Manifolds of mappings on cartesian products

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Abstract
Given smooth manifolds $M_1,\ldots,M_n$ (which may have a boundary or corners), a smooth manifold $N$ modeled on locally convex spaces and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$, we consider the set $C^\alpha(M_1 \times \cdots \times M_n, N)$ of all mappings $f: M_1 \times \cdots \times M_n \to N$ which are $C^\alpha$ in the sense of Alzaareer. Such mappings admit, simultaneously, continuous iterated directional derivatives of orders $\leq \alpha_j$ in the $j$th variable for $j \in \{1,\ldots,n\}$, in local charts. We show that $C^\alpha(M_1 \times \cdots \times M_n, N)$ admits a canonical smooth manifold structure whenever each $M_j$ is compact and $N$ admits a local addition. The case of non-compact domains is also considered.

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1 Introduction and statement of the results

As known from classical work by Eells [9], the set $C^\ell(M, N)$ of all $C^\ell$-maps $f: M \to N$ can be given a smooth Banach manifold structure for each $\ell \in \mathbb{N}_0$, compact smooth manifold $M$ and $\sigma$-compact finite-dimensional smooth manifold $N$. More generally, $C^\ell(M, N)$ is a smooth manifold for each $\ell \in \mathbb{N}_0 \cup \{\infty\},$

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locally compact, paracompact smooth manifold \( M \) with rough boundary in the sense of \( [15] \) (this includes finite-dimensional manifolds with boundary, and manifolds with corners as in \( [7, 8, 21] \)) and each smooth manifold \( N \) modeled on locally convex spaces such that \( N \) admits a local addition (a concept recalled in Definition \( 5.6 \)); see \( [16, 21, 22, 25, 4, 14] \) for discussions in different levels of generality, and \( [20] \) for manifolds of smooth maps in the convenient setting of analysis. For compact \( M \), the modeling space of \( C^\ell(M, N) \) around \( f \in C^\ell(M, N) \) is the locally convex space \( \Gamma_{C^\ell}(f^*(TN)) \) of all \( C^\ell \)-sections in the pullback bundle \( f^*(TN) \rightarrow M \), which can be identified with

\[
\Gamma_f := \{ \tau \in C^\ell(M, TN) : \pi_{TN} \circ \tau = f \};
\]

if \( M \) is not compact, the locally convex space of compactly supported \( C^\ell \)-sections of \( f^*(TN) \) is used. Let \( L \) be a smooth manifold modeled on locally convex spaces (possibly with rough boundary), and \( k \in \mathbb{N}_0 \cup \{\infty\} \). For compact \( M \), it is known from \( [4, \text{Proposition } 1.23 \text{ and Definition } 1.17] \) that a map

\[
g : L \rightarrow C^\ell(M, N)
\]

is \( C^k \) if and only if the corresponding map of two variables,

\[
g^\wedge : L \times M \rightarrow N, \ (x, y) \mapsto g(x)(y)
\]

is \( C^{k, \ell} \) in the sense of \( [4] \), i.e., a continuous map which in local charts admits up to \( \ell \) directional derivatives in the second variable, followed by up to \( k \) directional derivatives in the first variable, with continuous dependence on point and directions (see \( [2.11 \text{ and } 2.12] \) for details). We thus obtain a bijection

\[
\Phi : C^k(L, C^\ell(M, N)) \rightarrow C^{k, \ell}(L \times M, N), \ g \mapsto g^\wedge.
\]

As our first result, for compact \( L \) we construct a smooth manifold structure on \( C^{k, \ell}(L \times M, N) \) which turns \( \Phi \) into a \( C^\infty \)-diffeomorphism. More generally, analogous to the \( n = 2 \) case of \( C^{k, \ell} \)-maps, we consider \( N \)-valued \( C^\alpha \)-maps on an \( n \)-fold product \( M_1 \times \cdots \times M_n \) of smooth manifolds for any \( n \in \mathbb{N} \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}_0 \cup \{\infty\})^n \). With terminology explained presently, we get:

**Theorem 1.1** Given \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}_0 \cup \{\infty\})^n \), let \( M_j \) for \( j \in \{1, \ldots, n\} \) be a compact smooth manifold with rough boundary. Let \( N \) be a smooth manifold modeled on locally convex spaces such that \( N \) can be covered by local additions. Then \( C^\alpha(M_1 \times \cdots \times M_n, N) \) admits a smooth manifold structure which is canonical. The following hold for this canonical manifold structure:

(a) \( C^\alpha(M_1 \times \cdots \times M_n, N) \) can be covered by local additions. If \( N \) admits a local addition, then also \( C^\alpha(M_1 \times \cdots \times M_n, N) \) admits a local addition.

(b) Given \( m \in \mathbb{N} \) and \( \beta = (\beta_1, \ldots, \beta_m) \in (\mathbb{N}_0 \cup \{\infty\})^m \), let \( L_j \) be a compact smooth manifold with rough boundary for \( j \in \{1, \ldots, m\} \). Then canonical smooth manifold structures turn the bijection

\[
C^\beta(L_1 \times \cdots \times L_m, C^\alpha(M_1 \times \cdots \times M_n, N)) \rightarrow C^{\beta, \alpha}(L_1 \times \cdots \times L_m \times M_1 \times \cdots \times M_n, N)
\]

taking \( g \) to \( g^\wedge \) into a \( C^\infty \)-diffeomorphism.
The following terminology was used: We say that a smooth manifold $N$ can be covered by local additions if $N$ is the union of an upward directed family $(N_j)_{j \in J}$ of open submanifolds $N_j$ which admit a local addition. For instance, any (not necessarily paracompact) finite-dimensional smooth manifold has this property, e.g. the long line. We also used canonical manifold structures.

Note that if a map $f: L_1 \times \cdots \times L_m \times M_1 \times \cdots \times M_n \to N$ on a product of smooth manifolds with rough boundary is $C^{\beta, \alpha}$ with $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$ and $\beta \in (\mathbb{N}_0 \cup \{\infty\})^m$, then the map

$$f^\vee(x) := f(x, \cdot): M_1 \times \cdots \times M_n \to N$$

is $C^\alpha$ for each $x \in L_1 \times \cdots \times L_m$ (see [1, Lemma 3.3]).

**Definition 1.2** Let $N$ be a smooth manifold modeled on a locally convex space, $M_1, \ldots, M_n$ be finite-dimensional smooth manifolds with rough boundary and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$. A smooth manifold structure on $C^\alpha(M_1 \times \cdots \times M_n, N)$ is called pre-canonical if the following condition is satisfied for each $m \in \mathbb{N}$ and each $\beta \in (\mathbb{N}_0 \cup \{\infty\})^m$: If $L_j$ for $j \in \{1, \ldots, m\}$ is a smooth manifold with rough boundary modeled on locally convex spaces, then a map $g: L_1 \times \cdots \times L_m \to C^\alpha(M_1 \times \cdots \times M_n, N)$ is $C^\beta$ if and only if the map

$$g^\wedge: L_1 \times \cdots \times L_m \times M_1 \times \cdots \times M_n \to N$$

given by $g^\wedge(x_1, \ldots, x_m, y_1, \ldots, y_n) := g(x_1, \ldots, x_m)(y_1, \ldots, y_n)$ is $C^{\beta, \alpha}$. Thus

$$C^\alpha(L_1 \times \cdots \times L_m, C^\beta(M_1 \times \cdots \times M_n, N)) \to C^{\beta, \alpha}(L_1 \times \cdots \times L_m \times M_1 \times \cdots \times M_n, N),$$

$$g \mapsto g^\wedge$$

(1)

is a bijection. The manifold structure is called canonical if, moreover, its underlying topology is the compact-open $C^\alpha$-topology (as in Definition 3.4).

Canonical manifold structures are essentially unique whenever they exist, and so are pre-canonical ones (see Lemma 4.3(b) for details).

We address two further topics for not necessarily compact domains:

(i) We formulate criteria ensuring that $C^\alpha(M_1 \times \cdots \times M_n, G)$ admits a canonical smooth manifold structure (making the latter a Lie group), for a Lie group $G$ modeled on a locally convex space;

(ii) Manifold structures on $C^\alpha(M_1 \times \cdots \times M_n, N)$ which are modeled on certain spaces of compactly supported $TN$-valued functions, in the spirit of [21].

To discuss (i), we use a generalization of the regularity concept introduced by John Milnor [22] (the case $r = \infty$). If $G$ is a Lie group modeled on a locally...
convex space, with neutral element $e$, we write $\lambda_g : G \to G, x \mapsto gx$ for left translation with $g \in G$ and consider the smooth left action $G \times TG \to TG, (g, v) \mapsto g \cdot v := T\lambda_g(v)$ of $G$ on its tangent bundle. We write $\mathfrak{g} := TeG$ for the Lie algebra of $G$. Let $r \in \mathbb{N}_0 \cup \{\infty\}$. The Lie group $G$ is called $C^r$-semiregular if, for each $C^r$-curve $\gamma : [0, 1] \to \mathfrak{g}$, the initial value problem

$$\dot{\eta}(t) = \eta(t) \cdot \gamma(t), \quad \eta(0) = e$$

has a (necessarily unique) solution $\eta : [0, 1] \to G$. Write $\text{Evol}(\gamma) := \eta$. If, moreover, the map $\text{Evol} : C^r([0, 1], \mathfrak{g}) \to C^{r+1}([0, 1], G)$ is smooth, then $G$ is called $C^r$-regular (cf. [12]). We show:

**Theorem 1.3** Let $G$ be a $C^r$-regular Lie group modeled on a locally convex space with $r \in \mathbb{N}_0 \cup \{\infty\}$. For some $n \in \mathbb{N}$, let $M_1, \ldots, M_n$ be locally compact smooth manifolds with rough boundary and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$. For each $j \in \{1, \ldots, n\}$ such that $M_j$ is not compact, assume that $\alpha_j \geq r + 1$ and $M_j$ is $1$-dimensional with finitely many connected components. Then we have:

(a) $C^\alpha(M_1 \times \cdots \times M_n, G)$ admits a canonical smooth manifold structure;

(b) The canonical manifold structure from (a) makes $C^\alpha(M_1 \times \cdots \times M_n, G)$ a $C^r$-regular Lie group.

The Lie algebra of $C^\alpha(M_1 \times \cdots \times M_n, G)$ can be identified with the topological Lie algebra $C^\alpha(M_1 \times \cdots \times M_n, L(G))$ in a standard way (Proposition 6.6). Of course, we are most interested in the case that the non-compact $1$-dimensional factors are $\sigma$-compact and hence intervals, or finite disjoint unions of such. But we did not need to assume $\sigma$-compactness in the theorem, and thus $M_j$ with $\alpha_j \geq r + 1$ might well be a long line, or a long ray.

Disregarding the issue of being canonical, the Lie group structure on $C^\infty(M_1 \times \cdots \times M_n, G) = C^\alpha(M_1 \times \cdots \times M_n, G)$ with $\alpha_1 := \cdots := \alpha_n = \infty$ was first obtained in [24], for smooth manifolds $M_j$ without boundary which are compact or diffeomorphic to $\mathbb{R}$. The Lie group structure for $n = 1$ was first obtained in [2] for domains diffeomorphic to intervals, together with a sketch for the case $n = 2$ (assuming additional conditions, e.g. $\alpha_1 \geq r + 3$ and $\alpha_2 \geq r + 1$ if $M_1 = M_2 = \mathbb{R}$). Our approach differs: While the studies in [24] and [2] assume regularity of $G$ from the start to enforce exponential laws, and build it into a notion of Lie group structures on mapping groups that are “compatible with evaluations,” we take canonical and pre-canonical manifold structures as the starting point (independent of regularity) and combine them with regularity or compatibility with evaluations (adapted to $C^\alpha$-maps in Definition 6.2) only when needed.

As to topic (b), our constructions show:
**Theorem 1.4** Given \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}_0 \cup \{\infty\})^n \), let \( M_j \) for \( j \in \{1, \ldots, n\} \) be a paracompact, locally compact smooth manifold with rough boundary; abbreviate \( M := M_1 \times \cdots \times M_n \). Let \( N \) be a smooth manifold modeled on locally convex spaces such that \( N \) admits a local addition. Let \( \pi_{TN} : TN \to N \) be the canonical map. For \( f \in C^\alpha(M, N) \) and a compact subset \( K \subseteq M \), the set

\[
\Gamma_{f,K} := \{ \tau \in C^\alpha(M, TN) : \pi_{TN} \circ \tau = f \ \& \ \tau(x) = 0 \in T_{f(x)}N \text{ for all } x \in M \setminus K \}
\]

is a vector subspace of \( \prod_{x \in M} T_{f(x)}N \), and a locally convex space in the topology induced by \( C^\alpha(M, TN) \). Give \( \Gamma_f = \bigcup_K \Gamma_{f,K} \) the locally convex direct limit topology. Then \( C^\alpha(M, N) \) admits a unique smooth manifold structure modeled on the set \( \mathcal{E} := \{ \Gamma_f : f \in C^\alpha(M, N) \} \) of locally convex spaces such that, for each \( f \in C^\alpha(M, N) \) and local addition \( \Sigma : TN \supseteq U \to N \) of \( N \), the map

\[
\Gamma_f \cap C^\alpha(M, U) \to C^\alpha(M, N), \quad \tau \mapsto \Sigma \circ \tau
\]

is a \( C^\infty \)-diffeomorphism onto an open subset of \( C^\alpha(M, N) \).

In the case that \( n = 1, k = \infty \) and \( M := M_1 \) is a smooth manifold with corners, we recover the smooth manifold structure on \( C^\infty(M, N) \) discussed by Michor [21].

Using manifold structures on infinite cartesian products of manifolds making them “finite box products” (a concept recalled in Section 7), Theorem 1.4 turns into a corollary to Theorem 1.1.

In the case \( n = 1 \), for compact \( M \) and \( \ell \in \mathbb{N}_0 \cup \{\infty\} \), canonical manifold structures on \( C^\ell(M, N) \) as in Theorem 1.1 have already been considered in [4], in a weaker sense (fixing \( m = 1 \) in Definition 1.2). Parts of our discussion adapt arguments from [4] to the more difficult case of \( C^\infty \)-maps.

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## 2 Preliminaries and notation

We write \( \mathbb{N} := \{1, 2, \ldots\} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). If \( \alpha, \beta \in (\mathbb{N}_0 \cup \{\infty\})^n \) with \( n \in \mathbb{N} \), we write \( \alpha \leq \beta \) if \( \alpha_j \leq \beta_j \) for all \( j \in \{1, \ldots, n\} \). We let \( |\alpha| := \alpha_1 + \cdots + \alpha_n \in \mathbb{N}_0 \cup \{\infty\} \). As usual, \( \infty + k := \infty \) for all \( k \in \mathbb{N}_0 \cup \{\infty\} \). For \( j \in \{1, \ldots, n\} \), let \( e_j := (0, \ldots, 0, 1, 0, \ldots, 0) \in (\mathbb{N}_0)^n \) with 1 in the \( j \)th slot. We abbreviate “Hausdorff locally convex topological \( \mathbb{R} \)-vector space” as “locally convex space.”

We work in the setting of differential calculus going back to Andréé Bastiani [5] (see [10] [16] [17] [21] [22] [23] for discussions in varying generality), also known as Keller’s \( C^\infty_c \)-theory [19]. For \( C^\infty \)-maps, see [11] (cf. [3] and [15] for the case of two variables, \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^2 \)). We now introduce concepts for later use and collect basic facts. For proofs, see the appendix.
2.1 Consider locally convex spaces $E$, $F$ and a map $f: U \to F$ on an open subset $U \subseteq E$. Write

$$(D_y f)(x) := \frac{d}{dt} \bigg|_{t=0} f(x + ty)$$

for the directional derivative of $f$ at $x \in U$ in the direction $y \in E$, if it exists.

Let $k \in \mathbb{N}_0 \cup \{\infty\}$. If $f$ is continuous, the iterated directional derivatives

$$d^j f(x, y_1, \ldots, y_j) := (D_{y_j} \ldots D_{y_1} f)(x)$$

exist for all $j \in \mathbb{N}_0$ such that $j \leq k$, $x \in U$ and $y_1, \ldots, y_j \in E$, and the maps

$d^j f: U \times E^j \to F$ are continuous, then $f$ is called $C^k$. If $U$ may not be open, but has dense interior $U^o$ and is locally convex in the sense that each $x \in U$ has a convex neighbourhood in $U$, following [15] a map $f: U \to F$ is called $C^k$ if it is continuous, $f|_{U^o}$ is $C^k$ and $\partial f|_{U^o}$ has a continuous extension $\partial f : U \times E^j \to F$ for all $j \in \mathbb{N}_0$ with $j \leq k$. The $C^\infty$-maps are also called smooth.

Remark 2.2 If $E = \mathbb{R}^n$ and $U$ is relatively open in $[0, \infty[^n$, then $f$ as above is $C^k$ if and only if $f$ has a $C^k$-extension to an open set in $\mathbb{R}^n$ (see [13], cf. [17]).

2.3 Let $k \in \mathbb{N} \cup \{\infty\}$. A manifold with rough boundary modeled on a non-empty set $\mathcal{E}$ of locally convex spaces is a Hausdorff topological space $M$, together with a set $\mathcal{A}$ of homeomorphisms ("charts") $\phi : U_\phi \to V_\phi$ from an open subset $U_\phi \subseteq M$ onto a locally convex subset $V_\phi \subseteq E_\phi$ with dense interior for some $E_\phi \in \mathcal{E}$, such that $\phi \circ \psi^{-1}$ is $C^k$ for all $\phi, \psi \in \mathcal{A}$, the union $\bigcup_{\phi \in \mathcal{A}} U_\phi$ equals $M$, and $\mathcal{A}$ is maximal. If $k = 0$, assume in addition that $\phi(x) \in \partial V_\phi$ if and only if $\psi(x) \in \partial V_\psi$ for all $\phi, \psi \in \mathcal{A}$ with $x \in U_\phi \cap U_\psi$ (which is automatic if $k \geq 1$). Let $\partial M$ be the set of all $x \in M$ such that $\phi(x) \in \partial V_\phi$ for some (and hence any) chart $\phi$ around $x$. If $\mathcal{E}$ is a singleton, $M$ is called pure. If $M$ is a $C^k$-manifold with rough boundary and $\partial M = \emptyset$, then $M$ is called a $C^k$-manifold or a $C^k$-manifold without boundary, for emphasis. (See [15] for all of this in the pure case; cf. [1] for modifications in the general case).

2.4 All manifolds and Lie groups considered in the article are modeled on locally convex spaces which may be infinite-dimensional, unless the contrary is stated. Finite-dimensional manifolds need not be paracompact or $\sigma$-compact, unless stated explicitly. As we are interested in manifolds of mappings, consideration of pure manifolds would not be sufficient.

2.5 If $U$ is an open subset of a locally convex space $E$ (or a locally convex subset with dense interior), we identify its tangent bundle $TU$ with $U \times E$, as usual, with bundle projection $(x, y) \mapsto x$. If $M$ is a $C^k$-manifold with rough boundary and $f : M \to U$ a $C^k$-map with $k \geq 1$, we write $df$ for the second component of $Tf : TM \to TU = U \times E$. Thus $Tf = (f \circ \pi_{TM}, df)$, using the bundle projection $\pi_{TM} : TM \to M$.

2.6 If $G$ is a Lie group with neutral element $e$, we write $L(G) := T_e G$ (or $\mathfrak{g}$) for its tangent space at $e$, endowed with its natural topological Lie algebra structure.
If \( \psi: G \to H \) is a smooth homomorphism between Lie groups, we let \( L(\psi) := T_\psi: L(G) \to L(H) \) be the associated continuous Lie algebra homomorphism.

2.7 If \( G \) is a Lie group with Lie algebra \( \mathfrak{g} \) and \( I \) a non-degenerate interval with \( 0 \in I \), we define \( \delta^\ell(\eta) \) for \( \eta \in C^1(I, G) \) via \( \delta^\ell(\eta)(t) := \eta(t)^{-1} \eta(t) \), with \( \dot{\eta}(t) := T\eta(t, 1) \).

Lemma 2.8 Let \( k, r \in \mathbb{N}_0 \cup \{\infty\} \) with \( k \geq r \). If \( G \) is \( C^r \)-semiregular and \( \gamma \in C^k(I, \mathfrak{g}) \), then there exists a unique \( \eta \in C^1(I, \mathfrak{g}) \) such that \( \eta(0) = e \) and \( \delta^\ell(\eta) = \gamma \). Moreover, \( \eta \) is \( C^{k+1} \).

2.9 Let \( M \) be a smooth manifold (without boundary). A subset \( N \subseteq M \) is called a submanifold if, for each \( x \in N \), there exist a chart \( \phi: U_\phi \to V_\phi \subseteq M \) around \( x \) and a closed vector subspace \( F \subseteq E_\phi \) such that \( \phi(U_\phi \cap N) = V_\phi \cap F \).

2.10 Let \( M \) be a smooth manifold with rough boundary. A subset \( N \subseteq M \) is called a full submanifold if, for each \( x \in N \), there exists a chart \( \phi: U_\phi \to V_\phi \subseteq E_\phi \) of \( M \) around \( x \) such that \( \phi(U_\phi \cap N) \) is a locally convex subset of \( E_\phi \) with dense interior.

2.11 Let \( F \) and \( E_1, \ldots, E_n \) be locally convex spaces, \( U_j \subseteq E_j \) be an open subset for \( j \in \{1, \ldots, n\} \) and \( f: U \to F \) be a map on \( U := U_1 \times \cdots \times U_n \). Identifying \( E := E_1 \times \cdots \times E_n \) with \( E_1 \oplus \cdots \oplus E_n \), we can identify each \( E_j \) with a vector subspace of \( E \), and simply write \( D_y f(x) \) for a directional derivative with \( x \in U \), \( y \in E_j \) (rather than \( D_{(y_0, \ldots, y_0, \ldots)} f(x) \) with \( j \) zeros on the right-hand side). For \( y = (y_1, \ldots, y_k) \in E_j^k \), abbreviate
\[
D_y := D_{y_k} \cdots D_{y_1}.
\]

Let \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \). Following \[1\], we say that \( f \) is \( C^\alpha \) if \( f \) is continuous, the iterated directional derivatives
\[
d^{\beta} f(x, y_1, \ldots, y_n) := (D_{y_{n-1}} \cdots D_{y_1} f)(x)
\]
eexist for all \( \beta \in \mathbb{N}_0^n \) with \( \beta \leq \alpha \), \( x \in U \) and \( y_j = (y_{j,1}, \ldots, y_{j,\beta_j}) \in (E_j)^{\beta_j} \) for \( j \in \{1, \ldots, n\} \), and
\[
d^\beta f: U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \to F
\]
is continuous. If \( U_j \) may not be open but is a locally convex subset of \( E_j \) with dense interior, we say that \( f: U \to F \) is \( C^\alpha \) if \( f \) is continuous, \( f|_{U_\delta} \) is \( C^\alpha \) and \( d^\beta (f|_{U_\delta}) \) has a continuous extension \( d^\beta f: U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \to F \) for all \( \beta \in (\mathbb{N}_0)^n \) such that \( \beta \leq \alpha \).

