Abstract

It is shown that electromagnetic potentials convey physical information beyond that supplied by electric and magnetic fields alone, and are thus more fundamental. Observable physical properties can impose conditions on the selection of electromagnetic gauge (i.e. sets of potentials) that are explicit and restrictive. This is true both classically and quantum mechanically. The implication that the choice of gauge carries physical information is confirmed by exhibiting a set of potentials that describes fields correctly, but that violates physical constraints. The basic conclusions are that physical requirements place limits on acceptable gauges; and that potentials are more fundamental than fields in both classical and quantum physics, representing a major generalization of the quantum-only Aharonov–Bohm effect. These important properties are obscured if the dipole approximation is employed. The properties demonstrated here relate directly to conditions that exist in strong-field laser applications.

Keywords: electromagnetic gauge, strong fields, relativistic effects, potentials over fields, Aharonov–Bohm effect, unphysical potentials

1. Introduction

Electric and magnetic fields of electrodynamics can be represented by scalar and vector potentials. Particular derivatives of these potentials will generate the fields. These potentials are not unique, with alternative sets of potentials connected by a mathematical procedure called a gauge transformation. The conventional point of view is that physical processes depend only on the fields, and the potentials can be regarded as nothing more than useful auxiliary quantities [1]. An important exception to this rule is the Aharonov–Bohm effect [2, 3], in which a charged particle passing near a solenoid containing a magnetic field can be deflected by the potential that exists outside the solenoid in a field-free region. The effect is particular to quantum mechanics, and provides only a cautionary limitation to the general notion that fields are more basic than potentials.

It is shown here that such a ubiquitous phenomenon as a laser field imposes strong limitations on possible gauge transformations when the demand is made that the propagation property of the laser field must be sustained. This limitation applies both classically and quantum mechanically. It is further shown that if the laser field (a transverse field) impinges on a charged particle that is simultaneously subjected to a Coulomb binding potential (a longitudinal field), then the unique allowable gauge is the radiation gauge (also known as the Coulomb gauge) if all physical constraints are to be satisfied. This combination of transverse and longitudinal fields pervades atomic, molecular, and optical (AMO) physics. The common AMO practice of using the dipole approximation in the description of laser-induced effects amounts to replacing the transverse field of a laser by the fundamentally different longitudinal field, thus altering basic physical constraints.

A brief review of the basic features of gauge transformations is given in the next section. The standard requirement is that the generating function for the gauge transformation can be any scalar function that satisfies the homogeneous wave equation [1]. Section 3 considers the important case of a propagating field. This application applies to all transverse
fields, including laser fields. It is shown that the only possible departure from the familiar radiation gauge must be such that the 4-vector potential describing the field can have added to it only a contribution that depends on the light-cone coordinates appropriate to that field. This is an important supplement to the standard gauge requirements of electromodynamics.

Section 4 examines the more restrictive case where a charged particle is simultaneously subjected to a transverse field (like a laser field) and a longitudinal field (like a Coulomb binding potential). Since this combination of fields describes most AMO situations, the basic electrodynamic principles established here are directly applicable to strong-field laser experiments. The operative limitation in this case comes from the properties of the relativistic quantum equations of motion and the inferences that persist in the nonrelativistic limit. These considerations are not important for laser fields in the perturbative domain, but they are applicable under the conditions that exist when fields are strong enough to require nonperturbative methods. The reason is that laser fields propagate at the speed of light, thereby introducing relativistic considerations even into nominally nonrelativistic problems [4]. Relativistic conditions exist and must be properly accounted for in strong-field applications, since they signal the importance of the magnetic field in ways that are invisible within the dipole approximation.

Section 5 exhibits the general conclusion that potentials are more fundamental than the fields that are derived from them, by exhibiting two sets of potentials that describe exactly the same electric and magnetic fields; but one set satisfies all physical requirements, and the other set gives incorrect predictions for such basic matters as the propagation property, Lorentz symmetries, and the ponderomotive potential of a charged particle in the field. This marks a major generalization of the important Aharonov–Bohm effect, presently the sole practical example of the dominance of potentials over fields.

