AN A PRIORI ESTIMATE FOR THE SINGLY PERIODIC
SOLUTIONS OF A SEMILINEAR EQUATION

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Abstract. There exists an exponentially decreasing function $f$ such that any singly $2\pi$-periodic positive solution $u$ of $-\Delta u + u - u^p = 0$ in $[0, 2\pi] \times \mathbb{R}^{N-1}$ verifies $u(x_1, x') \leq f(\|x'\|)$. We prove that with the same period and with the same function $f$, any singly periodic positive solution of $-\varepsilon^2 \Delta u - u + u^p = 0$ in $[0, 2\pi] \times \mathbb{R}^{N-1}$ verifies $u(x_1, x') \leq f(\|x'\|/\varepsilon)$. We have a similar estimate for the gradient.

1. Introduction.

Let $N$ be an integer, $N \geq 2$, let $\varepsilon$ and $p$ be positive real numbers, $p > 1$. We study the equation

\begin{equation}
-\varepsilon^2 \Delta u + u - u^p = 0 \text{ in } S^1 \times \mathbb{R}^{N-1}
\end{equation}

where $S^1 = [0, 2\pi]$. We mean that $u$ is $2\pi$-periodic in $x_1$. We consider the positive solutions of (1.1), $u(x_1, x')$ ($x_1 \in S^1$ and $x' \in \mathbb{R}^{N-1}$) that tend to 0 as $\|x'\|$ tends to $\infty$, uniformly in $x_1$. It is known that these solutions are radial in $x'$ and decreasing in $\|x'\|$. This can be proved by an application of the moving plane method \cite{3, 7, 8}. The ground-state solution $w_0$, defined and radial on $\mathbb{R}^{N-1}$ is a particular solution which does not depend on $x_1$. In \cite{2}, Dancer proved the existence of positive solutions really depending on $x_1$ and $x'$. In \cite{11}, we studied the case $N = 2$ and we proved the following result:

\textbf{Theorem 1.1.} (i) The first continuum $\Sigma_1$ of positive bounded solutions even in $x_1$ and $x'$ of (1.1) bifurcating from $(\varepsilon_*, w_0(x'/\varepsilon*))$ is composed of $(\varepsilon_*, w_0(x'/\varepsilon*))$ and of all the solutions $(\varepsilon, z)$ of (1.1) such that $z > 0$, $z$ even in $x_1$ and $x_2$, $\lim_{x_2 \to \infty} z = 0$ and $\frac{\partial z}{\partial x_1} < 0$ in $[0, \pi] \times \mathbb{R}^+$. (ii) There exists a bounded subset $A$ of $L^\infty(S^1 \times \mathbb{R}^+)$ such that the set $\Sigma_1$ is entirely contained in $[0, \varepsilon_*) \times A$. (iii) For each $(\varepsilon, z) \in \Sigma_1$, $z$ is an isolated point of \{$v \in L^\infty(S^1 \times \mathbb{R}^+); v$ even in $x_1$ and $x'$; $(\varepsilon, v)$ solution of (1.1)\}. For every $\varepsilon > 0$, $\varepsilon < \varepsilon_*$, there exists a finite number of solutions $(\varepsilon, z)$ in $\Sigma_1$. (iv) There exists $\varepsilon_0$ such that for all $0 < \varepsilon < \varepsilon_0$ this continuum is a curve that has a one to one $C^1$ parameterization $\varepsilon \to (\varepsilon, z_\varepsilon)$.

In this paper we suppose that

\begin{equation}
1 < p < \frac{N + 2}{N - 2}
\end{equation}

If \( N = 2 \), this condition is \( p > 1 \).

We know, see [2], that the condition (1.2) for \( p \) is a necessary and sufficient condition to have the following property: There exists \( M > 0 \) such for all \( \varepsilon > 0 \), any positive solution \( u \) of (1.1) verifies

\[
\|u\|_{L^\infty} \leq M
\]

This property is related to the nonexistence of positive solutions for the equation

\[
-\Delta u - u^p = 0
\]

more precisely (1.4) \((v \geq 0), -\Delta v - v^p = 0 \) in \( \mathbb{R}^N \) \( \Rightarrow (v = 0) \)

(see Gidas and Spruck [4]). This paper is devoted to some a priori estimates for the solutions of (1.1).

