The algebra of Kleene stars of the plane and polylogarithms

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ABSTRACT

We extend the definition and study the algebraic properties of the polylogarithm \( L_\gamma \), where \( T \) is rational series over the alphabet \( X = \{ x_0, x_1 \} \) belonging to \( (\mathcal{C}(X) \cup \text{Crat}(\langle x_0 \rangle)) \cup \text{Crat}(\langle x_1 \rangle) \cup \mathcal{I}_X \).

Keywords

Algebraically independent ; Polylogarithms ; Transcendent.

1. Introduction

In all the sequel of this text,
1. We consider the differential forms
\[
\omega_0(z) = \frac{dz}{z} \quad \text{and} \quad \omega_1(z) = \frac{dz}{1-z}.
\]
We denote \( \Omega \) the cleft plane \( \mathbb{C} - \{ 0, \infty \} \cup \{ 1, +\infty \} \) and \( \lambda \) the rational fraction \( \frac{1}{1-z} \) belonging to the differential unitary ring \( \mathcal{U} := \mathbb{C}[z, z^{-1}, (1-z)^{-1}] \) with the differential operator \( \partial_z := d/dz \) and with the unitary element
\[
1_\Omega : \Omega \longrightarrow \mathbb{C}, \quad z \longrightarrow 1.
\]
2. We construct, over the alphabets
\[
X = \{ x_0, x_1 \}, \quad Y = \{ y_k \}_{k \geq 1} \quad \text{and} \quad Y_0 = Y \cup \{ y_0 \},
\]
totally ordered by \( x_0 < x_1 \) and \( y_0 > y_1 > \cdots \), respectively, the bialgebra
\[
(\mathcal{C}(X), \text{conc}, \Delta_{\mathcal{A}_X}, 1_X, \varepsilon), \quad (\mathcal{C}(Y), \text{conc}, \Delta_{\mathcal{A}_Y}, 1_Y, \varepsilon).
\]

These algebras, when endowed with their dual laws, are equipped with pure transcendence bases in bijection with the set of Lyndon words \( \mathcal{L}(X) \), \( \mathcal{L}(Y) \) and \( \mathcal{L}(Y_0) \) respectively.

Let us consider also the following morphism
\[
\pi_\mathcal{C} : (\mathcal{C} \oplus \mathcal{C}(X))_{x_1}, \text{conc} \longrightarrow (\mathcal{C}(Y)_s), \quad x_0^{r_1-1} x_1 \longrightarrow y_1 \cdots y_s,
\]
for \( r \geq 1 \) and, for any \( a \in \mathbb{C}, \pi_\mathcal{C}(a) = a \). The extension of \( \pi_\mathcal{C} \) over \( \mathcal{C}(X) \) is denoted by \( \pi_\mathcal{C} : \mathcal{C}(X) \longrightarrow \mathcal{C}(Y) \) satisfying, for any \( p \in \mathcal{C}(X) x_0, \pi_\mathcal{C}(p) = 0 \). Hence,
\[
\ker(\pi_\mathcal{C}) = \mathcal{C}(X) x_0 \quad \text{and} \quad \text{Im} (\pi_\mathcal{C}) = \mathcal{C}(Y).
\]

Let \( \pi_X \) be the inverse of \( \pi_\mathcal{C} \) :
\[
\pi_X : \mathcal{C}(Y) \longrightarrow \mathcal{C} \oplus \mathcal{C}(X) x_1,
\]
\[
y_1 \cdots y_s \longrightarrow x_0^{r_1-1} x_1 \cdots x_0^{r_s-1} x_1.
\]

The projectors \( \pi_X \) and \( \pi_\mathcal{C} \) are mutual adjoints :
\[
\forall p \in \mathcal{C}(X), \forall q \in \mathcal{C}(Y), \quad \langle \pi_\mathcal{C}(p) | q \rangle = \langle p | \pi_X(q) \rangle.
\]

In continuation of \([5, 7]\), the principal object of the present work is the polylogarithm well defined, for any \( r \)-uplet \( (s_1, \ldots, s_r) \in \mathbb{C}^r \), \( r \in \mathbb{N} \) and for any \( z \in \mathbb{C} \) such that \( |z| < 1 \), as follows
\[
\text{Li}_{s_1, \ldots, s_r}(z) := \sum_{n_1, \ldots, n_r \geq 0} \frac{z^{n_1}}{n_1! \cdots n_r!}.
\]

Then the Taylor expansion of the function \( (1-z)^{-1} \text{Li}_{s_1, \ldots, s_r}(z) \) is given by
\[
\frac{\text{Li}_{s_1, \ldots, s_r}(z)}{1-z} = \sum_{N \geq 0} H_{s_1, \ldots, s_r}(N) z^N,
\]
where the coefficient \( H_{s_1, \ldots, s_r} : \mathbb{N} \longrightarrow \mathbb{Q} \) is an arithmetic function, also called harmonic sum, which can be expressed as follows
\[
H_{s_1, \ldots, s_r}(N) := \sum_{N \geq n_1, \ldots, n_r \geq 0} \frac{1}{n_1! \cdots n_r!}.
\]

From the analytic continuation of polyzetas \([9, 24]\), for any \( r \geq 1 \), if \( (s_1, \ldots, s_r) \in \mathcal{H}_r \) satisfies \( 3 \) then \( 4 \), after a theorem by Abel, one obtains the polyzeata as follows
\[
\lim_{z \to 1} \text{Li}_{s_1, \ldots, s_r}(z) = \lim_{N \to \infty} H_{s_1, \ldots, s_r}(N) = \zeta(s_1, \ldots, s_r).
\]

This theorem is no more valid in the divergent cases (for \( (s_1, \ldots, s_r) \in \mathcal{H}_r \)) and require the renormalization of the corresponding divergent

2. With a little abuse of language, \( \pi_X \) is now considered as targeted to \( \mathcal{C}(X) \).

3. For \( r \geq 1 \), \( \zeta(s_1, \ldots, s_r) \) is as a meromorphic function on \( \mathcal{H}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r | \forall m = 1, \ldots, r, \Re(s_1) + \ldots + \Re(s_m) > 1 \} \).
polyzetas. It is already done for the corresponding case of polyzetas at positive multi-indices [3][4][20] and it is done [8][1][22] and completed in [5][7] for the case of polyzetas at positive multi-indices.

