Stability of circular geodesics in equatorial plane of Kerr spacetime

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Abstract We analyze the stability of circular geodesics for timelike as well as null geodesics of the Kerr BH spacetime with rotation parameter on the equatorial plane by Lyapunov stability analysis. Also, we verify the results of stability by presenting the phase portrait for both timelike and null geodesics. Further, by reviewing the Kosambi–Cartan–Chern (KCC) theory, we analyze the Jacobi stability for Kerr spacetime and present a comparative study of the methods used for stability analysis of geodesics.

1 Introduction
The generalized black hole (BH) solution in general relativity (GR) that includes spin along with mass is known as Kerr BH. This BH was observed as an exact solution of Einstein field equations whose mathematical description is given by Roy Kerr [1, 2]. This BH solution is different from Schwarzschild BH (SBH) solution in many aspects, such as it is spinning BH having two symmetries; one in time and the other along a certain axis. The Kerr metric is parameterized by its angular momentum and mass, while SBH metric is only parameterized by their mass, and it has only spherical symmetry [3–5]. The motion of test particles in curved spacetime under the influence of gravity is governed by geodesic equations [1, 6, 7]. As the mathematical formulation of geodesic equations in Kerr spacetime is nonlinear differential equation which are difficult to solve explicitly. To overcome this difficulty, we use the Lyapunov stability analysis of nonlinear system along with phase portraits [8–11]. A. M. Lyapunov first developed stability theory for nonlinear ordinary differential equations by characterizing the behavior of the dynamical systems trajectories on the basis of nearby solutions [12, 13].

The main aim of our analysis of dynamical system is that, whether or not the system has stable equilibria. We characterize an equilibrium as stable or unstable based on the behavior of solutions whose initial conditions are in the neighborhood of the equilibrium [14, 15]. As phase portraits combined with linear stability analysis can generally provide us a full picture of the dynamics. However, it becomes much more difficult in higher-dimensional spaces so one can use the Lyapunov stability analysis technique to determine the stability of an equilibrium point both near and far from the equilibrium point. The stability analysis carried out at each equilibrium point involves Lyapunov’s linear stability analysis and Jacobi stability analysis method [12, 16, 17]. Jacobi stability analysis is carried out by using Kosambi–Cartan–Chern (KCC) theory which is widely used to investigate the properties of dynamical system in terms of five geometrical invariants. The second KCC invariant which is also known as a deviation curvature tensor gives us the Jacobi stability of the trajectories which measures the robustness of the second-order differential equation [18]. The Jacobi stability studies the robustness of second-order differential equation which is analyzed by calculating deviation curvature tensor (second KCC invariant) by KCC theory [19–22]. The Lyapunov and Jacobi stability of circular orbits in the SBH spacetime has already been investigated in detail by Hossein [4]. However, various investigations regarding geodesics stability around BH spacetimes have been performed time and again in diverse context [10, 11, 23–26]. Here, our main objective is to perform the stability analysis of geodesics of Kerr BH with a different approach namely the Lyapunov and Jacobi stability analysis to have a more precise information regarding the stability of circular geodesics which are important from the view point of astrophysics.

The present paper is organized as follows. In Sect. 2, we review the basic mathematical concepts of the stability theory of the dynamical system. In Sect. 3, we introduce the metric of rotating BH and Euler–Lagrange equation is used to obtain geodesic equations. We have also studied the variation of effective potential with rotation parameter. In Sect. 4, the Lyapunov stability for...
timelike as well as null geodesics is calculated at the equilibrium point. Also, phase portrait in $r - p$ plane for timelike as well as null geodesics are presented in this section. In Sect. 5, we analyze the Jacobi stability of the system by using KCC theory. Finally, in Sect. 6, we discuss and conclude the obtained results.

2 Stability analysis of dynamical system

In this section, we first review the equilibrium point of the differential equation and its classification with its nature. We also review some mathematical concepts of stability of dynamical system used in this work [27]. As the dynamical system may be linear or nonlinear, some nonlinear systems can be very complicated and difficult to solve explicitly. To analyze such systems, we first linearize them at their equilibria and then construct a phase portrait to visualize the trajectories of the solutions of the system [28,29].

