Dissipative quantum backflow

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Dissipative backflow is studied in the context of open quantum systems. This theoretical analysis is carried out within two frameworks, the effective time-dependent Hamiltonian due to Caldirola-Kanai and the Caldeira-Leggett one where a master equation is used to describe the reduced density matrix in presence of dissipation and temperature of the environment. Two examples are considered, the free evolution of a superposition of two Gaussian wave packets and evolution under a constant field. Backflow is showed to be reduced with dissipation and temperature but never suppressed. The classical limit of backflow is also analyzed within the context of the classical Schrödinger equation and showed that it can be also observed. Backflow is also analyzed as an eigenvalue problem in the Caldirola-Kanai framework. In the free propagation case, eigenvalues are independent on mass, Planck constant, friction and duration of the backflow but, in the constant force case, eigenvalues depend on a factor which itself is a combination of all of them as well as the force constant.

Keywords: Backflow, Open quantum systems, Dissipation, Caldirola-Kanai model, Caldeira-Leggett master equation

I. INTRODUCTION

The so-called quantum backflow is a very interesting nonclassical effect which is quite counter-intuitive. It happens when a free particle described by a one-dimensional wave function located in the negative axis of the coordinate, and consisting entirely of positive momenta, displays a non-decreasing probability of remaining in the negative region during certain period of times. Since the first recognition by Allcock \cite{1} in 1969 when studying arrival times in quantum mechanics, not too much attention has been paid in the literature. The first systematic study of this effect is due to Bracken and Melloy \cite{2} emphasizing that quantum backflow merely reflects the structure of the Schrödinger equation. These authors showed that the highest probability which can flow back from positive to negatives values of the coordinate is around 0.04 for a superposition of two wave planes with positive momenta. Then they showed that the maximum amount of probability backflow that can occur in general over any finite time interval is about this value to be independent on the time interval, mass of the particle and Planck constant. Thus, a new dimensionless quantum number was then reported. Bracken and Melloy \cite{3} then studied the effect in the presence of a constant field and also relativistic particles obeying Dirac equation. They also showed that the probability flow can be regarded as an eigenvalue problem of the flux operator. Optimization numerical problems were reported by Penz et al. \cite{4}.

Quantum backflow is also connected to interference and quantum arrival times \cite{5}. This effect has been also related to superoscillations \cite{6} and weak values \cite{7}. More recently, Yearsley et al. \cite{8,9} have analyzed the classical limit and discussed some specific measurement models. Backflow can not take place for a single Gaussian wave packet but it occurs for states consisting of superpositions of Gaussian wavepackets \cite{8}. Albarelli et al. have addressed the notion of nonclassicality arising from the backflow effect and analyzed its relationship with the corresponding one on the negativity of the Wigner function \cite{10}. Very recently, it was also argued that the backflow under the presence of a constant field is mathematically equivalent to the problem of diffraction in time for particles initially confined to a semi-infinite line, expanding in free space \cite{11}. As far as we know, no experimental evidence of this effect has been yet reported.

On the other hand, apart from a work studying the arrival time problem in the framework of decoherent histories for a particle coupled to an environment \cite{12}, we are not aware of any other study carried out in the context of open quantum systems; in particular, how the friction and temperature can influence the backflow effect. In this work, we address this issue within two different frameworks, the Caldirola-Kanai \cite{13} and the Caldeira-Leggett \cite{14,15} approaches.

This work is organized as follows. In Section II, an effective time dependent Hamiltonian, the so-called Caldirola-Kanai Hamiltonian, is used to describe dissipation. This approach has been considered several times as an effective way to tackle such dissipative problems. It has been widely showed that this approach provides acceptable results
Free evolution of two Gaussian wavepackets and under the presence of a constant field are studied and analyzed. The classical limit is also proposed to be studied within the so-called classical Schrödinger equation [18]. In Section III, the backflow probability is considered as an eigenvalue problem with existing dissipation in the CK framework. In Section IV, the backflow effect is described within the Caldeira-Leggett approach where a master equation is used to describe the reduced density matrix in terms of friction and temperature of the environment. Free evolution of two Gaussian wavepackets as well as the dynamics under the presence of a constant field is again studied. In Section V, some numerical results are presented and discussed. Finally, in the last section, a summary and some conclusions are provided.

