Simultaneous Inference of Covariances

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Abstract: We consider asymptotic distributions of maximum deviations of sample covariance matrices, a fundamental problem in high-dimensional inference of covariances. Under mild dependence conditions on the entries of the data matrices, we establish the Gumbel convergence of the maximum deviations. Our result substantially generalizes earlier ones where the entries are assumed to be independent and identically distributed, and it provides a theoretical foundation for high-dimensional simultaneous inference of covariances.

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1. Introduction

Let \( X_n = (X_{ij})_{1 \leq i \leq n, 1 \leq j \leq m} \) be a data matrix whose \( n \) rows form independent samples from some population distribution with mean vector \( \mu_n \) and covariance matrix \( \Sigma_n \). High dimensional data increasingly occur in modern statistical applications in biology, finance and wireless communication, where the dimension \( m \) may be comparable to the number of observations \( n \), or even much larger than \( n \). Therefore, it is necessary to study the asymptotic behavior of statistics of \( X_n \) under the setting that \( m = m_n \) grows to infinity as \( n \) goes to infinity.

In many empirical examples, it is often assumed that \( \Sigma_n = I_m \), where \( I_m \) is the \( m \times m \) identity matrix, so it is important to perform the test

\[ H_0 : \Sigma_n = I_m \] (1)

before carrying out further estimation or inference procedures. Due to high dimensionality, conventional tests often do not work well or cannot be implemented. For example, when \( m > n \), the likelihood ratio test (LRT) cannot be used because the sample covariance matrix is singular; and even when \( m < n \), the LRT is drifted to infinity and lead to many false rejections if \( m \) is also large [Bai et al. 2008, Ledoit and Wolf 2002] found that the empirical distance test
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(Nagao, 1973) is not consistent when both \( m \) and \( n \) are large. The problem has been studied by several authors under the “large \( n \), large \( m \)” paradigm. Bai et al. (2009) and Ledoit and Wolf (2002) proposed corrections to the LRT and the empirical distance test respectively. Assuming that the population distribution is Gaussian with \( \mu_n = 0 \), Johnstone (2001) used the largest eigenvalue of the sample covariance matrix \( X_n^\top X_n \) as the test statistic, and proved that its limiting distribution follows the Tracy-Widom law (Tracy and Widom, 1994). Here we use the superscript \( \top \) to denote the transpose of a matrix or a vector.

His work was extended to the non-Gaussian case by Soshnikov (2002) and Péché (2009), where they assumed the entries of \( X_n \) are independent and identically distributed (i.i.d.) with sub-Gaussian tails.

Let \( x_1, x_2, \ldots, x_m \) be the \( m \) columns of \( X_n \). In practice, the entries of the mean vector \( \mu_n \) are often unknown, and are estimated by \( \bar{x}_i = (1/n) \sum_{k=1}^n X_{ki} \).

Write \( x_i - \bar{x}_i \) for the vector \( x_i - \bar{x}_i1_n \), where \( 1_n \) is the \( n \)-dimensional vector with all entries being one. Let \( \sigma_{ij} = \text{Cov}(X_{1i}, X_{1j}) \), \( 1 \leq i, j \leq m \), be the covariance function, namely, the \((i, j)\)th entry of \( \Sigma_n \). The sample covariance between columns \( x_i \) and \( x_j \) is defined as

\[
\hat{\sigma}_{ij} = \frac{1}{n} (x_i - \bar{x}_i)^\top (x_j - \bar{x}_j).
\]

In high-dimensional covariance inference, a fundamental problem is to establish an asymptotic distributional theory for the maximum deviation

\[
M_n = \max_{1 \leq i < j \leq m} |\hat{\sigma}_{ij} - \sigma_{ij}|.
\]

With such a distributional theory, one can perform statistical inference for structures of covariance matrices. For example, one can use \( M_n \) to test the null hypothesis \( H_0 : \Sigma_n = \Sigma(0) \), where \( \Sigma(0) \) is a pre-specified matrix. Here the null hypothesis can be that the population distribution is a stationary process so that \( \Sigma_n \) is Toeplitz, or that \( \Sigma_n \) has a banded structure.

It is very challenging to derive an asymptotic theory for \( M_n \) if we allow dependence among \( X_{11}, \ldots, X_{1m} \). Many of the earlier results assume that the entries of the data matrix \( X_n \) are i.i.d.. In this case \( \sigma_{ij} = 0 \) if \( i \neq j \). Jiang (2004) derived the asymptotic distribution of \( \hat{\sigma}_{ij} \).

**Theorem 1** (Jiang, 2004). Suppose \( X_{ij}, \ i, j = 1, 2, \ldots \) are independent and identically distributed as \( \xi \) which has variance one. Suppose \( \mathbb{E}|\xi|^{30-\epsilon} < \infty \) for any \( \epsilon > 0 \). If \( n/m \to c \in (0, \infty) \), then for any \( y \in \mathbb{R} \),

\[
\lim_{n \to \infty} P \left( nL_n^2 - 4 \log m + \log(\log m) + \log(8\pi) \leq y \right) = \exp \left( -e^{-y/2} \right).
\]

Jiang’s work has attracted considerable attention, and been followed by Li et al. (2010), Liu et al. (2008), Zhou (2007) and Li and Rosalsky (2006). Under the same setup that \( X_n \) consists of i.i.d. entries, these works focus on three
directions (i) reduce the moment condition; (ii) allow a wider range of \( p \); and (iii) show that some moment condition is necessary. In a recent article, Cai and Jiang (2011) extended those results in two ways: (i) the dimension \( p \) could grow exponentially as the sample size \( n \) provided exponential moment conditions; and (ii) they showed that the test statistic \( \max_{1 \leq i < j \leq m} |\hat{\sigma}_{i,j}| \) also converges to the Gumbel distribution if each row of \( X_n \) is Gaussian and is \( s_n \)-dependent. The latter generalization is important since it is one of the very few results that allow dependent entries.

In this paper we shall show that a self-normalized version of \( M_n \) converges to the Gumbel distribution under mild dependence conditions on the vector \((X_{11}, \ldots, X_{1m})\). Thus our result provides a theoretical foundation for high-dimensional simultaneous inference of covariances.

The rest of this article is organized as follows. We present the main result in Section 2. In Section 3, we use two examples on linear processes and nonlinear processes to demonstrate that the technical conditions are easily satisfied. We discuss three tests for the covariance structure using our main result in Section 4. The proof is given in Section 5, and some auxiliary results are collected in Section 6.

2. Main result

We consider a slightly more general situation where population distribution can depend on \( n \). Let \( X_n = (X_{n,k,i})_{1 \leq k \leq n, 1 \leq i \leq m} \) be a data matrix whose \( n \) rows are i.i.d. \( m \)-dimensional random vectors with mean \( \mu_n = (\mu_{n,i})_{1 \leq i \leq m} \) and covariance matrix \( \Sigma_n = (\sigma_{n,i,j})_{1 \leq i,j \leq m} \). Let \( x_1, x_2, \ldots, x_m \) be the \( m \) columns of \( X_n \). Let \( \bar{x}_i = (1/n) \sum_{k=1}^{n} X_{n,k,i} \), and write \( x_i - \bar{x}_i \) for the vector \( x_i - \bar{x}_i \). The sample covariance between \( x_i \) and \( x_j \) is defined as

\[
\hat{\sigma}_{n,i,j} = \frac{1}{n} (x_i - \bar{x}_i) \top (x_j - \bar{x}_j).
\]

It is unnatural to study the maximum of a collection of random variables which are on different scales, so we consider the normalized version \(|\hat{\sigma}_{n,i,j} - \sigma_{n,i,j}|/\tau_{n,i,j} \), where

\[
\tau_{n,i,j} = \text{Var}[(X_{n,1,i} - \mu_{n,i})(X_{n,1,j} - \mu_{n,j})].
\]

In practice, \( \tau_{n,i,j} \) are usually unknown, and can be estimated by

\[
\hat{\tau}_{n,i,j} = \frac{1}{n} |(x_i - \bar{x}_i) \circ (x_j - \bar{x}_j) - \hat{\sigma}_{n,i,j} \cdot 1_n|^2.
\]

where \( \circ \) denotes the Hadamard product defined as \( A \circ B := (a_{ij}b_{ij}) \) for two matrices \( A = (a_{ij}) \) and \( B = (b_{ij}) \) with the same dimensions. We thus consider

\[
M_n = \max_{1 \leq i < j \leq m} \frac{|\hat{\sigma}_{n,i,j} - \sigma_{n,i,j}|}{\sqrt{\hat{\tau}_{n,i,j}}}.
\]

(2)
Due to the normalization procedure, we can assume without loss of generality that \( \sigma_{n,i,i} = 1 \) and \( \mu_{n,i} = 0 \) for each \( 1 \leq i \leq m \).

Define the index set \( I_n = \{(i, j) : 1 \leq i < j \leq m\} \), and for \( \alpha = (i, j) \in I_n \), let \( X_{n,\alpha} := X_{n,1,i}X_{n,1,j} \). Define

\[
K_n(t, p) = \sup_{1 \leq i \leq m} \mathbb{E}\exp(t|X_{n,1,i}|^p),
\]

\[
M_n(p) = \sup_{1 \leq i \leq m} \mathbb{E}(|X_{n,1,i}|^p),
\]

\[
\tau_n = \inf_{1 \leq i < j \leq m} \tau_{n,i,j},
\]

\[
\gamma_n = \sup_{\alpha, \beta \in I_n \text{ and } \alpha \neq \beta} \text{Cor}(X_{n,\alpha}, X_{n,\beta}),
\]

\[
\gamma_n(b) = \sup_{\alpha \in I_n} \sup_{A \subseteq I_n, |A| = b} \inf_{\beta \in A} \text{Cor}(X_{n,\alpha}, X_{n,\beta}).
\]

We need the following technical conditions.

(A1). \( \liminf_{n \to \infty} \tau_n > 0 \).

(A2). \( \limsup_{n \to \infty} \gamma_n < 1 \).