2.12 Let \( M_1, \ldots, M_n \) be \( C^\infty \)-manifolds with rough boundary, \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \) and \( N \) a \( C^k \)-manifold with \( k \geq |\alpha| \). We say that a map \( f: M_1 \times \cdots \times M_n \to N \) is \( C^\alpha \) if, for each \( x = (x_1, \ldots, x_n) \in M_1 \times \cdots \times M_n \), there are charts \( \phi_j: U_j \to V_j \) for \( M_j \) around \( x_j \) for \( j \in \{1, \ldots, n\} \) and a chart \( \psi: U_\psi \to V_\psi \) for \( n \) around \( f(x) \) such that \( f(U_1 \times \cdots \times U_n) \subseteq U_\psi \) and
\[
\psi \circ f \circ (\phi_1 \times \cdots \times \phi_n)^{-1}: V_1 \times \cdots \times V_n \to V_\psi
\]
is $C^\alpha$. The latter then holds for any such charts, by the Chain Rule for $C^\alpha$-maps (as in [1, Lemma 3.16]).

2.13 Let $N$ and $M_1, \ldots, M_n$ be $C^\infty$-manifolds with rough boundary, $\sigma$ be a permutation of $\{1, \ldots, n\}$, and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$. If $f : M_{\sigma(1)} \times \cdots \times M_{\sigma(n)} \to N$ is $C^{\alpha \circ \sigma}$, then the map

$$M_1 \times \cdots \times M_n \to N, \quad (x_1, \ldots, x_n) \mapsto f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

is $C^\alpha$. This follows from Schwarz’ Theorem (in the form of [1, Proposition 3.5]).

We shall use simple facts:

**Lemma 2.14** Let $E_j$ for $j \in \{1, \ldots, n\}$ and $F$ be locally convex spaces, and $U_j \subseteq E_j$ be a locally convex subset with dense interior. Let $E := E_1 \times \cdots \times E_n$, $U := U_1 \times \cdots \times U_n$, $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$ and $f : U \to F$ be a map.

(a) If $Y \subseteq F$ is a closed vector subspace and $f(U) \subseteq Y$, then $f$ is $C^\alpha$ if and only if its co-restriction $f|Y : U \to Y$ is $C^\alpha$.

(b) If $F$ is the projective limit of a projective system $\left( (F_a)_{a \in A}, (\lambda_{a,b})_{a \leq b} \right)$ of locally convex spaces $F_a$ and continuous linear mappings $\lambda_{a,b} : F_b \to F_a$, with limit maps $\lambda_a : F \to F_a$, then $f$ is $C^\alpha$ if and only if $\lambda_a \circ f : U \to F_a$ is $C^\alpha$ for all $a \in A$.

**Lemma 2.15** Let $M$, $N$, and $L_1, \ldots, L_n$ be smooth manifolds with rough boundary, $F$ a locally convex space, $\psi : M \to F \times N$ be a $C^\infty$-diffeomorphism, and $f : L_1 \times \cdots \times L_n \to M$ be a map. Assume that $F$ is the projective limit of a projective system $\left( (F_a)_{a \in A}, (\lambda_{a,b})_{a \leq b} \right)$ of locally convex spaces $F_a$ and continuous linear mappings $\lambda_{a,b} : F_b \to F_a$, with limit maps $\lambda_a : F \to F_a$. For $a \in A$, let $M_a$ be a smooth manifold and $\rho_a : M \to M_a$ be a $C^\infty$-map. Assume that there exist $C^\infty$-maps $\psi_a : M_a \to F_a \times N$ making the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\psi} & F \times N \\
\downarrow \rho_a & & \downarrow \lambda_a \times \text{id}_N \\
M_a & \xrightarrow{\psi_a} & F_a \times N
\end{array}
$$

commute. Then $f$ is $C^\alpha$ if and only if $\rho_a \circ f$ is $C^\alpha$ for all $a \in A$.

2.16 If $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}_0 \cup \{\infty\})^n$ and $\beta = (\beta_1, \ldots, \beta_m) \in (\mathbb{N}_0 \cup \{\infty\})^m$, we shall write $(\alpha, \beta)$ as a shorthand for $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ and abbreviate $C^{(\alpha, \beta)}$ as $C^{\alpha, \beta}$. Likewise for higher numbers of multiindices.

Let $r \in \mathbb{N}_0 \cup \{\infty\}$, $E_1, \ldots, E_n$ and $F$ be locally convex spaces and $U_j$ be a locally convex subset of $E_j$ with dense interior, for $j \in \{1, \ldots, n\}$. We mention that a map $f : U_1 \times \cdots \times U_n \to F$ is $C^\alpha$ if and only if it is $C^\beta$ for all $\beta \in (\mathbb{N}_0 \cup \{\infty\})^n$ such that $|\beta| \leq r$. More generally, the following is known (as first formulated and proved in the unpublished work [18]):
Lemma 2.17 For \( i \in \{1, \ldots, n\} \), let \( E_i \) be a locally convex space of the form \( E_i = E_{i,1} \times \cdots \times E_{i,m_i} \) for some \( m_i \in \mathbb{N} \) and locally convex spaces \( E_{i,1}, \ldots, E_{i,m_i} \). Let \( U_{i,j} \) be a locally convex subset of \( E_{i,j} \) with dense interior for all \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, m_i\} \); define \( U_i := U_{i,1} \times \cdots \times U_{i,m_i} \). Let \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \). Then a map \( f: U_1 \times \cdots \times U_n \to F \) is \( C^\alpha \) if and only if \( f \) is \( C^{\beta_1, \ldots, \beta_n} \) for \( \beta = (\beta_1, \ldots, \beta_n) \in \prod_{i=1}^n (\mathbb{N}_0 \cup \{\infty\})^{m_i} \) such that \( |\beta_i| \leq \alpha_i \) for all \( i \in \{1, \ldots, n\} \).

3 The compact-open \( C^\alpha \)-topology

As a further preliminary, we introduce a topology on \( C^\alpha(M \times \cdots \times M, N) \) which parallels the familiar compact-open \( C^k \)-topology on \( C^k(M, N) \). Basic properties are recorded, with proofs in Appendix A.

As usual, \( T^0 M := M \), \( T^1 M := TM \) and \( T^k M := T(T^{k-1} M) \) for a smooth manifold \( M \) with rough boundary and integers \( k \geq 2 \) (see [15]).

3.1 In \( 6.2 - 6.10 \) \( M_1, \ldots, M_n \) will be smooth manifolds with rough boundary, and \( M := M_1 \times \cdots \times M_n \). In \( 3.3 - 3.9 \) we let \( N \) be a smooth manifold with rough boundary and \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \).

3.2 We define the \( \beta \)-tangent bundle of \( M \) as \( T^\beta M := T^\beta_1 M_1 \times \cdots \times T^\beta_n M_n \) for \( \beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{N}_0)^n \).

3.3 Let \( f: M \to N \) be a \( C^\alpha \)-map. For \( \beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{N}_0)^n \) with \( \beta \leq \alpha \), we define

\[
T^\beta(f): T^\beta(M) \to T^{|\beta|}N
\]

recursively, as follows: We first note that, by Lemma [A.1]

\[
T^{(0, \ldots, 0, \beta_n)} f: M_1 \times \cdots \times M_{n-1} \times T^\beta M_n \to T^\beta M_n,
\]

\((x_1, \ldots, x_{n-1}, v_n) \mapsto T^\beta_n (f(x_1, \ldots, x_{n-1}, .))(v_n)\) is a \( C^{(\alpha_1, \ldots, \alpha_{n-1}, 0)} \)-map if \( C^{(\alpha_1, \ldots, \alpha_{n-1}, 0)} \)-map \( g := T^{(0, \ldots, 0, \beta_n)} f: T^{(0, \ldots, 0, \beta_k, \ldots, \beta_n)} M \to T^{(0, \ldots, 0, \beta_k, \ldots, \beta_n)} M \) has already been constructed for \( k \in \{2, \ldots, n\} \), then the map

\[
T^{(0, \ldots, 0, \beta_{k-1}, \ldots, \beta_n)} f: T^{(0, \ldots, 0, \beta_{k-1}, \ldots, \beta_n)} M \to T^{(0, \ldots, 0, \beta_{k-1}, \ldots, \beta_n)} M
\]

taking \((x_1, \ldots, x_{k-2}, v_{k-1}, \ldots, v_n)\) to \( T^{\beta_{k-1}} (g(x_1, \ldots, x_{k-2}, ., v_{k-1}, \ldots, v_n))(v_{k-1})\) is a \( C^{(\alpha_1, \ldots, \alpha_{k-2}, 0, \ldots, 0)} \)-map (see Lemmas [2.13] and [A.1]).

Definition 3.4 The compact-open \( C^\alpha \)-topology on \( C^\alpha(M, N) \) is the initial topology with respect to the mappings

\[
T^\beta: C^\alpha(M, N) \to C(T^\beta M, T^{|\beta|}N), \quad f \mapsto T^\beta f
\]

for \( \beta \in (\mathbb{N}_0)^n \) with \( \beta \leq \alpha \), using the compact-open topology on \( C(T^\beta M, T^{|\beta|}N) \). Pushforwards and pullbacks are continuous.
Lemma 3.11 Let $F$ be a family of linear mappings $F_i: F \rightarrow F_i$ to locally convex spaces $F_i$, then the compact-open $C^\alpha$-topology makes $C^\alpha(M, F)$ a locally convex space.
(b) If $F$ is a locally convex space and $F = \prod_{i \in I} F_i$ for a family $(F_i)_{i \in I}$ of locally convex spaces, let $\pi_i: F \to F_i$ be the projection onto the $i$th component and $(\pi_i)^*: C^\alpha(M,F) \to C^\alpha(M,F_i)$. Then
\[
\Theta := ((\pi_i)^*)_{i \in I}: C^\alpha(M,F) \to \prod_{i \in I} C^\alpha(M,F_i)
\]
is an isomorphism of topological vector spaces.

(c) Assume that all of $M_1, \ldots, M_n$ are locally compact. Let $N_i$ be a smooth manifold with rough boundary for $i \in \{1, 2\}$ and $\pi_i: N_1 \times N_2 \to N_i$ be the projection onto the $i$th component. Using the compact-open $C^\alpha$-topology on sets of $C^\alpha$-maps, we get a homeomorphism
\[
\Psi := ((\pi_1)^*, (\pi_2)^*): C^\alpha(M, N_1 \times N_2) \to C^\alpha(M, N_1) \times C^\alpha(M, N_2).
\]
Using the multiplication $\mathbb{R} \times TN \to TN$, $(t,v) \mapsto tv$ with scalars, we have:

**Lemma 3.12** Let $M_1, \ldots, M_n$ be locally compact smooth manifolds with rough boundary, $M := \prod_{i \in I} M_i$ and $N := \prod_{i \in I} N_i$. If $F$ is a locally convex space. Abbreviate $N := \bigotimes_{i \in I} N_i$ and $N := \bigotimes_{i \in I} N_i$. Let $M$ be a smooth manifold with rough boundary. Then the map
\[
\mu: C^\alpha(M, \mathbb{R}) \times C^\alpha(M, TN) \to C^\alpha(M, TN)
\]
determined by $\mu(f, g)(x) := f(x)g(x)$ is continuous.

In [1], Exponential Laws were provided for function spaces on products of pure manifolds. The one we need remains valid for manifolds which need not be pure:

**Lemma 3.13** Let $N_1, \ldots, N_m$ and $M_1, \ldots, M_n$ be smooth manifolds with rough boundary (none of which needs to be pure). Let $\alpha \in \mathbb{N}_0 \cup \{\infty\}^m$, $\beta \in \mathbb{N}_0 \cup \{\infty\}^n$ and $E$ be a locally convex space. Abbreviate $N := \prod_{i \in I} N_i$ and $M := \prod_{i \in I} M_i$. If $f \in C^\alpha, \beta(N \times M, E)$, then $f \cdot g \in C^\beta(M, E)$.

We mention that the $C^\alpha$-topology on $C^\alpha(U, F)$ can be described more explicitly.

**Lemma 3.14** Let $E_j$ be a locally convex space for $j \in \{1, \ldots, n\}$ and $U_j \subseteq E_j$ be a locally convex subset with dense interior. Let $F$ be a locally convex space, $\alpha \in \mathbb{N}_0 \cup \{\infty\}^n$, and $U := U_1 \times \cdots \times U_n$. Then the compact-open $C^\alpha(U, F)$ is initial with respect to the maps
\[
d^\beta: C^\alpha(U, F) \to C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F), \quad f \mapsto d^\beta f
\]
for $\beta \in \mathbb{N}_0^n$ with $\beta \leq \alpha$, using the compact-open topology on the ranges.
4 (Pre-)Canonical manifold structures

In this section, we establish basic properties of canonical manifolds of mappings, and pre-canonical ones. We begin with examples.

Example 4.1 Let \( n \in \mathbb{N} \) and \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \).

(a) Let \( M_1, \ldots, M_n \) be locally compact smooth manifolds with rough boundary and \( E \) a locally convex space. Then \( C^\alpha(M_1 \times \cdots \times M_n, E) \) is a canonical manifold due to Lemma 3.13. The same holds for \( C^\alpha(M_1 \times \cdots \times M_n, N) \) if \( N \) is a smooth manifold diffeomorphic to \( E \), endowed with the \( C^\infty \)-manifold structure making \( \varphi_*: C^\alpha(M, N) \to C^\alpha(M, E) \) a diffeomorphism, where \( \varphi: E \to N \) is a \( C^\infty \)-diffeomorphism.

(b) Familiar examples of mapping groups turn out to be canonical, notably loop groups \( C^k(S^1, G) \) for \( G \) a Lie group, and certain Lie groups of the form \( C^k(\mathbb{R}, G) \) discussed in [224]. We extend these constructions in Section 6.

We will now establish general properties of canonical manifolds.

4.2 Conventions We denote by \( \alpha, \beta \) multiindices in \((\mathbb{N}_0 \cup \{\infty\})^n\) for some \( n \in \mathbb{N} \). Likewise we will usually adopt the shorthand \( M := M_1 \times M_2 \times \cdots \times M_n \) where the \( M_i \) are locally compact manifolds (possibly with rough boundary). If \( M \) is the domain of definition of the function space \( C^\alpha(M, N) \) we will assume that the number of entries of the multiindex \( \alpha \) coincides with the number of factors in the product \( M \).

Lemma 4.3 If \( C^\alpha(M, N) \) is endowed with a pre-canonical manifold structure, then the following holds:

(a) The evaluation map \( ev: C^\alpha(M, N) \times M \to N \), \( ev(\gamma, x) := \gamma(x) \) is \( C^{\infty,\alpha} \).

(b) Pre-canonical manifold structures are unique in the following sense: If we write \( C^\alpha(M, N) \) for \( C^\alpha(M, N) \) with another pre-canonical manifold structure, then \( id: C^\alpha(M, N) \to C^\alpha(M, N)' \), \( \gamma \mapsto \gamma \) is a \( C^\infty \)-diffeomorphism.

(c) Let \( S \subseteq N \) be a submanifold such that the set \( C^\alpha(M, S) \) is a submanifold of \( C^\alpha(M, N) \). Then the submanifold structure on \( C^\alpha(M, S) \) is pre-canonical.

Proof. (a) Since \( id: C^\alpha(M, N) \to C^\alpha(M, N) \) is \( C^\infty \) and \( C^\alpha(M, N) \) is endowed with a pre-canonical manifold structure, it follows that \( id^\wedge: C^\alpha(M, N) \times M \to N \), \( (\gamma, x) \mapsto id(\gamma)(x) = \gamma(x) = ev(\gamma, x) \) is \( C^{\infty,\alpha} \).

(b) The map \( f := id: C^\alpha(M, N) \to C^\alpha(M, N)' \) satisfies \( f^\wedge = ev \) where \( ev: C^\alpha(M, N) \times M \to N \) is \( C^{\infty,\alpha} \), by (a). Since \( C^\alpha(M, N)' \) is endowed with a pre-canonical manifold structure, it follows that \( f \) is \( C^\infty \). By the same reasoning, \( f^{-1} = id: C^\alpha(M, N)' \to C^\alpha(M, N) \) is \( C^\infty \).

(c) As \( C^\alpha(M, S) \) is a submanifold of \( C^\alpha(M, N) \), the inclusion \( \iota: C^\alpha(M, S) \to C^\alpha(M, N) \) is \( C^\infty \). Likewise, the inclusion map \( j: S \to N \) is \( C^\infty \). Let \( L = L_1 \times \cdots \times L_k \) be a product of smooth manifolds (possibly with rough boundary)
modeled on locally convex spaces and \( f : L \to C^\alpha(M, S) \) be a map. If \( f \) is \( C^\beta \), then \( \iota \circ f \) is \( C^\beta \), entailing that \((\iota \circ f)^\wedge : L \times M \to N, (x, y) \mapsto f(x)(y) \) is \( C^{\beta, \alpha} \). As the image of this map is contained in \( S \), which is a submanifold of \( N \), we deduce that \( f^\wedge = (\iota \circ f)^\wedge|_S \) is \( C^{\beta, \alpha} \). For the converse, assume that \( f^\wedge : L \times M \to S \) is \( C^{\beta, \alpha} \). Then also \((\iota \circ f)^\wedge = j \circ (f^\wedge) : L \times M \to N \) is \( C^{\beta, \alpha} \). Hence \( \iota \circ f : L \to C^\alpha(M, N) \) is \( C^\beta \) (the manifold structure on the range being pre-canonical). As \( \iota \circ f \) is a \( C^\beta \)-map with image in \( C^\alpha(M, S) \) which is a submanifold of \( C^{\alpha, \alpha}(M, N) \), we deduce that \( f \) is \( C^\beta \). \( \square \)

**Remark 4.4** Note that due to Lemma 4.3(a), the evaluation on a canonical manifold is a \( C^{\infty, \alpha} \)-map whence it is at least continuous. For a \( C^k \)-manifold \( M \) which is \( C^k \)-regular and a locally convex space \( E \neq \{0\} \), it is well known that for the compact-open \( C^k \)-topology the evaluation \( \operatorname{ev} : C^k(M, E) \times M \to E \) is continuous if and only if \( M \) is locally compact. A similar statement holds for the compact-open \( C^\alpha \)-topology. Using a chart for \( N \) and cut-off functions, we deduce that the evaluation of \( C^\alpha(M, N) \) is discontinuous if \( M \) fails to be locally compact, provided \( N \) is not discrete and \( M \) is \( C^{[\alpha]} \)-regular; then \( C^\alpha(M, N) \) cannot admit a canonical manifold structure.

We now turn to smoothness properties of the composition map.

**Lemma 4.5** Assume that \( C^{[\alpha]+s}(N, L) \), \( C^\alpha(M, N) \), and \( C^\alpha(M, L) \) are endowed with pre-canonical manifold structures. Then the composition map

\[
\operatorname{comp} : C^{[\alpha]+s}(N, L) \times C^\alpha(M, N) \to C^\alpha(M, L), \quad (f, g) \mapsto f \circ g
\]

is a \( C^{\infty, s} \)-map, for every \( s \in \mathbb{N}_0 \cup \{\infty\} \).

**Proof.** Since \( C^\alpha(M, L) \) is pre-canonical, \( \operatorname{comp} \) is \( C^{\infty, s} \) if and only if

\[
\operatorname{comp}^\wedge : C^{[\alpha]+s}(N, L) \times C^\alpha(M, N) \times M \to L, \quad (f, g, x) \mapsto f(g(x))
\]

is a \( C^{\infty, s, \alpha} \)-map. The formula shows that \( \operatorname{comp}^\wedge(f, g, x) = \operatorname{ev}(f, \operatorname{ev}(g, x)) \), where the outer evaluation map is \( C^{\infty, [\alpha]+s} \) and the inner one \( C^{\infty, \alpha} \), by Lemma 4.3(a), as \( C^{[\alpha]+s}(N, L) \) and \( C^\alpha(M, N) \) are pre-canonical manifolds. Using the chain rule \([1, \text{Lemma 3.16}]\), we deduce that \( \operatorname{comp}^\wedge \) is \( C^{\infty, s, \alpha} \). \( \square \)

**Corollary 4.6** If \( C^\alpha(M, N) \) and \( C^\alpha(M, L) \) are endowed with pre-canonical manifold structures, then the pushforward \( f_* : C^\alpha(M, N) \to C^\alpha(M, L) \), \( g \mapsto f \circ g \) is a \( C^s \)-map for every \( f \in C^{[\alpha]+s}(N, L) \).

**Corollary 4.7** Let \( C^{[\alpha]+s}(N, L) \) and \( C^\alpha(M, L) \) be endowed with pre-canonical manifold structures. For a \( C^s \)-map \( g : M \to N \) the pullback \( g^* : C^{[\alpha]+s}(N, L) \to C^\alpha(M, L) \), \( f \mapsto f \circ g \) is smooth for every \( s \in \mathbb{N}_0 \).

\(^2\text{Meaning that the topology on } M \text{ is initial with respect to } C^k(M, \mathbb{R}). \text{ This holds if } M \text{ is a regular topological space and all modeling spaces are } C^k \text{-regular, see } [13].\)
The chain rule also allows the following result to be deduced.

**Lemma 4.8** Let $C^\alpha(M, N)$ and $C^\alpha(L, N)$ be endowed with pre-canonical manifold structures where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $M = M_1 \times \cdots \times M_n$ and $L = L_1 \times \cdots \times L_n$. Assume that $g_i: L_i \to M_i$ is a $C^{\alpha_i}$-map for $i \in \{1, \ldots, n\}$. Then the pullback

$$g^*: C^\alpha(M, N) \to C^\alpha(L, N), \ f \mapsto f \circ (g_1 \times \cdots \times g_n)$$

with $g := g_1 \times \cdots \times g_n$ is smooth.

**Proof.** Due to the chain rule, the pullback $g^*$ makes sense. Since $C^\alpha(L, N)$ is pre-canonical, $g^*$ will be smooth if $((g^*)^\wedge: (f, \ell) \mapsto \text{ev}(f, \text{ev}((g_1 \times \cdots \times g_n), \ell))$ is a $C^{\infty, \alpha}$-map. Again, this is a consequence of Lemma 4.3 (a). $\blacksquare$

The key point was the differentiability of the evaluation map together with a suitable chain rule. Thus, by essentially the same proof, one obtains from the chain rule [1, Lemma 3.16] the following statement whose proof we omit.

**Proposition 4.9** Assume that all the manifolds of mappings occurring in the following are endowed with pre-canonical manifold structures. Further, we let $\beta = (\beta_1, \ldots, \beta_n) \in (N_0 \cup \{\infty\})^n$ such that for multiindices $\alpha^i \in (N_0 \cup \{\infty\})^{m_i}$, $i \in \{1, \ldots, n\}$ we have $\beta_i = |\alpha^i| + \sigma_i$ for some $\sigma_i \in N_0 \cup \{\infty\}$. Let now $N = \prod_{1 \leq i \leq n} N_i$ and $M^i := M_1^i \times \cdots \times M_{m_i}^i$, for certain locally compact manifolds $M^i_j$ with rough boundary (with $j \in \{1, \ldots, m_i\}$). Then for $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\alpha = (\alpha^1, \ldots, \alpha^n)$, the composition map

$$\prod_{1 \leq i \leq n} C^{\alpha^i}(M^i, N_i) \to C^\alpha(M^1 \times \cdots \times M^n, L)$$

$$(f, g_1, \ldots, g_n) \mapsto f \circ (g_1 \times \cdots \times g_n)$$

is a $C^{\infty, \sigma}$-map.

