GENERALIZATION AND DEFORMATION OF DRINFELD
QUANTUM AFFINE ALGEBRAS

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Abstract. Drinfeld gave a current realization of the quantum affine algebras as a Hopf algebra with a simple comultiplication for the quantum current operators. In this paper, we will present a generalization of such a realization of quantum Hopf algebras. As a special case, we will choose the structure functions for this algebra to be elliptic functions to derive certain elliptic quantum groups as a Hopf algebra, which degenerates into quantum affine algebras if we take certain degeneration of the structure functions.

1. Introduction.

Quantum groups as the very first noncommutative and noncocommutative Hopf algebras were discovered by Drinfeld[Dr1] and Jimbo[J1]. The standard definition of a quantum group is given as a deformation of a simple Lie algebra by the basic generators and the relations based on the data coming from the corresponding Cartan matrix. However, for the case of affine quantum groups, there is a different aspect of the theory.

For the undeformed affine Lie algebras, Garland showed [G] that the affine Kac-Moody algebra \( \hat{\mathfrak{g}} \) associated to a simple Lie algebra \( \mathfrak{g} \) admits a natural realization as a central extension of the corresponding loop algebra \( \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \). It is natural to expect similar realizations for the quantum affine algebras. The first approach was given by Faddeev, Reshetikhin and Takhtajan [FRT], who obtained a realization of the quantum loop algebra \( U_q(\mathfrak{g} \otimes [t, t^{-1}]) \) via a canonical solution of the Yang-Baxter equation depending on a parameter \( z \in \mathbb{C} \). This approach was completed by Reshetikhin and Semenov-Tian-Shansky [RS] by incorporating the central extension in the previous realization. However, in this approach, we can only have either the positive current operators or the negative current operators, which have complicated commutation relations.

On the other hand, Drinfeld [Dr2] gave another realization of the quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) and its special degeneration called the Yangian, which immediately is used in constructions of special representations of affine quantum algebras [FJ]. In [Dr2], Drinfeld only gave the realization of the quantum affine algebras as an algebra and as an algebra this realization is equivalent to the approach above [DF] through
certain Gauss decomposition for the case of $U_q(\hat{gl}(n))$. Certainly, the most important aspect of the structures of the quantum groups is its Hopf algebra structure, especially its comultiplication. However, if we extend the conventional comultiplication to the current operators, we acquire a very complicated formula which can not be written in a closed form with only the current operators. Drinfeld also gave the Hopf algebra structure for such a formulation in an unpublished note [DF]. The new comultiplication in this formulation, which we call the Drinfeld comultiplication, is simple. Recently in [DM] [DI], we used this new Hopf algebra structure to study vertex operators and zeros and poles of the quantum current operators.

In the Drinfeld realization of quantum affine algebras, the structure constants are certain rational functions $g_{ij}(z)$. In checking the Hopf algebra structure in this formulation, we notice that certain functional property of $g_{ij}(z)$ decides that this formulation gives us a Hopf algebra. This leads us to generalize this type of Hopf algebras. Namely, we can substitute $g_{ij}(z)$ by other functions that satisfy the functional property of $g_{ij}(z)$, to derive some new Hopf algebras.

In this paper, we will present our generalization of the Drinfeld realization of $U_q(\hat{sl}_n)$. We will show that $U_q(\hat{sl}_n)$ are simple examples of such a formulation. We will then choose the structure functions for this algebra to be certain elliptic functions. We derive an elliptic quantum groups as a Hopf algebra, which degenerates into quantum affine algebras if we choose special degeneration of the structure functions.

2.

The Drinfeld realization for the case of $U_q(\hat{sl}_n)$ [Dr2] as a Hopf algebra is presented in [DF] [DI], which leads us to define the following algebra.

Let $A = (a_{ij})$ be the Cartan matrix of type $A_{n-1}$. Let $g$ be the set of $\{g_{ij}(z), 0 < i,j < n\}$ which are analytic functions satisfying the following property that $g_{ij}(z) = g_{ji}(z^{-1})^{-1}$ and $\delta(z)$ be the distribution with support at 1.

**Definition 2.1.** $U_q(g, \hat{sl}_n)$ is an associative algebra with unit 1 and the generators: $x_i^+(z)$, $\varphi_i(z)$, $\psi_i(z)$, a central element $c$ and a nonzero complex parameter $q$. $\varphi_i(z)$ and $\psi_i(z)$ are invertible. In terms of the generating functions: the defining relations
Theorem 2.1. The algebra $U_q(g, \mathfrak{sl}_n)$ has a Hopf algebra structure, which are given by the following formulae.

