Finite Sample Smeariness of Fréchet Means and Application to Climate

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Abstract

Fréchet means on non-Euclidean spaces may exhibit nonstandard asymptotic rates rendering quantile-based asymptotic inference inapplicable. We show here that this affects, among others, all circular distributions whose support exceeds a half circle. We exhaustively describe this phenomenon and introduce a new concept which we call finite samples smeariness (FSS). In the presence of FSS, it turns out that quantile-based tests for equality of Fréchet means systematically feature effective levels higher than their nominal level which perseveres asymptotically in case of Type I FSS. In contrast, suitable bootstrap-based tests correct for FSS and asymptotically attain the correct level. For illustration of the relevance of FSS in real data, we apply our method to directional wind data from two European cities. It turns out that quantile based tests, not correcting for FSS, find a multitude of significant wind changes. This multitude condenses to a few years featuring significant wind changes, when our bootstrap tests are applied, correcting for FSS.

Key words and phrases: Fréchet means, smeariness, one- and two-sample tests, nonparametric asymptotic quantile based tests, bootstrap tests, directional data on circles and tori, wind directions.

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1 Introduction

Considering random variables $X_1, \ldots, X_n \sim X$ on a metric space $(M, d)$, generalizing the concept of the expected value, Fréchet (1948) proposed to consider minimizers of the expected squared distance:

$$\arg\min_{p \in M} F(p) \quad \text{where} \quad F(p) = \mathbb{E} \left[ d(p, X)^2 \right].$$

(1)

Under mild assumptions, Ziezold (1977) derived strong set-wise asymptotic consistency for

$$\arg\min_{p \in M} F_n(p) \quad \text{where} \quad F_n(p) = \frac{1}{n} d(p, X_j)^2$$

(2)

Under stronger conditions, among others that $M$ is a finite dimensional manifold, and under uniqueness of the minimizer $\mu$ of (1), called a Fréchet population mean, Bhattacharya and Patrangenaru (2005) derived, in a local chart $\phi : U \to \mathbb{R}^m$ near $\mu \in U \subset M$, a central limit theorem for a measurable selection $\hat{\mu}_n$ of (2), called a Fréchet sample mean, with a Gaussian limiting distribution and the usual rate of $n^{-1/2}$:

$$\sqrt{n} (\phi(\hat{\mu}_n) - \phi(\mu)) \xrightarrow{D} \mathcal{N}(0, \Sigma).$$

(3)

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While the covariance matrix Σ above reflects the asymptotic rescaled covariance of φ(\(\hat{\mu}_n\)), it is usually approximated by the covariance matrix \(\hat{\Sigma}_n\) of the sample \(\phi(X_1), \ldots, \phi(X_n)\), giving rise to the quantile-based one- and two-sample tests proposed by Bhattacharya and Patrangenaru (2005), see also e.g. Bhattacharya and Lin (2017). These tests rest on the approximation

\[
(\phi(\hat{\mu}_n) - \phi(\mu))^T \hat{\Sigma}_n^{-1} (\phi(\hat{\mu}_n) - \phi(\mu)) \rightarrow \chi^2_m.
\]

(4)

In order to assess the validity of this approximation, we consider the suitably rescaled quotient of Fréchet variances

\[
m_n := \frac{n\mathbb{E}[d(\mu, \hat{\mu}_n)^2]}{\mathbb{E}[d(\mu, X)^2]},
\]

(5)

which has been called by Pennec (2019) the modulation of the rate of convergence of the variance for sample size \(n\), which we will abbreviate as variance modulation. The following phenomena have been observed in the literature:

(A) \(m_n = 1\) for all \(n \in \mathbb{N}\),
(B) \(m_n = 0\) for all \(n > N\) where \(N\) is a suitable random positive integer (stickiness),
(C) \(\lim_{n \to \infty} m_n = \infty\) (smeariness).

Phenomenon (A) is the case on Euclidean spaces \((M, d)\) whenever second moments are finite. As we will show here, it is also the case if \((M, d)\) is a flat torus with sufficiently concentrated \(X\). For some nontrivial random variables on non-manifolds, using not a local chart but a suitable embedding, Phenomenon (B) has been observed by Barden et al. (2013, 2018) on the BHV spaces of Billera et al. (2001) for phylogenetic trees and it has been observed on related spaces by Hotz et al. (2013); Huckemann et al. (2015). Furthermore, Phenomenon (C) has been observed on the circle by Hotz and Huckemann (2015) with \(m_n\) of rate \(n^{\gamma-1}\) with arbitrary \(1 \leq \gamma \in \mathbb{N}\) and on spheres of arbitrary dimension by Eltzner and Huckemann (2019) with \(m_n\) of rate \(n^{2/3}\).

Notably, all but (A) render the approximation (4) invalid. At this point we remark that stickiness (B) is conceptually different from the opposite of smeariness (C), namely

(D) \(m_n > 0\) for all \(n \in \mathbb{N}\) and \(\lim_{n \to \infty} m_n = 0\),

which has been observed by Schötz (2019).

In this contribution we bring to attention two new phenomena:

(E) \(m_n > 1\) for all \(n \in \mathbb{N}\) and \(1 < \lim_{n \to \infty} m_n < \infty\)
(F) \(m_n > 1\) for all \(n \in \mathbb{N}\) and \(\lim_{n \to \infty} m_n = 1\),

and investigate them on the circle. Phenomenon (E) affects all (!) parametric distributions on the circle with nowhere vanishing density like the Fisher-von-Mises, the wrapped Gaussian, etc. In particular, it may render the approximation (4) invalid as dramatically illustrated in the left display of Table 1.

Phenomenon (F) affects all distributions on the circle whose support strictly exceeds a closed half circle as long as a neighborhood of the antipodal of their Fréchet mean carries no probability mass. While this phenomenon renders the approximation (4) asymptotically valid, for surprisingly high sample sizes, the approximation may still be far off.

In simulations, we see that in cases (E) and (F), \(m_n\) initially has a rate comparable to smeariness as in Phenomenon (C), in particular, the rate of \(\mathbb{E}[d(\hat{\mu}_n, \mu)^2]\) is strictly lower than the classical \(n^{-1}\). Equivalently, \(n \mapsto m_n\) starts off nonhorizontal as illustrated in Figures 1 and 2. Since in these cases, \(n \mapsto m_n\) is only asymptotically horizontal, we give these new phenomena the name finite sample smeariness (FSS) and distinguish between Type I (E) and Type II (F). Alternatively for these phenomena, the term lethargic means has been proposed.
Table 1: Empirical rejection probability based on 100 000 simulation runs of two-sample tests for equality of Fréchet means with nominal significance level 0.05 under the null hypothesis of a mixture of two equally weighted (β = 1/2) von-Mises distributions on the circle with antipodal means (the first with concentration parameter κ = 3 and the second with varying concentration parameter λ) as defined in (6). Left: the quantile-based Test 4.1(ii) resting on (4), Right: the bootstrap-based Test 4.4 with B = 1000.

| n     | λ   | 0  | 1/4 | 1/2 | 3/4 |
|-------|-----|----|-----|-----|-----|
|   100 | 0.320 | 0.447 | 0.582 | 0.689 |
| 1000  | 0.330 | 0.474 | 0.656 | 0.818 |
|10000  | 0.331 | 0.477 | 0.666 | 0.876 |

While under FSS, empirical levels of quantile-based tests can deviate strongly from their nominal level, we show for the circle that suitably designed bootstrap tests approximately keep their level, as illustrated in the right display of Table 1. In fact, we show in this work that the deviation of the quantile-based test perseveres as the sample size n tends to infinity whereas the bootstrap-based test asymptotically attains the correct level. In addition to simulations, in application to wind direction data we see that asymptotic quantile based two-sample tests, on the circle and a torus, due to FSS present in the data, give a wrong impression of the extent of extreme wind change events for two continental European cities. Using bootstrap tests which preserve the nominal level, extreme wind changes reduce to a few concise events, which in particular make the years 2003, 2005 and 2017, 2018 exceptional (the latter strongly), hinting towards an effect of recent climate change.

In the following Section 2 we introduce FSS on arbitrary metric spaces and give a test for the presence of FSS in data. Further, we explore FSS exhaustively on the circle and on tori. In Section 3 we investigate the asymptotics of circular Fréchet sample means and prove a consistency result for bootstrap Fréchet sample means under FSS. For tests to follow, we also verify that all moments of properly centered and scaled Fréchet sample means converge to the respective moment of the limit distribution. Section 4 explores the quantile based test by Bhattacharya and Patrangenaru (2005) and an implementation of the bootstrap test, which was also suggested by Bhattacharya and Patrangenaru (2005). This is followed by simulations in Section 5 and the application to wind direction data in Section 6. We conclude in Section 7 with an outlook to higher dimensions and list open problems that arise from our findings.

All of the more elaborate proofs are deferred to the supplement.

2 Finite Sample Smeariness

In the presence of finite sample smeariness (FSS) for finite sample sizes n, often considerably large, the rate of \( \phi(\hat{\mu}_n) - \phi(\mu) \), in the notation from (3), is not of order \( n^{-1/2} \) but similar to smeary rates as have been observed by Hotz and Huckemann (2015) on circles and by Eltzner and Huckemann (2019) on spheres of arbitrary dimension. In case of FSS, however, asymptotically the rate returns to \( n^{-1/2} \) but, possibly, with unanticipated asymptotic variance. In this section, we first give a general definition. Then, we describe this new phenomenon in detail on the circle and on flat tori and finally we propose a test for the presence of FSS in general data.

2.1 Definition of Finite Sample Smeariness

The following general definition assumes a metric space \((M, d)\) isometrically embedded in a finite dimensional Euclidean space, such that for every subset of \(M\), being compact is equivalent to being closed and bounded. Then Fréchet mean sets defined by (1) are compact because they are closed by continuity and
bounded due to
\[ d(\mu_1, \mu_2) \leq \mathbb{E}[d(\mu_1, X)] + \mathbb{E}[d(\mu_2, X)] \leq \sqrt{\mathbb{E}[d(\mu_1, X)^2]} + \sqrt{\mathbb{E}[d(\mu_2, X)^2]}. \]

If the set \( K \) of Fréchet means is the union of disjoint manifolds of same dimension, the uniform measure on \( K \) is the normed measure obtained from the Euclidean measure conditioned to \( K \). Thus, we can agree on the following.

**Agreement 2.1.** If sample Fréchet means defined by (2) are not unique and the mean set is sufficiently well behaved admitting a uniform measure, we agree that a measurable selection \( \hat{\mu}_n \) is drawn independently of \( X_1, \ldots, X_n \) from that uniform measure on the Fréchet sample mean set. Under uniqueness of the Fréchet population mean \( \mu \), \( \mathbb{E}[d(\hat{\mu}_n, \mu)^2] \) is then well defined and so is the variance modulation \( m_n \) from (5).

**Definition 2.2** (Finite sample smeariness). A random variable \( X \) on a metric space \((M, d)\) with unique Fréchet population mean \( \mu \) is finite sample smeary (FSS) if \( 1 < \sup_{n \in \mathbb{N}} m_n < \infty \). We speak of Type I FSS if \( \lim_{n \to \infty} m_n > 1 \) and of Type II FSS if \( \lim_{n \to \infty} m_n = 1 \).

### 2.2 Finite Sample Smeariness on Circles and Tori

The circle is the space \( \mathbb{S}^1 = [-\pi, \pi) \) with \(-\pi\) and \(\pi\) identified and the usual arc length distance
\[ d_{\mathbb{S}}(x, y) = \min \{|x - y|, 2\pi - |x - y|\}. \]

On the circle, we have the following characterization of the variance modulation \( m_n \).

**Theorem 2.3.** For arbitrary \( n \geq 2 \) suppose that \( X \) is a random variable on \( \mathbb{S}^1 \) with unique Fréchet mean \( \mu \) and support \( J \subseteq \mathbb{S}^1 \). Then \( m_n > 1 \) for all \( n \in \mathbb{N} \) under any of the two following conditions

(i) The interior of \( J \) contains a closed half circle,

(ii) \( J \) contains two antipodal points, each of which is assumed by \( X \) with positive probability.

Moreover, \( m_n = 1 \) for all \( n \in \mathbb{N} \) under any of the two following conditions

(iii) \( J \) is strictly contained in a closed half circle,

(iv) \( J \) is a closed half circle and one of its end points is assumed by \( X \) with zero probability.

Finally, suppose that \( X \) has a continuous density \( f \) near the antipode \( \overline{\nu} \) of \( \mu \).

(v) If \( f(\overline{\nu}) = 0 \) then \( \lim_{n \to \infty} m_n = 1 \),

(vi) if \( 0 < f(\overline{\nu}) < \frac{1}{2\pi} \) then \( \lim_{n \to \infty} m_n = \frac{1}{(1-f(\overline{\nu})^2)} > 1 \).

**Proof.** The cases (i)–(iv) follow at once from reducing to one of the cases of Lemma 3.6 in Supplement [A]. For the cases (v) and (vi) we note that \( \sup_{n \in \mathbb{N}} E[d(\hat{\mu}_n, \mu)^2] < \infty \) (Proposition 3.6) which asserts by [Billingsley 1995, Corollary to Theorem 25.12], that \( n \mathbb{E}[d(\hat{\mu}_n, \mu)^2] \) converges to the variance of the limiting distribution in the first assertion of Theorem 3.2. \( \square \)

As a consequence, we derive the following characterization of FSS on the circle.

**Corollary 2.4.** Let \( X \) be a random variable on \( \mathbb{S}^1 \) which has a continuous density near the antipode \( \overline{\nu} \) of the Fréchet population mean \( \mu \).

(i) Then, \( \mu \) is FSS of type I if and only if \( 0 < f(\overline{\nu}) < 1/(2\pi) \).

(ii) Further, \( \mu \) is FSS of type II if and only if \( f(\overline{\nu}) = 0 \) and condition (i) or (ii) of Theorem 2.3 holds.
Remark 2.5. In the above case of a continuous density \( f \) near the antipode \( \overline{0} \) of the Fréchet mean, \( f(\overline{0}) = (2\pi)^{-1} \) leads to smeariness as detailed in [Hotz and Huckemann (2013)] and \( f(\overline{0}) > (2\pi)^{-1} \) is not possible, cf. [Hotz and Huckemann (2013), Theorem 3.1(ii)].

Simple cases of FSS on the circle are illustrated in the following.

![Figure 1: Log-log plots of variance modulation curves](image)

\[ r \quad 0 \quad 0.1 \quad 0.2 \quad \pi/2 \]

Figure 1: Log-log plots of variance modulation curves \( n \mapsto \mathbf{m}_n \), for varying von Mises mixture distributions (vMmds) defined in [7] in black (\( r = 0 \)) and vMmds with a disk cut out of radius \( r = 0, 0.1, 0.2, \pi/2 \) (in fading gray) about the antipode of the mean, as defined in [7] based on 100 000 simulation runs for each sample size. In the left and middle display, the vMmds (black) feature Type I FSS, with the dotted line giving the asymptotic scaled variance, the two vMmds with an interval cut out (gray curves) of radius \( 0 < r < \pi/2 \) feature Type II FSS. In the right display, the vMmd (black) features smeariness, with theoretical scaled variance (dotted), the vMmd with an interval cut out (gray curves) of radius \( 0 < r < \pi/2 \) feature Type II FSS. Only the vMmds with an interval cut out of radius \( r = \pi/2 \) (light gray) features no FSS at all.

Example 2.6 (Von Mises mixtures). On \( S^1 \) we consider von Mises mixtures with antipodal modes with respect to arc length measure

\[
d\mathbb{P}^{\kappa,\beta,\lambda}_{vMm}(x) := \beta \exp(\kappa \cos(x)) \frac{\exp(\lambda \cos(x + \pi))}{2\pi I_0(\beta)} \, dx + (1 - \beta) \frac{\exp(\lambda \cos(x))}{2\pi I_0(\lambda)} \, dx \quad \text{for} \quad \kappa, \lambda \geq 0, \beta \in [0, 1], \tag{6}
\]

where \( I_0(\cdot) \) is the modified Bessel function of the first kind of order 0, e.g., [Mardia and Jupp (2000, p. 36)]. By symmetry, a von Mises mixture \( d\mathbb{P}^{\kappa,\beta,\lambda}_{vMm} \) attains either a unique mean at 0 or \( \pi \), or nonunique means at \( \{-t, t\} \) for some \( t \in (0, \pi) \). Furthermore, we define for \( r \geq 0 \) the function cutting out and mirroring a disk of radius \( r \) about \( -\pi \):

\[ \zeta^r : S^1 \to S^1, \quad p \mapsto \begin{cases} 
  p & \text{if } p \in [-\pi + r, \pi - r) \\
  p + \pi & \text{if } p \in [-\pi, -\pi + r) \\
  p - \pi & \text{if } p \in [\pi - r, \pi)
\end{cases} \]

For the von Mises mixture \( \mathbb{P}^{\kappa,\beta,\lambda}_{vMm} \) we then denote the push-forward measure under \( \zeta^r \) by

\[
\mathbb{P}^{\kappa,\beta,\lambda}_{vMm} := \zeta^r \mathbb{P}^{\kappa,\beta,\lambda}_{vMm}, \quad \tag{7}
\]

which preserves all mass except for that in the disk which is mirrored. Recall that by Theorem 2.3 all of the \( \mathbb{P}^{\kappa,\beta,\lambda}_{vMm} \) with unique mean \( \mu = 0 \) and \( r < \pi/2 - \varepsilon \) for some \( \varepsilon > 0 \) feature FSS (if they...
are not smearable themselves, which is of Type I if \( r = 0 \) and Type II if \( r > 0 \). For the parameters \((\kappa, \beta, \lambda) \in \{(3, 1, 0), (3, 0.5, 0.5), (3, 0.5, 0.8683)\}\) and varying choices for the parameter \( r \in \{0, 0.1, 0.2, \pi\} \) the respective variance modulation curves are depicted in Figure 7. Notably, in case of Type I FSS, the curve \( m_n \) rises from 1 to \((1 - 2\pi f(\pi))^{-2}\) whereas under Type II FSS, the curve first rises from 1 and eventually drops down to 1.

