MASS-ENERGY THRESHOLD DYNAMICS FOR THE FOCUSING NLS WITH A REPULSIVE INVERSE-POWER POTENTIAL

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Abstract. In this paper we study long time dynamics (i.e., scattering and blow-up) of solutions for the focusing NLS with a repulsive inverse-power potential and with initial data lying exactly at the mass-energy threshold, namely, when $E_V(u_0)M(u_0) = E_0(Q)M(Q)$. Moreover, we prove failure of the uniform space-time bounds at the mass-energy threshold.

1. Introduction

In this paper we consider the long time dynamics for the following nonlinear Schrödinger equation with a repulsive inverse-power potential

$$\begin{cases}
i \partial_t u + \Delta u - a|x|^{-\mu} u + |u|^2 u = 0, \\
u(0, x) = u_0 \in H^1(\mathbb{R}^3),
\end{cases} \tag{NLS_a}$$

where $u = u(t, x)$ is a complex-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, $1 < \mu < 2$ and $a > 0$. We define the operator $H = -\Delta + V(x)$, where

$$V(x) = a|x|^{-\mu} \text{ with } a > 0 \text{ and } 1 < \mu < 2.$$ We define the energy functional on $H^1(\mathbb{R}^3)$ as follows:

$$E_V(u) = \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 + \frac{1}{2} V(x)|u|^2 - \frac{1}{4}|u|^4 \, dx.$$ Note that $E_V$ is the generation Hamiltonian of $(NLS_a)$. The Cauchy problem for the present equation has been studied by Guo, Wang and Yao [8] (see also [10]), more precisely: for $u_0 \in H^1(\mathbb{R}^3)$, there exist $T_\ast = T(||u_0||_{H^1}) > 0$ and a unique solution $u \in C([0, T_\ast), H^1(\mathbb{R}^3))$ of the Cauchy problem $(NLS_a)$. Furthermore, the solution satisfies the conservation of energy and mass

$$E_V(u(t)) = E_V(u_0) \quad \text{and} \quad M(u(t)) = M(u_0),$$

for all $t \in [0, T_\ast)$, where

$$M(u) = \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 \, dx.$$ Scattering and blow-up for large date were studied for the NLS with a repulsive inverse-power potential in several papers in different contexts; see [4, 8–10, 12, 14, 16, 17] and references therein. In particular, the ground state solution of the free cubic nonlinear Schrödinger equation (i.e., $(NLS_a)$ with $a = 0$) plays an important role in the behavior (scattering/blow-up) of solutions for $(NLS_a)$. Recall that the ground state is the unique, radial, vanishing at infinity and positive solution of the following nonlinear elliptic equation

$$-\Delta Q + Q - Q^3 = 0.$$ \hfill (1.1)

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We defined the Sobolev space adapted to $H$ by
\[ \|u\|^2_{H^1_T} = \langleHu,u\rangle = \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2dx. \]
In [10], the authors have studied the global existence and scattering of solutions to (NLS$_a$) when the initial data has nonnegative virial functional $F_N$, where
\[ F_N(u) = 2\|\nabla u\|^2_{L^2} - \int_{\mathbb{R}^3} (x \cdot \nabla V)|u(x)|^2dx - \frac{\mu}{2}\|u\|^4_{L^4}. \]
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\[ P_V(u) = 2\|\nabla u\|^2_{L^2} - \int_{\mathbb{R}^3} (x \cdot \nabla V)|u(x)|^2dx - \frac{\mu}{2}\|u\|^4_{L^4}. \]
More specifically, we have the following result.

**Theorem 1.1** (Sub-threshold scattering, [8, 9]). Fix $1 < \mu < 2$ and $a > 0$. Let $u(t)$ be the corresponding solution to (NLS$_a$) with initial data $u_0 \in H^1(\mathbb{R}^3)$. If $u_0$ obeys
\[ E_V(u_0)M(u_0) < E_0(Q)M(Q) \quad \text{and} \quad P_V(u_0) \geq 0, \quad (1.2) \]
then the solution $u(t)$ exists globally and scatters in $H^1(\mathbb{R}^3)$.

In the case $\mu = 2$, Killip, Murphy, Visan and J. Zheng [12] proved a similar scattering result for $a > -\frac{1}{2}$.

The theorem above is a consequence of the fact that the solutions to (NLS$_a$) obeys the global spacetime bound
\[ \|u\|_{L^5_T,\infty(\mathbb{R} \times \mathbb{R}^3)} < C(E_V(u_0), M(u_0), E_0(Q), M(Q)), \quad (1.3) \]
for some $C : (0, E_0(Q)M(Q)) \to (0, \infty)$.

In our first result we show that Theorem 1.1 is sharp, i.e., the constant $C(\cdot)$ diverges as we approach the mass-energy threshold. Indeed,

**Theorem 1.2** (Failure of uniform space-time bounds at threshold.). Fix $1 < \mu < 2$ and $a > 0$. There exists a sequence of global solutions $u_n$ of (NLS$_a$) such that
\[ E_V(u_n)M(u_n) \not\to E_0(Q)M(Q) \quad \text{and} \quad P_V(u_n(0)) \to 0, \]
as $n \to \infty$ with
\[ \lim_{n \to \infty} \|u_n\|_{L^5_T,\infty(\mathbb{R} \times \mathbb{R}^3)} = \infty. \]

The purpose of this paper is to study the long time dynamics (i.e., scattering and blow-up) for (NLS$_a$) exactly at the mass-energy threshold, i.e., when $E_V(u_0)M(u_0) = E_0(Q)M(Q)$. We now state the main result of this paper.

**Theorem 1.3** (Threshold dynamics). Fix $1 < \mu < 2$ and $a > 0$. Let $u(t)$ be the corresponding solution to (NLS$_a$) with initial data $u_0 \in H^1(\mathbb{R}^3)$.

(i) If $u_0$ obeys
\[ E_V(u_0)M(u_0) = E_0(Q)M(Q) \quad \text{and} \quad P_V(u_0) \geq 0, \quad (1.4) \]
then the solution $u(t)$ to (NLS$_a$) is global and $u \in L^5_T(\mathbb{R} \times \mathbb{R}^3)$. Consequently, the solution $u$ scatters in both directions.

