Some statistical properties of the interaction between a two-level atom and three field modes

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Abstract

We consider the interaction between a two-level atom and a quantum system that consists of three electromagnetic fields. An analytic solution is provided for the wave function of a pairwise mutual interaction between a two-level atom and three modes using a frequency converter. SU(2) group generators are used to describe these field mode interactions. In addition, a canonical transformation is employed in order to convert the Hamiltonian model into a Jaynes–Cumming-like model, which is used to solve the Schrödinger equation. Statistical properties related to the atomic inversion, entanglement, and squeezing phenomena are discussed. Superstructure patterns and partial disentanglement, as well as squeezing swapping between quadratures, are displayed for selected parameters.

Keywords: interaction between coupled three waves and atom, non-classical properties, SU(2) algebra

(Some figures may appear in colour only in the online journal)

1. Introduction

In the field of quantum optics, the problem of determining the interaction between three electromagnetic fields has attracted substantial attention for more than half a century because it represents an important non-linear parametric interaction [1–9]. This interaction plays a significant role in multiple relevant physical phenomena, such as the stimulated and spontaneous emissions of radiation and coherent Raman and Brillouin scattering. In the optical realm, two different interactions occur: frequency conversion and parametric amplification. In a recent experiment, Bodani et al. produced an interaction between five modes in a non-linear crystal, which is suitable for generating a three-mode entangled state [10]. Concurrently, an experiment conducted by Geordiades et al. [11] generated squeezed light using non-degenerate parametric down conversion in order to excite a two-photon transition in atomic cesium. On the other hand, anti-Stokes radiation in coherent Raman and Brillouin scattering—as well as stimulated emission of radiation, excluding the up conversion of light signals in non-linear media—result from frequency conversion interactions. Furthermore, the combination of these types of interactions (parametric amplification and frequency conversion) led to the quantum non-demolition measurement [12–17]. This combination can also construct back-action evading amplifiers given different coupling parameters. This result prompted us to modify the Hamiltonian model and extend it to include interactions with two-level atoms [18].

The Hamiltonian model, which describes the interaction between three modes in the form of frequency conversion, is given by

\[
\frac{\dot{H}(t)}{\hbar} = \sum_{i=1}^{3} \alpha_i \left( \hat{a}_i^\dagger \hat{a}_i + \frac{1}{2} \right) + i\lambda_1 \left( \hat{a}_1^\dagger \hat{a}_2 \exp(i(\omega_1 - \omega_2)t) - \hat{a}_1 \hat{a}_2^\dagger \exp(-i(\omega_1 - \omega_2)t) \right)
\]

\[
+ i\lambda_2 \left( \hat{a}_1^\dagger \hat{a}_3 \exp(i(\omega_1 - \omega_3)t) - \hat{a}_1 \hat{a}_3^\dagger \exp(-i(\omega_1 - \omega_3)t) \right)
\]

\[
+ i\lambda_3 \left( \hat{a}_2^\dagger \hat{a}_2 \exp(i(\omega_1 - \omega_2)t) - \hat{a}_2 \hat{a}_2^\dagger \exp(-i(\omega_1 - \omega_2)t) \right),
\] (1)
in which $\hat{a}_i(\hat{a}_i^\dagger), j = 1, 2, 3, $ are annihilation (creation) operators that satisfy the commutation relation, $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \omega_i$, $i = 1, 2, 3$ are the field frequencies, and $\lambda_i, i = 1, 2, 3$ are coupling parameters for the interacting modes.

This work primarily studies the interaction between the present system and a two-level atom. This means that we injected three coupled waves of frequency conversion form into a cavity that contains a two-level atom. To the best of our knowledge, this type of Hamiltonian model has not been previously studied.

First, before we continue, we must remove time dependency from the Hamiltonian model (1). If we use the new time-dependent operators, $\hat{A}_j = \hat{a}_j \exp(-i \omega_j t)$, we can achieve a time-independent Hamiltonian model of the form

$$\frac{\hat{H}}{\hbar} = i \lambda_i (\hat{A}_i^\dagger \hat{A}_i - \hat{A}_i \hat{A}_i^\dagger) + i \lambda_j(\hat{A}_j^\dagger \hat{A}_j - \hat{A}_j \hat{A}_j^\dagger),$$