The final section is an overview of the essential results, including an appraisal of the practical consequences of the results arrived at here. A simple summary is that ‘physical intuition’ or ‘physical interpretation’ is dependent on the choice of electromagnetic gauge. The use of the dipole approximation severely limits those benefits and can lead to the adoption of physical pictures that do not match laboratory reality. A leading example is the ‘tunneling limit’ that envisions a very low frequency laser field as a nearly static electric field, in contrast to the actuality of a laser field that propagates at the velocity of light for all frequencies, and cannot possibly have a static limit. It is emphasized that some seriously misdirected criteria have been adopted, unchallenged, in strong-field physics.

2. Basic gauge transformation

For notational simplicity only vacuum conditions are considered, and Gaussian units are employed. The electric field \( \mathbf{E} \) and the magnetic field \( \mathbf{B} \) can be represented by the scalar potential \( \phi \) and the vector potential \( \mathbf{A} \) as

\[
\mathbf{E} = -\nabla \phi - \frac{1}{c} \partial_t \mathbf{A},
\]

\[
\mathbf{B} = \nabla \times \mathbf{A}.
\]

A gauge transformation of the \( \phi, \mathbf{A} \) potentials to a new set \( \tilde{\phi}, \tilde{\mathbf{A}} \) can be achieved with the scalar generating function \( \Lambda \) with the connection that

\[
\tilde{\phi} = \phi + \frac{1}{c} \partial_t \Lambda,
\]

\[
\tilde{\mathbf{A}} = \mathbf{A} - \nabla \Lambda.
\]

The substitution of equations (3) and (4) into (1) and (2) leaves the field expressions unchanged. The only constraint on \( \Lambda \) is that it should satisfy the homogeneous wave equation. This is to enable a decoupling of the equations for the scalar and vector potentials. Relativistic notation is useful. The 4-vector potential that encompasses both the scalar and 3-vector potentials is

\[
A^\mu : (\phi, \mathbf{A})
\]

and the basic spacetime 4-vector is

\[
x^\mu : (ct, \mathbf{r}).
\]

The expressions (3) and (4) are then subsumed into the single expression

\[
\tilde{\mathbf{A}}^\mu = A^\mu + \partial^\mu \Lambda
\]

subject to the constraint on \( \Lambda \) that it must satisfy

\[
\partial^\mu \partial_\mu \Lambda = 0.
\]

3. Gauge limitation for transverse fields

Now the important example is considered wherein \( A^\mu \) represents a transverse field, with the equivalent terminologies that it is a propagating field or a plane-wave field. Such a field propagates in vacuum with the speed of light \( c \), with the additional proviso of special relativity that this speed of propagation must be the same in all inertial frames of reference. This is equivalent to the statement (see, for example, \([5, 6]\)) that any occurrence of \( x^\mu \) in the potential \( A^\mu \) can only be in the form of the scalar product with the propagation 4-vector \( k^\mu \):

\[
\varphi \equiv k^\mu x_\mu,
\]

where

\[
k^\mu : \left( \frac{\omega}{c}, \mathbf{k} \right),
\]

\[
|\mathbf{k}| = \frac{\omega}{c}.
\]

In other words, \( k^\mu \) is a lightlike 4-vector with the 3-vector component \( \mathbf{k} \) in the propagation direction of the transverse field. The 4-vector potential \( A^\mu \) can depend on the spacetime 4-vector \( x^\mu \) only as \( A^\mu (\varphi) \), and the same condition must
apply to $\tilde{A}^\mu$. Therefore, equation (7) requires that $\partial^\nu A$ must also be a function of $\varphi$ alone. This requirement can be satisfied by the condition that $A$ depends on $x^\mu$ only as $A = A(\varphi)$, since then

$$\frac{d}{d\varphi} A(\varphi) = \frac{d}{d\varphi} A(\varphi) = k^\nu \tilde{N}(\varphi),$$

(12)

where $\tilde{N}$ is the total derivative of $A$ with respect to $\varphi$. The gauge-transformed 4-vector potential $\tilde{A}^\mu$ can therefore differ from the original $A^\mu$ only by a quantity that lies on the light-cone, since the gauge transformation condition (7) must be of the form

$$\tilde{A}^\mu = A^\mu + k^\nu \tilde{N}.$$  

(13)

The condition (13) is very restrictive. One important consequence is that the squared 4-vector potential is gauge invariant [7, 8], as follows from the light-cone condition

$$k^\mu k_\mu = 0$$  

(14)

and the transversality condition

$$k^\nu A_\mu = 0,$$  

(15)

so that equation (13) leads to

$$\tilde{A}^\mu \tilde{A}_\mu = A^\mu A_\mu.$$  

(16)

The ponderomotive potential of a charged particle in a transverse field is proportional to $A^\mu A_\mu$, meaning that this fundamentally important quantity [7, 8] is gauge invariant.

