Hardness of Minimum Barrier Shrinkage and Minimum Activation Path

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Abstract

In the Minimum Activation Path problem, we are given a graph $G$ with edge weights $w(\cdot)$ and two vertices $s, t$ of $G$. We want to assign a non-negative power $p: V \to \mathbb{R}_{\geq 0}$ to the vertices of $G$, so that the activated edges $\{uv \in E(G) \mid p(u) + p(v) \geq w(uv)\}$ contain some $s$-$t$-path, and minimize the sum of assigned powers. In the Minimum Barrier Shrinkage problem, we are given a family of disks in the plane and two points $x$ and $y$ lying outside the disks. The task is to shrink the disks, each one possibly by a different amount, so that we can draw an $x$-$y$ curve that is disjoint from the interior of the shrunken disks, and the sum of the decreases in the radii is minimized.

We show that the Minimum Activation Path and the Minimum Barrier Shrinkage problems (or, more precisely, the natural decision problems associated with them) are weakly NP-hard.

Keywords: activation path, activation network, barrier problem, NP-hardness

1 Introduction

Let $X$ be a subset of the plane, let $x$ and $y$ be points in $X$, and let $S$ be a family of shapes in the plane. An $x$-$y$ curve is a curve in $\mathbb{R}^2$ with endpoints $x$ and $y$. We say that $S$ separates $x$ and $y$ in $X$ if each $x$-$y$ curve contained in $X$ intersects some shape from $S$. Let $D(c, r)$ denote the open disk centered at $c$ with radius $r$.

In this work we show that the following two decision problems are weakly NP-hard. This means that in our reduction we will use numbers that are exponentially large, but have polynomial length when written in binary.

**Minimum Barrier Shrinkage.**

Input: a family $\{D(c_i, r_i) \mid i = 1, \ldots, n\}$ of $n$ open disks; two points $x, y \in \mathbb{R}^2$; a real number $C$.

Output: Whether there exist shrinking values $\delta_1, \ldots, \delta_n \geq 0$ such that their cost $\sum_i \delta_i$ is at most $C$ and the family of open disks $\{D(c_i, r_i - \delta_i) \mid i = 1, \ldots, n\}$ does not separate $x$ and $y$ in $\mathbb{R}^2$.

**Minimum Activation Path.**

Input: a graph $G = (V, E)$ with positive edge weights $w: E \to \mathbb{R}_{>0}$; two vertices $s$ and $t$ of $G$; a real number $C$.

Output: Whether there exists an assignment of powers $p: V \to \mathbb{R}_{\geq 0}$ to the vertices such

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that its cost $\sum_{v \in V} p(v)$ is at most $C$ and the active edges $E(p) = \{ uv \in E \mid p(u) + p(v) \geq w(uv) \}$ contain an $s$-$t$-path.

We next discuss the motivating and closest related work.

Minimum barrier shrinkage. Kumar, Lai and Arora [8] introduced the following barrier resilience problem in the plane. The input is a domain $X$, a family $D$ of disks, and two points $x$ and $y$ outside all the disks in $D$, everything in the plane. The task is to find an $x$-$y$ curve in $X$ that intersects as few disks of $D$ as possible, without counting multiplicities. An alternative statement is that we want to find a minimum cardinality subfamily $D' \subseteq D$ such that $D \setminus D'$ does not separate $x$ and $y$ in $X$. The intuition is that we have sensors detecting movements from $x$ to $y$, and we want to known how many sensors can suffer a total failure and still any agent moving from $x$ to $y$ within $X$ is detected by some of the remaining sensors.

Kumar, Lai and Arora [8] showed that the problem can be solved in polynomial time when the domain $X$ is a vertical strip bounded between two vertical lines $\ell$ and $\ell'$, the point $x$ lies above and the point $y$ lies below all disks of $D$. Let us call this scenario the rectangular scenario. The main insight is to consider the intersection graph $G$ defined by $D \cup \{ \ell, \ell' \}$ and to note that the solution is the maximum number of $\ell$-$\ell'$ internally vertex-disjoint paths in $G$. Thus, the problem can be solved in polynomial time by solving maximum flow problems. The same argument works for any family of shapes $S$, not just disks, as far as each shape of $S$ is connected.

Despite the claim in the preliminary version [7] of [8], we do not know whether the problem can be solved exactly in polynomial time when the domain $X$ is all of $\mathbb{R}^2$. In fact, we know that when $X = \mathbb{R}^2$ and the family $D$ of disks is replaced by some other family $S$ of shapes, the problem is NP-hard [16,11]. The difference between the strip and the whole plane is that in the former case we can use Menger’s theorem to relate the number of $\ell$-$\ell'$ paths in the intersection graph of $S \cup \{ \ell, \ell' \}$ to the $\ell$-$\ell'$ vertex connectivity, but no such statement applies to cycles that “separate” $x$ and $y$. The computational complexity of the barrier problem in the plane for (unit) disks and (unit) squares is a challenging open problem, and several approximation algorithms have been devised [2,4,6].

