THE GROMOV-ELIASHBERG THEOREM BY MICROLOCAL SHEAF THEORY

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Abstract. The Gromov-Eliashberg theorem says that the group of symplectomorphisms of a symplectic manifold is $C^0$-closed in the group of diffeomorphisms. This can be translated into a statement about the Lagrangian submanifolds which are graphs of symplectomorphisms. It is also known that such Lagrangian submanifolds are locally microsupports of sheaves. We explain how we can deduce the Gromov-Eliashberg theorem from the involutivity theorem of Kashiwara and Schapira which says that the microsupport of a sheaf is coisotropic.

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1. Introduction

In [10], D. Tamarkin gives a totally new approach for treating questions in symplectic geometry (especially classical problems of non-displaceability). His approach is based on the microlocal theory of sheaves, introduced and developed in [6, 7, 8]. In particular he remarks that in some situations it is possible to associate a sheaf with a
given Lagrangian submanifold of a cotangent bundle and then deduce properties of the Lagrangian submanifold from this sheaf. In this paper we follow this approach and explain how we can recover the Gromov-Eliashberg theorem by microlocal sheaf theory, using in particular the involutivity of the microsupport.

Let us briefly recall some facts of the microlocal theory of sheaves. We consider a real manifold $M$ of class $C^\infty$ and a commutative unital ring $k$ of finite global dimension. We let $\mathcal{D}^b(k_M)$ be the bounded derived category of sheaves of $k$-modules on $M$. In [8], the authors attach to an object $F$ of $\mathcal{D}^b(k_M)$ its microsupport, or singular support, $\text{SS}(F)$, a subset of $T^\ast M$, the cotangent bundle of $M$. By definition the microsupport is closed and conic for the action of $\mathbb{R}^+$ on $T^\ast M$. A deep result of [8] says that $\text{SS}(F)$ is involutive (or coisotropic). The initial motivation for this theorem comes from the theory of systems of linear PDE's because of its link with the propagation of singularities. (A microdifferential version of the involutivity theorem is given in [9] and an algebraic statement is given in [2].)

In [4] the authors prove the following result, inspired by [10]. Let $M$ be a manifold and set $\dot{T}^\ast M = T^\ast M \setminus M$. For $F \in \mathcal{D}^b(k_M)$ we also set $\dot{\text{SS}}(F) = \text{SS}(F) \cap \dot{T}^\ast M$. Let $I = [a, b]$ be an interval containing 0 and let $\psi: \dot{T}^\ast M \times I \to \dot{T}^\ast M$ be a homogeneous Hamiltonian isotopy. For $t \in I$ we let $\psi_t$ be the restriction of $\psi$ at time $t$ and we denote by $\Lambda_{\psi_t} \subset \dot{T}^\ast M^2$ the graph of $\psi_t$, twisted by the antipodal map. Hence $\Lambda_{\psi_t}$ is a conic Lagrangian submanifold. Then the main result of [4] says that there exists $K_t \in \mathcal{D}(k_M^2)$, for each $t \in I$, such that $\dot{\text{SS}}(K_t) = \Lambda_{\psi_t}$. We can also consider non homogeneous Hamiltonian isotopies by adding a variable: given a Hamiltonian isotopy $\varphi$ of $T^\ast M$, with compact support, we can define a homogeneous Hamiltonian isotopy $\psi$ of $\dot{T}^\ast (M \times \mathbb{R})$ which makes a commutative diagram with $\varphi$ and the map

$$\rho_M: T^\ast M \times T^\ast \mathbb{R} \to T^\ast M, \quad (x, s; \xi, \sigma) \mapsto (x; \xi/\sigma).$$

The Gromov-Eliashberg theorem (Theorem 1.1 below) says that, if a sequence of symplectic $C^1$ diffeomorphisms $\{\varphi_n\}_{n \in \mathbb{N}}$ of some symplectic manifold $(X, \omega)$ has a $C^0$ limit, says $\varphi_\infty$, and $\varphi_\infty$ is a $C^1$ diffeomorphism of $X$, then $\varphi_\infty$ is symplectic. The aim of this paper is to explain how it can be deduced from the involutivity theorem of [8].

The Gromov-Eliashberg theorem is in fact a local statement and we can assume that $X = \mathbb{R}^{2n}$, that $\varphi_n$ is the time 1 of a Hamiltonian isotopy and that the convergence occurs on some ball $B$ of $\mathbb{R}^{2n}$. We identify $\mathbb{R}^{2n}$ with $T^\ast \mathbb{R}^n$ and we add a variable to make the situation homogeneous. Then we can apply the results of [4] and we deduce that
there exists $K_n \in D^b(k_{\mathbb{R}^{2n+1}})$ such that $\hat{SS}(K_n) \subset T^* \mathbb{R}^{2n} \times \hat{T}^* \mathbb{R}$ and
\begin{equation}
\rho_{\mathbb{R}^{2n}}(\hat{SS}(K_n)) = \Lambda_{\varphi_n},
\end{equation}
where $\rho_{\mathbb{R}^{2n}}$ is defined in (1.1) and $\Lambda_{\varphi_n} \subset T^* \mathbb{R}^{2n}$ is the twisted graph of $\varphi_n$. We define $K$ by the distinguished triangle $\bigoplus_{n \in \mathbb{N}} K_n \to \prod_{n \in \mathbb{N}} K_n \to K \xrightarrow{\pm 1}$. Then $SS(K) \subset \bigcap_{k \in \mathbb{N}} \bigcup_{n \geq k} SS(K_n)$ and we have in particular
\begin{equation}
\rho_{\mathbb{R}^{2n}}(\hat{SS}(K)) \subset \Lambda_{\varphi_\infty}.
\end{equation}

Using this inclusion we can deduce from the involutivity theorem that $\Lambda_{\varphi_\infty}$ is coisotropic at any point $p$ which belongs to $\rho_{\mathbb{R}^{2n}}(SS(K))$. This means that it only remains to prove that (1.3) is in fact an equality. We do not prove it directly. We only prove that, for any given $p \in \Lambda_{\varphi_\infty}$, we can modify the $K_n$’s by a so called cut-off result of [8] and obtain another sheaf $K$ (depending on $p$) such that (1.3) still holds and moreover $p \in \rho_{\mathbb{R}^{2n}}(SS(K))$. Then the involutivity theorem applies at $p$. We obtain in this way that $\Lambda_{\varphi_\infty}$ is coisotropic at all points. Since it is a submanifold of dimension $2n$, it is Lagrangian, which means that $\varphi_\infty$ is a symplectic map.

Now we give a more precise idea of the proof. We first state the result in the following local form. Let $(E, \omega)$ be a symplectic vector space which we identify with $\mathbb{R}^{2n}$. We endow $E$ with the Euclidean norm of $\mathbb{R}^{2n}$. For $R > 0$ we let $B^E_R$ be the open ball of radius $R$ and center $0$. For a map $\psi: B^E_R \to E$ we set $\|\psi\|_{B^E_R} = \sup\{\|\psi(x)\|; x \in B^E_R\}$.

**Theorem 1.1** (Gromov-Eliashberg rigidity theorem, see [1, 3]). Let $R > 0$. Let $\varphi_n: B^E_R \to E, n \in \mathbb{N}$, and $\varphi_\infty: B^E_R \to E$ be $C^1$ maps. We assume
\begin{itemize}
  \item[(i)] $\varphi_n$ is a symplectic map, that is, $\varphi_n^*(\omega) = \omega$, for all $n \in \mathbb{N}$,
  \item[(ii)] $\|\varphi_n - \varphi_\infty\|_{B^E_R} \to 0$ when $n \to \infty$,
  \item[(iii)] $d\varphi_\infty: T_x E \to T_{\varphi_\infty(x)} E$ is an isomorphism, for all $x \in B^E_R$.
\end{itemize}
Then $\varphi_\infty|_{B^E_R}$ is a symplectic map.

**Main ingredients of the proof.** As seen above the essential ingredient of the proof is the involutivity theorem of [8]. The second ingredient is the main result of [4] which implies the existence of a “quantization” $K_n$ for $\varphi_n$, that is, an object $K_n \in D^b(k_{\mathbb{R}^{2n+1}})$ satisfying (1.2).

The third important tool is a “cut-off” result of [8]. We use the following statement. Let $V'$ be a vector space and $V = V' \times \mathbb{R}$, with coordinates $(x_1, \ldots, x_n)$. Let $\gamma_{c_2} \subset \gamma_{c_1} \subset V$ be two closed cones of the type $\gamma_c = \{x_n \geq c(a_1^2 + \cdots + x_{n-1}^2)^{1/2}\}$, with $c_2 > c_1 > 0$. Let
$B^V_R \subset V$ be the open ball of center $0$ and radius $R$. Let $F \in \mathcal{D}^b(k_{B^V_R})$ with a microsupport contained in the union of the polar cone $\gamma^{o_2}$ and the complement of $\gamma^{o_2}_c$:

\begin{equation}
\hat{\mathcal{S}}(F) \cap (B^V_R \times (\gamma^{o_2}_c \setminus \text{Int}(\gamma^{o_2}_c))) = \emptyset.
\end{equation}

The cut-off lemma says that we can decompose $F$ according to this decomposition of $\hat{\mathcal{S}}(F)$. More precisely, there exists $r$ such that $R > r > 0$ and the following holds. For any $F \in \mathcal{D}^b(k_{B^V_r})$ satisfying (1.4) there exists a distinguished triangle over the smaller ball $B^V_r$, $F_1 \oplus F_2 \to F|_{B^V_r} \to L \overset{+1}{\to}$, such that $\hat{\mathcal{S}}(L) = 0$, $\hat{\mathcal{S}}(F_1) = \hat{\mathcal{S}}(F) \cap (B^V_r \times \gamma^{o_2}_c)$ and $\hat{\mathcal{S}}(F_2) \cap (B^V_r \times \gamma^{o_2}_c) = \emptyset$.

We can use this decomposition to analyze $F$ and obtain some consequences on its cohomology. In particular we prove the following result (see Proposition 3.3). We use a notion of convex hull $\text{Conv}(S)$ for a subset $S \subset T^*M$ of a cotangent bundle: it is the union of the convex hulls in each fiber, that is, $\text{Conv}(S) = \bigsqcup_{x \in M} \text{Conv}(S \cap T^*_x M)$. Given $R, c_1, c_2$ as above, there exist non empty connected open subsets $W_1 \subset \cdots \subset W_4$ of $B^V_R \times \mathbb{R}$ such that: if $F \in \mathcal{D}^b(k_{B^V_R})$ satisfies (1.4), $\hat{\mathcal{S}}(F)$ is a Lagrangian submanifold, $F$ is simple (see Section 3) and the groups $\Gamma(W_4; F)$, $i = 1, \ldots, 4$, are distinct.

**Idea of the proof.** We prove that $T^*_p \Lambda_{\varphi_n}$ is coisotropic, for any given $p \in \Lambda_{\varphi_n}$. We work near $p$ and we approximate $\varphi_n$ by a globally defined Hamiltonian isotopy whose graph coincides with $\Lambda_{\varphi_n}$ near $p$. Hence we can assume that $\varphi_n$ is the time 1 of a Hamiltonian isotopy. By the main result of [4] there exists $K_n \in \mathcal{D}^b(k_{V^2 \times \mathbb{R}})$, for each $n$, such that $\rho_{V^2}(\hat{\mathcal{S}}(K_n)) = \Lambda_{\varphi_n}$. By the consequence of the cut-off result explained in the previous paragraph we can find connected open subsets $W_1 \subset \cdots \subset W_4$ of $V^2 \times \mathbb{R}$ and $L_n \in \text{Mod}(k_{V^2 \times \mathbb{R}})$ such that $\rho_{V^2}(\hat{\mathcal{S}}(L_n)) \subset \text{Conv}(\Lambda_{\varphi_n})$ near $p$ and the groups $\Gamma(W_i; L_n)$, $i = 1, \ldots, 4$, are distinct.

We define $L \in \text{Mod}(k_{V^2 \times \mathbb{R}})$ by $L = \text{coker}(\bigoplus_{n \in \mathbb{N}} L_n \to \prod_{n \in \mathbb{N}} L_n)$. Then $\rho_{V^2}(\hat{\mathcal{S}}(L))$ is contained in the limit of the $\text{Conv}(\Lambda_{\varphi_n})$. We can assume from the beginning that $\Lambda_{\varphi_n}$ is a section of the projection $T^*V^2 \to V^2$, near $p$. Then the limit of the $\text{Conv}(\Lambda_{\varphi_n})$ is $\Lambda_{\varphi_n}$.

We can see also that the groups $\Gamma(W_i; L)$, $i = 1, \ldots, 4$, are distinct, which implies that $L$ has a non trivial microsupport somewhere over $W_4$. Hence there exists $q = (y; \eta) \in \hat{\mathcal{S}}(L)$ such that $y \in W_4$. By the
The involutivity theorem we know that $\mathcal{SS}(L)$ is coisotropic at $q$. It follows that $\Lambda_{\varphi_\infty}$ is coisotropic at $q' = \rho V_2(q)$. Now $W_4$ can be made as small as we want so that $q'$ is arbitrarily close to $p$. It follows that $\Lambda_{\varphi_\infty}$ is coisotropic at $p$.

The proof is detailed in Section 11. Only Section 9 contains new results. The other sections are reminders of some notions on sheaves and results of [8]. The reader may also consult [12] for an introduction to the use of sheaves theory in symplectic geometry. The paper [11] gives another application of microlocal sheaf theory to the study of the $C^0$-rigidity of the Poisson bracket.

Acknowledgments. The idea of applying the involutivity theorem to the $C^0$-rigidity emerged after several discussions with Claude Viterbo, Pierre Schapira and Vincent Humilière (in particular about the paper [5]). It is a pleasure to thank them for their interest in this question.

2. Microlocal theory of sheaves

In this section, we recall some definitions and results from [8], following its notations with the exception of slight modifications. We consider a manifold $M$ of class $C^\infty$.

