POLYADIC SYSTEMS, REPRESENTATIONS AND QUANTUM GROUPS

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ABSTRACT. Polyadic systems and their representations are reviewed and a classification of general polyadic systems is presented. A new multiplace generalization of associativity preserving homomorphisms, a 'heteromorphism' which connects polyadic systems having unequal arities, is introduced via an explicit formula, together with related definitions for multiplace representations and multiactions. Concrete examples of matrix representations for some ternary groups are then reviewed. Ternary algebras and Hopf algebras are defined, and their properties are studied. At the end some ternary generalizations of quantum groups and the Yang-Baxter equation are presented.

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One of the most promising directions in generalizing physical theories is the consideration of higher arity algebras [Kerner [2000], in other words ternary and n-ary algebras, in which the binary composition law is substituted by a ternary or n-ary one [De Azcarraga and Izquierdo [2010]].

Firstly, ternary algebraic operations (with the arity \( n = 3 \)) were introduced already in the XIX-th century by A. Cayley in 1845 and later by J. J. Silvester in 1883. The notion of an n-ary group was introduced in 1928 by Dörnte [1929] (inspired by E. Nöther) and is a natural generalization of the notion of a group. Even before this, in 1924, a particular case, that is, the ternary group of idempotents, was used in Prüfer [1924] to study infinite abelian groups. The important coset theorem of Post explained the connection between n-ary groups and their covering binary groups [Post [1940]]. The next step in study of n-ary groups was the Gluskin-Hosszú theorem [Hosszú [1963], Gluskin [1965]]. Another definition of n-ary groups can be given as a universal algebra with additional laws [Dudek et al. [1977]] or identities containing special elements [Rusakov [1979]].

The representation theory of (binary) groups [Weyl [1946], Fulton and Harris [1991]] plays an important role in their physical applications [Cornwell [1997]]. It is initially based on a matrix realization of the group elements with the abstract group action realized as the usual matrix multiplication [Curtis and Reiner [1962], Collins [1990]]. The cubic and n-ary generalizations of matrices and determinants were made in Kapranov et al. [1994], Sokolov [1972], and their physical application appeared in Kawamura [2003], Rausch de Traubenberg [2008]. In general, particular questions of n-ary group representations were considered, and matrix representations derived, by the author Borowiec et al. [2006], and some general theorems connecting representations of binary and n-ary groups were presented in Dudek and Shahryari [2012]. The intention here is to generalize the above constructions of n-ary group representations to more complicated and nontrivial cases.

In physics, the most applicable structures are the nonassociative Grassmann, Clifford and Lie algebras [Löhmus et al. [1994], Lounesto and Ablamowicz [2004], Georgi [1999]], and so their higher arity generalizations play the key role in further applications. Indeed, the ternary analog of Clifford algebra was considered in Abramov [1995], and the ternary analog of Grassmann algebra [Abramov [1996]] was exploited to construct various ternary extensions of supersymmetry [Abramov et al. [1997]].

The construction of realistic physical models is based on Lie algebras, such that the fields take their values in a concrete binary Lie algebra [Georgi [1999]]. In the higher arity studies, the standard Lie bracket is replaced by a linear n-ary bracket, and the algebraic structure of the corresponding model is defined by the additional characteristic identity for this generalized bracket, corresponding to the Jacobi identity [De Azcarraga and Izquierdo [2010]]. There are two possibilities to construct the generalized Jacobi identity: 1) The Lie bracket is a derivation by itself; 2) A double Lie bracket vanishes, when antisymmetrized with respect to its entries. The first case leads to the so called Filippov algebras [Filippov [1985]] (or n-Lie algebra) and second case corresponds to generalized Lie algebras [Michor and Vinogradov [1996]] (or higher order Lie algebras).

The infinite-dimensional version of n-Lie algebras are the Nambu algebras [Nambu [1973], Takhtajan [1994]], and their n-bracket is given by the Jacobian determinant of n functions, the Nambu bracket, which in fact satisfies the Filippov identity [Filippov [1985]]. Recently, the ternary Filippov algebras were successfully applied to a three-dimensional superconformal gauge theory describing the effective worldvolume theory of coincident M2-branes of M-theory [Bagger and Lambert [2008a,b], Gustavsson [2009]]. The infinite-dimensional Nambu
bracket realization [Ho et al., 2008] gave the possibility to describe a condensate of nearly co-
incident $M2$-branes [Low, 2010].

From another side, Hopf algebras [Abe, 1980; Sweedler, 1969; Montgomery, 1993] play a fundamental role in quantum group theory [Kassel, 1995; Shnider and Sternberg, 1993]. Previously, their Von Neumann generalization was introduced in Duclić and Li [2001], Duclić and Sinel’Shchikov [2009], Li and Duclić [2002], their actions on the quantum plane were classified in Duclić and Sinel’Shchikov [2010], and ternary Hopf algebras were defined and studied in Duclić [2001], Borowiec et al. [2001].

The goal of this paper is to give a comprehensive review of polyadic systems and their representations. First, we classify general polyadic systems and introduce $n$-ary semigroups and groups. Then we consider their homomorphisms and multiplace generalizations, paying attention to their associativity. We define multiplace representations and multi-actions, and give examples of matrix representations for some ternary groups. We define and investigate ternary algebras and Hopf algebras, study their properties and give some examples. At the end we consider some ternary generalizations of quantum groups and the Yang-Baxter equation.

2. Preliminaries

Let $G$ be a non-empty set (underlying set, universe, carrier), its elements we denote by lowercase Latin letters $g_i \in G$. The $n$-tuple (or polyad) $g_1, \ldots, g_n$ of elements from $G$ is denoted by $(g_1, \ldots, g_n)$. The Cartesian product $\prod G \times \cdots \times G = G^\times_n$ consists of all $n$-tuples $(g_1, \ldots, g_n)$, such that $g_i \in G$, $i = 1, \ldots, n$. For all equal elements $g \in G$, we denote $n$-tuple (polyad) by power $(g^n)$. If the number of elements in the $n$-tuple is clear from the context or is not important, we denote it with one bold letter $(g)$, in other cases we use the power in brackets $(g^{(n)})$. We now introduce two important constructions on sets.

Definition 2-1. The $i$-projection of the Cartesian product $G^\times_n$ on its $i$-th “axis” is the map $Pr^{(n)}_i: G^\times_n \to G$ such that $(g_1, \ldots, g_i, \ldots, g_n) \mapsto g_i$.

Definition 2-2. The $i$-diagonal $\text{Diag}_n: G \to G^\times_n$ sends one element to the equal element $n$-tuple $g \mapsto (g^n)$.

The one-point set $\{\bullet\}$ can be treated as a unit for the Cartesian product, since there are bijections between $G$ and $G \times \{\bullet\}^\times_n$, where $G$ can be on any place. On the Cartesian product $G^\times_n$ one can define a polyadic (n-ary, n-adic, if it is necessary to specify $n$, its arity or rank) operation $\mu_n: G^\times_n \to G$. For operations we use small Greek letters and place arguments in square brackets $\mu_n [g]$. The operations with $n = 1, 2, 3$ are called unary, binary and ternary. The case $n = 0$ is special and corresponds to fixing a distinguished element of $G$, a “constant” $c \in G$, and it is called a 0-ary operation $\mu_0^{(c)}$, which maps the one-point set $\{\bullet\}$ to $G$, such that $\mu_0^{(c)}: \{\bullet\} \to G$, and formally has the value $\mu_0^{(c)} [\{\bullet\}] = c \in G$. The 0-ary operation “kills” arity, which can be seen from the following Bergman [1995]: the composition of $n$-ary and $m$-ary operations $\mu_n \circ \mu_m$ gives $(n + m - 1)$-ary operation by

\[ \mu_{n+m-1} [g, h] = \mu_n [g, \mu_m [h]] . \] (2.1)

Then, if to compose $\mu_n$ with the 0-ary operation $\mu_0^{(c)}$, we obtain

\[ \mu_{n-1}^{(c)} [g] = \mu_n [g, c] , \] (2.2)

\footnote{1We place the sign for the Cartesian product ($\times$) into the power, because the same abbreviation will also be used below for other types of product.}
because $g$ is a polyad of length $(n-1)$. So, it is necessary to make a clear distinction between the 0-ary operation $\mu_0^{(c)}$ and its value $c$ in $G$, as will be seen and will become important below.

**Definition 2-3.** A polyadic system $G$ is a set $G$ which is closed under polyadic operations.

We will write $G = \langle \text{set} | \text{operations} \rangle$ or $G = \langle \text{set} | \text{operations} | \text{relations} \rangle$, where “relations” are some additional properties of operations (e.g., associativity conditions for semigroups or cancellation properties). In such a definition it is not necessary to list the images of 0-ary operations (e.g. the unit or zero in groups), as is done in various other definitions. Here, we mostly consider concrete polyadic systems with one “chief” (fundamental) $n$-ary operation $\mu_n$, which is called polyadic multiplication (or $n$-ary multiplication).

**Definition 2-4.** A $n$-ary system $G_n = \langle G | \mu_n \rangle$ is a set $G$ closed under one $n$-ary operation $\mu_n$ (without any other additional structure).

Note that a set with one closed binary operation without any other relations was called a groupoid by Hausmann and Ore [Hausmann and Ore 1937] (see, also Clifford and Preston [1961]). However, nowadays the term “groupoid” is widely used in category theory and homotopy theory for a different construction with binary multiplication, the so-called Brandt groupoid [Brandt 1927] (see, also, Bruck [1966]). Alternatively, and much later on, Bourbaki [Bourbaki 1998] introduced the term “magma” for binary systems. Then, the above terms were extended to the case of one fundamental $n$-ary operation as well. Nevertheless, we will use some neutral notations “polyadic system” and “$n$-ary system” (when arity $n$ is fixed/known/important), which adequately indicates all of their main properties.

Let us consider the changing arity problem:

**Definition 2-5.** For a given $n$-ary system $\langle G | \mu_n \rangle$ to construct another polyadic system $\langle G | \mu_{n'} \rangle$ over the same set $G$, which has multiplication with a different arity $n'$.

The formulas (2.1) and (2.2) give us the simplest examples of how to change the arity of a polyadic system. In general, there are 3 ways:

(1) **Iterating.** Using composition of the operation $\mu_n$ with itself, one can increase the arity from $n$ to $n_{\text{iter}}$ (as in (2.1)) without changing the signature of the system. We denote the number of iterating multiplications by $\ell_\mu$, and use the bold Greek letters $\mu_\ell^n$ for the resulting composition of $n$-ary multiplications, such that

$$\mu_{n'} = \mu_\ell^n = \mu_n \circ (\mu_n \circ \ldots (\mu_n \times \text{id}^{\times(n-1)}) \ldots \times \text{id}^{\times(n-1)}),$$

(2.3)

where

$$n' = n_{\text{iter}} = \ell_\mu(n-1) + 1,$$

(2.4)

which gives the length of a polyad ($g$) in the notation $\mu_\ell^n [g]$. Without assuming associativity there many variants for placing $\mu_n$’s among id’s in the r.h.s. of (2.3). The operation $\mu_\ell^n$ is named a long product [Dörnte 1929] or derived [Dudek 2007].
(2) **Reducing (Collapsing).** Using \( n_c \) distinguished elements or constants (or \( n_c \) additional 0-ary operations \( \mu_0^{(c_i)} \), \( i = 1, \ldots, n_c \), one can decrease arity from \( n \) to \( n'_{\text{red}} \) (as in (2.2)), such that

\[
\mu'_{n'} = \mu_{n'}^{(c_1, \ldots, c_{n_c})} \overset{\text{def}}{=} \mu_\alpha \circ \left( \underbrace{\mu_0^{(c_1)} \times \ldots \times \mu_0^{(c_{n_c})}}_{n_c} \times \text{id} \times (n-n_c) \right),
\]

where

\[
n' = n_{\text{red}} = n - n_c,
\]

and the 0-ary operations \( \mu_0^{(c_i)} \) can be on any places.

(3) **Mixing.** Changing (increasing or decreasing) arity may be done by combining iterating and reducing (maybe with additional operations of different arity). If we do not use additional operations, the final arity can be presented in a general form using (2.4) and (2.6). It will depend on the order of iterating and reducing, and so we have two subcases:

(a) **Iterating → Reducing.** We have

\[
n' = n_{\text{iter} \rightarrow \text{red}} = \ell_\mu (n - 1) - n_c + 1.
\]

The maximal number of constants (when \( n'_{\text{iter} \rightarrow \text{red}} = 2 \)) is equal to

\[
n'^{\text{max}} = \ell_\mu (n - 1) - 1
\]

and can be increased by increasing the number of multiplications \( \ell_\mu \).

(b) **Reducing → Iterating.** We obtain

\[
n' = n_{\text{red} \rightarrow \text{iter}} = \ell_\mu (n - 1 - n_c) + 1.
\]

Now the maximal number of constants is

\[
n'^{\text{max}} = n - 2
\]

and this is achieved only when \( \ell_\mu = 1 \).

To give examples of the third (mixed) case we put \( n = 4, \ \ell_\mu = 3, \ n_c = 2 \) for both subcases of opposite ordering:

1. **Iterating → Reducing.** We can put

\[
\mu_8^{(c_1, c_2)} [g^{(8)}] = \mu_4 [g_1, g_2, g_3, \mu_4 [g_4, g_5, g_6, \mu_4 [g_7, g_8, c_1, c_2]]],
\]

which corresponds to the following commutative diagram

\[
\begin{array}{ccc}
G^{\times 8} & \xrightarrow{\ell} & G^{\times 8} \times \{\bullet\}^2 \xrightarrow{\text{id} \times 6 \times \mu_4} G^{\times 7} \xrightarrow{\text{id} \times 3 \times \mu_4} G^{\times 4} \\
\mu_8^{(c_1, c_2)} & & \mu_4 \\
\end{array}
\]

(2.12)

2. **Reducing → Iterating.** We can have

\[
\mu_4^{(c_1, c_2)} [g^{(4)}] = \mu_4 [g_1, c_1, c_2, \mu_4 [g_2, c_1, c_2, \mu_4 [g_3, c_1, c_2, g_4]]],
\]

(2.13)

\footnote{In [Dudek and Michalski (1984)] \( \mu_8^{(c_1, \ldots, c_{n_c})} \) is named a retract (which term is already busy and widely used in category theory for another construction).}
such that the diagram

\[
G^{\times 4} \xrightarrow{\epsilon} (G \times \{\bullet\})^{\times 3} \times G \xrightarrow{id^{\times 6} \times \mu_4} G^{\times 7} \xrightarrow{id^{\times 3} \times \mu_4} G^{\times 4} \xrightarrow{\mu_4} G
\]

is commutative.

It is important to find conditions where iterating and reducing compensate each other, i.e. they do not change arity overall. Indeed, let the number of the iterating multiplications \(\ell_\mu\) be fixed, then we can find such a number of reducing constants \(n_c^{(0)}\), such that the final arity will coincide with the initial arity \(n\). The result will depend on the order of operations. There are two cases:

1. **Iterating → Reducing.** For the number of reducing constants \(n_c^{(0)}\) we obtain from (2.4) and (2.6)

\[
n_c^{(0)} = (n-1) (\ell_\mu - 1),
\]

such that there is no restriction on \(\ell_\mu\).

2. **Reducing → Iterating.** For \(n_c^{(0)}\) we get

\[
n_c^{(0)} = \frac{(n-1) (\ell_\mu - 1)}{\ell_\mu},
\]

and now \(\ell_\mu \leq n - 1\). The requirement that \(n_c^{(0)}\) should be an integer gives two further possibilities

\[
n_c^{(0)} = \begin{cases} 
\frac{n-1}{2}, & \ell_\mu = 2, \\
\frac{n-2}{2}, & \ell_\mu = n - 1.
\end{cases}
\]

The above relations can be useful in the study of various \(n\)-ary multiplication structures and their presentation in special form is needed in concrete problems.

3. **Special Elements and Properties of Polyadic Systems**

Let us recall the definitions of some standard algebraic systems and their special elements, which will be considered in this paper, using our notation.

**Definition 3-1.** A zero of a polyadic system is a distinguished element \(z\) (and the corresponding 0-ary operation \(\mu_0(z)\)) such that for any \((n-1)\)-tuple (polyad) \(g \in G^{\times (n-1)}\) we have

\[
\mu_n [g, z] = z,
\]

where \(z\) can be on any place in the l.h.s. of (3.1).

There is only one zero (if its place is not fixed) which can be possible in a polyadic system. As in the binary case, an analog of positive powers of an element [POST [1940]] should coincide with the number of multiplications \(\ell_\mu\) in the iterating (2.3).

**Definition 3-2.** A (positive) polyadic power of an element is

\[
g^{(\ell_\mu)} = \mu_\mu^{\ell_\mu} [g^{\ell_\mu(n-1)+1}].
\]
**Definition 3-3.** An element of a polyadic system \( g \) is called \( \ell_\mu \)-nilpotent (or simply nilpotent for \( \ell_\mu = 1 \)), if there exist such \( \ell_\mu \) that
\[
g^{(\ell_\mu)} = z. \tag{3.3}\]

**Definition 3-4.** A polyadic system with zero \( z \) is called \( \ell_\mu \)-nilpotent, if there exists \( \ell_\mu \) such that for any \( (\ell_\mu (n - 1) + 1) \)-tuple (polyad) \( g \) we have
\[
\mu_n^{\ell_\mu}[g] = z. \tag{3.4}\]

Therefore, the index of nilpotency (number of elements whose product is zero) of an \( \ell_\mu \)-nilpotent \( n \)-ary system is \( (\ell_\mu (n - 1) + 1) \), while its polyadic power is \( \ell_\mu \).

**Definition 3-5.** A polyadic \((n\text{-ary})\) identity (or neutral element) of a polyadic system is a distinguished element \( e \) (and the corresponding \( 0 \)-ary operation \( \mu_0^{(e)} \)) such that for any element \( g \in G \) we have
\[
\mu_n[g,e^{n-1}] = g, \tag{3.5}\]
where \( g \) can be on any place in the l.h.s. of \( (3.5) \).

In binary groups the identity is the only neutral element, while in polyadic systems, there exist neutral polyads \( n \) consisting of elements of \( G \) satisfying
\[
\mu_n[g,n] = g, \tag{3.6}\]
where \( g \) can be also on any place. The neutral polyads are not determined uniquely. It follows from \( (3.5) \) that the sequence of polyadic identities \( e^{n-1} \) is a neutral polyad.

**Definition 3-6.** An element of a polyadic system \( g \) is called \( \ell_\mu \)-idempotent (or simply idempotent for \( \ell_\mu = 1 \)), if there exist such \( \ell_\mu \) that
\[
g^{(\ell_\mu)} = g. \tag{3.7}\]

Both zero and the identity are \( \ell_\mu \)-idempotents with arbitrary \( \ell_\mu \). We define (total) associativity as the invariance of the composition of two \( n \)-ary multiplications
\[
\mu_n^2[g,h,u] = \mu_n[g,\mu_n[h,u]] = \text{invariant} \tag{3.8}\]
under placement of the internal multiplication in r.h.s. with a fixed order of elements in the whole polyad of \((2n - 1)\) elements \( t^{(2n-1)} = (g,h,u) \). Informally, “internal brackets/multiplication can be moved on any place”, which gives \( n \) relations
\[
\mu_n \circ (\mu_n \times \text{id}^{(n-1)}) = \ldots = \mu_n \circ (\text{id}^{(n-1)} \times \mu_n). \tag{3.9}\]

There are many other particular kinds of associativity which were introduced in \textsc{Thurston} [1949] and studied in \textsc{Belousov} [1972], \textsc{Sokhatsky} [1997]. Here we will confine ourselves the most general, total associativity \( (3.8) \). In this case, the iteration does not depend on the placement of internal multiplications in the r.h.s of \( (2.3) \).

**Definition 3-7.** A polyadic semigroup \((n\text{-ary semigroup})\) is a \( n \)-ary system in which the operation is associative, or \( G_{\text{semigrp}}^n = \langle G \mid \mu_n \mid \text{associativity} \rangle \).

In a polyadic system with zero \( (3.1) \) one can have trivial associativity, when all \( n \) terms are \( (3.8) \) are equal to zero, i.e.
\[
\mu_n^2[g] = z \tag{3.10}\]
for any \((2n-1)\)-tuple \( g \). Therefore, we state that
Assertion 3.8. Any 2-nilpotent $n$-ary system (having index of nilpotency $(2n - 1)$) is a polyadic semigroup.

In the case of changing arity one should use in (3,10) not the changed final arity $n'$, but the "real" arity which is $n$ for the reducing case and $\ell_\mu (n - 1) + 1$ for all other cases. Let us give some examples.

