Welfare ordering of voting weight allocations*

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Abstract

This paper studies the allocation of voting weights in a committee representing groups of different sizes. We introduce a partial ordering of weight allocations based on stochastic comparison of social welfare. We show that when the number of groups is sufficiently large, this ordering asymptotically coincides with the total ordering induced by the cosine proportionality between the weights and the group sizes. A corollary is that a class of expectation-form objective functions, including expected welfare, the mean majority deficit and the probability of inversions, are asymptotically monotone in the cosine proportionality.

JEL classification: D70, D72

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1 Introduction

How is social welfare affected by the voting weights in a committee representing groups of different sizes? Studies in the literature have identified cases in which the weights proportional to the populations of the groups are optimal (Bàrbera and Jackson 2006 and Beisbart and Bovens 2007). However, those results do not tell us, for arbitrarily given two weight allocations, which one leads to higher welfare; whether increased proportionality implies welfare improvement; and, if it does, for what measure of proportionality. This paper extends the previous analysis by introducing a welfare ordering of weight allocations which is the intersection of various welfare indices employed in the literature, and relating it to a specific measure of weight-to-population proportionality.

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Our analysis is based on a standard model of two-stage voting in which the preferences of individuals are aggregated first within groups such as states or countries, and then across the groups through weighted majority voting by group representatives. We evaluate the weight allocation among the groups, at a prior stage where the individual preferences are random variables. Social welfare is measured by the (random) number of individuals who prefer the voting outcome. We define the stochastic welfare ordering to be the partial ordering of weight allocations that ranks allocation $a$ higher than allocation $b$ if the distribution of welfare under $a$ stochastically dominates that under $b$. This ordering is the intersection of the orderings induced by expectation-form objective functions, including expected welfare, the mean majority deficit, and the probability of inversions (see the discussion of the literature below).

The main result is that under certain assumptions about the distribution of preferences, when the number of groups is sufficiently large, the stochastic welfare ordering asymptotically coincides with the total ordering induced by cosine proportionality, a simple index measuring the proportionality between the weights and the populations (Theorem 1). It also follows, as a corollary, that the class of expectation-form social objectives mentioned above are asymptotically monotone in cosine proportionality, and that they thus asymptotically induce the same total ordering of weight allocations (Corollaries 1 and 2).

Relation to the literature. Barberà and Jackson (2006) study the design of two-stage voting and derive the optimal weights that maximize the total expected utility of individuals. Their result implies that under our assumptions, the optimal weights are proportional to the populations. Felsenthal and Machover (1999) introduce the quantity called the majority deficit, the gap between the size of the majority camp and the number of individuals preferring the voting outcome. They study the problem of minimizing the mean majority deficit. Beisbart and Bovens (2008) use the mean majority deficit to compare alternative systems of the US presidential election. A closely related index is the probability of inversions, i.e., the probability that the majority of individuals disagree with the social decision. This probability is computed under various specific settings by, for example, May (1948), Hinich et al. (1975), Feix et al. (2004), Kaniovski and Zaigraev (2018), and De Mouzon et al. (2020).

These papers concern either the optimal voting rule (including optimal weights), or the values of welfare indices under a specific weight allocation and/or population distribution. Our focus is on deriving an approximate expression of a welfare ordering over the entire set of weight allocations, which provides insight into the problem of how welfare under suboptimal weight allocations can be compared. Our result implies that the above

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1. Koppel and Diskin (2009) provide an axiomatic analysis of cosine proportionality.
2. Barberà and Jackson’s (2006) result also implies that under an alternative assumption on the preference distribution, the optimal weights are proportional to the square roots of the populations. See Section 5.2.
3. The above review of the literature is by no means exhaustive. In particular, there are papers in the literature that consider social objectives not listed above. For instance, Penrose (1946), Beisbart and Bovens (2007) and Kurz et al. (2017) study the optimal weights in terms of equity among individuals. Koriyama et al. (2013) study optimal weights in terms of the total utility when the utility of each individual is a concave function of the frequency with which the social decision matches his preference.
three examples of objective functions asymptotically induce the same ordering of weight allocations, which is represented by cosine proportionality. Corollary 2 also shows the asymptotic formulas of these functions.

2 Model

Social decision. Consider a society comprising $n$ groups labeled $i = 1, \cdots, n$. The population of group $i$ is $s_i \in [0, \bar{s}]$. The group is assigned a voting weight $a_i = a(s_i) \in [0, \bar{a}]$ as a function of population $s_i$. We call function $a : [0, \bar{s}] \to [0, \bar{a}]$ the weight allocation. We assume that $a$ is Borel-measurable and not almost everywhere zero.

