Quantum Bubble Dynamics in 2+1 Dimensional Gravity
I. Geometrodynamic Approach

by

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Abstract: The Dirac quantization of a 2+1 dimensional bubble is performed. The bubble consists of a string forming a boundary between two regions of space-time with distinct geometries. The ADM constraints are solved and the coupling to the string is introduced through the boundary conditions. The wave functional is obtained and the quantum uncertainty in the radius of the ring is calculated; this uncertainty becomes large at the Planck scale.

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1 Introduction

In recent years there has been considerable interest in 2+1 dimensional gravity, in part due to the fact that pure gravity can be exactly quantized in 2+1 dimensions and hence provides some insight into the more difficult problem of quantizing gravity in the physically realistic 3+1 dimensional case. The quantization can be performed in the first order Witten-Ashtekar formalism \[1, 2, 3\], as well as in the ADM or geometrodynamic formalism \[4\]; but coupling to matter complicates the problem \[5\]. The special case of point particles has been considered in some detail \[6\]. Here we consider another solvable model, namely a bubble consisting of a string forming the boundary between between an outer flat spacetime and an inner region with non-vanishing cosmological constant. It is possible to quantize this model non-perturbatively in both the ADM and first-order formalisms. In this paper we approach the quantization in a manner which parallels the ADM treatment of the ”bubble nucleation problem” in 3+1 dimensions \[7\]. The restriction to spherical symmetry in the latter makes this problem tractable. The analogue in 2+1 dimensions is to impose circular symmetry so that the string boundary between the inner and outer geometries is circular. A somewhat different but related problem, with no distortion of the spacetime metric due to the presence of the string, has also been treated classically; the motion of a circular string in fixed backgrounds of various types has been analyzed using the Einstein equations \[8\]. However, in our model, the string actively distorts the spacetime geometry by modifying the metric in the two regions, but with the components of the metric tensor continuous across the boundary. These distortions are determined by gauge fixing using the boundary conditions; thus the coupling with the string is introduced.
In the next section, a canonical form of the action is obtained and the ADM constraints and boundary conditions are derived. These constraints are solved for the canonical momenta in terms of the configuration space variables and we then proceed, in section 3, to Dirac quantization, ignoring factor-ordering problems. The general wave functional involves the gauge as well as the dynamical modes. After gauge fixing, the wave functional reduces to a wave function of the string radius, the only dynamical mode remaining. The behaviour of the quantized system is analyzed in section 4. Barrier penetration is observed in some cases and quantum uncertainty in the radius is calculated. As expected, these quantum effects are large for a string of radius of order of the Planck length. The results and possible extensions of the theory are discussed in section 5. In a following paper, the classical and quantum analysis of the first-order form of the model is developed, including the extension to the case where the string has non-radial modes.

2 Solution of the Constraints

We consider a string in a 2+1 dimensional Lorentzian spacetime $M^3$ with metric $g_{ab}$. We assume that the string is circular with variable radius $\hat{r}$ and string tension $\mu$. The string divides $M^3$ into an inner region $r < \hat{r}$ with cosmological constant $\lambda \neq 0$ and an outer region $r > \hat{r}$ with zero cosmological constant. The action functional for this system is:

$$S[g_{ab}, \hat{r}] = \frac{1}{16\pi G} \int_{M^3} d^3x \sqrt{-g} \left[ R - 2\lambda \theta(\hat{r} - r) \right] - \frac{\mu}{4\pi} \int_B d^2y \sqrt{-h},$$

(1)

where $G$ is the gravitational coupling constant, $g = \det g_{ab}$, $R$ is the Ricci scalar, $\theta(x)$ is the usual unit step function, $B$ is the trajectory of the string-and
hence here is topologically a cylinder- and finally $h_{ij}, i, j, ... = 0, 1$ is the metric induced on $B$ by $g_{ab}$. The $x^a$ are coordinates on $M^3$ and the $y^i$ are coordinates on $B$. It is convenient to choose polar coordinates $x^0 = t, x^1 = r, x^2 = \phi$, so that $B$ is given by $r = \hat{r}$ and $y^0 = t, y^1 = \phi$. In this case the ADM line-element can be written:

$$ds^2 = -N^2 dt^2 + L^2 (dr + M dt)^2 + R^2 d\phi^2,$$

where $N(t, r)$ is the lapse function, $M(t, r)$ is the radial component of the shift vector and the spatial metric is of the form diagonal $(L^2(t, r), R^2(t, r))$. Using the standard technique, the action can be written in $(2+1)$ form as:

