A NOTE ON REGULAR DE MORGAN SEMI-HEYTING ALGEBRAS

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Abstract. The purpose of this note is two-fold. Firstly, we prove that the variety $\text{RDMSH}_1$ of regular De Morgan semi-Heyting algebras of level 1 satisfies Stone identity and present (equational) axiomatizations for several subvarieties of $\text{RDMSH}_1$. Secondly, we give a concrete description of the lattice of subvarieties of the variety $\text{RDQDStSH}_1$ of regular dually quasi-De Morgan Stone semi-Heyting algebras that contains $\text{RDMSH}_1$. Furthermore, we prove that every subvariety of $\text{RDQDStSH}_1$, and hence of $\text{RDMSH}_1$, has Amalgamation Property. The note concludes with some open problems for further investigation.

1. Introduction

Semi-Heyting algebras were introduced by us in [12] as an abstraction of Heyting algebras. They share several important properties with Heyting algebras, such as distributivity, pseudocomplementedness, and so on. On the other hand, interestingly, there are also semi-Heyting algebras, which, in some sense, are “quite opposite” to Heyting algebras. For example, the identity $0 \rightarrow 1 \approx 0$, as well as the commutative law $x \rightarrow y \approx y \rightarrow x$, hold in some semi-Heyting algebras. The subvariety of commutative semi-Heyting algebras was defined in [12] and is further investigated in [13].

Quasi-De Morgan algebras were defined in [11] as a common abstraction of De Morgan algebras and distributive $p$-algebras. In [14], expanding semi-Heyting algebras by adding a dual quasi-De Morgan operation, we introduced the variety $\text{DQDSH}$ of dually quasi-De Morgan semi-Heyting algebras as a common generalization of De Morgan Heyting algebras (see [10] and [5]) and dually pseudocomplemented...
Heyting algebras (see [8]) so that we could settle an old conjecture of ours.

The concept of regularity has played an important role in the theory of pseudocomplemented De Morgan algebras (see [9]). Recently, in [15] and [16], we introduced and examined the concept of regularity in the context of DQDSH and gave an explicit description of (twenty five) simple algebras in the (sub)variety DQDStSH of regular dually quasi-De Morgan Stone semi-Heyting algebras of level 1. The work in [15] and [16] led us to conjecture that the variety RDMSH of regular De Morgan algebras satisfies Stone identity.

The purpose of this note is two-fold. Firstly, we prove that the variety RDMSH of regular De Morgan semi-Heyting algebras of level 1 satisfies Stone identity, thus settling the above mentioned conjecture affirmatively. As applications of this result and the main theorem of [15], we present (equational) axiomatizations for several subvarieties of RDMSH. Secondly, we give a concrete description of the lattice of subvarieties of the variety RDQDStSH of regular dually quasi-De Morgan Stone semi-Heyting algebras, of which RDMSH is a subvariety. Furthermore, we prove that every subvariety of RDQDStSH, and hence of RDMSH, has Amalgamation Property. The note concludes with some open problems for further investigation.

2. Dually Quasi-De Morgan Semi-Heyting Algebras

The following definition is taken from [12].

An algebra \( L = \langle L, \lor, \land, \to, 0, 1 \rangle \) is a semi-Heyting algebra if \( \langle L, \lor, \land, 0, 1 \rangle \) is a bounded lattice and \( L \) satisfies:

\( (SH1) \) \( x \land (x \to y) \approx x \land y \)
\( (SH2) \) \( x \land (y \to z) \approx x \land ((x \land y) \to (x \land z)) \)
\( (SH3) \) \( x \to x \approx 1. \)

Let \( L \) be a semi-Heyting algebra and, for \( x \in L \), let \( x^* := x \to 0. \) \( L \) is a Heyting algebra if \( L \) satisfies:

\( (SH4) \) \( (x \land y) \to y \approx 1. \)

\( L \) is a commutative semi-Heyting algebra if \( L \) satisfies:

\( (Co) \) \( x \to y \approx y \to x. \)

\( L \) is a Boolean semi-Heyting algebra if \( L \) satisfies:

\( (Bo) \) \( x \lor x^* \approx 1. \)

\( L \) is a Stone semi-Heyting algebra if \( L \) satisfies:

\( (St) \) \( x^* \lor x^{**} \approx 1. \)
Semi-Heyting algebras are distributive and pseudocomplemented, with $a^*$ as the pseudocomplement of an element $a$. We will use these and other properties (see [12]) of semi-Heyting algebras, frequently without explicit mention, throughout this paper.

