BOSE-EINSTEIN CONDENSATION
AND SPONTANEOUS SYMMETRY BREAKING

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After recalling briefly the connection between spontaneous symmetry breaking and off-diagonal long range order for models of magnets a general proof of spontaneous breaking of gauge symmetry as a consequence of Bose-Einstein Condensation is presented. The proof is based on a rigorous validation of Bogoliubov’s c-number substitution for the \( k = 0 \) mode operator \( a_0 \).

Keywords: Bose-Einstein condensation, gauge symmetry, c-number substitutions

1. Introduction

The connection between Bose-Einstein Condensation (BEC) and spontaneous gauge symmetry breaking (GSB) is often taken for granted, usually with the following argument that appeals to Bogoliubov’s pioneering work [1]: BEC means that in the thermodynamic limit the density \( V^{-1} \langle a_0^* a_0 \rangle \) of particles in the mode of lowest momentum \( k = 0 \) is

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[1]
different from zero. The spontaneous breaking of gauge symmetry, on the other hand, is associated with the non-vanishing of $V^{-1}\langle a_0 \rangle^2$ in the thermodynamic limit. Here it is important that the expectation value is understood in the sense of Bogoliubov’s ‘quasi averages’, i.e., with a symmetry breaking field that is taken to zero after the thermodynamic limit. By the Cauchy-Schwarz inequality, we always have $\langle a_0 \rangle^2 \leq \langle a_0^* a_0 \rangle$. Thus GSB implies BEC. On the other hand, if BEC holds, the macroscopic occupation of the zero mode is claimed to justify the replacement of the operators $a_0$ and $a_0^*$ by c-numbers and hence equating $\langle a_0 \rangle^2$ with $\langle a_0^* a_0 \rangle$. Therefore BEC and GSB are equivalent.

A rigorous justification of Bogoliubov’s c-number substitution is not a trivial matter, however, and the reasoning above is clearly not a mathematical proof. In a classic paper [2], that is less known in the physics community than it deserves, Ginibre established the validity of the substitution as far as the computation of the pressure in the thermodynamic limit is concerned. As Ginibre pointed out, it is not necessary to assume macroscopic occupancy of the zero mode (although the substitution is only a useful computational tool if this is the case.) In a recent paper [3] (see also [4], Appendix D) we presented a simple and general proof of the exactness of c-number substitutions in bosonic Hamiltonians in the thermodynamic limit, using the Berezin-Lieb inequality [5, 6, 7] (which was only derived after Ginibre’s paper appeared). We then proved the equivalence of GSB and BEC, where the expectation values are defined as quasi averages in the sense of Bogoliubov, and moreover that BEC in the standard sense without a symmetry breaking field implies BEC in the quasi average sense and hence GSB. This last statement, although intuitively plausible [8], does not seem to have been rigorously verified before. In brief:

$$\text{BEC} \rightarrow \langle \text{BEC}\rangle_{\text{qa}} \leftrightarrow \text{GSB}. \quad (1)$$

The present contribution to the Proceedings of the 21st Max Born Symposium gives an account of the proof. For comparison and for pedagogical reasons we start by recalling briefly the connection between spontaneous symmetry breaking and off-diagonal long range order for a model of a ferromagnet. An important difference between this case and that of a Bose gas is that the symmetry breaking term in the latter case does not commute with the Hamiltonian.

Another proof of $\langle \text{BEC}\rangle_{\text{qa}} \leftrightarrow \text{GSB}$, based on Ginibre’s original results about the c-number substitutions, was given by Sütő in [9]. A rather different approach, with focus on operator algebraic aspects and correlation inequalities, can be found in a 1982 paper by Fannes, Pulé and Verbeure [10]. see also a recent generalization to nonhomogeneous systems by Pulé, Verbeure and Zagrebnov [11]. In these papers the equilibrium states on the Weyl algebra for the infinite system and their decomposition into symmetry breaking states are studied. Some technical assumptions about these states are required in [10] and [11] but in Remark 2.1 in [11] it is claimed that the conclusions are valid under weaker assumptions. The algebraic approach sheds a different light on the problem and is complementary to our method where essentially the only prerequisites are the existence of the thermodynamic limit of the grand canonical pressure and average density.

