On the higher Riemann-Roch without denominators

A. Navarro *

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Abstract

We prove two refinements of the higher Riemann-Roch without denominators: a statement for regular closed immersions between arbitrary finite dimensional noetherian schemes, with no smoothness assumptions, and a statement for the relative cohomology of a proper morphism.

Introduction

Grothendieck’s Riemann-Roch theorem states that for $f: Y \to X$ a proper morphism between nonsingular quasi-projective varieties over a field $k$ the square

$$
\begin{array}{ccc}
K_0(Y) & \xrightarrow{f_!} & K_0(X) \\
\text{Td}(T_Y) \text{ch} & \downarrow & \text{Td}(T_X) \text{ch} \\
CH^\bullet(Y)_\mathbb{Q} & \xrightarrow{f_*} & CH^\bullet(X)_\mathbb{Q}
\end{array}
$$

commutes (cf. [BS58]). In other words, that for any element $a \in K_0(Y)$, the formula

$$
\text{Td}(T_X) \text{ch}(f_!(a)) = f_*(\text{Td}(T_Y) \text{ch}(a))
$$

holds in $CH^\bullet(X)_\mathbb{Q}$.

In a letter to Serre Grothendieck suggested that in the case $f$ is a closed immersion the Riemann-Roch should hold in $CH^\bullet(X)$, without the need of tensoring with $\mathbb{Q}$ (cf. [SGA6, 0 p. 70]). After the work on the Riemann-Roch at [SGA6] the question still remained as a conjecture known as the Riemann-Roch without denominators (cf. [SGA6, XIV.3]). Jouanolou proved this conjecture in [Jou70] for regular closed immersions between quasi-compact schemes over a field, and later for algebraic schemes (cf. [Ful98, §18]). More concretely, for any $a \in K_0(Y)$ denote $c_i(a)$ its $i$-th Chern class with values in the Chow ring and let $i: Y \to X$ be a regular closed immersion of codimension $d$ with normal vector bundle $N$. Jouanolou constructed polynomials with integral coefficients $P^d_q(X;Y)$, such that for any $a \in K_0(Y)$ the formula

$$
c_q(i_*(a)) = P^d_q(\text{rank}(a), c_1(a), \ldots, c_{q-d}(a); c_1(N), \ldots, c_{q-d}(N))
$$

holds in $CH^q(X)$. After the coming of higher $K$-theory, Gillet proved in [Gil81, 3] that this same formula holds for elements of higher $K$-theory and $i$ being a closed immersion between

*Institut für Mathematik, Universität Zürich, Switzerland
smooth schemes over a regular base. Finally, in [KY14] transferred Gillet’s proof and results into the motivic homotopy context.

In this note we show how to deduce the higher Riemann-Roch without denominators for regular immersions between arbitrary (possibly singular) finite dimensional noetherian schemes from the already known case of regular schemes. More concretely, the theorem we prove in 3.1 is the following:

**Theorem:** Let $i: Z \to X$ be a regular immersion of codimension $d$. Denote $\tilde{i}_*: K(Z) \to K(X)$ as well as $\tilde{i}_*: H_{\text{MH}}(Z, \mathbb{Z}) \to H_{\text{MH}}(X, \mathbb{Z})$ the refined Gysin morphisms, $c^Z_{q,r}: K_{r,Z}(X) \to H_{\text{MH},Z}(X, Z)$ the $r$-th Chern class with support on $Z$ (cf. [Jou70] and [2.4]) and $P^d_q(r, x_1, \ldots, y_1, \ldots)$ the polynomials with integers coefficients defined in [Jou70]. Then for any $a \in KH^r(Z)$ we have

$$c^Z_{q,r}(\tilde{i}_*(a)) = \tilde{i}_*(P^d_q(rk(a), c_{1,r}(a), \ldots, c_{q-d,r}(a); c_1(N_{Z/X}), \ldots, c_{q-d}(N_{Z/X}))).$$

Our proof lies on top of that of Kondo and Yasuda in [KY14]. More concretely, all classical computations are summarized in the motivic homotopy context by Kondo and Yasuda in Lemmas 3.2 and 3.3. Our proof relies on a explicit computation of the refined Gysin morphism (that is to say, direct image with support) of the zero section of a vector bundle in 2.7. Then, the characterization of direct images of [Nav18] and the deformation to the normal bundle allows to conclude.

