The Generalized Spike Process, Sparsity, and Statistical Independence

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Abstract

A basis under which a given set of realizations of a stochastic process can be represented most sparsely (the so-called best sparsifying basis (BSB)) and the one under which such a set becomes as less statistically dependent as possible (the so-called least statistically-dependent basis (LSDB)) are important for data compression and have generated interests among computational neuroscientists as well as applied mathematicians. Here we consider these bases for a particularly simple stochastic process called “generalized spike process”, which puts a single spike—whose amplitude is sampled from the standard normal distribution—at a random location in the zero vector of length $n$ for each realization.

Unlike the “simple spike process” which we dealt with in our previous paper and whose amplitude is constant, we need to consider the kurtosis-maximizing basis (KMB) instead of the LSDB due to the difficulty of evaluating differential entropy and mutual information of the generalized spike process. By computing the marginal densities and moments, we prove that: 1) the BSB and the KMB selects the standard basis if we restrict our basis search within all possible orthonormal bases in $\mathbb{R}^n$; 2) if we extend our basis search to all possible volume-preserving invertible linear transformations, then the BSB exists and is again the standard basis whereas the KMB does not exist. Thus, the KMB is rather sensitive to the orthonormality of the
transformations under consideration whereas the BSB is insensitive to that. Our results once again support the preference of the BSB over the LSDB/KMB for data compression applications as our previous work did.

1 Introduction

This paper is a sequel to our previous paper [3], where we considered the so-called best sparsifying basis (BSB), and the least statistically-dependent basis (LSDB) for the input data which are the realizations of a very simple stochastic process called the “spike process.” This process, which we will refer to as the “simple” spike process for convenience, puts a unit impulse (i.e., its amplitude is constant 1) at a random location in a zero vector of length \( n \). Here, the BSB is the basis in \( \mathbb{R}^n \) that best sparsifies the given input data, and the LSDB is the basis in \( \mathbb{R}^n \) that is the closest to the statistically independent coordinate system (regardless of whether such a coordinate system exists or not). In particular, we considered the BSB and LSDB chosen from all possible orthonormal transformations (i.e., \( O(n) \)) or all possible volume-preserving linear transformations (i.e., \( SL^\pm(n, \mathbb{R}) \), where any element in this set has its determinant \( \pm 1 \)).

In this paper, we consider the BSB and LSDB for a slightly more complicated process, the “generalized” spike process, and compare them with those of the simple spike process. The generalized spike process puts an impulse whose amplitude is sampled from the standard normal distribution \( \mathcal{N}(0, 1) \).

Our motivation to analyze the BSB and the LSDB for the generalized spike process stems from the work in computational neuroscience [17], [18], [2], [23] as well as in computational harmonic analysis [8]. The concept of sparsity and that of statistical independence are intrinsically different. Sparsity emphasizes the issue of compression directly, whereas statistical independence concerns the relationship among the coordinates. Yet, for certain stochastic processes, these two are intimately related, and often confusing. For example, Olshausen and Field [17], [18] emphasized the sparsity as the basis selection criterion, but they also assumed the statistical independence of the coordinates. For a set of natural scene image patches, their algorithm generated basis functions efficient to capture and represent edges of various scales, orientations, and positions, which are similar to the receptive field profiles of the neurons in our primary visual cortex. (Note the
criticism raised by Donoho and Flesia [9] about the trend of referring to these functions as “Gabor”-like functions; therefore, we just call them “edge-detecting” basis functions in this paper.) Bell and Sejnowski [2] used the statistical independence criterion and obtained the basis functions similar to those of Olshausen and Field. They claimed that they did not impose the sparsity explicitly and such sparsity emerged by minimizing the statistical dependence among the coordinates. These motivated us to study these two criteria. However, the mathematical relationship between these two criteria in the general case has not been understood completely. We wish to deepen our understanding of this intricate relationship. Therefore we chose to study such spike processes, which are much simpler than the natural scene images viewed as a high-dimensional stochastic process. It is important to use simple stochastic processes first since we can gain insights and make precise statements in terms of theorems. By these theorems, we now understand what are the precise conditions for the sparsity and statistical independence criteria to select the same basis for the spike processes, and the difference between the simple and generalized stochastic processes.

The organization of this paper is as follows. The next section specifies our notation and terminology. Section 3 defines how to quantitatively measure the sparsity and statistical dependence of a stochastic process relative to a given basis. Section 4 reviews the results on the simple spike process we obtained in [3]. Our main results are presented in Section 5 where we deal with the generalized spike process. We conclude with discussion in Section 6.

2 Notations and Terminology

Let us first set our notation and the terminology. Let \( \mathbf{X} \in \mathbb{R}^n \) be a random vector with some unknown probability density function (pdf) \( f_\mathbf{X} \). Let \( B \in \mathcal{D} \), where \( \mathcal{D} \) is the so-called basis dictionary. For very high dimensional data, we often use the wavelet packets and local Fourier bases as \( \mathcal{D} \) (see [20] and references therein for more about such basis dictionaries). In this paper, however, we use much more larger dictionaries: \( O(n) \) (the group of orthonormal transformations in \( \mathbb{R}^n \)) or \( \text{SL}^\pm(n, \mathbb{R}) \) (the group of invertible volume-preserving transformations in \( \mathbb{R}^n \), i.e., their determinants are \( \pm 1 \)). We are interested in searching a basis under which the original stochastic process becomes either the sparsest or the least statistically dependent among the bases in \( \mathcal{D} \). Let \( \mathcal{C}(B \mid \mathbf{X}) \) be a numerical measure of deficiency or cost
of the basis $B$ given the input stochastic process $X$. Under this setting, the best basis for the stochastic process $X$ among $D$ relative to the cost $C$ is written as $B_* = \arg \min_{B \in D} C(B | X)$.

We also note that log in this paper implies $\log_2$, unless stated otherwise. The $n \times n$ identity matrix is denoted by $I_n$, and the $n \times 1$ column vector whose entries are all ones, i.e., $(1, 1, \ldots, 1)^T$, is denoted by $1_n$.

