Quantization of Holomorphic Poisson structure
–related to Generalized Kähler structure–

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Abstract: It is known that holomorphic Poisson structures are closely related to theories of generalized Kähler geometry and bi-Hermitian structures. In this article, we introduce quantization of holomorphic Poisson structures which are closely related to generalized Kähler structures/bi-Hermitian structures. By resulting noncommutative product $\star$ obtained via quantization, we also demonstrate computations with respect to concrete examples.

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1 Introduction

There are results obtained in [1, 7, 8, 9], in which we can find relations among holomorphic Poisson structures, theories of generalized Kähler geometry and bi-Hermitian structures. Here we first note that generalized Kähler geometry has been developed in the framework of Courant algebroids. In 1990, the Courant bracket $[\cdot, \cdot]$ on $TM \oplus T^*M$, where $M$ is a manifold, was introduced by T. Courant [3]. A similar bracket was defined on the double of any Lie algebroid by Liu, Weinstein and Xu [19]. They proved that the double is not a Lie algebroid, but a more complicated object which is called a Courant algebroid today.

The original definition of Courant algebroid in the article [19], (see also Definition 2.1 of the present article), has five conditions which are including strange anomalies with respect to Jacobi identity and derivation rules, etc. In their article, they proposed a different definition with a slightly modification of the original bracket of the Courant algebroid. In fact, in Ph. D. thesis “Courant algebroids, derived brackets and even symplectic super manifolds” by D. Roytenberg [36], he showed the equivalence of original definition and new definition. Relating to this topics, note that K. Uchino [40] showed that two of

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1We recall the statement about this. See Theorems 2.12, 2.13 and 2.14.
the conditions and the relation (assuming the Leibniz rule) which are included in the original definition of Courant algebroid follow from the rest of the conditions. Moreover, descriptions of the Courant bracket via derived bracket was given (for more details, see articles Y. Kosmann-Schwarzbach [15] and D. Roytenberg [36]. See also [16, 17]).

On the other hand, “generalized complex structure” was introduced by N. Hitchin [10]. Roughly speaking, generalized almost complex structure \( J \) is defined as a section \( SO(TM \oplus T^*M) \) satisfying \( J^2 = -1 \), where \( SO \) is the special orthogonal group with respect to the symmetrization of the natural pairing \( \langle \cdot, \cdot \rangle \) between \( TM \) and \( T^*M \). It is known that a generalized almost complex structure \( J \) can be represented \( J = J_\Phi \) via kernel and complex conjugate-kernel of a nondegenerate pure spinor \( \Phi \). Furthermore, it is also shown that generalized almost complex structure \( J_\Phi \) is integrable, that is \( ker \Phi \) is involutive with respect to the Courant bracket, if and only if there exists a section \( E \in \Gamma(TM \oplus T^*M)^\mathbb{C} \) such that \( d\Phi = E \cdot \Phi \), where \( \cdot \) stands for natural spinor representation of \( CL(TM \oplus T^*M) \) on \( \wedge^* T^*M \) via the interior product and exterior product. Using frame-work Courant algebroids, theory of generalized complex/Kähler structure was developed by M. Gualtieri [7]. A generalized Kähler structure is a pair \( (J_1, J_2) \) of generalized complex (i.e. almost complex which is involutive with respect to the Courant bracket \([\cdot, \cdot]\)) structure s.t. \( J_1 J_2 = J_2 J_1 \) and \( (−J_1 J_2 , \cdot) \) is a positive definite symmetric form. As mentioned below, it is known that a generalized Kähler structure corresponds to a bi-Hermitian structure [7]. Furthermore, a complex manifold admitting bi-Hermitian structure has a holomorphic Poisson structure. Under this situation, in this article, we study quantization for holomorphic Poisson structures which appeared in generalized Kähler structures/bi-Hermitian structures. More precisely, we introduce a notion of quantization/formal symbol calculus, and study fundamental properties. By resulting noncommutative product \( \star \) from quantization using the above Poisson structure, we demonstrate computations with respect to concrete examples for \( \star \)-exponential functions. First, using holomorphic Poisson structures, we establish a notion of quantization/formal symbol calculus of projective schemes. This means that under a suitable condition, we construct deformations of ring structures on projective schemes. We also study a concrete example, that is, formal symbol calculus on a projective spaces \( \mathbb{C}P^n \), we demonstrate concrete computations of \( \star \)-exponential functions. Precisely speaking, when germs of all admissible functions of holomorphic Poisson structures contains all germs of structure sheaf, we construct deformations of ring structures for structure sheaves on a projective scheme. Using holomorphic Poisson structures, as will be seen later, which is a typical example of Dirac structure, we consider symbol calculus on a projective space. Since the structure sheaves on algebraic varieties

\[2\] The terminology “symbol calculus” is used in Fourier analysis and pseudo-differential operator in order to study partial differential equations [11, 12, 13, 37]. Especially, symbol calculus of pseudo-differential operator with respect to elliptic operator gives fruitful contribution to the index theorem. The notion of pseudo-differential operators was used to extend the class of possible deformation of an elliptic operator which has essential topological data of the base manifold.
have important and essential feature which plays crucial role to analyze their fundamental properties with respect to the base varieties, construction of symbol calculus using a holomorphic Poisson structure on the base variety is very interesting. We here state the main theorems of the present article.

**Theorem 1.1** Assume that \( Z = [z_0 : z_1 : \ldots : z_n] \) is the homogeneous coordinate system of \( \mathbb{CP}^n \), and \( \Lambda = \sum_{\alpha, \beta} \partial_{Z_\alpha} \Lambda^{\alpha, \beta} = \partial_{Z_{\beta_1}} \partial_{Z_{\beta_2}} \) defines a holomorphic skew-symmetric biderivation\(^3\) of order zero acting on the structure sheaf \( \mathcal{O}_{\mathbb{CP}^n} \) satisfying the Jacobi rule and an assumption below:

\[
(\partial_{Z_{\alpha_1}} \Lambda^{\alpha_1, \beta_1} \partial_{Z_{\beta_1}}) \cdots (\partial_{Z_{\alpha_k}} \Lambda^{\alpha_k, \beta_k} \partial_{Z_{\beta_k}}) = \partial_{Z_{\alpha_1} \cdots \alpha_k} \Lambda^{\alpha_1, \beta_1 \cdots \beta_k} \partial_{Z_{\beta_1} \cdots \beta_k},
\]

where

\[
f(Z) \partial_{Z_{\alpha_1} \cdots \alpha_k} (\text{resp. } \partial_{Z_{\beta_1} \cdots \beta_k} g(Z))
\]

means

\[
\partial_{Z_{\alpha_1}} \partial_{Z_{\alpha_2}} \cdots \partial_{Z_{\alpha_k}} f(Z) \quad (\text{resp. } \partial_{Z_{\beta_1}} \partial_{Z_{\beta_2}} \cdots \partial_{Z_{\beta_k}} g(Z)).
\]

Then, for any point “p” and germs \([f(Z)] \ [g(Z)]\) of the stalk \( \mathcal{O}_{\mathbb{CP}^n, p}[[z]] \)\(^4\)

\[
:= \left[ f(Z) \ast g(Z) \right] = \left[ \sum_{k=0}^{\infty} \frac{1}{k!} (\mu)^k \frac{\Lambda^{\alpha_1, \beta_1} \Lambda^{\alpha_2, \beta_2} \cdots \Lambda^{\alpha_k, \beta_k}}{\partial_{Z_{\alpha_1}} \partial_{Z_{\alpha_2}} \cdots \partial_{Z_{\alpha_k}} f(Z) \partial_{Z_{\beta_1}} \partial_{Z_{\beta_2}} \cdots \partial_{Z_{\beta_k}} g(Z)} \right]
\]

defines a non-commutative and associative ring structure, where \( \mu \) is a formal parameter.