Modeling the fact that sensors are less reliable further away from their placement, Cabello et al. [3] considered the problem of minimizing the total shrinkage of the disks such that there is an $x$-$y$ curve disjoint from the interior of the disks. This is precisely the problem minimum barrier shrinkage. Cabello et al. also provided an FPTAS for the rectangular scenario. The algorithm uses the connection to vertex-disjoint paths.

We believe that showing NP-hardness for the problem minimum barrier shrinkage is interesting because of the computational complexity of two closely related problems, the barrier resilience problem for $X = \mathbb{R}^2$ and the minimum barrier shrinkage problem in the rectangular scenario, are unknown.

Minimum activation path. Panigrahi [10] considered the minimum activation path and provided an algorithm with running time $O(\text{poly}(n, |D|))$. Here $n$ is the number of vertices of $G$ and $D$ is a set of possible values for the power assignments (the domain). Thus, in that scenario one considers power assignments $p : V \rightarrow D$.

Our weak NP-hardness result of the minimum activation path is consistent with the result of Panigrahi. In our reduction, we use large integer weights: they have a polynomial bit length, but they are exponentially large. Taking $D = \mathbb{Z} \cap [0, \max_{uv} w(uv)]$ in the algorithm of Panigrahi, one only gets a pseudopolynomial time algorithm for such instances. This is consistent with a weakly NP-hardness proof.

There is a rich literature on so-called Activation Network problems. The task is to assign powers $p(v)$ to the vertices of an edge-weighted graph $G = (V, E)$ so that the active edges $E(p) = \{ uv \in E \mid p(u) + p(v) \geq w(uv) \}$ satisfy certain property, such as for example spanning the whole graph. See
the survey by Nutov [2] for an overview of the area. It seems that the computational complexity of the Minimum Activation Path problem has not been set before.

Relation between the problems. We are not aware of any polynomial-time reduction from one problem to the other. Nevertheless, the NP-hardness proofs for both problems are very similar. The underlying connection between both problems is the following classical property: in a planar graph $G = (V, E)$, a set of edges $F \subseteq E$ is a minimum $s$-$t$ cut if and only if the dual edges $\{ e^* \mid e \in F \}$ form a shortest cycle separating the face $s^*$ from the face $t^*$ in the dual graph $G^*$. This relation does not directly provide a reduction even in the case of planar graphs, but does inspire the adaptation we make. Actually, our hardness proof for Minimum Barrier Shrinkage reuses components of the hardness proof for Minimum Activation Path.

Organization. It seems more convenient to present the NP-hardness of Minimum Activation Path first. We achieve this in Section 2. Then, in Section 3 we show that the Minimum Barrier Shrinkage problem is NP-hard.

2 Minimum activation path

In this section we show that the Minimum Activation Path problem is NP-hard.

2.1 Greedy solution in a path

Consider a graph $G$ and a path $\pi = v_0, \ldots, v_n$ in $G$. We define greedily a power assignment $p^*_\pi$ on the vertices of $G$ to activate $\pi$, in a way that power is pushed forward along $\pi$ as much as possible. Formally, the greedy power assignment along $\pi$ is

$$p^*_\pi(v) = \begin{cases} 0 & \text{if } v \text{ does not belong to } \pi \text{ or } v = v_0, \\ 0 & \text{if } v = v_i, i > 0 \text{ and } p^*_\pi(v_{i-1}) > w(v_{i-1}v_i), \\ w(v_{i-1}v_i) - p^*_\pi(v_{i-1}) & \text{if } v = v_i, i > 0 \text{ and } p^*_\pi(v_{i-1}) < w(v_{i-1}v_i). \end{cases}$$  \hfill (1)

For a power assignment $p$, let $\operatorname{cost}(p)$ denote the total cost of $p$, namely, the sum of the powers at the vertices. For path $\pi$, let $\operatorname{opt}(\pi)$ be the cost of the minimum cost power assignment that activates $\pi$. The following lemma tells that the greedy power assignment along $\pi$ has minimum cost to activate $\pi$.

Lemma 1. For each path $\pi$, $\operatorname{cost}(p^*_\pi) = \operatorname{opt}(\pi)$.

Proof. It is clear that $p^*_\pi$ activates all the edges of $\pi$. Let $p$ be another power assignment activating all edges of $\pi$. We have to show that $\operatorname{cost}(p^*_\pi) \leq \operatorname{cost}(p)$.

We can assume that $p(v) = 0$ at all vertices $v$ outside $\pi$. Otherwise, we change $p$ to have this property. This reassignment of power would decrease the cost and would keep activating the path $\pi$.

