THE PRO-CHERN-SCHWARZ-MACPHERSON CLASS IN BOREL-MOORE MOTIVIC HOMOLOGY

FANGZHOU JIN, PENG SUN, AND ENLIN YANG

ABSTRACT. We show that the zero-dimensional part of the pro-Chern-Schwarz-MacPherson class defined by Aluffi is equal to the pro-characteristic class in limit Borel-Moore motivic homology. A similar construction also produces a quadratic refinement of this class in the limit Borel-Moore Milnor-Witt homology.

CONTENTS

1. Introduction
2. The pro-characteristic class
3. Relation with the pro-CSM class
References

1. INTRODUCTION

1.1. Inspired by the Chern-Weil theory, Grothendieck shows how to define Chern classes for vector bundles in algebraic geometry in the group of algebraic cycles up to rational equivalence ([Gro58]), namely the Chow group, considered as an algebraic substitute of the cohomology ring of a complex manifold. As a particular case of this general construction, one can define the Chern class of a smooth algebraic variety \( X \) as the (total) Chern class of its tangent bundle:

\[
(1.1.1) \quad c(X) = c(T_X) \in CH_*(X).
\]

1.2. Over the field of complex numbers, the landmarking work of MacPherson ([Mac74]) extends the Chern class (1.1.1) to every (possibly singular) complex algebraic variety \( X \), in a way compatible with proper push-forward maps of constructible functions, and therefore answering affirmatively a conjecture of Deligne-Grothendieck. MacPherson’s original construction takes place in homology and uses transcendental methods, and later algebraic formulas have been found by González-Sprinberg and Verdier ([G-S81]), leading to what is now called the Chern-Schwarz-MacPherson class in the Chow group:

\[
(1.2.1) \quad c^{SM}(X) \in CH_*(X).
\]

Date: 27-9-2022.
1.3. In [Alu06], Aluffi gives a refinement of MacPherson’s class via an algebraic-geometric construction using resolution of singularities, by defining the pro-Chern-Schwarz-MacPherson class (abbreviated as pro-CSM class)

\[(1.3.1) \quad \hat{c}^{SM}(X) \in \widehat{CH}_*(X)\]

in the pro-Chow group of \(X\), that is, the limit of Chow groups of all possible compactifications of \(X\) ([Alu06, Def. 2.2]). This construction is briefly recalled in Theorem 3.2 below, and the functoriality of the pro-CSM class is extended to all morphisms of varieties ([Alu06, Thm. 5.2]).

1.4. The main purpose of this paper is to provide a category-theoretic approach to the 0-dimensional component \(\hat{c}^{SM}_0(X)\) of Aluffi’s pro-CSM class \(\hat{c}^{SM}(X)\) via motivic homotopy theory. In a previous work ([JY21]), the first and third-named authors defined a characteristic class in the Borel-Moore homology as a generalized trace map ([JY21, Def. 5.1.3])

\[(1.4.1) \quad C_X(\underline{F}) \in E^{BM}(X/k)\]

where \(\underline{F}\) is a constructible motivic spectrum over \(X\), and \(E^{BM}(X/k)\) is the Borel-Moore theory associated to a motivic spectrum (see 2.4), assuming resolution of singularities or inverting the characteristic (see 2.13). The class \(C_X(\underline{F})\) is defined by analogy with a construction due to Verdier in various categories of sheaves ([SGA5, Exp. III], [KS90, Def. 9.1.2], [AS07]).

For example, taking \(E\) to be the motivic Eilenberg-Mac Lane spectrum ([Spi18], see Example 2.11), we obtain a class \(C_X(\underline{F}) \in CH_0(X)\) in the Chow group of 0-cycles. In this case, the definition of this class is due to Olsson when \(k\) is an algebraically closed field ([Ols16]).

1.5. Similar to Aluffi’s definition, we define a pro-version of the Borel-Moore theory \(\hat{E}^{BM}(X/k)\) (Definition 2.7). However, as a substitute of Aluffi’s geometric approach, we use the six-functors formalism in motivic homotopy ([Ayo07]) to define two pro-versions of the class \(C_X(\underline{F})\), denoted as \(\hat{C}_X(\underline{F})\) and \(\widehat{C}_X(\underline{F})\) respectively (Definition 2.20), which are respectively the homological and compactly supported variants of the pro-characteristic class. Concretely, the class \(\hat{C}_X(\underline{F})\) (respectively \(\widehat{C}_X(\underline{F})\)) is the pro-class associated to the class \(C_{\underline{F}}(j_!\underline{F})\) (respectively \(C_{\underline{F}}(j!\underline{F})\)) for every compactification \(X \xrightarrow{j} \overline{X}\) of \(X\). These two classes are related by the local duality functor (Corollary 2.22), and satisfy push-forward (2.23) and additivity formulas (2.26).

