MODULI OF SINGULAR CURVES

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ABSTRACT. The purpose of this note is to prove that there is an algebraic stack \(\mathcal{U}\) parameterizing all curves. The curves that appear in the algebraic stack \(\mathcal{U}\) are allowed to be arbitrarily singular, non-reduced, disconnected, and reducible. We also prove the boundedness of the open substack of \(\mathcal{U}\) parameterizing geometrically connected curves with fixed arithmetic genus \(g\) and \(\leq e\) irreducible components. This is an expanded version of [Smy09, Appendix B].

1. INTRODUCTION

Fix a scheme \(S\). For an \(S\)-scheme \(T\), a \(T\)-curve is defined to be a proper, flat, and finitely presented morphism of algebraic spaces \(\pi: C \to T\), where the geometric fibers have dimension 1. By [Knud71, Theorem V.4.9], [Har77, Exercise III.5.8] and [EGA IV, 9.1.5], the geometric fibers of a \(T\)-curve are projective. Let \(\text{Sch}/S\) denote the category of \(S\)-schemes and define \(\mathcal{U}_S\) to be the \(\acute{e}tale\) stack over \(\text{Sch}/S\), which assigns to each \(S\)-scheme \(T\), the groupoid of \(T\)-curves.

It is tempting to restrict attention to \(T\)-curves \(\pi: C \to T\), where the map \(\pi\) is projective. Indeed, if \(T\) is an affine scheme, then any smooth \(T\)-curve \(X \to T\) with geometric fibers of genus \(g \neq 1\) is a projective \(T\)-scheme. In the case that \(g = 1\), there is an example due to M. Raynaud, which appears in [Ray70, XIII-3.1], of a family of elliptic curves, over an affine base, which is Zariski locally projective, but not projective. There is also an example due to D. Fulghesu, appearing in [Ful09, Example 2.3], of a proper algebraic 3-fold, fibered over a projective surface, which is a family of nodal curves of genus 0, with at most 2 nodes in each fiber, which is not a scheme. In particular, this family is not Zariski locally projective. Thus when parameterizing singular curves, the total spaces of the families are required to be algebraic spaces. We will prove the following:

**Theorem 1.1.** \(\mathcal{U}_S\) is an algebraic stack, locally finitely presented over \(S\), with quasi-compact and separated diagonal. There is an explicit, smooth cover of \(\mathcal{U}_S\) by Hilbert schemes of projective spaces.

We note that proofs of the algebraicity of \(\mathcal{U}_S\) have recently appeared in [dHS08, Prop. 2.3] and [Lun09] using Artin’s Criterion [Art74, Thm. 5.3]. We provide a proof logically independent of Artin’s Criterion [loc. cit.], by constructing an explicit presentation by Hilbert schemes of projective spaces. Theorem 1.1 and the corollaries that follow were used by [Smy09] in the production of alternate compactifications of \(M_{g,n}\).

**Corollary 1.2.** If \(C \to \text{Spec} k\) is a projective curve, then it has a versal deformation space defined by equations with integral coefficients.

We observe that Corollary 1.2 is a trivial corollary of Theorem 1.1 yet at face value it is entirely non-obvious. For example, if you were to consider a complex curve \(C \to \text{Spec} \mathbb{C}\), with defining equations in some embedding into \(\mathbb{P}^N\) having lots of transcendental terms, then you would certainly not expect the deformation theory to be governed by equations with integral co-efficients.

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coefficients. Since the versal deformation of a rigid curve is itself, we immediately obtain the following partial answer to a speculation of R. Vakil in \[Vak06\]:

**Corollary 1.3.** If \(C \to \text{Spec } k\) is a rigid, projective curve, then every singularity type of \(C\) is defined over \(\mathbb{Z}\).

The following corollaries show that from the construction of the algebraic stack \(\mathcal{U}_S\), one easily obtains fine moduli stacks of essentially every other moduli problem associated to curves.

**Corollary 1.4.** The stack \(\mathcal{U}_{S,n}\) whose objects are curves + \(n\) arbitrary sections is algebraic, locally finitely presented over \(S\), with quasi-compact and separated diagonal.

**Corollary 1.5.** One may impose any number of the following extra conditions on the morphisms in \(\text{Obj } C_S\) and still obtain an algebraic \(S\)-stack which is locally finitely presented over \(S\), with quasi-compact and separated diagonal:

1. geometric fibers are \(R_n\);
2. geometric fibers are \(S_n\);
3. geometric fibers are lci;
4. geometric fibers are Cohen-Macaulay;
5. geometric fibers are reduced with \(k\) connected components;
6. geometric fibers are reduced;
7. geometric fibers are reduced and connected;
8. geometric fibers are reduced, connected, and have \(e\) or fewer irreducible components;
9. geometric fibers are integral;
10. geometric fibers have arithmetic genus \(g\);
11. geometric fibers have quasi-finite automorphism group;
12. geometric fibers have no infinitesimal automorphisms;
13. any condition on a flat family of curves specified by a condition on an open function (e.g. a polynomial) in the cohomology groups on the fibers of a finite set of complexes of sheaves with coherent cohomology (possibly not flat), all of which respect pullback along the base (for example \(h^1(L_{X/T}) = 3\)).

In particular, (11) defines the largest substack of \(\mathcal{U}\) with quasi-finite diagonal, and (12) defines the largest Deligne-Mumford substack.

We also prove the following boundedness result, such a result was believed to exist, but there was no proof in the literature.

**Corollary 1.6.** For any fixed triple of integers \((g, n, e)\), the stack \(\mathcal{W}_{S,g,n,e}\) corresponding to geometrically connected, reduced curves of arithmetic genus \(g\), with \(n\) marked points, and \(e\) or fewer irreducible components is algebraic, finitely presented over \(S\), with quasi-compact and separated diagonal.