To study the polylogarithms at negative multi-indices, one relies on [5][7]

1. the (one-to-one) correspondence between the multi-indices \((s_1, \ldots, s_r) \in \mathbb{N}^r\) and the words \(y_{s_1} \ldots y_{s_r}\), defined over \(Y_0\).

2. indexing these polylogarithms by words \(y_{s_1} \ldots y_{s_r}\) :

\[
Li_{y_{s_1} \ldots y_{s_r}}(z) = \sum_{n_1 > \cdots > n_r > 0} n_1^{s_1} \cdots n_r^{s_r} z^{n_1}.
\]

In the same way, for polylogarithms at positive indices, one relies on [15][17]

1. the (one-to-one) correspondence between the combinatorial compositions \((x_1, \ldots, x_r)\) and the words \(x_0^{s_1-1}x_1 \cdots x_0^{s_r-1}x_r\) in \(X^*x_1 + 1\).

2. the indexing of these polylogarithms by words \(x_0^{s_1-1}x_1 \cdots x_0^{s_r-1}x_r\) :

\[
Li_{x_0^{s_1-1}x_1 \cdots x_0^{s_r-1}x_r}(z) = \sum_{n_1 > \cdots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \cdots n_r^{s_r}}.
\]

Moreover, one obtained the polylogarithms at positive indices as image by the following isomorphism of the shuffle algebra [1][9]

\[
Li_*: (\mathbb{C}(X), \cdot, 1x^* \cdot) \rightarrow (\mathbb{C}[Li_w]_{w \in X^*}, \cdot, 1\Omega),
\]

where \(\lambda \) is the closure by rational operations \(\mathbb{C}^{rat}(X)\), as follows

\[
\begin{align*}
S &= \sum_{n \geq 0} \langle S \, | \, x_0^{n} \rangle x_0^n + \sum_{k \geq 1} \sum_{w \in \{x_0^n \} \, x_0^k} \langle S \, | \, w \rangle w, \\
Li_{x_0^n}(z) &= \sum_{n \geq 0} \langle S \, | \, x_0^n \rangle \frac{\log^n \langle z \rangle}{n!} + \sum_{k \geq 1} \sum_{w \in \{x_0^n \} \, x_0^k} \langle S \, | \, w \rangle Li_w,
\end{align*}
\]

the morphism \(Li_*\) is no longer injective over \(\mathbb{C}^{rat}(X)\) but \(\mathbb{C}[Li_w]_{w \in X^*}\) are still linearly independent over \(\mathbb{C}\) [20][19].

**Example 1.**

1. \(\lambda = Li_{x_1} = Li_{x_1 \cdots x_0^{s_r-1}x_r}\).

2. \(\lambda = Li_{x_0^{s_1-1}x_1 \cdots x_0^{s_r-1}x_r}\).

3. \(\mathbb{C}[Li_{x_0}, Li_{x_1 \cdots x_0^{s_r-1}x_r}]\).

4. \(\mathbb{C}[Li_{x_0}, Li_{x_1 \cdots x_0^{s_r-1}x_r}] \subseteq \mathbb{C}[x_1, \ldots, x_r] \subseteq \mathbb{C}(X)\).

Let us consider also the differential and integration operators, acting on \(\mathcal{C}(Li_w)_{w \in X^*}\) [21] :

\[
\partial_t = \frac{d}{dz}, \quad \theta_t = z \frac{d}{dz}, \quad \theta_t = (1 - z) \frac{d}{dz},
\]

\(\forall f \in \mathcal{C}^*, \quad t_0(f) = \int_{t_0} f(s) \omega_0(s) \) and \(t_1(f) = \int_{t_1} f(s) \omega_1(s) \).

4. As follows defined on a superscript of the of Lyndon words, as pure transcendence basis, and extended by algebraic specialization [12][13].

5. \(\mathbb{C}^{rat}(X)\) is the closure by rational operations \(+, \cdot, \text{conc.}^*\) of \(\mathbb{C}(X)\), where, for \(S \in \mathbb{C}(X)\) such that \(\langle S \, | \, 1x^* \rangle = 0\), one has [1]

\[
S' = \sum_{k \geq 0} S_k.
\]

Here, the operator \(t_0\) is well-defined (as in definition [1] in section 2.2) then one can check easily [18][19][57]

1. The subspace \(\mathcal{C} \{ Li_w \}_{w \in X^*}\) is closed under the action of \(\{ \theta_t, \theta_1 \}\) and \(\{ t_0, t_1 \}\).

2. The operators \(\{ \theta_t, \theta_0, t_0, t_1 \}\) satisfy in particular,

\[
\theta_t + \theta_0 = [\theta_1, \theta_1] = \partial_t \quad \forall k \geq 0, \quad \theta_1 t_k = \text{Id}.
\]

3. \(\theta_0 t_0 + t_0 \theta_0\) are scalar operators with \(\mathcal{C} \{ Li_w \}_{w \in X^*}\), respectively with eigenvalues \(\lambda\) and \(1/\lambda\), i.e.

\[
(\theta_0 t_0) f = \lambda f \quad \text{and} \quad (\theta_1 t_0) f = (1/\lambda) f.
\]

4. Let \(w = y_{s_1} \cdots y_{s_r} \in Y^*\) (then \(\pi_X(w) = x_0^{s_1-1} \cdots x_0^{s_r-1}x_1\) and \(u = y_{s_1} \cdots y_{s_r} \in Y_0^r\). The functions \(Li_w, \lambda_{i_0}\) satisfy

\[
Li_w = (t_0^{s_1-1} \cdots t_0^{s_r-1} t_1)^i \Omega, \quad \lambda_{i_0} = (\theta_0^{s_1+1} \cdots \theta_0^{s_r+1} t_1)^i \Omega, \quad t_0 Li_{i_0}(w) = Li_{i_0}(\pi_X(w)), \quad t_1 Li_{i_0}(w) = Li_{i_0+1}(\pi_X(w)), \quad t_0 Li_{i_0+1}(w) = Li_{i_0}(\pi_X(w)), \quad t_1 Li_{i_0+1}(w) = Li_{i_0+2}(\pi_X(w)).
\]

Here, we explain the whole project of extension of \(Li_*\), study different aspects of it, in particular what is desired of \(i_0, i_1 = 0, 1\). The interesting problem in here is that what we do expect of \(i_1, i_2 = 0, 1\).

