Abstract

It is proved that (a) the solutions of the ideal magnetohydrodynamic equation, which describe the equilibrium states of a cylindrical plasma with purely poloidal flow and arbitrary cross sectional shape [G. N. Throumoulopoulos and G. Pantis, Plasma Phys. and Contr. Fusion 38, 1817 (1996)] are also valid for incompressible equilibrium flows with the axial velocity component being a free surface quantity and (b) for the case of isothermal incompressible equilibria the magnetic surfaces have necessarily circular cross section.
I. Introduction

In a recent paper [1] it is proved that, if the ideal MHD stationary flows of a cylindrical plasma with arbitrary cross sectional shape are purely poloidal, they must be incompressible. This property simplifies considerably the equilibrium problem, i.e. it turns out that the equilibrium is governed by an elliptic partial differential equation for the poloidal magnetic flux function $\psi$ which is amenable to several classes of analytic solutions. For an arbitrary flow, i.e. when the velocity has non vanishing axial and poloidal components, the equilibrium becomes much more complicated. With the adoption of a specific equation of state, e.g. isentropic magnetic surfaces [2], the symmetric equilibrium states in a two dimensional geometry are governed by a partial differential equation for $\psi$, which contains five surface quantities (i.e. quantities solely dependent on $\psi$), in conjunction with a nonlinear algebraic Bernoulli equation. The derivation of analytic solutions of this set of equations is difficult.

In the present note we study the equilibrium of a cylindrical plasma with incompressible flows and show that the incompressibility condition makes it possible to construct analytic equilibria, which constitute a generalization of the ones obtained in Ref. [1]. This is the subject of Sec. II. The special class of incompressible equilibria with isothermal magnetic surfaces is examined in Sec. III. Section IV summarizes our conclusions.

II. Equilibrium equations and analytic solutions

The ideal MHD equilibrium states of plasma flows are governed by the following set of equations, written in standard notations and convenient units:

\[ \nabla \cdot (\rho \mathbf{v}) = 0 \]  
\[ \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \mathbf{j} \times \mathbf{B} - \nabla P \]  
\[ \nabla \times \mathbf{E} = 0 \]  
\[ \nabla \times \mathbf{B} = \mathbf{j} \]  
\[ \nabla \cdot \mathbf{B} = 0 \]  
\[ \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0. \]

The system under consideration is a cylindrical plasma with flow and arbitrary cross sectional shape. For this configuration convenient coordinates are $\xi$, $\eta$ and $z$ with unit basis vectors $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, $\mathbf{e}_z$, where $\mathbf{e}_z$ is parallel to the axis of symmetry and $\xi$, $\eta$ are generalized coordinates pertaining to the poloidal cross section. The equilibrium quantities do not depend on $z$. The divergence free fields, i.e. the magnetic field $\mathbf{B}$, the current density density $\mathbf{j}$ and the mass flow $\rho \mathbf{v}$, can be
expressed in terms of the stream functions \( \psi(\xi, \eta) \), \( F(\xi, \eta) \), \( B_z(\xi, \eta) \) and \( v_z(\xi, \eta) \) as

\[
\mathbf{B} = B_z \mathbf{e}_z + \mathbf{e}_z \times \nabla \psi \\
\mathbf{j} = \nabla^2 \psi \mathbf{e}_z - \mathbf{e}_z \times \nabla B_z
\]

and

\[
\rho \mathbf{v} = \rho v_z \mathbf{e}_z + \mathbf{e}_z \times \nabla F.
\]

Constant \( \psi \) surfaces are the magnetic surfaces. Eqs. (1)-(6) can be reduced by means of certain integrals of the system, which are shown to be surface quantities. To identify two of these quantities, the time independent electric field is expressed by \( \mathbf{E} = -\nabla \Phi \) and the Ohm’s law (6) is projected along \( \mathbf{e}_z \) and \( \mathbf{B} \), respectively, yielding

\[
\mathbf{e}_z \cdot (\mathbf{e}_z \times \nabla F) \times (\mathbf{e}_z \times \nabla \psi) = 0 \tag{10}
\]

and

\[
\mathbf{B} \cdot \nabla \Phi = 0. \tag{11}
\]

Eqs. (10) and (11) imply that \( F = F(\psi) \) and \( \Phi = \Phi(\psi) \). Two additional surface quantities are found from the component of Eq. (3) perpendicular to a magnetic surface:

\[
\frac{B_z F'}{\rho} - v_z = \Phi', \tag{12}
\]

and from the component of the momentum conservation equation (2) along \( \mathbf{e}_z \):

\[
B_z - F' v_z \equiv X(\psi). \tag{13}
\]