II. THE CALDIROLA-KANAI FRAMEWORK

For the sake of simplicity, in one dimension, the classical Langevin equation reads as

\[ m \ddot{x} + \eta \dot{x} + \frac{\partial V}{\partial x} = F_r(t) \] (1)

where \( m \) is the mass of the particle, \( \eta \) is the damping constant and \( V(x) \) is the external potential. \( F_r(t) \) is a fluctuating force with the following properties, \( \langle F_r(t) \rangle = 0 \) and \( \langle F_r(t) F_r(t') \rangle = 2 \eta k_B T \delta(t-t') \) with \( T \) being the temperature of the environment and \( k_B \) the Boltzmann constant. By defining the relaxation constant \( \gamma \) as [14]

\[ \gamma = \frac{\eta}{2m} \] (2)

and neglecting the fluctuating force, the Langevin equation (1) can be derived from the time-dependent Lagrangian \( \mathcal{L}_{\text{CK}} \) can be derived from the time-dependent Lagrangian

\[ \mathcal{L}_{\text{CK}} = \left( \frac{1}{2} m \dot{x}^2 - V(x) \right) e^{2\gamma t} \] (3)

from which the Caldirola-Kanai (CK) Hamiltonian

\[ \mathcal{H}_{\text{CK}} = \frac{p_c^2}{2m} e^{-2\gamma t} + V(x) e^{2\gamma t} \] (4)

is obtained, \( p_c \) being the canonical momentum \( p_c = \frac{\partial \mathcal{L}_{\text{CK}}}{\partial \dot{x}} = m \dot{x} e^{2\gamma t} \). In quantum mechanics, one has to replace the canonical momentum \( p_c \) by \( -i\hbar \partial_x \). In this way, the time-dependent Schrödinger equation reads as

\[ i\hbar \frac{\partial}{\partial t} \psi(x,t) = \left[ -e^{-2\gamma t} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + e^{2\gamma t} V(x) \right] \psi(x,t). \] (5)

The probability current density \( j(x,t) \) fulfilling the continuity equation

\[ \frac{\partial |\psi(x,t)|^2}{\partial t} + \frac{\partial j(x,t)}{\partial t} = 0, \] (6)

is given by

\[ j(x,t) = \frac{\hbar}{m} \text{Im} \left\{ \psi^* \frac{\partial \psi}{\partial x} \right\} e^{-2\gamma t}. \] (7)

In the following we first show that backflow does not take place for a single Gaussian wave packet. Then we study the effect of superposition of two Gaussian wave packets both in free propagation and in the presence of a constant field. After, backflow is analyzed as an eigenvalue problem. Finally, the classical limit is considered from the so-called classical Schrödinger equation.

A. Free propagation of a Gaussian wave packet

For the initial momentum-space wave function

\[ \phi(p,0) = \frac{1}{(2\pi \sigma_p^2)^{1/4}} \exp \left[ -\frac{(p-p_0)^2}{4\sigma_p^2} \right] \] (8)
the configuration-space wavefunction reads

\[ \psi(x, 0) = \frac{1}{(2\pi)^{1/4}} \sqrt{\frac{2\sigma_p}{\hbar}} \exp \left[-\frac{\sigma_p^2 x^2}{2\hbar^2} + \frac{i p_0}{\hbar} x \right]. \] (9)

The solution of the CK equation [5] for free propagation of the Gaussian wave function [9] is given by [19]

\[ \psi(x, t) = \frac{1}{(2\pi s_t^2)^{1/4}} \exp \left[-\frac{\sigma_p}{2\hbar s_t} (x - x_t)^2 + \frac{p_0}{\hbar} (x - x_t) + \frac{i}{\hbar} A_{cl}(t) \right] \] (10)

where, \( s_t, x_t \) and \( A_{cl}(t) \) are the complex width, the center of the wavepacket which follows a classical trajectory and the classical action expressed as

\[ \begin{align*}
  s_t &= \frac{1}{2\sigma_p} \left( \hbar + \frac{2\sigma_p^2}{m} \frac{1 - e^{-2\gamma t}}{2\gamma} \right), \\
  x_t &= \frac{p_0}{m} \frac{1 - e^{-2\gamma t}}{2\gamma}, \\
  A_{cl,t} &= \frac{p_0^2}{2m} \frac{1 - e^{-2\gamma t}}{2\gamma},
\end{align*} \] (11a-b-c)

respectively.

With respect to Eq. (8), the probability of obtaining a negative value in a measurement of the momentum is given by

\[ \text{Prob}(p < 0) = \int_{-\infty}^{0} |\phi(p, 0)|^2 = \frac{1}{2} \text{erfc} \left[ \frac{p_0}{\sqrt{2\sigma_p}} \right]. \] (12)

By a proper choice of the initial parameters \( p_0 \) and \( \sigma_p \), one can easily make the value of \( \text{Prob}(p < 0) \) negligibly small. In this way, one makes sure that the wave packet [9] has only positive momentum components. From Eq. (10), one has that the probability that the particle remains in the half-space \( x < 0 \) is

\[ P(t) = \frac{1}{2} \text{erfc} \left[ \frac{x_t}{\sqrt{2\sigma_t}} \right] \] (13)

where

\[ \sigma_t = |s_t| = \frac{1}{2\sigma_p} \sqrt{\hbar^2 + \frac{4\sigma_p^4}{m^2} \left( \frac{1 - e^{-2\gamma t}}{2\gamma} \right)^2} \] (14)

being the width of the probability density \( |\psi(x, t)|^2 \). Noting that the complementary error function is a decreasing function of its argument and that the argument is an increasing function of time i.e., \( \frac{d}{dt} \left( \frac{x_t}{\sqrt{2\sigma_t}} \right) > 0 \) for \( p_0 > 0 \), it is concluded that \( P(t) \) is a decreasing function of time and thus the backflow effect is not expected to occur for a single Gaussian wave packet.

