Maslov $S^1$ Bundles and Maslov Data

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Abstract

In this paper we consider the Maslov $S^1$ bundles of a symplectic manifold $(M, \omega)$, which refer to both the determinant bundle (denoted by $\Gamma_J$) of the unitary frame bundle and the bundle $\Gamma_J^2 = \Gamma_J/\{\pm 1\}$. The symplectic action of a compact Lie group $G$ on $M$ can be lifted to group actions on the principal $S^1$ bundles $\Gamma_J$ and $\Gamma_J^2$. In this work we study the interplay between the geometry of the Maslov $S^1$ bundles and the dynamics of the group action on $M$. We show that when $M$ is a homogeneous $G$-space and the first real Chern class $c_1$ is nonvanishing, $\Gamma_J$ and $\Gamma_J^2$ are also homogeneous $G$-spaces. We also show that when $[\omega] = r \cdot c_1$ for some real number $r$, then the $G$ action is Hamiltonian and the Hamiltonians assume particular forms. In the end, we study a function called the $\beta$-Maslov data of a symplectic $S^1$ action with respect to a connection 1-form $\beta$ on $\Gamma_J^2$, which serves as the nonintegrable version of the notion of Maslov indices when $\Gamma_J^2$ is not a trivial bundle.

1 Introduction

1.1 Background and motivation

The notion of Maslov indices arised from the study of semi-classical approximations in quantum mechanics [9] in the phase space $(\mathbb{R}^{2n}, dx_i \wedge dy_i)$. It plays an important role in the problem of quantization as well as gives insights into the geometric aspect of classical dynamics [6]. In [1] Arnold proved that the Maslov index of a closed path $\tilde{\gamma}$ of Lagrangian planes in $\mathbb{R}^{2n}$ can be characterized either as the (signed) number of intersections of $\tilde{\gamma}$ with the Maslov cycle $\mathcal{M}_0$ induced by the Lagrangian subspace $E_0 = \mathbb{R}^n \times \{0\}$, or, equivalently, as the degree...
of the Maslov-Arnold map for \( \tilde{\gamma} \):

\[
m_{\tilde{\gamma}} : S^1 \xrightarrow{\tilde{\gamma}} \Lambda(n) \cong \mathbb{U}(n)/\mathbb{O}(n) \xrightarrow{\det^2} S^1,
\]

where \( \Lambda(n) \) is the Lagrangian Grassmannian of \( \mathbb{R}^{2n} \). Let \( \frac{\partial}{\partial \theta} = \frac{d}{d\theta} e^{2\pi i \theta} \) be the standard vector field on \( S^1 \) and \( d\theta \) be its dual 1-form. When \( \tilde{\gamma} \) is smooth, \( m_{\tilde{\gamma}} \) can be calculated as

\[
m_{\tilde{\gamma}} = \int_{\tilde{\gamma}} \tilde{\eta} = \int_\gamma d\theta,
\]

where \( \tilde{\eta} \) is the pullback of the canonical 1-form \( d\theta \) on \( S^1 \) via the Maslov-Arnold map\([4]\), and \( \gamma = \det^2(\tilde{\gamma}) \) is a loop in \( S^1 \).

In the general case where \((M, \omega)\) is a symplectic manifold, the Lagrangian Grassmannian will be replaced by the bundle of Lagrangian planes \( \Lambda_{pl} \), which consists of all the Lagrangian subspaces of the tangent spaces of \( M \)\([4, 2]\). Namely:

\[
\Lambda_{pl} = \bigcup_{p \in M} \{ \text{Lagrangian subspaces of } T_p M \}.
\]

A Maslov cycle \( \mathcal{M} \) consists of all the Lagrangian spaces that have nontrivial intersection with a Lagrangian subbundle \( \mathcal{L} \), and the Maslov index of a loop in \( \Lambda_{pl} \) is then defined in the same spirit as the number of intersections with \( \mathcal{M} \).

At the same time, \( \Lambda_{pl} \) can also be viewed as a structure associated to an almost complex structure \( J \) compatible with \( \omega \). In the case \((M, \omega) = (\mathbb{R}^{2n}, dx_i \wedge dy_i)\) the bundle of Lagrangian planes is a trivial bundle

\[
\mathbb{R}^{2n} \times \Lambda(n) = \mathbb{R}^{2n} \times \mathbb{U}(n)/\mathbb{O}(n),
\]

where the last space can be interpreted as the quotient space of \( \mathbb{R}^{2n} \times \mathbb{U}(n) \) by the \( \mathbb{O}(n) \) action. \( \mathbb{R}^{2n} \times \mathbb{U}(n) \) is the unitary frame bundle consisting of all the unitary frames of the tangent spaces of \( \mathbb{R}^{2n} \) with respect to the canonical almost complex structure

\[
J_0 = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}
\]

on the tangent bundle \( T\mathbb{R}^n \). The \( S^1 \) appearing at the end of the sequence \((1.1)\) as the codomain of \( m_{\tilde{\gamma}} \) should
be thought of as the $S^1$ principal bundle

$$\mathbb{R}^{2n} \times \mathbb{U}(n)/\mathbb{E}\mathbb{U}(n) \cong \mathbb{R}^{2n} \times S^1,$$

where $\mathbb{E}\mathbb{U}(n)$ is the subgroup of $\mathbb{U}(n)$ consisting of all the matrices with complex determinants equal to 1 or $-1$. For an arbitrary symplectic manifold $(M, \omega)$, an almost complex structure $J$ compatible with the symplectic structure $\omega$ can always be chosen, and the corresponding unitary frame bundle (associated to $J$)

$$\pi_{Fr^u_J} : Fr^u_J \to M$$

is a $\mathbb{U}(n)$–principal bundle (with $\mathbb{U}(n)$ acting from the right in our setting). Therefore we can define the associated fiber bundle

$$\pi_{\Lambda_J} : \Lambda_J = Fr^u_J/\mathcal{O}(n) \to M$$

as well as the $S^1$-principal bundle

$$\pi_{\Gamma^2_J} : \Gamma^2_J = Fr^u_J/\mathbb{E}\mathbb{U}(n) \to M,$$

which are the counterparts of $\mathbb{R}^{2n} \times \Lambda(n)$ and $\mathbb{R}^{2n} \times \mathbb{U}(n)/\mathbb{E}\mathbb{U}(n)$, respectively, in the general setting.

The space $\Lambda_J = Fr^u_J/\mathcal{O}(n)$ is indeed in one-one correspondence to $\Lambda_{pl}$. This is because each fiber $\pi_{\Lambda_J}^{-1}(p)$ of $\Lambda_J$ with $p \in M$ is exactly the Lagrangian Grassmannian of the symplectic vector space $(T_p M, \omega_p)$ with respect to the compatible linear complex structure $J|_{T_p M}$. In the text we will use interchangeably the spaces $\Lambda_J$ and $\Lambda_{pl}$. From the set-theoretic point of view, the advantage of thinking about $\Lambda_{pl}$ is that for any symplectic flow $\varphi$ on $M$, the tangent maps $\varphi'_t$ induces a flow on $\Lambda_{pl}$ in a natural way. However, the advantage of considering the space $\Lambda_J$ is that its relation with $\Gamma^2_J$ is straightforward.

Since $\mathcal{O}(n) \subset \mathbb{E}\mathbb{U}(n)$ and they are both closed subgroups of $\mathbb{U}(n)$, each $\mathcal{O}(n)$ orbit in $Fr^u_J$ lies in a single $\mathbb{E}\mathbb{U}(n)$ orbit, and there is a natural quotient map $det^2_J$ from $Fr^u_J/\mathcal{O}(n)$ to $Fr^u_J/\mathbb{E}\mathbb{U}(n)$, sending each $\mathcal{O}(n)$ orbit to its containing $\mathbb{E}\mathbb{U}(n)$ orbit. Viewed in proper local trivializations of the bundles $\Lambda_J = Fr^u_J/\mathcal{O}(n) \to M$ and $\Gamma^2_J = Fr^u_J/\mathbb{E}\mathbb{U}(n) \to M$, such a quotient map takes the form

$$U \times \mathbb{U}(n)/\mathcal{O}(n) \ni (p, [A]) \mapsto (p, det^2_J A) \in U \times S^1$$

with respect to some local charts over some open set $U$ of $M$. Due to this reason, we denote this quotient map
by

$$det_J^2 : \Lambda_J \to \Gamma_J^2.$$  

and the following composition

$$det_J^2 \circ \tilde{\gamma} : S^1 \to \Gamma_J^2$$

is then the Malsov-Arnold map for a closed path $\tilde{\gamma}$ in $\Lambda_J$. When the bundle admits a trivialization $\Gamma_J^2 \cong M \times S^1$ with respect to some global section $s$, such a trivialization leads to the definition of an index. To be precise, with the isomorphism from $\Gamma_J^2$ to $M \times S^1$ being denoted by $\text{tr}_s$, an index $m_s(\tilde{\gamma})$ for a closed path $\tilde{\gamma}$ of Lagrangian planes can be defined as the degree of the map defined in the following composition:

$$S^1 \xrightarrow{\tilde{\gamma}} \Lambda_J \xrightarrow{det_J^2} \Gamma_J^2 \xrightarrow{tc_s} M \times S^1 \xrightarrow{pr_{S^1}} S^1.$$

(1.4)

Note that a globally defined Lagrangian vector subbundle gives a global section $s$ to $\Lambda_J$ (or $\Lambda_{pl}$), and then the composition $s = det_J^2 \circ \sigma$ is a global section to $\pi_{\Gamma_J^2}$, which gives a trivialization $\Gamma_J^2 \cong M \times S^1$. Also note that this is always the case where $M$ is the cotangent bundle of some manifold, and the Lagrangian subbundle can be chosen as the vertical distribution. In this case, the index $m_s(\tilde{\gamma})$ defined above is exactly the Maslov index (with respect to $s$) in the usual sense.

The discussion above shows why we become interested in the principal $S^1$ bundle $\Gamma_J^2$ and the other related structures, $Fr_J^\nu$ and $\Lambda_J$, over a symplectic manifold $(M, \omega)$. In some cases it would be more convenient to consider the principal $S^1$ bundle $\Gamma_J \cong Fr_J^\nu / \text{SU}(n)$, which is related to $\Gamma_J^2$ by the relation $\Gamma_J^2 \cong \Gamma_J / \{ \pm 1 \}$. In the following text, we call $\pi_{\Gamma_J} : \Gamma_J \to M$ (or simply $\Gamma_J$) and $\pi_{\Gamma_J^2} : \Gamma_J^2 \to M$ Maslov $S^1$ bundles. Note that although from the set-theoretic point of view, the definitions of $Fr_J^\nu$, $\Lambda_J$ and $\Gamma_J$ depend on the choice of $J$, their bundle structures are independent of the choice of the compatible almost complex structure.