The above discussion shows that composition, pushforward, and pullback maps inherit differentiability and continuity properties. The following variant will be used in the construction process of canonical manifold structures.

**Proposition 4.10** Let $K$ be a compact smooth manifold such that $C^\alpha(K, M)$ and $C^\alpha(K, N)$ admit canonical manifold structures. If $\Omega \subseteq K \times M$ is an open subset and $f: \Omega \to N$ is a $C^{|\alpha| + k}$-map, then

$$\Omega' := \{\gamma \in C^\alpha(K, M): \text{graph}(\gamma) \subseteq \Omega\}$$

is an open subset of $C^\alpha(K, M)$ and

$$f_*: \Omega' \to C^\alpha(K, N), \ \gamma \mapsto f \circ (\text{id}_K, \gamma)$$

is a $C^k$-map.
Proof. By compactness of $K$, the compact-open topology on $C(K, M)$ coincides with the graph topology (see, e.g., [15, Proposition A.6.25]). Thus \{ $\gamma \in C(K, M)$: $\text{graph}(\gamma) \subseteq \Omega$ \} is open in $C(K, M)$. As a consequence, $\Omega'$ is open in $C^\omega(K, M)$. By Lemma 4.3(a), the evaluation $\text{ev}: C^\omega(K, M) \times K \to M$ is $C^{\infty, \omega}$ and hence $C^{k, \omega}$, whence also $C^\omega(K, M) \times K \to K \times M$, $(\gamma, x) \mapsto (x, \gamma(x))$ is $C^{k, \omega}$. Since $f$ is $C^{[a]+, \omega}$, the Chain Rule [11, Lemma 3.16] shows that

$$(f_\ast)^\gamma : \Omega' \times K \to N, \ (\gamma, x) \mapsto f_\ast(\gamma)(x) = f(x, \gamma(x))$$

is $C^{k, \omega}$. So $f_\ast$ is $C^k$, as the manifold structure on $C^\omega(K, N)$ is canonical. \hfill \Box

For later use we record several observations on stability of (pre-)canonical structures under pushforward by diffeomorphisms.

Lemma 4.11 Let $N_1$ and $N_2$ be smooth manifolds and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$, $\beta \in (\mathbb{N}_0 \cup \{\infty\})^m$.

(a) If $C^\omega(M, N_1)$ and $C^\omega(M, N_2)$ are endowed with (pre-)canonical manifold structures, then the smooth manifold structure on $C^\omega(M, N_1 \times N_2)$ which turns the bijection $C^\omega(M, N_1 \times N_2) \to C^\omega(M, N_1) \times C^\omega(M, N_2)$ sending a mapping to the pair of component functions into a $C^\omega$-diffeomorphism, is (pre-)canonical.

(b) If $\psi: N_1 \to N_2$ is a $C^\infty$-diffeomorphism and $C^\omega(M, N_2)$ is a (pre-)canonical manifold, then the smooth manifold structure on $C^\omega(M, N_1)$ turning the bijection

$$\psi_\ast : C^\omega(M, N_1) \to C^\omega(M, N_2), \ f \mapsto \psi \circ f$$

into a diffeomorphism is (pre-)canonical.

(c) Let $C^\omega(M, N)$ be endowed with a pre-canonical manifold structure and assume that both $C^{\beta}(L, C^\omega(M, N))$ and $C^{\beta, \omega}(L \times M, N)$ are smooth manifolds making the bijection

$$\Phi: C^{\beta, \omega}(L \times M, N) \to C^{\beta}(L, C^\omega(M, N)), \ f \mapsto f^\gamma$$

a $C^\infty$-diffeomorphism. Then $C^{\beta}(L, C^\omega(M, N))$ is pre-canonical if and only if $C^{\beta, \omega}(L \times M, N)$ is pre-canonical.

Proof. Let $L = L_1 \times \cdots \times L_m$ be a product of manifolds.

(a) A map $f = (f_1, f_2): L \to C^\omega(M, N_1) \times C^\omega(M, N_2)$ is $C^\beta$ if and only if $f_1$ and $f_2$ are $C^\beta$. As the manifold structures are (pre-)canonical, this holds if and only if $f_i^\gamma: L \times M \to M_i$ is $C^{\beta, \omega}$ for $i \in \{1, 2\}$. However, this holds if and only if $f^\gamma = (f_1^\gamma, f_2^\gamma)$ is $C^{\beta, \omega}$.

(b) A map $f: L \to C^\omega(M, N_1)$ is $C^\beta$ if and only if $\psi_\ast f$ is $C^\beta$. Since $C^\omega(M, N_2)$ is pre-canonical, this is the case if and only if $(\psi_\ast f)^\gamma = \psi_\ast f^\gamma$ is $C^{\beta, \omega}$. As $\psi$ is a smooth diffeomorphism we deduce from the chain rule that this is the case if and only if $f^\gamma$ is of class $C^{\beta, \omega}$. Thus $C^\omega(M, N_1)$ is pre-canonical. If $C^\omega(M, N_2)$ is even canonical, the $C^\omega$-topology is transported by
the diffeomorphism \( \psi \) to the \( C^\alpha \)-topology on \( C^\alpha(M, N_1) \). Hence the manifold \( C^\alpha(M, N_1) \) is also canonical in this case.

(c) By construction, a map \( f : K \to C^{\beta, \alpha}(L \times M, N) \) is of class \( C^\gamma \) (for some multiindex \( \gamma \)) if and only if \( \Phi \circ f = (f(\cdot))^\nu \) is \( C^\nu \) as a mapping to \( C^{\beta}(L, C^\alpha(M, N)) \). As \( C^\alpha(M, N) \) is pre-canonical, we observe that \( (\Phi \circ f)^\nu : K \times L \to C^\alpha(M, N) \) is \( C^{\gamma, \beta} \) if and only if \( ((\Phi \circ f)^\nu)^\lambda = f^\lambda : K \times L \times M \to N \) is a \( C^{\gamma, \beta, \alpha} \)-map. Hence \( C^{\beta, \alpha}(L \times M, N) \) is pre-canonical (i.e. \( f \) is \( C^\gamma \) and only if \( f^\lambda \) is \( C^{\gamma, \beta, \alpha} \) if and only if \( C^{\gamma}(L, C^\alpha(M, N)) \) is pre-canonical. \( \square \)

**Lemma 4.12** Fix \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \) and a permutation \( \sigma \) of \( \{1, \ldots, n\} \). Denote by \( \phi_\sigma : M_1 \times \cdots \times M_n \to Q := M_{\sigma(1)} \times \cdots \times M_{\sigma(n)} \) the diffeomorphism taking \((x_i)_{i=1}^n \) to \((x_{\sigma(i)})_{i=1}^n \).

(a) If \( C^{\alpha \sigma}(Q, N) \) and \( C^\alpha(M, N) \) are smooth manifolds such that the bijection

\[
\phi_\sigma^* : C^{\alpha \sigma}(Q, N) \to C^\alpha(M, N), \quad f \mapsto f \circ \phi_\sigma
\]

from \( \mathcal{C}^{\alpha \sigma}(Q, N) \) becomes a diffeomorphism, then \( C^\alpha(M, N) \) is (pre-)canonical if and only if \( C^{\alpha \sigma}(Q, N) \) is (pre-)canonical.

(b) If \( C^\alpha(M, N) \) and \( C^{\alpha \sigma}(Q, N) \) are endowed with pre-canonical manifold structures, then \( \phi_\sigma^* \) is a \( C^\infty \)-diffeomorphism.

(c) If \( \psi_i : L_i \to M_i \) is a smooth diffeomorphism for every \( i \in \{1, \ldots, n\} \) and \( C^\alpha(M, N) \) is (pre-)canonical, then the smooth manifold structure on \( C^\alpha(L, N) \) turning the bijection

\[
(\psi_1 \times \cdots \times \psi_n)^*: C^\alpha(M, N) \to C^\alpha(L, N)
\]

into a diffeomorphism is (pre-)canonical.

**Proof.** (a) Assume that \( C^\alpha(M, N) \) is (pre-)canonical. Then \( f : K \to C^{\alpha \sigma}(Q, N) \) is \( C^\beta \) if and only if \( \phi_\sigma^* \circ f \) is so. Now we deduce from \( C^\alpha(M, N) \) being pre-canonical that this is equivalent to \( (\phi_\sigma^* \circ f)^\lambda = f^\lambda \circ (\text{id}_K \times \phi_\sigma) : K \times M \to N \) being a \( C^{\beta, \alpha} \)-map. Exploiting the Theorem of Schwarz \( [1, \text{Proposition 3.5}] \), this is equivalent to \( f^\lambda \) being \( C^{\beta, \alpha \sigma} \). Thus \( C^{\alpha \sigma}(Q, N) \) is pre-canonical. The converse assertion for \( C^{\alpha \sigma}(M, N) \) follows verbatim by replacing \( \phi_\sigma \) with its inverse. Note that if one of the manifolds is even canonical, it follows directly from the definition of the \( C^\alpha \)-topology, Definition \( 5.4 \), that reordering the factors induces a homeomorphism of the \( C^\alpha \)- and \( C^{\alpha \sigma} \)-topology. Hence we see that one of the manifolds is canonical if and only if the other is so.

(b) Note that the inverse of \( \phi_\sigma^* \) is \( (\phi_\sigma^{-1})^* \) whence the situation is symmetric and it suffices to prove that \( \phi_\sigma^* \) (and by an analogous argument also its inverse) is smooth. As \( C^\alpha(M, N) \) is pre-canonical, smoothness of \( \phi_\sigma^* \) is equivalent to \( (\phi_\sigma^*)^\nu : C^{\alpha \sigma}(Q, N) \times M \to N, \ (f, m) \mapsto \text{ev}(f, \phi_\sigma(m)) \) being a \( C^{\infty, \alpha} \)-mapping. This follows from Lemma \( 13.1 \)(a), the chain rule, and Lemma \( 2.1 \).

(c) Replacing \( \phi_\sigma \) with \( \psi_1 \times \cdots \times \psi_n \), the argument is analogous to (b). If \( C^\alpha(M, N) \) is canonical, then the \( C^\alpha \)-topology pulls back to the \( C^\alpha \)-topology under the diffeomorphism, by Lemma \( 5.5 \). \( \square \)

An exponential law is available for pre-canonical smooth manifold structures.
**Proposition 4.13** Let $L_1, \ldots, L_m$ and $N$ be smooth manifolds with rough boundary, and $M_1, \ldots, M_n$ be locally compact smooth manifolds with rough boundary. Assume that $\mathcal{C}^\alpha(M, N)$ is endowed with a pre-canonical smooth manifold structure and also $\mathcal{C}^\beta(L, \mathcal{C}^\alpha(M, N))$ and $\mathcal{C}^{\beta,\alpha}(L \times M, N)$ are endowed with pre-canonical smooth manifold structures. Then the bijection 

$$\Phi: \mathcal{C}^{\beta,\alpha}(L \times M, N) \to \mathcal{C}^\beta(L, \mathcal{C}^\alpha(M, N))$$

from (1) is a $C^\infty$-diffeomorphism.

**Proof.** If we give $\mathcal{C}^\beta(L, \mathcal{C}^\alpha(M, N))$ the smooth manifold structure making $\Phi$ a $C^\infty$-diffeomorphism, then this structure is pre-canonical by Lemma 4.11(c). It therefore coincides with the given pre-canonical smooth manifold structure thereon, up to the choice of modeling spaces (Lemma 4.3(b)).

There is a natural identification of tangent vectors for pre-canonical manifolds, in good cases. If $\mathcal{C}^\alpha(M, N)$ is pre-canonical, an element $v \in T_f \mathcal{C}^\alpha(M, N)$ corresponds to an equivalence class of curves $\gamma^v: I \to \mathcal{C}^\alpha(M, N)$ on some open interval $I$ around 0 such that $\gamma^v(0) = f$ and $\dot{\gamma}^v(0) = v$. As $\mathcal{C}^\alpha(M, N)$ is pre-canonical, the map $\gamma^v_\cdot: I \times M \to N$ is $C^1$. $\alpha$-map. Hence $T\varepsilon_m(v) = T\varepsilon_m(\dot{\gamma}^v(0)) \in TN$ is $C^\alpha$ in $m \in M$, where we use the point evaluation $\varepsilon_m: \mathcal{C}^\alpha(M, N) \to N$, $f \mapsto f(m)$ at $m$. We thus obtain a map 

$$\Psi: TC^\alpha(M, N) \to C^\alpha(M, TN), \ v \mapsto (m \mapsto T\varepsilon_m(v)).$$

(2)

Under additional assumptions, one can show that $\Psi$ is a diffeomorphism, allowing tangent vectors $v \in TC^\alpha(M, N)$ to be identified with $\Psi(v)$. We will encounter a setting in which this statement becomes true in the next section (see Theorem 5.14).

## 5 Constructions for compact domains

We now construct and study manifolds of $C^\alpha$-mappings on compact domains. The results of this section subsume Theorem 1.1. They generalize constructions for $C^{k,\ell}$-functions in [4, Appendix A].

### 5.1 Let $N$ be a smooth manifold, $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$ and $M = M_1 \times \cdots \times M_n$ be a locally compact smooth manifold with rough boundary. If $\pi: E \to N$ is a smooth vector bundle over $N$ and $f: M \to N$ is a $C^\alpha$-map, then we define 

$$\Gamma_f := \{ \tau \in C^\alpha(M, E): \pi \circ \tau = f \}$$

with the topology induced by $C^\alpha(M, E)$. Pointwise operations turn $\Gamma_f$ into a vector space. Let us prove that $\Gamma_f$ is a locally convex space. To this end, we cover $N$ with open sets $(U_i)_{i \in I}$ on which the restriction $E|_{U_i} \cong U_i \times E_i$ (with $E_i$ a suitable locally convex space) is trivial. Combining continuity of $f$ and local compactness of $M$ we can find families $\mathcal{K}_j$ of full compact submanifolds of $M_j$ with the following properties: The interiors of the sets in $\mathcal{K}_j$ cover $M_j$. 

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There is a set $\mathcal{K} \subseteq \prod_{1 \leq j \leq n} K_j$ such that for every $K = K_1 \times \cdots \times K_n \in \mathcal{K}$ we have $f(K) \subseteq U_{i_K}$ for some $i_K \in I$ and the interiors of the submanifolds in $\mathcal{K}$ cover $M$. Hence we deduce from Lemma 5.7 that the map

$$\Psi: C^\alpha(M, E) \to \prod_{K \in \mathcal{K}} C^\alpha(K, E), \ \sigma \mapsto (\sigma|_K)_{K \in \mathcal{K}}$$

is a topological embedding. Now by construction $\Gamma f$ is contained in the open subset $\{G \in C^\alpha(M, E) \mid G(K) \subseteq \pi^{-1}(U_{i_K}), \forall K \in \mathcal{K}\}$. Restricting $\Psi$ to this subset we obtain a topological embedding

$$e: \Gamma f \to \prod_{K \in \mathcal{K}} C^\alpha(K, \pi^{-1}(U_{i_K})) \cong \prod_{K \in \mathcal{K}} C^\alpha(K, U_{i_K}) \times C^\alpha(K, E_{i_K}), \quad (3)$$

where the identification exploits Lemma 3.11 and the fact that pushforwards with smooth diffeomorphisms induce homeomorphisms of the $C^\alpha$-topology (see Lemma 3.5). The image of $e$ are precisely the mappings which coincide on the intersections of the compact sets $K$ (see 10 and the explanations there). Hence we can exploit that point evaluations are continuous on $C^\alpha(K, E_{i_K})$ by [2, Proposition 3.17] to see that the image of $e$ is a closed vector subspace of $\prod_{K \in \mathcal{K}} \{f|K\} \times C^\alpha(K, E_{i_K})$. As the space on the right hand side is locally convex, we deduce that the co-restriction of $e$ onto its image is an isomorphism of locally convex spaces. Thus $\Gamma f$ is a locally convex topological vector space.

We will sometimes write $\Gamma f(E)$ instead of $\Gamma f$ to emphasize the dependence on the bundle $E$.

The previous setup allows an essential Exponential Law to be deduced.

**Lemma 5.2** In the situation of 5.1 let $\beta \in (\mathbb{N}_0 \cup \{\infty\})^m$ and $g: L \to \Gamma f$ be a map, where $L_1, \ldots, L_m$ are smooth manifolds with rough boundary and $L := L_1 \times \cdots \times L_m$. Then $g$ is $C^\beta$ if and only if

$$g^\wedge: L \times M \to E, \quad (x, y) \mapsto g(x)(y)$$

is a $C^{\beta, \alpha}$-map.

**Proof.** With the notation as in 5.1 we identify $\Gamma f$ via $e$ with a closed subspace of the locally convex space $\prod_{K \in \mathcal{K}} C^\alpha(K, E_{i_K})$ (the identification will be suppressed in the notation). Thus Lemma 2.14 (n) implies that the map $g$ is $C^\beta$ if and only if the components $g_K: L \to C^\alpha(K, E_{i_K})$ are $C^\beta$-maps. By the Exponential Law [1 Theorem 4.4], the latter holds if and only if the mappings

$$(g_K)^\wedge: L \times K \to E_{i_K}, \quad (x, y) \mapsto g(x)(y)$$

are of class $C^{\beta, \alpha}$. Since the interiors of sets $K \in \mathcal{K}$ cover $M$, we deduce that this is the case if and only if $g^\wedge$ is of class $C^{\beta, \alpha}$. □
Remark 5.3 If all fibres of $E$ are Fréchet spaces and $K$ is $\sigma$-compact and locally compact, then $\Gamma_f$ is a Fréchet space; if all fibres of $E$ are Banach spaces, $K$ is compact, and $|\alpha|<\infty$, then $\Gamma_f$ is a Banach space. To see this, note that we can choose the family $K$ in 5.1 countable (resp., finite). Supressing again the identification,

$$\psi: \Gamma_f \to \prod_{j \in J} C^\alpha(K_j, F_j), \quad \tau \mapsto (\tau|_{K_j})_{j \in J}$$

is linear and a topological embedding with closed image. If all $F_j$ are Fréchet spaces, so is each $C^\alpha(K_j, F_j)$ (cf., e.g., [15]) and hence also $\Gamma_f$. If all $F_j$ are Banach spaces and $|\alpha|$ as well as $J$ is finite, then each $C^\alpha(K_j, F_j)$ is a Banach space (cf. loc. cit.) and hence also $\Gamma_f$.

Observe that the exponential law for $\Gamma_f$ gives this space the defining property of a pre-canonical manifold (and the only reason we do not call it pre-canonical is that it is only a subset of $C^\alpha(M, E)$). In particular, the proof of Lemma 4.3 (a) carries over and yields:

Lemma 5.4 In the situation of 5.1, the evaluation map

$$\text{ev}: \Gamma_f \times M \to E, \quad (\tau, x) \mapsto \tau(x)$$

is $C^{\infty, \alpha}$.

Lemma 5.5 Let $\pi_1: E_1 \to N$ and $\pi_2: E_2 \to N$ be smooth vector bundles over a smooth manifold $N$. Let $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^m$ and $f: M \to N$ be a $C^\alpha$-map on a product $M = M_1 \times \cdots \times M_n$ of smooth manifolds with rough boundary. Then the following holds:

(a) If $\psi: E_1 \to E_2$ is a mapping of smooth vector bundles over $\text{id}_M$, then $\psi \circ \tau \in \Gamma_f(E_2)$ for each $\tau \in \Gamma_f(E_1)$ and

$$\Gamma_f(\psi): \Gamma_f(E_1) \to \Gamma_f(E_2), \quad \tau \mapsto \psi \circ \tau$$

is a continuous linear map.

(b) $\Gamma_f(E_1 \oplus E_2)$ is canonically isomorphic to $\Gamma_f(E_1) \times \Gamma_f(E_2)$.

Proof. (a) If $\tau \in \Gamma_f(E_1)$, then $\psi \circ \tau: M \to E_2$ is $C^\alpha$ by the chain rule and $\pi_2 \circ \psi \circ \tau = \pi_1 \circ \tau = f$, whence $\psi \circ \tau \in \Gamma_f(E_2)$. Evaluating at points we see that the map $\Gamma_f(\psi)$ is linear; being a restriction of the continuous map $C^\alpha(M, \psi): C^\alpha(M, E_1) \to C^\alpha(M, E_2)$ (see Lemma 3.3), it is continuous.

(b) If $\rho_j: E_1 \oplus E_2 \to E_j$ is the map taking $(v_1, v_2) \in E_1 \times E_2$ to $v_j$ for $j \in \{1, 2\}$ and $\iota_j: E_j \to E_1 \oplus E_2$ is the map taking $v_j \in E_j$ to $(v_1, 0)$ and $(0, v_2)$, respectively, then

$$(\Gamma_f(\rho_1), \Gamma_f(\rho_2)): \Gamma_f(E_1 \oplus E_2) \to \Gamma_f(E_1) \times \Gamma_f(E_2)$$

is a continuous linear map which is a homeomorphism as it has the continuous map $(\sigma, \tau) \mapsto \Gamma_f(\iota_1)(\sigma) + \Gamma_f(\iota_2)(\tau)$ as its inverse. $\square$
Construction of the canonical manifold structure

Having constructed spaces of $C^\alpha$-sections as model spaces, we are now in a position to construct the canonical manifold structure on $C^\alpha(K, M)$, assuming that $M$ is covered by local additions and $K$ is compact.

Definition 5.6 Let $M$ be a smooth manifold. A local addition is a smooth map

$$\Sigma: U \to M,$$

defined on an open neighborhood $U \subseteq TM$ of the zero-section $0_M := \{0_p \in T_p M : p \in M\}$ such that $\Sigma(0_p) = p$ for all $p \in M$,

$$U' := \{ (\pi_{TM}(v), \Sigma(v)) : v \in U \}$$

is open in $M \times M$ (where $\pi_{TM}: TM \to M$ is the bundle projection) and the map

$$\theta := (\pi_{TM}, \Sigma): U \to U'$$

is a $C^\infty$-diffeomorphism. If $T_{0_p}(\Sigma|_{T_p M}) = id_{T_p M}$ for all $p \in M$, (4)
we say that the local addition $\Sigma$ is normalized.

Until Lemma 5.9, we fix the following setting, which allows a canonical manifold structure on $C^\alpha(K, M)$ to be constructed.

5.7 We consider a product $K = K_1 \times K_2 \times \cdots \times K_n$ of compact smooth manifolds with rough boundary, a smooth manifold $M$ which admits a local addition $\Sigma: TM \supseteq U \to M$, and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$.

