Inclination evolution of protoplanetary discs around eccentric binaries

J. J. Zanazzi* and Dong Lai*
Cornell Center for Astrophysics, Planetary Science, Department of Astronomy, Cornell University, Ithaca, NY 14853, USA

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ABSTRACT
It is usually thought that viscous torque works to align a circumbinary disc with the binary’s orbital plane. However, recent numerical simulations suggest that the disc may evolve to a configuration perpendicular to the binary orbit (‘polar alignment’) if the binary is eccentric and the initial disc–binary inclination is sufficiently large. We carry out a theoretical study on the long-term evolution of inclined discs around eccentric binaries, calculating the disc warp profile and dissipative torque acting on the disc. For discs with aspect ratio $H/r$ larger than the viscosity parameter $\alpha$, bending wave propagation effectively makes the disc precess as a quasi-rigid body, while viscosity acts on the disc warp and twist to drive secular evolution of the disc–binary inclination. We derive a simple analytic criterion (in terms of the binary eccentricity and initial disc orientation) for the disc to evolve towards polar alignment with the eccentric binary. When the disc has a non-negligible angular momentum compared to the binary, the final ‘polar alignment’ inclination angle is reduced from 90°. For typical protoplanetary disc parameters, the time-scale of the inclination evolution is shorter than the disc lifetime, suggesting that highly inclined discs and planets may exist orbiting eccentric binaries.

Key words: accretion, accretion discs – hydrodynamics – protoplanetary discs – binaries: general.

1 INTRODUCTION

To date, 11 transiting circumbinary planets have been detected around nine binary star systems (Doyle et al. 2011; Kostov et al. 2013, 2014, 2016; Orosz et al. 2012a,b; Schwamb et al. 2013; Welsh et al. 2012, 2015). All planets detected have orbital planes very well aligned with their binary orbital planes, with mutual binary–planet inclinations not exceeding 3°. The circumbinary planet detectability is a very sensitive function of the binary–planet inclination (Martin & Triaud 2015; Li, Holman & Tao 2016). If the mutual inclination is always small (≤5°), then the occurrence rate of circumbinary planets is comparable to that of planets around single stars, but if modest inclinations (≥5°) are common, the circumbinary planet occurrence rate may be much larger (Armstrong et al. 2014). For these reasons, it is important to understand if and how a binary aligns with its circumbinary disc from which these planets form.

Observations show that most circumbinary discs tend to be aligned with their host binary orbital planes. The gas rich circumbinary discs, HD 98800 B (Andrews et al. 2010), AK Sco (Czekala et al. 2015) and DQ Tau (Czekala et al. 2016), and the debris circumbinary discs, $\alpha$ CrB and $\beta$ Tri (Kennedy et al. 2016b), all have mutual disc–binary inclinations not exceeding 3°. However, there are some notable exceptions. The circumbinary disc around KH 15D is mildly misaligned with the binary orbital plane by ∼10°–20° (Chiang & Murray-Clay 2004; Winn et al. 2004; Capelo et al. 2012). Shadows (Marino et al. 2015) and gas kinematics (Casassus et al. 2015) of the discs in HD 142527 are consistent with a misalignment of ∼70° between the outer circumbinary disc and binary orbital plane (Lacour et al. 2016). The discs (circumbinary and two circumstellar) in the binary protostar IRS 43 are misaligned with each other and with the binary (Brinch et al. 2016). Most intriguingly, the debris disc around the eccentric ($e_0 = 0.77$) binary 99 Herculis may be highly inclined: By modelling the resolved images from Hershel, Kennedy et al. (2012a) strongly favour a disc orientation where the disc angular momentum vector is inclined to the binary orbital angular momentum vector by 90° (polar alignment). Kennedy et al. (2012a) also produced a model with a disc–binary inclination of 30°, which fits the observations, but this configuration is unlikely, since differential precession of dust due to the gravitational influence of the binary would rapidly destroy the disc.

Since star/binary formation takes place in turbulent molecular clouds (McKee & Ostriker 2007), the gas that falls on to the central protostellar core/binary and assembles on to the disc at different times may rotate in different directions (e.g. Bate, Bonnell & Bromm 2003, see also Bate, Lodato & Pringle 2010; Fielding et al. 2015). In this scenario, it is reasonable to expect a newly formed binary to be surrounded by a highly misaligned circumbinary disc, which forms as a result of continued gas accretion...
(Foucart & Lai 2013). The observed orientations of circumbinary discs then depend on the long-term inclination evolution driven by binary–disc interactions.

Foucart & Lai (2013, 2014) studied the warping and the dissipative torque driving the inclination evolution of a circumbinary disc, assuming a circular binary. Foucart & Lai (2013) considered an infinite disc and included the effect of accretion on to the binary, while Foucart & Lai (2014) considered a more realistic disc of finite size and angular momentum, which can precess coherently around the binary. It was shown that under typical protoplanetary conditions, both viscous torque associated with disc warping and accretion torque tend to damp the mutual disc–binary inclination on time-scale much shorter than the disc lifetime (a few Myr). By contrast, a circumstellar disc inside a binary can maintain large misalignment with respect to the binary orbital plane over its entire lifetime (Lubow & Ogilvie 2000; Foucart & Lai 2014). This is consistent with the observations that most circumbinary discs are nearly coplanar with their host binaries. On the other hand, the observed circumbinary disc misalignment (such as in KH 15D and IRS 43) can provide useful constraints on the uncertain aspects of the disc warp theory, such as non-linear effects (Ogilvie 2006) and parametric instabilities due to disc warping (Gammie, Goodman & Ogilvie 2000; Ogilvie & Latter 2013).

However, several recent numerical studies using smoothed particle hydrodynamics suggest that other outcomes may be possible for discs around eccentric binaries. Aly et al. (2015) showed that discs around binary black holes (which typically lie in the ‘viscous regime’ of disc warps, with the viscosity parameter $\alpha$ larger than the disc aspect ratio $H/r$; Papaloizou & Pringle 1983; Ogilvie 1999; see Section 3) and around eccentric binaries may be driven into polar alignment. Martin & Lubow (2017) found numerically that a circumbinary protoplanetary disc (typically in the bending wave regime, with $\alpha \lesssim H/r$; Papaloizou & Lin 1995; Lubow & Ogilvie 2000), inclined to an eccentric ($e_b = 0.5$) binary by 60$^\circ$, will evolve to a polar configuration. They suggested that this dynamical outcome arises from the combined influence of the gravitational torque on the disc from the binary and viscous torques from disc warping. They also proposed that 99 Herculis (with $e_b = 0.77$) followed such an evolution to end in the orbital configuration (polar alignment) observed today.

In this paper, we provide a theoretical analysis to the above numerical results. In particular, we generalize the study of Foucart & Lai (2014) to apply to circumbinary discs with arbitrary disc–binary inclinations and binary eccentricities. We derive the critical condition and calculate the time-scale for the disc to evolve towards polar alignment with the binary. In Section 2, we review the secular dynamics of a test particle around an eccentric binary. In Section 3, we calculate the disc warp profile and dissipative disc torques acting on the disc, and derive the requirements for the disc to evolve into polar alignment with the binary. Section 4 considers the situation when the circumbinary disc has a non-negligible angular momentum compared to the inner binary. In Section 5, we examine the back-reaction torque from the disc on the binary and the effect of gas accretion. We discuss our results in Section 6, and summarize in Section 7.

