The complexity of knapsack problems in wreath products

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Abstract

We prove new complexity results for computational problems in certain wreath products of groups and (as an application) for free solvable groups. For a finitely generated group we study the so-called power word problem (does a given expression \( u_1^{k_1} \cdots u_d^{k_d} \), where \( u_1, \ldots, u_d \) are words over the group generators and \( k_1, \ldots, k_d \) are binary encoded integers, evaluate to the group identity?) and knapsack problem (does a given equation \( u_1^{x_1} \cdots u_d^{x_d} = v \), where \( u_1, \ldots, u_d, v \) are words over the group generators and \( x_1, \ldots, x_d \) are variables, have a solution in the natural numbers). We prove that the power word problem for wreath products of the form \( G \wr \mathbb{Z} \) with \( G \) nilpotent and iterated wreath products of free abelian groups belongs to \( \text{TC}^0 \). As an application of the latter, the power word problem for free solvable groups is in \( \text{TC}^0 \). On the other hand we show that for wreath products \( G \wr \mathbb{Z} \), where \( G \) is a so called uniformly strongly efficiently non-solvable group (which form a large subclass of non-solvable groups), the power word problem is \( \text{coNP} \)-hard. For the knapsack problem we show \( \text{NP} \)-completeness for iterated wreath products of free abelian groups and hence free solvable groups. Moreover, the knapsack problem for every wreath product \( G \wr \mathbb{Z} \), where \( G \) is uniformly efficiently non-solvable, is \( \Sigma^p_2 \)-hard.

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1 Introduction

Since its very beginning, the area of combinatorial group theory [30] is tightly connected to algorithmic questions. The word problem for finitely generated (f.g. for short) groups lies at the heart of theoretical computer science itself. Dehn [8] proved its decidability for certain surface groups (before the notion of decidability was formalized). Magnus [31] extended this result to all one-relator groups. After the work of Magnus it took more than 20 years before Novikov [40] and Boone [4] proved the existence of finitely presented groups with an undecidable word problem (Turing tried to prove the existence of such groups but could only provide finitely presented cancellative monoids with an undecidable word problem).

Since the above mentioned pioneering work, the area of algorithmic group theory has been extended in many different directions. More general algorithmic problems have been studied and also the computational complexity of group theoretic problems has been investigated. In this paper, we focus on the decidability/complexity of two specific problems in group theory.
that have received considerable attention in recent years: the knapsack problem and the power word problem.

**Knapsack problems.** There exist several variants of the classical knapsack problem over the integers [20]. In the variant that is particularly relevant for this paper, it is asked whether a linear equation \( x_1 \cdot a_1 + \cdots + x_d \cdot a_d = b \), with \( a_1, \ldots, a_d, b \in \mathbb{Z} \), has a solution \( (x_1, \ldots, x_d) \in \mathbb{N}^d \). A proof for the NP-completeness of this problem for binary encoded integers \( a_1, \ldots, a_d, b \) can be found in [15]. In contrast, if the numbers \( a_i, b \) are given in unary notation then the problem falls down into the circuit complexity class TC^0 [9]. In the course of a systematic investigation of classical commutative discrete optimization problems in non-commutative group theory, Myasnikov, Nikolaev, and Ushakov [33] generalized the above definition of knapsack to any f.g. group \( G \): The input for the knapsack problem for \( G \) (KNAPSACK(\( G \)) for short) is an equation of the form \( g_1^{r_1} \cdots g_d^{r_d} = h \) for group elements \( g_1, \ldots, g_d, h \in G \) (specified by finite words over the generators of \( G \)) and pairwise different variables \( x_1, \ldots, x_d \) that take values in \( \mathbb{N} \) and it is asked whether this equation has a solution (in the main part of the paper, we formulate this problem in a slightly more general but equivalent way). In this form, KNAPSACK(\( \mathbb{Z} \)) is exactly the above knapsack problem for unary encoded integers studied in [9] (a unary encoded integer can be viewed as a word over a generating set \( \{t, t^{-1}\} \) of \( \mathbb{Z} \)). For the case where \( g_1, \ldots, g_d, h \) are commuting matrices over an algebraic number field, the knapsack problem has been studied in [1]. Let us emphasize that we are looking for solutions of knapsack equations in the natural numbers. One might also consider the variant, where the variables \( x_1, \ldots, x_d \) take values in \( \mathbb{Z} \). This latter version can be easily reduced to our knapsack version (with solutions in \( \mathbb{N} \)), but we are not aware of a reduction in the opposite direction.\(^1\)

Let us also mention that the knapsack problem is a special case of the more general rational subset membership problem [23].

We also consider a generalization of KNAPSACK(\( G \)): An exponent equation is an equation of the form \( g_1^{r_1} \cdots g_d^{r_d} = h \) as in the specification of KNAPSACK(\( G \)), except that the variables \( x_1, \ldots, x_d \) are not required to be pairwise different. *Solvability of exponent equations* for \( G \) (EXP_EQ(G) for short) is the problem where the input is a conjunction of exponent equations (possibly with shared variables) and the question is whether there is a joint solution for these equations in the natural numbers.

Let us give a brief survey over the results that were obtained for the knapsack problem in [33] and successive papers:

- Knapsack can be solved in polynomial time for every hyperbolic group [33]. Some extensions of this result can be found in [11, 26].
- There are nilpotent groups of class 2 for which knapsack is undecidable. Examples are direct products of sufficiently many copies of the discrete Heisenberg group \( H_3(\mathbb{Z}) \) [21], and free nilpotent groups of class 2 and sufficiently high rank [37]. In contrast, knapsack for \( H_3(\mathbb{Z}) \) is decidable [21]. It follows that decidability of knapsack is not preserved under direct products.
- Knapsack is decidable for every co-context-free group [21], i.e., groups where the set of all words over the generators that do not represent the identity is a context-free language. Lehnert and Schweitzer [22] have shown that the Higman-Thompson groups

\(^1\) Note that the problem whether a given system of linear equations has a solution in \( \mathbb{N} \) is NP-complete, whereas the problem can be solved in polynomial time (using the Smith normal form) if we ask for a solution in \( \mathbb{Z} \). In other words, if we consider the knapsack problem for \( \mathbb{Z}^n \) with \( n \) part of the input, then looking for solutions in \( \mathbb{N} \) seems to be more difficult than looking for solutions in \( \mathbb{Z} \).
are co-context-free.

- Knapsack belongs to \( \text{NP} \) for all virtually special groups (finite extensions of subgroups of graph groups) [24]. The class of virtually special groups is very rich. It contains all Coxeter groups, one-relator groups with torsion, fully residually free groups, and fundamental groups of hyperbolic 3-manifolds. For graph groups (also known as right-angled Artin groups) a complete classification of the complexity of knapsack was obtained in [28]: If the underlying graph contains an induced path or cycle on 4 nodes, then knapsack is \( \text{NP} \)-complete; in all other cases knapsack can be solved in polynomial time (even in LogCFL).

- Knapsack is \( \text{NP} \)-complete for every wreath products \( A \wr \mathbb{Z} \) with \( A \neq 1 \) f.g. abelian [12] (wreath products are formally defined in Section 3.2).

- Decidability of knapsack is preserved under finite extensions, HNN-extensions over finite associated subgroups and amalgamated free products over finite subgroups [24]. For a knapsack equation \( g_1^{n_1} \cdots g_d^{n_d} = h \) we may consider the set of all solutions \( \{(n_1, \ldots, n_d) \in \mathbb{N}^d \mid g_1^{n_1} \cdots g_d^{n_d} = g \text{ in } G\} \). In the papers [26, 21, 28] it turned out that in many groups the solution set of every knapsack equation is a semilinear set (see Section 2 for a definition). We say that a group is \textit{knapsack-semilinear} if for every knapsack equation the set of all solutions is semilinear and a semilinear representation can be computed effectively (the same holds then also for exponent equations). Note that in any group \( G \) the set of solutions on an equation \( g^x = h \) is periodic and hence semilinear. This result generalizes to solution sets of knapsack instances of the form \( g_1^{n_1} g_2^{n_2} = h \) (see Lemma 14), but there are examples of knapsack equations with three variables where solution sets (in certain groups) are not semilinear. Examples of knapsack-semilinear groups are graph groups [28] (which include free groups and free abelian groups), hyperbolic groups [26], and co-context free groups [21].

Moreover, the class of knapsack-semilinear groups is closed under finite extensions, graph products, amalgamated free products with finite amalgamated subgroups, HNN-extensions with finite associated subgroups (see [10] for these closure properties) and wreath products [12].

**Power word problems.** In the power word problem for a f.g. group \( G \) (\textsc{PowerWP}(G) for short) the input consists of an expression \( u_1^{n_1} u_2^{n_2} \cdots u_d^{n_d} \), where \( u_1, \ldots, u_d \) are words over the group generators and \( n_1, \ldots, n_d \) are binary encoded integers. The problem is then to decide whether the expression \( u_1^{n_1} u_2^{n_2} \cdots u_d^{n_d} \) evaluates to the identity in \( G \). The power word problem arises very naturally in the context of the knapsack problem: it allows us to verify a proposed solution for a knapsack equation with binary encoded numbers. The power word problem has been first studied in [27], where it was shown that the power word problem for f.g. free groups has the same complexity as the word problem and hence can be solved in logarithmic space. Other groups with easy power word problems are f.g. nilpotent groups and wreath products \( A \wr \mathbb{Z} \) with \( A \) f.g. abelian [27]. In contrast it is shown in [27] that the power word problem for wreath products \( G \wr \mathbb{Z} \), where \( G \) is either finite non-solvable or f.g. free, is \textsc{coNP}-complete. Implicitly, the power word problem appeared also in the work of Ge [13], where it was shown that one can verify in polynomial time an identity \( \alpha_1^{n_1} \alpha_2^{n_2} \cdots \alpha_d^{n_d} = 1 \), where the \( \alpha_i \) are elements of an algebraic number field and the \( n_i \) are binary encoded integers. Let us also remark that the power word problem is a special case of the compressed word

\footnote{Knapsack-semilinearity of co-context free groups is not stated in [21] but follows immediately from the proof for the decidability of knapsack.}
problem [25], which asks whether a grammar-compressed word over the group generators evaluates to the group identity.

**Main results.** In this paper, we are mainly interested in the problems \(\text{PowerWP}(G)\), \(\text{Knapsack}(G)\) and \(\text{ExpEq}(G)\) for the case where \(G\) is a wreath product. We start with the following result:

► **Theorem 1.** Let \(G\) be a f.g. nilpotent group. Then \(\text{PowerWP}(G \wr \mathbb{Z})\) is in \(\text{TC}^0\).

Theorem 1 generalizes the above mentioned result from [27] (for \(G\) abelian) in a nontrivial way. Our proof analyzes periodic infinite words over a nilpotent group \(G\). Roughly speaking, we show that one can check in \(\text{TC}^0\), whether a given list of such periodic infinite words pointwise multiplies to the identity of \(G\). We believe that this is a result of independent interest. We use this result also in the proof of the following theorem:

► **Theorem 2.** Let \(G\) be a finite nontrivial nilpotent group. Then \(\text{Knapsack}(G \wr \mathbb{Z})\) is NP-complete.

Next, we consider iterated wreath products. Fix a number \(r \geq 1\) and let us define the iterated wreath products \(W_0,r = \mathbb{Z}^r\) and \(W_{m+1},r = \mathbb{Z}^r \wr W_{m,r}\). By a famous result of Magnus [32] the free solvable group \(S_{m,r}\) of derived length \(r\) and rank \(m\) embeds into \(W_{m,r}\). Our main results for these groups are:

► **Theorem 3.** \(\text{PowerWP}(W_{m,r})\) and hence \(\text{PowerWP}(S_{m,r})\) belong to \(\text{TC}^0\) for all \(m \geq 0, r \geq 1\).

It was only recently shown in [35] that the word problem (as well as the conjugacy problem) for every free solvable group belongs to \(\text{TC}^0\). Theorem 3 generalizes this result (at least the part on the word problem).

► **Theorem 4.** \(\text{ExpEq}(W_{m,r})\) and hence \(\text{ExpEq}(S_{m,r})\) are NP-complete for all \(m \geq 0, r \geq 1\).

For the proof of Theorem 4 we show that if a given knapsack equation over \(W_{m,r}\) has a solution then it has a solution where all numbers are exponentially bounded in the length of the knapsack instance. Theorem 4 then follows easily from Theorem 3. For some other algorithmic results for free solvable groups see [34].

Finally, we prove a new hardness results for the power word problem and knapsack problem. For this we make use so-called uniformly strongly efficiently non-solvable groups (uniformly SENS groups) that were recently defined in [3]. Roughly speaking, a group \(G\) is uniformly SENS if there exist nontrivial nested commutators of arbitrary depth that moreover, are efficiently computable in a certain sense (see Section 6.1 for the precise definition). The essence of these groups is that they allow to carry out Barrington’s argument showing the \(\text{NC}^1\)-hardness of the word problem for a finite solvable group [2]. We prove the following:

► **Theorem 5.** Let the f.g. group \(G = \langle \Sigma \rangle\) be uniformly SENS. Then, \(\text{PowerWP}(G \wr \mathbb{Z})\) is coNP-hard.

This result generalizes a result from [27] saying that \(\text{PowerWP}(G \wr \mathbb{Z})\) is coNP-hard for the case that \(G\) is f.g. free or finite non-solvable.

► **Theorem 6.** Let the f.g. group \(G = \langle \Sigma \rangle\) be uniformly SENS. Then, \(\text{Knapsack}(G \wr \mathbb{Z})\) is \(\Sigma^p_2\)-hard.
Recall that for every nontrivial group $G$, Knapsack($G \times \mathbb{Z}$) is NP-hard [12].

In the main part we also state several corollaries of Theorem 5 and 6. For instance, we show that for the famous Thompson’s group $F$, PowerWP($F$) is coNP-complete and Knapsack($F$) is $\Sigma^p_2$-hard.

## 2 Preliminaries

**Complexity theory.** We assume some knowledge in complexity theory; in particular the reader should be familiar with the classes $\text{P}$, $\text{NP}$, and $\text{coNP}$. The class $\Sigma^p_2$ (second existential level of the polynomial time hierarchy) contains all languages $L \subseteq \Sigma^*$ for which there exists a polynomial $p$ and a language $K \subseteq \Sigma^* \# \{0,1\}^* \# \{0,1\}^*$ in $\text{P}$ (for a symbol $\# \notin \Sigma \cup \{0,1\}$) such that $x \in L$ if and only if $\exists y \in \{0,1\}^{p(|x|)} \forall z \in \{0,1\}^{|x|} : x \# y \# z \in K$.

The class $\text{TC}^0$ contains all problems that can be solved by a family of threshold circuits of polynomial size and constant depth. In this paper, $\text{TC}^0$ will always refer to the DLOGTIME-uniform version of $\text{TC}^0$. A precise definition is not needed for our work; see [42] for details.

All we need is that the following arithmetic operations on binary encoded integers belong to $\text{TC}^0$: iterated addition and multiplication (i.e., addition and multiplication of $n$ many $n$-bit numbers) and division with remainder.

For languages (or computational problems) $A, B_1, \ldots, B_k \subseteq \{0,1\}^*$ we write $A \in \text{TC}^0(B_1, \ldots, B_k)$ ($A$ is $\text{TC}^0$-Turing-reducible to $B_1, \ldots, B_k$) if $A$ can be solved by a family of threshold circuits of polynomial size and constant depth that in addition may also use oracle gates for the languages $B_1, \ldots, B_k$ (an oracle gate for $B_i$ yields the output 1 if and only if the string of input bits belongs to $B_i$).

**Arithmetic progressions.** An arithmetic progression is a tuple $p = (a + pi)_{0 \leq i \leq k}$ for some $a, p, k \in \mathbb{N}$ with $p \neq 0$. We call $a$ the offset, $p$ the period and $k + 1$ the length of $P$. The support of $p$ is $\text{supp}(p) = \{a + pi \mid 0 \leq i \leq k\}$. In computational problems we will represent the arithmetic progression $p$ by the triple $(a, p, k + 1)$, where the offset $a$ and the length $k + 1$ are represented in binary notation whereas the period $p$ is represented in unary notation (i.e., as the string $\#^p$ for some special symbol $\#$).

**Intervals.** A subset $B$ in a linear order $(A, \leq)$ is an interval if $a_1 \leq a_2 \leq a_3$ and $a_1, a_3 \in B$ implies $a_2 \in B$.

**Lemma 7.** Let $(A, \leq)$ be a linear order, let $\Omega$ be a finite set of colors and let $\beta : A \to 2^\Omega$ be a mapping such that $\{a \in A \mid \omega \in \beta(a)\}$ is an interval for each $\omega \in \Omega$. Then there exists a partition of $A$ into at most $O(|\Omega|)$ intervals $A_1, \ldots, A_k$ such that $|\beta(A_i)| = 1$ for all $1 \leq i \leq k$. Furthermore, if $A = [0,n]$ and each interval $\{a \in A \mid \omega \in \beta(a)\}$ is given by its endpoints (in binary encoding) we can compute the endpoints of the intervals $A_1, \ldots, A_k$ in $\text{TC}^0$.

**Proof.** We prove that there exists such a partition with at most $2|\Omega| + 1$ many intervals by induction on $|\Omega|$. The case $|\Omega| = 0$ is clear. Now let $\Omega = \Omega’ \cup \{\omega\}$ where $\omega \notin \Omega’$ and let $\beta’(a) = \beta(a) \cap \Omega’$, which still satisfies the condition from the lemma. By induction we obtain a partition of $A$ into at most $2|\Omega’| - 1$ intervals $A_1, \ldots, A_k$ such that $|\beta’(A_i)| = 1$ for all $1 \leq i \leq k$. Now consider the interval $A_0 = \{a \in A \mid \omega \in \beta(a)\}$. If $A_i$ is contained in $A_0$ or $A_i$ is disjoint from $A_0$ then $|\beta(A_i)| = |\beta’(A_i)| = 1$. Otherwise $A_i$ can be partitioned into the intervals $A_i \cap A_0$ and $A_i \setminus A_0$, which also satisfy $|\beta(A_i \cap A_0)| = |\beta(A_i \setminus A_0)| = 1$. Since there are at most two such intervals $A_i$ whose symmetric difference with $A_0$ is non-empty, at most two intervals are added in total.

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For the TC\(^0\)-statement we take a different approach. Let \( P \) be the set of all (at most \( 2|\Omega| \)) endpoints of the intervals \( \{ a \in A \mid \omega \in \beta(a) \} \) for \( \omega \in \Omega \) together with the minimum 0 and the maximum \( n \). We sort \( P \) in \( \text{TC}^0 \) [6], say \( P = \{ a_1, \ldots, a_m \} \) with \( a_1 < a_2 < \cdots < a_m \), and define the partition consisting of all singletons \( \{ a_i \} \) for \( 1 \leq i \leq m \) and all “gap” intervals \([a_{i-1}+1, a_i - 1]\) for \( 2 \leq i \leq m \) with \( a_{i-1} + 1 \leq a_i - 1 \). We clearly have \( |\beta(\{a_i\})| = 1 \).

Now consider \( a, b \in [a_{i-1}+1, a_i - 1] \) with \( a < b \) and assume that \( \beta(a) \neq \beta(b) \), i.e. there exists \( \omega \in \Omega \) with \( \omega \in \beta(a) \setminus \beta(b) \) (or \( \omega \in \beta(b) \setminus \beta(a) \)). Let \( d \) be the right endpoints of \( \{ c \in A \mid \omega \in \beta(c) \} \), which must satisfy \( a \leq d < b \). But then \( a_{i-1} + 1 \leq a \leq d < b \leq a_i - 1 \), and therefore \( a_{i-1} < d < a_i \), which is a contradiction.

**Semilinear sets.** Fix a dimension \( d \geq 1 \). All vectors will be column vectors. For a vector \( \mathbf{v} = (v_1, \ldots, v_d)^T \in \mathbb{Z}^d \) we define its norm \( \|\mathbf{v}\| := \max\{|v_i| \mid 1 \leq i \leq d\} \) and for a matrix \( M \in \mathbb{Z}^{c \times d} \) with entries \( m_{i,j} \) (\( 1 \leq i \leq c, 1 \leq j \leq d \)) we define the norm \( \|M\| = \max\{|m_{i,j}| \mid 1 \leq i \leq c, 1 \leq j \leq d\} \). Finally, for a finite set of vectors \( A \subseteq \mathbb{N}^d \) let \( \|A\| = \max\{|\mathbf{a}| \mid \mathbf{a} \in A\} \).