Some geometrical notions ([8 §4.2, §6.2]). For a locally closed subset $A$ of $M$, we denote by $\text{Int}(A)$ its interior and by $\overline{A}$ its closure.

We denote by $\pi_M : T^*M \to M$ the cotangent bundle of $M$. If $N \subset M$ is a submanifold, we denote by $T^*_N M$ its conormal bundle. We identify $M$ with $T^*_M M$, the zero-section of $T^*M$. We set $\dot{T}^*M = T^*M \setminus T^*_M M$ and we denote by $\dot{\pi}_M : \dot{T}^*M \to M$ the projection. We let $a_M : T^*M \to T^*M$ be the antipodal map $(x; \xi) \mapsto (x; -\xi)$. For a subset $A$ of $T^*M$ we set $A^a = a_M(A)$.

Let $f : M \to N$ be a morphism of real manifolds. It induces morphisms on the cotangent bundles:

$$ T^*M \xleftarrow{f^*_\dot{\pi}} M \times_N T^*N \xrightarrow{f_{\dot{\pi}}} T^*N. $$

We denote by $\Gamma_f \subset M \times N$ the graph of $f$. If $\varphi : T^*X \to T^*Y$ is a map between cotangent bundles we also consider the twisted graph

$$ \Lambda_{\varphi} = \Gamma_{a_Y \circ a_{\varphi}}. $$

The cotangent bundle $T^*M$ carries an exact symplectic structure. We denote the symplectic form by $\omega_M$. It is given in local coordinates $(x; \xi)$ by $\omega_M = \sum_i d\xi_i \wedge dx_i$. 
For a normed vector space \((E, ||.||)\), a point \(x \in E\) and \(r \geq 0\) we denote by \(B^E_{x,r}\) the open ball of radius \(r\) and center 0. If \(x = 0\), we usually write \(B^E_r\) for \(B^E_{0,r}\). For an open subset \(U \subset E\) and a continuous map \(\psi : U \to E\) we set
\[
\|\psi\|_U = \sup \{\|\psi(x)\| : x \in U\},
\]
\[
\|\psi\|_U^1 = \sup \{\|\psi(x)\|, \|d\psi_x(v)\| : x \in U, \|v\| = 1\}, \text{ if } \psi \text{ is } C^1.
\]

A subset of the cotangent bundle \(T^*M\) is called \(\mathbb{R}^+\)-conic (or conic) if it is invariant by the action of \((\mathbb{R}^+, \times)\) on \(T^*M\). We can turn nonconic subsets into conic ones by adding a variable and taking the inverse image by the following map \(\rho_M\).

We define \(\rho_M : T^*M \times \mathbb{R} \to T^*M\) by
\[
\rho_M(x, s; \xi, \sigma) = (x; \xi/\sigma).
\]
Finally we set \(T^*_{\sigma > 0}(M \times \mathbb{R}) = \{(x, s; \xi, \sigma) \in T^*(M \times \mathbb{R}) ; \sigma > 0\}\).

**Microsupport.** In this paper the coefficient ring \(k\) is assumed to be a field. This makes the description of simple sheaves easier (see Section 3). However the theory of microsupport works for a commutative unital ring of finite global dimension. We denote by \(\text{Mod}(k_M)\) the category of sheaves of \(k\)-vector spaces on \(M\). We denote by \(\mathcal{D}(k_M)\) (resp. \(\mathcal{D}^b(k_M)\)) the derived category (resp. bounded derived category) of \(\text{Mod}(k_M)\).

We recall the definition of the microsupport (or singular support) \(\text{SS}(F)\) of \(F \in \mathcal{D}^b(k_M)\), introduced by M. Kashiwara and P. Schapira in [6] and [7].

**Definition 2.1.** (see [8] Def. 5.1.2) Let \(F \in \mathcal{D}^b(k_M)\) and let \(p \in T^*M\). We say that \(p \notin \text{SS}(F)\) if there exists an open neighborhood \(U\) of \(p\) such that, for any \(x_0 \in M\) and any real \(C^1\)-function \(\phi\) on \(M\) satisfying \(d\phi(x_0) \in U\) and \(\phi(x_0) = 0\), we have \((\mathcal{R}\Gamma_{\{x;\phi(x) \geq 0\}}(F))_{x_0} \simeq 0\).

We set \(\text{SS}(F) = \text{SS}(F) \cap T^*M\).

In other words, \(p \notin \text{SS}(F)\) if the sheaf \(F\) has no cohomology supported by “half-spaces” whose conormals are contained in a neighborhood of \(p\). The following properties are easy consequences of the definition:

- \(\text{SS}(F)\) is closed and \(\mathbb{R}^+\)-conic,
- \(\text{SS}(F) \cap T^*_M = \pi_M(\text{SS}(F)) = \text{supp}(F)\),
- the triangular inequality: if \(F_1 \to F_2 \to F_3 \xrightarrow{+1} \) is a distinguished triangle in \(\mathcal{D}^b(k_M)\), then \(\text{SS}(F_2) \subset \text{SS}(F_1) \cup \text{SS}(F_3)\).
Example 2.2. (i) Let $F \in \mathcal{D}^b(k_M)$. Then $SS(F) = \emptyset$ if and only if $F \simeq 0$ and $SS(F) = \emptyset$ if and only if the cohomology sheaves $H^i(F)$ are local systems, for all $i \in \mathbb{Z}$.

(ii) If $N$ is a smooth closed submanifold of $M$ and $F = k_N$, then $SS(F) = T^*_NM$.

(iii) Let $\phi$ be $C^1$-function with $d\phi(x) \neq 0$ when $\phi(x) = 0$. Let $U = \{x \in M; \phi(x) > 0\}$ and let $Z = \{x \in M; \phi(x) \geq 0\}$. Then
\[
SS(k_U) = U \times_M T^*_M M \cup \{(x; \lambda d\phi(x)); \phi(x) = 0, \lambda \leq 0\},
SS(k_Z) = Z \times_M T^*_M M \cup \{(x; \lambda d\phi(x)); \phi(x) = 0, \lambda \geq 0\}.
\]

(iv) Let $\lambda$ be a closed convex cone with vertex at 0 in $E = \mathbb{R}^n$. Then $SS(k_\lambda) \cap T^*_0 E = \lambda^0$, the polar cone of $\lambda$, that is,
\[
\lambda^0 = \{\xi \in E^*; \langle v, \xi \rangle \geq 0 \text{ for all } v \in E\}.
\]

Functional operations. Let $M$ and $N$ be two manifolds. We denote by $q_i$ ($i = 1, 2$) the $i$-th projection defined on $M \times N$ and by $p_i$ ($i = 1, 2$) the $i$-th projection defined on $T^*(M \times N) \simeq T^*M \times T^*N$.

Definition 2.3. Let $f : M \to N$ be a morphism of manifolds and let $\Lambda \subset T^*N$ be a closed $\mathbb{R}^+$-conic subset. We say that $f$ is non-characteristic for $\Lambda$ if $f_!^{-1}(\Lambda) \cap T^*_M N \subset M \times_N T^*_N N$.

A morphism $f : M \to N$ is non-characteristic for a closed $\mathbb{R}^+$-conic subset $\Lambda$ of $T^*N$ if and only if $f_d : M \times_N T^*N \to T^*M$ is proper on $f_!^{-1}(\Lambda)$. In this case $f_d f_!^{-1}(\Lambda)$ is closed and $\mathbb{R}^+$-conic in $T^*M$.

We denote by $\omega_M$ the dualizing complex on $M$. Recall that $\omega_M$ is isomorphic to the orientation sheaf shifted by the dimension. We also use the notation $\omega_M/N$ for the relative dualizing complex $\omega_M \otimes f^{-1} \omega_N^{\otimes -1}$. We have the duality functors
\[
D_M(\bullet) = R\text{Hom}(\bullet, \omega_M), \quad D'_M(\bullet) = R\text{Hom}(\bullet, k_M).
\]

Theorem 2.4 (See [4, §5.4]). Let $f : M \to N$ be a morphism of manifolds, $F \in \mathcal{D}^b(k_M)$ and $G \in \mathcal{D}^b(k_N)$. Let $q_1 : M \times N \to M$ and $q_2 : M \times N \to N$ be the projections.

(i) We have
\[
SS(q_1^{-1}F \otimes q_2^{-1}G) \subset SS(F) \times SS(G),
SS(R\text{Hom}(q_1^{-1}F, q_2^{-1}G)) \subset SS(F)^a \times SS(G).
\]

(ii) We assume that $f$ is proper on $\text{supp}(F)$. Then $SS(Rf_! F) \subset f_! f_d^{-1} SS(F)$, with equality if $f$ is a closed embedding.
Corollary 2.5. Let $F, G \in \mathbb{D}^b(k_M)$.

(i) We assume that $SS(F) \cap SS(G)^a \subset T^*_M M$. Then $SS(F \otimes G) \subset SS(F) + SS(G)$.

(ii) We assume that $SS(F) \cap SS(G) \subset T^*_M M$. Then $SS(R\mathcal{H}om(F, G)) \subset SS(F)^a + SS(G)$.

Corollary 2.6. Let $I$ be a contractible manifold and let $p: M \times I \to M$ be the projection. If $F \in \mathbb{D}^b(k_{M \times I})$ satisfies $SS(F) \subset T^*M \times T^*_I I$, then $F \simeq p^{-1}R_*p_*F$.

The next result follows immediately from Theorem 2.4 (ii) and Example 2.2 (i). It is a particular case of the microlocal Morse lemma (see [8, Cor. 5.4.19]), the classical theory corresponding to the constant sheaves.

Corollary 2.7. Let $F \in \mathbb{D}^b(k_M)$, let $\phi: M \to \mathbb{R}$ be a function of class $C^1$ and assume that $\phi$ is proper on $supp(F)$. Let $a < b$ in $\mathbb{R}$ and assume that $d\phi(x) \notin SS(F)$ for $a \leq \phi(x) < b$. Then the natural morphisms $R\Gamma(\phi^{-1}([a, \infty)); F) \to R\Gamma(\phi^{-1}([-\infty, b[); F)$ and $R\Gamma_{\phi^{-1}([b, +\infty]}(M; F) \to R\Gamma_{\phi^{-1}([a, +\infty]}(M; F)$ are isomorphisms.

3. Simple sheaves on $\mathbb{R}$

Let $\Lambda \subset \tilde{T}^* M$ be a locally closed conic Lagrangian submanifold and let $p \in \Lambda$. Simple sheaves along $\Lambda$ at $p$ are defined in [8, Def. 7.5.4]. Here we only recall a characterization and some properties of simple sheaves. For $p \in T^*M$ we denote by $\mathbb{D}^b(k_M; p)$ the quotient of $\mathbb{D}^b(k_M)$ by the full triangulated subcategory formed by the $F$ such that $p \notin SS(F)$.

When $\Lambda$ is the conormal bundle to a submanifold $N \subset M$, that is, when the projection $\pi_M|_{\Lambda}: \Lambda \to M$ has constant rank, then an object $F \in \mathbb{D}^b(k_M)$ is simple along $\Lambda$ at $p$ if $F \simeq k_N[d]$ in $\mathbb{D}^b(k_M; p)$ for some shift $d \in \mathbb{Z}$. This means that there exist distinguished triangles $F' \to F \to L_1 \overset{+1}{\to}$ and $F' \to k_N \to L_2 \overset{+1}{\to}$ where $p \notin SS(L_i), i = 1, 2$.

If $SS(F)$ is contained in $\Lambda$ on a neighborhood of $\Lambda$, $\Lambda$ is connected and $F$ is simple at some point of $\Lambda$, then $F$ is simple at every point of $\Lambda$. 
Now we will describe the structure of the simple sheaves on \( \mathbb{R} \) with microsupport contained in the positive direction. We let \((s; \sigma)\) be the coordinates on \( T^* \mathbb{R} \) and we let \( T^* > 0 \mathbb{R} \) be the subset of \( T^* \mathbb{R} \) defined by \( \sigma > 0 \). We let \( I = [a, b] \) be an interval \((a \text{ and } b \text{ may be } \pm \infty)\). We recall that \( k \) is a field.

**Lemma 3.1.** Let \( \alpha, \beta \in I \) with \( \alpha < \beta \). Let \( F, G, H, L \in D^b(k_I) \). We assume that \( SS(L) = \emptyset \) and that we have a distinguished triangle

\[
(3.1) \quad F \oplus G \xrightarrow{u} H \oplus k_{[\alpha, \beta]} \xrightarrow{v} L \xrightarrow{+1},
\]

or

\[
(3.2) \quad H \oplus k_{[\alpha, \beta]} \rightarrow F \oplus G \rightarrow L \xrightarrow{+1}.
\]

Then we have a decomposition \( F \simeq H_1 \oplus k_{[\alpha, \beta]} \) or \( G \simeq H_1 \oplus k_{[\alpha, \beta]} \) for some \( H_1 \in D^b(k_I) \).

**Proof.** We give the proof when we have the distinguished triangle \((3.1)\).

The case \((3.2)\) is similar. Let \( i : k_{[\alpha, \beta]} \rightarrow H \oplus k_{[\alpha, \beta]} \) and \( p : H \oplus k_{[\alpha, \beta]} \rightarrow k_{[\alpha, \beta]} \) be the natural morphisms. Let \( p_F, p_G : F \oplus G \rightarrow F \oplus G \) be the projections to the factors \( F \) and \( G \) respectively.

Since \( L \) has constant cohomology sheaves, we have \( \text{Hom}(k_{[\alpha, \beta]}, L) = 0 \). Hence \( v \circ i = 0 \) and \( i \) factorizes through a morphism \( j : k_{[\alpha, \beta]} \rightarrow F \oplus G \). We set \( u_F = p \circ u \circ p_F \circ j \) and \( u_G = p \circ u \circ p_G \circ j \). Then \( u_F + u_G = \text{id}_{k_{[\alpha, \beta]}} \). Since \( \text{Hom}(k_{[\alpha, \beta]}, k_{[\alpha, \beta]}) = k \), we derive that a multiple of \( u_F \) or \( u_G \) must be \( \text{id}_{k_{[\alpha, \beta]}} \). It follows that \( k_{[\alpha, \beta]} \) is a direct summand of \( F \) or \( G \). \( \square \)

**Lemma 3.2.** We assume that \( 0 \in I \) and we set \( \Lambda = T^*_0 \mathbb{R} \cap T^* > 0 \mathbb{R} \). Let \( F \in D^b(k_I) \) be such that \( SS(F) = \Lambda \) and \( F \) is simple along \( \Lambda \). Then there exists \( M \in D^b(k) \) and \( d \in \mathbb{Z} \) such that \( F \simeq k_{[a, 0]}[d] \oplus M_1 \) or \( F \simeq k_{[0, b]}[d] \oplus M_1 \).

