Design of Optimal Guidance Laws with Multi-Constraints Considering Time-Varying Parameters

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This paper presents a design method for optimal guidance laws (OGLs) with multi-constraints when time-varying parameters are present in kinematic equations. This method uses cross-iteration to obtain time-varying parameter values and the piecewise function technology to solve the two-point boundary value problem. The proposed OGL can optimize the combination of landing angle, miss distance and control energy consumption. Ballistic simulations are conducted, and the landing angle using the OGL is twice more than using the proportional navigation law. The strike accuracy and damage effects increase because of the steep terminal trajectory. The time-varying parameters have no limit and require no simplification. Therefore, the proposed design method has a wide range of applications.

Key Words: Optimal Guidance Law, Time-Varying Parameter, Cross-Iteration, Landing Angle, Multi-Constraints

1. Introduction

The design of guidance laws against ground stationary targets usually considers three factors: miss distance, landing angle and control energy consumption. Miss distance expresses the accuracy in striking the target. Control energy consumption refers to the control ability of guided munitions and the decrease in flight speed. Landing angle relates to the damage effect and penetration capacity of guided munitions. Specifically, landing angle has important significance for satellite-guided munitions. The vertical measurement error of GPS is larger than its horizontal error.\textsuperscript{1,2} To reduce its effect on guidance accuracy, a near-vertical descent is required in the terminal guidance phase. The terminal flight-path angle is expected to be close, but not equal, to \(-90^\circ\) because the volume and cost constraints limit the control capability of guided munitions. Optimizing the combination of landing angle, miss distance and control energy consumption requires research on the optimal guidance law (OGL) with multi-constraints. In addition, OGLs serve as basis for the design of robust OGLs\textsuperscript{3,4} and neural network OGLs.\textsuperscript{5,6}

To design OGLs, kinematic equations should first be established. Without considering the landing angle constraint, kinematic equations can be expressed as the form without time-varying parameters.\textsuperscript{7–9} The OGL can be obtained after analytically solving the two-point boundary value problem. When the landing angle constraint is considered, the kinematic equations would contain time-varying parameters. Guidance laws should be expressed as the functional form of state variables before they can be used. Therefore, general numerical methods cannot be applied when designing OGLs.

If time-varying parameters can be replaced with constants, OGLs could be analytically solved. We present an OGL design method based on piecewise function technology when the time-varying parameters cannot be replaced with constants, but can be replaced with the function of time or time-to-go \(t_{go}\).\textsuperscript{10} In Ref. 10), the ratio of the flight speed change rate and flight speed is very small and can be replaced with zero. The remaining time-varying parameters are replaced with the function of \(t_{go}\).

When time-varying parameters cannot be replaced with constants or the functions of time or \(t_{go}\), OGLs cannot be designed even when using the above method. Solving this problem requires new methods. In this paper, we present the design method for OGLs with multi-constraints that do not require any restrictions and simplification for time-varying parameters in kinematic equations.

2. Kinematic Equations

Figure 1 depicts the homing guidance geometry. \(M\) and \(T\) denote the guided munition and target, respectively, and are regarded as particles that move in a vertical plane. Point \(M\), \(MF\) and \(r = MT\) are the pole, polar axis and munition-target distance, respectively. \(V\) is the munition velocity and \(\theta\) is its flight-path angle. Here, the guided munitions are used against stationary targets; thus, the target velocity is zero. The polar equations of motion in this homing problem are expressed as

\[
\dot{r} = -V \cos(q - \theta)
\]

\[
\dot{\theta} = V \sin(q - \theta)
\]