1.2 Maslov data: a nonintegrable version of Maslov indices

Generally speaking, the principal $S^1$ bundles $\Gamma_J$ and $\Gamma_J^2$ are not necessarily trivial, and the mapping $[1]$ is not available (this is the case for $S^2$, as we will explain later). However, the integration $[2]$ can still be defined by treating $\tilde{\eta}$ as the pullback of any connection 1-form of $\Gamma_J^2$ (or alternatively $\Gamma_J$) on $\Lambda_J$. Following such a perspective, we note that a global section of $\Gamma_J^2$ is just an integral manifold of a flat connection (which is an integrable horizontal distribution) on $\Gamma_J^2$. To generalize the notion of Maslov indices for the case where $\Gamma_J^2$
is a nontrivial bundle, we can replace the “integrable horizontal distribution” with simply “a connection”. We introduce a nonintegrable version of Maslov indices for smooth loops in $\Lambda_{pl}$ (or $\Lambda_{J}$) with respect to an arbitrary connection 1-form $\beta = \eta \cdot \frac{\partial}{\partial \theta}$ on $\Gamma^2_{J}$.

**Definition 1.** For any smooth loop $\bar{\gamma} : [0, 1] \to \Lambda_{J}$, define its Maslov data with respect to $\beta$ to be

$$\tilde{m}_{\beta}(\bar{\gamma}) = \int_{\bar{\gamma}} \tilde{\eta}$$

with $\tilde{\eta}$ being the pullback of $\eta$ on $\Lambda_{J}$ via $det_{J}^{2}$.

Note that whenever $\bar{\gamma}$ is a smooth loop in $\Lambda_{J}$,

$$\gamma = det_{J}^{2} \circ \bar{\gamma}$$

is a smooth loop in $\Gamma^2_{J}$. Moreover, it holds

$$\int_{\gamma} \eta = \int_{\bar{\gamma}} \tilde{\eta} = \tilde{m}_{\beta}(\bar{\gamma}), \tag{1.5}$$

and hence it will be convenient to consider $\gamma$ instead of $\bar{\gamma}$ and think of $\tilde{m}_{\beta}(\bar{\gamma})$ as a quantity of $\gamma$. We will use the symbol $m_{\beta}(\gamma)$ for the integration $\int_{\gamma} \eta$ and call it the Maslov data of $\gamma$ with respect to $\beta$.

When the horizontal distribution $\ker \beta$ has an integral manifold $S_{0}$ being a one-sheet covering over $M$, Definition 1 gives the Maslov indices in the usual sense with respect to the Maslov cycle $M_{0} = (det_{J}^{2})^{-1}(S_{0})$ (or to the section $\mathfrak{s}$). Note that this is the case in [4] where a dynamical system is considered in a vicinity of a connected submanifold $\Sigma$ in $M$ on which (the restriction of) the bundle of Lagrangian planes $\Lambda_{pl}$ can be trivialized as $\Sigma \times \Lambda(n)$. Also note that in general, $\ker \beta$ being flat does not necessary mean that the integral manifolds are single-sheeted coverings. A discussion taking care of a nontrivial bundle $\Gamma^2_{J}$ with flat connections would however be beyond the scope this work, and therefore we will not go into details for this issue.

**Remark 2.** We avoid using the term *index* just for emphasizing that the Maslov data is not necessarily a topological quantity, even when $\bar{\gamma}$ is a loop.
1.3 Purposes, Main Results and Layout

This work is intended to be an exploration of the interplay between dynamics on $M$ and the geometry of $\Gamma_J$ (and/or $\Gamma^2_J$), as well as an investigation into the notion of Maslov data proposed in Definition 1 as a generalized/nonintegrable version of ordinary Maslov indices.

The dynamics that we are interested in on the manifolds will be symplectic group actions. Note that given the flow of an arbitrary symplectic vector field $X : \mathbb{R} \times M \to M$, the tangent map $\varphi^*_t$ maps each Lagrangian space to another. Namely, $\varphi_*$ defines a flow on $\Lambda_{pl}$ which covers $\varphi$. Since each trajectory $\tilde{\gamma}$ of $\varphi_*$ is mapped to a path $\gamma$ in $\Gamma^2_J$ by $\gamma = det^2_J \circ \tilde{\gamma}$, it is natural to expect that, under some conditions, the composition

$$det^2_J \circ \varphi_* : \mathbb{R} \times \Lambda_{pl} \to \Gamma^2_J$$

factors through the space $\mathbb{R} \times \Gamma^2_J$ and induces a flow $\varphi_{\Gamma^2}$ on $\Gamma^2_J$. Such a factorization indeed exists when the vector field $X$ is an infinitesimal generator of a symplectic group action by a compact Lie group $G$. Based on this reason, we focus our study on compact and connected Lie group actions.

In the following we introduce the main results of this work. The simplest case where the Maslov bundles are non-trivial is given by $M = S^2$. Note that $S^2$ is a symplectic homogeneous space with a transitive symplectic $SO(3)$ action. Also note that $H^1_{dR}(S^2) = 0$ and hence any symplectic vector field on $S^2$ is Hamiltonian. A discussion of the $SO(3)$ action on $S^2$ (Section 4) shows that the lifted $SO(3)$ actions on the Maslov $S^1$ bundles $\Gamma^1_{S^2}$ and $\Gamma^2_{S^2}$ are transitive (Proposition 10), and the Hamiltonians for the infinitesimal generators can be expressed in terms of an invariant connection 1-form of the bundle $\Gamma^1_{S^2} \to S^2$ (Proposition 12). These properties are then extended to more general settings. To be specific, we obtain the following result for symplectic homogeneous spaces:

**Theorem 3.** Let $G$ be a compact Lie group acting transitively and symplectically on $M$. If the first real Chern class of the Maslov $S^1$ bundle $\Gamma_J$ is nonvanishing, then $\Gamma_J$ is also a homogeneous $G$-space.

We moreover get the following theorem which extends the result obtained in [8, 7, 3] about symplectic circle actions being Hamiltonian on monotone manifolds:

**Theorem 4.** Let $(M, \omega)$ be a symplectic manifold, and let $\pi_\Gamma : \Gamma_J \to M$ be its Maslov $S^1$ bundle with the first real Chern class $c_1 \in H^2_{dR}(M)$. If it satisfies $[\omega] = r \cdot c_1$ for some real number $r$, then any symplectic action $\Phi$ on $M$ by a compact Lie group $G$ is Hamiltonian. More specifically, for any $v \in g$, there exists a Hamiltonian
$H_v$ of $X_v$ such that $\pi^*_\Gamma(H_v) = r \cdot f^\beta(X_v)$ for some real number $r \neq 0$, with $\beta = f^\beta \cdot \frac{\partial}{\partial \theta}$ being a connection 1-form invariant under the lifted $G$-action $\Phi_\Gamma$.

**Remark 5.** $X_v$ and $X_v$ in the statement of Theorem 4 above stand for the infinitesimal generators associated to $v$ of the action $\Phi$ on $M$ and the lifted action $\Phi_\Gamma$ on $\Gamma_J$, respectively.

We also discuss the Maslov data of symplectic circle actions. When $G = S^1$, the orbits of $\Phi_\Gamma$ in $\Gamma_J$ covering the same orbit of $\Phi$ in $M$ have the same Maslov data. In other words, for $w, w' \in \pi^{-1}(p)$ with $\gamma_w(z) = \Phi_\Gamma^z(w)$ and $\gamma_{w'}(z) = \Phi_\Gamma^z(w')$, $m_\beta(\gamma_w) = m_\beta(\gamma_{w'})$ for any connection 1-form $\beta$ on $\Gamma_J$. Therefore, it defines a smooth function $Q_\theta$ (which we call the $\beta$-Maslov data of $\Phi$) on $M$ with $Q_\theta(p) = m_\beta(\gamma_w)$. Although with a different connection $\beta$ the function $Q_\theta$ may be different, its values at the fixed points of the action $\Phi$ turn out to be independent of $\beta$. When $[\omega] = r \cdot e^{\theta}$, $Q_\theta$ (after scaling) is a Hamiltonian of the action $\Phi$.

This paper is organized as follows.

In Section 2 we discuss some basic properties of Maslov indices from the perspective from the geometry of $\Gamma^2_J$. In Section 3, we recall some basics about principal $S^1$ bundles which will be used in the later discussion. In Section 4, we discuss the lift of the $G$-action $\Phi$ and study the case of $M = S^2$, which serves as an example in which the Maslov $S^1$ bundle $\Gamma_{S^2}$ is non-trivial. We explain that the lifted $SO(3)$ action acts transitively on $\Gamma_{S^2}$ (Proposition 10) and show that the Hamiltonians of its infinitesimal generators can be written down in terms of a connection 1-form which is invariant under the lifted $G$-action on $SO(3) \cong \Gamma_{S^2}$ (Proposition 12). In Section 5, we extend Proposition 10 and 12 to Theorem 3 and 4. In Section 6, we define the $\beta$-Maslov data of a symplectic $S^1$ action and discuss its properties, especially its values at the fixed points of the action.

### 1.4 Notations

In this paper we always assume $G$ to be a compact Lie group acting symplectically on $(M, \omega)$. Denote the Lie algebra of $G$ by $\mathfrak{g}$. The action of $G$ on $M$ is denoted by $\Phi$. The lifted actions on $\Gamma_J$ and $\Gamma^2_J$ are denoted by $\Phi_\Gamma$ and $\Phi_{\Gamma^2}$, respectively. For $v \in \mathfrak{g}$, $X_v$ is the corresponding infinitesimal generator of $\Phi$ on $M$, $\chi_v$ stands for the generator on $\Gamma_J$ (or $\Gamma^2_J$), and $X_v$ is the generator of the action $\Phi_\Gamma$ on $\Lambda_{pl}$ which is induced naturally by the tangent maps.

