Dehn surgery, the fundamental group and $SU(2)$

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1. Introduction

The main result of this paper, which is a companion to [13], is the following theorem.

**Theorem 1.** Let $K$ be a non-trivial knot in $S^3$, and let $Y_r$ be the 3-manifold obtained by Dehn surgery on $K$ with surgery-coefficient $r \in \mathbb{Q}$. If $|r| \leq 2$, then $\pi_1(Y_r)$ is not cyclic. In fact, there is a homomorphism $\rho : \pi_1(Y_r) \to SU(2)$ with non-cyclic image.

The statement that $Y_r$ cannot have cyclic fundamental group was previously known for all cases except $r = \pm 2$. The case $r = 0$ is due to Gabai [12], the case $r = \pm 1$ is the main result of [13], and the case that $K$ is a torus knot is analysed for all $r$ in [16]. All remaining cases follow from the cyclic surgery theorem of Culler, Gordon, Luecke and Schalen [2]. It is proved in [15] that $Y_2$ cannot be homeomorphic to $\mathbb{R}P^3$. If one knew that $\mathbb{R}P^3$ was the only closed 3-manifold with fundamental group $\mathbb{Z}/2\mathbb{Z}$ (a statement that is contained in Thurston’s geometrization conjecture), then the first statement in the above theorem would be a consequence. The second statement in the theorem appears to sharpen the result slightly. In any event, we have:

**Corollary 2.** Dehn surgery on a non-trivial knot cannot yield a 3-manifold with the same homotopy type as $\mathbb{R}P^3$. □

The proof of Theorem 1 provides a verification of the Property P conjecture that is independent of the results of the cyclic surgery theorem of [2]. Although the argument follows [13] very closely, we shall avoid making explicit use of instanton Floer homology and Floer’s exact triangle [11, 1]. Instead, we rely on the technique that forms just the first step of Floer’s proof from [11], namely the technique of “holonomy perturbations” for the instanton equations (see also the remark following Proposition 16 in [13]).

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2. Holonomy perturbations

This section is a summary of material related to the “holonomy perturbations” which Floer used in the proof of his surgery exact triangle for instanton Floer homology \[11\]. Similar holonomy perturbations were introduced for the 4-dimensional anti-self-duality equations in \[3\]; see also \[18\]. Our exposition is taken largely from \[1\] with only small changes in notation. Some of our gauge-theory notation is taken from \[14\].

Let \(Y\) be a compact, connected 3-manifold, possibly with boundary. Let \(w\) be a unitary line bundle on \(Y\), and let \(E\) be a unitary rank 2 bundle equipped with an isomorphism \(\psi : \text{det}(E) \rightarrow w\).

Let \(g_E\) denote the bundle whose sections are the traceless, skew-hermitian endomorphisms of \(E\), and let \(A\) be the affine space of \(SO(3)\) connections in \(g_E\). Let \(\mathcal{G}\) be the gauge group of unitary automorphisms of \(E\) of determinant 1 (the automorphisms that respect \(\psi\)). We write \(\mathcal{B}^w(Y)\) for the quotient space \(A/\mathcal{G}\).

A connection \(A\), or its gauge-equivalence class \([A] \in \mathcal{B}^w(Y)\), is irreducible if the stabilizer of \(A\) is the group \(\{\pm 1\} \subset \mathcal{G}\), and is otherwise reducible. The reducible connections are the ones that preserve a decomposition of \(g_E\) as \(\mathbb{R} \oplus L\), where \(L\) is an orientable 2-plane bundle; these connections have stabilizer either \(S^1\) or (in the case of the product connection) the group \(SU(2)\).

**Definition 3.** We write \(\mathcal{R}^w(Y) \subset \mathcal{B}^w(Y)\) for the space of \(\mathcal{G}\)-orbits of flat connections:

\[
\mathcal{R}^w(Y) = \{ [A] \in \mathcal{B}^w(Y) \mid F_A = 0 \}.
\]

This is the representation variety of flat connections with determinant \(w\).

We have the following straightforward fact:

**Lemma 4.** The representation variety \(\mathcal{R}^w(Y)\) is non-empty if and only if \(\pi_1(Y)\) admits a homomorphism \(\rho : \pi_1(Y) \rightarrow SO(3)\) with \(w_2(\rho) = c_1(w) \mod 2\). The representation variety contains an irreducible element if and only if there is such a \(\rho\) whose image is not cyclic.

If \(c_1(w) = 0 \mod 2\), then \(\mathcal{R}^w(Y)\) is isomorphic to the space of homomorphisms \(\rho : \pi_1(Y) \rightarrow SU(2)\) modulo the action of conjugation.