(2)

where $\hat{A}_i$ and $\hat{A}_i^\dagger$ have the same properties as $\hat{a}_i$ and $\hat{a}_i^\dagger$. Furthermore, if we define $\hat{J}_i, i = x, y, z$ to be

$$\hat{J}_x = i (\hat{A}_3^\dagger \hat{A}_2 - \hat{A}_2^\dagger \hat{A}_3), \quad \hat{J}_y = i (\hat{A}_3^\dagger \hat{A}_1 - \hat{A}_1^\dagger \hat{A}_3), \quad \hat{J}_z = i (\hat{A}_1^\dagger \hat{A}_2 - \hat{A}_2^\dagger \hat{A}_1),$$

(3)

then the following commutation relations are satisfied:

$$[\hat{J}_x, \hat{J}_y] = i \hat{J}_z, \quad [\hat{J}_y, \hat{J}_z] = i \hat{J}_x, \quad [\hat{J}_z, \hat{J}_x] = i \hat{J}_y.$$

(4)

It is easy to see that these operators generate a special unity, SU($2$) Lie algebra. If we substitute equation (3) into the Hamiltonian model (2), we produce

$$\frac{\hat{H}}{\hbar} = \lambda_1 \hat{J}_1 + \lambda_2 \hat{J}_2 + \lambda_3 \hat{J}_3.$$

(5)

In addition, if the coupled field modes are injected into a cavity, an interaction will occur between the modes and the atom. In this case, the Hamiltonian model has form

$$\frac{\hat{H}}{\hbar} = \omega_0 \hat{C}_i + g(\hat{c}_i + \hat{c}_i^\dagger)(\hat{\lambda}_i \hat{J}_i + \hat{\lambda}_j \hat{J}_j),$$

(6)

in which $\omega_0$ is the frequency of the atomic level differences and $g$ is the coupling between the atom and the coupled field modes that can be absorbed within coupling $\lambda$. Here, $\hat{c}_i$ and $\hat{c}_i^\dagger$ are the usual Pauli operators with the following properties:

$$[\hat{c}_i, \hat{c}_j] = \hat{c}_k, \quad [\hat{c}_i, \hat{c}_j^\dagger] = \pm 2 \hat{c}_k.$$

(7)

In order to discuss some statistical properties of the system, we must first determine the time-dependent dynamical operators. This is not an easy task for the Hamiltonian model (6) because of the existence of non-linear terms. Consequently, we must remove them or reduce their number, while ensuring that the physical properties of the system are unaffected. This process is demonstrated in the next section.

The rest of the paper is organized as follows. In section 3 we discuss atomic inversion. Section 4 addresses the degree of entanglement. Section 5 considers atomic and variance squeezing, and our conclusion is presented in section 6.

2. Time-dependent wave function

In this section, we obtain a time-dependent wave function, which allows us to derive and discuss some of the system’s statistical properties. In order to do this, let us first use Euler angles $\alpha, \beta$ and $\gamma$ to rotate operators $\hat{J}_i$, for $i = x, y, z$, from which we can remove angular momentum operator $\hat{J}_c$. Consider the following transformation:

$$\begin{bmatrix} \hat{J}_x \\ \hat{J}_y \\ \hat{J}_z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \hat{L}_x \\ \hat{L}_y \\ \hat{L}_z \end{bmatrix},$$

(8)

where $a_{ij}, i, j = 1, 2, 3, $ are given by

$$a_{11} = \cos \alpha \cos \beta - \cos \gamma \sin \beta \sin \alpha, \quad a_{12} = \cos \alpha \sin \beta + \cos \gamma \cos \beta \sin \alpha, \quad a_{13} = - \sin \gamma \sin \alpha, \quad a_{21} = - \sin \gamma \sin \beta, \quad a_{22} = - \sin \gamma \cos \beta \cos \alpha, \quad a_{23} = - \cos \gamma \sin \beta, \quad a_{31} = \cos \alpha \cos \beta + \cos \gamma \sin \beta \sin \alpha, \quad a_{32} = \cos \alpha \sin \beta - \cos \gamma \cos \beta \sin \alpha, \quad a_{33} = \cos \gamma \cos \alpha,$$

and $\alpha, \beta, \gamma$ are the well-known Euler angles. Now, if we let

$$\gamma = \tan^{-1} \left( \frac{\lambda_3}{\lambda_1 \sin \alpha + \lambda_2 \cos \alpha} \right),$$

(10)

then without the loss of generality, equation (6) assumes the following form:

$$\frac{\hat{H}}{\hbar} = \frac{\omega_0}{2} \hat{c}_i + (\hat{c}_i + \hat{c}_i^\dagger)(\lambda_1 \hat{L}_x + \lambda_2 \hat{L}_y + \lambda_3 \hat{L}_z),$$

