Continuity at a boundary point of solutions to quasilinear elliptic equations with generalized Orlicz growth and non-logarithmic conditions

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Abstract

We consider the Dirichlet problem for quasilinear elliptic equations with Musielak-Orlicz \((p,q)\)-growth and non-logarithmic conditions on the coefficients. A sufficient Wiener-type condition for the regularity of a boundary point is established.

Keywords: nonlinear elliptic equations; Musielak-Orlicz growth; non-logarithmic condition; boundary regularity; Wiener condition.

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1 Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) \((n \geq 2)\), \( \overline{\Omega} \) is the closure of \( \Omega \) in \( \mathbb{R}^n \), and \( \partial \Omega = \overline{\Omega} \setminus \Omega \) is the boundary of \( \Omega \). In this paper we study the regularity of boundary points \((x_0 \in \partial \Omega)\) for bounded solutions to the quasilinear elliptic equations with Musielak-Orlicz \((p,q)\)-growth and non-logarithmic Zhikov’s condition on the coefficients. In more detail this means that we consider equations of the form

\[
\text{div} \left( g(x,|\nabla u|) \frac{\nabla u}{|\nabla u|} \right) = 0, \quad x \in \Omega,
\]

where the function \( g(x,v) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+, \mathbb{R}_+ := [0, +\infty) \) satisfies the following assumptions:

\((g_0)\) \( g(\cdot, v) \in L^1(\Omega) \) for all \( v \in \mathbb{R}_+), \ g(x, \cdot) \) is continuous and increasing for all \( x \in \mathbb{R}^n), \lim_{v \to +0} g(x,v) = 0 \) and \( \lim_{v \to +\infty} g(x,v) = +\infty; \)

\((g_1)\) there exist \( c_1 > 0, \ q > 1 \) and \( b_0 > 0 \) such that

\[
\frac{g(x,w)}{g(x,v)} \leq c_1 \left( \frac{w}{v} \right)^{q-1},
\]

for all \( x \in \overline{\Omega} \) and for all \( w \geq v \geq b_0; \)

\((g_2)\) there exists \( p > 1 \) such that

\[
\frac{g(x,w)}{g(x,v)} \geq \left( \frac{w}{v} \right)^{p-1},
\]

for all \( x \in \overline{\Omega} \) and for all \( w \geq v > 0; \)
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there exist a non-decreasing function \( c_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) and a continuous, non-increasing function \( \lambda : (0, r_+) \to \mathbb{R}_+ \) such that for any ball \( B_r(x_0) \) centered at \( x_0 \in \Omega \), for all \( x_1, x_2 \in B_r(x_0) \) and for all \( 0 < r < v \leq K \), the following inequality holds:

\[
g(x_1, v/r) \leq c_2(K)e^{\lambda(r)}g(x_2, v/r). \tag{1.4}
\]

These assumptions are quite general and cover a wide class of elliptic equations with non-standard growth conditions. For example, the following functions:

1) variable exponent \( g(x, v) = v^{p(x)-1} \), cf. [1, 20, 21],

2) perturbed variable exponent \( g(x, v) = v^{p(x)-1}\ln(e + v) \), cf. [23, 35, 46, 47],

3) double phase \( g(x, v) = v^{p-1} + a(x)v^{q-1} \), \( 0 \leq a(x) \in L^\infty(\Omega) \), cf. [6, 8, 15, 16],

4) degenerate double phase \( g(x, v) = v^{p-1}(1 + b(x)\ln(1 + v)) \), \( 0 \leq b(x) \in L^\infty(\Omega) \), cf. [5, 7, 12],

5) double variable exponent \( g(x, v) = v^{p(x)-1} + v^{q(x)-1} \), cf. [13, 52, 60],

6) variable exponent double phase \( g(x, v) = v^{p(x)-1} + a(x)v^{q(x)-1} \), \( 0 \leq a(x) \in L^\infty(\Omega) \), cf. [38, 50]

satisfy assumptions \((g_0)-(g_3)\) if the exponents \( p, q, p(\cdot), q(\cdot) \) and the coefficients \( a(\cdot) \) and \( b(\cdot) \) are such that:

\[
\begin{align*}
& (i) \quad 1 < p < p(x) \leq q(x) < q < +\infty \quad \text{for all } x \in \Omega; \\
& (ii) \quad |p(x) - p(y)| + |q(x) - q(y)| \leq \frac{\lambda(|x - y|)}{\ln|x - y|}, \quad x, y \in \Omega, \quad x \neq y, \\
& \quad \quad \quad \text{the function } \frac{\lambda(r)}{\ln r} \text{ is non-decreasing on } (0, r_+), \lim_{r \to 0} \frac{\lambda(r)}{\ln r} = 0; \\
& (iii) \quad |a(x) - a(y)| \leq A|x - y|^\alpha e^{\lambda(|x - y|)}, \quad x, y \in \Omega, \quad x \neq y, \\
& \quad \quad \quad A > 0, \quad 0 < q - p \leq \alpha \leq 1, \\
& \quad \quad \quad \text{the function } r^\alpha e^{\lambda(r)} \text{ is non-decreasing on } (0, r_+), \lim_{r \to 0} r^\alpha e^{\lambda(r)} = 0; \\
& (iv) \quad |b(x) - b(y)| \leq \frac{Be^{\lambda(|x - y|)}}{\ln|x - y|}, \quad x, y \in \Omega, \quad x \neq y, \quad B > 0, \\
& \quad \quad \quad \text{the function } \frac{e^{\lambda(r)}}{\ln r} \text{ is non-decreasing on } (0, r_+), \lim_{r \to 0} \frac{e^{\lambda(r)}}{\ln r} = 0.
\end{align*}
\]

Recently, many studies have been devoted to questions of regularity under Zhikov’s [61] and Fan’s [20] logarithmic condition, when \( \lambda(r) \leq L < +\infty \) in [15, 17] (see e.g. [15, 16, 12, 15, 16, 21, 23, 35, 46, 47, 50, 57]), and now the elliptic theory has reached a completely satisfactory form (see original surveys [20, 41, 42, 48] and monograph [49]). In general, the logarithmic condition has significantly advanced the theory of function spaces with variable exponents [17, 19], which are an integral part of the generalized Orlicz (Musielak-Orlicz) spaces [27, 45]. Nonlinear differential equations in Musielak-Orlicz spaces is a fairly extensive topic for research (see the informative survey [14]). Here we only cite the recent papers [9, 10, 28, 30] on Hölder’s regularity and
Harnack’s inequality under the Musielak-Orlicz assumptions and logarithmic conditions in the elliptic framework.

The non-logarithmic case, when the function $\lambda(r)$ is unbounded in (1.4), (1.5)–(1.7), differs significantly from the logarithmic one. To our knowledge there are few regularity results in this direction. Zhikov [62] obtained a generalization of the logarithmic condition on the variable exponent $p(x) \geq p > 1$ which guarantees the density of smooth functions in the Sobolev space $W^{1,p(x)}(\Omega)$:

$$|p(x) - p(y)| \leq L \frac{|\ln |x-y||}{|\ln |x-y||}, \quad x, y \in \Omega, \quad x \neq y, \quad L < p/n. \quad (1.8)$$

Later Zhikov and Pastukhova [63] under the same condition proved higher integrability of the gradient of solutions to the $p(x)$-Laplace equation.

Interior continuity, continuity up to the boundary and Harnack’s inequality to the $p(x)$-Laplace equation were proved by Alkhutov and Krasheninnikova [3], Alkhutov and Surnachev [4] and Surnachev [55] under condition (1.5) on $p(\cdot)$ and the additional assumption

$$\int_0^R \exp \left(-C \exp(\beta \lambda(r))\right) \frac{dr}{r} = +\infty \quad (1.9)$$

with some constants $C > 0$, $\beta > 1$ depending only upon the data. Particularly, the function $\lambda(r) = L \ln \ln r^{-1}$, $L\beta < 1$ satisfies the above conditions. These results were generalized in [51,54] for a wide class of elliptic and parabolic equations with non-logarithmic generalized Orlicz growth, and improved in [25] for double phase elliptic equations (see type 3 and (1.6)) with $\lambda(r) = L \ln \ln r^{-1}$ (cf. [62,63] and (1.8)).

