Sub-Fourier characteristics of a $\delta$-kicked rotor resonance

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We experimentally investigate the sub-Fourier behavior of a $\delta$-kicked rotor resonance by performing a measurement of the fidelity or overlap of a Bose-Einstein condensate (BEC) exposed to a periodically pulsed standing wave. The temporal width of the fidelity resonance peak centered at the Talbot time and zero initial momentum exhibits an inverse cube pulse number ($1/N^3$) dependent scaling compared to a $1/N^2$ dependence for the mean energy width at the same resonance. A theoretical analysis shows that for an accelerating potential the width of the resonance in acceleration space depends on $1/N^3$, a property which we also verify experimentally. Such a sub-Fourier effect could be useful for high precision gravity measurements.

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The quantum $\delta$-kicked rotor (QDKR) has proved to be an excellent testing ground for theoretical and experimental studies of chaos in the classical and quantum domains [1]. An experimental version of this system in the form of the kicked particle is achieved by exposing cold atoms to $N$ pulses of an off-resonant standing wave of light [2]. Ever since its realization, the QDKR has continued to reveal a rich variety of effects including dynamical localization [3], quantum accelerator modes [4, 5], quantum ratchets [6, 10], and quantum resonances [2–4]. Such resonances appear for pulses separated by rational fractions of a characteristic time called the Talbot time and can be observed as sharp peaks in the mean energy of the system [11]. The width of these peaks has been found to scale as $1/N^2$, a sub-Fourier effect attributed to the non-linear nature of the QDKR and explained using a semi-classical picture [12]. Away from the resonances, dynamical localization sets in, characterized by the quantum suppression of classical momentum diffusion beyond a "quantum break time" [2]. This property, unique to quantum dynamics in the chaotic regime, was utilized to discriminate between two driving frequencies of the QDKR with sub-Fourier resolution [13].

High-precision measurements using quantum mechanical principles have been carried with atom interferometers for many years [14]. Such devices were used to determine the Earth’s gravitational acceleration [15], fine structure constant $\alpha$ [16], and the Newtonian constant of gravity [17]. The promise of the QDKR as a candidate for making these challenging measurements has begun to be realized [18]. Recently a scheme was proposed for measuring the overlap or fidelity between a near-resonant $\delta$-kicked rotor state and a resonant state via application of a tailored pulse at the end of a rotor pulse sequence [19]. It predicted a $1/N^3$ scaling of the temporal width of the fidelity peak. In this paper we report on the observation of such fidelity resonance peaks and their sub-Fourier nature. Figure 1 illustrates a plot of the fidelity (fraction of atoms in the zeroth order momentum state) vs pulse period obtained by the application of an overlap pulse at the end of five rotor kicks. For comparison we also plot the mean energy of the rotor sequence. It can be seen that even for relatively few kicks the fidelity peak is significantly narrower. We also investigated the sensitivity of this fidelity resonance to an accelerating rotor. As will be seen our calculations indicate that the width of the fidelity peak vs acceleration decreases at a sub-Fourier rate of $1/N^3$. We confirm this result with experiments.

The dynamics of a periodically kicked atom in the presence of a linear potential is described by the quantum $\delta$-kicked accelerator (QDKA) Hamiltonian

$$\hat{H} = \frac{\hat{P}^2}{2} + \frac{\eta}{\tau} \hat{X} + \phi_d \cos(\hat{X}) \sum_{t=1}^{N} \delta(t' - t\tau).$$  (1)

$\hat{P}$ is the momentum (in units of two photon recoils,
that an atom of mass $M$ acquires from short, periodic pulses of a standing wave with a grating vector $G = 2\pi\lambda/\lambda_G$ ($\lambda_G$ is the wavelength of the standing wave). $X$ is position in units of $G^{-1}$ and $\eta = MgT/hG$, $g'$ being its acceleration between pulses separated by $T$, the pulse period. $\phi_\delta = \Omega^2 \Delta t/8dL$ represents the kicking strength of a pulse of length $\Delta t$, $\Omega$ is the Rabi frequency and $\delta_\ell$ the detuning of the kicking laser from the atomic transition. $t'$ is the continuous time variable and $\tau = 2\pi T/T_{1/2}$ is the scaled pulse period. For the case $\eta = 0$, Eq. (1) reduces to the familiar QDKR Hamiltonian. Primary quantum resonances are seen for pulses separated by integer multiples of the half-Talbot time, $T_{1/2} = 2\pi M/hG^2$ or $\tau = 2\pi$. Adjacent momentum orders evolve integer multiples of 2 in phase by $\pi$ such a resonance, a “fidelity” test for the QDKR was performed at the end of the condensate. The kicking laser from the atomic transition. The resonances studied here appear for pulses separated by integer multiples of the half-Talbot time, $T_{1/2} = 2\pi M/hG^2$ or $\tau = 2\pi$. Adjacent momentum orders evolve integer multiples of 2 in phase by $\pi$ such a resonance, a “fidelity” test for the QDKR was proposed in Ref. [19]. In this scheme, a kick reversed in phase by $\pi$ and carrying a strength of $N\phi_\delta$ is applied at the end of the $N$ rotor kicks. The fidelity is then defined as $F = \langle |\beta U_r U_N |\beta \rangle \rangle^2$ where $U = \exp(-i\Omega^2 P^2) \exp[-i\phi_\delta \cos(X)]$ describes the one period evolution, $U_r = \exp(iN\phi_\delta \cos(X))$ is the overlap pulse and $\beta$ is the fractional part of the momentum. $F$ therefore gives the probability of the revival of the initial state and is measured by the fraction of atoms which have returned to the zero momentum state. Near a resonance at the Talbot time, $\tau = 4\pi$, the fidelity is