5.8 Manifold structure on $C^\alpha(K, M)$ if $M$ admits a local addition

For $f \in C^\alpha(K, M)$, let $\Gamma_f := \{ \tau \in C^\alpha(K, TM) : \pi_{TM} \circ \tau = f \}$ be the locally convex space constructed in 5.1. Then

$$O_f := \Gamma_f \cap C^\alpha(K, U)$$

is an open subset of $\Gamma_f$,

$$O'_f := \{ g \in C^\alpha(K, M) : (f, g)(K) \subseteq U' \}$$

is an open subset of $C^\alpha(K, M)$, and

$$\phi_f: O_f \to O'_f, \quad \tau \mapsto \Sigma \circ \tau$$

is a homeomorphism with inverse $g \mapsto \theta^{-1} \circ (f, g)$. By the preceding, if also $h \in C^\alpha(K, M)$, then $\psi := \phi_h^{-1} \circ \phi_f$ has an open (possibly empty) domain $D \subseteq \Gamma_f$ and is a smooth map $D \to \Gamma_h$ by Lemma 5.2 as $\psi^\wedge: D \times K \to TM$,

$$(\tau, x) \mapsto (\phi_h^{-1} \circ \phi_f)(\tau)(x) = \theta^{-1}(h(x), \Sigma(\tau(x))) = \theta^{-1}(h(x), \Sigma(\varepsilon(\tau, x)))$$

is a $C^\infty$-map (exploiting that the evaluation map $\varepsilon: \Gamma_f \times K \to TM$ is $C^\infty$, by Lemma 5.4). Hence $C^\alpha(K, M)$ endowed with the $C^\alpha$-topology has a smooth manifold structure for which each of the maps $\phi_f^{-1}$ is a local chart.
We now prove that the manifold structure on $C^\alpha(K,M)$ is canonical. Together with Lemma 5.3(b), this implies that the smooth manifold structure on $C^\alpha(K,M)$ constructed in 5.8 is independent of the choice of local addition.

**Lemma 5.9** The manifold structure on $C^\alpha(K,M)$ constructed in 5.8 is canonical.

**Proof.** We first show that the evaluation map $ev: C^\alpha_f(K,M) \times K \to M$ is $C^{\infty,\alpha}$. It suffices to show that $ev(\phi_f(\tau), x)$ is $C^{\infty,\alpha}$ in $(\tau, x) \in O_f \times K$ for all $f \in C^\alpha(K,M)$. This follows from

$$ev(\phi_f(\tau), x) = \Sigma(\tau(x)) = \Sigma(\epsilon(\tau, x)),$$

where $\epsilon: \Gamma_f \times K \to TM$, $\epsilon(x) \mapsto \tau(x)$ is $C^{\infty,\alpha}$ by Lemma 5.4. Now let $\beta \in (N_0 \cup \{\infty\})^m$ and $h: N \to C^\alpha(K,M)$ be a map, where $N = N_1 \times \cdots \times N_n$ is a product of smooth manifolds with rough boundary. If $h$ is $C^\beta$, then $h^\wedge = ev \circ (h \times \text{id}_K)$ is $C^{\beta,\alpha}$. Conversely, let $h^\wedge$ be a $C^{\beta,\alpha}$-map, then $h$ is continuous as a map to $C(K,M)$ with the compact-open topology (see [14, Proposition A.6.17]) and $h(x) = h^\wedge(x, \cdot) \in C^\alpha(K,M)$ for each $x \in N$. Given $x \in N$, let $f := h(x)$. Then $\psi_f: C(K,M) \to C(K,M) \times C(K,M) \cong C(K,M \times M)$, $g \mapsto (f, g)$ is a continuous map. Since $\psi_f(g)$ is $C^\alpha$ if and only if $g$ is $C^\alpha$, we see that $W := h^{-1}(O_f') = h^{-1}(\psi_f^{-1}(C^\alpha(K,U'))) = (\psi_f \circ h)^{-1}(C^\alpha(K,U'))$

$$= (\psi_f \circ h)^{-1}(C(K,U'))$$

is an open $x$-neighborhood in $N$. As the map $(\phi_f^{-1} \circ h|_W)^\wedge: W \times K \to TM,$

$$(y,z) \mapsto ((\phi_f)^{-1} \circ h|_W)^\wedge(y, z) = (\theta^{-1} \circ (f, h(y)))(z) = \theta^{-1}(f(z), h^\wedge(y, z))$$

is $C^{\beta,\alpha}$ by [14, Lemma 3.16], the map $\phi_f^{-1} \circ h|_W: W \to \Gamma_f$ (and hence also $h|_W$) is $C^k$, by Lemma 5.2. \hfill $\square$

**Proposition 5.10** Let $K = K_1 \times \cdots \times K_n$ be a product of compact smooth manifolds with rough boundary and $M$ be a manifold covered by local additions. For every $\alpha \in (N_0 \cup \{\infty\})^n$, the set $C^\alpha(K,M)$ can be endowed with a canonical manifold structure.

**Proof.** Let $(M_j, \Sigma_j)_{j \in J}$ be an upward directed family of open submanifolds $M_j$ with local additions $\Sigma_j$ whose union coincides with $M$. As $K$ is compact, we observe that the sets $C^\alpha(K,M_j) := \{ f \in C^\alpha(K,M) \mid f(K) \subseteq M_j \}$ are open in the $C^\alpha$-topology. Following Lemma 5.9 we can endow every $C^\alpha(K,M_j)$ with a canonical manifold structure. Now if $M_j \subseteq M_k$, Lemma 1.3(c) implies that also the submanifold structure induced by the inclusion $C^\alpha(K,M_j) \subseteq C^\alpha(K,M_k)$ is canonical. Thus uniqueness of canonical structures, Lemma 1.3(b), shows that the submanifold structure must coincide with the canonical structure constructed on $C^\alpha(K,M_j)$ via 5.8. As $C^\alpha(K,M) = \bigcup_{j \in J} C^\alpha(K,M_j)$ and each step of the ascending union is canonical, the same holds for the union. \hfill $\square$
The tangent bundle of the manifold of mappings

In the rest of this section, we identify the tangent bundle of $C^\alpha(K,M)$ as the manifold $C^\alpha(K,TM)$ (under the assumption that $K$ is compact and $M$ covered by local additions). To explain the idea, let us have a look at $C^\alpha(K,TM)$.

5.11 Consider a smooth manifold $M$ covered by local additions. Then also $TM$ is covered by local additions, cf. [4, A.11] for the construction. Thus for $K$ a compact manifold $C^\alpha(K,M)$ and $C^\alpha(K,TM)$ are canonical manifolds. If we denote by $\pi: TM \to M$ the bundle projection, Corollary 4.6 shows that the pushforward $\pi^*: C^\alpha(K,TM) \to C^\alpha(K,M)$ is smooth. The fibres of $\pi^*$ are the locally convex spaces $\pi^{-1}(f) = \Gamma_f$ from 5.1. We deduce that $\pi^*: C^\alpha(K,TM) \to C^\alpha(K,M)$ is a vector bundle (see Theorem 5.14 for a detailed proof).

We will first identify the fibres of the tangent bundle.

5.12 The tangent space $T_f C^\alpha(K,M)$ is given by equivalence classes $[t \mapsto c(t)]$ of $C^1$-curves $c: [-\varepsilon,\varepsilon] \to C^\alpha(K,M)$ with $c(0) = f$, where the equivalence relation $c \sim c'$ holds for two such curves if and only if $\dot{c}(0) = \dot{c}'(0)$. Since the manifold structure is canonical (Lemma 5.10) we see that $c$ is $C^1$ if and only if the adjoint map $c^\wedge: [-\varepsilon,\varepsilon] \times K \to N$ is a $C^1,\alpha$-map. The exponential law shows that the derivative of $c$ corresponds to the (partial) derivative of $c^\wedge$, i.e. the mapping $\Psi$ from (2) restricts to a bijection

$$\Psi_f: T_f C^\ell(K,M) \to \Gamma_f = \{ h \in C^\ell(K,TM) \mid \pi \circ h = f \},$$

$$[c] \mapsto (k \mapsto [t \mapsto c^\wedge(t,k)]).$$

We wish to glue the bijections on the fibres to identify the tangent manifold as the bundle from 5.11. To this end, we recall a fact from [4, Lemma A.14]:

5.13 If a manifold $M$ admits a local addition, it also admits a normalized local addition.

Hence we may assume without loss of generality that the local additions in the following are normalized. Moreover, we will write $\varepsilon_x: C^\alpha(K,M) \to M$ for the point evaluation in $x \in K$. Then the tangent bundle of $C^\alpha(K,M)$ can be described as follows.

**Theorem 5.14** Let $K = K_1 \times \cdots \times K_n$ be a product of compact smooth manifolds with rough boundary and $M$ be covered by local additions. Then

$$(\pi_{TM})_*: C^\ell(K,TM) \to C^\ell(K,M)$$

is a smooth vector bundle with fibre $\Gamma_f$ over $f \in C^\ell(K,M)$. For each $v \in T(C^\ell(K,M))$, we have $\Psi(v) := (T\varepsilon_x(v))_{x \in K} \in C^\alpha(K,TM)$ and the map (2),

$$\Psi: TC^\alpha(K,M) \to C^\alpha(K,TM), \quad v \mapsto \Psi(v)$$

is an isomorphism of smooth vector bundles (over the identity).
If we wish to emphasize the dependence on $M$, we write $\Psi_M$ instead of $\Psi$.

**Proof.** Since $M$ is covered by local additions, there is a family of open submanifolds (ordered by inclusion) $(M_j)_{j \in J}$ which admit local additions $\Sigma_j$. Now by compactness of $K$ the image of $f \in C^\infty(K, M)$ is always contained in some $M_j$ and similarly for $\tau \in \Gamma_f$ we then have $\tau(K) \subseteq \pi^{-1}(M_j) = T M_j$, where $\pi := \pi_{TM}$ is the bundle projection of $T M$. As the family $(M_j)_{j \in J}$ of open manifolds exhausts $M$, we have $C^\infty(K, M) = \bigcup_{j \in J} C^\infty(K, M_j)$ and all of these subsets are open. Hence it suffices to prove that $\Psi$ restricts to a bundle isomorphism for every $M_j$. In other words we may assume without loss of generality that $M$ admits a local addition $\Sigma$. Given $f \in C^\infty(K, M)$, the map $\phi_f : O_f \to O'_f \subseteq C^\infty(K, M)$ is a $C^\infty$-diffeomorphism with $\phi_f(0) = f$, whence $T \phi_f(0, \cdot) : \Gamma_f \to T_f(C^\infty(K, M))$ is an isomorphism of topological vector spaces. For $\tau \in \Gamma_f$, we have for $x \in K$

$$T_x T \phi_f(0, \tau) = T_x \left( [t \mapsto \Sigma \circ (\tau t)] = [t \mapsto \Sigma (\tau t(x))] \right) = [t \mapsto \Sigma|_{T_f(\tau)} (\tau t(x))] = T \Sigma|_{T_f(\tau)} (\tau t(x)) = \tau(x),$$

as $\Sigma$ is assumed normalized. Thus $\Psi(T \phi_f(0, \tau)) = \tau \in \Gamma_f \subseteq C^\infty(K, TM)$, whence $\Psi(v) \in \Gamma_f \subseteq C^\infty(K, TM)$ for each $v \in T_f(C^\infty(K, M))$ and $\Psi$ takes $T_f(C^\infty(K, M))$ bijectively and linearly onto $\Gamma_f$. Now the manifolds $T(C^\infty(K, M))$ and $C^\infty(K, TM)$ are the disjoint union of the sets $T_f(C^\infty(K, M))$ and $\Gamma_f = \pi^{-1}_\Sigma(\{f\})$, respectively, we see that $\Psi$ is a bijection. If we can show that $\Psi$ is a $C^\infty$-diffeomorphism, $\pi_* : C^\infty(K, TM) \to C^\infty(K, M)$ will be a smooth vector bundle over $C^\infty(K, M)$ (like $T(C^\infty(K, M))$). Finally, $\Psi$ will then be an isomorphism of smooth vector bundles over $id_M$.

For the proof we recall some results from the Appendix of [4]: Denote by $0 : M \to TM$ the zero-section and by $0_M := 0(M)$ its image. Let now $\lambda_\rho : T_\rho M \to TM$ be the canonical flip (given in charts by $(x, y, u, v) \mapsto (x, u, y, v)$) then [4] Lemma A.20(b)] yields a natural isomorphism $\Theta : TM \oplus TM \to \pi_{TM}^{-1}_M(0_M) \subseteq T^2 M, \Theta(v, w) = \kappa(T_\lambda_{\pi(v)}(v, w))$. On the level of function spaces $f \Theta$ induces a diffeomorphism (cf. [4] Lemma A.20(e))

$$\Theta_f : O_f \to O_{0 \circ f}, \quad \gamma \mapsto \Theta \circ (0 \circ f, \gamma).$$

Here for $f \in C^\infty(K, M)$ we have considered the composition $0 \circ f \in C^\infty(K, TM)$. Then the sets $S_f := T \phi_f(O_f \times \Gamma_f)$ form an open cover of $T(C^\infty(K, M))$ for $f \in C^\infty(K, M)$, whence the sets $\Psi(S_f)$ form a cover of $C^\infty(K, TM)$ by sets which are open as $\Psi(S_f) = (\phi_{0 \circ f} \circ \phi_f)(O_f \times \Gamma_f) = \phi_{0 \circ f}(O_{0 \circ f})$. Hence it suffices to prove that the bijective map $\Psi$ restricts to a $C^\infty$-diffeomorphism on these open sets. In other words it suffices to show that

$$\Phi \circ T \phi_f = \phi_{0 \circ f} \circ \Theta_f$$

\[3\]While the results in [4] were only established for the case of $C^{k, \ell}$-mappings, they carry over (together with their proofs) without any change to the more general case of the $C^\infty$-mappings considered here.
for each \( f \in C^\ell(K, M) \) (as all other mappings in the formula are smooth diffeomorphisms). Now
\[
T\phi_f(\sigma, \tau) = [t \mapsto \Sigma \circ (\sigma + t\tau)]
\]
for all \((\sigma, \tau) \in O_f \times \Gamma_f\), and thus we can rewrite \( \Psi(T\phi_f(\sigma, \tau)) \) as
\[
[(t \mapsto \Sigma(\sigma(x) + t\tau(x)))_{x \in K} = [(t \mapsto (\Sigma \circ \lambda_f(x))(\sigma(x) + t\tau(x)))_{x \in K} = (T(\Sigma \circ \lambda_f(x))((\sigma(x), \tau(x)))_{x \in K} = (\Sigma_{TM}((\kappa \circ T\lambda_f(x))(\sigma(x), \tau(x)))_{x \in K} = ((\Sigma_{TM} \circ \Theta_f)(\sigma, \tau)(x))_{x \in K} = (\phi_{\alpha_f} \circ \Theta_f)(\sigma, \tau).
\]
Thus the desired formula holds and shows that \( \Psi \) is a \( C^\infty \)-diffeomorphism. This concludes the proof. \( \square \)

**Remark 5.15** Assume that the local additions \( \Sigma: U_i \to M, \) covering \( M \) are normalized. Then the proof of Theorem \[5.14\] shows that
\[
\Psi \circ T\phi_f(0, \cdot): \Gamma_f \to C^\alpha(K, TM)
\]
is the inclusion map \( \tau \mapsto \tau \), for each \( f \in C^\alpha(K, M) \) (where \( \phi_f \) is as in \([5]\)).

Using canonical manifold structures, we have:

**Corollary 5.16** Let \( K = K_1 \times \cdots \times K_n \) be a product of compact smooth manifolds with rough boundary, \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \) and \( g: M \to N \) be a \( C^{[\alpha]+1} \)-map between smooth manifolds \( M \) and \( N \) covered by local additions. Then the tangent map of the \( C^1 \)-map
\[
g_*: C^\alpha(K, M) \to C^\alpha(K, N), \quad f \mapsto g \circ f
\]
is given by \( T(g_*) = \Psi^{-1}_{N} \circ (Tg)_* \circ \Psi_M \). For each \( f \in C^\alpha(K, M) \), we have \( \Psi_M(T_f(C^\alpha(K, M))) = \Gamma_f(TM), \) \( \Psi_N(T_{g_0f}(C^\alpha(K, N))) = \Gamma_{g_0f}(TN) \) and \( (Tg)_* \) restricts to the map
\[
\Gamma_f(TM) \to \Gamma_{g_0f}(TN), \quad \tau \mapsto Tg \circ \tau \quad (7)
\]
which is continuous linear and corresponds to \( T_f(g_*) \).

Moreover, the identification of the tangent bundle allows us to lift local additions (cf. \[3\] Remark A.17]).

**Lemma 5.17** Let \( K = K_1 \times \cdots \times K_n \) be a product of compact smooth manifolds with rough boundary, \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \) and \( M \) a manifold covered by local additions. Then the canonical manifold \( C^\alpha(K, M) \) is covered by local additions.

**Proof.** Consider first the case that \( M \) admits a local addition \( \Sigma: U \to M \) with \( \theta = (\pi_{TM}, \Sigma): U \to U' \subseteq M \times M \) the associated diffeomorphism. Since also \( TM \) admits a local addition, we have canonical manifold structures on \( C^\alpha(K, TM) \) and \( C^\alpha(K, M \times M) \cong C^\alpha(K, M) \times C^\alpha(K, M) \). Now \( K \) is compact,
whence $C^\alpha(K,U) \subseteq C^\alpha(K,TM)$ is an open submanifold, whence canonical by Lemma 4.3(c). In particular, $\Sigma: C^\alpha(K,U) \to C^\alpha(K,M)$ and $\theta: C^\alpha(K,U) \to C^\alpha(K,U') \subseteq C^\alpha(K,M \times M)$ are smooth by Corollary 4.6. As also the inverse of $\theta$ is smooth, we deduce that $\theta$ is again a diffeomorphism mapping $C^\alpha(K,U)$ to $C^\alpha(K,U')$ and we can identify the latter manifold with an open subset of $C^\alpha(K,M) \times C^\alpha(K,M)$ containing the diagonal. Hence we only need to verify that $0 \in T_f C^\alpha(K,TM)$ is mapped to $f$. However, using the point evaluation $\varepsilon_x(\Sigma_x(0_f)) = \Sigma(0(f(x))) = f(x)$ (where 0 is again the zero-section of $TM$), we obtain the desired equality pointwise and thus also on the level of functions. This proves that $C^\alpha(K,M)$ admits a local addition if $M$ admits a local addition.

If now $M$ is covered by open submanifolds $(M_j)_{j \in J}$ each admitting a local addition, it suffices to see that $C^\alpha(K,M_j)$ is an open submanifold of $C^\alpha(K,M)$ which admits a local addition by the above considerations. Thus $C^\alpha(K,M)$ is covered by the open submanifolds $(C^\alpha(K,M_j))_{j \in J}$ and as each of those admits a local addition, $C^\alpha(K,M)$ is covered by local additions. $\square$

**Proposition 5.18** Let $K = K_1 \times \cdots \times K_m$ and $L = L_1 \times \cdots \times L_n$ be products of compact manifolds with rough boundary and $M$ be a manifold covered by local additions. Fix $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n, \beta \in (\mathbb{N}_0 \cup \{\infty\})^m$. Then $C^{\beta,\alpha}(L \times K,M)$, $C^\alpha(K,M)$ and $C^{\beta}(L,C^\alpha(K,M))$ admit canonical manifold structures. Using these, the bijection $C^{\beta,\alpha}(L \times K,M) \to C^{\beta}(L,C^\alpha(K,M))$ is a $C^\infty$-diffeomorphism.

**Proof.** We apply Proposition 5.10 to obtain canonical manifold structures on $C^\alpha(K,M)$ and $C^{\beta,\alpha}(L \times K,M)$. By Lemma 5.17 $C^\alpha(K,M)$ is covered by local additions. Hence we may apply Proposition 5.10 again to obtain a canonical manifold structure on $C^{\beta}(L,C^\alpha(K,M))$. By Proposition 4.13 the bijection $C^{\beta,\alpha}(L \times K,M) \to C^{\beta}(L,C^\alpha(K,M))$ is a diffeomorphism. $\square$

### 6 Lie groups of Lie group-valued mappings

We now prove Theorem 1.3 starting with observations.

**Lemma 6.1** Let $M_1, \ldots, M_n$ be locally compact smooth manifolds with rough boundary, $G$ be a Lie group, and $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$. Setting $M := M_1 \times \cdots \times M_n$, the following holds:

(a) $C^\alpha(M,G)$ is a group.

(b) If a pre-canonical smooth manifold structure exists on $C^\alpha(M,G)$, then it makes $C^\alpha(M,G)$ a Lie group. Moreover, it turns the point evaluation $\varepsilon_x: C^\alpha(M,G) \to G, f \mapsto f(x)$ into a smooth group homomorphism for each $x \in M$.

**Proof.** (a) The group inversion $\iota: G \to G$ is smooth, whence $\iota \circ f$ is $C^\alpha$ for all $f \in C^\alpha(M,G)$ (by the Chain Rule Lemma 3.16, applied in local charts). Let
\( \mu : G \times G \to G \) be the smooth group multiplication and \( f, g \in C^\alpha(M, G) \). Then \( (f, g) : M \to G \times G \) is \( C^\alpha \) by [ Lemma 3.8]. By the Chain Rule, \( fg = \mu \circ (f, g) \) is \( C^\alpha \).

(b) The group inversion in \( C^\alpha(M, G) \) is the map \( C^\alpha(M, \iota) \) and hence smooth, by Corollary 4.14. Identifying \( C^\alpha(M, G) \times C^\alpha(M, G) \) with \( C^\alpha(M, G \times G) \) as a smooth manifold (as in Lemma 4.11(a)), the group multiplication of \( C^\alpha(M, G) \) is the map \( C^\alpha(M, \mu) \) and hence smooth. The group multiplication in \( C^\alpha(M, G) \) being pointwise, \( \varepsilon_x \) is a homomorphism of groups for each \( x \in M \). By Lemma 4.13(a), \( ev : C^\alpha(M, G) \times M \to G \) is \( C^{\infty, \alpha} \). Thus \( \varepsilon_x = ev(\cdot, x) \) is smooth. \( \square \)

Another concept is useful, with notation as in 2.6.

**Definition 6.2** Let \( M_1, \ldots, M_n \) be locally compact smooth manifolds with rough boundary, \( G \) be a Lie group, and \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \). For \( x \in M := M_1 \times \cdots \times M_n \), let \( \varepsilon_x : C^\alpha(M, G) \to G \) be the point evaluation. A smooth manifold structure on \( C^\alpha(M_1 \times \cdots \times M_n, G) \) making it a Lie group is said to be **compatible with evaluations** if \( \varepsilon_x \) is smooth for each \( x \in M \), we have \( \phi(v) := (L(\varepsilon_x))(v) \in C^\alpha(M, L(G)) \) for each \( v \in L(C^\alpha(M, G)) \), and the Lie algebra homomorphism

\[
\phi : L(C^\alpha(M, G)) \to C^\alpha(M, L(G)), \quad v \mapsto \phi(v)
\]

so obtained is an isomorphism of topological vector spaces.

**Remark 6.3** In the case that \( n = 1 \) and \( \alpha = \infty \), compatibility with evaluations was introduced in [24, Proposition 1.9 and page 19] (in different words), assuming that \( G \) is regular. Likewise, \( G \) is assumed regular in [16, Proposition 3.1], where the case \( n = 1, \alpha \in \mathbb{N}_0 \cup \{\infty\} \) is considered.

