ON THE NUMBER OF NON-INTERSECTING HEXAGONS IN 3-SPACE

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Abstract. Two hexagons in the space are said to intersect badly if the intersection of their convex hulls consists of at least one common vertex as well as an interior point. We are going to show that the number of hexagons on $n$ points in 3-space without bad intersections is $o(n^2)$, under the assumption that the hexagons are ‘fat’.

1. Introduction

The general problem of finding the maximum number of hyperedges in a geometric hypergraph in $d$-dimensional space with certain forbidden configurations (intersections) was considered by Dey and Pach in [1]. In this note we are interested in finding the maximum number of (convex planar) polygons on some vertex set of $n$ points in 3-space, where no two of them are allowed to intersect in certain ways.

It was asked by Gil Kalai and independently by Günter Ziegler what is the maximum number of triangles spanned by $n$ points such that any two are almost disjoint:

Definition 1 (Almost disjoint polygons). Two planar polygons in 3-space are said to be almost disjoint if they are either disjoint or their intersection consists of one common vertex.

The maximum number of pairwise almost disjoint triangles on $n$ points is bounded above by $O(n^2)$. Indeed, in a set of such triangles, any given point can be a vertex of at most $(n-1)/2$ triangles. It is not known whether the maximum number of such triangles is $o(n^2)$ or not. Károlyi and Solymosi constructed configurations with $\Omega(n^{3/2})$ almost disjoint triangles on $n$ points [2]. Finding sharper bounds seems like a very hard problem. In fact, it is not even known if the genus of a polytope on $n$-vertices can have order $n^2$. If so, there would be a configuration of order $n^2$ almost disjoint triangles on $n$ points. The best lower bound of the largest genus is $n \log n$, due to a construction of McMullen, Schulz and Wills [3]. For more details, we refer the interested readers to [5] where Ziegler gives a simplified construction providing the same bound. In this note, we study the maximum number of polygons without bad intersections, defined below.

Definition 2 (Badly intersecting polygons). Two planar polygons in 3-space are said to intersect badly if the intersection of their convex hulls consists of at least one common vertex as well as an interior point.

In what follows by polygons in space we mean planar polygons, i.e. the vertices are co-planar. We say that a collection of polygons has no bad intersections if no two of these polygons intersect badly. In such arrangements if two hexagons share a vertex then this is the only common point they have, but hexagons not sharing a vertex might intersect. In particular, they cannot share a diagonal and so the maximum number of $k$-gons ($k \geq 4$) without bad intersection is, again, $O(n^2)$. This trivial upper bound is actually sharp for quadrilaterals ($k = 4$). One can give a construction of $\Omega(n^2)$ quadrilaterals without a bad
intersection as follows: Suppose we are given \( n/2 \) points \( P_1, \ldots, P_{n/2} \) in general position (no three points collinear) on a plane \( \pi \). Fix any vector \( v \) not parallel to \( \pi \). Then the \( n \) points \( P_1, P_1 + v, \ldots, P_{n/2} + v \) are incident to \( \Theta(n^2) \) desired quadrilaterals with vertices \( P_i, P_i + v, P_i + v, P_i + v \), where \( 1 \leq i < j \leq n \).

When \( k = 6 \), we can show that the number of hexagons without bad intersections in 3-space is \( o(n^2) \), under an extra assumption on the ‘fatness’ of the hexagons defined below. We conjecture that Theorem 4 holds for any set of hexagons or even for pentagons.

**Definition 3 (Fat hexagons).** Let \( c \geq 1 \) and \( 0 < \alpha < \pi/2 \). A hexagon is \((c, \alpha)-fat\) if

1. it is convex;
2. the ratio of any two sides is bounded between \( 1/c \) and \( c \); and
3. it has three non-neighbour vertices having interior angles between \( \alpha \) and \( \pi - \alpha \).

Our main tool is the Triangle Removal Lemma of Ruzsa and Szemerédi, which states that for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that any graph on \( n \) vertices with at least \( \varepsilon n^2 \) pairwise edge-disjoint triangles has at least \( \delta n^3 \) triangles in total. See [4] for the original formulation of this result. The precise statement of our theorem is as follows.

