Fatigue crack growth threshold as a design criterion – statistical scatter and load ratio in the Kitagawa-Takahashi diagram

S.Kolitsch¹², H.-P. Gänser¹, J. Maierhofer¹, R. Pippan²

¹ Materials Center Leoben Forschung GmbH, Roseggerstraße 17, A-8700 Leoben
² Erich Schmid Institute of Materials Science, Austrian Academy of Sciences, and Department of Materials Physics, University of Leoben, Jahnstraße 12, A-8700 Leoben

E-mail: stefan.kolitsch@mcl.at

Abstract. Cracks in components reduce the endurable stress so that the endurance limit obtained from common smooth fatigue specimens cannot be used anymore as a design criterion. In such cases, the Kitagawa-Takahashi diagram can be used to predict the admissible stress range for infinite life, at a given crack length and stress range. This diagram is constructed for a single load ratio \( R \). However, in typical mechanical engineering applications, the load ratio \( R \) varies widely due to the applied load spectra and residual stresses. In the present work an extended Kitagawa-Takahashi diagram accounting for crack length, crack extension and load ratio is constructed. To describe the threshold behaviour of short cracks, a master resistance curve valid for a wide range of steels is developed using a statistical approach.

1. Introduction

Cracks reduce the endurable stress of components so that the endurance limit cannot be used anymore as a design criterion. In such cases, the Kitagawa-Takahashi (KT) diagram [4] is commonly used. In this diagram the admissible stress range depends on the crack length; this curve delimits the area of non-propagating cracks. By means of an intrinsic crack length \( a_{0,H} \) introduced by El Haddad [2], a transition from the threshold of long cracks to the endurance limit is given (Fig. 1). The intrinsic length \( a_{0,H} \) for the transition is calculated via

\[
a_{0,H} = \frac{1}{\pi} \left( \frac{\Delta K_{\text{th},\text{lc}}}{Y \cdot \Delta \sigma_e} \right)^2,
\]

(1)

and the admissible stress range depending on the intrinsic length can then be expressed as follows:

\[
\Delta \sigma_{\text{th},\text{lc}}(a_0) = \frac{\Delta K_{\text{th},\text{lc}}}{Y \cdot \pi \sqrt{a + a_{0,H}}}.
\]

(2)

Here, \( \Delta K_{\text{th},\text{lc}} \) denotes the fatigue crack growth threshold stress intensity factor range for long cracks, \( \Delta \sigma_e \) the endurance limit stress range of polished specimens without flaws, and \( Y \) is the geometry factor of the crack. It should be noted that the values \( \Delta K_{\text{th},\text{lc}} \) and \( \Delta \sigma_e \) depend on the load ratio \( R \) so that the curve is only valid for a single load ratio \( R \).
This approach can be used for materials where the crack closure is built up completely. For other materials where the crack closure is not built up, El Haddad’s description is not conservative anymore. For this reason Chapetti used an exponential function to describe the crack closure effect for short cracks in the KT-diagram [1]. Nevertheless, in this function preexisting notches are not considered. Similar work was done by Tabernig [9]. The goal of the present work is to extend these concepts and take different load ratios into account.

Nomenclature

- \( \Delta a \): crack extension length
- \( a_0 \): notch depth
- \( a_{0,H} \): fictitious intrinsic length scale following El Haddad
- \( a \): total crack length
- \( \Delta K \): stress intensity factor range
- \( \Delta K_{th} \): threshold of stress intensity factor range for crack propagation
- \( \Delta K_{th,eff} \): intrinsic (effective) threshold stress intensity factor range
- \( \Delta K_{th,lc} \): long crack growth threshold stress intensity factor range
- \( \Delta K_{th,lc}(R=0) \): long crack growth threshold stress intensity factor range for load ratio equal to zero
- \( f_{cl} \): function to describe the master resistance curve
- \( f_{cext} \): function to describe the standardized crack extension
- \( f_D \): tension and compression endurance strength factor
- \( K_D \): mean stress factor
- \( b_m \): material constant
- \( \Delta \sigma \): stress range
- \( \Delta \sigma_e \): endurance limit stress range of polished specimens without flaws
- \( \Delta \sigma_{th} \): threshold stress range for crack propagation
- \( \Delta \sigma_{th,lc} \): threshold stress range for crack propagation calculated using \( \Delta K_{th,lc} \)
- \( v_i \): weighting factors
- \( \Delta \sigma_{UTS} \): ultimate tensile stress
- \( M_\pi \): mean stress sensitivity
- \( K_D \): mean stress factor
- \( a_m \): material constant
- \( l_i \): fictitious length scales
- \( R \): load ratio
- \( l_0 \): weighting factor
- \( l_{90} \): fictitious length where 90% of the crack closure is done
- \( C \): weighting factor