**Theorem 1.2.** There exists a real number \( K \) independent of \( \varepsilon > 0 \) and of any solution \( u \) of (1.1), such that for all \( x = (x_1, x') \) in \( S^1 \times \mathbb{R}^{N-1} \), we have, with \( r' = \|x'\| \),

\[
u(x) \leq Ke^{-\frac{r'}{\varepsilon}}(\frac{r'}{\varepsilon})^{\frac{2-N}{2}}
\]

(1.5)

\[
\|\nabla u(x)\| \leq \frac{K}{\varepsilon} e^{-\frac{r'}{\varepsilon}}(\frac{r'}{\varepsilon})^{\frac{2-N}{2}}
\]

(1.6)

In [1], we have proved (1.5) for \( N = 2 \) but with a constant \( K \) depending on the solution \((\varepsilon, u)\) for \( \varepsilon \) greater than some \( \varepsilon > 0 \). Our proof extends easily for \( N \geq 2 \) and for the derivatives of \( u \). We have now to prove that \( K \) is independent from the solution \((\varepsilon, u)\), even when \( \varepsilon \) tends to 0.

In all what follows we will use \( \tilde{u}(x_1, x') = u(\varepsilon x_1, \varepsilon x') \) for \( (x_1, x') \in S^1/\varepsilon \times \mathbb{R}^{N-1} \). The notation \( \Delta' \) will stand for the Laplacian operator in \( \mathbb{R}^{N-1} \).

2. Proof of Theorem 1.2

We begin the proof by two propositions.

**Proposition 2.1.** Let \( v \) be a bounded solution of

\[
-\Delta v + v - v^p = 0 \quad \text{in} \quad \mathbb{R}^2
\]

(2.7)

Let us suppose that \( \frac{\partial v}{\partial x_i} \) is bounded and \( \frac{\partial v}{\partial x_i} \leq 0 \) in \( \mathbb{R}^2 \), for \( i = 1 \) or \( i = 2 \). Then \( v \) does not depend on the variable \( x_i \).

**Proof:** From Ghoussoub-Gui, [6], Theorem 1.1, there exist a function \( U \) and a vector \( a \in \mathbb{R}^2 \) such that \( v(x) = U(a.x) \). We have

\[-\|a\|^2U''(a.x) + (U - U^p)(a.x) = 0\]

If \( a_i \neq 0 \), \( U \) is monotone and bounded in \( \mathbb{R} \). The only possibility is that \( U \) is a constant function, equal to 0 or 1, so is \( v \).
Proposition 2.2. Let \((\varepsilon, u)\) be solutions of (1.1). Then \(\tilde{u}(x_1, r)\) tends to 0 as \(r\) tends to \(\infty\), uniformly with respect to \(x_1\) and to \(\varepsilon\) and \(u\).

