HANKEL DETERMINANTS OF A STURMIAN SEQUENCE

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Abstract. Let \( \tau \) be the substitution \( 1 \rightarrow 101 \) and \( 0 \rightarrow 1 \) on the alphabet \( \{0, 1\} \). The fixed point of \( \tau \) leading by 1, denoted by \( s \), is a Sturmian sequence. We first give a characterization of \( s \) using \( f \)-representation. Then we show that the distribution of zeros in the determinants induces a partition of integer lattices in the first quadrant. Combining those properties, we give the explicit values of the Hankel determinants \( H_{m,n} \) of \( s \) for all \( m \geq 0 \) and \( n \geq 1 \).

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1. Introduction

Let \( s = (s_j)_{j \geq 0} \) be an integer sequence. For all \( m \geq 0, n \geq 1 \), the \((m,n)\)-order Hankel matrix of \( s \) is

\[
M_{m,n} := (s_{m+i+j})_{0 \leq i,j \leq n-1} = \begin{pmatrix}
  s_m & s_{m+1} & \cdots & s_{m+n-1} \\
  s_{m+1} & s_{m+2} & \cdots & s_{m+n} \\
  \vdots & \vdots & \ddots & \vdots \\
  s_{m+n-1} & s_{m+n} & \cdots & s_{m+2n-2}
\end{pmatrix}.
\]

The \((m,n)\)-order Hankel determinant of \( s \) is \( H_{m,n} = \det M_{m,n} \).

Hankel determinants of automatic sequences have been widely studied, due to its application to the study of irrationality exponent of real numbers; see for example [1, 3, 5, 6, 7, 9, 14] and references therein. In 2016, Han [10] introduced the Hankel continued fraction which is a powerful tool for evaluating Hankel determinants. By using the Hankel continued fractions, Bugeaud, Han Wen and Yao [4] characterized the irrationality exponents of values of certain degree two Mahler functions at rational points. Recently, Guo, Han and Wu [8] fully characterized apwenian sequences, that is \( \pm 1 \) sequences whose Hankel determinants \( H_{0,n} \) satisfying \( H_{0,n}/2^{n-1} \equiv 1 \) (mod 2) for all \( n \geq 1 \).

However, the Hankel determinants of other low complexity sequences, such as Sturmian sequences, are rarely known. Kamae, Tamura and Wen [11] explicitly evaluated the Hankel determinants of the Fibonacci word. Tamura [12] extended this result to infinite words generated by

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the substitutions $a \to a^k b, b \to a$ ($k \geq 1$). In this paper, we study the Hankel determinants of the sequence generated by the substitution

$$
\tau : 1 \to 101, \ 0 \to 1.
$$

Denote by $s = (s_n)_{n \geq 0} = \lim_{n \to \infty} \tau^n(1)$ the fixed point of $\tau$. It follows from [13, Proposition 2.1] that $s$ is a Sturmian sequence.

We give the explicit values of Hankel determinants $H_{m,n}$ for the sequence $s$ for all $m \geq 0$ and $n \geq 1$. The distribution of first values of $H_{m,n}$ can be seen from Figure 1, where we use different colors to indicate different values of $H_{m,n}$. In particular, the white color indicates that the Hankel determinants are zero. The zeros of the Hankel determinants (together with the non-zero boundaries) form three types of parallelograms labelled by $U_{k,i}$, $V_{k,i}$ and $T_{k,i}$ (for detailed definitions, see Section 3). In fact, those parallelograms are disjoint and they tile the lattices in the first quadrant; see Proposition 3.1 in Section 3. This nice property allows to evaluate the Hankel determinants $H_{m,n}$ according to the parallelograms.

To state our main result, we need some notations. Let $U_k = \bigcup_{i \geq 1} U_{k,i}$, $V_k = \bigcup_{i \geq 1} V_{k,i}$ and $T_k = \bigcup_{i \geq 1} T_{k,i}$. The integer sequence $(f_n)_{n \geq 0}$ is given by $f_{2j} = |\tau_j(1)|$ and $f_{2j+1} = |\tau_j(10)|$ for all $j \geq 0$, where $|w|$ denotes the number of digits in the word $w$. To avoid repeating lengthy definitions, please see Section 2 for the truncated $f$-representation $\Phi_k(\cdot)$ and see Section 3 for the sequences $(\alpha_i)_{i \geq 1}$, $(\beta_i)_{i \geq 1}$ and $(\gamma_i)_{i \geq 1}$. The Hankel determinants $H_{m,n}$ for $(m,n)$ in parallelograms of type $U_{k,i}$ (resp. $V_{k,i}$ and $T_{k,i}$) are given in the following results.

**Theorem 1.1.** Let $k \geq 0$. For all $(m,n) \in U_k$,
(1) when \( n = f_{2k+3} - 1 \), \( H_{m,n} = (-1)^{k+1}(1) \frac{f_{2k+3}}{2} \cdot f_{2k+1} \);
(2) when \( n = f_{2k} \), \( H_{m,n} = (-1)^{k+1}(1) \frac{f_{2k+2}^{-3}}{2} \cdot f_{2k+1} \);
(3) when \( f_{2k} - 1 < n < f_{2k+3} - 1 \), if \( m + n = \alpha_i - 2\gamma_i \cdot f_{2k+2} + 1 \) or \( \beta_i + 2\gamma_i \cdot f_{2k+2} + 1 \) for some \( i \geq 1 \), then
\[
H_{m,n} = -(-1)^{k+1}(f_{2k+2} - 1)^{k+1}(1) \frac{f_{2k+2}^{-1} - n}{2} \cdot f_{2k+1} \;
\]
otherwise \( H_{m,n} = 0 \).

**Theorem 1.2.** Let \( k \geq 0 \). For all \( (m,n) \in V_k \),
(1) when \( n = f_{2k+2} - 1 \), \( H_{m,n} = (-1)^{k+1}(1) \frac{f_{2k+2}^{-1}}{2} \cdot f_{2k+1} \);
(2) when \( n = f_{2k} \), \( H_{m,n} = (-1)^{k+1}(1) \frac{f_{2k+2}^{-1}}{2} \cdot f_{2k+1} \);
(3) when \( f_{2k} < n < f_{2k+2} - 1 \), if \( m + n = \beta_i + 1 \) or \( \beta_i + f_{2k+2} + 1 \) for some \( i \geq 1 \), then
\[
H_{m,n} = -(-1)^{k+1}(f_{2k+2} - 1)^{k+1}(1) \frac{f_{2k+2}^{-1} - n}{2} \cdot f_{2k+1} \;
\]
otherwise \( H_{m,n} = 0 \).

**Theorem 1.3.** Let \( k \geq 0 \). For all \( (m,n) \in T_k \),
(1) when \( n = f_{2k+2} - 1 \), \( H_{m,n} = (-1)^{k+1}(1) \frac{f_{2k+2}^{-1}}{2} \cdot f_{2k+1} \);
(2) when \( n = f_{2k+1} \), \( H_{m,n} = (-1)^{k+1}(1) \frac{f_{2k+2}^{-1}}{2} \cdot f_{2k+1} \);
(3) when \( f_{2k} < n < f_{2k+2} - 1 \), if \( m + n = \gamma_i - f_{2k} + 1 \) or \( \gamma_i \) for some \( i \geq 1 \), then
\[
H_{m,n} = -(-1)^{k+1}(f_{2k+2}^{-1} - 1)^{k+1}(1) \frac{f_{2k+2}^{-1} - n}{2} \cdot f_{2k+1} \;
\]
otherwise \( H_{m,n} = 0 \).

The paper is organized as follows. In Section 2, we introduce the \( f \)-representation of positive integers and give a criterion (Proposition 2.3) to determine \( s_n \) according the \( f \)-representation of \( n \). This criterion leads us to the key ingredient (Theorem 2.4) in calculating the Hankel determinants. Then we introduce the truncated \( f \)-representation which is essential in describing the parallelograms. In Section 3, we show that the parallelograms \( U_{k,i}, V_{k,i} \) and \( T_{k,i} \) tile all the integer points in the first quadrant. In Section 4, we first show that the Hankel determinants vanish when \( (m,n) \) is inside a parallelogram of those three types. Next we show the relations of Hankel determinants \( H_{m,n} \) on the boundary of a parallelogram \( U_{k,i} \) (or \( V_{k,i}, T_{k,i} \)) for a given \( k \) and \( i \). Finally, for any \( k \geq 0 \), we describe the relation of values of Hankel determinants for parallelograms \( U_{k,i} \) for all \( i \geq 1 \). In Section 5, we give the expressions for Hankel determinants on the boundary of \( U_{k,i} \) (or \( V_{k,i}, T_{k,i} \)) for all \( k \geq 0 \). In the last section, we formulate and prove our main results.

### 2. Some properties of the sequence \( s \)

In this section, we first introduce the \( f \)-representation of positive integers according to the sequence \( (f_n)_{n \geq 0} \). By understanding the occurrences of 0s in the sequence \( s \), we prove a key result (Proposition 2.3) which can determine \( s_n \) according to the \( f \)-representation of \( n \). Then we give the essential result (Theorem 2.4). In subsection 2.3, we introduce the truncated \( f \)-representation which is useful in determining the parallelograms. In section 2.4, we investigate to some sub-sequences of \( (f_n)_{n \geq 0} \) which are need in evaluating the coefficients of the Hankel determinants. Then we characterize two sub-sequences of \( s \) which helps us understand \( H_{m,n} \).
2.1. **The occurrence of 0’s in s.** We introduce an auxiliary sequence \((f_n)_{n \geq 0}\) to determine the positions of 0’s. For all \(n \geq 0\), we define

\[
f_{2n} = |\tau^n(1)| \quad \text{and} \quad f_{2n+1} = |\tau^n(10)|.
\]

Then \(f_0 = 1, f_1 = 2\), and for all \(n \geq 0\),

\[
\begin{align*}
f_{2n+2} &= f_{2n} + f_{2n+1}, \\
f_{2n+3} &= f_{2n} + f_{2n+2}.
\end{align*}
\]

(1)

The first values are

\[(f_n)_{n \geq 0} = (1, 2, 3, 4, 7, 10, 17, 24, 41, 58, 99, 140, 239, 338, 577, 816, \ldots).\]

Since \((f_n)_{n \geq 0}\) is an increasing non-negative integer sequence, it is a numeration system in the following sense.

**Lemma 2.1 (Theorem 3.1.1 [2]).** Let \(u_0 < u_1 < u_2 < \ldots\) be an increasing sequence of integers with \(u_0 = 1\). Every non-negative integer \(N\) has exactly one representation of the form

\[
\sum_{i \leq r} a_i u_i \quad \text{where} \quad a_r \neq 0, \quad \text{and for} \quad i \geq 0, \quad \text{the digits} \quad a_i \quad \text{are non-negative integers satisfying the inequality}
\]

\[
a_0 u_0 + a_1 u_1 + \cdots + a_r u_r < u_{i+1}.
\]

**Proposition 2.2.** Every integer \(n \geq 0\) can be uniquely expressed as \(n = \sum_{0 \leq i \leq r} a_i f_i\) with \(a_i \in \{0, 1\}, a_r \neq 0\), and

\[
\begin{align*}
a_i a_{i+1} &= 0 \quad \text{for all} \quad 0 \leq i < r, \\
a_i a_{i+2} &= 0 \quad \text{for all even numbers} \quad 0 \leq i < r - 1.
\end{align*}
\]

(2)

**Proof.** Suppose \(a_i \in \{0, 1\}\). By Lemma 2.1, we only need to show that \(a_0 f_0 + a_1 f_1 + \cdots + a_t f_t < f_{t+1}\) for all \(t\) if and only if the condition (2) holds.

**The ‘only if’ part.** Suppose there is an index \(i\) such that \(a_i a_{i+1} = 1\). Then

\[
a_0 f_0 + a_1 f_1 + \cdots + a_i f_i + a_{i+1} f_{i+1} \geq f_i + f_{i+1} \geq f_{i+2},
\]

which is a contradiction for \(t = i + 1\). Suppose there is an even index \(i\) such that \(a_i a_{i+2} = 1\). Then

\[
a_0 f_0 + a_1 f_1 + \cdots + a_i f_i + a_{i+1} f_{i+1} + a_{i+2} f_{i+2} \geq f_i + f_{i+1} = f_{i+3},
\]

which is a contradiction for \(t = i + 2\).

**The ‘if’ part.** Suppose the condition (2) holds. When \(t\) is odd, the maximum possible value of \(a_0 f_0 + a_1 f_1 + \cdots + a_t f_t\) occurs when \(a_0 a_{t-1} \cdots a_0 = 1010 \ldots 10\), and this maximum value is \(f_1 + f_3 + \cdots + f_{t-2} + f_t = f_{t+1} - 1\). When \(t\) is even, the maximum possible value of \(a_0 f_0 + a_1 f_1 + \cdots + a_t f_t\) occurs when \(a_t a_{t-1} \cdots a_0 = 1001010 \ldots 10\). In this case, the maximum value is \(f_1 + f_3 + \cdots + f_{t-5} + f_{t-3} + f_t = f_{t+1} - 1\). \(\square\)

**Definition (f-representation).** Let \(n \geq 0\) be an integer. We call the representation \(n = \sum_{0 \leq i \leq r} a_i f_i\) in Proposition 2.2 the **f-representation** of \(n\). We also write \(n = \sum_{i=0}^{+\infty} a_i f_i\) where \(a_i = 0\) for all \(i > r\). In the case that we need to emphasize that \(a_i\) depends on \(n\), we write \(a_i = a_i(n)\) as a function of \(n\).

