WMAP constraints on Cardassian model

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We have investigated the constraints on the Cardassian model using the recent results from the Wilkinson Microwave Anisotropy Probe (WMAP) for the locations of the peaks of the Cosmic Microwave Background (CMB) anisotropy spectrum. We find that the model is consistent with the recent observational data for certain range of the model parameter \( n \) and the cosmological parameters. We find that the Cardassian model is favoured compared to ΛCDM model, for a higher spectral index \( n_s \approx 1 \) together with lower value of Hubble parameter \( h \leq 0.71 \). But for smaller values of \( n_s \), both ΛCDM and Cardassian are equally favoured. Also, irrespective of Supernova constraint, CMB data alone predicts current acceleration of the universe in this model. We have also studied the constraint on the \( \sigma_8 \), the rms density fluctuations at \( 8h^{-1}\) Mpc scale.

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I. INTRODUCTION

The recent era has been dedicated to precision cosmology. Different remarkable observations with great precision has set new direction in realising the details of the universal evolution. Over the last few years we have developed a firm broad outline about our universe to be spatially flat, homogeneous, isotropic on large scales and composing of radiation, ordinary matter including baryon and neutrinos, cold dark matter and a huge share of dark energy. There have been a host of phenomenological models in support of this picture, which sets strong limits on the cosmological parameters. But the latest Wilkinson Microwave Anisotropy Probe (WMAP)\textsuperscript{11} has set a new challenge for these models. WMAP has shown incredible precision in predicting different cosmological parameters including the positions of the peaks in CMB anisotropic spectrum. WAMP’s accurate determination of angular power spectrum along with other astronomical data sets like SNIa observations\textsuperscript{2, 3} and 2dF galaxy survey\textsuperscript{4}, put significant limit on the cosmological parameters, which should be defended by the various phenomenological models.

Since the significant observations regarding the SNIa\textsuperscript{2, 3}, several models both simple and complicated ones, have been used so far to justify the late time accelerating universe and at the same time proposing different ranges for the cosmological parameters. With ΛCDM among the simple and Quintessence\textsuperscript{6}, K-essence\textsuperscript{7} and several others\textsuperscript{8} as more complicated ones, each one aims in describing the universe in accordance to more precised observational agreement. One more thing common here is that these models accepts the idea of dark energy in any form of field or fluid. While there have been many other interesting ideas like the generalised Chaplygin gas\textsuperscript{9} or rolling tachyons\textsuperscript{10}, a different proposal has been put by Freese and Lewis\textsuperscript{11}. In this new approach the new cosmology is obtained by modifying the dynamics of the universe, namely, the Friedman equation, without seeking to another unknown component like dark energy.

They have proposed an universe composed of only radiation and matter(including baryon and cold dark matter) which expands in the speed up fashion if an empirical term, called Cardassian term, is added to the Friedman equation.

\[ H^2 = A \rho + B \rho^n, \]

where \( A = \frac{8\pi G}{3} \) and \( B \) and \( n \) are constants and are the parameters of the model. Here the energy density \( \rho \) contains only matter \( (\rho_m) \) and radiation \( (\rho_r) \), i.e, \( \rho = \rho_m + \rho_r \). Since at present \( \rho_m >> \rho_r \), \( \rho \) can be considered consisting of \( \rho_m \) only at present. The new term, dominates only recently at redshift \( \sim 1 \). To provide the requisite acceleration of the universe as the outcome of the dominance of this term, \( n \) should be \(< 2/3 \). The model has two main parameters \( n \) and \( B \).

