LIMITING BEHAVIOR OF FRACTIONAL STOCHASTIC
INTEGRO-DIFFERENTIAL EQUATIONS ON
UNBOUNDED DOMAINS

JI SHU
School of Mathematical Sciences and V.C. & V.R. Key Lab
Sichuan Normal University
Chengdu, Sichuan 610068, China

LINYAN LI
School of Mathematical Sciences
Sichuan Normal University
Chengdu, Sichuan 610068, China

XIN HUANG
Department of Basic Courses
Sichuan Vocational College of Finance and Economics
Chengdu, Sichuan 610101, China

JIAN ZHANG∗
School of Mathematical Sciences
University of Electronic Science and Technology of China
Chengdu, Sichuan 611731, China

(Communicated by Qi Lü)

ABSTRACT. We consider the dynamical behavior of fractional stochastic integro-
differential equations with additive noise on unbounded domains. The existence
and uniqueness of tempered random attractors for the equation in $\mathbb{R}^3$
are proved. The upper semicontinuity of random attractors is also obtained
when the intensity of noise approaches zero. The main difficulty is to show the
pullback asymptotic compactness due to the lack of compactness on unbounded
domains and the fact that the memory term includes the whole past history of
the phenomenon. We establish such compactness by the tail-estimate method
and the splitting method.

1. Introduction. In this paper, we investigate the asymptotic behavior of solutions
to the following stochastic integro-differential equation driven by additive noise
defined in the entire space $\mathbb{R}^3$:

$$
\frac{\partial u}{\partial t} + (-\Delta)^{\alpha} u + \lambda u + \int_0^\infty \mu(s)(-\Delta)^{\alpha} u(t-s)ds + f(u) = k(t,x) + \epsilon h(x) \frac{dW}{dt},
$$

$$
x \in \mathbb{R}^3, \ t > \tau,
$$

2020 Mathematics Subject Classification. Primary: 37L55, 60H15; Secondary: 35Q56.
Key words and phrases. Random attractor, stochastic integro-differential equation with mem-
ory, unbounded domains, asymptotic compactness, upper semicontinuity.

* Corresponding author: Jian Zhang.
where \( \tau \in \mathbb{R} \), \( \alpha \in (0, 1) \), \( \lambda > 0 \) and \( \epsilon > 0 \) are constants, \( u(x, t) \) is the unknown function, \( \mu \) is a decreasing and non-negative memory kernel, \( f \) is a nonlinear function satisfying certain conditions, \( k \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^3)) \) and \( h \in H^{2\alpha}(\mathbb{R}^3) \). \( W \) is a two-sided real-valued wiener process on a complete probability space. In the present case, the dynamics of \( u \) relies on the past history of the diffusion term, that is, \( \int_0^\infty \mu(s)(-\Delta)^\alpha u(t-s)ds \).

Fractional partial differential equations arise in a wide range of fields such as physics, biology, chemistry, etc., and some classical equations of mathematical physics have been postulated with fractional derivative to better describe complex phenomena, including the fractional Schrödinger equation [15, 21, 22], fractional Landau-Lifshitz equation [23], fractional Landau-Lifshitz-Maxwell equation [39], fractional Ginzburg-Landau equation [40] and fractional reaction-diffusion equation [17, 46].

As we know, the concept of pullback random attractor, which is a generalization of global attractor in deterministic systems (see [45]), was introduced in [1, 11, 12, 16, 41], and characterizes the long-time behavior of random dynamical systems perfectly. The random attractors for stochastic partial equations and stochastic lattice dynamical systems have been widely discussed by many authors, see, e.g., [3, 4, 43, 51] in the autonomous stochastic equations, and [25, 28, 46, 49, 48, 50, 53] in the non-autonomous case. In recent years, there are some results on random attractors for stochastic equations with the fractional Laplacian \((-\Delta)\alpha\) with \( \alpha \in (\frac{1}{2}, 1) \) and \( \alpha \in (0, 1) \) in [26, 27, 29, 31, 32, 33, 42, 44].

It is worth mentioning that many physical phenomena are better described if one considers in the equations of the model some terms which take into account the past history of the system. This is because that materials with memory have the property that the mathematical-physical description of their state at a given point of time includes such states in while the materials have been at earlier points of time. When \( \mu = 0 \), this is the case of no memory term and (1) reduces to a fractional stochastic reaction-diffusion equation with noise [2, 24, 32]. When \( \alpha = 1 \) and \( h(x) = 0 \), (1) is well-known and has been extensively discussed in [5, 6, 9, 18, 19]. For the fractional stochastic heat equations with memory term, few publications on existence of random attractors are given by [29, 31]. To the authors’ knowledge, there is no works dealing with the existence of random attractors for the fractional stochastic equations with white noise and memory on unbounded domains. Based on the motivation, in this paper we discuss the existence and upper semicontinuity of random attractors for (1).

The outline of this paper is as follows. In the next section, we recall some fundamental results on the existence of pullback attractors for non-autonomous random dynamical systems. In section 3, we get the existence of a continuous cocycle for the stochastic equation (1). In section 4, we derive a priori estimate on the solutions including the estimates on far-field values of solutions. In Section 5, we prove the existence and uniqueness of tempered attractors for (1). In the last section, we establish the upper semicontinuity of the attractors when the intensity of noise approaches zero.

2. Preliminaries. In this section, we first present some basic notions about pullback random attractors for non-autonomous random dynamical systems from [48, 49]. For the theory of pullback random attractors for autonomous random dynamical systems, the reader is referred to [11, 12, 16, 41].
Let \((X, \| \cdot \|_X)\) be a separable Banach space with the Borel \(\sigma\)-algebra \(\mathcal{B}(X)\), \((\Omega, \mathcal{F}, P)\) be a probability space and \((\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})\) be a metric dynamical system.

**Definition 2.1.** A mapping \(\Phi: \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X\) is called a continuouscocycle on \(X\) over \((\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})\) if for all \(\tau, \omega \in \Omega\) and \(t, s \in \mathbb{R}^+\), the following conditions are satisfied:

1. \(\Phi(\cdot, \cdot, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X\) is \((\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))\)-measurable mapping;
2. \(\Phi(0, \tau, \omega, \cdot)\) is the identity on \(X\);
3. \(\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_t \omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)\);
4. \(\Phi(t, \cdot, \cdot, \cdot) : X \to X\) is continuous.

**Definition 2.2.** Let \(D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}\) be a family of bounded nonempty subsets of \(X\). Such a family \(D\) is called tempered if for every \(b > 0\), \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\),

\[
\lim_{t \to -\infty} e^{bt} \|D(\tau + t, \theta_t \omega)\| = 0,
\]

where the norm \(\|D\|\) of a set \(D\) in \(X\) is given by \(\|D\| = \sup_{x \in D} \|x\|_X\).

**Definition 2.3.** Let \(\mathcal{D}\) be a collection of some families of nonempty subsets of \(X\) and \(K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\). Then \(K\) is called a \(\mathcal{D}\)-pullback absorbing set for \(\Phi\) if for all \(\omega \in \Omega\) and for every \(B \in \mathcal{D}\), there exists \(T = T(B, \omega) > 0\) such that

\[
\Phi(t, \tau - t, \theta_{-t} \omega, B(\tau - t, \theta_{-t} \omega)) \subseteq K(\tau, \omega), \quad \text{for all } t > T.
\]

If, in addition, for all \(\omega \in \Omega\), \(K(\tau, \omega)\) is a closed nonempty subset of \(X\) and \(K\) is measurable with respect to the \(P\)-completion of \(\mathcal{F}\) in \(\Omega\), then we say \(K\) is a closed measurable \(\mathcal{D}\)-pullback absorbing set for \(\Phi\).

**Definition 2.4.** Let \(\mathcal{D}\) be a collection of some families of nonempty subsets of \(X\) and \(\mathcal{A} = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\). Then \(\mathcal{A}\) is called a \(\mathcal{D}\)-pullback attractor of \(\Phi\) in \(X\) if the following conditions are satisfied:

1. For every \(\tau \in \mathbb{R}\), \(A(\cdot, \tau) : (\Omega, \mathcal{F}, P) \to 2^X\) is measurable, and \(A(\tau, \omega)\) is compact in \(X\);
2. \(\mathcal{A}\) is invariant, that is, for every \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\),

\[
\Phi(t, \tau, \omega, A(\tau, \omega)) = A(\tau + t, \theta_t \omega), t \geq 0;
\]
3. \(\mathcal{A}\) attracts every member of \(\mathcal{D}\) in \(X\), that is, given \(B \in \mathcal{D}\), \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\),

\[
\lim_{t \to \infty} \text{dist}_X(\Phi(t, \tau - t, \theta_{-t} \omega, B(\tau - t, \theta_{-t} \omega)), A(\tau, \omega)) = 0,
\]

where \(\text{dist}_X(\cdot, \cdot)\) denotes the Hausdorff semi-distance under the norm of \(X\), i.e., for two nonempty sets \(A, B \subset X\),

\[
\text{dist}_X(A, B) := \sup_{a \in A} \text{dist}_X(a, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_X.
\]

**Definition 2.5.** Let \(\mathcal{D}\) be a collection of some families of nonempty subsets of \(X\). Then \(\Phi\) is said to be \(\mathcal{D}\)-pullback asymptotically compact in \(X\) if for all \(\omega \in \Omega\), the sequence

\[
\{\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, x_n)\}_{n=1}^{\infty}
\]

has a convergent subsequence in \(X\), whenever \(t_n \to \infty\), and \(x_n \in B(\tau - t_n, \theta_{-t_n} \omega)\) with \(\{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\).
Definition 2.6. Let $Z$ and $\Lambda$ be metric spaces. A family of set $\{A_\epsilon\}_{\epsilon \in \Lambda}$ in $Z$ is said to be upper semi-continuous (u.s.c.) at $\epsilon_0 \in \Lambda$ if
\[
\lim_{\epsilon \to \epsilon_0} \text{dist}_Z(A_\epsilon, A_{\epsilon_0}) = 0.
\]

Proposition 1 ([48]). Let $D$ be an inclusion-closed collection of some families of nonempty subsets of $X$, and $\Phi$ be a continuous cocycle on $X$ over $(\Omega, F, P, (\theta_t)_{t \in \mathbb{R}})$. If $\Phi$ is $D$–pullback asymptotically compact in $X$ and has a closed measurable $D$–pullback absorbing set $K$ in $D$, then $\Phi$ has a unique $D$–pullback attractor $\Lambda \in D$ in $X$,
\[
\Lambda(\tau, \omega) = \bigcap_{r \geq 0} \bigcup_{t \geq r} \Phi(t, \tau - t, \theta_{-t}\omega, K(\tau - t, \theta_{-t}\omega)), \ \omega \in \Omega.
\]

Proposition 2 ([48]). Given $\epsilon \in \Lambda$, let $\Phi_\epsilon$ be a continuous cocycle on $X$ over $(\Omega, F, P, (\theta_t)_{t \in \mathbb{R}})$. Suppose that
(1) there exists $\epsilon_0 \in \Lambda$ such that for every $t \in \mathbb{R}^+$, $\omega \in \Omega$, $\epsilon_n \to \epsilon_0$, and $x_n, x \in X$ with $x_n \to x$,
\[
\lim_{n \to \infty} \Phi_{\epsilon_n}(t, \omega, x_n) = \Phi_{\epsilon_0}(t, \omega, x);
\]
(2) there exists a map $R_{\epsilon_0} : \Omega \to \mathbb{R}$ such that $B = \{ B(\omega) = \{ x \in X : \| x \|_X \leq R(\omega) \} : \omega \in \Omega \}$ belongs to $D_{\epsilon_0}$;
(3) for each $\epsilon \in \Lambda$, $\Phi_\epsilon$ has a $D_\epsilon$–pullback attractor $\Lambda_\epsilon$ and a $D_\epsilon$–pullback absorbing set $K_\epsilon \in D_\epsilon$ such that for all $\omega \in \Omega$,
\[
\limsup_{\epsilon \to \epsilon_0} \| K_\epsilon(\omega) \|_X \leq R_{\epsilon_0}(\omega);
\]
(4) for every $\omega \in \Omega$,
\[
\bigcup_{\epsilon \in \Lambda} \Lambda_\epsilon(\omega) \text{ is precompact in } X.
\]