Suppose now that \(Y\) is a closed oriented 3-manifold. The flat connections \(A \in \mathcal{A}\) are the critical points of the Chern-Simons function

\[
\text{CS} : \mathcal{A} \rightarrow \mathbb{R},
\]

\[
\text{CS}(A) = \frac{1}{4} \int_Y \text{tr}((A - A_0) \wedge (F_A + F_{A_0})),
\]

where \(A_0\) is a chosen reference point in \(\mathcal{A}\), and \(\text{tr}\) denotes the trace on 3-by-3 matrices. We define a class of perturbations of the Chern-Simons functional, the holonomy perturbations.
Let $D$ be a compact 2-manifold with boundary, and let $\iota: S^1 \times D \hookrightarrow Y$. Choose a trivialization of $w$ over the image of $\iota$. With this choice, each connection $A \in \mathcal{A}$ gives rise to a unique connection $\tilde{A}$ in $E|_{\text{im}(\iota)}$ with the property that $\det(\tilde{A})$ is the product connection in the trivialized bundle $w|_{\text{im}(\iota)}$. Thus $\tilde{A}|_{\text{im}(\iota)}$ is an $SU(2)$ connection. Given a smooth 2-form $\mu$ with compact support in the interior of $D$ and integral 1, and given a smooth class-function

$$\phi: SU(2) \to \mathbb{R},$$

we can construct a function

$$\Phi: \mathcal{A} \to \mathbb{R}$$

that is invariant under $G$ as follows. For each $z \in D$, let $\gamma_z$ be the loop $t \mapsto \iota(t, z)$ in $Y$, and let $\text{Hol}_{\gamma_z}(\tilde{A})$ denote the holonomy of $\tilde{A}$ along $\gamma_z$, as an automorphism of the fiber $E$ at the point $y = \iota(0, z)$. The class-function $\phi$ determines also a function on the group of determinant-1 automorphisms of the fiber $E_y$, and we set

$$\Phi(A) = \int_D \phi(\text{Hol}_{\gamma_z}(\tilde{A})){\mu}(z).$$

One can write down the equations for a critical point $A$ of the function $CS + \Phi$ on $\mathcal{A}$. They take the form

$$F_A = \phi'(H_A){\mu_Y},$$

where $H_A$ is the section of the bundle $\text{Aut}(E)$ over $\text{im}(\iota)$ obtained by taking holonomy around the circles, $\phi'$ is the derivative of $\phi$, regarded as a map from $\text{Aut}(E)$ to $g_{E}$, and $\mu_Y$ is the 2-form on $Y$ obtained by pulling back $\mu$ to $S^1 \times D$ and then pushing forward along $\iota$. (See [1].)

**Definition 5.** Given $\iota$ and $\phi$ as above, we write

$$\mathcal{R}_{\iota, \phi}^w(Y) = \{ [A] \in \mathcal{B}^w(Y) \mid F_A = \phi'(H_A){\mu_Y} \}.$$ 

This is the perturbed representation variety.

Now specialize to the case that $D$ is a disk, so $\iota$ is an embedding of a solid torus. Let

$$C = Y \setminus \text{im}(\iota)^0$$

be the complementary manifold with torus boundary. Let $z_0 \in \partial D$ be a basepoint, and let $a$ and $b$ be the oriented circles in $\partial C$ described by

$$a = \iota(S^1 \times \{z_0\})$$

$$b = \iota(\{0\} \times \partial D).$$

These are the “longitude” and “meridian” of the solid torus. We continue to suppose that $w$ is trivialized on $\text{im}(\iota)$ and hence on $\partial C$. So the restriction of $E$ to $\partial C$ is given the structure of an $SU(2)$ bundle. Given a connection $A$ on $g_{E}$ that is flat on $\partial C$, let $\tilde{A}$ be the corresponding flat $SU(2)$ connection in $E|_{\partial C}$. One can choose a determinant-1 isomorphism between the fiber of $E$
at the basepoint \( i(0, z_0) \) so that the holonomies of \( \tilde{A} \) around \( a \) and \( b \) become commuting elements of \( SU(2) \) given by

\[
\begin{align*}
\text{Hol}_a(\tilde{A}) &= \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} \\
\text{Hol}_b(\tilde{A}) &= \begin{bmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{bmatrix}.
\end{align*}
\]

The pair \( (\alpha(A), \beta(A)) \in \mathbb{R}^2 \) is determined by \( A \) up to the ambiguities

(a) adding integer multiples of \( 2\pi \) to \( \alpha \) or \( \beta \);

(b) replacing \( (\alpha, \beta) \) by \( (-\alpha, -\beta) \).

**Definition 6.** Let \( S \subset \mathbb{R}^2 \) be a subset of the plane with the property that \( S + 2\pi \mathbb{Z}^2 \) is invariant under \( s \mapsto -s \). Define the set

\[
\mathcal{R}^w(C \mid S) \subset \mathcal{R}^w(C)
\]
as

\[
\mathcal{R}^w(C \mid S) = \{ [A] \in \mathcal{R}^w(C) \mid (\alpha(A), \beta(A)) \in S + 2\pi \mathbb{Z}^2 \},
\]

where \( (\alpha(A), \beta(A)) \) are the longitudinal and meridional holonomy parameters, determined up to the ambiguities above.

One should remember that the choice of trivialization of \( w \) on \( \text{im}(i) \) is used in this definition, and in general the set we have defined will depend on this choice.

A class-function \( \phi \) on \( SU(2) \) corresponds to a function \( f : \mathbb{R} \to \mathbb{R} \) via

\[
f(t) = \phi \left( \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \right).
\]

The function \( f \) satisfies \( f(t) = f(t + 2\pi) \) and \( f(-t) = f(t) \). The following observation of Floer’s is proved as Lemma 5 in [1].