1.6. Our main result states that the 0-dimensional part of Aluffi’s pro-CSM class \(\hat{c}^{SM}_0(X)\) is equal to the compactly supported pro-characteristic class of the unit object \(\underline{1}_X\):

**Theorem 1.7** (see Theorem 3.4). For every \(k\)-scheme \(X\), we have

\[(1.7.1) \quad \hat{c}^{SM}_0(X) = \hat{C}_X(\underline{1}_X) \in \widehat{CH}_0(X).\]

The proof uses the properties of the class \(\hat{C}_X(\underline{F})\), as well as the computation of the Chern classes of the sheaf of differentials with logarithmic poles as in the proof of [Sil96, Thm. 3.1]. One can also take the spectrum \(\underline{E}\) in (1.4.1) to be the Milnor-Witt spectrum ([DF20], see Example 2.12), in which case the pro-characteristic class \(\hat{C}_X(\underline{1}_X)\) gives a quadratic refinement of the class \(\hat{c}^{SM}_0(X)\), see 3.5.

Note that in a recent work ([Azo22]), Azouri gives independently a construction of the compactly supported pro-characteristic class \(\hat{C}_X(\underline{1}_X)\), with an approach closer to Aluffi’s original work.
2. THE PRO-CHARACTERISTIC CLASS

2.1. Throughout the paper, all schemes are assumed separated of finite type over a field \( k \).

2.2. Let \( C \) be a symmetric monoidal category with unit object \( 1 \). If \( M \) is a (strongly) dualizable object in \( C \), denote by \( M^\vee \) its (strong) dual, and define the categorical Euler characteristic \( \chi(M) \) of \( M \) as the composition

\[
\chi(M) : 1 \to M^\vee \otimes M \cong M \otimes M^\vee \to 1
\]

considered as an endomorphism of the unit object \( 1 \) ([LMS86, III Def. 7.1]). By [LMS86, III Prop. 7.7], we have a canonical identification \( \chi(M) = \chi(M^\vee) \). In other words, the categorical Euler characteristic of a dualizable object agrees with that of its dual.

2.3. For every scheme \( X \), we denote by \( \text{SH}(X) \) the stable motivic homotopy category, which is endowed with a six functors formalism ([Ayo07]). The unit object of \( \text{SH}(X) \) is denoted as \( 1_X \). If \( i : Z \to X \) is a closed immersion with \( j : U \to X \) the open complement, there is a distinguished triangle in \( \text{SH}(X) \) called the localization triangle:

\[
j_! 1_U \to 1_X \to \text{res}_Z.
\]

2.4. If \( p : X \to S \) is a morphism of schemes and \( E \in \text{SH}(S) \) is a motivic spectrum, we define the Borel-Moore \( \mathbb{E} \)-homology group

\[
\mathbb{E}^{BM}(X/S) = \text{Hom}_{\text{SH}(S)}(1_S, p_* p^! E).
\]

2.5. For any proper morphism \( f : Y \to X \), there is a map

\[
f_* : \mathbb{E}^{BM}(Y/S) \to \mathbb{E}^{BM}(X/S).
\]

For any étale morphism \( f : Y \to X \) (more generally, any local complete intersection morphism of virtual relative dimension 0 endowed with a trivialization of the virtual tangent bundle), there is a map

\[
f^* : \mathbb{E}^{BM}(X/S) \to \mathbb{E}^{BM}(Y/S).
\]

2.6. Let \( f : X \to S \) be a morphism of schemes. We denote by \( \text{Cpt}(f) \) or \( \text{Cpt}(X/S) \) the category of compactifications of \( f \), such that

- (1) the objects are factorizations of \( f \) as

\[
X \xrightarrow{j} \overline{X} \xrightarrow{p} S
\]

with \( j \) an open immersion with dense image and \( p \) a proper morphism,
(2) morphisms from $X \xrightarrow{j} X' \to S$ to $X' \xrightarrow{p'} S$ are proper morphisms $X \xrightarrow{f} X'$ such that $j' = f \circ j$, $p = p' \circ f$, that is, such that there is a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow{f'} & & \downarrow{p'} \\
X' & \xrightarrow{p} & S
\end{array}
$$

(2.6.2)

The category $\text{Cpt}(f)$ is non-empty and cofiltered, see [Con07], [SGA4, Exp. XVII §3.2], [Stacks, Tag 0A TT].

**Definition 2.7.** We define the **limit Borel-Moore $E$-homology**

$$
\widehat{\text{E}}_{\text{BM}}(X/S) = \lim_{X \in \text{Cpt}(f)} \text{E}^{\text{BM}}(X/S).
$$

(2.7.1)

2.8. By definition, to determine a class $\alpha$ in $\widehat{\text{E}}_{\text{BM}}(X/S)$ amounts to determine, for every compactification $\overline{X}$ of $f$, a class $\alpha_{\overline{X}}$ in $\text{E}^{\text{BM}}(\overline{X}/S)$, compatible with proper push-forwards.