We denote by $\frac{\partial}{\partial \theta}$ the vector field on $S^1$ defined by $\frac{d}{d\theta} e^{2\pi i \theta}$. Note that, on a principal $S^1$ bundle $P$, there is a structural/inherent $S^1$ action. We also use the symbol $\frac{\partial}{\partial \theta}$ to denote the corresponding infinitesimal generator of this $S^1$ action.
2 Basic properties of Maslov indices

2.1 Lagrangian subbundles and Maslov indices

Let \( \mathcal{N} \) be a submanifold in \( M \) and \( \mathcal{L}_\mathcal{N} \subset TM \) be a Lagrangian subbundle of \( TM \) over \( \mathcal{N} \). That is, \( \mathcal{L}_\mathcal{N}|_b \) is a Lagrangian subspace for any \( b \in \mathcal{N} \). It is straightforward to see that \( \mathcal{L}_\mathcal{N} \) always induces a smooth section \( \sigma_\mathcal{N} \) from \( \mathcal{N} \) to \( \Lambda^+_J \cong \Lambda_{J} \) which assigns to each \( b \in \mathcal{N} \) the Lagrangian plane \( \mathcal{L}_\mathcal{N}|_b \). Moreover, when \( \mathcal{L}_\mathcal{N} \) is orientable, it also induces a section \( \tilde{\sigma}_\mathcal{N} \) from \( \mathcal{N} \) to \( Fr^u_J/\mathcal{SO}(n) \). We give a brief account of the latter case.

When \( \mathcal{L}_\mathcal{N} \) is orientable, we can first fix an orientation. For any \( b \in \mathcal{N} \), there is a neighbourhood \( U_b \) in \( \mathcal{N} \) on which it admits an ordered local frame \( f^*_U_b = (e_1, ..., e_n) \) of \( \mathcal{L}_\mathcal{N} \) which fits the orientation. With some modification, it can be extended to a(n) (ordered) unitary frame on \( U_b \) for \( TM \) with \( s_i = J e_i \). Then

\[
\sigma_b : U_b \ni x \mapsto f^*_U_b(x) \in Fr^u_J
\]

is a local section for \( Fr^u_J \). By composing with the quotient map \( q_{\mathcal{SO}(n)} \) from \( Fr^u_J \) to \( Fr^u_J/\mathcal{SO}(n) \) we get a local section \( \tilde{\sigma}_{s,b} \) from \( U_b \) to \( Fr^u_J/\mathcal{SO}(n) \). Namely,

\[
\tilde{\sigma}_{s,b} = q_{\mathcal{SO}(n)} \circ \sigma_b.
\]

The collection \( \{ U_b | b \in \mathcal{N} \} \) then constitutes an open cover of \( \mathcal{N} \) over each element of which there is a (smooth) section \( \sigma_{\mathcal{N},b} \). It remains to check that when \( U_b \cap U_{b'} \neq \emptyset \), \( \tilde{\sigma}_{s,b} \) agrees with \( \tilde{\sigma}_{s,b'} \) on \( U_b \cap U_{b'} \). Since both \( f^*_U_b \) and \( f^*_{U_{b'}} = (e'_1, ..., e'_n) \) can be taken as orthonormal frames with respect to the Euclidean metric \( g_J \) that fit the same orientation, at each \( x \in U_b \cap U_{b'} \), they are related by a matrix \( C \) from \( \mathcal{SO}(n) \) via

\[
(e'_1, ..., e'_n) = (e_1, ..., e_n) \cdot C.
\]

For \( f_{U_{b'}} = (e'_1, ..., e'_n, s'_1, ..., s'_n) \), it holds

\[
s'_i = J \cdot e'_i.
\]
and then it is straightforward to check that

\[(e'_1, ..., e'_n, s'_1, ..., s'_n) = (e_1, ..., e_n, s_1, ..., s_n) \cdot C.\]

It implies \(\tilde{\sigma}_{N,b}(x) = \tilde{\sigma}_{N,b}(x)\). Then we get \(\tilde{\sigma}_N\) by simply piecing the local sections \(\tilde{\sigma}_{N,b}\) together.

Now we consider the case \(N = M\). Suppose that \(S \subset M\) is a Lagrangian submanifold, and that \(\Gamma^2_J\) admits a global section \(s^2\) which induces a trivialization

\[tr_{s^2} : \Gamma^2_J \to M \times S^1.\]

Let \(\gamma\) be a loop on \(S\). Then \(\sigma_S \circ \gamma : t \mapsto T_{\gamma(t)}S\) is a loop in \(\Lambda_J\). Composed with the map \(det^2_J\) it becomes a loop in \(\Gamma^2_J\), and the Maslov index \(m_{s^2}(\gamma)\) is the degree of the following map

\[S^1 \xrightarrow{s^2 \circ \gamma} \Lambda_J \xrightarrow{det^2_J} \Gamma^2_J \xrightarrow{tr_{s^2}} M \times S^1 \xrightarrow{pr_{S^1}} S^1.\]

Now suppose that \(S\) is an orientable Lagrangian submanifold and that \(\Gamma_J\) admits a global section \(s\). Then we have a global section of \(\Gamma^2_J\), which is given by \(s^2 = q_{\pm 1}(s)\). Here \(q_{\pm 1} : \Gamma_J \to \Gamma^2_J/\{\pm 1\}\) is the quotient map. As a consequence, we have the following commutative diagram, which suggests that the Maslov index \(m_{s^2}(\gamma)\) should be an even number:

\[
\begin{array}{ccccccccc}
S^1 & \xrightarrow{\sigma_{s^2 \circ \gamma}} & F_{rJ} / SO(n) & \xrightarrow{det^2_J} & \Gamma_J & \xrightarrow{tr_{s^2}} & M \times S^1 & \xrightarrow{pr_{S^1}} & S^1 \\
\| & & q_{\pm 1} \downarrow & & q_{\pm 1} \downarrow & & q_{\pm 1} \downarrow & & \text{square} \\
S^1 & \xrightarrow{\sigma_{s^2 \circ \gamma}} & \Lambda_J & \xrightarrow{det^2_J} & \Gamma_J^2 & \xrightarrow{tr_{s^2}} & M \times S^1 & \xrightarrow{pr_{S^1}} & S^1
\end{array}
\] (2.1)

Remark 6. If \(\mathcal{L}\) is a Lagrangian subbundle of \(TM\) over \(M\), it induces a global section \(\sigma : M \to \Lambda_J\) by sending each \(b \in M\) to the fiber \(\mathcal{L}_b\) of \(\mathcal{L}\) above it. Then

\[s^2 := det^2_J \circ \sigma\]

is a global section to the bundle \(\Gamma^2_J\). Furthermore, if \(\mathcal{L}\) is orientable, then it also induces a trivialization for \(\Gamma_J\).
2.2 Maslov indices on simply connected spaces

Generally speaking, a loop $\tilde{\gamma}$ of Lagrangian planes (or its image $\gamma$ in $\Gamma^2_J$) will have different Maslov indices with respect to different global sections of $\Gamma^2_J$. However, when the manifold $M$ is simply connected, Maslov indices of a loop with respect to different sections/Maslov cycles will be identical. We give a brief explanation of this fact from the perspective of the geometry of the bundle $\Gamma^2_J$.

Suppose that $s$ and $s'$ are sections of $\Gamma^2$. In terms of the trivialization $\Gamma^2 \to M \times S^1$ with respect to $s'$, the section $s : M \to M \times S^1$ takes the form $s(p) = (p, \theta_p)$ with $\theta : p \mapsto \theta_p$ being a smooth map from $M$ to $S^1$.

For a loop $\gamma \subset \Gamma^2$, it takes the form $S^1 \ni z \mapsto (\lambda(z), \tau(z)) \in M \times S^1$ in the trivialization $\Gamma^2 \to M \times S^1$, and the form $S^1 \ni z \mapsto (\lambda(z), \tau(z)) \in M \times S^1$ in $\Gamma^2 \to M \times S^1$, with the maps $\lambda : S^1 \to M$, $\tau', \tau : S^1 \to S^1$.

The corresponding Maslov indices $m_{s'}(\gamma)$ and $m_s(\gamma)$ are then the degrees of the maps $\tau'$ and $\tau$, respectively. It holds that $\tau'(z) = \tau(z) \cdot \theta_{\lambda(z)}$. Since $M$ is simply connected, the mapping $z \mapsto \theta_{\lambda(z)}$ which is subject to the following factorization

\[ S^1 \xrightarrow{\lambda} M \xrightarrow{\theta} S^1 \]

has degree 0, and hence $\tau$ and $\tau'$ have the same degree.

3 Basics on Principal $S^1$-Bundles

For later purposes, we recall in the section basic properties of a principal $S^1$ bundle. A principal $S^1$-bundle consists of a total space $P$ on which there is a free $S^1$ action, its orbifold $B = P / S^1$, and the quotient map $\pi_P : P \to B$. To distinguish it from other group actions, in the rest of the paper we will call this $S^1$ action the inherent/structural $S^1$ action. For simplicity, we will also call $P$ the principal bundle when it is clear from the context.

A connection $H$ on the bundle $\pi_P : P \to B$ is a horizontal distribution invariant under the inherent $S^1$ action, and the tangent bundle $TP$ is split as a direct sum $H \oplus V$ with $V = \ker \pi_{P,*}$ being the vertical distribution. The Lie algebra of $S^1$ is $T_1S^1 = \mathbb{R} \cdot \frac{\partial}{\partial \theta}$ with $\frac{\partial}{\partial \theta} = \frac{d}{d\theta} \bigg|_{\theta=0} e^{i2\pi \theta}$. The following map

\[ \eta : P \times \left( \mathbb{R} \cdot \frac{\partial}{\partial \theta} \right) \to V \]
with
\[
\left( p, r \frac{\partial}{\partial \theta} \right) \mapsto \left. \frac{d}{d\theta} \right|_{\theta=0} (p \cdot e^{i2\pi r \theta})
\]
is a vector bundle isomorphism. For simplicity we also denote the vector \( \frac{d}{d\theta} \bigg|_{\theta=0} (p \cdot e^{i2\pi r \theta}) \) by \( r \frac{\partial}{\partial \theta} \). The connection 1-form associated to \( H \) is
\[
\alpha : TP \xrightarrow{pr_V} V \cong P \times (\mathbb{R} \cdot \frac{\partial}{\partial \theta}) \to \mathbb{R} \cdot \frac{\partial}{\partial \theta}
\]
with \( pr_V \) being the projection associated to the splitting \( TP = H \oplus V \) onto \( V \). \( \alpha \) is \( S^1 \)-invariant, and there is an \( S^1 \)-invariant 1-form \( f_\alpha \) such that \( \alpha = f_\alpha \cdot \frac{\partial}{\partial \theta} \). By abusing the terminology we also call \( f_\alpha \) a connection 1-form on \( P \).