We extend the operations of vector addition and multiplication of a vector by a matrix to sets of vectors in the obvious way. A linear subset of \( \mathbb{N}^d \) is a set of the form

\[
L = L(b, P) := b + P \cdot \mathbb{N}^k
\]

where \( b \in \mathbb{N}^d \) and \( P \in \mathbb{N}^{d \times k} \). We call a set \( S \subseteq \mathbb{N}^d \) semilinear, if it is a finite union of linear sets. Semilinear sets play a very important role in many areas of computer science and mathematics, e.g. in automata theory and logic. It is known that the class of semilinear sets is closed under Boolean operations and that the semilinear sets are exactly the Presburger definable sets (i.e., those sets that are definable in the structure \( (\mathbb{N},+) \)).

For a semilinear set \( S = \bigcup_{i=1}^k L(b_i, P_i) \), we call the tuple \( (b_1, P_1, \ldots, b_k, P_k) \) a semilinear representation of \( S \). The magnitude of the semilinear representation \( (b_1, P_1, \ldots, b_k, P_k) \) is \( \max\{|b_1|, \|P_1\|, \ldots, \|b_k\|, \|P_k\|\} \). The magnitude \( \|S\| \) of a semilinear set \( S \) is the minimal magnitude of all semilinear representations for \( S \).

**Lemma 8** ([16]). If \( M_1, \ldots, M_k \subseteq \mathbb{N}^d \) are semilinear sets with \( \|M_i\| \leq s \) then

\[
\left\| \bigcap_{i=1}^k M_i \right\| \leq (s \cdot k \cdot d + 1)^\Theta(k \cdot d).
\]

In the context of knapsack problems (which we will introduce in the next section), we will consider semilinear subsets as sets of mappings \( \nu \}: \{x_1, \ldots, x_d\} \to \mathbb{N} \) for a finite set of variables \( X = \{x_1, \ldots, x_d\} \). Such a mapping \( f \) can be identified with the vector \( (\nu(x_1), \ldots, \nu(x_d))^T \).

This allows to use all vector operations (e.g. addition and scalar multiplication) on the set \( \mathbb{N}^X \) of all mappings from \( X \) to \( \mathbb{N} \).

### 3 Groups

We assume that the reader is familiar with the basics of group theory. Let \( G \) be a group. We always write 1 for the group identity element. For \( g, h \in G \) we write \([g, h] := g^{-1}h^{-1}gh\) for the commutator of \( g \) and \( h \) and \( g^b \) for \( h^{-1}gh \). For subgroups \( A, B \) of \( G \) we write \([A, B]\) for the subgroup generated by all commutators \([a, b]\) with \( a \in A \) and \( b \in B \). The order of an element \( g \in G \) is the smallest number \( z > 0 \) with \( g^z = 1 \) and \( \infty \) if such a \( z \) does not exist.

The group \( G \) is torsion-free, if every \( g \in G \setminus \{1\} \) has infinite order.
We say that $G$ is finitely generated (f.g.) if there is a finite subset $\Sigma \subseteq G$ such that every element of $G$ can be written as a product of elements from $\Sigma$; such a $\Sigma$ is called a finite generating set for $G$. We also write $G = \langle \Sigma \rangle$. We then have a canonical morphism $h: \Sigma^* \to G$ that maps a word over $\Sigma$ to its product in $G$. If $h(w) = 1$ we also say that $w = 1$ in $G$. For $g \in G$ we write $|g|$ for the length of a shortest word $w \in \Sigma^*$ such that $h(w) = g$. This notation depends on the generating set $\Sigma$. We always assume that the generating set $\Sigma$ is symmetric in the sense that $a \in \Sigma$ implies $a^{-1} \in \Sigma$. Then, we can define on $\Sigma^*$ a natural involution $\neg$ by $(a_1 a_2 \cdots a_n)^\neg = a_n^{-1} \cdots a_2^{-1} a_1^{-1}$ for $a_1, a_2, \ldots, a_n \in \Sigma$. This allows to use the notations $[g, h] = g^{-1} h^{-1} g h$ and $g^h = h^{-1} g h$ also in case $g, h \in \Sigma^*$. In the following, when we say that we want to compute a homomorphism $h: G_1 = \langle \Sigma_1 \rangle \to G_2 = \langle \Sigma_2 \rangle$, we always mean that we compute the images $h(a)$ for $a \in \Sigma_1$.

A group $G$ is called orderable if there exists a linear order $\leq$ on $G$ such that $g \leq h$ implies $xgy \leq xhy$ for all $g, h, x, y \in G$ [39, 38]. Every orderable group is torsion-free (this follows directly from the definition) and has the unique roots property [41], i.e., $g^n = h^n$ implies $g = h$. The are numerous examples of orderable groups: for instance, torsion-free nilpotent groups, right-angled Artin groups, and diagram groups are all orderable.

### 3.1 Commensurable elements

Two elements $g, h \in G$ in a group $G$ are called commensurable if $g^x = h^y$ for some $x, y \in \mathbb{Z} \setminus \{0\}$. This defines an equivalence relation on $G$, in which the elements with finite order form an equivalence class. By [39, Corollary 1.2] commensurable elements in an orderable group commute.

**Lemma 9.** Let $G$ be an orderable group and let $U \subseteq G$ be a finite set of pairwise commensurable elements. Then $\langle U \rangle$ is a cyclic subgroup of $G$.

**Proof.** Recall that $G$ is torsion-free and has the unique roots property. We prove the lemma by induction on the size of $U$. The case $|U| = 1$ is obvious. Now assume that $|U| > 1$. By the above mentioned result from [39] $\langle U \rangle$ is abelian. Choose arbitrary elements $g, h \in U$ with $g \neq h$. Since $g$ and $h$ are commensurable, there exist $p, q \in \mathbb{Z} \setminus \{0\}$ with $g^p = h^q$. Since $G$ has the unique roots property, we can assume that $\gcd(p, q) = 1$. Hence, there exist $k, \ell \in \mathbb{Z}$ with $1 = kp + \ell q$. Consider the group element $a = g^k h^\ell$. We then have $g = g^{kp+\ell q} = g^{\ell k} h^q = a^q$ and similarly $h = a^p$. We therefore have $\langle g, h \rangle = \langle a \rangle$. Note that $a \neq 1$ since $(g, h) \neq 1$.

We next claim that every $b \in U \setminus \{g, h\}$ is commensurable to $a$. Since $g$ (resp., $h$) is commensurable to $b$, there exist $r, s, t, u \in \mathbb{Z} \setminus \{0\}$ with $g^r = b^s$ and $h^t = b^u$. We obtain $a^{rt} = g^{r t} h^{k r t} = b^{t s t + k r u}$. Finally, note that since $rt \neq 0$ and $G$ is torsion-free, we must have $t s t + k r u \neq 0$.

We have shown that $V = (U \setminus \{g, h\}) \cup \{a\}$ consists of pairwise commensurable elements. By induction, $\langle V \rangle$ is cyclic. Moreover, $\langle g, h \rangle = \langle a \rangle$ implies that $\langle U \rangle = \langle V \rangle$, which proves the lemma.

### 3.2 Wreath products

Let $G$ and $H$ be groups. Consider the direct sum $K = \bigoplus_{h \in H} G_h$, where $G_h$ is a copy of $G$. We view $K$ as the set $G^{(H)}$ of all mappings $f: H \to G$ such that $\text{supp}(f) := \{h \in H \mid f(h) \neq 1\}$ is finite, together with pointwise multiplication as the group operation. The set $\text{supp}(f) \subseteq H$ is called the support of $f$. The group $H$ has a natural left action on $G^{(H)}$ given by $h f(a) = f(h^{-1} a)$, where $f \in G^{(H)}$ and $h, a \in H$. The corresponding semidirect product $G^{(H)} \rtimes H$ is the (restricted) wreath product $G \wr H$. In other words:
Elements of $G \wr H$ are pairs $(f, h)$, where $h \in H$ and $f \in G(H)$.

The multiplication in $G \wr H$ is defined as follows: Let $(f_1, h_1), (f_2, h_2) \in G \wr H$. Then $(f_1, h_1)(f_2, h_2) = (f, h_1h_2)$, where $f(a) = f_1(a)f_2(h_1^{-1}a)$.

There are canonical mappings

$\sigma : G \wr H \rightarrow H$ with $\sigma(f, h) = h$ and

$\tau : G \wr H \rightarrow G^{(H)}$ with $\tau(f, h) = f$

In other words: $g = (\tau(g), \sigma(g))$ for $g \in G \wr H$. Note that $\sigma$ is a homomorphism whereas $\tau$ is in general not a homomorphism. Throughout this paper, the letters $\sigma$ and $\tau$ will have the above meaning, which of course depends on the underlying wreath product $G \wr H$, but the latter will always be clear from the context.

The following intuition might be helpful: An element $(f, h) \in G \wr H$ can be thought of as a finite multiset of elements of $G \setminus \{1_G\}$ that are sitting at certain elements of $H$ (the mapping $f$) together with the distinguished element $h \in H$, which can be thought of as a cursor moving in $H$. If we want to compute the product $(f_1, h_1)(f_2, h_2)$, we do this as follows: First, we shift the finite collection of $G$-elements that corresponds to the mapping $f_2$ by $h_1$: If the element $g \in G \setminus \{1_G\}$ is sitting at $a \in H$ (i.e., $f_2(a) = g$), then we remove $g$ from $a$ and put it to the new location $h_1a \in H$. This new collection corresponds to the mapping $f_2' : a \mapsto f_2(h_1^{-1}a)$. After this shift, we multiply the two collections of $G$-elements pointwise: If in $a \in H$ the elements $g_1$ and $g_2$ are sitting (i.e., $f_1(a) = g_1$ and $f_2'(a) = g_2$), then we put the product $g_1g_2$ into the location $a$. Finally, the new distinguished $H$-element (the new cursor position) becomes $h_1h_2$.

Clearly, $H$ is a subgroup of $G \wr H$. But also $G$ is a subgroup of $G \wr H$. We can identify $G$ with the set of all mappings $f \in G^{(H)}$ such that $\text{supp}(f) \subseteq \{1\}$. This copy of $G$ together with $H$ generates $G \wr H$. In particular, if $G = \langle \Sigma \rangle$ and $H = \langle \Gamma \rangle$ with $\Sigma \cap \Gamma = \emptyset$ then $G \wr H$ is generated by $\Sigma \cup \Gamma$. In this situation, we will also apply the above mappings $\sigma$ and $\tau$ to words over $\Sigma \cup \Gamma$. We will need the following embedding result:

**Lemma 10.** Given a unary encoded number $d$, one can compute in logspace an embedding of $G \wr Z$ into $G \wr Z$.

**Proof.** Let $G = \langle \Gamma \rangle$ and let $\Gamma_i$ ($0 \leq i \leq d - 1$) be pairwise disjoint copies of $\Gamma$, each of which generates a copy of $G$. For $G \wr Z$ we take the generating set $\{t, t^{-1}\} \cup \bigcup_{i=0}^{d-1} \Gamma_i$, where $t$ generates the right factor $Z$. We then obtain an embedding $h : G \wr Z \rightarrow G \wr Z$ by:

$h(t) = t^d$ and $h(t^{-1}) = t^{-d}$,

$h(a) = t^{ia}$ for $a \in \Gamma_i$.

This proves the lemma. ▶

In [34] it was shown that the word problem of a wreath product $G \wr H$ is $\text{TC}^0$-reducible to the word problems for $G$ and $H$. Let us briefly sketch the argument. Assume that $G = \langle \Sigma \rangle$ and $H = \langle \Gamma \rangle$. Given a word $w \in (\Sigma \cup \Gamma)^*$ one has to check whether $\sigma(w) = 1$ in $H$ and $\tau(w)(h) = 1$ in $H$ for all $h$ in the support of $\tau(w)$. One can compute in $\text{TC}^0$ the word $\sigma(w)$ by projecting $w$ onto the alphabet $\Gamma$. Moreover, one can enumerate the support of $\tau(w)$ by going over all prefixes of $w$ and checking which $\sigma$-values are the same. Similarly, one produces for a given $h \in \text{supp}(\tau(w))$ a word over $\Sigma$ that represents $\tau(w)(h)$.

**Lemma 11.** For $g_1, \ldots, g_k \in G \wr H$ we have $\tau(g_1 \cdots g_k) = \prod_{i=1}^k \tau(\sigma(g_1 \cdots g_{i-1}) g_i)$.

**Proof.** By definition of the wreath product we have (for better readability we write $\circ$ for the multiplication in $G$):

$\tau(g_1g_2)(h) = \tau(g_1)(h) \circ \tau(g_2)\sigma(g_1)^{-1}h = \tau(g_1)(h) \circ \tau(\sigma(g_1)g_2)(h)$
for all \( h \in H \) and therefore \( \tau(g_1g_2) = \tau(g_1) \circ \tau(g_1)g_2 \), which is the case \( k = 2 \). The general statement follows by induction.

Finally, we need the following result from [29]:

**Theorem 12** ([29]). If \( G \) and \( H \) are orderable then also \( G \wr H \) is orderable.3

### 3.3 Knapsack problem

Let \( G = \langle \Sigma \rangle \) be a f.g. group. Moreover, let \( X \) be a set of formal variables that take values from \( \mathbb{N} \). For a subset \( Y \subseteq X \), we use \( \mathbb{N}^Y \) to denote the set of maps \( \nu : Y \to \mathbb{N} \), which we call valuations. For valuations \( \nu \in \mathbb{N}^Y \) and \( \mu \in \mathbb{N}^Z \) such that \( Y \subseteq Z \) we say that \( \nu \) extends \( \mu \) (or \( \mu \) restricts to \( \nu \)) if \( \nu(x) = \mu(x) \) for all \( x \in Y \).

An exponent expression over \( G \) is an expression of the form \( E = v_0u_1^{x_1}v_1u_2^{x_2}v_2 \cdots u_d^{x_d}v_d \) with \( d \geq 1 \), words \( v_0, \ldots, v_d \in \Sigma^* \), non-empty words \( u_1, \ldots, u_d \in \Sigma^* \), and variables \( x_1, \ldots, x_d \). Here, we allow \( x_i = x_j \) for \( i \neq j \). If every variable \( x_i \) occurs at most once, then \( E \) is called a knapsack expression. Let \( X = \{x_1, \ldots, x_d\} \) be the set of variables that occur in \( E \). For a homomorphism \( h : G \to G' = \langle \Sigma' \rangle \) (that is specified by a mapping from \( \Sigma \) to \( \Sigma' \)), we denote with \( h(E) \) the exponent expression \( h(u_1)^{x_1}h(v_1)h(u_2)^{x_2}h(v_2) \cdots h(u_d)^{x_d}h(v_d) \). For a valuation \( \nu \in \mathbb{N}^Y \) such that \( X \subseteq Y \) (in which case we also say that \( \nu \) is a valuation for \( E \)), we define \( \nu(E) = u_1^{\nu(x_1)}v_1u_2^{\nu(x_2)}v_2 \cdots u_d^{\nu(x_d)}v_d \in \Sigma^* \). We say that \( \nu \) is a \( G \)-solution for \( E \) if \( \nu(E) = 1 \) in \( G \). With \( \text{sol}_G(E) \) we denote the set of all \( G \)-solutions \( \nu \in \mathbb{N}^X \) of \( E \). The length of \( E \) is defined as \( |E| = \sum_{i=1}^d |u_i| + |v_i| \). We define solvability of exponent equations over \( G \), \( \text{ExpEq}(G) \) for short, as the following decision problem:

**Input** A finite list of exponent expressions \( E_1, \ldots, E_n \) over \( G \).

**Question** Is \( \bigcap_{i=1}^n \text{sol}_G(E_i) \) non-empty?

The knapsack problem for \( G \), \( \text{KNAPSACK}(G) \) for short, is the following decision problem:

**Input** A single knapsack expression \( E \) over \( G \).

**Question** Is \( \text{sol}_G(E) \) non-empty?

It is easy to observe that the concrete choice of the generating set \( \Sigma \) has no influence on the decidability and complexity status of these problems.

We could also restrict to knapsack expressions of the form \( u_1^{x_1}u_2^{x_2} \cdots u_d^{x_d}v \) (but sometimes it will be convenient to allow nontrivial elements between the powers): for \( E = v_0u_1^{x_1}v_1u_2^{x_2}v_2 \cdots u_d^{x_d}v_d \) and

\[
E' = (v_0u_1v_1^{-1})^{x_1}(v_0v_1u_2v_1^{-1}v_0^{-1})^{x_2} \cdots (v_0 \cdots v_{d-1}\cdots v_0^{-1})^{x_d}v_0 \cdots v_{d-1}v_d
\]

we have \( \text{sol}_G(E) = \text{sol}_G(E') \).

For the knapsack problem in wreath products the following result has been shown in [12]:

**Theorem 13** ([12]). For every nontrivial group \( G \), \( \text{KNAPSACK}(G \wr \mathbb{Z}) \) is \( \text{NP-hard} \).

### 3.4 Knapsack-semilinear groups

The group \( G \) is called \( \text{knapsack-semilinear} \) if for every knapsack expression \( E \) over \( \Sigma \), the set \( \text{sol}_G(E) \) is a semilinear set of vectors and a semilinear representation can be effectively computed from \( E \). Since semilinear sets are effectively closed under intersection, it follows

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3 This holds only for the restricted wreath product; which is the wreath product construction we are dealing with.
that for every exponent expression $E$ over $\Sigma$, the set $\text{sol}_G(E)$ is semilinear and a semilinear representation can be effectively computed from $E$. Moreover, solvability of exponent equations is decidable for every knapsack-semilinear group. As mentioned in the introduction, the class of knapsack-semilinear groups is very rich. An example of a group $G$, where knapsack is decidable but solvability of exponent equations is undecidable is the Heisenberg group $H_3(\mathbb{Z})$ (which consists of all upper triangular $(3 \times 3)$-matrices over the integers, where all diagonal entries are 1), see [21]. In particular, $H_3(\mathbb{Z})$ is not knapsack-semilinear. In order to obtain a non-semilinear solution set, one needs a knapsack instance over $H_3(\mathbb{Z})$ with three variables. In fact, for two variables we have the following simple fact:

**Lemma 14.** Let $G$ be a group and $g_1, g_2, h \in G$ be elements.

(i) The solution set $S_1 = \{(x, y) \in \mathbb{Z}^2 \mid g_1^x g_2^y = 1\}$ is a subgroup of $\mathbb{Z}^2$. If $G$ is torsion-free and $(g_1, g_2) \neq \{1\}$ then $S_1$ is cyclic.

(ii) The solution set $S = \{(x, y) \in \mathbb{Z}^2 \mid g_1^x g_2^y = h\}$ is either empty or a coset $(a, b) + S_1$ of $S_1$ where $(a, b) \in S$ is any solution.

**Proof.** Clearly $(0, 0) \in S_1$, and if $g_1^x g_2^y = 1 = g_1^{x'} g_2^{y'}$ then also $g_1^{-x} g_2^{-y} = 1$. This shows the first part of statement (i). Now assume that $G$ is torsion-free and that $g_1 \neq 1$ (the case $g_2 \neq 1$ is analogous). If $(x, y), (x', y') \in S_1$ then $y'(x, y) - y(x', y') = (xy' - x'y, 0) \in S_1$ and hence $g_1^{x'y - x'y} = 1$. Since $G$ is torsion-free this implies that $xy' - x'y = 0$, i.e. $(x, y)$ and $(x', y')$ are linearly dependent, since det $(x, y') = 0$. This shows that $S_1$ is cyclic.

For (ii) let us assume that $S \neq \emptyset$ and take any solution $(a, b) \in S$, i.e. $g_1^a g_2^b = h$. We first show that $(a, b) + S_1 \subseteq S$. Take any $(x, y) \in S_1$, i.e. $g_1^x g_2^y = 1$. Then we obtain $g_1^{a+x} g_2^{b+y} = g_1^a g_1^x g_2^b g_2^y = g_1^a g_2^b = h$ and thus $(a + x, b + y) \in S$.

Finally we claim that $S \subseteq (a, b) + S_1$: Let $(x, y) \in S$, i.e. $g_1^x g_2^y = h$. Since $g_1^{-a} h g_2^{-b} = 1$, we get $g_1^{-a} g_2^{-b} = g_1^{-a} (g_1^x g_2^y) g_2^{-b} = g_1^{-a} g_2^{-b} = 1$ and therefore $(x - a, y - b) \in S_1$. Hence $S = (a, b) + S_1$.