2 TEST PARTICLE DYNAMICS

In preparation for later sections, we review the secular dynamics of a test particle surrounding an eccentric binary (Farago & Laskar 2010; Li, Zhou & Zhang 2014; Naoz et al. 2017). Consider a circular test mass with semi-major axis $r$ and orbital angular momentum unit vector $\hat{l}$, surrounding an eccentric binary with orbital angular momentum vector $\hat{l}_b$, eccentricity vector $\hat{e}_b$, semi-major axis $a_b$, total mass $M_b = M_1 + M_2$ (where $M_1, M_2$ are individual masses) and reduced mass $\mu_b = M_1 M_2 / M_b$. The orbit-averaged torque per unit mass on the test particle is (e.g. Liu, Muñoz & Lai 2015; Petrovich 2015)

$$ T_b = -r^2 \omega_b \left[ (1 - e_b^2) \hat{l} \cdot \hat{l}_b \hat{l}_b \times \hat{l} - 5(\hat{l} \cdot \hat{e}_b) \hat{e}_b \times \hat{l} \right], \tag{1} $$

where $n \simeq \sqrt{G M_b / r^3}$ is the test particle orbital frequency (mean-motion) and

$$ \omega_b = \frac{3 G \mu_b a_b^2}{4 r^2 n^2} \tag{2} $$

characterizes the precession frequency of the test particle around the binary. The torque $T_b$ in equation (1) is evaluated to the lowest order in $a_b / r$.

The time evolution of the test particle’s orbital angular momentum vector is given by

$$ \frac{d \hat{l}}{dt} = -\omega_b \left[ (1 - e_b^2) \hat{l} \cdot \hat{l}_b \hat{l}_b \times \hat{l} - 5(\hat{l} \cdot \hat{e}_b) \hat{e}_b \times \hat{l} \right]. \tag{3} $$

Equation (3) can be solved analytically (Landau & Lifshitz 1969; Farago & Laskar 2010; Li, Zhou & Zhang 2014), but the dynamics may be easily understood by analysing the energy curves. Equation (3) has an integral of motion

$$ \lambda = \left( 1 - e_b^2 \right) (\hat{l} \cdot \hat{l}_b)^2 - 5(\hat{l} \cdot \hat{e}_b)^2, \tag{4} $$

which is simply related to the quadrupole interaction energy (double-averaged over the two orbits) by (e.g. Tremaine, Touma & Namouni 2009; Tremaine & Yavetz 2014; Liu, Muñoz & Lai 2015)

$$ \Phi_{\text{quad}} = \frac{G \mu_b a_b^2}{8 r^3} \left( 1 - 6e_b^2 - 3 \Lambda \right). \tag{5} $$

To plot the energy curves, we set up the Cartesian coordinate system $(x, y, z)$, where $\hat{l}_b = \hat{x}$ and $\hat{e}_b = e_b \hat{z}$. We may write

$$ \hat{l} = (\sin I \sin \Omega, -\sin I \cos \Omega, \cos I) \tag{6} $$

where $I$ is the angle between $\hat{l}$ and $\hat{l}_b$, $\Omega$ is the test particle’s longitude of the ascending node (measured in the $x$-plane from the $x$-axis); similarly $I_e$ is the angle between $\hat{l}$ and $\hat{e}_b$, and $\Omega_e$ measures the longitude of the node in the $yz$-plane (see Fig. 1). In terms of $I$ and $\Omega$...
In terms of the variables $\Lambda = \sqrt{\Lambda_1^2 + \Lambda_2^2}$ and $\Omega = \Omega_1 - \Omega_2$, surrounded by a circular circumbinary disc with $I \sim \text{constant}$ and $\Omega$ circulating its full range of values ($0^\circ$–$360^\circ$), while $\Omega_1$ librates around $0^\circ$. When $\Lambda < 0$, $\hat{l}$ precesses around $\hat{e}_b$ with $I \sim \text{constant}$ and $\Omega_2$ circulating its full range of values ($-180^\circ$, $180^\circ$), while $\Omega_1$ librates around $90^\circ$. When $\Lambda > 0$, $\hat{l}$ precesses around $\hat{e}_b$ with $I \sim \text{constant}$ and $\Omega_1$ circulating its full range of values ($0^\circ$–$360^\circ$), while $\Omega_2$ librates around $90^\circ$. We assume $\Delta = 0.751$ (blue), $\Lambda = 0.348$ (green), $\Lambda = -0.110$ (magenta) and $\Lambda = -0.409$ (red). Only $\Omega_1$ and $\Omega_2$ in the range $[0^\circ$, $180^\circ]$ and $[-90^\circ$, $90^\circ]$ are shown. The energy curves for $\Omega$ in $[180^\circ$, $360^\circ]$ duplicate those of $[0^\circ$, $180^\circ]$, while the energy curves for $\Omega_2$ in $[90^\circ$, $270^\circ]$ duplicate those of $[-90^\circ$, $90^\circ]$.

In Fig. 2, we plot the test particle trajectories in the $I$–$\Omega$ (left-hand panel) and $I_e$–$\Omega_e$ (right-hand panel) planes for the binary eccentricity $e_b = 0.3$. The critical separatrix $\Lambda = 0$ is displayed in black in both plots. When $\Lambda > 0$, $\hat{l}$ precesses around $\hat{I}_b$ with $I \sim \text{constant}$ and $\Omega$ circulating the full range ($0^\circ$–$360^\circ$), while $\Omega_1$ librates around $0^\circ$. When $\Lambda < 0$, $\hat{l}$ precesses around $\hat{e}_b$ with $I \sim \text{constant}$ and $\Omega_2$ circulating the full range ($0^\circ$–$360^\circ$), while $\Omega_1$ librates around $90^\circ$ (see Fig. 1).

Thus, the test particle angular momentum axis $\hat{l}$ transitions from precession around $\hat{I}_b$ for $\Lambda > 0$ to precession around $\hat{e}_b$ for $\Lambda < 0$. Because the $\Lambda = 0$ separatrix has $\Omega \in [0^\circ$, $360^\circ]$ (Fig. 2), a necessary condition for $\hat{l}$ to precess around $\hat{e}_b$ is $I_{\text{crit}} < I < I_{\text{crit}}$, where

$$I_{\text{crit}} = \cos^{-1}\sqrt{\frac{5e_b^2}{1 + 4e_b^2}} = \tan^{-1}\sqrt{\frac{1 - e_b^2}{5e_b^2}}. \tag{8}$$

Fig. 2 clearly reveals the stable fixed points of the system. In terms of the variables ($\Omega$, $I$), the stable fixed points (where $dI/dt = d\Omega/dt = 0$) are $I = \pi/2$ and $\Omega = \pi/2$, $3\pi/2$, corresponding to $\hat{l} = \pm \hat{e}_b/e_b$. In terms of the variables ($\Omega_1$, $I_b$), the fixed points are $I_b = \pi/2$ and $\Omega_1 = 0$, $\pi$, corresponding to $\hat{l} = \pm \hat{I}_b$. We will see in Section 3 that in the presence of dissipation, the disc may be driven towards one of these fixed points.

### 3 Circumbinary Disc Dynamics

We now consider a binary (with the same parameters as in Section 2, see also Fig. 3) surrounded by a circular circumbinary disc with inner truncation radius $r_{\text{in}}$, outer truncation radius $r_{\text{out}}$, with unit angular momentum vector $\hat{l} = \hat{l}(r, t)$ and surface density $\Sigma = \Sigma(r)$. For concreteness, we adopt the surface density profile

$$\Sigma(r) = \Sigma_{\text{in}} \left(\frac{r_{\text{in}}}{r}\right)^{\alpha}. \tag{9}$$

We assume $r_{\text{in}} \ll r_{\text{out}}$ throughout this work. We could assume a more general surface density profile $\Sigma \propto r^{-\alpha}$, with $\alpha$ observationally

![Figure 3. Circumbinary disc setup. The binary has individual masses $M_1$ and $M_2$, with total mass $M_b = M_1 + M_2$ and reduced mass $\mu = M_1 M_2/M_b$, with orbital angular momentum vector $\hat{l}_b$ and eccentricity vector $\hat{e}_b$. The binary is surrounded by a circular circumbinary disc with unit orbital angular momentum $\hat{l} = \hat{l}(r, t)$, surface density $\Sigma = \Sigma(r)$ (equation 9) and inner (outer) truncation radii $r_{\text{in}}$ ($r_{\text{out}}$).](https://academic.oup.com/mnras/article-abstract/473/1/603/4157814/605)
constrained to lie in the range 0.5–1.5 (e.g. Weidenschilling 1977; Williams & Cieza 2011; Chiang & Laughlin 2013). A more general p will affect the disc mass (equation 11) and angular momentum (equation 12), as well as the precession (equation 28) and viscous (equation 47) rates, by factors of the order of unity.