Note that \( f_\alpha \big( \frac{\partial}{\partial \theta} \big) \equiv 1 \). The \( S^1 \)-invariance of \( f_\alpha \) implies \( \mathcal{L}_{\frac{\partial}{\partial \theta}} f_\alpha = 0 \). Then by Cartan’s formula, we get
\[
0 = \mathcal{L}_{\frac{\partial}{\partial \theta}} f_\alpha = \iota_{\frac{\partial}{\partial \theta}} df_\alpha + d \left( f_\alpha \left( \frac{\partial}{\partial \theta} \right) \right) = \iota_{\frac{\partial}{\partial \theta}} df_\alpha.
\]
With \( pr_H \) being the projection of \( H \oplus V \) onto \( H \), the equation above implies that
\[
df_\alpha(pr_H \cdot pr_H) = df_\alpha.
\]
Since \( df_\alpha \), \( H \) and \( V \) are all \( S^1 \)-invariant, this implies that there is a closed 2-form \( \Omega_\alpha \) on \( B \) such that \( \pi_P^*(\Omega_\alpha) = df_\alpha \). \( [\Omega_\alpha] \in H^2_{dR}(B) \) is independent of \( \alpha \) and is called the characteristic class of the bundle \( P \to B \).

By the definition of \( f_\alpha \), \( \ker f_\alpha = H \). According to the Frobenius integrability theorem, \( H \) is an integrable distribution if and only if \( f_\alpha \wedge df_\alpha = 0 \). In this case, such an identity actually requires \( df_\alpha \) to be 0. This is because if \( df_\alpha \neq 0 \), then there exist \( u, v \in H_p = \ker f_\alpha|_p \) such that \( df_\alpha(u, v) \neq 0 \), and then
\[
f_\alpha \wedge df_\alpha \left( \frac{\partial}{\partial \theta} \right) u, v = f_\alpha \left( \frac{\partial}{\partial \theta} \right) df_\alpha(u, v) = df_\alpha(u, v) \neq 0,
\]
yielding a contradiction. As a result, the corresponding 2-form \( \Omega_\alpha = 0 \), and hence \( [\Omega_\alpha] = 0 \).

Conversely, if \( [\Omega_\alpha] = 0 \), then there exists a 1-form \( \tau \) on \( B \) such that \( d\tau = \Omega_\alpha \). Let \( \alpha' = \alpha - \pi_{F,\delta}^*(\tau) \cdot \frac{\partial}{\partial \theta} \). Then \( \alpha' \) is also \( S^1 \)-invariant with
\[
\alpha' \left( r \frac{\partial}{\partial \theta} \right) = \alpha \left( r \frac{\partial}{\partial \theta} \right) = r \frac{\partial}{\partial \theta},
\]
namely, it is also a connection 1-form, and $\mathcal{H}' = \ker \alpha'$ defines another connection on $P \to B$. With $f_{\alpha'} \cdot \frac{\partial}{\partial \tau} = \alpha'$, it holds

$$f_{\alpha'} = f_{\alpha} - \pi^*_\Gamma_J(\tau).$$

Then

$$df_{\alpha'} = \pi^*_\Gamma_J(\Omega_\alpha) - \pi^*_\Gamma_J(d\tau) = 0,$$

implying $\mathcal{H}'$ to be integrable. Therefore, the characteristic class of a principal $S^1$-bundle is zero if and only if the bundle has an integrable connection.

4 Compact group actions on Maslov $S^1$ bundles

Let

$$\Phi : G \times M \to M$$

be a symplectic left action on $M$ by a compact Lie group $G$. Namely, $\Phi^{h'h} = \Phi^h \circ \Phi^h$ and $\Phi^h(\omega) = \omega$ for $h, h' \in G$. In this section, we first show that $\Phi$ can be lifted to a $G$-action on $\Gamma_J$. Namely, there is a $G$-action $\Phi_G$ on $\Gamma_J$ that covers $\Phi$. Moreover, $\Phi_G$ commutes with the inherent $S^1$ action on $\Gamma_J$. Then we study such a lifted $G$-action for the case where $M$ is a homogeneous $G$-space.

**Definition 7.** Let $P \to B$ be a principal $S^1$-bundle. A group action on the bundle $P \to B$ (or simply saying the bundle $P$) by $G$ is a $G$-action on the manifold $P$ such that it commutes with the inherent $S^1$-action.

4.1 $G$-actions on $Fr_J$ and $\Gamma_J$

The $G$-action $\Phi$ on $M$ can be lifted to a $G$-action on the bundle $Fr_J \to M$ by resorting to a $G$-equivariant almost complex structure $J$. In the following we briefly explain how this is done.

Let $g$ be an arbitrary Riemannian metric on $M$, and let $dh$ be a probability measure on $G$ which is invariant under the right translations. Define a new Riemannian metric $\tilde{g}$ with

$$\tilde{g}(u,v) = \int_G \Phi^{h'}(g)(u,v)dh$$

with $u, v \in T_xM$ and $x \in M$. Then $\tilde{g}$ is invariant under the (left) $G$-action $\Phi$. Let $\mathcal{A}$ be the vector bundle
isomorphism on $TM$ defined by
\[ \omega(u, \cdot) = \bar{g}(Au, \cdot) \quad (4.1) \]
and let
\[ \bar{J} = A^{-1}\sqrt{-A^2}. \quad (4.2) \]
Then $\bar{J}$ is an almost complex structure compatible with $\omega$, and $g\bar{J}(\cdot, \cdot) = \omega(\bar{J}\cdot, \cdot)$ defines a Riemannian metric.

For convenience we denote by $h_*$ the pushforward $\Phi_{h_*}$. It is a standard result that $\bar{J}$ commutes with $h_*$ for all $h \in G$, and $g\bar{J}$ is then invariant under the $G$-action. For completeness we give an argument in the following.

**Lemma 8.** Suppose that $\bar{g}$ is a Riemannian metric invariant under the $G$-action $\Phi$. Let $A$ and $\bar{J}$ be endomorphisms on $TM$ defined respectively by (4.1) and (4.2) above. Then for any $h \in G$, it holds
\[ h_* \circ \bar{J} = \bar{J} \circ h_. \quad (4.3) \]
and
\[ h^*g\bar{J}(\cdot, \cdot) = g\bar{J}(\cdot, \cdot). \quad (4.4) \]
As a consequence, $h_*$ maps the unitary frame bundle $Fr_J^u$ to itself.

**Proof.** Suppose that (4.3) holds, then
\[ h^*g\bar{J}(\cdot, \cdot) = \omega(\bar{J} \circ h_*, h_*\cdot) = \omega(h_* \circ \bar{J}, h_*\cdot) = g\bar{J}(\cdot, \cdot) \]
gives (4.4).

We start to prove (4.3) by looking at the bundle isomorphism $A$. Since $G$ acts symplectically on $M$, it holds
\[ \bar{g}(Ah_*u, h_*\cdot) = \omega(h_*u, h_*\cdot) \]
\[ = \omega(u, \cdot) \]
\[ = \bar{g}(Au, \cdot) \]
\[ = \bar{g}(h_*Au, h_*\cdot) \]

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Now that $\mathcal{A}$ is bijective and $u$ runs over $TM$, this implies

$$\mathcal{A}h_* = h_*\mathcal{A}.$$  

Consequently, $h_*$ also commutes with $\mathcal{A}^{-1}$ and $-\mathcal{A}^2$. For proving (4.3), it remains to show that

$$Bh_* = h_*B$$

with $B = \sqrt{-\mathcal{A}^2}$, i.e. $B^2 = -\mathcal{A}^2$.

Note that, at each $x$ in $M$, $B_x^2 = B^2|_{T_xM}$ and $B_x$ are both self-adjoint positive operators with respect to $\bar{g}$, and a vector $v \in T_xM$ is an eigenvector of $B_x^2$ with eigenvalue $\lambda^2$ if and only if it is an eigenvector of $B_x$ with eigenvalue $\lambda > 0$, i.e.

$$B_x^2(v) = \lambda^2 v \iff B_x(v) = \lambda v.$$  

Resorting to the commutativity, we have

$$B^2_{h_*}h_*(v) = h_*B_x^2(v) = \lambda^2 h_*(v),$$

and then

$$B_{h_*}h_*(v) = \lambda h_*(v).$$

The discussion above amounts to

$$B_x(v) = \lambda v \iff B_{h_*}h_*(v) = \lambda h_*(v).$$

Now let $\{e_1, ..., e_{2n}\}$ be a basis of $T_xM$ with each $e_i$ being an eigenvector of $B_x$ with eigenvalue $\lambda_i$. Then

$$h_*B_x(e_i) = \lambda_i h_*(e_i) = B_{h_*}h_*(e_i).$$

Since $\{e_1, ..., e_{2n}\}$ is a basis, this concludes the proof.

\[\square\]

The tangent maps of the actions by the elements in $G$ gives the following $G$-action on $Fr^n_\theta$:  

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\[ \Phi_\#: G \times Fr^u_J \rightarrow Fr^u_J \]

\[ \Phi_h^\#: (u_i, v_i) = (h_*(u_i), h_*(v_i)). \]

For simplicity, we write \( \Phi_h^\#(u_i, v_i) \) as \( h^\#(u_i, v_i) \). For any \( C \in U(n) \), it can also be checked that

\[ (h_*(u_1), ..., h_*(v_n)) \cdot C = h^\#((u_1, ..., v_n) \cdot C), \tag{4.5} \]

and hence \( \Phi^\# \) is a \( G \)-action on \( Fr^u_J \rightarrow M \). Due to \( \Phi^\# \), \( \Phi^\# \) induces a smooth \( G \)-action \( \Phi^\#_\Gamma \) on \( \Gamma^u_J = Fr^u_J / SU(n) \).