**Lemma 7.** Let \( f : \mathbb{R} \to \mathbb{R} \) correspond to \( \phi \) as above. Then restriction from \( Y \) to \( C \) gives rise to a bijection

\[
\mathcal{R}^w(C_{\alpha, \beta}) \to \mathcal{R}^w(C \mid \beta = -f'(\alpha)).
\]

We also have the straightforward fact:

**Lemma 8.** If \( g : \mathbb{R} \to \mathbb{R} \) is a smooth odd function with period \( 2\pi \), then there is a class-function \( \phi \) on \( SU(2) \) such that the corresponding function \( f \) satisfies \( f' = g \).
3. Removing flat connections by perturbation

Let us now take the case that $Y$ is a homology $S^1 \times S^2$, and let $w \to Y$ be a line-bundle with $c_1(w)$ a generator for $H^2(Y; \mathbb{Z}) = \mathbb{Z}$. Let $N \hookrightarrow Y$ be an embedded solid torus whose core is a curve representing a generator of $H_1(Y; \mathbb{Z})$, and let $C$ be the manifold with torus boundary

$$C = Y \setminus N^\circ.$$ 

By a “slope” we mean an isotopy class of essential closed curves on the torus $\partial C$. For each slope $s$, let $Y_s$ denote the manifold obtained from $C$ by Dehn filling with slope $s$: that is, $Y_s$ is obtained from $C$ by attaching a solid torus in such a way that curves in the class $s$ bound disks in the solid torus.

Proposition 9. Let $s$ be as above, and suppose $p/q \leq 2$.

Suppose that neither $\pi_1(Y_a)$ nor $\pi_1(Y_s)$ admits a homomorphism to $SU(2)$ with non-cyclic image. Then there is a holonomy-perturbation $(i, \phi)$ for the manifold $Y$ such that the perturbed representation variety $R^\phi_{\tau_0}(Y)$ is empty.

Proof. Fix a trivialization $\tau$ of $w$ over $N$. At this stage the choice is immaterial, because any two choices differ by an automorphism of $w$ that extends over all of $Y$. Write

$$Y_a = C \cup N_a,$$

$$Y_s = C \cup N_s,$$

where $N_a$ and $N_s$ are the solid tori from the Dehn surgery. The trivialization of $w$ over $\partial C$ allows us to extend $w$ to a line-bundle $w_a \to Y_a$ equipped with a trivialization $\tau_a$ over $N_a$, extending the given trivialization on $\partial C$. Note that $w_a$ is globally trivial on the homology 3-sphere $Y_a$, but the global trivialization differs from $\tau_a$ on the curve $b \subset \partial C$ by a map $b \to S^1$ of degree 1. This is because there is a surface $\Sigma \subset C$ with boundary $b$, and the original trivialization $\tau$ does not extend over $\Sigma$. The same remarks apply to $Y_s$.

On the manifold $Y_s$, in addition to constructing $w_a$ as above, we construct a different line bundle $\tilde{w}_s \to Y_s$ as follows. Let $\tilde{\tau}$ be the trivialization of $w|_{\partial C}$ with the property that $\tilde{\tau}\tau^{-1}$ is a map $\partial C \to S^1$ with degree $q$ on $b$ and degree 0 on $a$. Let $\tilde{w}_s$ be obtained by extending $w$ as a trivial bundle over $N_s$ extending the trivialization $\tilde{\tau}$.

If $p$ is odd, then $Y_s$ has $H^2(Y_s; \mathbb{Z}/2) = 0$. When $p$ is even, the construction of $\tilde{w}_s$ makes $c_1(\tilde{w}_s)$ divisible by 2. So in either case, elements of $R^\phi_{\tau_0}(Y_s)$ correspond to homomorphisms $\rho : \pi_1(Y_s) \to SU(2)$.

The following lemma is straightforward.
Lemma 10. Restriction to $C$ gives identifications
\[
R^w_a(Y_a) \to R^w(C | \alpha = 0) \\
R^w_s(Y_s) \to R^w(C | pa + q\beta = 0) \\
\tilde{R}^w_s(Y_s) \to R^w(C | pa + q\beta = q\pi)
\]

The manifold $C$ has $H_1(C;\mathbb{Z}) = \mathbb{Z}$, so the representation variety $R^w(C)$ contains reducibles. The next lemma describes their $\alpha$ and $\beta$ parameters.

Lemma 11. If $[A]$ is a reducible element of $R^w(C)$, then $(\alpha(A), \beta(A))$ lies on the line $\beta = \pi \mod 2\pi$.

