Quantum Dynamics for the Control of Atomic State by a 
Quantized Optical Ring Cavity

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Abstract

A generalized approach of the Born-Oppenheimer approximation is developed to analytically deal with the influence exercised by the spatial motion of atom’s mass-center on a two-level atom in an optical ring cavity with a quantized single-mode electromagnetic field. The explicit expressions of tunneling rate are obtained for various cases, such as that with initial coherent state and thermal equilibrium state at finite temperature. Therefore, the studies for Doppler and recoil effects of the spatial motion on the scheme controlling atomic tunneling should be reconsidered in terms of the initial momentum of atom’s mass center.

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1. Introduction

Many new developments in both the experimental and theoretical aspects of the so-called cavity quantum electrodynamics [1,2] have shown the possibility controlling the coherent tunneling of atomic states by a quantized or classical cavity field [3-7]. By immersing an atom into a cavity field with a proper strength and frequency, the tunneling rate can be enhanced or reduced to the values several orders of magnitude higher than the rate for the “bare” atom.

To understand the essence in the mechanism governing the control of atomic tunneling, we draw an obvious analogy in the basic quantum mechanics, the coherent tunneling phenomenon in a one-dimensional double well potential. It is well-known that the two lowest-energy states with a finite split of energy in this potential well possess even and odd parities respectively. Their symmetric and antisymmetric superpositions approximately represent the localizations of particle in the right and left wells separately. Due to the energy split, one of the two superposition states can evolve into another and then back to the original one. Such a coherent tunneling with a period determined by the energy split enjoys the general feature of quantum mechanics, but the phenomenon of localization does not appear in the usual case that the system is isolated as a closed system. To change the tunneling rate, a possible way is to immerse the system in a certain environment as an open system, but the control for a given goal can not be realized in this sense because of the random elements such as the Brownian motion exercised by the environment [8]. However, in the above-mentioned studies about the control of atomic tunneling, the random environment is replaced by the cavity field, which can be prepared in advance for a specific purpose to enhance or reduce the tunneling rate. Notice that all the investigations about atomic tunneling control have not concerned the motion of mass-center to our best knowledge.

In this paper, based on the generalized Born-Oppenheimer (BO) approximation developed by this author to separate the fast and slow dynamical variables for the spin-precession in an inhomogeneous magnetic field [9, 10], we present a delicate
study to analyse the influence of the motion of atomic mass-center on the tunneling and localization of atomic states in a cavity field. In fact, the spatial motion of atom (strictly speaking, its mass-center) plays a crucial role in many fashionable problems in so-called atom-optics, such as the diffraction and splitter of atom beam by a standing wave cavity field in connection with atom interferometer [11-15], the quantum nondemolition measurement by an optical ring cavity [16] and trapping and colling atoms with an adiabatically-decaying cavity mode [17,18]. In the limit of strong field, the dressed atomic eigenstates are obtained in accord with the generalized BO approximation. It is then proved that the higher order approximations mix them to cause the tunneling from one to another among them. The explicit expressions of tunneling rates are given to manifest the crucial role of the Doppler effect of spatial motion of atomic mass center in a locally-inhomogeneous cavity field.

2. The model

Consider the most simple case that a two-level atom moves along the optical axis $x$ in an optical ring cavity with a single-mode quantized electromagnetic field

$$E \sim a^\dagger e^{-ik\hat{x}} + ae^{ik\hat{x}}$$  \hspace{1cm} (1)

where $a^\dagger$ and $a$ are the creation and annihilation operators for the cavity mode respectively; $\hat{x}$ denotes the position operator conjugate to the momentum operator $\hat{p}$; $|1>$ and $|2>$ are the ground and excited states within the atom. According to Sleator and Wilkens [16], one write the Hamiltonian for the atom-cavity system with spatial motion

$$\hat{H} = \frac{\hat{p}^2}{2M} - \frac{\Delta}{2}(|1><1| - |2><2|) + \omega a^\dagger a + g(a^\dagger e^{-ik\hat{x}} + ae^{ik\hat{x}})(|1><2| + |2><1|)$$  \hspace{1cm} (2)

where $g$ is the atom-cavity coupling constant depending on the mode-volume and the atomic dipole matrix elements. For simplicity, we only consider the effect of the spatial motion of lower orders caused by the long-period cavity field with small $k$. 

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As pointed out in refs.[7], the rotation-wave approximation is only adequate to analyse the case of Jaynes-Cummings atom [19, 20] close to resonance and weak coupling, but the control of atomic tunneling requires the case far away from resonance with a proper coupling. Thus, it is necessary to develop an adaptable approximation method, which can work well in the present situation. Fortunately, the generalized BO approximation developed about four years ago for the induced gauge structure and Berry’s phase [9, 10] can be extended here as a systematically-analytical method to deal with the problem of the control for atomic tunneling. The present approach also recovers the adiabatic variational principle used in ref.[7] as its lowest order approximate result.