There are several interpretations for the origin of this new “Cardassian term” appearing in the Einstein’s equation (1)\textsuperscript{12}. To accommodate flatness with only matter and radiation in this new picture, the critical energy density has

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been modified and expressed as a function of the original one. Here the total density parameter at present $\Omega_0$ is defined as $\frac{\rho_m^0 + \rho_r^0}{\rho_c}$, where the redefined critical energy density $\rho_c$ is related to the actual one $\rho_{ca} (= 3H_0^2/8\pi G)$ as,

$$\rho_c = \rho_{ca} \times F(n, B)$$  \hspace{1cm} (2)

where

$$F(n, B) = \left[1 + \frac{B}{A} \rho_m^{n-1} (1 + \frac{\Omega_r}{\Omega_m})^{n-1}\right]^{-1}$$  \hspace{1cm} (3)

\(\Omega_m\) and \(\Omega_r\) are two parameters defined as \(\Omega_m = \frac{\rho_m^0}{\rho_c}\) and \(\Omega_r = \frac{\rho_r^0}{\rho_c}\) respectively.

Now, as matter and radiation evolves, equation (1) can be expressed as the following,

$$H^2 = A \rho_m^0 a^{-3} \left[1 + \frac{\Omega_r}{\Omega_m} a^{-1} + \frac{B}{A} \rho_m^{n} a^{-3n} \left(1 + \frac{\Omega_r}{\Omega_m} a^{-1}\right)^n\right]$$  \hspace{1cm} (4)

From equation (3) it is very straightforward to express \(B\) in terms of \(\Omega_r\) and \(\Omega_m\) to be

$$\frac{B}{A} \rho_m^{n-1} = \frac{1}{\Omega_m} \left(\frac{\Omega_r}{\Omega_m} - \Omega_m\right) (1 + \frac{\Omega_r}{\Omega_m})^{-n}.$$  \hspace{1cm} (5)

Substituting this expression in equation (4), equation (1) is finally recast in the following fashion

$$H^2 = \Omega_m^0 H_0^2 a^{-4} \left[a + \frac{\Omega_r}{\Omega_m} + a^{-4n+4} \left(1 - \frac{1}{1 + \frac{\Omega_r}{\Omega_m}}\right) \frac{\Omega_r}{\Omega_m} \right]$$  \hspace{1cm} (6)

One should note that the only model parameter appearing in the above expression is \(n\). (For details discussion please refer [13]) The other important point to notice is that the value \(n = 0\) corresponds to a ΛCDM model. This will be crucial when we shall talk later about the allowed region of the parameter space.

In one of our very recent papers [13] we have considered the constraints imposed on this model’s parameters from the CMB measurements like BOOMERanG and Archeop and the Supernova results. We found the model as a quite aspiring as an alternative to dark energy models with interesting limits on its parameter space (For other investigations in this regard please refer to [14]). Now in the light of the very precise measurements of WMAP it is certainly very much worthy to verify the consistency of the model and found the new restricted parameter space. We further study the constraints on \(\sigma_s\) in our model.

II. THE CONSTRAINTS FROM WMAP

The CMBR peaks arise from oscillation of the primeval plasma just before the universe becomes translucent. The oscillation of the tightly bound photon-baryon fluid is a result of the balance between the gravitational interaction and photon pressure and this oscillations gives rise to the peaks and troughs in the temperature anisotropic spectrum. Locations of the peaks at different angular momentum \(l\), resulted from this oscillation, depend on the acoustic scale \(l_A\) which in turn is related to the angular diameter distance to the last scattering \(D\) and the sound horizon at the last scattering \(s_l\), as $\frac{\pi D}{s_l}$. To a good approximation this ratio for \(l_A\) can also be represented by the simple expression [16]

$$l_A = \frac{\bar{\tau}_0 - \bar{\tau}_s}{\bar{c}_s \bar{\tau}_l}.$$  \hspace{1cm} (7)

where $\bar{\tau} (= \int a^{-1} dt)$ is the conformal time and the subscript 0 and \(ls\) represent the time at present and at the last scattering era respectively. \(\bar{c}_s\) is the average sound speed before last scattering defined as

$$\bar{c}_s \equiv \bar{\tau}_l^{-1} \int_0^{\bar{\tau}_s} \bar{c}_s d\bar{\tau},$$  \hspace{1cm} (8)

with

$$c_s^{-2} = 3 + \frac{9}{4} \frac{\rho_b}{\rho_r}.$$  \hspace{1cm} (9)