The purpose of this article is to establish a Wiener-type sufficient condition for the regularity of a boundary point $x_0 \in \partial\Omega$ for bounded solutions of Eq. (1.1) under assumptions (g0)–(g3). Boundary regularity in terms of the Sobolev variational capacity is a classical topic in the contemporary theory of PDE, which goes back to Lebesgue [33] and Wiener [58,59]. The papers [37], [40], [22], [53], [36], [31] provided the basic contribution to the boundary regularity for quasilinear elliptic equations with standard growth conditions. For the variable $p(x)$-growth, we cite [2] in the log-case, and [4] for non-logarithmic conditions (1.5), (1.9) on the exponent $p(\cdot)$. In the Orlicz case, the Wiener criterion for the regularity of Sobolev boundary point was established by Ki-Ahm Lee and Se-Chan Lee [34]. Under generalized Orlicz growth and logarithmic counterparts of (1.4), Harjulehto and Hästö [26] obtained the following capacity density condition for the regularity of a boundary point $x_0 \in \partial\Omega$: there exists $c \in (0,1)$ and $R > 0$ such that $\text{Cap}_G(B_\rho(x_0) \setminus \Omega, B_{2\rho}(x_0)) \geq c \text{Cap}_G(B_\rho(x_0), B_{2\rho}(x_0))$ for all $0 < \rho < R$, $G(\cdot, v) \approx g(\cdot, v)v$. This result was improved by Benyaiche and Khifi [11] to a full-fledged Wiener-type criterion: the point $x_0$ is regular if and only if for some $R > 0$,

$$\int_0^R g_{x_0}^{-1} \left(\frac{\text{Cap}_G(B_\rho(x_0) \setminus \Omega, B_{2\rho}(x_0))}{\rho^{n-1}}\right) d\rho = +\infty$$

where $g_{x_0}^{-1}(\cdot)$ denotes the inverse function to the function $g(x_0, \cdot)$.
In this paper, we show (see Theorem 2.4) that the corresponding regularity condition for bounded solutions to Eq. (1.1) under assumptions \((g_0)-(g_3)\) has the following form:

\[
\int_0^{g_{x_0}^{-1}} \Lambda(-C, 3n, \rho) \frac{\text{Cap}_G(B_{\rho/32}(x_0) \setminus \Omega, B_{\rho}(x_0))}{\rho^{n-1}} \, d\rho = +\infty,
\]

where \(C\) is a positive constant depending only upon the data, and

\[
\Lambda(c, \beta, \rho) = \exp \left( e^{e^{\beta\lambda(\rho)}} \right) \text{ for any } c, \beta \in \mathbb{R} \text{ and } \rho \in (0, r^*). \tag{1.10}
\]

Thus, we expand the previous results in several directions. Namely, the result of Alkhutov and Surnachev [4] extends from the \(p(x)\)-Laplace equation to equations with generalized Orlicz growth, and the results of Benyaiche and Khlifi [10, 11] extend to the non-logarithmic generalized Orlicz growth. In particular, our results are new for bounded solutions to equations of types 2, 4–6. The main results are set out in Section 2, and their proofs are contained in Sections 3–5.

## 2 Main Results

Before formulating the main results, we remind the definition of a weak solution to Eq. (1.1). Moreover, throughout the article, we use the well-known notation for sets in \(\mathbb{R}^n\), for function spaces and their elements etc. (see, for instance, [24, 27, 32]). We set

\[
G(x, v) = g(x, v)v \text{ for } x \in \mathbb{R}^n, \ v \geq 0 \tag{2.1}
\]

and write \(u \in W^{1, G}(\Omega)\) if \(u \in W^{1, 1}(\Omega)\) and \(\int_{\Omega} G(x, |\nabla u|) \, dx < +\infty\). We also need a class of functions \(W^{1,G}_0(\Omega) := W^{1,1}_0(\Omega) \cap W^{1,G}(\Omega)\).

**Definition 2.1** (definition of solutions). We say that a function \(u : \Omega \to \mathbb{R}\) is a bounded weak solution to Eq. (1.1) under hypotheses \((g_0)-(g_3)\) if \(u \in W^{1,G}(\Omega) \cap L^\infty(\Omega)\) and for any \(\varphi \in W^{1,G}_0(\Omega)\) the following integral identity holds:

\[
\int_{\Omega} g(x, |\nabla u|) \frac{\nabla u}{|\nabla u|} \nabla \varphi \, dx = 0. \tag{2.2}
\]

Further we also need the following definitions.

**Definition 2.2** (regular boundary points). We say that \(x_0 \in \partial \Omega\) is a regular boundary point of the domain \(\Omega\) for Eq. (1.1) if for every \(f \in C(\overline{\Omega}) \cap W^{1,G}(\Omega)\) and for every bounded weak solution \(u\) of Eq. (1.1) satisfying the condition \(u - f \in W^{1,G}_0(\Omega)\) the following equality holds

\[
\lim_{\Omega} u(x) = f(x_0).
\]

**Definition 2.3** (capacity). Let \(E\) be a compact subset of \(B_{\rho}(x_0)\), and let \(\mathfrak{M} = \mathfrak{M}(E, B_{\rho}(x_0))\) be the class of all functions \(v \in W^{1,G}_0(B_{\rho}(x_0))\) satisfying \(v \geq 1\) on \(E\) in the sense of \(W^{1,1}_0(B_{\rho}(x_0))\). Then the relative \(G\)-capacity of \(E\) is defined by

\[
\text{Cap}_G(E, B_{\rho}(x_0)) = \inf_{v \in \mathfrak{M}} \int_{B_{\rho}(x_0)} G(x, |\nabla \varphi|) \, dx. \tag{2.3}
\]
Now we can state our main result. In this case, we refer to the parameters $M = \text{ess sup}_\Omega |u|$, $n$, $p$, $q$, $c_1$, $c_2(M)$, $b_0$ as our structural data and write $\gamma$ for constants if they can be quantitatively determined a priori only in terms of the above quantities. The generic constant $\gamma$ may vary from line to line.

**Theorem 2.4** (Wiener type regularity condition). Let hypotheses $(g_0)$–$(g_3)$ be fulfilled. Then a boundary point $x_0 \in \partial \Omega$ is regular in the sense of Definition 2.4 if under notation $(1.10)$, $(2.1)$ and $(2.6)$ the following condition is satisfied:

$$
\int g_{x_0}^{-1} \left( \Lambda(-\gamma, 3n, \rho) \frac{\text{Cap}_G(B_{\rho/32}(x_0) \setminus \Omega, B_{\rho}(x_0))}{\rho^{n-1}} \right) d\rho = +\infty. \tag{2.4}
$$

Here we use the notation $g_{x_0}^{-1}(\cdot)$ for the inverse function to the function $g(x_0, \cdot)$.

The proof of Theorem 2.4 is carried out by contradiction (see Section 3) in the spirit of the classic paper by R. Gariepy and W. P. Ziemer [22]. In fact, Theorem 2.4 is a consequence of an integral growth estimate on the gradient of solutions (cf. [22, Theorem 2.1]). To state it we need additional notations. Let $u$ be a bounded weak solution of Eq. $(1.1)$ satisfying the condition $u - f \in W^{1,1}_0(\Omega)$. Next, let $x_0 \in \partial \Omega$ be a boundary point, and let $l \in \mathbb{R}$, $l \neq f(x_0)$. By the continuity of $f$ on $\partial \Omega$, there exists a sufficiently small radius $\rho > 0$ such that

$$
l \geq M_f(r) := \sup_{\partial \Omega \cap B_r(x_0)} f \quad \text{if} \quad l > f(x_0) \quad \text{and} \quad 0 < r \leq \rho,
$$

$$
l \leq m_f(r) := \inf_{\partial \Omega \cap B_r(x_0)} f \quad \text{if} \quad l < f(x_0) \quad \text{and} \quad 0 < r \leq \rho.
$$

We set $c_+ = \max \{c, 0\}$ for every $c \in \mathbb{R}$, and

$$
\Omega_{l,r} = \begin{cases} 
\{u > l\} \cap B_r(x_0) & \text{if} \quad l > f(x_0) \quad \text{and} \quad 0 < r \leq \rho, \\
\{u < l\} \cap B_r(x_0) & \text{if} \quad l < f(x_0) \quad \text{and} \quad 0 < r \leq \rho,
\end{cases} \tag{2.5}
$$

$$
u_l = \begin{cases} 
(u - l)_+ & \text{on} \ \Omega \quad \text{if} \quad l \geq M_f(\rho), \\
(l - u)_+ & \text{on} \ \Omega \quad \text{if} \quad l \leq m_f(\rho), \\
0 & \text{on} \ \mathbb{R}^n \setminus \Omega,
\end{cases} \quad M_f(\rho) = \text{ess sup}_{B_r(x_0)} u_l \quad \text{for} \quad 0 < r \leq \rho, \tag{2.6}
$$

$$
u_{l,\rho} = M_f(\rho) - u_l + 2b_0 \rho.
$$

**Theorem 2.5** (growth gradient estimate). Let hypotheses $(g_0)$–$(g_3)$ be fulfilled. Then, in terms of notation $(1.10)$, $(2.1)$, $(2.5)$ and $(2.6)$, the following inequality holds:

$$
\int_{\Omega_{l,\rho/16}} G(x, \nabla (u_{l,\rho} \eta)) \, dx 
\leq \Lambda(\gamma, 3n, \rho)(M_l(\rho) + 2b_0 \rho)\rho^{n-1} \left( x_0, \frac{M_l(\rho) - M_l(\rho/4) + 2b_0 \rho}{\rho} \right). \tag{2.7}
$$

Here $\eta \in C_0^\infty(B_{\rho/16}(x_0))$ is a function such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{3\rho/64}(x_0)$, $|\nabla \eta| \leq 64/\rho$.

