DERIVATIVE AND ENTROPY: THE ONLY DERIVATIONS FROM $C^1(\mathbb{R})$ TO $C(\mathbb{R})$

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Abstract. Let $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ be an operator satisfying the derivation equation

$$T(f \cdot g) = (T f) \cdot g + f \cdot (T g),$$

where $f, g \in C^1(\mathbb{R})$, and some weak additional assumption. Then $T$ must be of the form

$$(T f)(x) = c(x) f'(x) + d(x) f(x) \ln |f(x)|$$

for $f \in C^1(\mathbb{R})$, $x \in \mathbb{R}$, where $c, d \in C(\mathbb{R})$ are suitable continuous functions, with the convention $0 \ln 0 = 0$. If the domain of $T$ is assumed to be $C(\mathbb{R})$, then $c = 0$ and $T$ is essentially given by the entropy function $f \ln |f|$. We can also determine the solutions of the generalized derivation equation

$$T(f \cdot g) = (T f) \cdot (A_1 g) + (A_2 f) \cdot (T g),$$

where $f, g \in C^1(\mathbb{R})$, for operators $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ and $A_1, A_2 : C(\mathbb{R}) \to C(\mathbb{R})$ fulfilling some weak additional properties.

1. Solutions of the Leibniz rule equation

Derivations in algebra and differential geometry generalize the Leibniz formula for the derivative of products of functions, and there are many such examples. However, on the space $C^1(\mathbb{R})$ of continuously differentiable functions, which is the most basic space in analysis, there are only few examples of derivations $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ satisfying the Leibniz rule

$$(1) \quad T(f \cdot g) = (T f) \cdot g + f \cdot (T g), \quad f, g \in C^1(\mathbb{R}).$$

We do not assume a priori that $T : C^1(\mathbb{R}) \to C(\mathbb{R})$ is linear or continuous in any sense. At the same time, there is a very different solution of (1), namely $T f = f \log |f|$, the entropy function, which is essentially the only solution of (1) in the case that $T$ acts from all continuous functions $C(\mathbb{R})$ to $C(\mathbb{R})$. Note that if two operators $T_1$ and $T_2$ verify (1), so does $T_1 + T_2$.

As our first main result, we determine which of the operators satisfy the classical Leibniz product rule.

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Definition 1. For any \( c \in \mathbb{R} \), let \( S_c : C(\mathbb{R}) \to C(\mathbb{R}) \) denote the shift operator by \( c \). \( S_c f(x) := f(x + c), x \in \mathbb{R}, f \in C(\mathbb{R}) \). Clearly, \( S_c \) also maps \( C^1(\mathbb{R}) \) into itself. We call \( T : C^1(\mathbb{R}) \to C(\mathbb{R}) \) isotropic if \( T \) commutes with all shifts, \( S_c T = T S_c \) for all \( c \in \mathbb{R} \).

Theorem 1. Suppose \( T : C^1(\mathbb{R}) \to C(\mathbb{R}) \) satisfies the Leibniz rule

\[
T(f \cdot g) = (Tf) \cdot g + f \cdot (Tg), \quad f, g \in C^1(\mathbb{R}).
\]

Assume further that either \( T \) is isotropic or that for any \( x \in \mathbb{R} \), there exist \( \varepsilon_x > 0 \) and \( M_x < \infty \) such that \( \sup_{|a| \leq \varepsilon_x} |T(1 + a)(x)| \leq M_x \). Then there are continuous functions \( c, d \in C(\mathbb{R}) \) such that

\[
(Tf)(x) = c(x) f'(x) + d(x) f(x) \ln |f(x)| \quad f \in C^1(\mathbb{R}), \quad x \in \mathbb{R}.
\]

In the isotropic case, \( c \) and \( d \) are constant. If \( T \) also maps \( C^2(\mathbb{R}) \) into \( C^1(\mathbb{R}) \), also with respect to the isotropic condition, then \( Tf = c f' \).