$\frac{\rho_b}{\rho_r}$ is the ratio of baryon to photon energy density.
To calculate the conformal time today and at the last scattering, we use eqn.(6),

$$\tilde{\tau}_{ls} = \int_{0}^{\tilde{\tau}_{ls}} d\tilde{\tau} = \frac{1}{\Omega_{m}^{1/2} H_0} \int_{0}^{\tilde{a}_{ls}} \frac{da}{X(a)}$$  \hspace{1cm} (10)$$

and

$$\tilde{\tau}_{0} = \int_{0}^{\tilde{\tau}_{0}} d\tilde{\tau} = \frac{1}{\Omega_{m0}^{1/2} H_0} \int_{0}^{1} \frac{da}{X(a)}$$ \hspace{1cm} (11)$$

where $X(a) = \sqrt{(a + \Omega_{r} a_{0}) + a^{-4n+4} \left( \frac{1-\Omega_{r0} - \Omega_{m0}}{\Omega_{m0}} \right) \left( \frac{a + a_{0}}{1 + \frac{a_{0}}{\Omega_{m0}}} \right)^n}$.

Substituting the above expression in equation (9), we have the analytical expression for $l_A$ in case of this model

$$l_A = \frac{\pi}{c_s} \left[ \frac{\int_{0}^{1} \frac{da}{X(a)}}{\int_{0}^{\tilde{a}_{ls}} \frac{da}{X(a)}} - 1 \right]$$ \hspace{1cm} (12)$$

In an ideal photon-baryon fluid model, the simple analytic relation for the position of the m-th peak and the acoustic scale is $l_m = m l_A$. But this simplicity gets disturbed by the different driving and dissipative effects which in turn induces a phase shift to the original position [15]. This shift has been compensated by parametrizing the location of

FIG. 1: Contour plots of the first, second and third Doppler peak locations in the $(\Omega_m, h)$ plane for different values of $n$ and $n_s = 1$. Full and dashed contours correspond to the WMAP bound of first and second peak locations and the dot-dashed contour corresponds to the Boomerang bound on third peak location. The region within the dotted line corresponds to the $1-\sigma$ confidence region for $\Omega_m$ for the Supernova data. The allowed region is the intersection of all the contours.
the peaks and troughs by

\[ l_m \equiv l_A (m - \phi_m) \equiv l_A (m - \bar{\phi} - \delta \phi_m) \]  

(13)

where \( \bar{\phi} \) is the overall peak shift \( \equiv \phi_1 \) and \( \delta \phi_m \equiv \phi_m - \bar{\phi} \) is the relative shift of the \( m \)th peak. This parametrisation can be used to extract information about the matter content of the universe before last scattering. Although it is certainly very difficult to derive analytical relation between cosmological parameters and phase shifts, Doran and Lilley\[17\] have given certain fitting formulae which makes life very simple. These formulae do not have a prior and crucially depends on cosmological parameters like spectral index \( (n_s) \), baryon density \( (\omega_b = \Omega_b h^2) \), Hubble parameter \((h)\), ratio of radiation to matter at last scattering \((r_{ls})\) and also \( \Omega^d_{ls} \) as representative of dominating present dark energy density at the time of recombination. We use these formulae (given in Appendix) to specify the positions of the peaks in our model and finally constrain the model with the most recent WMAP results.

