The construction of observable algebra in field algebra of $G$-spin models determined by a normal subgroup

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Abstract: Let $G$ be a finite group and $H$ a normal subgroup. Starting from $G$-spin models, in which a non-Abelian field $\mathcal{F}_H$ w.r.t. $H$ carries an action of the Hopf $C^*$-algebra $D(H;G)$, a subalgebra of the quantum double $D(G)$, the concrete construction of the observable algebra $\mathcal{A}_{(H,G)}$ is given, as $D(H;G)$-invariant subspace. Furthermore, using the iterated twisted tensor product, one can prove that the observable algebra $\mathcal{A}_{(H,G)} = \cdots \rtimes H \rtimes \hat{G} \rtimes H \rtimes \hat{G} \rtimes \cdots$, where $\hat{G}$ denotes the algebra of complex functions on $G$, and $H$ the group algebra.

Keywords: twisted tensor product, field algebra, observable algebra, $C^*$-inductive limit

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1 Introduction

Let $G$ be a finite group with a unit $e$. The $G$-valued spin configuration on the 2-dimensional square lattices is the map $\sigma: \mathbb{Z}^2 \to G$ with Euclidean action functional

$$S(\sigma) = \sum_{(x,y)} f(\sigma_x^{-1}\sigma_y),$$

where the summation runs over the nearest neighbor pairs in $\mathbb{Z}^2$ and $f: G \to \mathbb{R}$ is a function of the positive type. This kind of classical statistical systems or the corresponding quantum field theories are called $G$-spin models, see [5, 9, 16]. Such models provide the simplest examples of lattice field theories exhibiting quantum symmetry. Generally, $G$-spin models with an Abelian group $G$ have a symmetry structure of $G \times \hat{G}$, where $\hat{G}$ is the Pontryagin dual of $G$. If $G$ is non-Abelian, the Pontryagin dual loses its meaning, and one usually considers the quantum double $D(G)$ of $G$, [4, 12]. Here $D(G)$ is defined as the crossed product of $C(G)$, the algebra of complex functions on $G$, and group algebra $\mathbb{C}G$ with respect to the adjoint action of the latter on the former. Then $D(G)$ becomes a Hopf $^*$-algebra of finite dimension [2, 10, 14]. As in the traditional quantum field theory, one can define a field algebra $\mathcal{F}$ associated with this models, which is a $C^*$-algebra generated by $\{\delta_g(x), \rho_h(l) : g \in G, h \in G, x \in \mathbb{Z}, l \in \mathbb{Z}+\frac{1}{2}\}$ subject to some relations [16]. There is a natural action of $D(G)$ on $\mathcal{F}$ so that $\mathcal{F}$ becomes a $D(G)$-module algebra. Under this action on $\mathcal{F}$, the observable

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algebra $A_G$ is obtained. In [14], Nill and Szlachányi pointed out $A_G = \cdots \rtimes G \rtimes \hat{G} \rtimes G \rtimes \hat{G} \rtimes G \rtimes \cdots$, where the crossed product is taken with respect to the natural left action of the latter factor on the former one. In this paper, we extend the result to a general situation.

Assume that $G$ is a finite group and $H$ is a normal subgroup of $G$. In our previous paper [17], we define a Hopf $C^*$-algebra $D(H;G)$, which is only a subalgebra of $D(G)$. Subsequently, we construct an algebra $\mathcal{F}_H$ in the field algebra $\mathcal{F}$ of $G$-spin models, which is a $C^*$-algebra generated by $\{\delta_g(x), p_h(l): g \in G, h \in H; x \in \mathbb{Z}, l \in \mathbb{Z} + \frac{1}{2}\}$, called the field algebra of $G$-spin models determined by $H$. There also exists a natural action of $D(H;G)$ on $\mathcal{F}_H$, such that $\mathcal{F}_H$ is a $D(H;G)$-module algebra whereas $\mathcal{F}$ is not. Then the observable algebra $A_{(H,G)}$, which is the set of fixed points of $\mathcal{F}_H$ under the action of $D(H;G)$ is obtained. In Section 2, we point out the concrete construction of the observable algebra $A_{(H,G)}$.

In Section 3, we identify $H$ with the group algebra $\mathbb{C}H$, and $\hat{G}$ the set of complex functions on $G$. We firstly construct iterated twisted tensor product algebras of three factors, i.e. $H \otimes R_{0,1} \hat{G} \otimes R_{1,2} H$ and $\hat{G} \otimes R_{1,2} H \otimes R_{0,1} \hat{G}$ by means of twisting maps $R_{0,1}$ and $R_{1,2}$, and then by induction build an iterated twisted product of any number of factors

$$A_{n,m} = A_n \otimes R_{n,n+1} A_{n+1} \otimes R_{n+1,n+2} \cdots A_{m-1} \otimes R_{m-1,m} A_m,$$

where $n, m \in \mathbb{Z}$ with $n < m$ and $A_i = \begin{cases} H, & \text{if } i \text{ is even} \\ \hat{G}, & \text{if } i \text{ is odd} \end{cases}$, which is a $C^*$-algebra of finite dimension. Let $n < n'$ and $m' < m$, one can show that $A_{n',m'} \subseteq A_{n,m}$, and then by the $C^*$-inductive limit of $A_{n,m}$, we obtain

$$\mathcal{A} = (C^*) \lim_{n < m} A_{n,m},$$

that is,

$$\mathcal{A} = \cdots H \otimes R_{0,1} \hat{G} \otimes R_{1,2} H \otimes R_{0,1} \hat{G} \otimes R_{1,2} H \otimes R_{0,1} \hat{G} \cdots.$$ 

Finally, we prove that there is a $C^*$-isomorphism between $C^*$-algebras $\mathcal{A}$ and $A_{(H,G)}$.