If \( T \) is defined on all continuous functions, it is given by the entropy function \( s:\mathbb{R} \):

Theorem 2. (a) If \( T : C(\mathbb{R}) \to C(\mathbb{R}) \) satisfies the Leibniz rule (1) for all \( f, g \in C(\mathbb{R}) \), there is a continuous function \( d \in C(\mathbb{R}) \) such that \( T \) has the form

\[
T f(x) = d(x) f(x) \ln |f(x)| \quad x \in \mathbb{R}.
\]

(b) If \( T : C^1(\mathbb{R}) \to C(\mathbb{R}) \) satisfies the Leibniz rule (1) for all \( f, g \in C^1(\mathbb{R}) \) and \( T(b) = 0 \) for all \( b \in \mathbb{R} \), there is a continuous function \( c \in C(\mathbb{R}) \) such that \( T \) has the form

\[
T f(x) = c(x) f'(x) \quad x \in \mathbb{R}.
\]

Taken together, the Leibniz product rule and the chain rule determine the derivative:

Proposition 3. Suppose \( T : C^1(\mathbb{R}) \to C(\mathbb{R}) \) satisfies the Leibniz rule and the chain rule functional equations

\[
T(f \cdot g) = (Tf) \cdot g + f \cdot (Tg),
\]

\[
T(f \circ g) = ((Tf) \circ g) \cdot (Tg),
\]

for all \( f, g \in C^1(\mathbb{R}) \). Then \( T \) is either identically 0 or the derivative, \( Tf = f' \).

Remarks.

(i) In the case of the chain rule functional equation there exist non-trivial examples from \( C(\mathbb{R}) \) into itself, cf. [2]. Thus the situation for the Leibniz rule functional equation is different where we have the non-trivial entropy function solution.

(ii) In Theorem 1, the continuity of a map \( T \) satisfying the Leibniz formula is not sufficient to guarantee that \( T \) is almost the derivative in the sense that \( Tf = c f' \).

(iii) Let \( C^k(\mathbb{R})_+ := \{ f \in C^k(\mathbb{R}) \mid f > 0 \} \) for \( k \in \mathbb{N} \). Then there are operators \( T : C^k(\mathbb{R})_+ \to C(\mathbb{R}) \) satisfying the Leibniz rule

\[
T(f \cdot g) = Tf \cdot g + f \cdot Tg, \quad f, g \in C^k(\mathbb{R})_+
\]

that are different from all of the above examples: simply take \( Tf = f \cdot (\log f)^{(k)} \). Since the \( k \)-th derivative is linear, \( (\log(fg))^{(k)} = (\log f)^{(k)} + (\log g)^{(k)} \).
(log \(g\))^{(k)}. Using this, the Leibniz rule follows easily. E.g. for \(k = 2\), we have \(Tf = f'' - (f')^2/f\).

(iv) Let \(C(\mathbb{R}_{>0})_+ := \{ f \in C(\mathbb{R}_{>0}) \mid f > 0 \}\) and \(H(x) := x \ln |x|\). Then the operation \(T\) defined by \(Tf(x) := H(f(x))/H(x)\) yields a map \(T : C(\mathbb{R}_{>0})_+ \to C(\mathbb{R})\) satisfying the Leibniz rule and the chain rule functional equations that is different from the derivative on the subset of positive \(C^1(\mathbb{R}_{>0})\) functions.

Looking at derivations from a more general point of view, let us search for an operation defined on \(C^1(\mathbb{R})\) or \(C(\mathbb{R})\) mimicking the logarithm operation in the sense of mapping the product of functions to the sum of its images. However, the logarithm \(\ln |f(x)|\) is not defined for functions \(f\) with zero values and, in fact, there is no such operation on all of \(C(\mathbb{R})\). Thus we allow some “tuning” by operators \(A_1, A_2 : C(\mathbb{R}) \to C(\mathbb{R})\) which we consider to be of “lower order” and consider the following “regularized” functional equation “of logarithmic type”

\[
T(f \cdot g) = (Tf) \cdot (A_1g) + (A_2f) \cdot (Tg) \quad f, g \in C^1(\mathbb{R}).
\]

(2)

Studying this equation, it turns out that the operators \(A_1\) and \(A_2\) may be only of a very restricted form. If there is a dependence on \(f'\) in \(Tf\), actually \(A_1 = A_2\) holds.

