The Euler class in the Simplicial de Rham Complex

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Abstract

We exhibit a cocycle in the simplicial de Rham complex which represents the Euler class. As an application, we construct a Lie algebra cocycle on $L\mathfrak{so}(4)$.

1 Introduction

For any Lie group $G$, we can define a simplicial manifold $\{NG(*)\}$ and a double complex $\Omega^*(NG(*))$ on it. In classical theory, it is well-known that the cohomology ring of the total complex $\Omega^*(NG)$ is isomorphic to $H^*(BG)$ where $BG$ is a classifying space of $G$, which is not a manifold in general [2] [5] [6].

In [4], Dupont introduced another double complex $A^{*,*}(NG)$ on $NG$ such that the cohomology ring of its total complex $A^*(NG)$ is also isomorphic to $H^*(BG)$. He used it to construct a homomorphism from $I^*(G)$, the $G$-invariant polynomial ring over Lie algebra $\mathcal{G}$, to $H^*(BG)$. By using Dupont’s method, in [8] the author exhibited cocycles in $\Omega^*(NG)$ which represent the Chern characters. In this paper, we will exhibit cocycles which represent the Euler classes.

Using a cocycle in $\Omega^*(NG)$, we can construct a cocycle in the local truncated complex $[\sigma_{<p}\Omega^*_{loc}(NG)]$ due to Brylinski [3]. Furthermore, we can obtain a Lie algebra cocycle of a free loop group $LG$. Following Brylinski’s idea, we will construct a Lie algebra 2-cocycle on $L\mathfrak{so}(4)$ using a cocycle in $\Omega^4(\mathsf{SO}(4))$. 
2 Review of the universal Chern-Weil Theory

In this section we recall the universal Chern-Weil theory following [5]. For any Lie group $G$, we have simplicial manifolds $\bar{N}G$, $NG$ and simplicial $G$-bundle $\gamma : \bar{N}G \to NG$ as follows:

$$\bar{N}G(q) = G \times \cdots \times G \ni (g_1, \ldots, g_{q+1})$$

$$NG(q) = G \times \cdots \times G \ni (h_1, \ldots, h_q) :$$

face operators $\varepsilon_i : NG(q) \to NG(q-1)$

$$\varepsilon_i(h_1, \ldots, h_q) = \begin{cases} (h_2, \ldots, h_q) & i = 0 \\ (h_1, \ldots, h_i h_{i+1}, \ldots, h_q) & i = 1, \ldots, q-1 \\ (h_1, \ldots, h_{q-1}) & i = q. \end{cases}$$

We define $\gamma : \bar{N}G \to NG$ as $\gamma(g_0, \ldots, g_q) = (g_0 g_1^{-1}, \ldots, g_{q-1} g_q^{-1})$.

For any simplicial manifold $X = \{X_*\}$, we can associate a topological space $\| X \|$ called the fat realization. It is well-known that $\| \gamma \|$ is the universal bundle $EG \to BG$ [7].

Now we introduce a double complex associated to a simplicial manifold.

**Definition 2.1.** For any simplicial manifold $\{X_*\}$ with face operators $\{\varepsilon_*\}$, we define a double complex as follows:

$$\Omega^{p,q}(X) := \Omega^q(X_p)$$

Derivatives are:

$$d' := \sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*, \quad d'' := (-1)^p \times \text{the exterior differential on } \Omega^*(X_p).$$

For $NG$ and $\bar{N}G$ the following holds [2] [5] [6].

**Theorem 2.1.** There exist ring isomorphisms

$$H(\Omega^*(\bar{N}G)) \cong H^*(EG), \quad H(\Omega^*(NG)) \cong H^*(BG).$$

Here $\Omega^*(\bar{N}G)$ and $\Omega^*(NG)$ mean the total complexes.
There is another double complex associated to a simplicial manifold.

**Definition 2.2** ([4]). A simplicial $n$-form on a simplicial manifold $\{X_p\}$ is a sequence $\{\phi^{(p)}\}$ of $n$-forms $\phi^{(p)}$ on $\Delta^p \times X_p$ such that

$$(\varepsilon^i \times \text{id})^* \phi^{(p)} = (\text{id} \times \varepsilon^i)^* \phi^{(p-1)}$$

on $\Delta^{p-1} \times X_p$.

Here $\varepsilon^i$ is the canonical $i$-th face operator of $\Delta^p$.