The locations of the first two acoustic peaks from the WMAP measurement\[1\] of the CMB temperature power spectrum are

\[ l_{p1} = 220.1 \pm 0.8; \]
\[ l_{p2} = 546 \pm 10; \]  

(14)

notice that all uncertainties are within 1\( \sigma \). The location for the third peak is given by BOOMERanG measurements\[18\]

\[ l_{p3} = 825^{+10}_{-13}. \]  

(15)

We have studied the locations of the first three acoustic peaks in the cosmological parameter space for the Cardassian model given by Eq. (6). The sole parameter of the model is \( n \). The cosmological parameter space we have investigated is given by \((\tau, n_s, h, w_b, \Omega_m)\). Throughout this paper, we have neglected the contribution from the spatial curvature.
and massive neutrinos, setting $\Omega_k = \Omega_r = 0$. We have also neglected the contribution from the gravitational wave in the initial fluctuation. Because of the rather tight WMAP constraint on $w_b$, ($w_b = 0.0224 \pm 0.0009$), we have assumed $w_b = 0.0224$ in our subsequent calculations. We have also assumed the optical depth of the last scattering $\tau = 0.11$ which is within the range of the WMAP bound, $\tau = 0.166^{+0.070}_{-0.071}$.

In Fig.1, we have plotted the contours of the first three acoustic peak locations, corresponding to the WMAP and BOOMERanG bounds given by Eqn.(14)-(15) in the $(\Omega_m, h)$ parameter space for different values of $n$ and for $n_s = 1$. In the same figure we have also shown the 1--σ bound on $\Omega_m$ by fitting our model with the SCP (Supernova Cosmology Project) data.

To obtain a fit to SCP data, we write the apparent magnitude as

$$m(z, \Omega_m, n, M) = 5\log_{10}D_l + M,$$

where $\mathcal{M} \equiv M - 5\log_{10}H_0 + 25$ and $D_l = H_0d_l$ is the dimensionless luminosity distance, given in ref [13]. The parameters $\Omega_m, n$ and $\mathcal{M}$ are determined by minimizing

$$\chi^2 = \sum \frac{(m_{\exp}(z_i) - m(z_i, \Omega_m, n, \mathcal{M}))^2}{\sigma_i^2}$$

where $\sigma_i^2$ is the error in $m_{\exp}(z_i)$. While computing $\Delta \chi^2 = \chi^2 - \chi_{\min}^2$ contours we choose the minimum of $\mathcal{M}$ at each point of $(\Omega_m, n)$. With a Gaussian distribution, we put the 1--σ (corresponding to 68.3%) bound on $\Omega_m$ for each set of $n$ across $\chi_{\min}^2$.

In Fig.2, we have plotted the same contours for the CMB peaks with $n_s = 0.97$ and Supernova bounds. For both $n_s = 1$ and $n_s = 0.97$, there is no 1--σ confidence region of $\Omega_m$ for SCP data within the assumed range (0.1 to 0.4) for $n \geq 0.4$.

In Table 1, we have given the the allowed region for $\Omega_m$ and $h$ coming from WMAP and BOOMERanG constraints on first three acoustic peak locations with and without the Supernova constraint.

In Fig.3, we have shown the same contours as in Fig.1 and Fig.2, but in $(\Omega_m, n)$ plane fixing $h = 0.71$. 

III. CONSTRAINTS ON $\sigma_8$

In this section we present the WMAP constraints on $\sigma_8$ which is related to the amplitude of the galaxy fluctuations for different values of the model parameter $n$. $\sigma_8$, the rms density fluctuations averaged over $8h^{-1}$ Mpc spheres, is determined by the COBE normalisation of the CMB power spectrum. This is done by essentially fixing the fluctuations at the last scattering for a given model.

Doran et.al [19] have given an estimate of the CMB-normalized $\sigma_8$-value for a very general class of dark energy models just from the knowledge of their “background solution” $[\Omega^d(a), \omega^d(a)]$ (here superscript “d” stands for dark
TABLE I: Limits on $\Omega_m$ and $h$ for different values of $n_s$ and $n$ from CMB constraints