The society makes a choice between two alternatives, denoted +1 and −1, through two voting stages. First, in each group, all members cast one vote for their preferred alternative. Then the weight of each group is divided between the alternatives in a ratio that depends on the groupwide voting result. The society chooses the alternative that receives most weights in total.

Formally, let $X_i \in [-1, 1]$ be the group- $i$ vote margin between alternatives +1 and -1 in the first voting stage (i.e., the share of group-$i$ votes for +1 minus that for -1). Let $r(X_i) \in [-1, 1]$ be the weight margin in the second stage (i.e., the share of group $i$’s weight for +1 minus that for -1). The function $r : [-1, 1] \to [-1, 1]$ represents the rule that converts the groupwide voting result into the division of the weight between the alternatives. We assume that $r$ is nondecreasing, odd and not identically zero. Examples of $r$ include winner-take-all $r(x) = \text{sign } x$ and proportional representation $r(x) = x$.

The social decision is +1 or −1, depending on the sign of the total weight margin $T = \sum a_i r(X_i)$:

$$D = \begin{cases} \text{sign } T & \text{if } T \neq 0 \\ \pm 1 \text{ equally likely} & \text{if } T = 0, \end{cases}$$

where we assume that a coin is flipped in case of a tie.

Random preferences. We consider the prior stage where individual preferences are random variables. This implies that the vote margins $X_i$ are random variables. We assume that the distribution of $X_i$’s satisfies the following:

**Assumption 1.** $X_1, X_2, \cdots$ are independent and identically distributed random variables, whose common distribution is nondegenerate and symmetric about 0.

This assumption captures a situation in which individual preferences are correlated within the groups, independent across the groups, and ex ante neutral with respect to the alternatives. It is well known that under this assumption, the perfectly proportional weight allocation (i.e., $a(s) \propto s$) maximizes the expected welfare. The object of this paper is to see
how this observation extends to welfare comparison of arbitrary weight allocations. See Section 5 for a discussion of the distributional assumption.

**Welfare and proportionality.** We measure social welfare by the (random) number of individuals who prefer the social decision minus those who do not. This equals $DS$, where $S = \sum s_i X_i$ is the total vote margin between alternatives $+1$ and $-1$. For convenience, we normalize it by dividing by the standard deviation $\sigma = \sqrt{\mathbb{E}(X_i^2) \times \sum s_i^2}$:

$$W = \frac{DS}{\sigma}.$$ 

We denote by $H_a$ the distribution of welfare $W$, which depends on the weight allocation $a$.

The *stochastic welfare ordering* is the partial ordering $\succeq$ on the set of weight allocations, defined as follows: for any two weight allocations $a$ and $b$, $a \succeq b$ if $H_a$ stochastically dominates $H_b$.

Apart from the welfare properties, weight allocations can also be evaluated in terms of the proportionality to the populations. There are various measures of proportionality. One such measure is *cosine proportionality*, defined by the cosine of the angle between the vector of populations in the groups and the vector of weights of the groups:

$$c(a) = \frac{\sum s_i a_i}{\sqrt{\sum s_i^2 \sum a_i^2}}.$$ 

**Large number of groups.** To obtain a clear analytical result, we focus on the asymptotic situation in which the number of groups is sufficiently large. We introduce the sequence of societies, indexed by $n = 1, 2, 3, \ldots$, in which the $n$th society consists of the $n$ groups $i = 1, \ldots, n$.

We claim that as $n \to \infty$, the welfare distribution $H_a$ converges in law to a nondegenerate distribution $H_a^*$. This claim is proved in the next section (Lemma 2). For now, let us take the existence of an asymptotic distribution as true, and define the *asymptotic stochastic welfare order* $\succeq^*$ as follows: for any two weight allocations $a$ and $b$, $a \succeq^* b$ if $H_a^*$ stochastically dominates $H_b^*$.

To define the asymptotic version of cosine proportionality, we need the following assumption on the sequence of group sizes.

**Assumption 2.** Let $\Psi^n$ be the distribution of group sizes in the $n$th society: $\Psi^n(s) = \#\{i \leq n | s_i \leq s\}/n$ for $s \in [0, \bar{s}]$. As $n \to \infty$, $\Psi^n$ weakly converges to a distribution $\Psi^*$, not degenerate at 0.