Moreover, \( \Phi^\#_\Gamma \) on \( \Gamma^u_J \) commutes with the \( S^1 \) action. To see this, note that

\[ \mathbb{U}(n)/SU(n) \ni \begin{bmatrix} C \end{bmatrix} \mapsto \det C(C) \in S^1 \]

is an isomorphism, and the \( S^1 \) action on \( \Gamma^u_J \) is given by

\[ [u_1, ..., v_n] \cdot e^{i\theta} := [(u_1, ..., v_n)] \cdot \begin{bmatrix} C \end{bmatrix} = [(u_1, ..., v_n) \cdot C] \]

with \( \det C(C) = e^{i\theta} \). Then

\[ \Phi^h_\Gamma([u_1, ..., v_n] \cdot e^{i\theta}) = \Phi^h_\Gamma([u_1, ..., v_n] \cdot C] = [(h_*(u_1), ..., h_*(v_n)) \cdot C] \]

\[ = \Phi^h_\Gamma([u_1, ..., v_n]) \cdot e^{i\theta} \]

Since the unitary structure on \( M \) is unique up to isomorphism, the discussion in this subsection amounts to the following proposition:

**Proposition 9.** Suppose that \( \Phi \) is a symplectic group action on \( (M, \omega) \) by a compact Lie group \( G \). Then there exists a \( G \)-action \( \Phi^\#_\# \) on \( Fr^u_J \rightarrow M \) that covers \( \Phi \). As a consequence, it induces \( G \)-actions \( \Phi^\#_\Gamma \) and \( \Phi^\#_\Gamma^2 \) on the bundles \( \Gamma^u_J \) and \( \Gamma^u_J^2 \), respectively. These actions covers \( \Phi \) and are covered by \( \Phi^\#_\# \).

### 4.2 An example: \( S^2 \)

We can get the idea of the main result in this section by looking into the case where \( M = S^2 \). This might be the simplest example in which the Maslov \( S^1 \) bundles are non-trivial.
Consider $S^2$ as an embedded submanifold in $\mathbb{R}^3$. At each point $p \in S^2$, the tangent space $T_p S^2$ is a subspace of $T_p \mathbb{R}^3$. Let $\bar{n}$ be the restriction of the vector field $x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$ on $S^2$. With $T_p \mathbb{R}^3$ being identified with $\mathbb{R}^3$, it holds $\bar{n}_p = p$. Then

$$\omega_{S^2} = \iota_{\bar{n}} dx \wedge dy \wedge dz.$$

is a symplectic structure on $S^2$. Viewing $u, v \in T_p S^2$ and as vectors in $\mathbb{R}^3$, it holds

$$\omega_{S^2}(u, v) = \bar{n} \cdot (u \times v).$$

With respect to $\omega_{S^2}$,

$$J_{S^2}(u) := -\bar{n} \times u$$

defines a compatible almost complex structure, and

$$g_{S^2}(\cdot, \cdot) := \omega_{S^2}(J_{S^2} \cdot, \cdot)$$

is the restriction of the standard Riemannian metric on $\mathbb{R}^3$ to $S^2$.

Denote by $\Gamma_{S^2}$ and $Fr_{S^2}^u$, respectively, the Maslov $S^1$ bundle $(\Gamma_f)$ and the unitary frame bundle $(Fr_f^u)$ of $(S^2, \omega_{S^2})$. Over each $p \in S^2$, each unitary frame can be denoted uniquely and distinctly as

$$(u, J_{S^2}(u); p) = (u, \bar{n}_p \times u; p) = (u, p \times u; p)$$

with $u \in \mathbb{R}^3$ tangent to $S^2$ at $p$. When written as a matrix $[u, p \times u, p]$ with $u$, $p \times u$ and $p$ being the columns, this is an element in $SO(3)$. More precisely, the following map

$$\text{mat} : Fr_{S^2}^u \ni (u, -J_{S^2}(u); p) \mapsto [u, p \times u, p] \in SO(3)$$

is a diffeomorphism from $Fr_{S^2}^u$ to $SO(3)$. Since $SU(1) = \{1\}$, $\Gamma_{S^2} = Fr_{S^2}^{id}$. Since $SO(3)$ acts symplectically on $(S^2, \omega_{S^2})$, there is an $SO(3)$-action on the bundle $\pi_{\Gamma_{S^2}} : \Gamma_{S^2} \to S^2$ which covers the action on $S^2$. In fact, this action is exactly given by

$$SO(3) \times \Gamma_{S^2} \xrightarrow{id \times \text{mat}} SO(3) \times SO(3) \xrightarrow{\text{multiply}} SO(3) \xrightarrow{\text{mat}^{-1}} \Gamma_{S^2}$$
Here $\mathbb{SO}(3) \times \mathbb{SO}(3) \xrightarrow{\text{multiply}} \mathbb{SO}(3)$ is simply the group multiplication $(A, B) \mapsto A \cdot B$, and hence the lifted group action is recognized as the left action of $\mathbb{SO}(3)$ on itself (identified as $\Gamma_{S^2}$). As a result, we have the following proposition.

**Proposition 10.** The lifted $\mathbb{SO}(3)$ action is transitive on $\Gamma_{S^2}$.

*Proof.* This is because the left action of $\mathbb{SO}(3)$ on itself is transitive. \hfill \square

Let $\alpha = f_\alpha \cdot \frac{\partial}{\partial \theta}$ be a connection 1-form on $\Gamma_{S^2}$. Since $S^2$ is simply connected and the principal bundle $\pi_{\Gamma_{S^2}} : \Gamma_{S^2} \to S^2$ is not trivial, there is no integrable connection on $\Gamma_{S^2}$ and hence $df_\alpha \neq 0$. Note that $df_\alpha$ descends to a nondegenerate closed 2-form $\Omega_\alpha$ on $S^2$, and since $H^2_{dR}(S^2) = \mathbb{R}$ it satisfies $[\Omega_\alpha] = r \cdot [\omega]$ for some real number $r \neq 0$. Moreover, we have the following proposition.

**Lemma 11.** There exists an $\mathbb{SO}(3)$-invariant connection 1-form $\bar{\alpha}$ on $\Gamma_{S^2}$ such that $df_\alpha = r \cdot \pi_{\Gamma_{S^2}}^*(\omega)$ with $r$ being some non-zero real number.

*Proof.* Starting with an arbitrary connection 1-form $\alpha$, for each element $v$ of the tangent bundle $T\Gamma_{S^2}$, define

$$f(v) := \int_{\mathbb{SO}(3)} f_\alpha(h \cdot v) dh$$

with $dh$ being a right invariant probability measure on $\mathbb{SO}(3)$. Then $f$ is both $S^1$-invariant and $\mathbb{SO}(3)$ invariant, and $f(\frac{\partial}{\partial \theta}) \equiv 1$. Therefore, $\bar{\alpha} := f \cdot \frac{\partial}{\partial \theta}$ defines a connection 1-form which is invariant under the $\mathbb{SO}(3)$-action. Then $df$ is also $\mathbb{SO}(3)$ invariant, and so is $\Omega_\alpha$ with $df = \pi_{\Gamma_{S^2}}^*(\omega)$. Since $\Omega_\alpha \neq 0$ and $S^2$ is 2 dimensional, there exists a non-zero real number $r$ such that $\Omega_\alpha = r \cdot \omega$. Note that $f = f_\alpha$ and this concludes the proof. \hfill \square

**Proposition 12.** There exists a connection 1-form $\bar{\beta} := f_\beta \cdot \frac{\partial}{\partial \theta}$ such that, for each $v \in \mathfrak{so}(3)$, the corresponding infinitesimal generator $X_v$ has a Hamiltonian function $H_v$ on $S^2$ satisfying $H_v \circ \pi_{\Gamma_{S^2}} = -\frac{1}{r} \cdot f_\beta(X_v)$ with some non-zero constant $r$.

*Proof.* Let $\bar{\alpha}$ be the connection 1-form in Lemma 11. For simplicity, we write the symplectic form $\omega_{S^2}$ as $\omega$ in the following discussion. Applying the averaging method again to $\bar{\alpha}$ with the $G$-action on $\Gamma_{S^2}$:

$$\bar{\beta}(v) := \int_G \bar{\alpha} \circ h_v(v) dh$$

with

$$(A, [u, p \times u, p]) \mapsto [Au, Ap \times Au, Ap].$$
with \(dh\) being a right invariant probability measure on \(G\). \(\bar{f}_\beta\) is then invariant under the \(G\)-action, and

\[
d\bar{f}(u,v) = \int_G h^*(df_\alpha)(u,v)dh
= r \cdot \int_G h^* \pi^*_\Gamma S_2(\omega)(u,v)dh
= r \cdot \int_G \pi^*_\Gamma S_2 h^* \omega(u,v)dh
= r \cdot \pi^*_\Gamma S_2(\omega)(u,v)
\]

By Cartan’s formula,

\[
d\left(f_\beta(\mathcal{X}_v)\right) = \mathcal{L}_{\mathcal{X}_v} f_\beta - \iota_{\mathcal{X}_v} df_\beta
= -r \cdot \omega(\pi_{\Gamma S_2,*}(\mathcal{X}_v), \pi_{\Gamma S_2,*} \cdot).
= -r \cdot \omega(\mathcal{X}_v, \pi_{\Gamma S_2,*})
\]

Note that the \(G\)-action commutes with the \(S^1\)-action on \(\Gamma S_2\), and hence \(\mathcal{X}_v\) is \(S^1\)-invariant. As a result, \(-\frac{1}{r} \cdot f_\beta(\mathcal{X}_v)\) is also \(S^1\)-invariant, and there is a function \(H_v\) on \(M\) such that

\[-\frac{1}{r} \cdot f_\beta(\mathcal{X}_v) = H_v \circ \pi_{\Gamma S_2}.\]

With the identity deduced above it yields

\[
dH_v \circ \pi_{\Gamma S_2,*} = \omega(\mathcal{X}_v, \pi_{\Gamma S_2,*} \cdot)
\]

and then

\[
dH_v = \omega(\mathcal{X}_v, \cdot).
\]

\[\square\]

5 Dynamics on the Maslov \(S^1\) Bundles

In this section we extend the properties obtained for \(S^2\), Proposition \([10]\) and \([12]\) to more general settings.
5.1 Extension for Proposition 10 to symplectic homogeneous \( G \)-spaces

In this subsections we extend Proposition 10 to the case where the group action \( \Phi \) by \( G \) is transitive on \( M \). Namely, \( M \) is a symplectic homogeneous \( G \)-space. For each \( p \in M \) and \( w \in \Gamma_J \), denote by \( G_p \) the isotropy group of the action \( \Phi \) at \( p \), and by \( G_w \) the isotropy group of \( \Phi_\Gamma \) at \( w \).