By invoking an unitary transformation similar to that in ref.[6]

\[
\hat{W}(x) = \exp(-ikxa^\dagger a)
\] 

one obtains an approximate effective Hamiltonian \(H_e = W^\dagger HW\) :

\[
\hat{H}_e = \frac{\hat{p}^2}{2M} + \frac{\Delta}{2}(|2><2| - |1><1|) + \hat{\Omega}(\hat{p})a^\dagger a + g(a + a^\dagger)(|1><2| + |2><1|)
\] 

(4)

with momentum-dependent frequency

\[
\hat{\Omega} = \Omega(\hat{p}) = \omega - \frac{k\hat{p}}{M}
\]

(5)

Here, the effective frequency was modified by the Doppler shift \(\frac{k\hat{p}}{M}\) and the nonlinear term \(k^2(a^\dagger a)^2\) appearing as the Kerr-like interaction has been neglected for the consideration of large-period cavity field.

3. Generalized BO approximation

To describe the tunneling and localization of the symmetric and antisymmetric superpositions of two atomic eigenstates

\[
|\pm> = \frac{1}{\sqrt{2}}(|1> \pm |2>)
\]

we make an ansatz

\[
|\psi> = \phi_+|+> + \phi_-|->
\] 

(6)

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for the eigenstate of the atom-cavity system by drawing an analogy to the original BO approximate expansion [9,10]. Here, the vector-valued coefficients \( \phi_\pm \) depend on both Fock space of the cavity field and the spatial variable of the atomic mass-center. They are imagined as the collective degree of freedom in the original BO approximation. Substituting \(|\psi>\) into the eigenvalue equation \( \hat{H}_e|\psi>=E|\psi>\), one can obtain an operator-valued matrix equation

\[
H\Phi + V\Phi = E\Phi
\]

with the definitions

\[
H = \begin{pmatrix} \hat{H}_+ & 0 \\ 0 & \hat{H}_- \end{pmatrix}, V = \frac{1}{2}\Delta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \Phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}
\]

where

\[
\hat{H}_\pm = \frac{p^2}{2M} + \frac{1}{2}\Omega(p)a^\dagger a \pm g(a + a^\dagger)
\]

As proved in the generalized approach of BO approximation [9,10], the usual stationary perturbation theory associated with the above representations (7-9) can result in the BO approximate solutions for the coefficient vector \( \Phi \) to any order of the perturbation \( V \). For instance, the second order solutions are constructed as

\[
\Phi^{[1]}_\gamma(n) = \sum_{n\neq m} \frac{<\Phi^{[0]}_\beta(m,p)|\Phi^{[0]}_\gamma(n,p)>}{2(E_n - E_m)} \Delta \Phi^{[0]}_\beta(m,p), \gamma \neq \beta = \pm
\]

from the first order ones

\[
\Phi^{[0]}_+(n,p) = \begin{pmatrix} |\eta_+(n)> \\ 0 \end{pmatrix} \otimes |p>, \Phi^{[0]}_-(n,p) = \begin{pmatrix} 0 \\ |\eta_-(n)> \end{pmatrix} \otimes |p>,
\]

with eigenvalues

\[
E_{\pm,n} = E_n = \frac{p^2}{2M} + (n - \alpha^2)\Omega(p), \alpha = \frac{g}{\Omega(p)}
\]

Here, \(|\eta_\pm(n)>\) are determined by the Hamiltonian (9) as the solutions

\[
|\eta_\pm(n)> = |\eta_\pm(n,\alpha) = D(\mp\alpha)|n>.
\]

of the eigen-equations

\[
\hat{H}_\pm|\eta_\pm(n)> \otimes |p> = E_n|\eta_\pm(n)> \otimes |p>;
\]
and
\[ D(z) = \exp[za^\dagger - z^*a] \]  

(14)
is the displace-operator of coherent state \(|z> = D(z)|0>\); \(|p>\) is a momentum eigenstate such that \(\hat{p}|p> = p|p>\); \(|n>\) is the Fock state. It has to be pointed out that the stationary states \(|\eta_\pm(n)>\) were even obtained with the adiabatic variational principle in ref.[7]. However, all the results in ref.[7] are only of the the special case for the general ones in present studies, which, in fact, are of first order in comparison with that in present studies.