The binary has orbital angular momentum

\[ L_b = \mu_b \sqrt{1 - e_b^2} GM_b a_{\text{orb}}, \]  

(10)

while the disc has mass

\[ M_d = 2\pi \int_{r_{\text{in}}}^{r_{\text{out}}} \Sigma r \, dr \simeq 2\pi \Sigma_{\text{in}} r_{\text{in}} r_{\text{out}} \]  

(11)

and angular momentum (assuming a small disc warp; see below)

\[ L_d = 2\pi \int_{r_{\text{in}}}^{r_{\text{out}}} \Sigma r^3 n \, dr \simeq \frac{2}{3} M_d \sqrt{G M_b r_{\text{out}}}, \]  

(12)

where \( n(r) \simeq \frac{1}{\sqrt{GM_b/r^3}} \). Comparing \( L_b \) to \( L_d \), we have

\[ \frac{L_d}{L_b} \simeq 0.067 \left( 1 - e_b^2 \right)^{-1/2} \left( \frac{M_d}{0.01 \mu_b} \right) \left( \frac{r_{\text{out}}}{100 a_{\text{in}}} \right)^{1/2}. \]  

(13)

Because \( L_d \gg L_b \) for typical circumbinary disc parameters, in this section we assume \( \hat{l}_d \) and \( e_b \) are fixed in time, neglecting the back-reaction torque on the binary from the disc. We discuss the system’s dynamics when \( L_d \) is non-negligible compared to \( L_b \) in Section 4 and the effects of the back-reaction torque on the binary from the disc in Section 5.

### 3.1 Qualitative discussion

Assuming the disc to be nearly flat, the time evolution of the disc unit angular momentum vector is given by

\[ \frac{d\hat{l}_d}{dr} = \left( \frac{T_b}{r^2 \kappa} \right), \]  

(14)

where \( T_b \) is given in equation (1), \( \hat{l}_d(t) \) is a suitably averaged unit angular momentum of the disc (see equation 24) and \( (\ldots) \) implies a proper average over \( r \) (see equation 27). When the disc is flat, the time evolution of \( \hat{l}_d \) is identical to that of a test particle (see discussion at the end of Section 3.2).

When \( \alpha \lesssim H/r \) (\( H \) is the disc scaleheight, \( \alpha \) is the viscosity parameter), the main internal torque enforcing disc rigidity and coherent precession comes from bending wave propagation (Papaloizou & Lin 1995; Lubow & Ogilvie 2000). As bending waves travel at 1/2 the sound speed, the wave crossing time is of the order of \( t_{\text{bw}} = 2r_e/c_s \). When \( t_{\text{bw}} \) is longer than the characteristic precession time \( a_{\text{in}}^{-1} \) (see equation 2), strong disc warps can be induced. In the extreme non-linear regime, disc breaking may be possible in circumbinary discs (Larwood & Papaloizou 1997; Facchini, Lodato & Price 2013; Nixon, King & Price 2013). To compare \( t_{\text{bw}} \) with \( a_{\text{in}}^{-1} \), we adopt the disc sound speed profile

\[ c_s(r) = H(r)n(r) = H \left( \frac{GM_b}{r_{\text{in}}} \right)^{1/2} / r, \]  

(15)

where \( H = H/r \). We find

\[ t_{\text{bw}} a_{\text{in}} = 0.94 \left( \frac{0.1}{H} \right) \left( \frac{4 \mu_b}{M_b} \right) \left( \frac{2 a_{\text{in}}}{M_b} \right)^{2} \left( \frac{r_{\text{in}}}{r} \right)^{2}. \]  

(16)

Thus, we expect that the small warp approximation should be valid everywhere in the disc except the inner-most region. Throughout this paper, we scale our results to \( h = 0.1 \). Real protoplanetary discs can have aspect ratios in the range \( h \sim 0.03–0.2 \) (e.g. Lynden-Bell & Pringle 1974; Chiang & Goldreich 1997; Williams & Cieza 2011). We normalize \( r_{\text{in}} \) to \( 2 a_{\text{in}} \), but note that the inner truncation radius of the disc depends non-trivially on the binary’s eccentricity (Miranda, Muñoz & Lai 2017).

Although the disc is flat to a good approximation, the interplay between disc twist/warp and viscous dissipation may modify the disc’s dynamics over time-scales much longer than \( a_{\text{in}}^{-1} \). When the external torque \( T_b \) is applied to the disc in the bending wave regime, the disc’s viscosity causes the disc to develop a small twist of the order of

\[ \frac{d\hat{l}_d}{\partial \ln r} \simeq \frac{4\alpha}{c_s^2} T_b, \]  

(17)

while the precession of bending waves from a non-Keplarian epicyclic frequency \( \kappa \) causes the disc to develop a small warp, of the order of

\[ \frac{d\hat{l}_d}{\partial \ln r} \simeq \frac{4}{c_s^2} \left( \frac{\kappa^2 - n^2}{2n^2} \right) \hat{l}_d \times T_b. \]  

(18)

The viscous twist (equation 17) interacts with the external torque, affecting the evolution of \( \hat{l}_d \) over the viscous time-scale. To an order of magnitude, we have

\[ \frac{d\hat{l}_d}{dr} \sim \left( \frac{T_b}{r^2 \kappa} \right) \frac{d\hat{l}_d}{\partial \ln r} \sim \left( \frac{4\alpha}{c_s^2} r^2 n \right) \alpha_0 \hat{l}. \]  

(19)

In the above estimate, we have assumed that the relevant misalignment angles (between \( \hat{l}_d \) and \( \hat{l}_b \), or between \( \hat{l}_d \) and \( e_b \)) are of the order of unity.

### 3.2 Formalism

The torque per unit mass on the disc from the inner binary is given by equation (1), with \( T_b = T_b(\rho_0, t) \). In addition, the gravitational potential from the binary induces a non-Keplarian angular frequency (Miranda & Lai 2015), with

\[ \kappa^2 - n^2 = -2\alpha_0 n f_b, \]  

(20)

where

\[ f_b = \frac{1}{2} \left\{ \left[ 3(\hat{l} \cdot \hat{l}_b)^2 - 1 \right] \left( \frac{1 + \frac{3}{2} e_b^2}{2} \right) - 15 e_b^2 (\hat{l} \times \hat{l}_b)^2 \right\}. \]  

(21)

When the Shakura–Sunae \( \alpha \)-viscosity parameter satisfies \( \alpha \lesssim H/r \), the disc lies in the bending wave regime (Papaloizou & Lin 1995; Lubow & Ogilvie 2000). Any warp induced by an external torque is smoothed by bending waves passing through the disc. Protoplanetary discs typically lie in the bending wave regime. The time evolution of \( \hat{l}(\rho_0, t) \) is governed by the equations (Lubow & Ogilvie 2000; see also Lubow, Ogilvie & Pringle 2002)

\[ \Sigma r^2 n \frac{d\hat{l}}{dt} = \frac{1}{r} \frac{\partial G}{\partial \rho_0} + \Sigma T_b, \]  

(22)

\[ \frac{\partial G}{\partial t} - \alpha_0 f_b \hat{l} \times G + \alpha z \Sigma G \frac{\Sigma r^2 n}{\partial \rho_0} \frac{d\hat{l}}{dt}, \]  

(23)

where \( G \) is the internal torque.