(11)

in which $\lambda_i$ and $\lambda_j$ are given by

$$\lambda_x = (\lambda_1 \cos \alpha - \lambda_2 \sin \alpha) \cos \beta - \lambda_3 \sin \beta \csc \gamma, \quad \lambda_y = (\lambda_1 \cos \alpha - \lambda_2 \sin \alpha) \sin \beta + \lambda_3 \cos \beta \csc \gamma.$$

(12)

As can be seen, dealing with equation (11) is not easy. Therefore, we apply rotating wave approximation. In order to do this, we rewrite the ladder operators in the following forms:

$$\hat{L}_x = \frac{1}{2}(\hat{L}_x + \hat{L}_x^\dagger), \quad \hat{L}_y = \frac{1}{2i}(\hat{L}_x - \hat{L}_x^\dagger).$$

(13)

These operators let us rewrite equation (11) as

$$\frac{\hat{H}}{\hbar} = \frac{\omega_0}{2} \hat{c}_i + (\hat{c}_i + \hat{c}_i^\dagger)(\lambda_1 \hat{L}_x + \lambda_2 \hat{L}_y + \lambda_3 \hat{L}_z).$$

(14)

Here, we applied rotating wave approximation, in which $\hat{L}_x$ and $\hat{L}_z$ satisfy the following commutation relations:
where \( m = -l - l + 1, ..., l - 1, l \) and \( \hat{L} \cdot |l; l \rangle = \hat{L} |l; l \rangle = 0 \).

It should also be noted that \( \hat{j}^2 = \hat{L}^2 \), and consequently, \( \hat{j}^2 |j; j \rangle = j(j + 1) |j; j \rangle \). Thus, we can conclude that \( |j; j \rangle = |l; l \rangle \), which implies that the eigenvalue of operator \( \hat{j}^2 \) equals the eigenvalue of operator \( \hat{L}^2 \). This means \( l = j \), and consequently, \( |\pm j; l \rangle = |\pm l; l \rangle \).

Using the properties of Pauli matrices, we can apply the evolution operator in order to derive the wave function for the present system. This gives us

\[
\frac{\hat{H}}{\hbar} = \begin{pmatrix}
\alpha_0 / 2 & \lambda \hat{L}_- \\
\lambda \hat{L} & -\alpha_0 / 2
\end{pmatrix}
\]

Therefore, the time-dependent evolution operator, \( \exp(-i\hat{H}t/\hbar) \), takes the form

\[
\exp\left(-i\frac{\hat{H}t}{\hbar}\right) = \begin{bmatrix}
\cos \hat{\mu}_t & -i \lambda \sin \hat{\mu}_t \\
-i \lambda \sin \hat{\mu}_t & \cos \hat{\mu}_t \\
\end{bmatrix}
\]

in which

\[
\hat{\mu}_t = \frac{\alpha_0^2}{4} + \lambda^2 \hat{g}_t, \quad i = 1, 2, \quad \hat{L} = \hat{L}_+ \hat{L}_-.
\]

In addition, let us consider the two-level atom. Initially, it exists in a superposition state \( |\theta, \phi \rangle \), which includes the excited state \( |e \rangle \), as well as the ground state \( |g \rangle \). This state is a combination of the upper and lower states and is defined to be

\[
|\theta, \phi \rangle = \cos \theta |e \rangle + \sin \theta \exp(i \phi) |g \rangle.
\]

where \( \theta \) is the atomic coherent angle and \( \phi \) is the relative phase of the atomic levels. This equation clearly indicates that the atom is in its excited state when \( \theta = 0 \) and its ground state when \( \theta = \pi / 2 \). On the other hand, we can use the generalized coherent state (Bloch state) in order to describe the state of the SU(2) algebra, which is given by

\[
|z; l \rangle = \frac{1}{(1 + |z|^2)^{1/4}} \sum_{m=-l}^{l} \frac{2l!}{(l-m)!(l+m)!} \nu^{l+m} |m; l \rangle,
\]

where \( z = \text{a complex number} \) and \( \nu = (z/|z|) \tan |z| \). However, in order to apply the present system, we must modify this state, creating a suitable representation of the rotated SU(2) system-level atom. Therefore, we use the transformation provided in equation (8) in order to rotate the Bloch state. After some simple algebraic manipulation, this produces the following form of the rotated state:

\[
|\eta; l \rangle = \left( \cos \eta + i \cos \alpha \sin \gamma \sin \eta \right)^{2l}
\times \sum_{m=-l}^{l} \sqrt{\frac{2l!}{(l-m)!(l+m)!}} \nu^{l+m} |m; l \rangle.
\]

