y_1, y_2) = 0 \) since otherwise, we can delete that node and keep only its left subtree. Let \( c \) be an integer such that for all nodes \((x, y_1, y_2)\), we have \( L(y_1, y_2) > 2^{-c} \).

Now color the internal nodes of \( T \) with \( c \) colors \( \{1, 2, \ldots c\} \) where a node \((x, y_1, y_2)\) is color \( i \) if \( \frac{1}{2^i} < L(y_1, y_2) \leq \frac{1}{2^{i-1}} \).

Note by Lemma 5.5, \( T \) does not contain any monochromatic, complete binary trees of color \( i \) with depth at least \( (i + 1) \cdot \text{Cdim}(H) \). By Lemma 5.6, this implies the total cost of \( T \) is at most

\[
\sum_{i=1}^{c} \frac{(i + 1) \cdot \text{Cdim}(H)}{2^{i-1}} \leq 6 \cdot \text{Cdim}(H)
\]

which completes the proof.

However, we cannot hope for any sort of converse to Theorem 5.4 as evidenced by the following example. Let \( \mathcal{X} = \mathcal{Y} = [0, 1] \) and let \( L(y_1, y_2) = |y_1 - y_2| \). Let \( H = \{1_x = c | c \in [0, 1]\} \) be the set of all indicator functions of points in \([0, 1]\). \( H \) has infinite covering dimension but its tree dimension is clearly just 1.

### 5.2 Loss Bounds using Tree Dimension

**Theorem 5.7.** In the full feedback model there exists an algorithm that incurs total loss \( O(\tau(H)) \). Furthermore, no algorithm can guarantee better than \( \tau(H)/2 \) loss.

First we prove that the algorithm below achieves the upper bound.

#### Algorithm 7 Contextual Binary Search

```
for t in 1, 2, \ldots , T do
    Adversary picks \( x_t \)
    Let \( S_t \) be the set of hypotheses consistent with the feedback so far
    For each \( \epsilon \) define \( A_{\epsilon,t} = \{y| y \in \mathcal{Y}, \tau(S_t \cap \{f| f(x_t) = y\}) \geq \tau(S_t) - \epsilon\} \)
    Choose \( y_t \in A_{\epsilon,t} \) for the smallest \( \epsilon \) such that \( A_{\epsilon,t} \neq \emptyset \)
```

**Lemma 5.8.** For any \( y_1, y_2 \in A_{\epsilon,t} \), \( L(y_1, y_2) \leq \epsilon \).

**Proof.** Assume for the sake of contradiction that this is false. Then we can construct a \( S_t \)-satisfiable \((\mathcal{X}, \mathcal{Y}, s_t)\)-tree with \((x_t, y_1, y_2)\) as its root node and cost bigger than \( \tau(S_t) \). ■

**Lemma 5.9.** For each \( t \), we have

\[
\tau(S_{t+1}) \leq \tau(S_t) - L(y_t, f_0(x_t))
\]

where \( L(y_t, f_0(x_t)) \) is the loss of the algorithm.
Proof. Since $\epsilon$ is the smallest value such that $A_{\epsilon,t}$ is non-empty, then for every $y \in A_{\epsilon,t}$ we must have $\tau(S_t \cap \{ f | f(x_i) = y \}) = \tau(S_t) - \epsilon$. So if $L(y_t, f_0(x_i)) \leq \epsilon$ we are done.

Consider now the case where $\ell := L(y_t, f_0(x_i)) > \epsilon$. For that case, we want to argue that $f_0(x_i) \notin A_{\ell,t}$ for any $\ell' < \ell$, since after we get the feedback, we will update $S_{t+1} = \{ f \in S_t | f(x_t) = f_0(x_i) \}$. Therefore $f_0(x_i) \notin A_{\ell,t}$ for all $\ell' < \ell$ implies that: $\tau(S_{t+1}) \leq \tau(S_t) - \ell$.

Pick any $\ell'$ with $\epsilon < \ell' < \ell$. To see that $f_0(x_i) \notin A_{\ell,t}$, observe that $y_t \in A_{\epsilon,t} \subseteq A_{\ell',t}$. If it were the case that $f_0(x_i) \in A_{\ell,t}$, we would have $\ell \leq \ell'$ by Lemma 5.8 contradicting the fact that $\ell' < \ell$.