|        | $n_s = 0.97$ | $n_s = 1.00$ |
|--------|-------------|-------------|
| $n$    | $\Omega_m$ | $h$         | $\Omega_m$ | $h$         |
| 0      | $\Omega_m \leq 0.35$ | $h \geq 0.65$ | $\Omega_m \leq 0.25$ | $h \geq 0.72$ |
| 0.1    | $0.13 \leq \Omega_m \leq 0.40$ | $0.60 \leq h \leq 0.78$ | $\Omega_m \leq 0.32$ | $h \geq 0.64$ |
| 0.2    | $\Omega_m \geq 0.15$ | $h \leq 0.72$ | $0.13 \leq \Omega_m \leq 0.35$ | $0.76 \geq h \geq 0.61$ |
| 0.4    | $\Omega_m \geq 0.18$ | $h \leq 0.66$ | $\Omega_m \geq 0.16$ | $h \leq 0.68$ |
| 0.5    | -           | -           | $\Omega_m \geq 0.19$ | $h \leq 0.62$ |

energy) and the $\sigma_8$-value of the ΛCDM model, with $\Omega_0^\Lambda = \Omega_0^d(\Lambda)$:

$$
\frac{\sigma_8(d)}{\sigma_8(\Lambda)} \approx (a_{eq})^3 \tilde{\tau}_0^d / 5 (1 - \Omega_0^\Lambda)^{-1+\omega^{-1}} 5 \sqrt{\tilde{\tau}_0(d)/\tilde{\tau}_0(\Lambda)}
$$

(18)

where $\tilde{\tau}_0$ is the conformal age of the universe, $a_{eq}$ is the scale factor at matter radiation equality given by,

$$
a_{eq} = \frac{\Omega_r 0}{\Omega_{m0}} = \frac{4.31 \times 10^{-3}}{h^2 (1 - \Omega_0^d)}.
$$

(19)

and $\omega$ is the effective equation of state which is an average value for $\omega^d$ during the time in which $\Omega^d$ is growing rapidly:

$$
\frac{1}{\omega} = \frac{\int_{\ln a_{eq}}^{\ln a_t} \Omega^d(a) / \omega^d(a) d \ln a}{\int_{\ln a_{eq}}^{\ln a_t} \Omega^d(a) d \ln a}.
$$

(20)

As shown in one of our previous papers [13], the extra term in equation 1 can be represented as dark energy component with slowly varying equation of state and in that case $\omega^d$ can be used as a fair approximation to $\tilde{\omega}$. In our case it is $n = 1 - \frac{1}{13}$.

$\Omega_{d}^{ef}$ is an average value for the fraction of dark energy during the matter dominated era, before $\Omega^d$ starts growing rapidly

$$
\Omega_{d}^{ef} = [\ln a_{eq} - \ln a_{tr}]^{-1} \int_{\ln a_{eq}}^{\ln a_{tr}} \Omega^d(a) d \ln a.
$$

(21)

For model without early quintessence, which is precisely our case, $\Omega_{d}^{ef}$ is zero.

In order to compute $\sigma_8$ for the ΛCDM model, we have used the standard definition

$$
\sigma_8^2 = \int_0^\infty \frac{dk}{k} \Delta^2(k) \left( \frac{3j_1(kr)}{kr} \right)^2,
$$

(22)

with $r = 8h^{-1}$ Mpc and

$$
\Delta^2(k) = \epsilon_H^2 \left( \frac{k}{H_0} \right)^3 T^2(k),
$$

(23)
TABLE II: Limits on $\sigma_8$ for different values of $n_s$ and $n$ and $\Omega_m = 0.25$

| $n_s$ = 1 | $n_s$ = 0.97 |
|-----------|---------------|
| $n$ | $\sigma_8$ | $\sigma_8$ |
| 0 | $0.83 \leq \sigma_8 \leq 0.85$ | $0.77 \leq \sigma_8 \leq 0.79$ |
| 0.1 | $0.78 \leq \sigma_8 \leq 0.80$ | $0.72 \leq \sigma_8 \leq 0.74$ |
| 0.2 | $0.73 \leq \sigma_8 \leq 0.75$ | $0.67 \leq \sigma_8 \leq 0.69$ |
| 0.3 | $0.67 \leq \sigma_8 \leq 0.68$ | $0.61 \leq \sigma_8 \leq 0.63$ |