Let $f: Y \to X$ be a morphism of schemes. Its inverse image in any reasonable cohomology theory fits into a long exact sequence relating groups called the relative cohomology of $f$, which we denote $K(f)$ in the case of $K$-theory. These groups generalize many classical cohomological constructions: the cohomology with proper support, the cohomology with support on a closed subscheme, and the reduced cohomology are the relative cohomology of a closed immersion, an open immersion and the projection over a base point respectively. We prove in Corollary 3.6 a formula without denominators for classes in $K(f)$ for $f$ a proper morphism.

The note is organized as follows: first we recall some basic notation of motivic homotopy theory, in section 2 we describe the Gysin morphism in our context and compute the case of the zero section of a vector bundle, finally we deduce the refinements of the Riemann-Roch.

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## 1 Preliminaries

In this section we recall the notation needed to describe the Gysin morphism constructed in [Nav18 § 2]. All schemes considered are finite dimensional and noetherian.

Let $X$ be a scheme. Denote $\text{SH}(X)$ the stable homotopy category of Voevodsky ([Voe98]). The category $\text{SH}(X)$, whose objects are called spectra, is triangulated and monoidal, we denote the unit by $\mathbb{1}_X$ and the $p$-shift functor by $[p]$ and $\mathbb{1}_X$ respectively. There is also Tate object, $\mathbb{1}_X(1)$ and we denote the $q$-twist functor given by $(q)$. Hence the $p$-shifted and $q$-twisted spectrum of a $E$ of $\text{SH}(X)$ is denoted $E[q][p]$.

The family of categories $\text{SH}(\_)$ satisfies Grothendieck’s six functor formalism ([Ayo07]). When we consider only schemes over a fixed base $S$, an object $E_S$ of $\text{SH}(S)$ (or simply $E$ if no confusion is possible) is called an absolute spectrum and for $f: Y \to S$ we denote
$$E_Y := f^* E_S.$$ In general, for any scheme $X$ a spectrum $E_X \in \operatorname{SH}(X)$ is also an absolute spectrum over $X$.

We say that a spectrum is a ring spectrum if it is an associative commutative unitary monoid in $\operatorname{SH}$. Ring spectra define cohomology theories with usual properties. More concretely, for any absolute ring spectrum $E$ in $\operatorname{SH}(S)$ we define the $E$-cohomology of an $S$-scheme $X$ to be

$$E^{p,q}(X) := \operatorname{Hom}_{\operatorname{SH}(X)}(\mathbb{I}_X, E_X(q)[p]).$$

We also denote $E(X) = \bigoplus_{p,q} E^{p,q}(X)$.

Let $E$ be an absolute ring spectrum, an absolute $E$-module is a spectrum $M$ in $\operatorname{SH}(S)$ together with a morphism of spectra $v: E \wedge M \to M$ in $\operatorname{SH}(S)$ satisfying the usual module condition (cf. [Nav18, 1.1]).

For any absolute ring spectrum $E$ with unit $e: S \to E_S$ there is a canonical class in $\eta \in E^{2,1}(\mathbb{P}^1)$ given by the composition $\mathbb{P}^1 \to \mathbb{I}_S(1)[2] \xrightarrow{e \otimes \text{Id}} E_S(1)[2]$. We define an orientation on $E$ to be a class $c_1 \in E^{2,1}(\mathbb{P}^\infty)$ such that for $i_1: \mathbb{P}^1 \hookrightarrow \mathbb{P}^\infty$ satisfies $i_1^!(c_1) = \eta$. We also say that $E$ is oriented. Let $(E, c_1)$ be an oriented absolute ring spectrum, then for any rank $n$ vector bundle $V \to X$ there are Chern classes $c_i(V) \in E^{2i,i}(X)$ for $0 < i \leq n$ satisfying the usual Whitney sum and functoriality property. We say that the orientation is additive if for two line bundles $L, L'$ we have $c_1(L \otimes L') = c_1(L) + c_1(L')$.

We denote $H(X)$ the unstable homotopy category of Morel and Voevodsky (cf. [MV99]), whose objects are called spaces. Its pointed version admits a pair of adjoint functors

$$\Sigma^\infty: H_*(X) \rightleftarrows \operatorname{SH}(X): \Omega^\infty$$

By analogy with classical topology, we denote the Hom sets of $H(X)$ by $[\_ , \_ ]$.