**Coproduct** $\Delta$

\begin{align*}
(0) & \quad \Delta(q^c) = q^c \otimes q^c, \\
(1) & \quad \Delta(x_i^+(z)) = x_i^+(z) \otimes 1 + \varphi_i(zq^c) \otimes x_i^+(zq^c), \\
(2) & \quad \Delta(x_i^-(z)) = 1 \otimes x_i^+(z) + x_i^-(zq^c) \otimes \psi_i(zq^c), \\
(3) & \quad \Delta(\varphi_i(z)) = \varphi_i(zq^{-c}) \otimes \varphi_i(zq^c), \\
(4) & \quad \Delta(\psi_i(z)) = \psi_i(zq^{c}) \otimes \psi_i(zq^{-c}),
\end{align*}

where $c_1 = c \otimes 1$ and $c_2 = 1 \otimes c$.

**Counit** $\varepsilon$

\begin{align*}
\varepsilon(q^c) = 1 & \quad \varepsilon(\varphi_i(z)) = \varepsilon(\psi_i(z)) = 1, \\
\varepsilon(x_i^+(z)) = 0.
\end{align*}
\textbf{Antipode } \ a

(0) \quad a(q^c) = q^{-c},

(1) \quad a(x_i^+(z)) = -\varphi_i(zq^{-\frac{q^c}{2}})^{-1}x_i^+(zq^{-c}),

(2) \quad a(x_i^-(z)) = -x_i^-(zq^{-c})\psi_i(zq^{-\frac{q^c}{2}})^{-1},

(3) \quad a(\varphi_i(z)) = \varphi_i(z)^{-1},

(4) \quad a(\psi_i(z)) = \psi_i(z)^{-1}.

Proof. For the comultiplication above we have that

(1) \quad \Delta(x_i^+(z)) = x_i^+(z) \otimes 1 + \varphi_i(zq^{-\frac{q^c}{2}}) \otimes x_i^+(zq^{c_1}),

(2) \quad \Delta(x_i^-(z)) = 1 \otimes x_i^-(z) + x_i^-(zq^{c_2}) \otimes \psi_i(zq^{-\frac{q^c}{2}}),

(3) \quad \Delta(\varphi_i(z)) = \varphi_i(zq^{-\frac{q^c}{2}}) \otimes \varphi_i(zq^{c_2}),

(4) \quad \Delta(\psi_i(z)) = \psi_i(zq^{-\frac{q^c}{2}}) \otimes \psi_i(zq^{c_1}).

Then,

\[\Delta\varphi_i(z)\Delta\psi_j(w)\Delta\varphi_i(z)^{-1}\Delta\psi_j(w)^{-1} = \varphi_i(zq^{-\frac{q^c}{2}})\psi_j(wq^{\frac{q^c}{2}})\varphi_i(zq^{-\frac{q^c}{2}})^{-1}\psi_j(wq^{-\frac{q^c}{2}})\varphi_i(zq^{-\frac{q^c}{2}})^{-1}\psi_j(wq^{-\frac{q^c}{2}})^{-1} = \frac{g_{ij}(z/wq^{-c_1}q^{-c_2})g_{ij}(z/wq^{c_2}q^{c_1})}{g_{ij}(z/wq^{c_1}q^{-c_2})g_{ij}(z/wq^{c_1}q^{c_2})} = \frac{g_{ij}(z/wq^{-c_1}c_2)}{g_{ij}(z/wq^{c_1}c_2)}\]

\[\Delta\varphi_i(z)\Delta x_j^+(w)\Delta\varphi_i(z)^{-1} = \varphi_i(zq^{-\frac{q^c}{2}}) \otimes \varphi_i(zq^{\frac{q^c}{2}})(x_j^+(w) \otimes 1 + \varphi_j(wq^{\frac{q^c}{2}}) \otimes x_j^+(wq^{c_1}))(\varphi_i(zq^{-\frac{q^c}{2}}) \otimes \varphi_i(zq^{\frac{q^c}{2}}))^{-1} = g_{ij}(z/wq^{-\frac{c_1+c_2}{2}})(x_j^+(w) \otimes 1 + \varphi_j(wq^{\frac{q^c}{2}}) \otimes x_j^+(wq^{c_1})).\]

