THERMODYNAMIC FORMALISM AND LARGE DEVIATION PRINCIPLE OF MULTIPLICATIVE ISING MODELS

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ABSTRACT. The aim of this study is three-fold. First, we investigate the thermodynamics of the Ising models with respect to 2-multiple Hamiltonians. This extends the previous results of [Chazotte and Redig, Electron. J. Probab., 2014] to $\mathbb{N}^d$. Second, we establish the large deviation principle (LDP) of the average $\frac{1}{N} \sum_{i \in \mathbb{N}} S_{iN}$, where $S_{iN}$ is a 2-multiple sum along a semigroup generated by $k$ numbers which are $k$ co-primes. This extends the previous results [Ban et al. Indag. Math., 2021] to a broad class of the long-range interactions. Finally, the results described above are generalized to the multidimensional lattice $\mathbb{N}^d$, $d \geq 1$.

1. INTRODUCTION

In this article, we investigate the thermodynamic formalism (e.g., Gibbs measures, entropy and pressure functions) and the large deviation principle of the multiplicative Ising models on $\mathbb{N}^d$, $d \geq 1$. Before presenting the main results, we provide the motives behind this study. Consider the lattice spin systems with Ising $\pm 1$ spins on $\mathbb{N}$. The Hamiltonian of the standard Ising model on each configuration $\sigma \in \{-1, +1\}^\mathbb{N}$ is

$$H(\sigma) = -\beta \left( \sum_{i \in \mathbb{N}} J \sigma_i \sigma_{i+1} + h \sum_{i \in \mathbb{N}} \sigma_i \right),$$

where the parameter $\beta$ is the inverse temperature, $J$ is the coupling strength and $h$ stands for the magnetic field. Note that the Hamiltonian $H(\sigma)$ is a nearest neighbor translation-invariant interaction. The thermodynamics of the Ising models, e.g., the existence of the Gibbs measure, phase transition problem, entropy and pressure functions have been studied in depth. The best reference on this topic is [16].

Motivated from multiple ergodic theory (we refer the reader to [11, 19] for detailed definitions and recent advances in multiple ergodic theory) and the previous works of Kifer [20], Kifer and Varadhan [21], Fan, Liao and Ma [13] on the non-conventional averages, Carinci et al. [7] and Chazottes and Redig [8] investigate the ‘multiplicative Ising models’. That is, they consider the Hamiltonian on each

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configuration $\sigma \in \{-1, +1\}^N$ as

$$H^{[m]}(\sigma) = -\beta \left( \sum_{i \in \mathbb{N}} J_{i} \sigma_{2i} + h \sum_{i \in \mathbb{N}} \sigma_{i} \right).$$

It should be noted that the Hamiltonian $H^{[m]}(\sigma)$ is a long-range non-translation invariant interaction. For $h = 0$, the existence and uniqueness of the Gibbs measure with respect to $H^{[m]}(\sigma)$ is constructed [8], which is multiplication invariant.

**Theorem 1.1** (Theorem 3.2 [8]). Let

$$\mu_N(\sigma_{2N}) = \frac{e^{-H_N(\sigma_{2N})}}{\sum_{\sigma_i = \pm 1, i \in \mathbb{N}_2} e^{-H_N(\sigma_{2N})}}$$

be the finite-volume probability measure corresponding to the Hamiltonian

$$H_N(\sigma_{2N}) = -\beta \sum_{i = 1}^{N} \sigma_i \sigma_{2i}.$$ 

Then

1. **Unique limit measure**: The measure $\mu_N$ have a unique weak limit denoted by $\mu_\infty$ which is the Gibbs measure.
2. **Independent Ising layers**: Under $\mu_\infty$, the $\{\tau_\ell = \sigma_{i_\ell}, \ell = 0, 1\}$ are independent and distributed according to the standard Ising model measure $\mu_{\infty}^{\text{Ising}}$ with a free boundary condition on the left.
3. **Multiplication invariance**: The measure $\mu_\infty$ is multiplication invariant, i.e., for all $i \in \mathbb{N}$, $\sigma = (\sigma_j)_{j=1}^{\infty}$ and $T_i \sigma = (\sigma_{ij})_{j=1}^{\infty}$ have the same distribution.

Carinci et al. [7] established the large deviation principle of the average $\frac{1}{N} S_N(\sigma) := \frac{1}{N} \sum_{i=1}^{N} \sigma_i \sigma_{2i}$ by calculating the rigorous formula of the free energy function

$$F_r(\beta) = F_{r, \infty}(\beta) = \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_r \left( e^{\beta S_N} \right),$$

where $\mathbb{P}_r$ is the product of Bernoulli measures with the parameter $r$ on two symbols $\{-1, +1\}$.

**Theorem 1.2** (Theorem 4.1 [7]). The explicit expression of the free energy function associated to the multiple sum $S_N$ is

$$F_r(\beta) = \log \left( r(1 - r) \right)^{\frac{1}{2}} \|v^T \cdot e_+\|_+ + G(\beta),$$

where

$$G(\beta) = \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{2^i} \log \left( 1 + \left( \frac{2 \cosh(h)}{\|v^T \cdot e_+\|^2} - 1 \right) \left( \frac{\Lambda_-}{\Lambda_+} \right)^i \right),$$

and $\Lambda_\pm = e^{\beta} \left( \cosh(h) \pm \sqrt{\sinh^2(h) + e^{-4\beta}} \right)$, $v^T = (e^{h/2}, e^{-h/2})$, $h = \frac{1}{2} \log(r/(1-r))$, $e_+ = \frac{w_+}{\|w_+\|}$ with $w_+ = (e^{-\beta}, e^{h+\beta} - \Lambda_+)^T$. 

$\mathbb{N}_{2N} = \{i \in \mathbb{N} : i \leq 2N\}$
Theorem \[1.2\] has been recently generalized to multidimensional lattice \(\mathbb{N}^d, d \geq 1\) \[2\].

**Theorem 1.3** (Theorem 3.2 [2]). For any \(d \geq 1\) and \(p = (p_1, ..., p_d) \in \mathbb{N}^d\), the following statements hold true.

1. The explicit expression of the free energy function associated to the multiple sum is
   \[
   S_{\mathbb{N}^1 \times \cdots \times \mathbb{N}^d}^p = \frac{1}{N_1 \cdots N_d} \sum_{i \in \mathbb{N}^1 \times \cdots \times \mathbb{N}^d} \sigma_1 \sigma_p
   \]
   is
   \[
   F_r(\beta) = \frac{2p_1 \cdots p_d - 1}{2p_1 \cdots p_d} \log(r(1 - r)) + p_1 \cdots p_d - \frac{1}{p_1 \cdots p_d} \log |v^T \cdot e_+|^2 + \log \Lambda_+ + \mathcal{G}(\beta),
   \]
   where \(\Lambda_{\pm}, v^T, h, e_+\) are defined as above and
   \[
   \mathcal{G}(\beta) = \sum_{\ell = 1}^\infty \frac{(p_1 \cdots p_d - 1)^2}{(p_1 \cdots p_d)^{\ell + 1}} \log \left(1 + \left(\frac{2 \cosh(h)}{|v^T \cdot e_+|^2} - 1\right) \left(\frac{\Lambda_-}{\Lambda_+}\right)^\ell\right).
   \]

2. The function \(F_r(\beta)\) is differentiable with respect to \(\beta \in \mathbb{R}\).

3. The multiple average satisfies a LDP with rate function given by
   \[
   I_r(x) = \sup_{\beta \in \mathbb{R}} (\beta x - F_r(\beta)).
   \]
   Furthermore, if \((F_r)'(\eta) = y\), then \(I_r(y) = \eta y - F_r(\eta)\).