In turn, the proof of Theorem 2.5 essentially relies on the weak Harnack inequality for solutions of Eq. $(1.1)$ at the boundary $\partial \Omega$. Let us state the corresponding result. In this case, we also use the notation

$$
\int_E f \, dx := |E|^{-1} \int_E f \, dx. \tag{2.8}
$$
for any measurable set $E \subset \mathbb{R}^n$ with $|E| \neq 0$ and $f \in L^1(E)$, where $|E|$ denotes the $n$-dimensional Lebesgue measure of $E$.

**Theorem 2.6** (boundary weak Harnack inequality). *Let hypotheses $(g_0)-(g_3)$ be fulfilled. Then, for every $0 < s < n/(n - 1)$, the following inequality holds in terms of notation $(1.10)$, $(2.6)$ and $(2.8)$:

$$
\left( \int_{B_{\rho/(s+\varepsilon)}} g^s(x_0, u_{l,\rho}/\rho) \, dx \right)^{1/s} \leq \Lambda(\gamma, 3n, \rho) g \left( x_0, \frac{M_l(\rho) - M_l(\rho/4) + 2b_0\rho}{\rho} \right).
$$

(2.9)

Here a positive constant $\gamma$ additionally depends on $s$.

**Remark 2.7.** The interior weak and strong Harnack’s inequalities for bounded solutions of Eq. $(1.1)$ under hypotheses $(g_0)-(g_3)$ were established in [51] (see also [9, 10] in the context of generalized Orlicz growth and logarithmic condition). But unlike [9, 10], the traditional Moser method [43, 44] is inapplicable for the proof of Theorem 2.6, since checking whether the logarithm of solutions belongs to the BMO space is a problem for the non-logarithmic condition $(1.4)$ with unbounded $\lambda(r)$. We use Trudinger’s arguments [56] adapted to Eq. $(1.1)$ near the boundary $\partial \Omega$ (see Section 4).

3 A sufficient condition for the regularity of a boundary point: proof of Theorem 2.4

Let $x_0 \in \partial \Omega$, $f \in C(\overline{\Omega}) \cap W^{1,G}(\Omega)$, and let $u$ be a bounded weak solution of Eq. $(1.1)$ satisfying the condition $u - f \in W_0^{1,G}(\Omega)$. We need to prove that $\lim_{\Omega^3 x \to x_0} u = f(x_0)$. This equality will be established if we show that $f(x_0) \leq \lim_{\rho \to 0} \text{ess inf}_{\Omega \setminus B_{\rho}(x_0)} u$ and $\lim_{\rho \to 0} \text{ess sup}_{\Omega \setminus B_{\rho}(x_0)} u \leq f(x_0)$. The proofs of the both inequalities are completely similar and we will prove only the second one. We argue by contradiction and assume that

$$
L := \lim_{\rho \to 0} \text{ess sup}_{\Omega \setminus B_{\rho}(x_0)} u > f(x_0).
$$

(3.1)

Let $f(x_0) < l < L$, then $\lim_{\rho \to 0} M_l(\rho) = L - l > 0$ and $M_l(\rho)$ is bounded away from zero for sufficiently small $\rho$: there exist constants $a, \rho_0 \in (0, 1)$ such that

$$
M_l(\rho) > a > 0 \quad \text{for all } \rho \in (0, \rho_0).
$$

(3.2)

Let us prove the following inequality:

$$
\text{Cap}_G(\overline{B}_{\rho/32}(x_0) \setminus \Omega, B_{\rho}(x_0)) \leq a^{1-q} \Lambda(\gamma, 3n, \rho)^{\rho^\eta - 1} g \left( x_0, \frac{M_l(\rho) - M_l(\rho/4) + 2b_0\rho}{\rho} \right)
$$

(3.3)

for arbitrary fixed $\rho \in (0, \rho_0)$. To do this, we fix a function $\eta \in C^\infty_0(B_{\rho/16}(x_0))$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B_{3\rho/64}(x_0)$ and $|\nabla \eta| \leq 64/\rho$. We set

$$
v_{l,\rho} = \frac{u_l,\rho,\eta}{M_l(\rho) + 2b_0\rho} \quad \text{and} \quad E_{l,\rho/32} = \overline{B}_{\rho/32}(x_0) \cap \{ x \in \mathbb{R}^n : u_l(x) = 0 \}.
$$
It is clear that \( E_{l,\rho}/32 \setminus \Omega \subset E_{l,\rho}/32 \) and \( v_{l,\rho} \in M(E_{l,\rho}/32, B_{\rho}(x_0)) \), therefore by Definition 2.3 the following inequality holds:

\[
\text{Cap}_G(E_{l,\rho}/32, B_{\rho}(x_0)) \leq \text{Cap}_G(E_{l,\rho}/32, B_{\rho}(x_0)) \leq \int_{B_{\rho}/16(x_0)} G(x, |\nabla v_{l,\rho}|) \, dx. \tag{3.4}
\]

The integral in (3.4) can be estimated as follows:

\[
\int_{\Omega_{l,\rho}/16} G(x, |\nabla v_{l,\rho}|) \, dx \leq a^{1-q} \Lambda(\gamma, 3n, \rho)^{-1} \frac{M_l(\rho) - M_l(\rho/4) + 2b_0\rho}{\rho}, \tag{3.5}
\]

that implies (3.3). Indeed, if \( M_l(\rho) + 2b_0\rho \geq 1 \), then by condition (g2) and (2.7) we have

\[
g \left( x, \frac{|\nabla (u_{l,\rho})|}{M_l(\rho) + 2b_0\rho} \right) \leq (M_l(\rho) + 2b_0\rho)^{1-p} g(x, |\nabla (u_{l,\rho})|) \quad \text{in } \Omega_{l,\rho}/16,
\]

and on the set \( \{ |\nabla (u_{l,\rho})| \geq b_0 \} \):

\[
g \left( x, \frac{|\nabla (u_{l,\rho})|}{M_l(\rho) + 2b_0\rho} \right) \leq c_1 (M_l(\rho) + 2b_0\rho)^{1-q} g(x, |\nabla (u_{l,\rho})|).
\]

and on the set \( \{ |\nabla (u_{l,\rho})| < b_0 \} \):

\[
g \left( x, \frac{|\nabla (u_{l,\rho})|}{M_l(\rho) + 2b_0\rho} \right) \leq g \left( x, \frac{b_0}{M_l(\rho) + 2b_0\rho} \right) \leq c_1 (M_l(\rho) + 2b_0\rho)^{1-q} g(x, b_0)
\]

\[
\leq c_1 (M_l(\rho) + 2b_0\rho)^{1-q} \left( M_l(\rho) - M_l(\rho/4) + 2b_0\rho \right)
\]

\[
\leq c_1 (M_l(\rho) + 2b_0\rho)^{1-q} g \left( x, \frac{M_l(\rho) - M_l(\rho/4) + 2b_0\rho}{\rho} \right).
\]

Using these relations, (2.7) and (3.2), we obtain

\[
\int_{\Omega_{l,\rho}/16} G(x, |\nabla v_{l,\rho}|) \, dx
\]

\[
= \int_{\Omega_{l,\rho}/16 \cap \{|\nabla v_{l,\rho}| \geq b_0 \}} G(x, |\nabla v_{l,\rho}|) \, dx + \int_{\Omega_{l,\rho}/16 \cap \{|\nabla v_{l,\rho}| < b_0 \}} G(x, |\nabla v_{l,\rho}|) \, dx
\]

\[
\leq c_1 (M_l(\rho) + 2b_0\rho)^{1-q} \int_{\Omega_{l,\rho}/16} G(x, |\nabla (u_{l,\rho})|) \, dx
\]

\[
+ \gamma e^{\lambda(\rho)} \rho^{n-1} (M_l(\rho) + 2b_0\rho)^{1-q} g \left( x, \frac{M_l(\rho) - M_l(\rho/4) + 2b_0\rho}{\rho} \right)
\]

\[
\leq a^{1-q} \Lambda(\gamma, 3n, \rho)^{-1} \frac{M_l(\rho) - M_l(\rho/4) + 2b_0\rho}{\rho},
\]
which again leads to (3.3). Thus, inequality (3.3), and hence (3.3), are completely proved.