An important caveat is that the use of the dipole approximation, a standard procedure in AMO physics, has the effect of losing altogether the propagation property of a laser field. The dipole approximation amounts to the replacement of the propagating, transverse field by a simple oscillatory electric field, with the significance that the basic condition of equation (13) is discarded. That is, equation (7) no longer leads to equation (13) when the dipole approximation is employed.

It is well-known from long experience in nuclear and high energy physics that calculations of the effects of plane-wave fields on charged particles can be successfully applied in the context of the radiation gauge. A convenient way to describe the radiation gauge is that it is the gauge within which a pure transverse field is described by the 3-vector potential $A$ alone, and a pure longitudinal field is described by a scalar potential $\phi$ (or $A^0$) alone.

4. Combined transverse and longitudinal fields

The ionization of an atom by a laser field typifies AMO processes. An atomic electron in a laser field experiences both the transverse field of the laser and the longitudinal field of the binding potential. The dipole approximation has been useful in AMO physics since it offers the simplifying property that the laser field is replaced by an oscillatory electric field, thus substituting another longitudinal field for the transverse field. That is, traditional AMO physics replaces a combination of a transverse laser field and a longitudinal binding potential by two longitudinal fields. This is true whether the so-called ‘length gauge’ is used, where the interaction Hamiltonian is of the form $\mathbf{r} \cdot \mathbf{E}(t)$ or by the gauge-equivalent [9] ‘velocity gauge’ where the interaction Hamiltonian contains $A(t) \cdot \mathbf{p}$.

When laser fields are very strong, the fact that the field propagates with the velocity of light becomes an important feature [4]. Replacement of the propagating field by the dipole-equivalent oscillatory electric field is no longer sufficient. Even when magnetic forces remain small, the complete neglect of the magnetic field removes all possibility of propagation. This has major importance in practical applications. For example, the laboratory detection of ‘above-threshold ionization’ (ATI) in 1979 [10], where the perturbation-theory dominance of the lowest order process gives way to the participation of higher orders of interaction with the applied field, caused a major sensation in the AMO community and triggered theoretical efforts lasting more than a decade (see the introductory remarks in [11]) to achieve some understanding of how this could happen in a dipole approximation context. By contrast, a theory based on the nonrelativistic limit of a relativistically formulated theory [4] provided an anticipatory prediction of all of the ATI features [12] in a theory paper prepared in advance of the observation of ATI. It is important to note that a nonrelativistic limit of a relativistic theory leads to analytical forms that resemble theories based on a priori employment of the dipole approximation, but the seemingly slight differences are nevertheless critical.

A laser-induced phenomenon that is unquestionably relativistic is the production of electron–positron pairs. It was predicted in 1971 [13] and confirmed in 1997 [14] that this was possible with a laser wavelength of the order of 1 μm at a focused field intensity of about $10^{18}$ W cm$^{-2}$. Many laboratories can now produce such intensities; it is simply a confirmation of the need to recognize the relativistic foundations of laser effects.

Reduction of a relativistic treatment to a nonrelativistic limit of the effects of combined transverse and longitudinal fields introduces a feature that had not been anticipated. Following the usual practice of neglecting effects of the spin of the electron, the universally employed relativistic description of the electron is the Klein–Gordon (KG) equation [15, 16]

$$\left[\left(\frac{i\hbar}{c}\partial_\mu - \frac{q}{c} A_\mu\right)\left(i\hbar \partial^\mu - \frac{q}{c} A^\mu\right) - m^2 c^2\right] \psi = 0.$$  

(17)

A separation of time and space parts gives the form

$$\left[\left(\frac{i\hbar}{c}\partial_\mu - \frac{q}{c} A^0\right)^2 - \left(-i\hbar \nabla - \frac{q}{c} \mathbf{A}\right)^2 - m^2 c^2\right] \psi = 0.$$  

(18)