Example 3.9. In the mixed (interacting-reducing) case with $n = 2$, $\ell_\mu = 3$, $n_c = 1$, we have a ternary system $\langle G \mid \mu_3 \rangle$ iterated from a binary system $\langle G \mid \mu_2, \mu_0^{(c)} \rangle$ with one distinguished element $c$ (or an additional 0-ary operation)

$$\mu_3^{(c)}[g, h, u] = (g \cdot (h \cdot (u \cdot c))), \quad (3.11)$$

where for binary multiplication we denote $g \cdot h = \mu_2[g, h]$. Thus, if the ternary system $\langle G \mid \mu_3^{(c)} \rangle$ is nilpotent of index 7 (see 3.4), then it is a ternary semigroup (because $\mu_3^{(c)}$ is trivially associative) independently of the associativity of $\mu_2$ (see, e.g., [BOROWIE ET AL. 2006]).

It is very important to find the associativity preserving conditions (constructions), where an associative initial operation $\mu_n$ leads to an associative final operation $\mu_n'$ during the change of arity.

Example 3.10. An associativity preserving reduction can be given by the construction of a binary associative operation using $(n - 2)$-tuple $c$ consisting of $n_c = n - 2$ different constants

$$\mu_2^{(c)}[g, h] = \mu_n[g, c, h]. \quad (3.12)$$

Associativity preserving mixing constructions with different arities and places were considered in [DUDEK AND MICHALSKI 1984], [MICHALSKI 1981], [SOKHATSY 1997].

Definition 3.11. An associative polyadic system with identity (3.5) is called a polyadic monoid.

The structure of any polyadic monoid is fixed [POP AND POP 2004]: it can be obtained by iterating a binary operation [CUPONA AND TRPENOVSKI 1961] (for polyadic groups this was shown in [DÖRSTE 1929]).

In polyadic systems, there are several analogs of binary commutativity. The most straightforward one comes from commutation of the multiplication with permutations.

Definition 3.12. A polyadic system is $\sigma$-commutative, if $\mu_n = \mu_n \circ \sigma$, or

$$\mu_n[g] = \mu_n[\sigma \circ g], \quad (3.13)$$

where $\sigma \circ g = (g_{\sigma(1)}, \ldots, g_{\sigma(n)})$ is a permutated polyad and $\sigma$ is a fixed element of $S_n$, the permutation group on $n$ elements. If (3.13) holds for all $\sigma \in S_n$, then a polyadic system is commutative.

A special type of the $\sigma$-commutativity

$$\mu_n[g, t, h] = \mu_n[h, t, g], \quad (3.14)$$

where $t$ is any fixed $(n - 2)$-polyad, is called semicommutativity. So for a $n$-ary semicommutative system we have

$$\mu_n[g, h^{n-1}] = \mu_n[h^{n-1}, g]. \quad (3.15)$$

If a $n$-ary semigroup $G^{\text{semigrp}}$ is iterated from a commutative binary semigroup with identity, then $G^{\text{semigrp}}$ is semicommutative.

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3 This construction is named the b-derived groupoid in [DUDEK AND MICHALSKI 1984].
Example 3.13. Let \( G \) be the set of natural numbers \( \mathbb{N} \), and the 5-ary multiplication is defined by
\[
\mu_5 [g] = g_1 - g_2 + g_3 - g_4 + g_5,
\]
then \( G^5 = (\mathbb{N}, \mu_5) \) is a semicommutative 5-ary monoid having the identity \( e_g = \mu_5 [g_5] = g \) for each \( g \in \mathbb{N} \). Therefore, \( G^5 \) is the idempotent monoid.

Another possibility is to generalize the binary mediability in semigroups
\[
(g_{11} \cdot g_{12}) \cdot (g_{21} \cdot g_{22}) = (g_{11} \cdot g_{21}) \cdot (g_{12} \cdot g_{22}),
\]
which, obviously, follows from binary commutativity. But for \( n \)-ary systems they are different. It is seen that the mediability should contain \((n + 1)\) multiplications, it is a relation between \( n \times n \) elements, and therefore can be presented in a matrix form. The latter can be achieved by placing the arguments of the external multiplication in a column.

Definition 3.14. A polyadic system is medial (or entropic), if \cite{Evans1963, Belousov1972}
\[
\mu_n \begin{bmatrix}
\mu_n [g_{11}, \ldots, g_{1n}]
\vdots
\mu_n [g_{n1}, \ldots, g_{nn}]
\end{bmatrix}
= \mu_n \begin{bmatrix}
\mu_n [g_{11}, \ldots, g_{n1}]
\vdots
\mu_n [g_{1n}, \ldots, g_{nn}]
\end{bmatrix}.
\]

For polyadic semigroups we use the notation \( G = \|g_{ij}\| \) and can present the mediability as follows
\[
\mu_n^n [G] = \mu_n^n [G^T],
\]
where \( G = \|g_{ij}\| \) is the \( n \times n \) matrix of elements and \( G^T \) is its transpose. The semicommutative polyadic semigroups are medial, as in the binary case, but, in general (except \( n = 3 \)) not vice versa \cite{Glazek1982}. A more general concept is \( \sigma \)-permutability \cite{Stojakovic1986}, such that the mediability is its particular case with \( \sigma = (1, n) \).

Definition 3.15. A polyadic system is cancellative, if
\[
\mu_n [g, t] = \mu_n [h, t] \implies g = h,
\]
where \( g, h \) can be on any place. This means that the mapping \( \mu_n \) is one-to-one in each variable. If \( g, h \) are on the same \( i \)-th place on both sides, the polyadic system is called \( i \)-cancellative.

The left and right cancellativity are \( 1 \)-cancellativity and \( n \)-cancellativity respectively. A right and left cancellative \( n \)-ary semigroup is cancellative (with respect to the same subset).

Definition 3.16. A polyadic system is called (uniquely) \( i \)-solvable, if for all polyads \( t, u \) and element \( h \), one can (uniquely) resolve the equation (with respect to \( h \)) for the fundamental operation
\[
\mu_n [u, h, t] = g
\]
where \( h \) can be on any \( i \)-th place.

Definition 3.17. A polyadic system which is uniquely \( i \)-solvable for all places \( i \) is called a \( n \)-ary (or polyadic) quasigroup.

It follows, that, if (3.21) uniquely \( i \)-solvable for all places, then
\[
\mu_n^{iu} [u, h, t] = g
\]
where \( h \) can be on any \( i \)-th place.

Definition 3.18. An associative polyadic quasigroup is called a \( n \)-ary (or polyadic) group.
The above definition is the most general one, but it is overdetermined. Much work on polyadic groups was done [Rusakov, 1998] to minimize the set of axioms (solubility not in all places [Post, 1940], Celakoski, 1977), decreasing or increasing the number of unknowns in determining equations [Galmak, 2003]), or construction in terms of additionally defined objects (various analogs of the identity and sequences [Usan, 2003]), as well as using not total associativity, but instead various partial ones [Sokolov, 1976], Sokhatsky, 1997, Yurevich, 2001].

In a polyadic group the only solution of (3.21) is called a querelement of \( g \) and denoted by \( \bar{g} \) (Dörnte, 1929), such that
\[
\mu_n [h, \bar{g}] = g,
\]
where \( \bar{g} \) can be on any place. So, any idempotent \( g \) coincides with its querelement \( \bar{g} = g \). It follows from (3.23) and (3.6), that the polyad
\[
n_g = (g^{n-2}\bar{g})
\]
is neutral for any element of a polyadic group, where \( \bar{g} \) can be on any place. If this \( i \)-th place is important, then we write \( n_{g;i} \). The number of relations in (3.23) can be reduced from \( n \) (the number of possible places) to only 2 (when \( g \) is on the first and last places [Dörnte, 1929], Timm, 1972, or on some other 2 places). In a polyadic group the Dörnte relations
\[
\mu_n [g, n_{j;i}] = \mu_n [n_{j;i}, g] = g
\]
hold true for any allowable \( i, j \). In the case of a binary group the relations (3.25) become \( g \cdot h \cdot h^{-1} = h \cdot h^{-1} \cdot g = g \).

The relation (3.23) can be treated as a definition of the unary querooperation
\[
\bar{\mu}_1 [g] = \bar{g},
\]
such that the diagram
\[
\begin{array}{ccc}
G^\times n & \xrightarrow{\mu_n} & G \\
\downarrow{id \times (n-1) \times \bar{\mu}_1} & & \downarrow{Pr_n} \\
G^\times n & \xleftarrow{Pr_1 \times \bar{\mu}_1 \times diag(n-1)} & G \times G
\end{array}
\]
commutes. Then, using the querooperation (3.26) one can give a diagrammatic definition of a polyadic group (cf. [Gleichgewicht and Głazek, 1967]).

**Definition 3.19.** A polyadic group is a universal algebra
\[
G_n^{grp} = \langle G \mid \mu_n, \bar{\mu}_1 \mid \text{associativity, Dörnte relations} \rangle,
\]
where \( \mu_n \) is a \( n \)-ary associative operation and \( \bar{\mu}_1 \) is the querooperation, such that the following diagram
\[
\begin{array}{ccc}
G^\times (n) & \xrightarrow{id \times (n-1) \times \bar{\mu}_1} & G^\times (n) \\
\downarrow{id \times diag(n-1)} & & \downarrow{Pr_1} \\
G \times G & \xleftarrow{Pr_2} & G \times G
\end{array}
\]
commutes, where \( \bar{\mu}_1 \) can be only on first and second places from the right (resp. left) on the left (resp. right) part of the diagram.

A straightforward generalization of the querooperation concept and corresponding definitions can be made by substituting in the above formulas (3.23)–(3.26) the \( n \)-ary multiplication \( \mu_n \) by iterating the multiplication \( \mu_\ell \mu_n \) (2.3) (cf. [Dudek, 1980] for \( \ell = 2 \)).
Definition 3-20. Let us define the querpower $k$ of $g$ recursively

$$
\bar{g}^{(k)} = \left(\bar{g}^{(k-1)}\right)^k,
$$

(3.30)

where $\bar{g}^{(0)} = g$, $\bar{g}^{(1)} = \bar{g}$, or as the $k$ composition $\bar{g}^k = \mu_1 \circ \mu_1 \circ \ldots \circ \mu_1$ of the queroperation (3.26).

For instance [GAL’MAK 2003], $\bar{g}^{(2)} = \mu_n^{n-3}$, such that for any ternary group $\bar{g}^{(2)} = \text{id}$, i.e. one has $\bar{g} = g$. Using the queroperation in polyadic groups we can define the negative polyadic power of an element $g$ by the following recursive relation

$$
\mu_n \left[ g^{(\ell_n-1)}, g^{n-2}, g^{(-\ell_n)} \right] = g,
$$

or (after use of (3.2)) as a solution of the equation

$$
\mu_n^{\ell_n} \left[ g^{\ell_n(n-1)}, g^{(-\ell_n)} \right] = g.
$$

It is known that the querpower and the polyadic power are mutually connected [DUDEK 1993]. Here, we reformulate this connection using the so called Heine numbers [HEINE 1878] or $q$-deformed numbers [KAC AND CHEUNG 2002]

$$
[k]_q = \frac{q^k - 1}{q - 1},
$$

(3.33)

which have the “nondeformed” limit $q \to 1$ as $[k]_q \to k$. Then

$$
\bar{g}^{(k)} = g^{\left\lfloor [k]_q \right\rfloor_{2-n}},
$$

(3.34)

which can be treated as follows: the querpower coincides with the negative polyadic deformed power with a “deformation” parameter $q$ which is equal to the “deviation” $(2 - n)$ from the binary group.

4. Homomorphisms of polyadic systems

Let $G_n = \langle G; \mu_n \rangle$ and $G'_n = \langle G'; \mu'_n \rangle$ be two polyadic systems of any kind (quasigroup, semigroup, group, etc.). If they have the multiplications of the same arity $n = n'$, then one can define the mappings from $G_n$ to $G'_n$. Usually such polyadic systems are similar, and we call mappings between them the equiary mappings.

Let us take $n + 1$ mappings $\varphi_i^{GG'} : G \to G'$, $i = 1, \ldots, n + 1$. An ordered system of mappings $\{ \varphi_i^{GG'} \}$ is called a homotopy from $G_n$ to $G'_n$, if [BELLOUSOV 1972]

$$
\varphi_{n+1}^{GG'}(\mu_n[g_1, \ldots, g_n]) = \mu_n \left[ \varphi_{G1}^{GG'}(g_1), \ldots, \varphi_{Gn}^{GG'}(g_n) \right], \quad g_i \in G.
$$

(4.1)

In general, one should add to this definition the “mapping” of the multiplications

$$
\mu_n \xrightarrow{\psi_{nn'}^{(\mu_n')}} \mu'_n.
$$

(4.2)

In such a way, homotopy can be defined as the extended system of mappings $\{ \varphi_i^{GG'}; \psi_{nn'}^{(\mu_n') \mu'_n} \}$. The corresponding commutative (eqiary) diagram is

$$
\begin{array}{ccc}
G & \xrightarrow{\varphi_{n+1}^{GG'}} & G' \\
\mu_n \downarrow & \ddots & \downarrow \mu'_n \\
G \times n & \xrightarrow{\varphi_{G1}^{GG'} \times \ldots \times \varphi_{Gn}^{GG'}} & (G') \times n
\end{array}
$$

(4.3)
The existence of the additional “mapping” \( \psi_{mn}(\mu'_{nn}) \) acting on the second component of \( (G; \mu_n) \) is tacitly implied. We will write/mention the “mappings” \( \psi_{mn}(\mu'_{nn}) \) manifestly, e.g.,

\[
\left\{ \varphi_{G'^{G'}}^{G,G'}, \psi_{mn}(\mu'_{nn}) \right\}
\]

\[ G_n \overset{\varphi_{G'^{G'}}^{G,G'}}{\rightarrow} G'_n, \tag{4.4} \]

only as needed. If all the components \( \varphi_{G'^{G'}}^{G,G'} \) of a homotopy are bijections, it is called an isotopy. In case of polyadic quasigroups Belousov [1972] all mappings \( \varphi_{G'^{G'}}^{G,G'} \) are usually taken as permutations of the same underlying set \( G = G' \). If the multiplications are also coincide \( \mu_n = \mu'_n \), then \( \{ \varphi_{G'^{G'}}^{G,G'}; \text{id} \} \) is called an autotopy of the polyadic system \( G_n \). Various properties of homotopy in universal algebras were studied, e.g. in Petrescu [1977], Halás [1994].

The homotopy, isotopy and autotopy are widely used equiary mappings in the study of polyadic quasigroups and loops, while their diagonal equiary counterparts (all \( \varphi_{G'^{G'}}^{G,G'} \) coincide), the homomorphism, isomorphism and automorphism, are more suitable in investigation of polyadic semigroups, groups and rings and their wide applications in physics. Usually, it is written about the latter between similar (equiary) polyadic systems: they “...are so well known that we shall not bother to define them carefully” Hobby and McKenzie [1988]. Nevertheless, we give a diagrammatic definition of the standard homomorphism between similar polyadic systems in our notation, which will be convenient to explain the clear way of its generalization.

A homomorphism from \( G_n \) to \( G'_n \) is given, if there exists a mapping \( \varphi^{G,G'} : G \rightarrow G' \) satisfying

\[
\varphi^{G,G'}(\mu_n [g_1, \ldots, g_n]) = \mu'_n \left[ \varphi^{G,G'}(g_1), \ldots, \varphi^{G,G'}(g_n) \right], \quad g_i \in G, \tag{4.5}\]

which means that the corresponding (equiary) diagram is commutative

\[
\begin{array}{ccc}
G & \xrightarrow{\varphi^{G,G'}} & G' \\
\mu_n & \downarrow & \mu'_n \\
G^{\times n} & \xrightarrow{\varphi^{G,G'} \times \mu_n} & (G')^{\times n}
\end{array} \tag{4.6}\]

Usually the homomorphism is denoted by the same one letter \( \varphi^{G,G'} \), while it would be more consistent to use for its notation the extended pair of mappings \( \left\{ \varphi_{G'^{G'}}^{G,G'}, \psi_{mn}(\mu'_{nn}) \right\} \). We will use both notations on a par.

We first mention a small subset of known generalizations of the homomorphism (for bibliography till 1982 see, e.g., Glazek and Gleichgewicht [1982b]) and then introduce a concrete construction for an analogous mapping which can change the arity of the multiplication (fundamental operation) without introducing additional (term) operations. A general approach to mappings between free algebraic systems was initiated in Fujisawa [1959], where the so-called basic mapping formulas for generators were introduced, and its generalization to many-sorted algebras was given in Vidal and Tur [2010]. In Novotný [2002] it was shown that the construction of all homomorphisms between similar polyadic systems can be reduced to some homomorphisms between corresponding mono-unary algebras Novotný [1990]. The notion of \( n \)-ary homomorphism is realized as a sequence of \( n \) consequent homomorphisms \( \varphi_i, i = 1, \ldots, n, \) of \( n \) similar polyadic systems

\[
G_n \overset{\varphi_1}{\rightarrow} G'_n \overset{\varphi_2}{\rightarrow} \ldots \overset{\varphi_{n-1}}{\rightarrow} G''_n \overset{\varphi_n}{\rightarrow} G'''_n \tag{4.7}\]

(generalizing Post’s \( n \)-adic substitutions Post [1940]) was introduced in Gal’mak [1998], and studied in Gal’mak [2001] and [2007].
The above constructions do not change the arity of polyadic systems, because they are based on the corresponding diagram which gives a definition of an equiary mapping. To change arity one has to:

1) add another equiary diagram with additional operations using the same formula (4.5), where both do not change arity;

2) use one modified (and not equiary) diagram and the underlying formula (4.5) by themselves, which will allow us to change arity without introducing additional operations.

The first way leads to the concept of weak homomorphism which was introduced in Goetz [1966], Marczewski [1966], Głązek and Michalski [1974] for non-indexed algebras and in Głązek [1980] for indexed algebras, then developed in Traczyk [1965] for Boolean and Post algebras, in Denecke and Wismath [2009] for coalgebras and F-algebras Denecke and Saengsura [2008] (see also Chung and Smith [2008]). To define the weak homomorphism in our notation we should incorporate into the polyadic systems \( \langle G; \mu_n \rangle \) and \( \langle G'; \mu'_n \rangle \) the following additional term operations of opposite arity \( \nu_n : G^{\times n} \to G \) and \( \nu'_n : G'^{\times n} \to G' \) and consider two equiary mappings between \( \langle G; \mu_n, \nu_n \rangle \) and \( \langle G'; \mu'_n, \nu'_n \rangle \).

A weak homomorphism from \( \langle G; \mu_n, \nu_n \rangle \) to \( \langle G'; \mu'_n, \nu'_n \rangle \) is given, if there exists a mapping \( \varphi^{GG'} : G \to G' \) satisfying two relations simultaneously

\[
\varphi^{GG'}(\mu_n[g_1, \ldots, g_n]) = \nu'_n \left[ \varphi^{GG'}(g_1), \ldots, \varphi^{GG'}(g_n) \right], \\
\varphi^{GG'}(\nu_n[g_1, \ldots, g_n]) = \mu'_n \left[ \varphi^{GG'}(g_1), \ldots, \varphi^{GG'}(g_n) \right],
\]

which means that two equiary diagrams commute

\[
\begin{array}{c}
G \xrightarrow{\varphi^{GG'}} G' \\
\mu_n \xrightarrow{\varphi^{GG'}} \mu'_n \\
G^{\times n} \xrightarrow{\left( \varphi^{GG'} \right)^{\times n}} (G')^{\times n}
\end{array}
\quad \quad
\begin{array}{c}
G \xrightarrow{\varphi^{GG'}} G' \\
\nu_n \xrightarrow{\varphi^{GG'}} \nu'_n \\
G^{\times n} \xrightarrow{\left( \varphi^{GG'} \right)^{\times n}} (G')^{\times n}
\end{array}
\]

If only one of the relations (4.8) or (4.9) holds, such a mapping is called a semi-weak homomorphism [Kolibiar 1984]. If \( \varphi^{GG'} \) is bijective, then it defines a weak isomorphism. Any weak epimorphism can be decomposed into a homomorphism and a weak isomorphism [Głązek and Michalski 1977], and therefore the study of weak homomorphisms reduces to weak isomorphisms (see also Czákány [1962], Mal’tcev [1957], Mal’tsev [1958]).