The strategy is to gradually transform $p$ into $p^*_\pi$ while keeping all edges of $\pi$ activated and without increasing the value of $\operatorname{cost}(p)$. The property $\operatorname{cost}(p^*_\pi) \leq \operatorname{cost}(p)$ is trivially correct if $p = p^*_\pi$. So assume $p \neq p^*_\pi$ and let $i$ be the smallest integer such that $p(v_i) \neq p^*_\pi(v_i)$. Because all edges are activated, and by construction of $p^*_\pi$, we must have $p(v_i) > p^*_\pi(v_i)$. Let $\Delta = p(v_i) - p^*_\pi(v_i) > 0$. There are two cases:

- Assume $i \leq n - 1$. Update $p$ by decreasing $p(v_i)$ by $\Delta$ and increasing $p(v_{i+1})$ by $\Delta$. Since each edge of $\pi$ is activated by $p^*_\pi$ and by $p$ before this transformation, each edge of $\pi$ is still activated by the new $p$. Moreover, $\operatorname{cost}(p)$ is unchanged.
• Assume $i = n$. Update $p$ by decreasing $p(v_n)$ by $\Delta$. Again, each edge of $\pi$ is still activated. The cost has decreased by $\Delta$.

This transformation does not increase the value of cost($p$). Moreover, the new power assignment coincides with $p^*_n$ on vertices $v_0, \ldots, v_i$. Thus, after a finite number of steps, $p = p^*_n$. This proves the lemma.

For the path $\pi = v_0, \ldots, v_n$, let $\varphi(\pi) = p^*_n(v_n) \geq 0$. That is, $\varphi(\pi)$ is the power assignment given by the greedy power assignment along $\pi$ to the final vertex. Since $p^*_n(v_n)$ depends on $p^*_n(v_{n-1})$, we have the following.

**Lemma 2.** Let $\pi$ be the path $v_0, \ldots, v_n$ and let $\pi'$ be the path $v_0, \ldots, v_n, u$. (Thus, $\pi'$ extends $\pi$ by an additional edge $v_nu$.) Then $\varphi(\pi') = \max\{0, w(v_nu) - \varphi(\pi)\}$ and $\text{opt}(\pi') = \text{opt}(\pi) + \varphi(\pi')$.

**Proof.** From the definition of the greedy power assignment along $\pi$ and $\pi'$, the power assignments $p^*_n$ and $p^*_{n'}$ differ only at vertex $u$. We have:

$$\varphi(\pi') = p^*_{n'}(u) = \max\{0, w(v_nu) - p^*_{n'}(v_n)\} = \max\{0, w(v_nu) - p^*_n(v_n)\} = \max\{0, w(v_nu) - \varphi(\pi)\}.$$ 

This proves the claim for $\varphi(\pi')$. Because of Lemma 1 for $\pi$ and $\pi'$ we also get

$$\text{opt}(\pi') = \text{cost}(p^*_{n'}) = \text{cost}(p^*_n) + p^*_{n'}(u) - p^*_n(u) = \text{opt}(\pi) + \varphi(\pi') - 0.$$ 

A consequence of Lemma 1 is the following integrality property.

**Lemma 3.** Assume that the weight function $w: E(G) \rightarrow \mathbb{R}_{\geq 0}$ takes only integer values, and that $C$ is also an integer. Then, for any $\alpha \in \{0, 1\}$, $\textsc{Minimum Activation Path}(G, w, s, t, C)$ has a positive answer if and only if $\textsc{Minimum Activation Path}(G, w, s, t, C + \alpha)$ has a positive answer.

**Proof.** Assume that $\textsc{Minimum Activation Path}(G, w, s, t, C + \alpha)$ is has a positive answer. Consider a power assignment $p$ corresponding to a feasible solution of minimum cost (at most $C + \alpha$); let $\pi$ be an $s$-$t$ path activated by $p$. Because of Lemma 1 we have $\text{cost}(p) = \text{opt}(\pi) = \text{cost}(p^*_n)$. From the inductive definition 1 of $p^*_n$, we see that $p^*_n$ assigns integral powers to all vertices, and thus $\text{cost}(p^*_n) = \sum_{v} p^*_n(v)$ is an integer, which is at most $C$. So $\textsc{Minimum Activation Path}(G, w, s, t, C)$ has a positive answer.

2.2 The reduction

Now we provide the reduction. The reduction is inspired by the reduction used to show that the restricted shortest path problem is NP-hard; this seems to be folklore and attributed to Megiddo by Garey and Johnson [5, Problem ND30]. We use the notation $[n] = \{1, \ldots, n\}$ and reduce from the following problem.

**SUBSET SUM**

Input: a sequence $a_1, \ldots, a_n$ of positive integers and a positive integer $b$.

Question: is there a set of indices $I \subseteq [n]$ such that $\sum_{i \in I} a_i = b$?

The problem **SUBSET SUM** is one of the standard weakly NP-hard problems that can be solved in pseudopolynomial time via dynamic programming [5, Section 4.2]. In particular, when the numbers $a_i$ are bounded by a polynomial in $n$, the problem can be solved in polynomial time.

Set $L$ to be an integer strictly larger than $2 \sum_{i \in [n]} a_i$. Then, for each $I \subseteq [n]$ we have $2 \sum_{i \in I} a_i < L$.

