The Taylor property in non-negative bilinear models

E. Gonçalves¹, C.M. Martins², N. Mendes-Lopes¹

¹CMUC, Dep. of Mathematics, University of Coimbra, Portugal
²Dep. of Mathematics, University of Coimbra, Portugal

Abstract. The aim of this paper is to discuss the presence of the Taylor property in the class of non-negative simple bilinear models. Considering strictly and weakly stationary models, we deduce autocorrelations of the process and of the square process and analyze the presence of the Taylor property considering several error process distributions. The relationship between the Taylor property and leptokurtosis of the corresponding bilinear process is discussed.

With the goal of extending this research to real valued bilinear models, a simulation study is developed in a class of such models with symmetrical innovations.

Keywords. Bilinear models; nonlinear time series; stationarity; Taylor property.

AMS Classification: 62M10

1 Introduction

The search for non-trivial empirical regularities in time series, usually called stylized facts, has been the subject of several studies in order to identify classes of time series models that conveniently capture such empirical properties. A stylized fact detected by Taylor (⁶) when he analyzed 40 returns series is known as the Taylor effect. He observed that, for most of the returns series, denoted by \( X_t \) for instant \( t \), the sample autocorrelations of the absolute returns, \( \hat{\rho}_{|X|}(n) = \hat{\text{corr}}(|X_t|, |X_{t-n}|) \), were larger than those of the squared returns, \( \hat{\rho}_{X^2}(n) = \hat{\text{corr}}(X_t^2, X_{t-n}^2) \), for \( n \in \{1, \ldots, 30\} \).

We point out that there is still little research on the theoretical counterpart on this empirical property due to the difficulty of handling the true autocorrelations of time series models. For example, this theoretical counterpart was studied by He and Teräsvirta (⁶) on conditionally Gaussian absolute value generalized ARCH (AVGARCH) models, assuring its pres-
ence for some of these models. More precisely, they called the theoretical relation $\rho_{X_1}(n) > \rho_{X^2}(n)$, $n \geq 1$, the Taylor property and concentrated their study on the autocorrelation of lag 1. More recently, Gonçalves, Leite and Mendes-Lopes (\cite{1}) studied the presence of the Taylor property in TAR$\mbox{ARCH}$ models, concluding that this property is satisfied when $n = 1$, for some first-order models. Generalizing these papers, Haas (\cite{2}) proposed a methodology for identifying the Taylor property in AVG$\mbox{ARCH}(1, 1)$ models at all lags.

Bilinear processes have also been proven to be suitable in financial and physical time series modeling, namely those presenting the Taylor effect. Therefore, it is obviously advisable to analyze the presence of the Taylor property in these processes. In this paper we consider the simple bilinear diagonal model

\begin{equation}
X_t = \beta X_{t-k} \varepsilon_{t-k} + \varepsilon_t, \quad k > 0,
\end{equation}

where $\beta$ is a real parameter and $(\varepsilon_t, t \in \mathbb{Z})$ an error process. We state sufficient conditions for the strict and weak stationarity of the processes $X = (X_t, t \in \mathbb{Z})$ and $X^2 = (X^2_t, t \in \mathbb{Z})$, and we derive expressions for the moments of $X$ up to the 4th order.

When dealing with bilinear models it is common to assume that $\varepsilon_t$, $t \in \mathbb{Z}$, are normally distributed. However, there has been considerable interest in non-negative time series models. For instance, Pereira and Scotto (\cite{5}) studied some properties of the simple first-order bilinear diagonal model $(k = 1)$ driven by exponentially distributed innovations.

In this paper, we analyze the presence of the Taylor property when $n = 1$ in the non-negative first-order bilinear time series model considering several distributions for the error process, which are chosen according to the kurtosis value as we have observed that the Taylor property is related with the value of this parameter.

Based on a simulation study, we also analyze the presence of the Taylor property in the class of real valued first-order bilinear diagonal models with symmetrical innovations.

\section{Stationarity of $X$ and $X^2$}

In this section we consider the simple bilinear model defined by (1) where $(\varepsilon_t, t \in \mathbb{Z})$ is a sequence of i.i.d. random variables. Let $\mu_i = E(\varepsilon_t^i)$, $i \in \mathbb{N}$.