Finally we remark that the Berezin-Lieb inequality used in our approach gives explicit
error bounds that allow us to go beyond [2] and make c-number substitutions for many k-modes at once, provided the number of modes is lower order than the particle number. This method, that was used in a different context in [12], has recently proved to be a powerful tool for solving other problems about the Bose gas [13, 14].

2. Symmetry Breaking and Off-diagonal Long Range Order in Magnets

The concept of spontaneous symmetry breaking means quite generally that a symmetry of the Hamiltonian or Lagrangian of a system is not present in the state under consideration (usually a ground state or a thermal equilibrium state). A Heisenberg ferromagnet with a Hamiltonian of the form

\[ H = H_0 - BM \quad \text{with} \quad H_0 = - \sum_{x \neq y \in \Lambda} J_{xy} \vec{S}_x \cdot \vec{S}_y \quad \text{and} \quad M = \sum_{x \in \Lambda} S_x^{(3)} \]  

(2)

provides a well known example to illustrate this phenomenon. Here \( \Lambda \subset \mathbb{Z}^d \) is a lattice cube of finite size \( |\Lambda| \) and periodic boundary conditions are assumed. The spin operators \( \vec{S}_x = (S_x^{(1)}, S_x^{(2)}, S_x^{(3)}) \) with \( [S^{(1)}, S^{(2)}] = iS^{(3)} \) etc act on a copy of \( \mathbb{C}^{2s+1} \) (with \( s \in \{\frac{1}{2}, 1, \ldots\} \)) at each lattice site and operators on different sites commute. The coefficients \( J_{x,y} \geq 0 \) measure the interactions between different sites and \( B \) denotes the strength of a magnetic field in the 3-direction. If \( B = 0 \) the Hamiltonian is invariant under the transformations \( \vec{S}_x \rightarrow R \vec{S}, R \in O(3) \). The last term in the Hamiltonian explicitly breaks this symmetry if \( B \neq 0 \).

The thermal equilibrium state at temperature \( T \) in the finite lattice is defined by

\[ \langle A \rangle_{B,T,\Lambda} = \frac{\text{Tr} A e^{-H/kT}}{\text{Tr} e^{-H/kT}} \]  

(3)

and the magnetization, i.e., the thermal average of the magnetic moment per spin, is

\[ m_{\Lambda}(B,T) = |\Lambda|^{-1} \langle M \rangle_{B,T,\Lambda}. \]  

(4)

The Gibbs free energy per spin,

\[ g_{\Lambda}(B,T) = -|\Lambda|^{-1}kT \ln \text{Tr} e^{-H/kT} \]  

(5)

is a concave function of \( B \) with

\[ m_{\Lambda}(B,T) = -\frac{\partial g_{\Lambda}(B,T)}{\partial B}. \]  

(6)

Since \( T \) may be regarded as fixed in the following we shall drop it in the notation, and in order to avoid irrelevant factors, choose energy units so that \( kT = 1 \).

For a finite \( \Lambda \), \( m_{\Lambda}(B) \) is continuous in \( B \), and for \( B = 0 \) we always have

\[ m_{\Lambda}(0) = 0 \]  

(7)
because the state is invariant under a unitarily implemented rotation that takes \( S_x^{(3)} \rightarrow -S_x^{(3)} \). On the other hand, the thermodynamic limit of the magnetization

\[
m(B) = \lim_{\Lambda \to \mathbb{Z}^d} m_{\Lambda}(B)
\]

may break the symmetry for sufficiently low \( T \), i.e., it is possible that

\[
m_\pm \equiv \lim_{B \to 0^+} m(B) \neq 0.
\]

Since the Hamiltonian is invariant under simultaneous flipping of the spins and \( B \) we always have \( m_- = -m_+ \), so if \( m_\pm \neq 0 \) the Gibbs free energy in the thermodynamic limit, \( g(B) = \lim_{\Lambda \to \mathbb{Z}^d} g_{\Lambda}(B) \), is not differentiable at \( B = 0 \). As a concave function it is, however, differentiable almost everywhere.