We construct a graph $G = G(a_1, \ldots, a_n, b)$ as follows (see Figure 1). $G$ will include vertices $s$, $t$, $u_1, \ldots, u_n$. Let us use the notation $u_0 = s$. For each $i \in [n]$, we put between $u_{i-1}$ and $u_i$ two 2-edge
paths, one path with weights $L + 2a_i$ and $L + 2a_i$, and the other path with weights $L + a_i$ and $L + 3a_i$, as we go from $u_{i-1}$ to $u_i$. Finally, we put the edge $u_nt$ with weight $2b$. This finishes the construction of $G$.

**Lemma 4.** There exists a path $\pi$ from $s$ to $u_n$ in $G$ with $\text{opt}(\pi) = c$ and $\varphi(\pi) = r$ if and only if there exists $I \subseteq [n]$ such that

$$r = 2 \sum_{i \in I} a_i \quad \text{and} \quad c = nL + 2 \sum_{i \in [n]} a_i + \sum_{i \in I} a_i.$$  

**Proof.** Consider the two 2-paths connecting $u_{i-1}$ to $u_i$. We refer to the path with weights $L + 2a_i$ the upper choice at $i$, and the path with weights $L + a_i$ and $L + 3a_i$ the lower choice at $i$. See Figure 2.

Assume that we have a path $\pi'$ that goes from $s = u_0$ to $u_{i-1}$ with $\varphi(\pi') \leq L$. Let $\pi'_u$ be the concatenation of $\pi'$ with the upper choice, and let $\pi'_l$ be the concatenation of $\pi'$ with the lower choice. Because of Lemma 2 we obtain that $\text{opt}(\pi'_u) = \text{opt}(\pi') + L + 2a_i$ and $\varphi(\pi'_u) = \varphi(\pi')$, while $\text{opt}(\pi'_l) = \text{opt}(\pi') + L + 3a_i$ and $\varphi(\pi'_l) = \varphi(\pi') + 2a_i$. See Figure 2. Here, the assumption $\varphi(\pi') \leq L$ has been important to ensure that in using Lemma 2 the maximum defining $\varphi(\cdot)$ is not at 0. It easily follows by induction on $i$ that, for each path $\pi'$ from $s = u_0$ to $u_i$, we indeed have $\varphi(\pi') \leq \sum_{j=1}^i 2a_i$, and thus the hypothesis is fulfilled for each $i \in [n]$.

The intuition here is that the lower choice has a larger cost, but keeps more power at the extreme of the prefix path for later use. See Figure 3 for a concrete example showing the values $\text{opt}(\pi')$ and $\varphi(\pi')$ for paths $\pi'$ from $s = u_0$ to $u_i$. It also helps understanding the idea behind the reduction.

Consider now a path $\pi$ from $s = u_0$ to $u_n$. Let $I$ be the set of indices $i \in [n]$ where the path takes the lower choice at $i$. From the previous discussion and a simple induction we have

$$\text{opt}(\pi) = \sum_{i \in [n] \setminus I} (L + 2a_i) + \sum_{i \in I} (L + 3a_i) = nL + \sum_{i \in I} 2a_i + \sum_{i \in I} a_i$$
Figure 3: Top: The graph $G$ for $n = 4$ with $a_1, \ldots, a_4 = 2, 3, 3, 2$ and $b = 7$, when we take $L = 22$. We have to decide whether there is an assignment of power with cost $nL + 2 \sum_{i} a_i + b = 115$ that activates some $s$-$t$ path. Bottom: pairs $(\text{opt}(\pi'), \varphi(\pi'))$ for all the $s$-$u_i$ paths $\pi'$.

and

$$\varphi(\pi) = 2 \sum_{i \in I} a_i \leq L.$$ 

Since all the paths from $s$ to $u_n$ must follow the upper or lower choice at each $i \in [n]$, the result follows.

**Lemma 5.** For any real numbers $A$ and $B$ we have

$$A + \max\{2B - 2A, 0\} \leq B \implies A = B.$$ 

**Proof.** If $A \leq B$, then $B - A \geq 0$ and the assumption implies $A + (2B - 2A) \leq B$, which implies $B \leq A$, and thus $A = B$. If $A > B$, then $B - A < 0$ and the assumption implies $A + 0 \leq B$, which implies $A \leq B$. Thus this cannot happen.

**Theorem 6.** The problem **Minimum Activation Path** is NP-hard.

**Proof.** We show that the instance for **Subset Sum** has a positive answer if and only if in the graph $G = G(a_1, \ldots, a_n, b)$ there is a power assignment with cost at most $C := nL + 2 \sum_{i \in [n]} a_i + b$ that activates some path from $s$ to $t$. 

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Assume that the exists a solution for the instance to the SUBSET SUM problem. This means that we have some $I \subseteq [n]$ such that $\sum_{i \in I} a_i = b$. Because of Lemma 4 there exists a path $\pi$ from $s = u_0$ to $u_n$ with optimal activation cost $\text{opt}(\pi) = nL + 2 \sum_{i \in [n]} a_i + \sum_{i \in I} a_i = C$ and $\varphi(\pi) = 2b$. Because of Lemma 5, this means that the power assignment $p^*_\pi$ has cost $\text{cost}(p^*_\pi) = C$, activates all edges of $\pi$, and assigns power $p^*_\pi(u_n) = \varphi(\pi) = 2b$ to vertex $u_n$. Such power assignment $p^*_\pi$ also activates the edge $u_nt$ because it has weight $2b = p^*_\pi(u_n)$. (In particular, the vertex $t$ gets power 0.)

