Gluino-Condensate (CIV-DV) Prepotential from its Whitham-Time Derivatives

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Abstract

We describe an expedient way to derive the CIV-DV prepotential in power series expansion in \(S_i\). This is based on integrations of equations for its derivatives \(\partial F/\partial T_m\) with respect to additional (Whitham) moduli \(T_m\). For illustrative purposes, we calculate explicitly the leading terms of the expansion and explicitly check some components of the WDVV equations to the leading order. Extension to any higher order is simple and algorithmic.

I. Introduction

This paper continues discussion of the CIV-DV prepotential \([1, 2, 3]\) from the perspective of the Seiberg-Witten theory \([4, 5]\), originated in \([6]\). We explained in \([6]\) that the full set of moduli in DV model is \(2n\)-dimensional (rather than \(n\)-dimensional), and the flat coordinates on it are

\[
S_k = \oint_{A_k} dS_{DV}, \quad T_k = \text{res}_\infty x^{-k} dS_{DV}, \quad k = 1, \ldots, n, \tag{1.1}
\]

where

\[
dS_{DV}(x) = Y(x) dx, \quad Y^2(x) = g^2 P_n^2(x) + f_{n-1}(x) = g^2 P_n^2(x) + 2gP_n(x) \sum_{i=1}^{n} \frac{\tilde{S}_i}{x - \alpha_i}, \tag{1.2}
\]

\[
P_n(x) = \prod_{i=1}^{n} (x - \alpha_i) = \sum_{m=0}^{n} u_m x^m, \quad u_n = 1, \quad u_m = (-)^{n-m} e_{n-m}(\alpha),
\]
and

\[ e_m(\alpha) = \sum_{i_1 < \ldots < i_m} \alpha_{i_1} \ldots \alpha_{i_m} \] (1.3)

are symmetric polynomials in \( \alpha \)'s. As \( dS_{DV} = (gP(x) + O(x^{-1}))dx \) at \( x \to \infty \), the Whitham times are obviously

\[ T_m = gu_{m-1}, \quad m = 1, \ldots, n. \] (1.4)

In [6, 7] an additional restriction \( T_n = gu_n - 1 = -g_n \sum_{i=1}^{n} \alpha_i = 0 \) was imposed on the moduli space, (so that only holomorphic differentials arise in discussion of the regularized DV model). However, as correctly pointed out in [8], this constraint can break some nice properties of the theory: in particular, it leads to complications with the WDVV equations [9, 10], which are observed in [7]. If \( T_n \) is included back into the set of moduli, the WDVV equations were shown in [8] to hold for the CIV-DV prepotential in the most straightforward way (with the residue formula, suggested in [6]).

Now, after the origin of difficulties in ref.[6] is identified and eliminated (and thus the whole approach is justified), we can continue developing the theory along the lines of that paper. The task of the present text is modest: we develop a calculus, based on expansion of \( dS_{DV} \) in power series in \( \tilde{S}_i \):

\[ dS_{DV}(x) = gP_n(x)dx + \sum_{i=1}^{n} \tilde{S}_i \frac{dx}{x - \alpha_i} - \frac{1}{2g} \sum_{i,j=1}^{n} \tilde{S}_i \tilde{S}_j \frac{dx}{(x - \alpha_i)(x - \alpha_j)} - \ldots \] (1.5)

Its main advantage is that contour integrals are substituted by residues at points \( x = \alpha_i \) and \( x = \infty \). The equations

\[ \frac{\partial F_{DV}}{\partial S_k} = \int_{B_k} dS_{DV} \] (1.6)

are difficult to handle. The problem is that particular terms of expansion (1.5) for \( dS_{DV} \) are singular at \( x = \alpha_i \) while \( dS_{DV} \) itself has no singularity at this point. The integral \( B_k \) in (1.6) actually goes between some large \( \Lambda \) and \( \tilde{\alpha}_i \), which is a root of \( Y^2(x) \), very close to \( \alpha_i \) at small \( \tilde{S}_i \): \( \tilde{\alpha}_i - \alpha_i \sim \sqrt{\tilde{S}_i} \). Thus after the integration of particular terms in (1.5), \( \tilde{S} \) can emerge in the denominator, and this breaks the naive structure of the power expansion in \( \tilde{S}_i \) and makes the evaluation of \( \partial F_{DV}/\partial S_i \) more sophisticated. See [11].

\[ ^{1}\text{In order to obtain this result, in eq.(35) of ref.} \]

\[ ^{1}\text{one should not transfer from} \sum_{dS_{DV}=0} \ldots \sum_{dP_{2n}=0} \ldots \text{(in this transition one should be careful about contributions from the zeroes of} R_{2n}) \text{and, accordingly, in (45) and (46) of} \]

\[ ^{1}\text{an algebra of polynomials modulo} P_{2n} \text{ (rather than} P'_{2n} \text{) should be considered. For such algebra to be closed, one needs polynomials of degree} 2n - 1 \text{ rather than} 2n - 2 \text{: the matching condition is always} \] (degree of polynomials that form the algebra) = (degree of polynomial which defines factorization) - 1. Still, the experimental result of [7], concerning the validity of WDVV equations for a whole family of the CIV-DV-like prepotentials with just three moduli \( S_1, S_2, T \) for \( n = 2 \), remains true. Moreover, by now we checked this fact with the help of Maple and it appears that, at least up to the order \( S^3 \) the WDVV equations are satisfied whenever \( \nu = \pm 1 \). Moreover, this result does not depend on the coefficients in front of the \( S^3, S^4 \) and \( S^5 \) terms. The result remains theoretically unexplained at this moment.
However, one can instead use a set of the remaining equations\(^2\)
\[
\frac{\partial F_{DV}}{\partial T_m} = \frac{1}{m} \text{res}_\infty (x^m - \Lambda^m) dS_{DV} = \frac{1}{m} \sum_{i=1}^{n} (\alpha_i^m - \Lambda^m) \tilde{S}_i
\]
(1.7)
to define the prepotential, which can be easily integrated for any given power of \(S_i\) and this procedure provides the CIV-DV prepotential as a power series in \(S_i\) with coefficients made from \(\alpha_i\),\(^3\)
\[
F_{DV}(S|\alpha) = g \sum_{i=1}^{n} S_i(W_{n+1}(\alpha_i) - W_{n+1}(\Lambda)) + \\
- \frac{1}{2} \left( \sum_{i=1}^{n} S_i \right)^2 \log \Lambda + \frac{1}{4} \sum_{i=1}^{n} S_i^2 (\log S_i - \frac{3}{2}) - \frac{1}{4} \sum_{i<j} (S_i^2 - 4S_iS_j + S_j^2) \log \alpha_{ij} + \\
+ \sum_{p=3}^{\infty} g^{2-p} F_p(S|\alpha),
\]
(1.8)
with \(W_{n+1}'(x) = P_n(x)\), i.e.
\[
W_{n+1}(x) = \sum_{k=0}^{n} \frac{u_k x^{k+1}}{k+1}.
\]
(1.9)
Actually, the two \(\alpha\)-independent terms in the second line of eq.(1.8) cannot be found from (1.7), but they can be easily fixed by other methods. Note also that in all orders of the \(\tilde{S}\)-expansion
\[
\sum_{i=1}^{n} \tilde{S}_i = \sum_{i=1}^{n} S_i
\]
(1.10)
since the sum of the integrals over all of the \(A_i\) contours is equal to residue at infinities, \(x = \infty\).