(A3). \( \gamma_n(b_n) : \log(b_n) = o(1) \) for any sequence \( (b_n) \) such that \( b_n \to \infty \).

(A3'). \( \gamma_n(b_n) = o(1) \) for any sequence \( (b_n) \) such that \( b_n \to \infty \), and

\[
\sum_{\alpha, \beta \in I_n} \text{Cor}(X_{n,\alpha}, X_{n,\beta})^2 = O(m^{4-\epsilon}) \text{ for some constant } \epsilon > 0.
\]

(A4). \( \log m = o(n^{p/(4+2p)}) \) and \( \limsup_{n \to \infty} K_n(t, p) < \infty \) for some constants \( t > 0 \) and \( 0 < p \leq 4 \).

(A4'). \( m = O(n^q) \) and \( \limsup_{n \to \infty} M_n(4q + 4 + \delta) < \infty \) for some constants \( q > 0 \) and \( \delta > 0 \).

The two conditions (A3) and (A3') require that the dependence among \( X_{n,\alpha}, \alpha \in I_n \), are not too strong. They are translations of (B1) and (B2) in Section 6.1 (see Remark 2 for some equivalent versions), and either of them will make our results valid. We use (A2) to get rid of the case where they may be lots of pairs \( (\alpha, \beta) \in I_n \) such that \( X_{n,\alpha} \) and \( X_{n,\beta} \) are perfectly correlated. Assumptions (A4) and (A4') connect the growth speed of \( m \) relative to \( n \) and the moment conditions. They are typical in the context of high dimensional covariance matrix estimation. Condition (A1) excludes the case that \( X_{n,\alpha} \) is a constant.

Theorem 2. Suppose that \( X_n = (X_{n,k,i})_{1 \leq k \leq n, 1 \leq i \leq m} \) is a data matrix whose \( n \) rows are i.i.d. \( m \)-dimensional random vectors, and whose entries have mean zero and variance one. Assume (A1), (A2), either of (A3) and (A3'), and either of (A4) and (A4'), then for any \( y \in \mathbb{R} \),

\[
\lim_{n \to \infty} P \left( nM_n^2 - 4 \log m + \log(\log m) + \log(8\pi) \leq y \right) = \exp \left( -e^{-y/2} \right).
\]
3. Examples

Except for (A4) and (A4′), which put conditions on every single entry of the random vector \((X_{n,1,i})_{1 \leq i \leq m}\), all the other conditions of Theorem 2 are related to the dependence among these entries, which can be arbitrarily complicated. In this section we shall provide examples which satisfy the four conditions (A1), (A2), (A3) and (A3′). Observe that if each row of \(X_n\) is a random vector with uncorrelated entries (specifically, the entries are independent), then all these conditions are automatically satisfied. They are also satisfied if the number of non-zero covariances is bounded.

3.1. Stationary Processes

Suppose \((X_{n,k,i}) = (X_{k,i})\), and each row of \((X_{k,i})_{1 \leq i \leq m}\) is distributed as a stationary process \((X_i)_{1 \leq i \leq m}\) of the form

\[ X_i = g(\epsilon_i, \epsilon_{i-1}, \ldots) \]

where \(\epsilon_i\)'s are i.i.d. random variables, and \(g\) is a measurable function such that \(X_i\) is well-defined. Let \((\epsilon'_i)_{i \in \mathbb{Z}}\) be an i.i.d. copy of \((\epsilon_i)_{i \in \mathbb{Z}}\), and \(X'_i = g(\epsilon_i, \ldots, \epsilon_{1}, \epsilon'_0, \epsilon_{-1}, \epsilon_{-2}, \ldots)\). Following Wu (2005), define the physical dependence measure of order \(p\) by

\[ \delta_p(i) = \|X_i - X'_i\|_p. \]

Define the squared tail sum

\[ \Psi_p(k) = \left[ \sum_{j=k}^{\infty} (\delta_p(i))^2 \right]^{1/2}, \]

and use \(\Psi_p\) as a shorthand for \(\Psi_p(0)\).

We give sufficient conditions for (A1), (A2), (A3) and (A3′) in the following lemma and leave its proof to the supplementary file.

**Lemma 3.**

(i) If \(0 < \Psi_4 < \infty\) and \(\text{Var}(X_i X_j) > 0\) for all \(i, j \in \mathbb{Z}\), then (A1) holds.

(ii) If in addition, \(|\text{Cor}(X_i X_j, X_k X_l)| < 1\) for all \(i, j, k, l\) such that they are not all the same, then (A2) holds.

(iii) Assume that the conditions of (i) and (ii) hold. If \(\Psi_p(k) = o(1/\log k)\) as \(k \to \infty\), then (A3) holds. If \(\sum_{j=0}^{m} (\Psi_4(j))^2 = O(m^{1-\delta})\) for some \(\delta > 0\), then (A3′) holds.

**Remark 1.** Let \(g\) be a linear function with \(g(\epsilon_i, \epsilon_{i-1}, \ldots) = \sum_{j=0}^{\infty} a_j \epsilon_{i-j}\), where \(\epsilon_j\) are i.i.d. with mean 0 and \(\mathbb{E}(|\epsilon_j|^p) < \infty\) and \(a_j\) are real coefficients with \(\sum_{j=0}^{\infty} a_j^2 < \infty\). Then the physical dependence measure \(\delta_p(i) = \|a_i\|\|\epsilon_0 - \epsilon'_0\|_p\). If \(a_i = i^{-\beta} \ell(i)\), where \(1/2 < \beta < 1\) and \(\ell\) is a slowly varying function, then \((X_i)\)
is a long memory process. Smaller $\beta$ indicates stronger dependence. Condition (iii) holds for all $\beta \in (1/2, 1)$. Moreover, if $a_i = i^{-1/2} (\log(i))^{-2}$, $i \geq 2$, which corresponds to the extremal case with very strong dependence $\beta = 1/2$, we also have $\Psi_p(k) = O((\log k)^{-3/2}) = o(1/\log k)$. So our dependence conditions are actually quite mild.

If $(X_i)$ is a linear process which is not identically zero, then the following regularity conditions are automatically satisfied: $\Psi_4 > 0$, $\text{Var}(X_i X_j) > 0$ for all $i, j \in \mathbb{Z}$, and $|\text{Cor}(X_i X_j, X_k X_l)| < 1$ for all $i, j, k, l$ such that they are not all the same.

### 3.2. Non-stationary Linear Processes

Assume that each row of $(X_{n,k,i})$ is distributed as $(X_{n,i})_{1 \leq i \leq m}$, which is of the form

$$X_{n,i} = \sum_{t \in \mathbb{Z}} f_{n,i,t} \epsilon_{i-t},$$

where $\epsilon_i$, $i \in \mathbb{Z}$ are i.i.d. random variables with mean zero, variance one and finite fourth moment, and the sequence $(f_{n,i,t})$ satisfies $\sum_{t \in \mathbb{Z}} f_{n,i,t}^2 = 1$. Denote by $\kappa_4$ the fourth cumulant of $\epsilon_0$. For $1 \leq i, j, k, l \leq m$, we have

$$\sigma_{n,i,j} = \sum_{t \in \mathbb{Z}} f_{n,i,i-t} f_{n,j,j-t},$$

$$\text{Cov}(X_{n,i} X_{n,j}, X_{n,k} X_{n,l}) = \text{Cum}(X_{n,i}, X_{n,j}, X_{n,k}, X_{n,l}) + \sigma_{n,i,k} \sigma_{n,j,l} + \sigma_{n,i,l} \sigma_{n,j,k},$$

where $\text{Cum}(X_{n,i}, X_{n,j}, X_{n,k}, X_{n,l})$ is the fourth order joint cumulant of the random vector $(X_{n,i}, X_{n,j}, X_{n,k}, X_{n,l})^\top$, which can be expressed as

$$\text{Cum}(X_{n,i}, X_{n,j}, X_{n,k}, X_{n,l}) = \sum_{t \in \mathbb{Z}} f_{n,i,i-t} f_{n,j,j-t} f_{n,k,k-t} f_{n,l,l-t} \kappa_4,$$

by the multilinearity of cumulants. In particular, we have

$$\text{Var}(X_i X_j) = 1 + \sigma_{n,i,j}^2 + \kappa_4 \cdot \sum_{t \in \mathbb{Z}} f_{n,i,t}^2 f_{n,j,t}^2.$$

Since $\kappa_4 = \text{Var}(\epsilon_0^2) - 2 (\mathbb{E} \epsilon_0^2)^2 \geq -2$, the condition

$$\kappa_4 > -2$$

guarantees (A1) in view of

$$\text{Var}(X_i X_j) \geq (1 + \sigma_{n,i,j}^2)(1 + \min\{\kappa/2, 0\}) \geq \min\{1, 1 + \kappa/2\} > 0.$$

To ensure the validity of (A2), it is natural to assume that no pairs $X_{n,i}$ and $X_{n,j}$ are strongly correlated, i.e.

$$\lim_{n \to \infty} \sup_{1 \leq i < j \leq m} \left| \sum_{t \in \mathbb{Z}} f_{n,i,i-t} f_{n,j,j-t} \right| < 1.$$
Lemma 4. The condition \(4\) suffices for (A2) if \(\epsilon_i\)'s are i.i.d. \(N(0, 1)\).