2.9. There is a canonical map

$$
\widehat{\text{E}}^{\text{BM}}(X/S) \to \text{E}^{\text{BM}}(X/S)
$$

induced by the pull-back map $j^*: \text{E}^{\text{BM}}(\overline{X}/S) \to \text{E}^{\text{BM}}(X/S)$ for every compactification $X \xrightarrow{j} \overline{X}$. In particular, if $X$ is proper over $k$, then $X = X \to k$ is an initial object in $\text{Cpt}(f)$, and the map (2.9.1) is an isomorphism.

2.10. For every morphism $f: Y \to X$, we have a map

$$
f_*: \widehat{\text{E}}^{\text{BM}}(Y/S) \to \widehat{\text{E}}^{\text{BM}}(X/S).
$$

(2.10.1)

Indeed, by [SGA4, XVII, Prop. 3.2.6], for every compactification $\overline{X}$ of $X$, there exists a compactification $\overline{Y}$ of $Y$, a proper morphism $\overline{f}: \overline{Y} \to \overline{X}$ and a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & \overline{Y} \\
\downarrow{f} & & \downarrow{\overline{f}} \\
X & \xrightarrow{\overline{f}} & \overline{X}
\end{array}
$$

(2.10.2)

By 2.8, we define the map (2.10.1) such that for every class $\alpha \in \widehat{\text{E}}^{\text{BM}}(Y/S)$, the direct image $(f_\ast \alpha) \in \widehat{\text{E}}^{\text{BM}}(X/S)$ is the class associated, for every compactification $\overline{X}$ of $X$, the class $(f_\ast \alpha)_{\overline{X}} = \overline{f}_\ast \alpha_{\overline{Y}}$, which is well-defined and does not depend on the choice of $\overline{Y}$. The map (2.10.1) is compatible with compositions of morphisms, see [Alu06, Lemma 2.4].

**Example 2.11.** For $E = \mathbb{H}$ the motivic Eilenberg-Mac Lane spectrum ([Spi18]) and $S$ the spectrum of a field $k$ of characteristic $p$, the Borel-Moore motivic homology group $\mathbb{H}^{\text{BM}}(X/k)[1/p]$ agrees with the Chow group of zero cycles on $X$:

$$
\mathbb{H}^{\text{BM}}(X/k) \simeq \text{CH}_0(X).
$$

(2.11.1)

It follows that the limit Borel-Moore motivic homology $\widehat{\mathbb{H}}^{\text{BM}}(X/k)$ agrees with the pro-Chow group of zero-cycles defined by Aluffi ([Alu06, Def. 2.2.2]):

$$
\widehat{\mathbb{H}}^{\text{BM}}(X/k) \simeq \widehat{\text{CH}}_0(X).
$$

(2.11.2)
Note the above statements can be enhanced to (pro-)Chow groups of cycles of any dimension (see for example [Jin16]), which we do not discuss here.

Example 2.12. For $E = H_{MW}Z$ the Milnor-Witt spectrum and $S$ the spectrum of an infinite perfect field $k$ ([DF20]), the Borel-Moore Milnor-Witt homology group $H_{MWZ}^{BM}(X/k)$ agrees with the Chow-Witt group of zero cycles on $X$, which is a quadratic refinement of the usual Chow group (see [DFJK21, §8]):

$$H_{MWZ}^{BM}(X/k) \cong \widehat{CH}_0(X).$$

(2.12.1)

The limit Borel-Moore Milnor-Witt homology agrees with the pro-Chow Witt group defined in a similar way

$$\widehat{H}_{MWZ}^{BM}(X/k) \cong \widehat{CH}_0(X).$$

(2.12.2)

2.13. From now on, we assume that $S$ is the spectrum of a field $k$, and that one of the following conditions holds:

- the field $k$ is perfect and satisfies resolution of singularities.
- we work with $\mathbb{Z}[1/p]$-coefficients, where $p$ is the exponential characteristic of $k$.

Let $E \in SH(k)$ be a motivic spectrum with a map $1_k \to E$.

2.14. Let $f : X \to k$ be the structure morphism and let $F \in SH_c(X)$ be a constructible motivic spectrum. In [JY21, Def. 5.1.3], we constructed a map

$$C_X(F) : 1_X \to Hom(F, F) \cong \delta^!(D_{X/k}(F) \boxtimes_k F) \to \delta^*(D_{X/k}(F) \boxtimes_k F)$$

$$= D_{X/k}(F) \boxtimes_k F \cong F \boxtimes_k D_{X/k}(F) \to f^!1_k \to f^!E$$

(2.14.1)

viewed as an element of $E^{BM}(X/k)$, where $\delta$ is the diagonal morphism of $f$. This class $C_X(F) \in E^{BM}(X/k)$ is called the ($E$-valued) characteristic class of $F$. If $X = k$, then $C_X(F)$ agrees with the categorical Euler characteristic of $F$ (see 2.2).