We may also consider the average of the square of the magnetic moment per spin, i.e.,

\[
m_\Lambda^2(B) \equiv \left\langle \frac{(M/|\Lambda|)^2}{B,T,\Lambda} \right\rangle = \left\langle |\Lambda|^{-2} \sum_{x,y \in \Lambda} S_x^3 S_y^3 \right\rangle_{B,T,\Lambda}
\]

with a corresponding thermodynamic limit \( m^2(B) \). By the Cauchy-Schwarz inequality we always have

\[
m^2(B) \geq (m(B))^2.
\]

Hence, spontaneous symmetry breaking in the sense that \( \lim_{B \to 0^+} m(B) \neq 0 \), implies off-diagonal long range order in the sense that

\[
\lim_{B \to 0^+} m^2(B) > 0.
\]

It is important to note, however, that it is a priori not clear that this implies \( m^2(0) > 0 \) for reasons explained further below.

On the other hand, a general argument due to Griffiths [15] shows that

\[
m^2(0) \leq \lim_{B \to 0^+} m^2(B) = \lim_{B \to 0^+} (m(B))^2.
\]

The argument is based on two facts. The first is the commutativity of the symmetry breaking term, \( BM \), with the other part, \( H_0 \), of the Hamiltonian. This allows a simultaneous diagonalization of the two parts. The second fact is Griffith’s lemma that we recall for completeness (for a proof see [15] or [16]):

**Griffith’s Lemma**[15] Let \( \mu_n \) be a sequence of probability measures on \( \mathbb{R} \) such that

\[
f(y) = \lim_{n \to \infty} \frac{1}{n} \ln \int e^{xy} d\mu_n(x)
\]

exists for all \( y \) in an interval around 0. Define

\[
a_\pm = \lim_{y \to 0^+} \frac{f(\pm y) - f(0)}{y}.
\]
Then there is a $c < 1$ such that, for any $\varepsilon > 0$,
\[
\int_{(a_+ - \varepsilon)n}^{(a_+ + \varepsilon)n} d\mu_n(x) = 1 - O(c^n).
\] (16)

In other words: The scaled measure, $d\mu_n(n\xi)$, is supported in $a_- \leq \xi \leq a_+$, with exponential accuracy, as $n \to \infty$.

To apply this lemma to the situation under consideration we take $x = B$, $f(B) = g(0) - g(B)$, $n = |A|$ and $d\mu_n(n\xi)$ the probability distribution at $B = 0$ for the magnetic moment per spin, $\xi = m = |A|^{-1}M$. Then $a_\pm = m_\pm$. Moreover, $f$ is a convex function of $B$ so the derivative with respect to $B$ is monotonously increasing. We can now conclude two things: 1) At points where $f$ is differentiable, i.e., for almost all $B \neq 0$, the probability distribution of the magnetization is a delta function concentrated at $m(B)$ and hence $m^2(B) = (m(B))^2$. 2) The probability distribution for the magnetization at $B = 0$ in the thermodynamic limit is concentrated in the interval $[m_-, m_+]$. Since by convexity of $f$ we have $m_+ \leq m(B)$ for all $B > 0$, the statements 1) and 2) together imply $m^2(0) \leq m^2(B)$ for $B > 0$. Altogether Eq. (13) is thus established. It is also clear that unless the probability distribution is concentrated at the end points of the interval $[m_-, m_+]$ the inequality $m^2(0) \leq m^2(B)$ is strict. This is exactly what happens for the Heisenberg ferromagnet because as remarked in [15] the probability distribution is a monotonously decreasing function of $|m|$.

3. Bosonic Hamiltonians

We start with the well-known Hamiltonian for bosons in a large box of volume $V$, expressed in terms of the second-quantized creation and annihilation operators $a_k, a_k^*$ satisfying the canonical commutation relations,
\[
H = \sum_k k^2 a_k^* a_k + \frac{1}{2V} \sum_{k,p,q} \nu(p) a_{k+p}^* a_{q-p}^* a_k a_q
\] (17)
(with $\hbar = 2m = 1$). Here, $\nu$ is the Fourier transform of the two-body potential $v(r)$. We assume that there is a bound on the Fourier coefficients $|\nu(k)| \leq \varphi < \infty$. The case of hard core potentials can be taken care of by cutting off the potential at some finite value that is taken to infinity at the end of the calculations.