$$S = \int dt \int dr L(L, R, \dot{\hat{r}}, \dot{\hat{r}}, L', R'),$$

$$\mathcal{L} := -N \left[ \frac{1}{4G} \left( \frac{R'}{L} \right)' - 4G \Pi R \Pi L + \frac{\lambda}{4G} \theta (\hat{r} - r) LR + \delta (\hat{r} - r) \frac{\hat{E}}{2L} \right]$$

$$- M \left[ -L \Pi' L + R \Pi' R + \delta (\hat{r} - r) \hat{p} \right] + \Pi L \dot{L} + \Pi R \dot{R} + \hat{p} \delta (\hat{r} - r),$$

where $L' = \partial L/\partial r$, $\dot{L} = \partial L/\partial t, \Pi L = \partial \mathcal{L}/\partial \dot{L}$, etc.; $\hat{p} = \partial \mathcal{L}/\partial \dot{\hat{r}}, \hat{E} = (4\hat{p}^2 + \mu^2 \hat{r}^2 \hat{L}^2)^{1/2}; \hat{L} = L(\hat{r})$, etc. In view of the circular symmetry, an integration over $\phi$ has been performed above.

It is easy to read off the fundamental Poisson brackets from the canonical form of the action. As indicated from the notation, $(L, \Pi_L), (R, \Pi_R)$ and $(\hat{r}, \hat{p})$ are canonically conjugate pairs. The lapse and shift, $N$ and $M$, play the role of Lagrange multipliers since $\Pi_N = \Pi_M = 0$, and enforce the constraints:

$$\mathcal{H}_t := \frac{1}{4G} \left( \frac{R'}{L} \right)' - 4G \Pi R \Pi L + \theta (\hat{r} - r) \frac{\lambda}{4G} LR + \delta (\hat{r} - r) \frac{\hat{E}}{2L} \approx 0,$$

$$\mathcal{H}_r := R' \Pi R - L \Pi' L + \delta (\hat{r} - r) \hat{p} \approx 0.$$
The constraints have the same form as in the 3+1 dimensional case, and they can be solved in the same manner \[7\]. Eliminating $\Pi_R$ we obtain from Eq.(\ref{eq:4}) and \ref{eq:5}:

\[ M' = \hat{\rho}\delta(\hat{r} - r), \tag{6} \]

where

\[ M := \begin{cases} 
-(8G)^{-1}(R'/L)^2 + 2G\Pi_L^2, & r > \hat{r} \\
-(8G)^{-1}[(R'/L)^2 + \lambda R^2] + 2G\Pi_L^2, & r < \hat{r} 
\end{cases} \tag{7} \]

and

\[ \hat{\rho} := \frac{\hat{E}}{2L^2} - 4G\frac{\hat{p}\Pi_L}{L}. \tag{8} \]

Eq.(\ref{eq:6}) has the form of Gauss’ law with the source density $\hat{\rho}$ concentrated at $r = \hat{r}$. Therefore, the parameter $M$ can be interpreted as the mass. Since $M' = 0$ for $r \neq \hat{r}$, $M$ is independent of $r$ in both regions. Moreover, $M(0) = 0$ since $R(r)$ is the radial coordinate with $R(0) = 0, R'(0) = 0$ and $\Pi_L = 0$. Therefore, $M = 0, r < \hat{r}$, and $M(\infty) = M = \text{constant}$. Hence Eq.(\ref{eq:7}) can be solved for $\Pi_L$ in the following form

\[ \Pi_L = \begin{cases} 
4(G)^{-1}[(R'/L)^2 + \lambda R^2]^{1/2}, & r < \hat{r} \\
4(G)^{-1}[(R'/L)^2 + 8GM], & r > \hat{r} 
\end{cases} \tag{9} \]

Moreover, Eq.(\ref{eq:8}) gives

\[ \Pi_R = (L/R')(\Pi_L'), \quad r \neq \hat{r}. \tag{10} \]

