THE CENTRAL LIMIT THEOREM FOR MONOTONE CONVOLUTION WITH APPLICATIONS TO FREE LÉVY PROCESSES AND INFINITE ERGODIC THEORY

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Abstract. Using free harmonic analysis and the theory of regular variation, we show that the monotonic strict domain of attraction for the standard arc-sine law coincides with the classical one for the standard normal law. This leads to the most general form of the monotonic central limit theorem and a complete description for the asymptotics of the norming constants. These results imply that the Lévy measure for a centered free Lévy process of the second kind cannot have a slowly varying truncated variance. In particular, the second kind free Lévy processes with zero means and finite variances do not exist. Finally, the method of proofs allows us to construct a new class of conservative ergodic measure preserving transformations on the real line $\mathbb{R}$ equipped with Lebesgue measure, showing an unexpected connection between free analysis and infinite ergodic theory.

1. Introduction

Denote by $\mathbb{C}^+ = \{z \in \mathbb{C} : \Re z > 0\}$ the complex upper half-plane, and let $F : \mathbb{C}^+ \to \mathbb{C}^+$ be an analytic map with $F(iy)/iy \to 1$ as $y \to \infty$. This paper aims to investigate the convergence properties of the iterations $F \circ F \circ \cdots \circ F$ through free probability tools and Karamata’s theory of regular variation.

The probabilistic framework for these iterations is built upon the theory of monotone convolution $\triangleright$, which was introduced by Muraki in [20, 21] according to his notion of monotonic independence. Denote by $\mathcal{M}$ the set of all Borel probability measures on $\mathbb{R}$. The convolution $\triangleright$ is an associative binary operation on $\mathcal{M}$ that corresponds to the addition of monotonically independent self-adjoint random variables. The monotonic independence is one of the five natural notions of independence in noncommutative probability [25, 6, 26, 22], and hence the corresponding monotone convolution becomes a fundamental object in this theory. The connection between the iteration of the function $F$ and monotone convolution is that there exists a unique measure $\mu \in \mathcal{M}$ such that the $n$-fold iteration $F^{\circ n} = F \circ F \circ \cdots \circ F$ of $F$ is...
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precisely the reciprocal Cauchy transform of the $n$-th monotone convolution power $\mu^{\bowtie n} = \mu \bowtie \mu \bowtie \cdots \bowtie \mu$ of $\mu$ for $n \geq 1$ (see Section 2).

In addition, monotone convolution also appears in the context of free probability theory. Indeed, by the subordination results of Biane and Voiculescu [11, 27], for any measures $\rho, \tau \in \mathcal{M}$ there exist unique measures $\sigma_1$ and $\sigma_2$ in $\mathcal{M}$ such that the free convolution $\rho \boxplus \tau = \rho \bowtie \sigma_1 = \tau \bowtie \sigma_2$. The measures $\sigma_1$ and $\sigma_2$ are interpreted in [11] as the Markov transitions for additive processes with free increments, and from this perspective Biane further introduced two natural classes of free Lévy processes: the free additive processes with stationary increment laws (the first kind) and the ones with stationary transition probabilities (the second kind). These two types of stationarity condition are not equivalent for free processes (see [11]).

The contribution of this paper is two-fold: noncommutative and classical aspects. First, on the noncommutative side, we examine the weak convergence of the measures $D_{1/B_n} \mu \bowtie D_{1/B_n} \mu \bowtie \cdots \bowtie D_{1/B_n} \mu$,

where $D_{1/B_n} \mu$ is the dilation of $\mu$ by a factor of $B_n^{-1} > 0$. This corresponds to the uniform convergence of the functions $B_n^{-1} F^{\bowtie n} (B_n z)$ on compact subsets of $\mathbb{C}^+$. This pattern of convergence has been considered in our previous work [29], where we initiated the investigation of strict domains of attraction relative to $\bowtie$. We proved in [29] that a law has a non-empty strict domain of attraction if and only if it is strictly stable. The current paper contributes to this study by characterizing the strict domain of attraction for a particular strictly stable law; namely, the standard arc-sine law $\gamma$ whose density is $\pi^{-1} (2 - x^2)^{-1/2}$ on the interval $(-\sqrt{2}, \sqrt{2})$. Here we discover that the monotonic strict domain of attraction of $\gamma$ coincides with the classical strict domain of attraction of the standard normal law $\mathcal{N}$ (Theorem 3.1). As a consequence, the monotonic central limit theorem (CLT) is equivalent to the classical CLT or to the free CLT.

In the same vein, we show that our CLT result can be applied to the study of free Lévy processes of the second kind (for short, FLP2). The second kind processes are less studied than are the first kind in the literature, mostly because their existence is hard to establish. (The only known examples of FLP2 to date are Biane’s $\bowtie$-strictly stable ones in [11].) In particular, the complete description for the Lévy measure associated to a FLP2, a question due to Biane [11, Section 4.7], is still not available at this point. Following Biane’s question, we show in this paper that the Lévy measure and every marginal law of a FLP2 cannot have slowly varying truncated variances, when one of the marginal laws is centered or does not have a finite mean (Theorem 3.7). This implies that it is not possible to construct a FLP2 with zero expectations.
and finite variances. For general processes with non-zero means, we have a law of large numbers (Theorem 3.6).

Secondly, on the classical side, we address the applications of our methods to infinite ergodic theory for inner functions on $\mathbb{C}$ $\mathbb{C}^+$. When the measure $\mu$ is singular relative to Lebesgue measure $\lambda$ on $\mathbb{R}$, the function $F$ is inner and its boundary restriction

$$T(x) = \lim_{y \to 0^+} F(x + iy) \in \mathbb{R}$$

is a measure preserving transformation on the measure space $(\mathbb{R}, B, \lambda)$, where $B$ is the $\sigma$-field of Borel measurable subsets of $\mathbb{R}$. A famous example of this type is Boole’s transformation: $T(x) = x - x^{-1}$ on $\mathbb{R}$. The main result here is a simple condition for the conservativity of $T$ in probabilistic terms (Theorem 3.9), which says that if $X_1, X_2, \cdots$ are i.i.d. according to $\mu$ and $B_n^{-1}(X_1 + X_2 + \cdots + X_n)$ tend weakly to the standard Gaussian with $\sum_{n=1}^{\infty} B_n^{-2} = \infty$, then the transformation $T$ is conservative. This means that for any $A \in B$ with $\lambda(A) > 0$, almost all points of $A$ will eventually return to $A$ under the iteration of $T$; or, in the sense of Poincaré’s recurrence theorem, that the relation $\liminf_{n \to \infty} d(f(x), f(T^n(x))) = 0$ holds for almost all $x \in \mathbb{R}$ and for any measurable $f$ taking values in a separable metric space $(Y, d)$. Moreover, various ergodic theorems now hold for the map $T$; for example, by Hopf’s ratio ergodic theorem, we obtain the $\lambda$-almost everywhere convergence

$$\lim_{n \to \infty} \frac{\sum_{j=0}^{n} f \circ T^{oj}(x)}{\sum_{j=0}^{n} g \circ T^{oj}(x)} = \frac{\int_X f \, d\lambda}{\int_X g \, d\lambda},$$

whenever $f, g \in L^1(\lambda)$ and $g > 0$ (cf. [2]). Thus, we get an a.e. convergence result for the iteration of $F$ on the boundary $\mathbb{R}$. Using Theorem 3.9, we also construct a new class of conservative and ergodic measure preserving transformations on $\mathbb{R}$, extending an old work of Aaronson [1]. (See Example 3.10.)

Finally, we would like to comment on the methods used in our proofs. The monotonic CLT for bounded summands was first shown by Muraki in [21], with a combinatorial proof (see also [24]). Our results go beyond the case of bounded variables and can treat variables with infinite variance. The proofs rely solely on the free harmonic analysis tools as in [9, 23] and the theory of regular variation [12], making no use of combinatorial methods. The key ingredient here is a connection between the asymptotic behavior of the norming constants $B_n$ and that of the measure $\sigma$ appeared in the Nevanlinna form of the function $F$ (see (2.2)). Indeed, this consideration also plays an important role in our construction of new conservative transformations. We would also like to mention that Anshelevich and Williams have recently proved a remarkable result on the equivalence between monotone and Boolean limit theorems in [4]. Their technique relies on the Chernoff product formula, which is different from
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the approach undertaken in this paper. Here the equivalence between the classical
and the monotonic CLT’s is proved directly, without any reference to the Boolean
theory.

This paper is organized into four sections. After collecting some preliminary ma-
terial in Section 2, we state our main results in Section 3 and present their proofs in
the last section.

2. Setting and Basic Properties

2.1. Monotone convolution. As shown by Franz in [14], given two measures \( \mu, \nu \in \mathcal{M} \) one can find two monotonically independent random variables \( X \) and \( Y \) distributed according to \( \mu \) and \( \nu \), respectively. The monotone convolution \( \mu \triangleright \nu \) of the measures \( \mu \) and \( \nu \) is defined as the distribution of \( X + Y \). In the same paper, Franz
also proved that the definition of the measure \( \mu \triangleright \nu \) does not depend on the partic-
ular realization of the variables \( X \) and \( Y \). (We refer to [14] for the details of this
construction.)