Assume now that there is a power assignment $p' \geq 0$ with cost at most $C$ that activates a path $\pi'$ from $s$ to $t$. Let $\pi$ be the restriction of $\pi'$ from $s$ to $u_n$. Because of Lemma 2 and using that the power assignment $p'$ activates $\pi'$, we have

$$\text{opt}(\pi) + \max\{2b - \varphi(\pi), 0\} = \text{opt}(\pi') \leq \text{cost}(p') \leq C. \quad (2)$$

Because of Lemma 4 there exists some $I \subseteq [n]$ such that

$$\text{opt}(\pi) = nL + 2 \sum_{i \in [n]} a_i + \sum_{i \in I} a_i \quad \text{and} \quad \varphi(\pi) = 2 \sum_{i \in I} a_i.$$

Substituting in (2), for such $I \subseteq [n]$ we have

$$nL + 2 \sum_{i \in [n]} a_i + \sum_{i \in I} a_i + \max\{2b - 2 \sum_{i \in I} a_i, 0\} \leq C = nL + 2 \sum_{i \in [n]} a_i + b.$$

This means that

$$\sum_{i \in I} a_i + \max\{2b - 2 \sum_{i \in I} a_i, 0\} \leq b.$$

Because of Lemma 5, we conclude that $\sum_{i \in I} a_i = b$, and the given instance to SUBSET SUM problem has a solution. \hfill \Box

### 3 Minimum Barrier Shrinkage

In this section, we show that the MINIMUM BARRIER SHRINKAGE problem is NP-hard. The structure of the proof is very similar to the proof given in Section 2.2 for the NP-hardness of the problem MINIMUM ACTIVATION PATH.

We first give the construction assuming that we can compute algebraic numbers to infinite precision. Then we explain how an approximate construction with enough precision suffices and can be computed in polynomial time.

The penetration depth of a pair $(D(c, r), D(c', r'))$ of open disks $D(c, r)$ and $D(c', r')$ is $r + r' - |c - c'|$, where $|c - c'|$ is the distance between the centers $c$ and $c'$. When no disk contains the center of the other disk, and they intersect, then the intersection $D(c, r) \cap D(c', r')$ is a lens of width equal to the penetration depth. See Figure 4. If we shrink the disks to $D(c, r - \delta)$ and $D(c', r' - \delta')$, the disks intersect if and only if $\delta + \delta'$ is strictly smaller than the penetration depth. (Recall that disks are taken as open sets.) Thus, the penetration depth equals the minimum total shrinking of the disks so that a curve can pass between the two disks.

We reduce again from SUBSET SUM. Consider an instance $I$ of SUBSET SUM given by a sequence $a_1, \ldots, a_n$ of positive integers and a positive integer $b$. Set $L$ to be an integer strictly larger than $2 \sum_{i \in [n]} a_i$. Then, for each $I \subseteq [n]$ we have $2 \sum_{i \in I} a_i < L$. Set $C = nL + 2 \sum_{i \in [n]} a_i + b$ and $\lambda = 10C$. We will construct an instance to MINIMUM BARRIER SHRINKAGE problem such that it has a solution if and only if the instance $I$ for SUBSET SUM has a solution.

Figure 5 shows the overall idea of the construction. Most of the action is happening around the filled (blue and green) disks. The remaining white disks create corridors to communicate from
one side to the other of the filled disks. To provide a feasible solution of cost at most $C$, we have to indicate how to shrink the disks for a total radius of at most $C$ and provide an $x$-$y$ curve in the plane that does not touch the (interior of the) shrunken disks.

In our construction, no point of the plane will be covered by more than two disks. In such a case, the $x$-$y$ curve can be described combinatorially by a sequence of pairs of disks such that, for each pair $(D, D')$, the curve passes between $D$ and $D'$, after shrinking.

If the penetration depth of two disks is at least $\lambda = 10C$, then, in any shrinking of the disks with total cost at most $C$, those two disks keep intersecting, which means that we cannot route the $x$-$y$ curve between those two disks. More precisely, the segment connecting the centers of such disks cannot be crossed by the $x$-$y$ curve. In the drawings we indicate this with a thick segment connecting the centers of the disks.

The main part to encode the instance, around the filled disks, consists of the following disks. See Figures 6 and 7.