5. Multiplace mappings of polyadic systems and heteromorphisms

Let us turn to the second way of changing the arity of the multiplication and use only one relation which we then modify in some natural manner. First, recall that in any set \( G \) there always exists the additional distinguished mapping, viz. the identity \( \text{id}_G \). We use the multiplication \( \mu_n \) with its combination of \( \text{id}_G \). We define an \((\ell_{id}-\text{intact})\) \( \ell \)-product for the polyadic system \( \langle G; \mu_n \rangle \) as

\[
\mu^{(\ell_{id})}_n = \mu_n \times (\text{id}_G)^{\times \ell_{id}}, \\
\mu^{(\ell_{id})}_n : G^{\times (n+\ell_{id})} \to G^{\times (1+\ell_{id})}.
\]

To indicate the exact \( i \)-th place of \( \mu_n \) in the r.h.s. of (5.1), we write \( \mu^{(\ell_{id})}_n (i) \), as needed. Here we use the \( \ell \)-product to generalize the homomorphism and consider mappings between polyadic systems of different arity. It follows from (5.2) that, if the image of the \( \ell \)-product is \( G \) alone, than \( \ell_{id} = 0 \).
Let us introduce a multiplace mapping \( \Phi^{(n,n')}_{k} \) acting as follows
\[
\Phi^{(n,n')}_{k} : G^{\times k} \rightarrow G'.
\] (5.3)

While constructing the corresponding diagram, we are allowed to take only one upper \( \Phi^{(n,n')}_{k} \), because of one \( G' \) in the upper right corner. Since we already know that the lower right corner is exactly \( G'^{\times n'} \) (as a pre-image of one multiplication \( \mu'^{n'}_{n} \)), the lower horizontal arrow should be a product of \( n' \) multiplace mappings \( \Phi^{(n,n')}_{k} \). So we can write a definition of a multiplace analog of homomorphisms which changes the arity of the multiplication using one relation.

**Definition 5-1.** A \( k \)-place heteromorphism from \( G_{n} \) to \( G'_{n'} \) is given, if there exists a \( k \)-place mapping \( \Phi^{(n,n')}_{k} \) (5.3) such that the following (arity changing or unequinary) diagram is commutative
\[
\begin{array}{ccc}
G^{\times k} & \xrightarrow{\Phi_{k}} & G' \\
\downarrow_{\mu^{(\ell_{id})}_{n}} & & \downarrow_{\mu'^{n'}_{n'}} \\
G^{\times kn'} & \xrightarrow{(\Phi_{k})^{\times n'}} & (G')^{\times n'}
\end{array}
\] (5.4)

and the corresponding defining equation (a modification of (5.5)) depends on the place \( i \) of \( \mu_{n} \) in (5.1).

For \( i = 1 \) a heteromorphism is defined by the formula
\[
\Phi^{(n,n')}_{k} \left( \begin{array}{c}
\mu_{n} \begin{bmatrix} g_{1}, \ldots, g_{n} \\
g_{n+1} \\
\vdots \\
g_{n+\ell_{id}} 
\end{bmatrix}
\end{array} \right) = \mu'^{n'}_{n'} \left[ \Phi^{(n,n')}_{k} \left( \begin{array}{c}
g_{1} \\
\vdots \\
g_{k}
\end{array} \right), \ldots, \Phi^{(n,n')}_{k} \left( \begin{array}{c}g_{k(n'-1)} \\
\vdots \\
g_{kn'}
\end{array} \right) \right].
\] (5.5)

The notion “heteromorphism” is motivated by Ellerman [2006, 2007], where mappings between objects from different categories were considered and called “chimera morphisms”. See, also, Pécs [2011].

In the particular case \( n = 3, n' = 2, k = 2, \ell_{id} = 1 \) we have
\[
\Phi^{(3,2)}_{2} \left( \begin{array}{c}
\mu_{3} \begin{bmatrix} g_{1}, g_{2}, g_{3} \\
g_{4}
\end{bmatrix}
\end{array} \right) = \mu'^{2}_{2} \left[ \Phi^{(3,2)}_{2} \left( \begin{array}{c}g_{1} \\
g_{2}
\end{array} \right), \Phi^{(3,2)}_{2} \left( \begin{array}{c}g_{3} \\
g_{4}
\end{array} \right) \right].
\] (5.6)

This formula was used in the construction of the bi-element representations of ternary groups Borowiec et al. [2006]. Consider the example.

**Example 5-2.** Let \( G = M_{2}^{\text{adig}}(\mathbb{K}) \), a set of antidiagonal \( 2 \times 2 \) matrices over the field \( \mathbb{K} \) and \( G' = \mathbb{K} \), where \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{H} \). The ternary multiplication \( \mu_{3} \) is a product of 3 matrices. Obviously, \( \mu_{3} \) is not derived from a binary multiplication. For the elements \( g_{i} = \begin{bmatrix} 0 & a_{i} \\
0 & b_{i}
\end{bmatrix}, i = 1, 2 \), we construct a 2-place mapping \( G \times G \rightarrow G' \) as
\[
\Phi^{(3,2)}_{2} \left( \begin{array}{c}
g_{1} \\
g_{2}
\end{array} \right) = a_{1}a_{2}b_{1}b_{2},
\] (5.7)

which is a heteromorphism, because it satisfies (5.5). Let us introduce a standard 1-place mapping by \( \varphi (g_{i}) = a_{i}b_{i}, i = 1, 2 \). It is important to note, that \( \varphi (g_{i}) \) is not a homomorphism, because the product \( g_{1}g_{2} \) belongs to diagonal matrices. Consider the product of mappings
\[
\varphi (g_{1}) \cdot \varphi (g_{2}) = a_{1}b_{1}a_{2}b_{2},
\] (5.8)
where the product (·) in l.h.s. is taken in ℂ. We observe that (5.2) and (5.3) coincide for the commutative field ℂ only (= ℚ, ℍ) only, and in this case we can have the relation between the heteromorphism \( \Phi^{(3,2)}_2 \) and the 1-place mapping \( \varphi \)

\[
\Phi^{(3,2)}_2 \left( \begin{array}{c}
g_1 \\
g_2 
\end{array} \right) = \varphi(g_1) \cdot \varphi(g_2),
\]

while for the noncommutative field \( \mathbb{K} (= \mathbb{Q} \text{ or } \mathbb{H}) \) there is no such relation.

A heteromorphism is called derived, if it can be expressed through a 1-place mapping (not necessary a homomorphism). So, in the above Example 5-2 the heteromorphism is derived (by formula (5.9)) for the commutative field \( \mathbb{K} \) and nonderived for the noncommutative \( \mathbb{K} \).

For arbitrary \( n \) a slightly modified construction 5.6 with still binary final arity, defined by \( n' = 2, k = n - 1, \ell_{id} = n - 2 \),

\[
\Phi^{(n,2)}_{n-1} \left( \begin{array}{c}
m_{[g_1, \ldots, g_{n-1}, h_1]} \\
h_2 \\
\vdots \\
h_{n-1}
\end{array} \right) = \mu'_2 \left[ \Phi^{(n,2)}_{n-1} \left( \begin{array}{c}
g_1 \\
\vdots \\
g_{n-1} 
\end{array} \right), \Phi^{(n,2)}_{n-1} \left( \begin{array}{c}
h_1 \\
\vdots \\
h_{n-1}
\end{array} \right) \right].
\]

was used in [Dudek 2007] to study representations of \( n \)-ary groups. However, no new results compared with [Borowiec et al. 2006] (other than changing 3 to \( n \) in some formulas) were obtained. This reflects the fact that a major role is played by the final arity \( n' \) and the number of \( n \)-ary multiplications in the l.h.s. of (5.3) and (5.10). In the above cases, the latter number was one, but can make it arbitrary below \( n \).

**Definition 5-3.** A heteromorphism is called a \( \ell_\mu \)-ple heteromorphism, if it contains \( \ell_\mu \) multiplications in the argument of \( \Phi^{(n,n')}_{k} \) in its defining relation.

According this definition the mapping defined by (5.5) is the \( 1_\mu \)-ple heteromorphism. So by analogy with (5.1–5.2) we define a \( \ell_\mu \)-ple \( \ell_{id} \)-intact \( id \)-product for the polyadic system \( \langle G; \mu_n \rangle \) as

\[
\mu^{(\ell_{\mu}, \ell_{id})}_n = (\mu_n)^{\ell_{\mu}} \times (id_G)^{\ell_{id}},
\]

\[
\mu^{(\ell_{\mu}, \ell_{id})}_n : G^{\times (n\ell_{\mu} + \ell_{id})} \to G^{\times (\ell_{\mu} + \ell_{id})}.
\]

**Definition 5-4.** A \( \ell_\mu \)-ple \( k \)-place heteromorphism from \( G_n \) to \( G'_n \) is given, if there exists a \( k \)-place mapping \( \Phi^{(n,n')}_{k} \) (5.3) such that the following unequary diagram is commutative

\[
\begin{array}{ccc}
G^{\times k} & \xrightarrow{\Phi_k} & G' \\
\mu^{(\ell_{\mu}, \ell_{id})}_n & \downarrow & \mu'_{n'} \\
G^{\times kn'} & \xrightarrow{(\Phi_k)^{\times n'}} & (G')^{\times n'}
\end{array}
\]

The corresponding main heteromorphism equation is

\[
\Phi^{(n,n')}_k \left( \begin{array}{c}
m_n [g_1, \ldots, g_n], \\
\vdots \\
m_n [g_n(\ell_{\mu} - 1), \ldots, g_n\ell_{\mu}], \\
g_{n\ell_{\mu} + 1}, \\
\vdots \\
g_{n\ell_{\mu} + \ell_{id}}
\end{array} \right) = \mu'_{n'} \left[ \Phi^{(n,n')}_k \left( \begin{array}{c}
g_1 \\
\vdots \\
g_k 
\end{array} \right), \ldots, \Phi^{(n,n')}_k \left( \begin{array}{c}
g_{k(n' - 1)} \\
\vdots \\
g_{kn'}
\end{array} \right) \right].
\]

\[
(5.14)
\]
Figure 1. Dependence of the final arity $n'$ through the number of heteromorphism places $k$ for the initial arity $n = 9$ with the fixed number of intact elements $\ell_{\text{id}}$ (left) and the fixed number of multiplications $\ell_\mu$ (right): $=1$ (solid curves), $=2$ (dash curves).

Obviously, we can consider various permutations of the multiplications on both sides, as further additional demands (associativity, commutativity, etc.), are introduced, which will be considered below. The commutativity of the diagram (5.13) leads to the system of equation connecting initial and final arities

$$kn' = n\ell_\mu + \ell_{\text{id}},$$

$$k = \ell_\mu + \ell_{\text{id}}. \tag{5.15}$$

Excluding $\ell_\mu$ or $\ell_{\text{id}}$, we obtain two arity changing formulas, respectively

$$n' = n - \frac{n - 1}{k}\ell_{\text{id}}, \tag{5.17}$$

$$n' = \frac{n - 1}{k}\ell_\mu + 1, \tag{5.18}$$

where $\frac{n - 1}{k}\ell_{\text{id}} \geq 1$ and $\frac{n - 1}{k}\ell_\mu \geq 1$ are integer.

As an example, the dependences $n'(k)$ for the fixed $\ell_\mu = 1, 2$ and $\ell_{\text{id}} = 1, 2$ with $n = 9$ are presented on Figure 1.

The following inequalities hold valid

$$1 \leq \ell_\mu \leq k, \tag{5.19}$$

$$0 \leq \ell_{\text{id}} \leq k - 1, \tag{5.20}$$

$$\ell_\mu \leq k \leq (n - 1)\ell_\mu, \tag{5.21}$$

$$2 \leq n' \leq n, \tag{5.22}$$

which are important for the further classification of heteromorphisms. The main statement follows from (5.22).

Proposition 5-5. The heteromorphism $\Phi_k^{(n,n')}$ defined by the general relation (5.14) always decreases the arity of polyadic multiplication.
Another important observation is the fact that only the id-product (5.11) with \( \ell_{id} \neq 0 \) leads to a change of the arity. In the extreme case, when \( k \) approaches its minimum, \( k = k_{\text{min}} = \ell_{\mu} \), the final arity approaches its maximum \( n'_{\text{max}} = n \), and the id-product becomes a product of \( \ell_{\mu} \) initial multiplications \( \mu_n \) without id’s, since now \( \ell_{id} = 0 \) in (5.14). Therefore, we call a heteromorphism defined by (5.14) with \( \ell_{id} = 0 \) a \( k (= \ell_{\mu}) \)-place homomorphism. The ordinary homomorphism (4.11) corresponds to \( k = \ell_{\mu} = 1 \), and so it is really a 1-place homomorphism. An opposite extreme case, when the final arity approaches its minimum \( n'_{\text{min}} = 2 \) (the final operation is binary), corresponds to the maximal value of \( k \), that is \( k = k_{\text{max}} = (n - 1) \ell_{\mu} \). The number of id’s now is \( \ell_{id} = (n - 2) \ell_{\mu} \geq 0 \), which vanishes, when the initial operation is binary as well. This is the case of the ordinary homomorphism (4.11) for both binary operations \( n' = n = 2 \) and \( k = \ell_{\mu} = 1 \). We conclude that:

Any polyadic system can be mapped into a binary system by means of the special \( k \)-place \( \ell_{\mu} \)-ple heteromorphism \( \Phi_{k}^{(n,n')} \), where \( k = (n - 1) \ell_{\mu} \) (we call it a binarizing heteromorphism) which is defined by (5.14) with \( \ell_{id} = (n - 2) \ell_{\mu} \).

In relation to the Gluskin-Hosszú theorem [GLUSKIN 1965] (any \( n \)-ary group can be constructed from the special binary group and its homomorphism) our statement can be treated as:

**Theorem 5.6.** Any \( n \)-ary system can be mapped into a binary system, using a suitable binarizing heteromorphism \( \Phi_{k}^{(n,2)} \). (5.14).

The case of 1-ple binarizing heteromorphism (\( \ell_{\mu} = 1 \)) corresponds to the formula (5.10). Further requirements (associativity, commutativity, etc.) will give additional relations between multiplications and \( \Phi_{k}^{(n,n')} \), and fix the exact structure of (5.14). Thus, we arrive to the following

**Proposition 5.7.** Classification of \( \ell_{\mu} \)-ple heteromorphisms:

1. \( n' = n'_{\text{max}} = n \implies \Phi_{k}^{(n,n')} \) is the \( \ell_{\mu} \)-place or multiplace homomorphism, i.e.,
   
   \[ k = k_{\text{min}} = \ell_{\mu}. \] (5.23)

2. \( 2 < n' < n \implies \Phi_{k}^{(n,n')} \) is the intermediate heteromorphism with
   
   \[ k = \frac{n - 1}{n' - 1} \ell_{\mu}. \] (5.24)

   In this case the number of intact elements is proportional to the number of multiplications
   
   \[ \ell_{id} = \frac{n - n'}{n' - 1} \ell_{\mu}. \] (5.25)

3. \( n' = n'_{\text{min}} = 2 \implies \Phi_{k}^{(n,2)} \) is the \((n - 1) \ell_{\mu}\)-place (multiplace) binarizing heteromorphism, i.e.,
   
   \[ k = k_{\text{max}} = (n - 1) \ell_{\mu}. \] (5.26)

In the extreme (first and third) cases there are no restrictions on the initial arity \( n \), while in the intermediate case \( n \) is “quantized” due to the fact that fractions in (5.17) and (5.18) should be integers.

Observe, that in the extreme (first and third) cases there are no restrictions on the initial arity \( n \), while in the intermediate case \( n \) is “quantized” due to the fact that fractions in (5.17) and (5.18) should be integer. In this way, we obtain the **Table 1** for the series of \( n \) and \( n' \) (we list only first ones, just for \( 2 \leq k \leq 4 \) and include the binarizing case \( n' = 2 \) for completeness).

Thus, we have established a general structure and classification of heteromorphisms defined by (5.14). The next important issue is the preservation of special properties (associativity, commutativity, etc.), while passing from \( \mu_n \) to \( \mu_{n'} \), which will further restrict the concrete shape of the main relation
### Table 1. “Quantization” of heteromorphisms

| \( k \) | \( \ell_\mu \) | \( \ell_{id} \) | \( n \) \( n' \) |
|-------|--------|--------|--------|
| 2     | 1      | 1      | \( n = 3, 5, 7, \ldots \) |
|       |        |        | \( n' = 2, 3, 4, \ldots \) |
| 3     | 1      | 2      | \( n = 4, 7, 10, \ldots \) |
|       |        |        | \( n' = 2, 3, 4, \ldots \) |
| 3     | 2      | 1      | \( n = 4, 7, 10, \ldots \) |
|       |        |        | \( n' = 3, 5, 7, \ldots \) |
| 4     | 1      | 3      | \( n = 5, 9, 13, \ldots \) |
|       |        |        | \( n' = 2, 3, 4, \ldots \) |
| 4     | 2      | 2      | \( n = 3, 5, 7, \ldots \) |
|       |        |        | \( n' = 2, 3, 4, \ldots \) |
| 4     | 3      | 1      | \( n = 5, 9, 13, \ldots \) |
|       |        |        | \( n' = 4, 7, 10, \ldots \) |

(5.14) for each choice of the heteromorphism parameters: arities \( n, n' \), places \( k \), number of intacts \( \ell_{id} \) and multiplications \( \ell_\mu \).

### 6. Associativity Quivers and Heteromorphisms

The most important property of the heteromorphism, which is needed for its next applications to representation theory, is the associativity of the final operation \( \mu'_{n'} \), when the initial operation \( \mu_n \) is associative. In other words, we consider here the concrete form of semigroup heteromorphisms. In general, this is a complicated task, because it is not clear from (5.14), which permutation in the l.h.s. should be taken to get an associative product in its r.h.s. for each set of the heteromorphism parameters. Straightforward checking of the associativity of the final operation \( \mu'_{n'} \) for each permutation in the l.h.s. of (5.14) is almost impossible, especially for higher \( n \). To solve this difficulty we introduce the concept of the associative polyadic quiver and special rules to construct the associative final operation \( \mu'_{n'} \).

**Definition 6-1.** A polyadic quiver of products is the set of elements from \( G_n \) (presented as several copies of some matrix of the elements glued together) and arrows, such that the elements along arrows form \( n \)-ary products \( \mu_n \).

For instance, for the 4-ary multiplication \( \mu_4 \{g_1, h_2, g_2, u_1\} \) (elements from \( G_n \) are arbitrary here) a corresponding 4-adic quiver will be denoted by \( \{g_1 \to h_2 \to g_2 \to u_1\} \), and graphically this 4-adic quiver is

\[
\begin{align*}
\mu_4 \{g_1, h_2, g_2, u_1\}.
\end{align*}
\]

Next we define polyadic quivers which are related to the main heteromorphism equation (5.14) in the following way:
1) the matrix of elements is the transposed matrix from the r.h.s. of (5.14), such that different letters correspond to their place in $\Phi_{n,n'}^{(\mu,\mu')}$ and the low index of an element is related to its position in the $\mu'_{n'}$ product;

2) the number of polyadic quivers is $\ell_{\mu}$, which corresponds to $\ell_{\mu}$ multiplications in the l.h.s. of (5.14);

3) the heteromorphism parameters $(n, n', k, \ell_{\text{id}}$ and $\ell_{\mu}$) are not arbitrary, but satisfy the arity changing formulas (5.17)-(5.18);

4) the intact elements will be placed after a semicolon.

In this way, a polyadic quiver makes a clear visualization of the main heteromorphism equation (5.14), and later on it will allow us to distinguish associativity preserving heteromorphisms by precise graphical rules.

For example, the polyadic quiver \{$g_1 \rightarrow h_2 \rightarrow g_2 \rightarrow u_1; h_1, u_2$\} corresponds to the unequiary heteromorphism with $n = 4$, $n' = 2$, $k = 3$, $\ell_{\text{id}} = 2$ and $\ell_{\mu} = 1$ is

\[
\begin{align*}
\Phi_{4}^{(4,2)} \left( \mu_4 \left[ g_1, h_2, g_2, u_1 \right] \right) &= \mu'_2 \left[ \Phi_{3}^{(4,2)} \left( g_1, h_1, u_1 \right), \Phi_{3}^{(4,2)} \left( g_2, h_2, u_2 \right) \right],
\end{align*}
\]

where the intact elements $h_1, u_2$ are boxed in squares. As it is seen from (6.2), the product $\mu'_2$ is not associative, if $\mu_4$ is associative. So, not all polyadic quivers preserve associativity.

**Definition 6-2.** An associative polyadic quiver is a polyadic quiver which ensures the final associativity of $\mu'_{n'}$ in the main heteromorphism equation (5.14), when the initial multiplication $\mu_n$ is associative.

One of the associative polyadic quivers which corresponds to the same heteromorphism parameters, as the non-associative quiver (6.2), is \{$g_1 \rightarrow h_2 \rightarrow u_1 \rightarrow g_2; h_1, u_2$\} which corresponds to

\[
\begin{align*}
\Phi_{3}^{(4,2)} \left( \mu_4 \left[ g_1, h_2, u_1, g_2 \right] \right) &= \mu'_2 \left[ \Phi_{3}^{(4,2)} \left( g_1, h_1, u_1 \right), \Phi_{3}^{(4,2)} \left( g_2, h_2, u_2 \right) \right],
\end{align*}
\]

Here we propose a classification of associative polyadic quivers and the rules of construction of the corresponding heteromorphism equations, and then use the heteromorphism parameters for the classification of $\ell_{\mu}$-ple heteromorphisms (5.24). In other words, we describe a consistent procedure for building the semigroup heteromorphisms.