Depending on the probability distribution near \( \pi \), also more complicated versions of increase and decrease may occur, as Example 2.7 and Figure 2 teach: every pair of bumps of the density near the antipode may result in a bump of \( m_n \). As before, however, it starts at 1 and eventually settles at \((1 - 2\pi f(\pi))^{-2}\), producing Type I FSS if \( f(\pi) > 0 \) and Type II FSS else.

**Example 2.7** (Relating antipodal density to variance modulation). To investigate the relationship between the variance modulation of intrinsic sample means \( n \mapsto m_n \) and the density \( f^{(t,w)} \) of \( X \) near the antipode of the intrinsic population mean \( \mu = 0 \) we consider a family of distributions for which the density near the antipode \( \pi \) is piecewise constant. Let \( l \in \mathbb{N} \), \( w = (w_1, \ldots, w_l) \in [0, 1]^l \), \( t = (t_1, \ldots, t_l) \in [0, \pi]^l \) with \( t_0 := 0 < t_1 < \cdots < t_l < \pi \) and define the distribution \( \mathbb{P}^{(t,w)}_U \) by

\[
d\mathbb{P}^{(t,w)}_U(x) := k \cdot d\delta_0(x) + \frac{1}{2\pi} \cdot f^{(t,w)}(x) dx \quad \text{with}
\]

\[
f^{(t,w)}(x) = \begin{cases}
  w_i & \text{if} \quad x \in [-\pi + t_i-1, -\pi + t_i) \cup (\pi - t_i, \pi - t_{i-1}] \text{ for some } i \in \{1, \ldots, l\}, \\
  0 & \text{if} \quad x \in [-\pi + t_i, \pi - t_i],
\end{cases}
\]

where \( k = k(t, w) > 0 \) is a normalization constant to ensure that \( \mathbb{P}^{(t,w)}_U \) is a probability measure. In Table 2 we list cases (a) – (e) of parameter choices considered.

| Type of FSS | 0    | 1    | 2    | 4    | 4    |
|-------------|------|------|------|------|------|
| \( l \)     | 1    | 2    | 4    | 2    | 4    |
| \( t \)     | 1.5  | (0.8, 2) | (0.1, 0.2, 0.5, 2) | (0.8, 2) | (0.1, 0.2, 0.5, 2) |
| \( w \)     | 0.5  | (0.5, 1) | (0.5, 0.8, 0.0, 1) | (0.0, 1) | (0.0, 0.85, 0.0, 1) |

Table 2: Selected values for the parameters \( l, t, w \) and their resulting type of FSS.
In all cases the population mean of $\mathbb{P}^{(t,w)}_U$ is unique and located at $\mu = 0$. Whenever the density at the antipode is strictly between zero and $1/2\pi$ we have FSS of Type I. Regardless of the type of FSS, every pair of bumps in the density near the antipode corresponds to a single bump in the variance modulation curve. Thus, in case of Type I FSS the rescaled Fréchet sample variance $nE[d(\hat{\mu}_n, \mu)^2]$ approaches asymptotically a value strictly above the Euclidean variance $E[d(X, \mu)^2]$, in case of Type II FSS it approaches asymptotically the Euclidean variance.

**Remark 2.8.** As Example 2.7 and Figure 3 teach, the variance modulation $m_n$ may be non-monotonic exhibiting different behaviours for different regimes of sample size.

The following example illustrates Theorem 2.3 with two point masses at or beyond the equator (case (ii)), or before (case (i)), with a possibly (in case (iii)) non-unique sample mean set, of which uniformly at random, a sample mean is selected.

**Example 2.9.** Letting $\varepsilon \in [-\pi/2, \pi/2]$ and $w \in (0, 1/4]$, define the circular distribution $\mathbb{P}^{(c,w)}_E$ by

$$d\mathbb{P}^{(c,w)}_E(x) := (1 - 2w) \cdot 1_{[-1/2,1/2]}(x)dx + w d\delta_{\pi/2 + \varepsilon}(x) + w d\delta_{-\pi/2 - \varepsilon}(x),$$

which assigns at least half the mass to $[-1/2,1/2]$ and the rest is evenly distributed close to the equator at $\pi/2 + \varepsilon$ and $-\pi/2 - \varepsilon$.

(i) For $\varepsilon < 0$ these distribution are supported in $(-\pi/2,\pi/2)$ and thus by Theorem 2.3 (i) feature no FSS.

(ii) For $\varepsilon > 0$, in contrast by Theorem 2.3 (iii), they always feature FSS, which is, by Theorem 3.2 (i) of Type II.

(iii) For $\varepsilon = 0$, by the same argument, they also feature FSS of Type II. Indeed, samples featuring only points at $\pm \pi/2$ and no others, occurring with positive probability, have nonunique sample means, namely one closer to 0 and one closer to $-\pi$. As agreed, of these with probability 1/2 the one closer to $-\pi$ is chosen which accounts for a higher variance than the Euclidean.

Each panel in Figure 3 illustrates the three cases, with larger effect on FSS the larger the weight $w$ of each of the point masses.

While we defined FSS for arbitrary metric spaces, we have investigated it only for the circle. These results extend at once to the $m$-torus $\mathbb{T}_m = (S^1)^m = \times_{i=1}^m [-\pi,\pi]$, $m \in \mathbb{N}$, equipped with the canonical product metric

$$d_{\mathbb{T}_m}(x, y) = \sum_{i=1}^m d(x^{(i)}, y^{(i)})^2, \quad x = \begin{pmatrix} x^{(1)} \\ \vdots \\ x^{(m)} \end{pmatrix}, \quad y = \begin{pmatrix} y^{(1)} \\ \vdots \\ y^{(m)} \end{pmatrix} \in \mathbb{T}_m.$$ 

Indeed, $\mu \in \mathbb{T}_m$ is a minimizer of the population Fréchet function (1) of a random variable $X$ on $\mathbb{T}_m$ if and only if all of its coordinates $\mu^{(i)}$ are minimizers of the population Fréchet functions of the marginals $X^{(i)}$ on the $i$-th circle, $i \in \{1, \ldots, m\}$.

**Remark 2.10.** In consequence, due to [Hotz and Huckemann 2013, Corollary 3], for a sample $X_1, \ldots, X_n$ on the $m$-torus $\mathbb{T}_m$ there are at most $n^m$ minimizers of the sample Fréchet function (4). For instance every one of the $n^m$ grid points

$$\left((-\pi + \frac{2\pi(j_1 - 1)}{2n}), \ldots, (-\pi + \frac{2\pi(j_m - 1)}{2n})\right), \quad j_1, \ldots, j_m \in \{1, \ldots, n\}$$

is a minimizer of the sample Fréchet function of $X_j = (-\pi + \frac{2\pi(j-1)}{n}) \in \mathbb{T}_m$, $j \in \{1, \ldots, n\}$, cf. Figure 4. Indeed, their $i$-th marginal $(1 \leq i \leq m)$ is given by the sample $X^{(i)}_j = (-\pi + \frac{2\pi(j-1)}{n})$, $j \in \{1, \ldots, n\}$ which has exactly $n$ minimizers of its sample Fréchet function which are equally spaced between the sample points.
Figure 3: Variance modulation curves, as in Figure 1, for the three types of distributions $P^E(\varepsilon,w)$ from Example 2.9 with weight $w$ of point mass at $\pm(\pi/2 + \varepsilon)$ based on 100,000 simulation runs for each sample size. Black: $\varepsilon = 0.5$, dark gray: $\varepsilon = 0.2$, gray: $\varepsilon = 0$, light gray: $\varepsilon = -0.2$. We have Type II FSS for all values of $\varepsilon \geq 0$ and no FSS (constant scaled variance) for $\varepsilon < 0$.

Figure 4: Six sample points (black circles) on the diagonal of the two-torus $\mathbb{T}^2$ and their equally spaced $6^2 = 36$ Fréchet sample means (empty squares), cf. Remark 3.5.

In particular, Theorem 2.3 yields the following characterization of FSS and the variance modulation for random variables on $\mathbb{T}^m$.

**Corollary 2.11.** For arbitrary $n \geq 2$ and $m \in \mathbb{N}$ let $X = (X^{(1)}, \ldots, X^{(m)})$ be a random variable on the torus $\mathbb{T}^m$ with unique Fréchet population mean $\mu$. For $i \in \{1, \ldots, m\}$ let the support of $X^{(i)}$ be $J^{(i)} \subseteq S^1$. Then $m_n > 1$ for all $n \in \mathbb{N}$ under any of the following two conditions

(i) there exists $i \in \{1, \ldots, m\}$ such that the interior of $J^{(i)}$ contains a closed half circle,
(ii) there exists \( i \in \{1, \ldots, m\} \) such that \( J^{(i)} \) contains two antipodal points, each of which are attained by \( X^{(i)} \) with positive probability

Moreover, \( m_n = 1 \) for all \( n \in \mathbb{N} \) under any of the following two conditions

(iii) for each \( i \in \{1, \ldots, m\} \) the support \( J^{(i)} \) is strictly contained in a closed half circle,

(iv) for each \( i \in \{1, \ldots, m\} \) the support \( J^{(i)} \) is contained in a closed half circle where one of the end points is a.s. not attained by \( X \).

Finally, suppose that each component \( X^{(i)} \) for \( i \in \{1, \ldots, m\} \) has near the antipode \( \pi^{(i)} \) of \( \mu^{(i)} \) a continuous density \( f^{(i)} \).

(v) If \( f^{(i)}(\pi^{(i)}) = 0 \) for all \( i \in \{1, \ldots, m\} \), then \( \lim_{n \to \infty} m_n = 1 \),

(vi) if \( 0 \leq f^{(i)}(\pi^{(i)}) < \frac{1}{2\pi} \) for all \( i \in \{1, \ldots, m\} \) with \( f^{(i)}(\pi^{(i)}) > 0 \) for at least one component then

\[
\lim_{n \to \infty} m_n = \frac{\sum_{k=1}^{m} \mathbb{E}[d^2(X^{(k)}(\hat{\mu}^{(i)}))]}{\sum_{i=1}^{m} \mathbb{E}[d^2(X^{(i)}, \mu^{(i)})]} > 1.
\]

As a consequence, we obtain a similar characterization as in Corollary 2.11 for FSS on the torus.

**Corollary 2.12.** Suppose \( X = (X^{(1)}, \ldots, X^{(m)}) \) be a random variable on \( \mathbb{T}^m \) and suppose for each \( i \in \{1, \ldots, m\} \) there exists a continuous density near the antipode \( \pi^{(i)} \) of the respective coordinate’s Fréchet population mean \( \mu^{(i)} \).

(i) Then, \( \mu \) is FSS of type I if and only if \( 0 \leq f^{(i)}(\pi^{(i)}) < \frac{1}{2\pi} \) for all \( i \in \{1, \ldots, m\} \) with \( f^{(i)}(\pi^{(i)}) > 0 \) for at least one component.

(ii) Further, \( \mu \) is FSS of type II if and only if \( f^{(i)}(\pi^{(i)}) = 0 \) for all \( i \in \{1, \ldots, m\} \) and condition (i) or (ii) of Corollary 2.11 holds.

For other spaces, an investigation of FSS is beyond the scope of this paper. Already for spheres, finding examples of smeary distributions is rather involved, and to date, only two-smeariness could be confirmed by [Eltzner and Huckemann (2019)](https://link.springer.com/article/10.1007/s10444-019-09554-1). It is, however, known for rather general manifolds that “smeariness begets FSS” as [Tn et al. (2021)](https://link.springer.com/article/10.1007/s10444-021-09690-4) constructed from every smeary random variable a random variable featuring FSS of Type I. On the other hand, for manifolds featuring a simply connected submanifold of constant positive sectional curvature, not exceeded by any sectional curvatures of the manifold, [Tn et al. (2021)](https://link.springer.com/article/10.1007/s10444-021-09690-4) have given examples of directional smeary random variables. In particular, [Eltzner et al. (2021)](https://link.springer.com/article/10.1007/s10444-021-09690-4) showed that every non trivial random variable on a sphere is FSS of Type I.

In the light of geometrical smeariness introduced by [Eltzner (2019)](https://link.springer.com/article/10.1007/s10444-019-09554-1) we conjecture that Type I FSS is present in all nondegenerate random variables on positively curved spaces. For such spaces, [Alsar (2009)](https://link.springer.com/article/10.1007/s10444-019-09554-1) has shown that the Hessian of the Fréchet function is smaller than its Euclidean equivalent, which is twice the identity matrix. Moreover, for very small sample sizes \( (n \approx 10) \) of highly concentrated random variables, [Pennesi (2019)](https://link.springer.com/article/10.1007/s10444-019-09554-1) explicitly derived \( m_n \sim 1 \) but well below what is predicted asymptotically by the CLT. We allow considerably larger sample sizes with no concentration constraints, however.

### 2.3 Testing for Finite Sample Smeariness on Metric Spaces

In order to heuristically assess the presence of FSS for a given sample \( X_1, \ldots, X_n \overset{i.i.d.}{\sim} X \) on a metric space \((M, d)\) consider random variables with unique population mean \( \mu \) and unique sample mean \( \hat{\mu}_n \) (in case of well behaved nonuniqueness, take a uniformly random selection as in Agreement 2.1) and set

\[
V := F(\mu), \quad \hat{V}_n := F_n(\hat{\mu}_n), \quad W := \mathbb{E}[d(\mu, X)^4], \quad \hat{W}_n := \frac{1}{n} \sum_{j=1}^{n} d(\hat{\mu}_n, X_j)^4, \quad (8)
\]
The Fréchet variance by Dubey and Müller (2019) teaches that

\[
\sqrt{n}(\hat{V}_n - V) \xrightarrow{D} N(0, W - V^2),
\]

where \( \hat{W}_n - \hat{V}_n^2 \) is an asymptotically unbiased estimator for \( W - V^2 \), that also satisfies \( \sqrt{n} \) asymptotic normality.

Further, let \( B > 0 \) be a large integer and \( \mu^*_b \) be a mean of an \( n \)-out-of-\( n \) bootstrap sample of \( X_1, \ldots, X_n \) for \( b = 1, \ldots, B \). Further, let \( \bar{\mu}^* \) be the mean of the \( \mu^*_1, \ldots, \mu^*_B \) and set

\[
\hat{V}^* := \frac{1}{B} \sum_{b=1}^{B} d(\bar{\mu}^*, \mu^*_b)^2, \quad \hat{W}^* := \frac{1}{B} \sum_{b=1}^{B} d(\bar{\mu}^*, \mu^*_b)^4.
\]

Then, under absence of smeariness and absence of FSS, \( \sqrt{B}(\hat{V}^* - V_n/n) \) is approximately normal with zero expectation and its variance can be estimated by \( \hat{W}^* - (\hat{V}^*)^2 \). Hence, in this case, with the standard-normal \( \alpha \)-quantile \( \phi_\alpha \) we expect that

\[
P \left\{ \frac{\hat{V}^*}{\hat{W}^*} > \frac{\hat{V}^*}{\hat{W}^*} + \phi_{1-\alpha} \sqrt{\frac{\hat{W}^* - (\hat{V}^*)^2}{B}} \right\} \approx \alpha.
\]

**Test 2.13** (For the Presence of FSS). For random variables \( X_1, \ldots, X_n \) on a metric space \( (M, d) \) with sample mean \( \hat{\mu}_n \), bootstrap means \( \mu^*_1, \ldots, \mu^*_B \), and their mean \( \bar{\mu}^* \), as above and the notation from [8]...
\( \hat{m}_n := \frac{n\hat{V}}{V_n} \) denotes the empirical (bootstrap-based) variance modulation for \( X_1, \ldots, X_n \). With the standard normal \( \alpha \)-quantile \( \phi_\alpha \) for \( \alpha \in (0,1) \) reject the absence of FSS for sample size \( n \) at nominal level \( \alpha \) if

\[
\hat{m}_n^* - 1 > h_{n,\alpha} \text{ where } h_{n,\alpha} = \frac{n\phi_\alpha}{\sqrt{B}} \left( \hat{W}^* - (\hat{V}^*)^2 \right).
\]

Figure 5 shows simulations for distributions \( \mathbb{P}^{(3,\beta,\lambda,r)}_{vMm} \) featuring Type I FSS for \( r = 0 \) as defined in (6) and Type II FSS for \( r > 0 \) as defined in (7). Indeed for \( r = \pi/2 \), the null hypothesis of absence of FSS is true and Test 2.13 keeps the level \( \alpha = 0.05 \). For Type II FSS, rejection probabilities are almost one within the regime of FSS (compare with Figure 1). Beyond that regime the null hypothesis is true again and detected with the correct size. For the von Mises mixture distribution (vMmd) consisting of only one von Mises distribution (\( \beta = 1 \), left panel, black) with almost no FSS visible in Figure 1, rejection probabilities increase visibly with sample size. As expected, the power of Test 2.13 increases with proximity to a smeary distribution (\( \beta = 0.5, \lambda = 0.8683, r = 0 \), right panel, black), and specifically in case of Type II FSS also with smaller hole size and proximity to a distribution featuring Type I FSS.

3 Asymptotics of Fréchet Means on the Circle and the Torus

In this section, we give with an exposition on the asymptotic behavior of Fréchet sample means on the circle. We then proceed with a proof on consistency of the bootstrap and afterwards show under the presence of FSS that moments of the Fréchet sample means and their bootstrapped versions converge to the respective moment of the limit distribution.

3.1 Central Limit Theorem and Bootstrap Consistency for Fréchet Means

We begin with the central limit theorem for Fréchet sample means on the circle under the following assumption.