**Proposition 1** Suppose that $\mu_4$ and $E(\ln |\varepsilon_t|)$ exist. If $\beta^2 \mu_2 < 1$ then the process $X$ is strictly and weakly stationary.
Proof. To prove the strict stationarity of process $X$, we start by proving that $X_t = Y_t$, a.s., with

$$Y_t = \varepsilon_t + \sum_{n=1}^{+\infty} T_n,$$

where, for each $n \in \mathbb{N}$, $T_n = T_n(t)$ is given by

$$T_n = \beta^n \varepsilon_{t-nk} \prod_{j=1}^{n} \varepsilon_{t-jk}.$$

Let us begin by verifying that the series $\sum_{n=1}^{+\infty} T_n$ is a.s. convergent. Using the ergodic theorem, we can assure that the limit $\lim_{n \to +\infty} \frac{1}{n} \ln |\beta^n \prod_{j=1}^{n} \varepsilon_{t-jk}|$ exists and that $\lim_{n \to +\infty} \frac{1}{n} \ln |\beta^n \prod_{j=1}^{n} \varepsilon_{t-jk}| = \ln |\beta| + E(\ln |\varepsilon_t|)$.

We can observe that $\lim_{n \to +\infty} \frac{1}{n} \ln |T_n| = \lim_{n \to +\infty} \frac{1}{n} \ln |\beta^n \prod_{j=1}^{n} \varepsilon_{t-jk}| + \frac{1}{n} \ln |\varepsilon_{t-nk}|$. Since $\lim_{n \to +\infty} \frac{1}{n} \ln |\varepsilon_{t-nk}| = 0$, we have

$$\lim_{n \to +\infty} \frac{1}{n} \ln |T_n| = \ln |\beta| + E(\ln |\varepsilon_t|).$$

On the other hand, the condition $\beta^2 \mu_2 < 1$ implies $2 \ln |\beta| < -\ln E(\varepsilon_t^2)$. Applying Jensen’s inequality to the random variable $\varepsilon_t^2$ and taking into account that $E(\ln |\varepsilon_t|) < +\infty$, we obtain $\gamma = \ln |\beta| + E(\ln |\varepsilon_t|) < 0$.

Consequently

$$\lim_{n \to +\infty} (a.s.) |T_n(t)|^{1/n} = \exp \gamma < 1,$$

which implies that the series $\sum_{n=1}^{+\infty} T_n$ is a.s. convergent; so $(Y_t, t \in \mathbb{Z})$ is a strictly stationary process, as it is a measurable function of the independent random variables $\varepsilon_s$, $s \leq t$. Moreover, it is easy to verify that the process $(Y_t, t \in \mathbb{Z})$ satisfies Equation (1).

This solution is the unique strictly stationary solution of (1). In fact, using (1) recursively, we obtain

$$X_t = \varepsilon_t + \sum_{i=1}^{n} T_i + \beta^{n+1} X_{t-(n+1)k} \prod_{j=0}^{n} \varepsilon_{t-(j+1)k}, \quad n = 0, 1, \ldots$$

with $\sum_{n=1}^{+\infty} T_n = 0$, for each $t \in \mathbb{Z}$, and taking limits, any strictly stationary solution of (1) satisfies

$$X_t = Y_t + \lim_{n \to +\infty} (a.s.) \beta^{n+1} X_{t-(n+1)k} \prod_{j=0}^{n} \varepsilon_{t-(j+1)k}.$$
Let $Z_n(t) = \beta^n X_{t-nk} \prod_{j=0}^{n-1} \epsilon_{t-(j+1)k}$. It is easy to verify that
\[
\lim_{n \to +\infty} (a.s.) \frac{1}{n} \ln |Z_n(t)| = \gamma < 0.
\]
Then
\[
\lim_{n \to +\infty} (a.s.) |Z_n(t)| = \lim_{n \to +\infty} (a.s.) \exp \left[ n \left( \frac{1}{n} \ln |Z_n(t)| \right) \right] = 0,
\]
which implies $\lim_{n \to +\infty} (a.s.) Z_n(t) = 0$. So, $(X_t, t \in \mathbb{Z})$ is strictly stationary, as $X_t = Y_t$, a.s.

To prove the weak stationarity, we now verify that $E(Y_t^2) < +\infty$. We have
\[
E(Y_t^2) = E \left( \epsilon_t + \sum_{i=1}^{+\infty} T_i \right)^2
\leq E(\epsilon_t^2) + 2 \sum_{i=1}^{+\infty} E(|\epsilon_t||T_i|) + \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} E(|T_i T_j|).
\]
(2)

Under the given conditions, each series in (2) is convergent. In fact, let us consider, for example, the series $\sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} E(|T_i T_j|)$.

For each $i, j \in \mathbb{N}$, we have
\[
E(|T_i T_j|) \leq |\beta|^{i+j} \left[ E \left( \epsilon_{t-ik}^4 \epsilon_{t-k}^2 \epsilon_{t-2k}^2 \cdots \epsilon_{t-(i-1)k}^2 \right) \right]^{1/2} \left[ E \left( \epsilon_{t-jk}^4 \epsilon_{t-k}^2 \epsilon_{t-2k}^2 \cdots \epsilon_{t-(j-1)k}^2 \right) \right]^{1/2} = \mu_4 \mu_2^{-1} \left[ (\beta^2 \mu_2)^{1/2} \right]^{i+j},
\]
using Schwarz’s inequality and the independence of the r.v.’s $\epsilon_t$, $t \in \mathbb{Z}$. As $(\beta^2 \mu_2)^{1/2} < 1$, the series is convergent.