Let us consider the first class of heteromorphisms (without intact elements $\ell_{\text{id}} = 0$ or intactless), that is $\ell_{\mu}$-ple (multiplace) homomorphisms. In the simplest case, associativity can be achieved, when all elements in a product are taken from the same row. The number of places $k$ is not fixed by
the arity relation \((5.17)\) and can be arbitrary, while the arrows can have various directions. There are \(2^k\) such combinations which preserve associativity. If the arrows have the same direction, this kind of mapping is also called a homomorphism. As an example, for \(n = n' = 3\), \(k = 2\), \(\ell_\mu = 2\) we have

\[
\Phi_2^{(3,3)} \left( \begin{array}{ccc}
\mu_3 & g_1 & g_2 & g_3 \\
\mu_3 & h_1 & h_2 & h_3
\end{array} \right) = \mu_3' \left[ \begin{array}{ccc}
\Phi_2^{(3,3)} & g_1 & h_1 \\
\Phi_2^{(3,3)} & g_2 & h_2 \\
\Phi_2^{(3,3)} & g_3 & h_3
\end{array} \right].
\]

(6.4)

Note that the analogous quiver with opposite arrow directions is

\[
\Phi_2^{(3,3)} \left( \begin{array}{ccc}
\mu_3 & g_1 & g_2 & g_3 \\
\mu_3 & h_1 & h_2 & h_3
\end{array} \right) = \mu_3' \left[ \begin{array}{ccc}
\Phi_2^{(3,3)} & h_1 & g_1 \\
\Phi_2^{(3,3)} & h_2 & g_2 \\
\Phi_2^{(3,3)} & h_3 & g_3
\end{array} \right].
\]

(6.5)

The latter mapping and the corresponding vertical quiver were used in constructing the middle representations of ternary groups [Borowiec et al. 2006].

For nonvertical quivers the main rule is the following: \textit{all arrows of an associative quiver should have direction from left to right or vertical, and they should not intersect}. Also, we start always from the upper left corner, because of the permutation symmetry of \((5.14)\), we can rearrange and rename variables in the necessary way.

An important class of intactless heteromorphisms (with \(\ell_{\mu_1} = 0\)) preserving associativity can be constructed using an analogy with the Post substitutions [Post 1940], and therefore we call it the Post-like associative quiver. The number of places \(k\) is now fixed by \(k = n - 1\), while \(n' = n\) and \(\ell_\mu = k = n - 1\). An example of the Post-like associative quiver with the same heteromorphisms parameters as in (6.4)-(6.5) is

\[
\Phi_2^{(3,3)} \left( \begin{array}{ccc}
\mu_3 & g_1 & g_2 & g_3 \\
\mu_3 & h_1 & h_2 & h_3
\end{array} \right) = \mu_3' \left[ \begin{array}{ccc}
\Phi_2^{(3,3)} & g_1 & h_1 \\
\Phi_2^{(3,3)} & g_2 & h_2 \\
\Phi_2^{(3,3)} & g_3 & h_3
\end{array} \right].
\]

(6.6)
This construction appeared in the study of ternary semigroups of morphisms [Chronowski
1994a, Chronowski and Novotný 1995]. Its \( n \)-ary generalization was used in the consider-
ation of polyadic operations on Cartesian powers [Gal’mak 2008], polyadic analogs of the
Cayley and Birkhoff theorems [Gal’mak 2001b, 2007] and special representations of \( n \)-groups
[Gleichgewicht et al. 1983, Wanke-Jakubowska and Wanke-Jerie 1984] (where the
\( n \)-group with the multiplication \( \mu_2 \) was called the diagonal \( n \)-group). Consider the follow-
ing example.

**Example 6.3.** Let \( \Lambda \) be the Grassmann algebra consisting of even and odd parts
\( \Lambda = \Lambda_0 \oplus \Lambda_1 \) (see e.g., Berezin [1987]). The odd part can be considered as a ternary
semigroup \( G_3^{(1)} = (\Lambda_1, \mu_3) \), its multiplication \( \mu_3 : \Lambda_1 \times \Lambda_1 \times \Lambda_1 \to \Lambda_1 \)
is defined by \( \mu_3 [\alpha, \beta, \gamma] = \alpha \cdot \beta \cdot \gamma \), where \( \cdot \) is
multiplication in \( \Lambda \) and \( \alpha, \beta, \gamma \in \Lambda_1 \), so \( G_3^{(1)} \) is nonderived and contains no unity. The even part can be
treated as a ternary group \( G_3^{(0)} = (\Lambda_0, \mu'_3) \) with the multiplication \( \mu'_3 : \Lambda_0 \times \Lambda_0 \times \Lambda_0 \to \Lambda_0 \), defined
by \( \mu'_3 [a, b, c] = a \cdot b \cdot c \), where \( a, b, c \in \Lambda_0 \), thus \( G_3^{(0)} \) is derived and contains unity. We introduce
the heteromorphism \( G_3^{(1)} \to G_3^{(0)} \) as a mapping (2-place homomorphism) \( \Phi_2^{(3,3)} : \Lambda_1 \times \Lambda_1 \to \Lambda_0 \) by the
formula

\[
\Phi_2^{(3,3)} \left( \begin{array}{c}
\alpha \\
\beta 
\end{array} \right) = \alpha \cdot \beta,
\]

where \( \alpha, \beta \in \Lambda_1 \). It is seen that \( \Phi_2^{(3,3)} \) defined by (6.7) satisfies the Post-like heteromorphism equation
(6.6), but not the “vertical” one (6.4), due to the anticommutativity of odd elements from \( \Lambda_1 \). In other
words, \( G_3^{(0)} \) can be treated as a nontrivial example of the “diagonal” semigroup of \( G_3^{(1)} \) (according
to the notation of Gleichgewicht et al. [1983], Wanke-Jakubowska and Wanke-Jerie
[1984]).

Note that for the number of places \( k \geq 3 \) there exist additional (to the above) associative
quivers having the same heteromorphism parameters. For instance, when \( n' = n = 4 \) and \( k = 3 \) we have
the Post-like associative quiver

\[
\Phi^{(4,4)} \left( \begin{array}{c}
\mu_4 [g_1, h_2, u_3, g_4] \\
\mu_4 [h_1, u_2, g_3, h_4] \\
\mu_4 [u_1, g_2, h_3, u_4] 
\end{array} \right) = \mu'_4 \left( \begin{array}{c}
\Phi_3^{(4,4)} (g_1, h_2, u_3, g_4) \\
\Phi_3^{(4,4)} (h_1, u_2, g_3, h_4) \\
\Phi_3^{(4,4)} (u_1, g_2, h_3, u_4) 
\end{array} \right).
\]

(6.8)
Also, we have one intermediate non-Post associative quiver

![Quiver Diagram](image)

\[(\Phi^{(4,4)}_4)_{\mu_4} \left( [g_1, h_1, u_1, s_1, t_1, v_1] \right) = \mu_4' \left( \Phi^{(4,4)}_3 \left( [g_1, h_1, u_1] , \Phi^{(4,4)}_3 \left( [g_2, h_2, u_2] , \Phi^{(4,4)}_3 \left( [g_3, h_3, u_3] , \Phi^{(4,4)}_3 \left( [g_4, h_4, u_4] \right) \right) \right) \right) \right].

A general method of constructing associative quivers for \( n' = n, \ell_{id} = 0 \) and \( k = n - 1 \) can be illustrated from the following more complicated example with \( n = 5 \). First, we draw \( k = n - 1 \) (\( = 4 \)) copies of element matrices. Then we go from the first element in the first column \( g_1 \) to the last element in this column \( g_{n'} (= g_5) \) by different \( k = n - 1 \) (\( = 4 \)) ways: by the vertical quiver, by the Post-like quiver (going to the second copy of the element matrix) and by the remaining \( k - 2 \) (\( = 2 \)) non-Post associative quivers, as

![Expanded Quiver Diagram](image)

Here we show, for short, only the quiver itself without the corresponding heteromorphism equation and only the arrows corresponding to the first product, while other arrows (starting from \( h_1, u_1, v_1 \)) are parallel to it, as in (6.9).

The next type of heteromorphisms (intermediate) is described by the equations (5.15, 5.25), it contains intact elements (\( \ell_{id} \geq 1 \)) and changes (decreases) arity \( n' < n \). For each fixed \( k \) the arities are not arbitrary and presented in TABLE 1. The first general rule is: the associative quivers are non-decreasing in both, vertical (from up to down) and horizontal (from left to right), directions. Second, if there are several multiplications (\( \ell_{\mu} \geq 2 \)), the corresponding associative quivers do not intersect.

Let us present some examples and start from the smallest number of heteromorphism places in \( \Phi_k^{(n,n')} \). For \( k = 2 \), the first (nonbinarizing \( n' \geq 3 \)) case is \( n = 5, n' = 3, \ell_{id} = 1 \) (see the first row
and second \(n/n'\) pair of Table I. The corresponding associative quivers are

\[
\begin{align*}
Φ_2^{(5,3)} & \left( \mu_5 [g_1, h_1, g_2, h_2, g_3] \right) \\
Φ_2^{(5,3)} & \left( \mu_5 [g_1, h_2, g_2, h_3, g_3] \right),
\end{align*}
\]

where we do not write the r.h.s. of the heteromorphism equation (5.14), because it is simply related to the transposed quiver matrix of the size \(n' \times k\).

More complicated examples can be given for \(k = 3\), which corresponds to the second and third lines of Table I. That is we can obtain a ternary final product \((n' = 3)\) by using one or two multiplications \(\ell_\mu = 1, 2\). Examples of the corresponding associative quivers are \((\ell_\mu = 1)\)

\[
\begin{align*}
Φ_3^{(4,3)} & \left( \mu_4 [g_1, u_2, h_2, g_3] \right) \\
Φ_3^{(4,3)} & \left( \mu_4 [h_1, u_2, g_3] \right),
\end{align*}
\]

respectively. Finally, for the case \(k = 4\) one can construct the associative quiver corresponding to the first pair of the last line in Table I. It has three multiplications and one intact element, and the
corresponding quiver is, e.g.,

![Diagram](image)

There are many other possibilities (using permutations and different variants of quivers) to obtain an associative final product \( \mu'_{n'} \) corresponding the same heteromorphism parameters, and therefore we do list them all. The above examples are sufficient to understand the rules of the associative quiver construction and obtain the polyadic semigroup heteromorphisms.

7. Multiplace representations of polyadic systems

Representation theory (see e.g. [Kirillov 1976]) deals with mappings from abstract algebraic systems into linear systems, such as, e.g. linear operators in vector spaces, or into general (semi)groups of transformations of some set. In our notation, this means that in the mapping of polyadic systems (4.4) the final multiplication \( \mu'_{n'} \) is a linear map. This leads to some restrictions on the final polyadic structure \( G'_{n'} \), which are considered below.

Let \( V \) be a vector space over a field \( \mathbb{K} \) (usually algebraically closed) and \( \text{End} V \) be a set of linear endomorphisms of \( V \), which is in fact a binary group. In the standard way, a linear representation of a binary semigroup \( G_2 = \langle G; \mu_2 \rangle \) is a (1-place) map \( \Pi_1 : G_2 \to \text{End} V \), such that \( \Pi_1 \) is a homomorphism

\[
\Pi_1 (\mu_2 [g, h]) = \Pi_1 (g) \ast \Pi_1 (h),
\]

where \( g, h \in G \) and \((\ast)\) is the binary multiplication in \( \text{End} V \) (usually, it is a (semi)group with multiplication as composition of operators or product of matrices, if a basis is chosen). If \( G_2 \) is a binary group with the unity \( e \), then we have the additional condition

\[
\Pi_1 (e) = \text{id}_V.
\]

We will generalize these known formulas to the corresponding polyadic systems along with the heteromorphism concept introduced above. Our general idea is to use the heteromorphism equation (7.1) instead of the standard homomorphism equation (7.1), such that the arity of the representation will be different from the arity of the initial polyadic system \( n' \neq n \).

Consider the structure of the final \( n' \)-ary multiplication \( \mu'_{n'} \) in (5.14), taking into account that the final polyadic system \( G'_{n'} \) should be constructed from \( \text{End} V \). The most natural and physically applicable way is to consider the binary \( \text{End} V \) and to put \( G'_{n'} = \text{der}_{n'} (\text{End} V) \), as it was proposed for the ternary case in [Borowiec et al. 2006]. In this way \( G'_{n'} \) becomes a derived \( n' \)-ary (semi)group of endomorphisms of \( V \) with the multiplication \( \mu'_{n'} : (\text{End} V)^{\times n'} \to \text{End} V \), where

\[
\mu'_{n'} [v_1, \ldots, v_{n'}] = v_1 \ast \ldots \ast v_{n'}, \quad v_i \in \text{End} V.
\]
Because the multiplication \( \mu' \) \((7.3)\) is derived and is therefore associative by definition, we may consider the associative initial polyadic systems (semigroups and groups) and the associativity preserving mappings that are the special heteromorphisms constructed in the previous section.

Let \( G_n = (G; \mu_n) \) be an associative \( n \)-ary polyadic system. By analogy with \((5.3)\), we introduce the following \( k \)-place mapping

\[
\Pi_k^{(n,n')} : G^{\times k} \to \text{End } V. \tag{7.4}
\]

A multiplace representation of an associative polyadic system \( G_n \) in a vector space \( V \) is given, if there exists a \( k \)-place mapping \((7.4)\) which satisfies the (associativity preserving) heteromorphism equation \((5.14)\), that is

\[
\Pi_k^{(n,n')} (\mu_n) \left( \begin{array}{c} g_1, \ldots, g_n \\ \vdots \\ g_{n\ell \mu + 1} \\ \vdots \\ g_{n\ell \mu + \ell \text{id}} \end{array} \right) = \prod_{i=1}^{n'} \Pi_k^{(n,n')} (\mu_{n' \ell \mu + \ell \text{id}}) \left( \begin{array}{c} g_{i g_{n(\ell \mu - 1)}} \ldots, g_{i n\ell \mu} \\ \vdots \\ g_{i n\ell \mu + 1} \\ \vdots \\ g_{i n\ell \mu + \ell \text{id}} \end{array} \right), \tag{7.5}
\]

and the following diagram commutes

\[
\begin{array}{c}
G^{\times k} \\
\mu_n (\ell \mu, \ell \text{id})
\end{array} \xrightarrow{\Pi_k} \text{End } V \\
\downarrow\text{(*)}^{n'} \\
G^{\times n'} \xrightarrow{(\Pi_k)^{\otimes n'}} (\text{End } V)^{\times n'}
\]

where \( \mu_n (\ell \mu, \ell \text{id}) \) is given by \((5.11)\), \( \ell \mu \) and \( \ell \text{id} \) are the numbers of multiplications and intact elements in the l.h.s. of \((7.5)\), respectively.

The exact permutation in the l.h.s. of \((7.5)\) is given by the associative quiver presented in the previous section. The representation parameters \((n, n', k, \ell \mu \text{ and } \ell \text{id})\) in \((7.5)\) are the same as the heteromorphism parameters, and they satisfy the same arity changing formulas \((5.17)\) and \((5.18)\). Therefore, a general classification of multiplace representations can be done by analogy with that of the heteromorphisms \((5.23)\)–\((5.26)\) as follows:

1. The hom-like multiplace representation which is a multiplace homomorphism with \( n' = n'_{\text{max}} = n \), without intact elements \( \ell \text{id} = \ell \text{id}^{(\text{min})} = 0 \), and minimal number of places

\[
k = k_{\text{min}} = \ell \mu. \tag{7.7}
\]

2. The intact element multiplace representation which is the intermediate heteromorphism with \( 2 < n' < n \) and the number of intact elements is

\[
\ell \text{id} = \frac{n - n'}{n' - 1} \ell \mu. \tag{7.8}
\]

3. The binary multiplace representation which is a binarizing heteromorphism \((5.26)\) with \( n' = n'_{\text{min}} = 2 \), the maximal number of intact elements \( \ell \text{id}^{(\text{max})} = (n - 2) \ell \mu \) and maximal number of places

\[
k = k_{\text{max}} = (n - 1) \ell \mu. \tag{7.9}
\]
The multiplace representations for \( n \)-ary semigroups have no additional defining relations, as compared with (7.5). In case of \( n \)-ary groups, we need an analog of the “normalizing” relation (7.2). If the \( n \)-ary group has the unity \( e \), then one can put

\[
\Pi_k^{(n,n')} \left( \begin{array}{l} e \\ \vdots \\ e \\ \end{array} \right) = \text{id}_V. \tag{7.10}
\]

If there is no unity at all, one can “normalize” the multiplace representation, using analogy with (7.2) in the form

\[
\Pi_1 (h^{-1} \ast h) = \text{id}_V, \tag{7.11}
\]

as follows

\[
\Pi_k^{(n,n')} \left( \begin{array}{l} \bar{h} \\ \vdots \\ \bar{h} \\ h \\ \vdots \\ h \\ \vdots \\ h \end{array} \right) = \text{id}_V, \tag{7.12}
\]

for all \( h \in G_n \), where \( \bar{h} \) is the querelement of \( h \). The latter ones can be placed on any places in the l.h.s. of (7.12) due to the Dörnte identities. Also, the multiplications in the l.h.s. of (7.5) can change their place due to the same reason.

A general form of multiplace representations can be found by applying the Dörnte identities to each \( n \)-ary product in the l.h.s. of (7.5). Then, using (7.12) we have schematically

\[
\Pi_k^{(n,n')} \left( \begin{array}{l} g_1 \\ \vdots \\ g_k \end{array} \right) = \Pi_k^{(n,n')} \left( \begin{array}{l} t_1 \\ \vdots \\ t_{\ell_{\mu}} \\ g \\ \vdots \\ g \end{array} \right), \tag{7.13}
\]

where \( g \) is an arbitrary fixed element of the \( n \)-ary group and

\[
t_a = \mu_n [g_{a1}, \ldots, g_{an-1}, \bar{g}], \quad a = 1, \ldots, \ell_{\mu}. \tag{7.14}
\]

This is the special shape of some multiplace representations, while the concrete formulas should be obtained in each case separately. Nevertheless, some conclusions can be drawn from (7.13). Firstly, the equivalence classes on which \( \Pi_k^{(n,n')} \) is constant are determined by fixing \( \ell_{\mu} + 1 \) elements, i.e. by the surface \( t_a = \text{const}, g = \text{const} \). Secondly, some \( k \)-place representations of a \( n \)-ary group can be reduced to \( \ell_{\mu} \)-place representations of its retract. In the case \( \ell_{\mu} = 1 \), multiplace representations of a \( n \)-ary group derived from a binary group correspond to ordinary representations of the latter (see Borowiec et al. [2006], Dudek [2007]).
Example 7.1. Let us consider the case of a binary multiplace representation \( \Pi_{2n-2}^{(n,2)} \) of \( n \)-ary group \( G_n = \langle G; \mu_n \rangle \) with two multiplications \( \ell_\mu = 2 \) defined by the associativity preserving equation

\[
\Pi_{2n-2}^{(n,2)} \left( \begin{array}{c}
\mu_n [g_1, u_1, \ldots, u_{n-2}, g_2] \\
\mu_n [h_1, v_1, \ldots, v_{n-2}, h_2]
\end{array} \right) = \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c}
g_1 \\
h_1
\end{array} \right) \ast \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c}
u_1 \\
h_2
\end{array} \right)
\]

(7.15)

and normalizing condition

\[
\Pi_{2n-2}^{(n,2)} \left( \begin{array}{c}
h \\
h \\
h \\
h \\
h \\
h
\end{array} \right) = \text{id}_Y,
\]

(7.16)

where \( h \in G_n \) is arbitrary. Using (7.15) and (7.16) for a general form of this \( (2n - 2) \)-place representation we have

\[
\Pi_{2n-2}^{(n,2)} = \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c}
g_1 \\
u_1 \\
\vdots \\
h_1 \\
v_1 \\
\vdots \\
u_{n-2}
\end{array} \right) \ast \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c}
h \\
h \\
h \\
h \\
h \\
h
\end{array} \right) = \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c}
g \\
\vdots \\
g \\
g \\
t_g \\
\vdots \\
t_g
\end{array} \right) \ast \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c}
h \\
\vdots \\
h \\
h \\
h \\
\vdots \\
h
\end{array} \right) = \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c}
g \\
\vdots \\
g \\
g \\
t_g \\
\vdots \\
t_g
\end{array} \right)
\]

(7.17)

The Dörnte identity applied to the first elements of the products in the r.h.s. of (7.17) together with associativity of \( \mu_n \) gives

\[
\left( \begin{array}{c}
\mu_n [g_1, u_1, \ldots, u_{n-2}, g] \\
\mu_n [g, t_g, h]
\end{array} \right) = \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c}
g \\
\vdots \\
g \\
g \\
t_g \\
\vdots \\
t_g
\end{array} \right) \ast \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c}
h \\
\vdots \\
h \\
h \\
h \\
\vdots \\
h
\end{array} \right)
\]

(7.18)
where
\[ t_g = \mu_n [\bar{g}, g_1, u_1, \ldots, u_{n-2}], \quad t_h = \mu_n [\bar{g}, h_1, v_1, \ldots, v_{n-2}] \tag{7.19} \]

Thus, the equivalent classes of the multiplace representation \( \Pi_{2n-2}^{(n,2)} \) are determined by the \((\ell_\mu + 1 = 3)\)-element surface
\[ t_g = \text{const}, \quad t_h = \text{const}, \quad g = \text{const.} \tag{7.20} \]

Note that the “\(\ell_\mu\)-place reduction” of a multiplace representation is possible not for all associativity preserving heteromorphism equations. For instance, if to exchange \(v_i \leftrightarrow v'_i\) in l.h.s. of (7.19), then the associativity remains, but the “\(\ell_\mu\)-place reduction”, analogous to (7.18) will not be possible.