We assume the following non-degeneracy condition.

**Definition 2.** Consider operators \(T : C^1(\mathbb{R}) \to C(\mathbb{R})\) and \(A : C(\mathbb{R}) \to C(\mathbb{R})\). We call \((T, A)\) non-degenerate if for every open interval \(J \subset \mathbb{R}\) and \(x \in J\), there exist functions \(g_1, g_2 \in C^1(\mathbb{R})\) with support in \(J\) such that \(z_i = \left(\frac{A g_i(x)}{T g_i(x)}\right) \in \mathbb{R}^2\) are two linearly independent vectors in \(\mathbb{R}^2\) for \(i = 1, 2\). We also require that for every \(x \in \mathbb{R}\), there is \(g \in C^1(\mathbb{R})\) with \(T g(x) = 0\) and \(A g(x) \neq 0\).

We also impose the following weak continuity assumption on the \(A_i\)'s.

**Definition 3.** We call a map \(A : C(\mathbb{R}) \to C(\mathbb{R})\) pointwise continuous provided that for any sequence \((f_n)_{n \in \mathbb{N}}\) of \(C^1(\mathbb{R})\) functions and \(f \in C(\mathbb{R})\), the uniform convergence of \(\lim_{n \to \infty} f_n = f\) on \(\mathbb{R}\) implies the pointwise convergence of \(\lim_{n \to \infty} (A f_n)(x) = (A f)(x)\) for all \(x \in \mathbb{R}\).

Our main result for the general Leibniz rule functional equation is as follows:

**Theorem 4.** Assume that \(T : C^1(\mathbb{R}) \to C(\mathbb{R})\) and \(A_1, A_2 : C(\mathbb{R}) \to C(\mathbb{R})\) are operators satisfying the functional equation

\[
T(f \cdot g) = (Tf) \cdot (A_1g) + (A_2f) \cdot (Tg)
\]

(2)

for all \(f, g \in C^1(\mathbb{R})\). Suppose that \((T, A_1)\) is non-degenerate and that \(A_1\) and \(A_2\) are pointwise continuous. Assume further that for any \(x \in \mathbb{R}\), there are \(\varepsilon_x > 0\) and \(M_x < \infty\) such that \(\sup_{|a| \leq \varepsilon_x} |T(1 + a)(x)| \leq M_x\). Assume also that for some \(x \in \mathbb{R}\), there is a function \(g \in C^1(\mathbb{R})\) with \(g(x) = 1\) and \(T g(x) \neq 0\). Then the operators \(A_1\) and \(A_2\) coincide, \(A := A_1 = A_2\), and \(A\) is determined uniquely up to some function \(p \in C(\mathbb{R})\) with \(\text{Im}(p) \subset [0, \infty)\),

\[
(Af)(x) = |f(x)|^{p(x)+1} \{\text{sgn}f(x)\}.
\]
Given this form of \( A \), any solution \( T \) of (2) is of the following type: there are continuous functions \( c, d \) with \( c \) not identically zero such that
\begin{equation}
(T f)(x) = [c(x) f'(x) |f(x)|^p(x) \operatorname{sgn} f(x) + d(x) (\ln |f(x)|) |f(x)|^{p(x)+1}] \{ \operatorname{sgn} f(x) \}.
\end{equation}
The factor \( \{ \operatorname{sgn} f(x) \} \) may be present or not simultaneously in \( A \) and in \( T \), yielding two different solutions. To avoid discontinuous functions in the range of \( T \), the factor has to be present if \( p(x) = 0 \).