B. Dissipative backflow for superposition of two Gaussian wave packets

Now consider the momentum representation of the initial state as

\[ \phi(p, 0) = N \frac{1}{(2\pi \sigma_p^2)^{1/4}} \left\{ \exp \left[-\frac{(p - p_{0a})^2}{4\sigma_p^2} \right] + \alpha e^{i\theta} \exp \left[-\frac{(p - p_{0b})^2}{4\sigma_p^2} \right] \right\}, \] (15)

where \( N \), the normalization constant, \( \alpha \) and \( \theta \) are all real numbers and

\[ N = \left( 1 + \alpha^2 + 2\alpha e^{-\frac{(p_{0a} - p_{0b})^2}{8\sigma_p^2}} \cos \theta \right)^{-1/2}. \] (16)
Thus, the free evolution of the probability density of the state given by Eq. (18) can be expressed as

$$|\psi(x, t)|^2 = N^2 \frac{1}{\sqrt{2\pi} \sigma_t} \exp \left\{ \frac{(x - x_{ta})^2}{2\sigma_t^2} \right\} + \alpha \exp \left\{ d_1 - \frac{(x - d_2(t))^2}{2\sigma_t^2} \right\} + \alpha^2 \exp \left\{ -\frac{(x - x_{tb})^2}{2\sigma_t^2} \right\}$$

(20)

where $x_{ta}$ and $x_{tb}$ are the centers of wave packets and

$$\begin{align*}
d_1 &= -i\theta - \frac{(p_{0a} - p_{0b})^2}{8\sigma_p^2} \\
d_2(t) &= \frac{x_{ta} + x_{tb}}{2} + \frac{\hbar (p_{0a} - p_{0b})}{4\sigma_p^2}
\end{align*}$$

(21a, 21b)

with $p_{0a}$ and $p_{0b}$ being the initial momenta and $\sigma_t$ is given by Eq. (14). From the probability density (20), one can easily write

$$P(t) = \frac{1}{2} N^2 \left\{ \text{erfc} \left( \frac{x_{ta}}{\sqrt{2}\sigma_t} \right) + \alpha^2 \text{erfc} \left( \frac{x_{tb}}{\sqrt{2}\sigma_t} \right) + 2\alpha e^{-(p_{0a} - p_{0b})^2/8\sigma_p^2} \cos \theta \left\{ \text{erfc} \left( \frac{d_2(t)}{\sqrt{2}\sigma_t} \right) \right\} + \sin \theta \left\{ \text{erfc} \left( \frac{d_2(t)}{\sqrt{2}\sigma_t} \right) \right\} \right\}$$

(22)

for the probability of remaining in the region $x < 0$. By noting the positive values for kick momenta, one observes that the arguments of the first two complementary error functions of the above equation are increasing functions of time. Thus, the last two terms are responsible for quantum backflow.

On the other hand, the evolution of the pure state (18) under Eq. (5) and in the presence of the linear potential

$$V(x) = -mgx$$

(23)

yields the same Eq. (22) for the probability of remaining in the region $x < 0$ but with the replacement of $x_t$ by $q_t$ where

$$q_t = \frac{p_0}{m} \frac{1 - e^{-2\gamma t}}{2\gamma} + g \frac{2\gamma t - 1 + e^{-2\gamma t}}{4\gamma^2}$$

(24)

is the classical trajectory in the presence of the constant force $mg$. 

State (15) is a superposition of two Gaussian wave packets in momentum space with the same width $\sigma_p$ but different centers $p_{0a}$ and $p_{0b}$. When a quantum system is described by Eq. (15) in momentum space, the probability for obtaining a negative value in a measurement of momentum is

$$\text{Prob}(p < 0) = \int_{-\infty}^{0} |\phi(p, 0)|^2$$

$$= \frac{1}{2} N^2 \left\{ \text{erfc} \left( \frac{p_{0a}}{\sqrt{2}\sigma_p} \right) + \alpha^2 \text{erfc} \left( \frac{p_{0b}}{\sqrt{2}\sigma_p} \right) + 2\alpha e^{-(p_{0a} - p_{0b})^2/8\sigma_p^2} \cos \theta \text{erfc} \left( \frac{(p_{0a} + p_{0b})}{2\sqrt{2}\sigma_p} \right) \right\}.$$ 

