REGULARITY OF SOLUTIONS TO THE NAVIER-STOKES EQUATIONS IN $\dot{B}^{-1}_{\infty, \infty}$

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Abstract. We prove that if $u$ is a suitable weak solution to the three dimensional Navier-Stokes equations from the space $L_{\infty}(0, T; \dot{B}^{-1}_{\infty, \infty})$, then all scaled energy quantities of $u$ are bounded. As a consequence, it is shown that any axially symmetric suitable weak solution $u$, belonging to $L_{\infty}(0, T; \dot{B}^{-1}_{\infty, \infty})$, is smooth.

1. Introduction

The main aim of this paper is to show that suitable weak solutions to the Navier-Stokes equations, whose $\dot{B}^{-1}_{\infty, \infty}$-norm is bounded, have the Type I singularities (or Type I blowups) only. To be more precise in the statement of our results, we need to define certain notions.

Definition 1.1. Let $\Omega$ be a domain in $\mathbb{R}^3$ and let $Q_T := \Omega \times ]0, T[$. It is said that a pair of functions $v$ and $q$ is a suitable weak solution to the Navier-Stokes equations in $Q_T$ if the following conditions are fulfilled:

(i) $v \in L_{\infty}(\delta, T; L^2_{2, loc}(\Omega)) \cap L^2_{2, loc}(\Omega)$, $q \in L^2_{3, 2}(\delta, T; L^3_{2, loc}(\Omega))$ for any $\delta \in ]0, T[$;

(ii) $v$ and $q$ satisfy the Navier-Stokes equations

$$\partial_t v + v \cdot \nabla v - \Delta v = -\nabla q, \quad \text{div} v = 0$$

in $Q_T$ in the sense of distributions;

(iii) for $Q(z_0, R) \subset \Omega \times ]0, T[$, the local energy inequality

$$\int_{B(z_0, R)} \varphi |v(x, t)|^2 dx + 2 \int_{t_0 - R^2}^{t} \int_{B(z_0, R)} \varphi |\nabla v|^2 dx d\tau \leq$$

$$\leq \int_{t_0 - R^2}^{t} \int_{B(z_0, R)} \left(|v|^2 (\partial_t \varphi + \Delta \varphi) + v \cdot \nabla \varphi (|v|^2 + 2q)\right) dx d\tau$$

holds for a.a. $t \in ]t_0 - R^2, t_0[$ and for all non-negative test functions $\varphi \in C^\infty_0(B(x_0, R) \times ]t_0 - R^2, t_0 + R^2[)$.

Let us introduce the following scaled energy quantities:

$$A(z_0, r) := \sup_{t_0 - r^2 < t < t_0} \frac{1}{r} \int_{B(z_0, r)} |v(x, t)|^2 dx, \quad E(z_0, r) := \frac{1}{r} \int_{Q(z_0, r)} |\nabla v|^2 dx dt,$$

$$C(z_0, r) := \frac{1}{r^2} \int_{Q(z_0, r)} |v|^3 dx dt, \quad D(z_0, r) := \frac{1}{r^2} \int_{Q(z_0, r)} |q|^3 dx dt,$$
\[ G(z_0, r) := \max\{A(z_0, r), E(z_0, r), C(z_0, r)\}, \]
\[ g(z_0, r) := \min\{A(z_0, r), E(z_0, r), C(z_0, r)\}. \]
Here, \( Q(z_0, r) := B(z_0, r) \times [t_0 - r^2, t_0] \) and \( B(x_0, r) \) is the ball of radius \( r \) centred at a point \( x_0 \in \mathbb{R}^3 \).

The important feature of the above quantities is that all of them are invariant with respect to the Navier-Stokes scaling.

Our main result is as follows.