(ii) If $u_0$ obeys
\[ E_V(u_0)M(u_0) = E_0(Q)M(Q) \quad \text{and} \quad P_V(u_0) < 0, \quad (1.5) \]
and $u_{0\bar{u}} \in L^2(\mathbb{R}^3)$ or $u_0$ is radially symmetric, then the solution $u$ to (NLS$_a$) blows up in both time directions.
In the case $a = 0$, a similar result was originally proven by Duyckaerts-Roudenko [7]. However, due to the presence of the potential, the method developed in [7] cannot be applied to (NLS$_a$). To overcome this problem, the proof of scattering result in Theorem 1.3 is based on the work of Miao, Murphy and Zheng [15]. An analogous result to Theorem 1.3 (i) for the NLS in the exterior of a convex obstacle was obtained by [6]. Recently, the same argument have been applied to the focusing NLS with a repulsive Dirac delta potential; see [3] for more details. On the other hand, our proof of blow-up result is based on the argument developed in [2,7,11]. For more details, we refer to Section 6.

Remark 1.4. Recently, in [4, Theorem 7.2], using the argument of Dodson and Murphy [5] the author shows that under condition (1.4) the corresponding solution $u(t)$ to Cauchy problem (NLS$_a$) either (i) scatters in $H^1(\mathbb{R}^3)$ forward in time, or (ii) there exist $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ with $t_n \rightarrow \infty$, $\{y_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$ with $|y_n| \rightarrow \infty$ and $\lambda, \theta \in \mathbb{R}$ so that $u(t_n, \cdot + y_n) \rightarrow e^{i\theta}Q$ strongly in $H^1(\mathbb{R}^3)$. As a consequence of the Theorem 1.3 (i), we can rule out the second possibility.

This present paper is organized as follows. In Section 2 we give some results that are necessary for later sections. In particular, the linear profile decomposition, the stability result to (NLS$_a$), localized Virial identities, and variational analysis of the ground state related to (1.1). In Section 3 we show that if the scattering result of Theorem 1.3 fails, then we can find a forward global solution $u \in C([0, \infty); H^1(\mathbb{R}^3))$ to (NLS$_a$) which satisfies that $\{u(t, \cdot + x_0(t)) : t \in [0, \infty)\}$ is pre-compact in $H^1(\mathbb{R}^3)$ for some function $x_0 : [0, \infty) \rightarrow \mathbb{R}$ (cf. Proposition 3.1). In Section 4 we discuss modulation (Proposition 4.4). In Section 5, using the result of Section 3 (Proposition 3.1) and adopting the method of Miao, Murphy and Zheng [15] we establish the scattering part of Theorem 1.3. Section 6 is devoted to the proof of the blow-up result given in Theorem 1.3. Finally, in Section 7 we prove Theorem 1.2.

Notations. Given two positive quantities $A$, $B$ we write $A \lesssim B$ or $B \gtrsim A$ to signify $A \leq CB$ for some positive constant $C > 0$. When $A \lesssim B \lesssim A$, we write $A \sim B$. Recall that $H := -\Delta + V(x)$, where $V(x) = a|x|^{-\mu}$ with $a > 0$. We write

$$
\|u\|_{H^1}^2 := \langle Hu, u \rangle = \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2 dx,
$$

and $\|u\|_{H^1 \cap L^2}^2 = \|u\|_{H^1}^2 + \|u\|_{L^2}^2$.

Throughout the paper, we will use the spaces $\dot{S}^s(I)$ for $s \geq 0$,

$$
\dot{S}^s(I) = L^\infty_t \dot{H}^s_x(I \times \mathbb{R}^3) \cap L^2_t \dot{H}^s_x(I \times \mathbb{R}^3), \quad S^s(I) = L^\infty_t H^s_x(I \times \mathbb{R}^3) \cap L^2_t H^s_x(I \times \mathbb{R}^3).
$$

Finally, for $f \in H^1(\mathbb{R}^3)$ we denote

$$
\delta(f) := \|Q\|_{H^1}^2 - \|f\|_{H^1}^2.
$$

2. Preliminaries

In this section we review the tools that will be needed in the proof of Theorems 1.2 and 1.3.

2.1. Cauchy problem and profile decomposition. First, we have the following result.

Proposition 2.1 (Theorem 1.1 in [8]). Fix $a > 0$ and $0 < \mu < 2$. Let $u_0 \in H^1(\mathbb{R}^3)$ and $u(t)$ be the corresponding solution of Cauchy problem (NLS$_a$). If $u$ is a global solution to (NLS$_a$) with $\|u\|_{L^\infty_t(\mathbb{R} \times \mathbb{R}^3)} < \infty$, then $u(t)$ scatters in $H^1$. 

Remar 2.2 (Existence of wave operators; Theorem 1.1 in [8]). From Theorem 1.1 (iii) in [8], we have that given \( \psi \in H^1(\mathbb{R}^3) \), there exist \( T > 0 \) and a solution \( v \in C([T, \infty), H^1(\mathbb{R}^3)) \) to (NLS) such that
\[
\|v(t) - e^{itH}\psi\|_{H^1} \to 0 \quad \text{as} \quad t \to \infty.
\]
A similar result holds in the negative time direction.

Proposition 2.3 (Linear profile decomposition: Lemma 2.12 in [8]). Let \( a > 0 \) and \( 1 < \mu < 2 \). Let \( \{\varphi_n\}_{n \in \mathbb{N}} \) be a bounded sequence in \( H^1(\mathbb{R}^3) \). Then, up to subsequence, we have the decomposition
\[
\varphi_n = \sum_{j=1}^{J} e^{it_j H_{\tau_{x_n}^j}} \psi_j + R_n^J, \quad \forall J \in \mathbb{N},
\]
where \( t_n \in \mathbb{R}, x_n \in \mathbb{R}^3, \psi_j \in H^1(\mathbb{R}^3) \setminus \{0\} \), and the following statements hold.

- for any fixed \( 1 \leq j \leq J \),
  - either \( t_n^j = 0 \) for any \( n \in \mathbb{N} \), or \( t_n^j \to \pm \infty \) as \( n \to \infty \);
  - either \( x_n^j = 0 \) for any \( n \in \mathbb{N} \), or \( |x_n^j| \to +\infty \) as \( n \to \infty \).
- orthogonality of the parameters: namely, for \( 1 \leq j \neq k \leq J \)
  \[|t_n^j - t_n^k| + |x_n^j - x_n^k| \to \infty \quad \text{as} \quad n \to \infty.\]
- asymptotic smallness property
  \[\forall \varepsilon > 0, \exists J = J(\varepsilon) \in \mathbb{N} \quad \text{such that} \quad \limsup_{n \to \infty} \|e^{-itH}R_n^J\|_{L^1_t L^\infty_x} < \varepsilon.\]
- asymptotic Pythagorean expansions: for any \( J \in \mathbb{N} \)
  \[\|\varphi_n\|_{L^2_t L^2_x}^2 = \sum_{j=1}^{J} \|\psi_j\|_{L^2_t L^2_x}^2 + \|R_n^J\|_{L^2_t L^2_x}^2 + o_n(1),\]
  \[\|\varphi_n\|_{H^1_t L^\infty_x}^2 = \sum_{j=1}^{J} \|\tau_{x_n}^j \psi_j\|_{H^1_t L^\infty_x}^2 + \|R_n^J\|_{H^1_t L^\infty_x}^2 + o_n(1).\]

