SPECIAL KÄHLER GEOMETRY

Does there Exist a Prepotential?

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Abstract. A symplectically invariant definition of special Kähler geometry, proposed in [1], is discussed. Certain aspects hereof are illustrated by means of Calabi-Yau moduli spaces.

1. Introduction

Special (Kähler) geometry [2] is, by definition, the geometry of vectormultiplet scalars in N = 2 supergravity. However, one would like to define this geometry only referring to these scalars, not to any other fields. Therefore one needs to know the most general way of coupling vectormultiplets to supergravity. Originally, supergravity actions for vectormultiplets were constructed using a holomorphic ‘prepotential’. As turned out later, duality transformations can lead to actions for which a prepotential does not exist [4]. In [1] a formulation (‘definition’) of special geometry was given which is manifestly invariant under duality transformations. It was proved that this formulation is equivalent to the original one, in the sense that it is always possible to perform a duality transformation such that a prepotential exists in the dual formulation of the theory. Moreover, it describes all presently known examples of special geometry. All constraints imposed in this definition have a nice physical interpretation (related to duality invariance and positivity of the kinetic energy), except for one constraint in the special case of only one vectormultiplet. This exception suggests that one could try to construct a more general supergravity theory for one vectormultiplet, one that could not be encoded in a holomorphic prepotential.

The aim of this contribution is to review this evolution, simplifying the discussion by omitting a lot of details and concentrating on the points essential for the interpretation of the constraints imposed in [1].
2. \( N = 2 \) supergravity coupled to vectormultiplets

The (abelian) theories we are going to consider describe a SUGRA multiplet (whose bosonic dynamical degrees of freedom are the graviton and a vector, the graviphoton) and \( n \) vectormultiplets (each a vector and a complex scalar). Thus we have \( n + 1 \) vectors (described by \( \mathcal{F}_I^{\mu\nu}, I = 0, \ldots, n \)) and \( n \) complex scalars \( z^\alpha, \alpha = 1, \ldots, n \).

The scalars are interpreted as coordinates on a manifold \( \mathcal{M} \). This manifold allows projective coordinates

\[
Z^I(z^\alpha) \sim e^{f(z^\alpha)} Z^I(z^\alpha).
\]

There exists a holomorphic function \( F(Z^I) \), homogeneous of second degree, named the ‘prepotential’, from which the following \( 2n+2 \) component vector can be defined:

\[
v \equiv \left( \begin{array}{c} Z^I \\ \partial F/\partial Z^I \end{array} \right) \equiv \left( \begin{array}{c} Z^I \\ F_I \end{array} \right).
\]

The kinetic term \( L_0 \) of the scalars is given by

\[
L_0 = -e g_{\alpha\bar{\beta}} \partial_\mu z^\alpha \partial_{\bar{\mu}} \bar{z}^\beta,
\]

where \( e \) is the determinant of the vierbein and \( g_{\alpha\bar{\beta}} \) is a Kähler metric with Kähler potential

\[
K = -\ln[i < \bar{v}, v>].
\]

The combination of equations of motion and Bianchi identities is invariant under symplectic duality transformations

\[
v \rightarrow \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) v
\]

\[
N \rightarrow (C + DN)(A + BN)^{-1},
\]
with \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n + 2, \mathbb{R}) \). These transformations leave \( g_{\alpha \bar{\beta}} \) invariant.

As an example, consider \( n = 1 \) and \( F(Z^0, Z^1) = -iZ^0Z^1 \), with \( Z^0 = 1 \) and \( Z^1 = z \). Thus
\[
v = \begin{pmatrix} Z^I \\ F_I \end{pmatrix} = \begin{pmatrix} 1 \\ z \\ -iz \\ -i \end{pmatrix},
\]
elevated to \( 2(z + \bar{z}) \) and \( g_{\bar{z}z} = (z + \bar{z})^{-2} \). Now consider the following symplectic transformation:
\[
v \rightarrow \tilde{v} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} v = \begin{pmatrix} 1 \\ i \\ -iz \\ z \end{pmatrix}
\]

It is clear that no prepotential exists for \( \tilde{v} \), since \( \tilde{v} \) is not of the form (1) (its lower part is independent of its upper one).