**Assumption 3.1.** Let \( X \) be a random element on \( S^1 \)

(i) with unique Fréchet population mean \( \mu = 0 \) and

(ii) that features a continuous density \( f \) on \( (\pi - \delta, \pi) \cup [-\pi, -\pi + \delta) \) with respect to the arc length measure for some \( \delta > 0 \) such that

\[
f(-\pi) = \lim_{x \searrow -\pi} f(x) = \lim_{x \nearrow \pi} f(x) < \frac{1}{2\pi}.
\]

**Theorem 3.2** (Central Limit Theorem on the Circle by \cite{McK12, Hot15}). Under Assumption 3.1 let \( n \in \mathbb{N} \) and consider i.i.d. random elements \( X_1, \ldots, X_n \sim X \) on \( S^1 \) with a measurable selection \( \hat{\mu}_n \) of Fréchet sample means. Then, for \( n \to \infty \), we have that

\[
\sqrt{n} \hat{\mu}_n \overset{D}{\to} \mathcal{N} \left( 0, \frac{\mathbb{E}[X^2]}{(1 - 2\pi f(-\pi))^2} \right),
\]

where \( \mathbb{E}[X^2] \) denotes Euclidean variance.

**Remark 3.3.** The case \( f(-\pi) = \frac{1}{2\pi} \) where the density \( f \) is \((k + 1)\)-times continuously differentiable near \(-\pi\) for \( k \in \mathbb{N} \), such that the first \( k \) derivatives vanish at \(-\pi\) whereas the derivative of order \( k + 1 \) at \(-\pi\) does not, has been investigated by \cite{Hot15}. In this setting, the convergence
rate for Fréchet sample means turns out to be of order $n^{-1/2}$, which is strictly slower than the standard $n^{-1}$-rate. The phenomenon of a slower convergence rate is known as smeariness. Moreover, it is not possible that the antipode of the Fréchet population mean exhibits a point mass or fulfills $f(-\pi) > \frac{1}{2\pi}$, cf. [Holz and Huckemann 2015, Theorem 1].

Notably, the limit distribution of the Fréchet sample mean depends on the behavior of the density near the antipode of the Fréchet population mean. In particular, in case the density near the antipode of the population mean does not vanish, approximating the distribution of $\sqrt{n}\hat{\mu}_n$ by a centered Gaussian with an estimated variance $\frac{1}{n} \sum_{i=1}^{n} d(\hat{\mu}_n, X_i)$ is not suited.

An alternative approach to imitate the law of scaled Fréchet means is by means of bootstrap methods. In the following, we show that the naïve $n$-out-of-$n$ bootstrap is indeed (asymptotically) consistent in approximating the distribution of Fréchet sample means. For this purpose, we denote by $BL_1(\mathbb{R})$ the space of bounded Lipschitz functions on $\mathbb{R}$ which are bounded by one and have Lipschitz modulus at most one. Further, we denote by $\xrightarrow{\mathbb{P}}$ the convergence in outer probability, cf. [van der Vaart and Wellner 1996].

**Theorem 3.4 (Consistency of Bootstrap).** **Under Assumption 3.2** let $n \in \mathbb{N}$ and consider i.i.d. random elements $X_1, \ldots, X_n \sim X$ on $\mathbb{S}^1$ with a measurable selection $\hat{\mu}_n$ of Fréchet sample means. Further, given $X_1, \ldots, X_n$ consider a bootstrap sample $X_1^*, \ldots, X_n^*$ with measurable selection $\hat{\mu}_n^*$ of its Fréchet sample mean. Then, for $n \to \infty$ we have

$$\sup_{h \in BL_1(\mathbb{R})} \left| \mathbb{E} \left[ h \left( \sqrt{n} \left( \hat{\mu}_n^* - \hat{\mu}_n \right) \right) \right] - \mathbb{E} \left[ h \left( \sqrt{n} \hat{\mu}_n \right) \right] \right| \xrightarrow{\mathbb{P}} 0.$$

**Proof.** For $x \geq 0$ the Fréchet function of the bootstrap realization is given by

$$F_n^*(x) = \frac{1}{n} \sum_{X_j^* \in [x-\pi, \pi]} (X_j^* - x)^2 + \frac{1}{n} \sum_{X_j^* < x - \pi} (X_j^* + 2\pi - x)^2 = \frac{1}{n} \sum_{j=1}^{n} (X_j^* - x)^2 + \frac{4\pi}{n} \sum_{X_j^* < x - \pi} (X_j^* - x + \pi).$$

With the Euclidean mean $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i^*$ of the bootstrap realizations we have for any $x \geq 0$ with $x - \pi \notin \{X_1^*, \ldots, X_n^*\}$ that

$$\frac{1}{2} \operatorname{grad} F_n^*(x) = x - \overline{X}_n^* - \frac{2\pi}{n} \sum_{X_j^* \leq x - \pi} 1. \quad (11)$$

We rewrite the sum in (11) as follows

$$- \frac{1}{n} \sum_{X_j^* \leq x - \pi} 1 = -f(-\pi)x + \frac{f(-\pi)x - \mathbb{P}(X \leq x - \pi)}{= A(x)} + \frac{1}{n} \left( \sum_{X_j^* \leq x - \pi} 1 \right) \quad \text{and} \quad \frac{1}{n} \left( \sum_{X_j^* \leq x - \pi} 1 \right) - \frac{1}{n} \left( \sum_{X_j^* \leq x - \pi} 1 \right) \quad \text{and} \quad \frac{1}{n} \left( \sum_{X_j^* \leq x - \pi} 1 \right).$$

Note that the function $A(x)$ is independent from samples $X_1, \ldots, X_n$ and bootstrap samples $X_1^*, \ldots, X_n^*$. Similarly we obtain for $x < 0$ with $x + \pi \notin \{X_1^*, \ldots, X_n^*\}$ that

$$\frac{1}{2} \operatorname{grad} F_n^*(x) = x - \overline{X}_n^* + \frac{2\pi}{n} \sum_{X_j^* > x + \pi} 1. \quad (12)$$
and we similarly rewrite
\[
\frac{1}{n} \sum_{X_j > x + \pi} 1 = -f(-\pi)x + \left[ f(-\pi)x + P(X > x + \pi) \right]
\]
\[
= A(x) + \left[ -P(X > x + \pi) + \frac{1}{n} \left( \sum_{X_j > x + \pi} 1 \right) \right] + \left[ \frac{1}{n} \left( \sum_{X_j > x + \pi} 1 \right) \right] \]
\[
= B_n(x) + \left[ \frac{1}{n} \left( \sum_{X_j > x + \pi} 1 \right) \right] = C_n(x).
\]

Note, we consider \( x - \pi \neq X_j^* \), thus the condition “\( X_j^* < x + \pi \)” (resp. “\( X_j^* > x - \pi \)” is equivalent to “\( X_j^* \leq x + \pi \)” (resp. “\( X_j^* \geq x - \pi \)” and likewise for analogous conditions with \( X_j \). The reason for our definition of \( B_n \) and \( C_n \) is to ensure that these functions are càdlàg on \([-\pi, \pi]\), i.e. right-continuous and limits from left exist. Plugging in \( x = \hat{\mu}_n^* \) into (11) and (12), which is well-defined since \( \hat{\mu}_n^* \neq X_j^* \) for each \( j = 1, \ldots, n \) \cite[Theorem 1(i)]{Hotz and Huckemann 2015}, asserts
\[
\overline{X}_n = (1 - 2\pi f(-\pi))\hat{\mu}_n + A(\hat{\mu}_n) + B_n(\hat{\mu}_n) + C_n(\hat{\mu}_n).
\]

Likewise, it follows for the Fréchet sample mean \( \hat{\mu}_n \) of \( X_1, \ldots, X_n \) that
\[
\overline{X}_n = (1 - 2\pi f(-\pi))\hat{\mu}_n + A(\hat{\mu}_n) + B_n(\hat{\mu}_n),
\]
which yields
\[
\sqrt{n} \left( \overline{X}_n - \overline{X}_n \right) = \sqrt{n} \left( 1 - 2\pi f(-\pi) \right) (\hat{\mu}_n - \hat{\mu}_n) + \sqrt{n} A(\hat{\mu}_n) - \sqrt{n} A(\hat{\mu}_n)
\]
\[
+ \sqrt{n} B_n(\hat{\mu}_n) - \sqrt{n} B_n(\hat{\mu}_n) + \sqrt{n} C_n(\hat{\mu}_n).
\]

By \cite[Theorem 23.4]{van der Vaart 2000} the bootstrap is consistent for the Euclidean sample mean, i.e.,
for \( n \to \infty \) we have
\[
\sup_{h \in BL_1(\mathbb{R})} \left| \mathbb{E} \left[ h \left( \sqrt{n} \left( \overline{X}_n - \overline{X}_n \right) \right) \right] \right| = \overline{X}_n \to 0.
\]

Upon defining for \( t \in \mathbb{R} \) the function \( H(t) = \min(2, |t|) \) we note for each \( h \in BL_1(\mathbb{R}) \) and any \( t, t' \in \mathbb{R} \) that the inequality \( |h(t + t') - h(t)| \leq H(t') \) holds since \( h \) is bounded by one and Lipschitz with modulus one. Further, it holds that \( H(t + t') \leq H(t) + H(t') \). Hence, we assert by (13) and (14) that
\[
\left| h \left( \sqrt{n} \overline{X}_n - \frac{1}{n} \sum_{X_j > x + \pi} 1 \right) \right| \leq H \left( \sqrt{n} A(\hat{\mu}_n) \right) + H \left( \sqrt{n} B_n(\hat{\mu}_n) \right),
\]
\[
\left| h \left( \sqrt{n} \left( \overline{X}_n - \overline{X}_n \right) \right) \right| \leq H \left( \sqrt{n} A(\hat{\mu}_n) \right) + H \left( \sqrt{n} B_n(\hat{\mu}_n) \right),
\]
\[
\leq H \left( \sqrt{n} A(\hat{\mu}_n) \right) + H \left( \sqrt{n} B_n(\hat{\mu}_n) \right) + H \left( \sqrt{n} C_n(\hat{\mu}_n) \right).
\]
Hence, we obtain by triangle inequality that

\[
\sup_{h \in \text{BL}_1(\mathbb{R})} \left| \mathbb{E} \left[ h \left( \sqrt{n}(1 - 2\pi f(-\pi)) (\hat{\mu}_n - \hat{\mu}_n) \right| X_1, \ldots, X_n \right] - \mathbb{E} \left[ h \left( \sqrt{n}(1 - 2\pi f(-\pi)) \hat{\mu}_n \right) \right] \right|
\]

\[
= \sup_{h \in \text{BL}_1(\mathbb{R})} \left| \mathbb{E} \left[ h \left( \sqrt{n}(1 - 2\pi f(-\pi)) (\hat{\mu}_n - \hat{\mu}_n) \right| X_1, \ldots, X_n \right] - \mathbb{E} \left[ h \left( \sqrt{n}(1 - 2\pi f(-\pi)) \hat{\mu}_n \right) \right] \right|
\]

\[
+ \sup_{h \in \text{BL}_1(\mathbb{R})} \left| \mathbb{E} \left[ h \left( \sqrt{n} (X_n^* - X_n) \right| X_1, \ldots, X_n \right] - \mathbb{E} \left[ h \left( \sqrt{n} \overline{X_n} \right) \right] \right|
\]

\[
\leq \mathbb{E} \left[ H \left( \sqrt{n} \mathcal{A}(\hat{\mu}_n) \right) + H \left( \sqrt{n} \mathcal{A}(\hat{\mu}_n) \right) + H \left( \sqrt{n} B_n(\hat{\mu}_n) \right) + H \left( \sqrt{n} C_n(\hat{\mu}_n) \right) \right| X_1, \ldots, X_n \]

\[
+ \sup_{h \in \text{BL}_1(\mathbb{R})} \left| \mathbb{E} \left[ h \left( \sqrt{n} (X_n^* - X_n) \right| X_1, \ldots, X_n \right] - \mathbb{E} \left[ h \left( \sqrt{n} \overline{X_n} \right) \right] \right|
\]

\[
+ \mathbb{E} \left[ H \left( \sqrt{n} \mathcal{A}(\hat{\mu}_n) \right) + H \left( \sqrt{n} B_n(\hat{\mu}_n) \right) \right] \xrightarrow{p^*} 0.
\]

The proof of the convergence in outer probability (17) is deferred to Proposition B.1 in Supplement B and employs empirical process theory as well as the notion of Donsker classes.

**Remark 3.5.** These results extend at once to the m-torus \( T^m = (\mathbb{S}^1)^m \) equipped with the canonical product metric as the Fréchet mean on the m-torus is given by the vector of circular Fréchet means for each component of the torus.

### 3.2 Moment Convergence of Fréchet Means

In order to employ the theory on the asymptotics of Fréchet sample means and bootstrap version for the formulation of Hotelling tests it is necessary to estimate the variance of the limit distribution. The following result guarantees that the naïve plug-in estimator for the covariance is indeed consistent. In fact, we prove that all moments of scaled Fréchet sample means and bootstrap variants converge to the corresponding moment of their limit distribution.

**Proposition 3.6.** Under Assumption 3.1 let \( n \in \mathbb{N} \) and consider i.i.d. random elements \( X_1, \ldots, X_n \sim X \) on \( \mathbb{S}^1 \) with a measurable selection \( \hat{\mu}_n \) of Fréchet sample means. Then for all \( p \geq 1 \) we have

\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left| \sqrt{n} \mathcal{d}(\hat{\mu}_n, \mu) \right| \right] = \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left| \sqrt{n} \hat{\mu}_n \right| \right] < \infty.
\]

Further, for \( Z \sim \mathcal{N}(0, \mathbb{E}[X^2]/(1 - 2\pi f(-\pi))^2) \) we have for any \( p \geq 1 \) as \( n \) tends to infinity that

\[
\mathbb{E} \left[ \left| \sqrt{n} \mathcal{d}(\hat{\mu}_n, \mu) \right|^p \right] = \mathbb{E} \left[ \left| \sqrt{n} \hat{\mu}_n \right|^p \right] \xrightarrow{p} \mathbb{E}[Z^p].
\]

The proof is stated in Supplement C and uses a moment convergence result for M-estimators relying on the theory by [Nishiyama (2010)](https://link.springer.com/article/10.1007/s00445-010-0319-0). Notably, explicit bounds for \( \mathbb{E} \left[ \left| \sqrt{n} \mathcal{d}(\hat{\mu}_n, \mu) \right|^p \right] \) and in case of more general M-estimators were derived by [Schötz (2019)](https://link.springer.com/article/10.1007/s00445-019-00766-2). Moreover, similar to the consistency result on moments of the empirical Fréchet sample mean, the moments of bootstrap-based Fréchet sample means are also consistent.
Proposition 3.7. Under Assumption 3.1 let \( n \in \mathbb{N} \) and consider i.i.d. random elements \( X_1, \ldots, X_n \sim X \) on \( S^1 \) with a measurable selection \( \hat{\mu}_n \) of Fréchet sample means. Further, consider a bootstrap sample \( X^*_1, \ldots, X^*_n \overset{i.i.d.}{\sim} \frac{1}{n} \sum_{i=1}^n \delta_{X_i} \), with measurable selection \( \hat{\mu}^*_n \) of its Fréchet sample mean. Then,

\[
\sup_{n \in \mathbb{N}} \mathbb{E} \left[ (\sqrt{n}d(\hat{\mu}^*_n, \hat{\mu}_n))^p \right] \leq \sup_{n \in \mathbb{N}} \mathbb{E} \left[ (\sqrt{n}(\hat{\mu}^*_n - \hat{\mu}_n))^p \right] < \infty
\]

for all \( p \geq 1 \) where the expectation is taken with respect to samples \( X_1, \ldots, X_n \overset{i.i.d.}{\sim} X \) and bootstrap based samples \( X^*_1, \ldots, X^*_n \overset{i.i.d.}{\sim} n^{-1} \sum_{i=1}^n \delta_{X_i} \). Further, for \( Z \sim N(0, \mathbb{E}[X^2]/(1 - 2\pi f(-\pi))^2) \) we have for any \( p \geq 1 \) as \( n \) tends to infinity that

\[
\mathbb{E} \left[ (\sqrt{n}(\hat{\mu}^*_n - \hat{\mu}_n))^p \mid X_1, \ldots, X_n \right] \xrightarrow{p} \mathbb{E}[Z^p].
\]

The proof is deferred to Supplement C and relies on a general result for conditional moment convergence of bootstrap \( M \)-estimators by [Kato (2011)].

4 One- and Two-Sample Tests for Fréchet Means under Finite Sample Smeariness

We begin with a brief review of the celebrated central limit theorem (CLT) by [Bhattacharya and Patrangenaru (2005)] for a \( m \)-dimensional manifold \( M \) and corresponding tests proposed therein. The CLT states that, under suitable conditions (further clarified in [Bhattacharya and Lin (2017); Eltzner and Huckemann (2019); Eltzner et al. (2021)], in a local chart \( \tilde{\phi} : U \to \mathbb{R}^m, U \subset M \), the fluctuation \( \phi(\hat{\mu}_n) - \phi(\mu) \) of the Fréchet sample mean \( \hat{\mu}_n \) about the Fréchet population mean \( \mu \), rescaled with the square root of sample size \( \sqrt{n} \), is asymptotically (for \( n \to \infty \)) Gaussian with zero mean and covariance given by

\[
\Sigma = H^{-1}C H^{-1},
\]

cf. [3]. Here, \( C \) is the population covariance of the gradient of the Fréchet function \( F \) from (1) at \( \mu \) in the local chart and \( H \) is twice the expected value of the Hessian of the Fréchet function at \( \mu \) in the local chart. One of the above mentioned conditions is that \( H \) be positive definite. In our language, this means that \( \tilde{\mu}_n \) is non-smear (i.e., on the circle we have the situation of Theorem 3.2 where \( C = \mathbb{E}[X^2] \) and \( H = 1 - 2\pi f(-\pi) \), as can be derived at once from by means of Equation (14)).