**Proposition 2.3.** For any integer \(n \geq 0\) with the f-representation \(\sum_{i=0}^{r} a_i(n) f_i\), we have \(s_n = 0\) if and only if \(a_0(n) = 1\).

**Proof.** One can verify directly that the result holds for all \(n < f_4 = 7\). Assume that the result holds for \(n < f_{2k}\) where \(k \geq 2\). We only need to prove it for all \(f_{2k} \leq n < f_{2k+2}\).
Suppose $f_{2k} \leq n < f_{2k+1}$. One has $a_{2k}(n) = 1$ and hence $a_0(n - f_{2k}) = a_0(n)$. Note that $f_{2k} = |τ^k(1)|$ and

$$s_0s_1 \ldots s_{f_{2k+2}-1} = τ^{k+1}(1) = τ^k(1)τ^k(0)τ^k(1).$$  \hspace{1cm} (3)

We see that $s_n$ is the $(n+1)$-th letter of $τ^{k+1}(1)$ and it is also the $(n+1 - f_{2k})$-th letter of $τ^k(0) = τ^{k-1}(1)$. Consequently, $s_n = s_{n-f_{2k}}$. Since

$$n - f_{2k} < f_{2k+1} - f_{2k} = f_{2k-2},$$

by the inductive assumption, we have $s_{n-f_{2k}} = 0$ if and only if $a_0(n-f_{2k}) = 1$. Therefore, $s_n = 0$ if and only if $a_0(n) = 1$.

Suppose $f_{2k+1} \leq n < f_{2k+2}$. In this case $a_{2k+1}(n) = 1$ and $a_0(n-f_{2k+1}) = a_0(n)$. Since $|τ^k(1)| = f_{2k+1}$, it follows from (3) that $s_n = s_{n-f_{2k+1}}$. Note that

$$n - f_{2k+1} < f_{2k+2} - f_{2k+1} = f_{2k}.$$

By the inductive assumption, $s_{n-f_{2k+1}} = 0$ if and only if $a_0(n-f_{2k+1}) = 1$ which implies the result also holds for all $f_{2k+1} \leq n < f_{2k+2}$. \hfill □

2.2. Comparing digits in the sequence $s$ with a fixed gap. We introduce the truncated $f$-representations (of positive integers) which are useful in telling two digits with a fixed gap in $s$ are equal or not.

**Definition.** (Truncated $f$-representation) Let $n \geq 0$ be an integer with the $f$-representation $\sum_{i=0}^{+\infty} a_i(n)f_i$. For all integers $k \geq 0$, the truncated $f$-representation of $n$ is

$$Φ_k(n) := \sum_{i=0}^{2k+2} a_i(n)f_i.$$

The next lemma gives a criterion that when two digits (with a fixed gap) in $s$ are equal by using their positions.

**Theorem 2.4.** Let $n \geq 0$ be an integer with the $f$-representation $\sum_{i=0}^{+\infty} a_i(n)f_i$. Then

(i) for all $k \geq 0$, $s_{n+f_k} \neq s_n$ if and only if $Φ_k(n) \in \{\frac{f_{2k}}{2}, \frac{f_{2k+1}}{2} - 1\}$;

(ii) for all $k \geq 1$, $s_{n+f_{k+1}} \neq s_n$ if and only if $Φ_k(n) \in \{\frac{f_{2k+3}}{2}, \frac{f_{2k+3}}{2} - 1, \frac{f_{2k+3}}{2} + f_{2k}, \frac{f_{2k+3}}{2} + f_{2k} - 1\}$.

**Proof.** (i) We prove by induction on $k$. When $k = 0$, by Proposition 2.2, there are only four possible values for $a_0(n)a_1(n)a_2(n)$. By Eq. (1), we have

| $Φ_0(n)$ | 0 | 1 | 2 | 3 |
|----------|---|---|---|---|
| $a_0(n)a_1(n)a_2(n)$ | 000 | 100 | 010 | 001 |
| $a_0(n + f_0)$ | 1 | 0 | 0 | 0 |

Then we see that $a_0(n) \neq a_0(n+f_0)$ if and only if $Φ_0(n) = 0$ or 1. The result holds for $k = 0$.

When $k = 1$, note that $Φ_1(n) \leq \sum_{i=0}^{4} f_i < f_5 = 10$. We see

| $Φ_1(n)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----------|---|---|---|---|---|---|---|---|---|---|
| $a_0(n) \ldots a_4(n)$ | 00000 | 10000 | 01000 | 00100 | 00010 | 01010 | 00101 | 00011 | 10001 | 01001 |
| $a_0(n + f_2)$ | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |

Then we have $a_0(n) \neq a_0(n+f_2)$ if and only if $Φ_1(n) = 1$ or 2, that is $f_2$ or $f_2-1$. The result also holds for $k = 1$.

Now assume that the result holds for all $0 \leq k < ℓ$ with $ℓ \geq 2$. We prove it for $k = ℓ$. Let $w = a_{2ℓ-2}(n)a_{2ℓ-1}(n) \ldots a_{2ℓ+2}(n)$ and $v = a_{2ℓ-2}(n + f_{2ℓ})a_{2ℓ-1}(n + f_{2ℓ}) \ldots a_{2ℓ+2}(n + f_{2ℓ})$. According to Proposition 2.2, $w$ can take only 10 different values. While $w \neq 01000$, one can
determine $v$ directly using Eq. (1); thus in these cases, $a_i(n + f_{2\ell}) = a_i(n)$ for all $0 \leq i \leq 2\ell - 3$; see Table 1. For instance, when $w = 10010$,

$$
\begin{align*}
n + f_{2\ell} &= \left( \sum_{i=0}^{2\ell-3} a_i(n)f_i + f_{2\ell-2} + f_{2\ell+1} + \sum_{i=2\ell+3}^{+\infty} a_i(n)f_i \right) + f_{2\ell} \\
&= \left( \sum_{i=0}^{2\ell-3} a_i(n)f_i + f_{2\ell-2} \right) + \left( f_{2\ell+2} + \sum_{i=2\ell+3}^{+\infty} a_i(n)f_i \right).
\end{align*}
$$

Hence one can see that $a_0(n + f_{2\ell}) = a_0(n)$.

When $w = 01000$, set $n' = \sum_{i=0}^{2\ell-2} a_i(n)f_i$. Then $a_0(n) = a_0(n')$ and

$$
n + f_{2\ell} = \left( n' + f_{2\ell-1} + \sum_{i=2\ell+3}^{+\infty} a_i(n)f_i \right) + f_{2\ell}
$$

$$
= (n' + f_{2\ell-4}) + f_{2\ell+1} + \sum_{i=2\ell+3}^{+\infty} a_i(n)f_i. \quad \text{(by Eq. 1)}
$$

Noticing that $n' + f_{2\ell-4} < f_{2\ell-2}$, we have $a_i(n + f_{2\ell}) = a_i(n' + f_{2\ell-4})$ for all $0 \leq i \leq 2\ell - 2$. In particular, $a_0(n + f_{2\ell}) = a_0(n' + f_{2\ell-4})$. By Proposition 2.3 and the inductive assumption,

$$
a_0(n + f_{2\ell}) \neq a_0(n) \iff a_0(n' + f_{2\ell-4}) \neq a_0(n')
$$

$$
\iff n' \in \left\{ \frac{f_{2\ell-3}}{2}, \frac{f_{2\ell-3}}{2} - 1 \right\}
$$

$$
\iff \Phi(\ell)(n) = n' + f_{2\ell-1} \in \left\{ \frac{f_{2\ell-3}}{2} + f_{2\ell-1}, \frac{f_{2\ell-3}}{2} + f_{2\ell-1} - 1 \right\}.
$$

By Eq. (1), we have

$$
\frac{f_{2\ell+1}}{2} = \frac{(f_{2\ell-2} + f_{2\ell})}{2}
$$

$$
= \frac{(f_{2\ell-2} + f_{2\ell-2} + f_{2\ell-1})}{2}
$$

$$
= f_{2\ell-2} + \frac{f_{2\ell-1}}{2}
$$

$$
= f_{2\ell-2} + f_{2\ell-4} + \frac{f_{2\ell-3}}{2}
$$

$$
= f_{2\ell-1} + \frac{f_{2\ell-3}}{2}.
$$

Then we obtain that

$$
a_0(n + f_{2\ell}) \neq a_0(n) \iff \Phi(\ell)(n) \in \left\{ \frac{f_{2\ell+1}}{2}, \frac{f_{2\ell+1}}{2} - 1 \right\}.
$$

It follows from Proposition 2.3 that the result holds for $k = \ell$.

(ii) For any $k \geq 1$, let $u = a_{2k}(n)a_{2k+1}(n)a_{2k+2}(n)$. It follows from Proposition 2.3 that $u \in \{100, 010, 001\}$. The proof is divided into the following three cases.

| $w$  | 00000 | 10000 | 01000 | 00100 | 00010 | 10010 | 01010 | 00001 | 10001 | 01001 |
|-----|-------|-------|-------|------|-------|-------|-------|-------|-------|-------|
| $v$ | 0010  | 0001  | ?     | 0101  | 00000 | 10000 | 01000 | 00001 | 10001 | 01001 |
• When \( u = 001 \), we also have \( a_{2k+3}(n) = a_{2k+4}(n) = 0 \). So

\[
\begin{align*}
n + f_{2k+1} &= \left( \sum_{i=0}^{2k-1} a_i(n)f_i + f_{2k+2} + \sum_{i=2k+5}^{\infty} a_i(n)f_i \right) + f_{2k+1} \\
&= \left( \sum_{i=0}^{2k-1} a_i(n)f_i + f_{2k-2} \right) + \left( f_{2k+3} + \sum_{i=2k+5}^{\infty} a_i(n)f_i \right).
\end{align*}
\]

Let \( n' = \sum_{i=0}^{2k-1} a_i(n)f_i \). Then \( a_0(n) = a_0(n') \). Since \( n' < f_{2k} \) and \( n' + f_{2k-2} < f_{2k+1} \), we have \( a_i(n') + f_{2k-2} = a_i(n + f_{2k+1}) \) for all \( 0 \leq i \leq 2k \). Thus

\[
a_0(n + f_{2k+1}) \neq a_0(n) \iff a_i(n' + f_{2k-2}) \neq a_0(n') \iff n' = \sum_{i=0}^{2k} a_i(n')f_i \in \left\{ \frac{f_{2k-1}}{2}, \frac{f_{2k-1} - 1}{2} \right\}
\]

where in the last step we use Theorem 2.4(i). By Eq. (1) and Eq. (4),

\[
\frac{f_{2k-1}}{2} + f_{2k+2} = \frac{f_{2k-1}}{2} + f_{2k+1} + f_{2k} = \frac{f_{2k+3}}{2} + f_{2k}.
\]

So when \( u = 001 \), \( a_0(n + f_{2k+1}) \neq a_0(n) \) if and only if

\[
\Phi_k(n) = n' + f_{2k+2} \in \left\{ \frac{f_{2k+3}}{2} + f_{2k}, \frac{f_{2k+3}}{2} + f_{2k} - 1 \right\}.
\]

• Suppose \( u = 010 \). Applying Eq. (1) twice, we obtain that

\[
2f_{2k+1} = f_{2k-2} + f_{2k} + f_{2k+1} = f_{2k-2} + f_{2k+2}.
\]

Then

\[
\begin{align*}
n + f_{2k+1} &= \left( \sum_{i=0}^{2k-1} a_i(n)f_i + f_{2k+1} + \sum_{i=2k+3}^{\infty} a_i(n)f_i \right) + f_{2k+1} \\
&= \left( \sum_{i=0}^{2k-1} a_i(n)f_i + f_{2k-2} \right) + \left( f_{2k+2} + \sum_{i=2k+3}^{\infty} a_i(n)f_i \right).
\end{align*}
\]

Let \( n' = \sum_{i=0}^{2k-1} a_i(n)f_i \). Using Theorem 2.4(i), the same argument as in the case \( u = 001 \) leads us to the fact that

\[
a_0(n + f_{2k+1}) \neq a_0(n) \iff n' = \sum_{i=0}^{2k} a_i(n')f_i \in \left\{ \frac{f_{2k-1}}{2}, \frac{f_{2k-1} - 1}{2} \right\} \\
\iff \Phi_k(n) = n' + f_{2k+1} \in \left\{ \frac{f_{2k+3}}{2}, \frac{f_{2k+3}}{2} - 1 \right\}.
\]

• When \( u = 100 \), we have \( a_{2k-2}(n) = a_{2k-1}(n) = 0 \). Then

\[
\begin{align*}
n + f_{2k+1} &= \left( \sum_{i=0}^{2k-3} a_i(n)f_i + f_{2k} + \sum_{i=2k+3}^{\infty} a_i(n)f_i \right) + f_{2k+1} \\
&= \left( \sum_{i=0}^{2k-3} a_i(n)f_i \right) + \left( f_{2k+2} + \sum_{i=2k+3}^{\infty} a_i(n)f_i \right)
\end{align*}
\]

which implies that \( a_0(n + f_{2k+1}) = a_0(n) \).
2.3. Integers with the same truncated f-representation. To apply Theorem 2.4, we need to investigate the integers of the same truncated f-representations. The following two lemmas (Lemma 2.5 and Lemma 2.7) serve for this purpose.