2.15. ([JY21, Cor. 5.1.8]) If $p : Y \to X$ is a proper morphism, then

$$p_* C_Y(F) = C_X(p_* F).$$

(2.15.1)

2.16. ([JY21, Example 5.1.16]) If $X$ is smooth of relative dimension $n$ over $k$, then

$$C_X(1_X) = e(T_{X/k})$$

(2.16.1)

where $e(T_{X/k})$ is the Euler class of the tangent bundle of $X$ ([DJK18, Def. 3.1.2]). If $E$ is an oriented motivic spectrum (for example $HZ$), then $e(T_{X/k})$ agrees with the top Chern class in the $E$-cohomology of $X$ ([DJK18, 4.4.3]).

2.17. ([JY21, Thm. 4.2.8]) For a distinguished triangle $F \to G \to H$ of constructible objects in $SH_c(X)$, we have

$$C_X(G) = C_X(F) + C_X(H).$$

(2.17.1)
2.18. ([BD17, Thm. 2.4.9], [EK20, Thm. 3.1.1]) For $f : X \to k$, denote by $\mathbb{D}_{X/k}$ the local duality functor
\begin{equation}
\mathbb{D}_{X/k} : \mathbf{SH}_c(X) \to \mathbf{SH}_c(X)
\end{equation}
\[M \mapsto \text{Hom}(M, f^! \mathbb{1}_k).\]
Then the Verdier duality holds, which states that the following canonical map of functors is invertible:
\begin{equation}
1 \to \mathbb{D}_{X/k} \circ \mathbb{D}_{X/k}.
\end{equation}

**Lemma 2.19.** If $\mathbb{F} \in \mathbf{SH}_c(X)$ is a constructible motivic spectrum, then we have
\begin{equation}
C_X(\mathbb{F}) = C_X(\mathbb{D}_{X/k}(\mathbb{F})).
\end{equation}

**Proof.** This follows from the fact that the class $C_X(\mathbb{F})$ can be interpreted as the categorical Euler characteristic of the object $\mathbb{F}$, which is a dualizable object in the category of correspondences over $X$ relative to $k$ whose dual is $\mathbb{D}_{X/k}(\mathbb{F})$ (see [LZ22]). The result then follows from Verdier duality and the fact that the categorical Euler characteristic of a dualizable object in a symmetric monoidal category agrees with that of its dual (see 2.2). \qed

**Definition 2.20.** Let $\mathbb{F} \in \mathbf{SH}_c(X)$ be a constructible motivic spectrum. To every compactification $X \to \overline{X} \to S \in \text{Cpt}(f)$, we associate two elements
\begin{equation}
(C_{X}(j_* \mathbb{F})) \text{ and } (C_{X}(j! \mathbb{F}))
\end{equation}
in $\mathbb{E}^{BM}(\overline{X}/k)$. By 2.8 and (2.15.1), the formation of these two families of elements associated to every compactification of $f$ determines two elements
\begin{equation}
\widehat{C_X}(\mathbb{F}) \text{ and } \widehat{C^c_X}(\mathbb{F})
\end{equation}
in $\mathbb{E}^{BM}(X/k)$, which we call the **pro-characteristic class** and **compactly supported pro-characteristic class** respectively.

2.21. Since the local duality functor (2.18.1) exchanges the functors $f_*$ and $f!$, we deduce from Lemma 2.19 the following relation between the two classes $\widehat{C}_{X}$ and $\widehat{C}^c_{X}$:

**Corollary 2.22.** If $\mathbb{F} \in \mathbf{SH}_c(X)$ is a constructible motivic spectrum, then we have
\begin{equation}
\widehat{C}_{X}(\mathbb{F}) = \widehat{C}^c_{X}(\mathbb{D}_{X/k}(\mathbb{F})).
\end{equation}

2.23. By 2.8 and (2.15.1), we have the following property: for every morphism $f : Y \to X$, we have
\begin{equation}
f_* \widehat{C}_{Y}(\mathbb{F}) = \widehat{C}_{X}(f_* \mathbb{F})
\end{equation}
and
\begin{equation}
f_* \widehat{C}^c_{Y}(\mathbb{F}) = \widehat{C}^c_{X}(f_* \mathbb{F}).
\end{equation}
In particular, for $f : X \to k$, the map $f_* : \mathbb{E}^{BM}(X/k) \xrightarrow{(2.10.1)} \mathbb{E}^{BM}(k/k) = [\mathbb{1}_k, \mathbb{E}]$ sends $\widehat{C}_{X}(\mathbb{F})$ (resp. $\widehat{C}^c_{X}(\mathbb{F})$) to the $\mathbb{E}$-valued Euler characteristic of $f_* \mathbb{F}$ (resp. the $\mathbb{E}$-valued Euler characteristic of $f^! \mathbb{F}$), see [JY21, 5.3.1].