We also have

**Theorem 1.2** Under the same assumptions and notations of Theorem 1.1, the product \( \ast \) induces globally defined non-commutative, associative product on the sheaf-cohomology space \( \sum_{k=0}^{\infty} H^0(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k))[[z]] \) where \( \mu \) can be specialized a scalar (for example \( \mu = 1 \)).

Using the product \( \ast \), we also have

**Theorem 1.3** Suppose that the same assumptions of Theorem 1.1. Let \( A[Z] := Z A^2 Z \) be a quadratic form with homogeneous degree 2. Then we have

\[
\frac{e^{A[Z]}}{e^{3/2}} = \det^{-1/2} \left( e^{\Lambda A + \Lambda \Lambda} \right) e^{1/\mu} \left( \frac{1}{2} \tan(\sqrt{\Lambda A}) \right)[Z]
\]

\[
\in \sum_{k=0}^{\infty} H^0(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k)[\mu, \mu^{-1}]).
\]

Note that our argument is able to apply to weighted projective spaces, for which one can find details elsewhere.

\(^3\)We use Einstein’s convention unless confusing.

\(^4\)Here \([z] \) denotes either \([\mu, \mu^{-1}] \) or \([\mu, \mu^{-1}]\), and we have to choose carefully in context.
2 Courant algebroid

In this section, we review fundamentals of Courant algebroid (See [19], [36]).

2.1 Courant algebroid, Lie bialgebroid, and Maurer-Cartan type equation

We start this subsection with:

Definition 2.1 A Courant algebroid is a vector bundle $E \to M$ equipped with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ on the bundle, a skew-symmetric bracket $[,]$ on the space $\Gamma(E)$ of all sections of $E$ and a bundle map $\rho : E \to \text{TP}$ such that the following properties are satisfied:

1. $[[[e_1, e_2], e_3]] + \text{c.p.} = DT(e_1, e_2, e_3), \forall e_1, e_2, e_3 \in \Gamma(E),$

2. $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)], \forall e_1, e_2 \in \Gamma(E),$

3. $[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - (e_1, e_2)Df, \forall e_1, e_2 \in \Gamma(E), \forall f \in C^\infty(M),$

4. $\rho \circ D = 0$, i.e. $(Df, Dg) = 0, \forall f, g \in C^\infty(M),$

5. $\rho(e)(h_1, h_2) = ([e, h_1] + D(e, h_1), h_2) + (h_1, [e, h_2] + D(e, h_2)), \forall e, h_1, h_2 \in \Gamma(E),$

where “c.p.” denotes cyclic permutation, $T(e_1, e_2, e_3)$ is the function on the base manifold $M$ defined by

$$T(e_1, e_2, e_3) = \frac{1}{3}([[e_1, e_2], e_3]) + \text{c.p.}, \quad (4)$$

and $D : C^\infty(M) \to \Gamma(E)$ is the map defined by $D = \frac{1}{2}\beta^{-1}\rho^*d_0$, where $\beta$ is the isomorphism between $E$ and $E^*$ given by the bilinear form. In other words,

$$(Df, e) = \frac{1}{2}\rho(e)f. \quad (5)$$

Note that Courant algebroid gives a non-trivial example of $L_\infty$-algebra. (cf. [36], [14].)

Next we recall Dirac structure.

Definition 2.2 Let $E$ be a Courant algebroid. A subbundle $L$ of $E$ is called isotropic if it is isotropic under the symmetric bilinear form $(\cdot, \cdot)$. $L$ is said to be almost Dirac if $L$ is maximally isotropic. Furthermore, $L$ is said to be Dirac if $L$ is almost Dirac and integrable (involutive), that is, $\Gamma(L)$ is closed under the bracket $[,]$. 

It is obvious that an integrable isotropic subbundle $L$ of a Courant algebroid $(E,\rho,\langle\cdot,\cdot\rangle)$ is a Lie algebroid with $\rho|_L$ and $\langle\cdot,\cdot\rangle$.

It is known that a Lie bialgebroid is a pair $(A,A^*)$ of vector bundles in duality, each of which is a Lie algebroid, such that the differential defined by one of them on the exterior algebra of its dual is a derivation of the Schouten bracket. Related to this definition, it is also known that

**Theorem 2.3 ([20])** If a pair of Lie algebroids $(A,A^*)$ in duality, then the differential $d$ is a derivation of $(\Gamma(A),\langle\cdot,\cdot\rangle)$ if and only if the differential $d_*$ is a derivation of $(\Gamma(A),\langle\cdot,\cdot\rangle)$.

Assume that both $A$ and $A^*$ are Lie algebroids over the base manifold $M$, with anchors $a$ and $a_*$ respectively. Let $E := A \oplus A^*$. We can naturally define non-degenerate bilinear forms in the following way:

$$(X_1 + \xi_1, X_2 + \xi_2)_\pm = \frac{1}{2}(\langle\xi_1, X_2\rangle \pm \langle\xi_2, X_1\rangle).$$

(6)

On $\Gamma(E)$, we can introduce a bracket by

$$\langle e_1, e_2 \rangle = \langle [X_1, X_2] + L\xi_1 X_2 - L\xi_2 X_1 - d_*(e_1, e_2)\rangle$$

$$+ \langle \xi_1, \xi_2 \rangle + L\xi_1 \xi_2 - L\xi_2 \xi_1 + d(e_1, e_2),$$

where $e_1 = X_1 + \xi_1, e_2 = X_2 + \xi_2$.

Furthermore, let $\rho : E \to TM$ be the bundle map defined by

$$\rho(X + \xi) = a(X) + a_*(\xi), \quad \forall X \in \Gamma(A), \xi \in \Gamma(A^*).$$

(8)

Finally, we define the operator $D$ by

$$D = d_* + d$$

(9)

where $d_* : \mathcal{C}^\infty(M) \to \Gamma(A)$ and $d : \mathcal{C}^\infty(M) \to \Gamma(A^*)$ are the usual differential operator associated to Lie algebroids. Under the above notations, it is known that

**Theorem 2.4 ([19])** If $(A,A^*)$ is a Lie bialgebroid, then $E = A \oplus A^*$ together with $([\cdot,\cdot],\rho,\langle\cdot,\cdot\rangle_\pm)$ is a Courant algebroid. Conversely, in a Courant algebroid $(E,\langle\cdot,\cdot\rangle,\rho,\langle\cdot,\cdot\rangle)$, suppose that $L_1, L_2$ are Dirac subbundle transversal to each other, i.e. $E = L_1 \oplus L_2$. Then, $(L_1, L_2)$ is a Lie bialgebroid, where $L_2$ is considered as the dual bundle of $L_1$ under the pairing $\langle\cdot,\cdot\rangle$. Hence, if $(A,A^*)$ is a Lie bialgebroid, then $(A^*,A)$ is also a Lie bialgebroid.

Next, we recall the fundamentals of Hamiltonian operators. Assume that $(A,A^*)$ is a Lie algebroid. Suppose that $H : A^* \to A$ is a bundle map, and denote by $A_H$ the graph of $H$, considered as a subbundle of $E = A \oplus A^*$, that is, $A_H = \{H\xi + \xi | \xi \in A^*\}$. Under these notations, one can prove the following important fact:

5When $(A,A^*) = (g,g^*)$, i.e. Lie bialgebra, the bracket reduces to the Lie bracket of Manin on the double $g \oplus g$. If $A = TM$, $A^* = T^*M$ the the above bracket takes the form:

$$[X_1 + \xi_1, X_2 + \xi_2] = [X_1, X_2] + \{L\xi_1 \xi_2 - L\xi_2 \xi_1 + d(e_1, e_2)\}.$$ This bracket is just known as the original Courant bracket.
Theorem 2.5 ([19]) \( A_H \) is a Dirac subbundle (i.e. maximally isotropic, and involutive with respect the Courant bracket \([\ , \]) if and only if \( H \) is skew-symmetric and satisfies the following Maurer-Cartan type equation:

\[
d_*H + \frac{1}{2}[H,H] = 0,
\]

where \( H \) is considered as a section of \( \wedge^2 A \).