For all \( k \geq 0 \), denote
\[
E_k' = \left\{ x \in \mathbb{N} : \Phi_k(x) = \frac{f_{2k+3}}{2} \right\} \quad \text{and} \quad E_k'' = \left\{ x \in \mathbb{N} : \Phi_k(x) = \frac{f_{2k+3}}{2} + f_{2k} \right\}.
\]

Let \( E_k = E_k' \cup E_k'' \).\( (x_j^{(k)})_{j \geq 1} \) where \( x_1^{(k)} < x_2^{(k)} < x_3^{(k)} < \ldots \). The first values of \( E_k \) are
\[
(x_j^{(k)})_{j \geq 1} = \left( \frac{f_{2k+3}}{2}, \frac{f_{2k+3}}{2} + f_{2k}, \frac{f_{2k+3}}{2} + f_{2k+3}, \frac{f_{2k+3}}{2} + f_{2k+3}, \frac{f_{2k+3}}{2} + f_{2k+5}, \frac{f_{2k+3}}{2} + f_{2k+5}, \frac{f_{2k+3}}{2} + f_{2k+7}, \ldots \right).
\]

Lemma 2.5. Let \( k \geq 0 \) and \( x \in E_k' \) with \( x = x_j^{(k)} \) for some \( j \geq 2 \). Then \( x - f_{2k+2} = x_{j-1}^{(k)} \in E_k \).

Proof. Let \( x \in E_k' \) with \( x = x_j^{(k)} \) for some \( j \geq 2 \). Note that \( \frac{f_{2k+3}}{2} + f_{2k} = f_{2k+2} + \frac{f_{2k+1}}{2} \). By Proposition 2.2, have \( a_{2k+3}(x) = a_{2k+4}(x) = 0 \). When \( 0 < b < f_{2k} - \frac{f_{2k-1}}{2} \), we see
\[
\Phi_k(x + b) = \frac{f_{2k+3}}{2} + f_{2k} + b < f_{2k+3}
\]
which implies that \( x + b \notin E_k \). When \( f_{2k} - \frac{f_{2k-1}}{2} \leq b < f_{2k+2} \),
\[
\Phi_k(x + b) = \frac{f_{2k+3}}{2} + f_{2k} + b - f_{2k+3} < \frac{f_{2k+3}}{2}.
\]
So \( x + b \notin E_k \). Since \( \Phi_k(x + f_{2k+2}) = \frac{f_{2k+3}}{2} \), we have \( x + f_{2k+2} = x_{j+1}^{(k)} \in E_k' \).

Let \( x \in E_k' \) with \( x = x_j^{(k)} \) for some \( j \geq 1 \). According to Proposition 2.2, \( a_{2k+3}(x)a_{2k+4}(x) = 0 \), 0 or 1, which can be divided into two sub-cases.

- \( a_{2k+3}(x)a_{2k+4}(x) = 0 \). For \( 0 < b \leq f_{2k} \), we see \( \Phi_k(x + b) = \Phi_k(x) + b = \frac{f_{2k+1}}{2} + b \). Thus \( x + f_{2k} = x_{j+1}^{(k)} \in E_k' \).
- \( a_{2k+3}(x)a_{2k+4}(x) = 1 \) or 0. For \( 0 < b < f_{2k} \), we see
\[
\Phi_k(x + b) = \Phi_k(x) + b = \frac{f_{2k+3}}{2} + b < \frac{f_{2k+3}}{2} + f_{2k}.
\]
Thus \( x + b \notin E_k \). For \( f_{2k} \leq b < f_{2k+2} \), we have
\[
\Phi_k(x + b) = \frac{f_{2k+3}}{2} + b - f_{2k+2} \in \left( \frac{f_{2k-1}}{2}, \frac{f_{2k+3}}{2} \right)
\]
which yields that \( x + b \notin E_k \). Noting that \( \Phi_k(x + f_{2k+2}) = \Phi_k(x) = \frac{f_{2k+3}}{2} \), we obtain that \( x + f_{2k+2} = x_{j+1}^{(k)} \in E_k' \).

From the above argument, we see that if \( x = x_j^{(k)} \in E_k' \) for some \( j \geq 2 \), then either \( x - f_{2k+2} = x_{j-1}^{(k)} \in E_k'' \) or \( x - f_{2k+2} = x_{j-1}^{(k)} \in E_k' \) with \( a_{2k+3}(x - f_{2k+2})a_{2k+4}(x - f_{2k+2}) \neq 0 \). The result holds. \( \square \)

Remark 2.6. From the proof of Lemma 2.5, we see the gaps between two adjacent elements in \( E_k \) are \( f_{2k} \) and \( f_{2k+2} \). That is \( x_j^{(k)} - x_{j+1}^{(k)} = f_{2k} \) or \( f_{2k+2} \) for all \( j \geq 1 \). Moreover, the gaps between two adjacent elements in \( E_k' \) are \( f_{2k+2} \) and \( f_{2k+3} \).
For all $k \geq 0$, let

$$F_k = \left\{ y \in \mathbb{N} : \Phi_k(y) = \frac{f_{2k+1}}{2} \right\} = (y^{(k)}_j)_{j \geq 1} \quad \text{and} \quad F'_k = \left\{ y \in \mathbb{N} : \Phi_{k+1}(y) = \frac{f_{2k+1}}{2} \right\}$$

where $y^{(k)}_1 < y^{(k)}_2 < y^{(k)}_3 < \ldots$. Write $F''_k = F_k - F'_k$. The first values of $F_k$ are

$$(y^{(k)}_j)_{j \geq 1} = \left( \frac{f_{2k+1}}{2}, \frac{f_{2k+1}}{2} + f_{2k+3}, \frac{f_{2k+1}}{2} + f_{2k+4}, \frac{f_{2k+1}}{2} + f_{2k+5}, \frac{f_{2k+1}}{2} + f_{2k+6}, \ldots \right).$$

**Lemma 2.7.** For any $y \in F_k$ with $y = y^{(k)}_j$ for some $j \geq 1$, we have

$$y^{(k)}_{j+1} = \begin{cases} y + f_{2k+3}, & \text{if } y \in F'_k; \\ y + f_{2k+2}, & \text{if } y \in F''_k. \end{cases}$$

**Proof.** We prove the result by giving the construction of $F_k$. It clear that $y^{(k)}_1 = \frac{f_{2k+1}}{2}$. Now suppose $y = y^{(k)}_j \in F_k$ where $j \geq 1$. According to Proposition 2.2, we see $a_{2k+3}(y)a_{2k+4}(y) = 00$, 01 or 01.

- $a_{2k+3}(y)a_{2k+4}(y) = 00$, i.e., $y \in F'_k$. Note that $\frac{f_{2k+1}}{2} = f_{2k-1} + \frac{f_{2k-3}}{2}$. For $0 < b < f_{2k+3} - \frac{f_{2k+1}}{2}$, we have $\Phi_k(y+b) = \frac{f_{2k+1}}{2} + b$, so $y + b \notin F_k$. For $f_{2k+3} - \frac{f_{2k+1}}{2} \leq b < f_{2k+3}$, $\Phi_k(y+b) = \frac{f_{2k+1}}{2} + b - f_{2k+3} < \frac{f_{2k+1}}{2}$, so $y + b \notin F_k$. Since $\Phi_k(y + f_{2k+3}) = \Phi_k(y)$, we obtain that $y + f_{2k+3} = y^{(k)}_{j+1} \in F_k - F'_k$.

- $a_{2k+3}(y)a_{2k+4}(y) = 10$ or 01. For $0 < b < f_{2k+2} - \frac{f_{2k+1}}{2}$, we have $\Phi_k(y+b) = \frac{f_{2k+1}}{2} + b$, so $y + b \notin F_k$. For $f_{2k+2} - \frac{f_{2k+1}}{2} \leq b < f_{2k+2}$, since $\Phi_k(y+b) = \frac{f_{2k+1}}{2} + b - f_{2k+2} < \frac{f_{2k+1}}{2}$, we also have $y + b \notin F_k$. It follows from $\Phi_k(y+f_{2k+2}) = \Phi_k(y)$ that $y + f_{2k+2} = y^{(k)}_{j+1} \in F_k$.

The result follows from the above two sub-cases. \qed

**Remark 2.8.** From the proof of Lemma 2.7, we see the gaps between two adjacent elements in $F_k$ are $f_{2k+2}$ and $f_{2k+3}$. That is $y^{(k)}_{j+1} - y^{(k)}_j = f_{2k+2}$ or $f_{2k+3}$ for all $j \geq 1$. Moreover, the gaps between two adjacent elements in $F''_k$ are $f_{2k+2}$ and $f_{2k+4}$.

### 2.4. Two subsequences of $s$. The subsequences $(s_{f_{2k+1}})_{k \geq 0}$ and $(s_{f_{2k+1}-1})_{k \geq 0}$ can be determined according to the parity of $k$; see Lemma 2.10. We start with an auxiliary lemma which concerns the parity of $\frac{f_{2k+1}}{2}$.

**Lemma 2.9.** For all $k \geq 0$,

1. $f_{2k} \equiv \begin{cases} 1, & \text{if } k \equiv 0 \text{ or } 3 \pmod{4}, \\ 3, & \text{if } k \equiv 1 \text{ or } 2 \pmod{4}, \end{cases}$
2. $f_{2k+1} \equiv \begin{cases} 2, & \text{if } k \text{ is even}, \\ 0, & \text{if } k \text{ is odd}, \end{cases}$
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Proof. (i) Note that \( f_0 = 1 \) and \( f_2 = 3 \). Since \( f_{2n} \) is odd for all \( n \geq 0 \), using Eq. (1) twice, we have for all \( k \geq 2 \),
\[
f_{2k} = f_{2k-2} + f_{2k-1} = 2f_{2k-2} + f_{2k-4} \equiv 2 + f_{2(k-2)} \pmod{4}.
\]
The result follows by induction on \( k \).

(ii) The initial value is \( f_1 = 2 \). Using Eq. (1) and the previous result (i), we have for all \( k \geq 1 \),
\[
f_{2k+1} = f_{2k} + f_{2k-2} \equiv \begin{cases} 2, & k \equiv 0, 2 \pmod{4}, \\ 0, & k \equiv 1, 3 \pmod{4}, \end{cases} \quad (\text{mod } 4)
\]
which is the desired result. \(\square\)

In the calculation of \( H_{m,n} \), we need to know \( s_n \) explicitly for some \( n \). The next lemma determines the values of two sub-sequences \( s \).

Lemma 2.10. For all \( k \geq 0 \),
\[
s_{\frac{f_{2k+1}}{2}} = \begin{cases} 1, & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even}, \end{cases} \quad \text{and} \quad s_{\frac{f_{2k+1}}{2} - 1} = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ 1, & \text{if } k \text{ is even}. \end{cases}
\]

Proof. By Eq. (1), we obtain that for all \( k \geq 0 \),
\[
\frac{f_{2k+1}}{2} = \frac{(f_{2k-2} + f_{2k})}{2} = \frac{(f_{2k-2} + f_{2k-2} + f_{2k-1})}{2} = f_{2k-2} + \frac{f_{2k-1}}{2} = \cdots = \sum_{i=0}^{k-1} f_{2i} + \frac{f_1}{2}.
\]

When \( k \) is odd,
\[
\frac{f_{2k+1}}{2} = f_{2k-2} + \frac{f_{2k-4}}{2} + \cdots + f_4 + f_2 + f_0 + \frac{f_1}{2} = f_{2k-1} + f_{2k-5} + \cdots + f_5 + f_1 \quad \text{(by Eq. (1))}
\]
\[
= \sum_{i=0}^{k-1} f_{4i+1}.
\]

When \( k \geq 2 \) is even,
\[
\frac{f_{2k+1}}{2} = f_{2k-2} + \frac{f_{2k-4}}{2} + \cdots + f_4 + f_2 + f_0 + \frac{f_1}{2} = f_{2k-1} + f_{2k-5} + \cdots + f_3 + \frac{f_1}{2} \quad \text{(by Eq. (1))}
\]
\[
= \sum_{i=0}^{\frac{k-2}{2}} f_{4i+3} + \frac{f_1}{2} = \sum_{i=0}^{\frac{k-2}{2}} f_{4i+3} + f_0.
\]

It follows from (6) and (7) that for all \( k \geq 0 \),
\[
a_0\left(\frac{f_{2k+1}}{2}\right) = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ 1, & \text{if } k \text{ is even}, \end{cases}
\]
and
\[
a_0\left(\frac{f_{2k+1}}{2} - 1\right) = \begin{cases} 1, & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even}. \end{cases}
\]

Then by Proposition 2.3, the result follows. \(\square\)
3. Partition of the Lattice

According to the values of the Hankel determinants of \( s \), we tile the integer lattice using the following parallelograms. Given a \( k \geq 0 \), write the elements in \( E_{k+1}', F_k'' \) and \( E_k' \) in ascending order as follows:

\[
E_{k+1}' = (\alpha_i)_{i \geq 1}, \quad F_k'' = (\beta_i)_{i \geq 1}, \quad E_k' = (\gamma_i)_{i \geq 1}.
\]

Moreover, let \( \beta_i = \beta_i' + f_{2k} \) for all \( i \geq 1 \). We define three different types of parallelograms: for \( i \geq 1 \),

\[
U_{k,i} = \{(m,n) \in \mathbb{N}^2 : f_{2k} \leq n < f_{2k+3}, \alpha_i - f_{2k+2} < n + m \leq \alpha_i \},
\]

\[
V_{k,i} = \{(m,n) \in \mathbb{N}^2 : f_{2k} \leq n < f_{2k+2}, \beta_i < n + m \leq \beta_i + f_{2k+1} \},
\]

\[
T_{k,i} = \{(m,n) \in \mathbb{N}^2 : f_{2k+1} \leq n < f_{2k+2}, \gamma_i - f_{2k} < n + m \leq \gamma_i \};
\]

see Figure 1. Let \( U_k = \bigcup_{i \geq 1} U_{k,i}, V_k = \bigcup_{i \geq 1} V_{k,i} \) and \( T_k = \bigcup_{i \geq 1} T_{k,i} \).