For mutually independent samples \( X_1, \ldots, X_n \overset{i.i.d.}{\sim} X \) and \( Y_1, \ldots, Y_n \overset{i.i.d.}{\sim} Y \) with population Fréchet means \( \mu(X) \) and \( \mu(Y) \), respectively, and \( \mu_0 \in M \), consider the hypotheses

\[
H^1_0 : \quad \mu(X) = \mu_0 \quad \text{for the one-sample test},
\]
\[
H^2_0 : \quad \mu(X) = \mu(Y) \quad \text{for the two-sample test}.
\]

Quantile based tests. With a local chart \( \tilde{\phi} : U \to \mathbb{R}^m \) where \( U \subset M \) contains \( \mu \) (for \( H^1_0 \)), or \( \mu(X) \) (for \( H^2_0 \)), respectively, [Bhattacharya and Patrangenaru (2005)] suggest to consider the following test statistics,

\[
T^1 = n \left( \phi(\hat{\mu}^*_n(X)) - \phi(\mu(X)) \right) ^T \tilde{H}_X \tilde{C}_X^{-1} \tilde{H}_X \left( \phi(\hat{\mu}^*_n(X)) - \phi(\mu(X)) \right),
\]
\[
T^2 = (n + m) \left( \phi(\hat{\mu}^*_n(X)) - \phi(\mu(X)) \right) ^T \tilde{H}_{X,Y} \tilde{C}_{X,Y}^{-1} \tilde{H}_{X,Y} \left( \phi(\hat{\mu}^*_n(X)) - \phi(\mu(X)) \right)
\]

for the one-sample test, and use \( \chi^2 \) as their asymptotic approximation under the respective null hypothesis. Here \( \tilde{H}_X, \tilde{C}_X, \tilde{C}_{X,Y} \) are the usual plug-in estimators of \( H \) and \( C \) based on the first sample, or the pooled sample, respectively, cf. also [Bhattacharya and Bhattacharya (2012) Section 5.4.1].

Test 4.1 [Bhattacharya and Patrangenaru (2005); Bhattacharya and Bhattacharya (2012)]. For \( 0 \leq \alpha \leq 1 \) and the \( 1 - \alpha \) quantile \( \chi^2_{m,1-\alpha} \) of the \( \chi^2_m \) distribution, reject at nominal level \( \alpha \)
(i) $H_0^1$, if $T^1 \geq \chi^2_{k,1-\alpha}$.

(ii) $H_0^2$, if $T^2 \geq \chi^2_{k,1-\alpha}$.

Asymptotically, while both tests are independent of the chart chosen, in general, they do not keep the level promised.

**Proposition 4.2.** Let $X, Y$ be random variables on $\mathbb{S}^1$ with unique Fréchet population means $\mu(X), \mu(Y)$, respectively, and where $X$ and $Y$ satisfy Assumption 3.2(ii) for respective densities $f(X), f(Y)$. Under Type I FSS $X$ or $Y$ the Tests 4.1 have asymptotically a size strictly higher than the nominal size.

**Proof.** With the notation of Theorem 3.2, if $M = \mathbb{S}^1$ is the circle, $\mu_0 = 0$, or $\mu(X) = 0 = \mu(Y)$, respectively, and $\delta$ the identity on $(-\pi, \pi) \subset \mathbb{S}^1$, plug-in estimators for $H^{-1}CH^{-1}$ are $\frac{1}{n} \sum_{j=1}^{n} X_j^2$, or $\frac{1}{n+m} \sum_{j=1}^{n} Y_j^2$, whereas $H^{-1}CH^{-1} = \mathbb{E}[X^2(1 - 2\pi f(-\pi))^{-2} + (1 - \delta)\mathbb{E}[Y^2(1 - 2\pi f(Y)(-\pi))^{-2}]$, with $n/(n+m) \to \delta$, respectively, by Theorem 3.2. Under Type I FSS Corollary 2.4 states that $f(X)(-\pi) > 0$ or $f(Y)(-\pi) > 0$. Hence, the plug-in estimators are asymptotically strictly smaller than the variance of the limit law and thus the true asymptotic size of Tests 4.1 is strictly higher than the nominal level.

**Remark 4.3.** On the circle, in case of Type II FSS, Tests 4.7 also have a true size higher than their nominal size in the range where the variance modulation $m_n$ larger than 1, in particular of the difference is substantial, as the following simulations in Section 7 show.

**Bootstrap based tests.** Tests based on a bootstrap principle have also been proposed by Bhattacharya and Patrangenaru (2005); Bhattacharya and Bhattacharya (2012) where $\Sigma = H^{-1}CH^{-1}$ is estimated by bootstrapping from the samples. As they have not provided details, in the following, we employ the bootstrap one- and two-sample tests by Eltzner and Huckemann (2017), for a given level $0 \leq \alpha \leq 1$.

Resample $B \in \mathbb{N}$ times $n$-out-of-$n$ from the sample $X_1, \ldots, X_n$ and let $\mu_b(X)^*$ be the corresponding Fréchet sample means for $b = 1, \ldots, B$. Mapping these sample means under $\phi$ to a Euclidean space yields the covariance estimate $\Sigma_b(\phi(X))$. From another round of resampling obtain new $\mu_b(\phi(X))^*$, set $T_b^1 = (\phi(\mu_b(\phi(X))^*) - \phi(\mu_b(\phi(X)))^T (\Sigma_b(\phi(X))^*)^{-1} (\phi(\mu_b(\phi(X))^*) - \phi(\mu_b(\phi(X))))$, $b \in \{1, \ldots, B\}$ and determine $e_{1-\alpha}$ such that

$$\frac{\sum_{b \in \{1, \ldots, B\}} T_b^1 \leq e_{1-\alpha}^1}{B} - 1 \leq 1 - \alpha \leq \frac{\sum_{b \in \{1, \ldots, B\}} T_b^1 \leq e_{1-\alpha}^1}{B}.$$ 

Further, set $T^1 = (\phi(\hat{\mu}_n(X)) - \phi(\mu_0))^T (\Sigma_b(\phi(X))^{-1}) (\phi(\hat{\mu}_n(X)) - \phi(\mu_0)).$

Similarly, using $m$-out-of-$m$ sampling with replacement from $Y_1, \ldots, Y_m$, obtain a bootstrap covariance estimate $\Sigma_b(\phi(Y))^*$. Now, set $\Sigma_B = \Sigma_b(\phi(X))^* + \Sigma_b(\phi(Y))^*$. This choice of $\Sigma_B$, and not using the pooled variance, proves to be more robust, cf. Huckemann and Eltzner (2020). Then, from another round of resampling obtain $\mu_b(\phi(X))^*$ and $\mu_b(\phi(Y))^*$, set $d_b^*(X) = \phi(\hat{\mu}_n(X))^* - \phi(\mu_b(\phi(X))^*$, $d_b^*(Y) = \phi(\hat{\mu}_n(Y))^* - \phi(\mu_b(\phi(Y))^*$, define $T_b^2 = (d_b^*(X)^* - d_b^*(Y)^*)^T A_B^{-1} (d_b^*(X)^* - d_b^*(Y)^*)$, $b \in \{1, \ldots, B\}$ and determine $f_{1-\alpha}$ such that

$$\frac{\sum_{b \in \{1, \ldots, B\}} T_b^2 \leq f_{1-\alpha}^2}{B} - 1 \leq 1 - \alpha \leq \frac{\sum_{b \in \{1, \ldots, B\}} T_b^2 \leq f_{1-\alpha}^2}{B}.$$ 

Further, set $T^2 = (\phi(\hat{\mu}_n(X)) - \phi(\hat{\mu}_m(Y)))^T A_B^{-1} (\phi(\hat{\mu}_n(X)) - \phi(\hat{\mu}_m(Y))).$

**Tests 4.4 (Bootstrap Based).** With the notation above, for $0 \leq \alpha \leq 1$, reject at level $\alpha$

(i) $H_0^1$, if $T^1 \geq e_{1-\alpha}^1$.

(ii) $H_0^2$, if $T^2 \geq f_{1-\alpha}^2$. 

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Again, asymptotically, both tests are independent of the chart chosen.

**Proposition 4.5.** Consider mutually independent samples $X_1, \ldots, X_n \overset{i.i.d.}{\sim} X$ and $Y_1, \ldots, Y_n \overset{i.i.d.}{\sim} Y$ on $S^1$, where $X$ and $Y$ have unique Fréchet population means $\mu(X) = 0$ and $\mu(Y)$. Further, assume that $X$ and $Y$ are not concentrated at $\mu(X)$ and $\mu(Y)$, respectively, and fulfill Assumption 3.1(ii) for respective densities $f(X), f(Y)$. Then it follows for $0 < \alpha < 1$ as $n, B \to \infty$

(i) under $\mu(X) = \mu_0$ that $P\left(T_{1,\ast} \geq e_{1-\alpha}^\ast \bigg| X_1, \ldots, X_n \right) \overset{p^*}{\longrightarrow} \alpha,$

(ii) and under $\mu(X) \neq \mu_0$ that $P\left(T_{1,\ast} \geq e_{1-\alpha}^\ast \bigg| X_1, \ldots, X_n \right) \overset{p^*}{\longrightarrow} 1.$

(iii) Moreover, under $\mu(X) = \mu(Y)$ that $P\left(T_{2,\ast} \geq f_{1-\alpha}^\ast \bigg| X_1, \ldots, X_n \right) \overset{p^*}{\longrightarrow} \alpha,$

(iv) whereas under $\mu(X) \neq \mu(Y)$ it holds $P\left(T_{2,\ast} \geq f_{1-\alpha}^\ast \bigg| X_1, \ldots, X_n \right) \overset{p^*}{\longrightarrow} 1.$

This shows that the bootstrap based test is indeed consistent under FSS of Type I and II for circular distributions which have a density near the antipode of the Fréchet population mean.

**Proof.** We only show Assertions (i) and (ii), the proof for the remaining assertions is analogous. Consider the identity on $(-\pi, \pi]$ and define for $t > 0$ the Lipschitz function

$$\Psi(t) : \mathbb{R} \to [-t, t], \Psi(t)(x) := \text{sign}(x) \min(|x|, t).$$

Suppose $\mu(X) = \mu_0 = 0$, then it follows for $n \to \infty$ that

$$\frac{1}{\mathbb{E}[n(\hat{\mu}_n^2)^2]} \to \frac{(1 - 2\pi f(\mu(X)))^2}{\mathbb{E}[X^2]},$$

cf. Proposition 3.6 This asserts that the Lipschitz function $g_n^{(i)}(x) := \Psi(t)(x^2(\mathbb{E}[n(\hat{\mu}_n^2)^2])^{-1})$ converges uniformly for $n \to \infty$ to $g(t)(x) := \Psi(t)(x^2(\mathbb{E}[\mu(X)])^{-2})$. Likewise, it holds for $n \to \infty$ that

$$\frac{1}{\mathbb{E}[n(\Sigma_B^{\ast})^2][X_1, \ldots, X_n]} \overset{p^*}{\longrightarrow} \frac{(1 - 2\pi f(\mu(X)))^2}{\mathbb{E}[X^2]},$$

cf. second assertion from Proposition 3.7 Moreover, as a consequence of the first part of Proposition 3.7 it follows using Markov’s inequality that $\text{Var}[\sqrt{n} (\hat{\mu}_n - \mu_n)]|X_1, \ldots, X_n]$ is stochastically bounded (see Relation [34] in Supplement C), therefore asserting that

$$\text{Var}[n(\Sigma_B^{\ast})^2][X_1, \ldots, X_n] = O_P(1/B).$$

Hence, for $t > 0$ the Lipschitz function $g_n^{(i)}(x) := \Psi(t)(x^2(\Sigma_B^{\ast})^{-1})$ converges uniformly for $n, B \to \infty$ conditional on $X_1, \ldots, X_n$ in outer probability to $g(t)$ This yields by Theorem 3.4 for $n, B \to \infty$ that

$$\sup_{h \in \text{B}L_1(\mathbb{R})} \left| \mathbb{E} \left[ h \circ g_n^{(i)}(\sqrt{n} (\hat{\mu}_n - \mu_n)) \bigg| X_1, \ldots, X_n \right] - \mathbb{E} \left[ h \circ g(t)(\sqrt{n} \mu_n) \right] \right| \overset{p^*}{\longrightarrow} 0.$$
In particular, for Assertion (ii) we note that $\Sigma_B^{(X)} \ast$ converges in outer probability conditioned on $X_1, \ldots, X_n$ to zero, whereas by assumption

$$\hat{\mu}_n^{(X)} - \mu_0 \xrightarrow{a.s.} \mu^{(X)} - \mu_0 \neq 0,$$

cf. Ziezold (1977). Hence, conditioned on $X_1, \ldots, X_n$ the test statistic $T^{(X)}$ diverges in outer probability to infinity and the assertion follows.

Remark 4.6. Simulations in Section 4 depicted in Figures 6 and 7 show that the Tests 4.4 keep the nominal level $\alpha = 0.05$ fairly well, in particular for Type I FSS. Upon (very) close inspection, for Type II FSS, the Tests 4.4 may be slightly too conservative, for Type I FSS too liberal. Investigating this effect and correcting for it is left for future research and beyond the scope of this paper.

Remark 4.7. Under uniqueness of means, the bootstrap based test is also asymptotically consistent on tori if the covariance of $X$ (one-sample) or the sum of covariances of $X$ and $Y$ (two-samples), respectively, is non-singular and each marginal distribution has a density near the antipode fulfilling Assumption 3.7(ii). This follows from the bootstrap consistency of the Fréchet sample mean on tori (Theorem 3.4 and Remark 3.5) and conditional convergence in outer probability of the covariance estimators $\Sigma_B^{(X)}$, and $A_B$ for $\phi(\hat{\mu}_n^{(X)})$ and $\phi(\hat{\mu}_n^{(Y)}) - \phi(\hat{\mu}_m^{(Y)})$, respectively, as $n, B \to \infty$ to the corresponding population covariance quantities.

5 Simulations

To assess the performance of the quantile-based and bootstrap-based tests under the presence of FSS we consider i.i.d. samples of some nonaltered (Type I FSS) von Mises mixtures, introduced in (6) and denoted by $P_{\kappa, \beta, \lambda}^{(0)}$, and altered (Type II FSS) von Mises mixtures, introduced in (7) and denoted by $P_{\kappa, \beta, \lambda, r}^{(0)}$. By Theorem 2.3 all of the $P_{\kappa, \beta, \lambda, r}^{(0)}$ with unique mean $\mu = 0$ and $r < \pi/2 - \varepsilon$ for some $\varepsilon > 0$ are FSS if they are not smeary themselves. For the following simulations we considered parameters as described in Table 3. All of them give unique means at $\mu = 0$, none of which is smeary. Only the nearby $P_{\kappa, \beta, \lambda, 0}^{(0)}$ is smeary for $\lambda \approx 0.8683$, (following the notation of Eltzner and Huckemann (2019), the order of smeariness is equal to 2, according to Theorem 3(ii) in Hotz and Huckemann (2015)).

In Figures 6 and 7 we compare the power functions at nominal level $\alpha = 0.05$ of the quantile tests (gray lines, Test 4.1) to the power functions of the bootstrap tests (black lines, Test 4.4). For the one-sample tests (Figure 6) we consider simulations of $P_{\kappa, \beta, \lambda, r}^{(0)}$ and rotated null hypotheses $\mu_0 \in [-\pi, \pi)$, for the two sample-tests (Figure 7) we consider two simulations of $P_{\kappa, \beta, \lambda, r}^{(0)}$ that are rotated with respect to one another by the angle $\phi \in [-\pi, \pi)$. Random variables from von Mises distributions were generated using the R-package circular (Lund et al., 2017).

| $r$ | $0$ | $0.1$ | $0.1$ | $0$ | $0.1$ |
|-----|-----|-----|-----|-----|-----|
| $\kappa$ | $3$ | $3$ | $3$ |
| $\beta$ | $1$ | $1/2$ | $1/2$ |
| $\lambda$ | $0$ | $0$ | $1/2$ |
| $\mu_0$ | $1.0$ | $3.7$ | $3.0$ | $7.2$ | $5.4$ |
| $m_{30}$ | $1.0$ | $3.7$ | $3.0$ | $7.2$ | $5.4$ |
| $m_{100}$ | $1.0$ | $4.2$ | $2.7$ | $11.6$ | $6.6$ |
| $m_{300}$ | $1.0$ | $4.4$ | $2.0$ | $15.9$ | $5.6$ |

Table 3: Parameters of $P_{\kappa, \beta, \lambda, r}^{(0)}$ with variance modulation $m_{n}$ of distributions considered for simulations on performance of Tests 4.1 and 4.4 and sample sizes $n \in \{30, 100, 300\}$. 

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Figure 6: Rejection probabilities for $H_0^I$: $\mu = \mu_0$ for varying $\mu_0 \in [-\pi, \pi)$ under quantile based tests (Test 4.1, gray) and bootstrap based tests for $B = 1000$ (Test 4.4, black) at nominal level 5% (dotted horizontal) based on 1000 simulation runs with one sample of size $n = 30$ (top row), $n = 100$ (middle row), and $n = 300$ (bottom row). The solid lines represent samples which were generated from mixed von Mises distributions $p_{vMm}(3,0.5,0,\gamma)$, i.e., $r = 0$. The dashed lines correspond to samples from $p_{vMm}(3,0.5,0.5,\gamma)$ where all elements closer to $-\pi$ than $r = 0.1$ were mirrored. Table 3 gives an overview of parameters.