Now, we can rewrite inequality (3.3) in the following form:

\[
g_{x_0}^{-1}\left(a^{q-1}\Lambda(-\gamma, 3n, \rho) \frac{\text{Cap}_G(\overline{B}_{\rho/32}(x_0) \setminus \Omega, B_\rho(x_0))}{\rho^{n-1}}\right) \leq M_l(\rho) - M_l(\rho/4) + 2b_0 \rho.
\]

Hence, setting

\[
w = g_{x_0}^{-1}\left(\Lambda(-\gamma, 3n, \rho) \frac{\text{Cap}_G(\overline{B}_{\rho/32}(x_0) \setminus \Omega, B_\rho(x_0))}{\rho^{n-1}}\right),
\]

\[
v = g_{x_0}^{-1}\left(a^{q-1}\Lambda(-\gamma, 3n, \rho) \frac{\text{Cap}_G(\overline{B}_{\rho/32}(x_0) \setminus \Omega, B_\rho(x_0))}{\rho^{n-1}}\right),
\]

and using (3.3), we derive the following:

\[
g_{x_0}^{-1}\left(\Lambda(-\gamma, 3n, \rho) \frac{\text{Cap}_G(\overline{B}_{\rho/32}(x_0) \setminus \Omega, B_\rho(x_0))}{\rho^{n-1}}\right) \leq a^{q-1} - \frac{M_l(\rho) - M_l(\rho/4) + 2b_0 \rho}{\rho}.
\]

It is readily verified that

\[
\int_0^1 \frac{M_l(\rho) - M_l(\rho/4) + 2b_0 \rho}{\rho} d\rho < +\infty,
\]

and therefore the previous inequality gives

\[
\int_0^{g_{x_0}^{-1}\left(\Lambda(-\gamma, 3n, \rho) \frac{\text{Cap}_G(\overline{B}_{\rho/32}(x_0) \setminus \Omega, B_\rho(x_0))}{\rho^{n-1}}\right)} d\rho < +\infty,
\]

which contradicts condition (2.4) of Theorem 2.6. Consequently, hypothesis (3.1) is not correct, and the inequality \( \lim_{\rho \to 0} \text{ess sup} \ u \leq f(x_0) \) is correct. The inequality \( f(x_0) \leq \lim_{\rho \to 0} \text{ess inf} \ u \) is proved similarly, which, together with the previous one, implies the equality \( f(x_0) = \lim_{\rho \to 0} \text{ess inf} \ u \). The proof is complete.

4 The weak Harnack inequality: proof of Theorem 2.6

In this section we prove Theorem 2.6. We assume that all the hypotheses and notation of this theorem are in force. For definiteness, we also assume that \( l \geq M_f(\rho) \) in (2.6). In the case where \( l \leq m_f(\rho) \), the proof is completely similar. We need some inequalities and several lemmas. First, we note simple analogues of Young’s inequality:

\[
g(x, a) b \leq \varepsilon g(x, a) a + g(x, b/\varepsilon) b \quad \text{if} \quad \varepsilon, a, b > 0, \ x \in \overline{\Omega}.
\]

In fact, if \( b \leq \varepsilon a \), then \( g(x, a) b \leq \varepsilon g(x, a) a \), and if \( b > \varepsilon a \), then since the function \( v \to g(x, v) \) is increasing we have that \( g(x, a) b \leq g(x, b/\varepsilon) b \), which proves inequality (4.1).

Next, we set

\[
\mathcal{G}(x, w) = \int_0^w g(x, v) dv \quad \text{for} \quad x \in \Omega, \ w > 0.
\]

The following inequalities hold:

\[
\mathcal{G}(x, w) \geq \gamma \mathcal{G}(x, w) \quad \text{for all} \quad x \in \overline{\Omega}, \ w \geq 2b_0,
\]

(4.3)
Continuity at a boundary point of solutions to quasilinear elliptic equations. . .

\[ G(x, w) \geq pG(x, w) \quad \text{for all } x \in \overline{\Omega}, \ w > 0. \quad (4.4) \]

Indeed, if \( x \in \overline{\Omega} \) and \( w \geq 2b_0 \), then by (1.2), (2.1) and (4.2), we have

\[ G(x, w) = \hat{w} \int_{0}^{w} g(x, v) dv \geq \frac{g(x, w)}{c_1 w^{q-1}} \int_{0}^{w} v^{q-1} dv \geq \frac{1 - 2^{-q}}{c_1 q} G(x, w), \]

which implies (4.3). Now, let \( x \in \overline{\Omega} \) and \( w > 0 \) be arbitrary, then by (1.3), (2.1) and (4.2) we obtain

\[ G(x, w) = \hat{w} \int_{0}^{w} g(x, v) dv \leq \frac{g(x, w)}{w^{p-1}} \int_{0}^{w} v^{p-1} dv = \frac{1}{p} g(x, w)w = \frac{1}{p} G(x, w), \]

which yields (4.4).

The rest of the lemmas in this section are successive stages in the proof of Theorem 2.6, which follows Trudinger’s strategy [56] adapted to Eq. (1.1) near the boundary \( \partial \Omega \) (see Remark 2.7).

**Lemma 4.1.** There exists positive constant \( \gamma \) depending only on the known data such that

\[ \exp \left( \int_{B_{\rho/2}(x_0)} \ln u_{t, \rho} \, dx \right) \leq \Lambda(\gamma, 3n, \rho) \left( M_t(\rho) - M_t(\rho/4) + 2b_0 \rho \right). \quad (4.5) \]

**Proof.** Let

\[ w = \ln \frac{\zeta}{u_{t, \rho}}, \quad (4.6) \]

where the function \( u_{t, \rho} \) is defined by (2.6) and the constant \( \zeta \) is defined by the condition

\[ (w)_{x_0, \rho/2} = \int_{B_{\rho/2}(x_0)} w \, dx = 0, \quad \text{i.e.} \quad \zeta = \exp \left( \int_{B_{\rho/2}(x_0)} \ln u_{t, \rho} \, dx \right). \quad (4.7) \]

Taking into account (2.6), (4.6) and (4.7), it is easy to see that inequality (4.5) is equivalent to the following estimate:

\[ \text{ess sup }_{B_{\rho/4}(x_0)} w \leq \gamma e^{3n\lambda(\rho)}. \quad (4.8) \]

The idea of using an upper bound of auxiliary logarithmic functions goes back to Moser [43,44] and has become a useful tool in the qualitative theory of partial differential equations [24,32]. In this paper, in order to prove (4.8), we use the approach of De Giorgi [18,32] in the spirit of our recent studies [51,54].

We fix \( \sigma \in (0, 1) \), and for any \( \rho/4 \leq r < r(1 + \sigma) \leq \rho/2 \) we take a function \( \zeta \in C^\infty_0(\bar{B}_{r(1+\sigma)}(x_0)) \), \( 0 \leq \zeta \leq 1, \ \zeta = 1 \) in \( B_r(x_0) \) and \( |\nabla \zeta| \leq (\sigma r)^{-1} \). Let

\[ k \geq \gamma e^{2n\lambda(\rho)} \left( \int_{B_{\rho/2}(x_0)} |w|^{\frac{n}{n-1}} \, dx \right)^{\frac{n-1}{n}} + 1. \quad (4.9) \]

From (4.3), (4.9) it follows that

\[ k > \ln \frac{\zeta}{M_t(\rho) + 2b_0 \rho} = \text{ess sup }_{\partial \Omega \cap B_{\rho}(x_0)} w. \quad (4.10) \]
We test (2.2) by the function
\[
\varphi = \begin{cases} 
\frac{u_{t,\rho}(w - k)_+}{G(x_0, u_{t,\rho}/\rho)} \zeta^q & \text{on } \Omega \cap B_{r(1+\sigma)}(x_0), \\
0 & \text{otherwise}.
\end{cases}
\]
\hfill (4.11)