3. Symplectically invariant definition

We define a special Kähler metric as a Kähler metric\(^1\) for which there exists a \( 2n + 2 \) component vector \( v(z^\alpha) \) such that (3) is satisfied and
\[
< v, D_\alpha v >= 0 ,
\]
where \( D_\alpha \equiv \partial_\alpha + (\partial_\alpha K) \) is covariant under \( v \rightarrow e^{f(z)} v \).

Using the notation \( v = \begin{pmatrix} Z^I \\ F_I \end{pmatrix} \), where \( F_I \) need not be the derivative of a prepotential, we can write down the following manifestly symplectically covariant form of \( N \):
\[
N_{IJ} = ( D_\alpha \bar{F}_I \quad F_I \quad ( D_\alpha \bar{Z}^J \quad Z^J )^{-1} .
\]

It was proved in [1] that this definition is equivalent to the prepotential-definition and that it includes all (presently) known supergravity actions.

Let us remark that the matrix \( \begin{pmatrix} D_\alpha \bar{Z}^J \quad Z^J \end{pmatrix} \) is always invertible, whereas the invertibility of \( \begin{pmatrix} D_\alpha Z^J \quad Z^J \end{pmatrix} \) is equivalent to the existence of a prepotential for \( v \).

\(^1\)We include positivity in the definition of a Kähler metric.
4. Interpretation of the constraint $< v, D_\alpha v > = 0$

The constraint (7) implies the positivity of the following matrix:

$$
\left( D_\alpha Z^I \right) i e^K (N - N^\dagger)_{IJ} \left( D_\beta \bar{Z}^J \ Z^J \right) = \begin{pmatrix} g_{\alpha \beta} & i < D_\alpha v, v > \\
< \bar{v}, D_\beta \bar{v} > & e^{-K} 
\end{pmatrix},
$$

which in turn implies both that \( \left( D_\alpha \bar{Z}^J \ Z^J \right) \) is invertible (guaranteeing that \( \mathcal{N} \) is well-defined) and, using the symmetry of \( \mathcal{N} \), that \( \text{Im} \, \mathcal{N} \) is negative (ensuring the positivity of the vector kinetic energy).

On the other hand, the symmetry of \( \mathcal{N} \) defined in (8) (needed for duality invariance) is equivalent to

$$
< D_\alpha v, D_\beta v > = 0 . \tag{10}
$$

Whereas (7) implies (10), the reverse only holds for \( n > 1 \).

This raises the question whether the constraint (7) can be relaxed for \( n = 1 \) to the combination of the positivity of (9), and (10). Solving this question means verifying whether it is possible to construct an \( N = 2 \) supergravity theory for a single vectormultiplet with a vector \( v \) satisfying both the positivity of (9) and (10) but not (7). This problem has not been solved yet.

5. Special geometry and Calabi-Yau moduli spaces

The low-energy description of a type II string theory compactified on a Calabi-Yau threefold is a \( d = 4, N = 2 \) supergravity theory. As such the (vector multiplet) scalar sector involved in such compactifications is expected to yield a special Kähler manifold. In this section we will outline how the various constraints of special Kähler geometry are realised in this concrete setting.

The interpretation of the special geometry relations goes basically via the following isomorphism

$$
H^3(X) \rightarrow \Sigma \\
\omega \mapsto v_\omega = \left( \int_{A^I} \omega \ \int_{B^J} \omega \right), \tag{11}
$$

where \( X \) is some generic CY threefold and \( \{ A^I, B^J \} \) a canonical basis of 3-cycles. \( \Sigma \) is a \( 2h^{2,1} + 2 \) complex dimensional vector space. The threeform cohomology may be endowed with the sesquilinear form \( Q \):

$$
Q(\eta, \omega) = i \int_X \eta \wedge \bar{\omega} ,
$$
which corresponds to a symplectic form \( i \langle v, \omega \rangle \) on \( \Sigma \). Note that a symplectic transformation on \( \Sigma \) correspond to a change of canonical homology basis. The form \( Q \) can be shown \([5]\) to enjoy the following properties:

1. It is block-diagonal in an appropriate cohomology basis. More explicitly, when expressed with respect to any cohomology basis adapted to the Hodge decomposition \( H^3 = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \), \( Q \) takes the following form
   \[
   \begin{pmatrix}
   H & 0 & 0 & 0 \\
   0 & -H_{\alpha\beta}^* & 0 & 0 \\
   0 & 0 & H_{\alpha\beta} & 0 \\
   0 & 0 & 0 & -H
   \end{pmatrix}
   \]
2. \( Q \) is positive-definite on \( H^{3,0} \oplus H^{1,2} \).