This method provides considerable simplification when evaluating higher corrections to the prepotential, making calculations comparably simple to those in matrix models (while straightforward evaluation involving contour integrals is incredibly sophisticated. See [11]). This allows us to check general arguments and proofs of [12, 6, 8] concerning the properties of the prepotential and the WDVV equations, with explicit and elementary calculations – for particular terms in the series expansion.

What is more important, this method provides a non-transcendental definition of particular terms \(F_p(S|\alpha)\) in the prepotential expansion, which are rational functions of \(\alpha\)’s and thus are expressed
\(^2\)Additional \(\Lambda\)-dependent term in this definition is important to reproduce the prepotential, associated with the spectral curve. If this term is omitted from (1.7), one obtains the planar-matrix-model prepotential, different from (1.8) by the absence of terms with \(\sum_{i=1}^{n} S_i W_{n+1}(\Lambda)\) and \(\sum_{i=1}^{n} S_i^2 \log \Lambda\). This difference is rarely important, but sometimes it is: for example the Ward identities in section VI. below are derived in the presence of the \(\Lambda\)-dependent terms.
\(^3\)Eq.(1.8) differs from analogous expressions in refs.[6, 11] by rescalings
\[
S_i = \frac{1}{2\pi i} S_i^{[11]}, \quad \frac{\partial F}{\partial S_i} = \frac{1}{2} \frac{\partial F^{[11]}}{\partial S_i^{[11]}}, \quad \text{i.e.} \quad F = \frac{1}{4\pi i} F^{[11]}
\]
This is because, in the present paper, we include the factor \((2\pi i)^{-1}\) into the definition of contour integrals in [11] and omit factor 2 in eqs.(1.6) and (1.7). (Such factor would appear in [11] to account for the fact that \(x = \infty\) describes two points on the two sheets of the hyperelliptic spectral curve. As it is omitted in [11], there should be no 2 in (1.6) as well).
through rational (rather than hyperelliptic) integrals. Such representation is useful for comparison to matrix-model calculations and instanton calculus [13].

The very simple form of the prepotential \( T \)-derivatives (1.7) deserves to be mentioned: the r.h.s. is just a linear function of \( \tilde{S}_i \) (though, of course, it becomes an infinite series when expressed in terms of the flat moduli \( S_i \))

In the next section, we derive a relation between \( \tilde{S}_i \) and \( S_i \) and sum rules for rational functions of \( \alpha_i \) associated with this. The \( T \)-derivatives of the roots \( \alpha_i \) are computed in section three, and consistency is checked on the second derivatives of the prepotential in section four. In section five, we describe how to obtain the prepotential from its \( T \)-derivatives, and in section six, \( \mathcal{L}_{-1} \) and \( \mathcal{L}_0 \)-constraints on \( F_{DV} \) are given. In section seven and eight, we discuss the WDVV equations of the prepotential, and check them explicitly to the leading order.

II. Relation between \( \tilde{S}_i \) and \( S_i \) and sum rules for rational functions of \( \alpha_i \)

Parameters \( \tilde{S}_i \) can be expressed through \( S_i \) by taking residues at \( x = \alpha_i \). In more detail, from (1.5):

\[
dS_{DV} = gP_n(x)dx + \sum_{i=1}^{n} \tilde{S}_i \frac{dx}{x-\alpha_i} - \frac{1}{2g} \sum_{j,k=1}^{n} \tilde{S}_j \tilde{S}_k \frac{dx}{(x-\alpha_j)(x-\alpha_k)P_n(x)} - \ldots \tag{2.1}
\]

Then

\[
S_i = \text{res}_{\alpha_i} dS_{DV} = \tilde{S}_i + \frac{1}{2g} \eta_{ijk} \tilde{S}_j \tilde{S}_k + \ldots \tag{2.2}
\]

or

\[
\tilde{S}_i = S_i - \frac{1}{2g} \eta_{ijk} S_j S_k + \ldots \tag{2.3}
\]

In order to evaluate the coefficients \( \eta_{ijk} \) one needs the following integrals. (As usual, the factor of \((2\pi i)^{-1}\) is included into the definition of contour integral.)

\[
\oint_{\alpha_i} \frac{dx}{(x-\alpha_j)(x-\alpha_k)P_n(x)} = \frac{1}{\alpha_{ij}\alpha_{ik}\Delta_i}, \text{ for } j,k \neq i;
\]

\[
\oint_{\alpha_i} \frac{dx}{(x-\alpha_i)(x-\alpha_j)P_n(x)} = \oint_{0} \frac{dz}{z^2(\alpha_{ij} + z)^2 \prod_{k \neq i,j}(\alpha_{ik} + z)} =
\]

\[
= -\frac{1}{\alpha_{ij}\Delta_i} \left( \frac{2}{\alpha_{ij}} + \sum_{k \neq i,j} \frac{1}{\alpha_{ik}} \right), \text{ for } i \neq j; \tag{2.4}
\]

\[
\oint_{\alpha_i} \frac{dx}{(x-\alpha_i)^2 P_n(x)} = \oint_{0} \frac{dz}{z^3 \prod_{j \neq i}(\alpha_{ij} + z)} = \frac{1}{\Delta_i} \left( \sum_{j \neq i} \frac{1}{\alpha_{ij}^2} + \sum_{j<k, j \neq i} \frac{1}{\alpha_{ij}\alpha_{ik}} \right)
\]

Here \( \alpha_{ij} = \alpha_i - \alpha_j, \Delta_i = \prod_{j \neq i} \alpha_{ij} \). Using the fact that the sums over all residues of the integrand should vanish, we obtain from (2.4) the following identities (sum rules):\(^4\)

\(^4\)This proof of \(\text{as} \) was suggested by V.Pestun. The first of these identities played an important role in ref. [11] (eq.(4.11) of that paper). The second identity can be used to convert the \( S_i^3 \) contributions to the prepotential, found in
\[ \sum_{k \neq i,j} \frac{1}{\alpha_{ik} \alpha_{jk} \Delta_k} = \frac{2}{\alpha_{ij}^2} \left( \frac{1}{\Delta_i} + \frac{1}{\Delta_j} \right) + \frac{1}{\alpha_{ij}} \sum_{k \neq i,j} \left( \frac{1}{\alpha_{ik} \Delta_i} - \frac{1}{\alpha_{jk} \Delta_j} \right), \]  

(2.5)

