CHARACTERS AND BLOCKS FOR FINITE-DIMENSIONAL REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

VYJAYANTHI CHARI AND ADRIANO A. MOURA

Introduction

In this paper we study the category $\mathcal{C}_q$ of finite–dimensional representations of a quantum loop algebra $U$. Our aim is to study and to put into a common representation theoretic framework, two kinds of characters which have been associated to an object of $\mathcal{C}_q$. One is the notion of $q$–characters defined in \cite{21} which is analogous in this context, to the usual notion of a character of a finite–dimensional representation of a simple Lie algebra. The other, is the notion of the elliptic character defined in \cite{17} which plays the role of the central character for representations of semi-simple Lie algebras. Both kinds of characters are needed in our situation, because the category $\mathcal{C}_q$ is not semi-simple and hence the problem of determining the blocks in this category becomes important.

The papers \cite{17} and \cite{21} use the universal $R$–matrix in fundamental ways to study the elliptic character and the $q$–character respectively. In particular, \cite{17} uses convergence properties of this matrix and hence, the main result of the paper describes the blocks in the case $|q| < 1$. Our methods, which avoid the use of the $R$–matrix, allows us to determine the blocks for all $q$ not a root of unity. One of the conjectures of \cite{21}, proved in \cite{20} (see also \cite{22}) is that the character of simple objects of $\mathcal{C}_q$ has a certain cone like form. We prove this result for the quantum affine algebras associated to a classical Lie algebra in a representation theoretic way rather than in a combinatorial fashion. We are actually able to prove a stronger version of their result, which allows us to give a formula for the $q$–characters of the fundamental representations in terms of the braid group action defined in \cite{5}.

We now describe the results of this paper. The algebra $U$ has a large commutative subalgebra $U(0)$ and the representations in $\mathcal{C}_q$ can be written as a sum of generalized eigenspaces for this subalgebra. The corresponding eigenvalues are known to be $n$–tuples of rational functions \cite{21}, where $n$ is the rank of the underlying finite–dimensional simple Lie algebra $\mathfrak{g}$. We define the $\ell$–weight lattice $P_q$ of $U$ to be the multiplicative subgroup consisting of the invertible elements in the ring of $n$–tuples of rational functions in an indeterminate $u$. It was proved in \cite{5} that the braid group of $\mathfrak{g}$ acts on $P_q$. Using this action, we define in Section 2 the notion of simple $\ell$–roots and the $\ell$–root lattice $Q_q$. It turns out that $Q_q$ is preserved by the braid group action. We then give generators and relations for the quotient group $\Xi_q = P_q/Q_q$. Our constructions make sense when $q = 1$ and the quotient group $\Xi_1$ is just the group of functions with finite support from $\mathbb{C}^\times$ to $P/Q$, where $P$ and $Q$ are the usual weight and root lattices of $\mathfrak{g}$. In Section 7 we prove that the blocks of $\mathcal{C}_q$ are in bijective correspondence with elements of $\Xi_q$. A somewhat unusual feature of this category, is that the tensor product of two blocks is contained in a single block.

Let $P^+_q$ be the monoid in $P_q$ generated by $n$–tuples of polynomials. It was proved in \cite{10} that elements of $P^+_q$ parametrize the isomorphism classes of the irreducible objects in $\mathcal{C}_q$. Motivated by this, we call the elements of $P^+_q$ the dominant $\ell$-weights. Given $\omega \in P^+_q$ one can define in a natural way the notion of an $\ell$–highest weight representation with $\ell$–highest weight $\omega$. In \cite{15} a family of universal $\ell$–highest weight module in $\mathcal{C}_q$, called the Weyl module $W(\omega)$ was constructed and it was conjectured
there and proved in the case of $\mathfrak{sl}_2$ that $W(\omega)$ was isomorphic to a tensor product of fundamental modules, or in other words, that $W(\omega)$ was isomorphic to a standard module. Using some deep results of Nakajima we deduce this conjecture for a general simple Lie algebra in section 6. We then prove that the $\ell$–weights of $W(\omega)$ and hence, those of any $\ell$–highest weight representations lie in the “cone” $\omega(Q^+_q)^{-1}$, here $Q^+_q$ is the monoid generated by the simple $\ell$–roots. This result plays an important role in Section 7, since it allows us to prove that the Weyl module has a well-defined elliptic character.

In sections 4 and 5, we study the $U(0)$–decomposition into generalized eigenspaces of objects in $C_q$. We call these the $\ell$–weight spaces and the corresponding eigenvalues the $\ell$–weights of that representation. We prove that the $\ell$–weights of any object of $C_q$ are in $P_q$ and that the $\ell$–weights determine the usual weights of objects in $C_q$ via a homomorphism $\text{wt} : P_q \to P$. To further describe the main result of Section 5, it is useful to compare it with the results of [20] and [21]. In those papers the authors developed the notion of a $q$–character for objects in $C_q$. In the language of this paper, they are the following element of $Z[P_q],$

$$\text{ch}_\ell(V) = \sum_{\varpi \in P_q} \text{dim}(V_{\varpi}) \cdot e(\varpi),$$

where $V_{\varpi}$ is the generalized eigenspace of $V$ corresponding to the eigenvalue $\varpi$, and $Z[P_q]$ is the integral group ring over $P_q$ with basis elements $e(\varpi)$. The elements $A_{i,c}$ of [21], where $i$ varies over the set of simple roots for $\mathfrak{g}$ and $c \in \mathbb{C}^\times$ turn out to correspond to the simple $\ell$–roots. In [20], a result corresponding to the main result of Section 5 of this paper, namely, that the $\ell$–weights of the irreducible representation $V(\omega)$ lie in the cone $\omega(Q^+_q)^{-1}$ was proved using combinatorial methods. Our methods on the other hand are purely representation theoretic. This allows us state more precise results on the $\ell$–weights of fundamental representations have a certain invariance under the braid group action analogous to the invariance of the set of weights under the Weyl group for finite–dimensional representations of simple Lie algebras. Interestingly enough, this invariance appears to be a very special property which fails if the Lie algebra is of exceptional type. Although there are a number of papers where $q$–characters have been studied, [22], [25], there are few explicit formulas available even for the fundamental representations, although there are some conjectures in [21] and there is a description of the $\ell$–characters in the $A_n$, $D_n$ case in terms of tableaux in [25]. As an application of our techniques, we write down the $q$–character of the fundamental representation corresponding to the adjoint representation when $\mathfrak{g}$ is of type $D_n$.

The more general case is studied in [7].

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1. Preliminaries

In this section, we recall the definition of quantum affine algebras and several results on the structure of these algebras.

1.1. Let $\mathfrak{g}$ be a complex finite–dimensional simple Lie algebra of rank $n$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Set $I = \{1, 2, \cdots, n\}$ and let $\{\alpha_i : i \in I\}$ (resp. $\{\omega_i : i \in I\}$) be the set of simple roots (resp. fundamental weights) of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let also $\check{\alpha}_i$ denote the simple co-roots. As usual, $Q$, (resp. $P$) denotes the root (resp. weight) lattice of $\mathfrak{g}$ and $Q^+, P^+$ the non–negative root and weight lattice respectively. Set $\Gamma = P/Q$. Let $A = (a_{ij})_{i,j \in I}$ be the $n \times n$ Cartan matrix of $\mathfrak{g}$ and let $h \in \mathbb{Z}$ be the dual Coxeter number of $\mathfrak{g}$. Fix non–negative integers $d_i$ such that the matrix $(d_i a_{ij})$ is symmetric. Assume that the nodes of the Dynkin diagram of $\mathfrak{g}$ are numbered as in Table 1 below and let $I_\bullet$ denote the subset of $I$ consisting of the shaded nodes.
1.2. Denote by $W$ the Weyl group of $\mathfrak{g}$, then $W$ is generated by simple reflections $\{s_i : i \in I\}$. For $w \in W$, let $\ell(w)$ denote the length of a reduced expression for $w$. Let $w_0$ denote the longest element of $W$, then $w_0$ defines a permutation of $I$, given by $w_0 \alpha_i = -\alpha_{w_i}$. Given $\lambda = \sum_{i \in I} \lambda_i \omega_i \in P^+$, let $I(\lambda) = \{i \in I : \lambda_i = 0\}$ and let $W(\lambda)$ be the subgroup of $W$ generated by $\{s_i : i \in I(\lambda)\}$. The following lemma is well-known [23].

**Lemma.** Let $\lambda = \sum_{i \in I} \lambda_i \omega_i \in P^+$. Then,

(i) $W(\lambda) = \{w \in W : w\lambda = \lambda\}$.

(ii) Each left (right) coset of $W(\lambda)$ in $W$ contains a unique element of minimal length. Denote by $W_\lambda$ the set of left coset representatives of minimal length.

(iii) Suppose that $w \in W_\lambda$ and that $w = s_j w'$ for some $w' \in W$ with $\ell(w') = \ell(w) - 1$. Then, $w' \in W_\lambda$.

\[\square\]

1.3. The next lemma is easily checked using the explicit formulas for the fundamental weights given in [23].

**Lemma.** Suppose that $\mathfrak{g}$ is of type $A_n$, $B_n$, $C_n$ or $D_n$.

(i) If $\lambda \in P^+$ is such that $\omega_i - \lambda \in Q^+$ for some $i \in I$, then either $\lambda = 0$ or $\lambda = \omega_j$ for some $j \in I$ with $j \leq i$.

(ii) Let $i, j \in I$ and assume that $i > j$. Then,

$$\omega_i - \omega_j - \alpha_j \notin Q^+, \quad \omega_i - \omega_j - 2\alpha_{j+1} \notin Q^+.$$  

\[\square\]

1.4. Let $q \in \mathbb{C}^\times$ and assume that $q$ is not a root of unity. For $r, m \in \mathbb{N}$, $m \geq r$, define complex numbers,

$$\begin{align*}
[m]_q &= q^m - q^{-m}, \\
[m]_q! &= [m]_q [m-1]_q \ldots [2]_q [1]_q, \\
\binom{m}{r}_q &= \frac{[m]_q!}{[r]_q! [m-r]_q!}.
\end{align*}$$

**Table 1**

- $A_n$: \[•\; 1 \; 2 \; \ldots \; n-1 \; n\]
- $B_n$: \[•\; 1 \; 2 \; \ldots \; n-1 \; n\]
- $C_n$: \[•\; 1 \; 2 \; \ldots \; n-1 \; n\]
- $D_n$, $n$ odd: \[\begin{array}{c}
\; 1 \; 2 \; \ldots \; n-2 \; n-1 \\
\; n\end{array}\]
- $D_n$, $n$ even: \[\begin{array}{c}
\; 1 \; 2 \; \ldots \; n-2 \; n-1 \\
\; n\end{array}\]
- $E_6$: \[• \; 1 \; 2 \; 3 \; 4 \; 5 \; 6\]
- $E_7$: \[• \; 1 \; 2 \; 3 \; 4 \; 5 \; 6 \; 7\]
- $E_8$: \[• \; 1 \; 2 \; 3 \; 4 \; 5 \; 6 \; 7 \; 8\]
- $F_4$: \[• \; 1 \; 2 \; 3 \; 4\]
- $G_2$: \[\begin{array}{c}
\; 1 \; 2\end{array}\]
Set \( q_i = q^{d_i} \) and \( [m]_i = [m]_{q_i} \). The quantum loop algebra \( U \) of \( \mathfrak{g} \) is the algebra with generators \( x^{\pm}_{i,r} \) (\( i \in I, r \in \mathbb{Z} \)), \( K_i^{\pm 1} \) (\( i \in I \)), \( h_{i,r} \) (\( i \in I, r \in \mathbb{Z} \setminus \{0\} \)) and the following defining relations:

\[
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \\
K_i h_{j,r} = h_{j,r} K_i, \\
K_i x_{j,r}^\pm K_i^{-1} = q_i^{a_{ij}} x_{j,r}^\pm,
\]

\[
[h_{i,r}, h_{j,s}] = 0, \quad [h_{i,r}, x_{j,s}^\pm] = \pm \frac{1}{r} [r a_{ij}] q_i^{\pm x_{j,r+s}^\pm},
\]

\[
x_{i,r+1}^\pm x_{j,s}^\pm - q_i^{\pm a_{ij}} x_{j,s}^\pm x_{i,r+1}^\pm = q_i^{\mp a_{ij}} x_{j,s+1}^\pm - x_{j,s+1}^\pm x_{i,r}^\pm
\]

\[
[x_{i,r}^+, x_{j,s}^-] = \delta_{i,j} \frac{\psi_{i,r+s}^+ - \psi_{i,r+s}^-}{q_i - q_i^{-1}},
\]

\[
\sum_{\pi \in \Sigma_m} \sum_{k=0}^m (-1)^k \left[ \begin{array}{c} m \\ k \end{array} \right] x_{i,r_1}^\pm \cdots x_{i,r_k}^\pm x_{i,r_{k+1}}^\pm \cdots x_{i,r_{m+1}}^\pm = 0, \quad \text{if } i \neq j,
\]

for all sequences of integers \( r_1, \ldots, r_m, \) where \( m = 1 - a_{ij}, \Sigma_m \) is the symmetric group on \( m \) letters, and the \( \psi_{i,r}^\pm \) are determined by equating powers of \( u \) in the formal power series

\[
\sum_{r=0}^{\infty} \psi_{i,r}^\pm u^{r} = K_i^{\pm 1} \exp \left( \pm (q_i - q_i^{-1}) \sum_{s=1}^{\infty} h_{i,\pm s} u^{s} \right).
\]