Let $A^{k,l}(X)$ be the set of all simplicial $(k + l)$-forms on $\Delta^p \times X_p$ which are expressed locally of the form

$$\sum a_{i_0 \cdots i_k j_1 \cdots j_l} (dt_{i_1} \wedge \cdots \wedge dt_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l})$$

where $(t_0, t_1, \ldots, t_p)$ are the barycentric coordinates in $\Delta^p$ and $x_j$ are the local coordinates in $X_p$. We define derivatives as:

$$d' := \text{the exterior differential on } \Delta^p$$

$$d'' := (-1)^k \times \text{the exterior differential on } X_p.$$  

Then $(A^{k,l}(X), d', d'')$ is a double complex and the following theorem holds.

**Theorem 2.2** ([4]). Let $A^*(X)$ denote the total complex of $A^{*,*}(X)$. A map $I_\Delta : A^*(X) \to \Omega^*(X)$ defined as $I_\Delta(\alpha) := \int_{\Delta^p} (\alpha|_{\Delta^p \times X_p})$ induces a natural ring isomorphism $I_\Delta : H(A^*(X)) \cong H(\Omega^*(X))$.

Let $G$ denote the Lie algebra of $G$. A connection on a simplicial $G$-bundle $\pi : \{E_p\} \to \{M_p\}$ is a sequence of 1-forms $\{\theta\}$ on $\{E_p\}$ with coefficients $G$ such that $\theta$ restricted to $\Delta^p \times E_p$ is a usual connection form.

There is a canonical connection $\theta \in A^1(NG)$ on $\gamma : NG \to NG$ defined as follows:

$$\theta|_{\Delta^p \times NG(p)} := t_0 \theta_0 + \cdots + t_p \theta_p.$$  

Here $\theta_i$ is defined as $\theta_i = \text{pr}^*_i \theta$ where $\text{pr}_i : \Delta^p \times NG(p) \to G$ is the projection into the $(i+1)$-th factor of $NG(p)$ and $\theta$ is the Maurer-Cartan form of $G$. We obtain also its curvature $\Omega \in A^2(NG)$ on $\gamma$ as:

$$\Omega|_{\Delta^p \times NG(p)} = d\theta|_{\Delta^p \times NG(p)} + \frac{1}{2} [\theta|_{\Delta^p \times NG(p)}, \theta|_{\Delta^p \times NG(p)}].$$  

Let $I^*(G)$ denote the ring of $G$-invariant polynomials on $G$. For $P \in I^k(G)$, we restrict $P(\Omega) \in A^{2k}(NG)$ to each $\Delta^p \times NG(p)$ and apply the usual Chern-Weil theory then we have $I_\Delta(P(\Omega)) \in \Omega^{2k}(NG)$. In this way we have a homomorphism $I^*(G) \to H(\Omega^*(NG))$ which maps $P \in I^*(G)$ to $[I_\Delta(P(\Omega))]$. 

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3 The Euler class in the double complex

In this section we exhibit a cocycle in $\Omega^*(NSO(2p))$ which represents the Euler class of the universal bundle $ESO(2p) \to BSO(2p)$. Throughout this section, $G$ means $SO(2p)$.

Recall that the polynomial on $\mathfrak{so}(2p)$ called Pfaffian is defined as follows:

$$\text{Pf}(A, \cdots, A) = \frac{1}{2^p p!} \sum_{\tau \in \mathfrak{S}_{2p}} \text{sgn}(\tau)a_{\tau(1)\tau(2)} \cdots a_{\tau(2p-1)\tau(2p)}.$$  

Here $a_{ij}$ is a $(i, j)$ entry of $A \in \mathfrak{so}(2p)$.

3.1 The cochain on the edge

We first give the cochain in $\Omega^{2p+1}(N\tilde{G}(1))$ which corresponds to the Euler class. This is given by integrating $\text{Pf}(\Omega|_{\Delta^1 \times N\tilde{G}(1)})$ along $\Delta^1$. Since $\Omega|_{\Delta^1 \times N\tilde{G}(1)} = -dt_1 \wedge (\theta_0 - \theta_1) - t_0 t_1 (\theta_0 - \theta_1)^2$, we can see $\text{Pf}(\Omega|_{\Delta^1 \times N\tilde{G}(1)})$ is equal to

$$\frac{1}{2^p p!} \sum_{\tau \in \mathfrak{S}_{2m}} \text{sgn}(\tau)((-dt_1 \wedge (\theta_0 - \theta_1) - t_0 t_1 (\theta_0 - \theta_1)^2)_{\tau(1)\tau(2)}$$

$$\cdots (-dt_1 \wedge (\theta_0 - \theta_1) - t_0 t_1 (\theta_0 - \theta_1)^2)_{\tau(2p-1)\tau(2p)}).$$