We shall work in the grand canonical ensemble and hence define $H_\mu = H - \mu N$ where $\mu$ is a chemical potential and $N = \sum_k a_k^* a_k$ the number operator that generates the gauge symmetry. We also add a gauge symmetry breaking term and define
\[
H_{\mu,\lambda} = H_\mu + \sqrt{V} (\lambda a_0 + \lambda^* a_0^*)
\] (18)
with a complex parameter $\lambda$ that we can take to be a real number without loss of generality. While $H_\mu$ commutes with $N$ this is no longer the case for $H_{\mu,\lambda}$ if $\lambda \neq 0$. 
We now come to the e-number substitution for \( a_0 \) and \( a_0^* \). Let \( z \) be a complex number, \(|z| = \exp\left(-|z|^2/2 + za_0^*\right) |0\rangle\) the coherent state vector in the \( a_0 \) Fock space and let \( \Pi(z) = |z\rangle\langle z| \) be the projector onto this vector. There are six relevant operators containing \( a_0 \) in \( H_{\mu,\lambda} \), which have the following expectation values \[17\] (called lower symbols)

\[
\begin{align*}
\langle z|a_0|z\rangle &= z, \\
\langle z|a_0^*a_0^*|z\rangle &= z^2, \\
\langle z|a_0^*a_0|z\rangle &= |z|^2.
\end{align*}
\]

Each also has an upper symbol, which is a function of \( z \) (call it \( u(z) \) generically) such that an operator \( F \) is represented as \( F = \int d^2z u(z)\Pi(z) \), where \( d^2z \equiv \pi^{-1} dxdy \) with \( z = x + iy \). These symbols are

\[
\begin{align*}
a_0 &\rightarrow z, \\
a_0^*a_0 &\rightarrow z^2, \\
a_0^* &\rightarrow z^*, \\
a_0^*a_0^* &\rightarrow z^{*2}, \\
a_0^*a_0a_0^* &\rightarrow |z|^4 - 1, \\
a_0^*a_0^*a_0^*a_0^* &\rightarrow |z|^4 - 4|z|^2 + 2.
\end{align*}
\]

We denote by \( H_{\mu,\lambda}'(z) \) and \( H_{\mu,\lambda}''(z) \) the operators obtained from \( H_{\mu,\lambda} \) by replacing the polynomials in \( a_0 \) and \( a_0^* \) by their lower and upper symbols respectively. These operators act on the Fock-space of all the modes other than the \( a_0 \) mode. One has

\[
H_{\mu}''(z) = H_{\mu}'(z) + \delta_{\mu}(z)
\]

with

\[
\delta_{\mu}(z) = \mu + \frac{1}{2V} \left[ (-4|z|^2 + 2)\nu(0) - \sum_{k \neq 0} a_k^*a_k (2\nu(0) + \nu(k) + \nu(-k)) \right].
\]

The partition functions corresponding to these operators are defined by

\[
\begin{align*}
e^{\beta V p(\mu,\lambda)} &\equiv \Xi(\mu,\lambda) = \text{Tr}_\mathcal{H} \exp[-\beta H_{\mu,\lambda}] \\
e^{\beta V p'(\mu,\lambda)} &\equiv \Xi'(\mu,\lambda) = \int d^2z \text{Tr}_{\mathcal{H}'} \exp[-\beta H_{\mu,\lambda}'(z)] \\
e^{\beta V p''(\mu,\lambda)} &\equiv \Xi''(\mu,\lambda) = \int d^2z \text{Tr}_{\mathcal{H}'} \exp[-\beta H_{\mu,\lambda}''(z)]
\end{align*}
\]

where \( \mathcal{H} \) is the full Hilbert (Fock) space and \( \mathcal{H}' \) is the Fock space without the \( a_0 \) mode. The functions \( p(\mu,\lambda) \), \( p'(\mu,\lambda) \) and \( p''(\mu,\lambda) \) are the corresponding finite volume pressures and we shall soon see that in the thermodynamic limit they all coincide. This follows from the inequalities

\[
\Xi'(\mu,\lambda) \leq \Xi(\mu,\lambda) \leq \Xi''(\mu,\lambda) \leq \Xi'(\mu + 2\bar{\varphi}/V,\lambda)e^{\beta(|\mu|+\bar{\varphi}/V)}
\]