3 Sparsity vs. Statistical Independence

Let us now define the measure of sparsity and that of statistical independence to evaluate a given basis (coordinate system).

3.1 Sparsity

Sparsity is a key property as a good coordinate system for compression. The true sparsity measure for a given vector $x \in \mathbb{R}^n$ is the so-called $\ell^0$ quasi-norm which is defined as

$$\|x\|_0 \triangleq \# \{i \in [1, n] : x_i \neq 0\},$$

i.e., the number of nonzero components in $x$. This measure is, however, very unstable for even small perturbation of the components in a vector. Therefore, a better measure is the $\ell^p$ norm:

$$\|x\|_p \triangleq \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}, \quad 0 < p \leq 1.$$

In fact, this is a quasi-norm for $0 < p < 1$ since this does not satisfy the triangle inequality, but only satisfies weaker conditions: $\|x + y\|_p \leq 2^{-1/p'}(\|x\|_p + \|y\|_p)$ where $p'$ is the conjugate exponent of $p$; and $\|x + y\|_p^p \leq \|x\|_p^p + \|y\|_p^p$. It is easy to show that $\lim_{p \downarrow 0} \|x\|_p^p = \|x\|_0$. See [8] for the details of the $\ell^p$ norm properties.

Thus, we can use the expected $\ell^p$ norm minimization as a criterion to find the best basis for a given stochastic process in terms of sparsity:

$$C_p(B | X) = E\|B^{-1}X\|_p^p,$$  \hspace{1cm} (1)
We propose to use the minimization of this cost to select the best sparsifying basis (BSB):

\[ B_p = \arg \min_{B \in \mathcal{D}} \mathcal{C}_p(B \mid X). \]

**Remark 3.1.** It should be noted that the minimization of the \( \ell^p \) norm can also be achieved for each realization. Without taking the expectation in (1), one can select the BSB \( B_p = B_p(x, \mathcal{D}) \) for each realization \( x \). We can guarantee that

\[
\min_{B \in \mathcal{D}} \mathcal{C}_p(B \mid X = x) \leq \min_{B \in \mathcal{D}} \mathcal{C}_p(B \mid X) \leq \max_{B \in \mathcal{D}} \mathcal{C}_p(B \mid X = x).
\]

For highly variable or erratic stochastic processes, however, \( B_p(x, \mathcal{D}) \) may significantly change for each \( x \) and we need to store more information of this set of \( N \) bases if we want to use them to compress the entire training dataset. Whether we should adapt a basis per realization or on the average is still an open issue. See [21] for more details.

### 3.2 Statistical Independence

The statistical independence of the coordinates of \( \mathbf{Y} \in \mathbb{R}^n \) means \( f_{\mathbf{Y}}(\mathbf{y}) = f_{Y_1}(y_1)f_{Y_2}(y_2) \cdots f_{Y_n}(y_n) \), where \( f_{Y_k} \) is a one-dimensional marginal pdf of \( f_{\mathbf{Y}} \). The statistical independence is a key property as a good coordinate system for compression and particularly modeling because: 1) damage of one coordinate does not propagate to the others; and 2) it allows us to model the \( n \)-dimensional stochastic process of interest as a set of 1D processes. Of course, in general, it is difficult to find a truly statistically independent coordinate system for a given stochastic process. Such a coordinate system may not even exist for a certain stochastic process. Therefore, the next best thing we can do is to find the least-statistically dependent coordinate system within a basis dictionary. Naturally, then, we need to measure the “closeness” of a coordinate system \( Y_1, \ldots, Y_n \) to the statistical independence. This can be measured by mutual information or relative entropy between the true pdf \( f_{\mathbf{Y}} \) and the product of its marginal pdf’s:

\[
I(\mathbf{Y}) \triangleq \int f_{\mathbf{Y}}(\mathbf{y}) \log \frac{f_{\mathbf{Y}}(\mathbf{y})}{\prod_{i=1}^n f_{Y_i}(y_i)} \, d\mathbf{y} = -H(\mathbf{Y}) + \sum_{i=1}^n H(Y_i),
\]
where $H(Y)$ and $H(Y_i)$ are the differential entropy of $Y$ and $Y_i$ respectively:

$$H(Y) = -\int f_Y(y) \log f_Y(y) dy$$

$$H(Y_i) = -\int f_{Y_i}(y_i) \log f_{Y_i}(y_i) dy_i.$$  

We note that $I(Y) \geq 0$, and $I(Y) = 0$ if and only if the components of $Y$ are mutually independent. See [7] for more details of the mutual information.

Suppose $Y = B^{-1}X$ and $B \in \text{GL}(n, \mathbb{R})$ with $\det B = \pm 1$. We denote such a set of matrices by $\text{SL}^\pm(n, \mathbb{R})$. Note that the usual $\text{SL}(n, \mathbb{R})$ is a subset of $\text{SL}^\pm(n, \mathbb{R})$. Then, we have

$$I(Y) = -H(Y) + \sum_{i=1}^n H(Y_i) = -H(X) + \sum_{i=1}^n H(Y_i),$$

since the differential entropy is invariant under such an invertible volume-preserving linear transformation, i.e.,

$$H(B^{-1}X) = H(X) + \log |\det B^{-1}| = H(X),$$

because $|\det B^{-1}| = 1$. Based on this fact, we proposed the minimization of the following cost function as the criterion to select the so-called least statistically-dependent basis (LSDB) in the basis dictionary context [20]:

$$C_H(B \mid X) = \sum_{i=1}^n H((B^{-1}X)_i) = \sum_{i=1}^n H(Y_i).$$

(2)