Theorem 1.1 and its corollaries will be proved in the subsequent sections.

2. Étale Local Projectivity

We will show that a \(T\)-curve \(C \to T\) is étale-locally projective. Note that this result is an immediate consequence of Artin approximation, but we provide an independent proof. An important preliminary observation is that \(\mathcal{U}_S\) is a limit preserving stack. That is, if \(\{X_j\}_{j \in J}\) is an inductive sequence of \(S\)-rings, we set \(A = \varinjlim_j A_j\), then the natural transformation:

\[
\varinjlim_j (\mathcal{U}_S)_{\text{Spec } A_j} \longrightarrow (\mathcal{U}_S)_{\text{Spec } A}
\]
is an equivalence of categories. In concrete terms, it means that if you have a Spec $A$-curve $X \to \text{Spec} A$, there is some $j \in J$ and a Spec $A_j$-curve $X_j \to \text{Spec} A_j$ such that $X_j \otimes_{A_j} A \to X$ is an isomorphism and that for any isomorphism of Spec $A$-curves $X \to Y$. Moreover, there is a $k \in J$ and Spec $A_k$-curves $X_k, Y_k$ together with an isomorphism of Spec $A_k$-curves $X_k \to Y_k$ such that this pulls back to the isomorphism of Spec $A$-curves $X \to Y$. This is a somewhat technical condition to verify, but it is very useful in the sense that it means the resulting moduli stack is locally of finite presentation, and it allows one to usually reduce arguments to the noetherian (even excellent) case.

The proof that $\mathcal{U}_S$ is limit preserving is standard, we will merely provide the references sufficient to prove the result. To obtain essential surjectivity, combine one of the reductions used in the proof of [LMB] Prop. 4.18 with [EGA] IV, 4.1.4] (for the dimension of fibers), [EGA] IV, 8.10.5(xii)] and [Knudsen] Thm. IV.3.1] (for the properness), and [EGA] IV, 11.2.6] (for the flatness). The techniques of [EGA] IV, 8.8.2.5] garner full faithfulness.

**Proposition 2.1.** Let $\pi : C \to S$ be a proper, finitely presented morphism of algebraic spaces. Let $s \in S$ be a closed point such that $\dim_{\pi(s)} C_s \leq 1$, then there is an étale neighbourhood $(U, \mathfrak{U})$ of $(S, \mathfrak{s})$ such that $C \times_S U \to U$ is projective.

**Proof.** The statement is local on $S$ for the étale topology and by standard limit methods, we reduce immediately to the following situation: $S = \text{Spec} R$, where $R$ is an excellent, strictly henselian local ring and $s \in S$ is the unique closed point.

First, assume that $C$ is a reduced scheme. Now, let $C_s \to s$ denote the special fiber of $C \to S$. Since $C_s$ is a proper scheme of dimension $1$ over a field, it is manifestly projective. Thus, it suffices to show that the map $\text{Pic}(C) \to \text{Pic}(C_s)$ is surjective. Indeed, one can then conclude that $C$ admits a line bundle $L$ such that the restriction to the central fiber is projective. By [EGA] III, 4.7.1], we deduce that $L$ is ample.

For this paragraph we utilize the arguments in [SGA4 1/2] Prop. 4.1.4]. Let $\mathcal{L}_s$ be a line bundle on $C_s$. Since $C_s \to \text{Spec} k(s)$ is a projective curve, to show that $\text{Pic}(C) \to \text{Pic}(C_s)$ is surjective, it suffices to treat the case where $\mathcal{L}_s = \mathcal{O}_{C_s}(-x_s)$, for some closed point $x_s \in C_s$. In an open neighborhood $U_x$ of $x_s \in C_s$, we have that $x_s = V(f_x) \cap U_x$, for some $f_x \in \mathcal{O}_{C_s}(U_x)$ which is not a $0$-divisor. Since $C_s \to C$ is a closed immersion, there is an open subscheme $U \subset C$ such that $U \cap C_0 = U_0$. By shrinking $U$, we may lift the equation $f_x \in \mathcal{O}_{C_0}(U_0)$ to $f \in \mathcal{O}_C(U)$ such that $f$ is not a $0$-divisor, and $V(f) \cap U \cap C_s = \{x_s\}$. In particular, the map $V(f) \cap U \to S$ is quasi-finite and separated. Since $S$ is local and strictly henselian, by [EGA] IV, 18.12.3], there is a decomposition $V(f) \cap U = V_1 \amalg V_2$, where $V_1 \to S$ is finite and contains $\pi^{-1}(s)$. Thus, by further shrinking $U$, we may assume that the map $V(f) \cap U \to S$ is finite. On $C$ we may now define an effective cartier divisor $D$ as $D \mid_{C \cap V(f) \cap U} = 0$ and $D \mid U = \text{div} f$. The cartier divisor $0(-D)$ has the property that $0_{C_s}(-D) = 0_{C_s}(-x_s)$. Since $C$ is reduced and noetherian, by [EGA] IV, 21.3.4], $\text{Pic}(C) \to \text{Pic}(C_s)$ is surjective.

If $C$ is a non-reduced scheme, and $C_{\text{red}}$ is the reduction, then we have shown that the morphism $C_{\text{red}} \to S$ is projective. Since $C$ is noetherian, if $\mathfrak{j}$ denotes the nilradical of $C$, then there is a $k$ such that $\mathfrak{j}^k = (0)$. Thus, it suffices to prove the following: if $i : C' \to C$ is a closed immersion over $S$, defined by a square 0 ideal $\mathfrak{j}$ such that $C'$ is projective, then $C$ is projective.