The calculation of monotone convolution of measures requires certain integral
transforms, which we now review. First, the Cauchy transform of a measure \( \mu \in \mathcal{M} \) is defined as
\[
G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z - t} d\mu(t), \quad z \in \mathbb{C}^+,
\]
so that the reciprocal Cauchy transform \( F_\mu = 1/G_\mu \) is an analytic self-map of \( \mathbb{C}^+ \).
The measure \( \mu \) is completely determined by its Cauchy transform \( G_\mu \), and hence by
the function \( F_\mu \). Given two measures \( \mu, \nu \in \mathcal{M} \), it was shown in [14] that
\[
F_{\mu \triangleright \nu}(z) = F_\mu(F_\nu(z)), \quad z \in \mathbb{C}^+.
\]
(See also [7] for measures with bounded support.)

Let \( j \) be a nonnegative integer. We write \( \mu^{ \triangleright j} \) for the \( j \)-th monotone convolution
power \( \mu \triangleright \mu \triangleright \cdots \triangleright \mu \) of a measure \( \mu \in \mathcal{M} \), with \( \mu^{ \triangleright 0} = \delta_c \). Here the notation \( \delta_c \) denotes
the Dirac point mass in a point \( c \in \mathbb{R} \). Analogously, if \( F \) is a map from a non-empty
set \( A \) into itself then the notation \( F^{ \circ j} \) denotes its \( j \)-fold iterate \( F \circ F \circ \cdots \circ F \), where
the case \( j = 0 \) means the identity function on \( A \).

For \( \mu \in \mathcal{M} \), we denote by \( D_b \mu \) the dilation of the measure \( \mu \) by a factor \( b > 0 \),
that is, \( D_b \mu(A) = \mu(b^{-1}A) \) for all Borel subsets \( A \subset \mathbb{R} \). At the level of reciprocal
Cauchy transforms, this means that
\[
F_{D_b \mu}(z) = bF_\mu(z/b), \quad z \in \mathbb{C}^+.
\]
Note that we have \( D_b(\mu \triangleright \nu) = D_b \mu \triangleright D_b \nu \) for any \( \mu, \nu \in \mathcal{M} \).
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Given probability measures \( \{ \mu_n \}_{n=1}^{\infty} \) and \( \mu \) on \( \mathbb{R} \), we say that \( \mu_n \) converges weakly to \( \mu \), written as \( \mu_n \Rightarrow \mu \), if
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f(t) \, d\mu_n(t) = \int_{-\infty}^{\infty} f(t) \, d\mu(t)
\]
for every bounded, continuous real function \( f \) on \( \mathbb{R} \). Note that the weak convergence \( \mu_n \Rightarrow \mu \) holds if and only if the relation \( \lim_{n \to \infty} F_{\mu_n}(z) = F_\mu(z) \) holds for every \( z \) in \( \mathbb{C}^+ \) (cf. [15]). Thus, if both \( \mu_n \Rightarrow \mu \) and \( \nu_n \Rightarrow \nu \) hold, then one has \( \mu_n \triangleright \nu_n \Rightarrow \mu \triangleright \nu \).

The weak convergence of measures needs tightness. Recall that a family \( F \) of positive Borel measures on \( \mathbb{R} \) is tight if
\[
\lim_{y \to +\infty} \sup_{\mu \in F} \mu(\{ t \in \mathbb{R} : |t| > y \}) = 0.
\]

We shall also mention that the tightness for probability measures can be characterized through the asymptotics of their reciprocal Cauchy transforms. More precisely, a family \( F \subset M \) is tight if and only if
\[
(2.1) \quad F_\mu(iy) = iy(1 + o(1)), \quad y > 0,
\]
uniformly for \( \mu \in F \) as \( y \to \infty \) (see [9] for the proof).

2.2. Functions of slow variation. Recall from [12] that a positive Borel function \( f \) on \( (0, \infty) \) is said to be regularly varying if for every constant \( c > 0 \), one has
\[
\lim_{x \to \infty} \frac{f(cx)}{f(x)} = c^d
\]
for some \( d \in \mathbb{R} \) (\( d \) is called the index of regular variation). A regularly varying function with index zero is said to be slowly varying. The notation \( R_d \) denotes the class of regularly varying functions with index \( d \).

The following properties of the class \( R_0 \) are important for our investigation. (See the book [12] for proofs.)

(P1). (Representation Theorem) A function \( f : (0, \infty) \to (0, \infty) \) belongs to \( R_0 \) if and only if it is of the form
\[
f(x) = c(x) \exp \left( \int_1^x \frac{\varepsilon(t)}{t} \, dt \right), \quad x \geq 1,
\]
where \( c(x) \) and \( \varepsilon(x) \) are measurable and \( c(x) \to c \in (0, \infty), \varepsilon(x) \to 0 \) as \( x \to \infty \).

(P2). If \( f \in R_0 \), then for all \( \varepsilon > 0 \) one has
\[
\lim_{x \to \infty} x^\varepsilon f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} x^{-\varepsilon} f(x) = 0.
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\((P3)\). If \(f \in R_0\), then there is a positive sequence \(\{B_n\}_{n=1}^{\infty}\) such that \(\lim_{n \to \infty} B_n = \infty\) and the limit

\[
\lim_{n \to \infty} nB_n^{-2} f(B_n x) = 1
\]

holds for each \(x > 0\).

\((P4)\). (Monotone Equivalents) If \(f \in R_0\) and \(d > 0\), then there exists a non-decreasing function \(g : (0, \infty) \to (0, \infty)\) with \(x^d f(x) \sim g(x)\) as \(x \to \infty\), that is, \(\lim_{x \to \infty} x^d f(x)/g(x) = 1\).

For a finite positive Borel measure \(\mu\) on \(\mathbb{R}\), we introduce the functions \(H_\mu, L_\mu : [0, \infty) \to [0, \infty)\) by

\[
H_\mu(x) = \int_{-x}^{x} t^2 \, d\mu(t) \quad \text{and} \quad L_\mu(x) = \int_{-\infty}^{\infty} \frac{t^2 x^2}{t^2 + x^2} \, d\mu(t).
\]

The mean and the second moment of \(\mu\) are defined in the usual way:

\[
m(\mu) = \int_{-\infty}^{\infty} t \, d\mu(t) \quad \text{and} \quad m_2(\mu) = \int_{-\infty}^{\infty} t^2 \, d\mu(t),
\]

provided that the above integrals converge absolutely. (We also use the somewhat abused notation \(m(\mu) = \infty\) or \(m_2(\mu) = \infty\) to indicate the divergence of these integrals.) The variance of \(\mu\) will be written as \(\text{var}(\mu)\).

Note that both \(H_\mu\) and \(L_\mu\) are continuous and non-decreasing functions. Also, we have that \(H_\mu(x), L_\mu(x) > 0\) for \(x\) large enough if and only if \(\mu \neq r\delta_0\), \(r \geq 0\). Moreover, the functions \(H_\mu(x)\) and \(L_\mu(x)\) are bounded if and only if the second moment \(m_2(\mu)\) exists, and in this case both functions tend to \(m_2(\mu)\) as \(x \to \infty\). It is also known that if \(H_\mu\) varies slowly, then the mean \(m(\mu)\) exists (see \[13\], Section VIII.9.). We should also mention that \(H_\mu \in R_0\) if and only if \(L_\mu \in R_0\), and in this case we have \(H_\mu(x) \sim L_\mu(x)\) as \(x \to \infty\) (see Proposition 3.3 in \[23\]).

Every analytic map from \(\mathbb{C}^+\) to \(\mathbb{C}^+ \cup \mathbb{R}\) has a unique Nevanlinna representation \[2\] Theorem 6.2.1]. In particular, the reciprocal Cauchy transform \(F_\mu\) of a measure \(\mu \in \mathcal{M}\) can be written as:

\[(2.2) \quad F_\mu(z) = z + a + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} \, d\sigma(t), \quad z \in \mathbb{C}^+,\]

where \(a \in \mathbb{R}\) and \(\sigma\) is a finite, positive Borel measure on \(\mathbb{R}\). This integral formula implies that \(\mathfrak{R} F_\mu(z) \geq 3z\). Moreover, this inequality is strict for every \(z \in \mathbb{C}^+\) unless the measure \(\mu\) is degenerate, i.e., \(\mu\) is a point mass. It is easy to verify that the monotone convolution \(\mu \triangleright \nu\) is always nondegenerate if \(\mu\) or \(\nu\) is not degenerate. Also, for an analytic map \(F : \mathbb{C}^+ \to \mathbb{C}^+\) with the property \(\lim_{y \to \infty} F(iy)/iy = 1\), the Nevanlinna form of \(-1/F\) shows that the function \(F\) is of the form \(F = F_\mu\) for a unique \(\mu \in \mathcal{M}\).
The following result is a summary of Lemma 3.5 and Propositions 3.6 and 3.7 in [23], and it will be used repeatedly in this paper.