In case, when it is possible, the corresponding \(\ell_\mu\)-place representation can be realized on the binary retract of \(G_n\) (for some special \(n\)-ary groups and \(\ell_\mu = 1\) see [Borowiec et al. 2006, Dudek 2007]). Indeed, let \(G_2^\text{ret} = \langle G, \otimes \rangle = \text{ret}_g \langle G; \mu_n \rangle\), where \(g_1 \otimes g_2 \stackrel{\text{def}}{=} \mu_n \left[ g_1, g, \ldots, g, g_2 \right], \ g_1, g_2, g \in G\) and we define the reduced \(\ell_\mu\)-place representation (now \(\ell_\mu = 2\)) through (7.18) as follows
\[ \Pi_{2n-2}^{\text{ret}g} \left( \begin{array}{c} g_1 \\ g_2 \end{array} \right) \stackrel{\text{def}}{=} \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c} g \\ g \end{array} \right), \tag{7.21} \]

From (7.17) and (7.16) we obtain
\[ \Pi_{2n-2}^{\text{ret}g} \left( \begin{array}{c} g_1 \\ g_2 \end{array} \right) \ast \Pi_{2n-2}^{\text{ret}g} \left( \begin{array}{c} g'_1 \\ g'_2 \end{array} \right) = \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c} g \\ g \end{array} \right) \ast \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c} g \\ g \end{array} \right) = \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c} g \\ g \end{array} \right), \tag{7.21a} \]
\[ = \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c} g \\ g \end{array} \right) = \Pi_{2n-2}^{(n,2)} \left( \begin{array}{c} g \\ g \end{array} \right). \tag{7.21b} \]
In the framework of our classification $\Pi_{2}^{\text{set,g}} \left( \begin{array}{c} g_{1} \\ g_{2} \end{array} \right)$ is a hom-like 2-place (binary) representation.

The above formulas describe various properties of multiplace representations, but they give no idea of how to build representations for concrete polyadic systems. The most common method of representation construction uses the concept of a group action on a set (see, e.g., [Kirillov (1976)]). Below we extend this concept to the multiplace case and use corresponding heteromorphisms, as it was done above for homomorphisms and representations.

8. Multiactions and G-spaces

Let $G_{n} = \langle G; \mu_{n} \rangle$ be a polyadic system and $X$ be a set. A (left) 1-place action of $G_{n}$ on $X$ is the external binary operation $\rho_{1}^{(n)} : G \times X \to X$ such that it is consistent with the multiplication $\mu_{n}$, i.e., composition of the binary operations $\rho_{1}^{(n)} \{ g | x \}$ gives the $n$-ary product, that is,

$$
\rho_{1}^{(n)} \{ \mu_{n} [g_{1}, \ldots, g_{n}] | x \} = \rho_{1}^{(n)} \{ g_{1\mu} \rho_{1}^{(n)} \{ g_{2\mu} \ldots \rho_{1}^{(n)} \{ g_{n\mu} | x \} \} \} \ldots \ , \ g_{1}, \ldots, g_{n} \in G, \ x \in X. \tag{8.1}
$$

If the polyadic system is a $n$-ary group, then in addition to (8.1) it is implied there exist such $e_{x} \in G$ (which may or may not coincide with the unity of $G_{n}$) that $\rho_{1}^{(n)} \{ e_{x} | x \} = x$ for all $x \in X$, and the mapping $x \mapsto \rho_{1}^{(n)} \{ e_{x} | x \}$ is a bijection of $X$. The right 1-place actions of $G_{n}$ on $X$ are defined in a symmetric way, and therefore we will consider below only one of them. Obviously, we cannot compose $\rho_{1}^{(n)}$ and $\rho_{1}^{(n')}$ with $n \neq n'$. Usually $X$ is called a $G$-set or $G$-space depending on its properties (see, e.g., [Husemöller et al. (2008)]).

The application of the 1-place action defined by (8.1) to the representation theory of $n$-ary groups gave mostly repetitions of the ordinary (binary) group representation results (except for trivial $b$-derived ternary groups) [Dudek and Shahryari (2012)]. Also, it is obviously seen that the construction (8.1) with the binary external operation $\rho_{1}$ cannot be applied for studying the most important regular representations of polyadic systems, when the $X$ coincides with $G_{n}$ itself and the action arises from translations.

Here we introduce the multiplace concept of action for polyadic systems, which is consistent with heteromorphisms and multiplace representations. Then we will show how it naturally appears when $X = G_{n}$ and apply it to construct examples of representations including the regular ones.

For a polyadic system $G_{n} = \langle G; \mu_{n} \rangle$ and a set $X$ we introduce an external polyadic operation

$$
\rho_{k} : G^{\times k} \times X \to X, \tag{8.2}
$$

which is called a (left) $k$-place action or multiaction. To generalize the 1-action composition (8.1), we use the analogy with multiplication laws of the heteromorphisms (5.14) and the multiplace representations (7.5) and propose (schematically)

$$
\rho_{k}^{(n)} \left( \begin{array}{c}
\mu_{n} [g_{1}, \ldots, g_{n}], \\
\vdots \\
\mu_{n} [g_{n(\ell_{\mu}-1)}, \ldots, g_{n\ell_{\mu}+1}] \\
g_{n\ell_{id}+1}, \\
\vdots \\
g_{n\ell_{\mu}+\ell_{id}}
\end{array} \right) \ell_{\mu} \times \left( \begin{array}{c}
g_{1} \\
\vdots \\
g_{k} \\
\vdots \\
g_{kn'}
\end{array} \right) \left( \begin{array}{c}
\rho_{k}^{(n)} \{ g_{1} \} \\
\vdots \\
\rho_{k}^{(n)} \{ g_{k} \} \\
\rho_{k}^{(n')} \{ g_{kn'} \}
\end{array} \right) \times \cdots \right) = \rho_{k}^{(n')}.
$$

The connection between all the parameters here is the same as in the arity changing formulas (5.17)-(5.18). Composition of mappings is associative, and therefore in concrete cases we can use...
the associative quiver technique, as it is described in the previous sections. If $G_n$ is $n$-ary group, then we should add to \[8.3\] the “normalizing” relations analogous with \[7.10\] or \[7.6\]. So, if there is a unity $e \in G_n$, then

$$
\rho_k^{(n)} \left\{ \begin{array}{c} e \\ \vdots \\ x \\ e \end{array} \right\} = x, \quad \text{for all } x \in X.
$$

(8.4)

In terms of the querelement, the normalization has the form

$$
\rho_k^{(n)} \left\{ \begin{array}{c} \bar{h} \\ \vdots \\ \ell_\mu \\ \bar{h} \\ h \\ \vdots \\ \ell_{id} \\ h \end{array} \right\} = x, \quad \text{for all } x \in X \text{ and for all } h \in G_n.
$$

(8.5)

The multiact $\rho_k^{(n)}$ is transitive, if any two points $x$ and $y$ in $X$ can be “connected” by $\rho_k^{(n)}$, i.e. there exist $g_1, \ldots, g_k \in G_n$ such that

$$
\rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ x \\ g_k \end{array} \right\} = y.
$$

(8.6)

If $g_1, \ldots, g_k$ are unique, then $\rho_k^{(n)}$ is sharply transitive. The subset of $X$, in which any points are connected by \[8.6\] with fixed $g_1, \ldots, g_k$ can be called the multiorbit of $X$. If there is only one multiorbit, then we call $X$ the heterogenous $G$-space (by analogy with the homogeneous one). By analogy with the (ordinary) 1-place actions, we define a $G$-equivariant map $\Psi$ between two $G$-sets $X$ and $Y$ by (in our notation)

$$
\Psi \left( \rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ x \\ g_k \end{array} \right\} \right) = \rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ \Psi (x) \\ g_k \end{array} \right\} \in Y,
$$

(8.7)

which makes $G$-space into a category (for details, see, e.g., [HUSEMÖLLER ET AL. [2008]]). In the particular case, when $X$ is a vector space over $\mathbb{K}$, the multiact \[6.2\] can be called a multi-$G$-module which satisfies \[5.4\] and the additional (linearity) conditions

$$
\rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ ax + by \\ g_k \end{array} \right\} = a \rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ x \\ g_k \end{array} \right\} + b \rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ y \\ g_k \end{array} \right\},
$$

(8.8)

where $a, b \in \mathbb{K}$. Then, comparing \[7.5\] and \[8.4\] we can define a multiplace representation as a multi-$G$-module by the following formula

$$
\Pi_{k}^{(n,n')} \left( \begin{array}{c} g_1 \\ \vdots \\ g_k \end{array} \right) (x) = \rho_k^{(n)} \left\{ \begin{array}{c} g_1 \\ \vdots \\ x \\ g_k \end{array} \right\}.
$$

(8.9)

In a similar way, one can generalize to polyadic systems many other notions from group action theory [KIRILLOV [1976]].
9. Regular Multiactions

The most important role in the study of polyadic systems is played by the case, when \( X = G_n \), and the multiaction coincides with the \( n \)-ary analog of translations [MAL’TCEV 1954], so called \( i \)-translations [BELOUSOV 1972]. In the binary case, ordinary translations lead to regular representations [KIRILLOV 1976], and therefore we call such an action a regular multiaction \( \rho^{\text{reg}}_k \). In this connection, the analog of the Cayley theorem for \( n \)-ary groups was obtained in [GAL’MAK 1986, 2001b]. Now we will show in examples, how the regular multiactions can arise from \( i \)-translations.

**Example 9-1.** Let \( G_3 \) be a ternary semigroup, \( k = 2 \), and \( X = G_3 \), then 2-place (left) action can be defined as

\[
\rho^{\text{reg}(3)}_2 \left\{ \begin{array}{c} g \\ h \\ \end{array} \right\} u \overset{\text{def}}{=} \mu_3 \left[ g, h, u \right].
\] (9.1)

This gives the following composition law for two regular multiactions

\[
\rho^{\text{reg}(3)}_2 \left\{ \begin{array}{c} g_1 \\ h_1 \\ \end{array} \right\} \rho^{\text{reg}(3)}_2 \left\{ \begin{array}{c} g_2 \\ h_2 \\ \end{array} \right\} u \overset{\text{def}}{=} \mu_3 \left[ g_1, h_1, \mu_3 \left[ g_2, h_2, u \right] \right]
= \mu_3 \left[ \mu_3 \left[ g_1, h_1, g_2 \right], h_2, u \right] = \rho^{\text{reg}(3)}_2 \left\{ \begin{array}{c} \mu_3 \left[ g_1, h_1, g_2 \right] \\ h_2 \\ \end{array} \right\} u.
\] (9.2)

Thus, using the regular 2-action \((9.1)\) we have, in fact, derived the associative quiver corresponding to \((5.10)\).

The formula \((9.1)\) can be simultaneously treated as a 2-translation [BELOUSOV 1972]. In this way, the following left regular multiaction

\[
\rho^{\text{reg}(n)}_k \left\{ \begin{array}{c} g_1 \\ \vdots \\ g_k \\ \end{array} \right\} h \overset{\text{def}}{=} \mu_n \left[ g_1, \ldots, g_k, h \right],
\] (9.3)

corresponds to \((5.10)\), where in the r.h.s. there is the \( i \)-translation with \( i = n \). The right regular multiaction corresponds to the \( i \)-translation with \( i = 1 \). The binary composition of the left regular multiactions corresponds to \((5.10)\). In general, the value of \( i \) fixes the minimal final arity \( n'_{\text{reg}} \), which differs for even and odd values of the initial arity \( n \).

It follows from \((9.3)\) that for regular multiactions the number of places is fixed

\[ k_{\text{reg}} = n - 1. \] (9.4)

and the arity changing formulas \((5.17)\)–\((5.18)\) become

\[ n'_{\text{reg}} = n - \ell_{\text{id}} \] (9.5)
\[ n'_{\text{reg}} = \ell_\mu + 1. \] (9.6)

From \((9.5)\)–\((9.6)\) we conclude that for any \( n \) a regular multiaction having one multiplication \( \ell_\mu = 1 \) is binarizing and has \( n - 2 \) intact elements. For \( n = 3 \) see \((9.2)\). Also, it follows from \((9.5)\) that for regular multiactions the number of intact elements gives exactly the difference between initial and final arities.

If the initial arity is odd, then there exists a special middle regular multiaction generated by the \( i \)-translation with \( i = (n + 1)/2 \). For \( n = 3 \) the corresponding associative quiver is \((5.5)\) and such 2-actions were used in [BOROWIEC ET AL. 2006] to construct middle representations of ternary
groups, which did not change arity \((n' = n)\). Here we give a more complicated example of a middle regular multiaction, which can contain intact elements and can therefore change arity.

**Example 9.2.** Let us consider 5-ary semigroup and the following middle 4-action

\[
\rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
g \in \mathbb{G} \\
\ell(h, u) \\
v \in \mathbb{G}
\end{array} \right\} = \mu_5 \left[ g, h, \frac{1}{s}, u, v \right]. \tag{9.7}
\]

Using (9.6) we observe that there are two possibilities for the number of multiplications \(\ell_\mu = 2, 4\). The last case \(\ell_\mu = 4\) is similar to the vertical associative quiver (6.5), but with a more complicated l.h.s., that is

\[
\rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
\mu_5 [g_1, h_1, g_2, g_3, v_1] \\
\mu_5 [h_3, g_4, h_4, g_5, h_5] \\
\mu_5 [v_5, u_1, v_4, v_1, u_1] \\
\mu_5 [v_2, u_2, v_1, u_1, v_1]
\end{array} \right\} = \rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
g_1 \\
h_1 \\
u_1
\end{array} \right\} \rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
g_2 \\
h_2 \\
v_2
\end{array} \right\} \rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
g_3 \\
h_3 \\
v_3
\end{array} \right\} \rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
g_4 \\
h_4 \\
v_4
\end{array} \right\} \rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
g_5 \\
h_5 \\
v_5
\end{array} \right\} \left\{ \begin{array}{c}
s
\end{array} \right\} \right\}. \tag{9.8}
\]

Now we have an additional case with two intact elements \(\ell_{id}\) and two multiplications \(\ell_\mu = 2\) as

\[
\rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
\mu_5 [g_1, h_1, g_2, g_3, v_1] \\
\mu_5 [h_3, v_3, u_2, v_2, u_1] \\
v_1
\end{array} \right\} = \rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
g_1 \\
h_1 \\
u_1
\end{array} \right\} \rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
g_2 \\
h_2 \\
v_2
\end{array} \right\} \rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
g_3 \\
h_3 \\
v_3
\end{array} \right\} \rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
g_4 \\
h_4 \\
v_4
\end{array} \right\} \rho^\text{reg(5)}_4 \left\{ \begin{array}{c}
g_5 \\
h_5 \\
v_5
\end{array} \right\} \left\{ \begin{array}{c}
s
\end{array} \right\} \right\}, \tag{9.9}
\]

with arity changing from \(n = 5\) to \(n'_{reg} = 3\). In addition to (9.9) we have 3 more possible regular multiactions due to the associativity of \(\mu_5\), when the multiplication brackets in the sequences of 6 elements in the first two rows and the second two ones can be shifted independently.

For \(n > 3\), in addition to left, right and middle multiactions, there exist intermediate cases. First, observe that the \(i\)-translations with \(i = 2\) and \(i = n - 1\) immediately fix the final arity \(n'_{reg} = n\). Therefore, the composition of multiactions will be similar to (9.7), but with some permutations in the l.h.s.

Now we consider some multiplace analogs of regular representations of binary groups [KIRILLOV 1976]. The straightforward generalization is to consider the previously introduced regular multiactions (9.3) in the r.h.s. of (8.9). Let \(\mathbb{G}_n\) be a finite polyadic associative system and \(\mathbb{K}\mathbb{G}_n\) be a vector space spanned by \(\mathbb{G}_n\) (some properties of \(n\)-ary group rings were considered in ZEKOVIĆ AND ARTAMONOV 1999, 2002). This means that any element of \(\mathbb{K}\mathbb{G}_n\) can be uniquely presented in the form \(w = \sum a_i \cdot h_i, a_i \in \mathbb{K}, h_i \in G\). Then, using (9.3) and (8.9) we define the \(i\)-regular \(k\)-place representation by

\[
\Pi^\text{reg(i)}_k \left( \begin{array}{c}
g_1 \\
\vdots \\
g_k
\end{array} \right) (w) = \sum a_i \cdot \mu_{k+1} [g_1 \cdots g_i h_l g_{i+1} \cdots g_k]. \tag{9.10}
\]

Comparing (9.3) and (9.10) one can conclude that all the general properties of multiplace regular representations are similar to those of the regular multiactions. If \(i = 1\) or \(i = k\), the multiplace
representation is called a right or left regular representation respectively. If $k$ is even, the representation with $i = k/2 + 1$ is called a middle regular representation. The case $k = 2$ was considered in [Borowiec et al., 2006] for ternary groups.

10. Multiplace representations of ternary groups

Let us consider the case $n = 3$, $k = 2$ in more detail, paying attention to its special peculiarities, which corresponds to the 2-place (bi-element) representations of ternary groups [Borowiec et al., 2006]. Let $V$ be a vector space over $\mathbb{K}$ and $\text{End} V$ be a set of linear endomorphisms of $V$. From now on we denote the ternary multiplication by square brackets only, as follows $\mu_3 (g_1, g_2, g_3) \equiv [g_1 g_2 g_3]$, and use the “horizontal” notation $\Pi \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \equiv \Pi (g_1, g_2)$. 

Definition 10-1. A left representation of a ternary group $G, [\ ]$ in $V$ is a map $\Pi^L : G \times G \to \text{End} V$ such that

$$\Pi^L (g_1, g_2) \circ \Pi^L (g_3, g_4) = \Pi^L ([g_1 g_2 g_3], g_4),$$

$$\Pi^L (g, \bar{g}) = \text{id}_V,$$  \hspace{1cm} (10.1)  \hspace{1cm} (10.2)

where $g, g_1, g_2, g_3, g_4 \in G$.

Replacing in (10.2) $g$ by $\bar{g}$ we obtain $\Pi^L (\bar{g}, g) = \text{id}_V$, which means that in fact (10.2) has the form $\Pi^L (\bar{g}, g) = \Pi^L (g, \bar{g}) = \text{id}_V$, $\forall g \in G$. Note that the axioms considered in the above definition are the natural ones satisfied by left multiplications $g \mapsto [abg]$. For all $g_1, g_2, g_3, g_4 \in G$ we have

$$\Pi^L ([g_1 g_2 g_3], g_4) = \Pi^L (g_1, [g_2 g_3 g_4]).$$

For all $g, h, u \in G$ we have

$$\Pi^L (g, h) = \Pi^L ([gu\bar{u}], h) = \Pi^L (g, u) \circ \Pi^L (\bar{u}, h)$$

and

$$\Pi^L (g, u) \circ \Pi^L (\bar{u}, \bar{g}) = \Pi^L (\bar{u}, g) \circ \Pi^L (g, u) = \text{id}_V,$$  \hspace{1cm} (10.3)  \hspace{1cm} (10.4)

and therefore every $\Pi^L (g, u)$ is invertible and $(\Pi^L (g, u))^{-1} = \Pi^L (\bar{u}, \bar{g})$. This means that any left representation gives a representation of a ternary group by a binary group [Borowiec et al., 2006]. If the ternary group is medial, then

$$\Pi^L (g_1, g_2) \circ \Pi^L (g_3, g_4) = \Pi^L (g_3, g_4) \circ \Pi^L (g_1, g_2),$$

i.e. the group so obtained is commutative. If the ternary group $\langle G, [\ ] \rangle$ is commutative, then also $\Pi^L (g, h) = \Pi^L (h, g)$, because

$$\Pi^L (g, h) = \Pi^L (g, h) \circ \Pi^L (g, \bar{g}) = \Pi^L ([g \ h \ g], \bar{g}) = \Pi^L ([h \ g \ g], \bar{g}) = \Pi^L (h, g) \circ \Pi^L (g, \bar{g}) = \Pi^L (h, g).$$

In the case of a commutative and idempotent ternary group any of its left representations is idempotent and $(\Pi^L (g, h))^{-1} = \Pi^L (g, h)$, so that commutative and idempotent ternary groups are represented by Boolean groups.