For a knapsack-semilinear group $G$ and a finite generating set $\Sigma$ for $G$ we define a growth function. For $n \in \mathbb{N}$ let $\text{Knap}(n)$ (resp., $\text{Exp}(n)$) be the finite set of all knapsack expressions (resp., exponent expression) $E$ over $\Sigma$ such that $\text{sol}_G(E) \neq \emptyset$ and $|E| \leq n$. We define the mapping $K_{G, \Sigma}: \mathbb{N} \to \mathbb{N}$ and $E_{G, \Sigma}: \mathbb{N} \to \mathbb{N}$ as follows:

$$K_{G, \Sigma}(n) = \max\{||\text{sol}_G(E)|| \mid E \in \text{Knap}(n)\}, \quad (1)$$

$$E_{G, \Sigma}(n) = \max\{||\text{sol}_G(E)|| \mid E \in \text{Exp}(n)\}. \quad (2)$$

Clearly, if $\text{sol}_G(E) \neq \emptyset$ and $||\text{sol}_G(E)|| \leq N$ then $E$ has a $G$-solution $\nu$ such that $\nu(x) \leq N$ for all variables $x$ that occur in $E$. Therefore, if $G$ has a decidable word problem and we have a computable bound on the function $K_{G, \Sigma}$ then we obtain a nondeterministic algorithm for $\text{KNAPSACK}(G)$: given a knapsack expression $E$ with variables from $X$ we can guess $\nu: X \to \mathbb{N}$ with $\sigma(x) \leq N$ for all variables $x$ and then verify (using an algorithm for the word problem), whether $\nu$ is indeed a solution.

Let $\Sigma$ and $\Sigma'$ be two generating sets for the group $G$. Then there is a constant $c$ such that $K_{G, \Sigma}(n) \leq K_{G, \Sigma'}(cn)$, and similarly for $E_{G, \Sigma}(n)$. To see this, note that for every $a \in \Sigma'$ there is a word $w_a \in \Sigma^*$ such that $a$ and $w_a$ represent the same element in $G$. Then we can choose $c = \max\{|w_a| \mid a \in \Sigma'\}$. Due to this fact, we do not need to specify the generating set $\Sigma$ when we say that $K_{G, \Sigma}$ (resp., $E_{G, \Sigma}$) is polynomially/exponentially bounded.

We will need the following simple lemma:

**Lemma 15.** Let $H$ be knapsack-semilinear and let $E = v_0(u_1^{k_1})^{x_1} v_1(u_2^{k_2})^{x_2} v_2 \cdots (u_d^{k_d})^{x_d} v_d$ be an exponent expression over $H$ where $k_1, \ldots, k_d \leq k$ and $|v_0 v_1 \cdots v_d| = n$. Then the magnitude of $\text{sol}_H(E)$ is $(n \cdot \max\{|K_H(n, k)| \} + 1)^{O(n)}$. 

Proof. Let $X = \{x_1, \ldots, x_d\}$ (some of the variables $x_i$ might be equal) and $Y = \{y_1, \ldots, y_d\}$ be a set of $d$ distinct variables. Then $\nu: X \rightarrow \mathbb{N}$ is a solution of $E = 1$ if and only if $\mu: Y \rightarrow \mathbb{N}$ is a solution of $E' = v_0u_1^{y_1}v_1u_2^{y_2}v_2 \cdots u_d^{y_d}v_d = 1$ where $\mu(y_i) = k_i\nu(x_i)$. Notice that $E'$ is a knapsack expression. Hence $\text{sol}_H(E)$ can be obtained as a projection of the intersection of $\text{sol}_H(E')$ with a semilinear set of magnitude $\leq k$ (it has to ensure that $\mu(y_i)$ is a multiple of $k_i$ and that $\mu(y_i)/k_i = \mu(y_j)/k_j$ whenever $x_i = x_j$). Therefore $\|\text{sol}_H(E)\| = (n \cdot \max\{K_H(n), k\} + 1)^O(n)$.

Important for us is also the following result from [12]:

▶ Theorem 16 ([12]). If $G$ and $H$ are knapsack-semilinear then also $G \wr H$ is knapsack-semilinear.

The proof of this result in [12] does not yield a good bound of $K_{G\wr H}(n)$ in terms of $K_G(n)$ and $K_H(n)$ (and similarly for the $E$-function). One of our main achievements will be such a bound for the special case that the left factor $G$ is f.g. abelian. For $E_G(n)$ we then have the following bound, which follows from well-known bounds on solutions of linear Diophantine equations [43]:

▶ Lemma 17. If $G$ is a f.g. abelian group then $E_G(n) \leq 2^nO(1)$.

3.5 Power word problem

A power word (over $\Sigma$) is a tuple $(u_1, k_1, u_2, k_2, \ldots, u_d, k_d)$ where $u_1, \ldots, u_d \in \Sigma^*$ are words over the group generators (called the periods of the power word) and $k_1, \ldots, k_d \in \mathbb{Z}$ are integers that are given in binary notation. Such a power word represents the word $u_1^{k_1}u_2^{k_2} \cdots u_d^{k_d}$. Quite often, we will identify the power word $(u_1, k_1, u_2, k_2, \ldots, u_d, k_d)$ with the word $u_1^{k_1}u_2^{k_2} \cdots u_d^{k_d}$. Moreover, if $k_1 = 1$, then we usually omit the exponent 1 in a power word. The power word problem for the f.g. group $G$, POWERWP($G$) for short, is defined as follows:

Input A power word $(u_1, k_1, u_2, k_2, \ldots, u_d, k_d)$.

Question Does $u_1^{k_1}u_2^{k_2} \cdots u_d^{k_d} = 1$ hold in $G$?

Due to the binary encoded exponents, a power word can be seen as a succinct description of an ordinary word. We have the following simple lemma:

▶ Lemma 18. If the f.g. group $G$ is knapsack-semilinear, $E_G(n)$ is exponentially bounded, and POWERWP($G$) belongs to NP then EXPEq($G$) belongs to NP.

Proof. Let us consider a list $E_1, \ldots, E_k$ of exponent expressions over the group $G$ and let $n = \sum_{i=1}^{k} |E_i|$ be the total input length. With Lemma 8 it follows that $\bigcap_{i=1}^{k} \text{sol}_G(E_i) \neq \emptyset$ if and only if there is some $\nu \in \bigcap_{i=1}^{k} \text{sol}_G(E_i)$ with $\nu(x) \leq 2^{nO(1)}$ for all variables $x$. We can therefore guess the binary encodings of all numbers $\nu(x)$ in polynomial time and then verify in polynomial time whether $\nu(E_i) = 1$ (which is an instance of POWERWP($G$)) for all $1 \leq i \leq k$.

4 Wreath products of nilpotent groups and $\mathbb{Z}$

The lower central series of a group $G$ is the sequence of groups $(G_i)_{i \geq 0}$ with $G_0 = G$ and $G_{i+1} = [G_i, G]$. The group $G$ is called nilpotent if there exists a $c \geq 0$ such that $G_c = 1$; in this case the minimal number $c$ with $G_c = 1$ is called the nilpotency class of $G$. In this section we prove Theorems 1 and 2 from the introduction. For the proofs of Theorems 1 and 2 we first have to consider periodic words over $G$ that were also used in [12].
4.1 Periodic words over groups

Let $G = \langle \Sigma \rangle$ be a f.g. group. Let $G^\omega$ be the set of all functions $f: \mathbb{N} \to G$, which forms a group by pointwise multiplication $(fg)(t) = f(t) \cdot g(t)$. A function $f \in G^\omega$ is periodic if there exists a number $d \geq 1$ such that $f(t) = f(t + d)$ for all $t \geq 0$. The smallest such number $d$ is called the period of $f$. If $f, g \in G^\omega$ has period $d$ and $g \in G^\omega$ has period $e$ then $fg$ has period at most $\text{lcm}(d,e)$. A periodic function $f \in G^\omega$ with period $d$ can be specified by its initial $d$ elements $f(0), \ldots, f(d-1)$ where each element $f(t)$ is given as a word over the generating set $\Sigma$. The periodic words problem $\text{PERIODIC}(G)$ over $G$ is defined as follows:

**Input** Periodic functions $f_1, \ldots, f_m \in G^\omega$ and a binary encoded number $T$.

**Question** Does the product $P = \prod_{i=1}^{m} f_i$ satisfy $f(t) = 1$ for all $t \leq T$?

The main result of this section is:

► **Theorem 19.** If $G$ is a f.g. nilpotent group then $\text{PERIODIC}(G)$ belongs to $\text{TC}^0$.

Previously it was proven that $\text{PERIODIC}(G)$ belongs to $\text{TC}^0$ if $G$ is abelian [12]. As an introduction let us reprove this result.

Let $\rho: G^\omega \to G^\omega$ be the shift-operator, i.e. $(\rho(f))(t) = f(t+1)$, which is a group homomorphism. For a subgroup $H$ of $G^\omega$, we denote by $H^{(n)}$ the smallest subgroup of $G^\omega$ that contains $\rho^n(H), \rho^2(H), \ldots, \rho^m(H)$. Note that $(H^{(m)})^{(n)} = H^{(m+n)}$ for any $m, n \in \mathbb{N}$. A function $f \in G^\omega$ satisfies a recurrence of order $d \geq 1$ if $\rho^d(f)$ is contained in the subgroup $\langle f \rangle^{(d-1)}$ of $G^\omega$. If $f$ has period $d$ then $f$ clearly satisfies a recurrence of order $d$.

Let us now consider the case that $G$ is abelian. Then, also $G^\omega$ is abelian and we use the additive notation for $G^\omega$. The following lemma is folklore:

► **Lemma 20** (cf. [17]). Let $G$ be a f.g. abelian group. If $f_1, \ldots, f_m \in G^\omega$ satisfy recurrences of order $d_1, \ldots, d_m \geq 1$ respectively, then $\sum_{i=1}^{m} f_i$ satisfies a recurrence of order $\sum_{i=1}^{m} d_i$.

**Proof.** Observe that $G^\omega$ is a $\mathbb{Z}[x]$-module with scalar multiplication

$$
\sum_{i=0}^{d} a_i x^i \cdot f \mapsto \sum_{i=0}^{d} a_i \rho^i(f).
$$

Then $f \in G^\omega$ satisfies a recurrence of order $d \geq 1$ if and only if there exists a monic polynomial $p \in \mathbb{Z}[x]$ of degree $d$ (where monic means that the leading coefficient is one) such that $pf = 0$. Therefore, if $p_1, \ldots, p_m \in \mathbb{Z}[x]$ such that $\deg(p_i) = d_i \geq 1$ and $p_i f_i = 0$ then $\prod_{i=1}^{m} p_i \sum_{j=1}^{m} f_j = \sum_{j=1}^{m} (\prod_{i=1}^{m} p_i) f_j = 0$. Since $\prod_{i=1}^{m} p_i$ is a monic polynomial of degree $d := \sum_{i=1}^{m} d_i$, $\sum_{i=1}^{m} f_i$ satisfies a recurrence of order $d$.

The above lemma implies that $\sum_{i=1}^{m} f_i = 0$ if and only if $\sum_{i=1}^{m} f_i(t) = 0$ for all $0 \leq t \leq d-1$, where $d$ is the sum of the periods of the $f_i$.

Let us now turn to the nilpotent case. For $n \in \mathbb{N}$, let $G^{\omega,n}$ be the subgroup of $G^\omega$ generated by all elements with period at most $n$. Then $G^{\omega,n}$ is closed under shift. The key fact for showing Theorem 19 is the following.

► **Proposition 21.** If $G$ is a f.g. nilpotent group, then there is a polynomial $p$ such that every element of $G^{\omega,n}$ satisfies a recurrence of order $p(n)$.

Let $H \leq G^\omega$ be a subgroup which is closed under shifting, i.e. $\rho(H) \subseteq H$. Since the shift is a homomorphism, the commutator subgroup $[H, H]$ is closed under shifting as well. We will work in the abelianization $H' = H/[H, H]$ where we write $\tilde{f}$ for the coset $f[H, H]$. We
also define $\rho: H' \to H'$ by $\rho(f) = \rho(\bar{f})$. This is well-defined since $fg^{-1} \in [H,H]$ implies $\rho(f)\rho(g)^{-1} = \rho(fg^{-1}) \in [H,H]$ and hence $\rho(\bar{f}) = \rho(g)$. As an abelian group $H'$ is a $\mathbb{Z}$-module and, in fact, $H'$ forms a $\mathbb{Z}[x]$-module using the shift-operator. By the above remark (see (3)) we have the following (where we use the multiplicative notation for $H'$):

> **Lemma 22.** $H'$ is a $\mathbb{Z}[x]$-module with the scalar multiplication $\sum_{i=0}^{d} a_i x^i \cdot \bar{f} \mapsto \prod_{i=0}^{d} \rho^i(\bar{f})^{a_i}$.

Our first step for proving Proposition 21 is to show that every element of $G^{\omega,n}$ satisfies a polynomial-order recurrence, modulo some element in $[G^{\omega,n}, G^{\omega,n}]$.

> **Lemma 23.** For every $f \in G^{\omega,n}$, we have $\rho^d(f) \in \langle f \rangle^{(d-1)}[G^{\omega,n}, G^{\omega,n}]$ for $d = n(n+1)/2$.

**Proof.** Suppose $f = f_1 \cdots f_m$ such that $f_1, \ldots, f_m \in G^\omega$ are elements of period $\leq n$. According to Lemma 22, we consider $G^{\omega,n}/[G^{\omega,n}, G^{\omega,n}]$ as a $\mathbb{Z}[x]$-module.

If $g \in G^\omega$ has period $p$ then $\rho^p(g)g^{-1} = 1$ and thus $(x^q - 1)\bar{g} = \rho^p(\bar{g})\bar{g}^{-1} = 1$. Define the polynomial $p(x) = \prod_{i=1}^{n}(x^i - 1) = \sum_{i=0}^{d} a_i x^i$ of degree $d = n(n+1)/2$ satisfying $a_d = 1$. Since all functions $f_1, \ldots, f_m$ have period at most $n$ we have $p\bar{f} = 1$. Written explicitly we have

$$1 = p\bar{f} = \prod_{i=0}^{d} \rho^i(\bar{f})^{a_i} = \prod_{i=0}^{d} \rho^i(f)^{a_i},$$

where the order in the product $\prod_{i=0}^{d} \rho^i(f)^{a_i}$ is arbitrary. Noticing that $a_d = 1$, we can write $\rho^d(f) = gh$ for some $g \in \langle f \rangle^{(d-1)}$ and $h \in [G^{\omega,n}, G^{\omega,n}]$, which has the desired form. ▶

The following lemma gives us control over the remaining factor from $[G^{\omega,n}, G^{\omega,n}]$.

> **Lemma 24.** Let $G$ be a group with nilpotency class $c$. Then $[G^{\omega,n}, G^{\omega,n}] \subseteq [G, G]^{\omega,n^2c}$.

**Proof.** We need the fact that the commutator subgroup $[F, F]$ of a group $F$ with generating set $\Gamma$ is generated by all left-normed commutators

$$[g_1, \ldots, g_k] := [[[g_1, g_2], g_3], \ldots], g_k$$

where $g_1, \ldots, g_k \in \Gamma \cup \Gamma^{-1}$ and $k \geq 2$, cf. [7, Lemma 2.6]. Therefore $[G^{\omega,n}, G^{\omega,n}]$ is generated by all left-normed commutators $[g_1, \ldots, g_k]$ where $k \geq 2$ and $g_1, \ldots, g_k \in G^\omega$ have period at most $n$. Furthermore, we can bound $k$ by $c$ since any left-normed commutator $[g_1, \ldots, g_{c+1}]$ is trivial (recall that $G$ is nilpotent of class $c$).

A left-normed commutator $[g_1, \ldots, g_k]$ with $2 \leq k \leq c$ and $g_1, \ldots, g_k$ periodic with period at most $n$ is a product containing at most $2k \leq 2c$ distinct functions of period at most $n$ (namely, the $g_1, \ldots, g_k$ and their inverses). Hence $[G^{\omega,n}, G^{\omega,n}]$ is generated by functions $g \in [G, G]^\omega$ of period at most $n^{2c}$. ▶

We are now ready to prove Proposition 21.

**Proof.** The proposition is proved by induction on the nilpotency class of $G$. If $G$ has nilpotency class $0$ then $G$ is trivial and the claim is vacuous. Now suppose that $G$ has nilpotency class $c \geq 1$. According to Lemma 23, we have $\rho^d(f) \in \langle f \rangle^{(d-1)}h$ for some $h \in [G^{\omega,n}, G^{\omega,n}]$. By Lemma 24, we have $[G^{\omega,n}, G^{\omega,n}] \subseteq [G, G]^{\omega,n^2c}$. Since the group $[G, G]$ has nilpotency class at most $c - 1$, we may apply induction. Thus, we know that

---

4 We could not find a proof for this fact in the literature, so let us provide the argument: Define $G_0 = G$ and $G_{i+1} = [G_i, G]$ and $H_0 = [G, G]$ and $H_{i+1} = [H_i, G, G]$. It suffices to show that $H_i \leq G_{i+1}$ for all $i \geq 0$. For $i = 0$ this is follows from the definition. For the induction step let $i > 0$. We get $H_i = [H_{i-1}, G, G] \subseteq [G_i, G] = G_{i+1}$.
\[ \rho^e(h) \in \langle h \rangle^{e-1} \] for some \( e = e(n^{2^n}) \). We claim that \( \rho^{d+e}(f) \in \langle f \rangle^{d+e-1} \). Note that
\[ \rho^{d+e}(f) \in \rho^e(\langle f \rangle^{d-1}h) \subseteq \rho^e(\langle f \rangle^{d-1})\rho^e(h) \subseteq \langle f \rangle^{d+e-1} \cdot \rho^e(h). \]

Therefore, it suffices to show that \( \rho^e(h) \in \langle f \rangle^{d+e-1} \). Since \( \rho^d(f) \in \langle f \rangle^{d-1}h \) we have \( h \in \langle f \rangle^d \) and thus \( \rho^e(h) \subseteq \langle h \rangle^{e-1} \subseteq \langle \langle f \rangle^d \rangle^{e-1} = \langle f \rangle^{d+e-1}. \]

\[ \text{Proof of Theorem 19.} \]
Given periodic functions \( f_1, \ldots, f_m \in G^\omega \) with maximum period \( n \), and a number \( T \in \mathbb{N} \). By Proposition 21 the product \( f = f_1 \cdots f_m \) satisfies a recurrence of order \( d \), where \( d \) is bounded polynomially in \( n \). Notice that \( f = 1 \) if and only if \( f(t) = 1 \) for all \( t \leq d - 1 \). Hence, it suffices to verify that \( f_1(t) \cdots f_m(t) = 1 \) for all \( t \leq \min\{d, T\} \). This can be accomplished by solving in parallel a polynomial number of instances of the word problem over \( G \), which is contained in \( \text{TC}^0 \) by [36].

\section{4.2 Proofs of Theorems 1 and 2}

Let us start with the proof of Theorem 1. The following result is from [27].

\[ \text{Proposition 25 ([27]).} \]
For every f.g. group \( G \), the problem \( \text{POWERWP}(G \wr \mathbb{Z}) \) belongs to \( \text{TC}^0(\text{PERIODIC}(G), \text{POWERWP}(G)) \).

The following proposition is from [12] (see the proof of Proposition 7.2 in [12]).

\[ \text{Proposition 26 ([12]).} \]
Let \( G \) be a f.g. group. There is a non-deterministic polynomial time Turing machine \( M \) that takes as input a knapsack expression \( E \) over \( G \wr \mathbb{Z} \) and outputs in each leaf of the computation tree the following data:
\[ = \text{an instance of EXP}\text{EQ}(G) \text{ and} \]
\[ = \text{a finite list of instances of PERIODIC}(G). \]
Moreover, the input expression \( E \) has a \( (G \wr \mathbb{Z}) \)-solution if and only if there is a leaf in the computation tree of \( M \) such that all instances that \( M \) outputs in this leaf are positive.

\[ \text{Proof of Theorem 1.} \]
By [27] the power word problem for a f.g. nilpotent group belongs to \( \text{TC}^0 \) and by Theorem 19, \( \text{PERIODIC}(G) \) belongs to \( \text{TC}^0 \). The theorem follows from Proposition 25.

\[ \text{Proof of Theorem 2.} \]
Let \( G \) be a finite nontrivial nilpotent group. By Theorem 13, knapsack for \( G \wr \mathbb{Z} \) is \( \text{NP} \)-hard. Moreover, \( \text{PERIODIC}(G) \) belongs to \( \text{TC}^0 \) and \( \text{EXP}\text{EQ}(G) \) belongs to \( \text{NP} \) (this holds for every finite group). Proposition 26 implies that \( \text{KNAPSACK}(G \wr \mathbb{Z}) \) belongs to \( \text{NP} \).

\section{5 Wreath products with abelian left factors}

In this section we prove Theorems 3 and 4. For this, we prove two transfer results. For a finitely generated group \( G = \langle \Sigma \rangle \) we define the \textit{power compressed power problem} \( \text{POWERPP}(G) \) as the following computational problem.

\textbf{Input} A word \( u \in \Sigma^* \) and a power word \( (v_1, k_1, \ldots, v_d, k_d) \) over \( \Sigma \).

\textbf{Output} A binary encoded number \( z \in \mathbb{Z} \) with \( u^z = v \) where \( v = v_1^{k_1} \cdots v_d^{k_d} \), or \( \text{no} \) if \( u^z = v \) has no solution.