It is the first parenthesis in the square bracket of equation (18) that is of interest here, since $A^0$ can represent the time part of the laser field 4-vector potential $A^\mu_{PW}$ (where the subscript PW stands for ‘plane-wave’) as well as a binding potential $V$ that may be present:

$$A^0 = A^\mu_{PW} + V.$$  

(19)
When $V$ is a Coulomb binding potential

$$V \sim 1/r,$$  \hspace{1cm} (20)

this singularity causes problems [17] in the reduction of the KG equation to the Schrödinger equation in the nonrelativistic limit [18]. The act of squaring indicated for the first term in equation (18) introduces a cross coupling $V\Delta_{PW}^0$, that is also singular at the origin of spatial coordinates, but it is an unacceptable term because the magnitude of this singular term depends upon the properties of the laser field. This unphysical behavior will not occur if

$$A^0_{PW} = 0,$$  \hspace{1cm} (21)

which corresponds to selection of the radiation gauge (also known as the Coulomb gauge) wherein longitudinal fields are represented by scalar potentials and transverse fields are represented by 3-vector potentials. If this gauge selection must be enforced in a relativistic problem, that gauge must also refer to the nonrelativistic limit.

The same considerations arising in the reduction of the KG equation to the Schrödinger equation in the nonrelativistic limit applies also to the reduction of the Dirac equation to the Pauli equation for a spin-$\frac{1}{2}$ particle. The is most easily seen from the second-order form of the Dirac equation [16, 19]:

$$\left[i\hbar \partial_\mu - \frac{e}{c} A_\mu \right] \left(i\hbar \partial^\mu - \frac{e}{c} A^\mu \right) + \frac{1}{2} \hbar c \sigma^{\mu\nu} F_{\mu\nu} - m^2 c^2 \psi = 0,$$  \hspace{1cm} (22)

$$\sigma^{\mu\nu} = \frac{1}{2i} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu),$$  \hspace{1cm} (23)

where the $\gamma^\mu$ are the standard Dirac matrices and $F^{\mu\nu}$ is the electromagnetic field tensor. This equation for a spin-$\frac{1}{2}$ particle is the same as the KG equation (17) for a spin-0 particle, but with the addition of a term representing the spin interaction. It presents the same dilemma in reduction to the Pauli equation as does the KG equation in reduction to the Schrödinger equation. The second-order Dirac equation has the dual advantages of its similarity to the KG equation, as well as lacking the Zitterbewegung problem [16, 20] of the first order Dirac equation in reduction to the nonrelativistic limit.

Elimination of the unphysical coupling of the laser field to a singular quantity means that

$$A^0_{PW} = 0,$$  \hspace{1cm} (24)

a feature of the radiation gauge, is a general requirement for attainment of the correct equation of motion. Explicitly, since equation (13) is a general requirement for a propagating field, and it has now been shown that the additional presence of a binding potential requires that any $A^0$ must also vanish, the condition (13) means that

$$A' = 0, \quad \Lambda = \text{constant}$$  \hspace{1cm} (25)

must hold true. That is, no departure from the radiation gauge for the transverse field is allowable if all field conditions are to be satisfied exactly.

None of the above reasoning arises if the dipole approximation is imposed from the outset. This does not mean that the dipole approximation is more convenient because of this; it means rather that the dipole approximation infers a measure of approximation beyond the usual interpretation. This is not consequential for fields that are perturbatively weak, but it is of fundamental importance when applied transverse fields are strong. A practical example of this has already been mentioned: the ATI phenomenon is perplexing within the dipole approximation [11], but it is obvious when propagating field considerations are a priori present [4, 12].

5. Potentials are more fundamental than fields

The strong constraints that have been found to apply to potentials, but without reference to the fields associated with those potentials, has immediate significance. That is, potentials have introduced essential probes into physical phenomena such as the propagation phenomenon, preservation of the ponderomotive energy, proper reduction to the Schrödinger equation, and so on. These properties become evident from the potentials, but not from the fields.

A specific simple (but fundamental) example is now given where two sets of potentials can be written for description of the same fields, where one set of potentials is acceptable, but the other is nonphysical. A monochromatic plane-wave field of constant amplitude can be described by the 4-potential

$$A^\mu(\varphi) = A^\mu_0 \cos \varphi,$$  \hspace{1cm} (26)

where the phase $\varphi$ is given in equation (9)

$$\varphi = k^\mu x_\mu = \omega t - \mathbf{k} \cdot \mathbf{r}$$  \hspace{1cm} (27)

and $A^\mu_0$ is a constant 4-vector amplitude. It is noted here that this 4-potential satisfies the Lorentz condition

$$\partial^\mu A_\mu = k^\mu A^\mu_0(\varphi) = 0$$  \hspace{1cm} (28)

due to the transversality condition of equation (15). (The Danish physicist Lorenz should not be confused with the Dutch physicist Lorentz.)