Then for each $\omega \in \Omega$,
\[
\text{dist}_X(\Lambda_\epsilon(\omega), \Lambda_{\epsilon_0}(\omega)) \to 0, \text{ as } \epsilon \to \epsilon_0.
\]

Next, we review some concepts and notations of the fractional derivative and fractional Sobolev space (see [14] for details). Let $\mathcal{S}$ be the Schwartz space of rapidly decaying $C^\infty$ functions on $\mathbb{R}^3$, then for $0 < \alpha < 1$, the integral fractional Laplace operator $(-\Delta)^\alpha$ is given by, for $u \in \mathcal{S}$,
\[
(-\Delta)^\alpha u(x) = -\frac{1}{2} C(\alpha) \int_{\mathbb{R}^3} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{3+2\alpha}} dy, \ x \in \mathbb{R}^3,
\]
where $C(\alpha)$ is a positive constant depending on $\alpha$ as given by
\[
C(\alpha) = \left( \int_{\mathbb{R}^3} \frac{1 - \cos(\xi_1)}{|\xi|^{3+2\alpha}} d\xi \right)^{-1}, \ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.
\]
In particular, it follows from [38] that for any $u \in \mathcal{S}$,
\[
(-\Delta)^\alpha u = \mathcal{F}^{-1}(|\xi|^{2\alpha}(\mathcal{F}u)), \ \xi \in \mathbb{R}^3,
\]
where $\mathcal{F}$ is the Fourier transform defined by
\[
(\mathcal{F}u)(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} u(x) dx, \ u \in \mathcal{S},
\]
and $\mathcal{F}^{-1}$ is the inverse Fourier transform.
For every $\alpha \in (0,1)$, denote by

$$H^\alpha(\mathbb{R}^3) = \{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2\alpha}} dxdy < \infty \}.$$ 

Then $H^\alpha(\mathbb{R}^3)$ is a fractional Sobolev space with inner product given by

$$(u,v)_{H^\alpha(\mathbb{R}^3)} = (u,v)_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3+2\alpha}} dxdy, \ u, v \in H^\alpha(\mathbb{R}^3).$$

For convenience, we will use the notation:

$$(u,v)_{H^\alpha(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{3+2\alpha}} dxdy, \ u, v \in H^\alpha(\mathbb{R}^3),$$

and

$$\|u\|_{H^\alpha(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2\alpha}} dxdy, \ u \in H^\alpha(\mathbb{R}^3).$$

Then for all $u \in H^\alpha(\mathbb{R}^3)$ we have

$$\|u\|^2_{H^\alpha(\mathbb{R}^3)} = \|u\|^2_{L^2(\mathbb{R}^3)} + \|u\|^2_{H^\alpha(\mathbb{R}^3)}.$$ 

In terms of (2.3), one can verify (see, e.g., [14])

$$\|u\|^2_{H^\alpha(\mathbb{R}^3)} = \|u\|^2_{L^2(\mathbb{R}^3)} + \frac{2}{C(\alpha)} \|(-\Delta)^{\frac{\alpha}{2}} u\|^2_{L^2(\mathbb{R}^3)} \quad \text{for all } u \in H^\alpha(\mathbb{R}^3), \tag{5}$$

and hence $\|u\|^2_{L^2(\mathbb{R}^3)} + \|(-\Delta)^{\frac{\alpha}{2}} u\|^2_{L^2(\mathbb{R}^3)}$ is an equivalent norm of $H^\alpha(\mathbb{R}^3)$. Similarly, from [14], we can define $H^{2\alpha}(\mathbb{R}^3)$ with $\alpha \in (0,1)$.

For treating with the memory term, we assume that $\mu(\infty) = 0$ and let

$$g(s) = -\mu'(s), \tag{6}$$

and $g$ is required to satisfy the following conditions:

$(b_1)$ $g(\cdot) \in C^1(\mathbb{R}^+ \cap L^1(\mathbb{R}^+), g(s) \geq 0, g'(s) \leq 0, \forall s \in \mathbb{R}^+$;

$(b_2)$ $g'(s) + \delta g(s) \leq 0, \forall s \in \mathbb{R}^+$ and some $\delta > 0$.

Notice that $(b_2)$ implies the exponential decay of $g(s)$. Nevertheless, it allows $g(s)$ to have a singularity at $s = 0$, whose order is less than 1, since $g(s)$ is a non-negative $L^1-$function. It also follows from $(b_2)$ that there exists $s^* > 0$ such that $g(s) > 0$ for $s \in [0,s^*)$ and $g(s) = 0$ for all $s \geq s^*$.

Let $L_2^g(\mathbb{R}^+, L^2(\mathbb{R}^3))$ be the Hilbert space of $L^2$-valued functions on $\mathbb{R}^+$, endowed with the inner product

$$(\eta_1, \eta_2)_{L_2^g(\mathbb{R}^+, L^2(\mathbb{R}^3))} = \int_0^\infty g(s) \int_{\mathbb{R}^3} \eta_1(s,x) \cdot \eta_2(s,x) dxds.$$

Similarly on $M = L_2^g(\mathbb{R}^+, \mathcal{H}^\alpha(\mathbb{R}^3))$ and $M_1 = L_2^g(\mathbb{R}^+, \mathcal{H}^{2\alpha}(\mathbb{R}^3))$, respectively, we have the inner products

$$(\eta_1, \eta_2)_M = \int_0^\infty g(s) \int_{\mathbb{R}^3} (-\Delta)^{\frac{\alpha}{2}} \eta_1(s,x) \cdot (-\Delta)^{\frac{\alpha}{2}} \eta_2(s,x) dxds,$$

and

$$(\eta_1, \eta_2)_{M_1} = \int_0^\infty g(s) \int_{\mathbb{R}^3} (-\Delta)^{\alpha} \eta_1(s,x) \cdot (-\Delta)^{\alpha} \eta_2(s,x) dxds,$$

where operator $(-\Delta)^{\frac{\alpha}{2}}$ and $(-\Delta)^{\alpha}$ are considered with respect to the spatial variable $x \in \mathbb{R}^3$.

In addition, we introduce the Hilbert spaces

$$\mathcal{H} = L^2(\mathbb{R}^3) \times L_2^g(\mathbb{R}^+, \mathcal{H}^\alpha(\mathbb{R}^3)).$$
and
\[ \mathcal{V} = \dot{H}^\alpha(\mathbb{R}^3) \times L_g^2(\mathbb{R}^+, \dot{H}^{2\alpha}(\mathbb{R}^3)), \]

which are endowed with the inner products, respectively,
\[ (\omega_1, \omega_2)_\mathcal{H} = (\psi_1, \psi_2)_{L^2(\mathbb{R}^3)} + (\varphi_1, \varphi_2)_M \]

and
\[ (\omega_1, \omega_2)_\mathcal{V} = (\psi_1, \psi_2)_{\dot{H}^{\alpha}(\mathbb{R}^3)} + (\varphi_1, \varphi_2)_M, \]

where \( \omega_i = (\psi_i, \varphi_i) \in \mathcal{H} \) for \( i = 1, 2 \). Then the norms induced on \( \mathcal{H} \) and \( \mathcal{V} \) are written as, respectively,
\[ \| (\psi, \varphi) \|^2_\mathcal{H} = \| \psi \|^2_{L^2(\mathbb{R}^3)} + \| \varphi \|^2_M, \]
\[ \| (\psi, \varphi) \|^2_\mathcal{V} = \| \psi \|^2_{\dot{H}^{\alpha}(\mathbb{R}^3)} + \| \varphi \|^2_M. \]

From now on, we denote by \( \| \cdot \| \) and \((\cdot, \cdot)\) the norm and the inner product in \( L^2(\mathbb{R}^3) \), respectively, and use the \( \| \cdot \|_{\varphi} \) to denote the norm in \( L^p(\mathbb{R}^3) \). Otherwise, the norm of a general Banach space \( X \) is written as \( \| \cdot \|_X \). The letters \( c \) and \( c_i \) denote positive constants which may be different from the context.

At last, we give two lemmas which are important in this paper (for details see [33, 38]).

**Lemma 2.7** ([33]). If \( f, g \in H^{2\alpha}(\mathbb{R}^3) \), then the following equation holds.
\[ \int_{\mathbb{R}^3} (-\Delta)^{\alpha} f \cdot g \, dx = \int_{\mathbb{R}^3} (-\Delta)^{\alpha_1} f \cdot (-\Delta)^{\alpha_2} g \, dx, \]

where \( \alpha_1 \) and \( \alpha_2 \) are nonnegative constants and satisfy \( \alpha_1 + \alpha_2 = \alpha \).

**Lemma 2.8** ([38]). Let \( g \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+) \) be a non-negative function and \( g(s_0) = 0 \) for some \( s_0 \in \mathbb{R}^+ \), then \( g(s) = 0 \) for every \( s > s_0 \). Let \( B_0, B, B_1 \) be three Banach spaces, with \( B_0 \) and \( B_1 \) reflexive, such that \( B_0 \rightarrow B \rightarrow B_1 \) the first injection being compact. Let \( \mathcal{M} \subset L_g^2(\mathbb{R}^+, B) \) satisfies
1. \( \mathcal{M} \) is bounded in \( L_g^2(\mathbb{R}^+, B_0) \cap H_g^1(\mathbb{R}^+, B_1) \);
2. \( \sup_{\eta \in \mathcal{M}} \| \eta(t) \|^2_B \leq h(s), \forall s \in \mathbb{R}^+ \) for some \( h \in L_g^1(\mathbb{R}^+) \).

Then \( \mathcal{M} \) is relatively compact in \( L_g^2(\mathbb{R}^+, B) \).

3. **Stochastic heat equations on \( \mathbb{R}^3 \).** In this section, we show that there is a continuous cocycle generated by the stochastic heat equation defined on \( \mathbb{R}^3 \):
\[ \frac{\partial u}{\partial t} + (-\Delta)^{\alpha} u + \lambda u + \int_0^\infty \mu(s)(-\Delta)^{\alpha} u(t-s) \, ds + f(u) = k(t, x) + \epsilon h(x) \frac{dW}{dt}, \]

where \( x \in \mathbb{R}^3, t > \tau, \)

and with initial condition
\[ u(t, x) = u_\tau(t, x), \quad x \in \mathbb{R}^3, t \leq \tau. \]

Throughout this paper, we assume the nonlinearity \( f \) is continuous and satisfies, for all \( u \in \mathbb{R} \),
\[ f(u)u \geq \beta_1 |u|^p - \beta_2, \]
\[ \frac{\partial f(u)}{\partial u} \geq -\beta_3, \]
\[ |f(u)| \leq \beta_4 (1 + |u|^{p-1}), \]
where \( \beta_i (i = 1, 2, 3, 4) \), \( p \geq 2 \) are positive numbers.