In what follows we will need also a slight generalization of these sum rules, with an extra factor of \( x^m \) in the integrand (with \( m \leq n \) to avoid contributions from \( x = \infty \)):

\[
\int_{\alpha_i} \frac{x^m dx}{(x - \alpha_i)(x - \alpha_j) P_n(x)} = \int_0^\infty \frac{(\alpha_i + z)^m dz}{z^2(\alpha_{ij} + z) \prod_{k \neq i,j} (\alpha_{ik} + z)} = 
\]

\[= - \frac{\alpha_i^m}{\alpha_{ij} \Delta_i} \left( \frac{2}{\alpha_{ij}} + \sum_{k \neq i,j} \frac{1}{\alpha_{ik}} \right) + \frac{ma_i^{m-1}}{\alpha_{ij} \Delta_i}, \text{ for } i \neq j; \]

\[
\int_{\alpha_i} x^m dx = \int_0^\infty \frac{(\alpha_i + z)^m dz}{z^3 \prod_{j,k \neq i} (\alpha_{ij} + z)} = 
\]

\[= \frac{\alpha_i^m}{\Delta_i} \left( \sum_{j \neq i} \frac{1}{\alpha_{ij}^2} + \sum_{j < k \atop j,k \neq i} \frac{1}{\alpha_{ij} \alpha_{ik}} \right) - \frac{ma_i^{m-1}}{\Delta_i} \sum_{j \neq i} \frac{1}{\alpha_{ij}} + \frac{m(m-1)\alpha_i^{m-2}}{2 \Delta_i} \]

and, as corollaries,

\[
\sum_{k \neq i,j} \frac{\alpha_k^m}{\alpha_{ik} \alpha_{jk} \Delta_k} = \frac{2}{\alpha_{ij}^2} \left( \frac{\alpha_i^m}{\Delta_i} + \frac{\alpha_j^m}{\Delta_j} \right) + \frac{1}{\alpha_{ij}} \sum_{k \neq i,j} \left( \frac{\alpha_k^m}{\alpha_{ik} \Delta_i} - \frac{\alpha_j^m}{\alpha_{jk} \Delta_j} \right) - \frac{ma_i^{m-1}}{\alpha_{ij} \Delta_i} \left( \frac{\alpha_i^{m-1}}{\Delta_i} - \frac{\alpha_j^{m-1}}{\Delta_j} \right), \]  

(2.7)

and

\[
\sum_{j \neq i} \frac{\alpha_j^m}{\Delta_j} = -\frac{\alpha_i^m}{\Delta_i} \left( \sum_{j \neq i} \frac{1}{\alpha_{ij}^2} + \sum_{j < k \atop j,k \neq i} \frac{1}{\alpha_{ij} \alpha_{ik}} \right) + 
\]

\[+ \frac{ma_i^{m-1}}{\Delta_i} \sum_{j \neq i} \frac{1}{\alpha_{ij}} - \frac{m(m-1)\alpha_i^{m-2}}{2 \Delta_i}. \]  

(2.8)

A simpler sum rule, associated with the integrand \( \frac{x^m dx}{(x - \alpha_i) P_n(x)} \), states:

\[
\sum_{j \neq i} \frac{\alpha_j^m}{\alpha_{ij} \Delta_j} = -\frac{\alpha_i^m}{\Delta_i} \sum_{j \neq i} \frac{1}{\alpha_{ij}} + \frac{ma_i^{m-1}}{\Delta_i}, \quad m < n \]  

(2.9)

Coming back to our problem, one can read from expressions for the coefficients \( \eta_{ijk} \) in [123]:

[123] as they pull \( \frac{1}{x} \) from inside the summation. This plays an important role for comparing, say, with matrix model calculations.
\[
\eta_{i;ii} = -\frac{1}{\Delta_i} \left( \sum_{j \neq i} \frac{1}{\alpha_{ij}^2} + \sum_{j < k \atop j, k \neq i} \frac{1}{\alpha_{ij}\alpha_{ik}} \right) \quad \text{eq. (2.5)} \equiv \sum_{j \neq i} \frac{1}{\alpha_{ij}^2\Delta_j},
\]
\[
\eta_{i;ij} = \frac{1}{\Delta_i} \left( \frac{2}{\alpha_{ij}^2} + \frac{1}{\alpha_{ij}} \sum_{k \neq i, j} \frac{1}{\alpha_{ik}} \right),
\]
\[
\eta_{i;jk} = -\frac{1}{\alpha_{ij}\alpha_{ik}\Delta_i}, \quad \text{for } j, k \neq i.
\]
These coefficients satisfy the sum rules
\[
\eta_{i;ii} = -\frac{1}{2} \sum_{j \neq i} \eta_{i;ij} = -\sum_{j \neq i} \eta_{j;ii},
\]
\[
\eta_{i;jj} = -\frac{1}{2} \left( \eta_{i;ij} + \sum_{k \neq i, j} \eta_{i;jk} \right) = -\frac{1}{2} \sum_{k \neq j} \eta_{i;jk}; \quad \text{(2.11)}
\]
\[
\sum_{i=1}^{n} \eta_{i;jk} = 0, \quad \forall j, k.
\]
To prove the last of these sum rules for \( j \neq k \) one should apply (2.5). This sum rule guarantees that
\[\sum_{i=1}^{n} \tilde{S}_i = \sum_{i=1}^{n} S_i \] to quadratic order in \( \tilde{S}^2 \).

\section*{III. \( T \)-derivatives of the roots \( \alpha_i \)}

To obtain \( \mathcal{F} \) as a function of \( S_i \) and \( T_m \), we need to express \( \alpha \)’s in (1.8) through \( T_m \). The modulus \( T_{m+1} = g u_m = (-)^{n-m} ge_{n-m}(\alpha) \) is a symmetric polynomial (of degree \( m \)) in \( \alpha_i \). Conversely, \( \alpha_i \) is a section of an \( n \)-dimensional bundle over the space of \( T_m \)’s, (that is, \( \alpha_i \)’s are obtained as solution of a system of algebraic equations and different \( \alpha_i \)’s are considered as different roots and get interchanged when \( T_m \)’s move around the singularities).

The \( T(u) \)-derivatives of \( \alpha \)’s can be obtained by inversion of the matrix
\[
\frac{\partial u_m}{\partial \alpha_i} = (-)^{n-m} e_{n-1-m}^{[i]}(\alpha) \quad \text{(3.1)}
\]
Here \( m = 0, \ldots, n-1, \ i = 1, \ldots, n, \) and \( e_{n}^{[i]}(\alpha) \) is a symmetric polynomial of degree \( m \) of the set of \( n-1 \) variables consisting of \( \alpha_j \) with \( j \neq i \). See \[6\] \[11\] for more details about the notations. Inverting (3.1), we obtain:
\[
\frac{\partial \alpha_i}{\partial u_m} = -\frac{\alpha_i^m}{\Delta_i}, \quad \text{(3.2)}
\]
Indeed, since \( \sum_{m=0}^{n-1} (-)^{n-m} e_{n-1-m}^{[i]}(\alpha)x^m = -\prod_{k \neq i} (x - \alpha_k) \), we have:

\[5\] As an example of (3.2), we have for \( n = 2 \):
\[
u_0 = \alpha_1 \alpha_2, \quad \nu_1 = -(\alpha_1 + \alpha_2),
\]
\[ \sum_{m=0}^{n-1} (-)^{n-m} e_{n-1-m}^{[i]} (\alpha) \cdot \left( -\frac{\alpha_m}{\Delta_j} \right) = \frac{1}{\Delta_j} \prod_{k \neq i} (\alpha_j - \alpha_k) = \delta_{ij}. \]  

