Moduli of mathematical instanton vector bundles with odd $c_2$ on projective space

A. S. Tikhomirov

Abstract. We study the moduli space $I_n$ of mathematical instanton vector bundles of rank 2 with second Chern class $n \geq 1$ on the projective space $\mathbb{P}^3$, and prove the irreducibility of $I_n$ for arbitrary odd $n \geq 1$.

Keywords: vector bundles, mathematical instantons, moduli space.

Dedicated to the memory of Andrei Nikolaevich Tyurin

§ 1. Introduction

By a mathematical $n$-instanton vector bundle (shortly, an $n$-instanton) on the 3-dimensional projective space $\mathbb{P}^3$ we mean an algebraic vector bundle $E$ of rank 2 on $\mathbb{P}^3$ with Chern classes

$$c_1(E) = 0, \quad c_2(E) = n, \quad n \geq 1,$$

such that the following conditions hold:

$$h^0(E) = h^1(E(-2)) = 0.$$

The set of isomorphism classes of $n$-instantons is denoted by $I_n$. This set is non-empty for all $n \geq 1$ (see, for example, [1], [2]). The condition $h^0(E) = 0$ for an $n$-instanton $E$ implies that $E$ is stable in the sense of Gieseker–Maruyama. Hence $I_n$ is a subset of the Gieseker–Maruyama moduli scheme $M_{\mathbb{P}^3}(2; 0, 2, 0)$ of semistable coherent torsion-free sheaves of rank 2 on $\mathbb{P}^3$ with Chern classes $c_1 = 0$, $c_2 = n$, $c_3 = 0$. By semicontinuity, the condition $h^1(E(-2)) = 0$ for $[E] \in I_n$ (called the instanton condition) implies that $I_n$ is a Zariski-open subset of $M_{\mathbb{P}^3}(2; 0, 2, 0)$, that is, $I_n$ is a quasiprojective scheme. It is called the moduli scheme (or moduli space) of mathematical $n$-instantons.

In this paper we study the problem of the irreducibility of the scheme $I_n$. This problem has an affirmative solution for small values of $n$: the cases $n = 1, \ldots, 5$ were settled in [3]–[7] respectively. Our aim is to prove the following result.

Theorem 1.1. When $n = 2m + 1$ for any $m \geq 0$, the moduli scheme $I_n$ of mathematical $n$-instantons is an integral scheme of dimension $8n - 3$.

The layout of the paper is as follows. In § 3 we recall a well-known relation between mathematical $n$-instantons and nets of quadrics in a fixed $n$-dimensional...
vector space $H_n$ over $k$. These nets are regarded as vectors in $S_n = S^2 H_n^\vee \otimes \wedge^2 V^\vee$, where $V = H^0(\mathcal{O}_{\mathbb{P}^3}(1))^\vee$, and the nets corresponding to $n$-instantons (which we call $n$-instanton nets) satisfy the so-called Barth conditions (see definition (3.12)), and they form a locally closed subset $MI_n$ of $S_n$ having the structure of a principal $(\text{GL}(n)/\{\pm 1\})$-bundle over $I_n$. Thus the problem of irreducibility of the moduli space $I_n$ reduces to the problem of the irreducibility of the space $MI_n$ of $n$-instanton nets of quadrics.

In §4 we study the linear algebra related to the direct sum decomposition $\xi: H_{m+1} \oplus H_m \sim H_{2m+1}$. Using a result from §12, we obtain formula (4.11), which plays a key role in what follows. The decomposition $\xi$ also enables us to relate $(2m+1)$-instantons $E$ to symplectic vector bundles $E_{2m+2}$ of rank $2m+2$ on $\mathbb{P}^3$ satisfying the vanishing conditions $h^0(E_{2m+2}) = h^2(E_{2m+2}(-2)) = 0$.

In §6 we introduce a new set $X_m$ as a locally closed subset of the vector space $S_{m+1} \oplus \Sigma_{m+1}$, where $\Sigma_{m+1} = \text{Hom}(H_m, H_{m+1}^\vee \otimes \wedge^2 V^\vee)$, defined by linear algebraic data that are somewhat similar to the Barth conditions. We prove that $X_m$ is isomorphic to an open dense subset $MI_{2m+1}(\xi)$ in $MI_{2m+1}$ determined by the choice of the direct sum decomposition $\xi$. (Here $X_m$ and $MI_{2m+1}(\xi)$ are regarded as reduced schemes.) This reduces the problem of the irreducibility of $I_{2m+1}$ to that of $X_m$.

The final step in the proof of Theorem 1.1 makes use of a scheme $Z_m$, which is introduced in §7 as a locally closed subscheme of the affine space $S_m^\vee \times \text{Hom}(H_m, H_{m+1}^\vee \otimes \wedge^2 V^\vee)$ defined by explicit equations (see (7.4)). In §7 we reduce the proof of Theorem 1.1 to proving that $Z_m$ (as a scheme) is an integral locally complete intersection in this affine space. This and other properties of $Z_m$ are stated in Theorem 7.2. The rest of the paper is devoted to the proof of Theorem 7.2.

This proof is by induction on $m$ and we begin in §8 by proving a part of the induction step (Proposition 8.1) using explicit computations in linear algebra. These computations are somewhat cumbersome. In Remark 8.3 we explain why they cannot be essentially simplified.

In §9 we use Proposition 8.1 to connect $Z_m$ with the so-called ’t Hooft instantons. A universal family of ’t Hooft extensions is described in §10 along with some related constructions. This enables us to complete the induction step of the proof in §11.

In the Appendix (§12) we prove two ‘general position’ results for nets of quadrics used in the text.

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§2. Notation and conventions

Our notation is mostly standard. The base field $k$ is assumed to be algebraically closed of characteristic 0. We identify algebraic vector bundles with locally free sheaves. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an algebraic variety or scheme $X$, then $n\mathcal{F}$ stands for the direct sum of $n$ copies of $\mathcal{F}$. We denote the $i$th cohomology group of $\mathcal{F}$ by $H^i(\mathcal{F})$, put $h^i(\mathcal{F}) := \dim H^i(\mathcal{F})$, and write $\mathcal{F}^\vee$ for the sheaf dual to $\mathcal{F}$, that is, $\mathcal{F}^\vee := \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. When $Z$ is a subscheme of $X$, the ideal
sheaf corresponding to $Z$ is denoted by $\mathcal{I}_{Z,X}$. If $X = \mathbb{P}^r$ and $t$ is an integer, then we write $\mathcal{F}(t)$ for the sheaf $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(t)$. The isomorphism class of a sheaf $\mathcal{F}$ is denoted by $[\mathcal{F}]$. Given any morphism $f: \mathcal{F} \rightarrow \mathcal{F}'$ of $\mathcal{O}_X$-sheaves and a $k$-vector space $U$ (resp. any homomorphism $f: U \rightarrow U'$ of $k$-vector spaces), we shall for brevity denote the induced morphism of sheaves $\text{id} \otimes f: U \otimes \mathcal{F} \rightarrow U \otimes \mathcal{F}'$ (resp. the induced morphism $f \otimes \text{id}: U \otimes \mathcal{F} \rightarrow U' \otimes \mathcal{F}$) again by $f$.

Throughout the paper $V$ is a fixed vector space of dimension 4 over $k$ and we put $\mathbb{P}^3 := P(V)$. We always let $u$ and $v$ denote the morphisms in the Euler exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} V \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{v} T_{\mathbb{P}^3}(-1) \rightarrow 0.$$

Given any $k$-vector spaces $U$, $W$ and an arbitrary vector $\varphi \in \text{Hom}(U, W \otimes \wedge^2 V^\vee) \subset \text{Hom}(U \otimes V, W \otimes V^\vee)$ regarded as a homomorphism $\varphi: U \otimes V \rightarrow W \otimes V^\vee$ or, equivalently, a homomorphism $\tilde{\varphi}: U \rightarrow W \otimes \wedge^2 V^\vee$, we write $\tilde{\varphi}$ for the composite

$$U \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\tilde{\varphi}} W \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\varepsilon} W \otimes \Omega_{\mathbb{P}^3}(2),$$

where $\varepsilon$ is the induced morphism in the exact triple $0 \rightarrow \wedge^2 \Omega_{\mathbb{P}^3}(2) \xrightarrow{\wedge^2 \nu^\vee} \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\varepsilon} \Omega_{\mathbb{P}^3}(2) \rightarrow 0$ obtained by passing to wedge squares in the dual Euler exact sequence. To simplify the notation, we sometimes omit the subscript $\mathbb{P}^3$ from the symbols for sheaves on $\mathbb{P}^3$ and write $\mathcal{O}, \Omega, \ldots$ instead of $\mathcal{O}_{\mathbb{P}^3}, \Omega_{\mathbb{P}^3}, \ldots$ respectively.

As above, for every integer $n \geq 1$ we let $H_n$ be a fixed $n$-dimensional vector space over $k$. (For example, one can take $k^n$ for $H_n$.)

The vector space $S^2 H_m^\vee \otimes \wedge^2 V^\vee$ (resp. $\text{Hom}(H_m, H_{m+1}^\vee \otimes \wedge^2 V^\vee)$) is denoted throughout by $\mathcal{S}_m$ (resp. $\Sigma_{m+1}$) for every $m \geq 1$. Given a $k$-vector space $U$ (resp. a direct sum $U \oplus U'$ of $k$-vector spaces), we shall use the same symbol $U$ (resp. $U \times U'$) to denote the affine space $\mathbf{V}(U^\vee) = \text{Spec}(\text{Sym}^* U^\vee)$ (resp. the direct product $\mathbf{V}(U^\vee) \times \mathbf{V}((U')^\vee)$ of affine spaces).

All schemes in this paper are Noetherian. By an irreducible scheme we understand a scheme whose underlying topological space is irreducible. An integral scheme is an irreducible reduced scheme. The dimension of a scheme is the maximum of the dimensions of its irreducible components. By a general point of an irreducible (not necessarily reduced) scheme $\mathcal{X}$ we mean any closed point belonging to an open dense subset of $\mathcal{X}$.

§ 3. Some generalities on instantons. The set $MI_n$

In this section we recall some well-known facts about mathematical instanton bundles (see, for example, [7]).

For a given $n$-instanton $E$ it follows from conditions (1.1), (1.2), the Riemann–Roch theorem and Serre duality that

\begin{align}
  h^1(E(-1)) = h^2(E(-3)) = n, & \quad h^1(E \otimes \Omega^1_{\mathbb{P}^3}) = h^2(E \otimes \Omega^2_{\mathbb{P}^3}) = 2n + 2, \\
  h^1(E) = h^2(E(-4)) = 2n - 2, & \quad (3.1) \\
  h^i(E) = h^i(E(-1)) = h^{3-i}(E(-3)) = h^{3-i}(E(-4)) = 0, & \quad i \not= 1, \\
  h^i(E(-2)) = 0, & \quad i \geq 0. \quad (3.2)
\end{align}
Furthermore, the condition $c_1(E) = 0$ yields an isomorphism $\wedge^2 E \cong \mathcal{O}_{\mathfrak{p}^3}$ and, therefore, a symplectic isomorphism $j: E \cong E^\vee$, which is uniquely defined up to a scalar factor. Consider a triple $(E, f, j)$, where $E$ is an $n$-instanton, $f$ is an isomorphism $H_n \cong H^2(E(-3))$, and $j: E \cong E^\vee$ is a symplectic structure on $E$. Note that $E$, being a stable rank-2 bundle, is simple, and so all the automorphisms $\varphi$ of $E$ have the form $\varphi = \lambda \text{id}$ for some $\lambda \in \mathbb{k}^*$. Imposing the condition of the compatibility of $\varphi$ with the symplectic structure $j$, that is, the condition $\varphi^\vee \circ j \circ \varphi = j$, we obtain $\lambda = \pm 1$. This suggests the following definition of equivalence of triples. Two triples $(E, f, j)$ and $(E', f', j')$ are equivalent if there is an isomorphism $g: E \cong E'$ such that $g_\ast \circ f = \lambda f'$, where $\lambda \in \{1, -1\}$ and $g_\ast: H^2(E(-3)) \cong H^2(E'(-3))$ is the induced isomorphism, and $j = g^\vee \circ j' \circ g$. We write $[E, f, j]$ for the equivalence class of a triple $(E, f, j)$. It follows easily from this definition that the set $F_{[E]}$ of all equivalence classes $[E, f, j]$ with given $[E]$ is a homogeneous space of the group $\text{GL}(n)/\{\pm \text{id}\}$.

Every class $[E, f, j]$ determines a point

$$A = A([E, f, j]) \in S^2 H_n^\vee \otimes \wedge^2 V^\vee$$

in the following way. Consider the exact sequences

$$0 \to \Omega^1_{\mathfrak{p}^3} \xrightarrow{i_1} \wedge^2 V^\vee \otimes \mathcal{O}_{\mathfrak{p}^3}(-1) \to \mathcal{O}_{\mathfrak{p}^3} \to 0, \quad (3.4)$$

$$0 \to \Omega^2_{\mathfrak{p}^3} \to \wedge^4 V^\vee \otimes \mathcal{O}_{\mathfrak{p}^3}(-2) \to \Omega^1_{\mathfrak{p}^3} \to 0,$n

$$0 \to \wedge^4 V^\vee \otimes \mathcal{O}_{\mathfrak{p}^3}(-4) \to \wedge^3 V^\vee \otimes \mathcal{O}_{\mathfrak{p}^3}(-3) \xrightarrow{i_2} \Omega^2_{\mathfrak{p}^3} \to 0$$

induced by the Koszul complex for $V^\vee \otimes \mathcal{O}_{\mathfrak{p}^3}(-1)^{ev} \to \mathcal{O}_{\mathfrak{p}^3}$. Twisting these sequences by $E$ and passing to cohomology, we see from (1.2), (3.1), (3.2) that $0 = h^0(E \otimes \Omega_{\mathfrak{p}^3}) = h^3(E \otimes \Omega^2_{\mathfrak{p}^3}) = h^2(E \otimes \Omega_{\mathfrak{p}^3})$ and there is a commutative diagram

$$0 \to H^2(E(-4)) \otimes \wedge^4 V^\vee \to H^2(E(-3)) \otimes \wedge^3 V^\vee \xrightarrow{i_2} H^2(E \otimes \Omega^2_{\mathfrak{p}^3}) \to 0$$

$$0 \leftarrow H^1(E) \leftarrow H^1(E(-1)) \otimes V^\vee \leftarrow H^1(E \otimes \Omega_{\mathfrak{p}^3}) \leftarrow 0$$

with exact rows, where $A' := i_1 \circ \partial^{-1} \circ i_2$. The Euler exact sequence (3.4) yields a canonical isomorphism $\omega_{\mathfrak{p}^3} \cong \wedge^4 V^\vee \otimes \mathcal{O}_{\mathfrak{p}^3}(-4)$, and every fixed isomorphism $\tau: \mathbb{k} \cong \wedge^4 V^\vee$ induces isomorphisms $\bar{\tau}: V \cong \wedge^3 V^\vee$ and $\tilde{\tau}: \omega_{\mathfrak{p}^3} \cong \mathcal{O}_{\mathfrak{p}^3}(-4)$. The point $A$ in (3.3) is now defined as the composite

$$A: H_n \otimes V \xrightarrow{\bar{\tau}} H_n \otimes \wedge^3 V^\vee \xrightarrow{f} H^2(E(-3)) \otimes \wedge^3 V^\vee \xrightarrow{A'} H^1(E(-1)) \otimes V^\vee$$

$$\xrightarrow{j} H^1(E(-1)) \otimes V^\vee \xrightarrow{\text{SD}} H^2(E(1) \otimes \omega_{\mathfrak{p}^3})^\vee \otimes V^\vee$$

$$\xrightarrow{\tilde{\tau}} H^2(E(-3))^\vee \otimes V^\vee \xrightarrow{f^\vee} H_n^\vee \otimes V^\vee,$$
where SD is the isomorphism of Serre duality. One checks that $A$ is a skew-symmetric map depending only on the class $[E, f, j]$ and independent of the choice of $\tau$, and that the point $A \in \wedge^2(H^\vee_n \otimes V^\vee)$ lies in the direct summand $S_n = S^2 H^\vee_n \otimes \wedge^2 V^\vee$ of the canonical decomposition

$$\wedge^2 (H^\vee_n \otimes V^\vee) = S^2 H^\vee_n \otimes \wedge^2 V^\vee \oplus \wedge^2 H^\vee_n \otimes S^2 V^\vee.$$  \hspace{1cm} (3.7)

Here $S_n$ is the space of nets of quadrics in $H_n$. Following [6], [8] and [9], we call $A$ the n-instanton net of quadrics corresponding to the data $[E, f, j]$.

We put $W_A := H_n \otimes V/ \ker A$. Using the chain of isomorphisms in (3.6), we can rewrite the diagram (3.5) in the form

$$\begin{array}{cccc}
0 & \longrightarrow & \ker A & \longrightarrow & H_n \otimes V & \xrightarrow{c_A} & W_A & \longrightarrow & 0 \\
& & \downarrow A & \searrow & q_A \\
0 & \longleftarrow & \ker A^\vee & \longleftarrow & H^\vee_n \otimes V & \xleftarrow{c_A^\vee} & W^\vee_A & \longleftarrow & 0
\end{array} \hspace{1cm} (3.8)$$

Here $\dim W_A = 2n + 2$ by (3.1) and $q_A: W_A \xrightarrow{\sim} W^\vee_A$ is the induced skew-symmetric isomorphism. An important property of $A = A([E, f, j])$ is that the induced morphism of sheaves

$$\begin{array}{cccc}
a_{\lambda}^\vee: W^\vee_A \otimes O_{\mathbb{P}^3} & \xrightarrow{c_{\lambda}^\vee} & H^\vee_n \otimes V \otimes O_{\mathbb{P}^3} & \xrightarrow{ev} & H^\vee_n \otimes O_{\mathbb{P}^3}(1)
\end{array} \hspace{1cm} (3.9)$$

is an epimorphism such that the composite $H_n \otimes O_{\mathbb{P}^3}(-1) \xrightarrow{a_{\lambda}} W_A \otimes O_{\mathbb{P}^3} \xrightarrow{q_A} W^\vee_A \otimes O_{\mathbb{P}^3} \xrightarrow{a_{\lambda}^\vee, e} H^\vee_n \otimes O_{\mathbb{P}^3}(1)$ is equal to zero, and $E = \ker (a_{\lambda}^\vee \circ q_A)/\text{Im} a_{\lambda}$. Thus $A$ determines a monad

$$M_A: 0 \rightarrow H_n \otimes O_{\mathbb{P}^3}(-1) \xrightarrow{a_{\lambda}} W_A \otimes O_{\mathbb{P}^3} \xrightarrow{a_{\lambda}^\vee \circ q_A} H^\vee_n \otimes O_{\mathbb{P}^3}(1) \rightarrow 0 \hspace{1cm} (3.10)$$

with cohomology sheaf

$$E = E(A) := \ker (a_{\lambda}^\vee \circ q_A)/\text{Im} a_{\lambda} \hspace{1cm} (3.11)$$

Note that, by passing to cohomology in the monad $M_A$ twisted by $O_{\mathbb{P}^3}(-3)$ and using (3.11), we get an isomorphism $f: H_n \xrightarrow{\sim} H^2(E(-3))$. Furthermore, since the form $q_A$ in $M_A$ is symplectic, there is a canonical isomorphism between $M_A$ and the dual monad, and this isomorphism induces a symplectic isomorphism $j: E \xrightarrow{\sim} E^\vee$. Thus the data $[E, f, j]$ can be recovered from the net $A$. This yields the following description of the moduli space $I_n$. Consider the set of n-instanton nets of quadrics

$$MI_n := \left\{ A \in S_n \right\} \hspace{1cm} (3.12)$$

(i) $\text{rk}(A): H_n \otimes V \rightarrow H^\vee_n \otimes V^\vee = 2n + 2$;

(ii) the morphism $a_{\lambda}^\vee: W^\vee_A \otimes O_{\mathbb{P}^3} \rightarrow H^\vee_n \otimes O_{\mathbb{P}^3}(1)$, defined by $A$ in (3.9) is surjective;

(iii) $h^0(E_2(A)) = 0$, where $E_2(A) := \ker (a_{\lambda}^\vee \circ q_A)/\text{im} a_{\lambda}$ and $q_A: W_A \xrightarrow{\sim} W^\vee_A$ is the symplectic isomorphism defined by $A$ in (3.8)
The conditions (i)–(iii) in (3.12) are called Barth’s conditions. They show that $MI_n$ has the natural structure of a locally closed subscheme in the vector space $S_n$. Moreover, it follows from (3.12) that there is a morphism $\pi_n : MI_n \to I_n$, $A \mapsto [E(A)]$, and this morphism is known to be a principal $(\text{GL}(H_n)/\{\pm \text{id}\})$-bundle in the étale topology (see [7]). By construction, the fibre $\pi_n^{-1}([E])$ over an arbitrary point $[E] \in I_n$ coincides with the homogeneous space $F[E]$ (described above) of the group $\text{GL}(H_n)/\{\pm \text{id}\}$. Hence the irreducibility of $(I_n)_{\text{red}}$ is equivalent to that of the scheme $(MI_n)_{\text{red}}$.

Definition (3.12) yields the following result.

**Theorem 3.1.** For every $n \geq 1$ the space $MI_n$ of $n$-instanton nets of quadrics is a locally closed subscheme in the vector space $S_n$. Near each point $A \in MI_n$ it is given by

$$\left( \frac{2n-2}{2} \right) = 2n^2 - 5n + 3$$

(3.13)

equations imposed by the rank condition (i) in (3.12).

It follows from (3.13) that

$$\dim_{[A]} MI_n \geq \dim S_n - (2n^2 - 5n + 3) = n^2 + 8n - 3$$

(3.14)
at every point $A \in MI_n$. On the other hand, deformation theory yields that $\dim_{[E]} I_n \geq 8n - 3$ for every $n$-instanton $E$. This agrees with (3.14) since $MI_n \to I_n$ is a principal $(\text{GL}(H_n)/\{\pm \text{id}\})$-bundle in the étale topology.

Let $S_n = \{ [E] \in I_n \mid \exists \text{ a line } l \subset \mathbb{P}^3 \text{ of maximal jump for } E, \text{ that is, a line } l \text{ with } h^0(E(-n)|_l) \neq 0 \}$. It is known [10] that $S_n$ is a closed subset of dimension $6n + 2$ in $I_n$, and $I_n$ is smooth along $S_n$. Since $\dim_{[E]} I_n \geq 8n - 3$ for every $[E] \in I_n$, it follows that the subset

$$I'_n := I_n \setminus S_n$$

(3.15)
is open and dense in $I_n$. Accordingly,

$$MI'_n := \pi_n^{-1}(I'_n)$$

(3.16)
is open and dense in $MI_n$, and we have an open embedding

$$MI'_n \hookrightarrow MI_n.$$  
(3.17)
Thus the irreducibility of $MI_n$ is equivalent to that of $MI'_n$. For technical reasons, we shall restrict ourselves to the set $MI'_n$ instead of $MI_n$.

**Remark 3.2.** There are smooth points in $I_n$ (see, for example, [2] or [10]). Hence there are smooth points in $MI_n$.

§ 4. The decomposition $H_{2m+1} \simeq H_{m+1} \oplus H_m$ and related constructions

4.1. A general-position result for $(2m+1)$-instanton nets. Given an integer $m \geq 3$ and a $(2m+1)$-instanton $[E] \in I'_{2m+1}$, we fix an isomorphism $f : H_{2m+1} \simeq H^2(E(-3))$ and write

$$H_{4m} := H^2(E(-4)), \quad W_{4m+4} := H^1(E \otimes \Omega_{\mathbb{P}^3})^\vee.$$  
(4.1)
(This agrees with equations (3.1) when $n = 2m + 1$.) Then the lower exact triple in (3.5) can be rewritten in the form

$$0 \to W_{4m+4}^\vee \to H^\vee_{2m+1} \otimes V^\vee \xrightarrow{\text{mult}} H^\vee_{4m} \to 0. \tag{4.2}$$

We now state a general-position result for $(2m + 1)$-instanton nets which will play an important role in what follows.

**Theorem 4.1.** Suppose that $m \geq 3$ and $E$ is a $(2m + 1)$-instanton, $[E] \in I'_{2m+1}$, endowed with an isomorphism $f: H_{2m+1} \xrightarrow{\cong} H^2(E(-3))$. We put $W_{4m+4} = H^1(E \otimes \Omega_{P^3})^\vee$ so that we have the monomorphism $W_{4m+4}^\vee \hookrightarrow H^\vee_{2m+1} \otimes V^\vee$ defined in (4.2). Then, for a general $m$-dimensional subspace $V_m$ of $H^\vee_{2m+1}$, we have

$$W_{4m+4}^\vee \cap V_m \otimes V^\vee = \{0\}. \tag{4.2}$$

The proof of Theorem 4.1 is rather technical. We postpone it till the end of the paper (see §12).

**4.2. The decomposition $H_{2m+1} \simeq H_{m+1} \oplus H_m$.** We fix an isomorphism

$$\xi: H_{m+1} \oplus H_m \xrightarrow{\cong} H_{2m+1} \tag{4.3}$$

and let

$$H_{m+1} \xleftarrow{i_{m+1}} H_{m+1} \oplus H_m \xrightarrow{i_m} H_m \tag{4.4}$$

be the inclusions of the direct summands. Given a $(2m + 1)$-instanton $E$, $[E] \in I'_{2m+1}$, we fix an isomorphism $f: H_{2m+1} \xrightarrow{\cong} H^2(E(-3))$ and a symplectic structure $j: E \xrightarrow{\cong} E^\vee$. The data $[E, f, j]$ determine a net of quadrics $A \in MI'_{2m+1}$ (see §3), and the exact triple (4.2) is naturally identified with the triple dual to $0 \to \ker A \to H_{2m+1} \otimes V \to W_A \to 0$ and fits into the diagram (3.8) for $n = 2m + 1$:

$$\begin{array}{cccccc}
0 & \to & \ker A & \to & H_{2m+1} \otimes V & \xrightarrow{c_A} W_A & \to & 0 \\
&&&&& \downarrow{A} & \cong & \downarrow{q_A} \\
0 & \leftarrow & \ker A^\vee & \leftarrow & H^\vee_{2m+1} \otimes V^\vee & \xleftarrow{c_A^\vee} W_A^\vee & \leftarrow & 0
\end{array} \tag{4.5}
$$

Consider the composite

$$j_{\xi, A}: H_{m+1} \otimes V \xleftarrow{i_{m+1}} H_{m+1} \otimes V \oplus H_m \otimes V \xrightarrow{\xi} H_{2m+1} \otimes V \xrightarrow{c_A} W_A. \tag{4.6}$$

Theorem 4.1 can be restated in this notation as follows.

**Theorem 4.1’.** Suppose that $m \geq 3$ and $A$ is an arbitrary net in $MI'_{2m+1}$. Then, for a general isomorphism $\xi: H_{2m+1} \xrightarrow{\cong} H_{m+1} \oplus H_m$, we have

$$\ker A \cap (\xi \circ i_{m+1})(H_{m+1} \otimes V) = \{0\}. \tag{4.7}$$

Equivalently, $j_{\xi, A}: H_{m+1} \otimes V \to W_A$ is an isomorphism.
Consider the direct sum decomposition corresponding to the isomorphism (4.3):
\[ \tilde{\xi} : S_{m+1} \oplus \Sigma_{m+1} \oplus S_{m} \xrightarrow{\sim} S_{2m+1}, \]
and let
\[
S_{2m+1} \twoheadrightarrow S_{m+1}, \quad A \mapsto A_1(\xi), \quad S_{2m+1} \twoheadrightarrow \Sigma_{m+1}, \quad A \mapsto A_2(\xi), \quad S_{2m+1} \twoheadrightarrow S_{m}, \quad A \mapsto A_3(\xi),
\]
be the projections onto the direct summands. By definition, \( A_1(\xi) \) coincides as a skew-symmetric homomorphism \( \tilde{H}_{m+1} \otimes V \to H_{m+1}^\vee \otimes V^\vee \) with the composite
\[
A_1(\xi) : H_{m+1} \otimes V \xrightarrow{j_{\xi,A}} W_A \xrightarrow{q_A} W_A^\vee \xrightarrow{j_{\xi,A}} H_{m+1}^\vee \otimes V^\vee.
\]
Hence Theorem 4.1' may be restated as follows.

**Theorem 4.1''.** Suppose that \( m \geq 3 \) and \( A \) is an arbitrary net of quadrics in \( MI'_{2m+1} \). Then, for a general isomorphism \( \xi \) in (4.3), the skew-symmetric homomorphism \( A_1(\xi) : H_{m+1} \otimes V \to H_{m+1}^\vee \otimes V^\vee \) is invertible.

Using notation in (4.9), we can now represent the net \( A \in S_{2m+1} \), regarded as a homomorphism \( A : H_{m+1} \otimes V \oplus H_m \otimes V \to H_{m+1}^\vee \otimes V^\vee \oplus H_m^\vee \otimes V^\vee \), by a \(((8m + 4) \times (8m + 4))\)-matrix of homomorphisms
\[
A = \begin{pmatrix}
A_1(\xi) & A_2(\xi) \\
A_2(\xi)^\vee & A_3(\xi)
\end{pmatrix}.
\]
This matrix is of rank \( 4m + 4 \) by Barth’s condition (i) in (3.12). On the other hand, Theorem 4.1'' yields that \( \text{rk} A_1(\xi) = 4m + 4 \), that is, the ranks of \( A \) and the submatrix \( A_1(\xi) \) coincide. Multiplying \( A \) on the left by the invertible matrix of homomorphisms
\[
\begin{pmatrix}
A_1(\xi)^{-1} & 0 \\
A_2(\xi)^\vee \circ A_1(\xi)^{-1} & \text{id}_{H^\vee_{m+1} \otimes V^\vee}
\end{pmatrix},
\]
we thus get the following relation between the matrices \( A_1(\xi), A_2(\xi) \) and \( A_3(\xi) \):
\[
A_3(\xi) = -A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi).
\]

**Remark 4.2.** The relation (4.11) means that \( A_3(\xi) \) is uniquely determined by the homomorphisms \( A_1(\xi) \) and \( A_2(\xi) \). This observation is used systematically in what follows.

For \( m \geq 1 \) let \( \text{Isom}_{2m+1} \) be the set of all isomorphisms \( \xi \) in (4.3). We put
\[
MI_{2m+1}(\xi) := \{ A \in MI'_{2m+1} \mid \text{the skew-symmetric homomorphism } A_1(\xi) \text{ in } (4.10) \text{ is invertible} \}, \quad \xi \in \text{Isom}_{2m+1}.
\]
In this notation we have the following result.

**Theorem 4.3.** For \( m \geq 3 \) the sets \( MI_{2m+1}(\xi), \xi \in \text{Isom}_{2m+1} \), are open dense subsets in \( MI'_{2m+1} \) and form an open covering of \( MI'_{2m+1} \).
Proof. Let $M$ be an arbitrary irreducible component of $MI_{2m+1}$. We consider the set $U := \{(A, \xi) \in M \times \text{Isom}_{2m+1} \mid A \in MI_{2m+1}(\xi)\}$ with projections $M \overset{p}{\to} U \overset{q}{\to} \text{Isom}_{2m+1}$. By construction, $U$ is open in $M \times \text{Isom}_{2m+1}$. Moreover, it follows from Theorem 4.1′′ that $p(U) = M$, whence $U$ is non-empty and, therefore, dense in $M \times \text{Isom}_{2m+1}$ since $M$ and $\text{Isom}_{2m+1}$ are irreducible. (Note that $\text{Isom}_{2m+1}$ is irreducible as a principal homogeneous space of $G = GL(2m+1)$.) Therefore $q(U)$ contains an open dense subset of $\text{Isom}_{2m+1}$. The group $G$ acts transitively by translations on $\text{Isom}_{2m+1}$ and the set $q(U)$ is clearly $G$-invariant. Hence $q(U)$ coincides with $\text{Isom}_{2m+1}$. □

Lemma 4.4. For $\xi \in \text{Isom}_{2m+1}$ and $A \in MI_{2m+1}(\xi)$ we put

$$B := A_1(\xi), \quad C := A_2(\xi).$$ (4.13)

Then the following assertions hold.

(i) If

$$\alpha_{\xi,A} := j_{\xi,A}^{-1} \circ a_A \circ \xi: (H_{m+1} \oplus H_m) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to H_{m+1} \otimes V \otimes \mathcal{O}_{\mathbb{P}^3}$$ (4.14)

is the subbundle morphism, then there is an epimorphism

$$\lambda_{\xi,A}: \text{coker}(B \circ \alpha_{\xi,A}) \to H_{m+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)$$ (4.15)

making the diagram

\begin{equation}
\begin{array}{ccc}
H_{m+1}^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\text{can}} & \text{coker}(B \circ \alpha_{\xi,A}) \\
\downarrow \quad u^\vee & & \downarrow \quad \lambda_{\xi,A} \\
H_{m+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & & \\
\end{array}
\end{equation}

commutative, where can is the canonical epimorphism.

(ii) If the diagram

\begin{equation}
\begin{array}{ccc}
0 & \xrightarrow{i_{m+1}} & (H_{m+1} \oplus H_m) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \\
& & \xrightarrow{B \otimes a_A} H_{m+1}^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\text{can}} \text{coker}(B \circ \alpha_{\xi,A}) \xrightarrow{\epsilon_{\xi,A}} 0 \\
& & \downarrow \quad B_{\alpha_{\xi,A}} \quad \downarrow \quad \tau_{\xi,A} \\
0 & \xrightarrow{v_0 B_{-1}} & H_{m+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \\
& & \xrightarrow{\text{can}} \quad \tau_{\xi,A} \\
& & \downarrow \quad \tau_{\xi,A} \\
& & \quad H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \\
\end{array}
\end{equation}

(4.17)
is commutative, where $\tau_{\xi, A}$ and $\varepsilon_{\xi, A}$ are the induced morphisms, then $\tau_{\xi, A}$ is a subbundle morphism fitting into the commutative diagram

\[
\begin{array}{ccc}
H_{m+1}^\vee \otimes V^\vee \otimes O_{\mathbb{P}^3} & \xrightarrow{\psi B^{-1}} & H_{m+1} \otimes T_{\mathbb{P}^3}(-1) \\
\downarrow \text{Cou} & & \downarrow \tau_{\xi, A} \\
H_m \otimes O_{\mathbb{P}^3}(-1) & \xrightarrow{} & H_m \otimes O_{\mathbb{P}^3}(-1)
\end{array}
\]

(4.18)

**Proof.** Consider the commutative diagram

\[
\begin{array}{cccc}
H_{2m+1} \otimes O(-1) & \xrightarrow{q^A} & W_A \otimes O & \xrightarrow{q^A} & W_A^\vee \otimes O & \xrightarrow{a^A} & H_{2m+1}^\vee \otimes O(1) \\
\downarrow \xi & \simeq & \downarrow j_{\xi, A} & \simeq & \downarrow j_{\xi, A}^\vee & \simeq & \downarrow \xi^\vee \\
(H_m \oplus H_m) \otimes O(-1) & \xrightarrow{\alpha_{\xi, A}} & H_{m+1} \otimes V \otimes O & \xrightarrow{B} & H_{m+1} \otimes V^\vee \otimes O & \xrightarrow{\alpha_{\xi, A}} & (H_m \oplus H_m)^\vee \otimes O(1) \\
\downarrow \iota_{m+1} & \simeq & \downarrow u & \simeq & \downarrow u^\vee & \simeq & \downarrow \iota_{m+1}^\vee \\
H_{m+1} \otimes O(-1) & \xrightarrow{a^A} & H_{m+1}^\vee \otimes O(1)
\end{array}
\]

(4.19)

Here the upper triple is the monad (3.10) for $n = 2m + 1$. This proves part (i).

Part (ii) is proved by standard diagram-chasing using (4.11), (4.13) and the diagram (4.17). □

**4.3. Remarks on ’t Hooft instantons.** We consider the set

\[
I_{2m+1}^H := \{ [E] \in I_{2m+1} \mid h^0(E(1)) \neq 0 \}
\]

of ’t Hooft instanton bundles and the corresponding set of ’t Hooft instanton nets

\[
MI_{2m+1}^H := \pi_n^{-1}(I_{2m+1}^H).
\]

Some well-known facts about $I_{2m+1}^H$ are collected in the following lemma (see [1], [2], [9], Proposition 2.2, [11]).

**Lemma 4.5.** Let $m \geq 1$. Then the following assertions hold.

(i) $I_{2m+1}^H$ is an irreducible $(10m + 9)$-dimensional subvariety in $I_{2m+1}$. Accordingly, $MI_{2m+1}^H$ is an irreducible $(4m^2 + 14m + 10)$-dimensional subvariety in $I_{2m+1}$.

(ii) $I_{2m+1}^{H^*} := I_{2m+1}^H \cap I_{2m+1}$ is a smooth open dense subset in $I_{2m+1}^H$ and

\[
h^0(E(1)) = 1, \quad [E] \in I_{2m+1}^{H^*}.
\]

(4.20)

(iii) $MI_{2m+1}^{H^*}$ is a smooth open dense subset in

\[
TH_{2m+1} := \left\{ A \in S_{2m+1} \mid A = \sum_{i=1}^{2m+2} h^2 \otimes w, \ h \in H_{2m+1}^\vee, \ w \in \wedge^2 V^\vee, \ w \wedge w = 0 \right\}.
\]
We now ask whether Theorem 4.3 holds for \( m = 1, 2 \). To do this, we fix an isomorphism \( \xi^0 \in \text{Isom}_{2m+1} \), \( \xi^0 : H_{m+1} \oplus H_m \xrightarrow{\sim} H_{2m+1} \), for \( m = 1, 2 \) and choose a basis \( \{ h_1, \ldots, h_{2m+1} \} \) in \( H_{2m+1}^\vee \) such that the vectors \( h_1, \ldots, h_{m+1} \) lie in \( H_{m+1}^\vee \) and the vectors \( h_{m+2}, \ldots, h_{2m+1} \) lie in \( H_m^\vee \). Let \( e_1, \ldots, e_4 \) be some fixed basis in \( V^\vee \). We define nets of quadrics \( A^{(m)} \in TH_{2m+1}, \ m = 1, 2 \), by putting

\[
A^{(1)} = h_1^2 \otimes (e_1 \wedge e_2 + e_3 \wedge e_4) + h_2^2 \otimes (e_1 \wedge e_3 + e_4 \wedge e_2),
\]

\[
A^{(2)} = h_1^2 \otimes (e_1 \wedge e_2 + e_3 \wedge e_4) + h_2^2 \otimes (e_1 \wedge e_3 + e_4 \wedge e_2) + h_3^2 \otimes (e_1 \wedge e_4 + e_2 \wedge e_3) \tag{4.21}
\]

Using the notation in (4.9), we can easily show that the homomorphisms

\[
A^{(m)}_1(\xi^0) : H_{m+1} \otimes V \to H_{m+1}^\vee \otimes V^\vee, \quad m = 1, 2,
\]

are invertible. On the other hand, for a given \( \xi \in \text{Isom}_{2m+1} \), the condition of invertibility of \( A_1(\xi) : H_{m+1} \otimes V \to H_{m+1}^\vee \otimes V^\vee \) is an open condition on the net \( A \in TH_{2m+1} \) and, respectively, on the net \( A \in S_{2m+1} \). Since the sets \( MI'_{2m+1} \) are irreducible for \( m = 1, 2 \) (see [7]), this together with Lemma 4.5 yields the following corollary.

**Corollary 4.6.** Theorem 4.3 holds for \( m = 1, 2 \).

Using (3.17), Theorem 4.3 and Corollary 4.6, we now get the following assertion.

**Corollary 4.7.** Let \( m \geq 1 \). Then the scheme \( (MI_{2m+1}(\xi))_{\text{red}} \) is an open dense subscheme in \( (MI_{2m+1})_{\text{red}} \) for every \( \xi \in \text{Isom}_{2m+1} \). In particular,

\[
\dim_A MI_{2m+1}(\xi) = \dim_A MI_{2m+1}, \quad A \in MI_{2m+1}(\xi), \quad \xi \in \text{Isom}_{2m+1}. \tag{4.22}
\]

§ 5. **Invertible nets of quadrics in \( S_{m+1} \) and symplectic bundles of rank \( 2m + 2 \)**

5.1. **The construction of symplectic bundles of rank \( 2m + 2 \) from invertible nets of quadrics in \( S_{m+1} \).** In this subsection we show that every invertible net of quadrics \( B \in S_{m+1} \) leads naturally to the construction of a symplectic vector bundle \( E_{2m+2}(B) \) of rank \( 2m + 2 \) on \( \mathbb{P}^3 \). We introduce more notation. Put

\[
S_{m+1}^0 := \{ B \in S_{m+1} \mid B : H_{m+1} \otimes V \to H_{m+1}^\vee \otimes V^\vee \text{ is an invertible homomorphism} \}. \tag{5.1}
\]

Then \( S_{m+1}^0 \) is an open dense subset of the vector space \( S_{m+1} \), and it is easy to see that the following conditions hold for every \( B \in S_{m+1}^0 \).