Now consider the gauge transformation generated by the function [21]

$$\Lambda = -A^0(\varphi) x_\mu.$$  \hspace{1cm} (29)

This gives the transformed 4-potential

$$\tilde{A}^\mu = -k^\mu (x^\nu A^\nu_0).$$  \hspace{1cm} (30)

It is readily verified that

$$\partial^\mu \partial_\mu \Lambda = 0,$$  \hspace{1cm} (31)

the sole condition normally required of the generating function of a gauge transformation [1]. Because $\tilde{A}^\mu$ was obtained from the $A^\mu$ of equation (26) by a gauge transformation, the electric and magnetic fields obtained from (30) are identical to those obtained from (26), as can be verified by direct computation. It is also true that $\tilde{A}^\mu$ is a Lorenz gauge, and it is
even true that $A^\mu$ is transverse because of the light-cone condition (14).

However, the vector potential $\tilde{A}^\mu$ given in equation (30) is not a physically acceptable gauge. It has the incorrect Lorentz transformation property of being lightlike rather than spacelike. It predicts that the all-important ponderomotive energy [7, 8] vanishes, because

$$\tilde{A}^\mu \tilde{A}_\mu = 0 \quad (32)$$

and it does not possess the basic property required by relativity that it depend on the spacetime 4-vector $x^\mu$ only in the combination $k^\mu x_\mu$ as demanded by the condition (9). All of these failures occur for the simple reason that the gauge transformation (29) that produced $\tilde{A}^\mu$ does not depend on $x^\mu$ solely in the form of the scalar product (9). Nevertheless, the unphysical nature of $\tilde{A}^\mu$ is not evident from the normal rules for performing a gauge transformation. Judged by prediction of the correct electric and magnetic fields, one would be justified in employing the $\tilde{A}^\mu$ of (30) as the gauge-equivalent version of (26). However, this seemingly safe conclusion based on the fields is incorrect because of the unphysical properties that are evident only by noting that the physical properties of the 4-potential (26) are different from those of the 4-potential (30).

It is not enough to know the fields; one must know the appropriate potentials.

### 6. Practical consequences

The focus of attention throughout this article is on the properties of propagating fields, with the specific case of laser fields as the most important practical example. It has been shown that when a laser field interacts with matter, so that bound charged particles are subjected simultaneously to both transverse and longitudinal fields, then the only formally acceptable electromagnetic gauge that can be employed is the radiation gauge (also called the Coulomb gauge). It has been remarked that this restriction is not of major importance when fields are perturbatively weak, but there is a great and growing interest in the effects of very strong laser fields. The practical models currently employed in strong-field applications are based on the dipole approximation, which amounts to treating the laser field as an oscillatory electric field, with no propagation property at all, and the results here obtained do not apply in the dipole context. Since the dipole approximation gives the appearance of introducing important conceptual and practical simplifications, and many successes have been achieved in this manner, it is natural to inquire about the practical consequences of the results shown above. That is a fundamental question, and a comprehensive answer is proposed.

Since laser fields are, in actuality, transverse, propagating fields, it is to be expected that physical understanding of practical consequences of laser interactions with matter should be based on the properties of propagating fields. The dipole approximation reduces the laser field to an oscillatory electric field, which is a longitudinal field that differs fundamentally from an actual laser field. One important example has already been mentioned. The ATI phenomenon, so startling and unexpected within the AMO community, is actually an obvious and commonplace consequence of all strong-field phenomena. For example, in the context of pair production by strong laser fields, one finds the 1971 comment [13]: ‘...an extremely high order process can be competitive with—and even dominate—the lowest order ... process’. In the context of strong-field bound-bound transitions, it was shown in 1970 that [22]: ‘...as the intensity gets very high, ... the lowest order process gets less probable ... (and) higher-order processes become increasingly important’. The 1980 strong-field approximation (SFA) paper demonstrates the basic aspects of ATI, including some that were not observed in the laboratory until much later. For example, the character of spectra generated by strong, circularly polarized fields, exhibiting a multi-peaked spectrum with a near-Gaussian envelope with the most probable order being significantly higher than the lowest order, was observed with astonishment in a 1986 experiment [23], but this was already predicted in 1980, and the 1980 theory was accurate in exhibiting [24] the explicit behavior found in the 1986 experiment. The reminder is important here that, although the 1980 paper superficially resembles dipole approximation theories, it is actually the nonrelativistic limit of a relativistic theory of laser-induced ionization [4, 25]. The distinction is vital.