**Example 1.1**

1. The $K$-theory oriented absolute ring spectrum $KGL$ is defined in [Voe98] (cf. also [Rio10]). It represents Weibel’s homotopy invariant $K$-theory for every scheme (cf. [CIs13, 2.15]), and therefore it represents Quillen’s algebraic $K$-theory for regular schemes. We denote the cohomology groups they define as $KH_i(\_ )$. Denote $Gr$ the infinite Grassmanian, the space $\mathbb{Z} \times Gr$ satisfies that $\Omega^\infty(KGL) = \mathbb{Z} \times Gr$ and therefore it also represents homotopy invariant $K$-theory. More concretely, denote $S^i$ the space given by the $i$-th simplicial sphere:

$$KH_i(X) = [S^i \wedge X, \mathbb{Z} \times Gr].$$

2. In [Spi12] Spitzweck defined the oriented absolute motivic cohomology ring spectrum $H_A$ with coefficients in $A$ for schemes over a Dedekind domain $S$. Motivic cohomology is the universal cohomology with additive orientation, hence there is a cycle class mapping towards more classical cohomologies (such as Betti or algebraic deRham) or towards more refined cohomologies such as absolute Hodge cohomology with integral coefficients.

3. Let $f: Y \to X$ be a morphism of schemes and $E$ be an absolute ring spectrum. The map $E_X \to f_* f^* E_Y$ in $\operatorname{SH}(X)$ fits into a distinguished square. Hence, the inverse image in the $E$-cohomology fits into a long exact sequence

$$\cdots \to E^{p-1,q}(Y) \to E^{p,q}(f) \to E^{p,q}(X) \xrightarrow{f^*} E^{p,q}(Y) \to \cdots$$

where the groups $E^{p,q}(f)$ are called relative $E$-cohomology of $f$ (cf. [Nav18, §1.2]). The spectrum representing them is an absolute $E$-module over $X$, so we can apply the
upcoming Corollary 2.5. However, the spectrum representing relative cohomology has, in general, bad functorial properties. In order to avoid this problem we require $f$ to be proper. We denote by $\text{hofib}(f)$ the space representing these groups in $\mathbf{H}(S)$, that is the homotopy fiber of $\Omega^\infty E_X \to f_* f^* \Omega^\infty E_X$.

2 Gysin morphism

For the rest of this note, let $(E, c_i)$ be an absolute oriented spectrum.

2.1 The $i$-th Chern class defines a natural transformation

$$c_i : K_0(-) \to \mathbb{E}^{2i,i}(-)$$

of presheaves of sets on $\text{Sm}/S$ which maps every locally free module to its $i$-th Chern class. Riou proved that we have an isomorphism

$$\text{Hom}(K_0(-), \mathbb{E}^{2i,i}(-)) \simeq [\mathbb{Z} \times \text{Gr}, \Omega^\infty E(i)[2i]]$$

(cf. [Rio10, 1.1.6]). Since $\mathbb{Z} \times \text{Gr}$ represents homotopy invariant $K$-theory (cf. [Cis13]) we also have $i$-th Chern class defined for elements of homotopy invariant $K$-theory satisfying that for any scheme $X$ the diagram

$$\begin{array}{ccc}
K(X) & \xrightarrow{c_i} & \mathbb{E}^{2i,i}(X) \\
\downarrow & & \downarrow \\
KH(X) & &
\end{array}$$

commutes. We also obtain higher Chern classes $c_{i,r} : KH_i(X) \to \mathbb{E}^{2i-r,i}(X)$.