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if the preferences of members of group $i$ are correlated via random variable $X_i$ so that conditional on $X_i$, each member prefers $+1$ with probability $(1 + X_i)/2$ and prefers $-1$ with probability $(1 - X_i)/2$, then the law of large numbers implies that $X_i$ is indeed the limit of the groupwide vote margin.

5Section 5.1 discusses the alternative definition of social welfare as the sum of utilities when preference intensities may vary across individuals.

6We say that a distribution $F$ stochastically dominates another distribution $G$ if $F(x) \leq G(x)$ for all $x$. 

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Under this assumption, the asymptotic cosine proportionality of weight allocation $a$ is defined as the limit of $c(a)$ as $n \to \infty$, which is

$$c^*(a) = \frac{\int sa(s)d\Psi^*(s)}{\sqrt{\int s^2d\Psi^*(s)} \int a(s)^2d\Psi^*(s)}$$

### 3 Asymptotic equivalence theorem

**Theorem 1.** The asymptotic stochastic welfare order coincides with the total order induced by asymptotic cosine proportionality. That is, for any weight allocations $a$ and $b$,

$$a \succeq^* b \iff c^*(a) \geq c^*(b).$$

We divide the proof of the theorem into the two lemmas below. We need some notations. Let $\tau = \sqrt{\mathbb{E} \left[ r(X_1)^2 \right] \sum a_i^2}$ be the standard deviation of the total weight margin $T = \sum a_i r(X_i)$. Recall also that $S = \sum s_i X_i$ denotes the total vote margin, and $\sigma = \sqrt{\mathbb{E} \left[ X_i^2 \right] \sum s_i^2}$ its standard deviation. Let $\rho$ be the correlation coefficient of $X_i$ and $r(X_i)$:

$$\rho = \text{Corr}(X_i, r(X_i)) = \frac{\mathbb{E} \left[ X_i r(X_i) \right]}{\sqrt{\mathbb{E} \left[ X_i^2 \right] \mathbb{E} \left[ r(X_i)^2 \right]}} > 0.$$  

The positiveness of $\rho$ can be checked by noting that function $r$ is an odd and nondecreasing function not identically zero, and that $X_i$ has a symmetric and nondegenerate distribution.

**Lemma 1.** As $n \to \infty$, the random vector $(S/\sigma, T/\tau)$ converges in law to the bivariate normal distribution with zero means, unit variances and correlation coefficient $\rho c^*(a)$.

**Proof of Lemma 1.** We use the superscript $n$ to clarify the number of groups. Fix two real numbers $\xi$ and $\eta$ not both zero, and let

$$Z^n = \xi \frac{S^n}{\sigma^n} + \eta \frac{T^n}{\tau^n}.$$  

The variance of $Z^n$ is

$$(\xi^n)^2 = \xi^2 + \eta^2 + 2\xi \eta \rho c^n(a),$$

where $c^n(a)$ denotes cosine proportionality.

By the Cramér-Wold device, the proof of the lemma reduces to showing that $S^n$ converges in law to the normal distribution with mean 0 and variance $\xi^2 + \eta^2 + 2\xi \eta \rho c^*(a)$. Since $c(a) \to c^*(a)$, we have

$$(\xi^n)^2 \to \xi^2 + \eta^2 + 2\xi \eta \rho c^*(a).$$ (1)

Thus it suffices to show that $Z^n/\xi^n$ converges in law to the standard normal distribution $N(0, 1)$.  

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Note that $Z^n$ is a sum of independent random variables:

$$Z^n = \sum_{i=1}^{n} R_i^n,$$

where $R_i^n = \xi s_i X_i + \eta a(s_i)r(X_i) \tau^n$.

To show that $Z^n/\xi^n$ converges in law to $N(0, 1)$, we apply the Lindeberg Central Limit Theorem to the triangular array $\{R_i^n : i \leq n; n = 1, 2, \cdots \}$. We need to check Lindeberg’s condition:

For any $\delta > 0$, \[
\sum_{i=1}^{n} \mathbb{E} \left( \left( \frac{R_i^n}{\xi^n} \right)^2 \mathbb{1}_{\left\{ \frac{R_i^n}{\xi^n} > \delta \right\}} \right) \to 0. \quad (2)
\]

In the definition of $R_i^n$, the numerators $s_i X_i$ and $a(s_i)r(X_i)$ are bounded, while the denominators $\sigma^n$ and $\tau^n$ tend to infinity ($\xi^n$ converges to the finite limit (I)). Thus, for any $\delta > 0$, there exists $n_0$ such that for all $n > n_0$ and all $i = 1, \cdots, n$, $\left( R_i^n / \xi^n \right)^2 < \delta$ almost surely, which implies that the sum in (2) is zero for all $n > n_0$.