$$F(\epsilon, \beta = 0) \approx J_0^2 \left( \frac{1}{12} N^3 \phi_\delta^2 \epsilon \right)$$

where $\epsilon = \tau - 4\pi$. The width of such a peak in $\epsilon$ therefore changes as $1/(N^3 \phi_\delta^2)$, displaying a stronger sub-Fourier dependence on the number of kicks than the mean energy.

Our experiment is performed by producing a BEC of 20000 Rb87 atoms in the $5S_{1/2}, F = 1, m_F = 0$ level in an optical trap [7, 8]. After being released from the trap, the condensate is exposed to a horizontal standing wave created by two beams of wavelength $\lambda = 780$ nm light detuned 6.8 GHz to the red of the atomic transition. The wave vector of each beam was aligned $\theta = 52^\circ$ to the vertical. This created a horizontal standing wave with a wavelength of $\lambda_G = \lambda/2\sin \theta$ and a corresponding Talbot time of 106.5 $\mu$s. Two acousto-optic modulators (AOMs) controlled the pulse lengths as well as the relative frequencies of the kicking beams enabling the control of the acceleration and initial momentum of the standing wave with respect to the condensate. The kicking pulse length was 0.8 $\mu$s with a $\phi_\delta \approx 0.6$. For the last kick the phase of one of the AOMs RF driving signal was changed by $\pi$ which shifted the standing wave by half a wavelength. In order to keep this final overlap pulse within the Raman-Nath regime we varied the intensity rather than the pulse length to create a kick strength of $N\phi_\delta$. This was done by adjusting the amplitudes of the RF waveforms driving the kicking pulse. Dephasing due to experimental instabilities made the reversal process inconsistent for $N > 6$. To reduce this effect, a $\pi$-shifted kicking sequence was adopted where each kick in the QDKR was shifted by half a wavelength with respect to the previous. This decreased the Talbot time by half and led to much improved results for larger number of kicks. Following the entire kicking sequence we waited 8 ms for the different momentum orders to separate before the atoms were absorption imaged.

From the time of flight images fidelity $F$ is measured as the fraction of atoms which have reverted back to the zero order momentum state, that is $F = P_0/\sum P_n$ where $P_n$ is the number of atoms in the $n^{th}$ momentum order. To facilitate the analysis of the data, all of the resonance widths ($\delta \epsilon$) were scaled to that at a reference kick number of $N = 4$. That is we define a scaled fidelity width $\Delta \epsilon = \delta \epsilon/\delta \epsilon_{N=4}$ for each scaled kick number $N_s = N/4$ and recover $\log \Delta \epsilon = -3 \log N_s$ using Eq. (2). For each kick, a scan is performed around the resonance time. To ensure the best possible fit of the central peak of the fidelity spectrum to a gaussian, the time is scanned between values which make the argument of $J_0^2$ of Eq. (2) $\approx 2.4$ so that the first side lobes are only just beginning to appear. Figure 2(a) plots the logarithm of the full width at half maximum for 4 to 9 kicks scaled to the fourth kick. A linear fit to the data gives a slope of $-2.73 \pm 0.19$ giving a reasonable agreement with the predicted value of -3 within the experimental error. As seen in the same figure, the results are close to the numerical simulations which take into account the finite width of the initial state of 0.068L [10]. We also compared the resonance widths of the kicked rotor mean energy ($\langle E \rangle$) to that of the fidelity widths. As in the fidelity, the plotted values $\Delta \langle E \rangle$ have been normalized to that of the fourth kick. On the log scale the width of each peak gets narrower with the kick number with a slope of $-1.93 \pm 0.21$ (Fig. 2(a)) in agreement with previous results [1, 21]. As a further test of Eq. (2), the variation in the widths of the fidelity and mean energy peaks were studied as a function of $\phi_\delta$. Figure 2(b) shows the fidelity with changes within $-1.96 \pm 0.3$, close to the predicted value of -2. This is again a faster scaling compared to the mean energy width which decreases with a slope of $-0.88 \pm 0.24$ (the theoretical value being -1).