Moreover, we have
\[\|\varphi_n\|_{L^4_t L^4_x}^4 = \sum_{j=1}^{J} \|e^{it_j H_{\tau_{x_n}^j}} \psi_j\|_{L^4_t L^4_x}^4 + \|R_n^J\|_{L^4_t L^4_x}^4 + o_n(1) \quad \forall J \in \mathbb{N}.
\]

For the following result, recall that for \( s \geq 0 \),
\[\dot{S}_V^s(I) = L_t^s H_{V}^s(I \times \mathbb{R}^3) \cap L_t^2 H_{V}^{s,6}(I \times \mathbb{R}^3), \quad S_V^s(I) = L_t^\infty H_{V}^s(I \times \mathbb{R}^3) \cap L_t^2 H_{V}^{s,6}(I \times \mathbb{R}^3).
\]

Lemma 2.4 (Stability; Lemma 2.3 in [8] and Theorem 4.10 in [10]). Fix \( a > 0 \) and \( 0 < \mu < 2 \). Let \( I \subset \mathbb{R} \) be a time interval containing \( t_0 \) and let \( \tilde{u} \) satisfy
\[(i\partial_t - H)\tilde{u} = -|\tilde{u}|^2\tilde{u} + e, \quad \tilde{u}(t_0) = \tilde{u}_0\]
on \( I \times \mathbb{R}^3 \) for some function \( e : I \times \mathbb{R} \to \mathbb{C} \). Fix \( u_0 \in H^1(\mathbb{R}^3) \) and suppose
\[\|\tilde{u}_0\|_{H^1} + \|u_0\|_{H^1} \leq E \quad \text{and} \quad \|\tilde{u}\|_{L^1_t L^\infty_x(I \times \mathbb{R}^3)} \leq L\]
for some \( E, L > 0 \). Assume the smallness conditions
\[\|u_0 - \tilde{u}_0\|_{H^1_x} \leq \varepsilon \quad \text{and} \quad \|\nabla \tilde{u} e\|_{N(I)} \leq \varepsilon,
\]
for some \( 0 < \varepsilon < \epsilon_0 = \epsilon_0(E,L) > 0 \). Here,
\[N(I) = L^1_t L^2_x(I \times \mathbb{R}^3) + L^\infty_t L^6_x(I \times \mathbb{R}^3) + L^2_t L^\infty_x(I \times \mathbb{R}^3).\]
Then there exists a unique solution $u$ to (NLS$_a$) with initial data $u_0$ at the time $t = t_0$ satisfying

$$\|u - \bar{u}\|_{S^1(t \times \mathbb{R}^3)} \leq C(E, L) \varepsilon \quad \text{and} \quad \|u\|_{S^1(t \times \mathbb{R}^3)} \leq C(E, L).$$

For $a \geq 0$, we define on $H^1(\mathbb{R}^3)$ the following functional (recall that $V(x) = a|x|^{-\mu}$):

$$S_V(f) = E_V(f) + \frac{1}{2}\|f\|_{L^2}^2 - \frac{1}{2}\|f\|_{H^1}^2 + \frac{1}{2}\|f\|_{L^2}^2 - \frac{1}{2}\|f\|_{H^1}^2, \quad \text{for } f \in H^1.$$  \hfill (2.1)

**Lemma 2.5** (Embedding nonlinear profiles; Lemma 2.13 in [8]). Fix $a > 0$ and $1 < \mu < 2$. Let $\{t_n\}_{n \in \mathbb{N}}$ satisfy $t_n \equiv 0$ or $t_n \to \pm \infty$, and let $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$ satisfy $|x_n| \to \infty$. Suppose $\phi \in H^1(\mathbb{R}^3)$ obeys

$$S_0(\phi) < S_0(Q) \quad \text{and} \quad P_0(\phi) \geq 0 \quad \text{if} \quad t_n \equiv 0$$

or

$$\frac{1}{2}\|\phi\|_{H^1} < S_0(Q) \quad \text{if} \quad t_n \to \pm \infty. \hfill (2.3)$$

Then for $n$ sufficiently large, there exists a global solution $v_n$ to (NLS$_a$) so that

$$v_n(0) = \phi_n \quad \text{and} \quad \|v_n\|_{S^1(\mathbb{R}^3)} \leq \|\phi\|_{H^1},$$

where

$$\phi_n(x) = e^{-it_n H_{ta_n}} \phi(x).$$

Furthermore, for any $\varepsilon > 0$ there exist $N = N(\varepsilon) \in \mathbb{N}$ and a smooth compactly supported function $\chi_\varepsilon \in C^\infty_c(\mathbb{R} \times \mathbb{R}^3)$ such that for $n \geq N$, we have

$$\|v_n(t,x) - \chi_\varepsilon(t + t_n, x - x_n)\|_{X(\mathbb{R} \times \mathbb{R}^3)} < \varepsilon, \hfill (2.4)$$

where

$$X \in \{L^{\frac{6}{5}}_{L^2}, L^{\frac{10}{7}}_{L^2}, L^{\frac{31}{21}}_{L^2}, L^{\frac{90}{71}}_{L^2}, H^{\frac{31}{21}}_{L^2}, \mathcal{S}^1_{L^2} \}.$$  \hfill  

**Lemma 2.6** (Hardy’s inequality, [17]). Fix $1 < p < \infty$ and $0 < \mu < 3$. Then, the following inequality holds

$$\int_{\mathbb{R}^3} \frac{|u(x)|^p}{|x|^\mu} \, dx \lesssim_p \|\nabla u\|_{L^p}^p.$$

In particular, if $0 < \mu < 2$, then we have that the embedding $H^1 \hookrightarrow L^2(\sqrt{\nabla} \, dx)$ is continuous.