Due to the presence of the memory term, from [31, 13], we introduce the new variables

\[
\begin{align*}
u^i(x, s) &= u(x,t) - s, \quad s \geq 0, \\
\eta^i(x, s) &= \int_0^s \nu^i(x, r) \, dr = \int_{t-s}^t u(x,r) \, dr, \quad s \geq 0.
\end{align*}
\]

By (6) and (12)-(13), the original system (7)-(8) can be transformed into the following equivalent system:

\[
\begin{align*}
\frac{\partial u}{\partial t} + (-\Delta)^{\alpha} u + \lambda u + \int_0^\infty g(s)(-\Delta)^{\alpha} \eta^i(s) \, ds + f(u) &= k(t, x) + \epsilon h(x) \frac{dW}{dt}, \\
x &\in \mathbb{R}^3, \quad t > \tau,
\end{align*}
\]

(14)

\[
\begin{align*}
\frac{\partial \eta^i}{\partial t} &= u - \frac{\partial \eta^i}{\partial s}, \quad x \in \mathbb{R}^3, \quad t > \tau, \quad s > 0
\end{align*}
\]

with initial condition

\[
\begin{align*}
u(x, t) &= u_{\tau}(x,t), \quad x \in \mathbb{R}^3, \quad t \leq \tau, \\
\eta^i(x, s) &= \eta_{\tau}(x,s), \quad x \in \mathbb{R}^3, \quad s > 0.
\end{align*}
\]

(15)

(16)

(17)

We denote \( \phi(t) = (u(t), \eta^i(t)) \) be the solution of (14)-(17).

The standard probability space \((\Omega, \mathcal{F}, P)\) will be used in this paper where \( \Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \} \), and \( \mathcal{F} \) is the Borel \( \sigma \)-algebra induced by the compact-open topology of \( \Omega \), and \( P \) is the Wiener measure on \((\Omega, \mathcal{F})\). Here we will identify \( W(t) \) with \( \omega(t) \), i.e., \( \omega(t) = W(t, \omega), \quad t \in \mathbb{R} \). For \( t \in \mathbb{R} \), let \( \theta_t : \Omega \to \Omega \) be the time shift given by: \( \theta_t \omega(t) = \omega(t + t) - \omega(t) \), for \( \omega \in \Omega \). Let \( \bar{\varepsilon} : \Omega \to \mathbb{R} \) be a random variable defined by:

\[
\bar{\varepsilon}(\omega) = -\lambda \int_0^\infty e^{\lambda s}(s) \, ds \quad \text{for} \quad \omega \in \Omega.
\]

Then \( \eta(t, \omega) = \bar{\varepsilon}(\theta_t \omega) \) is the unique stationary solution of the stochastic equation:

\[
dy + \lambda ydt = dW.
\]

Therefore, if we denote \( z(\omega)(x) = \bar{\varepsilon}(\omega)h(x) \), then the real-valued stochastic process \( z(\theta_t \omega)(x) = \bar{\varepsilon}(\theta_t \omega)h(x) \) is a solution to

\[
dz + \lambda zdW = h(x)dW.
\]

Note that the random variable \(|z(\omega)|\) is tempered and \( z(\theta_t \omega) \) is \( P \)-a.e. continuous. Moreover, from [1], there exists a tempered random variable \( r_0(\omega) > 0 \) such that

\[
|\bar{\varepsilon}(\omega)|^2 + |\bar{\varepsilon}(\omega)|^p + |(-\Delta)^{\alpha} \bar{\varepsilon}(\omega)|^2 + |(-\Delta)^{\alpha} \bar{\varepsilon}(\omega)|^2 + |(-\Delta)^{\alpha} \bar{\varepsilon}(\omega)|^p \leq r_0(\omega),
\]

then, it is easy to check that

\[
|z(\omega)|^2 + |z(\omega)|^p + |(-\Delta)^{\alpha} z(\omega)|^2 + |(-\Delta)^{\alpha} z(\omega)|^2 + |(-\Delta)^{\alpha} z(\omega)|^p \leq r(\omega),
\]

(18)

where \( r(\omega) \) is tempered, \( r(\theta_t \omega) \leq e^{\frac{\bar{\varepsilon}(\omega)}{\lambda}}r(\omega) \) and \( \sigma \) will be specified later.

From (18) we get, for \( P \)-a.e. \( \omega \in \Omega \),

\[
|z(\theta_t \omega)|^2 + |z(\theta_t \omega)|^p + |(-\Delta)^{\alpha} z(\theta_t \omega)|^2 + |(-\Delta)^{\alpha} z(\theta_t \omega)|^2 + |(-\Delta)^{\alpha} z(\theta_t \omega)|^p
\]

\[
\leq e^{\frac{\bar{\varepsilon}(\omega)}{\lambda}}r(\omega), \quad t \in \mathbb{R}.
\]

(19)

We now transform the stochastic equation (14) into a deterministic one. Put

\[
v(t, \tau, \omega, v_{\tau}) = u(t, \tau, \omega, u_{\tau}) - \epsilon z(\theta_t \omega),
\]

(20)
with \( v_\tau = u_\tau - \varepsilon z(\theta_\tau \omega) \). From (14)-(17) and (20), we obtain
\[
\frac{\partial v}{\partial t} + (-\Delta)^{\alpha} v + \lambda v + \int_{0}^{\infty} g(s) (-\Delta)^{\alpha} \eta'(s) ds + f(v + \varepsilon z) = k(t, x) - \varepsilon (-\Delta)^{\alpha} z,
\]
\( x \in \mathbb{R}^3, t > \tau, \) \( \quad \) (21)
\[
\frac{\partial \eta'}{\partial t} = v + \varepsilon z - \frac{\partial \eta'}{\partial s}, \quad x \in \mathbb{R}^3, t > \tau, s > 0 \) \( \quad \) (22)
with initial condition
\[
v(x, t) = v_\tau(x, t), \quad x \in \mathbb{R}^3, t \leq \tau, \) \( \quad \) (23)
\[
\eta'(x, s) = \eta_\tau(x, s), \quad x \in \mathbb{R}^3, s \geq 0. \) \( \quad \) (24)

Here we denote \( \varphi(t) = (v(t), \eta'(t)) \) be the solution of (21)-(21) with initial data \( \varphi_\tau = (v_\tau, \eta_\tau) \).

By a Galerkin method, one can show that if \( g \) satisfies (h1) - (h2) and \( f \) satisfies (9) - (11), then in the case of a bounded domain with Dirichlet boundary conditions, for \( \mathcal{P} - a.e. \omega \in \Omega \) and for all \( \varphi_\tau \), problem (21)-(21) is well posed. This was done in [29]. Then, following [36], one may take the domain to be a sequence of balls with radius approaching \( \infty \) to deduce the existence of a weak solution to (21)-(21) on \( \mathbb{R}^3 \). Furthermore, one may show that \( \varphi(t, \tau, \omega, \varphi_\tau) \) is measurable in \( \omega \in \Omega \) and continuous with respect to \( \varphi_\tau \) in \( \mathcal{H} \) for all \( t > \tau \).

We now define a continuous cocycle \( \Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathcal{H} \rightarrow \mathcal{H} \) for problem (7)-(8) by using (20). Given \( t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega \) and \( \phi_\tau \in \mathcal{H} \),
\[
\Phi(t, \tau, \omega, \phi_\tau) = \phi(t + \tau, \tau, \theta_{-\tau} \omega, \phi_\tau) = \varphi(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_\tau) + (\varepsilon z(\theta_\tau \omega), 0), \] \( \quad \) (25)
where \( \phi_\tau = \varphi_\tau - (\varepsilon z(\omega), 0) \).

From now on, we use \( \mathcal{D} \) to denote the collection of all tempered families of bounded nonempty subsets of \( \mathcal{H} \). For the external forcing \( k(x, t) \), we will assume that for every \( \tau \in \mathbb{R} \),
\[
\int_{-\infty}^{0} e^{\sigma s} \| k(s + \tau, \cdot) \|^2 ds < \infty, \] \( \quad \) (26)
and for every \( b > 0 \),
\[
\lim_{r \rightarrow \infty} e^{-br} \int_{-\infty}^{0} e^{\sigma s} \| k(s - r, \cdot) \|^2 ds = 0, \] \( \quad \) (27)
where \( \sigma = \min\{\lambda, \delta\} \).

4. Uniform estimates of solutions. In this section, we derive uniform estimates of solutions for problem (7)-(8) defined on \( \mathbb{R}^3 \). These estimates are needed to prove the existence of \( \mathcal{D} \)-pullback attractors. In this following, we first give the uniform estimates of solutions for problem (7)-(8) in \( \mathcal{H} \) and \( \mathcal{V} \) respectively.

Lemma 4.1. Suppose (h2), (9)-(11) and (26) hold. Then for \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \), there exists \( T_1(\tau, \omega, D) > 0 \) such that for all
t \geq T_1$, the solution $\phi$ of problem (7)-(8) with $\phi_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$ satisfies
\begin{align*}
\left\| \phi(\tau, \tau-t, \theta_{-t}\omega, \phi_{\tau-t}) \right\|^2_{\mathcal{H}} &+ \int_{-t}^0 e^{\sigma s} \left\| \phi(s, \tau-t, \theta_{-t}\omega, \eta_{\tau-t}) \right\|^2_{\mathcal{H}} ds
\end{align*}
\begin{align*}
&+ \int_{-t}^0 e^{\sigma s} \left\| (-\Delta)^{\frac{\alpha}{2}} v(s+\tau) \right\|^2_{\mathcal{H}} ds
\end{align*}
\begin{align*}
&+ \int_{-t}^0 e^{\sigma s} \left\| (-\Delta)^{\alpha} v(s+\tau) \right\|^2_{\mathcal{H}} ds
\end{align*}
\begin{align*}
&+ \int_{-t}^0 e^{\sigma s} \left\| u(s+\tau) \right\|^2_{\mathcal{H}} ds \leq R_{\epsilon}(\tau, \omega),
\end{align*}
where $R_{\epsilon}(\tau, \omega) = c_1 + c_1 \int_{-\infty}^0 e^{\sigma s}(1+\|k(s+\tau)\|^2) ds + c_1 (e^2 + \epsilon^p) r(\omega)$, $\sigma = \min\{\frac{\delta}{4}, \frac{1}{2}\}$ and $c_1$ is a positive constant independent of $\tau, \omega, \epsilon$ and $D$.

**Proof.** The proof can be found in [29], and we will not repeat the details here again. \qed

**Lemma 4.2.** Suppose $(h_2)$, (9)-(11) and (26) hold. Then for $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}$, there exists $T_2(\tau, \omega, D) \geq T_1(\tau, \omega, D)$ such that for all $t \geq T_2$, the solution $\phi$ of problem (7)-(8) with $\phi_{\tau-t} \in D(\tau-t, \theta_{-t}\omega)$ satisfies
\begin{align*}
\left\| \phi(\tau, \tau-t, \theta_{-t}\omega, \phi_{\tau-t}) \right\|^2_{\mathcal{H}} &+ \int_{-t}^0 e^{\sigma s} \left\| (-\Delta)^{\alpha} v(s+\tau) \right\|^2_{\mathcal{H}} ds
\end{align*}
\begin{align*}
&\leq c_2 + c_2 \int_{-\infty}^0 e^{\sigma s}(1+\|k(s+\tau)\|^2) ds + c_2 (e^2 + \epsilon^p) r(\omega) := \bar{R}_{\epsilon}(\tau, \omega),
\end{align*}
where $\sigma = \min\{\frac{\delta}{4}, \frac{1}{2}\}$ and $c_2$ is a positive number independent of $\tau, \omega, \epsilon$ and $D$.

**Proof.** The idea of proof is similar to that given in [29] and so we omit it. \qed

In order to establish the $D-$pullback asymptotic compactness of problem (7)-(8), we need to derive the uniform estimates on the tails of solutions. To that end, let $\rho$ be a smooth function defined on $\mathbb{R}^+$ such that $0 \leq \rho(s) \leq 1$ for all $s \in \mathbb{R}^+$, and
\begin{align*}
\rho(s) = \begin{cases} 
0, & \text{if } 0 \leq s \leq \frac{1}{2}, \\
1, & \text{if } s \geq 1.
\end{cases}
\end{align*}
(28)

Then there exists a positive constant $c$ such that $|\rho'(s)| \leq c$ for all $s \in \mathbb{R}^+$. Moreover, the cut-off function $\rho$ has the following property:

**Lemma 4.3** ([20]). Let $\rho$ be the smooth function defined by (28), then for every $y \in \mathbb{R}^3$, $k \in \mathbb{N}$ and $\alpha \in (0, 1)$ we have
\begin{align*}
\int_{\mathbb{R}^3} \frac{\left| \rho\left( \frac{|x|}{k} \right) - \rho\left( \frac{|y|}{k} \right) \right|^2}{|x-y|^{3+2\alpha}} dx \leq \frac{L}{k^{2\alpha}},
\end{align*}
(29)
where $L$ is a positive constant independent of $k \in \mathbb{N}$ and $y \in \mathbb{R}^3$.