Taking into account the equality $X_t = Y_t$, a.s., and the strict stationarity of the process $X$, we conclude that $E(X_t^2)$ exists and that $X$ is weakly stationary.

**Proposition 2** Suppose that $E(\ln |\epsilon_t|)$ and $\mu_8$ exist. If $\beta^4 \mu_4 < 1$ then the process $X^2$ is strictly and weakly stationary.

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**Proof.** The condition $\beta^4 \mu_4 < 1$ implies $\beta^2 \mu_2 < 1$, using Schwarz’s inequality, which implies the strict stationarity of $X$ and, consequently, of $X^2$. The proof of the weak stationarity of $X^2$ is analogous to the previous one. We have

$$E(Y_i^4) \leq E(\varepsilon_i^4) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} E(|T_i T_j T_p T_q|) + 4 \sum_{i=1}^{\infty} E(\varepsilon_i^4 | T_i)$$

$$+ 4 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{p=1}^{\infty} E(|\varepsilon_i^4 | T_j T_p) + 6 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} E(\varepsilon_i^2 | T_j).$$

Let us consider, for example, the series $\sum\sum\sum\sum E(|T_i T_j T_p T_q|)$, which is a sum of series of the types

(i) $\sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=p+1}^{\infty} E(|T_i T_j T_p T_q|)$

(ii) $\sum_{i=1}^{\infty} \sum_{p=1}^{\infty} E(T_i^2 T_p^2)$

(iii) $\sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \sum_{p=1}^{\infty} E(|T_i T_j| T_p^2)$.

Concerning (i), as $j > i$ and $q > p$, we have

$$E(|T_i T_j T_p T_q|) = E([|T_i T_j|] | T_p T_q)$$

$$\leq \left[ E(\varepsilon_i^4 \varepsilon_j^4 \varepsilon_{i-l-k}^4 \varepsilon_{l-l-(i-1)k}^4 \varepsilon_{l-j-l-ik}^2 \varepsilon_{l-l-(j-1)k}^2)\right]^{1/2}$$

$$\left[ E(\varepsilon_{l-pk}^4 \varepsilon_{l-1-p-1-k}^4 \varepsilon_{l-1-(p-1)k}^2 \varepsilon_{l-qk}^2 \varepsilon_{l-1-(q-1)k}^2)\right]^{1/2},$$

using Schwarz’s inequality.

Taking into account the independence of the random variables $\varepsilon_t$, we have, for $i, j \in \mathbb{N}$, $j > i$,

$$E(\varepsilon_{i-l-ik}^4 \varepsilon_{i-l-jk}^4 \varepsilon_{l-l-(i-1)k}^2 \varepsilon_{l-j-l-ik}^2 \varepsilon_{l-l-(j-1)k}^2) = \mu_1^{i+1} \mu_2^{j-i+1}.$$
\[
\leq \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=p+1}^{\infty} |\beta|^i + j + p + q \left( \mu_4^{i+p+2} \mu_2^{j-i+q-p+2} \right)^{1/2} \\
= \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \sum_{p=1}^{\infty} \sum_{q=p+1}^{\infty} \mu_2 \mu_4 \left[ \left( \beta^4 \mu_4 \right)^{1/2} \right]^{i+p} \left[ \left( \beta^2 \mu_2 \right)^{1/2} \right]^{(j+q)-(i+p)}.
\]

As \((\beta^4 \mu_4)^{1/2} < 1\) and \((\beta^2 \mu_2)^{1/2} < 1\), the series in (i) is convergent. The convergence of the series (ii) and (iii) is proved in a similar way. Then we conclude that \(E(X_t^4) < +\infty, t \in \mathbb{Z}\). As the process \(X^2\) is strictly stationary and \(E(X_t^4)\) exists, then it is weakly stationary.

### 3 Moments up to the 4th order

Under the same conditions of Section 2, we now evaluate the moments up to the 4th order of the process \(X_t\) given by (1) where \((\varepsilon_t, t \in \mathbb{Z})\) is a sequence of i.i.d. random variables, and \(\mu_i = E(\varepsilon_t^i), i \in \mathbb{N}\).