In fact, the answers are:

- it is a section of \(\theta_0, i_0 = 0, 1\); (i.e., takes primitives for the corresponding differential operators).

- it extends \(t_i, i = 0, 1\) (defined on \(C \{ Li_w \}_{w \in X^*}\), and very surprisingly, although not coming directly from Chen calculus, they provide a group-like generating series).

We will use this construction to extend \(Li_*\) to \(\mathcal{C} \{ Li_w \}_{w \in X^*}\) and, after that, we extend it to a much larger rational algebra.

2. Background

2.1 Standard topology on \(\mathbb{H}(\Omega)\)

The algebra \(\mathbb{H}(\Omega)\) is that of analytic functions defined over \(\Omega\). It is endowed with the topology of compact convergence whose semifinors are indexed by compact subsets of \(\Omega\), and defined by

\[
p_{K}(f) = \| f \|_{K} = \text{sup}_{z \in K} | f(z) |.
\]

Of course,

\[
p_{K_1 \cup K_2} = \text{sup} (p_{K_1}, p_{K_2}).
\]

and therefore the same topology is defined by extracting a fundamental subset of semifinors, here it can be choosen denumerable. As \(\mathbb{H}(\Omega)\) is complete with this topology it is a Frechet space and even, as

\[
p_{K}(fg) \leq p_{K}(f)p_{K}(g),
\]

it is a Frechet algebra (even more, as \(p_{K}(1_{\Omega}) = 1\) a Frechet algebra with unit).

With the standard topology above, an operator \(\phi \in \text{End}(\mathbb{H}(\Omega))\) is continuous iff (with \(K_i\) compacts of \(\Omega\))

\[
(\forall K_2)(\exists K_1)(\exists M_{12} > 0)(\forall f \in \mathbb{H}(\Omega))(\| \phi(f) \|_{K_2} \leq M_{12} \| f \|_{K_1}).
\]
2.2 Study of continuity of the sections $\theta_i$ and $t_i$

Then, $\mathcal{C}(\text{Li}_w, w \in X^r)$ and $\mathcal{H}(\Omega)$ being closed under the operators $\theta_i, t_i$, we will first build sections of them, then see that they are continuous and, propose (discontinuous) sections more adapted to renormalisation and the computation of associators.

For $z_0 \in \Omega$, let us define $t_{\theta_i}^{z_0} \in \text{End}(\mathcal{H}(\Omega))$ by

$$t_{\theta_i}^{z_0}(f) = \int_{z_0}^z f(s) \theta_i(s), \quad t_{t_i}^{z_0}(f) = \int_{z_0}^z f(s) t_i(s).$$

It is easy to check that $t_{\theta_i}^{z_0} = Id_{\mathcal{H}(\Omega)}$ and that they are continuous on $\mathcal{H}(\Omega)$ for the topology of compact convergence.

Hence the operators $t_{\theta_i}^{z_0}$ are also well defined on $\mathcal{C}(\text{Li}_w, w \in X^r)$ and it is easy to check that

$$t_{\theta_i}^{z_0}(\mathcal{C}(\text{Li}_w, w \in X^r)) \subset \mathcal{C}(\text{Li}_w, w \in X^r).$$

Due to the decomposition of $\mathcal{H}(\Omega)$ into a direct sum of closed subspaces $\mathcal{H}(\Omega) = \mathcal{H}_{\text{cont}} + \mathcal{H}_{\text{disc}}$, it is not hard to see that the graphs of $\theta_i$ are closed, thus, the $\theta_i$ are also continuous.

Much more interesting (and adapted to the explicit computation of associators) we have the operator $t_i$ (without superscripts), mentioned in the introduction and (rigorously) defined by means of a $C$-basis of $\mathcal{C}(\text{Li}_w, w \in X^r)$.

As $\mathcal{C}(\text{Li}_w, w \in X^r) = \mathcal{C}(\text{Li}_w, w \in X^r) \otimes C$ and $\mathcal{P}_m$ can be approached by two limits. For continuity, we should have “equality of the limits of the image-sequences” which is not the case.

$$z = \sum_{n \geq 0} \log^n(z) n!,$$

$$z = \sum_{n \geq 1} (-1)^m \log^n(1 - z).$$

Let then

$$f_n = \sum_{0 \leq m \leq n} \frac{\log^m(z)}{m!} \text{ and } g_n = \sum_{1 \leq m \leq n} (-1)^{m+1} \text{log}^m(1 - z),$$

we have $f_n, g_n \in \mathcal{C}(\text{Li}_w, w \in X^r)$ and $t_0(f_n) = f_{n+1} - 1$. Hence, one has $\lim(t_0(f_n)) = z - 1$. On the other hand

$$\lim t_0(g_n) = \int_0^z g_n(s) \theta_i(s) = \int_0^z \lim t_0(g_n) \theta_i(s) = \int_0^z s \theta_i(s) = z.$$

The exchange of the integral with the limit above comes from the fact that the operator

$$\phi \mapsto \int_0^z \phi(s) \theta_i(s),$$

is continuous on the space $\mathcal{H}(\mathcal{H}(\Omega) \cup B(0,1))$ of analytic functions $f \in \mathcal{H}(\mathcal{H}(\Omega) \cup B(0,1))$ such that $f(0) = 0$ ($B(0,1)$ is the open ball of center $0$ and radius $1$).

3. Algebraic extension of Li to $\mathcal{C}(\mathcal{Y}(X), ..., \mathcal{Y}(X))$

We will use several times the following lemma which is characteristic.

**Lemma 1.** Let $(\mathcal{A}, d)$ be a commutative differential ring without zero divisor, and $R = \ker(d)$ be its subring of constants. Let $z \in \mathcal{A}$ such that $d(z) = 1$ and $S = \{e_A \mid A \in I\}$ be a set of eigenfunctions of $d$ all different (I $\subset R$) i.e.

i. $e_A \neq 0$,

ii. $d(e_A) = \alpha e_A$, $\alpha \in I$.