All the algebras in this paper will be unital associative algebras over the complex field $\mathbb{C}$. The unadorned tensor product $\otimes$ will stand for the usual tensor product over $\mathbb{C}$. For general results on Hopf algebras we refer to the books of Abe [1] and Sweedler [15]. We shall adopt their notations, such as $S$, $\Delta$, $\varepsilon$ for the antipode, the comultiplication and the counit, respectively. Also we shall use Sweedler-type notation

$$\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}.$$ 

2 The structure of the observable algebra with respect to $H$

In this section, suppose that $G$ is a finite group with a normal subgroup $H$, we will give the algebraic generators for the observable algebra with respect to a normal group $H$. First, we recall some definitions in $G$-spin models with respect to $H$ which will be needed in the sequel [17].
Definition 2.1. \textsuperscript{17} \( D(H;G) \) is the crossed product of \( C(H) \) and group algebra \( \mathbb{C}G \), where \( C(H) \) denotes the set of complex functions on \( H \), with respect to the adjoint action of the latter on the former.

Using the linear basis elements (\( h, g \)) of \( D(H;G) \), the structure maps are given by

\[
\begin{align*}
(h_1, g_1)(h_2, g_2) &= \delta_{h_1g_1, h_2} h_1 g_1 g_2, & \text{(multiplication)} \\
\Delta(h, g) &= \sum_{\ell \in H} (\ell, g) \otimes (\ell^{-1} h, g), & \text{(coproduct)} \\
\varepsilon(h, g) &= \delta_{h, e}, & \text{(counit)} \\
S(h, g) &= (g^{-1} h^{-1} g, g^{-1}), & \text{(antipode)} \\
(h, g)^* &= (g^{-1} h, g^{-1}), & \text{(*-operation)},
\end{align*}
\]

where \( \delta_{g,h} = \begin{cases} 1, & \text{if } g = h \\ 0, & \text{if } g \neq h \end{cases} \). In \textsuperscript{17}, we have shown \( D(H;G) \) is a Hopf \( C^* \)-algebra, with a unique element \( z_{(H,G)} = \frac{1}{|G|} \sum_{g \in G} (e, g) \), called an integral, satisfying for any \( a \in D(H;G) \),

\[
az_{(H,G)} a = \varepsilon(a) z_{(H,G)}.
\]

As in the traditional case, one can define the local quantum field algebra as follows.

Definition 2.2. \textsuperscript{17} The local field \( \mathcal{F}_{H,\text{loc}} \) determined by \( H \) is an associative algebra with a unit \( I \) generated by \( \{ \delta_g(x), \rho_h(l) : g \in G, h \in H; x \in \mathbb{Z}, l \in \mathbb{Z} + \frac{1}{2} \} \) subject to

\[
\begin{align*}
\sum_{g \in G} \delta_g(x) &= I = \rho_e(l), \\
\delta_{g_1}(x)\delta_{g_2}(x) &= \delta_{g_1, g_2} \delta_{g_1}(x), \\
\rho_{h_1}(l)\rho_{h_2}(l) &= \rho_{h_1 h_2}(l), \\
\delta_{g_1}(x)\delta_{g_2}(x') &= \delta_{g_2}(x') \delta_{g_1}(x), \\
\rho_h(l)\delta_g(x) &= \begin{cases} \delta_g(x) \rho_h(l), & l < x, \\ \rho_g(x) \delta_h(l), & l > x, \end{cases} \\
\rho_{h_1}(l)\rho_{h_2}(l') &= \begin{cases} \rho_{h_2}(l') \rho_{h_1 h_2}(l), & l > l', \\ \rho_{h_1 h_2 h_1^{-1}}(l') \rho_{h_1}(l), & l < l' \end{cases}
\end{align*}
\]

for \( x, x' \in \mathbb{Z}; l, l' \in \mathbb{Z} + \frac{1}{2} \) and \( h_1, h_2 \in H, g_1, g_2 \in G \).

The *-operation is defined on the generators as \( \delta_g^*(x) = \delta_g(-x) \), \( \rho_h^*(l) = \rho_{h^{-1}}(l) \) and is extended to an involution on \( \mathcal{F}_{H,\text{loc}} \). In this way, \( \mathcal{F}_{H,\text{loc}} \) becomes a unital *-algebra. Using the \( C^* \)-inductive limit, \( \mathcal{F}_{H,\text{loc}} \) can be extended to a \( C^* \)-algebra \( \mathcal{F}_H \), called the field algebra of \( G \)-spin models determined by a normal subgroup \( H \). There is an action \( \gamma \) of \( D(H;G) \) on \( \mathcal{F}_H \) in the following. For \( x \in \mathbb{Z}; l \in \mathbb{Z} + \frac{1}{2} \) and \( h \in H, g \in G \), set

\[
\begin{align*}
(h, g)\delta_f(x) &= \delta_h, \delta_g f(x), & \forall f \in G, \\
(h, g)\rho_l(t) &= \delta_h, \rho_{gt}(l), & \forall t \in H.
\end{align*}
\]

One can check that \( \mathcal{F}_H \) is a \( D(H;G) \)-module algebra \textsuperscript{17}.