which we now prove. The first step is

\[
\Xi(\mu,\lambda) \geq \Xi'(\mu,\lambda),
\]
which is a consequence of the following two facts: The completeness property of coherent states, $\int d^2z \Pi(z) = \text{Identity}$, and

$$\langle z \otimes \phi | e^{-\beta H_{\mu,\lambda}(z)} | z \otimes \phi \rangle \geq e^{-\beta \langle z \otimes \phi | H_{\mu,\lambda}(z) | z \otimes \phi \rangle} = e^{-\beta \langle \phi | H_{\mu,\lambda}(z) | \phi \rangle}, \quad (26)$$

where $\phi$ is any normalized vector in $\mathcal{H}'$. This is Jensen’s inequality for the expectation value of a convex function (like the exponential function) of an operator. To prove (25) we take $\phi$ in (26) to be one of the normalized eigenvectors of $H'_{\mu,\lambda}(z)$, in which case

$$\exp\{\langle \phi | -\beta H'_{\mu,\lambda}(z) | \phi \rangle\} = \langle \phi | \exp\{-\beta H'_{\mu,\lambda}(z)\} | \phi \rangle.$$ We then sum over all such eigenvectors (for a fixed $z$) and integrate over $z$. The left side is then $\Xi(\mu, \lambda)$, while the right side is $\Xi'(\mu, \lambda)$.

The second inequality is the Berezin-Lieb inequality [5, 6, 7]

$$\Xi(\mu, \lambda) \leq \Xi''(\mu, \lambda). \quad (27)$$

Its proof is the following. Let $|\Phi_j\rangle \in \mathcal{H}$ denote the complete set of normalized eigenfunctions of $H_{\mu,\lambda}$. The partial inner product $|\Psi_j(z)| = \langle z | \Phi_j \rangle$ is a vector in $\mathcal{H}'$ whose square norm, given by $c_j(z) = \langle \Psi_j(z) | \Psi_j(z) \rangle_{\mathcal{H}'}$, satisfies $\int d^2z c_j(z) = 1$. By using the upper symbols, we can write

$$(\Phi_j | H_{\mu,\lambda} | \Phi_j) = \int d^2z \langle \Psi_j(z) | H''_{\mu,\lambda}(z) | \Psi_j(z) \rangle = \int d^2z \langle \Psi'_j(z) | H''_{\mu,\lambda}(z) | \Psi'_j(z) \rangle c_j(z),$$

where $|\Psi'_j(z)\rangle$ is the normalized vector $c_j(z)^{-1/2}\Psi_j(z)$. To compute the trace, we can exponentiate this to write $\Xi(\mu, \lambda)$ as

$$\sum_j \exp \left\{-\beta \int d^2z c_j(z) \langle \Psi'_j(z) | H''_{\mu,\lambda}(z) | \Psi'_j(z) \rangle \right\}.$$ Using Jensen’s inequality twice, once for functions and once for expectations as in (26), $\Xi(\mu, \lambda)$ is less than

$$\sum_j \int d^2z c_j(z) \exp \left\{ \langle \Psi'_j(z) | -\beta H''_{\mu,\lambda}(z) | \Psi'_j(z) \rangle \right\} \leq \sum_j \int d^2z c_j(z) \langle \Psi'_j(z) | \exp \left\{ -\beta H''_{\mu,\lambda}(z) \right\} | \Psi'_j(z) \rangle.$$

Since $\text{Tr} \Pi(z) = 1$, the last expression can be rewritten as

$$\int d^2z \sum_j \langle \Phi_j | \Pi(z) \otimes \exp \left\{ -\beta H''_{\mu,\lambda}(z) \right\} | \Phi_j \rangle = \Xi''(\mu, \lambda)$$

and (27) is proved.
For the last inequality in (23) we have to bound $\delta \mu(z)$ in (20). This is easily done in terms of the total number operator whose lower symbol is $N'(z) = |z|^2 + \sum_{k \neq 0} a_k^* a_k$. In terms of the bound $\varphi$ on $\nu(p)$

$$|\delta \mu(z)| \leq 2\varphi(N'(z) + \frac{1}{V} + |\mu|).$$

(28)

Consequently, $\Xi''(\mu, \lambda)$ and $\Xi'(\mu, \lambda)$ are related by the inequality

$$\Xi''(\mu, \lambda) \leq \Xi'(\mu + 2\varphi/V, \lambda) e^{\beta(|\mu| + \varphi/V)}.$$  

(29)

Closely related to this point is the question of relating $\Xi(\mu, \lambda)$ to the maximum value of the integrand in (23), which is $\max_z \text{Tr}_{H'} \exp[-\beta H''_{\mu,\lambda}(z)] = \exp(\beta V p_{\text{max}})$. This latter quantity is often used in discussions of the $z$ substitution problem, e.g., in refs. [2, 18].