Then the family $(e_A)_{A \in I}$ is linearly free over $R[z]^R$.

**Proof.** If there is no non-trivial $R[z]$-linear relation, we are done. Otherwise let us consider relations

$$\sum_{j=1}^N P_j(z) e_A = 0,$$

6. Here $R[z]$ is understood as ring adjunction i.e. the smallest subring generated by $R \cup \{z\}$. 


with $P_j \in R[t]_{j \neq 0}$ for all $j$ (packed linear relations). We choose one minimal w.r.t. the triplet
\[
[N, \deg(P_j), \sum_{j < N} \deg(P_j)],
\]
lexicographically ordered from left to right.  

Remark that $d(P(z)) = P'(z)$ (because $d(z) = 1$), we apply the operator $d - \alpha d^2$ to both sides of (6) and get
\[
\sum_{j=1}^N \left( P'_j(z) + (\alpha_j - \alpha) P_j(z) \right) e_{\alpha_j} = 0.
\]
Minimality of $R$ implies that (5) is trivial i.e.
\[
P'_j(z) = 0; \quad (\forall j \in \mathbb{N} - 1)(P'_j(z) + (\alpha_j - \alpha) P_j(z) = 0).
\]
Now relation (4) implies
\[
\prod_{1 \leq j \leq N-1} (\alpha_j - \alpha_j) \left( \sum_{j=1}^N P_j(z) e_{\alpha_j} \right) = 0,
\]
which, because $\alpha$ has no zero divisor, is packed and has the same associated triplet $\{0\}$ as $R[\alpha]$. From (7), we see that for all $k < N$
\[
\prod_{1 \leq j \leq N-1} (\alpha_j - \alpha_j) P_k(z) = \prod_{1 \leq j \leq N-1} (\alpha_j - \alpha) P_k(z),
\]
so, if $N > 2$, we would get a relation of lower triplet $\{0\}$. This being impossible, we get $N = 1$ and (7) boils down to $P_k(z) e_{\alpha_k} = 0$ which, as $\alpha$ has no zero divisor, implies $P_k(z) = 0$, contradiction.

Then the $\{e_{\alpha} \mid \alpha \in \mathbb{Z}\}$ are $\mathbb{Z}^\mathbb{N}$-linear independent. \qed

**Remark 1.** If $\alpha$ is a $\mathbb{Q}$-algebra or only of characteristic zero (i.e., $n_\alpha = 0 \Rightarrow n = 0$), then $d(z) = 1$ implies that $z$ is transcendental over $R$.

First of all, let us prove

**Lemma 2.** Let $k$ be a field of characteristic zero and $Z$ an alphabet. Then $(k[|Z|], \cup, 1_{Z^*})$ is a $k$-algebra without zero divisor.

**Proof.** Let $B = (b_{ij})_{i,j \in I}$ be an ordered basis of $L^2(Z)$ and $(b_{ij})_{i,j \in I}$ the divided corresponding PBW basis. One has
\[
\Delta(\alpha) = \sum_{a + b = a} B(\alpha) \otimes B(\alpha) = \sum_{a + b = a} B(\alpha) \otimes B(\alpha).
\]
Hence, if $S, T \in (k[|Z|], \cup, 1_{Z^*})$, considering
\[
(S \otimes T) \frac{B(\alpha) \otimes 1}{a_1} = \sum_{a + b = a} \langle S \otimes T | \Delta(\alpha) \rangle \frac{B(\alpha) \otimes 1}{a_1} = \sum_{a + b = a} \langle S \otimes T | \frac{B(\alpha) \otimes 1}{a_1} \rangle (T | \frac{B(\alpha) \otimes 1}{a_1} ) ,
\]
we see that $(k[|Z|], \cup, 1_{Z^*}) \simeq (k[[\mathbb{Z}]], \cup, 1_{Z^*})$ (commutative algebra of formal series) which has no zero divisor. \qed

**Lemma 3.** Let $\alpha$ be a $\mathbb{Q}$-algebra (associative, unital, commutative) and $z$ an indeterminate, then $e^z \in \mathbb{Q}[[z]]$ is transcendental over $\mathbb{Q}[z]$.

**Proof.** This is straightforward consequence of Remark 1. Note that this can be rephrased as $[z, e^z]$ are algebraically independent over $\mathbb{Q}[z]$. \qed

**Proposition 1.** Let $Z = \{z_i \mid i \in \mathbb{N}\}$ be an alphabet, then $[e^z, e^z]$ is algebraically independent on $\mathbb{C}[Z]$ within $\mathbb{C}[|Z|]$.

**Proof.** Using Lemma 1 one can prove by recurrence that
\[
[e^z, e^z, \ldots, e^z, z_1, \ldots, z_k],
\]
are algebraically independent. This implies that $Z \cup \{e^z \mid z \in Z\}$ is an algebraically independent set and, by restriction $Z \cup \{e^z, e^z\}$ is the proposition. \qed

**Corollary 1.** i. The family $\{e^z, x^z, x^z\}_{x \in Z}$ is algebraically independent over $(\mathbb{C}[X], \cup, 1_X)$ within $(\mathbb{C}[X])^{\text{rat}, \cup, 1_X}$.

ii. $(\mathbb{C}[X], \cup, 1_X) \cup \{x^z, x^z\}$ is a free module over $(\mathbb{C}[X], \cup, 1_X)$, the family $\{x^z, x^z\}_{x \in Z}$ is a $(\mathbb{C}[X], \cup, 1_X)$-basis of it.

iii. As a consequence, $(\mathbb{C}[X], \cup, 1_X) \cup \{x^z, x^z\}_{x \in Z}$ is a $(\mathbb{C}[X], \cup, 1_X)$-basis of it.