(17)
C. The classical limit

A quite natural way to study the classical limit is through the quantum-classical transition wave equation originally proposed by Richardson et al. in the context of conservative systems [18]. They started from what is known as the classical Schrödinger equation. This transition wave equation is governed by a continuous parameter covering the two regimes as being the two extreme cases, the classical one leading to the classical Schrödinger equation. Recently, an extension to open quantum systems has been carried out in the CK framework [16]. We have shown that Eq. (5) can be written as

\[ i\hbar \frac{\partial}{\partial t} \tilde{\psi}(x,t) = \left[ -e^{-2\gamma t} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + e^{2\gamma t} V(x) \right] \tilde{\psi}(x,t) \]  

where \( \hbar = \frac{\sqrt{\epsilon}}{m} \) (scaled Planck constant) with \( 0 \leq \epsilon \leq 1; \epsilon = 1 \) is for the quantum regime and \( \epsilon = 0 \) for the classical regime. This parameter gives us an idea of the degree of quantumness of the dynamical regime. This equation is also called scaled linear Schrödinger equation and the corresponding wave function which depends on this parameter can be seen as a transition wave function. With this procedure, the classical limit is reached in a continuous way.

In this context, the scaled probability current density \( \tilde{j}(x,t) \) fulfills the continuity equation

\[ \frac{\partial}{\partial t} |\tilde{\psi}(x,t)|^2 + \frac{\partial \tilde{j}(x,t)}{\partial x} = 0, \]  

with

\[ \tilde{j}(x,t) = \frac{\hbar}{m} \text{Im} \left\{ \tilde{\psi}^* \frac{\partial \tilde{\psi}}{\partial x} \right\} e^{-2\gamma t}. \]  

The amount of backflow in the time interval \([0,\tau]\) for the scale dynamical regime is then

\[ \Delta P \equiv \tilde{P}(\tau) - P(0) = -\int_0^\tau dt \tilde{j}(0,t). \]  

Moreover, it is clearly seen that the \( K \) operator defined in Eq. (35) has also to be rewritten in terms of \( \epsilon, \tilde{K} \). Then, the corresponding eigenvalue equation (39) also depends on the scale parameter. However, Eq. (47) is still independent on the scaled Planck constant. Thus, in the classical limit, the classical Schrödinger equation still displays the backflow effect.

III. PROBABILITY FLOW AS AN EIGENVALUE PROBLEM IN THE CK FRAMEWORK

A. Free evolution

Bracken and Melloy [2] constructed an eigenvalue equation for the probability flow. Here, following the same steps as the previous authors, we are going to build a similar eigenvalue equation but in the dissipative CK context. From the continuity equation (5), one writes that

\[ \frac{d}{dt} \int_{-\infty}^0 dx |\psi(x,t)|^2 = \int_{-\infty}^0 dx \frac{\partial j(x,t)}{\partial x} = -j(0,t) \]  

where in the second equality we have used the square-integrability of the wave function. Thus, for the backflow in the time interval \([0,\tau]\) one obtains that

\[ \Delta P \equiv P(\tau) - P(0) = -\int_0^\tau dt \tilde{j}(0,t) \]  

the wave function at the instant \( t \) being related via the following integral equation to the initial wave function

\[ \psi(x,t) = \int_{-\infty}^\infty dx' G(x,t; x', 0) \psi(x', 0) = \int_{-\infty}^\infty dx' G(x,t; x', 0) \frac{1}{\sqrt{2\pi\hbar}} \int_0^\infty dp' e^{ip'x'/\hbar} \phi(p') \]
where in the second equality the Fourier transform of the initial wave function involves only positive momenta and 
\( G(x, t; x', 0) \) is the propagator for free evolution \[19\],

\[
G(x, t; x', 0) = \sqrt{\frac{m}{2\pi i\hbar \tau(t)}} \exp \left[ \frac{im(x - x')^2}{2\hbar \tau(t)} \right]
\]  \hspace{1cm} (32)

with

\[
\tau(t) = \frac{1 - e^{-2\gamma t}}{2\gamma}.
\]  \hspace{1cm} (33)

From Eqs. (30), (31) and (32), we have that

\[
\Delta P = \int_0^\infty \int_0^\infty dp dq \, \phi^*(p) K(p, q) \phi(q)
\]  \hspace{1cm} (34)

where

\[
K(p, q) = -\frac{p + q}{4\pi m\hbar} \int_0^\tau dt \exp \left[ i\frac{p^2 - q^2}{2m\hbar} \tau(t) \right]
\]

\[
= -\frac{1}{\pi p - q} \exp \left[ i\tau(\tau) \frac{p^2 - q^2}{4m\hbar} \right] \sin \left( \frac{\tau(\tau) p^2 - q^2}{4m\hbar} \right).
\]  \hspace{1cm} (35)