**Remark 2.24.** The formulas (2.23.1) and (2.23.2) are formulas related to push-forward maps, expressed in terms of (motivic) sheaves. If $k = \mathbb{C}$ is the field of complex numbers, it is plausible that one may translate the push-forward formulas in the language of constructible functions, and recover [Alu06, Thm. 5.2] (see [Ill15]).
2.25. The canonical map \( \hat{\mathbb{B}}_{BM}(X/k) \xrightarrow{(2.9.1)} \mathbb{B}_{BM}(X/k) \) sends both classes \( \hat{C}_X(F) \) and \( \hat{C}_X^c(F) \) to the class \( C_X(F) \) in \( (2.14.1) \). This follows from 2.8 and the fact that the class \( C_X(F) \) is compatible with pullbacks by étale morphisms ([JY21, Rem. 3.1.7]). In particular, if \( X \) is proper over \( k \), then by 2.9, under the isomorphism \( (2.9.1) \) we have identifications

\[
(2.25.1) \quad \hat{C}_X(F) = \hat{C}_X^c(F) = C_X(F).
\]

2.26. We deduce from 2.17 the following property: for a distinguished triangle \( \mathbb{F} \to \mathbb{G} \to \mathbb{H} \) of constructible objects in \( \mathbf{SH}_c(X) \), we have

\[
(2.26.1) \quad \hat{C}_X(\mathbb{G}) = \hat{C}_X(\mathbb{F}) + \hat{C}_X(\mathbb{H}),
\]

\[
(2.26.2) \quad \hat{C}_X^c(\mathbb{G}) = \hat{C}_X^c(\mathbb{F}) + \hat{C}_X^c(\mathbb{H}).
\]

3. Relation with the pro-CSM class

3.1. Recall Aluffi’s construction of the (zero-dimensional part of the) pro-CSM class:

**Theorem 3.2** ([Alu06, Prop. 4.3]). Assume that \( k \) is a perfect field which satisfies resolution of singularities. Then there is a unique way to associate a class \( \hat{c}^{SM}(X) \in \hat{C}_0(H_*(X)) \) for every \( k \)-scheme \( X \), such that

1. (Additivity for stratifications) If \( X \) has a finite stratification into locally closed subschemes \( X = \sqcup_{i=1}^n U_i \), with each \( U_i \) smooth and irreducible, then we have

\[
(3.2.1) \quad \hat{c}^{SM}(X) = \sum_{i=1}^n \iota_i^* \hat{c}^{SM}(U_i) \in \hat{C}_0(H_*(X)),
\]

where \( \iota_i : U_i \to X \) is the inclusion.

2. If \( X = \overline{X} - D \xrightarrow{j} \overline{X} \) with \( \overline{X} \) smooth and proper and \( D \) a simple-normal crossing divisor in \( \overline{X} \), then the image of \( \hat{c}^{SM}(X) \) in \( CH_*(\overline{X}) \) is given by the top Chern class of the dual of the sheaf of differential forms with logarithmic poles along \( D \) ([Del70, Def. 3.1]):

\[
(3.2.2) \quad j_* \hat{c}^{SM}(X) = c(\Omega^{\log}_X/\log D) \in \hat{C}_0(H_*(\overline{X}) = CH_*(\overline{X}).
\]

3.3. Consider the 0-dimensional part \( \hat{c}^{SM}_0(X) \in \hat{C}_0(H_0(X)) \) of the pro-CSM class \( \hat{c}^{SM}(X) \). We now prove the main theorem of the paper:

**Theorem 3.4.** For every \( k \)-scheme \( X \), the class \( \hat{c}^{SM}_0(X) \) agrees with the compactly supported pro-characteristic class \( \hat{C}_X^c(\mathbb{1}_X) \) (Definition 2.20) in \( \hat{C}_0(H_0(X)) \):

\[
(3.4.1) \quad \hat{c}^{SM}_0(X) = \hat{C}_X^c(\mathbb{1}_X)
\]

via the isomorphism \( \hat{C}_0(H_0(X)) \simeq \hat{H}^{BM}_0(X/k) \) in \( (2.11.2) \).

**Proof.** The class \( \hat{C}_X^c(\mathbb{1}_X) \) also satisfies the additivity for stratifications by \( (2.2.2) \) and the localization triangle \( (2.3.1) \). By Aluffi’s Theorem 3.2, it suffices to show that if \( X = \overline{X} - D \xrightarrow{j} \overline{X} \) with \( \overline{X} \) smooth and proper of dimension \( n \) and \( D \) a simple-normal crossing divisor in \( \overline{X} \), then one has

\[
(3.4.2) \quad C_{\overline{X}}(j_! \mathbb{1}_X) = c_n(\Omega^{\log}_X/\log D) \in CH_0(\overline{X}).
\]
We use the computation of the \( c_n(\Omega^1_X(k \log D)^\vee) \) as in the proof of [Sil96, Thm. 3.1]. Let \( D_1, \ldots, D_m \) be the irreducible components of \( D \). We have a short exact sequence of sheaves ([Del70, 3.3.2.2])

\[
0 \to \Omega^1_X(k) \to \Omega^1_X(k \log D) \to \bigoplus_{i=1}^m \mathcal{O}_{D_i} \to 0
\]

from which we deduce an equality of total Chern classes

\[
c(\Omega^1_X(k \log D)) = c(\Omega^1_X(k)) \prod_{i=1}^m c(\mathcal{O}_{D_i}).
\]