We set:

$$P^k_\tau := (\theta_0 - \theta_1)^2_{\tau(1)\tau(2)} \cdots (\theta_0 - \theta_1)^2_{\tau(2k-3)\tau(2k-2)} (\theta_0 - \theta_1)_{\tau(2k-1)\tau(2k)}$$

$$(\theta_0 - \theta_1)^2_{\tau(2k+1)\tau(2k+2)} \cdots (\theta_0 - \theta_1)^2_{\tau(2p-1)\tau(2p)}.$$  

Then the following equation holds.

$$\int_{\Delta^1} \text{Pf}(\Omega|_{\Delta^1 \times N\tilde{G}(1)}) = (-1)^p \frac{1}{2^p p!} \left( \int_0^1 (t_0 t_1)^{p-1} dt_1 \right) \sum_{\tau \in \mathfrak{S}_{2p}} \sum_{k=1}^{p} \text{sgn}(\tau)P^k_\tau.$$  

Now we obtain the cochain in $\Omega^{2p-1}(N\tilde{G}(1))$.

**Proposition 3.1.** The cochain $\mu_p$ in $\Omega^{2p-1}(N\tilde{G}(1))$ which corresponds to the Euler class is given as follows:

$$\mu_1 = (-1)^p \frac{1}{2^p p!} \frac{1}{2p-1} \sum_{\tau \in \mathfrak{S}_{2p}} \sum_{k=1}^{p} \text{sgn}(\tau)P^k_\tau.$$  

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Here $P_k^\tau$ is defined as:

$$P_k^\tau := \left( h^{-1} dh \right)_{(1)\tau(2)} \cdots \left( h^{-1} dh \right)_{(2k-1)\tau(2k-2)} \left( h^{-1} dh \right)_{(2k-1)\tau(2k)} \left( h^{-1} dh \right)_{(2k+1)\tau(2k+2)} \cdots \left( h^{-1} dh \right)_{(2p-1)\tau(2p)}.$$

**Proof.** This follows from the equation

$$\int_0^1 (t_0 t_1)^{p-1} dt_1 = \frac{1}{2p-1} c_{p-1} \cdot p$$

and

$$\gamma^* \sum_{\tau \in \mathcal{S}_{2p}} \text{sgn}(\tau) P_k^\tau = \sum_{\tau \in \mathcal{S}_{2p}} \text{sgn}(\tau) \bar{P}_k^\tau.$$

As a special case of Proposition 3.1, we obtain the following theorem.

**Theorem 3.1.** In the case of $G = SO(2)$, the cocycle $E_{1,1}$ in $\Omega^2(NG)$ which represents the Euler class of $ESO(2) \to BSO(2)$ is given as follows:

$$E_{1,1} = \frac{1}{4\pi} \left( -(h^{-1} dh)_{12} + (h^{-1} dh)_{21} \right) \in \Omega^1(SO(2)).$$

If we write an element $h$ in $SO(2)$ as

$$h = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

then the equation

$$h^{-1} dh = \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix}$$

holds, so we obtain

$$E_{1,1} = \frac{1}{4\pi} (2d\theta) = \frac{d\theta}{2\pi}.$$

**3.2 The cochain in $\Omega^p(NG(p))$**

In $\Omega^p(NG(p))$, $\Omega|_{\Delta^p \times NG(p)}$ is equal to $-\sum_{i=1}^p dt_i \wedge (\theta_0 - \theta_i) - \sum_{0 \leq i < j \leq p} t_i t_j (\theta_i - \theta_j)^2$, so the cochain $\int_{\Delta^p} \text{Pf}(\Omega|_{\Delta^p \times NG(p)})$ in $\Omega^p(NG(p))$ which corresponds to the Euler class is given as follows:

$$\frac{1}{2^{2p-p} p!} \sum_{\tau \in \mathcal{S}_{2p}} \text{sgn}(\tau) \left( -\sum_{i=1}^p dt_i \wedge (\theta_0 - \theta_i) \right)_{(1)\tau(2)} \cdots \left( -\sum_{i=1}^p dt_i \wedge (\theta_0 - \theta_i) \right)_{(2p-1)\tau(2p)}.$$
Now
\[ dt_i \wedge (\theta_0 - \theta_i) = dt_i \wedge \{ (\theta_0 - \theta_1) + (\theta_1 - \theta_2) + \cdots + (\theta_{i-1} - \theta_i) \} \]

and for any differential forms \( \alpha, \beta, \gamma \) and any integer \( 0 \leq k, l, x \leq p \), the equation
\[
\alpha \wedge (dt_i \wedge (\theta_x - \theta_{x+1})_p^{(2k-1)\tau_p}) \wedge (dt_j \wedge (\theta_x - \theta_{x+1})_p^{(2l-1)\tau_p}) \wedge \gamma = \]
\[-\alpha \wedge (dt_j \wedge (\theta_x - \theta_{x+1})_p^{(2k-1)\tau_p}) \wedge (dt_i \wedge (\theta_x - \theta_{x+1})_p^{(2l-1)\tau_p}) \wedge \gamma \]
holds, so the terms of these forms cancel with each other in \( \text{Pf}(\Omega|_{\Delta^p \times NG(p)}) \).