Notice that if \( G \) is torsion-free then \( u^z = v \) has at most one solution whenever \( u \neq 1 \).

We say that a group \( G = \langle \Sigma \rangle \) is \textit{tame with respect to commensurability}, or short \textit{c-tame}, if there exists a number \( d \in \mathbb{N} \) such that for all commensurable elements \( g, h \in G \) having infinite order there exist numbers \( s, t \in \mathbb{Z} \setminus \{0\} \) such that \( g^s = h^t \) and \( |s|, |t| \leq O(|g| + |h|)^d \).
Theorem 27. Let $H$ and $A$ be f.g. groups where $A$ is abelian and $H$ is c-tame and torsion-free. Then $\text{PowerPP}(A \wr H)$ is $\text{TC}^0$-reducible to $\text{PowerPP}(H)$.

Later, we will show how to derive Theorem 3 from Theorem 27. For Theorem 4 we need the following transfer theorem (recall the definition of an orderable group from Section 3 and the definition of the function $E_G(n)$ from (2) in Section 3.4):

Theorem 28. Let $H$ and $A$ be f.g. groups where $A$ is abelian and $H$ is orderable and knapsack-semilinear. If $E_H(n)$ is exponentially bounded then so is $E_{A \wr H}(n)$.

Using Theorem 3 and 28 we can prove Theorem 4: let us fix an iterated wreath product $W = W_{m,r}$ for some $m \geq 0$, $r \geq 1$ (recall that $W_{m,r} = \mathbb{Z}^r$ and $W_{m+1,r} = \mathbb{Z}^r \wr W_{m,r}$). Since $\mathbb{Z}^m$ is orderable, Theorem 12 implies that $W$ is orderable. Moreover, by Theorem 16, $W$ is also knapsack-semilinear. Since by Lemma 17, $E_A(n)$ is exponentially bounded for every f.g. abelian group $A$, it follows from Theorem 28 that $E_W(n)$ is exponentially bounded as well. By Theorem 3 and Lemma 18, $\text{ExpEq}(W)$ belongs to $\text{NP}$. Finally, $\text{NP}$-hardness of $\text{ExpEq}(W)$ follows from the fact that the question whether a given system of linear Diophantine equations with unary encoded numbers has a solution in $\mathbb{N}$ is $\text{NP}$-hard.

Before we start the proofs of Theorems 27 and 28 we show some simple normalization results and introduce the concept of a progression in a torsion-free group.

Normalization. Consider a wreath product $G = A \wr H$, where $A$ is abelian. We will show how to bring an exponent expression (resp., a power word) into a particular form that will be useful later.

An exponent expression $E = v_0 u_1^v_1 v_2^u_2^x v_3^x \cdots u_d^x v_d$ over $G$ is normalized if

(i) $u_i \in AH$ for all $1 \leq i \leq d$ (here $AH$ is $\{ah \mid a \in A, h \in H\}$)
(ii) $v_i \in H$ for all $0 \leq i \leq d$, and
(iii) $v_0 = 1$.

By the following lemma we can assume normalized exponent expressions in order to prove Theorem 28.

Lemma 29. Let $E$ be an exponent expression over $G = A \wr H$ of length $n$ and assume that $H$ is knapsack-semilinear. There exists a normalized exponent expression $E'$ such that

$\|\text{sol}_G(E)\| = (2n + 1)\|\text{sol}_G(E')\| + O(n)$ and $|E'| \leq O(n^2)$.

Proof. Note that by Theorem 16 also $G$ is knapsack-semilinear. Property (iii) can always be established by conjugating with $v_0$. Hence we can focus on properties (i) and (ii).

We first explain how to achieve property (i) for a given power $u^x$. Since $u$ is given as a word over the generators of $A$ and $H$ we can factorize $u$ as $u = g_0 g_1 \cdots g_\ell$ where $g_0 \in H$ and $g_1, \ldots, g_\ell \in AH$. Let us write $\sigma_{i,j} = \sigma(g_i \cdots g_j)$ for $i \leq j$ and $\sigma_{i,j} = 1$ for $i > j$. Then, for every $x$ we have

$$u^x = \left( \prod_{i=1}^{\ell} (\sigma_{0,i-1}^{-1} g_i \sigma_{i+1,\ell})^x \sigma(u)^{-x} \right) \sigma(u)^x$$

$$= \left( \prod_{i=1}^{\ell} \sigma_{0,i-1}^{-1} (g_i \sigma_{i+1,\ell} \sigma_{0,i-1}^{-1})^x \sigma_{0,i-1}^{-1} \sigma(u)^{-x} \right) \sigma(u)^x \quad (4)$$

Notice that $g_i \sigma_{i+1,\ell} \sigma_{0,i-1}^{-1} \in AH$ and $\sigma(u) \in H \subseteq AH$. Let $\tilde{u}$ be the expression from (4), which has length $O(|u|^2)$.
Now let $E = v_0u_1^{x_1}v_1u_2^{x_2}v_2 \cdots u_d^{x_d}v_d$ be an exponent expression over $G$. We construct the exponent expression

$$E' = v_0u_1v_1u_2 \cdots u_dv_d.$$  

We have $\text{sol}_G(E) = \text{sol}_G(E')$. Notice that the $E$ and $E'$ use the same variables and that the length of $E'$ is bounded by $O(n^2)$.

For condition (ii) from the lemma observe that every element $v \in G$ in (5) that occurs between two consecutive powers or before (after) the first (last) power is given as a word over the generators of $A$ and $H$, say $v = a_1h_1a_2h_2 \cdots a_kh_k$ where $a_j \in A$, $h_j \in H$, $1 \leq j \leq k$. We replace $v$ by the expression $\hat{v} = a_1^yh_1a_2^yh_2 \cdots a_k^yh_k$ for a fresh variable $y$ and enforce $y = 1$ by a semilinear constraint. Applying this to every such word $v$ in (5) yields an exponent expression $E''$ with at most $n + 1$ variables and length $O(n^2)$.

We have $\text{sol}_G(E) = \text{sol}_G(E') = \pi(\text{sol}_G(E'') \cap C)$ where $C$ is the semilinear constraint saying that $y = 1$, and $\pi$ is the projection to the original variables of $E$. By Lemma 8 we have

$$||\text{sol}_G(E)|| = ||\text{sol}_G(E'') \cap C|| \leq (2(n + 1)||\text{sol}_G(E'')|| + 1)^{O(n)}.$$

This concludes the proof. ▷

A power word $(u_1, k_1, \ldots, u_d, k_d)$ over $G$ is normalized if $u_i \in AH$ for all $1 \leq i \leq d$.

► **Lemma 30.** From a given power word over $G$ one can compute in $\text{TC}^0$ a normalized power word that evaluates to the same group element of $G$.

**Proof.** We apply the same construction as in the proof of Lemma 29 (where of course the variables in the exponents are replaced by the numbers from the power word). The new variable $y$ in the above proof is of course replaced by the exponent 1. Finally, notice that the transformation from $E$ to $E''$ can be carried out in $\text{TC}^0$. ▷

**Progressions.** A progression over a torsion-free group $H$ is a non-empty finite sequence $p = (p_i)_{0 \leq i \leq k}$ of the form $p_i = ab^i$ where $a, b \in H$. We call $a$ the offset, $b$ the period\(^5\) and define $\text{supp}(p) = \{p_i \mid 0 \leq i \leq k\}$. The length of $p$ is $|p| = k + 1$ and its endpoints are $p_0$ and $p_k$. A progression whose period is nontrivial is called a ray. Since $H$ is torsion-free, all entries of a ray are pairwise distinct. Two rays are parallel if their periods are commensurable (see Section 3.1).

► **Lemma 31.** If two rays $p$ and $q$ are not parallel then $|\text{supp}(p) \cap \text{supp}(q)| \leq 1$.

**Proof.** Let $p_i = ab^i$ and $q_j = gh^j$. Suppose that $|\text{supp}(p) \cap \text{supp}(q)| \geq 2$. Then there are numbers $i \neq i'$ and $j \neq j'$ such that $ab^i = gh^j$ and $ab^{i'} = gh^{j'}$. This implies $b^{i-i'} = h^{j-j'}$, which means that $p$ and $q$ are parallel – a contradiction. ▷

### 5.1 Proofs of Theorem 27

In this section we prove Theorem 27. In Section 5.2 we then deduce Theorem 3 from Theorem 27.

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\(^5\) A progression of length two or more has a unique period; progressions of length one are assigned a fixed but arbitrary period.
5.1.1 Reducing $\text{PowerWP}(A \upharpoonright H)$ to $\text{PowerPP}(H)$

For the rest of this section we fix a finitely generated abelian group $A = \langle \Gamma \rangle$ and a finitely generated torsion-free group $H = \langle \Sigma \rangle$.

A power-compressed ray over $H$ is a triple $(u, v, \ell)$ where $u$ is a power word over $\Sigma$, $v \in \Sigma^*$ is a word with $v \neq 1$ in $H$ and $\ell \in \mathbb{N}$ is a binary encoded number. Such a power-compressed ray $(u, v, \ell)$ defines the ray $(uv^t)_{0 \leq t \leq \ell}$. We will identify the triple with the ray itself. Define the intersection set $\text{Int}(p, q)$ of two rays $p, q$ by

$$\text{Int}(p, q) = \{i \in [0, |p| - 1] | \exists j \in [0, |q| - 1]; p_i = q_j\}.$$

If $p, q$ are parallel rays and $H$ is c-tame then one can reduce the computation of $\text{Int}(p, q)$ to $\text{PowerPP}(H)$.

Lemma 32. If $H$ is c-tame and torsion-free then the following problem is $\text{TC}^0$-reducible to $\text{PowerPP}(H)$: given two parallel power-compressed rays $p, q$ over $H$, decide whether $\text{Int}(p, q)$ is non-empty and, if so, compute an arithmetic progression $s$ such that $\text{Int}(p, q) = \text{supp}(s)$.

Proof. Suppose that $p = (ab^i)_{0 \leq i \leq k}$ and $q = (gh^j)_{0 \leq j \leq \ell}$. By c-tameness there exists $s, t \in \mathbb{Z} \setminus \{0\}$ such that $\{(i, j) \in \mathbb{Z}^2 | b^i = h^j\} = \{(s, t)\}$ and $|s|, |t|$ are polynomially bounded in $|b| + |h|$. We compute the unary encodings of such numbers $s, t$ by checking all identities $b^s = h^t$ for $|s|, |t| \leq (|b| + |h|)^{(1)}$ (the word problem of $H$ is a special case of $\text{PowerPP}(H)$).

Since $b \neq 1 \neq h$ and $H$ is torsion-free we must have $s \neq 0 \neq t$. We can enforce $t > 0$ by inverting the generator $(s, t)$ if necessary. Since $ab^i = gh^j$ is equivalent to $b^ih^{-i} = a^{-1}g$, Lemma 14 implies that $\{(i, j) \in \mathbb{Z}^2 | ab^i = gh^j\}$ is either empty or a coset of $\langle (s, t) \rangle$. Therefore, if $ab^i = gh^j$ has any solution, then it has a solution $(i, j) \in \mathbb{Z}^2$ where $0 \leq j \leq t - 1$. For all $0 \leq t_0 \leq t - 1$ we solve the $\text{PowerPP}(H)$-instance $ab^i = gh^{t_0}$. If there is no solution for any $0 \leq t_0 \leq t - 1$ we can conclude $\text{Int}(p, q) = \emptyset$. Otherwise let $0 \leq t_0 \leq t - 1$ and $s_0 \in \mathbb{Z}$ with $ab^{s_0} = gh^{t_0}$. We obtain the integer $s_0$ in binary encoding. Then $\text{Int}(p, q)$ is the projection of the first component of the set

$$\{(s_0, t_0) + ((s, t))\} \cap [0, k] \times [0, \ell].$$

Next we compute the interval $Y = \{y \in \mathbb{Z} | 0 \leq t_0 + yt \leq \ell\}$. Since $t > 0$ we have

$$y \in Y \iff 0 \leq t_0 + yt \leq \ell \iff \frac{t_0}{t} \leq y \leq \frac{\ell - t_0}{t}.$$

Hence the endpoints of $Y$ are $y_1 = \lceil -t_0/t \rceil$ and $y_2 = \lfloor (\ell - t_0)/t \rfloor$, which can be computed in $\text{TC}^0$ since integer division is in $\text{TC}^0$ (here, we only need the special case, where we divide by a unary encoded integer). If $y_1 > y_2$ then $Y$ is empty and also $\text{Int}(p, q)$ is empty. Otherwise, we compute $\text{Int}(p, q)$ using the fact that

$$\text{Int}(p, q) = \{s_0 + sy | y \in Y\} \cap [0, k].$$

We transform $Y = [y_1, y_2]$ into the arithmetic progression $(s_0 + sy)_{y_1 \leq y \leq y_2}$ and intersect it with the interval $[0, k]$.  

Lemma 33. Let $H$ be torsion-free. Then the following problem is $\text{TC}^0$-reducible to $\text{PowerPP}(H)$: given a power word $u$ over $\Gamma \cup \Sigma$ representing an element in $A \upharpoonright H$, and a power word $v$ over $\Sigma$ representing an element in $H$, compute a power word for $\tau(u)(v)$.
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Proof. Let \( u = u_1^{k_1} \cdots u_d^{k_d} \). By Lemma 30 we normalize \( u \) in \( \text{TC}^0 \) so that for every \( i \), \( u_i = a_i \sigma(u_i) \) for some \( a_i \in A \). By Lemma 11 we have

\[
\tau(u)(v) = \sum_{i=1}^{d} \tau(u_1^{k_1}) \cdots u_i^{k_i}(v) = \sum_{i=1}^{d} \tau(u_i^{k_i}) \sigma(u_1^{k_1}) \cdots u_{i-1}^{k_{i-1}} v).
\]

Hence it suffices to compute \( \tau(u_i^{k_i}) \) for the power word \( v_i = \sigma(u_i-1) \cdots u_i^{-1} v_i \). If \( \sigma(u_i) = 1 \) then \( \tau(u_i^{k_i})(v_i) = k_i a_i \) if \( v_i = 1 \) and \( \tau(u_i^{k_i})(v_i) = 0 \) otherwise. If \( \sigma(u_i) \neq 1 \) we compute a solution \( x \in \mathbb{Z} \) for \( \sigma(u_i)x = v_i \) (this is an instance of \( \text{POWERPP}(H) \)). If there is no solution or \( x < 0 \) or \( x \geq k_i \) then \( \tau(u_i^{k_i})(v_i) = 0 \); otherwise \( \tau(u_i^{k_i})(v_i) = a_i \).

If \( s \) is an arithmetic progression and \( a \in A \) then we define \( f_{s,a} : \mathbb{N} \rightarrow A \) by \( f_{s,a}(t) = a \) if \( t \in \text{supp}(s) \) and \( f_{s,a}(t) = 0 \) otherwise.

Lemma 34. Given a unary encoded number \( b \in \mathbb{N} \) and finite multiset \( M \) of pairs \((s,a)\), where \( s \) is an arithmetic progression and \( a \in \Gamma^* \) is an element of \( A \), we can compute in \( \text{TC}^0 \) the following:

\[ \text{the set } T = \{ t \in \mathbb{N} | \sum_{(s,a) \in M} f_{s,a}(t) \neq 0 \} \text{ in case } |T| < b, \]

\[ \perp \text{ in case } |T| \geq b. \]

Proof. Let \( s = (d + ie)_{0 \leq i \leq \ell} \) be an arithmetic progression with \( d \in \mathbb{N} \) and \( e \in \mathbb{N} \setminus \{0\} \). We define the interval \( I(s) = [d, d + \ell e] \) and for \( a \in A \) we define the mapping \( g_{s,a} : \mathbb{N} \rightarrow A \) by

\[
g_{s,a}(t) = \begin{cases} a, & \text{if } t \equiv d \mod e, \\ 0, & \text{otherwise.} \end{cases}
\]

It is easy to verify that \( f_{s,a}(t) = g_{s,a}(t) \) for all \( t \in I(s) \). Let \( n \in \mathbb{N} \) be the maximal number in any interval \( I(s) \) for \((s,a) \in M \). By Lemma 7 we can compute in \( \text{TC}^0 \) a partition \( \mathcal{J} \) of \([0, n]\) into intervals such that for all \( J \in \mathcal{J} \) and all \((s,a) \in M \) we have either \( J \subseteq I(s) \) or \( J \cap I(s) = \emptyset \). In the following we will show how to either compute \( T \cap J \) or establish that \( |T \cap J| \geq b \) for all \( J \in \mathcal{J} \). Then the statement follows because if \( |T \cap J| \geq b \) for some \( J \in \mathcal{J} \) then \( |T| \geq b \); otherwise we can compute \( T = \bigcup_{J \in \mathcal{J}} (T \cap J) \) and return \( T \) if \( |T| < b \).

Let \( J \in \mathcal{J} \) be arbitrary. Then for all \( t \in J \) we have

\[
\sum_{(s,a) \in M} f_{s,a}(t) = \sum_{(s,a) \in M, J \subseteq I(s)} f_{s,a}(t) + \sum_{(s,a) \in M, J \cap I(s) = \emptyset} f_{s,a}(t) = \sum_{(s,a) \in M_J} g_{s,a}(t),
\]

where \( M_J = \{(s,a) \in M | J \subseteq I(s)\} \). Recall that \( g_{s,a}(t) = a \) if \( t \) is congruent to the offset of \( s \) modulo its period, and otherwise \( g_{s,a}(t) = 0 \). Hence, for a given input \( t \in \mathbb{N} \) we can compute the value \((6)\) in \( \text{TC}^0 \) (input as well as output are binary encoded). Let \( p \) be the sum of all periods of all arithmetic progressions \( s \) occurring in \( M \), which is linear in the input size. If \( |J| < bp \) then we can compute \( T \cap J \) in \( \text{TC}^0 \). If \( |J| \geq bp \) and \( j = \min J \) we compute

\[
T_J = \{ t \in [j, j + bp - 1] | \sum_{(s,a) \in M} f_{s,a}(t) \neq 0 \},
\]

which is a subset of \( T \). If \( |T_J| \geq b \), we have established \( |T| \geq b \) and we can output \( \perp \). If \( |T_J| < b \) then \( [j, j + bp - 1] \setminus T_J \) contains at least \( bp - (b - 1) = bp - 1 \) many elements and \( [j, j + bp - 1] \setminus T_J \) is a disjoint union of at most \( b \) intervals. Hence there exists an interval \( I \subseteq [j, j + bp - 1] \) containing at least \( p \) elements. This implies that

\[
0 = \sum_{(s,a) \in M} f_{s,a}(t) = \sum_{(s,a) \in M_J} g_{s,a}(t)
\]
for all \( t \in I \). Since \( \sum_{(s,a) \in M_J} g_{s,a} \) satisfies a recurrence of order at most \( p \) by Lemma 20 we know that in fact \( \sum_{(s,a) \in M_J} g_{s,a} = 0 \). By (6) we have \( \sum_{(s,a) \in M} f_{s,a}(t) = 0 \) for all \( t \in J \), and thus we can output \( T \cap J = \emptyset \). This concludes the proof.  

We now come to the main reduction of this subsection:

**Proposition 35.** If the group \( H \) is \( c \)-tame and torsion-free then \( \text{PowerWP}(A \cup H) \in \text{TC}^0(\text{PowerPP}(H)) \).  

**Proof.** Take a power word \( u = u_1^{k_1} \cdots u_d^{k_d} \) over \((\Gamma \cup \Sigma)^*\). By Lemma 30 we normalize \( u \) in \( \text{TC}^0 \) so that for every \( i \), \( u_i = a_i \sigma(u_i) \) for some \( a_i \in \Gamma^* \). To test \( u = 1 \) we need to verify both \( \sigma(u) = 1 \) and \( \tau(u) = 0 \). The former equation is an instance of \( \text{PowerWP}(H) \).

For \( 1 \leq i \leq d \) we define \( \hat{u}_i = \sigma(u_1^{k_1} \cdots u_{i-1}^{k_{i-1}} u_i^{k_i}) \). By Lemma 11 we know that \( \tau(u) = \sum_{i=1}^d \tau(\hat{u}_i) \).

For \( 1 \leq i \leq d \) and \( 0 \leq k < k_i \) we define

\[
\sigma(i,k) = \sigma(u_1^{k_1} \cdots u_i^{k_i-1} u_i^{k_i}).
\]

Notice that \( p_i = (\sigma(i,k))_{0 \leq k < k_i} \) is a power-compressed progression, and, since \( u \) is normalized, we have \( \text{supp}(\tau(\hat{u}_i)) \subseteq \text{supp}(p_i) \) (we have equality if \( a_i \neq 0 \)). Hence it suffices to test whether \( \tau(u)(h) = 0 \) for all \( h \in \text{supp}(p_i) \) and \( i \in [1,d] \).

