Mixing Times in Quantum Walks on Two-Dimensional Grids

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Mixing properties of discrete-time quantum walks on two-dimensional grids with torus-like boundary conditions are analyzed, focusing on their connection to the complexity of the corresponding abstract search algorithm. In particular, an exact expression for the stationary distribution of the coherent walk over odd-sided lattices is obtained after solving the eigenproblem for the evolution operator for this particular graph. The limiting distribution and mixing time of a quantum walk with a coin operator modified as in the abstract search algorithm are obtained numerically. On the basis of these results, the relation between the mixing time of the modified walk and the running time of the corresponding abstract search algorithm is discussed.

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I. INTRODUCTION

The concept of quantum walk is analogous to that of classical random walks, with the walker replaced by a quantum particle and coherent superpositions playing a key role [1]. There are discrete-time and a continuous-time versions of the quantum walk—the former defined by Aharonov et al. [2] and the latter, by Farhi and Gutmann [3]. Similarly to random walks, which have been used as a basis for classical algorithms that outperform their deterministic counterparts [4], quantum walks have also been used as a basis for quantum algorithms that outperform their classical correspondents [5–8].

Aharonov et al. [9] have presented important results on the theory of quantum walks on graphs, several of them concerning mixing-time properties. In particular, they have analyzed odd-sided N-cycles and showed that the quantum walk converges to a stationary distribution in time $O(n \log n)$, almost quadratically faster than the classical walk. They have also obtained bounds on mixing-times for general graphs. Mixing times on hypercubes were properly addressed by Marquezino et al. [11], who presented an analytical expression for the stationary distribution of a coherent discrete-time quantum walk on the hypercube, a Cayley graph with important algorithmic applications [5]. The stationary distribution is the first step in the calculation of the mixing time. An important graph is missing in the mixing-time picture: the two-dimensional grid. This grid, with torus-like boundary conditions, is the natural extension of the $N$-cycle to two dimensions. The quantum search in this graph is one of the first examples of abstract search algorithms [11, 12], a general framework for developing and analyzing quantum walk searches on graphs.

In this paper, we consider the mixing time of a discrete-time quantum walk on torus-like 2D-grids, and analyze the relation between these properties and the complexity of the corresponding abstract search. First, the eigenvector problem of the evolution operator for
the quantum walk on the two-dimensional torus is solved. This is an essential result for the analysis of many mathematical properties of the quantum walk. Then, the limiting probability distribution of the coherent walk in odd-sided lattices is derived, for an initial condition localized at the origin in a uniform superposition of coin states. In this case, the limiting probability distribution is found to have a maximum at the origin. The mixing time has been numerically calculated and we show that the walk mixes in time \( O(\sqrt{N \log N}) \), where \( N \) is the total number of vertices.

A quantum walk in which the coin is modified according to the prescription of the abstract search algorithm has also been considered. Its limiting distribution and its mixing time have been numerically obtained. These results imply a relation between the mixing time of the modified walk and the complexity of the corresponding abstract search algorithm. In previous works, the complexity of the quantum search algorithm was considered to be related only to the hitting time, while the mixing time was considered to be related to data sampling [13]. We show how previous knowledge of the limiting distribution of the quantum walk may be used to estimate the running time of an abstract search algorithm.

The paper is organized as follows. In Section II, we consider the coherent quantum walk on the two-dimensional finite grid with torus-like boundary conditions and solve the eigenvector problem of its evolution operator. In Section II B, the limiting distribution of the coherent walk for the case of odd-sided lattices is derived. In Section III A, the results of numerical simulations used to estimate the mixing time for the coherent evolution are presented. In Section III B, the relation between the mixing time and the running time of the abstract search algorithm on the two-dimensional grid is discussed. In Section IV, the main results are summarized and our conclusions are presented.