See Theorem 6.1, Definition 6.2 and Corollary 6.3 in [19] for details.

**Remark** Note that the equation (10) in Theorem 2.5 plays essential roles in order to discuss deformation theory of generalized Kähler structure.

Now we recall new definition of Courant algebroid (cf. [19] and [36]).

**Definition 2.6** A Courant algebroid is a vector bundle \( E \to M \) equipped with a non-degenerate symmetric bilinear form \((\ , \)\) on the bundle, a (not necessarily skew-symmetric) bracket \([\ , \] \) on the space \( \Gamma(E) \) of all sections of \( E \) and a bundle map \( \rho : E \to TP \) such that the following properties are satisfied:

1. \([e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_1, [e_2, e_3]], \forall e_1, e_2, e_3 \in \Gamma(E),\)
2. \(\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)], \forall e_1, e_2 \in \Gamma(E),\)
3. \([e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2, \forall e_1, e_2 \in \Gamma(E), \forall f \in \mathcal{C}^\infty(M),\)
4. \([e, e] = \frac{1}{2}D\langle e, e \rangle = 0, \forall f, g \in \mathcal{C}^\infty(M),\)
5. \(\rho(e)(h_1, h_2) = ([e, h_1] + [e, h_2], h_2) + (h_1, [e, h_2]) \forall e, h_1, h_2 \in \Gamma(E),\)

where

**Proposition 2.7** Relation between original Courant bracket \([\ , \]) and new Courant bracket \([\ , \] \) is given in the following way:

\([e_1, e_2] = [e_1, e_2] + \frac{1}{2}D\langle e_1, e_2 \rangle.\)

Conversely,

\([e_1, e_2] = \frac{1}{2}([e_1, e_2] - [e_2, e_1]).\)

Notice that the notion of a Dirac subbundle remains unchanged when switch to the new definition of a Courant algebroid. (See [36].)

**Example** There are so many examples of Lie bialgebroids, and then Courant algebroids. For instance, Poisson manifolds, Nijenhuis manifolds, Poisson-Nijenhuis manifolds, objects constructed by solutions of classical Yang-Baxter equations, etc. See [15] [16] [17] [19] [20].
2.2 Dirac structure and generalized complex structure

Let $M$ be a complex manifold. Define sheaves on $M$ by

$$TM_p = \text{sheaf of germs of holomorphic vector fields around } p$$
$$T^*M_p = \text{sheaf of germs of holomorphic one forms around } p$$

We use symmetric and skew-symmetric bilinear operations on a sheaf $T^*M := TM \oplus T^*M$ as

$$\langle (X, \xi), (Y, \eta) \rangle := \frac{1}{2} \{ \xi(Y) \pm \eta(X) \}$$
and

$$[(X, \xi), (Y, \eta)] := ([X, Y], L_X \eta - L_Y \xi - i_Y d\langle (X, \xi), (Y, \eta) \rangle)$$
for all $(X, \xi), (Y, \eta) \in TM$. Here $L_X$ means the Lie derivative by $X$ and $i_Y$ stands for the interior product by $Y$. As mentioned in the previous section, a subbundle $D \subset TM$ is called a Dirac structure if the following conditions are satisfied:

1. $(D1)$ $\langle \cdot, \cdot \rangle|_D = 0$;
2. $(D2)$ rank$(D)$ is equal to dim$_C(M)$;
3. $(D3)$ $[TM, TM] \subset TM$.

A complex manifold $M$ together with holomorphic Dirac structure $D \subset TM$ is called a holomorphic Dirac manifold and denoted by $(M, D)$.

The followings are well-known.

**Example 2.1** Assume that $M$ be a complex manifold with a holomorphic presymplectic form $\omega$. Then the 2-form $\omega$ induces the bundle map

$$\omega^\flat : \Gamma(M) \to \Omega^1(M); \ X \mapsto i_X \omega,$$

where $\Gamma(X)$ denotes the space consisting of all germs of holomorphic vector fields, and $\Omega^1(M)$ denotes the space consisting of all germs of holomorphic 1-differential forms. Then one can obtain the subbundle $\text{graph}(\omega^\flat)$ in $TM$ as

$$\text{graph}(\omega^\flat)_m := \{(X_m, i_X m \omega_m) \in T_m M \oplus T^*_m M | X_m \in T_m M \} \quad (m \in M)$$
and can verify that $\text{graph}(\omega^\flat)$ satisfies the conditions $(D1)$-$(D3)$. Therefore, $(M, \text{graph}(\omega^\flat))$ defines a Dirac structure.

The following is also a typical example.

**Example 2.2** Assume that $M$ be a complex manifold with a holomorphic Poisson structure $\varphi$. Then the Poisson structure $\varphi$ induces the bundle map

$$\varphi^\# : \Omega^1(M) \to \Gamma(M); \ \alpha \mapsto \{ \bullet \mapsto \varphi(\bullet, \alpha) \},$$
where $\Gamma(X)$ and $\Omega^1(M)$ are as above. Then one can obtain the subbundle $\text{graph}(\varphi^\#)$ in $TM$ as

$$\text{graph}(\varphi^\#)_m := \{(\varphi^\#(x), \xi_m) \in T_x M \oplus T^*_m M \mid \xi_m \in T^*_m M\} \quad (m \in M)$$

and can verify that $\text{graph}(\varphi^\#)$ satisfies the conditions (D1)-(D3). Therefore, $(M, \text{graph}(\varphi^\#))$ defines a Dirac structure.

For each point $m$, a holomorphic Dirac structure $D \subset TM$ defines two natural projections as:

$$\rho_m := pr_1|_{D_m} : D_m \to T_x M \quad \text{and} \quad \rho^*_m := pr_2|_{D_m} : D_m \to T^*_x M.$$  

**Proposition 2.8** As for the above projections, we obtain the followings:

$$\ker \rho = D \cap (\{0\} \oplus T^* M) \quad \text{and} \quad \ker \rho^* = D \cap (TM \oplus \{0\}) \cdots (\ast 1)$$

and

$$\text{Im} \rho = (D \cap (\{0\} \oplus T^* M))^\circ \quad \text{and} \quad \text{Im} \rho^* = (D \cap (TM \oplus \{0\}))^\circ \cdots (\ast 2)$$

where the symbol $\circ$ stands for the annihilator.

To the end of this section, we recall the fundamental facts related to generalized complex geometry: Based on the original Courant bracket of the direct sum $TM = TM \oplus T^* M$ over a manifold $M$. The fiber bundle of the direct sum $TM$ admits an indefinite metric $\langle \cdot, \cdot \rangle$ by which we obtain the fiber bundle $SO(TM)$ with fiber the special orthogonal group. As mentioned above, an almost generalized complex structure $\mathcal{J}$ is defined as a section of the fiber bundle $SO(TM)$ with $\mathcal{J}^2 = -1$, which gives rise to the decomposition $TM \oplus \mathbb{C} = \mathcal{L}_{\mathcal{J}} \oplus \bar{\mathcal{L}}_{\mathcal{J}}$, where $\mathcal{L}_{\mathcal{J}}$ (resp. $\bar{\mathcal{L}}_{\mathcal{J}}$) $\mathcal{J}$-eigenspace (resp. $\overline{\mathcal{J}}$-eigenspace). Almost generalized complex structures form an orbit of the action of the real Clifford group of the real Clifford algebra bundle $CL$ with respect to $(TM, \langle \cdot, \cdot \rangle)$.

**Definition 2.9** A generalized complex structure is an almost generalized complex structure which is integrable/involutive with respect to the original Courant bracket. A generalized Kähler structure is a pair $(\mathcal{J}_0, \mathcal{J}_1)$ consisting of commuting generalized complex structures such that $G := -\mathcal{J}_0 \mathcal{J}_1$ is a generalized metric.