With the help of straightforward substitutions of eq.(11) into eq.(10), it is not difficult to calculate the coefficient \(\phi_\pm\) to be of second order
\[ |\phi_\pm[^1](n,p) >= |\eta_\pm(n) > \otimes | \pm > + \sum_{m \neq n} \Delta_{mn}^\pm (p) |\eta_\mp(m) > \otimes | \mp > \otimes |p> \]  

(15)

where
\[ \Delta_{mn}^\pm (p) = \frac{\Delta}{2\Omega(p)} \frac{<\mp |\eta_\mp(m)|\eta_\pm(n)>}{m-n} = \frac{\Delta \cdot F(m,n)}{2\Omega(p)(m-n)} \]

\[ \equiv \frac{\Delta \sqrt{m!n!} \cdot M_{\min(m,n)}^{m-n}}{2(\omega - \frac{kp}{M})} \sum_{i=0} e^{-|\alpha|^2} (\mp 2\alpha)^{m+n-2l} (m-l)! (m+l)! (m-n)! \]  

(16)

Notice that \(|\phi_\pm(n,p)>\) are degenerate for a fixed energy \(E_n\). In the view of BO approximation, the first term in r.h.s. of eq.(15) describes the adiabatic process and leads to an approximately- stationary evolution for the atomic localization states \(|\pm>\), that is to say, the evolving state starting from \(|+>\) (or \(|->\)) only differs from this initial state in a phase factor with an invariant norm. In this sense, one need to neglect the second term in the r.h.s. of eq.(15) whose norm proportional to \(|\Delta_{mn}^\pm|\) under the adiabatic conditions
\[ |\Delta_{mn}^\pm| \sim \frac{\Delta}{|\omega - \frac{kp}{M}|} \ll 1. \]  

(17)

This is just the stationary condition for the evolution of \(|\pm>\) In fact, the second term in the r.h.s. of eq.(15) represents the non-adiabatic effects of evolution. It is quite interesting that whether the states \(|\pm>\) are stationary or not must depend on the initial momentum of atom in the cavity, exactly speaking, the velocity and
the direction of spatial motion of atomic mass-center. Therefore, as shown in next section, the spatial motion of the atomic mass-center must exercise an observable effect— the Doppler effects on the tunneling and localization of atomic state.

4. Control of atomic tunneling at zero temperature

Recently, Plata and Gomez Lorent showed that the existence of cavity field under certain conditions may decrease the effective energy-difference between the dressed states of \( |1 \rangle \) and \( |2 \rangle \) so that they approach degeneracy in presence of the cavity field [7]. In the present sense, the dressed states of \( |1 \rangle \) and \( |2 \rangle \) are modified by the spatial variable of atom besides the cavity mode and then can also approach degeneracy only for the suitable momentum state of the atom’s mass center. Therefore, they evolve according to Schrodinger equation with the approximately-equal phases to realize the localization for the tunneling between the dressed states \( |+ \rangle \) and \( |− \rangle \) in presence of certain quantized cavity. During this process, the dressed localization states are approximately stationary. Indeed, the discussion in the section 3 demonstrates that the first order dressed states

\[
|n, α, ±, p \rangle = |\eta_\pm (n) > \otimes |± > \otimes |p >
\]

like the approximate eigenstates of first order for \( H_e \) possess the approximately-equal energies for the effective Hamiltonian (4).

To analyse dynamics quantitatively for the problems mentioned above, the solutions (15) are transformed back to the original representation

\[
|\psi_\pm^{(1)} (n, p) \rangle = \hat{W} (x) |\phi_\pm^{(1)} (n, p) \rangle = |\psi_\pm^{(0)} (n, p) \rangle + \sum_{m \neq n} \Delta^\pm_{mn}(p) |\psi_\pm^{(0)} (n, p) \rangle
\]

where

\[
|\psi_\pm^{(0)} (n, p) \rangle = \hat{W} |n, α, ±, p \rangle = |\eta_\pm (n, \alpha e^{ikx}) > \otimes |± > \otimes |p + nk >.
\]

The above expression manifests that, under the first order approximation, the approximately-
stationary states are $|\psi^0_{\pm}(n, p)\rangle$. Unlike those for the case without spatial motion effect, the above stationary states are not only dressed by the cavity field, but also accompanied with the momentum shifts for the different components to $|n\rangle$. Imaging $|\psi^0_{\pm}(n, p)\rangle$ as the right and left localization states in one-dimensional double-well potential, one can consider the tunneling problem between $|\pm\rangle$ and $|\mp\rangle$.