Here, we have

\[
\xi = \sqrt{A_0} \left( \cos \gamma \cos \alpha - i \sin \alpha \right) \sin \eta \exp(-i\beta), \quad \text{and} \quad A_0 = \left( \cos \eta + i \cos \alpha \sin \gamma \sin \eta \right)^2.
\]

Now we can construct the wave function for the system when \( t > 0 \):

\[
|\psi(t)\rangle = A(t)|\eta; l, e \rangle + B(t)|\eta; l, g \rangle,
\]

in which

\[
A(t) = \left[ \cos \left( \cos \eta - i \frac{\alpha_0}{2\beta_1} \sin \theta \right) \sin \theta - i e^{i\phi} \sin \left( \sin \frac{\beta_1}{\beta_2} \right) \sin \theta \right],
\]

\[
B(t) = \left[ e^{i\phi} \left( \cos \eta - i \frac{\alpha_0}{2\beta_2} \sin \theta \right) \sin \theta - i \lambda \sin \left( \sin \frac{\beta_1}{\beta_2} \right) \sin \theta \right].
\]

In the following sections, we employ this result in order to explore some statistical properties of our system.

3. Atomic inversion

Collapse and revival form one of the fundamental phenomena in quantum optics. This non-classical phenomenon is related to the interaction between atoms and fields and is typically observed in atomic systems when interactions between atomic subsystems and cavity fields occur. In particular, it represents the difference in the probabilities of finding a two-level atom in an excited state and a ground state. Mathematically, it can be described using the expected value of the Pauli operator, \( \hat{\sigma}_z(t) \). This means that in order to discuss the revival and collapse phenomenon, we must calculate (\( \hat{\sigma}_z(t) \)), which can be done using equations (24), (25), and (22), together with equation (25). Applying these equations gives us the following.
Here, we have

\[ \begin{align*}
\delta &= (\cos^2 \eta + \cos^2 \alpha \sin^2 \gamma \sin^2 \eta), \\
\phi_0 &= \tan^{-1} \left( \frac{y}{x} \right), \\
\phi_1 &= (\cos \alpha \sin \gamma \tan \eta + \sec \gamma \tan \alpha).
\end{align*} \]

(27)

As we can see, analysing the behaviour of this function is not easy. Therefore, we perform numerical calculations and plot our results in figure 1 in order to examine the effect of some of the involved parameters. In figures 1(a) and (b), we fixed the value of ratio \( \omega_0/\lambda \) to 0.31. We also considered \( \phi = \beta = \pi/2 \) and \( \alpha = \eta = \pi/4 \) in order to examine the behaviour of the function for different values of quantum number \( l \) and atomic angle \( \theta \).

For example, we consider the case in which \( l = 5 \) and \( \theta = \pi/6 \) and observe fluctuations around zero between –0.58 and 0.58. In this case, the function displays significantly short quiescent periods, which occur after each revival period. This can be seen in figure 1(a). These revivals have superstructures, which result from the cumulative interference between...
the involved sinusoidal functions. When the value of quantum number \( l \) is doubled, a similar behavior is observed. However, an increase in the number of fluctuations is also observed, in addition to an interference between the patterns, which signifies the breaking up of the superstructure shapes and the appearance of new quiescent periods. Figure 1(b) displays these behaviors. Our results indicate that an increase in the value of quantum number \( l \) produces fast oscillations and a strong correlation between the two systems. A slight change in the value of the ratio, \( \omega/\lambda = 0.35 \), results in an increase in the amplitude, and consequently, the range of the fluctuations. Moreover, the atomic angle appears to increase the number of revival periods, which can be observed in figure 1(c). An increase in the amplitude of the fluctuations of the revival periods is apparent when we understand that the atom’s initial state was an excited state. Furthermore, we also perceive that the superstructure shape of the function remains unchanged, even though the quiescent period diminishes. This behavior is exhibited in figure 1(d).

4. Degree of entanglement

The possible existence of non-classical correlations between distinct quantum systems creates a fundamental difference between quantum and classical physics. The behavior responsible for this phenomenon is called entanglement. Therefore, in order to continue our examination, we devote this section to the phenomenon of entanglement. Quantum entanglement has multiple applications. For example, it is a resource for communication, information processing, and quantum computing [20], such as in the investigation of quantum teleportation [21, 22], dense coding [23], decoherence in quantum computers, and the evaluation of quantum cryptographic schemes [24]. Some authors have proposed physically motivated postulates based on the operational formulation of quantum mechanics in order to characterize entanglement, as in [25–29].