1.5. For \( i \in I \), set

\[
h_i^\pm(u) = \sum_{k=1}^{\infty} \frac{q_i^k h_i,\pm k}{|k|_i} u^k,
\]

and define elements \( P_{i,\pm k}, i \in I, k \in \mathbb{Z}, k \geq 0 \), by the generating series,

\[
P_{i,\pm k}(u) = \sum_{k=0}^{\infty} P_{i,\pm k} u^k = \exp(-h_i^{\pm}(u)).
\]

Let \( U^{\pm}(0) \) be the subalgebra of \( U \) generated by the elements \( h_{i,\pm k} \) \( i \in I, k \in \mathbb{Z}, k > 0 \). It is easy to see that \( U^{\pm}(0) \) are commutative subalgebras of \( U \) and that monomials in the \( h_{i,\pm k}, i \in I, k \in \mathbb{Z} \) form a basis of \( U^{\pm}(0) \). Notice also that \( U^{\pm}(0) \) is also the subalgebra generated by the elements \( P_{i,\pm k}, i \in I, k \in \mathbb{Z}, k > 0 \). The subalgebra of \( U \) generated by \( x_{i,0}^\pm \) and \( K_i^{\pm 1}, i \in I \) is isomorphic to the quantized enveloping algebra \( U^{q_{i,0}} \) of \( \mathfrak{g} \). For each \( i \in I \), the subalgebra \( U_{i,0} \) of \( U \) generated by the elements \( x_{i,r}^\pm, K_i^{\pm 1}, h_{i,s}, r \in \mathbb{Z}, s \neq 0 \) is isomorphic to the quantum loop algebra of \( \mathfrak{sl}_2 \).

1.6. It is well–known that \( U \) has the structure of a Hopf algebra, let \( \Delta : U \to U \otimes U \) be the comultiplication. Although, explicit formulas for the comultiplication on the generators \( x_{i,k}^\pm, h_{i,r}, \) are not known, the next proposition proved in \( \mathfrak{g} \) Proposition 5.3, \( \mathfrak{g} \) Proposition 5.4 gives partial information that suffices for our purposes.

Define subspaces \( X^{\pm} \) of \( U \), by

\[
X^{\pm} = \sum_{i \in I, r \in \mathbb{Z}} C x_{i,r}^{\pm}.
\]
Proposition. Modulo $UX^- \otimes UX^+$, we have
\[ \Delta(h_{i,s}) = h_{i,s} \otimes 1 + 1 \otimes h_{i,s} \quad (s \in \mathbb{Z}_+, s > 0), \]
\[ \Delta(P_{i,r}) = \sum_{m=0}^{r} P_{i,r-m} \otimes P_{i,m} \quad (r \in \mathbb{Z}_+, r > 0). \]

2. Braid group actions, the $\ell$–weight lattice and the $\ell$–root lattice

In this section we introduce the notion of $\ell$–integral weights and $\ell$–roots. These are certain multiplicative subgroups of the ring of algebra homomorphisms $\text{Hom}(U(0), C)$.

2.1. Set
\[ \mathcal{A} = \{ f \in C[[u]] : f(0) = 1 \}, \]
where $C[[u]]$ is the ring of formal power series in an indeterminate $u$. Clearly $\mathcal{A}$ is a group under multiplication. Given $f \in \mathcal{A}$ and $r \in \mathbb{Z}^+$, we let $f_r$ denote the coefficient of $u^r$ in $f$. Let
\[ \mathcal{A} = \mathcal{A} \cap C(u). \]
Then $\mathcal{A}$ is a free subgroup of $\mathcal{A}$ with generators $\{ 1 - au : a \in C^\times \}$. Given $f^+ \in C[u]$ with $f^+(0) = 1$, define $f^- \in C[u]$ by
\[ f^-(u) = u^{\deg f^+} f^+(u^{-1})/(u^\deg f^+ f^+(u))|_{u=0}. \]
Given $\varpi^+ = f^+/g^+ \in \mathcal{A}$ set $\varpi^- = f^-/g^-$. Define an injective group homomorphism $\iota : \mathbb{A}^n \rightarrow \text{Hom}(U(0), C)$ by extending,
\[ \iota(\varpi)(P^\pm_i(u)) = \varpi^\pm_i, \]
where $\varpi = (\varpi_1^+, \ldots, \varpi_n^+) \in \mathbb{A}^n$ and the equality is one of power series. We call $\iota(\mathbb{A}^n)$ the $\ell$–integral weight lattice of $U$ and henceforth denote it by $P_q$ in what follows we shall identify $P_q$ and $\mathbb{A}^n$ and denote elements of $P_q$ as $n$–tuples of elements from $\mathcal{A}$. For $i \in I$ and $a \in C^\times$, let $\omega_{i,a} \in P_q$ denote the element whose $i^{th}$ entry is $1 - au$ and all other entries 1. We call these the $\ell$–fundamental weights. It is obvious that $P_q$ is generated freely as an abelian group by the elements $\omega_{i,a}$, $i \in I$, $a \in C^\times$. Let $P_q^+$ denote the monoid generated by 1 and the elements $\omega_{i,a}$, $i \in I$, $a \in C^\times$, clearly $P_q^+$ is isomorphic to the monoid in $\mathbb{A}^n$ consisting of $n$–tuples of polynomials with constant term one and we call such elements $\ell$–dominant weights.

Definition. Let $\text{wt} : P_q \rightarrow P$ be the group homomorphism defined by extending,
\[ \text{wt}(\omega_{i,a}) = \omega_i. \]

2.2. We now define an action of a braid group on $P_q$. Let $B$ be the group generated by elements $T_i$ ($i \in I$) with defining relations:
\[ T_i T_j = T_j T_i, \quad \text{if } a_{ij} = 0, \]
\[ T_i T_j T_i = T_i T_j T_i, \quad \text{if } a_{ij} a_{ji} = 1, \]
\[ (T_i T_j)^2 = (T_j T_i)^2, \quad \text{if } a_{ij} a_{ji} = 2, \]
\[ (T_i T_j)^3 = (T_j T_i)^3, \quad \text{if } a_{ij} a_{ji} = 3, \]
where $i, j \in \{1, 2, \ldots, n\}$. The next proposition is a reformulation of [5, Proposition 3.1] and can be easily checked.
Proposition. The following formulas define an action of $B$ on $P_q$: let $\mathfrak{w} = (\mathfrak{w}_1, \cdots, \mathfrak{w}_n) \in P_q$, then $T_i(\mathfrak{w})$ is defined by,

\[
(T_i(\mathfrak{w}))_j = \mathfrak{w}_j, \quad \text{if } a_{ji} = 0,
\]

\[
(T_i(\mathfrak{w}))_j = \mathfrak{w}_j(u)\mathfrak{w}_i(qu), \quad \text{if } a_{ji} = -1,
\]

\[
(T_i(\mathfrak{w}))_j = \mathfrak{w}_j(u)\mathfrak{w}_i(q^2 u)\mathfrak{w}_i(qu), \quad \text{if } a_{ji} = -2,
\]

\[
(T_i(\mathfrak{w}))_j = \mathfrak{w}_j(u)\mathfrak{w}_i(q^3 u)\mathfrak{w}_i(q^2 u)\mathfrak{w}_i(qu), \quad \text{if } a_{ji} = -3,
\]

\[
(T_i(\mathfrak{w}))_i = \frac{1}{\mathfrak{w}_i(q_i^2 u)}.
\]

\[
2.3. \quad \text{For } i \in I, \text{ set }
\]

\[
\alpha_{i,a} = (T_i(\mathfrak{w}_{i,a}))^{-1}\mathfrak{w}_{i,a}.
\]

Clearly we have $\alpha_{i,a} \in P_q$ and we let $Q_q$ be the subgroup of $P_q$ generated by the $\alpha_{i,a}$. We call $\alpha_{i,a}$ the $\ell$–simple roots and $Q_q$ the $\ell$–root lattice and let $Q_q^+$ be the monoid generated by $\alpha_{i,a}, i \in I, a \in C^\times$, and $Q_q^- = (Q_q^+)^{-1}$. Given $f \in C[u]$, say $f = (1 - a_{1u}) \cdots (1 - a_{ru}), a_1, \cdots, a_r \in C^\times$, set

\[
\alpha_{i,f} = \prod_{m=1}^r \alpha_{i,a_m},
\]

and

\[
\alpha_{i,f/g} = \alpha_{i,f}(\alpha_{i,g})^{-1}.
\]

Finally, given $\mathfrak{w} \in A^n$, set

\[
\alpha_{i,\mathfrak{w}} = \alpha_{i,(\mathfrak{w})_i}.
\]

It is now clear that Proposition 2.2 is equivalent to,

\[
(2.1) \quad T_i(\mathfrak{w}) = \mathfrak{w}(\alpha_{i,\mathfrak{w}})^{-1}, \quad \forall \mathfrak{w} \in A^n.
\]

In particular, we have

Proposition. The action of the braid group on $P_q$ preserves $Q_q$. \hfill \square

2.4. We list the simple roots for the various classical Lie algebras below for the reader’s convenience. If $g$ is of type $A_n$, then

\[
\alpha_{i,a}(u) = \omega_{i-1, aq^2}^{-1}\omega_{i, aq}\omega_{i, aq^2}\omega_{i+1, aq^2}, \quad i \in I
\]

where we understand $\omega_{-1, a} = \omega_{n+1, a} = 1$.

If $g$ is of type $B_n$, then

\[
\alpha_{i,a} = (\omega_{i-1, aq^2})^{-1}\omega_{i, a}\omega_{i, aq^4}(\omega_{i+1, aq^2})^{-1}, \quad i \in I, \quad i \neq n - 1, n,
\]

\[
\alpha_{n-1,a} = (\omega_{n-2, aq^2})^{-1}\omega_{n-1, a}\omega_{n-1, aq}(\omega_{n, aq}\omega_{n, aq}^{-1})^{-1},
\]

\[
\alpha_{n,a} = (\omega_{n-1, aq})^{-1}\omega_{n, a}\omega_{n, aq^2}.
\]

If $g$ is of type $C_n$, then

\[
\alpha_{i,a} = (\omega_{i-1, aq})^{-1}\omega_{i, a}\omega_{i, aq^4}(\omega_{i+1, aq})^{-1}, \quad i \in I, \quad i \neq n,
\]

\[
\alpha_{n,a} = (\omega_{n-1, aq}\omega_{n-1, aq}^{-1})^{-1}\omega_{n, a}\omega_{n, aq^4}.
\]
If \( \mathfrak{g} \) is of type \( \mathfrak{D}_n \), then
\[
\alpha_{i,a} = (\omega_{i-1,aq})^{-1} \omega_{i,a} \omega_{i,aq}^2 (\omega_{i+1,aq})^{-1}, \quad i \in I, \ i \neq n - 2, n - 1, n,
\]
\[
\alpha_{n-2,a} = (\omega_{n-3,aq})^{-1} \omega_{n-2,a} \omega_{n-2,aq}^2 (\omega_{n-1,aq} \omega_{n,aq})^{-1},
\]
\[
\alpha_{n-1,a} = (\omega_{n-2,aq})^{-1} \omega_{n-1,a} \omega_{n-1,aq}^2,
\]
\[
\alpha_{n,a} = (\omega_{n-2,aq})^{-1} \omega_{n,a} \omega_{n,aq}^2.
\]