We here recall the definition of bi-Hermitian structure:

**Definition 2.10** A quadruplet $(h, J^+, J^-, b)$ is said to be a bi-Hermitian structure if $h$ is a Riemannian metric, $J^+, J^-$ are complex structures, and $b$ is a real 2-form satisfying:

1. The Riemannian metric $h$ is Hermitian metric with respect to $J^+$ (resp. $J^-$).
2. Let $\omega_{\pm}$ be the fundamental 2-forms with respect to $J^\pm$. Then,
\[ d_+ \omega_+ = -d_- \omega_- = db, \]
where $\partial_+, \bar{\partial}_+$ are operators defined by complex structures $J^\pm$, and we put $d_\pm := \sqrt{-1}(\partial_+ - \partial_+)$. Then we have

**Theorem 2.11** [7] A bi-Hermitian structure gives a generalized Kähler structure. Conversely, a generalized Kähler structure gives a bi-Hermitian structure.

Furthermore, it is known that

**Theorem 2.12** [3] A complex manifold admitting bi-Hermitian structure has a holomorphic Poisson structure.

Note that on a complex surfaces, Poisson structures are non-trivial section of anti-canonical bundle $K^{-1}$. Therefore it is well-known about holomorphic Poisson structures and complex Poisson surfaces. Moreover, it is known that

**Theorem 2.13** [8, 9] A Poisson-Kähler manifold has non-trivial bi-Hermitian structures.

Summing up, through generalized Kähler structures, there are close relations between holomorphic structures and bi-Hermitian structures.

### 3 Main results:

In this section, we show the main results and its super version.

#### 3.1 Proofs of Theorems 1.1, 1.2 and 1.3

In order to give of proofs of main results, we begin with a slight modification of the standard theory of scheme. Let $S = \oplus_{n=0}^{\infty} S_n$ be a graded commutative ring. Then, $S_0$ is obviously commutative and $S$ is an $S_0$-algebra. It is well-known that the homogeneous ideal $S_+ := \oplus_{n=1}^{\infty} S_n$ is called the irrelevant ideal. Then we have

**Proposition 3.1** A graded commutative ring $S$ is noetherian if and only if $S_0$ is noetherian and $S$ is finitely generated by $S_1$ as an $S_0$-algebra.

It is known that a projective scheme

\[ \text{Proj}(S) := \{ p : \text{a homogeneous prime ideal} \mid \neg(S_+ \subset p) \} \]

admits the canonical scheme structure in the following way: Set

\[ D_+(f) := \{ p \in \text{Proj}(S) \mid \neg(f \in p) \}, \]
for any homogeneous element \( f \in S_d \) with degree \( d \), then the family \( \{ D_+(f) \} \) forms a basis of open sets. Hence it gives the canonical topology \( \mathcal{O}_{\text{Proj}(S)} \) (that is, the Zariski topology) for \( \text{Proj}(S) \). Note that \( -(f \in p) \) means \( f(\text{point}_p) \neq 0 \), intuitively. We also set

\[
\Gamma(D_+(f), \mathcal{O}_{\text{Proj}(S)}) := \{ g/f^m \mid g \in S_m, \ m \geq 0 \},
\]

\[
\mathcal{O}_{\text{Proj}(S)} : \mathcal{O}_{\text{Proj}(S)} \ni D_+(f) \mapsto \Gamma(D_+(f), \mathcal{O}_{\text{Proj}(S)}) \in \text{Mod}.
\]  

(11)

The functor above is the structure sheaf. We remark that when \( g \in S_0 \), we easily see that \( g/1 = fg/f \ (fg \in S_d) \). Hence we may consider \( g/f^m \ (m \geq 1) \) instead of \( g/f^m \ (m \geq 0) \). We obtain that R.H.S. of (11) is a part of degree 0 of localization \( S_f \) of \( S \) by a product closed set \( \{ f^t \}_{t=0,1,2,...} \). We denote it by \((S_f)_0 \) or \( S_f \). Strictly speaking, for any homogeneous element \( f \) with \( \text{deg}(f) = d \),

\[
(S_f)_0 := S_f := \{ g/f^m \mid g \in S_{md}, \ m \geq 0 \}.
\]

Hence summing up what mentioned above, we have

**Proposition 3.2** As for \( D_+(f) \),

\[
(D_+(f), \mathcal{O}_{\text{Proj}(S)}|_{D_+(f)}) \cong \text{Spec}(S_f).
\]  

(12)

Thus, \( \text{Proj}(S) \) is obtained by glueing of affine schemes. It indicates that \((\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)})\) is a scheme in the genuine sense.

Next we consider cohomology of quasi-coherent sheaf over \( \text{Proj}(S) \). Assume that a graded ring \( S \) is generated by \( S_1 \) as an \( S_0 \)-algebra. For instance

\[
S = R[z_0, z_1, \ldots, z_n], \quad S_0 = R, \quad S_1 = \{ a \in S \mid \text{deg}(a) = 1 \}.
\]

As for a quasi-coherent sheaf\(^6\) \( \mathcal{F} \), we set

\[
\mathcal{F}(m)[[z]] := \mathcal{F} \otimes_{\mathcal{O}_{\text{Proj}(S)}} \mathcal{O}_{\text{Proj}(S)}(m)[[z]],
\]

and define

\[
\Gamma_*(\mathcal{F}) := \oplus_{m \in \mathbb{Z}} \Gamma(X, \mathcal{F}(m))[[z]], \quad \text{deg}(a) := m, \quad (\forall a \in \Gamma(X, \mathcal{F}(m))).
\]  

(13)

Then we see that \( \Gamma_*(\mathcal{F}[[z]]) \) is a graded \( \Gamma(\mathcal{O}_{\text{Proj}(S)}[[z]]) \)-module. For any element \( f \in S_d \), we set \( \alpha_d(f) := a/1 \). Then it is well-known that

**Proposition 3.3** The map \( \alpha_d \) obtained above defines a homomorphism

\[
\alpha_d : S_d[[z]] \ni a \mapsto a/1 \in S(d)_f = \Gamma(D_+(f), \mathcal{O}_{\text{Proj}(S)}(d)[[z]]).
\]  

(14)

A family \( \{ \alpha_d(f) \}_f \) : homogeneous induces a module homomorphism

\[
\alpha_d : S_d[[z]] \rightarrow \Gamma(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)}(d)[[z]]).
\]  

(15)

---

\(^6\) A sheaf \( \mathcal{F} \) is quasi-coherent if and only if there is a pre-sheaf exact sequence \( \mathcal{O}^0_{U_j} \rightarrow \mathcal{O}^1_{U_j} \rightarrow \mathcal{F} \rightarrow 0 \). A sheaf \( \mathcal{F} \) is coherent if and only if there is a pre-sheaf exact sequence \( \mathcal{O}^0_{U_j} \rightarrow \mathcal{F} \rightarrow 0 \) \((n \in \mathbb{N})\).
Hence, using the module homomorphisms \( \{\alpha_d\} \), a graded ring homomorphism

\[
\alpha := \bigoplus_{n=0}^{\infty} \alpha_d : S = \bigoplus_{n=0}^{\infty} S_d[z] \to \Gamma(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)}[z])
\]

(16)

can be defined for any quasi-coherent sheaf \( \mathcal{F} \). Thus, \( \Gamma(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)}[z]) \) admits a graded \( S[[z]] \)-module structure.

**Definition 3.4** We denote the pair \((\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)}[z])\) by \( \text{Proj}(S)[[z]] \).