Proof. If $[A]$ is a reducible element of $R^w(C)$, then $A$ is a flat $SO(3)$ connection on $C$ with cyclic holonomy. The holonomy around $b$ is the identity element of $SO(3)$ because $b$ bounds the surface $\Sigma$ in $C$. So the corresponding $SU(2)$ connection $\tilde{A}$ on $E|_b$ (regarding $E|_b$ as an $SU(2)$ bundle using $\tau$) has holonomy $\pm 1$ in $SU(2)$. It follows that $\beta$ is $0$ or $\pi$ mod $2\pi$. We can equip $w$ on $C$ with a connection $\theta$ which respects the trivialization $\tau$ on $\partial C$ and whose curvature $F_\theta$ integrates to $-2\pi i$ on $\Sigma$. The $SU(2)$ connection $\tilde{A}$ can be uniquely extended to a $U(2)$ connection $\hat{A}$ on all of $E|_C$, in such a way that the associated $SO(3)$ connection is $A$ and such that the induced connection on $\det(E) = w$ is $\theta$. The connection reduces $E$ to a sum of line bundles, both of which have curvature $F_\theta/2$. The holonomy of these line bundles on $b$ is given by
\[
\exp \int_{\Sigma} (F_\theta/2) = -1.
\]
So $\beta = \pi \mod 2\pi$ as claimed. This completes the proof of the lemma.

If we suppose that the homology-sphere $Y_a$ has a fundamental group with no non-trivial homomorphisms to $SU(2)$, then $R^w_a(Y_a)$ consists of a single reducible element. By the previous two lemmas, the $\alpha$ and $\beta$ parameters of this connection lie on the two line $\alpha = 0$ and $\beta = \pi$. So it is the point
\[
v_a = (0, \pi)
\]
mod $2\pi\mathbb{Z}^2$. Similarly the $\alpha$ and $\beta$ parameters of the reducible elements in $R^w_s(Y_s)$ lie on the line $pa + q\beta = \pi \mod 2\pi$ and the line $\beta = \pi$. So they are represented by the points
\[
v_{s,k} = (2\pi k/p, \pi)
\]
mod $2\pi\mathbb{Z}^2$. The next lemma is a standard result, from [11] of [1]. We supply the proof for completeness.

Lemma 12. Suppose $\pi_1(Y_a)$ admits no non-trivial homomorphisms to $SU(2)$. For any neighborhood $W$ of $(0, \pi)$, let us write
\[
W^* = W \cap \{\beta \neq \pi\}.
\]
Then there exists a symmetric neighborhood $W$ of $(0,\pi)$ such that 

$$R_w(C \mid W) = \emptyset.$$ 

**Proof.** The space $R_w(Y_a)$ consists of a single point, represented by the $SO(3)$ connection $A_a$ with trivial holonomy. By the one-to-one correspondence from Lemma 10, it follows that $R_w(C \mid (0,\pi))$ consists of a single point $[A]$ represented by an $SO(3)$ connection which trivializes $g_E$. We need only show that a neighborhood of $[A]$ in $R_w(C)$ consists entirely of reducibles. Equivalently, writing $\pi$ for $\pi_1(C)$, we can study a neighborhood of the trivial homomorphism $\rho_1: \pi \to SO(3)$ and show that it consists of reducible connections.

The deformations of $\rho_1$ are governed by $H^1(\pi; \mathbb{R}^3) = H^1(C) \otimes \mathbb{R}^3$, which is a copy of $\mathbb{R}^3$. It will be sufficient to exhibit a 1-parameter deformation of $\rho$ realizing any given vector in this $H^1$ as its tangent vector and consisting entirely of reducibles. This is straightforward. Given $\xi \in \mathfrak{so}(3)$, we can consider the 1-parameter family of connections in the trivial $SO(3)$ bundle given by the connection 1-forms $t\xi\eta$, where $\eta$ is a closed 1-form with period 1 on $C$ and $t \in \mathbb{R}$.

We need one more lemma before completing the proof of Proposition 9.

**Lemma 13.** For any $S$, there is a one-to-one correspondence between $R_w(C \mid S)$ and $R_w(C \mid S')$, where $S'$ is the translate $S + (\pi,0)$.

**Proof.** Let $\epsilon$ be an automorphism of the $U(2)$ bundle $E \to C$ whose determinant is a function $C \to S^1$ which has degree 1 on the curve $a$. (The automorphism $\epsilon$ does not belong to the gauge group $G$, because elements of $G$ have determinant 1.) The element $\epsilon$ acts on the space of flat connections $A$ in $A(C)$, and gives rise to a bijective self-map of the space $R_w(C)$: 

$$\bar{\epsilon}: R_w(C) \to R_w(C).$$

This map restricts to a bijection $\bar{\epsilon}: R_w(C \mid S) \to R_w(C \mid S')$.

We can now conclude the proof of the proposition. Suppose that $\pi_1(Y_a)$ admits only the trivial homomorphism to $SU(2)$, and that the only homomorphisms $\rho: \pi_1(Y_a) \to SU(2)$ are those with cyclic image. Let $L \subset \mathbb{R}^2$ be the closed line segment 

$$L = \{ (\alpha,\beta) \mid \alpha = 0, -\pi \leq \beta \leq \pi \}$$

and let $L^*$ be the open line-segment obtained by removing the endpoints. Let $L_+^*$ and $L_-^*$ be the translates of this line segment by the vectors $(\pi,0)$ and $(-\pi,0)$. By Lemmas 10 and 11, the hypothesis on $\pi_1(Y_a)$ means that 

$$R_w(C \mid L^*) = \emptyset.$$ 

By Lemma 13, we therefore have 

$$R_w(C \mid L_\pm^*) = \emptyset.$$
FIGURE 1. The set $S$, for $p/q = 5/3$. The $(\alpha, \beta)$ parameters of reducible elements of $R^w(C)$ lie on the dashed lines.