**Lemma 6.4** Let \( M_1, \ldots, M_n \) and \( N_1, \ldots, N_m \) be locally compact smooth manifolds with rough boundary, \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^m, \beta \in (\mathbb{N}_0 \cup \{\infty\})^n, M := M_1 \times \cdots \times M_n, N := N_1 \times \cdots \times N_m, \) and \( G \) be a Lie group. Assume that \( C^\beta(M, G) \) is endowed with a pre-canonical smooth manifold structure which is compatible with evaluations and that \( C^\alpha(N, C^\beta(M, G)) \), whose definition uses the latter structure, is endowed with a pre-canonical smooth manifold structure which is compatible with evaluations. Endow \( C^{\alpha, \beta}(N \times M, G) \) with the smooth manifold structure turning the bijection

\[
\Phi : C^{\alpha, \beta}(N \times M, G) \to C^\alpha(N, C^\beta(M, G)), \quad f \mapsto f^\vee
\]

into a \( C^{\infty} \)-diffeomorphism. Then the preceding smooth manifold structure on \( C^{\alpha, \beta}(N \times M, G) \) is pre-canonical and compatible with evaluations.

**Proof.** By Lemma 4.11(c), the \( C^\infty \)-manifold structure on \( C^{\alpha, \beta}(N \times M, G) \) is pre-canonical, whence the latter is a Lie group. The \( C^{\infty} \)-diffeomorphism \( \Phi \) is a homomorphism of groups. Hence

\[
L(\Phi) : L(C^{\alpha, \beta}(N \times M, G)) \to L(C^\alpha(N, C^\beta(M, G)))
\]
is an isomorphism of topological Lie algebras. Consider the point evaluations \( \varepsilon_x : C^\alpha(N, C^\beta(M, G)) \to C^\beta(M, G) \), \( \varepsilon_{(x,y)} : C^\alpha,\beta(N \times M, G) \to G \) and \( \varepsilon_y : C^\beta(M, G) \to G \) for \( x \in N, y \in M \). By hypothesis, we have isomorphisms of topological Lie algebras

\[
\Psi : L(C^\beta(M, G)) \to C^\beta(M, L(G)), \quad w \mapsto (L(\varepsilon_y)(w))_{y \in M}
\]

and \( \Theta : L(C^\alpha(N, C^\beta(M, G))) \to C^\alpha(N, L(C^\beta(M, G))), \quad v \mapsto (L(\varepsilon_x)(v))_{x \in N} \). Then also

\[
\Psi_* : C^\alpha(N, L(C^\beta(M, G))) \to C^\alpha(N, C^\beta(M, L(G))), \quad f \mapsto \Psi \circ f
\]
is an isomorphism of topological Lie algebras and so is

\[
\Xi : C^\alpha(N, C^\beta(M, L(G))) \to C^\alpha,\beta(N \times M, L(G)), \quad f \mapsto f^\wedge,
\]
by the Exponential Law (Lemma 3.13). Hence

\[
\phi := \Xi \circ \Psi_* \circ \Theta \circ L(\Phi) : L(C^\alpha,\beta(N \times M, G)) \to C^\alpha,\beta(M \times N, L(G))
\]
is an isomorphism of topological Lie algebras. Regard \( v \in L(C^\alpha,\beta(N \times M, G)) \) as a geometric tangent vector \([\gamma]\) for a smooth curve \( \gamma : [-\varepsilon, \varepsilon] \to C^\alpha,\beta(N \times M, G) \) with \( \gamma(0) = e \). Then \( L(\Phi)(v) = [\Phi \circ \gamma] \) and \( \Theta(L(\Phi)(v)) = ([\varepsilon_x \circ \Phi \circ \gamma])_{x \in N} =: g \). Thus

\[
\phi(v)(x, y) = \Xi \circ \Psi_* \circ \Theta \circ L(\Phi)(v)(x, y) = \Psi((\varepsilon_x \circ \Phi \circ \gamma))(x, y) = L(\varepsilon_y)([\varepsilon_x \circ \Phi \circ \gamma]) = [\varepsilon_x \circ \varepsilon_y \circ \Phi \circ \gamma] = [t \mapsto \varepsilon_x(\varepsilon_y(\Phi(\gamma(t))))] = [t \mapsto \gamma(t)(x, y)] = L(\varepsilon(x, y))(\gamma) = L(\varepsilon(x, y))(v).
\]

We deduce that \( (L(\varepsilon(x,y))(v))_{(x,y) \in N \times M} = \phi(v) \in C^\alpha,\beta(N \times M, L(G)) \). Since \( \phi \) is an isomorphism of topological Lie algebras, the Lie group structure on \( C^\alpha,\beta(N \times M, G) \) is compatible with evaluations.

**Lemma 6.5** Let \( M_1, \ldots, M_n \) be locally compact smooth manifolds with rough boundary, \( M := M_1 \times \cdots \times M_n \), \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \), and \( G \) be a Lie group. Assume that \( C^\alpha(M, G) \) is endowed with a pre-canonical smooth manifold structure which is compatible with evaluations. If the Lie group \( G \) is \( C^r \)-regular for some \( r \in \mathbb{N}_0 \cup \{\infty\} \), then also the Lie group \( C^\alpha(M, G) \) is \( C^r \)-regular.

**Proof.** Consider the smooth evolution map \( \text{Evol} : C^r([0, 1], \mathfrak{g}) \to C^{r+1}([0, 1], G) \), where \( \mathfrak{g} := L(G) \). For \( x \in M \), let \( \varepsilon_x : C^\alpha(M, G) \to G, \quad f \mapsto f(x) \) be evaluation at \( x \). By hypothesis, \( \phi : L(C^\alpha(M, G)) \to C^\alpha(M, \mathfrak{g}), \quad v \mapsto (L(\varepsilon_x)(v))_{x \in M} \) is an isomorphism of topological Lie algebras. Then also

\[
\phi_* : C^r([0, 1], L(C^\alpha(M, G))) \to C^r([0, 1], C^\alpha(M, \mathfrak{g})), \quad f \mapsto \phi \circ f
\]
is an isomorphism of topological Lie algebras. By Example 4.1, the smooth manifold structures on all of the locally convex spaces \( C^r([0, 1], C^\alpha(M, \mathfrak{g})), \quad C^{\alpha, r}([0, 1], \mathfrak{g}), \quad C^\alpha(M, C^r([0, 1], \mathfrak{g})) \)
are canonical. By Lemma 3.13 the Lie algebra homomorphism
\[ \psi: C^r([0, 1], C^\alpha(M, g)) \to C^{r,\alpha}([0, 1] \times M, g), \ f \mapsto f^\wedge \]
is an isomorphism of topological Lie algebras. Flipping the factors \([0, 1]\) and
\(M\) (with Lemma 4.12(b)) and using the Exponential Law again, we obtain an
isomorphism of topological Lie algebras
\[ \theta: C^{r,\alpha}([0, 1] \times M, g) \to C^\alpha(M, C^r([0, 1], g)) \]
determined by \(\theta(f)(x)(t) = f(t, x)\). By Theorem 1.1, \(C^{r+1}([0, 1], C^\alpha(M, G))\) has
a canonical smooth manifold structure. Using Lemma 4.11(c), Lemma 4.12(a),
and Lemma 4.11(c) in turn, we can give
\(C^\alpha(M, C^{r+1}([0, 1], G))\) a pre-canonical
smooth manifold structure making the map
\[ \beta: C^\alpha(M, C^{r+1}([0, 1], G)) \to C^{r+1}([0, 1], C^\alpha(M, G)) \]
determined by \(\beta(f)(t)(x) = f(x)(t)\) a \(C^\infty\)-diffeomorphism. The structures being
pre-canonical,
\[ \text{Evol}_*: C^\alpha(M, C^r([0, 1], g)) \to C^\alpha(M, C^{r+1}([0, 1], G)), \ f \mapsto \text{Evol} \circ f \]
is smooth. Hence also \(E := \beta \circ \text{Evol}_* \circ \theta \circ \psi \circ \phi_*\) is smooth as a map
\[ C^r([0, 1], L(C^\alpha(M, G))) \to C^{r+1}([0, 1], C^\alpha(M, G)). \]

It remains to show that \(E\) is the evolution map of \(C^\alpha(M, G)\). As the \(L(\varepsilon_x)\)
separate points on \(h := L(C^\alpha(M, G))\) for \(x \in M\), it suffices to show that
\(\varepsilon_x \circ E(\gamma) = \text{Evol}(L(\varepsilon_x) \circ \gamma)\) for all \(\gamma \in C^r([0, 1], h)\) and \(x \in M\) (see [12
Lemma 10.1]). Note that \((\phi \circ \gamma)(t)(x) = L(\varepsilon_x)(\gamma(t))\), whence
\[ ((\psi \circ \theta)(\phi \circ \gamma))(x)(t) = L(\varepsilon_x)(\gamma(t)) \]
and \((\text{Evol}_*(((\psi \circ \theta)(\phi \circ \gamma)))(x) = \text{Evol}(((\psi \circ \theta)(\phi \circ \gamma))(x)) = \text{Evol}(L(\varepsilon_x) \circ \gamma)\). So
\((\varepsilon_x \circ E(\gamma))(t) = (\text{Evol}_* \circ \theta \circ \psi \circ \phi_*)(\gamma)(x)(t) = \text{Evol}(L(\varepsilon_x) \circ \gamma)(t). \quad \square \)

We establish Theorem 1.3 in parallel with the first conclusion of the following
proposition, starting with two basic cases:

Case 1: The manifolds \(M_1, \ldots, M_n\) are compact;

Case 2: \(M\) is 1-dimensional with finitely many connected components.

**Proposition 6.6** In Theorem 1.3 the Lie group structure on \(C^\alpha(M, G)\) is
compatible with evaluations, writing \(M := M_1 \times \cdots \times M_n\). Moreover, there
is a unique canonical pure smooth manifold structure on \(C^\alpha(M, G)\) which is
modeled on \(C^\alpha(M, L(G))\).
The final assertion is clear: Starting with any canonical structure on \( C^\alpha(M,G) \) and a chart \( \phi: U_\phi \to V_\phi \to E_\phi \) around the constant map \( e \), using left translations (which are \( C^\infty \)-diffeomorphisms) we can create charts around every \( f \in C^\alpha(M,G) \) which are modeled on the given \( E_\phi \). We can therefore select a subatlas making \( C^\alpha(M,G) \) a pure smooth manifold. Since \( E_\phi \) is isomorphic to \( L(C^\alpha(M,G)) \), which is isomorphic to \( E := C^\alpha(M,L(G)) \) as a locally convex space (by compatibility with evaluations), we can replace \( E_\phi \) with \( E \). The pure canonical structure modeled on \( E \) is unique, since \( \text{id}_{C^\alpha(M,G)} \) is a \( C^\infty \)-diffeomorphism for any two canonical structures (cf. Lemma 4.3(b)).

**Lemma 6.7** Let \( M_1, \ldots, M_n \) be compact smooth manifolds with rough boundary, \( G \) be a Lie group and \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \). Abbreviate \( M := M_1 \times \cdots \times M_n \). Then \( C^\alpha(M,G) \) admits a canonical smooth manifold structure which is compatible with evaluations.

**Proof.** By Theorem 6.1, \( C^\alpha(M,G) \) admits a canonical smooth manifold structure. Let \( \theta: M \to G \) be the constant map \( x \mapsto e \). By Theorem 5.14 the diffeomorphism \( (\tau_x)_{x \in M} \) maps \( L(C^\alpha(M,G)) = T_0(C^\alpha(M,G)) \) onto \( \Gamma_\theta = \{ \tau \in C^\alpha(M,TG): \pi_{TG} \circ \tau = \theta \} = C^\alpha(M,L(G)) \).

By Lemma 6.3 \( C^\alpha(M,TG) \) induces on \( C^\alpha(M,L(G)) \) the compact-open \( C^\alpha \)-topology. Thus, the Lie group structure on \( C^\alpha(M,G) \) is compatible with evaluations. For the last assertion, see Lemma 6.3. \( \square \)

**Lemma 6.8** Let \( M \) be a 1-dimensional smooth manifold with rough boundary, such that \( M \) has only finitely many connected components (which need not be \( \sigma \)-compact). Let \( r \in \mathbb{N}_0 \cup \{\infty\} \), \( G \) be a \( C^r \)-regular Lie group, and \( k \in \mathbb{N} \) such that \( k \geq r + 1 \). Then \( C^k(M,G) \) admits a canonical smooth manifold structure which makes it a \( C^r \)-regular Lie group and is compatible with evaluations.

**Proof.** We first assume that \( M \) is connected. Let \( \mathfrak{g} := L(G) \) be the Lie algebra of \( G \). If \( N \) is a full submanifold of \( M \), we write \( \Omega^1_{C^k-1}(N,\mathfrak{g}) \subseteq C^{k-1}(TN,\mathfrak{g}) \) for the locally convex space of \( \mathfrak{g} \)-valued 1-forms on \( N \), of class \( C^{k-1} \). Using the Maurer-Cartan form

\[
\kappa: TG \to \mathfrak{g}, \quad v \mapsto \pi_{TG}(v)^{-1}.v,
\]

a \( \mathfrak{g} \)-valued 1-form

\[
\delta_N(f) := \kappa \circ T f \in \Omega^1_{C^k-1}(N,\mathfrak{g})
\]

can be associated to each \( f \in C^k(N,G) \), called its left logarithmic derivative. Fix \( x_0 \in M \). For every \( \sigma \)-compact, connected, full submanifold \( N \subseteq M \) such that \( x_0 \in N \), there exists a \( C^\infty \)-diffeomorphism \( \psi: I \to N \) for some non-degenerate interval \( I \subseteq \mathbb{R} \), such that \( 0 \in I \) and \( \psi(0) = x_0 \). Then the diagram

\[
\begin{align*}
C^k(N,G) & \xrightarrow{\delta_N} \Omega^1_{C^k-1}(N,\mathfrak{g}) \\
\psi^* \downarrow & \quad \downarrow \theta \\
C^k(I,G) & \xrightarrow{\delta_I} C^{k-1}(I,\mathfrak{g}),
\end{align*}
\]
is commutative, where \( \psi^* : C^k(N, G) \to C^k(I, G) \), \( f \mapsto f \circ \psi \) and the vertical map \( \theta \) on the right-hand side, which takes \( \omega \) to \( \omega \circ \psi \), are bijections. For each \( \omega \in \Omega^1_{C^k-1}(N, g) \), there is a unique \( f \in C^k(N, G) \) such that \( f(x_0) = e \) and \( \delta_N(f) = \omega \): In fact, Lemma \( \ref{lemma:psi} \) yields a unique \( \eta \in C^k(I, G) \) with \( \eta(0) = e \) and \( \delta^k(\eta) = \theta(\omega) \); then \( f := (\psi^*)^{-1}(\eta) \) is as required. We set \( \text{Evol}_N(\omega) := f \).

If \( \omega \in \Omega^1_{C^k-1}(M, g) \), we have \( \text{Evol}_L(\omega|TL) = \text{Evol}_N(\omega|TN)|_L \) for all \( \sigma \)-compact, connected open submanifolds \( N, L \) of \( M \) such that \( L \subseteq N \). As such submanifolds \( N \) form a cover of \( M \) which is directed under inclusion, we can define \( f : M \to G \) piecewise via \( f(x) := \text{Evol}_N(\omega|TN)(x) \) if \( x \in N \) and obtain a well-defined \( C^k \)-map \( f : M \to G \) such that \( \delta_M(f) = \omega \). Thus

\[
\delta_M(C^k(M, G)) = \Omega^1_{C^k-1}(M, g),
\]

which is a submanifold of \( \Omega^1_{C^k-1}(M, g) \). Let \( K \) be the set of all connected, compact full submanifolds \( K \subseteq M \) such that \( x_0 \in K \). By the preceding, \( \delta_K(C^k(K, G)) = \Omega^1_{C^k-1}(K, g) \), which is a submanifold of \( \Omega^1_{C^k-1}(K, g) \). Since

\[
M = \bigcup_{K \in K} K^o,
\]

[16] Theorem 3.5] provides a smooth manifold structure on \( C^k(M, g) \) which makes it a \( C^r \)-regular Lie group, is compatible with evaluations, and turns

\[
\psi : C^k(M, G) \to \Omega^1_{C^k-1}(M, g) \times G, \ f \mapsto (\delta_M(f), f(x_0))
\]

into a \( C^\infty \)-diffeomorphism. It remains to show that the smooth manifold structure is canonical. To prove the latter, we first note that \( K \) is directed under inclusion. In fact, if \( K_1, K_2 \in K \), then \( K_1 \cup K_2 \) is contained in a \( \sigma \)-compact, connected open submanifold \( N \) of \( M \) (a union of chart domains diffeomorphic to convex subsets of \( \mathbb{R} \), around finitely many points in the compact set \( K_1 \cup K_2 \)). Pick a \( C^\infty \)-diffeomorphism \( \psi : I \to N \) as above. Then \( \psi^{-1}(K_1) \) and \( \psi^{-1}(K_2) \)

are compact intervals containing 0, whence so is their union. Thus \( K_1 \cup K_2 \) is a connected, compact full submanifold of \( N \) and hence of \( M \).

For \( K, L \in K \) with \( K \subseteq L \), let \( r_{K,L} : \Omega^1_{C^k-1}(L, g) \to \Omega^1_{C^k-1}(K, g) \) be the restriction map. As a consequence of Lemma \[3.7\] and \[8\],

\[
\Omega^1_{C^k-1}(M, g) = \lim_{\leftarrow K \in K} \Omega^1_{C^k-1}(K, g)
\]

holds as a locally convex space, using the restriction maps \( r_K : \Omega^1_{C^k-1}(M, g) \to \Omega^1_{C^k-1}(K, g) \) as the limit maps. For \( K \in K \), let \( \rho_K : C^k(M, G) \to C^k(K, G) \) be the restriction map; endow \( C^k(K, G) \) with its canonical smooth manifold structure (as in Lemma \[6.7\]), which is compatible with evaluations (the “ordinary” Lie group structure in \[16\]). Then

\[
\psi_K : C^k(K, G) \to \Omega^1_{C^k-1}(K, g) \times G, \ f \mapsto (\delta_K(f), f(x_0))
\]

is a \( C^\infty \)-diffeomorphism (see \[16\] proof of Theorem 3.5]). Note that \( \rho_K = \psi_K^{-1} \circ (r_K \times \text{id}_G) \circ \psi \) is smooth on \( C^k(M, G) \), using the above Lie group structure
making \( \psi \) a \( C^\infty \)-diffeomorphism. Let \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^m \), \( L_1, \ldots, L_m \) be smooth manifolds with rough boundary, \( L := L_1 \times \cdots \times L_m \) and \( f: L \to C^k(M, G) \) be a map. If \( f \) is \( C^\alpha \), then also \( \rho_K \circ f \) is \( C^\alpha \). Since \( C^k(K, G) \) is canonical, the map

\[
f^\wedge|_{L \times K} = (\rho_j \circ f)^\wedge: L \times K \to G
\]

is \( C^{\alpha,k} \). Using (3), we deduce that \( f^\wedge \) is \( C^{\alpha,k} \). If, conversely, \( f^\wedge \) is \( C^{\alpha,k} \), then \( (\rho_K \circ f)^\wedge = f^\wedge|_{L \times K} \) is \( C^{\alpha,k} \). The smooth manifold structure on \( C^k(K, G) \) being canonical, we deduce that \( \rho_K \circ f \) is \( C^\alpha \). The hypotheses of Lemma 2.15 being satisfied with \( A := K, C^k(M, G) \) in place of \( M, M_K := C^k(K, G), F := \Omega^1_{C^k-1}(M, g), F_K := \Omega^1_{C^k-1}(K, g), \) and \( N := G \), we see that \( f \) is \( C^\alpha \). The smooth manifold structure on \( C^k(M, G) \) is therefore pre-canonical. The topology on the projective limit \( \Omega^1_{C^k-1}(M, g) \) is initial with respect to the limit maps \( r_K \), whence the topology on \( \Omega^1_{C^k-1}(M, g) \times G \) is initial with respect to the maps \( r_K \times id_G \). Since \( \psi \) is a homeomorphism, we deduce that the topology \( \mathcal{O} \) on the Lie group \( C^k(M, G) \) is initial with respect to the maps \( (r_K \times id_G) \circ \psi = \psi \circ \rho_K \). Since \( \psi \circ \rho_K \) is a homeomorphism, \( \mathcal{O} \) is initial just as well with respect to the family \( (\rho_K)_{K \in K} \).

But also the compact-open \( C^k \)-topology \( \mathcal{T} \) on \( C^k(M, G) \) is initial with respect to this family of maps (see Lemma 3.7), whence \( \mathcal{O} = \mathcal{T} \) and \( C^k(M, G) \) is canonical.

If \( M \) has finitely many components \( M_1, \ldots, M_n \), we give \( C^k(M, G) \) the smooth manifold structure turning the bijection

\[
\rho: C^k(M, G) \to \prod_{j=1}^n C^k(M_j, G), \quad f \mapsto (f|_{M_j})_{j=1}^n
\]

into a \( C^\infty \)-diffeomorphism. Let \( \rho_j \) be its \( j \)th component. Since \( \rho \) is a homeomorphism for the compact-open \( C^k \)-topologies (cf. Lemma 3.7) and an isomorphism of groups, the preceding smooth manifold structure makes \( C^k(M, G) \) a Lie group and is compatible with the compact-open \( C^k \)-topology. As each of the Lie groups \( C^k(M_j, G) \) is \( C^\alpha \)-regular, also their direct product (and thus \( C^k(M, G) \)) is \( C^\alpha \)-regular. Since \( \rho = (\rho_j)_{j=1}^n \) is an isomorphism of Lie groups,

\[
(L(\rho_1), \ldots, L(\rho_n)): L(C^k(M, G)) \to L(C^k(M_1, G)) \times \cdots \times L(C^k(M_n, G))
\]

is an isomorphism of topological Lie algebras. For \( x \in M_j \), the point evaluation \( \varepsilon_x: C^k(M, G) \to G \) is smooth, as the point evaluation \( \bar{\varepsilon}_x: C^k(M_j, G) \to G \) is smooth and \( \varepsilon_x = \bar{\varepsilon}_x \circ \rho_j \). We know that \( \phi_j(v) := (L(\bar{\varepsilon}_x(v))x \in M_j \in C^k(M_j, g) \) for all \( v \in L(C^k(M_j, G)) \) and that \( \phi_j: L(C^k(M_j, G)) \to C^k(M_j, g) \) is an isomorphism of topological Lie algebras. For each \( v \in L(C^k(M, G)) \), we have

\[
(L(\varepsilon_x(v))x \in M_j) = (L(\bar{\varepsilon}_x(L(\rho_j)(v))))x \in M_j = \phi_j(L(\rho_j)(v)) \in C^k(M_j, g)
\]

for \( j \in \{1, \ldots, n\} \), whence \( \phi(v) := (L(\varepsilon_x(v))x \in M) \in C^k(M, g) \). Let us show that the Lie algebra homomorphism \( \phi: L(C^k(M, G)) \to C^k(M, g) \) is a homomorphism. Lemma 3.7 entails that the map

\[
r = (r_j)_{j=1}^n: C^k(M, g) \to \prod_{j=1}^n C^k(M_j, g), \quad f \mapsto (f|_{M_j})_{j=1}^n
\]

is
is a homeomorphism. By the preceding, \( r \circ \phi = (\phi_1 \times \cdots \times \phi_n) \circ (L(\rho_j))^{j=1}_n \) is a homeomorphism, whence so is \( \phi \). Thus, the Lie group structure on \( C^k(M,G) \) is compatible with evaluations. If \( \alpha, L = L_1 \times \cdots \times L_m \) and \( f : L \to C^k(M,G) \) are as above and \( f \) is \( C^\alpha \), then \( f^\wedge \) is \( C^{\alpha,k} \) by the above argument. If, conversely, \( f^\wedge \) is \( C^{\alpha,k} \), then \( f^\wedge|_{L \times M_j} \) is \( C^{\alpha,k} \), whence \( (f^\wedge|_{L \times M_j})^\vee = \rho_j \circ f \) is \( C^\alpha \) for all \( j \in \{1, \ldots, n\} \). As a consequence, \( \rho \circ f \) is \( C^\alpha \) and thus also \( f \). We have shown that the smooth manifold structure on \( C^k(M,G) \) is pre-canonical and hence canonical, as compatibility with the compact-open \( C^k \)-topology was already established.

\[ \square \]

Another lemma is useful.