The skew normal distribution\footnote{See, e.g., Theorem 27.2 in Billingsley (2008).} with parameter $\lambda$, denoted $SN(\lambda)$, is the distribution with the density function

$$2\phi(x)\Phi(\lambda x)$$

for $x \in \mathbb{R}$, where $\phi$ and $\Phi$ denote the density function and the distribution function of the standard normal distribution $N(0, 1)$.

**Lemma 2.** As $n \to \infty$, social welfare $W \to W^*$ in law, where $W^*$ has the skew normal distribution $SN(\lambda_a)$ with parameter $\lambda_a = \rho c^*(a) / \sqrt{1 - \rho^2 c^*(a)^2}$. The distribution $SN(\lambda_a)$ is stochastically increasing in the asymptotic cosine proportionality $c^*(a)$.

**Proof of Lemma**\footnote{See Azzalini and Valle (1996) for basic properties of the skew normal distribution.} Welfare $W$ can be written as

$$W = \left( \text{sign} \frac{T}{\tau} + L \mathbb{1}_{\{T=0\}} \right) \times \frac{S}{\sigma},$$

where $L$ is a tie-breaking variable that takes values $\pm 1$ with equal probabilities and is independent of all other variables. By Lemma (S/\sigma, T/\tau) converges in law to a random vector $(S^*, T^*)$ that has the bivariate normal distribution with zero means, unit variances and correlation coefficient $\rho c^*(a)$. Since $T^* \neq 0$ almost surely and the function $f(s, t) = s \text{sign} t$ is continuous except on a set of probability zero, $W$ converges in law to the random variable $W^* = S^* \text{sign} T^*$.

We now check that the distribution of $W^*$ is $SN(\lambda_a)$ as stated in the lemma, by conditioning on the sign of $T^*$. The conditional distribution of $W^*$ given $T^* > 0$ equals the conditional distribution of $S^*$ given $T^* > 0$. By Proposition 2 in Azzalini and Valle (1996), the latter conditional distribution is the skew normal distribution $SN(\lambda_a)$ with the parameter $\lambda_a$ specified in the lemma. Similarly, the conditional distribution of $W^*$ given $T^* < 0$ is also $SN(\lambda_a)$.
For the last sentence of the lemma, it suffices to show that the skew normal distribution $SN(\lambda)$ is stochastically increasing in $\lambda$. Let $H(x; \lambda)$ be the distribution function of $SN(\lambda)$:

$$H(x; \lambda) = \int_{-\infty}^{x} 2\phi(y)\Phi(\lambda y)dy.$$ 

We show that $\frac{\partial H}{\partial \lambda} < 0$. The derivative is $\frac{\partial H}{\partial \lambda} = \int_{-\infty}^{x} 2y\phi(y)\phi(\lambda y)dy$. This expression is proportional to the conditional expectation $E(Y|Y \leq x)$, where $Y$ is a random variable having a density function proportional to $\phi(y)\phi(\lambda y)$. Obviously $Y$ is normally distributed with mean 0. Thus $\int_{-\infty}^{x} 2y\phi(y)\phi(\lambda y)dy \propto E(Y|Y \leq x) < E(Y) = 0$ and therefore $\frac{\partial H}{\partial \lambda} < 0$. □

4 Expectation-form objective functions

We discuss the implications of the asymptotic equivalence theorem to social objective functions of the form $E[f(W)]$ for a nondecreasing function $f$.

To deal with the limits of expectations, we restrict the class of functions $f$. We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ has a square exponential bound if there exist constants $\alpha > 0$ and $\beta \in (0, 1)$ such that

$$|f(w)| \leq \exp \left( \alpha + \beta w^2 \right)$$

for all $w \in \mathbb{R}$. This condition is fairly weak. It allows $f(w)$ to increase as fast as $e^{\beta w^2}$ when $w \to \infty$ and decrease as fast as $-e^{\beta w^2}$ when $w \to -\infty$, for some $\beta \in (0, 1)$.

**Corollary 1.** Let $f$ be any nondecreasing function with a square exponential bound.

(i) As $n \to \infty$, $E[f(W)] \to E[f(W^*)]$, where $W^*$ has the skew normal distribution $SN(\lambda_0)$ with parameter $\lambda_0 = \rho c^*(a)\sqrt{1 - \rho^2c^*(a)^2}$.