Eqs. Eq.(\ref{eq:4}) and Eq.(\ref{eq:10}) represent the complete solution of the constraints, in both regions. The coupling to the string is specified by the boundary conditions. As in ref. 7, we require that $R(r)$ and $L(r)$ be continuous, but the derivatives are discontinuous at $r = \hat{r}$. Moreover, $\Pi_L$ and $\Pi_R$ are discontinuous since the string momentum, $\hat{p}$, also contributes to the momentum balance.
These considerations applied to Eq. (4) and Eq. (5) lead to the following boundary conditions

\[ \Delta \tilde{\Pi}_L = -\dot{\hat{p}}/\hat{L} \]  
\[ \Delta \hat{R}' = -2G\hat{E} \]  

where \( \Delta \tilde{\Pi}_L = \lim_{\epsilon \to 0^+} \Pi_L(\hat{r} + \epsilon) - \Pi_L(\hat{r} - \epsilon) \), etc.

The quantities \( L(r) \) and \( R(r) \) represent the gauge degrees of freedom whereas \( \hat{r} \) represents the dynamical degree of freedom. The arbitrariness of \( L(r) \) and \( R(r) \) represents the reparameterization invariance of classical gravity. The gauge fixing to determine the radial parameterization will be considered in section 4.

### 3 Quantization

Having solved the constraints, we can proceed with Dirac quantization. However, we will ignore the quantum ordering problem. The Poisson brackets are now replaced with commutators for quantum operators; e.g., \([L^{op}(r'), \Pi^{op}_L(r)] = i\hbar\delta(r' - r)\). In the coordinate representation, \( \Pi^{op}_L \to (\hbar/i)\delta/\delta L \), where \( \delta f/\delta L \) represents the functional derivative of functional, \( f \). Therefore, the wave function, \( \psi(L, R, \hat{r}) \) is the solution of the following equations

\[ (\hbar/i)\delta\psi/\delta L = \Pi_L\psi; \quad (\hbar/i)\delta\psi/\delta R = \Pi_R\psi \]  

where \( \Pi_L \) and \( \Pi_R \) are given by Eq.(9) and Eq.(10). Let

\[ \psi = \exp(iS/t). \]  

Then Eq.(13) reduce to

\[ \delta S/\delta L = \Pi_L; \quad \delta S/\delta R = \Pi_R. \]
The following solution of the above simultaneous equation was found, which can be verified by a direct substitution:

\[ S = \int_0^\hat{r} dr S_- + \int_{\hat{r}}^{\infty} dr S_+; \] (16)

where

\[ S_\pm := \frac{1}{4G} \left[ Q_\pm + \frac{R'}{2} \ln \frac{Q_\pm - R'}{Q_\pm + R'} \right]_\pm, \] (17)

with

\[ Q_+ := [(R'/L)^2 + 8GM]^{1/2}; \quad Q_- := [(R'/L)^2 + \lambda R'^2]^{1/2}. \] (18)

Here \([+\, -]\) refer respectively to the regions \(r > \hat{r}, [r < \hat{r}]\).

It follows from Eq.(18) that \(S_+\) is always real and \(S_-\) is real if \(\lambda > 0\) (de Sitter space-time). In the latter case, the probability density

\[ P = \psi^* \psi, \] (19)

is a constant (= 1, ) independent of \(L, R\) or \(\hat{r}\). This represents an unbounded motion where the string has the same probability for any radius, \(\hat{r}\). However bound states can exist if \(\lambda < 0\) (anti-de Sitter space time). In such a case one or more classical turning points may exist, whenever

\[ R'(r)^2 = -\lambda [R(r)L(r)]^2. \] (20)

The bound states will be analysed in the next section.

The wave functional \(\psi\) in Eq.(14) is in WKB form. This is due to the operator ordering implied by Eq.(13), which is essentially equivalent to a WKB approximation with arbitrary \(L(r)\) and \(R(r)\). Eq.(14) represents an infinite set of wave functionals for pure gravity. However, if one fixes the gauge to satisfy the boundary condition, which represent the coupling to the string, a particular wave function can be found for the whole system. This will be done in the next section.
Here we will specialize to the case of an anti-de Sitter space-time in the inner region \( r < \hat{r} \). In the absence of the string, the appropriate choice of gauge would be

\[
R(r) = r \\
L(r) = \begin{cases} 
1, & r > \hat{r} \\
(1 - \lambda r^2)^{-1/2}, & r < \hat{r}
\end{cases}
\]

We will consider a two parameter gauge which modifies the above space-time such that the parameters are completely determined by the boundary conditions. An appropriate choice is

\[
R(r) = r[1 - \beta \exp(-\alpha|r - \hat{r}|)] \\
L(r) = \begin{cases} 
1 - \beta \exp[-\alpha(r - \hat{r})], & r > \hat{r} \\
(1 - \beta)(1 - \lambda \gamma^2)^{-1/2} + \beta \exp[\alpha(r - \hat{r})], & r < \hat{r}
\end{cases}
\]

where \( \alpha \) is positive and it represents the range of the coupling to the string, and \( \beta \) represents the strength of that coupling. Note that Eq.(22) satisfies the limiting behaviour, Eq.(21), as \( r \to \infty \) or as \( \beta \to 0 \).