- For $i = 0, \ldots, n$, a disk $D_i$ of radius $4\lambda$ centered at $((4\lambda) \cdot 2i, 0)$;
- for each $i \in [n]$, a disk $D'_i$ of radius $\lambda$ centered at $((4\lambda) \cdot (2i - 1), 0)$;
- for each $i \in [n]$, a disk $A_i$ (for above) of radius $3\lambda$ placed such that the center is above the $x$-axis, and the penetration depth of $(A_i, D_{i-1})$ and $(A_i, D_i)$ is $L + 2a_i$; this means that the distance between center($A_i$) and center($D_{i-1}$) is $7\lambda - (L + 2a_i)$, and the distance between center($A_i$) and center($D_i$) is $7\lambda - (L + 2a_i)$;
- for each $i \in [n]$, a disk $B_i$ (for below) of radius $3\lambda$ placed such that the center is below the $x$-axis, the penetration depth of $(B_i, D_{i-1})$ is $L + a_i$ and the penetration depth of $(B_i, D_i)$ is $L + 3a_i$; this means that the distance between center($B_i$) and center($D_{i-1}$) is $7\lambda - (L + a_i)$ and the distance between center($B_i$) and center($D_i$) is $7\lambda - (L + 3a_i)$;
- a disk $A_{n+1}$ of radius $3\lambda$ placed such that the center is above the $x$-axis, the $x$-coordinate of center($A_{n+1}$) is $(4\lambda) \cdot (2n + 1)$, and the penetration depth of $(A_{n+1}, D_n)$ is $2b$; $A_{n+1}$ is one of the green disks in the figures;
- a disk $B_{n+1}$ of radius $3\lambda$ placed such that the center is below the $x$-axis, the $x$-coordinate of center($B_{n+1}$) is $(4\lambda) \cdot (2n + 1)$, and the penetration depth of $(B_{n+1}, D_n)$ is $2b$; $B_{n+1}$ is another of the green disks in the figures;
- for each $i \in [n + 1]$, a disk $A'_i$ of radius $3\lambda$ centered at $((4\lambda) \cdot (2i - 1), 8\lambda)$ and a disk $B'_i$ of radius $3\lambda$ centered at $((4\lambda) \cdot (2i - 1), -8\lambda)$.

For $i \in [n]$, the block $\mathbb{B}_i$ consists of the disks $D_{i-1}$, $D_i$, $D'_i$, $A_i$, $A'_i$, $B_i$ and $B'_i$. We also define the block $\mathbb{B}_{n+1}$ as the group of disks $D_n$, $A_{n+1}$, $A'_{n+1}$, $B_{n+1}$ and $B'_{n+1}$. Note that the blocks $\mathbb{B}_i$ and $\mathbb{B}_{i+1}$, for $i \in [n]$, share the disk $D_i$.

![Diagram](image-url)
Figure 5: Basic idea of the construction for $n = 4$. All the shrinking of disks and the decisions on how to route the $x$-$y$ curve are happening around the (blue and green) filled disks. The thick lines will not be crossed by any $x$-$y$ curve that is disjoint from the shrunken disks in a solution with the desired cost.

For each $i \in [n+1]$, we make a path of disks of radius $3\lambda$, starting from $A'_i$ and finishing with $B'_i$, where any two consecutive disks have penetration depth at least $3\lambda$. The disks in these paths are pairwise disjoint for different indices $i$, and disjoint from the rest of the construction. The disks in each such path can be centered along a 5-link axis-parallel path, and it uses $O(i)$ disks. See Figure 5. We denote the path of disks for the index $i \in [n+1]$ by $\Pi_i$. For later use, we place a point $y_i$ in the “tunnel” between the paths $\Pi_i$ and $\Pi_{i+1}$. See Figure 5.

Lemma 7. For each $i \in [n]$, the disks $D'_i$, $A_i$ and $B_i$ are pairwise disjoint. Moreover, the penetration depth of the pairs $(D_{i-1}, D'_i)$, $(D_i, D'_i)$, $(A_i, A'_i)$ and $(B_i, B'_i)$ is at least $\lambda$. For $B_{n+1}$, the disks $A_{n+1}$ and $B_{n+1}$ are disjoint and the penetration depth of the pairs $(A_{n+1}, A'_{n+1})$ and $(B_{n+1}, B'_{n+1})$ is at least $\lambda$.

Proof. We consider only the case $i \in [n]$. The arguments for $B_{n+1}$ are similar. The penetration depth of the pairs $(D_{i-1}, D'_i)$ and $(D_i, D'_i)$ is $\lambda$ by construction.

Consider the disk $\tilde{A}_i$ of radius $3\lambda$ centered at $((4\lambda) \cdot (2i - 1), 5\lambda)$ and the disk $\tilde{B}_i$ of radius $3\lambda$ centered at $((4\lambda) \cdot (2i - 1), -5\lambda)$. See Figure 5. We will compare $B_i$ to $\tilde{B}_i$; note that they have the same size, just a different placement. The argument for $A_i$ is the same.
The penetration depth of the pairs \((\tilde{B}_i, D_{i-1})\) and \((\tilde{B}_i, D_i)\) is

\[
3\lambda + 4\lambda - \sqrt{(5\lambda)^2 + (4\lambda)^2} = (7 - \sqrt{41})\lambda \approx 0.59687\lambda,
\]

while the penetration depth of \((\tilde{B}_i, B'_i)\) is exactly 3λ. The disk \(D'_i\) is at distance \(\lambda\) from \(\tilde{B}_i\).

