THE PROJECTIVE INDECOMPOSABLE MODULES FOR THE
RESTRICTED ZASSENHAUS ALGEBRAS IN
CHARACTERISTIC 2

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Abstract. It is shown that for the restricted Zassenhaus algebra \( \mathfrak{W} = \mathfrak{W}(1, n) \), \( n > 1 \), defined over an algebraically closed field \( F \) of characteristic 2 any projective indecomposable restricted \( \mathfrak{W} \)-module has maximal possible dimension \( 2^n - 1 \), and thus is isomorphic to some induced module \( \text{ind}_{\mathfrak{W}}(F(\mu)) \) for some torus of maximal dimension \( t \). This phenomenon is in contrast to the behavior of finite-dimensional simple restricted Lie algebras in characteristic \( p > 3 \) (cf. \cite[Thm. 6.3]{zassenhaus}).

1. Introduction

Let \( \mathfrak{L} \) be a finite-dimensional restricted Lie algebra defined over an algebraically closed field \( F \) of characteristic \( p > 0 \), and let \( t \subseteq \mathfrak{L} \) be a torus of maximal dimension. Then, as \( \mathfrak{u}(t) \) - the restricted universal enveloping algebra of \( t \) - is a commutative and semi-simple associative \( F \)-algebra, every irreducible restricted \( t \)-module is 1-dimensional and also projective. Moreover, there exists a canonical one-to-one correspondence between the isomorphism classes of irreducible restricted \( t \)-modules and the set

\[
(1.1) \quad t^{\circ} = \text{span}_{\mathbb{F}_p} \{ t \in t \mid t^{[p]} = t \} ^*,
\]

where \( \mathbb{F}_p \subseteq F \) denotes the prime field, and \( ^* = \text{Hom}_{\mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p) \). For \( \mu \in t^{\circ} \) let \( F(\mu) \) denote the corresponding irreducible restricted \( t \)-module. Then

\[
(1.2) \quad P(\mu) = \text{ind}_t^L(F(\mu)) = \mathfrak{u}(\mathfrak{L}) \otimes_{\mathfrak{u}(t)} F(\mu)
\]

is a projective restricted \( \mathfrak{L} \)-module (cf. \cite[Prop. 2.3.10]{zassenhaus}), but in general it will be decomposable. Indeed, it has been shown in \cite[Thm. 6.3]{zassenhaus} that for \( p > 3 \) the finite-dimensional restricted Lie algebra \( \mathfrak{L} \) is solvable if, and only if, \( P(0) \) is indecomposable. The authors conclude their paper with the remark, that for \( p = 2 \) the “if” part of the assertion is false, e.g., for \( p = 2 \) and \( \mathfrak{L} \) equal to the restricted Zassenhaus algebra \( \mathfrak{W}(1, 2) \) one has that \( P(0) \) is indecomposable and coincides with the projective cover of the trivial \( \mathfrak{W}(1, 2) \)-module. However, \( \mathfrak{W}(1, 2) \) is simple and non-abelian. The main purpose of this note is to extend this result to all restricted Zassenhaus algebras \( \mathfrak{W}(1, n) \), \( n > 1 \), in characteristic 2 and to all projective restricted modules \( P(\mu), \mu \in t^{\circ} \).

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**Theorem.** Let $\mathfrak{M} = \mathfrak{M}(1, n)$, $n \geq 2$, be a restricted Zassenhaus Lie algebra defined over an algebraically closed field of characteristic 2, and let $t \subseteq \mathfrak{M}$ be a torus of maximal dimension. Then $P(\mu)$ is indecomposable for all $\mu \in t^0$. In particular, all projective indecomposable restricted $\mathfrak{M}(1, n)$-modules have maximal possible dimension $2^{2^n-1}$, and $\mathfrak{M}(1, n)$ has maximal 0-p.i.m.