### 2.2. Variational analysis

First, we recall here some well-known properties of the ground state. We have the following sharp Gagliardo-Nirenberg inequality,

$$\|f\|_{L^4}^4 \leq C_{GN}\|\nabla f\|_{L^2}^2 \|f\|_{L^2}^2, \hfill (2.5)$$

where

$$C_{GN} = \frac{\|Q\|_{L^4}}{\|\nabla Q\|_{L^2}^2 \|Q\|_{L^2}}. \hfill (2.6)$$

It is well-known that the ground state $Q$ satisfies the Pohozaev’s identities

$$E_0(Q) = \frac{1}{2}\|Q\|_{L^2}^2 = \frac{1}{2}\|\nabla Q\|_{L^2}^2 = \frac{1}{2}\|Q\|_{L^4}^4. \hfill (2.7)$$

Moreover, by straightforward calculations we deduce

$$\left[ C_{GN}\|Q\|_{L^2} \right]^{-\frac{4}{3}} = \frac{1}{4}\|Q\|_{L^4}^{-\frac{4}{3}}. \hfill (2.8)$$

For $a \geq 0$, we define the following variational problem (recall that $V(x) = a|x|^{-\mu}$):

$$d_a := \inf\{S_V(\phi) : \phi \in H^1(\mathbb{R}^3) \setminus \{0\}, P_V(\phi) = 0\}, \hfill (2.9)$$

where the functional $S_V$ is given by (2.1).

For the proof of the following lemma see [8, Lemmas 3.5 and 3.6]
Lemma 2.7. Fix $a > 0$ and $0 < \mu < 2$. Then $d_a$ is never attained for any $a > 0$. Moreover, $d_a = S_0(Q)$.

Lemma 2.8. Fix $a > 0$ and $0 < \mu < 2$. Assume that $u_0 \in H^1(\mathbb{R}^3)$ satisfies

$$E_V(u_0) = E_0(Q), \quad M(u_0) = M(Q) \quad \text{and} \quad P_V(u_0) \geq 0. \quad (2.10)$$

Then the corresponding solution $u(t)$ to (NLS$_a$) is global and satisfies

$$P_V(u(t)) > 0 \quad \text{for all} \quad t \in \mathbb{R}. \quad (2.11)$$

Moreover, we have

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^1}^2 \sim S_0(Q) \quad (2.12)$$

$$\|u(t)\|_{H^1}^2 < \|Q\|_{H^1}^2 \quad \text{for all} \quad t \in \mathbb{R}. \quad (2.13)$$

Proof. Notice that by (2.10) we get $S_V(u(t)) = S_0(Q)$, where the functional $S_V$ is given by (2.1). By contradiction, suppose that there exists $t_0 \in \mathbb{R}$ so that $P_V(u(t_0)) = 0$. Then $u(t_0)$ is a minimizer of $d_a$ (cf. (2.9)), which is a contradiction with Lemma 2.7. Thus,

$$P_V(u(t)) > 0 \quad \text{for all} \quad t \in \text{the existence time}. \quad (2.14)$$

Next, notice that $2S_0(Q) \leq \|u(t)\|_{H^1}^2$ for all $t$ in the existence time. On the other hand, by using (2.14) we infer that (recall that $0 < \mu < 2$)

$$\|u(t)\|_{H^1}^2 + 3\|u(t)\|_{L^2}^2 < \|u(t)\|_{H^1}^2 + 3\|u(t)\|_{L^2}^2 + P_V(u(t)) \leq 6S_V(u(t)) = 6S_0(Q)$$

for all $t$ in the existence time, which implies that $u$ is global and satisfies (2.12). Finally, as $6S_0(Q) = \|Q\|_{H^1}^2 + 3\|Q\|_{L^2}^2$ and $M(Q) = M(u_0)$, by inequality above we obtain (2.13).

This completes the proof of lemma. \hfill \Box

Lemma 2.9. Fix $a > 0$ and $0 < \mu < 2$. Assume that $u_0 \in H^1(\mathbb{R}^3)$ satisfies

$$E_V(u_0) = E_0(Q), \quad M(u_0) = M(Q) \quad \text{and} \quad P_V(u_0) < 0. \quad (2.15)$$

Then the corresponding solution $u(t)$ to (NLS$_a$) satisfies $P_V(u(t)) < 0$ for all $t$ in the existence time. Furthermore,

$$\|u(t)\|_{H^1}^2 > \|Q\|_{H^1}^2 \quad \text{for all} \quad t \in \text{the existence time}. \quad (2.16)$$

Proof. Following the same argument as Lemma 2.8, we can prove $P_V(u(t)) < 0$ for all $t$ in the existence time.

Next, suppose that $\|u(t_0)\|_{H^1}^2 \leq \|Q\|_{H^1}^2$, for some $t_0$. Then, as $M(u(t_0)) = M(Q)$, by the Gagliardo-Nirenberg inequality and (2.5)-(2.7) we obtain

$$\frac{1}{2}P_V(u(t_0)) \geq \|\nabla u(t_0)\|_{L^2}^2 + \frac{a}{2} \int_{\mathbb{R}^3} \frac{|u(x, t_0)|}{|x|} dx - \frac{3C_G}{4} \|\nabla u(t_0)\|_{L^2}^2 \|u(t_0)\|_{L^2}^2 \geq \|\nabla u(t_0)\|_{L^2}^2 \left(1 - \frac{3C_G}{4} \frac{\|\nabla u(t_0)\|_{L^2}^2 \|u(t_0)\|_{L^2}^2}{\|Q\|_{L^2}^2 \|Q\|_{L^2}^2}\right) + \frac{a}{2} \int_{\mathbb{R}^3} \frac{|u(x, t_0)|}{|x|} dx \geq \|\nabla u(t_0)\|_{L^2}^2 \left(1 - \frac{3C_G}{4} \frac{\|Q\|_{L^2}^2 \|Q\|_{L^2}^2}{\|Q\|_{L^2}^2 \|Q\|_{L^2}^2}\right) + \frac{a}{2} \int_{\mathbb{R}^3} \frac{|u(x, t_0)|}{|x|} dx$$

which is a contradiction. This proves the lemma. \hfill \Box
2.3. Virial identities.

\[ w_R(x) = R^2 \phi \left( \frac{x}{R} \right) \quad \text{and} \quad w_\infty(x) = |x|^2, \]

(2.17)

where \( \phi \) is a real-valued and radial function so that

\[ \phi(x) = \begin{cases} |x|^2, & |x| \leq 1 \\ 0, & |x| \geq 2, \end{cases} \quad \text{with} \quad |\partial^a \phi(x)| \lesssim |x|^{2-|a|}. \]

We introduce the localized virial functional

\[ I_R[u] = 2 \Im \int_{\mathbb{R}^3} \nabla w_R(x) \cdot \nabla u(t,x) \overline{u(t,x)} \, dx. \]

We need the following lemma; see e.g., [8].