II. COHERENT WALK ON A TWO-DIMENSIONAL GRID

A coined quantum walk in a \( \sqrt{N} \times \sqrt{N} \) grid with periodical boundary conditions has a Hilbert space \( \mathcal{H}_C \otimes \mathcal{H}_P \), where \( \mathcal{H}_C \) is the 4-dimensional coin subspace and \( \mathcal{H}_P \) the \( N \)-dimensional position subspace. A basis for \( \mathcal{H}_C \) is the set \{\( |d, s\rangle \)\} for \( 0 \leq d, s \leq 1 \) and \( \mathcal{H}_P \) is spanned by the set \{\( |x, y\rangle \)\} with \( 0 \leq x, y \leq \sqrt{N} \), where we assume integer \( \sqrt{N} \). A generic state of the quantum walk is

\[
|\Psi(t)\rangle = \sum_{d,s=0}^{\sqrt{N}-1} \sum_{x,y=0}^{\sqrt{N}-1} \psi_{d,s;x,y}(t)|d,s\rangle|x,y\rangle.
\]  

(1)

The evolution operator for one step of the walk is

\[
U = S \cdot (C \otimes I),
\]

(2)

where \( I \) is the identity in \( \mathcal{H}_P \), \( S \) is the shift operator

\[
S = \sum_{d,s=0}^{1} \sum_{x,y=0}^{\sqrt{N}-1} |d, s + 1\rangle \langle d, s| \otimes |x + (-1)^s \delta_{d,0}, y + (-1)^s \delta_{d,1}\rangle \langle x, y|
\]

(3)

and \( C \) is a unitary coin operation in \( \mathcal{H}_C \). Notice that the binary sum \( \oplus \) in \( S \) inverts the direction, as required by the abstract search algorithm [11]. We shall consider the Grover coin, \( C = G \), given by

\[
G = 2|u\rangle \langle u| - I,
\]

(4)

where \( |u\rangle = \frac{1}{\sqrt{2}} \sum_{d,s=0}^{1} |d, s\rangle \) is the uniform superposition in \( \mathcal{H}_C \).
A. Eigenproblem for $U$

The analysis of the problem is simplified in Fourier space. Since the two-dimensional grid with periodic boundary conditions is a Cayley graph of $\mathbb{Z}^2 \sqrt{N}$, we use the Fourier transform on this group, which has a basis spanned by the $N$ kets

$$|k_x, k_y\rangle = \frac{1}{\sqrt{N}} \sum_{x,y=0}^{\sqrt{N}-1} \omega^{xk_x + yk_y} |x, y\rangle,$$

where $\omega = e^{2\pi i \sqrt{N}}$. The components of the evolution operator in the Fourier space are

$$\langle d, s, k'_x, k'_y | U | d', s', k_x, k_y\rangle = \omega^{(-1)^s (\delta_{d0}k_x + \delta_{d1}k_y)} G_{d,s\oplus 1; d', s', \delta_{k_x, k'_x} \delta_{k_y, k'_y}}. \quad (6)$$

For each $k_x, k_y$, we define a reduced evolution operator in the coin subspace given by

$$\tilde{G}_{d,s; d', s'} = \omega^{(-1)^s (\delta_{d0}k_x + \delta_{d1}k_y)} G_{d,s\oplus 1; d', s', \delta_{k_x, k'_x} \delta_{k_y, k'_y}}. \quad (7)$$

which is a $4 \times 4$ matrix that can be diagonalized. The eigenvectors of $U$ are tensor products of the eigenvectors of $\tilde{G}$ and $|k_x, k_y\rangle$.