The proof of the theorem will be arranged in two major steps. First, it will be shown that every restricted Zassenhaus algebra $\mathfrak{M}(1, n)$ has a direct sum decomposition

$$\mathfrak{M}(1, n) = t_0 \oplus \mathfrak{B},$$

where $t_0$ is a torus of maximal dimension, and $\mathfrak{B}$ is a certain restricted Lie subalgebra of $\mathfrak{M}(1, n)$. In the second step it will be shown that $\mathfrak{F}$ and $L = W(1, n)^{(1)} = \text{soc}_\mathfrak{M}(\mathfrak{M}(1, n))$ are the only isomorphism types of irreducible restricted $\mathfrak{M}(1, n)$-modules. From this fact and [3, Lemma 6.1] one concludes that $\mathfrak{M}(1, n)$ has maximal 0-p.i.m., i.e., the projective cover $P_L$ of the trivial $\mathfrak{M}(1, n)$-module has maximal possible dimension $2^{2^n-1}$, and thus must be isomorphic to $P(0)$ (cf. (1.2)). An elementary calculation then shows that the projective cover $P_L$ of the restricted $\mathfrak{M}(1, n)$-module $L$ must be isomorphic to $P(\mu)$ for any $\mu \in t^0 \setminus \{0\}$, i.e., any projective indecomposable restricted $\mathfrak{M}(1, n)$-module has the maximal possible dimension $2^{2^n-1}$ (cf. Remark 3.3). As a by-product we may also conclude that the restricted Lie subalgebra $\mathfrak{B}$ has properties similar to the Borel subalgebra in a classical restricted Lie algebra (cf. Remark 2.2 Thm 3.4(b)).

2. **Restricted Lie algebras**

From now on we will assume that $\mathfrak{F}$ is a field of characteristic $p > 0$.

2.1. **Simple algebras and simple restricted Lie algebras.** A (restricted) $\mathfrak{F}$-Lie algebra $\mathfrak{L}$ is said to be simple if any (restricted) Lie ideal $\mathfrak{I}$ of $\mathfrak{L}$ coincides either with $\mathfrak{L}$ or with 0. The following fact is straightforward and its easy proof is left to the reader (cf. [3, §6]).

**Fact 2.1.** Let $\mathfrak{F}$ be a field of characteristic $p > 0$.

(a) Let $\mathfrak{L}$ be a non-abelian, finite-dimensional, simple restricted $\mathfrak{F}$-Lie algebra, and let $L = \text{soc}_\mathfrak{L}(\mathfrak{L})$ denote the socle of the restricted $\mathfrak{L}$-module $\mathfrak{L}$ which action is given by the adjoint representation. Then $L$ is a minimal Lie ideal of $\mathfrak{L}$ and a non-abelian, simple $\mathfrak{F}$-Lie algebra. Moreover, $\mathfrak{L}$ coincides with the minimal restricted $p$-envelope $\text{env}_p(L)$ of $L$.

(b) Let $L$ be a non-abelian, finite-dimensional, simple $\mathfrak{F}$-Lie algebra. Then $\text{env}_p(L)$ is a finite-dimensional, simple restricted $\mathfrak{F}$-Lie algebra, and $L = \text{soc}_{\text{env}_p(L)}(\text{env}_p(L))$.

2.2. **Generalized Borel subalgebras.** Let $\mathfrak{L}$ be a finite-dimensional restricted Lie algebra over an algebraically closed field $\mathfrak{F}$ of characteristic $p > 0$. A proper restricted Lie subalgebra $\mathfrak{B}$ of $\mathfrak{L}$ will be said to be a generalized Borel subalgebra, if

(i) the unipotent radical $\text{rad}_u(\mathfrak{B})$ of $\mathfrak{B}$ is non-trivial, i.e., $\text{rad}_u(\mathfrak{B}) \neq 0$;

(ii) for any irreducible restricted $\mathfrak{L}$-module $S$, $[S] \in \text{Irr}(\mathfrak{L})$, the restricted $\mathfrak{B}/\text{rad}_u(\mathfrak{B})$-module $S^{\text{rad}_u(\mathfrak{B})}$ is irreducible;