2.2 Let $Z \to X$ be a closed immersion and denote $T = S^r \wedge X/X - Z$, then we denote

$$c^Z_{i,r} : KH_{i,r}(X) := [T, \mathbb{Z} \times \text{Gr}] \to [T, \Omega^\infty E(i)[2i]] =: \mathbb{E}^{2i-r,i}_Z(X).$$

In addition, it follows from the construction that these Chern classes with support are functorial. As before, we obtain a commutative diagram

$$\begin{array}{ccc}
K_Z(X) & \xrightarrow{c^Z_{i,r}} & \mathbb{E}^{2i-r,i}_Z(X) \\
\downarrow & & \downarrow \\
KH_Z(X) & &
\end{array}$$

for the $i$-th Chern class with support. Let $i : Z \to X$ be a closed immersion, we denote by $i_* : E_Z(X) \to E(X)$ the natural morphism of forgetting support. We have $i_* \circ (c^Z_{i,r}) = c_{i,r}$. We are ready to state the construction of the refined Gysin morphism obtained in [Nav18, §2].
Theorem 2.3 Let \((E, \zeta_1)\) be an absolute oriented ring spectrum, there exist a unique way of assigning to any regular immersion a morphism \(\tilde{i}: E(Z) \to E_X(X)\) such that:

1. **Normalization:** if \(i\) is of codimension one then \(\tilde{i}_*(a) = a \cdot \zeta_1^c(L_Z)\), where \(L_Z\) denotes the line bundle given by the dual of the sheaf of ideals of regular functions vanishing on \(Z\).

2. **Key formula:** Consider the blow-up square

   \[
   \begin{array}{ccc}
   E & \xrightarrow{j} & B_Z X \\
   \downarrow{\pi'} & & \downarrow{\pi} \\
   Z & \xrightarrow{i} & X
   \end{array}
   \]

   and denote \(K = \pi'^*N_{Z/X}/N_{E/B_Z X}\) the excess vector bundle. Then \(\pi^*(\tilde{i}_*(a)) = \tilde{j}_*(\pi'^*(a))\) for all \(a \in E(Z)\).

These Gysin morphism satisfies also the following properties:

2'. **Excess intersection formula:** Consider a cartesian square

   \[
   \begin{array}{ccc}
   P & \xrightarrow{j} & X' \\
   \downarrow{\pi'} & & \downarrow{\pi} \\
   Z & \xrightarrow{i} & X
   \end{array}
   \]

   where both \(i\) and \(j\) are regular immersions of codimension \(n\) and \(m\) respectively. Denote \(K = \pi'^*N_{Z/X}/N_{P/X'}\) the excess vector bundle, then

   \[
   \pi^*\tilde{i}_*(a) = \tilde{j}_*(c_{n-m}(K) \cdot \pi'^*(a))
   \]

3. **Functoriality:** Let \(j: Y \to Z\) be a regular immersion, then \((ij)_* = \tilde{i}_*\tilde{j}_*\).

4. **Projection formula:** The Gysin morphism is \(E(X)\)-linear. In other words,

   \[
   a \cdot \tilde{i}_*(b) = \tilde{i}_*(a \cdot \pi^*(b)) \quad \forall \ a \in E(X) , \ b \in E(Z).
   \]

The direct image \(i_*\) in Thomason’s algebraic \(K\)-theory satisfies the excess intersection formula as well as the normalization (cf. [Tho93]). As a consequence, after a base change injective in cohomology the Gysin morphism is expressed in terms of Chern classes. Hence, applying an analogous argument as in [21] to KGL instead of \(E\) and the normalization of the preceding result we deduce that for a regular immersion \(i: Z \to X\) the square

\[
\begin{array}{ccc}
K(Z) & \xrightarrow{i_*} & K(X) \\
\downarrow & & \downarrow \\
KH(Z) & \xrightarrow{i_*} & KH(X)
\end{array}
\]

also commutes.
Definition 2.4 Let $i: Z \to X$ be a regular closed immersion. We define the **Gysin morphism** to be $i^* := i_! \circ \bar{i} : E(Z) \to E(X)$. If $M$ is an absolute $E$-module, we define a refined Gysin morphism and its non-refined version in the $M$-cohomology by the formulas $\bar{i}_*(m) := m \cdot \bar{i}^*(1)$ and $i_*(m) := i_!(\bar{i}_*(m))$, for all $m \in M(Z)$.

It is clear that the above theorem implies analogue formulas for the nonrefined Gysin morphism and also for modules. The reader may find a more elaborated proof of the following statement for modules and proper morphisms in [NN17, 2.5].

Corollary 2.5 Let $E$ be an absolute oriented ring spectrum and $M$ be an absolute $E$-module. The Gysin morphism, both in the $E$-cohomology and $M$-cohomology, satisfy the analogue of the properties 1, 2', 3, 4 of the previous theorem.

