Disproof of the List Hadwiger Conjecture

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Abstract

The List Hadwiger Conjecture asserts that every $K_t$-minor-free graph is $t$-choosable. We disprove this conjecture by constructing a $K_{3t+2}$-minor-free graph that is not $4t$-choosable for every integer $t \geq 1$.

1 Introduction

In 1943, Hadwiger [6] made the following conjecture, which is widely considered to be one of the most important open problems in graph theory; see [26] for a survey\(^1\).

**Hadwiger Conjecture.** Every $K_t$-minor-free graph is $(t-1)$-colourable.

The Hadwiger Conjecture holds for $t \leq 6$ (see [3, 6, 17, 18, 28]) and is open for $t \geq 7$. In fact, the following more general conjecture is open.

**Weak Hadwiger Conjecture.** Every $K_t$-minor-free graph is $ct$-colourable for some constant $c \geq 1$.

It is natural to consider analogous conjectures for list colourings\(^2\). First, consider the

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\(^1\)MSC: graph minors 05C83, graph coloring 05C15

\(^2\)A list-assignment of a graph $G$ is a function $L$ that assigns to each vertex $v$ of $G$ a set $L(v)$ of colours. $G$ is $L$-colourable if there is a colouring of $G$ such that the colour assigned to each vertex $v$ is in $L(v)$. $G$ is $k$-choosable if $G$ is $L$-colourable for every list-assignment $L$ with $|L(v)| \geq k$ for each vertex $v$ of $G$. The choice number of $G$ is the minimum integer $k$ such that $G$ is $k$-choosable. If $G$ is $k$-choosable then $G$ is also $k$-colourable—just use the same set of $k$ colours for each vertex. Thus the choice number of $G$ is at least the chromatic number of $G$. See [32] for a survey on list colouring.

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choosability of planar graphs. Erdős et al. [5] conjectured that some planar graph is not 4-choosable, and that every planar graph is 5-choosable. The first conjecture was verified by Voigt [27] and the second by Thomassen [25]. Incidentally, Borowiecki [1] asked whether every $K_t$-minor-free graph is $(t-1)$-choosable, which is true for $t \leq 4$ but false for $t = 5$ by Voigt’s example. The following natural conjecture arises (see [10, 30]).

**List Hadwiger Conjecture.** Every $K_t$-minor-free graph is $t$-choosable.

The List Hadwiger Conjecture holds for $t \leq 5$ (see [7, 20, 31]). Again the following more general conjecture is open.

**Weak List Hadwiger Conjecture.** Every $K_t$-minor-free graph is $ct$-choosable for some constant $c \geq 1$.

In this paper we disprove the List Hadwiger Conjecture for $t \geq 8$, and prove that $c \geq 4/3$ in the Weak List Hadwiger Conjecture.

**Theorem 1.** For every integer $t \geq 1$,

(a) there is a $K_{3t+2}$-minor-free graph that is not $4t$-choosable.

(b) there is a $K_{3t+1}$-minor-free graph that is not $(4t-2)$-choosable,

(c) there is a $K_{3t}$-minor-free graph that is not $(4t-3)$-choosable.

Before proving Theorem 1, note that adding a dominant vertex to a graph does not necessarily increase the choice number (as it does for the chromatic number). For example, $K_{3,3}$ is 3-choosable but not 2-choosable. Adding one dominant vertex to $K_{3,3}$ gives $K_{1,3,3}$, which again is 3-choosable [16]. In fact, this property holds for infinitely many complete bipartite graphs [16]; also see [19].

**2 Proof of Theorem 1**

Let $G_1$ and $G_2$ be graphs, and let $S_i$ be a $k$-clique in each $G_i$. Let $G$ be a graph obtained from the disjoint union of $G_1$ and $G_2$ by pairing the vertices in $S_1$ and $S_2$ and identifying each pair. Then $G$ is said to be obtained by *pasting* $G_1$ and $G_2$ on $S_1$ and $S_2$. The following lemma is well known.

**Lemma 2.** Let $G_1$ and $G_2$ be $K_t$-minor-free graphs. Let $S_i$ be a $k$-clique in each $G_i$. Let $G$ be a pasting of $G_1$ and $G_2$ on $S_1$ and $S_2$. Then $G$ is $K_t$-minor-free.

*Proof.* Suppose on the contrary that $K_{t+1}$ is a minor of $G$. Let $X_1, \ldots, X_{t+1}$ be the corresponding branch sets. If some $X_i$ does not intersect $G_1$ and some $X_j$ does not intersect $G_2$, then no edge joins $X_i$ and $X_j$, which is a contradiction. Thus, without loss of generality, each $X_i$ intersects $G_1$. Let $X'_i := G_1[X_i]$. Since $S_1$ is a clique, $X'_i$ is connected. Thus $X'_1, \ldots, X'_{t+1}$ are the branch sets of a $K_{t+1}$-minor in $G_1$. This contradiction proves that $G$ is $K_t$-minor-free.  


Let $K_{r \times 2}$ be the complete $r$-partite graph with $r$ colour classes of size $2$. Let $K_{1, r \times 2}$ be the complete $(r + 1)$-partite graph with $r$ colour classes of size $2$ and one colour class of size $1$. That is, $K_{r \times 2}$ and $K_{1, r \times 2}$ are respectively obtained from $K_{2r}$ and $K_{2r+1}$ by deleting a matching of $r$ edges. The following lemma will be useful.