As an immediate consequence, when \(\epsilon_i\)'s are i.i.d. \(N(0, 1)\), we have
\[
\ell := \limsup_{n \to \infty} \inf_{\rho \in \mathbb{R}} \inf_{\ast} \text{Var} (X_{n,i} X_{n,j} - \rho X_{n,k} X_{n,l}) > 0,
\]
where \(\inf_{\ast}\) is taken over all \(1 \leq i, j, k, l \leq m\) such that \(i < j, k < l\) and \((i, j) \neq (k, l)\). Observe that when \(\epsilon_i\)'s are i.i.d. \(N(0, 1)\),
\[
\text{Var} (X_{n,i} X_{n,j} - \rho X_{n,k} X_{n,l}) = 2 \sum_{t \in \mathbb{Z}} (f_{n,i,i-t} f_{n,j,j-t} - \rho f_{n,k,k-t} f_{n,l,l-t})^2 + \sum_{s \leq t} (f_{n,i,i-t} f_{n,j,j-s} + f_{n,i,i-s} f_{n,j,j-t} - \rho f_{n,k,k-t} f_{n,l,l-s} - \rho f_{n,k,k-s} f_{n,l,l-t})^2;
\]
and when \(\epsilon_i\)'s are arbitrary variables, the variance is given by the same formula with the number 2 in \(5\) being replaced by \(2 + \kappa_4\). Therefore, if \(3\) holds, then
\[
\limsup_{n \to \infty} \inf_{\rho \in \mathbb{R}} \text{Var} (X_{n,i} X_{n,j} - \rho X_{n,k} X_{n,l}) \geq \min \{1, 1 + \kappa_4/2\} \cdot \ell > 0,
\]
which implies (A2) holds. To summarize, we have shown that \(3\) and \(4\) suffice for (A2).

Now we turn to Conditions (A3) and (A3'). Set
\[
h_n(k) = \sup_{1 \leq i \leq m} \left( \sum_{|t| = \lfloor k/2 \rfloor} f_{n,i,t}^2 \right)^{1/2},
\]
where \(|x| = \max\{y \in \mathbb{Z} : y \leq x\}\) for any \(x \in \mathbb{E}\), then we have
\[
|\sigma_{n,i,j}| \leq 2h_n(0)h_n(|i - j|) = 2h_n(|i - j|).
\]
Fixing a subset \(\{i, j\}\), for any integer \(b > 0\), there are at most \(8b^2\) subsets \(\{k, l\}\) such that \(\{k, l\} \subset B(i; b) \cup B(j; b)\), where \(B(x; r)\) is the open ball \(\{y : |x - y| < r\}\). For all other subsets \(\{k, l\}\), we have
\[
|\text{Cov}(X_{n,i} X_{n,j}, X_{n,k} X_{n,l})| \leq (4 + 2\kappa_4)h_n(b),
\]
and hence (A3) holds if we assume \(h_n(k_n) \log k_n = o(1)\) for any positive sequence \((k_n)\) such that \(k_n \to \infty\). (A3') holds if we assume
\[
\sum_{k=1}^{m} |h_n(k)|^2 = O(m^{1-\delta}).
\]
for some \(\delta > 0\), because
\[
|\text{Cov}(X_{n,i} X_{n,j}, X_{n,k} X_{n,l})| \leq 2\kappa_4h_n(|i - j|) + 2h_n(|i - k|) + 2h_n(|i - l|).
\]
The asymptotic distribution given in Theorem 2 has several applications. One of them is in high dimensional covariance matrix regularization, because Theorem 2 implies a uniform convergence rate for all sample covariances. Recently, Cai and Liu (2011) explored this direction, and proposed a thresholding procedure for sparse covariance matrix estimation, which is adaptive to the variability of each individual entry. Their method is superior to the uniform thresholding approach studied by Bickel and Levina (2008).

Testing structures of covariance matrices is also a very important statistical problem. As mentioned in the introduction, when the data dimension is high, conventional tests often cannot be implemented or do not work well. Let \( \Sigma_n \) and \( R_n \) be the covariance matrix and correlation matrix of the random vector \( (X_{n,1}, i \leq i \leq m) \) respectively. Two types of tests have been studied under the large \( n \), large \( m \) paradigm. Chen et al. (2010), Bai et al. (2009), Ledoit and Wolf (2002) and Johnstone (2001) considered the test
\[
H_0 : \Sigma_n = \Sigma_0; \quad (6)
\]
and Liu et al. (2008), Schott (2005), Srivastava (2005) and Jiang (2004) studied the problem of testing for complete independence
\[
H_0 : R_n = I_m. \quad (7)
\]
Their testing procedures are all based on the critical assumption that the entries of the data matrix \( X_n \) are i.i.d., while the hypotheses themselves only require the entries of \( (X_{n,1}, i \leq i \leq m) \) to be uncorrelated. Evidently, we can use \( M_n \) in (2) to test (7), and we only require the uncorrelatedness for the validity of the limiting distribution established in Theorem 2 as long as the mild conditions of the theorem are satisfied. On the other hand, we can also take the sample variances into consideration, and use the following test statistic
\[
M_n' = \max_{1 \leq i \leq j \leq m} \frac{\hat{\sigma}_{n,i,j} - \sigma_{n,i,j}}{\sqrt{\hat{\tau}_{n,i,j}}},
\]
to test the identity hypothesis (6), where \( \sigma_{n,i,j} = I\{i = j\} \). It is not difficult to verify that \( M_n' \) has the same asymptotic distribution as \( M_n \) under the same conditions with the only difference being that we now have to take sample variances into account as well, namely, the index set \( I_n \) in Section 2 is redefined as \( I_n = \{(i, j) : 1 \leq i < j \leq m\} \). Clearly, we can also use \( M_n' \) to test \( H_0 : \Sigma_n = \Sigma_0 \) for some known covariance matrix \( \Sigma_0 \).

By checking the proof of Theorem 2, it can be seen that if instead of taking the maximum over the set \( I_n = \{(i, j) : 1 \leq i < j \leq m\} \), we only take the maximum over some subset \( A_n \subset I_n \) whose cardinality \( |A_n| \) converges to infinity, then the maximum also has the Gumbel type convergence with normalization constants which are functions of the cardinality of the set \( A_n \). Based on this observation, we are able to consider three more testing problems.
4.1. Test for stationarity

Suppose we want to test whether the population is a stationary time series. Under the null hypothesis, each row of the data matrix \( X_n \) is distributed as a stationary process \((X_i)_{1 \leq i \leq m}\). Let \( \gamma_l = \text{Cov}(X_0, X_l) \) be the autocovariance at lag \( l \). In principle, we can use the following test statistic

\[
\tilde{T}_n = \max_{1 \leq i \leq j \leq m} \left| \frac{\hat{\sigma}_{n,i,j} - \gamma_{i-j}}{\hat{\tau}_{n,i,j}} \right|
\]

The problem is that \( \gamma_l \) are unknown. Fortunately, they can not only be estimated, but also be estimated with higher accuracy

\[
\hat{\gamma}_{n,l} = \frac{1}{nm} \sum_{k=1}^{n} \sum_{i=|l|+1}^{n} (X_{n,k,i-|l|} - \hat{\mu}_n)(X_{n,k,i} - \hat{\mu}_n),
\]

where \( \hat{\mu}_n = \frac{1}{nm} \sum_{k=1}^{n} \sum_{i=1}^{m} X_{n,k,i} \), and we are lead to the test statistic

\[
T_n = \max_{1 \leq i \leq j \leq m} \left| \frac{\hat{\sigma}_{n,i,j} - \hat{\gamma}_{i-j}}{\hat{\tau}_{n,i,j}} \right|
\]

Using similar arguments of Theorem 2 of Xiao and Wu (2011), under suitable conditions, we have

\[
\max_{0 \leq l \leq m-1} |\hat{\gamma}_{n,l} - \gamma_l| = O_P(\sqrt{\log m/nm}).
\]

Therefore, the limiting distribution for \( M_n \) in Theorem 2 also holds for \( T_n \).

4.2. Test for bandedness

In time series and longitudinal data analysis, it can be of interest to test whether \( \Sigma_m \) has the banded structure. The hypothesis to be tested is

\[
H_0 : \sigma_{n,i,j} = 0 \text{ if } |i - j| > B,
\]

where \( B = B_n \) may depend on \( n \). Cai and Jiang (2011) studied this problem under the assumption that each row of the data matrix \( X_n \) is a Gaussian random vector. They proposed to use the maximum sample correlation outside the band

\[
\tilde{T}_n = \max_{|i-j| > B} \frac{\hat{\sigma}_{n,i,j}}{\sqrt{\hat{\sigma}_{n,i,i} \hat{\sigma}_{n,j,j}}}
\]

as the test statistic, and proved that \( T_n \) also has the Gumbel type convergence provided that \( B_n = o(m) \) and several other technical conditions hold.

Apparently, our Theorem 2 can be employed to test (8). If all the conditions of the theorem are satisfied, the test statistic

\[
T_n = \max_{|i-j| > B_n} \frac{|\hat{\sigma}_{n,i,j}|}{\sqrt{\hat{\tau}_{n,i,j}}}
\]

has the same asymptotic distribution as \( M_n \) as long as \( B_n = o(m) \). Our theory does not need the normality assumption.
4.3. Assess the tapering procedure

Bandit and tapering are commonly used regularization procedures in high dimensional covariance matrix estimation. Convergence rates were first obtained by [Bickel and Levina (2008a)], and later on improved by [Cai et al. (2010)]. Let us introduce a weaker version of the latter result. Suppose each row of $X_n$ is distributed as the random vector $X = (X_i)_{1 \leq i \leq m}$ with mean $\mu$ and covariance matrix $\Sigma = (\sigma_{ij})$. Let $K_0, K$ and $t$ be positive constants, and $\mathcal{C}_n(K_0, K, t)$ be the class of $m$-dimensional distributions which satisfy the following conditions

$$\max_{|i-j|=k} |\sigma_{ij}| \leq K k^{-1+\eta} \quad \text{for all } k; \quad (9)$$

$$\lambda_{\text{max}}(\Sigma) \leq K_0;$$

$$P \left[ |v^T (X - \mu)| > x \right] \leq e^{-tx^2/2} \quad \text{for all } x > 0 \text{ and } \|v\| = 1;$$

where $\lambda_{\text{max}}(\Sigma)$ is the largest eigenvalue of $\Sigma$. For a given even integer $1 \leq B \leq m$, define the tapered estimate of the covariance matrix $\hat{\Sigma}_{n,B_n} = (\hat{\sigma}_{n,i,j})$, where the weights correspond to a flat top kernel and are given by

$$w_{ij} = \begin{cases} 1, & \text{when } |i-j| \leq B_n/2, \\ 2 - 2 |i-j|/B_n, & \text{when } B_n/2 < |i-j| \leq B_n, \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 5 [Cai et al., 2010].** If $m \geq n^{1/(2\eta+1)}$, $\log m = o(n)$ and $B_n = n^{1/(2\eta+1)}$, then there exists a constant $C > 0$ such that

$$\sup_{\mathcal{C}_n} \mathbb{E} \left[ \lambda(\hat{\Sigma}_{n,B_n} - \Sigma) \right]^2 \leq C n^{-2\eta/(2\eta+1)} + C \log m n^{-1}.$$