1) The morphism \( \tilde{B} : H_{m+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to H_{m+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \) induced by the homomorphism \( B : H_{m+1} \otimes V \to H_{m+1}^\vee \otimes V^\vee \) is a subbundle morphism, that is,

\[
E_{2m+2}(B) := \text{coker}(\tilde{B}) \tag{5.2}
\]
is a vector bundle of rank $2m + 2$ on $\mathbb{P}^3$. This follows from the diagram

$$
\begin{array}{c}
0 \\
\downarrow \\
H_{m+1} \otimes O_{\mathbb{P}^3}(-1) \\
\downarrow \\
H_{m+1} \otimes \Omega_{\mathbb{P}^3}(1) \overset{e}{\to} E_{2m+2}(B) \to 0
\end{array}
\quad \begin{array}{c}
0 \\
\downarrow \\
H_{m+1} \otimes V \otimes O_{\mathbb{P}^3} \\
\downarrow \\
H_{m+1} \otimes V^\vee \otimes O_{\mathbb{P}^3}
\end{array}
$$

(5.3)

2) The homomorphism $\sharp B: H_{m+1} \to H_{m+1}^\vee \otimes \wedge^2 V^\vee$ induced by $B: H_{m+1} \otimes V \to H_{m+1}^\vee \otimes V^\vee$ is injective. This follows from the commutative diagram that extends the upper horizontal triple in (5.3):

$$
\begin{array}{c}
0 \\
\downarrow \\
H_{m+1} \otimes T_{\mathbb{P}^3}(-2) \\
\downarrow \\
H_{m+1} \otimes T_{\mathbb{P}^3}(-2)
\end{array}
\quad \begin{array}{c}
0 \\
\downarrow \\
H_{m+1} \otimes O_{\mathbb{P}^3} \\
\downarrow \\
H_{m+1} \otimes O_{\mathbb{P}^3}(1)
\end{array}
\quad \begin{array}{c}
0 \\
\downarrow \\
E_{2m+2}(B)^\vee \\
\downarrow \\
E_{2m+2}(B)^\vee(1) \\
\downarrow \\
0
\end{array}
$$

(5.4)

where $w$ is the morphism induced by the morphism $v$ by the Euler exact sequence in (5.3). The diagram (5.4) yields an isomorphism

$$\text{coker}(\sharp B) \simeq H^0(E_{2m+2}(B)(1)).$$

(5.5)

3) Using the diagram (5.3) and the snake lemma, we get an isomorphism

$$\theta: E_{2m+2}(B) \overset{\sim}{\to} E_{2m+2}(B)^\vee,$$

(5.6)

which is symplectic:

$$\theta^\vee = -\theta$$

since the homomorphism $B: H_{m+1} \otimes V \to H_{m+1}^\vee \otimes V^\vee$ is skew-symmetric. The isomorphism $\theta$ together with the upper triple in (5.3) and its dual triple fits into
the commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & H_{m+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \overset{\tilde{B}}{\to} & H_{m+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \overset{\epsilon}{\to} & E_{2m+2}(B) & \to & 0 \\
0 & \to & H_{m+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \overset{B \circ u}{\to} & H_{m+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \overset{v \circ B^{-1}}{\to} & H_{m+1} \otimes T_{\mathbb{P}^3}(-1) & \to & 0 \\
& & & & & & & (5.7)
\end{array}
\]

\[
\begin{array}{cccccc}
H_{m+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \to & H_{m+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \to & 0 \\
\end{array}
\]

Note that the upper horizontal triple in (5.3) yields the equations

\[h^0(E_{2m+2}(B)) = h^i(E_{2m+2}(B)(-2)) = 0, \quad i \geq 0. \quad (5.8)\]

5.2. The relation between \((2m + 1)\)-instantons and symplectic bundles of rank \(2m + 2\). Suppose that \(m \geq 1, \xi \in \text{Isom}_{2m+1}, \text{ and } A \in MI_{2m+1}(\xi)\). In this subsection we relate an instanton vector bundle \(E(A)\) to a symplectic vector bundle \(E_{2m+2}(B)\) of rank \(2m + 2\) for \(B = A_1(\xi)\). We shall prove that \(E(A)\) is the cohomology sheaf of the monad (5.12) determined by the data \((\xi, A)\) with middle term \(E_{2m+2}(B)\) (see Lemma 5.1).

Indeed, since \(\xi \in \text{Isom}_{2m+1}\), the homomorphism \(B: H_{m+1} \otimes V \to H_{m+1}^\vee \otimes V^\vee\) lies in \(S_{m+1}^0\) by definition. Hence by Lemma 4.4 we have the diagram (4.18). This diagram together with (5.7) yields that \(\tilde{B}^\vee \circ \tau_{\xi, A} = 0\) (note that we have \(\text{im}(C \circ u) \subset H_{m+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1)\) in (4.18) since \(C \in \Sigma_{m+1}\)). Hence there is a morphism

\[\rho_{\xi, A}: H_m \otimes \mathcal{O}(-1) \to E_{2m+2}(B) \quad (5.9)\]

such that \(\tau_{\xi, A} = e^\vee \circ \theta \circ \rho_{\xi, A}\). Since \(\tau_{\xi, A}\) is a subbundle morphism, so is \(\rho_{\xi, A}\). Moreover, the diagrams (4.18) and (5.7) yield a commutative diagram

\[
\begin{array}{cccccc}
H_{m+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) & \overset{\epsilon}{\to} & E_{2m+2}(B) \\
\downarrow & & \downarrow \rho_{\xi, A} & & \downarrow e^\vee \circ \theta \\
H_m \otimes \mathcal{O}(-1) & \overset{\tau_{\xi, A}}{\to} & \tilde{C} & \overset{\tilde{C}}{\leftarrow} & H_{m+1} \otimes T_{\mathbb{P}^3}(-1) \\
H_{m+1}^\vee \otimes V^\vee \otimes \mathcal{O} & \overset{v \circ B^{-1}}{\to} & H_{m+1} \otimes T_{\mathbb{P}^3}(-1) \\
\end{array}
\]

\[ (5.10) \]
Diagrams (5.7) and (5.10) yield a commutative diagram

\[
\begin{align*}
H_m \otimes \mathcal{O}(-1) \xrightarrow{\widetilde{C}} H_m^{\vee} \otimes \Omega^1_{\mathbb{P}^3} \xrightarrow{e} E_{2m+2}(B) & \xleftarrow{e} H_m^{\vee} \otimes \mathcal{O}(1) \\
\xrightleftharpoons{\rho_{\xi,A} \circ \theta} D_C & \xrightleftharpoons{\rho_{\xi,A} \circ \theta \circ e} H_m^{\vee} \otimes \mathcal{O}(1) \xrightarrow{\rho_{\xi,A}} E_{2m+2}(B) \xleftarrow{e} H_m^{\vee} \otimes \mathcal{O}(1) \\
H_m^{\vee} \otimes \mathcal{O}(1) & \xrightarrow{\widetilde{C}} H_m^{\vee} \otimes \mathcal{O}(1) \xrightarrow{e} E_{2m+2}(B) \xleftarrow{e} H_m^{\vee} \otimes \mathcal{O}(1) \\
\end{align*}
\]

(5.11)

where \( D_C := -\widetilde{C}^\vee \circ B^{-1} \circ \widetilde{C} = -u^\vee \circ (C^\vee \circ B^{-1} \circ C) \circ u \) is the zero map. Indeed, it follows from (4.11) and (4.13) that \( D_C = p_2(A_3(\xi)) \), where \( p_2 : \wedge^2 (H_n^\vee \otimes V^\vee) \to \wedge^2 H_n^\vee \otimes S^2 V^\vee \) is the projection onto the second direct summand in (3.7). Since \( A_3(\xi) \) belongs to the first direct summand in (3.7) by (4.9), we have \( D_C = 0 \). Thus we get a monad

\[
0 \to H_m \otimes \mathcal{O}(-1) \xrightarrow{\rho_{\xi,A} \circ \theta} E_{2m+2}(B) \xrightarrow{\rho_{\xi,A} \circ \theta} H_m^{\vee} \otimes \mathcal{O}(1) \to 0
\]

(5.12)

whose cohomology sheaf

\[
E_2(\xi, A) := \ker(\rho_{\xi,A}^\vee \circ \theta) / \operatorname{Im} \rho_{\xi,A}
\]

is a vector bundle since \( \rho_{\xi,A} \) is a subbundle morphism. Furthermore, by (5.8) the monad (5.12) implies that \( E_2(\xi, A) \) is a \((2m+1)\)-instanton:

\[
[E_2(\xi, A)] \in I_{2m+1}.
\]

(5.14)

Lemma 5.1. We have \( E_2(\xi, A) \simeq E(A) \), where the sheaf \( E(A) \) is defined by (3.11).

Proof. We apply diagram-chasing using formula (4.11) to the diagrams (4.17)–(4.19), (5.3), (5.4) and (5.7). \( \square \)

§ 6. The scheme \( X_m \). An isomorphism between \( X_m \) and an open subset of \((MI_{2m+1})_{\text{red}}\)

In this section we introduce a locally closed subset \( X_m \) of the affine space \( S_{m+1} \times \Sigma_{m+1} \) and prove in Theorem 6.1 that this subset, regarded as a reduced scheme, is isomorphic to the reduced scheme \((MI_{2m+1}(\xi))_{\text{red}}\) for every \( \xi \in \text{Isom}_{2m+1} \).
The set $X_m$ is defined as follows:

$$X_m := \begin{cases} (B, C) \in S_{m+1}^0 \times \Sigma_{m+1} & \text{if } (C^\vee \circ B^{-1} \circ C : H_m \otimes V \to H_m^\vee \otimes V^\vee) \in S_m; \\
\text{(i) the map } (H_{m+1} \oplus H_m) \otimes \mathcal{O} \xrightarrow{(B,C)_{\text{can}}} H_{m+1}^\vee \otimes V^\vee \otimes \mathcal{O}(1) \text{ is a subbundle morphism; } \\
\text{(ii) the map } H_{m+1}^\vee \otimes \wedge^2 V^\vee \xrightarrow{\text{can}} H_{m+1}^\vee \otimes \wedge^2 V^\vee / \text{Im}(\mathcal{O}_1) \approx H^0(E_{2m+2}(B)(1)) \\
\text{yields a subbundle morphism } \\
\rho_{B,C} : H_m \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to E_{2m+2}(B), \\
\text{that is, } \rho_{B,C} \text{ is epimorphic and the sheaf } E_2(B, C) := \text{Ker}(\mathcal{O}_1) / \text{Im}(\rho_{B,C}) \\
is locally free \end{cases}$$

By definition, $X_m$ is a locally closed subset of $S_{m+1}^0 \times \Sigma_{m+1}$. Hence it has the natural structure of a reduced scheme.

Note that in condition (iii) of (6.1), we put $\mathcal{O}_1 := \rho_{B,C} \circ \theta$, where $\theta : E_{2m+2}(B) \xrightarrow{\sim} E_{2m+2}(B)$ is the natural symplectic structure on $E_{2m+2}(B)$ defined by (5.6).

**Theorem 6.1.** If $m \geq 1$ and $\xi \in \text{Isom}_{2m+1}$, then the following assertions hold.

(i) There is an isomorphism of reduced schemes

$$f_m : (MI_{2m+1}(\xi))_{\text{red}} \xrightarrow{\sim} X_m, \quad A \mapsto (A_1(\xi), A_2(\xi)). \quad (6.2)$$

(ii) The inverse isomorphism is given by

$$g_m : X_m \xrightarrow{\sim} (MI_{2m+1}(\xi))_{\text{red}}, \quad (B, C) \mapsto \tilde{\xi}(B, C, -C^\vee \circ B^{-1} \circ C). \quad (6.3)$$

**Proof.** (i) We first prove that the image of the map

$$f_m : (MI_{2m+1}(\xi))_{\text{red}} \to S_{m+1}^0 \times \Sigma_{m,m+1}, \quad A \mapsto (A_1(\xi), A_2(\xi)),$$

lies in $X_m$, that is, it satisfies conditions (i)–(iii) in (6.1). Indeed, (i) holds automatically since (4.9) and (4.11) give

$$-C^\vee \circ B^{-1} \circ C = -A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) = A_3(\xi) \in S^2 H_m^\vee \otimes \wedge^2 V^\vee.$$

We now claim that the morphism $\rho_{B,C}$ introduced in part (iii) of (6.1) coincides by definition with the morphism $\rho_{\xi,A}$ introduced in (5.9). Indeed, the upper triangle

\[^1\text{Here we use the decomposition (4.8) fixed by a choice of } \xi.\]
in (5.10) (twisted by $\mathcal{O}(1)$) and the lower part of (5.4) fit into the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H_{m+1} \otimes \mathcal{O} & \xrightarrow{\delta_B} & H_{m+1}^\vee \otimes \wedge^2 V^\vee \otimes \mathcal{O} & \xrightarrow{\text{can}} & H^0(E_{2m+2}(B)(1)) \otimes \mathcal{O} & \rightarrow & 0 \\
0 & \rightarrow & H_{m+1} \otimes \mathcal{O} & \xrightarrow{\delta_C} & H_{m+1}^\vee \otimes \Omega(2) & \xrightarrow{e} & E_{2m+2}(B)(1) & \rightarrow & 0
\end{array}
\]

(6.4)

where the composite $\widetilde{C} = \text{can} \circ \delta C$ is defined by condition (iii) in (6.1). Hence,

\[
\rho_{B,C} = \rho_{\xi,A}.
\]

(6.5)

Since $\rho_{\xi,A}$ is a subbundle morphism, condition (iii) of (6.1) holds and, moreover, $\widetilde{C}$ is also a subbundle morphism. Thus the lower part of the diagram (6.4) shows that

\[
(\widetilde{B}, \widetilde{C}): (H_{m+1} \oplus H_m) \otimes \mathcal{O} \rightarrow H_{m+1}^\vee \otimes \Omega(2)
\]

is a subbundle morphism, and so is its composite with the subbundle morphism $v^\vee: H_{m+1}^\vee \otimes \Omega(2) \hookrightarrow H_{m+1}^\vee \otimes V \otimes \mathcal{O}(1)$. By definition, this composite is equal to $(B, C) \circ u$. Thus condition (ii) of (6.1) holds.

This shows that $f_m((MI_{2m+1}(\xi))_{\text{red}})$ lies in $X_m$. Finally, the equality $g_m \circ f_m = \text{id}$ follows directly from (4.9) and (4.11).

(ii) We first prove that the image of the map\(^2\)

\[
g_m: X_m \rightarrow S_{2m+1}, \quad (B, C) \mapsto (B, C, C^\vee \circ B^{-1} \circ C),
\]

(6.6)

lies in $(MI_{2m+1}(\xi))_{\text{red}}$. Indeed, the subbundle morphism

\[
\mathcal{A} := (B, C) \circ u: (H_{m+1} \oplus H_m) \otimes \mathcal{O} \rightarrow H_{m+1}^\vee \otimes V^\vee \otimes \mathcal{O}(1)
\]

and its dual morphism extend to the following sequence, which is exact on the left and on the right:

\[
0 \rightarrow (H_{m+1} \oplus H_m) \otimes \mathcal{O}(-1) \xrightarrow{\mathcal{A}} H_{m+1}^\vee \otimes V^\vee \otimes \mathcal{O} \xrightarrow{\mathcal{A}^\vee \circ B^{-1}} (H_{m+1} \oplus H_m)^\vee \otimes \mathcal{O}(1) \rightarrow 0.
\]

(6.7)

Furthermore, by definition, we have $\mathcal{A}^\vee \circ B^{-1} \circ \mathcal{A} = u^\vee \circ A \circ u$, where $A$ is the matrix

\[
\begin{pmatrix}
B & C \\
-C^\vee & -C^\vee \circ B^{-1} \circ C
\end{pmatrix}.
\]

Since condition (i) of (6.1) holds, we can regard $A$ as an element of $S_m$ in view of the decomposition (4.8). It follows that $u^\vee \circ A \circ u = 0$, whence (6.7) is a monad. We claim that its cohomology bundle

\[
E(B, C) := \ker(\mathcal{A}^\vee \circ B^{-1})/ \text{Im} \mathcal{A}
\]

\(^2\)Here we identify the triple $(B, C, C^\vee \circ B^{-1} \circ C)$ with a point of $S^2H_{2m+1}^\vee \otimes \wedge^2 V^\vee$ by means of the decomposition (4.8).
is a \((2m+1)\)-instanton, thus giving the desired inclusion \(g(X_m) \subset (MI_{2m+1}(\xi))_{\text{red}}\). Indeed, consider the diagram (4.17), where we denote \(B \circ \alpha_{\xi,A}\) by \(A\). Put \(G := \text{coker} \mathcal{A}\) and replace the symbols \(\tau_{\xi,A}\) and \(\varepsilon_{\xi,A}\) by \(\tau_{B,C}\) and \(\varepsilon_{B,C}\) respectively:

\[
\begin{array}{cccccccc}
H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{A} & H_{m+1}^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{\text{can}} & G & \xrightarrow{\varepsilon_{B,C}} & 0 \\
0 \xrightarrow{i_{m+1}} & (H_{m+1} \oplus H_m) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{B \circ u} & H_{m+1}^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} & \xrightarrow{v \circ B^{-1}} & H_{m+1} \otimes T_{\mathbb{P}^3}(-1) & \xrightarrow{\tau_{B,C}} & 0 \\
& & & & & H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1)
\end{array}
\]

In this notation (5.7) becomes the display of the anti-self-dual monad

\[
0 \to H_{m+1} \otimes \mathcal{O}(-1) \xrightarrow{B \circ u} H_{m+1}^\vee \otimes \mathcal{O} \xrightarrow{u^\vee} H_{m+1}^\vee \otimes \mathcal{O}(1) \to 0 \quad (6.9)
\]

with symplectic cohomology sheaf \(E_{2m+2}(B)\):

\[
E_{2m+2}(B) = \ker(u^\vee)/\text{Im}(B \circ u). \quad (6.10)
\]

Moreover, arguing as in (5.9), (5.10), we have a subbundle morphism

\[
\rho_{B,C} : H_m \otimes \mathcal{O}(-1) \to E_{2m+2}(B) \quad (6.11)
\]

such that

\[
\tau_{B,C} = e^\vee \circ \theta \circ \rho_{B,C}, \quad (6.12)
\]

where \(\theta : E_{2m+2}(B) \xrightarrow{\sim} E_{2m+2}(B)\) is the symplectic structure on \(E_{2m+2}(B)\). Moreover, as in (5.8), we have

\[
h^0(E_{2m+2}(B)) = h^i(E_{2m+2}(B)(-2)) = 0, \quad i \geq 0. \quad (6.13)
\]

Furthermore, from the anti-self-dual monads (6.7) and (6.9) one can recover the anti-self-dual monad (5.12), which by (6.5) becomes a monad

\[
0 \to H_m \otimes \mathcal{O}(-1) \xrightarrow{\rho_{B,C}} E_{2m+2}(B) \xrightarrow{\rho_{B,C}^\vee \circ \theta} H_{m}^\vee \otimes \mathcal{O}(1) \to 0 \quad (6.14)
\]

with cohomology sheaf

\[
E(B,C) = \ker(\rho_{B,C}^\vee \circ \theta)/\text{Im}(\rho_{B,C}). \quad (6.15)
\]

We now see from (6.13) and (6.14) that \(h^0(E(B,C)) = h^i(E(B,C)(-2)) = 0, \ i \geq 0\), that is, \(E(B,C)\) is a \((2m+1)\)-instanton.

Thus \(\text{im} \ g_m \subset I_{2m+1}(\xi)\). The equality \(f_m \circ g_m = \text{id}\) follows directly from (6.2) and (6.3). □
§ 7. The scheme $Z_m$. Reduction of the irreducibility of $X_m$ to that of $Z_m$. Proof of the main theorem

7.1. The scheme $\hat{Z}_m$ and the open subset $Z_m$. In this subsection we define a new set $Z_m$ as a locally closed subset of some vector space (see (7.5)) and endow it with a natural scheme structure. Then we state Theorem 7.2 on the irreducibility of $Z_m$. This theorem plays a key role in the proof of irreducibility of $I_{2m+1}$ to be given in § 7.2. The proof of Theorem 7.2 begins in § 8 and ends in § 11.

We put

$$\Lambda_m := \wedge^2 H_m^\vee \otimes S^2 V^\vee, \quad \Phi_m := \text{Hom}(H_m, H_m^\vee \otimes \wedge^2 V^\vee),$$

(7.1)

$$(S_m^\vee)^0 := \{D \in S_m^\vee \mid \text{the homomorphism } D: H_m^\vee \otimes V^\vee \to H_m \otimes V \text{ is invertible}\}.$$  

(7.2)

Note that $(S_m^\vee)^0$ is an open dense subset of $S_m^\vee$ and there is a canonical isomorphism

$$S_m^0 \xrightarrow{\sim} (S_m^\vee)^0, \quad A \mapsto A^{-1}.$$  

(7.3)

We consider the sets

$$\hat{Z}_m := \left\{(D, \varphi) \in S_m^\vee \times \Phi_m \mid \Theta(D, \varphi) := \varphi^\vee \circ D \circ \varphi : H_m \otimes V \to H_m^\vee \otimes V^\vee \right\},$$

(7.4)

$$Z_m := \hat{Z}_m \cap (S_m^\vee)^0 \times \Phi_m$$

(7.5)

(here we understand a point $D \in S_m^\vee$ as a homomorphism $H_m^\vee \otimes V^\vee \to H_m \otimes V$), and let $Z_m$ be the closure of $Z_m$ in $S_m^\vee \times \Phi_m$. By definition, $Z_m$ is an open subset of $\hat{Z}_m$ and an open dense subset of $\overline{Z}_m$.

Note that there is a standard decomposition

$$\wedge^2 (H_m^\vee \otimes V^\vee) = S_m \oplus \Lambda_m$$

with induced projection

$$q_m : \wedge^2 (H_m^\vee \otimes V^\vee) \to \Lambda_m$$

onto the second summand and with a morphism

$$h : S_m^\vee \times \Phi_m \to \Lambda_m, \quad (D, \varphi) \mapsto q_m(\Theta(D, \varphi)).$$

By the definition of $\hat{Z}_m$ we have

$$\hat{Z}_m = h^{-1}(0).$$  

(7.7)

Clearly, the point $(0, 0)$ belongs to $\hat{Z}_m$, that is, $\hat{Z}_m$ is non-empty.

We adopt the following convention. The set $\hat{Z}_m$ is endowed with the scheme structure of a scheme-theoretic fibre $h^{-1}(0)$ of the morphism $h$, and $Z_m$ accordingly inherits the structure of an open subscheme of $\hat{Z}_m$. 
Remark 7.1. It follows from (7.7) that \( \hat{Z}_m \) can be regarded as the zero scheme \( (h^*s_{\text{taut}})_0 \) of the section \( h^*s_{\text{taut}} \) of the trivial vector bundle \( \Lambda_m \otimes \mathcal{O}_{S_m \times \Phi_m} \), where \( s_{\text{taut}} \) is the tautological section of the trivial vector bundle \( \Lambda_m \otimes \mathcal{O}_{\Lambda_m} \) of rank \( \dim \Lambda_m = 5m(m-1) \) over \( \Lambda_m \). This yields the following estimate for the dimension of \( \hat{Z}_m \) at each point \( z \in \hat{Z}_m \):

\[
\dim_z \hat{Z}_m = \dim h^{-1}(0) \geq \dim (S_m^\vee \times \Phi_m) - \dim \Lambda_m = 3m(m+1) + 6m^2 - 5m(m-1) = 4m(m+2).
\]  

(7.8)

In particular, if \( Z_m \) is non-empty, then

\[
\dim_z Z_m \geq 4m(m+2), \quad z \in Z_m.
\]  

(7.9)

The following properties of \( Z_m \) will be used in the next subsection.

Theorem 7.2. (i) \( Z_m \) is an integral scheme and a locally complete intersection of dimension \( 4m(m+2) \).

(ii) The natural morphism \( p_m : Z_m \to (S_m^\vee)_0, (D, \varphi) \mapsto D \), is surjective.

The proof of Theorem 7.2 begins in §8 and ends in §11.

7.2. Proof of the main theorem. In this subsection we give a proof of Theorem 1.1. We put

\[
\tilde{X}_m := \{(D, C) \in (S_{m+1}^\vee)_0 \times \Sigma_{m+1} \mid (C^\vee \circ D \circ C : H_m \otimes V \to H_m^\vee \otimes V \vee) \in S_m \}. \tag{7.10}
\]

The set \( \tilde{X}_m \) has the natural structure of the closed subscheme in \( (S_{m+1}^\vee)_0 \times \Sigma_{m+1} \) given by the equations

\[
C^\vee \circ D \circ C \in S_m. \tag{7.11}
\]

Since conditions (ii) and (iii) in (6.1) are open and \( X_m \) is non-empty (see Theorem 6.1), it follows immediately from (7.3) that \( X_m \) is a non-empty open subset of \( (\tilde{X}_m)_{\text{red}} \):

\[
\emptyset \neq X_m \overset{\text{open}}{\hookrightarrow} (\tilde{X}_m)_{\text{red}}. \tag{7.12}
\]

We fix a direct sum decomposition

\[
H_{m+1} \cong H_m \oplus k.
\]

Under this isomorphism, every homomorphism

\[
C \in \Sigma_{m+1} = \text{Hom}(H_m, H_{m+1}^\vee) \otimes \wedge^2 V \vee, \quad C : H_m \otimes V \to H_{m+1}^\vee \otimes V \vee, \tag{7.13}
\]

can be represented as a homomorphism

\[
C : H_m \otimes V \to H_{m+1}^\vee \otimes k \vee \otimes V \vee, \tag{7.14}
\]

that is, as a matrix of homomorphisms

\[
C = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \tag{7.15}
\]
where
\[ \varphi \in \text{Hom}(H_m, H_m^{\vee}) \otimes \wedge^2 V^{\vee} = \Phi_m, \quad \psi \in \Psi_m := \text{Hom}(H_m, k^{\vee}) \otimes \wedge^2 V^{\vee}. \] (7.16)
Every homomorphism \( D \in (S_{m+1}^{\vee})^0 \subset S_{m+1}^{\vee} = S^2 H_{m+1} \otimes \wedge^2 V \subset \text{Hom}(H_{m+1}^{\vee} \otimes V^{\vee}, H_{m+1} \otimes V) \) can accordingly be written as a matrix of homomorphisms
\[
D = \begin{pmatrix} D_1 & \lambda \\ -\lambda^{\vee} & \mu \end{pmatrix},
\] (7.17)
where
\[
D_1 \in S_{m}^{\vee} \subset \text{Hom}(H_{m}^{\vee} \otimes V^{\vee}, H_{m} \otimes V), \quad \lambda \in L_{m} := \text{Hom}(k^{\vee}, H_{m}) \otimes \wedge^2 V, \quad \mu \in M_{m} := \text{Hom}(k, k) \otimes \wedge^2 V.
\] (7.18)
It follows from (7.15) and (7.17) that the homomorphism
\[
C^{\vee} \circ D \circ C \colon H_m \otimes V \to H_{m}^{\vee} \otimes V^{\vee}, \quad C^{\vee} \circ D \circ C \in \wedge^2 (H_{m}^{\vee} \otimes V^{\vee}),
\]
can be represented in the form
\[
C^{\vee} \circ D \circ C = \varphi^{\vee} \circ D_1 \circ \varphi + \varphi^{\vee} \circ \lambda \circ \psi - \psi^{\vee} \circ \lambda^{\vee} \circ \varphi + \psi^{\vee} \circ \mu \circ \psi.
\] (7.19)

We now use the following proposition, whose proof is postponed till §12.

**Proposition 7.3.** Let \( m \geq 1 \). Then, for every point \( D \in (S_{m+1}^{\vee})^0 \) and a general choice of the decomposition \( H_{m+1} \sim H_m \oplus k \), the induced homomorphism \( D_1 \) in the matrix \( D \) of homomorphisms in (7.17) is non-degenerate.

By (7.15)–(7.18) we have
\[
S_{m+1}^{\vee} \times \Sigma_{m+1} = S_{m}^{\vee} \times \Phi_m \times \Psi_m \times L_{m} \times M_{m}.
\]
Therefore, in accordance with Proposition 7.3 we fix a decomposition \( H_{m+1} \sim H_m \oplus k \) such that \( \widetilde{X}'_m = \widetilde{X}_m \cap (S_{m}^{\vee})^0 \times \Phi_m \times \Psi_m \times L_{m} \times M_{m} \) is an open dense subset of \( \widetilde{X}_m \). For the sake of simplicity, we redenote the sets \( \widetilde{X}'_m \) and \( \widetilde{X}'_m \cap X_m \) by \( \widetilde{X}_m \) and \( X_m \) respectively, and let \( \overline{X}_m \) be the closure of \( X_m \) in \( \widetilde{X}_m \). There are well-defined morphisms
\[
\tilde{p}_m : \widetilde{X}_m \to L_{m} \times M_{m}, \quad (D_1, \varphi, \psi, \lambda, \mu) \mapsto (\lambda, \mu),
\]
\[
p_m := \tilde{p}_m|_{\overline{X}_m} : \overline{X}_m \to L_{m} \times M_{m}.
\]

Let \( \mathcal{X} \) be an arbitrary irreducible component of \( X_m \) and let \( \overline{\mathcal{X}} \) be its closure in \( \overline{X}_m \). We fix a point \( z = (D_1, \varphi, \psi, \lambda, \mu) \in \mathcal{X} \) not lying in the components of \( X_m \) different from \( \mathcal{X} \). Consider the morphism
\[
f : \mathbb{A}^1 \to \overline{\mathcal{X}}, \quad t \mapsto (D_1, t^2 \varphi, t\psi, t\lambda, t^2 \mu), \quad f(1) = z.
\] (7.20)
(It is well defined because of (7.19).) By definition, the point \( f(0) = (D_1, 0, 0, 0, 0) \) lies in the fibre \( p_m^{-1}(0, 0) \). Hence \( p_m^{-1}(0, 0) \cap \overline{\mathcal{X}} \neq \emptyset \). In other words,
\[
\rho^{-1}(0, 0) \neq \emptyset, \quad \text{where } \rho := p_m|_{\overline{\mathcal{X}}}. \] (7.21)
Then it follows from (7.19) and the definition of $\tilde{X}_m$ that
\[ \tilde{p}_m^{-1}(0,0) = \{(D_1, \varphi, \psi) \in (S_m^\vee)^0 \times \Phi_m \times \Psi_m \mid \varphi^\vee \circ D_1 \circ \varphi \in S_m \}. \] (7.22)
Comparing (7.22) with the definition (7.5) of $Z_m$, we get a set-theoretic equality
\[ \tilde{p}_m^{-1}(0,0) = Z_m \times \Psi_m, \] (7.23)
whence
\[ \rho^{-1}(0,0) \subseteq \tilde{p}_m^{-1}(0,0) \subseteq \tilde{p}_m^{-1}(0,0) = Z_m \times \Psi_m. \] (7.24)
In particular, it follows from (7.23) and Theorem 7.2 that
\[ \dim \rho^{-1}(0,0) \leq \dim \tilde{p}_m^{-1}(0,0) \leq \dim Z_m + \dim \Psi_m = 4m(m+2) + 6m = 4m^2 + 14m. \] (7.25)
Hence, in view of (7.21),
\[ \dim \overline{X} \leq \dim \rho^{-1}(0,0) + \dim L_m + \dim M_m \leq 4m^2 + 14m + 6m + 6 = 4m^2 + 20m + 6. \] (7.26)
On the other hand, it follows from formula (3.14) for $n = 2m+1$, equality (4.22) and Theorem 6.1, (ii) that for every point $x \in X$ with $A := g_m(x) \in MI_{2m+1}(\xi)$ we have
\[ 4m^2 + 20m + 6 = (2m + 1)^2 + 8(2m + 1) - 3 \leq \dim_A MI_{2m+1}(\xi) = \dim \overline{X}. \] (7.27)
Comparing (7.26) with (7.27), we see that all the inequalities in (7.25)–(7.27) are equations. In particular,
\[ \dim \rho^{-1}(0,0) = \dim (Z_m \times \Psi_m) = \dim \overline{X} - \dim (L_m \times M_m). \] (7.28)
Since the scheme $Z_m$ is integral by Theorem 7.2 and, therefore, $Z_m \times \Psi_m$ is also integral, we see that (7.24) and (7.28) yield isomorphisms of integral schemes
\[ \rho^{-1}(0,0) \cong \tilde{p}_m^{-1}(0,0) \cong \tilde{p}_m^{-1}(0,0) \cong Z_m \times \Psi_m. \] (7.29)
We now state the following lemma, whose proof is left to the reader.

**Lemma 7.4.** Let $f: X \to Y$ be a morphism of reduced schemes, where $Y$ is a smooth integral scheme. Assume that there is a closed point $y \in Y$ such that the following conditions hold for every irreducible component $X'$ of $X$.

(a) $\dim f^{-1}(y) = \dim X' - \dim Y$.

(b) The scheme-theoretic inclusion of fibres $(f|_{X'})^{-1}(y) \subset f^{-1}(y)$ is an isomorphism of integral schemes.

Then the following assertions hold.

(i) There is an open subset $U$ of $Y$ containing the point $y$ such that the morphism $f|_{f^{-1}(U)}: f^{-1}(U) \to U$ is flat.

(ii) $X$ is an integral scheme.

(iii) $X$ is smooth at every smooth point of $f^{-1}(y)$. 


Applying assertions (i), (ii) of Lemma 7.4 for $X = X_m$, $X' = X$, $Y = L_m \times M_m$, $y = (0, 0)$, $f = p_m$ and using (7.28) and (7.29), we obtain that $X_m$ is an integral scheme of dimension $4m^2 + 20m + 6$.

Then it follows from Corollary 4.7 and Theorem 6.1 that $(MI_{2m+1})_{\text{red}}$ is irreducible and has dimension $4m^2 + 20m + 6 = n^2 + 8n - 3$ for $n = 2m + 1$, that is, the inequality (3.14) becomes an equality. This together with Theorem 3.1 yields that $MI_{2m+1}$ is a locally complete intersection in the vector space $S_{2m+1}$. We now use the following simple lemma, whose proof is left to the reader.

**Lemma 7.5.** Let $\mathcal{X}$ be an irreducible locally complete intersection subscheme of a smooth integral scheme $\mathcal{Y}$ such that $\mathcal{X}$ is smooth at some point. Then $\mathcal{X}$ is integral.

Applying Lemma 7.5 with $\mathcal{X} = MI_{2m+1}$, $\mathcal{Y} = S_{2m+1}$ and using Remark 3.2, we obtain that $MI_{2m+1}$ is integral. Since $\pi_{2m+1}: MI_{2m+1} \rightarrow I_{2m+1}$, $A \mapsto [E(A)]$, is a principal $(\text{GL}(H_{2m+1})/\{\pm \text{id}\})$-bundle in the étale topology (see §3), it follows that $I_{2m+1}$ is an integral scheme of dimension $16m + 5 = 8n - 3$ for $n = 2m + 1$. This completes the proof of Theorem 1.1. □

**Remark 7.6.** Consider the natural projections $p_1: X_m \rightarrow L_m \times M_m \times \Psi_m$, $p_2: X_m \rightarrow S_m \times L_m \times M_m \times \Psi_m \simeq S_{m+1} \times \Psi_m$ and $p: X_m \rightarrow S_{m+1} \times \Psi_m \xrightarrow{\text{pr}_1} S_{m+1}$. It follows from (7.29) that $p_1^{-1}(0, 0, 0) \simeq Z_m$. On the other hand, Theorem 7.2 shows that the projection $p': Z_m \xrightarrow{p_m} (S_m^0)^0 \simeq S_m^0 \hookrightarrow S_m$ is dominant, whence the fibre $(p')^{-1}(D_1)$ is an integral scheme of dimension $\dim Z_m - \dim S_m = m(m + 5)$ for a general point $D_1 \in S_m$. This fibre coincides with $p_2^{-1}(D_1, 0, 0, 0)$ because of the equality $p_1^{-1}(0, 0, 0) \simeq Z_m$. Thus we have

$$\dim p_2^{-1}(D_1, 0, 0, 0) = 5m(m + 1) = 4m^2 + 20m + 6 - \left(\frac{3}{2}(m + 1)(m + 2) + 6m\right)$$

$$= \dim X_m - \dim (S_m \times \Psi_m).$$

Therefore, using Lemma 7.4 with $X = X', Y = L_m \times M_m$, $y = (D_1, 0, 0, 0)$, $f = p_2$, we obtain that $p_2$ is a dominant morphism. Hence,

$$p: X_m \rightarrow S_{m+1}, \quad (D, \varphi) \mapsto D,$$

is a dominant morphism.

**§ 8. The study of $Z_m$. Beginning of the proof of Theorem 7.2**

In this section we start the proof of Theorem 7.2 on the irreducibility of $Z_m$. The case $m = 1$ is treated in §8.1. Then we obtain explicit equations for $Z_m$ given a fixed decomposition of $H_m$ into the direct sum of $H_{m-1}$ and $k$. In §8.2 we state the main result of this section, Proposition 8.1, which is a part of the induction step in the proof of Theorem 7.2. (The rest of the proof of Theorem 7.2 will be given in the last subsection of §11.) In §§8.3–8.5 we study the explicit equation for $Z_m$ in detail and, as a result, give a proof of Proposition 8.1.
8.1. Explicit equations for $Z_m$ in $(S^\vee_m)^0 \times \Phi_m$. We proceed to the proof of the irreducibility of $Z_m$ by induction on $m$. Clearly, $\Lambda_m = 0$ for $m = 1$, whence the equations $\{\Theta_1(D_1, \varphi_1) \in S_1\}$ for $Z_1$ in $(\wedge^2(k^\vee \otimes V^\vee))^0$ are empty and scheme-theoretically we have

$$Z_1 = (\wedge^2(k^\vee \otimes V^\vee))^0 \times \Phi_1 \xrightarrow{\text{open}} \mathbb{A}^{12}.$$

Thus $Z_1$ is integral, being an open dense subset of $\mathbb{A}^{12}$.

We now fix an isomorphism

$$H_{m-1} \oplus k \sim H_m, \quad ((a_1, \ldots, a_{m-1}), a_m) \mapsto (a_1, \ldots, a_m). \quad (8.1)$$

Under this isomorphism, every homomorphism

$$\varphi : H_m \otimes V \to H_m^\vee \otimes V^\vee, \quad \varphi \in \Phi_m = \text{Hom}(H_m, H_m^\vee \otimes \wedge^2 V^\vee), \quad (8.2)$$

can be represented as a homomorphism

$$\varphi : H_{m-1} \otimes V \oplus k \otimes V \to H_{m-1}^\vee \otimes V^\vee \oplus k^\vee \otimes V^\vee, \quad (8.3)$$

that is, as a matrix of homomorphisms

$$\varphi = \begin{pmatrix} \varphi_{m-1} & \chi \\ \psi & \theta \end{pmatrix}, \quad (8.4)$$

where

$$\varphi_{m-1} \in \Phi_{m-1} = \text{Hom}(H_{m-1}, H_{m-1}^\vee \otimes \wedge^2 V^\vee),$$

$$\psi \in \Psi_{m-1} := \text{Hom}(H_{m-1}, k^\vee \otimes \wedge^2 V^\vee), \quad (8.5)$$

$$\chi \in \text{Hom}(k, H_{m-1}^\vee \otimes \wedge^2 V^\vee) = \Psi_{m-1}, \quad \theta \in B_\alpha := \text{Hom}(k, k^\vee \otimes \wedge^2 V) = S_1.$$

Accordingly, every homomorphism

$$D \in S^\vee_m \subset \text{Hom}(H_m^\vee \otimes V^\vee, H_m \otimes V) \quad (8.6)$$

can be represented as a matrix of homomorphisms

$$D = \begin{pmatrix} D_{m-1} & a \\ -a^\vee & \alpha \end{pmatrix}, \quad (8.7)$$

where

$$D_{m-1} \in S^\vee_{m-1} \subset \text{Hom}(H_{m-1}^\vee \otimes V^\vee, H_{m-1} \otimes V),$$

$$a \in \text{Hom}(k^\vee, H_{m-1} \otimes \wedge^2 V) = \Psi^\vee_{m-1}, \quad \alpha \in B_\alpha := \text{Hom}(k^\vee, k \otimes \wedge^2 V). \quad (8.8)$$

Note that formulae (8.5)–(8.8) yield isomorphisms

$$S^\vee_m \xrightarrow{\sim} B_\alpha \times \Psi^\vee_{m-1} \times S^\vee_{m-1}, \quad \Phi_m \xrightarrow{\sim} \Phi_{m-1} \times \Psi_{m-1} \times \Psi_{m-1} \times B_\theta \quad (8.9)$$
and hence an isomorphism
\[
S_m^\vee \times \Phi_m \cong B_\theta \times B_\alpha \times \Psi_m \times S_{m-1}^\vee \times \Phi_{m-1} \times \Psi_{m-1} \times \Psi_{m-1},
\]
\[(D, \varphi) \mapsto (\theta, \alpha, a, D_{m-1}, \varphi_{m-1}, \psi, \chi). \tag{8.10}\]

It follows from (8.4) and (8.7) that the homomorphism
\[
\Theta(D, \varphi) := \varphi^\vee \circ D \circ \varphi : H_m \otimes V \to H_m^\vee \otimes V^\vee, \quad \Theta(D, \varphi) \in \wedge^2(H_m^\vee \otimes V^\vee),
\]
can be represented as a matrix of homomorphisms
\[
\Theta(D, \varphi) = \begin{pmatrix}
\Theta_1(D, \varphi) & b(D, \varphi) \\
-b(D, \varphi) & \beta(D, \varphi)
\end{pmatrix}, \tag{8.11}
\]
where
\[
\Theta_1(D, \varphi) := \varphi_{m-1}^\vee \circ D_{m-1} \circ \varphi_{m-1} + \varphi_{m-1}^\vee \circ a \circ \psi - \psi^\vee \circ a^\vee \circ \varphi_{m-1} + \psi^\vee \circ a \circ \psi
\in \wedge^2(H_{m-1}^\vee \otimes V^\vee) \subset \text{Hom}(H_{m-1}^\vee \otimes V^\vee, H_{m-1} \otimes V),
\]
\[
b(D, \varphi) := \varphi_{m-1}^\vee \circ D_{m-1} \circ \chi + \varphi_{m-1}^\vee \circ a \circ \theta - \psi^\vee \circ a^\vee \circ \chi + \psi^\vee \circ a \circ \theta \tag{8.12}
\in \text{Hom}(H_{m-1} \otimes V, k^\vee \otimes V^\vee),
\]
\[
\beta(D, \varphi) := \chi^\vee \circ D_{m-1} \circ \chi + \chi^\vee \circ a \circ \theta - \theta^\vee \circ a^\vee \circ \chi + \theta^\vee \circ a \circ \theta \in B_\theta.
\]

In this notation, \(Z_m\) can be described as follows:
\[
Z_m = \left\{ (D, \varphi) \in (S_m^\vee)^0 \times \Phi_m \middle| \begin{array}{l}
(i) \Theta_1(D, \varphi) \in S_{m-1}; \\
(ii) b(D, \varphi) \in \Psi_{m-1}
\end{array} \right\}. \tag{8.13}
\]

(Note that the condition \(\beta(D, \varphi) \in S_1\) is empty here.)

Thus we have the following explicit equations for \(Z_m\) in the open subset \((S_m^\vee)^0 \times \Phi_m\) of the variety \(S_m^\vee \times \Phi_m\), where \(S_m^\vee \times \Phi_m\) is regarded as a direct product \(B_\theta \times B_\alpha \times \Psi_m \times S_{m-1} \times \Phi_{m-1} \times \Psi_{m-1} \times \Psi_{m-1}\) according to (8.10):
\[
\Theta_1(D, \varphi) := \varphi_{m-1}^\vee \circ D_{m-1} \circ \varphi_{m-1} + \varphi_{m-1}^\vee \circ a \circ \psi - \psi^\vee \circ a^\vee \circ \varphi_{m-1} + \psi^\vee \circ a \circ \psi \in S_{m-1}, \tag{8.14}
\]
\[
b(D, \varphi) := \varphi_{m-1}^\vee \circ D_{m-1} \circ \chi + \varphi_{m-1}^\vee \circ a \circ \theta - \psi^\vee \circ a^\vee \circ \chi + \psi^\vee \circ a \circ \theta \in \Psi_{m-1}. \tag{8.15}
\]

Equations (8.14), (8.15) will be used systematically in the next few subsections.