We are also interested in \( \Gamma^*(\mathcal{F}[[z]])_f \). As for any element \( f \in S_d \), and \( x \in \Gamma(\text{Proj}(S), \mathcal{F}[[z]])_f \), we see \( x/f^n \in \Gamma(\text{Proj}(S), \mathcal{F}[[z]])_f \). We denote the restriction of \( x \) to \( D_+(f) \) by \( x|_{D_+(f)} \). Then matching the degrees of \( x|_{D_+(f)} \) and \( (\alpha_d|_{D_+(f)})^n \) we get \( x|_{D_+(f)}/(\alpha_d|_{D_+(f)})^n \).

\[
\beta(f) : \Gamma^*(\text{Proj}(S), \mathcal{F}[[z]])_f \ni x \mapsto \frac{x|_{D_+(f)}}{(\alpha_d|_{D_+(f)})^n} \in \Gamma(D_+(f), \mathcal{F}[[z]]).
\]

(17)

As similarly for \( \alpha \), for any homogeneous element \( g \in S_e \), we obtain a diagram:

\[
\begin{array}{ccc}
\Gamma^*(\mathcal{F}[[z]])_f & \longrightarrow & \Gamma(D_+(f), \mathcal{F}[[z]]) \\
& \downarrow & \\
\Gamma^*(\mathcal{F}[[z]])_{fg} & \longrightarrow & \Gamma(D_+(fg), \mathcal{F}[[z]])
\end{array}
\]

By a similar argument as above, we define an \( \text{Proj}(S)[[z]] \)-module homomorphism

\[
\beta_F : \Gamma^*(\mathcal{F}[[z]]) \to \mathcal{F}[[z]].
\]

(18)

**Proposition 3.5** Assume that a graded ring \( S \) is generated by \( S_1 = \{f_1, f_2, \ldots, f_\ell\} \) \( (\exists \ell \in \mathbb{Z}_{\geq 0}) \) as \( S_0 \)-algebra. Then we see

(i) If \( S \) is a domain then the map \( \alpha \) induced in (16) is injective.

(ii) If \( \{f_i\}_{i=1,2,\ldots,\ell} \) are all prime ideals, then the map \( \alpha \) is an isomorphism.

(iii) When \( S = k[z_0, z_1, z_2, \ldots, z_n] \), then the map \( \alpha \) is an isomorphism.
Proof. It is almost obvious. However I also give a sketch of the proof for convenience.

Injectivity of α. For any element \( a \in S_\mu[[\mu]] \), if \( \alpha_m(a) = 0 \), then we have

\[
\alpha(a)(D_+(f_i)) = (a/f_i^{m_i}) \cdot (f_i^{m_i}/1) = 0,
\]

by the definition of the map \( \alpha_m \). Hence we see that \( a/f_i^{m_i} = 0 \) (\( \forall i = 1, 2, \ldots, n \)).

Hence there exists a positive integer \( N >> 0 \) such that \( f_i^N a = 0 \) (\( \forall i = 1, 2, \ldots, n \)). Since \( S[[\mu]] \) is generated by \( \{f_1, f_2, \ldots, f_n\} \) as an \( S_\mu[[\mu]] \)-algebra, and is a domain, we have \( a = 0 \). This shows that the map \( \alpha \) is an injective map.

Surjectivity of α. We also assume that \( (f_i) \) is a prime ideal (\( \forall i = 1, 2, \ldots, n \)).

For any element \( h \in \Gamma(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)}(m[[\mu]])) \), we can write

\[
h|_{D_+(f_i)} = \frac{b_i}{f_i^{m_i}} \cdot (f_i^{m_i}/1) \in \Gamma(D_+(f_i), \mathcal{O}_{\text{Proj}(S)}(m_1)f_i, b_i \in S_m.
\]

We can assume that \( m_i \geq m \). Note again that \( D_+(f_i) \cap D_+(f_j) = D_+(f_i, f_j) \).

We also have the following:

Proposition 3.6 Assume that \( \mathcal{F}[[z]] \) is a quasi-coherent sheaf \( \text{Proj}(S)[[[z]] \)-module. Then the homomorphism \( \beta_\mathcal{F} \) induced in \( (10) \) is an isomorphism. Furthermore, when \( S = k[z_0, z_1, \ldots, z_n] \) via \( (19) \), we see

\[
H^0(\text{Proj}(k[z_0, z_1, \ldots, z_n][[z]]) = \left\{ \begin{array}{ll}
0 & (\text{if } m < 0), \\
k[z_0, z_1, \ldots, z_n][m] & (\text{o.w. } m \geq 0).
\end{array} \right.
\]

Definition 3.7 Assume that \( \Lambda \) is a holomorphic skew-biderivation satisfying Jacobi rule. Then, \( \ast \) is called symbol calculus on \( \text{Proj}(k[z_0, z_1, \ldots, z_n]) \) if \( (\text{Proj}(k[z_0, z_1, \ldots, z_n][[z]], \ast) \) has an associatiive algebra sheaf structure such that

\[
f(Z) \ast g(Z) = f \cdot g + \left( \frac{\mu}{2} \right) \Lambda^\alpha \partial_{Z_\alpha} f(z)g(Z) + \cdots,
\]

where \( f, g \) stands for germs of structure sheaf and \( Z = [z_0, z_1, \ldots, z_n] \).

We are now in the position to give proofs of Theorems 1.1, 1.2 and 1.3. Under the assumption 11, it is easy to check

\[
f(Z) \sum_{k=0}^\infty \frac{1}{k!} \left( \frac{\mu}{2} \right)^k \Lambda^\alpha_1 \frac{\partial}{\partial Z_{\alpha_1}} \cdots \frac{\partial}{\partial Z_{\alpha_k}} \Lambda^\beta_k \frac{\partial}{\partial Z_{\beta_k}} g(Z) = \sum_{k=0}^\infty \frac{1}{k!} \left( \frac{\mu}{2} \right)^k \Lambda^\alpha_1 \Lambda^\alpha_2 \cdots \Lambda^\alpha_k \partial_{Z_{\alpha_1}} f(Z) \partial_{Z_{\beta_2}} \cdots \partial_{Z_{\beta_k}} g(Z).
\]

Then the right hand side of \( (22) \) coincides with the asymptotic expansion formula for product of the Weyl type pseudo-differential operators. Thus, it shows Theorem 1.1.
As seen in the previous argument, for sheaf cohomology of projective space, we obtain that

$$\sum_{k=0}^{\infty} H^0(\mathbb{C}P^n, \mathcal{O}_{\mathbb{C}P^n}(k)) \cong \sum_{k=0}^{\infty} \mathbb{C}[Z]_k,$$

where $\mathbb{C}[Z]_k$ stands for the space of homogeneous polynomials of degree $k \in \mathbb{Z}_{\geq 0}$. Then a direct computation using (22) shows that $\mu$ can be specialized a scalar.

Finally we show Theorem 1.3. We would like to compute exponentials having the following form $f(Z) = g(t)e^{1/\mu Q[Z](t)}$ with respect to $\star$ for quadratic polynomials under a quite general setting. Let $Z = [z^1 : \ldots : z^n]$, $A[Z] := ZA'Z$, where $A \in \text{Sym}(n, \mathbb{C})$, i.e. $A$ is an $n \times n$-complex symmetric matrix. In order to compute the exponential $F(t) := e^{1/\mu A[Z]}$ with respect to the Weyl type product formula, we treat the following evolution equation:

$$\partial_t F = \frac{1}{\mu} A[Z] \ast F,$$

with an initial condition

$$F_0 = e^{1/\mu B[Z]},$$

where $B \in \text{Sym}(n, \mathbb{C})$.

As seen above, since the coefficients of bivectors are note constant, our setting might be seen rather different one from the situations considered in the appendix of the present article. However, to compute exponentials, we can use similar methods with Cayley transform for homogeneous coordinates systematically, as will be seen below:

Under the assumption $F(t) = g \cdot e^{1/\mu Q[Z]}$ ($g = g(t)$, $Q = Q(t)$), we would like to find a solution of the equations (24) and (25).