(ii) The limit $E[f(W^*)]$ is a nondecreasing function of the asymptotic cosine proportionality $c^*(a)$.

**Proof:** (i) We use superscript $n$ to indicate the number of groups. By Lemma 1, $W_n \to W^*$ in law. Since $f$ is nondecreasing, the set of its discontinuity points $\Delta$ has Lebesgue measure 0. Since $W^*$ is continuously distributed, $P\{W^* \in \Delta\} = 0$. Thus, by Slutsky’s theorem, $f(W^*) \to f(W^*)$ in law.

If, in addition, the sequence of random variables $\{f(W^n)\}$ is uniformly integrable (UI), then statement (i) follows. To show that $\{f(W^n)\}$ is UI, it suffices to show that the dominating sequence $\{Y^n\} := \{\exp(\alpha + \beta(W^n)^2)\}$ is UI. Without loss of generality, assume $\alpha = 0$. A sufficient condition for uniform integrability is boundedness in $L^p$ for some $p > 1$, i.e., that there is a constant $K < \infty$ with $E[(Y^n)^p] < K$ for all $n$.

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*See e.g. [Ferguson (2017)].
*\[\text{A sequence } \{X^n\} \text{ of random variables is called uniformly integrable if for any } \epsilon > 0, \text{ there exists } x > 0 \text{ such that } E(|X^n|1_{(|X^n| > x)}) < \epsilon \text{ for all } n. \text{ See e.g. } [\text{Williams (1991)}].\]
To check the above sufficient condition, choose $p > 1$ so that $p\beta \in (0, 1)$. Note that
\[
\mathbb{E} [(Y^n)^p] = \int_0^\infty \mathbb{P} \{(Y^n)^p > y\} \, dy \leq 1 + \int_1^\infty \mathbb{P} \{(Y^n)^p > y\} \, dy.
\]
Rewrite $\mathbb{P} \{(Y^n)^p > y\} = \mathbb{P} \left\{ \left( \frac{W^n}{p\beta} \right)^2 > \frac{\log y}{p\beta} \right\}$ for $y > 1$. By definition, $\left( W^n \right)^2 = \left( \sum_{i=1}^n s_i X_i / \sigma \right)^2$, the square of a sum of independent variables with mean $0$, where the sum is normalized to unit variance. We can thus apply Hoeffding’s inequality to get
\[
\mathbb{P} \left\{ \left( \frac{W^n}{p\beta} \right)^2 > \frac{\log y}{p\beta} \right\} \leq 2y^{-\frac{1}{p\beta}}.
\]
Thus, recalling that $p\beta \in (0, 1)$,
\[
\int_1^\infty \mathbb{P} \{(Y^n)^p > y\} \, dy \leq \int_1^\infty 2y^{-\frac{1}{p\beta}} \, dy = \frac{2p\beta}{1 - p\beta}.
\]
Therefore $\mathbb{E} [(Y^n)^p] \leq (1 + p\beta) / (1 - p\beta) < \infty$ for all $n$.

(ii) This follows directly from the second sentence in Lemma 2. □

**Examples.** We provide three examples of expectation-form objective functions. The simplest one is the *expected welfare*:

\[
u(a) = \mathbb{E}[W],
\]
which corresponds to the linear function $f(w) = w$.

A related index is the *mean majority deficit*, which was introduced by Felsenthal and Machover (1999). It is the expected difference between the size of the majority camp, $|S| + \sum s_i / 2$, and the number of individuals who prefer the social decision, $(DS + \sum s_i) / 2$; that is, $\mathbb{E} \left( \frac{|S| - DS}{\sigma} \right)$. We redefine it by dividing by $\sigma$:

\[
\delta(a) = \frac{1}{2} \mathbb{E} \left( \frac{|S|}{\sigma} - W \right).
\]

Note that the distribution of $|S| / \sigma$ does not depend on the weight allocation $a$. Thus the negative of the mean majority deficit $-\delta(a)$ can be represented as the expectation $\mathbb{E} [f(W)]$ for the positive affine function $f(w) = (w - E(|S|) / \sigma) / 2$.