We see that it is the parameter $\eta$ that decides the convergence rate under the operator norm. After such a tapering procedure has been applied, it is important to ask whether it is appropriate, and in particular, whether (9) is satisfied. We propose to use

$$T_n = \max_{|i-j| > B_n} \frac{|\hat{\sigma}_{n,i,j}|}{\sqrt{\hat{\tau}_{n,i,j}}}$$

as the test statistic. According to the observation made at the beginning of Section 4, if the conditions of Theorem 4 are satisfied, then

$$T'_n = \max_{|i-j| > B_n} \frac{|\hat{\sigma}_{n,i,j} - \sigma_{i,j}|}{\sqrt{\hat{\tau}_{n,i,j}}}$$

has the same limiting law as $M_n$. On the other hand, (9) implies that

$$\max_{|i-j| > B_n} |\sigma_{i,j}| = O \left( n^{-(1+\eta)/(2\eta+1)} \right),$$

so $T_n$ has the same limiting distribution as $T'_n$ if we further assume $\log m = o \left( n^{2/(4\eta+2)} \right)$. 
5. Proof

The proofs of Theorem 2 under (A4) and (A4') are very similar, and they share a common Poisson approximation step, which we will formulate in Section 5.1 under a more general context, where the limiting distribution of the maximum of sample means is obtained. Since the proof under (A4') is more involved, we provide the detailed proof under this assumption in Section 5.2 and point out in Section 5.3 how it can be adapted to give a proof under (A4).

5.1. Maximum of Sample Means: An Intermediate Step

In this section we provide a general result on the maximum of sample means. Let $Y_n = (Y_{n,k,i})_{1 \leq k \leq n, i \in I_n}$ be a data matrix whose $n$ rows are independent and identically distributed, and whose entries have mean zero and variance one, where $I_n$ is an index set with cardinality $|I_n| = s_n$. For each $i \in I_n$, let $y_i$ be the $i$-th column of $Y_n$, $y_i = (1/n) \sum_{k=1}^{n} Y_{n,k,i}$. Define

$$ W_n = \max_{i \in I_n} |y_i|, \quad (10) $$

Let $\Sigma_n$ be the covariance matrix of the $s_n$-dimensional random vector $(Y_{n,1,i})_{i \in I_n}$.

**Lemma 6.** Assume $\Sigma_n$ satisfies either (B1) or (B2) of Section 6.1 and $\log s_n = o(n^{1/3})$. Suppose there is a constant $C > 0$ such that $Y_{n,k,i} \in B(1,Ct_n)$ for each $1 \leq k \leq n$, $i \in I_n$, with

$$ t_n = \frac{\sqrt{n} \delta_n}{(\log s_n)^{3/2}}, $$

where $(\delta_n)$ is a sequence of positive numbers such that $\delta_n = o(1)$ and $(\log s_n)^3/n = o(\delta_n)$, and the definition of the collection $B(d, \tau)$ is given in [27]. Then

$$ \lim_{n \to \infty} P \left( n W_n^2 - 2 \log s_n + \log(\log s_n) + \log \pi \leq 0 \right) = \exp \left( -e^{-z^2/2} \right). \quad (11) $$

**Proof.** For each $z \in \mathbb{R}$, let $z_n = a_{2n} z/2 + b_{2n}$. Let $(Z_{n,i})_{i \in I_n}$ be a mean zero normal random vector with covariance matrix $\Sigma_n$. For any subset $A = \{i_1, i_2, \ldots, i_d\} \subset I_n$, let $y_A = \sqrt{n} (\bar{y}_{i_1}, \bar{y}_{i_2}, \ldots, \bar{y}_{i_d})$ and $Z_A = (Z_{i_1}, Z_{i_2}, \ldots, Z_{i_d})$.

By Lemma 8 we have for $\theta_n = \delta_n^{1/2}/\sqrt{(\log s_n)}$ that

$$ P(|y_A| > z_n) \leq P(|Z_A| > z_n - \theta_n) + C_d \exp \left\{ -\theta_n \right\} C_d \exp \left\{ -C_d \log s_n \delta_n^{1/2} \right\} $$

Therefore,

$$ \sum_{A \subset I_n, |A| = d} P(|y_A| > z_n) \leq \sum_{A \subset I_n, |A| = d} P(|Z_A| > z_n - \theta_n) + C_d s_n^{d} \exp \left\{ -(\log s_n) \delta_n^{1/2} \right\}. $$
Similarly, we have
\[
\sum_{A \subset I, |A|=d} P(|y_A| > z_n) \\
\geq \sum_{A \subset I, |A|=d} P(|Z_A| > z_n + \theta_n) - C_d s_n^d \exp \left\{ - (\log s_n) \delta_n^{-1/2} \right\}.
\]

Since \((z_n \pm \theta_n)^2 = 2 \log s_n - \log(\log s_n) - \log \pi + o(1)\), by Lemma 7, we know
\[
\lim_{n \to \infty} \sum_{A \subset I, |A|=d} P(|Z_A| > z_n \pm \theta_n) = e^{-dz/2} d!,
\]
and hence
\[
\lim_{n \to \infty} \sum_{A \subset I, |A|=d} P(|y_A| > z_n) = e^{-dz/2} d!.
\]
The proof is complete in view of Lemma 9.

5.2. Proof under (A4')

We divide the proof into three steps. The first one is a truncation step, which will make the Gaussian approximation result Lemma 8 and the Bernstein inequality applicable, so that we can prove Theorem 2 under the assumption that all the involved mean and variance parameters are known. In the next two steps we show that plugging in estimated mean and variance parameters does not change the limiting distribution.

**Step 1: Truncation** For notational simplicity we let \( q = p/(4 + 2p) \). Define
\[
\hat{X}_{n,k,i} = X_{n,k,i} I \left\{ |X_{n,k,i}| \leq n^{1/(4+2p)} \right\},
\]
and define \( \hat{M}_n \) similarly as \( M_n \) with \( X_{n,k,i} \) being replaced by its truncated version \( \hat{X}_{n,k,i} \). Since \( \log m = o(n^q) \), we have
\[
P \left( M_n \neq M_n \right) \leq \sum_{k=1}^n \sum_{i=1}^m P \left[ |X_{n,k,i}| > n^{1/(4+2p)} \right]
\leq nm K_n(t, p) \exp \left\{ -tr^{p/(4+2p)} \right\}
= K_n(t, p) \exp \{-tn^q + \log m + \log n\} = o(1).
\]
Therefore, in the rest of the proof, it suffices to consider \( \hat{X}_{n,k,i} \). For notational simplicity, we still use \( \hat{X}_{n,k,i} \) to denote its centered version with mean zero.
Define $\tilde{\sigma}_{n,i,j} = \mathbb{E}(\tilde{X}_{n,1,i}\tilde{X}_{n,1,j})$, and $\tilde{\tau}_{n,i,j} = \text{Var}(\tilde{X}_{n,1,i}\tilde{X}_{n,1,j})$. Set

$$M_{n,1} = \max_{1 \leq i < j \leq m} \frac{1}{\tilde{\tau}_{n,i,j}} \left| \frac{1}{n} \sum_{k=1}^{n} \tilde{X}_{n,k,i} \tilde{X}_{n,k,j} - \tilde{\sigma}_{n,i,j} \right|;$$

$$M_{n,2} = \max_{1 \leq i < j \leq m} \frac{1}{\tilde{\tau}_{n,i,j}} \left| \frac{1}{n} \sum_{k=1}^{n} \tilde{X}_{n,k,i} \tilde{X}_{n,k,j} - \sigma_{n,i,j} \right|.$$

Elementary calculations show that

$$\max_{1 \leq i \leq j \leq m} |\tilde{\sigma}_{n,i,j} - \sigma_{n,i,j}| \leq C \exp\left\{ -tn^q/2 \right\}, \quad \text{and} \quad (13)$$

$$\max_{\alpha, \beta \in I_n} |\text{Cov}(\tilde{X}_{n,\alpha}, \tilde{X}_{n,\beta}) - \text{Cov}(X_{n,\alpha}, X_{n,\beta})| \leq C \exp\left\{ -tn^q/2 \right\}. \quad (14)$$

By (14), we know the covariance matrix of $(\tilde{X}_{n,\alpha})_{\alpha \in I_n}$ satisfies either (B1) or (B2) if $\Sigma_n$ satisfies (B1) or (B2) correspondingly. On the other hand, we have by elementary calculation that there exists a constant $C_p > 0$ such that

$$\limsup_{n \to \infty} \max_{\alpha \in I_n} \mathbb{E} \exp\left\{ C_p t |\tilde{X}_{n,\alpha}|^{p/2} \right\} < \infty.$$

It follows that when $0 < p < 2$, for each integer $r \geq 3$

$$\mathbb{E}|\tilde{X}_{n,\alpha}|^r \leq \mathbb{E}|\tilde{X}_{n,\alpha}|^{p/2} \cdot \left( 4n^2/(4+2p) \right)^{r(1-p/2)} \leq \left( 4n^2/(4+2p) \right)^{r(1-p/2)} r! (C_p t)^{-r} \mathbb{E} \exp\{C_p t|X_{n,\alpha}|^{p/2}\}.$$

Therefore,

$$\mathbb{E}_0\tilde{X}_{n,\alpha} \in \mathcal{B}\left[ 1, C \frac{\sqrt{n}}{n^{2p/(4+2p)}} \right].$$

When $2 \leq p \leq 4$, it is easily seen that $\mathbb{E}_0\tilde{X}_{n,\alpha} \in \mathcal{B}(1, C)$. Since $\log m = o(n^q)$, we know all the conditions of Lemma 6 are satisfied, and hence

$$\lim_{n \to \infty} P\left( nM_{n,1}^2 - 4 \log m + \log(\log m) + \log(8\pi) \leq y \right) = \exp\left( -e^{-y/2} \right). \quad (15)$$

Combining (13) and (14), we know the preceding equation (15) also holds with $M_{n,1}$ being replaced by $M_{n,2}$.