Let \( R = \{ i \in [1,d] \mid \sigma(u_i) \neq 1 \} \) and define the equivalence relation \( \parallel \) on \( R \) by \( i \parallel j \) if and only if \( \sigma(u_i) \) and \( \sigma(u_j) \) are commensurable, or equivalently if the rays \( p_i \) and \( p_j \) are parallel. For all \( i,j \in R \) with \( i \parallel j \) we compute \( \text{Int}(p_i,p_j) \) as an arithmetic progression \( s_{i,j} \). By Lemma 32 this can be accomplished by a \( \text{TC}^0 \)-reduction to \( \text{PowerPP}(H) \). If \( t \in \text{Int}(p_i,p_j) \) then \( \sigma(i,t) \in \text{supp}(p_j) \) and therefore \( f_{s_{i,j},a_j}(t) = a_j = \tau(\hat{u}_j)(\sigma(i,t)) \).

Hence we have shown that

\[
f_{s_{i,j},a_j}(t) = \tau(\hat{u}_j)(\sigma(i,t)), \quad \text{for all } 0 \leq t \leq k_i - 1.
\]

For all \( i \in R \) we define

\[
T_i = \{ t \in [0,k_i - 1] \mid \sum_{i,j} \tau(\hat{u}_j)(\sigma(i,t)) \neq 0 \}.
\]

By Lemma 34 for all \( i \in R \) we can either compute the set \( T_i \) or conclude that \( |T_i| \geq d + 1 \).

If there exists \( i \in R \) with \( |T_i| \geq d + 1 \) then we claim that \( \tau(u) \neq 0 \). We say that an index \( j \in [1,d] \) crosses \( t \in T_i \) if \( i \parallel j \) and \( \sigma(i,t) \in \text{supp}(p_j) \) (note that if \( j \notin R \) then \( i \parallel j \) holds).

Notice that a single index \( j \in [1,d] \) can cross at most one element \( t \in T_i \) since otherwise \( |\text{supp}(p_i) \cap \text{supp}(p_j)| \geq 2 \), which contradicts Lemma 31. This implies that \( T_i \) contains at most \( d \) crossed elements and therefore at least one uncrossed element, say \( t \in T_i \). Since \( \sigma(i,t) \notin \text{supp}(p_j) \supseteq \text{supp}(\tau(\hat{u}_j)) \) for all \( j \in [1,d] \) with \( i \parallel j \) we obtain

\[
\tau(u)(\sigma(i,t)) = \sum_{j=1}^d \tau(\hat{u}_j)(\sigma(i,t)) = \sum_{i \parallel j} \tau(\hat{u}_j)(\sigma(i,t)) + \sum_{i \not\parallel j} \tau(\hat{u}_j)(\sigma(i,t)) \neq 0,
\]

which shows the claim.

In the other case we have computed all sets \( T_i \) for \( i \in R \). Using Lemma 33 we test in \( \text{TC}^0 \) whether

\[
\tau(u)(\sigma(i,t)) = 0, \quad \text{for all } t \in T_i \text{ and } i \in R
\]

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and whether
\[ \tau(u)(\sigma(i, 0)) = 0, \quad \text{for all } i \in [1, d] \setminus R \]
holds. If any of the equalities in (10) and (11) does not hold we know that \( \tau(u) \neq 0 \).

Otherwise we can verify that \( \tau(u) = 0 \): Let \( h \in \bigcup_{i \leq d} \text{supp}(p_i) \). If \( h = \sigma(i, t) \) for some \( i \in [1, d] \setminus R \) or \( h = \sigma(i, t) \) for some \( t \in T_i \) and \( i \in R \) we are done by (10) and (11). Now assume the contrary. Then we know \( \tau(\hat{u}_j)(h) = 0 \) for all \( j \in [1, d] \setminus R \). We have
\[ \tau(u)(h) = \sum_{j \in R} \tau(\hat{u}_j)(h) + \sum_{j \in [1, d] \setminus R} \tau(\hat{u}_j)(h) = \sum_{C \in \mathcal{H}} \sum_{j \in C} \tau(\hat{u}_j)(h), \]
and we claim that \( \sum_{j \in C} \tau(\hat{u}_j)(h) = 0 \) for all \( \mathcal{H} \)-classes \( C \). Consider a \( \mathcal{H} \)-class \( C \). If \( h = \sigma(i, t) \) for some \( i \in C \) and \( t \in [0, k_i - 1] \) then
\[ \sum_{j \in C} \tau(\hat{u}_j)(h) = \sum_{i \in \mathcal{H}} \tau(\hat{u}_j)(\sigma(i, t)) = 0, \]
since \( t \notin T_i \). Otherwise \( h \notin \{\sigma(i, t) | i \in C, t \in [0, k_i - 1]\} = \bigcup_{i \in C} \text{supp}(p_i) \), and therefore \( \tau(\hat{u}_j)(h) = 0 \) for all \( j \in C \). By (12) we conclude that \( \tau(u) = 0 \). ▷

5.1.2 Reducing PowerPP(\( A \wr H \)) to PowerWP(\( A \wr H \)) and PowerPP(\( H \))

Lemma 36. If the finitely generated group \( H \) is torsion-free then PowerPP(\( A \wr H \)) belongs to TC⁰(PowerWP(\( A \wr H \)), PowerPP(\( H \))).

Proof. We want to solve \( u^x = v \) in \( A \wr H \), where \( u \in (\Gamma \cup \Sigma)^* \) and \( v \) is a power compressed word, namely \( v = v_1^{k_1} \cdots v_d^{k_d} \) with binary encoded integers \( k_j \) and \( v_j \in (\Gamma \cup \Sigma)^* \). We check whether \( \sigma(u) = 1 \), which is an instance of PowerWP(\( H \)), and make a case distinction:

Case 1. \( \sigma(u) \neq 1 \): Since \( H \) is torsion-free the equation \( \sigma(u)^x = \sigma(v) \) has at most one solution. We can solve it using the oracle for PowerPP(\( H \)). If \( \sigma(u)^x = \sigma(v) \) has no solution then also \( u^x = v \) has no solution. Otherwise we obtain a binary encoded \( z \in \mathbb{Z} \) with \( \sigma(u)^x = \sigma(v) \). It remains to check whether \( u^x = v \) in \( A \wr H \), i.e. whether \( v_1^{k_1} \cdots v_d^{k_d} u^{−z} = 1 \) in \( A \wr H \). This is an instance of PowerWP(\( A \wr H \)).

Case 2. \( \sigma(u) = 1 \): We first check whether \( \sigma(v) = 1 \) in \( H \), which is an instance of PowerWP(\( H \)). If \( \sigma(v) \neq 1 \) then we output no. Now assume that \( \sigma(u) = \sigma(v) = 1 \). We can compute \( \text{supp}(\tau(u)) \) as well as \( \Gamma \)-words for all \( \tau(u)(h) \) (\( h \in \text{supp}(\tau(u)) \)) in TC⁰ by going over all prefixes of the word \( u \) (see Section 3.2).

Since \( \sigma(u) = \sigma(v) = 1 \), the equation \( u^x = v \) is equivalent to \( x \cdot \tau(u) = \tau(v) \). The f.g. abelian group \( A \) can be written as \( A = \mathbb{Z}^m \times B \) for some finite abelian group \( B \) and \( m \in \mathbb{N} \). If \( \tau(u)(h) \in B \) for all \( h \in \text{supp}(\tau(u)) \) then \( u \) has order at most \( |B| \). Hence, there is a \( z \in \mathbb{Z} \) with \( u^z = v \) if and only if there is \( 0 \leq z < |B| \) with \( u^z = v \). Using the oracle for PowerWP(\( A \wr H \)) we can check all such \( z \) in parallel. Now assume that there is \( h \in \text{supp}(\tau(u)) \) such that \( \tau(u)(h) = (a, b) \) for some \( a = \mathbb{Z}^m \setminus \{0\} \). From the word for \( \tau(u)(h) \) we can compute the vector \( a \) in unary notation. Moreover, using Lemma 33 we compute in TC⁰ a power word for \( \tau(v)(h) \in A \) in TC⁰. Let \( \tau(v)(h) = (b, c) \). From the computed power word we can compute (using simple arithmetic) the binary encoding of the vector \( b \).

Every solution \( z \) for \( u^z = v \) has to satisfy \( z \cdot a = b \). The only candidate for this is \( z = b_i/a_i \) (recall that integer division is in TC⁰) where \( a_i \) is a non-zero entry of the vector \( a \neq 0 \) and \( b_i \) is the corresponding entry of \( b \). If \( z \) is not an integer, then \( u^z = v \) has no solution. Otherwise, if \( z \in \mathbb{Z} \), we check whether \( u^z = v \) using the oracle for PowerWP(\( A \wr H \)). ▷
5.2 Proof of Theorem 3

In this section we deduce Theorem 3 from Theorem 27. Recall the definition of the iterated wreath products $W_{m,r}$. Every $W_{m,r}$ is orderable, and hence is torsion-free and has the unique roots property, see Section 3. The main point is that all groups $W_{m,r}$ are c-tame. We start with the following easy lemma which covers the case $m = 0$ (i.e., $W_{m,r} = \mathbb{Z}^r$).

**Lemma 37.** For vectors $a, b \in \mathbb{Z}^n \setminus \{0\}$ there exist numbers $s, t \in \mathbb{Z}$ with $|s| \leq \|b\|$, $|t| \leq \|a\|$ such that $\{(x, y) \in \mathbb{Z}^2 \mid xa = yb\} = \{(s, t)\}$.

**Proof.** If $a$ and $b$ are linearly independent over $\mathbb{Q}$ then $(0, 0)$ is the only rational and hence integer solution. Otherwise there exist coprime integers $s, t \in \mathbb{Z} \setminus \{0\}$ with $sa = tb$. Since $s$ divides all entries in $b$ we must have $|s| \leq \|b\|$, and similarly for $t$ and $a$. Observe that $xa = yb$ is equivalent to $tx = sy$.

Therefore, every vector in $\{(s, t)\}$ is a solution. Conversely, if $tx = sy$, then $s$ divides $x$ (due to coprimality of $s$ and $t$) and thus $(x, y) = \frac{s}{t}(s, t)$ is an integer multiple of $(s, t)$. Hence the solution set is $\{(s, t)\}$. ◀

Recall the definition of the iterated wreath products $W_{m,r}$. Every $W_{m,r}$ is orderable, and hence is torsion-free and has the unique roots property, see Section 3.

**Proposition 38.** For all $r \geq 1$, $m \geq 0$ the groups $W_{m,r}$ and $S_{m,r}$ are c-tame.

**Proof.** It suffices to show the statement for $W_{m,r}$. We fix the rank $r$ and the prove the claim by induction on $m$. If $m = 0$ then $W_{0,r} = \mathbb{Z}^r$ and the statement follows from Lemma 37. Now assume that $m \geq 1$ and let $u, v$ be words over the generators of $W_{m,r}$ with $u \neq 1 \neq v$ in $W_{m,r}$. Let $U = \{(x, y) \in \mathbb{Z}^2 \mid u^x = v^y\}$ and $V = \{(x, y) \in \mathbb{Z}^2 \mid \sigma(u)^x = \sigma(v)^y\}$. By Lemma 14 the sets $U, V$ are subgroups of $\mathbb{Z}^2$ and $U \leq V$.

**Case 1.** $\sigma(u) \neq 1 \neq \sigma(v)$: Then $V$ is cyclic by Lemma 14, say $V = \{(s, t)\}$. By the induction hypothesis, $|s|$ and $|t|$ are polynomially bounded in $|\sigma(uv)| \leq |uv|$. Since $U \leq V$ we can write $U$ as $U = \{a \cdot (s, t)\}$ for some $a > 0$. We obtain the identity $u^{as} = v^a$ in $W_{m,r}$. Since $W_{m,r}$ has the unique roots property we get $u^s = v^s$ and hence $U = V = \{(s, t)\}$. Since $|s|$ and $|t|$ are polynomially bounded in $|uv|$, we are done.

**Case 2.** $\sigma(u) = 1$ or $\sigma(v) = 1$ in $W_{m-1,r}$. If exactly one of these projections is 1, then we only have the trivial solution $(0, 0)$ for the equation $u^x = v^y$, since $W_{m-1,r}$ and $W_{m,r}$ are torsion-free. If $\sigma(u) = \sigma(v) = 1$ then $\tau(u) = 0 \neq \tau(v)$ by $u \neq 1 \neq v$. The equation $u^x = v^y$ is equivalent to $x \cdot \tau(u) = y \cdot \tau(v)$ in the abelian group $(\mathbb{Z}^r)^{(W_{m-1,r})}$. Since the absolute values of the integers that appear in the images of $\tau(u)$ and $\tau(v)$ are linearly bounded by $|uv|$ we can conclude the proof with Lemma 37. ◀

We can now show Theorem 3:

**Proof of Theorem 3.** We will prove by induction on $m \in \mathbb{N}$ that $\text{POWERPP}(W_{m,r})$ and hence also $\text{POWERWP}(W_{m,r})$ belongs to $\text{TC}^0$. If $m = 0$ then $\text{POWERPP}(W_{0,r})$ is the problem of solving a system of $r$ linear equations $a_i x = b_i$ where $a_i$ is given in unary encoding and $b_i$ is given in binary encoding for $1 \leq i \leq r$. Since integer division belongs to $\text{TC}^0$ (here, we only have to divide by the unary encoded integers $a_i$) this problem can be solved in $\text{TC}^0$. The inductive step follows from Theorem 27 and the fact that all groups $W_{m,r}$ are c-tame (Proposition 38) and torsion-free. ◀
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5.3 Proof of Theorem 28

For the rest of this section fix the groups $H$ and $A$ from Theorem 28. Hence, $A$ is f.g. abelian and $H$ is orderable and knapsack-semilinear with $E_H(n) = 2^{n^{O(1)}}$. By Theorem 16, also $A \wr H$ is knapsack-semilinear.

The main idea for the proof of Theorem 28 is to describe the solution set $\text{sol}_{A \wr H}(E)$ for a given exponential expression $E$ by a Presburger formula (Section 5.3.3). This formula is an exponentially long disjunction of existential Presburger formulas. For bounding the magnitude of the solution set, the disjunction (leading to a union of semilinear sets) as well as the existential quantifiers (leading to a projection of a semilinear set) have no influence. The remaining formula is a polynomially large conjunction of semilinear constraints of exponential magnitude. With Lemma 8 we then obtain an exponential bound on the magnitude of the solution set.

A crucial fact is that our Presburger formula for $\text{sol}_{A \wr H}(E)$ does not involve quantifier alternations. This is in contrast to the Presburger formulas constructed in [12] for showing that the class of knapsack-semilinear groups is closed under wreath products. We can avoid quantifier alternations since we restrict to wreath products $A \wr H$ with $A$ abelian. Let us also remark that we do not have to algorithmically construct the Presburger formula for the solution set. Only its existence is important, which yields an exponential bound on the size of a solution.

Before we construct the Presburger formula for the set of solutions, we first have to introduce a certain decomposition of solutions that culminates in Proposition 44.

5.3.1 Decomposition into packed bundles

In this section, we will only work with the orderable group $H$. A bundle $P$ is a finite multiset of progressions over $H$. A refinement of a progression $p = (p_i)_{0 \leq i \leq m}$ is a bundle $\{ (p_i)_{m_{k-1} \leq i \leq m_k - 1} \mid 1 \leq k \leq \ell \}$ for some $0 = m_0 < m_1 < \cdots < m_\ell = m + 1$. A bundle $Q$ is a refinement of a bundle $P$ if one can decompose $Q = \bigcup_{p \in P} Q_p$ such that each $Q_p$ is a refinement of $p$. We emphasize that a union of bundles is always understood as the union of multisets, and that $|Q|$ (for a bundle $Q$) refers to the size of $Q$ as a multiset.

Two progressions $p_1, p_2$ are disjoint if $\text{supp}(p_1) \cap \text{supp}(p_2) = \emptyset$. Two bundles $P, Q$ are disjoint if any two progressions $p \in P$, $q \in Q$ are disjoint. A bundle $P$ is stacking if there exists $h \in H$ such that $\text{supp}(p) = \{h\}$ for all $p \in P$.

Lemma 39. For every bundle $P$ there exists a refinement $Q$ of $P$ of size $|Q| = \mathcal{O}(|P|^3)$ and a partition $Q = \bigcup_k Q_k$ into pairwise disjoint subbundles $Q_k$ such that each bundle $Q_k$ consists of parallel rays or is stacking.

Proof. Let $S$ be the union of all intersections $\text{supp}(p) \cap \text{supp}(q)$ of size one over all $p, q \in P$, which contains at most $|P|^2$ elements. We refine each progression $p = (p_i)_{0 \leq i \leq m}$ into progressions $p^{(j)}$ such that $|\text{supp}(p^{(j)})| = 1$ or $\text{supp}(p^{(j)}) \cap S = \emptyset$ as follows. Define the following relation on $[0, m]$:

Let $i_1 \sim i_2$ if either (i) there exists $h \in S$ such that $p_i = h$ for all $i_1 \leq i \leq i_2$ or (ii) $p_i \notin S$ for all $i_1 \leq i \leq i_2$. Notice that this defines an equivalence relation, which partitions $[0, m]$ into at most $2|S| + 1 \leq 2|P|^2 + 1$ many intervals and in that way yields a refinement $Q_p$ of $p$ of size $2|P|^2 + 1$. Let $Q$ be the union of all bundles $Q_p$ over all $p \in P$, which contains $\mathcal{O}(|P|^3)$ many progressions. Notice that $S$ is still the union of all intersections $\text{supp}(p) \cap \text{supp}(q)$ of size one over all $p, q \in Q$. Therefore any two progressions $p, q \in Q$ with $|\text{supp}(p) \cap \text{supp}(q)| = 1$ satisfy $|\text{supp}(p)| = |\text{supp}(q)| = 1$ and $\text{supp}(p) = \text{supp}(q)$.

Finally we define the subbundles $Q_k$. Two $p, q \in Q$ are bundled together if
1. \(|\text{supp}(p)|, |\text{supp}(q)| = 1\) and \(\text{supp}(p) = \text{supp}(q)\), or
2. \(|\text{supp}(p)|, |\text{supp}(q)| \geq 2\) and \(p, q\) are parallel.

Let us verify that any two progressions \(p, q \in Q\) which are not in the same bundle have disjoint supports. As observed above, if \(|\text{supp}(p) \cap \text{supp}(q)| = 1\) then \(\text{supp}(p) = \text{supp}(q)\), which would mean that \(p, q\) are in the same bundle. If \(|\text{supp}(p) \cap \text{supp}(q)| \geq 2\) then \(p\) and \(q\) are parallel rays by Lemma 31, which contradicts the fact that they are in different bundles.

A ray \(p = (ab^i)_{0 \leq i \leq m}\) is packed into a ray \(q = (gh^j)_{0 \leq j \leq \ell}\) if \(b = h^d\) for some \(d \in \mathbb{Z} \setminus \{0\}\) and \(\text{supp}(p) = \text{supp}(q) \cap a(b)\). Intuitively, this means that \(p\) is contained in \(q\) and \(p\) cannot be extended in \(q\). More explicitly, the latter condition states that \(i \in \mathbb{Z}\) and \(ab^i \in \text{supp}(q)\) implies \(i \in [0, m]\) (we call this the maximality condition). A bundle \(P\) of rays is packed into \(q\) if every \(p \in P\) is packed into \(q\).

\[\text{Lemma 40.}\] Let \(p = (ab^i)_{0 \leq i \leq m}, q = (gh^j)_{0 \leq j \leq \ell}\) be rays with \(b = h^d\) for some \(d \in \mathbb{Z} \setminus \{0\}\).

(i) \(\{i \in \mathbb{Z} \mid ab^i \in \text{supp}(q)\}\) is an interval.

(ii) \(p\) is packed into the ray \(q\) if and only if \(a, ab^m \in \text{supp}(q)\) and \(ab^{-1}, ab^{m+1} \in \text{supp}(q)\) where \(\text{supp}(q) = \{gh^j \mid j \in \mathbb{Z} \setminus [0, \ell]\}\).

(iii) If a bundle \(P\) is packed into \(q\) then \(P\) is packed into a subray \(q'\) of \(q\) whose endpoints are endpoints of rays in \(P\).