Figure 1. Kitagawa-Takahashi diagram, showing the area of non-propagating cracks according to El Haddad.
2. Build-up of crack closure
In Fig 2 the combination of the notch depth and the crack length is illustrated.

![Diagram of build-up of crack closure](image)

**Figure 2.** Schematic illustration of the proposed mechanical model: emanating from a deep sharp notch $a_0$, a crack of extension $\Delta a$ grows. Only on this crack extension $\Delta a$ the build-up of crack closure is possible [6].

The stress intensity factor range is defined by

$$\Delta K = Y \cdot \Delta \sigma \cdot \sqrt{\pi \cdot a},$$

where the total crack length

$$a = a_0 + \Delta a$$

is the sum of notch depth $a_0$ and crack extension $\Delta a$.

For the description of the crack closure effect only the crack extension $\Delta a$ is important and not the total crack length. To consider also the notch depth, Maierhofer et al. [5] describe the crack resistance curve by a double exponential function (Eq (5)). The different closure effects and their contributions to the total effect are shown schematically in Fig. 3, the plasticity-induced and roughness-induced closure contributions being represented each separately by an exponential function with a characteristic length scale $l_i$.

$$\Delta K_{\text{th}} = \Delta K_{\text{th,eff}} + (\Delta K_{\text{th,lc}} - \Delta K_{\text{th,eff}}) \cdot \left[1 - \sum_{i=1}^{2} v_i \cdot \exp\left(-\frac{\Delta a}{l_i}\right)\right]$$

**Figure 3.** Illustration of the crack resistance curve caused by two different closure mechanisms; each closure mechanism is built up completely over a specific crack extension (described by the fictitious length scales $l_i$) [5]
3. Cyclic fracture mechanics experiments

Constant amplitude load increasing tests [5] have been conducted on single edge notched bending (SENB) specimens for five different materials to obtain the cyclic crack resistance curves shown in Fig. 4. Most of the materials have been tested for two different load ratios $R = -1$ and $R = 0.1$. By using the experimental data points and the experimentally determined threshold for long cracks $\Delta K_{\text{th,lc}}$, the parameters $\nu_i$ and $l_i$ in Eq (5) can be fitted and an estimate for the crack resistance curve can be calculated (the intrinsic threshold is for all iron alloys $\Delta K_{\text{th,eff}} \approx 2.5 \text{ MPa}\sqrt{\text{m}}$; this is also clearly visible from Fig. 4). Those estimates are shown by the solid and dashed lines in Fig. 4.

It can be seen that Eq (5) describes the experimentally observed behavior well. However, the contributions from plasticity-induced and roughness-induced crack closure are different between the materials and are not comparable to each other. The threshold for long cracks changes due to the load ratio, but also due to differences in the closure mechanisms. This means that the resistance curve can be described by an exponential function whose parameters have to be determined separately for each material and load ratio.

![Figure 4. Crack resistance curve for $\Delta K_{\text{th}}$: experimental data points and analytically estimation curve from Eq. (5).](image)

4. Proposal for resistance curves for $\Delta K_{\text{th}}$

To design a more general diagram to compare the resistance curves of each material, the curves have to be converted to a standardized form. To this purpose, we relate the threshold increase due to closure effects at a given crack extension, $\Delta K_{\text{th}} - \Delta K_{\text{th,eff}}$, to the total threshold increase due to closure for long cracks, $\Delta K_{\text{th,lc}} - \Delta K_{\text{th,eff}}$. This crack closure factor is thus calculated by

$$f_{\text{cl}} = \frac{\Delta K_{\text{th}} - \Delta K_{\text{th,eff}}}{\Delta K_{\text{th,lc}} - \Delta K_{\text{th,eff}}}.$$  \hspace{1cm} (6)