8.2. Part of the induction step in the proof of Theorem 7.2. We first introduce more notation. We put
\[
(\wedge^2 V^\vee)^0 := \{ a \in \wedge^2 V \mid a : V^\vee \to V \text{ is an isomorphism} \},
\]
\[
(\wedge^2 V^\vee)^1 := \{ a \in \wedge^2 V^\vee \mid a : V \to V^\vee \text{ is an isomorphism} \}.
\]

Consider the projective space \(P(\wedge^2 V^\vee)\) and the Grassmannian \(G = G(1, 3) \subset P(\wedge^2 V^\vee)\) (the Plücker embedding). Take any points \(a \in (\wedge^2 V^\vee)^0\) and \(b \in (\wedge^2 V^\vee)^0\) such that the corresponding points \(\langle a^{-1} \rangle\) and \(\langle b \rangle\) in \(P(\wedge^2 V^\vee)\) are distinct. The
Suppose that \( P^1(a,b) := \text{Span}((a^{-1}), (b)) \) through these points intersects the quadric \( G \) in two points \( y_1, y_2 \), not necessarily distinct. Let \( \mathbb{P}^1_i(a,b), i = 1,2 \), be the lines in \( \mathbb{P}^3 \) corresponding to the points \( y_1, y_2 \). If \( y_1 \) and \( y_2 \) are distinct, then the lines \( \mathbb{P}^1_1(a,b) \) and \( \mathbb{P}^1_2(a,b) \) are disjoint. We put

\[
L(a,b) := \mathbb{P}^1_1(a,b) \sqcup \mathbb{P}^1_2(a,b).
\]

Note that there are natural isomorphisms \( S^\vee_1 \simeq \wedge^2 V \) and \( \Phi^\vee_1 \simeq \wedge^2 V^\vee \). For every \( m \geq 2 \) we have the induced isomorphisms

\[
U_S := \bigoplus_{i=1}^m (S^\vee_1)^{(i)} \simeq \bigoplus_1^m \wedge^2 V, \quad U_\Phi := \bigoplus_{i=1}^m (\Phi^\vee_1)^{(i)} \simeq \bigoplus_1^m \wedge^2 V^\vee,
\]

where \( (S^\vee_1)^{(i)} \) and \( (\Phi^\vee_1)^{(i)} \) are copies of \( S^\vee_1 \) and \( \Phi^\vee_1 \) respectively. Every isomorphism

\[
h : \underbrace{H_1 \oplus \cdots \oplus H_1}_m \cong H_m
\]

induces embeddings \( U_S \hookrightarrow S^\vee_m \) and \( U_\Phi \hookrightarrow \Phi_m \) and, therefore, an embedding

\[
\tau_h : U_S \times U_\Phi \hookrightarrow S^\vee_m \times \Phi_m.
\]

Moreover, the set

\[
W_{S\Phi} := \{ ((D^{(1)}, \ldots, D^{(m)}), (\varphi^{(1)}, \ldots, \varphi^{(m)})) \in U_S \times U_\Phi \mid \text{the subsets } L(D^{(i)}, \varphi^{(i)}) \text{ in } \mathbb{P}^3, 1 \leq i \leq m, \text{ are well defined, pairwise disjoint and do not lie on a quadric} \}
\]

is clearly an open subset of \( U_S \times U_\Phi \).

Our aim in this section is to prove the following proposition, which is a part of the induction step \( m-1 \sim m \) in the proof of Theorem 7.2.

**Proposition 8.1.** Suppose that \( m \geq 2 \) and that \( Z_{m-1} \) satisfies the conclusions of Theorem 7.2. Then there is an irreducible component \( Z \) of \( Z_m \) such that the following assertions hold.

(i) If \( Z_m = Z \cup Y \) is the decomposition of \( Z_m \) into components, then \( Z^0 := Z \setminus (Z \cap Y) \) is an integral subscheme and a locally complete intersection in \( (S^\vee_m)^0 \times \Phi_m \).

(ii) We have \( \dim Z = 4m(m+2) \), and the natural projection \( p_m|_Z : Z \to (S^\vee_m)^0 \), \( (D, \varphi) \mapsto D \), is dominant.

(iii) There is an isomorphism \( h \) of the form (8.18) such that we have \( Z \cap \tau_h(W_{S\Phi}) \neq \emptyset \) in the notation (8.19) and (8.20).

Before proving Proposition 8.1, we make some preliminary remarks.

We first consider the case \( m = 2 \). Then \( D_{m-1} = D_1 \in \wedge^2 V \), \( \varphi_{m-1} = \varphi_1 \in \wedge^2 V^\vee \) and \( a, \alpha \in \wedge^2 V \), \( \psi, \chi, \theta \in \wedge^2 V^\vee \). Hence equations (8.14) become empty, and equations (8.15) take the form

\[
(\varphi_1 \circ D_1 - \psi_1 \circ a) \circ \chi - (\varphi_1 \circ a - \psi_1 \circ \alpha) \circ \theta \in \wedge^2 V^\vee.
\]
For a general point \( x = (D_1, \varphi_1, \psi, a, \alpha) \in (\wedge^2 V)^0 \times (\wedge^2 V^\vee)^4 \), the system (8.21) of linear equations with respect to the pair \((\chi, \theta) \in (\wedge^2 V^\vee)^2\) has maximal rank equal to 10. Thus the space \( F_x \) of solutions of this system is a subspace of dimension 2 in \((\wedge^2 V^\vee)^2\). This means that there is a component \( Z \) in \( Z_2 \) with projection \( p_Z : Z \to (\wedge^2 V)^0 \times (\wedge^2 V^\vee)^4 \), \((D_1, \varphi_1, \psi, a, \alpha, \chi, \theta) \mapsto (D_1, \varphi_1, \psi, a, \alpha)\), and smooth fibre \( F_x = p_Z^{-1}(x) \) of dimension 2. Therefore \( \dim Z \leq \dim((\wedge^2 V)^0 \times (\wedge^2 V^\vee)^4) + 2 = 32 \). On the other hand, since (8.21) is a system of 10 equations for \( Z_2 \) in \((S^\vee_2)^0 \times \Phi_2\), it follows that \( Z \), being an irreducible component of \( Z_2 \), has dimension \( \geq \dim((S^\vee_2)^0 \times \Phi_2) - 10 = 42 - 10 = 32 \). Hence \( \dim Z = 32 \) and \( p_Z \) is dominant. As a corollary, the projection \( p_2|Z : Z \to (S^\vee_2)^0 \), \((D_1, \varphi_1, \psi, a, \alpha, \chi, \theta) \mapsto (D_1, a, \alpha)\), is also dominant. Moreover, let \( Z^0 \) be the complement in \( Z \) of the intersection of \( Z \) with the union of the other components of \( Z_2 \). Since the fibre \( F_x \) is smooth and \( p_Z(Z) \) is also smooth, being an open dense subset of \((\wedge^2 V)^0 \times (\wedge^2 V^\vee)^4\), we see that \( Z^0 \) is generically reduced as an open subscheme of \( Z_2 \).

We now use the following remark.

**Remark 8.2.** Let \( \tilde{X} \) be a locally closed subscheme in an affine space \( \mathbb{A}^M \) locally defined by \( N \) equations, \( \mathcal{X} \) an irreducible component of \( \tilde{X} \), and \( \mathcal{X}^0 \) the complement in \( \mathcal{X} \) of its intersection with the union of the other components of \( \tilde{X} \). Suppose that \( \mathcal{X}^0 \) is generically reduced as an open subscheme of \( \tilde{X} \) and that \( \dim \mathcal{X} = M - N \). Then \( \mathcal{X}^0 \) is an integral locally complete intersection subscheme of \( \mathbb{A}^M \).

Applying Remark 8.2 to the case \( \tilde{X} = Z_2 \), \( \mathcal{X} = Z \), \( M = 42 \), \( N = 10 \), \( \mathbb{A}^{42} = (\wedge^2 V)^0 \times (\wedge^2 V^\vee)^6 \), we obtain from what was said above that parts (i), (ii) of Proposition 8.1 hold for \( Z \). An explicit calculation shows that part (iii) also holds for \( Z \). This proves Proposition 8.1 for \( m = 2 \).

We now proceed to prove Proposition 8.1 for \( m \geq 3 \). Note that by hypothesis \( Z_{m-1} \) is an integral subscheme of \((S^\vee_{m-1})^0 \times \Phi_{m-1}\) such that \( \dim Z_{m-1} = 4(m^2 - 1) \) and the natural projection \( p_{m-1} : Z_{m-1} \to (S^\vee_{m-1})^0 \), \((D_{m-1}, \varphi_{m-1}) \mapsto D_{m-1}\), is surjective:

\[
p_{m-1}(Z_{m-1}) = (S^\vee_{m-1})^0. \tag{8.22}
\]

Since \( \dim(S^\vee_{m-1})^0 = 3m(m-1) \) and, therefore, \( \dim Z_{m-1} - \dim(S^\vee_{m-1})^0 = (m-1)(m+4) \), it follows that the set

\[
(S^\vee_{m-1})^{\text{int}} := \{ D_{m-1} \in (S^\vee_{m-1})^0 \mid \text{the fibre } p_{m-1}^{-1}(D_{m-1}) \text{ is integral of dimension } (m-1)(m+4) \} \tag{8.23}
\]

is an open dense subset of \((S^\vee_{m-1})^0\). Accordingly,

\[
Z_{m-1}^{\text{int}} := p_{m-1}^{-1}((S^\vee_{m-1})^{\text{int}}) \tag{8.24}
\]

is an open dense subset of \( Z_{m-1} \).

Using (8.10) and the embedding \( Z_m \hookrightarrow S^\vee_m \times \Phi_m \), we consider the projections

\[
\text{pr}_m : S^\vee_m \times \Phi_m \to B_\theta \times B_\alpha \times \Psi_{m-1}^\vee \times S^\vee_{m-1},
\]

\[
\text{pr}_m : (D, \varphi) = (\theta, a, D_{m-1}, \varphi_{m-1}, \psi, \chi) \mapsto (\theta, a, D_{m-1}), \tag{8.25}
\]

\[
\pi_m := \text{pr}_m|_{Z_m} : Z_m \to B_\theta \times B_\alpha \times \Psi_{m-1}^\vee \times S^\vee_{m-1}.
\]
We now consider the fibre \( \pi_{m-1}(y^0) \) of the projection \( \pi_m \) over the point

\[
y^0 := (\theta^0, \alpha^0, 0, D_{m-1}) \in B_\theta \times B_\alpha \times \Psi_{m-1}^\vee \times (S_{m-1}^\vee)^0, \quad (8.26)
\]

where\(^3\)

\[
\alpha^0 = (p_{ij}) \in \wedge^2 V^\vee \simeq B_\alpha, \quad \theta^0 = (q_{ij}) \in \wedge^2 V^\vee \simeq B_\theta, \quad p_{ij}, q_{ij} \in k. \quad (8.27)
\]

Note that, by the definition of \( \pi_m \), the fibre \( \pi_{m-1}(y^0) \) naturally lies in \( \Phi_{m-1} \times \Psi_{m-1} \times \Psi_{m-1}^\vee \):

\[
\pi_{m-1}(y^0) \subset \Phi_{m-1} \times \Psi_{m-1} \times \Psi_{m-1}^\vee. \quad (8.28)
\]

Thus, substituting (8.26) in (8.14) and (8.15), we obtain equations for \( \pi_{m-1}(y^0) \) as a subscheme in \( \Phi_{m-1} \times \Psi_{m-1} \times \Psi_{m-1}^\vee \), regarded as equations in the variables \( \varphi_{m-1}, \chi \) and \( \psi \):

\[
\varphi_{m-1}^\vee \circ D_{m-1} \circ \varphi_{m-1} + \psi^\vee \circ \alpha^0 \circ \psi \in S_{m-1}, \quad (8.29)
\]

\[
\varphi_{m-1}^\vee \circ D_{m-1} \circ \chi + \psi^\vee \circ \alpha^0 \circ \theta^0 \in \Psi_{m-1}. \quad (8.30)
\]

For an arbitrary point \( y^0 \) in (8.26), where \( D_{m-1} \in (S_{m-1}^\vee)^0 \), we consider the set

\[
F(\theta^0, \alpha^0, D_{m-1}) := \pi_{m-1}(y^0) \cap \{ \chi = \psi = 0 \}. \quad (8.31)
\]

It follows from (8.29) that

\[
F(\theta^0, \alpha^0, D_{m-1}) \simeq \{ \varphi_{m-1} \in \Phi_{m-1} \mid \varphi_{m-1}^\vee \circ D_{m-1} \circ \varphi_{m-1} \in S_{m-1} \}. \quad (8.32)
\]

Hence,

\[
\bigcup_{D_{m-1} \in (S_{m-1}^\vee)^0} F(\theta^0, \alpha^0, D_{m-1}) = \{ (\theta^0, \alpha^0) \} \times Z_{m-1}.
\]

Moreover, it follows from definition (8.23) that if \( D_{m-1} \in (S_{m-1}^\vee)^{\text{int}} \), then the set \( F(\theta^0, \alpha^0, D_{m-1}) \) is irreducible, has dimension \((m - 1)(m + 4)\) and, by (8.10), (8.24) and (8.31),

\[
\bigcup_{D_{m-1} \in (S_{m-1}^\vee)^{\text{int}}} F(\theta^0, \alpha^0, D_{m-1}) = \{ (\theta^0, \alpha^0, 0) \} \times Z_{m-1}^{\text{int}} \times \{ (0, 0) \}. \quad (8.33)
\]

\[\textbf{8.3. Proof of Proposition 8.1: the case of odd } m; \text{ first computations.} \]

In this subsection we prove Proposition 8.1 in the case of odd \( m, m = 2p + 1, p \geq 1 \).

We fix decompositions

\[
H_{m-1} \simeq H_2 \oplus \cdots \oplus H_2, \quad p \quad H_2 \simeq H_1 \oplus H_1. \quad (8.34)
\]

\(^3\)Here and in what follows we use a fixed basis \( e_1, \ldots, e_4 \) of \( V \) in order to interpret points of \( \wedge^2 V \) and \( \wedge^2 V^\vee \) as skew-symmetric \( 4 \times 4 \) matrices.
Using these decompositions, we consider the points $D^\Delta_{m-1} \in (S^\vee_{m-1})^0$ and $\varphi^\Delta_{m-1} \in \Phi_{m-1}$ given by the matrices\(^4\)

\[
D^\Delta_{m-1} := D_2 \oplus \cdots \oplus D_p, \quad \varphi^\Delta_{m-1} = \varphi^\Delta_{m-1}(N,a,d,f,g) := \varphi_2 \oplus \cdots \oplus \varphi_p, \quad (8.35)
\]

where

\[
D_2 = D' \oplus D'' \in S^\vee_2, \quad D' = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \in \wedge^2 V, \quad D'' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \wedge^2 V; \quad (8.36)
\]

\[
\varphi_2 = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \in \Phi_2, \quad \varphi_{11} = \begin{pmatrix} 1 & -1 \\ -N & N \end{pmatrix}, \quad \varphi_{22} = \begin{pmatrix} 1 & -N \\ -1 & N \end{pmatrix}, \quad N \in k; \quad (8.37)
\]

\[
\varphi_{12} = \begin{pmatrix} 1 & -g & f \\ -f & g & f \end{pmatrix}, \quad \varphi_{21} = \begin{pmatrix} 1 & -g & f \\ -a & g & f \end{pmatrix} \in \wedge^2 V^\vee, \quad a,d,f,g \in k.
\]

One easily checks that

\[
(\varphi^\Delta_{m-1})^\vee \circ D^\Delta_{m-1} \circ \varphi^\Delta_{m-1} \in S_{m-1}; \quad (8.38)
\]

whence the point $(D^\Delta_{m-1}, \varphi^\Delta_{m-1}) \in S^\vee_{m-1} \times \Phi_{m-1}$ lies in $\hat{Z}_{m-1}$. Moreover, since $D^\Delta_{m-1} \in (S^\vee_{m-1})^0$, we have

\[
(D^\Delta_{m-1}, \varphi^\Delta_{m-1}) \in Z_{m-1}. \quad (8.39)
\]

It follows from (8.38) that the equations (8.29) hold automatically for any $\psi \in \Psi_{m-1}$. Substituting the data $(\theta^0, \alpha^0, D^\Delta_{m-1}, \varphi^\Delta_{m-1})$ in (8.30), we now obtain equations for the variables $\chi, \psi$:

\[
(\varphi^\Delta_{m-1})^\vee \circ D^\Delta_{m-1} \circ \chi + \psi^\vee \circ \alpha^0 \circ \theta^0 \in \Psi_{m-1}. \quad (8.40)
\]

We put

\[
W(\theta^0, \alpha^0, D^\Delta_{m-1}, \varphi^\Delta_{m-1}) := \{ (\chi, \psi) \in \Psi_{m-1} \times \Psi_{m-1} \mid (\chi, \psi) \text{ satisfies (8.40)} \}. \quad (8.41)
\]

Since the equations (8.40) for $\chi, \psi$ are linear, it follows that $W(\theta^0, \alpha^0, D^\Delta_{m-1}, \varphi^\Delta_{m-1})$ is a subspace of the vector space $\Psi_{m-1} \times \Psi_{m-1} \simeq \Psi_{m-1} \oplus \Psi_{m-1}$.

\(^4\)Here and everywhere below, empty matrix entries stand for zeros. We also use the standard notation $A = A_1 \oplus \cdots \oplus A_n$ for the direct sum $A$ of matrices $A_1, \ldots, A_n$. By definition, this is a block matrix with diagonal blocks $A_1, \ldots, A_n$ and the other blocks 0.
To find the dimension of $W(\theta^0, \alpha^0, D^\Delta_{m-1}, \varphi^\Delta_{m-1})$, we use the decompositions (8.34) to represent $\chi$ and $\psi$ as $p$-tuples:

$$\chi = (\chi_1, \ldots, \chi_p), \quad \psi = (\psi_1, \ldots, \psi_p), \quad \psi_k, \chi_k \in \Psi_2, \quad k = 1, \ldots, p,$$

where

$$\chi_k = (X_k, Y_k), \quad \psi_k = (A_k, B_k), \quad X_k, Y_k, A_k, B_k \in \wedge^2 V^\vee$$

and

$$X_k = (x_{ij}^{(k)}), \quad Y_k = (y_{ij}^{(k)}), \quad A_k = (a_{ij}^{(k)}), \quad B_k = (b_{ij}^{(k)})$$

are skew-symmetric $4 \times 4$ matrices. Substituting $D^\Delta_{m-1}$ and $\varphi^\Delta_{m-1}$ from (8.35) in the system of equations (8.40), we can rewrite this system in the form

$$\varphi_2^\vee \circ D_2 \circ \chi_k + \psi_2^\vee \circ \alpha^0 \circ \theta^0 \in \Psi_2, \quad k = 1, \ldots, p.$$  

(8.45)

Substituting $D_2$, $\varphi_2$ and $\theta^0$ from (8.36), (8.37) and (8.27) in the system (8.45) and writing

$$x_1^{(k)} = x_{12}^{(k)}, \quad x_2^{(k)} = x_{34}^{(k)}, \quad x_3^{(k)} = x_{13}^{(k)}, \quad x_4^{(k)} = x_{14}^{(k)}, \quad x_5^{(k)} = x_{23}^{(k)}, \quad x_6^{(k)} = x_{24}^{(k)},$$

$$x_7^{(k)} = y_{12}^{(k)}, \quad x_8^{(k)} = y_{34}^{(k)}, \quad x_9^{(k)} = y_{13}^{(k)}, \quad x_{10}^{(k)} = y_{14}^{(k)}, \quad x_{11}^{(k)} = y_{23}^{(k)}, \quad x_{12}^{(k)} = y_{24}^{(k)},$$

$$x_{13}^{(k)} = a_{12}^{(k)}, \quad x_{14}^{(k)} = a_{34}^{(k)}, \quad x_{15}^{(k)} = a_{13}^{(k)}, \quad x_{16}^{(k)} = a_{14}^{(k)}, \quad x_{17}^{(k)} = x_{23}^{(k)}, \quad x_{18}^{(k)} = x_{24}^{(k)},$$

$$x_{19}^{(k)} = b_{12}^{(k)}, \quad x_{20}^{(k)} = b_{34}^{(k)}, \quad x_{21}^{(k)} = b_{13}^{(k)}, \quad x_{22}^{(k)} = b_{14}^{(k)}, \quad x_{23}^{(k)} = b_{23}^{(k)}, \quad x_{24}^{(k)} = b_{24}^{(k)},$$

we can rewrite (8.45) in the form

$$\sum_{j=1}^{24} m_{ij} x_j^{(k)} = 0, \quad i = 1, \ldots, 20, \quad k = 1, \ldots, p.$$  

(8.46)

where $M := (m_{ij})$ is a $20 \times 24$ matrix, whose entries depend on $N, a, d, f, g, p_{ij}, q_{ij}$.

A direct computation of the matrix $M = (m_{ij})$ for

$$N = 101, \quad a = 4, \quad d = 6, \quad f = 2, \quad g = 5,$$

$$p_{12} = -9, \quad p_{13} = -2, \quad p_{14} = -4, \quad p_{23} = 6, \quad p_{24} = -3, \quad p_{34} = -7,$$

$$q_{12} = -4, \quad q_{13} = -4, \quad q_{14} = -2, \quad q_{23} = 3, \quad q_{24} = -7, \quad q_{34} = 8$$

(8.47) (8.48)

now shows that $M$ is the upper left block submatrix,

$$M = \begin{pmatrix}
M_{11} & M_{12} & M_{13} & 0 \\
M_{21} & M_{22} & 0 & M_{23}
\end{pmatrix}.$$  

(8.49)

of the block matrix $\tilde{M}$ which will be described in (8.79)–(8.84) below. It follows from (8.49) and (8.80)–(8.84) by explicit computation that

$$\text{rk } M = 20.$$  

(8.50)
Since the matrix of the system (8.46) is a direct sum of \( p \) copies of \( \mathbf{M} \), we find that its rank is given by
\[
p \text{rk} \mathbf{M} = 20p = 10(m - 1). \tag{8.51}
\]

We now write \( \varphi_{m-1} \) and \( \alpha, \theta \) for the matrices obtained by substituting the data (8.47) in the matrix \( \varphi^\Delta_{m-1} \) in (8.35) and the data (8.48) in the matrices \( \alpha^0 \) and \( \theta^0 \) in (8.27), respectively. Let \( R(\theta^0, \alpha^0, D^\Delta_{m-1}, \varphi^\Delta_{m-1}) \) be the rank of the system of linear equations (8.40) as a function of \( \theta^0, \alpha^0, D^\Delta_{m-1}, \varphi^\Delta_{m-1} \). Then we can rewrite (8.51) in the form
\[
R(\theta, \alpha, D^\Delta_{m-1}, \varphi_{m-1}) = 10(m - 1). \tag{8.52}
\]

Note that \( (D^\Delta_{m-1}, \varphi_{m-1}) \in Z_{m-1} \) by (8.39), and \( Z^\text{int}_{m-1} \) is an irreducible open dense subset of \( Z_{m-1} \) by (8.24). Moreover, since the maximal value of \( R(\theta^0, \alpha^0, D_{m-1}, \varphi_{m-1}) \) is equal to \( 10(m - 1) \), the condition \( R(\theta^0, \alpha^0, D_{m-1}, \varphi_{m-1}) = 10(m - 1) \) imposed on the point \( (D_{m-1}, \varphi_{m-1}) \in Z_{m-1} \) is open. Therefore (8.52) yields the following remark.

**Remark 8.3.** The set
\[
(Z^\text{int}_{m-1})^0 := \{(D_{m-1}, \varphi_{m-1}) \in Z^\text{int}_{m-1} \mid R(\theta, \alpha, D_{m-1}, \varphi_{m-1}) = 10(m - 1)\}
\]
is open and dense in \( Z^\text{int}_{m-1} \) and, therefore, in \( Z_{m-1} \). By (8.22), it follows that there is an open dense subset \( (S^V_{m-1})^* \) of \( (S^\text{int}_{m-1})^* \) such that, for \( D_{m-1} \in (S^V_{m-1})^* \), the set
\[
F(\theta, \alpha, D_{m-1})^0 := F(\theta, \alpha, D_{m-1}) \cap (Z^\text{int}_{m-1})^0,
\]
where \( F(\theta^0, \alpha^0, D_{m-1}) \) is defined by (8.31), is an integral scheme of dimension \( (m - 1)(m + 4) \) and an open and dense subset of \( F(\theta, \alpha, D_{m-1}) \).

For \( D_{m-1} \in (S^V_{m-1})^* \) we put
\[
\mathbf{F} := \pi_{m-1}^{-1}(\theta, \alpha, 0, D_{m-1}), \quad F = F(D_{m-1}) := F(\theta, \alpha, D_{m-1}) = \mathbf{F} \cap \{\chi = \psi = 0\}.
\]
It follows from Remark 8.3 in analogy with (8.33) that \( \bigcup_{D_{m-1} \in (S^V_{m-1})^*} F(D_{m-1}) \) is open and dense in \( \{\theta, \alpha\} \times Z^\text{int}_{m-1} \times \{(0, 0)\} \), whence
\[
\bigcup_{D_{m-1} \in (S^V_{m-1})^*} F(D_{m-1}) = \{(\theta, \alpha, 0)\} \times \hat{Z}_{m-1} \times \{(0, 0)\}, \tag{8.53}
\]
where the closure is taken in \( S^V_m \times \hat{\Phi}_m \) and we use the isomorphism (8.10).

Take an arbitrary point \( D_{m-1} \in (S^V_{m-1})^* \). By Remark 8.3, the set \( F = F(D_{m-1}) \) is an integral scheme of dimension \( (m - 1)(m + 4) \) and contains an open dense subset \( F^0 \) such that, for every point \( w = (\theta, \alpha, D_{m-1}, \varphi_{m-1}) \in F^0 \), we have
\[
R(w) := R(\theta, \alpha, D_{m-1}, \varphi_{m-1}) = 10(m - 1).
\]
Fix a point \( w \in F^0 \) which is smooth on \( F \). We now want to calculate the dimension of the tangent space \( T_w F \).
Note that by (8.28) we can regard $F$ as a subset of $\Phi_{m-1} \times \Psi_{m-1} \times \Psi_{m-1}$. Therefore the equations for $T_wF$ are obtained by differentiating the equations (8.29) and (8.30) at the point $w$:

$$d\varphi_{m-1}^\lor \circ D_{m-1} \circ \varphi_{m-1} + (\varphi_{m-1}')^\lor \circ D_{m-1} \circ d\varphi_{m-1} |_{\varphi_{m-1}} \in S_{m-1},$$

$$\text{(8.54)}$$

The equations (8.54) coincide with the equations obtained by differentiating (at the point $w$) the equations $\varphi_{m-1}^\lor \circ D_{m-1} \circ \varphi_{m-1} \in S_{m-1}$, which determine $F$ as a subscheme of $\Phi_{m-1}$. Since $w$ is a smooth point of $F^0$, the equations (8.54) determine the tangent space $T_wF^0 = T_wF$ as a subspace of $T_{\varphi_{m-1}}\Phi_{m-1}$ and

$$\dim T_wF = \dim F = (m-1)(m-4).$$

On the other hand, equations (8.55) coincide with equations (8.30) after the identification of $((\chi_0, d\psi_0)_{\theta})$ with $(\chi, \psi)$, that is, they are equations for the subspace $W(w) = W(\theta, \alpha, D_{m-1}, \varphi_{m-1})$ of $\Psi_{m-1} \oplus \Psi_{m-1}$. Hence,

$$\dim W(w) = \dim(\Psi_{m-1} \oplus \Psi_{m-1}) - R(w) = 12(m-1) - 10(m-1) = 2(m-1).$$

By (8.56) we have

$$\dim_w F \leq \dim T_w F = \dim T_w F + \dim W(w) = (m-1)(m+4) + 2(m-1) = m^2 + 5m - 6. \quad \text{(8.57)}$$

Since $D_{m-1} \in (S_{m-1}^\lor)^0$ and $\alpha \in S_0^\lor$ (see (8.27)), it follows that $D = D_{m-1} \oplus \alpha \in (S_{m}^\lor)^0$, whence

$$w \in Z_m. \quad \text{(8.58)}$$

Moreover, $\dim(B_\theta \times B_\alpha \times \Psi_{m-1}^\lor \times S_{m-1}^\lor) = \dim(B_\theta \times S_{m}^\lor) = 6 + 3m(m+1) = 3m^2 + 3m + 6$. Calculating the dimension of the fibres of the projection $\pi_m: Z_m \to B_\theta \times B_\alpha \times \Psi_{m-1}^\lor \times S_{m-1}^\lor \simeq B_\theta \times S_{m}^\lor$ and using (8.57), we obtain that

$$\dim_w Z_m \leq \dim_w F + \dim(B_\theta \times S_{m}^\lor) \leq (m^2 + 5m - 6) + (3m^2 + 3m + 6) = 4m(m+2).$$

Comparing this with (7.9), we see that these inequalities are actually equalities. In particular, $\dim_w Z_m = 4m(m+2)$, $\dim_w F = \dim T_w F = m^2 + 5m - 6$ and $\dim \pi_m(Z_m) = (3m^2 + 3m + 6) = \dim(B_\theta \times S_{m}^\lor)$. This together with part (iii) of Lemma 7.4 shows that there is a unique irreducible component $Z$ of $Z_m$ passing through $w$ and possessing the following properties.

(i) $\dim Z = 4m(m+2)$ and the schemes $Z_m$ and $Z$ are smooth at $w$. Hence, in the notation of Proposition 8.1 (i), $Z^0$ is an integral locally complete intersection subscheme of $(S_{m}^\lor)^0 \times \Phi_m$. (Here we use Remark 8.2.)

(ii) $\pi_m(Z)$ is dense in $B_\theta \times S_{m}^\lor$ and, accordingly, $p_m(Z) = \text{pr}_S(\pi_m(Z))$ is dense in $S_{m}^\lor$, where $\text{pr}_S: B_\theta \times S_{m}^\lor \to S_{m}^\lor$ is the projection. This proves parts (i) and (ii) of Proposition 8.1.

Moreover, by Remark 8.3 we have $F = F(D_{m-1}) \subset Z$ for $D_{m-1} \in (S_{m-1}^\lor)^*$. Hence formula (8.53) yields the inclusion

$$\{ (\theta, \alpha, 0) \} \times \hat{Z}_{m-1} \times \{(0,0)\} \subset \bar{Z},$$

$$\text{(8.59)}$$
where \( \overline{Z} \) is the closure of \( Z \) in \( S_m^\vee \times \Phi_m \). In particular, arguing as in (8.58) and using (8.39), we have

\[
w^0 := (\theta, \alpha, 0, D_m^\Delta, \varphi_m^\Delta, 0, 0) \in Z.
\] (8.60)

8.4. Proof of Proposition 8.1: the case of odd \( m \); last computations. In this subsection we prove part (iii) of Proposition 8.1 for odd \( m \). To do this, we consider the following modification of the data (8.35)-(8.37):

\[
\begin{align*}
D_m^\Delta(c, f, g) &:= D_2(c, f_1, g_1) \oplus \cdots \oplus D_2(c, f_p, g_p), \\
\varphi_m^\Delta(\varepsilon, f, g) &:= \varphi_2(\varepsilon, f_1, g_1) \oplus \cdots \oplus \varphi_2(\varepsilon, f_p, g_p),
\end{align*}
\] (8.61)

where

\[
D_2(c, f_i, g_i) = \begin{pmatrix} D'(c, f_i, g_i) & D'' \end{pmatrix} \in S_2^\vee,
\]

\[
D'(c, f_i, g_i) = \begin{pmatrix} 1 & -1 & c g_i \\ -c f_i & 1 \end{pmatrix}, \quad i = 1, \ldots, p,
\]

\[
D'' = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \in \wedge^2 V;
\]

\[
\varphi_2(\varepsilon, f_i, g_i) = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21}(\varepsilon, f_i, g_i) & \varphi_{22} \end{pmatrix} \in \Phi_2,
\]

\[
\varphi_{11} = \begin{pmatrix} 1 & -1 \\ -N & N \end{pmatrix}, \quad \varphi_{22} = \begin{pmatrix} 1 & -N \\ N & N \end{pmatrix},
\] (8.63)

\[
\varphi_{12}(\varepsilon, f_i, g_i) = \begin{pmatrix} \varepsilon f_i \\ -\varepsilon g_i \\ -\varepsilon f_i \end{pmatrix}, \quad \varphi_{21}(\varepsilon, f_i, g_i) = \begin{pmatrix} \varepsilon a & \varepsilon f_i \\ -\varepsilon a & \varepsilon g_i \\ -\varepsilon f_i & -\varepsilon d \end{pmatrix} \in \wedge^2 V^\vee,
\]

\[
c, \varepsilon, N, a, d, f_i, g_i \in k, \quad i = 1, \ldots, p,
\]

and where \( f = (f_1, \ldots, f_p), \ g = (g_1, \ldots, g_p) \in k^p \). One easily checks that

\[
(\varphi_m^\Delta(\varepsilon))^\vee \circ D_m^\Delta(c, f, g) \circ \varphi_m^\Delta(\varepsilon) \in S_{m-1}, \quad \text{whence the point}
\]

\[
(D_m^\Delta(c, f, g), \varphi_m^\Delta(\varepsilon, f, g)) \in S_{m-1}^\vee \times \Phi_{m-1}
\]

lies in \( \overline{Z}_{m-1} \). Moreover, since \( D_m^\Delta(0, f, g) = D_m^\Delta \in (S_{m-1}^\vee)^0 \) and \( (S_{m-1}^\vee)^0 \) is open in \( S_{m-1}^\vee \), we see that for any \( f, g \in k^p \) there is an open dense subset \( U(f, g) \) of \( k \) such that \( D_m^\Delta(c, f, g) \in (S_{m-1}^\vee)^0, \ c \in U(f, g) \). Hence \( (D_m^\Delta(c, f, g), \varphi_m^\Delta(\varepsilon, f, g)) \in Z_{m-1} \) for \( c \in U(f, g) \). Since \( Z_{m-1} \) is closed in \( (S_{m-1}^\vee)^0 \times \Phi_{m-1} \), it follows that

\[
(D_m^\Delta(c, f, g), \varphi_m^\Delta(\varepsilon, f, g)) \in Z_{m-1}, \quad c, \varepsilon \in k, \quad f, g \in k^p.
\] (8.64)
In particular, taking $c = 1$ and $e = 0$ in (8.61)–(8.63), we see that the point
\[
    w(f, g, \theta^0, \alpha^0) := (D^\Delta_{m-1}(1, f, g) \oplus \alpha^0, \varphi^\Delta_{m-1}(0, f, g) \oplus \theta^0),
\]

is the image of the point
\[
    (\langle D'(1, f_1, g_1), \ldots, D'(1, f_p, g_p), D'', \ldots, D''', \alpha^0, (\varphi_{11}, \ldots, \varphi_{11}, \varphi_{22}, \ldots, \varphi_{22}, \theta^0) \rangle)
\]
in $U_S \times U_\Phi$ under the embedding $\tau_h : U_S \times U_\Phi \hookrightarrow S_m \times \Phi_m$, which is defined (up to a permutation of the direct summands) as in (8.18), (8.19) by means of the isomorphism
\[
    h : \bigoplus H_1 \oplus \cdots \oplus H_m \cong H_m, \quad m = 2p + 1, \tag{8.65}
\]
determined by the decompositions (8.34).

On the other hand, by (8.10) and (8.64) we have $w(f, g, \theta, \alpha) \in \{(\theta, \alpha, 0)\} \times \hat{Z}_{m-1} \times \{(0, 0)\}$, whence $w(f, g, \theta, \alpha) \in Z$ in view of (8.59) and (8.60). Thus,
\[
    w(f, g, \theta, \alpha) \in Z \cap \tau_h(U_S \times U_\Phi), \quad (f, g, \theta, \alpha) \in k^p. \tag{8.66}
\]

Note that $D^\Delta_{m-1}(1, 0, 0) = D^\Delta_{m-1}$. Hence it follows from the definition of $w(f, g, \theta^0, \alpha^0)$ that the point $w(0, 0, \theta, \alpha)$ lies in $Z_m$ (compare (8.60)). Since the condition $w(f, g, \theta^0, \alpha^0) \in Z_m$ on the point $(f, g, \theta^0, \alpha^0) \in k^{2p} \times \wedge^2 V^\vee \times \wedge^2 V$ is open, we obtain from (8.66) that there is an open dense subset $U \subset k^{2p} \times \wedge^2 V^\vee \times \wedge^2 V$ such that
\[
    w(f, g, \theta^0, \alpha^0) \in Z \cap \tau_h(U_S \times U_\Phi), \quad (f, g, \theta^0, \alpha^0) \in U. \tag{8.67}
\]

Next, for general $f_i, g_i \neq 0$ we easily see that the points $D'(0, f_i, g_i), D'(1, f_i, g_i)$ lie in $(\wedge^2 V)^0$ and, moreover, the projective plane $\text{Span}(\langle D'(0, f_i, g_i)^{-1}, D'(1, f_i, g_i)^{-1}, \varphi_{11} \rangle)$ in $P(\wedge^2 V^\vee)$ intersects the Grassmannian $G = G(1, 3)$ along a smooth conic. It follows that, in the notation (8.16), the sets $L(D'(1, f_1, g_1)^{-1}, \varphi_{11})$ and $L(D'(1, f_2, g_2)^{-1}, \varphi_{11})$ are well defined and disjoint for a general choice of $f_1, g_1, f_2, g_2 \in k$. This can be restated as follows. Using the notation (8.17) and taking the projection onto the direct summand
\[
    \text{pr}_{ij} : U_S \times U_\Phi \to ((S_i^\vee)_{(i)} \oplus (S_j^\vee)_{(j)}) \times ((\Phi_1)_{(i)} \oplus (\Phi_1)_{(j)}) \simeq (S_i^\vee \oplus S_j^\vee) \times (\Phi_1 \oplus \Phi_1),
\]
we define for every pair $(i, j), 1 \leq i < j \leq m$, an open dense subset $W_{ij}$ of $U_S \times U_\Phi$ by putting
\[
    W_{ij} := \text{pr}_{ij}^{-1}(\{(D_1, D_2), (\varphi_1, \varphi_2) \in (S_i^\vee \oplus S_j^\vee) \times (\Phi_1 \oplus \Phi_1) \mid \text{the subsets } L(D_1^{-1}, \varphi_1) \text{ and } L(D_2^{-1}, \varphi_2) \text{ of } \mathbb{P}^3 \text{ are well defined,}
    \text{disjoint and do not lie on a quadric}\}).
\]

Then (8.67) can be restated in the following way:
\[
    Z \cap \tau_h(W_{12}) \neq \varnothing. \tag{8.68}
\]
Now since the set $\text{Isom}_m$ of all isomorphisms $h$ in (8.65) is a connected principal homogeneous space of $\text{GL}(H_m)$, it follows from (8.68) that $Z \cap \tau_h(W_{ij}) \neq \emptyset$ for a general $h \in \text{Isom}_m$ and any pair $(i, j)$, $1 \leq i < j \leq m$. Since we have $W_{S\Phi} = \bigcap_{1 \leq i < j \leq m} W_{ij}$ by definition (8.20) of the set $W_{S\Phi}$, we deduce that $Z \cap \tau_h(W_{S\Phi}) \neq \emptyset$. This completes the proof of Proposition 8.1 for odd $m$.