Let us now focus on the simplest case that the atom is initially in the “left” dressed state $|\psi^0_{+}(n, p)\rangle$, in which the cavity is in the displace Fock state $|\eta_{+}(n, \alpha e^{ik}\rangle >$ while the atom in the “left” state with the momentum shift $p + nk$. Under the second approximation, the wavefunction of cavity-atom system at $t$ is

$$|\Psi_n(t)\rangle = \hat{W}(x)\{\exp(-i\frac{\hat{p}^2t}{2M} + i\alpha^2\Omega t) 
\times[e^{-i\omega t}|\phi^{[1]}_{+}(n, p)\rangle - \sum_{m\neq n}\Delta^+_{mn}(p)e^{-i\omega t}|\phi^{[1]}_{-}(n, p)\rangle]\}$$

$$= \exp[-i(p + ka^+a)t + i\alpha^2\Omega t][e^{-i\omega t}|\psi^0_{+}(n, p)\rangle + \sum_{m\neq n}\Delta^+_{mn}(p + ka^+a)[e^{-i\omega t} - e^{-i\omega t}]|\psi^0_{-}(m, p)\rangle]$$

(20)

which gives the probability of the transition from $|\psi^0_{+}(n, p)\rangle$ to $|-\rangle$

$$P_n = \sum_{m\neq n}4|\Delta^+_{mn}|^2\sin^2[\frac{1}{2}(m - n)\omega t]$$

$$= \sum_{m\neq n}\frac{\Delta^2 F(m, n)^2}{(\omega - \frac{\alpha^2\Omega}{\Delta^2})^2(m - n)^2}\sin^2[\frac{1}{2}(m - n)\omega t].$$

(21)

Obviously, if one ignores the second term proportional to $\Delta^\pm_{m,n}$ in eq.(20), the atomic state $|+\rangle$ is approximately stationary for

$$\sum_{n,p} |<\psi^0_{-}\rangle n, p|\Psi_m(t)\rangle|^2 \sim 0$$

It is also observed that the tunneling rate from a dressed state of $|+\rangle$ to $|-\rangle$ can be controlled by using the cavity field with suitable frequency $\omega$ and preparing the atom with a proper initial momentum. If the atom moves along the direction opposite to the wave vector $\vec{k}$ in high frequency cavity field, the tunneling rate (21) tends to be very small and then the atomic dressed state $|\psi^0_{\pm}(p)\rangle$ tends to be well localized in $|+\rangle$. Preparing different initial state, e.g., the lower frequency cavity
and the atomic momentum along \( \vec{k} \), the localization will be broken and the tunneling process will be enhanced.

To complete a dynamical description of the tunneling control, one must also specify the different initial conditions for the problem. Since various initial conditions for the single-mode field have become experimentally realizable, it is useful to obtain the different formulas of the corresponding tunneling rates. For a general pure-state distribution of the cavity field

\[ |c> = \sum C_n |\eta+(n)> , \]  

the tunneling rate is a superposition of those oscillations with different frequencies \((m - n)\omega\)

\[ P = \sum \sum |C_n|^2 \frac{\Delta^2 \cdot F(m, n)^2}{(\omega - \frac{\delta k}{M})^2(m - n)^2} \sin^2 \left[ \frac{\omega(m - n)}{2} \right] \]  

When the coherent state \(|Z>\) is taken as the initial state for the cavity, the distributions \(C_n\) are specified as

\[ C_n(z) = <\eta+(n)|z> = e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\alpha+z|^2} \frac{(\alpha + z)^n}{\sqrt{n!}} \]  

where \(|z|^2\) denotes the mean photon number. If the initial state has definite photon number, that is \(|m>\), then one has the specific distribution

\[ C_n = C_{n;m}(\alpha) = <\eta+(n)|m> = <n|D(\alpha)|m> = \sum_{l=0}^{\text{Min}(n,m)} \frac{\sqrt{n!m!}}{(n-l)!(m-l)!} e^{-\frac{1}{2}|\alpha|^2} \alpha^{n+m-2l} ; \]  

5. **The cases with finite temperature**

In this section, we turn to discuss the influence of the temperature \(T\) of the cavity on the dynamics of tunneling control. In this sense, the cavity is supposed in thermal equilibrium and then the corresponding mixed state described by Bose-Einstein photon number distribution

\[ \rho_c(0) = \sum_{n=0}^{\infty} \frac{1}{\Omega} e^{-n\beta\omega} |n><n| \]  