(3.3)

IV. Second derivatives of the prepotential: consistency of eq. (1.7)

From (1.7), (1.4) and (2.3) we obtain:

\[ \frac{\partial F_{DV}}{\partial u_m} = \frac{g}{m + 1} \text{res}_\infty (x^{m+1} - \Lambda^{m+1}) dS_{DV} = \frac{g}{m + 1} \sum_{i=1}^{n} \tilde{S}_i (\alpha_i^{m+1} - \Lambda^{m+1}) = \]

\[ = \frac{1}{m + 1} \left( g \sum_{i=1}^{n} \tilde{S}_i (\alpha_i^{m+1} - \Lambda^{m+1}) - \frac{1}{2} \sum_{i,j,k=1}^{n} \eta_{ijk} S_j S_k \alpha_i^{m+1} \right) \]

(4.1)

Substituting explicit expressions (2.10) for \( \eta_{i,j,k} \), we obtain for the \( S^2 \)-term:

\[ -\frac{1}{2(m + 1)} \sum_{i,j,k=1}^{n} \eta_{ijk} \alpha_i^{m+1} S_j S_k \]

\[ - \frac{1}{m + 1} \sum_{j \neq i} S_i S_j \left( \frac{2}{\alpha_i^{2}} + \frac{1}{\alpha_{ij}} \sum_{k \neq i,j} \frac{1}{\alpha_{ik}} \right) \frac{\alpha_i^{m+1}}{\Delta_i} + \frac{1}{2(m + 1)} \sum_{i \neq j \neq k \neq i} S_j S_k \frac{\alpha_i^{m+1}}{\alpha_{ij} \alpha_{ik} \Delta_i} \]

(4.2)

Since eq. (2.7) was used, (4.2) is valid only for \( m \leq n \).

Now we are ready to check the consistency of equation (4.1), i.e. the property of the matrix \( \frac{\partial^2 F_{DV}}{\partial u_l \partial u_m} \) being symmetric. To the first order in \( S \), the contribution is indeed symmetric under an exchange of \( l \) and \( m \):

\[ \sum_{i=1}^{n} S_i \alpha_i^m \frac{\partial \alpha_i}{\partial u_l} \]  

\[ \text{eq. (3.2)} \]

\[ - \sum_{i=1}^{n} S_i \alpha_i^{l+m} \frac{\partial \alpha_l}{\partial u_i} \]  

(4.3)

Similarly, to the second order in \( S_i \), from the second term in the r.h.s. of (4.2) we find:

\[ \alpha_{1,2} = -\frac{1}{2} (u_1 \pm \sqrt{u_1^2 - 4u_0}), \quad \Delta_1 = -\Delta_2 = \alpha_{12} = \mp \sqrt{u_1^2 - 4u_0}, \]

\[ \frac{\partial \alpha_{1,2}}{\partial u_0} = \pm \frac{1}{\sqrt{u_1^2 - 4u_0}} = -\frac{1}{\Delta_{1,2}}, \quad \frac{\partial \alpha_{1,2}}{\partial u_1} = -\frac{1}{2} \left( 1 \pm \frac{u_1}{\sqrt{u_1^2 - 4u_0}} \right) = -\frac{\alpha_{1,2}}{\Delta_{1,2}} \]
\[-\frac{\partial}{\partial u_l} \sum_{i<j} \frac{S_i S_j}{\alpha_{ij}} \left( \frac{\alpha^m_i}{\Delta_i} - \frac{\alpha^m_j}{\Delta_j} \right) = \]
\[= \sum_{i<j} S_i S_j \left[ -\frac{1}{\alpha_{ij}} \left( \frac{\alpha^l_i}{\Delta_i} - \frac{\alpha^l_j}{\Delta_j} \right) \left( \frac{\alpha^m_i}{\Delta_i} - \frac{\alpha^m_j}{\Delta_j} \right) + \right. \]
\[+ \frac{1}{\alpha_{ij}} \left( \frac{m \alpha^{l+m-1}_i}{\Delta_i^2} - \frac{m \alpha^{l+m-1}_j}{\Delta_j^2} \right) - \]
\[\left. \frac{\alpha^m_i}{\alpha_{ij} \Delta_i} \sum_{k \neq i} \frac{1}{\alpha_{ik}} \left( \frac{\alpha^l_i}{\Delta_i^2} - \frac{\alpha^l_k}{\Delta_k^2} \right) + \frac{\alpha^m_j}{\alpha_{ij} \Delta_j} \sum_{k \neq j} \frac{1}{\alpha_{jk}} \left( \frac{\alpha^l_j}{\Delta_j^2} - \frac{\alpha^l_k}{\Delta_k^2} \right) \right] = \text{(4.4)} \]
\[= \sum_{i<j} S_i S_j \left[ -\frac{1}{\alpha_{ij}} \left( \frac{\alpha^l_i}{\Delta_i^2} - \frac{\alpha^l_j}{\Delta_j^2} \right) \left( \frac{\alpha^m_i}{\Delta_i^2} - \frac{\alpha^m_j}{\Delta_j^2} \right) + \right. \]
\[+ \frac{l + m}{\alpha_{ij}} \left( \frac{\alpha^{l+m-1}_i}{\Delta_i^2} - \frac{\alpha^{l+m-1}_j}{\Delta_j^2} \right) - \frac{2}{\alpha_{ij}} \left( \frac{\alpha^{l+m}_i}{\Delta_i^2} \sum_{k \neq i} \frac{1}{\alpha_{ik}} - \frac{\alpha^{l+m}_j}{\Delta_j^2} \sum_{k \neq j} \frac{1}{\alpha_{jk}} \right) \]
which is obviously symmetric under an exchange of \( l \) and \( m \).

As for the first term in (4.2), it should first be transformed with the help of (2.8):
\[-\frac{1}{2(m+1)} \sum_{i,j \neq i} \frac{S^2_i}{\alpha_{ij}^2} \left( \frac{\alpha^{m+1}_i - \alpha^{m+1}_j}{\Delta_j} \right) = \frac{1}{2} \sum_i \frac{S^2_i}{\Delta_i} \left( \alpha_i^m \sum_{j \neq i} \frac{1}{\alpha_{ij}} - \frac{m}{2} \alpha_i^{m-1} \right) \quad \text{(4.5)} \]