**Lemma 3 ([8, 29]).** $K_{r \times 2}$ is $K_{3r/2+1}$-minor-free, and $K_{1, r \times 2}$ is $K_{3r/2+2}$-minor-free.

**Proof of Theorem 1.** Our goal is to construct a $K_p$-minor-free graph and a non-achievable list assignment with $q$ colours per vertex, where the integers $p, q$ and $r$ and a graph $H$ are defined in the following table. Let $\{v_1w_1, \ldots, v_rw_r\}$ be the deleted matching in $H$. By Lemma 3, the calculation in the table shows that $H$ is $K_p$-minor-free.

| case | $p$  | $q$  | $r$  | $H$                     |
|------|------|------|------|-------------------------|
| (a)  | $3t+2$ | $4t$  | $2t+1$ | $K_{r \times 2}$ $\lfloor \frac{3r}{2} \rfloor + 1 = 3t + 2 = p$ |
| (b)  | $3t+1$ | $4t-2$ | $2t$  | $K_{r \times 2}$ $\lfloor \frac{3r}{2} \rfloor + 1 = 3t + 1 = p$ |
| (c)  | $3t$    | $4t-3$ | $2t-1$ | $K_{1, r \times 2}$ $\lfloor \frac{3r}{2} \rfloor + 2 = 3t = p$ |

For each vector $(c_1, \ldots, c_r) \in [1, q]^r$, let $H(c_1, \ldots, c_r)$ be a copy of $H$ with the following list assignment. For each $i \in [1, r]$, let $L(w_i) := [1, q + 1] \setminus \{c_i\}$. Let $L(u) := [1, q]$ for each remaining vertex $u$. There are $q + 1$ colours in total, and $|V(H)| = q + 2$. Thus in every $L$-colouring of $H$, two non-adjacent vertices receive the same colour. That is, $\text{col}(v_i) = \text{col}(w_i)$ for some $i \in [1, r]$. Since each $c_i \not\in L(w_i)$, it is not the case that each vertex $v_i$ is coloured $c_i$.

Let $G$ be the graph obtained by pasting all the graphs $H(c_1, \ldots, c_r)$, where $(c_1, \ldots, c_r) \in [1, q]^r$, on the clique $\{v_1, \ldots, v_r\}$. The list assignment $L$ is well defined for $G$ since $L(v_i) = [1, q]$. By Lemma 2, $G$ is $K_p$-minor-free. Suppose that $G$ is $L$-colourable. Let $c_i$ be the colour assigned to each vertex $v_i$. Thus $c_i \in L(v_i) = [1, q]$. Hence, as proved above, the copy $H(c_1, \ldots, c_r)$ is not $L$-colourable. This contradiction proves that $G$ is not $L$-colourable. Each vertex of $G$ has a list of $q$ colours in $L$. Therefore $G$ is not $q$-choosable. (It is easily seen that $G$ is $q$-degenerate\(^3\), implying $G$ is $(q+1)$-choosable.)

Note that this proof was inspired by the construction of a non-4-choosable planar graph by Mirzakhani [15].

### 3 Conclusion

Theorem 1 disproves the List Hadwiger Conjecture. However, list colourings remain a viable approach for attacking Hadwiger’s Conjecture. Indeed, list colourings provide potential routes around some of the known obstacles, such as large minimum degree, and lack of exact structure theorems; see [10, 11, 30, 31].

\(^3\)A graph is $d$-degenerate if every subgraph has minimum degree at most $d$. Clearly every $d$-degenerate graph is $(d + 1)$-choosable.
The following table gives the best known lower and upper bounds on the maximum choice number of $K_t$-minor-free graphs. Each lower bound is a special case of Theorem 1. Each upper bound (except $t = 5$) follows from the following degeneracy results. Every $K_3$-minor-free graph (that is, every forest) is 1-degenerate. Dirac [4] proved that every $K_4$-minor-free graph is 2-degenerate. Mader [14] proved that for $t \leq 7$, every $K_t$-minor-free graph is $(2t - 5)$-degenerate. Jørgensen [9] and Song and Thomas [21] proved the same result for $t = 8$ and $t = 9$ respectively. Song [22] proved that every $K_{10}$-minor-free graph is 21-degenerate, and that every $K_{11}$-minor-free graph is 25-degenerate. In general, Kostochka [12, 13] and Thomason [23, 24] independently proved that every $K_t$-minor-free graph is $O(t \sqrt{\log t})$-degenerate.

| $t$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | ⋯ | $t$ |
|-----|---|---|---|---|---|---|---|----|----|---|---|
| lower bound | 2 | 3 | 5 | 6 | 7 | 9 | 10 | 11 | 13 | ⋯ | $\frac{4}{3}t - c$ |
| upper bound | 2 | 3 | 5 | 8 | 10 | 12 | 14 | 22 | 26 | ⋯ | $O(t \sqrt{\log t})$ |

The following immediate open problems arise:

- Is every $K_6$-minor-free graph 7-choosable?
- Is every $K_6$-minor-free graph 6-degenerate?
- Is every $K_6$-minor-free graph 6-choosable?

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