Since the penetration depth of \((\tilde{B}_i, D_{i-1})\) and \((\tilde{B}_i, D_i)\) is at most \(L + 3a_i \leq C = \lambda/10\), these penetration depths are smaller than the penetration depths of \((\tilde{B}_i, D_{i-1})\) and \((\tilde{B}_i, D_i)\), namely, between 0 and 0.59687\(\lambda\). See Figure 9. As can be seen on the figure (and proved by a slightly involved computation), this implies that \(B_i\) and \(D'_i\) are disjoint, and that the disk \(B'_i\) contains the center of \(B_i\). The latter fact implies that the penetration depth of \((B_i, B'_i)\) is at least 3\(\lambda\).

From Lemma 7 we conclude that, in any solution with cost under \(\lambda = 10C\), the \(x\)-\(y\) curve cannot cross the segments connecting center\((D_{i-1})\) and center\((D_i)\), the segments connecting center\((A_i)\) and center\((A'_i)\), nor the segments connecting center\((B_i)\) and center\((B'_i)\), for each \(i \in [n+1]\). Furthermore, it cannot cross the path \(\Pi_i\) connecting \(A'_i\) to \(B'_i\), for each \(i \in [n+1]\). This implies that, at each block \(B_i\), we have to decide whether the \(x\)-\(y\) curve goes above (crossing \(A_i\) before shrinking) or below (crossing \(B_i\) before shrinking). See Figure 10 for one such choice.

So, in a nutshell, the strategy is to reformulate the problem in terms of graphs, and to note that the instance is equivalent to the Minimum Activation Path in that graph. Let \(X_i = A_i\) or \(X_i = B_i\), depending on the choice of how to route the \(x\)-\(y\) curve. If \(X_i = A_i\), then the \(x\)-\(y\) curve, after shrinking the disks, passes between \(D_{i-1}\) and \(A_i\), and also between \(D_i\) and \(A_i\). If \(X_i = B_i\), then the \(x\)-\(y\) curve, after shrinking the disks, passes between \(D_{i-1}\) and \(B_i\), and also between \(D_i\) and \(B_i\).
Figure 7: The blocks $\mathbb{B}_n$ and $\mathbb{B}_{n+1}$; the penetration depths are not to scale.

Note that we can assume that the $x$-$y$ curve passes between two disks at most once. Moreover, for each disk $D_i$, the $x$-$y$ curve passes between $D$ and another disk at most twice. Once we decide the combinatorial routing of the $x$-$y$ curve, that is, once we select $X_1, \ldots, X_n, X_{n+1}$, then greedily shrinking the disks gives an optimal solution, similarly to Lemma 1: it pays off to push the shrinking towards disks that are crossed later by the $x$-$y$ curve. That is, to pass between $D_1$ and $X_1$, it pays off to do not shrink $D_1$, as it is never crossed again, and shrink $X_1$ just enough to pass in between. Similarly, it pays off to shrink $D_2$ to pass between $D_2$ and $X_1$, because $X_1$ will not be crossed again later on. In general, to pass between $D_{i-1}$ and $X_i$ it pays off to reduce $X_i$ just enough to pass between them, taking into account how much $D_{i-1}$ was already shrunken, and to pass between $X_i$ and $D_i$ it pays off to reduce $D_i$ just enough to pass between them, taking into account how much $X_i$ was already reduced.

Let $D = \mathbb{D}(I)$ be the set of all disks in the constructed instance.

**Lemma 8.** The instance $I = (a_1, \ldots, a_n, b)$ to SUBSET SUM has a solution if and only if the instance $(\mathbb{D}, x, y, C)$ to MINIMUM BARRIER SHRINKAGE has a positive answer, where $C = nL + 2 \sum_{i \in [n]} a_i + b$. Furthermore, for any $\alpha \in [0, 1)$, MINIMUM BARRIER SHRINKAGE$(\mathbb{D}, x, y, C)$ has a positive answer if and only if MINIMUM BARRIER SHRINKAGE$(\mathbb{D}, x, y, C + \alpha)$ has a positive answer.

**Proof.** We construct a graph $G'$ as follows. We make a node for each connected component of $\mathbb{R}^2 \setminus \bigcup \mathbb{D}$ that may be crossed by the $x$-$y$ curve after shrinking disks for a cost strictly smaller than $\lambda = 10C$. This means that we have the following nodes in the graph:

- a node for the cell containing $x$, which we call $x$ also;
- a node for the cell containing $y$, which we call $y$ also;
- a node called $\alpha_i$ for the region bounded between the disks $D_{i-1}, D_i, A_i$ ($i \in [n]$);
- a node called $\beta_i$ for the region bounded between the disks $D_{i-1}, D_i, B_i$ ($i \in [n]$);
\[ \text{Figure 8: Disks } \tilde{A}_i \text{ and } \tilde{B}_i \text{ considered in the proof of Lemma 7.} \]

- a node for the cell that contains \( y_i \) (\( i \in [n] \)), that is, the tunnel bounded by \( \Pi_i \) and \( \Pi_{i+1} \); we call the node \( y_i \) also.