**Proof.** Chase denominators and use a theorem by Radford 23 with $Z = \mathcal{L}_2^\mathcal{N}(X)$. \qed

**Corollary 2.** There exists a unique morphism $\mu$, from $(\mathbb{C}[X], \cup, 1_X) \cup \{x^z, x^z\}$ to $\mathcal{H}$ defined by

i. $\mu(w) = \text{Li}_w$ for any $w \in X^*$.

ii. $\mu(x^z) = z$.

iii. $\mu((-x^z))^* = 1/z$.

iv. $\mu((x^z)^* = 1/(1 - z)$.

**Definition 2.** We call $\text{Li}_w$ the morphism $\mu$.

Remark that the image of $(\mathbb{C}[X], \cup, 1_X) \cup \{x^z, x^z\}$ by $\text{Li}_w(\cup, 1_X)$ (sect. 3) is exactly $\mathcal{H}[\text{Li}_w]_{w \in X^*}$. And the operator $\text{Li}_w$ defined by means of $\text{Li}_w$ is the same as the one defined by tensor decomposition. We have a diagram as follows

\[
\begin{array}{cccc}
(\mathbb{C}[X], \cup, 1_X) & \xrightarrow{\text{Li}_w} & \mathbb{C}[\text{Li}_w]_{w \in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}[X], \cup, 1_X) & \xrightarrow{\text{Li}_w^{(1)}} & \mathbb{C}[\text{Li}_w]_{w \in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}[X], \cup, 1_X) & \xrightarrow{\text{Li}_w^{(2)}} & \mathcal{H} \\
\end{array}
\]

**Diagram 1.** Arrows and spaces obtained in this project (so far). Among horizontal arrows only $\text{Li}_w$ is into (and is an isomorphism) $\text{Li}_w^{(1)}$ and $\text{Li}_w^{(2)}$ are not into (for example, the image of the non-zero element $x^z x^z - x^z + 1$ is zero, see Example 7). The image of $\text{Li}_w^{(1)}$ is presumably

\[
\text{Im}(\text{SpC}(X)) \subset \mathbb{C}[\text{Li}_w]_{w \in X^*} \simeq \mathbb{C}[\text{Li}_w]_{w \in X^*} \simeq \mathbb{C}[\text{Li}_w]_{w \in X^*} \simeq \mathbb{C}[\text{Li}_w]_{w \in X^*}.
\]

**4.** Extension to $\mathbb{C}[X] \cup \mathbb{C}[\text{Li}_w]_{w \in X^*}$

**4.1 Study of the shuffle algebra $\mathbb{C}[X]$**

Indeed, the set $(a_0 x_0 + a_1 x_1)^* \cup a_0, a_1 \in \mathbb{C}$ is a shuffle monoid as
\[
(a_0 x_0 + a_1 x_1)^* \cup (b_0 x_0 + b_1 x_1)^* = ((a_0 x_0 + b_0 x_0) + (a_1 x_1 + b_1 x_1)).
\]
As there are many relations between these elements (as a monoid it is isomorphic to $\mathbb{Z}^2$, hence far from being free), we study here the vector space
\[
\text{SpC}(X) = \text{span}_{\mathbb{C}} \langle (a_0 x_0 + a_1 x_1)^* \rangle_{a_0, a_1 \in \mathbb{C}}.
\]
It is a shuffle sub-algebra of \((\mathbb{C}^\text{rat})\langle x_0 \rangle, (\mathbb{C}^\text{rat})\langle x_1 \rangle\) which will be called \textit{star of the plane}. Note that it is also a shuffle sub-algebra of the algebra \((\mathbb{C}^\text{exch})\langle X \rangle, \ldots, X^p \rangle\) of exchangeable series. We can give the

\textbf{Definition 3.} A series is said exchangeable iff whenever two words have the same multidegree (here bidegree) then they have the same coefficient within it. Formally for all \(u, v \in X^*
\)

\((\forall x \in X)(|u_x| = |v_x|) \implies (S \mid u) = (S \mid v)\).

On the other hand, for any \(S \in \mathbb{C}^\langle \langle X \rangle \rangle\), we can write

\[ S = \sum_{n=0}^{\infty} P_n, \]

where \(P_n \in \mathbb{C}[X]\) such that \(\deg P_n = n\) for every \(n \geq 0\). Then \(S\) is called exchangeable iff \(P_n\) are symmetric by permutations of places for every \(n \in \mathbb{N}\). If \(S\) is written as above then we can write

\[ P_n = \sum_{i=0}^{n} a_{n,i} x_0^{a_0} \cdots x_1^{a_1}. \]

\textbf{Definition 4.} Let \(S \in \mathbb{C}^\langle \langle X \rangle \rangle\) (resp. \(\mathbb{C}^\langle X \rangle\)) and let \(P \in \mathbb{C}^\langle X \rangle\) (resp. \(\mathbb{C}^\langle \langle X \rangle \rangle\)). The left and right residual of \(S\) by \(P\) are respectively the formal power series \(P \triangleleft S\) and \(S \triangleright P\) in \(\mathbb{C}^\langle \langle X \rangle \rangle\) defined by

\[ (P \triangleleft S \mid w) = (S \mid wP) \quad (\text{resp.} \quad (S \triangleright P \mid w) = (S \mid Pw)). \]

For any \(S \in \mathbb{C}^\langle \langle X \rangle \rangle\) (resp. \(\mathbb{C}^\langle X \rangle\)) and \(P, Q \in \mathbb{C}^\langle \langle X \rangle \rangle\) (resp. \(\mathbb{C}^\langle X \rangle\)), we straightforwardly get

\[ P \triangleleft (Q \triangleleft S) = PQ \triangleleft S, \]
\[ (S \triangleright P) \triangleright Q = S \triangleright PQ, \]
\[ (P \triangleleft S) \triangleright Q = P \triangleleft (S \triangleright Q). \]

In case \(x, y \in X\) and \(w \in X^*\), we get

\[ x \triangleleft (w y) = \delta^x_w \quad \text{and} \quad x w y = \delta^y_x w. \]

\textbf{Theorem 1.} Let \(\delta \in \mathcal{D}^\text{exch}(\mathbb{C}^\langle X \rangle)\), and let \(\delta^x = (\delta^x_1 \cdots \delta^x_1)\) be the family of \((\mathbb{C}^\langle X \rangle)\) derived from \(\delta\). Then the family \((\delta^x)^n / n!\) is summable and its sum, denoted \(\exp(n\delta)\), is a one-parameter group of automorphisms of \((\mathbb{C}^\langle X \rangle)\).