Now differentiation over \( u_l \) gives:
\[-\frac{\partial}{\partial u_l} \left[ \frac{1}{2} \sum_i \frac{S^2_i}{\Delta_i} \left( \alpha_i^m \sum_{j \neq i} \frac{1}{\alpha_{ij}} - \frac{m}{2} \alpha_i^{m-1} \right) \right] = \]
\[= \frac{1}{2} \sum_i \frac{S^2_i}{\Delta_i} \left[ -\frac{m \alpha^{l+m-1}_i}{\Delta_i^2} \sum_{j \neq i} \frac{1}{\alpha_{ij}} + \frac{m(m-1) \alpha^{l+m-2}_i}{\Delta_i^2} \right. + \]
\[+ \frac{1}{\Delta_i} \left( \alpha_i^m \sum_{j \neq i} \frac{1}{\alpha_{ij}} - \frac{m}{2} \alpha_i^{m-1} \right) \sum_{j \neq i} \frac{1}{\alpha_{ij}} \left( \frac{\alpha^l_i}{\Delta_i} - \frac{\alpha^l_j}{\Delta_j} \right) + \]
\[+ \frac{\alpha_i^m}{\Delta_i} \sum_{j \neq i} \frac{1}{\alpha_{ij}} \left( \frac{\alpha^l_i}{\Delta_i} - \frac{\alpha^l_j}{\Delta_j} \right) \right] \quad \text{(4.6)} \]
and an application of (2.9) to the second line and (2.8) to the third line finally provides a symmetric expression (under an exchange of \( l \) and \( m \)): 

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\[
\frac{1}{2} \sum_i \frac{S_i^2}{\Delta_i^2} \left[ -m \alpha_i^{l+m-1} \sum_{j \neq i} \frac{1}{\alpha_{ij}} + \frac{m(m-1)}{2} \alpha_i^{l+m-2} + 2 \alpha_i^{l+m} \left( \sum_{j \neq i} \frac{1}{\alpha_{ij}} \right)^2 - m \alpha_i^{l+m-1} \sum_{j \neq i} \frac{1}{\alpha_{ij}} - l \alpha_i^{l+m-1} \sum_{j \neq i} \frac{1}{\alpha_{ij}} + \frac{lm}{2} \alpha_i^{l+m-2} \right]
\]

\[= \sum_i \frac{S_i^2}{\Delta_i^2} \left[ \alpha_i^{l+m} \left( \sum_{j \neq i} \frac{1}{\alpha_{ij}} \right)^2 - (l + m) \alpha_i^{l+m-1} \sum_{j \neq i} \frac{1}{\alpha_{ij}} + \frac{(l + m)^2 - (l + m)}{4} \alpha_i^{l+m-2} \right] \tag{4.7} \]

V. Prepotential from its \(T\)-derivatives

After consistency of the system (4.1) is checked, one can integrate these equations to obtain the prepotential. Instead one can just check that the \(u_m\)-derivative of (1.8) is indeed equal to the r.h.s. of eq.(4.1):

\[
\frac{\partial F_{DV}}{\partial u_m} = g \sum_{i=1}^n S_i \left( \frac{\partial W_{n+1}(\alpha_i)}{\partial u_m} - \frac{\partial W_{n+1}(\Lambda)}{\partial u_m} \right) - \frac{1}{4} \sum_{j < k} (S_i^2 + S_j^2 - 4S_jS_k) \frac{1}{\alpha_{jk}} \frac{\partial \alpha_{jk}}{\partial u_m} + \ldots \tag{5.1} \]

The check for the linear terms in \(S_i\) is trivial: one should just take into account that \(\partial W_{n+1}(x)/\partial u_m = x^{m+1}/(m + 1)\) and \(W_{n+1}'(\alpha_i) = P_n(\alpha_i) = 0\). As for the quadratic terms in \(S_i\), comparison with (4.1) implies that

\[
\frac{1}{\alpha_{jk}} \frac{\partial \alpha_{jk}}{\partial u_m} = - \frac{1}{m + 1} \sum_{i=1}^n \eta_{i;jk} \alpha_i^{m+1} = \frac{1}{4} \sum_{k \neq j} \frac{1}{\alpha_{jk}} \frac{\partial \alpha_{jk}}{\partial u_m} = \frac{1}{2} \left( \eta_{i;jj} \alpha_i^{m+1} + \sum_{i \neq j} \eta_{i;jj} \alpha_i^{m+1} \right), \quad \text{for } j < k; \tag{5.2} \]

Consistency of these two relations is ensured by the second sum rule (2.11). Relations can be checked with the help of identities (2.7) and (2.8) respectively, after explicit expressions are substituted for \(\eta_{i;jk}\) from (2.10) and for \(\partial \alpha_{jk}/\partial u_m = -\alpha_j^m/\Delta_j + \alpha_k^n/\Delta_k\).

These calculations are straightforwardly generalized to the higher order contributions \(F_p\) to the prepotential in \(S_i\): one should just consider next terms in the expansion (1.5) and repeat all the steps of the above procedure.
VI. \(\mathcal{L}_{-1}\) and \(\mathcal{L}_0\)-constraints on \(\mathcal{F}_{DV}\)

The usual way to derive Ward identities (see, for example, [14]) is to shift integration variables without changing the integral. This shift changes the shape of the integrand, which is equivalent to a certain change of its parameters (a shift along the moduli space), so that invariance of the integral provides a differential equation for it. In our present case this reasoning can be applied to the integrals (1.1), (1.6) and (1.7).

The freedom to shift the coordinate \(x\) on the spectral surface by \(x \rightarrow x + \epsilon\) is equivalent to

\[
\alpha_i \rightarrow \alpha_i + \epsilon, \quad u_{k-1} \rightarrow u_{k-1} - \epsilon ku_k, \quad \Lambda \rightarrow \Lambda + \epsilon;
\]

\[
S_i \rightarrow S_i, \quad \frac{\partial \mathcal{F}_{DV}}{\partial S_i} \rightarrow \frac{\partial \mathcal{F}_{DV}}{\partial S_i}.
\] (6.1)

This implies a constraint on \(\mathcal{F}_{DV}(S|T)\):

\[
\mathcal{L}_{-1} \mathcal{F}_{DV}(S|T) = \frac{\partial}{\partial \Lambda} \mathcal{F}_{DV}(S|T)
\] (6.2)

where

\[
\mathcal{L}_{-1} = \sum_{k=1}^{n} ku_k \frac{\partial}{\partial u_{k-1}} = gn \frac{\partial}{\partial T_n} + \sum_{k=1}^{n-1} kT_{k+1} \frac{\partial}{\partial T_k}
\] (6.3)

The CIV-DV prepotential indeed satisfies this constraint, for which the presence of the term \(W_{n+1}(\Lambda) \sum_{i=1}^{n} S_i\) is essential however. Note that

\[
\left(\mathcal{L}_{-1} - \frac{\partial}{\partial \Lambda}\right) W_{n+1}(\Lambda) = \left(\mathcal{L}_{-1} - \frac{\partial}{\partial \Lambda}\right) W_{n+1}(\alpha_i) = -u_0 = -\frac{1}{g} T_1.
\] (6.4)

Since \(u_n = 1 = \text{const}\), the derivative with respect to \(T_n = gu_{n-1}\) appears in the constraint (6.2) with a moduli-independent coefficient. This implies that the dependence of the prepotential on \(T_n\) is easy to restore once its dependence on all other moduli is known. However, since the constraint is inhomogeneous in \(T\)'s, the \(T_n\)-dependence can not be simply ignored – the constraint (6.2) does not commute with the \(T\)-derivatives and using it to eliminate the \(T_n\)-dependence changes, say, the form of the WDVV equations.