**Lemma 2.10.** Let \( R \in [1, \infty] \). Suppose \( u(t) \) solves (NLS\(_a\)). Then we have

\[ \frac{d}{dt} I_R[u] = F_{R,V}[u(t)], \]

(2.18)

where

\[ F_{R,V}[u] := \int_{\mathbb{R}^3} \left( -\Delta \Delta w_R \right) |u|^2 - \Delta [w_R(x)] |u|^4 + 4 \Re \overline{u_j u_k \partial_{jk} [w_R]} dx \]

\[- 2 \int_{\mathbb{R}^3} |u|^2 \nabla w_R \cdot \nabla V dx \]

\[ = F_{R,0}[u] - 2 \int_{\mathbb{R}^3} |u|^2 \nabla w_R \cdot \nabla V dx. \]

In particular, when \( R = \infty \) we have \( F_{\infty,V}[u] = 4P_V(u) \).

The proofs of the next two lemmas are very similar to the ones in [15, Lemmas 2.9 and 2.10].

**Lemma 2.11** (Lemmas 2.9 in [15]). Consider \( R \in [1, \infty], \theta \in \mathbb{R} \) and \( y \in \mathbb{R} \). Then we have

\[ F_{R,0}[e^{i\theta} Q(\cdot - y)] = 0. \]

**Lemma 2.12** (Lemmas 2.10 in [15]). Let \( R \in [1, \infty], \chi : I \to \mathbb{R}, \theta : I \to \mathbb{R}, \) \( y : I \to \mathbb{R} \). Then if \( u \) is a solution to (NLS\(_a\)) on an interval \( I \) we have

\[ \frac{d}{dt} I_R[u] = F_{\infty,0}[u(t)] \]

\[ + F_{R,V}[u(t)] - F_{\infty,0}[u(t)] \]

\[- \chi(t) \{ F_{R,0}[e^{i\theta(t)} Q(\cdot - y(t))] - F_{\infty,0}[e^{i\theta(t)} Q(\cdot - y(t))] \}, \]

(2.19)

(2.20)

for all \( t \in \mathbb{R} \).

We need the following Cauchy-Schwarz inequality; a similar inequality is obtained in [7, Claim 5.4]; see also [11, Lemma 2.4] and [2, Lemma 2.2].

**Lemma 2.13.** Fix \( a > 0 \). Let \( f \in H^1(\mathbb{R}^3) \) such that \( |x|f \in L^2(\mathbb{R}^3) \). If

\[ M(f) = M(Q) \quad \text{and} \quad E_V(f) = E_0(Q), \]

(2.21)

then

\[ \left( \Im \int_{\mathbb{R}^3} (x \cdot \nabla f) \overline{f} dx \right)^2 \lesssim |\delta(f)|^2 \int_{\mathbb{R}^3} |x|^2 |f|^2 dx. \]

**Proof.** Given \( f \in H^1(\mathbb{R}^3) \) and \( \lambda \in \mathbb{R} \) we see that (cf. (2.5))

\[ \|f\|_{L^4}^4 \leq C_{GN} \|e^{i\lambda|x|^2} f\|_{L^1}^3 \|f\|_{L^2}. \]
As
\[ \|e^{i\lambda|x|^2} f\|_{L^2}^2 = 4\lambda^2 \int_{\mathbb{R}^3} |x|^2 |f|^2 \, dx + 4\lambda \text{Im} \int_{\mathbb{R}^3} (x \cdot \nabla f) \overline{f} \, dx + \int_{\mathbb{R}^3} |\nabla f|^2 \, dx + \int_{\mathbb{R}^3} V(x) |f|^2 \, dx \]
we get
\[
4\lambda^2 \int_{\mathbb{R}^3} |x|^2 |f|^2 \, dx + 4\lambda \text{Im} \int_{\mathbb{R}^3} (x \cdot \nabla f) \overline{f} \, dx + \int_{\mathbb{R}^3} |\nabla f|^2 \, dx
+ \int_{\mathbb{R}^3} V(x) |f|^2 \, dx - \left( \frac{\|f\|_{L^4}^4}{C_{GN} \|f\|_{L^2}^2} \right) \geq 0.
\]
Since the left-hand side of inequality above is a quadratic polynomial in \( \lambda \), it follows that the discriminant of this polynomial is non-positive, which implies
\[
\left( \text{Im} \int_{\mathbb{R}^3} (x \cdot \nabla f) \overline{f} \, dx \right)^2 \leq \int_{\mathbb{R}^3} |x|^2 |f|^2 \, dx \left( \|f\|_{H^1}^2 - \left( \frac{\|f\|_{L^4}^4}{C_{GN} \|f\|_{L^2}^2} \right)^{\frac{1}{2}} \right).
\] (2.22)

Next, by using the fact that \( E_V(f) = E_0(Q) \) (cf. (2.21)), it is clear that \( \|f\|_{L^4} = \|Q\|_{L^4} - 2\delta(f) \). But then, since \( M(f) = M(Q) \), it follows
\[
\|f\|_{H^1}^2 - \left( \frac{\|f\|_{L^4}^4}{C_{GN} \|f\|_{L^2}^2} \right)^{\frac{1}{4}} = \|Q\|_{H^1}^2 - \delta(f) - \left( \frac{\|Q\|_{L^4}^4 - 2\delta(f)}{C_{GN} \|Q\|_{L^2}^2} \right)^{\frac{1}{4}}.
\]
On the other hand, Taylor expansion and (2.8) implies
\[
\left( \frac{\|Q\|_{L^4}^4 - 2\delta(f)}{C_{GN} \|Q\|_{L^2}^2} \right)^{\frac{1}{4}} = \left( \|Q\|_{L^4}^2 - \frac{\delta}{4} \|Q\|_{L^2}^2 \delta(f) + O(\delta(f)^2) \right) \frac{\|Q\|_{L^2}^2}{C_{GN} \|Q\|_{L^2}^2} = \|Q\|_{L^4}^2 - \delta(f) + O(\|f\|^2).
\]
Thus, combining identities above we obtain
\[
\|f\|_{H^1}^2 - \left( \frac{\|f\|_{L^4}^4}{C_{GN} \|f\|_{L^2}^2} \right)^{\frac{1}{4}} = O(\|f\|^2),
\]
hence, by (2.22) we obtain
\[
\left( \text{Im} \int_{\mathbb{R}^3} x \cdot \nabla f \, dx \right)^2 \leq C \|\delta(f)\|^2 \int_{\mathbb{R}^3} |x|^2 |f|^2 \, dx.
\]
This completes the proof of lemma.