One direction is not hard. It is the inequality (used in ref. [2])

$$\Xi(\mu, \lambda) \geq \max_z \text{Tr}_{H'} \exp[-\beta H''_{\mu,\lambda}(z)],$$

(30)

and the proof is the same as the proof of (25), except that this time we replace the completeness relation for the coherent states by the simple inequality

$$\geq \Pi(z)$$

for any fixed number $z$.

For the other direction, split the integral defining $\Xi''(\mu, \lambda)$ into a part where $|z|^2 < \xi$ and $|z|^2 \geq \xi$. Thus,

$$\Xi''(\mu, \lambda) \leq \xi \max_z \text{Tr}_{H'} \exp[-\beta H''_{\mu,\lambda}(z)] + \frac{1}{\xi} \int \int \text{Tr}_{H'} \exp[-\beta H''_{\mu,\lambda}(z)].$$

(31)

Dropping the condition $|z|^2 \geq \xi$ in the last integral and using $|z|^2 \leq N'(z) = N''(z)+1$, we see that the last term on the right side of (31) is bounded above by $\xi^{-1} \Xi''(\mu, \lambda)[V \rho''(\mu, \lambda) + 1]$, where $\rho''(\mu, \lambda)$ denotes the density in the $H''_{\mu,\lambda}$ ensemble. Optimizing over $\xi$ leads to

$$\Xi''(\mu, \lambda) \leq 2[V \rho''(\mu, \lambda) + 1] \max_z \text{Tr}_{H'} \exp[-\beta H''_{\mu,\lambda}(z)].$$

(32)

Note that $\rho''(\mu, \lambda)$ is order one, since $p''(\mu, \lambda)$ and $p'(\mu, \lambda)$ agree in the thermodynamic limit (and are convex in $\mu$), and we assume that the density in the original ensemble is finite. By (28), $H''_{\mu,\lambda} \geq H'_\mu + 2\varphi/V - |\mu| - \varphi/V$, and it follows from (27), (32) and (30) that $p_{\text{max}}$ agrees with the true pressure $p$ in the thermodynamic limit. Their difference, in fact, is at most $O(\ln V/V)$. This is the result obtained by Ginibre in [2] by more complicated arguments, under the assumption of super-stability of the interaction, and without the explicit error estimates obtained here.

To summarize the situation so far, we have four expressions for the grand-canonical pressure and they are all equal in the thermodynamic limit

$$p(\mu, \lambda) = p'(\mu, \lambda) = p''(\mu, \lambda) = p_{\text{max}}(\mu, \lambda)$$

(33)

when $(\mu, \lambda)$ is not a point at which the density can be infinite.
The expectation values $\langle a_0^* a_0 \rangle_{\mu, \lambda}$ and $\langle a_0 \rangle_{\mu, \lambda}$ are obtained by integrating $(|z|^2 - 1)$ and $z$, respectively, with the weight $W_{\mu, \lambda}(z)$, given by

$$W_{\mu, \lambda}(z) \equiv \Xi(\mu, \lambda)^{-1} \text{Tr}_{H'} \langle z \mid \exp\{-\beta H_{\mu, \lambda}\}\rangle_z.$$

We will show that for almost every $\lambda$, the density $W_{\mu, \lambda}(\sqrt{V})$ converges in the thermodynamic limit to a $\delta$-function at the point $\zeta_{\text{max}} = \lim_{V \to \infty} z_{\text{max}}/\sqrt{V}$, where $z_{\text{max}}$ maximizes the partition function $\text{Tr}_{H'} \exp\{-\beta H_{\mu, \lambda}(z)\}$.

