Nonlinear localized modes in $\mathcal{PT}$-symmetric Rosen-Morse potential well

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We report the existence and properties of localized modes described by nonlinear Schrödinger equation with complex $\mathcal{PT}$-symmetric Rosen-Morse potential well. Exact analytical expressions of the localized modes are found in both one dimensional and two-dimensional geometry with self-focusing and self-defocusing Kerr nonlinearity. Linear stability analysis reveals that these localized modes are unstable for all real values of the potential parameters although corresponding linear Schrödinger eigenvalue problem possesses unbroken $\mathcal{PT}$-symmetry. This result has been verified by the direct numerical simulation of the governing equation. The transverse power flow density associated with these localized modes has also been examined.

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I. INTRODUCTION

Recently, there has been considerable amount of attention to theoretical and experimental investigation of light propagation in parity-time ($\mathcal{PT}$) symmetric optical media [1–14]. The interest in study of such $\mathcal{PT}$-symmetric optical materials has its roots in quantum mechanics: the paraxial equation of diffraction is mathematically equivalent to that of quantum Schrödinger equation. Quantum mechanics requires that the spectrum of every physical observable should be real, which of course are satisfied by Hermitian operators. However, Bender and Boettcher [15] pointed out that some non-Hermitian Hamiltonians with $\mathcal{PT}$-symmetry can also exhibit an entirely real spectrum and may constitute unitary quantum systems without violating any of the axioms of quantum mechanics. Moreover, it has been shown that for a $\mathcal{PT}$-symmetric complex Hamiltonian, there may exist a threshold above which its eigenvalues are not real but become complex, and the system undergoes a phase transition because of spontaneous $\mathcal{PT}$-symmetry breaking. In general the action of the parity $\hat{P}$ and time $\hat{T}$ operators is defined as $\hat{p} \rightarrow -\hat{p}$, $\hat{x} \rightarrow -\hat{x}$ and $\hat{p} \rightarrow -\hat{p}$, $\hat{x} \rightarrow \hat{x}$, $\hat{i} \rightarrow -\hat{i}$, respectively. A Hamiltonian with a complex $\mathcal{PT}$-symmetric potential requires that the real part of the potential must be even function of position and the imaginary part should be odd [16]. In optics such complex $\mathcal{PT}$-symmetric structure can be designed through a judicious designs that involve both optical gain/loss regions and the process of index guiding [4–6]. In such settings, complex refractive index distribution $n(x) = n_R(x) + i n_I(x)$ plays the role of an optical potential so that the index guiding $n_R(x)$ and the gain/loss profile $n_I(x)$ satisfy $n_R(x) = n_R(-x)$ and $n_I(x) = -n_I(-x)$, respectively. Unusual exotic phenomena like $\mathcal{PT}$ phase transition, band merging, double refraction, non-reciprocity [5, 20, 21], and unidirectional invisibility [17–19] etc have been reported to exist in.

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linear $\mathcal{PT}$-symmetric complex optical media. Spontaneous $\mathcal{PT}$-symmetry breaking has been experimentally observed in active or passive $\mathcal{PT}$ dimers \cite{2,4} and periodic lattices \cite{1}. These findings, in turn, have stimulated considerable research activity in the non-linear $\mathcal{PT}$-symmetric systems as well.

