LEVI-CIVITA CONNECTIONS OF FLAG MANIFOLDS

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Abstract. For any flag manifold $G/T$ we obtain an explicit expression of its Levi-Civita connection with respect to any invariant Riemannian metric.

1. Introduction

Let $G/T$ be a flag manifold, where $T$ is a maximal torus of a compact semi-simple Lie group $G$. In this case we obtain an explicit formula of its Levi-Civita connection (in terms of the root decomposition for the Lie algebra $\mathfrak{g}$ of $G$) with respect to any invariant Riemannian metric. It is possible to realize this formula, for example, in the case of any classical simple Lie group $G$. In this paper it is done for $SU(n)$.

This result may prove useful in solving different problems. For instance, it enables us to determine whether a given metric $f$-structure $(f,g)$ on $G/T$ belongs to the main classes of generalized Hermitian geometry (see, for example, [2] and [1]).

2. Levi-Civita connections of flag manifolds

In this paper we consider a flag manifold $G/T$, where $T$ is a maximal torus of a compact semi-simple Lie group $G$. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the corresponding Lie algebras of $G$ and $T$. $G/T$ is a reductive homogeneous space, its reductive decomposition being $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is an orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ with respect to the Killing form $B$ of $\mathfrak{g}$. Denote by $\mathfrak{g}^C$ and $\mathfrak{t}^C$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$. Then $\mathfrak{t}^C$ is a Cartan subalgebra of $\mathfrak{g}^C$ and we denote by $R$ the root system of $\mathfrak{g}^C$ with respect to $\mathfrak{t}^C$. In this way we have the root decomposition

$$\mathfrak{g}^C = \mathfrak{t}^C \oplus \sum_{\alpha \in R} \mathfrak{g}^\alpha. \tag{1}$$

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Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ be a basis of $R$. Denote by $R^+$ the set of all positive roots and by $R^-$ the set of all negative roots. In this paper the following notation will be used:

$$|\alpha| = \begin{cases} \alpha, & \text{if } \alpha \in R^+, \\ -\alpha, & \text{if } \alpha \in R^- \end{cases}$$

Recall that we can consider the lexicographic order on $R$: $\gamma = \sum_{i=1}^n \gamma_i \alpha_i$ is said to be greater than $\delta = \sum_{i=1}^n \delta_i \alpha_i$ ($\gamma > \delta$) if the first nonzero coefficient $\gamma_k - \delta_k$ in the decomposition $\gamma - \delta = \sum_{i=1}^n (\gamma_i - \delta_i) \alpha_i$ is positive. If $\gamma - \delta \in R$ then $\gamma > \delta$ if and only if $\gamma - \delta \in R^+$.

It is well-known that in the case under consideration the reductive complement $\mathfrak{m}$ can be decomposed into the direct sum of 2-dimensional $\text{Ad}(T)$-modules $\mathfrak{m}^\alpha$ which are mutually non-equivalent:

$$\mathfrak{m} = \sum_{\alpha \in R^+} \mathfrak{m}^\alpha, \text{ where } \mathfrak{m}^\alpha = \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}.$$ 

Therefore, any invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$ on $G/T$ is given by

$$g = \langle \cdot, \cdot \rangle = \sum_{\alpha \in R^+} c_\alpha \langle \cdot, \cdot \rangle|_{\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}}, \tag{2}$$

where $c_\alpha > 0$, $\langle \cdot, \cdot \rangle$ is the negative of the Killing form $B$ of the Lie algebra $\mathfrak{g}$.

In this paper we will need the following result.

**Theorem 1.** Let $(M, g)$ be a Riemannian manifold, $M = G/H$ a reductive homogeneous space with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then the Levi-Civita connection with respect to $g$ can be expressed in the form

$$\nabla_X Y = \frac{1}{2}[X, Y]_\mathfrak{m} + U(X, Y), \tag{3}$$

where $U$ is a symmetric bilinear mapping $\mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ defined by the formula

$$2g(U(X, Y), Z) = g(X, [Z, Y]_\mathfrak{m}) + g([Z, X]_\mathfrak{m}, Y), \quad X, Y, Z \in \mathfrak{m}. \tag{4}$$

We can consider (4) as an equation of variable $U$. Let us try to solve this equation in the case of an arbitrary flag manifold $G/T$.