**Step 2: Effect of Estimated Means**

Set $\bar{X}_{n,i} = (1/n) \sum_{k=1}^{n} \tilde{X}_{n,k,i}$. Define

$$M_{n,3} = \max_{1 \leq i < j \leq m} \frac{1}{\tilde{\tau}_{n,i,j}} \left| \frac{1}{n} \sum_{k=1}^{n} (\tilde{X}_{n,k,i} - \bar{X}_{n,i})(\tilde{X}_{n,k,j} - \bar{X}_{n,j}) - \sigma_{n,i,j} \right|.$$
In this step we show that (15) also holds for $M_{n,3}$. Observe that

$$|M_{n,3} - M_{n,2}| \leq \max_{1 \leq i < j \leq m} \frac{1}{\sqrt{\bar{\tau}_{n,i,j}}} \leq \max_{1 \leq i \leq m} |\bar{X}_{n,i}|^2 \left( \min_{1 \leq i < j \leq m} \frac{\bar{\tau}_{n,i,j}}{\tilde{\tau}_{n,i,j}} \right)^{-1/2}.$$  

Since each $X_{n,k,i}$ is bounded by $2n^{1/(4+2p)}$, by Bernstein’s inequality we have for any constant $K > 0$,

$$\max_{1 \leq i \leq m} P \left( |\bar{X}_{n,i}| > 2K \sqrt{\frac{\log m}{n}} \right) \leq C \exp \left\{ - \frac{2K^2 n \log m}{Cn + 2K \sqrt{n \log m} \cdot 2n^{1/(4+2p)}} \right\} \leq Cm^{-K^2/C},$$

and hence

$$\max_{1 \leq i \leq m} |\bar{X}_{n,i}| = O_p \left( \sqrt{\frac{\log m}{n}} \right),$$

which together with (14) implies that

$$|M_{n,3} - M_{n,2}| = O_p \left( \frac{\log m}{n} \right) = o_p \left( \sqrt{\frac{1}{n \log m}} \right).$$

Therefore, (16) also holds for $M_{n,3}$.

**Step 3: Effect of Estimated Variances** Denote by $\hat{\sigma}_{n,i,j}$ the estimate of $\bar{\sigma}_{n,i,j}$

$$\hat{\sigma}_{n,i,j} = \frac{1}{n} \sum_{k=1}^{n} (\bar{X}_{n,k,i} - \bar{X}_{n,i})(\bar{X}_{n,k,j} - \bar{X}_{n,j}).$$

In the definition of $\hat{M}_n$, $\tilde{\tau}_{n,i,j}$ is unknown, and is estimated by

$$\hat{\tau}_{n,i,j} = \frac{1}{n} \sum_{k=1}^{n} \left[ (\bar{X}_{n,k,i} - \bar{X}_{n,i})(\bar{X}_{n,k,j} - \bar{X}_{n,j}) - \hat{\sigma}_{n,i,j} \right]^2$$

In this step we show that (15) holds for $\hat{M}_n$. Since

$$n \left| M_{n,3}^2 - \hat{M}_n^2 \right| \leq nM_{n,3}^2 \max_{1 \leq i < j \leq m} |1 - \hat{\tau}_{n,i,j}/\tilde{\tau}_{n,i,j}|,$$

it suffices to show that

$$\max_{1 \leq i < j \leq m} |\hat{\tau}_{n,i,j} - \tilde{\tau}_{n,i,j}| = o_p(1/ \log m).$$

Set

$$\hat{\tau}_{n,i,j,1} = \frac{1}{n} \sum_{k=1}^{n} \left[ (\bar{X}_{n,k,i} - \bar{X}_{n,i})(\bar{X}_{n,k,j} - \bar{X}_{n,j}) - \hat{\sigma}_{n,i,j} \right]^2$$

$$\hat{\tau}_{n,i,j,2} = \frac{1}{n} \sum_{k=1}^{n} \left( \bar{X}_{n,k,i} - \bar{X}_{n,k,j} \right)^2.$$
Observe that

\[ \hat{\tau}_{n,i,j,1} - \hat{\tau}_{n,i,j} = (\hat{\sigma}_{n,i,j} - \tilde{\sigma}_{n,i,j})^2 \]

which in together with (15) implies that

\[ \max_{1 \leq i < j \leq m} |\hat{\tau}_{n,i,j,1} - \hat{\tau}_{n,i,j}| = O_P(\log m/n). \] (18)

Note that \( \tilde{X}_{n,k,i,j} \) are uniformly bounded according to the truncation (12), so

\[ (\tilde{X}_{n,k,i} \tilde{X}_{n,k,j} - \tilde{\sigma}_{n,i,j})^2 \leq \frac{64}{n^{4/(4+2p)}}. \]

By Bernstein’s inequality, we have

\[ \max_{1 \leq i < j \leq m} \mathbb{P} \left( |\hat{\tau}_{n,i,j,2} - \hat{\tau}_{n,i,j}| \geq 2n^{-q} \right) \leq \exp \left\{ - \frac{2n^{2(1-q)}}{Cn + 2n^{1-q} \cdot 128n^{4/(4+2p)}/3} \right\} \leq \exp \left( -n^q/100 \right), \]

and it follows that

\[ \max_{1 \leq i < j \leq m} |\hat{\tau}_{n,i,j,2} - \hat{\tau}_{n,i,j}| = O_P(n^{-q}). \] (19)

In view of (18), (19), and the assumption \( \log m = o(n^q) \), we know to show (17), it remains to prove

\[ \max_{1 \leq i < j \leq m} |\hat{\tau}_{n,i,j,1} - \hat{\tau}_{n,i,j,2}| = o_P(1/\log m). \] (20)

Elementary calculations show that

\[ \max_{1 \leq i < j \leq m} |\hat{\tau}_{n,i,j,1} - \hat{\tau}_{n,i,j,2}| \leq 4h_{n,1}^2 h_{n,2} + 3h_{n,1}^4 + 4h_{n,1}^{1/2} h_{n,2} h_{n,1} + 2h_{n,3} h_{n,1}^2, \]

where

\[ h_{n,1} = \max_{1 \leq i \leq m} |\hat{X}_{n,i}| \]

\[ h_{n,2} = \max_{1 \leq i \leq m} \frac{1}{n} \sum_{k=1}^{n} \hat{X}_{n,k,i}^2 \]

\[ h_{n,3} = \max_{1 \leq i \leq j \leq m} \left| \frac{1}{n} \sum_{k=1}^{n} \hat{X}_{n,k,i} \hat{X}_{n,k,j} - \hat{\sigma}_{n,i,j} \right| \]

\[ h_{n,4} = \hat{\tau}_{n,i,j,2}. \]

By (16), we know \( h_{n,1} = O_P(\sqrt{\log m/n}) \). By (19) we have \( h_{n,4} = O_P(1) \). Combining (12) and the Bernstein’s inequality, we can show that

\[ h_{n,3} = O_P \left( \sqrt{\log m/n} \right). \]
As an immediate consequence, we know \( h_{n,2} = O_P(1) \). Therefore,
\[
\max_{1 \leq i < j \leq m} |\hat{\tau}_{n,i,j,1} - \hat{\tau}_{n,i,j,2}| = O_P \left( \sqrt{\log m/n} \right),
\]
and (20) holds by using the assumption \( \log m = o(n^q) = o(n^{1/3}) \). The proof of Theorem 2 under (A4) is now complete.

5.3. Proof under (A4)

We follow the proof in Section 5.2 and point out necessary modifications to make it work under (A4). If not specified, all the notations have the same definitions as in Section 5.2. For notational simplicity, we let \( p = 4(1 + q) + \delta \).

**Step 1: Truncation** We truncate \( X_{n,k,i} \) by
\[
\tilde{X}_{n,k,i} = X_{n,k,i} I \left\{ |X_{n,k,i}| \leq n^{1/4}/\log n \right\},
\]
then
\[
P \left( \tilde{M}_n \neq M_n \right) \leq nm \mathcal{M}_n(p) n^{-p/4} (\log n)^p \leq C \mathcal{M}_n(p) n^{-\delta/4} (\log n)^p = o(1).
\]

Therefore, in the rest of the proof, it suffices to consider \( \tilde{X}_{n,k,i} \). For notational simplicity, we still use \( \tilde{X}_{n,k,i} \) to denote its centered version with mean zero.

Elementary calculations show that
\[
\max_{1 \leq i \leq m} |\tilde{\sigma}_{n,i,j} - \sigma_{n,i,j}| \leq C n^{-(p-2)/4} (\log n)^{p-2}, \quad \text{and}
\]
\[
\max_{\alpha,\beta \in I_n} \left| \text{Cov}(\tilde{X}_{n,\alpha}, \tilde{X}_{n,\beta}) - \text{Cov}(X_{n,\alpha}, X_{n,\beta}) \right| \leq C n^{-(p-4)/4} (\log n)^{p-4}. \tag{21}
\]

By (21), we know the covariance matrix of \( (\tilde{X}_{n,\alpha})_{\alpha \in I_n} \) satisfies either (B1) or (B2) if \( \Sigma_n \) satisfies (B1) or (B2) correspondingly. Since
\[
E_0 \tilde{X}_{n,\alpha} \in \mathcal{B} \left[ 1, 8\sqrt{n}/(\log n)^2 \right],
\]
we know all the conditions of Lemma 6 are satisfied, and hence (15) holds for \( M_{n,1} \). Combining (21) and (22), we know (15) also holds with if we replace \( M_{n,1} \) by \( M_{n,2} \).