Another index that has been extensively studied in the literature is the *probability of inversion* (or the *probability of the referendum paradox*). It is the probability that the majority of individuals dislike the social decision:

\[
p(a) = \mathbb{P}\{W < 0\}.
\]

The complementary probability $1 - p(a)$ is the expectation $\mathbb{E} [f(W)]$ for the nondecreasing step function $f = \mathbb{1}\{w \geq 0\}$.
For finite $n$, these three indices induce different complete extensions of the stochastic welfare ordering $\succeq$ over weight allocations. Since the functions $f$ corresponding to these indices have square exponential bounds, Corollary 1 implies that they are asymptotically monotone in cosine proportionality. In fact, Lemma 2 allows us to derive the limiting monotone functions explicitly.

**Corollary 2.** As $n \to \infty$,

(i) $u(a) \to \sqrt{\frac{2}{\pi}} \rho c^*(a)$.

(ii) $\delta(a) \to \frac{1 - \rho c^*(a)}{\sqrt{2\pi}}$.

(iii) $p(a) \to \frac{\arccos(\rho c^*(a))}{\pi}$.

**Proof.** (i) The mean of the skew normal distribution $SN(\lambda)$ is $\sqrt{2/\pi \cdot \lambda} \cdot \sqrt{1 + \lambda^2}$ Plugging $\lambda_a$ from Lemma 2 yields (i).

(ii) In the definition of $\delta(a)$, the variable $|S|/\sigma$ converges in law to the standard half-normal distribution, whose mean is $\sqrt{2/\pi}$. Combining this with (i) proves (ii).

(iii) The limit probability of inversion is

$$P\{W^* < 0\} = \int_{-\infty}^{0} 2\phi(x)\Phi(\lambda_a x)dx = \frac{1}{\pi} \arctan \frac{1}{\lambda_a}$$

Noting that $1/\lambda_a = \tan \theta$ for $\theta \in [0, \pi/2]$ satisfying $\cos \theta = \rho c^*(a)$ proves (iii). \hfill \Box

## 5 Extensions

### 5.1 Preference intensities

We have shown Theorem 1 based on the definition of social welfare as the (normalized) number of individuals who prefer the social decision, which ignores cardinal differences between individual preferences. This section discusses an extension of the model for which the theorem is valid, even if heterogeneous preference intensities are incorporated in the definition of social welfare.

Let $m_i$ denote the population of group $i$ and label its members by $1, \ldots, m_i$. We denote by $U_{ik}$ the utility member $k$ of group $i$ obtains from alternative $+1$, and fix the utility from alternative $-1$ to be 0. The member thus votes for $+1$ or $-1$, depending on whether $U_{ik}$ is positive or negative. We assume that $U_{ik}$ has the following form:

$$U_{ik} = \Theta_i + \epsilon_{ik}.$$ 

The term $\Theta_i$ summarizes group-specific factors affecting the utility, while $\epsilon_{ik}$ summarizes individual-specific factors. We make the following assumptions on the distribution of these factors:

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4 Hinich et al. (1975) derives a similar limit formula of $p(a)$, assuming equally sized groups.

5 See, e.g., Azzalini and Valle (1996).

6 The second equality follows, e.g., from Formula 1.010.3 in Owen (1980).
variables: (i) All $\Theta_i$ and $\epsilon_{ik}$ ($i = 1, 2, \ldots; k = 1, 2, \ldots$) are independent. (ii) All $\Theta_i$ have a common symmetric and nondegenerate distribution $G_{\Theta}$ on $[-\theta, \theta]$. (iii) All $\epsilon_{ik}$ have a common symmetric and nondegenerate distribution $G_{\epsilon}$ on $[-\bar{\epsilon}, \bar{\epsilon}]$.

Consider the large-population limit as the absolute size of each group $m_i$ tends to infinity, and the relative size of group $i$ normalized by group 1’s size, $m_i/m_1$, tends to the constant $s_i \in [0, \delta]$. Then, almost surely, the group-$i$ average utility of alternative $+1$, $\sum_{k=1}^{m_i} U_{ik}/m_i$, converges to $\Theta_i$, while the group-$i$ vote margin in favor of alternative $+1$, $\sum_{k=1}^{m_i} \text{sign } U_{ik}/m_i$, converges to $2G_{\epsilon}(\Theta_i) - 1$. The first part of this claim follows from a direct application of the law of large numbers. The second part can be proved as follows: by the law of large numbers, $\sum_{k=1}^{m_i} \text{sign } U_{ik}/m_i \to E[\text{sign } U_{ik}|\Theta_i] = P\{U_{i1} > 0|\Theta_i\} - P\{U_{i1} < 0|\Theta_i\} = 2G_{\epsilon}(\Theta_i) - 1$ almost surely.