Since we are dealing with bounded solutions, then this function and all other test functions used in the article belong to \(W_0^{1,G}(\Omega)\). This is a consequence of conditions \((g_0), (g_1)\), the result of Marcus and Mizel [39, Theorem 2] and the notion of the weak inequality on the boundary \(\partial \Omega\) [22, 37]. So, after substitution (4.11) into (2.2), we have
\[
\int_{A_{k,r(1+\sigma)}} \frac{G(x,|\nabla u|)}{G(x_0, u_{t,\rho}/\rho)} \zeta^q dx
+ \int_{A_{k,r(1+\sigma)}} \frac{G(x,|\nabla u|)}{G(x_0, u_{t,\rho}/\rho)} \left\{ \frac{G(x_0, u_{t,\rho}/\rho)}{G(x_0, u_{t,\rho}/\rho)} - 1 \right\} (w - k)_+ \zeta^q dx
\leq \frac{\gamma}{\sigma} \int_{A_{k,r(1+\sigma)}} \frac{g(x,|\nabla u|)}{G(x_0, u_{t,\rho}/\rho)} \frac{u_{t,\rho}}{\sigma \rho \zeta} (w - k)_+ \zeta^{q-1} dx,
\]
where \(A_{k,r(1+\sigma)} = \Omega \cap B_{r(1+\sigma)}(x_0) \cap \{ w > k \} \) and the embedding \(A_{k,r(1+\sigma)} \subset \Omega_{l,r(1+\sigma)} \) is true due to (4.10) (see (2.5) for the definition of \(\Omega_{l,r(1+\sigma)}\)). By (4.4), the value in curly brackets is estimated from below as follows:
\[
\frac{G(x_0, u_{t,\rho}/\rho)}{G(x_0, u_{t,\rho}/\rho)} - 1 \geq p - 1,
\]
and therefore
\[
\int_{A_{k,r(1+\sigma)}} \frac{G(x,|\nabla u|)}{G(x_0, u_{t,\rho}/\rho)} \zeta^q dx + (p - 1) \int_{A_{k,r(1+\sigma)}} \frac{G(x,|\nabla u|)}{G(x_0, u_{t,\rho}/\rho)} (w - k)_+ \zeta^q dx
\leq \gamma \int_{A_{k,r(1+\sigma)}} \frac{g(x,|\nabla u|)}{G(x_0, u_{t,\rho}/\rho)} \frac{u_{t,\rho}}{\sigma \rho \zeta} (w - k)_+ \zeta^{q-1} dx.
\]
We use inequality (4.11) with \(a = |\nabla u|\), \(b = \frac{u_{t,\rho}}{\sigma \rho \zeta}\) and sufficiently small \(\varepsilon > 0\), and then (4.3) with \(w = u_{t,\rho}/\rho\), to estimate from above the right-hand side of (4.13):
\[
\gamma \int_{A_{k,r(1+\sigma)}} \frac{g(x,|\nabla u|)}{G(x_0, u_{t,\rho}/\rho)} \frac{u_{t,\rho}}{\sigma \rho \zeta} (w - k)_+ \zeta^q dx
\leq \frac{p - 1}{2} \int_{A_{k,r(1+\sigma)}} \frac{G(x,|\nabla u|)}{G(x_0, u_{t,\rho}/\rho)} (w - k)_+ \zeta^q dx
\]
So, from (4.14) we obtain
\[
\int_{A_{k,r}(1+\sigma)} \frac{G(x,|\nabla u|)}{\mathcal{G}(x, u_{l,p}/\rho)} \zeta^q \, dx \leq \gamma \sigma^{-q} e^{\lambda(\rho)} \int_{A_{k,r}(1+\sigma)} (w - k)_+ \, dx. \tag{4.15}
\]

To estimate the term on the left-hand side of (4.15), we use (4.1) with \(\varepsilon = 1, a = u_{l,p}/\rho, \ b = |\nabla u|\), assumption (g3), the definitions of the functions \(G, \mathcal{G}, w\) (see equalities (2.1), (4.2) and (4.6), respectively) and (4.4):
\[
\int_{A_{k,r}(1+\sigma)} |\nabla w| \zeta^q \, dx = \int_{A_{k,r}(1+\sigma)} \frac{|\nabla u|}{u_{l,p}} \frac{g(x, u_{l,p}/\rho)}{g(x, u_{l,p}/\rho)} \zeta^q \, dx
\leq \frac{1}{\rho} \left| A_{k,r}(1+\sigma) \right| + \frac{1}{\rho} \int_{A_{k,r}(1+\sigma)} \frac{G(x, |\nabla u|)}{\mathcal{G}(x, u_{l,p}/\rho)} \zeta^q \, dx.
\tag{4.16}
\]
Collecting (4.15) and (4.16), we obtain
\[
\int_{A_{k,r}(1+\sigma)} |\nabla w| \zeta^q \, dx \leq \frac{\gamma e^{2\lambda(\rho)}}{\sigma^q} \left( \frac{1}{\rho} \left| A_{k,r}(1+\sigma) \right| + \int_{A_{k,r}(1+\sigma)} (w - k)_+ \, dx \right).
\]
From this, using Sobolev’s embedding theorem, standard iteration arguments (see e.g. Section 2, Theorem 5.3]) and condition (4.9) on \(k\), we obtain that
\[
\text{ess sup}_{B_{\rho/4}(x_0)} w \leq \gamma e^{2\lambda(\rho)} \left( \int_{B_{\rho/2}(x_0)} |w|^{\frac{n}{n-1}} \, dx \right)^\frac{n-1}{n} + 1. \tag{4.17}
\]

In order to estimate the right-hand side of (4.17) we use the Poincaré inequality. By our choice of \(\sigma\) in (4.7) we have
\[
\left( \int_{B_{\rho/2}(x_0)} |w|^{\frac{n}{n-1}} \, dx \right)^\frac{n-1}{n} = \left( \int_{B_{\rho/2}(x_0)} |w - (w)_{x_0,p/2}|^{\frac{n}{n-1}} \, dx \right)^\frac{n-1}{n}
\leq \gamma \rho^{1-n} \int_{B_{\rho/2}(x_0)} |\nabla w| \, dx. \tag{4.18}
\]
Next, similarly to (4.16), we have
\[
\int_{B_{\rho/2}(x_0)} |\nabla w| \, dx \leq \int_{B_p(x_0)} |\nabla w| \zeta^q \, dx \leq \gamma \rho^{n-1} + \gamma e^{\lambda(\rho)} \int_{\Omega, \rho} \frac{G(x, |\nabla u|)}{\mathcal{G}(x, u_{l,p}/\rho)} \zeta^q \, dx, \tag{4.19}
\]
where we have \(\zeta \in C_0^\infty(B_p(x_0)), 0 \leq \zeta \leq 1, \ \zeta = 1\) in \(B_{\rho/2}(x_0)\), and \(|\nabla \zeta| \leq 2/\rho\). In addition, testing (2.2) by
\[
\varphi = \begin{cases} \frac{u_{l,p}}{\mathcal{G}(x, u_{l,p}/\rho)} - \frac{M_l(\rho) + 2b_0 \rho}{\mathcal{G}(x, M_l(\rho) \rho^{-1} + 2b_0)} \zeta^q & \text{on } \Omega \cap B_p(x_0), \\
0 & \text{otherwise}, \end{cases}
\]
similarly to (4.15), we obtain
\[
\int_{\Omega, \rho} \frac{G(x, |\nabla u|)}{\mathcal{G}(x, u_{l,p}/\rho)} \zeta^q \, dx \leq \gamma \rho^n e^{\lambda(\rho)}. \tag{4.20}
\]
Now, collecting (4.17)–(4.20), we arrive at the required inequality (4.8). The proof of the lemma is complete.
Lemma 4.2. There exists a positive number $\delta_0 = \delta_0(\rho)$ depending only on the data and $\rho$, such that

$$\left( \int_{B_{\rho/4}(x_0)} u_{l,\rho}^\delta \, dx \right)^{1/\delta_0} \leq \Lambda(\gamma, 2n, \rho) \exp \left( \int_{B_{\rho/2}(x_0)} \ln u_{l,\rho} \, dx \right). \tag{4.21}$$

Proof. Let's fix $\sigma \in (0, 1)$ and for any $\rho/4 \leq r < r(1 + \sigma) \leq \rho/2$ consider the function $\zeta \in C^\infty_0(B_{r(1+\sigma)}(x_0))$, $0 \leq \zeta \leq 1$, $\zeta = 1$ in $B_r(x_0)$, $|\nabla \zeta| \leq (\sigma r)^{-1}$. We define

$$v = \ln \frac{u_{l,\rho}}{\zeta}, \quad v_\mu = \max\{v, \mu\}, \quad \mu > 0. \tag{4.22}$$

Testing (2.2) by

$$\varphi = \begin{cases} \frac{\sigma - 1}{\mu} \left( \frac{u_{l,\rho}}{G(x_0, u_{l,\rho}/\rho)} - \frac{M_l(\rho) + 2b_0\rho}{G(x_0, M_l(\rho)^{-1} + 2b_0)} \right) \zeta^\theta & \text{on } \Omega \cap B_{r(1+\sigma)}(x_0), \\ 0 & \text{otherwise}, \end{cases}$$

where $s \geq 1$, $\theta \geq q$, and using (4.12), we have

$$\begin{align*}
(p - 1) \int_{\Omega_{1, r(1+\sigma)}} & \frac{G(x, |\nabla u|)}{G(x_0, u_{l,\rho}/\rho)} v_\mu^{s-1} \zeta^\theta \, dx \\
& \leq (s - 1) \int_{\{v > \mu\} \cap \Omega_{1, r(1+\sigma)}} \frac{G(x, |\nabla u|)}{G(x_0, u_{l,\rho}/\rho)} v_\mu^{s-2} \zeta^\theta \, dx \\
& \quad + \frac{\gamma \theta}{\zeta} \int_{\Omega_{1, r(1+\sigma)}} \frac{g(x, |\nabla u|)}{G(x_0, u_{l,\rho}/\rho)} \frac{u_{l,\rho}}{\sigma \rho} \zeta v_\mu^{s-1} \zeta^\theta \, dx.
\end{align*}$$