2.6 In order to fix notation, we recall the theory of Thom classes. Let $V$ be a vector bundle of rank $n$ on a scheme $X$. We define the **Thom space** of $V$ as

$$Th(V) = V/V - 0 \cong \bar{V}/\mathbb{P}(V) \text{ in } H(X),$$

where $\bar{V} = \mathbb{P}(V \oplus 1)$. Its $E$-cohomology fits into a long exact sequence

$$\ldots \to E^{**}(Th(V)) \xrightarrow{\pi^*} E^{**}(\bar{V}) \to E^{**}(V) \to \ldots$$

where, from the projective bundle theorem, the third arrow is always a split epimorphism. Let $0 \to H \to V \to O_{\bar{V}}(1) \to 0$ be the dual of the canonical short exact sequence. We call the **Thom class** of $V$ to the class $t(V) := c_n(H) = \sum_{i=0}^{n} (-1)^i c_i(V) \cdot x^i \in E^{2i,i}(\bar{V})$,

where $x = c_1(O_{\bar{V}}(-1))$ and the equality follows from Cartan-Whitney formula. Since $t(V)$ is zero in $E(\mathbb{P}(V))$, we call the **refined Thom class** to the unique element

$$\bar{t}(V) \in E(Th(V)) \simeq E_X(\bar{V}) = E_X(V)$$

such that $\pi^*(\bar{t}(V)) = t(V)$.

Any short exact sequence of vector bundles $0 \to V' \to V \to V'' \to 0$ induces an isomorphism

$$E(\text{Th}(V')) \otimes E(\text{Th}(V'')) \xrightarrow{\sim} E(\text{Th}(V)).$$

(cf. [Dég14, 2.4.8]). This pairing satisfies that

$$\bar{t}(V') \otimes \bar{t}(V'') \mapsto \bar{t}(V). \quad (2)$$

Let $i: Z \to X$ be closed immersion between regular schemes. The deformation to the normal bundle gives an isomorphism $E_Z(X) \simeq E(\text{Th}(N)) = E_Z(N)$, where $N$ denotes the normal bundle and $Z$ is considered as a closed subscheme through the zero section. It follows from the unicity of direct image restricted to regular schemes that $\bar{i}_*(1) = t(N)$. Let us now treat the general case:
Theorem 2.7 Let $X$ be a finite dimensional noetherian scheme and $V \to X$ be a vector bundle. Denote $\bar{V} = \mathbb{P}(V \oplus 1)$ the projective completion and $s : X \to \bar{V}$ the zero section. Then $s_*(1) = \mathfrak{t}(V)$ in $E(\text{Th}(\bar{V})) \simeq E_X(\bar{V})$ or, equivalently,

$$s_*(1) = \mathfrak{t}(V) \quad \text{in} \quad E(V).$$

**Proof:** First observe that if $V$ is trivial the result follows from the normalization of the Gysin morphism. Now, applying Jouanolou's trick we can assume $X$ is affine. As a consequence $[V] = [V']$ in $K_0(X)$ if and only if they are stably equivalent. In other words, $V \oplus r\mathcal{O} \simeq V' \oplus r'\mathcal{O}$ for some $r, r' \in \mathbb{N}$. Hence, any relation $[V] = [V'] + [V'']$ is given by a short exact sequence $0 \to V' \oplus r'\mathcal{O} \to V \oplus r\mathcal{O} \to V'' \oplus r''\mathcal{O} \to$ for some $r, r', r'' \in \mathbb{N}$. Now, for any $r$ and any $V$ we have a natural short exact sequence $0 \to V \to V \oplus r\mathcal{O} \to r\mathcal{O} \to 0$. Thanks to the the additivity given by (2) the statement $\mathfrak{t}(V + r\mathcal{O})$, for any $r \in \mathbb{N}$, is equivalent to that of $\mathfrak{t}(V)$. Hence, we can reduce to the case of line bundles.