$T(\kappa)$ is the matter transfer function describing the processing of the initial fluctuations, for which we use the result

$$T(q) = \frac{\ln(1 + 2.34q)}{2.34q} [1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{-1/4},$$

with $q = \frac{n}{H_0} \text{Mpc}$ and $\Gamma$ is.

$$\Gamma = \Omega_m h \exp \left[ -\Omega_b \left( 1 + \frac{\sqrt{2} \Gamma}{\Omega_m} \right) \right].$$

$\delta_H$ is the density perturbation at horizon crossing and a fit to the four-year COBE data gives

$$10^5 \delta_H(n_s, \Omega_m) = 1.94 \Omega_m^{-0.785 - 0.05 \ln \Omega_m} \exp[-0.95(n_s - 1) - 0.169(n_s - 1)^2]$$

assuming that there is no gravitational waves and also no reionization. Taking in to account the reionization effects, the above expression has been corrected by Griffiths and Liddle:

$$\frac{\delta_H(\tau)}{\delta_H(\tau = 0)} = 1 + 0.76\tau - 1.96\tau^2 + 1.46\tau^3,$$

where $\tau$ is the optical depth. The above correction is reliable upto $\tau = 0.5$. As mentioned earlier we have used $\tau = 0.11$ in our calculation.

In Table II, we show the constraint on $\sigma_8$ for different values of $n$ and $n_s$, normalizing the calculation at $\Omega_m = 0.25$.

IV. DISCUSSION

The first result is that the WMAP constraint on the first two acoustic peak locations together with the BOOMERanG bound on the third peak location restricts the model parameter $n$ as $n \leq 0.5$ for $n_s = 1$ and $n \leq 0.4$ for $n_s = 0.97$. One should note that to have current acceleration in this model, $n \leq 0.66$ is required, which means that CMB data alone predicts a late time accelerating universe. When one puts the Supernova constraint on top of this, the bound on $n$ becomes $n \leq 0.3$ for both values of $n_s$. Another interesting feature of our analysis is that in order to distinguish between a $\Lambda$CDM model and a Cardassian model, one has to impose a low value of $h$ (e.g. $h \leq 0.71$) and higher value of $n_s$ ($n_s \approx 1$). This has also been shown in Fig.3 where we have plotted the contours of the peak locations in the $(\Omega_m, n)$ plane fixing $h = 0.71$. It is clear that for $n_s = 1$, the allowed region does not include $n = 0$ which is precisely the $\Lambda$CDM case. But for $n_s = 0.97$ it includes $n = 0$. In other words, for low values of $h$ together with $n_s$ closer to 1, it appears that the $\Lambda$CDM model is more disfavoured than the Cardassian model. On the other hand for smaller values of $n_s$, $\Lambda$CDM and Cardassian model are equally favoured. An interesting point to note here is that WMAP Constraint on $n_s$ is $n_s = 0.99 \pm 0.04$. Hence although this remains still an unresolved issue, the fact that $n_s$ plays an important role in determining the nature of dark energy is one very interesting outcome of this investigation. We have shown it in the context of Cardassian model (the fact, that $n_s$ plays an important role in determining the nature of dark energy, has also been shown recently by Barreiro et.al in the context of early quintessence model), but it would be very interesting to study this feature in a model independent way.
We have also calculated the WMAP bound on $\sigma_8$ in our model for different values of $n$ and $n_s$ normalizing the calculation at $\Omega_m = 0.25$. Table II shows that $\sigma_8$ gets lower for higher value of $n$ and lower value of $n_s$. Considering the fact that the joint WMAP/Large scale structure data set indicate a suppressed clustering power at small scale $\Omega$, resulting a lower $\sigma_8$ value, one can conclude that lower value of $n_s$ and higher value of $n$ is favoured so far structure formation is concerned.