In order to optimize \( \Delta P \) in Eq. (34) under the constraint

\[
\int_0^\infty dp |\phi(p)|^2 = 1
\]  \hspace{1cm} (36)

the method of Lagrange multipliers is used to construct the functional

\[
I[\phi, \phi^*] = \int_0^\infty \int_0^\infty dp dq \, \phi^*(p) K(p, q) \phi(q) - \lambda \int_0^\infty dp \, \phi(p)^* \phi(p)
\]  \hspace{1cm} (37)

where \( \lambda \) is the Lagrange multiplier. From the optimization condition \( \delta I = I[\phi, \phi^* + \delta \phi^*] - I[\phi, \phi^*] = 0 \), the eigenvalue equation is then expressed as

\[
\int_0^\infty dq \, K(p, q) \phi(q) = \lambda \phi(p)
\]  \hspace{1cm} (38)

which in its standard form should be written as

\[
\int_0^\infty dq \, K(p, q) \phi_\lambda(q) = \lambda \phi_\lambda(p).
\]  \hspace{1cm} (39)

From the property

\[
K^*(p, q) = K(q, p)
\]  \hspace{1cm} (40)

we prove, in the following, that eigenvalues are real and eigenfunctions corresponding to distinct eigenvalues are orthogonal i.e., the integral operator \( K \) is Hermitian. By multiplying the complex-conjugated eigenfunction \( \phi_{\lambda}^*(p) \) in the eigenvalue equation corresponding to the eigenvalue \( \lambda' \), multiplying \( \phi_{\lambda}(p) \) in the complex-conjugated eigenvalue equation corresponding to the eigenvalue \( \lambda^* \), subtracting the resulting equations and finally integrating over \( p \), one has that

\[
\int_0^\infty dp \int_0^\infty dq \left\{ \phi_{\lambda}^*(p) K(p, q) \phi_{\lambda}(q) - \phi_{\lambda}(p) K^*(p, q) \phi_{\lambda}^*(q) \right\} = (\lambda - \lambda^*) \int_0^\infty dp \, \phi_{\lambda}^*(p) \phi_{\lambda}(p).
\]  \hspace{1cm} (41)

Now, by using Eq. (40) and interchanging \( q \) by \( p \) and viceversa in the second term of the left-hand-side, this term will be the same as the first one. Thus, we have

\[
(\lambda - \lambda^*) \int_0^\infty dp \, \phi_{\lambda}^*(p) \phi_{\lambda}(p) = 0.
\]  \hspace{1cm} (42)
If we set $\lambda' = \lambda$, $\lambda - \lambda' = 0$ revealing that the eigenvalues are real. Then,

$$\int_0^\infty dp \, \phi^*_\lambda(p) \phi_\lambda(p) = 0$$

(43)

which for $\lambda' \neq \lambda$ yields the orthogonality of eigenfunctions corresponding to different eigenvalues.

From Eqs. (38) and (44) and noting the constraint (36) one has

$$\Delta p = \lambda$$

(44)

and concludes that $-1 \leq \Delta p \leq 1$ with

$$-1 \leq \lambda \leq 1.$$  

(45)

After these general considerations, we now come back to Eq. (35). By setting

$$p = 2 \sqrt{\frac{m \hbar}{\tau(\tau)}} \, u$$

(46a)

$$q = 2 \sqrt{\frac{m \hbar}{\tau(\tau)}} \, v$$

(46b)

$$\phi(p) = e^{iu^2} \phi(u)$$

(46c)

$$\phi(q) = e^{iv^2} \phi(v)$$

(46d)

and using Eq. (35), the eigenvalue equation (38) can be written as

$$\frac{1}{\pi} \int_0^\infty dv \, \sin\left(\frac{u^2 - v^2}{u - v}\right) \phi(v) = -\lambda \phi(u).$$

(47)

From this equation, it is clearly seen that the eigenvalues are also independent of mass, Planck constant $\hbar$, friction coefficient $\gamma$ and backflow interval $\tau$. The same is true for the conservative case.

### B. Constant force

For the linear potential (23), the propagator of CK is given by [19],

$$G(x,t;x',0) = \exp \left[ \frac{im}{\hbar} g \left( x \left( \frac{e^{2\gamma t} - 2\gamma t - 1}{2\gamma(1 - e^{-2\gamma t})} + x' \left( \frac{e^{-2\gamma t} + 2\gamma t - 1}{2\gamma(1 - e^{-2\gamma t})} \right) \right) - \frac{im}{2\hbar} g^2 \left( \frac{e^{2\gamma t} + e^{-2\gamma t} - 4\gamma^2 t^2 - 2}{2\gamma^3(1 - e^{-2\gamma t})} \right) \right] G_l(x,t;x',0)$$

(48)

where $G_l(x,t;x',0)$ is the propagator [32] for the free propagation in the CK framework. Following the same steps as in the previous subsection, the eigenvalue equation (39) still holds but now one has that

$$K(p,q) = -\frac{1}{\pi} \frac{1}{p - q} \exp \left[ i \left( \tau(\tau) \frac{p^2 - q^2}{4m\hbar} - \frac{(p - q)g \tau(\tau) - \tau}{2\hbar} \right) \right] \sin \left( \frac{\tau(\tau) \frac{p^2 - q^2}{4m\hbar} - \frac{(p - q)g \tau(\tau) - \tau}{2\hbar}}{2\gamma} \right)$$