9
where
\[ \beta = \frac{1}{kT}, \quad \Omega = (1 - e^{-\beta \omega})^{-1}. \]
Expressing the Fock states \( |n\rangle \) in terms of the displaced Fock states \( |\eta+(n, \alpha e^{-ikt})\rangle \), one have an initial state for the atom-cavity system
\[
\rho(0) = \sum_{n=0}^{\infty} \sum_{m,m' \neq n} \frac{1}{\Omega} e^{-n\beta \omega} C'_{m,n} C'^*_{m',n}|\psi^{[0]}_+(m, p - mk)\rangle \langle \psi^{[0]}_+(m', p - m'kh)| (27)
\]
where \( C'_{m,n} = C_{m,n}(\alpha e^{ikx}) \) is defined by eq.(25) and the initial state of atomic mass-center is chosen as \( |+\rangle \otimes |p\rangle \). Then, one can write the density matrix for the atom-cavity system at time \( t \).
\[
\rho(t) = U(t) \rho(0) U(t)^\dagger = \sum_{n=0}^{\infty} \sum_{m,m' \neq n} \frac{1}{\Omega} e^{-n\beta \omega} C'_{m,n} C'^*_{m',n}|\Psi'_m(t)\rangle \langle \Psi''_m(t)| (28)
\]
where
\[
|\Psi'_m(t)\rangle = |\Psi_m(t)\rangle |p_{-p-mk}\rangle
\]
is given by eq.(20) and \( U(t) \) is the evolution operator.

Then, the tunneling rate at temperature \( T \) is obtained from eq.(28) as the probability of finding atom in the state \( |-\rangle \):
\[
P(T, t) = Tr(|-\rangle \langle -| \rho(t)) = \sum \frac{1}{\Omega} e^{-n\beta \omega} P_n(t) (29)
\]
where
\[
P_n(t) = Tr(<-|U(t)|n, +, p><n, +, p|U(t)|-) = \sum_{m,m'} C'_{m,n} C'^*_{m',n} \sum_{l=0} \sum_{m \neq l} C'_{m,n} \Delta_k^l(p - mk)[e^{-i\omega t} - e^{-i\omega t}]^2 (30)
\]
is the transition probability for \( n \)'th channel switched on by the existence of the thermal cavity field where
\[
|n, +, p\rangle = |n > \otimes |+\rangle \otimes |p\rangle
\]

Obviously, if we have many identical atoms in the cavity with single-mode radiation in thermal equilibrium with the wells at temperature \( T \), the tunneling rate will
increase as $T$ becomes higher and then the thermal perturbation must enhances the tunneling. Conversely, at the lower temperature, the tunneling rate is suppressed and then the localization of state $|\phi_+(p)\rangle$ is easily realized. Therefore, the experiment to control tunneling and localization should be well carried out at lower temperature. This is a trivial but very useful observation.

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References

[1] P.Meystre, Phys.Rep, 219(1992),2435
[2] S.Haroches, D.Kleppner, Phys. Today 1989,24
[3] W.A.Lin, L.E.Ballentine, Phys. Rev.Lett.65(1990), 2927
[4] F.Grossmann, T.Dittrich, P.Jung, P.Haggi, Phys.Rev.Lett. 67(1991),516
[5] R.Bavli, H.Metiu, Phys.Rev.Lett. 69(1992),1986
[6] H.Holthaus, Phys.Rev.Lett. 69(1992),1596
[7] J.Plata, J.M.Gomez Lorente, Phys.Rev.A.48(1993),782 ibd.45(1992)R6958.
[8] L.H.Yu, C.P.Sun,Phys.Rev.A,49(1994),592 and refs therein
[9] C.P.Sun. M.L.Ge. Phys.Rev.D.,38(1990),1349
[10] C.P.Sun. M.L.Ge. Q.Xiao Commun.Theor.Phys. 13(1990),63
[11] E.Arimondo, A.Bambini, S.Stenholm, Opt.Comnn. 37(1981),103
[12] S.Glasgow, P.Meystre, M.Wilkens,E.M.Wright, Phys.Rev.A.43(1991),2455
[13] M.Lindberg, Appl.Phys. B.54(1992), 476
[14] T.Sleator, T.Pfau, V.Balykin, O Carnal, J.Mlynek, Phys.Rev.Lett.68 (1992),1996
[15] D.W.Keith, C.R.Ekstrom, Q.A.Turchette, D.E.Prichard, Phys.Rev.lett. 66 (1991),2693
[16] T.Sleator, M.Wilkens, Phys.Rev.A.48(1993),3286
[17] M.Kasevich,S.Chu,Phys.Rev.Lett.67(1991),181
[18] T.Zaugg, M.Wilkens, P.Meystre, G.Lents,Opt.Commun.97(1993),189
[19] E.T.Jaynes, F.W.Comnings, Proc.IEEE,51(1963),89
[20] B.W.Shore, P.L.Knight, J.Mod.Opt.,40(1993),1195