**Remark.** The elements \( \alpha_{i,a} \) are essentially the elements \( A_{i,a} \) defined in [21].

**2.5.** Let \( w \in W \) and assume that \( w = s_{i_1} s_{i_2} \cdots s_{i_k} \) is a reduced expression for \( w \). Set \( T_w = T_{i_1} \cdots T_{i_k} \). It is well–known that \( T_w \) is independent of the choice of the reduced expression for \( w \). Given \( \varpi \in \mathfrak{A}^n \) and \( w \in W \), we have
\[
T_w \varpi = T_{i_1} T_{i_2} \cdots T_{i_k} \varpi = ((T_w \varpi)_1, \ldots, (T_w \varpi)_n).
\]

**Lemma.**

(i) For all \( w \in W \), and \( \varpi \in \mathcal{P}_q \), we have
\[
\text{wt}(T_w \varpi) = w(\text{wt}(\varpi)).
\]

(ii) Suppose that \( \varpi, r \in \mathcal{P}_q \), \( r = 1, 2 \) are such that \( \text{wt}(\varpi, r) \in \mathbb{P}^+ \), \( r = 1, 2 \). Then
\[
T_{w_1} \varpi_1 = T_{w_2} \varpi_2 \implies \text{wt}(\varpi_1) = \text{wt}(\varpi_2).
\]

Further if we set \( \lambda = \text{wt}(\varpi_1) \) then
\[
w_1^{-1}w_2 \in W(\lambda).
\]

**Proof.** It suffices to check the result when \( T_w = T_j \) and \( \varpi = \omega_{i,a} \) for some \( i, j \in I \) and \( a \in \mathbb{C}^\times \). But this is now immediate from Proposition 2.6. To prove (ii), notice that (i) implies that
\[
w_1(\text{wt}(\varpi_1)) = w_2(\text{wt}(\varpi_2)).
\]

Since \( \text{wt}(\varpi_r) \in \mathbb{P}^+ \), this implies that \( \text{wt}(\varpi_1) = \text{wt}(\varpi_2) \) and hence also that \( w_1^{-1}w_2 \in W(\lambda) \). \( \square \)

**2.6.** The following proposition can be deduced from the results of [3], using some representation theory. However, an elementary case by case proof is sketched below.

**Proposition.** Let \( \omega \in \mathcal{P}_q^+ \). Then, \( (T_{w_0} \omega)^{-1} \in \mathcal{P}_q^+ \). More precisely, we have
\[
(T_{w_0} \omega)_i = ((\omega)_{w_0} (q^h \omega))^{-1}.
\]

**Proof.** Assume first that \( \mathfrak{g} \) is of type \( \mathfrak{A}_n \), then \( w_0 \) has a reduced expression of the form
\[
w_0 = s_1 \cdots s_n s_1 \cdots s_{n-1} \cdots s_1 s_2 s_1.
\]
Proceed by induction on \( n \), noting that induction obviously begins at \( n = 1 \). Since
\[
w'_0 = s_1 \cdots s_{n-1} \cdots s_1 s_2 s_1
\]
is the reduced expression for the longest element of \( \mathfrak{A}_{n-1} \), we can assume by induction that
\[
T_{w'_0} (\omega)_j = (\omega)_{n-j} (q^h \omega)^{-1}, \quad 1 \leq j \leq n - 1,
\]
\[
T_{w'_0} (\omega)_n = (\omega)_n (q^{n-1} \omega)_n.
\]
A simple computation now gives the result. The proof for the other Lie algebras is similar working with the reduced expressions for \( w_0 \) given in [3, Section 6]. \( \square \)
2.7.

**Lemma.** Suppose that \( \varpi, \varpi' \in \mathcal{P}_q \) are such that \( \varpi' = T_w \varpi \) for some \( w \in W \). Then
\[
\varpi(\varpi')^{-1} \in \mathcal{Q}_q.
\]
Moreover if \( \varpi \in \mathcal{P}_q^\pm \), then
\[
\varpi(\varpi')^{-1} \in \mathcal{Q}_q^\pm.
\]

**Proof.** It suffices to prove the lemma when \( w = s_i \) and \( \varpi = \omega_{j,a} \) for some \( i, j \in I \). If \( j = i \), the result is immediate from the definition of \( \alpha_{i,a}(u) \). If \( j \neq i \), then the result is immediate since \( T_i(\omega_{j,a}) = \omega_{j,a} \). \( \square \)

2.8.

**Lemma.** Given \( \varpi \in \mathcal{P}_q \) there exists \( \omega \in \mathcal{P}_q^+ \) such that \( \omega(\varpi)^{-1} \in \mathcal{Q}_q^+ \).

**Proof.** It clearly suffices to prove the lemma in the case when \( \varpi = \omega_{i,a}^{-1} \) for some \( i \in I, a \in \mathbb{C}^\times \). By Proposition 2.6 we see that \( T_{w_0}(\omega_{i,a})^{-1} = \omega_{w_0,i,aq}^{-1} \). The result is now immediate from Lemma 2.7. \( \square \)

3. The group \( \mathcal{P}_q/\mathcal{Q}_q \)

In this section we give a set of generators and relations for the quotient group \( \mathcal{P}_q/\mathcal{Q}_q \) in the case when \( g \) is of type \( A_n, B_n, C_n \) or \( D_n \). The case of the exceptional algebras is postponed to the appendix.

3.1.

**Proposition.**

(i) Assume that \( g \) is of type \( A_n, B_n, C_n \) or \( D_m \), where \( m \) is odd. The group \( \mathcal{P}_q/\mathcal{Q}_q \) is isomorphic to the (additive) abelian group \( \Xi_q \) with generators \( \{ \chi_a : a \in \mathbb{C}^\times \} \) and relations:
\[
\sum_{r=0}^{n} \chi_{aq^{n+1-2r}} = 0, \quad \text{if } g = A_n,
\]
\[
\chi_a + \chi_{aq^{n-2}} = 0, \quad \text{if } g = B_n,
\]
\[
\chi_a + \chi_{aq^{n+2}} = 0, \quad \text{if } g = C_n,
\]
\[
\chi_a + \chi_{aq^2} + \chi_{aq^{2m-2}} + \chi_{aq^{2m}} = 0, \quad \text{if } g = D_m,
\]
for all \( a \in \mathbb{C}^\times \).

(ii) If \( g \) is of type \( D_m \) with \( m \) even, then \( \mathcal{P}_q/\mathcal{Q}_q \) is isomorphic to the (additive) abelian group with generators \( \{ \chi_a^\pm : a \in \mathbb{C}^\times \} \) and relations:
\[
\chi_a^+ + \chi_{aq^{2m-2}} = 0, \quad \chi_a^- + \chi_{aq^2} + \chi_{aq^{2m-2}} + \chi_{aq^{2m}} = 0,
\]
for all \( a \in \mathbb{C}^\times \).

The proposition is proved in the rest of the section.
3.2. We begin with the following Lemma whose proof is an obvious computation.

Lemma.

(i) Suppose that $g$ is of type $A_n$ or $C_n$ (resp. $B_n, D_n$). Then, for all $i \in I$, (resp $i \neq n$, $i \neq n-1, n$) we have,

$$\omega_{1,aq^{i-1}} \omega_{1,aq^{i-1}} = \omega_{i,a} \left( \prod_{j=1}^{i-1} \alpha_{j,aq^{i-j-2}} \right).$$

Further,

$$\omega_{1,aq^{-n-1}} \omega_{n,a} = \prod_{j=1}^{n} \alpha_{j,aq^{-n+j-2}}, \text{ if } g = A_n,$$

and

$$\omega_{1,aq^{n+i}} \omega_{1,aq^{n-1}} = \left( \prod_{j=1}^{n-1} \alpha_{j,aq^{-j}} \alpha_{j,aq^{-n+j-2}} \right) \alpha_{n,aq^{-2}}, \text{ if } g = C_n.$$

(ii) Assume that $g$ is of type $B_n$. Then,

$$\omega_{n,aq^{2i-1}} \omega_{n,aq^{2i-1}} = \omega_{n-1,a} \left( \prod_{j=n-i+1}^{n-1} \prod_{r=0}^{j-(n-i+1)} \alpha_{j,aq^{2(n-i-r)+4r}} \right) \left( \prod_{r=0}^{i-1} \alpha_{n,aq^{-2i+4r}} \right).$$

Further,

$$\omega_{n,aq^{2n-1}} \omega_{n,aq^{2n-1}} = \left( \prod_{j=1}^{n-1} \prod_{r=0}^{j-1} \alpha_{j,aq^{2j+4r}} \right) \left( \prod_{r=0}^{n-1} \alpha_{n,aq^{-2n+1+4r}} \right).$$

(iii) Assume that $g$ is of type $D_n$. Then, for $j = 1, 2, \ldots, \lfloor (n-1)/2 \rfloor$,

$$\omega_{n,aq^{2j-1}} \omega_{n,aq^{2j-1}} = \omega_{n-2j,a} \left( \prod_{k=n-2j+1}^{n-2} \prod_{r=0}^{k-(n-2j+1)} \alpha_{k,aq^{n-2j+k+2r}} \right) \times \left( \prod_{r=0}^{j-2} \alpha_{n-1,aq^{-2j+3+4r}} \right) \left( \prod_{r=0}^{j-1} \alpha_{n,aq^{-2j+1+4r}} \right).$$

Also, for $j = 1, 2, \ldots, \lfloor (n-2)/2 \rfloor$,

$$\omega_{n-1,aq^{2j}} \omega_{n,aq^{2j}} = \omega_{n-2j-1,a} \left( \prod_{k=n-2j}^{n-2} \prod_{r=0}^{k-(n-2j)} \alpha_{k,aq^{n-2j-1-k+2r}} \right) \left( \prod_{r=0}^{j-1} \alpha_{n-1,aq^{-2j+2+4r}} \alpha_{n,aq^{-2j+4r}} \right).$$

Similar formulas hold interchanging $n$ and $n-1$ on the left hand side. In addition, if $n$ is odd, we have

$$\omega_{n,aq^{2j}} \omega_{n,a} \omega_{n,aq^{2n+4}} = \omega_{n-1,a} \left( \prod_{k=1}^{n-2k-1} \prod_{r=0}^{k} \alpha_{k,aq^{3-n+k-2r}} \right) \left( \prod_{r=0}^{j-1} \alpha_{n-1,aq^{6-2n+4r}} \right) \times \left( \prod_{r=0}^{n-2} \alpha_{n,a} \prod_{r=0}^{n,aq^{-2n+4r}} \right),$$
The inductive step is now easily completed using Lemma 3.2(i) again. Thus we have a homomorphism
\[ \omega_{n-1,aq^{n-1}}\omega_{n,aq^{-(n-1)}} = \left( \prod_{k=1}^{n-2k-1} \prod_{r=0}^{k} \alpha_{k,aq^{-k+2r}} \right) \left( \prod_{r=0}^{n-3} \alpha_{n-1,aq^{3-n+4r}} \right), \]
and
\[ \omega_{n,a}\omega_{n,aq^2}\omega_{n,aq^{2n-2}}\omega_{n,aq^{2n}} = \left( \prod_{k=1}^{n-2k-1} \prod_{r=0}^{k} \alpha_{k,aq^{-k+2r}} \right) \left( \prod_{r=0}^{n-3} \alpha_{n-1,aq^{3-n+4r}} \right) \times \left( \prod_{r=0}^{n-1} \alpha_{n,aq^{2r}} \right). \]

If \( n \) is even we have
\[ \omega_{n,aq^{n-1}}\omega_{n,aq^{-(n-1)}} = \left( \prod_{k=1}^{n-2k-1} \prod_{r=0}^{k} \alpha_{k,aq^{-k+2r}} \right) \left( \prod_{r=0}^{n-3} \alpha_{n-1,aq^{3-n+4r}} \right) \left( \prod_{r=0}^{n-2} \alpha_{n,aq^{2r}} \right), \]
and
\[ \omega_{n-1,a}\omega_{n-1,aq^2}\omega_{n,aq^{2n-2}}\omega_{n,aq^{2n}} = \left( \prod_{k=1}^{n-2k-1} \prod_{r=0}^{k} \alpha_{k,aq^{-k+2r}} \right) \left( \prod_{r=0}^{n-2} \alpha_{n-1,aq^{2r}} \alpha_{n,aq^{2r+2r}} \right). \]

3.3. We can now prove Proposition 3.1.

Proof. Assume first that \( g \) is of type \( A_n \). We claim that the assignment \( \chi_a \rightarrow \omega_{1,a} \) defines a homomorphism \( \tau : \Xi_q \rightarrow P_q/Q_q \). For this, it is enough to check that for all \( a \in C^\times \),
\[ (3.1) \]
\[ \prod_{r=0}^{n} \omega_{1,aq^{n+1-2r}} \in Q_q^+. \]

By using Lemma 3.2(i) repeatedly we see that
\[ \omega_{n,q} = \left( \prod_{r=0}^{n-1} \omega_{1,aq^{-2r}} \right) (\varpi)^{-1}, \]
for some \( \varpi \in Q_q^+ \). Hence to prove (3.1) it suffices to observe from Lemma 3.2(i) that
\[ \omega_{1,q^{-n-1}}\omega_n = \prod_{j=1}^{n} \alpha_{j,q^{-n-j-2}}. \]

To see that this is an isomorphism of groups, consider first the homomorphism \( P_q \rightarrow \Xi_q \) given by mapping
\[ \omega_{i,a} \mapsto \sum_{r=0}^{i-1} \chi_{aq^{-r+i+1}}. \]

We claim that \( Q_q \) is in the kernel of this map. For this it is enough to prove that \( \alpha_{i,a} \) is in the kernel. We prove this by induction on \( i \). If \( i = 1 \), then the result follows since \( \omega_{2,aq} = \omega_{1,aq^2}\omega_{1,a}^{-1}. \) The inductive step is now easily completed using Lemma 3.2(i) again. Thus we have a homomorphism \( P_q/Q_q \rightarrow \Xi_q \) which is clearly an inverse of \( \tau \) and we are done.