From equation (16), we see that \( t_{\text{bw}} < a_{\text{in}}^{-1} \) for standard circumbinary disc parameters, so the disc should be only mildly warped. We may therefore expand

\[ \hat{l}(\rho_0, t) = \hat{l}_d(t) + \hat{l}_1(\rho_0, t) + \cdots, \]  

(24)
\[ G(r, t) = G_0(r, t) + G_1(r, t) + \cdots \]  

(25)

where \( \hat{I}_d \) is the unit vector along the total angular momentum of the disc, \( |\hat{I}_d| \ll |\hat{I}_d| = 1 \) (see equations 31–32 below). As we will see, the internal torque \( G_1(r, t) \) maintains the rigid body dynamical evolution of \( \hat{I}_d \), while \( G_0(r, t) \) maintains the warp profile \( \hat{I}_l \). Perturbative expansions to study warped disc structure and time evolution have been taken by Lubow & Ogilvie (2000, 2001) and Foucart & Lai (2014). Inserting (24) into equation (22), integrating over \( rdr \) and using the zero torque boundary condition

\[ G_0(r_{in}, t) = G_0(r_{out}, t) = 0, \]  

(26)

we find that the leading order time evolution of \( \hat{I} \) is given by

\[ \frac{d\hat{I}}{dt} = -\bar{\omega}_b \left[ \left( 1 - e_b^2 \right) (\hat{I}_d \cdot \hat{I}_b) \hat{I}_b \times \hat{I}_d - 5(\hat{I}_d \cdot \hat{e}_b)\hat{e}_b \times \hat{I}_d \right] \tag{27} \]

Here,

\[ \bar{\omega}_b = \frac{2\pi}{L_d} \int_{r_{in}}^{r_{out}} \Sigma r^3 \Omega_{ob} dr \lesssim \frac{9 G M_2 \alpha_b^2}{16 r_{out}^2 \sqrt{G M b_{out}}} \]

\[ = 4.97 \times 10^{-5} \left( \frac{2 \alpha_b}{r_{in}} \right)^2 \left( \frac{4 \mu_b}{M_b} \right) \]

\[ \times \left( \frac{M_b}{2 \mu_b} \right)^{1/2} \left( \frac{r_{out}}{100 \mu_b} \right)^{-3/2} \left( \frac{2\pi}{yr} \right) \]  

(28)

is the characteristic precession frequency of the rigid disc. Equation (27) is equivalent to equation (3), if one replaces \( \bar{\omega}_b \) with \( \omega_b \), and the disc dynamics reduce to those of a test particle with \( \hat{I} = \hat{I}_d \) when \( c_s \to \infty \).

### 3.3 Disc warp profile

With \( \hat{I}_d \) determined with boundary condition (26), we may solve for \( G_0(r, t) \):

\[ G_0(r, t) = g_b \left[ \left( 1 - e_b^2 \right) (\hat{I}_d \cdot \hat{I}_b) \hat{I}_b \times \hat{I}_d - 5(\hat{I}_d \cdot \hat{e}_b)\hat{e}_b \times \hat{I}_d \right] \tag{29} \]

where

\[ g_b(r) = \int_{r_{in}}^{r} \Sigma r^3 n \left( \omega_b - \bar{\omega}_b \right) dr'. \]  

(30)

With the leading order terms for \( I \) and \( G \), we may solve for \( \hat{I}_1 \). We impose the normalization condition,

\[ \int_{r_{in}}^{r_{out}} \Sigma r^3\Omega l(r, t) dr = 0, \]  

(31)

so that \( \hat{I}_d \) is the unit vector along the total angular momentum of the disc, or

\[ \hat{I}_d(t) = \frac{2\pi}{L_d} \int_{r_{in}}^{r_{out}} \Sigma r^3 \Omega l(r, t) dr. \]  

(32)

Inserting equation (29) into equation (23) and integrating, we obtain

\[ l_1(r, t) = (l_1)_{\text{warp}} + (l_1)_{\text{warp}} \]

\[ + S \bar{\omega}_b g_b(1 - e_b^2)(\hat{I}_d \cdot \hat{e}_b) \left[ (\hat{I}_d \cdot \hat{e}_b)\hat{e}_b \hat{I}_d - (\hat{I}_d \cdot \hat{e}_b)\hat{e}_b \hat{I}_d \right], \]  

(33)

where

\[ (l_1)_{\text{warp}} = V_b \left[ \left( 1 - e_b^2 \right) (\hat{I}_d \cdot \hat{I}_b) \hat{I}_b \times \hat{I}_d - 5(\hat{I}_d \cdot \hat{e}_b)\hat{e}_b \times \hat{I}_d \right] \]  

(34)

and

\[ (l_1)_{\text{warp}} = \bar{\omega}_b \tau_b \left( 1 - e_b^2 \right) (\hat{I}_d \cdot \hat{I}_b) \]

\[ \times \left[ (1 - e_b^2)(\hat{I}_d \cdot \hat{I}_b)\hat{I}_b \times (\hat{I}_b \times \hat{I}_d) - 5(\hat{I}_d \cdot \hat{e}_b)\hat{e}_b \times (\hat{e}_b \times \hat{I}_d) \right] \]

\[ + S \bar{\omega}_b g_b(1 - e_b^2)(\hat{I}_d \cdot \hat{e}_b) \left[ (\hat{I}_d \cdot \hat{e}_b)\hat{e}_b \hat{I}_d - (\hat{I}_d \cdot \hat{e}_b)\hat{e}_b \hat{I}_d \right] \]

\[ - W_{bb, f} \]

\[ \times \left[ (1 - e_b^2)(\hat{I}_d \cdot \hat{I}_b)\hat{I}_b \times (\hat{I}_b \times \hat{I}_d) - 5(\hat{I}_d \cdot \hat{e}_b)\hat{e}_b \times (\hat{e}_b \times \hat{I}_d) \right]. \]  

(35)

Here,

\[ \tau_b(r) = \frac{\int_{r_{in}}^{r} \frac{g_b}{\Sigma c_r^2 r^3 n} dr'}{-\bar{\omega}_b}, \]  

(36)

\[ V_b(r) = \frac{\int_{r_{in}}^{r} \frac{\alpha g_b}{\Sigma c_r^2 r^3 n} dr'}{-V_{bb}}, \]  

(37)

\[ W_{bb}(r) = \frac{\int_{r_{in}}^{r} \frac{\alpha g_b}{\Sigma c_r^2 r^3 n} dr'}{-W_{bb0}}, \]  

(38)

and

\[ \tau_{bb} = \frac{2\pi}{L_d} \int_{r_{in}}^{r_{out}} \Sigma r^3 n \left( \int_{r_{in}}^{r} \frac{g_b}{\Sigma c_r^2 r^3 n} dr' \right) dr, \]  

(39)

\[ V_{bb}(r_{in}) - V_{bb}(r_{out}) \approx -0.258 \]

\[ \times \left( \frac{\alpha}{0.01} \right) \left( \frac{0.1}{h} \right)^2 \left( \frac{4 \mu_b}{M_b} \right) \left( \frac{2 \alpha_b}{r_{in}} \right)^2 \left( \frac{50 r_{in}}{r_{out}} \right)^{3/2}, \]  

(43)

\[ W_{bb0}(r_{in}) - W_{bb0}(r_{out}) \approx -0.108 \]

\[ \times \left( \frac{0.1}{h} \right)^2 \left( \frac{4 \mu_b}{M_b} \right) \left( \frac{2 \alpha_b}{r_{in}} \right)^2. \]  

(44)

The third term in equation (33) arises from the fact that \( \hat{I}_d \cdot \hat{e}_b \) and \( \hat{I}_d \times \hat{I}_b \) are not constant in time, and is dynamically unimportant. Although it is straightforward to compute the integrals in equations (36)–(38), this calculation is tedious and unilluminating. Instead, we notice that over most of the region in the integrals, the internal torque radial function \( g_b(r) \) is of the order of

\[ g_b(r) \sim \Sigma r^3 n \omega_b. \]  

(42)

Evaluating the warp functions and using the fact that \( r_{in} \ll r_{out} \), we obtain the approximate expressions

\[ \bar{\omega}_b \left[ r_b(r_{in}) - r_{bb}(r_{out}) \right] \approx -0.108 \]

\[ \times \left( \frac{0.1}{h} \right)^2 \left( \frac{4 \mu_b}{M_b} \right) \left( \frac{2 \alpha_b}{r_{in}} \right)^2 \left( \frac{50 r_{in}}{r_{out}} \right)^{3/2}, \]  