**Step 2: Effect of Estimated Means** Using Bernstein’s inequality, we can show
\[
\max_{1 \leq i \leq m} |\tilde{X}_{n,i}| = O_P \left( \sqrt{\frac{\log n}{n}} \right),
\]
which implies that
\[
|M_{n,3} - M_{n,2}| = O_P \left( \frac{\log n}{n} \right)
\]
and hence (15) also holds for \( M_{n,3} \).
Step 3: Effect of Estimated Variances

It suffices to show that

\[
\max_{1 \leq i < j \leq m} |\hat{\tau}_{n,i,j} - \tilde{\tau}_{n,i,j}| = o_P(1/\log n). \tag{23}
\]

Using (15), we know

\[
\max_{1 \leq i < j \leq m} |\hat{\tau}_{n,i,j,1} - \tilde{\tau}_{n,i,j}| = O_P(\log n/n). \tag{24}
\]

Since

\[
\left(\hat{X}_{n,k,i} - \tilde{\sigma}_{n,i,j}\right)^2 \leq 64n/(\log n)^4.
\]

By Corollary 1.6 of Nagaev (1979) (with \(x = n/(\log n)^2\) and \(y = n/[2(\log n)^3]\) in their inequality (1.22)), we have

\[
\max_{1 \leq i < j \leq m} P\left(|\hat{\tau}_{n,i,j,2} - \tilde{\tau}_{n,i,j}| \geq (\log n)^{-2}\right) \leq \left[\frac{Cn}{n(\log n)^{-2} \cdot [n(\log n)^{-3}/2]^{\eta^2}}\right]^{\log n} \leq \left[\frac{C(\log n)^{5}}{n^{\eta^1}}\right]^{\log n},
\]

and it follows that

\[
\max_{1 \leq i < j \leq m} |\hat{\tau}_{n,i,j,2} - \tilde{\tau}_{n,i,j}| = O_P\left((\log n)^{-2}\right). \tag{25}
\]

In view of (24), (25), we know to show (23), it remains to prove

\[
\max_{1 \leq i < j \leq m} |\hat{\tau}_{n,i,j,1} - \tilde{\tau}_{n,i,j,2}| = o_P(1/\log n). \tag{26}
\]

We know \(h_{n,1} = O_P(\sqrt{\log n/n})\) and \(h_{n,4} = O_P(1).\) Using the Bernstein’s inequality, we can show that

\[
h_{n,3} = O_P\left(\sqrt{\log n/n}\right),
\]

and it follows that \(h_{n,2} = O_P(1).\) Therefore,

\[
\max_{1 \leq i < j \leq m} |\hat{\tau}_{n,i,j,1} - \tilde{\tau}_{n,i,j,2}| = O_P\left(\sqrt{\log n/n}\right),
\]

and (26) holds. The proof of Theorem 2 under (A4) is now complete.

6. Some auxiliary results

In this section we provide a normal comparison principle and a Gaussian approximation result, and a Poisson convergence theorem.
6.1. A normal comparison principle

Suppose for each \( n \geq 1 \), \((X_{n,i})_{i \in \mathcal{I}_n}\) is a Gaussian random vector whose entries have mean zero and variance one, where \( \mathcal{I}_n \) is an index set with cardinality \(|\mathcal{I}_n| = s_n\). Let \( \Sigma_n = (r_{n,i,j})_{i,j \in \mathcal{I}_n} \) be the covariance matrix of \((X_{n,i})_{i \in \mathcal{I}_n}\). Assume that \( s_n \to \infty \) as \( n \to \infty \).

We impose either of the following two conditions.

**(B1)** For any sequence \((b_n)\) such that \( b_n \to \infty \), \( \gamma(n, b_n) = o\left(\frac{1}{\log b_n}\right) \); and \( \limsup_{n \to \infty} \gamma_n < 1 \).

**(B2)** For any sequence \((b_n)\) such that \( b_n \to \infty \), \( \gamma(n, b_n) = o(1) \);

\[
\sum_{i \neq j \in \mathcal{I}_n} r_{n,i,j}^2 = O\left(s_n^{2-\delta}\right) \text{ for some } \delta > 0; \text{ and } \limsup_{n \to \infty} \gamma_n < 1.
\]

where

\[
\gamma(n, b_n) := \sup_{i \in \mathcal{I}_n} \sup_{A \subset \mathcal{I}_n, |A| = b_n} \inf_{j \in A} |r_{n,i,j}| \quad \text{and} \quad \gamma_n := \sup_{i,j \in \mathcal{I}_n; i \neq j} |r_{n,i,j}|.
\]

**Lemma 7.** Assume either (B1) or (B2). For a positive real number \( z_n \), define

\[
A'_{n,i} = \{|X_{n,i}| > z_n\} \quad \text{and} \quad Q'_{n,d} = \sum_{A \subset \mathcal{I}_n, |A| = d} P\left(\bigcap_{i \in A} A'_{n,i}\right).
\]

If \( z_n \) satisfies that \( z_n^2 = 2 \log s_n - \log \log s_n - \log \pi + 2 + o(1) \), then for all \( d \geq 1 \),

\[
\lim_{n \to \infty} Q'_{n,d} = \frac{e^{-dz}}{d!},
\]

Lemma 7 is a refined version of Lemma 20 in [Xiao and Wu (2011)](https://www.jstor.org/stable/41586883), so we omit the proof and put the details in a supplementary file.

**Remark 2.** The conditions imposed on \( \gamma(n, b_n) \) seem a little involved. We have the following equivalent versions. Define

\[
G_n(t) = \max_{i \in \mathcal{I}_n} \sum_{j \in \mathcal{I}_n} I\{|r_{n,i,j}| > t\}.
\]

Then (i) \( \gamma(n, b_n) = o(1) \) for any sequence \( b_n \to \infty \) if and only if the sequence \([G_n(t)]_{n \geq 1}\) is bounded for all \( t > 0 \); and (ii) \( \gamma(n, b_n)(\log b_n) = o(1) \) for any sequence \( b_n \to \infty \) if and only if \( G_n(t_n) = \exp\{o(1/t_n)\} \) for any positive sequence \( (t_n) \) converging to zero.
6.2. A Gaussian approximation result

For a positive integer \( d \), let \( \mathcal{B}_d \) be the Borel \( \sigma \)-field on the Euclidean space \( \mathbb{R}^d \). For two probability measures \( P \) and \( Q \) on \( (\mathbb{R}^d, \mathcal{B}_d) \) and \( \lambda > 0 \), define the quantity

\[
\pi(P, Q; \lambda) = \sup_{A \in \mathcal{B}_d} \left\{ \max \left[ P(A) - Q(A^\lambda), Q(A) - P(A^\lambda) \right] \right\},
\]

where \( A^\lambda :\) is the \( \lambda \)-neighborhood of \( A \)

\[
A^\lambda := \left\{ x \in \mathbb{R}^d : \inf_{y \in A} |x - y| < \lambda \right\}.
\]

For \( \tau > 0 \), let \( \mathcal{B}(d, \tau) \) be the collection of \( d \)-dimensional random variables which satisfy the multivariate analogue of the Bernstein’s condition. Denote by \( \langle x, y \rangle \) the inner product of two vectors \( x \) and \( y \).

\[
\mathcal{B}(d, \tau) = \{ \xi : \mathbb{E}\xi = 0, \text{ and } \mathbb{E}\langle \xi, t \rangle^2 \leq \frac{1}{2} m! \tau^{m-2} \| u \|^m \mathbb{E}\langle \xi, t \rangle^2 \}
\]

(27)

The following Lemma on the Gaussian approximation is taken from Zaitsev (1987).

**Lemma 8.** Let \( \tau > 0 \), and \( \xi_1, \xi_2, \ldots, \xi_n \in \mathbb{R}^d \) be independent random vectors such that \( \xi_i \in \mathcal{B}(d, \tau) \) for \( i = 1, 2, \ldots, n \). Let \( S = \xi_1 + \xi_2 + \ldots + \xi_n \), and \( \mathcal{L}(S) \) be the induced distribution on \( \mathbb{R}^d \). Let \( \Phi \) be the Gaussian distribution with the zero mean and the same covariance matrix as that of \( S \). Then for all \( \lambda > 0 \)

\[
\pi[\mathcal{L}(S), \Phi; \lambda] \leq c_{1,d} \exp \left( -\frac{\lambda}{c_{2,d} \tau} \right),
\]

where the constants \( c_{j,d}, j = 1, 2 \) may be taken in the form \( c_{j,d} = c_j \tau^{5/2} \).

6.3. Poisson approximation: moment method

**Lemma 9.** Suppose for each \( n \geq 1 \), \( (A_{n,i})_{i \in I_n} \) is a finite collection of events. Let \( I_{A_{n,i}} \) be the indicator function of \( A_{n,i} \), and \( W_n = \sum_{i \in I} I_{A_{n,i}} \). For each \( d \geq 1 \), define

\[
Q_{n,d} = \sum_{A \subset I_n, |A| = d} P\left( \bigcap_{i \in A} A_{n,i} \right).
\]

Suppose there exists a \( \lambda > 0 \) such that

\[
\lim_{n \to \infty} Q_{n,d} = \lambda^d / d! \text{ for each } d \geq 1.
\]

Then

\[
\lim_{n \to \infty} P(W_n = k) = \lambda^k e^{-\lambda} / k! \text{ for each } k \geq 0.
\]
Observe that for each $d \geq 1$, the $d$-th factorial moment of $W_n$ is given by
\[ \mathbb{E}[W_n(W_n - 1) \cdots (W_n - d + 1)] = d! \cdot Q_{n,d}, \]
so Lemma 10 is essentially the moment method. The proof is elementary, and we omit details.