To this end, we recall the exponential sequence on $C$:

$$0 \longrightarrow \mathfrak{j} \longrightarrow \mathfrak{o}_{C}^X \longrightarrow i_* \mathfrak{o}_{C'}^X \longrightarrow 1.$$  

By taking the long exact sequence of cohomology, we see that the obstruction to lifting a line bundle on $C'$ to a line bundle on $C$ lies in the cohomology group $H^2(C', \mathfrak{j})$. Since, $C'$ is a projective $S$-curve, we have that $H^2(C', \mathfrak{j}) = 0$. Consequently, we deduce that $\text{Pic} C \to \text{Pic} C'$ is
surjective. Hence, we may lift an ample bundle on $C'$ to a line bundle on $C$, and any such lift must be ample.

We now treat the case where $C$ is an algebraic space, and it remains to show that it is a scheme. By [LMB Thm. 16.6], there is a finite and surjective $S$-map $\tilde{C} \rightarrow C$, where $\tilde{C}$ is a scheme. Since $\tilde{C}$ is a proper $S$-scheme, with special fiber of dimension $\leq 1$, we may conclude that $C$ is a projective $S$-scheme. In particular, $C$ has the Chevalley-Kleiman property (i.e. every finite set of points is contained in an open affine). Since $S$ is excellent, we may apply [Kol08 Cor. 48] to conclude that $C$ has the Chevalley-Kleiman property, thus is a scheme. □

3. Representability of the Diagonal

In this section, we will prove that the diagonal morphism $\Delta : \mathcal{U}_S \rightarrow \mathcal{U}_S \times_S \mathcal{U}_S$ is representable, locally of finite presentation, separated and quasicompact. M. Artin, in [Art74], calls this relative representability and as we will see, it is an essential and natural part of the proof of algebraicity of $\mathcal{U}_S$. Fix an $S$-scheme $T$ and let $g_1 : C_1 \rightarrow T$, $g_2 : C_2 \rightarrow T$ be two $T$-curves. We form the 2-cartesian diagram:

$$
\begin{array}{ccc}
T \times_{s_1, \mathcal{U}_S, s_2} T & \longrightarrow & T \times_S T \\
\downarrow & & \downarrow \phi_{(s_1, s_2)} \\
\mathcal{U}_S & \longrightarrow & \mathcal{U}_S \times_S \mathcal{U}_S
\end{array}
$$

where the $s_i$ are the induced maps to $\mathcal{U}_S$ defined by the $T$-curve $g_i$. The 2-fiber product, $T \times_{s_1, \mathcal{U}_S, s_2} T$, is isomorphic to the $(\mathbf{Sch}/T)_{/\mathcal{U}_S}$-sheaf of isomorphisms $\text{Isom}_{T}(g_1, g_2)$. That is, the sections over a $T$-scheme $\phi : T' \rightarrow T$ are $T'$-isomorphisms $\phi : \phi^* g_1 \rightarrow \phi^* g_2$.

To prove that $\Delta$ is representable, quasi-compact and separated, we must show that the sheaf $\text{Isom}_{T}(g_1, g_2)$ is an algebraic space which is quasi-compact and separated over $T$. Also, there is a $(\mathbf{Sch}/T)_{/\mathcal{U}_S}$-sheaf $\text{Hom}_{T}(g_1, g_2)$ whose sections over a morphism $\phi : T' \rightarrow T$ are the $T'$-morphisms $\phi : \phi^* g_1 \rightarrow \phi^* g_2$. One observes that $\text{Isom}_{T}(g_1, g_2)$ is a subsheaf of $\text{Hom}_{T}(g_1, g_2)$.

We recall the definition of the Hilbert functor for a $T$-scheme $X \rightarrow T$: let $T' \rightarrow T$ be a morphism of schemes, let $\text{Hilb}_{T'/T}(X)$ be the set of isomorphism classes of closed subschemes $Z \rightarrow X \times_T T'$ which are flat, proper, and finitely presented over $T'$. Clearly, $\text{Hilb}_{T'/T} : (\mathbf{Sch}/T)_{/\mathcal{U}_S} \rightarrow \text{Sets}$ is a sheaf.

There is a natural transformation $\Gamma : \text{Hom}_{T}(g_1, g_2) \rightarrow \text{Hilb}_{C_1 \times_T C_2/T}$ which associates to any $T'$-morphism $f : C_1 \times_T T' \rightarrow C_2 \times_T T'$ its graph $\Gamma_f$.

Lemma 3.1. Suppose that $g_1 : C_1 \rightarrow T$, $g_2 : C_2 \rightarrow T$ are objects of $\mathcal{C}_S$, then the $(\mathbf{Sch}/T)_{/\mathcal{U}_S}$-sheaves $\text{Hom}_{T}(g_1, g_2)$ and $\text{Isom}_{T}(g_1, g_2)$ are both representable by finitely presented and separated algebraic $T$-spaces.

Proof. By Proposition 2.1, there is an étale surjection $\phi : U \rightarrow T$ such that for $i = 1, 2$, the pullbacks, $g_{i, U} : C_i \times_T U \rightarrow U$, are projective, flat and finitely presented. The inclusions $\text{Isom}_{U_i}(g_{1, U}, g_{2, U}) \subseteq \text{Hom}_{U_i}(g_{1, U}, g_{2, U}) \subseteq \text{Hilb}_{(C_1 \times_T U \times_C C_2, T) \times_T U/U}$ are representable by finitely-presented open immersions. Indeed, $U_S$ is limit preserving, so we may assume that $T$ is noetherian. The first inclusion is covered by [EGA II, 4.6.7(ii)] (without any dimension hypotheses on the fibers of $C_i$ over $T$). We observe that the assertion for the second inclusion follows from the first. Indeed, the latter inclusion is given by the graph homomorphism and it has image those families of closed subschemes of $(C_1 \times_T C_2) \times_T U$ for which projection onto the first factor is an isomorphism, which as we have already seen is an open condition.