We begin with obtaining an important preliminary result. Consider $X_\gamma \in \mathfrak{g}^\gamma, Y_\delta \in \mathfrak{g}^\delta, \gamma, \delta \in R$. In the view of (2), (4) takes the following form:
Obviously, the right-hand side of this equation is equal to
\[
\sum_{\alpha \in R^+} c_\alpha ((X_\gamma)_{\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}}, [Z, Y_\delta]_{\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}})
+ \sum_{\alpha \in R^+} c_\alpha ([Z, X_\gamma]_{\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}}, (Y_\delta)_{\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}}).
\]

\[\text{(5)}\]

Let \( Z = \sum_{\alpha \in R} Z_\alpha \), where \( Z_\alpha = Z_{\mathfrak{g}^\alpha} \). Note that
\[
[Z, Y_\delta]_m = \left( \sum_{\alpha \in R} Z_\alpha, Y_\delta \right)_m = \sum_{\alpha \in R} [Z_\alpha, Y_\delta]_m = \sum_{\alpha, \alpha + \delta \in R} [Z_\alpha, Y_\delta],
\]
and, evidently, \([Z_\alpha, Y_\delta] = [Z, Y_\delta]_{\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}}\). It is easy to see that
\[
(X_\gamma, [Z, Y_\delta]_{\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}}) = \left( X_\gamma, \left( \sum_{\alpha, \alpha + \delta \in R} [Z_\alpha, Y_\delta] \right)_{\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}} \right).
\]

If \( \left( \sum_{\alpha, \alpha + \delta \in R} [Z_\alpha, Y_\delta] \right)_{\mathfrak{g}^{-\gamma}} \neq 0 \), then there exists such \( \alpha \in R \) that \( \alpha + \delta = -\gamma \). In other words, \( \alpha = -\gamma - \delta \in R \). Therefore,
\[
c_{|\gamma|}(X_\gamma, [Z, Y_\delta]_{\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}}) = \begin{cases} 
0, & \text{if } \gamma + \delta \notin R, \\
\left( X_\gamma, [Z_{-\gamma - \delta}, Y_\delta] \right), & \text{if } \gamma + \delta \in R.
\end{cases}
\]

Arguing as above, one can prove that
\[
c_{|\delta|}([Z, X_\gamma]_{\mathfrak{g}^\delta \oplus \mathfrak{g}^{-\delta}}, Y_\delta) = \begin{cases} 
0, & \text{if } \gamma + \delta \notin R, \\
\left( [Z_{-\delta - \gamma}, X_\gamma], Y_\delta \right), & \text{if } \gamma + \delta \in R.
\end{cases}
\]

Hence, if \( \gamma + \delta \notin R \), \( \mathbf{4} \) is transformed into
\[
2g(U(X_\gamma, Y_\delta), Z) = 0
\]
for any \( Z \in \mathfrak{m} \). Thus, if \( \gamma + \delta \notin R \), then \( U(X_\gamma, Y_\delta) = 0 \).

If \( \gamma + \delta \in R \), then \( \mathbf{5} \) is equivalent to
\[
2 \sum_{\alpha \in R^+} c_\alpha (U(X_\gamma, Y_\delta)_{\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}}, Z_{\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}})
= c_{|\gamma|}(X_\gamma, [Z_{-\gamma - \delta}, Y_\delta]) + c_{|\delta|}([Z_{-\delta - \gamma}, X_\gamma], Y_\delta).
\]
By the properties of the Killing form we obtain

\[
\sum_{\alpha \in R^+ \atop \alpha \neq [\gamma + \delta]} (2c_\alpha U(X_\gamma, Y_\delta)_\alpha \oplus g^{-\alpha}, Z_{g^\alpha \oplus g^{-\alpha}}) + (2c_{[\gamma + \delta]} U(X_\gamma, Y_\delta)_{g^{[\gamma + \delta]} \oplus g^{-[\gamma + \delta]}} - c_{[\gamma]} [Y_\delta, X_\gamma] - c_{[\delta]} [X_\gamma, Y_\delta], Z_{g^{[\gamma + \delta]} \oplus g^{-[\gamma + \delta]}}) = 0.
\]