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Supplementary file of
Simultaneous Inference of Covariances

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In this document we give the proofs of Lemma 3, Lemma 4 and Lemma 7 of the main article.

Proof of Lemma 3. Assume $X_i$ has mean zero and variance one. Let $\gamma_k = E(X_0 X_k)$ be the autocovariance of lag $k$. Then by Proposition 8, Eq. (34) of Xiao and Wu (2011), we know

$$|\gamma_k| \leq \Psi_2 \cdot \Psi_2(|k|). \quad \text{(S.1)}$$

(i) Since $\Psi_4 < \infty$, we know for any $\eta > 0$, there exists a $N_1 > 0$ such that $|\gamma_k| < \eta$ when $k \geq N_1$. For $j \leq k$, define $\tilde{X}_{k,j} = g(\varepsilon_k, \ldots, \varepsilon_{j+1}, \varepsilon'_j, \varepsilon'_{j-1}, \ldots)$, where $(\varepsilon'_i)_{i \in \mathbb{Z}}$ is an i.i.d. copy of $(\varepsilon_i)_{i \in \mathbb{Z}}$. By Eq. (38) of Xiao and Wu (2011), we know there exists a $N_2 > 0$ such that when $k \geq N_2$, $\|X_k - \tilde{X}_{k-j}\|_4 \leq \eta$. Set $N = \max\{N_1, N_2\}$, when $k \geq N$, we have

$$\operatorname{Var}(X_0 X_k) = E(X_0^2 X_k^2) - \gamma_k^2 = E(X_0^2 X_k^2) + E\left[X_0^2(X_k^2 - X_{k,j}^2)\right] - \gamma_k^2 \geq 1 - \eta^2 - 2\|X_0\|_4^2 \cdot \eta.$$

Therefore, (A1) holds because $\eta$ can be arbitrarily small.

(ii) We need to show that

$$\sup_{j \geq 0, 0 \leq k \leq l, (0,j) \neq (k,l)} \operatorname{Cor}(X_0 X_j, X_k X_l) < 1.$$

It suffices to show that for some $N > 0$

$$\sup_{j \geq 0, 0 \leq k \leq l, (0,j) \neq (k,l), j+k+l \geq N} \operatorname{Cor}(X_0 X_j, X_k X_l) < 1.$$

If $j+k+l \geq N$, then the set $\{0, j, k, l\}$ can be partitioned into two non-empty subsets $B_1$ and $B_2$ whose distance is no less than $N/6$. We only consider this type of partitions. If there is a partition such that
We first consider (A3). Note that

\[ |\text{Cov}(X_0X_j, X_kX_l)| = |\mathbb{E}(X_0X_jX_kX_l) - \gamma_j\gamma_{l-k}| \leq \eta. \]

If for any partition both \( B_1 \) and \( B_2 \) has cardinality two, there are two sub-cases. (a) \( j < k \leq l \) and \( k - j \geq N/6. \) For any \( \eta > 0, \) when \( N \) is large enough, we have

\[ |\text{Cov}(X_0X_j, X_kX_l)| = |\mathbb{E}[X_0X_j(X_kX_l - X_{k,j}X_{l,j})]| \leq \eta. \]

(b) \( \min\{j, l\} - k \geq N/6. \) As in (i), for any \( \eta > 0, \) when \( N \) is large enough, we have \( \text{Var}(X_0X_j) \geq 1 - \eta, \) \( \text{Var}(X_kX_l) \geq 1 - \eta, \) and \( |\gamma_j\gamma_{l-k}| < \eta. \) On the other hand, the condition \( \Psi_4 > 0 \) guarantees that the process is non-deterministic, and hence \( \gamma := \sup_{t \geq 1} |\gamma_t| < 1. \) It follows that when \( N \) is large enough

\[ |\mathbb{E}(X_0X_jX_kX_l)| = |\mathbb{E}[X_0X_jX_kX_l] + \mathbb{E}[X_0X_k(X_jX_l - X_{j,k}X_{l,k})]| \leq \gamma + \eta. \]

Therefore,

\[ |\text{Cor}(X_0X_j, X_kX_l)| \leq (\gamma + 2\eta)/(1 - \eta) < 1 \]

when \( \eta \) is small enough. The proof of (ii) is now complete.

(iii) We first consider (A3). Note that

\[ \text{Cov}(X_iX_j, X_kX_l) = \text{Cum}(X_i, X_j, X_k, X_l) + \gamma_{i-k}\gamma_{j-l} + \gamma_{i-l}\gamma_{j-k}, \]

where \( \text{Cum}(X_i, X_j, X_k, X_l) \) is the fourth order joint cumulant of the random vector \( (X_i, X_j, X_k, X_l)^\top. \)

Fix a subset \( \{i, j\} \), for any integer \( b > 0, \) there are at most \( 8b^2 \) subsets \( \{k, l\} \) such that \( \{k, l\} \subset B(i; b) \cup B(j; b), \) where \( B(x; r) \) is the open ball \( \{y : |x - y| < r\}. \) For all other subsets \( \{k, l\}, \) by (S.1), we have

\[ |\gamma_{i-k}\gamma_{j-l} + \gamma_{i-l}\gamma_{j-k}| \leq C\Psi_4(b). \]

On the other hand, using similar arguments as Theorem 21 of Xiao and Wu (2011), we can show that

\[ |\text{Cum}(X_i, X_j, X_k, X_l)| \leq C\Psi_4([b/2]). \]

Therefore, if \( \Psi_4(k) = o(1/\log k) \) as \( k \to \infty, \) then (A3) holds.

Now we turn to (A3'). Write

\[ \text{Cov}(X_iX_j, X_kX_l) = \mathbb{E}(X_iX_jX_kX_l) - \gamma_{i-j}\gamma_{k-l}. \]

By (S.1), it is easily seen that

\[ \sum_{1 \leq i, j, k, l \leq m} \gamma_{i-j}\gamma_{k-l}^2 = O(m^4-2\delta). \]
It then suffices to show

\[ \sum_{1 \leq i \leq j \leq k \leq l \leq m} \left[ \mathbb{E}(X_i X_j X_k X_l) \right]^2 = O(m^{4-\delta}), \]

which is true because by Eq. (38) of Xiao and Wu (2011)

\[ \left[ \mathbb{E}(X_i X_j X_k X_l) \right]^2 = \left[ \mathbb{E}(X_i X_j X_k (X_l - X_{1,k})) \right]^2 \leq 12 \| X_0 \|_4^4 [\Psi_4(l - k)]^2. \]

The proof of Lemma 3 is now complete. \( \square \)

We now give the proof of Lemma 4.

**Proof of Lemma 4.** Suppose \( (Y_1, Y_2, Y_3, Y_4) \) has a joint normal distribution. We can write \( Y_i = \alpha_i^\top Z \), where \( Z \) is a fourth dimensional standard Gaussian random vector. For any \( 0 < \nu < 1 \), define the subset of \( \mathbb{R}^{16} \),

\[ D_\nu = \{ (\alpha_1^\top, \alpha_2^\top, \alpha_3^\top, \alpha_4^\top) : |\alpha_i|^2 = 1 \text{ and } |\alpha_i^\top \alpha_j| \leq 1 - \nu \text{ for } 1 \leq i \neq j \leq 4. \} \]

Since \( \text{Cor}(Y_1 Y_2, Y_3 Y_4) \) is a continuous function on \( D_\nu \), and \( D_\nu \) is compact, the maximum correlation is attained at some point in \( D_\nu \).

On the other hand, elementary calculation shows that \( \text{Cor}(Y_1 Y_2, Y_3 Y_4) = 1 \) if and only if \( Y_1, Y_2, Y_3, Y_4 \) are all perfectly correlated. The proof is now complete. \( \square \)

The proof of Lemma 7 is a refined version of that of Lemma 20 in Xiao and Wu (2011). We need the following bounds on normal tail probabilities, which are taken from Lemma 19 of Xiao and Wu (2011).

Denote by \( \varphi_d((r_{ij}); x_1, \ldots, x_d) \) the density of a \( d \)-dimensional multivariate normal random vector \( X = (X_1, \ldots, X_d)^\top \) with mean zero and covariance matrix \( (r_{ij}) \), where we always assume \( r_{ii} = 1 \) for \( 1 \leq i \leq d \) and \( (r_{ij}) \) is nonsingular. Let

\[ Q_d((r_{ij}); z) = \int_z^\infty \cdots \int_z^\infty \varphi_d((r_{ij}), x_1, \ldots, x_d) \, dx_d \cdots dx_1. \]

**Lemma S.1.** For every \( z > 0 \), \( 0 < s < 1 \), \( d \geq 1 \) and \( \epsilon > 0 \), there exists positive constants \( C_d \) and \( \epsilon_d \) such that for \( 0 < \epsilon < \epsilon_d \)

1. if \( |r_{ij}| < \epsilon \) for all \( 1 \leq i < j \leq d \), then

\[ Q_d((r_{ij}); z) \leq C_d f_d(\epsilon, 1/z) \exp \left\{ - \left( \frac{d}{2} - C_d \epsilon \right) z^2 \right\} \]  \( \text{(S.2)} \)

where \( f_{2k}(x, y) = \sum_{l=0}^{k} x^l y^{2(k-l)} \) and \( f_{2k-1}(x, y) = \sum_{l=0}^{k-1} x^l y^{2(k-l)-1} \) for \( k \geq 1 \);

2. if for all \( 1 \leq i < j \leq d + 1 \) such that \( (i, j) \neq (1, 2) \), \( |r_{ij}| \leq \epsilon \), then

\[ Q_{d+1}((r_{ij}); z) \leq C_d \exp \left\{ - \left( \frac{(1 - |r_{12}|)^2 + d}{2} - C_d \epsilon \right) z^2 \right\}. \]  \( \text{(S.3)} \)