**Proposition 3.1.** The parallelograms \( \{U_{k,i}\}, \{V_{k,i}\} \) and \( \{T_{k,i}\} \) introduce a partition of pairs of positive integers. Namely, \( \mathbb{N} \times \mathbb{N} \geq 1 = \bigcup_{k \geq 0} (U_k \cup V_k \cup T_k) \) where \( \cup \) denotes the disjoint union.

**Proof.** Let \( m \geq 0 \) and \( n \geq 1 \) be two integers. Since \( (f_k)_{k \geq 0} \) and \( (\gamma_k)_{k \geq 1} \) are two increasing unbounded non-negative integer sequences, there exist \( k \geq 0 \) and \( \ell \geq 1 \) such that \( f_{2k} \leq n < f_{2k+2} \) and \( \gamma_{\ell-1} < n + m \leq \gamma_{\ell} \) where \( \gamma_0 := 0 \). The result clearly holds when \( \ell = 1 \). Now we assume that \( \ell \geq 2 \). From the proof of Lemma 2.5 we see that \( \gamma_{\ell} - \gamma_{\ell-1} = f_{2k+2} \) or \( f_{2k+3} \) for all \( \ell \geq 2 \). When \( \gamma_{\ell} - f_{2k} < n + m \leq \gamma_{\ell} \), we have

\[
(m,n) \in \begin{cases}
U_{k-1}, & \text{if } f_{2k} \leq n < f_{2k+1}; \\
T_k, & \text{if } f_{2k+1} \leq n < f_{2k+2};
\end{cases}
\]

see also Figure 2. When \( \gamma_{\ell-1} < n + m \leq \gamma_{\ell} - f_{2k} \), we have the following two cases:

**Case 1:** \( \gamma_{\ell} - \gamma_{\ell-1} = f_{2k+2} \). In this case, we shall verify that \( (m,n) \in V_k \). To do this, we only need to show that \( \gamma_{\ell-1} - f_{2k} \in E_k'' \). Since \( \gamma_{\ell-1} \in E_k' \), we have \( \Phi_k(\gamma_{\ell-1}) = \frac{f_{2k+3}}{2} \) and \( \Phi_k(\gamma_{\ell-1} - f_{2k}) = \frac{f_{2k+3}}{2} - f_{2k} = \frac{f_{2k+1}}{2} \). So \( (\gamma_{\ell-1} - f_{2k}) \in F_k' \). Suppose on the contrary that \( (\gamma_{\ell-1} - f_{2k}) \in E_k'' \). Then \( \Phi_{k+1}(\gamma_{\ell-1} - f_{2k}) = \frac{f_{2k+1}}{2} \) and \( \Phi_{k+1}(\gamma_{\ell-1}) = \frac{f_{2k+3}}{2} \). This implies \( \Phi_k(\gamma_{\ell-1} + f_{2k+2}) = \frac{f_{2k+3}}{2} \) and \( (\gamma_{\ell-1} + f_{2k+2}) \in E_k' \). Note that in this case \( \gamma_{\ell} = \gamma_{\ell-1} + f_{2k+2} \). We conclude that \( \gamma_{\ell} \notin E_k'' \) which is a contradiction. Hence, \( (\gamma_{\ell-1} - f_{2k}) \in F_k' \). The result follows.

**Case 2:** \( \gamma_{\ell} - \gamma_{\ell-1} = f_{2k+3} \). We assert that, in this case, \( \gamma_{\ell-1} - f_{2k} \in F_k' \). Since \( \gamma_{\ell-1} \in E_k' \), we have \( \Phi_k(\gamma_{\ell-1}) = \frac{f_{2k+3}}{2} \). Consequently, \( \Phi_k(\gamma_{\ell-1} - f_{2k}) = \frac{f_{2k+3}}{2} \) and \( (\gamma_{\ell-1} - f_{2k}) \in F_k \). Suppose \( (\gamma_{\ell-1} - f_{2k}) \in E_k'' \). Then \( \Phi_k(\gamma_{\ell-1}) = \frac{f_{2k+3}}{2} \) and \( \Phi_{k+1}(\gamma_{\ell-1}) = \frac{f_{2k+3}}{2} \). It follows that \( \Phi_k(\gamma_{\ell-1} + f_{2k+3}) = \frac{f_{2k+3}}{2} + f_{2k} \). Since \( \gamma_{\ell-1} + f_{2k+3} = \gamma_{\ell} \), we obtain that \( \gamma_{\ell} \notin E_k'' \) which is a
contradiction. Now we have $\gamma_{l-1} - f_{2k} \in F'_k$. This yields that $\Phi_{k+1}^m(\gamma_{l-1}) = \frac{f_{2k+3}}{2}$. Observing that $\Phi_{k+1}^m(\gamma - f_{2k}) = \Phi_{k+1}^m(\gamma_{l-1} + f_{2k+2}) = \frac{f_{2k+5}}{2}$, we see $\gamma - f_{2k} \in E'_{k+1}$. So $(m, n) \in U_k$. □

4. Relations of Hankel determinants

In this section, we use the Theorem 2.4 to show the determinant value inside $U_k$, $V_k$, $T_k$ is 0. For some integer $k \geq 0$, we prove the relationship between the determinant value of the boundary of $U_k$, $V_k$, $T_k$. We assert that as long as we know one value of $U_k(V_k$ or $T_k$), we can know all its values.

4.1. Inside the parallelograms. The Hankel determinant $H_{m,n}$ vanishes if $(m, n)$ is not on the boundary of any parallelogram $U_{k,i}$, $V_{k,i}$ or $T_{k,i}$.

Lemma 4.1. Let $m \geq 1$ and $n \geq 0$ be two integer.

(i) If $(m, n)$ is inside $V_{k,i}$ for some $k \geq 0$ and $i \geq 1$, i.e.,

$$\begin{cases}
f_{2k} + 1 \leq n < f_{2k+2} - 1, \\
\beta_i + 1 < n + m \leq \beta_i + f_{2k+1} - 1,
\end{cases}$$

then $H_{m,n} = 0$.

(ii) If $(m, n)$ is inside $T_{k,i}$ for some $k \geq 0$ and $i \geq 1$, i.e.,

$$\begin{cases}
f_{2k+1} + 1 \leq n < f_{2k+2} - 1, \\
\gamma_i - f_{2k} + 1 < n + m \leq \gamma_i - 1,
\end{cases}$$

then $H_{m,n} = 0$.

(iii) If $(m, n)$ is inside $U_{k,i}$ for some $k \geq 0$ and $i \geq 1$, i.e.,

$$\begin{cases}
f_{2k} + 1 \leq n < f_{2k+3} - 1, \\
\alpha_i - f_{2k+2} + 1 < n + m \leq \alpha_i - 1,
\end{cases}$$

then $H_{m,n} = 0$.

Proof. Let $A_{m+i}$ be the $i$-th row of $H_{m,n}$. Then

$$H_{m,n} = \det \begin{pmatrix} s_m & s_{m+1} & \cdots & s_{m+n-1} \\
&s_{m+1} & s_{m+2} & \cdots & \cdots \\
&\vdots & \ddots & \ddots & \vdots \\
&s_{m+n-1} & \cdots & \cdots & s_{m+2n-2} \end{pmatrix} = \det \begin{pmatrix} A_m \\
A_{m+1} \\
\vdots \\
A_{m+n-1} \end{pmatrix}.$$

(i) When $m \leq \beta'_i + 1$, recall that $\beta'_i = \beta_i - f_{2k} \in F''_k$. Since $n \leq f_{2k+2} - 2$, by Lemma 2.7, we have $\Phi_k(\beta'_i + j) \neq \frac{f_{2k+1}}{2}$ or $\frac{f_{2k+3}}{2} - 1$ for all $1 \leq j \leq n$. Then it follows from Theorem 2.4(i) that

$$A_{\beta'_i+1} = \left(s_{\beta'_i+1}, s_{\beta'_i+2}, \ldots, s_{\beta'_i+n}\right) = (s_{\beta_i+1}, s_{\beta_i+2}, \ldots, s_{\beta_i+n}) = A_{\beta_i+1},$$

which gives $H_{m,n} = 0$. When $m > \beta'_i + 1$, note that $n + m \leq \beta'_i + f_{2k+2} - 1$. By Lemma 2.7, we have $\Phi_k(m + j) \neq \frac{f_{2k+1}}{2}$ or $\frac{f_{2k+3}}{2} - 1$ for all $1 \leq j \leq n$. Then it follows from Theorem 2.4(i) that

$$A_{m} = (s_m, s_{m+1}, \ldots, s_{m+n-1})$$

$$= (s_m + f_{2k}, s_m + f_{2k+1}, \ldots, s_m + f_{2k+n-1}) = A_{m+f_{2k}}.$$

So $H_{m,n} = 0$. 


Recall that $\gamma_i \in E_k'$ and by Lemma 2.5, $\gamma_i$ and $\gamma_i - f_{2k+2}$ are adjacent elements in $E_k$.

Let

$$r = \begin{cases} 
\gamma_i - f_{2k+2} + 1 - m, & \text{if } m \leq \gamma_i - f_{2k+2} + 1, \\
0, & \text{if } m > \gamma_i - f_{2k+2} + 1.
\end{cases}$$

Combining Lemma 2.5 and Theorem 2.4(ii), we have $A_{m+r} = A_{m+r+f_{2k+1}}$ which means $H_{m,n} = 0$.

Recall that $\alpha_i \in E_{k+1}'$ and by Lemma 2.5, $\alpha_i$ and $\alpha_i - f_{2k+4}$ are adjacent elements in $E_{k+1}$. When $m \leq \alpha_i - f_{2k+4} + n < \alpha_i - f_{2k+2} - 1$. By Theorem 2.4(ii), we have

$$A_{\alpha_i - f_{2k+4} + 1} = (s_{\alpha_i - f_{2k+4} + 1}, s_{\alpha_i - f_{2k+4} + 2}, \ldots, s_{\alpha_i - f_{2k+4} + n})$$

$$= (s_{\alpha_i - f_{2k+4} + 1}, s_{\alpha_i - f_{2k+4} + 2}, \ldots, s_{\alpha_i - f_{2k+4} + n}) = A_{\alpha_i - f_{2k+2} + 1}.$$

Thus $H_{m,n} = 0$. When $m > \alpha_i - f_{2k+4} + 1$, since $n + m - 1 \leq \alpha_i - 2$, by Theorem 2.4(ii), we obtain that

$$A_m = (s_m, s_{m+1}, \ldots, s_{m+n-1})$$

$$= (s_{m+f_{2k+3}}, s_{m+f_{2k+3}+1}, \ldots, s_{m+f_{2k+3}+n-1}) = A_{m+f_{2k+3}}$$

which also implies $H_{m,n} = 0$. \hfill \Box

4.2. Determinants on the horizontal edges of the parallelograms. We first deal with the Hankel determinants $H_{m,n}$ on the horizontal edges with $n = f_{2k}$ and $f_{2k+1}$ where $k \geq 0$.

**Lemma 4.2.** Let $k \geq 0$ and $i \geq 1$.

(i) (Bottom edge of $V_{k,i}$) $H_{\beta_i'+r, f_{2k}} = H_{\beta_i'+1, f_{2k}}$ for all $1 \leq r \leq f_{2k+1}$.

(ii) (Bottom edge of $U_{k,i}$) $H_{\alpha_i - f_{2k+3}+r, f_{2k}} = H_{\alpha_i - f_{2k}, f_{2k}}$ for all $1 \leq r \leq f_{2k+2}$.

(iii) (Bottom edge of $T_{k,i}$) $H_{\gamma_i - f_{2k+3}+r, f_{2k+1}} = (-1)^{r+1}H_{\gamma_i - f_{2k+1}, f_{2k+1}}$ for all $1 \leq r \leq f_{2k}$ with $\gamma_i - f_{2k+2} + r \geq 0$.