We set:
\[ \varphi_s := h_1 \cdots h_{s-1} dh_s h_s^{-1} \cdots h_1^{-1}. \]
Then we can check that \( \gamma^* \varphi_s = g_1(\theta_{s-1} - \theta_s) g_1^{-1} \) hence we obtain the following proposition.

**Proposition 3.2.** The cochain \( \mu_p \) in \( \Omega^p(NG(p)) \) which corresponds to the Euler class is given as follows:

\[
\mu_p = \left( -1 \right)^{\frac{p(p+1)}{2}} \frac{1}{2^p \pi^p (p!)^2} \sum_{\sigma \in S_p} \sum_{\tau \in S_{2p}} \text{sgn}(\tau) \text{sgn}(\sigma) (\varphi_{\sigma(1)})_{\tau(1)\tau(2)} \cdots (\varphi_{\sigma(p)})_{\tau(2p-1)\tau(2p)}.\]

Using Proposition 3.1 and Proposition 3.2, we obtain the cocycle which represents the Euler class of \( ESO(4) \rightarrow BSO(4) \) in \( \Omega^4(NG(4)) \).

**Theorem 3.2.** In the case of \( G = SO(4) \), the cocycle which represents the Euler class of \( ESO(4) \rightarrow BSO(4) \) in \( \Omega^4(NG) \) is the sum of the following \( E_{1,3} \) and \( E_{2,2} \):

\[
E_{1,3} = \frac{1}{192 \pi^2} \sum_{\tau \in S_4} \text{sgn}(\tau) \left( (h^{-1} dh)_{\tau(1)} (h^{-1} dh)_{\tau(2)} (h^{-1} dh)_{\tau(3)} (h^{-1} dh)_{\tau(4)} \right) + (h^{-1} dh)_{\tau(1)} (h^{-1} dh)_{\tau(2)} (h^{-1} dh)_{\tau(3)} (h^{-1} dh)_{\tau(4)} \right) \]

\[0 \]
\[\uparrow d' \]
\[E_{1,3} \in \Omega^3(SO(4)) \quad \xrightarrow{d'} \quad \Omega^3(SO(4) \times SO(4)) \]
\[\uparrow d' \]
\[E_{2,2} \in \Omega^2(SO(4) \times SO(4)) \quad \xrightarrow{d'} \quad 0 \]

\[E_{1,3} = \frac{1}{192 \pi^2} \sum_{\tau \in S_4} \text{sgn}(\tau) \left( (h^{-1} dh)_{\tau(1)} (h^{-1} dh)_{\tau(2)} (h^{-1} dh)_{\tau(3)} (h^{-1} dh)_{\tau(4)} \right) + (h^{-1} dh)_{\tau(1)} (h^{-1} dh)_{\tau(2)} (h^{-1} dh)_{\tau(3)} (h^{-1} dh)_{\tau(4)} \right) \]
3.3 The cocycle in $\Omega^{p+q}(NG(p - q))$

Repeating the same argument in section 3.2, we obtain a cocycle in $\Omega^{p+q}(NG(p-q))$.

We set:

$$R_{ij} := (\varphi_i + \varphi_{i+1} + \cdots + \varphi_{j-1})^2 \quad (1 \leq i < j \leq p - q + 1).$$

**Theorem 3.3.** The cocycle in $\Omega^{p+q}(NG(p - q))$ (0 ≤ q ≤ p − 1) which represents the Euler class of $ESO(2p) \to BSO(2p)$ is

$$\sum_{\sigma \in S_{p-q}, \tau \in S_2} \sum_{\tau \in S_{p-q}} (T^\tau_\sigma \cdot (R_{i1j1})_{\tau(1)} \cdot (\varphi_{\sigma(1)})_{\tau(3)} \cdot (\varphi_{\sigma(2)})_{\tau(4)}$$

$$\cdots \cdot (R_{i2j2})_{\tau(2p-3)} \cdot (\varphi_{\sigma(p-q)})_{\tau(2p-1)} \cdot (\varphi_{\sigma(2p)})_{\tau(2p)})$$

where $R_{ij}$ (1 ≤ i < j ≤ p − q + 1) are put q-times between $\varphi_{\sigma(l)}$ and $\varphi_{\sigma(l+1)}$, or the edge in $\varphi_{\sigma(1)} \cdots \varphi_{\sigma(p-q)}$ permitting overlaps and $\sum$ means the sum of all such forms. $T^\tau_\sigma$ is defined as:

$$T^\tau_\sigma = \sgn(\tau) \sgn(\sigma) \left( \frac{(-1)^{p+q(q-1)}}{2^{2p} \pi^p p!} \int_{\Delta_{p-q}} \prod_{i < j} \left( t_{i-1} t_{j-1} \right)^{r_{ij}} d t_1 \wedge \cdots \wedge d t_{p-q} \right)$$

where $r_{ij}$ means the number of $R_{ij}$ in each form.