Since \( m^\alpha \) is orthogonal to \( m^\beta \) with respect to the Killing form of \( g \) \((\alpha, \beta \in R^+, \alpha \neq \beta)\), we have

\[
(2 \sum_{\alpha \in R^+ \atop \alpha \neq [\gamma + \delta]} c_\alpha U(X_\gamma, Y_\delta)_{m^\alpha} + 2c_{[\gamma + \delta]} U(X_\gamma, Y_\delta)_{m^{[\gamma + \delta]}} - (c_{[\gamma]} - c_{[\delta]}) [Y_\delta, X_\gamma]) = 0
\]

for any \( Z \in m \). This yields that

\[
2 \sum_{\alpha \in R^+ \atop \alpha \neq [\gamma + \delta]} c_\alpha U(X_\gamma, Y_\delta)_{m^\alpha} + 2c_{[\gamma + \delta]} U(X_\gamma, Y_\delta)_{m^{[\gamma + \delta]}} - (c_{[\gamma]} - c_{[\delta]}) [Y_\delta, X_\gamma]
\]

(and, consequently, any of its projections onto \( m^\alpha, \alpha \in R^+ \)) is equal to 0. We have proved the following result.

**Lemma 1.** Let \( G/T \) be a flag manifold with the root decomposition \((7)\). Then for any \( X_\gamma \in g^\gamma, Y_\delta \in g^\delta \), where \( \gamma, \delta \in R \), we have

\[
U(X_\gamma, Y_\delta) = \begin{cases} \frac{c_{[\gamma]} - c_{[\delta]}}{2c_{[\gamma + \delta]}} [Y_\delta, X_\gamma], & \text{if } \gamma + \delta \in R, \\ 0, & \text{if } \gamma + \delta \notin R. \end{cases} \tag{6}
\]

This lemma enables us to obtain the similar expression for \( U(X, Y) \) in the case of any \( X = \sum_{\alpha \in R} X_\alpha \) and \( Y = \sum_{\beta \in R} Y_\beta \) in \( m \). As \( U \) is bilinear, application of \((6)\) gives us

\[
U(X, Y) = \sum_{\alpha, \beta \in R} U(X_\alpha, Y_\beta) = \sum_{\alpha, \beta \atop \alpha + \beta \in R} \frac{c_{[\alpha]} - c_{[\beta]}}{2c_{[\alpha + \beta]}} [Y_\beta, X_\alpha]. \tag{7}
\]

For any \( \alpha, \beta \in R \) such that \( \alpha + \beta \in R \) we group together terms with the coefficient \( \frac{c_{[\alpha]} - c_{[\beta]}}{2c_{[\alpha + \beta]}} \). In this way we obtain the sum of the following summands

\[
\frac{c_{[\alpha]} - c_{[\beta]}}{2c_{[\alpha + \beta]}} Z^\beta_\alpha,
\]

where

\[
Z^\beta_\alpha = [Y_\beta, X_\alpha] + [X_\beta, Y_\alpha] + [Y_\alpha, X_\beta] + [X_\beta, Y_\alpha], \quad \alpha, \beta \in R. \tag{8}
\]
However, $Z^\beta_\alpha = Z^{-\beta}_{-\alpha} = Z^\alpha_{-\beta} = Z^{-\alpha}_{-\beta}$, which implies that there is a need to restrict the range of $\alpha$ and $\beta$. Certainly, (7) is equivalent to

$$U(X, Y) = \frac{1}{4} \sum_{\alpha, \beta \in \mathbb{R}} \frac{c_{|\alpha|} - c_{|\beta|}}{2c_{|\alpha + \beta|}} Z^\beta_\alpha,$$

but this formula is definitely not the most convenient since there are repetitions of summands. Luckily, it is easy to establish a condition which makes it possible to select one pair of roots out of four pairs $(\alpha, \beta), (\beta, \alpha), (-\alpha, -\beta), (-\beta, -\alpha)$.

**Lemma 2.** For any $\alpha, \beta \in \mathbb{R}$ there exists only one pair $(a_1, a_2) \in \{(\alpha, \beta), (\beta, \alpha), (-\alpha, -\beta), (-\beta, -\alpha)\}$ such that $|a_1| < a_2$.

**Proof.** The condition $|a_1| < a_2$ presupposes that $a_2 \in \mathbb{R}^+$. Obviously, $|a_1| < a_2$ if and only if $-a_2 < a_1 < a_2$.

Such a pair can be chosen as follows.

