Direct experimental verification of quantum commutation relations for Pauli operators

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Quantum theory associates each observable physical quantity with Hermitian operator acting on Hilbert space of states of a given physical system [1]. A fundamental property of operators representing different quantities such as position, momentum or angular momentum of a particle is that they do not mutually commute, which gives rise to peculiar quantum effects like Heisenberg uncertainty relations. Although predictions of quantum theory have been corroborated by countless experiments, direct observation of the non-commutativity of the underlying operators has eluded us. Recently, testing of commutation rules for bosonic creation and annihilation operators based on the combination of single-photon addition [2] and subtraction [3–5] has been reported [6–8].

Besides non-commutativity of bosonic creation and annihilation operators, commutation rules of Pauli operators are also fundamental and important. The Pauli operators were originally introduced to describe Cartesian components of electron spin. More generally, they form, together with the identity operator I, a complete basis in the space of operators acting on a two-level quantum system, a qubit in the language of quantum information theory [9]. Pauli operators are both Hermitian and unitary, which can be represented by $2 \times 2$ matrices,

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1)$$

They satisfy fundamental commutation relations characteristic of Lie algebra su(2),

$$[X, Y] = 2iZ, \quad [Y, Z] = 2iX, \quad [Z, X] = 2iY, \quad (1)$$

where $[A, B] = AB - BA$. Any two different Pauli operators anti-commute, which means that

$$\{X, Y\} = \{Y, Z\} = \{Z, X\} = 0, \quad (2)$$

where $\{A, B\} = AB + BA$. The nonzero commutator implies that, e.g., $ZX \neq XZ$, the overall operation depends on the order of $X$ and $Z$. By combining Eqs. (1) and (2) we find that $ZX = -XZ = iY$. It follows that, in contrast to the case of bosonic creation and annihilation operators [6], it is impossible to demonstrate the non-commutativity of $Z$ and $X$ by applying the two different sequences of operations $ZX$ and $XZ$ to some input single qubit state $|\psi\rangle$. The output states $ZX|\psi\rangle = iY|\psi\rangle$ and $XZ|\psi\rangle = -iY|\psi\rangle$ differ only by an overall phase that is not directly observable. Even applying the operation to a part of an entangled two-qubit state does not help.

In this letter, we report on the direct experimental verification of commutation relations for Pauli operators. Our optical scheme combines two programmable quantum gates [10–12] and an auxiliary maximally entangled two-photon Bell state [13, 14] to directly implement the (anti)-commutator of two Pauli operators. We have completely characterized the commutators by quantum process tomography [15–17]. Our work directly reveals the peculiar algebraic structure underlying quantum theory and represents realization of an advanced quantum information processor [10].

We work with optical qubits encoded into polarization states of single photons whose Hilbert space is spanned by the basis states $|H\rangle$ and $|V\rangle$, representing linearly horizontally and vertically polarized photon, respectively. In Dirac notation we have $X = |H\rangle\langle V| + |V\rangle\langle H|$, $Y = i|V\rangle\langle H| - i|H\rangle\langle V|$ and $Z = |H\rangle\langle H| - |V\rangle\langle V|$. As the Pauli operators are unitary, they can be deterministically implemented by (a sequence of) optical wave-plates [18]. Our main tool is a programmable quantum gate [10–11] where operation on signal qubit is controlled by the state of program qubit. The employed linear optical gate [12] [19,20] consists of a polarizing beam splitter (PBS), where signal and program photons interfere, and the projection of the output program photon onto diagonally linearly polarized state $|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$. If the program photon is prepared in state $|D\rangle$, the polarization state of the signal photon will be unchanged and hence identity operation is applied. However, if we prepare program photon in orthogonal state $|A\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$, the operation $Z$ will be applied to signal photon. The success probability of the gate is $\frac{1}{4}$ and does not depend on the
input state of signal photon.

In our experiment we combine two programmable quantum gates with intermediate unitary operation $U$ on the polarization state of signal photon (see inset in Fig. 1). By preparing the two program photons in the maximally entangled singlet polarization state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|H\rangle|V\rangle - |V\rangle|H\rangle)$ we obtain the following transformation of state of signal photon,

$$\frac{1}{4\sqrt{2}}(ZUI - IUZ) = \frac{1}{4\sqrt{2}}[Z, U],$$

which is, up to a constant prefactor, equal to the commutator $C_{Z,U} = [Z, U]$. By varying $U$ we can thus directly test various commutation relations. Moreover, by preparing the two-photon program state in the triplet Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|D\rangle|A\rangle + |A\rangle|D\rangle) = \frac{1}{\sqrt{2}}(|H\rangle|V\rangle + |V\rangle|H\rangle)$ we realize anti-commutator of $Z$ and $U$.

The experimental setup is shown in Fig. 1. We first generate two pairs of entangled photons by spontaneous down conversion. The photons pass through the half-wave and quarter-wave plates (HWPs and QWPs) and are superposed on the PBSs (see Fig. 1) to implement the desired quantum gates. To achieve good spatial and temporal overlap, the photons are spectrally filtered ($\Delta \lambda_{FWHM} = 3.2$ nm) and detected by the fiber-coupled single-photon detectors [13].