(43)

\[ V_{bb}(r_{in}) - V_{bb}(r_{out}) \approx -0.258 \]

\[ \times \left( \frac{\alpha}{0.01} \right) \left( \frac{0.1}{h} \right)^2 \left( \frac{4 \mu_b}{M_b} \right) \left( \frac{2 \alpha_b}{r_{in}} \right)^2, \]  

(44)

\[ W_{bb0}(r_{in}) - W_{bb0}(r_{out}) \approx -0.108 \]

\[ \times \left( \frac{0.1}{h} \right)^2 \left( \frac{4 \mu_b}{M_b} \right) \left( \frac{2 \alpha_b}{r_{in}} \right)^2. \]  

(45)

In Fig. 4, we plot the rescaled warp functions \( \bar{W} = r_b/r_b(r_{in}) - r_{bb}(r_{out}) \), \( V_b = V_{bb}(r_{in}) - V_{bb}(r_{out}) \) and \( W_{bb} = W_{bb0}(r_{in}) - W_{bb0}(r_{out}) \).
the viscous dissipation from disc twisting affects the evolution of $\dot{I}_d$ according to

$$\frac{d\dot{I}_d}{dr}_{\text{visc}} = \gamma_\Lambda \left( (1 - \epsilon_i^0) (\dot{L}_d, \dot{I}_d) \dot{I}_d \times (\dot{L}_b \times \dot{I}_d) \right) - 5 \left( \dot{L}_b \cdot \dot{I}_d \right) \dot{L}_b \times (e_b \times \dot{I}_d),$$

(50)

where $\Lambda$ is given by equation (4), except we replace $\dot{L}$ by $\dot{I}_d$:

$$\Lambda = (1 - \epsilon_i^0) (\dot{I}_d, \dot{I}_d)^2 - 5(\dot{L}_d \cdot e_b)^2.$$  

(51)

Equation (50) is the main result of our technical calculation. We see

$$\frac{d}{dr} \dot{I}_d = \gamma_\Lambda (\dot{I}_d, \dot{I}_d) \left( (1 - \epsilon_i^0) - \Lambda \right);$$

(52)

$$\frac{d}{dr} (e_b \cdot \dot{I}_d) \bigg|_{\text{visc}} = -\gamma_\Lambda (\dot{I}_d, e_b) \left[ \Lambda + 5\epsilon_i^0 \right].$$

(53)

Because $-5\epsilon_i^0 < \Lambda < (1 - \epsilon_i^0)$ (equation 51), Equations (52)–(53) show that the system has two different end-states depending on the initial value for $\Lambda$:

(i) $\Lambda > 0$: The viscous torque (50) pushes $\dot{I}_d$ towards $\dot{L}_d$. The final state of $\dot{I}_d$ is alignment (if $\dot{L}_d \cdot \dot{I}_d > 0$ initially) or anti-alignment (if $\dot{L}_d \cdot \dot{I}_d < 0$ initially) with $\dot{L}_d$.

(ii) $\Lambda < 0$: The viscous torque (50) pushes $\dot{I}_d$ towards $e_b$. The final state of $\dot{I}_d$ is alignment (or anti-alignment) with $e_b$.

Fig. 5 shows several examples of the results for the evolution of disc orientation, obtained by integrating the time evolution of $\dot{I}_d$, including gravitational (equation 27) and viscous (equation 50) torques. On the left-hand panels, we plot the disc inclination $I$ with time, for the binary eccentricities indicated. We choose the initial $\Omega(0) = 90^\circ$ for all cases, so that $I < I_{\text{crit}}$ ($I > I_{\text{crit}}$) corresponds exactly to $\Lambda > 0$ ($\Lambda < 0$) (see equations 4 and 8). Thus, we expect $I \rightarrow 0^\circ$ when $I < I_{\text{crit}}$, $I \rightarrow 90^\circ$ when $I_{\text{crit}} < I < I_{\text{crit}}$ and $I \rightarrow 180^\circ$ when $I > 180^\circ - I_{\text{crit}}$. On the right-hand panels of Fig. 5, we plot the disc trajectories on the $I-\Omega$ plane (equation 6 with $\dot{L} \rightarrow \dot{I}$). Again, we see when $I < I_{\text{crit}}$ ($\Lambda > 0$), $\dot{I}_d$ aligns with $\dot{L}_d$, while when $I > I_{\text{crit}}$ ($\Lambda < 0$), $\dot{I}_d$ aligns with $e_b$, as expected.

4 SECULAR DYNAMICS WITH MASSIVE INCLINED OUTER BODY

Sections 2 and 3 neglected the circumbinary disc’s angular momentum, a valid assumption as long as $L_d \ll L_b$ (equation 13). When $L_d \gtrsim L_b$, the non-zero disc angular momentum will change the locations of the fixed points of the system, and hence may affect its dynamical evolution over viscous times-scales.

Consider the setup of Section 2, except we now include the outer body’s mass $m$ and angular momentum $L = m\sqrt{GM_b}\dot{L}$. The evolution equations for $\dot{L}$, $j_b = \sqrt{1 - \epsilon_i^0}L_b$ and $e_b$ are (Liu, Muñoz & Lai 2015; equations 17–19)

$$\frac{d\dot{L}}{dr} = -\alpha_b \left[ (j_b \cdot \dot{L})j_b \times \dot{L} - 5 \left( e_b \cdot \dot{L} \right) e_b \times \dot{L} \right],$$

(54)

$$\frac{d}{dr} \left( j_b \right) = J_{\text{orb}} \left[ (j_b \cdot \dot{L})j_b \times \dot{L} - 5(e_b \cdot \dot{L})e_b \times \dot{L} \right],$$

(55)

$$\frac{de_b}{dr} = J_{\text{orb}} \left[ (j_b \cdot \dot{L})e_b \times \dot{L} + 2j_b \times e_b - 5(e_b \cdot \dot{L})j_b \times \dot{L} \right],$$

(56)
Figure 5. Time evolution of the disc orientation for two binary eccentricities $e_b$ as indicated. Left-hand panels: disc inclination $I$ (the angle between $\hat{l}_d$ and $\hat{l}_b$) as a function of time. The black dashed lines mark $I_{\text{crit}}$ ($55^\circ$ for $e_b = 0.3$ and $31^\circ$ for $e_b = 0.6$) and $180^\circ - I_{\text{crit}}$. Right-hand panels: disc trajectories on the $I$--$\Omega$ plane (where $\Omega$ is the longitude of the ascending node of the disc). The black solid curves mark the $\Lambda = 0$ separatrix. Initial values are $I(0) = 20^\circ$ (blue), $I(0) = 40^\circ$ (green), $I(0) = 60^\circ$ (magenta), $I(0) = 80^\circ$ (red) and $I(0) = 160^\circ$ (cyan), with $\Omega(0) = 90^\circ$ for all trajectories. The other parameters are $M_b = 2M_\odot$, $\mu_b = 0.5M_\odot$, $a_b = 1a_\odot$, $r_{\text{in}} = 2a_\odot$, $r_{\text{out}} = 100a_\odot$, $\alpha = 0.01$ and $h = 0.1$.

where

$$J = \frac{L}{L_b/\sqrt{1 - e_b^2}} = \frac{\mu}{\mu_b} \left( \frac{M_b + m}{M_b} \right)^{1/2} \left( \frac{r}{a_b} \right)^{1/2},$$

(57)

$$\omega_b = \frac{3}{4} \left( \frac{m}{\mu} \right) \left( \frac{\mu_b}{M_b} \right)^{1/2} \left( \frac{a_b}{r} \right)^{7/2} \sqrt{\frac{GM_b}{a_b}},$$

(58)

and $\mu = mM_b/(m + M_b)$. Equation (58) reduces to equation (2) when $m \to 0$. The conservations of total quadrupole potential energy (see equation 5) and total angular momentum yield two constants of motion (e.g. Liu, Muñoz & Lai 2015; Anderson, Lai & Storch 2017)

$$\Psi = 1 - 6e_b^2 - 3 \left( 1 - e_b^2 \right) \cos^2 I + 15e_b^2 \sin^2 I \sin^2 \Omega,$$