Let $L$ be a line bundle, we have $\mathfrak{t}(L) = c_1(L) - c_1(\mathcal{O}_L(-1))$ and $s_*(1) = c_1(\mathcal{I}^*)$, where $\mathcal{I}$ stands for the sheaf of ideals of the zero section in $\bar{L}$. This sheaf may be computed explicitly: the composition $\mathcal{O}_\bar{L}(-1) \to \mathcal{L} \to \mathcal{O} \to \mathcal{L} \to \mathcal{O}$ of the canonical morphism and the projection is an isomorphism out of the zero section, which induces $L^* \otimes \mathcal{O}_\bar{L}(-1) \simeq \mathcal{I} \to \mathcal{O}$.

Consider the canonical short exact sequence

$$0 \to \mathcal{O}_\bar{L}(-1) \to L \oplus \mathcal{O} \to Q \to 0$$

where $Q$ is the canonical quotient bundle. Taking second exterior product it induces $L = \bigwedge^2(L \oplus \mathcal{O}) = Q \otimes \mathcal{O}_\bar{L}(-1)$ so that $Q = L \otimes \mathcal{O}_L(1) = \mathcal{I}^*$.

To finish the proof note that $c_1(L \oplus \mathcal{O}) = c_1(L)$. Also by the additivity of Chern classes $c_1(L \oplus \mathcal{O}) = c_1(Q) + c_1(\mathcal{O}(-1))$ and we conclude. 

\[\Box\]

### 3 The two results

Denote $P_d^q(\xi, c_1, \ldots, c_{q-d}; c'_1, \ldots, c'_{q-d})$ the universal polynomial with integer coefficients defined in [Jou70, §1]. We are ready to prove the first result of this note.

**Theorem 3.1 (Riemann-Roch without denominators)** Let $i : Z \to X$ be a regular immersion of codimension $d$. Then for any $q > 0$ and any $a \in K_r(Z)$ we have in $H^{2q-r}_\mathcal{M,Z}(X, Z(q))$ that

$$c_{q,r}^Z(i)(a) = i(P_d^q(\text{rk}(a), c_1,r(a), \ldots, c_{q-d,r}(a); c_1(N_{Z/X}, \ldots, c_{q-d}(N_{Z/X}))). \quad (3)$$

**Proof:** Thanks to the remark of [2,22] it is enough to prove the result for homotopy invariant $K$-theory.

For convenience, denote $P_d^q(a, E) = P_d^q(\text{rk}(a), c_1,r(a), \ldots, c_{q-d,r}(a); c_1(E), \ldots, c_{q-d}(E))$ for any vector bundle $E$ on $Z$. We consider the deformation to the projective closure of the normal bundle. That is to say, consider the commutative diagram

$$\begin{array}{c}
\mathbb{P}^1 \xrightarrow{\mathfrak{i}_0} X' \xrightarrow{i_1} X \\
\mathfrak{s}_0 \downarrow \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow 1 \\
Z \xrightarrow{1} \mathbb{A}^1 \xrightarrow{1} Z
\end{array}$$

(4)
where $X' = B_{Y \times \{0\}}\mathbb{A}^1_X$. For $U = X' - \mathbb{A}^1_Z$, taking motivic cohomology in the deformation diagram we obtain

$$
\begin{array}{c}
H_\mathcal{M}(U, \mathbb{Z}) \\
\downarrow h \\
H_\mathcal{M}(\bar{N}, \mathbb{Z}) \\
\downarrow p_1 \\
\downarrow p_i \\
H_\mathcal{M}(Z, \mathbb{Z})
\end{array}
\begin{array}{c}
\xrightarrow{i^*_0} \\
\xrightarrow{i^*_i} \\
\xrightarrow{i^*_p} \\
\xrightarrow{i^*_q} \\
\xrightarrow{i^*_r} \\
\xrightarrow{i^*_s}
\end{array}
\begin{array}{c}
H_\mathcal{M, A}_1(X', \mathbb{Z}) \\
H_\mathcal{M, A}_1^1(X', \mathbb{Z}) \\
H_\mathcal{M}(X, \mathbb{Z}) \\
\sim H_\mathcal{M}(\mathbb{A}^1_Z, \mathbb{Z}) \\
H_\mathcal{M}(Z, \mathbb{Z})
\end{array}
$$

where $h = j^*\iota_0$ and $\bar{s}$ is injective (since it has a retract).