8.5. Proof of Proposition 8.1: the case of even $m$. The proof of Proposition 8.1 in the case of even $m$, $m = 2p + 4$, $p \geq 0$, is completely parallel to that given above for odd $m$. As in (8.34), we fix decompositions

$$H_{m-1} \simeq H_3 \oplus H_2 \oplus \cdots \oplus H_2, \quad H_2 \simeq H_1 \oplus H_1, \quad H_3 \simeq H_1 \oplus H_1 \oplus H_1.$$  

(8.69)

As in (8.35), we consider the points $D^\Delta_{m-1} \in (S^\vee_{m-1})^0$ and $\varphi^\Delta_{m-1} \in \Phi_{m-1}$ which are given in terms of these decompositions by the following matrices with diagonal blocks:

$$D^\Delta_{m-1} := D_3 \oplus \underbrace{D_2 \oplus \cdots \oplus D_2}_p, \quad \varphi^\Delta_{m-1} = \varphi^\Delta_{m-1}(N, a, d, f, g, \lambda) := \varphi_3 \oplus \varphi_2 \oplus \cdots \oplus \varphi_2,$$

(8.70)

$$D_3 = D_2 \oplus D', \quad \varphi_3 = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \lambda \varphi_{21} \\ \varphi_{31} & \lambda \varphi_{12} & \varphi_{11} \end{pmatrix} \in \Phi_3, \quad \lambda \in k,$$

(8.71)

where $D_2$, $D'$ and $\varphi_2$, $\varphi_{ij}$, $i, j = 1, 2$, are given by (8.36), (8.37) and

$$\varphi_{13} = (r_{ij}) \in \wedge^2 V^\vee, \quad \varphi_{31} = (s_{ij}) \in \wedge^2 V^\vee,$$

(8.72)

with $r_{ij}, s_{ij} \in k$ satisfying additional relations

$$r_{i3} + r_{i4} = s_{i3} + s_{i4}, \quad i = 1, 2.$$

(8.73)

We now proceed along the same lines as before. In particular, it follows from (8.37) and (8.70)–(8.73) that the relations (8.38) and (8.39) hold for the point $(D^\Delta_{m-1}, \varphi^\Delta_{m-1})$. Therefore, as above, the relations (8.29) hold for any $\psi \in \Psi_{m-1}$. Substituting the data $(\theta^0, \alpha^0, D^\Delta_{m-1}, \varphi^\Delta_{m-1})$ from (8.27) and (8.70)–(8.72) in (8.30), we obtain the following equations for $\chi, \psi$:

$$(\varphi^\Delta_{m-1})^\vee \circ D^\Delta_{m-1} \circ \chi + \psi^\vee \circ \alpha^0 \circ \theta^0 \in \Psi_{m-1}.$$  

(8.74)

We now use the decompositions (8.69) to represent $\chi$ and $\psi$ as $(p + 1)$-tuples (compare (8.42))

$$\chi = (\chi_0, \ldots, \chi_p), \quad \psi = (\psi_0, \ldots, \psi_p), \quad \psi_0, \chi_0 \in \Psi_3, \quad \psi_k, \chi_k \in \Psi_2, \quad k = 1, \ldots, p,$$

(8.75)

\footnote{Note that we start with $m = 4$ since the case $m = 2$ has already been treated in §8.2.}
where $\chi_k = (X_k, Y_k)$, $\psi_k = (A_k, B_k)$, $k = 1, \ldots, p$, are the same matrices of variables as in (8.43), and $\chi_0 = (X_0, Y_0, Z_0)$, $\psi_0 = (A_0, B_0, C_0)$, $X_0, Y_0, Z_0, A_0, B_0, C_0 \in A^2 V^\vee$, that is,

\[
\begin{align*}
X_0 &= (x_{ij}^{(0)}), & Y_0 &= (y_{ij}^{(0)}), & Z_0 &= (z_{ij}^{(0)}), \\
A_0 &= (a_{ij}^{(0)}), & B_0 &= (b_{ij}^{(0)}), & C_0 &= (c_{ij}^{(0)})
\end{align*}
\]

are skew-symmetric $4 \times 4$ matrices of variables. Using the same notation for the variables $x_1^{(k)}, \ldots, x_{24}^{(k)}$, $k = 1, \ldots, p$, as in (8.46) and introducing new variables $x_1^{(0)}, \ldots, x_{36}^{(0)}$ by the formulae

\[
\begin{align*}
x_1^{(0)} &= x_{12}^{(0)}, & x_2^{(0)} &= x_{34}^{(0)}, & x_3^{(0)} &= x_{13}^{(0)}, & x_4^{(0)} &= x_{14}^{(0)}, & x_5^{(0)} &= x_{23}^{(0)}, & x_6^{(0)} &= x_{24}^{(0)}, \\
x_7^{(0)} &= y_{12}^{(0)}, & x_8^{(0)} &= y_{34}^{(0)}, & x_9^{(0)} &= y_{13}^{(0)}, & x_{10}^{(0)} &= y_{14}^{(0)}, & x_{11}^{(0)} &= y_{23}^{(0)}, & x_{12}^{(0)} &= y_{24}^{(0)}, \\
x_{13}^{(0)} &= z_{12}^{(0)}, & x_{14}^{(0)} &= z_{34}^{(0)}, & x_{15}^{(0)} &= z_{13}^{(0)}, & x_{16}^{(0)} &= z_{14}^{(0)}, & x_{17}^{(0)} &= z_{23}^{(0)}, & x_{18}^{(0)} &= z_{24}^{(0)}, \\
x_{19}^{(0)} &= a_{12}^{(0)}, & x_{20}^{(0)} &= a_{34}^{(0)}, & x_{21}^{(0)} &= a_{13}^{(0)}, & x_{22}^{(0)} &= a_{14}^{(0)}, & x_{23}^{(0)} &= a_{23}^{(0)}, & x_{24}^{(0)} &= a_{24}^{(0)}, \\
x_{25}^{(0)} &= b_{12}^{(0)}, & x_{26}^{(0)} &= b_{34}^{(0)}, & x_{27}^{(0)} &= b_{13}^{(0)}, & x_{28}^{(0)} &= b_{14}^{(0)}, & x_{29}^{(0)} &= b_{23}^{(0)}, & x_{30}^{(0)} &= b_{24}^{(0)}, \\
x_{31}^{(0)} &= c_{12}^{(0)}, & x_{32}^{(0)} &= c_{34}^{(0)}, & x_{33}^{(0)} &= c_{13}^{(0)}, & x_{34}^{(0)} &= c_{14}^{(0)}, & x_{35}^{(0)} &= c_{23}^{(0)}, & x_{36}^{(0)} &= c_{24}^{(0)},
\end{align*}
\]

we rewrite the system (8.74), in a similar way to (8.46), in the form

\[
\sum_{j=1}^{36} \tilde{m}_{ij} x_{ij}^{(0)} = 0, \quad \sum_{j=1}^{24} m_{ij} x_{ij}^{(k)} = 0, \quad i = 1, \ldots, 20, \quad k = 1, \ldots, p.
\] (8.77)

We now compute the matrices $\mathbf{M} = (m_{ij})$ and $\tilde{\mathbf{M}} = (\tilde{m}_{ij})$ for the values (8.47), (8.48) of the variables $N, a, d, f, g, p_{ij}, q_{ij}$ in (8.37) and (8.70) and, accordingly, for the following values of $\lambda, r_{ij}, s_{ij}$ in (8.71) and (8.72) satisfying (8.73):

\[
\begin{align*}
\lambda &= -2, & r_{12} &= 3, & r_{13} &= 7, & r_{14} &= -2, & r_{23} &= 4, & r_{24} &= -6, & r_{34} &= -8, \\
s_{12} &= -8, & s_{13} &= -3, & s_{14} &= 8, & s_{23} &= -2, & s_{24} &= 0, & s_{34} &= -5.
\end{align*}
\] (8.78)

This computation yields that $\mathbf{M}$ is the block matrix (8.49), and $\tilde{\mathbf{M}}$ is the following block matrix:

\[
\tilde{\mathbf{M}} = \begin{pmatrix} 
M_{11} & M_{12} & M_{13} & M_{14} & 0 & 0 \\
M_{21} & M_{22} & M_{23} & 0 & M_{14} & 0 \\
M_{31} & M_{32} & M_{33} & 0 & 0 & M_{14}
\end{pmatrix}
\] (8.79)

with blocks

\[
\begin{align*}
M_{11} &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 100 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 0 & 0 \\
0 & 0 & 0 & 0 & 100 & 0 \\
0 & 0 & 0 & 0 & 0 & 100
\end{pmatrix}, & M_{12} &= \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -5 & -2 & 0 \\
0 & 0 & 2 & 2 & 0 & -2 \\
0 & 0 & 5 & 0 & -2 & -5 \\
5 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, & M_{14} &= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{align*}
\] (8.80)
\[ M_{21} = \begin{pmatrix} 0 & 0 & 0 & -5 & 2 & 0 \\ 0 & 0 & 0 & -5 & 2 & 0 \\ -2 & -2 & 0 & 0 & 0 & 0 \\ -5 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & -100 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (8.81) \]

\[ M_{13} = \begin{pmatrix} 0 & 0 & -6 & -4 & -2 & -7 \\ 0 & 0 & 6 & -4 & -2 & 7 \\ -7 & -7 & -5 & 0 & 0 & 0 \\ 2 & 2 & 0 & -5 & 0 & 0 \\ -4 & -4 & 0 & 0 & -5 & 0 \\ 6 & 6 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & -12 & -8 \\ 0 & 0 & -8 & 0 & 14 & 0 \\ 0 & 0 & -4 & -14 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 & -4 \end{pmatrix}, \quad M_{23} = \begin{pmatrix} 0 & 0 & 0 & 10 & -4 & 0 \\ 0 & 0 & 0 & 10 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 \\ 10 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix}, \quad (8.82) \]

\[ M_{31} = \begin{pmatrix} 0 & 0 & 0 & 2 & 8 & 3 \\ 0 & 0 & 0 & 2 & 8 & -3 \\ 3 & 3 & -13 & 0 & 0 & 0 \\ -8 & -8 & 0 & -13 & 0 & 0 \\ 2 & 2 & 0 & 0 & -13 & 0 \\ 0 & 0 & 0 & 0 & -13 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 & -6 & 0 \\ 0 & 0 & 16 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{pmatrix}, \quad M_{32} = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -10 & 0 & 4 \\ 0 & 0 & 0 & 10 & -4 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (8.83) \]

\[ M_{33} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{3} = \begin{pmatrix} -20 & 0 & 20 & 66 & -5 & -47 \\ 0 & 3 & 40 & -38 & -37 & -57 \\ 57 & -47 & -82 & -38 & -22 & 0 \\ 37 & 5 & 7 & -79 & 0 & -22 \\ -38 & 66 & -28 & 0 & -62 & -38 \\ 40 & -20 & 0 & -28 & 7 & -59 \\ 0 & 0 & 0 & 0 & 0 & 40 \\ 0 & -76 & -76 & 0 & -114 & 0 \\ 44 & 0 & -10 & -94 & 0 & 0 \\ 0 & -14 & 0 & 80 & 0 & -74 \end{pmatrix}, \quad (8.84) \]

We now have \( \text{rk} \hat{M} = 20 \) as in (8.50). An explicit computation using (8.79)–(8.84) yields that \( \text{rk} \hat{\tilde{M}} = 30 \). Since the matrix of the linear system (8.77) is a direct sum of \( \hat{\tilde{M}} \) and \( p \) copies of \( M \), we see that its rank is equal to

\[ \text{rk} \hat{\tilde{M}} + p \text{rk} M = 30 + 20p = 10(m - 1). \quad (8.85) \]

We write \( R(\theta^0, \alpha^0, D_{m-1}^\Delta, \varphi_{m-1}^\Delta) \) for the rank of the linear system (8.74) (which is equivalent to the system (8.77)) regarded as a function of \( \theta^0, \alpha^0, D_{m-1}^\Delta, \varphi_{m-1}^\Delta \).
It follows from (8.85) that one can choose the values $\varphi_{m-1}, \alpha, \theta$ of the variables $\varphi_{m-1}^\Delta, \alpha^0, \theta^0$ respectively in such a way that, as in (8.52), we have

$$R(\theta, \alpha, D_{m-1}^\Delta, \varphi_{m-1}) = 10(m - 1).$$

(8.86)

Repeating the arguments in § 8.3 and using (8.86), we get the inclusions (8.59) and (8.60) for the data $\theta, \alpha, D_{m-1}^\Delta, \varphi_{m-1}$ chosen above.

Finally, using (8.70)–(8.72), we can appropriately modify the matrices (8.61)–(8.63) in such a way that, arguing as in § 8.4 and using the inclusions (8.59) and (8.60), we get $Z \cap \tau_h(W_{S\Phi}) \neq \emptyset$. This completes the proof of Proposition 8.1 for even $m$.

Remark 8.4. When computing the rank of the linear system (8.30), one might try to simplify the expressions for the matrices $\varphi_2$ in (8.37). For example, one might take $\varphi_{12} = \varphi_{21} = 0$ in order to perform the computations simultaneously for odd and even values of $m$. However, under these constraints, computational experiments with arbitrary values of the parameters $N, p_{ij}, q_{ij}$ give at best the value $9(m - 1)$ for the rank of the system (8.30), which is insufficient for further arguments. In the case of even $m$, one might also try to simplify the expression for the matrix $\varphi_3$ in (8.71), say, by putting $\varphi_{13} = \varphi_{31} = 0$, which is sufficient for equations (8.29) to hold. However, computational experiments with arbitrary values of the parameters $N, p_{ij}, q_{ij}, a, d, f, g, \lambda$ give at best the value 29 for the rank of $\tilde{M}$, which is again insufficient.

§ 9. The geometric meaning of $Z_m$.

Relation of $Z_m$ to ’t Hooft instantons

9.1. A property of the component $Z$ of the scheme $Z_m$. In this subsection we prove an openness property for the component $Z$ of $Z_m, m \geq 3$, introduced in Proposition 8.1 (see Lemma 9.2 below).

Take an arbitrary point $D \in (S_m^\chi)^0$. In the notation (6.10) we have a symplectic bundle $E_{2m}(D^{-1})$ of rank $2m$ (see (5.2) and (5.6), where we replace $2m + 2$ by $2m$ and put $B = D^{-1}$) and a natural epimorphism

$$c_D : H_m^\vee \otimes \wedge^2 V^\vee \to W_{5m} := H_m^\vee \otimes \wedge^2 V^\vee / \im(\xi(D^{-1})) \simeq H^0(E_{2m}(D^{-1})(1)), \dim W_{5m} = 5m.$$  

We now take an arbitrary point $z = (D, \varphi) \in Z_m$. Here the morphism $\varphi$, regarded as a homomorphism $\tilde{\varphi} : H_m \to H_m^\vee \otimes \wedge^2 V^\vee$, determines a diagram

$$
\begin{array}{c}
0 \to H_m \xrightarrow{\iota(D^{-1})} H_m^\vee \otimes \wedge^2 V^\vee \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
0 \to H_m^\vee \otimes \wedge^2 V^\vee \xrightarrow{c_D} W_{5m} \to 0
\end{array}
$$

(9.1)

The lower horizontal triple in (9.1) yields the diagram

$$
\begin{array}{c}
0 \to H_m \otimes O_{P^3} \xrightarrow{\iota(D^{-1})} H_m^\vee \otimes \wedge^2 V^\vee \otimes O_{P^3} \xrightarrow{c_D} W_{5m} \otimes O_{P^3} \to 0 \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
0 \to H_m \otimes O_{P^3} \xrightarrow{D^{-1}} H_m^\vee \otimes O_{P^3}(2) \xrightarrow{c_D} E_{2m}(D^{-1})(1) \to 0
\end{array}
$$

(9.2)
Moreover, the diagrams (9.1) and (9.2) determine a composite

\[ s_z : H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s(z)} W_{5m} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{ev} E_{2m}(D^{-1}). \tag{9.3} \]

Note that the relation \( \varphi^\vee \circ D \circ \varphi \in S_m \), which follows from the definition of \( Z \), can be rewritten in the form

\[ t^i s_z \circ s_z = 0, \tag{9.4} \]

where \( t^i s_z := s_z^\vee \circ \theta \) and \( \theta : E_{2m}(D^{-1}) \xrightarrow{\sim} E_{2m}(D^{-1})^\vee \) is the symplectic structure on \( E_{2m}(D^{-1}) \) defined in (5.6). Hence we have an anti-self-dual complex

\[ 0 \to H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s_z} E_{2m}(D^{-1}) \xrightarrow{t^i s_z} H_m' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0. \tag{9.5} \]

According to part (iii) of Proposition 8.1, we take a point

\[ z = (D, \varphi) \in Z \cap \tau_h(W_{S\Phi}), \tag{9.6} \]

where \( h \) is a fixed decomposition (8.18), and consider the induced decompositions

\[ D = D_1 \oplus \cdots \oplus D_m, \quad \varphi := \varphi_1 \oplus \cdots \oplus \varphi_m, \quad (D_i, \varphi_i) \in (\wedge^2 V^\vee)^0 \times (\wedge^2 V)^0, \tag{9.7} \]

such that

\[ L := \bigsqcup_{i=1}^m L(D_i, \varphi_i) \tag{9.8} \]

is a disjoint union of 2m lines in \( \mathbb{P}^3 \) not lying on a quadric. Moreover, for this point \( z \) we have

\[ E_{2m}(D^{-1}) = \bigoplus_{i=1}^m E_2(D_i^{-1}), \tag{9.9} \]

where \( E_2(D_i^{-1}) \), \( i = 1, \ldots, m \), are null-correlation bundles of rank 2.

In terms of the decomposition (8.18), the diagrams (9.1) and (9.2) decompose into direct sums of \( m \) diagrams of the form

\[ \begin{array}{ccccccccccc}
0 & \to & k & \xrightarrow{\gamma} & \wedge^2 V^\vee & c_{D_1} & \xrightarrow{s_i(z)} & W_5(i) & \to & 0 \\
0 & \xrightarrow{\gamma^i(D_i^{-1})} & \wedge^2 V^\vee & \xrightarrow{c_{D_i}} & W_5(i) & \xrightarrow{ev} & 0 \\
0 & \xrightarrow{\gamma^i(D_i^{-1})} & \wedge^2 V^\vee & \xrightarrow{c_{D_i}} & W_5(i) & \xrightarrow{ev} & 0 \\
0 & \xrightarrow{\gamma^i(D_i^{-1})} & \Omega_{\mathbb{P}^3}(2) & \xrightarrow{c_{D_i}} & E_2(D_i^{-1})(1) & \to & 0
\end{array} \tag{9.10} \]

for \( i = 1, \ldots, m \), in which we have substituted \( k \) for \( H_1 \) and put \( W_{5(i)} := \wedge^2 V^\vee / \text{im}(\gamma^i(D_i^{-1} : k \to \wedge^2 V^\vee)) \), \( \dim W_{5(i)} = 5, i = 1, \ldots, m. \)
Note that the decomposition (8.18) induces a decomposition of the complex (9.5) into a direct sum of $m$ complexes

$$0 \to \mathcal{O}_{	ext{Pr}}(-1) \overset{s_i}{\to} E_2(D_i^{-1}) \overset{t_s}{\to} \mathcal{O}_{	ext{Pr}}(1) \to 0, \quad i = 1, \ldots, m. \quad (9.12)$$

Here the sections $s_i$, $0 \neq s_i \in H^0(E_2(D_i^{-1})(1)) \simeq W_{5(i)}$, regarded as homomorphisms $k \to W_{5(i)}$, coincide by construction with the monomorphisms $s_i(z)$ in the diagram (9.10). Hence the homomorphism $s(z)$ in the diagram (9.1) is also injective, being the direct sum of the monomorphisms $s_i(z)$. This means that $\text{im}(\sharp \varphi) \cap \text{im}(\sharp(D^{-1})) = \{0\}$, that is,

$$z \in ((S_m^\vee)^0 \times \Phi_m)^* := \{(D, \varphi) \in (S_m^\vee)^0 \times \Phi_m \mid \text{the homomorphism } \sharp \varphi: H_m \to H_m^\vee \otimes \wedge^2 V^\vee \text{ is injective and } \text{im}(\sharp \varphi) \cap \text{im}(\sharp(D^{-1})) = \{0\}\}. \quad (9.13)$$

It follows from the definition of $L$ and the construction of the morphisms $s_z, s_i, i = 1, \ldots, m$ (see (9.3), (9.8) and (9.12)) that the complexes (9.5) and (9.12) are exact except at the terms on the right, and

$$\text{coker}(t_s) = \mathcal{O}_L(1), \quad \text{coker}(t_s) = \mathcal{O}_{L(D_i, \varphi_i)}(1), \quad (s_i)_0 = L(D_i, \varphi_i), \quad i = 1, \ldots, m. \quad (9.14)$$

**Remark 9.1.** For an arbitrary embedding $j: H_{m-1} \hookrightarrow H_m$ and an arbitrary point $z \in (S_m^\vee)^0 \times \Phi_m$ there is an induced morphism of sheaves

$$s_z(j): H_{m-1} \otimes \mathcal{O}_{	ext{Pr}}(-1) \overset{j}{\to} H_m \otimes \mathcal{O}_{	ext{Pr}}(-1) \overset{s_z}{\to} E_2m(D^{-1}). \quad (9.15)$$

Let $e_1, \ldots, e_m$ be a basis of $H_m$ compatible with the decomposition (8.18). We put

$$H_{m-1} := \text{Span}(e_1, \ldots, e_{m-1}).$$

Consider the monomorphism

$$j_0: H_{m-1} \hookrightarrow H_m, \quad e_i \mapsto e_i + e_{i+1}, \quad i = 1, \ldots, m - 1. \quad (9.16)$$

Since $L$ is a disjoint union of pairs of lines $L(D_i, \varphi_i), i = 1, \ldots, m$, it follows from (9.14)–(9.16) that $s_z(j_0)$ is a subbundle morphism, that is,

$$\text{coker}(t_{s_z(j_0)}) = 0. \quad (9.17)$$

Given any monomorphism $j: H_{m-1} \hookrightarrow H_m$, we consider the following conditions on a point $z = (D, \varphi) \in Z$.

(I) The composite $s_z(j) = s_z \circ j: H_{m-1} \otimes \mathcal{O}_{	ext{Pr}}(-1) \to E_2m(D^{-1})$ is a subbundle morphism.

(II) $s_z: H_m \otimes \mathcal{O}_{	ext{Pr}}(-1) \to E_2m(D^{-1})$ is an injective morphism of sheaves (but not a subbundle morphism).

Note that (I) and (II) are open conditions on the point $z \in Z_m$. Condition (I) holds for the point $z$ in (9.6) and the embedding $j_0$ by (9.17). Condition (II) holds for this point $z$ by (9.14). Since the set $((S_m^\vee)^0 \times \Phi_m)^*$ defined in (9.13) is open and dense in $(S_m^\vee)^0 \times \Phi_m$, we get the following result.
Lemma 9.2. There is a monomorphism $j: H_{m-1} \hookrightarrow H_m$ such that the set

$$Z_m(j) := \{ z = (D, \varphi) \in Z_m \cap ((S'_m)^0 \times \Phi_m)^* \mid z \text{ satisfies conditions (I) and (II)} \}$$

is a non-empty open subset of $Z_m$ and, accordingly, the set

$$Z(j) := Z \cap Z_m(j)$$

is an open dense subset of $Z$. The same holds for a general monomorphism $j: H_{m-1} \hookrightarrow H_m$. In other words, the sets

$$P(H_m^\vee)^* := \{ j \in P(H_m^\vee) \mid Z_m(j) \neq \emptyset \}, \quad P(H_m^\vee)^{**} := \{ j \in P(H_m^\vee) \mid Z(j) \neq \emptyset \}$$

are open and dense in $P(H_m^\vee)$.

9.2. The relation between $Z$ and 't Hooft instantons. The morphism $\lambda_j: Z_m \to S_{2m-1}$. In this subsection we relate the open subset $Z_m(j)$ of $Z_m$ introduced in Lemma 9.2 to 't Hooft instantons (see Lemma 9.3).

Suppose that $j \in P(H_m^\vee)^*$. Take an arbitrary point $z = (D, \varphi) \in Z_m(j)$ such that the symplectic vector bundle $E_{2m}(D^{-1})$ satisfies the diagrams (9.1), (9.2). Then the morphism $s_z$ of sheaves defined in (9.3) is injective (see condition (II)). Moreover, $s_z$ satisfies (9.4), which clearly implies that

$$t s_z(j) \circ s_z(j) = 0 \quad (9.18)$$

for the subbundle morphism $s_z(j)$. Thus we obtain a monad

$$0 \to H_{m-1} \otimes O_{\mathbb{P}^3}(-1) \xrightarrow{s_z(j)} E_{2m}(D^{-1}) \xrightarrow{t s_z(j)} H_m^\vee \otimes O_{\mathbb{P}^3}(1) \to 0. \quad (9.19)$$

We deduce from the diagram (9.2) that $h^i(E_{2m}(D^{-1})(-2)) = 0$, $i \geq 0$, whence the cohomology sheaf of the monad (9.19) is an instanton bundle

$$E_2(z, j) := \text{Ker}(t s_z(j))/\text{Im}(s_z(j)), \quad [E_2(z, j)] \in I_{2m-1}. \quad (9.20)$$

We now consider the subvariety $I_{2m-1}^H \subset I_{2m-1}$ of 't Hooft instanton bundles (see § 4.3).

Lemma 9.3. Suppose that $j \in P(H_m^\vee)^*$ and $z = (D, \varphi)$ is an arbitrary point in $Z_m(j)$. Then the bundle $E_2(z, j)$ is a 't Hooft instanton bundle, that is, $[E_2(z, j)] \in I_{2m-1}^H$. Moreover, there is an exact triple

$$0 \to O_{\mathbb{P}^3}(-1) \xrightarrow{s} E_2(z, j) \to \mathcal{I}_{L, \mathbb{P}^3}(1) \to 0, \quad (9.21)$$

where $L = (s)_0$. Furthermore, if the scheme $L$ does not lie on a quadric, then

$$h^0(E_2(z, j)(1)) = 1, \quad h^1(E_2(z, j)(1)) = 6m - 10, \quad h^2(E_2(z, j)(1)) = 0. \quad (9.22)$$
\textbf{Proof.} Consider the complexes (9.5) and (9.19) and put

\[ \mathcal{H}_{m-1} := H_{m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1), \quad \mathcal{H}_m := H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1), \]

\[ \mathcal{K}_{m+1} := \text{coker } s_z(j), \quad \mathcal{K}_m := \text{coker } s_z. \]

The complexes (9.5) and (9.19) are anti-self-dual. Hence they extend to a commutative diagram

\[ \begin{array}{cccccccccccc}
\mathcal{O}_{\mathbb{P}^3}(-1) & \xrightarrow{s_z} & E_{2m}(D^{-1}) & \xrightarrow{\tau} & \mathcal{K}_{m+1} \\
\mathcal{O}_{\mathbb{P}^3}(1) & \xrightarrow{t_{s_z}} & 0 & \xrightarrow{\gamma} & \mathcal{H}_m^\vee & \xrightarrow{\delta} & \mathcal{H}_{m-1}^\vee \\
\mathcal{H}_m & \xrightarrow{t_{s_z}} & \mathcal{H}_m^\vee & \xrightarrow{j^\vee} & \mathcal{H}_{m-1}^\vee & \xrightarrow{j^\vee} & 0 \\
\mathcal{H}_{m-1} & \xrightarrow{s_z(j)} & E_{2m}(D^{-1}) & \xrightarrow{\alpha} & \mathcal{K}_m \\
\end{array} \]

(9.23)

where \( \alpha, \beta, \gamma, \delta \) and \( \tau \) are the induced morphisms. In this diagram we have \( \beta \circ \alpha = 0 \) and \( j^\vee \circ \gamma \circ \beta = \delta \). Hence \( \delta \circ \alpha = 0 \). It follows that \( \alpha \) factors through \( \tau \), that is, \( \alpha = \tau \circ u_z \) for some injective morphism \( u_z : \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow E_2(z,j) \). This morphism \( u_z \) is a non-zero section \( u_z \in H^0(E_2(z,j)(1)) \). Hence \( E_2(z,j) \) is a ’t Hooft instanton bundle.

Since the triple (9.5) is a part of the diagram (9.23), the triple (9.21) is obtained from this diagram by standard diagram-chasing in view of the first equation in (9.14). If \( L \) does not lie on a quadric, then (9.22) follows from (9.21) and the Riemann–Roch theorem. \( \square \)

We fix an isomorphism

\[ \xi : H_m \oplus H_{m-1} \xrightarrow{\sim} H_{2m-1}, \quad \xi \in \text{Isom}_{2m-1}, \]

(9.24)

and suppose that \( j \in P(H_m^\vee) \). Then there is a morphism

\[ \lambda_j : Z_m \rightarrow S_{2m-1}, \quad z = (D, \varphi) \mapsto A_j = \tilde{\xi}(D^{-1}, \varphi \circ j, -(\varphi \circ j)^\vee \circ D \circ (\varphi \circ j)). \]

(9.25)

Together with the isomorphism (9.24) we fix an isomorphism

\[ \eta : H_m \oplus H_m \xrightarrow{\sim} H_{2m}, \quad \eta \in \text{Isom}_{2m}, \]

(9.26)

and, in analogy with (9.25), consider the morphism

\[ \lambda_\eta : Z_m \rightarrow S_{2m}, \quad z = (D, \varphi) \mapsto A = \tilde{\eta}(D^{-1}, \varphi, -\varphi^\vee \circ D \circ \varphi). \]

(9.27)
By construction, $\lambda_\eta$ is an embedding: $\lambda_\eta^{-1}(A) = (A_1(\eta)^{-1}, A_2(\eta))$. Note that for every $z \in Z_m$ the rank of the $8m \times 8m$ matrix $A = \lambda_\eta(z)$ is equal to the rank of the matrix $D^{-1} = \text{pr}(A) : H_m \otimes V \xrightarrow{\cong} H_m^\vee \otimes V^\vee$, where $\text{pr} : S_{2m} \to S_m$ is the projection induced by the inclusion $i_\eta : H_m \hookrightarrow H_{2m}$ of the first summand in (9.26). Therefore, putting $W_{4m} := H_m \otimes V$ and using the isomorphism $i_\eta^\vee : \text{im} A \cong W_{4m}$, we obtain a complex $0 \to H_{2m} \otimes O_{\mathbb{P}^3}(-1) \boxtimes O_{Z_m} \xrightarrow{s} W_{4m} \otimes O_{\mathbb{P}^3 \times Z_m} \xrightarrow{t_s} H^\vee_{2m-1} \otimes O_{\mathbb{P}^3} (1) \boxtimes O_{Z_m} \to 0$. Moreover, the inclusion

$$j : H_{2m-1} \xrightarrow{\xi} H_m \oplus H_{m-1} \xrightarrow{id \oplus j} H_m \oplus H_m \xrightarrow{\eta} H_{2m}$$

(9.28)

induces a diagram of complexes

$$
\begin{array}{llllll}
0 & \to & H_{2m-1} \otimes O_{\mathbb{P}^3}(-1) \boxtimes O_{Z_m} & \xrightarrow{s(j)} & W_{4m} \otimes O_{\mathbb{P}^3 \times Z_m} & \xrightarrow{t_s(j)} & H^\vee_{2m-1} \otimes O_{\mathbb{P}^3} (1) \boxtimes O_{Z_m} & \to 0 \\
0 & \to & H_{2m} \otimes O_{\mathbb{P}^3}(-1) \boxtimes O_{Z_m} & \xrightarrow{s} & W_{4m} \otimes O_{\mathbb{P}^3 \times Z_m} & \xrightarrow{t_s} & H^\vee_{2m} \otimes O_{\mathbb{P}^3} (1) \boxtimes O_{Z_m} & \to 0 \\
\end{array}
$$

(9.29)

where $s(j) := s \circ j$. Given an arbitrary point $z \in Z_m$, we consider the restriction of the diagram (9.29) to $\mathbb{P}^3 \times \{z\}$:

$$
\begin{array}{llllll}
0 & \to & H_{2m-1} \otimes O_{\mathbb{P}^3}(-1) & \xrightarrow{s_z(j)} & W_{4m} \otimes O_{\mathbb{P}^3} & \xrightarrow{t_s(j)} & H^\vee_{2m-1} \otimes O_{\mathbb{P}^3} (1) & \to 0 \\
0 & \to & H_{2m} \otimes O_{\mathbb{P}^3}(-1) & \xrightarrow{s_z} & W_{4m} \otimes O_{\mathbb{P}^3} & \xrightarrow{t_s_z} & H^\vee_{2m} \otimes O_{\mathbb{P}^3} (1) & \to 0 \\
\end{array}
$$

(9.30)

and put

$$Z^0_m := \{z \in Z_m \mid \text{the lower complex (9.30) is exact on the left and in the middle}\},$$

$$Z^1_m(j) := \{z \in Z_m \mid \text{the cohomology sheaf } E_2(z, j) := \ker(t_s) / \text{im}(s_z) \text{ satisfies the conditions (9.22)}, j \in P(H^\vee_m)\}.$$  

(9.31)

Clearly, $Z^0_m$ is an open subset of $Z_m$. Moreover, $Z^1_m(j)$ is an open subset of $Z_m$ for every $j \in P(H^\vee_m)$ by semicontinuity. In particular, the following lemma holds.

**Lemma 9.4.** For every $j \in P(H^\vee_m)^\ast$ the set $Z^1(j) := Z(j) \cap Z^1_m(j)$ is open and dense in $Z$.

**Proof.** By (9.6), (9.8) and Lemma 9.2 one can find a point $z \in Z(j)$ and a section $s \in H^0(E(z, j)(1))$ such that the scheme $L = (s)_0$ does not lie on a quadric and, therefore, $E_2(z, j)$ satisfies (9.22). Note that the last two equations in (9.22) follow from the first, the triple (9.21) and the Riemann–Roch theorem. Consider the morphism

$$Z(j) \to I_{2m-1}^{1H}, \quad z \mapsto [E_2(z, j)].$$


That \( Z^1(j) \) is open and dense in \( Z \) now follows from the irreducibility of \( Z(j) \) since the set
\[
I_{2m-1}^{H^*} = \{ [E] \in I_{2m-1}^H \mid h^0(E(1)) = 1 \}
\]
is open and dense in \( I_{2m-1}^H \) (Lemma 4.5, (ii)). □

**Remark 9.5.** Suppose that \( j \in P(H_m^\vee)^* \).

(i) It follows directly from the definition of \( Z_m(j) \) that \( Z_m(j) \subset Z_m^0 \cap Z_m^1(j) \).

(ii) For \( z \in Z_m(j) \) the upper complex in (9.30) is a monad constructed from the net of quadrics \( A_j \) defined in (9.25), and its cohomology sheaf coincides with the 't Hooft bundle \( E_2(z,j) \) defined in (9.20).

Therefore the morphism \( \lambda_j \) defined in (9.25) satisfies the condition
\[
\lambda_j(Z_m(j)) \subset MI_{2m-1}^H(\xi). \tag{9.32}
\]
Moreover, for every point \( z = (D, \varphi) \in Z_m \), it follows from the fact that \( D \) is invertible that the net of quadrics \( A_j \) regarded as a homomorphism
\[
A_j : H_{2m-1} \otimes V \xrightarrow{\sim} H_{2m-1}^\vee \otimes V^\vee
\]
satisfies the equation \( \text{rk} A = 4m \).

**9.3. Pointwise description of the fibres of \( \lambda_j : Z_m \to S_{2m-1} \).** Let \( j \in P(H_m^\vee) \) be such that \( Z_m^0 \cap Z_m^1(j) \neq \emptyset \). (For example, this holds for \( j \in P(H_m^\vee)^* \); see Remark 9.5, (i).) In this subsection we describe the fibres of the morphism \( \lambda_j : Z_m \to S_{2m-1} \) at the points of the open set \( Z_m^0 \cap Z_m^1(j) \). A precise statement will be given in Lemma 9.6 below.

To obtain the result on the fibres, we note that for every point \( z = (D, \varphi) \in Z_m^0 \cap Z_m^1(j) \) the upper complex in (9.30) determines a complex (9.19) (which is a monad if \( z \in Z_m(j) \)) with cohomology sheaf \( E_2(z,j) \) at the middle term. The display of this complex twisted by \( \mathcal{O}_{\mathbb{P}^3}(1) \) is of the form
\[
\begin{array}{c}
E_2(z,j)(1) \\
\downarrow \\
H_{m-1} \otimes \mathcal{O}_{\mathbb{P}^3} \\
\xrightarrow{s_{z}(j)} E_{2m}(D^{-1})(1) \\
\xrightarrow{\varepsilon} \text{coker}(s_{z}(j)) \\
\downarrow \\
H_{m-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(2)
\end{array} \tag{9.33}
\]

Since \( z \in Z_m^1(j) \), we have \( h^0(E_2(z,j)(1)) = 1 \). Therefore, passing to sections in the diagram (9.33), we get a well-defined homomorphism
\[
b(z,j) := h^0(t_{s_{z}(j)}): h^0(E_{2m}(D^{-1})(1)) \xrightarrow{h^0(\varepsilon)} h^0(\text{coker}(s_{z}(j))) \xrightarrow{\text{can}} h^0(\text{coker}(s_{z}(j)))/h^0(E_2(z,j)(1)) \simeq \mathbb{k}^{4m} \hookrightarrow H_{m-1}^\vee \otimes S^2V^\vee. \tag{9.34}
\]
Consider the epimorphism \( c_D : H_m^\vee \otimes \Lambda^2 V^\vee \to H^0(E_{2m}(D^{-1})(1)) \) defined at the beginning of §9.1 and put

\[
V(z, j) := c^{-1}_D(\ker b(z, j)). \tag{9.35}
\]

It follows immediately from (9.34) that

\[
V(z, j) \simeq k^{2m}. \tag{9.36}
\]

**Lemma 9.6.** Let \( j \in P(H_m^\vee) \) be such that \( Z^0_m \cap Z^1_m(j) \neq \emptyset \). For every point \( z \in Z^0_m \cap Z^1_m(j) \) the fibre of the morphism \( \lambda_j : Z_m \to S_{2m-1} \) passing through the point \( z \) is a reduced scheme which is an open subset of the affine space \( V(z, j) \) defined in (9.35):

\[
\lambda_j^{-1}(\lambda_j(z)) \overset{\text{open}}{\hookrightarrow} V(z, j) \simeq \Lambda^{2m}, \quad \dim \lambda_j^{-1}(\lambda_j(z)) = 2m. \tag{9.37}
\]

**Proof.** Consider the spaces \( \Lambda_m = \Lambda^2 H_m^\vee \otimes S^2 V^\vee \) and \( \Lambda_{m-1} = \Lambda^2 H_{m-1}^\vee \otimes S^2 V^\vee \) with the projections \( q_m : \Lambda^2(H_m^\vee \otimes V^\vee) \to \Lambda_m \) and \( q_{m-1} : \Lambda^2(H_{m-1}^\vee \otimes V^\vee) \to \Lambda_{m-1} \) (compare (7.1) and (7.6)). Fix a monomorphism \( j_k : k \hookrightarrow H_m \) such that \( j(H_{m-1}) \cap k = \{0\} \). Thus we have a direct sum decomposition of \( H_m \) and embeddings of the direct summands:

\[
H_m = H_{m-1} \oplus k, \quad H_{m-1} \overset{j}{\hookrightarrow} H_m \overset{j_k}{\hookrightarrow} k. \tag{9.38}
\]

This decomposition induces a direct sum decomposition of \( \Lambda \) and projections:

\[
\Lambda_m = \Lambda_{m-1} \oplus \text{Hom}(k, H_{m-1}^\vee \otimes S^2 V^\vee),
\]

\[
\Lambda_{m-1} \overset{\text{pr}}{\twoheadrightarrow} \Lambda_m \overset{\text{pr}'}{\twoheadrightarrow} \text{Hom}(k, H_{m-1}^\vee \otimes S^2 V^\vee).
\]

Then the equations of \( Z_m \) in \( (S_m^\vee)^0 \times \Phi_m \) take the form

\[
A := q_m(\varphi^\vee \circ D \circ \varphi) = 0. \tag{9.39}
\]

We now consider the diagram (5.11) twisted by \( O_{\mathbb{P}^3}(1) \), where we replace \( m \) by \( m - 1 \), put \( B = D^{-1} \) and take \( s_z(j) \) instead of \( p_{\xi, A} \) and, accordingly, \( \varphi \circ j \) instead of \( \widetilde{C} \). Passing to sections in this diagram and, accordingly, to sections in diagram (9.33), we see that the condition

\[
0 = \text{pr}'(A) := q_{m-1}((\varphi \circ j)^\vee \circ D \circ (\varphi \circ j)) = b(z, j) \circ e(z)
\]

holds automatically, where \( e(z) \) is the homomorphism \( e(z) = h^0(s_z(j)) : H_{m-1} \to H^0(E_{2m}(B)(1)) \). (Clearly, the vanishing of \( \text{pr}'(A) \) is equivalent to saying that \( \sharp \varphi \circ j \) embeds \( H_{m-1} \) in \( V(z, j) \).) Therefore the equations (9.39) are equivalent to the equations

\[
\text{pr}''(A) := b(z, j) \circ c(z) \circ \sharp \varphi \circ j_k = 0,
\]

which in view of definition (9.35) mean that

\[
\sharp \varphi |_k \subset V(z, j).
\]
Since the point $\lambda_j(z)$ is given and determines the points $D$ and $\varphi \circ j$ (see (9.25)), it follows that the point $(D, \varphi) \in \lambda_j^{-1}(\lambda_j(z))$ is determined by the data $\varphi|_k$. Hence the inclusion above shows that $\lambda_j^{-1}(\lambda_j(z)) \simeq V(z, j)$.

This argument can be illustrated by the diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & V(z, j) & \rightarrow & H_m & \rightarrow & H_m \otimes \wedge^2 V^\vee & \rightarrow & H_{m-1}^\vee \otimes S^2 V^\vee \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \ker b(z, j) & \rightarrow & H^0(E_{2m}(D^{-1})(1)) & \rightarrow & H_{m-1}^\vee \otimes S^2 V^\vee \\
\end{array}
\]

The lemma is proved. ∎

§ 10. The universal family of 't Hooft extensions with $c_2 = 2m - 1$ and related constructions

In this section we construct a complete $(10m-1)$-dimensional family $T$ of 't Hooft $(2m-1)$-instantons and their degenerations (we call these degenerations 't Hooft sheaves). These sheaves are realized as extensions of sheaves of rank 1, so that $T$ is the base of the universal family of such extensions, which are referred to as 't Hooft extensions. The family $T$ and other constructions in this section will be used in the next section to prove the coincidence of $Z_m$ with the variety $Z$ considered in §§ 8, 9. This will complete the proof of Theorem 7.2.

10.1. 't Hooft $(2m-1)$-bundles and the variety $H$.

We consider the subvariety $I_{2m-1}^H \subset I_{2m-1}$ of 't Hooft $(2m-1)$-instantons. We first recall the following properties of 't Hooft instantons $[E] \in I_{2m-1}^H$, $m \geq 1$ (see [1] and [2]).

(i) $h^0(E(1)) \leq 2$.
(ii) For every section $s$, $0 \neq s \in H^0(E(1))$, the zero scheme $Z_s = \langle s \rangle$ lies locally on a smooth surface.
(iii) $(Z_s)_{\text{red}}$ is a disjoint union of lines, say $l_1, \ldots, l_r$, $1 \leq r \leq 2m$, and we have $O_{Z_s} = \bigoplus_{i=1}^r O_{Z_i}$, where every scheme $Z_i$, $1 \leq i \leq r$, has a filtration by subschemes $l_i = Z_{1i} \subset Z_{2i} \subset \cdots \subset Z_{mi,i} = Z_i$ for some $m_i \geq 1$. Here $\text{Supp}(Z_{ji}) = l_i$ and if $m_i \geq 2$, then

\[
O_{Z_{j-1,i}} = O_{Z_{ji}}/O_{l_i}, \quad 2 \geq j \geq m_i.
\]

For every integer $d \geq 1$ we consider the Hilbert scheme $H_d := \text{Hilb}^dG$ of 0-dimensional subschemes of length $d$ in the Grassmannian $G = G(1, 3)$ of lines
in \( \mathbb{P}^3 \). Let \( \Gamma_{\mathcal{H}} \subset G \times \mathcal{H} \) be the universal family with the projections \( G \xrightarrow{p_d} \Gamma_{\mathcal{H}} \xrightarrow{q_d} \mathcal{H} \). Given a point \( x \in \mathcal{H} \), we denote the corresponding 0-dimensional subscheme \( p_d(q_d^{-1}(x)) \) in \( G \) by \( \mathcal{Y}_x \). A point \( x \in \mathcal{H} \) is said to be curvilinear if one can find an integer \( b \geq 1 \), a partition \( d = d_1 + \cdots + d_b, d_i \geq 1 \), and points \( x_i \in \mathcal{H}_{d_i}, 1 \leq i \leq b \), with the following properties.

(a) For every \( i, 1 \leq i \leq b \), the subscheme \( \mathcal{Y}_{x_i} \subset G \) is isomorphic to \( \text{Spec}(k[t]/(t^{d_i+1})) \).

(b) \( \mathcal{Y}_x \) is a disjoint union: \( \mathcal{Y}_x = Y_{x_1} \sqcup \cdots \sqcup Y_{x_b} \).