Taking the degree \( n \) part of (3.4.4) gives us

\[
c_n(\Omega^1_X(k \log D)) = c_n(\Omega^1_X(k)) + \sum_{i=1}^n \sum_{j_1, \ldots, j_m = i} c_{n-i}(\Omega^1_X(k)) c_{j_1}(\mathcal{O}_{D_1}) \cdots c_{j_m}(\mathcal{O}_{D_m}).
\]

From the short exact sequence

\[
0 \to \mathcal{O}_X(-D_i) \to \mathcal{O}_X \to \mathcal{O}_{D_i} \to 0
\]

we deduce

\[
1 = c(\mathcal{O}_X) = c(\mathcal{O}_{D_i}) c(\mathcal{O}_X(-D_i)) = c(\mathcal{O}_{D_i})(1 + c_1(-D_i)),
\]

and therefore \( c_j(\mathcal{O}_{D_i}) = c_1(D_i)^j \). Since taking the dual of a bundle changes its \( s \)-th Chern class by a sign \((-1)^s\), we obtain

\[
(-1)^n c_n(\Omega^1_X(k \log D)) = (-1)^n c_n(\Omega^1_X(k)) + \sum_{i=1}^n \sum_{j_1, \ldots, j_m = i} c_{n-i}(\Omega^1_X(k)) c_{j_1}(\mathcal{O}_{D_1}) \cdots c_{j_m}(\mathcal{O}_{D_m}).
\]

On the other hand, denoting by \( \iota : D \to \mathcal{X} \) the inclusion, the localization triangle

\[
\mathcal{X} \to \mathcal{X} \to \mathcal{O}_D \to \mathcal{O}_D
\]

with 2.17, (2.15.1) and (2.16.1) show that

\[
C_{\mathcal{X}}(\iota) = C_{\mathcal{X}}(\mathcal{X}) - C_{\mathcal{X}}(\mathcal{X} \to \mathcal{X} \to \mathcal{O}_D) = c_n(T_{\mathcal{X}/k}) - \iota_* C_D(\mathcal{X}).
\]

Comparing (3.4.8) and (3.4.10), we are reduced to show that

\[
-\iota_* C_D(\mathcal{X}) = \sum_{i=1}^n \sum_{j_1, \ldots, j_m = i} c_{n-i}(T_{\mathcal{X}/k}) c_1(-D_1)^{j_1} \cdots c_1(-D_m)^{j_m}.
\]

For each \( i \), we have the short exact sequence

\[
0 \to N_{D_i} \mathcal{X} \to T_{\mathcal{X}/D_i} \to T_{D_i} \to 0
\]

and the equivalence \( N_{D_i} \mathcal{X} \simeq \mathcal{O}_X(D_i)|_{D_i} \), we deduce

\[
c(T_{\mathcal{X}/D_i}) = c(N_{D_i} \mathcal{X}) c(T_{D_i}) = c(\mathcal{O}_X(D_i)|_{D_i}, c(T_{D_i})) = (1 + c_1(D_i)) c(T_{D_i})
\]
and consequently

\[(3.4.14) \quad c_d(T_X)_{|D_i} = c_d(T_{D_i}) - c_{d-1}(T_{D_i})c_1(-D_i)_{|D_i}.\]

We now prove (3.4.11) by induction on the number of irreducible components \(m\).

- If \(m = 1\), \(D = D_1\) is smooth, in which case we have

\[
\sum_{l=1}^{n} c_{n-l}(T_{X/k})c_1(-D_1)_{|D_1}^{l-1} = \sum_{l=1}^{n} c_{n-l-1}(T_{D_1})c_1(-D_1)_{|D_1}^l - \sum_{l=1}^{n-1} c_{n-l-1}(T_{D_1})c_1(-D_1)_{|D_1}^l
\]

\[(3.4.15) = \sum_{l=0}^{n-1} c_{n-l-1}(T_{D_1})c_1(-D_1)_{|D_1}^l - \sum_{l=1}^{n-1} c_{n-l-1}(T_{D_1})c_1(-D_1)_{|D_1}^l = c_{n-1}(T_{D_1}) \quad \text{by (2.16.1)} \]

which implies (3.4.11) using the projection formula \(\iota_\ast \iota^* \alpha = c_1(D) \cap \alpha \) ([Ful98, Prop. 2.6 (c)])

- Assume that (3.4.11) holds when \(D\) has \(m - 1\) branches. We first split the right-hand side of (3.4.11) into two sums, according to whether the last index \(j_m\) is 0 or not:

\[
\sum_{l=1}^{n} \sum_{j_1 + \cdots + j_m = l} c_{n-l}(T_{X/k})c_1(-D_1)^{j_1} \cdots c_1(-D_m)^{j_m}
\]

\[(3.4.16) = \sum_{l=1}^{n} \sum_{j_1 + \cdots + j_{m-1} = l} c_{n-l}(T_{X/k})c_1(-D_1)^{j_1} \cdots c_1(-D_{m-1})^{j_{m-1}}
\]

\[+ \sum_{l=1}^{n} \sum_{j_1 + \cdots + j_m = l \atop j_m \geq 1} c_{n-l}(T_{X/k})c_1(-D_1)^{j_1} \cdots c_1(-D_m)^{j_m}.
\]