3. Compactness properties

**Proposition 3.1.** Fix \( a > 0 \) and \( 1 < \mu < 2 \). Suppose Theorem 1.3 (i) fails for some \( a > 0 \). Then we can find a forward global solution \( u \in C([0, \infty); H^1(\mathbb{R}^3)) \) to (NLS\(_a\)) which satisfies
\[
E_V(u_0) = E_0(Q), \quad M(u_0) = M(Q) \quad \text{and} \quad P_V(u_0) \geq 0, \quad \text{(3.1)}
\]
\[
\|u\|_{L^2}^2 \in C([0, \infty) \times \mathbb{R}^3) = \infty, \quad \text{(3.2)}
\]
and there exists a function \( x_0 : [0, \infty) \to \mathbb{R} \) so that \( \{u(t, \cdot + x_0(t)) : t \in [0, \infty)\} \) is pre-compact in \( H^1(\mathbb{R}^3) \).

Before the proof of Proposition 3.1, we need the following lemma.

**Lemma 3.2.** Suppose that Theorem 1.3 (i) holds for any \( a > 0 \) with the condition (1.4) replaced by (3.1). Then we can prove the same conclusion in Theorem 1.3 (i) (for any \( a > 0 \)) with the original condition (1.4).
Proof. Let $a > 0$. Suppose that $u_0 \in H^1(\mathbb{R}^3)$ such that
\[ E_V(u_0)M(u_0) = E_0(Q)M(Q) \quad \text{and} \quad P_V(u_0) \geq 0. \]
and assume that Theorem 1.3 (i) is true with the condition (3.1).

Writing $\lambda = \frac{M(u_0)}{M(Q)}$, $v_0(x) := \lambda u_0(\lambda x)$, $v(t, x) = \lambda u(\lambda^2 t, \lambda x)$ and
\[ V_\lambda(x) = \lambda^2 V(\lambda x) = \lambda^2 - \mu a|x|^{-\mu} \]
we obtain from (1.4),
\[ E_{V_\lambda}(v_0) = E_0(Q), \quad M(v_0) = M(Q) \quad \text{and} \quad P_{V_\lambda}(v_0) = \lambda P_V(u_0) \geq 0. \]
Notice also that the function $v$ satisfies
\[ i\partial_t v + \Delta v - \lambda^2 - \mu a|x|^{-\mu} v + |v|^2 v = 0. \]
Since $\lambda^2 - \mu a > 0$, by hypothesis we infer $v \in L^2_t L^2_x(\mathbb{R} \times \mathbb{R}^3)$, which implies that $u \in L^2_t L^5_x(\mathbb{R} \times \mathbb{R}^3)$. Therefore, we obtain that $u$ scatters in $H^1(\mathbb{R}^3)$. \qed

Proof of Proposition 3.1. We follow the outline of [15, Proposition 3.1]. Suppose that Theorem 1.3 (i) fails. Lemma 3.2 implies that there exists $u_0 \in H^1(\mathbb{R}^3)$ so that
\[ E_V(u_0) = E_0(Q), \quad M(u_0) = M(Q) \quad \text{and} \quad P_V(u_0) \geq 0, \]
Moreover,
\[ \|u\|_{L^5_t L^3(0, \infty) \times \mathbb{R}^3} = \infty, \]
where $u$ is the corresponding forward-global solution to (NLS$_a$) with initial data $u_0$. By Lemma 2.8 we see that $\|u(t)\|_{V^1} \lesssim_Q 1$ for all $t \in \mathbb{R}$. Now we show that there exists a parameter $\tau_0 : [0, \infty) \to \mathbb{R}$ such that $\{u(t, \cdot + \tau_0(t)) : t \in [0, \infty)\}$ is precompact in $H^1(\mathbb{R}^3)$.

By [15, Subsection 3.2], it is enough to show that if $\{\tau_n\}_{n \in \mathbb{N}}$ is an arbitrary sequence so that $\tau_n \to \infty$, then there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^3$ such that $u(\tau_n, x + x_n)$ converges strongly in $H^1(\mathbb{R}^3)$.

Linear profile decomposition (cf. Lemma 2.3), implies, up to subsequence, that
\[ u_n = u(\tau_n) = \sum_{j=1}^J e^{it\tau_n^j H_{x^n_j}} \psi_j + R_n^J, \]
and the properties in the statement hold. We set $\psi_j := e^{it\tau_n^j H_{x^n_j}} \psi_j$.

We claim that $J^* = 1$. Indeed, first assume $J^* = 0$. By the profile decomposition (cf. Lemma 2.3) we get $\|e^{-itH}u(\tau_n)\|_{L^5_t L^3(\mathbb{R} \times \mathbb{R}^3)} \to 0$ as $n \to \infty$. But then, from stability (cf. Lemma 2.4), we get $\|u\|_{L^5_t L^3([\tau_n, \infty) \times \mathbb{R}^3)} \lesssim 1$ for large $n \in \mathbb{N}$, which is a contradiction with the definition of $u$.

Next, suppose $J^* \geq 2$. By Lemma 2.3 we have the following for any $0 \leq J \leq J^*$,
\[
\lim_{n \to \infty} \left( \sum_{j=1}^J M(\psi_j^n) + M(R_j^n) \right) = \lim_{n \to \infty} M(u_n) = M(u_0) = M(Q),
\]
\[
\lim_{n \to \infty} \left( \sum_{j=1}^J E_V(\psi_j^n) + E_V(R_j^n) \right) = \lim_{n \to \infty} E_V(u_n) = E_V(u_0) = E_0(Q),
\]
\[
\limsup_{n \to \infty} \left( \sum_{j=1}^J \|\psi_j^n\|^2_{H^1} + \|R_j^n\|^2_{H^1} \right) \leq \limsup_{n \to \infty} \|u(\tau_n)\|^2_{H^1} \leq \|Q\|^2_{H^1}.\]
Here we also have used Lemma 2.8 in the last inequality. In particular,
\[
\lim_{n \to \infty} \sum_{j=1}^{J} S_V(\psi_n^j) + S_V(R_n^J) = \lim_{n \to \infty} S_V(S_n) = S_V(u_0) = S_0(Q).
\] (3.3)

It is not hard to show that \(\lim \inf_{n \to \infty} E_V(\psi_n^j) > 0\) (see, e.g., [15, (3.13)]). In particular, \(\lim \inf_{n \to \infty} S_V(\psi_n^j) > \|\psi^j\|_{L^2}^2 > 0\). Thus, as \(J^* \geq 2\), there exists \(\delta > 0\) so that
\[
M(\psi_n^j)E_V(\psi_n^j) \leq M(Q)E_0(Q) - \delta,
\] (3.4)
\[
\|\psi_n^j\|_{L^2} \|\psi_n^j\|_{\dot{H}^1} \leq \|Q\|_{L^2} \|Q\|_{\dot{H}^1} - \delta,
\] (3.5)
\[
S_V(\psi_n^j) \leq S_0(Q) - \delta,
\] (3.6)

for sufficiently large \(n\).