Now, we can define the LSDB as

$$B_{LSDB} = \arg \min_{B \in \mathcal{D}} C_H(B \mid X).$$

We were informed that Pham [19] had proposed the minimization of the same cost (2) earlier. We would like to point out the main difference between our work [20] and Pham’s. We used the basis libraries such as wavelet packets and local Fourier bases that allow us to deal with datasets with large dimensions such as face images whereas Pham used more general dictionary $\text{GL}(n, \mathbb{R})$. In practice, however, the numerical optimization (2) clearly becomes more difficult in his general case particularly if one wants to use this for high dimensional datasets.
Closely related to the LSDB is the concept of the *kurtosis-maximizing basis* (KMB). This is based on the approximation of the marginal differential entropy \( H(Y_i) \) by higher order moments/cumulants using the Edgeworth expansion and was derived by Comon \[6\]:

\[
H(Y_i) \approx -\frac{1}{48} \kappa(Y_i) = -\frac{1}{48} (\mu_4(Y_i) - 3\mu_2^2(Y_i)) \tag{3}
\]

where \( \mu_k(Y_i) \) is the \( k \)th central moment of \( Y_i \), and \( \kappa(Y_i) / \mu_2^2(Y_i) \) is called the *kurtosis* of \( Y_i \). See also Cardoso \[5\] for a nice exposition of the various approximations to the mutual information. Now, the KMB is defined as follows:

\[
B_\kappa = \arg \min_{B \in \mathcal{D}} \mathcal{C}_\kappa(B \mid X) = \arg \max_{B \in \mathcal{D}} \sum_{i=1}^{n} \kappa(Y_i), \tag{4}
\]

where \( \mathcal{C}_\kappa(B \mid X) = -\sum_{i=1}^{n} \kappa(Y_i) \). We note that the LSDB and the KMB are tightly related, yet can be different. After all, \( \mathcal{B} \) is simply an approximation to the entropy up to the fourth order cumulant. We also would like to point out that Buckheit and Donoho \[4\] independently proposed the same measure as a basis selection criterion, whose objective was to find a basis under which an input stochastic process looks maximally “non-Gaussian.”

### 4 Review of Previous Results on the Simple Spike Process

In this section, we briefly summarize the results of the simple spike process, which we obtained previously. See \[3\] for the details and proofs.

An \( n \)-dimensional *simple spike process* generates the standard basis vectors \( \{e_j\}_{j=1}^{n} \subset \mathbb{R}^n \) in a random order, where \( e_j \) has one at the \( j \)th entry and all the other entries are zero. One can view this process as a unit impulse located at a random position between 1 and \( n \).

\footnote{Note that there is a slight abuse of the terminology; We call the kurtosis-maximizing basis in spite of maximizing unnormalized version (without the division by \( \mu_2^2(Y_i) \)) of the kurtosis.}
4.1 The Karhunen-Loève Basis

The Karhunen-Loève basis of this process is not unique and not useful because of the following theorem.

**Proposition 4.1.** The Karhunen-Loève basis for the simple spike process is any orthonormal basis in $\mathbb{R}^n$ containing the “DC” vector $\mathbf{1}_n = (1, 1, \ldots, 1)^T$.

This theorem reminds us of non-Gaussianity of the simple spike process.

4.2 The Best Sparsifying Basis

As for the BSB, we have the following result:

**Theorem 4.2.** The BSB with any $p \in [0, 1]$ for the simple spike process is the standard basis if $\mathcal{D} = O(n)$ or $\text{SL}^\pm(n, \mathbb{R})$.

4.3 Statistical Dependence and Entropy of the Simple Spike Process

Before considering the LSDB of this process, let us note a few specifics about the simple spike process. First, although the standard basis is the BSB for this process, it clearly does not provide the statistically independent coordinates. The existence of a single spike at one location prohibits spike generation at other locations. This implies that these coordinates are highly statistically dependent.

Second, we can compute the true entropy $H(X)$ for this process unlike other complicated stochastic processes. Since the simple spike process selects one possible vector from the standard basis vectors of $\mathbb{R}^n$ with uniform probability $1/n$, the true entropy $H(X)$ is clearly $\log n$. This is one of the rare cases where we know the true high-dimensional entropy of the process.

4.4 The LSDB among $O(n)$

For $\mathcal{D} = O(n)$, we have the following theorem.

**Theorem 4.3.** The LSDB among $O(n)$ is the following:
for $n \geq 5$, either the standard basis or the basis whose matrix representation is

\[
\frac{1}{n} \begin{bmatrix}
    n-2 & -2 & \cdots & -2 & -2 \\
    -2 & n-2 & \ddots & -2 \\
    \vdots & \ddots & \ddots & \vdots \\
    -2 & \cdots & n-2 & -2 \\
    -2 & -2 & \cdots & -2 & n-2 \\
\end{bmatrix};
\] (5)

for $n = 4$, the Walsh basis, i.e.,

\[
\frac{1}{2} \begin{bmatrix}
    1 & 1 & 1 & 1 \\
    1 & -1 & 1 & -1 \\
    1 & -1 & -1 & 1 \\
\end{bmatrix};
\]

for $n = 3,

\[
\begin{bmatrix}
    \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\
    \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\
    \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 \\
\end{bmatrix};
\] and

for $n = 2, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$ and this is the only case where the true independence is achieved.

Remark 4.4. Note that when we say the basis is a matrix as above, we really mean that the column vectors of that matrix form the basis. This also means that any permuted and/or sign-flipped (i.e., multiplied by $-1$) versions of those column vectors also form the basis. Therefore, when we say the basis is a matrix $A$, we mean not only $A$ but also its permuted and sign-flipped versions of $A$. This remark also applies to all the propositions and theorems below, unless stated otherwise.

Remark 4.5. There is an important geometric interpretation of (5). This matrix can also be written as:

\[
B_{HR(n)} \triangleq I_n - 2\frac{1_n}{\sqrt{n}} \frac{1_n^T}{\sqrt{n}}.
\]

In other words, this matrix represents the Householder reflection with respect to the hyperplane \(\{y \in \mathbb{R}^n | \sum_{i=0}^{n} y_i = 0\}\) whose unit normal vector is $1_n/\sqrt{n}$. 