Direct computations give

$$\text{L.H.S. of (24)} = g' e^{1/\mu Q[Z]} + \frac{1}{\mu} Q'[Z] e^{1/\mu Q[Z]},$$

$$\text{R.H.S. of (24)} = \frac{1}{\mu} A[Z] \cdot F + \frac{i\hbar}{2} A^{i_{1}j_{1}} \partial_{i_{1}} \frac{1}{\mu} A[Z] \cdot \partial_{j_{1}} F$$

$$- \frac{\hbar^2}{2 \cdot 4} A^{i_{1}j_{1}} A^{i_{2}j_{2}} \partial_{i_{1}i_{2}} \frac{1}{\mu} A[Z] \partial_{j_{1}j_{2}} F$$

(26)

where $A = (A_{ij}), \Lambda = (A^{ij})$ and $Q = (Q_{ij})$. Comparing the coefficient of $\mu^{-1}$ gives

$$Q'[Z] = A[Z] - 2 \mu A A Q[Z] - Q A A Q[Z].$$

(27)

7See the article [22] and in [31, 35] for original methods. Quillen’s method employing the Cayley transform is very useful to compute superconnection character forms and supertrace of Dirac-Laplacian heat kernels.
Applying \( \Lambda \) by left and setting \( q := \Lambda Q \) and \( a := \Lambda A \), we easily obtain

\[
\Lambda Q' = \Lambda A + \Lambda QAA - \Lambda A\Lambda Q - \Lambda QAAAQ
\]

\[= (1 + q)a(1 - q). \tag{28}
\]

As to the coefficient of \( \mu^0 \), we have

\[
g' = \frac{1}{2} \Lambda_{ij} \Lambda_{j2} A_{i1} g_{Qi,j2}
\]

\[= -\frac{1}{2} \text{tr}(aq) \cdot g, \tag{29}
\]

where “\text{tr}” means the trace. Summing up, we have

**Proposition 3.8** The equation (27) is rewritten by

\[
\partial_t q = (1 + q)a(1 - q), \tag{30}
\]

\[
\partial_t g = -\frac{1}{2} \text{tr}(aq) \cdot g. \tag{31}
\]

In order to solve the equations (30) and (31), we now recall the “Cayley transform.”

**Proposition 3.9** Set

\[
C(X) := \frac{1 - X}{1 + X} \tag{32}
\]

if \( \det(1 + X) \neq 0 \). Then

1. \( X \in \text{sp}_\Lambda(n, \mathbb{R}) \iff \Lambda X \in \text{Sym}(n, \mathbb{R}) \),

and then \( C(X) \in \text{Sp}_\Lambda(n, \mathbb{R}) \), where

\[
\text{sp}_\Lambda(n, \mathbb{R}) := \{ g \in GL(n, \mathbb{R}) | g^t \Lambda g = \Lambda \},
\]

\[
\text{Sp}_\Lambda(n, \mathbb{R}) := \text{Lie}(\text{Sp}_\Lambda(n, \mathbb{R})).
\]

2. \( C^{-1}(g) = \frac{1 - g}{1 + g} \), (the “inverse Cayley transform”).

3. \( e^{2\sqrt{-1}a} = c(-\sqrt{-1}\tan(a)) \).

4. \( \log a = 2\sqrt{-1} \arctan(\sqrt{-1}C^{-1}(g)) \).

5. \( \partial_t q = (1 + q)a(1 - q) \iff \partial_t C(q) = -2aC(q) \).

**Proof** We only show the assertion 5.

\[
C(q)' = \left(\frac{1 - q}{1 + q}\right)'
\]

\[= (1 + q)^{-1}(-q)' + (1 + q)^{-1}(-q)'(1 + q)^{-1}(1 - q)
\]

\[= -a(1 - q) - a(1 - q)(1 + q)^{-1}(1 - q)
\]

\[= -a\left(1 + \frac{1 - q}{1 + q}\right)(1 - q)
\]

\[= -2aC(q). \tag{33}
\]
This completes the proof.

Solving the above equation 5 in Proposition 3.9 we have
\[ C(q) = e^{-2at}C(b), \]
where \( b = AB \) and then
\[ q = C^{-1}(e^{-2at} \cdot C(b)) = C^{-1}(C(-\sqrt{-1} \tan(\sqrt{-1}at)) \cdot C(b)). \]
Hence, according to the inverse Cayley transform, we can get \( Q \) in the following way.

**Proposition 3.10**
\[ Q = -\Lambda \cdot C^{-1}(C(-\sqrt{-1} \tan(\sqrt{-1}AAt)) \cdot C(AB)). \] (34)

Next we compute the amplitude coefficient part \( g \). In order to find a solution, we consider the following.
\[ g' = -\frac{1}{2} \text{tr}(aq) \cdot g \] (35)

Thus, we have

**Proposition 3.11**
\[ g = \det \left( \frac{e^{at}(1 + b) + e^{-at}(1 - b)}{2} \right). \] (36)

**Proof** First we replace
\[ g' = -\frac{1}{2} \text{tr}(aq) \cdot g \] (37)
by
\[ (\log g)' = -\frac{1}{2} \text{tr}(aq). \] (38)

We also have
\[
\text{tr}\left\{ \log \left( \frac{e^{at}(1 + b) + e^{-at}(1 - b)}{2} \right) \right\}' \\
= \text{tr}\left\{ e^{at}(1 + b) - e^{-at}(1 - b) \right\} \\
= \text{tr}(aq).
\]

Combining this formula with (38), we obtain
\[
(\log g)' = -\frac{1}{2} \text{tr}\left\{ \log \left( \frac{e^{at}(1 + b) + e^{-at}(1 - b)}{2} \right) \right\}' \\
= -\frac{1}{2} \log \left\{ \det \left( \frac{e^{at}(1 + b) + e^{-at}(1 - b)}{2} \right) \right\}'.
\]
This completes the proof.
Setting \( t = 1, \ a = \Lambda A \) and \( b = 0 \), we get

**Proposition 3.12**

\[
e^{\frac{1}{\tau} A[Z]} = \det -\frac{1}{2} \left( e^{\tau AA} + e^{-\tau AA} \right) \cdot e^{\frac{1}{2\tau} \frac{\Lambda^{-1}}{\sqrt{-1}} \tan(\sqrt{-1} \tau AA)] [Z]. \tag{39}
\]

As usual, using the Čech resolution, we can compute the sheaf-cohomology \( \sum_{k=0}^{\infty} H^0(\mathbb{C}P^n, \mathcal{O}_{\mathbb{C}P^n}[\mu, \mu^{-1}]) \). Combining it with Proposition 3.12 we see

\[
e^{\frac{1}{\tau} A[Z]} \in \sum_{k=0}^{\infty} H^0(\mathbb{C}P^n, \mathcal{O}_{\mathbb{C}P^n}[\mu, \mu^{-1}])
\]

This completes the proof of Theorem 1.3.

Suppose that \( \beta \) is a holomorphic Poisson structure Then it satisfies the Maurer-Cartan equation in Theorem 2 (cf. [19]) and the adjoint action of \( e^{\beta \tau} \) on \( J \) in the sense of [9] induces an analytic family of deformations of generalized complex structures. We write it by \( J_{\beta \tau} = \text{Ad}(e^{\beta \tau}) J \). Under this situation, the following was proved.

**Theorem 3.13 ([9])** Let \( \beta \) be a holomorphic Poisson structure on a compact Kähler manifold \( X \). Then we have a family of generalized Kähler structures denoted by \( \{ J_{\beta \tau}, \psi_{\tau} \} \).

Here we call a parameter \( \tau \) a deformation parameter. Then by Theorems 1.1, 1.2 and 1.3, we have the following.

**Theorem 3.14** Under the same assumptions as above with condition (7), we have a family of associative product \( \tau_{\tau} \) with deformation parameter \( \tau \) by quantizing a family of holomorphic Poisson structures \( \{ \beta \tau \} \). Furthermore, a family of star exponentials \( \{ e^{\frac{1}{\tau} A[Z]}_{\tau} \} \) exists. More precisely,

\[
e^{\frac{1}{\tau} A[Z]}_{\tau} = \det -\frac{1}{2} \left( e^{\tau AA} + e^{-\tau AA} \right) \cdot e^{\frac{1}{2\tau} \frac{\Lambda^{-1}}{\sqrt{-1}} \tan(\sqrt{-1} \tau AA)] [Z]. \tag{40}
\]

We also obtain

**Corollary 3.15** The differences of deformation parameter \( \tau \) can be detected in the sheaf-cohomology \( \sum_{k=0}^{\infty} H^0(\mathbb{C}P^n, \mathcal{O}_{\mathbb{C}P^n}[\mu, \mu^{-1}]) \).