**Proposition 2.1.** Let \( \mu \) be a nondegenerate probability measure on \( \mathbb{R} \) whose reciprocal Cauchy transform \( F_\mu \) is given by (2.2). Then:

1. \( H_\mu \in \mathbb{R}_0 \) if and only if \( L_\sigma \in \mathbb{R}_0 \) or \( L_\sigma(x) = 0 \) for every \( x \geq 0 \). In this case we have the mean \( m(\mu) = m(\sigma) - a \) and
   \[
   H_\mu(x) - m(\mu)^2 \sim L_\sigma(x) + \sigma(\mathbb{R}) \quad (x \to \infty).
   \]

2. If \( L_\sigma \in \mathbb{R}_0 \), then it follows that
   \[
   \lim_{x \to \infty} \frac{1}{L_\sigma(x)} \int_{-\infty}^{\infty} \frac{|t|^3 x^2 + x^2}{t^2 + x^2} d\sigma(t) = 0.
   \]

3. The measure \( \mu \) has finite variance if and only if the measure \( \sigma \) does. In this case the variance of \( \mu \) is equal to \( m_2(\sigma) + \sigma(\mathbb{R}) \).

3. **Main Results**

3.1. **Central limit theorems.** Let \( \{X_n\}_{n=1}^\infty \) be a sequence of classically independent real-valued random variables with common distribution \( \mu \in \mathcal{M} \). The distribution of the scaled sum

\[
\frac{X_1 + X_2 + \cdots + X_n}{C_n}
\]

is the probability law \( D_{1/C_n} \mu^\ast n \), where \( \mu^\ast n \) denotes the \( n \)-fold classical convolution power of the measure \( \mu \). The classical CLT asserts that there exists a sequence \( C_n > 0 \) such that the sequence \( D_{1/C_n} \mu^\ast n \) converges weakly to the standard normal law \( \mathcal{N} \) as \( n \to \infty \) if and only if \( H_\mu \in \mathbb{R}_0 \) and the mean \( m(\mu) = 0 \) (see [13]). Our first result here is a monotonic analogue of the above CLT, in which the limiting distribution is the standard arc-sine law \( \gamma \) with the reciprocal Cauchy transform

\[
F_\gamma(z) = \sqrt{z^2 - 2}, \quad z \in \mathbb{C}^+.
\]

Here, and in the sequel, the branch of the square root function is chosen so that it is analytic in \( \mathbb{C} \setminus [0, +\infty) \) and \( \sqrt{-1} = i \).

**Theorem 3.1.** (General Monotone CLT) Let \( \mu \) be a nondegenerate probability measure on \( \mathbb{R} \). Then the following assertions are equivalent:

1. There exists a sequence \( B_n > 0 \) such that the measures \( D_{1/B_n} \mu^\ast n \) converge weakly to the arc-sine law \( \gamma \) as \( n \to \infty \).
2. The function \( H_\mu \) is slowly varying and the mean of the measure \( \mu \) is zero.

A priori, the norming constants \( B_n \) and \( C_n \) in the monotonic and the classical CLT’s could be different. Here we would like to emphasize that we can actually
choose the same constants for both limit theorems. More precisely, in the proof of Theorem 3.1 we take \( B_n = C_n \) to be the classical cutoff constants
\[
(3.1) \quad \inf \{ y > 0 : nH_\mu(y) \leq y^2 \}.
\]
(See (4.1) below for details.)

Let \( \nu \in \mathcal{M} \) with \( \nu \neq \delta_0 \). We say that a probability measure \( \mu \) belongs to the strict domain of attraction of the law \( \nu \) (relative to \( \vartriangleright \), and we write \( \mu \in D_\vartriangleright[\nu] \)) if the weak convergence \( D_{1/B_n} \mu^{\times n} \Rightarrow \nu \) holds for some \( B_n > 0 \). The strict domains of attraction relative to the convolutions \( \ast \) and \( \boxplus \) are defined analogously.

For any probability law \( \mu \), Theorem 3.1 shows the equivalence of \( D_{1/B_n} \mu^{\times n} \Rightarrow \gamma \) and \( D_{1/B_n} \mu^{\ast n} \Rightarrow N \). By the free CLT \cite{23}, this equivalence extends to \( D_{1/B_n} \mu^{\boxplus n} \Rightarrow S \), where \( S \) denotes the standard semicircle law. We record this consequence formally in the following

**Corollary 3.2.** One has that \( D_\vartriangleright[\gamma] = D_\ast[N] = D_\boxplus[S] \).

**Remark 3.3.** Given a measure \( \mu \in \mathcal{M} \) and a sequence \( B_n > 0 \), it was shown in Theorem 4.3 of \cite{29} that if the measures \( D_{1/B_n} \mu^{\times n} \) converge weakly to a nondegenerate law \( \nu \in \mathcal{M} \), then there exists a unique \( \alpha \in (0, 2] \) such that the norming sequence \( B_n = n^{1/\alpha} f(n) \) for some \( f \in R_0 \). In fact, the correspondence between the function \( f(x) \) and the sequence \( B_n \) is given by \( f(x) = [x]^{-1/\alpha} B_{[x]} \), where \([x]\) means the integral part of \( x \). Furthermore, by Theorem 3.4 of \cite{29}, the limit law \( \nu \) must be \( \vartriangleright \)-strictly stable with the index \( \alpha \). The arc-sine law \( \gamma \) represents the class of strictly stable laws of index 2 in monotone probability, as the normal law does in classical probability.

Thus, the sequence \( B_n \) in Theorem 3.1 is necessarily of the form \( \sqrt{n} b_n \), where \( b_n \) is a slowly varying sequence. The usual form of the CLT corresponds to the case when \( b_n \) is a constant sequence, and we have

**Theorem 3.4.** (Monotone CLT) Let \( \mu \) be any nondegenerate probability measure on \( \mathbb{R} \), and let \( a \in \mathbb{R} \) and \( b > 0 \). Then the following statements are equivalent:

1. The weak convergence \( D_{1/\sqrt{nb}} (\mu \triangleright \delta_{-a})^{\times n} \Rightarrow \gamma \) holds.
2. The measure \( \mu \) has finite variance.

If (1) and (2) are satisfied, then the constants \( a \) and \( b \) can be chosen as \( a = m(\mu) \) and \( b = \text{var}(\mu) \).

The proof of our results will be presented in the next section. Here we would like to illustrate its main idea through the following example.
Example 3.5. Suppose $\mu = (\delta_0 + \delta_1)/2$. By taking $B_n = \sqrt{n}/2$ and $a = 1/2$, we obtain that

$$F_n(z) = F_{D_1/B_n(\mu \triangleright \delta_n)}(z) = z - \frac{1}{nz}, \quad z \in \mathbb{C}^+,$$

for every $n \geq 1$. To see that the measures

$$\mu_n = D_{1/B_n}(\mu \triangleright \delta_n)^{\otimes n} = [D_{1/B_n}(\mu \triangleright \delta_n)]^{\otimes n}$$

converge weakly to the law $\gamma$, we need to show the pointwise convergence of the iterations $F_n^\otimes(z)$ to the function $\sqrt{z^2 - 2}$ in an appropriate domain. To overcome the difficulty of computing the iteration of $F_n$, we introduce the following conjugacy functions: $\psi_1(z) = z^2$ and $\psi_2(z) = \sqrt{z}$. Note that $\psi_2(-y^2) = iy$ for all $y > 0$ and $\psi_2 \circ \psi_1(z) = z$ for every $z \in \mathbb{C}^+$. For $z = -y^2$, $y > 1$, we observe that

$$F_{\mu_n}(\sqrt{z})^2 = \psi_1 \circ F_n^\otimes \circ \psi_2(z)$$

$$= (\psi_1 \circ F_n \circ \psi_2)^{\otimes n}(z)$$

$$= \left(z - \frac{2}{n} + \frac{1}{n^2z}\right)^{\otimes n}$$

$$= z - 2 + \frac{1}{n^2} \sum_{j=0}^{n-1} \frac{1}{(\psi_1 \circ F_n \circ \psi_2)^{\otimes j}(z)} = z - 2 + O\left(\frac{1}{n}\right).$$

Hence we have $F_{\mu_n}(iy) = \sqrt{(iy)^2 - 2 + O(1/n)}$ for $y > 1$, which implies that $\lim_{n \to \infty} F_{\mu_n}(z) = \sqrt{z^2 - 2}$ for $z = iy$, $y > 1$, as desired.

3.2. Applications to free processes. Let us now consider a $\triangleright$-convolution semigroup $\{ \mu_t : t \geq 0 \}$ of probability measures on $\mathbb{R}$, that is, $\mu_0 = \delta_0$ and other $\mu_t$’s are nondegenerate for $t > 0$, $\mu_s \triangleright \mu_t = \mu_{s+t}$ for all $s, t \geq 0$, and the map $t \mapsto \mu_t$ is weakly continuous. The CLT holds in this case, as well as the law of large numbers.

