UPPER BOUNDS FOR STEKLOV EIGENVALUES ON SURFACES

ALEXANDRE GIROUARD AND IOSIF POLTEROVICH

Abstract. We give explicit isoperimetric upper bounds for all Steklov eigenvalues of a compact orientable surface with boundary, in terms of the genus, the length of the boundary, and the number of boundary components. Our estimates generalize a recent result of Fraser–Schoen, as well as the classical inequalities obtained by Hersch–Payne–Schiffer, whose approach is used in the present paper.

1. Introduction

1.1. Steklov spectrum. Let \( \Sigma \) be a compact orientable surface with boundary, and let \( \Delta \) be the Laplace–Beltrami operator associated with a Riemannian metric on \( \Sigma \). The Steklov eigenvalue problem on \( \Sigma \) is given by:

\[
\Delta u = 0 \text{ in } \Sigma, \quad \partial_n u = \sigma u \text{ on } \partial \Sigma,
\]

where \( \partial_n \) denotes the outward normal derivative. The spectrum of the Steklov problem is discrete and its eigenvalues form a sequence

\[
0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \sigma_3 \leq \cdots \rightarrow \infty,
\]

where each eigenvalue is repeated according to its multiplicity \([2]\). The eigenfunctions \( \phi_k \), \( k = 0, 1, 2 \ldots \) can be chosen to form an orthogonal basis of \( L^2(\partial \Sigma) \). Note that the eigenfunction \( \phi_0 \) corresponding to \( \sigma_0 = 0 \) is constant.

The Steklov eigenvalues coincide with the eigenvalues of the Dirichlet-to-Neumann map \( \Lambda \). If the boundary \( \partial \Sigma \) is smooth, it is a pseudodifferential elliptic operator \( \Lambda : C^\infty(\partial \Sigma) \to C^\infty(\partial \Sigma) \) of order one \([20]\), defined by

\[
\Lambda(f) = \partial_n H f,
\]

2010 Mathematics Subject Classification. 58J50, 35P15, 35J25.

Key words and phrases. Steklov problem, Riemannian surface, eigenvalue inequalities.
where $Hf$ is the harmonic extension of $f$ to the interior of $\Sigma$ (i.e. $\Delta(Hf) = 0$ on $\Sigma$). The Dirichlet-to-Neumann map has important applications to inverse problems \cite{6, 19}.

1.2. Main results. Isoperimetric inequalities for Steklov eigenvalues have been actively studied for more than fifty years \cite{21, 22, 3, 15, 12}. In particular, a number of recent papers are concerned with the Steklov spectrum on manifolds with boundary \cite{9, 10, 14, 17}. The following estimate on the first nontrivial Steklov eigenvalue on a surface with boundary was proved by Fraser and Schoen \cite{10}:

\begin{equation}
\sigma_1 L \leq 2\pi (\gamma + l) .
\end{equation}

Here $L$ is the length of the boundary, $\gamma$ is the genus of the surface and $l$ the number of boundary components. For simply connected planar domains, inequality (1.1) is sharp and was proved by Weinstock in \cite{21}.

The goal of this note is to generalize (1.1) to higher eigenvalues. We prove

**Theorem 1.2.** Let $\Sigma$ be a compact orientable surface of genus $\gamma$, such that the boundary $\partial \Sigma$ has $l$ connected components of total length $L$. Then

\begin{equation}
\sigma_k L \leq 2\pi (\gamma + l) k
\end{equation}

for any integer $k \geq 1$.

In fact, Theorem 1.2 is a special case (set $p = q$ below) of the following result:

**Theorem 1.4.** Under the assumptions of Theorem 1.2

\begin{equation}
\sigma_p \sigma_q L^2 \leq \begin{cases} 
\pi^2 (\gamma + l)^2 (p + q)^2 & \text{if } p + q \text{ is even,} \\
\pi^2 (\gamma + l)^2 (p + q - 1)^2 & \text{if } p + q \text{ is odd,}
\end{cases}
\end{equation}

for any pair of integers $p, q \geq 1$.