Let $P_1$ be the line

$$P = \{ p\alpha + q\beta = q\pi \}$$

and let $P_2 = P_1 - (0, 2\pi)$. The hypothesis on $\pi_1(Y_s)$ means that $R^w(C \mid P_i)$ consists only of reducibles, lying over the points on $P_i$ where $\beta = \pi \mod 2\pi$. Let $S \subset \mathbb{R}^2$ be the piecewise-linear arc with vertices at the points

- $z_1 = (-\pi, 0)$
- $z_2 = (-\pi, -(1 - p/q)\pi)$
- $z_3 = (0, -\pi)$
- $z_4 = (0, \pi)$
- $z_5 = (\pi, (1 - p/q)\pi)$
- $z_6 = (\pi, 0)$.

Figure 1 shows the set $S$ in the case $p/q = 5/3$. Because $p/q \leq 2$, the set is contained in the region $-\pi \leq \beta \leq \pi$. If $p/q = 2$, then $S$ has four points on the lines $\beta = \pm \pi$; otherwise it has just two.

Let $S^*$ be the complement in $S$ of the points whose $\beta$ coordinates are $\pm \pi$. Given any symmetric neighborhood $U$ of $S$, let $U^*$ similarly stand for

$$U^* = U \setminus \{ \beta = \pm \pi \}. \quad (2)$$

We know that $R^w(C \mid S^*) = \emptyset$, because $S$ is entirely contained in the union of $L$, $L_{\pm \pi}$ and the two lines $P_1$, $P_2$. From Lemma 12 and the compactness of $R^w(C)$, it follows that there is a symmetric neighborhood $U$ of $S$ such that

$$R^w(C \mid U^*) = \emptyset. \quad (3)$$

We now observe that, given any neighborhood $U$ of $S$, we can find a smooth odd function $g$ with period $2\pi$ such that the graph of $-g$ on the interval $[-\pi, \pi]$
is entirely contained in $U^*$. By Lemma 4 and Lemma 8 there exists a $\phi$ such that
\[
\mathcal{R}^w_{\iota,\phi}(Y) = \mathcal{R}^w(C \mid \beta = -g(\alpha)).
\]
The right hand side is empty because it is contained in the empty set (3). This finishes the proof of the proposition.

We can reformulate the result of Proposition 9 in the special case that $Y_a$ is $S^3$ as follows.

**Corollary 14.** Let $K$ be a knot in $S^3$ and let $Y_r$ be the manifold obtained by Dehn surgery with coefficient $r \in \mathbb{Q}$. Let $Y_0$ be the manifold obtained by 0-surgery, and let $v \to Y_0$ be a line bundle whose first Chern class is a generator of $H^2(Y_0;\mathbb{Z})$. Suppose $\pi_1(Y_r)$ admits no homomorphism $\rho$ to $SU(2)$ with non-cyclic image. Then, if $0 < r < 2$, the manifold $Y_0$ admits a holonomy deformation $(\iota,\phi)$ so that $\mathcal{R}^w_{\iota,\phi}(Y_0)$ is empty.

4. **Proof of the theorem**

(i) **A stretching argument**

Let $X$ be a closed, oriented 4-manifold containing a connected, separating 3-manifold $Y$. Let $g_1$ be metric on $X$ that is cylindrical on a collar region $[-1,1] \times Y$ containing $Y$ in $X$. For $L > 0$, let $X_L \cong X$ be the manifold obtained from $X$ by removing the piece $[-1,1] \times Y$ and replacing it with $[-L,L] \times Y$. There is a metric $g_L$ on $X_L$ that contains a cylindrical region of length of $2L$ and agrees with the original metric on the complement of the cylindrical piece.

Let $v \to X$ be a line bundle, let $E \to X$ be a unitary rank-2 bundle with $\det(E) = v$, and form the configuration space $B^v(X,E)$ of connections in $\mathfrak{g}_E$ modulo determinant-1 gauge transformations of $E$, as we did in the 3-dimensional case. In dimension 4, the bundle $E$ is not determined up to isomorphism by $v$ alone, so we include it in our notation. Inside $B^v(X,E)$ is the moduli space of anti-self-dual connections,

\[
M^v(X,E) = \{ [A] \in B^v(X,E) \mid F_A^+ = 0 \}.
\]

For each $L > 0$, we also have a moduli space

\[
M^v(X_L,E) \subset B^v(X_L,E).
\]

(We do not take the trouble to introduce the additional notation $v_L$ and $E_L$ for the corresponding bundles on $X_L$.)