In the nonlinear domain, a novel class of one and two dimensional localized modes were found to exist below and above the phase transition point \cite{22} and the interplay between the Kerr nonlinearity and the $\mathcal{PT}$ threshold was investigated \cite{6}. Subsequently, nonlinear modes are studied in complex $\mathcal{PT}$-symmetric periodic \cite{27,32}, Gaussian \cite{33}, Bessel \cite{23}, Scarf-II \cite{31}, and harmonic \cite{35} potentials, as well as in a harmonic trap with a rapidly decaying $\mathcal{PT}$-symmetric imaginary component \cite{34}. Stable localized modes in a $\mathcal{PT}$-symmetric slab waveguide with distributed gain and loss are found in \cite{36}. Existence of optical solitons in $\mathcal{PT}$-symmetric nonlinear couplers with gain/loss \cite{37,38}, gap solitons in $\mathcal{PT}$-symmetric optical lattices \cite{39} and optical defect modes in $\mathcal{PT}$-symmetric potentials \cite{40} are also reported. Stable 1D and 2D bright spatial solitons are found to exist in defocusing Kerr media with $\mathcal{PT}$-symmetric Scarf II potentials \cite{24}. Also, it has been found that the gray solitons in $\mathcal{PT}$-symmetric potentials can be stable \cite{41}. However the existence of nonlinear localized modes in yet another important potential e.g. complex $\mathcal{PT}$-symmetric Rosen-Morse well has not been reported so far. The complex Rosen-Morse potential well is characterized by the same real component as the complex Scarf-II potential, however, its imaginary component is different. In fact, in contrast with the real component, the imaginary potential component doesn’t vanish asymptotically, rather it tends to a finite value. This is the reason why the phenomenon of spontaneous breakdown of $\mathcal{PT}$-symmetry is elusive in such system \cite{25,26}. Nevertheless, the bound state energy eigenvalues of $\mathcal{PT}$-symmetric Rosen-Morse potential well undergoes a shift from negative to positive domain for certain range of parameters which controls the strength of the potential.

In this paper, we investigate the propagation of nonlinear beam in a single $\mathcal{PT}$ waveguide cell which is characterized by the nonlinear Schrödinger equation with complex Rosen-Morse potential well. Specifically, the existence of the spatial localized modes have been reported in both one-dimensional and two-dimensional settings with self-focusing and self-defocusing Kerr nonlinearity. We have shown, with the help of linear stability analysis of the one-dimensional localized modes, that though the spontaneous breakdown of $\mathcal{PT}$-symmetry does not occur in complex Rosen-Morse well the localized modes corresponding to nonlinear Schrödinger equation are always unstable. This linear instability has been verified by direct numerical simulation of the governing equation. The transverse power flow density associated with these nonlinear localized modes has also been examined.

II. LOCALIZED MODES IN $\mathcal{PT}$-SYMMETRIC COMPLEX ROSEN-MORSE WELL

A. Mathematical Model

We consider optical wave propagation in a Kerr nonlinear $\mathcal{PT}$-symmetric potential. In this case, $(1+1)$-dimensional optical beam propagation along longitudinal $z$ direction is governed by the following non-linear Schrödinger like equation \cite{6,10}:

\[
i \frac{\partial \Psi}{\partial z} + \frac{\partial^2 \Psi}{\partial x^2} + \left[V(x) + iW(x)\right]\Psi + \sigma|\Psi|^2\Psi = 0.\]  \( (2.1) \)
Here \( \Psi(x, z) \) is slowly varying complex electric field envelop, \( x \) is the transverse co-ordinate, and \( \sigma = \pm 1 \) represent the self-focusing and self-defocusing nonlinearity respectively. \( V(x) \) and \( W(x) \) are the real and imaginary parts of the complex PT-symmetric potential such that \( V(-x) = V(x) \) and \( W(-x) = -W(x) \). Physically, \( V(x) \) is responsible for the bending and slowing down of light, and \( W(x) \) can lead to either amplification (gain) or absorption (loss) of light within an optical material.

The optical beam propagation in a single PT cell is important to understand light self-trapping in complex optical lattices. In order to investigate the optical beam propagation in a single PT cell, we consider the complex PT-symmetric Rosen-Morse potential well as

\[
V(x) = -a(a + 1) \text{sech}^2 x, \\
W(x) = 2b \tanh x,
\]

(2.2)

where \( a \) and \( b \) characterizes the strength of the real and imaginary parts of the potential, respectively. Both the real and imaginary part of this potential are shown in figure 1b for the potential parameters \( a = .75 \) and \( b = .8 \). The linear Schrödinger eigenvalue problem for the potential (2.2) has been thoroughly studied in [25, 26]. It has been shown that all the bound state energy eigenvalues corresponding to the linear Schrödinger equation of complex Rosen-Morse well are real so that the spontaneous breakdown of PT-symmetry never occur. However, the energy eigenvalues

\[
\lambda_n = -(a - n)^2 + \frac{b^2}{(a - n)^2}, \quad n = 0, 1, ..., n_{\text{max}} < a.
\]