With increasing scale of FSS (see Table 3) we see that the quantile tests (gray) become more and more liberal while the bootstrap tests (black) maintain the correct level. In particular, the quantile based tests perform poorly in the presence of considerable Type I FSS, in consequence of the scaled variance of intrinsic sample means $nE[d(\hat{\mu}_n, \mu)]^2$ being larger than the Euclidean variance $\sigma^2$. Upon very close inspection, in case of Type II FSS (dashed lines, $r > 0$) we see that the bootstrap tests may be slightly too conservative. Conversely, in case of Type I FSS (solid lines, $r = 0$) the bootstrap tests may be slightly to liberal. Both effects may be due to a systematic bias, cf. Remark 4.6.
Figure 7: Rejection probabilities for \( H_0^2 \) over angle \( p \in [\pi, \pi] \) between \( \mu(X) \) and \( \mu(Y) \) under quantile based test (Test 4.1, gray) and bootstrap based test for \( B = 1000 \) (Test 4.4, black) at nominal level 5% (dotted horizontal) based on 1000 simulation runs with two samples from the same distribution of size \( n = 30 \) (top row), \( n = 100 \) (middle row), and \( n = 300 \) (bottom row) but where the latter sample is rotated by \( p \). The solid lines represent samples which were generated from mixed von Mises distributions \( \mathcal{M}_3^3,1,0,r \), i.e., \( r = 0 \). The dashed lines correspond to samples from \( \mathcal{M}_3^3,0.5,0.5,r \) where all elements closer to \( -\pi \) than \( r = 0.1 \) were mirrored.
Figure 8: Histograms of Fréchet means of daily wind direction data by meteoblue AG (2020) for Basel (top) and Göttingen (bottom) for the years 2000 to 2019. The x-axis is divided into 32 segments with labels N: “North”, E: “East”, S: “South”, and W: “West”, indicating the average direction of wind origin.
6 Assessing Significant Change of Wind Direction

In application of our methods dealing with FSS, we analyze wind data from Basel and Göttingen (the city of the authors’ institution) provided by meteoblue AG (2020). For our purpose, we consider daily Fréchet mean wind directions for the years 2000 to 2019 giving for each city 20 samples of two-dimensional circular data of size \( n = 365 \). The respective daily wind directions are illustrated in Figure 8. To assess a possible effect of climate change, we test for a significant change in wind direction.

For each of these 40 samples we computed the estimated variance modulation \( \hat{m}_n^* \) from (10) with \( B = 1000 \), cf. Table 4. Remarkably, for both cities all of the scales of FSS statistically indicate presence of FSS (Test 4.4 rejects the absence of FSS with the lowest \( p = B^{-1} \) possible) except for Göttingen in the year 2017.

Table 4: Empirical variance modulation \( \hat{m}_n^* \) for \( n = 365 \) from (10) for daily Fréchet mean wind directions in Basel and Göttingen for the years 2000 to 2019. All of these values indicate presence of FSS (Test 4.4 rejects absence of FSS with the lowest \( p \)-value possible: \( p = 10^{-3} \)) except for Göttingen in the year 2017.

In consequence, we expect that the quantile based two-sample Test 4.1 will feature a considerably high error of the first kind as compared to the bootstrap based Test 4.4. For each series of 20 \( \times \) 19 / 2 = 190 tests at nominal level \( \alpha = 0.05 \) we performed a Benjamini-Hochberg correction, cf. Figure 9.

The first series of tests is performed on the two-torus \( T^2 = S^1 \times S^1 \) for Basel and Göttingen jointly (left panel of Figure 9). According to the quantile Test 4.1 a majority of years seem to be significantly different from a great number of others. Applying the bootstrap Test 4.4 for \( B = 1000 \) we see this
“noise” disappearing, leaving only the years 2003, 2005, 2017 and 2018 as moderately exceptional, where 2018 appears to be more exceptional than the other three.

Looking only at Basel (now testing only on $S^1$, middle panel of Figure 9), Test 4.1 seems to suggest very clearly that the years 2003, 2017 and 2018 are exceptional. Test 4.4 however, clarifies the picture: 2017 is not exceptional and 2003 and 2018 are only moderately exceptional.

Comparison with Göttingen only (again testing on $S^1$, right panel of Figure 9) shows the “noise” from Test 4.1 is rooted in Göttingen. Again, Test 4.4 clarifies the picture: only the years 2005, 2017 are mildly exceptional. The year 2018 remains more prominently exceptional. The year 2010, which appeared almost as strongly exceptional as year 2018 for Test 4.1 is no longer exceptional for Test 4.4. Notably, 2017 is also exceptional through absence of FSS, cf. Table 4.

In conclusion, using the bootstrap test which preserves the nominal level, the years 2003 and 2018 appear exceptional along with 2005 and 2017. These findings fall well into the climatological context of central European heat waves linked to exceptional wind constellations (Kornhuber et al. (2019) identify a recurrent wave-7 wind pattern for the years 2003, 2006, 2015 and 2018). While the 2003 heat wave occurred in most of Europe, the 2018 heat wave manifested in a climatic dipole: hot and dry north of the Alps, comparably cool and moist across large parts of Mediterranean, cf. Buras et al. (2020). This is in agreement with our finding that the wind anomaly of 2018 was more prominent for Göttingen (in the northern part of Germany), rather than for Basel (edging the southern border of Germany) which is barely north of the Alps.

In a debate quantifying climate change, its anthropogenic component and future costs, linking to changes of wind patterns (e.g. McInnes et al. (2011)), our new inferential tools for cyclic data presented here, warrant a more detailed application in future work.

7 Discussion and Outlook

In this contribution, we have investigated two manifolds with codimension one cut loci, namely circles and tori and found FSS manifesting in two different types. We expect similar findings for other manifolds with codimension one cut loci, such as real projective spaces, modeling projective shapes, say, as in Mardia and Patrangenaru (2005); Hotz et al. (2016). We conjecture that this is, using the language of Eltzner (2019), a consequence of topological smeariness. On manifolds with higher codimension cut loci, in the language of Eltzner (2019), for instance on arbitrary spheres, there is the different phenomenon of geometrical smeariness. We conjecture that this leads to Type I FSS only.

Moreover, we have shown that the proposed bootstrap tests are asymptotically consistent and demonstrated in simulations that they preserve the level fairly well for reasonable sample sizes. Inspired by recent findings by Zhilova (2020) on the non-asymptotic accuracy of the bootstrap for Euclidean sample means for the finite sample regime, we deem it possible to derive similar non-asymptotic results for the bootstrap Fréchet sample mean. This issue is beyond the scope of this work and left for future work.

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Appendix

A Lemmata for the Proof of Theorem 2.3.

Lemma A.1. Consider $X_1, \ldots, X_n \overset{iid}{\sim} X$ on $\mathbb{S}^1 = [-\pi, \pi]$ with Euclidean sample and Fréchet sample mean $\overline{X}_n$ and $\hat{\mu}_n$, respectively. Then

(i) there is $j \in \{-n, \ldots, n\}$ such that $\hat{\mu}_n = \overline{X}_n + \frac{2\pi j}{n}$, and

(ii) $|\hat{\mu}_n| \leq |\overline{X}_n|$.

Furthermore, assume that $X$ has a unique population mean $\mu = 0$ and that $\hat{\mu}_n$ is a measurable selection of Fréchet samples means. Then

(iii) if $P(|\hat{\mu}_n| \neq |\overline{X}_n|) > 0$ then $m_n > 1$.

Proof. The first assertion can be found in Hotz and Huckemann (2015, proof of Corollary 4). For convenience we reproduce the short argument. For arbitrary fixed $\mu \in \mathbb{S}^1$ consider the index sets $I_0 = \{j : -\pi \leq X_j - \mu \leq \pi\}$, $I_1 = \{j : X_j - \mu > \pi\}$ and $I_2 = \{j : X_j - \mu < -\pi\}$. Then

$$
\sum_{j=1}^{n} d(X_j, \mu)^2 = \sum_{j \in I_0} (X_j - \mu)^2 + \sum_{j \in I_1} (X_j - 2\pi - \mu)^2 + \sum_{j \in I_2} (X_j + 2\pi - \mu)^2
$$

$$
= \sum_{j=1}^{n} (X_j - \mu)^2 - 4\pi \sum_{j \in I_1} (X_j - \mu - \pi) + 4\pi \sum_{j \in I_2} (X_j - \mu + \pi)
$$

(18)

This is minimized by

$$
\hat{\mu}_n = \overline{X}_n + 2\pi \frac{|I_2| - |I_1|}{n},
$$

(19)

yielding the first assertion.

For the second assertion, we consider the above introduced $I_0, I_1, I_2$, now with $\hat{\mu}_n$ instead of $\mu$.

Assume first that $\hat{\mu}_n < 0 \leq \overline{X}_n$. Then $I_2 = \emptyset$. Setting $m = |I_1|$ we have thus by (19) that $0 < \overline{X}_n - \hat{\mu}_n = \frac{2\pi m}{n}$, whence by (18).

$$
\sum_{j=1}^{n} d(X_j, \hat{\mu}_n)^2 = \sum_{j=1}^{n} (X_j - \overline{X}_n + \overline{X}_n - \hat{\mu}_n)^2 - 4\pi \sum_{j \in I_1} (X_j - \pi - \overline{X}_n + \overline{X}_n - \hat{\mu}_n)
$$

$$
= \sum_{j=1}^{n} (X_j - \overline{X}_n)^2 + n \left( \frac{2\pi m}{n} \right)^2 + 4\pi \sum_{j \in I_1} (\overline{X}_n + \pi - X_j) - 4\pi m \frac{2\pi m}{n}
$$

$$
= \sum_{j=1}^{n} (X_j - \overline{X}_n)^2 + 4\pi \sum_{j \in I_1} (\overline{X}_n + \pi - X_j) - 4\pi m \frac{\overline{X}_n - \hat{\mu}_n}{2}
$$

$$
= \sum_{j=1}^{n} (X_j - \overline{X}_n)^2 + 4\pi \sum_{j \in I_1} \left( \frac{\overline{X}_n + \hat{\mu}_n}{2} + \pi - X_j \right).
$$

Note that $\pi - X_j \geq 0$ always. Hence if $-\overline{X}_n < \hat{\mu}_n$ then $\sum_{j=1}^{n} d(X_j, \hat{\mu}_n)^2 > \sum_{j=1}^{n} (X_j - \overline{X}_n)^2 \geq \sum_{j=1}^{n} d(X_j, \overline{X}_n)^2$, so that $\hat{\mu}_n$ cannot be a Fréchet mean. In consequence we have shown that $\hat{\mu}_n < 0 \leq \overline{X}_n$ implies $|\hat{\mu}_n| \geq |\overline{X}_n|$.

Next assume that $0 \leq \hat{\mu}_n \leq \overline{X}_n$. Then $I_1 = \emptyset$ and setting $m = |I_2| \geq 0$ we infer from (19) that $0 \leq \overline{X}_n - \hat{\mu}_n = -\frac{2\pi m}{n}$ and thus $m = 0$ implying $\hat{\mu}_n = \overline{X}_n$.

With the above, this yields the second assertion.
Since $m_n = n\mathbb{E}[d(\mu_n,0)^2]/\mathbb{E}[d(X,0)^2]$, the last assertion follows at once from
\[
\mathbb{E}[d(X,0)^2] = n\mathbb{E}[|\hat{\mu}_n|^2] < n\mathbb{E}[|\mu_n|^2] = n\mathbb{E}[d(\mu_n,0)^2],
\]
where the inequality is due to $1 = \mathbb{P}\{|\hat{\mu}_n| = |\hat{X}_n|\} + \mathbb{P}\{|\hat{\mu}_n| > |\hat{X}_n|\}$ by (ii), since the second probability is positive by hypothesis.

**Lemma A.2.** Let $n \geq 2$, $a \in (0,\pi/2]$ and consider $X_1, \ldots, X_n \overset{i.i.d.}{\sim} X$ on $\mathbb{S}^1$ with unique population Fréchet mean $\mu = 0$.

(i) If $\mathbb{P}\{X_1, \ldots, X_m \in [a,a+\varepsilon] \text{ and } X_{m+1}, \ldots, X_n \in (a-\pi, a-\pi]\} > 0$ for some $0 \leq \varepsilon < \min\{a, \pi-a, \frac{\pi}{n}\}$ and $m \in \mathbb{N}$ with $1 \leq m \leq n-1$, then $m_n > 1$.

(ii) If $\mathbb{P}\{a-\pi < X \leq a\} = 1$ or $\mathbb{P}\{a-\pi \leq X < a\} = 1$, then $m_n = 1$.

**Proof.** For (i) we show that under the assumption,
\[
\mathbb{P}\{|X_n| < |\hat{\mu}_n|\} > 0 \tag{20}
\]
whence $\mathbb{P}\{|\hat{\mu}_n| \neq |\hat{X}_n|\} > 0$ and in consequence of Lemma A.1 (iii), $m_n > 1$.

Let $A = \{X_1, \ldots, X_m \in [a,a+\varepsilon] \text{ and } X_{m+1}, \ldots, X_n \in (a-\pi, a-\pi]\}$ with $\mathbb{P}(A) > 0$. By hypothesis on $\varepsilon$ we have $(a-\pi, a+\varepsilon) \subset (-\pi, \pi)$ yielding that for all $\omega \in A$,
\[
\hat{X}_n = \left\{ \frac{n-m}{n} \frac{a}{n} - \frac{m}{n} \frac{a}{n} \frac{\pi}{n} + \frac{m}{n} \frac{\varepsilon}{n}, \pi \leq \frac{n-m}{n} \frac{\varepsilon}{n} \varepsilon \right\},
\]
i.e.
\[
|X_n| < \max\left\{ a - \frac{n-m}{n} \pi + \frac{m}{n} \pi, \frac{n-m}{n} \pi + \frac{m}{n} \pi - a \right\}. \tag{21}
\]

Note that $\varepsilon < \frac{1}{n-1} \pi = \frac{n-(n-1)}{n-1} \pi \leq \frac{n-m}{m} \pi$ for all $0 < m < n$ and $\varepsilon < \frac{1}{n-1} \pi \leq \frac{m}{n-m} \pi$ for all $1 \leq m < n$ ensure that
\[
a - \frac{n-m}{n} \pi + \frac{m}{n} \pi \leq \frac{\pi}{2} \quad \text{and} \quad \frac{n-m}{n} \pi + \frac{m}{n} \pi - a < \pi - a \tag{22}
\]
for all $1 \leq m \leq n-1$. On the other hand, we have a measurable subset $B$ of $A$ with $\mathbb{P}(B) > 0$ such that $\hat{\mu}_n \in [-\pi, a-\pi] \cup (a, \pi)$ for all $\omega \in B$ (we have $B = A$ unless $A = \{X_1, \ldots, X_n = a\}$ and $X_{m+1}, \ldots, X_n = \pi - a\}$, then $0 < \mathbb{P}(B) = \mathbb{P}(A)/2$ according to our agreement how to draw measurable selections from nonunique means).

In case of $\hat{\mu}_n \in [-\pi, a-\pi]$ this means $|\hat{\mu}_n| > \pi - a \geq \frac{\pi}{2}$, which, for all $\omega \in B$, is greater than any of the bounds for $|X_n|$ given by (21) and (22), yielding (20).

In case of $\hat{\mu}_n \in (a, \pi)$, we have for all $\omega \in B$ that (the first inequality is due to the fact that unwrapping to the tangent line at $\hat{\mu}_n$, its origin is the Euclidean mean)
\[
\hat{\mu}_n > \frac{n-m}{n} \frac{a}{n} + \frac{m}{n} \frac{a}{n} \frac{\pi}{n} = a + \frac{n-m}{n} \frac{\pi}{n} - a,
\]
which is larger than the upper line of (21) due to the upper line of (22). It is also larger than the lower line of (21) since $\varepsilon < a < \frac{n}{n-1} a \leq \frac{n-m}{n-m} a$ for all $1 \leq m \leq n-1$ yields
\[
a + \frac{n-m}{n} \frac{\pi}{n} - a = \left( \frac{n-m}{n} \frac{\pi}{n} - a \right) = 2a - 2 \frac{n-m}{n} \varepsilon > 0.
\]

In conclusion, $|\hat{\mu}_n| > |X_n|$ for all $\omega \in B$ which in conjunction with $\mathbb{P}(B) > 0$ yields again (20).

In case (ii) we infer $\hat{\mu}_n = \hat{X}_n$ from (19) since with the notation there, $I_1 = \emptyset = I_n$. Hence, taking into account $n\mathbb{E}[d(0,\hat{X}_n)^2] = \mathbb{E}[d(0,X)^2], m_n = 1$. \qed
B Lemmata and Propositions for Proof of Theorem 3.4

In this section, we prove that the terms from (17) converge in outer probability to zero and thus complete the proof on consistency of the bootstrap for circular Fréchet means.

Throughout this section, under Assumption 3.1, we construct a sample $X_1, \ldots, X_n \overset{i.i.d.}{\sim} X$ with a measurable selection $\hat{\mu}_n$ of its Fréchet sample means, as well as a bootstrap sample $X_1^*, \ldots, X_n^* \overset{i.i.d.}{\sim} \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$ with a measurable selection $\hat{\mu}_n^*$ of its Fréchet sample means. For convention we denote the underlying probability measure of $X$ by $\mathbb{P}$, the empirical measure of $X_1, \ldots, X_n$ by $\hat{\mathbb{P}}_n$ and the bootstrap empirical measure of $X_1^*, \ldots, X_n^*$ by $\mathbb{P}_n^*$. Further, recall that we define for $t \in \mathbb{R}$ the function $H(t) := \min(2, |t|)$ which satisfies for any $h \in \mathbb{B}_L(\mathbb{R})$ and $t' \in \mathbb{R}$ the inequality $|h(t + t') − h(t)| \leq H(t')$. Further, let us recall the càdlàg functions $A, B_n, C_n : [−\pi, \pi] \to \mathbb{R}$ which are defined for $x \in [−\pi, \pi)$ by

$$
A(x) := \begin{cases} 
    f(−\pi)x − \mathbb{P}(X \leq x − \pi) & \text{if } x \geq 0, \\
    f(−\pi)x + \mathbb{P}(X > x + \pi) & \text{if } x < 0,
\end{cases} \quad (23)
$$

$$
B_n(x) := \begin{cases} 
    \mathbb{P}(X \leq x − \pi) − \frac{1}{n} \left( \sum_{i=1}^n 1_{X_i \leq x − \pi} \right) & \text{if } x \geq 0, \\
    −\mathbb{P}(X > x + \pi) + \frac{1}{n} \left( \sum_{i=1}^n 1_{X_i > x + \pi} \right) & \text{if } x < 0,
\end{cases} \quad (24)
$$

$$
C_n(x) := \begin{cases} 
    \frac{1}{n} \left( \sum_{i=1}^n 1_{X_i \leq x − \pi} \right) − \frac{1}{n} \left( \sum_{i=1}^n 1_{X_i > x + \pi} \right) & \text{if } x \geq 0, \\
    \frac{1}{n} \left( \sum_{i=1}^n 1_{X_i \leq x − \pi} \right) + \frac{1}{n} \left( \sum_{i=1}^n 1_{X_i > x + \pi} \right) & \text{if } x < 0.
\end{cases} \quad (25)
$$

Proposition B.1. For $n \to \infty$, we have

$$
\mathbb{E} \left[ H(\sqrt{n}A(\hat{\mu}_n)) + H(\sqrt{n}A(\bar{\mu}_n)) + H(\sqrt{n}B_n(\hat{\mu}_n)) + H(\sqrt{n}B_n(\bar{\mu}_n)) \right] \to 0.
$$

Proof. We prove in Proposition B.4 for $n \to \infty$ that the terms (26) involving $B_n$ and $C_n$ (conditionally) converge (in outer probability) to zero. For the terms involving the function $A$ we show (conditional) convergence (in outer probability) to zero in Proposition B.6.