Let us now describe the eigenspectrum of $\tilde{G}$. If $k_x = 0$ and $k_y = 0$, the eigenvalue 1 is three-fold degenerate and the eigenvectors are $\frac{1}{2} (1, -1, 0, 0)^T$, $\frac{1}{2} (0, 0, 1, -1)^T$ and $|u\rangle \equiv \frac{1}{2} (1, 1, 1, 1)^T$, where $(\ldots)^T$ is a column vector. The eigenvector with eigenvalue $-1$ is $\frac{1}{2} (1, 1, -1, -1)^T$. If $k_x \neq 0$ or $k_y \neq 0$, the eigenvalues are $\pm 1$ and $e^{\pm i\theta}$ with $\theta$ defined by

$$\cos \theta = \frac{1}{2} \left[ \cos \left( \frac{2\pi k_x}{\sqrt{N}} \right) + \cos \left( \frac{2\pi k_y}{\sqrt{N}} \right) \right]. \quad (8)$$

In this case, the eigenvectors of $\tilde{G}$ with eigenvalue $+1$ are

$$|\nu_{k_x, k_y}^{+1}\rangle = \frac{1}{4 \sin(\theta/2)} \begin{bmatrix} \omega^{k_x} (\omega^{k_y} - 1) \\ 1 - \omega^{k_y} \\ \omega^{k_y} (1 - \omega^{k_x}) \\ \omega^{k_x} - 1 \end{bmatrix}, \quad (9)$$

and those with eigenvalue $-1$ are

$$|\nu_{k_x, k_y}^{-1}\rangle = \frac{1}{4 \cos(\theta/2)} \begin{bmatrix} -\omega^{k_x} (1 + \omega^{k_y}) \\ - (1 + \omega^{k_y}) \\ \omega^{k_y} (1 + \omega^{k_x}) \\ 1 + \omega^{k_x} \end{bmatrix}. \quad (10)$$

Finally, the eigenvectors with eigenvalues $e^{i\theta}$ are

$$|\nu_{k_x, k_y}^{+\theta}\rangle = \frac{i}{2\sqrt{2} \sin \theta} \begin{bmatrix} e^{-i\theta} - \omega^{k_x} \\ e^{-i\theta} - \omega^{-k_x} \\ e^{-i\theta} - \omega^{k_y} \\ e^{-i\theta} - \omega^{-k_y} \end{bmatrix}. \quad (11)$$
and the eigenvectors with eigenvalue $e^{-i\theta}$ are obtained by replacing $\theta \to -\theta$ in Eq. (11). Note that the eigenvectors are normalized, $|\nu^{\pm\theta}_{k_x,k_y}\rangle$, form an orthonormal basis for the reduced space and have a real constant component on the uniform state, $\langle \nu^{\pm\theta}_{k_x,k_y} | u \rangle = 1/\sqrt{2}$.

We take the state
\begin{equation}
|\Psi(0)\rangle = |u\rangle |x = 0, y = 0\rangle
\end{equation}
as the initial condition, i.e., the walker starts localized at the point $(0,0)$ and uniformly distributed in the coin subspace. In the eigenbasis, this initial condition is given by
\begin{equation}
|\Psi(0)\rangle = \frac{1}{\sqrt{N}} |u\rangle |k_x = 0, k_y = 0\rangle + \frac{1}{\sqrt{2N}} \sum_{(k_x,k_y) \neq (0,0)}^{\sqrt{N}-1} \left( |\nu^{+\theta}_{k_x,k_y}\rangle + |\nu^{-\theta}_{k_x,k_y}\rangle \right) |k_x,k_y\rangle.
\end{equation}

Applying $U^t$ on $|\Psi(0)\rangle$ we obtain the state of the quantum walk after $t$ steps,
\begin{equation}
|\Psi(t)\rangle = \frac{1}{\sqrt{N}} |u\rangle |0,0\rangle + \frac{1}{\sqrt{2N}} \sum_{(k_x,k_y) \neq (0,0)}^{\sqrt{N}-1} \left( e^{i\theta t} |\nu^{+\theta}_{k_x,k_y}\rangle + e^{-i\theta t} |\nu^{-\theta}_{k_x,k_y}\rangle \right) |k_x,k_y\rangle.
\end{equation}

Having solved the eigenvector problem for the evolution operator, we proceed to the calculation of the limiting distribution for the quantum walk.