   Now consider a family of CY threefolds with varying complex structure and fixed Kähler class. In the generic case \( h^{2,1} \) complex parameters \( z^\alpha \) specify one particular member in this family. Furthermore fix a complex structure on this space of moduli such that the unique holomorphic threeform \( \Omega \) on \( X \) depends only holomorphically on these moduli \( z^\alpha \). As small variations of a \((p, q)\) form give at most \((p \pm 1, q \mp 1)\) forms, we find
   \[
   \partial_\alpha \Omega = -\kappa_\alpha \Omega \oplus \Omega_\alpha ,
   \]\
   \( \in H^{3,0} \oplus H^{2,1} \).

Thus we may define \( D_\alpha \Omega \equiv [\partial_\alpha + \kappa_\alpha] \Omega \). Moreover this set can be shown to constitute a basis for \( H^{2,1} \). We define further \( D_\alpha v \equiv \partial_\alpha v = 0 \). Under the mapping \((11)\) the threeforms \( \{\Omega, \Omega_\alpha\} \) are mapped to a corresponding set \( \{v, D_\alpha v\} \) of symplectic vectors in \( \Sigma \), with the mentioned properties induced.

Putting things together we are now in a position to state the main results of this whole setup. After specifying some arbitrary canonical homology basis we define \( v \) to be the image of the unique holomorphic threeform \( \Omega \) under the mapping \((11)\). This object \( v \) is the one central object in section 3. Furthermore \( D \) is defined as in the above paragraph.

The various properties of \( Q \) may now be properly translated into relations satisfied by \( v \) and \( D_\alpha v \). The quantity \(-i\langle v, \bar{v} \rangle\) is strictly positive as is correspondingly the matrix entry \( H \) in the bilinear form \( Q \) on cohomology. As a result we may define a real Kähler potential through \(-i\langle v, \bar{v} \rangle = e^{-K}\). It is an easy exercise to check that the Kähler potential thus defined appears in the derivative \( D_\alpha = \partial_\alpha + (\partial_\alpha K) \), precisely as in section 3. The positivity of \( Q \) on \( H^{1,2} \) is equivalent to the positivity of the metric which we define to be \( g_{\alpha\beta} = -i\langle D_\alpha v, D_\beta \bar{v} \rangle \). This metric coincides with the one obtained from the Kähler potential, \( g_{\alpha\beta} = \partial_\alpha \partial_\beta K \). The derivative \( D \) is covariant under holomorphic rescalings \( v \to e^{f(z)} v \), which in turn correspond to Kähler
transformations on the moduli space. From the block-diagonal form of $Q$ in the specified basis the following relations are immediate consequences:

\begin{align}
\langle v, D_\alpha v \rangle &= 0, \\
\langle v, D_\alpha \bar{v} \rangle &= 0.
\end{align}

As to the invertibility of the matrix \( \left( D_\alpha \bar{Z}^J Z^J \right) \) in section 3, first notice that it corresponds to \( \left( \int A^J \bar{\Omega} \int A_J \Omega \right) \) in the present context. The positivity of $Q$ on $H^{3,0} \oplus H^{1,2}$ guarantees its invertibility [1]. Replacing $H^{1,2}$ with $H^{2,1}$, no similar statement can be made, as a result of which \( \left( D_\alpha Z^J \bar{Z}^J \right) \), or equivalently \( \left( \int A_J \Omega_\alpha \int A_J \Omega \right) \), may not be invertible. Via an appropriate symplectic rotation on $\Sigma$ it is always possible to make it invertible, though [1]. A prepotential formulation of the special geometry of the CY moduli space is absent if the above matrix is non-invertible. This non-invertibility is due to a particular choice of 3-cycles.

Concerning section 4 it is obvious that in the present case the stronger constraint \( \langle v, D_\alpha v \rangle \) is always obeyed, as is implied by $Q$ and its properties once more.

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