2.1 Classifying equilibria with stability

In a differential equation, an equilibrium point is a constant solution of the equation. For a system of autonomous ODE’s

$$\dot{X} = f(X),$$

a point $X^* \in \mathbb{R}^n$ will be an equilibrium point if $f(X^*) = 0$. This equilibrium point is called hyperbolic if all the eigen values of the Jacobian matrix evaluated at equilibrium point have a negative real part; otherwise, the point is non-hyperbolic. If $\lambda_1$ and $\lambda_2$ are the eigen values of the Jacobian matrix $J$ evaluated at the equilibrium point $X^*$ of the two-dimensional system of differential equation, then the nature and stability of equilibrium points are classified as shown in Table 1.

Boehmar et al. [16] broadly represented the phase portraits of a two-dimensional differential equation for the various cases of the eigenvalues of the Jacobian matrix of the system.

2.2 Jacobi stability via KCC theory

To investigate the properties of the dynamical systems, a powerful mathematical theory was proposed by the Kosambi–Cartan–Chern (KCC) [30–32]. In this theory, the properties of any dynamical system are described in terms of five geometrical invariants with the second one giving the Jacobi stability of the system. Consider a system of second-order differential equation [33]

$$\frac{d^2 x^i}{dt^2} + 2G^i(X, Y) = 0; i = 1, 2, 3, ..., n,$$

where $G^i(X, Y)$ are smooth functions defined in a local system of coordinates on TM(tangent bundle of real, smooth n-dimensional manifold M) with $X = (x^1, x^2, ..., x^n)$, $Y = (y^1, y^2, ..., y^n)$.

The second KCC invariant which is also known as a deviation curvature tensor of the system of second-order differential equation (2) is given by

$$P_j^i = -2 \frac{\partial G^i}{\partial y^j} - 2G^jG^i_{jl} + y^j \frac{\partial N^i_j}{\partial x^l} + N^i_j N^i_l,$$

where $G^i_{jl} = \frac{\partial N^i_j}{\partial y^l}$ is called the Berwald connection and $N^i_j = \frac{\partial G^i}{\partial y^j}$ defines the coefficient of a nonlinear connection N on the tangent bundle TM.

The trajectories of given system of second-order differential equations are Jacobi stable if and only if the real parts of the eigenvalues of the deviation curvature tensor $P_j^i$ are strictly negative everywhere, and Jacobi unstable otherwise. Geometrically, the trajectories of the system are bunching together if they are Jacobi stable and dispersing if they are Jacobi unstable.

| Eigen values of Jacobian matrix | Types of equilibrium point | Stability |
|---------------------------------|---------------------------|-----------|
| $\lambda_1 \leq \lambda_2 < 0$ | Node(sink)                | Stable    |
| $\lambda_1 \geq \lambda_2 > 0$| Node(source)              | Unstable  |
| $\lambda_1 < 0 < \lambda_2$  | Saddle                    | Unstable  |
| $\lambda_{1,2} = a \pm ib$ with $a < 0$ | Spiral                   | Stable    |
| $\lambda_{1,2} = a \pm ib$ with $a > 0$ | Spiral                   | Unstable  |
| $\lambda_{1,2} = a \pm ib$ with $a = 0$ | Center                   | Stable    |
3 The Kerr BH spacetime

The Kerr BH spacetime is an exact solution of Einstein’s field equations in vacuum describing an axisymmetric, non-static, asymptotically flat gravitational field [34,35]. In Boyer–Lindquist coordinates, the geometry of Kerr metric is given as follows

\[ ds^2 = g_{ab} dx^a dx^b = -(1 - \frac{2M}{\rho^2}) dt^2 + \left(\frac{4Mr \sin^2 \theta}{\rho^2}\right) d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right) \sin^2 \theta d\phi^2. \]  

(4)

where \( \rho^2 = r^2 + a^2 \cos^2 \theta \), and \( \Delta = r^2 - 2Mr + a^2 \).