Similarly, rescaling of \(x\), \(x \lambda \rightarrow x\), equivalent to

\[
\alpha_i \rightarrow \lambda \alpha_i, \quad u_k \rightarrow \lambda^{n-k} u_k, \quad \Lambda \rightarrow \lambda \Lambda;
\]

\[
S_i \rightarrow \lambda^{n+1} S_i, \quad \frac{\partial \mathcal{F}_{DV}}{\partial S_i} \rightarrow \lambda^{n+1} \frac{\partial \mathcal{F}_{DV}}{\partial S_i},
\] (6.5)

provides the \(\mathcal{L}_0\)-constraint on \(\mathcal{F}_{DV}\):

\[
\mathcal{L}_0 \mathcal{F}_{DV}(S|T) = \frac{\partial}{\partial \log \Lambda} \mathcal{F}_{DV}(S|T) - (n + 1) \left(2 - \sum_i S_i \frac{\partial}{\partial S_i}\right) \mathcal{F}_{DV}(S|T)
\] (6.6)

\(^6\)Of course, the prepotential satisfies this and, what is more, each individual term does. For example, for

\[
S_i W_{n+1}(\Lambda) = S_i \sum_{k=0}^{n} \frac{u_k}{k+1} \Lambda^{k+1}
\]
where
\[ \mathcal{L}_0 = - \sum_{k=0}^{n-1} (n-k) u_k \frac{\partial}{\partial u_k} = \sum_{k=1}^{n} (n+1-k) T_k \frac{\partial}{\partial T_k} \] (6.7)

From eq. (1.7) we obtain
\[ \mathcal{L}_0 \mathcal{F}_{DV}(S|T) = -g \sum_{i=1}^{n} \left( \frac{1}{n-k+1} \sum_{k=0}^{n-k} u_k (\alpha_i^{k+1} - \Lambda_i^{k+1}) \tilde{S}_i \right) \] (6.8)

At the special Seiberg-Witten (\(N = 2\) supersymmetric) point, where \(Y^2(x) = g^2 (P_n(x)^2 - \Lambda_{N=2}^n)\), i.e.
\[ \tilde{S}_i = -\frac{g \Lambda_{N=2}^n}{2 \Delta_i} \] (6.9)

and
\[ \sum_i \alpha_i^{k+1} = \text{res}_{\Delta_i} x^{k+1} dx \left( k_n - u_{n-1} \delta_{k,n-1} \right) \] (6.10)

The term \(W_{n+1}(\Lambda) \sum_{i=1}^{n} S_i\) does not contribute, because at the SW point \(\sum_{i=1}^{n} S_i = 0\).

VII. Third derivatives of the prepotential and WDVV equations

Generic proof of the WDVV equations [10],
\[ \tilde{F}_I \tilde{F}_J^{-1} \tilde{F}_K = \tilde{F}_K \tilde{F}_J^{-1} \tilde{F}_I, \] (7.1)
consists of deriving residue formula for the third derivatives,
\[ (\tilde{F}_I)_{JK} = \frac{\partial F}{\partial \mu_I \partial \mu_J \partial \mu_K} = \sum_{dS=0 \text{ or } dS/dS=0}^{\text{res}} \frac{dW_I dW_J dW_K}{ddS}, \] (7.2)

we have
\[ \sum_{k=0}^{n} \frac{n-k}{k+1} u_k \Lambda_i^{k+1} = -\Lambda W_{n+1}(\Lambda) + (n+1)(2W_{n+1}(\Lambda) - W_{n+1}(\Lambda)). \]

\(^7\)Note that, for the sake of simplicity, we have changed our notation from the usual one: our \(u_k\) is conventional \(u_{n-k}\) and, in particular, our \(u_{n-2}\) is conventional \(u_2\).
Here \( \{ \mu_i \} = \{ S_i, T_i \} \) is the set of flat moduli, \( dW_I \) are the corresponding canonical one-differentials, in DV model, and the two-differential \( ddS \sim dS_{DV}(x) d \log Y(x) = dY(x) dx \). See [10, 6] for further details. The proof of such kind for the CIV-DV prepotential was discussed in [6] (for regularized DV model) and [3], and we do not repeat it here. Instead we check the validity of the WDVV equations to the leading order in the expansion in \( S \) (or \( g^{-1} \)) with the help of explicit formulas for \( F_{DV}(S, T) \) from the previous sections.

Because of the existence of two types of moduli, \( S_i \) and \( T_i \), the \( 2n \times 2n \) matrices of the prepotential third derivatives naturally possess the block form \(^8\):

\[
\tilde{F}_{S_i} = \begin{pmatrix}
\frac{1}{2S_i} \tilde{\Pi}_i & \tilde{B}^{(i)} \\
\tilde{B}^{(i)} & g \tilde{C}^{(i)}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2S_i} \delta_{ij} \delta_{ik} & \tilde{B}^{(i)}_{jl} \\
\tilde{B}^{(i)}_{mk} & g \tilde{C}^{(i)}_{ml}
\end{pmatrix}
\]

\[
\tilde{F}_{ur} = \begin{pmatrix}
\tilde{\mathcal{B}}_r & g \tilde{C}_r \\
g \tilde{C}_r & 0
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
0 & g \tilde{C}_r
\end{pmatrix} \begin{pmatrix}
\tilde{B}_r & I \\
I & 0
\end{pmatrix} = \begin{pmatrix}
(I)_{jk} & g (\tilde{C}_r)_{jl} \\
g (\tilde{C}_r)_{mk} & 0
\end{pmatrix}
\]

Tilde denotes a transposition of matrices. We neglect here all the contributions with higher powers of \( g^{-1} \) (i.e. with higher powers of \( S \)) in each block. If not the singular \( 1/S \) item, these would be the values of matrices for \( S = 0 \). The entries are:

\[
(\mathcal{B}_m)_{ij} = \mathcal{B}^{(i)}_{jm} = \frac{1}{2} \delta_{ij} \sum_{k \neq i} \left( \frac{\alpha_i^m \alpha_k^m}{\Delta_i} - \frac{\alpha_k^m}{\Delta_k} \right) \frac{1}{\alpha_m} - \frac{1}{\alpha_m} \delta_{ij} \left( \frac{\alpha_i^m}{\Delta_i} - \frac{\alpha_j^m}{\Delta_j} \right)
\]

\[
(\tilde{\mathcal{C}}_m)_{il} = \tilde{\mathcal{C}}^{(i)}_{lm} = -\frac{\alpha_i^{l+m}}{\Delta_i}
\]

Matrices \( \mathcal{B}_m \) and \( \mathcal{C}_m \) are symmetric, but \( \mathcal{B}^{(i)} \) and \( \tilde{\mathcal{C}}_m \) are not. The matrix \( \tilde{F}_{ur} \) (in variance with \( \tilde{F}_{S_i} \)) can be easily inverted:

\[
\tilde{F}^{-1}_{ur} = \begin{pmatrix}
I & 0 \\
0 & \frac{1}{g} \tilde{C}^{-1}_r
\end{pmatrix} \begin{pmatrix}
0 & I \\
I & -\tilde{B}_r
\end{pmatrix} \begin{pmatrix}
I & 0 \\
0 & \frac{1}{g} \tilde{C}^{-1}_r
\end{pmatrix}
\]

\[
\tilde{F}^{-1}_{ur} = \begin{pmatrix}
I & 0 \\
0 & \frac{1}{g} \tilde{C}^{-1}_r
\end{pmatrix} \begin{pmatrix}
\frac{1}{g} \tilde{C}^{-1}_r \\
\frac{1}{g} \tilde{C}^{-1}_r - \frac{1}{g^2} \tilde{B}_r \tilde{C}^{-1}_r
\end{pmatrix}
\]

Matrix \( \tilde{C}_r \) in its turn is decomposed into diagonal and \( r \)-independent matrices:

\(^8\)In accordance with [14] the \( T_m \) derivatives are actually taken with respect to \( g \mu_{m-1} \).