Note that the method employed in this subsection can be extended to the case of the base variety is a weighted projective space (cf. [29]).
3.2 Super-twistor version

Consider the following diagram:

\[
\begin{align*}
((x^{\alpha,\dot{\alpha}}, [\pi_1 : \pi_2]) & \in M := \mathbb{C}^4 \times \mathbb{CP}^1 \\
\Pi_1 & \rightarrow \Pi_2 \\
([z_1 : \ldots : z_4]) & \in \mathbb{CP}^3 \\
(x^{\alpha,\dot{\alpha}}) & \in \mathbb{C}^4
\end{align*}
\]

where \(x^{\alpha,\dot{\alpha}}\) are even variables. We set

\[
(x^{\alpha,\dot{\alpha}}) := (x^{1,1}, x^{1,2}, x^{2,1}, x^{2,2}),
\]

\[
([z_1 : \ldots : z_4]) := ([x^{\alpha,1}; x^{\alpha,2}; \pi_1 : \pi_2]).
\]

Here we use Einstein’s convention (we will often omit \(\sum\) unless there is a danger of confusion). We call \(([z_1 : \ldots : z_4])\) the homogeneous coordinate system of \(\mathbb{CP}^3\).

1. The relations \(\widehat{\alpha}, \widehat{\beta} = \hat{1}, \hat{2}\)

\[
[z^{\dot{\alpha}}, z^{\dot{\beta}}] = \hbar D^{\alpha,\dot{\alpha},\beta,\dot{\beta}} \pi_\alpha \pi_\beta,
\]

where \(z^{\dot{1}} := z_1, z^{\dot{2}} := z_2\), give a globally defined non-commutative associative product \(#\) on \(\mathbb{CP}^3\), where \((D^{\alpha,\dot{\alpha},\beta,\dot{\beta}})\) is a skew symmetric matrix.

2. Let \(A[Z]\) be a homogeneous polynomial of \(z^{\dot{1}} = z_1 = x^{\alpha,1} \pi_\alpha, z^{\dot{2}} = z_2 = x^{\alpha,2} \pi_\alpha\) with degree 2. Then a sharp exponential function \(e^{\frac{1}{\hbar} A[Z]}\) gives a “function” on \(\mathbb{CP}^3\).

More precisely,

**Theorem 3.16** Assume that \(\Lambda := \Lambda\) and \(A[Z]\) a homogeneous polynomial of \(z^{\dot{1}} = x^{\alpha,1} \pi_\alpha, z^{\dot{2}} = x^{\alpha,2} \pi_\alpha\) with degree 2. Then a sharp exponential function \(e^{\frac{1}{\hbar} A[Z]}\) gives a cohomology class of \(\mathbb{CP}^3\) with coefficients in the sheaf \(\sum_{k=0}^{\infty} O_{\mathbb{CP}^3}(k)\).

\(^8\)Here \([,]\) denotes the commutator bracket.
where $\Pi$ denotes the chiral projection, we can consider non-anti-commutative deformation of super twistor space.

In order to give a brief explanation, we recall the definition of super twistor manifold ([18, 39, 41]).

**Definition 3.17** $(3|N)$-dimensional complex super manifold $Z$ is said to be a super twistor space if the following conditions $(1) - (3)$ are satisfied.

1. $p : Z \to \mathbb{C}P^1$ is a holomorphic fiber bundle.
2. $Z$ has a family of holomorphic section of $p$ whose normal bundle is isomorphic to $\mathcal{O}_{\mathbb{C}P^1}(1) \oplus \mathcal{O}_{\mathbb{C}P^1}(1) \oplus C^N \otimes \Pi \mathcal{O}_{\mathbb{C}P^1}(1)$.
3. $Z$ has an anti-holomorphic involution $\sigma$ being compatible with $(1)$, $(2)$ and $\sigma$ has no fixed point.

We define $\mathbb{C}P^3_{3|N} = (\mathbb{C}P^3, \mathcal{O}_{\mathbb{C}P^3|N}, \alpha^0)$.

Let $f(z|\xi; \alpha')$ be a local section defined in the following manner:

$$f(z|\xi; \alpha') = \sum_{k=0}^{N} \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq N} f_{i_1i_2 \cdots i_k}(z) \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k}$$

where $f_{i_1 \cdots i_k}(z)$ is a homogeneous element of $z = [z_1 : z_2 : z_3 : z_4]$ with homogeneous degree $(-k)$ on $\mathbb{C}P^3$. Then we can introduce a structure sheaf $\mathcal{O}_{\mathbb{C}P^3|N, \ast \alpha'}$, whose local section is given by $f(z|\xi; \alpha')$.

Under these notations, we can introduce a ringed space denoted by $\mathbb{C}P^3_{3|N} = (\mathbb{C}P^3, \mathcal{O}_{\mathbb{C}P^3|N, \ast \alpha'})$. We shall call it non-anti-commutative complex projective super space.

As for the non-anti-commutative deformed product $\ast$ associated with the non-anti-commutative complex projective super space, we have commutation relations of local coordinate functions:
(1) Let \((z_1, z_2, \pi_1, \pi_2|\xi^1, \ldots, \xi^N)\) be a local coordinate system of \(\mathcal{P}^{3|N}\), where \(\mathcal{P}^{3|N}\) denotes the non-anti-commutative open super twistor space. Then
\[
\{\xi^i, \xi^j\}_* = \alpha^* C^{i\alpha,j\beta} \pi_\alpha \pi_\beta, \quad (0 \text{ o.w.})
\]

(2) A local coordinate system \((z_1, z_2, \pi_1, \pi_2|\xi^1, \ldots, \xi^N)\) of \(\mathbb{C}^{3|N}\) satisfies
\[
\{\xi^i, \xi^j\}_* = \alpha^* C^{i\alpha,j\beta} \pi_\alpha \pi_\beta, \quad (0 \text{ o.w.})
\]

These arguments leads us quantization of body-part and soul-part of super-twister spaces. Here we do not explain more the notion and notations which appeared above and do not give the proof of them. For details, see [39].

4 Appendix: Riccati-type equation appeared in \(*\)-transcendental elements

Star-exponential/transcendental elements are important objects in deformation quantization (cf. [2, 4, 6, 14, 38, 42]). In the present section, for convenience, we explain about transcendally extended Weyl algebras which gave a crew for our study. See [22, 21, 30, 31, 32, 33, 34], about relating topics.

Let \(\mathcal{G}(n)\) (resp. \(\mathfrak{A}(n)\)) be the spaces of complex symmetric matrices (resp. skew-symmetric matrices). Note that \(\mathfrak{M}(n)=\mathcal{G}(n) \oplus \mathfrak{A}(n)\). For an arbitrary fixed \(n \times n\)-complex matrix \(\Lambda \in \mathfrak{M}(n)\), we define a product \(*_\Lambda\) on the space of polynomials \(\mathbb{C}[u]\) by the formula
\[
f *_\Lambda g = f e^{\frac{1}{\hbar}\left(\sum \bar{\partial}_{u_1} \Lambda_{ij} \partial_{u_j} \right)} g = \sum_k \left(\frac{i\hbar}{2}\right)^k \Lambda_{i_1 j_1} \cdots \Lambda_{i_k j_k} \partial_{u_{i_1}} \cdots \partial_{u_{i_k}} f \partial_{u_{j_1}} \cdots \partial_{u_{j_k}} g. \tag{43}
\]

\(\partial_{u_i}\) acts as a derivation of the algebra \((\mathbb{C}[u], *_\Lambda)\) in the biderivation \(\sum \bar{\partial}_{u_i} \Lambda_{ij} \partial_{u_j}\). Then \((\mathbb{C}[u], *_\Lambda)\) is an associative algebra. Remark that it is not necessary commutative, in general. If the matrix \(\Lambda = (\Lambda_{ij})\) is symmetric, then the obtained algebra is commutative and it is isomorphic to the standard polynomial algebra with \(\hbar\).