Let $(\iota,\phi)$ be data for a holonomy perturbation for the bundle $E|_Y$. Following [10] [11] [4], we shall use $\phi$ also to perturb the anti-self-duality equations on $X_L$. We use $\iota$ to embed $[-L,L] \times S^1 \times D$ into $X_L$, and let $\mu_X$ be the 2-form on the cylindrical part $[-L,L] \times Y$ obtained by pulling back $\mu$ from $D$ and pushing forward using this embedding. We choose a trivialization of $v = \det(E)$ on the image of the embedding so that each $SO(3)$ connection in $\mathfrak{g}_E$ determines...
uniquely an \( SU(2) \) connection. For each \( A \), the holonomy around the circles defines, as before, a section \( H_A \) over \([-L, L] \times \text{im}(\iota)\) of the bundle \( \text{Aut}(E) \), and we obtain

\[
\phi'(H_A) \in C^\infty([-L, L] \times \text{im}(\iota); \mathfrak{g}_E).
\]

For \( L > 1 \), let \( \beta : X_L \to [0, 1] \) be a smooth cut-off function, supported in \([-L, L] \times Y\) and equal to 1 on \([-L + 1, L - 1] \times Y\). On \( X_L \), the perturbed anti-self-duality equation is the equation

\[
F_A^+ + \beta \phi'(H_A) \mu^+ = 0.
\]

We define the corresponding moduli space:

\[
M^v_\phi(X_L, E) = \{ [A] \in B^v(X_L, E) \mid \text{equation (4) holds} \}.
\]

**Proposition 15.** Let \( w = v|_Y \). Suppose that there is a holonomy perturbation on \( Y \) such that the perturbed representation variety \( R^v_{w, \iota, \phi}(Y) \) is empty. Then for each \( E \) with determinant \( v \) on \( X \), there exists an \( L_0 \) such that \( M^v_\phi(X_L, E) \) is also empty, for all \( L \geq L_0 \).

**Proof.** The proof is some subset of a standard discussion of holonomy perturbations and compactness in Floer homology theory (see [11, 1, 4]). Suppose on the contrary that we can find \([A_i]\) in \( M^v_\phi(X_L, E)\) for an increasing, unbounded sequence of lengths \( L_i \). We start as usual with the fact that the quantity

\[
\mathcal{E}(A_i) = \int_{X_{L_i}} \text{tr}(F_{A_i} \wedge F_{A_i})
\]

\[
= \|F_{A_i}^{-}\|^2 - \|F_{A_i}^{+}\|^2
\]

is independent of \( i \) and depends only on the Chern numbers of the bundle \( E \). (The norms are \( L^2 \) norms.) We write this quantity as the sum of three terms:

\[
\mathcal{E}(A_i) = \mathcal{E}(A_i \mid X^1) + \mathcal{E}(A_i \mid X^2) + \mathcal{E}(A_i \mid X^3),
\]

where

\[
X^1 = X_{L_i} \setminus \{[-L_i, L_i] \times Y\}
\]

\[
X^2 = \{[-L_i, -L_i + 1] \times Y\} \cup \{[L_i - 1, L_i] \times Y\}
\]

\[
X^3_i = [-L_i + 1, L_i - 1] \times Y.
\]

Only the third piece has a geometry which depends on \( i \). From the equation (4), we have

\[
\mathcal{E}(A_i \mid X^1) \geq 0
\]

because \( \beta \) is zero on \( X^1 \). The second term in equation (4) is pointwise uniformly bounded, so

\[
\mathcal{E}(A_i \mid X^2) \geq -C_2
\]

where \( C_2 \) is independent of \( i \). Because the sum of the three terms is constant, we deduce that

\[
\mathcal{E}(A_i \mid X^3) \leq K,
\]
where $K$ is independent of $i$.

To understand the term $\mathcal{E}(A_i | X^3_i)$ better, one must reinterpret (4). On $X^3_i$, the function $\beta$ is 1. Identify $E$ on this cylinder with the pull-back of a bundle $E_Y \to Y$, and choose a gauge representative $A_i$ for $[A_i]$ in temporal gauge. Write

$$A_i(t) = A_i|_{\{t\} \times Y}, \quad (-L_i + 1 \leq t \leq L_i - 1).$$

Thus $A_i(t)$ becomes a path in the space of connections $\mathcal{A}(Y; E_Y)$. The equation (4) is equivalent on $X^3_i$ to the condition that $A_i(t)$ solves the downward gradient flow equation for the perturbed Chern-Simons functional on $\mathcal{A}(Y; E_Y)$:

$$\frac{d}{dt} A_i(t) = -\text{grad}(\text{CS} + \Phi).$$

In particular, $\text{CS} + \Phi$ is monotone decreasing along the path (or constant). The function $|\Phi|$ is a bounded function on $\mathcal{A}(Y; E_Y)$: we can write

$$|\Phi| \leq K'.$$

The change in CS is equal to the quantity $-\mathcal{E}$: that is,

$$\text{CS}(A_i(-L_i + 1)) - \text{CS}(A_i(L_i - 1)) = \mathcal{E}(A_i | X^3_i) \leq K.$$

So from the bound on $|\Phi|$ we obtain

$$(\text{CS} + \Phi)(A_i(-L_i + 1)) - (\text{CS} + \Phi)(A_i(L_i - 1)) \leq K + 2K'.$$

Now let $\delta > 0$ be given. Because $\text{CS} + \Phi$ is decreasing and the total drop is bounded by $K + 2K'$, we can find intervals

$$(a_i, b_i) \subset [-L_i + 1, L_i + 1]$$

of length $\delta$, so that the drop in $\text{CS} + \Phi$ along $(a_i, b_i)$ tends to zero as $i$ goes to infinity. Because the equation is a gradient-flow equation, this means

$$\lim_{i \to \infty} \int_{a_i}^{b_i} \|\text{grad}(\text{CS} + \Phi)(A_i(t))\|_{L^2(Y)}^2 dt = 0.$$