\[\Delta\varphi_i(z)\Delta x_j^-(w)\Delta\varphi_i(z)^{-1} = \varphi_i(zq^{-\frac{q^c}{2}}) \otimes \varphi_i(zq^{\frac{q^c}{2}})(1 \otimes x_j^-(w) + x_j^-(wq^{c_2}) \otimes \psi_j(wq^{\frac{q^c}{2}}))(\varphi_i(zq^{-\frac{q^c}{2}}) \otimes \varphi_i(zq^{\frac{q^c}{2}}))^{-1} = 1 \otimes x_j^-(w)g_{ij}(z/wq^{\frac{c_1+c_2}{2}})(x_j^-(wq^{c_2}) \otimes \psi_j(wq^{\frac{q^c}{2}}))g_{ij}(z/wq^{\frac{c_1+3c_2}{2}})g_{ij}(z/wq^{\frac{3c_1+c_2}{2}}) = \Delta x_j^-(w)g_{ij}(z/wq^{\frac{c_1+3c_2}{2}})^{-1}.\]

The relation between \(\Delta\psi_i(z)\) and \(\Delta x_i^{\pm}(w)\) can be proved with the same method demonstrated above.
\[
[x_i^+(z) \otimes 1 + \varphi_i(zq^{\frac{1}{2c}}) \otimes x_i^+(zq^{c_1}),
1 \otimes x_i^-(w) + x_i^- (wq^{c_2}) \otimes \psi_i(wq^{\frac{1}{2}})] = \]
\[
[x_i^+(z) \otimes 1, x_i^- (wq^{c_2}) \otimes \psi_i(wq^{\frac{1}{2}})] + [\varphi_i(zq^{\frac{1}{2}}) \otimes x_i^+(zq^{c_1}), 1 \otimes x_i^-(w)] + [\varphi_i(zq^{\frac{1}{2}}) \otimes x_i^+(zq^{c_1}), x_i^- (wq^{c_2}) \otimes \psi_i(wq^{\frac{1}{2}})].
\]

It is easy to show that the last term above is 0, therefore we have that
\[
[\Delta x_i^+(z), \Delta x_i^-(w)] = \]
\[
\frac{\delta(z/wq^{-c_1-c_2}) \psi_i(zq^{\frac{1}{2}+\frac{1}{2c_2}}) - \delta(z/wq^{c_1-c_2}) \varphi_i(zq^{\frac{1}{2}}) \otimes \psi_i(wq^{\frac{1}{2}})}{q - q^{-1}} \]
\[
\frac{\varphi_i(zq^{\frac{1}{2}}) \otimes (\delta(z/wq^{c_1-c_2}) \psi_i(wq^{\frac{1}{2}+\frac{1}{2c_2}}) - \delta(z/wq^{c_1+c_2}) \varphi_i(zq^{\frac{1}{2}+\frac{1}{2c_2}}))}{q - q^{-1}} = \]
\[
\frac{\delta(z/wq^{-c_1-c_2}) \psi_i(zq^{\frac{1}{2}+\frac{1}{2c_2}}) \otimes \psi_i(zq^{-\frac{1}{2}+\frac{1}{2c_2}}) - \delta(z/wq^{c_1+c_2}) \varphi_i(zq^{\frac{1}{2}+\frac{1}{2c_2}}) \otimes \varphi_i(zq^{\frac{1}{2}+\frac{1}{2c_2}})}{q - q^{-1}}.
\]
\[
\Delta x_i^+(z) \Delta x_j^-(w) = \]
\[
(x_i^+(z) \otimes 1 + \varphi_i(zq^{\frac{1}{2}}) \otimes x_i^+(zq^{c_1}))(x_j^+(w) \otimes 1 + \varphi_j(wq^{\frac{1}{2}}) \otimes x_j^+(wq^{c_1})) = \]
\[
g_{ij}(z/w)x_j^+(w)x_i^+(z) \otimes 1 + \varphi_j(wq^{\frac{1}{2}})x_i^+(z) \otimes x_j^+(wq^{c_1})g_{ji}(w/z)^{-1} + x_j^+(w)\varphi_i(zq^{\frac{1}{2}}) \otimes x_i^+(zq^{c_1})g_{ij}(z/w) + g_{ij}(z/w)\varphi_j(wq^{\frac{1}{2}})\varphi_i(zq^{\frac{1}{2}}) \otimes x_j^+(wq^{c_1})x_i^+(zq^{c_1}) = \]
\[
g_{ij}(z/w)\Delta x_i^+(z)\Delta x_j^-(w). \]