The above paragraph reveals that a propagating field theory, since it models the actual laser field, can produce results that are more insightful and more successful than theories based on an oscillating-electric-field model. Furthermore, the 1980 SFA theory is actually easier to apply than the dipole approximation versions of the SFA.

A recent example is instructive. In very precise spectrum measurements in an ionization experiment with circularly polarized light, it was found to be possible to detect the effects of radiation pressure on the photoelectrons [26]. Attempts to provide a theoretical explanation for the effect in a dipole approximation context proved to be extremely difficult and inconclusive [11, 26, 27]. This is not surprising. Radiation pressure arises from photon momentum that does not exist in a dipole approximation theory. In the context of a transverse field description, the most probable kinetic energy of a photoelectron released by a strong, circularly polarized field is just the ponderomotive energy $U_p$. The number of photons above-threshold needed to produce such a photoelectron is $n = U_p/\hbar\omega$. Each photon carries a momentum of $\hbar\omega/c$, with all photon momenta aligned in the direction of propagation of the laser field, so the field-induced momentum in the propagation direction is just $U_p/c$; and this is independent of the atom being ionized when the field is strong. This is precisely what the laboratory measurements reveal [26, 28]. Transverse field concepts produce insightful and quantitatively accurate results as shown in the span of three sentences given above, as contrasted with three journal articles [11, 26, 27].

The seeming simplicity of dipole approximation methods is actually counter-productive in strong fields, as shown by
the ATI and radiation pressure examples. The dipole approximation can lead to complication rather than simplicity. Perhaps the most consequential of all misconceptions that arise from dependence on a dipole approximation model of laser effects is the matter of low frequency behavior [29, 30]. The oscillatory electric fields that arise from a dipole approximation theory approach a constant electric field as the frequency declines. This limit (sometimes called the tunneling limit) has been applied as a test of the accuracy of theoretical models. For example, a textbook on the subject of strong laser field effects altogether rejects models based on transverse fields, since they do not approach the tunneling limit [31]. Another example is a paper that assesses the accuracy of analytical approximations based on their behavior as the field frequency approaches zero [32]. However, actual transverse fields in vacuum always propagate at the speed of light independently of frequency. There is no limit possible in which a real propagating field becomes a static field. The effect on strong-field theory of this zero-frequency misconception continues to the present. It is related to the equally problematic concept that the final arbiters of validity is to be found in the exact numerical solution of the Schrödinger equation, generally referred to as time-dependent Schrödinger equation (TDSE). Since TDSE as customarily employed is based on the dipole approximation, it reinforces the critically misleading concept that there is a zero-frequency limit of laser effects equivalent to a constant electric field.

It is difficult to conceive of a notion more consequential for an entire field of inquiry than this reliance on the criterion that laser-induced effects have a zero-frequency limit equivalent to that of a constant electric field. When joined with the equally difficult concept, championed by Yang [33] and others [34], that the scalar potential known as the length gauge (accurate for a scalar field like a longitudinal field) can somehow be a privileged gauge for the description of a vector field like the transverse field of a laser beam, the discipline of strong-field physics is laboring under a burden of misdirected criteria. This has stood nearly unchallenged since the 1979 observation of ATI [10]. The scrutiny provided by an approach based on the radiation gauge creates the necessary challenge.