That is,

$$V^{-1} \delta_{\langle a_0^* a_0 \rangle_{\mu, \lambda}} = V^{-1} |\langle a_0 \rangle_{\mu, \lambda}|^2 = V^{-1} |z_{\text{max}}|^2$$

in the thermodynamic limit. This holds for those $\lambda$ where the pressure in the thermodynamic limit is differentiable; since $p(\mu, \lambda)$ is convex (upwards) in $\lambda$ this is true almost everywhere. The right and left derivatives exist for every $\lambda$ and hence the quasi average

$$\lim_{\lambda \to 0^+} \lim_{V \to \infty} V^{-1} |\langle a_0 \rangle_{\mu, \lambda}|^2$$

exists.

If we could replace $W_{\mu, \lambda}(z)$ by $W_{\mu, 0}(z)e^{-\beta \lambda \sqrt{V}(z+z^*)}$, the convergence to a $\delta$-function would follow from Griffiths’ argument in the same way as for the magnetic model in the previous section. Because $[H, a_0] \neq 0$, $W_{\mu, \lambda}$ is not of this product form. However, the weight for $\Xi''(\mu, \lambda)$, which is

$$W_{\mu, \lambda}(z) \equiv \Xi''(\mu, \lambda)^{-1} \text{Tr}_{H'} \exp\{-\beta H_{\mu, \lambda}(z)\},$$

does have the right form. In the following we shall show that the two weights are equal apart from negligible errors.

Equality (33) holds also for all $\lambda$, i.e., $p(\mu, \lambda) = p''(\mu, \lambda) = p''(\mu, \lambda)$ in the thermodynamic limit. In fact, since the upper and lower symbols agree for $a_0$ and $a_0^*$, the error estimates above remain unchanged. (Note that since $\sqrt{V} |a_0 + a_0^*| \leq \delta (N + \frac{1}{2}) + V/\delta$ for any $\delta > 0$, $p(\mu, \lambda)$ is finite for all $\lambda$ if it is finite for $\lambda = 0$ in a small interval around $\mu$.) At any point of differentiability with respect to $\lambda$, Griffiths’ Lemma [15] (see the previous section), applied to the partition function $\Xi''(\mu, \lambda)$, implies that $W_{\mu, \lambda}'(\sqrt{V})$ converges to a $\delta$-function at some point $\hat{\zeta}$ on the real axis as $V \to \infty$. (The original Griffiths argument can easily be extended to two variables, as we have here. Because of radial symmetry, the derivative of the pressure with respect to $\text{Im} \lambda$ is zero at any non-zero real $\lambda$.) Moreover, by comparing the derivatives of $p''$ and $p''_{\text{max}}$ we see that $\hat{\zeta} = \lim_{V \to \infty} z_{\text{max}}/\sqrt{V}$, since $z_{\text{max}}/\sqrt{V}$ is contained in the interval between the left and right derivatives of $p''_{\text{max}}(\mu, \lambda)$ with respect to $\lambda$.

We shall now show that the same is true for $W_{\mu, \lambda}$. To this end, we add another term to the Hamiltonian, namely $\epsilon F \equiv \epsilon V \int d^2 z \{I(z)f(zV^{-1/2})\}$, with $\epsilon$ and $f$ real. If $f(\zeta)$ is a nice function of two real variables with bounded second derivatives, it is then easy to see that the upper and lower symbols of $F$ differ only by a term of order 1. Namely, for some $C > 0$ independent of $z_0$ and $V$,

$$\left| V \int d^2 z \langle |z| z_0 \rangle^2 \left( f(zV^{-1/2}) - f(zV^{-1/2}) \right) \right| \leq C.$$
Hence, in particular, \( p(\mu, \lambda, \epsilon) = p''(\mu, \lambda, \epsilon) \) in the TL. Moreover, if \( f(\zeta) = 0 \) for \( |\zeta - \hat{\zeta}| \leq \delta \), then the pressure is independent of \( \epsilon \) for \( |\epsilon| \) small enough (depending only on \( \delta \)). This can be seen as follows. We have

\[
p''(\mu, \lambda, \epsilon) - p''(\mu, \lambda, 0) = \frac{1}{\beta V} \ln \left( e^{-\beta \epsilon V f(zV^{-1/2})} \right),
\]

where the last expectation is in the \( H''_\mu \) ensemble at \( \epsilon = 0 \). The corresponding distribution is exponentially localized at \( z/\sqrt{V} = \hat{\zeta} \) by Griffiths’ Lemma, and therefore the right side of (35) goes to zero in the thermodynamic limit for small enough \( \epsilon \). In particular, the \( \epsilon \)-derivative of the thermodynamic limit pressure at \( \epsilon = 0 \) is zero.