Assume next that \( g \) is of type \( B_n \). Define a group homomorphism \( P_q \rightarrow \Xi_q \) by extending
\[ \omega_{i,a} \mapsto \chi_{aq^{2n-2i-1}} + \chi_{a,q^{-2n+2i+1}} \]
for \( i < n \), \( \omega_{n,a} \mapsto \chi_a. \)

Using Lemma 3.2(ii) we see by an induction starting at \( n \) that \( \alpha_{j,a} \) is the kernel of this map and hence we get a homomorphism from \( P_q/Q_q \rightarrow \Xi_q \). To see that this map is an isomorphism it suffices to show
as in the case of $A_n$ that the assignment $\chi_a \mapsto \omega_{n,a}$ defines a homomorphism $\Xi_q \to P_q/Q_q$. For this, it is enough to show that

$$\omega_{n,a} \omega_{n,aq^{n-2}} \in Q_q^+,$$

which is just the second statement in Lemma 3.2(ii).

If $g$ is of type $C_n$, then we show by using Lemma 3.2(i) that the map $\chi_a \mapsto \omega_{1,a}$ gives an isomorphism between $\Xi_q$ and $P_q/Q_q$. We omit the details. □

**Remark.** If $q = 1$, note that $\Xi_1$ is isomorphic to the group of functions from $C^\times \to \Gamma$ with finite support (cf. [4]).

4. The $\ell$–weights of finite–dimensional representations

4.1. Given a $U$-module $V$ and $\mu = \sum_i \mu_i \omega_i \in P$, set

$$V_\mu = \{v \in V : K_i.v = q_i^{\mu_i}v, \ \forall \ i \in I\}.$$  

We say that $V$ is a module of type 1 if

$$V = \bigoplus_{\mu \in P} V_\mu.$$  

Analogous definitions hold for representations of $U^{fin}$. Recall from [20] that for every $\lambda \in P^+$, there exists a unique (up to isomorphism) irreducible finite–dimensional representation of $U^{fin}$ which we denote by $V(\lambda)$. Let $C_q$ be the abelian category consisting of type 1 finite–dimensional representations of $U$. We set

$$\text{wt}(V) = \{\mu \in P : V_\mu \neq 0\},$$

and given $v \in V_\mu$ we set $\text{wt}(v) = \mu$.

4.2. **Definition.** Let $V$ be a $U$-module. We say that $\varpi \in A^n$ is an $\ell$–weight of $V$ if there exists a non–zero element $v \in V$ such that

$$(P_{i,r} - (\varpi_i)r)^N v = 0, \quad N \equiv N(i,r,v) \in Z_+,$$

for all $i \in I$ and $r \in Z^+$ and we call $v$ an $\ell$–weight vector in $V$ with $\ell$–weight $\varpi$. Denote the subspace consisting of all $\ell$–weight vectors with $\ell$–weight $\varpi$ by $V_{\varpi}$.

**Remark.** We shall see later in the section, (see Proposition 4.10), that the generalized eigenspaces for the action of the $P_{i,-r}$, $i \in I$, $r \in Z_+$, are actually determined uniquely by those of the $P_{i,r}$, $r \in Z_+$.

4.3. The following lemma is trivially established.

**Lemma.** Let $V \in C_q$. We have

$$V = \bigoplus_{\varpi \in A^n} V_{\varpi}, \quad V_\mu = \bigoplus_{\varpi \in A^n} V_{\varpi} \cap V_\mu.$$

□

Denote by $\text{wt}_\ell(V)$ the set of $\ell$–weights of $V$ and define $\text{wt}_\ell(v)$ in the obvious way. It is obvious from the definition of $\ell$–weights that any morphism between objects of $C_q$ preserves $\ell$–weight spaces.
4.4. We now study the behavior of \( \ell \)-weights under tensor products. This is essentially the same proof given in [21], we include it here for completeness.

**Lemma.** Let \( V_r \in \mathcal{C}_q \), \( r = 1, 2 \) and let \( v_{i,r}, 1 \leq j \leq \text{dim}(V_r) \) be a basis of \( V_r \), \( r = 1, 2 \) such that \( \text{wt}(v_{i,r}) \) and \( \text{wt}_\ell(v_{i,r}) \) are defined and assume that if \( j < j^\prime \) then \( \text{wt}(v_{i,r}) - \text{wt}(v_{i,r}^\prime) \in Q^+ \). Then the \( \ell \)-weight vectors of \( V_1 \otimes V_2 \) are of the form

\[
v_{i,1} \otimes v_{j,2} + \text{(terms in } \bigoplus (V_1)_{\nu_1} \otimes (V_2)_{\nu_2},)
\]

where the direct sum is over \( \nu_r \in \text{wt}(V_r), r = 1, 2 \) and \( \nu_2 - \text{wt}(v_2) \in Q^+ \). The corresponding \( \ell \)-weight is \( \text{wt}_\ell(v_{i,1})\text{wt}_\ell(v_{j,2}). \) In particular,

\[
\text{wt}_\ell(V_1 \otimes V_2) = \text{wt}_\ell(V_1)\text{wt}_\ell(V_2).
\]

**Proof.** It is easy to see from Proposition 1.6 that the matrices of the action of \( P_{i,s}, i \in I, s \in \mathbb{Z}_+ \) on \( V_1 \otimes V_2 \) with respect to the basis \( v_{i,1} \otimes v_{j,2} \) are simultaneously upper triangular with diagonal entries given by \( \text{wt}_\ell(v_{i,1})\text{wt}_\ell(v_{j,2}). \) The result is now immediate. \( \square \)

4.5. We need several results on irreducible representations of \( U \). We begin with the definition of an \( \ell \)-highest weight module.

**Definition.** We say that a \( U \)-module \( V \in \mathcal{C}_q \) is \( \ell \)-highest weight with \( \ell \)-highest weight \( \varpi \in \mathcal{P}_q \) if there exists a non-zero vector \( 0 \neq v \in V \) such that \( V = \mathcal{U}v \) and,

\[
(4.1) \quad x^+_{i,v} v = 0, \quad P^\pm_i(u) v = (\varpi)_i^\pm v, \quad K^\pm_1 v = q^{\pm \text{wt}(\varpi)}(\alpha_i)v, \quad (x^-_{i,v})^{\text{wt}(\varpi)}(\alpha_i)v = 0,
\]

for all \( i \in I, r \in \mathbb{Z} \). The element \( v \) is called the \( \ell \)-highest weight vector.

The following lemma is standard.

**Lemma.** Any \( \ell \)-highest weight module has a unique irreducible quotient which is also a highest weight module with the same highest weight.

4.6. The following was proved in [9, 10].

**Theorem.**

(i) Any irreducible module in \( \mathcal{C}_q \) is \( \ell \)-highest weight.

(ii) There exists a bijective correspondence between elements of \( \mathcal{P}_q^+ \) and isomorphism classes of irreducible modules in \( \mathcal{C}_q \). \( \square \)

**Corollary.** Let \( V \in \mathcal{C}_q \) be a highest weight module with highest weight \( \varpi \). Then \( \varpi \in \mathcal{P}_q^+ \). \( \square \)

4.7. Given \( \varpi \in \mathcal{P}_q^+ \), let \( V(\varpi) \in \mathcal{C}_q \) be an element in the corresponding isomorphism class, and let \( v_\varpi \) be the \( \ell \)-highest weight vector. We note the following simple consequence of Theorem 4.6.

**Lemma.** Let \( \varpi \in \mathcal{P}_q^+ \).

(i) We have \( V(\varpi)_{\text{wt} \varpi} = C v_\varpi \).

(ii) If \( v \in V(\varpi) \) is such that \( x^+_{i,k} v = 0 \) for all \( k \leq n, k \in \mathbb{Z} \), then \( v = cv_\varpi \) for some \( c \in C \).

(iii) Let \( V \in \mathcal{C}_q, \varpi = (\varpi_1, \ldots, \varpi_n) \in \mathcal{C}_q \) and assume that \( v \in V(\varpi), v \neq 0, \) is such that \( x^+_{i,k} v = 0 \) for some \( i \in I \) and all \( k \in \mathbb{Z} \). Then, \( \varpi_i \in Cu \). \( \square \)
4.8. We now consider the case when \( \mathfrak{g} = \mathfrak{sl}_2 \). Given \( m > 0 \), let \( \omega_a(m) \in \mathcal{P}_q^+ \) be the polynomial
\[
\omega_a(m) = (1 - aq^{m-1}u)(1 - aq^{m-3}u) \cdots (1 - aq^{-m+1}u),
\]
and set \( \omega_a(0) = 1 \). The following result was proved in [13].

**Theorem.** Suppose that \( V \in \mathcal{C}_q \) is an \( \ell \)-highest weight module with highest weight \( \omega = \prod_{r=1}^k (1 - a_r u) \) where \( a_r \in \mathbb{C}^\times \) is such that \( a_r/a_r \neq q^{r'} \) if \( r' < r \). Then \( V \) is a quotient of \( V(\omega_{a_1}(1)) \otimes \cdots \otimes V(\omega_{a_k}(1)) \).

\( \square \)

4.9.

**Proposition.**

(i) There exists an isomorphism of \( \mathcal{U}^{fin} \)-modules
\[
V(\omega_a(m)) \cong V(m\omega_1).
\]

(ii) The eigenvalues of \( P^\pm(u) \) on \( V(\omega_a(m)) \) are \( \varpi_{a,r}(m)^\pm \), \( 0 \leq r \leq m \), where
\[
\varpi_{a,r}(m)^\pm = \omega_{aq^{-r}(m-r)^\pm}(\omega_{aq^{m-r+2}(r)^\pm})^{-1}.
\]

In particular \( \varpi_{a,m} = \varpi_{aq^{-1}m}(m^{-1}) \).

(iii) Any irreducible \( \mathcal{U} \)-module is isomorphic to a tensor product \( V(\omega_{a_1}(m_1)) \otimes \cdots \otimes V(\omega_{a_r}(m_r)) \) where
\[
a_k/a_s \neq q^{p(m_k+m_s-2p)}, \quad 0 \leq p < \min\{m_k, m_s\},
\]
for some \( a_1, \ldots, a_r \in \mathbb{C}^\times \) and \( m_1, \ldots, m_r \in \mathbb{Z}^+ \).

**Proof.** Parts (i) and (iii) were proved in [9]. Part (ii) was proved in a slightly different form in [21]. We include a proof here for the reader’s convenience.