**Proof.** (i) Let $A_j = (s_{\beta_i'+j}, s_{\beta_i'+j+1}, \ldots, s_{\beta_i'+j+f_{2k-1}})$. Then for $1 \leq j < f_{2k+1}$,

$$H_{\beta_i'+j, f_{2k}} = \det \begin{pmatrix} A_j \\ A_{j+1} \\ \vdots \\ A_{f_{2k}+j-1} \end{pmatrix}$$

and

$$H_{\beta_i'+j+1, f_{2k}} = \det \begin{pmatrix} A_{j+1} \\ A_{j+2} \\ \vdots \\ A_{f_{2k}+j} \end{pmatrix}.$$
Suppose \( U \) value which depends only on Lemma 4.3.

For \( 1 \leq j < f_{2k+2} \),

\[
H_{y+j, f_{2k}} = \det \begin{pmatrix} B_j \\ B_{j+1} \\ \vdots \\ B_{f_{2k}+j-1} \end{pmatrix} \quad \text{and} \quad H_{y+j+1, f_{2k}} = \det \begin{pmatrix} B_{j+1} \\ B_{j+2} \\ \vdots \\ B_{f_{2k}+j} \end{pmatrix}.
\]

Since \( j + f_{2k} - 1 \leq f_{2k+3} - 2 \), by Lemma 2.7 and Theorem 2.4(i),

\[
B_j = (s_{y+j}, s_{y+j+1}, \ldots, s_{y+j+f_{2k}-1}) = (s_{y+j+f_{2k}}, s_{y+j+1+f_{2k}}, \ldots, s_{y+j+2f_{2k}-1}) = B_{f_{2k}+j}.
\]

Therefore, for \( 1 \leq j < f_{2k+2} \),

\[
H_{y+j, f_{2k}} = (-1)^{f_{2k}+1} H_{y+j+1, f_{2k}} = H_{y+j+1, f_{2k}}
\]

where the last equality follows from Lemma 2.9(i).

(iii) Recall that \( \gamma_i \in E_k' \). By Lemma 2.5, \( g := \gamma_i - f_{2k+2} \in E_k \). Write

\[
A_{y+j} = (s_{g+y}, s_{g+y+1}, \ldots, s_{g+y+f_{2k+1}-1}).
\]

For \( 1 \leq r < f_{2k} \),

\[
H_{g+r, f_{2k}+1} = \left| \begin{array}{ccc} A_{g+r} & \cdots & A_{g+r+1} \\ \vdots & \ddots & \vdots \\ A_{g+r+f_{2k+1}-1} & \cdots & A_{g+r+f_{2k+1}} \end{array} \right| \quad \text{and} \quad H_{g+r+1, f_{2k}+1} = \left| \begin{array}{ccc} A_{g+r+1} & \cdots & A_{g+r+2} \\ \vdots & \ddots & \vdots \\ A_{g+r+f_{2k+1}} & \cdots & A_{g+r+f_{2k+1}} \end{array} \right|.
\]

By Theorem 2.4, \( A_{g+r} = A_{g+r+f_{2k+1}} \). Then using Lemma 2.9, for all \( 1 \leq r < f_{2k} \),

\[
H_{g+r, f_{2k}+1} = (-1)^{f_{2k}+1} H_{g+r+1, f_{2k}+1} = -H_{g+r+1, f_{2k}+1}
\]

and

\[
H_{g+r+1, f_{2k}+1} = (-1)^{f_{2k}} H_{g+r+f_{2k}+1} = (-1)^{1+r} H_{g+r+f_{2k}, f_{2k}+1}.
\]

In fact, for all \( i \geq 1 \), the Hankel determinants on the bottom of \( U_{k,i} \) and \( V_{k,i} \) take the same value which depends only on \( k \). The following lemma helps us to connect the determinants on the bottom of \( U_{k,*} \) and \( V_{k,*} \).

**Lemma 4.3.** Let \( k \geq 0 \) and \( i \geq 1 \). If \( \gamma_{i+1} - \gamma_i = f_{2k+3} \), then \( H_{i+1, f_{2k+1}, f_{2k} = H_{i+1, f_{2k+1}, f_{2k}} \). If \( \gamma_{i+1} - \gamma_i = f_{2k+2} \), then \( H_{i+1, f_{2k+1}, f_{2k+1}} = H_{i+1, f_{2k+1}, f_{2k+1}} \).

**Proof.** Suppose \( \gamma_{i+1} - \gamma_i = f_{2k+3} \). Then \( \Phi_k(\gamma_i + f_{2k}) = f_{2k+3} + f_{2k} \). Since \( 3f_{2k} = f_{2k+2} + f_{2k-1} < f_{2k+3} \), by Theorem 2.4(ii), we have

\[
\begin{pmatrix} s_{\gamma_i + f_{2k}+1} & \cdots & s_{\gamma_i + 3f_{2k-1}} \\ s_{\gamma_i + f_{2k}+1} & \cdots & s_{\gamma_i + 3f_{2k-1}} \\ \vdots & \ddots & \vdots \\ s_{\gamma_i + f_{2k}} & \cdots & s_{\gamma_i + 3f_{2k-1}} \end{pmatrix} = \begin{pmatrix} s_{\gamma_i + f_{2k}+1} & \cdots & s_{\gamma_i + 3f_{2k-1}} \\ s_{\gamma_i + f_{2k}+1} & \cdots & s_{\gamma_i + 3f_{2k-1}} \\ \vdots & \ddots & \vdots \\ s_{\gamma_i + f_{2k}} & \cdots & s_{\gamma_i + 3f_{2k-1}} \end{pmatrix}.
\]

Therefore

\[
H_{\gamma_i + f_{2k}+1, f_{2k}} = \begin{pmatrix} s_{\gamma_i + f_{2k}+1} & \cdots & s_{\gamma_i + 3f_{2k}} \\ \vdots & \ddots & \vdots \\ s_{\gamma_i + f_{2k}+1} & \cdots & s_{\gamma_i + 3f_{2k}-1} \end{pmatrix} = \begin{pmatrix} s_{\gamma_i + f_{2k}+1} & \cdots & s_{\gamma_i + 3f_{2k}-1} \\ \vdots & \ddots & \vdots \\ s_{\gamma_i + f_{2k}+1} & \cdots & s_{\gamma_i + 3f_{2k}-1} \end{pmatrix}.
\]

When \( \gamma_{i+1} - \gamma_i = f_{2k+2} \), we have \( \Phi_k(\gamma_i) = f_{2k+3} \). By Theorem 2.4(ii),

\[
\begin{pmatrix} s_{\gamma_i + 1} & \cdots & s_{\gamma_i + 2f_{2k}} \\ \vdots & \ddots & \vdots \\ s_{\gamma_i + 1} & \cdots & s_{\gamma_i + 2f_{2k}-1} \end{pmatrix} = \begin{pmatrix} s_{\gamma_i + 1} & \cdots & s_{\gamma_i + 2f_{2k}} \\ \vdots & \ddots & \vdots \\ s_{\gamma_i + 1} & \cdots & s_{\gamma_i + 2f_{2k}-1} \end{pmatrix}.
\]
So \( H_{\gamma_1+1,f_{2k}} = H_{\gamma_1+1-f_{2k+1},f_{2k}} \).

Next we give the connection between \( T_{k,i} \) and \( T_{k,i+1} \).

**Lemma 4.4.** For all \( i \geq 1 \), \( H_{\gamma_i-f_{2k+1},f_{2k+1}} = H_{\gamma_i+1-f_{2k+1},f_{2k+1}} \).

**Proof.** If \( \gamma_{i+1} - \gamma_i = f_{2k+3} \), then \( \Phi_{k+1}(\gamma_i) = \frac{f_{2k+3}}{2} \) and \( \Phi_{k+1}(\gamma_i + f_{2k+1}) < \frac{f_{2k+5}}{2} \). By Theorem 2.4(ii), we have
\[
(s_{\gamma_1-f_{2k+1}} \cdots s_{\gamma_i+f_{2k+1}-2}) = \left( s_{\gamma_1-f_{2k+1}+f_{2k+3}} \cdots s_{\gamma_i+f_{2k+2}+f_{2k+3}} \right)
\]
Consequently, \( H_{\gamma_i-f_{2k+1},f_{2k+1}} = H_{\gamma_i+1-f_{2k+1},f_{2k+1}} \).

If \( \gamma_i+1 - \gamma_i = f_{2k+2} \), then \( \Phi_{k+1}(\gamma_i) = \frac{f_{2k+2}}{2} + f_{2k+3} \) or \( \frac{f_{2k+2}}{2} + f_{2k+4} \). By Theorem 2.4(i), we have
\[
(s_{\gamma_1-f_{2k+1}} \cdots s_{\gamma_i+f_{2k+1}-2}) = \left( s_{\gamma_1-f_{2k+1}+f_{2k+2}} \cdots s_{\gamma_i+f_{2k+1}+2+f_{2k+2}} \right)
\]
Consequently, \( H_{\gamma_i-f_{2k+1},f_{2k+1}} = H_{\gamma_i+1-f_{2k+1},f_{2k+1}} \).

According to Lemma 4.3 and Lemma 4.4, the values of the determinants on the bottom edges of \( U_{k,i} \) and \( V_{k,i} \) only depends on \( k \). We improve Lemma 4.2 to the following proposition.

**Proposition 4.5.** Let \( k \geq 0 \). For all \( i \geq 1 \),

(i) (Bottom edges of \( U_{k,i} \) and \( V_{k,i} \)) for all \( 1 \leq r \leq f_{2k+1} \) and \( 1 \leq r' \leq f_{2k+2}, \)
\[
H_{\alpha_1-f_{2k+1}+r,f_{2k}} = H_{\alpha_1-f_{2k+1}+r,f_{2k}}
\]
(ii) (Bottom edge of \( T_{k,i} \)) \( H_{\gamma_i-f_{2k+2}+r,f_{2k+1}} = (-1)^{r+1} H_{\gamma_i-f_{2k+1}+f_{2k+1}} \) for all \( 1 \leq r \leq f_{2k+1} \) with \( \gamma_i - f_{2k+2} + r \geq 0 \).

**Proof.** Since \( \alpha_i - f_{2k+2} \in E'_{k} \) and \( \beta_i + f_{2k} \in E'_{k} \), Lemma 4.3 shows that the values of two determinants on the bottom edge of two adjacent parallelograms in \( \{U_{k,j}\}_{j \geq 1} \cup \{V_{k,j}\}_{j \geq 1} \) are the same. Then Lemma 4.2 implies the result (i). The result (ii) follows from Lemma 4.2(iii) and Lemma 4.4. \( \square \)

### 4.3. On the boundary of \( U_{k,i} \).

**Lemma 4.6.** Let \( k \geq 0 \) and \( i \geq 1 \). For all \( 0 \leq r \leq f_{2k+2} - 1 \) with \( \alpha_i - f_{2k+4} + 2 + r \geq 0 \),

(i) (Right edge of \( U_{k,i} \)) \( \alpha_i-f_{2k+3}+1+r,f_{2k+3}-1=r \)
\[
(-1)^{r+1} H_{\alpha_i-f_{2k+3}+1,f_{2k+3}-1}
\]
(ii) (Left edge of \( U_{k,i} \)) \( \alpha_i-f_{2k+4}+2+r,f_{2k+3}-1=r \)
\[
(-1)^{r+1} H_{\alpha_i-f_{2k+3}+1,f_{2k+3}-1}
\]
(iii) (Upper edge of \( U_{k,i} \)) \( \alpha_i-f_{2k+4}+2+r,f_{2k+3}-1=r \)
\[
(-1)^{r} H_{\alpha_i-f_{2k+3}+1,f_{2k+3}-1}
\]

**Proof.** Write \( y = \alpha_i - f_{2k+3} \). Recall that \( \alpha_i \in E'_{k+1} \). So \( \Phi_{k+1}(y) = \frac{f_{2k+1}}{2} \) and \( y \in E'_{k} \).