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\( \check{C}_r = \mathcal{A}^r \check{C}; \quad \check{A} = \text{diag}(\alpha_i), \ i.e. \ \check{A}_{ij} = \delta_{ij} \alpha_i^r ; \quad \check{C}_{im} = -\frac{\alpha^m_i}{\Delta_i} \) (7.8)

Both are easy to invert (use (3.3) in the case of \( \check{C} \)):

\[ \check{C}_r^{-1} = \check{C}^{-1} \check{A}^{-r}; \quad \check{A}^{-r}_{ij} = \delta_{ij} \alpha_i^{-r}; \quad (\check{C}^{-1})_{mi} = (-)^{n-m} c_{n-1-m}(\alpha). \] (7.9)

**VIII. WDVV equations, explicit check (in the leading order only)**

For illustrative purposes we use the results of the previous section to check explicitly the WDVV equations (7.1) to the leading order in power series in \( S \). As we saw in the previous section, matrices \( \check{F}_{ur}(S = 0) \) are especially easy to invert and we will exploit this. Actually, it is enough to check (7.1) for any particular \( J \) but for all \( I \) and \( K \). Equations for other \( J \) follow automatically. See [10]. Indices \( I, K \) still correspond to either \( S_k \) or \( T_k \) however.

The simplest case is when all the three moduli \( \mu_{I,J,K} \) in (7.1) are Whitham times \( T_{r+1} = gu_r \):

\[ \check{F}_{ur_1} \check{F}_{ur_2}^{-1} \check{F}_{ur_3} = \check{F}_{ur_1} \check{F}_{ur_2}^{-1} \check{F}_{ur_3} \] (8.1)

In the leading order, we can \( S_i = 0 \) in (8.1), while some entries of \( \check{F}_{S_j} \) are singular when \( S = 0 \). This means that one can use (8.5) and (8.7) in (8.1), which becomes a condition for the matrix being symmetric

\[
\begin{pmatrix}
\check{B}_{r_1} & g \check{C}_{r_1} \\
g \check{C}_{r_1} & 0
\end{pmatrix}
\begin{pmatrix}
0 & \frac{1}{g} \check{C}_{r_2}^{-1} \\
\frac{1}{g} \check{C}_{r_2}^{-1} & 0
\end{pmatrix}
\begin{pmatrix}
\check{B}_{r_3} & g \check{C}_{r_3} \\
g \check{C}_{r_3} & 0
\end{pmatrix}
= \begin{pmatrix}
\check{B}_{r_1} & g \check{C}_{r_1} \\
g \check{C}_{r_1} & 0
\end{pmatrix}
\begin{pmatrix}
\frac{1}{g} \check{C}_{r_2}^{-1} \check{C}_{r_3}^{-1} & 0 \\
0 & \check{C}_{r_2}^{-1} \check{C}_{r_3}^{-1}
\end{pmatrix}
= \begin{pmatrix}
\check{B}_{r_1} \check{C}_{r_2}^{-1} \check{C}_{r_3}^{-1} + \check{C}_{r_1} \check{C}_{r_2}^{-1} \check{B}_{r_3} - \check{C}_{r_1} \check{C}_{r_2}^{-1} \check{B}_{r_3} \check{C}_{r_2}^{-1} \check{C}_{r_3}^{-1} & g \check{C}_{r_1} \check{C}_{r_2}^{-1} \check{C}_{r_3}^{-1} \\
g \check{C}_{r_1} \check{C}_{r_2}^{-1} \check{C}_{r_3}^{-1} & 0
\end{pmatrix} \]

We need to check that

\[ \check{C}_{r_1} \check{C}_{r_2}^{-1} \check{C}_{r_3}^{-1} = \check{C}_{r_3} \check{C}_{r_2}^{-1} \check{C}_{r_1} \] (8.3)

as well as

\[
\check{B}_{r_1} \check{C}_{r_2}^{-1} \check{C}_{r_3}^{-1} + \check{C}_{r_1} \check{C}_{r_2}^{-1} \check{B}_{r_3} - \check{C}_{r_1} \check{C}_{r_2}^{-1} \check{B}_{r_3} \check{C}_{r_2}^{-1} \check{C}_{r_3}^{-1} = \]

\[ \check{C}_{r_1} \check{C}_{r_2}^{-1} \check{B}_{r_1} + \check{B}_{r_1} \check{C}_{r_2}^{-1} \check{C}_{r_1} - \check{C}_{r_1} \check{C}_{r_2}^{-1} \check{B}_{r_1} \check{C}_{r_2}^{-1} \check{C}_{r_1} \] \hspace{1cm} (8.4)

This is straightforward (with the aid of explicit formulas from the previous section). Indeed, from (7.8)

\[ \check{C}_{r_1} \check{C}_{r_2}^{-1} = \check{C}_{r_2}^{-1} \check{C}_{r_1} = \mathcal{A}^{r_1-r_2}, \] (8.5)
so that the entries in (8.3) are obviously identical:

\[ \tilde{\mathcal{C}}_{r_1} \tilde{\mathcal{C}}_{r_2}^{-1} \tilde{\mathcal{C}}_{r_3} = \mathcal{A}^{r_1 - r_2 + r_3} \tilde{\mathcal{C}} = \tilde{\mathcal{C}}_{r_3} \tilde{\mathcal{C}}_{r_2}^{-1} \tilde{\mathcal{C}}_{r_1} \]  

(8.6)

As for (8.2), it reduces to

\[ \mathcal{B}_{r_1} \mathcal{A}^{r_3 - r_2} + \mathcal{A}^{r_1 - r_2} \mathcal{B}_{r_3} - \mathcal{A}^{r_1 - r_2} \mathcal{B}_{r_2} \mathcal{A}^{r_3 - r_2} = \]

\[ = \mathcal{A}^{r_3 - r_2} \mathcal{B}_{r_1} + \mathcal{B}_{r_3} \mathcal{A}^{r_1 - r_2} - \mathcal{A}^{r_3 - r_2} \mathcal{B}_{r_2} \mathcal{A}^{r_1 - r_2} \]  

(8.7)

which is easily checked by substituting explicit expressions for \( \mathcal{B} \) from (8.6).

