Partial differential equations

Applications of Bourgain–Brézis inequalities to fluid mechanics and magnetism

Applications des inégalités de Bourgain–Brézis à la mécanique des fluides et au magnétisme

Sagun Chanillo\textsuperscript{a}, Jean Van Schaftingen\textsuperscript{b}, Po-Lam Yung\textsuperscript{c}

\textsuperscript{a} Department of Mathematics, State University of New Jersey, Rutgers, NJ 08854, USA
\textsuperscript{b} Institut de recherche en mathématique et en physique, Université catholique de Louvain, chemin du Cyclotron 2 bte L7.01.01, 1348 Louvain-la-Neuve, Belgium
\textsuperscript{c} Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong

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\section*{A B S T R A C T}

As a consequence of inequalities due to Bourgain–Brézis, we obtain local-in-time well-posedness for the two-dimensional Navier–Stokes equation with velocity bounded in spacetime and initial vorticity in bounded variation. We also obtain spacetime estimates for the magnetic field vector through improved Strichartz inequalities.

\section*{R É S U M É}

À partir d’inégalités de Bourgain–Brézis, nous démontrons le caractère bien posé localement dans le temps des équations de Navier–Stokes avec vitesse bornée en espace-temps et un tourbillon initial à variation bornée. Nous obtenons également des estimations en espace-temps pour le champ magnétique grâce à des inégalités de Strichartz améliorées.

\section*{1. Incompressible Navier–Stokes flow}

Let \( \mathbf{v}(x, t) \in \mathbb{R}^2 \) be the velocity and \( p(x, t) \) be the pressure of a fluid of viscosity \( \nu > 0 \) at position \( x \in \mathbb{R}^2 \) and time \( t \in \mathbb{R} \), governed by the incompressible two-dimensional Navier–Stokes equation:

\[
\begin{align*}
\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} &= \nu \Delta \mathbf{v} - \nabla p, \\
\nabla \cdot \mathbf{v} &= 0,
\end{align*}
\]

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E-mail addresses: chanillo@math.rutgers.edu (S. Chanillo), Jean.VanSchaftingen@uclouvain.be (J. Van Schaftingen), plyung@math.cuhk.edu.hk (P.-L. Yung).

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When the viscosity coefficient $\nu$ degenerates to zero, (1) becomes the Euler equation. In two spatial dimensions, the vorticity of the flow is a scalar, defined by
\[ \omega = \partial_{x_1} v_2 - \partial_{x_2} v_1 \]
where we wrote $\mathbf{v} = (v_1, v_2)$. In the sequel, when we consider the Navier–Stokes equation, without loss of generality we set the viscosity coefficient $\nu = 1$.

The vorticity associated with the incompressible Navier–Stokes flow in two dimensions propagates according to the equation
\[ \omega_t - \Delta \omega = -\nabla \cdot (\omega \mathbf{v}). \tag{2} \]
This follows from (1) by taking the curl of both sides. We express the velocity $\mathbf{v}$ in the Navier–Stokes equation in terms of the vorticity through the Biot–Savart relation:
\[ \mathbf{v} = (-\Delta)^{-1}(\partial_{x_2} \omega, -\partial_{x_1} \omega). \tag{3} \]
This follows formally by differentiating $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$, and using that $\nabla \cdot \mathbf{v} = 0$.

Our theorem states:

**Theorem 1.** Consider the two-dimensional vorticity equation (2) and an initial vorticity $\omega_0 \in W^{1,1}(\mathbb{R}^2)$ at time $t = 0$. If
\[ \|\omega_0\|_{W^{1,1}(\mathbb{R}^2)} \leq A_0, \]
then there exists a unique solution to the vorticity equation (2) for all time $t \leq t_0 = C/A_0^2$, such that
\[ \sup_{t \leq t_0} \|\omega(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \leq cA_0. \]

Moreover, the solution $\omega$ depends continuously on the initial data $\omega_0$, in the sense that if $\omega_0^{(i)}$ is a sequence of initial data converging in $W^{1,1}(\mathbb{R}^2)$ to $\omega_0$, then the corresponding solutions $\omega^{(i)}$ to the vorticity equation (2) satisfy
\[ \sup_{t \leq t_0} \|\omega^{(i)}(\cdot, t) - \omega(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \to 0 \]
as $i \to \infty$.