Different methods can be used to discuss entangling. One method involves the von Neumann entropy of the state, \( \hat{\rho}(t) \), which is defined to be

\[
S = -\text{Tr}(\hat{\rho} \ln \hat{\rho}), \quad \hat{\rho} = |\psi(t)\rangle \langle \psi(t)|,
\]

(28)

where \( |\psi(t)\rangle \) is the state given by equation (24). Another method considers purity \( P(t) = \text{Tr}(\hat{\rho}^2(t)) \), in which \( \hat{\rho}(t) \) is the reduced density operator for the subsystem. Given a two-level subsystem, we can use the following linear entropy:

\[
\Lambda(t) = \frac{1}{2}(1 - f^2(t)),
\]

(29)

in which \( f(t) \) is the well-known Bloch sphere radius defined by

\[
f(t) = \sqrt{\langle \hat{\sigma}_z(t) \rangle^2 + \langle \hat{\sigma}_y(t) \rangle^2 + \langle \hat{\sigma}_x(t) \rangle^2},
\]

(30)

where the simple qubit state is successfully represented, up to an overall phase, by a point on the sphere with coordinates that are the expected values of the atomic set of operators. Here, we must calculate the expected values of both \( \langle \hat{\sigma}_x(t) \rangle \) and \( \langle \hat{\sigma}_y(t) \rangle \). However, first we need to calculate the expected value of \( \langle \hat{\sigma}_x(t) \rangle \). In this case, we have

\[
\langle \hat{\sigma}_x(t) \rangle = \frac{1}{2} e^{i\phi} \delta^{(2)} \sum_{m=-l}^{l} \frac{2l!}{(l-m)!(l+m)!} \xi_1^{(2l+2m)} \sin 2\theta 
\]

\[
	imes \left( \cos \mu_1 t + \frac{i\alpha_0}{2\mu_1} \sin \mu_1 t \right) \left( \cos \mu_2 t + \frac{i\alpha_0}{2\mu_2} \sin \mu_2 t \right)
\]

\[
+ \frac{2^2}{2} \delta^{(2)} \sum_{m=-l}^{l} \frac{2l!}{(l-m-1)!(l+m+1)!} \xi_2^{(2l+2m)}
\]

\[
	imes e^{-\frac{\phi}{2} + \frac{\phi}{2} \sin \mu_1 t + \frac{\xi_1}{\mu_1} \sin \mu_2 t + \frac{\xi_2}{\mu_2} \sin \mu_2 t}
\]

\[
- \frac{i\lambda}{2} \delta^{(2)} \sum_{m=-l}^{l} \frac{2l!}{(l-m-1)!(l+m+1)!} \xi_2^{(2l+2m)}
\]

\[
	imes e^{-\frac{i\phi}{2} - \frac{i\phi}{2} \sin \mu_1 t - \frac{\xi_1}{\mu_1} \sin \mu_2 t - \frac{\xi_2}{\mu_2} \sin \mu_2 t}
\]

\[
+ \frac{i\lambda}{2} \delta^{(2)} \sum_{m=-l}^{l} \frac{2l!}{(l-m)!} \xi_2^{(2l+2m)}
\]

\[
	imes e^{-\frac{i\phi}{2} + \frac{i\phi}{2} \sin \mu_1 t + \frac{\xi_1}{\mu_1} \sin \mu_2 t + \frac{\xi_2}{\mu_2} \sin \mu_2 t}
\]

\[
(31)
\]

Thus, the definition of \( \hat{\sigma}_x \) allows us to write the expected values of \( \hat{\sigma}_x \) and \( \hat{\sigma}_y \). Simple calculations provide the following results.

\[
\langle \hat{\sigma}_x(t) \rangle = \frac{1}{2} e^{i\phi} \delta^{(2)} \sum_{m=-l}^{l} \frac{2l!}{(l-m)!(l+m)!} \xi_1^{(2l+2m)} \sin 2\theta 
\]

\[
\times \left( \cos \mu_1 t \cos \mu_2 t - \frac{\alpha_0^2}{4} \sin \mu_1 t \sin \mu_2 t \right)
\]

\[
+ \frac{1}{2} e^{i\phi} \delta^{(2)} \sum_{m=-l}^{l} \frac{2l!}{(l-m-1)!(l+m+1)!} \xi_2^{(2l+2m)}
\]

\[
\times \cos 2(\phi_0 + \phi_1) \sin \mu_1 t \sin \mu_2 t
\]

\[
\times \sin \mu_1 t \sin \mu_2 t
\]

\[
+ \frac{\lambda}{2} \delta^{(2)} \sum_{m=-l}^{l} \frac{2l!}{(l-m)!} \xi_2^{(2l+2m)} \cos 2\theta
\]

\[
\times \left( \cos \mu_1 t \sin \phi_1 - \phi_0 + \frac{\alpha_0}{2\mu_1} \sin \mu_1 t \cos (\phi_1 - \phi_0) \right) \sin \mu_2 t
\]

\[
\times \mu_2 t
\]

\[
\times \sin \mu_2 t
\]

\[
(32)
\]