The resonances studied here appear for pulses separated by the Talbot time and an initial momentum state of $\beta = 0$. Away from this resonant state, phase changes in the amplitudes of the different momentum orders lead to a fidelity which depends on the initial momentum as, $F(\epsilon = 0, \beta) = J_0^2 (2\pi \phi_\delta N(N+1)\beta)$ [19]. The peak width in $\beta$ space is thus expected to change as $1/[N(N+1)]$. 

$$\text{(2)}$$
FIG. 3. Variation of the fidelity peak width around $\beta=0$ as a function of kick number $N(N+1)/20$ scaled to the 4th kick. The straight line is a linear fit to the data with a slope of $-0.92 \pm 0.06$. Errorbars as in Fig. 2(b).

FIG. 4. (a) Momentum width of the reversed zeroth order state as a function of kick number. Errorbars are an average over three experiments. (b) Optical density plots for the initial state (red,solid) and kick numbers 2 (magenta,dot-dashed), 4 (black,dotted), and 6 (blue,dashed) after summation of the time-of-flight image along the axis perpendicular to the standing wave.
coefficients $c_n$ are $c_n(\epsilon, \beta, \eta) = \langle n + \beta | \hat{U}^{\dagger} \hat{g}_{n} \ldots \hat{U}^{\dagger} \hat{g}_{1} | \beta \rangle$. $\hat{U}_{g_{n}} = \exp[-\frac{i}{\hbar}(\hat{p} + \eta \hat{q} + \frac{\eta}{2})^{2}] \exp[-i\hat{\Theta}(\hat{X})]$ is the $t$th kick evolution operator in the freely falling frame obtained after a gauge transformation of the Hamiltonian $\hat{H}$ which restores the conservation of quasimomentum $\beta$ [20]. Close to the resonances, we have $F(\epsilon, \beta, \eta) \approx | \sum_{n} J_{n}^{2}(N \Theta_{n}) \exp(-i\Theta_{n})^{2} |^{2}$, where $\Theta_{n} = \frac{\partial \epsilon}{\partial \beta} \epsilon + \frac{\partial \hat{\Theta}}{\partial \eta} \beta + \frac{\partial \hat{\Theta}}{\partial \eta} \eta$ describes the effect of deviations from resonance on the coefficients $c_{n}$. Using a procedure detailed in Ref. [19], one can show that $\frac{\partial \Theta}{\partial \eta}(\epsilon = \beta = \eta = 0) = \frac{\partial \Theta}{\partial \eta}(0, 0.0) = -4\pi \eta N^{2}/\beta$, where we have kept terms in $N^{2}$. Finally we arrive at the fidelity in the presence of acceleration,

$$F(\eta, \epsilon = \beta = 0) = J_{0}^{2} \left( \frac{4\pi}{3} N^{3} \hat{\phi} d\eta \right).$$

Thus the width of such a peak centered at the resonant zero acceleration should drop as $1/N^{3}$. In order to verify the above result, the standing wave was accelerated during the application of the pulses. This acceleration was scanned across the resonant zero value and readings of the fidelity collected. Figure 5 plots the experimental data for 4 to 9 kicks, where the widths of the peaks decrease with a slope of $-3.00 \pm 0.23$ in excellent agreement with the theory.

In conclusion, we performed experimental measurements of the fidelity widths of a $\delta$-kicked rotor state near a quantum resonance. The widths of these peaks centered at the Talbot time decreased at a rate of $N^{-2.73}$ comparable to the predicted exponent of $-3$. By comparison, the mean energy widths was found to reduce only as $N^{-1.93}$. Furthermore, the fidelity peaks in momentum space changed as $(N(N + 1))^{-0.32}$, also consistent with theory. The reversal process used in the fidelity experiments was found to lead to a cooling effect, whereby the momentum distribution of the final zeroth order state decreased by 25% from the initial width at the end of a 9th reversal kick, as a result of the velocity selection by the final pulse. The sub-Fourier dependencies of the mean energy and fidelity observed here are characteristic of the dynamical quantum system that is the QDKR [10]. The narrower resonances of the fidelity scheme could be exploited in locating the resonance frequency with a resolution below the limit imposed by the fourier relation. This could be used to determine the photon recoil frequency ($\omega_{r} = E_{c}/\hbar$) which together with the photon wavelength enables measurement of the fine structure constant with a high degree of precision [10][11]. We also demonstrated a $N^{-3}$ dependence of the resonance width in acceleration space in accordance with the extended theory. The sensitivity of an atom interferometer based graviometer scales as the square of the loop time, hence the pursuit of large area interferometers to improve accuracy [12]. The possibilities offered by a process like the fidelity measure which is responsive to gravity with the cube of the ‘time’ $N$ becomes evident here. With future refinements this scheme could therefore serve as a highly sensitive measure of the local gravity.

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