(i) For \( 0 \leq r < f_{2k+2} \), let \( A_{y+r+r} \) be the \( j \)-th column of \( M_{y+1+r,f_{2k+3}-1} \). Applying Lemma 2.7 and Theorem 2.4(i), we see \( s_{y+r+r+1} = s_{y+r+r+1} f_{2k} \) for \( 1 \leq r \leq f_{2k+1} - r - 2 \) and \( s_{y+f_{2k+3}-1} \neq s_{y+f_{2k+3}-1} f_{2k} \). Then Proposition 2.3 and Lemma 2.10 yields \( s_{y+f_{2k+3}-1} = s_{y+f_{2k+3}-1} f_{2k} = (-1)^{k} \). Therefore,
\[
A_{y+r+1} - A_{y+r+f_{2k}} = \begin{pmatrix}
\frac{s_{y+r+1}}{s_{y+r+2}} & \frac{s_{y+r+2}}{s_{y+r+2}} \\
\vdots & \vdots \\
\frac{s_{y+f_{2k+3}-2}}{s_{y+f_{2k+3}-1}} & \frac{s_{y+f_{2k+3}-1}}{s_{y+f_{2k+3}-1}}
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]
and

\[ H_{y+1+r,f_{2k+3}-1-r} = \begin{vmatrix} A_{y+r+1} & A_{y+r+2} & \cdots & A_{y+f_{2k+3}-1} \\ (A_{y+r+1} - A_{y+r+f_{2k}}) & A_{y+r+2} & \cdots & A_{y+f_{2k+3}-1} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^k & \cdots & \cdots & (-1)^k \\ 0_{f_{2k+3}-2-r} & M_{y+r+2,f_{2k+3}-r-2} & \cdots & \cdots & \cdots \end{vmatrix} \]

\[ = (-1)^{k+r} (-1)^{1+f_{2k+3}-1-r} H_{y+r+2,f_{2k+3}-r-2} \]

\[ = (-1)^{k+r} H_{y+r+2,f_{2k+3}-r-2} \tag{8} \]

where in the last equality we apply Lemma 2.9 and \( 0_{i,j} \) denotes the \( i \times j \) zero matrix. It follows from Eq. (8) that

\[ H_{y+1,f_{2k+3}-1} = (-1)^k H_{y+2,f_{2k+3}-2} = (-1)^k (-1)^{k+1} H_{y+3,f_{2k+3}-3} = (-1)^k (-1)^{k+1} \cdots (-1)^{k+r-1} H_{y+r,f_{2k+3}-1-r} \]

\[ = (-1)^r (-1)^{r+1} H_{y+1+r,f_{2k+3}-1-r}. \]

(ii) Let \( B_{y-f_{2k+2}+1+r+j} \) be the \( j \)-th row of \( M_{y-f_{2k+2}+2+r,f_{2k+3}-1-r} \). Combining Lemma 2.7, Theorem 2.4(i), Proposition 2.3 and Lemma 2.10, a similar argument as above yields

\[ H_{y-f_{2k+2}+2+r,f_{2k+3}-1-r} = \begin{vmatrix} B_{y-f_{2k+2}+1+r+1} \\ \vdots \\ B_y \\ \vdots \\ B_{y+f_{2k}} \\ B_{y+f_{2k}+1} \end{vmatrix} = \begin{vmatrix} B_{y-f_{2k+2}+1+r+1} \\ \vdots \\ B_y \\ \vdots \\ B_{y+f_{2k}} \\ B_{y+f_{2k}+1} \end{vmatrix} \]

\[ = (-1)^{k+r} H_{y-f_{2k+2}+2+(r+1),f_{2k+3}-1-(r+1)}. \tag{9} \]

Applying Eq. (9), we have

\[ H_{y-f_{2k+2}+2+r,f_{2k+3}-1-r} = (-1)^{k+r} (-1)^{k+r+1} \cdots (-1)^{k+f_{2k+2}-2} H_{y+1,f_{2k}} \]

\[ = (-1)^{(2k+r)f_{2k+2}-3} H_{y+1,f_{2k}} \]

\[ = (-1)^{(2k+r)f_{2k+2}-3} H_{y+f_{2k+2},f_{2k}} \] (by Lemma 4.2(ii))

\[ = (-1)^r (-1)^{r+1} H_{y+1,f_{2k+3}-1}. \] (by Lemma 4.6(iii))

(iii) Let \( C_j \) be the \( j \)-th column of \( M_{y-f_{2k+2}+2+r,f_{2k+3}-1} \). Then

\[ H_{y-f_{2k+2}+2+r,f_{2k+3}-1} = \det(C_1,C_2,\ldots,C_{f_{2k+3}-1}) \]

\[ = \det(C_1,C_2,\ldots,C_{f_{2k+3}-1-r},C_1',C_2',\ldots,C_r') \]

where \( C_p^r = C_{f_{2k+3}-1-r+p} - C_{f_{2k+3}-1-r+p-f_{2k}} \) for \( 1 \leq p \leq r \). According to Lemma 2.7, Theorem 2.4(i), we have \( s_{y+\ell} = s_{y+\ell+f_{2k}} \) for all \( 1 \leq \ell \leq f_{2k+3}-2 \) and \( f_{2k+3}+1 \leq \ell \leq f_{2k+3}+r-2 \). By Proposition 2.3 and Lemma 2.10, we obtain that \( s_{y+f_{2k+3}-1+f_{2k}} = s_{y+f_{2k+3}-1} = (-1)^{k+1} \) and \( s_{y+f_{2k+3}+f_{2k}} - s_{y+f_{2k+3}} = (-1)^k \). Thus

\[ (C_1',C_2',\ldots,C_r') = \begin{pmatrix} 0_{f_{2k+3}-1-r,r} \\ X \end{pmatrix} \]
where $X$ is the $r \times r$ matrix

$$
\begin{pmatrix}
0 & \cdots & 0 & (-1)^{k+1} \\
\vdots & \ddots & \ddots & (-1)^k \\
\vdots & \ddots & \ddots & \vdots \\
(-1)^{k+1} & (-1)^k & \cdots & 0
\end{pmatrix}.
$$

Expanding by the last $r$ columns, we have

$$
H_{y-f_{2k+2}+2+r,f_{2k+3}-1} = \det \left( \begin{array}{c}
M_{y-f_{2k+2}+2+r,f_{2k+3}-1-r} \\
0_{f_{2k+3}-1-r,r} \\
X
\end{array} \right)
$$

$$
= (-1)^{(k+1)r}(r-1)^{r+1} H_{y-f_{2k+2}+2+r,f_{2k+3}-1-r}
$$

$$
= (-1)^{(k+1)r}(r-1)^{r+1} (-1)^{r+k} \frac{r(r+1)}{2} H_{y+1,f_{2k+3}-1}
$$

(by Lemma 4.6(ii))

$$
= (-1)^r H_{y+1,f_{2k+3}-1}.
$$

**Remark 4.7.** From Proposition 4.5(i) and Lemma 4.6, Hankel determinants on the boundary of $U_{k,i}$ can be determined by $H_{\alpha_1-f_{2k+3}+1,f_{2k+3}-1} = H_{f_{2k+4}+1,f_{2k+3}-1}$ (the upper right corner of $U_{k,1}$).

4.4. On the boundary of $V_{k,i}$.

**Lemma 4.8.** Let $k \geq 0$ and $i \geq 1$. For all $0 \leq r \leq f_{2k+1} - 1$,

(i) (Left edge of $V_{k,i}$) $H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r} = (-1)^k(1) \frac{r(r+1)}{2} H_{\beta'_i-f_{2k+1}+2,f_{2k+2}-1}$,

(ii) (Right edge of $V_{k,i}$) $H_{\beta'_i+1+r,f_{2k+2}-1-r} = (-1)^k(1) \frac{r(r+1)}{2} H_{\beta'_i+1,f_{2k+2}-1}$,

(iii) (Upper edge of $V_{k,i}$) $H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1} = H_{\beta'_i+1,f_{2k+2}-1}$.

**Proof.** (i) Denote by $A_{\beta'_i-f_{2k+1}+1+r}$ the $j$-th row of $M_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r}$. Then

$$
H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r} = \det \left( \begin{array}{c}
A_{\beta'_i-f_{2k+1}+2+r} \\
\vdots \\
A_{\beta'_i+f_{2k}}
\end{array} \right) = \det \left( \begin{array}{c}
A_{\beta'_i-f_{2k+1}+2+r} \\
\vdots \\
A_{\beta'_i+f_{2k}}
\end{array} \right) = \det \left( \begin{array}{c}
A_{\beta'_i+f_{2k}} - A_{\beta'_i}
\end{array} \right).
$$

From Lemma 2.7, Theorem 2.4 and Lemma 2.10, we have

$$
A_{\beta'_i+f_{2k}} - A_{\beta'_i} = ((-1)^k, 0, \cdots, 0).
$$

For $0 \leq r \leq f_{2k+1} - 1$,

$$
H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r} = \left( \begin{array}{c}
* \\
(-1)^k M_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r} \\
0_{f_{2k+2}-2-r}
\end{array} \right)
$$

$$
= (-1)^k(1)^{r+1} H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r}
$$

$$
= (-1)^k(1)^{r+1} H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r}
$$

(by Lemma 2.9(i))

Thus

$$
H_{\beta'_i-f_{2k+1}+2+r,f_{2k+2}-1-r} = (-1)^{k+1+r-1} (-1)^{k+1+r-2} \cdots (-1)^{k+1} H_{\beta'_i+2-f_{2k+1},f_{2k+2}-1}
$$

$$
= (-1)^r(k+1) \frac{r(r+1)}{2} H_{\beta'_i+2-f_{2k+1},f_{2k+2}-1}.
$$
(ii) Let $B_{\beta_1'+r+j}$ be the $j$-th column of $M_{\beta_1'+1+r, f_{2k+2}-1-r}$.

\[
H_{\beta_1'+1+r, f_{2k+2}-1-r} = \det \begin{pmatrix}
    s_{\beta_1'+1+r} & s_{\beta_1'+2+r} & \cdots & s_{\beta_1'+f_{2k+2}-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    s_{\beta_1'+f_{2k+2}-1} & s_{\beta_1'+f_{2k+2}} & \cdots & s_{\beta_1'+2f_{2k+2}-3-r}
\end{pmatrix}
\]

\[
= \det (B_{\beta_1'+1+r}, B_{\beta_1'+2+r}, \ldots, B_{\beta_1'+f_{2k+2}-1})
\]

Recall that $\beta_1' \in F_k'$. By Lemma 2.7, $\beta_1'$ and $\beta_1' + f_{2k+2}$ are adjacent elements in $F_k$. It follows from Theorem 2.4(i) and Lemma 2.10 that

\[
H_{\beta_1'+1+r, f_{2k+2}-1-r} = \det \begin{pmatrix}
    0_{f_{2k+2}-2-r,1} & M_{\beta_1'+1+(r+1), f_{2k+2}-1-(r+1)} \ast
\end{pmatrix}
\]

\[
= (-1)^k \cdot (-1)^{f_{2k+2}-1-r} H_{\beta_1'+1+(r+1), f_{2k+2}-1-(r+1)}
\]

\[
= (-1)^{k+1+r} H_{\beta_1'+1+(r+1), f_{2k+2}-1-(r+1)}. \quad \text{(by Lemma 2.9(i))}
\]

Hence $H_{\beta_1'+1+r, f_{2k+2}-1-r} = (-1)^{k(r+1)} (-1)^{f_{2k+2}-1} H_{\beta_1'+1, f_{2k+2}-1}$.

(iii) Let $C_{\beta_1'-f_{2k+1}+1+r, j}$ be the $j$-th column of $M_{\beta_1'-f_{2k+1}+2+r, f_{2k+2}-1}$. Then

\[
H_{\beta_1'-f_{2k+1}+2+r, f_{2k+2}-1} = \det \begin{pmatrix}
    C_{\beta_1'-f_{2k+1}+2+r} & C_{\beta_1'-f_{2k+1}+3+r} & \cdots & C_{\beta_1'+f_{2k+1}+r}
\end{pmatrix}
\]

where $C_p = C_{\beta_1'+f_{2k}+p} - C_{\beta_1'+p}$ for $1 \leq p \leq r$. By Lemma 2.7 and Theorem 2.4(i), we have $s_{\beta_1'+\ell} = s_{\beta_1'+\ell+f_{2k}}$ for $1 \leq \ell \leq f_{2k+2} - 2$ and $f_{2k+2} + 1 \leq \ell \leq r + f_{2k+2} - 1$. Moreover, by Proposition 2.3 and Lemma 2.10, we have $s_{\beta_1'+f_{2k}+f_{2k+2}+1} - s_{\beta_1'+f_{2k+2}+1} = (-1)^{k+1}$ and $s_{\beta_1'+f_{2k}+f_{2k+2}} - s_{\beta_1'+f_{2k+2}} = (-1)^k$. Thus

\[
(C_1' \cdots C_r') = \begin{pmatrix}
    0_{f_{2k+2}-1-r, r} \\
\end{pmatrix}
\]

where $X$ is the $r \times r$ matrix

\[
\begin{pmatrix}
    0 & \cdots & 0 & (-1)^{k+1} \\
    \vdots & \ddots & \vdots & \vdots \\
    (-1)^{k+1} & \ddots & \ddots & \vdots \\
    (-1)^k & \cdots & (-1)^k & 0
\end{pmatrix}
\]

Now expanding $H_{\beta_1'-f_{2k+1}+2+r, f_{2k+2}-1}$ by its last $r$ columns, we obtain that for $0 \leq r \leq f_{2k+1} - 1$,

\[
H_{\beta_1'-f_{2k+1}+2+r, f_{2k+2}-1} = \det \begin{pmatrix}
    M_{\beta_1'-f_{2k+1}+2+r, f_{2k+2}-1-r} & 0_{f_{2k+2}-1-r, r} \\
\ast & X
\end{pmatrix}
\]

\[
= (-1)^{k(r+1)} (-1)^{f_{2k+2}-1-r} H_{\beta_1'-f_{2k+1}+2+r, f_{2k+2}-1-r}
\]

\[
= H_{\beta_1'-f_{2k+1}+2, f_{2k+2}-1}. \quad \text{(by Lemma 4.8(i))}
\]

\[\square\]

**Remark 4.9.** From Proposition 4.5(i) and Lemma 4.8, Hankel determinants on the boundary of $V_{k,1}$ can be determined by $H_{f_{2k+1}+1, f_{2k+2}+1, f_{2k}}$ (the lower left corner of $V_{k,1}$).
4.5. On the boundary of $T_{k,1}$.