Theorem 3.6. Let $\{ \mu_t : t \geq 0 \}$ be a $\triangleright$-convolution semigroup.

(1) If there is a time parameter $t_0 > 0$ such that $\mu_{t_0} \in D_0[\gamma]$, then one can find a positive function $B(t) \in R_{1/2}$ such that $D_{1/B(t)}(\mu_t) \Rightarrow \gamma$ as $t \to \infty$. In particular, we have $\mu_t \in D_0[\gamma]$ for every $t > 0$.

(2) If the mean $a = m(\mu_{t_0})$ exists for some $t_0 > 0$, then $D_{1/t}(\mu_t) \Rightarrow \delta_{a/t_0}$ as $t \to \infty$.

A free additive process (in law) is a family $(Z_t)_{t \geq 0}$ of random variables with the following properties: (i) For each $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, the increments

$$Z_{t_0}, Z_{t_1} - Z_{t_0}, Z_{t_2} - Z_{t_1}, \ldots, Z_{t_n} - Z_{t_{n-1}}$$

are freely independent in the sense of Voiculescu [28]. (ii) For any $t$ in $[0, \infty)$, the distribution of $Z_{s+t} - Z_t$ converges weakly to $\delta_0$ as $s \to 0$. (iii) The distribution of $Z_0$ is $\delta_0$ and that of other $Z_t$’s are nondegenerate.
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Given such a process \((Z_t)_{t \geq 0}\), let \(\mu_t\) be the distribution of \(Z_t\) and \(\mu_{s,t}\) be the distributions of \(Z_t - Z_s\) whenever \(0 \leq s < t\). Clearly, these laws satisfy \(\mu_0 = \delta_0\),

\[
\mu_t = \mu_s \boxplus \mu_{s,t} \quad \text{and} \quad \mu_{s,u} = \mu_{s,t} \boxplus \mu_{t,u}, \quad 0 \leq s < t < u,
\]

and \(\mu_t \to \delta_0\) as \(t \to 0\). Conversely, given family \(\{\mu_t : t \geq 0\} \cup \{\mu_{s,t} : 0 \leq s < t\}\) in \(\mathcal{M}\) with the above properties, there exists a free additive process \((Z_t)_{t \geq 0}\) such that the distributions of \(Z_t\) and \(Z_t - Z_s\) are \(\mu_t\) and \(\mu_{s,t}\), respectively (see [11]).

For a free additive process \((Z_t)_{t \geq 0}\) with marginal laws \(\mu_t\), Biane’s subordination result shows that there exist unique measures \(\sigma_{s,t} \in \mathcal{M}\) such that \(\mu_t = \mu_s \triangleright \sigma_{s,t}\) and \(\sigma_{s,t} \triangleright \sigma_{t,u} = \sigma_{s,u}\) for all \(0 \leq s < t < u\).

A free additive process \((Z_t)_{t \geq 0}\) is said to be a **free Lévy process of the second kind** (FLP2) if \(\sigma_{s+t,h,t+h} = \sigma_{s,t}\) for all \(0 \leq s < t\) and \(h \geq 0\). Thus, the marginal laws \(\mu_t = \sigma_{0,t}\) of such a process form a \(\triangleright\)-convolution semigroup, and their reciprocal Cauchy transforms \(F_{\mu_t}\) form a **composition semigroup** of analytic self-maps on \(\mathbb{C}^+\).

It is well-known that for a composition semigroup \(\{F_t\}_{t \geq 0}\) of analytic self-maps on \(\mathbb{C}^+\) with \(F_0(z) = z\), the infinitesimal generator

\[
\varphi(z) = \lim_{\varepsilon \to 0^+} \frac{F_{\varepsilon}(z) - z}{\varepsilon}, \quad z \in \mathbb{C}^+,
\]

of \(\{F_t\}_{t \geq 0}\) exists and is unique [10]. The function \(\varphi : \mathbb{C}^+ \to \mathbb{C}^+ \cup \mathbb{R}\) is analytic with the property \(\lim_{y \to \infty} \varphi(iy)/iy = 0\), and hence it can be written as

\[
\varphi(z) = a + \int_{-\infty}^{\infty} \frac{1 + xz}{x - z} \, d\rho(x).
\]

In [11] Biane showed that a measure \(\rho\) corresponds to the semigroup \(\{F_t\}_{t \geq 0}\) associated with a FLP2 if and only if for each \(t > 0\) the function \(\varphi \circ F_t^{-1}\) has an analytic continuation to \(\mathbb{C}^+\), with values in \(\mathbb{C}^+\). He called such a measure \(\rho\) the **Lévy measure** of FLP2 and raised the question of finding a direct description for \(\rho\).

We have the following result.

**Theorem 3.7.** Let \((Z_t)_{t \geq 0}\) be a FLP2 with marginal laws \(\mu_t\), and let \(\rho\) be the Lévy measure of the semigroup \(F_{\mu_t}\). Suppose that there is a time parameter \(t_0 > 0\) such that the mean \(m(\mu_{t_0}) = 0\) or \(m(\mu_{t_0}) = \infty\). Then

1. For every \(t > 0\), the function \(H_{\mu_t}\) is not slowly varying.
2. The function \(H_{\rho}\) is not slowly varying.

In particular, we have the second moments \(m_2(\rho) = \infty\) and \(m_2(\mu_t) = \infty\) for \(t > 0\).

**Remark 3.8.** Every marginal law \(\mu_t\) in a free additive process can be written as a free convolution of infinitesimal probability measures; for instance, we have

\[
\mu_t = \mu_{0,t/n} \boxplus \mu_{t/n,2t/n} \boxplus \cdots \boxplus \mu_{(n-1)t/n,t}.
\]
Here the infinitesimality of the array \( \{ \mu_{kt/n,(k+1)t/n} : n \geq 1, 0 \leq k \leq n - 1 \} \) is guaranteed by the stochastic continuity of the process \((Z_t)_{t \geq 0}\) (cf. Remark 5.5 of [5]). It follows that each measure \( \mu_t \) is \( \mathbb{H} \)-infinitely divisible (cf. [8]).

3.3. Applications to ergodic theory of inner functions. The general framework for infinite ergodic theory consists of a \( \sigma \)-finite measure space \((X, \mathcal{F}, \nu)\), \( \nu(X) \neq 0 \), and a measure preserving transformation \( T : X \to X \). Thus, the map \( T \) is measurable with respect to the \( \sigma \)-field \( \mathcal{F} \) and \( \nu(T^{-1}A) = \nu(A) \) for every set \( A \in \mathcal{F} \). The notation \( T^{-1}A \) means the pre-image \( \{ x \in X : Tx \in A \} \), and we write inductively \( T^{-n}A = T^{-1}(T^{-(n-1)}A) \) for \( n \geq 2 \). As usual, the map \( T \) is said to be ergodic if for every set \( A \in \mathcal{F} \) such that \( T^{-1}A = A \), either \( \nu(A) = 0 \) or \( \nu(X \setminus A) = 0 \).

The key to understanding the recurrence behavior of the map \( T \) lies in the study of its conservativity, a notion that can be traced back to E. Hopf’s early work [16]. We say that \( T \) is conservative if for every set \( W \in \mathcal{F} \) such that \( \{ T^{-n}W \}_{n=0}^{\infty} \) are pairwise disjoint, necessarily \( \nu(W) = 0 \). For a conservative dynamical system \((X, \mathcal{F}, \nu, T)\) and a non-null set \( A \in \mathcal{F} \), one has the occupation time \( \sum_{j=0}^{\infty} I_A \circ T^j(x) = \infty \) a.e. on \( A \) (i.e., the trajectory \( \{ T^j(x) \}_{j=0}^{\infty} \) returns to the set \( A \) infinitely often). The concept of conservativity plays no role in finite measure spaces; for if \( \nu(X) < \infty \), then any measure preserving map \( T \) on \( X \) will be conservative. We refer to the book of Aaronson [2] for the basics of infinite ergodic theory.

An inner function on \( \mathbb{C}^+ \) is an analytic map \( F : \mathbb{C}^+ \to \mathbb{C}^+ \) for which the limits

\[
T(x) = \lim_{y \to 0^+} F(x + iy) \in \mathbb{R}
\]

exist for almost every \( x \in \mathbb{R} \), relative to Lebesgue measure \( \lambda \) on \( \mathbb{R} \). The measurable map \( T : \mathbb{R} \to \mathbb{R} \) (defined modulo nullsets) is called the boundary restriction of \( F \) to \( \mathbb{R} \).

For \( \mu \in \mathcal{M} \), we recall that

\[
F_\mu(z) = z + a + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} d\sigma(t).
\]

It is known that the function \( F_\mu \) is inner if and only if \( \sigma \) is singular with respect to \( \lambda \) (cf. Chapter 6 of [2]). Clearly, this happens if and only if \( \mu \) is singular. Moreover, for a singular measure \( \mu \), Letac [17] has shown that the boundary restriction \( T \) of \( F_\mu \) is a measure preserving transformation of the measure space \((\mathbb{R}, \mathcal{B}, \lambda)\), and hence is an object of ergodic theory. (The symbol \( \mathcal{B} \) here denotes the Borel \( \sigma \)-field on \( \mathbb{R} \).)