The aim of this paper is to study the multiplicative Ising models on \(\mathbb{N}^d\) and the objective is three-fold.

1. The thermodynamics of the Ising model with respect to 2-multiple Hamiltonian: We establish the thermodynamics formalism of the multiplicative Ising models on multidimensional lattice \(\mathbb{N}^d\) with respect to the Hamiltonian \(\sum_{i \in \mathbb{N}} \sigma_i \sigma_{2i}\) (call it 2-multiple Hamiltonian). Theorem 3.3 extends Theorem 1.1 to \(\mathbb{N}^d\) and the formula of the Kolmogorov-Sinai (KS) entropy with respect to the limiting measure constructed in Theorem 3.3 is presented in Theorem 3.9.

2. The large deviation principle of the average \(\frac{1}{N} S_N^G\): Let \(k \geq 1\) and \(p_1, \ldots, p_k\) be co-primes, and \(G = \langle p_1, \ldots, p_k \rangle = \{1 = \ell_1 < \ell_2 < \cdots\}\) be a semigroup generated by \(p_1, \ldots, p_k\). The rigorous formula for the free energy function and the large deviation principle with respect to the average \(S_N^G(\sigma) := \sum_{i=1}^N \sigma_{\ell_{i(1)}}, \sigma_{\ell_{i(1)}+1}\) on \(\mathbb{N}\) is given in Theorem 4.1. Clearly, 2-multiple Hamiltonian is a special case of the \(S_N^G\) with \(G = \langle 2 \rangle\). Therefore, Theorem 4.1 can be seen as a generalization of Theorem 1.2 to a broad class of the long-range non-translation invariant interactions.

3. The \(\mathbb{N}^d\) version of the aforementioned results: The \(\mathbb{N}^d\) version of Theorem 4.1 (or the generalization of Theorem 1.3) are presented in Theorems 4.4 and 4.5. We remark that Theorem 4.4 provides a general formula for \(F_r(\beta)\). Meanwhile, Theorem 4.5 presents an explicit formula for \(F_r(\beta)\) if the sum of \(F_r(\beta)\) is along some specific direction. In this circumstance, we call \(F_r(\beta)\) the directional free energy
function which is defined in Section 4.2. We emphasize that the calculation of the rigorous formula for $F_r(\beta)$ along other directions is extremely difficult since the independent sublattices according to the constrains $G$ are quite hard to analysis. The notation $G$ represents the semigroup generated by the vectors $p_1, ..., p_k \in \mathbb{N}^d$, the formal definition can be found in Section 2. And the generalizations of Theorems 3.3 and 3.9 are given in Theorems 3.6 and 3.11 respectively.

We remark that Peres et al. [22] computed the Hausdorff and Minkowski dimensions of the set

$$X_G^G_{\Omega} = \{ (x_k)_{k=1}^{\infty} \in \Sigma_m : x |_{iG} \in \Omega \text{ for all } i, \gcd(i, G) = 1 \},$$

where $\Omega \subseteq \Sigma_m := \{0, \ldots, m-1\}^N$ and $x |_{iG} := (x |_{it_k})_{k=1}^{\infty}$. The work of [22] concerns the decomposition of the lattices $\mathbb{N}^d$ into independent sublattices according to the multiple constraints $G$ and calculates its density among the entire lattice $\mathbb{N}$. The study can be regarded as a study of the thermodynamic formalism of $X_G^G_{\Omega}$ with the potential function $S^G_N$ in $\mathbb{N}^d, d \geq 1$. In addition, the challenge of the current problem is to look at the same issue as [22] in $\mathbb{N}^d, d \geq 1$. The results can be summarized in the following table.

| Thermodynamics and LDP | Limit measure and KS entropy | Formula of the free energy function |
|------------------------|-----------------------------|-----------------------------------|
| 2-multiple, $d = 1$    | $\frac{S}{N_1 \times \cdots \times N_d}, d \geq 1$ | $S^G_{N_1 \times \cdots \times N_d}, d \geq 1$ |
| Theorem 1.1            | Theorem 3.3                  | Theorem 4.4 and 4.5               |
| Theorem 3.4            | Theorem 3.9                  |                                   |
| Theorem 3.3            | Theorem 3.6                  |                                   |
| Theorem 3.4            | Theorem 3.11                 |                                   |

2. Preliminaries

In this section, we provide necessary materials and results on the decomposition of the multidimensional lattice $\mathbb{N}^d$ into independent sublattices and calculate their densities.

Let $p_1, ..., p_k \in \mathbb{N}^d$ with $p_i = (p_{i1}, ..., p_{id})$ for all $1 \leq i \leq k$. Denote by $\mathcal{I}_{p_1, ..., p_k} = \{(i_1, ..., i_d) \in \mathbb{N}^d : p_{i1} \nmid i_1 \text{ or } ... \text{ or } p_{jd} \nmid i_d \text{ for all } 1 \leq j \leq k\}$, $\mathcal{M}_{p_1, ..., p_k} = \{ (p_{i1}^{t_{i1}} \cdots p_{ik}^{t_{ik}}, ..., p_{jd}^{t_{jd}} \cdots p_{kd}^{t_{kd}}) : \ell_j \geq 0 \text{ for all } 1 \leq j \leq k\}$ and

$$\mathcal{M}_{p_1, ..., p_k}(i) = \{(i_1^{t_{i1}} \cdots i_k^{t_{ik}}, ..., i_d^{t_{jd}} \cdots i_d^{t_{kd}}) : \ell_j \geq 0 \text{ for all } 1 \leq j \leq k\}$$

is the lattice $\mathcal{M}_{p_1, ..., p_k}$ that starts at $i$. We also need the following definitions.

**Definition 2.1.** For $N_1, ..., N_d$ and $M_1, ..., M_d \geq 1$, let

1. $S^{(j)} = \langle p_{ij}, ..., p_{kj} \rangle = \{ 1 = \ell_1^{(j)} < \ell_2^{(j)} < \cdots \}$ for all $1 \leq j \leq d$. 

The following lemmas are \( N^d \) version of Lemmas 2.4 and 2.7 in \cite{1}.

**Lemma 2.2.** For \( p_1, \ldots, p_k \in \mathbb{N}^d \) with \( \gcd(p_{is}, p_{js}) = 1 \) for all \( 1 \leq i \neq j \leq k \) and \( 1 \leq s \leq d \),

\[
\mathbb{N}^d = \bigcup_{i \in I_{p_1, \ldots, p_k}} \mathcal{M}_{p_1, \ldots, p_k}(i).
\]

**Proof.** We first claim that for all \( i \neq i' \in I_{p_1, \ldots, p_k} \), \( \mathcal{M}_{p_1, \ldots, p_k}(i) \cap \mathcal{M}_{p_1, \ldots, p_k}(i') = \emptyset \).

If we suppose not, then there exist \( i \neq i' \in I_{p_1, \ldots, p_k} \) such that \( \mathcal{M}_{p_1, \ldots, p_k}(i) \cap \mathcal{M}_{p_1, \ldots, p_k}(i') \neq \emptyset \). Without loss of generality, we may assume \( \ell_i < \ell_i' \).

Then by the \( \ell_i p_{is} - \ell_i' p_{is} = \ell_i' p_{js} - \ell_i p_{js} \) for all \( 1 \leq i \neq j \leq k \) and \( 1 \leq s \leq d \), we have \( p_{is} | \ell_i' \) for all \( 1 \leq s \leq d \). This contradicts with \( i' \in I_{p_1, \ldots, p_k} \).