When deriving the uniform constant estimates on the tails of solutions in $\mathcal{H}$, we need the following estimate:

**Lemma 4.4.** Suppose that $(h_2)$, (9)-(11) and (26) hold. Then for $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}$, there exists $T_3(\tau, \omega, D) > 0$ such that for all $t \geq T_3$,
\begin{align*}
\int_0^\infty g(s) (\| \frac{\partial \eta_{\xi+\tau}}{\partial \xi} \|^2 + \| \frac{\partial \eta_{\xi+\tau}}{\partial \xi} \|^2) ds < \infty,
\end{align*}
where $\xi \in (-t, 0)$. 

Proof. By (12)-(13) we find that

\[
\eta^{\xi+\tau}(t, \theta_{-\tau}, \eta_{\tau-\xi})(s) = \begin{cases} \int_0^s u(\xi + \tau - r, \tau - t, \theta_{\tau-\xi+\xi}, u_{\tau-\xi})dr, & 0 < s \leq t, \\ \int_0^t u(\xi + \tau - r, \tau - t, \theta_{\tau-\xi+\xi}, u_{\tau-\xi})dr, & s > t. \end{cases}
\]

Then

\[
\frac{\partial \eta^{\xi+\tau}(t, \theta_{-\tau}, \eta_{\tau-\xi})(s)}{\partial s} = \begin{cases} u(\xi + \tau - s, \tau - t, \theta_{\tau-\xi+\xi}, u_{\tau-\xi}), & 0 < s \leq t, \\ 0, & s > t. \end{cases}
\]

which together with Lemma 4.1 we have

\[
\int_0^\infty g(s)\|\frac{\partial \eta^{\xi+\tau}}{\partial s}\|^2ds = \int_0^\infty g(s)\|u(\xi + \tau - s)\|^2ds \leq c_3(\tau, \omega)\|g\|_{L^1(\mathbb{R}^+)}, \tag{30}
\]

where \(c_3(\tau, \omega)\) is a positive number depending on \(\tau\) and \(\omega\).

On the other hand, by (22) and (30) we get

\[
\int_0^\infty g(s)\|\frac{\partial \eta^{\xi+\tau}}{\partial \xi}\|^2ds = \int_0^\infty g(s)\|u - \frac{\partial \eta^{\xi+\tau}}{\partial s}\|^2ds \leq c_4(\tau, \omega)\|g\|_{L^1(\mathbb{R}^+)} \tag{31}
\]

From (30)-(31), the proof is completed. \(\square\)

The uniform estimates on the tails of solutions in \(\mathcal{H}\) are given below.

**Lemma 4.5.** Suppose that \((h_2), (9)-(11)\) and (26) hold. Let \(\tau \in \mathbb{R}, \omega \in \Omega\) and \(D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}\), then for every \(\varepsilon > 0\), there exists \(T(\tau, \omega, D, \varepsilon) > 0\), \(K(\tau, \omega, \varepsilon) \geq 1\) such that for all \(t \geq T\) and \(k \geq K\), the solution \(\varphi\) of problem (21)-(24) with \(\varphi_{\tau-\xi} \in D(\tau - t, \theta_{-\tau}, \omega)\) satisfies

\[
\int_{|x| \geq k} |\varphi(\tau, \tau - t, \theta_{-\tau}, \omega, v_{\tau-\xi})|^2dx + \frac{C(\alpha)}{2} \int_0^\infty g(s) \int_{|x| \geq k} \frac{(|\varphi(\tau - t, \theta_{-\tau}, \eta_{\tau-\xi})(x) - \eta^\tau(y)|^2)}{|x - y|^{3+2\alpha}}dxdyds \leq \varepsilon.
\]

**Proof.** Multiplying (21) by \(\rho(\frac{|x|}{k})v\), we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho(\frac{|x|}{k})|v|^2dx + \int_{\mathbb{R}^3} (-\Delta)^{\alpha} \rho(\frac{|x|}{k})vdx + \lambda \int_{\mathbb{R}^3} \rho(\frac{|x|}{k})|v|^2dx \\
+ \int_0^\infty g(s) \int_{\mathbb{R}^3} (-\Delta)^{\alpha} \eta^\tau \rho(\frac{|x|}{k})vdxds \\
= - \int_{\mathbb{R}^3} f(u)\rho(\frac{|x|}{k})vdx + \int_{\mathbb{R}^3} k(t, x)\rho(\frac{|x|}{k})vdx - \epsilon \int_{\mathbb{R}^3} (-\Delta)^{\alpha} z\rho(\frac{|x|}{k})vdx. \tag{32}
\]
For the second term on the left-hand side of (32), by (32) we get
\[
- \int_{\mathbb{R}^3} (-\Delta)\alpha v\rho\left(\frac{|x|}{k}\right) dx
= -\frac{C(\alpha)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) v(x) - \rho\left(\frac{|y|}{k}\right) v(y) |v(x) - v(y)| \frac{dxdy}{|x - y|^{3+2\alpha}}
\]
\[
= -\frac{C(\alpha)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) v(x) - v(y) \frac{dxdy}{|x - y|^{3+2\alpha}}
\]
\[
- \frac{C(\alpha)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) v(x) - v(y) \frac{dxdy}{|x - y|^{3+2\alpha}}
\]
\[
\leq -\frac{C(\alpha)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) v(x) - v(y) \frac{dxdy}{|x - y|^{3+2\alpha}}
\]
\[
+ \frac{C(\alpha)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) v(x) - v(y) \frac{dxdy}{|x - y|^{3+2\alpha}}
\]
\[
\leq -\frac{C(\alpha)}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) v(x) - v(y) \frac{dxdy}{|x - y|^{3+2\alpha}}
\]
\[
+ \int_0^{\infty} g(s) \int_{\mathbb{R}^3} \left(-\Delta\right)^\alpha \eta^t \rho\left(\frac{|x|}{k}\right) \eta^t \frac{dxds}{\mathbb{R}^3},
\]
For the last term on the left-hand side of (32), by (22) we have
\[
- \int_0^{\infty} \frac{\partial}{\partial t} g(s) \int_{\mathbb{R}^3} \left(-\Delta\right)^\alpha \eta^t \rho\left(\frac{|x|}{k}\right) \eta^t \frac{dxds}{\mathbb{R}^3}
\]
\[
= -\frac{C(\alpha)}{2} \int_0^{\infty} g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) \eta^t(x) \frac{\partial}{\partial t} \frac{dxdy}{|x - y|^{3+2\alpha}}
\]
\[
= -\frac{C(\alpha)}{2} \int_0^{\infty} g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) \eta^t(x) \frac{\partial}{\partial t} \frac{dxdy}{|x - y|^{3+2\alpha}}
\]
\[
- \frac{C(\alpha)}{2} \int_0^{\infty} g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) \eta^t(x) \frac{\partial}{\partial t} \frac{dxdy}{|x - y|^{3+2\alpha}}
\]
\[
\leq -\frac{C(\alpha)}{2} \int_0^{\infty} g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) \eta^t(x) \frac{\partial}{\partial t} \frac{dxdy}{|x - y|^{3+2\alpha}}
\]
\[
+ \int_0^{\infty} g(s) ||\eta^t||_{\mathbb{R}^3} ds
\]
\[
\leq -\frac{C(\alpha)}{2} \int_0^{\infty} g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) \eta^t(x) \frac{\partial}{\partial t} \frac{dxdy}{|x - y|^{3+2\alpha}}
\]
\[
+ \int_0^{\infty} g(s) ||\eta^t||_{\mathbb{R}^3} ds
\]
Similar to (35), by (h2) and (29) we have

\[- \int_{0}^{\infty} g(s) \int_{\mathbb{R}^3} (-\Delta)^{\alpha} \eta^t \rho \left( \frac{|x|}{k} \right) \frac{\partial \eta^t}{\partial s} dxds \leq - \frac{C(\alpha)}{4} \int_{0}^{\infty} g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho \left( \frac{|x|}{k} \right) (\eta^t(x) - \eta^t(y))^2}{|x-y|^{3+2\alpha}} dxdyds \]
\[+ \frac{C(\alpha)}{2} ck^{-\alpha} \int_{0}^{\infty} g(s) \| \frac{\partial \eta^t}{\partial s} \|^2 ds + ck^{-\alpha} \| \eta^t \|^2_M. \]  

(36)

By Young’s inequality we have

\[\int_{0}^{\infty} g(s) \int_{\mathbb{R}^3} (-\Delta)^{\alpha} \eta^t \rho \left( \frac{|x|}{k} \right) cxdxds \]
\[\leq c \int_{0}^{\infty} g(s) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) (-\Delta)^{\alpha} \eta^t |x|^2 dxds + ce^2 \int_{0}^{\infty} g(s) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) |x|^2 dxds. \]  

(37)

By (34)-(37) we obtain

\[- \int_{0}^{\infty} g(s) \int_{\mathbb{R}^3} (-\Delta)^{\alpha} \eta^t \rho \left( \frac{|x|}{k} \right) vdxds \leq - \frac{C(\alpha)}{4} \int_{0}^{\infty} \frac{d}{dt} g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho \left( \frac{|x|}{k} \right) (\eta^t(x) - \eta^t(y))^2}{|x-y|^{3+2\alpha}} dxdyds \]
\[- \frac{C(\alpha)}{4} \delta \int_{0}^{\infty} g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho \left( \frac{|x|}{k} \right) (\eta^t(x) - \eta^t(y))^2}{|x-y|^{3+2\alpha}} dxdyds \]
\[+ \frac{C(\alpha)}{2} ck^{-\alpha} \int_{0}^{\infty} g(s) \left( \| \frac{\partial \eta^t}{\partial t} \|^2 + \| \frac{\partial \eta^t}{\partial s} \|^2 \right) ds \]
\[+ c \int_{0}^{\infty} g(s) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) (-\Delta)^{\alpha} \eta^t |x|^2 dxds + ck^{-\alpha} \| \eta^t \|^2_M \]
\[+ ce^2 \int_{0}^{\infty} g(s) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) |x|^2 dxds. \]  

(38)

For the first term on the right-hand side of (32), by (9) and (11) we get

\[- \int_{\mathbb{R}^3} f(u) \rho \left( \frac{|x|}{k} \right) vdx \]
\[= - \int_{\mathbb{R}^3} f(u) \rho \left( \frac{|x|}{k} \right) udx + c \int_{\mathbb{R}^3} f(u) \rho \left( \frac{|x|}{k} \right) zdx \]
\[\leq - \beta_1 \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) |u|^p dx + \beta_2 \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) dx + \epsilon \beta_4 \int_{\mathbb{R}^3} |u|^{p-1} \rho \left( \frac{|x|}{k} \right) |z|dx \]
\[+ \epsilon \beta_4 \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) |z|dx \]
\[\leq - \frac{\beta_1}{2} \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) |u|^p dx + c \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) dx + c(\epsilon^2 + \epsilon^p) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) (|z|^2 + |z|^p) dx. \]  

(39)

By Young’s inequality we get

\[\int_{\mathbb{R}^3} k(t,x) \rho \left( \frac{|x|}{k} \right) vdx \leq \frac{\lambda}{4} \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) |v|^2 dx + c \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) (k(t,x))^2 dx, \]  