**Theorem 1.2.** Let \( \Omega = \mathbb{R}^3 \). Assume that a pair \( v \) and \( q \) is a suitable weak solution to the Navier-Stokes equations in \( Q_T \). Moreover, it is supposed that

\[ v \in L_\infty(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)). \]

Then, for any \( z_0 \in \mathbb{R}^3 \times (0, T) \), we have the estimate

\[ \sup_{0 < r < r_0} G(z_0, r) \leq c [r_0^{\frac{1}{2}} + \|v\|_{L_\infty(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3))}^2 + \|v\|_{L_\infty(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3))}^6], \]

where \( r_0 \leq \frac{1}{2} \min\{1, t_0\} \) and \( c \) depends on \( C(z_0, 1) \) and \( D(z_0, 1) \) only.

Let us recall one of definitions of the norm in the space \( \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3) = \{ f \in S' : \|f\|_{\dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)} < \infty \} \), which is the following:

\[ \|f\|_{\dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)} := \sup_{t > 0} t^{\frac{1}{2}} \|w\|_{L_\infty(\mathbb{R}^3)}, \]

where \( S' \) is the space of tempered distributions, \( w \) is the solution to the Cauchy problem for the heat equation with initial datum \( f \).

**Definition 1.3.** Assume that \( z_0 = (x_0, t_0) \) is a singular point of \( v \), i.e., there is no parabolic vicinity of \( z_0 \) where \( v \) is bounded. We call \( z_0 \) Type I singularity (or Type I blowup) if there exists a positive number \( r_1 \) such that

\[ \sup_{0 < r < r_1} g(z_0, r) < \infty. \]

According to Definition 1.3, any suitable weak solution, satisfying assumption (1.1), has Type I singularities only. In particular, arguments, used in paper [21], show that axially symmetric suitable weak solutions to the Navier-Stokes equations have no Type I blowups. This is an improvement of what has been known so far, see papers [14] and [21], where condition (1.1) is replaced by stronger one

\[ v \in L_\infty(0, T; BMO^{-1}(\mathbb{R}^3)). \]

Regarding other regularity results on axially symmetric solutions to the Navier-Stokes equations, we refer to papers [2, 3, 4, 5, 9, 10, 11, 12, 15, 16, 17, 18, 19, 20, 22, 23, 24].

Another important consequence is that the smallness of \( \|v\|_{L_\infty(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3))} \) implies regularity, see also [1, 7].

**2. PROOF OF THE MAIN RESULT**

In this section, Theorem 1.2 is proved. First, we recall the known multiplicative inequality, see [6].
Proof. It follows from Lemma 2.1 that for all $u \in \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$, the following is valid:

\begin{equation}
\|u\|_{L^4(\mathbb{R}^3)} \leq c\|u\|_{\dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla u\|_{L^2(\mathbb{R}^3)},
\end{equation}

where $\dot{H}^1(\mathbb{R}^3)$ is a homogeneous Sobolev space.

In fact, a weaker version of (2.1) with $\|u\|_{L^4(\mathbb{R}^3)}$ instead of $\|u\|_{L^4(\mathbb{R}^3)}$ is needed. Here, $L^{1, \infty}(\mathbb{R}^3)$ is a weak Lebesgue space. An elementary proof of a weaker inequality is given in [13].

The second auxiliary statement is about cutting-off in the space $\dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3)$.

Lemma 2.2. Let $u \in \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3)$ and $\phi \in C_0^\infty(\mathbb{R}^3)$. Then

\[ \|u\phi\|_{\dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3)} \leq c(|\text{spt}\ \phi|)\|u\|_{\dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3)}. \]

We have not found out a proof of Lemma 2.2 in the literature and presented it in Appendix. Our proof is elementary and based on typical PDE’s arguments. A scaled version of the previous lemma is as follows.

Lemma 2.3. For any $u \in \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3) \cap \dot{H}^1(B(2))$, the estimate

\begin{equation}
\|u\|_{L^4(\mathbb{R}^3)} \leq c\|u\|_{\dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3)}^{\frac{1}{2}} \|\nabla u\|_{L^2(B(2))}.
\end{equation}

is valid for a universal constant $c$. Moreover, if $u \in \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3) \cap \dot{H}^1(B(x_0, 2R))$, then

\begin{equation}
\|u\|_{L^4(\mathbb{R}^3)} \leq c\|u\|_{\dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3)}^{\frac{1}{2}} \left(\|\nabla u\|_{L^2(B(2R))} + \frac{1}{R} \|u\|_{L^2(B(2R))}\right)^{\frac{1}{2}},
\end{equation}

with a universal constant $c$.