We now prove that if formula (3) holds for $\mathcal{M}(\mathbb{A}^1_Z)$ then it also holds for $i: Z \to X$. Since $v^*N_{\mathbb{A}^1_Z/X'} = N_{Z/X}$ the refined versions of the excess intersection formula applied to the right square and the functoriality of higher Chern classes gives

$$
c^q_{r,t}(v^*(b)) = c^q_{r,t}i^*_i(i(b)) = i^*_1c^1_{q,r}q_i(b)
$$

and

$$
i^*_1(i(P^d_q(b, N_{\mathbb{A}^1_Z/X'}))) = i((P^d_q(v^*(b), N_{Z/X})))
$$

for $b \in KH(\mathbb{A}^1_Z)$ such that $v^*(b) = a$. The last and the first term respectively are elements that theorem state that coincide.

Chasing the diagram we have that if $a \in H_\mathcal{M}(X', \mathbb{Z})$ has $h(a) = 0$ and $i^*_0(a) = 0$ then $b = 0$. Since $b(c^1_{q,r}\bar{i}(a)) = c^1_{q,r}j^*\bar{i}(a) = 0$ and $h = j^*\iota_0 = 0$ the last case left to prove is formula (3) for the zero section $s: Z \to \bar{N}$ of the projective completion of the normal bundle. This is the case treated in the literature when $Z$ is smooth.

First recall that in [2.3] we observed that $H_\mathcal{M}(\text{Th}(N), \mathbb{Z}) \simeq H_\mathcal{M, Z}(\bar{N}, \mathbb{Z}) \to H_\mathcal{M}(\bar{N}, \mathbb{Z})$ is injective, so it is enough to prove formula (3) in $H_\mathcal{M}(\bar{N}, \mathbb{Z})$.

We summarize in two lemmas computations which involve arguments in $K_0(\bar{N})$ that do not need the smoothness assumption (cf. [KY14], 4.3 and 4.4).

**Lemma 3.2** With the previous notations, denote $Q$ the canonical quotient bundle of $\bar{N}$ and $t^{KH}(N)$ and $t(N)$ the Thom class in $KH$ and $H_\mathcal{M}$ respectively. Then for any $b \in KH_r(\bar{N})$ we have

$$
c^q_{r,t}(b \cdot t^{KH}(N)) = P^d_q(b, Q) \cdot t(N).
$$

**Lemma 3.3** Let $p: \bar{N} \to Z$ be the projection. For any $a \in KH_r(Z)$ we have

$$
p^*(P^d_q(a, N)) \cdot t(N) = P^d_q(p^*a, Q) \cdot t(N).
$$

To conclude recall that the fundamental class of the zero section coincide with the Thom class (Theorem [2.7]). From here, the projection formula gives $s_*(a) = p^*(a) \cdot t^{KH}(N)$ and the analogous formula for motivic cohomology. With them, we conclude

$$
c^q_{r,t}(i_*(a)) = c^q_{r,t}(p^*(a) \cdot t(N)) = i_*(P^d_q(a, N)).$$

8
Recall that motivic cohomology is universal among cohomology theories such that Chern classes have an additive formal group law. Hence, we deduce the following result:

**Corollary 3.4** Let \((E, c_1)\) be an absolute ring spectrum with an additive orientation. Let \(i: Z \to X\) be a regular immersion of codimension \(d\), then for any \(q > 0\) and any \(a \in K_r(Z)\) we have in \(E^{2r-q}(X)\) that

\[
c_{q,r}(i_*(a)) = i_*(P^q_d(rk(a), c_{1,r}(a), \ldots, c_{q-d,r}(a); c_1(N_{Z/X}), \ldots, c_{q-d}(N_{Z/X}))).
\]

\(\square\)

**3.5** Recall that the \(i\)-th Chern class defines a map \(c_i: Z \times \text{Gr}_X \to \Omega^\infty E(i)[2i]\) in \(H_*(X)\). Let \(f: T \to X\) be a proper morphism, the \(i\)-th Chern class fits into a commutative diagram

\[
\begin{array}{c}
\text{hofib}_{Z \times \text{Gr}_X}(g) \\
\downarrow \downarrow \\
\text{hofib}_Z(g) \\
\end{array}
\begin{array}{c}
Z \times \text{Gr}_X \\
\downarrow \\
\Omega^\infty E(i)[2i] \\
\end{array}
\begin{array}{c}
f_i(f^*Z \times \text{Gr}_X) \\
\downarrow \\
f_i(f^*\Omega^\infty E(i)[2i])
\end{array}
\]

so that there exist a Chern class for the relative cohomology of \(g\), which we still denote \(c_i\). An analogous argument hold for the rank function. We deduce from the construction that these classes are functorial.