The total utility from alternative $+1$ is $\sum s_i \{2G_{\epsilon}(\Theta_i) - 1\}$, and hence the total utility from the social decision is $D \times \sum s_i \{2G_{\epsilon}(\Theta_i) - 1\}$. If we redefine social welfare $W$ to be (a normalization of) the total utility, then Theorem 1 holds. The proof only requires us to redefine $X_i := \Theta_i$, $S := \sum s_i \{2G_{\epsilon}(\Theta_i) - 1\}$, and $T := \sum a(s_i)r(\Theta_i)$; then Lemmas 1 and 2 hold with correlation coefficient $\rho := \text{Corr} \{2G_{\epsilon}(\Theta_i) - 1, r(\Theta_i)\}$, and the theorem follows.

5.2 Independent preferences

Assumption 1 captures a situation in which individual preferences are correlated within the groups. This section discusses how the results in this paper change under the alternative assumption that the preferences are independent within the groups.

We use the same basic model of preferences as in Section 5.1, but assume that $\Theta_i \equiv 0$ and $\epsilon_{ik} = \pm 1$ equally likely. Thus, each individual independently prefers alternative $+1$ or $-1$ with equal probabilities. Consider the large-population limit as $m_1 \to \infty$ and $m_i/m_1 \to s_i$ for each group $i$. The following lemma is a direct consequence of the central limit theorem.

**Lemma 3.** The group-$i$ vote margin in favor of alternative $+1$ multiplied by $\sqrt{m_i}$, i.e., $\sqrt{m_i} \sum_{k=1}^{m_i} \epsilon_{ik}/m_i$, converges in law to $X_i := N_i \sqrt{N_i}$, where $N_i$ ($i = 1, \ldots, n$) are independent and have the standard normal distribution $N(0, 1)$.

Note the following two differences from the case with intragroup correlations of preferences: first, $X_i$ are not identically distributed, since the variances are smaller for larger groups; second, the support of $X_i$ is $\mathbb{R}$, not $[-1, 1]$.

We provide an analog of Theorem 1 for this independent-preference model, but not in full generality. We only consider two specific representation rules: winner-take-all $r(x) = \text{sign } x$ and proportional representation $r(x) = x$.

**Theorem 2.** The following statements hold in the independent-preference model:

(i) Under winner-take-all, as the number of groups $n \to \infty$, the asymptotic stochastic welfare ordering $\succsim$ coincides with the ordering induced by the following index:

$$c_{\text{sqt}}^*(a) = \frac{\int \sqrt{s}a(s) d\Psi^*(s)}{\sqrt{\int sd\Psi^*(s) \int a(s)^2 d\Psi(s)}}.$$
(ii) Under proportional representation, for any number \( n \) of groups, the stochastic welfare ordering \( \geq \) coincides with the ordering induced by the following index:

\[
\hat{c}_{\text{sqrt}}(a) = \frac{\sum \alpha(s_i)}{\sqrt{\sum \sqrt{s_i}} \sum \frac{\alpha(s_i)^2}{s_i}}.
\]

**Proof.** The total vote margin and the total weight margin in favor of alternative +1 are

\[
S = \sum_{i=1}^{n} \sqrt{s_i} N_i, \quad T = \sum_{i=1}^{n} \alpha(s_i) r(X_i).
\]

The variance of \( S \) is \( \sigma^2 = \sum s_i \). We begin with the proof of part (ii).

**Proof of (ii).** Under proportional representation, \( r(X_i) = X_i = \frac{N_i}{\sqrt{n}}. \) The variance of \( T \) is \( \tau^2 = \sum_{i=1}^{n} \frac{\alpha(s_i)^2}{s_i}. \) Since a sum of independent and normally distributed random variables is again normally distributed, it follows that for any \( n, \) \( \left( \frac{S}{\sigma}, \frac{T}{\tau} \right) \) has the centered normal distribution with unit variances and correlation coefficient \( \hat{c}_{\text{sqrt}}(a). \) The same argument as in the proof of Lemma 2 shows that welfare \( W \) has the skew normal distribution with parameter \( \lambda = \frac{\hat{c}_{\text{sqrt}}(a)}{\sqrt{1 - \hat{c}_{\text{sqrt}}(a)^2}} \), and hence that \( W \) is stochastically increasing in \( \hat{c}_{\text{sqrt}}(a). \)