(iii) the mapping $\chi_{\mathfrak{B}} : \text{Irr}(\mathfrak{L}) \to \text{Irr}(\mathfrak{B}/\text{rad}_u(\mathfrak{B}))$ given by $\chi_{\mathfrak{B}}([S]) = [S^{\text{rad}_u(\mathfrak{B})}]$ is injective.
Proof. Put (3.6)

\[ W = W(1, n) = \text{span}_F \{ e_j \mid -1 \leq j \leq p^n - 2 \} \]

In particular, one has

(3.3)

\[ [e_i, e_j] = \begin{cases} c_{ij} \cdot e_{i+j} & \text{if } -1 \leq i \leq p^n - 2, \\ 0 & \text{otherwise,} \end{cases} \]

where \( c_{ij} = \binom{i+j+1}{n} - \binom{i+j}{n} \). It is well known that if \( \mathbb{E} \subset \mathbb{F} \) is the finite field with \( p^n \) elements then one has a (non-canonical) isomorphism

(3.4)

\[ L(\mathbb{E}) \cong W(1, n) \]

(cf. [4].) If \( p > 2 \), then \( W(1, n) \) is a simple Lie algebra (cf. [6] Thm. 4.2.4)); while if \( p = 2 \) and \( n > 1 \) then

(3.5)

\[ W(1, n) = \text{span}_F \{ e_j \mid -1 \leq j \leq 2^n - 3 \} \]

is simple (cf. [4]). The minimal \( p \)-envelope \( \mathfrak{M} = \mathfrak{M}(1, n) = \text{env}_p(W(1, n)) \) of the Zassenhaus algebra \( W(1, n) \) coincides with the derivation algebra of \( W(1, n) \) (cf. [2], [5] Thm. 7.1.2) and is also called the restricted Zassenhaus algebra, i.e., identifying \( W(1, n) \) with its image in \( \text{Der}(W(1, n)) \) one has

(3.6)

\[ \mathfrak{M}(1, n) = \bigoplus_{1 \leq k \leq n-1} \mathbb{F} \cdot \partial^{[p^k]} \oplus W(1, n). \]

The following result holds.

Proposition 3.1. If \( p = 2 \) and \( n \geq 2 \), then \( \mathfrak{M}(1, n) \) is isomorphic to the minimal 2-envelope of the simple Lie algebra \( W(1, n) \).

Proof. Put \( \mathfrak{M} = \mathfrak{M}(1, n) = \text{Der}(W(1, n)) \) and consider the canonical injective map

(3.7)

\[ \alpha: W(1, n) \twoheadrightarrow W(1, n) \xrightarrow{\text{ad}_W} \mathfrak{M}. \]

As \( C_{\mathfrak{M}}(\partial) = \text{span}_F \{ \partial^{[j]} \mid 0 \leq j \leq n - 1 \} \), one has that \( C_{\mathfrak{M}}(A) = 0 \) for \( A = \text{im}(\alpha) \). Hence, by Fact [2, Theorem 9], \( \mathfrak{K} = (\partial)_p \) - the restricted Lie subalgebra of \( \mathfrak{M} \) generated by \( A \) - is a minimal \( p \)-envelope of \( W(1, n) \) (cf. [5] Thm. 1.1.7)). For simplicity we assume that \( \alpha \) is given by inclusion. By construction, \( \bigoplus_{1 \leq k \leq n-1} \mathbb{F} \cdot \partial^{[p^k]} \oplus W(1, n) \) is
In particular, \( x_{2n-1} = e_{2n-2} \) (cf. [2 §1]). Thus \( \mathfrak{h} = \mathfrak{m} \), and this yields the claim. \( \square \)

3.1. **Toral complements.** We define the restricted Lie subalgebra \( \mathfrak{B} \) of \( \mathfrak{M}(1, n) \) by

\[
\mathfrak{B} = \text{span}_F \{ e_j \mid 0 \leq j \leq 2^n - 2 \},
\]

i.e., \( \mathfrak{B} \subseteq W(1, n) \). This subalgebra has the following property.