(The prime denotes differentiation with respect to \( \psi \)). Solving the set of Eqs. (12) and (13) for \( B_z \) and \( v_z \), one obtains

\[
B_z = \frac{X(\psi) \rho - F'(\psi) \Phi'(\psi)}{\rho - (F'(\psi))^2} \tag{14}
\]

and

\[
v_z = \frac{F'(\psi) X(\psi) - \Phi'(\psi)}{\rho - (F'(\psi))^2}. \tag{15}
\]

With the aid of Eqs. (14)-(15), the components of Eq. (2) along \( \mathbf{B} \) and perpendicular to a magnetic surface, respectively, are put in the form

\[
\mathbf{B} \cdot \left[ \nabla \left( \frac{v_z^2}{2} + v_z \Phi' \right) + \frac{\nabla P}{\rho} \right] = 0 \tag{16}
\]
and
\[ \nabla \cdot \left[ \left( 1 - \frac{(F')^2}{\rho} \right) \nabla \psi \right] + \frac{F'' F' |\nabla \psi|^2}{\rho} + \frac{B_z \nabla B_z \cdot \nabla \psi}{|\nabla \psi|^2} + \rho \frac{\nabla \psi}{|\nabla \psi|^2} \cdot \left[ \nabla \left( \frac{(F')^2 |\nabla \psi|^2}{2\rho^2} \right) + \frac{\nabla P}{\rho} \right] = 0. \] (17)

It is pointed out here that Eqs. (16) and (17) are valid for any equation of state for the plasma.

In order to reduce further the equilibrium equations, we employ the incompressibility condition
\[ \nabla \cdot \mathbf{v} = 0. \] (18)

Then Eq. (1) implies that the density is a surface quantity,
\[ \rho = \rho(\psi), \] (19)

and, consequently, Eqs. (14) and (15) yield
\[ B_z = B_z(\psi), \quad v_z = v_z(\psi). \] (20)

With the use of Eqs. (13) and (20), Eq. (16) can be integrated yielding an expression for the pressure, i.e.
\[ P = P_s(\psi) - \frac{(F')^2}{2\rho} |\nabla \psi|^2. \] (21)

We note here that, unlike in static equilibria, in the presence of flow magnetic surfaces do not coincide with isobaric surfaces because Eq. (2) implies that \( \mathbf{B} \cdot \nabla P \) in general differs from zero. In this respect, the term \( P_s(\psi) \) is the static part of the pressure which does not vanish when \( F' \) is set to zero; Eqs. (14), (15) and (17) have a singularity when
\[ \frac{(F')^2}{\rho} = 1. \] (22)

On the basis of Eq. (3) for \( \rho \mathbf{v} \) and the definitions \( v_{Ap}^2 \equiv \frac{|\nabla \psi|^2}{\rho} \) for the Alfvén velocity associated with the poloidal magnetic field and the Mach number \( M^2 \equiv \frac{v^2}{v_{Ap}^2} \), Eq. (22) can be written as \( M^2 = 1 \).

Assuming now \( \frac{(F')^2}{\rho} \neq 1 \), and inserting Eq. (21) into Eq. (17), the latter reduces to the elliptic differential equation
\[ \left[ 1 - \frac{(F')^2}{\rho} \right] \nabla^2 \psi + \frac{F''}{\rho} \left( \frac{F' \rho'}{2 \rho} - F'' \right) |\nabla \psi|^2 + \left( P_s + \frac{B_z^2}{2} \right)' = 0. \] (23)
The absence of any hyperbolic regime in Eq. (23) can be understood by noting that, as is well known from the gas dynamics, the flow must be compressible to allow the equilibrium differential equation to depart from ellipticity. Eq. (23) does not contain the axial velocity $v_z$ and is identical to the equation governing cylindrical equilibria with purely poloidal flow [1]. With the use of the ansatz $\frac{\rho'}{\rho} = 2 \frac{F''}{F'}$, which implies that $(\frac{F'}{\rho})^2 \equiv M^2 = \text{const.}$, Eq. (23) reduces to

$$\nabla^2 \psi + \frac{1}{1 - M^2} \left( P_s + \frac{B_z^2}{2} \right)' = 0. \tag{24}$$

This is similar in form to the equation governing static equilibria; the only explicit reminiscence of flow is the presence of $M_c$. Eq. (24) can be linearized for several choices of $P_s + \frac{B_z^2}{2}$ and a variety of analytic solutions of the linearized equation can be derived. In particular, the exact solutions for a circular cylindrical plasma obtained in Ref. [1] are also valid for incompressible equilibrium flows with a free axial velocity $v_z(\psi)$.

The singularity $M_c^2 = 1$ is the limit at which the confinement can be assured by the axial current $\nabla^2 \psi$ alone. For $M_c^2 > 1$ the derivative of $B_z^2/2$ must partly compensate for the pressure gradient.