(iv) If \(p\) is packed into \(q\) then \(\text{supp}(p) = \{gh^j \mid 0 \leq j \leq \ell, j \equiv t \mod d\}\) for some unique remainder \(0 \leq t < |d|\).

\[\text{Proof.}\] For point (i) consider integers \(i_1 \leq i \leq i_2\) and assume that \(ab^{i_1}, ab^{i_2} \in \text{supp}(q)\), i.e. there exist \(j_1, j_2 \in [0, \ell]\) with \(ab^{i_1} = gh^{j_1}\) and \(ab^{i_2} = gh^{j_2}\). Hence \(ah^{i_1} = gh^{j_1}\) and \(ah^{i_2} = gh^{j_2}\). From this we obtain

\[h^{j_2-j_1} = (gh^{j_1})^{-1}gh^{j_2} = (ah^{i_1})^{-1}ah^{i_2} = h^{d(i_2-i_1)}\]

and therefore \(j_2 - j_1 = d(i_2 - i_1)\) (since \(h\) has infinite order). We claim that \(ab^t = ab^{i_1+t-i_1} = gh^{j_1+d(i_2-i_1)}\) belongs to \(\text{supp}(q)\): If \(d > 0\) then \(j_1 \leq j_1 + d(i - i_1) \leq j_1 + d(i_2 - i_1) = j_2\), thus, \(ab^t \in \text{supp}(q)\). The case \(d < 0\) is symmetric.

For point (ii), if \(p\) is packed into \(q\) then \(a \in \text{supp}(q)\) by definition, i.e. \(a = gh^d\) for some \(j \in [0, \ell]\). Therefore \(ab^{-1} = gh^{-1}h^{-1} = gh^{j-d}\) and, since \(ab^{-1} \notin \text{supp}(q)\), we deduce that \(j - d \notin [0, \ell]\) by the maximality condition. Similarly \(ab^m \in \text{supp}(q)\) by definition, i.e. \(ab^m = gh^{j'}\) for some \(j' \in [0, \ell]\). Therefore \(ab^{m+1} = gh^{j'+d}\) and, since \(ab^{m+1} \notin \text{supp}(q)\) we know that \(j' + d \notin [0, \ell]\).

For the direction from right to left assume that \(a, ab^m \in \text{supp}(q)\) and \(ab^{-1}, ab^{m+1} \in \text{supp}(q)\). From point (i) we get \(\text{supp}(p) \subseteq \text{supp}(q)\). Moreover, if \(ab^t \in \text{supp}(q)\) for some \(i \in \mathbb{Z} \setminus [0, m]\) then point (i) would imply \(ab^{-1} \in \text{supp}(q)\) or \(ab^{m+1} \in \text{supp}(q)\), which is a contradiction.

For point (iii) suppose that \(q\) is an endpoint of \(q\) which is not the endpoint of any ray \(p \in P\). If \(q'\) is obtained by removing \(q\) from \(q\) then the property from point (ii) is preserved for \(q'\) since \(\text{supp}(q) \subseteq \text{supp}(q')\). Hence we can remove endpoints of \(q\) until the desired property is satisfied.

For point (iv) assume that \(p\) is packed into \(q\). Hence, we have \(\text{supp}(p) = \text{supp}(q) \cap a(b) = \text{supp}(q) \cap a(h^d)\). There exists \(s \in [0, m]\) with \(a = gh^s\). Let \(t = s \mod d\). It suffices to show

\[\{gh^j \mid 0 \leq j \leq \ell, j \equiv t \mod d\} = \{gh^j \mid 0 \leq j \leq \ell\} \cap a(h^d)\.]
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First, consider some \( j \in [0, \ell] \) with \( j \equiv t \pmod{d} \). We have to show that \( gh^j \in a(h^d) \). Since \( j \equiv t \pmod{d} \) we have \( j \equiv s \pmod{d} \). Let \( j = s + rd \) for some \( r \in \mathbb{Z} \). We obtain \( gh^j = gh^{s + rd} = gh^{s}h^rd = ah^rd \).

For the other inclusion let \( j \in [0, \ell] \) such that \( gh^j \in a(h^d) \), i.e. \( gh^j = ah^{di} \) for some \( i \in \mathbb{Z} \). We have to show that \( j \equiv t \pmod{d} \). Since \( a = gh^s \) we have \( gh^j = ah^{di} = gh^{s + di} \), i.e. \( j = s + di \). Hence, \( j \equiv s \pmod{d} \), and therefore \( j \equiv t \pmod{d} \).

The remainder is clearly unique since \( h \) is nontrivial.

\[ \text{Lemma 41.} \]

Let \( h \in H \) and let \( P \) be a bundle of parallel rays whose periods are contained in \( \langle h \rangle \). Then there exist a refinement \( Q \) of \( P \) of size \( |Q| = O(|P|^2) \) and a partition \( Q = \bigcup_k Q_k \) into pairwise disjoint subbundles \( Q_k \) such that each subbundle \( Q_k \) is packed into a ray with period \( h \).

\[ \text{Proof.} \]

For every ray \( p \in P \) there exists a left coset \( g(h) \) which contains \( \text{supp}(p) \). Therefore we can split \( P \) into disjoint bundles \( P_{g(h)} = \{ p \in P \mid \text{supp}(p) \subseteq g(h) \} \) and treat each bundle \( P_{g(h)} \) individually.

Consider a left coset \( K \) of \( \langle h \rangle \) in \( H \) and suppose that \( \text{supp}(p) \subseteq K \) for all \( p \in P \). Define the linear order \( \leq_h \) on \( K \) by \( h_1 \leq_h h_2 \) if \( h_1 h^d = h_2 \) for some \( d \in \mathbb{N} \). Define \( \beta : K \to 2^P \) by

\[ \beta(g) = \{ p \in P \mid \exists g_1, g_2 \in \text{supp}(p): g_1 \leq_h g \leq_h g_2 \}. \]

Intuitively, \( \beta(g) \) contains all rays \( p \) that cover the element \( g \). The mapping \( \beta \) satisfies the condition of Lemma 7 and hence we obtain a partition \( J = \{ J_1, \ldots, J_k \} \) of \( K \) into at most \( O(|P|^2) \) many intervals (with respect to \( \leq_h \)) and subsets \( P_J \subseteq P \) such that \( \beta(J) = \{ P_J \} \) for all \( J \in J \).

For \( p \in P \) and \( J \in J \) define the restriction \( p|_J \) to those entries \( p_i \in J \). Notice that, if \( p|_J \) is non-empty, then it is a subray of \( p \) since the natural order on \( p \) respects \( \leq_h \) or \( \geq_h \), i.e. either \( i \leq j \) implies \( p_i \leq_h p_j \) or it implies \( p_i \geq_h p_j \), depending on whether the period of \( p \) is a positive or a negative power of \( h \). Furthermore, if \( p \in P \setminus P_J \) then \( p \notin \beta(g) \) for all \( g \in J \) and thus \( p|_J \) is empty.

For every \( J \in J \) let \( Q_J \) be the bundle containing all non-empty restrictions \( p|_J \) for \( p \in P_J \). Then \( Q = \bigcup_{J \in J} Q_J \) is a refinement of \( P \) and the subbundles \( Q_J \) are pairwise disjoint. Its size is bounded by \( |Q| \leq |P| |J| = O(|P|^2) \). It remains to prove that every bundle \( Q_J \) is packed into a ray with period \( h \). Consider an interval \( J \in J \). If \( P_J = \emptyset \) then \( Q_J \) is empty and the claim is vacuous. If \( P_J \) contains some ray \( p' \) then for all \( g \in J \) there exist \( g_1, g_2 \in \text{supp}(p') \) with \( g_1 \leq_h g \leq_h g_2 \). Since \( \text{supp}(p') \) is finite also \( J \) must be finite. Therefore we can write \( J = \{ gh^j \mid 0 \leq j \leq \ell \} \) for some \( g \in J \) and \( \ell \in \mathbb{N} \). We naturally view \( J \) as the ray \( q_J = (gh^j)_{0 \leq j \leq \ell} \). We claim that for all \( p \in P_J \) the restriction \( p|_J \) is packed into \( q_J \).

Suppose that \( p = (ab^i)_{0 \leq i \leq m} \) and that \( p|_J = (ab^i)_{s \leq i \leq t} \) for some \( 0 \leq s \leq t \leq m \). First observe that \( b \) is a power of \( h \), say \( b = h^d \) for \( d \in \mathbb{Z} \setminus \{0\} \), and that \( \text{supp}(p), p < J \leq \text{supp}(q_J) \). Let \( ab^i \in \text{supp}(q_J) = J \) be an arbitrary element with \( i \in \mathbb{Z} \), and thus \( p \in P_J = \beta(ab^i) \). It follows that there exist \( ab^{i_1}, ab^{i_2} \in \text{supp}(p) \) with \( 0 \leq i_1, i_2 \leq m \) and \( ab^{i_1} \leq_h ab^{i_2} \leq_h ab^{i_2} \). Therefore either \( ab^{i_1} \leq_h ab^{i_2} \leq_h ab^{i_2} \) or \( ab^{i_1} \geq_h ab^{i_2} \geq_h ab^{i_2} \). This implies \( ab^i \in \text{supp}(p) \) and thus \( ab^i \in \text{supp}(p|_J) \). This concludes the proof.

### 5.3.2 From knapsack to bundles

Fix a normalized exponent expression \( E = u_1^{i_1}v_1u_2^{i_2}v_2 \cdots u_d^{i_d}v_d \) over \( A \setminus H \) for the rest of this section where \( |E| \leq n \). Let \( X = \{ x_1, \ldots, x_d \} \) be the set of variables appearing in \( E \). Since
u_1, \ldots, u_d \in \mathcal{A} \mathcal{H} \text{ there exist (unique) elements } a_1, \ldots, a_d \in A \text{ such that } u_r = a_r \sigma(u_r) \text{ for all } 1 \leq r \leq d. \text{ For } 1 \leq r \leq d \text{ and a fresh variable } y \notin X \text{ we define the exponent expression}

\[ E_r(y) = u_1^{2^r} v_1 \cdots u_{r-1}^{2^{r-1}} v_{r-1} u_r^p. \]  

Let \( 1 \leq r \leq d, \nu \in \mathbb{N}^X \text{ and } k \in \mathbb{N}. \) With \( \nu[y/k] \) we denote the valuation that extends \( \nu \) by \( \nu[y/k](y) = k. \) We define \( \sigma_r(k, \nu) = \nu[y/k](\sigma(E_r(y))) \) and \( \tau_r = \sigma_r(k, \nu). \) Given \( 0 \leq s \leq t \leq \nu(x_r) - 1 \) we define \( \sigma_r(k, s, t) = (\sigma_r(k, \nu))_{s \leq k \leq t} \) and \( \tau_r(r, s, t) = \sum_{k=s}^t \tau_r(k). \) Notice that \( \sigma_r(k, s, t) \) is a progression with period \( \nu(x_r) \) and \( \tau_r = \text{supp}(\sigma_r(k, s, t)). \) If \( r \in R, \) i.e. \( \sigma(u_r) \neq 1, \) and \( h \in \text{supp}(\sigma_r(k, s, t)) \) then there exists exactly one index \( s \leq k_h \leq t \text{ such that } h = \sigma_r(k, k_h), \) and

\[ \tau_r(r, s, t)(h) = \sum_{k=s}^t \tau_r(k)(\sigma_r(k, h)) = \tau_r(k, h)(\sigma_r(k, h)) = a_r. \]  

For a valuation \( \nu \in \mathbb{N}^X \) we define a \( \nu \)-decomposition to be a set \( D \subseteq [1, d] \times \mathbb{N}^2 \) such that \( \{ \{r \} \times [s, t] | (r, s, t) \in D \} \) is a partition of \( \{ \{r \} | 1 \leq r \leq d, 0 \leq k \leq \nu(x_r) - 1 \}. \)

\textbf{Lemma 42.} For all \( \nu \in \mathbb{N}^X \text{ and } \nu \)-decompositions \( D \) we have \( \tau(E_r) = \sum_{(r, s, t) \in D} \tau_r(r, s, t). \)

\textbf{Proof.} First we observe that

\[ \sum_{(r, s, t) \in D} \tau_r(r, s, t) = \sum_{(r, s, t) \in D} \sum_{s \leq k \leq t} \tau_r(k, s, t) = \sum_{1 \leq r \leq d} \sum_{0 \leq k \leq \nu(x_r) - 1} \tau_r(k). \]

For all \( 1 \leq r \leq d \) and \( 0 \leq k \leq \nu(x_r) - 1 \) we have

\[ \tau_r(k) = \tau(\sigma_r(k, r)) = \tau(\sigma_r(\nu(x_r), v_1^{\nu(x_r)} u_1^{\nu(x_r-1)} v_{r-1} u_r^k)) = \tau(\sigma_r(\nu(x_r), v_1^{\nu(x_r)} u_1^{\nu(x_r-1)} v_{r-1} u_r) \nu(x_r) = 1) \]

where the last equality follows from \( u_r = a_r \sigma(u_r). \) Then the statement follows easily from Lemma 11. \hfill \( \Box \)

Let \( R = \{ r \in [1, d] | \sigma(u_r) \neq 1 \} \) and define the equivalence relation \( \| \) on \( R \) by \( r_1 \| r_2 \) if \( \sigma(u_{r_1}) \) and \( \sigma(u_{r_2}) \) are commensurable.

\textbf{Lemma 43. Let } C \text{ be a } \|\text{-class and let } U = \{ \sigma(u_r) | r \in C \}. \text{ Then there exist numbers } a_r, \beta_r \in \mathbb{Z} \text{ for } r \in C \text{ with } |a_r|, |\beta_r| \leq (|C| + 1) \cdot K_H(n)^p \text{ and an element } h_C \in H \text{ such that } h_C = \prod_{r \in C} \sigma(u_r)^{a_r} \text{ and } h_C^{\beta_r} = \sigma(u_r^{-1}) \text{ for all } r \in C. \)

\textbf{Proof.} By Lemma 9, the group \( \langle U \rangle \) is cyclic of infinite order. Let \( \psi : \langle U \rangle \to \mathbb{Z} \) be any isomorphism. First we ensure that all numbers \( |\psi(\sigma(u_r)| \) are bounded exponentially. Fix an element \( s \in C \) and let \( C' = C \setminus \{ s \}. \) Because of commensurability for each \( r \in C \) there exist numbers \( p_r, q_r \in \mathbb{Z} \setminus \{ 0 \} \) such that \( p_r \cdot \psi(\sigma(u_s)) = q_r \cdot \psi(\sigma(u_s)) \) (we can take \( p_s = q_s = 1 \)).

We can assume that the numbers \( p_r \) and \( q_r \) are coprime for all \( r \in C. \) Let \( \lambda \in \mathbb{N} \) be the
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By Lemma 42 we know

\[ p_r \cdot \delta \cdot \lambda / q_r = \psi(\sigma(u_r)) \]

for \( r \in C' \). Hence every number in \( \psi([U]) = Z \) is divided by \( \delta \) which implies \( |\delta| = 1 \). Since \( \sum_{r \in C} |\sigma(u_r)| \) is bounded by \( n \) we can further assume that \( |p_r| \) and \( |q_r| \) are bounded by \( K_H(n) \) for all \( r \in C \). Let us define \( \beta_r := \psi(\sigma(u_r)) \) for all \( r \in C \). We get

\[ |\beta_r| = |\psi(\sigma(u_r))| \leq |p_r| \cdot \lambda \leq K_H(n)^n. \]

Since \( 1 \in \psi([U]) \) there exist numbers \( \alpha_r \in Z \) for \( r \in C \) such that

\[ \psi \left( \prod_{r \in C} \sigma(u_r)^{\alpha_r} \right) = \sum_{r \in C} \alpha_r \cdot \psi(\sigma(u_r)) = 1. \]

By the standard bounds [43] there exists such a solution where

\[ |\alpha_r| \leq (|C| + 1) \cdot \max \{|\psi(\sigma(u_r))| \mid r \in C\} \leq (|C| + 1) \cdot K_H(n)^n. \]

Finally we set \( h_C = \psi^{-1}(1). \)

By Lemma 43 there exist numbers \( \alpha_r, \beta_r \in Z \) for \( r \in R \) and elements \( h_C \in H \) for all \(|\cdot\)-classes \( C \) such that the following holds (recall that by assumption \( E_H(n) \) is exponentially bounded):

- \( |\alpha_r|, |\beta_r| \leq 2^{n|\cdot|} \) for all \( r \in R \),
- \( h_C = \prod_{r \in C} \sigma(u_r)^{\alpha_r} \) for all \(|\cdot\)-classes \( C \),
- \( h_C^\beta = \sigma(u_r) \) for all \(|\cdot\)-classes \( C \) and \( r \in C \).

**Proposition 44.** A valuation \( \nu \in \mathbb{N}^X \) satisfies \( \nu(E) = 1 \) if and only if \( \nu(\sigma(E)) = 1 \) and there exists a \( \nu \)-decomposition \( D \) of size \( O(n^6) \) and a partition \( \{D_1, \ldots, D_m\} \) of \( D \) such that for all \( 1 \leq i \leq m \) we have:

- \( \sum_{(r,s,t) \in D_i} \tau_\nu(r,s,t) = 0 \) and
- the bundle \( Q_i = \{ \sigma_\nu(r, s, t) \mid (r, s, t) \in D_i \} \) is stacking or \( Q_i \) is a bundle of parallel rays that is packed into a ray with period \( h_C \) for some \(|\cdot\)-class \( C \).

**Proof.** By Lemma 42 we know

\[ \tau(\nu(E)) = \sum_{(r,s,t) \in D} \tau_\nu(r,s,t) = \sum_{1 \leq i \leq m} \sum_{(r,s,t) \in D_i} \tau_\nu(r,s,t) \]

since \( \{D_1, \ldots, D_m\} \) forms a partition of \( D \). Then the direction from right to left is easy since (16) implies \( \nu(\tau(E)) = 0 \), and together with \( \nu(\sigma(E)) = 1 \) we get \( \nu(E) = 1 \).

Conversely, if \( \nu(E) = 1 \) then clearly \( \nu(\sigma(E)) = 1 \). Let us define the bundle

\[ P = \{ \sigma_\nu(r, 1, \nu(x_r)) \mid 1 \leq r \leq d \}. \]

The period of \( \sigma_\nu(r, 1, \nu(x_r)) \) is \( \sigma(u_r) \), and if \( r \in R \), then \( \sigma(u_r) \in \langle h_r \rangle \), where \( [r] \) denotes the \(|\cdot\)-class of \( r \). By Lemma 39 and Lemma 41 there exists a refinement \( Q \) of \( P \) of size \( O(n^6) \) \( \leq Q(n^6) \) and a partition \( Q = \bigcup_{i=1}^m Q_i \) into pairwise disjoint subbundles \( Q_i \) such that each bundle \( Q_i \) is stacking or is packed into a ray with period \( h_C \) for some \(|\cdot\)-class \( C \). The bundles \( Q \) and \( Q_1, \ldots, Q_m \) induce a \( \nu \)-decomposition \( D \) and a partition \( \{D_1, \ldots, D_m\} \) of \( D \) such that

\[ Q_i = \{ \sigma_\nu(r, s, t) \mid (r, s, t) \in D_i \}. \]
for all $1 \leq i \leq m$. By (16) we can derive $\sum_{i=1}^{m} \sum_{(r,s,t) \in D_i} \tau_v(r, s, t) = 0$. We claim that the summands $\sum_{(r,s,t) \in D_i} \tau_v(r, s, t)$ have disjoint supports and thus each summand must be equal to 0. Observe that

$$\text{supp} \left( \sum_{(r,s,t) \in D_i} \tau_v(r, s, t) \right) \subseteq \bigcup_{(r,s,t) \in D_i} \text{supp}(\tau_v(r, s, t)) \subseteq \bigcup_{(r,s,t) \in D_i} \text{supp}(\sigma_v(r, s, t))$$

where the second inclusion follows from (14). The claim follows from the fact that the subbundles $Q_i$ are pairwise disjoint.

### 5.3.3 Constructing the formulas

A **bundle descriptor** is a set $\theta = \{ (r_1, y_1, z_1), \ldots, (r_m, y_m, z_m) \}$ where $1 \leq r_i \leq d$ for all $1 \leq i \leq m$ and $y_1, z_1, \ldots, y_m, z_m \notin X$ are $2m$ distinct fresh variables. We define $V_0 = \{ y_1, z_1, \ldots, y_m, z_m \}$ and the extended set of variables $X_\theta = X \cup V_0$. A **$\theta$-valuation** is a valuation $\nu \in \mathbb{N}^{X_\theta}$ such that $\nu(y_i) \leq \nu(z_i) \leq \nu(x_{r_i})$ for all $1 \leq i \leq m$. We will use the numbers $\alpha_i$ and $\beta_r$ ($r \in R$) constructed in the previous subsection.

**Lemma 45.** Let $\theta = \{ (r_1, y_1, z_1), \ldots, (r_m, y_m, z_m) \}$ be a bundle descriptor. There exists an existential Presburger formula $\text{Stack}_\theta$ with free variables over $X_\theta$ such that a $\theta$-valuation $\nu \in \mathbb{N}^{X_\theta}$ satisfies $\text{Stack}_\theta$ if and only if

1. The bundle $\{ \sigma_v(r_i, \nu(y_i), \nu(z_i)) \mid 1 \leq i \leq m \}$ is stacking, and
2. $\sum_{i=1}^{m} \tau_v(r_i, \nu(y_i), \nu(z_i)) = 0$.