(49)

with the same property (40). Then, the eigenvalue equation can be written as

$$\frac{1}{\pi} \int_0^\infty dv \, \sin\left(\frac{u^2 - v^2 - \eta(u - v)}{u - v}\right) \phi(v) = -\lambda \phi(u).$$

(50)

where $u$ and $v$ are defined through Eqs. (46a) and (46b) but here

$$\begin{cases}
\phi(p) = e^{-i(u^2 - \xi u)} \phi(u) \\
\phi(q) = e^{-i(v^2 - \xi v)} \phi(v)
\end{cases}$$

(51a)

(51b)

with

$$\xi = \frac{g}{2 \sqrt{\frac{m}{\hbar \tau(\tau)}}} \frac{\tau(\tau) - \tau}{2\gamma}.$$  

(52)

Note that in the limit $\gamma \to 0$ one has $\xi = -\frac{g}{2 \sqrt{\tau^{3/2}}}$ which is the result reported in [3]. Eq. (50) shows that in the presence of the constant force $mg$, the backflow depends on the parameter $\xi$ which itself depends on mass, Planck constant, force constant, friction coefficient and the duration of backflow according to Eq. (52).
IV. THE CALDERIA-LEGGETT APPROACH

In this Section, we are going to introduce the temperature of the environment through the so-called Calderia-Leggett (CL) equation. The master equation for the reduced density matrix $\rho$ in the coordinate representation and at high-temperatures is expressed as \[ [14, 15]

$$
\frac{\partial \rho(x, x', t)}{\partial t} = \left[ -\frac{\hbar}{2m} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2} \right) - \gamma(x-x') \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right) + \frac{V(x) - V(x')}{\hbar} - \frac{D}{\hbar^2} (x-x')^2 \right] \rho(x, x', t)
$$

(53)

where $D = 2m\gamma k_B T$ plays the role of the diffusion coefficient. One way of solving this equation is by defining the new variables $R = \frac{x+x'}{2}$ and $r = x-x'$, taking the partial Fourier transform with respect to the coordinate $R$ and solving the resulting equation by the method of characteristics [20]. Finally, the inverse Fourier transform is applied to obtain the density matrix in the configuration space. From the CL equation (53), one can easily reaches the continuity equation,

$$
\frac{\partial \rho(x, x', t)}{\partial t} \bigg|_{x' = x} + \frac{\partial j(x, t)}{\partial x} = 0,
$$

(54)

with

$$
j(x, t) = \frac{\hbar}{m} \text{Im} \left\{ \frac{\partial \rho(x, x', t)}{\partial x} \bigg|_{x' = x} \right\}.
$$

(55)

Diagonal elements of the density matrix are interpreted as probability density.

A. Free evolution of a Gaussian wave packet

The free evolution of the pure state [9] under Eq. (53) yields

$$
\rho(x, t) = \rho(x, x, t) = \frac{1}{\sqrt{2\pi w_t}} \exp \left[ -\frac{(x-x_t)^2}{2w_t^2} \right],
$$

(56)

for the diagonal elements of the density matrix where

$$
w_t = \frac{1}{2\sigma_p} \sqrt{\hbar^2 + \frac{4\sigma_p^2}{m^2} \left( 1 - e^{-2\gamma t} \right)^2 + \frac{4\gamma t + 4e^{-2\gamma t} - 3 - e^{-4\gamma t}}{2m^2\gamma^3} \sigma_p^2 D},
$$

(57)

is the width of the probability density. Comparison of Eq. (14) and Eq. (57) reveals that the probability density in the CL model has a temperature dependence. For a given friction $\gamma$, the last term under the square root increases with time. As discussed in the CK framework, here too the backflow cannot take place for a single Gaussian wavepacket.

B. Evolution of a superposition of two Gaussian wave packets

The time evolution of the superposition of two Gaussian wavepackets (18)

$$
\rho_0(x, x') = \psi_0(x)\psi_0^*(x') = N^2[a_{0a}(x)a_{0a}^*(x') + 2\alpha \text{ Re}[e^{-i\theta} a_{0a}(x)a_{0b}^*(x') + \alpha^2 a_{0b}(x)a_{0b}^*(x')]]
$$

(58)

under the CL master equation (53) can easily be obtained by noting the linearity of this equation. One can separately evolve each term of the density matrix and then combine the resulting solutions properly i.e., according to Eq. (58). From the procedure outlined at the beginning of this section, one has that

$$
\rho_{ab}(R, r, t) = \frac{1}{\sqrt{2\pi w_t}} \exp \left[ -\frac{(R-a_{1}(r, t))^2}{2w_t^2} \right],
$$

(59)

for the evolution of the cross term $\psi_{0a}(x)\psi_{0b}^*(x')$ under the CL equation where $w_t$ is given by Eq. (57). From Eq. (55), the corresponding probability current density can be now written as

$$
\frac{\partial a_0}{\partial t} - \frac{\partial a_1}{\partial t} \left( 1 - 2 \frac{\partial a_1}{\partial r} \right) \rho_{ab}(x, 0, t)
$$

(60)
In this way, one sees that the probability density, diagonal elements of the density matrix, is obtained from Eq. (20) replacing $\sigma$ by $\gamma$. Thus, from Eq. (17), the probability for finding the particle with a negative momentum is maximal for the non-dissipative dynamics [10]. Note that for $t < 0$, the sign of the last term under the square root is negative. Thus, when this term dominates, this leads to an imaginary width which is not acceptable.