It remains to show that the equality holds. For \( i \in \mathbb{N}^d \), there exist \( \ell_{js} \geq 0 \) for all \( 1 \leq s \leq d \) and \( 1 \leq j \leq k \) such that

\[
(i_1, \ldots, i_d) = \left( \ell_i \prod_{j=1}^{k} \frac{p_{js} \ell_{js}}{p_{js}} \right) = (\ell_i' \prod_{j=1}^{k} \frac{p_{js} \ell_{js}}{p_{js}}) = (i_1', \ldots, i_d'),
\]

where \( p_{js} \nmid \ell_{js} \) for all \( 1 \leq j \leq k \) and for all \( 1 \leq s \leq d \). Take \( \ell_j = \min \{ \ell_{js} : 1 \leq s \leq d \} \) for all \( 1 \leq j \leq k \). Then, we have \( (i_1, \ldots, i_d) = \mathcal{M}_{p_1, \ldots, p_k}(\frac{i_1}{\prod_{j=1}^{k} p_{js}}, \ldots, \frac{i_d}{\prod_{j=1}^{k} p_{js}}) \) and \( \mathcal{M}_{p_1, \ldots, p_k}(\frac{i_1'}{\prod_{j=1}^{k} p_{js}}, \ldots, \frac{i_d'}{\prod_{j=1}^{k} p_{js}}) \in I_{p_1, \ldots, p_k} \). The converse is then clear. \( \Box \)

**Lemma 2.3.** For \( M_i, N_i \geq 1 \) for all \( 1 \leq i \leq d \), we have the following assertions.

1. \( |\mathcal{J}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}| = \prod_{i=1}^{d} \left( \left( \frac{N_i}{\ell_i M_i} \right) - \left( \frac{N_i}{\ell_i M_i + 1} \right) \right) \).

2. \( \lim_{N_1, \ldots, N_d \to \infty} \frac{|\mathcal{K}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}|}{|\mathcal{J}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}|} = \prod_{i=1}^{k} \left( 1 - \frac{1}{p_{1i} p_{2i} \cdots p_{di}} \right) \).

3. \( \lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \sum_{M_1=1}^{N_1} \ldots \sum_{M_d=1}^{N_d} \frac{1}{|\mathcal{K}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}|} \log a_{M_1, \ldots, M_d} = \sum_{M_1, \ldots, M_d=1}^{\infty} \lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} |\mathcal{K}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}| \log a_{M_1, \ldots, M_d} \).

**Proof.**
1. Since \( i \in J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \) if and only if \( \ell_{M_j}^{(i)}(j) \leq N_j < \ell_{M_j+1}^{(i)}(j) \) for all \( 1 \leq j \leq d \). It follows that \( N_j / \ell_{M_j}^{(i)}(j) < \ell_{M_j+1}^{(i)}(j) \leq N_j / \ell_{M_j}^{(i)}(j+1) \) for all \( 1 \leq j \leq d \). Therefore

\[
|J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}| = \prod_{i=1}^d \left( \frac{N_i}{\ell_{M_i}^{(i)}(1)} - \frac{N_i}{\ell_{M_i}^{(i)}(d+1)} \right).
\]

2. Let the complement of \( T_{P_1, \ldots, P_k} \) be

\[
T_{P_1, \ldots, P_k}^c = \bigcup_{j=1}^k S_j = \bigcup_{j=1}^k \{ p_{js} : p_s \text{ for all } 1 \leq s \leq d \}.
\]

Since \( \gcd(p_{i1}, p_{js}) = 1 \) for all \( 1 \leq i \neq j \leq k \) and \( 1 \leq s \leq d \), we have for any \( 1 \leq \ell \leq k \),

\[
\bigcap_{w=1}^\ell S_{jw} = \left\{ i : \prod_{w=1}^\ell p_{jw} \mid i_s \text{ for all } 1 \leq s \leq d \right\}.
\]

Then the inclusion–exclusion principle infers that

\[
\lim_{N_1, \ldots, N_d \to \infty} \frac{|K_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}|}{|J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}|} = \lim_{N_1, \ldots, N_d \to \infty} 1 - \frac{|J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \cap T_{P_1, \ldots, P_k}|}{|J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}|} = \lim_{N_1, \ldots, N_d \to \infty} 1 - \left( \sum_{n=1}^{k} \frac{|J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \cap S_n|}{|J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}|} \right)
\]

\[
- \sum_{1 \leq n_1 \neq n_2 \leq k} \frac{|J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \cap S_{n_1} \cap S_{n_2}|}{|J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}|} + \sum_{1 \leq n_1 \neq n_2 \neq n_3 \leq k} \frac{|J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \cap S_{n_1} \cap S_{n_2} \cap S_{n_3}|}{|J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}|} - \cdots + (-1)^{k-1} \frac{|J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \cap S_{1} \cap \cdots \cap S_{k}|}{|J_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}|}
\]

\[
= \prod_{i=1}^k \left( 1 - \frac{1}{p_{i1}p_{i2} \cdots p_{id}} \right).
\]

3. Define

\[
\hat{K}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} = \begin{cases} |K_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}|, & \text{if } M_j \leq N_j \text{ for all } 1 \leq j \leq d, \\ 0, & \text{otherwise.} \end{cases}
\]

Then

\[
\lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \sum_{M_1=1}^{N_1} \cdots \sum_{M_j=1}^{N_j} |K_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d}| \log a_{M_1, \ldots, M_d}
\]

\[
= \lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \sum_{M_1, \ldots, M_d=1}^{\infty} \hat{K}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \log a_{M_1, \ldots, M_d}.
\]
We claim that \( \sum_{M_1, \ldots, M_d = 1}^{\infty} \frac{\hat{K}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \log a_{M_1, \ldots, M_d}}{N_1 \cdots N_d} \) converges uniformly in \( N_1, \ldots, N_d \) by Weierstrass M-test with
\[
\left| \frac{\hat{K}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \log a_{M_1, \ldots, M_d}}{N_1 \cdots N_d} \right| \leq \frac{\left| \mathcal{J}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \right| \log a_{M_1, \ldots, M_d}}{N_1 \cdots N_d} \leq \prod_{i=1}^{d} \left( \frac{1}{\ell_{M_i}^i} - \frac{1}{\ell_{M_i+1}^i} \right) \log a_{M_1, \ldots, M_d}
\]
for all \( N_1, \ldots, N_d \geq 1 \) and
\[
\sum_{M_1, \ldots, M_d = 1}^{\infty} \prod_{i=1}^{d} \left( \frac{1}{\ell_{M_i}^i} - \frac{1}{\ell_{M_i+1}^i} \right) \log a_{M_1, \ldots, M_d}
\]
whenever \( a_{M_1, \ldots, M_d} \leq C^{M_1 \cdots M_d} \) for all \( M_1, \ldots, M_d \geq 1 \).