We proceed by induction on \( m \). If \( m = 1 \), let \( v_0, v_1 \) be the basis for \( V(\omega_a(1)) \), where \( v_0 \) is the \( \ell \)-highest weight vector. It is now a simple computation to check, using the formulas in [9] to check that the eigenvalues of \( P^\pm(u) \) on \( v_1 \) are \( \varpi_{aq^{-1}m}(1)^{-1} = (1 - aq^{2}u)^{-1} \). Assume now that we know the result for all \( s < m \). By [9], we know that there exists a short exact sequence,
\[
0 \to V(\omega_{aq^{-2}(m-2)}) \to V(\omega_{aq^{-2}(m-1)}) \to V(\omega_{aq^{-1}(m-1)}) \to V(\omega_{aq^{-1}(m)}) \to 0.
\]
Let \( v_0, v_1 \) be a basis for \( V(\omega_{aq^{-2}(m-1)}) \) and \( w_0, \ldots, w_{m-1} \) a basis for \( V(\omega_{aq^{-1}(m-1)}) \). Then, the elements \( w_j \otimes v_0 \), \( 0 \leq j \leq m-1 \) and the element \( w_{m-1} \otimes v_1 \) all must have non-zero projection onto \( V(\omega_{aq^{-1}(m)}) \). For otherwise, applying \( x^+_0 \) repeatedly we find that \( w_0 \otimes v_0 \notin V(\omega_{aq^{-1}(m-2)}) \) which is impossible.

On the other hand, using the formulas in Proposition 4.8, we see that
\[
P(u)(w_j \otimes v_0) = (\varpi_{a,m-1}^{j} \varpi_{aq^{-2}(m-1)}^{0})(w_j \otimes v_0) = (\varpi_{aq^{-1},m}^{j})(w_j \otimes v_0), \quad 0 \leq j \leq j - 1
\]
and
\[
P(u)(w_{m-1} \otimes v_1) = \frac{1}{\omega_{aq^{-1}}}(w_{m-1} \otimes v_1) = (\varpi_{aq^{-1},m}^{m})(w_{m-1} \otimes v_0).
\]
The result follows. \( \square \)

4.10. We now assume that \( \mathfrak{g} \) is an arbitrary simple Lie algebra.

**Proposition.** Let \( V \in \mathcal{C}_q \).

(i) Suppose that \( 0 \neq v \in \mathcal{V}_{\varpi} \) is such that \( x^+_i v = 0 \) for some \( i \in I \) and all \( k \in \mathbb{Z} \). Then \( \varpi_{i,k} \in \mathcal{C}[u] \).

(ii) For all \( \varpi \in \mathcal{W}(V) \), we have \( \varpi \in \mathcal{P}_q \) and \( V_{\varpi} \subset V_{\varpi} \). In particular,
\[
\mathcal{W}(V) = \left\{ \mathcal{W}(\varpi) : \varpi \in \mathcal{W}(V) \right\}.
\]

(iii) The eigenvalues of the elements \( P_{i,r} \), \( i \in I, r \in \mathbb{Z} \), on \( V_{\varpi} \) are given by \( \varpi^{−} \in \mathcal{P}_q \).
Proof. It clearly suffices to prove the proposition when \( g = sl_2 \). Since \( V \) is finite-dimensional it has a Jordan–Holder series and hence we may assume without loss of generality that \( V = V(\omega) \) for some \( \omega \in \mathcal{P}_q^+ \). The proof of (i) is now immediate since Corollary 4.4(i) implies that \( v = cv_\omega \) for some \( c \in \mathbb{C}^\times \). To prove the other parts notice that by Proposition 4.9(ii) and Lemma 4.4 it suffices to consider the case of the representations \( V(\omega_a(r)) \), \( r \in \mathbb{Z}_+, a \in \mathbb{C}^\times \). But this is exactly part (iii) of Proposition 4.9.

4.11. Let \( \mathbb{Z}[P_q] \) be the integral group ring over \( \mathcal{P}_q \) and let \( e(\varpi), \varpi \in \mathcal{P}_q \) be a basis of the group ring.

Definition. Given \( V \in \mathcal{C}_q \), let \( ch_\ell(V) \in \mathbb{Z}[P_q] \) be defined by

\[
ch_\ell(V) = \sum_{\varpi \in \mathcal{P}_q} \dim(V\varpi) \cdot e(\varpi).
\]

In [21] it was proved that the \( q \)-character of \( V \), which was defined using the \( R \)-matrix, is just \( ch_\ell(V) \). It is quite clear, as observed in [21], that \( ch_\ell \) is additive and by Lemma 4.4 multiplicative.

5. Braid group invariance of \( \ell \)-weights of fundamental representations

Throughout this section we assume that \( g \) is of classical type. The representations \( V(\omega_{i,a}), i \in I, a \in \mathbb{C}^\times \) are called the fundamental \( \ell \)-highest weight representations. The main result of this section is the following theorem. Recall from Section 1, the subsets \( W_\lambda \subset W \) defined for elements \( \lambda \in P^+ \).

5.1. Theorem. Let \( i \in I, a \in \mathbb{C}^\times \) and assume that \( \varpi \in \text{wt}_\ell(V(\omega_{i,a})) \) is such that \( \text{wt}(\varpi) = \lambda \in P^+ \).

(i) Let \( w' = s_j w \in W_\lambda \) for some \( j \in I \) with \( \ell(s_j w) = \ell(w) + 1 \). Then, \((T_w \varpi)_j \in \mathbb{C}[u]\). In particular,

\[
\varpi \in \omega_{i,a} Q^+_q \implies T_w \varpi \in \omega_{i,a} Q^+_q.
\]

(ii) For all \( w \in W_\lambda \) we have

\[
\dim(V(\omega_{i,a})) = \dim(V(\omega_{i,a})_{\text{T}_w \varpi}),
\]

and

\[
T_w(\text{wt}_\ell(V(\omega_{i,a}))) = \text{wt}_\ell(V(\omega_{i,a})_{\varpi}).
\]

(iii) Suppose that \( \varpi' \neq \omega_{i,a} \). There exists \( \varpi' = (\varpi'_1, \ldots, \varpi'_n) \in \text{wt}_\ell(V(\omega_{i,a})) \) and \( j \in I \) with \( \varpi'_{j} = (1 - cu)(1 - c'u) \), and

\[
\varpi = \varpi'(\alpha_{j,c})^{-1}.
\]

If \( c' \neq cq^{-2} \) then \( \varpi'(\alpha_{j,c'})^{-1} \in \text{wt}_\ell(V(\omega_{i,a})) \) and if \( c = c' \) then \( \dim(V(\omega_{i,a})) \geq 2 \).

5.2. Corollary. We have \( \text{wt}_\ell(V(\omega_{i,a})) \subset \omega_{i,a} Q^+_q \).

Proof. By part (i) of the theorem, it suffices to prove the corollary for \( \varpi \in \text{wt}_\ell(V(\omega_{i,a})) \) with \( \text{wt}(\varpi) = \lambda \in P^+ \). We proceed by induction on \( \text{ht}(\omega_i - \lambda) \), with induction obviously beginning when \( \omega_i = \lambda \). Let \( \varpi' \) be as in part (iii) of the theorem, so that \( \text{wt}(\varpi') = \lambda + \alpha_j \). Choose \( w \in W \) and \( \mu \in P^+ \) with \( w\mu = \lambda + \alpha_j \). Then \( \mu \geq \lambda \) and by part (ii), we have

\[
\varpi' = T_w \varpi''
\]

for some \( \varpi'' \in \text{wt}_\ell(V(\omega_{i,a})) \) with \( \text{wt}(\varpi'') = \mu \). By the induction hypothesis \( \varpi'' \in \omega_{i,a} Q^+_q \) and hence by part (i) again \( \varpi' \in \omega_{i,a} Q^+_q \). The corollary is proved. \( \square \)
Remark. The corollary was proved [20] by combinatorial methods for all simple Lie algebras. On the other hand, Theorem 5.1 is not true for the exceptional algebras since there is no suitable analog of Lemma 1.3 available for those algebras. In particular, it follows that Theorem 5.1 is stronger than the Corollary. The theorem is also obviously false for an arbitrary irreducible representation since the evaluation representations of $U(sl_2)$ of Section 4 are counterexamples.

5.3. Corollary. We have

$$ch_\ell(V(\omega_{i,a})) = \sum_{\varpi \in s \cdot \varpi = \mu \in P^+} \dim(V_{\varpi}) \left( \sum_{w \in W_\mu} e(T_w \varpi) \right).$$

We remark here that the preceding results are analogous to the following well-known result for finite-dimensional representations $V$ of simple Lie algebras: for all $\mu \in P$ with $V_\mu \neq 0$, and $w \in W$, we have

$$\dim(V_\mu) = \dim(V_{w\mu}).$$

5.4. Before proving Theorem 5.1 we note some consequences of it in computing $q$-characters (or $\ell$-characters) of the fundamental representations, see also [28]. Thus, suppose that $i \in I$ is such that the $U^{fin}$-representation $V(\omega_i)$ is miniscule. In that case there exist no weights $\mu \in P^+$ such that $\mu \leq \omega_i$ and it is known that $V(\omega_{i,a}) \cong V(\omega_i)$ as $U^{fin}$-modules. Hence, the $q$-character of the fundamental representation is of the form,

$$ch_\ell(V(\omega_{i,a})) = \sum_{w \in W_i} e(T_w \omega_{i,a}).$$

5.5. Suppose next that $g$ is of type $D_n$, and that $V = V(\omega_{2,a})$. Let $\varpi_j$ be defined by

$$\varpi_j = (\omega_{j-1,aq^{j+1}})^{-1} \omega_{j-1,aq^{n-j-3}} \omega_{j,aq^j} (\omega_{j,aq^{n-3-j}})^{-1}, \quad \text{for } 1 \leq j \leq n-2,$$

$$= (\omega_{j,aq^{n-3}})^{-1} \omega_{j,aq^{j+1}}, \quad \text{for } j = n-1,n.$$ We claim that

$$ch_\ell(V) = \sum_{w \in W_{2,a}} e(T_w \omega_{2,a}) + \sum_{j \neq n-2} e(\varpi_j) + 2e(\varpi_{n-2}).$$

To prove the claim, set

$$w_j = s_{j-1} \cdots s_1 s_j s_{j+1} \cdots s_{n-2} s_n \cdots s_{n-2} \cdots s_1,$$

and observe that $w_j \in W_{2,a}$ with $w_j \omega_2 = \alpha_j$. A straightforward calculation shows that

$$T_{w_j} \omega_{2,a} = (\omega_{j-1,aq^{j+1}})^{-1} \omega_{j,aq^j} (\omega_{j,aq^{n-3-j}})^{-1}, \quad \text{if } j \leq n-3,$$

$$= (\omega_{n-3,aq^{j+1}})^{-1} (\omega_{n-2,aq^{j+1}})^{-1} \omega_{n-1,aq^{n-3}} \omega_{n,aq^{n-1}}^{-1}, \quad \text{if } j = n-2,$$

$$= (\omega_{n-2,aq^{j+1}})^{-1} \omega_{j,aq^{n-j}} \omega_{j,aq^{n-3}}, \quad \text{if } j = n-1,n.$$ It follows from Theorem 5.1 that

$$wt_\ell(V) = \{T_w \omega_{2,a} : w \in W_{2,a} \} \cup \{ \varpi_j : 1 \leq j \leq n \},$$

and that

$$\dim(V_{\varpi_j}) \geq 1, \quad j \neq n-2, \quad \dim(V_{\varpi_{n-2}}) > 1.$$ On the other hand, we know from [19] that $V \cong V(\omega_2) \oplus C$ as $U^{fin}$-modules. In particular, $\dim(V_0) = n+1$. Hence it follows that

$$\dim(V_{\varpi_j}) = 1, \quad j \neq n-2, \quad \dim(V_{\varpi_{n-2}}) = 2.$$
and our claim is proved.

5.6. We shall need the following result which holds for all simple Lie algebras \( g \).

**Proposition.** Let \( V \in C_q, \mathfrak{v} = (\mathfrak{v}_1, \cdots, \mathfrak{v}_n) \in P_q \) and suppose that there exists \( v \in V_{\mathfrak{v}}, v \neq 0 \), and \( j \in I \) such that

\[
x_{j,s}^+ v = 0, \quad \forall s \in \mathbb{Z}.
\]

(i) Then, \( \mathfrak{v}_j \in C[u] \) is of degree \( wt(\mathfrak{v})(\hat{\alpha}_j) \) and

\[
(x_{j,0}^{-})^{wt(\mathfrak{v})(\hat{\alpha}_j)} v \in V_{\mathfrak{v}_j} \quad .
\]

(ii) Write \( \mathfrak{v}_j \) as a product of polynomials \( \prod_{r=1}^{k} \omega_a_r(m_r) \) as in Proposition 4.9. Then \( \mathfrak{v}(\alpha_{j,a_r}q^{m_r-1})^{-1} \in wt(V) \) for all \( 1 \leq r \leq k \). Furthermore, for all \( s \in \mathbb{Z} \), we have

\[
x_{j,s}^- v = \sum_{r=1}^{k} \sum_{t=0}^{m_r-1} V_{\mathfrak{v} \alpha_{j,a_r}q^{m_r-1}}
\]

and

\[
\dim V_{\mathfrak{v} \alpha_{j,a_r}q^{m_r-1}} \geq \# \{ 1 \leq s \leq k : a_r = a_s \}.
\]

Analogous statements hold if \( x_{j,s}^- v = 0 \) for all \( s \in \mathbb{Z} \).