By the transitivity of the \( G \)-action, \( M \) is diffeomorphic to \( G/G_{p_0} \) for any fixed point \( p_0 \) through the map

\[
F_{p_0} : G/G_{p_0} \ni [h] \mapsto h \cdot p_0 \in M.
\]

Let \( w_0 \) be a point in \( \pi_{\Gamma_J}^{-1}(p_0) \). Then \( G_{w_0} \subset G_{p_0} \). Since \( \Phi_\Gamma \) on \( \Gamma_J \) covers \( \Phi \) on \( M \), \( G_{p_0} \) acts on the fiber \( \pi_{\Gamma_J}^{-1}(p_0) \). That is, \( h \cdot w \in \pi_{\Gamma_J}^{-1}(p_0) \) for any \( h \in G_{p_0} \) and \( w \in \pi_{\Gamma_J}^{-1}(p_0) \). It turns out that \( G_w = G_{w_0} \) for any \( w \in \pi_{\Gamma_J}^{-1}(p_0) \), and that the \( G_{p_0} \)-action on \( \pi_{\Gamma_J}^{-1}(p_0) \) induces a homomorphism to \( S^1 \) with \( G_{w_0} \) being the kernel.

**Proposition 13.** There exists a Lie group homomorphism \( \phi_p \) from \( G_p \) to \( S^1 \) with \( G_w \) being the kernel such that, for \( w \in \pi_{\Gamma_J}^{-1}(p) \) and \( h \in G_p \), it holds

\[
h \cdot w = w \cdot \phi_p(h).
\]

Moreover, the family \( \{\phi_p\} \) is \( G \)-related in the sense that

\[
\phi_{h' \cdot p} \circ \text{Ad}_{h'} = \phi_p
\]

with \( p \in M \) and \( h' \in G \).

**Proof.** We start by arguing with the fixed points \( p_0 \) and \( w_0 \). Since \( S^1 \) acts transitively and freely on \( \pi_{\Gamma_J}^{-1}(p_0) \), for each \( h \in G_{p_0} \) there exists a unique \( z_h \in S^1 \) such that \( h \cdot w_0 = w_0 \cdot z_h \). Then

\[
w \cdot z_{h' \cdot h} = (h' \cdot h) \cdot w_0 = w_0 \cdot z_h \cdot z_{h'} = w_0 \cdot (z_h \cdot z_{h'}),
\]

and hence \( z_{h' \cdot h} = z_h \cdot z_{h'} \). Define \( \phi_{p_0} \) with \( \phi_{p_0}(h) = z_h \). Then this is a homomorphism, and \( h \cdot w_0 = w_0 \) if and only if \( \phi_{p_0}(h) = 1_G \). Hence \( \ker \phi_{p_0} = G_{w_0} \). Note that \( \pi_{\Gamma_J}^{-1}(p_0) \) is diffeomorphic to \( S^1 \) via \( \theta_{w_0} : w_0 \cdot z \mapsto z \). Since \( \phi_{p_0}(h) = \theta_{w_0} \circ \Phi_\Gamma'(w_0) \), \( \phi_{p_0} \) is a smooth map, and therefore it is a Lie group homomorphism.

The assignment of \( z_h \) to \( h \) is actually independent of the choice of \( w_0 \). Namely, the identity \( h \cdot w = w \cdot z_h \)
holds for all \( w \in \pi_{\Gamma_j}^{-1}(p_0) \). To see this, first note that there exists \( z \in S^1 \) such that \( w_0 \cdot z = w \), and then

\[
h \cdot w = h \cdot (w_0 \cdot z) = (h \cdot w_0) \cdot z = w_0 \cdot z_h \cdot z = w \cdot z_h.
\]

Moreover, the argument above holds for any point \( p \), and then the homomorphism \( \phi_p \) can be constructed in the same way.

It remains to show that \( G \) acts on the family \( \{ \phi_p \} \) according to Eq.5.1. Let \( w \) be a point on the fiber \( \pi_{\Gamma_j}^{-1}(p) \). Then \( w' = h' \cdot w \in \pi_{\Gamma_j}^{-1}(h' \cdot p) \). For any \( h \in G_p \),

\[
h'hh'^{-1} \cdot w' = h'hw = h' \cdot (w \cdot \phi_p(h)) = (h' \cdot w) \cdot \phi_p(h) = w' \cdot \phi_p(h),
\]

which implies \( \phi_{h' \cdot p}(h'hh'^{-1}) = \phi_p(h) \) and concludes the proof.  

Due to (5.1) in Proposition 13 above, for any \( p, p' \in M \), \( \text{Im} \phi_p = \text{Im} \phi_{p'} \), and we denote this subgroup by \( S^1_\phi \). Since \( G_p \) is compact, \( S^1_\phi = \phi_p(G_p) \) is a compact subgroup of \( S^1 \). Therefore it is either a finite cyclic group \( \{ e^{\frac{2\pi i l}{k}} \mid l = 0, \ldots, k - 1 \} \), or \( S^1 \) itself.

For each element \( v \) of the Lie algebra \( \mathfrak{g} \) of \( G \), denote by \( X_v \) the infinitesimal generator of \( \Phi_\Gamma \) in the direction \( v \). That is, \( X_v(p) = \frac{d}{dt} \exp(tv) \cdot p \) with \( t \mapsto \exp(tv) \) being the one parameter subgroup of \( G \) generated by \( v \). We claim:

**Proposition 14.** \( S^1_\phi \) is either \( S^1 \) or a finite subgroup of \( S^1 \). If \( S^1_\phi \) is finite, then the infinitesimal generators \( X_v \) span an integrable connection \( \mathcal{D} \) on the bundle \( \Gamma_j \) with the orbits \( G \cdot w \) being the maximal connected integral manifolds.

**Proof.** As an orbit of a compact Lie group action, \( G \cdot w \) is an embedded submanifold in \( \Gamma_j \), and then its dimension is no larger than \( 2n + 1 \). If \( \dim G \cdot w = 2n + 1 \), \( G \cdot w \) is an open set in \( \Gamma_j \) and then \( G \cdot w \cap \pi_{\Gamma_j}^{-1}(p) \) is open in \( \pi_{\Gamma_j}^{-1}(p) \) (and then it is the whole \( \pi_{\Gamma_j}^{-1}(p) \) since it is open and compact for any \( p \in M \)). Now suppose that \( \dim G \cdot w \leq 2n + 1 \). Since \( G \) acts transitively on \( M \) and \( \pi_\Gamma \circ \Phi_\# = \Phi \circ \pi_\Gamma \), it holds

\[
\pi_{\Gamma_j}(T_w(G \cdot w)) = T_{\pi_{\Gamma_j}(w)}M,
\]
and hence \( \dim G \cdot w \geq 2n \). Together this yields \( \dim G \cdot w = 2n \). Due to the commutativity of \( \Phi_\Gamma \) and the \( S^1 \) action on \( \Gamma_J \), \((G \cdot w) \cdot z = G \cdot (w \cdot z)\) with \( z \in S^1 \). This means \((G \cdot w) \cdot z\) is exactly the \( G \)-orbit through \( w \cdot z \), and the \( S^1 \) action maps orbits to orbits. Since

\[
T_w(G \cdot w) = \text{span}\{X_v \mid v \in g\} = \mathcal{D}_w,
\]

\( \mathcal{D} \) is a \( 2n \) dimensional distribution invariant under the \( S^1 \) action, and \( \pi_{\Gamma,*}(\mathcal{D}_w) = T_{\pi(w)}M \).

Therefore, from Proposition 14 we can deduce:

**Theorem 15.** Let \( G \) be a compact Lie group acting transitively and symplectically on \( M \). If the characteristic class of the Maslov \( S^1 \) bundle \( \Gamma_J \) is non-zero, then \( \Gamma_J \) is also a homogeneous \( G \)-space.

**Proof.** If \( \Phi_\Gamma \) does not act transitively on \( \Gamma_J \), then \( S^1 \) should be a finite subgroup of \( S^1 \). According to Proposition 14 \( \mathcal{D} \) would be an integrable connection, and then the characteristic class would be zero, which violates the condition.

5.2 Extension for Proposition 12 to the case \([\omega] = r \cdot c_\Gamma\)

The argument for Proposition 11 can actually be applied to any symplectic action on a symplectic manifold \((M, \omega)\) satisfying \([\omega] = r \cdot c_\Gamma\) for \( r \in \mathbb{R} \). Instead of directly giving a proof, we introduce some notions and formalize the demonstration in such a way that the relevant structures are better illustrated.

Recall that a symplectic \( G \)-action on \( M \) is Hamiltonian if and only if there is a smooth map (called a momentum map)

\[
F : M \to g^*
\]

such that for each \( h \in G \) and \( v \in g \), \( F \) is \( G \)-coadjoint equivariant, i.e.

\[
F_{h \cdot x} = \text{Ad}^*_h F_x,
\]

and the mapping

\[
F^v : M \ni x \mapsto F_x(v) \in \mathbb{R}
\]
defines a Hamiltonian for \( \mathcal{X}_v \) with \( v \in \mathfrak{g} \).

**Remark 16.** In the rest of this subsection the symbol “\( \Gamma \)” stands for both the spaces \( \Gamma_J \) and \( \Gamma_J^2 \), since the same argument works for both of them.

**Definition 17.** An \( S^1 \)-invariant 1 form \( \eta \) on the bundle \( \pi \Gamma : \Gamma \to M \) is called a symplectic potential if

\[
d\eta = -\pi_\Gamma^* P(\omega).
\]

It is straightforward to check that averaging a symplectic potential \( \eta \) with the (lifted) \( G \) action \( \Phi_\Gamma \) gives a \( G \)-invariant symplectic potential \( \bar{\eta} \). We have the following lemma.

**Lemma 18.** Let \( \bar{\eta} \) be a \( G \)-invariant symplectic potential. For each \( p \in \Gamma \), let \( \mu_p \) be the element in \( \mathfrak{g}^* \) with \( \mu_p(v) = \bar{\eta}(\mathcal{X}_v)(p) \) for \( v \in \mathfrak{g} \), where \( \mathcal{X}_v \) is the infinitesimal generator of \( \Phi_\Gamma \) corresponding to \( v \). Then the map \( \mu \) from \( \Gamma \) to \( \mathfrak{g}^* \) defined by

\[
\mu : p \mapsto \mu_p
\]

is a momentum map for \( G \).