**Proof.** (i) Let \( p = (0, 1) \in T^* \mathbb{R} \). By definition we have \( F \simeq k_0[\delta] \) in \( D^b(k_M; p) \) for some \( \delta \in \mathbb{Z} \). The functor \( (R\Gamma_{[0, +\infty]}(-))_0 \) vanishes on the \( F \) with \( p \notin SS(F) \), by definition of the microsupport. Hence it is well-defined in \( D^b(k_M; p) \) and we find \( (R\Gamma_{[0, +\infty]}F)_0 \simeq k[\delta] \). The image of \( 1 \in k \) by this isomorphism gives a morphism \( v : k_{[0, \varepsilon]}[-\delta] \rightarrow F|_J \) defined on some neighborhood \( J = ]-\varepsilon, \varepsilon[ \) of 0. Then, defining \( L \) on \( J \) and \( u : L \rightarrow k_{[0, \varepsilon]}[-\delta] \) by the distinguished triangle \( L \xrightarrow{u} k_{[0, \varepsilon]}[-\delta] \xrightarrow{v} F|_J \xrightarrow{+1} \), we have \( SS(L) \subset T^*_J \).

(ii) If \( u = 0 \), we obtain \( F|_J \simeq L \oplus k_{[0, \varepsilon]}[-\delta] \). If \( u \neq 0 \), then we can decompose \( L \simeq k_J[-\delta] \oplus L' \) (by splitting \( u_x \) for some \( x \in ]0, \varepsilon[ \)) so that \( u \) is induced by the projection \( L \rightarrow k_J[-\delta] \) composed with \( k_J \rightarrow k_{[0, \varepsilon]} \). We deduce \( F|_J \simeq L' \oplus k_{[-\varepsilon, 0]}[1-\delta] \). Since \( F \) is constant outside 0 we deduce the lemma. \( \square \)
We let $A = \{s_1, \ldots, s_k\}$ be a finite subset of $I$ and we set $\Lambda = (\bigsqcup_{i=1}^k T_s I) \cap T_{\sigma > 0} \mathbb{R}$.

**Proposition 3.3.** (i) Let $F \in \mathcal{D}^b(k_I)$ be such that $\text{SS}(F) = \Lambda$ and $F$ is simple along $\Lambda$. Then, up to reordering the indices of the $s_i$’s, there exists an isomorphism

$$F \cong L \oplus \bigoplus_{i=1}^l k_{[s_{2i-1}, s_{2i}]}[d_i] \oplus \bigoplus_{i=2l+1}^m k_{[a,s_i]}[d_i] \oplus \bigoplus_{i=m+1}^k k_{[s_i,b_i]}[d_i];$$

for some integers $d_i$ and some $L \in \mathcal{D}^b(k_{\mathbb{R}})$ with constant cohomology sheaves.

(ii) Using the notations of (3.3) we set $S^\infty(F) = \{s_i; i = 2l+1, \ldots, k\}$. Then $S^\infty(F)$ only depends on $F$. Moreover, for any distinguished triangle $F \to F' \to F'' \to$ where $\text{SS}(F') = \emptyset$, we have $F'$ simple along $\Lambda$ and $S^\infty(F') = S^\infty(F)$.

**Proof.** (i-a) Let us first assume that $F$ is concentrated in degree 0. Let us proceed by induction on $k = |A|$. The case $k = 1$ is given by Lemma 3.2. If $k > 1$, let $s_1 < s_2$ be the first two elements of $A$. By Lemma 3.2 we have either $F|_{[a,s_2]} \cong L \oplus k_{[a,s_1]}$ or $F|_{[a,s_2]} \cong L \oplus k_{[s_1,s_2]}$, for some constant sheaf $L$ on $[a, s_2]$.

(i-b) If $F|_{[a,s_2]} \cong L \oplus k_{[a,s_1]}$, then this decomposition immediately extends to $F \cong k_{[a,a_1]} \oplus G$, where $G$ satisfies the same hypothesis as $F$ with $A$ replaced by $A \setminus \{s_1\}$. Then the induction hypothesis gives the result.

(i-c) Now we assume that $F|_{[a,s_2]} \cong L \oplus k_{[s_1,s_2]}$ and we let $u : k_{[s_1,s_2]} \to F$ be the morphism induced by this decomposition. Let $s \in (A \setminus \{s_1\}) \cup \{b\}$ be maximal such that there exists a monomorphism $v : k_{[s_1,s]} \to F$ extending $u$. We define $G$ by the exact sequence

$$0 \to k_{[s_1,s]} \to F \to G \to 0.$$

Using Lemma 3.2 around $s$, we see that $G$ satisfies the same hypothesis as $F$, with $A$ replaced by $A \setminus \{s_1, s\}$ (if $s$ is one of the $s_i$’s) or $A \setminus \{s_1\}$ (if $s = b$). By the induction hypothesis $G$ is a sum of sheaves of the type $k_{[s',s'']}$, with $s', s'' \in A \cup \{a, b\}$ and $s'' \neq s_1$. We remark that $\text{Ext}^1(k_{[x,y]}, k_{[z,w]}) \cong 0$ if $x \neq w$. Hence the exact sequence (3.3) splits and we obtain the result.

(i-d) For a general $F$ we deduce from Lemma 3.2 that each $H^i F$ satisfies the same hypothesis as $F$, with $A$ replaced by some subset of $A$. Hence we know the structure of $H^i F$ by (i-a)-(i-c). We deduce easily that
Ext^p(H^iF, H^jF) \simeq 0 \text{ for all } i, j \text{ and } p \geq 2. \text{ This implies that } F \simeq \bigoplus_i H^iF[-i] \text{ and we obtain the result.}

(ii) Lemma 3.4 implies that \( F \) and \( F' \) have the same direct summands \( k_{s_2, s_1} \), \( i = 1, \ldots, l \). This is equivalent to (ii). \( \square \)

4. THE INVOLUTIVITY THEOREM

The main tool in our proof of the Gromov-Eliashberg theorem is the involutivity theorem of [8]. This is a deep result originally inspired by the similar theorem for the characteristic variety of a system of linear PDE’s (see loc. cit. for historical comments on this point).

We first recall a general definition of involutivity given in [8]. Let \( X \) be a manifold and let \( x \in X \). We denote by \( C_x(S) \subset T_xX \) the tangent cone of \( S \) at \( x \). In case \( X \) is a vector space this is the set of \( v \in X \simeq T_xX \) which can be written \( v = \lim_{n \to \infty} c_n(x_n - x) \), for some sequences \( \{c_n\}_{n \in \mathbb{N}} \) and \( \{x_n\}_{n \in \mathbb{N}} \) with \( c_n \in \mathbb{R}^+, x_n \in S \) satisfying \( x = \lim_{n \to \infty} x_n \). For two subsets \( S_1, S_2 \subset X \), we also have \( C_x(S_1, S_2) \subset T_xX \) (see [8]). In case \( X \) is a vector space this is the set of \( v \) which can be written \( v = \lim_{n \to \infty} c_n(x_n^1 - x_n^2) \), for some sequences \( \{c_n\}_{n \in \mathbb{N}} \) and \( \{x_n^i\}_{n \in \mathbb{N}} \) with \( c_n \in \mathbb{R}^+, x_n^i \in S_i, i = 1, 2 \), satisfying \( x = \lim_{n \to \infty} x_n^i \).

If \((E, \omega)\) is a symplectic vector space and \( A \subset E \) we set \( A^\perp = \{v \in E; \omega(v, w) = 0, \text{ for all } w \in A\} \).

**Definition 4.1** (Def. 6.5.1 of [8]). Let \((X, \omega)\) be a symplectic manifold and let \( S \) be a locally closed subset of \( X \). For a given \( p \in S \) we say that \( S \) is coisotropic (or involutive) at \( p \) if \((C_p(S, S))^{\perp}_{\omega_p} \subset C_p(S)\).

**Theorem 4.2** (Thm. 6.5.4 of [8]). Let \( M \) be a manifold and \( F \in D^b(k_M) \). Then \( SS(F) \) is coisotropic.

We will use the following results when we apply Theorem 4.2 in Section 11.

**Lemma 4.3.** Let \( X \) be a symplectic manifold and let \( S \subset S' \) be locally closed subsets of \( X \). Let \( p \in S \). We assume that \( S \) is coisotropic at \( p \). Then \( S' \) is also coisotropic at \( p \).

**Proof.** We have the inclusions
\[
(C_p(S', S'))^{\perp}_{\omega_p} \subset (C_p(S, S))^{\perp}_{\omega_p} \subset C_p(S) \subset C_p(S').
\]

**Proposition 4.4.** Let \( M \) be a manifold and \( S \subset T^*M \) a locally closed subset. We recall the map \( \rho_M^*: T^*M \times \dot{T}^*R \to T^*M \) defined in (2.4). Let \( p \in S \) and \( q \in \rho_M^{-1}(p) \). Then \( S \) is coisotropic at \( p \) if and only if \( \rho_M^{-1}(S) \) is coisotropic at \( q \).
Lemma 5.1. Let $A \subset V$ be a closed convex subset. We assume that $\text{Conv}(A) = \{ \langle x; \xi \rangle \mid \sigma \in A \}$ is proper for all $x \in X$, that is, $\text{Conv}(A) \cap T_x^*X = \{ \langle x; \xi \rangle \mid \sigma \in A \}$ contains no line. Then we have $d \rho_q(X, S; \Xi, \Sigma) = (X; \Xi)$ and we deduce $C_q(S') = C_p(S) \times T_{(s_0,1)}T^*\mathbb{R}$ and $C_q(S', S'') = C_p(S, S) \times T_{(s_0,1)}T^*\mathbb{R}$. Now the result follows easily.

5. Bounds for microsupports

For a real vector space $V$ and $A \subset V$, we denote by $\text{Conv}(A)$ the convex hull of $A$. Let $X$ be a manifold. For $\Lambda \subset T^*X$ we define $\text{Conv}(\Lambda) \subset T^*X$ by $\text{Conv}(\Lambda) \cap T_x^*X = \text{Conv}(\Lambda \cap T_x^*X)$, for all $x \in X$. In general it can happen that $\Lambda$ is closed but not $\text{Conv}(\Lambda)$. We leave the following result to the reader.

Lemma 5.1. Let $\Lambda \subset T^*X$ be a closed conic subset. We assume that $\text{Conv}(\Lambda \cap T_x^*X)$ is proper for all $x \in X$, that is, $\text{Conv}(\Lambda \cap T_x^*X)$ contains no line. Then $\text{Conv}(\Lambda)$ is closed.

We will use the following cut-off result (other similar and more precise results are recalled in Section 8). Let $V$ be a vector space and let $\gamma \subset V$ be a closed convex cone. Let $V\gamma$ be the space $V$ endowed with the topology whose open sets are the usual open subsets $\Omega$ of $V$ such that $\Omega + \gamma = \Omega$. Let $\text{Mod}(k_{V\gamma})$ be the category of sheaves of $k$-vector spaces on $V\gamma$ and $\text{D}^b(k_{V\gamma})$ its bounded derived category. The identity map induces a continuous map $\phi_\gamma: V \to V\gamma$.

Proposition 5.2 (see §3.5 and Prop. 5.2.3 of [8]). The inverse image $\phi_\gamma^{-1}: \text{D}^b(k_{V\gamma}) \to \text{D}^b(k_V)$ induces an equivalence between $\text{D}^b(k_{V\gamma})$ and the full subcategory of $\text{D}^b(k_V)$ consisting of the $F$ such that $\text{SS}(F) \subset V \times \gamma^{\text{op}}$.

The following consequence was pointed out to the author by Pierre Schapira.

Proposition 5.3. Let $F \in \text{D}^b(k_X)$ be such that $\text{Conv}(\text{SS}(F) \cap T_x^*X)$ is proper for all $x \in X$. Then, for all $n \in \mathbb{Z}$, the microsupports of $\tau_{\leq n}F$, $\tau_{\geq n}F$, and $H^n(F)$ are contained in $\text{Conv}(\text{SS}(F))$.

Proof. (i) We set $\Lambda = \text{SS}(F)$. Let $G \in \text{D}^b(k_X)$ be one of $\tau_{\leq n}F$, $\tau_{\geq n}F$, or $H^n(F)$. We prove that $\text{SS}(G) \cap T_x^*X$ is contained in $\text{Conv}(\Lambda \cap T_x^*X)$ for any $x \in X$. Since this is a local problem around $x_0$ we may
assume that $X$ is an open subset of a vector space, say $V$, and that $T^*X \simeq X \times V^*$. We also assume that $x_0 = 0$.

(ii) We let $\gamma \subset V$ be a closed convex proper cone of $V$ such that $\gamma^{oa}$ is a neighborhood of $\text{Conv}(\Lambda \cap T^*_0 X) \setminus \{0\}$. We can find a neighborhood of 0, say $U$, such that $\Lambda \cap T^* U \subset U \times \gamma^{oa}$. We choose coordinates $(x_1, \ldots, x_n)$ on $V$ such that $(0, \ldots, 0, -1) \in \gamma$. We choose an open convex cone $\gamma'$ such that $\gamma'$ is proper and $\gamma \setminus \{0\} \subset \gamma'$. For $\varepsilon > 0$ we define $\gamma'_\varepsilon = ((0, \ldots, 0, \varepsilon) + \gamma') \cap \{x_n \geq -\varepsilon\}$. Then $\gamma'_\varepsilon$ is a neighborhood of 0 in $V$ and, for $\varepsilon$ small enough, we have $\gamma'_\varepsilon \subset U$. Moreover $\text{SS}(k_{\gamma'_\varepsilon}) \subset \gamma'_\varepsilon \times \gamma^{oa}$.