Here, the specific angular momentum (Kerr parameter) is defined as \( a = J/M \), where \( J \) is angular momentum and \( M \) is the mass of the BH. The coordinates \((t, r, \theta, \phi)\) used in this geometry are called Boyer–Lindquist coordinate. To analyze the stability of geodesics in Kerr BH spacetime on the equatorial plane, we fixed \( \theta = \pi/2 \), for which \( \rho = r \) and \( d\theta = 0 \). Thus, the geometry of Kerr metric is reduced to,

\[ ds^2 = -(1 - \frac{2M}{r}) dt^2 - \frac{4aM}{r} dr d\phi + \frac{r^2}{\Delta} d\phi + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right) d\phi^2. \]  

(5)

From the Euler–Lagrange equation, the generalized momentum for the geodesics of Kerr BH is as follows

\[ i = \frac{1}{\Delta} \left[ (r^2 + a^2 + \frac{2Ma^2}{r}) \epsilon - \frac{2Mae}{r} \right], \]  

(6)

\[ \dot{\phi} = \frac{1}{\Delta} \left[ (1 - \frac{2M}{r}) l + \frac{2Mae}{r} \right], \]  

(7)

\[ \dot{r} = \sqrt{\epsilon^2 + K - \frac{2KM}{r} - \frac{a^2(\epsilon^2 + K)}{r^2} + \frac{2M(l - ae)^2}{r^3}}, \]  

(8)

where,

\[ K = \begin{cases} -1, & \text{for timelike geodesics} \\ 0, & \text{for null geodesics} \end{cases} \]  

(9)

Here, \( \epsilon \) and \( l \) represent the energy and angular momentum per unit mass of the particle moving around BH, respectively.

Now, from Eq. (8) we obtain, radial equation as

\[ \dot{r}^2 = \epsilon^2 + K - \frac{2KM}{r} - \frac{a^2(\epsilon^2 + K)}{r^2} + \frac{2M(l - ae)^2}{r^3}. \]  

(10)

This equation can be written in the following form

\[ \frac{\epsilon^2}{2} = \frac{\dot{r}^2}{2} + V(r), \]  

(11)

where,

\[ V(r) = -\frac{K}{2} + \frac{KM}{r} + \frac{a^2(\epsilon^2 + K)}{2r^2} - \frac{M(l - ae)^2}{r^3}. \]  

(12)

is the expression of effective potential for Kerr BH spacetime. By putting the values of \( K \) in Eq. (12), we can obtained the effective potential expression for timelike and null geodesics. The graphical representation of effective potential is shown in the figure given below for the various values of \( \epsilon \).

To study the behavior of the test particle near BH, Fig. 1a, b shows graphical representation of effective potential for timelike and null geodesics for various value of Kerr parameter ‘\( a \)’. The same is also presented for SBH case with limit \( a = 0 \). In each curve of Fig. 1a, we see that there are two extreme points in which one is maxima and other is minima. Thus, stable circular orbits exist correspond to point of minima, while in each curve of Fig. 1b there is no minima which indicates no stable circular orbits exists in case of null geodesics.

4 Lyapunov stability analysis

The derivative of (11) w.r.t. affine parameter \( \tau \) gives us a one-dimensional geodesics equation

\[ \ddot{r} = -V'(r). \]  

(13)
Then, Eq. (13) can be transformed into the system of first-order differential equation in $r - p$ space as follows,

$$\begin{cases}
\dot{r} = p, \\
\dot{p} = -V'(r).
\end{cases} \quad (14)$$

To apply linearization process, let $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ be the vector field $f(r, p) = (p, -V'(r))$ for the system (14).

Clearly, the equilibrium point of this system is the points $(r_*, 0)$, where $r_*$ is the solution of the equation $V'(r) = 0$. Now, the Jacobian matrix for the above system at the any point $(r, p)$ is defined as,

$$J = \frac{\partial f}{\partial x}(r, p) = \begin{bmatrix} 0 & 1 \\ -V''(r) & 0 \end{bmatrix}. \quad (15)$$

Now, the eigenvalues obtained from the characteristic equation of this Jacobian matrix at the equilibrium point $(r_*, 0)$ are given as,

$$\lambda = \pm \sqrt{-V''(r_*)}. \quad (16)$$

Thus, the equilibrium point is a saddle point when $V''(r_*) < 0$ and is a possible center when $V''(r_*) > 0$. The Lyapunov function for the system is defined as

$$E = \frac{p^2}{2} + V(r), \quad (17)$$

and its Hessian matrix is as follows

$$H = \frac{\partial^2 E}{\partial r \partial p} = \begin{bmatrix} V''(r_*) & 0 \\ 0 & 1 \end{bmatrix}. \quad (18)$$