Conversely, the operators \( T \) and \( A \) defined by the previous formulas satisfy (2), with \( A_1 = A_2 \colon := A \).

Remarks.

(i) If in the conditions of Theorem 4 the last condition is not satisfied, that is, for all \( x \in \mathbb{R} \) and \( g \in C^1(\mathbb{R}) \) with \( g(x) = 1 \) always \( T g(x) = 0 \) holds in a solution of type (3), \( c \) is identically zero. In this case (3) still provides a solution of (2), but there are two further families of solutions \( T \) of (2) with \( A_i \) not necessarily multiplicative. This is investigated in [3].

(ii) The pointwise continuity of \( A_1, A_2 \in C(\mathbb{R}) \) is needed to ensure that these operators are local operators depending on \( f(x) \) but not on \( f'(x) \), as in the case of the operator \( T \), see Proposition 6 below.

Simple additional conditions will guarantee that \( T \) has a very specific form:

**Corollary 5.** Suppose the assumptions of Theorem 4 are satisfied.

(i) Assume further that \( T(4) = 4T(2) \) for the constant functions \( 4 \) and \( 2 \) and that, if for a certain \( x \), \( T(2)(x) = 0 \) holds, \( T(2) \text{Id}(x) = 2T(\text{Id})(x) \neq 0 \). Then the "lower order" operators \( A_i \) are given by \( A_1 f = A_2 f = f \) and
\begin{equation}
(T f)(x) = c(x) f'(x) + d(x) f(x) \ln |f(x)| \quad x \in \mathbb{R}
\end{equation}
where \( c, d \in C(\mathbb{R}) \) are continuous functions, with \( c, d \) not both identically zero.

(ii) If \( T \) also maps \( C^\infty(\mathbb{R}) \) into \( C^\infty(\mathbb{R}) \), there is \( n \in \mathbb{N} \) and a non-zero \( C^\infty(\mathbb{R}) \)-function \( c \) such that \( T \) has the form \((T f)(x) = c(x) f'(x) f(x)^n\).

2. Sketch of the Proof of Theorem 1

We indicate the main steps of the Proof of Theorem 1, which has to be modified in the case of Theorem 4. Corollary 5 is an easy consequence of the formulas in Theorem 4. The first step in the Proof of Theorem 1 is to show that the operator \( T \) is a local operator, i.e. determined pointwise. We show

**Proposition 6.** Assume \( T : C^1(\mathbb{R}) \to C(\mathbb{R}) \) satisfies the Leibniz rule
\begin{equation}
T(f \cdot g) = (T f) \cdot g + f \cdot (T g) \quad f, g \in C^1(\mathbb{R})
\end{equation}
Then there is a function \( F : \mathbb{R}^3 \to \mathbb{R} \) such that for all \( f \in C^1(\mathbb{R}) \) and all \( x \in \mathbb{R} \)
\begin{equation}
(T f)(x) = F(x, f(x), f'(x)).
\end{equation}

**Proof.**

(a) Choosing successively \( f = 0 \) and \( f = 1 \) in (1) yields that \( T(0) = T(1) = 0 \).

(b) Localization on intervals: Let \( J \subset \mathbb{R} \) be an open interval and consider any two functions \( f_1, f_2 \in C^1(\mathbb{R}) \) with \( f_1|_J = f_2|_J \). Then \( f_1 \cdot g = f_2 \cdot g \) holds for any function
$g \in C^1(\mathbb{R})$ with $\text{supp}(g) \subset J$, that is $g(x) = 0$ for $x \notin J$. The functional equation (1) yields
\begin{align*}
Tf_1 \cdot g + f_1 \cdot Tg &= T(f_1 \cdot g) = Tf_2 \cdot g = Tf_2 \cdot g + f_2 \cdot Tg,
\end{align*}
and
\begin{align*}
(Tf_1(x) - Tf_2(x)) \cdot g(x) = (f_2(x) - f_1(x)) \cdot Tg(x) = 0, \quad x \in J.
\end{align*}
Choose any $g$ supported in $J$ with $g(x) \neq 0$ to conclude that $Tf_1(x) = Tf_2(x)$.