**Proposition 3** If \(\beta^4 \mu_4 < 1\) and \(\mu_8\) exists then the \(n\)th moment of \(X_t\), \(n \leq 4\), can be expressed as

\[
E(X_t^n) = \sum_{i=0}^{n} \binom{n}{i} \beta^{n-i} \mu_i E(X_t^{n-i} \varepsilon_t^{n-i}),
\]

where

\[
E(X_t^n \varepsilon_t^n) = \frac{1}{1 - \beta^n \mu_n} \sum_{i=1}^{n} \binom{n}{i} \beta^{n-i} \mu_{n+i} E(X_t^{n-i} \varepsilon_t^{n-i}), \quad n \leq 4.
\]

**Proof.** For \(n \leq 4\), we have

\[
E(X_t^n) = \sum_{i=0}^{n} \binom{n}{i} \beta^{n-i} E\left[ \varepsilon_t^i (X_{t-k} \varepsilon_{t-k})^{n-i} \right]
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \beta^{n-i} \mu_i E(X_t^{n-i} \varepsilon_t^{n-i}),
\]

since the process \((X_t \varepsilon_t, t \in \mathbb{Z})\) is strictly stationary due to the fact that \(X_t \varepsilon_t\) is a measurable function of \(\varepsilon_t, \varepsilon_{t-1}, \ldots\). Now we need to evaluate \(E(X_t^n \varepsilon_t^n)\),
$n \leq 4.$

\[
E(X_t^n \varepsilon_t^n) = \sum_{i=0}^{n} \binom{n}{i} \beta^{n-i} E \left[ \varepsilon_t^i (X_{t-k} \varepsilon_{t-k})^{n-i} \varepsilon_t^n \right]
\]

\[
= \sum_{i=0}^{n} \binom{n}{i} \beta^{n-i} E \left( \varepsilon_t^{n+i} \right) E \left( X_t^{n-i} \varepsilon_t^{n-i} \right)
\]

\[
= \beta^n \mu_n E(X_t^n \varepsilon_t^n) + \sum_{i=1}^{n} \binom{n}{i} \beta^{n-i} \mu_{n+i} E(X_t^{n-i} \varepsilon_t^{n-i}).
\]

Then

\[
E(X_t^n \varepsilon_t^n) = \frac{1}{1 - \beta^n \mu_n} \sum_{i=1}^{n} \binom{n}{i} \beta^{n-i} \mu_{n+i} E(X_t^{n-i} \varepsilon_t^{n-i}).
\]

It is easy to verify that $E(X_t \varepsilon_t) = \mu_2/(1 - \beta \mu_1)$. Recursively, we obtain $E(X_t^n \varepsilon_t^n), n = 1, 2, 3,$ and, finally, we achieve $E(X_t^n), n \leq 4$.

We note that $\beta^4 \mu_4 < 1$ implies $|\beta^n \mu_n| < 1$, $n = 1, 2, 3$, using Schwarz’s inequality.

4 The Taylor property in first-order non-negative bilinear models

In this section we consider the first-order non-negative bilinear model

\[(3) \quad X_t = \beta X_{t-1} \varepsilon_{t-1} + \varepsilon_t, \quad t \in \mathbb{Z}, \]

where $\beta > 0$ and $(\varepsilon_t, t \in \mathbb{Z})$ is a sequence of non-negative i.i.d. random variables.

We assume that $E(\ln \varepsilon_t)$ and $\mu_8$ exist and that $\beta^4 \mu_4 < 1$ in order to guarantee that both processes, $X$ and $X^2$, are strictly and weakly stationary.

In this context, the Taylor property for $n = 1$ establishes that $\rho_X(1) > \rho_{X^2}(1)$, where $\rho_X(1)$ and $\rho_{X^2}(1)$ denote, respectively, the autocorrelations of lag 1 of the processes $X$ and $X^2$. In order to obtain these autocorrelations, it is enough to evaluate $E(X_t X_{t-1})$ and $E(X_t^2 X_{t-1}^2)$ since we derived $E(X_t^i)$, $i = 1, 2, 3, 4,$ in the previous section. Using (3) and the stationarity of the involved processes, we have

\[
E(X_t X_{t-1}) = \beta E(X_t^2 \varepsilon_t) + E(X_{t-1} \varepsilon_t)
\]

\[
= \beta E(\beta^2 X_{t-1}^2 \varepsilon_{t-1}^2 \varepsilon_t + 2 \beta X_{t-1} \varepsilon_{t-1} \varepsilon_t^2 + \varepsilon_t^3) + E(X_{t-1} \varepsilon_t).
\]
Taking into account the independence of the random variables $\varepsilon_t$, \( t \in \mathbb{Z} \), and the strict stationarity of the related processes, we have \( E(X_t^2) = \mu_1 E(X_t^2) \) and \( E(X_{t-1}\varepsilon_{t-1}^2) = \mu_2 E(X_t\varepsilon_t) \). Then

\[
E(X_tX_{t-1}) = \beta^3 \mu_1 E(X_t^2\varepsilon_t^2) + 2\beta^2 \mu_2 E(X_t\varepsilon_t) + \mu_1 E(X_t) + \beta \mu_3.
\]