(2.3)

begin to shift from negative to the positive domain when the strength of the non-Hermiticity is increased. In fact all the energy eigenvalues become positive whenever \( \sqrt{|b|} > a^2 \). Here, we search for stationary solution of the nonlinear equation (2.1) in the form \( \Psi(x, z) = \phi(x) e^{i\lambda z} \), where \( \lambda \) is the real propagation constant, and the complex function \( \phi(x) \) satisfies following equation

\[
\frac{d^2 \phi}{dx^2} - [a(a + 1) \text{sech}^2 x - 2ib \tanh x] \phi + \sigma |\phi|^2 \phi = \lambda \phi.
\]

(2.4)

In the following we report the existence and linear stability of the localized modes of the above nonlinear equation (2.4) for both the self-focusing and self-defocusing cases.

B. Analytical solutions and their linear stability

1. Self-focusing case (\( \sigma = 1 \))

For \( \sigma = 1 \), equation (2.4) is found to admit an exact analytical expression of the localized mode of the form

\[
\phi(x) = \sqrt{a^2 + a + 2} \ \text{sech} x \ e^{ibx},
\]

(2.5)

where \( \lambda = 1 - b^2 \). In figure 1b, the real and imaginary parts of this spatial soliton have been shown for \( a = .75 \) and \( b = .8 \) and \( \lambda = .36 \). To focus on the properties of this non-linear solution, we examine following three quantities: the transverse power flow density (Poynting vector) \( S \) across the beam, the power \( P \), and the linear stability of these localized modes. For the nonlinear modes given in equation (2.5), the Poynting vector
FIG. 1: (Color online) (a) Real and Imaginary parts of the Rosen-Morse Potential Well; (b) Real and Imaginary parts of the localized modes $\phi(x)$ in the self-focusing medium; (c) The transverse power (Poynting vector) $S(x)$; (d) Power $P(a)$ as a function of potential parameter $a$. In (a), (b) and (c) we have considered $a = 0.75, b = 0.8, \lambda = 0.36$ and $\sigma = 1$.

$S = \frac{i}{2}(\phi_x^2 - \phi^2) = b(a^2 + a + 2) \text{sech}^2 x$ depends on the sign of the strength, $b$, of the imaginary part of the potential. It may be both negative and positive for negative and positive values of $b$ respectively. However, we consider only positive values of $b$ in which case $S$ is everywhere positive and the power flow in the $\mathcal{PT}$ cell is in one direction, i.e. from the gain towards loss domain. For $a = 0.75, b = 0.8$, the transverse power flow is shown in figure 1(c). For the localized modes (2.5) the power $P$ is calculated as

$$P(a, b) = \int_{-\infty}^{\infty} |\phi(x)|^2 dx = 2(a^2 + a + 2).$$

Clearly, the power is independent of the parameter $b$. In figure 1(d) we have shown the power $P$ as a function of the real potential strength parameter $a$. It remains positive for all values of the parameter $a$. The power increases with the increasing absolute value of the parameter $a$ and becomes minimum for $a = -0.5$. At $a = -0.5$, the amplitude of the localized modes is also minimum.

In order to determine the linear stability properties of the self-trapped localized modes obtained here, we consider small perturbation to the solution $\Psi(x, z)$, of the form

$$\Psi(x, z) = \phi(x)e^{i\lambda z} + \left\{ [f(x) + g(x)] e^{\eta z} + [f^*(x) - g^*(x)] e^{\eta^* z} \right\} e^{i\lambda z}$$

(2.7)

where $f(x)$ and $g(x)$ are infinitesimal perturbation eigen-functions such that $|f|, |g| \ll |\phi|$, $\eta$ stands for the perturbation growth rate. By linearizing the equation (2.1) around the localized solution $\phi(x)$, we find that the functions $f$ and $g$ satisfy the following eigenvalue problem

$$\begin{pmatrix} 0 & \hat{\mathcal{L}}_1 \\ \hat{\mathcal{L}}_2 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = -i\eta \begin{pmatrix} f \\ g \end{pmatrix}$$