Set

\[
\mathcal{A}_{(H,G)} = \{ F \in \mathcal{F}_H : a(F) = \varepsilon(a)(F), \forall a \in D(H;G) \}.
\]
We call it an observable algebra related to $H$ in the field algebra $\mathcal{F}$ of $G$-spin models. Furthermore, one can show that $\mathcal{A}_{(H,G)}$ is a nonzero $C^*$-subalgebra of $\mathcal{F}_H$, and

$$\mathcal{A}_{(H,G)} = \{ F \in \mathcal{F}_H : z_{(H,G)}(F) = F \} \equiv z_{(H,G)}(\mathcal{F}_H).$$

Now, we will discuss the concrete construction of $\mathcal{A}_{(H,G)}$. In order to do this, for $g \in G$, $x \in \mathbb{Z}$, and $l \in \mathbb{Z} + \frac{1}{2}$, put

$$v_g(x) = \sum_{h \in G} \varrho_{gh^{-1}}(x - \frac{1}{2})\delta_h(x)\varrho_{gh^{-1}}(x + \frac{1}{2}),$$

$$w_g(l) = \sum_{h \in G} \delta_h(l - \frac{1}{2})\delta_{gh}(l + \frac{1}{2}).$$

**Theorem 2.1.** The observable algebra $\mathcal{A}_{(H,G)}$ related to $H$ is a unital $C^*$-subalgebra of $\mathcal{F}_H$ generated by

$$\left\{ v_h(x), w_g(l) : h \in H, g \in G, x \in \mathbb{Z}, l \in \mathbb{Z} + \frac{1}{2} \right\}.$$

**Proof.** Since $z_{(G,H)} = z_{(G,H)}^{z_{(G,E)}}$, where $z_{(G,E)}^{z_{(G,E)}}$ is the unique integral of Hopf algebra $D(G, \{e\})$, we have that

$$\gamma_{z_{(G,E)}}(\mathcal{F}_H) = \gamma_{z_{(G,E)}^{z_{(G,E)}}}(\mathcal{F}_H) = \gamma_{z_{(G,E)}}(\mathcal{F}_H).$$

$\gamma_{z_{(G,E)}}$ is the projection to operators with trivial twist ([16]). Hence, $\gamma_{z_{(G,E)}}(\mathcal{F}_H)$ is generated by

$$\{ \delta_g(x), v_h(x) : g \in G, h \in H, x \in \mathbb{Z} \}.$$ Moreover, we can obtain that $\gamma_{z_{(G,E)}}(\mathcal{F}_H)$ is generated by

$$\left\{ w_g(l), v_h(x) : g \in G, h \in H, x \in \mathbb{Z}, l \in \mathbb{Z} + \frac{1}{2} \right\},$$

since

$$\gamma_{z_{(G,E)}}(\delta_g(1)\delta_{g_2(2)} \cdots \delta_{g_n(n)}) = \frac{1}{|G|} w_{g_1^{-1}g_2}(\frac{3}{2})w_{g_2^{-1}g_3}(\frac{5}{2}) \cdots w_{g_{n-1}^{-1}g_n}(n - \frac{1}{2}).$$

Now let us consider $\gamma_{z_{(G,E)}}|_{\mathcal{F}_H}$, the restriction of $\gamma_{z_{(G,E)}}$ on $\mathcal{F}_H$, and $\gamma_{z_{(H,G)}}$ as projections on $\mathcal{F}_H$, we have that

$$\gamma_{z_{(G,E)}}|_{\mathcal{F}_H} \gamma_{z_{(H,G)}} = \gamma_{z_{(H,G)}} \gamma_{z_{(G,E)}}|_{\mathcal{F}_H} = \gamma_{z_{(H,G)}},$$

then

$$\gamma_{z_{(H,G)}} \leq \gamma_{z_{(G,E)}}|_{\mathcal{F}_H},$$

which implies $\gamma_{z_{(H,G)}}(\mathcal{F}_H) \subseteq \gamma_{z_{(G,E)}}(\mathcal{F}_H)$. Again, for $g \in G, h \in H, x \in \mathbb{Z}, l \in \mathbb{Z} + \frac{1}{2}$,

$$\gamma_{z_{(H,G)}}(w_g(l)) = w_g(l), \quad \gamma_{z_{(H,G)}}(v_h(x)) = v_h(x).$$

Hence, $\gamma_{z_{(H,G)}}(\mathcal{F}_H)$ is generated by

$$\left\{ v_h(x), w_g(l) : h \in H, g \in G, x \in \mathbb{Z}, l \in \mathbb{Z} + \frac{1}{2} \right\}.$$
3 The characterization of the observable algebra $A_{(H,G)}$

In this section, we identify $H$ with the group algebra $\mathbb{C}H$, and $\hat{G}$ the set of complex functions on $G$. We will construct iterated twisted tensor product algebras of any number of factors. To do this, let us recall briefly the concept of a twisted tensor product of algebras (3).

**Definition 3.1.** Let $A$ and $B$ be two unital associative algebras. Suppose that $R : \mathcal{B} \otimes A \to A \otimes \mathcal{B}$ is a linear map such that

$$R \circ (id_\mathcal{B} \otimes m_A) = (m_A \otimes id_\mathcal{B}) \circ (id_A \otimes R) \circ (R \otimes id_A),$$

$$R \circ (m_B \otimes id_A) = (id_A \otimes m_B) \circ (R \otimes id_B) \circ (id_B \otimes R),$$

then $m_R = (m_A \otimes m_B) \circ (id_A \otimes R \otimes id_B)$ is an associative product on $A \otimes B$. Here, $m_A$ and $m_B$ denote the multiplication in algebras $A$ and $B$, respectively. In this case, we call $R$ a twisting map, and $(A \otimes \mathcal{B}, m_R)$ a twisted tensor product of $A$ and $\mathcal{B}$, which has $A \otimes \mathcal{B}$ as underlying vector space endowed with the multiplication $m_R$, simply denoted by $A \otimes \mathcal{B}$. Using a Sweedler-type notation, we denote by $R(b \otimes a) = a_R \otimes b_R$ for $a \in A$ and $b \in B$.