Using an analogous procedure, we obtain

\[
E(X_t^2X_{t-1}^2) = \beta^4 E_1 + 2\beta^3 E_2 + 2\beta^2 \mu_1 E_3 + 4\beta^2 \mu_1 E_4 + \beta^2 E_5 + 2\beta \mu_1 E_6 + \beta^2 \mu_2 E(X_t^2 \varepsilon_t^2) + 2 \beta \mu_1 \mu_2 E(X_t \varepsilon_t) + \mu_2^2,
\]

where

\[
E_1 = E(X_t^2X_{t-1}^2) = \beta^2 \mu_2 E(X_t^4\varepsilon_t^2) + 2\beta \mu_3 E(X_t^3\varepsilon_t^3) + \mu_4 E(X_t^2\varepsilon_t^2)
\]

\[
E_2 = E(X_t^2X_{t-1}^3\varepsilon_{t-1}) = \beta \mu_3 E(X_t^4\varepsilon_t^3) + 2\beta \mu_4 E(X_t^2\varepsilon_t^2) + \mu_5 E(X_t \varepsilon_t)
\]

\[
E_3 = E(X_tX_{t-1}^3 \varepsilon_{t-1}^1) = \beta \mu_1 E(X_t^2\varepsilon_t^3) + \mu_2 E(X_t^2\varepsilon_t^2)
\]

\[
E_4 = E(X_t^2X_{t-1}^2 \varepsilon_{t-1}) = \beta \mu_2 E(X_t^2\varepsilon_t^2) + \mu_3 E(X_t \varepsilon_t)
\]

\[
E_5 = E(X_t^2 \varepsilon_t^4) = \beta \mu_4 E(X_t^2\varepsilon_t^2) + 2\beta \mu_5 E(X_t \varepsilon_t) + \mu_6
\]

\[
E_6 = E(X_t \varepsilon_t^3) = \beta \mu_3 E(X_t \varepsilon_t) + \mu_4.
\]

Finally, the results of the previous section allow us to obtain the values of \( E(X_tX_{t-1}) \) and \( E(X_t^2X_{t-1}^2) \) in terms of the moments of \( \varepsilon_t \).

In the following, we investigate the presence of the Taylor property in Model (3), considering some non-negative distributions for the error process, namely, the uniform distribution in \( [0, \alpha] \), the exponential distribution in \( [0, +\infty] \) with mean \( \alpha \), and the Pareto distribution with density

\[
f(x) = \frac{\nu \alpha^\nu}{x^{\nu+1}} 1_{[0, \infty]}(x), \text{ for } \nu = 12 \text{ and } \nu = 9.
\]

In all cases, \( \alpha \) is a non-negative parameter and the condition \( E(\|\ln \varepsilon_t\|) < +\infty \) is satisfied.

The choice of these distributions takes into account the fact that the Taylor property seems to be related with the kurtosis value of the process. In this sense, we choose four distributions with significantly different behavior as regards their tails. We point out that the uniform and exponential distributions have constant kurtosis values, while the kurtosis of the Pareto distribution depends on the value of the parameter \( \nu \). Consequently, valid comparisons may be made separately between the first two distributions, uniform and exponential, and then between the two referred Pareto distributions.

We also point out that, in all cases, the condition \( \beta^4 \mu_4 < 1 \) and the values of \( \rho_X(1) \) and \( \rho_X(2) \) can be written in terms of \( r = \alpha \beta \).
In each case, we also present the value of the kurtosis of the process $X$ given by (4.1), which also depends on $r = \alpha \beta$, as well as the corresponding graphic representation as a function of $r$.

**Error process with uniform distribution in $]0, \alpha[**

In this case, the condition $\beta^4 \mu_4 < 1$ is equivalent to $0 < r < \sqrt[4]{5} \simeq 1.495$ and we obtain

\[
\begin{align*}
\rho_X(1) &= \frac{r(-180 + 120r - 51r^2 - 4r^3 + r^4)}{-180 + 180r - 177r^2 + 12r^3 + 7r^4}, \\
\rho_{X^2}(1) &= \frac{r \cdot N_u(r)}{12 D_u(r)},
\end{align*}
\]

with

\[
\begin{align*}
N_u(r) &= -604800 - 480600r - 155700r^2 - 257400r^3 - 2490r^4 + 48525r^5 \\
&\quad - 6270r^6 + 6810r^7 + 10620r^8 + 11384r^9 + 4012r^{10} - 586r^{11} \\
&\quad + 94r^{12} - 53r^{13} + 6r^{14}, \\
D_u(r) &= 50400 + 12600r + 35700r^2 + 40200r^3 + 13490r^4 + 14015r^5 + 8360r^6 \\
&\quad - 5210r^7 - 5999r^8 - 2407r^9 - 720r^{10} + 114r^{11} + 177r^{12} - 8r^{13}.
\end{align*}
\]