We first give a one-sided version of Lemma 7 and its proof, then we show how it implies Lemma 7.
Lemma S.2. Assume either (B1) or (B2). For a positive real number $z_n$, define the event $A_{n,i}$ and $Q_{n,d}$ as

$$A_{n,i} = \{ X_{n,i} > z_n \} \quad \text{and} \quad Q_{n,d} = \sum_{A \subseteq \mathcal{I}_n, |A| = d} P \left( \bigcap_{i \in A} A_{n,i} \right).$$

If $z_n$ satisfies that $z_n^2 = 2 \log s_n - \log \log s_n - \log(4\pi) + 2z + o(1)$, then for all $d \geq 1$

$$\lim_{n \to \infty} Q_{n,d} = \frac{e^{-dz}}{d!}.$$

Proof. The following facts about normal tail probabilities are well-known:

$$P(X_1 \geq x) \leq \frac{1}{\sqrt{2\pi x}} e^{-x^2/2} \quad \text{for} \quad x > 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{P(X_1 \geq x)}{(1/x)(2\pi)^{-1/2} \exp \{-x^2/2\}} = 1,$$  \hfill (S.4)

By the assumption on $z_n$, if for each $n$, $X_{n,i}, i \in \mathcal{I}_n$ are i.i.d., then by (S.3),

$$\lim_{n \to \infty} Q_{n,d} = \lim_{n \to \infty} \binom{n}{d} Q_{n,d}(I_d, z_n) = \lim_{n \to \infty} \binom{n}{d} \left( \frac{1}{(2\pi)^d/2\pi d!} \exp \left\{ -\frac{dz_n^2}{2} \right\} \right) = \frac{e^{-dz}}{d!}.$$

When the $X_{n,i}$’s are dependent, the result is still trivially true when $d = 1$. Now we deal with the $d \geq 2$ case. Suppose $(b_n)$ is a sequence of positive numbers which converges to infinity. For each subset $J$ of $\mathcal{I}_n$ with cardinality $|J| = d$, we define an undirected graph $\mathcal{G}(J)$ by identifying each $i \in J$ with a node and saying $i$ and $j$ are adjacent if $|r_{n,i,j}| > \gamma(n, b_n)$. Suppose the graph $\mathcal{G}(J)$ has $d-s$ connected components $B_1, \ldots, B_{d-s}$. If $s \geq 1$, assume w.l.o.g. that $|B_1| \geq 2$. Pick $k_0, k_1 \in B_1$, and $k_p \in B_p$ for $2 \leq p \leq d-s$, and set $K = \{k_0, k_1, k_2, \ldots, k_{d-s}\}$. Define $Q_J = P(\cap_{h \in J} A_h)$ and $Q_K$ similarly, then $Q_J \leq Q_K$. By (S.3) of Lemma S.1 there exists a number $M > 1$ depending on $d$ and the sequences $(\gamma_n)$ and $(b_n)$, such that when $n \geq M$,

$$Q_K \leq C_{d-s} \exp \left\{ - \left( \frac{(1-\gamma_n)^2 + d-s}{2} - C_{d-s} \gamma(n, b_n) \right) \frac{z_n^2}{2} \right\} \leq C_{d-s} \exp \left\{ - \left( \frac{d-s}{2} - \frac{(1-\gamma_n)^2}{3} \right) \frac{z_n^2}{2} \right\}.$$

Note that $z_n^2 = 2 \log s_n - \log \log s_n + O(1)$. Pick $b_n = \lfloor s_n^\alpha \rfloor$ for some $\alpha < (1-\gamma_n)^2/3d$. For any $1 \leq a \leq d-1$, since there are at most $O\left( b_n^{\frac{a-\alpha}{3}} \right)$ subsets $J \subseteq \mathcal{I}_n$ such that $|J| = d$ and the graph $\mathcal{G}(L)$ has $d-a$ connected components, we know the sum of $Q_J$ over these $J$ is dominated by

$$C_{d-a} \exp \left\{ \log s_n \left( (d-a) + \frac{2(d-1)(1-\gamma_n)^2}{3d} - (d-a) - \frac{2(1-\gamma_n)^2}{3} \right) \right\}$$

when $n$ is large enough, which converges to zero. Therefore, it remains to consider all the subsets $J \subseteq \mathcal{I}_n$ such that the graph $\mathcal{G}(J)$ has no edges

Let $J \subset \mathcal{I}_n$ be a subset such that $|J| = d$, and $|r_{n,i,j}| < \gamma(n, b_n)$ for all pairs $i, j$ such that $i, j \in J$ and $i \neq j$, and $\mathcal{J}(d, b_n)$ be the collection of all such subsets. Let $(r_{ij})_{i, j \in J}$ be the $d$-dimensional covariance matrix of $X_J := (X_{n,i})_{i \in J}$. There exists a matrix $R_J = \theta(r_{ij})_{i, j \in J} + (1-\theta)I_d$ for some $0 < \theta < 1$ such that

$$Q_J - Q_d(I_d, z_n) = \sum_{h,l \in J, h < l} \frac{\partial Q_d}{\partial r_{hl}} [R_J; z_n] r_{hl}.$$
Let \( R_H, H = J \setminus \{h, l\} \), be the correlation matrix of the conditional distribution of \( X_H \) given \( X_h \) and \( X_l \).

By \( \text{S2} \) of Lemma \( \text{S1} \), for \( n \) large enough

\[
\frac{\partial Q_d}{\partial r_{hd}}[R_J; z_n] \leq C \exp \left\{ -\frac{z_n^2}{1 + |r_{n,h,l}|} \right\} \cdot Q_d - 2 (R_K; (1 - 3\gamma(n, b_n)) z_n)
\]

\[
\leq C C_d f_{d-2} (\gamma(n, b_n), 1/z_n) \exp \left\{ -\left( \frac{d}{2} - 2C_{d-2} \gamma(n, b_n) \right) (1 - 3\gamma(n, b_n))^2 z_n^2 \right\}
\]

\[
\times \exp \left\{ - \left( \frac{d}{2} - (2C_{d-2} + 3(d - 2)) \gamma(n, b_n) - |r_{n,h,l}| \right) z_n^2 \right\}
\]

\[
\leq C_d f_{d-2} (\gamma(n, b_n), 1/z_n) \exp \left\{ - \left( \frac{d}{2} - C_d \gamma(n, b_n) \right) z_n^2 \right\}.
\]

It follows that

\[
\sum_{J \in \mathcal{J}(d, b_n)} |Q_J - Q_d(I_d; z_n)| \leq C_d f_{d-2} (\gamma(n, b_n), 1/z_n)
\]

\[
\times \sum_{J \in \mathcal{J}(d, b_n)} \sum_{i,j \in J, i \neq j} \exp \left\{ - \left( \frac{d}{2} - C_d \gamma(n, b_n) \right) z_n^2 \right\} |r_{n,i,j}|
\]

\[
\leq C_d f_{d-2} (\gamma(n, b_n), 1/z_n) s_n^{d-2}
\]

\[
\times \sum_{i,j \in \mathcal{I}_n} \exp \left\{ - \left( \frac{d}{2} - C_d \gamma(n, b_n) \right) z_n^2 \right\} |r_{n,i,j}|,
\]

where the sum \( \sum_{i,j \in \mathcal{I}_n} \) is over all the pair \((i, j)\) such that \( |r_{n,i,j}| \leq \gamma(n, b_n) \). Under the assumption (B1), we have

\[
\sum_{J \in \mathcal{J}(d, b_n)} |Q_J - Q_d(I_d; z_n)| \leq C_d f_{d-2} (\gamma(n, b_n), 1/z_n) (\log s_n)^{d/2} \gamma(n, b_n) \exp \left\{ C_d \gamma(n, b_n)(\log s_n) \right\}
\]

Since \( \lim_{n \to \infty} \gamma(n, b_n) \log b_n = 0 \), it also holds that \( \lim_{n \to \infty} \gamma(n, b_n) \log s_n = 0 \). Note that \( \lim_{n \to \infty} (\log s_n)^{1/2}/z_n = 2^{-1/2} \), it follows that \( \lim_{n \to \infty} f_{d-2} (\gamma(n, b_n), 1/z_n) (\log s_n)^{d/2 - 1} = 2^{-d/2+1} \). Therefore, the term in (S.6) converges to zero, and the theorem holds under (B1).

Alternatively, if (B2) is true, from (S.5) we have

\[
\sum_{J \in \mathcal{J}(d, b_n)} |Q_J - Q_d(I_d; z_n)| \leq C_d f_{d-2} (\gamma(n, b_n), 1/z_n) s_n^{-2} (\log s_n)^{d/2} \sum_{i,j \in \mathcal{I}_n} \exp \left\{ C_d \gamma(n, b_n)(\log s_n) \right\} |r_{n,i,j}|
\]

\[
\leq C_d f_{d-2} (\gamma(n, b_n), 1/z_n) s_n^{-1} (\log s_n)^{d/2} \exp \left\{ C_d \gamma(n, b_n)(\log s_n) \right\} \left( \sum_{i,j \in \mathcal{I}_n} r_{n,i,j}^2 \right)^{1/2}
\]

\[
\leq C_d s_n^{-\delta/2} (\log s_n) \exp \left\{ C_d \gamma(n, b_n)(\log s_n) \right\} = o(1),
\]

5
and the proof is complete.

Now we give the proof of Lemma 7.

Proof of Lemma 7. In the proof of Theorem S.2, the upper bounds on $Q_J$ and $|Q_J - Q(I_d; z_n)|$ are expressed through the absolute values of the covariances, so we can obtain the same bounds for probabilities of the form $P(\cap_{1 \leq i \leq d}\{(-1)^{a_i}X_{t_i} \geq z_n\})$ for any $(a_1, \ldots, a_d) \in \{0, 1\}^d$. Based on this observation, Lemma 7 is an immediate consequence of Lemma S.2.

References

Han Xiao and Wei Biao Wu. Asymptotic inference of autocovariances of stationary processes. *preprint*, available at [http://arxiv.org/abs/1105.3423](http://arxiv.org/abs/1105.3423) 2011.