Choosing $\mu$ from the condition $\frac{s}{\mu} = \frac{p - 1}{2}$ and using inequalities (4.1), (4.13) and conditions (g1) and (g3) similarly to the derivation of (4.15), from the previous we obtain

$$\int_{\Omega_{1, r(1+\sigma)}} \frac{G(x, |\nabla u|)}{G(x_0, u_{l,\rho}/\rho)} v_\mu^{s-1} \zeta^\theta \, dx \leq \frac{\gamma \theta^2 e^{2\lambda(\rho)}}{\sigma^q} \int_{B_{r(1+\sigma)}(x_0)} v_\mu^{s-1} \zeta^{\theta-q} \, dx. \tag{4.23}$$

Estimating from below the term on the left-hand side of (4.24), similarly to (4.16), we obtain

$$\begin{align*}
\int_{B_{r(1+\sigma)}(x_0)} & |\nabla v_\mu| v_\mu^{s-1} \zeta^\theta \, dx \\
& \leq \frac{\gamma \theta^2 e^{2\lambda(\rho)}}{\sigma^q} \rho \int_{B_{r(1+\sigma)}(x_0)} v_\mu^{s-1} \zeta^{\theta-q} \, dx \leq \frac{\gamma \theta^2 e^{2\lambda(\rho)}}{\sigma^q} \rho \int_{B_{r(1+\sigma)}(x_0)} v_\mu^{s-1} \zeta^{\theta-q} \, dx.
\end{align*}$$

Using Sobolev’s embedding theorem from this we have

$$\int_{B_r(x_0)} v_\mu^{\frac{an}{n-1}} \, dx \leq \left( \frac{\gamma s e^{2\lambda(\rho)}}{\sigma^q} \right)^{\frac{n}{n-1}} \int_{B_{r(1+\sigma)}(x_0)} v_\mu^{s} \, dx \tag{4.24}.$$
Then inequality (4.24) can be rewritten in the form
\[ y_{j+1} \leq \left( \gamma^{2^{js_j}} e^{2\lambda(\rho)} \right)^{1/s_j} y_j, \quad j = 0, 1, 2, \ldots . \] 

In addition, by (4.6), (4.22), (4.18)–(4.20), for \( j = 0 \), we have
\[ y_0 \leq \gamma \mu_j + \gamma \left( \int_{B_{\rho/2}(x_0)} |u|^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \leq \gamma e^{2\lambda(\rho)}. \] 

Iterating (4.26) and taking into account (4.27), for \( j = 0, 1, 2, \ldots \), we have
\[ y_{j+1} \leq \gamma y_j \left( \sum_{i=0}^{s_j} \frac{1}{s_i} \right)^{2^{js_j}} \sum_{i=0}^{s_j} i^{\frac{n-1}{n}} \exp \left( 2\lambda(\rho) \sum_{i=0}^{s_j} i \right) \leq \gamma e^{2\lambda(\rho)}. \] 

Let \( m \in \mathbb{N} \) be arbitrary, then there exists \( j \geq 1 \) such that \( s_{j-1} < m \leq s_j \). Using Hölder’s inequality, from (4.25), (4.28) we obtain
\[ \int_{B_{\rho/4}(x_0)} \frac{u^m}{m!} dx \leq \int_{B_{\rho/4}(x_0)} \frac{v^m}{m!} dx \leq \gamma \frac{y_j^m}{m!} \leq \gamma e^{2m\lambda(\rho)} \leq \gamma e^{2nm\lambda(\rho)} . \]

Choosing \( \delta_0 = \delta_0(\rho) \) from the condition
\[ \delta_0 = \frac{1}{2\gamma} e^{-2n\lambda(\rho)}, \] 
from the previous we have
\[ \int_{B_{\rho/4}(x_0)} \frac{(\delta_0 v)^m}{m!} dx \leq \gamma 2^{-m}, \]
which implies that
\[ \int_{B_{\rho/4}(x_0)} e^{\delta_0 v} dx \leq \int_{B_{\rho/4}(x_0)} e^{\delta_0 v} dx \leq \sum_{m=0}^{\infty} \int_{B_{\rho/4}(x_0)} \frac{(\delta_0 v)^m}{m!} dx \leq 2\gamma. \]

From this, since \( e^{\delta_0 v} = (u_{t,\rho} / \mathcal{X})^{\delta_0} \) we have
\[ \left( \int_{B_{\rho/4}(x_0)} u_{t,\rho}^{\delta_0} dx \right)^{1/\delta_0} \leq (2\gamma)^{1/\delta_0} \mathcal{X} \leq \Lambda(\gamma, 2n, \rho) \mathcal{X}, \]
that together with (4.7) yields the desired inequality (4.21). This completes the proof of the lemma.

The next lemma is a simple consequence of Lemmas 4.1 and 4.2.

**Lemma 4.3.** The following inequality holds:
\[ \left( \int_{B_{\rho/4}(x_0)} g^s (x_0, ut, / \mathcal{X}) dx \right)^{1/s_0} \leq \Lambda(\gamma, 3n, \rho) g \left( x_0, \frac{M(\rho) - M(\rho/4) + 2b_0 \rho}{\rho} \right), \] 
where
\[ \delta_0 = \delta_0/(q - 1), \] 
and \( \delta_0 \) is defined by (4.29).
Proof. By condition \((g_1)\) we have

\[
\int_{B_{\rho/4}(x_0)} g^{\delta_1}(x_0, u_{l,\rho}/\rho) dx \leq c_1^\delta_1 \int_{B_{\rho/4}(x_0)} \left(\frac{u_{l,\rho}}{M_l(\rho) - M_l(\rho/4) + 2b_0\rho}\right)^{\delta_0} dx.
\]

By Lemmas 4.1 and 4.2, the right-hand side of this inequality is estimated from above as follows:

\[
\int_{B_{\rho/4}(x_0)} \left(\frac{u_{l,\rho}}{M_l(\rho) - M_l(\rho/4) + 2b_0\rho}\right)^{\delta_0} dx \leq \Lambda(\gamma, 3n, \rho),
\]

which proves the lemma.

To complete the proof of Theorem 2.6 we need the following lemma.

**Lemma 4.4** (inverse Hölder inequality). Let \(\delta_1 \leq s < n/(n-1)\), where the number \(\delta_1\) is defined by (4.29) and (4.31). Then the following inequality holds:

\[
\left( \int_{B_{\rho/8}(x_0)} g^s(x_0, u_{l,\rho}/\rho) dx \right)^{1/s} \leq \Lambda(\gamma, 2n + 1, \rho) \left( \int_{B_{\rho/4}(x_0)} g^{\delta_1}(x_0, u_{l,\rho}/\rho) dx \right)^{1/\delta_1}. \tag{4.32}
\]

**Proof.** We set \(\psi(x, w) = w^{-1}G(x, w)\) for \(x \in \Omega, w > 0\), and note that by (4.33) and (4.34) we have

\[
g(x, w) \leq \gamma \psi(x, w) \quad \text{for all} \quad x \in \Omega, w \geq 2b_0, \tag{4.34}
\]

\[
\psi(x, w) \leq \frac{1}{p} g(x, w) \quad \text{for all} \quad x \in \Omega, w > 0, \tag{4.35}
\]

which gives

\[
\psi'_w(x, w) \leq \gamma \frac{\psi(x, w)}{w} \quad \text{for all} \quad x \in \Omega, w \geq 2b_0, \tag{4.36}
\]

\[
\psi'_w(x, w) = \frac{g(x, w) - \psi(x, w)}{w} \geq (p-1) \frac{\psi(x, w)}{w} \quad \text{for all} \quad x \in \Omega, w > 0. \tag{4.37}
\]

We need a Caccioppoli-type inequality for negative powers of \(\psi(x_0, u_{l,\rho}/\rho)\). To establish it, we fix \(\sigma \in (0, 1)\) and \(r > 0\) such that \(\rho/8 \leq r < r(1 + \sigma) \leq \rho/4\), and take a function \(\zeta \in C_0^\infty(B_{r(1+\sigma)}(x_0))\), \(0 \leq \zeta \leq 1\), \(\zeta = 1\) in \(B_r(x_0)\), \(|\nabla \zeta| \leq (\sigma r)^{-1}\). Testing (2.2) by

\[
\varphi = \begin{cases} 
\left[\psi^{-\tau}(x_0, u_{l,\rho}/\rho) - \psi^{-\tau}(x_0, \rho^{-1}M_l(\rho) + 2b_0)\right] \zeta^\theta & \text{on} \ \Omega \cap B_r(1+\sigma)(x_0), \\
0 & \text{otherwise},
\end{cases}
\]

where \(0 < \tau < 1, \ \theta \geq q\), and using (4.37) and the properties of \(\zeta\), we obtain the following inequality:

\[
(p-1)\tau \int_{\Omega \cap B_r(1+\sigma)} \psi^{-\tau}(x_0, u_{l,\rho}/\rho) G(x, |\nabla u|) u_{l,\rho}^\theta \zeta^\theta dx \\
\leq \frac{\gamma \theta}{\sigma \rho} \int_{\Omega \cap B_r(1+\sigma)} \psi^{-\tau}(x_0, u_{l,\rho}/\rho) g(x, |\nabla u|) \zeta^{\theta-1} dx,
\]
which by (4.1), (g₁), (g₃) and (4.31) implies the following:

\[
\int_{\Omega_{t,r}(1+\sigma)} \psi^{-\tau} (x_0, u_{l,\rho}/\rho) \frac{G(x,|\nabla u|)}{u_{l,\rho}} \zeta^{\theta} \, dx \\
\leq \frac{\gamma}{\rho} \int_{B_{t,r}(1+\sigma)(x_0)} \psi^{1-\tau} (x_0, u_{l,\rho}/\rho) \zeta^{\theta-q} \, dx.
\] (4.38)