From Figure 1(a), we can see that the Taylor property is present for values of $r$ in the interval $]1.1868987, \sqrt[4]{5}[$. So, for a fixed $\alpha$, the Taylor property is achieved for parameterizations of Model (3) such that

\[
\beta \in \left[\frac{1.1868987}{\alpha}, \frac{\sqrt[4]{5}}{\alpha}\right],
\]

Figure 1: Graphs from $\rho_X(1) - \rho_{X^2}(1)$ (a) and $K_u(r)$ (b), with $0 < r < \sqrt[4]{5}$.
where the value 1.1868987 was obtained with an approximation error inferior to $5 \times 10^{-9}$.

For Model (3) with such an error process, the kurtosis is given by

$$K_u(r) = \frac{-3(-3 + r^2)}{7(-4 + r^3)(-5 + r^4)} N_u^*(r) - 3,$$

where

$$N_u^*(r) = 907200 - 1814400r + 4284000r^2 - 4510800r^3 + 3254460r^4 - 2030520r^5 + 1973540r^6 - 185700r^7 + 371005r^8 - 236308r^9 + 78747r^{10} - 11496r^{11} - 2030520r^{12} + 78747r^{13}.$$  

$$D_u^*(r) = (-180 + 180r - 177r^2 + 12r^3 + 7r^4)^2.$$

From Figure 1(b), we observe that the kurtosis of this model is an increasing function of $r$ and, for large values of the kurtosis, the Taylor property occurs.

Error process with exponential distribution with mean $\alpha$ (in $[0, +\infty]$)

The condition $\beta^4 \mu_4 < 1$ is now equivalent to $0 < r < \frac{1}{\sqrt{24}} \approx 0.4518$. In this case,

$$\rho_X(1) = \frac{2r(2 - 3r + 7r^2 - 6r^3 + 2r^4)}{1 - 2r + 19r^2 - 20r^3 + 6r^4}$$

$$\rho_{X^2}(1) = 2r \frac{N_e(r)}{D_e(r)},$$

with

$$N_e(r) = -5 - 80r + 65r^2 - 112r^3 - 1184r^4 - 5774r^5 + 10848r^6 + 12720r^7 - 9408r^8 - 17880r^9 - 16272r^{10} + 52992r^{11} + 9216r^{12} - 46656r^{13} + 17280r^{14}$$

$$D_e(r) = -5 + 2r - 21r^2 - 602r^3 - 9060r^4 + 11126r^5 + 13252r^6 - 26448r^7 + 16368r^8 + 13896r^9 - 12192r^{10} + 13824r^{11} - 12672r^{12} + 4032r^{13}.$$  

So, when the errors are exponentially distributed with mean $\alpha$, Model (3) presents the Taylor property for parameterizations such that

$$\beta \in \left[0, \frac{0.0695566}{\alpha} \cup \frac{0.1437879}{\alpha} \cup \frac{1}{\sqrt{24} \alpha} \right],$$

where the values 0.0695566 and 0.1437879 were obtained with an approximation error inferior to $5 \times 10^{-8}$. This conclusion is illustrated in Figure 2(a).
In Figure 2(b), we have the graphic representation of the kurtosis of model (3) with exponential errors, which is given by

$$K_e(r) = \frac{-3(-1 + 2r^2)}{(-1 + 6r^3)(-1 + 24r^4)} \frac{N_e^*(r)}{D_e^*(r)} - 3,$$

where

$$N_e^*(r) = 3 - 12r + 52r^2 - 134r^3 + 11815r^4 - 36752r^5 + 44802r^6 + 1062r^7 - 42648r^8 + 17028r^9 + 12240r^{10} + 5616r^{11} - 17280r^{12} + 6048r^{13},$$

$$D_e^*(r) = (1 - 2r + 19r^2 - 20r^3 + 6r^4)^2.$$
with

\[
N_{p12}(r) = -7043652000 - 5638479000r - 1900482000r^2 - 6228372150r^3 \\
-306469280r^4 + 2622844140r^5 + 24533447400r^6 \\
+19854650865r^7 + 11360213480r^8 - 1634041620r^9 \\
-3023582480r^{10} + 23037530976r^{11} + 7650162960r^{12} \\
-1121587456r^{13} + 2802615552r^{14}
\]

\[
D_{p12}(r) = -58697100 + 14229600r - 142425360r^2 - 468153840r^3 \\
-218936564r^4 + 536116224r^5 + 616017864r^6 \\
+374454192r^7 + 130906149r^8 - 805701976r^9 \\
-15605040r^{10} + 401099652r^{11} \\
-245871648r^{12} + 48736320r^{13}.
\]

As can be seen in Figure 3(a), the Taylor property is now achieved for all considered parameterizations of Model (3).