The last is to check the cubic relations between $\Delta x_i^+(z) + \Delta x_j^-(w)$ for $a_{ij} = -1$. Therefore we need to show that
\[
\Delta x_i^+(z_1)\Delta x_j^+(z_2)\Delta x_j^-(w) - (h_{ij}(z_1/w, z_2/w))\Delta x_i^+(z_1)\Delta x_j^+(w)\Delta x_i^-(z_2) + \Delta x_j^+(w)\Delta x_i^+(z_1)\Delta x_i^+(z_2) + \{z_1 \leftrightarrow z_2\} = 0
\]
\[
(x_i^+(z_1) \otimes 1 + \varphi_i(z_1q^{\frac{1}{2}}) \otimes x_i^+(z_1q^{c_1}))(x_j^+(z_2) \otimes 1 + \varphi_i(z_2q^{\frac{1}{2}}) \otimes x_j^+(z_2q^{c_1}))(x_j^-(w) \otimes 1 + \varphi_j(wq^{\frac{1}{2}}) \otimes x_j^+(wq^{c_1})) = \]
\[
x_i^+(z_1)x_j^+(z_2)x_j^-(w) \otimes 1 + x_i^+(z_1)x_j^-(z_2)\varphi_j(wq^{\frac{1}{2}}) \otimes x_j^+(wq^{c_1}) + x_i^+(z_1)\varphi_i(z_2q^{\frac{1}{2}})x_j^+(w) \otimes x_j^+(z_2q^{c_1}) + x_i^+(z_1)\varphi_i(z_2q^{\frac{1}{2}})x_j^+(w) \otimes x_j^+(z_2q^{c_1}) + x_i^+(z_1)\varphi_i(z_2q^{\frac{1}{2}})x_j^+(w) \otimes x_j^+(z_2q^{c_1})x_j^+(wq^{c_1}) + \varphi_i(z_1q^{\frac{1}{2}})x_i^+(z_2)x_j^+(w) \otimes x_j^+(z_1q^{c_1}) + \varphi_i(z_1q^{\frac{1}{2}})x_i^+(z_2)x_j^+(wq^{c_1}) + \varphi_i(z_1q^{\frac{1}{2}})x_i^+(z_2)x_j^+(w) \otimes x_j^+(z_1q^{c_1})x_j^+(wq^{c_1}) + \varphi_i(z_1q^{\frac{1}{2}})x_i^+(z_2)x_j^+(w) \otimes x_j^+(z_1q^{c_1})x_j^+(wq^{c_1}) +
\[
\varphi_i(z_1 q^{\frac{1}{2}})\varphi_i(z_2 q^{\frac{1}{2}}) x^+_i(w) \otimes x^+_i(z_1 q^{\epsilon}) x^+_i(z_2 q^{\epsilon}) + \\
\varphi_i(z_1 q^{\frac{1}{2}}) \varphi_i(z_2 q^{\frac{1}{2}}) \varphi_j(w q^{\frac{1}{2}}) \otimes x^+_i(z_1 q^{\epsilon}) x^+_i(z_2 q^{\epsilon}) x^+_j(w q^{\epsilon}).
\]

Note that
\[
h_{ij}(z_1/w, z_2/w) = h_{ij}(z_2/w, z_1/w).
\]

Thus we have that
\[
\Delta x^+_i(z_1) \Delta x^+_j(z_2) \Delta x^+_j(w) - (h_{ij}(z_1/w, z_2/w)) \Delta x^+_i(z_1) \Delta x^+_j(w) \Delta x^+_i(z_2) + \\
\Delta x^+_j(w) \Delta x^+_i(z_1) \Delta x^+_i(z_2) + \{z_1 \leftrightarrow z_2\} = \\
\{\varphi_j(w q^{\frac{1}{2}}) x^+_i(z_2) x^+_i(z_1) \otimes x^+_j(w q^{\epsilon}) + x^+_j(w) x^+_i(z_2) \varphi_i(z_1 q^{\frac{1}{2}}) \otimes x^+_i(z_1 q^{\epsilon}) + \\
x^+_j(w) \varphi_i(z_2 q^{\frac{1}{2}}) x^+_i(z_1) \otimes x^+_j(z_2 q^{\epsilon}) + x^+_j(w) \varphi_i(z_2 q^{\frac{1}{2}}) \varphi_i(z_1 q^{\frac{1}{2}}) \otimes x^+_i(z_2 q^{\epsilon}) x^+_i(z_1 q^{\epsilon}) + \\
\varphi_j(w q^{\frac{1}{2}}) x^+_i(z_2) \varphi_i(z_1 q^{\frac{1}{2}}) \otimes x^+_j(w q^{\epsilon}) x^+_i(z_1 q^{\epsilon}) + \\
\varphi_j(w q^{\frac{1}{2}}) \varphi_i(z_2 q^{\frac{1}{2}}) x^+_i(z_1) \otimes x^+_j(w q^{\epsilon}) \otimes x^+_i(z_2 q^{\epsilon}) \} \times \\
\{g_{ii}(z_1/z_2) g_{ij}(z_1/w) g_{ij}(z_2/w) - h_{ij}(z_1/w, z_2/w) g_{ii}(z_1/z_2) g_{ij}(z_1/w) + g_{ii}(z_1/z_2) + \\
g_{ij}(z_1/w) g_{ij}(z_2/w) - h_{ij}(z_2/w, z_1/w) g_{ij}(z_2/w) + 1\}.
\]