(iii) We set $F' = F \otimes k_{\gamma'_\varepsilon}$. Then $F'$ is isomorphic to $F$ on $\text{Int} \gamma'_\varepsilon$ and $\text{SS}(F') \subset V \times \gamma^{oa}$ by Corollary 2.5. Hence, by Proposition 5.2 there exists $F_1 \in \mathcal{D}^{b}(k_{\gamma'_\varepsilon})$ such that $F' \simeq \phi^{-1}(F_1)$. Then $G|_{\text{Int} \gamma'_\varepsilon} \simeq \phi^{-1}(G_1)|_{\text{Int} \gamma'_\varepsilon}$, where $G_1$ is one of $\tau_{\leq n} F_1$, $\tau_{\geq n} F_1$ or $H^{\nu}(F_1)$. By Proposition 5.2 again we deduce $\text{SS}(G|_{\text{Int} \gamma'_\varepsilon}) \subset V \times \gamma^{oa}$. Since $\gamma^{oa}$ is an arbitrary small neighborhood of $\text{Conv}(\Lambda \cap T^*_0 X) \setminus \{0\}$, we obtain that $\text{SS}(G) \cap T^*_0 X$ is contained in $\text{Conv}(\Lambda \cap T^*_0 X)$, as required.

**Proposition 5.4.** Let $X$ be a manifold and $a \leq b \in \mathbb{Z}$. Let $\{K_n\}_{n \in \mathbb{N}}$ be a family of objects of $\mathcal{D}^{b}(k_X)$ such that $H^i K_n = 0$ for $i \not\in [a, b]$ and for all $n \in \mathbb{N}$. We define $K \in \mathcal{D}^{b}(k_X)$ by the distinguished triangle

$$
\bigoplus_{n \in \mathbb{N}} K_n \rightarrow \prod_{n \in \mathbb{N}} K_n \rightarrow K \xrightarrow{1}.
$$

Then we have $\text{SS}(K) \subset \bigcap_{k \in \mathbb{N}} \bigcup_{n \geq k} \text{SS}(K_n)$.

**Proof.** For any $k \in \mathbb{N}$ we also have a distinguished triangle $\bigoplus_{n \geq k} K_n \rightarrow \prod_{n \geq k} K_n \rightarrow K \xrightarrow{1}$. We can check, similarly as in [8, Ex. V.7], that $\text{SS}(\bigoplus_{n \geq k} K_n) \subset \bigcup_{n \geq k} \text{SS}(K_n)$ and $\text{SS}(\prod_{n \geq k} K_n) \subset \bigcup_{n \geq k} \text{SS}(K_n)$. We conclude by the triangular inequality for the microsupport.

6. Approximation of symplectic maps

Let $(E, \omega)$ be a symplectic vector space which we identify with $\mathbb{R}^{2n}$. We endow $E$ with the Euclidean norm of $\mathbb{R}^{2n}$.

**Lemma 6.1.** Let $R > r$ and $\varepsilon$ be positive numbers. Let $\varphi: B^E_R \rightarrow E$ be a symplectic map of class $C^1$. Then there exists $R' > r$ and a symplectic map $\psi: B^E_{R'} \rightarrow E$ which is of class $C^\infty$ such that $\|\varphi - \psi\|_{B^E_{R'}} \leq \varepsilon$.

**Proof.** We set $r_1 = (R + r)/2$ and we choose a (non symplectic) map $\varphi': B^E_R \rightarrow E$ of class $C^\infty$ such that $\|\varphi - \varphi'\|_{B^E_R} \leq \varepsilon$ (we use the norm (2.3)). We set $\omega' = \varphi'^* (\omega)$. We have $\omega - \omega' = (\varphi - \varphi')^* \omega$. Hence, if we consider $\omega$ and $\omega'$ as maps from $E$ to $\wedge^2 E$ and we endow $\wedge^2 E$ with the standard norm, we have $\|\omega - \omega'\| \leq \varepsilon$. We conclude by Proposition 5.4.
with the Euclidean structure induced by \( E \), we have \( \|\omega - \omega'\|_{B^1_{E_1}} \leq C\varepsilon \),
where the constant \( C \) only depends on \( n \).

We set \( r_2 = (r_1 + r)/2 \). By Moser’s argument for the Darboux theorem we can find a flow \( \Phi: B^E_{r_2} \times [0,1] \to E \) such that \( \Phi_t(B^E_{r_2}) \subset B^E_{r_2} \)
for all \( t \in [0,1] \) and \( \omega|_{B^E_{r_2}} = \Phi_t^*(\omega)|_{B^E_{r_2}} \). The flow \( \Phi \) is the flow of a vector field \( X_t \) which satisfies \( \iota_{X_t}(\omega_t) = -\sigma \) over \( B^E_{r_1} \), where \( \omega_t = t\omega' - (1-t)\omega \) and \( d\sigma = \omega' - \omega \). We can assume that \( \sigma \) satisfies the bound \( \|\sigma\|_{B^E_{r_1}} \leq C'\|\omega' - \omega\|_{B^E_{r_1}} \) for some \( C' > 0 \) only depending on \( r_1 \).

Hence \( X_t \) satisfies \( \|X_t\|_{B^E_{r_1}} \leq C''\varepsilon \), for some constant \( C'' > 0 \) and all \( t \in [0,1] \).

We may assume from the beginning that \( C''\varepsilon < r_1 - r_2 \). Hence \( \Phi_1(B^E_{r_2}) \subset B^E_{r_1} \) and we have \( \|\Phi_1 - \text{id}\|_{B^E_{r_2}} \leq C''\varepsilon \). The map \( \psi = \varphi' \circ \Phi_1: B^E_{r_2} \to E \) is a symplectic map such that \( \|\varphi - \psi\|_{B^E_{r_2}} \leq (1+C'')\varepsilon \), which gives the lemma (up to replacing \( \varepsilon \) by \( \varepsilon/(1+C'') \)). \( \square \)

**Proposition 6.2.** Let \( R > r \) and \( \varepsilon > 0 \) positive numbers. Let \( \varphi: B^E_{R} \to E \) be a symplectic map of class \( C^1 \). Then there exists a Hamiltonian isotopy \( \Phi: E \times \mathbb{R} \to E \) of class \( C^\infty \) and a compact subset \( C \subset E \) such that \( \|\varphi - \Phi_1\|_{B^E_{R}} \leq \varepsilon \) and \( \Phi_t|_{E \setminus C} = \text{id}_{E \setminus C} \) for all \( t \in \mathbb{R} \).

**Proof.** (i) By Lemma 6.1 we may assume that \( \varphi \) is of class \( C^\infty \). Composing with a translation in \( E \) and a symplectic linear map, we may also assume that \( \varphi(0) = 0 \) and \( d\varphi_0 = \text{id}_E \).

We first show that there exists a symplectic map \( \varphi': B^E_{R} \to E \) such that \( \|\varphi - \varphi'\|_{B^E_{R}} \leq \varepsilon \) and \( \varphi' \approx \text{id}_E \) near 0.

(ii) We choose an isomorphism \( E \cong V \times V^* \) and then \( E^2 \cong T^*W \cong W \times W^* \), where \( W = V \times V^* \) is the product of first and last factors in \( E^2 = V \times V^* \times V \times V^* \) (of course \( W \cong E \)). We have a natural isomorphism \( s: W \cong W^* \) given by switching the factors \( V \) and \( V^* \).

For a function \( f: W \to \mathbb{R} \) we write \( \Lambda_f = \{(x,df_x); x \in W \} \). We can find two balls centered around 0, \( B^W_{r_1} \) in \( W \) and \( B^{W^*}_{r_2} \) in \( W^* \) and a function \( f \) defined on \( B^W_{r_1} \) such that \( \Lambda_{\varphi} \cap (B^W_{r_1} \times B^{W^*}_{r_2}) = \Lambda_f \cap (B^W_{r_1} \times B^{W^*}_{r_2}) \). We have \( df_0 = 0 \) and we assume \( f(0) = 0 \). We choose \( \eta > 0 \) and \( f' \) defined on \( B^W_{\eta} \) such that

- \( \|f - f'|_{B^W_{\eta}} \leq \varepsilon \),
- \( f'(x,\xi) = (x,\xi) \), hence \( df' = s \), on \( B^E_{\eta} \),
- \( f' \approx f \) outside of \( B^W_{2\eta} \).

Then we can define a symplectic map \( \varphi': B^E_{R} \to E \) such that \( \Lambda_{\varphi'} = \Lambda_{\varphi} \) outside of \( B^{W}_{4\eta} \times B^{W^*}_{2\eta} \) and \( \Lambda_{\varphi'} \cap (B^{W}_{4\eta} \times B^{W^*}_{2\eta}) = \Lambda_f \cap (B^{W}_{4\eta} \times B^{W^*}_{2\eta}) \). We have \( \|\varphi - \varphi'\|_{B^E_{R}} \leq \varepsilon \) and \( \varphi'|_{B^E_{R}} = \text{id}_E \), as claimed in (i).
(iii) We will prove that $\varphi'|_{B^E}$ is the time 1 of some Hamiltonian isotopy. We define $U \subset E \times [0, +\infty[$ by $U = \{(x,t); \|x\| < R/t\}$ and $\psi: U \to E$ by $\psi(x,t) = t^{-1}\varphi'(tx)$. We let $U' = \{((\psi(x,t),t); (x,t) \in U\}$ be the image of $U$ by $\psi \times id_{\mathbb{R}}$. Then $U'$ is contractible and we can find $h: U' \to \mathbb{R}$ such that $\psi$ is the Hamiltonian flow of $h$.

We define $U_0 \subset U$ by $U_0 = \{(x,t); t > 0, \|x\| < \eta/t\}$. Since $\varphi'|_{B^E} = id_E$, we have $\psi(x,t) = x$ for all $(x,t) \in U_0$. Hence $U_0 \subset U'$. Moreover $h$ is constant on $U_0$. We can assume $h|_{U_0} = 0$ and extend $h$ by 0 to a $C^\infty$ function defined on $U'' = [-\infty, 0] \cup U'$.

We set $Z = \{(\psi(x,t),t); t \in [0,1]\}$ and $\|x\| \leq r\}$. We remark that $Z \cap U_0 = (\overline{B}^E \times [0,1]) \cap U_0$. Hence $Z$ is relatively compact in $U''$. We choose a compact subset $C \subset E$ such that $Z \subset C \times [0,1]$ and a $C^\infty$ function $g: E \times \mathbb{R} \to \mathbb{R}$ such that $g = h$ on $Z$ and $g = 0$ outside of $C \times [0,2]$. Then the Hamiltonian isotopy $\Phi$ defined by $g$ has compact support contained in $C$ and satisfies $\Phi_1 = \varphi'$ on $B^E_r$. This proves the proposition.

When we apply Proposition 6.2 above we have no control on the extension of the map outside the ball $B^E_r$. In the following lemmas we check that, up to a linear transformation, the “interesting part” of the graph can be moved near the zero section of $T^*E^2$ and the “extended part” far away from the zero section (see (6.2) and (6.5)).

**Lemma 6.3.** Let $R_0 > 0$ and let $\varphi: B^E_{R_0} \to E$ be a map of class $C^1$ such that $\varphi(0) = 0$ and $d\varphi_0$ is an isomorphism. We set $V = \mathbb{R}^n$. Then we can find symplectic linear isomorphisms $u: T^*V \cong E$, $v: E \to T^*V$ and positive numbers $r_0, A$ such that, setting $\psi = v \circ \varphi \circ u: u^{-1}(B^E_{R_0}) \to T^*V$, we have

(6.1) $\psi$ is defined on $B^V_{r_0} \times B^V_{A r_0}$;

(6.2) $\Gamma_\psi \cap (B^V_{r} \times B^V_{A r}) \subset B^V_{r} \times B^V_{A r}$ for any $r_0 \geq r > 0$;

(6.3) the projection $\Gamma_\psi \cap (B^V_{r_0} \times B^V_{A r_0}) \to B^V_{r_0}$ is a diffeomorphism.

**Proof.** We first choose a symplectic isomorphism $u: T^*V \cong E$ arbitrarily and we set $L = d\varphi_0(V \times \{0\})$. Since $L$ is of dimension $n$, we can find a Lagrangian subspace $L' \subset E$ such that $L \cap L' = \{0\}$. We choose a symplectic diffeomorphism $v: E \to T^*V$ such that $v(L') = V \times \{0\}$.

We set $\psi = v \circ \varphi \circ u$, $W = T_{(0,0)}\Gamma_\psi \subset T^*V$ and

$W_1 = \{((x_1; 0), (\psi_0(x_1); 0)); x_1 \in V\}$, $W_2 = \{((d\psi_0)^{-1}(x_2; 0), (x_2; 0)); x_2 \in V\}$. We also set $p_i(x_1, x_2; \xi_1, \xi_2) = x_i$, $i = 1, 2$. Then $W = W_1 \oplus W_2$, $p_1(W_1) = V$ and $p_2(W_2) = V$. Hence $\pi_{V^2}: T^*V^2 \to V^2$ maps $W$ onto $V^2$. 

Lemma 6.4. Let $\Gamma$ satisfy (6.2). We choose $0 < r < r_0 / 4$ and $0 < \varepsilon < A r / (A + 1)$. Then, for any map $\psi: U \rightarrow T^* V$ satisfying
\[
\text{(6.4)} \quad d(\psi(p), \psi_1(p)) < \varepsilon \quad \text{for all } p \in U,
\]
we have
\[
\text{(6.5)} \quad \Gamma_\psi \cap (B_r^V \times B_{A r_0 / 2}^{2 V^*}) \subset \Gamma_{\psi_1|U} \cap \Gamma_\psi(\varepsilon) \cap (B_r^V \times B_{2 A r}^{2 V^*}),
\]
where $\Gamma_\psi(\varepsilon) = \{p \in T^* E^2; d(p, \Gamma_\psi) < \varepsilon\}$.