Concerning the kurtosis of this model, it is given by

\[
K_{p12}(r) = \frac{-2(-5 + 6r^2)}{49(-3 + 4r^3)(-2 + 3r^4)} \frac{N_{p12}^*(r)}{D_{p12}^*(r)} - 3,
\]

where

\[
N_{p12}^*(r) = 599933276250 - 261789066000r + 4970166270300r^2 \\
-5546727078200r^3 + 59041720498845r^4 - 161234870633760r^5 \\
+126074334149694r^6 + 2238307939140r^7 + 25296348317400r^8 \\
-57875913071352r^9 - 89078826937116r^{10} + 180941306693040r^{11}
\]

Figure 3: Graphs from $\rho_X(1) - \rho_X^2(1)$ (a) and $K_{p12}(r)$ (b), with $0 < r < \frac{4\sqrt{2}}{3}$. 
\[
D_{p12}(r) = (36300 - 79200r + 219255r^2 - 171160r^3 + 29472r^4)^2.
\]

Error process with Pareto density \( f(x) = \frac{9\alpha^9 x^{10 - 11\alpha}}{\Gamma(11\alpha)} I_{[0, +\infty]}(x) \)

We have
\[
\beta^4 \mu_4 < 1 \iff 0 < r < \frac{\sqrt{5}}{9} \approx 0.863 \text{ and}
\]
\[
\rho_X(1) = \frac{8r(15680 - 27720r + 39564r^2 - 27864r^3 + 6561r^4)}{47040 - 105840r + 343119r^2 - 315504r^3 + 73791r^4}
\]
\[
\rho_{X^2}(1) = \frac{r N_{p9}(r)}{48 D_{p9}(r)},
\]

with
\[
N_{p9}(r) = -67737600 - 83339200r + 19038600r^2 - 88401600r^3
-148138920r^4 - 511287075r^5 + 1466330040r^6 + 1499354145r^7
-1537629480r^8 - 1966005837r^9 - 602608896r^{10}
+3869347563r^{11} - 61620912r^{12} - 2818841796r^{13} + 1179090432r^{14}
\]
\[
D_{p9}(r) = -627200 + 235200r - 1650600r^2 - 8601600r^3 - 13809280r^4
+31729095r^5 + 27010080r^6 - 23002305r^7 - 21773448r^8
-24182469r^9 + 58517640r^{10} + 9248823r^{11}
-50143536r^{12} + 19665504r^{13}.
\]

The Taylor property is also present for all considered parameterizations of Model (3), as it is illustrated in Figure 4(a), and we point out that the magnitude of the difference \( \rho_X(1) - \rho_{X^2}(1) \) is greater in this case than in the case \( \nu = 12 \).

The kurtosis of Model (3) is now given by
\[
K_{p9}(r) = \frac{7 - 9r^2}{9(-2 + 3r^2)(-5 + 9r^4)} \frac{N_{p9}^*(r)}{D_{p9}^*(r)} - 3,
\]

where
\[
N_{p9}^*(r) = 62449049600 - 281020723200r + 532657440000r^2 - 582241598400r^3
+25718506014670r^4 - 92872063045440r^5 + 100396353649230r^6
-63377116367257r^7 - 8536591340550r^8 - 41782534519365r^9
-62336742758694r^{10} + 195729014255481r^{11}
-145385404543008r^{12} + 35664808109193r^{13}
\]
\[
D_{p9}^*(r) = (15680 - 35280r + 114373r^2 - 105168r^3 + 24597r^4)^2.
\]
We observe that the kurtosis of the process $X$ is greater when $\nu = 9$ than when $\nu = 12$, corresponding to an analogous relation between the kurtosis of the respective error processes. In these two examples, it is seen again how the Taylor property emerges when the process $X$ is leptokurtic.

As regards the Pareto distribution, graphic representations for several values of $\nu$ suggest that the difference $\rho_X(1) - \rho_{X^2}(1)$ tends to zero as $\nu$ tends to infinity (corresponding to decreasing values of the kurtosis of the Pareto distribution). This situation is illustrated in Figure 5 and strongly contributes to conjecture that the Taylor property and leptokurtosis are highly related in time series.
5 The Taylor property in the case of symmetrically distributed errors: simulation study