We therefore derive the proof that the comultiplication is an algebra homomorphism.

We can prove in the same manner the relation between \(\Delta x^-_i(z)\) and \(\Delta x^-_j(w)\).

Let \(M\) be the operator from \(U_q(g, sl_n) \otimes U_q(g, sl_n)\) to \(U_q(g, sl_n)\) defined by algebra multiplication. We can check that \(M(1 \otimes \epsilon) \Delta = \text{id}\).

\[
M(1 \otimes a) \Delta(x^+_i(z)) = \\
M(1 \otimes a)(x^+_i(z) \otimes 1 + \varphi_i(z q^{\frac{1}{2}}) \otimes x^+_i(z q^{\epsilon})) = \\
x^+_i(z) - \varphi_i(z q^{\frac{1}{2}})(\varphi_i(z q^{\frac{1}{2}}))^{-1} x^+_i(z) = \\
0 = \epsilon(x^+_i(z))
\]

Similarly we can check all the other relations to show that
\[
M(a \otimes 1) = \epsilon.
\]

Thus, we prove that the comultiplication, the counit and the antipode give a Hopf algebra structure.

Strictly speaking, \(U_q(g, sl_n)\) is not an algebra. This concept, which we call a functional algebra, has already been used before [S], etc. For our algebra \(U_q(g, sl_n)\), each generating functions gives an operator at any point \(C\), or if we restrict our generating functions at any point, we get an associative algebra. Certainly, this is not the case at the poles of the structure functions \(g_{ij}(z)\). A functional algebra as a generalization of algebra is similar to the generalization from the concept of a number to a function. Furthermore, we can extend the definition of this functional algebra to any abelian
Lie group. We simply substitute $C^\infty$ at 0 or generalize in the same way. It is an algebra and Hopf algebra. We will present similar

Definition 2.2. The algebra $U_q(g, sl_n)$ is an associative algebra with unit 1 and the generators: $a_i(l), \bar{b}_i(l), x_i^\pm(l)$, for $i = i, \ldots, n - 1, l \in \mathbb{Z}$ and a central element $c$. Let $z$ be a formal variable and

$$x_i^\pm(z) = \sum_{l \in \mathbb{Z}} x_i^\pm(l) z^{-l}$$

$$\varphi_i(z) = \sum_{m \in \mathbb{Z}} \varphi_i(m) z^{-m} = \exp[ \sum_{m \in \mathbb{Z}_{<0}} a_i(m) z^{-m} ] \exp[ \sum_{m \in \mathbb{Z}_{<0}} \bar{a}_i(m) z^{-m} ]$$

and

$$\psi_i(z) = \sum_{m \in \mathbb{Z}} \psi_i(m) z^{-m} = \exp[ \sum_{m \in \mathbb{Z}_{<0}} \bar{b}_i(m) z^{-m} ] \exp[ \sum_{m \in \mathbb{Z}_{<0}} b_i(m) z^{-m} ]$$

In terms of the formal variables $z$, $w$, $z_1$ and $z_2$, the defining relations are

$$a_i(l)a_j(m) = a_j(m)a_i(l),$$

$$b_i(l)b_j(m) = b_j(m)b_i(l),$$

$$\varphi_i(z)\varphi_j(w)\varphi_i(z)^{-1}\varphi_j(w)^{-1} = g_{ij}(z/wq^{-c}) g_{ij}(z/wq^c),$$

$$g_{ij}(z/wq^{-c}) g_{ij}(z/wq^c),$$

$$\varphi_i(z)x_j^\pm(w)\varphi_i(z)^{-1} = g_{ij}(z/wq^{-c})^{\pm1} x_j^\pm(w),$$