(59)

$$K = \sqrt{1 - e_b^2 \cos I - \frac{e_b^2}{2J}}.$$

(60)

For a given $K$, one may solve equation (60) to get $e_b^2 = e_b^2(I)$. Assuming $0 \leq I \leq \pi/2$ and requiring $0 \leq e_b < 1$, we obtain

$$e_b^2 = 2J^2 \left[ \cos I \sqrt{\left( \frac{2K}{J} + \frac{1}{J^2} \right) + \cos^2 I - \left( \frac{K}{J} + \cos^2 I \right)} \right].$$

(61)
Equation (59) then gives $\Psi = \Psi(I, \Omega)$. When $J \sim K^{-1} \ll 1$, equation (61) reduces to

$$e_b^2 \approx -2 K J = \text{constant},$$

while when $J \gg 1$, equation (61) becomes

$$e_b^2 \approx 1 - \frac{K^2}{\cos I}.$$  

The fixed points of the system in the $I$--$\Omega$ plane are determined by

$$\frac{\partial \Psi}{\partial \Omega} = \frac{\partial \Psi}{\partial \Omega} = 0.$$  

The condition $\partial \Psi/\partial \Omega = 0$ gives $\Omega = \pi/2$ and $3\pi/2$, as before (see Section 2). For arbitrary $J$, one must numerically solve $\partial \Psi/\partial I|_{\Omega=\pi/2,3\pi/2} = 0$ to calculate the fixed points $I = I_{fp} > 0$ ($I = 0$ is always a fixed point of the system). However, when $J \ll 1$, one may show analytically that (as found in Section 2)

$$I_{fp} \approx \pi/2,$$  

while when $J \gg 1$,

$$I_{fp} \approx \cos^{-1} \sqrt{\frac{3(1 - e_b^2)}{5}}.$$  

where $e_b^2 = e_b(0)$. Notice $I_{fp}$ is the Lidov--Kozai critical inclination when $J \gg 1$ and $e_b(0) = 0$ (Lidov 1962; Kozai 1962).

Fig. 6 plots trajectories of the system in the $I$--$\Omega$ and $e_b$--$\Omega$ planes. When $J \ll 1$, the system’s dynamics reduce to that discussed in Section 2, with $I_{fp} \approx 90^\circ$ (black x’s), $e_b \approx e_b(0)$, and trajectories above and below $I = 90^\circ$ are symmetric. As $J$ increases in magnitude, $I_{fp}$ decreases, $e_b$ begins to oscillate and the inclination symmetry above and below $I = 90^\circ$ is lost. Although different trajectories may cross in the $I$--$\Omega$ plane, each still has a unique $\Psi$ value (equation 59), since the binary’s $e_b$ value differs from equation (61) when $I > \pi/2$. When $J \gg 1$, the system’s dynamics approaches the classic Lidov--Kozai regime (Lidov 1962; Kozai 1962). The fixed point $I_{fp}$ of the system approaches equation (66), with $e_b$ reaching large values when $I(0) > I_{fp}$, and with trajectories symmetric above and below $I = 90^\circ$.

Fig. 7 plots $I_{fp}$ as a function of $J^{-1}$, computed with equation (64) with $\Omega = \pi/2$. The binary eccentricity $e_b = e_b(0)$ takes values as indicated.
Nevertheless, Figs 6 and 7 show there exist highly inclined fixed points for any value of $J$. For $J \lesssim 0.1$, the system may evolve into near polar alignment, with $l_{\theta b}$ somewhat less than 90°.

5 TORQUE ON BINARY AND EFFECT OF ACCRETION

In the previous sections, we have studied the evolution of the disc around a binary with fixed $I_b$ and $e_b$. Here, we study the back-reaction torque on the binary from the disc. First, we consider a circular binary. The viscous back-reaction torque on the binary from the disc is (equation 46)

$$\frac{dL_b}{dt} = -\frac{dL_d}{dt}.\tag{67}$$

In addition, accretion on to the binary from the disc adds angular momentum to the binary’s orbit:

$$\frac{dL_b}{dt} = \frac{\lambda M}{b} \sqrt{GM_b r_{\text{in}}} \frac{d\hat{t}}{dt}.\tag{68}$$

Here, $M$ is the mass accretion rate on to the binary, $\lambda \sim 1$ (e.g. Miranda, Muñoz & Lai 2017), and we have assumed $\hat{t}(r_{\text{in}}, t) \approx \hat{t}(t)$ (see below). The torques (67) and (68) are equivalent to those considered in Foucart & Lai (2013), except we give different power-law prescriptions for $\Sigma = \Sigma(r)$ and $H = H(r)$. For discs in steady state, we have

$$M \approx 3\pi \alpha h^2 \Sigma_{\text{in}} r_{\text{in}}^2 L_{\text{in}}.\tag{69}$$

Using equations (46) (with $e_b = 0$), (67) and (68), we obtain the net disc–binary alignment time-scale for small angle between $\hat{L}_d$ and $\hat{L}_b$:

$$t_{\text{align}} = 1 + (1 + \eta) \frac{L_d}{L_b},\tag{70}$$

where

$$\eta = \frac{\lambda M \sqrt{GM_b r_{\text{in}}}}{L_d} \approx 0.031 \lambda \left(\frac{\mu}{0.1}\right)^4 \left(\frac{r_{\text{in}}}{2\mu_h}\right)^4 \left(\frac{M_h}{4\mu_h}\right)^2.$$

measures the strength of the accretion torque to the viscous torque on the binary ($\eta/\lambda = f^{-1}$, $\lambda = g$ in the notation of Foucart & Lai 2013). Since $\hat{L}(r_{\text{in}}, t) \neq \hat{L}(r_{\text{out}}, t)$, the disc angular momentum loss through accretion causes $\hat{L}_d$ to change with time:

$$\frac{d\hat{L}_d}{dt} = -\frac{\lambda M \sqrt{GM_b r_{\text{in}}}}{L_d} \left\{\hat{L}(r_{\text{in}}, t) - \hat{L}_d \right\}.\tag{72}$$

Because the magnitude of the tilt of $\hat{L}(r_{\text{in}}, t)$ from $\hat{L}_d$ is of the order of

$$\left|\frac{\hat{L}(r_{\text{in}}, t) - \hat{L}_d}{\partial \ln r}\right|_{\text{warp}},\tag{73}$$

we find

$$\frac{d\hat{L}_d}{dt} \approx -\frac{4\pi \lambda M \sqrt{GM_b r_{\text{in}}}}{c^2 L_d} \left(\frac{n^2}{2m^2}\right) \hat{L}_d \times T_b.\tag{74}$$

Detailed calculation shows that the accretion torque (74) is always much less than the viscous torque (19) on the disc. We relegate the calculation and discussion of the accretion torque (74) to the Appendix.

For eccentric binaries, the back-reaction from the disc is $dL_b/dt = -dL_d/dt$ (equation 46). But this is not sufficient for determining the evolution of $e_b$ and $I_b$. In addition, how accretion affects the binary eccentricity is also uncertain (e.g. Rafikov 2016; Miranda, Muñoz & Lai 2017). Nevertheless, as long as $L_b > \approx L_d$, the time-scale for the disc–binary inclination evolution should be of the order of $\gamma_b^{-1}$, with an estimate given by equation (48).

6 DISCUSSION

6.1 Theoretical uncertainties

Our theoretical analysis of discs around binaries assumes a linear disc warp. However, we find that at the inner disc region, $|\partial I/\partial \ln r|$ reaches $\approx 0.1$ for a wide range of binary and disc parameters. Inclusion of weakly non-linear warps in equations (22)–(23) may introduce new features in the disc warp profile (Ogilvie 2006). In addition, disc warps of this magnitude may interact resonantly with inertial waves in the disc, leading to a parametric instability that may excite turbulence in the disc (Gammie, Goodman & Ogilvie 2000; Ogilvie & Latter 2013). An investigation of these effects is outside the scope of this paper, but their inclusion is unlikely to change the direction of disc–binary inclination evolution (alignment versus polar alignment).