Hence $f_1|_J = f_2|_J$ implies that $Tf_1|_J = Tf_2|_J$.

(c) Localization on intervals implies pointwise localization: Take $x_0 \in \mathbb{R}$ and $f \in C^1(\mathbb{R})$. Let $g$ be the tangent line approximation to $f$ at $x_0$,
\begin{align*}
g(x) := f(x_0) + f'(x_0)(x - x_0), \quad x \in \mathbb{R}
\end{align*}
and
\begin{align*}
h(x) := \begin{cases} f(x) & x < x_0 \\ g(x) & x \geq x_0 \end{cases}.
\end{align*}
By construction, $h \in C^1(\mathbb{R})$. For $I_1 = (-\infty, x_0)$ and $I_2 = (x_0, \infty)$, $f|_{I_1} = h|_{I_1}$ and $h|_{I_2} = g|_{I_2}$. By (b), $(Tf)|_{I_1} = (Th)|_{I_1}$ and $(Th)|_{I_2} = (Tg)|_{I_2}$. Since $Tf$, $Th$ and $Tg$ are continuous, both equalities also hold on $I_1$ and $I_2$. Thus for $x_0 \in I_1 \cap I_2$
\begin{align*}
(Tf)(x_0) = (Th)(x_0) = (Tg)(x_0).
\end{align*}
Since $g$ only depends on $x_0$, $f(x_0)$ and $f'(x_0)$, this means that there is a function $F : \mathbb{R}^3 \to \mathbb{R}$ with $(Tf)(x_0) = F(x_0, f(x_0), f'(x_0))$.

The second step is to analyze the function $F$ in Proposition 6. For this, we use the following lemma.

**Lemma 7.** Suppose $H_x : \mathbb{R} \to \mathbb{R}$ is a family of additive functions defined for any $x \in \mathbb{R}$, $H_x(a + b) = H_x(a) + H_x(b)$ for all $a, b \in \mathbb{R}$, such that
(i) $H_x(x)$ is a continuous function of $x \in \mathbb{R}$ and
(ii) $H_x(b)$ is a continuous function of $x \in \mathbb{R}$ for any fixed $b \in \mathbb{R}$.
Then $H_x$ is linear and there exists $d \in C(\mathbb{R})$ such that for all $x, b \in \mathbb{R}$, $H_x(b) = d(x) \cdot b$.

**Proof of Lemma 7**
For any $0 \neq x, b \in \mathbb{R}$ such that $x/b = r \in \mathbb{Q}$ is rational, $H_x(r \cdot b) = r \cdot H_x(b)$, and hence,
\begin{align*}
H_x(b) = (H_x(x)/x) \cdot b =: d(x) \cdot b.
\end{align*}
Fixing $b$, both sides are continuous functions of $x \in \mathbb{R}$. Therefore the equality extends from the dense set of all $x \in \mathbb{R}$ such that $b/x$ is rational to all $x \in \mathbb{R} : H_x$ is linear, $H_x(b) = d(x) \cdot b$ and $d \in C(\mathbb{R})$.

**Sketch of the Proof of Theorem 1**
(a) Let $x \in \mathbb{R}$, $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}$ be arbitrary. Choose functions $f, g \in C^1(\mathbb{R})$ such that $f(x) = \alpha_0, f'(x) = \alpha_1, g(x) = \beta_0, g'(x) = \beta_1$. The functional equation (1) means in terms of $(Tf)(x) = F(x, f(x), f'(x))$ that
\begin{align*}
F(x, \alpha_0 \beta_0, \alpha_1 \beta_0 + \alpha_0 \beta_1) = F(x, \alpha_0, \alpha_1) \beta_0 + F(x, \beta_0, \beta_1) \alpha_0.
\end{align*}
Further, as in the proof of Proposition 6, since \( T1 = T0 = 0 \), for any \( x \) one has \( F(x, 0, 0) = F(x, 1, 0) = 0 \). Choosing \( \beta_0 = 1 \) in (4) yields

\[
F(x, \alpha_0, \alpha_1 + \alpha_0 \beta'_1) = F(x, \alpha_0, \alpha_1) + F(x, 1, \beta'_1) \alpha_0.
\]