Observe that the multiplication $m_R$ in the twisted product $A \otimes \mathcal{B}$ of algebras $A$ and $\mathcal{B}$ can be given in the following form:

$$(a \otimes b)(a' \otimes b') = aa'_{R} \otimes b_{R}b',$$

where, as already mentioned, the Sweedler-type notation for the twisting map $R$ has been used, i.e., $R(b \otimes a) = a_R \otimes b_R$ for $a \in A$ and $b \in B$.

**Example 3.1.** (1) Consider the usual flip $\tau : \mathcal{B} \otimes A \to A \otimes \mathcal{B}$ defined by

$$\tau(b \otimes a) = a \otimes b.$$

It is obvious that $\tau$ satisfies all conditions for the twisting map, and then it give rise to the standard tensor product of algebras $A \otimes \mathcal{B}$.

(2) Suppose that $M$ is a Hopf algebra over $\mathbb{C}$, and $B$ is a (left) $M$-module algebra, that is, $B$ is an algebra which is a left $M$-module such that $m \cdot (ab) = \sum (m_{(1)} \cdot a)(m_{(2)} \cdot b)$ and $m \cdot 1_B = \varepsilon(m)1_B$, for all $a, b \in B, m \in M$.

Let the map $R : M \otimes B \to B \otimes M$ be defined by

$$R(m \otimes b) = \sum (m_{(1)} \cdot b) \otimes m_{(2)}.$$

One can show that $R$ is a twisting map, and then obtain the algebra $B \otimes_R M$, which is $B \otimes M$ as a vector space with multiplication

$$(a \otimes m)(b \otimes n) = \sum (m_{(1)} \cdot b) \otimes m_{(2)}n.$$

From the definition of the smash product, it is easy to see that $B \otimes_R M$ coincides with the ordinary smash product $B\#M$ introduced in [6].
In order to study the construction of iterated twisted tensor products, we consider three twisted tensor products \( A \otimes_{R_1} B, B \otimes_{R_2} C \) and \( A \otimes_{R_3} C \), and the maps

\[
T_1 : C \otimes (A \otimes_{R_1} B) \to (A \otimes_{R_3} B) \otimes C
\]
defined by \( T_1 = (id_A \otimes R_2) \circ (R_3 \otimes id_B) \) and

\[
T_2 : (B \otimes_{R_2} C) \otimes A \to A \otimes (B \otimes_{R_2} C)
\]
defined by \( T_2 = (R_1 \otimes id_C) \circ (id_B \otimes R_3) \) associated to \( R_1, R_2 \) and \( R_3 \). The following lemma states a sufficient and necessary condition ensuring that both \( T_1 \) and \( T_2 \) are twisting maps.

**Lemma 3.1.** The following statements are equivalent:

1. \( T_1 \) is a twisting map.
2. \( T_2 \) is a twisting map.
3. The maps \( R_1, R_2, R_3 \) satisfy the following compatibility condition (called the hexagon equation):

\[
(id_A \otimes R_2) \circ (R_3 \otimes id_B) \circ (id_C \otimes R_1) = (R_1 \otimes id_C) \circ (id_B \otimes R_3) \circ (R_2 \otimes id_A).
\]

Moreover, if all the three conditions are satisfied, then \( A \otimes_{T_2} (B \otimes_{R_2} C) \) and \( (A \otimes_{R_1} B) \otimes_{T_1} C \) are equal. In this case, we will denote this algebra by \( A \otimes_{R_1} B \otimes_{R_2} C \), which is called the iterated twisted tensor product.

Now, we consider the group algebra \( CG \), endowed with a comultiplication \( \triangle(g) = g \otimes g \), a counit \( \epsilon(g) = 1 \), antipode \( S(g) = g^{-1} \) and \( g^* = g^{-1} \) for all \( g \in G \), such that \( CG \) is a \( C^* \)-Hopf algebra. From now on, we use \( G \) for \( CG \). The dual of \( G \) is \( \hat{G} \), with \( \triangle(\delta_g) = \sum_{t \in G} \delta_t \otimes \delta_{t^{-1}g} \), \( \epsilon(\delta_g) = \delta_{g,e} \), \( S(\delta_g) = \delta_{g^{-1}} \) and \( \delta^*_g = \delta_g \) for all \( g \in G \). There is a natural pairing between \( G \) and \( \hat{G} \) given by

\[
\langle, \rangle : G \otimes \hat{G} \to \mathbb{C}, \quad g \otimes \delta_s \mapsto \langle g, \delta_s \rangle \equiv \delta_s(g),
\]

\[
\langle, \rangle : \hat{G} \otimes G \to \mathbb{C}, \quad \delta_s \otimes g \mapsto \langle \delta_s, g \rangle \equiv \delta_s(g).
\]

Associated to this pairing we have the natural action of \( G \) on \( \hat{G} \) and that on \( \hat{G} \) on \( G \) given by the Sweedler’s arrows:

\[
g \to \delta_s : = \sum_{(\delta_s)} \delta_s(1) \langle g, \delta_s(2) \rangle = \delta_{sg^{-1}},
\]

\[
\delta_s \to g : = \sum_{(g)} g(1) \langle \delta_s, g(2) \rangle = g\delta_s(g),
\]

where \( \triangle(\delta_s) = \sum_{(\delta_s)} \delta_s(1) \otimes \delta_s(2) = \sum_{t \in G} \delta_t \otimes \delta_{t^{-1}s} \), and \( \triangle(g) = \sum_{(g)} g(1) \otimes g(2) = g \otimes g \).