We put \( \mathcal{H}^{\text{curv}} := \{ x \in \mathcal{H} \mid \mathcal{Y}_x \text{ is curvilinear} \} \). It is well known (and easily verifiable) that \( \mathcal{H}^{\text{curv}} \) is a smooth open \((4d)\)-dimensional subscheme of \( \mathcal{H} \). Let \( \Gamma \subset \mathbb{P}^3 \times G \) be the graph of incidence with the projections \( \mathbb{P}^3 \xrightarrow{p} \Gamma \xrightarrow{q} G \). The following lemma can be deduced from the properties (i)–(iii) above.

**Lemma 10.1.** For any \( [E] \in I^{H}_{2m-1} \) and \( s, 0 \neq s \in H^0(E(1)) \), there is a curvilinear point \( x = x([E], s) \in \mathcal{H}^{\text{curv}} \) such that \( Z_s = p(q^{-1}(\mathcal{Y}_x)) \) and the scheme structure on \( Z_s \) coincides with the structure given by the formula

\[
\mathcal{O}_{Z_s} = p_* q^* \mathcal{O}_{\mathcal{Y}_x}.
\]

**Proof.** Since in view of (ii) the support of the scheme \( Z_s \) is a disjoint union of lines, it follows from the definition of the curvilinear scheme that it is enough to consider the case when \( Z_s \) is a single line, say, \( l \) with a non-reduced scheme structure. This means that there is a filtration of the scheme \( Z_s \) by subschemes

\[
l = Z_1 \subset Z_2 \subset \cdots \subset Z_{2m} = Z_s, \quad m \geq 2,
\]

such that the following triples are exact (see (10.1)):

\[
0 \to \mathcal{O}_l \to \mathcal{O}_{Z_2} \to \mathcal{O}_l \to 0, \ldots, 0 \to \mathcal{O}_l \to \mathcal{O}_{Z_{2m}} \to \mathcal{O}_{Z_{2m-1}} \to 0.
\]

Using the first triple in (10.4), property (ii) and the Ferrand construction ([12], §1), we see that \( \mathcal{O}_l \) is a quotient sheaf of the co-normal sheaf \( N_{l/\mathbb{P}^3} \simeq 2\mathcal{O}_{\mathbb{P}^3} \) and that the surjection \( N_{l/\mathbb{P}^3} \to \mathcal{O}_l \) determines a double structure on \( l \) which coincides with the structure of the scheme \( Z_2 \). It follows from this surjection that \( Z_2 \) lies as a scheme on a smooth quadric (\( Q \) say) passing through \( l \). We choose homogeneous coordinates \((x_0 : x_1 : x_2 : x_3)\) on \( \mathbb{P}^3 \) with the following properties.

1) \( l = \{ x_2 = x_3 = 0 \}, Q = \{ x_0x_2 - x_1x_3 = 0 \} \).

2) Let \( \mathbb{P}^3 = U_0 \cup U_1 \) be the open covering of \( \mathbb{P}^3 \) by the sets \( U_i = \{ x_i \neq 0 \}, i = 0, 1 \). Then the ideal of the scheme \( Z_2 \cap U_i \) in \( k[U_i] \) is generated by the elements \( x_2/x_0 \) and \( (x_3/x_0)^2 \) for \( i = 0 \) and by the elements \( x_3/x_1 \) and \( (x_2/x_1)^2 \) for \( i = 1 \).

Let \( S_1, \ldots, S_c \) be smooth quasiprojective surfaces in \( \mathbb{P}^3 \) such that the sets \( Z_{(k)} := Z_s \cap S_k, k = 1, \ldots, c, \) form an open covering of \( Z_s \). (Such surfaces exist by (ii).)

We put \( Z_{(ik)} := Z_{(k)} \cap U_i, i = 0, 1, k = 1, \ldots, c \). Then (i)–(iii) and 1), 2) yield the following property.

3) For \( k = 1, \ldots, c \) the ideal \( I_{Z_{(ik)}} \) of the scheme \( Z_{(ik)} \) in \( \mathcal{O}[U_i \cap S_k] \) is generated by the elements \( (x_3/x_0)^{2m+1} \) for \( i = 0 \) and by the elements \( (x_2/x_1)^{2m+1} \) for \( i = 1 \).
In view of 1), the elements \( x_3/x_0 \in \mathcal{O}[Z_{(0k)}] \) and \( x_2/x_1 \in \mathcal{O}[Z_{(1k)}] \) coincide on \( \mathcal{O}[Z_{(0k)} \cap Z_{(1k)}] \), \( k = 1, \ldots, c \). Therefore we have well-defined homomorphisms

\[ k[t]/(t^{2m+1}) \to \mathcal{O}[Z_{(ik)}], \quad 1 \mod(t^{2m+1}) \mapsto 1 \mod I_{Z_{(ik)}}, \]

\[ t \mod(t^{2m+1}) \mapsto (x_3/x_0) \mod I_{Z_{(ok)}}, \quad i = 0, \]

and

\[ t \mod(t^{2m+1}) \mapsto (x_2/x_1) \mod I_{Z_{(1k)}}, \quad i = 1, \]

which coincide on \( Z_{(0k)} \cap Z_{(1k)} \). This determines a morphism \( \pi_Z: Z_s \to \text{Spec}(k[t]/(t^{2m+1})) \). We put \( \tau_i := \text{Spec}(k[t]/(t^{i+1})), i = 0, \ldots, 2m \). It follows from the definition of \( \pi_Z \) and the exact triples (10.4) that the (nilpotent) ideal sheaf \( \mathcal{I}_i := \mathcal{I}_{\tau_{i-1},\tau_i} \subset \mathcal{O}_{\tau_i}, i = 2, \ldots, 2m \), admits an isomorphism, mult: \( \mathcal{I}_i \otimes_{\mathcal{O}_{\tau_i}} \mathcal{O}_{Z_i} \xrightarrow{\sim} \mathcal{I}_{Z_i}, a \otimes 1 \mapsto \pi_{Z_i}^*(a) \). Hence, by [13], Lemma 2.13, the morphism \( \pi_Z \) is a flat family of lines over \( \tau_{2m} \), so that it determines an embedding \( \tau_{2m} = \text{Spec}(k[t]/(t^{2m+1})) \to G \), that is, a curvilinear point \( x \in H_{2m} \) such that \( p: q^{-1}(Y_x) \xrightarrow{\sim} Z_s \) is an isomorphism. \( \square \)

**Remark 10.2.** It is easy to see that the set \( \mathcal{H} := \{ x \in H_{2m}^{\text{curv}} | x = x([E], s) \text{ for some } [E] \in H_{2m-1}^H \text{ and } s, 0 \neq s \in H^0(E(1)) \} \), is open and dense in \( H_{2m}^{\text{curv}} \). In particular, \( \mathcal{H} \) is a smooth integral \((8m)\)-dimensional scheme.

**Remark 10.3.** The set \( H_{2m}^{\text{curv}} \) contains an open dense subset \( \mathcal{H}_0 = \{ x \in H_{2m}^{\text{curv}} | Y_x \text{ is a reduced scheme} \} \). We put \( D_0^* := \{ x \in \mathcal{H}_0 | \text{ among the lines of the set } p(q^{-1}(Y_x)) \text{ there are intersecting lines} \} \) and let \( D^* \) be the closure of \( D_0^* \) in \( H_{2m}^{\text{curv}} \). We easily see that \( D^* \) is an irreducible divisor in \( H_{2m}^{\text{curv}} \), and it follows easily from Lemma 10.1 that \( \mathcal{H} = H_{2m}^{\text{curv}} \setminus D^* \).

**Lemma 10.4.** (i) Let \( H \) be the closure (regarded as an integral scheme) of the scheme \( \mathcal{H} \) in the Hilbert scheme \( H_{2m} \). Then the closure \( D_H \) of the divisor \( D^* \) in \( H \) is an irreducible Weil divisor in \( H \).

(ii) Consider the set \( H_s = H \setminus H_{2m}^{\text{curv}} = \{ x \in H \mid \text{the scheme } Y_x \text{ contains a point } y \text{ such that } \dim T_y(Y_x) \geq 2 \} \). Then \( \text{codim}_H H_s \geq 2 \) and \( H_s \subset D_H \). Thus, \( \mathcal{H} = H \setminus D_H \).

**Proof.** Part (i) follows from Remark 10.3. Next, given an arbitrary point \( x \in G \) and any two-dimensional subspace \( U \) of the tangent space \( T_x G \), we write \( Y(x, U) \) for the 0-dimensional subspace of length 3 in \( G \) which is supported at \( x \) and isomorphic to \( \text{Spec}(k[u, v]/(u, v)^2) \) and satisfies \( T_x Y(x, U) = U \). Put \( H_{(3)} = \{ Y \in \text{Hilb}^2 G | Y = Y(x, U) \text{ for some } x \in G \text{ and } U \in \text{Gr}(2, T_x G) \} \). We have \( \dim H_{(3)} = \dim G + \dim \text{Gr}(2, 4) = 8 \). It is easy to see that \( H_s = \{ Y \in H | Y \text{ contains a subscheme } Y' \in \dim H_{(3)} \} \) and, moreover, the set \( H_s^* = \{ Y \in H_s | Y \text{ is the disjoint union of a subscheme } Y' \in \dim H_{(3)} \text{ and some scheme } Y'' \in H_{2m-3}^{\text{curv}} \} \) is open and dense in \( H_s \). Hence \( \dim H_s = \dim H_{(3)} + \dim H_{2m-3}^{\text{curv}} = 8 + 4(2m - 3) = 4(2m - 1), \) so that \( \text{codim}_H H_s = 8m - 4(2m - 1) = 4 \geq 2 \). Finally, an elementary verification shows that \( H_s^* \subset D_H \), whence \( H_s \subset D_H \). This together with Remark 10.3 yields the equality \( \mathcal{H} = H \setminus D_H \). \( \square \)
Remark 10.5. Consider the closure \( H_{(2)} \) of the set \( \mathcal{H}(2) := \{ x \in \mathcal{H} \mid x = x([E], s) \} \) for some vector bundle \([E] \in I_{2m-1}^H, h^0(\mathcal{L}(1)) = 2 \) in \( H \). Then \( H_{(2)} \) (resp. \( \mathcal{H}(2) \)) is a closed subscheme of \( H \) (resp. of \( \mathcal{H} \)), and it is well known (see, for example, [1]) that the condition \( x = x([E], s) \in \mathcal{H}(2) \) on the point \( x \in \mathcal{H} \) is equivalent to the condition that the scheme \( Z_s = (s)_0 \) lies on a smooth quadric in \( \mathbb{P}^3 \). This is in turn equivalent to saying that the 0-dimensional subscheme \( \mathcal{Y}_x \) of \( G \) lies in a projective plane \( \mathbb{P}^2 \) in \( \mathbb{P}^5 = \text{Span}(G) \) that intersects \( G \) along a smooth conic, that is, in a general projective plane in \( \mathbb{P}^5 \). It follows that \( \dim \mathcal{H}(2) = \text{length}(\mathcal{Y}_x) + \dim G(2, \mathbb{P}^5) = 2m + 9 \). Accordingly,

\[
\text{codim}_H H_{(2)} = \text{codim}_H \mathcal{H}(2) = 8m - (2m + 9) = 6m - 9 > 2, \quad m \geq 2. \tag{10.5}
\]

Remark 10.6. Consider the set \( D^\tau := \{ x \in \mathcal{H} \mid \text{codim}(\mathcal{Y}_x) = 2m - 1 \} \). In other words, \( x \in D^\tau \) if \( \text{Supp} \mathcal{Y}_x = z_1 \cup z_2 \cup \cdots \cup z_{2m-1} \) and the scheme \( \mathcal{Y}_x \) is reduced at the points \( z_2, \ldots, z_{2m-1} \) and has length 2 at the point \( z_1 \). Put \( D_0^\tau := D^\tau \setminus (\mathcal{H}(2) \cap D^\tau) \). It is easy to see that \( D^\tau := \overline{D^\tau} = \overline{D_0^\tau} \) is an irreducible divisor in \( H \) (the closures here are taken in \( H \)) and

\[
\mathcal{H}_0 = \mathcal{H} \setminus (\mathcal{H} \cap D^\tau), \quad \text{codim}_H (D^\tau \setminus D_0^\tau) = \text{codim}_H (D^\tau \setminus D_0^\tau) \geq 2. \tag{10.6}
\]

Consider the universal family \( \Gamma_H \subset G \times H \) with the projections \( G \leftarrow \Gamma_H \rightarrow H \) and regard the scheme \( \Gamma = \Gamma \times_H \Gamma_H \) as a subscheme of \( \mathbb{P}^3 \times G \times H \) with projection \( \Gamma \rightarrow \Gamma_H \). Let \( L \) be the scheme-theoretic image of the morphism \( \gamma := p_{13}[\Gamma : \mathcal{H} \rightarrow \mathbb{P}^3 \times H] \) in the sense of [14], Ch. II, Exercise 3.11, (d), where \( p_{13} : \mathbb{P}^3 \times G \times H \rightarrow \mathbb{P}^3 \times H \) is the projection. We consider the projections \( \pi : \mathbb{P}^3 \times H \rightarrow H \) and \( p := \pi \circ p_{13} : \mathbb{P}^3 \times G \times H \rightarrow H \) and put \( \mathcal{I}_L := \mathcal{I}_{L, \mathbb{P}^3 \times H}, \mathcal{I}_H := \mathcal{I}_{H, \mathbb{P}^3 \times G \times H}. \) For any \( m, n \in \mathbb{Z} \) we put \( O(m, 0) := O_{\mathbb{P}^3}(m) \boxtimes O_H \) and, given an arbitrary sheaf \( F \) on \( \mathbb{P}^3 \times G \times H \) (resp. an arbitrary sheaf \( G \) on \( \mathbb{P}^3 \times H \)), define \( F(m, n) := F \boxtimes O_{\mathbb{P}^3}(m) \boxtimes O_G(n) \boxtimes O_H \) (resp. \( G(m) := G \boxtimes O_{\mathbb{P}^3}(m) \boxtimes O_H \)). We also put \( \Gamma_t := \mathbb{P}^3 \times G \times \{ t \} \cap \Gamma, L_t := \mathbb{P}^3 \times \{ t \} \cap L \) and \( \mathcal{I}_{\Gamma_t} := \mathcal{I}_{\Gamma_t, \mathbb{P}^3 \times G} \) for an arbitrary point \( t \in H \).

It follows directly from Lemma 10.1 that \( p_{13} : \Gamma \times_H \mathcal{H} \rightarrow L \times_H \mathcal{H} \) is an isomorphism of reduced schemes. In particular, for any point \( t \in \mathcal{H} \), the scheme \( L_t \), being a subscheme of \( \mathbb{P}^3 \), has Hilbert polynomial \( \chi(O_{\mathcal{L}}(n)) = 2m(n + 1) \). Hence there is a morphism \( g : \mathcal{H} \rightarrow \text{Hilb}^{2m(n+1)} \mathbb{P}^3 \), \( t \mapsto \{ L_t \} \), which is a locally closed embedding.

Lemma 10.7. The embedding \( g : \mathcal{H} \rightarrow \text{Hilb}^{2m(n+1)} \mathbb{P}^3 \) extends to an isomorphism \( g : H \cong g(H) \), \( t \mapsto \{ L_t \} \).

Proof. Since the sheaf \( O_{\mathbb{P}^3}(1) \boxtimes O_G(1) \) is ample on \( \mathbb{P}^3 \times G \), for all \( n \gg 0 \) and any \( t \in \mathcal{H} \) Serre’s theorem yields the restriction epimorphism

\[
e_n, n(t) : H^0(O_{\mathbb{P}^3}(n)) \otimes H^0(O_G(n)) \twoheadrightarrow H^0(O_{\mathbb{P}^3}(n) \boxtimes O_G(n)) \cong k^{2m(n+1)}
\]

and the vanishing of the cohomology groups \( H^i(O_{\mathcal{L}}(n), n) \), \( i > 0 \). By construction we have \( \mathcal{Y}_t = q^{-1}(t) \), where \( \mathcal{Y}_t = u(u^{-1}(t)) \) is a 0-dimensional subscheme in \( G \), whence we have a (non-canonical) isomorphism of sheaves \( O_{\mathcal{L}}(n, n) \cong O_{\mathcal{L}}(n) \).
Therefore the epimorphism $e_{n,n}(t)$ and the vanishing of $H^i(O_{\Gamma_t}(n,n))$, $i > 0$, induce a restriction epimorphism $e_n(t) : H^0(O_{\mathbb{P}^3}(n) \otimes O_G) \to H^0(O_{\Gamma_t}(n)) \cong k^{2m(n+1)}$. Passing to the cohomology of the exact triple $0 \to I_t(n) \to O_{\mathbb{P}^3}(n) \otimes O_G \to O_{\Gamma_t}(n) \to 0$, we see that $H^i(O_{\Gamma_t}(n)) = 0$, $i > 0$, and that $r_n := h^0(I_t(n)) = h^0(O_{\mathbb{P}^3}(n)) - 2m(n + 1)$ is independent of $t \in H$. We also have a monomorphism $H^0(I_t(n)) \to H^0(O_{\mathbb{P}^3}(n))$. Since $I_t(n) = I_t(n) \otimes k(t)$ and the sheaf $I_t(n)$ is $H$-flat in view of the $H$-flatness of $O_H(n)$, we deduce using the theorem on change of base ([14], Ch. III, §7.11) that $p_*(I_t(n))$ is a locally free $O_H$-sheaf of rank $r_n$ and $R^0p_*(I_t(n)) = 0, i > 0$.

On the other hand, it follows from the definition of the scheme $L$ and the projection formula that $p_{13*}(I_t(n)) = I_L(n)$. This and the above equalities, together with the spectral sequence $E_2^{3,0} = R^3p_*(R^3p_{13*}(I_t(n))) \Rightarrow R^0p_*(I_t(n))$, yield that $\pi_*(L(n)) = p_*(I_t(n))$, $R^i\pi_*(I_t(n)) = 0, i > 0$. Therefore, applying the functor $R^0\pi_*$ to the exact triple $0 \to I_L(n) \to O_{\mathbb{P}^3}(n) \otimes O_H \to O_L(n) \to 0$, we get an exact triple

$$0 \to p_*(I_t(n)) \to H^0(O_{\mathbb{P}^3}(n)) \otimes O_H \to \pi_*(O_L(n)) \to 0$$

and the equalities $R^i\pi_*(O_L(n)) = 0, i > 0$. Restricting the last triple to an arbitrary point $t \in H$, we obtain an exact triple

$$H^0(I_t(n)) \to H^0(O_{\mathbb{P}^3}(n)) \to \pi_*(O_L(n)) \otimes k(t) \to 0.$$ 

Since $\varphi(t)$ is injective, it follows that $\dim_{k(t)}(\pi_*(O_L(n)) \otimes k(t)) = 2m(n + 1)$ is independent of $t$. Since $H$ is an integral scheme, we obtain using [15], Russian p. 51, that the sheaf $\pi_*(O_L(n))$ is locally free of rank $2m(n + 1)$. Using [16], Lecture 7, Corollary 3, we see that $O_L(n)$ is a flat $O_H$-sheaf, and the equations $R^i\pi_*(I_L(n)) = 0, i > 0$, together with the above-mentioned theorem on change of base yield that $h^0(O_L(n)) = 2m(n + 1)$. Thus the morphism $g : H \to \text{Hilb}^{2m(n+1)} \mathbb{P}^3$, $t \mapsto \{L_t\}$, is well defined.

That $g$ is an isomorphism may be verified pointwise in the following way. For any point $t \in H$ we consider the projection $\gamma_t := \gamma|\Gamma_t : \Gamma_t \to L_t$. Let $y_t$ be the point of $\text{Hilb}^{2m(n+1)} \mathbb{P}^3$ corresponding to the subscheme $L_t$ of $\mathbb{P}^3$, that is, $y_t = g(t)$. Here $p_t$ is a finite birational morphism. Hence we have $q(p_t^{-1})(\mathbb{P}^2 \cap L_t) = \gamma_t$ for a general plane $\mathbb{P}^2$ in $\mathbb{P}^3$, so that the point $t = \{\gamma_t\} \in H$ is uniquely determined by the point $y_t$. By construction, the map $y_t \mapsto t$ is inverse to $g$ and is a morphism. $\square$

Remark 10.8. In view of Lemma 10.7 we shall always identify $H$ with $g(H)$ and put $g = \text{id}_H$. Thus $L \hookrightarrow \mathbb{P}^3 \times H$ is the universal family over $H$, and $\gamma : \Gamma \to L$ is a finite birational morphism.

10.2. The family $T$ of ‘t Hooft extensions. Description of the projection $T \to H$. We consider the projections $\pi : \mathbb{P}^3 \times H \to H$ and $p : \Gamma \to L \to H$. Using the notation in §10.1, we put $A_i := \mathcal{E}xt^2_\pi(I_L(1), O(-1,0))$, $i \geq 0$, $A := A_1$ and $T := \text{Proj}(A^\vee)^{\vee} \to H$.

Proposition 10.9. (i) We have the following isomorphisms of sheaves on $H$:

$$A \cong \pi_* \mathcal{E}xt^2(\gamma_* O_{\Gamma}(1,0), O(-1,0)) \cong p_* O_{\Gamma}(0,1) \cong v_* u^* O_G(1). \quad (10.7)$$
Hence $A$ is a locally free $\mathcal{O}_H$-sheaf of rank $2m$ and, therefore, $\rho: T \to H$ is a projective bundle with fibre $\mathbb{P}^{2m-1}$. Moreover, the construction of the sheaf $A$ commutes with the base change.

(ii) For an arbitrary divisor $X \in |\mathcal{O}_G(1)|$ we consider the divisor $X_H = \nu(u^{-1}(X))$ on $H$ and its complement $H_X = H \setminus X_H$. Then we have the following isomorphism of sheaves on $\mathbb{P}^3 \times H_X$:

$$\mathcal{E}xt^2(\gamma_*\mathcal{O}_\Gamma(1,0), \mathcal{O}(-1,0))|_{\mathbb{P}^3 \times H_X} \simeq \gamma_*\mathcal{O}_\Gamma|_{\mathbb{P}^3 \times H_X}. \quad (10.8)$$

Moreover, $\mathbb{P}^3 \times H_X$ carries an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes B \xrightarrow{\alpha_X} \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{W}_{4m} \xrightarrow{\beta_X} \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes B^\vee \xrightarrow{\text{ev}} \gamma_*\mathcal{O}_\Gamma|_{\mathbb{P}^3 \times H_X} \to 0, \quad (10.9)$$

where $B := A^\vee|_{H_X}$ and $\mathcal{W}_{4m}$ is a sheaf of rank $4m$ on $H_X$. The sequence $(10.9)$ determines a section $A_X \in H^0(S^2B^\vee \otimes \wedge^2V^\vee)$ as a composite morphism

$$A_X: B \otimes V \xrightarrow{H^0(\alpha_X)^\vee} \mathcal{W}_{4m} \xrightarrow{H^0(\beta_X)} B^\vee \otimes V^\vee. \quad (10.10)$$

Furthermore, $A_X$ is a family of nets of quadrics of rank $4m$ with base $H_X$. By construction, the nets $A_X|_{H_X \cap H_X'}$ and $A_X'|_{H_X \cap H_X'}$, coincide for any two divisors $X, X' \in |\mathcal{O}_G(1)|$:

$$A_X|_{H_X \cap H_X'} = A_X'|_{H_X \cap H_X'}. \quad (10.11)$$

and therefore determine a net of quadrics on $H$:

$$A \in H^0(S^2A \otimes \wedge^2V^\vee). \quad (10.12)$$

Proof. (i) For any point $t \in H$, the morphism $\gamma_t := \gamma|_{\Gamma_t}: \Gamma_t \to L_t$ is a finite birational morphism and, by the projection formula,

$$\gamma_{t*}(\mathcal{O}_\Gamma(1,0)|_{\Gamma_t}) \otimes \mathcal{O}_{\mathbb{P}^3}(-3) \boxtimes \mathcal{O}_H = \gamma_{t*}(\mathcal{O}_\Gamma(-2,0)|_{\Gamma_t}).$$

In view of the Leray spectral sequence, the number $h^1(\gamma_{t*}(\mathcal{O}_\Gamma(-2,0)|_{\Gamma_t})) = h^1(\mathcal{O}_\Gamma(-2,0)|_{\Gamma_t}) = 2m$ is independent of $t$, and $h^2(\mathcal{O}_\Gamma(-2,0)|_{\Gamma_t}) = 0$. Hence the sheaf $R^1\pi_*(\gamma_*\mathcal{O}_\Gamma(-2,0))$ commutes with the base change. Using the projection formula PF, Serre duality SD (see [17]), and the Leray spectral sequence $L$ for the composite $\pi \circ \gamma = p$, we obtain

$$\mathcal{E}xt^2(\gamma_*\mathcal{O}_\Gamma(1,0), \mathcal{O}(-1,0))|_{\mathbb{P}^3 \times H} \simeq \mathcal{E}xt^2(\gamma_*\mathcal{O}_\Gamma(-2,0), \omega_{\mathbb{P}^3} \boxtimes \mathcal{O}_H)$$

$$\simeq (R^1\pi_*(\gamma_*\mathcal{O}_\Gamma(-2,0)))^\vee \simeq (R^1p_*\mathcal{O}_\Gamma(-2,0))^\vee. \quad (10.13)$$

Since $p$ coincides with the composite $\Gamma \xrightarrow{\nu} \Gamma_H \xrightarrow{u} H$ and we have $\omega_{\mathbb{P}^3} \simeq \mathcal{O}_\Gamma(-2,1)$, a standard computation shows that

$$(R^1p_*\mathcal{O}_\Gamma(-2,0))^\vee \simeq p_*\mathcal{O}_\Gamma(0,1) \simeq v_*u^*\mathcal{O}_G(1) \quad (10.14)$$

is a locally free $\mathcal{O}_H$-sheaf of rank $2m$. Since $\mathcal{E}xt^k_\pi(\mathcal{O}(1,0), \mathcal{O}(-1,0)) = 0$ for $k = 1, 2$, we have

$$\mathcal{E}xt^1_\pi(\mathcal{I}_L(1), \mathcal{O}(-1,0)) \simeq \mathcal{E}xt^2_\pi(\mathcal{O}_L(1), \mathcal{O}(-1,0)) \simeq \mathcal{E}xt^3_\pi(\mathcal{O}_L(1), \mathcal{O}(-1,0)).$$
On the other hand, since codim_{\mathbb{P}^3 \times H} L = 2, the spectral sequence

\[ E_2^{j,k} = R^j \pi_* \mathcal{E}x t^k (\mathcal{O}_L(1), \mathcal{O}(-1,0)) \Rightarrow \mathcal{E}x t^*_\pi (\mathcal{O}_L(1), \mathcal{O}(-1,0)) \]
degenerates and yields an isomorphism

\[ \mathcal{E}x t^2_\pi (\mathcal{O}_L(1), \mathcal{O}(-1,0)) \simeq \pi_* \mathcal{E}x t^2 (\mathcal{O}_L(1), \mathcal{O}(-1,0)). \]
Hence \( A \simeq \pi_* \mathcal{E}x t^2 (\mathcal{O}_L(1), \mathcal{O}(-1,0)) \). Moreover, we have an exact triple

\[ 0 \to \mathcal{O}_L(1) \xrightarrow{c} \gamma_* \mathcal{O}_\Gamma(1,0) \to \text{coker}(c) \to 0, \]
in which, by construction, codim_{\mathbb{P}^3 \times H} \text{Supp}(\text{coker}(c)) = 4. Therefore,

\[ \mathcal{E}x t^2(\mathcal{O}_L(1), \mathcal{O}(-1,0)) = \mathcal{E}x t^2(\gamma_* \mathcal{O}_\Gamma(1,0), \mathcal{O}(-1,0)) \]
and, as in the computations above, we have

\[ \mathcal{E}x t^2(\gamma_* \mathcal{O}_\Gamma(1,0), \mathcal{O}(-1,0)) = \pi_* \mathcal{E}x t^2(\gamma_* \mathcal{O}_\Gamma(1,0), \mathcal{O}(-1,0)). \]
Hence \( A \simeq \pi_* \mathcal{E}x t^2(\gamma_* \mathcal{O}_\Gamma(1,0), \mathcal{O}(-1,0)) \), and we obtain (10.7) in view of (10.13) and (10.14). The computations above also show that the construction of the sheaf \( A \) commutes with the base change.

(iii) For an arbitrary point \( t \in H \) we consider the sheaf \( \gamma_{t*} \mathcal{O}_\Gamma \), and put \( B_t := (H^0(\gamma_{t*} \mathcal{O}_\Gamma))^\vee \). By definition, the sheaf \( \gamma_{t*} \mathcal{O}_\Gamma \), has a filtration as in (10.4), with factors of the form \( \mathcal{O}_{l_i}, i = 1, \ldots, 2m \), where the \( l_i \) are (not necessarily distinct) lines in \( \mathbb{P}^3 \). Therefore the evaluation map \( B_t^\vee \otimes \mathcal{O}_\mathbb{P}^3 \xrightarrow{ev_t} \gamma_{t*} \mathcal{O}_\Gamma(1,0) \) is surjective.

Consider the sheaf \( \mathcal{G} := \ker(B_t^\vee \otimes \mathcal{O}_\mathbb{P}^3(1) \to \gamma_{t*} \mathcal{O}_\Gamma(1,0)) \). Using the above filtration of the sheaf \( \gamma_{t*} \mathcal{O}_\Gamma \), we obtain by an easy computation that the sheaf \( \mathcal{G} \) is 0-regular in the sense of Mumford–Castelnuovo ([16], Lecture 14) and \( h^0(\mathcal{G}) = 4m \). Thus \( \mathcal{G} \) is globally generated. We put \( W_{4m}(t) := H^0(\mathcal{G}) \) and let \( K_t \) be the kernel of the evaluation map \( W_{4m}(t) \otimes \mathcal{O}_\mathbb{P}^3(1) \xrightarrow{ev_t} \mathcal{G}(1) \). A computation similar to the above shows that \( \text{rk}(K_t) = h^0(K_t) = 2m \) and the sheaf \( K_t \) is 0-regular in the sense of Mumford–Castelnuovo and, therefore, globally generated. Putting \( B'_t := H^0(K_t) \), we thus obtain a surjective evaluation morphism \( B'_t \otimes \mathcal{O}_\mathbb{P}^3 \twoheadrightarrow K_t \). This epimorphism is an isomorphism since the ranks of both sheaves coincide. Hence we obtain an exact sequence

\[ (A): 0 \to B'_t \otimes \mathcal{O}_\mathbb{P}^3(-1) \xrightarrow{\alpha_t} W_{4m}(t) \otimes \mathcal{O}_\mathbb{P}^3 \xrightarrow{\beta_t} B_t^\vee \otimes \mathcal{O}_\mathbb{P}^3(1) \xrightarrow{ev_t(1)} \gamma_{t*} \mathcal{O}_\Gamma(1,0) \to 0. \]  

(Applying the functor \( \mathcal{H}om(\cdot, \mathcal{O}_\mathbb{P}^3) \) to this sequence, we get the dual sequence

\[ (A)^\vee: 0 \to B_t \otimes \mathcal{O}_\mathbb{P}^3(-1) \xrightarrow{\beta_t^\vee} W_{4m}(t) \otimes \mathcal{O}_\mathbb{P}^3 \xrightarrow{\alpha_t^\vee} (B'_t)^\vee \otimes \mathcal{O}_\mathbb{P}^3(1) \xrightarrow{ev_t(1)^{-1}} \mathcal{E}x t^2(\gamma_{t*} \mathcal{O}_\Gamma(1,0), \mathcal{O}_\mathbb{P}^3(-1)) \otimes \mathcal{O}_\mathbb{P}^3(1) \to 0. \]

In particular, this yields an isomorphism \( (B'_t)^\vee \simeq H^0(\mathcal{E}x t^2(\gamma_{t*} \mathcal{O}_\Gamma(1,0), \mathcal{O}_\mathbb{P}^3(-1))) \).
On the other hand, arguing as in (i), we see that the isomorphisms of sheaves in (10.7) commute with the base change, that is,
\[ H^0(\mathcal{E}xt^2(\gamma_{t*}\mathcal{O}_{\Gamma_1}(1,0),\mathcal{O}_{\mathbb{P}^3}(-1)))) = H^0(\mathcal{E}xt^2(\gamma_*\mathcal{O}_{\Gamma}(1,0),\mathcal{O}(-1,0)) \otimes k(t)) = \pi_*\mathcal{E}xt^2(\gamma_*\mathcal{O}_{\Gamma}(1,0),\mathcal{O}(-1,0)) \otimes k(t) \simeq p_*\mathcal{O}_{\Gamma}(0,1) \otimes k(t) = H^0(\gamma_{t*}\mathcal{O}_{\Gamma_1}) = B_t^\vee. \]
This yields an isomorphism \((B_t')^\vee \simeq B_t^\vee\), which extends to an isomorphism between the complexes \((A)\) and \((A)^\vee\). In particular, this isomorphism implies that
\[ \mathcal{E}xt^2(\gamma_{t*}\mathcal{O}_{\Gamma_1}(1,0),\mathcal{O}_{\mathbb{P}^3}(-1))) \simeq \gamma_{t*}\mathcal{O}_{\Gamma_1}. \] (10.16)
Moreover, a standard verification using the isomorphism \((B_t')^\vee \simeq B_t^\vee\) shows that the composite
\[ B_t \otimes V \xrightarrow{H^0(\alpha^\vee)} W_{4m}(t) \xrightarrow{H^0(\beta)} B_t^\vee \otimes V^\vee \]
is anti-self-dual, that is, \(A_t \in \wedge^2(B_t^\vee \otimes V^\vee)\). Here the condition \(\beta_t \circ \alpha_t = 0\) implies that \(A_t\) is contained in the direct summand \(S^2B_t^\vee \otimes \wedge^2V^\vee\) of the space \(\wedge^2(B_t^\vee \otimes V^\vee)\), that is, \(A_t\) is a net of quadrics. Moreover, since by construction \(H^0(\beta): W_{4m}(t) \to B_t^\vee \otimes V^\vee\) is an injective homomorphism, it follows that \(A_t\) is a net of quadrics of rank \(4m\).

We now remark that the isomorphism \(\mathcal{O}_G(1)|_{\mathbb{G} \setminus X} \simeq \mathcal{O}_G \setminus X\) implies an isomorphism \(p_*\mathcal{O}_\Gamma(0,1)|_{\mathbb{H}_X} \simeq p_*\mathcal{O}_\Gamma|_{\mathbb{H}_X}\). Hence the isomorphisms (10.16) for \(t \in \mathbb{H}_X\) globalize to the isomorphism (10.8). The complexes \((A)\) and \((A)^\vee\) accordingly globalize to isomorphic complexes \((A)_X\) and \((A)^\vee_X\), where \((A)_X\) is the complex
\[ 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes B^\vee \xrightarrow{\alpha_X} \mathcal{O}_{\mathbb{P}^3} \boxtimes W_{4m} \xrightarrow{\beta_X} \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes B^\vee \xrightarrow{\psi_X} \gamma_*\mathcal{O}_{\Gamma}|_{\mathbb{P}^3 \times \mathbb{H}_X} \to 0, \]
in which
\[ B^\vee = p_*\mathcal{O}_\Gamma|_{\mathbb{H}_X}, \]
and the isomorphism \(B^\vee \simeq (B')^\vee\) follows from (10.8). The nets \(A_t, t \in \mathbb{H}_X\), accordingly globalize to the section \(A_X \in H^0(S^2B^\vee \otimes \wedge^2V^\vee)\) defined in (10.10).

We finally note that the equation (10.11) follows because, by construction, for any point \(t \in \mathbb{H}_X \cap \mathcal{H}_0\), where \(\mathbb{H}_X \cap \mathcal{H}_0\) is dense in \(\mathbb{H}_X\), the net of quadrics
\[ A_X \otimes k(t): (A \otimes k(t))^\vee \otimes V \to (A \otimes k(t)) \otimes V^\vee \]
is a corollary of the Serre–Grothendieck duality \(\text{Ext}^2(\gamma_{t*}\mathcal{O}_{\Gamma_1},\mathcal{O}_{\mathbb{P}^3}) \simeq H^0(\gamma_{t*}\mathcal{O}_{\Gamma_1})^\vee \simeq H^0(\mathcal{O}_{\Gamma_1})^\vee\), the equality \(\text{Ext}^2(\gamma_{t*}\mathcal{O}_{\Gamma_1},\mathcal{O}_{\mathbb{P}^3}) = H^0(\mathcal{E}xt^2(\gamma_{t*}\mathcal{O}_{\Gamma_1},\mathcal{O}_{\mathbb{P}^3}))\), and the fact that the isomorphism (10.7) commutes with the base change. Hence this net does not depend on the choice of a basis in \(H^0(\mathcal{O}_{\Gamma_1})^\vee \simeq A \otimes k(t)\). \(\square\)

We now consider the variety \(Y := \text{Isom}(A, H^\vee_{2m} \otimes \mathcal{O}_\mathbb{H})\) with the projection \(\zeta: Y \to \mathbb{H}\). Under the universal isomorphism \(\psi_Y: \zeta^* A \xrightarrow{\simeq} H^\vee_{2m} \otimes \mathcal{O}_Y\), the net \(A \in H^0(S^2A \otimes \wedge^2V^\vee)\) constructed in (10.12) induces on \(Y\) a net
\[ \mathcal{A}_Y = \psi_Y(\zeta^* A) \in H^0(S^2H^\vee_{2m} \otimes \wedge^2V^\vee \otimes \mathcal{O}_Y) = H^0(S^2m \otimes \mathcal{O}_Y). \] (10.17)
This net determines a GL($H_{2m}$)-equivariant morphism

$$f_Y: Y \to S_{2m}$$

(10.18)
such that $A_Y = f^*_Y A_{S_{2m}}$, where $A_{S_{2m}}: O_{S_{2m}} \to S_{2m} \otimes O_{S_{2m}}$ is the universal section.

We now consider the universal epimorphism $\varepsilon: \rho^* A^\vee \to O_T(1)$ over the scheme $T$, where $O_T(1)$ is the Grothendieck sheaf. Let $T \xrightarrow{\hat{\pi}} \mathbb{P}^3 \times T \xrightarrow{\rho} \mathbb{P}^3 \times H$ be the induced projections. Since $\pi$ and $\hat{\pi}$ are projections onto the factors, we have the base change isomorphism

$$O_T(1) \otimes \rho^* A_k \cong \mathcal{E}xt^k_{\hat{\pi}}(\hat{\pi}^* I_L(1), \hat{\pi}^* O(-1, 0) \otimes \hat{\pi}^* O_T(1)), \quad k \geq 0,$$

and, therefore, a spectral sequence

$$(S): E_2^{j,k} = H^j(O_T(1) \otimes \rho^* A_k) \Rightarrow \mathcal{E}xt^k(\hat{\pi}^* I_L(1), \hat{\pi}^* O(-1, 0) \otimes \hat{\pi}^* O_T(1)).$$

Since $\text{codim}_{\mathbb{P}^3 \times H} L = 2$, we have $A_0 = \pi_* \mathcal{H}om(O(1), O(-1, 0)) = \pi_* O(-2, 0) = 0$, and the spectral sequence $(S)$ yields an isomorphism

$$\psi: H^0(O_T(1) \otimes \rho^* A) \cong \mathcal{E}xt^1(\hat{\pi}^* I_L(1), \hat{\pi}^* O(-1, 0) \otimes \hat{\pi}^* O_T(1)).$$

We consider the composite

$$s: O_T \xrightarrow{\rho^*(id)} \rho^*(A^\vee \otimes A) = \rho^*(A^\vee) \otimes \rho^* A \xrightarrow{\varepsilon \times \text{id}} O_T(1) \otimes \rho^* A$$

and put $\xi = \psi(s)$. The element $\xi$ determines an extension

$$0 \to \hat{\pi}^* O(-1, 0) \otimes \hat{\pi}^* O_T(1) \to E_T \to \hat{\pi}^* I_L(1) \to 0$$

(10.19)
as the universal family of extensions of $O_{\mathbb{P}^3}$-sheaves with base $T$. We call this family the universal family of $t$-Hoot fre extensions. For an arbitrary $(O_{\mathbb{P}^3 \times T})$-sheaf $G$ we shall also use the notation $G(k) = G \otimes O_{\mathbb{P}^3}(k) \boxtimes O_T$.

**Remark 10.10.** Consider the divisor $D_T = \rho^{-1}(D_H)$ in $T$ and its complement $T_{\mathcal{H}} := T \setminus D_T$. By Lemma 10.4 and Proposition 10.9 (i), $D_T$ is an irreducible Weil divisor in $T$ and $T_{\mathcal{H}} = \rho^{-1}(\mathcal{H})$. We take an arbitrary point $y \in \mathcal{H}$ and consider the subscheme $L_y = L \cap (\mathbb{P}^3 \times \{y\})$ and the vector space $U_y := \text{Ext}^1(I_{L_y, \mathbb{P}^3}(1), O_{\mathbb{P}^3}(-1))$. By definition of the scheme $T$, the fibre $\rho^{-1}(y)$ is isomorphic to $P(U_y) \simeq \mathbb{P}^{2m-1}$. If $y \in \mathcal{H}_0$, then $L_y = \cup_{1 \leq i \leq 2m} l_i$ is a disjoint union of $2m$ lines $l_1, \ldots, l_{2m}$, and a standard computation yields an isomorphism

$$\varphi_y: U_y \cong \bigoplus_{1 \leq i \leq 2m} H^0(I_{l_i}) \cong k^{2m}, \quad \xi \mapsto (x_1(\xi), \ldots, x_{2m}(\xi)).$$

We consider the open dense subset $T_{\mathcal{H}_0} = \rho^{-1}(\mathcal{H}_0)$ of $T$ and the divisor $D_{\mathcal{H}_0} := \{ t = (y, k\xi) \mid y \in \mathcal{H}_0 \text{ and } \xi \in U_y \text{ satisfies the condition that not all the } x_i(\xi), i = 1, \ldots, 2m, \text{ are non-zero} \}$ in this subset. Let $D_T'$ be the closure of $D_{\mathcal{H}_0}$ in $T$. One easily sees that $D_T'$ is an irreducible Weil divisor in $T$. Moreover, a direct computation using formulae (10.4) and Nakayama’s lemma shows that the open dense subset $T^* = \{ t \in T \mid E_T|_{\mathbb{P}^3 \times \{t\}} \text{ is a locally free sheaf on } \mathbb{P}^3 \}$ of $T$ coincides with $T \setminus (D_T \cup D_T')$. 