Outline of proof. We begin in Lemma B.2 with a characterization of conditional convergence in outer probability which will be used throughout the rest of this section. Afterwards, in Lemma B.3 we verify that the bootstrap Fréchet sample mean conditionally converges to the Fréchet population mean as the sample size $n$ tends to infinity. In Proposition B.4 we show that the terms in (26) with $B_n$ and $C_n$ (conditionally) converge to zero. For this purpose, we show in Proposition B.4 that the (random) functions $\sqrt{n}B_n, \sqrt{n}C_n$ converge weakly in the space of càdlàg functions on $[−\pi, \pi)$ to a tight limit process with small fluctuation close to 0. In conjunction with (conditional) convergence in probability of $\hat{\mu}_n$ (Ziezold 1977) and $\bar{\mu}_n$ (Lemma B.3) to $\mu = 0$ as well as our characterization of conditional convergence (Lemma B.2) the claim follows. To show (conditional) convergence of the terms in (26) involving $A$ we notice by Assumption B.1 that $A(x) = o(x)$ for $x \to 0$. To make use of this fact for our convergence analysis, we show a (conditional) tightness property for $\sqrt{n}\mu_n$ and $\sqrt{n}\bar{\mu}_n$ given $X_1, \ldots, X_n$. For this purpose, we first show in Lemma B.5 that the Euclidean bootstrap sample mean is conditionally tight. Afterwards, we prove in Proposition B.6 the required (conditional) tightness property for the Fréchet means and verify the convergence of the terms in (26) involving $A$.

Lemma B.2. For a random variable $Y_n$ on $\mathbb{R}$ that is measurable w.r.t. $X_1, \ldots, X_n$ we have

$$
\mathbb{E} \left[ H(Y_n) \big| X_1, \ldots, X_n \right] \to 0 \quad \text{for } n \to \infty
$$

if and only if for all $\epsilon > 0,$

$$
\mathbb{P} \left( |Y_n| \geq \epsilon \big| X_1, \ldots, X_n \right) \to 0 \quad \text{for } n \to \infty.
$$

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Proof. Suppose for all $\varepsilon > 0$ holds $\mathbb{P}(|Y_n| \geq \varepsilon |X_1, \ldots, X_n) \xrightarrow{\mathbb{P}^*} 0$. Fix $\varepsilon > 0$. Define
\[ K_n(\varepsilon) := \{ \mathbb{P}(|Y_n| \geq \varepsilon |X_1, \ldots, X_n) \geq \varepsilon \} \]
and set $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon)$ it holds that $\mathbb{P}^*(K_n(\varepsilon)) \leq \varepsilon$. Conditioned on $K_n(\varepsilon/4)^c$ it follows for $n \geq N(\varepsilon/4)$ that $\mathbb{P}(|Y_n| \geq \varepsilon/4 |X_1, \ldots, X_n) \leq \varepsilon/4$, thus by definition of $H(z) = \min(2, |z|)$
\[
\mathbb{E} \left[ H(Y_n) | X_1, \ldots, X_n \right] = \mathbb{E} \left( H(Y_n) 1 \left( |Y_n| < \frac{\varepsilon}{4} \right) |X_1, \ldots, X_n \right) + \mathbb{E} \left[ H(Y_n) 1 \left( |Y_n| \geq \frac{\varepsilon}{4} \right) |X_1, \ldots, X_n \right] \\
\leq \frac{\varepsilon}{4} + 2\mathbb{P} \left( |Y_n| \geq \frac{\varepsilon}{4} |X_1, \ldots, X_n \right) \leq \frac{3}{4}\varepsilon < \varepsilon.
\]
Consequently, for $n \geq N(\varepsilon/4)$ we assert that
\[
\mathbb{P}^* \left( \mathbb{E} \left[ H(Y_n) | X_1, \ldots, X_n \right] \geq \varepsilon \right) \\
\leq \mathbb{P}^* \left( K_n(\varepsilon/4)^c \cap \left\{ \mathbb{E} \left[ H(Y_n) | X_1, \ldots, X_n \right] \geq \varepsilon \right\} \right) \leq \frac{\varepsilon}{4} \leq \varepsilon,
\]
which proves $\mathbb{E} [H(Y_n)|X_1,\ldots,X_n] \xrightarrow{\mathbb{P}^*} 0$. For the converse, assume there exists $\varepsilon > 0$ such that $\mathbb{P}^*(K_n(\varepsilon)) > \varepsilon$ for all $n \in \mathbb{N}$. Then it follows conditioned on $K_n(\varepsilon)$ for each $n \in \mathbb{N}$ that
\[
\mathbb{E} \left[ H(Y_n) | X_1, \ldots, X_n \right] \geq \mathbb{E} \left[ H(Y_n) 1(|Y_n| \geq \varepsilon) |X_1, \ldots, X_n \right] \geq \varepsilon \mathbb{P}(|Y_n| \geq \varepsilon |X_1, \ldots, X_n) > \varepsilon^2.
\]
Hence, we have for all $n \in \mathbb{N}$ that
\[
\mathbb{P}^* \left( \mathbb{E} \left[ H(Y_n) | X_1, \ldots, X_n \right] \geq \varepsilon^2 \right) \geq \mathbb{P}^*(K_n(\varepsilon)) > \varepsilon.
\]
which shows that $\mathbb{E} [H(Y_n)|X_1,\ldots,X_n]$ does not converge in outer probability to zero.

For the next results we use concepts of empirical process theory. In particular, we employ the notion of general weak convergence on Banach spaces, bracketing numbers, and Donsker function classes. For an introduction to these concepts we refer to Van der Vaart & Wellner (1996) and Van der Vaart (1998).

First, we establish consistency of the bootstrap sample mean $\hat{\mu}_n$ to the population mean $\mu$.

**Lemma B.3.** There exists for each $\varepsilon > 0$ an $N \in \mathbb{N}$ such that $\mathbb{P}^*(\mathbb{P}(\hat{\mu}^*_n \geq \varepsilon |X_1, \ldots, X_n) \geq \varepsilon) \leq \varepsilon$ for all $n \geq N$.

*Proof.* By uniqueness of the Fréchet population mean $\mu = 0$ in conjunction with continuity of the Fréchet function $F$ it follows for any $\varepsilon > 0$ that there exists $\delta > 0$ such that $F^{-1}([F(\mu), F(\mu) + \delta]) \subseteq [-\varepsilon, \varepsilon]$. Hence, for any continuous function $G: \mathbb{S}^1 \to \mathbb{R}$ such that $\sup_{x \in \mathbb{S}^1} |F(x) - G(x)| = \|F - G\|_\infty < \delta/3$ we see for $x \notin [-\varepsilon, \varepsilon]$ that
\[
G(x) \geq F(x) - \frac{\delta}{3} \geq F(\mu) + \frac{2\delta}{3} \geq G(\mu) + \frac{\delta}{3},
\]
which yields that $\arg\min_{x \in \mathbb{S}^1} G(x) \in [-\varepsilon, \varepsilon]$.

Next, we show for sufficiently large $n$ that the bootstrap empirical Fréchet function $F^*_n$ is close to $F$ in supremum norm with high probability and thus the bootstrap empirical Fréchet mean $\hat{\mu}^*_n$ is close to $\mu = 0$. For this purpose, consider the function class $\mathcal{F} := \{ d^2(:,x) : x \in \mathbb{S}^1 \}$ and denote by $\ell^\infty(\mathcal{F})$ the Banach space of bounded linear functionals on $\mathcal{F}$ equipped with supremum norm. Note that the probability measure $\mathbb{P}$ associated to $X$, the empirical measure $\mathbb{P}_n$ of $X_1, \ldots, X_n$ and the bootstrap empirical measure $\mathbb{P}^*_n$ of $X^*_1, \ldots, X^*_n$ are all contained in $\ell^\infty(\mathcal{F})$ since $\mathcal{F}$ is uniformly bounded. In particular, each probability measure is assigned to its corresponding Fréchet function.

Further, note that there exists $M > 0$ such that for all $x, y, z \in [-\pi, \pi]$ it holds that
\[
|d^2(y, x) - d^2(z, x)| \leq M|y - z|.
\]
Thus, we see conditioned on $\sqrt{n}(F_n^* - F_n)$ converges weakly in the space of continuous functions on $\mathbb{S}^1$, denoted by $C(\mathbb{S}^1)$, for $n \to \infty$, to a tight limit. Additionally, since $\hat{F}$ is $\mathbb{P}$-Donsker, it follows by (van der Vaart and Wellner, 1996, Theorem 3.6.1) for the bootstrap empirical Fréchet function that

$$\sup_{h \in \text{BL}_1(C(\mathbb{S}^1))} \left| \mathbb{E} \left[ h\left(\sqrt{n}(F_n^* - F_n)\right) \right] - \mathbb{E} \left[ h\left(\sqrt{n}(F_n - F)\right) \right] \right| \xrightarrow{\mathbb{P}^*} 0,$$

where $\text{BL}_1(C(\mathbb{S}^1))$ denotes the space of 1-Lipschitz functions mapping from $C(\mathbb{S}^1)$ to $\mathbb{R}$ which are bounded by one.

To prove the assertion consider $\varepsilon > 0$ and let $\delta > 0$ such that $F^{-1}([F(\mu), F(\mu) + \delta]) \subseteq [-\varepsilon, \varepsilon]$. By weak convergence of $\sqrt{n}(F_n - F)$ there exists $M > 0$ and $N_1 \in \mathbb{N}$ such that for $n \geq N_1$ holds $\mathbb{P}(\sqrt{n}(F_n - F)\|_\infty \geq M) \leq \varepsilon/3$. Define the Lipschitz function $h(f) = \min(\max(||f||_\infty - M, 0), 1)$ for $f \in C([-\pi, \pi])$ which satisfies

$$1(||f||_\infty \geq M + 1) \leq h(f) \leq 1(||f||_\infty \geq M). \tag{27}$$

Then it holds that $\mathbb{E}[h(\sqrt{n}(F_n - F))] \leq \varepsilon/3$ and there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$ follows

$$\mathbb{P}^*\left( \mathbb{E} \left[ h\left(\sqrt{n}(F_n^* - F_n)\right) \right] X_1, \ldots, X_n - \mathbb{E} \left[ h\left(\sqrt{n}(F_n - F)\right) \right] \right) \geq \frac{\varepsilon}{3} \leq \frac{\varepsilon}{3}.$$

Define $N_3 := [9(M + 1)^2/\delta^2]$, then Inequality $(27)$ asserts for $n \geq \max(N_1, N_2, N_3)$ on the complement $(K_n^{(1)})^c$ that

$$\mathbb{P}\left( \|F_n^* - F_n\|_\infty \geq \frac{\delta}{3} \right) X_1, \ldots, X_n = \mathbb{P}\left( \sqrt{n}\|F_n^* - F_n\|_\infty \geq \frac{\delta}{3} \sqrt{n} X_1, \ldots, X_n \right) \leq \mathbb{P}\left( \sqrt{n}\|F_n - F\|_\infty \geq M + 1 \right) X_1, \ldots, X_n \leq \mathbb{E} \left[ h\left(\sqrt{n}(F_n^* - F_n)\right) \right] X_1, \ldots, X_n \leq \mathbb{E} \left[ h\left(\sqrt{n}(F_n - F)\right) \right] + \varepsilon \leq \frac{2}{3} \varepsilon$$

In particular, conditioned on $\sqrt{n}\|F_n - F\|_\infty < M \cap (K_n^{(1)})^c$ it holds for $n \geq \max(N_1, N_2, N_3)$ that

$$\mathbb{P}\left( \|F_n^* - F\|_\infty \geq \frac{2\delta}{3} \right) X_1, \ldots, X_n \leq \mathbb{P}\left( \|F_n^* - F\|_\infty \geq \frac{\delta}{3} \right) X_1, \ldots, X_n + \mathbb{P}\left( \|F_n - F\|_\infty \geq \frac{\delta}{3} \right) X_1, \ldots, X_n \leq \frac{2}{3} \varepsilon + 0.$$

Recall by our first part of the proof that $\|F_n^* - F\|_\infty \leq \delta$ implies by continuity of $F_n^*$ that $\tilde{\mu}_n^* \in [-\varepsilon, \varepsilon]$. Thus, we see conditioned on $\sqrt{n}\|F_n - F\|_\infty < M \cap (K_n^{(1)})^c$ for $n \geq \max(N_1, N_2, N_3)$ that

$$\mathbb{P}\left( \tilde{\mu}_n^* \geq \varepsilon \right) X_1, \ldots, X_n \leq \frac{2}{3} \varepsilon < \varepsilon.$$

Concluding, the assertion follows since for all $n \geq \max(N_1, N_2, N_3)$ holds

$$\mathbb{P}^*\left( \mathbb{P}\left( \tilde{\mu}_n^* \geq \varepsilon \right) \right) \geq \mathbb{P}\left( \sqrt{n}\|F_n - F\|_\infty < M \right) + \mathbb{P}^*\left( (K_n^{(1)})^c \cap (K_n^{(1)})^c \cap \left\{ \tilde{\mu}_n^* \geq \varepsilon \right\} \right) \leq \frac{2}{3} \varepsilon + 0 \leq \varepsilon. \quad \Box$$
Proposition B.4. With the functions $B_n$ and $C_n$ defined in (24) and (25), respectively we have for $n \to \infty$ that

$$\mathbb{E} \left[ H(\sqrt{n}B_n(\hat{\mu}_n)) \right] + \mathbb{E} \left[ H(\sqrt{n}B_n(\hat{\mu}_n)) + H(\sqrt{n}B_n(\hat{\mu}_n^*)) + H(\sqrt{n}C_n(\hat{\mu}_n^*)) \right] X_1, \ldots, X_n \xrightarrow{p} 0.$$ 

Proof. We first show that $\sqrt{n}B_n$ and $\sqrt{n}C_n$ are stochastic processes which converge for $n \to \infty$ to a tight Gaussian process. For this limit process we show that it exhibit a small fluctuation near zero. We then prove the claim by exploiting the consistency of the Fréchet sample mean and its bootstrap version.

Step 1. Derivation of limit process. Define the function class

$$\mathcal{G} := \left\{ -1 \left( \cdot \in [-\pi, t - \pi) \right), t \in [0, \pi] \right\} \cup \left\{ 1 \left( \cdot \in (t + \pi, \pi) \right), t \in [-\pi, 0) \right\}.$$ 

Notably, the Banach space $\ell^\infty(\mathcal{G})$ contains the probability measures $\mathbb{P}, \mathbb{P}_n,$ and $\mathbb{P}_n^*$ for each $n \in \mathbb{N}$. In particular, it holds for $t \in [0, \pi]$ that

$$(\mathbb{P}_n - \mathbb{P})(-1(\cdot \in [-\pi, t - \pi))) = B_n(t), \quad (\mathbb{P}_n^* - \mathbb{P}_n)(-1(\cdot \in [-\pi, t - \pi))) = C_n(t)$$

and likewise for $t \in (-\pi, 0)$ we obtain

$$(\mathbb{P}_n - \mathbb{P})(1(\cdot \in (t + \pi, \pi))) = B_n(t), \quad (\mathbb{P}_n^* - \mathbb{P}_n)(1(\cdot \in (t + \pi, \pi))) = C_n(t).$$

Hence, the empirical process $\sqrt{n}(\mathbb{P}_n - \mathbb{P})$ and the bootstrap empirical process $\sqrt{n}(\mathbb{P}_n^* - \mathbb{P}_n)$ can be identified in $\ell^\infty(\mathcal{G})$ with the (random) functions $\sqrt{n}B_n$ and $\sqrt{n}C_n$ on $[-\pi, \pi]$. Notably, for each $n \in \mathbb{N}$ the functions $B_n$ and $C_n$ are càdlàg on $[-\pi, \pi]$, i.e. right-continuous and limits from left exist. In the following we denote the space of such functions equipped with supremum norm by $D([-\pi, \pi])$. Since $\mathcal{G}$ is the union of two Donsker classes (Van der Vaart & Wellner, 1998, Example 2.5.4) it follows that $\mathcal{G}$ is also a Donsker class. Hence, $\sqrt{n}B_n$ converges weakly in $D([-\pi, \pi])$ to a tight centered Gaussian process $\mathcal{G}_B$. The covariance structure of $\mathcal{G}_B$ is characterized by

$$\text{Cov} [\mathcal{G}_B(t)\mathcal{G}_B(s)] = \begin{cases} \mathbb{P}(X \leq t - \pi) \land \mathbb{P}(X \leq s - \pi) - \mathbb{P}(X \leq t - \pi)\mathbb{P}(X \leq s - \pi) & \text{if } t, s \geq 0, \\ \mathbb{P}(X > t + \pi) \land \mathbb{P}(X > s + \pi) - \mathbb{P}(X > t + \pi)\mathbb{P}(X > s + \pi) & \text{if } t, s < 0, \\ \mathbb{P}(X \leq t - \pi)\mathbb{P}(X > s + \pi) & \text{if } t \geq 0, s < 0. \end{cases}$$