\[ x \triangleleft (y^n) = \delta^n_x(y) \quad \text{and} \quad x y^n = \delta^n_y(x). \]

\textbf{Theorem 2.} Let \(L\) be a Lie series. Let \(\delta^L_1\) and \(\delta^L_2\) be defined respectively by

\[ \delta^L_1(P) := P \triangleleft L \quad \text{and} \quad \delta^L_2(P) := L \triangleright P. \]

Then \(\delta^L_1\) and \(\delta^L_2\) are locally nilpotent derivations of \((\mathbb{C}^\langle X \rangle)\).

\[ \exp(t \delta^L_1)P = P \triangleleft \exp(tL) \quad \text{and} \quad \exp(t \delta^L_2)P = \exp(tL) \triangleright P. \]

It is not hard to see that the algebra \(\mathbb{C}^\langle X \rangle \otimes_{\mathbb{C}^\text{rat}} (\langle x_0 \rangle) \otimes_{\mathbb{C}^\text{rat}} (\langle x_1 \rangle)\) is closed by the shuffle derivations \(\delta^L_1, \delta^L_2\). In particular, on it,

\[ \forall S \in \mathbb{C}^\langle \langle X \rangle \rangle, \quad (\delta^L_1(S) \mid w) = (S \mid x w). \]

Moreover, one has

\[ (\alpha \delta^L_1 + \beta \delta^L_2)(a_0 x_0 + a_1 x_1)^n = (\alpha a_0 + \beta a_1)(a_0 x_0 + a_1 x_1)^n, \]

from this we get that the family \((a_0 x_0 + a_1 x_1)^n\) is linearly free over \(\mathbb{C}\)

\[ SP_C(X) = \bigoplus (a_0 x_0 + a_1 x_1)^n. \]

We can get an arrow of \(L^r_1\) type \((SP_C(\langle x_0 \rangle), \ldots, \langle x_1 \rangle) \rightarrow \mathcal{X}(\Omega)\) by sending

\[ (a_0 x_0 + a_1 x_1)^n \mapsto \exp(x^n (1-z)^{-a}). \]

In particular, for any \(n \in \mathbb{N}_+,\) one has

\[ L^r_1(\ldots, 0, (z) = L^n(\ldots, (z), (z)). \]

This arrow is a morphism for the shuffle product.

\textbf{4.2. Study of the algebra} \(\mathbb{C}^\langle X \rangle \otimes_{\mathbb{C}^\text{rat}} (\langle x_0 \rangle) \otimes_{\mathbb{C}^\text{rat}} (\langle x_1 \rangle)\)

We will start by studying the one-letter shuffle algebra, i.e. \((\mathbb{C}^\text{rat}((\langle x \rangle))\) and use two times Lemma\[\] above.

Let us now consider \(\mathcal{A} = \mathbb{C}^\text{rat}((\langle x \rangle))\), \(x \in \mathbb{C}\) endowed with \(d = \delta^x_1\) (which is a derivation for the shuffle) and \(z = x\). We are in the conditions of Lemma\[\] and then \((\mathbb{C}^\langle X \rangle)\) is \(\mathbb{C}[x]\)-linearly free which amounts to say that

\[ B_0 = (x^{1-k} \alpha x^k)_{k \in \mathbb{N}, \alpha \in \mathbb{C}}. \]

is \(\mathbb{C}\)-linearly free in \(\mathbb{C}^\langle x \rangle\).

To see that it is a basis, it suffices to prove that \(B_0\) is (linearly) generating.

Consider that

\[ \mathbb{C}^\text{rat}((\langle x \rangle)) = \{ P/Q \mid P,\!Q \in \mathbb{C}(\langle x \rangle), Q(0) \neq 0 \}. \]

then, as \(\mathbb{C}\) is algebraically closed, we have a basis

\[ B_1 \cup B_2 = \{ x^j \}_{j \geq 0} \cup \{ (\alpha x)^j \}_{\alpha \in \mathbb{C}, j \geq 1}, \]

and it suffices to see that we can generate \(B_2\) by elements of \(B_0\), which s a consequence of the two identities

\[ x \alpha \cdots ((\alpha x)^j)^{n+j} = \sum_{j=1}^{n+1} a(n, j)(\alpha x)^{n+j} \quad \text{with} \quad a(n, n+1) \neq 0, \]
\[ x^k \alpha \cdots (\alpha x)^j = \frac{1}{k!}(x^{k-1} \alpha \cdots (\alpha x)). \]

Now, we use again Lemma\[\] with

\[ \mathcal{A} = \mathbb{C}^\text{rat}((\langle x_0 \rangle)) \otimes_{\mathbb{C}^\text{rat}} (\langle x_1 \rangle) \subset \mathbb{C}(\langle x_0 \rangle, \langle x_1 \rangle), \]

hence without zero divisor (see Lemma\[\]), endowed with \(d = \delta^x_1\), then \(\langle x_0 \rangle \otimes_{\mathbb{C}^\text{rat}} (\alpha x)^n\) is linearly free over \(R = \mathbb{C}^\text{rat}((\langle x_0 \rangle)).\) It is easily seen, using a decomposition like

\[ S = \sum_{p \geq 0, q \geq 0} (S \mid x_0^p x_1^q) x_0^p x_1^q. \]

9. For any words \(u, v \in X^*\), if \(u = v\) then \(\delta^u_1 = 1\).
10. \(\phi \in \text{End}(V)\) is said to be locally nilpotent iff, for any \(v \in V\), there exists \(N \in \mathbb{N}\) s.t. \(\phi^N(v) = 0\).
11. i.e., \(\Delta_1(L) = L \otimes 1 + 1 \otimes L\).
12. These operators are, in fact, the shifts of Harmonic Analysis and therefore defined as adjoints of multiplication, i.e.