1.3. Discussion. It follows from Weyl’s law for eigenvalues of the Dirichlet-to-Neumann operator that the linear dependence on $k$ in (1.3) is optimal. For simply connected planar domains, the inequalities (1.5) were obtained by Hersch, Payne, and Schiffer in \cite{16}. In \cite{12} we proved that in this case (here $\gamma = 0$, $l = 1$) the estimates (1.3) are sharp for all $k \geq 1$. We do not expect (1.3) to be sharp for other values of $\gamma$ and $l$ (cf. \cite{10} Theorem 2.5); see also Question 1.8 below.

The proof of Theorem 1.4 combines the methods of \cite{10} and \cite{16}. Following \cite{10}, we use a version of Ahlfors Theorem \cite{11} proved by Gabard \cite{11}, according to which any Riemannian surface of genus $\gamma$ with $l$ boundary components can be represented as a proper conformal
branched cover of a disk $\mathbb{D}$ with degree at most $\gamma + l$. Properness of the covering map implies that the boundary $\partial \Sigma$ is mapped to the circle $S^1$. It is essential in this proof since the test functions for the variational characterization of the eigenvalues $\sigma_k$ are built from the eigenfunctions of a certain one-dimensional problem on $S^1$. This approach was suggested by Hersch, Payne and Schiffer in [16].

The analogue of the estimate (1.1) for the first nonzero Laplace eigenvalue $\lambda_1$ on a closed surface $\Sigma$ (without boundary) is the Yang–Yau inequality [23]:

\begin{equation}
\lambda_1 \frac{\text{Area}(\Sigma)}{d} \leq 8\pi d,
\end{equation}

where $d$ is the degree of a conformal branch covering of $\Sigma$ over a sphere. It was observed in [8] that one could take $d \leq \lfloor \gamma + 3/2 \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part.

For higher eigenvalues of the Laplacian on surfaces, no explicit estimates like (1.3) are known. However, with an implicit constant such a bound was proved by Korevaar in [18] using a different approach. The analogue of Korevaar’s result for Steklov eigenvalues on surfaces was obtained in [7] (see also [13, Section 5.3] and [17, Example 1.3]): there exists a universal constant $C$ such that

\begin{equation}
\sigma_k \leq C (\gamma + 1)k, \quad k = 1, 2, 3, \ldots
\end{equation}

Note that the bound (1.7) does not depend on the number of boundary components of $\partial \Sigma$, which makes it a sharper estimate than (1.3) for $l$ large enough. Another interesting development of Korevaar’s method for both Laplace and Steklov eigenvalues can be found in [14] where $\lambda_k$ and $\sigma_k$ are bounded by a linear combination of $k$ and $\gamma$ (instead of its product). However, the constants in [14] are also implicit.

Let us conclude by an open question. It was proved in [5] that there exists a sequence of closed surfaces $\Sigma_n$ of genera $\gamma_n \to \infty$ such that

\[ \lim_{n \to \infty} \frac{\lambda_1(\Sigma_n) \text{Area}(\Sigma_n)}{\gamma_n} = \infty. \]

Moreover, it was subsequently shown in [4] that one can choose a sequence of surfaces with $\gamma_n = n$ and $\lambda_1(\Sigma_n) \text{Area}(\Sigma_n)$ growing linearly as $n \to \infty$. Therefore, the dependence on the genus $\gamma$ in the Yang–Yau inequality (1.6) is optimal up to a multiplicative constant.

**Question 1.8.** Is there a sequence $\Sigma_n$ of surfaces with boundary of genera $\gamma_n \to \infty$ such that $\sigma_1(\Sigma_n)L(\partial \Sigma_n) \to \infty$ as $n \to \infty$? If yes, is it possible to achieve linear growth?
Acknowledgments. The authors would like to thank Bruno Colbois for fruitful discussions. Research of I.P. was supported by NSERC, FQRNT and the Canada Research Chairs program.

2. Proof of Theorem 1.4

2.1. Reduction to the circle. Let \((\phi_k)_{k=0}^{\infty} \subset L^2(\partial \Sigma)\) be a complete orthonormal system of eigenfunctions of the Dirichlet–to–Neumann map. It is well known that if a function \(f \in C^\infty(\Sigma)\) satisfies

\[
\int_{\partial \Sigma} f \phi_j \quad \text{for } j = 0, 1, 2, \ldots, k - 1,
\]

then

\[
\sigma_k \leq R_\Sigma(f) := \frac{\int_\Sigma |\nabla f|^2}{\int_{\partial \Sigma} f^2}.
\]

The proof of Theorem 1.4 is based on the approach of [16]. We construct test functions using linear combinations of harmonic oscillators on \(S^1\), extend them harmonically to the disk and then lift to a branched cover representation of \(\Sigma\). Using sufficiently many harmonic oscillators, one can ensure the existence of a linear combination satisfying the orthogonality conditions (2.1).