Proof of Theorem 5.7. The upper bound follows directly from the previous lemma. For the lower bound, consider an $\mathcal{H}$-tree with cost $\tau(\mathcal{H})$ that is $\mathcal{H}$-satisfiable. We can ensure that all leaves have the same depth $d$ (by adding nodes of the form $(x, y, y)$). Now the adversary chooses a leaf uniformly at random. If the sequence of nodes from the root to the leaf are

$$(x_1, y_{i1}, y_{i2}), \ldots, (x_d, y_{i1}, y_{i2})$$

then the adversary presents the inputs $x_1, x_2, \ldots, x_d$ in that order to the learner. Since the loss function satisfies the triangle inequality, the expected loss of any learner is at least $\tau(\mathcal{H})/2$ so we are done.

Remark. Algorithm 7 assumes that the set $\{ \epsilon; A_{\epsilon,t} \neq \emptyset \}$ has a minimum, which is always the case if the $H$ is finite. For infinite $\mathcal{H}$ the minimum might not exist. In such case, choose $\epsilon = \inf\{ \epsilon; A_{\epsilon,t} \neq \emptyset \}$ and choose $y_t \in A_{\epsilon+g_t,t}$ for $g_t = 1/2^t$. Theorem 5.7 can be easily adapted to provide a bound of $\tau(\mathcal{H}) + \sum_t g_t \leq \tau(\mathcal{H}) + 1$.

5.3 Separating binary and full feedback

With binary feedback we can no longer obtain loss bounds that depend only on tree dimension. To see this, consider the following example:

Let $\mathcal{H}$ be the set of all functions $f : [n] \rightarrow \{0, 1/n, \ldots, (n-1)/n\}$. There are $n^n$ such functions. Now for each, slightly perturb the outputs (i.e. $f(i) = j/n + \epsilon$) so that for every $f_1, f_2$ and $i$, $f_1(i) \neq f_2(i)$. Let $L(y_1, y_2) = |y_1 - y_2|$. The tree dimension of this class is $O(1)$. However, clearly any algorithm must incur $\Omega(n)$ loss in expectation with binary feedback.

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A Efficient implementations for linear contextual search

In the main paper, we remarked that we can efficiently implement Algorithms 3 and 4 as long as we can solve the problem of (approximately) computing the minimum distance from a point to the convex set $S_t$. This is a convex problem, so the guarantee of cutting plane methods (like the ellipsoid algorithm) tells us that (given an initial ellipsoid $E \supseteq S_t$), it is possible to compute an $\epsilon$-optimal solution in time

$$O\left( T \cdot \text{poly}(d) \cdot \log \left( \frac{\text{Vol}(E)}{\epsilon \text{Vol}(S_t)} \right) \right).$$

We can take $E$ to be the unit ball; then this is algorithm is efficient as long as $\text{Vol}(S_t)$ is never too small (anything at least $\exp(-\text{poly}(T))$ is fine). Here we present a simple modification of Algorithm 3 that makes sure that the volume of $S_t$ stays large enough throughout by preserving a small ball around $v_0$. 

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Initialize $S_1 = (1 + 1/T^4)B$ to make sure that $B(v_0; 1/T^4) \subseteq S_1$. Instead of guessing $y_t = m_t \pm 1/T$ we can add a small perturbation $\delta_t$ sampled from the uniform distribution over $[0, 1/T^2]$. Guess according to:

$$y_t = m_t \pm (1/T + \delta_t)$$

The total additional loss from the perturbation is $O(1)$ and it doesn’t affect the analysis in Section 3.1. The advantage of this perturbation is that the probability that the cut passes through the ball of radius $1/T^4$ around $v_0$ is at most $1/T^2$ per period. So with probability $1 - 1/T$, $B(v_0, 1/T^4) \subseteq S_t$ for all periods $t$. It follows that $\text{Vol}(S_t) \geq \frac{1}{T^d} \text{Vol}(B(0,1))$, and with this, the convex minimization problem can be solved efficiently.

The same idea also works for the $O(d \log d)$ algorithm in Section 3.5. For this algorithm, the same idea works: we query $m_t \pm (u_i + \delta_t)$ again with $\delta_t \sim [0, 1/T^2]$. The analysis in Section 3.5 works when $u_i \geq 1/T$. It suffices to note that the total loss from rounds when $u_i \leq 1/T$ is $O(d)$. 