Let \(E_m = D_1 \cup \cdots \cup D_{m-1}\) with \(\phi_m : E_m \to X\) the inclusion. By induction hypothesis, the first term on the right-hand side of (3.4.16) can be rewritten as

\[
(3.4.17) \quad \sum_{l=1}^{n} \sum_{j_1 + \cdots + j_{m-1} = l} c_{n-l}(T_{X/k})c_1(-D_1)^{j_1} \cdots c_1(-D_{m-1})^{j_{m-1}} = -\phi_m \ast C_{E_m}(\mathbb{1}_{E_m}).
\]
Concerning the second term on the right-hand side of (3.4.16), we proceed as in the case \( m = 1 \):

\[
\sum_{l=1}^{n} \sum_{j_1 + \ldots + j_m = l} c_{n-l}(T_{\mathcal{X}/k})|_{D_m} c_1(-D_1)|_{D_m}^{j_1} \cdots c_1(-D_{m-1})|_{D_m}^{j_{m-1}}
\]

(3.4.14)

\[
\sum_{l=1}^{n} \sum_{j_1 + \ldots + j_m = l} c_{n-l-1}(T_{D_m/k}) c_1(-D_1)|_{D_m}^{j_1} \cdots c_1(-D_{m-1})|_{D_m}^{j_{m-1}}
\]

(3.4.18)

\[
\sum_{l=0}^{n-1} \sum_{j_1 + \ldots + j_m = l} c_{n-l-1}(T_{D_m/k}) c_1(-D_1)|_{D_m}^{j_1} \cdots c_1(-D_{m-1})|_{D_m}^{j_{m-1}}
\]

where in the last equality we use the induction hypothesis again, with \( \psi_m : E_m \cap D_m \to D_m \) is the inclusion. Denote by \( t_m : D_m \to \mathcal{X} \) the inclusion. Combining (3.4.16), (3.4.17) and (3.4.18), we are reduced to show the following equality between characteristic classes:

(3.4.19) \[ \psi_m^* C_{E_m \cap D_m} (\mathbb{1}_{E_m \cap D_m}) = C_{D_m} (\mathbb{1}_{D_m}) - \phi_m^* C_{E_m} (\mathbb{1}_{E_m}) \]

The formula (3.4.19) follows from 2.17, the localization triangle (2.3.1) and a standard Mayer-Vietoris argument, which finishes the proof.

\[ \square \]

3.5. By Theorem 3.4 and Example 2.12, for \( \mathbb{E} = H_{MWZ} \), our compactly supported pro-characteristic class

(3.5.1) \[ \hat{C}^\chi_X (\mathbb{1}_X) \in \tilde{H}_{MWZ}^{BM} (X/k) \]

provides a quadratic refinement of the class \( \hat{c}_0^{SM} (X) \in \tilde{C}H_0(X) \).

**Remark 3.6.** M. Levine communicated to us the following remark: if the motivic spectrum \( \mathbb{E} \) is not oriented (for example \( \mathbb{E} = H_{MWZ} \)), then the direct analog of (3.4.2) between \( C_X (\mathbb{1}_{\mathcal{X}}) \) and the Euler class \( e(\Omega^1_{\mathcal{X}/k} (\log D)^\vee) \) fails to hold in general. This can already be seen at the level of the categorical Euler characteristic: for example, for the compactification \( \mathbb{G}_m \to \mathbb{P}^1 \), we have an isomorphism of sheaves \( \Omega^1_{\mathbb{P}^1} (\log D) = \mathcal{O}_{\mathbb{P}^1} \), while the homological Euler characteristic and the compactly supported Euler characteristic of \( \mathbb{G}_m \) can be computed respectively: \( \chi (\mathbb{G}_m) = 1 - \langle -1 \rangle \), \( \chi^c (\mathbb{G}_m) = \langle -1 \rangle - 1 \) (this can be seen, for example, from the fact that the motive of \( \mathbb{G}_m \) is isomorphic to \( 1 \oplus 1(1)[1] \), see [JY21, 5.1.12]).
REFERENCES

[AS07] A. Abbes, T. Saito, The characteristic class and ramification of an l-adic étale sheaf, Invent. Math. 168 (2007), no. 3, 567-612.  

[Alu06] P. Aluffi, Limits of Chow groups, and a new construction of Chern-Schwartz-MacPherson classes, Pure Appl. Math. Q. 2 (2006), no. 4, Special Issue: In honor of Robert D. MacPherson. Part 2, 915-941.  

[Ayo07] J. Ayoub, Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique, Astérisque No. 314-315 (2007).  

[Azo22] R. Azouri, Motivic characteristic classes for singular spaces, arXiv:2208.14440.  