References

[1] Jackson J D 1975 Classical Electrodynamics 2nd edn (New York: Wiley)
[2] Ehrenberg W and Siday R E 1949 The refractive index in electron optics and the principles of dynamics Proc. R. Soc. B 62 8
[3] Aharonov Y and Bohm D 1959 Significance of potentials in the quantum theory Phys. Rev. 115 485
[4] Reiss H R 1990 Complete Keldysh theory and its limiting cases Phys. Rev. A 42 1476
[5] Schwinger J 1951 On gauge invariance and vacuum polarization Phys. Rev. 82 664
[6] Sarachik E S and Schappert G T 1970 Classical theory of the scattering of intense laser radiation by free electrons Phys. Rev. D 1 2738
[7] Reiss H R 2012 On a modified electrodynamics J. Mod. Opt. 59 1371
Reiss H R 2013 J. Mod. Opt. 60 687
[8] Reiss H R 2014 Mass shell of strong-field quantum electrodynamics Phys. Rev. A 89 022116
[9] Gérbert-Mayer M 1931 Über Elementarakte mit zwei Quantensprüngen Ann. Phys., Lpz. 9 273–94
[10] Agostini P, Fabre F, Mainfray G, Petit G and Rahman N K 1979 Free-free transitions following six-photon ionization of xenon atoms Phys. Rev. Lett. 42 1127
[11] Chelkowski S, Bandrauk A D and Corkum P B 2014 Photon momentum sharing between an electron and an ion in photoionization from one-photon (photoelectric effect) to multiphoton absorption Phys. Rev. Lett. 113 263005
[12] Reiss H R 1980 Effect of an intense electromagnetic field on a weakly bound system Phys. Rev. A 22 1786
[13] Reiss H R 1971 Production of electron pairs from a zero-mass state Phys. Rev. Lett. 26 1072
[14] Burke D L et al 1997 Postion production in multiphoton light-by-light scattering Phys. Rev. Lett. 79 1626
[15] Pauli W and Weisskopf V F 1934 Über die Quantisierung der skalaren relativistischen Wellengleichung Helv. Phys. Acta. 7 709
[16] Schweber S S 1962 An Introduction to Relativistic Quantum Field Theory (New York: Harper & Row)
[17] Schoene A Y 1979 On the nonrelativistic limits of the Klein–Gordon and Dirac equations J. Math. Anal. Appl. 71 36
[18] Schiff L I 1968 Quantum Mechanics (New York: McGraw-Hill)
[19] Feynman R P and Gell-Mann M 1958 Theory of the Fermi interaction Phys. Rev. 109 193
[20] Bjorken J D and Drell S D 1964 Relativistic Quantum Mechanics (New York: McGraw-Hill)
[21] Reiss H R 1979 Field intensity and relativistic considerations in the choice of gauge in electrodynamics Phys. Rev. A 19 1140
[22] Reiss H R 1970 Atomic transitions in intense fields and the breakdown of perturbation calculations Phys. Rev. Lett. 25 1149
[23] Bucksbaum P H, Bashkansky M, Freeman R R, McLlrath T J and DiMauro L F 1986 Suppression of multiphoton ionization with circularly polarized coherent light Phys. Rev. Lett. 56 2590
[24] Reiss H R 1987 Spectrum of atomic electrons ionized by an intense field J. Phys. B: At. Mol. Phys. 20 L79
[25] Reiss H R 1990 Relativistic strong-field photoionization J. Opt. Soc. Am. B 7 574
[26] Smeenk C T L, Arissian L, Zhou B, Mysyrowicz A, Villeneuve D M, Staudte A and Corkum P B 2011 Partitioning of the linear photon momentum in multiphoton ionization Phys. Rev. Lett. 106 193002
[27] Chelkowski S, Bandrauk A D and Corkum P B 2015 Photon-momentum transfer in multiphoton ionization and in time-resolved holography with photoelectrons Phys. Rev. A 92 051401(R)
[28] Reiss H R 2013 Relativistic effects in nonrelativistic ionization Phys. Rev. A 87 033421
[29] Reiss H R 2008 Limits on tunneling theories of strong-field ionization Phys. Rev. Lett. 101 043002
Reiss H R 2008 Phys. Rev. Lett. 101 159901(E)
[30] Reiss H R 2014 The tunneling model of laser-induced ionization and its failure at low frequencies J. Phys. B: At. Mol. Opt. Phys. 47 204006
[31] Joachain C J, Kylstra N J and Potvliege R M 2012 Atoms in Intense Laser Fields (Cambridge: Cambridge University Press)
[32] Bauer J 2006 Low-frequency–high-intensity limit of the Keldysh–Faisal–Reiss theory Phys. Rev. A 73 023421
[33] Yang K-H 1976 Gauge transformations and quantum mechanics: I. Gauge invariant interpretation of quantum mechanics *Ann. Phys., NY* **101** 62

[34] Lamb W E, Schlicher R R and Scully M O 1987 Matter–field interaction in atomic physics and quantum optics *Phys. Rev. A* **36** 2763