Basing on inequality (4.38), we organize Moser-type iterations for the function \(\psi (x_0, u_{l,\rho}/\rho)\). To do this, we fix \(0 < t < n/(n-1)\) and \(\vartheta \geq nq/(n-1)\), then by Sobolev’s inequality and by (4.36) and (4.35), we obtain

\[
\left( \int_{B_{t,r}(1+\sigma)(x_0)} \psi^t (x_0, u_{l,\rho}/\rho) \zeta^{\theta} \, dx \right)^{\frac{n-1}{n}} \\
\leq \gamma t \int_{B_{t,r}(1+\sigma)(x_0)} \left| \nabla \left[ \psi \frac{t(n-1)}{n} (x_0, u_{l,\rho}/\rho) \zeta^{\vartheta(n-1)/n} \right] \right| \, dx \\
\leq \gamma t \int_{\Omega_{t,r}(1+\sigma)} \psi^{\frac{t(n-1)-1}{n}} (x_0, u_{l,\rho}/\rho) \frac{g(x_0, u_{l,\rho}/\rho)}{u_{l,\rho}} |\nabla u| \zeta^{\vartheta(n-1)/n} \, dx \\
+ \frac{\gamma \vartheta}{\sigma \rho} \int_{B_{t,r}(1+\sigma)(x_0)} \psi^{\frac{t(n-1)}{n}} (x_0, u_{l,\rho}/\rho) \zeta^{\vartheta(n-1)/n-1} \, dx.
\] (4.39)

Using (g₃), (4.1), (4.34) and (4.35) with \(\tau = 1 - t(n-1)/n\) and \(\theta = \vartheta(n-1)/n\), we estimate the first term on the right-hand side of (4.39) as follows:

\[
\int_{\Omega_{t,r}(1+\sigma)} \psi^{\frac{t(n-1)-1}{n}} (x_0, u_{l,\rho}/\rho) \frac{g(x_0, u_{l,\rho}/\rho)}{u_{l,\rho}} |\nabla u| \zeta^{\vartheta(n-1)/n} \, dx \\
\leq \gamma e^{\lambda(\rho)} \int_{\Omega_{t,r}(1+\sigma)} \psi^{\frac{t(n-1)}{n}-1} (x_0, u_{l,\rho}/\rho) \frac{g(x, u_{l,\rho}/\rho)}{u_{l,\rho}} |\nabla u| \zeta^{\vartheta(n-1)/n} \, dx \\
\leq \gamma e^{\lambda(\rho)} \int_{\Omega_{t,r}(1+\sigma)} \psi^{\frac{t(n-1)}{n}-1} (x_0, u_{l,\rho}/\rho) \frac{G(x,|\nabla u|)}{u_{l,\rho}} \zeta^{\vartheta(n-1)/n} \, dx \\
+ \frac{\gamma e^{\lambda(\rho)}}{\rho} \int_{B_{t,r}(1+\sigma)(x_0)} \psi^{\frac{t(n-1)}{n}-1} (x_0, u_{l,\rho}/\rho) \frac{g(x, u_{l,\rho}/\rho)}{u_{l,\rho}} \zeta^{\vartheta(n-1)/n} \, dx \\
\leq \gamma \frac{\vartheta}{\sigma \rho} \left[ 1 - \frac{t(n-1)}{n} \right]^{\vartheta(n-1)/n} \int_{B_{t,r}(1+\sigma)(x_0)} \psi^{\frac{t(n-1)}{n}} (x_0, u_{l,\rho}/\rho) \zeta^{\vartheta(n-1)/n-1} \, dx.
\] (4.40)

Combining (4.39), (4.40), we arrive at

\[
\left( \int_{B_{t}(x_0)} \psi^t (x_0, u_{l,\rho}/\rho) \, dx \right)^{\frac{n-1}{n}} \\
\leq \gamma \frac{\vartheta}{\sigma \rho} \left[ 1 - \frac{t(n-1)}{n} \right]^{\vartheta(n-1)/n} \int_{B_{t,r}(1+\sigma)(x_0)} \psi^{\frac{t(n-1)}{n}} (x_0, u_{l,\rho}/\rho) \, dx,
\] (4.41)

for \(0 < t < n/(n-1), \vartheta \geq nq/(n-1)\).

Now, let \(\delta_1 \leq s < n/(n-1)\), and let \(j\) be a non-negative integer number such that

\[
s \left( \frac{n-1}{n} \right)^{j+1} \leq \delta_1 \leq s \left( \frac{n-1}{n} \right)^j.
\] (4.42)
Setting in (4.11) \( \vartheta = nq, r = r_i = \frac{\rho}{s} (2-2^{-i}), r(1+\sigma) = r_{i+1}, B_i = B_r(x_0) \) and \( t = t_i = s (\frac{n-1}{n})^i \) for \( i = 0, 1, \ldots, j + 1 \), we have
\[
\left( \int_{B_i} \psi^{t_i} (x_0, u_{l,\rho}/\rho) \, dx \right)^{1/t_i} \leq \left[ \gamma 2^q \left( 1 - \frac{n-1}{n} s \right) - q e^{2\lambda(\rho)} \right]^{1/t_i+1} \left( \int_{B_{i+1}} \psi^{t_{i+1}} (x_0, u_{l,\rho}/\rho) \, dx \right)^{1/t_{i+1}}.
\]
Iterating this relation and using Hölder’s inequality, we obtain the following:
\[
\left( \int_{B_{j+1}} \psi^s (x_0, u_{l,\rho}/\rho) \, dx \right)^{1/s} = \left( \int_{B_0} \psi^{t_0} (x_0, u_{l,\rho}/\rho) \, dx \right)^{1/t_0} \\
\leq \prod_{i=0}^j \left[ \gamma 2^q e^{2\lambda(\rho)} \left( 1 - \frac{n-1}{n} s \right) - q \right]^{1/t_{i+1}+1} \left( \int_{B_{j+1}} \psi^{t_{i+1}} (x_0, u_{l,\rho}/\rho) \, dx \right)^{1/t_{i+1}} \\
\leq 2^{q \sum_{i=0}^j 1/t_{i+1}} \left[ e^{2\lambda(\rho)} \left( 1 - \frac{n-1}{n} s \right) - q \right] \sum_{i=0}^j \left( \int_{B_{j+1}} \psi^{t_{i+1}} (x_0, u_{l,\rho}/\rho) \, dx \right)^{1/t_{i+1}},
\]
and by (4.42), (4.31) and (4.29) we have
\[
\sum_{i=0}^j \frac{1}{t_{i+1}} \leq \frac{1}{\delta_1} \frac{n}{n-1} \sum_{i=0}^\infty \left( \frac{n-1}{n} \right)^i = \frac{n^2}{\delta_1 (n-1)},
\]
\[
\sum_{i=0}^j \frac{i}{t_{i+1}} \leq j \sum_{i=0}^j \frac{1}{t_{i+1}} \leq \frac{\gamma (\lambda(\rho) + 1)}{\delta_1}.
\]
From this, recalling the definition of \( \delta_1 \) (see again (4.31) and (4.29)), we arrive at the required inequality (4.32). This completes the proof of the lemma.

Combining Lemmas 4.3 and 4.4 we obtain that
\[
\left( \int_{B_{j+1}} \psi^s (x_0, u_{l,\rho}/\rho) \, dx \right)^{1/s} \leq \Lambda(\gamma, 3n, \rho) g \left( x_0, \frac{M_1(\rho) - M_1(\rho/4) + 2b_0\rho}{\rho} \right),
\]
for \( 0 < s < n/(n-1) \), which proves Theorem 2.6.