Under the presence of the accelerating constant force $mg$, one simply has

\begin{align}
\begin{array}{l}
a_0(r, t) = -\frac{(p_{0a} - p_{0b})^2}{8\sigma_p^2} - \left[ \frac{\sigma_p^2 e^{-4\gamma t}}{2h^2} - \frac{1 - e^{-4\gamma t}}{4h^2\gamma} D \right] r^2 + i \frac{e^{-2\gamma t} p_{0a} + p_{0b}}{2h} r \\
a_1(r, t) = \frac{\bar{r}_{ta} + \bar{r}_{tb}}{2} + i \left[ \frac{\hbar(p_{0a} - p_{0b})}{4\sigma_p^2} + \left( \frac{\sigma_p^2 e^{-2\gamma t}(1 - e^{-2\gamma t})}{2m\gamma} + \frac{1 - e^{-2\gamma t}(2 - e^{-2\gamma t})}{4hm\gamma^2} D \right) r \right].
\end{array}
\end{align}

V. NUMERICAL CALCULATIONS

All numerical calculations in this section are carried out for the initial state given by Eq. (18) working in a system of units where $\hbar = m = 1$ with $\sigma_p = 0.05$, $p_0 = 0.3$, $\delta = 1.1$, $\alpha = 1.9$ and $\theta = \pi$. With these values, the amount of backflow is maximal for the non-dissipative dynamics [10]. Thus, from Eq. (17), the probability for finding the particle with a negative momentum is $\approx 7.72 \times 10^{-10}$. This means that the initial state in the configuration space is practically constructed by superposition of plane waves with positive momenta.

Figure 1 (left panel) displays the dissipative quantum backflow. In this figure, we have plotted the probability for remaining in the region $x < 0$ versus time for different values of the friction coefficient $\gamma$: $\gamma = 0$ (black), $\gamma = 0.025$ (red), $\gamma = 0.05$ (green), $\gamma = 0.1$ (blue), $\gamma = 0.2$ (magenta), $\gamma = 0.3$ (brown) and $\gamma = 0.4$ (cyan). This figure shows that there are some time intervals where the probability for remaining in the negative semi-infinite region increases with time. This is the hallmark of the backflow effect. This is better observed in the right panel of this figure where a zoom of this type of region is displayed. As observed, the difference between the minima and maxima is reduced by friction and thus the backflow probability. After Eq. (30), during the backflow intervals where the probability increases, the probability current is negative as clearly observed in Figure 2 where the probability current density at the origin versus time, $j(0, t)$, for four different values of friction coefficient is plotted.

According to Eq. (30), in order to compute the different backflow probabilities, the time intervals which contain the negative peaks of the probability current should be identified by finding the zeros of this current. Alternatively,
another method proposed in [10] is based on the numerical integration of the negative part of the probability current, $0.5(|j(0, t)| - j(0, t))$, over a time interval which contains the most prominent negative peak and does not require the exact knowledge of the zeros of the current. This quantity is plotted in the left panel of Figure 3. The maximum amplitude of such a temporary increase of the probability has been used to quantify this effect [10],

$$\beta = \sup_{t_1 < t_2} [P(t_2) - P(t_1)]$$

(63)

where $(t_1, t_2)$ represents the different time intervals where the backflow occurs. Using this method, we have computed the backflow amount $\beta$ for different values of friction coefficient and plotted in the right panel of Figure 3. This amount decreases with friction after a certain plateau at low friction values.

Figure 4 highlights the quantum backflow when a constant force is present. In this figure, this effect is studied for non-dissipative and dissipative dynamics separately for different values of the force constant in the CK framework. The corresponding backflow probability for remaining in the region $x < 0$ versus time under the presence of a constant force for the initial state (18) and for two values of the friction coefficient, $\gamma = 0$ (left panel) and $\gamma = 0.1$ (right panel), is plotted. Different values of the force constant are analyzed: $g = 0$ (black curve), $g = 0.1$ (red curve), $g = 0.2$ (green curve) and $g = 0.3$ (blue curve). It is seen in both cases, non-dissipative and dissipative dynamics, that the backflow probability reduces with the force constant.
FIG. 4: The probability for remaining in the region \( x < 0 \) versus time under the presence of a constant force in the CK framework for the initial state \( |18\rangle \) for two values of the friction coefficient: \( \gamma = 0 \) (left panel) and \( \gamma = 0.1 \) (right panel). Different values of the force constant are analyzed: \( g = 0 \) (black curve), \( g = 0.1 \) (red curve), \( g = 0.2 \) (green curve) and \( g = 0.3 \) (blue curve).