Thus,
\[
\begin{align*}
&\lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \sum_{M_1 = 1}^{N_1} \cdots \sum_{M_d = 1}^{N_d} \left| \hat{K}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \log a_{M_1, \ldots, M_d} \right| \\
&= \lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \sum_{M_1, \ldots, M_d = 1}^{\infty} \hat{K}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \log a_{M_1, \ldots, M_d} \\
&= \sum_{M_1, \ldots, M_d = 1}^{\infty} \lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \hat{K}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \log a_{M_1, \ldots, M_d} \\
&= \sum_{M_1, \ldots, M_d = 1}^{\infty} \lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \left| \hat{K}_{N_1 \times \cdots \times N_d; M_1, \ldots, M_d} \log a_{M_1, \ldots, M_d} \right|
\end{align*}
\]

The proof is complete. \( \square \)

Before state the main results, we need more definitions for semigroup \( \mathbf{G} \) which is generated by vectors \( \mathbf{p}_1, \ldots, \mathbf{p}_k \in \mathbb{N}^d \). For \( d, k \geq 1 \), \( \mathbf{p}_i = (p_{i1}, \ldots, p_{id}) \in \mathbb{N}^d \), \( 1 \leq i \leq k \) and for each \( 1 \leq j \leq d \), \( \gcd(p_{ij}, p_{ij'}) = 1 \) for all \( 1 \leq i \neq i' \leq k \). Let \( \mathbf{G} = \langle \mathbf{p}_1, \ldots, \mathbf{p}_k \rangle = \{ 1 = l_{1}^{(j)} \prec l_{1}^{(j')} \prec l_{2}^{(j)} \prec l_{2}^{(j')} \cdots \} \) be the semigroup generated by \( \mathbf{p}_1, \ldots, \mathbf{p}_k \) with an order \( \prec^{(j)} \) such that \( \mathbf{G} \) follows \( S^{(j)} \) for each \( 1 \leq j \leq d \). In particular, when \( k = 1 \), \( \mathbf{G} = \langle \mathbf{p}_1 \rangle = \{ l_n = \mathbf{p}_1^{n-1} \}_{n=1}^{\infty} \).

### 3. The infinite-volume limit \( \mu_\infty \) on \( \mathbb{N}^d \) and Kolmogorov-Sinai entropy

In this section, we prove the \( \mathbb{N}^d \) version of Theorem 3.1 in [8] for layer stationarity and multiplication invariance (Theorems 3.3 and 3.4), and then obtain the limit measures (Theorems 3.5 and 3.6). We also give the \( \mathbb{N}^d \) version of Lemma 3.4 in [8] (Lemmas 3.8 and 3.10) to obtain the Kolmogorov-Sinai entropies (Theorems 3.9 and 3.11).

\( \mathbf{p}_1^{n-1} = (p_{11}^{n-1}, \ldots, p_{1d}^{n-1}) \)
3.1. Existence and invariance of the limit measure on \( \mathbb{N}^d \).

3.1.1. 2-multiple Hamiltonian. Let \( p_1 = (p_1, \ldots, p_d) \in \mathbb{N}^d \) and \( \tau^i = \{ \tau^i_\ell = \sigma_{p_1^i(\ell)} \}_{\ell=0}^\infty \) for each \( i \in \mathcal{I}_{p_1} \). Then we have following result which is the \( \mathbb{N}^d \) version of Theorem 3.1 in [8].

**Theorem 3.1.** Suppose that the \( \{ \tau^i : i \in \mathcal{I}_{p_1} \} \) form an i.i.d. stationary processes, then the distribution of the corresponding \( \sigma \) is multiplication invariant.

**Proof.** We are going to show that for every finite collection of points \( v_1, \ldots, v_k \in \mathbb{N}^d \), and \( m \in \mathbb{N}^d \) the joint distribution of

\[
(\sigma_{m \cdot v_1}, \ldots, \sigma_{m \cdot v_k})
\]

coincides with that of

\[
(\sigma_{v_1}, \ldots, \sigma_{v_k}).
\]

First, we write \( v_j = i_j \cdot p_1^{\ell_j} \) with \( i_j \in \mathcal{I}_{p_1} \) for all \( 1 \leq j \leq k \) and \( m = i \cdot p_1^\ell \) with \( i \in \mathcal{I}_{p_1} \). Denote \( \ell_j^\prime : p_1^{\ell_j} = i \cdot l_j \) with \( l_j \in \mathcal{I}_{p_1} \) for all \( 1 \leq j \leq k \).

Then by the definition of \( \tau \), we have to prove that the joint distribution of

\[
\tau_{\ell_j + \ell_j^\prime}, \quad j \in \{1, \ldots, k\}
\]

coincides with that of

\[
\tau_{\ell_j^\prime}, \quad j \in \{1, \ldots, k\}.
\]

Denote \( \{i_1, \ldots, i_k\} = \{i_{n_1}, \ldots, i_{n_M}\} \) where \( i_{n_i} \neq i_{n_j} \) for all \( 1 \leq i \neq j \leq M \). For \( 1 \leq w \leq M \), define

\[
X^w = \left( \tau_{\ell_j + \ell_j^\prime}^i : 1 \leq j \leq k, i_j = i_{n_w} \right)
\]

and

\[
Y^w = \left( \tau_{\ell_j}^i : 1 \leq j \leq k, i_j = i_{n_w} \right).
\]

Then, by the independence of the different layers, the joint distribution of \( \tau_{\ell_j + \ell_j^\prime}, j \in \{1, \ldots, k\} \) and \( \tau_{\ell_j^\prime}, j \in \{1, \ldots, k\} \) coincide with the joint distribution of \( \bigotimes_{w=1}^{M} X^w \) and \( \bigotimes_{w=1}^{M} Y^w \) respectively, where \( \bigotimes \) denotes independent joining.

Therefore, it remains to show that for each \( 1 \leq w \leq M \), the distributions of \( X^w \) and \( Y^w \) coincide. Since the layers \( \tau_j^i \) and \( \tau_{\ell_j}^i \) are i.i.d., we have \( X^w \) and \( Y^w \) have the same distribution. The proof is complete. \( \square \)

**Remark 3.2.** We remark that for \( i, i_1 \neq i_2 \in \mathcal{I}_{p_1} \), if \( i \cdot i_1 = i_1 \cdot p_1^{\ell_1} \) and \( i \cdot i_2 = i_2 \cdot p_1^{\ell_2} \) with \( i_1, i_2 \in \mathcal{I}_{p_1} \), then \( i_1 \neq i_2 \). When \( \ell_1 = \ell_2 \), the statement easily true. If we assume \( \ell_1 > \ell_2 \) and \( i_1 = i_2 \), then \( i \cdot i_1 = i_1 \cdot i_2 \cdot p_1^{\ell_1-\ell_2} \). That gives \( i_1 = i_2 \cdot p_1^{\ell_1-\ell_2} \) which contradicts with \( i_1 \in \mathcal{I}_{p_1} \).

As a consequence of Theorem 3.1 with the existence of the infinite-volume limit in each layer of \( \tau \) spins we have the following results.
Theorem 3.3. Let
\[ \mu_{N_1 \times \ldots \times N_d}(\sigma_{N_1 \times \ldots \times N_d}) = \frac{e^{-H_{N_1 \times \ldots \times N_d}(\sigma_{N_1 \times \ldots \times N_d})}}{\sum_{\sigma_{ij}(0) = \pm 1, i \in N_{N_1 \times \ldots \times N_d}} e^{-H_{N_1 \times \ldots \times N_d}(\sigma_{N_1 \times \ldots \times N_d})} } \]
be the finite-volume probability measure corresponding to the Hamiltonian
\[ H_{N_1 \times \ldots \times N_d}(\sigma_{N_1 \times \ldots \times N_d}) = -\beta \sum_{i=1}^{N_1} \cdots \sum_{i_d=1}^{N_d} \sigma_{ij(0)} \sigma_{ij(0)+1}, \]
where \( j(i) \) is the unique number such that \( i = (k_1, \ldots, k_d) \cdot j(i) \) with \( (k_1, \ldots, k_d) \in \mathcal{I}_{p_1}. \)

Then
1. Unique limit measure: The measure \( \mu_{N_1 \times \ldots \times N_d} \) have a unique weak limit (as \( N_1, \ldots, N_d \to \infty \)) denoted by \( \mu_{\infty} \) which is Gibbs.
2. Independent Ising layers: Under \( \mu_{\infty} \), the \( \tau^i, i \in \mathcal{I}_{p_1}, \) are independent and distributed according to the standard Ising model measure \( \mu_{\infty}^{\text{Ising}} \) with a free boundary condition on the left.
3. Multiplication invariance: The measure \( \mu_{\infty} \) is multiplication invariant.