From the existence of the Hilbert scheme for finitely presented projective morphisms, we make the following two observations:
(1) \( \text{Hom}_T(g_1, g_2) \times_T U \simeq \text{Hom}_U(g_1, U, g_2, U) \) is representable by a separated and locally of finite type \( S \)-scheme. In particular, the morphism \( \text{Hom}_U(g_1, U, g_2, U) \to \text{Hom}_T(g_1, g_2) \) is étale and surjective.

(2) The map \( \text{Hom}_U \times_{\text{Hom}_T} \text{Hom}_U \to \text{Hom}_U \times_{U} \text{Hom}_U \) is a closed immersion. Indeed, this is simply the locus where two separated morphisms of schemes agree.

Putting these together, one concludes that \( \text{Hom}_T(g_1, g_2) \) is representable by a separated and locally of finite type algebraic \( T \)-space. Since finitely presented open immersions are local for the étale topology, we deduce the corresponding result for \( \text{Isom}_T(g_1, g_2) \).

All that remains is to verify that \( \text{Hom}_T(g_1, g_2) \) is quasiprojective in the case that the \( g_i \) are projective. We plagiarize the argument of [dHS08] and include it for completeness only. Let the étale topology, we deduce the corresponding result for \( \text{Isom}_T(g_1, g_2) \).

4. Existence of a Smooth Cover

To construct a smooth cover of the stack \( \mathcal{U}_S \) by a scheme, we need to understand the deformation theory of singular curves. A good introduction to deformation theory is contained in [Ser06] and [FGI+05]. Our setup will be slightly different than what appears in those sources, however.

Throughout, we assume that \( k \) is an \( S \)-field, not necessarily algebraically closed. Let \( \text{Art}_k \) denote the category with objects \( (A, \iota) \), where \( A \) is a local artinian \( S \)-algebra, with maximal ideal \( m_A \), and an \( S \)-map \( \iota : A \to k \). The map \( \iota \) automatically induces an isomorphism of \( S \)-fields \( \iota : A/m_A \to k \). The morphisms in \( \text{Art}_k \) are the obvious ones. If \( X \) is a \( k \)-scheme, define the functor of \( S \)-deformations \( \text{Def}_X : \text{Art}_k \to \text{Sets} \) as follows. For \( (A, \iota) \in \text{Art}_k \), \( \text{Def}_X(A, \iota) \) is the set of isomorphism of classes of cartesian diagrams:

\[
\begin{array}{c}
\text{Spec} k \\
\downarrow \\
\text{Spec} A/m_A \\
\downarrow \\
\text{Spec} A
\end{array}
\quad \to 
\begin{array}{c}
X \\
\downarrow \\
X \otimes_k A/m_A \\
\downarrow \\
X'
\end{array}
\]

where \( X' \to \text{Spec} A \) is flat. Note that if we have a morphism of deformations \( X' \to X'' \), then since we necessarily have an isomorphism \( X' \otimes_k A/m_A \to X'' \otimes_k A/m_A \), then \( X' \to X'' \) is an isomorphism by the flatness over \( A \).

Let \( Y \) be an \( S \)-scheme, if \( j : X \to Y \otimes_S k \) (when the context is clear, we will henceforth write \( X \subset Y \otimes_S k \)) is a closed immersion of \( k \)-schemes, then define the embedded deformation
functor Def_{X\subset Y\otimes S^k} : Art_{k}^{\text{opp}} \to \text{Sets} as follows. For \((A,1) \in \text{Art}_k\), Def_{X\to Y\otimes S^k}(A,1) is the set of isomorphism of classes of cartesian diagrams:

\[
\begin{array}{ccc}
X & \rightarrow & X \otimes_k A/m_A \\
\downarrow & & \downarrow \\
Y & \rightarrow & Y \otimes_S A
\end{array}
\]

where \(X' \to \text{Spec} A\) is flat. The same argument as before shows that any map \(X' \to X''\) of embedded deformations is an isomorphism. There is an obvious natural transformation Def_{X\subset Y\otimes S^k} \to \text{Def}_X\) given by forgetting the embedding into \(Y\). Given \((A,1) \in \text{Art}_k\), we can define a functor \(\text{Spec}(A,1) : \text{Art}_{k}^{\text{opp}} \to \text{Sets}\) as \((A',1') \mapsto \text{Hom}_{\text{Art}_k}((A,1),(A',1'))\). Note that the Yoneda Lemma immediately implies that a map \(\text{Spec}(A,1) \to F\), where \(F\) is a functor \(F : \text{Art}_{k}^{\text{opp}} \to \text{Sets}\), is equivalent to an element of \(F(A,1)\).

A natural transformation of functors from \(\text{Art}_{k}^{\text{opp}} \to \text{Sets}\), \(F \to G\) is said to be \textbf{formally smooth} if for any surjection \((A,1) \to (A_0,1_0)\) and any diagram

\[
\begin{array}{ccc}
\text{Spec}(A_0,1_0) & \rightarrow & F \\
\downarrow & & \downarrow \\
\text{Spec}(A,1) & \rightarrow & G
\end{array}
\]

we may fill in the dashed arrow so that it commutes. Note that if \(f : (A,1) \to (A_0,1_0)\) is a surjection, then it may be factored into a sequence of surjections:

\[
(A,1) = (A_n,1_n) \to (A_{n-1},1_{n-1}) \to \cdots \to (A_1,1_1) \to (A_0,1_0)
\]

where \(m_{A_i}, \ker(A_i \to A_{i-1}) = 0\) (this is immediate from the Jordan-Hölder Theorem). We call such morphisms \textit{small extensions} and note that any such morphism has square 0 kernel.