**B. Limiting Distribution**

Let $P(x,y,t)$ be the probability to find the walker at a vertex $(x,y)$ of the grid at time $t$. As mentioned in the introduction, this probability depends on the initial condition and, as is typical of unitary evolutions, it does not converge to a stationary distribution. However, the time-averaged distribution $\bar{P}(x,y,T) \equiv \frac{1}{T} \sum_{t=0}^{T-1} P(x,y,t)$ always converges as $T$ goes to infinity. The limiting or stationary distribution is then defined in terms of the average distribution,
\begin{equation}
\pi(x,y) \equiv \lim_{T \to \infty} \bar{P}(x,y,T).
\end{equation}

Using Theorem 3.4 from Ref. 9, the coefficients of the initial condition expressed in the eigenbasis of $U$, Eq. (13), and the fact that $\langle \nu^{+\theta}_{k'_x,k'_y} | \nu^{-\theta}_{k_x,k_y}\rangle = \langle \nu^{-\theta}_{k'_x,k'_y} | \nu^{+\theta}_{k_x,k_y}\rangle$, we obtain a simple expression for the limiting distribution,
\begin{equation}
\pi(x,y) = \frac{1}{N^2} + \frac{1}{N^2} \sum_{(k'_x,k'_y) \neq (0,0)} \langle \nu^{\theta}_{k'_x,k'_y} | \nu^{\theta}_{k_x,k_y}\rangle \omega^{x(k_x-k'_x)+y(k_y-k'_y)}.
\end{equation}

When $\theta(k'_x,k'_y) = \theta(k_x,k_y)$ we have
\begin{equation}
\langle \nu^{\theta}_{k'_x,k'_y} | \nu^{\theta}_{k_x,k_y}\rangle = \frac{1 - 2 \cos^2 \theta(k_x,k_y) + \cos \theta(k_x-k'_x,k_y-k'_y)}{2 \sin^2 \theta(k_x,k_y)}.
\end{equation}
Analyzing all cases such that \( \theta(k'_x, k'_y) = \theta(k_x, k_y) \), the expression for the limiting distribution, for odd \( \sqrt{N} \), has the explicit form

\[
\pi(x, y) = \frac{1}{N} + \frac{2}{N^2} \sum_{k_x=1}^{\sqrt{N}-1} \frac{1}{3 + \cos \tilde{k}_x} \left\{ \left[ (x - y) \cos \tilde{k}_x + \omega^{k_x(x+y)} \right] \left( 1 + \cos \tilde{k}_x \right) + 2 \left( \omega^{2k_xx} + \omega^{2k_xy} \right) \right\} + \\
\frac{1}{2N^2} \sum_{\substack{k_x, k_y=1 \\
 k_y \notin \{k_x, \sqrt{N} - k_x\}}}^{\sqrt{N}-1} \frac{1}{\sin^2 \theta} \left\{ \left[ \cos (\tilde{k}_x - \tilde{k}_y) - \cos 2\theta \right] \omega^{(k_x-k_y)x+(k_y-k_x)y} + \cos \tilde{k}_x - \cos \tilde{k}_y \right\}^{2 \omega^{2k_xx+2k_yy} + \left[ \cos (\tilde{k}_x + \tilde{k}_y) - \cos 2\theta \right] \omega^{(k_x+k_y)x+(k_x+k_y)y} }{ }\
\left[ \cos^2 \tilde{k}_x - \cos 2\theta \right] \omega^{2k_xx} \left[ \cos^2 \tilde{k}_y - \cos 2\theta \right] \omega^{2k_yy} + \right.
\]

(18)

where \( \tilde{k}_x = 2\pi k_x / \sqrt{N} \) and \( \tilde{k}_y = 2\pi k_y / \sqrt{N} \).