Step 2. Fluctuation of limit process near zero. In order to obtain concentration bounds for $\mathcal{G}_B$ we express it in terms of a standard brownian bridge. For this purpose, consider a standard Wiener process $\mathbb{W}$ on $[0, 1]$ and define the Brownian bridge $\mathbb{B}$ by $\mathbb{B}(t) := \mathbb{W}(t) - t\mathbb{W}(1)$. Defining the Gaussian process $\mathcal{G}_B$ on $[\pi, \pi]$ for $t \in [-\pi, \pi]$ by

$$\mathcal{G}_B(t) := \begin{cases} \mathbb{B}(\mathbb{P}(X \leq t - \pi)) & \text{if } t \in [0, \pi), \\ \mathbb{B}(1 - \mathbb{P}(X > t + \pi)) & \text{if } t \in [-\pi, 0). \end{cases}$$

we observe by a straight forward computation using $\mathbb{E}[\mathbb{B}(t)] = 0$ and $\text{Cov}[\mathbb{B}(t)\mathbb{B}(s)] = \min(t, s) - ts$ for all $t, s \in [0, 1]$ that $\mathcal{G}_B$ is centered and exhibits the same covariance structure as $\mathcal{G}_B$. Thus, $\mathcal{G}_B$ is a
cadlag modification of $G_B$. Consequently, it follows for $0 < \delta < \pi$ and $L > 0$ that
\[
P\left( \sup_{t \in [0, \delta]} |G_B(t)| \geq L \right) = \mathbb{P}\left( \sup_{t \in [0, \delta]} |\hat{G}_B(t)| \geq L \right)
\leq \mathbb{P}\left( \sup_{t \in [0, \delta]} \mathbb{W}\left( \mathbb{P}(X \leq t - \pi) \right) \right) \geq \frac{L}{2} + \mathbb{P}\left( \sup_{t \in [0, \delta]} |\mathbb{P}(X \leq t - \pi)\mathbb{W}(1)| \geq \frac{L}{2} \right)
\leq 2 \mathbb{E}\left[ \mathbb{W}\left( \mathbb{P}(X \leq \delta - \pi) \right) \right] + \frac{2 \mathbb{P}(X \leq \delta - \pi) \mathbb{E}[|\mathbb{W}(1)|]}{L}
\leq 2 \sqrt{\mathbb{E}[\mathbb{P}(X \leq \delta - \pi) + \mathbb{P}(X \leq \delta - \pi)]} \leq \frac{4 \sqrt{2}}{L} \sqrt{\mathbb{P}(X \leq \delta - \pi)}.
\]
Here, ($\ast$) follows by Doob’s inequality for sub-martingales [Le Gall 2016, Proposition 3.15] in conjunction with Markov’s inequality and ($\ast\ast$) holds by the formula for expected values of half-normal distributed random variables. Likewise, it follows by symmetry of the Brownian bridge $(B(t))_{t \in [0,1]} \overset{D}{=} (\mathbb{B}(1-t))_{t \in [0,1]}$ that
\[
P\left( \sup_{t \in [-\delta, 0]} |G_B(t)| \geq L \right) \leq \frac{4 \sqrt{2}}{L \sqrt{\pi}} \sqrt{\mathbb{P}(X > -\delta + \pi)}.
\]
As the unique Fréchet population mean is located at $\mu = 0$ the antipodal point exhibits no point mass. Hence, there exists for any $\varepsilon > 0$ a sufficiently small $\delta > 0$ such that
\[
P\left( \sup_{t \in [-\delta, \delta]} |G_B(t)| \geq \frac{\varepsilon}{2} \right) \leq \frac{\varepsilon}{3}.
\]

**Step 3. Convergence of $E[H(\sqrt{n}B_n(\hat{\mu}_n))]$ to zero.** W.l.o.g. choose $\varepsilon \leq 1$. Further, define the function
\[
h(f) := \min\left( \max\left( \sup_{x \in [-\delta, \delta]} |f(x)| - \varepsilon/2, 0 \right), \varepsilon \right),
\]
which is $1$-Lipschitz with respect to supremum norm on $[-\pi, \pi]$ and bounded by one since $\varepsilon \leq 1$. Notably, it holds for each $f \in D([-\pi, \pi])$ that
\[
\varepsilon 1\left( \sup_{x \in [-\delta, \delta]} |f(x)| \geq \varepsilon \right) \leq h(f) \leq \varepsilon 1\left( \sup_{x \in [-\delta, \delta]} |f(x)| \geq \varepsilon/2 \right)
\]
and thus (28) asserts $E[h(G_B)] \leq \varepsilon^2/3$. By weak convergence of $\sqrt{n}B_n$ to $G_B$ in $D([-\pi, \pi])$ there exists some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$ holds
\[
|E[h(\sqrt{n}B_n)] - E[h(G_B)]| \leq \frac{\varepsilon^2}{3}.
\]
Consequently, for $n \geq N_1$ we obtain that
\[
P\left( \sup_{t \in [-\delta, \delta]} |\sqrt{n}B_n(t)| \geq \varepsilon \right) \leq \frac{E[h(\sqrt{n}B_n)]}{\varepsilon} \leq \frac{E|h(G_B)|}{\varepsilon} + \frac{|E[h(\sqrt{n}B_n)] - E[h(G_B)]|}{\varepsilon} \leq \frac{2}{3} \varepsilon.
\]
Furthermore, since $\hat{\mu}_n \overset{p}{\rightarrow} 0$ there exists some $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ holds
\[
P(|\hat{\mu}_n| \geq \delta) \leq \frac{\varepsilon}{3}.
\]

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Combining these assertions, we conclude for \( n \geq \max(N_1, N_2) \) that
\[
\mathbb{P}(|\sqrt{n}B_n(\hat{\mu}_n)| \geq \varepsilon) \leq \mathbb{P}(|\hat{\mu}_n| \geq \delta) + \mathbb{P} \left( \sup_{t \in [-\delta, \delta]} |\sqrt{n}B_n(t)| \geq \varepsilon \right) \leq \frac{1}{3} \varepsilon + \frac{2}{3} \varepsilon = \varepsilon,
\]
which yields for \( n \to \infty \) that \( \mathbb{E} [H(\sqrt{n}B_n(\hat{\mu}_n))] \to 0 \) and thus also in outer probability.

**Step 4. Conditional convergence of \( \mathbb{E}[H(\sqrt{n}B_n(\hat{\mu}_n))|X_1, \ldots, X_n] \) in outer prob. to zero.** We employ a similar strategy as in Step 3. Note that we cannot simply infer that \( G \) is Donsker, thus by Van der Vaart & Wellner (1998, Theorem 3.6.1) it follows for \( n \to \infty \) that
\[
\mathbb{E} \left[ \mathbb{E}[H(\sqrt{n}B_n(\hat{\mu}_n))|X_1, \ldots, X_n] \right] - \mathbb{E} [E(GH)] \to 0.
\]

Hence, for \( n \geq \max(N_1, N_2) \) we obtain that
\[
\mathbb{P}^* \left( \mathbb{P}(|\sqrt{n}B_n(\hat{\mu}_n)| \geq \varepsilon|X_1, \ldots, X_n) \geq \varepsilon \right) \leq \mathbb{P}(|\hat{\mu}_n| \geq \delta) + \mathbb{P} \left( \sup_{t \in [-\delta, \delta]} |\sqrt{n}B_n(t)| \geq \varepsilon \right) + \mathbb{P} \left( \{|\hat{\mu}_n| < \delta\} \cap \{\sup_{t \in [-\delta, \delta]} |\sqrt{n}B_n(t)| < \varepsilon\} \cap \{\mathbb{P}(\sqrt{n}B_n(\hat{\mu}_n) \geq \varepsilon|X_1, \ldots, X_n) \geq \varepsilon\} \right)
\leq \frac{\varepsilon}{3} + \frac{2}{3} \varepsilon + 0 = \varepsilon,
\]
which yields by Lemma B.2 that \( \mathbb{E}[H(\sqrt{n}B_n(\hat{\mu}_n))|X_1, \ldots, X_n] \mathbb{P}^* \to 0 \) for \( n \to 0 \).

**Step 5. Conditional convergence of \( \mathbb{E}[H(\sqrt{n}B_n(\hat{\mu}_n))|X_1, \ldots, X_n] \) in outer prob. to zero.** Given \( \varepsilon > 0 \) there exists by Lemma B.3 some \( N_3 \in \mathbb{N} \) such that for all \( n \geq N_3 \) holds
\[
\mathbb{P}^* \left( \mathbb{P}(|\hat{\mu}_n| \geq \delta|X_1, \ldots, X_n) \geq \varepsilon \right) \leq \frac{\varepsilon}{3}.
\]
Hence, for \( n \geq N_3 \) we see conditioned on \( (K^{(2)}_n)^c \cap \{\sup_{t \in [-\delta, \delta]} |\sqrt{n}B_n(t)| < \varepsilon\} \) that
\[
\mathbb{P} \left( |\sqrt{n}B_n(\hat{\mu}_n^n)| \geq \varepsilon|X_1, \ldots, X_n \right) \leq \mathbb{P}(|\hat{\mu}_n^n| \geq \delta|X_1, \ldots, X_n) + \mathbb{P} \left( \sup_{t \in [-\delta, \delta]} |\sqrt{n}B_n(t)| \geq \varepsilon|X_1, \ldots, X_n \right)
\leq \frac{\varepsilon}{3} + 0 < \varepsilon.
\]
Consequently, it follows for \( n \geq \max(N_1, N_3) \) that
\[
\mathbb{P}^* \left( \mathbb{P}(|\sqrt{n}B_n(\hat{\mu}_n^n)| \geq \varepsilon|X_1, \ldots, X_n) \geq \varepsilon \right) \leq \mathbb{P}^*(K^{(2)}_n) + \mathbb{P} \left( \sup_{t \in [-\delta, \delta]} |\sqrt{n}B_n(t)| \geq \varepsilon \right) + \mathbb{P}^* \left( (K^{(2)}_n)^c \cap \{\sup_{t \in [-\delta, \delta]} |\sqrt{n}B_n(t)| < \varepsilon\} \cap \{\mathbb{P}(|\sqrt{n}B_n(\hat{\mu}_n^n)| \geq \varepsilon|X_1, \ldots, X_n) \geq \varepsilon\} \right)
\leq \frac{1}{3} \varepsilon + \frac{2}{3} \varepsilon + 0 = \varepsilon,
\]
hence by Lemma B.2 the assertion that \( \mathbb{E}[H(\sqrt{n}B_n(\hat{\mu}_n^n))|X_1, \ldots, X_n] \mathbb{P}^* \to 0 \) holds.

**Step 6. Conditional convergence of \( \mathbb{E}[H(\sqrt{n}C_n(\hat{\mu}_n^n))|X_1, \ldots, X_n] \) in outer prob. to zero.** To prove this claim, we recall that the function class \( \mathcal{G} \) is Donsker, thus by Van der Vaart & Wellner (1998, Theorem 3.6.1) it follows for \( n \to \infty \) that
\[
\sup_{h \in \mathbb{B}_1(D([\tau, \pi]))} \mathbb{E} \left[ \mathbb{E}[H(\sqrt{n}C_n)|X_1, \ldots, X_n] - \mathbb{E}[h(G_H)] \right] \mathbb{P}^* \to 0.
\]
Hence, for the previously defined function $h$ from $[\text{28}]$ there exists $N_4 \in \mathbb{N}$ such that for all $n \geq N_4$ holds

$$ P^* \left( \mathbb{E} \left[ h(\sqrt{n}C_n) \Big| X_1, \ldots, X_n \right] - \mathbb{E} \left[ h(G_B) \right] \geq \frac{\varepsilon^2}{6} \right) \leq \frac{2}{3} \varepsilon. $$

Consequently, for $n \geq N_4$ it follows conditioned on $(K_n^{(3)})^c$ by definition of $h$ and the choice of $\delta$, recall $[\text{28}]$, that

$$ P \left( \sup_{t \in [\varepsilon, \delta]} |\sqrt{n}C_n(t)| \geq \varepsilon \Big| X_1, \ldots, X_n \right) \leq \mathbb{E} \left[ h(\sqrt{n}C_n) \Big| X_1, \ldots, X_n \right] \leq \mathbb{E} \left[ h(G_B) \right] + \frac{\varepsilon}{6} \leq \frac{\varepsilon}{2}. $$

Moreover, for $n \geq \max(N_2, N_4)$ it follows conditioned on $(K_n^{(2)})^c \cap (K_n^{(3)})^c$ that

$$ P \left( |\sqrt{n}C_n(\hat{\mu}_n^*)| \geq \varepsilon \Big| X_1, \ldots, X_n \right) \leq P \left( \hat{\mu}_n^* \geq \delta \Big| X_1, \ldots, X_n \right) + P \left( \sup_{t \in [\varepsilon, \delta]} |\sqrt{n}C_n(t)| \geq \varepsilon \Big| X_1, \ldots, X_n \right) \leq \frac{1}{3} \varepsilon + 2 \varepsilon < \varepsilon. $$

Consequently, it follows that for $n \geq \max(N_3, N_4)$ that

$$ P^* \left( P \left( |\sqrt{n}C_n(\hat{\mu}_n^*)| \geq \varepsilon \Big| X_1, \ldots, X_n \right) \geq \varepsilon \right) \leq P^* (K_n^{(2)}) + P^* (K_n^{(3)}) + P^* \left( (K_n^{(2)})^c \cap (K_n^{(3)})^c \cap \left\{ P \left( |\sqrt{n}C_n(\hat{\mu}_n^*)| \geq \varepsilon \Big| X_1, \ldots, X_n \right) \geq \varepsilon \right\} \right) \leq \frac{1}{3} \varepsilon + 2 \varepsilon + 0 = \varepsilon, $$

which finishes the proof by Lemma B.2.

To prove that the terms in $[\text{26}]$ involving the function $A$ tend to zero, we show that $\sqrt{n}\hat{\mu}_n^*$ is conditionally tight given $X_1, \ldots, X_n$. To this end, we required in our outline of proof that the Euclidean bootstrap sample mean $\sqrt{n}\bar{X}_n^*$ be conditionally tight. For convenience we give the following lemma, which we believe is part from mathematical folklore.

**Lemma B.5.** For any $\varepsilon > 0$ there exist $M > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$,

$$ P^* \left( P \left( |\sqrt{n}X_n^*| \geq M \Big| X_1, \ldots, X_n \right) \geq \varepsilon \right) \leq \varepsilon. $$

**Proof.** Indeed, by tightness of $\sqrt{n}\bar{X}_n$ for any $\varepsilon > 0$ there exists some $M > 0$ and $N_1 \in \mathbb{N}$ such that $P(|\sqrt{n}\bar{X}_n| \geq M) \leq \varepsilon/4$ for all $n \geq N_1$. For this $M$ define the set $K_n^{(1)} := \{ |\sqrt{n}\bar{X}_n| \geq M \}$ and the function $h(x) := \min(\max(|x - M, 0), 1)$ which is 1-Lipschitz and bounded for all $x \in \mathbb{R}$ through

$$ 1(|x| \geq M + 1) \leq h(x) \leq 1(|x| \geq M), $$

thus $h \in \text{BL}_1(\mathbb{R})$. Hence, by $[\text{20}]$ we assert for $n \geq N_1$ that $\mathbb{E}[h(\sqrt{n}\bar{X}_n)] \leq P(|\sqrt{n}\bar{X}_n| \geq M) \leq \varepsilon/4$. By consistency of the bootstrap for the Euclidean sample mean $\text{van der Vaart} (2000)$ Theorem 23.4 there exists some $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$ holds

$$ P^* \left( \mathbb{E} \left[ h \left( \sqrt{n}(X_n^* - \bar{X}_n) \right) \right] \geq \frac{\varepsilon}{4} \right) \leq \frac{\varepsilon}{4}. $$
In particular, for \( n \geq \max(N_1, N_2) \) we thus obtain conditioned on \((K_n^{(2)})^c\) that
\[
\mathbb{P}^* \left( |\sqrt{n}(\bar{X}_n - \bar{X})| \geq M + 1 \bigg| X_1, \ldots, X_n \right) \leq \mathbb{E} \left[ h \left( \sqrt{n}(\bar{X}_n - \bar{X}) \right) \bigg| X_1, \ldots, X_n \right] \\
\leq \mathbb{E} \left[ h \left( \sqrt{n}\bar{X}_n \right) \right] + \frac{\varepsilon}{4} \leq \mathbb{P}(|\sqrt{n}\bar{X}_n| \geq M) + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2}.
\]
Consequently, for \( n \geq \max(N_1, N_2) \) it follows conditioned on \((K_n^{(1)})^c \cap (K_n^{(2)})^c\) that
\[
\mathbb{P} \left( |\sqrt{n}\bar{X}_n| \geq 2(M + 1) \bigg| X_1, \ldots, X_n \right) \\
\leq \mathbb{P} \left( |\sqrt{n}\bar{X}_n| \geq M + 1 \bigg| X_1, \ldots, X_n \right) + \mathbb{P} \left( |\sqrt{n}(\bar{X}_n - \bar{X})| \geq M + 1 \bigg| X_1, \ldots, X_n \right) \\
\leq 1 \left( |\sqrt{n}\bar{X}_n| \geq M + 1 \right) + \mathbb{P} \left( |\sqrt{n}(\bar{X}_n - \bar{X})| \geq M + 1 \bigg| X_1, \ldots, X_n \right) \\
\leq 0 + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} < \varepsilon,
\]
which yields for \( n \geq \max(N_1, N_2) \) that
\[
\mathbb{P}^* \left( \mathbb{P} \left( |\sqrt{n}\bar{X}_n| \geq 2(M + 1) \bigg| X_1, \ldots, X_n \right) \geq \varepsilon \right) \\
\leq \mathbb{P}^* \left( (K_n^{(1)})^c \right) + \mathbb{P}^* \left( (K_n^{(2)})^c \right) + \mathbb{P}^* \left( (K_n^{(1)})^c \cap (K_n^{(2)})^c \right) + \mathbb{P} \left( |\sqrt{n}\bar{X}_n| \geq 2(M + 1) \bigg| X_1, \ldots, X_n \right) \geq \varepsilon \right) \\
\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 0 + \varepsilon = \varepsilon,
\]
and concludes the proof by Lemma [B.2].

\[ \square \]

**Proposition B.6.** Assume the random element \( X \) fulfills Assumption [3.7]. With the function \( A \) defined in [29] we have
(i) \( \mathbb{E} [H(\sqrt{n}A(\hat{\mu}_n))] \to 0 \) and \( \mathbb{E} [H(\sqrt{n}A(\hat{\mu}_n))] |X_1, \ldots, X_n| \xrightarrow{P^*} 0 \) for \( n \to \infty \).
(ii) \( \sqrt{n}\hat{\mu}_n \) is conditionally tight given \( X_1, \ldots, X_n \), i.e. for any \( \varepsilon > 0 \) there exist \( \varepsilon > 0 \) and \( N \in \mathbb{N} \) such that \( \mathbb{P}^* \left( \mathbb{P} \left( |\sqrt{n}\hat{\mu}_n| \geq M \bigg| X_1, \ldots, X_n \right) \geq \varepsilon \right) \leq \varepsilon \) for all \( n \geq N \),
(iii) and \( \mathbb{E} [H(\sqrt{n}A(\hat{\mu}_n))] |X_1, \ldots, X_n| \xrightarrow{P^*} 0 \) for \( n \to \infty \).