Proof. Let $(x, \xi) \in T^* V$ and $(y_1; \eta_1) = \psi_1(x; \xi)$. We assume that $q = (x, y_1; \xi, \eta_1)$ is in the LHS of (6.5).

(i) We have in particular $(x; \xi) \in B_r^V \times B_{A r_0 / 2}^{2 V^*}$. Hence $(x; \xi) \in U$ and $q \in \Gamma_{\psi_1|U}$. Moreover (6.4) gives $q \in \Gamma_\psi(\varepsilon)$.

(ii) It remains to prove that $(\xi, \eta_1) \in B_{2 A r}^{2 V^*}$. Let us write $(y; \eta) = \psi(x; \xi)$. By (6.4) we have $d(y_1, y) < \varepsilon$ and $d(\eta_1, \eta) < \varepsilon$. Hence $(x, y) \in B_{r + \varepsilon}^V$ and $(\xi, \eta) \in B_{A r_0 / 2 + \varepsilon}^{2 V^*}$. The hypothesis on $r$ and $\varepsilon$ implies
\[
r + \varepsilon < r_0, \quad A r_0 / 2 + \varepsilon < A r_0, \quad A (r + \varepsilon) + \varepsilon < 2 A r.
\]
By (6.2) we deduce $(\xi, \eta) \in B_{A (r + \varepsilon)}^{2 V^*}$. Since $d(\eta_1, \eta) < \varepsilon$, we obtain $(\xi, \eta_1) \in B_{A (r + \varepsilon) + \varepsilon}^{2 V^*}$ and this proves the result. \hfill \Box

7. Degree of a continuous map

We recall the definition of the degree of a continuous map. Let $M, N$ be two oriented manifolds of the same dimension, say $d$. We assume that $N$ is connected. We have a morphism $H^d_c(M; \mathbb{Z}_M) \rightarrow \mathbb{Z}$ and an isomorphism $H^d_c(N; \mathbb{Z}_N) \cong \mathbb{Z}$. Let $f: M \rightarrow N$ be a proper continuous map. Applying $H^d_c(N; \cdot)$ to the morphism $\mathbb{Z}_N \rightarrow R f_* f^{-1} \mathbb{Z}_N \cong R f_! \mathbb{Z}_M$ we find
\[
\mathbb{Z} \cong H^d_c(N; \mathbb{Z}_N) \rightarrow H^d_c(M; \mathbb{Z}_M) \rightarrow \mathbb{Z}.
\]
The degree of $f$, denoted $\text{deg } f$, is the image of 1 by this morphism.

Lemma 7.1. Let $M, N$ be two oriented manifolds of dimension $d$. We assume that $N$ is connected.

(i) Let $f: M \rightarrow N$ be a proper continuous map and let $V \subset N$ be a connected open subset. Then $\text{deg } f = \text{deg } (f|_{f^{-1}(V)}): f^{-1}(V) \rightarrow V$. 

(ii) Let $I \subset \mathbb{R}$ be an interval. Let $U \subset M \times I$, $V \subset N \times I$ be open subsets and let $f: U \to V$ be a continuous map which commutes with the projections $U \to I$ and $V \to I$. We set $U_t = U \cap (M \times \{t\})$, $V_t = V \cap (N \times \{t\})$ and $f_t = f|_{U_t}: U_t \to V_t$, for all $t \in I$. We assume that $f$ is proper and that $V$ and all $V_t$, $t \in I$, are non empty and connected. Then $\deg f = \deg f_t$, for all $t \in I$.

Proof. (i) and (ii) follow respectively from the commutative diagrams

$$
\begin{array}{cccc}
Z \xrightarrow{\sim} H^d_c(N; \mathbb{Z}_V) & \longrightarrow & H^d_c(M; \mathbb{Z}_{f^{-1}(V)}) & \longrightarrow & Z \\
\downarrow & & \downarrow \quad & & \downarrow \\
Z \xrightarrow{\sim} H^d_c(N; \mathbb{Z}_N) & \longrightarrow & H^d_c(M; \mathbb{Z}_M) & \longrightarrow & Z,
\end{array}
$$

$$
\begin{array}{cccc}
Z \xrightarrow{\sim} H^d_c(V; \mathbb{Z}_V) & \longrightarrow & H^d_c(U; \mathbb{Z}_U) & \longrightarrow & Z \\
\downarrow & & \downarrow \quad & & \downarrow \\
Z \xrightarrow{\sim} H^{d+1}_c(V; \mathbb{Z}_V) & \longrightarrow & H^{d+1}_c(U; \mathbb{Z}_U) & \longrightarrow & Z.
\end{array}
$$

Proposition 7.2. Let $B_R$ be the open ball of radius $R$ in $\mathbb{R}^d$. Let $U, V \subset \mathbb{R}^d$ be open subsets and let $f: U \to B_R$, $g: V \to B_R$ be proper continuous maps. We assume that there exists $r < R$ such that $f^{-1}(\overline{B_r}) \subset U \cap V$, and that $d(f(x), g(x)) < r/2$, for all $x \in U \cap V$. Then $\deg f = \deg g$.

Proof. (i) We define $h: (U \cap V) \times [0, 1] \to \mathbb{R}^{d+1}$ by $h(x, t) = (tf(x) + (1-t)g(x), t)$. Let us prove that $h^{-1}(\overline{B_r} \times [0, 1])$ is compact. Since $f^{-1}(\overline{B_r})$ is compact and contained in $U \cap V$, it enough to prove that $h^{-1}(\overline{B_r/2} \times [0, 1]) \subset f^{-1}(\overline{B_r}) \times [0, 1]$. Let $(x, t) \in (U \cap V) \times [0, 1]$ be such that $\|h(x, t)\| \leq r/2$. Since $h(x, t)$ belongs to the line segment $[f(x), g(x)]$ which is of length $< r/2$, we deduce $f(x) \in B_r$, as required.

(ii) We define $W = h^{-1}(B_{r/2} \times [0, 1])$, $W_t = W \cap (\mathbb{R}^d \times \{t\})$ for $t \in [0, 1]$ and $h'_t = h|_{W_t}: W_t \to B_{r/2}$. By (i) $h|_W: W \to B_{r/2} \times [0, 1]$ is proper. Hence Lemma [7.1] (ii) implies that $\deg h'_0 = \deg h'_1$. We conclude with Lemma [7.1] (i) which implies $\deg h'_0 = \deg g$ and $\deg h'_1 = \deg f$. □

8. Cut-off

In this section we recall several results of [8] which are called "(dual) (refined) cut-off lemmas". In loc. cit. the statements of the refined cut-off lemmas are given around a point. With stronger hypothesis on the microsupports they hold on a fixed neighborhood of a given point, which is needed in the next section. We include the part of the proof
which is concerned with this claim but it is actually the same as in loc. cit.

Let $V$ be a vector space and let $\gamma \subset V$ be a closed convex cone (with vertex at 0). We denote by $\gamma^a = -\gamma$ its opposite cone and by $\gamma^o \subset V^*$ its polar cone (see (2.5)). We also define $\tilde{\gamma} = \{(x, y) \in V^2; x - y \in \gamma\}$. Let $q_i: V^2 \to V$, $i = 1, 2$, be the projection to the $i$th factor. The following functors are introduced in [8]:

(8.1) \[ P_\gamma: D^b(k_V) \to D^b(k_V), \quad F \mapsto Rq_{2*}(k_\gamma \otimes q_1^{-1}F), \]

(8.2) \[ Q_\gamma: D^b(k_V) \to D^b(k_V), \quad F \mapsto Rq_{2*}(R\mathcal{H}om(k_{\tilde{\gamma}}, q_1^*F)). \]

For $\gamma = \{0\}$ we have $P_{\{0\}}(F) \simeq Q_{\{0\}}(F) \simeq F$. Hence the inclusion $\{0\} \subset \gamma$ induces morphisms of functors $u_\gamma: P_\gamma \to \text{id}$ and $v_\gamma: \text{id} \to Q_\gamma$. If $F$ has compact support, Corollary 2.3 and Theorem 2.4 (ii) give, for any $x \in V$ and using the identification $T^*V = V \times V^* = V \times T^*_x V$,

(8.3) \[ \text{SS}(P_\gamma(F)) \cap T^*x V \subset q_2(\text{SS}(F) \cap \text{SS}(k_{x+\gamma})^a), \]

(8.4) \[ \text{SS}(Q_\gamma(F)) \cap T^*x V \subset q_2(\text{SS}(F) \cap \text{SS}(k_{x+\gamma^a})), \]

where $q_2: T^*V \to T^*_x V$ is the projection.

For a given subset $\Omega$ of $T^*V$, a morphism $a: F \to G$ in $D^b(k_V)$ is said to be an isomorphism on $\Omega$ if $\text{SS}(C(a)) \cap \Omega = \emptyset$, where $C(a)$ is given by the distinguished triangle $F \xrightarrow{a} G \to C(a) \xrightarrow{+1}$.

**Proposition 8.1** (see [8] Prop. 5.2.3 and Lem. 6.1.5). We assume that $\gamma$ is proper and $\text{Int}(\gamma) \neq \emptyset$. For any $F \in D^b(k_V)$ we have $\text{SS}(P_\gamma(F)) \cup \text{SS}(Q_\gamma(F)) \subset V \times \gamma^a$. Moreover the morphisms $u_\gamma(F): P_\gamma(F) \to F$ and $v_\gamma(F): F \to Q_\gamma(F)$ are isomorphisms on $V \times \text{Int}(\gamma^a)$.

In order to obtain a local statement from Proposition 8.1 we will use Lemmas 8.2 and 8.4 below. Let $\gamma \subset V$ be a closed convex proper cone. For $x \in V$ we define $S_x^\gamma \subset T^*V$ by

(8.5) \[ S_x^\gamma = (\text{SS}(k_{x+\gamma})^a \cup \text{SS}(k_{x+\gamma^a})) \setminus \{x\} \times \text{Int}(\gamma^a), \]

where the second equality follows from Example 2.2. Hence we have

(8.6) \[ \text{SS}(k_{x+\gamma})^a \cup \text{SS}(k_{x+\gamma^a}) \subset S_x^\gamma \cup \{x\} \times \text{Int}(\gamma^a). \]

**Lemma 8.2.** Let $F \in D^b(k_V)$ be such that $\text{supp}(F)$ is compact and let $W \subset V$ be an open subset such that, for any $x \in W$

(8.7) \[ S_x^\gamma \cap \text{SS}(F) = \emptyset. \]
Then \( v_\gamma(F) \circ u_\gamma(F)|_W : P_\gamma(F)|_W \to Q_\gamma(F)|_W \) is an isomorphism on \( T^*W \) and we have a distinguished triangle in \( D^b(k_W) \)

\[
(8.8) \quad P_\gamma(F)|_W \oplus G \to F|_W \to L^{+1},
\]

where \( L, G \in D^b(k_W) \) satisfy \( SS(L) = \emptyset \) and \( SS(G) \cap (W \times \gamma^{oa}) = \emptyset \).

**Proof.** (i) We define \( L \in D^b(k_W) \) by the distinguished triangle

\[
(8.9) \quad P_\gamma(F)|_W \xrightarrow{v_\gamma(F) \circ u_\gamma(F)} Q_\gamma(F)|_W \xrightarrow{a} L^{+1}.
\]

The formulas (8.3), (8.4), (8.6) and (8.7) give \( SS(L) \subset W \times \text{Int}(\gamma \circ a) \).

On the other hand Proposition 8.1 implies that \( v_\gamma(F) \circ u_\gamma(F) \) is an isomorphism on \( W \times \text{Int}(\gamma \circ a) \). Hence \( SS(L) \cap (W \times \text{Int}(\gamma^{oa})) = \emptyset \) and we find \( SS(L) = \emptyset \). This proves the first assertion.

(ii) We define \( G \) and \( F' \) by the distinguished triangles

\[
(8.10) \quad G \to F|_W \xrightarrow{v_\gamma(F)} Q_\gamma(F)|_W \xrightarrow{a} L^{+1},
\]

\[
(8.11) \quad F' \to F|_W \xrightarrow{a \circ v_\gamma(F)} L^{+1}.
\]

Since \( v_\gamma(F) \) is an isomorphism on \( V \times \text{Int}(\gamma^{oa}) \) we have \( SS(G) \cap (W \times \gamma^{oa}) = \emptyset \). By the octahedral axiom we deduce from (8.9)-(8.11) the distinguished triangle

\[
G \to F' \to P_\gamma(F)|_W \xrightarrow{c} G[1],
\]

where \( c = b \circ (v_\gamma(F) \circ u_\gamma(F)) \). By the triangle (8.10) we have \( b \circ v_\gamma(F) = 0 \), hence \( c = 0 \). It follow that \( F' \simeq G \oplus P_\gamma(F)|_W \) and (8.11) gives (8.8). \( \square \)

Now we write \( V = V' \times \mathbb{R} \), where \( V' = \mathbb{R}^{n-1} \). We take coordinates \( x = (x', x_n) \) on \( V \) and we endow \( V' \) and \( V \) with the natural Euclidean structure. For \( c > 0 \) we let \( \gamma_c \subset V \) be the cone

\[
(8.12) \quad \gamma_c = \{ (x', x_n) \in V; \ x_n \leq -c\|x'\| \}.
\]

For a conic subset \( C \subset V^* \) and \( \varepsilon > 0 \) we define \( C(\varepsilon) \subset V^* \) by

\[
(8.13) \quad C(\varepsilon) = \{ \xi \in V^*; \ d(\xi, C) < \varepsilon\|\xi\| \}.
\]