Below, we use the notation $B_{O(n)}$ for the LSDB among $O(n)$ to distinguish it from the LSDB among $GL(n, \mathbb{R})$, which is denoted by $B_{GL(n)}$. So, for example, for $n \geq 5$, $B_{O(n)} = I_n$ or $B_{HR(n)}$.

4.5 The LSDB among $GL(n, \mathbb{R})$

As discussed in [3], for the simple spike process, there is no important distinction in the LSDB selection from $GL(n, \mathbb{R})$ and from $SL^\pm(n, \mathbb{R})$. Therefore, we do not have to treat these two cases separately. On the other hand, the generalized spike process in Section 5 requires us to treat $SL^\pm(n, \mathbb{R})$ and $GL(n, \mathbb{R})$ differently due to the continuous amplitude of the generated spikes.

We now have the following curious theorem:

**Theorem 4.6.** The LSDB among $GL(n, \mathbb{R})$ with $n > 2$ is the following basis pair (for analysis and synthesis respectively):

$$B^{-1}_{GL(n)} = \begin{bmatrix}
  a & a & \cdots & \cdots & \cdots & \cdots & \cdots & a \\
  b_2 & c_2 & b_2 & \cdots & \cdots & \cdots & \cdots & b_2 \\
  b_3 & b_3 & c_3 & b_3 & \cdots & \cdots & \cdots & b_3 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  b_{n-1} & \cdots & \cdots & b_{n-1} & c_{n-1} & b_{n-1} \\
  b_n & \cdots & \cdots & \cdots & b_n & c_n 
\end{bmatrix}, \quad (6)$$

$$B_{GL(n)} = \begin{bmatrix}
  (1 + \sum_{k=2}^{n} b_k d_k) / a & -d_2 & -d_3 & \cdots & -d_n \\
  -b_2 d_2 / a & d_2 & 0 & \cdots & 0 \\
  -b_3 d_3 / a & 0 & d_3 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \vdots \\
  -b_n d_n / a & 0 & \cdots & 0 & d_n 
\end{bmatrix}, \quad (7)$$

where $a, b_k, c_k$ are arbitrary real-valued constants satisfying $a \neq 0, b_k \neq c_k$, and $d_k = 1/(c_k - b_k), k = 2, \ldots, n$.

If we restrict ourselves to $D = SL^\pm(n, \mathbb{R})$, then the parameter $a$ must satisfy:

$$a = \pm \prod_{k=2}^{n} (c_k - b_k)^{-1}.$$
Remark 4.7. The LSDB such as (3) and the LSDB pair (4), (5) provide us with further insight into the difference between sparsity and statistical independence. In the case of (3), this is the LSDB, yet does not sparsify the spike process at all. In fact, these coordinates are completely dense, i.e., $C_0 = n$. We can also show that the sparsity measure $C_p$ gets worse as $n \to \infty$. More precisely, we have the following proposition.

Proposition 4.8.

\[
\lim_{n \to \infty} C_p \left( B_{HR(n)} \mid X \right) = \begin{cases} 
\infty & \text{if } 0 \leq p < 1; \\
3 & \text{if } p = 1.
\end{cases}
\]

It is interesting to note that this LSDB approaches to the standard basis as $n \to \infty$. This also implies that

\[
\lim_{n \to \infty} C_p \left( B_{HR(n)} \mid X \right) \neq C_p \left( \lim_{n \to \infty} B_{HR(n)} \mid X \right).
\]

As for the analysis LSDB (3), the ability to sparsify the spike process depends on the values of $b_k$ and $c_k$. Since the parameters $a$, $b_k$ and $c_k$ are arbitrary as long as $a \neq 0$ and $b_k \neq c_k$, let us put $a = 1$, $b_k = 0$, $c_k = 1$, for $k = 2, \ldots , n$. Then we get the following specific LSDB pair:

\[
B_{GL(n)}^{-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & \cdots & I_{n-1} \\ 0 & \cdots & \end{bmatrix}, \quad B_{GL(n)} = \begin{bmatrix} 1 & -1 & \cdots & -1 \\ 0 & \cdots & I_{n-1} \\ \end{bmatrix}.
\]

This analysis LSDB provides us with a sparse representation for the simple spike process (though this is clearly not better than the standard basis). For $Y = B_{GL(n)}^{-1} X$,

\[
C_p = E \left[ \| Y \|_p^p \right] = \frac{1}{n} \times 1 + \frac{n-1}{n} \times 2 = 2 - \frac{1}{n}, \quad 0 \leq p \leq 1.
\]

Now, let us take $a = 1$, $b_k = 1$, $c_k = 2$ for $k = 2, \ldots , n$ in (3) and (4). Then we get

\[
B_{GL(n)}^{-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & \end{bmatrix}, \quad B_{GL(n)} = \begin{bmatrix} n & -1 & \cdots & -1 \\ -1 & \cdots & I_{n-1} \\ \end{bmatrix}.
\]
The sparsity measure of this process is:

\[ C_p = \frac{1}{n} \times n + \frac{n - 1}{n} \times \{(n - 1) + 2^p\} = n + (2^p - 1) \left( 1 - \frac{1}{n} \right), \quad 0 \leq p \leq 1. \]

Therefore, the spike process under this analysis basis is completely dense, i.e., \( C_p \geq n \) for \( 0 \leq p \leq 1 \) and the equality holds if and only if \( p = 0 \). Yet this is still the LSDB.

Finally, from Theorems 4.3 and 4.6, we can prove the following corollary:

**Corollary 4.9.** There is no invertible linear transformation providing the statistically independent coordinates for the spike process for \( n > 2 \).