We shall fix a singular measure \( \mu \) in \( \mathcal{M} \). The ergodic theory for the inner function \( F_\mu \) was studied thoroughly in Aaronson’s work [1] (see also [2]), where he proved that the conservativity of the boundary restriction \( T \) implies the ergodicity of \( T \), and that
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$T$ is conservative if and only if

\[ \sum_{n=1}^{\infty} \frac{1}{F_\mu(z)} = \infty \]

for some $z \in \mathbb{C}^+$. Moreover, this condition is independent of the choice of $z$. With this criterion, Aaronson further showed that if the measure $\mu$ is compactly supported and $m(\mu) = 0$, then $T$ is conservative. (See also [18] for a different approach to this result.) In the case of unbounded support, he proved that if $\mu$ is symmetric (i.e., $\mu(A) = \mu(-A)$ for all $A \in \mathcal{B}$) and the function $H_\mu$ is regularly varying with index $d$ for $d \in (0, 1]$, then $T$ is conservative.

Our next result gives a probabilistic criterion for the conservativity of $T$, in which the measure $\mu$ is not assumed to be compactly supported or symmetric. Note that the condition (3.5) below does not involve the iterations of $F_\mu$.

**Theorem 3.9.** Let $\mu$ be a singular probability measure in the set $\mathcal{D}_s[\mathcal{N}]$, and let $\{B_n\}_{n=1}^{\infty}$ be a positive sequence for which the classical CLT holds for $\mu$. If

\[ \sum_{n=1}^{\infty} \frac{1}{B_n} = \infty, \]

then the boundary restriction of $F_\mu$ is conservative (and hence ergodic).

We conclude this section by showing some examples of conservative transformations. It is easy to see from Proposition 2.1 that a measure $\mu \in \mathcal{M}$ has finite variance and $m(\mu) = 0$ if and only if the Nevanlinna form of $F_\mu$ can be rewritten as:

\[ F_\mu(z) = z + \int_{-\infty}^{\infty} \frac{1}{t-z} d\rho(t), \]

where $\rho$ is a finite positive Borel measure on $\mathbb{R}$. Moreover, one has $\rho(\mathbb{R}) = \text{var}(\mu)$.

**Example 3.10.** (a) If the measure $\mu$ has finite variance, then Theorem 3.4 implies that $B_n \sim \sqrt{n \text{var}(\mu)}$ as $n \to \infty$. So, the condition (3.5) is always satisfied in this case. In particular, the generalized Boole transformation

\[ T(x) = x + \sum_{n=1}^{\infty} \frac{p_n}{t_n - x} \]

is conservative and ergodic, whenever $t_n \in \mathbb{R}$ and $p_n > 0$ are sequences such that $\sum_{n=1}^{\infty} p_n < \infty$. Boole’s original transformation $x \mapsto x - x^{-1}$ was proved to be ergodic by Adler and Weiss in [3]. In case $\{t_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ are finite sequences, the ergodicity of $T$ is due to Li and Schweiger [19].

(b) $(d = 0)$ In the case of infinite variance, the simplest way to construct a conservative transformation is to discretize a law from the domain of attraction of the normal
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law $\mathcal{N}$. Let $\nu$ be a probability measure with $m(\nu) = 0$ and $\nu(\{t \in \mathbb{R} : |t| > x\}) = x^{-2}$. Then the measure $\nu$ has infinite variance and satisfies the classical CLT with the norming constants $B_n = \sqrt{n \log n}$ (see (3.1)). Let $\sigma$ be the atomic probability measure drawn from the law $\nu$:

$$\sigma = \sum_{k \in \mathbb{Z}} p_k \delta_k,$$

where $p_k = \nu([k, k+1))$ for all $k \in \mathbb{Z}$. It follows that the measure $\sigma$ also satisfies the classical CLT with the same constants $B_n$.

Now, let $\mu$ be the probability measure defined via the formula:

$$F_\mu(z) = z + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} d\sigma(t),$$

Since $m(\sigma) = 0$ and $L_\sigma \in \mathbb{R}_0$, we know $m(\mu) = 0$ and $H_\mu \in \mathbb{R}_0$ from Proposition 2.1; in other words, $\mu \in \mathcal{D}_*[\mathcal{N}]$.

We now claim that the same sequence $\{B_n\}_{n=1}^{\infty}$ serves as the norming constants for the CLT of the measure $\mu$. Indeed, the sequence $B_n$ satisfies $n B_n^{-2} H_\sigma(B_n) \sim 1$ ($n \to \infty$). This is equivalent to the relation

$$n B_n^{-2} [L_\sigma(B_n) + \sigma(\mathbb{R})] \sim 1 \quad (n \to \infty),$$

which is exactly the criterion of selecting the norming constants in the Monotone CLT for the measure $\mu$ (see (4.1)). It follows that the sequence $B_n$ can be used as the norming constants in the classical CLT for $\mu$.

Therefore, by Theorem 3.9, the boundary restriction

$$T(x) = x + \sum_{k \in \mathbb{Z}} \frac{1 + kx}{k - x} p_k$$

is conservative and ergodic.

4. The Proofs

4.1. Proof of Theorem 3.1. Fix a nondegenerate measure $\mu \in \mathcal{M}$. Suppose that the function $H_\mu$ is slowly varying (and hence $m(\mu)$ exists). Assume further that $m(\mu) = 0$. Let us first specify the positive sequence $\{B_n\}_{n=1}^{\infty}$ that will be used to prove the weak convergence of the measures

$$\mu_n = D_{1/B_n} \mu_n \wedge n \geq 1.$$

By Proposition 2.1 (1), the function $F_\mu$ has the Nevanlinna form

$$F_\mu(z) = z + \int_{-\infty}^{\infty} \frac{1 + t^2}{t - z} d\sigma(t), \quad z \in \mathbb{C}^+,$$
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where the function $L_\sigma \in R_0$ or $L_\sigma = 0$. The case $L_\sigma = 0$ implies that the measure
\( \sigma \) takes the form $r \delta_0$ for some $r > 0$, and hence the measure $\mu = (\delta_{-\sqrt{\sigma}} + \delta_{\sqrt{\sigma}})/2$. Then this case reduces to Example 3.5. Thus, we may and do assume that $L_\sigma \neq 0$ and $L_\sigma \in R_0$. Since measures with finite variance will be treated in Theorem 3.4, we confine ourselves to the case of infinite variance; that is, $L_\sigma(x) \to \infty$ as $x \to \infty$.

Since the function $L_\sigma(x) + \sigma(\mathbb{R})$ is slowly varying, (P3) implies that there exists a sequence $B_n > 0$ such that $\lim_{n \to \infty} B_n = +\infty$ and the relation
\[
(4.1) \quad nB_n^{-2} [L_\sigma (B_n y) + \sigma (\mathbb{R})] \sim 1 \quad (n \to \infty)
\]
holds for each $y > 0$. (The constants $B_n$ as in (3.4) do satisfy these conditions, see Feller’s book [13, Section IX.8.].

For notational convenience, we set $F_n(z) = F_{B_1/B_n \mu}(z)$ and write
\[
F_n(z) = z + \frac{1}{B_n} \int_{-\infty}^\infty \frac{1 + t^2}{t - B_n z} \, d\sigma(t), \quad z \in \mathbb{C}^+.
\]
For every $z = x + iy \in \mathbb{C}^+$ with $|x| < y$, one has that
\[
\left| \frac{1}{B_n} \int_{-\infty}^\infty \frac{1 + t^2}{t - B_n z} \, d\sigma(t) \right| \leq \frac{1}{B_n} \int_{-\infty}^\infty \frac{1 + t^2}{|t - B_n iy| |t - B_n z|} \, d\sigma(t) \leq \frac{2}{B_n} \int_{-\infty}^\infty \frac{1}{\sqrt{t^2 + B_n^2 y^2}} \, d\sigma(t) \leq \frac{2}{B_n^2 y} \int_{-\infty}^\infty \left[ 1 + \frac{t^2 B_n y}{t^2 + B_n^2 y^2} (|t| + B_n y) \right] \, d\sigma(t).
\]
Hence we have, for such $z$’s and $n \geq 1$, that
\[
(4.2) \quad |F_n(z) - z| \leq \frac{2}{B_n^2 y} \left[ L_\sigma (B_n \mathbb{R}) + \sigma (\mathbb{R}) + \int_{-\infty}^\infty \frac{|t|^3 B_n \mathbb{R}}{t^2 + B_n^2 (\mathbb{R})^2} \, d\sigma(t) \right].
\]

**Lemma 4.1.** There exists $N = N(\mu) > 0$ such that if $n \geq N$, then the estimate
\[
(4.3) \quad | F_n^{\circ j} (iy) - iy | \leq \frac{10 j}{n}
\]
holds uniformly for $y > 10$ and for any integer $j$ between 0 and $n$. In particular, the sequence $\{\mu_n\}_{n=1}^\infty$ is tight.