3.1.2. Generalization of 2-multiple Hamiltonian to \( S_{N_1 \times \ldots \times N_d}^G \) on \( \mathbb{N}^d \). Let \( \tau^i = \{ \tau_k^i = \sigma_{ij(i)}(\sigma_{ij(i)})_{k=1}^\infty \} \) for each \( i \in \mathcal{I}_{p_1, \ldots, p_k}. \) For each \( 1 \leq i \leq d, \gcd(p_{j_i}, p_{j'_{k}}) = 1 \) for all \( 1 \leq j \neq j' \leq k. \)

Theorem 3.4. Suppose that the \( \{ \tau^i : i \in \mathcal{I}_{p_1, \ldots, p_k} \} \) form an i.i.d. stationary processes, then the distribution of the corresponding \( \sigma \) is multiplication invariant.

Proof. We have to show that for every finite collection of points \( v_1, \ldots, v_N \in \mathbb{N}^d \) and \( m \in \mathbb{N}^d \), the joint distribution of
\[ (\sigma_{m \cdot v_1}, \ldots, \sigma_{m \cdot v_N}) \]
coincides with that of
\[ (\sigma_{v_1}, \ldots, \sigma_{v_N}). \]
First, we write \( v_j = i_j \cdot p_1^{\ell_{1j}} \cdots p_k^{\ell_{kj}} \) with \( i_j \in \mathcal{I}_{p_1, \ldots, p_k} \) for all \( 1 \leq j \leq N \) and \( m = i \cdot p_1^{m_1} \cdots p_k^{m_k} \) with \( i \in \mathcal{I}_{p_1, \ldots, p_k} \). Note that \( i'_j \cdot p_1^{\ell_{1j}'} \cdots p_k^{\ell_{kj}'} = i \cdot i_j \) with \( i'_j \in \mathcal{I}_{p_1, \ldots, p_k} \) for all \( 1 \leq j \leq k. \) For convenience, we denote \( \ell_j = (\ell_{j1}, \ldots, \ell_{jk}) \), \( \ell'_j = (\ell'_{j1}, \ldots, \ell'_{jk}) \) for all \( 1 \leq j \leq k \) and \( m = (m_1, \ldots, m_k) \).

Then by the definition of \( \tau \), we have to prove that the joint distribution of \( \tau_{\ell_j + m + \ell'_j}, j \in \{1, \ldots, k\} \)
coincides with that of
\[ \tau_{\ell_j'}, j \in \{1, \ldots, k\}. \]
Denote \( \{i_1, \ldots, i_k\} = \{i_{n_1}, \ldots, i_{n_M}\} \) where \( i_{n_i} \neq i_{n_j} \) for all \( 1 \leq i \neq j \leq M. \) For \( 1 \leq w \leq M \), we define
\[ X^w = \left( \tau_{\ell_j + m + \ell'_j} : 1 \leq j \leq k, i_j = i_{n_w} \right) \]
and

\[ Y^w = \left( \tau^i_j : 1 \leq j \leq k, i_j = i_{n_w} \right). \]

Then, by the independence of the different layers, the joint distribution of \( \tau^i_j + \mu_j \), \( j \in \{1, \ldots, k\} \) and \( \tau^i_j, j \in \{1, \ldots, k\} \) coincide with the joint distribution of \( \otimes_{w=1}^M X^w \) and \( \otimes_{w=1}^M Y^w \) respectively.

Therefore, it remains to show that for each \( 1 \leq w \leq M \), the distributions of \( X^w \) and \( Y^w \) coincide. Since the layers \( \tau^i_j \) and \( \tau^i_j \) are i.i.d., we have that \( X^w \) and \( Y^w \) have the same distribution. The proof is complete. \( \square \)

**Remark 3.5.** If \( i, i_1 \neq i_2 \in I_p, \ldots, p_k \) and \( i \cdot i_1 = i_1' \cdot i_1 \cdot i_2 \cdot i_2' = i_2' \cdot i_1' \cdot p_k \) with \( i_1, i_2 \in I_p, \ldots, p_k \), then \( i_1 \neq i_2 \). When \( (\ell'_{i_1}, \ldots, \ell'_{i_k}) = (\ell'_{i_1}, \ldots, \ell'_{i_k}) \), the statement clearly true. If \( (\ell'_{i_1}, \ldots, \ell'_{i_k}) \neq (\ell'_{i_1}, \ldots, \ell'_{i_k}) \) and \( i_1 = i_2 \), without loss of generality, we may assume \( \ell'_{i_1} > \ell'_{i_2} \). This implies \( i \cdot i_1 \cdot c_1 = i \cdot i_2 \cdot p_1 \cdot \ell'_{i_1} \cdot c_2 \) where \( c_1, c_2 \) are the vectors only multipliable by \( p_1 \) for \( 2 \leq i \leq k \). Then the coprime property of \( p_1 \) and \( p_2 \), \( 2 \leq i \leq k \) gives \( p_1 \mid i_1 \) which contradicts \( i_1 \in I_p, \ldots, p_k \).

As a consequence of Theorem 3.1 with the existence of the infinite-volume limit in each layer of \( \tau \) spins we have the following theorem.

**Theorem 3.6.** Let

\[ \mu_{N_1 \times \cdots \times N_d} (i) = e^{-H_{N_1 \times \cdots \times N_d}} e^{-H_{N_1 \times \cdots \times N_d}} \]

be the finite-volume probability measure corresponding to the Hamiltonian

\[ H_{N_1 \times \cdots \times N_d} (i) = -\beta \sum_{i_{k_1}=1}^{N_1} \cdots \sum_{i_{k_d}=1}^{N_d} \sigma(i_{j(1)}^i) \sigma(i_{j(d)}^i), \]

where \( j(1) \) is the unique number such that \( i = (k_1, \ldots, k_d) : i_{j(1)}^{(j)} \) with \( (k_1, \ldots, k_d) \in I_p, \ldots, p_k \). Then

1. **Unique limit measure**: The measure \( \mu_{N_1 \times \cdots \times N_d} \) have a unique weak limit with respect to order \( \prec \) denoted by \( \mu^{(j)} \) which is Gibbs.
2. **Independent Ising layers**: Under \( \mu^{(j)} \), the \( \tau^i, i \in I_p, \ldots, p_k \) are independent and distributed according to the standard Ising model measure \( \mu^{(j)} \) with a free boundary condition on the left.
3. **Multiplication invariance**: The measure \( \mu^{(j)} \) is multiplication invariant.

**Remark 3.7.** We remark that the order \( \prec \) leads \( \mu_{N_1 \times \cdots \times N_d} \) to be Markov on each independent layer \( \tau^i, i \in I_p, \ldots, p_k \).