Differentiating Eq. (2) with respect to time and substituting it with Eq. (1), we obtain
Let $x_1(t) = q - \theta_j$, $x_2(t) = \dot{x}_1(t) = \ddot{q}$ and $u(t) = \dot{\theta}$, $\dot{\theta}_f$ is the expected flight-path angle in landing moment $t_f$, $\dot{\theta}_j = \theta(t_j)$, and the expression of $u(t)$ is the guidance law that we wish to obtain. Therefore, from Eq. (3), we have

$$
\dot{x}_1(t) = x_2(t)
$$  
(4)

$$
\dot{x}_2(t) = \left(\frac{V}{r} - 2\frac{\dot{r}}{r}\right)x_2(t) + \frac{\dot{r}}{r}u(t)
$$  
(5)

Based on Eqs. (4) and (5), the kinematic equations can be written as

$$
\dot{X}(t) = A(t)X(t) + B(t)u(t)
$$  
(6)

where

$$
X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} 0 & 1 \\ \frac{V}{\dot{r}} - 2\frac{\dot{r}}{r} & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ \dot{r} \end{bmatrix}. 
$$

Initial conditions are $t = t_0$, $x_1(t_0) = q(t_0) - \theta_j$, and $x_2(t_0) = \dot{q}(t_0)$.

3. OGL Design

In the guided process, the control energy consumption should be as small as possible, the terminal trajectory should be as steep as possible and the landing angle should be close to 90°. Therefore, the performance index can be defined as follows

$$
J = \frac{1}{2}X^T(t_f)CX(t_f) + \frac{1}{2}\int_{t_0}^{t_f} R(t)u^2(t)dt
$$  
(7)

where

$$
C = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}, \quad R(t) = [c_3].
$$

c_1, c_2 and $c_3$ will be discussed below. A variety of calculation methods are available for $t_f$ and the most frequently used is

$$
t_f \equiv t_0 - \frac{r(t_f)}{\dot{r}(t_f)}
$$

Based on the optimal control theory, the OGL for this system is

$$
u(t) = K(t)X(t)
$$  
(8)

where

$$
K(t) = -R^{-1}(t)B^T(t)Y(t)W^{-1}(t)
$$  
(9)

Here, $K(t)$ is the guidance law coefficient matrix that would be obtained by the following solution procedure. $Y(t)$ and $W(t)$ satisfy the differential Eqs. (10) and (11) as follows

$$
\dot{W}(t) = A(t)W(t) - B(t)R^{-1}(t)B^T(t)Y(t)
$$  
(10)

$$
\dot{Y}(t) = -Q(t)W(t) - A^T(t)Y(t)
$$  
(11)

where

$$
Q(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

and the terminal conditions of $W(t)$ and $Y(t)$ are

$$
W(t_f) = I \quad (12)
$$

$$
Y(t_f) = C
$$  
(13)

where $I$ is the second order unit matrix. Let

$$
Z(t) = \begin{bmatrix} W(t) \\ Y(t) \end{bmatrix}
$$

$$
F(t) = \begin{bmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -Q(t) & -A^T(t) \end{bmatrix}
$$

$Z(t)$ can be calculated using the following recursive formula.

$$
Z_{m-1} = \left(I + \frac{1}{2}hF_m\right)^{-1}Z(t_f)
$$  
(16)

$$
Z_{k-1} = \left(I + \frac{1}{2}hF_k\right)^{-1}\left(I - \frac{1}{2}hF_k\right)Z_k
$$  
(17)

Here, $Z_k$ is the block pulse function coefficient of $Z(t)$, $k = 2, 3, \cdots, m$, $m$ is the number of block pulse functions and $h$ is the block pulse width. $F_k$ is the mean value of $F(t)$ in the interval $[(k-1)h, kh]$. From Eq. (14),

$$
W_k(i, j) = Z_k(i, j)
$$

$$
Y_k(i, j) = Z_k(2 + i, j)
$$

Based on Eq. (9),

$$
K(t) = \begin{bmatrix} k_1(t) & k_2(t) \end{bmatrix} \equiv -\sum_{k=1}^{m} R_k^{-1}B_k^T Y_k W_k^{-1} \phi_k(t)
$$  
(18)