**Moduli of mathematical instanton vector bundles**
Remark 10.11. By construction, \( E_T = E_T|_{\mathbb{P}^3 \times T} \) is the family of ‘t Hooft \((2m-1)\)-instantons with base \( T' \), and the modular morphism \( f_T : T' \rightarrow T_{2m-1}^{H}, t \mapsto [E_T|_{\mathbb{P}^3 \times \{t\}}] \), is surjective. Since \( \dim T' = \dim H + 2m - 1 = 10m - 1 = \dim I_{2m-1}^{H} \), we see that \( f_T \) is generically finite. Moreover, putting

\[
T^{**} = \{ t \in T'| h^0((E_T|_{\mathbb{P}^3 \times \{t\}})(1)) = 1, \ h^1((E_T|_{\mathbb{P}^3 \times \{t\}})(1)) = 6m - 10, \ h^{2,2}((E_T|_{\mathbb{P}^3 \times \{t\}})(1)) = 0 \},
\]

we have \( T' \setminus T^{**} = \rho^{-1}(H(2)) \cap T' \) (see [1]), and it follows from Remark 10.5 that \( \text{codim}_{T'}(T' \setminus T^{**}) > 2, \ m \geq 2. \)

We consider the following universal exact triple of bundles on \( T \):

\[
0 \rightarrow \mathcal{O}_T(-1) \rightarrow \rho^* A \xrightarrow{\psi} A_T \rightarrow 0, \tag{10.20}
\]

where \( A_T \) is the universal quotient bundle of rank \( 2m - 1 \). The net \( A \in H^0(S^2 A \otimes \wedge^2 V') \) in (10.12) determines a net on \( T \):

\[
A_T = e_T(\rho^* A) \in H^0(S^2 A_T \otimes \wedge^2 V'). \tag{10.21}
\]

We now consider the variety \( M := \text{Isom}(A_T, H_{2m-1} \otimes \mathcal{O}_T) \) with projection \( \tau : M \rightarrow T \). The universal isomorphism \( \psi_M : \tau^* A_T \simeq H_{2m-1} \otimes \mathcal{O}_M \) and the net \( A_T \) in (10.21) determine a net on \( M \):

\[
\mathcal{A}_M = \psi_M(\tau^* A_T) \in H^0(S^2 H_{2m-1} \otimes \wedge^2 V' \otimes \mathcal{O}_M) = H^0(S_{2m-1} \otimes \mathcal{O}_M). \tag{10.22}
\]

This net yields a GL(\( H_{2m-1} \))-equivariant morphism

\[
f_M : M \rightarrow S_{2m-1} \tag{10.23}
\]

such that \( \mathcal{A}_M = f_M^* \mathcal{A}_{S_{2m-1}} \), where \( \mathcal{A}_{S_{2m-1}} : \mathcal{O}_{S_{2m-1}} \rightarrow S_{2m-1} \otimes \mathcal{O}_{S_{2m-1}} \) is the universal section.

We note that the monomorphism \( j : H_{2m-1} \hookrightarrow H_{2m} \) corresponding to a point \( j \in P(H_{m}') \) (see (9.28)) determines an exact triple

\[
0 \rightarrow k \xrightarrow{j} H_{2m}' \xrightarrow{j'} H_{2m-1}' \rightarrow 0 \tag{10.24}
\]

(here and below we use for simplicity the notation \( j \) for the left morphism in the triple). Therefore every point \( y \in Y \), regarded as an isomorphism of vector spaces, \( \psi_y : A(y) := A \otimes k(\zeta(y)) \simeq H_{2m} \), together with the triple (10.24) determines an embedding \( \psi_y^* \circ j : k \hookrightarrow A(y) \) and an isomorphism \( \psi(y, j) : A(y)/(\psi_y^* \circ j)(k) \simeq H_{2m-1} \). Thus we have a morphism \( \mu_j : Y \rightarrow M, y \mapsto ((\psi_y^* \circ j)(k), \psi(y, j)) \). This morphism is surjective with fibre \( \mathbb{A}_{2m} \) and, by construction, \( \zeta = \rho \circ \tau \circ \mu_j \). It follows from the definition of the nets \( \mathcal{A}_Y \) and \( \mathcal{A}_M \) that \( \tilde{j}(\mathcal{A}_Y) = \mu_j^* \mathcal{A}_Y \), where \( \tilde{j} : S_{2m} \rightarrow S_{2m-1} \) is the projection induced by the monomorphism \( j \). Therefore the morphisms \( f_Y \) and \( f_M \) (defined in (10.18) and (10.23)) fit into the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f_Y} & S_{2m} \\
\mu_j \downarrow & & \downarrow j \\
M & \xrightarrow{f_M} & S_{2m-1}
\end{array} \tag{10.25}
\]
Moreover, the sections $\mathcal{A}_Y$ and $\mathcal{A}_M$ determine a commutative diagram of complexes on $\mathbb{P}^3 \times Y$ (an exact analogue of the diagram (9.29))

\[
\begin{align*}
H_{2m-1} \otimes O_{\mathbb{P}^3}(-1) \otimes O_Y & \xrightarrow{s_Y^{(j)}} W_{4m} \otimes O_{\mathbb{P}^3} \times Y \xrightarrow{t_Y^{(j)}} H_{2m-1} \otimes O_{\mathbb{P}^3} \otimes O_Y \rightarrow 0 \\
H_{2m} \otimes O_{\mathbb{P}^3}(-1) \otimes O_Y & \xrightarrow{s_Y} W_{4m} \otimes O_{\mathbb{P}^3} \times Y \xrightarrow{t_Y} H_{2m} \otimes O_{\mathbb{P}^3} \otimes O_Y \xrightarrow{\text{can}_Y} p_Y \cdot O_{\Gamma_Y}(0,1) \\
O_{\mathbb{P}^3} \otimes O_Y & \xrightarrow{j} Y
\end{align*}
\]

(10.26)

Here $\Gamma_Y := \Gamma \times_H Y$, $p_Y : \Gamma_Y \to Y$ is the projection, $\text{can}_Y$ is the canonical surjection, and $\xi_Y(j)$ is the composite. Note that the lower complex in the diagram (10.26) is exact. By construction, it is the relative version of the complex (10.15).

We put $M^* := \tau^{-1}(T^*)$. In view of Remark 10.11 we obtain the following lemma.

**Lemma 10.12.** The variety $\overline{MI} = f_M(M)$ contains the variety $M^{1H}_{2m-1}$ of ‘t Hooft $(2m - 1)$-instanton nets as an open dense subset. There is a $\text{GL}(H_{2m-1})$-invariant commutative diagram

\[
\begin{align*}
M^* & \xrightarrow{f_M} M^{1H}_{2m-1} \\
\tau & \\
T^* & \xrightarrow{f_{T^*}} I^{H}_{2m-1}
\end{align*}
\]

(10.27)

§ 11. Completion of the proof of Theorem 7.2

In this section we finish the proof of Theorem 7.2 using the universal family of ‘t Hooft extensions and related constructions from § 10.

11.1. Maps of components of $Z_m$ to $T$. Using the notation in Remarks 10.3, 10.5 we put $H'_0 := \{x \in H_0 \setminus H_{(2)} \mid \text{among the lines of the set } p(q^{-1}(Y_x)) \text{ there are at most two that intersect} \}$. Clearly, $H'_0$ is an open dense subset of $H$, and $D^0_H := D_H \cap H'_0 = \{x \in H_0 \mid \text{among the lines of the set } p(q^{-1}(Y_x)) \text{ there are precisely two that intersect} \}$. It is easy to see that $(H'_0)^* := H'_0 \setminus D^0_H$ is an open dense subset of $H'_0$. Next, using the notation in Remark 10.6, we put $H_0 := H'_0 \cup D^0_H$.

It follows directly from (10.6) and the definition of $D_H$ that

\[
codim_{H \setminus D_H}(H \setminus (D_H \cup H_0)) \geq 2.
\]

(11.1)

We use the notation in § 10.2 to consider the scheme $\Gamma \subset \mathbb{P}^3 \times G \times H$ with projection $\gamma : \Gamma \to \mathbb{P}^3 \times H$ and regard the sheaf $\gamma_* O_\Gamma$ as a family of sheaves $\{\gamma_{t*} O_{\Gamma_t}\}_{t \in H}$ on $\mathbb{P}^3$. As already mentioned in the proof of Proposition 10.9 (ii), each sheaf $\gamma_{t*} O_{\Gamma_t}$ in this family has a filtration with factors of the form $O_{l_i}$, $i = 1, \ldots, 2m$, where the $l_i$ are the lines in $\mathbb{P}^3$ determined by the point $t \in H$. Therefore $\gamma_{t*} O_{\Gamma_t}$ is a Gieseker-semistable $O_{\mathbb{P}^3}$-sheaf with Hilbert polynomial $P_m(n) := \chi(\gamma_{t*} O_{\Gamma_t}(n)) = 2m(n + 1)$.
For an arbitrary point \( t \in \mathbf{H}_0' \), the sheaf \( \gamma_{t*}\mathcal{O}_{\Gamma_t} \) is of the form

\[
\gamma_{t*}\mathcal{O}_{\Gamma_t} = \bigoplus_{i=1}^{2m} \mathcal{O}_{t_i}.
\] (11.2)

This sheaf is 0-regular in the sense of Mumford–Castelnuovo and, therefore, the evaluation morphism \( e_t : \mathbb{k}^{2m} \otimes \mathcal{O}_{\mathbb{P}^3} \to \gamma_{t*}\mathcal{O}_{\Gamma_t} \) is surjective. Hence the evaluation morphism \( e_t : \mathbb{k}^{M_n} \otimes \mathcal{O}_{\mathbb{P}^3}(-n) \to \gamma_{t*}\mathcal{O}_{\Gamma_t} \) is also surjective for \( n \gg 0 \), where \( M_n = 2mh^0(\mathcal{O}_{\mathbb{P}^3}(n)) = 2m(n+3)_3 \), and its class \([e_t]\) may be regarded as a point of the Quot-scheme \( \text{Quot}(\mathbb{k}^{M_n} \otimes \mathcal{O}_{\mathbb{P}^3}(-n), P_m) \). Let \( Q \) be the irreducible component of the scheme \( \text{Quot}(\mathbb{k}^{M_n} \otimes \mathcal{O}_{\mathbb{P}^3}(-n), P_m) \) containing the point \([e_t]\). We put \( Q^{ss} := \{ [\mathbb{k}^{M_n} \otimes \mathcal{O}_{\mathbb{P}^3}(-n) \to F] \in Q \mid \text{the sheaf } F \text{ is semistable} \} \). Consider the irreducible component \( Y_m \) containing the point \([F_t]\) in the Gieseker–Maruyama moduli scheme of semistable sheaves on \( \mathbb{P}^3 \) with Hilbert polynomial \( P_m \), where \( F_t := \gamma_{t*}\mathcal{O}_{\Gamma_t} \). It is known ([13], Ch. 4) that \( Y_m = Q^{ss}/G \), where \( G = \text{GL}(M_n) \) acts on \( Q \) via its action on \( \mathbb{k}^{2m} \). We put \( K_t := \ker(\mathbb{k}^{M_n} \otimes \mathcal{O}_{\mathbb{P}^3} \to F_t) \). Then it is known that \( T_{[e_t]}Q = \text{Hom}(K_t, F_t) \), and an elementary computation yields \( \dim T_{[e_t]}Q = M_n^2 + 6m = \dim Q \), that is, \( Q^{ss} \) is non-singular at the point \([e_t]\). On the other hand, a similar computation yields that \( \text{Stab}_G[e_t] = \text{Aut}(F_t) = (k^*)^{2m} \), whence the canonical projection \( Q^{ss} \to Y_m \) is a smooth morphism with fibre \( \text{GL}(M_n)/(k^*)^{2m} \) in the neighbourhood of the point \([e_t]\). Hence \( Y_m \) has dimension 8m and is smooth at every point \([F_t]\) for \( t \in \mathbf{H}_0' \). We similarly obtain that \( Y_m \) has dimension 8m and is smooth at every point \([F_t]\) for \( t \in D_0^r \). Moreover, the family \( \gamma_{*}\mathcal{O}_{\Gamma} \) determines a morphism \( \psi : \mathbf{H} \to Y_m, \ t \mapsto [\gamma_{t*}\mathcal{O}_{\Gamma_t}] \). It follows directly from the definition of \( \mathbf{H}_0 \) that the restriction \( \psi_0 := \psi|_{\mathbf{H}_0} : \mathbf{H}_0 \to Y_m^0 := \psi(\mathbf{H}_0) \) is bijective. Since \( Y_m^0 \) is smooth, we obtain that the morphism \( \psi_0 \) is an isomorphism, that is, we have an inverse morphism \( \psi_0^{-1} : Y_m^0 \to \mathbf{H}_0 \).

Given an arbitrary point \( z \in Z_m \) and any monomorphism \( j : H_{m-1}^0 \to H_m \), we consider the diagram (9.30) and put \( C_z := \text{coker}(t_{s_z}) \). If \( z \in Z_{m}^0 \) (see definition (9.31)), then a direct computation yields that \( \chi(C_z(n)) = P_m(n) \). Thus the set \( Z_{m}^{ss} := \{ z \in Z_{m}^0 \mid [C_z] \in Y_{m}^{ss} \} \) is well defined and open in \( Z_m \).

Remark 11.1. The set \( Z_{m}^{ss} \) is non-empty. Indeed, the point \( z \in Z \) (defined in (9.6)) lies in \( Z_{m}^{ss} \cap Z \) by (9.8) and (9.14). Moreover, there is a morphism

\[
h_m : Z_{m}^{0} \to \mathbf{H}_0, \quad z \mapsto \psi_0^{-1}([C_z]).\]

Let \( Z_c \) be an arbitrary irreducible component of \( Z \) satisfying, in the notation (9.31), the condition

\[
Z_{c}^1(j) := Z_{m}^1(j) \cap Z_c \neq \emptyset \] (11.3)

for some monomorphism \( j : H_{m-1} \to H_m \) and the condition

\[
Z_{c}^0 := Z_{m}^{0} \cap Z_c \neq \emptyset. \] (11.4)

We put \( P(H_m)_c := \{ j \in P(H_m^\vee) \mid Z_{m}^1(j) \neq \emptyset \} \). By construction, \( P(H_m)_c \) is an open subset of \( P(H_m^\vee) \), and condition (11.3) means that

\[
P(H_m^\vee)_c \neq \emptyset. \] (11.5)
We assume that, for an arbitrarily chosen monomorphism \( j \in P(H^\vee_m)_c \), there is a finite birational morphism \( \sigma: \tilde{Z}_c \to Z_c \) such that
\[
\text{codim}_{Z_c}(Z_c \setminus \sigma(\tilde{Z}_c)) \geq 2. \tag{11.6}
\]
Let \( j \in P(H^\vee_m)_c \), so that \( Z_c^0 \cap Z_c^1(j) \) is a non-empty (and hence open and dense) subset of \( Z_c \). Therefore,
\[
\tilde{Z} = \tilde{Z}(j) := \sigma^{-1}(Z_c^0 \cap Z_c^1(j)) \tag{11.7}
\]
is an open dense set in \( \tilde{Z}_c \) with projections
\[
h := (h_m \circ \sigma)|_{\tilde{Z}}: \tilde{Z} \to H_0, \quad \sigma := \sigma|_{\tilde{Z}}: \tilde{Z} \to Z_c, \quad \lambda := \lambda_j \circ \sigma: \tilde{Z} \to S_{2m-1}. \tag{11.8}
\]
Consider the variety \( \tilde{\Gamma} := \Gamma \times_H \tilde{Z} \) with projection \( \tilde{p}: \tilde{\Gamma} \to \tilde{Z} \).

**Proposition 11.2.** Let \( Z_c \) be an irreducible component of \( Z \) satisfying conditions (11.4) and (11.5), let \( \sigma: \tilde{Z}_c \to Z_c \) be a finite birational morphism satisfying (11.6), and let \( j \in P(H^\vee_m)_c \). Consider the variety \( \tilde{Z} \) defined in (11.7). Then the following assertions hold.

(i) There is a morphism \( \theta: \tilde{Z} \to T_0 := \rho^{-1}(H_0) \) depending on \( j \) such that
\[
h = \rho \circ \theta. \tag{11.9}
\]

(ii) There are isomorphisms of sheaves on \( \tilde{Z} \):
\[
\tilde{A} := \tilde{p}_*O_{\tilde{\Gamma}} \simeq h^*A \simeq H_{2m} \otimes O_{\tilde{Z}}. \tag{11.10}
\]

(iii) The exact triple \( 0 \to O_{\tilde{Z}} \stackrel{j}{\to} H^\vee_{2m} \otimes O_{\tilde{Z}} \to H^\vee_{2m-1} \otimes O_{\tilde{Z}} \to 0 \) is obtained by applying the functor \( \theta^* \) to the triple (10.20). In particular, we have isomorphisms of sheaves \( \theta^*A_T \simeq H_{2m-1} \otimes O_{\tilde{Z}}, h^*A \simeq H_{2m} \otimes O_{\tilde{Z}} \) and, therefore, isomorphisms of varieties
\[
\tilde{M} := \text{Isom}(\theta^*A_T, H_{2m-1} \otimes O_{\tilde{Z}}) \simeq M \times T \tilde{Z}, \quad \tilde{Y} := \text{Isom}(h^*A, H_{2m} \otimes O_{\tilde{Z}}) \simeq Y \times_H \tilde{Z} \tag{11.11}
\]
and equalities of composite morphisms
\[
f_M \circ \theta_M = \tilde{\lambda} \circ \tau_{\tilde{Z}}, \quad f_Y \circ \theta_Y = \tilde{\sigma} \circ \zeta_{\tilde{Z}}, \tag{11.12}
\]
where \( \tilde{M} \xrightarrow{\theta_M} \tilde{Z} \) and \( \tilde{Y} \xrightarrow{\theta_Y} \tilde{Z} \) are the projections, and \( f_Y, f_M \) are taken from (10.18), (10.23) respectively.

**Proof.** (i) We distinguish two cases: either \( h(\tilde{Z}) \not\subset D^0_H \) or \( h(\tilde{Z}) \subset D^0_H \).

Suppose that \( h(\tilde{Z}) \not\subset D^0_H \). Applying the functor \( (\text{id}_{p^3} \times \tilde{\sigma})^* \) to the diagram (9.29), we get
\[
\begin{array}{ccccccccc}
0 & \to & H_{2m-1} \otimes O_{p^3}(-1) \otimes O_{\tilde{Z}} & \xrightarrow{s_{\tilde{Z}}(j)} & W_{4m} \otimes O_{p^3 \times \tilde{Z}} & \xrightarrow{t_{s_{\tilde{Z}}(j)}} & H^\vee_{2m-1} \otimes O_{p^3(1)} \otimes O_{\tilde{Z}} & \to & 0\\
& & \downarrow j & & \downarrow & & \downarrow j^\vee & & \\
0 & \to & H_{2m} \otimes O_{p^3}(-1) \otimes O_{\tilde{Z}} & \xrightarrow{s_{\tilde{Z}}} & W_{4m} \otimes O_{p^3 \times \tilde{Z}} & \xrightarrow{t_{s_{\tilde{Z}}}} & H^\vee_{2m} \otimes O_{p^3(1)} \otimes O_{\tilde{Z}} & \to & 0
\end{array} \tag{11.13}
\]
This diagram induces a section $s$ of the sheaf $\tilde{E} \otimes \mathcal{O}_{\mathbb{P}^3(1)} \boxtimes \mathcal{O}_{\tilde{Z}}$, where $\tilde{E} := \ker(t(s_{\tilde{Z}}))//\text{im}(s_{\tilde{Z}})$. This section extends to an exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3(-1)} \otimes \mathcal{O}_{\tilde{Z}} \to E \to \mathcal{I}_{L, \mathbb{P}^3(1)} \otimes \mathcal{O}_{\mathbb{P}^3(1)} \otimes \mathcal{O}_{\tilde{Z}} \to 0. \quad (11.14)$$

Here, by the assumption that $h(\tilde{Z}) \subset \mathbb{H}_0 \setminus \mathbf{D}_H^0$, the scheme $L$ is a reduced subscheme of $\mathbb{P}^3 \times \tilde{Z}$ such that, for every point $z \in \tilde{Z}$, the scheme $L_z = L \cap \mathbb{P}^3 \times \{z\}$ is of the form $L_z = l_1 \cup \cdots \cup l_{2m}$, where $l_1, \ldots, l_{2m}$ are reduced lines in $\mathbb{P}^3$. It is easy to see that $\chi(L_z(n)) = P_m(n)$. By the universality of the extension $(10.19)$, the triple $(11.14)$ determines a morphism $\theta: \tilde{Z} \to T$ such that the triple $(11.14)$ is obtained from $(10.19)$ by applying the functor $(\text{id}_{\mathbb{P}^3} \times \theta)^*$ (see [18]). Here, by construction, $\theta$ satisfies $(11.9)$ and $\text{im} \theta \subset \rho^{-1}(\mathbb{H}_0 \setminus \mathbf{D}_H^0) \subset T_0$, so that $\theta$ is the desired morphism.

Suppose that $h(\tilde{Z}) \subset \mathbf{D}_H^0$. Applying the functor $(\text{id}_{\mathbb{P}^3} \times \tilde{\sigma})^*$ to the diagram $(9.29)$, we get the diagram $(11.13)$. This diagram induces a section $s$ of the sheaf $\tilde{E} \otimes \mathcal{O}_{\mathbb{P}^3(1)} \boxtimes \mathcal{O}_{\tilde{Z}}$, where $\tilde{E} := \ker(t(s_{\tilde{Z}}))//\text{im}(s_{\tilde{Z}})$, and this section extends to an exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3(-1)} \otimes \mathcal{O}_{\tilde{Z}} \to E \to \mathcal{I}_{L, \mathbb{P}^3(1)} \otimes \mathcal{O}_{\mathbb{P}^3(1)} \otimes \mathcal{O}_{\tilde{Z}} \to 0. \quad (11.14)$$

Here, by the assumption that $h(\tilde{Z}) \subset \mathbf{D}_H^0$, the scheme $\tilde{L}$ is a reduced subscheme of $\mathbb{P}^3 \times \tilde{Z}$ such that, for any point $z \in \tilde{Z}$, the scheme $\tilde{L}_z = \tilde{L} \cap \mathbb{P}^3 \times \{z\}$ is of the form $\tilde{L}_z = l_1 \cup l_2 \cup \cdots \cup l_{2m}$, where $l_1, \ldots, l_{2m}$ are reduced lines in $\mathbb{P}^3$ and, moreover, $l_1$ and $l_2$ intersect each other in a point, say, $x(z)$. It is easy to see that $\chi(\tilde{L}_z(n)) = P_m(n) - 1$. Moreover, there is a unique scheme $L$ which contains $\tilde{L}$ such that, for any point $z \in \tilde{Z}$, the scheme $L_z = L \cap \mathbb{P}^3 \times \{z\}$ has Hilbert polynomial $\chi(L_z(n)) = P_m(n)$, and which is uniquely determined by the property that $\ker(O_{L_z} \to O_{L_z}) = k(x(z))$. In other words, $L_z$ is a scheme with an embedded point $x(z)$. The embedding $\tilde{L} \hookrightarrow L$ induces an embedding of the ideal sheaves $\mathcal{I}_{L, \mathbb{P}^3(1)} \hookrightarrow \mathcal{I}_{\tilde{L}, \mathbb{P}^3(1)} \otimes \mathcal{O}_{\tilde{Z}}$. We put $E := \varepsilon^{-1}(\mathcal{I}_{\tilde{L}, \mathbb{P}^3(1)} \otimes \mathcal{O}_{\mathbb{P}^3(1)} \otimes \mathcal{O}_{\tilde{Z}})$. In this notation, the triple $(11.14)$ is again exact and, as above, it determines a desired morphism $\theta: \tilde{Z} \to T$. In this case, by construction, we have $\text{im} \theta \subset \rho^{-1}(\mathbf{D}_H^0) \subset T_0$.

(ii) The isomorphisms $(11.10)$ follow from the definition of the variety $\tilde{\Gamma}$ since the construction of the sheaf $A \simeq p_* \mathcal{O}_T(0,1)$ commutes with the base change (Proposition 10.9 (i)).

(iii) The assertion on the isomorphism of triples follows from part (ii) and the construction of the sheaf $A_T$. The other assertions in (iii) follow. \(\square\)

11.2. Description of the morphism $\lambda_j$ for components of $Z_m$. Let $Z_c$ be an arbitrary irreducible component of $Z_m$ satisfying conditions $(11.4)$ and $(11.5)$. In this subsection, for an arbitrary $j \in P(H_m^\vee)$, we construct a variety $\tilde{Z}_c$ and a finite birational morphism $\sigma: \tilde{Z}_c \to Z_c$ satisfying condition $(11.6)$ and prove that the morphism $\lambda = \lambda_j \circ \sigma: \tilde{Z}_c \to \mathbf{S}_{2m-1}$ is smooth as a morphism onto its image (see Proposition 11.3).

Let $\tilde{v}: Z_c^\nu \to Z_c$ be the normalization of $Z_c$. We put $Z_c^\ast := Z_c^\nu \setminus \text{Sing} Z_c^\nu$. By construction, there is a finite birational morphism

$$\nu = \tilde{v}|_{Z_c^\ast}: Z_c^\ast \to \nu(Z_c^\ast), \quad \text{codim}_{Z_c}(Z_c \setminus \nu(Z_c^\ast)) \geq 2. \quad (11.15)$$
Over $Z^*_c$ we have the following commutative diagram of complexes obtained from (9.29) by applying the functor $(\text{id}_{p^3} \times i)^*$, where $i$ is the composite $Z^*_c \xrightarrow{\nu} Z_c \hookrightarrow Z_m$:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H_{2m-1} \otimes O_{p^3}(-1) \otimes O_{Z^*_c} & \xrightarrow{s(j)} & W_{4m} \otimes O_{p^3 \times Z^*_c} & \xrightarrow{t^*_s(j)} & H_{2m-1}^\vee \otimes O_{p^3}(-1) \otimes O_{Z^*_c} & \rightarrow & 0 \\
0 & \rightarrow & H_{2m} \otimes O_{p^3}(-1) \otimes O_{Z^*_c} & \xrightarrow{s} & W_{4m} \otimes O_{p^3 \times Z^*_c} & \xrightarrow{t^*_s} & H_{2m}^\vee \otimes O_{p^3}(-1) \otimes O_{Z^*_c} & \rightarrow & 0
\end{array}
\]

(11.16)

In accordance with (11.7), we consider the open dense subset $\tilde{Z} = \sigma^{-1}(Z^0_c \cap Z^1_c(j))$ of $Z^*_c$. By Proposition 11.2 (i) there is a morphism $\theta: \tilde{Z} \rightarrow T_0$ such that $\rho \circ \theta = h$, where the morphism $h$ is given by the formula (11.8).

Let $\text{pr}_{2*}: \mathbb{P}^3 \times \tilde{Z} \rightarrow \tilde{Z}$ be the projection. Applying the functor $\text{pr}_{2*}$ to the left square of the diagram (11.16) twisted by $O_{p^3}(-1) \otimes O_{\mathcal{Z}}$, we obtain a flag $U_{\tilde{Z}}$ of vector subbundles of the trivial bundle $W_{4m} \otimes V^\vee \otimes O_{\mathcal{Z}}$:

\[
U_{\tilde{Z}} = \left\{ H_{2m-1} \otimes O_{\mathcal{Z}} \hookrightarrow H_{2m} \otimes O_{\mathcal{Z}} \rightarrow W_{4m} \otimes V^\vee \otimes O_{\mathcal{Z}} \right\}.
\]

Consider the flag variety $\mathcal{F} := \mathcal{F}(2m-1, 2m, W_{4m} \otimes V^\vee)$ and the universal property of flags $U_{\mathcal{F}} = \{ S_{2m-1} \hookrightarrow S_{2m} \hookrightarrow W_{4m} \otimes V^\vee \otimes O_{\mathcal{F}} \}$ on $\mathcal{F}$. By the universal property of $\mathcal{F}$ there is a morphism $f' : \tilde{Z} \rightarrow \mathcal{F}$ such that $U_{\tilde{Z}} = (f')^*(U_{\mathcal{F}})$. In particular, $H_{2m-1} \otimes O_{\mathcal{Z}} = (f')^*(S_{2m-1})$, $H_{2m} \otimes O_{\mathcal{Z}} = (f')^*(S_{2m})$. Since $\tilde{Z}$ is an open dense subset of $Z^*_c$, the morphism $f'$ extends to a rational morphism $f': Z^*_c \dashrightarrow \mathcal{F}$. Let $\tilde{Z}^*_c$ be the closure of the graph of $f'$ in $Z^*_c \times \mathcal{F}$, and let $Z^*_c \xrightarrow{\tilde{\sigma}} \tilde{Z}^*_c \xrightarrow{f'} \mathcal{F}$ be the projections. Since $Z^*_c$ is normal, there is an open dense subset $\tilde{Z}^*_c$ of $\tilde{Z}^*_c$ such that $\tilde{\sigma}|_{\tilde{Z}^*_c} : \tilde{Z}^*_c \xrightarrow{\sim} \tilde{\sigma}(\tilde{Z}^*_c)$ is an isomorphism and codim$_{Z^*_c}(Z^*_c \setminus \tilde{\sigma}(\tilde{Z}^*_c)) \geq 2$.

In view of (11.15), this yields a finite birational morphism

\[
\sigma := \nu \circ \tilde{\sigma} : \tilde{Z}^*_c \rightarrow Z_c, \quad \text{codim}_{Z^*_c}(Z^*_c \setminus \sigma(\tilde{Z}^*_c)) \geq 2. \quad (11.17)
\]

We put $f := \tilde{f}|_{\tilde{Z}^*_c}$, $S_{2m-1} := f^*S_{2m-1}$, $S_{2m} := f^*S_{2m}$. The equality $U_{\tilde{Z}} = (f')^*(U_{\mathcal{F}})$ extends to a diagram of morphisms of $O_{Z^*_c}$-sheaves

\[
\begin{array}{ccccccccc}
H_{2m-1} \otimes O_{Z^*_c} & \xrightarrow{\iota} & S_{2m-1} & \xrightarrow{w} & W_{4m} \otimes V^\vee \otimes O_{Z^*_c} \\
\downarrow j & & \downarrow j & & \downarrow j \\
H_{2m} \otimes O_{Z^*_c} & \xrightarrow{\iota'} & S_{2m} & \xrightarrow{w'} & W_{4m} \otimes V^\vee \otimes O_{Z^*_c}
\end{array}
\]

(11.18)

where $w$ and $w'$ are the tautological embeddings of subbundles. The upper row of (11.18) induces a diagram whose upper row coincides with the result of applying the functor $(\text{id}_{p^3} \times \sigma)^*$ to the upper row of (9.29):

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H_{2m-1} \otimes O_{p^3}(-1) \otimes O_{Z^*_c} & \xrightarrow{s'(j)} & W_{4m} \otimes O_{p^3 \times Z^*_c} & \xrightarrow{t^*_s'(j)} & H_{2m-1}^\vee \otimes O_{p^3}(-1) \otimes O_{Z^*_c} & \rightarrow & 0 \\
0 & \rightarrow & O_{p^3}(-1) \otimes S_{2m-1} & \xrightarrow{w} & W_{4m} \otimes O_{p^3 \times Z^*_c} & \xrightarrow{t^*_w} & O_{p^3}(1) \otimes S_{2m-1}^\vee & \rightarrow & 0
\end{array}
\]

(11.19)
Here \( s'(j) := \left( \text{id}_{\mathcal{P}^3} \times \sigma \right)^* s(j) \). By construction, \( \tilde{Z}_c \) contains an open dense subset \( \tilde{\sigma}^{-1}(\tilde{Z}) \cong \tilde{Z} \) (11.20) such that the restriction of the diagram (11.19) to \( \mathbb{P}^3 \times \tilde{Z} \) is an isomorphism of complexes. In what follows we identify \( \tilde{\sigma}^{-1}(\tilde{Z}) \) with \( \tilde{Z} \).

Consider the lower complex in (11.19) and put \( K := \text{coker} \, w, \, E_w := \ker t \, w / \text{im} \, w, \, \mathcal{O}(1, 0) := \mathcal{O}_{\mathcal{P}^3}(1) \otimes \mathcal{O}_{\tilde{Z}_c}, \, K(1, 0) := \mathcal{K} \otimes \mathcal{O}(1, 0), \, E_w(1, 0) := E_w \otimes \mathcal{O}(1, 0). \) Twisting the lower complex in (11.19) by \( \mathcal{O}(1, 0) \), we obtain two exact triples:

\[
0 \to \mathcal{O}_{\mathcal{P}^3} \otimes S_{2m-1} \to W_{4m} \otimes \mathcal{O}(1, 0) \to K(1, 0) \to 0, \\
0 \to E_w(1, 0) \to K(1, 0) \to \mathcal{O}_{\mathcal{P}^3}(2) \otimes S_{2m-1}^\vee.
\]

Let \( \text{pr}_2: \mathbb{P}^3 \times \tilde{Z}_c \to \tilde{Z}_c \) be the projection. Applying the functor \( R^\bullet \, \text{pr}_2 \) to the triples above, we get exact triples

\[
0 \to S_{2m-1} \overset{w}{\to} W_{4m} \otimes V^\vee \otimes \mathcal{O}_{\tilde{Z}_c} \to \text{pr}_2^* K(1, 0) \to 0, \\
0 \to \text{pr}_2^* E_w(1, 0) \to \text{pr}_2^* K(1, 0) \to S_{2m-1}^\vee \otimes S_{2m-1}^\vee \otimes \mathcal{O}_{\tilde{Z}_c}.
\]

Since \( w \) in the first triple is a subbundle morphism, the sheaf \( \text{pr}_2^* K(1, 0) \) is locally free. Therefore it follows from the second triple that the sheaf \( \text{pr}_2^* E_w(1, 0) \) is reflexive, being the kernel of a morphism of locally free sheaves. Since the morphism \( \tilde{Z} \to \tilde{Z}_c \) is birational and the open subset \( Z_c(j) \) of \( Z_c \) is dense, we obtain by the base change that \( \text{pr}_2^* E_w(1, 0)|_{\tilde{Z}} \) is a sheaf of rank 1. Hence \( \text{pr}_2^* E_w(1, 0) \) is an invertible sheaf.

Note that the decomposition (9.24) induces a commutative diagram of complexes

\[
\begin{array}{ccc}
0 & \longrightarrow & H_m \otimes \mathcal{O}(-1, 0) \\
\downarrow & & \downarrow \text{id} \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{P}^3}(-1) \otimes S_{2m-1} \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & W_{4m} \otimes \mathcal{O}_{\mathcal{P}^3 \times \tilde{Z}_c} \\
\downarrow \text{id}_{W_{4m}} & & \downarrow \text{id}_{W_{4m}} \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{P}^3}(1) \otimes S_{2m-1}^\vee \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & H_m \otimes \mathcal{O}(1, 0) \\
\downarrow \text{id} & & \downarrow \text{id} \\
0 & \longrightarrow & \mathcal{O}_{\mathcal{P}^3}(1) \otimes S_{2m-1}^\vee \\
\end{array}
\]

(11.21)

The upper complex of diagram (11.21) is a monad and, by construction, \( E_{2m} := \ker t \, \mathfrak{w}_0 / \text{im} \, \mathfrak{w}_0 \) is a locally free sheaf of rank \( 2m \) such that, for every point \( z' \in \tilde{Z}_c \), the sheaf \( E_{2m} \otimes k(z') \) on \( \mathbb{P}^3 \) is a vector bundle \( E_{2m}(D^{-1}) \), where \( (D, \varphi) = \sigma(z') \). Accordingly, \( \iota_0 = \text{id}_{\mathcal{O}_{\mathcal{P}^3}} \otimes i_0 \), where \( i_0: H_m \otimes \mathcal{O}_{\tilde{Z}_c} \to S_{2m-1} \) is a subbundle morphism, so that \( S_{m-1} := S_{2m-1} / H_m \otimes \mathcal{O}_{\tilde{Z}_c} \) is a locally free \( \mathcal{O}_{\tilde{Z}_c} \)-sheaf. Thus the diagram (11.21) induces the complex

\[
0 \to \mathcal{O}_{\mathcal{P}^3}(-1) \otimes S_{m-1} \overset{w_1}{\longrightarrow} E_{2m} \overset{i_0 \otimes i_0}{\longrightarrow} \mathcal{O}_{\mathcal{P}^3}(1) \otimes S_{m-1}^\vee \to 0,
\]

where the sheaf \( \ker t \, \mathfrak{w}_1 / \text{im} \, \mathfrak{w}_1 \) coincides with the sheaf \( E_w \). Note that the display (D) of this complex twisted by \( \mathcal{O}(1, 0) \), when restricted to \( \mathbb{P}^3 \times \{ \tilde{z} \}, \, \tilde{z} \in \tilde{Z} \),
coincides by construction with the display (9.33) for the point \( z = \sigma(\bar{z}) \). Applying the functor \( R^* \text{pr}_2^* \) to the display (D), we get an exact triple

\[
0 \rightarrow S'_m \xrightarrow{\zeta} W_{5m} \xrightarrow{\tau} S^\vee_{m-1} \otimes S^2 V^\vee \otimes \mathcal{O}_{\overline{Z}_c},
\]

(11.22) where \( W_{5m} = \text{pr}_2^* \mathcal{E}_2(1, 0) \) is a locally free sheaf of rank \( 5m \) and the sheaf \( S'_m \) is defined as the extension \( 0 \rightarrow S_{m-1} \rightarrow S'_m \rightarrow \text{pr}_2^* \mathcal{E}_w(1, 0) \rightarrow 0 \). Since the sheaf \( \text{pr}_2^* \mathcal{E}_w(1, 0) \) is invertible, \( S'_m \) is a locally free sheaf of rank \( m \). Note that we have the universal subbundle morphism on \( (S^\vee_m)^0 \):

\[
b_{\text{univ}} : H_m \otimes \mathcal{O}(S^\vee_m)^0 \hookrightarrow H_m^\vee \otimes \wedge^2 V^\vee \otimes \mathcal{O}(S^\vee_m)^0
\]

(this is a relative version of the homomorphism \( \delta(D^{-1}) : H_m \hookrightarrow H_m^\vee \otimes \wedge^2 V^\vee \)) and the following triple is exact by construction:

\[
0 \rightarrow H_m \otimes \mathcal{O}_{\overline{Z}_c} \xrightarrow{b} H_m^\vee \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\overline{Z}_c} \rightarrow W_{5m} \rightarrow 0,
\]

where \( b \) is the pullback of the morphism \( b_{\text{univ}} \) under the composite of the projections \( \overline{Z}_c \xrightarrow{\sigma} Z \) and \( Z \rightarrow (S^\vee_m)^0, (D, \varphi) \hookrightarrow D \). This triple together with the triple (11.22) fits into a diagram of morphisms of bundles

\[
\begin{array}{cccccc}
H_m \otimes \mathcal{O}_{\overline{Z}_c} & \xrightarrow{\zeta} & W_{5m} & \xrightarrow{\tau} & H^\vee_{m-1} \otimes S^2 V^\vee \otimes \mathcal{O}_{\overline{Z}_c} \\
V_{2m} & \xrightarrow{\lambda} & H_m^\vee \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\overline{Z}_c} & \xrightarrow{\lambda} & H^\vee_{m-1} \otimes S^2 V^\vee \otimes \mathcal{O}_{\overline{Z}_c} \\
S'_m & \xrightarrow{\zeta} & W_{5m} & \xrightarrow{\tau} & H^\vee_{m-1} \otimes S^2 V^\vee \otimes \mathcal{O}_{\overline{Z}_c}
\end{array}
\]

(11.23)

where \( V_{2m} \) is a subbundle of rank \( 2m \) in \( H^\vee_m \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\overline{Z}_c} \). The restriction of the diagram (11.23) to the point \( \bar{z} \in \overline{Z} \) is by construction equal to the right-hand side of the diagram (9.40) for \( z = \sigma(\bar{z}) \). Since \( \overline{Z}_c \) is irreducible, we get the following proposition.

**Proposition 11.3.** Let \( Z_c \) be an arbitrary irreducible component of \( Z_m \) satisfying conditions (11.4) and (11.5). Take \( j \in P(H^\vee_m)_c \). Then one can find a smooth variety \( \overline{Z}_c \), which depends on \( j \), and a finite birational morphism \( \sigma : \overline{Z}_c \rightarrow Z_c \) satisfying condition (11.6) such that the fibre of the morphism \( \lambda = \lambda_j \circ \sigma : \overline{Z}_c \rightarrow S_{2m-1} \) has the following description:

\[
\lambda^{-1}(\lambda(\bar{z})) = \sigma^{-1}(\lambda_j^{-1}(\lambda(\bar{z}))) \overset{\text{open}}{\hookrightarrow} V_{2m} \otimes k(\bar{z}) \cong \mathbb{A}^{2m}, \quad \bar{z} \in \overline{Z}_c.
\]

(11.24)

**11.3. The intersection of \( Z \) with other components of \( Z_m \).** In this subsection we show that \( Z \) has no common divisors with other components of \( Z_m \) (see Proposition 11.8 in the end of the subsection).

We first note that by Lemma 9.4 and Remark 11.1 the variety \( Z_c = Z \) and the element \( j \in P(H^\vee_m)^* \) satisfy the hypotheses of Propositions 11.2, 11.3. Writing
Since \( \dim Z' := \tilde{Z}_c, \sigma' := \sigma, \) and \( Z_b := \tilde{Z} \) in these propositions, we arrive at the following conclusions.

(i) There are a smooth variety \( Z' \) and a finite birational morphism \( \sigma' : Z' \to Z \) (both depend on \( j \)) such that

\[
\text{codim}_Z (Z \setminus \sigma'(Z')) \geq 2 \tag{11.25}
\]

and the fibre of the morphism \( \tilde{\lambda} = \lambda_j \circ \sigma' : Z' \to S_{2m-1} \) satisfies (11.24).

(ii) The set \( Z_b = (\sigma')^{-1}(Z \cap Z_{m0}^0 \cap Z_{m1}^1 (j)) \) is open and dense in \( Z' \), and one can find a morphism \( \theta_b : Z_b \to T_0 \) and varieties \( M_b := Z_b \times_T M \) and \( Y_b := Z_b \times_H Y \) with projections \( Z_b \xrightarrow{\tau_b} M_b \xrightarrow{\delta_b} M \) and \( Z_b \xrightarrow{\zeta_b} Y_b \xrightarrow{\delta_b} Y \) satisfying the relations \( f_M \circ \delta_b = \tilde{\lambda} \circ \tau_b \) and \( f_Y \circ \delta_b = \tilde{\sigma} \circ \zeta_b \).