\[ \forall S \in \mathbb{C}^\langle \langle x \rangle \rangle, \quad (\delta^L_1(S) \mid w) = (S \mid x w). \]
that \( \mathbb{C}^\text{rat}\langle x_0 \rangle = \ker(d) \) and one obtains then that the arrow

\[
\mathbb{C}^\text{rat}\langle x_0 \rangle \otimes_{\mathbb{C}} \mathbb{C}^\text{rat}\langle x_1 \rangle \to \mathbb{C}^\text{rat}\langle x_0 \rangle \oplus \mathbb{C}^\text{rat}\langle x_1 \rangle \subseteq \mathbb{C}^\text{rat}\langle x_0, x_1 \rangle
\]

is an isomorphism. Hence, \((\alpha_0, \alpha_1, x_0) \otimes \langle \alpha_1, x_1 \rangle \in \mathbb{C}^\text{rat} \langle x_0, x_1 \rangle \) is a \( \mathbb{C} \)-basis of \( \mathfrak{A} = \mathbb{C}^\text{rat}\langle x_0 \rangle \oplus \mathbb{C}^\text{rat}\langle x_1 \rangle \). In order to extend \( L_{i*} \) to \( \mathfrak{A} \) we send

\[
T(k_0, k_1, \alpha_0, \alpha_1) = x_0^{k_0} \alpha_0 \oplus x_1^{k_1} \alpha_1,
\]

to \( \log^0(z)^{k_0} \log^1((1/(1-z))^{k_1}) \). and see that the constructed arrow follows multiplication given by

\[
T(j_0, j_1, \alpha_0, \alpha_1) T(k_0, k_1, \beta_0, \beta_1) = T(j_0 + k_0, j_1 + k_1, \alpha_0 + \beta_0, \alpha_1 + \beta_1).
\]

Using, once more, Lemma 1 one gets

**Proposition 2.** The family \( \{ (\alpha_0 x_0)^\ast \oplus (\alpha_1 x_1)^\ast \}_{\alpha \in \mathbb{C}} \) is a \((\mathbb{C}(X), \cup, 1, 0)\)-basis of \( \mathbb{C}(X) \mathbb{C}^\text{rat}\langle x_0 \rangle \mathbb{C}^\text{rat}\langle x_1 \rangle \), then we have \( \mathbb{C}(X) \mathbb{C}^\text{rat}\langle x_0 \rangle \mathbb{C}^\text{rat}\langle x_1 \rangle \) and \( \mathbb{C}(X) \mathbb{S}_\mathbb{C}(X) \) to \( \text{Im}(L_{i\ast}^{(2)}) \).

\[
\begin{align*}
\mathbb{C}(X) \mathbb{C}^\text{rat}\langle x_0 \rangle \mathbb{C}^\text{rat}\langle x_1 \rangle & = \mathbb{C}(X) \mathbb{C}^\text{rat}\langle x_0 \rangle \mathbb{C}^\text{rat}\langle x_1 \rangle \\
& = \mathbb{C}(X) \mathbb{S}_\mathbb{C}(X),
\end{align*}
\]

\[\text{PROOF.}\] We will use a multi-parameter consequence of Lemma 1.

**Lemma 4.** Let \( Z \) be an alphabet, and \( k \) a field of characteristic zero. Then, the family \( \{ e^{\alpha x} \}_{\alpha \in k} \subseteq \mathbb{C}[Z] \) is linearly independent over \( k[Z] \).

This proves that, in the shuffle algebra the elements

\[
\{ (\alpha_0 x_0)^\ast \oplus (\alpha_1 x_1)^\ast \}_{\alpha_0, \alpha_1 \in \mathbb{C}}
\]

are linearly independent over \( \mathbb{C}(X) \subseteq \mathbb{C}[\mathbb{Z}^n(X)] \) within \( \mathbb{C}(X), \cup, 1, X^\ast \).

Now \( L_{i\ast}^{(2)} \) is well-defined and this morphism is not into from

\[
\begin{align*}
\mathbb{C}(X) \mathbb{C}^\text{rat}\langle x_0 \rangle \mathbb{C}^\text{rat}\langle x_1 \rangle & = \mathbb{C}(X) \mathbb{C}^\text{rat}\langle x_0 \rangle \mathbb{C}^\text{rat}\langle x_1 \rangle \\
& = \mathbb{C}(X) \mathbb{S}_\mathbb{C}(X),
\end{align*}
\]

\[\text{PROPOSITION 3.}\] Let \( L_{i\ast}^{(1)} : \mathbb{C}(X)[x_0, x_1, (-x_0)^\ast] \to \mathfrak{H}(\Omega) \) then

\[
\text{i. } \text{Im}(L_{i\ast}^{(1)}) = \mathfrak{H}(\text{Lin}_i)_{w X}\text{.}
\]

\[
\text{ii. } \ker(L_{i\ast}^{(1)}) = \text{the ideal generated by } x_0 \cup x_1 - x_1 \cup x_0 + X^\ast \text{.}
\]

\[\text{PROOF.}\] As \( \mathbb{C}(X)[x_0, x_1, (-x_0)^\ast] \) admits \( \{ (x_0)^{i_0} \cup \cdots \cup (x_1)^{i_1} \}_{i_0, \cdots, i_1 \in \mathbb{N}} \) as a basis for its structure of \( \mathbb{C}(X) \)-module, it suffices to remark

\[
L_{i\ast}^{(1)}(x_0)^{i_0} \cup \cdots \cup (x_1)^{i_1}(z) = z^k \times \frac{1}{(1-z)^l}
\]

is a generating system of \( \mathfrak{H} \).

First of all, we recall the following lemma

**Lemma 5.** Let \( M_1 \) and \( M_2 \) be \( k \)-modules (\( K \) is a unitary ring). Suppose \( \phi : M_1 \to M_2 \) is a linear mapping. Let \( N \subseteq \ker(\phi) \) be a submodule. If there is a system of generators in \( M_1 \), namely \( \{ g_i \}_{i \in I} \), such that

1. For any \( i \in I \setminus J, g_i \equiv \sum_{j \in J} c_i^j g_j \mod N, (c_i^j \in K; \forall j \in J) \);
2. \( \{ \phi(g_j) \}_{j \in J} \) is \( K \)-free in \( M_2 \);
then \( N = \ker(\phi) \).