Clearly, this matrix is positive definite when $V''(r) > 0$, and thus at a possible center $(r_*, 0)$ the Lyapunov function has a local minimum. Thus, the equilibrium point $(r_*, 0)$ is said to be

$$\begin{cases}
\text{Lyapunov stable if, } V''(r_*) > 0, \\
\text{Lyapunov unstable if, } V''(r_*) < 0.
\end{cases} \quad (19)$$

4.1 For timelike geodesics

In this section, we analyze the Lyapunov stability of equilibrium point for timelike geodesics and nature of equilibrium point is shown by corresponding phase-portrait. By substituting $K = -1$ in the equation (12), we get the expression of effective potential for timelike geodesics equation as,

$$V(r) = \frac{1}{2} - \frac{M}{r} + \frac{l^2 - a^2 (e^2 - 1)}{2r^2} - \frac{M (l - ae)^2}{r^3}. \quad (20)$$
Table 2 Calculation of $V''(r)$ at the equilibrium point for the timelike geodesics for different values of Kerr parameter ‘$a$’ fixing $M = 1$, $e = 1$ and $l = 4$

| S. no. | $a$       | $r^−_a$ | $r^+_a$ | $V''(r^−_a)$ | $V''(r^+_a)$ |
|-------|-----------|---------|---------|--------------|--------------|
| 1     | 0 (SBH)   | 4       | 12      | -0.03125     | 0.0003858024 |
| 2     | 0.2       | 3.45247 | 12.5475 | -0.0640159   | 0.000366925  |
| 3     | 0.5       | 2.77985 | 13.2202 | -0.174834    | 0.00034179   |
| 4     | 0.8       | 2.23112 | 13.7689 | -0.465616    | 0.000321014  |

The derivative of Eq. (20) w.r.t. ‘$r$’ is obtained as

$$V'(r) = \frac{M}{r^2} - \frac{l^2 - a^2 (e^2 - 1)}{r^3} + \frac{3M(l - ae)^2}{r^4},$$  

(21)

and the second derivative of Eq.(20) is then

$$V''(r) = -\frac{2M}{r^3} + \frac{3(l^2 - a^2 (e^2 - 1))}{r^4} - \frac{12M(l - ae)^2}{r^5}.$$  

(22)

Therefore, equilibrium points can be obtained by solving the equation $V'(r) = 0$; thus, we get

$$r^\pm_a = \frac{l^2 - a^2 (e^2 - 1) \pm \sqrt{(l^2 - a^2 (e^2 - 1))^2 - 12M^2(l - ae)^2}}{2M}.$$  

(23)

Clearly, there are two equilibrium points $(r^−_a, 0)$ and $(r^+_a, 0)$ in the case of timelike geodesics for which $V''(r)$ is calculated to analyze the Lyapunov stability and shown in Table 2 for the different values of ‘$a$’.

As, $\lambda = \pm \sqrt{-V''(r_a)}$ therefore from Table 2, we can see that at the equilibrium point $(r^−_a, 0)$ the eigenvalues are real, distinct and opposite in sign. So $(r^−_a, 0)$ is a saddle point which is Lyapunov unstable for all the values of specific angular momentum ‘$a$’, while at the equilibrium point $(r^+_a, 0)$ the eigenvalues are purely imaginary, so $(r^+_a, 0)$ is center which is Lyapunov stable for all the values of ‘$a$’.

To visualize the nature of equilibrium points of timelike geodesics, the phase portraits are depicted for different values of Kerr parameter ‘$a$’ as shown in Fig. 2

Figure 2 depicts the phase portrait in $r - p$ plane for stable and unstable circular orbits for timelike geodesics in Kerr spacetime for various values of Kerr parameter ‘$a$’. For each values of ‘$a$’, we observe that $V''(r^-_a) < 0$ and $V''(r^+_a) > 0$ hence equilibrium point $(r^-_a, 0)$ is an unstable saddle point, while $(r^+_a, 0)$ is a stable center.