3.2. **Kolmogorov-Sinai entropy** with respect to the limit measure on \( \mathbb{N}^d \).
3.2.1. 2-multiple Hamiltonian. Recall that $\mathcal{L}_{N_1 \times \ldots \times N_d}(i) = \mathcal{M}_{p_1}(i) \cap \mathcal{N}_{N_1 \times \ldots \times N_d}$ is the subset of $\mathcal{M}_{p_1}(i)$, which belongs to the $\mathcal{N}_{N_1 \times \ldots \times N_d}$ lattice. Then we have results as follows.

**Lemma 3.8.** Let $\phi : \mathbb{N} \to \mathbb{R}$ be a measurable function such that there exist $C' > 0$ and $r > 0$ such that $|\phi(n)| \leq C'n^r$ for all $n \in \mathbb{N}$. Then we have

$$
\lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \sum_{i \in \mathcal{I}_p} \phi(\mathcal{L}_{N_1 \times \ldots \times N_d}(i)) = \sum_{\ell=1}^{\infty} \frac{(p_1 \cdots p_d - 1)^2}{(p_1 \cdots p_d)^{\ell+1}} \phi(\ell).
$$

**Proof.** The proof is a direct consequence of Lemma 2.3. More precisely,

$$
\lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \sum_{i \in \mathcal{I}_p} \phi(\mathcal{L}_{N_1 \times \ldots \times N_d}(i)) = \lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \sum_{i=1}^{K(N_1, \ldots, N_d)} \left(1 - \frac{1}{p_1 \cdots p_d}\right) \left(\frac{N_1 \cdots N_d}{(p_1 \cdots p_d)^{\ell-1}} - \frac{N_1 \cdots N_d}{(p_1 \cdots p_d)^{\ell}}\right) \phi(\ell)
$$

$$
= \sum_{\ell=1}^{\infty} \frac{(p_1 \cdots p_d - 1)^2}{(p_1 \cdots p_d)^{\ell+1}} \phi(\ell),
$$

where the last equality holds by Weierstrass M-test with $|\phi(\ell)| \leq C'\ell^r$ and $K(N_1, \ldots, N_d)$ is the maximum cardinality of $\mathcal{L}_{N_1 \times \ldots \times N_d}(i)$ for all $i \in \mathcal{I}_p$. □

**Theorem 3.9.** The explicit formula for the KS entropy of $\mu_\infty$ is

$$
\sum_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \mathbb{E}_{\mu_\infty} \log \mu_\infty(\sigma_{N_1 \times \ldots \times N_d}) = -\sum_{k=1}^{\infty} \frac{(p_1 \cdots p_d - 1)^2}{(p_1 \cdots p_d)^{k+1}} \mathbb{E}_{\mu_\infty} \log \mu_\infty^{\text{sing}}(\tau_0, \ldots, \tau_{k-1}).
$$

3.2.2. Generalization of 2-multiple Hamiltonian to $S_{N_1 \times \ldots \times N_d}^G$ on $\mathbb{N}^d$. For each $1 \leq j \leq d$ and $i \in \mathcal{I}_{p_1, \ldots, p_d} \cap \mathcal{N}_{N_1 \times \ldots \times N_d}$, let $\mathcal{L}_{N_1 \times \ldots \times N_d}^{(j)}(i)$ be the subset of $\mathcal{M}_{p_1, \ldots, p_d}(i)$ which satisfies $i_j p_1^{t_1} \cdots p_d^{t_d} \leq N_j$. Then the similar result of Lemma 3.8 is obtained.

**Lemma 3.10.** Let $\phi : \mathbb{N} \to \mathbb{R}$ be a measurable function such that there exist $C' > 0$ and $r > 0$ such that $|\phi(n)| \leq C'n^r$ for all $n \in \mathbb{N}$. Then we have

$$
\lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \sum_{i \in \mathcal{I}_{p_1, \ldots, p_d}} \phi(\mathcal{L}_{N_1 \times \ldots \times N_d}^{(j)}(i)) = \prod_{i=1}^{k} \left(1 - \frac{1}{p_1 \cdots p_d}\right) \sum_{M_1, \ldots, M_d=1}^{\infty} \prod_{i=1}^{d} \left(\frac{1}{\ell_i^{(M_i)}} - \frac{1}{\ell_i^{(M_i+1)}}\right) \phi(M_j).
$$

**Proof.** Applying the same argument of the Lemma 3.8 with Lemma 2.3, the proof is complete. □
By Lemma 3.10, we have the following result.

**Theorem 3.11.** For each $1 \leq j \leq d$, the explicit formula for the KS entropy of $\mu^\infty$ is

$$-\lim_{N_1, \ldots, N_d \to \infty} \frac{1}{N_1 \cdots N_d} \mathbb{E}_{\mu^\infty} \log \mu^\infty(\sigma_{N_1, \ldots, N_d})$$

$$= -C \sum_{k=1}^\infty \left( \frac{1}{\ell_k(j)} - \frac{1}{\ell_k(j+1)} \right) \mathbb{E}_{\mu^\text{Ising}} \log \mu^\text{Ising}(\tau_0, \ldots, \tau_k-1),$$

where $C$ is defined in Theorem 4.5.

### 4. Free energy functions and large deviation principle

In this section, we obtain the generalization of Theorem 1.2 (Theorem 4.1), and we consider two types of generalizations of Theorem 1.3 (Theorems 4.4 and 4.5).

#### 4.1. Generalization of 2-multiple Hamiltonian to $S_N^G$ on $\mathbb{N}$.

Let $k \geq 1$, $p_1, p_2, \ldots, p_k$ be co-primes, and $G = \langle p_1, p_2, \ldots, p_k \rangle = \{ 1 = \ell_1 < \ell_2 < \cdots \}$ be a semigroup generated by $p_1, p_2, \ldots, p_k$. Denote $\gamma(G) = \sum_{i=1}^\infty \frac{1}{\ell_i}$. The LDP of $S_N^G = \sum_{i=1}^N \sigma_{j(i)} \sigma_{j(i)+1}$, $j(i)$ is the unique number such that $i = i' \ell_j$ and $p_n \nmid i'$ for all $1 \leq n \leq k$, is presented in the following theorem.

**Theorem 4.1.** The following statements hold true.

1. The explicit expression of the free energy function associated to the multiple sum $S_N^G$ is

$$F_r(\beta) = \frac{1 + \gamma(G)^{-1}}{2} \log \left( r(1 - r) \right) + \gamma(G)^{-1} \log |v^T \cdot e_+|^2 + \log \Lambda_+ + \mathcal{G}(\beta),$$

where

$$\mathcal{G}(\beta) = \gamma(G)^{-1} \sum_{k=1}^\infty \left( \frac{1}{\ell_k} - \frac{1}{\ell_{k+1}} \right) \log \left( 1 + \left( \frac{2 \cosh(h)}{|v^T \cdot e_+|^2} - 1 \right) \left( \frac{\Lambda_-}{\Lambda_+} \right)^k \right).$$

2. The function $F_r(\beta)$ is differentiable with respect to $\beta \in \mathbb{R}$.

3. The multiple average $S_N^G$ satisfies a LDP with rate function given by

$$I_r(x) = \sup_{\beta \in \mathbb{R}} (\beta x - F_r(\beta)).$$

Furthermore, if $(F_r)'(\eta) = y$, then $I_r(y) = \eta y - F_r(\eta)$.