The next two results are to be considered folklore in this generality. For similar statements, with stronger hypotheses, see for example [Ser06, Prop. 3.2.9] and [FGI05, Cor. 8.5.32].

**Theorem 4.1.** Suppose that \(X\) is a projective \(k\)-scheme, with \(h^2(\mathcal{O}_X) = 0\). Consider an embedding \(X \hookrightarrow \mathbb{P}^N\) such that \(h^1(X,\mathcal{O}_X(1)) = 0\), then \(\text{Def}_{X \subset \mathbb{P}^N} \to \text{Def}_X\) is formally smooth.

We will prove this in a moment. The following is a variant of [FGI05, Thm. 8.5.31], with a supplied proof.

**Proposition 4.2.** Let \(X\) be a proper \(k\)-scheme and consider an embedding \(j : X \hookrightarrow Y \otimes_S k\), where \(Y\) is a smooth \(S\)-scheme, then if \(H^1(X,j^*T_{Y\otimes S^k/k}) = 0\), \(\text{Def}_{X \subset Y\otimes S^k} \to \text{Def}_X\) is formally smooth.
Proof. Fix a small extension \((A_1, 1) \to (A_0, 1_0)\) and let \(K = A_1/m_{A_1} = A/m_{A_0}\). Consider a diagram

\[
\begin{array}{ccc}
X & \longrightarrow & Y \otimes_S k \\
\downarrow & & \downarrow \\
X \otimes_S k & \longrightarrow & Y_K := Y \otimes_S k \\
\downarrow & & \downarrow \\
X_0 & \longrightarrow & Y_{A_0} := Y \otimes_S A_0 \\
\downarrow & & \downarrow \\
X_1 & \longrightarrow & Y_{A_1} := Y \otimes_S A \\
\end{array}
\]

where \([X_1] \in \text{Def}_X(A_1), [X_0] \to Y_{A_0} \in \text{Def}_{X \otimes_S A_0}(A_0)\), and each restrict to \([X_0] \in \text{Def}_X(A_0)\).

To show that \(\text{Def}_{X \otimes_S k} \to \text{Def}_X\) is formally smooth, it suffices to construct a map \(X_1 \to Y_{A_1}\), since any such map is automatically a closed immersion. Indeed, the morphism is affine by using Serre’s Criterion, and by the Nakayama Lemma for modules over an artinian ring, it is a closed immersion, because it is a closed immersion modulo a nilpotent ideal.

For \(i = 0, 1\), let \(S_i = \text{Spec } A_i\), and consider the composition of morphisms \(X_0 \xrightarrow{f} Y_{A_1} \to S_1\).

By \([III] 2.1.2\) there is a distinguished triangle:

\[
f^*L_{Y_{A_1}/S_1} \longrightarrow L_{X_0/S_1} \longrightarrow L_{X_0/Y_{A_1}}.
\]

Note that since the closed immersion \(S_0 \to S_1\) is defined by a square 0 sheaf of ideals \(I\), it is supported on \(\text{Spec } K\) and hence \(S_0\). Let \(s_0 : X_0 \to S_0\) be the structure map, taking the long exact sequence associated to \(\text{Hom}_{X_0}(\cdot, s_0^*I)\), gives an exact sequence:

\[
\text{Ext}^1_{X_0}(L_{X_0/Y_{A_1}}, s_0^*I) \longrightarrow \text{Ext}^1_{X_0}(L_{X_0/S_1}, s_0^*I) \longrightarrow \text{Ext}^1_{X_0}(f^*L_{Y_{A_1}/S_1}, s_0^*I).
\]

Since \(Y_{A_1} \to S_1\) is smooth, then \(L_{Y_{A_1}/S_1} \cong \Omega_{Y_{A_1}/S_1}\), by \([III] 3.3\). In particular, if \(\sigma : Y_{A_0} \to Y_{A_1}\) is the inclusion and \(\sigma : Y_{A_0} \to Y_{A_1}\) is the induced map from \(S_0 \to S_1\), then \(\sigma_0 = f\) and

\[
f^*L_{Y_{A_1}/S_1} \cong f^*\Omega_{Y_{A_1}/S_1} \cong \sigma_*\sigma^*\Omega_{Y_{A_1}/S_1} \cong \sigma_*\Omega_{Y_{A_0}/S_0}
\]

since differentials pullback along the base. By \([EGA 0_{III}, 12.3.5]\) and \([EGA 0_{I}, 5.4]\), since \(\sigma_*\Omega_{Y_{A_0}/S_0}\) is locally of finite rank and \(s_0^*I\) is coherent, then

\[
\text{Ext}^1_{X_0}(f^*L_{Y_{A_1}/S_1}, s_0^*I) \cong \text{Ext}^1_{X_0}(X_0, \text{Hom}_{X_0}(\sigma_*\sigma^*\Omega_{Y_{A_0}/S_0}, s_0^*I)) \\
\cong \text{Ext}^1_{X_0}(X_0, \text{Hom}_{X_0}(\sigma^*\Omega_{Y_{A_0}/S_0}, \Omega_{X_0}), s_0^*I) \\
\cong \text{Ext}^1_{X_0}(X_0, \sigma_*T_{Y_{A_0}/S_0}, s_0^*I) \\
\cong \text{Ext}^1_{X_0}(X_0, \sigma_*T_{Y_{A_0}/S_0}/s_0^{-1}s_0 I) \\
\cong \text{Ext}^1_{X_0}(X_0, \sigma_*T_{Y_{A_0}/S_0}/s_0 I, s_0 I)
\]