(40)
By (32)-(33), (38)-(41) we obtain

\[
\begin{align*}
&- \epsilon \int_{\mathbb{R}^3} (-\Delta)^{\alpha} \rho \left( \frac{|x|}{k} \right) v dx 
\leq \frac{\lambda}{4} \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) |v|^2 dx + c\epsilon^2 \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) \rho \left( \frac{|x|}{k} \right) (-\Delta)^{\alpha} z^2 dx. \quad (41)
\end{align*}
\]

and

\[
\begin{align*}
&\frac{d}{dt} \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) |v|^2 dx + \frac{C(\alpha)}{2} \frac{d}{dt} \int_0^\infty g(s) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) \left( \frac{\partial \eta(t)}{\partial x} \right)^2 dx dy ds \\
&+ \lambda \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) |v|^2 dx + C(\alpha) \int_0^\infty g(s) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) \left( \frac{\partial \eta(t)}{\partial y} \right)^2 dx dy ds \\
&+ C(\alpha) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) (v(x) - v(y))^2 dx dy \\
&\leq c_5 k^{-\alpha} \|v\|_{H^\alpha(\mathbb{R}^3)}^2 + c_5 k^{-\alpha} \|\eta\|_M^2 + \frac{c_5 k^{-\alpha}}{\lambda} \int_0^\infty g(s) \left( \| \frac{\partial \eta(t)}{\partial t} \|^2 + \| \frac{\partial \eta(t)}{\partial s} \|^2 \right) ds \\
&+ c_5 \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) (1 + |k(t, x)|^2) dx + c_5 \int_0^\infty g(s) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) (-\Delta)^{\alpha} \eta^2 dx ds \\
&+ c_5 \epsilon^2 \int_0^\infty g(s) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) |z|^2 dx ds + c_5 (\epsilon^2 + \epsilon^p) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) (|z|^2 + |z|^p + |(-\Delta)^{\alpha} z|^2) dx.
\end{align*}
\]

Since \(\alpha \in (0, 1)\) and \(c_5\) is independent of \(k\), for any given \(\epsilon > 0\), there exists \(K(\epsilon) \geq 1\) such that for all \(k \geq K\),

\[
\begin{align*}
&c_5 k^{-\alpha} \|v\|_{H^\alpha(\mathbb{R}^3)}^2 + c_5 k^{-\alpha} \|\eta\|_M^2 + c_5 k^{-\alpha} \int_0^\infty g(s) \left( \| \frac{\partial \eta(t)}{\partial t} \|^2 + \| \frac{\partial \eta(t)}{\partial s} \|^2 \right) ds \\
&\leq \epsilon \|v\|_{H^\alpha(\mathbb{R}^3)}^2 + \epsilon \|\eta\|_M^2 + \epsilon \int_0^\infty g(s) \left( \| \frac{\partial \eta(t)}{\partial t} \|^2 + \| \frac{\partial \eta(t)}{\partial s} \|^2 \right) ds.
\end{align*}
\]

In addition, by the definition of \(\rho\) we find that

\[
\begin{align*}
&c_5 \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) (1 + |k(t, x)|^2) dx + c_5 \int_0^\infty g(s) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) (-\Delta)^{\alpha} \eta^2 dx ds \\
&+ c_5 \epsilon^2 \int_0^\infty g(s) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) |z|^2 dx ds + c_5 (\epsilon^2 + \epsilon^p) \int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) (|z|^2 + |z|^p + |(-\Delta)^{\alpha} z|^2) dx \\
&\leq c_5 \left( \frac{1}{\lambda} + \int_{|z| \geq \frac{1}{2} k} |z|^2 dx \right) + c_5 \int_0^\infty g(s) \int_{|z| \geq \frac{1}{2} k} |(-\Delta)^{\alpha} \eta|^2 dx ds \\
&+ c_5 \epsilon^2 \int_0^\infty g(s) \int_{|z| \geq \frac{1}{2} k} |z|^2 dx ds + c_5 (\epsilon^2 + \epsilon^p) \int_{|z| \geq \frac{1}{2} k} (|z|^2 + |z|^p + |(-\Delta)^{\alpha} z|^2) dx.
\end{align*}
\]
Since $\sigma = \min\{\frac{\lambda}{2}, \frac{\delta}{4}\}$, by (42)-(44) we get, for $k \geq K_1$,

$$
\frac{d}{dt} \left( \int_{\mathbb{R}^3} \rho\left( \frac{|x|}{k} \right) |v|^2 \, dx \right) + \frac{C(\alpha)}{2} \int_0^\infty g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho\left( \frac{|x|}{k} \right)(\eta^r(x) - \eta^r(y))^2}{|x - y|^{3+2\alpha}} \, dx \, dy \, ds \\
+ \sigma \left( \int_{\mathbb{R}^3} \rho\left( \frac{|x|}{k} \right) |v|^2 \, dx \right) + \frac{C(\alpha)}{2} \int_0^\infty g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho\left( \frac{|x|}{k} \right)(\eta^r(x) - \eta^r(y))^2}{|x - y|^{3+2\alpha}} \, dx \, dy \, ds \\
+ C(\alpha) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho\left( \frac{|x|}{k} \right)(v(x) - v(y))^2}{|x - y|^{3+2\alpha}} \, dx \, dy \\
\leq \varepsilon \|v\|^2_{H^\alpha(\mathbb{R}^3)} + \varepsilon \|\eta^r\|^2_M + \varepsilon \int_0^\infty g(s)(\|\frac{\partial \eta^r}{\partial t}\|^2 + \|\frac{\partial \eta^r}{\partial s}\|^2) \, ds \\
+ c_5 \int_{|x| \geq \frac{1}{k}} (1 + |k(t, x)|)^2 \, dx + c_5 \int_0^\infty g(s) \int_{|x| \geq \frac{1}{k}} |(-\Delta)^\alpha \eta^r|^2 \, dxds \\
+ c_5 \varepsilon^2 \int_0^\infty g(s) \int_{|x| \geq \frac{1}{k}} |z|^2 \, dxds \\
+ c_5 (\varepsilon^2 + \varepsilon^p) \int_{|x| \geq \frac{1}{k}} (|z|^2 + |z|^p + |(-\Delta)^\alpha z|^2) \, dx.
$$

By the Gronwall inequality, we get for each $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $k \geq K_1$,

$$
\int_{\mathbb{R}^3} \rho\left( \frac{|x|}{k} \right)|v(\tau, \tau - t, \omega, v_{\tau - t})|^2 \, dx \\
+ \frac{C(\alpha)}{2} \int_0^\infty g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho\left( \frac{|x|}{k} \right)(\eta^r(\tau - t, \omega, v_{\tau - t})(x) - \eta^r(y))^2}{|x - y|^{3+2\alpha}} \, dx \, dy \, ds \\
\leq e^{-\sigma t} \left( \int_{\mathbb{R}^3} \rho\left( \frac{|x|}{k} \right)|v_{\tau - t}|^2 \, dx \right) \\
+ \frac{C(\alpha)}{2} \int_0^\infty g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho\left( \frac{|x|}{k} \right)(\eta_{\tau - t}(x) - \eta_{\tau - t}(y))^2}{|x - y|^{3+2\alpha}} \, dx \, dy \, ds \\
+ \varepsilon \int_{\tau - t}^\tau e^{\sigma(\tau - \xi)} \|v(\xi)\|^2_{H^\alpha(\mathbb{R}^3)} \, d\xi + \varepsilon \int_{\tau - t}^\tau e^{\sigma(\tau - \xi)} \|\eta^r\|^2_M \, d\xi \\
+ \varepsilon \int_{\tau - t}^\tau e^{\sigma(\tau - \xi)} \int_0^\infty g(s)(\|\frac{\partial \eta^r}{\partial \xi}\|^2 + \|\frac{\partial \eta^r}{\partial s}\|^2) \, ds \, d\xi \\
+ c_5 \int_{\tau - t}^\tau e^{\sigma(\tau - \xi)} \int_{|x| \geq \frac{1}{k}} (1 + |k(\xi, x)|)^2 \, dx \, d\xi \\
+ c_5 \int_{\tau - t}^\tau e^{\sigma(\tau - \xi)} \int_0^\infty g(s) \int_{|x| \geq \frac{1}{k}} |(-\Delta)^\alpha \eta| \, dx \, d\xi \\
+ c_5 \varepsilon^2 \int_{\tau - t}^\tau e^{\sigma(\tau - \xi)} \int_0^\infty g(s) \int_{|x| \geq \frac{1}{k}} |z(\theta_\xi \omega)|^2 \, dx \, d\xi \\
+ c_5 (\varepsilon^2 + \varepsilon^p) \int_{\tau - t}^\tau e^{\sigma(\tau - \xi)} \int_0^\infty g(s) \int_{|x| \geq \frac{1}{k}} (|z(\theta_\xi \omega)|^2 + |z(\theta_\xi \omega)|^p + |(-\Delta)^\alpha z(\theta_\xi \omega)|^2) \, dx \, d\xi.
$$
and then replacing $\omega$ by $\theta - \tau \omega$, after change of variables, we obtain, for all $k \geq K_1$,
\[
\int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) |v(\tau - t, \theta - \tau \omega, v_{\tau-t})|^2 dx
+ \frac{C(\alpha)}{2} \int_0^\infty g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) \left| \eta_{\tau-t}(x) - \eta_{\tau-t}(y) \right|^2 \frac{dxdyds}{|x-y|^{3+2\alpha}}
\leq e^{-\sigma t} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) |v_{\tau-t}|^2 dx
+ \frac{C(\alpha)}{2} \int_0^\infty g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) \left| \eta_{\tau-t}(x) - \eta_{\tau-t}(y) \right|^2 \frac{dxdyds}{|x-y|^{3+2\alpha}}
\leq \varepsilon.
\]

We now estimate the first term on the right-hand side of (47). Since $\phi_{\tau-t} \in D(\tau - t, \theta - \tau \omega)$, $D \in \mathcal{D}$ and $z(\omega)$ is tempered, we see that there exists $T_4(\tau, \omega, D, \varepsilon) > 0$ such that for all $t \geq T_4$,
\[
e^{-\sigma t} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) |v_{\tau-t}|^2 dx
+ \frac{C(\alpha)}{2} \int_0^\infty g(s) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \rho\left(\frac{|x|}{k}\right) \left| \eta_{\tau-t}(x) - \eta_{\tau-t}(y) \right|^2 \frac{dxdyds}{|x-y|^{3+2\alpha}}
\leq \varepsilon.
\]

By Lemma 4.1, Lemma 4.2 and Lemma 4.4, we find that there exists $T_5(\tau, \omega, D, \varepsilon) \geq T_4$ such that for all $t \geq T_5$,
\[
\varepsilon \int_0^\infty e^{\sigma \xi} \|v(\xi + \tau)\|_{H^\alpha(\mathbb{R}^3)}^2 d\xi + \varepsilon \int_0^\infty e^{\sigma \xi} \|\eta_{\xi+\tau}^{\xi+\tau}\|_{M}^2 d\xi
+ \varepsilon \int_0^\infty e^{\sigma \xi} \int_0^\infty g(s) \left( \|\nabla \eta_{\xi+\tau}^{\xi+\tau}\|_{L^2}^2 + \|\frac{\partial \eta_{\xi+\tau}^{\xi+\tau}}{\partial s}\|_{L^2}^2 \right) ds d\xi
\leq \varepsilon c_0(\tau, \omega),
\]
where $c_0(\tau, \omega)$ is a positive number depending on $\tau$ and $\omega$. By (26), we find that
\[
\int_{-\infty}^0 e^{\sigma \xi} (1 + \|k(\xi + \tau)\|^2) d\xi < \infty,
\]
which implies that there exists $K_2(\tau, \varepsilon)$ such that for all $k \geq K_2$,
\[
c_5 \int_{-\infty}^0 e^{\sigma \xi} \int_{|x| \geq \frac{1}{2} k} (1 + |k(\xi + \tau, x)|^2) dx d\xi \leq \varepsilon.
\]
Suppose pullback absorbing set. estimates on the solutions of (21)-(24), we first obtain the existence of tempered Existence of random attractors.