Additionally, we can write the expected value of \( \hat{\sigma}_y(t) \) in the following form.
\[ \langle \hat{\sigma}_z(t) \rangle = \frac{\alpha_0}{4} \delta^{2l} \sum_{m=-l}^{l} \frac{2l!}{(l-m)!(l+m)!} \left| \xi \right|^{2l+2m} \]

\[ \times \left( \frac{\sin \mu_1 t}{\mu_1} \cos \mu_2 t + \frac{\sin \mu_2 t}{\mu_2} \cos \mu_1 t \right) \sin 2\theta \]

\[ - \frac{1}{2} \delta^{2l} \sum_{m=-l}^{l} \frac{2l!}{(l-m)!(l+m+1)!} \left| \xi \right|^{2l+2m} \]

\[ \times \sin 2(\phi_1 + \phi_0) \frac{\sin \mu_1 t}{\mu_1} \sin \mu_2 t - \frac{\sin \mu_2 t}{\mu_2} \sin \mu_1 t \sin 2\theta \]

\[ - \lambda \delta^{2l} \sum_{m=-l}^{l} \frac{2l!}{(l-m+1)!(l+m)!} \left| \xi \right|^{2l+2m+1} \cos^2 \theta \]

\[ \times \left( \cos \mu_1 t \cos(\phi_1 - \phi_0) - \frac{\alpha_0}{2} \frac{\sin \mu_1 t}{\mu_1} \sin(\phi_1 - \phi_0) \right) \sin \mu_2 t \]

\[ + \lambda \delta^{2l} \sum_{m=-l}^{l} \frac{2l!}{(l-m-1)!(l+m+1)!} \left| \xi \right|^{2l+2m+1} \sin^2 \theta \]

\[ \times \left( \cos \mu_2 t \cos(\phi_1 + \phi_0) + \frac{\alpha_0}{2} \frac{\sin \mu_2 t}{\mu_2} \sin(\phi_1 + \phi_0) \right) \sin \mu_1 t \]

\[ \frac{\mu_1}{\mu_2} \]

(33)

Here, the atomic phase has been dropped. In order to address the degree of entanglement, we must examine the behaviour of function \( \Lambda(t) \). However, because this expression is complicated, we plot different values of the involved parameters in figure 2. For instance, values \( \alpha_0/\lambda = 0.31, \alpha = \eta = \pi/4 \) and \( \beta = \pi/3, l = 5 \) have been fixed and considered in Figures 2(a) and (b), in which \( \theta = 0 \) and \( \theta = \pi/4 \), respectively. The function is observed to fluctuate, displaying both entanglement and disentanglement. We also see partial entanglement with rapid fluctuations, which becomes pronounced in the middle of the considered time period, as can be seen in figure 2(a). Moreover, it should be noted that when transition energy increases, as compared with the coupling parameter (not presented here), the function’s amplitude increases, and consequently, disentanglement also increases. Furthermore, the rapid fluctuations reduce the function value and shift the partial entanglement during the prolonged period. On the other hand, given a superposition state and \( \theta = \pi/4 \), the partial entanglement increases after a short time period, and the function fluctuates between 0.27 and 0.5. In addition, the maximum value of disentanglement occurs when \( \tau = 11 \), which is displayed in figure 2(b). Note that if \( \alpha_0/\lambda = 0.77 \) (not displayed here), no essential change is observed. However, when \( \alpha_0/\lambda = 0.38 \), the situation is somewhat different. For example, given \( \theta = \pi/4 \), the function fluctuates between 0.3 and 0.5, and the interference between the patterns reduces when compared with the previous two cases. This behaviour is exhibited in figure 2(c). In this case, the second period of maximum entanglement occurs in the middle of the considered time period, as is shown in figure 2(a). Lastly, we let \( \theta = \pi/6 \) for ratio \( \alpha_0/\lambda = 0.38 \).

Here, the function displays partial entanglement at 0.19, after the onset of the interaction. This signifies that the function’s amplitude decreases, and consequently, the value of partial entanglement increases. Moreover, the function increases the rapid fluctuations, as well as the interference between the pattern, which can be observed in figure 2(d).