For real numbers \( c, c', \delta, \varepsilon \) and \( r \in [0, +\infty] \) such that \( c' > c > 0 \), \( \delta \geq 0 \) and \( \varepsilon > 0 \) we define a locally closed subset \( Z^{r,\delta,\varepsilon}_c \) of \( V \) and a subset \( W^{r,\delta,\varepsilon}_c,c' \)
of $Z_r^c$ by

\[(8.14) \quad Z_r^c = (B_r^{V^*} \times \mathbb{R}) \setminus \left((0, -\delta) + \gamma_c \right) \cup \left((0, \delta) + \text{Int} \gamma^0_c\right),\]

\[(8.15) \quad W_{r,c,c'}^{\epsilon,\delta} = \{ \{x \in Z_r^c, \{0, \delta\}, (0, -\delta)\} \cap (x + Z_r^{\infty,0}) = \emptyset,\]

\[I := \partial Z_r^{\infty,0} \cap \partial(x + Z_r^{\infty,0}) \subset B_r^{V^*} \times \mathbb{R} \text{ and }\]

\[\forall y \in I, \left((T_y^*V \cap (S_y^{\gamma_c'})) \subset (T_y^*V \cap \text{SS}(k_{Z_r^c,\delta}))(\varepsilon)\right).\]

\[\text{Lemma 8.3. (i) For any given } c, c', r, \delta, \epsilon > 0 \text{ the subset } W_{r,c,c'}^{\epsilon,\delta} \text{ is open in } V.\]

\[\text{(ii) For any given } c, r, \epsilon > 0 \text{, setting } c' = c + \epsilon/2 \text{ and choosing } \delta < \epsilon r/2,\]

\[\text{we have } W_{r,c,c'}^{\epsilon,\delta} \neq \emptyset.\]

**Proof.** The first assertion is clear on the definition \ref{8.15}. For the second claim we check easily that \((0, 0) \in W_{r,c,c'}^{\epsilon,\delta}.\)

**Lemma 8.4.** Let $c_2 > c_1 > 0$ and $r > 0$ be given. We choose $c, \epsilon > 0$ such that, using the notation \ref{8.13}, we have \((\partial \gamma^c_2)(\varepsilon) \subset \gamma^c_{c_2} \setminus \text{Int}(\gamma^c_{c_1}).\)

We choose $\delta > 0$ and $c' > c$. We set $U = B_r^{V^*} \times \mathbb{R}$. Then, for all $F \in D^0(k_U)$ satisfying

\[(8.16) \quad \text{SS}(F) \cap (U \times (\gamma^c_{c_2} \setminus \text{Int}(\gamma^c_{c_1}))) = \emptyset,\]

we have $S_x^{\gamma_{c'}} \cap \text{SS}(F \otimes k_{Z_r^{e,\delta}}) = \emptyset$, for all $x \in W_{r,c,c'}^{r,\delta,\epsilon}.$

**Proof.** Let $x \in W_{r,c,c'}^{r,\delta,\epsilon}$ be a given point. Let us assume that there exists a point $(y; \eta) \in S_x^{\gamma_{c'}} \cap \text{SS}(F \otimes k_{Z_r^{e,\delta}}).$ We set

\[I = \partial Z_r^{\infty,0} \cap \partial(x + Z_r^{\infty,0}) = \partial Z_r^{\infty,0} \cap \pi_V(S_x^{\gamma_{c'}})\]

as in \ref{8.15}.

(i) We first prove that we must have $y \in I$. Since $y \in \pi_V(S_x^{\gamma_{c'}})$ we have to check that $y \in \partial Z_r^{e,\delta}$. We remark that the set of $y' \in V$ such that $(y'; \eta) \in S_x^{\gamma_{c'}}$ is a line through $x$. This line meets $\partial(x + Z_r^{\infty,0})$ at some
point \( y_1 \), necessarily distinct from \((0, \pm \delta)\). Then the definition of \( W_{c,c'}^{r,\delta,\varepsilon} \) in (8.15) implies \(-\eta \in (T_{y_1}^* V \cap \text{SS}(k_{Z_{c,\varepsilon}}))\langle \varepsilon \rangle\). Since \( y_1 \neq (0, \pm \delta) \) we deduce \( \eta \in (\partial \gamma_{c_1}^{\alpha})\langle \varepsilon \rangle \subset \gamma_{c_2}^{\alpha} \setminus \text{Int}(\gamma_{c_1}^{\alpha}) \). By (8.16) we have in particular (8.17)

\[
(y; \eta) \notin \text{SS}(F).
\]

This implies that \( T_y^* V \cap \text{SS}(k_{Z_{c,\varepsilon}}) \neq 0 \). We deduce \( y \in \partial Z_{c,\varepsilon}^{\delta} \) and then \( y \in I \).

(ii) Since \( y \in \pi_V(S_{c,c'}^\varepsilon) \) we have \( y \neq (0, \pm \delta) \), hence \( T_y^* V \cap \text{SS}(k_{Z_{c,\varepsilon}}) \subset \partial \gamma_c \subset \gamma_{c_2}^{\alpha} \setminus \text{Int}(\gamma_{c_1}^{\alpha}) \). We deduce from (8.16) that \( T_y^* V \cap \text{SS}(F) \cap \text{SS}(k_{Z_{c,\varepsilon}})^{\circ} \) is contained in the zero section. Then Corollary 2.3 gives

\[
T_y^* V \cap \text{SS}(F \otimes k_{Z_{c,\varepsilon}}) \subset T_y^* V \cap \text{SS}(F) \cap \text{SS}(k_{Z_{c,\varepsilon}})^{\circ}.
\]

Hence \( \eta \) can be written \( \eta = \eta_1 + \eta_2 \), with \( \eta_1 \in T_y^* V \cap \text{SS}(F) \) and \( \eta_2 \in T_y^* V \cap \text{SS}(k_{Z_{c,\varepsilon}}) \). By (8.17) we have \( \eta_2 \neq 0 \). Since \( \partial Z_{c,\varepsilon} \) is a smooth hypersurface around \( y \), we have in fact \( T_y^* V \cap \text{SS}(k_{Z_{c,\varepsilon}}) = \mathbb{R}_{\geq 0}\eta_2 \). As in (i) the definition of \( W_{c,c'}^{r,\delta,\varepsilon} \) gives \(-\eta \in (\mathbb{R}_{\geq 0}\eta_2)\langle \varepsilon \rangle\). Since \( (\mathbb{R}_{\geq 0}\eta_2)\langle \varepsilon \rangle \) is convex, it follows that

\[
-\eta_1 = -\eta + \eta_2 \in (\mathbb{R}_{\geq 0}\eta_2)\langle \varepsilon \rangle \subset (\partial \gamma_c^{\alpha})\langle \varepsilon \rangle \subset \gamma_{c_2}^{\alpha} \setminus \text{Int}(\gamma_{c_1}^{\alpha}).
\]

This contradicts (8.16) and proves the lemma. \( \square \)

**Proposition 8.5.** Let \( c_2 > c_1 > 0 \) and \( r > 0 \) be given. Then there exists \( r_1 > 0 \) such that, for any \( s \in \mathbb{R} \), we have, setting \( U = B^V_r \times \mathbb{R} \) and \( B = B^V_{(0,s),r_1} \): for all \( F \in \mathcal{D}^b(k_U) \) satisfying (8.16), there exist \( F_1, F_2, L \in \mathcal{D}^b(k_B) \) and a distinguished triangle in \( \mathcal{D}^b(k_B) \)

\[
F_1 \oplus F_2 \rightarrow F|_B \rightarrow L \rightarrow F_1 \oplus F_2 \rightarrow F|_B.
\]

such that \( \hat{SS}(F_1) = \hat{SS}(F) \cap (B \times \gamma_{c_1}^{\alpha}) \), \( SS(F_2) = \hat{SS}(F) \setminus \hat{SS}(F_1) \) and \( SS(L) \subset T_B^* B \). In particular \( F_1 \rightarrow F|_B \) is an isomorphism on \( B \times \gamma_{c_1}^{\alpha} \) and \( F_2 \rightarrow F|_B \) is an isomorphism on \( B \times (V^* \setminus \text{Int}(\gamma_{c_1}^{\alpha})) \).

**Proof.** Since the statement is invariant by translation in the factor \( \mathbb{R} \) of \( V = V' \times \mathbb{R} \) we can assume \( s = 0 \). We choose \( c, \varepsilon \) such that \( (\partial \gamma_c^{\alpha})\langle \varepsilon \rangle \subset \gamma_{c_2}^{\alpha} \setminus \text{Int}(\gamma_{c_1}^{\alpha}) \), as in Lemma 8.4. We set \( c' = c + \varepsilon/2 \). We remark that \( c_1 < c' < c_2 \). By Lemma 8.3 we can choose \( \delta > 0 \) so that \( W_{c,c'}^{r,\delta,\varepsilon} \neq \emptyset \).

We choose \( r_1 \) such that \( B := B^V_{(0,0),r_1} \subset W_{c,c'}^{r,\delta,\varepsilon} \).

By Lemma 8.4 we can apply Lemma 8.2 to \( F \otimes k_{Z_{c,\varepsilon}} \) with \( \gamma = \gamma_{c'} \).

We obtain the distinguished triangle (8.3) with \( F \) replaced by \( F \otimes k_{Z_{c,\varepsilon}} \).

Then we set \( F_1 = P_{\gamma_{c'}}(F \otimes k_{Z_{c,\varepsilon}})|_B \) and \( F_2 = G|_B \). The proposition follows. \( \square \)
9. Non constant groups of sections

In this section we use Proposition 8.5 to give conditions on a sheaf which imply that it has non isomorphic cohomology groups over some open sets. These open sets only depend on the microsupport. This will be used in the proof of the Gromov-Eliashberg Theorem to insure the non triviality of the microsupport of a sheaf (see Proposition 9.3).

We still use the notations $V' = \mathbb{R}^{n-1}$ and $V = V' \times \mathbb{R}$ of Section 8. We also let $\gamma_c \subset V$ be the cone defined in (8.12) for any $c > 0$.

**Proposition 9.1.** Let $c_2 > c_1 > 0$ and $r > 0$ be given and let $r_1 > 0$ be given by Proposition 8.5. We set $U = B_{r'} \times \mathbb{R}$. Let $F \in D^b(k_U)$ and $S_1 = SS(F) \cap (U \times \gamma_{c_1})$, $S_2 = SS(F) \setminus S_1$.

We assume that $F$ satisfies (8.16) and that there exist $a < b$ such that

1. $F|_{\{0\} \times \mathbb{R}} \simeq k_{[a,b]} \oplus F'$ for some $F' \in D^b(k_{\mathbb{R}})$,
2. $S_i \cap T_{(0, a)}^* V = \emptyset$ or $S_i \cap T_{(0, b)}^* V = \emptyset$, for $i = 1$ and $i = 2$.

Then $b - a \geq r_1$.

**Proof.** If $b - a < r_1$, the ball $B$ of center $(0, (a + b)/2)$ and radius $r_1$ contains $a$ and $b$. Proposition 8.5 implies that we have a distinguished triangle $F_1 \oplus F_2 \to F|_B \to L \xrightarrow{\pm 1} \to$, where $SS(L) \subset T_{B}^* B$ and $SS(F_i) \subset S_i$, $i = 1, 2$.

Then the hypothesis (i) and Lemma 3.1 implies that $k_{[a,b]}$ is a direct summand of $F'|_{B \cap \{0\} \times \mathbb{R}}$ where $F' = F_1$ or $F' = F_2$. Hence $SS(F')$ meets both $T_{(0, a)}^* V$ and $T_{(0, b)}^* V$ outside the zero-section, which contradicts the hypothesis (ii). □

Proposition 9.1 will be used together with Lemma 9.2 below. It says that we can extend sections over a segment to some neighborhood of this segment which only depends on the microsupport. For given $c, r, s > 0$ with $r < s/c$, we define $U_{c, r, s} \subset V$ by

(9.1) $U_{c, r, s} = \{(x', x_n) \in V; \|x'\| < r \text{ and } -s < x_n < s - c\|x'\|\}$.

The Euclidean structure on $V'$ gives an identification $V' \simeq V'^*$ and we can define $T^{* \text{out}} V' = \{(x'; \xi') \in T^* V'; \xi' = \lambda x', \lambda \geq 0\}$. We also set
Proof. Following. For any there exists \( F \) \( \in \mathbb{D}^b(k_U) \) we have
\[
\begin{align*}
(9.2) & \quad SS(k_{U_{r,s}^c}) \cap \pi_V^{-1}(C_R^+) \subset \partial \gamma^{oa}_c \cap (T^*_{\sigma,0}V' \times T^*_R \mathbb{R}), \\
(9.3) & \quad SS(k_{U_{r,s}^c}) \cap \pi_V^{-1}(C_R^-) \subset T^*_V V' \times \{ (-s; \sigma); \sigma \geq 0 \}.
\end{align*}
\]

**Lemma 9.2.** Let \( c, r, s > 0 \) be given with \( r < s/c \). Let \( U \subset V \) be a neighborhood of \( \overline{U_{r,s}^c} \) and let \( F \in \mathbb{D}^b(k_U) \) be such that \( SS(F) \subset \overline{U \times \text{Int}(\gamma^{oa}_c)} \). Then the morphism \( \Gamma(\overline{U_{r,s}^c}; F) \to \Gamma([-s, s]; F|\{0\} \times [-2s, 2s]) \) is an isomorphism.

Proof. We choose \( R > r \) such that \( \overline{U_{r,s}^c} \subset U \) and we define \( G \in \mathbb{D}^b(k_{V'}) \) by \( G = Rq_1(R \Gamma_{U_{r,s}^c}(F)) \), where \( q_1: V \to V' \) is the projection. By \( (9.2) \), \( (9.3) \) and the hypothesis on \( SS(F) \) we obtain that \( SS(F) \cap SS(k_{U_{r,s}^c}) \) is contained in the zero section over \( C_R^+ \) and \( C_R^- \). By Corollary \( 2.7 \) and Theorem \( 2.4 \) (ii) we deduce
\[
\begin{align*}
SS(G) \cap \pi_V^{-1}(B_{R}^+ \setminus \{0\}) \subset \{ (x'; \xi'); \exists (x', x_n; \xi_1', \xi_{1,n}) \in SS(k_{U_{r,s}^c})^o, \xi' = \xi_1' + \xi_2', \xi_{1,n} + \xi_{2,n} = 0 \},
\end{align*}
\]
If \( x_n \geq 0 \) in \( (9.4) \), then \( (9.2) \) gives \( \xi_{1,n} = -c^{-1}||\xi'|| \) and \( \xi_1' = \lambda x' \), for some \( \lambda \leq 0 \). The hypothesis on \( SS(F) \) gives \( \xi_{2,n} \geq c^{-1}||\xi_2'|| \). We deduce \( ||\xi_1'|| \geq ||\xi_2'|| \) and then \( \langle x', \xi' \rangle \leq \lambda ||x'||^2 + ||x'|| ||\xi_2'|| \leq 0 \).