where $\phi_k(t)$ is a block pulse function, $k = 1, 2, \cdots, m$, $R_k$ is the mean value of $R(t)$ in the interval $[(k-1)h, kh]$ and $B_k$ is the mean value of $B(t)$ in the interval $[(k-1)h, kh]$. In Eq. (7), $c_1$ is the weight of the difference between the line-of-sight angle and the expected terminal flight-path angle, $c_2$ is the weight of the line-of-sight angle rate and
c_3 is the weight of the control energy consumption. Considering the constraints of the landing angle, miss distance and control energy consumption, c_1, c_2 and c_3 should all assume non-zero values. These weights are determined depending on the strength of each constraint.

Proportional navigation law (PNL) is the type of OGL that only considers the constraints of miss distance and control energy, without regard for that of the landing angle; i.e., c_1 = 0, c_2 \neq 0 and c_3 \neq 0, in Eq. (7). We can gradually increase the value of c_1 to similarly increase the strength of the landing angle constraint. When c_1 assumes a small value, the landing angle constraint is weak. Using the OGL (A) designed with the small c_1, the landing angle slightly increases, and the trajectory slightly deviates from that using PNL. If c_1 is sufficiently small, the two trajectories will be close enough. The time-varying parameter values required by the OGL (A) design can be obtained by simulation using PNL. Then, we slightly increase the value of c_1 to design OGL (B). The simulation using OGL (A) can be used to obtain the time-varying parameter values required by the OGL (B) design. Finally, we can increase c_1 to 1, and the OGL can be designed to meet the requirements of the landing angle, miss distance and control energy consumption. The OGL design is considered complete when two adjacent guidance law coefficient curves coincide. Figure 2 depicts the design process, and section 4 describes the simulation.

4. Simulation Results

In a ground coordinate system, let the initial velocity of the guided munition be 800 m/s, and its initial flight-path angle \( \theta_0 \) be 30°. The initial coordinates of the guided munition are (0, 0), and the target coordinates are (20000, 0).

Figure 2 depicts the design process, and section 4 describes the simulation.

Figure 3. \( k_1 \) history vs. \( t_{go} \).

Figure 4. \( k_2 \) history vs. \( t_{go} \).
mate the expected optimal trajectory. Lines 7 and 8 in Fig. 5 coincide, and line 8 in Figs. 3 and 4 can be considered as the final OGL coefficient.

In Fig. 6, line 0 is the flight-path angle history using PNL, and lines 1 to 8 are the flight-path angle histories using lines 1 to 8 in Figs. 3 and 4, respectively. The absolute values of the end of the lines are the landing angles. The landing angles increase as the iteration number increases.

Figure 7 depicts the performance index histories based on Eq. (7), in which $c_1$, $c_2$ and $c_3$ all have the value of one, using the OGL coefficients in Figs. 3 and 4. As $c_1$ increases, the performance index values decrease. In the 8th iteration, performance indexes reach the extreme value, indicating that the guidance law is optimal.

Table 1 indicates the landing angles, miss distances and performance index values using each guidance law. As $c_1$ increases, the landing angles likewise increase, and the optimal landing angle is twice more than that using PNL. As the steepness of the terminal trajectories increase, the firing accuracy improves. The performance index values indicate the approximation process of the final OGL.

5. Conclusions

This paper presented a design method for OGLs with multi-constraints when time-varying parameters are present in kinematic equations. The proposed method does not limit the time-varying parameters nor require their simplification. Therefore, this method has a very wide range of applications. The designed OGL was able to satisfy the constraints of landing angle, miss distance and control energy consumption. The landing angle using the OGL was twice more than that obtained when using PNL. As the landing angles increased, the miss distances decreased, indicating that strike accuracy and damage effects increased. In the iterative design process, the performance index value continuously decreased and eventually reached the minimum value, indicating that the obtained guidance law was optimal.

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