[BD17] M. Bondarko, F. Déglise, Dimensional homotopy t-structures in motivic homotopy theory, Adv. Math. 311 (2017), 91-189.  

[Con07] B. Conrad, Deligne’s notes on Nagata compactifications, J. Ramanujan Math. Soc. 22 (2007), no. 3, 205-257.  

[DF20] F. Déglise, J. Fasel, The Milnor-Witt motivic ring spectrum and its associated theories, in Milnor-Witt Motives, to appear in Mem. Am. Math. Soc.  

[DFJK21] F. Déglise, J. Fasel, F. Jin, A. Khan, On the rational motivic homotopy category, J. Ec. Polytech. Math. 8 (2021), 533-583.  

[DJK18] F. Déglise, F. Jin, A. Khan, Fundamental classes in motivic homotopy theory, J. Eur. Math. Soc. 23 (2021), no. 12, 3935-3993.  

[Del70] P. Deligne, Équations différentielles à points singuliers réguliers, Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin-New York, 1970.  

[EK20] E. Elmanto, A. Khan, Perfection in motivic homotopy theory, Proc. Lond. Math. Soc. 120 (2020), no. 1, 28-38.  

[Fu98] W. Fulton, Intersection theory. Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 2. Springer-Verlag, Berlin, 1998.  

[G-S81] G. González-Sprinberg, L’obstruction locale d’Euler et le théorème de MacPherson, in Caractéristique d’Euler-Poincaré: Séméinaire E. N. S. 1978-1979 (Astérisque 82-83), 7-32. Soc. Math. France, Paris, 1981.  

[Gro58] A. Grothendieck, La théorie des classes de Chern, Bull. Soc. Math. France 86 (1958), 137-154.  

[Ill15] L. Illusie, From Pierre Deligne’s secret garden: looking back at some of his letters, Jpn. J. Math. (3) 10, No. 2, 237-248 (2015).  

[Jin16] F. Jin, Borel-Moore motivic homology and weight structure on mixed motives, Math. Z. 283 (2016), no. 3-4, 1149-1183.  

[JJY21] F. Jin, E. Yang, Küneth formulas for motives and additivity of traces, Adv. Math. 376 (2021), Article ID 107446.  

[K-S90] M. Kashiwara, P. Schapira, Sheaves on manifolds, With a chapter in French by Christian Houzel. Grundlehren der Mathematischen Wissenschaften 292. Springer-Verlag, Berlin, 1990.  

[LMS86] L. G. Lewis, J. P. May, M. Steinberger, Equivariant stable homotopy theory, With contributions by J. E. McClure. Lecture Notes in Mathematics, 1213. Springer-Verlag, Berlin, 1986.  

[LZ22] Q. Lu, W. Zheng, Categorical traces and a relative Lefschetz–Verdier formula, Forum of Mathematics, Sigma, 10, E10, 2022.  

[Mac74] R. MacPherson, Chern classes for singular algebraic varieties, Ann. of Math. (2) 100 (1974), 423-432.  

[Ols16] M. Olsson, Motivic cohomology, localized Chern classes, and local terms, Manuscripta Math. 149 (2016), no. 1-2, 1-43.  

[SGA4] M. Artin, A. Grothendieck, J.-L. Verdier, Théorie des topos et cohomologie étale des schémas, in: Séminaire de Géométrie Algèbrique du Bois-Marie 1963–1964 (SGA 4). Dirigé par M. Artin, A.Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, in: Lecture Notes in Mathematics, vol. 269, 270, 305, Springer-Verlag, Berlin-New York, 1972-1973.  

[SGA5] A. Grothendieck, Cohomologie l-adique et fonctions L, Séminaire de géométrie algèbrique du Bois-Marie 1965-66 (SGA 5). Avec la collaboration de I. Bucur, C. Houzel, L. Illusie, J.-P. Jouanolou, et J.-P. Serre. Springer Lecture Notes, Vol. 589. Springer-Verlag, Berlin-New York, 1977.  

[Sil96] R. Silvotti, On a conjecture of Varchenko, Invent. Math. 126 (1996), no. 2, 235-248.  

[Spi18] M. Spitzweck, A commutative $\mathbb{P}^1$-spectrum representing motivic cohomology over Dedekind domains, Mém. Soc. Math. Fr. (N.S.) No. 157 (2018), 110 pp.  

[Stacks] The Stacks project, available at https://stacks.math.columbia.edu/, 2020.
School of Mathematical Sciences, Tongji University, Siping Road 1239, 200092 Shanghai, China
Email address: fangzhoujin@tongji.edu.cn
URL: https://fangzhoujin.github.io/

School of Mathematics, Hunan University, 410082 Changsha, China
Email address: sunpeng@hnu.edu.cn
URL: http://math.hnu.edu.cn/info/1028/3587.htm

School of Mathematical Sciences, Peking University, No.5 Yiheyuan Road Haidian District,, Beijing 100871, P.R.China
Email address: yangenlin@math.pku.edu.cn
URL: https://www.math.pku.edu.cn/teachers/yangenlin/ely.htm