Furthermore, $\text{Stack}_\theta$ defines a semilinear set with magnitude $2^{(n+m)^{O(1)}}$.

**Proof.** Let $\nu$ be a $\theta$-valuation. For better readability we define $s_i = \nu(y_i)$ and $t_i = \nu(z_i)$ for $1 \leq i \leq m$. By definition $\{ \sigma_v(r_i, s_i, t_i) \mid 1 \leq i \leq m \}$ is stacking if and only if there exists $h \in H$ such that for all $1 \leq i \leq m$ we have $\{ \sigma_v(r_i, k) \mid s_i \leq k \leq t_i \} = \{ h \}$. Since $\sigma_v(r_i, k) = \sigma_v(r_i, s_i) \sigma(u_{r_i})^{k-s_i}$, this is equivalent to the statement that for each $1 \leq i \leq m$, we have (i) $\sigma_v(r_i, s_i) = \sigma_v(r_i, s_i)$ and $s_i = t_i$ or (ii) $\sigma(u_{r_i}) = 1$, i.e. $r_i \notin R$. Under condition 1. from the lemma, condition 2. is equivalent to

$$0 = \sum_{i=1}^{m} \tau_v(r_i, s_i, t_i)(h) = \sum_{i=1}^{m} \sum_{k=s_i}^{t_i} \tau_v(r_i, k)(h) = \sum_{i=1}^{m} \sum_{k=s_i}^{t_i} a_{r_i} = \sum_{i=1}^{m} (t_i - s_i + 1) \cdot a_{r_i},$$

where $h \in H$ is the unique element with $\sigma_v(r_i, k) = h$ for all $1 \leq i \leq m$, $s_i \leq k \leq t_i$. This description can be directly expressed as the following formula (we use the exponent expressions from (13)):

$$\text{Stack}_\theta = \bigwedge_{i=1}^{m} (\sigma(E_{r_i}(y_i)) = \sigma(E_{r_i}(y_i)) \land (y_i = z_i \lor r_i \notin R)) \land \bigwedge_{i=1}^{m} (z_i - y_i + 1) \cdot a_{r_i} = 0. \quad (17)$$

It consists of $m$ exponent equations over $H$ of length $O(n)$, identities between variables, and an exponent equation over $A$ of length $O(mn)$. By Lemma 17 and Lemma 8 the semilinear set defined by $\text{Stack}_\theta$ has magnitude $2^{O(n+m)^{O(1)}}$.

**Lemma 46.** Let $\theta = \{ (r_1, y_1, z_1), \ldots, (r_m, y_m, z_m) \}$ be a bundle descriptor such that $r_1, \ldots, r_m \in C$ for some $\|C\|$-class $C$. There exists an existential Presburger formula $\text{Pack}_\theta$ with free variables over $X_\theta$ such that a $\theta$-valuation $\nu \in \mathbb{N}^{X_\theta}$ satisfies $\text{Pack}_\theta$ if and only if there exists a ray $q$ with period $h_C$ such that

1. The bundle $\{ \sigma_v(r_i, \nu(y_i), \nu(z_i)) \mid 1 \leq i \leq m \}$ is packed into $q$, and
2. \( \sum_{i=1}^{m} \tau_{\nu}(r_{i}, \nu(y_{i}), \nu(z_{i})) = 0. \)

Furthermore \( \text{Pack}_{\theta} \) defines a semilinear set with magnitude \( 2^{(n+m)O(1)} \).

**Proof.** Let \( \nu \) be an \( \theta \)-valuation and again define \( s_{i} = \nu(y_{i}) \) and \( t_{i} = \nu(z_{i}) \) for \( 1 \leq i \leq m \). By Lemma 40(iii) we can restrict the choice of the left endpoint \( q_{0} \) of the ray \( q \) to the set \( \{ \sigma_{\nu}(r_{i}, s_{i}), \sigma_{\nu}(r_{i}, t_{i}) \mid 1 \leq i \leq m \} \). We guess an index \( r_{0} \in \{ r_{1}, \ldots, r_{m} \} \), a variable \( y_{0} \in V_{\theta} \) and a length \( \ell \in \mathbb{N} \), and verify that the ray \( q = (q_{i})_{0 \leq j \leq \ell} \) with period \( h_{C} \) and \( q_{0} = \sigma_{\nu}(r_{0}, \nu(y_{0})) \) satisfies the two conditions. The formula \( \text{Pack}_{\theta} \) has the form

\[
\bigvee_{r_{0}, y_{0}} \exists x \geq 0 : (\phi_{1} \land \phi_{2}).
\]  

where \( \phi_{1} \) and \( \phi_{2} \) are constructed in the following, stating that conditions 1. and 2., respectively, hold for the ray \( q \). Note that the variable \( x \) in (18) stands for the value \( \ell \) in the ray \( q = (q_{i})_{0 \leq j \leq \ell} \).

**Condition 1.** By Lemma 40(ii) we can express that \( \sigma_{\nu}(r_{i}, s_{i}, t_{i}) \) is packed into \( q \) by stating that \( \sigma_{\nu}(r_{i}, s_{i}) \) and \( \sigma_{\nu}(r_{i}, t_{i}) \) belong to \( \text{supp}(q) \):

\[
\exists y \leq x : \sigma(E_{r_{0}}(y_{0})) h_{C}^{y} = \sigma(E_{r_{i}}(y_{i}))
\land \exists y \leq x : \sigma(E_{r_{0}}(y_{0})) h_{C}^{y} = \sigma(E_{r_{i}}(z_{i}))
\]  

and that \( \sigma_{\nu}(r_{i}, s_{i}) \sigma(u_{r_{i}})^{-1} \) and \( \sigma_{\nu}(r_{i}, t_{i}) \sigma(u_{r_{i}}) \) belong to \( \text{supp}(q) \):

\[
(\exists y < 0 : \sigma(E_{r_{0}}(y_{0})) h_{C}^{y} = \sigma(E_{r_{i}}(y_{i})) \sigma(u_{r_{i}})^{-1} \lor
\exists y > x : \sigma(E_{r_{0}}(y_{0})) h_{C}^{y} = \sigma(E_{r_{i}}(y_{i})) \sigma(u_{r_{i}})^{-1})
\land (\exists y < 0 : \sigma(E_{r_{0}}(y_{0})) h_{C}^{y} = \sigma(E_{r_{i}}(z_{i})) \sigma(u_{r_{i}}) \lor
\exists y > x : \sigma(E_{r_{0}}(y_{0})) h_{C}^{y} = \sigma(E_{r_{i}}(z_{i})) \sigma(u_{r_{i}})).
\]  

Using the representation \( h_{C} = \prod_{r \in C} \sigma(u_{r})^{\alpha_{r}} \) we can write the term \( h_{C}^{y} \) as \( \prod_{r \in C} \sigma(u_{r})^{\alpha_{r}y} \) (recall that the \( \sigma(u_{r}) \) for \( r \in C \) pairwise commute). The formula \( \phi_{1} \) is the conjunction of the above formulas (19) and (20) for all \( 1 \leq i \leq m \).

**Condition 2.** Next we will express condition 2. under the assumption that condition 1. already holds. Let \( 1 \leq i \leq m \). Since \( \sigma_{\nu}(r_{i}, s_{i}, t_{i}) \) is packed into \( q \) and \( \sigma(u_{r_{i}}) = h_{C}^{y_{i}} \), by Lemma 40(iv) there exists a unique number \( 0 \leq \gamma_{i} < \beta_{r_{i}} \) such that we have

\[
\text{supp}(\sigma_{\nu}(r_{i}, s_{i}, t_{i})) = \{ q_{j} \mid 0 \leq j \leq \ell, j \equiv \gamma_{i} \pmod{\beta_{r_{i}}} \},
\]  

which is equivalent (due to condition 1.) to

\[
q_{\gamma_{i}} h_{C}^{\gamma_{i}} \in \text{supp}(\sigma_{\nu}(r_{i}, s_{i}, t_{i})) = \{ \sigma_{\nu}(r_{i}, k) \mid s_{i} \leq k \leq t_{i} \}.
\]  

Since we have the bound \( \beta_{r_{i}} \) we can guess and verify these numbers \( \gamma_{i} \). Consider a tuple \( \gamma = (\gamma_{1}, \ldots, \gamma_{m}) \in \mathbb{N}^{m} \). For \( 1 \leq i \leq m \) we define the function \( f_{\gamma,i} : \mathbb{N} \rightarrow A \) by

\[
f_{\gamma,i}(j) = \begin{cases} a_{r_{i}}, & \text{if } j \equiv \gamma_{i} \pmod{\beta_{r_{i}}}, \\ 0, & \text{otherwise}. \end{cases}
\]  

By (15) and (21) we have for all \( 0 \leq j \leq \ell \):

\[
\tau_{\nu}(r_{i}, s_{i}, t_{i})(q_{j}) = \begin{cases} a_{r_{i}}, & \text{if } q_{j} \in \text{supp}(\sigma_{\nu}(r_{i}, s_{i}, t_{i})) \\ 0, & \text{otherwise} \end{cases} = f_{\gamma,i}(j)
\]  

where \( \gamma_{i} \) and \( \beta_{r_{i}} \) are constructed in the following, stating that conditions 1. and 2., respectively, hold for the ray \( q \). Note that the variable \( x \) in (18) stands for the value \( \ell \) in the ray \( q = (q_{i})_{0 \leq j \leq \ell} \).
Hence, for all $0 \leq j \leq \ell$ we have
\[
\sum_{i=1}^{m} \tau_{\nu}(r_{i}, s_{i}, t_{i})(q_{j}) = \sum_{i=1}^{m} f_{\gamma,i}(j) =: f_{\gamma}(j).
\]

Since $f_{\gamma,i}$ is $\beta_{r_{i}}$-periodic, Proposition 21 implies that the number
\[
b_{\gamma} := \sup\{ j \in \mathbb{N} \mid f_{\gamma}(j') = 0 \text{ for all } 0 \leq j' \leq j \}
\]
is either infinite or bounded polynomially in $\max\{ \beta_{r_{i}} \mid 1 \leq i \leq m \} \leq 2^{n^{O(1)}}$. Hence, with (21) and (25) we get
\[
\sum_{i=1}^{m} \tau_{\nu}(r_{i}, s_{i}, t_{i}) = 0 \iff \forall j \in [0, \ell] : \sum_{i=1}^{m} \tau_{\nu}(r_{i}, s_{i}, t_{i})(q_{j}) = 0
\]
\[
\iff \forall j \in [0, \ell] : f_{\gamma}(j) = 0
\]
\[
\iff \ell \leq b_{\gamma}.
\]

The formula $\phi_{2}$ can now be defined by guessing the numbers $\gamma_{i}$ (bounded by $\beta_{r_{i}} - 1$), verifying them using (22) and testing that $\ell$ is at most $b_{\gamma}$:
\[
\phi_{2} = \bigvee_{\gamma} \left( x \leq b_{\gamma} \land \bigwedge_{i=1}^{m} \exists z_{i} \exists y_{i} \leq y \leq z_{i} : (\sigma(E_{r_{i}}(y_{0})) h_{C}^{z} = \sigma(E_{r_{i}}(y)) \land z = \gamma_{i}) \right)
\]

Notice that at the atomic level the formula $\text{Pack}_{\theta}$ consists of (in)equalities and exponent equations over $H$ (see (18), (19), (20) and (26)). The exponent equations over $H$ define semilinear sets with magnitude $2^{n^{O(1)}}$ by Lemma 15 (we need the exponents $k_{i}$ in Lemma 15 because of the exponents $\alpha_{r}$ in $h_{C}$). The coefficients in the (in)equalities are also bounded by $2^{n^{O(1)}}$. By pushing conjunctions inside we can transform $\text{Pack}_{\theta}$ into a disjunction of existential formulas of size with $O(n + m)$ many variables and conjunctions of length $O(m)$. By Lemma 8 the semilinear set defined by $\text{Pack}_{\theta}$ has magnitude $2^{(n+m)^{O(1)}}$. \hfill ▶

**Proof of Theorem 28.** We express the statement from Proposition 44 using Lemma 45 and Lemma 46. First we guess the total number $k = O(n^{q})$ of progressions. Let $Y_{k} = \{ y_{1}, z_{1}, \ldots, y_{k}, z_{k} \}$ be a set of $2k$ distinct variables. We then guess a set $\Theta$ of bundle descriptors such that $\{ Y_{\theta} \mid \theta \in \Theta \}$ forms a partition of $Y_{k}$. In particular, the size of $\Theta$ is bounded by $k = O(n^{q})$. The final formula then is:
\[
\sigma(E) = 1 \land \bigwedge_{k, \theta} \exists y_{1} \exists z_{1} \cdots \exists y_{k} \exists z_{k} \left( \text{Decomp} \land \bigwedge_{\theta \in \Theta} (\text{Stack}_{\theta} \lor \text{Pack}_{\theta}) \right)
\]

Here the formula $\text{Pack}_{\theta}$ should be interpreted as false if the $r_{i}$-values in $\theta$ are not contained in a common $||$-class. The formula $\text{Decomp}$ expresses that for all $1 \leq r \leq m$ the set $\{ [\nu(y), \nu(z)] \mid (r, y, z) \in \theta \in \Theta \}$ constitutes a partition of $[1, \nu(x_{r})]$, which is a semilinear constraint with constant magnitude. \hfill ▶

## 6 Wreath products with difficult knapsack and power word problems

In this section we will prove Theorems 5 and 6 and present some applications. We start with a formal definition of uniformly SENS groups [3].
6.1 Strongly efficiently non-solvable groups

Let us fix a f.g. group $G = \langle \Sigma \rangle$. Following [3] we need the additional assumption that the generating set $\Sigma$ contains the group identity $1$. This allows to pad words over $\Sigma$ to any larger length without changing the group element represented by the word. One also says that $\Sigma$ is a standard generating set for $G$. The group $G$ is called strongly efficiently non-solvable (SENS) if there is a constant $\mu \in \mathbb{N}$ such that for every $d \in \mathbb{N}$ and $v \in \{0,1\} \leq d$ there is a word $w_{d,v} \in \Sigma^*$ with the following properties:

- $|w_{d,v}| = 2^{\mu d}$ for all $v \in \{0,1\}^d$,
- $w_{d,v} = [w_{d,v0}, w_{d,v1}]$ for all $v \in \{0,1\} < d$ (here we take the commutator of words),
- $w_{d,v} \neq 1$ in $G$.

The group $G$ is called uniformly strongly efficiently non-solvable if, moreover,

- given $v \in \{0,1\}^d$, a binary number $i$ with $\mu d$ bits, and $a \in \Sigma$ one can decide in linear time on a random access Turing-machine whether the $i$-th letter of $w_{d,v}$ is $a$.

Here are examples for uniformly SENS groups; see [3] for details:

- finite non-solvable groups (more generally, every f.g. group that has a finite non-solvable quotient),
- f.g. non-abelian free groups,
- Thompson’s group $F$,
- weakly branched self-similar groups with a f.g. branching subgroup (this includes several famous self-similar groups like the Grigorchuk group, the Gupta-Sidki groups and the Tower of Hanoi groups).

6.2 Applications of Theorems 6

Recall that Theorem 6 states that $\text{Knapsack}(G \wr \mathbb{Z})$ is $\Sigma_2^p$-hard for every uniformly SENS group $G$. Before we prove this results we show some applications.

▶ Corollary 47. For the following groups $G$, $\text{Knapsack}(G \wr \mathbb{Z})$ is $\Sigma_2^p$-complete:

- finite non-solvable groups,
- f.g. non-abelian free groups,
- Thompson’s group $F$,
- weakly branched self-similar groups with a f.g. branching subgroup (this includes several famous self-similar groups like the Grigorchuk group, the Gupta-Sidki groups and the Tower of Hanoi groups).

Proof. Finite non-solvable groups and f.g. non-abelian free groups are uniformly SENS [3]. By Theorem 6, $\text{Knapsack}(G \wr \mathbb{Z})$ is $\Sigma_2^p$-hard. It remains to show that $\text{Knapsack}(G \wr \mathbb{Z})$ belongs to $\Sigma_2^p_2$. According to Proposition 26, it suffices to show that $\text{Periodic}(G)$ and $\text{ExpEq}(G)$ both belong to $\Sigma_2^p$. The problem $\text{Periodic}(G)$ belongs to $\text{coNP}$ (since the word problem for $G$ can be solved in polynomial time) and $\text{ExpEq}(G)$ belongs to $\text{NP}$. For a finite group this is clear. If $G$ is hyperbolic, then one can reduce $\text{ExpEq}(G)$ to the existential fragment of Presburger arithmetic using [26].

Theorem 6 can be also applied to Thompson’s group $F$. This is one of the most well studied groups in (infinite) group theory due to its unusual properties, see e.g. [5]. It can be defined in several ways; let us just mention the following finite presentation: $F = \langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^{-1}] \rangle$. Thompson’s group $F$ is uniformly SENS [3] and contains a copy of $F \wr \mathbb{Z}$ [14]. Theorem 6 yields

▶ Corollary 48. The knapsack problem for Thompson’s group $F$ is $\Sigma_2^p$-hard.

We conjecture that the knapsack problem for $F$ is in fact $\Sigma_2^p$-complete.

---

6 A hyperbolic group is non-elementary if it is not virtually cyclic. Every non-elementary hyperbolic group contains a non-abelian free group.
6.3 Proof of Theorems 6

We prove Theorem 6 in two steps. The second step works for every f.g. group $G$. Fix this group $G$ and let $\Sigma$ be a standard generating set for $G$. Let $\overline{X} = (X_1, \ldots, X_n)$ be a tuple of boolean variables. We identify $\overline{X}$ with the set $\{X_1, \ldots, X_n\}$ when appropriate. A $G$-program with variables from $\overline{X}$ is a sequence

\[ P = (X_1, a_1, b_1)(X_2, a_2, b_2) \cdots (X_\ell, a_\ell, b_\ell) \in (\overline{X} \times \Sigma \times \Sigma)^*. \]

The length of $P$ is $\ell$. For a mapping $\alpha : \overline{X} \to \{0, 1\}$ (called an assignment) we define $P(\alpha) \in G$ as the group element $c_1c_2 \cdots c_\ell$, where $c_j = a_j$ if $X_j = 1$ and $c_j = b_j$ if $X_j = 0$ for all $1 \leq j \leq \ell$. We define the following computational problem $\exists\forall$-SAT($G$):

**Input** A $G$-program $P$ with variables from $\overline{X} \cup \overline{Y}$, where $\overline{X}$ and $\overline{Y}$ are disjoint.

**Question** Is there an assignment $\alpha : \overline{X} \to \{0, 1\}$ such that for every assignment $\beta : \overline{Y} \to \{0, 1\}$ we have $P(\alpha \cup \beta) = 1$ (we write $\exists \overline{X} \forall \overline{Y} : P = 1$ for this)?

**Lemma 49.** Let the f.g. group $G = (\Sigma)$ be uniformly SENS. Then, $\exists\forall$-SAT($G$) is $\Sigma^p_2$-hard.

**Proof.** We prove the lemma by a reduction from the following $\Sigma^p_2$-complete problem: given a boolean formula $F = F(\overline{X}, \overline{Y})$ in disjunctive normal form, where $\overline{X}$ and $\overline{Y}$ are disjoint tuples of boolean variables, does the quantified boolean formula $\exists \overline{X} \forall \overline{Y} : F$ hold? Let us fix such a formula $F(\overline{X}, \overline{Y})$. We can write $F$ as a fan-in two boolean circuit of depth $O(\log |F|)$.

By [3, Remark 34] we can compute in logspace from $F$ a $G$-program $P$ over the variables $\overline{X} \cup \overline{Y}$ of length polynomial in $|F|$ such that for every assignment $\gamma : \overline{X} \cup \overline{Y} \to \{0, 1\}$ the following two statements are equivalent:

\[ F(\gamma(\overline{X}), \gamma(\overline{Y})) \text{ holds}. \]

\[ P(\gamma) = 1 \text{ in } G. \]

Hence, $\exists \overline{X} \forall \overline{Y} : F$ holds if and only if $\exists \overline{X} \forall \overline{Y} : P = 1$ holds.

**Lemma 50.** For every f.g. nontrivial group $G$, $\exists\forall$-SAT($G$) is logspace many-one reducible to Knapsack($G \wr \mathbb{Z}$).