**Proof.** By Lemma 4.7 and Proposition 4.10, we see that \( x_{j,s}^- v = 0 \) and \( \mathfrak{v}(\alpha)(\hat{\alpha}_j) = \deg \mathfrak{v}_j \). Further, \( (x_{j,0}^-)^{wt(\mathfrak{v})(\hat{\alpha}_j)} v \neq 0 \). Suppose first that \( v \) is actually an eigenvector with eigenvalue \( \mathfrak{v} \). It follows from Lemma 4.7(iii) and Proposition 4.10(iii), that \( (x_{j,0}^-)^{wt(\mathfrak{v})(\hat{\alpha}_j)} v \) is an eigenvector for \( P_j^+(u) \) with eigenvalue \( \mathfrak{v}(\alpha_{j,a_r}q^{m_r-1})^{-1} \). To compute the eigenvalues corresponding to \( P_j^+(u) \) it suffices in view of (1.1) to compute the eigenvalues for \( h_{i,r} \) for all \( i \in I \) and \( r \in \mathbb{Z}, r \neq 0 \). To simplify our notation, we set for all \( i \in I, r \in \mathbb{Z}, r > 0 \),

\[
h_{i,r} = - \frac{q^r}{|r|_q}, \quad k = wt(\mathfrak{v})(\hat{\alpha}_j).
\]

Using the relations in \( U \), we see that

\[
[h_{i,r}, (x_{j,s}^-)^k] = \left[ \frac{r_{a_j}j}{|r|_q} \right]_{ij} [h_{j,r}, (x_{j,s}^-)^k] = \left[ \frac{r_{a_j}j}{|r|_q} \right]_{ij} [h_{j,r}, (x_{j,s}^-)^k],
\]

which gives,

\[
\hat{h}_{i,r}(x_{j,s}^-)^k = (x_{j,s}^-)^k \hat{h}_{i,r} + [h_{i,r}, (x_{j,s}^-)^k] = (x_{j,s}^-)^k \hat{h}_{i,r} + \left[ \frac{r_{a_j}j}{|r|_q} \right]_{ij} [h_{j,r}, (x_{j,s}^-)^k].
\]

Writing \( \ln(\mathfrak{v}_i) = \sum_{r \geq 1} \mathfrak{v}_{i,r} u^r \), for some \( \mathfrak{v}_{i,r} \in C \), we find from (1.1) that,

\[
\hat{h}_{i,r} v = \mathfrak{v}_{i,r} v, \quad i \in I, r \in \mathbb{Z}, r > 0.
\]

On the other hand, we have already observed that

\[
\hat{h}_{j,r}(x_{j,s}^-)^k v = (\ln(\mathfrak{v}(\alpha_{j,a_r}q^{m_r-1})))_{i,r} (x_{j,s}^-)^k v.
\]

Since \( \mathfrak{v}_j \) is a polynomial of degree \( k \), write

\[
\mathfrak{v}_j(u) = \prod_{t=1}^{k} (1 - b_t u)
\]
for some $b_1, \cdots, b_k \in \mathbb{C}^\times$. This means that
\[
(ln(\varpi_j(q_j^2u)))_r = \left(\sum_{i=1}^k b_i q_i^{2r}\right)/r.
\]
It is now a simple checking to see that
\[
\tilde{h}_{i,r}(x_{j,s}^-)^k v = \left(\varpi_{i,r} + \frac{q_j^{r[a_{ij}]}j}{r[r]_j} \sum_{i=1}^k b_i^r\right) (x_{j,s}^-)^k v
\]
for all $i \in I$ and $r \in \mathbb{Z}$, $r > 0$. Using (1.1) again we see that,
\[
(x_{j,0}^-)^{wt(\varpi)}(\alpha_j)v = \in V_{T_j\varpi}.
\]
For the second statement, observe first that, by Theorem 1.8 and Proposition 1.9, $x_{j,s}^- v$ is a sum of eigenvectors for $P_j^+(u)$ with eigenvalues $(1 - a_r q^{m_r - 1 - 2l+2} u)^{-1}(1 - a_r q^{m_r - 1 - 2l} u)^{-1} \varpi_j$ where $\varpi_j = \prod_{r=1}^k \omega_r(m_r)$ and $l = 0, 1, \cdots, m_r - 1$. But now, a calculation identical to the preceding one gives the result. The statement on the dimensions follows from Lemma 1.4, Theorem 1.8 and Proposition 1.9.

It remains to consider the case when $v$ is a generalized eigenvector for the action of the $P_j(u)$. Clearly, we can choose a Jordan basis $v_1, \cdots, v_m$ of $U(0) v \subset V_{\varpi}$ simultaneously for the action of the $\tilde{h}_{i,k}$, i.e.,
\[
\tilde{h}_{i,k} v_t \in \oplus_{s \subset t} C v_t + \varpi v_t
\]
for all $1 \leq t \leq m$. We proceed by induction on $t$, the case $t = 1$ is dealt with above. Let $V_t$ be the $U^j$-module generated by $v_t$. Then $V_t \subset V_{t+1}$ and the image of $v_{t+1}$ in the quotient $V_{t+1}/V_t$ is an $\ell$-highest weight vector for $U^j$. In particular, it follows that
\[
\tilde{h}_{j,r}(x_{j,s}^-)^k v_t \in \oplus_{s \subset t} C(x_{j,s}^-)^k v_{t'} + \tilde{\omega}_{j,r}(x_{j,s}^-)^k v_t
\]
where $\tilde{\omega}_{j,r} = (ln(\varpi_j(q_j^2u)))_r$. Using the inductive hypothesis and (5.3) we get
\[
\tilde{h}_{i,r}(x_{j,s}^-)^k v_t \in \oplus_{s \subset t} C(x_{j,s}^-)^k v_{t'} + \left(\varpi_{i,r} + \frac{q_j^{r[a_{ij}]}j}{r[r]_j} \sum_{i=1}^k b_i^r\right) (x_{j,s}^-)^k v_t
\]
It follows that $(x_{j,s}^-)^k v_t \in V_{T_j\varpi}$. The second statement is proved similarly. \hfill \square

5.7. We now prove Theorem 5.1.

Proof. Notice that by Lemma 1.3 we can assume that either $\lambda = 0$ or $\lambda = \omega_r$ for some $r \in I$. Assume first that $\lambda = \omega_r$ and let $w' = s_j w$ where $\ell(w') = \ell(w) + 1$. It follows that $w^{-1} \alpha_j \in Q^+$ and hence $(w \omega_r, \alpha_j) = (\omega_r, (w)^{-1} \alpha_j) > 0$. In particular, this means that $(w)^{-1} \alpha_j - \alpha_r \in Q^+$ and hence by Lemma 1.3 we see that
\[
\omega_i - (\omega_r + (w)^{-1} \alpha_j) \notin Q^+
\]
This in turn implies that
\[
\omega_r + (w)^{-1} \alpha_j \notin wt(V(\omega_i,a)),
\]
or equivalently that $w \omega_r + \alpha_j \notin wt(V(\omega_i,a))$. Thus,
\[
x_{j,a}^s V(\omega_i,a) t_{w^s} \omega = 0, \quad \forall \, s \in \mathbb{Z}.
\]
A similar argument proves that
\[
x_{j,0}^s V(\omega_i,a) t_{w^s} \omega = 0.
\]
The first statement in part (i) is immediate from Lemma 1.4 while the second follows from equation (2.1) and the fact that $T_{w'} = T_j T_w$. To prove (ii), note that Proposition 5.6 implies that the map $(x_{j,0}^s)^{w \omega_r, \alpha_j} : V(\omega_i,a) t_{w^s} \omega \rightarrow V(\omega_i,a) t_{w^s} \omega$ is an isomorphism of vector spaces. Since, $\dim V(\omega_i,a) \lambda =
dim $V(\omega_{i,a})_{\omega_\lambda}$ th second statement in (ii) follows as well. If $\lambda = 0$ there is nothing to prove in (i) and (ii) since $W_\lambda = \{ e \}$.

To prove (iii), choose a non-zero element $v \in V(\omega_{i,a})_\varpi$, $j \in I$, and $s \in \mathbb{Z}$ such that $v = x^-_{j,a} v'$. Then $v' \in V(\omega_{i,a})_{\lambda + \alpha_j} \neq 0$ and by Lemma 1.3 we see that

$$x^+_{j,a} V(\omega_{i,a})_{\lambda + \alpha_j} = 0.$$ 

Since $(\lambda + \alpha_j)(\alpha_j) = 2$, it follows from Lemma 1.7 that $\varpi_j$ is a polynomial of degree two for all $\varpi' = (\varpi_1, \cdots, \varpi_n) \in \text{wt}(V(\omega_{i,a}))$ with $\text{wt}(\varpi') = \lambda + \alpha_j$. Furthermore, since $v' \in \oplus_{\text{wt}\varpi' = \lambda + \alpha_j} V(\omega_{i,a})_{\varpi'}$, it follows from Proposition 5.8 that

$$\varpi = (\varpi')^{-1}(\alpha_{j,c})^{-1},$$

for some $\varpi' \in \mathcal{P}_q$ with $\text{wt}\varpi' = \lambda + \alpha_j$ and $c \in \mathbb{C}^\times$ satisfying $\varpi'_c(c^{-1}) = 0$.

To prove the final statement in (iii), let $v'' \in V(\omega_{i,a})_{\varpi''}$, where $\text{wt}\varpi'' = \lambda + \alpha_j$ and $\varpi'' = (1 - cu)(1 - c'u)$. Since $x^+_{j,a} v'' = 0$ it follows from Proposition 1.9 and Proposition 5.8 that if $c \neq c'q^2$, $V(\omega_{i,a})_{\varpi'(\alpha_{j,c})^{-1}} \neq 0$ and $V(\omega_{i,a})_{\varpi'(\alpha_{j,c'})^{-1}} \neq 0$. The other statement is proved similarly.

6. On the tensor product structure of Weyl modules

6.1. In this section we establish the conjecture in [14] on the structure of finite-dimensional Weyl modules using some deep results of Nakajima. This allows us to prove the following generalization of Corollary 6.1 and [20, Theorem 4.1].

**Theorem.** Let $V \in \mathcal{C}_q$ be an $\ell$–highest weight representation with highest weight $\omega \in \mathcal{P}_q^+$. We have

$$\text{wt}_\ell(V) \subset \omega \mathcal{Q}_q^-.$$

6.2. We begin by recalling the definition and some results on Weyl modules. Thus, let $W(\omega)$ be the $U$–module generated by an element $v_\omega$ satisfying the relations:

$$x^+_i v_\omega = 0, \quad P_i^\pm(u) v_\omega = (\omega^\pm_i v_\omega, \quad K_i^\pm_{1} v_\omega = q^\pm_{\text{wt}(\omega)_{(\alpha_i)}} v_\omega, \quad (x^-_{i,r})^{\text{wt}(\omega)_{(\alpha_i)}} v_\omega = 0,$$

(6.1)

The following result was proved in [14].

**Proposition.**

(i) For all $\omega \in \mathcal{P}_q^+$, we have $W(\omega) \in \mathcal{C}_q$.

(ii) Any $\ell$–highest weight representation in $\mathcal{C}_q$ is a quotient of $W(\omega)$ for some $\omega \in \mathcal{P}_q^+$.

6.3. We shall also need, the following result proved in [11, 14, 29].

**Theorem.** Let $k \in \mathbb{Z}$, $k \geq 1$ and let $i_1, \cdots, i_k \in I$, $a_1, \cdots, a_k \in \mathbb{C}^\times$. The $U$–module $V(\omega_{i_1,a_1}) \otimes \cdots \otimes V(\omega_{i_k,a_k})$ is cyclic on the tensor product of the $\ell$–highest weight vectors if for all $1 \leq s' < s \leq k$ we have $a_s \neq a_{s'} q^r$ for any $r \in \mathbb{Z}$, $r > 0$.

The following corollary is now immediate.