$$\psi_i(z)x_j^\pm(w)\psi_i(z)^{-1} = g_{ij}(w/zq^{\pm c})^{\pm1} x_j^\pm(w),$$

$$[x_i^\pm(z), x_j^\mp(w)] = \frac{\delta_{ij}}{q - q^{-1}} \left\{ \delta(z/wq^{-c})\psi_i(wq^{\pm c}) - \delta(z/wq^c)\varphi_i(zq^{\pm c}) \right\},$$

$$G_{ij}^\pm(z/w)x_i^\pm(z)x_j^\pm(w) = G_{ij}^\pm(z/w)x_j^\pm(w)x_i^\pm(z),$$

$$[x_i^\pm(z), x_j^\pm(w)] = 0 \quad \text{for } a_{ij} = 0,$$

$$x_i^\pm(z_1) x_i^\pm(z_2) x_j^\pm(w) - (h_{ij}(z_1/w, z_2/w)) x_i^\pm(z_1) x_j^\pm(w) x_i^\pm(z_2)$$

$$+ x_j^\pm(w) x_i^\pm(z_1) x_i^\pm(z_2) + \{z_1 \leftrightarrow z_2\} = 0, \quad \text{for } a_{ij} = -1$$

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where
\[
    h_{ij}(z_1/w, z_2/w) = \frac{(g_{ii}(z_1/z_2) + 1)(g_{ij}(z_1/w)g_{ij}(z_2/w) + 1)}{g_{ij}(z_2/w) + g_{ii}(z_1/z_2)g_{ij}(z_1/w)}
\]

and by \( g_{ij}(z) \) we mean the Laurent expansion of \( g_{ij}(z) \) in a region \( r_{ij,1} > |z| > r_{ij,2}. \)

**Theorem 2.2.** The algebra \( U_q(g, \mathfrak{sl}_n) \) has a Hopf algebra structure. The formulas for the coproduct \( \Delta \), the counit \( \varepsilon \) and the antipode \( a \) are the same as given in Theorem 2.1.

The proof is almost the same as the proof for Theorem 2.1, except that one has to be careful with the expansion direction of the structure functions \( g_{ij}(z) \) and \( \delta(z) \), for the reason that the relations between \( x_i^±(z) \) and \( x_j^±(z) \) are different from the case of the functional algebra above. If we modify the relation between \( x_i^±(z) \) and \( x_j^±(z) \) in Definition 2.1 to be the same as in Definition 2.2., this will make a subtle difference on the algebra. One problem that appears in such a formulation is that the commutation relations above can involve infinite expressions, which requires certain topology to define our algebras. However, it is well-defined in the case of the highest weight representations.

**Example 2.1.** Let \( g(z) \) be a an analytic function such that \( g(z^{-1}) = -z^{-1}g(z) \). Let \( g_{ij}(z) = q^{-a_{ij}} \frac{g_i(q^{a_{ij}}z)}{g_i(q^{-a_{ij}}z)}. \) Then \( g_{ij}(z) = g_{ij}(z^{-1})^{-1}. \) With these \( g_{ij}(z) \), we can define an algebra \( U_q(g, \mathfrak{sl}_n) \) by Definition 2.2.

For this algebra, we obtain its vector representation.

Let \( V = \bigoplus_{i=0}^{n-1} \mathbb{C}[i] \) be a n-dimensional space, where \( \{i\} \) is its standard basis. Let \( V_z = V \otimes \mathbb{C}[z, z^{-1}] \), where \( z \) is a formal variable.

**Lemma 2.3.** There exists an n-dimensional representation of \( U_q(g, \mathfrak{sl}_n) \) on \( V \). The
action of the current operators is given by the following:

\[ x_i^+(w).|j\rangle = c_i^+ \delta_{ij} \delta(\frac{w}{q^i z})|i - 1\rangle, \]

\[ x_i^-(w).|j\rangle = c_i^- \delta_{i-1,j} \delta(\frac{w}{q^i z})|i\rangle, \]

\[
\begin{cases}
\varphi_i(w).|i - 1\rangle = q^{-1}g(q^2 \frac{w}{q^i z}) \sum_{k \geq 0} (\frac{w}{q^i z})^k|i - 1\rangle \\
\varphi_i(w).|i\rangle = qg(q^{-2} \frac{w}{q^i z}) \sum_{k \geq 0} (\frac{w}{q^i z})^k|i\rangle \\
\varphi_i(w).|j\rangle = g(q^2 \frac{w}{q^i z}) \sum_{k \geq 0} (\frac{w}{q^i z})^k|j\rangle \quad (j \neq i, i - 1) \\
\psi_i(w).|i - 1\rangle = qg(q^{-2} \frac{q^i z}{w}) \sum_{k \geq 0} (\frac{q^i z}{w})^k|i - 1\rangle, \\
\psi_i(w).|i\rangle = q^{-1}g(q^2 \frac{q^i z}{w}) \sum_{k \geq 0} (\frac{q^i z}{w})^k|i\rangle, \\
\psi_i(w).|j\rangle = g(\frac{q^i z}{w}) \sum_{k \geq 0} (\frac{q^i z}{w})^k|j\rangle \quad (j \neq i, i - 1)
\end{cases}
\]