The range of stable circular orbit for the different values of ‘$a$’ in the case of timelike geodesics is given in Table 3.
Table 3  Range of stable circular orbits for timelike geodesics for various values of \( a \)

| S. no. | \( a \)     | Range of stable circular orbits |
|--------|------------|---------------------------------|
| 1      | 0 (SBH)   | \( r_a^+ > 6 \)                 |
| 2      | 0.2       | \( r_a^+ > 5.28447 \)           |
| 3      | 0.5       | \( r_a^+ > 4.08432 \)           |
| 4      | 0.8       | \( r_a^+ > 2.44271 \)           |

Table 4  Calculation of \( V''(r) \) at the equilibrium point for the null geodesics for different values of Kerr parameter \( a \) when \( M = 1, e = 1 \) and \( l = 4 \)

| S. no. | \( a \)     | \( r_\ast \) | \( V''(r_\ast) \) |
|--------|------------|-------------|-------------------|
| 1      | 0 (SBH)   | 3           | -0.197531         |
| 2      | 0.2       | 2.71429     | -0.294039         |
| 3      | 0.5       | 2.33333     | -0.506045         |
| 4      | 0.8       | 2           | -0.96             |

4.2 For null geodesics

By setting \( K = 0 \) in Eq. (12), the expression of effective potential for null geodesics is obtained as,

\[
V(r) = \frac{l^2 - a^2 e^2}{2r^2} - \frac{M (l - ae)^2}{r^3},
\]

(24)

The derivative of Eq. (24) w.r.t. \( r \) is given as

\[
V'(r) = -\frac{l^2 - a^2 e^2}{r^3} + \frac{3M (l - ae)^2}{r^4},
\]

(25)

and the second derivative is given as

\[
V''(r) = \frac{3 (l^2 - a^2 e^2)}{r^4} - \frac{12 (l - ae)^2}{r^5}.
\]

(26)

By solving the equation \( V'(r) = 0 \), we obtain

\[
r_\ast = \frac{3M (l - ae)}{l + ae}.
\]

(27)

Here, we can see that in the case of null geodesics, we have only one equilibrium point \((r_\ast, 0)\) for each values of Kerr parameter \( a \), the value of \( V''(r) \) is calculated and shown in Table 4 given below.

From Table 4, since \( V''(r_\ast) < 0 \), i.e., \( V \) is maximum at \( r_\ast \), therefore, from Eq. (16) we see that the eigenvalues of the Jacobian matrix at \((r_\ast, 0)\) are real, distinct and opposite in sign so the equilibrium point \((r_\ast, 0)\) is saddle point which is Lyapunov unstable for all values of \( a \). Therefore, in the case of null geodesics there are no stable circular orbits for any value of Kerr parameter \( a \). To visualize the complete nature of equilibrium point of null geodesics, phase portrait is depicted in Fig. 3 for different values of Kerr parameter \( a \). Figure 3 depicts that there is only one extreme point in case of null geodesics for each values of Kerr parameter \( a \) which is unstable saddle point. No stable circular orbits exist in this case.

5 Jacobi stability

The second-order differential equation corresponding to the system of Eq. (14) is given as

\[
\ddot{r} + V'(r) = 0.
\]

(28)

By using the expression of effective potential of Kerr spacetime given in Eq. (12), we get

\[
\ddot{r} - \frac{KM}{r^2} - \frac{l^2 - a^2 (e^2 + K)}{r^3} + \frac{3M (l - ae)^2}{r^4} = 0.
\]

(29)
Comparing the above equation with general second-order differential equation (2) used in KCC theory, we have

\[
G^1 (r, p) = \frac{1}{2} \left( -\frac{KM}{r^2} - \frac{l^2 - a^2 (e^2 + K)}{r^3} + \frac{3M (l - ae)^2}{r^4} \right). \tag{30}
\]

The derivative of Eq. (30) with respect to \( r \) is obtained as

\[
\frac{\partial G^1}{\partial r} = \frac{1}{2} \left[ \frac{2KM}{r^3} - 3 \frac{l^2 - a^2 (e^2 + K)}{r^4} \right] + 12 \frac{M (l - ae)^2}{r^5}. \tag{31}
\]

The nonlinear connection associated with this system is obtained as

\[
N^1_1 = \frac{\partial G^1}{\partial p} = 0, \tag{32}
\]

and the Berwald connection is obtained as

\[
G^1_{11} = \frac{\partial N^1_1}{\partial p} = 0. \tag{33}
\]

Finally, the second KCC invariant is given by the equation

\[
P^1_1 (r, p) = -2 \frac{\partial G^1}{\partial r} - 2G^1G^1_{11} + p \frac{\partial N^1_1}{\partial r} + N^1_1 N^1_1. \tag{34}
\]