We have an expression for $\text{grad}\Phi$ as a uniformly bounded form, so

$$\limsup_{i \to \infty} \int_{a_i}^{b_i} \|F_{A_i(t)}\|_{L^2(Y)}^2 dt \leq \delta J$$

for some constant $J$ depending on $\phi$. So given any $\epsilon > 0$, we can find a $\delta > 0$ and a sequence of intervals $(a_i, b_i)$ of length $\delta$ so that

$$\int_{(a_i, b_i) \times Y} |F_{A_i}|^2 \text{dvol} \leq \epsilon.$$
for all $i \geq i_0$. We now regard the $A_i$ as connections on the fixed cylinder $(0, \delta) \times Y$. At this point, if $\epsilon$ is smaller than the threshold for Uhlenbeck’s gauge fixing theorem on the 4-ball, we can find 4-dimensional gauge transformations on the cylinder so that, after applying these gauge transformations and passing to a subsequence, the connections converge in $C^\infty$ on compact subsets. (See for example [4, section 5.5].)

If $A$ is the limiting connection on $(0, \delta) \times Y$, in temporal gauge, then the function $CS + \Phi$ is constant along the path $A(t)$. It follows that $A(t)$ is constant and is a critical point of $CS + \Phi$. This tells us that $[A(t)]$ belongs to the perturbed representation variety $\mathcal{R}_w^{\omega, \phi}(Y)$, which we were supposing to be empty.

The proposition above has the following corollary for the Donaldson polynomial invariants. (Our notation and conventions for these invariants is taken from [14].)

**Corollary 16.** Let $X$ be an admissible 4-manifold in the sense of [14], so that its Donaldson polynomial invariants $D^\nu_X$ are defined. (For example, suppose $H_1(X; \mathbb{Z})$ is zero and $b_1^+(X)$ is greater than 1.) Then, under the assumptions of the previous proposition, the polynomial invariants are identically zero, regarded as a map

$$D^\nu_X : \mathbb{A}(X) \to \mathbb{Z}.$$  

**Proof.** The definition of $D^\nu_X$ involves first choosing a Riemannian metric on $X$ so that the moduli spaces $M^\nu(X, E)$ are smooth submanifolds of $B^\nu(X, E)$, containing no reducibles and cut out transversely by the equations. If $X$ is admissible, then this can always be done, by changing the metric inside a ball in $X$. The value of the invariant is then defined as a signed count of the intersection points between $M^\nu(X, E)$ and some specially-constructed finite-codimension submanifolds of $B^\nu(X, E)$. This part of the construction of $D^\nu_X$ involves only transversality arguments, which can be carried out equally with $M^\nu(X_L, E)$ in place of $M^\nu(X, E)$, for any fixed $L$. That the signed count is independent of the choices made, in the unperturbed setting, is a consequence of the compactness theorem for the moduli space. The Uhlenbeck compactification works the same way for $M^\nu(X_L, E)$ as it does for the unperturbed anti-self-duality equations (see [4] for example); so the Donaldson invariants can be defined using the perturbed moduli spaces. Each moduli space is empty once $L$ is large enough, so the invariants are zero.

**(ii) Concluding the proof**

The rest of the argument is essentially the same as the proof of the main theorem in [13]. Let $K$ be a knot in $S^3$ that is a counterexample to Theorem [14]; we will obtain a contradiction.

The manifold $Y_0$ obtained by zero-surgery admits a taut foliation and is not $S^3 \times S^2$, by the results of [12]. The following proposition is proved in [13] using the results of [7] and [8].
Proposition 17. Let $Y$ be a closed orientable 3-manifold admitting an oriented taut foliation. Suppose $Y$ is not $S^1 \times S^2$. Then $Y$ can be embedded as a separating hypersurface in a closed symplectic 4-manifold $(X, \Omega)$. Moreover, we can arrange that $X$ satisfies the following additional conditions.

(a) The first homology $H_1(X; \mathbb{Z})$ vanishes.

(b) The euler number and signature of $X$ are the same as those of some smooth hypersurface in $\mathbb{CP}^3$, whose degree is even and not less than 6.

(c) The restriction map $H^2(X; \mathbb{Z}) \to H^2(Y; \mathbb{Z})$ is surjective.

(d) The manifold $X$ contains a tight surface of positive self-intersection number, and a sphere of self-intersection $-1$.

We apply this proposition to the manifold $Y_0$, to obtain an $X$ with all of the above properties. Using the results of [8], it was shown in [13] that a 4-manifold satisfying these conditions satisfies Witten’s conjecture relating the Seiberg-Witten and Donaldson invariants. (See [13, Conjecture 5 and Corollary 7] for an appropriate statement of Witten’s conjecture in this context.) Because $X$ is symplectic, its Seiberg-Witten invariants are non-trivial by [14]. For the same reason, $X$ has Seiberg-Witten simple type. From Witten’s conjecture, it follows that the Donaldson invariants $D^v_X$ are non-trivial, for all $v$ on $X$.