For some values of \( (x, y) \), it is possible to simplify Eq. (18) and achieve simple results. An interesting point to be considered is the initial site, which was assumed to be \( (x_0, y_0) = (0, 0) \), without loss of generality. It is straightforward to show that

\[
\pi(0, 0) = \frac{4N - 8\sqrt{N} + 5}{N^2}
\]

and, for \( N \gg 1 \), we obtain \( \pi(0, 0) \approx 4/N \), which is also the maximum of the limiting distribution. This distribution is shown in Fig. 1 for a grid of dimensions 41 \times 41, with the initial site shifted to \( (x_0, y_0) = (20, 20) \) for better visualization.

It is well known that the behavior of quantum walks on even lattices may be different than that observed for odd lattices [9, 14]. The analysis of the walk over odd lattices is sufficient for the main objective of this paper, namely to study the relation between mixing time and the complexity of quantum-walk based search algorithms. Our numerical simulations show that the limiting distribution for even lattices present two peaks, instead of the single peak on the initial site as observed for the odd lattice. Now that we have the limiting distribution for the quantum walk on the two-dimensional grid, we consider its mixing time and its relation to the abstract search problem.

### III. MIXING TIMES AND ABSTRACT SEARCH

In this subsection we consider the mixing time for a coherent evolution. The rate at which the average probability distribution of a quantum walk approaches its asymptotic distribution is captured by the following definition [9].
Figure 1: Left panel: Limiting distribution for quantum walk in two-dimensional grid with $\sqrt{N} = 41$, obtained from Eq. (18) with the initial condition (12). Right panel: contour plot for the same distribution.

Definition III.1 The average mixing time $M_\epsilon$ of a quantum Markov chain to a reference distribution $\pi$ is

$$M_\epsilon = \min \{ T \mid \forall t \geq T, \| \tilde{P}_t - \pi \| \leq \epsilon \},$$

where $\|A - B\| \equiv \sum_x |A(x) - B(x)|$ is the total variation distance between the two distributions.

An alternative definition captures the first instant in which the walk is $\epsilon$-close to the reference distribution $\pi$,

Definition III.2 The instantaneous mixing time $I_\epsilon$ of a quantum Markov chain is

$$I_\epsilon = \min \{ t \mid \| P_t - \pi \| \leq \epsilon \}.$$ 

Both mixing times depend on the initial condition of the quantum walk.

For a quantum walk in a generic graph with arbitrary initial condition, an upper bound for the total variation distance to the asymptotic distribution $\pi(x, y)$ was derived by Aharonov et al. [9],

$$\| \tilde{P}(x, y, T) - \pi(x, y) \| \leq \frac{\pi}{T\Delta} \left[ \log \left( \frac{Nd}{2} + 1 \right) \right],$$

(20)

where $N$ is the number of vertices, $d$ is the degree of each vertex and $\Delta$ is the minimum separation between distinct eigenvalues of $U$.

A. Mixing time in the two-dimensional grid

Let us now consider the particular case of a two-dimensional cartesian grid, for which $d = 4$. The value of $\Delta$ is the minimum value of $|e^{i\theta(k_x, k_y)} - e^{i\theta(k'_x, k'_y)}|$ for $k_x, k_y$ and $k'_x, k'_y$ in the range $[0, \sqrt{N} - 1]$. For large $N$ and small values of $k_x, k_y, k'_x$ and $k'_y$, we obtain

$$\Delta \approx \frac{\sqrt{2\pi}}{\sqrt{N}} \left| \sqrt{k_x^2 + k_y^2} - \sqrt{k'_x^2 + k'_y^2} \right|.$$

(21)
The minimum value of $\Delta$ is obtained taking values of $k_x$, $k_y$, $k'_x$ and $k'_y$ such that $\theta(k_x, k_y) \neq \theta(k'_x, k'_y)$ and