(2.8)

where $\hat{\mathcal{L}}_1 = \partial_{xx} + (V + iW) + \sigma|\phi|^2 - \lambda$ and $\hat{\mathcal{L}}_2 = \partial_{xx} + (V + iW) + 3\sigma|\phi|^2 - \lambda$. The linear stability of the localized modes $\phi(x)$ depends on the nature of the eigenvalue $\eta$. The $\mathcal{PT}$-symmetric nonlinear localized mode is unstable if $\eta$ has any positive real part, because for $\Re(\eta) > 0$ the corresponding perturbed nonlinear eigenmodes (2.7) would grow exponentially with $z$. The eigenvalues $\eta$ can be obtained by solving equation (2.8) with the help of several numerically techniques [29]. In this paper we have used Fourier collocation method.
Our numerical investigations corresponding to the nature of the eigenvalue $\eta$ reveal that $\eta$ never becomes purely imaginary. It has always non vanishing positive real part for all real values of potential parameters $a$ and $b$. This implies that the nonlinear localized modes obtained here are always unstable. The results of linear stability analysis are corroborated by direct numerical simulations of Eq. (2.1) using the solution (2.5) as initial condition i.e. $\Psi(x,0) = \phi(x)$. In figure 2, we have shown the localized modes $\phi(x)$, unstable intensity profiles $|\Psi(x,z)|^2$ and corresponding linear stability spectra for very small and large values of the parameter $b$. As expected, localized modes which are predicted to be unstable fail to maintain their original shapes. The reason behind such instability of the localized modes is that unlike the real part, the imaginary part of the Rosen-Morse potential well does not vanish asymptotically. Therefore gain/loss remains in the system even far from the place of localization and any small fluctuations of the field is amplified/absorbed, eventually leading to instability.

![Image 1](image1.png)

**FIG. 2:** (Color online) (a) Plots of the real (solid blue curve) and imaginary (dotted-dashed red curve) parts of $\phi(x)$ and $|\phi(x)|^2$ (solid black curve) in self focusing medium; (b) The evolution of field intensity $|\Psi(x,z)|^2$; (c) Numerically computed stability spectra corresponding to the figure 2(a). In all these cases we have considered $a = .1, b = .03, \lambda = .999$ and $\sigma = 1$. (d), (e) and (f) Plots of the same quantities as in figure 2(a), (b) and (c) respectively, for $a = .1, b = 3, \lambda = −8, \sigma = 1$.

It is worth mentioning here that unlike the $PT$-symmetric Scarf II potential (for which the solitons are stable below the certain critical value of the imaginary potential component and become unstable above this critical value), localized modes in the complex Rosen-Morse potential well, discussed here, are linearly unstable for all real values of non-Hermiticity parameter $b$. This result is valid in spite of the fact that the
Hamiltonian corresponding to the linearized version of equation (2.4) possesses unbroken \( PT \)-symmetry (all energy eigenvalues are real). Nevertheless, the parameters range \( \sqrt{|b|} > a^2 \), for which the transition from the negative to positive energy corresponding to the linear Schrödinger eigenvalue problem takes place, does not affect the instability of the localized modes. Only the magnitude of the localized modes differs.

2. Self-defocusing case \( (\sigma = -1) \)

Optical beam propagation in nonlinear self-defocusing Kerr medium is governed by the equation (2.1) for \( \sigma = -1 \) and its corresponding stationary solutions satisfy equation (2.4). Like the self-focusing case, here equation (2.4) admits an exact solution \( \phi(x) = \sqrt{-(a^2 + a + 2)} \text{sech} x e^{ibx} \). Note that these non-linear modes in self-defocusing case are very similar to those in the self-focusing case. Only the amplitude of the localized modes are different. Nevertheless, like the self-focusing case here also the localized modes are unstable for all \( a, b \in \mathbb{R} \). In figures 3(a), and 3(b), we have shown such nonlinear modes and corresponding unstable intensity evolution for the parameter values \( a = 1, b = 4, \lambda = .84 \) and \( \sigma = -1 \). Numerical solution of the eigenvalue problem (2.8) has been plotted in figure 3(c) which also implies that the the corresponding modes are linearly unstable.