Finally, the velocity vector $\mathbf{v}$ defined by the Biot–Savart relation (3) solves the 2-dimensional incompressible Navier–Stokes equation (1), and satisfies
\[ \sup_{t \leq t_0} \|\mathbf{v}(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} + \sup_{t \leq t_0} \|\nabla \mathbf{v}(\cdot, t)\|_{L^2(\mathbb{R}^2)} \leq cA_0. \]

Via the Gagliardo–Nirenberg inequality, we can conclude from our theorem that
\[ \sup_{0 \leq t \leq t_0} \|\omega(\cdot, t)\|_{L^p(\mathbb{R}^2)} \leq C, \quad 1 \leq p \leq 2. \]

In particular, this is enough to apply Theorem II of Kato [8] to express the velocity vector in the Navier–Stokes equation (1) in terms of the vorticity via the Biot–Savart relation displayed above.

In [7,8], it was proved that under the hypothesis that the initial vorticity is a measure, there is a global solution that is well-posed to the vorticity and Navier–Stokes equation; see also an alternative approach in Ben-Artzi [1], and a stronger uniqueness result in Brézis [4]. The velocity constructed then satisfies the estimate [8, (0.5)]:
\[ \|\mathbf{v}(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \leq C t^{-\frac{1}{2}}, \quad t \to 0. \tag{4} \]

In contrast, in Theorem 1 we have $\mathbf{v} \in L^\infty L_x^\infty, x \in \mathbb{R}^2$, though we are assuming that the initial vorticity has bounded variation, that is, its gradient is a measure.

The estimate (4) is indeed sharp as can be seen by the famous example of the Lamb–Oseen vortex [9], which consists of an initial vorticity $\omega_0 = \alpha_0 \delta_0$, a Dirac mass at the origin of $\mathbb{R}^2$ with strength $\alpha_0$. The constant $\alpha_0$ is called the total circulation of the vortex. A unique solution to the vorticity equation (2) can be obtained by setting
\[ \omega(x, t) = \frac{\alpha_0}{4\pi t} e^{-\frac{x^2}{4t}}, \quad \mathbf{v}(x, t) = \frac{\alpha_0}{2\pi} \frac{(-x_2, x_1)}{|x|^2} \left(1 - e^{-\frac{x^2}{4t}}\right). \]

It can be seen from the identities above that
\[ \|\omega(\cdot, t)\|_{W^{1,1}(\mathbb{R}^2)} \sim \|\mathbf{v}(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \sim c t^{-\frac{1}{2}}, \quad t \to 0. \]
Hence the assumption that the initial vorticity is a measure cannot yield an estimate like in Theorem 1. Thus to get uniform-in-time, $L^\infty$ space bounds all the way to $t = 0$, we need a stronger hypothesis and one such is vorticity in BV (bounded variation).

It is also helpful to further compare our result with that of Kato [8], who establishes in (0.4) of his paper that given that the initial vorticity is a measure, one has for the vorticity at further time

$$\|\nabla \omega(\cdot,t)\|_{L^q(\mathbb{R}^2)} \leq C t^{\frac{1}{2} - \frac{1}{2q}}, \quad 1 < q \leq \infty.$$ 

In contrast, we obtain uniform-in-time bounds for $q = 1$, as opposed to singular bounds for $q > 1$ when $t \to 0$.

It is an open question whether there is a global version of Theorem 1 of our paper.

In order to prove Theorem 1, we rely on a basic proposition that follows from the work of Bourgain and Brézis [2,3]. A part of this proposition also holds in three dimensions. Recall that if $\mathbf{v}(x,t) \in \mathbb{R}^3$ is the velocity of a fluid at a point $x \in \mathbb{R}^3$ at time $t$, then the vorticity of $\mathbf{v}$ is defined by

$$\omega = \nabla \times \mathbf{v}.$$

Under the assumption that the flow is incompressible, the Biot–Savart relation reads

$$\mathbf{v} = (-\Delta)^{-1}(\nabla \times \omega).$$

**Proposition 2.**

(a) **Consider the velocity $\mathbf{v}$ in three spatial dimensions. Assume that $\mathbf{v}$ satisfies the Biot–Savart relation (5). Then at any fixed time $t$,**

$$\|\mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} + \|\nabla \mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^3)} \leq C \|\nabla \times \omega(\cdot,t)\|_{L^2(\mathbb{R}^3)}$$

where $C$ is a constant independent of $t$, $\mathbf{v}$, and $\omega$.