**Proof.** (i) By assumption on the density \( f \) near \( -\pi \) it follows that \( A(x) = o(x) \), i.e. for each \( \gamma > 0 \) there exists a \( \delta > 0 \) such that for all \( |x| \leq \delta \) follows \( |A(x)| \leq \gamma|x| \). Let \( \varepsilon > 0 \) and consider \( M > 0 \) such that for all \( n \in \mathbb{N} \) holds \( \mathbb{P}(|\sqrt{n}\hat{\mu}_n| \geq M) \leq \varepsilon/4 \), by convergence in distribution of \( \sqrt{n}\hat{\mu}_n \) (Theorem [3.3]) such an \( M > 0 \) exists. Moreover, set \( \gamma = \varepsilon/(2M) > 0 \) and define \( N_1 = \left[ M^2/\delta^2(\gamma) \right] \in \mathbb{N} \), then it follows for all \( n \geq N_1 \) in case \( |\sqrt{n}\hat{\mu}_n| < M \) that \( |\hat{\mu}_n| < M/\sqrt{N_1} \leq \delta(\gamma) \) and hence
\[ \sqrt{n}|A(\hat{\mu}_n)| \leq \sqrt{n}\frac{\varepsilon}{2M} |\hat{\mu}_n| \leq \frac{\varepsilon}{2}. \]
Consequently, for \( n \geq N_1 \) it holds by \( H(x) = \min(2, |x|) \) that
\[
\mathbb{E} \left[ H(\sqrt{n}A(\hat{\mu}_n)) \right] = \mathbb{E} \left[ H(\sqrt{n}A(\hat{\mu}_n)) 1(\sqrt{n}\hat{\mu}_n \geq M) \right] + \mathbb{E} \left[ H(\sqrt{n}A(\hat{\mu}_n)) 1(\sqrt{n}\hat{\mu}_n < M) \right] \\
\leq 2\frac{\varepsilon}{4} + \mathbb{E} \left[ H \left( \frac{\varepsilon}{2} \right) 1(\sqrt{n}\hat{\mu}_n < M) \right] \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\]
which implies \( \mathbb{E} \left[ H(\sqrt{n}A(\hat{\mu}_n)) \right] \to 0 \) for \( n \to \infty \). Moreover, conditioning on \( \{\sqrt{n}\hat{\mu}_n < M\} \) asserts for \( n \geq N_1 \) that \( \mathbb{P}(\sqrt{n}A(\hat{\mu}_n)) \geq \varepsilon |X_1, \ldots, X_n| = 0 \), hence
\[
\mathbb{P}^* \left( \mathbb{P} \left( \sqrt{n}A(\hat{\mu}_n) \geq \varepsilon \bigg| X_1, \ldots, X_n \right) \geq \varepsilon \right) \\
\leq \mathbb{P}^* \left( \sqrt{n}A(\hat{\mu}_n) \geq \varepsilon \right) + \mathbb{P}^* \left( \{\sqrt{n}\hat{\mu}_n < M\} \cap \left\{ \mathbb{P} \left( \sqrt{n}A(\hat{\mu}_n) \geq \varepsilon \bigg| X_1, \ldots, X_n \right) \geq \varepsilon \right\} \right) \leq \frac{\varepsilon}{4} + 0 \leq \varepsilon,
\]
and
\[
\mathbb{P}^* \left( \mathbb{P} \left( \sqrt{n}A(\hat{\mu}_n) \geq \varepsilon \bigg| X_1, \ldots, X_n \right) \geq \varepsilon \right) \\
\leq \mathbb{P}^* \left( \sqrt{n}A(\hat{\mu}_n) \geq \varepsilon \right) + \mathbb{P}^* \left( \{\sqrt{n}\hat{\mu}_n < M\} \cap \left\{ \mathbb{P} \left( \sqrt{n}A(\hat{\mu}_n) \geq \varepsilon \bigg| X_1, \ldots, X_n \right) \geq \varepsilon \right\} \right) \leq \frac{\varepsilon}{4} + 0 \leq \varepsilon,
\]
which proves by Lemma B.2 Assertion (i).

(ii) Showing this assertion relies on Relation (13) which states

\[ \bar{X}_n = (1 - 2\pi f(-\pi))\hat{\mu}_n + A(\hat{\mu}_n) + B_n(\hat{\mu}_n) + C_n(\hat{\mu}_n). \]  

(31)

We employ conditional tightness of the quantities \( \bar{X}_n, B_n(\hat{\mu}_n), C_n(\hat{\mu}_n) \) given \( X_1, \ldots, X_n \). In particular, there exists by Lemma B.3 for \( \varepsilon > 0 \) some \( M' > 0 \) and \( N_2 \in \mathbb{N} \) such that for all \( n \geq N_2 \) holds

\[ P^* \left( P \left( \left| \frac{\sqrt{n}}{\pi f} \right| \geq M' \right| X_1, \ldots, X_n \right) \geq \frac{\varepsilon}{5} \right) \leq \frac{\varepsilon}{5}. \]

Additionally, by Proposition B.3 in conjunction with Lemma B.2 there exists some \( N_3 \in \mathbb{N} \) such that for all \( n \geq N_3 \) holds

\[ P^* \left( P \left( \left| \frac{\sqrt{n}}{\pi f} B_n(\hat{\mu}_n) \right| \geq \frac{\varepsilon}{5} \right| X_1, \ldots, X_n \right) \leq \frac{\varepsilon}{5}, \quad P^* \left( P \left( \left| \frac{\sqrt{n}}{\pi f} C_n(\hat{\mu}_n) \right| \geq \frac{\varepsilon}{5} \right| X_1, \ldots, X_n \right) \leq \frac{\varepsilon}{5} \]

Furthermore, consider \( \delta > 0 \) such that for each \( x \in (-\delta, \delta) \) follows \( |A(x)| \leq (1 - 2\pi f(-\pi))x/2 \) and select by Lemma B.3 some \( N_4 \in \mathbb{N} \) large enough such that for \( n \geq N_4 \) holds

\[ P^* \left( P \left( \left| \frac{\sqrt{n}}{\pi f} \right| \geq \delta \right| X_1, \ldots, X_n \right) \geq \frac{\varepsilon}{5} \right) \leq \frac{\varepsilon}{5}. \]

Note, if \( |\hat{\mu}_n| \leq \delta \) it follows that the condition \((1 - 2\pi f(-\pi))|\sqrt{n}\hat{\mu}_n| \geq L\), given some \( L > 0 \), implies

\[ \sqrt{n}|(1 - 2\pi f(-\pi))\hat{\mu}_n + A(\hat{\mu}_n)| \geq \frac{L}{2}. \]

Define \( M := 6 \max(M', \varepsilon/5)/(1 - 2\pi f(-\pi)) \), then it follows conditioned on \( \bigcap_{i=1}^4 (K^{(i)}_n)^c \) for all \( n \geq \max(N_2, N_3, N_1) \) that

\[ P \left( \left| \frac{\sqrt{n}}{\pi f} \right| \geq \frac{\varepsilon}{5} \right| X_1, \ldots, X_n \bigg| X_1, \ldots, X_n \right) = P \left( \left| \frac{\sqrt{n}}{\pi f} \right| \geq \frac{\varepsilon}{5} \right| X_1, \ldots, X_n \right) \]

\[ \leq P \left( \left| \frac{\sqrt{n}}{\pi f} \right| \geq \frac{\varepsilon}{5} \right| X_1, \ldots, X_n \right) + P \left( \left| \frac{\sqrt{n}}{\pi f} \right| \geq \frac{\varepsilon}{5} \right| X_1, \ldots, X_n \right) \]

\[ \leq \frac{4}{5} + \frac{4}{5} \leq \frac{4}{5} \varepsilon < \varepsilon. \]

Consequently, it follows for all \( n \geq \max(N_1, N_2, N_3) \) that

\[ P^* \left( \left| \frac{\sqrt{n}}{\pi f} \right| \geq \frac{\varepsilon}{5} \right| X_1, \ldots, X_n \right) \leq P^* \left( K^{(1)}_n \right) + P^* \left( K^{(2)}_n \right) + P^* \left( K^{(3)}_n \right) + P^* \left( K^{(4)}_n \right) \]

\[ + P^* \left( \bigcap_{i=1}^4 (K^{(i)}_n)^c \right) \leq \frac{4}{5} \varepsilon < \varepsilon, \]

which yields by Lemma B.2 Assertion (ii).
Concluding, we obtain for

\[ \Pi^* \left( \mathbb{P} \left( |\sqrt{n} \mu_n^*| \geq M \right) \right) \geq \frac{\varepsilon}{2} =: K_n^{(5)} \]

Let \( \delta > 0 \) such that for all \( |x| \leq \delta \) holds \( |A(x)| \leq \varepsilon |x|/(2M) \) and choose \( N_0 := |M^2/\delta^2| \). Then it follows for all \( n \geq N_0 \) that the condition \( \sqrt{n} |\mu_n^*| \leq M \) implies \( |\mu_n^*| < M/\sqrt{N_0} \leq \delta \). Hence, under \( \sqrt{n} |\mu_n^*| \leq M \) it holds for \( n \geq N_0 \) that

\[ \sqrt{n} |A(\mu_n^*)| \leq \sqrt{n} \frac{\varepsilon}{2M} |\mu_n^*| \leq \frac{\varepsilon}{2}. \]

Consequently, conditioned on \((K_n^{(5)})^c\) it follows for \( n \geq \max(N_5, N_6) \) that

\[ \Pi^* \left( \mathbb{P} \left( |\sqrt{n} A(\mu_n^*)| \leq \varepsilon \right) \right) \leq \Pi^* \left( \mathbb{P} \left( |\sqrt{n} A(\mu_n^*)| \geq \varepsilon \right) \right) \leq \frac{\varepsilon}{2} + 0 < \varepsilon. \]

Concluding, we obtain for \( n \geq \max(N_4, N_5) \) that

\[ \Pi^* \left( \mathbb{P} \left( \frac{|\sqrt{n} A(\mu_n^*)|}{\varepsilon} \geq \varepsilon \right) \right) \leq \Pi^* \left( K_n^{(5)} \right) + \Pi^* \left( (K_n^{(5)})^c \cap \left\{ \mathbb{P} \left( \frac{|\sqrt{n} A(\mu_n^*)|}{\varepsilon} \geq \varepsilon \right) \geq \varepsilon \right\} \right) \leq \frac{\varepsilon}{2} + 0 \leq \varepsilon, \]

which yields Assertion (iii) by Lemma B.2 and finishes the proof. \( \square \)

**C Proofs for Moment Convergence of Fréchet Sample Means and Bootstrap Version**

**Proof of Proposition 3.6.** Consider the function class \( M := \{m_{\theta} := -d(\cdot, \theta)^2; \theta \in S^1\} \) and define \( M : S^1 \to \mathbb{R}, \theta \mapsto \mathbb{E}[m_{\theta}(X)] \). By reverse triangle inequality it follows for all \( x, \theta, \theta' \in S^1 \)

\[ |m_{\theta}(x) - m_{\theta'}(x)| = | -d(\theta, x)^2 + d(\theta', x)^2 | = |d(\theta, x) + d(\theta', x)||d(\theta, x) - d(\theta', x)| \leq 2\pi d(\theta, \theta') \leq 2\pi |\theta - \theta'| \]

which asserts that \( M \) is continuous. Notably, \( M \) coincides with the negative Fréchet function, hence by assumption \( \mu = 0 \) is the unique maximizer of \( M \). By assumption there exists \( \delta > 0 \) such that \( X \) has a continuous density \( f \) on \( [-\pi, -\pi + \delta] \cup [\pi - \delta, \pi] \) which is bounded by \((1 - \kappa)/(2\pi)\) for some \( \kappa \in (0, 1] \).

By Equation (3) in [Hotz and Huckemann (2015)] it holds for each \( \theta \in [0, \delta] \) that

\[ M(\theta) - M(0) = -\theta^2 + 4\pi \int_{-\pi}^{-\pi+\theta} (-\pi + \theta - x)f(x)dx \leq -\theta^2 + 4\pi \int_{-\pi}^{-\pi+\theta} (-\pi + \theta - x) \left( \frac{1 - \kappa}{2\pi} \right) dx \]

\[ = -\theta^2 + \theta^2 (1 - \kappa) = -\kappa \theta^2 = -\kappa d(0, \theta)^2 = -\kappa \theta^2. \]

Likewise, it holds for each \( \theta \in [-\delta, 0] \) that

\[ M(\theta) - M(0) \leq -\kappa d^2(0, \theta) = -\kappa \theta^2. \]

Since \( \mu = 0 \) is the unique maximizer of \( M \) there exists by continuity of \( M \) a sufficiently small \( \varepsilon > 0 \) such that \( M^{-1}([M(0) - \varepsilon, M(0)]) \subseteq [-\delta, \delta] \). Therefore, selecting \( \kappa' > 0 \) sufficiently small such that \( -\kappa' \theta^2 \geq -\varepsilon \) for all \( \theta \in [-\pi, -\delta] \cup [\delta, \pi] \) implies for all \( \theta \in S^1 \) that

\[ M(\theta) - M(0) \geq -\min(\kappa, \kappa') d(\theta, 0)^2 = -\min(\kappa, \kappa') \theta^2. \]
Moreover, for a sample $X_1, \ldots, X_n \overset{i.i.d.}{\sim} X$ define $M_n: S^1 \to \mathbb{R}, \theta \mapsto \frac{1}{n} \sum_{i=1}^n m_{\theta}(X_i)$. Then it follows by Lipschitz property (32) that the constant function $(2\pi)^p$ is integrable for any $p \geq 1$ with respect to the law of $X$ according to van der Vaart and Wellner [1996, Theorem 2.14.1 and Example 3.2.12] that Assumption (2) in Nishiyama [2010] is fulfilled for $\phi(\gamma) = \gamma$. The first assertion follows at once by Nishiyama [2010] since the Fréchet sample mean $\hat{\mu}_n$ is a maximizer of $M_n$ and therefore satisfies $M_n(\hat{\mu}_n) \geq M_n(0) \geq M_n(0) - 1/n$. The second assertion follows by $d(x,0) = |x|$ for each $x \in S^1$, Theorem 3.2(i) and (Billingsley, 1995, Corollary to Theorem 25.12).

Proof of Proposition 3.7. Recall that the function class $\mathcal{M} = \{m_\theta : \theta \in S^1\}$ from the proof of Proposition 3.6 satisfies by (32) that $|m_\theta(x) - m_{\theta'}(x)| \leq 2\pi d(\theta, \theta') \leq 2\pi |\theta - \theta'|$ for all $\theta, \theta' \in S^1$, where $(2\pi)^p$ is integrable with respect to the law of $X$ for any $p \geq 1$. Hence, the first assertion follows at once from $d(x,y) \leq |x-y|$ for all $x,y \in S^1$ and (Kato, 2011, Theorem 2.2 and Remark 2.3).

For the second assertion we show for $p \geq 1$ that

$$\mathbb{E} \left[ \left| \sqrt{n}(\hat{\mu}_n - \mu_0) \right|^p |X_1, \ldots, X_n \right] = O_P(1).$$

Let $\epsilon > 0$, then it follows for $M > 0$ by Markov’s inequality that

$$\mathbb{P} \left( \mathbb{E} \left[ \left| \sqrt{n}(\hat{\mu}_n - \mu_0) \right|^p |X_1, \ldots, X_n \right] \geq M \right) \leq \frac{\mathbb{E} \left[ \left| \sqrt{n}(\hat{\mu}_n - \mu_0) \right|^p |X_1, \ldots, X_n \right]}{M} \leq \frac{\sup_{n \in \mathbb{N}} \mathbb{E} \left[ \left| \sqrt{n}(\hat{\mu}_n - \mu_0) \right|^p \right]}{M},$$

where the upper bound is finite by the first part. Hence, $M$ can be chosen large enough such that the upper bound is smaller than $\epsilon$, thus giving (34). The assertion then follows by (Kato, 2011, Lemma 2.1) since $p \geq 1$ is arbitrary and $\sqrt{n}(\hat{\mu}_n - \mu_0)$ converges weakly conditioned on $X_1, \ldots, X_n$ to the same limit as $\sqrt{n}(\hat{\mu}_n - \mu_0)$ (cf. Theorem 3.4), i.e. the distribution of $Z$. 

\[\square\]