5. Similarly, by Lemma 4.2, we see that there exists $K_3(\tau, \omega, \epsilon)$ such that for all $k \geq K_3$,

$$c_5 \int_{-t}^{0} e^{\sigma \xi} \int_{0}^{\infty} g(s) \int_{|x| \geq \frac{1}{2} k} |(-\Delta)\alpha \xi + \tau|^2 dx ds d\xi \leq \epsilon. \quad (51)$$

By (19), there is $K_4(\omega, \epsilon)$ such that for all $k \geq K_4$,

$$c_5 \epsilon^2 \int_{-t}^{0} e^{\sigma \xi} \int_{0}^{\infty} g(s) \int_{|x| \geq \frac{1}{2} k} |z(\theta \xi \omega)|^2 dx ds d\xi$$

$$+ c_5 (\epsilon^2 + \epsilon^p) \int_{-t}^{0} e^{\sigma \xi} \int_{|x| \geq \frac{1}{2} k} (|z(\theta \xi \omega)|^2 + |z(\theta \xi \omega)|^p + |(-\Delta)\alpha z(\theta \xi \omega)|^2) dx ds d\xi \leq \epsilon. \quad (52)$$

It follows from (47)-(52) that for all $k \geq \max\{K_1, K_2, K_3, K_4\}$ and $t \geq T_5$,

$$\int_{\mathbb{R}^3} \rho \left( \frac{|x|}{k} \right) |v(\tau, \tau - t, \theta_{-\tau} \omega, v_{-\tau})|^2 dx$$

$$+ \frac{C(\alpha)}{2} \int_{\mathbb{R}^3} g(s) \int_{\mathbb{R}^3} \int_{|x| \geq \frac{1}{2} k} \rho \left( \frac{|x|}{k} \right) (\eta^2(\tau - t, \theta_{-\tau} \omega, \eta_{-\tau})(x) - \eta^2(y))^2 |x - y|^{3+2\alpha} dxdyds \leq \epsilon (4 + c_0(\tau, \omega)), \quad (53)$$

which shows that for all $k \geq \max\{K_1, K_2, K_3, K_4\}$ and $t \geq T_5$,

$$\int_{|x| \geq k} |v(\tau, \tau - t, \theta_{-\tau} \omega, v_{-\tau})|^2 dx$$

$$+ \frac{C(\alpha)}{2} \int_{\mathbb{R}^3} g(s) \int_{|x| \geq k} (\eta^2(\tau - t, \theta_{-\tau} \omega, \eta_{-\tau})(x) - \eta^2(y))^2 |x - y|^{3+2\alpha} dxdyds \leq \epsilon (4 + c_0(\tau, \omega)), \quad (54)$$

the proof is completed.

5. Existence of random attractors. In this section we prove the existence of tempered random pullback attractors for problem (7)-(8). Based on the uniform estimates on the solutions of (21)-(24), we first obtain the existence of tempered pullback absorbing set.

**Lemma 5.1.** Suppose $(h_2),(9)-(11)$ and (27) hold. Then the continuous cocycle $\Phi$ associated with problem (7)-(8) has a closed measurable $\mathcal{D}$–pullback absorbing set $K_\epsilon \in \mathcal{D}$, which is given by, for $\epsilon > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$K_\epsilon(\tau, \omega) = \{ \phi \in \mathcal{H} : ||\phi||^2_{\mathcal{H}} \leq R(\tau, \omega) \},$$

where $R(\tau, \omega)$ is the same number as in Lemma 4.1.

**Proof.** By (25) and Lemma 4.1 we have $\Phi(t, \tau - t, \theta_{-\tau} \omega, \phi_{-\tau}) \in K_\epsilon(\tau, \omega)$ for all $t \geq T_1$, and here $K_\epsilon$ absorbs all elements of $\mathcal{D}$. By (27), after some calculations, one can check that $K_\epsilon = \{ K_\epsilon(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \}$ is tempered, i.e., $K_\epsilon \in \mathcal{D}$. Note that $R(\tau, \omega)$ is measurable in $\omega \in \Omega$, and so is $K(\tau, \omega)$, which completes the proof.

Given $k > 0$, let $O_k = \{ x \in \mathbb{R}^3 : |x| < k \}$ and $O^c_k = \mathbb{R}^3 \setminus O_k$. Based on Lemma 4.5, we can establish the following uniform estimates on the tails of solutions of problem (7)-(8).
Lemma 5.2. Suppose $(h_2)$, (9)-(11) and (26) hold. Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$. Then for every $\varepsilon > 0$, there exist $T(\tau, \omega, D, \varepsilon) > 0$ and $K(\tau, \omega, \varepsilon) \geq 1$ such that for all $t \geq T$ and $k \geq K$, 
\[
\int_{|x| \geq k} |u(t, \tau - t, \theta_{-t} \omega, u_{t-\tau})|^2 dx \leq \varepsilon,
\]
and for every $\varepsilon > 0$, there exist $T(\tau, \omega, D, \varepsilon) > 0$ and $K(\tau, \omega, \varepsilon) \geq 1$ such that for all $t \geq T$ and $k \geq K$, 
\[
\int_{|x| \geq k} |u(t, \tau - t, \theta_{-t} \omega, u_{t-\tau})|^2 dx \leq \varepsilon.
\]

Proof. By (20) and Lemma 4.5, we have that for all $t \geq T$ and $k \geq K$, 
\[
\int_{|x| \geq k} |u(t, \tau - t, \theta_{-t} \omega, u_{t-\tau})|^2 dx \leq \varepsilon.
\]
On the other hand, by Lemma 4.5, we get that for $t \geq T$ and $k \geq K$, 
\[
\frac{C(\alpha)}{2} \int_0^\infty g(s) \int_{\mathbb{R}^3} \left| \frac{\eta^\tau(t, \theta_{-t} \omega, \eta_{t-\tau})(x) - \eta^\tau(y)}{|x-y|^{3+2\alpha}} \right|^2 dx dy ds \leq \varepsilon.
\]
interchange $x$ and $y$ in (56) we obtain, 
\[
\frac{C(\alpha)}{2} \int_0^\infty g(s) \int_{\mathbb{R}^3} \left| \frac{\eta^\tau(t, \theta_{-t} \omega, \eta_{t-\tau})(x) - \eta^\tau(y)}{|x-y|^{3+2\alpha}} \right|^2 dx dy ds \leq \varepsilon.
\]
By (56)-(57) we get 
\[
\frac{C(\alpha)}{2} \int_0^\infty g(s) \int_{\mathbb{R}^3} \left| \frac{\eta^\tau(t, \theta_{-t} \omega, \eta_{t-\tau})(x) - \eta^\tau(y)}{|x-y|^{3+2\alpha}} \right|^2 dx dy ds \leq \varepsilon.
\]
From (55) and (58), the desired estimates follow immediately.

In this following, we prove the asymptotic compactness of the cocycle $\Phi$ associated with problem (7)-(8) in $\mathcal{H}$.

Lemma 5.3. Suppose $(h_2)$, (9)-(11) and (27) hold. Then the continuous cocycle $\Phi$ associated with problem (7)-(8) is $\mathcal{D}$-pullback asymptotically compact in $\mathcal{H}$, that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D \in \mathcal{D}$, $t_n \to \infty$ and $\phi_{0,n} \in D(\tau - t_n, \theta_{-t_n} \omega)$, the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, \phi_{0,n})$ has a convergent subsequence in $\mathcal{H}$.

Proof. By Lemma 5.2 we find that for every $\varepsilon > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there exist $K(\tau, \omega, \varepsilon) \geq 1$ and $N_1(\tau, \omega, D, \varepsilon) \geq 1$ such that for all $n \geq N_1$, 
\[
\|\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, \phi_{0,n})\|_{L^2(\mathcal{O}_k) \times L^2(\mathbb{R}^+, \mathcal{H}^\alpha(\mathcal{O}_k)} \leq \varepsilon.
\]

By Lemma 4.2 there exists $N_2(\tau, \omega, D, \varepsilon) \geq N_1$ such that 
\[
\|u(t_n, \tau - t_n, \theta_{-t_n} \omega, u_{0,n})\|_{\mathcal{H}^\alpha(\mathcal{O}_k)} \leq c_\tau(t, \omega),
\]
where $c_\tau(t, \omega)$ is a positive constant. Then the compact embedding $H^\alpha(\mathcal{O}_k) \hookrightarrow L^2(\mathcal{O}_k)$ together with (59) implies $\{u(t_n, \tau - t_n, \theta_{-t_n} \omega, u_{0,n})\}_{n=1}^{\infty}$ has a finite covering in $L^2(\mathbb{R}^3)$ of balls of radii less than $\varepsilon$ (see [4]).

On the other hand, by the way of decomposing the solution of (21)-(24) and Lemma 2.10, we get there exists $N_3 \geq N_2$ such that for all $n \geq N_3$, $\eta^{t_n}(\tau - t_n, \theta_{-t_n} \omega, \eta_{0,n})$ is compact in $L^2(\mathbb{R}^+, \mathcal{H}^\alpha(\mathcal{O}_k))$ (see [5, 31]), which together with (59) shows that $\{\eta^{t_n}(\tau - t_n, \theta_{-t_n} \omega, \eta_{0,n})\}_{n=1}^{\infty}$ has a finite covering in $L^2(\mathbb{R}^+, \mathcal{H}^\alpha(\mathbb{R}^3))$ of balls of radii less than $\varepsilon$.

The proof is completed.
Now we are in position to give the existence of random attractors for problem (7)-(8) in $\mathcal{H}$.

**Theorem 5.4.** Suppose $(h_2)$, (9)-(11) and (27) hold. Then the cocycle $\Phi$ of problem (7)-(8) has a unique $\mathcal{D}$--pullback attractor $A_\epsilon = \{A_\epsilon(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $\mathcal{H}$.

**Proof.** This is an immediate consequence of Proposition 2.7 based on Lemma 5.1 and Lemma 5.3. \qed

6. **Upper semicontinuity of random attractors.** In this section, we prove the upper semicontinuity of random attractors for problem (7)-(8) on $\mathbb{R}^3$ when $\epsilon \to 0$. Throughout this section, we assume $0 \leq \epsilon \leq 1$, and write the cocycle of problem (7)-(8) as $\Phi_\epsilon$ to indicate its dependence on $\epsilon$. Then $\Phi_\epsilon$ has a tempered pullback attractor $A_\epsilon$ by Theorem 5.4, and has a tempered pullback absorbing set $K_\epsilon$ by Lemma 5.1. Given $\tau \in \mathbb{R}$, $\omega \in \Omega$, let

$$ K(\tau, \omega) = \{\phi \in \mathcal{H} : \|\phi\|^2_{\mathcal{H}} \leq R(\tau, \omega)\}, $$

where $R(\tau, \omega)$ is given by

$$ R(\tau, \omega) = c + c \int_{-\infty}^{0} e^{\sigma s}(1 + \|k(s + \tau)\|^2)ds + cr(\omega). $$

By Lemma 5.1 and (60)-(61) we have $K_\epsilon(\tau, \omega) \subseteq K(\tau, \omega)$ for every $0 < \epsilon \leq 1$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$. This implies that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$ \bigcup_{0 < \epsilon \leq 1} A_\epsilon(\tau, \omega) \subseteq \bigcup_{0 < \epsilon \leq 1} K_\epsilon(\tau, \omega). $$

When $\epsilon = 0$, the stochastic equation (7)-(8) reduces to a deterministic one:

$$ \frac{\partial u}{\partial t} + (-\Delta)^\alpha u + \lambda u + \int_{0}^{\infty} \mu(s)(-\Delta)^\alpha u(t-s)ds + f(u) = k(t,x), x \in \mathbb{R}^3, t > \tau, $$

with initial condition

$$ u(t,x) = u_\tau(t,x), \quad x \in \mathbb{R}^3, \ t \leq \tau. $$

Similar to (7)-(8), one can prove that problem (63)-(64) generates a continuous cocycle $\Phi_0$ in $\mathcal{H}$. Moreover, $\Phi_0$ has a unique tempered pullback attractor $A_0 = \{A_0(\tau) : \tau \in \mathbb{R}\} \in \mathcal{H}$ and has a tempered pullback absorbing set $K_0$ where $K_0$ is given by

$$ K_0 = \{K_0(\tau) = \{\phi \in \mathcal{H} : \|\phi\|^2_{\mathcal{H}} \leq R_0(\tau)\} : \tau \in \mathbb{R}\}, $$

where $R_0(\tau)$ is the constant:

$$ R_0(\tau) = c + c \int_{-\infty}^{0} e^{\sigma s}(1 + \|k(s + \tau)\|^2)ds. $$

By Lemma 5.1 and (65)-(66) we have, for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$ \limsup_{\epsilon \to 0} \|K_\epsilon(\tau, \omega)\| \leq \|K_0(\tau)\|, $$

which will be used for proving the upper semicontinuity of $A_\epsilon$.