**Theorem 4.** For any \( c \geq 1 \) and \( 0 < \alpha < \pi/2 \) numbers there is a function \( F(c, \alpha)(n) \),

\[
\frac{F(c, \alpha)(n)}{n^2} \to 0 \text{ as } n \to \infty,
\]

such that any family of \((c, \alpha)-fat\) hexagons in 3-space on \( n \) points without bad intersections has size at most \( F(c, \alpha)(n) \).

Note that, since almost disjoint hexagons don’t intersect badly, the same upper bound, \( o(n^2) \), holds for pairwise almost disjoint hexagons.

2. **Proof of Theorem**

Suppose there are \( \varepsilon n^2 \) \((c, \alpha)-fat\) hexagons on \( n \) vertices in 3-space. We will show that two of these hexagons intersect badly.

To reduce the dimension of the ambient space, we project these hexagons onto a random plane such that a positive fraction is \((c', \alpha')-fat\). Indeed, if we project a \((c, \alpha)-fat\) hexagon \( H \) to a plane making an angle at most \( \theta < \pi/2 \) with the plane containing \( H \), some simple calculations show that the projected hexagon is \((c', \alpha')-fat\), where

\[
c' = \frac{c}{\cos \theta} \quad \text{and} \quad \alpha' = \cos^{-1} \left( \frac{\cos \alpha + \sin^2 \theta}{\cos^2 \theta} \right).
\]

The existence of badly intersecting hexagons relies on a similar-slope property. This can be described quantitatively by the difference of two angles of inclination. To this end, let \( \phi > 0 \) be the smallness of such difference which is to be determined later.

We choose from the \((c', \alpha')\)-fat projected hexagons the most popular family consisting of \( \varepsilon' n^2 \) hexagons, which have inscribed triangles of similar shapes and orientations.

More precisely, let us enumerate by any order the projected hexagons as \( \{H_i\} \) and label their vertices as \( A_i, B_i, C_i, D_i, E_i, F_i \), oriented counter-clockwise, where \( B_i, D_i, F_i \) are the three non-neighbour vertices having angles between \( \alpha' \) and \( \pi - \alpha' \).

There exists a positive fraction of these hexagons so that for any \( i, j \), the inclined angles of the diagonals \( A_i C_i \) and \( A_j C_j \) differ by at most \( \phi \). Similarly the same property holds true for the diagonals \( C_i E_i \) and \( E_i A_i \) in yet a sub-collection of \( \varepsilon' n^2 \) hexagons.
We define $G$ to be the graph whose vertices are the $n$ projected points and whose edges are from the triangles formed by the vertices $A_i, C_i, E_i$ chosen above. Then, $G$ contains $\varepsilon' n^2$ edge-disjoint triangles. An application of the Triangle Removal Lemma yields a triangle $T$ whose edges come from three different hexagons, say $H_1$, $H_2$ and $H_3$. For each $i = 1, 2, 3$, let $T_i$ be the triangle $A_iC_iE_i$.

We are ready to study the intersection properties of these three hexagons in the 3-space. In other words, we now 'unproject' the $n$ points.

Two of the triangles, say $T_1$ and $T_2$, lie on the same side of $T$ and let $T_1$ be the triangle making a larger angle with $T$. Then, as shown in Figure 1, the hexagon $H_2$ intersects badly with the triangle $T_1$, and hence with the hexagon $H_1$, as long as the three non-neighbour vertices $B_1, D_1, F_1$ lie outside of the triangle $T$ on the plane of projection, which is guaranteed if we choose

$$\phi < \tan^{-1}\left(\frac{\sin \alpha'}{\varepsilon' + \cos \alpha'}\right),$$

the right hand side being a lower bound of the six angles $B_1A_1C_1$ etc. under the $(\varepsilon', \alpha')$-fatness assumption. This completes the proof.

![Figure 1](image_url)

**Figure 1.** The triangle $T_1 = AC_1E_1$ and the hexagon $H_2 = AB_2C_2D_2E_2F_2$ intersect badly. Here the triangle $T$ is $AC_2E_1$.

**References**

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