More sophisticated is the check of another subset of eqs. (7.1),

\[ \mathcal{F}_{S_i} = \mathcal{F}_{u r_2} \mathcal{F}_{u r_3} \]  

(8.8)

At the l.h.s. we have in our approximation:

\[ \left( \frac{1}{2 S_i} \tilde{\Pi}_i \tilde{\mathcal{B}}^{(i)} \right) \left( \begin{array}{ccc} 0 & \frac{1}{2} \tilde{\mathcal{C}}_{r_2}^{-1} & \frac{1}{2} \tilde{\mathcal{C}}_{r_2}^{-1} \tilde{\mathcal{C}}_{r_3} \end{array} \right) \left( \begin{array}{c} \tilde{\mathcal{B}}_{r_3} \ g \tilde{\mathcal{C}}_{r_3} \end{array} \right) = \]

\[ = \left( \frac{1}{2 S_i} \tilde{\Pi}_i \tilde{\mathcal{B}}^{(i)} \right) \left( \begin{array}{ccc} \tilde{\mathcal{C}}_{r_2}^{-1} \tilde{\mathcal{C}}_{r_3} & 0 \end{array} \right) \]

\[ = \left( \frac{1}{2 S_i} \tilde{\Pi}_i \tilde{\mathcal{C}}_{r_2}^{-1} \tilde{\mathcal{C}}_{r_3} + O(g^{-1}) \right) \left( \begin{array}{c} \tilde{\mathcal{B}}^{(i)} \tilde{\mathcal{C}}_{r_2}^{-1} \tilde{\mathcal{C}}_{r_3} \end{array} \right) \]  

(8.9)

WDVV equations do not really imply, however, that the entire matrix at the r.h.s. of (8.9) is symmetric. This is because some entries of \( \mathcal{F}_{S_i} \) are singular at \( S_i = 0 \), and therefore some linear-in-\( S \) contributions to \( \mathcal{F}_{u r_2} \) and \( \mathcal{F}_{u r_3} \) (in particular, those denoted by zeroes in (8.9)) will contribute even in the limit \( S = 0 \). It is easy to see that such contributions can arise at the upper-right corner of (8.9).

Thus one should check that the matrices at the upper-left and lower-right corners are symmetric only. The first one is,

\[ \tilde{\Pi}_i \tilde{\mathcal{C}}_{r_2}^{-1} \tilde{\mathcal{C}}_{r_3} = \tilde{\Pi}_i \mathcal{A}^{r_3 - r_2} \]  

(8.10)

This is trivially symmetric under an interchange of \( j \) and \( l \):

\[ (\tilde{\Pi}_i \mathcal{A}^{r_3 - r_2})_{jl} = \delta_{jl} \delta_{il} \mathcal{A}^{r_3 - r_2} \]  

(8.11)

Whether the second matrix

\[ \tilde{\mathcal{C}}^{(i)} \tilde{\mathcal{C}}_{r_2}^{-1} \tilde{\mathcal{C}}_{r_3} \]  

(8.12)
is symmetric is a little less trivial to check, since now (8.5) is unapplicable and one should use explicit expression (7.9) for \( \tilde{\mathcal{C}}_{-1} \):

\[
(\tilde{\mathcal{C}}^{(i)} \tilde{\mathcal{C}}^{-1}_{r_2} \tilde{\mathcal{C}}_{r_3})_{km} = \sum_{l} (\tilde{\mathcal{C}}^{(i)})_{kl} (\tilde{\mathcal{C}}^{-1}_{r_2} \tilde{\mathcal{C}}_{r_3})_{lm} = \sum_{l} (\tilde{\mathcal{C}}^{(i)})_{kl} (\tilde{\mathcal{C}}^{-1}_{r_2} \tilde{\mathcal{C}}_{r_3})_{lm} = \\
= \sum_{l,j} \frac{\alpha^{k+l}_{i,j}}{\Delta_j} \epsilon_{n-l-1}(\alpha)(-)^l \alpha^{r_3-r_2}_{j} \alpha^{m}_{j} \Delta_j
\]

Summing over \( l \) gives \( \delta_{ij} \Delta_i \) and the entire expression becomes equal to

\[
\frac{\alpha^{k+m+r_3-r_2}_{i}}{\Delta_i}
\]  

which is obviously symmetric under an interchange of \( k \) and \( m \).

As already mentioned, the remaining condition for the matrix (8.9) being symmetric reads

\[
\tilde{\mathcal{B}}^{(i)} \tilde{\mathcal{C}}^{-1}_{r_2} \tilde{\mathcal{C}}_{r_3} - \tilde{\mathcal{C}}_{r_3} \tilde{\mathcal{C}}^{-1}_{r_2} \tilde{\mathcal{B}}^{(i)} + \left( \tilde{\mathcal{C}}_{r_4} \tilde{\mathcal{C}}^{-1}_{r_2} \tilde{\mathcal{B}}_{r_3} - \tilde{\mathcal{B}}_{r_3} \tilde{\mathcal{C}}^{-1}_{r_2} \tilde{\mathcal{C}}_{r_4} \right) C^{-1}_{r_2} C^{-1} = - \frac{1}{2S_i} \tilde{\Pi}_i \tilde{\varepsilon},
\]  

and this depends on the \( S \)-linear contributions to the upper-right corner of \( \tilde{\mathcal{F}}^{-1}_{u_{r_2}} \tilde{\mathcal{F}}_{u_{r_3}} \), denoted by \( \tilde{\varepsilon} \) at the r.h.s. of (8.15). Manipulations similar to (8.13) transform the matrix element \( jm \) of the l.h.s. of (8.15) to

\[
-\frac{1}{2} \delta_{ij} \sum_{k \neq i} \frac{\alpha^{r_2}_{i}}{\alpha^{r_3}_{k} \Delta_k} (\alpha^{r_3-r_2}_{k} \alpha^{r_3-r_2}_{i} \alpha^{m-r_2}_{k} \alpha^{m-r_2}_{i}).
\]  

If the sum rule (2.9) could be applied to this sum over \( k \), one would immediately conclude that it vanishes. However, one of the contributions to (8.16) involves \( \alpha^{m+r_3-r_2}_{k} \), whose exponent is \( m+r_3-r_2 \). This last factor can be either negative or exceed \( n-1 \). Whenever this happens, eq. (2.9) is violated by contributions from residues at zero or infinity, i.e. the l.h.s. of (8.15) does not vanish and in this case the r.h.s. should not vanish as well. Since the proof of this statement takes us beyond the leading approximation, we do not go into further details here.

The last subset of eqs. (7.1),

\[
\tilde{\mathcal{F}}_{S_i} \tilde{\mathcal{F}}^{-1}_{u_{r_2}} \tilde{\mathcal{F}}_{S_k} = \tilde{\mathcal{F}}_{S_k} \tilde{\mathcal{F}}^{-1}_{u_{r_2}} \tilde{\mathcal{F}}_{S_i}
\]  

can be analyzed in a similar way.

As already mentioned, the WDVV identities with any \( \tilde{\mathcal{F}}^{-1}_{u_{S_j}} \) standing in place of \( \tilde{\mathcal{F}}^{-1}_{u_{r_2}} \) are not independent and do not require a separate validation. See [10].

**IX. Acknowledgements**

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