For every \( n \in \mathbb{Z} \), take \( A_n : = H \) if \( n \) is even and \( A_n : = \hat{G} \) if \( n \) is odd, and define the maps:

\[
R_{2n,2n+1} : A_{2n+1} \otimes A_{2n} \to A_{2n} \otimes A_{2n+1}, \quad \delta_g \otimes h \mapsto \sum_{(\delta_g)} (\delta_g(1) \to h) \otimes \delta_{g(2)} = h \otimes \delta_{g^{-1}h},
\]

\[
R_{2n-1,2n} : A_{2n} \otimes A_{2n-1} \to A_{2n-1} \otimes A_{2n}, \quad h \otimes \delta_g \mapsto \sum_{(h)} (h(1) \to \delta_g) \otimes h(2) = \delta_{gh^{-1}} \otimes h,
\]

\[
R_{i,j} : A_j \otimes A_i \to A_i \otimes A_j, \quad x_j \otimes x_i \mapsto x_i \otimes x_j, \quad \text{if } j - i \geq 2.
\]
It is clear that all of them are twisting maps.

**Proposition 3.1.** $R_{0,1}, R_{1,2}, R_{0,2}$ and $R_{1,2}, R_{2,3}, R_{1,3}$ are compatible, respectively.

**Proof.** It suffices to show that $R_{0,1}, R_{1,2}, R_{0,2}$ are compatible. To do this, apply the left-hand side of the hexagon equation to a generator $h_2 \otimes \delta_g \otimes h_1$ of $A_2 \otimes A_1 \otimes A_0$, we get

\[
(id_H \otimes R_{1,2}) \circ (R_{0,2} \otimes id_G) \circ (id_H \otimes R_{0,1}) (h_2 \otimes \delta_g \otimes h_1) = (id_H \otimes R_{1,2}) (h_2 \otimes h_1 \otimes \delta_{h_1^{-1}g}) = h_1 \otimes \delta_{h_1^{-1}gh_2^{-1}} \otimes h_2.
\]

On the other hand, for the right hand side we obtain that

\[
(R_{0,1} \otimes id_H) \circ (id_G \otimes R_{0,2}) \circ (R_{1,2} \otimes id_H) (h_2 \otimes \delta_g \otimes h_1) = (R_{0,1} \otimes id_H) (\delta_{gh_2^{-1}} \otimes h_1 \otimes h_2) = h_1 \otimes \delta_{h_1^{-1}gh_2^{-1}} \otimes h_2.
\]

Now, we have shown $R_{0,1}, R_{1,2}$ and $R_{0,2}$ are compatible. Similarly, one can prove $R_{1,2}, R_{2,3}$ and $R_{1,3}$ are compatible. \qed

It follows from Proposition 3.1 and Lemma 3.1 that one can construct the algebras $A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$ and $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$, in which the multiplications can be given, respectively, by the formulas

\[
(h_1 \otimes \delta_{g_1} \otimes f_1)(h_2 \otimes \delta_{g_2} \otimes f_2) = h_1h_2 \otimes \delta_{g_2h_1^{-1}g_1} \delta_{g_2f_1} \otimes f_1f_2,
\]

\[
(\delta_{g_1} \otimes h_1 \otimes \delta_{s_1})(\delta_{g_2} \otimes h_2 \otimes \delta_{s_2}) = \delta_{g_1g_2h_1^{-1}} \otimes h_1h_2 \otimes \delta_{h_2^{-1}s_1}\delta_{s_2}.
\]

Moreover, we can define a map

\[
\theta: A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2 \longrightarrow A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2
\]

\[
\theta(h_1 \otimes \delta_g \otimes h_2) \longrightarrow h_1^{-1} \otimes \delta_{h_1gh_2} \otimes h_2^{-1}.
\]

Then $\theta$ satisfies the following properties: for any $x, y \in A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$,

\[
\theta(\theta(x)) = x, \quad \theta(xy) = \theta(y)\theta(x).
\]

Thus $A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$ is a *-algebra by means of

\[
(h_1 \otimes \delta_g \otimes h_2)^* = \theta(h_1 \otimes \delta_g \otimes h_2)
\]

for $h_1 \otimes \delta_g \otimes h_2 \in A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$.

Similarly, for $\delta_g \otimes h \otimes \delta_s \in A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$, set

\[
(\delta_g \otimes h \otimes \delta_s)^* = \delta_{gh} \otimes h^{-1} \otimes \delta_{hs}.
\]

Then $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$ is also a *-algebra.

Moreover, we have the following proposition.
Proposition 3.2. $A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$ and $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$ are $C^*$-algebras.

Proof. Let $\mathcal{H} = L^2(\hat{G}, h)$ be a Hilbert space with inner product

$$\langle \varphi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\phi(g)},$$

and $\mathcal{H}_{0,2} \doteq \mathcal{H} \otimes \mathcal{H}$.