The following definition is taken from [14].

**Definition 2.1.** An algebra $L = \langle L, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a semi-Heyting algebra with a dual quasi-De Morgan operation or dually quasi-De Morgan semi-Heyting algebra (DQDSH-algebra, for short) if $\langle L, \vee, \wedge, \rightarrow, 0, 1 \rangle$ is a semi-Heyting algebra, and $L$ satisfies:

(a) $0' \approx 1$ and $1' \approx 0$
(b) $(x \wedge y)' \approx x' \vee y'$
(c) $(x \vee y)''' \approx x'' \vee y''$
(d) $x'' \leq x$.

Let $L \in \text{DQDSH}$. Then $L$ is a dually Quasi-De Morgan Stone semi-Heyting algebra ($\text{DQDStSH}$-algebra) if $L$ satisfies (St). $L$ is a De Morgan semi-Heyting algebra or symmetric semi-Heyting algebra ($\text{DMSH}$-algebra) if $L$ satisfies:

(St) $x'' \approx x$.

$L$ is a dually pseudocomplemented semi-Heyting algebra ($\text{DPCSH}$-algebra) if $L$ satisfies:

(PC) $x \vee x' \approx 1$.

The varieties of $\text{DQDSH}$-algebras, $\text{DQDStSH}$-algebras, $\text{DMSH}$-algebras and $\text{DPCSH}$-algebras are denoted, respectively, by $\text{DQDSH}$, $\text{DQDStSH}$, $\text{DMSH}$ and $\text{DPCSH}$. Furthermore, $\text{DMcmSH}$ denotes the subvariety of $\text{DMSH}$ defined by the commutative identity (Co), and $\text{DQDBSH}$ denotes the one defined by (Bo).

If the underlying semi-Heyting algebra of a $\text{DQDSH}$-algebra is a Heyting algebra we denote the algebra by $\text{DQDH}$-algebra, and the corresponding variety is denoted by $\text{DQDH}$. In the sequel, $a^{**}$ will be denoted by $a^+$, for $a \in L \in \text{DQDSH}$. The following lemma will often be used without explicit reference to it. Most of the items in this lemma were proved in [14], and the others are left to the reader.

**Lemma 2.2.** Let $L \in \text{DQDSH}$ and let $x, y, z \in L$. Then

(i) $1'' = 1$
(ii) $x \leq y$ implies $x' \geq y'$
(iii) $(x \wedge y)^* = x^* \wedge y^*$
(iv) $x'' = x'$
(v) $(x \vee y)' = (x'' \vee y'')'$
(vi) \((x \lor y)' = (x'' \lor y)\)^

(vii) \(x \leq (x \lor y) \rightarrow x\)

(viii) \(x \land [(x \lor y) \rightarrow z] = x \land z\). 

Next, we describe some examples of \textbf{DQDSH}-algebras by expanding the semi-Heyting algebras given in Figure 1. These will play a crucial role in the rest of the note.

\[
\begin{array}{c|cc}
2 : & 1 & 0 & 1 \\
\hline
& 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
\end{array}
\quad \begin{array}{c|cc}
2 : & 1 & 0 & 1 \\
\hline
& 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
L_1 : & a & 0 & 1 \\
\hline
& 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
\end{array}
\quad \begin{array}{c|ccc}
L_2 : & a & 0 & 1 \\
\hline
& 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
L_3 : & a & 0 & 1 \\
\hline
& 0 & 1 & 1 \\
0 & 0 & 1 & a \\
\end{array}
\quad \begin{array}{c|ccc}
L_4 : & a & 0 & 1 \\
\hline
& 0 & 1 & 1 \\
0 & 0 & 1 & a \\
\end{array}
\]

\[
\begin{array}{c|ccc}
L_5 : & a & 0 & 1 \\
\hline
& 0 & 1 & a \\
0 & 0 & 1 & 1 \\
\end{array}
\quad \begin{array}{c|ccc}
L_6 : & a & 0 & 1 \\
\hline
& 0 & 1 & a \\
0 & 0 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
L_7 : & a & 0 & 1 \\
\hline
& 0 & 1 & a \\
0 & 0 & 1 & a \\
\end{array}
\quad \begin{array}{c|ccc}
L_8 : & a & 0 & 1 \\
\hline
& 0 & 1 & a \\
0 & 0 & 1 & a \\
\end{array}
\]