because \(s_0 : X_0 \to S_0\) is flat. Noting that the coherent sheaf \(s_0^*\Omega_{Y_{A_0}/S_0}\) is flat over the artinian local scheme \(S_0\) and \(H^1(X_0, \sigma_*T_{Y_{A_0}/S_0}) = 0\), then \([Har77, Exercise III.11.8]\) implies that \(\text{Ext}^1_{X_0}(f^*L_{Y_{A_1}/S_1}, s_0^*I) = 0\). Thus, \(\text{Ext}_{X_0}(f^*L_{Y_{A_1}/S_1}, s_0^*I) = 0\).
We now apply III.7.1 to observe that our original exact sequence (together with the vanishing result proved above) provides a surjection:

\[
\text{Exal}_Y (\mathcal{O}_X, s_0) \longrightarrow \text{Exal}_Y (\mathcal{O}_{X_0}, s_0) \longrightarrow 0.
\]

In particular, \([X_0, s_0] \) is an element of \(\text{Exal}_Y (\mathcal{O}_X, s_0)\) and so there is an \(\mathcal{O}_{Y_1}\)-extension of \(\mathcal{O}_{X_0}\) by \(s_0\) corresponding to \(X_A\), (indeed, it is given by the sheaf of algebras \(\mathcal{O}_{X_A}\)). Hence, there is a map of sheaves of algebras \(\mathcal{O}_{Y_1} \to \mathcal{O}_{X_0}\) and consequently a morphism of schemes \(X_1 \to Y_1\), which extends \(X_0 \to Y_1\).

\[\square\]

**Proof of Theorem 1.1.** Since the Euler sequence is exact, we may pull it back and dualize it to obtain an exact sequence:

\[
0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \oplus (\mathcal{N} + 1) \longrightarrow \mathcal{V}^* \mathcal{T}_{\mathbb{P}^1} \longrightarrow 0.
\]

Taking the long exact sequence of cohomology, we arrive at the following segment:

\[
\text{H}^1(\mathcal{O}_X(1)) \oplus (\mathcal{N} + 1) \longrightarrow \text{H}^1(X, \mathcal{V}^* \mathcal{T}_{\mathbb{P}^1}) \longrightarrow \text{H}^2(\mathcal{O}_X).
\]

The assumptions ensure that \(\text{H}^1(X, \mathcal{V}^* \mathcal{T}_{\mathbb{P}^1}) = 0\). An application of Proposition 4.2 proves the result.

\[\square\]

With the relevant deformation theory in place, we can now complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** We have shown that \(U_S\) is a limit preserving stack over \((\text{Sch}/S)_{\text{fet}}\). In \(\text{[I]}\), we proved that the diagonal is represented by finitely presented, quasi-compact, and separated algebraic \(S\)-spaces. Thus, to show that \(U_S\) is an algebraic \(S\)-stack, it remains to construct an \(S\)-scheme \(U\) together with a smooth, surjective \(S\)-morphism \(U \to U_S\).

To prove the existence of a smooth cover, we consider a geometric point \(\text{Spec} k \to U_S\). This corresponds to \(C \to \text{Spec} k\), for some 1-dimensional projective scheme \(C\). Pick a \(k\)-embedding \(C \to \mathbb{P}^M_k\) such that \(\text{H}^1(\mathcal{O}_C(1)) = 0\). Let \(U_{C/k}\) denote an affine open neighbourhood of the point \(x : \text{Spec} k \to \text{Hilb} \mathbb{P}^M_{S/k}\) and take \(V_{C/k} \to U_{C/k}\) to denote the universal family. By Cohomology and Base Change \(\text{[Har77]}\), Thm. 12.11, we may replace \(U_{C/k}\) and \(V_{C/k}\) changes also by an affine open subset containing the image of \(x\) such that \(\text{h}^1(V_{C/k, v}, \mathcal{O}_{V_{C/k, v}}(1)) = 0\) and all fibers are flat of dimension 1 for all points \(v \to U_{C/k}\).

In particular, we obtain a finitely presented morphism \(U_{C/k} \to U_S\). We will now proceed to show that this morphism is in fact smooth. That is, if \(T \to U_S\) is an \(S\)-morphism, then \(p_T : T' := T \times_{U_S} U_{C/k} \to T\) is smooth. Since \(U_S\) is limit preserving, we may assume that \(T\) is noetherian. By taking a faithfully flat étale cover of \(T\), by Proposition 2.1, we may assume that the family of curves corresponding to the map \(T \to U_S\) is projective. Thus, the map \(p_T\) is a map of schemes. To show that the map \(p_T\) is smooth, by \(\text{[EGA]}\), IV, 17.14.2, it suffices to fix \(t' \in T',\) let \(t = p_T(t') \in T\) and we need to show that there is an arrow completing the diagram:

\[
\begin{array}{ccc}
\text{Spec} A/I & \longrightarrow & T' \\
\downarrow & & \downarrow \\
\text{Spec} A & \longrightarrow & U_{C/k}
\end{array}
\]

where \(A\) is an artin local ring, \(I \triangleleft A\) is an ideal, with the closed point of \(\text{Spec} A\) mapping to \(t\), and the residue field of \(A/I\) is the same as the residue field of \(t', K\). Let the \(\text{Spec} A\)-curve \(C_A \to \text{Spec} A\) denote that induced by the morphism \(\text{Spec} A \to U_S\). Then the 2-commutativity of
the diagram implies that there is a \( \text{Spec} A/I \)-embedding \( C_A \otimes (A/I) \to \mathbb{P}^M_{A/I} \). Let \( C_K \to \text{Spec} K \) denote the \( K \)-curve \( C_A \otimes_{A,K} K \). We have a map \( u : \text{Spec} K \to U_{C/K} \) and a \( K \)-isomorphism \( V_{C/K,u} \to C_K \), thus \( h^1(C_K, 0_{C_K}(1)) = 0 \). By Theorem \[4.1\] the map \( \text{Def}_{C_K \subset \mathbb{P}^M_K} \to \text{Def}_{C_K} \) is formally smooth, hence a map \( \text{Spec} A \to U_{C/K} \) exists completing the given diagram.