As was shown in [11], there exists a proper conformal branched cover \(\psi : \Sigma \to \mathbb{D}\) of degree \(d \leq \gamma + l\). Because \(\psi\) is proper, it takes the boundary \(\partial \Sigma\) to the circle \(S^1 = \partial \mathbb{D}\). The restriction of \(\psi\) to each connected component of \(\partial \Sigma\) is a covering map of \(S^1\). Let \(ds\) be the Riemannian measure on \(\partial \Sigma\). We define the push-forward measure \(d\mu = \psi_* ds\) on the circle \(S^1\), and introduce the “mass parameter”

\[m(\theta) = \int_0^\theta d\mu(\theta).\]

In particular, \(d\mu = m'(\theta)d\theta\) is absolutely continuous with respect to the Lebesgue measure \(d\theta\), and the length of the boundary \(\partial \Sigma\) is given by

\[L = m(2\pi) = \int_{S^1} d\mu.\]

Given a smooth periodic function \(h : \mathbb{R} \to \mathbb{R}\) with period \(L\), define \(f : S^1 \to \mathbb{R}\) by

\[f(\theta) = h(m(\theta)).\]
The function $f$ admits a unique harmonic extension $u$ to the disk $\mathbb{D}$. Because the disk is simply connected, this function has a unique harmonic conjugate $v$ normalized in such a way that

$$\int_{S^1} v \, d\mu = 0. \quad (2.3)$$

Let the functions $\alpha, \beta : \Sigma \to \mathbb{R}$ be defined by

$$\alpha = u \circ \psi \quad \text{and} \quad \beta = v \circ \psi. \quad (2.4)$$

Recall that the map $\psi$ is a $d$-fold conformal branched covering of $\mathbb{D}$. It follows from conformal invariance of the Dirichlet energy in two dimensions (see also [23]) that

$$\int_\Sigma |\nabla \alpha|^2 = d \int_\mathbb{D} |\nabla u|^2, \quad \int_\Sigma |\nabla \beta|^2 = d \int_\mathbb{D} |\nabla v|^2. \quad (2.5)$$

Moreover, the Cauchy–Riemann equations imply that these two quantities are equal. Integration by parts gives

$$\int_\Sigma |\nabla \alpha|^2 \int_\Sigma |\nabla \beta|^2 = d^2 \left( \int_{S^1} v \partial_r v \right)^2. \quad (2.6)$$

The Cauchy–Riemann equations also imply the pointwise equality

$$\partial_r v = -\partial_\theta u = -f'(\theta) = -h'(m(\theta))m'(\theta). \quad (2.7)$$

Applying the Cauchy–Schwarz inequality to the measure $d\mu = m'(\theta) d\theta$ leads to:

$$\left( \int_{S^1} v \partial_r v \right)^2 \leq \int_{S^1} v^2(\theta) \, d\mu(\theta) \int_{S^1} h'(m(\theta))^2 \, d\mu(\theta). \quad (2.8)$$

At the same time,

$$\int_{\partial \Sigma} \alpha^2 \, d\theta = \int_{S^1} f^2 \, d\mu \quad \text{and} \quad \int_{\partial \Sigma} \beta^2 \, d\theta = \int_{S^1} v^2 \, d\mu. \quad (2.9)$$

Estimating the product of the Rayleigh quotients $R_\alpha := R_\Sigma(\alpha)$ and $R_\beta := R_\Sigma(\beta)$ using the relations (2.6), (2.7) and (2.9), we notice that $\int_{S^1} v^2(\theta) \, d\mu(\theta)$ cancels out on the right–hand side. This is the key trick
in the method introduced in [16]. Namely, we obtain the following bound:

\[ R(\alpha)R(\beta) \leq d^2 \frac{\int_{S^1} h'(m(\theta))^2 \, d\mu(\theta)}{\int_{S^1} h(m(\theta))^2 \, d\mu(\theta)} = d^2 R_L(h). \]