**Proof.** Let us fix a $G$-program

\[ P = (Z_1, a_1, b_1)(Z_2, a_2, b_2) \cdots (Z_\ell, a_\ell, b_\ell) \in ((\overline{X} \cup \overline{Y}) \times \Sigma \times \Sigma)^* \]

(27)

where $\overline{X}$ and $\overline{Y}$ are disjoint sets of variables. Let $m = |\overline{X}|$ and $n = |\overline{Y}|$. We want to construct a knapsack expression $E$ over $G \wr \mathbb{Z}$ which has a solution if and only if there is an assignment $\alpha : \overline{X} \to \{0, 1\}$ such that $P(\alpha \cup \beta) = 1$ for every assignment $\beta : \overline{Y} \to \{0, 1\}$. Let us choose a generator $t$ for $\mathbb{Z}$. Then $\Sigma \cup \{t, t^{-1}\}$ generates the wreath product $G \wr \mathbb{Z}$. First, we compute in logspace the $m + n$ first primes $p_1, \ldots, p_{m+n}$ and fix a bijection $p : \overline{X} \cup \overline{Y} \to \{p_1, \ldots, p_{m+n}\}$. Moreover, let $M = \prod_{i=1}^{m+n} p_i$.

Roughly speaking, the plan is as follows. Each assignment $\alpha : \overline{X} \to \{0, 1\}$ will correspond to a valuation $\nu$ for our expression $E$. The resulting element $\nu(E) \in G \wr \mathbb{Z}$ then encodes the value $P(\alpha \cup \beta)$ for each $\beta : \overline{Y} \to \{0, 1\}$ in some position $s \in \{0, M - 1\}$. To be precise, to each $s \in \{0, M - 1\}$, we associate the assignment $\beta_s : \overline{Y} \to \{0, 1\}$ where $\beta_s(\tau) = 1$ if and only if $s \equiv 0 \mod p(\tau)$. Then, $\tau(\nu(E))(s)$ will be $P(\alpha \cup \beta_s)$. This means, $\nu(E) = 1$ implies that $P(\alpha \cup \beta) = 1$ for all assignments $\beta : \overline{Y} \to \{0, 1\}$.

Our expression implements this as follows. For each $i = 1, \ldots, \ell$, it walks to the right to some position $M' \geq M$ and then walks back to the origin. On the way to the right, the behavior depends on whether $Z_i$ is an existential or a universal variable. If $Z_i$ is existential, we either place $a_i$ at every position (if $\alpha(Z_i) = 1$) or $b_i$ at every position (if $\alpha(Z_i) = 0$). If $Z_i$
The complexity of knapsack problems in wreath products

is universal, we place \( a_i \) in the positions divisible by \( p(Z_i) \); and we place \( b_i \) in the others. That way, in position \( s \in [0, M - 1] \), the accumulated element will be \( P(\alpha \cup \beta_s) \).

We define \( I_3 = \{ i \in [1, \ell] \mid Z_i \in X \} \) and \( I_4 = \{ i \in [1, \ell] \mid Z_i \in Y \} \). For an existentially quantified variable \( X \in X \) let \( I_X = \{ i \in [1, \ell] \mid X = Z_i \} \) be the set of those positions in the \( G \)-program \( P \), where the variable \( X \) is queried. Moreover, let us write \( q_i \) for the prime number \( p(Z_i) \). We compute for every \( i \in I_3 \) the words (over the wreath product \( G \wr \mathbb{Z} \))

\[
\nu = (a_i \ell)^{q_i} \quad \text{and} \quad v_i = (b_i \ell)^{q_i}
\]

and for every \( i \in I_4 \) the word

\[
w_i = a_i \ell (b_i \ell)^{q_i - 1}.
\]

Let us now consider the knapsack expression

\[
E_1 = \prod_{i=1}^{\ell} f_i(t^{-1})^{z_i}
\]

with

\[
f_i = \begin{cases} w_i & \text{if } i \in I_3, \\ v_i & \text{if } i \in I_4. \end{cases}
\]

The idea is that in \( E_1 \), for each \( i \in [1, \ell] \), we go to right with \( f_i \) and then we go back to the origin with \( t^{-1}(t^{-1})^{z_i} \). If \( Z_i \) is existential, we use \( f_i = u_i^{x_i}v_i^{x'_i} \) to either place \( a_i \) at every position or \( b_i \) at every position. If \( Z_i \) is universal, we use \( w_i \) to place \( a_i \) at positions divisible by \( q_i = p(Z_i) \) and \( b_i \) at the others. Note that the expression itself cannot guarantee that, e.g., \( (i) \ (t^{-1})^{z_i} \) moves exactly onto the origin or \( (ii) \) that we either use only \( u_i \) or only \( v_i \) for each \( i \in I_3 \). Therefore, we ensure these properties temporarily by imposing additional linear equations (Claim 1). In a second step, we shall extend \( E_1 \) to get an expression in which a solution will automatically satisfy these linear equations (Claim 2).

**Claim 1**: \( \exists X \forall Y : P = 1 \) holds if and only if there exists a \((G \wr \mathbb{Z})\)-solution \( \nu \) for \( E_1 \) with the following properties:

(a) \( q_i \cdot \nu(y_i) = \nu(z_i) + 1 \) for all \( i \in I_4 \),

(b) \( q_i \cdot \nu(x_i) + \nu(x'_i) = \nu(z_i) + 1 \) for all \( i \in I_3 \),

(c) \( \nu(z_i) = \nu(z_j) \) for all \( i, j \in [1, \ell] \) with \( i \neq j \),

(d) \( \nu(x_i) = 0 \) or \( \nu(x'_i) = 0 \) for all \( i \in I_3 \),

(e) for all \( X \in X \) and all \( i, j \in I_X \) we have: \( \nu(x_i) = 0 \) if and only if \( \nu(x_j) = 0 \).

**Proof of Claim 1**: Assume first that \( \exists X \forall Y : P = 1 \) holds. Let \( \alpha : X \to \{0, 1\} \) be an assignment such that for every assignment \( \beta : Y \to \{0, 1\} \), we have \( P(\alpha \cup \beta) = 1 \) in \( G \).

We have to find a \((G \wr \mathbb{Z})\)-solution for \( E_1 \) such that the above properties (a)-(d) hold. For this, we set:

\[
\begin{align*}
\nu(z_i) &= M - 1 \quad \text{for all } i \in [1, \ell], \\
\nu(y_i) &= M/q_i \quad \text{for all } i \in I_4, \\
\nu(x_i) &= M/q_i \quad \text{and } \nu(x'_i) = 0 \quad \text{for all } i \in I_X, \ X \in X \text{ such that } \alpha(X) = 1, \\
\nu(x'_i) &= M/q_i \quad \text{and } \nu(x_i) = 0 \quad \text{for all } i \in I_X, \ X \in X \text{ such that } \alpha(X) = 0.
\end{align*}
\]

Then, clearly, (a)-(e) hold. It remains to verify that \( \nu \) is a \((G \wr \mathbb{Z})\)-solution for \( E_1 \). Let \( h = \tau(\nu(E_1)) \in G(\mathbb{Z}) \) and \( k = \sigma(\nu(E_1)) \in \mathbb{Z} \). We have \( k = 0 \) and \( h(s) = 1 \) for all \( s \in \mathbb{Z} \setminus [0, M - 1] \). Moreover, for every \( s \in [0, M - 1] \) we have \( h(s) = c_1 c_2 \ldots c_\ell \) where

\[
c_i = \begin{cases} a_i & \text{if } (i \in I_4 \text{ and } s \equiv 0 \mod q_i) \text{ or } (i \in I_X, X \in X \text{ and } \alpha(X) = 1) \\ b_i & \text{if } (i \in I_4 \text{ and } s \not\equiv 0 \mod q_i) \text{ or } (i \in I_X, X \in X \text{ and } \alpha(X) = 0). \end{cases}
\]
Here, the \( a_i \) and \( b_i \) are from (27). Hence, there is an assignment \( \beta_s : \{0, 1\}^\ell \to \{0, 1\} \) such that \( h(s) = P(\alpha \cup \beta_s) \). Thus, \( h(s) = 1 \) for all \( s \in [0, M - 1] \), which implies that \( \nu(E_1) = 1 \) in \( G \). 

For the other direction, assume that \( \nu \) is a \((G\ {\mathbb Z})\)-solution for \( E_1 \) such that the properties (a)–(e) hold. Let \( M' = \nu(z_1) + 1 > 0 \). We then have \( M' = \nu(z_i) + 1 \) for all \( i \in [1, \ell] \) by property (c). By properties (a) and (b), \( M' \) is divisible by the first \( m + n \) primes. This implies that \( M' \) is a multiple of \( M \) and thus \( M' \geq M \).

Let us define an assignment \( \alpha : \{0, 1\}^\ell \to \{0, 1\} \) as follows, where \( i \in I_\beta \):

\[
\alpha(Z_i) = \begin{cases} 
0 & \text{if } \nu(x_i) = 0 \\
1 & \text{if } \nu(x'_i) = 0 
\end{cases}
\]

By properties (d) and (e) this defines indeed an assignment \( \alpha : \{0, 1\}^\ell \to \{0, 1\} \). Moreover, for every position \( s \in [0, M' - 1] \) we define the assignment \( \beta_s : \{0, 1\}^\ell \to \{0, 1\} \) by \( \beta_s(Y) = 1 \) if \( s \equiv 0 \mod p(Y) \) and \( \beta_s(Y) = 0 \) otherwise. By the Chinese remainder theorem, for every \( \beta : \{0, 1\}^\ell \to \{0, 1\} \) there exists \( s \in [0, M' - 1] \) with \( \beta = \beta_s \). Moreover, the construction of \( E_1 \) implies that \( \nu(E_1) \) writes \( P(\alpha \cup \beta_s) \) into position \( s \). Since \( \nu(E_1) = 1 \) in \( G \) we have \( P(\alpha \cup \beta_s) = 1 \) for all \( s \in [0, M' - 1] \), i.e., \( P(\alpha \cup \beta) = 1 \) for all assignments \( \beta : \{0, 1\}^\ell \to \{0, 1\} \). We have shown Claim 1.

In the rest of the proof we construct a knapsack expression \( E_2 \) such that each of the variables from \( E_1 \) also occurs in \( E_2 \). Moreover, the following properties will hold:

- Every \((G\ {\mathbb Z})\)-solution of \( E_1 \) that satisfies the properties (a)–(e) extends to a \((G\ {\mathbb Z})\)-solution of \( E_2 \).
- Every \((G\ {\mathbb Z})\)-solution of \( E_2 \) restricts to a \((G\ {\mathbb Z})\)-solution of \( E_1 \) that satisfies the properties (a)–(e).

This implies that \( E_2 \) has a \((G\ {\mathbb Z})\)-solution if and only if \( E_1 \) has a \((G\ {\mathbb Z})\)-solution that satisfies the properties (a)–(e) if and only if \( \exists xyF : P = 1 \) holds.

Let \( g \in G \) be any nontrivial element. To construct \( E_2 \) it is convenient to work in a wreath product \((g)^d \times G\ {\mathbb Z})\) for some \( d \), whose unary encoding can be computed (in logspace) from the input formula \( \exists X\forall Y : F \). By Lemma 10 we can compute in logspace an embedding of \((g)^d \times G\ {\mathbb Z})\) into \( G \). Let \( \zeta \) be the canonical embedding of \((g)\) into \((g)^d\) that maps \( g \) to \((1, \ldots, 1, g, 1, \ldots, 1)\), where in the latter, \( g \) appears in the \( i \)-th coordinate. We assume that the coordinates are numbered from 0 to \( d - 1 \). In the following, we write \( g_i \) for \( \zeta(g) \). We set \( d = 2\ell + 1 \).

We then define the following knapsack expression \( E_2 = E_{2,1}E_{2,2} \) where \( z, z' \) and \( \tilde{X}, \tilde{X}' \) for all \( X \in \tilde{X} \) appear as fresh variables:

\[
E_{2,1} = \prod_{X \in \tilde{X}} \prod_{i \in \ell_X} g_{t+i}^t \prod_{X \in \tilde{X}} \prod_{i \in \ell_X} g_{t+i}'^t \prod_{X \in \tilde{X}} \prod_{i \in \ell_X} g_{t+i}'^t \prod_{X \in \tilde{X}} \prod_{i \in \ell_X} g_{t+i}'^t
\]
\[
E_{2,2} = \{ \prod_{i=1}^\ell f_i g_i^{-1} t^{-1}(i^{-1})^z g_i^{-1} \text{ with } f_i = \begin{cases} 
u_i^z g_i^{-1} v_i^z & \text{if } i \in I_3, \\
u_i^z g_i^{-1} v_i^z & \text{if } i \in I_4. 
\end{cases} \}
\]

The idea of the construction is that the \( g_i \) implement pebbles that can be put on different positions in \( Z \). At the end all pebbles have to be recollected. Note that we only use the pebbles \( g_0, g_1, \ldots, g_\ell \) and \( g_{\ell+i} \) for \( i \in I_3 \); hence we could reduce the dimension \( 2\ell + 1 \) to \( \ell + 1 + |I_3| \) but this would make the indexing slightly more inconvenient.

**Claim 2**: Every \((G\ {\mathbb Z})\)-solution of \( E_1 \) that satisfies the properties (a)–(e) extends to a \((G\ {\mathbb Z})\)-solution of \( E_2 \).
Proof of Claim 2: Let $\nu$ be a $(G \wr \mathbb{Z})$-solution of $E_1$ that satisfies the properties (a)–(e). Let $M' = \nu(z_1) + 1 > 0$. Hence, $M' = \nu(z_i) + 1$ for all $i \in [1, \ell]$. We then extend $\nu$ to the fresh variables in $E_2$ by:
- $\nu(z) = \nu(z') = M' - 1$,
- for all $X \in \mathbb{X}$ such that $x_i = 0$ for some (and hence all) $i \in I_X$, we set $\nu(\tilde{X}') = 1$ and $\nu(X) = 0$,
- for all $X \in \mathbb{X}$ such that $\nu(z_i') = 0$ for some (and hence all) $i \in I_X$, we set $\nu(\tilde{X}') = 0$ and $\nu(X) = 1$.

It is easy to check that this yields indeed a $(G \wr \mathbb{Z})$-solution of $E_2$.

Claim 3: Every $(G \wr \mathbb{Z})$-solution of $E_2$ restricts to a $(G \wr \mathbb{Z})$-solution of $E_1$ that satisfies the properties (a)–(e).

Proof of Claim 3: Fix a $(G \wr \mathbb{Z})$-solution $\nu$ of $E_2$. First of all, we must have $\nu(z) = \nu(z')$; otherwise the pebble $g_0$ will not be recollected. Let $M' = \nu(z) + 1 > 0$. The word $\nu(E_{2,1})$ leaves pebbles $g_1, \ldots, g_\ell$ at positions 0 and $M'$ (it also leaves powers of the pebbles $g_{\ell+i}$ — we will deal with those later) and puts the cursor back to position 0. With the word $\nu(E_{2,1})$ the pebbles at positions 0 and $M'$ have to be recollected. This happens only if $\nu(z_i) = M' - 1$ for all $i \in [1, \ell]$, $q_i \cdot \nu(y_i) = M'$ for all $i \in I_\nu$, and $q_i \cdot (\nu(x_i) + \nu(x_i')) = M'$ for all $i \in I_\nu$. Hence, conditions (a)–(c) hold.

Conditions (d) and (e) are enforced with the pebbles $g_{\ell+i}$ for $i \in I_\nu$. Consider an existentially quantified variable $X \in \mathbb{X}$. The word $\nu(E_{2,1})$ leaves for every $i \in I_X$ the “pebble powers” $g_{\ell+i}^{\nu(X)}$ and $g_{\ell+i}^{\nu(\tilde{X})}$ at positions 0 and $M' > 0$, respectively. With the word $\nu(E_{2,2})$ exactly one pebble $g_{\ell+i}$ is recollected. Therefore, exactly one of the following two cases has to hold:
- $g^{\nu(X)} = 1$ and $g^{\nu(\tilde{X})} = g$ in $G$,
- $g^{\nu(X)} = g$ and $g^{\nu(\tilde{X})} = 1$ in $G$.

Assume first that $g^{\nu(X)} = 1$ and $g^{\nu(\tilde{X})} = g$ in $G$. Then $\nu(E_{2,1})$ places the pebble $g_{\ell+i}$ at position $M'$ (and it places this pebble at no other position) for all $i \in I_X$. In order to recollect this pebble with $\nu(E_{2,2})$ we must have $\nu(x_i) = M'/q_i = M'/(p(X))$ and $\nu(x_i') = 0$ for all $i \in I_X$. If $g^{\nu(X)} = g$ and $g^{\nu(\tilde{X})} = 1$ in $G$ then we must have $\nu(x_i') = M'/q_i = M'/(p(X))$ and $\nu(x_i) = 0$ for all $i \in I_X$. This shows that (d) and (e) holds and concludes the proof of Claim 3 and hence the proof of the lemma.

Theorem 6 is now a direct corollary of Lemmas 49 and 50.

6.4 Wreath product with difficult power word problems

In [27] it was shown that $\text{POWERWP}(G \wr \mathbb{Z})$ is $\text{coNP}$-complete in case $G$ is a finite non-solvable group or a f.g. free group. The proof in [27] immediately generalizes to the case were $G$ is uniformly SENS. This yields Theorem 5. Alternatively, one can prove Theorem 5 by showing the following two facts:
- $\forall\text{-SAT}(G)$ (the question whether for a given $G$-program $P$, $P(\alpha) = 1$ for all assignments) is $\text{coNP}$-hard if $G$ is uniformly SENS.
- $\forall\text{-SAT}(G)$ is logspace many-one reducible to $\text{POWERWP}(G \wr \mathbb{Z})$.

The proofs for these facts are in fact simplifications of the proofs for Lemmas 49 and 50.

We can also show that for a large class of groups the power word problem is contained in $\text{coNP}$. Fix a f.g. group $G = \langle \Sigma \rangle$. With $\text{WP}(G, \Sigma)$ we denote the set of all words $w \in \Sigma^*$ such that $w = 1$ in $G$ (the word problem for $G$ with respect to $\Sigma$). We say that $G$ is $\text{co-context-free}$.
if $\Sigma^* \setminus \text{WP}(G, \Sigma)$ is context-free (the choice of $\Sigma$ is not relevant for this), see [18, Section 14.2] for more details.

\textbf{Theorem 51.} The power word problem for a co-context-free group $G$ belongs to coNP.

\textbf{Proof.} Let $G = \langle \Sigma \rangle$ and let $(u_1, k_1, u_2, k_2, \ldots, u_d, k_d)$ be the input power word, where $u_i \in \Sigma^*$. We can assume that all $k_i$ are positive. We have to check whether $u_1^{k_1} u_2^{k_2} \cdots u_d^{k_d}$ is trivial in $G$. Let $L$ be the complement of $\text{WP}(G, \Sigma)$, which is context-free. Take the alphabet $\{a_1, \ldots, a_d\}$ and define the morphism $h : \{a_1, \ldots, a_d\}^* \to \Sigma^*$ by $h(a_i) = u_i$. Consider the language $K = h^{-1}(L) \cap a_1^* a_2^* \cdots a_d^*$. Since the context-free languages are closed under inverse morphisms and intersections with regular languages, $K$ is context-free too. Moreover, from the tuple $(u_1, u_2, \ldots, u_d)$ we can compute in polynomial time a context-free grammar for $K$: Start with a push-down automaton $M$ for $L$ (since $L$ is a fixed language, this is an object of constant size). From $M$ one can compute in polynomial time a push-down automaton $M'$ for $h^{-1}(L)$: when reading the symbol $a_i$, $M'$ has to simulate (using $\varepsilon$-transitions) $M$ on $h(a_i)$. Next, we construct in polynomial time a push-down automaton $M''$ for $h^{-1}(L) \cap a_1^* a_2^* \cdots a_d^*$ using a product construction. Finally, we transform $M''$ back into a context-free grammar. This is again possible in polynomial time using the standard triple construction. It remains to check whether $a_1^* a_2^* \cdots a_d^* \not\in L(G)$. This is equivalent to $(k_1, k_2, \ldots, k_d) \not\in \Psi(L(G))$, where $\Psi(L(G))$ denotes the Parikh image of $L(G)$. Checking $(k_1, k_2, \ldots, k_d) \in \Psi(L(G))$ is an instance of the uniform membership problem for commutative context-free languages, which can be solved in $\text{NP}$ according to [19]. This implies that the power word problem for $G$ belongs to coNP. \hfill $\blacksquare$

Let us remark that the above context-free language $K$ was also used in [21] in order to show that the knapsack problem for a co-context-free group is decidable.

\textbf{Theorem 52.} For Thompson’s group $F$, the power word problem is coNP-complete.

\textbf{Proof.} The upper bound follows from Theorem 51 and the fact that $F$ is co-context-free [22]. The lower bound follows from Theorem 5 and the facts that $F$ is uniformly SENS and that $F \not\leq \text{Z}$. \hfill $\blacksquare$

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