Here, we use notation for the ball centred at the origin $B(R) = B(0, R)$ and $B = B(1)$.

\begin{proof}
It follows from Lemma 2.1 that for all $\phi \in C_0^\infty(\mathbb{R}^3),

\[ \|u\phi\|_{L^4(\mathbb{R}^3)} \leq c\|u\phi\|_{\dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3)}^{\frac{1}{2}} \|u\phi\|_{\dot{H}^1(\mathbb{R}^3)}. \]

Taking a cut-off function $\phi$ such that $\phi = 1$ in $B$, $\phi = 0$ out of $B(2)$, and $0 \leq \phi \leq 1$ for $1 \leq |x| \leq 2$, we get inequality (2.2) from Lemma 2.2.

To prove inequality (2.3), one can use scaling and shift $x = x_0 + Ry$, $x \in B(x_0, 2R)$, $y \in B(2)$ in (2.2). \qed

In order to prove the main result, we need the following auxiliary inequalities for $C(z_0, r)$.

Lemma 2.4. For any $0 < r \leq R < \infty$, we have

\begin{equation}
C(z_0, r) \leq c\|u\|_{L^4_{\infty}(0, T; \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3))}^{\frac{1}{2}} \left(A^{\frac{1}{2}}(z_0, 2r) + E^{\frac{1}{2}}(z_0, 2r)\right),
\end{equation}

and

\begin{equation}
C(z_0, r) \leq c\|u\|_{L^4_{\infty}(0, T; \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3))}^{\frac{1}{2}} \left(R \right)^{\frac{1}{2}} \left(A^{\frac{1}{2}}(z_0, R) + E^{\frac{1}{2}}(z_0, R)\right).
\end{equation}
Similarly, (2.6), (2.7) and Young’s inequality with an arbitrary positive constant $\delta$, we can derive

$$C(z_0, 2r) \leq c\delta (A(z_0, \rho) + E(z_0, \rho)) + c\delta^{-3} \|u\|_{L^6((0,T;\mathcal{B}^{1,\infty}_x(\mathbb{R}^3))}^3 \left( \frac{\rho}{r} \right)^3 .$$

Similarly,

$$C^\frac{2}{3}(z_0, 2r) \leq c\delta (A(z_0, \rho) + E(z_0, \rho)) + c\delta^{-1} \|u\|_{L^6((0,T;\mathcal{B}^{1,\infty}_x(\mathbb{R}^3))}^2 \left( \frac{\rho}{r} \right)^3 ,$$

and

$$\left( \frac{\rho}{r} \right)^2 C(z_0, \frac{\rho}{2}) \leq c\delta (A(z_0, \rho) + E(z_0, \rho)) + c\delta^{-3} \|u\|_{L^6((0,T;\mathcal{B}^{1,\infty}_x(\mathbb{R}^3))}^6 \left( \frac{\rho}{r} \right)^8 .$$

This completes the proof of inequality (2.4).

Now we are going to justify our main result.

Proof of Theorem 1.2. From the local energy inequality, it follows that, for any $0 < r < \infty$,

$$A(z_0, r) + E(z_0, r) \leq c \left( C^\frac{2}{3}(z_0, 2r) + C(z_0, 2r) + D(z_0, 2r) \right) .$$

For the pressure $q$, we have the decay estimate

$$D(z_0, r) \leq c \left( \frac{r}{R} D(z_0, R) + \left( \frac{R}{r} \right)^2 C(z_0, R) \right) ,$$

which is valid for any $0 < r < R < \infty$.