We will use \(\psi_n^j\) to build approximate solutions to (NLS_n) under three cases: \(x_n^j \equiv 0\) and \(t_n^j \equiv 0\); \(x_n^j \equiv 0\) and \(t_n^j \to \pm \infty\); and \(|x_n^j| \to +\infty\). For \(j\) such that \(x_n^j \equiv 0\) and \(t_n^j \to \pm \infty\), we use perturbation argument (cf. Lemma 2.8]). This implies that \(\psi_n^j \equiv 0\) for \(j\) such that \(|x_n^j| \to +\infty\). Again, from (3.4)-(3.5) we have that the solution is global and satisfies uniform space-time bounds. In either case, we set
\[
v_n^j(t, x) = v^j(t + t_n^j, x)
\]

Finally, for \(j\) such that \(|x_n^j| \to +\infty\), we have that (cf. [8, Lemma 2.7 and (2.20)])
\[
\lim_{n \to \infty} \|\psi_n^j\|_{\dot{H}^1}^2 = \|\psi^j\|_{\dot{H}^1}^2 > 0.
\]

In particular,
\[
M(\psi^j)E_0(\psi^j) \leq M(Q)E_0(Q) - \delta,
\] (3.7)
Next, if \(t_n^j \equiv 0\), we get
\[
\|\psi^j\|_{L^2} \|\psi^j\|_{\dot{H}^1} \leq \|Q\|_{L^2} \|Q\|_{\dot{H}^1} - \delta,
\]
which implies, by (3.7), that \(P_0(\psi^j) \geq 0\) for \(n\) large (cf. [1, p. 636]). Moreover, in this case \(t_n^j \equiv 0\), we also have (cf. (3.6))
\[
S_0(\psi^j) = \lim_{n \to \infty} S_V(\psi_n^j) \leq S_0(Q) - \delta.
\]

Therefore, when \(t_n^j \equiv 0\), we obtain that \(\psi^j\) satisfies the condition (2.2).

On the other hand, if \(t_n^j \to \pm \infty\), we get
\[
\frac{1}{2}\|\psi^j\|_{\dot{H}^1}^2 \leq \lim_{n \to \infty} S_V(\psi_n^j) \leq S_0(Q) - \delta,
\]

where we have used that the nonlinear part of \(S_V\) tends to zero as \(n \to \infty\) (cf. [8, Lemma 2.8]). This implies that \(\psi^j\) satisfies the condition (2.3). Thus, by Lemma 2.5 we obtain a solution \(v_n^j\) to (NLS_n) with \(v_n^j(0) = \psi_n^j\) obeying the global space-time bounds.

Now the idea of the proof is approximate
\[
\text{NLS}_V(t)u_n \approx \sum_{j=1}^{J} v_n^j(t) + e^{-itH}R_n^J,
\]
under tree cases \(x_n^j \equiv 0\) and \(t_n^j \equiv 0\); \(x_n^j \equiv 0\) and \(t_n^j \to \pm \infty\) and \(|x_n^j| \to +\infty\), and we use perturbation argument (cf. Lemma 2.4) to obtain a contradiction to (3.2).
With this in mind, we set
\[ u_n^J(t, x) := \sum_{j=1}^{J} v_n^j(t, x) + e^{-itH} R_n^J. \]
First, we note for each \( J \),
\[ \|u_n^J(0) - u_n\|_{\dot{H}^1} \to 0, \quad \text{as } n \to \infty. \] (3.8)
Moreover, by using the same argument to [15, see proof of (3.15)-(3.16)] we have
\[
\begin{align*}
\limsup_{n \to \infty} \sup_{J} \|u_n^J(0)\|_{H^1} + \|u_n^J\|_{L_t^1 L_x^2(\mathbb{R} \times \mathbb{R}^3)} & \leq 1 \\
\limsup_{n \to \infty} \|\nabla \psi (i\partial_t - H)u_n^J + |u_n^J|^2 u_n^J\|_{N(\mathbb{R})} & = 0.
\end{align*}
\]
By estimates above and (3.8), Lemma 2.4 implies that \( u \in L_t^6 L_x^\infty(\mathbb{R} \times \mathbb{R}^3) \), which is a contradiction to (3.2).
Therefore \( J' = 1 \). In particular, we obtain
\[ u(\tau_n) = e^{it\nabla H} \tau_n \psi + R_n \]
with \( \lim_{n \to \infty} \|R_n\|_{H^1} = 0. \) Notice that if \( |\tau_n| \to \infty \), then we have a contradiction to the non-scattering of \( u \) by the standard argument. Thus, \( u(\tau_n, x + x_n) = \psi(x) + R_n(x + x_n) \), hence \( u(\tau_n, \cdot + x_n) \) strongly converges to \( \psi \) in \( H^1(\mathbb{R}^3) \). This completes the proof of proposition.

\[ \square \]

4. Modulation analysis

Through this section, we assume that \( u(t) \) is a solution to (NLS\(_u\)) with
\[
\begin{align*}
E_V(u_0) &= E_0(\psi) \quad \text{and} \quad M(u_0) = M(\psi). \quad (4.1)
\end{align*}
\]
For \( \delta_0 > 0 \) small, we define (Recall that \( \delta(t) := \delta(u(t)) \))
\[ I_0 = \{ t \in [0, \infty) : |\delta(u(t))| < \delta_0 \quad \text{for } t \in \text{the domain existence of } u \}, \]
where \( u(t) \) is the corresponding solution to Cauchy problem (NLS\(_u\)).

**Lemma 4.1.** For any \( \varepsilon > 0 \), there exists \( \delta_0 = \delta_0(\varepsilon) > 0 \) small such that if \( |\delta(u(t))| < \delta_0 \), then there exists \( (\theta_0(t), y_0(t)) \in \mathbb{R} \times \mathbb{R}^3 \) so that
\[ \|u(t) - e^{it\theta_0(t)} Q(\cdot - y_0(t))\|_{H^1} < \varepsilon. \] (4.2)