(b) **Consider the velocity $\mathbf{v}$ in two spatial dimensions. Assume that $\mathbf{v}$ satisfies the Biot–Savart relation (3). Then at any fixed time $t$,**

$$\|\mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)} + \|\nabla \mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq C \|\omega(\cdot,t)\|_{L^1(\mathbb{R}^2)}$$

where $C$ is a constant independent of $t$, $\mathbf{v}$ and $\omega$.

We remark that in two dimensions, by the Poincaré inequality, it follows from $\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)} < \infty$, that $\mathbf{v}$ lies in $VMO(\mathbb{R}^2)$, i.e. has vanishing mean oscillation.

**Proof of Proposition 2.** Note that

$$\nabla \cdot (\nabla \times \omega) = 0.$$ 

Thus we can immediately apply the result of Bourgain–Brézis [3] (see also [2,5,10]) to the Biot–Savart formula (5) and get the desired conclusions in part (a).

To consider the 2-dimensional flow, note that $(-\partial_y \omega, \partial_x \omega)$ is a vector field in $\mathbb{R}^2$ with vanishing divergence. In view of the two-dimensional Biot–Savart relation (3), we can then use the two-dimensional Bourgain–Brézis result [3], and we obtain (b). \(\square\)

We note further that the proposition applies to both the Euler (inviscid) or the Navier–Stokes (viscous) flow.

**Proof of Theorem 1.** Now set $K_t$ for the heat kernel in two dimensions, i.e.

$$K_t(x) = \frac{1}{4\pi t} e^{-|x|^2/4t}.$$ 

Rewriting (2) as an integral equation for $\omega$ using Duhamel’s theorem, where $\omega_0$ is the initial vorticity, we have

$$\omega(x,t) = K_t \ast \omega_0(x) + \int_0^t \partial_x K_{t-s} \ast [\mathbf{v} \omega(x,s)] \, ds$$

where $\mathbf{v}$ is given by (3).

We apply a Banach fixed point argument to the operator $T$ given by

$$T \omega(x,t) = K_t \ast \omega_0(x) + \int_0^t \partial_x K_{t-s} \ast [\mathbf{v} \omega(x,s)] \, ds,$$

as

$$\|T \omega_0 - T \omega_1\|_{\operatorname{BV}^1(\mathbb{R}^2)} \leq C \|\mathbf{v}\|_{L^2(\mathbb{R}^2)} \|\omega_0 - \omega_1\|_{L^2(\mathbb{R}^2)}$$

for all $\omega_0, \omega_1 \in \operatorname{BV}^1(\mathbb{R}^2)$.
where again \( v \) is given by (3). Let us set

\[
E = \left\{ g \mid \sup_{0 \leq t \leq t_0} \| g(\cdot, t) \|_{W^{1,1}(\mathbb{R}^2)} \leq A \right\}.
\]

We will first show that \( T \) maps \( E \) into itself, for \( t_0 \) chosen as in the theorem. Differentiating (7) in the space variable once, we get

\[
(T \omega(x, t))_x = K_t \ast f_0(x) + \int_0^t \partial_x K_{t-s} \ast (v_x \omega) \, ds + \int_0^t \partial_x K_{t-s} \ast (\omega x) \, ds.
\]

Here we denote by \( f_0 \) the spatial derivative of the initial vorticity \( \omega_0 \). Using Young’s convolution inequality, we have

\[
\| (T \omega(\cdot, t))_x \|_{L^1(\mathbb{R}^2)} \leq \| f_0 \|_{L^1(\mathbb{R}^2)} + C \int_0^t (t-s)^{-1/2} \| v_x \omega \|_{L^1(\mathbb{R}^2)} + \| \omega x \|_{L^1(\mathbb{R}^2)} \, ds.
\]

Now we apply Proposition 2(b) to each of the terms on the right. For the first term, we have, by Cauchy–Schwarz,

\[
\| v_x \omega \|_{L^1(\mathbb{R}^2)} \leq C \| \nabla v \|_{L^2(\mathbb{R}^2)} \| \omega \|_{L^2(\mathbb{R}^2)}.
\]