$$\sqrt{k_x^2 + k_y^2} \approx \sqrt{k'_x^2 + k'_y^2}.$$ 

We can make a good guess by exploring the structure of the above equation. With those guesses we can try to find an upper bound for the average mixing time, $M_\epsilon$. It is easy to obtain the value $O(1/\sqrt{N})$ for $\Delta$. Therefore, a good guess for $M_\epsilon$ is $O\left(\frac{\sqrt{N \log N}}{\epsilon}\right)$. By performing a numerical analysis, we have succeeded in obtaining better bounds regarding the dependence on $N$. Plotting $M_\epsilon$ against $\sqrt{N \log N}$ for several values of $\epsilon$ we obtain straight lines as shown in the right panel of Fig. 2. These results show that $M_\epsilon$ is proportional to $1/\epsilon^c$, where $c$ is approximately 1. Therefore, the numerical data strongly suggests that

$$M_\epsilon = \Theta\left(\frac{\sqrt{N \log N}}{\epsilon^c}\right).$$ \hspace{1cm} (22)

In the left panel of Fig. 2, we have the total variation distance to both the uniform and the stationary distribution. For long times, the variation distance to the stationary distribution decays approximately as $\sim 1/t$ while the corresponding distance to the uniform distribution remains essentially constant.

![Figure 2: Left panel: total variation distance to both the uniform and the stationary distributions of the quantum walk on the two-dimensional grid with flip-flop shift as a function of the time step. Right panel: mixing time to the stationary distribution as a function of the input size.](image)

The classical random walk on a two-dimensional grid increases with the size of the lattice as $\Theta(N)$—see, for instance, chapters 3 and 5 of Aldous and Fill [15]. Comparing this result with Eq. (22), we observe that the quantum walk mixes almost quadratically faster than its classical counterpart on the same lattice. The faster mixing rates observed in the quantum case is one of the main advantages of using quantum algorithms over their classical equivalents.

Until now, we have analyzed the standard quantum walk on the torus, without direct algorithmic applications. In the next section, we investigate the behavior of a quantum walk with a modified evolution operator which makes it useful to to mark a searched vertex of the grid.
B. Connection with abstract search algorithms

In this section we discuss the relation between the mixing time and the running time of the abstract search algorithm on the two-dimensional grid. The running time of search algorithms is usually associated with the notion of hitting time \[16\], which is defined as the first time a given vertex is reached. On the other hand, the mixing time is usually associated with data sampling \[13\]. We argue that those measures are related and both may be used to estimate the running time of abstract search algorithms.

The abstract search algorithm is a search framework introduced by AKR \[11\] based on a modified quantum walk. The standard quantum walk is driven by the evolution operator given by Eq. (2). The modified operator is

\[
U' = S \cdot C',
\]

where \(C'\) is given by

\[
C' = -I \otimes |x_0, y_0\rangle\langle x_0, y_0| + G \otimes \left(I - |x_0, y_0\rangle\langle x_0, y_0|\right).
\]

(23)

The modified operator \(C'\) applies the coin \(-I\) if the vertex is the target \(|x_0, y_0\rangle\), otherwise it applies Grover’s coin operation \(G\). The effect of \(C'\) is to mark the searched vertex with a relative phase. With this new evolution operator, AKR have shown that a quantum walker departing from the uniform distribution will be at the marked vertex after \(O(\sqrt{N \log N})\) steps with probability \(O(1/\log N)\). The time complexity of AKR’s algorithm is \(O(\sqrt{N \log N})\), after using the method of amplitude amplification \[17\]. Tulsi \[12\] has improved this result and have shown a method to reach the marked vertex with probability \(O(1)\). The time complexity of Tulsi’s algorithm is \(O(\sqrt{N \log N})\).

The effect of the modified evolution operator is to increase the probability of finding the walker in the marked vertex at some specific steps. The best point to stop is the one at which the probability is maximum. AKR described a method to find the running time without calculating directly the point of maximum and without using the notion of hitting time. On the other hand, Szegedy \[16\] has described a general method to obtain the hitting time. Szegedy’s method, however, requires a change on the evolution operator, which does not have the same time complexity of AKR’s method for searching one vertex on the two-dimensional grid.