Denoting \( \beta_1 = \alpha_0 \beta'_1 \), which for \( \alpha_0 \neq 0 \) is arbitrary as well, this means that for all \( x, \alpha_0 \neq 0, \alpha_1 \) and \( \beta_1 \),

\[
F(x, \alpha_0, \alpha_1 + \beta_1) = F(x, \alpha_0, \alpha_1) + F(x, 1, \beta_1/\alpha_0) \alpha_0
\]

\[
= F(x, \alpha_0, \beta_1) + F(x, 1, \alpha_1/\alpha_0) \alpha_0,
\]

using the symmetry in \((\alpha_1, \beta_1)\) on the left hand side. Taking differences, we get that

\[
F(x, \alpha_0, \alpha_1) - F(x, 1, \alpha_1/\alpha_0) \alpha_0 = F(x, \alpha_0, \beta_1) - F(x, 1, \beta_1/\alpha_0) \alpha_0.
\]

This equality implies that both sides are independent of the variables \( \alpha_1 \) and \( \beta_1 \). Choose \( \beta_1 = 0 \) to find using \( F(x, 1, 0) = 0 \) that

\[
F(x, \alpha_0, \alpha_1) = F(x, 1, \alpha_1/\alpha_0) \alpha_0 + F(x, \alpha_0, 0).
\]

Moreover, for any \( x \in \mathbb{R} \), equation (4) with \( \alpha_0 = \beta_0 = 1 \) implies that \( F(x, 1, \cdot) \) is additive, that is

\[
F(x, 1, \alpha_1 + \beta_1) = F(x, 1, \alpha_1) + F(x, 1, \beta_1).
\]

We will demonstrate our method under the additional assumption that for some \( x \in \mathbb{R} \) there is \( g \in C^1(\mathbb{R}) \) such that \( g(x) = 1 \) and \( Tg(x) \neq 0 \). This yields that \( F(x, 1, \cdot) \) is not the zero function. The degenerate case \( F(x, 1, \cdot) = 0 \) identically will not be considered in this note; it is treated in [3] and leads to the same result in the situation of Theorem 1. However, a similarly degenerate case in the more general Theorem 4 would yield two additional families of solutions, cf. [3]. Now, taking \( \alpha_1 = \beta_1 = 0 \) in (4) yields

\[
F(x, \alpha_0 \beta_0, 0) = F(x, \alpha_0, 0) \beta_0 + F(x, \beta_0, 0) \alpha_0.
\]

Dividing by \( \alpha_0 \beta_0 \), we find that for any non-zero \( \alpha_0, \beta_0 \in \mathbb{R} \),

\[
\frac{F(x, \alpha_0 \beta_0, 0)}{\alpha_0 \beta_0} = \frac{F(x, \alpha_0, 0)}{\alpha_0} + \frac{F(x, \beta_0, 0)}{\beta_0}.
\]

This means that for any \( x \), \( \Phi(x, \alpha_0) := F(x, \alpha_0, 0)/\alpha_0 \) satisfies

\[
\Phi(x, \alpha_0, \beta_0) = \Phi(x, \alpha_0) + \Phi(x, \beta_0), \quad \alpha_0, \beta_0 \in \mathbb{R}.
\]