**Theorem 3.4.** In the case of $G = SO(6)$, the cocycle which represents the Euler class in $\Omega^6(NG)$ is the sum of the following $E_{1,5}, E_{2,4}$ and $E_{3,3}$:

$$E_{1,5} \in \Omega^5(G) \xrightarrow{d'} \Omega^5(NG(2))$$

$$E_{2,4} \in \Omega^4(NG(2)) \xrightarrow{d'} \Omega^4(NG(3))$$

$$E_{3,3} \in \Omega^3(NG(3)) \xrightarrow{d'} 0$$

$$E_{2,2} = \frac{-1}{64\pi^2} \sum_{\tau \in S_4} \sgn(\tau) \left( (h_1^{-1} d h_1)_{\tau(1)}(d h_2 h_2^{-1})_{\tau(2)}(h_3^{-1} d h_3)_{\tau(3)}(h_4^{-1} d h_4)_{\tau(4)} \right) + (d h_2 h_2^{-1})_{\tau(1)}(h_1^{-1} d h_1)_{\tau(2)}(h_3^{-1} d h_3)_{\tau(3)}(h_4^{-1} d h_4)_{\tau(4)}.$$
\[ E_{1,5} = \frac{-1}{2^6 \cdot 180 \pi^3} \sum_{\tau \in G_6} \text{sgn}(\tau) \left( (h^{-1}dh)^2 \tau(1) \tau(2) (h^{-1}dh) \tau(3) \tau(4) (h^{-1}dh) \tau(5) \tau(6) \right. \\
+ (h^{-1}dh) \tau(1) \tau(2) (h^{-1}dh)^2 \tau(3) \tau(4) (h^{-1}dh) \tau(5) \tau(6) \\
+ (h^{-1}dh) \tau(1) \tau(2) (h^{-1}dh) \tau(3) \tau(4) (h^{-1}dh)^2 \tau(5) \tau(6) \bigg) \]

\[ E_{2,4} = \frac{1}{2^6 \cdot 6 \cdot 4! \pi^3} \sum_{\tau \in G_6} \text{sgn}(\tau) \cdot \left( (h_1^{-1}dh_1) \tau(1) \tau(2) (dh_2 h_2^{-1}) \tau(3) \tau(4) \right. \\
\left( 2h_1^{-1}dh_1 h_1^{-1}dh_1 + 2dh_2 h_2^{-1}dh_2 h_2^{-1} + h_1^{-1}dh_1dh_2 h_2^{-1} + dh_2 h_2^{-1} h_1^{-1}dh_1 \right) \tau(5) \tau(6) \\
+ (h_1^{-1}dh_1) \tau(1) \tau(2) \left( 2h_1^{-1}dh_1 h_1^{-1}dh_1 + 2dh_2 h_2^{-1}dh_2 h_2^{-1} \\
+ h_1^{-1}dh_1dh_2 h_2^{-1} + dh_2 h_2^{-1} h_1^{-1}dh_1 \right) \tau(3) \tau(4) (dh_2 h_2^{-1}) \tau(5) \tau(6) \\
+ \left( 2h_1^{-1}dh_1 h_1^{-1}dh_1 + 2dh_2 h_2^{-1}dh_2 h_2^{-1} + h_1^{-1}dh_1dh_2 h_2^{-1} + dh_2 h_2^{-1} h_1^{-1}dh_1 \right) \tau(1) \tau(2) \\
\left. \cdot (h_1^{-1}dh_1) \tau(3) \tau(4) (dh_2 h_2^{-1}) \tau(5) \tau(6) \right) \\
-(dh_2 h_2^{-1}) \tau(1) \tau(2) (h_1^{-1}dh_1) \tau(3) \tau(4) \cdot \left( 2h_1^{-1}dh_1 h_1^{-1}dh_1 + 2dh_2 h_2^{-1}dh_2 h_2^{-1} + h_1^{-1}dh_1dh_2 h_2^{-1} + dh_2 h_2^{-1} h_1^{-1}dh_1 \right) \tau(5) \tau(6) \\
-(dh_2 h_2^{-1}) \tau(1) \tau(2) \left( 2h_1^{-1}dh_1 h_1^{-1}dh_1 + 2dh_2 h_2^{-1}dh_2 h_2^{-1} \\
+ h_1^{-1}dh_1dh_2 h_2^{-1} + dh_2 h_2^{-1} h_1^{-1}dh_1 \right) \tau(3) \tau(4) (h_1^{-1}dh_1) \tau(5) \tau(6) \]
\[-\left(2h_1^{-1}dh_1h_1^{-1}dh_1 + 2dh_2h_2^{-1}dh_2h_2^{-1} + h_1^{-1}dh_1dh_2h_2^{-1} + dh_2h_2^{-1}h_1^{-1}dh_1\right)_{\tau(1)\tau(2)} \cdot \left(dh_2h_2^{-1}\right)_{\tau(3)\tau(4)}(h_1^{-1}dh_1)_{\tau(5)\tau(6)}\).