where we set $| - 1\rangle = |n\rangle = 0$ and $c_i^\pm$ are constants satisfying

\[ c_i^+ c_i^- = \frac{qg(q^{-2})}{q - q^{-1}}. \]

**Case I.** Let $g(z) = 1 - z$; $\varphi_i(m) = \psi_i(-m) = 0$ for $m \in \mathbb{Z}_{> 0}$ and $\varphi_i(0)\psi_i(0) = 1$. Then this algebra is $U_q(\mathfrak{sl}_n)$.

In this case, the comultiplication structure requires certain completion on the tenor space. Nevertheless, if we consider any two highest weight representations, which can be defined as for the case of $U_q(\mathfrak{sl}_n)$, this comultiplication is well-defined. For the case of $U_q(\mathfrak{sl}_n)$, this is used [DM] to study the poles and zeros of those current operators for integrable representations. In [DI], this is also used to study the corresponding vertex operators, where we derive simple bosonization formulas for vertex operators.

**Case II** Let $\theta_p(z) = \prod_{j > 0}(1 - p^j)(1 - p^{j-1}z)(1 - p^j z^{-1})$ be the Jacobi’s theta function. $\theta_p(z^{-1}) = -z^{-1}\theta_p(z) = \theta_p(pz)$. We will call this algebra $U_q(\theta, \mathfrak{sl}_n)$. We take the expansion of $g_{ii}(z)$ in the region $|q^2| > |z| > |q^2 p|$ and $g_{ij}(z)$ in the region $|q| > |z| > |qp|$.

This algebra is related to the algebra in [FO] [FF], especially the subalgebra of $U_q(\theta, \mathfrak{sl}_n)$ generated by $x_i^+(z)$, although the cubic relations and the Hopf algebra structure were not given for their cases. The subalgebras generated by $\psi(z)$ and $x_i^-(z)$ with a comultiplication for certain curves were given in [ER], where they did
not have the full Hopf algebra structure, for example the antipode. As an algebra, $U_q(\theta, \mathfrak{sl}_2)$ is also given in [ER].

If we take the limit that $p$ goes to zero, we would recover $U_q(\hat{\mathfrak{sl}}_n)$. Thus $U_q(\theta, \mathfrak{sl}_2)$ gives a deformation of $U_q(\hat{\mathfrak{sl}}_n)$. One question to ask would concern functions $h_{ij}(z_1/w, z_2/w)$, which in the case of $U_q(\mathfrak{sl}_n)$ is $q + q^{-1}$. We note that the functions $h_{ij}(z_1/w, z_2/w)$ are not constant. However we maintain this function should have certain geometrical meaning.

Let $Q = \bigoplus_{j=1}^{n-1} \mathbb{Z} \alpha_j$ be the root lattice of $\mathfrak{sl}_n$ and $\Lambda_j = \Lambda_j - \Lambda_0$ be the classical part of the $i$-th fundamental weight.

Let Heisenberg algebra be an algebra generated by \{\(a_{i,k}\) | \(1 \leq i \leq n-1, \ k \in \mathbb{Z}\setminus\{0\}\} satisfying:

\[
[a_{i,k}, a_{j,l}] = -[b_{i,k}, b_{j,l}] = \delta_{k+l,0} \left\{ \frac{[\alpha_i, \alpha_j]k[k]}{k(1 - p^k)} \right\},
\]

\[
[a_{i,k}, b_{j,l}] = 0
\]

Let us define the group algebra $\mathbb{C}(q)[P]$. Let $P$ be the weight lattice of $\mathfrak{sl}_n$. We fix our free basis $\alpha_2, \ldots, \alpha_{n-1}, \overline{\Lambda}_{n-1}$. They satisfy

\[
e^{\alpha_i}e^{\alpha_j} = (-1)^{\langle \alpha_i, \alpha_j \rangle}e^{\alpha_j}e^{\alpha_i}, \quad 2 \leq i, j \leq n-1,
\]

\[
e^{\alpha_i}e^{\overline{\Lambda}_{n-1}} = (-1)^{\delta_i,n-1}e^{\overline{\Lambda}_{n-1}}e^{\alpha_i}, \quad 2 \leq i \leq n-1.
\]