\[
\begin{array}{c|ccc}
L_9 : & a & 0 & 1 \\
\hline
& 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\end{array}
\quad \begin{array}{c|ccc}
L_{10} : & a & 0 & 1 \\
\hline
& 0 & 1 & 0 \\
0 & 0 & 1 & a \\
\end{array}
\]
Let $\mathbf{2}^e$ and $\overline{\mathbf{2}}^e$ be the expansions of the semi-Heyting algebras 2 and $\overline{\mathbf{2}}$ (shown in Figure 1) by adding the unary operation $'$ such that $0' = 1$, $1' = 0$.

Let $L_i^d$, $i = 1, \ldots, 10$, denote the expansion of the semi-Heyting algebra $L_i$ (shown in Figure 1) by adding the unary operation $'$ such that $0' = 1$, $1' = 0$, and $a' = 1$.

Let $L_i^d$, $i = 1, \ldots, 10$, denote the expansion of $L_i$ (in Figure 1) by adding the unary operation $'$ such that $0' = 1$, $1' = 0$, and $a' = a$.

We let $C_{10}^{dp} := \{L_i^{dp} : i = 1, \ldots, 10\}$ and $C_{10}^{dm} := \{L_i^{dm} : i = 1, \ldots, 10\}$. We also let $C_{20} := C_{10}^{dm} \cup C_{10}^{dp}$.

Each of the three 4-element algebras $D_1$, $D_2$ and $D_3$ has its lattice reduct as the Boolean lattice with the universe $\{0, a, b, 1\}$, $b$ being the complement of $a$, has the operation $\rightarrow$ as defined in Figure 1, and has the unary operation $'$ defined as follows: $a' = a$, $b' = b$, $0' = 1$, $1' = 0$. For the variety $V(D_1, D_2, D_3)$ generated by $\{D_1, D_2, D_3\}$, it was shown in [14] that $V(D_1, D_2, D_3) = DQDBSH$.

The following is a special case of Definition 5.5 in [14]. Let $x'^{rs} := x^{2(rs)}$. Note that $x^{2(rs)} \leq x$ in a DMSH-algebra.

**DEFINITION 2.3.** The subvariety DMSH$_1$ of level 1 of DMSH is defined by the identity: $x \land x'^{rs} \land x^{2(rs)} \approx x \land x'^{rs}$, or equivalently, by the identity:

$$(L1) \ (x \land x'^{rs})'^{rs} \approx x \land x'^{rs}.$$
It follows from [14] that the variety $\text{DMSH}_1$, is a discriminator variety. We note here that the algebras described above in Figure 1 are actually in $\text{DMSH}_1$.

3. Regular De Morgan Semi-Heyting algebras of level 1

Recall that $a^+ := a^{='*}$ in $L \in \text{DMSH}_1$.

**DEFINITION 3.1.** Let $L \in \text{DMSH}_1$. Then $L$ is regular if $L$ satisfies the following identity:

(R) $x \land x^+ \leq y \lor y^*$.

The variety of regular $\text{DMSH}_1$-algebras will be denoted by $\text{RDMSH}_1$.

In the rest of this section, $L$ denotes an $\text{RDMSH}_1$-algebra and $x, y \in L$. The following lemmas lead us to prove that $\text{RDMSH}_1$ satisfies (St).

**LEMMA 3.2.** $(x \lor x^*)^* = x' \land x^*$.

*Proof.*

\[
x' \land x^* = x' \land x''^{*} \\
= (x' \land x'')^{*}_{*} \quad \text{by (L1)} \\
= (x'' \lor x''^{*})^* \\
= (x \lor x^{*})^*, \quad \text{since} \ x'' = x.
\]

\[\square\]

**LEMMA 3.3.** $x \lor x^* \lor x^* = 1$.

*Proof.*

\[
x \lor x^* \lor x^* = (x^* \land x' \land x^*)^* \quad \text{by (DM)} \\
= [(x^* \land (x \lor x^*))^*] \quad \text{by Lemma 3.2} \\
= (x^* \land 0)^* \quad \text{by Lemma 2.2 (viii)} \\
= 0^* \\
= 1.
\]

\[\square\]

**LEMMA 3.4.** We have

\[
x \land (x^+ \lor y \lor y^*) = x \land (y \lor y^*).
\]
Proof. \[ x \land (y \lor y^*) = x \land [(x \land x^+) \lor (y \lor y^*)] \text{ by (R)} \]
\[ = (x \land x^+) \lor [x \land (y \lor y^*)] \]
\[ = x \land [x^+ \lor y \lor y^*]. \]
\[ \square \]

Lemma 3.5. Let \( x \neq 1 \). Then \( x \leq x' \).