**Proof of (i).** Under winner-take-all, \( r(X_i) = \text{sign} N_i. \) The variance of \( T \) is now \( \tau^2 = \sum \alpha(s_i)^2. \) We first show that \( \left( \frac{S}{\sigma}, \frac{T}{\tau} \right) \) converges in law to the centered normal distribution with unit variances and correlation coefficient \( \sqrt{2/\pi} \hat{c}_{\text{sqrt}}(a). \) As in the proof of Lemma 1 for any two real numbers \( \alpha \) and \( \beta \) not both zero, let

\[
Z^n = \alpha \frac{S^n}{\sigma^n} + \beta \frac{T^n}{\tau^n} = \sum_{i=1}^{n} R^n_i, \quad \text{where} \quad R^n_i = \alpha \sqrt{s_i} N_i \sqrt{n} + \beta \frac{\alpha(s_i) \text{sign} N_i}{\tau^n}.
\]

The variance of \( Z^n \) is \( (\xi^n)^2 = \alpha^2 + \beta^2 + 2 \sqrt{\frac{\alpha}{\tau^n}} \alpha \beta \hat{c}_{\text{sqrt}}(a), \) where we define \( \xi_{\text{sqrt}}(a) := \sum \sqrt{s_i} a(s_i)/\sqrt{\sum s_i} \sum a(s_i)^2. \) By the Cramér-Wold Device, it suffices to show that \( \frac{S^n}{\xi^n} \) converges in law to the standard normal distribution. To do this, we use the Lyapunov Central Limit Theorem (Theorem 27.3 in [Billingsley, 2008]). We need to check Lyapunov’s condition:

\[
\frac{1}{(\xi^n)^2} \sum_{i=1}^{n} \mathbb{E}[(R^n_i)^4] \to 0.
\]

Minkowski’s inequality implies that for all \( i = 1, \cdots, n, \)

\[
\left[ \mathbb{E} \left( (R^n_i)^4 \right) \right]^\frac{1}{4} \leq \frac{|\alpha| \sqrt{s_i}}{\sigma^n} \left[ \mathbb{E}(N_i^4) \right]^\frac{1}{4} + \frac{|\beta| \bar{a}}{\tau^n} =: K_n.
\]

Since both \( \sigma^n \) and \( \tau^n \) are \( O(\sqrt{n}) \), \( K_n = O(1/\sqrt{n}). \) Noting that \( \zeta^n \) converges to a positive

\[\text{[14]} \text{The computation of the variance uses the fact that } \mathbb{E}(Z_i \text{ sign } Z_i) = \mathbb{E}(|Z_i|) = \sqrt{2/\pi}. \]

\[\text{11} \]
finite limit, we have
\[
\frac{1}{(\xi^n)^4} \sum_{i=1}^{n} \mathbb{E}[(R_i^n)^4] \leq \frac{nK_n^4}{(\xi^n)^4} \to 0.
\]
This completes the proof for asymptotic normality of \(\left(\frac{\mathcal{S}}{r}, \frac{\mathcal{T}}{r}\right)\). The remaining step that the limit distribution of \(W\) is stochastically increasing in \(c_{\text{sqr}}(a)\) is the same as in the proof of Lemma\(\ref{lemma:asymptotic_normality}\). \(\square\)

Theorem\(\ref{theorem:asymptotic_normality}\) shows that in the independent-preference model, the (asymptotic) stochastic welfare ordering of weight allocations differs from the ordering induced by plain cosine proportionality, and depends on the representation rule \(r\). Under winner-take-all, the stochastic welfare ordering is asymptotically equivalent to the cosine between the weights of the groups and the square roots of the populations in the groups. This is consistent with what is known in the literature as the “square root law,” which states that in a model with independent preferences, under winner-take-all, the weights proportional to the square roots of populations are optimal in terms of social welfare, and also in terms of equity among individuals (e.g., Barberà and Jackson \(2006\), Beisbart and Bovens \(2007\), and Penrose \(1946\)). Under proportional representation, the equivalence theorem holds for any number of groups, but with a somewhat complicated index: the cosine proportionality between the square roots of the populations \((\sqrt{s_i})\) and the weights divided by the square roots of the populations \((a_i/\sqrt{s_i})\). Interestingly, if the total weight \(\sum a_i\) is fixed, this index is ordinally equivalent to (the negative of) the well-known Saint-Laguè index of disproportionality: \(\sum a_i^2/s_i\) (see, e.g., Balinski and Young \(2010\)).

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