\[E_{3,3} = \frac{1}{2^6 \cdot 6^2 \pi^3} \sum_{\tau \in \mathcal{S}_6} \text{sgn}(\tau) \cdot\]

\[
\left((h_1^{-1}dh_1)_{\tau(1)\tau(2)}(dh_2h_2^{-1})_{\tau(3)\tau(4)}(h_2dh_3h_3^{-1}h_2^{-1})_{\tau(5)\tau(6)}
- (dh_2h_2^{-1})_{\tau(1)\tau(2)}(h_1^{-1}dh_1)_{\tau(3)\tau(4)}(h_2dh_3h_3^{-1}h_2^{-1})_{\tau(5)\tau(6)}
- (h_1^{-1}dh_1)_{\tau(1)\tau(2)}(h_2dh_3h_3^{-1}h_2^{-1})_{\tau(3)\tau(4)}(dh_2h_2^{-1})_{\tau(5)\tau(6)}
+ (h_2dh_3h_3^{-1}h_2^{-1})_{\tau(1)\tau(2)}(h_1^{-1}dh_1)_{\tau(3)\tau(4)}(dh_2h_2^{-1})_{\tau(5)\tau(6)}
+ (dh_2h_2^{-1})_{\tau(1)\tau(2)}(h_2dh_3h_3^{-1}h_2^{-1})_{\tau(3)\tau(4)}(h_1^{-1}dh_1)_{\tau(5)\tau(6)}
- (h_2dh_3h_3^{-1}h_2^{-1})_{\tau(1)\tau(2)}(dh_2h_2^{-1})_{\tau(3)\tau(4)}(h_1^{-1}dh_1)_{\tau(5)\tau(6)}\right).
\]

4 The cocycle in a local truncated complex

We recall the filtered local simplicial de Rham complex due to Brylinski [3].

**Definition 4.1** ([3]). The filtered local simplicial de Rham complex \(F^p\Omega^\ast_{\text{loc}}(NG)\) over a simplicial manifold \(NG\) is defined as follows:

\[F^p\Omega^\ast_{\text{loc}}(NG) = \begin{cases} 
\lim_{V \subset G^r} \Omega^\ast(V) & \text{if } s \geq p \\
0 & \text{otherwise.}
\end{cases}\]

Let \(F^p\Omega^\ast(NG)\) be a filtered complex

\[F^p\Omega^\ast(NG) = \begin{cases} 
\Omega^\ast(NG(r)) & \text{if } s \geq p \\
0 & \text{otherwise}
\end{cases}\]
and \([\sigma_{<p}\Omega^*(NG)]\) a truncated complex

\[
[\sigma_{<p}\Omega^{r,s}(NG)] = \begin{cases} 0 & \text{if } s \geq p \\ \Omega^s(NG(r)) & \text{otherwise.} \end{cases}
\]

Then there is an exact sequence:

\[0 \to F^p\Omega^*(NG) \to \Omega^*(NG) \to [\sigma_{<p}\Omega^*(NG)] \to 0\]

which induces a boundary map \(\beta : H^l(NG, [\sigma_{<p}\Omega^*_{loc}]) \to H^{l+1}(NG, [F^p\Omega^*_{loc}]).\)

Let \(\mu_1 + \cdots + \mu_p, \mu_{p-q} \in \Omega^{p+q}(NG(p-q))\) be a cocycle in \(\Omega^{2p}(NG).\) Using this cocycle, we can construct a cocycle \(\eta\) in \([\sigma_{<p}\Omega^*_{loc}(NG)]\) in the following way.