**Proof** In this proof, we omit the indices of the sequences. Let us suppose, by contradiction, that there exist a sequence \((a, b)\) in \(\mathbb{R}^N\), with \(||b||\) tending to \(+\infty\), a real positive number \(\varepsilon_2\) and solutions \((\varepsilon, u)\) of (1.1) such that \(\tilde{u}(a, b) \geq \varepsilon_2\). We can suppose \(\varepsilon_2 < 1\). For every solution \((\varepsilon, u)\), we have that \(\lim_{r \to +\infty} \tilde{u}(x_1, r) = 0\), uniformly in \(x_1\). So, for every \(\varepsilon_1 \in ]0, \varepsilon_2[\), there exists a sequence, \(\tilde{b}, ||\tilde{b}|| \geq ||b||\), such that \(\tilde{u}(a, \tilde{b}) = \varepsilon_1\). We define a sequence, still denoted by \(b\), with \(||b||\) tending to \(\infty\), such that \(\tilde{u}(a, b) = \varepsilon_2\). As \(\tilde{u}\) is radial in \(x'\), let us define
\[
v(x_1, r) = \tilde{u}(x_1 + a, r + ||b||) \text{ for } r \geq -||b||
\]
and
\[
\bar{v}(x_1, r) = \tilde{u}(x_1 + a, r + ||\tilde{b}||) \text{ for } r \geq -||\tilde{b}||
\]
The function \(v\) verifies
\[
-v_{x_1x_1} - v_{rr} - \frac{N - 2}{r + ||b||} v_r + v - v^p = 0
\]
and \(\bar{v}\) verifies a similar equation. It is standard that the both sequences \(v\) and \(\bar{v}\) tend uniformly on the compact sets of \(\mathbb{R}^2\) to limits, which will be denoted respectively by \(z\) and \(\bar{z}\). But \(z\) and \(\bar{z}\) are positive, bounded and non increasing in the variable \(r\) and they are periodic in \(x_1\). Moreover, \(z\) and \(\bar{z}\) verify
\[
-z_{x_1x_1} - z_{rr} + z - z^p = 0 \text{ in } \mathbb{R}^2
\]
By Proposition 2.1, \(z\) and \(\bar{z}\) depend only on \(x_1\). By Kwong, [9], if they are not constant functions, they oscillate indefinitely as \(x_1\) tends to \(\infty\), around the solution 1. As \(0 < \varepsilon_1 < \varepsilon_2 < 1\), then \(z\) and \(\bar{z}\) are not constant solutions. So \(z\) and \(\bar{z}\) oscillate infinitely around 1, too. The function \(h = z - 1\) and the function \(\bar{h} = \bar{z} - 1\) verify respectively the equations
\[
(2.8) \quad h'' + h(-1 + \frac{z^p - 1}{z - 1}) = 0 \quad \text{and} \quad \bar{h}'' + \bar{h}(-1 + \frac{\bar{z}^p - 1}{\bar{z} - 1}) = 0
\]
As \(z \geq \bar{z}\) and \(z(0) > \bar{z}(0)\), we have from the ordinary differential equations theory that \(z > \bar{z}\). It is easy to see that \(-1 + \frac{z^p - 1}{z - 1} > -1 + \frac{\bar{z}^p - 1}{\bar{z} - 1}\). By the Sturm Theory (see Ince, quoted in [9], Lemma 1), applied to the equations (2.8), there exists at least a zero of \(\bar{z} - 1\) between any two consecutive zeroes of \(\bar{z} - 1\). But there exist pairs \((\alpha, \beta)\) of zeroes of \(\bar{z} - 1\) such that \(\bar{z} > 1\) in \(]\alpha, \beta[\). Thus \(z > 1\) in \(]\alpha, \beta[\). We get a contradiction. We infer that the sequence \((a, b)\), in the beginning of this proof, doesn’t exist. We have proved the proposition.

We will need the following lemma

**Lemma 2.1.** There exists \(M\), such that for all solution \((\varepsilon, u)\) of (1.1)
\[
(2.9) \quad ||\nabla \tilde{u}||_{L^\infty(\frac{S_1}{\varepsilon} \times \mathbb{R}^N - 1)} \leq M
\]
Proof Let \((a, b) \in (S^1/\varepsilon) \times \mathbb{R}^{N-1}\). We set \(v(x_1, x') = \tilde{u}(x_1 + a, x' + b)\). It verifies 
\[-\Delta v + v - v^p = 0\] in \((S^1/\varepsilon) \times \mathbb{R}^{N-1}\). Moreover, we have \(\|v\|_\infty \leq M\), for a constant \(M\) independent from \(\varepsilon\). By standard elliptic arguments, \([5]\), \(\nabla v\) is bounded on the compact sets of \(\mathbb{R}^N\). So, there exists \(M\), independent from \(\varepsilon\), such that \(\|\nabla v(0, 0)\| \leq M\). This proves (2.9).

Proof of Theorem 1.2 We define

\[h(r') = \int_0^{2\pi/\varepsilon} \tilde{u}(x_1, r')dx_1\]

There exists a constant \(C\), independent from the solution \((\varepsilon, u)\), such that \(\|h\|_{L^\infty(\mathbb{R}^{N-1})} \leq C\). Since \(u \to 0\), uniformly in \(x_1\), as \(r'\) tends to \(\infty\), then for all \(\eta < 1\), there exists \(X > 0\) such that for all \(r' > X\) and for all \(\varepsilon\) we have for all solution \((\varepsilon, u)\) and for all \(x_1 \in \frac{S^1}{\varepsilon}\).

(2.10) \(\tilde{u}^{p-1}(x_1, r') < \eta\)

Integrating (1.1) with respect to \(x_1\), we find for \(r' > X\)

(2.11) \(h_{rr} + ((N - 2)/r')h_r > (1 - \eta)h\)

Let us multiply (2.11) by \(h_r\), we obtain that the function \(h_r^2 - (1 - \eta)h^2\) is non increasing. Moreover, it tends to 0 as \(r'\) tends to \(\infty\). We get \(h_r + \sqrt{1 - \eta}h \leq 0\), for \(r' > X\). So there exists \(C\) such that for all \(r'\)

(2.12) \(h(r') \leq \frac{C}{\varepsilon} e^{-\sqrt{1 - \eta}r'}\)

Let us remark that the constant \(C\) is independent from the choice of the solution \((\varepsilon, u)\).