Proof. Since \( x \neq 1 \), we have \( x \land x^* = 0 \) by (L1). So,
\[ x \land x' = (x \land x') \lor (x \land x^*) \]
\[ = x \land (x' \lor x^*) \]
\[ = x \land (x^+ \lor x' \lor x^*) \text{ by Lemma 3.4} \]
\[ = x \land 1 \text{ by Lemma 3.3} \]
So, \( x \leq x' \). \[ \square \]

Lemma 3.6. Let \( x^* \neq 0 \). Then \( x \lor x^* = 1 \).

Proof. Since \( x^* \neq 0 \), we have \( x^{*'} \neq 1 \), so \( x^{*'} \leq x^* \) by Lemma 3.3 and (DM), implying \( x \lor x^* = 1 \) by Lemma 3.3. \[ \square \]

Theorem 3.7. Let \( L \in \text{RDMSH}_1 \). Then \( L \models x^* \lor x^{**} \approx 1 \).

Proof. Let \( a \in L \). If \( a^* = 0 \), Then the theorem is trivially true. So, we can assume that \( a^* \neq 0 \). Then \( a \lor a^* = 1 \), in view of the preceding lemma. The conclusion is now immediate. \[ \square \]

Recall from [14] that the subvariety \( \text{DMSH}_2 \) of level 2 of \( \text{DMSH} \)
is defined by the identity: \( x \land x^{*'} \land x^{2(*)} \approx x \land x^{*'} \land x^{2(*)} \land x^{3(*)} \), or equivalently, by the identity:

\[ \text{(L2)} \ (x \land x^{*'})^{2(*)} \approx (x \land x^{*'})^{(*)}. \]

Remark 3.8. The above theorem fails in \( \text{RDMSH}_2 \), as the following example shows:
4. Applications

Let $V(K)$ denote the variety generated by the class $K$ of algebras. The following corollary is immediate from Theorem 3.7 and Corollary 3.4(a) of [16], and hence is an improvement on Corollary 3.4(a) of [16].

**COROLLARY 4.1.** We have

(a) $\text{RDMSh}_1 = \text{RDMStSh}_1 = V(C_{10}) \lor V(D_1, D_2, D_3)$

(b) $\text{RDMSH}_1 = \text{RDMStH}_1 = V(L_{dm}^1) \lor V(D_1)$

(c) $\text{RDMcmSh}_1 = V(L_{dm}^9, L_{dm}^{10}, D_1) = V(L_{dm}^1) \lor V(D_1)$.

Let $L \in \text{DMSH}_1$. We say $L$ is pseudocommutative if $L \models (x \to y)^* = (y \to x)^*$.

**COROLLARY 4.2.** Let $V$ be a subvariety of $\text{RDMSh}_1$. Then $V$ is pseudocommutative iff $V = V(L_{dm}^9, L_{dm}^{10}, D_1)$.

Proof. It suffices, in view of (a) of the preceding corollary, to verify that $L_{dm}^9, L_{dm}^{10}$, and $D_1$ satisfy the pseudocommutative law, while the rest of the simples in $\text{RDMSH}_1$ do not. \[\square\]

The proofs of the following corollaries are similar.

**COROLLARY 4.3.** The variety $V(L_{dm}^9, L_{dm}^{10}, D_1)$ is also defined, modulo $\text{RDMSh}_1$, by

$x^* \to y^* \approx y^* \to x^*$.

**COROLLARY 4.4.** The variety $V(L_{dm}^1, L_{dm}^2, L_{dm}^3, L_{dm}^4, D_2, D_3)$ is defined, modulo $\text{RDMSh}_1$, by
\[ (0 \to 1)^+ \to (0 \to 1)^{**} \approx 0 \to 1. \]

It was proved in [14] that \( V(D_1, D_2, D_3) = DQDBSH \). Here are some more bases for \( V(D_1, D_2, D_3) \).