To bring the length scales over which the build-up of closure occurs into a standardized form, the crack extension $\Delta a$ is rendered dimensionless via division by the crack extension $\Delta a$ where 90% of crack closure is reached, denoted by $l_{90}$:
\[ f_{cext} = \frac{\Delta a}{l_{0}}. \]  
\[ (7) \]

In Fig. 5 the standardized form for the resistance curves of all experimental data is plotted with the standardized range of the stress intensity factors and the normalized crack length. Based on this proposal the material and physical behavior of the different material is standardized and they are comparable. A simple analytical approximation of this curve is given, similarly as in Eq (5), by

\[ \frac{\Delta K_{th} - \Delta K_{th,eff}}{\Delta K_{th,lc} - \Delta K_{th,eff}} = 1 - e^{\frac{\Delta a}{l_{0}} + C}, \]  
\[ (8) \]

\[ f_{a}(f_{cext}) = 1 - e^{\left( \frac{1}{l_{0}} \right) \Delta a + C}, \]  
\[ (9) \]

where \( l_{0} \) and \( C \) denote constants to fit the experimental data points.

![Figure 5](image_url)

**Figure 5.** Standardized resistance curve of \( \Delta K_{th} \) obtained from Fig. 4 by normalizing the individual data points via Eqns (6) and (7).

To provide a general resistance curve for \( \Delta K_{th} \) of all tested materials a statistical analysis of the standardized experimental data is necessary.

5. Statistical approach

By determining the constants \( l_{0} \) and \( C \) by means of statistical regression, a general resistance curve can be obtained. A linear model for the data is obtained by transforming Eq (8) into

\[ -\ln \left( 1 - \frac{\Delta K_{th} - \Delta K_{th,eff}}{\Delta K_{th,lc} - \Delta K_{th,eff}} \right) = \frac{\Delta a}{l_{0}} + l_{0} + C, \]  
\[ (10) \]
The linear regression of $y = -\ln(1 - f_{cl})$ vs. $f_{ext}$ to the mean as well as the lines corresponding to ±1 coefficient of variation (COV) of the regression parameters $1/l_0$ and $C$ are plotted in Fig. 6.

$$y(f_{ext}) = -\ln(1 - f_{cl}) = \frac{1}{l_0} \cdot f_{ext} + C. \quad (11)$$

The general resistance curve obtained by back-transformation is plotted in Fig. 7. The restriction of this fitting curve is that it does not start anymore at zero. This means that the closure estimate is not conservative for very short cracks in particular. However, such very short crack extensions are of minor importance for damage tolerance issues, given the current detection limits in non-destructive testing.
6. Consideration of load ratio $R$

Using the proposed resistance curve for $\Delta K_{th}$, the Kitagawa-Takahashi diagram can be calculated for different materials. In addition, the influence of the load ratio $R$ is estimated by the simple bi-linear approximation shown in Fig 8:

$$\Delta K_{th,lc}(R) = \begin{cases} \frac{\Delta K_{th,lc}(R=0) \cdot (1-R)}{\Delta K_{th,eff}} & \text{for } \Delta K_{th,lc}(R=0) \cdot (1-R) > \Delta K_{th,eff} \\ \Delta K_{th,lc}(R=0) \cdot (1-R) & \text{for } \Delta K_{th,lc}(R=0) \cdot (1-R) \leq \Delta K_{th,eff} \end{cases}$$ (12)

Here $\Delta K_{th,lc}(R=0)$ denotes the range of the stress intensity factor for long cracks at a load ratio $R = 0$.