**Proposition 3.2.** Let \( F \) be an algebraically closed field of characteristic \( p \).

(a) Let \( \mathfrak{t} \subseteq \mathfrak{M}(1, n) \) be any torus of maximal dimension. Then \( \mathfrak{t} \) has dimension \( n \).

(b) There exists a torus \( \mathfrak{t}_0 \subseteq \mathfrak{M}(1, n) \) of maximal dimension such that

\[
\mathfrak{M}(1, n) = \mathfrak{t}_0 \oplus \mathfrak{B}.
\]

**Proof.** (a) For \( p > 2 \) this has been shown in [3 Thm. 7.6.3]. Hence we may assume that \( F \) is an algebraically closed field of characteristic 2. Define \( s = e_{-1} + e_{2^n-2} \in W(1, n) \). Then one concludes easily by induction that

\[
\begin{align*}
(3.10) \quad s^2 &= (\partial_2^k + e_{2^n-2}-1) \quad \text{for } k \in \{1, \ldots, n-1\}, \\
(3.11) \quad s^{2^n} &= \partial^{2^n} + e_{-1} + e_{2^n-2}.
\end{align*}
\]

In particular, the minimal polynomial of \( \text{ad}_W(s) \) divides the separable polynomial \( T^{2^n} - T \in F[T] \). Thus \( \text{ad}_W(s) \) is semi-simple and the elements \( \text{ad}_W(s)^i \in \text{Der}(W(1, n)) = \mathfrak{M}(1, n), \ i \in \{0, \ldots, n-1\} \), are linearly independent. Hence \( \mathfrak{t}_0 = \text{span}_F \{ \text{ad}(s)^i \mid i \in \{0, \ldots, n-1\} \} \) is a torus of \( \mathfrak{M}(1, n) \) of dimension \( n \), i.e., the maximal dimension of a torus \( MT(\mathfrak{M}(1, n)) \) of \( \mathfrak{M}(1, n) \) must be greater or equal to \( n \). Since \( O(1, n) \) and \( O(n, 1) \) are isomorphic \( F \)-algebras, there exists an injective homomorphism of restricted Lie algebras \( i : \mathfrak{M}(1, n) \rightarrow \mathfrak{M}(n, 1) \). Thanks to [5 Thm 1.2.7, Cor. 7.5.2] this yields \( MT(\mathfrak{M}(1, n)) \leq MT(\mathfrak{M}(n, 1)) \) = \( n \). So we may conclude that \( MT(\mathfrak{M}(1, n)) = n \).

(b) For \( p = 2 \) the just mentioned argument shows that \( \mathfrak{t}_0 \oplus \mathfrak{B} = \mathfrak{M}(1, n) \). For \( p > 2 \) one may identify \( W(1, n) \) with \( L(E) \) and define \( \mathfrak{t}_0 = \text{span}_F \{ y_0^j \mid 0 \leq j \leq n-1 \} \) (cf. [3 Thm. 7.6.3]). \( \square \)

3.2. **The restricted subalgebra \( \mathfrak{B} \).** Note that \( e_0 \in \mathfrak{B} \) is a toral element. Moreover, as \( \text{rad}_u(\mathfrak{B}) = \text{span}_F \{ e_j \mid 1 \leq j \leq p^n - 2 \} \), one has

\[
\mathfrak{B} = \mathfrak{s} \oplus \text{rad}_u(\mathfrak{B}), \tag{3.12}
\]

where \( \mathfrak{s} = F \cdot e_0 \). For \( \lambda \in \{0, \ldots, p-1\} \) we denote by \( F[\lambda] \) the 1-dimensional restricted \( \mathfrak{B} \)-module satisfying

\[
\begin{align*}
(3.13) \quad e_0 \cdot z &= \lambda \cdot z \quad \text{for } z \in F[\lambda].
\end{align*}
\]

In particular, \( x \cdot z = 0 \) for all \( x \in \text{rad}_u(\mathfrak{B}) \) and \( z \in F[\lambda] \). For our purpose the following property will turn out to be useful.