**Proof.** First of all, by (P1), there exists $N_1 = N_1(\mu)$ such that the estimate
\[
\frac{L_\sigma (B_n y) + \sigma (\mathbb{R})}{L_\sigma (B_n) + \sigma (\mathbb{R})} = \frac{c(B_n y)}{c(B_n)} \exp \left( \int_{B_n}^{B_n y} \frac{\varepsilon(t)}{t} \, dt \right) \leq 2 \exp \left( \int_{B_n}^{B_n y} \frac{1}{t} \, dt \right) = 2 y
\]
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holds for any \( y > 10 \) and \( n \geq N_1 \). Then (4.11) shows that there is further a \( N_2 > N_1 \) so that

\[
L_{\sigma}(B_n) + \sigma(\mathbb{R}) \leq \frac{5B_n^2}{4n}, \quad n \geq N_2.
\]

Finally, by Proposition 2.1 (2), we can find the desired \( N > N_2 \) such that

\[
\int_{-\infty}^{\infty} \frac{|t|^3 B_n y}{t^2 + B_n^2 y^2} d\sigma(t) \leq L_{\sigma}(B_n y)
\]

for any \( n \geq N \) and for \( y > 10 \).

Combining these inequalities with (4.2), we obtain that

(4.4)

\[
|F_n(z) - z| \leq \frac{10}{n}
\]

for any \( n \geq N \) and for any \( z \) in the truncated cone

\[
\Gamma_{10} = \{x + iy \in \mathbb{C}^+ : |x| < y, y > 10\}.
\]

In particular, the complex numbers \( F_n(iy) \) lie in the cone \( \Gamma_{10} \) for \( n \geq N \) and \( y > 10 \). For such \( n \)'s and \( y \)'s we now make use of (4.4) to get

\[
\left| F_n^{o2}(iy) - iy \right| \leq \left| F_n(F_n(iy)) - F_n(iy) \right| + \left| F_n(iy) - iy \right| \leq \frac{20}{n},
\]

which implies further that \( F_n^{o2}(iy) \in \Gamma_{10} \). Proceeding inductively, we obtain that

\[
\left| F_n^{oj}(iy) - iy \right| \leq \left| F_n(F_n^{o(j-1)}(iy)) - F_n^{o(j-1)}(iy) \right| + \left| F_n^{o(j-1)}(iy) - iy \right| \leq \frac{10j}{n}
\]

for any integer \( 0 \leq j \leq n \), whence the estimate (4.3) holds.

Finally, it follows from (4.3) that

\[
F_{\mu_n}(iy) = F_{\mu_n}^{on}(iy) = iy(1 + o(1))
\]

uniformly in \( n \) as \( y \to \infty \), and this establishes the tightness of \( \{\mu_n\}_{n=1}^{\infty} \). \( \square \)

Next, let us recall the conjugacy functions appeared in Example 3.5: \( \psi_1(z) = z^2 \) and \( \psi_2(z) = \sqrt{z} \). We write

\[
\psi_1 \circ F_n \circ \psi_2(z) = F_n(\sqrt{z})^2 = z + R(\psi_2(z)), \quad z \in \mathbb{C} \setminus [0, +\infty),
\]

where the function \( R : \mathbb{C}^+ \to \mathbb{C} \) is given by

\[
R(w) = \frac{2w}{B_n} \int_{-\infty}^{\infty} \frac{1 + t^2}{t - B_n w} d\sigma(t) + \left[ \frac{1}{B_n} \int_{-\infty}^{\infty} \frac{1 + t^2}{t - B_n w} d\sigma(t) \right]^2.
\]

We require the following result.
Lemma 4.2. We shall have
\[ \lim_{n \to \infty} \sum_{j=0}^{n-1} R \left( F_n^{\omega j}(iy) \right) = -2, \quad 10 < y < 11. \]

Proof. Fix \( y \in (10, 11) \). Denoting \( w_j = F_n^{\omega j}(iy) \) for \( j = 0, 1, \cdots, n - 1 \), it follows from (4.3) that every \( w_j \) is in the set \( \Gamma = \{ u + iv : |u| < v, 10 < v < 21 \} \) whenever \( n \geq N \). Moreover, since \( \Gamma \subset \Gamma_{10} \) the estimate (4.4) shows that
\[
\sum_{j=0}^{n-1} \left[ \frac{1}{B_n} \int_{-\infty}^{\infty} \frac{1 + t^2}{t - B_n w_j} \, d\sigma(t) \right]^2 = n \cdot O \left( \frac{1}{n^2} \right) = o(1) \quad (n \to \infty).
\]

Thus, we only need to prove that
\[
\sum_{j=0}^{n-1} w_j \int_{-\infty}^{\infty} \frac{1 + t^2}{t - B_n w_j} \, d\sigma(t) = -1 + o(1) \quad (n \to \infty).
\]

By virtue of (4.1), this amounts to showing that
\[
\sum_{j=0}^{n-1} \left[ \frac{w_j}{B_n} \int_{-\infty}^{\infty} \frac{1 + t^2}{t - B_n w_j} \, d\sigma(t) + \frac{1}{B_n^2} (L_\sigma(B_n y) + \sigma(\mathbb{R})) \right] = o(1)
\]
as \( n \to \infty \).

Note that
\[
\frac{w_j}{B_n} \int_{-\infty}^{\infty} \frac{1 + t^2}{t - B_n w_j} \, d\sigma(t) + \frac{1}{B_n^2} (L_\sigma(B_n y) + \sigma(\mathbb{R})) = \frac{1}{B_n} \int_{-\infty}^{\infty} \left[ \frac{(1 + t^2) B_n w_j}{t - B_n w_j} + \frac{t^2 B_n^2 y^2}{t^2 + B_n^2 y^2} + 1 \right] \, d\sigma(t) = \frac{1}{B_n} \int_{-\infty}^{\infty} \frac{t^3(1 + B_n^2 y^2) + t^4 B_n w_j + t B_n^2 y^2}{(t^2 + B_n^2 y^2)(t - B_n w_j)} \, d\sigma(t) = \frac{1}{B_n^2} \int_{-\infty}^{\infty} \frac{t^3 B_n y}{t^2 + B_n^2 y^2} \left[ \frac{1}{B_n y(t - B_n w_j)} + \frac{B_n y}{t - B_n w_j} + \frac{w_j t}{y(t - B_n w_j)} \right] \, d\sigma(t) + \frac{1}{B_n^2} \int_{-\infty}^{\infty} \frac{t B_n y}{t^2 + B_n^2 y^2} \left[ \frac{B_n y}{t - B_n w_j} \right] \, d\sigma(t).
\]

Meanwhile, observe that
\[
\left| \frac{B_n y}{t - B_n w_j} \right| \leq \frac{y}{\Im w_j} = \frac{y}{\Im F_n^{\omega j}(iy)} \leq 1
\]
and
\[
\left| \frac{t}{t - B_n w_j} \right| \leq \sqrt{1 + \left( \Re w_j \frac{\Im w_j}{\Im w_j} \right)^2} < \sqrt{2}.
\]
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for any $n \geq N$, $0 \leq j \leq n - 1$, and $t \in \mathbb{R}$. Thus, when $n$ is large enough such that $B_n > 1$, we have

$$
\sum_{j=0}^{n-1} \frac{w_j}{B_n} \int_{-\infty}^{\infty} \frac{1 + t^2}{t - B_n w_j} \, d\sigma(t) + \frac{1}{B_n^2} (L_\sigma(B_n y) + \sigma(\mathbb{R})) \leq \frac{7n}{B_n} \int_{-\infty}^{\infty} \frac{|t|^3 B_n y}{t^2 + B_n^2 y^2} \, d\sigma(t) + \frac{n}{B_n^2} \int_{-\infty}^{\infty} \frac{|t| B_n y}{t^2 + B_n^2 y^2} \, d\sigma(t).
$$

By (4.1) and Proposition 2.1 (2), we deduce that

$$
\lim_{n \to \infty} F_{\mu_n}(iy) = \sqrt{(iy)^2 - 2}, \quad 10 < y < 11.
$$

Since $\{\mu_n\}_{n=1}^\infty$ is a tight sequence, the above equation determines uniquely the limit function $F_\gamma(z) = \sqrt{z^2 - 2}$, and hence determines uniquely the weak limit $\gamma$ of the sequence $\{\mu_n\}_{n=1}^\infty$. Therefore, the full sequence $\{\mu_n\}_{n=1}^\infty$ converges to the law $\gamma$. □

We now consider the converse of the central limit theorem. Suppose $\mu$ is a distribution in the set $\mathcal{D}_c[\gamma]$, that is, there exist norming constants $B_n > 0$ for which the measures

$$
\mu_n = D_{1/B_n} \mu_n^{\gamma_n}, \quad n \geq 1,
$$
converge weakly to the law $\gamma$ as $n \to \infty$. Clearly, the measure $\mu$ must be nondegenerate. We shall prove that the function $L_\sigma$ is slowly varying, for it follows from Proposition 2.1 (1) that $H_\mu \in \mathbb{R}_0$.