**Lemma 6.9** Let \( N_1, \ldots, N_m \) and \( M_1, \ldots, M_n \) be locally compact smooth manifolds with rough boundary, \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^m \), \( \beta \in (\mathbb{N}_0 \cup \{\infty\})^n \), and \( G \) be a Lie group. Abbreviate \( N := N_1 \times \cdots \times N_m \) and \( M := M_1 \times \cdots \times M_n \). Assume that \( C^\beta(M,G) \) has a pre-canonical smooth manifold structure, using which \( C^\alpha(N,C^\beta(M,G)) \) has a canonical smooth manifold structure. Endow \( C^{\alpha,\beta}(N \times M,G) \) with the pre-canonical smooth manifold structure turning

\[
\Phi : C^{\alpha,\beta}(N \times M,G) \to C^\alpha(N,C^\beta(M,G)), \ f \mapsto f^\vee
\]

into a \( C^\infty \)-diffeomorphism. Assume that there exists a family \((K_i)_{i \in I}\) of compact full submanifolds \( K_i \) of \( N \) whose interiors cover \( N \), with the following properties:

(a) For each \( i \in I \), we have \( K_i = K_{i,1} \times \cdots \times K_{i,m} \) with certain compact full submanifolds \( K_{i,\ell} \subseteq N_i \); and

(b) \( C^\beta(M,C^\alpha(K_i,G)) \) admits a canonical smooth manifold structure for each \( i \in I \), using the canonical smooth manifold structure on \( C^\alpha(K_i,G) \) provided by Theorem 1.1.

Then the pre-canonical manifold structure on \( C^{\alpha,\beta}(N \times M,G) \) is canonical.

**Proof.** Let \( \mathcal{O} \) be the topology on \( C^{\alpha,\beta}(N \times M,G) \), equipped with its pre-canonical smooth manifold structure. Using Theorem 1.1 for \( i \in I \) we endow \( C^\alpha(K_i,C^\beta(M,G)) \) with a canonical smooth manifold structure; the underlying topology is the compact-open \( C^\alpha \)-topology. The given smooth manifold structure on \( C^\alpha(N,C^\beta(M,G)) \) being canonical, its underlying topology is the compact-open \( C^\alpha \)-topology, which is initial with respect to the restriction maps

\[
\rho_i : C^\alpha(N,C^\beta(M,G)) \to C^\alpha(K_i,C^\beta(M,G))
\]

for \( i \in I \). We have bijections

\[
C^\alpha(K_i,C^\beta(M,G)) \cong C^{\alpha,\beta}(K_i \times M,G) \cong C^{\beta,\alpha}(M \times K_i,G) \cong C^\beta(M,C^\alpha(K_i,G))
\]

using in turn the Exponential Law (in the form 1), a flip in the factors (cf. Lemma 1.12(a)), and again the Exponential Law. If, step by step, we transport the smooth manifold structure from the left to the right, we obtain a pre-canonical smooth manifold structure in each step (see Lemmas 1.11(c) and
As pre-canonical structures are unique, the pre-canonical structure obtained on \(C^\beta(M, C^\alpha(K_i, G))\) must coincide with the canonical structure which exists by hypothesis. Hence, using this canonical structure, the map

\[
\Psi_i : C^\alpha(K_i, C^\beta(M, G)) \to C^\beta(M, C^\alpha(K_i, G))
\]
determined by \(\Psi(f)(y)(x) = f(x)(y)\) is a \(C^\infty\)-diffeomorphism. Let \(L_k\) be the set of compact full submanifolds of \(M_k\) for \(k \in \{1, \ldots, n\}\). Write \(L_1 \times \cdots \times L_n =: J\). If \(j \in J\), then \(j = (L_{j,1}, \ldots, L_{j,n})\) with certain compact full submanifolds \(L_{j,k} \subseteq M_k\); we define \(L_j \coloneqq L_{j,1} \times \cdots \times L_{j,n}\). By Lemma 4.7, the topology on \(C^\beta(M, C^\alpha(K_i, G))\) is initial with respect to the restriction maps

\[
r_{i,j} : C^\beta(M, C^\alpha(K_i, G)) \to C^\beta(L_j, C^\alpha(K_i, G)),
\]
using the compact-open \(C^\alpha\)-topology on the range which underlies the canonical smooth manifold structure given by Theorem 1.1. Let \(\Theta_{i,j}\) be the composition of the bijections

\[
C^\beta(L_j, C^\alpha(K_i, G)) \to C^\beta(L_j \times K_i, G) \to C^{\alpha, \beta}(K_i \times L_j, G);
\]
thus \(\Theta_{i,j}(f)(x, y) = f(y)(x)\). As each of the domains and ranges admits a canonical smooth manifold structure (by Theorem 1.1), all of the maps have to be homeomorphisms (see Proposition 4.13 and Lemma 4.12(b)). Thus \(\Theta_{i,j}\) is a homeomorphism. By transitivity of initial topologies, \(O\) is initial with respect to the mappings

\[
\rho_{i,j} := \Theta_{i,j} \circ r_{i,j} \circ \Psi_i \circ \rho_i \circ \Phi \text{ for } i \in I \text{ and } j \in J,
\]
which are the restriction maps \(C^{\alpha, \beta}(N \times M, G) \to C^{\alpha, \beta}(K_i \times L_j, G)\). Also the compact-open \(C^{\alpha, \beta}\)-topology on \(C^{\alpha, \beta}(N \times M, G)\) is initial with respect to the maps \(\rho_{i,j}\), and hence coincides with \(O\). The given pre-canonical smooth manifold structure on \(C^{\alpha, \beta}(N \times M, G)\) therefore is canonical.

**Lemma 6.10** Let \(M_1, \ldots, M_n\) be locally compact, smooth manifold with rough boundary, \(M := M_1 \times \cdots \times M_n\) \(\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n\), and \(G\) be a Lie group. Assume that the group \(C^\alpha(M, G)\) is endowed with a smooth manifold structure which makes it a Lie group and is compatible with evaluations. Let \(\sigma\) be a permutation of \(\{1, \ldots, n\}\) and \(Q := M_{\sigma(1)} \times \cdots \times M_{\sigma(n)}\). Consider \(\phi_{\sigma} : M \to Q, x \mapsto x \circ \sigma\). Then the smooth manifold (and Lie group) structure on the group \(C^{\alpha_{\sigma}}(Q, G)\) making the bijective group homomorphism

\[
(\phi_{\sigma})^* : C^{\alpha_{\sigma}}(Q, G) \to C^\alpha(M, G), f \mapsto f \circ \phi_{\sigma}
\]
a \(C^\infty\)-diffeomorphism is compatible with evaluations.

**Proof.** The map \(\psi : C^{\alpha_{\sigma}}(Q, L(G)) \to C^\alpha(M, L(G)), f \mapsto f \circ \phi_{\sigma}\) is an isomorphism of topological vector spaces, by Example 4.1 and Lemma 4.12(b). Write \(\varepsilon_{y} : C^{\alpha_{\sigma}}(Q, G) \to G\) for the point evaluation at \(y \in Q\) and \(\varepsilon_{x} : C^\alpha(M, G) \to G\) for the point evaluation at \(x \in M\). The action of \(\phi_{\sigma}\) on \(G\) is compatible with the group structure on \(G\). Therefore, the map \(\psi\) is a \(C^\infty\)-diffeomorphism.
$G$ for the point evaluation at $x \in M$. For $v \in L(C^{\alpha}(M,G))$, let $\phi(v) := (L(\varepsilon_x(v)))_{x \in M}$. Then $\varepsilon_x \circ (\phi_\sigma)^* = \bar{\varepsilon}_{\phi_\sigma(x)}$. As a consequence,

$$\phi(v) := (L(\bar{\varepsilon}_y)(v))_{y \in Q} = (\psi^{-1} \circ \phi \circ L((\phi_\sigma)^*))(v) \in C^\alpha \sigma(Q,L(G))$$

for all $v \in L(C^\alpha \sigma(Q,G))$. Moreover, $\bar{\phi} = (\psi^{-1})^* \circ \phi \circ L((\phi_\sigma)^*)$ is an isomorphism of topological vector spaces, being a composition of such.

**Proof of Theorem 1.3 and Proposition 6.6.** Step 1. We first assume that $M_j$ is 1-dimensional with finitely many components for all $j \in \{1, \ldots, n\}$, and prove the assertions by induction on $n$. The case $n = 1$ was treated in Lemma 6.7. We may therefore assume that $n \geq 2$ and assume that the conclusions hold for $n - 1$ factors. We abbreviate $k := \alpha_1, \beta := (\alpha_2, \ldots, \alpha_n)$, and $L := M_2 \times \cdots \times M_n$. By the inductive hypothesis, $C^\beta(L,G)$ admits a canonical smooth manifold structure which makes it a $C^r$-regular Lie group and is compatible with evaluations. By the induction base, $C^k(M_1, C^\beta(L,G))$ admits a canonical smooth manifold structure making it a $C^r$-regular Lie group. Since $C^\beta(L,G)$ is canonical, the group homomorphism

$$\Phi: C^{k,\beta}(M_1 \times L,G) \to C^k(M_1, C^\beta(L,G)), \quad f \mapsto f^\psi$$

is a bijection (see (1.13)). We endow $C^{\alpha}(M,G) = C^{k,\beta}(M_1 \times L,G)$

with the smooth manifold structure turning $\Phi$ into a $C^\infty$-diffeomorphism. By Lemma 6.4 this structure is pre-canonical, makes $C^{\alpha}(M,G)$ Lie group, and is compatible with evaluations. The Lie group $C^{\alpha}(M,G)$ is $C^r$-regular, as $\Phi$ is an isomorphism of Lie groups. Let $C_1, \ldots, C_\ell$ be the connected components of $M_1$. Let $K$ be the set of compact, full submanifolds $K$ of $M_1$. Then the interiors $K^0$ cover $M_1$ (as the interiors of connected, compact full submanifolds cover each connected component of $M_1$, by the proof of Lemma 6.8). Now $C^k(K,G)$ admits a canonical smooth manifold structure making it a $C^r$-regular Lie group, by Lemma 6.4. Thus $C^\beta(L,C^k(K,G))$ admits a canonical smooth manifold structure, by the inductive hypothesis. By Lemma 6.9 the pre-canonical smooth manifold structure on $C^{\alpha}(M,G)$ is canonical.

Step 2 (the general case). Let $M_1, \ldots, M_n$ be arbitrary. Using Lemma 4.12(a), we may re-order the factors and assume that there exists an $m \in \{0, \ldots, n\}$ such that $M_j$ is compact for all $j \in \{1, \ldots, n\}$ with $j \leq m$, while $M_j$ is 1-dimensional with finitely many components for all $j \in \{1, \ldots, n\}$ such that $j > m$. If $m = 0$, we have the special case just settled. If $m = n$, then all conclusions hold by Lemma 6.7. We may therefore assume that $1 \leq m < n$. We abbreviate $K := M_1 \times \cdots \times M_m$ and $N := M_{m+1} \times \cdots \times M_n$. Let $\gamma := (\alpha_1, \ldots, \alpha_m)$ and $\beta := (\alpha_{m+1}, \ldots, \alpha_n)$. By Step 1, $C^\beta(N,G)$ admits a canonical smooth manifold structure which makes it a $C^r$-regular Lie group and is compatible with evaluations. By Lemma 6.7 $C^\gamma(K,C^\beta(N,G))$ admits a canonical smooth manifold structure which makes it a $C^r$-regular Lie group and is compatible
with evaluations. We give \( C^\alpha(M,G) = C^{\gamma,\beta}(K \times N,G) \) the smooth manifold structure making the bijection

\[
\Phi: C^{\gamma,\beta}(K \times N,G) \to C^\alpha(K,C^\beta(N,G)), \quad f \mapsto f^\gamma
\]
a \( C^\infty \)-diffeomorphism. By Lemma 6.4, this smooth manifold structure is pre-canonical, makes \( C^\alpha(M,G) \) a Lie group, and is compatible with evaluations. The Lie group \( C^\alpha(M,G) \) is \( C^\ast \)-regular as \( \Phi \) is an isomorphism of Lie groups. Now \( C^\gamma(K,G) \) admits a canonical smooth manifold structure, which makes it a \( C^\gamma \)-regular Lie group (Lemma 6.7). By Step 1, \( C^\beta(N,C^\gamma(K,G)) \) admits a canonical smooth manifold structure. The pre-canonical smooth manifold structure on \( C^\alpha(M,G) \) is therefore canonical, by Lemma 6.9. □

The following result complements Theorem 1.3. Under a restrictive hypothesis, it provides a Lie group structure without recourse to regularity.

**Proposition 6.11** Let \( M_1, \ldots, M_n \) be locally compact smooth manifolds with rough boundary, \( \alpha \in (\mathbb{N}_0 \cup \{ \infty \})^k \) and \( G \) be a Lie group that is \( C^\infty \)-diffeomorphic to a locally convex space \( E \). Abbreviate \( M := M_1 \times \cdots \times M_n \). Then \( C^\alpha(M,G) \) admits a canonical \( C^\infty \)-manifold structure, which is compatible with evaluations. If \( G \) is \( C^\ast \)-regular for some \( r \in \mathbb{N}_0 \cup \{ \infty \} \), then also \( C^\alpha(M,G) \) is \( C^r \)-regular.

**Proof.** By Example 4.1 \( H := C^\alpha(M,G) \) admits a canonical smooth manifold structure and this structure makes it a Lie group (see Lemma 6.1). Let \( \psi: G \to E \) be a \( C^\infty \)-diffeomorphism such that \( \psi(e) = 0 \). Abbreviating \( g := L(G) \) and \( h := L(H) \), the map \( \alpha := d\psi|_g: g \to E \) is an isomorphism of topological vector spaces. Then also \( \phi := \alpha^{-1} \circ \psi: G \to E \) is a \( C^\infty \)-diffeomorphism such that \( \phi(e) = 0 \); moreover, \( d\phi|_g = \text{id}_g \). Now

\[
\phi_*: C^\alpha(M,G) \to C^\alpha(M,\mathfrak{g}), \quad f \mapsto \phi \circ f
\]
is a \( C^\infty \)-diffeomorphism, and thus \( \beta := d(\phi_*)|_h: h \to C^\alpha(M,\mathfrak{g}) \) is an isomorphism of topological vector spaces. For \( x \in M \), let \( \varepsilon_x: H \to G \) and \( e_x: C^\alpha(M,\mathfrak{g}) \to \mathfrak{g} \) be the respective point evaluation at \( x \). We show that \( \beta(v) = (L(\varepsilon_x)(v))_{x \in M} \) for each \( v \in \mathfrak{h} \), whence the Lie group structure on \( H \) is compatible with evaluations. Regard \( v = [\gamma] \) as a geometric tangent vector. As \( L(\varepsilon_x)(v) \in \mathfrak{g} \), we have

\[
L(\varepsilon_x)(v) = d\phi(L(\varepsilon_x)(v)) = d(\phi \circ \varepsilon_x)(v) = d\bigg|_{t=0} (\phi \circ \varepsilon_x \circ \gamma)(t) = \frac{d}{dt}\bigg|_{t=0} (\phi_\ast \circ \gamma)(t) = d(\phi_\ast)(v)(x),
\]

since \( (\phi \circ \varepsilon_x \circ \gamma)(t) = \phi(\gamma(t)(x)) \). For the final assertion, see Lemma 6.5. □

7 Manifolds of maps with finer topologies

We now turn to manifold structures on \( C^\alpha(M,N) \) for non-compact \( M \), which are modeled on suitable spaces of compactly supported \( C^\alpha \)-functions. Notably,
a proof for Theorem\textsuperscript{14} will be provided. Such manifold structures need not be compatible with the compact-open $C^\infty$-topology, and need not be pre-canonical. But we can essentially reduce their structure to the case of canonical structures for compact domains, using box products of manifolds as a tool. We recall pertinent concepts from\textsuperscript{14}.

7.1 If $I$ is a non-empty set and $(M_i)_{i \in I}$ a family of $C^\infty$-manifolds modeled on locally convex spaces, then the fine box topology $\mathcal{O}_R$ on the cartesian product $P := \prod_{i \in I} M_i$ is defined as the final topology with respect to the mappings

$$\Theta_\phi : \bigoplus_{i \in I} V_i := \left( \bigoplus_{i \in I} E_i \right) \cap \prod_{i \in I} V_i \to P, \quad (x_i)_{i \in I} \mapsto (\phi_i^{-1}(x_i))_{i \in I}, \quad (9)$$

for $\phi := (\phi_i)_{i \in I}$ ranging through the families of charts $\phi_i : U_i \to V_i \subseteq E_i$ of $M_i$ such that $0 \in V_i$; here $E_\phi := \bigoplus_{i \in I} E_i$ is endowed with the locally convex direct sum topology, and the left-hand side $V_\phi$ of (9), which is an open subset of $E_\phi$, is endowed with the topology induced by $E_\phi$. Let $U_\phi := \Theta_\phi(V_\phi)$. Thus

$$U_\phi = \left\{ (y_i)_{i \in I} \in \prod_{i \in I} U_i : y_i \neq \phi_i^{-1}(0) \text{ for only finitely many } i \in I \right\}.$$ 

Note that the projection $p_i : P \to M_i$ is continuous for each $i \in I$, entailing that the fine box topology is Hausdorff. In fact, using the continuous linear projection $\pi_i : E_\phi \to E_i$ onto the $i$th component, we deduce from the continuity of $p_i \circ \Theta_\phi = \phi_i^{-1} \circ \pi_i|_{V_\phi}$ for each $\phi$ that $p_i$ is continuous.

7.2 Let $\phi$ be as before and $\psi$ be an analogous family of charts $\psi_i : R_i \to S_i \subseteq F_i$. If $\phi_i^{-1}(0) = \psi_i^{-1}(0)$ for all but finitely many $i \in I$, then

$$(\Theta_\phi)^{-1}(U_\phi \cap U_\psi) = \bigoplus_{i \in I} \phi_i(U_i \cap R_i),$$

which is an open 0-neighbourhood in $\bigoplus_{i \in I} E_i$. The transition map

$$(\Theta_\phi)^{-1} \circ \Theta_\psi : \bigoplus_{i \in I} \psi_i(U_i \cap R_i) \to \bigoplus_{i \in I} \phi_i(U_i \cap R_i), \quad (x_i)_{i \in I} \mapsto ((\phi_i \circ \psi_i^{-1})(x_i))_{i \in I}$$

is $C^\infty$ (as follows from\textsuperscript{11} Proposition 7.1) and in fact a $C^\infty$-diffeomorphism, and hence a homeomorphism, since $\Theta_\psi^{-1} \circ \Theta_\phi$ is the inverse map. If $\phi_i^{-1}(0) \neq \psi_i^{-1}(0)$ for infinitely many $i \in I$, then $(\Theta_\phi)^{-1}(U_\phi \cap U_\psi) = \emptyset$ and the transition map trivially is a homeomorphism. Using a standard argument, we now deduce that $U_\phi = \Theta_\phi(V_\phi)$ is open in $(P, \mathcal{O}_R)$ for all $\phi$ and $\Theta_\phi$ is a homeomorphism onto its image (see, e.g.,\textsuperscript{15} Exercise A.3.1)). By the preceding, the maps $\Phi_\phi := (\Theta_\phi|_{U_\phi})^{-1} : U_\phi \to V_\phi \subseteq E_\phi$ are smoothly compatible and hence form an atlas for a $C^\infty$-manifold structure on $P$. Following\textsuperscript{14}, we write $P^b$ for $P$, endowed with the topology $\mathcal{O}_R$, and the smooth manifold structure just described, and call $P^b$ the fine box product.

Some auxiliary results are needed. We use notation as in\textsuperscript{5,8} and Theorem\textsuperscript{14}.
Lemma 7.3  Let $M := M_1 \times \cdots \times M_n$ be a product of locally compact smooth manifolds with rough boundary, $N$ be a smooth manifold, $\alpha \in (\mathbb{N}_0 \cup \{\infty\})^n$ and $f \in C^\alpha(M,N)$.

(a) If $M_1, \ldots, M_n$ are compact, then the following bilinear map is continuous:

$$C^\alpha(M,\mathbb{R}) \times \Gamma_f \to \Gamma_f, \ (h, \tau) \mapsto h\tau \text{ with } (h\tau)(x) = h(x)\tau(x).$$

(b) If $M_1, \ldots, M_n$ are paracompact, $L \subseteq M$ is a compact subset and $K := K_1 \times \cdots \times K_n$ with compact full submanifolds $K_j \subseteq M_j$ for $j \in \{1, \ldots, n\}$, then the linear map $\Gamma_f \times L \to \Gamma_{f|K}, \ \tau \mapsto \tau|_K$ is continuous.

(c) If $M_1, \ldots, M_n$ are paracompact, $K := K_1 \times \cdots \times K_n$ with compact full submanifolds $K_j \subseteq M_j$ for $j \in \{1, \ldots, n\}$ and $L \subseteq K$ be compact. Then

$$r: \Gamma_f \times L \to \Gamma_{f|K}, L, \tau \mapsto \tau|_K$$

is an isomorphism of topological vector spaces.

Proof.  (a) The bilinear map is a restriction of the continuous mapping $\mu: C^\alpha(M,\mathbb{R}) \times C^\alpha(M,TN) \to C^\alpha(K,TN)$ from Lemma 3.12.

(b) The map is a restriction of the restriction map $C^\alpha(M,TN) \to C^\alpha(K,TN)$, which is continuous (see Remark 3.6).