One can check that in the universal polynomials \(P^d_q(\xi, c_1, \ldots, c_{q-d}; c'_1, \ldots, c'_{q-d})\) the elements \(c'_1, \ldots, c'_{q-d}\) always appear multiplied by elements \(\xi, c_1, \ldots, c_{q-d}\) (cf. \text{[Lev98, \S3.2]} for example, where the reference denotes \(Q_{d,q-d}\) instead of \(P^d_q\)). Hence, for any \(m \in KH(f)\) and any vector bundle \(m\) the element \(P^d_q(m, E)\) belongs to \(H_M(f, Z)\).

**Corollary 3.6** Let \(i: Z \to X\) be a regular immersion of codimension \(d\) and \(g: T \to X\) be a proper morphism. Then for any \(q > 0\) and any \(m \in K_r(f_z)\) we have in \(H_{M,Z}(f, Z)\) that

\[
c_{q,r}(i_*(m)) = i_*(P^d_q(rk(m), c_{1,r}(m), \ldots, c_{q-d,r}(m); c_1(N_{Z/X}), \ldots, c_{q-d}(N_{Z/X}))).
\]

**Proof:** Taking fiber product of \(T \to X\) along the deformation to the projective closure (cf. diagram \(\square\)) we obtain the diagram

\[
\begin{array}{c}
\mathbb{N}_T \\
\downarrow \downarrow \\
T_Z \\
\end{array}
\begin{array}{c}
\to T' \\
\downarrow \downarrow \\
\to \mathbb{A}^1_T
\end{array}
\begin{array}{c}
\to T \\
\downarrow \downarrow \\
T_Z
\end{array}
\]

where \(T' = B_{T \times \{0\}}\mathbb{A}^1_T\) with proper morphisms \(f': T' \to X', f_z: T_Z \to Z\) and \(f: \mathbb{N}_T \to \mathbb{N}\). We obtain a commutative diagram

\[
\begin{array}{c}
H_M(g, Z) \\
\downarrow h \\
H_M(g_z, Z) \\
\downarrow s \\
\end{array}
\begin{array}{c}
\rightarrow H_M(g', Z) \\
\downarrow i' \\
\rightarrow H_M(g, Z)
\end{array}
\begin{array}{c}
\rightarrow H_M(g_{\mathbb{A}^1_T}, Z) \\
\downarrow i_* \\
\rightarrow H_M(g_z, Z)
\end{array}
\begin{array}{c}
\rightarrow H_M(g_{\mathbb{A}^1_T}, Z) \\
\downarrow v_* \\
\rightarrow H_M(g_z, Z)
\end{array}
\begin{array}{c}
\rightarrow H_M(g, Z)
\end{array}
\]

\(\square\)
The arguments from Theorem 3.1 transfer to the relative cohomology. Indeed, $H^*_M(f_! Z, Z) \simeq H^*_M(f_! Z, Z)$, the excess intersection formula holds for the Gysin morphism (Corollary 2.5). Chern classes are functorial, and localization holds for relative cohomology. Hence, we only have to prove the formula for the direct image $s_!$.

Recall from Definition 2.4 that the direct image in relative cohomology is defined by multiplying by the same fundamental class as in cohomology. In this case by the Thom classes by $t^K H(N)$ and $t(N)$. We conclude by recalling once again the computations from [KY14, 4.3 and 4.4]. □

**Corollary 3.7** Let $(E, c_1)$ be an absolute ring spectrum with an additive orientation. Let $i: Z \to X$ be a regular immersion of codimension $d$ and $f: T \to X$ be a proper morphism. Then for any $q > 0$ and any $m \in K_r(f_!)$ we have in $E(f)$ that

$$c_{q,r}(i_!(m)) = i_!(P_0^d(rk(m), c_{1,r}(m), \ldots, c_{q-d,r}(m); c_{1}(N_{Z/X}), \ldots, c_{q-d}(N_{Z/X})))$$

□

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