**Proof.** For the smoothness of \( \mu \), it suffices to show that the map \( \bar{\mu} : \Gamma \times \mathfrak{g} \to \mathbb{R} \) defined by \( \bar{\mu}(p, v) = \mu_p(v) \) is smooth. Observe that \( \bar{\mu} \) factors as

\[
\bar{\mu} : \Gamma \times \mathfrak{g} \xrightarrow{\mathcal{X}} T\Gamma \xrightarrow{\bar{\eta}} \mathbb{R}
\]

with \( \mathcal{X} \) being the map from \( P \times \mathfrak{g} \) to \( TP \) sending \( (p, v) \) to \( \mathcal{X}_v(p) \). The factorization below shows smoothness of \( \mathcal{X} \):

\[
\mathcal{X} : \Gamma \times \mathfrak{g} \xrightarrow{\sigma} T\Gamma \times TG \cong T(\Gamma \times G) \xrightarrow{D\Phi_\Gamma} TT.
\]

Here \( \sigma \) is the map sending \( (p, v) \in \Gamma \times \mathfrak{g} \) to \( (0_p, v) \in T_p\Gamma \times T_1G \) with 0 being the zero section from \( \Gamma \) to \( TT \), and \( D\Phi_\Gamma \) is the tangent map of \( \Phi_\Gamma \). Hence \( \mathcal{X} \) is smooth. As a consequence, \( \bar{\mu} \) is also smooth.

Since \( \bar{\eta} \) and \( \mathcal{X}_v \) are both \( S^1 \)-invariant on \( \Gamma \), the function \( \bar{\eta}(\mathcal{X}_v) \) is constant along each \( S^1 \)-fiber, and hence there exists a function \( H_v \) on \( M \) such that \( H_v \circ \pi = \bar{\eta}(\mathcal{X}_v) \). Check that \( 0 = \mathcal{L}_{\mathcal{X}_v}(\bar{\eta}) = d\iota_{\mathcal{X}_v} \bar{\eta} + \iota_{\mathcal{X}_v} d\bar{\eta} \) and then it holds

\[
d(\bar{\eta}(\mathcal{X}_v)) = -\iota_{\mathcal{X}_v} d\bar{\eta} = \iota_{\mathcal{X}_v} \pi_{\mathfrak{p}}^*(\omega),
\]

implying \( \iota_{\mathcal{X}_v} \omega = dH_v \), where \( \mathcal{X}_v = \pi_{\Gamma,*}(\mathcal{X}_v) \) is the generator of \( \Phi \) corresponding to \( v \).
It remains to show that \( \mu \) is \( G \)-coadjoint equivariant. Note that for \( h \in G \) and \( p \in \Gamma \),

\[
h_*(X_{v \mid p}) = X_{Ad_{h \cdot p}}(h \cdot p).
\]

Then

\[
\mu_{h \cdot p}(v) = \tilde{\eta}(X_{v \mid h \cdot p})
\]

\[
= \tilde{\eta}|_{h \cdot p}(X_{v \mid h \cdot p})
\]

\[
= h^* \tilde{\eta}|_{h \cdot p}(X_{Ad_{h \cdot p}}(v)(p))
\]

\[
= h^* \tilde{\eta}|_{h \cdot p}(X_{Ad_{h \cdot p}}(v)(p))
\]

\[
= \mu_p \circ Ad_{h \cdot p}(v)
\]

and this concludes the proof.

We give the following remark for later reference.

**Theorem 19.** If \([\omega] = r \cdot c_G \) with \( c_G \) being the first real Chern class of \( \Gamma \) and \( r \in \mathbb{R} \), then any symplectic action \( \Phi \) on \( M \) by a compact Lie group \( G \) is Hamiltonian. Moreover, if \( r \neq 0 \), then there exists a connection 1-form \( \beta = f_\beta \cdot \frac{d}{d\theta} \) with some nonzero real number \( r \) such that \( \frac{1}{r} f_\beta \) is a \( G \)-invariant symplectic potential, and then \( \frac{1}{r} f_\beta(X_v) \) is an Hamiltonian for \( X_v \).

**Proof.** If the homology class \([\omega] = 0\), then there exists a 1-form on \( M \) with \( d\tau = \omega \). It is straightforward to check that \( \pi_\tau^*(\tau) \) is an \( S^1 \)-invariant 1-form on \( \Gamma \) and is a symplectic potential. If \([\omega] \neq 0\), then there exists a non-zero constant \( r \) such that \([r \cdot \omega] = [\Omega_\alpha] \) with \( \pi_\tau^*(\Omega_\alpha) = df_\alpha \) for a connection 1-form \( \alpha = f_\alpha \cdot \frac{d}{d\theta} \). Then it holds \( r \cdot \omega = \Omega_\alpha + d\tau \) for some 1-form \( \tau \) on \( M \). Let \( f_\beta = f_\alpha + \pi_\tau^*(\tau) \) and then \( df_\beta = r \cdot \pi_\tau^*(\omega) \). Hence \( \frac{1}{r} f_\beta \) is a symplectic potential on the bundle \( \Gamma \to M \).

The conclusion then follows from Lemma 18.

### 5.3 Conservation laws

When \( \eta \) (or \( \beta \)) is invariant under the action \( \Phi_{\Gamma^2} \), it is always true that \( \eta(X_v) \) is constant along the flow of \( X_v \), i.e. \( \mathcal{L}_{X_v} \eta(X_v) = 0 \). This is because \( X_v \) and \( \eta \) are invariant under the pushforward and pullback of the flow of \( X_v \), respectively. Moreover, since \( X_v \) and \( \eta \) are also invariant under the inherent \( S^1 \) action of the bundle \( \Gamma^2 \), there
always exists a function $Q_v$ on $M$ such that $\eta(X_v) = Q_v \circ \pi_{T^2}$. Since $\pi_{T^2}$ is a submersion, the smoothness of $\eta(X_v)$ implies the smoothness of $Q_v$. Check that $d\eta(X_v) = dQ_v \circ \pi_{T^2} \cdot X_v = dQ_v(X_v)$, meaning $Q_v$ is invariant under the flow $\varphi_v$ of $X_v$. This implies in particular that, if the trajectory of the one-parameter subgroup $\exp(tv)$ is dense in $G$, then $Q_v$ is invariant under the $G$-action.

6 $\beta$-Maslov data for Symplectic $S^1$ Actions

In this section we consider the case $G = S^1$. For a connection 1-form $\beta = \eta \cdot \frac{\partial}{\partial \theta}$ on $\Gamma^2$, the $\beta$-Maslov data $Q_\beta$ of the $S^1$ action $\Phi$ is the function on $M$ defined by

$$Q_\beta(p) = m_\beta(\gamma_w) = \int_{\gamma_w} \eta$$

with $w \in \pi_{T^2}^{-1}(p)$ and $\gamma_w(z) = \Phi^*_z(w)$ for $z \in S^1$. Note that the integral in (6.1) is independent of the choice of $w$ on $\pi_{T^2}^{-1}(p)$ and hence $Q_\beta$ is a well-defined function on $M$. To see this, note that for $w, w' \in \pi_{T^2}^{-1}(p)$, there exists an element $z \in S^1$ such that $w' = w \cdot z$ and then $\gamma_w' = \gamma_w \cdot z$, and $\int_{\gamma_w} \eta = \int_{\gamma_w'} \eta$ follows from the fact that $\eta$ is invariant under the inherent $S^1$ action on $\Gamma^2$ (since $\beta$ is a connection 1-form).

For simplicity we denote by $\varphi$ the period-1 flow of the circle action, namely, $\frac{d}{dt} \varphi^t = X_\beta$.

6.1 The Maslov index of the $S^1$ action

In this subsection we look at the case when $\Gamma^2$ is a trivial bundle.

Let $s$ be a global section of $\Gamma^2$. For each Lagrangian plane $w$, $\tilde{\gamma}_w(z) = \Phi^*_z(w)$ with $z \in S^1$ is a loop in $\Lambda_{pl}$ and hence has a Maslov index $m_s(\tilde{\gamma}_w)$ with respect to $s$. Note that $m_s(\tilde{\gamma}_w)$ equals to the degree of the map obtained by the following composition

$$S^1 \cong [0,1]/\{0,1\} \xrightarrow{\Phi_s(w)} \Lambda_{pl} \xrightarrow{det^2} \Gamma^2 \xrightarrow{tr} M \times S^1 \xrightarrow{pr_2} S^1,$$

which, together with the connectedness of $\Lambda_{pl}$, implies that $m_s(\tilde{\gamma}_w)$ is actually independent of $w$. Namely, all the orbits in $\Lambda_{pl}$ of the action $\Phi_s$ have the same Maslov index.

Remark 20. Note that the construction above is equivalent to looking at the degree $m_s(\gamma_w)$ of the following
map

\[ S^1 \cong [0, 1]/\{0, 1\} \xrightarrow{\phi^{1}_{\Gamma^2}(w)} \Gamma^2_{\mathcal{J}} \xrightarrow{\text{tr} \times} M \times S^1 \xrightarrow{pr_{S^1}} S^1 \]  

(6.3)

with \( \gamma_w(z) = \Phi_{\Gamma^2}(w) \).

We show further that when the action has a fixed point, then \( m_s(\gamma_w) \) is also independent of the choice of \( s \), and hence we can simply talk about the Maslov index of the flow with respect to \( s \) and denote it by \( m_s(\Phi) \).

Let \( p \) be an arbitrary fixed point. Then on the fiber \( \pi_{-1}^{-1}(p) \), the vector field takes the form as \( X_\theta(w) = a_w \partial / \partial \theta \) for all \( w \in \pi_{-1}^{-1}(p) \). Since \( X_\theta \) is invariant under the inherent \( S^1 \) action, it holds \( a_w \equiv k_p \) on \( \pi_{-1}^{-1}(p) \). Let \( \alpha = f_\alpha \cdot \partial / \partial \theta \) be the connection 1-form that has (the image of) \( s \) as an integral manifold of its kernel. Now that \( m_s(\gamma_w) \) is independent of the choice of \( w \), we particularly choose \( w \in \pi_{-1}^{-1}(p) \). Then it yields

\[ m_s(\gamma_w) = \int_0^1 f_\alpha(X_\theta) \circ \gamma_w(t) dt = k_p. \]  

(6.4)

and our argument is concluded by the fact that \( k_p \) is independent of \( s \).