(c) For each $x$ in the open subset $M \setminus K$ of $M$, there exist compact full submanifolds $K_{x,j} \subseteq M_j$ for $j \in \{1, \ldots, n\}$ such that $K_x := K_{x,1} \times \cdots \times K_{x,n} \subseteq M \setminus K$ and $x \in K^o_x$. Lemma 3.7 implies that the compact-open $C^\alpha$-topology on $\Gamma_f \times L$ is initial with respect to the restriction maps $\rho: \Gamma_f \times L \to C^\alpha(K,TN)$ and $\rho_x : \Gamma_f \times L \to C^\alpha(K_x,TN)$ for $x \in M \setminus K$. As each $\rho_x$ is constant (its value is the function $K_x \ni y \mapsto 0_{f(y)} \in T_{f(y)}N$), it can be omitted without affecting the initial topology. The topology on $\Gamma_{f,K}$ is therefore initial with respect to $\rho$, and hence also with the co-restriction $r$ of $\rho$. Thus $r$ is a topological embedding and hence an homeomorphism, as $r(\tau) = \sigma$ can be achieved for $\sigma \in \Gamma_{f,K}$. If we define $r: M \to TN$ piecewise via $r(x) := \sigma(x)$ if $x \in K$, $\tau(x) := \rho(y) \in T_{f(y)}N$ if $x \in M \setminus L$. Being linear, $r$ is an isomorphism of topological vector spaces. \(\square\)

Proof of Theorem 4.3. For $j \in \{1, \ldots, n\}$, let $(K_{j,i})_{i \in I_j}$ be a locally finite family of compact, full submanifolds $K_{j,i}$ of $M_j$ whose interiors cover $M_j$. Let $I := I_1 \times \cdots \times I_n$. Then the sets $K_i := K_{1,i} \times \cdots \times K_{n,i}$ form a locally finite family of compact full submanifolds of $M$ whose interiors cover $M$, for $i = (i_1, \ldots, i_n) \in I$. The map

$$\rho : C^\alpha(M,N) \to \prod_{i \in I} C^\alpha(K_i,N), \ f \mapsto (f|_{K_i})_{i \in I}$$

is injective with image

$$\text{im}(\rho) = \left\{ (f_i)_{i \in I} \in \prod_{i \in I} C^\alpha(K_i,N) : (\forall i,j \in I) (\forall x \in K_i \cap K_j) f_i(x) = f_j(x) \right\}. \ (10)$$

In fact, the inclusion “$\subseteq$” is obvious. If $(f_i)_{i \in I}$ is in the set on the right-hand side, then a piecewise definition, $f(x) := f_i(x)$ if $x \in K_i$, gives a well-defined
function \( f : M \to N \) which is \( C^\alpha \) since \( f|_{(K_i)^o} = f_i|_{(K_i)^o} \) is \( C^\alpha \) for each \( i \in I \). Then \( \rho(f) = (f_i)_{i \in I} \).

For each \( i \in I \), endow \( C^\alpha(K_i, N) \) with the canonical smooth manifold structure, as in Theorem [10] modeled on the set \( \{ \Gamma_f : f \in C^\alpha(K_i, N) \} \) of the locally convex spaces \( \Gamma_f := \{ \tau \in C^\alpha(K_i, TN) : \pi_{TN} \circ \tau = f \} \) for \( f \in C^\alpha(K_i, N) \).

Let \( \Sigma : TN \cong U \to N \) be a local addition for \( N \); as in Section 3 write \( U' := \{(\pi_{TN}(v), \Sigma(v)) : v \in U \} \) and \( \theta := (\pi_{TN}|_U, \Sigma) : U \to U' \). For \( f \in C^\alpha(K_i, N) \), consider \( O_f := \Gamma_f \cap C^\alpha(K_i, U) \), \( O'_f := \{ g \in C^\alpha(K_i, N) : (f, g) \in C^\alpha(K_i, U') \} \), and \( \phi_f : O_f \to O'_f, \tau \mapsto \Sigma \circ \tau \) as in Section 5. For \( f \in C^\alpha(M, N) \), let \( \Gamma_f \) be the set of all \( \tau \in C^\alpha(M, TN) \) such that \( \pi_{TN} \circ \tau = f \) and

\[
\{ x \in M : \tau(x) \neq 0, f(x) \in T_f(x) N \}
\]

is relatively compact in \( M \). Define \( O_f := \Gamma_f \cap C^\alpha(M, U) \) and let \( O'_f \) be the set of all \( g \in C^\alpha(M, N) \) such that

\[
(f, g) \in C^\alpha(M, U') \quad \text{and} \quad g|_{M \setminus K} = f|_{M \setminus K}
\]

for some compact subset \( K \subseteq M \).

Then \( \phi_f : O_f \to O'_f, \tau \mapsto \Sigma \circ \tau \) is a bijection with \( (\phi_f)^{-1}(g) = \theta^{-1} \circ (f, g) \). The linear map

\[
s : \Gamma_f \to \bigoplus_{i \in I} \Gamma_{f|_{K_i}} : \tau \mapsto (\tau|_{K_i})_{i \in I}
\]

is continuous on \( \Gamma_{f,L} \) for each compact subset \( L \subseteq M \) (see Lemma [7.3(b)] and hence continuous on the locally convex direct limit \( \Gamma_f \). As above, we see that

\[
\text{im}(s) = \{(\tau_i)_{i \in I} \in \bigoplus_{i \in I} \Gamma_{f|_{K_i}} : (\forall i, j \in I)(\forall x \in K_i \cap K_j) \tau_i(x) = \tau_j(x)\},
\]

which is a closed vector subspace of \( \bigoplus_{i \in I} \Gamma_{f|_{K_i}} \). We now show that \( s \) is a homeomorphism onto its image. In fact, \( s \) admits a continuous linear left inverse. To see this, pick a \( C^\infty \)-partition of unity \( (h_i)_{i \in I} \) on \( M \) subordinate to \( (K_i^o)_{i \in I} \); then \( L_i := \text{supp}(h_i) \) is a closed subset of \( K_i \) and thus compact. The multiplication operator \( \beta_i : \Gamma_{f|_{K_i}} \to \Gamma_{f|_{K_i,L_i}}, \tau \mapsto h_i \tau \) is continuous linear (by Lemma [7.3(a)]). Moreover, the restriction operator \( s_i : \Gamma_{f,L_i} \to \Gamma_{f|_{K_i},L_i} \) is an isomorphism of topological vector spaces (Lemma [7.3(c)]). Thus \( s_i^{-1} \circ \beta_i : \Gamma_{f|_{K_i}} \to \Gamma_{f,L_i} \subseteq \Gamma_f \) is a continuous linear map. By the universal property of the locally convex direct sum, also the linear map

\[
\sigma : \bigoplus_{i \in I} \Gamma_{f|_{K_i}} \to \Gamma_f, (\tau_i)_{i \in I} \mapsto \sum_{i \in I} (s_i^{-1} \circ \beta_i)(\tau_i)
\]

is continuous. We easily verify that \( \sigma \circ s = \text{id}_{\Gamma_f} \).

Abbreviate \( \phi_i := (\phi_{f|_{K_i}})^{-1} \) and \( \phi := (\phi_i)_{i \in I} \). We now use the \( C^\infty \)-diffeomorphism

\[
\Theta_\phi : \bigoplus_{i \in I} O_{f|_{K_i}} \to U_\phi, (\tau_i)_{i \in I} \mapsto (\phi_i^{-1}(\tau_i))_{i \in I} = (\Sigma \circ \tau_i)_{i \in I}
\]

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from \( \mathcal{L} \) the inverse of which is the chart

\[
\Phi_\phi : U_\phi \rightarrow \bigoplus_{i \in I} O_{f|\kappa_i}, \quad (g_i)_{i \in I} \mapsto (\phi_i(g_i))_{i \in I}
\]

of \( \prod_{i \in I} c^\alpha(K, N) \) around \((f|\kappa_i)_{i \in I}\). For \((\tau_i)_{i \in I} \in \bigoplus_{i \in I} O_{f|\kappa_i}\), we have

\[
\Theta_\phi((\tau_i)_{i \in I}) \in \text{im}(\rho) \iff (\tau_i)_{i \in I} \in \text{im}(s).
\]

In fact, for \( i, j \in I \) and \( x \in K_i \cap K_j \) we have \( \Sigma(\tau_i(x)) = \Sigma(\tau_j(x)) \) if and only if \( \tau_i(x) = \tau_j(x) \), from which the assertion follows in view of (10) and (11). Thus

\[
\Phi_\phi(\text{im}(\rho) \cap U_\phi) = \text{im}(s) \cap \bigoplus_{i \in I} O_{f|\kappa_i},
\]

showing that \( \text{im}(\rho) \) is a submanifold of \( \prod_{i \in I} c^\alpha(K, N) \). Let

\[
\Psi_\phi : \text{im}(\rho) \cap U_\phi \rightarrow \text{im}(s) \cap \bigoplus_{i \in I} O_{f|\kappa_i}, \quad (g_i)_{i \in I} \mapsto \Phi_\phi((g_i)_{i \in I})
\]

be the corresponding submanifold chart for \( \text{im}(\rho) \). Then

\[
\rho(O_f') = \text{im}(\rho) \cap U_\phi \quad \text{and} \quad s(O_f) = \text{im}(s) \cap \bigoplus_{i \in I} O_{f|\kappa_i}.
\]

Hence \( (\phi_f)^{-1} = s^{-1} \circ \Psi_\phi \circ \rho|_{O_f'} : O_f' \rightarrow O_f \) is a chart for the smooth manifold structure on \( c^\alpha(M, N) \) modeled on \( \mathcal{E} \) (the set of all \( \Gamma_f \)) which makes \( \rho : c^\alpha(M, N) \rightarrow \text{im}(\rho) \) a \( C^\infty \)-diffeomorphism. Note that the smooth manifold structure on \( c^\alpha(M, N) \) which is modeled on \( \mathcal{E} \) and makes \( \rho \) a \( C^\infty \)-diffeomorphism is uniquely determined by these properties. Thus, it is independent of the choice of \( \Sigma \). On the other hand, the \( (\phi_f)^{-1} \) form a \( C^\infty \)-atlas for a given local addition \( \Sigma \). As the definition of the \( \phi_f \) does not involve the cover \((K_i)_{i \in I}\), the smooth manifold structure just constructed is independent of the choice of \((K_i)_{i \in I}\). \( \square \)

### A Details for Sections 2 and 3

In this appendix, we provide proofs for preliminaries in Sections 2 and 3.

**Proof of Lemma 2.8.** The right-hand side \((t, y) \mapsto y: \gamma(t) \) of the differential equation \( \dot{y}(t) = y(t), \gamma(t) \) is \( C^k \), whence its solution \( \eta \) will be \( C^{k+1} \), if it exists.

To verify existence and uniqueness of \( \eta \), we may assume that \( I \) is a non-degenerate compact interval with initial point 0 or endpoint 0, since \( I \) is covered by such intervals. Thus, let \( I \) be a line segment joining 0 and \( \tau \neq 0 \). Define \( \xi : [0, 1] \rightarrow G \) via \( \xi(t) := \tau \gamma(\tau t) \). By the Chain Rule, a \( C^1 \)-function \( \eta : I \rightarrow G \) with \( \eta(0) = \varepsilon \) satisfies \( \delta^k \eta = \gamma \) if and only if \( \theta : [0, 1] \rightarrow G, \ t \mapsto \eta(\tau t) \) satisfies \( \delta^k \theta = \xi \). The assertion now follows from the case \( I = [0, 1] \), which holds by \( C^1 \)-semiregularity. \( \square \)
Proof for Lemma 2.14 (a) Let $\lambda: Y \to F$ be the inclusion map, which is continuous linear and thus smooth. If $f|Y$ is $C^\alpha$, then also $f = \lambda \circ f|Y$ is $C^\alpha$, by the Chain Rule [1, Lemma 3.16]. Conversely, assume that $f$ is $C^\alpha$ and $f(U) \subseteq Y$. It suffices to deduce that $f|Y$ is $C^\alpha$ if $\alpha \in (N_0)^n$. The proof is by induction on $|\alpha|$, and establishes in parallel that $d^\beta(f|Y) = (d^\beta f)|Y$ for all $\beta \leq \alpha$. If $|\alpha| = 0$, the conclusion holds since $f|Y$ is continuous. If $|\alpha| \geq 1$, let $j \in \{1, \ldots, n\}$ be minimal with $\alpha_j > 0$. Then $d^\beta(f|Y)$ exists for all $\beta \leq \alpha$ such that $\beta_j \leq \alpha_j - 1$, and equals $(d^\beta f)|Y$. If $\beta \leq \alpha$ with $\beta_j = \alpha_j$, let $x \in U^\alpha$ and $y_i \in E^{\beta_i}$ for $i \in \{j, \ldots, n\}$. Then all difference quotients needed to define $d^\beta f(x, 0, \ldots, 0, y_j, y_{j+1}, \ldots, y_n)$

are linear combinations of function values of $d^{\beta - e_j} f$ and hence in $Y$. Since $Y$ is closed, the limit $d^\beta f(x, 0, \ldots, 0, y_j, y_{j+1}, \ldots, y_n)$ is in $Y$ as well, and this remains valid for $x \in U$, by density of $U^\alpha$ in $U$. Thus $(d^\beta f)|Y$ is a continuous function which extends $d^\beta f|U^\alpha$. We deduce that $f|Y$ is $C^\alpha$ and $d^\beta f|Y = (d^\beta f)|Y$.

(b) If $f$ is $C^\alpha$, then also $\lambda_a \circ f$, using that $\lambda_a$ is continuous linear and thus smooth. Conversely, assume that $\lambda_a \circ f$ is $C^\alpha$ for all $a \in A$. Then $Y := \{(x_a)_{a \in A} \in \prod_{a \in A} F_a : (\forall a \leq b) x_a = \lambda_{a,b}(x_b)\}$

is a closed vector subspace of $\prod_{a \in A} F_a$ and the map $\lambda: F \to Y, \ x \mapsto (\lambda_a(x))_{a \in A}$

is an isomorphism of topological vector spaces. Let $\rho_a: Y \to F_a$ be the projection onto the $a$th component. Since $\rho_a \circ \lambda \circ f = \lambda_a \circ f$ is $C^\alpha$ for all $a \in A$, the map $\lambda \circ f$ is $C^\alpha$ to $\prod_{a \in A} F_a$ by [1, Lemma 3.8]. By (a), $\lambda \circ f$ is $C^\alpha$ also as a map to $Y$. Thus $f = \lambda^{-1} \circ (\lambda \circ f)$ is $C^\alpha$. □

Proof of Lemma 2.15. If $f$ is $C^\alpha$, then $\rho_a \circ f$ is $C^\alpha$ for each $a \in A$, the map $\rho_a$ being smooth. Assume that, conversely, $\rho_a \circ f$ is smooth for each $a \in A$. Write $\psi = (\psi_1, \psi_2)$ with $\psi_1: M \to F$ and $\psi_2: M \to N$. Since $\psi_a$ is smooth, $\psi_a \circ \rho_a \circ f = (\lambda_a \circ \id_N) \circ \psi_2 \circ f$ is $C^\alpha$, whence so is its second component $\psi_2 \circ f$ (see [1, Lemma 3.8]). Also the first component $\lambda_a \circ \psi_1 \circ f$ is $C^\alpha$ for each $a \in A$, whence $\psi_1 \circ f$ is $C^\alpha$ by Lemma 2.14(b). Hence $\psi \circ f$ is $C^\alpha$, by [1, Lemma 3.8], and hence so is $f = \psi^{-1} \circ (\psi \circ f)$. □

Proof of Lemma 2.17. The proof is by induction on $m := m_1 + \cdots + m_n$. If $m = n$, there is nothing to show. Assume that $m > n$. After a permutation of $E_1, \ldots, E_n$, we may assume that $m_n \geq 2$ (cf. Lemma 2.13). Let $(\beta_1, \ldots, \beta_{n-1}) \in \prod_{i=1}^{n-1} (N_0 \cup \{\infty\})^{m_i}$, $\beta_n = (\beta_{n,1}, \ldots, \beta_{n,m_n-1}) \in (N_0 \cup \{\infty\})^{m_n-1}$ such that $|\beta_i| \leq \alpha_i$ for all $i \in \{1, \ldots, n\}$. Abbreviate $\beta'_n := (\beta_{n,1}, \ldots, \beta_{n,m_n-2})$. For all $k, \ell \in N_0$ such that $k + \ell \leq \beta_{n,m_n-1}$, the map $f$ is $C^{\beta_1, \ldots, \beta_{n-1}, \beta'_n, k, \ell}$. Hence

$$f: \prod_{i=1}^{n-1} \prod_{j=1}^{m_i} U_{i,j} \times U_{n,1} \times \cdots \times U_{n,m_n-2} \times (U_{n,m_n-1} \times U_{n,m_n}) \to F$$

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is \( C^{\beta_1, \ldots, \beta_n} \), by [11 Lemma 3.12]. By the inductive hypothesis, \( f \) is \( C^\alpha \). □

The following lemma fills in the details for 3.3.

**Lemma A.1** Let \( M_1, \ldots, M_n \) and \( N \) be smooth manifolds with rough boundary, \( M := M_1 \times \cdots \times M_n \) and \( f : M \to N \) be a \( C^\alpha \)-map with \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \). Then \( f(\bar{x}, \cdot) : M_n \to N \) is \( C^\alpha \) for each \( \bar{x} \in M := M_1 \times \cdots \times M_{n-1} \) and

\[ h_k : M_1 \times \cdots \times M_{n-1} \times T^k(M_n) \to T^k N, \quad (\bar{x}, v) \mapsto T^k(f(\bar{x}, \cdot))(v) \]

is a \( C^{\alpha-k\varepsilon_n} \)-map for all \( k \in \mathbb{N}_0 \) such that \( k \leq \alpha_n \).

**Proof.** We show by induction on \( k_0 \in \mathbb{N} \) that the conclusion holds with \( k \leq k_0 \) for all functions as described in the lemma, for all \( \alpha \) with \( \alpha_n \geq k_0 \). Using local charts, we may assume that \( U_j := M_j \) is a locally convex subset of a locally convex space \( E_j \) for all \( j \in \{1, \ldots, n\} \) and \( N \) a locally convex subset of a locally convex space \( F \); thus \( f \) is a map \( U := U_1 \times \cdots \times U_n \to F \). The case \( k_0 = 0 \) being trivial as \( h_0 = f \) is \( C^\alpha \). Let \( 1 \leq k_0 \leq \alpha_n \) now. Then

\[ d^{\varepsilon_n} f : U_1 \times \cdots \times U_n \times E_n \to F \]

is a \( C^{(\alpha-\varepsilon_n,0)} \)-map. Being linear in the final argument, \( d^{\varepsilon_n} f \) is \( C^{\alpha-\varepsilon_n} \) as a map

\[ U_1 \times \cdots \times U_{n-1} \times (U_n \times E_n) \to F \]

of \( n \) variables, i.e., as a map on the domain \( T^{\varepsilon_n} U = U_1 \times U_{n-1} \times TU_n \) (see [11 Lemma 3.11]). Let \( \text{pr}_1 : TU_n = U_n \times E_n \to U_n \) be the projection onto the first component. Then \( g := f \circ \text{id}U_1 \times \cdots \circ \text{id}U_{n-1} \times \text{pr}_1 : U_1 \times \cdots \times U_{n-1} \times TU_n \to F \) is \( C^\alpha \) by the Chain Rule [11 Lemma 3.16], and hence \( C^{\alpha-\varepsilon_n} \). Thus \( h_1 = (g, d^{\varepsilon_n} f) \) is \( C^{\alpha-\varepsilon_n} \), by [11 Lemma 3.8]. By the inductive hypothesis, the maps

\[ U_1 \times \cdots \times U_{n-1} \times T^j(TU_n) \to T^j(TF), \quad (\bar{x}, v) \mapsto T^j(h_1(\bar{x}, \cdot))(v) \]

are \( C^{\alpha-\varepsilon_n-j\varepsilon_n} \) for all \( j \in \{0, \ldots, k_0 - 1\} \). It only remains to observe that this map equals \( h_{j+1} \). □

**Proof of Lemma 3.3.** (a) For \( \beta \in \mathbb{N}_0^n \) with \( \beta \leq \alpha \), consider the maps

\[ T^\beta : C^\alpha(M, N) \to C(T^\beta M, T^{|\beta|} N), \quad f \mapsto T^\beta f \]

and \( \tau_\beta : C^\alpha(M, L) \to C(T^\beta N, T^{|\beta|} L), \quad f \mapsto T^\beta f \). Going through the recursive construction of \( T^\beta(g \circ f) \) in [13] for \( f \in C^\alpha(M, N) \) and making repeated use of the functoriality of \( T \), we see that

\[ T^\beta(g \circ f) = T^{|\beta|} g \circ T^\beta f. \quad (12) \]

Thus \( \tau_\beta \circ C^\alpha(M, g) = C(T^\beta M, T^{|\beta|} g) \circ T^\beta \), which is a continuous map by [15] Lemma A.6.3. The topology on \( C^\beta(M, L) \) being initial with respect to the maps \( \tau_\beta \), we deduce that \( C^\alpha(M, g) \) is continuous.

(b) For \( \beta \in \mathbb{N}_0^n \) with \( \beta \leq \alpha \), consider the maps \( T^\beta : C^\alpha(M, N) \to C(T^\beta M, T^{|\beta|} N) \),
$f \mapsto T^\beta f$ and $\tau_\beta: C^\alpha(L, N) \to C(T^\beta L, T^\beta N)$, $f \mapsto T^\beta f$. Going through the recursive construction of $T^\beta(f \circ g)$ in $\mathbf{3}$ for $f \in C^\alpha(M, N)$ and making repeated use of the functoriality of $T$, we see that

$$T^\beta(f \circ g) = (T^\beta f) \circ h_\beta$$

(13)

with $h_\beta := T^\beta g_1 \times \cdots \times T^\beta g_n$. Thus $\tau_\beta \circ C^\alpha(g, N) = C(h_\beta, T^\beta N) \circ T^\beta$, which is a continuous map by [15, Lemma A.6.9]. The topology on $C^\alpha(L, N)$ being initial with respect to the maps $\tau_\beta$, we deduce that $C^\alpha(g, N)$ is continuous. □

**Proof of Lemma 3.7** By definition, the compact-open $C^\alpha$-topology $\mathcal{O}$ on $C^\alpha(M, N)$ is initial with respect to the maps $\tau_\beta: C^\alpha(M, N) \to C(T^\beta M, T^\beta N)$, $f \mapsto T^\beta f$ for $\beta \in (N_0)^n$ such that $\beta \leq \alpha$. As the interiors $(T^\beta K_i)$ cover $T^\beta M$, the compact-open topology on $C(T^\beta M, T^\beta N)$ is initial with respect to the restriction maps $\rho_{\beta,i}: C(T^\beta M, T^\beta N) \to C(T^\beta K_i, T^\beta N)$, by [15, Lemma A.6.11]. By transitivity of initial topologies, $\mathcal{O}$ is initial with respect to the mappings $\rho_{\beta,i} \circ \tau_\beta$. Let $\rho_i: C^\alpha(M, N) \to C^\alpha(K_i, N)$ the restriction map. The compact-open $C^\alpha$-topology on $C^\alpha(K_i, N)$ being initial with respect to the mappings $\tau_{\beta,i}: C^\alpha(K_i, N) \to C(T^\beta K_i, T^\beta N)$, $f \mapsto T^\beta f$, we deduce from

$$\rho_{\beta,i} \circ \tau_\beta = \tau_{\beta,i} \circ \rho$$

that $\mathcal{O}$ is initial with respect to the maps $\rho_i$. □.

**Proof of Lemma 3.8** The case $n = 1$ is well known. The general case follows as $T^\beta S = T^\beta S_1 \times \cdots \times T^\beta S_n$ and $T^\beta M = T^\beta M_1 \times \cdots \times T^\beta M_n$. □.