By convexity in \( \epsilon \), this implies that the derivative of \( p \) at finite volume, given by

\[
V^{-1} \langle F \rangle_{\mu, \lambda} = \int \frac{d^2z}{2} f(zV^{-1/2}) W_{\mu, \lambda}(z),
\]

goes to zero in the thermodynamic limit. Since \( f \) was arbitrary, \( V \int |\xi - \hat{\zeta}| \geq \delta \frac{d^2\zeta}{2} W_{\mu, \lambda}(\xi \sqrt{V}) \rightarrow 0 \) as \( V \rightarrow \infty \). This holds for all \( \delta > 0 \), and therefore proves the statement.

Our method also applies to the case when the pressure is not differentiable in \( \lambda \) (which is the case at \( \lambda = 0 \) in the presence of BEC). In this case, the resulting weights \( W_{\mu, \lambda} \) and \( W''_{\mu, \lambda} \) need not be \( \delta \)-functions, but as in the argument for (13) in the previous section Griffiths’ method implies that they are, for \( \lambda \neq 0 \), supported on the real axis between the right and left derivative of \( p \) and, for \( \lambda = 0 \), on a disc (due to the gauge symmetry) with radius determined by the right derivative at \( \lambda = 0 \). Convexity of the pressure as a function of \( \lambda \) thus implies that in the thermodynamic limit the supports of the weights \( W_{\mu, \lambda} \) and \( W''_{\mu, \lambda} \) for \( \lambda \neq 0 \) lie outside of this disc. Hence \( \langle n_0 \rangle_{\lambda} \) is monotone increasing in \( \lambda \) in the thermodynamic limit. In combination with (36) this implies in particular that

\[
\lim_{V \rightarrow \infty} \frac{1}{V} \langle a_0^* a_0 \rangle_{\mu, \lambda = 0} \leq \lim_{\lambda \rightarrow 0} \lim_{V \rightarrow \infty} \frac{1}{V} \langle a_0^* a_0 \rangle_{\mu, \lambda} = \lim_{\lambda \rightarrow 0} \lim_{V \rightarrow \infty} \frac{1}{V} \langle |a_0| \rangle_{\mu, \lambda}^2.
\]

Hence (1) is established.

We note that by Eq. (36) spontaneous symmetry breaking (in the sense that the right side of (36) is not zero) always takes place whenever there is BEC is the usual sense, i.e., without explicit gauge breaking (meaning that the left side of (36) is non-zero). Note, however, that a non-vanishing of the right side of (36) does not a priori imply a non-vanishing of the left side. I.e., it is a priori possible that BEC only shows up after introducing an explicit gauge-breaking term to the Hamiltonian. While it is expected on physical grounds that positivity of the right side of (36) implies positivity of the left side, a rigorous proof is lacking, so far. In the example of the Heisenberg magnet discussed in the previous section, equality in (36) does not generally hold, but still both sides are non-vanishing in the same parameter regime.

To illustrate what could arise mathematically, in principle, consider a weight function of the form

\[
W''_{\mu, \lambda = 0}(\sqrt{V} \zeta) \equiv w_V(\zeta) = \begin{cases} 
V^2 - V + 1/V & \text{for } |\zeta| \leq 1/V \\
1/V & \text{for } 1/V \leq |\zeta| \leq 1 \\
0 & \text{for } |z| > 1.
\end{cases}
\]

(37)
This distribution converges for $V \to \infty$ to a $\delta$-function at $\zeta = 0$, and consequently there is no BEC at $\lambda = 0$. On the other hand, it is easy to see that the weight function $w_V(\zeta)e^{-\beta \lambda V \zeta}$ (with an appropriate normalization factor) converges, for any $\lambda > 0$, to a $\delta$-function at $\zeta = -1$ as $V \to \infty$, and hence there is spontaneous symmetry breaking. An open problem for the mathematician is to prove that examples like (37) do not occur in realistic bosonic systems.

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