### 5 The Generalized Spike Process

In [10], Donoho et al. analyzed the following generalization of the simple spike process in terms of the KLB and the rate distortion function. This process first picks one coordinate out of \( n \) coordinates randomly as before, but then the amplitude of this single spike is picked according to the standard normal distribution \( N(0,1) \). The pdf of this process can be written as follows:

\[ f_{X}(x) = \frac{1}{n} \sum_{i=1}^{n} \left( \prod_{j \neq i} \delta(x_j) \right) g(x_i), \quad (8) \]

where \( \delta(\cdot) \) is the Dirac delta function, and \( g(x) = (1/\sqrt{2\pi}) \cdot \exp(-x^2/2) \), i.e., the pdf of the standard normal distribution. Figure 1 shows this pdf for \( n = 2 \). Interestingly enough, this generalized spike process shows rather different behavior (particularly in the statistical independence) from the simple spike process in Section 4. We also note that our proofs here are rather analytical compared to those for the simple spike process presented in [3], which have more combinatorial flavor.

#### 5.1 The Karhunen-Loève Basis

We can easily compute the covariance matrix of this process, which is proportional to the identity matrix. In fact, it is just \( I_n/n \). Therefore, we have the following proposition, which was also stated without proof by Donoho et al. [10]:

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Proposition 5.1. The Karhunen-Loève basis for the generalized spike process is any orthonormal basis in $\mathbb{R}^n$.

Proof. Let us first compute the marginal pdf of (8). By integrating out all $x_i, i \neq j$, we can easily get:

$$f_{X_j}(x_j) = \frac{1}{n} g(x_j) + \frac{n-1}{n} \delta(x_j).$$

Therefore, we have $E[X_j] = 0$. Now, if $X_i$ and $X_j$ cannot be simultaneously nonzero, therefore,

$$E[X_i X_j] = \delta_{ij} E[X_j^2] = \frac{1}{n} \delta_{ij},$$

since the variance of $X_j$ is 1. Therefore, the covariance matrix of this process is, as announced, $I_n/n$. Therefore, any orthonormal basis is the KLB. \qed

In other words, the KLB for this process is less restrictive than that for the simple spike process (Proposition 4.1), and the KLB is again completely useless for this process.
5.2 Marginal distributions and moments under $\text{SL}^\pm(n, \mathbb{R})$

Before analyzing the BSB and LSDB, we need some background work. First, let us compute the pdf of the process relative to a transformation $Y = B^{-1}X$, $B \in \text{SL}^\pm(n, \mathbb{R})$. In general, if $Y = B^{-1}X$, then

$$f_Y(y) = \frac{1}{|\det B^{-1}|} f_X(By).$$

Therefore, from (8), and the fact $|\det B| = 1$, we have

$$f_Y(y) = \frac{1}{n} \sum_{i=1}^{n} \left( \prod_{j \neq i} \delta(r_i^T y) \right) g(r_i^T y), \quad (9)$$

where $r_i^T$ is the $j$th row vector of $B$. As for its marginal pdf, we have the following lemma:

**Lemma 5.2.**

$$f_{Y_j}(y) = \frac{1}{n} \sum_{i=1}^{n} g(y; |\Delta_{ij}|), \quad j = 1, \ldots, n, \quad (10)$$

where $\Delta_{ij}$ is the $(i, j)$th cofactor of matrix $B$, and $g(y; \sigma) = g(y/\sigma)/\sigma$ represents the pdf of the normal distribution $\mathcal{N}(0, \sigma^2)$.

In other words, one can interpret the $j$th marginal pdf as a mixture of Gaussians with the standard deviations $|\Delta_{ij}|$, $i = 1, \ldots, n$. Figure 2 shows several marginal pdf’s for $n = 2$. As one can see from this figure, it can vary from a very spiky distribution to a usual normal distribution depending on the rotation angle of the coordinate.

*Proof.* Let us rewrite (9) as

$$f_Y(y) = \frac{1}{n} \sum_{i=1}^{n} \delta(r_1^T y) \cdots \delta(r_{i-1}^T y) \delta(r_i^T y) \cdots \delta(r_{i+1}^T y) \cdots \delta(r_n^T y) g(r_i^T y). \quad (11)$$

The $j$th marginal pdf can be written as

$$f_{Y_j}(y_j) = \int f_Y(y_1, \ldots, y_n) dy_1 \cdots dy_{j-1} dy_{j+1} \cdots dy_n.$$
Figure 2: The marginal pdf’s of the generalized spike process \((n = 2)\). All the pdf’s shown here are projections of the 2D pdf in Figure \(\square\) onto the rotated 1D axis. The axis angle in the top row is 0.088 rad., which is close to the the first axis of the standard basis. The axis angle in the bottom row is \(\pi/4\) rad., i.e., 45 degree rotation, which gives rise to the exact normal distribution. The other axis angles are equispaced angles between these two.
Consider the $i$th term in the summation of (11) and integrate it out with respect to $y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n$:
\[
\int \delta(r^T_i y) \cdots \delta(r^T_{i-1} y) \delta(r^T_{i+1} y) \cdots \delta(r^T_n y) g(r^T_i y) dy_1 \cdots dy_{j-1} dy_{j+1} \cdots dy_n.
\]
(12)

We use a change of variable formula to integrate this. Let $r^T_k y = x_k, k = 1, \ldots, n$, and let $b_\ell$ be the $\ell$th column vector of $B$. The relationship $B y = x$ can be rewritten as follows:

\[
B^{(i,j)} y^{(i)} + y_j b_j^{(i)} = x^{(i)},
\]

where $B^{(i,j)}$ is the $(n-1) \times (n-1)$ matrix by removing $i$th row and $j$th column, and the vectors with superscripts indicate the length $n-1$ column vectors by removing the elements whose indices are specified in the parentheses. This means that

\[
y^{(j)} = (B^{(i,j)})^{-1} (x^{(i)} - y_j b_j^{(i)}).
\]

Thus,

\[
dy^{(j)} = dy_1 \cdots dy_{j-1} dy_{j+1} \cdots dy_n = \frac{1}{\mid \det B^{(i,j)} \mid} dx^{(i)} = \frac{1}{\mid \Delta_{ij} \mid} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n.
\]

Let us now express $r^T_i y = x_i$ in terms of $y_j$ and $x$.