The Gagliardo–Nirenberg inequality applies as \( \omega \in E \) and so \( \omega(\cdot, t) \in L^1(\mathbb{R}^2) \) and so,

\[
\| \omega \|_{L^1(\mathbb{R}^2)} \leq C \| \nabla \omega \|_{L^1(\mathbb{R}^2)},
\]

and to \( \| \nabla v \|_{L^2(\mathbb{R}^2)} \) we apply Proposition 2(b). Similarly, for the second term,

\[
\| \omega x \|_{L^1(\mathbb{R}^2)} \leq \| \nabla v \|_{L^2(\mathbb{R}^2)} \| \omega \|_{L^2(\mathbb{R}^2)}.
\]

Again we apply Proposition 2(b) to \( \| v \|_{L^\infty(\mathbb{R}^2)} \). Hence in all we have,

\[
\| (T \omega)_x \|_{L^1(\mathbb{R}^2)} \leq \| f_0 \|_{L^1(\mathbb{R}^2)} + C \int_0^t (t-s)^{-1/2} \| \nabla \omega \|_{L^1(\mathbb{R}^2)}^2 \, ds.
\]

Thus setting \( \| f_0 \|_{L^1(\mathbb{R}^2)} = \| \omega_0 \|_{W^{1,1}(\mathbb{R}^2)} \leq A_0 \), we get for \( t \leq t_0 \) and since \( \omega \in E \),

\[
\| \nabla (T \omega)(\cdot, t) \|_{L^1(\mathbb{R}^2)} \leq A_0 + C_{t_0}^{1/2} A^2.
\]

Next from Young’s convolution inequality it follows from (7) that

\[
\| T \omega(\cdot, t) \|_{L^1(\mathbb{R}^2)} \leq A_0 + \int_0^t (t-s)^{-1/2} \| \omega(\cdot, s) \|_{1} \, ds.
\]

But by Proposition 2(b) again,

\[
\| \omega \|_{1} \leq \| \nabla \|_{\infty} \| \omega \|_{1} \leq cA^2.
\]

Thus

\[
\| T \omega(\cdot, t) \|_{1} \leq A_0 + ct^{1/2} A^2.
\]

So, adding the estimates for \( T \omega \) and \( \nabla (T \omega) \), we have:

\[
\sup_{t \leq t_0} \| T \omega(\cdot, t) \|_{W^{1,1}(\mathbb{R}^2)} \leq 2A_0 + c t_0^{1/2} A^2.
\]

By choosing \( A \) so that \( A_0 = A/8 \) and \( t < t_0 = C/A_0^2 \), we can assure that if \( \omega \in E \), then

\[
\sup_{t \leq t_0} \| (T \omega)(\cdot, t) \|_{W^{1,1}(\mathbb{R}^2)} \leq \frac{A}{2}.
\]

Thus \( T \omega \in E \), if \( \omega \in E \). If we establish that \( T \) is a contraction, then we are done.

Next we observe that the estimates in Proposition 2(b) are linear estimates. That is

\[
\| v_1 - v_2 \|_{\infty} + \| \nabla v_1 - \nabla v_2 \|_2 \leq C \| \omega_1 - \omega_2 \|_{W^{1,1}(\mathbb{R}^2)}.
\]

We easily can see from the computations above, that we have
\[
\sup_{t \leq t_0} \| T \omega_1 - T \omega_2 \|_{W^{1,1}(\mathbb{R}^3)} \leq C A_0^{1/2} \sup_{t \leq t_0} \| \omega_1 - \omega_2 \|_{W^{1,1}(\mathbb{R}^3)}.
\]

By the choice of \( t_0 \), it is seen that \( T \) is a contraction. Thus using the Banach fixed-point theorem on \( E \), we obtain our operator \( T \) has a fixed point and so the integral equation (6) has a solution in \( E \). The remaining part of our theorem follows easily from Proposition 2(b). □

We note in passing an estimate in \( \mathbb{R}^3 \) from Proposition 2(a) above for the Navier–Stokes or the Euler flow:

\[
\sup_{t>0} \| \mathbf{v} \|_{L^3(\mathbb{R}^3)} + \sup_{t>0} \| \nabla \mathbf{v} \|_{L^{3/2}(\mathbb{R}^3)} \leq C \sup_{t>0} \| \nabla \times \omega \|_{L^1(\mathbb{R}^3)}. \tag{8}
\]