**Lemma 4.10.** Let $k \geq 0$ and $i \geq 1$.

(i) (Left edge of $T_{k,1}$) For all $0 \leq r \leq f_{2k} - 1$ with $\gamma_i - f_{2k+3} + 2 + r \geq 0$,

$$H_{\gamma_i - f_{2k+3} + 2 + r, f_{2k+2} - 1 - r} = (-1)^{r+1} (\frac{r+1}{2}) H_{\gamma_i - f_{2k+3} + 2, f_{2k+2} - 1}.$$

(ii) (Right edge of $T_{k,1}$) For all $0 \leq r \leq f_{2k} - 1$ with $\gamma_i - f_{2k+2} + 1 + r \geq 0$,

$$H_{\gamma_i - f_{2k+2} + 1 + r, f_{2k+2} - 1 - r} = (-1)^{r} (\frac{r}{2}) H_{\gamma_i - f_{2k+2} + 1, f_{2k+2} - 1}.$$

(iii) (Upper edge of $T_{k,1}$) For all $0 \leq r \leq f_{2k} - 1$ with $\gamma_i - f_{2k+3} + 2 + r \geq 0$,

$$H_{\gamma_i - f_{2k+3} + 2 + r, f_{2k+2} - 1} = H_{\gamma_i - f_{2k+3} + 2, f_{2k+2} - 1}.$$

**Proof.** To shorten the notation, write $x = \gamma_i - f_{2k+3} + 2$ and $x' = \gamma_i - f_{2k+2} + 1$.

(i) Let $\max\{0, -x\} \leq \ell \leq f_{2k} - 1$ and let $A_j$ be the $j$th row of $H_{x+\ell, f_{2k+2} - 1 - \ell}$. By Theorem 2.4(ii) and Lemma 2.10, we see

$$A_{f_{2k+2} + 1 - \ell} - A_{f_{2k+2} - 1} = ((-1)^{k+1} 0 \ldots 0).$$

Then for $\max\{0, -x\} \leq \ell \leq f_{2k} - 1$,

$$H_{x+\ell, f_{2k+2} - 1 - \ell} = \det \begin{pmatrix} A_1 & A_2 & \cdots & A_{f_{2k+2} - 2 - \ell} & A_{f_{2k+2} - 1 - \ell} \\ A_1 & A_2 & \cdots & A_{f_{2k+2} - 2 - \ell} & A_{f_{2k+2} - 1 - \ell} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{f_{2k+2} - 2 - \ell} & A_{f_{2k+2} - 1 - \ell} & \cdots & A_1 & A_{f_{2k+2} - 1} \end{pmatrix} = (-1)^{k+1} \cdot \begin{pmatrix} M_{x+\ell+1, f_{2k+2} - 1 - (\ell+1)} \\ \vdots \\ M_{x+\ell+1, f_{2k+2} - 1 - (\ell+1)} \end{pmatrix}.$$

Applying the above equality $r$ times, one has

$$H_{x+r, f_{2k+2} - 1 - r} = (-1)^{k+r-1}((-1)^{k+r-2} \ldots (-1)^{k-1} H_{x, f_{2k+2} - 1} = (-1)^{r} (\frac{r}{2}) H_{\gamma_i - f_{2k+2} + 1, f_{2k+2} - 1}.$$

(ii) Let $\max\{0, -x'\} \leq \ell \leq f_{2k} - 1$ and let $B_j$ be the $j$th column of $H_{x'+\ell, f_{2k+2} - 1 - \ell}$. By Theorem 2.4(ii) and Lemma 2.10, we see

$$B_1 - B_1 + f_{2k+1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{k+1} \end{pmatrix}.$$

Therefore, for $\max\{0, -x'\} \leq \ell \leq f_{2k} - 1$,

$$H_{x'+\ell, f_{2k+2} - 1 - \ell} = \det \begin{pmatrix} B_1 & B_2 & \cdots & B_{f_{2k+2} - 1 - \ell} \\ B_1 - B_1 + f_{2k+1} & B_2 & \cdots & B_{f_{2k+2} - 1 - \ell} \\ \vdots & \vdots & \ddots & \vdots \\ B_{f_{2k+2} - 2 - \ell} & B_{f_{2k+2} - 1} & \cdots & B_1 \end{pmatrix} = (-1)^{k+1}((-1)^{k+1} H_{x'+\ell+1, f_{2k+2} - 1 - (\ell+1)}) = (-1)^{k+1} H_{x'+\ell+1, f_{2k+2} - 1 - (\ell+1)}.$$
Applying the above equality \( r \) times, one has

\[
H_{x+r,f_{2k+2}-1-r} = (-1)^{k+r-1}(-1)^{k+r-2} \cdots (-1)^k H_{x',f_{2k+2}-1} \\
= (-1)^r(-1)^{\frac{r(r-1)}{2}} H_{x',f_{2k+2}-1}.
\]

(iii) Let \( \max\{0,-x\} \leq r \leq f_{2k} - 1 \) and let \( C_j \) be the \( j \)th column of \( M_{x+r,f_{2k+2}-1} \). Then

\[
H_{x+r,f_{2k+2}-1} = \det (C_1 \ C_2 \ \cdots \ C_{f_{2k+2}-1})
\]

where \( C'_p = C_{f_{2k+2}-r-1+p} - C_{f_{2k}-r-1+p} \) for \( 1 \leq p \leq r \). Note that

\[
(C_1' \ \cdots \ C_r') = \begin{pmatrix}
  s_{1-r,f_{2k+1}} & \cdots & s_{1-r,f_{2k+r}} \\
  \vdots & \ddots & \vdots \\
  s_{1-r+f_{2k+1}-1} & \cdots & s_{1-r+f_{2k+1}+r-2}
\end{pmatrix} - \begin{pmatrix}
  s_{1-r,f_{2k+2}+1} & \cdots & s_{1-r,f_{2k+2}+r} \\
  \vdots & \ddots & \vdots \\
  s_{1-r+f_{2k+1}+1-r} & \cdots & s_{1-r+f_{2k+1}+r-2}
\end{pmatrix}.
\]

By Lemma 2.5 and Theorem 2.4(ii), for \( 1 \leq q \leq f_{2k+2} - 2 \) and \( f_{2k+2} + 1 \leq q \leq f_{2k+2} + r - 2 \),

\[s_{1-r,f_{2k+q}+q} = s_{1-r,f_{2k+2}+q}.\]

Moreover, by Lemma 2.10, \( s_{1-r+f_{2k+1}+r-1} = (-1)^k \) and \( s_{1-r+f_{2k+1}} = s_{1-r} = (-1)^k+1 \). Then

\[
(C_1' \ \cdots \ C_r') = \begin{pmatrix}
  0 & \cdots & 0 & (-1)^k \\
  \vdots & \ddots & \vdots & \vdots \\
  (-1)^k & \cdots & (-1)^k+1 & \cdots & 0
\end{pmatrix}.
\]

Therefore,

\[
H_{x+r,f_{2k+2}-1} = \det \left( M_{x+r,f_{2k+2}-1-r} \begin{pmatrix}
  0 & \cdots & 0 & (-1)^k \\
  \vdots & \ddots & \vdots & \vdots \\
  (-1)^k & \cdots & (-1)^k+1 & \cdots & 0
\end{pmatrix}ight)
\]

\[
= (-1)^k(-1)^{\frac{r(r-1)}{2}} H_{x',f_{2k+2}-1-r}
\]

\[= H_{x,f_{2k+2}-1}. \quad \text{(by Lemma 4.10(i))}
\]

Remark 4.11. From Proposition 4.5(ii) and Lemma 4.10, Hankel determinants on the boundary of \( T_{k,i} \) can be determined by \( H_{f_{2k+2}+f_{k+1},f_{k+1}} \) (the lower left corner of \( T_{k,2} \)).

5. Evaluating the Hankel determinants

In section 4, we show that for any \( k \geq 0 \), to know all the determinants on the boundary of \( U_{k,i} \) (resp. \( V_{k,i}, T_{k,i} \)) for all \( i \), it is enough to know the value of one determinant on the boundary \( U_{k,i} \) (resp. \( V_{k,i} \) or \( T_{k,i} \)) for some \( i \). In this section, for each \( i \), we shall give the expression of a determinant on the boundary \( U_{k,i} \) (resp. \( V_{k,i} \) or \( T_{k,i} \)) for all \( k \).

The next result allows us to determine the determinant on the lower left corner of \( U_{k,i} \) by using the determinants on the boundary of \( U_{k-1,*} \) and \( T_{k-1,*} \).

Lemma 5.1. (Lower left corner of \( U_{k,i} \)) For all \( k \geq 1 \) and \( i \geq 1 \),

\[
H_{a_i-f_{2k+3}+1,f_{2k}} = (-1)^k (H_{a_i-f_{2k+3}+2,f_{2k}-1} - H_{a_i-f_{2k+3}+1,f_{2k}-1}).
\]
Proof. Let $y = \alpha_1 - f_{2k+3}$ and let $A_j$ be the $j$th column of $H_{y+1, f_{2k}}$. Then

$$H_{y+1, f_{2k}} = \begin{vmatrix} s_{y+1} & s_{y+2} & \cdots & s_{y+f_{2k}} \\
\vdots & \vdots & \ddots & \vdots \\
s_{y+f_{2k}} & s_{y+f_{2k}+1} & \cdots & s_{y+2f_{2k}-1} \end{vmatrix} = \det (A_1 \ A_2 \ \cdots \ A_{f_{2k}-1}).$$

Recall that $\alpha_i \in E_{k+1}$. Then $\Phi_{k+1}(y) = \frac{f_{2k+1}}{2}$ and $\Phi_{k-1}(y + f_{2k}) = \frac{f_{2k-1}}{2}$. This implies $y \in F'_k$ and $y + f_{2k} \in F_{k-1}$. By Lemma 2.7 and Theorem 2.4(i), the fact $y + f_{2k} \in F_{k-1}$ yields that $s_{y+\ell} = s_{y+f_{2k-2}+\ell}$ for $1 \leq \ell \leq f_{2k} - 2$. By Lemma 2.10, $s_{y+f_{2k}} - s_{y+f_{2k}+f_{2k}-1} = (-1)^{k+1}$ and $s_{y+f_{2k}} - s_{y+f_{2k}+f_{2k}-1} = (-1)^k$. So

$$H_{y+1, f_{2k}} = \det \begin{vmatrix} 0 & s_{y+2} & \cdots & s_{y+f_{2k}} \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{k+1} & s_{y+f_{2k}-1} & \cdots & s_{y+2f_{2k}-3} \\
(-1)^k & s_{y+f_{2k}+1} & \cdots & s_{y+2f_{2k}-1} \end{vmatrix} = (-1)^{k}(1)^{f_{2k}}H_{y+2, f_{2k}-1} + (-1)^{k+1}(1)^{f_{2k}}X \quad (10)$$

where

$$X = \begin{vmatrix} s_{y+2} & \cdots & s_{y+f_{2k}} \\
\vdots & \ddots & \vdots \\
s_{y+f_{2k}-1} & \cdots & s_{y+2f_{2k}-3} \\
s_{y+f_{2k}+1} & \cdots & s_{y+2f_{2k}-1} \end{vmatrix} = (-1)^{f_{2k}-2} \begin{vmatrix} s_{y+f_{2k}+1} & \cdots & s_{y+2f_{2k}-1} \\
\vdots & \ddots & \vdots \\
\vdots & \cdots & \vdots \end{vmatrix}.$$ 

Since $y + f_{2k} \in F'_k$, by Lemma 2.7 and Theorem 2.4(i), we see

$$(s_{y+f_{2k}+1} \ \cdots \ s_{y+2f_{2k}-1}) = (s_{y+1} \ \cdots \ s_{y+f_{2k}-1})$$

and $X = (-1)^{f_{2k}-2}H_{y+1, f_{2k}-1}$. Then the result follows from Eq. (10) and Lemma 2.9. \hfill \Box

Now we show how to obtain the determinant on the lower left corner of $T_{k,i}$ by using determinants on the boundary of $U_{k,i-1}$ and $U_{k-1,i+1}$.