**Corollary.** There exist $i_1, \cdots, i_k \in I$, $a_1, \cdots, a_k \in \mathbb{C}^\times$ such that $V(\omega_{i_1,a_1}) \otimes \cdots \otimes V(\omega_{i_k,a_k})$ is a quotient of $W(\omega)$ where,

$$\omega = \prod_{k: i_k = j} (1 - a_k u).$$
In particular,
\[ \dim W(\omega) \geq \prod_k \dim(V(\omega_{i_k,a_k})). \]

\[ \square \]

6.4. As a result, to prove Theorem 5.1 it now suffices to prove that the \( \ell \)-weights of \( W(\omega) \) lie in \( \omega Q_q^- \). This is immediate from Lemma 4.4 and Theorem 4.1 and the following Theorem.

**Theorem.** The module \( W(\omega) \) is isomorphic to a tensor product of fundamental representations. In particular if \( i_1, \cdots, i_k \in I \), \( a_1, \cdots, a_k \in \mathbb{C}^* \) are such that \( V = V(\omega_{i_1,a_1}) \otimes \cdots \otimes V(\omega_{i_k,a_k}) \) is cyclic on the tensor product of highest weight vectors, then \( V \cong W(\omega) \) where \( \omega \) defined as in Corollary 6.3.

**Remark.** This theorem was conjectured in [14] where it was proved when \( g = \mathfrak{sl}_2 \). In the general case, the proof we give is deduced easily from some deep results of Nakajima [27] in the simply–laced case and of Beck and Nakajima in [4] in the general case.

**Proof.** Given \( \lambda \in P^+ \), let \( \tilde{W}(\lambda) \) be the \( U \)-module generated by an element \( v_\lambda \) satisfying,
\[ x_{i,r}^+ v_\lambda = 0, \quad K_i^{\pm 1} v_\lambda = q^{\pm \lambda_i} v_\lambda, \quad (x_{i,r}^-)^{\lambda_i+1} v_\lambda = 0. \]

It is not hard to see that \( \tilde{W}(\lambda) \) can also be regarded as a \( U(0) \)-module by right multiplication. Further, it was shown in [14] that
\[ P_{i,\pm r} v_\lambda = 0, \quad i \in I, \ |r| > \lambda_i, \quad P_{i,\lambda_i} P_{i,-\lambda_i} v_\lambda = v_\lambda. \]

The quotient of \( U(0) \) by the ideal \( I(\lambda) \) generated by the elements, \( P_{i,r} \), \( i \in I, \ |r| > \lambda_i \) and \( P_{i,\lambda_i} P_{i,-\lambda_i} - 1 \) can be identified with the ring \( \Sigma \) of symmetric functions in the variables \( t_{i,r}, i \in I, 1 \leq r \leq \lambda_i \) and \( t_{i,\lambda_i}^{-1} \) [14], [27]. It was proved in [27, Proposition 14.1.2] for the simply–laced case, and in [4, Section 4] for the general case, that \( U(0)v_\lambda \) is free as a module for \( \Sigma \) of rank
\[ \prod_{i \in I} \dim V(\omega_{i,1})^\otimes \lambda_i. \]

Given \( \omega \in P_q^+ \) with \( wt \omega = \lambda \), let \( m_\omega \) be the maximal ideal in \( U(0) \) generated by the elements \( P_{i,\pm r} - ((\omega^\pm)_r) \). The module \( W(\omega) \) is clearly a quotient of \( \tilde{W}(\lambda) \), and in fact as vector spaces, we have
\[ W(\omega) \cong \tilde{W}(\lambda)/m_\omega, \]
and hence
\[ \dim(W(\omega)) = \prod_{i \in I} \dim V(\omega_{i,1})^\otimes \lambda_i. \]

The result now follows from Corollary 6.3. The second statement is immediate. \[ \square \]

7. Block Decomposition of \( \mathcal{C}_q \)

It is by now well–known that the category \( \mathcal{C}_q \) is not semisimple. In this section we use the results of the previous section to describe the blocks in \( \mathcal{C}_q \). To do this, we redefine the notion of elliptic characters introduced in [14] as elements of \( \Xi_q \). This allows us to state and prove the main result of [14] for generic \( q \).

7.1. We begin by recalling the definition of the blocks of an abelian category.

Recall that that two objects \( V_r \in \mathcal{C}_q \), \( r = 1, 2 \) are linked if there does not exist a splitting of \( \mathcal{C}_q \) into a direct sum of abelian categories \( \mathcal{C}_r \), such that \( V_r \in \mathcal{C}_r \) for \( r = 1, 2 \). It is not hard to see that linking defines an equivalence relation on \( \mathcal{C}_q \) and a block is an equivalence class of this relation.
7.2. We now define the notion of an elliptic character of a representation.

**Definition.** The elliptic character of an irreducible representation $V(\omega)$ of $U_q$ is the element $\chi_{\omega} = \omega \in P_q/Q_q \cong \Xi_q$. A finite dimensional representation $V$ of $U_q$ is said to have elliptic character $\chi \in \Xi_q$ if every irreducible constituent of $V$ has elliptic character $\chi$. Let $C_\chi$ be the subcategory of $C_q$ consisting of representations with elliptic character $\chi$.

**Remark.** Recall in section 3.1 that we use additive notation when working with $\Xi_q$, rather than the multiplicative notation induced from $P_q$.

**Theorem.**

(i) Every indecomposable object in $C_q$ has a well defined elliptic character.

(ii) Any two simple objects in $C_\chi$ are linked.

(iii) The categories $C_\chi$ are the blocks of $C_q$.

We prove the theorem in the rest of the section.

7.3. The following proposition plays an important role in the proof of Theorem 7.2.

**Proposition.**

(i) For all $\omega \in P_q^+$, we have $W(\omega) \in C_{\chi_\omega}$.

(ii) $C_{\chi_1} \otimes C_{\chi_2} \subset C_{\chi_1 + \chi_2}$.

**Proof.** To prove (i), note that if $V(\omega')$ is a constituent of $W(\omega)$, then by Theorem 6.1 we must have $\omega' \in \omega Q_q - q.$ It follows that $\chi_{\omega} = \chi_{\omega'}$. The proof of the second part is similar. It is enough to prove that if $V(\omega_r) \in C_{\chi_r}$, $r = 1, 2$, then $V = V(\omega_1) \otimes V(\omega_2) \in C_{\chi_1 + \chi_2}$. Suppose that $V(\omega)$ is an irreducible constituent of $V$. In particular $\omega \in \text{wt}_\ell(V) \subset \text{wt}_\ell(V_1) \cup \text{wt}_\ell(V_2)$. By Theorem 5.2 we know that $\text{wt}_\ell(V_r) \subset \omega_r Q_q$ for $r = 1, 2$. Hence $\omega \in \omega_1 \omega_2 Q_q$. Together with Proposition 3.1, it immediately implies that

\[ \chi_{\omega} = \omega = \omega_1 \omega_2 = \chi_{\omega_1} + \chi_{\omega_2}. \]

\[ \square \]

7.4. We can now prove Theorem 7.2(i) by methods similar to those used in [6] for affine algebras. Namely, one proves the following:

**Lemma.**

(i) Let $U \in C_\chi$. Let $\omega_0 \in P_q^+$ be such that $\chi \neq \chi_{\omega_0}$. Then $\text{Ext}^1_{C_q}(U, V(\omega_0)) = 0$.

(ii) Assume that $V_j \in C_{\chi_j}$, $j = 1, 2$ and that $\chi_1 \neq \chi_2$. Then $\text{Ext}^1_{C_q}(V_1, V_2) = 0$.

**Proof.** Since $\text{Ext}^1_{C_q}$ is an additive functor, to prove (i) it suffices to consider the case when $U$ is indecomposable. Consider an extension,

\[ 0 \rightarrow V(\omega_0) \rightarrow V \rightarrow U \rightarrow 0 \]

We prove by induction on the length of $U$ that the extension is trivial. Suppose first that $U = V(\omega)$ for some $\omega \in P_q^+$. Then $V_\omega \neq 0$ and one of the following must hold,

(i) $\text{wt}_\omega \subset \text{wt}_\omega 0$, or

(ii) $\text{wt}_\omega 0 - \text{wt}_\omega \notin Q^+ \setminus \{0\}$. 


We can always assume (by taking duals if necessary) that we are in case (ii). Since \( \text{wt}(V(\omega_0)) \subset \text{wt} \omega_0 - Q^+ \), it follows that
\[
x_{i,k}^+ V_{i,k} \omega = 0, \quad \forall \ i, k \in \mathbb{Z}.
\]

Thus there exists an element \( 0 \neq v \in V_\omega \) which is a common eigenvector for the action of \( P_i(u) \) with eigenvalue \( \omega \) and hence \( Uv \) is a quotient of \( W(\omega) \). In particular \( Uv \in C_{\chi,\omega} \). Notice that either
\[
V(\omega_0) \subset Uv \quad \text{or} \quad Uv \cap V(\omega_0) = 0.
\]
If \( \chi \omega \neq \chi \omega_0 \) then the second possibility must hold and so
\[
V \cong V(\omega_0) \oplus Uv.
\]

This shows that induction begins.

Now assume that \( U \) is indecomposable with length \( \ell > 1 \) and that we know the result for all modules with length strictly smaller than \( \ell \). Let \( U_1 \) be a proper non–trivial submodule of \( U \) and consider the short exact sequence,
\[
0 \to U_1 \to U \to U_2 \to 0
\]
Since \( \text{Ext}^1_{C_q}(U_j, V(\omega_0)) = 0 \) for \( j = 1, 2 \) by the induction hypothesis, the result follows by using the exact sequence \( \text{Ext}^1_{C_q}(U_2, V(\omega_0)) \to \text{Ext}^1_{C_q}(U, V(\omega_0)) \to \text{Ext}^1_{C_q}(U_1, V(\omega_0)) \). Part (ii) is now immediate by using a similar induction on the length of \( V_2 \).

7.5. The proof of Theorem 7.2 (i) is completed as follows. Let \( V \) be an indecomposable \( U \)-module. We prove that there exists \( \chi \in \Xi_q \) such that \( V \in C_\chi \) by an induction on the length of \( V \). If \( V = V(\omega) \) is irreducible it follows from the definition of \( C_{\chi,\omega} \). If \( V \) is reducible, let \( V(\omega_0) \) be an irreducible subrepresentation of \( V \) and let \( U \) be the corresponding quotient. In other words, we have an extension
\[
0 \to V(\omega_0) \to V \to U \to 0
\]
Write \( U = \bigoplus_{j=1}^r U_j \) where each \( U_j \) is indecomposable. By the inductive hypothesis, there exist \( \chi_j \in \Xi_q \) such that \( U_j \in C_{\chi_j}, 1 \leq j \leq r \). Suppose that there exists \( j_0 \) such that \( \chi_{j_0} \neq \chi \omega_0 \). Lemma 7.4 implies that
\[
\text{Ext}^1_{C_q}(U, V(\omega_0)) \cong \bigoplus_{j=1}^r \text{Ext}^1_{C_q}(U_j, V(\omega_0)) \cong \bigoplus_{j \neq j_0} \text{Ext}^1_{C_q}(U_j, V(\omega_0)).
\]

In other words, the exact sequence \( 0 \to V(\omega_0) \to V \to U \to 0 \) is equivalent to one of the form
\[
0 \to V(\omega_0) \to V' \oplus U_{j_0} \to \bigoplus_{j \neq j_0} U_j \to 0
\]
where
\[
0 \to V(\omega_0) \to V' \to \bigoplus_{j \neq j_0} U_j \to 0
\]
is an element of \( \bigoplus_{j \neq j_0} \text{Ext}^1_{C_q}(U_j, V(\omega_0)) \). But this contradicts the fact that \( V \) is indecomposable. Hence \( \chi_j = \chi \omega_0 \) for all \( 1 \leq j \leq r \) and \( V \in C_{\chi,\omega_0} \).

7.6. We now prove Theorem 7.2 (ii). The idea is similar to the one used in [17], although, again, with the theory of \( \ell \)-lattices the proof is simpler, uniform and works for generic \( q \). The proof we give depends on Proposition 3.4 which has only been stated so far for the classical Lie algebras. The proof of part (ii) of Theorem 7.2 for the exceptional Lie algebras is postponed to the appendix after we state the analog of Proposition 3.1 for these algebras.

We first consider the cases \( q = A_n, B_n, C_n, D_n \), where, in the case of \( D_n \), we assume that \( n \) is odd. Thus let \( i_* \) be the unique element in \( I_* \). From now on we denote by \( V(a) \) the irreducible fundamental representation \( V(\omega_{i_*,a}) \).