Assertion 10-2. Let $\langle G, [\ ] \rangle = \text{der} (G, \circ)$ be a ternary group derived from a binary group $\langle G, \circ \rangle$, then there is one-to-one correspondence between representations of $\langle G, \circ \rangle$ and left representations of $\langle G, [\ ] \rangle$. 

Indeed, because \((G,[])\) is a derivation of \((G,\circ)\), then \(g \circ h = [geh]\) and \(\overline{e} = e\), where \(e\) is unity of the binary group \((G,\circ)\). If \(\pi \in \text{Rep}(G,\circ)\), then (as it is not difficult to see) \(\Pi^L(g,h) = \pi(g) \circ \pi(h)\) is a left representation of \((G,[])\). Conversely, if \(\Pi^L\) is a left representation of \((G,[])\) then \(\pi(g) = \Pi^L(g,e)\) is a representation of \((G,\circ)\). Moreover, in this case \(\Pi^L(g,h) = \pi(g) \circ \pi(h)\), because we have

\[
\Pi^L(g,h) = \Pi^L([e,he]) = \Pi^L([geh],e) = \Pi^L(g,e) \circ \Pi^L(h,e) = \pi(g) \circ \pi(h).
\]

Let \((G,[])\) be a ternary group and \((G \times G,\ast)\) be a semigroup used in the construction of left representations. According to Post \[\text{Post} 1940\] one says that two pairs \((a,b), (c,d)\) of elements of \(G\) are equivalent, if there exists an element \(g \in G\) such that \([a,bg] = [cdg]\). Using a covering group we can see that if this equation holds for some \(g \in G\), then it holds also for all \(g \in G\). This means that

\[
\Pi^L(a,b) = \Pi^L(c,d) \iff (a,b) \sim (c,d),
\]

i.e.

\[
\Pi^L(a,b) = \Pi^L(c,d) \iff [abg] = [cdg]
\]

for some \(g \in G\). Indeed, if \([abg] = [cdg]\) holds for some \(g \in G\), then

\[
\Pi^L(a,b) = \Pi^L(a,b) \circ \Pi^L(g,\overline{f}) = \Pi^L([abg], \overline{g}) = \Pi^L([cdg], \overline{f}) = \Pi^L(c,d) \circ \Pi^L(g,\overline{f}) = \Pi^L(c,d).
\]

By analogy we can define

**Definition 10-3.** A right representation of a ternary group \((G,[])\) in \(V\) is a map \(\Pi^R : G \times G \to \text{End} \ V\) such that

\[
\Pi^R(g_3,g_4) \circ \Pi^R(g_1,g_2) = \Pi^R([g_2g_3g_4]),
\]

\[
\Pi^R(g,\overline{f}) = \text{id}_V,
\]

where \(g, g_1, g_2, g_3, g_4 \in G\).

From (10.5)-(10.6) it follows that

\[
\Pi^R(g,h) = \Pi^R(g,[u,\overline{w}h]) = \Pi^R([u,\overline{w}],h) \circ \Pi^R(g,u).
\]

From \((10.5)-(10.6)\) it follows that

\[
\Pi^R(g,h) = \Pi^R(g,[u,\overline{w}h]) = \Pi^R([u,\overline{w}],h) \circ \Pi^R(g,u).
\]

It is easy to check that \(\Pi^R(g,h) = (\Pi^L(g,h))^{-1}\). So it is sufficient to consider only left representations (as in the binary case). Consider the following example of a group algebra ternary generalization \[\text{Borowiec et al.} 2006\].

**Example 10-4.** Let \(G\) be a ternary group and \(\mathbb{K}G\) be a vector space spanned by \(G\), which means that any element of \(\mathbb{K}G\) can be uniquely presented in the form \(t = \sum_{i=1}^n k_i h_i, k_i \in \mathbb{K}, h_i \in G, n \in \mathbb{N}\) (we do not assume that \(G\) has finite rank). Then left and right regular representations are defined by

\[
\Pi^L_{\text{reg}}(g_1,g_2) t = \sum_{i=1}^n k_i [g_1g_2h_i],
\]

\[
\Pi^R_{\text{reg}}(g_1,g_2) t = \sum_{i=1}^n k_i [h_i g_1g_2].
\]

Let us construct the middle representations as follows.

**Definition 10-5.** A middle representation of a ternary group \((G,[])\) in \(V\) is a map \(\Pi^M : G \times G \to \text{End} \ V\) such that

\[
\Pi^M(g_3,g_3) \circ \Pi^M(g_2,h_2) \circ \Pi^M(g_1,h_1) = \Pi^M([g_2g_3g_1], [h_1h_2h_3]),
\]

\[
\Pi^M(g,h) \circ \Pi^M(\overline{g},\overline{h}) = \Pi^M(\overline{g},\overline{h}) \circ \Pi^M(g,h) = \text{id}_V.
\]
It can be seen that a middle representation is a ternary group homomorphism $\Pi^M : G \times G^{op} \to \text{der End } V$. Note that instead of (10.11) one can use $\Pi^M (g, h) \circ \Pi^M (\overline{g}, \overline{h}) = id_V$ after changing $g$ to $\overline{g}$ and taking into account that $g = \overline{\overline{g}}$. In the case of idempotent elements $g$ and $h$ we have $\Pi^M (g, h) = id_V$, which means that the matrices $\Pi^M$ are Boolean. Thus all middle representation matrices of idempotent ternary groups are Boolean. The composition $\Pi^M (g_1, h_1) \circ \Pi^M (g_2, h_2)$ is not a middle representation, but the following proposition nevertheless holds.

Let $\Pi^M$ be a middle representation of a ternary group $\langle G, [\ ] \rangle$, then, if $\Pi^L_u (g, h) = \Pi^M (g, u) \circ \Pi^M (h, \overline{u})$ is a left representation of $\langle G, [\ ] \rangle$, then $\Pi^L_u (g, h) \circ \Pi^L_{u'} (g', h') = \Pi^L_{u'} ([ghu'], h')$, and, if $\Pi^R_u (g, h) = \Pi^M (u, h) \circ \Pi^M (\overline{u}, g)$ is a right representation of $\langle G, [\ ] \rangle$, then $\Pi^R_u (g, h) \circ \Pi^R_{u'} (g', h') = \Pi^R_{u'} ([hg'h'])$. In particular, $\Pi^L_u (\Pi^R_u)$ is a family of left (right) representations.

If a middle representation $\Pi^M$ of a ternary group $\langle G, [\ ] \rangle$ satisfies $\Pi^M (g, \overline{g}) = id_V$ for all $g \in G$, then it is a left and a right representation and $\Pi^M (g, h) = \Pi^M (h, g)$ for all $g, h \in G$. Note that in general $\Pi^M_{reg} (g, \overline{g}) \neq id$. For regular representations we have the following commutation relations

$$
\Pi^L_{reg} (g_1, h_1) \circ \Pi^R_{reg} (g_2, h_2) = \Pi^R_{reg} (g_2, h_2) \circ \Pi^L_{reg} (g_1, h_1).
$$

Let $\langle G, [\ ] \rangle$ be a ternary group and let $\langle G \times G, [\ ]' \rangle$ be a ternary group used in the construction of the middle representation. In $\langle G, [\ ] \rangle$, and consequently in $\langle G \times G, [\ ]' \rangle$, we define the relation

$$(a, b) \sim (c, d) \iff [aub] = [cud]$$

for all $u \in G$. It is not difficult to see that this relation is a congruence in the ternary group $\langle G \times G, [\ ]' \rangle$. For regular representations $\Pi^M_{reg} (a, b) = \Pi^M_{reg} (c, d)$ if $(a, b) \sim (c, d)$. We have the following relation

$$a \sim a' \iff a = [\overline{g}a'g] \text{ for some } g \in G$$

or equivalently

$$a \sim a' \iff a' = [ga\overline{g}] \text{ for some } g \in G.$$

It is not difficult to see that it is an equivalence relation on $\langle G, [\ ] \rangle$, moreover, if $\langle G, [\ ] \rangle$ is medial, then this relation is a congruence.

Let $\langle G \times G, [\ ]' \rangle$ be a ternary group used in a construction of middle representations, then

$$(a, b) \approx (a', b') \iff a' = [ga\overline{g}] \quad \text{and} \quad b' = [hhb\overline{b}] \text{ for some } (g, h) \in G \times G$$

is an equivalence relation on $\langle G \times G, [\ ]' \rangle$. Moreover, if $\langle G, [\ ] \rangle$ is medial, then this relation is a congruence. Unfortunately, however it is a weak relation. In a ternary group $\mathbb{Z}_3$, where $[ghu] = (g - h + u) \mod 3$ we have only one class, i.e. all elements are equivalent. In $\mathbb{Z}_4$ with the operation $[ghu] = (g + h + u + 1) \mod 4$ we have $a \approx a' \iff a = a'$. However, for this relation the following statement holds. If $(a, b) \approx (a', b')$, then

$$\text{tr } \Pi^M (a, b) = \text{tr } \Pi^M (a', b').$$

We have $\text{tr } (AB) = \text{tr } (BA)$ for all $A, B \in \text{End } V$, and

$$
\begin{align*}
\text{tr } \Pi^M (a, b) &= \text{tr } \Pi^M ([ga\overline{g}], [hb\overline{h}]) = \text{tr } (\Pi^M (g, \overline{h}) \circ \Pi^M (a', b') \circ \Pi^M (g, h)) \\
&= \text{tr } (\Pi^M (g, \overline{h}) \circ \Pi^M (\overline{g}, h) \circ \Pi^M (a', b')) = \text{tr } (id_V \circ \Pi^M (a'b')) = \text{tr } \Pi^M (a', b').
\end{align*}
$$

In our derived case the connection with standard group representations is given by the following. Let $(G, \odot)$ be a binary group, and the ternary derived group as $\langle G, [\ ] \rangle = \text{der } (G, \odot)$. There is one-to-one correspondence between a pair of commuting binary group representations and a middle
ternary derived group representation. Indeed, let $\pi, \rho \in \text{Rep}(G, \otimes)$, $\pi(g) \circ \rho(h) = \rho(h) \circ \pi(g)$ and $\Pi^L \in \text{Rep}(G, [\cdot])$. We take

$$\Pi^M(g,h) = \pi(g) \circ \rho(h^{-1}), \quad \pi(g) = \Pi^M(g,e), \quad \rho(g) = \Pi^M(e,\overline{g}).$$

Then using (10.10) we prove the needed representation laws.

Let $\langle G, [\cdot] \rangle$ be a fixed ternary group, $\langle G \times G, [\cdot] \rangle$ a corresponding ternary group used in the construction of middle representations, $((G \times G)^*, \otimes)$ a covering group of $\langle G \times G, [\cdot] \rangle$, $(G \times G, \otimes) = \text{ret}_{(a,b)}(G \times G, \langle \cdot \rangle)$. If $\Pi^M(a,b)$ is a middle representation of $\langle G, [\cdot] \rangle$, then $\pi$ defined by

$$\pi(g,h,0) = \Pi^M(g,h), \quad \pi(g,h,1) = \Pi^M(g,h) \circ \Pi^M(a,b)$$

is a representation of the covering group $\text{Post}$ (1940). Moreover

$$\rho(g,h) = \Pi^M(g,h) \circ \Pi^M(a,b) = \pi(g,h,1)$$

is a representation of the above retract induced by $(a,b)$. Indeed, $(\overline{a}, \overline{b})$ is the identity of this retract and $\rho(\overline{a}, \overline{b}) = \Pi^M(\overline{a}, \overline{b}) \circ \Pi^M(a,b) = \text{id}_V$. Similarly

$$\rho((g,h) \otimes (u,u)) = \rho((g,h), (a,b), (u,u))) = \rho([gau], [ubh]) = \Pi^M([gau], [ubh]) \circ \Pi^M(a,b)$$

$$= \Pi^M(g,h) \circ \Pi^M(a,b) \circ \Pi^M(u,u) \circ \Pi^M(a,b) = \rho(g,h) \circ \rho(u,u)$$

But $\tau(g) = (g, \overline{g})$ is an embedding of $(G, [\cdot])$ into $\langle G \times G, [\cdot] \rangle$. Hence $\mu$ defined by $\mu(g,0) = \Pi^M(g,\overline{g})$ and $\mu(g,1) = \Pi^M(g,\overline{g}) \circ \Pi^M(a,\overline{a})$ is a representation of a covering group $G^*$ for $(G, [\cdot])$ (see the Post theorem $\text{Post}$ (1940) for $a = c$). On the other hand, $\beta(g) = \Pi^M(g,\overline{g}) \circ \Pi^M(a,\overline{a})$ is a representation of a binary retract $\langle G, \cdot \rangle = \text{ret}_a(G, [\cdot])$. Thus $\beta$ can induce some middle representation of $(G, [\cdot])$ (by the Gluskin-Hosszú theorem $\text{Gluskin}$ (1965)).

Note that in the ternary group of quaternions $\langle K, [\cdot] \rangle$ (with norm 1), where $[ghu] = ghu(-1) = -ghu$ and $gh$ is the multiplication of quaternions $(-1)$ is a central element we have $\overline{1} = -1, -\overline{1} = 1$ and $\overline{g} = g$ for others. In $\langle K \times K, [\cdot] \rangle$ we have $(a,b) \sim (-a,-b)$ and $(a,-b) \sim (-a,b)$, which gives 32 two-element equivalence classes. The embedding $\tau(g) = (g, \overline{g})$ suggest that $\Pi^M(i,i) = \pi(i) \neq \pi(-i) = \Pi^M(-i, -i)$. Generally $\Pi^M(a,b) \neq \Pi^M(-a,-b)$ and $\Pi^M(a,-b) \neq \Pi^M(-a,a,b)$.

The relation $(a,b) \sim (c,d) \iff [abg] = [cdg]$ for all $g \in G$ is a congruence on $(G \times G, \ast)$. Note that this relation can be defined as ”for some $g$”. Indeed, using a covering group we can see that if $[abg] = [cdg]$ holds for some $g$ then it holds also for all $g$. Thus $\pi^L(a,b) = \Pi^L(c,d) \iff (a,b) \sim (c,d)$. Indeed

$$\Pi^L(a,b) = \Pi^L(a,b) \circ \Pi^L(g,\overline{g}) = \Pi^L([g b g], \overline{g})$$

$$= \Pi^L([c d g], \overline{g}) = \Pi^L(c,d) \circ \Pi^L(g,\overline{g}) = \Pi^L(c,d).$$

We conclude, that every left representation of a commutative group $\langle G, [\cdot] \rangle$ is a middle representation. Indeed,

$$\Pi^L(g,h) \circ \Pi^L(g,h) = \Pi^L([g h g], \overline{h}) = \Pi^L([g \overline{g} h], \overline{h}) = \Pi^L(h, \overline{h}) = \text{id}_V$$

and

$$\Pi^L(g_1, g_2) \circ \Pi^L(g_3, g_4) \circ \Pi^L(g_5, g_6) = \Pi^L([[g_1g_2g_3g_4g_5], g_6]) = \Pi^L([g_1g_2g_3g_4g_5], g_6)$$

$$= \Pi^L([g_1g_3g_2g_4g_5], g_6) = \Pi^L([g_1g_3g_5g_4g_2], g_6) = \Pi^L([g_1g_3g_5], [g_4g_2g_6]) = \Pi^L([g_1g_3g_5], [g_6g_4g_2]).$$

Note that the converse holds only for the special kind of middle representations such that $\Pi^M(g,\overline{g}) = \text{id}_V$. Therefore,
**Assertion 10.6.** There is one-one correspondence between left representations of $\langle G, [\ ] \rangle$ and binary representations of the retract $\text{ret}_a(G, [\ ])$.

Indeed, let $\Pi^L(g, a)$ be given, then $\rho(g) = \Pi^L(g, a)$ is such representation of the retract, as can be directly shown. Conversely, assume that $\rho(g)$ is a representation of the retract $\text{ret}_a(G, [\ ])$ and $\Pi^L(g, a) = \rho(g) \circ \rho(h)^{-1}$, then $\Pi^L(g, h) \circ \Pi^L(u, u) = \rho(g) \circ \rho(h)^{-1} \circ \rho(u) \circ \rho(\pi)^{-1} = \rho(g \oplus (h)^{-1} \circ \oplus u) \circ \rho(\pi)^{-1} = \rho([g\ a \ [\pi\ h\ \pi]\ a\ u]) \circ \rho(\pi)^{-1} = \rho([g\ h\ g]) \circ \rho(\pi)^{-1} = \Pi^L([g\ h\ u], u)$.

11. **Matrix representations of ternary groups**

Here we give several examples of matrix representations for concrete ternary groups. Let $G = \mathbb{Z}_3 \ni \{0, 1, 2\}$ and the ternary multiplication be $[ghu] = g - h + u$. Then $[ghu] = [uhg]$ and $\overline{0} = 0, \overline{1} = 1, \overline{2} = 2$, therefore $(G, [\ ])$ is an idempotent medial ternary group. Thus $\Pi^L(g, h) = \Pi^R(h, g)$ and

\[\Pi^L(a, b) = \Pi^L(c, d) \iff (a - b) = (c - d) \mod 3. \tag{11.1}\]

The calculations give the left regular representation in the manifest matrix form

\[\Pi^L_{\text{reg}}(0, 0) = \Pi^L_{\text{reg}}(2, 2) = \Pi^L_{\text{reg}}(1, 1) = \Pi^R_{\text{reg}}(0, 0) = \Pi^R_{\text{reg}}(2, 2) = \Pi^R_{\text{reg}}(1, 1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [1] \oplus [1] \oplus [1], \tag{11.2}\]

\[\Pi^L_{\text{reg}}(2, 0) = \Pi^L_{\text{reg}}(1, 2) = \Pi^L_{\text{reg}}(0, 1) = \Pi^R_{\text{reg}}(2, 1) = \Pi^R_{\text{reg}}(1, 0) = \Pi^R_{\text{reg}}(0, 2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \sqrt{3} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \frac{1}{2} \sqrt{3} \sqrt{3} \end{pmatrix} \oplus \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \sqrt{3} \frac{1}{2} \sqrt{3} \end{pmatrix}, \tag{11.3}\]

\[\Pi^L_{\text{reg}}(2, 1) = \Pi^L_{\text{reg}}(1, 0) = \Pi^L_{\text{reg}}(0, 2) = \Pi^R_{\text{reg}}(2, 0) = \Pi^R_{\text{reg}}(1, 2) = \Pi^R_{\text{reg}}(0, 1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \sqrt{3} \\ \frac{1}{2} & \frac{1}{2} \frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \sqrt{3} \frac{1}{2} \sqrt{3} \end{pmatrix} \oplus \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \sqrt{3} \frac{1}{2} \sqrt{3} \end{pmatrix}. \tag{11.4}\]

Consider next the middle representation construction. The middle regular representation is defined by

\[\Pi^M_{\text{reg}}(g_1, g_2) t = \sum_{i=1}^{n} k_i [g_1h_i g_2]. \]

For regular representations we have

\[\Pi^M_{\text{reg}}(g_1, h_1) \circ \Pi^R_{\text{reg}}(g_2, h_2) = \Pi^R_{\text{reg}}(h_2, h_1) \circ \Pi^M_{\text{reg}}(g_1, g_2), \tag{11.5}\]

\[\Pi^M_{\text{reg}}(g_1, h_1) \circ \Pi^L_{\text{reg}}(g_2, h_2) = \Pi^L_{\text{reg}}(g_1, g_2) \circ \Pi^M_{\text{reg}}(h_2, h_1). \tag{11.6}\]
For the middle regular representation matrices we obtain
\[
\Pi_{\text{reg}}^M(0, 0) = \Pi_{\text{reg}}^M(1, 2) = \Pi_{\text{reg}}^M(2, 1) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\]
\[
\Pi_{\text{reg}}^M(0, 1) = \Pi_{\text{reg}}^M(1, 0) = \Pi_{\text{reg}}^M(2, 2) = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
\[
\Pi_{\text{reg}}^M(0, 2) = \Pi_{\text{reg}}^M(2, 0) = \Pi_{\text{reg}}^M(1, 1) = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

The above representation \(\Pi_{\text{reg}}^M\) of \(\langle \mathbb{Z}_3, [\cdot] \rangle\) is equivalent to the orthogonal direct sum of two irreducible representations
\[
\Pi_{\text{reg}}^M(0, 0) = \Pi_{\text{reg}}^M(1, 2) = \Pi_{\text{reg}}^M(2, 1) = [1] \oplus \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix},
\]
\[
\Pi_{\text{reg}}^M(0, 1) = \Pi_{\text{reg}}^M(1, 0) = \Pi_{\text{reg}}^M(2, 2) = [1] \oplus \begin{bmatrix}
1 & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & 1
\end{bmatrix},
\]
\[
\Pi_{\text{reg}}^M(0, 2) = \Pi_{\text{reg}}^M(2, 0) = \Pi_{\text{reg}}^M(1, 1) = [1] \oplus \begin{bmatrix}
1 & \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -1
\end{bmatrix},
\]

i.e. one-dimensional trivial [1] and two-dimensional irreducible. Note, that in this example \(\Pi^M(g, g) = \Pi^M(g, g) \neq \text{id}_V\), but \(\Pi^M(g, h) \circ \Pi^M(g, h) = \text{id}_V\), and so \(\Pi^M\) are of the second degree.