Consider the map $\pi_{0,2}: A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2 \to \text{End} \mathcal{H}_{0,2}$ be given by

$$(\pi_{0,2}(g^{(0)})\psi)(g_0, g_2) = \psi(g_0 g^{(0)}, g_2)$$

$$(\pi_{0,2}(g^{(2)})\psi)(g_0, g_2) = \psi(g_0, g_2 g^{(2)})$$

$$(\pi_{0,2}(\delta_g)\psi)(g_0, g_2) = \delta_g(g_0^{-1} g_2) \psi(g_0, g_2),$$

where $\delta_g \in A_1$, $g^{(0)} \in A_0$, $g^{(2)} \in A_2$ and $g_0, g_2 \in G$. One can show that $(\pi_{0,2}, \mathcal{H}_{0,2})$ is a faithful $^*$-representation of $A_{0,2}$.

For $h_1 \otimes \delta_g \otimes h_2 \in A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2$, set

$$\|h_1 \otimes \delta_g \otimes h_2\| = \|\pi_{0,2}(h_1 \otimes \delta_g \otimes h_2)\|,$$

then $(A_0 \otimes_{R_{0,1}} A_1 \otimes_{R_{1,2}} A_2, \|\cdot\|)$ is a $C^*$-algebra of finite dimension.

As to $A_{1,3}$, we define faithful $^*$-representation $(\pi_{1,3}, \mathcal{H}_{1,3})$ of $A_{1,3}$, where $\mathcal{H}_{1,3} = \mathcal{H} \otimes \mathcal{H}$ with $\mathcal{H} = L^2(G, \delta_e)$, $\delta_e \in \hat{G}$ being the Haar measure on the group algebra $G$. Hence, $A_1 \otimes_{R_{1,2}} A_2 \otimes_{R_{2,3}} A_3$ is a $C^*$-algebra.

In the following, by induction, we will construct an iterated twisted tensor product of any number of factors.

Proposition 3.3. The three maps $R_{i,j}, R_{j,k}$ and $R_{i,k}$ are compatible for any $i < j < k$.

Proof. Let us distinguish among several cases:

If $j - i \geq 2$ and $k - j \geq 2$, all three maps are just usual flips, and thus $R_{i,j}, R_{j,k}$ and $R_{i,k}$ are compatible.

If $j - i = 1$ and $k - j \geq 2$, then we have that both $R_{i,k}$ and $R_{j,k}$ are usual flips. Hence, $R_{i,j}, R_{j,k}$ and $R_{i,k}$ are compatible. Indeed, for any $x \otimes y \otimes z \in A_k \otimes A_j \otimes A_i$, we have

$$(id_{A_k} \otimes R_{i,k}) \circ (R_{i,k} \otimes id_{A_j}) \circ (id_{A_k} \otimes R_{i,j})(x \otimes y \otimes z)$$

$$= (id_{A_k} \otimes R_{i,k}) \circ (R_{i,k} \otimes id_{A_j})(x \otimes z_{R_{i,j}} \otimes y_{R_{i,j}})$$

$$= (id_{A_k} \otimes R_{i,k})(z_{R_{i,j}} \otimes x \otimes y_{R_{i,j}})$$

$$= z_{R_{i,j}} \otimes y_{R_{i,j}} \otimes x,$$

and

$$(R_{i,j} \otimes id_{A_k}) \circ (id_{A_j} \otimes R_{i,k}) \circ (R_{j,k} \otimes id_{A_i})(x \otimes y \otimes z)$$

$$= (R_{i,j} \otimes id_{A_k}) \circ (id_{A_j} \otimes R_{i,k})(y \otimes x \otimes z)$$

$$= (R_{i,j} \otimes id_{A_k})(y \otimes z \otimes x)$$

$$= z_{R_{i,j}} \otimes y_{R_{i,j}} \otimes x.$$
Moreover, for any three twisting maps, if two of them are usual flips, then these three twisting maps are compatible. This statement implies $R_{i,j}, R_{j,k}$ and $R_{i,k}$ are compatible for $j - i \geq 2$ and $k - j = 1$.

If $j - i = 1$ and $k - j = 1$, then $R_{i,j}, R_{j,k}$ and $R_{i,k}$ are compatible, the proof of which is the same as that of $R_{0,1}, R_{1,2}, R_{0,2}$ and $R_{1,2}, R_{2,3}, R_{1,3}$ are compatible. □

Assume that we have $n$ algebras $A_1, A_2, \ldots, A_n$ with a twisting map $R_{i,j} : A_j \otimes A_i \to A_i \otimes A_j$ for any $i < j$, and such that for every $i < j < k$ the maps $R_{i,j}, R_{j,k}$ and $R_{i,k}$ are compatible. Define now for any $i < n - 1$ the map

$$T^n_{i-1,n} : (A_{n-1} \otimes \cdots \otimes A_i) A_i \to A_i \otimes (A_{n-1} \otimes \cdots \otimes A_n)$$

given by $T^n_{i-1,n} = (R_{i,n-1} \otimes A_n) \circ (A_{n-1} \otimes R_{i,n})$, which are twisting maps for each $i \in \mathbb{Z}$, as the maps $R_{i,n-1}, R_{i,n}$ and $R_{n-1,n}$ are compatible (Lemma 3.1). Moreover, for every $i < j < n - 1$, the compatibility for $R_{i,j}, R_{i,n-1}, R_{j,n-1}$ and $R_{i,j}, R_{i,n}, R_{j,n}$ implies that for $R_{i,j}, T^n_{i-1,n}$ and $T^n_{j-1,n}$. Hence, we can apply the induction hypothesis to the $n - 1$ algebras $A_1, A_2, \ldots, A_{n-2}$ and $A_{n-1} \otimes R_{n-1,n} A_n$, and can construct the twisted product of these $n - 1$ factors, then we obtain the $C^*$-algebra

$$A_1 \otimes R_{1,2} \cdots \otimes R_{n-3,n-2} A_{n-2} \otimes T^n_{i-1,n} A_i = (A_{n-1} \otimes \cdots \otimes A_n).$$