Note that, for any other constant symmetric matrix \(K\), we can also define a new product \(*_{\Lambda + K}\) by the formula
\[
f *_{\Lambda + K} g = f e^{\frac{1}{\hbar}\left(\sum \bar{\partial}_{u_i} \Lambda_{ij} \partial_{u_j} \right)} g = \sum_k \left(\frac{i\hbar}{2}\right)^k (\Lambda + K)_{i_1 j_1} \cdots (\Lambda + K)_{i_k j_k} \partial_{u_{i_1}} \cdots \partial_{u_{i_k}} f \partial_{u_{j_1}} \cdots \partial_{u_{j_k}} g. \tag{44}
\]

where we used the following formula:
\[
e^{\frac{i\hbar}{\hbar}\left(\sum \bar{\partial}_{u_i} \partial_{u_i} \right)} = f e^{\frac{1}{\hbar}\left(\sum K_{ij} \partial_{u_i} \partial_{u_j} \right)} g = f *_{\Lambda + K} g. \tag{45}
\]
Before discussing the next topics, we fix notations. Set \( \Lambda = K + J \) where \( K, J \) are the symmetric part and the skew part of \( \Lambda \), respectively. Then the commutator is \([u, u_j] = i\hbar J_{ij} \). We also use

\[
u = (u_1, u_2, \cdots, u_{2m}) = (u, v), \quad u = (u_1, \cdots, u_m), \quad v = (v_1, \cdots, v_m). \tag{46}
\]

and use the standard skew-symmetric matrix \( J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \).

Note that according to the choice of \( K = 0, K_0, -K_0 \) where

\[
(0, K_0, -K_0) = \left( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix} \right).
\]

When \( K = 0 \) (resp. \( K_0, K_0 \)), the ordering is called Weyl ordering (resp. normal ordering, anti-normal ordering). For each ordered expression, the product formulas are given respectively by the following formulas:

\[
\begin{align*}
&f(u) e_{s,*} g(u) = f \exp \left( \frac{i\hbar}{2} \left( \overrightarrow{D}_s \cdot \overleftarrow{D}_u \right) \right) g, \quad \text{(Moyal product formula)} \\
&f(u) e_{s,k} g(u) = f \exp \left( i\hbar \left( \overrightarrow{D}_s \cdot \overleftarrow{D}_u \right) \right) g, \quad \text{(\( \Psi DO \)-product formula)} \\
&f(u) e_{s,-k} g(u) = f \exp \left( -i\hbar \left( \overrightarrow{D}_s \cdot \overleftarrow{D}_u \right) \right) g, \quad \text{(\( \overline{\Psi DO} \)-product formula)}
\end{align*}
\]

where \( \overrightarrow{D}_s \cdot \overleftarrow{D}_u = \sum_i (\overrightarrow{D}_s \cdot \overleftarrow{D}_u) \) and \( \overrightarrow{D}_s \cdot \overleftarrow{D}_u = \sum_i \overrightarrow{D}_s \cdot \overleftarrow{D}_u \).

For \( f(u) \in Hol(\mathbb{C}^{2m}) \), the direct calculation via the product formula (43) by using Taylor expansion gives the following:

\[
e^{\frac{i\hbar}{2} (a,u)_* f(u) = e^{\frac{i\hbar}{2} (a,u)_k} f(u) = e^{\frac{i\hbar}{2} (a,u)_k} f(u + \frac{i\hbar}{2} a(K+J)), \\
(f(u) e_{s,k} g(u) = f(u + \frac{i\hbar}{2} a(-K+J))e^{-\frac{i\hbar}{2} (a,u)}
\]

as natural extension of the product formula. This also gives the associativity of computations involving two functions of exponential growth and a holomorphic function.

We use notation \( \cdot \cdot_{*k} \) which stands for the ordering (\( K \)-ordered expression parameter) for elements of Weyl algebra \( W_h(2m) \). For instance, we write

\[
\cdot \cdot_{*k} = u_i u_j + \frac{i\hbar}{2} (K+J)_{ij}, \quad \text{etc.}
\]

Set \( H_s = au^2 + bv^2 + 2cu = \frac{1}{2} (u+u+v+u) \), and \( c^2 - ab = D \). It is easy to see that \( H_s = 0 = au^2 + bv^2 + 2cu = \text{Weyl ordered expression for } e^{t(au^2 + bv^2 + 2cu = t)}. For the purpose, we set \( e^{t(au^2 + bv^2 + 2cu = t)} : 0 = F(t, u, v) \) and consider the real analytic solution of the evolution equation

\[
\frac{\partial}{\partial t} F(t, u, v) = (au + bv + 2cu)^* F(t, u, v), \quad F(0, u, v) = 1. \tag{49}
\]

By the product formula (44), we have

\[
(au^2 + bv^2 + 2cu)^* F(t, u, v) = \left\{ (au + bv + 2cu)F + h i \{(bv + cu)\overrightarrow{D}_u F - (au + cv)\overrightarrow{D}_u F \} - \frac{\hbar^2}{4} \{b\overrightarrow{D}_u^2 F - 2c\overrightarrow{D}_u F + a\overrightarrow{D}_u^2 F \}. \right.
\]

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By using a function $f(x)$ of one variable
\[ e^{t(au^2+bv^2+2cuv)} \] for $t(au^2+bv^2+2cuv)$, we get a simplified form
\[
\begin{align*}
& (au^2+bv^2+2cuv) *_0 f_t(au^2+bv^2+2cuv) \\
= & (au^2+bv^2+2cuv)f_t(au^2+bv^2+2cuv) \\
& - \hbar^2(ab-c^2)(f''_t(au^2+bv^2-2cuv) \\
& + f'_t(au^2+bv^2+2cuv)(au^2+bv^2+2cuv)),
\end{align*}
\]
where $x = au^2 + bv^2 + 2cuv$. Then we obtain the equation (we call this Riccati-type equation)
\[
\frac{d}{dt}f_t(x) = x f_t(x) + \hbar^2 D(f'_t(x) + xf''_t(x))
\]
where $D = c^2 - ab$ is the discriminant of $H_*$. Set $f_t(x) = g(t)e^{h(t)x}$. Substituting this into (50), we obtain
\[
\{ g'(t) - Dh^2g(t)h(t) + xg(t)\{ h'(t) - 1 - Dh^2h(t)^2 \} e^{h(t)x} = 0.
\]
Hence, $h'(t) - 1 - Dh^2h(t)^2 = 0$. Thus, we obtain
\[
h(t) = \frac{1}{\hbar \sqrt{D}} \tan(h(\sqrt{D})t).
\]
Next, solving
\[
g'(t) - g(t)D\hbar^2 \frac{1}{\hbar \sqrt{D}} \tan(h(\sqrt{D})t) = 0,
\]
we have $g(t) = \frac{1}{\cos(h(\sqrt{D})t)}$. Summing up what mentioned above, we have the solution of the differential equation (50) with the initial function 1 in the following way:

**Theorem 4.1** The Weyl ordered expression of the $*$-exponential function $e^{t(au^2+bv^2+2cuv)}$ is given by
\[
_:e^{t(au^2+bv^2+2cuv)}_0 = \frac{1}{\cos(h\sqrt{D}t)} \exp \left( \frac{1}{\hbar \sqrt{D}} \tan(h\sqrt{D}t)(au^2+bv^2+2cuv) \right)
\]
where $\frac{1}{h\sqrt{D}} \tan(h\sqrt{D}t) = t$ in the case $D=0$. 

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