By Lemma 4.2 we find that, for every $0 < \epsilon \leq 1$ and $\omega \in \Omega$, there exists $T > 0$ such that for all $t \geq T$,

$$ \|\Phi_\epsilon(t - t_{0}, \theta_{-t_{0}} \omega, A_\epsilon(t_{0}, \theta_{-t_{0}} \omega))\|^2_{\mathcal{H}} \leq \tilde{R}_\epsilon(\tau, \omega). $$
Lemma 6.1. Suppose respectively. If \( \beta \) where

\[
\| \phi \|_{\nu}^2 \leq R_\epsilon(\tau, \omega) \quad \text{for all } \phi \in A_\epsilon(\tau, \omega) \quad \text{with} \quad 0 < \epsilon \leq 1.
\]

We will use (69) to prove the precompactness of the union of \( A_\epsilon \) in \( \mathcal{H} \) for \( 0 < \epsilon \leq 1 \).

We also need the convergence of solutions of problem (7)-(8) as \( \epsilon \to 0 \). To that end, we further assume the nonlinearity \( f \) satisfies:

\[
| \frac{\partial f(s)}{\partial s} | \leq \beta_\epsilon (1 + |s|^{p-2}), \quad s \in \mathbb{R},
\]

where \( \beta_\epsilon \) is a positive number.

Lemma 6.1. Suppose (h₂), (9)-(11) and (70) hold. Let \( \phi_\epsilon(t; \tau, \omega, \phi_{\epsilon, \tau}) \) and \( \phi(t, \tau, \phi_{\tau}) \) be the solutions of (7)-(8) and (63)-(64) with initial data \( \phi_{\epsilon, \tau} \) and \( \phi_{\tau} \), respectively. If \( \lim_{\epsilon \to 0} \phi_{\epsilon, \tau} = \phi_{\tau} \) in \( \mathcal{H} \), then for any \( t \geq \tau \) and \( \omega \in \Omega \),

\[
\lim_{\epsilon \to 0} \phi_\epsilon(t; \tau, \omega, \phi_{\epsilon, \tau}) = \phi(t, \tau, \phi_{\tau}).
\]

Proof. Let \( \varphi_\epsilon \) be the solutions of (21)-(24) and \( \varphi \) be the solutions of (21)-(24) with \( \epsilon = 0 \), respectively. Set \( \tilde{\varphi} = (\tilde{v}, \tilde{\eta}^t) = (v_\epsilon - v, \eta_\epsilon^t - \eta^t) \). Then we get

\[
\begin{align*}
\frac{\partial \tilde{v}}{\partial t} + (-\Delta)^{\alpha} \tilde{v} + \lambda \tilde{v} + \int_0^\infty g(s)(-\Delta)^{\alpha} \eta^t(s) ds + f(u_\epsilon) - f(u) \\
= -\epsilon (-\Delta)^{\alpha} z, \quad x \in \mathbb{R}^3, \quad t > \tau,
\end{align*}
\]

(71)

\[
\frac{\partial \eta^t}{\partial t} = \tilde{v} + \epsilon z - \frac{\partial \eta^t}{\partial s}, \quad x \in \mathbb{R}^3, \quad t > \tau, \quad s > 0.
\]

(72)

From (71) we obtain

\[
\frac{1}{2} \frac{d}{dt} \| \tilde{v} \|^2 + \|(-\Delta)^{\frac{\alpha}{2}} \tilde{v} \|^2 + \lambda \| \tilde{v} \|^2 + \int_0^\infty g(s) \int_{\mathbb{R}^3} (-\Delta)^{\alpha} \eta^t \cdot \tilde{v} dx ds
\]

\[
= -\int_{\mathbb{R}^3} (f(u_\epsilon) - f(u)) \tilde{v} dx - \epsilon (\epsilon (-\Delta)^{\alpha} z, \tilde{v}).
\]

(73)

By (h₂) and (72) we get

\[
\int_0^{\infty} g(s) \int_{\mathbb{R}^3} (-\Delta)^{\alpha} \eta^t \cdot \tilde{v} dx ds \geq \frac{1}{2} \frac{d}{dt} \| \eta^t \|^2_{M} + \frac{\delta}{4} \| \eta^t \|^2_{M} - c \epsilon^2 \| (-\Delta)^{\frac{\alpha}{2}} z \|^2.
\]

(74)

For the nonlinear term in (73), by (10) and (70) we have

\[
-\int_{\mathbb{R}^3} (f(u_\epsilon) - f(u)) \tilde{v} dx
\]

\[
= -\int_{\mathbb{R}^3} \frac{\partial f(s)}{\partial s} (u_\epsilon - u) \tilde{v} dx
\]

\[
= -\int_{\mathbb{R}^3} \frac{\partial f(s)}{\partial s} \tilde{v}^2 dx - \epsilon \int_{\mathbb{R}^3} \frac{\partial f(s)}{\partial s} \tilde{v} dx
\]

\[
\leq \beta_3 \| \tilde{v} \|^2 + \epsilon \beta_\epsilon \int_{\mathbb{R}^3} (1 + (|u_\epsilon| + |u|)^{p-2}) \tilde{v}^2 dx
\]

\[
\leq \beta_3 \| \tilde{v} \|^2 + \epsilon \epsilon + \epsilon \epsilon (\| u_\epsilon \|^p + \| u \|^p + \| z \|^p).
\]

(75)

By Young’s inequality,

\[
- \epsilon \epsilon ((-\Delta)^{\alpha} z, \tilde{v}) \leq \frac{\epsilon}{2} \| (-\Delta)^{\alpha} z \|^2 + \frac{1}{2} \| \tilde{v} \|^2.
\]

(76)
It follows from (73)-(76) that
\[
\frac{d}{dt} \|\tilde{\varphi}\|_{H_\omega}^2 \leq c \|\tilde{\varphi}\|_{H_\omega}^2 + \epsilon c + \epsilon \|u_\epsilon\|_{p}^p + \|u\|_{p}^p + c c e^\frac{1}{2}(|t|) r(\omega). 
\] (77)
Solving (77) on \([\tau, \tau + T]\) we get, for all \(t \in [\tau, \tau + T]\), \(T > 0\),
\[
\|\tilde{\varphi}(t, \tau, \omega, \tilde{\varphi}_\tau)\|_{H_\omega}^2 \leq c \|\tilde{\varphi}\|_{H_\omega}^2 + \epsilon c + \epsilon \int_{\tau}^{t} (\|u_\epsilon(s, \tau, \omega, u_\epsilon, \tau)\|_{p}^p + \|u(s, \tau, u_\epsilon)\|_{p}^p) ds + c r(\omega). 
\] (78)

By (9) and Lemma 4.1, we find that there exists \(c(\tau, \omega, T) > 0\) such that for all \(0 < \epsilon < 1\) and \(t \in [\tau, \tau + T]\),
\[
\|\varphi_\epsilon(t, \tau, \omega, \varphi_{\epsilon, \tau})\|_{H_\omega}^2 + \int_{\tau}^{t} \|u_\epsilon(s, \tau, \omega, u_\epsilon, \tau)\|_{p}^p ds \leq c + c \|\varphi_{\epsilon, \tau}\|_{H_\omega}^2 + c r(\omega). 
\] (79)
Similarly, by (63)-(64) for \(\epsilon = 0\), we can also get that
\[
\int_{\tau}^{t} \|u(s, \tau, u_\epsilon)\|_{p}^p ds \leq c + c \|\varphi_{\tau}\|_{H_\omega}^2. 
\] (80)

By (78)-(80) we find that
\[
\|\varphi_\epsilon(t, \tau, \omega, \varphi_{\epsilon, \tau}) - \varphi(t, \tau, \varphi_{\tau})\|_{H_\omega}^2 \leq c \|\varphi_{\epsilon, \tau} - \varphi_{\tau}\|_{H_\omega}^2 + c \epsilon c + \epsilon \|\varphi_{\epsilon, \tau}\|_{H_\omega}^2 + c \|\varphi_{\tau}\|_{H_\omega}^2 + c r(\omega). 
\] (81)

Due to \(\varphi_{\epsilon, \tau} = \varphi_\epsilon, \tau - (\epsilon, 0)\), we get from (6.22) that if \(\lim_{\epsilon \to 0} \varphi_{\epsilon, \tau} = \varphi_{\tau}\) then
\[
\lim_{\epsilon \to 0} \varphi_{\epsilon}(t, \tau, \omega, \varphi_{\epsilon, \tau}) = \varphi(t, \tau, \varphi_{\tau}), 
\]
which together with (20), the proof is completed.

We now prove the compactness of pullback attractors.

**Lemma 6.2.** Suppose \((h_2), (9), (11)\) and (27) hold. Then for every \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\), the union \(\cup_{0 \leq \epsilon \leq 1} \mathcal{A}_\epsilon(\tau, \omega)\) is precompact in \(\mathcal{H}\).

**Proof.** We only need to show that, for every \(\epsilon > 0\), \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\), the set \(\cup_{0 \leq \epsilon \leq 1} \mathcal{A}_\epsilon(\tau, \omega)\) has a finite covering of balls of radii less than \(\epsilon\). Let \(B\) be the random set given in (60). By Lemma 5.2, we find that there exist \(T(\tau, \omega, \epsilon) > 0\) and \(K(\tau, \omega, \epsilon) \geq 1\) such that for all \(t \geq T\) and \(0 < \epsilon \leq 1\),
\[
\int_{|x| \geq k} |u_\epsilon, \tau - t, \theta_{-t} \omega, u_{\epsilon, \tau - t}|^2 dx 
+ \frac{C(\alpha)}{2} \int_{0}^{\infty} g(s) \int_{\mathbb{R}^3 \times \mathbb{R}^3 \setminus \mathcal{O}_k \setminus \mathcal{O}_k} \frac{(\eta_\epsilon^\tau(t, \theta_{-t}, \omega, \eta_{\epsilon, \tau - t}))(x) - \eta_\epsilon^\tau(y)(y))^2}{|x - y|^{3+2\alpha}} dxdyds \leq \frac{\epsilon}{2}, 
\] (82)
where \((u_\epsilon, \tau - t, \eta_{\epsilon, \tau - t}) \in B(\tau - t, \theta_{-t}, \omega)\). By (62) and (82), we get from the invariance of \(\mathcal{A}_\epsilon\) that, for each \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\),
\[
\int_{|x| \geq k} |u(x)|^2 dx + \frac{C(\alpha)}{2} \int_{0}^{\infty} g(s) \int_{\mathbb{R}^3 \times \mathbb{R}^3 \setminus \mathcal{O}_k \setminus \mathcal{O}_k} \frac{(\eta^\epsilon(x) - \eta^\epsilon(y))^2}{|x - y|^{3+2\alpha}} dxdyds \leq \frac{\epsilon}{2}, 
\] (83)
for all \((u, \eta^\epsilon) \in \mathcal{A}_\epsilon(\tau, \omega)\) with \(0 < \epsilon \leq 1\).