If \( x_n \geq 0 \) in \( (9.4) \), then \( (9.3) \) and the hypothesis on \( SS(F) \) give \( \xi' = 0 \).

We conclude that \( SS(G) \cap T^*(V' \setminus \{0\}) \subset \{ (x'; \xi') \in T^*V'; \langle x', \xi' \rangle \leq 0 \} \). By Corollary \( 2.7 \) we deduce that \( \Gamma(B_{R}^+; G) \to \Gamma(B_{R}^+; G) \) is an isomorphism for any \( 0 < r_1 < r_2 < R \). For \( r_2 = r \) and \( r_1 \to 0 \) we find the isomorphism of the lemma. \( \square \)

**Proposition 9.3.** Let \( c_2 > c_1 > 0 \) and \( r > 0 \) be given. We set \( U = B_{r}^+ \times \mathbb{R} \). Then there exist non empty connected open subsets \( W_i \), \( i = 1, \ldots, 4 \), of \( U \) such that \( \overline{W_i} \subset W_{i+1} \), for \( i = 1, \ldots, 3 \), which satisfy the following. For any \( F \in \mathbb{D}^b(k_U) \) such that
\[
\begin{align*}
(1)  & \quad F \text{ satisfies } (8.16), \\
(2)  & \quad \Lambda = SS(F) \text{ is a Lagrangian submanifold of } \hat{T}^*U \text{ and } F \text{ is simple along } \Lambda, \\
(3)  & \quad \text{setting } \Lambda_1 = \Lambda \cap (U \times \gamma^{oa}_{c_1}), \text{ the natural map } \Lambda_1/\mathbb{R}_{>0} \to B_{r}^+ \text{ is proper and of degree } 1,
\end{align*}
\]
there exists \( F_1 \in \text{Mod}(k_{W_4}) \) such that
\[
\begin{align*}
(1)  & \quad SS(F_1) \subset T^*W_1 \cap T^*_{\sigma>0}(V' \times \mathbb{R}) \text{ and } SS(F_1) \subset \text{Conv}(\Lambda_1), \\
(2)  & \quad \text{there exists } u \in \Gamma(W_4; F_1) \text{ such that } u|_{W_3} \neq 0 \text{ and } u|_{W_2} = 0 \text{ or there exists } v \in \Gamma(W_2; F_1) \text{ such that } v|_{W_1} \neq 0 \text{ and } v \text{ is not in the image of } \Gamma(W_3, F_1) \to \Gamma(W_2; F_1).
\end{align*}
\]
Remark 9.4. The conclusion (b) of the proposition also holds with \( \Gamma(W_i; F_1) \) replaced by \( H^0(W_i; F') \), where \( F' \) is obtained from \( F \) by a shift in the derived category and a vertical translation in \( V = V' \times \mathbb{R} \) (see part (C) of the proof). We introduce \( F_1 \) because we want to be sure to stay in the bounded derived category when we use this result in paragraph 11.4. It is likely that the properties of the microsupport hold for unbounded complexes. In this case the introduction of \( F_1 \) would be useless.

Proof. (A) Let \( r_1 \) be given from \( r, c_1, c_2 \) by Proposition 8.3. We set \( D_1 = ]-r_1/8, -r_1/16[, D_2 = ]-r_1/4, 0[, D_3 = ]-r_1/2, r_1/2[, D_4 = ]-r_1, r_1[. \) For any \( x \in V' \) and \( i = 1, \ldots, 4 \), we set \( D_{i,x} = \{x\} \times D_i \). We choose \( t_i, t_1 > 0 \) and \( 0 < \rho < \min\{t_i/c_1\} \) such that \( W_{i,x} := (x, t_i) + U_{\rho, t_i} \) satisfies \( W_{i,x} \cap \{x\} \times \mathbb{R} = D_{i,x} \). Now we can choose \( \varepsilon > 0 \) and non empty open subsets \( W_i, i = 1, \ldots, 4 \), of \( U \) such that \( W_i \subset W_{i+1} \) and, for all \( \|x\| < \varepsilon \),

\[
(9.5) \quad D_{1,x} \subset W_1 \subset W_2 \subset W_{2,x}, \quad D_{3,x} \subset W_3 \subset W_4 \subset W_{4,x}.
\]

(B) We can find \( x \in V' \) arbitrarily close to 0 such that \( \Lambda \) intersects \( T_x^* V' \times T^* \mathbb{R} \) transversally. Since \( \Lambda_1 / \mathbb{R}_{>0} \to B_{r'}^* \) is proper, \( \Lambda_1 \cap (T_x^* V' \times T^* \mathbb{R}) \) consists of finitely many half lines, all contained in \( T_x^* V' \times T^* I \) for some bounded interval \( I = ]a, b[. \) Then \( \Lambda \cap (T_x^* V' \times T^* I) \) is also a finite set of half lines and we write

\[
\Lambda \cap (T_x^* V' \times T^* I) = \bigcup_{k=1}^{l} \{(x, s_k; \sigma \xi_k, \sigma); \sigma \neq 0\}.
\]

Near a point \( (x, s_i; \xi_i, 1) \) the Lagrangian \( \Lambda \) is the conormal bundle to a smooth hypersurface, say \( X_i \). If \( s_i = s_j \), we have \( \xi_i \neq \xi_j \) and the hypersurfaces \( X_i \) and \( X_j \) are transversal at \( (x, s_i) \). Then, by moving \( x \) a little bit we can make \( s_i \) and \( s_j \) distinct, which we will assume from now on (for all pairs \( i \neq j \)). We set \( D = \{x\} \times I \subset V' \times I \), \( S = \pi_{V' \times \mathbb{R}}(\Lambda) \cap D = \{s_1, \ldots, s_l\} \) and \( S_1 = \pi_{V' \times \mathbb{R}}(\Lambda_1) \cap D \).

(C) We remark that \( F|_D \) is simple. By Proposition 3.3 it follows that \( F|_D \) is a sum of sheaves of the type \( k_{a,s_i}[1], k_{[s_i,b]}[1] \) or \( k_{s_i,s_j}[1] \) up to shift, where \( s_i, s_j \in S \). Since \( \Lambda_1 \to B_{r'}^* \) has degree 1, the set \( S_1 \) is of odd order. Hence there exist \( s \in S_1 \) and \( d \in \mathbb{Z} \) such that \( F|_D[d] \) has a direct summand isomorphic to \( k_{a,s_i}[1], k_{[s_i,b]}[1], k_{s_i,s_j}[1] \) or \( k_{s_i,s_j}[1] \), with \( s' \in S \setminus S_1 \).

In case the summand is \( k_{s_i,s_j}[1] \), \( k_{[s_i,b]}[1] \) or \( k_{s_i,s_j}[1] \), Proposition 9.3 gives \( |s' - s| \geq r_1 \) (recall that \( r_1 \) is given by Proposition 8.3). We could also have assumed in (B) that \( S_1 \subset ]a + r_1, b - r_1[. \) Hence in any case we obtain the following. We set \( F' = T_s^{-1} F[d], \) where \( T_s : V' \times \mathbb{R} \to V' \times \mathbb{R} \) is
the translation \((v', t)\mapsto(v', t + s)\). Then \(F'|_{D_{t,x}}\) has a direct summand isomorphic to \(k_{[-r_1,0]}\) or \(k_{[0,r_1]}\). We have \((x, 0)\in \pi_{V'\times R}(\Lambda_1)\) and \((x, 0)\not\in \pi_{V\times R}(A \setminus A_1)\).

(D) Let us set \(B = B_{r_1/2}^{V\times R}\). For \(x\) close enough to 0 we have \(B \subset B_{(x, 0), r_1}^{V\times R}\). By Proposition 8.5, we have a distinguished triangle

\[
F'_1 \oplus F'_2 \to F'|_B \to L \xrightarrow{+1},
\]

where \(F'_1, F'_2, L \in D^b(k_B)\) satisfy \(SS(L) \subset T^*_R B\), \(SS(S'_i) = \Lambda_1 \cap T^*_B\) and \(SS(S'_i) \cap \Lambda_1 = \emptyset\). By Lemma 3.1, we obtain that \(F'_1|_{D_{t,x}}\) or \(F'_2|_{D_{t,x}}\) has a direct summand isomorphic to \(k_{[-r_1,0]}\) or \(k_{[0,r_1]}\). Since \((x, 0)\not\in \pi_{V\times R}(A \setminus A_1)\) we deduce that \(F'_1|_{D_{t,x}}\) has a direct summand isomorphic to \(k_{[0,r_1]}\) or \(k_{[-r_1,0]}\).

(E) We define \(F_1 = H^0(F'_1)\) \(\in Mod(k_W)\). Then \(F_1|_{D_{t,x}}\) also has a direct summand isomorphic to \(k_{[0,r_1]}\) or \(k_{[-r_1,0]}\). Hence the conclusion (b) of the corollary holds with \(\Gamma(W_i; F_1)\) replaced by \(\Gamma(D_{t,x}; F_1|_{D_{t,x}})\). By Lemma 9.2 and the inclusions \((9.5)\) we deduce that (b) also holds as stated.

By Proposition 5.3, we have \(SS(F_1) \subset Conv(SS(F'_1))\), which gives the assertion (a) of the corollary. \(\Box\)

10. Quantization

In this section, we recall the main result of [4]. Let \(N\) be a manifold and \(I\) an open interval of \(\mathbb{R}\) containing 0. We consider a homogeneous Hamiltonian isotopy \(\Psi: \tilde{T}^*N \times I \to \tilde{T}^*N\) of class \(C^\infty\). For \(t \in I\), \(p \in \tilde{T}^*N\) we set \(\Psi_t(p) = \Psi(p, t)\). Hence \(\Psi_0 = \text{id}_{\tilde{T}^*N}\) and, for each \(t \in I\), \(\Psi_t\) is symplectically diffeomorphic such that \(\Psi_t(x; \lambda \xi) = \lambda \cdot \Psi_t(x; \xi)\), for all \((x; \xi) \in \tilde{T}^*N\) and \(\lambda > 0\). We let \(\Lambda_{\Psi_t} \subset \tilde{T}^*N^2\) be the twisted graph of \(\Psi_t\). We can see that there exists a unique conic Lagrangian submanifold \(\Lambda_{\Psi_t} \subset \tilde{T}^*(N^2 \times I)\) such that \(\Lambda_{\Psi_t} = i_t \circ i_{t,\pi}^{-1}(\Lambda_{\Psi_t})\), for all \(t \in I\), where \(i_t\) is the embedding \(N^2 \times \{t\} \to N^2 \times I\). We let \(D^b(k_{N^2 \times I})\) be the full subcategory of \(D(k_{N^2 \times I})\) formed by the \(F\) such that \(F|_C \in D^b(k_C)\) for all compact subsets \(C \subset N^2 \times I\).

**Theorem 10.1** (Theorem 4.3 of [4]). There exists a unique \(K_{\Psi} \in D^b(k_{N^2 \times I})\) such that \(SS(K_{\Psi}) \subset \Lambda_{\Psi}\) and \(K_{\Psi}|_{N^2 \times \{0\}} \isom k_{\Delta}\). Moreover \(K_{\Psi}\) is simple along \(\Lambda_{\Psi}\) and both projections \(\text{supp}(K_{\Psi}) \to N \times I\) are proper.

The fact that \(K\) is simple along \(\Lambda_{\Psi}\) is not explicitly stated in [4] but it follows from the claim \(K_{\Psi}|_{N^2 \times \{0\}} \isom k_{\Delta}\).
We reduce the case of non homogeneous Hamiltonian isotopies to the homogeneous framework by adding one variable as follows. Let $M$ be a connected manifold and let $\Phi: T^*M \times I \to T^*M$ be a Hamiltonian $C^\infty$ isotopy. We assume that $\Phi$ has compact support, that is, there exists a compact subset $C \subset T^*M$ such that $\Phi(p, t) = p$ for all $p \in T^*M \setminus C$ and all $t \in I$. We let $(s; \sigma)$ be the coordinates on $T^*\mathbb{R}$ and we recall the map $\rho_M: T^*M \times T^*\mathbb{R} \to T^*M$, $((x; \xi), (s; \sigma)) \mapsto (x; \xi/\sigma)$ defined in \cite{2.4}. By \cite[Prop. A.6]{3} there exists a homogeneous Hamiltonian isotopy $\Psi: \tilde{T}^*(M \times \mathbb{R}) \times I \to \tilde{T}^*(M \times \mathbb{R})$ whose restriction to $T^*M \times T^*\mathbb{R} \times I$ gives the commutative diagram

$$
\begin{array}{ccc}
T^*M \times \tilde{T}^*\mathbb{R} \times I & \xrightarrow{\Psi} & T^*M \times \tilde{T}^*\mathbb{R} \\
\rho_M \times \text{id}_I \downarrow & & \downarrow \rho_M \\
T^*M \times I & \xrightarrow{\Phi} & T^*M.
\end{array}
$$

(10.1)

Moreover there exists a $C^\infty$-function $u: (T^*M) \times I \to \mathbb{R}$ such that

$$
\Psi((x; \xi), (s; \sigma), t) = ((x'; \xi'), (s + u(x, \xi/\sigma, t); \sigma)),
$$

(10.2)

where $(x'; \xi'/\sigma) = \Phi_t(x, \xi/\sigma)$. In particular, if $(x; \xi/\sigma) \notin C$, we have $(x'; \xi') = (x; \xi)$. Since $\Psi_t$ is symplectic, we deduce $d(u|_{T^*M \times \{t\}}) = 0$ outside $C$. It follows that if $\Omega$ is a connected component of $T^*M \setminus C$, then there exists $\nu_\Omega: I \to \mathbb{R}$ such that

$$
\Psi((x; \xi), (s; \sigma), t) = ((x; \xi), (s + \nu_\Omega(t); \sigma)), \quad \text{for all } (x; \xi) \in \Omega.
$$

(10.3)

**Corollary 10.2.** Let $\Phi: T^*M \times I \to T^*M$ be a Hamiltonian $C^\infty$ isotopy with compact support. Then, for any $t \in I$, there exist a closed conic connected Lagrangian submanifold $\Lambda_t \subset \tilde{T}^*(M^2 \times \mathbb{R})$ and $K_t \in D^b(k_{M^2 \times \mathbb{R}})$ such that

(i) $\rho_{M^2}: T^*M^2 \times T^*\mathbb{R} \to T^*M^2$ induces a diffeomorphism between $(\Lambda_t \cap (T^*M^2 \times \tilde{T}^*\mathbb{R}))/\mathbb{R}^\times$ and $\Lambda_{\Phi_t}$, the twisted graph of $\Phi_t$,

(ii) $\text{SS}(K_t) = \Lambda_t$ and $K_t$ is simple along $\Lambda_t$.