By the penultimate condition on $X$ in Proposition 17, we can choose $v \to X$ so that $c_1(v)$ restricts to a generator of $H^2(Y_0; \mathbb{Z})$. Write $w = v|_{Y_0}$. If $K$ is a counterexample to Theorem 1, then Corollary 14 tells us there is a holonomy perturbation $\phi$ such that

$$R^w_{\gamma, \phi}(Y_0) = \emptyset.$$  

Corollary 16 then tells us that $D^v_X$ is zero. This is the contradiction. □

(iii) Further remarks

An analysis of the proof of Theorem 1 reveals that it proves a slightly stronger result (stronger, that is, if one is granted the results of [12]). For example, we can state:

Theorem 18. Let $N$ be an embedded solid torus in an irreducible closed 3-manifold $Y$ with $H_1(Y) = \mathbb{Z}$. Let $C = Y \setminus N^c$ be the complementary manifold with torus boundary.

Then there is at most one Dehn filling of $C$ which yields a homotopy sphere. Indeed, for all but one slope, the fundamental group of the manifold obtained by Dehn filling admits a non-trivial homomorphism to $SU(2)$. □

The point here is that the original hypothesis need not be that $K$ is a non-trivial knot in $S^3$. What one wants is that zero-surgery on $K$ should be an irreducible homology $S^1 \times S^2$; and if we make this our hypothesis, then we can also consider the case that $K$ is a knot in (for example) a homotopy sphere.

One can also ask whether there is a non-trivial extension of Theorem 1 to other integer surgeries. The results of [15] show that surgery with coefficient 3
or 4 on a non-trivial knot cannot be a lens space. It would be interesting to know whether the fundamental groups of $Y_3$ and $Y_4$ must admit homomorphisms to $SU(2)$ with non-abelian image when $K$ is non-trivial. Surgery with coefficient +5 on the right-handed trefoil produces a lens space, so one does not expect to extend Theorem further in the direction of integer surgeries without additional hypotheses. Dunfield [5] has provided an example of a non-trivial knot in $S^3$ for which the Dehn filling $Y_{37/2}$ has a fundamental group which is not cyclic but admits no homomorphism to $SU(2)$ (or even $SO(3)$) with non-abelian image. (The knot is the $(-2,3,7)$ pretzel knot, for which $Y_{18}$ and $Y_{19}$ are both lens spaces [9].) This example shows that the property of having cyclic fundamental group and the property of admitting no cyclic homomorphic image in $SU(2)$ are in general different for 3-manifolds obtained by Dehn surgery.

References

[1] P. J. Braam and S. K. Donaldson, *Floer’s work on instanton homology, knots and surgery*, The Floer memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 195–256.

[2] M. Culler, C. McA. Gordon, J. Luecke, and P. B. Shalen, *Dehn surgery on knots*, Ann. of Math. (2) **125** (1987), no. 2, 237–300.

[3] S. K. Donaldson, *The orientation of Yang-Mills moduli spaces and 4-manifold topology*, J. Differential Geom. **26** (1987), no. 3, 397–428.

[4] ———, *Floer homology groups in Yang-Mills theory*, Cambridge Tracts in Mathematics, vol. 147, Cambridge University Press, Cambridge, 2002, With the assistance of M. Furuta and D. Kotschick.

[5] N. M. Dunfield, Private communication.

[6] Y. M. Eliashberg, *Few remarks about symplectic filling*, arXiv:math.SG/0311459, 2003.

[7] Y. M. Eliashberg and W. P. Thurston, *Confoliations*, University Lecture Series, no. 13, American Mathematical Society, 1998.

[8] P. M. N. Feehan and T. G. Leness, *A general SO(3)-monopole cobordism formula relating Donaldson and Seiberg-Witten invariants*, arXiv:math.DG/0203047, 2003.

[9] R. Fintushel and R. J. Stern, *Constructing lens spaces by surgery on knots*, Math. Z. **175** (1980), no. 1, 33–51.

[10] A. Floer, *An instanton-invariant for 3-manifolds*, Comm. Math. Phys. **118** (1988), no. 2, 215–240.

[11] ———, *Instanton homology and Dehn surgery*, The Floer memorial volume, Progr. Math., vol. 133, Birkhäuser, Basel, 1995, pp. 77–97.
[12] D. Gabai, *Foliations and the topology of 3-manifolds. III*, J. Differential Geom. 26 (1987), no. 3, 479–536.

[13] P. B. Kronheimer and T. S. Mrowka, *Witten’s conjecture and Property P*, arXiv:math.GT/0311489.

[14] _____, *Embedded surfaces and the structure of Donaldson’s polynomial invariants*, J. Differential Geom. 41 (1995), no. 3, 573–734.

[15] P. B. Kronheimer, T. S. Mrowka, P. S. Ozsvath, and Z. Szabo, *Monopoles and lens space surgeries*, arXiv:math.GT/0310164.

[16] L. Moser, *Elementary surgery along a torus knot*, Pacific J. Math. 38 (1971), 737–745.

[17] C. H. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. 1 (1994), no. 6, 809–822.

[18] Clifford Henry Taubes, *Casson’s invariant and gauge theory*, J. Differential Geom. 31 (1990), no. 2, 547–599.