We take a contractible open set \(U \subset G\) containing 1. Using the same argument in [5], we can construct mappings \(\{\sigma_l : \Delta^l \times U \to U\}_{0 \leq l}\) inductively with the following properties:
(1) \(\sigma_0(pt) = 1;\)
(2) \(\sigma_l(\varepsilon^j(t_0, \ldots , t_{l-1}); h_1, \ldots , h_l) = \begin{cases} \sigma_{l-1}(t_0, \ldots , t_{l-1}; \varepsilon_j(h_1, \ldots , h_l)) & \text{if } j \geq 1 \\ h_1 \cdot \sigma_{l-1}(t_0, \ldots , t_{l-1}; h_2, \ldots , h_l) & \text{if } j = 0. \end{cases}\)

We define mappings \(\{f_{m,q} : \Delta^q \times U^{m+q-1} \to G^m\}\) as

\[f_{m,q}(t_0, \ldots , t_q; h_1, \ldots , h_{m+q-1}) := (h_1, \ldots , h_{m-1}, \sigma_q(t_0, \ldots , t_q; h_m, \ldots , h_{m+q-1})).\]

A \((2p-m-q)\)-form \(\beta_{m,q}\) on \(U^{m+q-1}\) is defined as \(\beta_{m,q} = (-1)^m \int_{\Delta^q} f_{m,q}^* \mu_m.\)

Then we define the cochain \(\eta\) as the sum of following \(\eta_l\) on \(U^{2p-1-l}\) for \(0 \leq l \leq p-1:\)

\[\eta_l := \sum_{m+q=2p-l, \ p \geq m \geq 1} \beta_{m,q}.\]

**Theorem 4.1** [3][8]. \(\eta := \eta_0 + \cdots + \eta_{p-1}\) is a cocycle in \([\sigma_{<p}\Omega^*_{loc}(NG)]\) whose cohomology class is mapped to \([\mu_1 + \cdots + \mu_p]\) in \(H^{2p}(NG, [F^p\Omega^*_{loc}])\) by a boundary map \(\beta : H^{2p-1}(NG, [\sigma_{<p}\Omega^*_{loc}]) \to H^{2p}(NG, [F^p\Omega^*_{loc}]).\)

**Proof.** See [8].
5 Construction of a Lie algebra cocycle

For any Lie group $G$, let $C^\infty_{loc}(G^p, \mathbb{R})$ denote the group of germs at $(1, \cdots, 1)$ of smooth functions $G^p \to \mathbb{R}$ and $H^p_{loc}(G, \mathbb{R})$ denote the cohomology group of the following complex:

$$\cdots \to C^\infty_{loc}(G^p, \mathbb{R}) \xrightarrow{\delta := \sum_{i=0}^p (-1)^i \varepsilon_i^*} C^\infty_{loc}(G^{p+1}, \mathbb{R}) \to \cdots$$

Brylinski constructed a natural cochain map $\phi : C^p_{loc}(G, \mathbb{R}) \to C^p(\mathcal{G}, \mathbb{R})$ as follows:

$$\phi(c)(\xi_1, \cdots, \xi_p) := \left[ \frac{\partial^p}{\partial y_1 \cdots \partial y_p} \sum_{\rho \in \mathcal{S}_p} \text{sgn}(\rho) (\exp(y_{\rho(1)} \xi_{\rho(1)}), \cdots, \exp(y_{\rho(p)} \xi_{\rho(p)})) \right]_{y_i=0}$$

where $C^p(\mathcal{G}, \mathbb{R})$ is the space of smooth alternating multilinear maps $\mathcal{G} \to \mathbb{R}$ and $\xi_i \in \mathcal{G}$. For example, if we take $\delta c \in C^\infty_{loc}(G^2, \mathbb{R})$ and set $X_{\rho(i)} := \exp(y_{\rho(i)} \xi_{\rho(i)})$ then

$$\phi(\delta c)(\xi_1, \xi_2) \equiv \left[ \frac{\partial^2}{\partial y_1 \partial y_2} \sum_{\rho \in \mathcal{S}_2} \text{sgn}(\rho) (\delta c(X_{\rho(1)}, X_{\rho(2)}) \right]_{y_i=0}$$

$$= \left[ \frac{\partial^2}{\partial y_1 \partial y_2} \sum_{\rho \in \mathcal{S}_2} \text{sgn}(\rho) (c(X_{\rho(2)}) - c(X_{\rho(1)}X_{\rho(2)} + c(X_{\rho(1)}) \right]_{y_i=0}$$

$$= \left[ \frac{\partial^2}{\partial y_1 \partial y_2} (-c(X_{1}X_{2} - X_{2}X_{1})) \right]_{y_i=0} = (d(\phi(c)))(\xi_1, \xi_2).$$

Let $LU$ be the free loop space of a contractible open set $U \subset SO(4)$ containing 1 and $ev : LU \times S^1 \to U$ be the evaluation map, i.e. for $\gamma \in LU$ and $\theta \in S^1$, $ev(\gamma, \theta)$ is defined as $\gamma(\theta)$. Then $\int_{S^1} \text{ev}^* \eta_1 \in \Omega^1(U^2)$ to a cochain in $\Omega^0(LU^2)$. This cochain defines a cohomology class in local cohomology group $H^2_{loc}(LSO(4), \mathbb{R})$. So as an application of Theorem 3.2, we can obtain a cocycle in $\phi(\int_{S^1} \text{ev}^* \eta_1) \in C^2(\mathfrak{so}(4), \mathbb{R})$.