The greatest challenge in cosmology today is to unravel the nature of dark energy. Although the WMAP observations has put stringent constraint on many cosmological parameters, but there remains still some uncertainty in some important parameters like $n_s$, $\sigma_8$, $\Omega_m$ which plays very crucial roles in determining the nature of the dark energy. Hence the future observations with stronger constraints on these parameters hold promise to uncover the nature of dark energy.

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**V. APPENDIX**

For completeness, we repeat here the formulas used in our search for parameter space. These fitting formulae are quoted from the cited literature.[17]

We assume the standard recombination history and define the redshift of decoupling $z_{ls}$ as the redshift at which the optical depth of Thompson scattering is unity. A useful fitting formula for $z_{ls}$ is given by [23]:

$$z_{ls} = 1048 \left[ 1 + 0.00124w_b^{-0.738} \right] \left[ 1 + g_1 w_m^2 \right],$$  \hspace{1cm} (28)

where

$$g_1 = 0.0783w_b^{-0.238} \left[ 1 + 39.5w_b^{0.763} \right]^{-1}, \quad g_2 = 0.56 \left[ 1 + 21.1w_b^{1.81} \right]^{-1},$$

$w_b \equiv \Omega_b h^2$ and $w_m \equiv \Omega_m h^2$.

And the ratio of radiation to matter at last scattering $r_{ls} = \rho_r(z_{ls})/\rho_m(z_{ls}) = 0.0416w_m^{-1} \left( z_{ls}/10^3 \right).$  \hspace{1cm} (29)

The overall phase shift $\varphi$ (which is basically the phase shift of the first peak) is parametrised by the following formula

$$\varphi = (1.466 - 0.466n_s) \left[ a_1 r_{ls}^{a_2} + 0.291\bar{\Omega}^\varphi_{ls} \right],$$ \hspace{1cm} (30)

where $a_1$ and $a_2$ are given by

$$a_1 = 0.286 + 0.626 w_b,$$  \hspace{1cm} (31)

$$a_2 = 0.1786 - 6.308 w_b + 174.9 w_b^2 - 1168 w_b^3,$$ \hspace{1cm} (32)

and $\bar{\Omega}^\varphi_{ls}$ is the average fraction of dark energy before last scattering which is zero for our case.

The relative shift of the second peak ($\delta \varphi_2$) is given by

$$\delta \varphi_2 = c_0 - c_1 r_{ls} - c_2 r_{ls}^{-c_3} + 0.05 (n_s - 1),$$ \hspace{1cm} (33)

with

$$c_0 = -0.1 + 0.213 - 0.123\bar{\Omega}^d_{ls} \times \exp \left\{ -52 - 63.6\bar{\Omega}^d_{ls} w_b \right\}$$ \hspace{1cm} (34)

$$c_1 = 0.063 \exp \left\{ -3500 w_b^2 \right\} + 0.015$$ \hspace{1cm} (35)

$$c_2 = 6 \times 10^{-6} + 0.137 (w_b - 0.07)^2$$ \hspace{1cm} (36)

$$c_3 = 0.8 + 2.3\bar{\Omega}^d_{ls} + (70 - 126\bar{\Omega}^d_{ls}) w_b.$$ \hspace{1cm} (37)
For the third peak,
\[ \delta \varphi_3 = 10 - d_1 \delta \varphi_2^2 + 0.08 (n_s - 1), \]
with
\[ d_1 = 9.97 + (3.3 - 3\bar{\Omega}^d_{ls}) \omega_b \]
\[ d_2 = 0.0016 - 0.0067\bar{\Omega}^d_{ls} + (0.196 - 0.22\bar{\Omega}^d_{ls}) \omega_b + (2.25 + 2.779\bar{\Omega}^d_{ls}) \times 10^{-5}, \]
The overall shifts for the second and the third peak is \( \delta \phi_2 \) and \( \delta \phi_3 \) respectively.

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