III. LOCALIZED MODES IN TWO-DIMENSIONS

Finally, we discuss the formation of nonlinear localized modes in two-dimensional Rosen-Morse potential. The two-dimensional generalization of the equation (2.1), with the self-focusing nonlinearity, is given by [22]

\[
\frac{\partial \Psi}{\partial z} + \nabla^2 \Psi + [V(x, y) + iW(x, y)]\Psi + |\Psi|^2 \Psi = 0,
\]

where \( \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the two-dimensional Laplacian. The two-dimensional complex Rosen-Morse potential well, which obey the \( PT \)-symmetric requirements \( V(-x, -y) = V(x, y) \) and \( W(-x, -y) = -W(x, y) \), can be

FIG. 3: (Color online) (a) Plots of the real (solid blue curve) and imaginary (dotted-dashed red curve) parts of the localized modes \( \phi(x) \) in self-defocusing media; (b) Unstable intensity evolution \( |\Psi(x, z)|^2 \) corresponding to the figure 3a; (c) Plot of the corresponding stability spectra obtained numerically. In all these cases we have considered potential parameters \( a = 1, b = 4, \lambda = .84 \) and \( \sigma = -1 \).
considered as
\[
V(x, y) = 2(\text{sech}^2 x + \text{sech}^2 y) - (a^2 + a + 2) \text{sech}^2 x \text{ sech}^2 y
\]
\[
W(x, y) = 4b(\tanh x + \tanh y).
\]  
(3.2)

The stationary solutions of the equation (3.1) can be assumed in the form
\[
Ψ(x, y, z) = φ(x, y) e^{iλz+iθ(x,y)}
\]  
(3.3)

where φ(x, y) and the phase θ(x, y) satisfy the following two equations
\[
∇^2φ - |∇θ|^2φ + V(x, y)φ + φ^3 = λφ,
\]
\[
φ∇^2θ + 2∇θ.∇φ + W(x, y)φ = 0
\]  
(3.4)

respectively.

A nonlinear solution to equation (3.4) that satisfies φ → 0 as (x, y) → ±∞ is obtained as
\[
φ(x, y) = \sqrt{a^2 + a + 2} \ \text{sech} x \ \text{sech} y,
\]  
(3.5)

with the phase θ(x, y) = b(x + y) and the propagation constant is given by λ = 2 - 4b^2. Figures 4(a), (b) show the real and imaginary parts of the 2D Rosen-Morse potential well. Two-dimensional soliton |φ(x, y)|^2 is shown in figure 4(c). In all these cases we have considered a = 1.75 and σ = 1. To understand the internal structure of the two dimensional self-trapped modes, we calculate the two-dimensional transverse power flow vector \( \vec{S} = b(a^2 + a + 2) (\text{sech}^2 x, \text{sech}^2 y) \). In figure 4(d), we have shown such 2D transverse power flow for a = 1.25, b = .5λ = 1, σ = 1, which indicates energy exchange from gain towards loss regions.

IV. SUMMARY

To summarize, we have investigated the existence and properties of nonlinear localized modes in a single \( \mathcal{PT} \) waveguide cell characterized by the nonlinear Schrödinger equation with complex Rosen-Morse potential.
well. The closed form expressions for the localized modes in such one- and two-dimensional self-focusing and self-defocusing Kerr nonlinear media are obtained. The transverse power flow density is shown to remain positive for some parameter values which indicates that power flow is in a single direction, mainly from gain towards loss regions. However, linear stability analysis of the one-dimensional solitons reveals that these solitons are unstable over the whole range of the potential parameter in spite of the fact that corresponding linear Schrödinger eigenvalue problem possesses unbroken $PT$-symmetry. The main reason behind such instability is that unlike the real part, the imaginary part of the complex Rosen-Morse potential well does not vanish asymptotically. Therefore any small fluctuation in the field intensity is amplified (or absorbed) which leads to the instability. The results presented here definitely encourage one to search for the stable localized modes (if any) of the nonlinear Schrödinger equation with the $PT$-symmetric Rosen-Morse potential well in the presence of higher-order/competing or other nonlinearities.

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