By (69) we see that \(\cup_{0 \leq \epsilon \leq 1} \mathcal{A}_\epsilon(\tau, \omega)\) is bounded in \(\tilde{H}^\alpha(\mathcal{O}_k) \times L^2_g(\mathbb{R}^+; \tilde{H}^{2\alpha}(\mathcal{O}_k))\). Moreover, \(H^\alpha(\mathcal{O}_k) \hookrightarrow L^2(\mathcal{O}_k)\) is compact and \(\eta^\epsilon\) is compact in \(L^2_g(\mathbb{R}^+; \tilde{H}^{2\alpha}(\mathcal{O}_k))\), then the set \(\cup_{0 \leq \epsilon \leq 1} \mathcal{A}_\epsilon(\tau, \omega)\) has a finite covering of balls of radii less than \(\frac{\epsilon}{2}\) in
This along with (83) shows that \( \bigcup_{0<\varepsilon<1} A_{\varepsilon}(\tau, \omega) \) has a finite covering of balls of radii less than \( \varepsilon \) in \( \mathcal{H} \).

The proof is completed. \( \square \)

We are now in a position to present the upper semicontinuity of pullback attractors for problem (7)-(8).

**Theorem 6.3.** Suppose (h2), (9)-(11) and (27) hold. Then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\lim_{\varepsilon \to 0} \text{dist}_{\mathcal{H}}(A_{\varepsilon}(\tau, \omega), A_0(\tau)) = 0.
\]

**Proof.** Let \( \varepsilon_n \to 0 \) and \( \phi_{0,n} \to \phi_0 \) in \( \mathcal{H} \), by Lemma 6.1 we find that for \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\Phi_{\varepsilon_n}(t, \tau, \omega, \phi_{0,n}) \to \Phi(t, \tau, \phi_0) \quad \text{in} \quad \mathcal{H},
\]

Then the upper semicontinuity of pullback attractors follows from Proposition 2.8 immediately based on (67), (84) and Lemma 6.2. \( \square \)

**Acknowledgments.** The first author is supported by joint research project of Laurent Mathematics Center of Sichuan Normal University and National-Local Joint Engineering Laboratory of System Credibility Automatic Verification, the funding of V.C. & V.R. Key Lab of Sichuan Province. The fourth author is supported by the National Natural Science Foundation of China(No.11871138). The authors would like to thank the reviewers for their helpful comments.

**REFERENCES**

[1] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, Berlin, 1998.

[2] Q. Bai, J. Shu, L. Li and H. Li, Dynamical behavior of non-autonomous fractional stochastic reaction-diffusion equations, *J. Math. Anal. Appl.*, 485 (2020), 123833.

[3] P. W. Bates, H. Lisei and K. Lu, Attractors for stochastic lattice dynamical systems, *Stoch. Dyn.*, 6 (2006), 1–21.

[4] P. W. Bates, K. Lu and B. Wang, Random attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differential Equations*, 246 (2009), 845–869.

[5] T. Caraballo, J. Real and I. D. Chueshov, Pullback attractors for stochastic heat equations in materials with memory, *Discrete Contin. Dyn. Syst. Ser. B*, 9 (2008), 525–539.

[6] T. Caraballo, I. D. Chueshov, P. Marin-Rubio and J. Real, Existence and asymptotic behaviour for stochastic heat equations with multiplicative noise in materials with memory, *Discrete Contin. Dyn. Syst.*, 18 (2007), 253–270.

[7] T. Caraballo, J. A. Langa, V. S. Melnik and J. Valero, Pullback attractors of nonautonomous and stochastic multivalued dynamical systems, *Set–Valued Anal.*, 11 (2003), 153–201.

[8] T. Caraballo, G. Lukaszewicz and J. Real, Pullback attractors for asymptotically compact nonautonomous dynamical systems, *Nonlinear Anal.*, 64 (2006), 484–498.

[9] M. Conti, V. Pata and M. Squassina, Singular limit of differential systems with memory, *Indiana Univ. Math. J.*, 55 (2006), 169–215.

[10] H. Crauel and P. E. Kloeden, Nonautonomous and random attractors, *Jahresber. Dtsch. Math.-Ver.*, 117 (2015), 173–206.

[11] H. Crauel and F. Flandoli, Attractors for random dynamical systems, *Probab. Theory Related Fields*, 100 (1994), 365–393.

[12] H. Crauel, A. Debussche and F. Flandoli, Random attractors, *J. Dynam. Differential Equations*, 9 (1997), 307–341.

[13] C. M. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Ration. Mech. Anal.*, 37 (1970), 297–308.

[14] E. DiNezza, G. Palatucci and E. Valdinoci, Hitchhikers guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, 136 (2012), 521–573.

[15] J. Dong and M. Xu, Space-time fractional Schrödinger equation with time-independent potentials, *J. Math. Anal. Appl.*, 344 (2008), 1005–1017.
[16] F. Flandoli and B. Schmalfuss, Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise, *Stochastics Stochastics Rep.*, 59 (1996), 21–45.

[17] C. Gal and M. Warma, Reaction-diffusion equations with fractional diffusion on non-smooth domains with various boundary conditions, *Discrete Contin. Dyn. Syst.*, 36 (2016), 1279–1319.

[18] C. Giorgi, V. Pata and A. Marzocchi, Asymptotic behavior of a semilinear problem in heat conduction with memory, *Nonlinear Differential Equations Appl.*, 5 (1998), 333–354.

[19] M. Grasselli and V. Pata, Uniform attractors of nonautonomous dynamical systems with memory, in *Progress in Nonlinear Differential Equations and Their Applications*, Vol. 50, Birkhäuser, Basel, 2002, 155–178.

[20] A. Gu, D. Li, B. Wang and H. Yang, Regularity of random attractors for fractional stochastic reaction-diffusion equations on $\mathbb{R}^n$, *J. Differential Equations*, 264 (2018), 7094–7137.

[21] B. Guo, Y. Han and J. Xin, Existence of the global smooth solution to the period boundary value problem of fractional nonlinear Schrödinger equation, *Appl. Math. Comput.*, 204 (2008), 468–477.

[22] B. Guo and Z. Huo, Global well-posedness for the fractional nonlinear Schrödinger equation, *Comm. Partial Differential Equations*, 36 (2011), 247–255.

[23] B. Guo and M. Zeng, Solutions for the fractional Landau-Lifshitz equation, *J. Math. Anal. Appl.*, 361 (2010), 131–138.

[24] B. Guo and G. Zhou, Ergodicity of the stochastic fractional reaction diffusion equation, *Nonlinear Anal.*, 109 (2014), 1–22.

[25] C. Guo, J. Shu and X. Wang, Fractal dimension of random attractors for non-autonomous fractional stochastic Ginzburg-Landau equations, *Acta Math. Sin.(Engl. Ser.)*, 36 (2020), 318–336.

[26] Y. Lan and J. Shu, Fractal dimension of random attractors for non-autonomous fractional stochastic Ginzburg-Landau equations with multiplicative noise, *Dyn. Syst.*, 34 (2019), 274–300.

[27] Y. Lan and J. Shu, Dynamics of non-autonomous fractional stochastic Ginzburg-Landau equations with continuous nonlinearity, *Asymptot. Anal.*, 44 (2005), 111–130.

[28] L. Liu and T. Caraballo, Well-posedness and dynamics of a fractional stochastic integro-differential equation, *Phys. D*, 355 (2017), 45–57.

[29] H. Lu, P. W. Bates, J. Xin and M. Zhang, Asymptotic behavior of stochastic fractional power dissipative equations on $\mathbb{R}^n$, *Nonlinear Anal.*, 128 (2015), 176–198.

[30] H. Lu, P. W. Bates, S. Lu and M. Zhang, Dynamics of 3-D fractional complex Ginzburg-Landau equation, *J. Differential Equations*, 259 (2015), 5276–5301.

[31] G. Lv and J. Duan, Martingale and weak solutions for a stochastic nonlocal Burgers equation on finite intervals, *J. Math. Anal. Appl.*, 449 (2017), 176–194.

[32] G. Lv, H. Gao, J. Wei and J. Wu, BMO and Morrey Campanato estimates for stochastic convolutions and Schauder estimates for stochastic parabolic equations, *J. Differential Equations*, 266 (2019), 2666–2717.

[33] F. Morillas and J. Valero, Attractors for reaction-diffusion equations in $\mathbb{R}^n$ with continuous nonlinearity, *Asymptot. Anal.*, 44 (2005), 111–130.

[34] L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 13 (1959), 115–162.

[35] V. Pata and A. Zucchi, Attractors for a damped hyperbolic equation with linear memory, *Adv. Math. Sci. Appl.*, 11 (2001), 505–529.

[36] X. Pu and B. Guo, Global weak solutions of the fractional Landau-Lifshitz-Maxwell equation, *J. Math. Anal. Appl.*, 372 (2010), 86–98.

[37] X. Pu and B. Guo, Well-posedness and dynamics for the fractional Ginzburg-Landau equation, *Appl. Anal.*, 92 (2013), 318–334.

[38] B. Schmalfuss, Forward cocycle and attractors of stochastic differential equations, in *International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behavior*, Technische Universität, (1992), 185–192.
[42] T. Shen and J. Huang, Well-posedness and dynamics of stochastic fractional model for nonlinear optical fiber materials, *Nonlinear Anal.*, **110** (2014), 33–46.

[43] J. Shu, Random attractors for stochastic discrete Klein-Gordon-Schrödinger equations driven by fractional Brownian motions, *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), 1587–1599.

[44] J. Shu, P. Li, J. Zhang and O. Liao, Random attractors for the stochastic coupled fractional Ginzburg-Landau equation with additive noise, *J. Math. Phys.*, **56** (2015), 102702.

[45] R. Temam, *Infinite Dimension Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1997.

[46] B. Wang, Asymptotic behavior of non-autonomous fractional stochastic reaction-diffusion equations, *Nonlinear Anal.*, **158** (2017), 60–82.

[47] B. Wang, Upper semicontinuity of random attractors for non-compact random systems, *Electron J. Differential Equations*, **139** (2009), 1–18.

[48] B. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differential Equations*, **253** (2012), 1544–1583.

[49] B. Wang, Existence and upper-semicontinuity of attractors for stochastic equations with deterministic non-autonomous terms, *Stoch. Dyn.*, **14** (2014), 1450009. 1–31.

[50] B. Wang, Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Discrete Contin. Dyn. Syst.*, **34** (2014), 269–300.

[51] X. Wang, S. Li and D. Xu, Random attractors for second-order stochastic lattice dynamical systems, *Nonlinear Anal.*, **72** (2010), 483–494.

[52] X. Wang, K. Lu and B. Wang, Wong-Zakai approximations and attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differential Equations*, **264** (2018), 378–424.

[53] S. Zhou, Random exponential attractor for stochastic reaction-diffusion equation with multiplicative noise in $\mathbb{R}^3$, *J. Differential Equations*, **263** (2017), 6347–6383.

Received May 2020; revised July 2020.

E-mail address: shuji@sicnu.edu.cn
E-mail address: 2430410871@qq.com
E-mail address: huangxinnv@163.com
E-mail address: zhangjian@sina.com