The isomorphism classes of morphisms \( Y \to \mathcal{U} \), where \( Y \) is an affine open subscheme of \( H_N \) for some \( N \), form a set. Take \( \mathcal{U} \) to be the disjoint union over all those such \( Y \to \mathcal{U} \) which are smooth. By the above, \( \mathcal{U} \to \mathcal{U} \) is smooth and surjective.

5. Proofs of the Corollaries

In this section, we run through the proofs of the corollaries.

Proof of Corollary \[1.4\] First, we show that the universal curve \( \mathcal{U}_{S,1} \) is an algebraic stack, which is locally of finite type over \( S \). This is obvious: one has the forgetful morphism \( \mathcal{U}_{S,1} \to \mathcal{U}_S \) (given by forgetting the section of the family) and this morphism is representable. Indeed, for an affine geometrically fibral, so we may work over a faithfully flat étale extension of the base \( \text{Spec} S \).

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Moreover, it suffices to consider those families of curves which are projective over an affine noetherian base. For the remainder, we fix an object \( f : C \to T \) of \( \mathcal{U}_S \), where \( T \) is the spectrum of the noetherian ring \( A \) and we assume that \( f \) is projective.

In this section, we run through the proofs of the corollaries.

Proof of Corollary \[1.5\] In all cases, we need to check that if \( C \to T \) is a \( T \)-curve, then the locus in \( T \) which satisfies the condition has a natural scheme structure. Cases (1)-(9) are all open conditions, by the results of \[EGA\ IV, \S 12.2\].

For the remainder of the cases, we may reduce to the noetherian case as follows: we will reduce to the case of a noetherian base and a projective family. Since \( \mathcal{U} \) is limit preserving, \( f \) factors as \( \text{Spec} A \to \text{Spec} A_0 \to \mathcal{U} \), where \( A_0 \) is of finite type over \( \mathbb{Z} \). All the conditions are geometrically fibral, so we may work over a faithfully flat étale extension of the base \( \text{Spec} A_0 \).

Hence, it suffices to consider those families of curves which are projective over an affine noetherian base. For the remainder, we fix an object \( f : C \to T \) of \( \mathcal{U}_S \), where \( T \) is the spectrum of the noetherian ring \( A \) and we assume that \( f \) is projective.

(10) \[Har77\ Cor. III.9.3\] shows this condition is open and closed.

(11) Follows immediately from Chevalley’s semicontinuity Theorem.

(12) It suffices to show that if \( C \to S \) is an object of \( \mathcal{U}_S \), where \( S \) is the spectrum of a noetherian ring \( A \), then \( S \times_S S \to S \) being unramified is an open condition on \( S \). This will follow from the more general assertion: let \( p : G \to S \) be a locally of finite type, group algebraic space, with \( S \) noetherian, then if \( \pi : \mathfrak{U} \) is a geometric point and the group scheme \( G_{\pi} \to \pi \) is unramified, then there is an open subscheme \( \mathfrak{U} \) of \( \mathfrak{U} \) such that \( G_{\mathfrak{U}} \to \mathfrak{U} \) is unramified. Observe that we can find an open subspace \( W \) containing \( G_{\pi} \subset G \) such that \( p|_W : W \to S \) is unramified. Let \( e : S \to G \) be the identity section and \( t : W \to G \) the immersion, then the fiber product \( \mathfrak{U} = W \times_G S \) is an open subscheme of \( S \). Moreover, for any geometric point \( \pi \to \mathfrak{U} \) we have \( G_{\pi} \to \pi \) is unramified on an open subscheme of the identity and by using translations in this group, we can cover it by unramified open subschemes.

(13) If the complex has flat cohomology over the base, then it is immediate from cohomology and base change. If the cohomology is not flat, take a flattening stratification (the morphism is projective, so these exist), then apply the earlier case. In this situation, you obtain a locally closed substack (as opposed to an open substack).
To prove Corollary 1.6 it remains to show that the stack is quasicompact. We proceed to prove the relevant boundedness results. The following argument is due to F. Van Der Wyck.

**Lemma 5.1.** Let \( k \) be an algebraically closed field and \( S \) a reduced curve singularity, then \( S \) may be embedded in an affine space of dimension \( \leq (\delta_S + 1)^2 \).

**Proof.** First, suppose that \( S \) is unibranched, then it is a finitely generated subalgebra of \( k[[t]] \). Let \( f_1, \ldots, f_r \) denote a set of generators and we may assume that the degree of each \( f_i \) is distinct. Let \( M \) denote the semigroup generated by the degrees of the \( f_i \). Observe that if \( n = \min \{ \deg f_i \} \), then \( n \in M \). In particular, it follows that there are at most \( n - 2 \) other generators (by inspection of the residues), since the \( \deg f_i \) are all distinct. Note that \( n \leq \delta_S \) and so the embedding dimension for a unibranched singularity is \( \leq \delta_S + 1 \).