**Proof.** (i) Let $\Psi$ be the homogeneous Hamiltonian isotopy introduced in the diagram (10.1). We see on (10.2) that $\Psi$ preserves the variable $\sigma$, that is, $\Lambda_{\Psi} = \text{im} q_{\sigma}$ is contained in $\Sigma := \{\sigma + \sigma' = 0\}$. Let us define $q: (M \times \mathbb{R})^2 \times I \to M^2 \times \mathbb{R} \times I$, $(x, s, x', s', t) \mapsto (x, x', s - s', t)$. Then $\Sigma = \text{im} q_d$ and the quotient map to the symplectic reduction of $\Sigma$ is $q_{\pi}$. Hence we can write $\Lambda_{\Psi} = q_d q_{\pi}^{-1}(\Lambda)$, where $\Lambda \subset \tilde{T}^*(M^2 \times \mathbb{R})$ is given by $\Lambda = q_{\pi}^{-1}(\Lambda_{\Phi})$. Now we set $\Lambda_t = i_{t,d} i_{t,\pi}^{-1}(\Lambda)$, where $i_t: M^2 \times \mathbb{R} \times \{t\} \to M^2 \times \mathbb{R} \times I$ is the embedding, and (i) follows from the diagram (10.1).
(ii) Let $K_{\Psi} \in D^b(k_{(M \times \mathbb{R})^2})$ be given by Theorem 10.1. Let us first check that $K_{\Psi, t} := K_{\Psi}|_{(M \times \mathbb{R})^2 \times \{ t \}} \in D^b(k_{(M \times \mathbb{R})^2})$ for any $t \in I$.

Let $C \subset T^* M$ be a compact subset such that $\Phi(p, t) = p$ for $p$ outside $C$. We remark that we may enlarge $C$ so that $\Omega = T^* M \setminus C$ has a single connected component when $M$ is not the circle $S^1$, and two components, say $\Omega_{\pm}$, corresponding to $\pm \xi > 0$, when $M = S^1$. We set $Z = \pi_M(C)$, $U = M \setminus Z$ and $q_t = q|_{(M \times \mathbb{R})^2 \times \{ t \}}$. By (10.3) and the unicity of $K_{\Psi}|_{(U \times \mathbb{R})^2 \times I}$ we find

$$K_{\Psi, t}|_{(U \times \mathbb{R})^2} \simeq \begin{cases} q_t^{-1}k_{\Delta \times \{ v\Omega(t) \}}, & \text{if } M \neq S^1, \\ q_t^{-1}k_{\Delta \times [v\Omega_-(t), v\Omega_+(t)]}, & \text{if } M = S^1. \end{cases}$$

We also have $\Lambda_{\Psi} \cap T^*(Z \times U \times \mathbb{R}^2) = \emptyset$. Hence $K_{\Psi}$ is locally constant on $Z \times U \times \mathbb{R}^2 \times I$. We deduce $K_{\Psi}|_{Z \times U \times \mathbb{R}^2 \times I} \simeq 0$ since this holds at $t = 0$. In the same way $K_{\Psi}|_{U \times Z \times \mathbb{R}^2 \times I} \simeq 0$.

We conclude that $K_{\Psi, t}|_{(M \times \mathbb{R}^2) \times \mathbb{R}^2}$ is a bounded complex. Since $Z^2$ is compact we have $K_{\Psi, t} \in D^b(k_{(M \times \mathbb{R}^2)})$ as claimed.

Now, by (i) and Corollary 2.6 there exists $K_t \in D^b(k_{M^2 \times \mathbb{R}})$ such that $K_{\Psi, t} \simeq q^{-1}K_t$. Then $K_t$ satisfies (ii). \hfill \Box

Now we apply Proposition 9.3 to the quantization given by Corollary 10.2 in the following situation. Let $V = \mathbb{R}^n$ be a vector space and let $r, A > 0$. We will consider the following hypothesis on a symplectic map $\varphi: T^* V \to T^* V$:

$$\begin{align*}
&\begin{cases}
(i) \text{ there exists a } C^\infty \text{ Hamiltonian isotopy with compact support, } \Phi: T^* V \times I \to T^* V, \text{ such that } \varphi = \Phi_1, \\
(ii) \text{ } \Lambda_{\varphi} \cap (B^V_r \times B^V_{3A_r}) \subset B^V_r \times B^V_{3A_r}, \\
(iii) \text{ setting } \Lambda^1_{\varphi} = \Lambda_{\varphi} \cap (B^V_r \times B^V_{3A_r}), \text{ the map } \Lambda^1_{\varphi} \to B^V_r \\
\text{induced by the projection to the base is of degree 1.}
\end{cases}
\end{align*}$$

Corollary 10.3. Let $A, r > 0$ be given. There exist non empty connected open subsets $W_i$, $i = 1, \ldots, 4$, of $B^V_r \times \mathbb{R}$ such that $\overline{W_i} \subset W_{i+1}$, for $i = 1, \ldots, 3$, which satisfy the following. For any symplectic map $\varphi: T^* V \to T^* V$ satisfying (10.4), there exists $L \in \text{Mod}(k_{W_1})$ such that

(i) $\text{SS}(L) \subset T^* W_4 \cap T^*_{>0}(V^2 \times \mathbb{R})$ and $\rho_{V^2}(\text{SS}(L)) \subset \text{Conv}(\Lambda^1_{\varphi})$,
(ii) there exists $u \in \Gamma(W_4; L)$ such that $u|_{W_3} \neq 0$ and $u|_{W_2} = 0$ or there exists $v \in \Gamma(W_2; L)$ such that $v|_{W_1} \neq 0$ and $v$ is not in the image of $\Gamma(W_3; L) \to \Gamma(W_2; L)$.

Proof. By Corollary 10.2 there exist a conic closed Lagrangian submanifold $\Lambda \subset \hat{T}^*(V^2 \times \mathbb{R})$ and $K \in D^b(k_{V^2 \times \mathbb{R}})$ such that
(a) \( \rho_{V^2}: T^*V^2 \times \mathbb{T}^* \mathbb{R} \to T^*V^2 \) induces a diffeomorphism between 
\((\Lambda \cap (T^*V^2 \times \mathbb{T}^* \mathbb{R}))/\mathbb{R}^\times \) and \( \Lambda_\varphi \),
(b) \( \text{SS}(K) = \Lambda \) and \( K \) is simple along \( \Lambda \).

We set \( c_1 = (3Ar)^{-1} \) and \( c_2 = (2Ar)^{-1} \). We apply Proposition \ref{prop:existence} with \( c_1, c_2, r \) and \( V' = V^2 \). We obtain open subsets \( W_i \) of \( B_{V^2}^r \times \mathbb{R} \), \( i = 1, \ldots, 4 \). The hypothesis \((10.4)-(ii)\) implies that \( K \) satisfies \((8.16)\) and the hypothesis \((10.4)-(iii)\) implies that \( K \) satisfies \((iii)\) of Proposition \ref{prop:existence}. Hence the proposition gives \( L \in \text{Mod}(k_{W_4}) \) satisfying the required properties.

\[ \square \]

11. **Proof of the Gromov-Eliashberg theorem**

We use the notations in the statement of Theorem 1.1.

11.1. Up to a translation in \( E \) it is enough to prove that \( d\varphi_\infty|_0 \) is a symplectic linear map and we work near 0. We can also assume that \( \varphi_\infty(0) = 0 \). By Lemma \ref{lem:identification} we can identify \( E \) with \( T^*V \) for some vector space \( V \) and assume that \( \varphi_\infty \) satisfies \((6.2)\) and \( (6.3) \) for some \( A, r_0 > 0 \).

11.2. By Proposition \ref{prop:hamiltonian isotopy}, for each \( n \in \mathbb{N} \) we can find a \( C^\infty \) Hamiltonian isotopy \( \Phi_n: E \times \mathbb{R} \to E \) and a compact subset \( C_n \subset E \) such that \( \|\varphi_n - \Phi_n,1\|_{B_{V^2}^r} \leq 1/n \) and \( \Phi_n,t|_{E\setminus C_n} = \text{id}_{E\setminus C_n} \) for all \( t \in \mathbb{R} \). We still have \( \|\Phi_n,1 - \varphi_\infty\|_{B_{V^2}^r} \to 0 \) when \( n \to \infty \) and we may assume that \( \varphi_n = \Phi_n,1 \). We choose \( 0 < r < r_0 \) (arbitrarily small). By Lemma \ref{lem:identification} there exists \( N_r \in \mathbb{N} \) such that

\[ \Lambda_{\varphi_n} \cap (B_r^{V^2} \times B_{3Ar}^{V^2}) \subset B_r^{V^2} \times B_{2Ar}^{V^2}, \quad \text{for all } n \geq N_r. \]

11.3. We set \( \Lambda_n^1 = \Lambda_{\varphi_n} \cap (B_r^{V^2} \times B_{3Ar}^{V^2}) \). By Proposition \ref{prop:restriction} the map \( \Lambda_n^1 \to B_r^{V^2} \) has degree 1, for \( n \geq N_r \). Let \( W_i, i = 1, \ldots, 4 \), be the non empty connected open subsets of \( B_r^{V^2} \times \mathbb{R} \) given by Corollary \ref{cor:restriction}. We apply Corollary \ref{cor:restriction} to \( \varphi_n \) and we obtain \( L_n \in \text{Mod}(k_{W_4}) \) such that

(i) \( \text{SS}(L_n) \subset T^*W_4 \cap T_{\sigma_n=0}^*(V^2 \times \mathbb{R}) \) and \( \rho_{V^2}(\text{SS}(L_n)) \subset \text{Conv}(\Lambda_n^1) \),
(ii-a) there exists \( u_n \in \Gamma(W_4; L_n) \) such that \( u_n|_{W_3} \neq 0 \) and \( u_n|_{W_2} = 0 \),
(ii-b) or there exists \( v_n \in \Gamma(W_2; L_n) \) such that \( v_n|_{W_1} \neq 0 \) and \( v_n \) is not in the image of \( \Gamma(W_3; L_n) \to \Gamma(W_2; L_n) \).

One of the two possibilities (ii-a) or (ii-b) occurs infinitely many times. Up to taking a subsequence we will assume that (ii-a) holds for all \( n \in \mathbb{N} \) (the other case being similar).
11.4. We define $L_\infty \in \text{Mod}(k_{W_4})$ by the exact sequence

$$0 \to \bigoplus_{n \geq N_r} L_n \to \prod_{n \geq N_r} L_n \to L_\infty \to 0.$$ 

The sections $u_n \in \Gamma(W_4; L_n)$ define $u \in \Gamma(W_4; L_\infty)$. By (ii-a) in §11.3 we have $u|_{W_2} = 0$. Since $W_3$ is compact we have $\Gamma(W_3; \bigoplus_{n \geq N_r} L_n) \simeq \bigoplus_{n \geq N_r} \Gamma(W_3; L_n)$. If $u = 0$, we deduce that $(u_n)_{n \geq N_r}|_{W_3}$ belongs to $\bigoplus_{n \geq N_r} \Gamma(W_3; L_n)$, which implies $u_n|_{W_3} = 0$ for $n$ big. But this contradicts (ii-a) and it follows that $u \neq 0$. Hence $L_\infty$ is not locally constant and $\dot{SS}(L_\infty) \neq \emptyset$.

11.5. Proposition 5.4 gives $\dot{SS}(L_\infty) \subset \rho^{-1}_{V^2}(\bigcap_{k \geq N_r} \bigcup_{n \geq k} \text{Conv}(\Lambda^1_n))$. By hypothesis $\{\Lambda^1_n\}_{n \in \mathbb{N}}$ converges to $\Lambda^\infty_1 := \Lambda^\infty_{\varphi,0} \cap (B_{V^2} \times B_{3V^2}^{3V^2})$. Since $\varphi_\infty$ satisfies (6.3), the set $\Lambda^\infty_1$ is a section of the projection to the base $T^*(B_{V^2}) \to B_{V^2}$. Hence $\{\text{Conv}(\Lambda^1_n)\}_{n \in \mathbb{N}}$ also converges to $\Lambda^1_\infty$ and we obtain $\dot{SS}(L_\infty) \subset \rho^{-1}_{V^2}(\Lambda^\infty_{\varphi,0}) \cap \pi^{-1}_{V^2 \times \mathbb{R}}(W_4)$.

Let us choose $p \in \dot{SS}(L_\infty)$. By the involutivity Theorem and Proposition 11.4 we obtain that $\Lambda^\infty_{\varphi,0}$ is coisotropic at $\rho_{V^2}(p)$. Since $\rho_{V^2}(p) \in B_{V^2} \times B_{3V^2}^{3V^2}$ and $r$ can be chosen arbitrarily small, we deduce that $\Lambda^\infty_{\varphi,0}$ is coisotropic at 0, as required.

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