We shall need the following result.
Proposition. Let $a_1, \ldots, a_k \in \mathbb{C}^\times$ and let $\sigma$ be any permutation of $\{1, \ldots, k\}$. Then, the modules $V(a_1) \otimes \cdots \otimes V(a_k)$ and $V(a_{\sigma(1)}) \otimes \cdots \otimes V(a_{\sigma(k)})$ are linked.

(ii) Given $\omega \in \mathcal{P}_q^+$, there exists a set (possibly not unique) $S_\omega = \{a_1, \ldots, a_k\} \subset \mathbb{C}^\times$ such that $W(\omega)$ is a subquotient of $V(a_1) \otimes \cdots \otimes V(a_k)$ and hence $W(\omega)$ and $W(\omega')$ are linked to it.

Proof. To prove (i), note first that since the Grothendieck ring of the category of finite-dimensional representations is commutative, it follows that the modules $V(a_1) \otimes \cdots \otimes V(a_k)$ and $V(a_{\sigma(1)}) \otimes \cdots \otimes V(a_{\sigma(k)})$ have the same irreducible constituents for all permutations $\sigma$ of $\{1, 2, \ldots, k\}$. Hence to show that they are linked it suffices to prove that there exists a permutation $\tau$ such that $V(a_{\tau(1)}) \otimes \cdots \otimes V(a_{\tau(k)})$ is indecomposable. But this is clear using Theorem 6.3 which implies that there exists a permutation $\sigma$ of $\{1, 2, \ldots, k\}$ such that $V(a_{\tau(1)}) \otimes \cdots \otimes V(a_{\tau(k)})$ is cyclic on the tensor product of highest weight vectors and hence indecomposable. By Theorem 6.3 it suffices to prove (ii) when $\omega = \omega_{i,a}$ for some $i \in I$, $a \in \mathbb{C}^\times$. But this follows from Theorem 6.1, Proposition 7.5, Theorem 8.2. \hfill □

Corollary. Let $\omega, \omega' \in \mathcal{P}_q^+$ and assume that $S_\omega = \{a_1, \ldots, s_k\}$, $S_{\omega'} = \{a'_1, \ldots, a'_l\}$. Then $V(\omega)$ and $V(\omega')$ are linked iff $V(a_1) \otimes \cdots \otimes V(a_k)$ and $V(a'_1) \otimes \cdots \otimes V(a'_l)$ are linked.

As a consequence to prove Theorem 7.7(ii) it suffices to show that, if the modules $V(a_1) \otimes \cdots \otimes V(a_k)$ and $V(b_1) \otimes \cdots \otimes V(b_s)$ have the same elliptic character, then they are linked.

7.7. The next result identifies minimal sets $S_e$, where $e$ is the identity element in $\mathcal{P}_q^+$. Notice that in this case the associated irreducible representation is the trivial one.

Proposition. For all $a \in \mathbb{C}^\times$ we can take,

$$S_e(a) = \{a, aq^2, \ldots, aq^{2n}\}, \quad \text{if } g = A_n,$$

$$S_e(a) = \{a, aq^{4n-2}\}, \quad \text{if } g = B_n,$$

$$S_e(a) = \{a, aq^{2n+2}\}, \quad \text{if } g = C_n,$$

$$S_e(a) = \{a, aq^2, aq^{2n-2}, aq^{2n}\}, \quad \text{if } g = D_n.$$ 

Proof. It was proved in Proposition 5.1 that the dual $V(\omega)^*$ is isomorphic to $V(\omega^*)$ where $(\omega^*)_i = (\omega)_{w_{01}(q_i^k)}u).$ This proves the statements for $B_n$ and $C_n$. For $A_n$, it suffices to prove that

$$V(\omega_{i,q_i^{k-i}}) \subset V(\omega_{i,q_i^k}) \otimes V(\omega_{i-1,q_i^k}), \quad 1 \leq i \leq n.$$ 

But this follows from Lemma 5.2. The proof for $D_n$ is similar and we omit the details. \hfill □

7.8. Given $a \in \mathbb{C}^\times$, let $\omega_a \in \mathcal{P}_q^+$ be the element defined by

$$(\omega_a)_j = 1, \quad j \neq i, \quad (\omega_a)_i = \prod_{a_j \in S_e(a)} (1 - a_j u).$$

The following is now immediate.

Corollary. Let $V' \in \mathcal{C}_q$, then $V'$ is linked to $V' \otimes W(\omega_a)$. 
7.9. Recall from Section 3.1 that the group $\tilde{\Xi}_q$ is isomorphic to the quotient of the free group $\tilde{\Xi}$, generated by elements $\chi_a$, by the subgroup generated by

$$\kappa_a = \sum_{a_j \in S_e(a)} \chi_{a_j}$$

for all $a \in C^\times$. We can now complete the proof of Theorem 7.2(ii). Suppose that the modules $V(a_1) \otimes \cdots \otimes V(a_k)$ and $V(b_1) \otimes \cdots \otimes V(b_s)$ have the same elliptic character. Then, in $\tilde{\Xi}_q$, we have

$$\sum_{r=1}^k \chi_{a_r} - \sum_{r=1}^s \chi_{b_r} = \sum_{r=1}^m \kappa_{c_r}$$

for some $c_r \in C^\times$ and integers $m_r$. We can assume that there exists $p'$ such that $m_r \leq 0$ if $1 \leq r \leq p'$ and $m_r > 0$ otherwise. Now we have,

$$\sum_{r=1}^k \chi_{a_r} + \sum_{r=p'}^s (-m_r)\kappa_{c_r} = \sum_{r=1}^s \chi_{b_r} + \sum_{r=p'+1}^m m_r\kappa_{c_r}.$$

Since this is an equality in a free group, it follows that we have an equality of sets with multiplicities,

$$\{a_1, \ldots, a_k\} \cup \bigcup_{r=p}^s S_e(c_r) = \{b_1, \ldots, b_s\} \cup \bigcup_{r=p'+1}^m S_e(c_r),$$

where the multiplicity of $S_e(c_r)$ is $-m_r$ if $r \leq p'$, and $m_r$ if $r > p'$.

This now gives,

$$V(a_1) \otimes \cdots \otimes V(a_k) \sim V(a_1) \otimes \cdots \otimes V(a_k) \otimes (\otimes_{r=1}^{p'} W(\omega_{c_r}) \otimes (-m_r))$$

$$\sim V(b_1) \otimes \cdots \otimes V(b_s) \otimes (\otimes_{r=p'+1}^s W(\omega_{c_r}) \otimes m_r)$$

$$\sim V(b_1) \otimes \cdots \otimes V(b_s)$$

where $\sim$ stands for linking relation. The proof of Theorem 7.2(ii) is immediate.

7.10. We now consider the $\mathfrak{g} = D_n$ for even $n$. Thus, set $V_{-}(a) = V(\omega_{n-1,a})$ and $V_{+}(a) = V(\omega_{n,a})$. We state the analogue of Proposition 7.9 which is proved in a similar way.

**Proposition.**

(i) Let $a_1, \ldots, a_k \in C^\times, \varepsilon_1, \ldots, \varepsilon_k \in \{+, -, 0\}$, and $\sigma$ be a permutation of $\{1, \ldots, k\}$. Then the modules $V_{\varepsilon_1}(a_1) \otimes \cdots \otimes V_{\varepsilon_k}(a_k)$ and $V_{\varepsilon_{\sigma(1)}}(a_{\sigma(1)}) \otimes \cdots \otimes V_{\varepsilon_{\sigma(k)}}(a_{\sigma(k)})$ are linked.

(ii) Given $\omega \in P_q$, there exists a (non-unique) pair of sets $S_{\omega} = (\{a_1, \ldots, a_k\}, \{\varepsilon_1, \ldots, \varepsilon_k\})$, where $a_j \in C^\times$ and $\varepsilon_j \in \{+, -, 0\}$, such that $W(\omega)$ is a subquotient of $U = V_{\varepsilon_1}(a_1) \otimes \cdots \otimes V_{\varepsilon_k}(a_k)$. In particular $W(\omega)$ and $V(\omega)$ are linked to $U$.

The analogue of Corollary 7.6 is immediate.

7.11. Given $a \in C^\times$, define the sets

$$S_{e,0}(a) = \{a, aq^{2n-2}\}, \quad S_{e,1}(a) = \{a, aq^{2}\}$$

and, for $k \in \{0, +, -\}$, let $\omega_{a,k} \in P_q$ be given by

$$(\omega_{a,k})_j = 1, \quad j \notin I.$$
The next Proposition is proved exactly like Proposition \ref{prop:trivial_representation_linked}.

**Proposition.** The trivial representation is linked to \( W(\omega_{a,k}) \).

**Corollary.** Let \( V' \in C_q \), then \( V' \) is linked to \( V' \otimes W(\omega_{a,k}) \).

**7.12.** We complete the proof of Theorem \ref{thm:block_decomposition} ii) as in section 7.9 using Proposition \ref{prop:trivial_representation_linked}.

### 8. Appendix: Exceptional Algebras

We now consider the problem of determining the block decomposition when \( g \) is one of the exceptional algebras.

**8.1.** Thus, let \( i = 1 \) and \( V = V(\omega_{1,a}) \). We start observing that Proposition and Corollary 7.6 holds for the exceptional algebras.

**8.2.** Given \( a \in \mathbb{C}^\times \), define sets \( S_{e,k}(a) \), where \( k = 1, 2 \), when \( E_6, E_7, F_4, G_2 \), \( k = 1, 2, 3 \), when \( g = E_8 \), as follows.

\[
\begin{align*}
S_{e,1}(a) &= \{ a, aq^8, aq^{16} \}, \\
S_{e,2}(a) &= \{ a, aq^2, aq^4, aq^{12}, aq^{14}, aq^{16} \}, & \text{if } g &= E_6, \\
S_{e,1}(a) &= \{ a, aq^{18} \}, \\
S_{e,2}(a) &= \{ a, aq^2, aq^2, aq^{14}, aq^{24}, aq^{26} \}, & \text{if } g &= E_7, \\
S_{e,1}(a) &= \{ a, aq^{30} \}, \\
S_{e,2}(a) &= \{ a, aq^{20}, aq^{30} \}, & \text{if } g &= E_8, \\
S_{e,3}(a) &= \{ a, aq^{12}, aq^{24}, aq^{36}, aq^{48} \}, & \text{if } g &= E_8, \\
S_{e,1}(a) &= \{ a, aq^{18} \}, \\
S_{e,2}(a) &= \{ a, aq^{12}, aq^{24} \}, & \text{if } g &= F_4, \\
S_{e,1}(a) &= \{ a, aq^{12} \}, \\
S_{e,2}(a) &= \{ a, aq^8, aq^{16} \}, & \text{if } g &= G_2,
\end{align*}
\]

Now, define \( \omega_{a,k} \in \mathcal{P}_q^+ \) by

\[
(\omega_{a,k})_j = 1, \quad j \neq 1, \\
(\omega_{a,k})_1 = \prod_{a_j \in S_{e,k}(a)} (1 - a_j u).
\]

Similarly to Proposition \ref{prop:trivial_representation_linked} we have the following.

**Proposition.** The trivial representation is linked to \( W(\omega_{a,k}) \).

**Corollary.** Let \( V' \in C_q \), then \( V' \) is linked to \( V' \otimes W(\omega_{a,k}) \).

**8.3.** Now we complete the statement of Proposition \ref{prop:trivial_representation_linked} for the exceptional algebras.

**Proposition.** Assume that \( g \) is of type \( E_6, E_7, E_8, F_4 \) or \( G_2 \). The group \( \mathcal{P}_q/\mathbb{Q}_q \) is isomorphic to the (additive) abelian group \( \Xi_q \) with generators \( \{ \chi_a : a \in \mathbb{C}^\times \} \) and relations:

\[
\sum_{a_j \in S_{e,k}(a)} \chi_{a_j} = 0
\]

for all \( a \in \mathbb{C}^\times \) and \( k = 1, 2, 3 \), for \( g \neq E_8 \) and \( k = 1, 2, 3 \), for \( g = E_8 \).
The idea of the proof is the same as Proposition 3.1 and requires a long, but straightforward, case by case checking.

8.4. The rest of the proof of Theorem 7.2 (ii) and (iii) for the exceptional algebras is similar to section 8.3 using Proposition 8.3.

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Department of Mathematics, University of California, Riverside, CA 92521.

E-mail address: chari@math.ucr.edu, adrianoam@math.ucr.edu