\[
r^T_i y = \begin{pmatrix} r^{(j)}_i \end{pmatrix}^T y^{(j)} + b_{ij} y_j = \begin{pmatrix} r^{(j)}_i \end{pmatrix}^T (B^{(i,j)})^{-1} (x^{(i)} - y_j b_j^{(i)}) + b_{ij} y_j = \begin{pmatrix} r^{(j)}_i \end{pmatrix}^T (B^{(i,j)})^{-1} x^{(i)} + y_j \left( b_{ij} - \left( \begin{pmatrix} r^{(j)}_i \end{pmatrix}^T (B^{(i,j)})^{-1} b_j^{(i)} \right) \right) = \begin{pmatrix} r^{(j)}_i \end{pmatrix}^T (B^{(i,j)})^{-1} x^{(i)} + \frac{y_j}{\Delta_{ij}} \det B = \begin{pmatrix} r^{(j)}_i \end{pmatrix}^T (B^{(i,j)})^{-1} x^{(i)} + \frac{y_j}{\Delta_{ij}},
\]

\[(\ast)\]
where (*) follows from the following lemma whose proof is shown in Appendix A.

**Lemma 5.3.** For any $B = (b_{ij}) \in \text{GL}(n, \mathbb{R})$,

$$b_{ij} - \left( r_i^{(j)} \right)^T (B^{(i,j)})^{-1} b_j^{(i)} = \frac{1}{\Delta_{ij}} \text{det } B, \quad 1 \leq i, j \leq n.$$ 

Now, let us go back to the integration (12). Thanks to the property of the delta function with Equation (13), we have

$$\int \cdots \int \delta(x_1) \cdots \delta(x_{i-1}) \delta(x_{i+1}) \cdots \delta(x_n) g(r_i^T y) \frac{1}{\Delta_{ij}} dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n$$

$$= \frac{1}{\Delta_{ij}} g(\pm y_j / \Delta_{ij})$$

$$= g(y_j; |\Delta_{ij}|),$$

where we used the fact that $g(\cdot)$ is an even function. Therefore, we can write the $j$th marginal distribution as announced in (10).

Let us now compute the moments of $Y_i$, which will be used later. We use the fact that this is a mixture of $n$ Gaussians each of which has mean 0 and variance $|\Delta_{ij}|^2$. Therefore, it is obvious to have $E[Y_i] = 0$ for all $i = 1, \ldots, n$. Now we have the following lemma for the moments.

**Lemma 5.4.**

$$E[|Y_j|^p] = \frac{\Gamma(p)}{n2^{p/2-1}\Gamma(p/2)} \sum_{i=1}^{n} |\Delta_{ij}|^p, \quad \text{for all } p > 0. \quad (14)$$

**Proof.** We have:

$$E[|Y_j|^p] = \frac{1}{n} \sum_{i=1}^{n} \int_{-\infty}^{\infty} |y|^p g(y; |\Delta_{ij}|) dy$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sqrt{\frac{2}{\pi}} |\Delta_{ij}|^p \Gamma(1 + p) D_{-p}(0)$$

by Gradshteyn and Ryzhik [11, Formula 3.462.1], where $D_{-p}(\cdot)$ is Whittaker’s function as defined by Abramowitz and Stegun [1, pp.687]:

$$D_{-a-1/2}(0) = U(a, 0) = \frac{\sqrt{\pi}}{2a^{2+1/4} \Gamma(a/2 + 3/4)}.$$
Thus, putting $a = p + 1/2$ to the above equation yields:

$$D_{-1-p}(0) = \frac{\sqrt{\pi}}{2^{1/2+p/2} \Gamma(1 + p/2)}.$$  

Therefore, we have

$$E[|Y_j|^p] = \frac{1}{n} \sum_{i=1}^{n} |\Delta_{ij}|^p \frac{\Gamma(1 + p)}{2^{p/2} \Gamma(1 + p/2)}$$

$$= \frac{1}{n} \sum_{i=1}^{n} |\Delta_{ij}|^p \frac{\Gamma(p)}{2^{p/2-1} \Gamma(p/2)}$$

$$= \frac{\Gamma(p)}{n 2^{p/2-1} \Gamma(p/2)} \sum_{i=1}^{n} |\Delta_{ij}|^p,$$

as we desired.  

\[ \square \]

5.3 The Best Sparsifying Basis

As for the BSB, after all, there is no difference between the generalized spike process and the simple spike process.

**Theorem 5.5.** The BSB with any $p \in [0, 1]$ for the generalized spike process is the standard basis if $D = O(n)$ or $\text{SL}^\pm(n, \mathbb{R})$.

**Proof.** Let us first consider the case $p \in (0, 1]$. Then, using Lemma 5.4, the cost function (2) can be rewritten as follows:

$$C_p(B \mid x) = \sum_{j=1}^{n} E[|Y_j|^p] = \frac{\Gamma(p)}{n 2^{p/2-1} \Gamma(p/2)} \sum_{i=1}^{n} \sum_{j=1}^{n} |\Delta_{ij}|^p.$$  

Let us now define a matrix $\tilde{B} \triangleq (\Delta_{ij})$. Then $\tilde{B} \in \text{SL}^\pm(n, \mathbb{R})$ since

$$B^{-1} = \frac{1}{\det B} (\Delta_{ji}) = \pm (\Delta_{ji}),$$

and $B^{-1} \in \text{SL}^\pm(n, \mathbb{R})$. Therefore, this reduces to

$$C_p(B \mid x) = \frac{\Gamma(p)}{n 2^{p/2-1} \Gamma(p/2)} \sum_{i=1}^{n} \sum_{j=1}^{n} |\tilde{b}_{ij}|^p = C_p(\tilde{B} \mid x).$$
This means that our problem now becomes the same as Theorem 1 in [3] (or Theorem 4.2 in this paper) by replacing $B$ by $\tilde{B}$. Thus, it asserts that the $\tilde{B}$ must be the identity matrix $I_n$ or its permuted or sign flipped versions. Suppose $\Delta_{ij} = \delta_{ij}$. Then, $B^{-1} = \pm (\Delta_{ji}) = \pm I_n$, which implies that $B = \pm I_n$. If $(\Delta_{ji})$ is any permutation matrix, then $B^{-1}$ is just that permutation matrix or its sign flipped version. Therefore, $B$ is also a permutation matrix or its sign flipped version.