**Proof.**
1. By the $\mathbb{N}^d$ version of Lemma 2.7 in [1] and the similar argument of Theorem 3.2 in [2], we obtain

$$F_r(\beta) = B \sum_{k=1}^{\infty} \left( \frac{1}{\ell_k} - \frac{1}{\ell_{k+1}} \right) \log (r(1-r))^{\frac{k+1}{r}} Z(\beta, h, k+1)$$

$$= B \sum_{k=1}^{\infty} \left( \frac{1}{\ell_k} - \frac{1}{\ell_{k+1}} \right) \log (r(1-r))^{\frac{k+1}{r}}$$

$$+ B \sum_{k=1}^{\infty} \left( \frac{1}{\ell_k} - \frac{1}{\ell_{k+1}} \right) \log \left( |v^T \cdot e_+|^2 \Lambda_k + (2 \cosh(h) - |v^T \cdot e_+|^2) \Lambda_k \right)$$

$$= B \frac{1 + \gamma(G)}{2} \log (r(1-r)) + B \log |v^T \cdot e_+|^2 + B \gamma(G) \log \Lambda_+ + G(\beta)$$

$$= \frac{1 + \gamma(G)}{2} \log (r(1-r)) + \gamma(G)^{-1} \log |v^T \cdot e_+|^2 + \log \Lambda_+ + G(\beta),$$

where the constant $B = \prod_{k=1}^{\infty} \left( 1 - \frac{1}{\ell_k} \right) = \gamma(G)^{-1}$,

$$Z(\beta, h, k+1) = v^T \left[ e^{\beta h} e^{\beta h} e^{\beta h} \right] v$$

and

$$G(\beta) = \gamma(G)^{-1} \sum_{k=1}^{\infty} \left( \frac{1}{\ell_k} - \frac{1}{\ell_{k+1}} \right) \log \left( 1 + \left( \frac{2 \cosh(h)}{|v^T \cdot e_+|^2} - 1 \right) \left( \frac{\Lambda_-}{\Lambda_+} \right)^k \right).$$

2. The proof is similar to the (2) of Theorem 3.2 in [2]. More precisely, we are going to show the sum

$$\sum_{k=1}^{\infty} \left( \frac{1}{\ell_k} - \frac{1}{\ell_{k+1}} \right) \left[ \log \left( 1 + \left( \frac{2 \cosh(h)}{|v^T \cdot e_+|^2} - 1 \right) \left( \frac{\Lambda_-}{\Lambda_+} \right)^k \right) \right]$$

coverge uniformly with respect to $\beta \in \mathbb{R}$, where the notation $'$ stays for the derivative with respect to $\beta$. Then, we apply the same reasoning on

$$\left[ \log \left( 1 + \left( \frac{2 \cosh(h)}{|v^T \cdot e_+|^2} - 1 \right) \left( \frac{\Lambda_-}{\Lambda_+} \right)^k \right) \right]$$

and then apply the Weierstrass M-test, and the proof is complete.

3. The proof is a direct consequence of Theorem 4.1 (2) and Gärtner-Ellis Theorem [2].

\[ \square \]

**Corollary 4.2.**

1. For $k = 1$, $p_1 \geq 1$, $G = \langle p_1 \rangle = \{p_1^{i-1} : i \in \mathbb{N} \}$ and $\gamma(S) = \frac{p_1 - 1}{p_1}$. The explicit expression of the free energy function associated to the multiple sum $S_k^k$ is

$$F_r(\beta) = \frac{2p_1 - 1}{2p_1} \log (r(1-r)) + \frac{2(p_1 - 1)}{p_1} \log |v^T \cdot e_+| + \log \Lambda_+ + G(\beta),$$

where

$$G(\beta) = \frac{p_1 - 1}{p_1} \sum_{k=1}^{\infty} \frac{p_1 - 1}{p_1^k} \log \left( 1 + \left( \frac{2 \cosh(h)}{|v^T \cdot e_+|^2} - 1 \right) \left( \frac{\Lambda_-}{\Lambda_+} \right)^k \right).$$
2. In addition, \( p_1 = 2 \), \( G = \langle 2 \rangle = \{ \ell_i = 2^{i-1} : i \in \mathbb{N} \} \) and \( \gamma(S) = 2 \). The explicit expression of the free energy function associated to the multiple sum \( S^G \) is

\[
F_r(\beta) = \frac{3}{4} \log(r(1 - r)) + \log |v^T \cdot e_+| + \log \Lambda_+ + G(\beta),
\]

where

\[
G(\beta) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2^k} \log \left( 1 + \left( \frac{2 \cosh(h)}{|v^T \cdot e_+|^2} - 1 \right) \left( \frac{\Lambda_-}{\Lambda_+} \right)^k \right).
\]

Which is coincides with the result in Theorem 1.2.

Example 4.3. Figure 1 illustrates the free energy function for different \( r \in (0, 1) \) which is obtained from Theorem 4.1 by truncating the sum to the first 100 terms. The graph is obtained for the case \( d = 1 \) with \( p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11 \), \( G = \langle 2, 3, 5, 7, 11 \rangle \) and \( \gamma(G) = 2 \). The explicit expression of the free energy function associated to the multiple sum \( S^G \) is

\[
F_r(\beta) = \frac{3}{4} \log(r(1 - r)) + \log |v^T \cdot e_+| + \log \Lambda_+ + G(\beta),
\]

where

\[
G(\beta) = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{2^k} \log \left( 1 + \left( \frac{2 \cosh(h)}{|v^T \cdot e_+|^2} - 1 \right) \left( \frac{\Lambda_-}{\Lambda_+} \right)^k \right).
\]

Which is coincides with the result in Theorem 1.2.

4.2. Generalization of 2-multiple Hamiltonian to \( S^G_{N_1 \times \cdots \times N_d} \) on \( \mathbb{N}^d \). Recall that \( S^{(j)} = \langle p_{j1}, \ldots, p_{jk} \rangle = \{ 1 = \ell_{1}^{(j)} < \ell_{2}^{(j)} < \cdots \} \) is the semigroup generated by \( p_{j1}, \ldots, p_{jk} \). And \( G = \langle p_1, \ldots, p_k \rangle = \{ 1 = \ell_{1}^{(j)} \prec \ell_{2}^{(j)} \prec \cdots \} \) be the semigroup generated by \( p_1, \ldots, p_k \) with an order \( \prec^{(j)} \) such that \( G \) follows \( S^{(j)} \) for each \( 1 \leq j \leq d \). The LDP of \( S^G_{N_1 \times \cdots \times N_d} = \sum_{i_1=1}^{N_1} \cdots \sum_{i_d=1}^{N_d} \sigma_{j_{(i)}}^{(1)} \sigma_{j_{(i)}}^{(2)} \cdots \sigma_{j_{(i)}}^{(d)} \) \( \mathbf{j}(i) \) is the unique number such that \( i = (k_1, \ldots, k_d) \cdot \mathbf{j}(i) \) with \( (k_1, \ldots, k_d) \in \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_d} \), is described by the following energy function.

The directional free energy function is defined as following. For \( 1 \leq j \leq d \),

\[
F_r(\beta) := \lim_{N_1, \ldots, N_d \to \infty} \log E_r \left( e^{\beta \sum_{i=1}^{N_1} \cdots \sum_{i_d=1}^{N_d} \sigma_{j_{(i)}}^{(1)} \sigma_{j_{(i)}}^{(2)} \cdots \sigma_{j_{(i)}}^{(d)}} \right).
\]
The following result takes the sum over all elements $i$ if it is less than the maximum element on its layer intersection with the $N_1 \times \cdots \times N_d$ lattice (in the $\prec$ sense). That is, we take the sum over all $i \in M_{p_1, \ldots, p_d}(j(i))$ if $i \prec j(i)$.