Notice that the maps $R_{n-2,n-1}, R_{n-1,n}$ and $R_{n-2,n-1}$ are compatible, which implies that

$$A_{n-2} \otimes T^n_{i-1,n} (A_{n-1} \otimes R_{n-1,n} A_i) = (A_{n-2} \otimes R_{n-2,n-1} A_{n-1}) \otimes T^n_{i-1,n} A_n.$$ 

Now, we get the $C^*$-algebra

$$A_1 \otimes R_{1,2} \cdots \otimes R_{n-3,n-2} A_{n-2} \otimes R_{n-2,n-1} A_{n-1} \otimes R_{n-1,n} A_n.$$ 

In particular, for any $n, m \in \mathbb{Z}$ with $n < m$, we can define the $C^*$-algebras of finite dimension

$$A_{n,m} : = A_n \otimes R_{n,n+1} A_{n+1} \otimes R_{n+1,n+2} \cdots A_{m-1} \otimes R_{m-1,m} A_m.$$ 

If $n < n'$ and $m' < m$, then one can check that $A_{n',m'} \subseteq A_{n,m}$, with the map $i : A_{n',m'} \to A_{n,m}$ defined by

$$i(x_{n'} \otimes x_{n'+1} \otimes \cdots \otimes x_{m'}) = 1_{A_n} \otimes \cdots \otimes 1_{A_{n'-1}} \otimes x_{n'} \otimes x_{n'+1} \otimes \cdots \otimes x_{m'} \otimes 1_{A_{m'+1}} \otimes \cdots \otimes 1_{A_m}$$

is a $C^*$-algebra homomorphism preserving the norm, where we use $1_{A_i}$ for the identity in $A_i$ for $i \in \mathbb{Z}$.

We write $A$ for the $C^*$-inductive limit of $A_{n,m}$ with $n, m \in \mathbb{Z}$:

$$A : = \bigcup_{n < m} A_{n,m}.$$ 

That is

$$A : = \cdots H \otimes R_{2,-1} \bigoplus H \otimes R_{0,1} \bigoplus H \otimes R_{1,2} \bigoplus H \otimes R_{2,3} \cdots,$$

where the dots include a $C^*$-inductive limit procedure.

The following theorem is the main result of this paper, which gives a characterization of $A_{(H,G)}$, the observable algebra related to a normal subgroup $H$ of $G$. 


Theorem 3.1. The iterated crossed product $\mathcal{A}$ is $C^*$-isomorphic to $\mathcal{A}_{(H,G)}$.

Proof. For a finite interval $\Lambda$, let

$$\mathcal{A}_H(\Lambda) = \langle v_h(x), w_g(l) : h \in H, g \in G, x \in \Lambda \cap \mathbb{Z}, l \in \Lambda \cap (\mathbb{Z} + \frac{1}{2}) \rangle.$$ 

It follows from Theorem 2.1 that

$$\mathcal{A}_{(H,G)} = \bigcup_\Lambda \mathcal{A}_H(\Lambda),$$

where the union is taken over finite intervals $\Lambda$ and the bar denotes uniform closure.

Consider the map

$$\Phi_{0,1} : A_{0,2} = H \otimes_{R_{0,1}} \hat{G} \otimes_{R_{1,2}} H \rightarrow \mathcal{A}_H(\Lambda,1) = \langle v_h(0), w_g(\frac{1}{2}), v_f(1) : h, f \in H, g \in G \rangle$$

defined on a generator $h \otimes \delta_g \otimes f$ of $H \otimes_{R_{0,1}} \hat{G} \otimes_{R_{1,2}} H$ by

$$\Phi_{0,1}(h \otimes \delta_g \otimes f) = v_h(0)w_g(\frac{1}{2})v_f(1)$$

Observe that

$$v_{h_1}(0)w_{g_1}(\frac{1}{2})v_{f_1}(1)v_{h_2}(0)w_{g_2}(\frac{1}{2})v_{f_2}(1) = v_{h_1}(0)w_{g_1}(\frac{1}{2})v_{h_2}(0)v_{f_1}(1)w_{g_2}(\frac{1}{2})v_{f_2}(1)$$

or

$$v_{h_1}(0)v_{h_2}(0)w_{g_1}(\frac{1}{2})v_{f_1}(1)v_{h_2}(0)w_{g_2}(\frac{1}{2})v_{f_2}(1)$$

where we use the commutation relations of the $v, w$ generators

$$v_{h_1}(x)v_{h_1}(x) = v_{h_1}h_2(x),$$

$$w_{g_1}(l)w_{g_2}(l) = \delta_{g_1,g_2}w_{g_1}(l),$$

$$v_h(x)w_g(x + \frac{1}{2}) = w_{gh^{-1}}(x + \frac{1}{2})v_h(x),$$

$$v_h(x)w_g(x - \frac{1}{2}) = w_{gh^{-1}}(x - \frac{1}{2})v_h(x),$$

other pairs of $v$ and/or $w$ fields commute.