Set $a_1 = \alpha$, $a_2 = \beta$. If $a_2 \in \mathbb{R}^-$, set $a_1$ equal to $-a_1$ and $a_2$ equal to $-a_2$. Thus we have $a_2 \in \mathbb{R}^+$. Now let us check if $a_1 < a_2$. If this condition is not satisfied, set $a_2$ equal to $a_1$ and $a_1$ equal to $a_2$. It remains to verify if $a_1 > -a_2$. If this is true, the desired pair $(a_1, a_2)$ is obtained, otherwise we choose $(-a_2, -a_1)$.

The uniqueness of this pair can be proved as follows. Without loss of generality, suppose that $|\alpha| < \beta$, that is, $-\beta < \alpha < \beta$. Then $(\beta, \alpha)$ satisfies $\beta > \alpha$ and for $(-\alpha, -\beta)$ we have $-\alpha > -\beta$ which means that these two pairs do not satisfy the stipulated condition. The pair $(-\beta, -\alpha)$ should satisfy $\alpha < -\beta < -\alpha$ and this contradicts the assumption made above. \qed

In the view of this lemma we have

$$U(X, Y) = \sum_{\alpha, \beta, \alpha + \beta \in \mathbb{R}, \ |\alpha| < \beta \in \mathbb{R}^+} \frac{c_{|\alpha|} - c_{|\beta|}}{2c_{|\alpha + \beta|}} Z^\beta_\alpha,$$

(9)

($Z^\beta_\alpha$ is determined by means of (8)).

Consider different cases for $\alpha, \beta \in \mathbb{R}$. $\beta$ always belongs to $\mathbb{R}^+$ and $\alpha$ can be selected from both $\mathbb{R}^+$ and $\mathbb{R}^-$.

If $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}^+$ then the conditions $\alpha + \beta \in \mathbb{R}$ and $|\alpha| < \beta$ can be replaced by the conditions $\alpha + \beta \in \mathbb{R}^+$ and $\alpha < \beta$ respectively.

If $\alpha \in \mathbb{R}^-$, $\beta \in \mathbb{R}^+$ then $|\alpha| < \beta$ is equivalent to $-\alpha < \beta$. If $\alpha + \beta \in \mathbb{R}$ then $-\alpha < \beta$ can be substituted for the condition $\alpha + \beta \in \mathbb{R}^+$. 

Therefore, the right-hand side of (9) is transformed into
\[
\sum_{\alpha,\beta,\alpha+\beta \in \mathbb{R}^+, \alpha < \beta} \frac{c_{\alpha} - c_{\beta}}{2c_{\alpha+\beta}} Z_\alpha^\beta + \sum_{\alpha,\beta,\alpha+\beta \in \mathbb{R}^+, \alpha < \beta} \frac{c_{\alpha} - c_{\beta}}{2c_{\alpha+\beta}} Z_\alpha^\beta = \sum_{\alpha,\beta,\alpha+\beta \in \mathbb{R}^+, \alpha < \beta} \frac{c_{\alpha} - c_{\beta}}{2c_{\alpha+\beta}} Z_\alpha^\beta + \sum_{\alpha,\beta,\alpha+\beta \in \mathbb{R}^+, \alpha < \beta} \frac{c_{\alpha} - c_{\beta}}{2c_{\alpha+\beta}} Z_\alpha^\beta.
\]

Thus, the following theorem is proved.

**Theorem 2.** Let \( G/T \) be a flag manifold with the root decomposition \( (\mathbb{I}) \). Then for any \( X, Y \in \mathfrak{m} \) we have
\[
U(X,Y) = \sum_{\alpha,\beta,\alpha+\beta \in \mathbb{R}^+, \alpha < \beta} \frac{c_{\alpha} - c_{\beta}}{2c_{\alpha+\beta}} Z_\alpha^\beta + \sum_{\alpha,\beta,\alpha+\beta \in \mathbb{R}^+, \alpha < \beta} \frac{c_{\alpha} - c_{\beta}}{2c_{\alpha+\beta}} Z_\alpha^\beta, \tag{10}
\]

where \( Z_\alpha^\beta = [Y_\beta, X_\alpha] + [X_\beta, Y_\alpha] + [Y_\beta, X_\alpha] + [X_\beta, Y_\alpha], \alpha, \beta \in \mathbb{R}. \)

3. Examples

As an example, let us consider the flag manifold \( G/T = SU(n+1)/T \) \((n \geq 2)\), where \( T \) is a maximal torus of \( SU(n+1) \).