### III. Equilibria with isothermal magnetic surfaces

For fusion plasmas the thermal conduction along $B$ is fast compared to the heat transport perpendicular to a magnetic surface and therefore equilibria with isothermal magnetic surfaces are of particular interest. The plasma is also assumed to obey the ideal gas law $P = R\rho T$. For this kind of equilibria, Eq. (21) implies that $|\nabla \psi|$ is a surface quantity and consequently from Eq. (17) it turns out that $\nabla^2 \psi$ is a surface quantity as well. Thus, the incompressible, $T = T(\psi)$ equilibria satisfy the set of equations

$$|\nabla \psi|^2 = (g(\psi))^2 \tag{25}$$

and

$$\nabla^2 \psi = f(\psi) \tag{26}$$

Eqs. (23) and (26) imply that, on a magnetic surface the modulus of the vector $\nabla \psi$, which is perpendicular to this (arbitrary) magnetic surface, and $\nabla^2 \psi$, related to the variation of $|\nabla \psi|$, are constants. Therefore, one could speculate that magnetic surfaces are restricted to be circular. This conjecture can be proved as follows.
The coordinates \( \xi, \eta \) and \( z \) are specified to be the Cartesian coordinates \( x, y, z \). With the introduction of the quantities \( p = \partial \psi / \partial x, q = \partial \psi / \partial y, r = \partial^2 \psi / \partial x^2 \) and \( t = \partial^2 \psi / \partial y^2 \), Eqs. (25) and (26) are written in the form

\[
p^2 + q^2 = g^2 \tag{27}\]

and

\[
r + t = f. \tag{28}\]

The set of Eqs. (27) and (28) can be integrated by applying a procedure suggested by Palumbo [3]. Accordingly, considering the functions \( p \) and \( q \) which are functions of \( x \) and \( y \) as functions of \( x \) and \( \psi(x, y) \) one has

\[
r = \left. \frac{\partial p}{\partial x} \right|_y + \left. \frac{\partial p}{\partial \psi} \right|_y \tag{29}\]

and

\[
t = q \left. \frac{\partial q}{\partial \psi} \right|_y. \tag{30}\]

(It is noted here that a surface function \( \zeta = \zeta(x, y) \equiv \zeta(\psi) \) can be employed instead of \( \psi \)). With the aid of Eqs. (27), (29) and (30), Eq. (28) reduces to

\[
\left. \frac{\partial p}{\partial x} \right|_\psi = f - gg' \quad \text{and consequently} \quad p = x(f - gg') + h(\psi). \tag{31}\]

On a magnetic surface it holds that \( d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \equiv 0 \), and therefore

\[
\left( \left. \frac{dy}{dx} \right|_\psi \right)^2 = \left( \frac{p^2}{q^2} \right) = \frac{\left[ x(f - gg') + h \right]^2}{g^2 - \left[ x(f - gg') + h \right]^2}. \tag{32}\]

Introducing the new quantities \( a(\psi) \equiv f - gg' \), \( X \equiv ax + h \) and \( Y \equiv ay \), Eq. (32) is put in the form

\[
\left( \frac{dY}{dX} \right)^2 = \frac{X^2}{g^2 - X^2}. \tag{33}\]

Eq. (33) describes a circle on the \((x, y)\) plane with radius \( |g| \) centred at \((-h/a, 0)\).

**IV. Conclusions**

It was proved that the ideal MHD equilibrium states of a cylindrical plasma with incompressible flows and arbitrary cross section shape satisfy an elliptic partial differential equation [Eq. (23)], which is identical to the equation governing cylindrical equilibria with purely poloidal flow; the axial flow velocity is
a free surface quantity. This equation permits the construction of several classes of analytic solutions. In particular, the exact equilibrium solutions for a circular cylindrical plasma and purely poloidal flow $\Pi$ are also valid for the present case. In addition, it was proved that the magnetic surfaces of isothermal incompressible equilibria must have circular cross section.

It is interesting to investigate symmetric incompressible equilibria in geometries representing more realistically the magnetic confinement systems, e.g. axisymmetric and straight helically symmetric configurations. In this respect it may be noted here that, as proved in Ref. [4], the special class of axially symmetric, incompressible, $\beta_p = 1$, MHD equilibria with purely poloidal velocity does not exist; the only possible stationary equilibria of this kind are of cylindrical shape.

Acknowledgments

This work was conducted during a visit by one of the authors (G.N.T.) to Max-Planck Institute für Plasmaphysik, Garching. The hospitality provided at the said institute is appreciated. G.N.T. acknowledges support by EURATOM (Mobility Contract No 131-83-7 FUSC). One of the authors (H.T.) would like to thank Prof. D. Pfirsch for a useful discussion.

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