![Figure 8. Dependency of long crack threshold and the load ratio.](image)
Due to the linear connection between the threshold for long cracks and the load ratio, the range for any stress intensity factor can be calculated. Combining Eq (8) with the master resistance curve, Eq (9), one obtains the threshold stress intensity factor range as a function of crack extension \( \Delta a \) and load ratio \( R \),

\[
\Delta K_{th}(\Delta a, R) = \left[ \Delta K_{th,eff} + \left[ \Delta K_{th,lc}(R,a_0) - \Delta K_{th,eff} \right] \cdot \gamma \right] \cdot \frac{1}{\Delta K_{th,eff}} \quad \text{for} \quad \Delta K_{th,lc}(R) > \Delta K_{th,eff}
\]

\[
\Delta K_{th}(\Delta a, R) = \left[ \Delta K_{th,eff} + \left[ \Delta K_{th,lc}(R,a_0) - \Delta K_{th,eff} \right] \cdot \gamma \right] \cdot \frac{1}{\Delta K_{th,eff}} \quad \text{for} \quad \Delta K_{th,lc}(R) \leq \Delta K_{th,eff}
\]

(13)

The admissible applied stress range \( \Delta \sigma \) is then obtained by equating the crack growth resistance to the applied load,

\[
\Delta K_{th}(\Delta a, a_0) = \Delta \sigma \cdot Y \cdot \sqrt{\pi \cdot (a_0 + \Delta a)}
\]

(14)

The admissible stress range \( \Delta \sigma \) thus depends on three variables, \( \Delta a \), \( a_0 \) and \( R \), and is calculated as follows:

\[
\Delta \sigma(\Delta a, R, a_0) = \min \left( \frac{\Delta K_{th}(\Delta a, R)}{Y \cdot \sqrt{\pi \cdot (a_0 + \Delta a)}}, \Delta \sigma_c(R) \right)
\]

(15)

Not only \( \Delta \sigma \), also the stress endurance limit \( \Delta \sigma_c \) depends on the stress ratio \( R \)[3]:

\[
\Delta \sigma_c(R) = 2 \cdot \sigma_{UTS} \cdot f_D \cdot K_D(R)
\]

(16)

Here, \( \sigma_{UTS} \) denotes the ultimate tensile stress of the material, \( f_D \) the endurance limit factor depending on the material class, and \( K_D(R) \) the mean stress factor accounting for the mean stress sensitivity of the material. The mean stress sensitivity is considered by using the Haigh-diagram (Fig. 9).

![Figure 9. Haigh diagram](image)
The mean stress factor depending on the stress ratio can be calculated as follows [3]:

\[
K_D(R) = \begin{cases} 
\frac{1}{1 + M_\sigma \cdot (1 + R)/(1 - R)} & \text{for } R < 0 \\
\frac{1}{(1 + M_\sigma) \cdot [3 + M_\sigma \cdot (1 + R)/(1 - R)]} & \text{for } 0 < R < 0.5 \\
\frac{3 + M_\sigma}{3 \cdot (1 + M_\sigma)^2} & \text{for } R > 0.5
\end{cases}
\]  
(17)

\(M_\sigma\) describes the mean stress sensitivity depending on the ultimate tensile stress \(\sigma_{\text{UTS}}\) with constants \(a_m, b_m\) depending on the material class [3]:

\[
M_\sigma = a_m \cdot 10^{-3} \cdot \frac{\sigma_{\text{UTS}}}{\text{MPa}} + b_m
\]  
(18)

So the stress range can be calculated for different stress ratios. In Fig. 10 the stress range \(\Delta \sigma(\Delta a, a_0, R)\) is plotted for a constant initial notch depth \(a_0 = 0.1\) mm depending on the crack extension \(\Delta a\). The stress decreases with increasing load ratio \(R\).

**Figure 10.** Admissible stress range \(\Delta \sigma\) vs. crack extension \(\Delta a\) for different load ratios \(R\) and a constant initial notch depth \(a_0 = 0.1\) mm.

Overall, Eq (15) describes for a given constant load ratio \(R\) a three-dimensional surface [6], which is shown in Fig 11. This surface represents the stress range depending on the crack extension \(\Delta a\), the initial crack (notch) depth \(a_0\) and is plotted for different load ratios \(R\). A different load ratio \(R\) will create an offset and scaling of the failure surface which is shown for four different load ratios in Fig. 11.
7. Conclusion
A master resistance curve for fatigue crack closure has been proposed; its parameters have been determined for medium-strength steels by statistical regression. With this standardized resistance curve, a Kitagawa-Takahashi diagram depending on the initial notch/crack depth $a_0$ and the crack extension $\Delta a$ can be computed for any load ratio $R$. This extended Kitagawa-Takahashi diagram is readily applicable to the damage tolerant design of components.

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