In this case \( \mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C}) \). The root system of \( SU(n+1) \) with respect to \( \mathfrak{t} \) is
\[
R = A_n = \{ \varepsilon_i - \varepsilon_j \mid i \neq j, 1 \leq i, j \leq n+1 \},
\]
its basis being
\[
\{ \alpha_i = \varepsilon_i - \varepsilon_{i+1} \} \text{ for } i \leq n.
\]
The set of all positive roots in this case is
\[
R^+ = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n \}.
\]

An arbitrary positive root \( \alpha = \varepsilon_i - \varepsilon_j \), where \( i < j \), is decomposed into the sum of basis vectors as follows:
\[
\alpha = \varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j.
\]
It is easy to see that \( \alpha = \varepsilon_i - \varepsilon_j < \beta = \varepsilon_k - \varepsilon_l \) \((\alpha, \beta \in R^+)\) if and only if \( i > k \).

Take \( \alpha = \varepsilon_i - \varepsilon_j, \beta = \varepsilon_k - \varepsilon_l \in R^+ \), where \( i < j, k < l \).
\[
\alpha + \beta \in R^+ \text{ if and only if either } i < j = k < l \text{ (hence } \alpha + \beta = \varepsilon_i - \varepsilon_l \text{), or } k < i = l < j \text{ (hence } \alpha + \beta = \varepsilon_k - \varepsilon_j \text{). Note that in the first case } \alpha > \beta \text{ and in the second case } \beta > \alpha.
\]
\[
\beta - \alpha \in R^+ \text{ if and only if either } i = k < j < l \text{ (hence } \beta - \alpha = \varepsilon_j - \varepsilon_i \text{), or } k < i < j = l \text{ (hence } \beta - \alpha = \varepsilon_k - \varepsilon_i \).
It is not difficult to show that $Z_\alpha^\beta = [X_\alpha^\beta, Y_{\alpha'}] + [Y_\alpha^\beta, X_{\alpha'}]$ for any $\alpha, \beta \in R^+$.

Therefore, in the case of $SU(n+1)/T_{\text{max}}$ ($n \geq 2$) (10) takes form

$$U(X, Y) = \sum_{1 \leq i < j < k \leq n+1} \frac{c_{\varepsilon_j - \varepsilon_k} - c_{\varepsilon_i - \varepsilon_j}}{2c_{\varepsilon_i - \varepsilon_k}} ([X^\varepsilon_i - \varepsilon_j, Y^\varepsilon_j - \varepsilon_k] + [Y^\varepsilon_i - \varepsilon_j, X^\varepsilon_j - \varepsilon_k])$$

$$+ \sum_{1 \leq i < j < k \leq n+1} \frac{c_{\varepsilon_i - \varepsilon_j} - c_{\varepsilon_i - \varepsilon_k}}{2c_{\varepsilon_j - \varepsilon_k}} ([X^\varepsilon_i - \varepsilon_k, Y^\varepsilon_i - \varepsilon_j] + [Y^\varepsilon_i - \varepsilon_k, X^\varepsilon_i - \varepsilon_j])$$

$$+ \sum_{1 \leq i < j < k \leq n+1} \frac{c_{\varepsilon_j - \varepsilon_k} - c_{\varepsilon_i - \varepsilon_k}}{2c_{\varepsilon_i - \varepsilon_j}} ([X^\varepsilon_j - \varepsilon_k, Y^\varepsilon_j - \varepsilon_i] + [Y^\varepsilon_j - \varepsilon_k, X^\varepsilon_j - \varepsilon_i]).$$

(11)

As a particular case, let us consider the flag manifold $SU(3)/T_{\text{max}}$. The set of all positive roots is

$R^+ = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_1 - \varepsilon_3, \alpha_3 = \varepsilon_2 - \varepsilon_3\}$.

In order to obtain a more compact formula denote $c_{\alpha_i}$ by $c_i$ and $m^{\alpha_i}$ by $m_i$. We also agree to write $X_i$ instead of $X_m$.

Therefore, in the case of $SU(3)/T_{\text{max}}$, using the notations introduced above, we can rewrite (11) as follows:

$$U(X, Y) = \frac{c_3 - c_2}{2c_1} ([X_2, Y_3] + [Y_2, X_3])$$

$$+ \frac{c_3 - c_1}{2c_2} ([X_1, Y_3] + [Y_1, X_3]) + \frac{c_2 - c_1}{2c_3} ([X_1, Y_2] + [Y_1, X_2]).$$

Actually, this result is well-known (see, for example, [4]).
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