Now we compute this cocycle. We define:

$$a := \int_{S^1} \text{ev}^* \int_{\Delta^2} f^*_{1,2} E_{1,3}, \quad b := \int_{S^1} \text{ev}^* \int_{\Delta^1} f^*_{2,1} E_{2,2}, \quad c := \int_{S^1} \text{ev}^* \eta_1$$
then $c(\gamma_1, \gamma_2) = a(\gamma_1, \gamma_2) + b(\gamma_1, \gamma_2)$ for $\gamma_1, \gamma_2 \in LU$. Recall that
\[
f_{1,2}(t_0, t_1, t_2; \gamma_1(\theta), \gamma_2(\theta)) = \sigma_2(t_0, t_1, t_2; \gamma_1(\theta), \gamma_2(\theta))
\]
\[
f_{2,1}(t_0, t_1, t_2; \gamma_1(\theta), \gamma_2(\theta)) = (\gamma_1(\theta), \sigma_1(t_0, t_1; \gamma_2(\theta))).
\]
In this case we can take:
\[
\gamma_i(\theta) = \exp(y_i \xi_i(\theta))
\]
\[
\sigma_1(t_0, t_1; \exp(y_2 \xi_2(\theta))) := \exp(t_1 y_2 \xi_2(\theta))
\]
\[
\sigma_2(t_0, t_1, t_2; \exp(y_1 \xi_1(\theta)), \exp(y_2 \xi_2(\theta))) := \exp((1 - t_0)y_1 \xi_1(\theta)) \exp(t_2 y_2 \xi_2(\theta))
\]
where $\xi_i \in L\mathfrak{so}(4)$. By observing the coefficient of $y_1 y_2$, we see $\phi(a(\gamma_1, \gamma_2)) = 0$.

We define a map $\beta_{\gamma_1, \gamma_2} : S^1 \times \Delta^1 \to SO(4) \times SO(4)$ as follows:
\[
\beta_{\gamma_1, \gamma_2}(\theta; t_0, t_1) := (\gamma_1(\theta), \sigma_1(t_0, t_1; \gamma_2(\theta))).
\]
Then $b(\gamma_1, \gamma_2) = \int_{S^1 \times \Delta^1} \beta_{\gamma_1, \gamma_2}^* E_{2,2}$ and up to $O(|y_1|^2)$ and $O(|y_2|^2)$,
\[
\frac{\partial \beta_{\gamma_1, \gamma_2}}{\partial \theta} = \left( y_1 \frac{\partial \xi_1(\theta)}{\partial \theta}, t_1 y_2 \frac{\partial \xi_2(\theta)}{\partial \theta} \right), \quad \frac{\partial \beta_{\gamma_1, \gamma_2}}{\partial t_1} = (0, y_2 \xi_2(\theta)).
\]
Therefore
\[
\left[ \frac{\partial^2}{\partial y_1 \partial y_2} b(\gamma_1, \gamma_2) \right]_{y_i = 0} = -\frac{1}{128 \pi^2} \sum_{\tau \in \mathcal{S}_4} \text{sgn}(\tau) \int_0^1 \left( \frac{\partial \xi_1(\theta)}{\partial \theta} \right)_{\tau(1)\tau(2)} \xi_2(\theta)_{\tau(3)\tau(4)} d\theta.
\]
Now we obtain the following theorem.

**Theorem 5.1.** There exists a Lie algebra 2-cocycle $\alpha$ on $L\mathfrak{so}(4)$ which is expressed as follows:
\[
\alpha(\xi_1, \xi_2) := -\frac{1}{128 \pi^2} \sum_{\tau \in \mathcal{S}_4} \left( \text{sgn}(\tau) \cdot
\int_0^1 \left( \left( \frac{\partial \xi_1(\theta)}{\partial \theta} \right)_{\tau(1)\tau(2)} \xi_2(\theta)_{\tau(3)\tau(4)} - \left( \frac{\partial \xi_2(\theta)}{\partial \theta} \right)_{\tau(1)\tau(2)} \xi_1(\theta)_{\tau(3)\tau(4)} \right) d\theta \right).
\]
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