Consider a rational morphism \( \theta' : Z' \to T \) such that \( \theta'|_{Z_b} \) coincides with the morphism \( \theta_b \). Since \( Z' \) is normal, there is a subset \( B \) of \( Z' \) with the conditions

\[
\text{codim}_{Z'} B \geq 2, \quad Z_a := Z' \setminus B \supset Z_b
\]

such that \( \theta_a := \theta'|_{Z_a} : Z_a \to T \) is a morphism and \( \theta_a|_{Z_b} = \theta_b \). The condition (11.25) and the last inequality imply that

\[
\text{codim}_Z (Z \setminus \sigma_a(Z_a)) \geq 2, \quad \sigma_a := \sigma'|_{Z_a}. \tag{11.26}
\]

Consider the varieties \( M_a := Z_a \times_T M \) and \( Y_a := Z_a \times_H Y \) with projections \( Z_a \xrightarrow{\tau_a} M_a \xrightarrow{\delta_a} M \) and \( Z_a \xrightarrow{\zeta_a} Y_a \xrightarrow{\delta_a} Y \). Here the equality \( f_M \circ \delta_a = \tilde{\lambda} \circ \tau_a \) by construction is the equality \( f_M \circ \delta_a|_{M_a} = \tilde{\lambda} \circ \tau_a|_{M_b} \). Since \( M_b \) is an open dense subset of \( M_a \) and \( M_a \) and \( S_{2m-1} \) are integral reduced schemes, it follows from the above equality of morphisms that the morphisms \( \tilde{\lambda} \circ \tau_a \) and \( f_M \circ \delta_a \) also coincide. Moreover, \( \tilde{\sigma} \circ \zeta_a \) and \( f_Y \circ \delta_a \) coincide for similar reasons. Thus,

\[
\tilde{\lambda} \circ \tau_a = f_M \circ \delta_a, \quad \tilde{\sigma} \circ \zeta_a = f_Y \circ \delta_a. \tag{11.27}
\]

The equalities (11.27) yield that

\[
\tilde{\lambda}(Z_a) = (\tilde{\lambda} \circ \tau_a)(M_a) = (f_M \circ \delta_a)(M_a) \subset f_M(M) = \overline{Mf}, \tag{11.28}
\]

\[
\tilde{\sigma}(Z_a) = (\tilde{\sigma} \circ \zeta_a)(M_a) = (f_Y \circ \delta_a)(M_a) \subset f_Y(Y). \tag{11.29}
\]

These embeddings together with diagram (10.25) fit into the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f_Y} & f_Y(Y) \\
\mu_j & & \sigma_a(Z_a) \\
\downarrow & & \downarrow \lambda_j \\
M & \xrightarrow{f_M} & \overline{Mf} \\
\end{array}
\] \hspace{1cm} \lambda(Z_a) \quad (11.30)

\textbf{Remark 11.4.} Since \( \dim Y = \dim H + \dim GL(H_{2m}) = 8m + 4m^2 = \dim Z \) and \( \sigma_a(Z_a) \) is dense in \( Z \) by (11.26), it follows from the inclusion \( \sigma_a(Z_a) \subset f_Y(Y) \) and the irreducibility of \( Y \) that \( \overline{f_Y(Y)} = \overline{Z} \), where both closures are taken in \( S_{2m} \). (We recall that \( Z \) lies in \( S_{2m} \) by (9.27).)
We can deduce the following result from (11.28) in view of Lemma 10.12.

**Lemma 11.5.** Let \( j \in P(H'_m)^* \). Then, in the notation introduced above, the image of the variety \( Z_a \) under the morphism \( \tilde{\lambda} \) is a dense subset of the variety \( \tilde{M} \) defined in Lemma 10.12. Therefore \( \theta_a(Z_a) \) is dense in \( \mathbf{T} \).

**Proof.** Indeed, \( \tilde{\lambda}(Z_a) \) contains the dense subset \( \lambda_j(Z(j)) \subset M_{12m-1}^H(\xi) \subset M_{12m-1}^H \) by (9.32). Since \( \dim Z' = \dim Z = 4m(m+2) \) and \( \dim M_{12m-1}^H = \dim \text{GL}(2m-1) + \dim M_{12m-1}^H = 4m^2 + 6m \), it follows from (11.24) that \( \dim \tilde{\lambda}(Z_a) = \dim Z_a - 2m = 4m(m+2) - 2m = \dim M_{12m-1}^H \), whence \( \tilde{\lambda}(Z_a) \) is dense in \( M_{12m-1}^H \) and, therefore, in \( \tilde{M} \). Using the equality \( \tilde{\lambda} = f_M \circ \tilde{\theta}_a \) and the irreducibility of \( M \), we obtain that \( \tilde{\theta}_a(M_a) \) is dense in \( M \) and, therefore, the set \( \theta_a(Z_a) = \theta_a(\tau_a(M_a)) = \tau(\tilde{\theta}_a(M_a)) \) is dense in \( \mathbf{T} \). \( \square \)

Consider the following subsets of \( Z_a \):

\[
Z := \theta_a^{-1}(\mathbf{T}^*), \quad D := \theta_a^{-1}(\mathbf{D}_T), \quad D' := \theta_a^{-1}(\mathbf{D}'_T).
\]

We have \( Z_a = Z \cup D \cup D' \) by Remark 10.10, and \( Z \) is dense in \( Z_a \) by Lemma 11.5. It follows from the equality \( \lambda \circ \tau_a = f_M \circ \tilde{\theta}_a \) that

\[
Z = \tilde{\lambda}^{-1}(f_M(\tau^{-1}(\mathbf{T}^*))), \quad D = \tilde{\lambda}^{-1}(f_M(\tau^{-1}(\mathbf{D}_T))), \quad D' = \tilde{\lambda}^{-1}(f_M(\tau^{-1}(\mathbf{D}'_T))).
\]

We note that the morphism \( \rho: \mathbf{T} \to \mathbf{H} \) is a projective bundle by Proposition 10.9 (i). Therefore, using (11.31), Proposition 11.3, Lemma 11.5 and Remark 10.10, we conclude that if \( D \) and \( D' \) are non-empty, then they are irreducible divisors in \( Z_a \).

**Corollary 11.6.** If \( D \) is a divisor in \( Z_a \) lying in \( Z_a \setminus Z \), then one of the following cases holds.

(i) \( D = D \) and \( (\rho \circ \theta_a)(D) \) is dense in \( \mathbf{D}_H \).

(ii) \( D = D' \) and \( \theta_a(D) \) is dense in \( \mathbf{D}'_T \).

**Proof.** Indeed, one of the following cases holds: \( D = D \) or \( D = D' \). It suffices to consider the first since a similar argument works in the second.

In accordance with (11.31) we put \( D_M := \tau^{-1}(\mathbf{D}_T), D_S := f_M(\mathbf{D}_M), D_S := D_S \cap \tilde{\lambda}_j(Z_b) \), so that \( D = \tilde{\lambda}^{-1}(D_S) \). We obtain from the definition of \( D_M \) that \( \dim D_M = \dim D_S + (2m-1)^2 = 4m^2 + 6m - 1 \). On the other hand, since \( D = \dim D = \dim Z - 1 = 4m(m+2) - 1 \) and \( \tilde{\lambda} \) is a smooth morphism with 2m-dimensional fibre, we have \( \dim D_S = 4m^2 + 6m - 1 = \dim M - 1 \). It follows that \( D_S \) is dense in \( D_S \) and, therefore, \( D_M := (f_M(\mathbf{D}_M))^{-1}(D_S) \) is dense in \( D_M \). Since \( \tau: D_M \to \mathbf{D}_T \) is a principal \( \text{GL}(2m-1) \)-bundle, there is an open dense subset \( \mathbf{D}'_T \) of \( \mathbf{D}_T \) such that \( \tau^{-1}(\mathbf{D}'_T) \cap D_M \) is dense in \( D_M \). It follows from the equalities \( \tau \circ \tilde{\theta}_a = \theta_a \circ \tau_a \) and \( f_M \circ \tilde{\theta}_a = \tilde{\lambda} \circ \tau_a \) that \( \tau_a^{-1}(\mathbf{D}'_T) \subset \tau_a^{-1}(D) \), so that \( \tilde{\theta}_a^{-1}(\mathbf{D}'_T) \subset D \). Therefore \( \theta_a(D) \) is dense in \( \mathbf{D}_T \). Since \( \rho: \mathbf{D}'_T \to \mathbf{D}'_H \) is a projective bundle, we obtain that \( (\rho \circ \theta_a)(D) \) is dense in \( \mathbf{D}_H \). \( \square \)
Lemma 11.7. Let $Z_c$ be an irreducible component of $Z_m$ which is different from $Z$ and such that $Z_c \cap Z$ contains a Weil divisor $D_c$ in $Z$. Then $Z_c$ satisfies the conditions (11.4) and (11.5).

Proof. Let $D$ be an arbitrary irreducible component of $\sigma^{-1}_a(D_c)$. By (11.26), $D$ is a divisor in $Z_a$. In view of Corollary 11.6, two cases can occur: $D \subset Z \cup D'$ and $D = D$. We shall consider both of them.

1. Suppose that $D \subset Z \cup D'$. In this case, since $\dim D = \dim Z - 1 = 4m(m+2) - 1$, it follows from Proposition 11.3 that $\dim \lambda(D) \geq 4m^2 + 6m - 1$ and, by (11.27), we have

$$\dim (f_M \circ \theta_a)(\tau^{-1}_a(D)) \geq \dim \lambda(D) \geq 4m^2 + 6m - 1.$$ 

Hence $\dim \theta_a(D) = \dim (\tau \circ f_M \circ \theta_a)(\tau^{-1}_a(D)) \geq 4m^2 + 6m - 1 - \dim \Sigma \{ \text{fibre of the projection } \tau \} = 4m^2 + 6m - 1 - (2m - 1)^2 = 10m - 2 = \dim T - 1$. Accordingly, $D_a := (\rho \circ \theta_a)(D)$ satisfies the inequality $\dim D_a \geq \dim H - 1$. Since $D \not\subset D$, we see that $D_a$ is not contained in $D_H$. Therefore the last inequality together with (11.1) shows that $D^*_a := D_a \cap H_0$ is an open dense subset of $D_a$. Hence $D^* := D \cap (\rho \circ \theta_a)^{-1}(D^*_a)$ is an open dense subset of $D$.

Take an arbitrary point $\tilde{z} \in D^*$ and put $t = (\rho \circ \theta_a)(\tilde{z})$. Since $\sigma_a(D) \subset f_Y(Y)$ by diagram (11.30), there is a point $y \in Y$ such that $z := f_Y(y) = \tilde{z}$ and $\rho \circ \zeta(y) = t$. We write $C(y)$ for the diagram (10.26) restricted to $\mathbb{P}^3 \times \{ y \}$. In this diagram, let $\xi_y(j) : O_{\mathbb{P}^3} \rightarrow \gamma_t^* O_{\Gamma^t}$ be the restriction (twisted by $O_{\mathbb{P}^3}(-1)$) of the morphism $\xi_Y(j)$ to $\mathbb{P}^3 \times \{ y \}$. (Here $\gamma_t$ and $\Gamma_t$ are as in (10.15).) Let $E_y$ be the cohomology sheaf at the middle term of the upper complex in the diagram $C(y)$. As in the diagram (9.23), the embedding $j : k \hookrightarrow H^y_{2m}$ determines a section $s_y : O_{\mathbb{P}^3} \rightarrow E_y(1)$.

If $\xi_y(j)$ is an epimorphism, then standard diagram-chasing shows that, first, the upper complex in $C(y)$ is right exact (so that $E_y$ is a ‘t Hooft bundle) and, second, $\text{coker}(s_y) = I_{L_t, \mathbb{P}^3}(2)$, where $L_t = \gamma_t(\Gamma_t)$. We note that since $t \in D^*_a \subset H_0$, the definition of $H_0$ shows that $L_t$ does not lie on a quadric. Therefore $h^0(E_y(1)) = 1$. However, $E_y$ coincides by construction with the bundle $E_2(z, y)$ in the notation of Lemma 9.3. This means that the point $z = \sigma_a(\tilde{z}) \in D \subset Z_c$ lies in $Z_m(j) \cap Z_{m0} \cap Z_{m1}(j)$, that is, $Z_c$ satisfies (11.4) and (11.5).

Assume that $\xi_y(j)$ is not an epimorphism. Note that the lower complex in $C(y)$ depends only on the point $t = (\rho \circ \theta_a)(\tilde{z})$. Since $t \in D^* \subset H^0$, an elementary computation shows (see the end of Remark 10.10) that the morphism $\xi_y(j') : O_{\mathbb{P}^3} \rightarrow \gamma_{t'}^* O_{\Gamma_t}$ is surjective for a general embedding $j' : k \hookrightarrow H^y_{2m}$. Therefore, replacing $j$ by $j'$ in the diagram (10.26) and arguing as above, we obtain that $z \in Z_m(j') \cap Z_{m0} \cap Z_{m1}(j')$. Hence $Z_c$ also satisfies (11.4) and (11.5).

2. Suppose that $D = D$. In this case, comparing the dimensions of the schemes in the diagram (11.30), we obtain that $(\rho \circ \theta_a)(D)$ is dense in $D_H$. In particular, since $D_H^1$ is dense in $D_H$ (see § 11.1), the set $D^* = \theta^{-1}_a(D_H^1)$ is dense in $D$. Take an arbitrary point $\tilde{z} \in D^*$ and, as above, put $t = (\rho \circ \theta_a)(\tilde{z})$. By the diagram (11.30) we again have $\sigma_a(D) \subset f_Y(Y)$, whence there is a point $y \in Y$ such that $z := f_Y(y) = \tilde{z}$ and $(\rho \circ \zeta)(y) = t$. Let $C(y)$ be the diagram obtained by restricting the diagram (10.26) to $\mathbb{P}^3 \times \{ y \}$. With our previous definitions of the morphism $\xi_y(j) : O_{\mathbb{P}^3} \rightarrow \gamma_{t'}^* O_{\Gamma_t}$ and the sheaf $E_y$, we again obtain, as in the diagram (9.23), that the monomorphism $j : k \hookrightarrow H^y_{2m}$ determines a section $s_y : O_{\mathbb{P}^3} \rightarrow E_y(1)$. As before, we
may assume that $j$ is a general monomorphism. For a general $j$, as in the proof of Proposition 11.2 (the case when $h(\tilde{Z}) \subset D^0_H$), we have $\text{coker}(s_y) = I_{L_t, p^3(2)}$, where $L_t = \gamma_t(\Gamma_t)$ is a reduced scheme with Hilbert polynomial $P_m(n) - 1$. This scheme does not lie on a quadric, whence we still have $h^0(E_y) = 1$ although the sheaf $E_y$ is not locally free at the point of intersection of the unique pair of intersecting lines in $L_t$. Therefore, as in case 1, we obtain that $z \in Z_{m0}^{00} \cap Z_{m1}^1(j)$, so that $Z_c$ satisfies (11.4) and (11.5). □

We now prove the main result of this subsection.

**Proposition 11.8.** There is no irreducible component $Z_c$ in $Z_m$ other than $Z$ such that $Z_c \cap Z$ contains a Weil divisor in $Z$.

**Proof.** Assume the opposite: there is an irreducible component $Z_c$ in $Z_m$ which is distinct from $Z$ and whose intersection with $Z$ contains a Weil divisor $D_c$ in $Z$. By Lemma 11.7, $Z_c$ satisfies (11.4) and (11.5), whence the hypotheses of Propositions 11.2, 11.3 hold for $Z_c$. Therefore we may argue for $Z_c$ in the same way as for $Z$ above.

First, by Proposition 11.3, for a fixed element $j \in P(H^\vee_m)_c$ there is a finite birational morphism $\sigma: \tilde{Z}_c \to Z_c$ of a smooth variety $\tilde{Z}_c$ such that condition (11.6) holds and the fibre of the morphism $\tilde{\lambda} = \sigma \circ \lambda_j: \tilde{Z}_c \to Z_c$ admits the description (11.24). Second, the set $\tilde{Z} = \sigma^{-1}(Z_c \cap Z_{m0}^{00} \cap Z_{m1}^1(j))$ is open and dense in $\tilde{Z}_c$ by Proposition 11.2, and one can find a morphism $\theta: \tilde{Z} \to T_0$ and varieties $\tilde{M} := \tilde{Z} \times_T M$ and $\tilde{Y} := \tilde{Z} \times_H Y$ with projections $\tilde{Z} \stackrel{\tau}{\to} \tilde{M} \to Y$ and $\tilde{Z} \stackrel{\zeta}{\to} \tilde{Y} \to Y$ such that we have $f_M \circ \theta_M = \tilde{\lambda} \circ \tau_{\tilde{Z}}$ and $f_Y \circ \theta_Y = \tilde{\sigma} \circ \zeta_{\tilde{Z}}$.

Putting $Z_a = \tilde{Z}$, $M_a = \tilde{M}$, $Y_a = \tilde{Y}$, $\tilde{\theta}_a = \theta_M$, $\tilde{\theta}_a = \theta_Y$, $\tau_a = \tau_{\tilde{Z}}$, $\zeta_a = \zeta_{\tilde{Z}}$ in the formulae (11.27), we see that the above relations for the composite morphisms coincide with the relations (11.27). Therefore we have the equalities (11.27)–(11.29) and the diagram (11.30). Moreover, we have $\dim \tilde{Z} = \dim Z_c \geq 4m(m+2)$ by (7.9). As in Remark 11.4, we deduce from this that $\tilde{f}_Y(\tilde{Y}) = \tilde{Z}_c$, where both closures are taken in $S_{2m}$. However, we have $\tilde{f}_Y(\tilde{Y}) = \tilde{Z}$ by Remark 11.4, whence $\tilde{Z}_c = \tilde{Z}$. Thus $Z_c = \tilde{Z}$ contrary to assumption. □

**11.4. Completion of the proof of Theorem 7.2.** We shall prove that $Z_m$ is irreducible, and the surjectivity of the projection $p_m: Z \to (S^\vee_m)^0$, $(D, \varphi) \mapsto D$, will follow from the proof. First, $Z_m$ contains the irreducible component $\tilde{Z}$ introduced in Proposition 8.1. Suppose that there is another irreducible component $Z'$ of $Z_m$.

Let $b: \Phi_m \setminus \{0\} \to P(\Phi_m)$, $\varphi \mapsto \langle \varphi \rangle$, be the canonical projection and $b := \text{id} \times b: (S^\vee_m)^0 \times (\Phi_m \setminus \{0\}) \to (S^\vee_m)^0 \times P(\Phi_m)$ the induced projection. The equations for $Z_m$ in $(S^\vee_m)^0 \times \Phi_m$ (see (7.4), (7.5)) are homogeneous with respect to the affine coordinates in $\Phi_m$. Therefore there are closed irreducible subsets $\tilde{Z}$ and $\tilde{Z}'$ and a closed subset $Z_m$ of $(S^\vee_m)^0 \times P(\Phi_m)$ such that $\tilde{Z} = b^{-1}(\tilde{Z}) \cup p_m(Z) \times \{0\}$ and $\tilde{Z}' = b^{-1}(\tilde{Z}') \cup p_m(Z') \times \{0\}$, $Z_m = b^{-1}(Z_m) \cup p_m(Z_m) \times \{0\}$. Moreover, by construction, $\tilde{Z}$ and $\tilde{Z}'$ are irreducible components of $Z_m$.

Take an arbitrary point

$$y = (D_0, \langle \varphi \rangle) \in \tilde{Z}' \setminus \tilde{Z}' \cap \tilde{Z}$$ (11.32)
and consider the projective space \( \mathbb{P} = \{ D_0 \} \times P(\mathbf{F}_m) \), \( \dim \mathbb{P} = 6m^2 - 1 \). By definition, the sets \( Z_m(D_0) = Z_m \cap \mathbb{P} \) and \( Z^r(D_0) = Z^r \cap \mathbb{P} \) are closed subsets of \( \mathbb{P} \) such that

\[
y \in Z^r(D_0) \subset Z_m(D_0)
\]

and, by Remark 7.1, we have \( \text{codim}_{\mathbb{P}} Z_m(D_0) \leq 5m(m - 1) \), that is,

\[
\dim_{\mathbb{P}} Z_m(D_0) \geq m^2 + 5m - 1, \quad m \geq 1.
\]

By definition, \( Z_m(D_0) \) is given in \( \mathbb{P} \) by \( 5m(m - 1) \) global equations of the form \( \varphi^\vee \circ D_0 \circ \varphi \in S_m^r \), which are homogeneous equations of degree 2 with respect to the coordinates in \( \Phi \).

**Lemma 11.9.** Let \( X \) be the set-theoretic complete intersection of \( r = 5m(m - 1) \) hyperquadrics in the projective space \( \mathbb{P} \) of dimension \( N = 6m^2 - 1 \). Then \( X \) is connected.

**Proof.** Consider the vector space \( V := H^0(\mathcal{O}_\mathbb{P}(2)) \) and the Grassmannian \( G := G(r, V) \). Let \( i : S \to V \otimes \mathcal{O}_G \) be the tautological subbundle. We consider the composite morphism \( s : S \boxtimes \mathcal{O}_\mathbb{P} \xrightarrow{i} V \otimes \mathcal{O}_{G \times \mathbb{P}} \xrightarrow{ev} \mathcal{O}_G \boxtimes \mathcal{O}_\mathbb{P}(2) \) on \( G \times \mathbb{P} \), where \( ev \) is the evaluation morphism. Let \( \Gamma = (s)_0 \) be the scheme of zeros of the section \( s \) with the projections \( G \xleftarrow{\mathbb{P}} \Gamma \xrightarrow{q} \mathbb{P} \). By construction, the fibre \( q^{-1}(y) \) is isomorphic to \( G(r, H^0(\mathcal{I}_{x, \mathbb{P}}(2))) \) for an arbitrary point \( x \in \mathbb{P} \), whence \( q \) is a smooth projective morphism and, therefore, is of positive dimension. The fibre \( p^{-1}(y) \) over a general point \( y \in G \) is a complete intersection of \( r \) hyperquadrics, has codimension \( r \) in \( \mathbb{P} \) and, therefore, is of positive dimension. It is well known and easily provable that this complete intersection is connected. Thus, in view of the Stein decomposition (see [14], Ch. III, § 11.5), the fibre \( p^{-1}(y) \) over an arbitrary point \( y \in G \), being the set-theoretic complete intersection of \( r \) hyperquadrics, is connected. □

Applying Lemma 11.9 to \( X = Z_m(D_0) \), we obtain in view of (11.34) that the set \( Z_m(D_0) \) is connected.

The morphism \( pr_1 : Z \to (S_m^r)^0, (D, \varphi) \mapsto D \) is dominant by Proposition 8.1 (ii). Hence the induced projective morphism \( pr : Z \to (S_m^r)^0, (D, \langle \varphi \rangle) \mapsto D \) is also dominant and, therefore, surjective since \( Z \) is closed in \( (S_m^r)^0 \times P(\mathbf{F}_m) \). In particular, the set \( Z(D_0) = Z \cap \mathbb{P} \) is a non-empty closed subset of \( Z_m(D_0) \). We also obtain from (11.32) that \( y \in Z_m(D_0) \setminus Z(D_0) \). Therefore, since the set \( Z_m(D_0) \) is connected, it contains an irreducible component (call it \( Z^0(D_0) \)) distinct from \( Z(D_0) \) and intersecting \( Z(D_0) \). Let \( Z^0 \) be an irreducible component of \( Z_m \) that contains \( Z^0(D_0) \) and, therefore, is distinct from \( Z(D_0) \). Then we have

\[
Z \cap Z^0 \neq \emptyset.
\]

**Remark 11.10.** It follows from the surjectivity of the projective morphism \( pr : Z \to (S_m^r)^0 \) that the projection \( p_m = pr_1 : Z \to (S_m^r)^0 \) is also surjective.

We put \( Z^0 = b^{-1}(Z^0) \cup p_m(Z^0) \times \{0\} \) and let \( Z_c \) be the union of all irreducible components of \( Z_m \) distinct from \( Z \). By construction, \( Z_c \supset Z^0 \). Hence, by (11.35), there is a point

\[
z_0 = (D, \varphi) \in Z \cap Z_c, \quad \varphi \neq 0.
\]
We endow $S'_m$ and $\Phi$ with affine coordinates $x_i$, $1 \leq i \leq M := 3m(m + 1)$, and $y_j$, $1 \leq j \leq 6m^2$, respectively, and complete the affine space $S'_m \times \Phi$ to a projective space $\mathbb{P}^N$, $N := 3m(3m + 1)$, with homogeneous coordinates $w_0, \ldots, w_N$ such that $x_i = w_i/w_0$, $1 \leq i \leq M$, $y_j = w_{M+j}/w_0$, $1 \leq j \leq 6m^2$. Let $\overline{Z}_m$, $\overline{Z}$ and $\overline{Z}_c$ be the closures in $\mathbb{P}^N$ of the sets $Z_m$, $Z$ and $Z_c$ respectively. Since the equations $\varphi \circ D_0 \circ \varphi \in S'_m$ of the set $Z_m$ in $(S'_m)^0 \times \Phi$ are homogeneous of degree 1 in the coordinates $x_i$ and of degree 2 in the coordinates $y_j$, the equations of $\overline{Z}_m$ are homogeneous of degree 3 in the coordinates $w_i$ in $\mathbb{P}^N$. Let $F_1 = \ldots = F_r = 0$ be these equations. We choose general hyperplanes $\Pi_1, \ldots, \Pi_r$ in $\mathbb{P}^N$ such that

$$z_0 \not\in \Pi_1 \cup \ldots \cup \Pi_r, \quad \Pi_1 \cap \ldots \cap \Pi_r =: \mathbb{P}^{N-r} \not\subset \overline{Z} \cup \overline{Z}_c,$$

(11.37)

where $\mathbb{P}^{N-r}$ is a subspace of codimension $r$ in $\mathbb{P}^N$. We write $\Pi_i = \{L_i = 0\}$, $1 \leq i \leq r$. Let $Y$ be the subset of $\mathbb{P}^N$ given by the equations $F_1L_1 = \ldots = F_rL_r = 0$. By (11.37), $Y$ contains the subspace $\mathbb{P}^{N-r}$ and the variety $\overline{Z}$ as irreducible components and contains $\overline{Z}_c$ as a component. We denote the closure of $Y \setminus \overline{Z}$ in $\mathbb{P}^N$ by $Y_0$.

Consider the Veronese embedding $v_4 : \mathbb{P}^N \hookrightarrow \mathbb{P}^{N_1} := P(H^0(\mathcal{O}_{\mathbb{P}^N}(4))^\vee)$. We identify $\mathbb{P}^N$ with $v_4(\mathbb{P}^N)$ and hence identify $\overline{Z}$, $\overline{Z}_c$, $Y$ and $\mathbb{P}^{N-r}$ with their images under $v_4$. Consider the subspace $\mathbb{P}^{N_2} := \text{Span}(\overline{Z})$ in $\mathbb{P}^{N_1}$, and let $\sigma : \mathbb{P}^{N_1} \to \mathbb{P}^{N_2}$ be the blow-up of $\mathbb{P}^{N_1}$ along $\mathbb{P}^{N_2}$. We put $\mathbb{P}^{N_3} := \sigma^{-1}(\mathbb{P}^N)$. The linear projection $\mathbb{P}^{N_1} \dashrightarrow \mathbb{P}^{N_3} := P(\mathcal{W}^\vee)$, where $\mathcal{W} := H^0(\mathcal{I}_{\overline{Z}, \mathbb{P}^N}(4))$, extends to a morphism $\pi : \mathbb{P}^{N_1} \to \mathbb{P}^{N_3}$. We put $G := G(r, \mathcal{W})$ and let $\Gamma \subset \mathbb{P}^{N_3} \times \mathbb{P}^N$ be the graph of incidence with the projections $G \to \mathbb{P}^N$. We also put $\overline{X} := \mathbb{P}^{N_1} \times_{\mathbb{P}^{N_3}} \Gamma$ and let $\pi_X : \overline{X} \to \Gamma$ be the projection. By construction, the exceptional divisor $D$ of the blow-up $\sigma$ is isomorphic to $\mathbb{P}^{N_2} \times \mathbb{P}^{N_3}$ and contains $\overline{Z} \times \mathbb{P}^{N_3}$. Therefore $\overline{X}$ contains $\overline{Z} \times \Gamma$. We consider the closure $X$ of the set $U = \overline{X} \setminus (\overline{Z} \times \Gamma)$ in $\mathbb{P}^{N_1} \times_{\mathbb{P}^{N_3}} \Gamma$, and let $\overline{P} \overset{\pi_X}{\to} \overline{X}$ $\overset{\pi_X}{\to} \Gamma$ and $q_X := q \circ \pi_X : X \to G$ be the projections. For an arbitrary point $y \in G$, let $X_y$ be the image of the fibre $q_X^{-1}(y)$ under the morphism $\sigma_y := \sigma \circ q_X^{-1}(y)$. By construction, $X$ is irreducible and all the fibres of the projection $q_X$ have dimension $\geq N - r$.

Let $y_0 \in G$ be the point corresponding to the equations $F_1L_1 = \ldots = F_rL_r = 0$. By construction, $X_{y_0}$ contains the set $Y_0$ described above, and is contained in $Y_0 \cup \overline{Z}$:

$$Y_0 \subset X_{y_0} \subset Y_0 \cup \overline{Z}.$$  

(11.38)

Since $Y_0$ contains the subspace $\mathbb{P}^{N-r}$ (defined in (11.37)) as a component, we see that $\overline{P}^{N-r} := (\sigma_{y_0})^{-1}(\mathbb{P}^{N-r})$ is a component of the fibre $q_X^{-1}(y_0)$ and has dimension $N - r$ because the morphism $\sigma_{y_0} : \overline{P}^{N-r} \to \mathbb{P}^{N-r}$ is birational. Therefore the fibre $q_X^{-1}(y)$ over a general point $y \in G$ is irreducible and has dimension $N - r$. Here $\sigma_y : q_X^{-1}(y) \to X_y$ is a birational morphism and, by construction, $X_y \cup \overline{Z}$ is the complete intersection of $r$ hypersurfaces of degree 4 in $\mathbb{P}^N$. It has codimension $r$ in $\mathbb{P}^N$. Hence, by the principle of connectedness of locally complete intersections in codimension 1 (see [19]), we have

$$\dim_z(Z \cap X_y) = \dim Z - 1, \quad z \in Z \cap X_y.$$  

(11.39)
We now take a general point \( y \in G \) and an arbitrary point \( z(y) \) of the variety \( X_y \setminus (X_y \cap Y_0) \). Let \( \mathbb{P}^1 \) be the line through \( z_0 \) and \( z(y) \) in \( \mathbb{P}^N \), and let \( \tilde{C} \) be the proper transform of \( \mathbb{P}^1 \) under the birational projection \( \sigma \circ p_X : X \to \mathbb{P}^N \). Since \( \mathbb{P}^1 \) is smooth, the map \( \nu = (\sigma \circ p_X|_{\tilde{C}})^{-1} : \mathbb{P}^1 \to \tilde{C} \) is an isomorphism. Consider the composite \( \varphi : \mathbb{P}^1 \to \tilde{C} \) for some subspaces \( x \) such that \( x \in U \). Since \( x \) is irreducible, there is a neighbourhood \( C \) of \( z_0 \) in \( \mathbb{P}^1 \) such that, for all points \( z \in C \setminus \{z_0\} \), the fibres \( X_y \) over \( y = \varphi(z) \) are irreducible, have dimension \( N - r \) and satisfy (11.39). Consider the scheme \( X_C := X \times_C C \) and the natural embeddings of schemes \( X_C \subseteq \tilde{T} := C \times \mathbb{Z} \cup X_C \subseteq C \times \mathbb{P}^N \). By construction, the fibre of the projection \( p_{\tilde{T}} : \tilde{T} \to C \) over each point \( y \in C \) is the set-theoretic intersection of \( r \) hypersurfaces of degree 4 in \( \{y\} \times \mathbb{P}^N \). Let \( T \) be the unique irreducible component of \( X_C \) for which the projection \( p_T := p_{\tilde{T}}|_{T} : T \to C \) is dominant. We note that \( \tilde{Z} \), being a component of the fibre \( p_{\tilde{T}}^{-1}(z_0) \), is an integral scheme in view of the integrality of \( Z \). Since \( \tilde{Z} \) is the fibre of the projection \( p_{\tilde{T}}|_{C \times \mathbb{Z}} \) over \( y_0 \), it follows that the fibre \( T_{z_0} := p_{\tilde{T}}^{-1}(z_0) \) does not contain \( \tilde{Z} \). Now, by construction, \( T_{z_0} \subset X_{y_0} \). Therefore \( T_{z_0} \subset Y_0 \) by (11.38). Moreover, it follows from the construction of the line \( \mathbb{P}^1 \) in \( \mathbb{P}^N \) that \( z_0 \in T_{z_0} \). On the other hand, by hypothesis, the fibre \( X_y = p_{\tilde{T}}^{-1}(z) \) for \( y = \varphi(z) \) satisfies (11.39) for every \( z \in C \setminus \{z_0\} \). Therefore \( \text{codim}_{\mathbb{Z}}(\tilde{Z} \cap T_{z_0}) = 1 \) if \( z_0 \in T_{z_0} \cap Y_0 \), and there is \( \text{codim}_{\mathbb{Z}}(\tilde{Z} \cap Z_0) = 1 \) and \( z_0 \in \tilde{Z} \cap Y_0 \). Finally, it follows from conditions (11.37) and the equations \( F_1 L_1 = \cdots = F_r L_r = 0 \) that \( z_0 \not\in Y \), where \( Y \) is any irreducible component of \( Y_0 \) not contained in \( \tilde{Z} \). Hence \( \text{codim}_{\mathbb{Z}}(\tilde{Z} \cap Z_c) = 1 \) and, in particular, \( \text{dim}_{\mathbb{Z}}(\tilde{Z} \cap Z_c) = \text{dim} \tilde{Z} - 1 \). Since \( z_0 \in Z \cap Z_c \) by (11.36), it follows that \( \text{dim}_{\mathbb{Z}}(Z \cap Z_c) = \text{dim} Z - 1 \). Thus the intersection \( Z \cap Z_c \) contains a Weil divisor in \( Z \) contrary to Proposition 11.8. Hence \( Z_m \) is irreducible.

The surjectivity of the morphism \( p_m : Z_m \to (S^\vee_m)^0 \) was already mentioned in Remark 11.10. Theorem 7.2 is proved.

\section{Appendix: two general-position results}

In this section we prove Theorem 4.1 and Proposition 7.3.

\subsection{Proof of Theorem 4.1}

We first recall some definitions and standard facts from the theory of determinantal varieties.

\begin{definition}
Let \( U \) and \( U' \) be vector spaces of dimensions \( m \) and \( n \) respectively, where \( m \geq n \). Consider the projective space \( \mathbb{P}(U \otimes U') \). We say that a point \( x \in \mathbb{P}(U \otimes U') \) has rank \( r \) (and write \( \text{rk}(x) = r \)) if the following conditions hold.

i) There are unique subspaces \( U_r(x) \subset U \) and \( U'_r(x) \subset U' \) of dimensions \( \dim U_r(x) = \dim U'_r(x) = r \) such that \( x \in \mathbb{P}(U_r(x) \otimes U'_r(x)) \).

ii) There are no subspaces \( \tilde{U} \subset U \) and \( \tilde{U}' \subset U' \) of dimensions \( \dim \tilde{U} = \dim \tilde{U}' < r \) such that \( x \in \mathbb{P}(\tilde{U} \otimes \tilde{U}') \).

The following lemma is well known (see, for example, [20]).

\begin{lemma}
Each point \( x \in \mathbb{P}(U \otimes U') \) has a uniquely determined rank \( \text{rk}(x) \), \( 1 \leq \text{rk}(x) \leq n \). Moreover, if \( x \in \mathbb{P}(U \otimes U') \) has rank \( \text{rk}(x) = r \) and \( x \in W \otimes W' \) for some subspaces \( W \subset U \) and \( W' \subset U' \), then the subspaces \( U_r(x) \subset U \) and
$U'_r(x) \subset U'$ of dimensions $\dim U_k(x) = \dim U'_k(x) = r$ from Definition 12.1 (i) are such that $U_r(x) \subset W$ and $U'_r(x) \subset W'$.

Proof of Theorem 4.1. By Definition 12.1 with $U = H_{2m+1}^\vee$ and $U' = V^\vee$, each point $x \in P(H_{2m+1}^\vee \otimes V^\vee)$ has rank$^6$ in the interval $1 \leq \rk(x) \leq \dim V^\vee = 4$. Thus,

$$P(W_{4m+4}^\vee) = \bigcup_{r=1}^4 Z_r,$$

where

$$Z_r := \{ x \in P(W_{4m+4}^\vee) \mid \rk(x) = r \}, \quad 1 \leq r \leq 4,$$

are locally closed subsets of $P(W_{4m+4}^\vee)$. Consider the Grassmannian

$$G := G(m, H_{2m+1}^\vee)$$

and the locally closed subsets

$$\Sigma_r := \{ V_m \in G \mid V_m \supset U_r(x) \text{ for some point } x \in Z_r \}, \quad 1 \leq r \leq 4. \quad (12.2)$$

By Lemma 12.2, the condition $x \in Z_r \cap P(V_m \otimes V^\vee)$ means that $x \in Z_r \cap P(U_r \otimes V^\vee)$ for some $r$-dimensional subspace $U_r = U_r(x) \subset V_m$. This together with $(12.1)$ and $(12.2)$ shows that

$$\{ V_m \in G \mid P(V_m \otimes V^\vee) \cap P(W_{4m+4}^\vee) \neq \emptyset \} = \bigcup_{r=1}^4 \Sigma_r.$$

Thus the theorem is equivalent to the assertion that $\bigcup_{r=1}^4 \Sigma_r \subset G$. To prove the theorem, it suffices to show that

$$\dim \Sigma_r < \dim G, \quad 1 \leq r \leq 4. \quad (12.3)$$

We now proceed to prove these inequalities for $r = 4, 3, 2, 1$.

Case $r = 4$. We put $\Gamma_4 := \{ (x, U) \in P(W_{4m+4}^\vee) \times G(4, H_{2m+1}^\vee) \mid \rk(x) = 4, U = U_4(x) \}$, and let $P(W_{4m+4}^\vee) \xleftarrow{p_4} \Gamma_4 \xrightarrow{q_4} G(4, H_{2m+1}^\vee)$ be the projections. We have $p_4(\Gamma_4) = Z_4$ by construction, and the projection $p_4 : \Gamma_4 \to Z_4$ is bijective by Definition 12.1 (i). Hence,

$$\dim q_4(\Gamma_4) \leq \dim \Gamma_4 = \dim Z_4 \leq \dim P(W_{4m+4}^\vee) = 4m + 3.$$

By construction we have the graph of incidence

$$\Pi_4 = \{ (U, V_m) \in q_4(\Gamma_4) \times \Sigma_4 \mid U \subset V_m \}$$

with the surjective projections $q_4(\Gamma_4) \xleftarrow{pr_1} \Pi_4 \xrightarrow{pr_2} \Sigma_4$ and the fibre

$$\pr_1^{-1}(U) \cong G(m - 4, H_{2m+1}^\vee/U). \quad (12.4)$$

---

$^6$Throughout this proof, the rank of a point $x$ of a given subspace of $P(H_{2m+1}^\vee \otimes V^\vee)$ is understood as the rank of $x$ as a point in $P(H_{2m+1}^\vee \otimes V^\vee)$. 
over an arbitrary point $U \in q_4(\Gamma_4)$. (Indeed, the condition $U \subset V_m \subset H_{2m+1}^\vee$ means that $V_m/U \in G(m-4, H_{2m+1}^\vee/U)$. Hence,

$$\dim \Sigma_4 \leq \dim \Pi_4 = \dim q_4(\Gamma_4) + \dim G(m-4, H_{2m+1}^\vee/U)$$

$$\leq 4m + 3 + (m-4)(m+1) = m(m+1)-1 = \dim G - 1 < \dim G,$$

that is, (12.3) holds for $r = 4$.