6.2 Observational implications

In Section 3.4, we showed that the viscous torque associated with disc twist/warp tends to drive the circumbinary disc axis $\hat{L}_d$ towards $\pm \hat{L}_b$ (alignment or anti-alignment) when $\Lambda > 0$, and towards $\pm \hat{e}_b$ (polar alignment) when $\Lambda < 0$. Note that $I_{\text{crit}} < I < 180^\circ$–$I_{\text{crit}}$ is a necessity, but not sufficient condition for polar alignment of the disc (equation 8). An extreme example is when $\Omega = 0^\circ$, since $\Lambda \geq 0$ for all inclinations $I$. Because the circumbinary disc probably formed in a turbulent molecular cloud, the disc is unlikely to have a preferred $\Omega$ when it forms. The condition for polar alignment ($\Lambda < 0$) requires $\Omega$ to satisfy

$$\sin^2 \Omega > \frac{1 - e_b^2}{2e_b^2} \frac{1}{\tan^2 I} = \frac{\tan^2 I_{\text{crit}}}{\tan^2 I}.\tag{75}$$

Assuming a uniform distribution of $\Omega$-values from 0 to $2\pi$, the probability of the disc to polar align is (for given $I, e_b$)

$$P_{\text{polar}}(I, e_b) = 1 - 2\Omega_{\text{min}}/\pi,\tag{76}$$

where

$$\Omega_{\text{min}}(I, e_b) = \begin{cases} \pi/2 \quad &|\sin I| \leq |\sin I_{\text{crit}}| \\ \sin^{-1} \left(\frac{\tan I_{\text{crit}}}{|\tan I|}\right) \quad &\text{otherwise}. \end{cases}\tag{77}$$

We define the inclination $I_{\text{polar}}$ through $P_{\text{polar}}(I_{\text{polar}}, e_b) = 0.5$. Solving for $I_{\text{polar}}$, we obtain

$$I_{\text{polar}} = \tan^{-1} \sqrt{2(1 - e_b^2)/5e_b^2}.\tag{78}$$

In Fig. 8, we plot contours of constant $P_{\text{polar}}$ in the $I$–$e_b$ space. The $P_{\text{polar}} = 0$ curve (black) traces out $I_{\text{crit}}$ (equation 8), while the $P_{\text{polar}} = 0.5$ curve (red) traces out $I_{\text{polar}}$ (equation 78). When $I < I_{\text{crit}}$, alignment of $\hat{I}$ with $\hat{b}$ is inevitable. When $I > I_{\text{polar}}$, alignment of $\hat{I}$ with $\hat{e}_b$ is probable.

Table 1 lists a number of circumbinary systems with highly eccentric binaries. With the exception of 99 Herculis, all the binaries
Figure 8. Contour plot of the probability of polar alignment $P_{\text{polar}}$ (equation 76) as a function of disc inclination $I$ and binary eccentricity $e_b$. Contours of constant $P_{\text{polar}}$ are labelled as indicated. The $P_{\text{polar}} = 0$ line (black) traces out $\beta_{\text{crit}}$ (equation 8), while the $P_{\text{polar}} = 0.5$ line (red) traces out $\beta_{\text{polar}}$ (equation 78).

Table 1. Binary eccentricities $e_b$, with their inclinations $\beta_{\text{crit}}$ (equation 8) and $\beta_{\text{polar}}$ (equation 78), for the selected eccentric binaries with circumbinary discs. With the exception of the debris disc around 99 Herculis, all binaries have circumbinary discs aligned with the binary orbital plane within a few degrees. Binary eccentricities are from Kennedy et al. (2012a) (99 Herculis), Tomkin & Popper (1986) ($\alpha$ CrB), Pourbaix (2000) ($\beta$ Tri), Czekala et al. (2016) (DQ Tau), Alencar et al. (2003) (AK Sco) and Boden et al. (2005) (HD 98800 B).

| Binary      | $e_b$  | $\beta_{\text{crit}}$ | $\beta_{\text{polar}}$ |
|-------------|--------|------------------------|------------------------|
| 99 Herculis | 0.77   | 20°                     | 28°                     |
| $\alpha$ CrB| 0.37   | 48°                     | 58°                     |
| $\beta$ Tri | 0.43   | 43°                     | 53°                     |
| DQ Tau      | 0.57   | 33°                     | 42°                     |
| AK Sco      | 0.47   | 40°                     | 50°                     |
| HD 98800 B  | 0.78   | 20°                     | 27°                     |

listed have discs coplanar with the binary orbital plane within a few degrees. We also list $\beta_{\text{crit}}$ (equation 8) and $\beta_{\text{polar}}$ (equation 78) for these systems. We do not list the binaries KH 15D (Chiang & Murray-Clay 2004; Winn et al. 2004; Capelo et al. 2012) and HD 142527B (Casassus et al. 2015; Marino et al. 2015; Lacour et al. 2016) since the orbital elements of these binaries are not well constrained. However, both binaries appear to have significant eccentricities (Chiang & Murray-Clay 2004; Lacour et al. 2016).

Since planets form in gaseous circumbinary discs, planets may form with orbital planes perpendicular to the binary orbital plane if the binary is sufficiently eccentric. Such planets may be detectable in transit surveys of eclipsing binaries due to nodal precession of the planet’s orbit.

The twist and warp calculated in Section 3.3 is non-negligible. Further observations of (gaseous) circumbinary discs may be able to detect such warps (Juhász & Facchini 2017), further constraining the orientation and dynamics of circumbinary disc systems.

7 SUMMARY

Using semi-analytic theory, we have studied the warp and long-term evolution of circumbinary discs around eccentric binaries. Our main results and conclusions are listed below.

(i) For protoplanetary discs with dimensionless thickness $H/r$ larger than the viscosity parameter $\alpha$, bending wave propagation effectively couples different regions of the disc, making it precess as a quasi-rigid body. Without viscous dissipation from disc warping, the dynamics of such a disc is similar to that of a test particle around an eccentric binary (Sections 2 and 3.2).

(ii) When the binary is eccentric and the disc is significantly inclined, the disc warp profile exhibits new features not seen in previous works. The disc twist (equation 34) and warp (equation 35) have additional contributions due to additional torques on the disc when the binary is eccentric.

(iii) Including the dissipative torque from warping, the disc may evolve to one of two states, depending on the initial sign of $\Lambda$ (equation 4; Section 3.4). When $\Lambda$ is initially positive, the disc angular momentum vector aligns (or anti-aligns) with the binary orbital angular momentum vector. When $\Lambda$ is initially negative, the disc angular momentum vector aligns with the binary eccentricity vector (polar alignment). Note that $\Lambda$ depends on both $I$ (the disc–binary inclination) and $\Omega$ (the longitude of ascending node of the disc). Thus, for a given $e_b$, the direction of inclination evolution depends not only on the initial $I(0)$, but also on the initial $\Omega(0)$.

(iv) When the disc has a non-negligible angular momentum compared to the binary, the system’s fixed points are modified (Section 4). The disc may then evolve to a state of near polar alignment, with the inclination somewhat less than 90°.

(v) The time-scale of evolution of the disc–binary inclination angle (see equations 52–53) depends on various disc parameters (see equation 48), but is in general less than a few Myr. This suggests that highly inclined discs and planets may exist around eccentric binaries.