We have experimentally characterized the commutator $C_{Z,U}$ by full quantum process tomography [15–17] for the following five different $U$: $X$, $Y$, $\frac{X+Y}{\sqrt{2}}$, $\frac{X-Y}{\sqrt{2}}$, and $H$, where $H = \frac{X+Z}{\sqrt{2}}$ denotes a Hadamard operation. We have reconstructed the completely positive map $\chi$ that describes the transformation of density matrix $\rho$,

$$\rho_{\text{out}} = \chi(\rho_{\text{in}}) = C_{Z,U}\rho_{\text{in}}C_{Z,U}^\dagger.$$

We can see that $\chi$ contains complete information on the commutator $C$. According to Choi-Jamiolkowski isomorphism [21] [22], $\chi$ can be represented by a positive semidefinite operator on a Hilbert space of two qubits. This operator possesses an intuitive and appealing physical interpretation: it is proportional to a density matrix of a two-qubit state obtained by sending through the channel $\chi$ one part of a maximally entangled Bell state $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|H\rangle|H\rangle + |V\rangle|V\rangle)$. We have probed each commutator by six different input states forming three mutually unbiased bases $\{|H\rangle, |V\rangle\}$, $\{|D\rangle, |A\rangle\}$, and $\{|R\rangle, |L\rangle\}$, where $|R\rangle$ and $|L\rangle$ denote right- and left-hand circularly polarized states. Each output polarization state was fully characterized by measurements in those three bases. From the collected data the operator $\chi$ was reconstructed by means of a standard maximum likelihood estimation algorithm [23].

Examples of the results are shown in Fig. 2a where we plot the real and imaginary parts of the reconstructed $\chi$ for three different $U$. For all five tested $U$ the theory predicts that $C_{Z,U} = KV$ where $V$ is a unitary operation and $K$ is normalization prefactor. Therefore the corresponding $\chi$ should be proportional to the density matrix of a pure maximally entangled state. In particular, for $U = X$ we expect $\chi_{\text{th}} = 8|\Psi^-\rangle\langle\Psi^-|$ and for $U = Y$ we have $\chi_{\text{th}} = 8|\Psi^+\rangle\langle\Psi^+|$, where $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$. The experimental results shown in Fig. 2a are in very good agreement with these theoretical predictions and the fidelity of the reconstructed commutators, defined as normalized overlap of $\chi$ and $\chi_{\text{th}}$, reads $F_X = 0.912 \pm 0.008$ and $F_Y = 0.873 \pm 0.009$. The operator $\chi$ is normalized such that $\text{Tr}(\chi) = \text{Tr}(C^{\dagger}C) = 2|K|^2$. However, the normalization factor $K$ cannot be determined solely from the tomographic data without some reference.

We have therefore performed additional calibration measurements. We have introduced a temporal delay between the signal and program photons so that their wave-packets did not overlap on the PBSs and they behaved as independent entities. To ensure the calibration data and signal data are obtained under identical circumstances, we have carried out an independent calibration for each $U$. With this calibration data at hand and taking into account that the success probability of each programmable quantum gate is $\frac{1}{4}$ we can normalize the data and fix $\text{Tr}(\chi)$ and $|K|$. As can be seen in Table I, the experimentally determined factors $K$ coincide with theoretical prediction within the statistical error. The
fidelities of reconstructed $\chi$ are also shown in Table I. All fidelities exceed 0.8 which indicates good agreement between experimental observations and theory for all five tested commutators.

The matrix $\chi$ actually characterizes the commutator only up to a phase factor. Indeed, two different commutators $C_1$ and $C_2 = e^{i\phi}C_1$ would be represented by the same $\chi$, because $C_1 \rho C_1^\dagger = C_2 \rho C_2^\dagger$. However, the phase $\phi$ does play an important role in the commutation relations. Although we can’t directly measure the overall phase, we can verify the relative phase relations between two commutators. Consider the commutators $C_{Z,X} = 2iY$ and $C_{Z,Y} = -2iX$. From the measurements reported so far we can infer that, with high fidelity, $C_{Z,X} = e^{i\phi_X}2Y$ and $C_{Z,Y} = e^{i\phi_Y}2X$ where the phases $\phi_X$, $\phi_Y$ remain undetermined. We next choose $U = \frac{1}{\sqrt{2}}(X + Y)$. By linearity, but without making any further assumptions, we have $C_{Z,U} = [Z,U] = \sqrt{2}(e^{i\phi_X}Y + e^{i\phi_Y}X)$, hence $C_{Z,U}$ depends on $\phi_X - \phi_Y$. We have experimentally determined this commutator and the result is shown in the right column of Fig. 2a. Notice the nonzero imaginary part of the matrix $\chi$. The reconstructed $\chi$ exhibits a high overlap (fidelity $0.913 \pm 0.009$) with the maximally entangled state $|\Psi_U\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - i|VH\rangle)$. One can easily check that $|\Psi_U\rangle = \frac{1}{\sqrt{2}}(Y - X) \otimes I|\Phi^+\rangle$. The relative phase factor $-1$ between Pauli operators $X$ and $Y$ is consistent with the theoretically expected relationship $e^{i(\phi_X - \phi_Y)} = -1$. The measurement thus corroborates the expected phase relationship between the commutators $C_{Z,X}$ and $C_{Z,Y}$.