Figure 3: Quantum walk in two-dimensional grid with \(\sqrt{N} = 41\) and a modified coin used to search for a marked node. Left panel: Probability distribution after \(t = 80\) steps, corresponding to the instant of maximum probability at the marked node. Right panel: stationary distribution approximated with \(T = 10^4\) simulation steps.

The left panel of Fig. 3 displays the probability distribution for the case \(\sqrt{N} = 41\) after 80 steps, which corresponds to the first maximum of the probability at the marked vertex.
The running time is the instantaneous mixing time using as reference distribution the one with maximum probability at the marked vertex and taking a small value of $\epsilon$. This fact cannot be used to estimate the running time, because there are no clues about how to find that specific probability distribution beforehand. On the other hand, there is a method to find \textit{a priori} the limiting distribution. Hence, an interesting question is whether the limiting distribution can be used to estimate the running time. The right panel of Fig. 3 displays the stationary distribution. Note the remarkable similarity between the distribution with maximum probability and the stationary distribution.

The first thing we need to check is how the mixing time of the modified walk scales with $N$. The right panel of Fig. 4 strongly suggests that replacing $C$ by $C'$ defined in Eq. (23) does not alter the scaling of the average mixing time, since for $N \gg 1$ it scales as in eq. (22). There is a small oscillation in the mixing time function for large $N$, related to the way the modified walk approaches the limiting distribution. In the the left panel of Fig. 4, we have the total variation distance to both the uniform and the stationary distribution. For long times, the variation distance approaches the stationary distribution with a strong oscillation and several local minima, corresponding to the steps when the instantaneous probability distribution has a maximum at the searched node.

![Figure 4](image_url)

**Figure 4:** Left panel: total variation distance to both the uniform and the stationary distributions of the quantum walk in two-dimensional grid with modified coin used to search for a marked node. Right panel: mixing time to the stationary distribution as a function of the input size.

This behavior is remarkably different from the one observed in the last section for the Grover walk on the two-dimensional grid (Fig. 2). In that case the mixing time increases without oscillations and the total variation distance decays almost as a power law.

**IV. DISCUSSION**

In this paper, the mixing properties of a discrete-time quantum walk on the finite two-dimensional grid with torus-like boundary conditions has been considered in detail. This particular topology is quite interesting due to its algorithmic applications. The relation between mixing time and the time complexity of the corresponding abstract search algorithm has also been investigated.

The eigenvector problem of the evolution operator for the coherent quantum walk has been solved and the corresponding stationary distribution has been found analytically for the
particular case of odd lattices and a localized initial condition with a uniform superposition of coin states. The stationary distribution is not uniform and has a maximum at the origin—for which we have also provided a simplified expression. According to our numerical simulations, the mixing time $M_\varepsilon$ on the two-dimensional grid with $N$ vertices increases as $O\left(\frac{\sqrt{N \log N}}{\varepsilon^c}\right)$, i.e., almost quadratically faster than the mixing time of the classical random walk, which is $\Theta(N)$. The value of $c$ is approximately 1. The faster mixing rates on the quantum case is an advantage of using quantum-walk based algorithms over the classical equivalents.

We have also considered a quantum walk in which the coin is modified according to the prescription of the abstract search algorithm. We have numerically calculated its limiting distribution and its mixing time. Our numerical simulations show that the mixing time $M_\varepsilon$ for this particular walk also increases as $O\left(\frac{\sqrt{N \log N}}{c}\right)$, where $c$ is approximately 1. This mixing time of the modified quantum walk corresponds to the time complexity of marking a searched vertex on the torus using the abstract search algorithm [12]. This result establishes a relation between the mixing time of the modified walk and the running time of the corresponding abstract search algorithm—which previously has been associated only to the hitting time. This relation may be useful for the complexity analysis of quantum-walk based algorithms.

In order to obtain the exact running time of the quantum algorithm, one should also consider the constants that have been absorbed by the asymptotic notation used to represent the mixing time. As a future work, it would be interesting to investigate the mixing time of search algorithms on other graphs and compare the corresponding constants.

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