On the other hand,

$$(h_1 \otimes \delta_{g_1} \otimes f_1)(h_2 \otimes \delta_{g_2} \otimes f_2) = h_1h_2 \otimes \delta_{g_1}^{-1} \otimes \delta_{g_2}^{-1} \otimes f_1f_2.$$

Thus, $\Phi_{0,1}$ is an algebra homomorphism. And we have that

$$(\Phi_{0,1}(h \otimes \delta_g \otimes f))^* = (v_h(0)w_g(\frac{1}{2})v_f(1))^*$$

or

$$v_{h^{-1}}(1)v_{f^{-1}}(1)w_{g}(\frac{1}{2})$$

or

$$v_{h^{-1}}(1)w_{h_2}(\frac{1}{2})v_{f^{-1}}(1)$$

or

$$\Phi_{0,1}(h^{-1} \otimes \delta_{gf} \otimes f^{-1})$$

or

$$\Phi_{0,1}((h \otimes \delta_g \otimes f)^*)$$

where we use the properties: $w_g(l)$ is a self-adjoint projection in $\mathcal{A}_H(\Lambda)$, and $v_h(x)$ is a unitary element in $\mathcal{A}_H(\Lambda)$ for any $l \in \Lambda \cap (\mathbb{Z} + \frac{1}{2}), x \in \Lambda \cap \mathbb{Z}$. Hence, $\Phi_{0,1}$ is a $C^*$-homomorphism. By Theorem 2.1.7 in [13], we know $\Phi_{0,1}$ is norm-decreasing.
Also, $\Phi_{0,1}$ is bijective, which together with the open mapping theorem yields that $\Phi_{0,1}$ is a $C^*$-isomorphism between $C^*$-algebras $A_{0,2}$ and $A_H(\Lambda_{0,1})$.

By induction, we can build a $C^*$-isomorphism $\Phi_{-n,m}$ between $A_{-2n,2m}$ and $A_H(\Lambda_{-n,m})$, for any $n, m \in \mathbb{Z}$.

Now, because of the last relation we can define a $C^*$-isomorphism

$$\Phi: \bigcup_{n<m} A_{-2n,2m} \rightarrow \bigcup_{n<m} A_H(\Lambda_{-n,m})$$

by $\Phi|_{A_{-2n,2m}} = \Phi_{-n,m}$. Since each $\Phi_{-n,m}$ is an isometry, then $\Phi$ is an isometry, and $\Phi$ can therefore be extended to the map of $\bigcup_{n<m} A_{-2n,2m}$ onto $\bigcup_{n<m} A_H(\Lambda_{-n,m})$ by continuity. Since all the operations in the definition of a $C^*$-algebra are norm continuous, this extended map is an isomorphism. Hence, $\bigcup_{n<m} A_{-2n,2m}$ is $C^*$-isomorphic to $\bigcup_{n<m} A_H(\Lambda_{-n,m})$.

Finally, the uniqueness of the $C^*$-inductive limit (\[\Pi\]) implies that $A = \bigcup_{n<m} A_{-2n,2m}$ and $A_{(H,G)} = \bigcup_{n<m} A_H(\Lambda_{-n,m})$. As a result, $A$ is $C^*$-isomorphic to $A_{(H,G)}$.

\[\square\]

**Remark 3.1.** From Theorem 3.1, one can see that for a normal subgroup $H$ of $G$ the observable algebra related to $H$ in the field algebra $F$ of $G$-spin models $A_{(H,G)}$ can be defined as

$$A_{(H,G)} = \cdots H \otimes_{R_{-2,-1}} \widetilde{G} \otimes_{R_{-1,0}} H \otimes_{R_{0,1}} \widetilde{G} \otimes_{R_{1,2}} H \otimes_{R_{2,3}} \widetilde{G} \cdots.$$ 

In particular, take $G$ as a normal subgroup of $G$, then we have the observable algebra $A_G$ (\[\ref{14}\]) can also be expressed as

$$A_G = \cdots G \otimes_{R_{-2,-1}} \widetilde{G} \otimes_{R_{-1,0}} G \otimes_{R_{0,1}} \widetilde{G} \otimes_{R_{1,2}} G \otimes_{R_{2,3}} \widetilde{G} \cdots.$$ 

What is more, we can get $A_{(H,G)} \subseteq A_G$, from the above expressions of $A_{(H,G)}$ and $A_G$, which is different from $A_G \subseteq A_{(G,H)}$ (\[\ref{S}\]).

**Remark 3.2.** Notice that the linear map $\varphi: G \otimes \widetilde{H} \rightarrow \widetilde{H}$ defined naturally by

$$\varphi_g(\delta_h) = g \rightarrow \delta_h = \sum_{t \in H} \delta_t \delta_{t^{-1}h}(g) = \begin{cases} \delta_{hg^{-1}}, & \text{if } g \in H \\ 0, & \text{if } g \in G/H \end{cases}$$ 

is not a left action of $G$ on $\widetilde{H}$. Thus, $\widetilde{H} \otimes_{R_{1,2}} G$ can not be defined for any subgroup $H$ of $G$, and then

$$\cdots \widetilde{G} \otimes_{R_{-2,-1}} \widetilde{H} \otimes_{R_{-1,0}} G \otimes_{R_{0,1}} \widetilde{H} \otimes_{R_{1,2}} G \otimes_{R_{2,3}} \widetilde{H} \cdots$$

can not be defined naturally. However, the observable algebra $A_{(G,H)}$ in the field algebra $F$ is well defined (\[\ref{S}\]), which can be obtained as the fixed point algebra

$$A_{(G,H)} = \mathcal{F}^{D(G;H)} \equiv \{ F \in \mathcal{F}: a(F) = \varepsilon(a)F, \forall a \in D(G;H) \}.$$ 

Here $D(G;H)$ denotes the crossed product of $C(G)$ and $CG$ with respect to the adjoint to the action of the latter on the former, $\varepsilon$ is the counit of $D(G;H)$, and $\mathcal{F}$ is the field algebra of $G$-spin models.

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