**Corollary 4.5.** Each of the following identities is a base for the variety \( V(D_1, D_2, D_3) \) modulo \( RDMSH_1 \):

1. \( x \to y \approx y^* \to x^* \) (Law of contraposition)
2. \( x \lor (y \to z) \approx (x \lor y) \to (x \lor z) \)
3. \( [(x \lor (x \to y^*)) \to (x \to y^*)] \lor (x \lor y^*) = 1 \).

**Corollary 4.6.** The variety \( V(L_{dm}^1, L_{dm}^2, L_{dm}^3, L_{dm}^4, L_{dm}^5, D_1, D_2, D_3) \) is defined, modulo \( RDMSH_1 \), by

\[ x \to y^* \approx y \to x^*. \]

**Corollary 4.7.** The variety \( V(L_{dm}^7, L_{dm}^8, L_{dm}^9, L_{dm}^{11}, D_1, D_2, D_3) \) is defined, modulo \( RDMSH_1 \), by

\[ x \lor (x \to y) \approx x \lor [(x \to y) \to 1]. \]

**Corollary 4.8.** The variety \( V(L_{dm}^7, L_{dm}^8, D_2) \) is defined, modulo \( RDMSH_1 \), by

1. \( x \lor (x \to y) \approx x \lor [(x \to y) \to 1] \)
2. \( (0 \to 1)^{**} \approx 1. \)

**Corollary 4.9.** The variety \( V(2^e, L_{dm}^7, L_{dm}^8, L_{dm}^9, L_{dm}^{11}) \) is defined, modulo \( RDMSH_1 \), by

1. \( x \lor (x \to y) \approx x \lor [(x \to y) \to 1] \)
2. \( x^{*'} \approx x^{**}. \)

**Corollary 4.10.** The variety \( V(2^e, L_{dm}^9, L_{dm}^{11}) \) is defined, modulo \( RDMSH_1 \), by

1. \( x \lor (x \to y) \approx x \lor [(x \to y) \to 1] \)
2. \( x^{*'} \approx x^{**} \)
3. \( (0 \to 1)^* \lor (0 \to 1)^* \approx 1. \)

**Corollary 4.11.** The variety \( V(L_{dm}^9, L_{dm}^{11}) \) is defined, modulo \( RDMSH_1 \), by

1. \( x \lor (x \to y) \approx x \lor [(x \to y) \to 1] \)
2. \( x^{*'} \approx x^{**} \)
3. \( (0 \to 1)^* \approx 1. \)

**Corollary 4.12.** The variety \( V(L_{dm}^1, L_{dm}^2, L_{dm}^3, L_{dm}^4, L_{dm}^5, L_{dm}^6, L_{dm}^7, L_{dm}^8) \) is defined, modulo \( RDMSH_1 \), by
COROLLARY 4.13. The variety $V(L_{dm_1}^1, L_{dm_2}^2, L_{dm_3}^3, L_{dm_4}^4, D_2)$ is defined, modulo $RDMSH_1$, by
\begin{align*}
(1) & \; (0 \to 1)^* \approx 1 \\
(2) & \; (0 \to 1)^{**} \approx 1.
\end{align*}

COROLLARY 4.14. The variety $V(L_{dm_1}^1, L_{dm_3}^3, D_1, D_2, D_3)$ is defined, modulo $RDMSH_1$, by
\begin{align*}
(1) & \; y \to (y \to x) \approx (x \lor y) \to x \\
(2) & \; (0 \to 1)^* \approx 1.
\end{align*}

COROLLARY 4.15. The variety $V(L_{dm_1}^1, L_{dm_3}^3, D_2)$ is defined, modulo $RDMSH_1$, by
\begin{align*}
(1) & \; y \to (x \to y) \approx (x \lor y) \to x \\
(2) & \; (0 \to 1)^{**} \approx 1 \\
(3) & \; (0 \to 1)^* \approx 1.
\end{align*}

COROLLARY 4.16. The variety $V(L_{dm_1}^1, L_{dm_2}^2, L_{dm_3}^3, D_1, D_2, D_3)$ is defined, modulo $RDMSH_1$, by
\begin{align*}
y \lor (y \to (x \lor y)) & \approx (0 \to x) \lor (x \to y).
\end{align*}

COROLLARY 4.17. The variety $V(L_{dm_1}^1, L_{dm_2}^2, L_{dm_8}^3, D_1, D_2, D_3)$ is defined, modulo $RDMSH_1$, by
\begin{align*}
x \lor [y \to (0 \to (y \to x))] & \approx x \lor y \lor (y \to x).
\end{align*}

$V(D_2)$ was axiomatized in [14]. Here are some more bases for it.