Now suppose there are \( r \) branches, then \( S \subseteq \prod_{i=1}^r k[[t_i]] \). Note that the \( \delta \) of a branch is bounded by \( \delta_S \) and the number of branches is bounded by \( \delta_S + 1 \). The former is obvious, the latter clear from the observation that \( S \) doesn’t contain the elements \( t_1 \) or \( (t_1, \ldots, t_r) \) and there are \( r + 1 \) of these. Hence, the embedding dimension is bounded by \( (\delta_S + 1)^2 \). \( \square \)

The following argument had inputs from D. Smyth and R. Vakil.

**Theorem 5.2.** Suppose that \( C \) is a connected curve with \( \leq e \) irreducible components with arithmetic genus \( g \), then there is an embedding \( C \hookrightarrow \mathbb{P}^{N_{g,e}} \) such that \( \deg C \leq D_{g,e} \), where

\[
\begin{align*}
N_{g,e} &= (g + e)^2 + 1, \\
D_{g,e} &= 2e(g + e - 1)(g + e) + e^2.
\end{align*}
\]

**Proof.** We first determine \( D_{g,e} \). Let \( C_{sm} \subset C \) denote the smooth locus, then \( C_{sm} \) is a disjoint union \( \coprod_{i=1}^e W_i \). For each \( i = 1, \ldots, e \), take \( p_i \in W_i \). Let \( Z \) be the divisor \( p_1 + \cdots + e \) and let \( \mathcal{L} = \mathcal{O}(Z) \), then \( \deg \mathcal{O}(Z) = e \). It suffices to find some \( m = m(\mathcal{L}) \) (depending only on \( g, e \)) such that \( \mathcal{L}^m \) is very ample. Indeed, we would then have \( D_{g,e} = me \). We need to show that \( \mathcal{L}^m \) separates points and tangent vectors. Thus, it remains to show that for any \( c \in C \):

\[
H^1(C, \mathcal{L}^m(-c)) = H^1(C, \mathcal{L}^m(-2c)) = 0.
\]

Using the standard exact sequence relating these two ideal sheaves, the vanishing of the former is determined by the vanishing of the latter.

Let \( \pi : \tilde{C} \to C \) be the normalization map and let \( c \in C \). Take \( c_1, \ldots, c_r \) to denote the points of the fiber \( \pi^{-1}(c) \) and let \( \delta_c \) be the \( \delta \)-invariant of \( c \), then there is an exact sequence

\[
0 \longrightarrow \pi_* \mathcal{O}_{\tilde{C}}(-2\delta_c(c_1 + \cdots + c_r)) \longrightarrow \mathcal{O}(-2c) \longrightarrow \mathcal{E} \longrightarrow 0,
\]

with \( \mathcal{E} \) supported only on \( c \). Twisting this exact sequence by \( \mathcal{L}^m \) (for some \( m \) yet to be determined) and taking the exact sequence of cohomology we obtain:

\[
H^1(C, \mathcal{L}^m \otimes \pi_* \mathcal{O}_{\tilde{C}}(-2\delta_c(c_1 + \cdots + c_r))) \longrightarrow H^1(C, \mathcal{L}^m(-2c)) \longrightarrow 0.
\]

Since \( \pi_* \) is exact, we obtain from the projection formula:

\[
H^1(C, \mathcal{L}^m \otimes \pi_* \mathcal{O}_{\tilde{C}}(-2\delta_c(c_1 + \cdots + c_r))) = H^1(\tilde{C}, (\pi^* \mathcal{L})^m(-2\delta_c(c_1 + \cdots + c_r))).
\]

Taking \( m_c = 2r_c\delta_c + e \) and applying [Har77] Cor. IV.3.3 and Exercise III.7.1 furnishes us with:

\[
H^1(\tilde{C}, (\pi^* \mathcal{L})^{m_c}(-2\delta_c(c_1 + \cdots + c_r))) = 0.
\]

We may conclude that \( H^1(C, \mathcal{L}^{m_c}(-2c)) = 0 \).
Hence, if we take \( m \), the dimension of every singularity of \( C \), we see that the number of branches over each singular point is bounded by \( g + e \). Continuing with these ideas, an application of Lemma 5.1 implies that the embedding \( \delta \) has dimension bounded by \( (g + e)^2 \).

Using our line bundle \( L^m \), we produce an embedding \( C \hookrightarrow \mathbb{P}^m \) and let \( \text{Sec}(C) \) be the secant variety of \( C \), this has dimension bounded by \( 3 \). Take \( \text{Tan}(C) \) to denote the tangent variety, then this has dimension bounded by \( (g + e)^2 + 1 \) by the above bound on embedding dimension. By choosing a point \( P \) not in \( \text{Sec}(C) \cup \text{Tan}(C) \) and projecting from it, we obtain \( N_{g,e} \leq (g + e)^2 + 1 \). Note that the degree of the embedding \( C \hookrightarrow \mathbb{P}^N \), remains \( \leq D_{g,e} \).

\[ \text{Proof of Corollary 1.6} \]

From the proof of Corollary 1.4, it suffices to show that \( \mathcal{M}_{g,e,0} \) is quasicompact. Let \( W_{g,e} \) denote the Hilbert scheme of curves in \( \mathbb{P}^{N_{g,e}} \), whose fibers are embedded curves of arithmetic genus \( g \), with less than \( e \) irreducible components and of degree \( \leq D_{g,e} \).

Note that \( W_{g,e} \) is quasicompact, since the component of the Hilbert scheme \( \text{oh} \) corresponding to a fixed Hilbert polynomial is projective and the Hilbert polynomial of a curve is completely determined by its degree and genus.

Let \( W_{g,e} \rightarrow W_{g,e} \) denote the universal family, then we have an induced morphism \( W_{g,e} \rightarrow \mathcal{M}_{g,e,0} \). By Theorem 5.2, this map is surjective. We conclude that \( \mathcal{M}_{g,e,0} \) is quasicompact.

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