Hence \( \Phi(x, 1) = \Phi(x, 1^2) = 2\Phi(x, 1) \), \( \Phi(x, 1) = 0 \) and \( 0 = \Phi(x, 1) = \Phi(x, (-1)^2) = 2\Phi(x, -1) \), so that \( \Phi(x, -1) = 0 \). Thus \( \Phi(x, -\alpha_0) = \Phi(x, \alpha_0) \) and with \( H(x, s) := \Phi(x, e^s) \) for \( s \in \mathbb{R} \), we have that \( H(x, \cdot) \) is additive and \( F(x, \alpha_0, 0) = H(x, \ln |\alpha_0|) \alpha_0 \). Using (5) we get

\[
F(x, \alpha_0, \alpha_1) = \left[ (F(x, 1, \alpha_1/\alpha_0) + H(x, \ln |\alpha_0|)) \right] \alpha_0,
\]

where both \( F(x, 1, \cdot) \) and \( H(x, \cdot) \) are additive but not yet known to be continuous and linear. For the operator \( T \), formula (6) means that for any \( C^1 \)-function \( f \) and any point \( x \) with \( f(x) \neq 0 \) one has

\[
(Tf)(x) = \left[ F(x, 1, f'(x)/f(x)) + H(x, \ln |f(x)|) \right] f(x).
\]

(b) The continuity of \( Tf \) implies that

\[
F(x, 1, f'(x)/f(x)) + H(x, \ln |f(x)|)
\]
is a continuous function of $x$ for any function $f \in C^1(\mathbb{R})$ at any point $x$ with $f(x) \neq 0$. We apply this to three functions: $f_1(x) = \exp(b)$, $f_2(x) = \exp(bx)$ for any fixed $b \in \mathbb{R}$ and $f_3(x) = \exp(x^2/2)$. We conclude that $H(x, b), F(x, 1, b) + H(x, bx), F(x, 1, x) + H(x, x^2/2)$ are continuous functions of $x$.

In the case where we assume that for any $x \in \mathbb{R}$ there are $\varepsilon_x > 0$ and $M_x < \infty$ such that $\sup_{|a| \leq \varepsilon_x} |T(1 + a)(x)| \leq M_x$, the additive function $H(x, \cdot)$ is bounded in a neighborhood of $1$ since $H(x, 1 + a) = T(1 + a)(x)$. This implies that $H(x, \cdot)$ is linear, $H(x, b) = d(x) \cdot b$, cf. Aczel [1], with $d$ being continuous since $T(e^b)$ is a continuous function. Using the continuity of the above three functions, we find that $F(x, 1, b)$ and $F(x, 1, x)$ are continuous in $x \in \mathbb{R}$ for any $b$. Hence, by Lemma 7, $F(x, 1, \cdot)$ is linear, $F(x, 1, b) = c(x) \cdot b$, with $c \in C(\mathbb{R})$.

In the other case when we assume that $T$ is isotropic, i.e. commutes with shifts, the representing function $F(x, 1, f'(x)/f(x)) + H(x, \ln |f(x)|)$ does not depend on the first variable $x$. Using the function $\exp(-bx)$ evaluated at $-x$ and replacing the first argument, we find that $F(-x, 1, b) + H(-x, -bx) = F(x, 1, b) - H(x, bx)$ is also continuous, and hence both summands $F(x, 1, b)$ and $H(x, bx)$ are continuous in $x$ for any $b$. Again by Lemma 7, $H(x, b) = d(x) \cdot b$ is linear with $d \in C(\mathbb{R})$; actually, $d$ is constant in view of the isotropy assumption.

The Proof of Theorem 2 does not require local boundedness of $(T(1 + a))(x)$ or local isotropicity of $T$ since only one of the terms in $F(x, 1, \alpha_1/\alpha_0) + H(x, \ln |\alpha_0|)$ shows up and Lemma 7 is directly applicable. Proposition 3 follows from Theorem 2 (b) in the degenerate case when $T$ is zero on all bounded functions and from Theorem 1 of [2] otherwise. The resulting formulas then have to satisfy the second functional equation, as well, and this is possible only for the derivative $T = D$ or $T = 0$.

References

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