When the errors are symmetrically distributed, the autocorrelation function of \( X^2 \) for model (1) verifies \( \rho_{X^2}(1) = 0 \), if \( k > 1 \) (Martins, [6]). So, in this case, the property \( \rho_{|X|}(1) > \rho_{X^2}(1) \) is equivalent to \( \rho_{|X|}(1) > 0 \). However, the autocorrelation function of the process \((|X_t|, t \in \mathbb{Z})\) is not available when the error process is allowed to assume negative values. To investigate the presence of the Taylor property in Model (3) with symmetrically distributed errors, we perform a simulation study considering the simple first-order bilinear diagonal model with an i.i.d. error process \((\varepsilon_t, t \in \mathbb{Z})\) with four symmetrical distributions with unit variance, namely, the uniform distribution in \([-\sqrt{3}, \sqrt{3}]\), the standard normal distribution, and the distribution of a variable \( \varepsilon = \sqrt{\frac{\nu-2}{\nu}} Y \), where \( Y \) has a Student distribution with \( \nu \) degrees of freedom (\( \nu = 30 \) and \( \nu = 9 \)). In each case, the condition \( E(\ln|\varepsilon_t|) < +\infty \) is satisfied and parameterizations that satisfy \( \beta^4 \mu_4 < 1 \) are considered in the simulations. For each value of the parameter \( \beta \) and each one of the considered distributions, we generate 500 observations according to the corresponding model and obtain the 95% confidence intervals for the probability that such a model satisfies the Taylor property. The results appear in Table II (where NA means “Not Applicable”, due to the fact that the corresponding value of \( \beta \) does not satisfy the condition \( \beta^4 \mu_4 < 1 \)). The special values 0.69, 0.74, 0.75 and 0.863 are the greatest values of \( \beta \) such that \( \beta^4 \mu_4 < 1 \) for each one of the considered distributions.

We can observe that the Taylor property seems to be present for high values of \( \beta \) and that this presence increases with the kurtosis of the error process, as we have established and observed in non-negative bilinear models.

The confidence intervals corresponding to small values of \( \beta \) do not allow us to infer about the presence of the Taylor property, as they certainly correspond to values of \( \beta \) for which the difference \( \rho_X(1) - \rho_{X^2}(1) \) is close to zero.
Table 1: 95% confidence intervals for the probability that the model with symmetrical innovations presents the Taylor property.

| $\beta$ | $U \left( \sqrt{3}, \sqrt{3} \right)$ | $N(0,1)$ | $Y = \frac{1}{14} Y, Y \sim T(30)$ | $Y = \frac{1}{7} Y, Y \sim T(9)$ |
|---------|-----------------------------------|-----------|-----------------------------------|-----------------------------------|
| 0.01    | [0.373,0.627]                      | [0.459,0.708] | [0.459,0.708]                     | [0.476,0.724]                     |
| 0.05    | [0.357,0.610]                      | [0.373,0.627] | [0.373,0.627]                     | [0.407,0.660]                     |
| 0.1     | [0.140,0.360]                      | [0.292,0.541] | [0.214,0.453]                     | [0.260,0.506]                     |
| 0.2     | [0,0]                             | [0.0,105]  | [0.0,049]                         | [0.0,049]                         |
| 0.3     | [0,0]                             | [0,0]      | [0,0]                             | [0.0,079]                         |
| 0.4     | [0,0]                             | [0,0]      | [0,0]                             | [0,0]                             |
| 0.5     | [0,0]                             | [0.155,0.379] | [0.292,0.541]                     | [0.699,0.901]                     |
| 0.6     | [0,0]                             | [0.566,0.801] | [0.603,0.831]                     | [0.781,0.953]                     |
| 0.69    | [0,0]                             | [0.802,0.965] | [0.802,0.965]                     | [0.951,1]                         |
| 0.74    | [0,0.079]                         | [0.847,0.987] | [0.870,0.996]                     | NA                                |
| 0.75    | [0.004,0.130]                     | [0.847,0.987] | NA                                | NA                                |
| 0.863   | [0.566,0.801]                     | NA         | NA                                | NA                                |

6 Conclusions

The studies presented here show that bilinear models are able to reproduce the Taylor effect. They also reinforce the connection of the Taylor property to leptokurtic models which has been observed in the few theoretical studies developed until now. In fact, He and Teräsvirta (3), Gonçalves, Leite and Mendes-Lopes (1) and Haas (2) show the presence of this property in some conditional heteroskedastic models, which are leptokurtic processes. Moreover, all the cases considered in this paper, also show that, when the Taylor property occurs, the model is leptokurtic.

We still observe that leptokurtosis is not enough to induce the Taylor property. Examples of bilinear models that are leptokurtic but do not have the Taylor property are $X_t = X_{t-1} + \varepsilon_t$, where $\varepsilon_t$ is uniformly distributed in $[0,1]$, and $X_t = 0.5X_{t-1} + \varepsilon_t$, where $\varepsilon_t$ is exponentially distributed with mean 0.2. This is in line with the simulation results of He and Teräsvirta (3) suggesting that the Taylor property is not present for the standard GARCH(1,1) process with normal errors.

In conclusion, our study allows to conjecture that a general assessment of the Taylor property in the bilinear process is strongly dependent on its tails weight.
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