**Proposition 3.3.** Let \( F \) be an algebraically closed field of characteristic \( p > 0 \), let \( \mathfrak{W} = \mathfrak{M}(1, n) \) and \( \mathfrak{B} \) be as described above. Then one has isomorphisms of restricted \( \mathfrak{W} \)-modules

\[
\begin{align*}
(3.14) \quad \text{ind}_{\mathfrak{B}}^{\mathfrak{W}}(F[-2]) &\simeq W, \\
(3.15) \quad \text{ind}_{\mathfrak{B}}^{\mathfrak{W}}(F[1]) &\simeq W^* = \text{Hom}_F(W, F).
\end{align*}
\]
Proof. Note that by (3.3)

\[(3.16) \quad [e_0, e_{p^n-2}] = ((p^n - 1) - 1) \cdot e_{p^n-2} = -2 \cdot e_{p^n-2},\]

and \([e_k, e_{p^n-2}] = 0\) for \(k > 0\). Hence one has an isomorphism of restricted \(\mathcal{B}\)-modules \(\alpha : F[-2] \to F \cdot e_{p^n-2}, \alpha(1) = e_{p^n-2}\). Let \(\alpha_* : \text{ind}_E^F(F[-2]) \to W\) be the induced homomorphism of restricted \(\mathcal{M}\)-modules. From (3.3) one concludes that \(\text{ind}_E^F(F[-2]) \cdot e_{p^n-2} = W\). Hence \(\alpha_*\) is surjective. As

\[(3.17) \quad \dim(W) = p^n = \dim(\text{ind}_E^F(F[-2])\),\]

one concludes that \(\ker(\alpha_*) = 0\), i.e., \(\alpha_*\) is also injective.

Let \(\theta \in W^*\) be given by \(\theta(e_k) = \delta_{-1,k}, k \in \{-1, \ldots, p^n - 2\}\). Then, by (3.3),

\[(3.18) \quad (e_0 \cdot \theta)(\partial) = -[e_0, \partial] = \partial,\]

and \((e_0 \cdot \theta)(y) = 0\) for \(y \in \mathcal{B}\), i.e., \(e_0 \cdot \theta = \theta\). As \([W, \text{rad}_E(\mathcal{B})] \subseteq \mathcal{B}\), one concludes that \(x \cdot \theta = 0\) for all \(x \in \text{rad}_E(\mathcal{B})\), i.e., one has an isomorphism of restricted \(\mathcal{B}\)-modules \(\beta : F[1] \to F \cdot \theta\). Let \(\beta_* : \text{ind}_E^F(F[1]) \to W^*\) denote the induced homomorphism of restricted \(\mathcal{M}\)-modules. One verifies easily that \(\{ \partial^k \cdot \theta \mid 0 \leq k \leq p^n - 1\}\) is a basis of \(W^*, \mathcal{B} \cdot \theta = W^*\) and hence \(\beta_*\) is surjective. As \(\dim(W^*) = \dim(\text{ind}_E^F(F[-2])\), \(\beta_*\) must be also surjective. This yields the claim. \(\square\)

3.3. Projective indecomposable modules. Proposition 3.2 has the following consequence for \(p = 2\).

**Theorem 3.4.** Let \(F\) be an algebraically closed field of characteristic 2, let \(n \geq 2\), and let \(\mathcal{M} = \mathcal{M}(1, n)\).

(a) \(\text{Irr}(\mathcal{M}(1, n)) = \{ [F], [L] \}, \) where \(F\) denotes the 1-dimensional trivial \(\mathcal{M}\)-module, and \(L = \text{soc}_E(\mathcal{M}) = W(1, n)^{(1)}\).

(b) \(\mathcal{B}\) is a generalized Borel subalgebra of \(\mathcal{M}(1, n)\).