Let \(R > 0\) be a given positive real number. We use a Harnack inequality (\([2]\), Theorem 9.20) to get that there exists a constant \(C\) independent from \(y\) and from \(\varepsilon\) such that

(2.13) \(\sup_{B_R(y)} \tilde{u} \leq C \int_{B_{2R}(y)} \tilde{u} dx_1 dx' \leq C \int_{\|x' - y'\| \leq 2R} \int_{y_1 - R}^{y_1 + R} \tilde{u}(x_1, x')dx_1dx'\)

that gives

\(\sup_{B_R(y)} \tilde{u} \leq C \int_{\|x' - y'\| \leq 2R} h(||x'||)dx'.\)

Finally, using (2.12), for all \(\eta \in ]0, 1[\) there exists \(C\), independent from the solution \((\varepsilon, u)\), such that,

(2.14) \(\tilde{u}(y) \leq \frac{C}{\varepsilon} e^{-\eta\|y'\|}\)
For the remainder of the proof, we will need the Green function for the equation. We have

\begin{equation}
G(x, x') = \sum_{j=0}^{\infty} \frac{k_j^{N-3}}{\varepsilon^{N-1}} g\left(\frac{k_j}{\varepsilon} x'\right) \cos(j x_1)
\end{equation}

where \(k_j = \sqrt{1 + \varepsilon^2 j^2}\) and \(g\) is the Green function for the operator \(-\Delta' + I\) in \(\mathbb{R}^n\), \(n = N - 1\), with the null limit at infinity. It is recalled in [3] that

\begin{equation}
0 < g(r) \leq C e^{-r (1 + r)^{(n-3)/2}} \quad \text{for } n \geq 2 \quad \text{and} \quad g(r) = \frac{1}{2} e^{-r} \quad \text{for } n = 1
\end{equation}

We will need the following estimate, valid for all \(\eta \in [0, 1]\).

\begin{equation}
\int_{\mathbb{R}^{N-1}} g\left(\|y' - x'\|\right) e^{-\eta \|y'\|} dy' \leq C e^{-\eta \|x'\|}
\end{equation}

which is an easy consequence of (2.16). For all function \(f\), that is \(2\pi\)-periodic in \(x_1\), the solution of

\begin{equation}
-\varepsilon^2 \Delta u + u = f \quad \text{in } \mathbb{R}^N
\end{equation}

that is \(2\pi\)-periodic in \(x_1\) and that tends to 0, as \(\|x'\|\) tends to \(\infty\) is \(u = G* f\). If \(f\) is positive, then \(u\) is positive, by the maximum principle. So \(G\) is positive. Moreover we can use (2.15) to verify that

\begin{equation}
\int_{S^1} G(x, x') dx_1 = \frac{2\pi}{\varepsilon^{N-1}} g\left(\frac{x'}{\varepsilon}\right)
\end{equation}

Let us prove that for all \(\eta \in [0, 1]\), there exists \(C\), independent from \(x_1\) and from \((\varepsilon, u)\) such that

\begin{equation}
\tilde{u}(x_1, x') \leq C e^{-\eta \|x'\|}
\end{equation}

It is clear by (2.14) that for all solution \((\varepsilon, u)\) and all \(\eta \in [0, 1]\), the function \(\tilde{u} e^{\eta \|x'\|}\) belongs to \(L^\infty(\mathbb{R}^N)\). We set

\begin{equation}
K(\eta) = \|\tilde{u} e^{\eta \|x'\|}\|_{\infty}
\end{equation}

We use the Green function \(G\) to get

\begin{equation}
u(x_1, x') = \int_{S^1 \times \mathbb{R}^{N-1}} G(y_1 - x_1, y' - x') u(y_1, y') dy_1 dy'
\end{equation}

and (2.18) gives

\begin{equation}
\tilde{u}(x_1, x') \leq 2\pi K\left(\frac{\eta}{p}\right)^p \int_{\mathbb{R}^{N-1}} g\left(\|y' - x'\|\right) e^{-\eta \|y'\|} dy'
\end{equation}