**Proof of Lemma 3.9** The inclusion map $\lambda: S \to N$ is smooth. By Lemma 3.8, the inclusion map $T^\beta \lambda: T^\beta |S| \to T^\beta |N|$ is a topological embedding, for each $\beta \in (N_0)^n$ such that $\beta \leq \alpha$. Thus $(T^\beta \lambda)_*: C(T^\beta M, T^\beta |S|) \to C(T^\beta M, T^\beta |N|)$ is a topological embedding for the compact-open topologies (see, e.g., [15, Lemma A.6.5]). The compact-open $C^\alpha$-topology $\mathcal{O}$ on $C^\alpha(M, S)$, which is initial with respect to the maps $\tau_{\beta,S}: C^\alpha(M, S) \to C(T^\beta M, T^\beta |S|)$, $f \mapsto T^\beta f$ is therefore also initial with respect to the mappings $(T^\beta \lambda)_* \circ \tau_{\beta,S}$. The compact-open $C^\alpha$-topology on $C^\alpha(M, N)$ is initial with respect to the maps $\tau_{\beta,N}: C^\alpha(M, N) \to C(T^\beta M, T^\beta |N|)$, $f \mapsto T^\beta f$. As $(T^\beta \lambda)_* \circ \tau_{\beta,S} = \tau_{\beta,N} \circ \lambda_*$, we see that the topology $\mathcal{O}$ is initial with respect to the inclusion map $\lambda_*: C^\alpha(M, S) \to C^\alpha(M, N)$. Thus $\mathcal{O}$ is the induced topology. □.

**Proof of Lemma 3.10** For each $k \in N_0$, $T^k F = F^{2k}$ is a locally convex space. For each $\beta \in (N_0)^n$ such that $\beta \leq \alpha$, the map

$$T^\beta: C^\alpha(M, F) \to C(T^\beta M, T^\beta F), \quad f \mapsto T^\beta f$$

is linear. In fact, $T^k: C^k(N, F) \to C(T^k N, T^k F)$ is linear for each smooth manifold $N$ with rough boundary [15, proof of Proposition 4.1.11] and $k \in N_0$, establishing linearity if $n = 1$. If $n \geq 2$, the preceding entails that $T^{0,\ldots,0,\beta_n}(f) = T^{\beta_n}(f(x_1, \ldots, x_{n-1}, \cdot))(v_n)$ is linear in $f$ for all $x_j \in M_j$ for $j \in \{1, \ldots, n-1\}$ and $v_n \in T^{\beta_n}M_n$, showing that $T^{0,\ldots,0,\beta_n}f$ is linear in $f$. Likewise, $g$ and
$T^{(0,\ldots,0,\beta_{k-1},\ldots,\beta_n)}f$ is linear in $f$ in the recursive construction in 3.3 which gives the assertion for $n \geq 2$. Thus

$$C^\alpha(M,F) \to \prod_{\beta \leq \alpha} C(T^\beta M, T^{[\beta]} F), \quad f \mapsto (T^\beta f)_{\beta \leq \alpha}$$

is a linear map. It is a homeomorphism onto its image, which is a locally convex space. Hence also $C^\alpha(M,F)$ is a locally convex space. □

**Proof of Lemma 3.11.** (a) For each $k \in \mathbb{N}_0$, the topology on $T^k F = F^{2k}$ is initial with respect to the linear maps $T^k \lambda_i = \lambda_i^{2k} : F^{2^k} \to F^{2^k}$. For each $\beta \in \mathbb{N}_0$ with $\beta \leq \alpha$, the compact-open topology on $C(T^\beta M, T^{[\beta]} F)$ is therefore initial with respect to the mappings

$$C(T^\beta M, T^{[\beta]} \lambda_i) : C(T^\beta M, T^{[\beta]} F) \to C(T^\beta M, T^{[\beta]} F_i)$$

for $i \in I$, see [15] Lemma A.6.4. Thus, the compact-open $C^\alpha$-topology $O$ on $C^\alpha(M,F)$ is initial with respect to the maps $C(T^\beta M, T^{[\beta]} \lambda_i) \circ T^\beta$ with $T^\beta : C^\alpha(M,F) \to C(T^\beta M, T^{[\beta]} F)$. As $T^\beta(\lambda_i \circ f) = (T^{[\beta]} \lambda_i) \circ (T^\beta f)$, writing $\tau_{i,\beta}(g) := T^\beta g$ for $g \in C^\alpha(M,F_i)$ we have

$$C(T^\beta M, T^{[\beta]} \lambda_i) \circ T^\beta = \tau_{i,\beta} \circ C^\alpha(M,\lambda_i).$$

The topology on $C^\alpha(M,F_i)$ being initial with respect to the mappings $\tau_{i,\beta} : C^\alpha(M,F_i) \to C(T^\beta M, T^{[\beta]} F_i)$ for $\beta \leq \alpha$, we deduce that $O$ is initial with respect to the mappings $C^\alpha(M,\lambda_i) = (\lambda_i)_*$. (b) By [11] Lemma 3.8, the linear map $\Theta$ is a bijection. The topology on $F$ being initial with respect to the maps $pr_i$, (a) shows that the topology on $C^\alpha(M,F)$ is initial with respect to the maps $(pr_i)_*$ and hence makes $\Theta$ a topological embedding. Hence $\Theta$ is a homeomorphism, being bijective. (c) By [11] Lemma 3.8, $\Psi$ is a bijection. By Lemma 3.5 $\Psi$ is continuous. To see that $\Psi^{-1}$ is continuous, we prove its continuity at a given element $(f_1, f_2)$ in $C^\alpha(M,N_1) \times C^\alpha(M,N_2)$. For $x \in M$, pick a chart $\phi_{x,i} : U_{x,i} \to V_{x,i} \subseteq E_{x,i}$ of $N_i$ around $f_i(x)$, for $i \in \{1,2\}$. There exist compact full submanifolds $K_{x,j}$ of $M_j$ for $j \in \{1,\ldots,n\}$ such that $K_x := K_{x,1} \times \cdots \times K_{x,n} \subseteq (f_1,f_2)^{-1}(U_{x,1} \times U_{x,2})$ and $x \in K_x^2$. By Lemma 3.7 the topology on $C^\alpha(M,N_1 \times N_2)$ is initial with respect to the restriction maps

$$\rho_x : C^\alpha(M,N_1 \times N_2) \to C^\alpha(K_x,N_1 \times N_2).$$

It thus suffices to show that $\rho_x \circ \Psi^{-1}$ is continuous at $(f_1,f_2)$ for all $x \in M$. Now $\rho_x \circ \Psi^{-1} = \Psi^{-1}_x \circ (\rho_{x,1} \times \rho_{x,2})$ using the continuous restriction maps $\rho_{x,i} : C^\alpha(M,N_i) \to C^\alpha(K_x,N_i)$ for $i \in \{1,2\}$ and the map

$$\Psi_x : C^\alpha(K_x,N_1 \times N_2) \to C^\alpha(K_x,N_1) \times C^\alpha(K_x,N_2)$$

taking a function to its pair of components. Thus, it suffices to show that $\Psi^{-1}_x$ is continuous at $(f_1|_{K_x}, f_2|_{K_x})$. Now $f_1|_{K_x}$ is contained in the open subset
$C^\alpha(K_x,U_{x,i})$ of $C^\alpha(K_x,N_i)$, on which the latter induces the compact-open $C^\alpha$-topology, by Lemma 3.11. The map $\Psi^{-1}$ takes this set onto $C^\alpha(M,U_{x,1} \times U_{x,2})$, on which $C^\alpha(M,N_1 \times N_2)$ induces the compact-open $C^\alpha$-topology. It thus suffices to show that $\Psi_x^{-1}$ is continuous at $(f_1|_{K_x}, f_2|_{K_x})$ as a map

$$C^\alpha(K_x,U_{x,1}) \times C^\alpha(K_x,U_{x,2}) \to C^\alpha(K_x,U_{x,1} \times U_{x,2}).$$

Now $(\phi_{x,i}) : C^\alpha(K_x,U_{x,i}) \to C^\alpha(K_x,V_{x,i})$ is a homeomorphism for $i \in \{1, 2\}$ and also $(\phi_{x,1} \times \phi_{x,2}) : C^\alpha(K_x,U_{x,1} \times U_{x,2}) \to C^\alpha(K_x,V_{x,1} \times V_{x,2})$ is a homeomorphism, by Lemma 3.16. It thus suffices to show that the mapping $(\phi_{x,1} \times \phi_{x,2}) \circ \Psi_x^{-1} \circ ((\phi_{x,1})_\ast \times (\phi_{x,2})_\ast)^{-1}:

$$C^\alpha(K_x,V_{x,1}) \times C^\alpha(K_x,V_{x,2}) \to C^\alpha(K_x,V_{x,1} \times V_{x,2})$$

is continuous. But this mapping is a restriction of the homeomorphism $C^\alpha(K_x,E_{x,1}) \times C^\alpha(K_x,E_{x,2}) \to C^\alpha(K_x,E_{x,1} \times E_{x,2})$ discussed in (b).

**Proof of Lemma 3.12.** The scalar multiplication $\sigma : \mathbb{R} \times TN \to TN$ being smooth, the map $\sigma_\ast : C^\alpha(M,\mathbb{R} \times TN) \to C^\alpha(M,TN)$, $h \mapsto \sigma \circ h$ is continuous (see Lemma 3.13). Hence $\mu = \sigma_\ast \circ \Psi^{-1}$ is continuous, using the homeomorphism $\Psi : C^\alpha(M,\mathbb{R} \times TN) \to C^\alpha(M,\mathbb{R} \times T(M,TN))$ from Lemma 3.11.

**Proof of Lemma 3.13.** Let $(U_i)_{i \in I}$ be the family of pairwise distinct connected components of $N$ and $(V_j)_{j \in J}$ be the family of components of $M$. Then

$$r : C^\alpha(M,E) \to \prod_{j \in J} C^\beta(V_j,E), \ f \mapsto (f|_{V_j})_{j \in J}$$

is a bijective linear map; by Lemma 3.7 it is a homeomorphism. Likewise,

$$\rho : C^{\alpha,\beta}(N \times M,E) \to \prod_{(i,j) \in I \times J} C^{\alpha,\beta}(U_i \times V_j,E), \ f \mapsto (f|_{U_i \times V_j})_{(i,j) \in I \times J}$$

and $R : C^\alpha(N,C^\beta(M,E)) \to \prod_{i \in I} C^\alpha(U_i,C^\beta(M,E))$, $f \mapsto (f|_{U_i})_{i \in I}$ are isomorphisms of topological vector spaces. By Lemma 3.13, the mapping $C^\alpha(U_i,r) : C^\alpha(U_i,C^\beta(M,E)) \to C^\alpha(U_i,\prod_{j \in J} C^\beta(V_j,E))$ is an isomorphism of topological vector spaces and so is the map

$$\Theta_i : C^\alpha(U_i,\prod_{j \in J} C^\beta(V_j,E)) \to \prod_{j \in J} C^\alpha(U_i,C^\beta(V_j,E))$$

taking a map to its family of components (see Lemma 3.11(b)). Hence

$$\Xi := \prod_{i \in I} \Theta_i \circ \prod_{i \in I} C^\alpha(U_i,r) \circ R : C^\alpha(N,C^\beta(M,E)) \to \prod_{(i,j) \in I \times J} C^\alpha(U_i,C^\beta(V_j,E))$$

is an isomorphism of topological vector spaces. By [1, Theorem B], the map $\Phi_{i,j} : C^{\alpha,\beta}(U_i \times V_j,E) \to C^\alpha(U_i,C^\beta(V_j,E))$, $f \mapsto f^j$ is linear and a topological embedding, whence so is

$$\Psi := \prod_{(i,j) \in I \times J} \Phi_{i,j} : \prod_{(i,j) \in I \times J} C^{\alpha,\beta}(U_i \times V_j,E) \to \prod_{(i,j) \in I \times J} C^\alpha(U_i,C^\beta(V_j,E)).$$
Evaluating at $x \in N$ and then in $y \in M$ (say $x \in U_i$ and $y \in V_j$), we verify that
$$f^\gamma = (\Xi^{-1} \circ \Psi \circ \rho)(f)$$
for all $f \in C^{\alpha,\beta}(N \times M, E)$, whence $f^\gamma \in C^\alpha(N, C^\beta(M, E))$ and $\Phi$ makes sense as a map to the latter space. We have a commutative diagram
$$
\begin{array}{ccc}
C^{\alpha,\beta}(N \times M, E) & \xrightarrow{\Phi} & C^\alpha(N, C^\beta(M, E)) \\
\rho \downarrow & & \downarrow \Xi \\
\prod_{i,j} C^{\alpha,\beta}(U_i \times V_j, E) & \xrightarrow{\Psi} & \prod_{i,j} C^\alpha(U_i, C^\beta(V_j, E))
\end{array}
$$
where the vertical arrows are homeomorphisms and $\Psi$ is a topological embedding. Hence $\Phi$ is a topological embedding. If $M$ is locally compact, then so are the $V_j$, whence each of the maps $\Phi_{i,j}$ is a homeomorphism by [1, Theorem 4.4] and hence also $\Psi$. Then also $\Phi = \Xi^{-1} \circ \Psi \circ \rho$ is a homeomorphism. □

**Proof of Lemma 3.14.** Let $O$ be the compact-open $C^\alpha$-topology on $C^\alpha(U, F)$ and $T$ be the initial topology with respect to the maps
$$d^\beta : C^\alpha(U, F) \to C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F)$$
for $\beta \in \mathbb{N}_0^n$ such that $\beta \leq \alpha$. We claim that, for each $\beta$ as before, there exist a continuous linear map $\lambda_\beta : T^{[\beta]}F \to F$ and $C^\infty$-maps $\theta_{\beta,j} : U_j \times E_j^{\beta_j} \to T^{\beta}U_j$ for $j \in \{1, \ldots, n\}$ such that, for all $f \in C^\alpha(U, F),$ $d^\beta f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (\lambda_\beta \circ T^\beta f)(\theta_{\beta,1}(x_1, y_1), \ldots, \theta_{\beta,n}(x_n, y_n))$ (14)
holds for all $(x_1, \ldots, x_n) \in U$ and $(y_1, \ldots, y_n) \in E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}$. Consider the map $\pi_\beta : U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \to (U \times E_1^{\beta_1}) \times \cdots \times (U_n \times E_n^{\beta_n})$, $(x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto ((x_1, y_1), \ldots, (x_n, y_n))$.
If the claim is true, setting $\theta_\beta := \theta_{\beta,1} \times \cdots \times \theta_{\beta,n}$ we have
$$d^\beta = C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, \lambda_\beta) \circ C(\pi_\beta, T^{[\beta]}F) \circ C(\theta_\beta, T^{[\beta]}F) \circ T^\beta,$$
which is a continuous $C(U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n}, F)$-valued function on $(C^\alpha(U, F), O)$ by Lemmas A.5.3 and A.5.9 in [15]. Thus $T \subseteq O$. The claim is established by induction on $|\beta|$. If $|\beta| = 1$, then $\beta = e_j$ for some $j \in \{1, \ldots, n\}$. Using $pr_2 : F \times F \to F$, $(v, w) \mapsto w$, we have
$$d^{e_j} f(x_1, \ldots, x_n, y_1, \ldots, y_n) = (pr_2 \circ T^{e_j} f)(\theta_{e_j,1}(x_1, y_1), \ldots, \theta_{e_j,n}(x_n, y_n))$$
with $\theta_{e_j,j}(x_j, y_j) := (x_j, y_j)$ for all $(x_j, y_j) \in U_j \times E_j$ and $\theta_{e_j,i}(x_i, y_i) := x_i$ if $i \neq j$ and $(x_i, y_i) \in U_i \times E_i^{\beta_i} = U_i \times \{0\}$. Assume the claim holds for $\beta$; thus $d^{\beta} f$ is of the form (13). Let $k \in \{1, \ldots, n\}$ be minimal with $\beta_k \neq 0$. For $j \in \{1, \ldots, k\}$, $(x_1, \ldots, x_n) \in U$, $y_j = (v, w) \in E_j^{\beta_j} \times E_j$ and $y_i \in E_i^{\beta_i}$ if $i \neq j$, we then have
$$d^{\beta + e_j} f(x_1, \ldots, x_n, y_1, \ldots, y_n)$$
$$= (d\lambda_\beta \circ T^{\beta + e_j} f)(\theta_{\beta + e_j,1}(x_1, y_1), \ldots, \theta_{\beta + e_j,n}(x_n, y_n))$$
\[45\]
of the desired form with \( \theta_{\beta+e_j,i} := \theta_{\beta,i} \) for \( i \neq j \) and 
\[ \theta_{\beta+e_j,j}(x_j, v, w) := T \theta_{\beta,j}(x, v, w, 0). \]

To see that \( \mathcal{O} \subseteq \mathcal{T} \), we show that, for each \( \beta \in \mathbb{N}_0^n \) such that \( \beta \leq \alpha \), there exist \( m_\beta \in \mathbb{N} \), multindices \( \gamma_{\beta,a} \leq \beta \) for \( a \in \{1, \ldots, m_\beta\} \), continuous linear functions \( \lambda_{\beta,a} : F \to T^{|\beta|}F \) and smooth functions \( \xi_{\beta,a,j} : T^{|\beta|}U_j \to E_j^{\gamma_{\beta,a}} \) such that

\[
T^\beta f(y_1, \ldots, y_n) = \sum_{a=1}^{m_\beta} \lambda_{\beta,a}(d^{\gamma_{\beta,a}} f(\theta_{1,\beta_1}(y_1), \ldots, \theta_{n,\beta_n}(y_n), \xi_{\beta,a,1}(y_1), \ldots, \xi_{\beta,a,n}(y_n))) \tag{15}
\]

for all \( (y_1, \ldots, y_n) \in \prod_{j=1}^n T^{|\beta|}U_j = T^\beta U \), where

\[
\theta_{j,k} : T^k U_j = U_j \times E_j^{2^{k-1}} \to U_j
\]
is the projection onto the first component for \( j \in \{1, \ldots, n\} \) and \( k \in \mathbb{N}_0 \) (if we identify \( U_j \times E_j^0 \) with \( U_j \) for \( k = 0 \)). The map \( \Xi_{\beta,a} : T^\beta U \to U \times E_1^{\beta_1} \times \cdots \times E_n^{\beta_n} \)
\( (y_1, \ldots, y_n) \mapsto (\theta_{1,\beta_1}(y_1), \ldots, \theta_{n,\beta_n}(y_n), \xi_{\beta,a,1}(y_1), \ldots, \xi_{\beta,a,n}(y_n)) \) is \( C^\infty \) and

\[
T^\beta = \sum_{a=1}^{m_\beta} C(T^\beta U, \lambda_{\beta,a}) \circ C(\Xi_{\beta,a}, F) \circ d^{\gamma_{\beta,a}}
\]
is a continuous \( C(T^\beta U, T^{|\beta|}F) \)-valued function on \( (C^\alpha(U,F), \mathcal{T}) \); so \( \mathcal{O} \subseteq \mathcal{T} \).

The proof is by induction on \( |\beta| \). If \( |\beta| = 1 \), then \( \beta = e_j \) for some \( j \) and

\[
T^{e_j} f(y_1, y_2) = (\lambda_1 \circ f)(\theta_{1,\beta_1}(y_1), \ldots, \theta_{n,\beta_n}(y_n))
+ (\lambda_2 \circ d^{\gamma_{\beta,a}} f)(\theta_{1,\beta_1}(y_1), \ldots, \theta_{n,\beta_n}(y_n), \text{pr}_2(y_j))
= (\lambda_1 \circ f)(\theta_{1,\beta_1}(y_1), \ldots, \theta_{n,\beta_n}(y_n), \xi_{\beta,1,1}(y_1), \ldots, \xi_{\beta,1,n}(y_n))
+ (\lambda_2 \circ d^{\gamma_{\beta,a}} f)(\theta_{1,\beta_1}(y_1), \ldots, \theta_{n,\beta_n}(y_n), \xi_{\beta,2,1}(y_1), \ldots, \xi_{\beta,2,n}(y_n))
\]

with \( \xi_{\beta,1,i}(y_i) := 0 \in E_j^0 \) for \( i \in \{1, \ldots, n\} \), \( \xi_{\beta,2,i}(y_i) := \text{pr}_2(y_j) \) and \( \xi_{\beta,2,i}(y_i) := 0 \in E_j^0 \) for \( i \neq j \), using \( \text{pr}_2 : \mathcal{T}U_j = U_j \times E_j \to E_j \), \( \lambda_1 : F \to F \times F \), \( v \mapsto (v, 0) \) and \( \lambda_2 : F \to F \times F \), \( v \mapsto (0, v) \). Note that we identified \( U \) with \( U \times (E_1^0 \times \cdots \times (E_n^0)). \) If \( \beta \leq \alpha \) with \( |\beta| \geq 1 \) is given, let \( k \in \{1, \ldots, n\} \) with \( \beta_k \geq 1 \) be minimal. Let \( j \in \{1, \ldots, k\} \) and assume that \( \beta' := \beta + e_j \leq \alpha \). Write \( \beta'_1, \ldots, \beta'_n \) for the components of \( \beta' \). Consider the continuous linear map \( \lambda_1 : T^{|\beta'|}F \to T^{|\beta'|}F \times T^{|\beta'|}F \), \( v \mapsto (v, 0) \) and define \( \lambda_2 \) analogously. Keeping the other variables fixed and differentiating in the \( y_j \)-variable, (15) implies that

\[
T^{\beta+e_j} f(y_1, \ldots, y_n) = \sum_{a=1}^{m_\beta} \lambda_1(\lambda_{\beta,a}(d^{\gamma_{\beta,a}} f(\theta_{1,\beta_1}(y_1), \ldots, \theta_{n,\beta_n}(y_n), \xi_{\beta',a,1}(y_1), \ldots, \xi_{\beta',a,n}(y_n))))
+ \sum_{a=1}^{m_\beta} \lambda_2(\lambda_{\beta,a}(d^{\gamma_{\beta,a}} f(\theta_{1,\beta_1}(y_1), \ldots, \theta_{n,\beta_n}(y_n), \xi_{\beta',a,1}(y_1), \ldots, \xi_{\beta',a,n}(y_n))))
+ \sum_{a=1}^{m_\beta} \lambda_2(\lambda_{\beta,a}(d^{\gamma_{\beta,a}+e_j} f(\theta_{1,\beta_1}(y_1), \ldots, \theta_{n,\beta_n}(y_n), \xi_{\beta',a,1}(y_1), \ldots, \xi_{\beta',a,n}(y_n))))
\]

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for all \((y_1, \ldots, y_n) \in T^{\beta+e_j}U_j\), where \(\xi_{\beta', a, j}(y_j) := \xi_{\beta, a, j}(\text{pr}_1(y_j))\), \(\xi_{\beta', a, i}(y_i) := \xi_{\beta, a, i}(y_i)\) for \(i \neq j\), \(\eta_{\beta', a, j}(y_j) := d\xi_{\beta, a, j}(y_j)\), \(\eta_{\beta', a, i}(y_i) := \xi_{\beta, a, i}(y_i)\) for \(i \neq j\), \(\xi_{\beta', a, j}(y_j) := (\xi_{\beta, a, j}(\text{pr}_1(y_j)), d\eta_{\beta, j, i}(y_j))\) and \(\xi_{\beta', a, i}(y_i) := \xi_{\beta, a, i}(y_i)\) for \(i \neq j\), using the map \(\text{pr}_1\): \(T^{\beta+1}U_j = T^{\beta}U_j \times T^{\beta}E_j \to T^{\beta}U_j\). Thus also \(T^{\beta+e_j}f\) is of the desired form. □

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