Consider a more complicated example of left representations. Let \(G = \mathbb{Z}_4 \ni \{0, 1, 2, 3\}\) and the ternary multiplication be
\[
[ghu] = (g + h + u + 1) \mod 4.
\]

We have the multiplication table
\[
[g, h, 0] = \begin{pmatrix}
1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1 \\
3 & 0 & 1 & 2 \\
0 & 1 & 2 & 3
\end{pmatrix}, \quad [g, h, 1] = \begin{pmatrix}
2 & 3 & 0 & 1 \\
3 & 0 & 1 & 2 \\
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0
\end{pmatrix},
\]
\[
[g, h, 2] = \begin{pmatrix}
3 & 0 & 1 & 2 \\
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1
\end{pmatrix}, \quad [g, h, 3] = \begin{pmatrix}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1 \\
3 & 0 & 1 & 2
\end{pmatrix}.
\]

Then the skew elements are \(\overline{0} = 3, \overline{1} = 2, \overline{2} = 1, \overline{3} = 0\), and therefore \((G, [\cdot])\) is a (non-idempotent) commutative ternary group. The left representation is defined by the expansion \(\Pi_{\text{reg}}^L(g_1, g_2) t = \sum_{i=1}^n k_i [g_1 g_2 h_i]\), which means that (see the general formula (9.10))
\[
\Pi_{\text{reg}}^L(g, h) |u> = |ghu>.
\]
Analogously, for right and middle representations

\[ \Pi^R_{\text{reg}}(g, h) |u> = |[ugh]> , \quad \Pi^M_{\text{reg}}(g, h) |u> = |[guh]> . \]

Therefore \(|[ghu]> = |[ugh]> = |[guh]> \) and

\[ \Pi^L_{\text{reg}}(g, h) = \Pi^R_{\text{reg}}(g, h) |u> = \Pi^M_{\text{reg}}(g, h) |u> , \]

so \( \Pi^L_{\text{reg}}(g, h) = \Pi^R_{\text{reg}}(g, h) = \Pi^M_{\text{reg}}(g, h) \). Thus it is sufficient to consider the left representation only.

In this case the equivalence is \( \Pi^L(a, b) = \Pi^L(c, d) \iff (a + b) = (c + d) \mod 4 \), and we obtain the following classes

\[
\begin{align*}
\Pi^L_{\text{reg}}(0, 0) &= \Pi^L_{\text{reg}}(1, 3) = \Pi^L_{\text{reg}}(2, 2) = \Pi^L_{\text{reg}}(3, 1) = \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = [1] \oplus [-1] \oplus [-i] \oplus [i],
\end{align*}
\]

\[
\begin{align*}
\Pi^L_{\text{reg}}(0, 1) &= \Pi^L_{\text{reg}}(1, 0) = \Pi^L_{\text{reg}}(2, 3) = \Pi^L_{\text{reg}}(3, 2) = \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = [1] \oplus [-1] \oplus [-1] \oplus [-1],
\end{align*}
\]

\[
\begin{align*}
\Pi^L_{\text{reg}}(0, 2) &= \Pi^L_{\text{reg}}(1, 1) = \Pi^L_{\text{reg}}(2, 0) = \Pi^L_{\text{reg}}(3, 3) = \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = [1] \oplus [-1] \oplus [i] \oplus [-i],
\end{align*}
\]

\[
\begin{align*}
\Pi^L_{\text{reg}}(0, 3) &= \Pi^L_{\text{reg}}(1, 2) = \Pi^L_{\text{reg}}(2, 1) = \Pi^L_{\text{reg}}(3, 0) = \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = [1] \oplus [-1] \oplus [1] \oplus [1].
\end{align*}
\]

It is seen that, due to the fact that the ternary operation (11.7) is commutative, there are only one-dimensional irreducible left representations.

Let us “algebralize” the above regular representations in the following way. From (10.1) we have, for the left representation

\[ \Pi^L_{\text{reg}}(i, j) \circ \Pi^L_{\text{reg}}(k, l) = \Pi^L_{\text{reg}}(i, [jkl]) \],

where \([jkl] = j - k + l\), \(i, j, k, l \in \mathbb{Z}_3\). Denote \( \gamma^L_i = \Pi^L_{\text{reg}}(0, i), \ i \in \mathbb{Z}_3 \), then we obtain the algebra with the relations

\[ \gamma^L_i \gamma^L_j = \gamma^L_{i+j}. \] (11.9)

Conversely, any matrix representation of \( \gamma^L_i \gamma^L_j = \gamma^L_{i+j} \) leads to the left representation by \( \Pi^L(i, j) = \gamma^L_{j-i} \). In the case of the middle regular representation we introduce \( \gamma^M_k = \Pi^M_{\text{reg}}(k, l), \ k, l \in \mathbb{Z}_3 \), then we obtain

\[ \gamma^M_i \gamma^M_j = \gamma^M_{i+j}. \] (11.10)

In some sense (11.10) can be treated as a ternary analog of the Clifford algebra. As before, any matrix representation of (11.10) gives the middle representation \( \Pi^M(k, l) = \gamma_{k+l} \).
12. Ternary algebras and Hopf algebras

Let us consider associative ternary algebras [CARLSSON 1980], [DE AZCARRAGA AND IZQUIERDO 2010]. One can introduce an autodistributivity property \([xyz] \cdot ab = [xab] \cdot yab \cdot zab\) (see DUDEK 1993). If we take 2 ternary operations \(\{, , \}\) and \([, , ]\), then distributivity is given by \([xyz] \cdot ab = \{xab\} \cdot \{yab\} \cdot \{zab\}\). If \((+ )\) is a binary operation (addition), then left linearity is

\[\left( [x + z] \right) \cdot ab = [xab] + [zab]. \tag{12.1}\]

By analogy one can define central (middle) and right linearity. Linearity is defined, when left, middle and right linearity hold simultaneously.

**Definition 12-1.** An associative ternary algebra is a triple \((A, \mu_3, \eta^{(3)})\), where \(A\) is a linear space over a field \(\mathbb{K}\), \(\mu_3\) is a linear map \(A \otimes A \otimes A \rightarrow A\) called ternary multiplication \(\mu_3(a \otimes b \otimes c) = [abc]\) which is ternary associative \([abc] \cdot de = [a \cdot bcd] \cdot e = [ab] \cdot [cde]\) or

\[\mu_3 \circ (\mu_3 \otimes \text{id} \otimes \text{id}) = \mu_3 \circ (\text{id} \otimes \mu_3 \otimes \text{id}) = \mu_3 \circ (\text{id} \otimes \text{id} \otimes \mu_3). \tag{12.2}\]

There are two types [DUPLIJ 2001] of ternary unit maps \(\eta^{(3)} : \mathbb{K} \rightarrow A\):

1) One strong unit map

\[\mu_3 \circ (\eta^{(3)} \otimes \eta^{(3)} \otimes \text{id}) = \mu_3 \circ (\eta^{(3)} \otimes \text{id} \otimes \eta^{(3)}) = \mu_3 \circ (\text{id} \otimes \eta^{(3)} \otimes \eta^{(3)}) = \text{id}; \tag{12.3}\]

2) Two sequential units \(\eta^{(3)}_1\) and \(\eta^{(3)}_2\) satisfying

\[\mu_3 \circ \left(\eta^{(3)}_1 \otimes \eta^{(3)}_2 \otimes \text{id}\right) = \mu_3 \circ \left(\eta^{(3)}_1 \otimes \text{id} \otimes \eta^{(3)}_2\right) = \mu_3 \circ \left(\text{id} \otimes \eta^{(3)}_1 \otimes \eta^{(3)}_2\right) = \text{id}; \tag{12.4}\]

In first case the ternary analog of the binary relation \(\eta^{(2)}(x) = x1\), where \(x \in \mathbb{K}\), \(1 \in A\), is

\[\eta^{(3)}(x) = [x, 1, 1] = [1, 1, x] = [x, 1, 1]. \tag{12.5}\]

Let \((A, \mu_A, \eta_A), (B, \mu_B, \eta_B)\) and \((C, \mu_C, \eta_C)\) be ternary algebras, then the ternary tensor product space \(A \otimes B \otimes C\) is naturally endowed with the structure of an algebra. The multiplication \(\mu_{AB \otimes BC}\) on \(A \otimes B \otimes C\) reads

\[\left(a_1 \otimes b_1 \otimes c_1\right) \cdot \left(a_2 \otimes b_2 \otimes c_2\right) = [a_1 a_2 a_3] \otimes [b_1 b_2 b_3] \otimes [c_1 c_2 c_3], \tag{12.6}\]

and so the set of ternary algebras is closed under taking ternary tensor products. A ternary algebra map (homomorphism) is a linear map between ternary algebras \(f : A \rightarrow B\) which respects the ternary algebra structure

\[f([xyz]) = [f(x), f(y), f(z)], \tag{12.7}\]

\[f(1_A) = 1_B. \tag{12.8}\]

Let \(C\) be a linear space over a field \(\mathbb{K}\).

**Definition 12-2.** A ternary comultiplication \(\Delta^{(3)}\) is a linear map over a field \(\mathbb{K}\) such that

\[\Delta^{(3)} : C \rightarrow C \otimes C \otimes C. \tag{12.9}\]

In the standard Sweedler notations [SWEEDELE 1969] \(\Delta^{(3)}(a) = \sum \alpha_i \otimes a_i' \otimes a_i''\). Consider different possible types of ternary coassociativity [DUPLIJ 2001], [BOROWIEC ET AL. 2001].

1) A standard ternary coassociativity

\[(\Delta^{(3)} \otimes \text{id} \otimes \text{id}) \circ \Delta^{(3)} = (\text{id} \otimes \Delta^{(3)} \otimes \text{id}) \circ \Delta^{(3)} = (\text{id} \otimes \text{id} \otimes \Delta^{(3)}) \circ \Delta^{(3)}. \tag{12.10}\]
(2) A nonstandard ternary $\Sigma$-coassociativity (Gluskin-type positional operatives)
\[
(\Delta_3 \otimes \text{id} \otimes \text{id}) \circ \Delta_3 = (\text{id} \otimes (\sigma \circ \Delta_3) \otimes \text{id}) \circ \Delta_3,
\]
where $\sigma \circ \Delta_3(a) = \Delta_3(a) = a_{(\sigma(1))} \otimes a_{(\sigma(2))} \otimes a_{(\sigma(3))}$ and $\sigma \in \Sigma \subset S_3$.

(3) A permutational ternary coassociativity
\[
(\Delta_3 \otimes \text{id} \otimes \text{id}) \circ \Delta_3 = \pi \circ (\text{id} \otimes \Delta_3 \otimes \text{id}) \circ \Delta_3,
\]
where $\pi \in \Pi \subset S_5$.

A ternary comediaity is defined as
\[
(\Delta_3 \otimes \Delta_3 \otimes \Delta_3) \circ \Delta_3 = \sigma_{\text{medial}} \circ (\Delta_3 \otimes \Delta_3 \otimes \Delta_3) \circ \Delta_3,
\]
where $\sigma_{\text{medial}} = (123456789) \in S_9$. A ternary counit is defined as a map $\varepsilon(3) : C \to \mathbb{K}$. In general, $\varepsilon(3) \neq \varepsilon(2)$, satisfying one of the conditions below. If $\Delta_3$ is derived, then maybe $\varepsilon(3) = \varepsilon(2)$, but another counits may exist. There are two types of ternary counits:

1. Standard (strong) ternary counit
\[
(\varepsilon(3) \otimes \varepsilon(3) \otimes \text{id}) \circ \Delta_3 = (\varepsilon(3) \otimes \text{id} \otimes \varepsilon(3)) \circ \Delta_3 = (\text{id} \otimes \varepsilon(3) \otimes \varepsilon(3)) \circ \Delta_3 = \text{id}, \quad (12.11)
\]

2. Two sequential (polyadic) counits $\varepsilon_1(3)$ and $\varepsilon_2(3)$
\[
(\varepsilon_1(3) \otimes \varepsilon_2(3) \otimes \text{id}) \circ \Delta = (\varepsilon_1(3) \otimes \text{id} \otimes \varepsilon_2(3)) \circ \Delta = (\text{id} \otimes \varepsilon_1(3) \otimes \varepsilon_2(3)) \circ \Delta = \text{id}, \quad (12.12)
\]

Below we will consider only the first standard type of associativity (12.10). The $\sigma$-cocommutativity is defined as $\sigma \circ \Delta_3 = \Delta_3$.

**Definition 12.3.** A ternary coalgebra is a triple $(C, \Delta_3, \varepsilon(3))$, where $C$ is a linear space and $\Delta_3$ is a ternary comultiplication (12.9) which is coassociative in one of the above senses and $\varepsilon(3)$ is one of the above counits.

Let $(A, \mu(3))$ be a ternary algebra and $(C, \Delta_3)$ be a ternary coalgebra and $f, g, h \in \text{Hom}_\mathbb{K} (C, A)$. Ternary convolution product is
\[
[f, g, h]_* = \mu(3) \circ (f \otimes g \otimes h) \circ \Delta_3 \quad \text{(12.13)}
\]
or in the Sweedler notation $[f, g, h]_* (a) = \left[ f(a_{(1)}) g(a_{(2)}) h(a_{(3)}) \right]$.

**Definition 12.4.** A ternary coalgebra is called derived, if there exists a binary (usual, see e.g. Sweedler [1969]) coalgebra $\Delta_2 : C \to C \otimes C$ such that
\[
\Delta_3,\text{der} = (\text{id} \otimes \Delta_2) \otimes \Delta_2. \quad (12.14)
\]

**Definition 12.5.** A ternary bialgebra $B$ is $(B, \mu(3), \eta(3), \Delta_3, \varepsilon(3))$ for which $(B, \mu(3), \eta(3))$ is a ternary algebra and $(B, \Delta_3, \varepsilon(3))$ is a ternary coalgebra and they are compatible
\[
\Delta_3 \circ \mu(3) = \mu(3) \circ \Delta_3 \quad \text{(12.15)}
\]

One can distinguish four kinds of ternary bialgebras with respect to a “being derived” property:

1. A $\Delta$-derived ternary bialgebra
\[
\Delta_3 = \Delta_3,\text{der} = (\text{id} \otimes \Delta_2) \circ \Delta_2 \quad \text{(12.16)}
\]

2. A $\mu$-derived ternary bialgebra
\[
\mu^{(3)} = \mu^{(3)} \circ \Delta_2 = (\mu^{(2)} \otimes \text{id}) \quad \text{(12.17)}
\]

3. A derived ternary bialgebra is simultaneously $\mu$-derived and $\Delta$-derived ternary bialgebra.
A non-derived ternary bialgebra which does not satisfy (12.16) and (12.17).
Possible types of ternary antipodes can be defined by analogy with binary coalgebras.

**Definition 12-6.** A skew ternary antipode is

$$
\mu^{(3)} \circ (S_{skew}^{(3)} \otimes \text{id} \otimes \text{id}) \circ \Delta_3 = \mu^{(3)} \circ (\text{id} \otimes S_{skew}^{(3)} \otimes \text{id}) \circ \Delta_3 = \mu^{(3)} \circ (\text{id} \otimes \text{id} \otimes S_{skew}^{(3)}) \circ \Delta_3 = \text{id}. \quad (12.18)
$$

If only one equality from (12.18) is satisfied, the corresponding skew antipode is called left, middle or right.

**Definition 12-7.** Strong ternary antipode is

$$
\left( \mu^{(2)} \otimes \text{id} \right) \circ (\text{id} \otimes S_{strong}^{(3)} \otimes \text{id}) \circ \Delta_3 = 1 \otimes \text{id}, \quad \left( \text{id} \otimes \mu^{(2)} \right) \circ (\text{id} \otimes \text{id} \otimes S_{strong}^{(3)}) \circ \Delta_3 = \text{id} \otimes 1,
$$

where 1 is a unit of algebra.

If in a ternary coalgebra the relation

$$
\Delta_3 \circ S = \tau_{13} \circ (S \otimes S \otimes S) \circ \Delta_3 \quad (12.19)
$$

holds true, where \( \tau_{13} = \binom{123}{321} \), then it is called skew-involutive.

**Definition 12-8.** A ternary Hopf algebra \((H, \mu^{(3)}, \eta^{(3)}, \Delta_3, \varepsilon^{(3)}, S^{(3)})\) is a ternary bialgebra with a ternary antipode \(S^{(3)}\) of the corresponding above type.

Let us consider concrete constructions of ternary comultiplications, bialgebras and Hopf algebras. A ternary group-like element can be defined by \(\Delta_3 (g) = g \otimes g \otimes g\), and for 3 such elements we have

$$
\Delta_3 ([g_1 g_2 g_3]) = \Delta_3 (g_1) \Delta_3 (g_2) \Delta_3 (g_3). \quad (12.20)
$$

But an analog of the binary primitive element (satisfying \(\Delta^{(2)} (x) = x \otimes 1 + 1 \otimes x\)) cannot be chosen simply as \(\Delta_3 (x) = x \otimes e \otimes e + e \otimes x \otimes e + e \otimes e \otimes x\), since the algebra structure is not preserved. Nevertheless, if we introduce two idempotent units \(e_1, e_2\) satisfying “semiorthogonality” \([e_1 e_1 e_2] = 0\), \([e_2 e_2 e_1] = 0\), then

$$
\Delta_3 (x) = x \otimes e_1 \otimes e_2 + e_2 \otimes x \otimes e_1 + e_1 \otimes e_2 \otimes x, \quad (12.21)
$$

and now \(\Delta_3 ([x_1 x_2 x_3]) = [\Delta_3 (x_1) \Delta_3 (x_2) \Delta_3 (x_3)]\). Using \(\varepsilon (x) = 0\), \(\varepsilon (e_{1,2}) = 1\), and \(S^{(3)} (x) = -x, S^{(3)} (e_{1,2}) = e_{1,2}\), one can construct a ternary universal enveloping algebra in full analogy with the binary case (see e.g. [Kassel 1995]).