COROLLARY 4.18. Each of the following identities is an equational base for $V(D_2)$, mod $RDMH_1$:
\begin{align*}
(1) & \; y \to [0 \to (y \to x)] \approx y \lor (y \to x) \\
(2) & \; x \lor (y \to z) \approx (x \lor y) \to (x \lor z) \\
(3) & \; x \lor [y \to (y \to x)^*] \approx x \lor y \lor (y \to x) \\
(4) & \; [\{x \lor (x \to y^*)] \to (x \to y^*) \lor x \lor y^* \approx 1.
\end{align*}

$V(D_1)$ was axiomatized in [14]. Here are more bases for it.

COROLLARY 4.19. Each of the following identities is an equational base for $V(D_1)$, mod $RDMcmSH_1$:
\begin{align*}
(1) & \; y \to [0 \to (y \to x)] \approx y \lor (y \to x) \\
(2) & \; x \lor (y \to z) \approx (x \lor y) \to (x \lor z) \\
(3) & \; x \lor [y \to (y \to x)^*] \approx x \lor y \lor (y \to x) \\
(4) & \; [\{x \lor (x \to y^*)] \to (x \to y^*) \lor x \lor y^* \approx 1.
\end{align*}
\[ y \lor (y \rightarrow (x \lor y)) \approx (0 \rightarrow x) \lor (x \rightarrow y) \]
\[ x \lor [y \rightarrow (y \rightarrow x)] \approx x \lor y \lor (y \rightarrow x) \]
\[ \{x \lor (x \rightarrow y^*) \rightarrow (x \rightarrow y^*)\} \lor x \lor y^* \approx 1 \]
\[ x \lor (y \rightarrow z) \approx (x \lor y) \rightarrow (x \lor z). \]

**COROLLARY 4.20.** The variety \( V(\text{L}^{\text{dm}}_1, \text{L}^{\text{dm}}_2, \text{L}^{\text{dm}}_3, \text{D}_1, \text{D}_2, \text{D}_3) \) is defined, mod \( \text{RDMSH}_1 \), by
\[ x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z). \]

**COROLLARY 4.21.** The variety \( V(\text{C}^{\text{dm}}_{10}) \) is defined, mod \( \text{RDMSH}_1 \), by
\[ x \land x' \leq y \lor y' \text{ (Kleene identity)}. \]

**COROLLARY 4.22.** The variety \( V(\text{L}^{\text{dm}}_{10}) \) is defined, mod \( \text{RDMSH}_1 \), by
\[ x \land x' \leq y \lor y' \text{ (Kleene identity)} \]
\[ x \rightarrow y \approx y \rightarrow x. \]

5. **Lattice of subvarieties of \( \text{RDQDStSH}_1 \)**

We now turn to describe the lattice of subvarieties of \( \text{RDQDStSH}_1 \) which contains \( \text{RDMSH}_1 \) in view of Theorem 3.7. For this purpose we need the following theorem which is proved in [15].

**THEOREM 5.1.** Let \( \text{L} \in \text{RDQDStSH}_1 \). Then TFAE:
\[ (1) \text{L} \text{ is simple} \]
\[ (2) \text{L} \text{ is subdirectly irreducible} \]
\[ (3) \text{L} \in \{\text{2}^e, \bar{\text{2}}^e\} \cup \text{C}_{20} \cup \{\text{D}_1, \text{D}_2, \text{D}_3\}. \]

Let \( \mathcal{L} \) denote the lattice of subvarieties of \( \text{RDQDStSH}_1 \). \( \text{T} \) denotes the trivial variety, and, for \( n \) a positive integer, \( \text{B}_n \) denotes the \( n \)-atom Boolean lattice. We also let \( 1 + \text{B} \) denote the lattice obtained by adding a new least element 0 to the Boolean lattice \( \text{B} \).