After testing the commutation relations, we have proceeded to verify the anti-commutation properties of Pauli operators. For this purpose, we have changed the two-photon program state to $|\Phi^-\rangle$ and performed complete tomographic characterization of the anti-commutator $A_{Z,U} = \{Z,U\}$ for four different operators $I$, $Z$, $H$, and $\frac{1}{\sqrt{2}}(Y + Z)$. Examples of results of tomographic reconstruction are given in Fig. 2b. Since $A_{Z,I} = 2Z$ and $A_{Z,Z} = 2I$, the corresponding $\chi$ are proportional to density matrices of Bell states $|\Phi^-\rangle$ and $|\Phi^+\rangle$, respectively. The experimental results shown in Fig. 2b are in good agreement with theory, as witnessed by high fidelities of the reconstructed completely positive maps, $F_{A_{Z,I}} = 0.910 \pm 0.008$ and $F_{A_{Z,Z}} = 0.897 \pm 0.009$. The reconstructed $\chi$ representing $A_{Z,H}$ and $A_{Z,Z}$ are very similar, which is not surprising because $\{Z,H\} = \sqrt{2}I$. The difference is only in the normalization of $\chi$ that was determined from calibration measurements and reflects the different amplitudes of $A_{Z,Z}$ and $A_{Z,H}$. All four experimentally evaluated normalization factors $|K|$ and fidelities are summarized in Table II.

Finally, we directly observed the anti-commutativity of

| $U$  | $F$   | $K_{\text{calib}}$ | $K_{\text{th}}$ |
|------|-------|-----------------|-----------------|
| $X$  | 0.912 ± 0.008 | 1.98 ± 0.03 | 2.00 |
| $Y + Z \sqrt{2}$ | 0.897 ± 0.009 | 1.98 ± 0.03 | 2.00 |
| $Y$  | 0.852 ± 0.016 | 1.39 ± 0.04 | 1.41 |

| $U$  | $F$   | $K_{\text{calib}}$ | $K_{\text{th}}$ |
|------|-------|-----------------|-----------------|
| $I$  | 0.910 ± 0.008 | 1.97 ± 0.03 | 2.00 |
| $Z$  | 0.897 ± 0.009 | 1.95 ± 0.03 | 2.00 |
| $Y + Z \sqrt{2}$ | 0.901 ± 0.011 | 1.36 ± 0.03 | 1.41 |
| $H$  | 0.869 ± 0.011 | 1.39 ± 0.03 | 1.41 |
directly demonstrate the anti-commutativity of parameterized by the rotation angle of HWP. The dips directly demonstrate the anti-commutativity of quantum operators corresponding to generators. In this way we can directly probe the non-commutativity of quantum operators corresponding to two different Pauli operators. Since e.g., \( \{ Z, X \} = 0 \), we should not observe any four-photon coincidences when \( U = X \) and program photons are prepared in state \( |\Psi^-\rangle \). By rotating the central half-wave plate by angle \( \alpha \) we set \( U = \cos(2\alpha)Z + \sin(2\alpha)X \) and measured the dependence of total number of coincidences on \( \alpha \), that in theory should be proportional to \( \cos^2(2\alpha) \). The results are plotted in Fig. 3a together with a fit by a sinusoidal function.

We can clearly see the dip in coincidences at \( \alpha \approx 45^\circ \) which is a direct manifestation of the anti-commutativity of \( Z \) and \( X \). The visibility of the dip obtained from the fit reads \( V = 84.6 \pm 0.5\% \). We have also tested the relation \( \{ Z, Y \} = 0 \). The results were very similar and are not presented here. Finally, we have directly checked that \( Z \) commutes with itself, \( \{ Z, Z \} = 0 \). We prepared program photons in state \( |\Psi^-\rangle \) and scanned \( \alpha \) over the interval \([-45^\circ, 45^\circ]\). The result can be seen in Fig. 3b with the visibility \( V = 88.7 \pm 0.5\% \).

The coincidence dips plotted in Fig. 3 are slightly shifted with respect to the theoretically expected positions. We attribute this effect to the imperfections of the optical elements employed in the experiment. Another source of experimental errors is noise in the state generation and the imperfect overlap of photons on the PBSs, which reduces the visibility of four-photon interference.

In summary, we have devised and implemented a linear optical scheme that enables direct observation and testing of quantum commutation relations for Pauli operators. In this way we can directly probe the non-commutativity of quantum operators corresponding to different physical quantities, which is one of the cornerstones of quantum physics. The demonstrated scheme also represents an advanced programmable quantum gate, where the type of operation (commutation or anti-commutation) is decided by the state of two-photon program register. By altering the program state, a whole class of operations can be realized, including all linear combinations of \( \{ Z, U \} \) and \( \{ Z, U \} \). Thus, besides being of fundamental interest, our work may also find applications in quantum information processing.

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