First we determine \( \beta \) using the continuity of the metric at \( r = \hat{r} \). \( R(r) \) is already continuous, and the continuity of \( L(r) \) gives

\[
\beta = [1 - (1 - \lambda \hat{r}^2)^{1/2}][1 - 2(1 - \lambda \hat{r}^2)^{1/2}]^{-1}.
\]

Note that \( \beta \to 0 \) if \(-\lambda \hat{r}^2 \to 0 \) and \( \beta \to 1/2 \) if \(-\lambda \hat{r}^2 \to \infty \). Next, we combine the boundary conditions Eq.(21) and Eq.(22) by eliminating \( \hat{p} \). The result is

\[
(\triangle \hat{R})^2 = 4G^2 \hat{L}^2[4(\triangle \hat{\Pi}_L)^2 + \mu^2 \hat{R}^2].
\]
Substitution of Eq. (24) into Eq. (24) gives an algebraic equation for \( \alpha \) in terms of \( \hat{r} \) and \( \beta(\hat{r}) \). Note that \( \alpha \) and \( \beta \) are completely determined by the string parameters \( \hat{r} \) and \( \mu \). An examination of limiting cases can reveal the main qualitative features of the classical motion implied by Eq. (24).

First consider the case with a small mass and cosmological constant, so that \( 4MG/(R'/L)^2 \ll 1 \) and \( \lambda R^2/(R'/L)^2 \ll 1 \). Note that \( R'/L = 1 \) for a flat space.) In such a case Eq. (24) reduces to a quadratic equation. We are looking for a real positive root which is given by

\[
\alpha = \left[-b - (b^2 - 4ac)^{1/2}\right]/2a, \quad (25)
\]

when \( a = -4[4MG + 2\lambda \hat{r}^2(1-\beta)^2 + G^2 \mu^2 \hat{r}^2] \beta \hat{r}, \quad b = 4[4MG - 2\lambda \hat{r}^2(1-\beta)^2](1-\beta)\beta, \quad \text{and} \quad c = 4G^2 \mu^2(1-\beta)^2 \hat{r} \). If the \( \lambda \hat{r}^2 \) term is small, \( a < 0, \quad b > 0, \quad c > 0 \). Therefore \( \alpha \geq 0 \) in such a case. To take a closer look at the situation, consider the case where the string tension is dominant. Then Eq. (25) reduces to

\[
\alpha \hat{r} = \beta(1-\beta) \approx -2/\lambda \hat{r}^2 >> 1. \quad (26)
\]

That is, the distortion of the space-time produced by the string has a very short range.

According to Eq. (20), there is no classical turning point in the limit considered above. The more interesting case of a bound state occurs if there is a classical turning point at \( r = r_0 \); this is possible if \( r_0 < \hat{r} \). If we further assume that \( r_0 \) is close to \( \hat{r} \), then Eq. (24) has a single root given by

\[
\alpha = \frac{1 + \beta}{\beta} \left[(G^2 \mu^2 - \lambda/4)(1-\beta)^2 \hat{r}^2 + 2GM\right]^{1/2} \quad (27)
\]

\[
r_0 = \left(\sqrt{-\lambda} + G\mu\right)^{-1}. \quad (28)
\]

Again one obtains Eq. (26) if the tension term is dominant in Eq. (28). Therefore, Eq. (26) gives a reasonable estimate of the range of interaction for a small
cosmological constant. (There is also the possibility of multiple classical turning points, with a probability of quantum tunneling between different classical regions. We will not consider such cases here.)