**Theorem 4.4.** For $1 \leq j \leq d$, the explicit formula of the directional energy function associated to the multiple sum $S_{N_1, \ldots, N_d}^G$ converges to

$$F_r(\beta) = \prod_{i=1}^k \left(1 - \frac{1}{p_{i1} \cdots p_{id}}\right) \sum_{k_1, \ldots, k_d=1}^d \prod_{i=1}^d \left(\frac{1}{\ell^{(i)}_{k_i}} - \frac{1}{\ell^{(j)}_{k_i} + 1}\right) \log(r(1-r))^{\frac{b_{k_1, \ldots, k_d} + 1}{2}} Z(\beta, h, b_{k_1, \ldots, k_d} + 1),$$

where $b_{k_1, \ldots, k_d}$ is the number of the elements in $G$ less than or equal to the maximum element (with the order $\prec$) in the $\ell^{(1)}_{k_1} \times \cdots \times \ell^{(d)}_{k_d}$ lattice.

**Proof.** The proof is directly by the Lemmas 2.2 and 2.3 with the observation $|b_{k_1, \ldots, k_d}| \leq k_j$.

Due to the explicit expression of Theorem 4.4 is difficult to obtain, we consider the following type directional free energy function which takes the sum over all $i \in I_{p_1, \ldots, p_d} \cap N_{N_1, \ldots, N_d}$ layers with the layer members $i \cdot p_{j_1} \cdots p_{j_k}$ satisfy $i_j p_{j_1} \cdots p_{j_k} \leq N_j$. Then we have following result.

**Theorem 4.5.** For $1 \leq j \leq d$,

1. The explicit expression of the directional energy function associated to the multiple sum $S_{N_1, \ldots, N_d}^G$ is

$$F_r(\beta) = \frac{C + C \gamma(S^{(j)}) \log(r(1-r)) + C \log|v^T \cdot e_+|^2 + C \gamma(S^{(j)}) \log \Lambda_+ + G(\beta)}{2},$$

where $C = \prod_{i=1}^k \left(1 - \frac{1}{p_{i1} \cdots p_{id}}\right) \Pi_{1 \leq i \neq j \leq d} \gamma(S^{(i)})$ and

$$G(\beta) = C \sum_{k=1}^\infty \left(\frac{1}{\ell^{(j)}_{k+1}} - \frac{1}{\ell^{(j)}_{k+1}}\right) \log \left(1 + \frac{2 \cosh(h)}{|v^T \cdot e_+|^2} - 1\right) \left(\frac{\Lambda_+}{\Lambda_+}\right)^k.$$

2. The function $F_r(\beta)$ is differentiable with respect to $\beta \in \mathbb{R}$.

3. The multiple average $S_{N_1, \ldots, N_d}^{G_{N_1, \ldots, N_d}}$ satisfies a LDP with rate function given by

$$I_r(x) = \sup_{\beta \in \mathbb{R}} (\beta x - F_r(\beta)).$$

Furthermore, if $(F_r)'(\eta) = y$, then $I_r(y) = \eta y - F_r(\eta)$.

**Proof.**

1. By Lemmas 2.2 and Lemma 2.3 with the similar process of Theorem 3.2 in [2], we have

$$F_r(\beta) = \lim_{N_1, \ldots, N_d \to \infty} \log \mathbb{E}_r \left(\beta \sum_{j=1}^{N_1} \cdots \sum_{j=1}^{N_d} \sigma_{j(i)} \sigma_{j(j)} \sigma_{j(j+1)} \right)$$

$$= \prod_{i=1}^k \left(1 - \frac{1}{p_{i1} \cdots p_{id}}\right) \sum_{k_1=1}^\infty \cdots \sum_{k_d=1}^d \prod_{i=1}^d \left(\frac{1}{\ell^{(i)}_{k_i}} - \frac{1}{\ell^{(j)}_{k_i} + 1}\right) \log(r(1-r))^{\frac{k_j + 1}{2}} Z(\beta, h, k_j + 1).$$
Then replace $k_j$ by $k$, we obtain

$$F_r(\beta) = C \sum_{k=1}^{\infty} \left( \frac{1}{\ell_k^{(j)}} - \frac{1}{\ell_{k+1}^{(j)}} \right) \log(r(1-r))^{\frac{k+1}{2}} Z(\beta, h, k + 1)$$

$$= \frac{C + C\gamma(S^{(j)})}{2} \log(r(1-r)) + C \log |v^T \cdot e_+|^2 + C\gamma(S^{(j)}) \log A + G(\beta),$$

where $C = \prod_{i=1}^{k} \left( 1 - \frac{1}{p_1 \cdots p_d} \right) \prod_{1 \leq i \neq j \leq d} \gamma(S^{(i)})$ and

$$G(\beta) = C \sum_{k=1}^{\infty} \left( \frac{1}{\ell_k^{(j)}} - \frac{1}{\ell_{k+1}^{(j)}} \right) \log \left( 1 + \left( \frac{2 \cosh(h)}{|v^T \cdot e_+|^2} - 1 \right) \left( \frac{\Lambda}{\Lambda_{+}} \right)^k \right).$$

2. By a similar argument of Theorem 4.1 (2), we have $F_r(\beta)$ is differentiable with respect to $\beta \in \mathbb{R}$.

3. The proof is a direct consequence of Theorem 4.5 (2) and Gärtner-Ellis Theorem [2].

**Corollary 4.6.** For $k = 1, p_1 = (p_1 \cdots p_d, 1, \ldots, 1) \in \mathbb{N}^d$ and $j = 1$, then $C = \left( 1 - \frac{1}{p_1 \cdots p_d} \right), S^{(1)} = (p_1 \cdots p_d) = \{ \ell_n^{(1)} = (p_1 \cdots p_d)^{(n-1)} \}_{n=1}^{\infty}$ and $\gamma(S^{(1)}) = p_1 \cdots p_d / p_1 \cdots p_d - r$.

These give the directional energy function associated to the sum $S_{N_1 \times \cdots \times N_d}$ is

$$F_r(\beta) = \frac{2p_1 \cdots p_d - 1}{2p_1 \cdots p_d} \log(r(1-r)) + \frac{p_1 \cdots p_d - 1}{p_1 \cdots p_d} \log |v^T \cdot e_+|^2 + \log A + G(\beta),$$

where

$$G(\beta) = \sum_{\ell=1}^{\infty} \left( \frac{p_1 \cdots p_d - 1}{(p_1 \cdots p_d)^{\ell+1}} \right)^2 \log \left( 1 + \left( \frac{2 \cosh(h)}{|v^T \cdot e_+|^2} - 1 \right) \left( \frac{\Lambda}{\Lambda_{+}} \right)^\ell \right).$$

Which is coincides with the formula in Theorem 1.3.
Example 4.7. Figure 2 shows the free energy behaviour for the multidimensional case \( d = 2 \) with \( j = 1, p_1 = (2, 3), p_2 = (3, 5), p_3 = (5, 7), p_4 = (7, 11), p_5 = (11, 2), \) \( S^{(1)} = S^{(2)} = (2, 3, 5, 7, 11), \gamma(S^{(1)})^{-1} = \gamma(S^{(2)})^{-1} = \frac{16}{77} \) and \( C = (1 - \frac{1}{7})(1 - \frac{1}{11})(1 - \frac{1}{13})(1 - \frac{1}{17})(1 - \frac{1}{39})(1 - \frac{1}{77}) = \frac{2261}{660}. \)

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