*Proof of Theorem 3.1 (1) implies (2).* By Remark 3.3, the sequence $B_n$ is of the form $B_n = \sqrt{n} f(n)$, where $f$ is a slowly varying function on $(0, \infty)$. By (P4), every regularly varying function with positive index is asymptotically equivalent to a nondecreasing function at infinity. Hence, replacing $B_n$ by its monotone equivalent if necessary, we can assume that $\{B_n\}_{n=1}^\infty$ is an increasing sequence.

Let $\varepsilon \in (0, 1/2)$ be arbitrary. The weakly convergent sequence $\{\mu_n\}_{n=1}^\infty$ is tight. By (2.1), there exists $\beta = \beta(\varepsilon, \mu) \geq 1$ such that for every $y > \beta$, we have

$$\left| \frac{1}{B_j} F_\mu^j(iB_j y) - iy \right| = \left| F_\mu^j(iy) - iy \right| \leq \varepsilon y, \quad j \geq 1.$$  

Then, since $\{B_n\}_{n=1}^\infty$ is monotonic, we deduce, for such $y$’s and any $n > 1$, that

$$\left| F_\mu^j(iB_n y) - iB_n y \right| \leq \varepsilon B_n y, \quad 0 \leq j \leq n - 1. \quad (4.5)$$

We write the function $F_\mu$ in the form: $F_\mu(z) = z + a + A(z)$, where

$$A(z) = \int_{-\infty}^{\infty} \frac{1 + tz}{t - z} d\sigma(t), \quad z \in \mathbb{C}^+.$$  

It follows from (4.5) and a straightforward calculation that

$$\left| A \left( F_\mu^j(iB_n y) \right) - A(iB_n y) \right| \leq 2\varepsilon \Re \left( F_\mu^j(iB_n y) \right), \quad 0 \leq j \leq n - 1.$$  

Since

$$B_n^{-1} \sum_{j=0}^{n-1} A \left( F_\mu^j(B_n z) \right) = F_\mu_n(z) - z - nB_n^{-1} a,$$

we obtain

$$|F_\mu_n(iy) - iy - nB_n^{-1} a - nB_n^{-1} A(iB_n y)| \leq 2\varepsilon \Re [F_\mu_n(iy) - iy] \quad (4.6)$$

for any $y > \beta$ and for $n \geq 1$. This implies that

$$\left| (1 - 2\varepsilon)f_n(y) \right| \leq nyU_\sigma(B_n y) \leq (1 + 2\varepsilon)f_n(y), \quad (4.7)$$

where the functions $f_n$ and $U_\sigma$ are defined as

$$f_n(y) = \Re F_\mu_n(iy) - y, \quad y > 0,$$

and

$$U_\sigma(x) = \int_{-\infty}^{\infty} \frac{1 + t^2}{x^2 + t^2} d\sigma(t), \quad x > 0.$$
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We note for further reference that \( f_n(y) \to \sqrt{y^2 + 2} - y \) as \( n \to \infty \) for each \( y > 0 \), and that the function \( U_\sigma(x) \) is decreasing in \( x \). Also, both \( f_n \) and \( U_\sigma \) are positive functions because \( \mu \) is nondegenerate.

Observe that \( x^2 U_\sigma(x) \sim L_\sigma(x) + \sigma(\mathbb{R}) \) as \( x \to \infty \). Therefore, we need to show that the function \( U_\sigma \) is regularly varying with index \(-2\); that is, for any fixed \( c > 0 \) we shall prove that

\[
\lim_{x \to \infty} U_\sigma(x)^{-1} U_\sigma(cx) = c^{-2}.
\]

We proceed as follows. First, since \( B_n \leq B_{n+1} \), for any large \( x > 0 \) we can choose a positive integer \( n = n(x,y) \) such that

\[
B_n y \leq x < B_{n+1} y.
\]

Moreover, the monotonicity of \( U_\sigma \) and (4.7) imply

\[
 \frac{1 - 2\varepsilon}{1 + 2\varepsilon} \left( \frac{n}{n + 1} \right) f_{n+1}(cy) \leq U_\sigma(cx) \leq \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \left( \frac{n + 1}{n} \right) f_n(cy),
\]

Secondly, by letting \( x \to \infty \) (hence \( n \to \infty \)), we obtain that

\[
\left( \frac{1 - 2\varepsilon}{1 + 2\varepsilon} \right) c^{-2} \left( \frac{y + \sqrt{y^2 + 2}}{y + \sqrt{y^2 + 2/c^2}} \right) \leq \inf_{x \to \infty} \frac{U_\sigma(cx)}{U_\sigma(x)}
\]

and

\[
\limsup_{x \to \infty} \frac{U_\sigma(cx)}{U_\sigma(x)} \leq \left( \frac{1 + 2\varepsilon}{1 - 2\varepsilon} \right) c^{-2} \left( \frac{y + \sqrt{y^2 + 2}}{y + \sqrt{y^2 + 2/c^2}} \right)
\]

for every \( y > \beta \). Finally, by first letting \( y \to \infty \) and then \( \varepsilon \to 0 \), we have that

\[
c^{-2} \leq \liminf_{x \to \infty} \frac{U_\sigma(cx)}{U_\sigma(x)} \leq \limsup_{x \to \infty} \frac{U_\sigma(cx)}{U_\sigma(x)} \leq c^{-2},
\]

whence the desired result for \( U_\sigma \) follows. Consequently, we have \( H_{\mu} \in R_0 \).

The last thing which needs to be proved is that \( m(\mu) = 0 \). To this purpose, we examine the real part of \( A(iB_ny) \) in (4.6) for \( y = 2\beta \). We obtain that

\[
\frac{n}{B_n} \left| a - \int_{-\infty}^{\infty} \frac{(4B_n^3\beta^2 - 1)t}{4B_n^3\beta^2 + t^2} d\sigma(t) \right| \leq (1 + 2\varepsilon) |F_{\mu,}(2\beta i) - 2\beta i| = O \left( \frac{1}{\beta} \right).
\]

Note that \( B_n = \sqrt{n}f(n) \) with \( f \in R_0 \). Since the function \( f \) grows slower than any power at infinity (see (P2)), the above estimate and the dominated convergence theorem imply \( a = m(\sigma) \). Hence, by Proposition 2.1 (1), the measure \( \mu \) has zero expectation. \( \square \)

4.2. Proof of Theorem 3.4. By Proposition 2.1 (3), if the measure \( \mu \) has finite variance, say, \( \text{var}(\mu) = b > 0 \), then the constants \( B_n \) can be taken as \( B_n = \sqrt{n}(m_2(\sigma) + \sigma(\mathbb{R})) = \sqrt{nb} \) in order to satisfy the condition (4.1). Since the measure
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\( \mu \triangleright \delta_{-a} \) is simply a translation of \( \mu \), the function \( H_{\mu \triangleright \delta_{-a}} \) is slowly varying if and only if the function \( H_{\mu} \) is. Thus, the proof of Theorem 3.4 is merely a word-for-word translation of the proof of Theorem 3.1; only this time the key estimate (4.2) and Lemma 4.1 are easier to obtain because \( (1 + t^2) d\sigma(t) \) is a finite measure.

4.3. Proof of Theorem 3.6. We first focus on the proof of Theorem 3.6 (1). Suppose that \( \{ \mu_t : t \geq 0 \} \) is a \( \triangleright \)-convolution semigroup, and that there is a \( t_0 > 0 \) such that \( \mu_{t_0} \in \mathcal{D}_b[\gamma] \). We aim to construct a positive function \( B \in R_{1/2} \) for which the weak convergence \( D_{1/B(t)} \mu_t \rightharpoonup \gamma \) \( (t \to \infty) \) holds.

To this purpose, we let \( t_n \) be any positive sequence so that \( \lim_{n \to \infty} t_n = \infty \) and write

\[
\nu = \mu_{t_0} \quad \text{and} \quad \nu_n = \mu_{t_0(t_n/t_0-[t_n/t_0])},
\]

where \([x]\) denotes again the integral part of \( x \). By the semigroup property, we have

\[
\mu_{t_n} = \mu_{t_0(t_n/t_0)+t_0(t_n/t_0-[t_n/t_0])} = \nu_n \mu_{t_0}, \quad n \geq 1.
\]

Proof of Theorem 3.6 (1). Since \( \nu \in \mathcal{D}_b[\gamma] \), there exists a positive sequence \( C_n \) such that \( D_{1/C_n} \nu^{t_n} \rightharpoonup \gamma \) along the set of positive integers. By Remark 3.3, the function \( C(x) = C[x] \) belongs to the class \( R_{1/2} \).