By substituting the values of \( \frac{\partial G^1}{\partial r} \), \( N^1_1 \) and \( G^1_{11} \) in Eq. (34), the second KCC invariant is obtained as

\[
P^1_1 (r, p) = -\frac{2KM}{r^3} - 3 \frac{l^2 - a^2 (e^2 + K)}{r^4} + 12 \frac{M (l - ae)^2}{r^5}. \tag{35}
\]

At the equilibrium point \((r_*, 0)\), the second KCC invariant is reduced as

\[
P^1_1 (r_*, 0) = -\frac{2KM}{r_*^3} - 3 \frac{l^2 - a^2 (e^2 - 1)}{r_*^4} + 12 \frac{M (l - ae)^2}{r_*^5}. \tag{36}
\]

For the timelike geodesics, inserting \( K = -1 \) in Eq. (36) we get

\[
P^1_1 (r_*, 0) = \frac{2M}{r_*^3} - 3 \frac{l^2 - a^2 (e^2 - 1)}{r_*^4} + 12 \frac{M (l - ae)^2}{r_*^5}. \tag{37}
\]

which can be rewritten as following form

\[
P^1_1 (r_*, 0) = \frac{(r_* - A) (r_* - B)}{r_*^3}, \tag{38}
\]
Thus, in the case of null geodesics the equilibrium point is Jacobi unstable.

From this inequality, we observed that the point \( (r_*, 0) \) is Jacobi stable if \( r_* < A \) or \( r_* > B \); otherwise, the point is Jacobi unstable.

For the null geodesics, inserting \( K = 0 \) in Eq. (36) we get

\[
P_1^1 (r_*, 0) = -\frac{3 (l^2 - a^2 e^2)}{r_*^4} + \frac{12M(l - ae)^2}{r_*^5} < 0
\]

Thus, the point \( (r_*, 0) \) is Jacobi stable if \( P_1^1 (r_*, 0) < 0 \), i.e.,

\[
\frac{(r_* - A)(r_* - B)}{r_*^5} < 0
\]

From this inequality, we obtained that the point \( (r_*, 0) \) is Jacobi stable if either \( r_* < A \) or \( r_* > B \); otherwise, the point is Jacobi unstable.

Thus, in the case of null geodesics the equilibrium point \( (r_*, 0) \) is Jacobi stable if \( r_* > \frac{4M(l - ae)}{l + ae} \), otherwise the point is Jacobi unstable.

6 Conclusions

In this paper, we have studied the stability of timelike as well as null circular geodesics in background of Kerr BH spacetime on the equatorial plane by using Lyapunov stability and the Jacobi stability analysis. We have analyzed the effect of Kerr parameter (specific angular momentum) ‘a’ of the BH in the region of stable circular orbit by using effective potential and phase portrait analysis. The linear stability analysis is performed by the linearization of the dynamical system via the Jacobian matrix of a nonlinear system at the equilibrium point. The KCC theory is used to examine Jacobi stability which shows that the trajectories of the dynamical system bunch together or disperse when approaching the equilibrium point. In the present paper, we see that in the case of timelike geodesics there are two equilibrium points \( (r_*^-, 0) \) and \( (r_*^+, 0) \) out of which the equilibrium point \( (r_*^-, 0) \) is a Lyapunov unstable saddle point and the another equilibrium point \( (r_*^+, 0) \) is stable center. We also calculate the range of stable circular geodesics for the various values of specific angular momentum ‘a’ for the particle of unit energy per unit mass and observed that as the value of Kerr parameter increases from 0 to 1 the range of stable circular orbits expand as shown in Table 3. But in the case of null geodesics, there are no stable circular orbits as there is only one equilibrium point \( (r_*, 0) \) which is a Lyapunov unstable saddle point. Further, we investigate the Jacobi stability for timelike as well as null geodesics and obtained the condition for Jacobi stable equilibrium point. In the case of timelike geodesics, the equilibrium point \( (r_*, 0) \) is Jacobi stable if either \( r_* < A \) or \( r_* > B \); otherwise, the point is Jacobi unstable and in the case of null geodesics, the equilibrium point \( (r_*, 0) \) is Jacobi stable if \( r_* > \frac{4M(l - ae)}{l + ae} \), otherwise, the point is Jacobi unstable.

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