Remark 21. According to the discussion above we have \( X_\theta = k_p \partial / \partial \theta \) on the fiber \( \pi_{-1}^{-1}(p) \).

We call \( k_p \) defined by (6.3) the local Maslov index of the \( S^1 \) action at the fixed point \( p \). Note that, on the one hand this definition does not depend on the triviality of \( \Gamma^2_{\mathcal{J}} \), and on the other hand it equals to the Maslov index of the restricted action on an invariant Daboux chart of \( p \). The discussion above leads to the following proposition.

Proposition 22. When the bundle \( \Gamma^2_{\mathcal{J}} \) is trivial and the \( S^1 \) action \( \Phi \) has fixed points, then for any fixed points \( p, p' \) and sections \( s, s' \), it holds

\[ k_p = k_{p'} = m_s(\Phi) = m_{s'}(\Phi). \]  

6.2 The \( \beta \)-Maslov data \( Q_\beta \)

Now we take a look that the function \( Q_\beta \) defined in (6.3) As is noted in Remark 21 on the fiber over a fixed point \( p \), it holds \( X_\theta = k_p \partial / \partial \theta \) and then \( \eta(X_\theta) \equiv k_p \) on the loop the loop \( \gamma_w(z) = \Phi_{\Gamma^2}(w) \) with \( w \in \pi_{-1}^{-1}(p) \). Hence

\[ Q_\beta(p) := \int_{\gamma_w} \eta = \int_0^1 \eta(X_\theta) \gamma_w(t) dt = k_p. \]  

(6.5)

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In particular, if $\beta$ is invariant under $\Phi_{T^2}$, then $\eta(\mathcal{X}_\theta)$ is constant along any orbit of $\Phi_{T^2}$ and it holds that $\pi_{T^2}(Q_\beta) = \eta(\mathcal{X}_\theta)$.

The following remark comes as an observation on Eq. (6.5).

**Remark 23.** Although the construction of $Q_\beta$ depends on the choice of $\beta$, its values at a fixed point of the $S^1$ action do not. In particular, if the action has fixed points with different local Maslov indices, then $Q_\beta$ is never constant.

Note that for some invariant neighbourhood a fixed point $p$ of the symplectic $S^1$ action, a Darboux chart $(U, dq_i \wedge dp_i) \xrightarrow{\phi} (M, \omega)$ can be chosen such that viewed in the chart the restricted $S^1$ action is linearized as $(e^{2m_1 \pi i t}, \ldots, e^{2m_n \pi i t})$.

We denote this linearized $S^1$ action on $U$ by $L_\Phi$. We call $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ the resonance type of the fixed point $p$. From the structure of $Q_\beta$ we have the following proposition.

**Proposition 24.** At the fixed point $p$ of resonant type $(m_1, \ldots, m_n)$, the local Maslov index is

$$k_p = Q_\beta(p) = 2 \cdot \sum_{i=1}^n m_i. \quad (6.6)$$

**Proof.** Since $\phi$ is a symplectomorphism, it induces a bundle isomorphism $\phi_*$ from $\Gamma^2_U$ to $\Gamma^2_{\bar{U}} |_{\bar{U}}$ with $\bar{U} = \phi(U)$. Here $\Gamma^2_U$ is the Maslov $S^1$ bundle for $(U, dq_i \wedge dp_i)$ and it is isomorphic to $U \times S^1$. The lifted $S^1$ action $L_\Phi$ on $\Gamma^2_U$ and the action $\Phi_{T^2}$ on $\Gamma^2_{\bar{U}} |_{\bar{U}}$ are topologically conjugate via $\phi_*$. Then it is straightforward to check that the local Maslov index of $L_\Phi$ at $0 \in U$ is the same as that of $\Phi_{T^2}$ at $p$ (both equal to $k_p$). Due to Proposition 24, this index equals the ordinary Maslov index of the linearized flow $(e^{2m_1 \pi i t}, \ldots, e^{2m_n \pi i t})$ on $(U, dq_i \wedge dp_i)$, which can be computed directly via the Maslov-Arnold map and is equal to $2 \cdot \sum_{i=1}^n m_i$. \hfill $\Box$

The following corollary is a consequence of Theorem 19 and Proposition 24.

**Corollary 25.** If $[\omega] = r \cdot c_1$ and $\beta$ is invariant under $\Phi_{T^2}$, $h_\theta = r \cdot Q_\beta$ is an Hamiltonian for $\Phi$. Then $h_\theta$ takes the same value at all its critical points of the same resonance type. Moreover, the critical values of $h_\theta$ lie in the lattice $r \cdot \mathbb{Z}$.

While we can tell directly from Eq. (6.6) that the local Maslov index at a fixed point is an even number, it can also be observed from the bundle structures of $\Gamma$ and $\Gamma^2$. At a fixed point $p$, since $\dot{\gamma}_\theta(t) = \mathcal{X}_\theta = k_p \frac{\partial}{\partial \theta}$ and
\( \gamma_w(0) = \gamma_w(1) = w \), the number \( k_p \) is an integer and it counts how many rounds the orbit \( \gamma_w \) winds around the fiber \( \pi^{-1}_{\Gamma^2}(p) \cong S^1 \). Note that \( \Phi \) is also lifted to the action \( \Phi_{\Gamma} \) on \( \Gamma_J \) and it satisfies

\[
q_\pm \circ \Phi_{\Gamma} = \Phi_{\Gamma^2} \circ q_\pm, \tag{6.7}
\]

where \( q_\pm : \Gamma_J \to \Gamma_J^2 \) is the map in the commutative diagram 2.1 (in Subsection 2.1). From Eq. (6.7) we can tell that the closed orbit(s) of \( \Phi_{\Gamma} \) has (have) nontrivial winding along the fiber \( \pi^{-1}_{\Gamma^2}(p) \) if and only if the closed orbit(s) of \( \Phi_{\Gamma^2} \) has (have) nontrivial winding along the fiber \( \pi^{-1}_{\Gamma^2}(p) \), and that when \( \Phi_{\Gamma} \) goes one round, \( \Phi_{\Gamma^2} \) goes twice. Therefore, \( k_p \) is always an even number.

It is then straightforward to deduce the following corollary from the results and discussion above, as well as the theorem of Delzant’s polytopes (for references about Delzant’s polytopes, see [10] or [5]).

**Corollary 26.** (A Delzant picture) Suppose that \( G = T^l \). Then there is a smooth map \( Q_\beta = (Q_\beta^{(1)}, \ldots, Q_\beta^{(l)}) \) from \( M \) to \( \mathbb{R}^l \) such that \( Q \) maps the fixed points of the \( T^l \) action to the lattice \( 2 \cdot \mathbb{Z}^l \subset \mathbb{R}^l \). If \( M \) is compact and \( [\omega] = r \cdot c_{T^l} \), then the image \( \text{Im}Q_\beta \) is a polytope with the vertices taking the form \((k_1^i, \ldots, k_l^i)\) with \( k_j^i \) being local Maslov indices at \( p \).

**Proposition 27.** Suppose that \( c_{T^l} = 0 \) and \( \Phi \) is a symplectic \( S^1 \) action on \( M \). Then all the local Maslov indices at the fixed points are equal.

**Proof.** Since \( c_{T^l} = 0 \), there is a connection 1-form \( \alpha = f_\alpha \cdot \frac{\partial}{\partial \theta} \) on \( \Gamma_J^2 \) such that \( df_\alpha = 0 \). For any two fixed points \( p_0, p_1 \) of \( \Phi \), choose \( w_0 \in \pi^{-1}_{\Gamma^2}(p_0) \) and \( w_1 \in \pi^{-1}_{\Gamma^2}(p_1) \). Let \( \lambda : [0, 1] \to \Gamma_J^2 \) be a smooth path from \( w_0 \) to \( w_1 \). Define a map \( h \) from \( S^1 \times [0, 1] \) to \( \Gamma_J^2 \) by

\[
h(z, t) = \Phi_{\Gamma^2} \circ \lambda(t).
\]

Let \( \tilde{f}_\alpha \) be the pulled-back 1-form of \( f_\alpha \) on \( S^1 \times [0, 1] \), i.e. \( \tilde{f}_\alpha = h^*(f_\alpha) \). Then it holds that \( d\tilde{f}_\alpha = h^*(df_\alpha) = 0 \).

By Stokes’ formula,

\[
\int_{S^1 \times \{0\}} \tilde{f}_\alpha = \int_{S^1 \times \{1\}} \tilde{f}_\alpha. \tag{6.8}
\]

Note that for \( i \in \{0, 1\} \), it holds \( h_*\left( \frac{\partial}{\partial \theta} \right)_{(z, i)} = X_\theta \left( \Phi_{\Gamma^2}(w_i) \right) \). As a consequence, we have

\[
\int_{S^1 \times \{i\}} \tilde{f}_\alpha = \int_0^1 f_\alpha \left( \frac{\partial}{\partial \theta} \right)_{(e^{2\pi it}, i)} \, dt = \int_0^1 f_\alpha (X_\theta) \bigg|_{\gamma_{w_i}(t)} \, dt = k_p.
\]

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with \( \gamma_{w_i}(t) = \Phi_{T_2}^{2\pi i} (w_i) \). Together with Eq.(6.8) it concludes the proof.

The following two remarks can be obtained by averaging the Liouville 1-form of a cotangent bundle, but they also appear as consequences of the discussion above about the local Maslov indices. Note that when \( M \) is a cotangent bundle, it holds that \( c_T = [\omega] = 0 \), and the Maslov \( S^1 \) bundles are trivial.

**Remark 28.** Suppose that \( M \) is a connected cotangent bundle and \( h_\theta \) is a Hamiltonian of the \( S^1 \) action. Since \( k_p = k_{p'} \) for any fixed points, \( h_\theta \) has at most 1 critical value.

**Remark 29.** Consider the case where \( M \) is a cotangent bundle and \( \dim M = 4 \). Note that in a neighbourhood of a fixed point \( p \), the \( S^1 \) action can be linearized in a Daboux chart as \( t \mapsto (e^{2\pi i mt}, e^{-2\pi i nt}) \). When \( k_p = 0 \), we have \( m - n = 0 \), and hence the fixed points are all 1 : −1 resonances.

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