One of the most important examples of noncocommutative Hopf algebras is the well known Sweedler Hopf algebra [Sweedler 1969] which in the binary case has two generators \(x\) and \(y\) satisfying

$$
\mu^{(2)} (x, x) = 1, \quad (12.22)
$$

$$
\mu^{(2)} (y, y) = 0, \quad (12.23)
$$

$$
\sigma_{+}^{(2)} (xy) = -\sigma_{-}^{(2)} (xy). \quad (12.24)
$$

It has the following comultiplication

$$
\Delta_2 (x) = x \otimes x, \quad (12.25)
$$

$$
\Delta_2 (y) = y \otimes x + 1 \otimes y, \quad (12.26)
$$
counit $\varepsilon^{(2)}(x) = 1$, $\varepsilon^{(2)}(y) = 0$, and antipod $S^{(2)}(x) = x$, $S^{(2)}(y) = -y$, which respect the algebra structure. In the derived case a ternary Sweedler algebra is generated also by two generators $x$ and $y$ obeying [DUPLIJ 2001]

$$
\mu^{(3)}(x, e, x) = \mu^{(3)}(e, x, x) = \mu^{(3)}(x, x, e) = e,
$$
(12.27)

$$
\sigma^{(3)}_+(\{yey\}) = 0,
$$
(12.28)

$$
\sigma^{(3)}_-(\{xey\}) = -\sigma^{(3)}_-(\{xey\}).
$$
(12.29)

The derived Hopf algebra structure is given by

$$
\Delta_3(x) = x \otimes x \otimes x, 
$$
(12.30)

$$
\Delta_3(y) = y \otimes x \otimes x + e \otimes y \otimes x + e \otimes e \otimes y,
$$
(12.31)

$$
\varepsilon^{(3)}(x) = \varepsilon^{(2)}(x) = 1,
$$
(12.32)

$$
\varepsilon^{(3)}(y) = \varepsilon^{(2)}(y) = 0,
$$
(12.33)

$$
S^{(3)}(x) = S^{(2)}(x) = x,
$$
(12.34)

$$
S^{(3)}(y) = S^{(2)}(y) = -y,
$$
(12.35)

and it can be checked that (12.30)-(12.34) are algebra maps, while (12.34) are antialgebra maps. To obtain a non-derived ternary Sweedler example we have the following possibilities: 1) one “even” generator $x$, two “odd” generators $y_{1,2}$ and one ternary unit $e$; 2) two “even” generators $x_{1,2}$, one “odd” generator $y$ and two ternary units $e_{1,2}$. In the first case the ternary algebra structure is (no summation, $i = 1, 2$)

$$
[x_{i}x_{j}x_{k}] = e, 
$$
(12.36)

$$
[y_{i}y_{j}y_{k}] = 0,
$$
(12.37)

$$
\sigma^{(3)}_+(\{y_{i}x_{j}x_{k}\}) = \sigma^{(3)}_+(\{x_{i}y_{j}x_{k}\}) = 0, 
$$
(12.38)

$$
[x_{i}e_{j}x_{k}] = -[x_{i}y_{j}e_{k}],
$$
(12.39)

$$
[e_{i}y_{j}x_{k}] = -[y_{i}x_{j}e_{k}],
$$
(12.40)

$$
\sigma^{(3)}_+(\{y_{1}x_{1}x_{2}\}) = -\sigma^{(3)}_-(\{y_{1}x_{1}x_{2}\}).
$$
(12.41)

The corresponding ternary Hopf algebra structure is

$$
\Delta_3(x) = x \otimes x \otimes x, \Delta_3(y_{1,2}) = y_{1,2} \otimes x \otimes x + e_{1,2} \otimes y_{2,1} \otimes x + e_{1,2} \otimes e_{2,1} \otimes y_{2,1},
$$
(12.42)

$$
\varepsilon^{(3)}(x) = 1, \varepsilon^{(3)}(y_{i}) = 0,
$$
(12.43)

$$
S^{(3)}(x) = x, \quad S^{(3)}(y_{i}) = -y_{i}.
$$
(12.44)

In the second case we have for the algebra structure

$$
[x_{i}x_{j}x_{k}] = \delta_{ij}\delta_{jk}\delta_{ik}e_{i}, \quad [y_{i}y_{j}y_{k}] = 0,
$$
(12.45)

$$
\sigma^{(3)}_+(\{y_{i}x_{j}x_{k}\}) = 0, \quad \sigma^{(3)}_+(\{x_{i}y_{j}x_{k}\}) = 0,
$$
(12.46)

$$
\sigma^{(3)}_+(\{y_{1}x_{1}x_{2}\}) = 0, \quad \sigma^{(3)}_-(\{y_{1}x_{1}x_{2}\}) = 0,
$$
(12.47)
and the ternary Hopf algebra structure is \[ \Delta_3(x_i) = x_i \otimes x_i \otimes x_i, \]
\[ \Delta_3(y) = y \otimes x_1 \otimes x_1 + e_1 \otimes y \otimes x_2 + e_1 \otimes e_2 \otimes y, \] (12.48)
\[ \varepsilon^{(3)}(x_i) = 1, \] (12.49)
\[ \varepsilon^{(3)}(y) = 0, \] (12.50)
\[ S_3^{(3)}(x_i) = x_i, \] (12.51)
\[ S_3^{(3)}(y) = -y. \] (12.52)

13. Ternary Quantum Groups

A ternary commutator can be obtained in different ways. Let us consider the simplest version called a Nambu bracket (see e.g. [2001]). We will consider the simplest version called a Nambu bracket (see e.g. [2001]). Let us introduce two maps \( \omega^{(3)}: A \otimes A \otimes A \to A \otimes A \otimes A \) by

\[ \omega_+^{(3)}(a \otimes b \otimes c) = a \otimes b \otimes c + b \otimes c \otimes a + c \otimes a \otimes b, \] (13.1)
\[ \omega_-^{(3)}(a \otimes b \otimes c) = b \otimes a \otimes c + c \otimes b \otimes a + a \otimes c \otimes b. \] (13.2)

Thus, obviously \( \mu^{(3)} \circ \omega^{(3)} = \sigma^{(3)} \circ \mu^{(3)} \), where \( \sigma^{(3)} \in S_3 \) denotes a sum of terms having even and odd permutations respectively. In the binary case \( \omega_+^{(2)} = \text{id} \otimes \text{id} \) and \( \omega_-^{(2)} = \tau \) is the twist operator \( \tau: a \otimes b \rightarrow b \otimes a \), while \( \mu^{(2)} \circ \omega^{(2)} \) is permutation \( \sigma^{(2)}(ab) = ba \). So the Nambu product is \( \omega^{(3)} = \omega_+^{(3)} - \omega_-^{(3)} \), and the ternary commutator is \( [\cdot, \cdot, \cdot] = \sigma^{(3)} - \sigma^{(3)}, \) or [2001].

\[ [a, b, c]_N = [abc] + [bca] + [cab] - [cba] - [acb] - [bac] \] (13.3)

An abelian ternary algebra is defined by the vanishing of the Nambu bracket \( [a, b, c]_N = 0 \) or ternary commutation relation \( \sigma_+^{(3)} = \sigma_-^{(3)} \). By analogy with the binary case a deformed ternary algebra can be defined by

\[ \sigma_+^{(3)} = q \sigma_-^{(3)} \text{ or } [abc] + [bca] + [cab] = q ([cba] + [acb] + [bac]), \] (13.4)

where multiplication by \( q \) is treated as an external operation.

Let us consider a ternary analog of the Woronowicz example of a bialgebra construction, which in the binary case has two generators satisfying \( xy = qyx \) (or \( \sigma_+^{(2)}(xy) = q \sigma_-^{(2)}(xy) \)), then the following coproducts

\[ \Delta_2(x) = x \otimes x \] (13.5)
\[ \Delta_2(y) = y \otimes x + 1 \otimes y \] (13.6)

are algebra maps. In the derived ternary case using (13.4) we have

\[ \sigma_+^{(3)}([xy]) = q \sigma_-^{(3)}([xy]), \] (13.7)

where \( e \) is the ternary unit and ternary coproducts are

\[ \Delta_3(e) = e \otimes e \otimes e, \] (13.8)
\[ \Delta_3(x) = x \otimes x \otimes x, \] (13.9)
\[ \Delta_3(y) = y \otimes x \otimes x + e \otimes y \otimes x + e \otimes e \otimes y, \] (13.10)
which are ternary algebra maps, i.e. they satisfy
\[
\sigma_+^{(3)} ([\Delta_3 (x) \Delta_3 (e) \Delta_3 (y)]) = q \sigma_-^{(3)} ([\Delta_3 (x) \Delta_3 (e) \Delta_3 (y)]).
\] (13.11)

Let us consider the group \( G = SL(n, \mathbb{K}) \). Then the algebra generated by \( a_i^j \in SL(n, \mathbb{K}) \) can be endowed with the structure of a ternary Hopf algebra (see, e.g., [Madore 1995] for the binary case) by choosing the ternary coproduct, counit and antipode as (here summation is implied)
\[
\Delta_3 (a_i^j) = a_i^k \otimes a_i^l \otimes a_i^j = a_i^j \otimes a_i^k \otimes a_i^l,
\] (13.12)
This antipode is a skew one since from (12.18) it follows that
\[
\mu^{(3)} \circ (S^{(3)} \otimes id \otimes id) \circ \Delta_3 (a_i^j) = S^{(3)} (a_i^j) a_i^k a_i^j = (a_i^j)^{(-1)} a_i^k a_i^j = \delta_i^j a^i = a_i^j.
\] (13.13)

This ternary Hopf algebra is derived since for \( \Delta^{(2)} = a_j^i \otimes a_k^i \) we have
\[
\Delta_3 = (id \otimes \Delta^{(2)}) \otimes \Delta^{(2)} (a_i^j) = (id \otimes \Delta^{(2)}) (a_i^k \otimes a_i^j) = a_i^j \otimes \Delta^{(2)} (a_i^j) = a_i^j \otimes a_i^k \otimes a_i^j.
\] (13.14)

In the most important case \( n = 2 \) we can obtain the manifest action of the ternary coproduct \( \Delta_3 \) on components. Possible non-derived matrix representations of the ternary product can be done only by four-rank \( n \times n \times n \times n \) twice covariant and twice contravariant tensors \{\( a_{ij}^{kl} \}\}. Among all products the non-derived ones are only the following: \( a_{ij}^{kl} b_{ik}^{mn} c_{jk}^{mn} \) and \( a_{ij}^{kl} b_{jk}^{mn} c_{ik}^{mn} \) (where \( o \) is any index). So using e.g. the first choice we can define the non-derived Hopf algebra structure by
\[
\Delta_3 (a^{kl}_{ij}) = a_{ij}^{\mu \rho} \otimes a_{ij}^{\nu \sigma} \otimes a_{ij}^{\sigma \rho} = \frac{1}{2} (\delta_i^j \delta_k^l + \delta_i^l \delta_k^j).
\] (13.15)

and the skew antipod \( s^{ij}_{kl} = S^{(3)} (a_{ij}^{kl}) \) which is a solution of the equation \( s_{ij}^{\mu \rho} a_{ij}^{\nu \sigma} = \delta^\rho_j \delta_i^\mu \delta^\sigma_l \).

Next consider ternary dual pair \( k^* (G) \) (push-forward) and \( \mathcal{F} (G) \) (pull-back) which are related by \( k^* (G) \cong \mathcal{F} (G) \) (see e.g. [Kogorodski and Soibelman 1998]). Here \( k^* (G) = \text{span} (G) \) is a ternary group algebra (\( G \) has a ternary product \([\ ]_G \) or \( \mu^{(3)}_G \) ) over a field \( k \). If \( u \in k^* (G) \) then \( k (G) \) becomes a ternary algebra. Define a ternary coproduct \( \Delta_3 : k (G) \otimes k (G) \otimes k (G) \) by
\[
\Delta_3 (u) = u_i x_i \otimes x_i \otimes x_j
\] (13.16)
(derived and associative), then \( \Delta_3 ([uwv]_k) = [\Delta_3 (u) \Delta_3 (v) \Delta_3 (w)]_k \), and \( k (G) \) is a ternary bialgebra. If we define a ternary antipod by \( S^{(3)}_k = u_i \bar{x}_i \), where \( \bar{x}_i \) is a skew element of \( x_i \), then \( k (G) \) becomes a ternary Hopf algebra.

In the dual case of functions \( \mathcal{F} (G) : \{ \varphi : G \to k \} \) a ternary product \([\ ]_\mathcal{F} \) or \( \mu^{(3)}_\mathcal{F} \) (derived and associative) acts on \( \psi (x, y, z) \) as
\[
\left( \mu^{(3)}_\mathcal{F} \psi \right) (x, y, z) = \psi (x, x, x),
\] (13.19)
and so \( \mathcal{F} (G) \) is a ternary algebra. Let \( \mathcal{F} (G) \ctimes \mathcal{F} (G) \ctimes \mathcal{F} (G) \cong \mathcal{F} (G \times G \times G) \), then we define a ternary coproduct \( \Delta_3 : \mathcal{F} (G) \ctimes \mathcal{F} (G) \ctimes \mathcal{F} (G) \) as
\[
\left( \Delta_3 \varphi \right) (x, y, z) = \varphi ([xyz]_\mathcal{F}),
\] (13.20)
which is derive and associative. Thus we can obtain \( \Delta_3 \left( [\varphi_1 \varphi_2 \varphi_3]_F \right) = [\Delta_3 (\varphi_1) \Delta_3 (\varphi_2) \Delta_3 (\varphi_3)]_F \), and therefore \( \mathcal{F}(G) \) is a ternary bialgebra. If we define a ternary antipod by

\[
S^{(3)}_F (\varphi) = \varphi (\bar{x}),
\]

(13.21)

where \( \bar{x} \) is a skew element of \( x \), then \( \mathcal{F}(G) \) becomes a ternary Hopf algebra.

Let us introduce a ternary analog of the \( R \)-matrix [DUPLIJ 2001]. For a ternary Hopf algebra \( H \) we consider a linear map \( R^{(3)} : H \otimes H \otimes H \rightarrow H \otimes H \otimes H \).

**Definition 13-1.** A ternary Hopf algebra \( (H, \mu^{(3)}, \eta^{(3)}, \Delta_3, \varepsilon^{(3)}, S^{(3)}) \) is called quasifiveangular\(^4\) if it satisfies

\[
\begin{align*}
(\Delta_3 \otimes \text{id} \otimes \text{id}) &= R^{(3)}_{145} R^{(3)}_{245} R^{(3)}_{345}, \\
(\text{id} \otimes \Delta_3 \otimes \text{id}) &= R^{(3)}_{125} R^{(3)}_{145} R^{(3)}_{135}, \\
(\text{id} \otimes \text{id} \otimes \Delta_3) &= R^{(3)}_{125} R^{(3)}_{124} R^{(3)}_{123},
\end{align*}
\]

(13.22) (13.23) (13.24)

where as usual the index of \( R \) denotes action component positions.

Using the standard procedure (see, e.g., [KASSEL 1995], [CHARI AND PRESSLEY 1996], [MAJID 1995]) we obtain a set of abstract ternary quantum Yang-Baxter equations, one of which has the form [DUPLIJ 2001]

\[
R^{(3)}_{243} R^{(3)}_{342} R^{(3)}_{125} R^{(3)}_{145} R^{(3)}_{135} = R^{(3)}_{123} R^{(3)}_{132} R^{(3)}_{145} R^{(3)}_{245} R^{(3)}_{345},
\]

(13.25)

and others can be obtained by corresponding permutations. The classical ternary Yang-Baxter equations form a one parameter family of solutions \( R(t) \) can be obtained by the expansion

\[
R^{(3)} (t) = e \otimes e \otimes e + rt + \mathcal{O} (t^2),
\]

(13.26)

where \( r \) is a ternary classical \( R \)-matrix, then e.g. for (13.25) we have

\[
\begin{align*}
& r_{342} r_{125} r_{145} r_{135} + r_{243} r_{125} r_{145} r_{135} + r_{243} r_{342} r_{125} r_{135} + r_{243} r_{342} r_{125} r_{145} \\
& = r_{132} r_{145} r_{245} r_{345} + r_{123} r_{145} r_{245} r_{345} + r_{123} r_{132} r_{245} r_{345} + r_{123} r_{132} r_{145} r_{345} + r_{123} r_{132} r_{145} r_{245}.
\end{align*}
\]

For three ternary Hopf algebras \( \left( H_{A,B,C}, \mu^{A,B,C}, \eta^{A,B,C}, \Delta^{A,B,C}, \varepsilon^{A,B,C}, S^{A,B,C} \right) \) we can introduce a non-degenerate ternary “pairing” (see, e.g., [CHARI AND PRESSLEY 1996] for the binary case) \( \langle \cdot, \cdot \rangle^{(3)} : H_A \times H_B \times H_C \rightarrow \mathbb{K}, \) trilinear over \( \mathbb{K} \), satisfying [DUPLIJ 2001]

\[
\begin{align*}
\langle \eta^{(3)}_A (a) , b, c \rangle^{(3)} &= \langle a, \varepsilon^{(3)}_B (b), c \rangle^{(3)}, & \langle a, \eta^{(3)}_B (b) , c \rangle^{(3)} &= \langle \varepsilon^{(3)}_A (a) , b, c \rangle^{(3)}, \\
\langle b, \eta^{(3)}_B (b) , c \rangle^{(3)} &= \langle a, b, \varepsilon^{(3)}_C (c) \rangle^{(3)}, & \langle a, b, \eta^{(3)}_C (c) \rangle^{(3)} &= \langle a, \varepsilon^{(3)}_B (b) , c \rangle^{(3)}, \\
\langle a, b, \eta^{(3)}_C (c) \rangle^{(3)} &= \langle \varepsilon^{(3)}_A (a) , b, c \rangle^{(3)}, & \langle \eta^{(3)}_A (a) , b, c \rangle^{(3)} &= \langle a, b, \varepsilon^{(3)}_C (c) \rangle^{(3)},
\end{align*}
\]

\(^4\) The reason for such notation is clear from (13.25).
\[ \left\langle \mu^{(3)}_A (a_1 \otimes a_2 \otimes a_3), b, c \right\rangle^{(3)} = \left\langle a_1 \otimes a_2 \otimes a_3, \Delta_B^{(3)} (b), c \right\rangle^{(3)}, \]
\[ \left\langle \Delta_A^{(3)} (a), b_1 \otimes b_2 \otimes b_3, c \right\rangle^{(3)} = \left\langle a, \mu_B^{(3)} (b_1 \otimes b_2 \otimes b_3), c \right\rangle^{(3)}, \]
\[ \left\langle a, \mu_B^{(3)} (b_1 \otimes b_2 \otimes b_3), c \right\rangle^{(3)} = \left\langle a, b_1 \otimes b_2 \otimes b_3, \Delta_C^{(3)} (c) \right\rangle^{(3)}, \]
\[ \left\langle a, \Delta_B^{(3)} (b), c_1 \otimes c_2 \otimes c_3 \right\rangle^{(3)} = \left\langle a, b, \mu_C^{(3)} (c_1 \otimes c_2 \otimes c_3) \right\rangle^{(3)}, \]
\[ \left\langle a, b, \mu_C^{(3)} (c_1 \otimes c_2 \otimes c_3) \right\rangle^{(3)} = \left\langle \Delta_A^{(3)} (a), b, c_1 \otimes c_2 \otimes c_3 \right\rangle^{(3)}, \]
\[ \left\langle a_1 \otimes a_2 \otimes a_3, b, \Delta_C^{(3)} (c) \right\rangle^{(3)} = \left\langle \mu_A^{(3)} (a_1 \otimes a_2 \otimes a_3), b, c \right\rangle^{(3)}, \]
\[ \left\langle \mu_A^{(3)} (a_1 \otimes a_2 \otimes a_3), b, c \right\rangle^{(3)} = \left\langle a, \mu_A^{(3)} (b_1 \otimes b_2 \otimes b_3), \Delta_B^{(3)} (c) \right\rangle^{(3)}, \]
\[ \left\langle a, \mu_A^{(3)} (b_1 \otimes b_2 \otimes b_3), \Delta_B^{(3)} (c) \right\rangle^{(3)} = \left\langle a, b, \mu_C^{(3)} (c_1 \otimes c_2 \otimes c_3) \right\rangle^{(3)}, \]
\[ \left\langle a, b, \mu_C^{(3)} (c_1 \otimes c_2 \otimes c_3) \right\rangle^{(3)} = \left\langle \Delta_A^{(3)} (a), b, c_1 \otimes c_2 \otimes c_3 \right\rangle^{(3)}, \]
\[ \left\langle \Delta_A^{(3)} (a), b, c_1 \otimes c_2 \otimes c_3 \right\rangle^{(3)} = \left\langle a_1 \otimes a_2 \otimes a_3, b, \Delta_C^{(3)} (c) \right\rangle^{(3)} = \left\langle \mu_A^{(3)} (a_1 \otimes a_2 \otimes a_3), b, c \right\rangle^{(3)}, \]

where \( a, a_i \in H_A, b, b_i \in H_B \). The ternary “paring” between \( H_A \otimes H_A \otimes H_A \) and \( H_B \otimes H_B \otimes H_B \) is given by \( \left\langle a_1 \otimes a_2 \otimes a_3, b_1 \otimes b_2 \otimes b_3 \right\rangle^{(3)} = \left\langle a_1, b_1 \right\rangle^{(3)} \left\langle a_2, b_2 \right\rangle^{(3)} \left\langle a_3, b_3 \right\rangle^{(3)} \). These constructions can naturally lead to ternary generalizations of the duality concept and the quantum double which are key ingredients in the theory of quantum groups. \[ \text{Drinfeld} \ [1987], \text{Kassel} \ [1995], \text{Majid} \ [1995]. \]

14. Conclusions

In this paper we presented a review of polyadic systems and their representations, ternary algebras and Hopf algebras. We have classified general polyadic systems and considered their homomorphisms and their multiplace generalizations, paying attention to their associativity. Then, we defined multiplace representations and multiactions and have given examples of matrix representations for some ternary groups. We defined and investigated ternary algebras and Hopf algebras, and have given some examples. We then considered some ternary generalizations of quantum groups and the Yang-Baxter equation.

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