Let us define \( B(x) = C(x)/\sqrt{t_0} \) for \( x > 0 \). The function \( B(x) \) is in the class \( R_{1/2} \), and we write

\[
(4.8) \quad D_{1/B(t_n)} \mu_{t_n} = \left( D_{B(t_n/t_0)/B(t_0)} D_{1/B(t_0)} \nu_n \mu_{t_0} \right) \triangleright D_{1/B(t_n)} \nu_n.
\]

Since \( \{\nu_n\}_{n=1}^\infty \subset \{\mu_t : 0 \leq t \leq t_0\} \), the stochastic continuity of the semigroup \( \{\mu_t : t \geq 0\} \) implies that the family \( \{\nu_n\}_{n=1}^\infty \) is tight. Then (P2) shows that \( D_{1/B(t_n)} \nu_n \rightharpoonup \delta_0 \) as \( n \to \infty \). Now, the desired weak convergence \( D_{1/B(t_n)} \mu_{t_n} \rightharpoonup \gamma \) follows from (4.8) and

\[
\lim_{n \to \infty} B(t_n)^{-1} B(t_n/t_0) = 1/\sqrt{t_0}.
\]

Finally, every \( \mu_t \) belongs to the set \( \mathcal{D}_b[\gamma] \) simply because \( \mu_t^{t_n} = \mu_{nt} \) for \( n \geq 1 \). \( \square \)

Theorem 3.6 (2) follows from a similar consideration based on the law of large numbers: \( D_{1/B} \nu^{t_n} \rightharpoonup \delta_0 \) (see Theorem 5.1 of [29]). We omit the details.

4.4. Proof of Theorem 3.7. Consider the marginal laws \( \mu_t \) of a FLP2, and let \( \rho \) be the Lévy measure of the corresponding semigroup \( \{F_{\mu_t}\}_{t \geq 0} \). Suppose there is a \( t_0 > 0 \) such that \( m(\mu_{t_0}) = 0 \) or \( m(\mu_{t_0}) = \infty \).

In the sequel we write \( F_t = F_{\mu_t} \) for each \( t \geq 0 \) and denote by \( \sigma_t \) the measure associated to the Nevanlinna form of \( F_t \).

\[\text{For a WLLN of non-identical summands, see the article arXiv:1304.1230 (added April 20, 2013).}\]
By calculus, we have
\[(4.9) \quad F_t(z) - z = \int_0^t \varphi(F_s(z)) \, ds, \quad t \geq 0, \quad z \in \mathbb{C}^+,\]
where the function \(\varphi\) is the infinitesimal generator of \(\{F_t\}_{t \geq 0}\) as in (3.2).

**Lemma 4.3.** For any fixed \(t > 0\), we shall have
\[
\frac{1}{t} (\Im F_t(iy) - y) \sim \int_0^1 \Im \varphi(F_s(iy)) \, ds \quad (y \to \infty).
\]
When \(t = 0\), we have
\[
\Im \varphi(iy) \sim \int_0^1 \Im \varphi(F_s(iy)) \, ds \quad (y \to \infty).
\]

**Proof.** Let \(t \geq 0\) be given. By a change of variable in (4.9), we need to show that
\[
\int_0^1 \Im \varphi(F_s(iy)) \, ds \sim \int_0^1 \Im \varphi(F_s(iy)) \, ds \quad (y \to \infty).
\]
Indeed, denoting \(d\nu(x) = (1 + x^2) \, d\rho(x)\) as in the Nevanlinna representation (3.3) of the generator \(\varphi\), the estimate
\[
|\Im \varphi(F_{ts}(iy)) - \Im \varphi(F_s(iy))| \leq \int_{-\infty}^\infty \frac{1}{x - F_{ts}(iy)} - \frac{1}{x - F_s(iy)} \, d\nu(x) \\
\leq \int_{-\infty}^\infty \frac{|F_{ts}(iy) - F_s(iy)|}{|x - F_{ts}(iy)| \, |x - F_s(iy)|} \, d\nu(x) \\
\leq \varepsilon(y) \Im \varphi(F_s(iy))
\]
holds for all \(s \in [0, 1]\) and \(y > 0\), where the bound
\[
\varepsilon(y) = \sup_{0 \leq s \leq 1} \left[ y^{-1} |F_{ts}(iy) - F_s(iy)| \left(1 + y^{-1} |F_{ts}(iy) - F_s(iy)|\right) \right].
\]
Also, since the family \(\{\mu_u\}_{0 \leq u \leq T}\) is tight for any finite time \(T > 0\), (2.1) implies
\[
\lim_{y \to \infty} \varepsilon(y) = 0.
\]
Therefore, by integrating the above estimate with respect to \(s\), we get
\[
\left| \int_0^1 \Im \varphi(F_{ts}(iy)) \, ds - \int_0^1 \Im \varphi(F_s(iy)) \, ds \right| \leq \varepsilon(y) \int_0^1 \Im \varphi(F_s(iy)) \, ds,
\]
whence the desired result follows. \(\square\)

We now prove Theorem 3.7.

**Proof of Theorem 3.7 (1).** Assume the contrary that we can find a \(t > 0\) such that \(H_{\mu_t} \in R_0\).
Proposition 2.1 then shows that \( L_{\sigma_t} \in R_0 \) or \( L_{\sigma_t} = 0 \). In addition, by Lemma 4.3, we have
\[
t_0^{-1} (\Re F_0(iy) - y) \sim t^{-1} (\Re F(iy) - y) \quad (y \to \infty),
\]
or, in other words,
\[
\int_{-\infty}^{\infty} \frac{(1 + x^2)y^2}{x^2 + y^2} d\sigma_{t_0}(x) \sim \frac{t_0}{t} \int_{-\infty}^{\infty} \frac{(1 + x^2)y^2}{x^2 + y^2} d\sigma_t(x) \quad (y \to \infty).
\]
If \( L_{\sigma_t} = 0 \), then the measure \( \sigma_{t_0} \) has finite second moment. Hence, \( m_2(\mu_{t_0}) < \infty \) by Proposition 2.1. If \( L_{\sigma_t} \in R_0 \), then so does the function \( L_{\sigma_{t_0}} \). Thus, by Proposition 2.1 again, we have \( H_{\mu_{t_0}} \in R_0 \).

Clearly, the above conclusion gives a contradiction in the case \( m(\mu_{t_0}) = \infty \). On the other hand, if \( m(\mu_{t_0}) = 0 \), then Theorems 3.1 and 3.6 imply that \( D_{1/B(s)} \mu_s \Rightarrow \gamma \) as \( s \to \infty \) for some function \( B(s) > 0 \). We know from Remark 3.8 that every marginal law in a free additive process is \( \boxplus \)-infinitely divisible. So, in this case the law \( \gamma \) must be \( \boxplus \)-infinitely divisible for being a weak limit of \( \boxplus \)-infinitely divisible measures. This is, however, a contradiction because one can verify that the inverse of the function \( F_\gamma \) (relative to composition) cannot be extended analytically to \( \mathbb{C}^+ \) (cf. Theorem 5.10 of [9]). Therefore, none of the functions \( H_{\mu_t} \) shall be slowly varying, proving Theorem 3.7 (1).

**Proof of Theorem 3.7 (2).** The second part of Lemma 4.3 implies
\[
\int_{-\infty}^{\infty} \frac{(1 + x^2)y^2}{x^2 + y^2} dp(x) \sim \int_{-\infty}^{\infty} \frac{(1 + x^2)y^2}{x^2 + y^2} d\sigma_1(x) \quad (y \to \infty).
\]
Since the function \( H_{\mu_t} \) is not slowly varying, the function \( L_{\sigma_t} \) is not slowly varying neither. Thus, the above asymptotic equivalence shows that the function \( H_{\rho} \) cannot be slowly varying. \( \square \)

4.5. **Proof of Theorem 3.9.** We begin by noticing that the measure \( \mu \) is also in the set \( D_\gamma[\gamma] \), and that we may (and do) assume \( \mu_n = D_{1/B_n} \mu_{n} \Rightarrow \gamma \). In view of Aaronson’s condition (3.4), we seek for a better control on the summands \(-\Re G_{\mu_n}(z)\).
We will do this for \( z = i \).

**Proof of Theorem 3.9.** Since \( \gamma \) is Lebesgue absolutely continuous, Theorem 3.1 implies
\[
\lim_{n \to \infty} \mu_n([-1, 1]) = \gamma([-1, 1]) = 1/2.
\]
In other words, one has
\[
\mu_{n}([-B_n, B_n]) \sim 1/2 \quad (n \to \infty).
\]
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Also, it is easy to see that
\[
\sum_{n=1}^{\infty} \frac{1}{F_{\mu}^n(i)} = \sum_{n=1}^{\infty} -G_{\mu^n}(i)
\]
\[
= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{1 + t^2} \, d\mu^n(t)
\]
\[
\geq \sum_{n=1}^{\infty} \int_{|t| \leq B_n} \frac{1}{1 + t^2} \, d\mu^n(t) \geq \sum_{n=1}^{\infty} \frac{1}{1 + B_n^2} \mu^n([-B_n, B_n]).
\]

Clearly, if the sequence $B_n^{-1}$ is not square summable, then
\[
\sum_{n=1}^{\infty} \frac{1}{F_{\mu}^n(i)} = \infty.
\]

Therefore, the boundary restriction of $F_\mu$ is conservative. \square

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