5 Growth estimate on the gradient: proof of Theorem 2.5

Throughout this section we assume that the hypotheses and notation of Theorem 2.5 are in force. For definiteness, we assume that \( l \geq M_1(\rho) \) in (2.6). In the case where \( l \leq m_1(\rho) \) in (2.6), the proof is completely similar. We first prove the following inequality:
\[
\int_{\Omega_{l,\rho/8}} G(x, |\nabla u|) \zeta^q \, dx \leq \Lambda(\gamma, 3n, \rho) M_1(\rho) \rho^{n-1} g \left( x_0, \frac{M_1(\rho) - M_1(\rho/4) + 2b_0\rho}{\rho} \right), \quad (5.1)
\]
where \( \zeta \in C_0^\infty (B_{\rho/8}(x_0)), 0 \leq \zeta \leq 1, \zeta = 1 \) in \( B_{\rho/16}(x_0) \), and \( |\nabla \zeta| \leq 16/\rho \). For this purpose, we test (2.2) by \( \varphi = u_{l,\rho} \zeta^q \) and obtain
\[
\int_{\Omega_{l,\rho/8}} G(x, |\nabla u|) \zeta^q \, dx \leq \gamma \frac{M_1(\rho)}{\rho} \int_{\Omega_{l,\rho/8}} g(x, |\nabla u|) \zeta^q \, dx.
\]
The integral on the right-hand side of (5.2) is estimated by Young’s inequality (4.1) with $a = |\nabla u|$, $b = \zeta^{-1}$ and
\[
\varepsilon = \frac{\rho}{u_t,\rho} \frac{\psi^\tau(x_0, M_l(\rho) - M_l(\rho/4) + 2b_0\rho)}{\psi^\tau(x_0, u_t,\rho/\rho)} ,
\]
(5.3)
where the function $\psi$ is defined in (4.33), and
\[
0 < \tau < \frac{1}{(q - 1)(n - 1)} .
\]
(5.4)
Thus, it follows from (5.2) that
\[
\int_{\Omega_t,\rho/s} G(x, |\nabla u|) \zeta^q dx \leq I_1 + I_2 ,
\]
(5.5)
where
\[
I_1 = \gamma \frac{M_l(\rho)}{\rho} \int_{\Omega_t,\rho/s} \varepsilon G(x, |\nabla u|) \zeta^q dx ,
\]
(5.6)
\[
I_2 = \gamma \frac{M_l(\rho)}{\rho} \int_{\Omega_t,\rho/s} g\left(x, \frac{1}{\varepsilon \zeta}\right) \zeta^{q-1} dx .
\]

Let’s estimate $I_2$ from above. Using (1.2) with $w = (\varepsilon \zeta)^{-1}$, $v = \varepsilon^{-1}$ and taking into account (5.3), we obtain
\[
I_2 \leq \gamma \frac{M_l(\rho)}{\rho} \int_{\Omega_t,\rho/s} g(x, \varepsilon^{-1}) dx
\]
\[
= \gamma \frac{M_l(\rho)}{\rho} \int_{\Omega_t,\rho/s} g\left(x, \frac{u_{t,\rho}}{\rho} \frac{\psi^\tau(x_0, u_{t,\rho}/\rho)}{\psi^\tau(x_0, M_l(\rho) - M_l(\rho/4) + 2b_0\rho)}\right) dx .
\]

The resulting integral is estimated using condition (g3) and inequality (1.2) with
\[
w = \frac{u_{t,\rho}}{\rho} \frac{\psi^\tau(x_0, u_{t,\rho}/\rho)}{\psi^\tau(x_0, M_l(\rho) - M_l(\rho/4) + 2b_0\rho)} , \quad v = \frac{u_{t,\rho}}{\rho} ,
\]
that yields
\[
I_2 \leq \gamma \frac{e^{\lambda(\rho)} M_l(\rho)}{\rho} \int_{B_{\rho/s}} \psi^\tau(q-1)\left(x_0, \frac{u_{t,\rho}/\rho}{\psi^\tau(q-1)}\left(x_0, \frac{M_l(\rho) - M_l(\rho/4) + 2b_0\rho}{\rho}\right)\right) g(x_0, u_{t,\rho}/\rho) dx .
\]

Hence, taking into account (4.34) and (4.35), we derive the estimate
\[
I_2 \leq \gamma \frac{e^{\lambda(\rho)} M_l(\rho)}{\rho} g^{\tau(q-1)}\left(x_0, \frac{M_l(\rho) - M_l(\rho/4) + 2b_0\rho}{\rho}\right) \int_{B_{\rho/s}} g(x_0, u_{t,\rho}/\rho)^{1+\tau(q-1)} dx .
\]
By (5.4) we have $1 + \tau(q - 1) < n/(n - 1)$. Therefore, it is possible to apply the weak Harnack inequality (2.9) to obtain
\[
I_2 \leq \Lambda(\gamma, 3n, \rho) M_l(\rho) (n-1) g\left(x_0, \frac{M_l(\rho) - M_l(\rho/4) + 2b_0\rho}{\rho}\right) .
\]
(5.7)
In order to estimate $I_1$, note that by virtue of (5.3) and (5.6) we have the equality
\[
I_1 = \gamma M_l(\rho) \psi^\tau\left(x_0, \frac{M_l(\rho) - M_l(\rho/4) + 2b_0\rho}{\rho}\right) \int_{\Omega_t,\rho/s} \psi^{-\tau}(x_0, u_{t,\rho}) g(x, |\nabla u|) u_{t,\rho} \zeta^q dx .
\]
Hence, applying successively (4.38), (4.35) and the weak Harnack inequality (2.9), we derive the inequality
\[
I_1 \leq \gamma \frac{e^{\lambda(\rho)}M_1(\rho)}{\rho} \left(x_0, \frac{M_1(\rho) - M_1(\rho/4) + 2b_0\rho}{\rho}\right) \int_{B_{\rho/8}(x_0)} \psi^{1-\tau}(x_0, u_{l,\rho}/\rho) \, dx
\leq \gamma \frac{e^{\lambda(\rho)}M_1(\rho)}{\rho} \left(x_0, \frac{M_1(\rho) - M_1(\rho/4) + 2b_0\rho}{\rho}\right) \int_{B_{\rho/8}(x_0)} g^{1-\tau}(x_0, u_{l,\rho}/\rho) \, dx
\leq \Lambda(\gamma, 3n, \rho)M_1(\rho)\rho^{n-1}g \left(x_0, \frac{M_1(\rho) - M_1(\rho/4) + 2b_0\rho}{\rho}\right).
\]
Combining inequalities (5.5), (5.7) and (5.8), we arrive at (5.1).

We are now in a position to complete the proof of the inequality (2.7). Let \( \eta \in C_0^\infty(B_{\rho/16}(x_0)) \) be a function such that \( 0 \leq \eta \leq 1, \eta = 1 \) in \( B_{3\rho/64}(x_0) \) and \( |\nabla \eta| \leq 64/\rho \), then
\[
\int_{\Omega_{l,\rho/16}} G(x, |\nabla(u_{l,\rho}\eta)|) \, dx \leq \int_{\Omega_{l,\rho/16}} G(x, |\nabla u| + 64u_{l,\rho}/\rho) \, dx
\leq \int_{\Omega_{l,\rho/16}} g(x, |\nabla u| + 64u_{l,\rho}/\rho) \, dx.
\]
In addition, due to (1.2) the following holds on the set \( \Omega_{l,\rho/16} \):
\[
g(x, |\nabla u| + 64u_{l,\rho}/\rho) \leq g(x, 2\max\{|\nabla u|, 64u_{l,\rho}/\rho\})
\leq c_1 128^{q-1}g(x, \max\{|\nabla u|, u_{l,\rho}/\rho\})
\leq c_1 128^{q-1}(g(x, |\nabla u|) + g(x, u_{l,\rho}/\rho)).
\]
Substituting (5.10) into (5.9), expanding the brackets and splitting the terms \( g(x, u_{l,\rho}/\rho)|\nabla u| \) and \( g(x, |\nabla u|)u_{l,\rho}/\rho \) by Young’s inequality (4.1), we obtain
\[
\int_{\Omega_{l,\rho/16}} G(x, |\nabla(u_{l,\rho}\eta)|) \, dx \leq \gamma \left( \int_{\Omega_{l,\rho/16}} G(x, |\nabla u|) \, dx + \int_{\Omega_{l,\rho/16}} G(x, u_{l,\rho}/\rho) \, dx \right).
\]
The first integral on the right-hand side of (5.11) is estimated by using (5.1):
\[
\int_{\Omega_{l,\rho/16}} G(x, |\nabla u|) \, dx \leq \int_{\Omega_{l,\rho/16}} G(x, |\nabla u|) \zeta^q \, dx
\leq \Lambda(\gamma, 3n, \rho)M_1(\rho)\rho^{n-1}g \left(x_0, \frac{M_1(\rho) - M_1(\rho/4) + 2b_0\rho}{\rho}\right),
\]
and the second one, by using condition (g3) and the weak Harnack inequality (2.9):
\[
\int_{\Omega_{l,\rho/16}} G(x, u_{l,\rho}/\rho) \, dx \leq \gamma \rho^{-1}(M_1(\rho) + 2b_0\rho)e^{\lambda(\rho)} \int_{B_{\rho/8}(x_0)} g(x_0, u_{l,\rho}/\rho) \, dx
\leq \Lambda(\gamma, 3n, \rho)(M_1(\rho) + 2b_0\rho)\rho^{n-1}g \left(x_0, \frac{M_1(\rho) - M_1(\rho/4) + 2b_0\rho}{\rho}\right).
\]
Collecting (5.11)–(5.13), we arrive at (2.7). The proof is complete.
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