(c) Let \(t_0\) be a torus of maximal dimension of \(\mathcal{M}\) satisfying \(\mathcal{M} = t_0 \oplus \mathcal{B}\) (cf. (3.3)). Then \(L^0 = 0\). In particular, \(\mathcal{M}(1, n)\) is a restricted Lie algebra with maximal 0-p.i.m., i.e., if \(P_\mathcal{M}\) is the projective cover of the 1-dimensional trivial \(\mathcal{M}\)-module, then one has \(P_\mathcal{M} \simeq \text{ind}_{t_0}^\mathcal{M}(F(0))\). Hence

\[(3.19) \quad \dim(P_\mathcal{M}) = 2^{\dim(\mathcal{M})} - \dim(t_0) = 2^{2n-1} - 1 .\]

(d) Let \(P_L\) denote the projective cover of the irreducible \(\mathcal{M}\)-module \(L\). Then \(\dim(P_L) = 2^{2n-1} - 1\). In particular, \(P_L \simeq \text{ind}_{t_0}^\mathcal{M}(F(\mu))\) for any non-trivial irreducible \(t_0\)-module \(F(\mu), \mu \in t_0^0 \setminus \{0\}\).

Proof. (a) As \(\mathcal{B}/\text{rad}_a(\mathcal{B}) \simeq \mathfrak{s}\), one has \(\text{Irr}(\mathcal{B}) = \{ [F(0)], [F(1)] \}\). Let \(S\) be an irreducible restricted \(\mathcal{M}\)-module, and let \(\Sigma = \text{soc}_E(\text{res}_E(S))\). Then \(\Sigma\) contains either a restricted \(\mathcal{B}\)-submodule isomorphic to \(F[0]\) or \(F[1]\) or both. As

\[(3.20) \quad \text{Hom}_E(F[\lambda], \text{res}_E(S)) \simeq \text{Hom}_E(\text{ind}_E^F(F[\lambda]), S), \quad \lambda \in \{0, 1\},\]

Proposition 3.3 implies that \(S\) is either a homomorphic image of \(\text{ind}_E^F(F[0])\), in which case \(S \simeq W/L \simeq F\), or \(S\) is a homomorphic image of \(\text{ind}_E^F(F[1])\), in which case \(S \simeq L^*\). Hence \(\text{Irr}(\mathcal{M}) = \{ [F], [L^*] \}\). As \(L\) is an irreducible restricted \(\mathcal{M}\)-module, one has \(L \simeq L^*\) by dimension reasons.

(b) As \(\mathcal{B}/\text{rad}_a(\mathcal{B}) \simeq \mathfrak{s}\) is a torus, \(M_{\text{rad}_a(\mathcal{B})}\) is a semi-simple restricted \(\mathcal{B}/\text{rad}_a(\mathcal{B})\)-module for any finite-dimensional \(\mathcal{M}\)-module \(M\). Hence

\[(3.21) \quad M_{\text{rad}_a(\mathcal{B})} = \text{soc}_E(\text{res}_E(M)).\]
From (3.21) one concludes that one has isomorphisms of restricted \( \mathfrak{B} \)-modules \( F^{rad, \mathfrak{B}}(\mathfrak{B}) \simeq F[0] \), and - as \( L \simeq L^* \) - also \( L^{rad, \mathfrak{B}}(\mathfrak{B}) \simeq F[1] \). In particular, the mapping \( \chi_\mathfrak{B} = [\cdot^{rad, \mathfrak{B}}]_\mathfrak{B} : \text{Irr}(\mathfrak{B}) \to \text{Irr}(\mathfrak{B}/rad_\mathfrak{B}(\mathfrak{B})) \) is injective showing that \( \mathfrak{B} \) is a generalized Borel subalgebra.