For $\alpha = m_2\alpha_2 + \cdots m_{n-1}\alpha_{n-1} + m_{n}\overline{\Lambda}_{n-1}$, we set

\[
e^{\alpha} = e^{m_2\alpha_2} \cdots e^{m_{n-1}\alpha_{n-1}}e^{m_{n}\overline{\Lambda}_{n-1}}.
\]

Note that the following equations hold.

\[
\overline{\Lambda}_i = -\alpha_{i+1} - 2\alpha_{i+2} - \cdots - (n - i - 1)\alpha_{n-1} + (n - i)\overline{\Lambda}_{n-1},
\]

\[
\alpha_1 = -2\alpha_2 - 3\alpha_3 - \cdots - (n - 1)\alpha_{n-1} + n\overline{\Lambda}_{n-1}.
\]

Set

\[
\mathcal{F}_i := \mathbb{C}(q)[[p^{1/2}]][a_{j,-k}, b_{j,-k}(1 \leq j \leq n-1, \ k \in \mathbb{Z}_{>0})] \otimes \mathbb{C}(q)[Q]e^{\overline{\Lambda}_i} \ (0 \leq i \leq n-1).
\]

This gives the Fock space.
The action of operators $b_{j,k}, a_{j,k}, \partial_\alpha, e^\alpha$ $(1 \leq j \leq n - 1, \alpha \in \overline{Q})$ is given by

\[
\begin{align*}
 a_{j,k} \cdot f \otimes e^\beta &= \begin{cases} 
 a_{j,k} f \otimes e^\beta & k < 0 \\
 [a_{j,k}, f] \otimes e^\beta & k > 0 
\end{cases}, \\
 b_{j,k} \cdot f \otimes e^\beta &= \begin{cases} 
 b_{j,k} f \otimes e^\beta & k < 0 \\
 [b_{j,k}, f] \otimes e^\beta & k > 0 
\end{cases}, \\
 \partial_\alpha \cdot f \otimes e^\beta &= (\alpha, \beta) f \otimes e^\beta, \\
e^\alpha \cdot f \otimes e^\beta &= f \otimes e^\alpha e^\beta.
\end{align*}
\]

**Proposition 2.4.** The following gives a highest weight representation on $\mathcal{F}_i$.

\[
\begin{align*}
 x^+_j(z) &\mapsto \exp\left[\pm \sum_{k>0} \frac{a_{j,-k} q^{\frac{k}{2}} z^k - b_{j,-k} (pq)^{\frac{k}{2}} z^{-k}}{[k]} \right] \exp\left[\mp \sum_{k>0} \frac{a_{j,k} q^{\frac{k}{2}} z^k - b_{j,k} (pq)^{\frac{k}{2}} z^{-k}}{[k]} \right] z^{-k}, \\
 \varphi_j(z) &\mapsto \exp\left[-(q - q^{-1}) \sum_{k>0} a_{j,-k} z^k \right] \exp\left[(q - q^{-1}) \sum_{k>0} b_{j,k} p^{k/2} z^{-k} \right] q^{-\partial_\alpha_j}, \\
 \psi_j(z) &\mapsto \exp\left[-(q - q^{-1}) \sum_{k>0} b_{j,-k} p^{k/2} z^k \right] \exp\left[(q - q^{-1}) \sum_{k>0} a_{j,k} z^{-k} \right] q^\partial_\alpha_j.
\end{align*}
\]

The extension of our construction to other simply laced cases is straightforward. At this moment, we know very little about these algebras. We do not know when this algebra would be zero. If we use Drinfeld’s double construction to double the subalgebra generated by $x^+_j(z)$ and $\varphi_i(z)$, this will give the whole algebra. From the point of view of double construction, an immediate question is what the universal $R$-matrix is [Dr2]. The answer to this question will shed insight towards understanding this algebra. Another problem is certainly its representation theory, which hopefully should relate to various problem in mathematics and physics. In this paper, we suggest a new structure inspired by the ideas of Drinfeld’s current realization of affine quantum groups. It is still a very small step since a complete understanding of this construction, why it yields a Hopf algebra and its implication are not yet achieved. We hereby suggest that a geometric approach towards the realization of the representations of the algebras should be very helpful. [GRV][Go]

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