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REFERENCES

Alencar S. H. P., Melo C. H. F., Dullemond C. P., Andersen J., Batalha C., Vaz L. P. R., Mathieu R. D., 2003, A&A, 409, 1037
Aly H., Dehnen W., Nixon C., King A., 2015, MNRAS, 449, 65
Anderson K. R., Lai D., Storchi N. I., 2017, MNRAS, 467, 3066
Andrews S. M., Czekala I., Wilner D. J., Espaillat C., Dullemond C. P., Hughes A. M., 2010, ApJ, 710, 462
Armstrong D. J., Osborn H. P., Brown D. J. A., Faedi F., Gómez Maqueo Chew Y., Martin D. V., Pollacco D., Udry S., 2014, MNRAS, 444, 1873
Bate M. R., Bonnell I. A., Bromm V., 2003, MNRAS, 339, 577
Bate M. R., Lodato G., Pringle J. E., 2010, MNRAS, 401, 1505

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APPENDIX: ACCRETION TORQUES

If the inner disc is not coplanar with the outer disc, accretion will change the disc angular momentum vector over time. We parametrize this accretion torque according to

\[
\frac{d\mathbf{L}_{\text{acc}}}{dt} = -\lambda M \sqrt{GM_{\text{bin}} r_{\text{in}}} \mathbf{r}_{\text{in}} \times \mathbf{L}_{\text{out}}.
\]

(A1)

where \(\mathbf{L}_{\text{acc}}\) is the accretion torque on to the binary, and \(\lambda \sim 1\) parameterizes the angular momentum loss from the disc to the binary. The time evolution of \(\mathbf{G}\), as well as the pericentre precession induced by the non-Keplerian angular frequency, warps the inner edge of the disc in the direction of the binary orbital plane \([\mathbf{f}_{\text{in}}]\), and equation (35). Inserting \(\mathbf{f}(r, t) = \mathbf{f}_{\text{in}} + \mathbf{f}_{\text{circ}}\) in equation (A1), we obtain

\[
\frac{d\mathbf{L}_{\text{acc}}}{dt} = \gamma_\lambda \delta_{\text{in}} \mathbf{r}_{\text{in}} \Lambda \left[ (1 - e_\Lambda^2) (\hat{\mathbf{r}}_{\text{out}} \cdot \mathbf{L}_{\text{in}}) \times (\hat{\mathbf{r}}_{\text{in}} \times \mathbf{L}_{\text{in}}) \\
- 5(\hat{\mathbf{r}}_{\text{in}} \cdot \mathbf{e}_b) \mathbf{L}_{\text{in}} \times (\mathbf{e}_b \times \mathbf{L}_{\text{in}})ight] + \gamma_b \mathbf{W}_{\text{in}} \mathbf{f}_{\text{in}} \left[ (1 - e_\Lambda^2) (\hat{\mathbf{r}}_{\text{in}} \cdot \mathbf{L}_{\text{out}}) \times (\hat{\mathbf{r}}_{\text{in}} \times \mathbf{L}_{\text{out}}) \\
- 5(\hat{\mathbf{r}}_{\text{in}} \cdot \mathbf{e}_b) \mathbf{L}_{\text{out}} \times (\mathbf{e}_b \times \mathbf{L}_{\text{out}}) \right],
\]

(A2)

where

\[
\gamma_\lambda = \frac{\lambda M \sqrt{GM_{\text{bin}} r_{\text{in}}}}{L_{\text{out}}} = \frac{9}{2} \frac{\lambda}{a} h^2 n(r_{\text{out}})
\]

\[
= 3.18 \times 10^{-7} \lambda \left( \frac{\alpha}{0.01} \right) \left( \frac{h}{0.1} \right)^2
\times \left( \frac{M_{\odot}}{2M_{\odot}} \right)^{1/2} \left( \frac{100 \text{ au}}{r_{\text{out}}} \right)^{3/2} \left( \frac{2\pi}{\text{yr}} \right). \]

(A3)

We have assumed the disc to be in a steady state, so

\[
\dot{M} \simeq 3\pi ah^2 \Sigma_{\text{in}} \sqrt{GM_{\odot} r_{\text{in}}}. \]

(A4)

Equation (A2) agrees with the rough magnitude and direction of the accretion torque estimated in equation (74). Since

\[
\frac{d}{dt}(\hat{\mathbf{r}}_{\text{in}} \cdot \mathbf{L}_{\text{in}}) = \gamma_\ \delta_{\text{in}} \mathbf{r}_{\text{in}} \Lambda \left[ (1 - e_\Lambda^2) - \Lambda \right] \]

\[
+ \gamma_b \mathbf{W}_{\text{in}} \mathbf{f}_{\text{in}} (1 - e_\Lambda^2) - \Lambda \right]
\]

(A5)

\[
\frac{d}{dt}(\hat{\mathbf{r}}_{\text{in}} \cdot \mathbf{e}_b) = -\gamma_\lambda \delta_{\text{in}} \mathbf{r}_{\text{in}} \Lambda \left[ \Lambda + 5e_b^2 \right] \\
- \gamma_b \mathbf{W}_{\text{in}} \mathbf{f}_{\text{in}} (5e_b^2),
\]

(A6)

the radial functions \(\delta_{\text{in}}(r_{\text{in}})\), \(\mathbf{W}_{\text{in}}(r_{\text{in}}) < 0\) and \(f_b \sim \Lambda\), equations (A5)–(A6) drive the disc one of two ways depending on the rough value of \(\Lambda\):

(i) \(\Lambda \gtrsim 0\): The accretion torque (A2) pushes \(\hat{\mathbf{r}}_{\text{in}}\) away from \(\hat{\mathbf{r}}_{\text{in}}\).

(ii) \(\Lambda \lesssim 0\): The accretion torque (A2) pushes \(\hat{\mathbf{r}}_{\text{in}}\) away from \(\mathbf{e}_b\).

From equation (27), it may be shown that there are no fixed points near the \(\Lambda \equiv 0\) separatrix. Therefore, the accretion torque drives the disc to a trajectory near the \(\Lambda \equiv 0\) separatrix.
Fig. A1 plots the examples considered in Fig. 2 with accretion torques (equation 80). We take $h$ and $\alpha$ to be significantly higher than our canonical values of $\alpha = 0.01$ and $h = 0.1$ so that accretion torques affect the dynamical evolution of the circumbinary disc (equation 81). In the left-hand panels of Fig. A1, we plot the disc inclination with time, for the binary eccentricities indicated. The trajectories that start at $I(0) = 20^\circ$, $40^\circ$ and $80^\circ$ all evolve towards the prograde separatrix, which nutates around $I \sim 50^\circ$ when $e_b = 0.3$, and $I \sim 40^\circ$ when $e_b = 0.6$. The trajectories that start at $I(0) = 60^\circ$ both evolve to the retrograde separatrix, which nutates around $I \sim 130^\circ$ when $e_b = 0.3$, and $I \sim 140^\circ$ when $e_b = 0.6$. On the right-hand panels, we plot the disc trajectories on the $I-\Omega$ plane, for the binary eccentricities indicated. All disc trajectories evolve towards the $\Lambda \approx 0$ separatrix.

The relative strength of the viscous to the accretion torques from disc warping is given by the ratio

$$\frac{|\gamma_a|}{|\gamma_b W_{\Omega b}(r_{in})|} \approx 300 \lambda^{-1} \left( \frac{0.1}{h} \right)^2.$$  \hspace{1cm} (A7)

As long as $|\gamma_b| \gg |\gamma_a W_{\Omega b}(r_{in})|$, the viscous torque dominates, and $\dot{I}_d$ aligns with either $I_b$ or $e_b$, depending on the sign of $\Lambda$ (Section 3.4).
When $|\gamma_b| \lesssim |\gamma_a W_{bb}(r_{in})|$, the accretion torques may dominate, and $I_d$ may be driven to the seperatrix $\Lambda \approx 0$.

Fig. A2 is identical to Fig. 5, except we include viscous (equation 50) and accretion (equation 80) torques with $\alpha = 0.01$, $h = 0.1$ and $\lambda = 1$. Because $|\gamma_b| \gg |W_{bb}(r_{in})\gamma_a|$, the viscous torque dominates the disc’s dynamics. As a result, Fig. A2 is almost indistinguishable from Fig. 5. Only for unrealistically hot protoplanetary discs with $h \gtrsim 0.5$ may accretion torques significantly affect the disc evolution over viscous time-scales.

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