Here

\[ R_L(h) := \frac{\int_0^L h'(m)^2 \, dm}{\int_0^L h(m)^2 \, dm} \]

is the Rayleigh quotient of a uniform circular string of length \( L \). Its eigenmodes are well known. Let \( h_k : \mathbb{R} \to \mathbb{R}, \ k = 0, 1, 2, \ldots \), be defined by \( h_0 = 1 \) and

\[ h_k(m) = \begin{cases} \cos \left( \frac{2n\pi m}{L} \right) & \text{if } k = 2n - 1, \\ \sin \left( \frac{2n\pi m}{L} \right) & \text{if } k = 2n. \end{cases} \]

for \( k \geq 1 \). Clearly,

\[ R_L(h_k) = \left( \frac{2\pi n}{L} \right)^2 \quad \text{for } k = 2n \text{ or } k = 2n - 1. \]

This leads to

\[ R(\alpha)R(\beta) \leq \left( \frac{\pi d}{L} \right)^2 \begin{cases} k^2 & \text{if } k = 2n, \\ (k+1)^2 & \text{if } k = 2n - 1. \end{cases} \tag{2.9} \]

2.2. Construction of test-functions. The rest of the argument is almost exactly the same as in [16]. We present it below for the sake of completeness. Let \( N = p + q - 1 \). Consider a function

\[ f = \sum_{k=1}^N c_k f_k, \quad (c_k \in \mathbb{R}), \tag{2.10} \]

where the functions \( f_k : S^1 \to \mathbb{R} \) are defined by \( f_k(\theta) = h_k(m(\theta)) \). The functions \( f_k \) are \( d\mu \)-orthogonal to each other, and hence linearly independent. The harmonic extensions \( u_k \) of \( f_k \) are also linearly independent, because taking the harmonic extension is a linear and injective operation. For the same reason, the harmonic conjugates \( v_k \) are linearly independent as well. Moreover, since by definition \( f_0 = 1 \), \( f_k \) are \( d\mu \)-orthogonal to constants for all \( k = 1, 2, 3, \ldots \), and hence \( \int_{\partial \Sigma} \alpha_k = 0 \) for all \( k \geq 1 \), where \( \alpha_k = u_k \circ \psi \). At the same time, by the normalization \( \int_{\partial \Sigma} \beta_k = 0 \) for all \( k \geq 1 \), where \( \beta_k = v_k \circ \psi \). Let

\[ u = \sum_{k=1}^N c_k u_k \quad \text{and} \quad v = \sum_{k=1}^N c_k v_k. \]
As before, these functions are lifted to $\alpha = u \circ \psi$ and $\beta = v \circ \psi$.

In order to use $u$ and $v$ in the variational characterization \((2.2)\) for $\sigma_p$ and $\sigma_q$ respectively, they have to satisfy the orthogonality conditions \((2.1)\):

$$
\int_{\partial \Sigma} \alpha \phi_k = 0 \quad \text{for } k = 1, \ldots, p - 1
$$

$$
\int_{\partial \Sigma} \beta \phi_k = 0 \quad \text{for } k = 1, \ldots, q - 1
$$

These $N - 1$ linear constraints can be resolved for some choice of $N$ constants $c_1, \ldots, c_N$. It follows from \((2.9)\) that

$$
\sigma_p \sigma_q \leq R(\alpha) R(\beta) \leq d^2 R_L(h),
$$

where $h = \sum_{k=1}^{N} c_k h_k$. We conclude by observing that

$$
R_L(h) \leq R_L(h_N) = \left( \frac{\pi d}{L} \right)^2 \begin{cases} 
N^2 & \text{if } N \text{ is even,} \\
(N + 1)^2 & \text{if } N \text{ is odd.}
\end{cases}
$$

$$
= \left( \frac{\pi d}{L} \right)^2 \begin{cases} 
(p + q - 1)^2 & \text{if } p + q \text{ is odd,} \\
(p + q)^2 & \text{if } p + q \text{ is even.}
\end{cases}
$$

Recalling that $d \leq \gamma + l$ completes the proof of Theorem 1.4. \qed

\section*{References}

[1] Lars L. Ahlfors. Open Riemann surfaces and extremal problems on compact subregions. \textit{Comment. Math. Helv.}, 24:100–134, 1950.

[2] Catherine Bandle. \textit{Isoperimetric inequalities and applications}, volume 7 of \textit{Monographs and Studies in Mathematics}. Pitman, Boston, Mass., 1980.