Assume that $0 < r \leq \frac{\rho}{4} < \rho \leq 1$. Combining (2.7) and (2.6), we find

$$A(z_0, r) + E(z_0, r) + D(z_0, r) \leq$$

$$\leq c \left( C^\frac{2}{3}(z_0, 2r) + C(z_0, 2r) + \left( \frac{\rho}{r} \right)^2 C(z_0, \frac{\rho}{2}) + \frac{r}{\rho} D(z_0, \frac{\rho}{2}) \right) .$$

Now, let us estimate each term on the right hand side of the last inequality. From (2.4), (2.5), and Young’s inequality with an arbitrary positive constant $\delta$, we can derive

$$(\frac{\rho}{r})^2 C(z_0, \frac{\rho}{2}) \leq c\delta (A(z_0, \rho) + E(z_0, \rho)) + c\delta^{-3} \|u\|_{L^6((0,T;\mathcal{B}^{1,\infty}_x(\mathbb{R}^3))}^6 \left( \frac{\rho}{r} \right)^8 .$$

Proof. Obviously, (2.5) easily follows from (2.4). So, we need to prove the first inequality only. By the Hölder inequality, we have

$$\|u(\cdot, t)\|_{L^3(B(x_0, r))} \leq cr^{\frac{3}{4}} \|u(\cdot, t)\|_{L^{4,\infty}(B(x_0, r))}$$

and thus, by (2.3),

$$C(z_0, r) = \frac{1}{r^2} \int_{z_0 - r^2}^{z_0} \|u(\cdot, t)\|^3_{L^3(B(x_0, r))} dt$$

$$\leq c \frac{1}{r^2} \|u\|^2_{L^6((0,T;\mathcal{B}^{1,\infty}_x(\mathbb{R}^3))} \left( \int_{t_0 - (2r)^2}^{t_0} \|\nabla u(\cdot, t)\|^2_{L^2(B(x_0, 2r))} dt + \frac{1}{r^2} \|u(\cdot, t)\|_{L^2(B(x_0, 2r))}^2 dt \right)$$

$$\leq c \frac{1}{r^2} \|u\|^2_{L^6((0,T;\mathcal{B}^{1,\infty}_x(\mathbb{R}^3))} \left( \int_{t_0 - (2r)^2}^{t_0} \|\nabla u(\cdot, t)\|^2_{L^2(B(x_0, 2r))} dt + \frac{\sup_{-(2r)^2 < t < t_0} \|u(\cdot, t)\|_{L^2(B(x_0, 2r))}^2 dt + \frac{\delta}{r} D(z_0, \frac{\rho}{2}) \right) .$$

This completes the proof of inequality (2.4). \qed
Denote $E(r) = A(z_0, r) + E(z_0, r) + D(z_0, r)$. By a simple inequality $D(z_0, \rho/2) \leq cD(z_0, \rho)$,

$$E(r) \leq c(\delta + \frac{r}{\rho})E(\rho) + c\left\{ \|u\|^2_{L^\infty(0,T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))} \left( \frac{\rho}{r} \right) \delta^{-1} + \|u\|^6_{L^\infty(0,T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))} \left[ \left( \frac{\rho}{r} \right)^3 + \left( \frac{\rho}{r} \right)^8 \right] \delta^{-3} \right\}.$$ 

Letting $r = \theta \rho$ and $\delta = \theta$ and picking up $\theta$ such that $2c\theta^{1/2} \leq 1$, we find

$$E(\theta r) \leq \theta^{1/2}E(\rho) + c\left\{ \|u\|^2_{L^\infty(0,T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))} \theta^{-2} + \|u\|^6_{L^\infty(0,T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))} \theta^{-11} \right\}.$$ 

Standard iteration gives us that for $0 < r \leq \frac{1}{2}$,

$$E(r) \leq c\left( r^{1/2}E(1) + \|u\|^2_{L^\infty(0,T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))} + \|u\|^6_{L^\infty(0,T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))} \right).$$

Taking into account (2.4), we get in addition that

$$C(z_0, r) \leq c\left( r^{1/2}E(1) + \|u\|^6_{L^\infty(0,T; \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))} \right).$$