By (2.17), we infer that there exists a constant \(C\), independent from \((\varepsilon, u)\), such that

\begin{equation}
K(\eta) \leq C K\left(\frac{\eta}{p}\right)^p
\end{equation}

Now, let \(\tau = (\tau_1, \tau')\) be such that the function \(\tilde{u}(x + \tau) e^{\eta \|x' + \tau'\|}\) attains its maximal value at \(x = 0\). The existence of \(\tau\) is provided by (2.14). Let us suppose that \(K(\eta)\)
tends to $\infty$. We claim that $\|\tau'\|$ tends to infinity. Let us prove this claim. Let $\alpha$ be a positive real number, that will be chosen later. We set

$$v(x) = \tilde{u}(\alpha x + \tau)e^{\eta \|\alpha x' + \tau'\|}/K(\eta)$$

It verifies

$$-\Delta v + \left(1 + \eta^2 + \frac{(N-2)\eta}{\|\alpha x' + \tau'\|}\right)\alpha^2 v$$

$$= K(\eta)^{p-1}\alpha^2 e^{-(p+1)\eta\|\alpha x' + \tau'\|}p^p + \frac{2\eta\alpha^2}{K(\eta)}\sum_{i=2}^{N} \frac{\partial \tilde{u}}{\partial x_i}(\alpha x + \tau)(\frac{\alpha x_i + \tau_i}{\|\alpha x' + \tau'\|}e^{\eta\|\alpha x' + \tau'\|})$$

If $\|\tau'\|$ were bounded, we would choose $\alpha$ that tends to $0$ such that $K(\eta)^{p-1}\alpha^2 e^{-(p+1)\eta\|\alpha x' + \tau'\|}$ tends to $1$. By Lemma 2.1 and by standard results, $v$ would tend to a limit $\tau$, uniformly in the compact sets of $\mathbb{R}^N$. Then, $\tau$ would verify $-\Delta \tau - \tau^p = 0$ while $0 \leq \tau \leq 1$ and $\tau(0) = 1$. This is impossible by (1.4). So, if we suppose that $K(\eta)$ tends to $\infty$, then $\|\tau'\|$ tends to $\infty$. Let $\tilde{\tau} = (\tilde{\tau}_1, \tilde{\tau}_2)$ be such that $K(\frac{\tilde{\tau}}{p}) = \tilde{u}(\tilde{\tau})e^{\frac{\tilde{\tau}}{p}\|\tau'\|}$. We have $K(\frac{\tilde{\tau}}{p})^p = \tilde{u}^p(\tilde{\tau})e^{p\|\tau'\|}$, that gives

$$\tag{2.22} K\left(\frac{\tilde{\tau}}{p}\right)^p \leq K(\eta)\tilde{u}^{p-1}(\tilde{\tau})$$

Then (2.21) and (2.22) give

$$\tag{2.23} K(\eta) \leq CK\left(\frac{\tilde{\tau}}{p}\right)^p \leq CK(\eta)\tilde{u}^{p-1}(\tilde{\tau})$$

Consequently, if $K(\eta)$ tends to $\infty$, then $K(\frac{\tilde{\tau}}{p})$ tends to $\infty$, too. Then, $\|\tilde{\tau}'\| \to \infty$. By Proposition 2.2 we have $\tilde{u}(\tilde{\tau}) \to 0$. Then (2.23) gives a contradiction. So, we have proved that for all $\eta \in ]0, 1[$, $K(\eta)$ is bounded, independently from $(\varepsilon, u)$. We have (2.19). Now, let us choose $\eta$ such that $\eta p > 1$. In [3], it is proved that for $b > 1$ and for $N - 1 \geq 2$

$$\tag{2.24} \int_{\mathbb{R}^{N-1}} g(\|x' - y\|)e^{-b\|y\|}dy' \leq C\|x'\|^{\frac{2-N}{2}}e^{-\|x'\|}$$

We can use (2.16) to prove that the estimate (2.24) is valid also for $N = 2$. Now we use (2.20), (2.18) and (2.24) to obtain (1.5) with $K$ independent from $(\varepsilon, u)$. Now, let us estimate the gradient of $u$. We have, for $i = 1, ..., N$

$$\tag{2.25} \frac{\partial u}{\partial x_i}(x_1, x') = p \int_{S^1 \times \mathbb{R}^{N-1}} G(y_1 - x_1, y' - x')(u^{p-1}\frac{\partial \tilde{u}}{\partial x_i})(y_1, y')dy_1dy'$$