Finally, let us consider the case $p = 0$. Then, any linear invertible transformation except the identity matrix or its permuted or sign-flipped versions clearly increases the number of nonzero elements after the transformation. Therefore, the BSB with $p = 0$ is also a permutation matrix or its sign flipped version.

This completes the proof of Theorem 5.5.

5.4 The LSDB/KMB among $O(n)$

As for the LSDB/KMB, we can see some difference from the simple spike process.

Let us now consider a more specific case of $D = O(n)$. So far, we have been unable to prove the following conjecture.

**Conjecture 5.6.** The LSDB among $O(n)$ is the standard basis.

The difficulty is the evaluation of the sum of the marginal entropies (2) for the pdf’s of the form (10). However, a major simplification occurs if we consider the KMB instead of the LSDB, and we can prove the following:

**Theorem 5.7.** The KMB among $O(n)$ is the standard basis.

**Proof.** Because $E[Y_j] = 0$ and $E[Y_j^2] = \frac{1}{n} \sum_{i=1}^n \Delta_{ij}^2$ for all $j$, the fourth order central moment of $Y_j$ can be written as $\mu_4(Y_j) = \frac{3}{n} \sum_{i=1}^n \Delta_{ij}^4$, and consequently the cost function in (4) becomes

$$C_\kappa(B | X) = \frac{3}{n} \sum_{j=1}^n \sum_{i=1}^n \Delta_{ij}^4 - \frac{1}{n} \left( \sum_{i=1}^n \Delta_{ij}^2 \right)^2.$$  \hspace{1cm} (15)

Note that this is true for any $B \in SL^\pm(n, \mathbb{R})$. If we restrict our basis search within $O(n)$, another major simplification occurs because we have the following special relationship between $\Delta_{ij}$ and the matrix element $b_{ji}$ of $B \in O(n)$:

$$B^{-1} = \frac{1}{\det B} (\Delta_{ji}) = B^T.$$  

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In other words,
\[
\Delta_{ij} = (\det B)b_{ij} = \pm b_{ij}.
\]

Therefore, we have
\[
\sum_{i=1}^{n} \Delta_{ij}^2 = \sum_{i=1}^{n} b_{ij}^2 = 1.
\]

Inserting this into (15), we get the following simplified cost for \( D = O(n) \):
\[
\mathcal{C}_\kappa(B | X) = -\frac{3}{n} \left( 1 - \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta_{ij}^4 \right).
\]

This means that the KMB can be rewritten as follows:
\[
B_\kappa = \arg \max_{B \in O(n)} \sum_{i,j} b_{ij}^4.
\]

(16)

Let us note that the existence of the maximum is guaranteed because the set \( O(n) \) is compact and the cost function \( \sum_{i,j} b_{ij}^4 \) is continuous.

Now, let us consider a matrix \( P = (p_{ij}) = (b_{ij}^2) \). Then, from the orthonormality of columns and rows of \( B \), this matrix \( P \) belongs to a set of doubly stochastic matrices \( S(n) \). Since doubly stochastic matrices obtained by squaring the elements of \( O(n) \) consist of a proper subset of \( S(n) \), we have
\[
\max_{B \in O(n)} \sum_{i,j} b_{ij}^4 \leq \max_{P \in S(n)} \sum_{i,j} p_{ij}^2.
\]

Now, we prove that such \( P \) must be an identity matrix or its permuted version.
\[
\max_{P \in S(n)} \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij}^2 \leq \sum_{j=1}^{n} \left( \max_{P \in S(n)} \sum_{i=1}^{n} p_{ij}^2 \right)
= \sum_{j=1}^{n} 1
= n,
\]

where the first equality follows from the fact that maxima of the radius of the sphere \( \sum_{i} p_{ij}^2 \) subject to \( \sum_{i} p_{ij} = 1, p_{ij} \geq 0 \) occur only at the vertices of
that simplex, i.e., $p_j = e_{\sigma(j)}$, $j = 1, \ldots, n$ where $\sigma(\cdot)$ is a permutation of $n$ items. That is, the column vectors of $P$ must be the standard basis vectors. This implies that the matrix $B$ corresponding to $P = I_n$ or its permuted version must be either $I_n$ or its permuted and/or sign-flipped version.

5.5 The LSDB/KMB among $\text{SL}^\pm(n, \mathbb{R})$

If we extend our search to this more general case, we have the following theorem.

**Theorem 5.8.** The KMB among $\text{SL}^\pm(n, \mathbb{R})$ does not exist.

**Proof.** The set $\text{SL}^\pm(n, \mathbb{R})$ is not compact. Therefore, there is no guarantee that the cost function $\mathcal{C}_\kappa(B \mid X)$ has a minimum value on this set. One can in fact consider a simple counter-example, $B = \text{diag}(a, a^{-1}, 1, \ldots, 1)$, where $a$ is any nonzero real scalar. Then, one can show that $\mathcal{C}_\kappa(B \mid X) = -(a^4 + a^{-4} + n - 2)$, which tends to $-\infty$ as $a \uparrow \infty$.

As for the LSDB, we do not know whether the LSDB exists among $\text{SL}^\pm(n, \mathbb{R})$ at this point, although we believe that the LSDB is the standard basis (or its permuted/sign-flipped versions). The negative result in the KMB does not imply the negative result in the LSDB.

6 Discussion

Unlike the simple spike process, the BSB and the KMB (an alternative to the LSDB) selects the standard basis if we restrict our basis search within $O(n)$. If we extend our basis search to $\text{SL}^\pm(n, \mathbb{R})$, then the BSB exists and is again the standard basis whereas the KMB does not exist.