**Lemma 5.2.** (Lower left corner of $T_{k,i}$) For all $k \geq 1$ and $i \geq 2$,

$$H_{\gamma_i, f_{2k+2}+1, f_{2k+1}+1} = (-1)^{k}(H_{\gamma_i, f_{2k+2}+2, f_{2k+1}+1} + H_{\gamma_i, f_{2k+2}+1, f_{2k+1}+1}).$$

Proof. Let $y = \gamma_i - f_{2k+2}$ and let $A_j$ be the $j$th column of $H_{y+1, f_{2k+1}}$. Then

$$H_{y+1, f_{2k+1}} = \begin{vmatrix} s_{y+1} & s_{y+2} & \cdots & s_{y+f_{2k+1}} \\
\vdots & \vdots & \ddots & \vdots \\
s_{y+f_{2k+1}+1} & s_{y+f_{2k+1}+2} & \cdots & s_{y+2f_{2k+1}+1} \end{vmatrix} = \det (A_1 \ A_2 \ \cdots \ A_{f_{2k+1}-1}).$$

Recall that $\gamma_i \in E'_k$. By Lemma 2.5, $y \in E_k$ and $\Phi_k(y + f_{2k+1}) = \frac{f_{2k+1}-1}{2}$. This implies $y + f_{2k+1} \in F_k$. By Lemma 2.7 and Theorem 2.4(i), the fact $y + f_{2k+1} \in F_k$ yields that $s_{y+\ell} = s_{y+f_{2k}+\ell}$ for $1 \leq \ell \leq f_{2k+1} - 2$. By Lemma 2.10, $s_{y+f_{2k+1}+1} - s_{y+f_{2k+1}+f_{2k+1}} = (-1)^k$ and $s_{y+f_{2k+1}} - s_{y+f_{2k+1}+f_{2k+1}} = (-1)^{k+1}$. So

$$H_{y+1, f_{2k+1}} = \det (A_1 \ A_2 \ \cdots \ A_{f_{2k+1}-1}).$$
Theorem 5.3. (Upper right corner of $U_{k,i}$) Let $k \geq 1$. Then

\[ H_{f_{2k+1}+1, f_{2k+3}-1} = (-1)^{k+1} \frac{f_{2k+1}}{2}. \]

Proof. We can check directly that the result holds for $k = 1, 2$. Now suppose $k \geq 3$. Let $h_k = H_{f_{2k+1}+1, f_{2k+3}-1}$. Then

\[ h_k = (-1)^k f_{2k+2-1} f_{2k+2-1} \frac{f_{2k+2-1} + f_{2k+2-2}}{2} - f_{2k+2} f_{2k+1} \]

(by Lemma 4.6(i))

\[ = (-1)^{k+1} \frac{f_{2k+2-1}}{2} H_{f_{2k+1}+1, f_{2k}} \]

(by Lemma 2.9 and Lemma 4.2(ii))

\[ = (-1)^{k+1} \frac{f_{2k+2-1}}{2} \left( H_{f_{2k+1}+2, f_{2k+1}} - H_{f_{2k+1}+1, f_{2k}} \right). \]

(by Lemma 5.1)

Applying Lemma 4.6(i) for $k - 1^1$ and $r = f_{2k-2},$

\[ H_{f_{2k+1}+1, f_{2k+1}} = (-1)^{k+1} \frac{f_{2k+2-1}}{2} H_{f_{2k+1}+1, f_{2k}} + f_{2k+2} f_{2k+1} \]

(by Lemma 2.9)

Applying Lemma 4.10(i) for $k - 1$ and $r = f_{2k-2},$

\[ H_{f_{2k+1}+1, f_{2k+1}} = (-1)^{k+1} \frac{f_{2k+2-1}}{2} \left( H_{f_{2k+1}+2, f_{2k-1}} + H_{f_{2k+1}+1, f_{2k}} \right) \]

(by Lemma 2.9)

\[ = (-1)^{k+1} \frac{f_{2k+2-1}}{2} \left( \alpha_{k-2} + H_{f_{2k+1}+1, f_{2k}} \right). \]

(by Lemma 4.6(iii))
Combining previous equations, we have

\[ h_k = (-1)^{\frac{f_{2k}+2}{2}}(-1)^k (H_{\alpha_1-f_{2k+3}+2}, f_{2k-1}) - H_{\alpha_1-f_{2k+3}+1}, f_{2k-1}) \]

\[ = (-1)^{\frac{f_{2k}+2}{2}}(-1)^k (-1)^{\frac{f_{2k}+2}{2}} (-1)^k (h_{k-2} - 2h_{k-1}) \]

The initial values are \( h_1 = 2, h_2 = -5 \). The result follows from the recurrence relation of \( h_k \) and its initial values.

\[ \square \]

**Corollary 5.4.** (Lower left corner of \( V_{k,1} \)) For all \( k \geq 1 \),

\[ H_{\frac{f_{2k+3}+2}{2}} + f_{2k+3} + 1, f_k = (-1)^k + \frac{f_{2k+2}+1}{2} \frac{f_{2k+1}}{2}. \]

**Proof.** By Proposition 4.5,

\[ H_{\frac{f_{2k+3}+2}{2}} + f_{2k+3} + 1, f_k = H_{\frac{f_{2k}+2}{2}} - f_{2k}, f_{2k} \]

\[ = (-1)^{\frac{f_{2k}+2}{2}}(-1)^k (-1)^{\frac{f_{2k}+2}{2}} \frac{H_{\frac{f_{2k}+1}{2}}}{1}, f_{2k+3} - 1 \]

(by Lemma 4.6(i))

\[ = (-1)^{\frac{f_{2k}+2}{2}} \frac{H_{\frac{f_{2k}+1}{2}}}{1}, f_{2k+3} - 1 \]

(by Lemma 4.6(iii))

\[ = (-1)^k + \frac{f_{2k+2}+1}{2} \frac{f_{2k+1}}{2}. \]

(by Lemma 5.3)

\[ \square \]

**Corollary 5.5.** (Lower left corner of \( T_{k,2} \)) For all \( k \geq 1 \), \( H_{\frac{f_{2k+3}+2}{2}} + f_{2k+1}, f_{2k+1} = f_{2k} \).

**Proof.** From Lemma 5.2, we have

\[ H_{\frac{f_{2k+3}+2}{2}} + f_{2k+1}, f_{2k+1} = (-1)^k (H_{\frac{f_{2k}+2}{2}} + f_{2k+2}, f_{2k+1} - 1) + H_{\frac{f_{2k}+2}{2}} + f_{2k+3} + f_{2k+1} - 1). \]  

(by Lemma 4.6(iii))

Note that \( H_{\frac{f_{2k}+2}{2}} + f_{2k+3} + f_{2k+1} - 1 \) is on the upper left corner of \( U_{k-1,2} \). By Lemma 4.6(iii),

\[ H_{\frac{f_{2k}+3}{2}} + f_{2k+3} + f_{2k+1} - 1 = H_{\frac{f_{2k}+3}{2}} + f_{2k+3} + f_{2k+1} - 1 \]

According to Proposition 4.5 and Lemma 4.6, the determinants on the upper left corner of \( U_{k-1,1} \) and \( U_{k-1,1} \) are equal. Namely, \( H_{\frac{f_{2k}+3}{2}} + f_{2k+3} + f_{2k+1} - 1 = H_{\frac{f_{2k}+3}{2}} + f_{2k+3} + f_{2k+1} - 1 \). Therefore,

\[ H_{\frac{f_{2k}+3}{2}} + f_{2k+3} + f_{2k+1} - 1 = H_{\frac{f_{2k}+3}{2}} + f_{2k+3} + f_{2k+1} - 1 \]

It follows from Lemma 4.6(i) and Lemma 2.9 that

\[ H_{\frac{f_{2k}+3}{2}} + f_{2k+1}, f_{2k+1} = (-1)^k 2f_{2k} (-1)^{\frac{f_{2k}+2}{2}} \frac{H_{\frac{f_{2k}+1}{2}}}{1}, f_{2k+3} - 1 \]

(by Lemma 4.6(i))

\[ = -\frac{H_{\frac{f_{2k}+1}{2}}}{1}, f_{2k+3} - 1 \]

(by Lemma 5.3)

\[ \square \]
6. Proof of Theorem 1.1, 1.2 and 1.3

Proof of Theorem 1.1. Suppose \((m, n) \in U_{k_i}\) for some \(i\). Then \(\alpha_i - f_{2k+2} < n + m \leq \alpha_i\) and \(m = \alpha_i - f_{2k+2} + 1 - n + r\) where \(0 \leq r < f_{2k+2}\).

Case 1: \(n = f_{2k+3} - 1\). Applying Lemma 4.6(i), Proposition 4.5 and then Lemma 4.6(i) again, we have

\[
H_{\alpha_i-f_{2k+3}+1, f_{2k+3}-1} = H_{\alpha_i-f_{2k+3}+1, f_{2k+3}-1} = (-1)^{k+1} \frac{f_{2k+1}}{2}. \tag{15}
\]

where the last equality follows from Theorem 5.3. Since \(r = \alpha_i - m - n = \frac{f_{2k+2}}{2} - \Phi_{k+1}(m+n)\), by Lemma 4.6(iii),

\[
H_{m,n} = H_{\alpha_i-f_{2k+2}+r, f_{2k+3}-1} = (-1)^{k+1} \frac{f_{2k+2}}{2} \Phi_{k+1}(m+n) f_{2k+1} \frac{f_{2k+1}}{2}
\]

where the last equality follows from Eq. (15).

Case 2: \(n = f_{2k}\). By Proposition 4.5, we have

\[
H_{m,n} = (-1)^{f_k} (-1)^{\frac{f_k-1}{2}} H_{\alpha_i-f_{2k+3}+1, f_{2k+3}-1} \tag{by Lemma 4.6(iii)}
\]

where \(\ell = f_{2k+3} - 1 - f_{2k}\) is even by Lemma 2.9.

Case 3: \(f_{2k} < n < f_{2k+3} - 1\). If \(m + n = \alpha_i\) (or \(\alpha_i - f_{2k+2} + 1\), then applying Lemma 4.6(i) (or Lemma 4.6(ii)) and then Eq. (15), we have

\[
H_{m,n} = (-1)^{f_k} (-1)^{\frac{f_k-1}{2}} H_{\alpha_i-f_{2k+3}+1, f_{2k+3}-1} \tag{by Theorem 5.3}
\]

where \(\ell = f_{2k+3} - 1 - n\). If \(\alpha_i - f_{2k+2} + 1 < m + n < \alpha_i\), then Lemma 4.1 yields \(H_{m,n} = 0\). \(\Box\)

Proof of Theorem 1.2. Suppose \((m, n) \in V_{k_i}\) for some \(i\). Then \(\beta_i < n + m \leq \beta_i + f_{2k+1}\).

Case 1: \(n = f_{2k+2} - 1\). By Lemma 4.8(iii) & (i), we have

\[
H_{m,n} = H_{\beta_i'-f_{2k+2}+2, f_{2k+2}-1} = (-1)^{f_{2k+2}-1} f_{2k+1} f_{2k+2} \tag{16}
\]

According to Proposition 4.5(i) and Corollary 5.4,

\[
H_{\beta_i'+1, f_{2k}} = H_{\beta_i'+1, f_{2k}} = (-1)^{f_{2k+2}-1} f_{2k+1} \frac{f_{2k+1}}{2}. \tag{17}
\]

The result follows from Eq. (16) and Eq. (17).

Case 2: \(n = f_{2k}\). By Proposition 4.5, \(H_{m,n} = H_{\alpha_i-f_{2k}, f_{2k}}\). Then the result follows from Theorem 1.1(ii).

Case 3: \(f_{2k} < n < f_{2k+2} - 1\). If \(m + n = \beta_i + 1\) (or \(\beta_i + f_{2k+1}\), then by Lemma 4.8(i) (or Lemma 4.8(ii)), we have

\[
H_{m,n} = (-1)^{f_{2k+2}-n} f_{2k+1} f_{2k+2} \tag{by Theorem 5.3}
\]
If \( \beta_i + 1 < m + n < \beta_i + f_{2k+1} \), then Lemma 4.1 shows \( H_{m,n} = 0 \).

**Proof of Theorem 1.3.** Suppose \((m,n) \in T_{k,i}\) for some \(i\). Then \( \gamma_i - f_{2k} < n + m \leq \gamma_i \).

**Case 1:** \( n = f_{2k+2} - 1 \). By Lemma 4.10(iii) \& (i),

\[
H_{m,n} = H_{\gamma_i - f_{2k+3}, f_{2k+2} - 1} = (-1)^{(f_{2k+2} - 1)/2}(-1)^{(f_{2k+2} - 2)/2}H_{\gamma_i - f_{2k+2}, f_{2k+1}} = (-1)^{(f_{2k+2} - 2)/2}H_{\gamma_i - f_{2k+2} + 1, f_{2k+1}}.
\]

where the last equality follows from Lemma 2.9(i). Using Proposition 4.5(ii) and Corollary 5.5,

\[
H_{\gamma_i - f_{2k+2} + 1, f_{2k+1}} = H_{\gamma_i - f_{2k+2} + 1, f_{2k+1}} = f_{2k}. \tag{19}
\]

The result follows from Eq. (18) and Eq. (19).

**Case 2:** \( n = f_{2k+1} \). Write \( m = \gamma_i - f_{2k+2} + r \) with \( 1 \leq r \leq f_{2k} \). By Proposition 4.5(ii) and Corollary 5.5,

\[
H_{m,n} = (-1)^{r+1}H_{\gamma_i - f_{2k+2} + 1, f_{2k+1}} = (-1)^{r+1}f_{2k}.
\]

Note that \( \gamma_i = m + n + f_{2k} - r \) and \( 1 \leq r \leq f_{2k} \). Consequently,

\[
f_{2k+3}/2 = \Phi_k(\gamma_i) = \Phi_k(m + n + f_{2k} - r) = \Phi_k(m + n) + f_{2k} - r
\]

which gives \( r = \Phi_k(m + n) - f_{2k+1} \). Then the result follows.

**Case 3:** \( f_{2k} < n < f_{2k+2} - 1 \). If \( m + n = \gamma_i - f_{2k} + 1 \) \((\text{or } \gamma_i)\), then by Lemma 4.10(i) \((\text{or } \text{Lemma 4.10(ii)})\),

\[
H_{m,n} = (-1)^{(f_{2k+2} - 1 - n)/2}(-1)^{(f_{2k+2} - 2 - n)/2}H_{f_{2k+2} - n, f_{2k}}.
\]

If \( \gamma_i - f_{2k} + 1 < m + n < \gamma_i \), then Lemma 4.1 yields \( H_{m,n} = 0 \).

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