(c) As \( \mathfrak{W} \cong t_0 \oplus \mathfrak{B} \), one has

\[
U = \text{res}_{t_0}^{\mathfrak{W}}(\text{ind}_{t_0}^{\mathfrak{W}}(F[0])) \cong u(t_0) \otimes_{\mathfrak{B}} F[0] \cong u(t_0),
\]

i.e., \( U \) is a free \( u(t_0) \)-module of rank 1. As \( \text{ind}_{t_0}^{\mathfrak{W}}(F[0]) \) is isomorphic to the \( \mathfrak{W} \)-module \( W \), one has a short exact sequence of \( t_0 \)-modules

\[
0 \longrightarrow \text{res}_{t_0}^{\mathfrak{W}}(L) \longrightarrow U \longrightarrow F(0) \longrightarrow 0,
\]

where \( F(0) \) denotes the 1-dimensional trivial \( t_0 \)-module, i.e., \( \text{res}_{t_0}^{\mathfrak{W}}(L) \) is isomorphic to the augmentation ideal \( \ker(\varepsilon : u(t_0) \to F) \) of \( u(t_0) \). Hence \( L^{t_0} = 0 \). Since any non-trivial, irreducible \( \mathfrak{W} \)-module must be isomorphic to \( L \), one concludes from [3, Lemma 6.1] that the projective cover \( P_\mathfrak{W} \) of the 1-dimensional irreducible \( \mathfrak{W} \)-module \( F \) must be isomorphic to \( \text{ind}_{t_0}^{\mathfrak{W}}(F[0]) \).

(d) As one has an isomorphism \( u(\mathfrak{W}) \cong P_\mathfrak{W} \oplus \dim(L) \cdot P_L \) of \( \mathfrak{W} \)-modules, one concludes that

\[
\dim(P_L) = \frac{2^{2n+1} - 2^{2^n - 1}}{2^n - 1} = \frac{2^{2n-1}(2^n - 1)}{2^n - 1} = 2^{2^n - 1} - 1.
\]

If \( F(\mu) \) is a non-trivial irreducible \( t_0 \)-module, then \( \text{ind}_{t_0}^{\mathfrak{W}}(F(\mu)) \) is projective, and has dimension equal to \( 2^{2^n - 1} - 1 \). As \( F(\mu) \) is isomorphic to a direct summand of \( L \) (cf. (3.23)), \( P_L \) is a homomorphic image of \( \text{ind}_{t_0}^{\mathfrak{W}}(F(\mu)) \). Since both \( \mathfrak{W} \)-modules have the same dimension, they must be isomorphic. This completes the proof.

Remark 3.5. Let \( F \) be an algebraically closed field of characteristic \( p \), let \( \mathfrak{L} \) be a finite-dimensional restricted Lie algebra, and let \( t \subseteq \mathfrak{L} \) be a torus of maximal dimension. If \( P \) is a projective indecomposable restricted \( \mathfrak{L} \)-module, then \( P \) is isomorphic to the projective cover \( P_\mathfrak{L} \) for some irreducible restricted \( \mathfrak{L} \)-module \( S \). Let \( F(\mu) \) be an irreducible \( t \)-submodule of \( \text{res}\_t^\mathfrak{L}(S) \). Then - as \( \text{Hom}\_t^\mathfrak{L}(\text{ind}\_t^\mathfrak{L}(F(\mu)), S) \neq 0 \) - \( P \) is isomorphic to a direct summand of \( P_\mathfrak{L}(\mu) = \text{ind}\_t^\mathfrak{L}(F(\mu)) \) which is a projective restricted \( \mathfrak{L} \)-module for any \( \mu \in t^\mathfrak{L} \). Hence

\[
\dim(P) \leq p^{\text{dim}(\mathfrak{L}) - MT(\mathfrak{L})},
\]

This shows that for \( \mathfrak{L} = \mathfrak{W}(1, n), n > 1, \) and \( p = 2 \), equality holds in (3.25) for any projective indecomposable restricted \( \mathfrak{L} \)-module \( P \). Therefore, for the restricted Lie algebra \( \mathfrak{W} \) all p.i.m.s have maximal possible dimension.

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