This completes the proof of Theorem 1.2. \qed

**Appendix A. Proof of Lemma 2.2.**

We let $w(\cdot, t) = S(t)f(\cdot)$ and $w_\varphi(\cdot, t) = S(t)\varphi f(\cdot)$, where $S(t)$ is a solution operator of the Cauchy problem for the heat equation with the initial data $f$ and $\varphi f$, respectively. Then $u := w \varphi - w_\varphi$ satisfies the equation

$$\partial_t u - \Delta u = -2\text{div}(\nabla \varphi w) + w \Delta \varphi$$

and the initial condition $u(\cdot, 0) = 0$. A unique solution to the problem is as follows:

$$u(x, t) = I + J,$$

where

$$I = -\int_0^t \int_{\mathbb{R}^3} \Gamma(x - y, t - \tau)(2\text{div}(\nabla \varphi w))(y, \tau)dyd\tau,$$

$$J = \int_0^t \int_{\mathbb{R}^3} \Gamma(x - y, t - \tau)(w \Delta \varphi)(y, \tau)dyd\tau,$$

and $\Gamma$ is the heat kernel.

Let us evaluate $I$. We abbreviate

$$A := \|f\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} = \sup_{t > 0} \sqrt{7}\|w(\cdot, t)\|_{L^\infty(\mathbb{R}^3)}$$

and $\Omega = \text{spt} \varphi$. Then we have

$$\sqrt{7}|I| \leq 2A\sqrt{t} \int_0^t \frac{1}{\sqrt{\Omega}} \frac{1}{(4\pi(t - \tau))^{3/2}} \exp \left\{ -\frac{|x - y|^2}{4(t - \tau)} \right\} \frac{|x - y|}{t - \tau} dyd\tau \leq$$

$$\leq cA \int_0^t \sqrt{\frac{t}{\tau(t - \tau)^2}} \int_{\Omega} \exp \left\{ -\frac{|x - y|^2}{4(t - \tau)} \frac{|x - y|}{\sqrt{t - \tau}} \right\} dyd\tau =$$
\[ = cA \left[ \int_0^{\frac{t}{2}} \ldots + \int_{\frac{t}{2}}^t \ldots \right] = cA(I_1 + I_2). \]

Regarding \( I_1 \), consider first the case \( 0 < t < 1 \). By the standard change of variables, we have
\[
I_1 \leq cAC_0 \int_0^{\frac{t}{2}} \sqrt{\frac{t}{\tau}} \frac{1}{\sqrt{t - \tau}} d\tau
\]
with
\[
C_0 = \int_{\mathbb{R}^3} \exp\{-|u|^2\}|u|du.
\]
And thus \( I_1 \leq cA \). In the second case \( t \geq 1 \),
\[
I_1 \leq c|\Omega| \int_0^{\frac{t}{2}} \sqrt{\frac{t}{\tau}} \frac{1}{(t - \tau)^2} d\tau \leq c|\Omega| \frac{1}{t} \leq c|\Omega|.
\]

Now, let us evaluate \( I_2 \). Obviously,
\[
I_2 \leq c \int_0^{t} \frac{1}{(t - \tau)^2} \int_{\Omega} \exp \left\{ -\frac{|x - y|^2}{4(t - \tau)} \right\} \frac{|x - y|}{\sqrt{t - \tau}} dy d\tau.
\]
Make change of variables \( \vartheta = t - \tau \), then
\[
I_2 \leq c \int_0^{\infty} \frac{1}{\vartheta^2} \int_{\Omega} \exp \left\{ -\frac{|x - y|^2}{4\vartheta} \right\} \frac{|x - y|}{\sqrt{\vartheta}} dy d\vartheta =
\]
\[
= c \int_0^{\infty} \ldots + c \int_1^{\infty} \ldots = J_1 + J_2.
\]
For \( J_1 \), we have
\[
J_1 \leq cC_0 \int_0^{\frac{1}{\sqrt{\vartheta}}} d\vartheta \leq cC_0.
\]
Finally, \( J_2 \) is bounded as follows:
\[
J_2 \leq c \int_1^{\infty} \frac{1}{\vartheta^2} d\vartheta \leq c|\Omega|.
\]

The quantity \( J \) is estimated in the same way. Lemma 2.2 is proved.

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