Case $r = 3$. Consider the projection $f_3: Z_3 \to P(V^\vee) = \mathbb{P}^3$, $x \mapsto V_3(x)$, where the pair $(U_3(x), V_3(x))$ of 3-dimensional spaces $U_3(x) \subset H_{2m+1}^\vee$ and $V_3(x) \subset V^\vee$ is uniquely determined by the point $x$ via the condition $x \in P(U_3(x) \otimes V_3(x))$ since $\text{rk}(x) = 3$ (see Definition 12.1 and Lemma 12.2). Given a 3-dimensional subspace $V_3 \subset V^\vee$, we put

$$\Sigma_3(V_3) = \{ V_m \in G \mid V_m \supset U_3(x) \text{ for some point } x \in f_3^{-1}(V_3) \}. \quad (12.5)$$

Comparing this with (12.2) for $r = 3$, we get

$$\Sigma_3 = \bigcup_{V_3 \subset V^\vee} \Sigma_3(V_3). \quad (12.6)$$

Note that \textit{a priori} $f_3$ is not necessarily surjective. Therefore,

$$\dim \Sigma_3 \leq \dim \Sigma_3(V_3) + 3. \quad (12.7)$$

We want to estimate the dimension of $\Sigma_3(V_3)$ for an arbitrary 3-dimensional subspace $V_3$ of $V^\vee$. This subspace determines a commutative diagram

$$\begin{array}{ccccccc}
0 & \to & F & \to & \Omega_{\mathbb{P}^3} & \to & I_z(-1) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \to & V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \to & \mathcal{O}_{\mathbb{P}^3}(-1) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & I_z & \to & \mathcal{O}_{\mathbb{P}^3} & \to & k_z & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array} \quad (12.8)$$

where $z = P(\ker: V \to V_3^\vee)$ is a point in $\mathbb{P}^3$ and the sheaf $F$ has an $\mathcal{O}_{\mathbb{P}^3}$-resolution $0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to 3\mathcal{O}_{\mathbb{P}^3}(-2) \to F \to 0$. Twisting this resolution by the vector bundle $E$ and passing to cohomology, we obtain the equalities $H^1(F \otimes E) \simeq H^2(E(-3)) = H_{2m+1}^\vee$, $H^2(F \otimes E) = 0$. Accordingly, passing to cohomology in the diagram (12.8) twisted by $E$ and using the equalities above and the obvious
formulae \( H^0(E \otimes k_z) \simeq k^2 \) and \( H^1(E \otimes k_z) = 0 \), we get a diagram

\[
\begin{array}{ccccccc}
0 & \to & H_{2m+1} & \to & W_{4m+4}^\vee & \to & H^1(E \otimes \mathcal{I}_z(-1)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_{2m+1}^\vee \otimes V_3 & \xrightarrow{\lambda} & H_{2m+1}^\vee \otimes V^\vee & \to & H_{2m+1}^\vee & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \uparrow{\text{mult}} & & \\
k^2 \searrow & \to & H^1(E \otimes \mathcal{I}_z) & \to & H^4_{4m} & \to & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]

(12.9)

Here the composite \( \varepsilon := \text{mult} \circ \lambda \) is surjective. Hence, putting \( W_{2m+3}(V_3) := \ker \varepsilon \), where \( \dim W_{2m+3}(V_3) = 2m + 3 \), we obtain a commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & W_{2m+3}(V_3) & \to & W_{4m+4}^\vee & \to & H_{2m+1}^\vee & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_{2m+1}^\vee \otimes V_3 & \xrightarrow{\lambda} & H_{2m+1}^\vee \otimes V^\vee & \to & H_{2m+1}^\vee & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \uparrow{\text{mult}} & & \\
H_{4m}^\vee & \to & H_{4m}^\vee & \to & 0 & & 0 & & \\
\end{array}
\]

This yields the relation

\[ W_{2m+3}(V_3) = H_{2m+1}^\vee \otimes V_3 \cap W_{4m+4}^\vee, \]

(12.10)

where the intersection is taken in \( H_{2m+1}^\vee \otimes V^\vee \). We put

\[ Z_3(V_3) := \{ x \in P(W_{2m+3}(V_3)) \mid \text{rk}(x) = 3 \}. \]

Relation (12.10) and Lemma 12.2 yield a bijection

\[ Z_3(V_3) \xrightarrow{\sim} f_3^{-1}(V_3). \]

(12.11)

Consider the graph of incidence \( \Gamma_3(V_3) := \{(x, U) \in Z_3(V_3) \times G(3, H_{2m+1}^\vee) \mid U = U_3(x)\} \) with the projections \( Z_3(V_3) \xrightarrow{p_3} \Gamma_3(V_3) \xrightarrow{q_3} G(3, H_{2m+1}^\vee) \). By Lemma 12.2 we have \( p_3(\Gamma_3(V_3)) = Z_3(V_3) \) and the projection \( p_3: \Gamma_3(V_3) \to Z_3(V_3) \) is a bijection. Hence,

\[ \dim q_3(\Gamma_3(V_3)) \leq \dim \Gamma_3(V_3) = \dim Z_3(V_3) \leq \dim P(W_{2m+3}(V_3)) = 2m + 2. \]

(12.12)
Consider the graph of incidence
\[ \Pi_3(V_3) = \{(U, V_m) \in q_3(\Gamma_3(V_3)) \times \Sigma_3(V_3) \mid U \subset V_m \} \]
with the projections \( q_3(\Gamma_3(V_3)) \xrightarrow{pr_1} \Pi_3(V_3) \xrightarrow{pr_2} \Sigma_3(V_3) \) and the fibre
\[ \text{pr}_1^{-1}(U) \simeq G(m - 3, H_{2m+1}^\wedge / U) \tag{12.13} \]
over an arbitrary point \( U \in q_3(\Gamma_3(V_3)) \) (compare \( (12.4) \)). The projection \( \Pi_3(V_3) \xrightarrow{pr_2} \Sigma_3(V_3) \) is surjective by \( (12.11) \). Hence, using \( (12.12) \), we obtain
\[ \dim \Sigma_3(V_3) \leq \dim \Pi_3(V_3) = \dim q_3(\Gamma_3(V_3)) + \dim G(m - 3, H_{2m+1}^\wedge / U) \]
\[ \leq 2m + 2 + (m - 3)(m + 1) = m^2 - 1. \]
This together with \( (12.7) \) and the assumption \( m \geq 3 \) yields that \( \dim \Sigma_3 \leq m^2 + 2 = \dim G + 2 - m < \dim G \). Hence \( (12.3) \) holds for \( r = 3 \).

Before proceeding to the case \( r = 2 \), we make a digression on the properties of jumping lines of \( E \). These properties will be stated in Lemma 12.3 and used below. We introduce some notation. For every line \( l \subset \mathbb{P}^3 \) we have \( E[l] \simeq \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(-d) \), where \( d \) is a well-defined non-negative integer called the \textit{jump} of \( E[l] \) and denoted by \( d_E(l) \). The line \( l \) is called a \textit{jumping line} of \( d \) of the bundle \( E \). We put \( G_{2,4} := G(2, V^\vee), J_k(E) := \{l \in G_{2,4} \mid d_E(l) \geq k\}, \) and \( J_k^*(E) := J_k(E) \setminus J_{k+1}(E), k \geq 0 \). Since \( h^0(E[l]) \) is a semicontinuous function of \( l \in G_{2,4} \), we see that \( J_k(E) \) (resp. \( J_k^*(E) \)) is a closed (resp. locally closed) subset of \( G_{2,4}, k \geq 0 \). Moreover, by the Grauert–Mülich theorem ([21], Ch. II, §2.1.4, Corollary 2), \( J_0^*((E) \) is an open dense subset of \( G_{2,4} \). Since \( E \in I_{2m+1} \), we have
\[ J_{2m+1}(E) = \emptyset, \tag{12.14} \]
whence
\[ J_{2m-1}(E) = J_{2m-1}^*(E) \cup J_{2m}^*(E). \tag{12.15} \]

\textbf{Lemma 12.3.} Suppose that \( E \in I_{2m+1} \). Then
1) \( \dim J_{2m-1}(E) \leq 1 \),
2) \( \dim J_k^*(E) \leq 3 \) for \( 1 \leq k \leq 2m - 2 \).

\textit{Proof.} To prove part 1), assume the opposite: \( \dim J_{2m-1}(E) \geq 2 \). Take any irreducible surface \( S \subset J_{2m-1}(E) \), and let \( D \) be the degree of \( S \) with respect to the sheaf \( \mathcal{O}_{G_{2,4}}(1) \). We fix an integer \( r \geq 5 \) and an arbitrary irreducible curve \( C \) in the linear series \( \mathcal{O}_{G_{2,4}}(r) \). Then the degree \( deg C \) of \( C \) with respect to the sheaf \( \mathcal{O}_{G_{2,4}}(1) \) is equal to \( Dr \), whence \( deg C \geq 5 \). By Lemma 6 in [22], there are two distinct lines \( l_1, l_2 \subset C \) that intersect each other in \( \mathbb{P}^3 \). Let \( \mathbb{P}^2 \) be the projective plane spanned by \( l_1 \) and \( l_2 \) in \( \mathbb{P}^3 \). The exact triple \( 0 \to E(-2)|_{\mathbb{P}^2} \to E|_{\mathbb{P}^2} \to E|_{l_1 \cup l_2} \to 0 \) yields an exact sequence
\[ H^0(E|_{\mathbb{P}^2}) \to H^0(E|_{l_1 \cup l_2}) \to H^1(E(-2)|_{\mathbb{P}^2}). \tag{12.16} \]
Moreover, since \( [E] \in I_{2m+1} \), we have \( h^0(E(-1)) = h^1(E(-2)) = 0 \). Hence the exact triple
\[ 0 \to E(-2) \to E(-1) \to E(-1)|_{\mathbb{P}^2} \to 0 \]
yields that
\[ H^0(E(-1)|_{\mathbb{P}^2}) = 0. \]  
(12.17)

We now assume that \( h^0(E|_{\mathbb{P}^2}) > 0 \). Then a section \( s, 0 \neq s \in H^0(E|_{\mathbb{P}^2}) \), determines an injection \( \mathcal{O}_{\mathbb{P}^2} \hookrightarrow E|_{\mathbb{P}^2} \). This injective morphism and (12.17) show that the zero scheme \( Z \) of the section \( s \) is 0-dimensional and the injection \( s \) extends to a triple \( 0 \to \mathcal{O}_{\mathbb{P}^2} \overset{\delta}{\to} E|_{\mathbb{P}^2} \to \mathcal{I}_Z \to 0 \). Hence we have
\[ h^0(E|_{\mathbb{P}^2}) \leq 1. \]  
(12.18)

It follows from (12.17), the Riemann–Roch theorem and Serre duality for the vector bundle \( E(-1)|_{\mathbb{P}^2} \) that \( h^1(E(-2)|_{\mathbb{P}^2}) = 2m + 1 \). Hence, using (12.16) and (12.17), we obtain that
\[ h^0(E|_{l_1 \cup l_2}) \leq 2m + 2. \]  
(12.19)

On the other hand, put \( x := l_1 \cap l_2 \). Since by construction \( l_1, l_2 \in J_{2m-1}(E) \), it follows from (12.15) that either
\[ E|_{l_i} \simeq \mathcal{O}_{\mathbb{P}^2}(2m - 1) \oplus \mathcal{O}_{\mathbb{P}^2}(1 - 2m), \]
or
\[ E|_{l_i} \simeq \mathcal{O}_{\mathbb{P}^2}(2m) \oplus \mathcal{O}_{\mathbb{P}^2}(-2m), \]
whence \( h^0(E \otimes \mathcal{I}_{x,l_i}) \geq 2m - 1 \), \( i = 1, 2 \). This clearly implies that
\[ h^0(E|_{l_1 \cup l_2}) \geq h^0(E \otimes \mathcal{I}_{x,l_1 \cup l_2}) \geq h^0(E \otimes \mathcal{I}_{x,l_1}) + h^0(E \otimes \mathcal{I}_{x,l_2}) = 4m - 2. \]
Comparing these inequalities with (12.19), we see that \( 2m + 2 \geq 4m - 2 \), that is, \( m \leq 2 \). This contradicts the assumption that \( m \geq 3 \). Part 1) is proved.

Part 2) is an immediate corollary of the Grauert–Mülich theorem. ☐

**Case** \( r = 2 \). Here our notation and arguments are parallel to those for \( r = 3 \). We consider a morphism
\[ f_2: Z_2 \to G_{2,4}, \quad x \mapsto V_2(x), \]
where the pair \((U_2(x), V_2(x))\) of 2-dimensional vector spaces \( U_2(x) \subset H^0_{2m+1} \) and \( V_2(x) \subset V^\vee \) is uniquely determined by the point \( x \) via the condition \( x \in P(U_2(x) \otimes V_2(x)) \) since \( \text{rk}(x) = 2 \) (see Lemma 12.2).

By (12.14) we may assume that \( l \in J_k^*(E) \) for some \( k, \ 0 \leq k \leq 2m \), that is,
\[ h^0(E|l) = 2, \quad h^1(E|l) = 0, \quad l \in J_0^*(E), \]
and
\[ h^0(E|l) = k + 1, \quad h^1(E|l) = k - 1, \quad l \in J_k^*(E), \quad 1 \leq k \leq 2m. \]  
(12.20)

For \( 1 \leq k \leq 2m \) and a given subspace \( V_2 \in J_k^* \) we put
\[ \Sigma_{2,k}(V_2) = \{ V_m \in G \mid V_m \supset U_2(x) \text{ for some point } x \in f_2^{-1}(V_2) \}. \]  
(12.21)
In analogy with (12.6), we have

\[ \Sigma_2 = \bigcup_{k=0}^{2m} \bigcup_{V_2 \in J^*_k} \Sigma_{2,k}(V_2). \]

Hence, in view of Lemma 12.3,

\[ \dim \Sigma_2 \leq \max_{V_2 \in J^*_k, 0 \leq k \leq 2m} \left( \dim \Sigma_{2,k}(V_2) + \dim J^*_k \right). \] (12.22)

We now want to estimate the dimension of \( \Sigma_{2,k}(V_2) \) for every 2-dimensional subspace \( V_2 \in J^*_k, \ 0 \leq k \leq 2m \). This subspace determines a commutative diagram

\[
\begin{array}{cccccccccc}
0 & \rightarrow & \mathcal{O}_{\mathbb{P}^3}(-2) & \rightarrow & \Omega_{\mathbb{P}^3} & \rightarrow & F & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \rightarrow & V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \rightarrow & V'_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{I}_l & \rightarrow & \mathcal{O}_{\mathbb{P}^3} & \rightarrow & \mathcal{O}_l & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & & & 0 & & & & 0
\end{array}
\] (12.23)

where \( V'_2 := V^\vee / V_2, \ l = P((V'_2)^\vee) \) is a line in \( \mathbb{P}^3 \), and \( F := \text{coker} \ s \). Passing to cohomology in the diagram (12.23) twisted by \( E \), we obtain a diagram

\[
\begin{array}{cccccccccc}
0 & \rightarrow & \mathcal{O}_{\mathbb{P}^3}(-2) & \rightarrow & \Omega_{\mathbb{P}^3} & \rightarrow & F & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \rightarrow & V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \rightarrow & V'_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{I}_l & \rightarrow & \mathcal{O}_{\mathbb{P}^3} & \rightarrow & \mathcal{O}_l & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & & & 0 & & & & 0
\end{array}
\]

\[
\begin{array}{cccccccccc}
0 & \rightarrow & \mathcal{O}_{\mathbb{P}^3}(-2) & \rightarrow & \Omega_{\mathbb{P}^3} & \rightarrow & F & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \rightarrow & V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \rightarrow & V'_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{I}_l & \rightarrow & \mathcal{O}_{\mathbb{P}^3} & \rightarrow & \mathcal{O}_l & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & & & 0 & & & & 0
\end{array}
\]

\[
\begin{array}{cccccccccc}
0 & \rightarrow & H^{0}(E|\mathcal{I}_l) & \rightarrow & H^{1}(E \otimes F) & \rightarrow & \mathcal{W}_{4m+4} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^{0}_{2m+1} \otimes V_2 & \rightarrow & H^{0}_{2m+1} \otimes V^\vee & \rightarrow & H^{0}_{2m+1} \otimes V'_2 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^{0}(E|\mathcal{I}_l) & \rightarrow & H^{1}(E \otimes \mathcal{I}_l) & \rightarrow & H^{0}_{4m} & \rightarrow & \mathcal{W}_{4m} & \rightarrow & \mathcal{W}_{4m} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^{1}(E \otimes \mathcal{I}_l) & \rightarrow & H^{1}(E|\mathcal{I}_l) & \rightarrow & 0 & \rightarrow & 0
\end{array}
\] (12.24)
We first assume that \(1 \leq k \leq 2m\). (The case \(k = 0\) will be treated below.) Then the equations (12.20) and the diagram (12.24) yield the diagram

\[
\begin{array}{ccccccc}
0 & \to & W_{k+1}(V_2) & \to & W_{4m+4}^\vee & \to & \ker \varepsilon_2 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H_{2m+1}^\vee \otimes V_2 & \to & H_{2m+1}^\vee \otimes V^\vee & \to & H_{2m+1}^\vee \otimes V_2' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow_{\text{mult}} & & \downarrow & & \downarrow_{\varepsilon_2} \\
0 & \to & \ker \varepsilon_1 & \to & H_{4m}^\vee & \xrightarrow{\varepsilon_1} & H^1(E|l) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

where we put \(W_{k+1}(V_2) := H^0(E|l)\). By (12.20) we have \(\dim W_{k+1}(V_2) = k + 1\), \(\dim \ker \varepsilon_1 = 4m - k + 1\), and \(\dim \ker \varepsilon_2 = 4m - k + 3\). This diagram yields the equality (compare (12.10))

\[W_{k+1}(V_2) = H_{2m+1}^\vee \otimes V_2 \cap W_{4m+4}^\vee,\]  

where the intersection is taken in \(H_{2m+1}^\vee \otimes V^\vee\). We put

\[Z_{2,k}(V_2) := \{x \in P(W_{k+1}(V_2)) \mid \rk(x) = 2\}.
\]

The equality (12.25) and Lemma 12.2 yield a bijection

\[Z_{2,k}(V_2) \xrightarrow{\sim} f_2^{-1}(V_2).\]  

Consider the graph of incidence \(\Gamma_{2,k}(V_2) := \{(x, U) \in Z_{2,k}(V_2) \times G(2, H_{2m+1}^\vee) \mid U = U_2(x)\}\) with the projections \(Z_{2,k}(V_2) \xrightarrow{p_2} \Gamma_{2,k}(V_2) \xrightarrow{q_2} G(2, H_{2m+1}^\vee)\). By construction, \(p_2(\Gamma_{2,k}(V_2)) = Z_{2,k}(V_2)\) and the projection \(p_2 : \Gamma_{2,k}(V_2) \to Z_{2,k}(V_2)\) is a bijection. Hence,

\[\dim q_2(\Gamma_{2,k}(V_2)) \leq \dim \Gamma_{2,k}(V_2) = \dim Z_{2,k}(V_2) \leq \dim P(W_{k+1}(V_2)) = k.\]  

Consider the graph of incidence

\[\Pi_{2,k}(V_2) = \{(U, V_m) \in q_2(\Gamma_{2,k}(V_2)) \times \Sigma_{2,k}(V_2) \mid U \subset V_m\}\]

with the projections \(q_2(\Gamma_{2,k}(V_2)) \xleftarrow{\pr_1} \Pi_{2,k}(V_2) \xrightarrow{\pr_2} \Sigma_{2,k}(V_2)\) and the fibre

\[\pr_1^{-1}(U) \simeq G(m - 2, H_{2m+1}^\vee/U)\]

over an arbitrary point \(U \in q_2(\Gamma_{2,k}(V_2))\) (compare (12.4) and (12.13)). The projection \(\Pi_{2,k}(V_2) \xrightarrow{\pr_2} \Sigma_{2,k}(V_2)\) is surjective by (12.26). Hence, using (12.27), we obtain that

\[
\dim \Sigma_{2,k}(V_2) \leq \dim \Pi_{2,k}(V_2) = \dim q_2(\Gamma_{2,k}(V_2)) + \dim G(m - 2, H_{2m+1}^\vee/U)
\leq k + (m - 2)(m + 1) = m^2 - m - 2 + k
\leq \dim G - (2m - k + 2), \quad 1 \leq k \leq 2m.
\]  

(12.28)
We now consider the case \( k = 0 \). Then \( h^0(E|l) = 2 \) and, accordingly, \( \dim q_2(\Gamma_{2,0}(V_2)) \leq \dim \Gamma_{2,0}(V_2) = \dim Z_{2,0}(V_2) \leq \dim P(W_1(V_2)) = 1 \) instead of (12.27). Arguing as above, we obtain for \( k = 0 \) that

\[
\dim \Sigma_{2,0}(V_2) \leq 1 + (m - 2)(m + 1) = m^2 - m - 1 = \dim G - (2m + 1).
\]

This inequality together with (12.28), (12.22), Lemma 12.3 and the assumption \( m \geq 3 \) yields that \( \dim \Sigma_2 < \dim G \). Hence (12.3) holds for \( r = 2 \).

**Case \( r = 1 \).** Here our notation and arguments are similar to those used in the previous cases \( r = 4, 3, 2 \). Consider the projection \( f_1: Z_1 \to P(V^\vee) = (\mathbb{P}^3)^\vee \), \( x \mapsto V_1(x) \), where the pair \((U_1(x), V_1(x))\) of 1-dimensional spaces \( U_1(x) \subset H_{2m+1}^\vee \) and \( V_1(x) \subset V^\vee \) is uniquely determined by the point \( x \) via the condition \( x \in P(U_1(x) \otimes V_1(x)) \) since \( \text{rk}(x) = 1 \) (see Lemma 12.2). Given a subspace \( V_1 \in (\mathbb{P}^3)^\vee \), we put

\[
\Sigma_1(V_1) := \{V_m \in G \mid V_m \supset U_1(x) \text{ for some point } x \in f_1^{-1}(V_1)\}.
\]

Then, in analogy with (12.6), we have

\[
\Sigma_1 = \bigcup_{V_1 \in (\mathbb{P}^3)^\vee} \Sigma_1(V_1).
\]

(12.29)

Hence,

\[
\dim \Sigma_1 \leq \dim \Sigma_1(V_1) + 3.
\]

(12.30)

We want to estimate the dimension of \( \Sigma_1(V_1) \) for every 1-dimensional subspace \( V_1 \) of \( V^\vee \). The subspace \( V_1 \) determines a commutative diagram

\[
\begin{array}{cccc}
0 & 0 & \Omega_{p3} & \Omega_{p3} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & V_1 \otimes \mathcal{O}_{p3}(-1) & V^\vee \otimes \mathcal{O}_{p3}(-1) & V_3 \otimes \mathcal{O}_{p3}(-1) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{O}_{p3}(-1) & \mathcal{O}_{p3}(-1) & \mathcal{O}_{p3}(-1) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0
\end{array}
\]

(12.31)

We note that the point \( V_1 \in (\mathbb{P}^3)^\vee \) determines a projective plane \( P(V_1) \) in \( \mathbb{P}^3 \) and put \( B(E) := \{V_1 \in (\mathbb{P}^3)^\vee \mid h^0(E|_{P(V_1)}) \neq 0\} \). It is known that \( \dim B(E) \leq 2 \) for \( m \geq 1 \) (see [3]). Moreover, by (12.18), we have

\[
h^0(E|_{P(V_1)}) = 1, \quad V_1 \in B(E).
\]

(12.32)
Passing to cohomology in the diagram (12.31) twisted by $E$ and using the equation $h^0(E) = 0$ for $[E] \in I_{2m+1}$, we get a diagram

\[
\begin{array}{cccccc}
0 & & H^0(E|P(V_1)) & & \\
\downarrow & & \downarrow & & \\
W^\vee_{4m+4} & \longrightarrow & W^\vee_{4m+4} & \longrightarrow & \\
\downarrow & & \downarrow & & \\
0 & \longrightarrow & H^\vee_{2m+1} \otimes V_1 & \longrightarrow & H^\vee_{2m+1} \otimes V_1 & \longrightarrow & H^\vee_{2m+1} \otimes V_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^0(E|P(V_1)) & \longrightarrow & H^\vee_{2m+1} & \longrightarrow & H^1(E|P(V_1)) & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\] (12.33)

Let $V_1 \in B(E)$. Putting $W(V_1) := \ker (\text{mult} \circ \lambda) = H^0(E|P(V_1))$, where $\dim W_1(V_1) = 1$ by (12.32), we obtain the following commutative diagram from (12.33):

\[
\begin{array}{cccccc}
0 & & 0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_1(V_1) & \longrightarrow & W^\vee_{4m+4} & \longrightarrow & W^\vee_{4m+4}/W_1(V_1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^\vee_{2m+1} \otimes V_1 & \longrightarrow & H^\vee_{2m+1} \otimes V_1 & \longrightarrow & H^\vee_{2m+1} \otimes V_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^\vee_{2m+1}/W_1(V_1) & \longrightarrow & H^\vee_{4m} & \longrightarrow & H^1(E|P^2(V_1)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]

It yields the relation

\[
W_1(V_1) = H^\vee_{2m+1} \otimes V_1 \cap W^\vee_{4m+4},
\] (12.34)

where the intersection is taken in $H^\vee_{2m+1} \otimes V^\vee$. We put

\[
Z_1(V_1) := \begin{cases} 
\emptyset & \text{if } V_1 \notin B(E), \\
P(W_1(V_1)) = \{\text{pt}\} & \text{if } V_1 \in B(E).
\end{cases}
\]

The relation (12.34) and Lemma 12.2 yield a bijection

\[
Z_1(V_1) \overset{\cong}{\rightarrow} f_1^{-1}(V_1), \quad V_1 \in (\mathbb{P}^3)^\vee.
\] (12.35)
Consider the graph of incidence \( \Gamma_1(V_1) := \{(x, U) \in Z_1(V_1) \times P(H^\vee_{2m+1}) \mid U = U_1(x)\} \) with the projections \( Z_1(V_1) \overset{pr_1}{\rightarrow} \Gamma_1(V_1) \overset{pr_2}{\rightarrow} P(H^\vee_{2m+1}) \). By construction, \( p_1(\Gamma_1(V_1)) = Z_1(V_1) \) and the projection \( p_2: \Gamma_1(V_1) \rightarrow Z_1(V_1) \) is a bijection. Hence,
\[
\dim q_1(\Gamma_1(V_1)) \leq \dim \Gamma_1(V_1) = \dim Z_1(V_1) \leq 0. \tag{12.36}
\]

Consider the graph of incidence
\[
\Pi_1(V_1) = \{(U, V_m) \in q_1(\Gamma_1(V_1)) \times \Sigma_1(V_1) \mid U \subset V_m \}
\]
with the projections \( q_1(\Gamma_1(V_1)) \overset{pr_1}{\leftarrow} \Pi_1(V_1) \overset{pr_2}{\rightarrow} \Sigma_1(V_1) \) and the fibre
\[
pr_1^{-1}(U) \simeq G(m - 1, H^\vee_{2m+1}/U)
\]
over an arbitrary point \( U \in q_1(\Gamma_1(V_1)) \). The projection \( \Pi_1(V_1) \overset{pr_2}{\rightarrow} \Sigma_1(V_1) \) is surjective by (12.35). Hence, using (12.36), we have
\[
\dim \Sigma_1(V_1) \leq \dim \Pi_1(V_1) = \dim q_1(\Gamma_1(V_1)) + \dim G(m - 1, H^\vee_{2m+1}/U)
\]
\[
\leq 0 + (m - 1)(m + 1) = m^2 - 1.
\]

This together with (12.30) and the assumption \( m \geq 3 \) yields that \( \dim \Sigma_1 \leq m^2 + 2 = \dim G + 2 - m < \dim G \). Hence (12.3) holds for \( r = 1 \). Theorem 4.1 is proved. \( \square \)

### 12.2. Proof of Proposition 7.3.

Before proving Proposition 7.3, we give some auxiliary arguments. For every point \( B \in S_{m+1} \) we denote the induced homomorphism by \( B: S^2 H_{m+1} \rightarrow \wedge^2 V^\vee \). Consider a morphism of affine varieties
\[
b: H_{m+1} \times S_{m+1} \rightarrow \wedge^2 V^\vee, \quad (h, B) \mapsto \hat{B}(h \otimes h). \tag{12.37}
\]

Fix a basis \( e_1, e_2, e_3, e_4 \) in \( V \). Then the point \( B \in S_{m+1} \) (regarded as a homomorphism \( B: H_{m+1} \otimes V \rightarrow H^\vee_{m+1} \otimes V^\vee \)) can be represented as a skew-symmetric block matrix
\[
B = \begin{pmatrix}
0 & A_{12} & A_{13} & A_{14} \\
-A_{12} & 0 & A_{23} & A_{24} \\
-A_{13} & -A_{23} & 0 & A_{34} \\
-A_{14} & -A_{24} & -A_{34} & 0
\end{pmatrix}, \tag{12.38}
\]
where \( A_{ij} \in S^2 H^\vee_{m+1}, \ 1 \leq i < j \leq 4 \). Here we regard the \( A_{ij} \) as quadratic forms
\[
H_{m+1} \rightarrow k, \quad x \mapsto A_{ij}(x), \ 1 \leq i < j \leq 4, \tag{12.39}
\]
on \( H_{m+1} \). Hence we have corresponding quadrics in the projective space \( P(H_{m+1}) \simeq \mathbb{P}^m \):
\[
Q_{ij}(B) := \{\langle x \rangle \in P(H_{m+1}) \mid A_{ij}(x) = 0\}, \quad 1 \leq i < j \leq 4. \tag{12.40}
\]

Let \( K \subset \wedge^2 V^\vee \) be the cone of decomposable tensors, \( K = \{w \in \wedge^2 V^\vee \mid \rk(w: V \rightarrow V^\vee) \leq 2\} \). For \( m \geq 1 \) we put
\[
M_{m+1} := \{B \in S_{m+1} \mid b(H_{m+1} \times \{B\}) \subset K\}. \tag{12.41}
\]
By construction, $M_{m+1}$ is a closed subset of $S_{m+1}$. We regard it as a reduced subscheme of $S_{m+1}$.

We first consider the cases $m = 0, 1, 2$. The following assertions are proved by direct computation.

(i) $M_1, M_2$ and $M_3$ are irreducible and, moreover,

$$M_1 = K, \quad M_{m+1} \subset S_{m+1} \setminus (S_{m+1})^0, \quad \text{codim}_{m+1} M_{m+1} = 2, \quad m = 1, 2. \tag{12.42}$$

(ii) The set $M_3^* := \{B \in M_3 \mid Y_3(B) := Q_{13}(B) \cap Q_{23}(B) \text{ is a quadruple of distinct points in the projective plane } P(H_3)\}$ is open in $M_3$.

Proceeding to the case $m \geq 3$, we put

$$S_{m+1}^* := \{B \in S_{m+1} \mid Y_{m+1}(B) := Q_{13}(B) \cap Q_{23}(B) \text{ is an integral scheme of codimension 2 in the projective space } P(H_{m+1})\}. \tag{12.43}$$

Since $m \geq 3$, the set $S_{m+1}^*$ is open and dense in $S_{m+1}$.

**Lemma 12.4.** Suppose that $m \geq 3$ and $B \in S_{m+1}^* \cap M_{m+1}$. Then $B \notin S_{m+1}^0$.

*Proof.* We represent the point $B \in S_{m+1}^* \cap M_{m+1}$ as a matrix (12.38). Using the notation in (12.39) for an arbitrary point $x \in H_{m+1}$, we get a skew-symmetric $4 \times 4$ matrix with entries in $k$:

$$B(x) = \begin{pmatrix} 0 & A_{12}(x) & A_{13}(x) & A_{14}(x) \\ -A_{12}(x) & 0 & A_{23}(x) & A_{24}(x) \\ -A_{13}(x) & -A_{23}(x) & 0 & A_{34}(x) \\ -A_{14}(x) & -A_{24}(x) & -A_{34}(x) & 0 \end{pmatrix}. \tag{12.44}$$

The condition $B \in M_{m+1}$ means by definition that the matrix $B(x)$ is degenerate, that is, its Pfaffian is identically equal to zero as a polynomial function on $H_{m+1}$:

$$A_{12}(x)A_{34}(x) - A_{13}(x)A_{24}(x) + A_{14}(x)A_{23}(x) \equiv 0, \quad x \in H_{m+1}. \tag{12.45}$$

Since $B \in S_{m+1}^*$, it follows from (12.40) and (12.43) that the quadrics $Q_{13}(B)$ and $Q_{23}(B)$ are irreducible and their intersection $Y := Y_{m+1}(B)$ is an integral scheme of codimension 2 in $P(H_{m+1})$. In this case it follows from (12.45) that either $Q_{12}(B) \supset Y$, or $Q_{34}(B) \supset Y$. Suppose, for example, that $Q_{34}(B) \supset Y_{m+1}(B)$. This means that $A_{34}(x) \in H^0(I_{Y,p_m}(2))$. Passing to the spaces of sections in the exact triple

$$0 \to \mathcal{O}_{p_m}(-2) \to 2\mathcal{O}_{p_m} \xrightarrow{A_{13}(x), A_{23}(x)} I_{Y,p_m}(2) \to 0,$$

we obtain that $A_{34}(x) = \alpha A_{13}(x) + \beta A_{23}(x)$ for some $\alpha, \beta \in k$. Substituting this in (12.45), we have

$$A_{13}(x)(\alpha A_{12}(x) - A_{24}(x)) + A_{23}(x)(\beta A_{12}(x) + A_{14}(x)) \equiv 0.$$

Since $Q_{13}$ and $Q_{23}$ are irreducible quadrics, this identity means that one of the following sets of relations holds.

(i) $A_{23} = \lambda A_{13}$, $A_{24} - \alpha A_{12} = \lambda(\beta A_{12} + A_{14})$ for some $\lambda \in k$.

(ii) $\beta A_{12} + A_{14} = \mu A_{13}$, $A_{24} - \alpha A_{12} = \mu A_{23}$ for some $\mu \in k$. 

Substituting the relations (i) in (12.38) and putting $\gamma = \alpha + \lambda \beta$, we see that

$$
B = \begin{pmatrix}
0 & A_{12} & A_{13} & A_{14} \\
-A_{12} & 0 & \lambda A_{13} & \gamma A_{12} + \lambda A_{14} \\
-A_{13} & -\lambda A_{23} & 0 & \gamma A_{13} \\
-A_{14} & -\gamma A_{12} - \lambda A_{14} & -\gamma A_{13} & 0
\end{pmatrix}.
$$

(12.46)

Multiplying the first block-column of $B$ by $\lambda$ and adding it to the fourth block-column, and then repeating this procedure for block-rows, we obtain the matrix

$$
B' = \begin{pmatrix}
0 & A_{12} & A_{13} & A_{14} \\
-A_{12} & 0 & \lambda A_{13} & \lambda A_{14} \\
-A_{13} & -\lambda A_{13} & 0 & 0 \\
-A_{14} & -\lambda A_{14} & 0 & 0
\end{pmatrix},
$$

(12.47)

which is degenerate. Hence $B$ is degenerate. A similar computation with the set of relations (ii) also yields that $B$ is degenerate. \( \square \)

It follows from Lemma 12.4 that, for every irreducible component $M'_{m+1}$ of $M_{m+1}$,

$$
1 \leq \text{codim}_{S_{m+1}} M'_{m+1} \leq 2, \quad m \geq 3.
$$

(12.48)

Indeed, the lemma yields that $S^*_{m+1} \cap S^0_{m+1} \cap M_{m+1} = \emptyset$. Since $S^*_{m+1} \cap S^0_{m+1}$ is an open dense subset of $S_{m+1}$, we have $M_{m+1} \neq S_{m+1}$, that is, $1 \leq \text{codim}_{S_{m+1}} M_{m+1}$. On the other hand since $K$ is a non-empty divisor in $S^2 V'$, we see that $b^{-1}_{m+1}(K)$ is a non-empty divisor in $H_{m+1} \times S_{m+1}$. Since $M_{m+1}$ is non-empty (in fact $\{0\} \in M_{m+1}$), a computation of dimensions of the fibres of the natural projection $b^{-1}_{m+1}(K) \to S_{m+1}$ shows that $\text{codim}_{S_{m+1}} M'_{m+1} \leq 2$ for every irreducible component $M'_{m+1}$ of $M_{m+1}$. This proves (12.48).

**Lemma 12.5.** Suppose that $m \geq 3$ and $M'_{m+1}$ is an arbitrary irreducible component of $M_{m+1}$. Then $S^*_{m+1} \cap M'_{m+1} \neq \emptyset$. Therefore $S^*_{m+1} \cap M'_{m+1}$ is an open dense subset of $M'_{m+1}$.

**Proof.** 1) We first consider the case $m = 3$. Choose coordinates $x_1, \ldots, x_4$ in $H_4$, and let $H_1, H_3$ be the subspaces of $H_4$ given by the equations $x_1 = x_2 = x_3 = 0$ and $x_4 = 0$ respectively. The direct sum decomposition $H_4 = H_1 \oplus H_3$ induces an inclusion $S_1 \oplus S_3 \hookrightarrow S_4$ of the direct summand. Regarding this inclusion as an embedding $S_1 \times S_3 \hookrightarrow S_4$ of the affine space, we obtain from (12.42) and definition (12.41) that

$$
M_3 = (\{0\} \times S_3) \cap M_4, \quad K = (S_1 \times \{0\}) \cap M_4.
$$

(12.49)

This together with (12.48) and the irreducibility of $M_3$ (see property (i) above) shows that, for every irreducible component $M'_4$ of $M_4$,

$$
M_3 = (\{0\} \times S_3) \cap M'_4, \quad K = (S_1 \times \{0\}) \cap M'_4.
$$

(12.50)

Note that (12.42) and (12.48) yield the equality

$$
\text{codim}_{S_4} M'_4 = 2.
$$

(12.51)
We take an arbitrary point $B' \in M_3^*$ and let $A_{i3}(B')(x_1, x_2, x_3)$ be the quadratic forms on $H_3$ corresponding to the entries $A_{i3}(B')$, $i = 1, 2$, of the matrix $B'$. Then the set $Y_3(B')$ is given in the projective space $P(H_3)$ by the equations $A_{i3}(B')(x_1, x_2, x_3) = 0$, $i = 1, 2$. We now take an arbitrary point $B'' \in S_1 \simeq \mathbb{P}^2$ and, using (12.38), regard $B''$ as a skew-symmetric matrix $(a_{ij}(B''))$. Then the point $B := (B', B'') \in S_1 \times S_3$ determines a scheme $Y_4(B)$ (see (12.43)), which is given in the projective space $P(H_4)$ by the equations

$$A_{i3}(B')(x_1, x_2, x_3) - a_{i3}(B''_x)x^2_{i} = 0, \quad i = 1, 2. \quad (12.52)$$

Consider the sets $U' = \{(B', B'') \in S_1 \times S_3 \mid Y_3(B') = Q_1(B') \cap Q_23(B')$ is a quadruple of distinct points in the plane $P(H_3)\}$ and $U'' = \{(B', B'') \in S_1 \times S_3 \mid a_{i3}(B'') \neq 0, \quad i = 1, 2\}$. These are open dense subsets of $S_1 \times S_3$, and it follows from (12.50) and property (ii) that the set $M''_4 := M_4' \cap U' \cap U''$ is open and dense in $M'_4$. Now, for an arbitrary point $B = (B', B'') \in M''_4$, we can rewrite equations (12.52) in the form

$$A(x_1, x_2, x_3) := A_{13}(B')(x_1, x_2, x_3)a_{23}(B'') - A_{23}(B')(x_1, x_2, x_3)a_{13}(B'') = 0, \quad A_{13}(B')(x_1, x_2, x_3) - a_{13}(B'')x^2 = 0. \quad (12.53)$$

Consider the conic $C(B) = \{A(x_1, x_2, x_3) = 0\}$ in $\mathbb{P}^2$. Then $M''_4 = \{B \in M'_4 \mid C(B)$ is an integral scheme$\}$ is an open dense subset of $M''_4$. By construction, the set $\{A_{13}(B')(x_1, x_2, x_3) = 0\} \cap C(B)$ coincides with the set $Y_3(B')$, which is by definition a quadruple of distinct points in $\mathbb{P}^2$. Therefore the equations (12.53) that define $Y_4(B)$ show that $Y_4(B)$ is a double covering of the conic $C(B)$ ramified at $Y_3(B')$. Hence $Y_4(B)$ is an integral elliptic curve of degree 4 in $\mathbb{P}^3$. In other words, $B \in S^*_4 \cap M'_4$. This means that $M''_4 \subset S^*_4 \cap M'_4$, whence $S^*_4 \cap M'_4$ is open and dense in $M'_4$.

2) The argument for $m \geq 4$ is similar to the above. We choose coordinates $x_1, \ldots, x_{m+1}$ in $H_{m+1}$ and let $H_{m-3}$ and $H_4$ be the subspaces of $H_{m+1}$ given by the equations $x_1 = \cdots = x_4 = 0$ and $x_5 = \cdots = x_{m+1} = 0$ respectively. The direct-sum decomposition $H_{m+1} = H_{m-3} \oplus H_4$ induces an inclusion $S_1 \hookrightarrow S_{m+1}$ of the direct summand. Regarding this inclusion as an embedding $S_4 \hookrightarrow S_{m+1}$ of the affine space, we obtain from definition (12.41) in analogy with (12.50) that

$$M_4 = S_4 \cap M_{m+1}. \quad (12.54)$$

Let $M_{m+1}'$ be an arbitrary irreducible component of $M_{m+1}$. It follows from (12.48), (12.51) and (12.54) that the set $(M'_4)^* = S^*_4 \cap M'_4$ is an open dense subset of $M_4'$ for every irreducible component $M'_4$ of $S_4 \cap M_{m+1}'$. By definition, every point $B \in (M'_4)^*$ is such that $Y_4(B)$ is an integral irreducible curve of degree 4 in $\mathbb{P}^3$. Then the construction of the embedding $S_4 \hookrightarrow S_{m+1}$ shows that, for such a point $B$ regarded as a point of $M'_{m+1}$, the scheme $Y_{m+1}(B)$ is a cone in $P(H_{m+1})$ over $Y_4(B)$. Therefore $Y_{m+1}(B)$ is an integral subscheme of codimension 2 in $P(H_{m+1})$, whence $B \in S^*_{m+1}$. This means that $S^*_{m+1} \cap M'_{m+1}$ is an open dense subset of $M'_{m+1}$. \(\square\)
Corollary 12.6. For every $m \geq 0$ we have $M_{m+1} \subset S_{m+1} \setminus S^0_{m+1}$.

Proof. For $m \leq 2$ this follows from (12.42). Suppose that $m \geq 3$ and let $M'_{m+1}$ be an arbitrary irreducible component of $M_{m+1}$. By Lemma 12.4 we have $S^*_m \cap M'_{m+1} \subset S_{m+1} \setminus S^0_{m+1}$. Since $S_{m+1} \setminus S^0_{m+1}$ is a closed irreducible subset of $S_{m+1}$ and, by Lemma 12.5, $S^*_m \cap M'_{m+1}$ is an open dense subset of the irreducible set $M'_{m+1}$, we have $M'_{m+1} \subset S_{m+1} \setminus S^0_{m+1}$. □

Proof of Proposition 7.3. Suppose that $D \in (S^\vee_{m+1})^0$. Hence $D$ is a non-degenerate homomorphism $D: H^\vee_{m+1} \otimes V \rightarrow H_{m+1} \otimes V$. Assume that the composite $j_D := j^\vee \circ D \circ j: H^\vee_m \otimes V^\vee \rightarrow H^\vee_{m+1}$ is degenerate for every monomorphism $j: H^\vee_m \hookrightarrow H^\vee_{m+1}$. To derive a contradiction, we regard $j$ in the dual way as a monomorphism $j_k: k \hookrightarrow H^\vee_{m+1}$. Consider the non-degenerate homomorphism $B := D^{-1}: H^\vee_{m+1} \otimes V \rightarrow H^\vee_m \otimes V^\vee$ and the induced skew-symmetric homomorphism $j_B := j_k^\vee \circ B \circ j_k: V \cong k \otimes V \rightarrow k^\vee \otimes V^\vee \cong V^\vee$. Then degeneracy of $j'_D$ is equivalent to that of $j_B$. As above, $B$ is represented by a skew-symmetric matrix (12.38). In this notation, the degeneracy of $j_B$ for every monomorphism $j_k: k \hookrightarrow H^\vee_{m+1}$ just means that the skew-symmetric $4 \times 4$ matrix $B(x)$ in (12.44) is degenerate for every vector $x \in H^\vee_{m+1}$, that is, by definition we have $B \in M_{m+1}$. Then the matrix $B$ is degenerate by Corollary 12.6. This contradiction proves the proposition. □

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**A. S. Tikhomirov**
Yaroslavl' State Pedagogical University
*E-mail: astikhomirov@mail.ru*

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