Although the generalized spike process is a simple stochastic process, we have the following important interpretation. Consider a stochastic process generating a basis vector randomly selected from some fixed orthonormal basis and multiplied by a scalar varying as the standard normal distribution at a time. Then, both that basis itself is the BSB and the KMB among $O(n)$. Theorems 5.5 and 5.7 claim that once we transform the data to the generalized spikes, one cannot do any better than that both in sparsity and independence within $O(n)$. Of course, if one extends the search to nonlinear transformations, then it becomes a different story. We refer the reader to our recent articles [14], [15], for the details of a nonlinear algorithm.
The results of this paper further support our conclusion of the previous paper: dealing with the BSB is much simpler than the LSDB. To deal with statistical dependency, we need to consider the probability law of the underlying process (e.g., entropy or the marginal pdf’s) explicitly. That is why we need to consider the KMB instead of the LSDB to prove the theorems. Also in practice, given a finite set of training data, it is a nontrivial task to reliably estimate the marginal pdf’s. Moreover, the LSDB unfortunately cannot tell how close it is to the true statistical independence; it can only tell that it is the best one (i.e., the closest one to the statistical independence) among the given set of possible bases. In order to quantify the absolute statistical dependence, we need to estimate the true high-dimensional entropy of the original process, $H(X)$, which is an extremely difficult task in general. We would like to note, however, a recent attempt to estimate the high-dimensional entropy of the process by Hero and Michel [12], which uses the minimum spanning trees of the input data and does not require to estimate the pdf of the process. We feel that this type of techniques will help assessing the absolute statistical dependence of the process under the LSDB coordinates. Another interesting observation is that the KMB is rather sensitive to the orthonormality of the basis dictionary whereas the BSB is insensitive to that. Our previous results on the simple spike process (e.g., Theorems 4.3, 4.6) also suggest the sensitivity of the LSDB to the orthonormality of the basis dictionary. This may restrict and discourage us to develop a new basis or a new basis dictionary that optimize the statistical independence.

On the other hand, the sparsity criterion neither requires estimating the marginal pdf’s nor reveals the sensitivity to the orthonormality. Simply computing the expected $\ell^p$ norms suffices. Moreover, one can even adapt the BSB for each realization rather than for the whole realizations, which is impossible for the LSDB, as we discussed in [3], [22], [21].

These observations, therefore, suggest that the pursuit of sparse representations should be encouraged rather than that of statistically independent representations, if we believe that mammalian vision systems were evolved and developed by the principle of data compression. This is also the viewpoint indicated by Donoho [8].

Finally, there are a few interesting generalizations of the spike processes, which need to be addressed in the near future. We need to consider a stochastic process that randomly throws in multiple spikes to a single realization. If one throws in more and more spikes to one realization, the standard basis is getting worse in terms of sparsity. Also, we can consider various rules to
throw in multiple spikes. For example, for each realization, we can select the locations of the spikes statistically independently. This is the simplest multiple spike process. Alternatively, we can consider a certain dependence in choosing the locations of the spikes. The ramp process of Yves Meyer analyzed by the wavelet basis is such an example; each realization of the ramp process generates a small number of spikes in the wavelet coefficients in the locations determined by the location of the discontinuity of the process. See \cite{1, 10, 16, 22} for more about the ramp process.

Unless very special circumstances, it would be extremely difficult to find the BSB of a complicated stochastic process (e.g., natural scene images) that truly converts its realizations to the spike process. More likely, a theoretically and computationally feasible basis that sparsifies the realizations of a complicated process well (e.g., curvelets for the natural scene images \cite{1}) may generate expansion coefficients that may be viewed as an amplitude-varying multiple spike process. In order to tackle this scenario, we certainly need to: 1) identify interesting, useful, and simple enough specific stochastic processes; 2) develop the BSB adapted to such specific processes; and 3) deepen our understanding of the amplitude-varying multiple spike process.

Acknowledgment

I would like to thank the fruitful discussions with Dr. Motohico Mulase and Dr. Roger Wets, of UC Davis. This research was partially supported by NSF DMS-99-73032, DMS-99-78321, and ONR YIP N00014-00-1-046.

A Proof of Lemma 5.3

Proof. Let us consider the following system of linear equations:

\[ B^{(i,j)} z^{(j)} = b_j^{(i)}, \]
where \( z^{(j)} = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)^T \in \mathbb{R}^{n-1}, j = 1, \ldots, n \). Using Cramer’s rule (e.g., [13, pp.21]), we have, for \( k = 1, \ldots, j-1, j+1, \ldots, n \),

\[
    z_k^{(j)} = \frac{1}{\det B^{(i,j)}} \det \begin{bmatrix} b_i^{(i)} & \cdots & b_k^{(i)} & b_{k-1}^{(i)} & \cdots & b_n^{(i)} \end{bmatrix}
\]

\[
\begin{aligned}
    &\stackrel{(a)}{=} (-1)^{|k-j|-1} \frac{B^{(i,k)}}{B^{(i,j)}} \\
    &\stackrel{(b)}{=} (-1)^{|k-j|-1} \frac{\Delta_{ik}/(-1)^{i+k}}{\Delta_{ij}/(-1)^{i+j}} \\
    &\quad = -\frac{\Delta_{ik}}{\Delta_{ij}},
\end{aligned}
\]

where (a) follows from the \((|k-j|-1)\) column permutations to move \( b_j^{(i)} \) located at the \( k \)th column to the \( j \)th column of \( B^{(i,j)} \), and (b) follows from the definition of the cofactor. Hence,

\[
    b_{ij} - \left( r_i^{(j)} \right)^T (B^{(i,j)})^{-1} b_j^{(i)} = b_{ij} - \left( r_i^{(j)} \right)^T z^{(j)}
\]

\[
\begin{aligned}
    &= b_{ij} + \frac{1}{\Delta_{ij}} \sum_{k \neq j} b_{ik} \Delta_{ik} \\
    &= \frac{1}{\Delta_{ij}} \sum_{k=1}^n b_{ik} \Delta_{ik} \\
    &= \frac{1}{\Delta_{ij}} \det B.
\end{aligned}
\]

This completes the proof of Lemma 5.3.

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