**Proof.** Suppose \( P \in \ker(\phi) \). Then \( P = \sum_{j \in J} p_j \phi(g_j) \). From the fact that \( \{ \phi(g_j) \}_{j \in J} \) is \( K \)-free on \( M_2 \), we obtain \( p_j = 0 \) for any \( j \in J \). This implies that \( N = \ker(\phi) \).

**Figure 1.** Rewriting \( \mathbb{H} \) of \( \{ w \cdot (x_0^{i_0} \cup \cdots \cup (x_1)^{i_1}) \}_{i_0, \cdots, i_1 \in \mathbb{N}}, w \in \mathbb{C}(X) \).

\[
L_{\ast} \mathbb{C}(X)[x_0, x_1, (-x_0)^\ast]. \quad M_2 = \mathfrak{H}(\Omega), \quad N = \mathfrak{H},
\]

the families and indices

\[
\begin{align*}
\{ g_i \} & = \{ w \cdot (x_1)^{i_0} \cup \cdots \cup (x_1)^{i_1} \}_{w, n, m} \in I, \\
I & = X^\ast \times N \times \mathbb{Z}, \\
J & = (X^\ast \times N \times \{ 0 \}) \cup (X^\ast \times \{ 0 \} \times \mathbb{Z}),
\end{align*}
\]

we have the second point of proposition 3.

Of course, we also have \( (x_0 \cup x_1 - x_1 \cup x_0 + X^\ast) \subseteq \ker(L_{i\ast}^{(2)}) \), but the converse is conjectural.

**5. Applications on polylogarithms**

Let us consider also the following morphisms \( \mathfrak{S} \) and \( \Theta \) of algebras \( \mathbb{C}(X) \to \text{End}(\mathfrak{H}(\text{Lin}_i)) \) defined by

\[
\begin{align*}
\mathfrak{S}(w) & = \text{Id} \quad \mathfrak{S}(w) = \text{Id, if } w = 1 X^\ast, \\
\mathfrak{S}(w) & = \mathfrak{H}(w), \quad \mathfrak{S}(w) = \mathfrak{H}(w) \text{, if } w = \varphi x_1, x_1 \in X^\ast, w \in X^\ast.
\end{align*}
\]

For any \( n \geq 0 \) and \( u \in X^\ast, f, g \in \mathfrak{H}(\text{Lin}_i)_{w \in X^\ast} \), one has

\[
\partial^n_{x_i} = \sum_{w \in X^\ast} \mu \circ (\Theta \otimes \Theta)(A_{\lambda} \cdot w),
\]

\[
\Theta(u)(f g) = \mu \circ \Theta(\Theta)(A_{\lambda} \cdot u) \circ (f \otimes g).
\]

By extension to complex coefficients, we obtain

\[
\begin{align*}
\mathfrak{H} \cong & (\mathbb{C}(\mathbb{H}(X)), \cup, 1, \mathbb{H}_X, \mathbb{H}_X), \\
\mathfrak{H}_X & \cong (\mathbb{C}(\mathbb{H}(X)), \cup, 1, \mathbb{H}_X, \mathbb{H}_X).
\end{align*}
\]

Hence.

\[\text{14. In figure } \square \text{ (w, l, k) codes the element } w \cdot (x_0^{i_0} \cup \cdots \cup (x_1)^{i_1}) \cup (x_1)^{i_1}.\]
THEOREM 3 (DERIVATIONS AND AUTOMORPHISMS).
Let $P, Q \in \mathcal{C}(X)$ (resp. $\mathcal{C}[x_0, (-x_0)^*] \subset \mathcal{C}(X)$), $T \in \mathcal{L}ie_{\mathbb{C}}(\langle X \rangle)$ (resp. $\mathcal{L}ie_{\mathbb{C}}(X)$). Then $\Theta(T)$ is a derivation in $\mathcal{C}(\mathcal{L}_w)_{w \in \mathbb{C}X}$, where $\exp(\Theta(T))$ is then one-parameter group of automorphisms of $\langle \exp(\mathcal{L}_w)_{w \in \mathbb{C}X} \rangle$.

PROOF. Because $\exp\{P \cdot Q - Q \cdot P\}/Q = 0$ and $\Theta(T) = (\exp(\mathcal{L}_w)_{w \in \mathbb{C}X} \cdot Q) - (Q \cdot \exp(\mathcal{L}_w)_{w \in \mathbb{C}X})$, then $\Theta(T)(\exp(\mathcal{L}_w)_{w \in \mathbb{C}X}) = \exp(\mathcal{L}_w)_{w \in \mathbb{C}X}$.

THEOREM 4 (EXTENSION OF $\mathcal{L}_w$).

The following map is surjective
$$(\mathcal{C}[x_0]_{w \in \mathbb{C}X} \cup \mathcal{C}((-x_0)^*) \cup \mathcal{C}(X), \ldots, 1) \to \langle \exp(\mathcal{L}_w)_{w \in \mathbb{C}X} \rangle,$$
$T \mapsto \mathcal{S}(T)_{1 \Omega}$.

One has, for any $u \in Y^*$,
$$\mathcal{L}_w \cdot u = \theta^0(\theta_0)_{1 \Omega} \cdot \mathcal{L}_w = \theta^0(\lambda)(\mathcal{L}_w) = \sum_{k_i = 0} \left( \frac{1}{1 - z^k} \right)_{i=1}^\infty \mathcal{L}_w.$$

Hence, successively [8],
$$\mathcal{L}_w = \sum_{k_i = 0} \left( \frac{1}{1 - z^k} \right)_{i=1}^\infty \mathcal{L}_w.$$

Due to surjectivity of $\mathcal{L}_w$, from $\mathcal{C}[x_0]_{w \in \mathbb{C}X} \cup \mathcal{C}((-x_0)^*) \cup \mathcal{C}(X)$ to $\langle \exp(\mathcal{L}_w)_{w \in \mathbb{C}X} \rangle$, one also has
$$\mathcal{L}_w \cdot u = \mathcal{G}(R)_{1 \Omega}.$$

where $R$ is the following exchangeable rational series
$$R = \sum_{k_i = 0} \left( \frac{1}{1 - z^k} \right)_{i=1}^\infty \mathcal{L}_w.$$
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