2. Magnetism

We next turn to our results on magnetism. We denote by \( \mathbf{B}(x, t) \) and \( \mathbf{E}(x, t) \) the magnetic and electric field vectors at \( (x, t) \in \mathbb{R}^3 \times \mathbb{R} \). Let \( \mathbf{j}(x, t) \) denote the current density vector. The Maxwell equations imply

\[
\begin{align*}
\nabla \cdot \mathbf{B} &= 0, \tag{9} \\
\partial_t \mathbf{B} + \nabla \times \mathbf{E} &= 0, \tag{10} \\
\partial_t \mathbf{E} - \nabla \times \mathbf{B} &= -\mathbf{j}. \tag{11}
\end{align*}
\]

Differentiating (10) in \( t \) and using (11), together with the vector identity \( \nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \Delta \mathbf{B} \) and (9), one obtains an inhomogeneous wave equation for \( \mathbf{B} \):

\[
\mathbf{B}_{tt} - \Delta \mathbf{B} = \nabla \times \mathbf{j}. \tag{12}
\]

The right side of (12) satisfies the vanishing divergence condition

\[
\nabla \cdot (\nabla \times \mathbf{j}) = 0
\]

for any fixed time \( t \). Thus an improved Strichartz estimate, namely Theorem 1 in [6], applies. We point out that the Bourgain–Brézis inequalities play a key role in the proof of Theorem 1 in [6]. We conclude easily:

**Theorem 3.** Let \( \mathbf{B} \) satisfy (12) and let \( \mathbf{B}(x, 0) = \mathbf{B}_0, \partial_t \mathbf{B}(x, 0) = \mathbf{B}_1 \) denote the initial data at time \( t = 0 \). Let \( s, k \in \mathbb{R} \). Assume \( 2 \leq q \leq \infty, 2 < q \leq \infty \) and \( 2 \leq r < \infty \). Let \( (q, r) \) satisfy the wave compatibility condition

\[
\frac{1}{q} + \frac{1}{r} \leq \frac{1}{2},
\]

and assume the following scale invariance condition is verified:

\[
\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - s = \frac{1}{q} + 1 - k.
\]

Then, for \( \frac{1}{q} + \frac{1}{q'} = 1 \), we have

\[
\| \mathbf{B} \|_{L^q_t L^r_x} + \| \mathbf{B} \|_{C^0_t H^s_x} + \| \partial_t \mathbf{B} \|_{C^0_t H^{s-1}_x} \leq C(\| \mathbf{B}_0 \|_{H^s_x} + \| \mathbf{B}_1 \|_{H^{s-1}_x} + \| (-\Delta)^{k/2}(\nabla x j) \|_{L^q_t L^r_x}).
\]

The main point in the theorem above is that we have \( L^1 \) norm in space on the right side.

**References**

[1] M. Ben-Artzi, Global solutions of two-dimensional Navier–Stokes and Euler equations, Arch. Ration. Mech. Anal. 128 (4) (1994) 329–358.

[2] J. Bourgain, H. Brézis, New estimates for the Laplacian, the div-curl, and related Hodge systems, C. R. Acad. Sci. Paris, Ser. I 338 (7) (2004) 539–543.

[3] J. Bourgain, H. Brézis, New estimates for elliptic equations and Hodge type systems, J. Eur. Math. Soc. 9 (2) (2007) 277–315.

[4] H. Brézis, Remarks on the preceding paper by M. Ben-Artzi: “Global solutions of two-dimensional Navier–Stokes and Euler equations”, Arch. Ration. Mech. Anal. 128 (4) (1994) 359–360.

[5] S. Chanillo, J. Van Schaftingen, P.-L. Yung, Variations on a proof of a borderline Bourgain–Brézis Sobolev embedding theorem, to appear in Chin. Ann. Math. Ser. B.

[6] S. Chanillo, P.-L. Yung, An improved Strichartz estimate for systems with divergence free data, Commun. Partial Differ. Equ. 37 (2) (2012) 225–233.

[7] Y. Giga, T. Miyakawa, H. Osada, Two-dimensional Navier–Stokes flow with measures as initial vorticity, Arch. Ration. Mech. Anal. 104 (3) (1988) 223–250.

[8] T. Kato, The Navier–Stokes equation for an incompressible fluid in \( \mathbb{R}^2 \) with a measure as the initial vorticity, Differ. Integral Equ. 7 (3–4) (1994) 949–966.

[9] C.W. Oseen, Über Wirbelbewegung in einer reibenden Flüssigkeit, Ark. Mat. Astron. Fys. 7 (1912) 1–13.

[10] J. Van Schaftingen, Estimates for \( L^1 \)-vector fields, C. R. Acad. Sci. Paris, Ser. I 339 (3) (2004) 181–186.