**THEOREM 5.2.** \( \mathcal{L} \cong (1 + \text{B}_9) \times (1 + \text{B}_5) \times \text{B}_9 \).
Proof. Let $S_1 := \{L_{dm}^i : i = 1, 2, 3, 4\} \cup \{L_{dp}^i : i = 1, 2, 3, 4\} \cup \{D_2\}$, $S_2 := \{L_{dm}^i : i = 9, 10\} \cup \{L_{dp}^i : i = 9, 10\} \cup \{D_1\}$, and $S_3 := \{L_{dm}^i : i = 5, 6, 7, 8\} \cup \{L_{dp}^i : i = 5, 6, 7, 8\} \cup \{D_3\}$. Observe that each of the simples in $S_1$ contains $2^e$. Let us first look at the interval $[V(2^e), V(S_1)]$. Since each algebra in $S_1$ is an atom in this interval, we can conclude that the interval is a 9-atom Boolean lattice; thus the interval $[T, V(S_1)]$ is isomorphic to $1 + B_9$. Similarly, since each of the simples in $S_2$ contains $\overline{2}^e$, it is clear that the interval $[T, V(S_2)]$ is isomorphic to $1 + B_5$. Likewise, since each of the simples in $S_3$ has only one subalgebra, namely the trivial algebra, the interval $[T, S_3]$ is isomorphic to $B_9$. Observe that the intersection of the subvarieties $V(S_1)$, $V(S_2)$ and $V(S_3)$ is $T$ and their join is $RDQDSH_1$ in $L$. It, therefore, follows that $L$ is isomorphic to $(1 + B_9) \times (1 + B_5) \times B_9$. □

COROLLARY 5.3. The lattice of subvarieties of $RDMSH_1$ is isomorphic to $(1 + B_5) \times (1 + B_3) \times B_5$.

COROLLARY 5.4. The lattice of subvarieties of $RDPCSH_1$ is isomorphic to $(1 + B_4) \times (1 + B_2) \times B_4$.

Similar formulas can be obtained for other subvarieties of $RDQDSH_1$.

6. Amalgamation

We now examine the Amalgamation Property for subvarieties of the variety $RDQDStSH_1$. For this purpose we need the following theorem from [3].

THEOREM 6.1. Let $K$ be an equational class of algebras satisfying the Congruence Extension Property, and let every subalgebra of each subdirectly irreducible algebra in $K$ be subdirectly irreducible. Then $K$ satisfies the Amalgamation Property if and only if whenever $A$, $B$, $C$ are subdirectly irreducible algebras in $K$ with $A$ a common subalgebra of $B$ and $C$, the amalgam $(A; B, C)$ can be amalgamated in $K$.

THEOREM 6.2. Every subvariety of $RDQDStSH_1$ has the Amalgamation Property.

Proof. It follows from [14] that $RDQDStSH_1$ has CEP. Also, it follows from Theorem 5.1 that every subalgebra of each subdirectly irreducible (= simple) algebra in $RDQDStSH_1$ is subdirectly irreducible. Therefore, in each subvariety $V$ of $RDQDStSH_1$, we need only consider
an amalgam \((A : B, C)\), where \(A, B, C\) are simple in \(\text{RDQDStSH}_1\) and \(A\) a subalgebra of \(B\) and \(C\). Then it is not hard to see, in view of the description of simples in \(\text{RDQDStSH}_1\) given in Theorem 5.1 that \((A : B, C)\) can be amalgamated in \(V\).

\[\square\]

7. Concluding Remarks and Open Problems

We know from [14] that every simple algebra in \(\text{RDQDH}_1\) is quasiprimal.

Of all the 25 simple algebras in \(\text{RDQDStSH}_1\), \(2^e, \bar{2}^e,\) and \(L_i, i = 5, 6, 7, 8,\) and \(D_3\) are primal algebras and the rest are semiprimal algebras. We now mention some open problems for further research.

Problem 1: For each variety \(V(L)\), where \(L\) is a simple algebra in \(\text{RDMSH}_1\) (except \(V(2^e)\)), find a Propositional Calculus \(P(V)\) such that the equivalent algebraic semantics for \(P(V)\) is \(V(L)\) (with 1 as the designated truth value, using \(\rightarrow\) and \(\not\) as implication and negation respectively). (For the variety \(V(2^e)\), the answer is, of course, well known: Classical Propositional Calculus.)

We think such (many-valued) logics will be of interest in computer science and in switching circuit theory.

Problem 2: Describe simples in the variety of pseudocommutative \(\text{RDQDStSH}_1\)-algebras.

Problem 3: Find equational bases for the remaining subvarieties of \(\text{RDMSH}_1\).

Problem 4: Let \(\text{RDmsStSH}_1\) denote the subvariety of \(\text{DQDStSH}_1\) defined by: \((x \lor y) \approx x' \land y'\). Describe simples in \(\text{RDmsStSH}_1\).

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