FIG. 5: The probability for remaining in \( x < 0 \) versus time in free propagation of the state \( |18\rangle \) under CL equation for different values of friction coefficients: \( \gamma = 0.1 \) (top left panel) and \( \gamma = 0.5 \) (top right panel) for different values of temperature, \( k_B T = 1 \) (black), \( k_B T = 2 \) (red), \( k_B T = 5 \) (green) and \( k_B T = 10 \) (blue). In the bottom panels we have focused on the first backflow interval.

The influence of the temperature is studied here within the CL framework. In Figure 5, the probability of finding the particle in the region \( x < 0 \) for free propagation under the CL equation for two values of friction coefficient (0.1 and 0.5) and several temperatures (\( k_B T = 1 \) (black), \( k_B T = 2 \) (red), \( k_B T = 5 \) (green) and \( k_B T = 10 \) (blue)) is plotted in the top two panels. This figure shows two intervals where the mentioned probability increases i.e., the backflow occurs. A close-up of the first time interval is displayed in the bottom two panels. As it is apparent, the amount and duration of this first backflow decreases with temperature for a given friction coefficient. Furthermore, for \( \gamma = 0.1 \), the duration and amount of the first backflow are respectively \( \approx 0.657143 \) and \( \approx 0.002705 \) for \( k_B T = 1 \) while the corresponding values for \( k_B T = 10 \) are \( \approx 0.385714 \) and \( \approx 0.001756 \).

Finally, in Figure 6 is depicted the probability for remaining in the negative part of \( x \)-axis for dissipative dynamics with \( \gamma = 0.1 \) at the temperature \( k_B T = 1 \) (left top panel) and \( k_B T = 10 \) (right top panel) for different values of force constant, \( g = 0 \) (black), \( g = 0.01 \) (red), \( g = 0.02 \) (green) and \( g = 0.03 \) (blue). This figure shows that there is a time interval where the probability for remaining in the negative semi-infinite region increases with time. This is better seen in the bottom panels where a zoom of this interval is plotted. Backflow decreases with increasing acceleration \( g \) for a given temperature. Furthermore, by comparing left and right panels for a given \( g \), one observes that the temperature decreases the duration and amount of backflow. For instance, when \( g = 0.03 \) we have \( \approx 0.628571 \) and
FIG. 6: The probability for remaining in $x < 0$ versus time in the CL model for the pure state $|18\rangle$ for $\gamma = 0.1$ for different temperatures: $k_B T = 1$ (top left panel) and $k_B T = 10$ (top right panel) for different values of acceleration, $g = 0$ (black), $g = 0.01$ (red), $g = 0.02$ (green) and $g = 0.03$ (blue). In the bottom panels we have focused on the region around the maximum.

$\approx 0.002631$ for $k_B T = 1$ respectively, and $\approx 0.385714$ and $\approx 0.001734$ for $k_B T = 10$.

VI. SUMMARY AND DISCUSSION

In this work, we have studied the backflow effect, that is, negative fluxes for positive-momentum wavepackets for an open quantum system in the CK and CL frameworks. In the CK approach, only friction is taken into account through an effective time-dependent Hamiltonian. In the CL approach, the quantum system under study is coupled to a boson reservoir, i.e., to an infinite number of quantum oscillators at thermal equilibrium. In this approach, both system and its environment is described by a single Hamiltonian. The equation of motion for the reduced density matrix describing the system of interest is obtained by tracing over the degrees of freedom of the environment. In this way, Caldeira and Leggett obtained the master equation [53] in the high-temperature limit. In both CK and CL approaches, the equation of motion for the quantum state is linear. The other approaches that one could choose are the logarithmic non-linear Schrödinger equations [16, 22, 23] and the stochastic Schrödinger equation [24, 25]. In these approaches the linearity is not fulfilled and the form of the initial superposed state is not preserved under such dynamics.

As far as we know, this is the first study of dissipative quantum backflow. For a superposition of Gaussian wavepackets with a negligible negative momentum contribution, the backflow also takes place as it has been already reported for the non-dissipative dynamics in free propagation. We have found that the backflow decreases with increasing the parameters of the environment i.e., friction $\gamma$ and temperature $T$ and also observed that it is never suppressed. The constant force $mg \geq 0$ behaves against backflow. Furthermore, we have studied backflow as an eigenvalue problem in the context of CK framework and concluded that eigenvalues are independent of mass, Planck’s constant, friction and duration of the backflow in free propagation. However, the dynamics under the presence of a constant force is again dependent on those parameters through a dimensionless quantity. Finally, the classical limit has been briefly discussed within a more natural context which is that of the classical Schrödinger equation.

Acknowledgement

SVM acknowledges support from the University of Qom and SMA support from the Ministerio de Ciencia, Inno-
vacación y Universidades (Spain) under the Project FIS2017-83473-C2-1-P.

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