[3] Friedemann Brock. An isoperimetric inequality for eigenvalues of the Stekloff problem. \textit{Z. Angew. Math. Mech.}, 81(1):69–71, 2001.

[4] Robert Brooks and Eran Makover. Riemann surfaces with large first eigenvalue. \textit{J. Anal. Math.}, 83:243–258, 2001.

[5] Peter Buser. On the bipartition of graphs. \textit{Discrete Appl. Math.}, 9(1):105–109, 1984.

[6] Alberto-P. Calderón. On an inverse boundary value problem. In \textit{Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)}, pages 65–73. Soc. Brasil. Mat., Rio de Janeiro, 1980.

[7] Bruno Colbois, Ahmad El Soufi, and Alexandre Girouard. Isoperimetric control of the Steklov spectrum. \textit{J. Funct. Anal.}, 261(5):1384–1399, 2011.

[8] Ahmad El Soufi and Saïd Ilias. Le volume conforme et ses applications d’après Li et Yau. In \textit{Séminaire de Théorie Spectrale et Géométrie, Année 1983–1984}, pages VII.1–VII.15. 1984.

[9] José F. Escobar. An isoperimetric inequality and the first Steklov eigenvalue. \textit{J. Funct. Anal.}, 165(1):101–116, 1999.

[10] Ailana Fraser and Richard Schoen. The first Steklov eigenvalue, conformal geometry, and minimal surfaces. \textit{Adv. Math.}, 226(5):4011–4030, 2011.
[11] Alexandre Gabard. Sur la représentation conforme des surfaces de Riemann à bord et une caractérisation des courbes séparantes. Comment. Math. Helv., 81(4):945–964, 2006.

[12] Alexandre Girouard and Iosif Polterovich. On the Hersch-Payne-Schiffer estimates for the eigenvalues of the Steklov problem. Funktsional. Anal. i Prilozhen., 44(2):33–47, 2010.

[13] Alexander Grigor’yan, Yuri Netrusov, and Shing-Tung Yau. Eigenvalues of elliptic operators and geometric applications. In Surveys in differential geometry. Vol. IX, Surv. Differ. Geom., IX, pages 147–217. Int. Press, Somerville, MA, 2004.

[14] Asma Hassannezhad. Conformal upper bounds for the eigenvalues of the Laplacian and Steklov problem. Journal of Functional Analysis, 261(12):3419–3436, 2011.

[15] Antoine Henrot, Gérard A. Philippin, and Abdessamad Safoui. Some isoperimetric inequalities with application to the Stekloff problem. J. Convex Anal., 15(3):581–592, 2008.

[16] Joseph Hersch, Lawrence E. Payne, and Menahem M. Schiffer. Some inequalities for Stekloff eigenvalues. Arch. Rational Mech. Anal., 57:99–114, 1975.

[17] Gerasim Kokarev. Variational aspects of Laplace eigenvalues on Riemannian surfaces. Preprint (2011): arXiv:1103.2448.

[18] Nicholas Korevaar. Upper bounds for eigenvalues of conformal metrics. J. Differential Geom., 37(1):73–93, 1993.

[19] Matti Lassas, Michael Taylor, and Gunther Uhlmann. The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary. Comm. Anal. Geom., 11(2):207–221, 2003.

[20] Michael E. Taylor. Partial differential equations. II, volume 116 of Applied Mathematical Sciences. Springer-Verlag, New York, 1996.

[21] Robert Weinstock. Inequalities for a classical eigenvalue problem. J. Rational Mech. Anal., 3:745–753, 1954.

[22] Lewis Wheeler and Cornelius O. Horgan. Isoperimetric bounds on the lowest nonzero Stekloff eigenvalue for plane strip domains. SIAM J. Appl. Math., 31(2):385–391, 1976.

[23] Paul C. Yang and Shing-Tung Yau. Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 7(1):55–63, 1980.

Institut de Mathématiques de Neuchâtel, rue Émile-Argand 11, 2000 Neuchâtel, Suisse.

E-mail address: alexandre.girouard@unine.ch

Département de mathématiques et de statistique, Université de Montréal, C. P. 6128, Succ. Centre-Ville, Montréal, Québec, H3C 3J7, Canada

E-mail address: iossif@dms.umontreal.ca