Returning to the quantum behaviour of the system, it follows from Eq.(19) that

\[ P = \exp[-2\text{Im}(S/\hbar)], \]

(29)
gives the quantum probability of classical barrier penetration. As noted earlier, ImS can only result from the term in \( S_- \). Separating the real and imaginary parts in Eq.(16), we obtain for the case of one classical turning point, \( r_0 \).

\[
S_- = \frac{1}{4G} \int_0^{r_0} dr \left[ Q_- - R' \ln \frac{Q_- + R'}{\sqrt{-\lambda R L}} + \frac{i}{4G} \int_{r_0}^\hat{r} dr \left[ Q - R' \cos^{-1} \left( \frac{R'}{\sqrt{-\lambda R L}} \right) \right] \right],
\]

(30)

where \( \hat{Q} := (-\lambda R^2 L^2 - R^2)^{1/2} \). For small \( -\lambda \), Eq.(29) and Eq.(30) give

\[
P(\hat{r}) = \exp[-(1 - \beta)\sqrt{\lambda} r (\hat{r}^2 - r_0^2)/4\ell_0^2],
\]

(31)

where \( \hat{r}^2 = \hbar G \) with \( \ell_0 = \text{Planck length} ) \). Therefore, the barrier is determined by Planck length, which is characteristic of quantum gravity. The resulting uncertainty in the radius of the string, \( \Delta \hat{r} \), is easy to obtain from Eq.(21).

\[
\Delta r \approx \left( \frac{2}{\sqrt{-\lambda}} \right) (\ell_0/\hat{r})^2.
\]

(32)

It follows that \( \Delta r << (\lambda)^{-1/2} \) if \( \hat{r} >> \ell_0 \), but \( \Delta \hat{r} \sim (\lambda)^{-1/2} >> \hat{r} \) if \( \hat{r} \sim \ell_0 \). That is, the radius of a string is completely uncertain in this limit. This is to be expected, since the quantum fluctuation of space-time are expected to be large on a Planck scale.

The form chosen for the radial parameterization, Eq.(22), is quite appropriate for the simple distortion of space-time considered here. The general
solution is valid for other choices of gauge as long as the boundary conditions are satisfied. Of course, the complete gauge fixing also requires the choice of “time slice” to define the dynamics. In the spirit of this work, this can be done at the classical level by solving the classical equation of motion that follow from Eq.(4). However, this is a separate problem which is outside the scope of this work.

5 CONCLUSIONS

We have considered some new features of the problem of a circular string, which form the boundary of a two component space-time. Unlike the previous works [8], we assume an active role for the string in the distortion of background space-time. The model is simple enough so that the constraints can be explicitly solved, and one can proceed with Dirac quantization. The general wave functional, \( \psi (L(r), R(r), \hat{r}) \), is valid for arbitrary gauge choice as long as the boundary conditions that represent the coupling to the string are satisfied. This is illustrated with a two-parameter gauge fixing; it leads to a \( \psi \) which depends only on the true dynamical degree of freedom, \( \hat{r} \). The quantum behaviour of the system is examined in some limiting cases. In particular, the quantum uncertainty in the radius of the ring, \( \Delta \hat{r} \), is calculated in the limit of large tension in the string. If \( \hat{r} \sim \ell_0 \), it is found that \( \Delta \hat{r} >> \hat{r} \), i.e. a very “fuzzy” string. As expected, the quantum fluctuations of space-time are large in this limit, and the notion of a sharp circular string breaks down.

There are several interesting aspects of the problem which require further work. For example, it is possible that more than one classical turning points exist for our choice of gauge, or for other choices. In that case quantum tunneling could occur between different classical regions. In addition, the choice
of “time slice” to study the time evolution of the system should be interesting. As noted earlier, this will require a solution of classical equations of motion. In such a case, it should be possible to obtain a canonical transformation [4] relating the time-dependent results to our time independent results. However, our immediate interest lies in the first order approach [1, 2, 3], where the emphasis is on the topological aspects. This is the subject of the following paper.

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The action and classical equations of motion for pure gravity are independent of $G$. However, in a quantum theory, $S$ should have the dimensions of $[\text{energy}] \cdot [\text{time}]$, if the wave function is to have the form $\psi \simeq \exp(iS/\hbar)$. (See Eq.(I4).) Therefore the gravitational coupling constant $G$ in Eq.(I) should have the dimensions of $[G]/[\text{length}]$ in a 2+1 dimensional theory. The appropriate choice of length scale in this problem is the radius of the string separating the two regions of different curvature in space. This choice is also suggested by the topological invariants associated with the theory.

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