We put an edge between two nodes whenever we can pass from one region to the other passing between two disks with penetration strictly below \( \lambda = 10C \). See Figure 11 for the resulting graph, \( G' \). This graph \( G' \) is essentially the graph \( G(a_1, \ldots, a_n, b) \) used in Section 2.2. (The only difference is that, in \( G' \), we have two parallel edges from \( y_n \) to \( y_i \), instead of a single edge.)

We assign a weight to each edge of \( G' \) equal to the penetration depth of the pair of disks that separate the cell. For example, the edges \( y_{i-1} \alpha_i \) and \( \alpha_i y_i \) have weight \( L + 2a_i \) (\( i \in [n] \)), the edge \( \beta_i y_i \) has weight \( L + 3a_i \) (\( i \in [n] \)), and the two parallel edges \( y_n y_i \) have weight \( 2b \).

There is a simple correspondence between power assignments \( p(\cdot) \) that give a feasible solution for MINIMUM ACTIVATION PATH(\( G', x, y, C \)) and the reduction in radii for feasible solutions for MINIMUM BARRIER SHRINKAGE(\( D, x, y, C \)), as follows:

- the decrease in radius of \( D_i \) corresponds to the power \( p(y_i) \) (\( i \in [n] \));
- the decrease in radius of \( A_i \) corresponds to the power \( p(\alpha_i) \) (\( i \in [n] \));
- the decrease in radius of \( B_i \) corresponds to the power \( p(\beta_i) \) (\( i \in [n] \));
- the decrease in radius of \( D_0 \) corresponds to the power \( p(x) \);
- we may assume that at most one of the disks \( A_{n+1} \) and \( B_{n+1} \) is shrunken; the decrease in radius of \( A_{n+1} \) or \( B_{n+1} \), whichever is larger, corresponds to the power \( p(y) \);
- we may assume that all other disks are not shrunken.

This correspondence transforms feasible solutions for MINIMUM ACTIVATION PATH(\( G', x, y, C \)) into feasible solutions for MINIMUM BARRIER SHRINKAGE(\( D, x, y, C \)), and conversely. So the instances MINIMUM ACTIVATION PATH(\( G', x, y, C \)) and MINIMUM BARRIER SHRINKAGE(\( D, x, y, C \)) are equivalent.
Theorem 9. The Minimum Barrier Shrinkage problem is NP-hard.

Proof. Consider an instance \((a_1, \ldots, a_n, b)\) for Subset Sum and the associated instance \((D, x, y, C)\) for Minimum Barrier Shrinkage constructed above, with \(C = nL + 2 \sum_{i\in[n]} a_i + b\).

The centers of the disks in \(D \setminus \{A_1, \ldots, A_{n+1}, B_1, \ldots, B_{n+1}\}\) are integers bounded by \(O(\lambda) = O(nL)\). For each \(i \in [n+1]\), we compute the centers of the disks \(A_i, B_i\) up to a precision of at least \(\epsilon = \frac{1}{6(n+1)}\). Thus, the coordinates of the centers are multiples of \(\epsilon\). Let \(\hat{A}_i, \hat{B}_i\) be the resulting disks; they have the same radius, \(3\lambda\), but have been displaced by at most \(\epsilon\) with respect to the original position in the construction. Let \(\hat{D}\) be the set of disks obtained from \(D\), where each \(A_i, B_i\) are replaced with \(\hat{A}_i, \hat{B}_i\) \((i \in [n+1])\).

We consider instances of Minimum Barrier Shrinkage. If the instance \((D, x, y, C)\) is positive, then the instance \((\hat{D}, x, y, C + 1/3)\) is also positive (because each of the \(2(n+1)\) disks are moved by at most \(\epsilon\), so the total displacement is at most \(1/3\)), which implies that the instance \((D, x, y, C + 2/3)\) is also positive (by the same argument), which in turn also implies that the instance \((D, x, y, C)\) is positive (by Lemma 3). So, the instances \((D, x, y, C)\) and \((\hat{D}, x, y, C + 1/3)\) are equivalent.

Scaling all values in the construction of \(\hat{D}\) (coordinates and radii) by \(1/\epsilon\), we get a construction where the disks have centers with integer coordinates, the radii are integers, and the whole construction can be constructed in polynomial time.

Remark. A similar statement can be done for axis-parallel squares. For this we have to place the overlapping squares in such a way that the overlap region, an axis-parallel rectangle, has width
Figure 10: The red $x$-$y$ curve shows the type of decisions that have to be made to make a feasible solution. In this example, we have to decide 5 times independently whether the $x$-$y$ curve is routed above or below. Note that the curve can be routed to pass through each $y_i$, if desired.

equal to the value we want to encode ($L + a_i, L + 2a_i, L + 3a_i$ or $2b$). In such a case we do not run into the numerical issues with the centers because all the coordinates can be taken directly to be integers.

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Figure 11: The combinatorially different $x$-$y$ curves can be encoded in a graph, denoted $G'$. This is essentially the same graph $G(a_1, \ldots, a_n, b)$ used in Section 2.2 but with a different drawing; see Figure 1.

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