Since $\frac{\partial \tilde{u}}{\partial x_i}$ is bounded and $u \leq Ce^{-r'/\varepsilon}$, that gives

$$|\frac{\partial u}{\partial x_i}(x_1, x')| \leq \frac{C}{\varepsilon} \int_{S^1 \times \mathbb{R}^{N-1}} G(y_1 - x_1, y' - x')e^{-(p-1)\|y'\|/\varepsilon}dy_1dy'$$

and (2.18) gives

$$\tag{2.26} |\frac{\partial \tilde{u}}{\partial x_i}(x_1, x')| \leq C \int_{\mathbb{R}^{N-1}} g(\|y' - x\|)e^{-(p-1)\|y\|}dy'$$
Now, the proof is more easy if \( p > 2 \) than if \( p < 2 \). If \( p > 2 \), we deduce directly \( (1.6) \) from \( (2.24) \) and \( (2.26) \). If \( 1 < p < 2 \), we deduce from \( (2.17) \) and \( (2.26) \) that

\[
\left| \frac{\partial \tilde{u}}{\partial x_i}(x_1, x') \right| \leq Ce^{-(p-1)||x'||}
\]

Iterating this process, we get an integer \( k \) such that

\[
\left| \frac{\partial \tilde{u}}{\partial x_i}(x_1, x') \right| \leq Ce^{-k(p-1)||x'||}
\]

with \( k(p-1) < 1 \) and \( (k+1)(p-1) \geq 1 \). If \( (k+1)(p-1) > 1 \), we get \( (1.6) \) and the proof is complete. If \( (k+1)(p-1) = 1 \), we get

\[
\left| \frac{\partial \tilde{u}}{\partial x_i}(x_1, x') \right| \leq C \int_{\mathbb{R}^{N-1}} g(||y' - x'||) e^{-||y'||} dy'
\]

If \( N \geq 3 \), we have

\[
\int_{\mathbb{R}^{N-1}} g(||y' - x'||) e^{-||y'||} dy' \leq C \int_{\mathbb{R}^{N-1}} e^{-||x'-y'||(N-4)/2/||x'-y'||^{N-3}} dy'
\]

We can write the integral in the right hand member of this inequality as \( I = I_1 + I_2 \) and

\[
I_1 = \int_{||z|| \leq ||x'||} e^{-||z||-||z+x'||(1 + ||z||)(N-4)/2/||z||^{N-3}} dz
\]

and

\[
I_2 = \int_{||z|| \geq ||x'||} e^{-||z||-||z+x'||(1 + ||z||)(N-4)/2/||z||^{N-3}} dz
\]

We obtain, as \( ||x'|| \) tends to \( \infty \),

\[
I_1 \leq e^{-||x'||} \int_{0}^{||x'||} (1 + s)^{N-4} s ds \quad \text{and} \quad I_2 \leq e^{||x'||} \int_{||x'||}^{+\infty} e^{-2s(1 + s)^{N-4}} s ds
\]

These integrals are both less than \( Ce^{-||x'||} ||x'||^{N/2} \). Thus, if \( N \geq 3 \) we have obtained that

\[
(2.27) \quad \left| \frac{\partial \tilde{u}}{\partial x_i}(x_1, x') \right| \leq Ce^{-||x'||} ||x'||^{N/2}
\]

If \( N = 2 \), we have, when \( ||x'|| \) tends to \( \infty \)

\[
\int_{\mathbb{R}} g(||y' - x'||) e^{-||y'||} dy' \leq C \int_{\mathbb{R}} e^{-||x'-y'||-||y'||} dy' \leq C ||x'|| e^{-||x'||}
\]

In any case, we get that there exists \( b \in [0, 1] \), with \( b + p - 1 > 1 \) and such that

\[
\left| \frac{\partial \tilde{u}}{\partial x_i}(x_1, x') \right| \leq Ce^{-b||x'||}
\]

Using this estimate in \( (2.23) \) and thanks to \( (2.24) \), we get \( (1.6) \), for \( 1 < p < 2 \). If \( p = 2 \), \( (2.26) \) is \( (2.27) \) and we deduce \( (2.28) \) again. This ended the proof of Proposition \( 1.2 \).
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