### 5. Squeezing phenomenon

This section addresses entropy squeezing, as well as atomic variance. The squeezing phenomenon is one of the most important phenomena that reflect non-classical behaviour in quantum systems. As is well known, in this case, the discussion of quantum fluctuations depends on the concept of uncertainty relations. This means that we must apply entropic uncertainty relations for two-level systems rather than Heisenberg uncertainty relations. This condition has previously been discussed by multiple authors [30–34]. In an even \( N \)-dimensional Hilbert space, the optimal entropic uncertainty relation for sets containing \( N + 1 \) complementary observables with non-degenerate eigenvalues can be described using the following inequality [35–38]:

\[ \sum_{i=1}^{N+1} H(\hat{\sigma}_i(t)) \geq \frac{N}{2} \ln \left( \frac{N}{2} \right) + \ln \left( 1 + \frac{N}{2} \right) \]

(34)

where \( H(\hat{\sigma}_i(t)) \) is the Shannon information entropy of variable \( \hat{\sigma}_i(t) \). Because the uncertainty relation of entropy can be used as a general criterion for squeezing in the entropy of atomic variables, given a two-level atom with \( N = 2 \), we have \( 0 \leq H(\hat{\sigma}_i(t)) \leq \ln 2 \). Consequently, equation (34) confirms that the information entropies of operators \( \hat{\sigma}_x(t), \hat{\sigma}_y(t) \) and \( \hat{\sigma}_z(t) \) satisfy inequality

\[ H(\hat{\sigma}_x(t)) + H(\hat{\sigma}_y(t)) + H(\hat{\sigma}_z(t)) \geq 2 \ln 2 \]

(35)

Therefore, if we define \( \delta H(\hat{\sigma}_i(t)) = \exp(\lambda H(\hat{\sigma}_i(t))) \), then we can write

\[ \delta H(\hat{\sigma}_x(t)), \delta H(\hat{\sigma}_y(t)) \geq 4\delta H(\hat{\sigma}_z(t)) \]

(36)

and the corresponding Shannon information entropies are given by

\[ H(\hat{\sigma}_i(t)) = - \sum_{j=1}^{N} P_j(\hat{\sigma}_j(t)) \ln P_j(\hat{\sigma}_j(t)) \]

where \( P_j(\hat{\sigma}_j(t)) \) is the probability distribution given \( N \) possible outcomes for the measurement of operator \( \hat{\sigma}_j(t) \). Thus, in order to obtain the information entropies of the atomic operators for a two-level atom with \( N = 2 \), we can use the following equation:

\[ H(\hat{\sigma}_i(t)) = - \frac{1}{2} \left( (1 + \langle \hat{\sigma}_i(t) \rangle) \ln \frac{1}{2} (1 + \langle \hat{\sigma}_i(t) \rangle) \right) + \left( 1 - \langle \hat{\sigma}_i(t) \rangle \right) \ln \frac{1}{2} (1 - \langle \hat{\sigma}_i(t) \rangle) \]

(37)

Although the uncertainty relation for the entropy of any quantum system that interacts with a two-level atom can be used as a general criterion to squeeze atomic variables, we refer to research that presents the probability representation.
of quantum mechanics and analyse experimental possibilities in order to confirm the uncertainty relations for position and momentum \([30, 31]\), as well as for the entropic uncertainty relation \([32–34]\). This may open the door for more applications in the future.

Squeezing of atomic variables can be defined using the entropy uncertainty relation, equation (36), and is called entropy squeezing. Fluctuations in the components (\(\varepsilon = x\) or \(y\)) of the atomic dipole are said to be squeezed in entropy if the information entropy, \(H(\hat{\alpha}(t))\) of \(\hat{\alpha}(t)\), satisfies the following condition:

\[
\delta H(\hat{\alpha}(t)) - \frac{2}{\sqrt{\delta H(\hat{\alpha}(t))}} < 0, \quad \varepsilon = x, y \quad (39)
\]

Given fixed values \(l = 2\), \(\eta = \pi/4\), and \(\phi = 0\), we consider the case in which \(\alpha = \pi/4\), \(\beta = \pi/3\), \(\omega_0/\lambda = 0.37\), and the atom is in its excited state, \(\theta = 0\). Squeezing occurs twice in the second quadrature, \(E_y(t)\), but is absent from the first quadrature, \(E_x(t)\). Squeezing is observed once after a short time period, whereas the second occurrence happens after a long time period, producing a squeezing amount less than that of the previous period. Figure 3(a) displays this behaviour.

If \(\alpha = \pi/3\), and we use ratio \(\omega_0/\lambda = 0.83\), squeezing occurs once in the first quadrature, \(E_x(t)\), and has a greater amount than that of the previous case, as is shown in figure 3(b). When the atom is in a ground state, \(\theta = \pi/2\), and \(\omega_0/\lambda = 0.94\), although \(\alpha\) and \(\beta\) have equivalent values to those of the first case, considerably more squeezing is observed in \(E_y(t)\), which can be seen in figure 3(c). On the other hand, the squeezing phenomenon grows more pronounced and appears after the onset of the interaction in the first quadrature, \(E_x(t)\), when \(\theta = \pi/6\) and \(\omega_0/\lambda = 0.38\). Therefore, we can conclude that squeezing in the entropy is affected by a change in the involved angles as well as the value of ratio \(\omega_0/\lambda\).

Now we are ready to turn our attention to variance squeezing, in which fluctuations in the component, \(\Delta \hat{\alpha}_x\), of the atomic dipole are said to be squeezed if \(\hat{\alpha}_x\) satisfies the following condition:
Therefore, in order to examine variance squeezing, given fixed values $l = 5$ and $\eta = \pi/4$, but different values of the involved parameters, we plot the results in figure 4. For example, given excited state $\theta = 0$ for $\alpha = \beta = \pi/4$ and ratio $\omega_0/\lambda = 0.44$, squeezing is observed in the first quadrature, $V_x$, and absent from the second quadrature, $V_y$. Figure 4(a) reveals this behaviour. If we increase the ratio and let $\omega_0/\lambda = 1.6$, the squeezing disappears from the first quadrature and occurs only in the second, $V_y$, which is displayed in figure 4(b).

However, if we change the value of the atomic angle to $\theta = \pi/6$, the squeezing is exchanged between the two quadratures, $V_x$ and $V_y$. In addition, squeezing is observed in the first quadrature but is pronounced in the second quadrature, $V_y$, as can be seen in figure 4(c). The same conclusion can be made when $\theta = \pi/2$ (ground state); however, the amount of squeezing is less than that of the previous case. Figure 4(d) shows this in greater detail.

6. Conclusion

This paper considers the interaction between a two-level atom and three field modes. The interaction is non-linear, assuming a form of frequency conversion, in which two modes mutually interact with an atom. In this study, SU(2) group generators were used to describe this interaction. The particular choice of Euler angles allowed us to transform the generators into a new set of operators, in which $L_z$ was removed. Consequently, applying rotating wave approximation converted the Hamiltonian model into a Jaynes–Cummings-like model. This allowed us to determine the wave function when the atom was initially in a superposition state and the SU(2) system was initially in a Bloch coherent state. Some statistical properties of

$$V(\hat{d}_j) = \langle \Delta \hat{d}_j - \sqrt{\langle \hat{d}_j^2 \rangle} \rangle < 0, \quad \epsilon = x \quad \text{or} \quad y. \quad (40)$$

Figure 3. Entropy squeezing subject to scaled time $\tau$ for a two-level atom. The atom initializes the interaction in a superposition state and the SU(2) quantum system in a modified Bloch generalized coherent state for fixed values of $\eta = \pi/4$ and atomic phase angle $\phi = 0$, but for different values of the other parameters, as is indicated in each figure.
the system were discussed. Specifically, we examined atomic inversion and determined that collapse periods became quiescent periods, which diminish when the parameter, \( \frac{\omega_0}{\lambda} \), is altered. Superstructure shapes produced by interference between the patterns were also observed. These shapes break up when the coherent angle, \( \theta \), is changed.

In addition, linear entropy was used to discuss the entanglement between the atom and field system. Partial entanglement, as well as maximum entanglement, were demonstrated and depended on the value of parameters \( \frac{\omega_0}{\lambda} \) and \( \theta \). No sign of disentanglement was produced during the interaction time. Furthermore, two types of squeezing were considered: entropy squeezing and variance squeezing. Atomic variables are squeezed in entropy squeezing during specific time periods of the interaction. Squeezing occurred only in one quadrature and could be transferred from one quadrature to another by changing angle \( \theta \). However, given the proper choices of angle \( \theta \) and \( \frac{\omega_0}{\lambda} \), variance squeezing resulted in two quadratures. Therefore, we can conclude that this system exhibits a rich variety of phenomena.

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Figure 4. Variance squeezing per scaled time \( \tau \) for a two-level atom. The atom initializes the interaction in a superposition state and the SU(2) quantum system in a modified Bloch generalized coherent state for fixed values of \( l = 5 \), \( \eta = \pi/4 \), and atomic phase angle \( \phi = 0 \), but for different values of the other parameters, as is indicated in each figure.
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