**Proof.** We argue by contradiction. Thus, suppose that there exist \( \varepsilon > 0 \) small and a sequence of times \( \{t_n\} \subset \mathbb{R} \) with
\[ |\delta(u(t_n))| \to 0, \quad \inf_{\theta \in \mathbb{R}} \inf_{y \in \mathbb{R}^3} \|u(t_n) - e^{i\theta} Q(\cdot - y)\|_{H^1} \geq \varepsilon. \] (4.3)
By using (4.1) we see that (recall that \( |\delta(u(t_n))| \to 0 \))
\[ S_V(u(t_n)) = S_0(Q) \quad \text{and} \quad N_V(u(t_n)) \to N_0(Q) = 0, \]
where \( N_V \) is the Nehari functional,
\[ N_V(f) = \|\nabla f\|^2_{L^2} + \|f\|^2_{L^2} + \int_{\mathbb{R}^3} V(x)|f|^2 dx - \|f\|_{L^4}^4 \quad \text{for } f \in H^1(\mathbb{R}^3). \]
But then, we infer that \( N_0(u(t_n)) \leq 0 \) for \( n \) sufficiently large, which implies that \( \{u(t_n)\} \) is a minimizing sequence of problem
\[ S_0(Q) = \inf \{ S_0(f) : f \in H^1(\mathbb{R}^3) \setminus \{0\}, N_0(f) \leq 0 \}. \]
From [13, Proposition 3.12] we have that there exists \( (\theta_n, y_n) \in \mathbb{R}^4 \) such that \( e^{i\theta_n} u(t_n, \cdot + y_n) \to Q \) in \( H^1(\mathbb{R}^3) \). However, this leaves a contradiction to (4.3). \( \square \)
Remark 4.2. Let $R \geq 1$. If $\delta_0$ is sufficiently small in Lemma 4.1 we can assume that
\[ |y_0(t)| \geq R \quad \text{for } t \in \mathbb{R}. \] (4.4)
Indeed, if (4.4) is false, then there exists a sequence $\{t_n\}$ such that
\[ |\delta(t_n)| \to 0 \quad \text{and} \quad |y_0(t_n)| \leq R \quad \text{for all } n \in \mathbb{N}. \] (4.5)
Moreover, by using (4.2) we see that (see proof of Lemma 4.1)
\[ e^{-i\delta_0(t_n)}u(t_n, \cdot) + y_0(t_n)) \to Q \quad \text{in } H^1(\mathbb{R}^3). \] (4.6)
In particular, we get $E_V(u(t_n)) = E_0(Q) = \lim_{n \to \infty} E_0(u(t_n))$, which implies by (4.5) we get
\[ \int_{\mathbb{R}^3} V(x)|u(t_n, x)|^2 dx \to 0 \quad \text{as } n \to \infty. \]
But then, again by (4.6) we have
\[ \int_{\mathbb{R}^3} V(x)|Q(- y_0(t_n))|^2 dx \to 0 \quad \text{as } n \to \infty, \]
which is a contradiction because the sequence $\{y_0(t_n)\}$ is bounded.

By Lemma 4.1 and an application of implicit function theorem we obtain the following result.

Lemma 4.3. If $\delta_0 > 0$ is sufficiently small, then there exist two functions $\theta : I_0 \to \mathbb{R}$ and $y : I_0 \to \mathbb{R}^3$ so that
\[ \|u(t) - e^{i\theta(t)}Q(- y(t))\|_{H^1} \ll 1. \] (4.7)
Writing $g(t) := g_1(t) + ig_2(t) = e^{-i\theta(t)}[u(t) - e^{i\theta(t)}Q(- y(t))]$, we have that $g$ satisfies
\[ \langle g_2(t), Q(- y(t)) \rangle = \langle g_1(t), \partial_x Q(- y(t)) \rangle \equiv 0 \quad (j = 1, 2, 3). \] (4.8)

Proof. The proof is the same as in [15, Lemma 5.3]. \qed

Proposition 4.4 (Modulation). Fix $\alpha > 0$ and $0 < \mu < 2$. Suppose that $u(t)$ is a solution to (NLS) obeying (4.1) and $|\delta(0)| = |\delta(u_0)| > 0$. Then, there exist $\delta_0 > 0$ sufficiently small and two functions $\theta : I_0 \to \mathbb{R}$ and $y : I_0 \to \mathbb{R}^3$ so that $u(t)$ admits the decomposition
\[ u(t, x) = e^{i\theta(t)}[g(t) + Q(x - y(t))] \quad \text{for all } t \in I_0, \] (4.9)
and the following holds:
\[ \frac{|e^{-2|\theta(t)|} + |y'(t)||}{|g(t)|^2} + \left[ \int_{\mathbb{R}^3} V(x)|u(t, x)|^2 dx \right]^{\frac{1}{2}} \lesssim |\delta(t)| \sim \|g(t)\|_{H^1} \quad \text{for all } t \in I_0. \] (4.10)
Furthermore, letting $g = \alpha Q(- y) + h$ and $g = g_1 + ig_2$, where
\[ \alpha = \frac{\langle g_1(- y), \Delta Q \rangle}{\langle Q, \Delta Q \rangle} \in \mathbb{R}, \]
and we have
\[ |\alpha(t)| \sim |\delta(t)|, \quad \|h(t)\|_{H^1} \sim |\delta(t)| \quad \text{and} \]
\[ |\alpha'(t)| \lesssim |\delta(t)| \quad \text{for } t \in I_0. \] (4.11)
Proof. With Lemma 4.3 and Remark 4.2 in hand, the proof of (4.10) and (4.11) is essentially the same as in [15, Proposition 5.1].

The proof of estimate (4.12) is similar to that given in [7, Lemma 4.3] (see also [15, Lemma 5.6]). Indeed, by (4.9) we have

\[ h(t, x) = e^{-\theta(t)}[u(t) - e^{\theta(t)}(1 + \alpha(t))Q(x - y(t))]. \]

Let \( h = h_1 + ih_2 \). Then we have the following orthogonality relations (see proof of Lemma 5.4 in [15]),

\[ \langle h_1, \Delta Q(\cdot - y) \rangle = \langle h_2, Q(\cdot - y) \rangle = \langle h_1, \partial_t Q(\cdot - y) \rangle = 0 \]

for \( j = 1, 2, 3 \). In particular, by (4.10)-(4.11) we get \( \langle \partial_t h_1, Q(\cdot - y) \rangle \lesssim |\delta(t)| \).

Now, using the equation (NLS) and (1.1) we derive the equation

\[
\begin{align*}
    i\partial_t h &+ \Delta h - \theta' h - V e^{-\theta u} - \theta'(1 + \alpha)Q(x - y) + i\alpha' Q(x - y) \\
    + i(1 + \alpha)g' \cdot \nabla Q(x - y) + (1 + \alpha)Q(x - y) + f(e^{-i\theta} u) - (1 + \alpha)f(Q(x - y)) = 0,
\end{align*}
\]

where \( f(z) = z|z|^2 \). Then, multiplying Eq. (4.14) with \( Q(\cdot - y) \), taking integral and imaginary part, by estimates (4.10)-(4.11), it is not difficult to show that (see proof of Lemma 5.6 in [15])

\[ |\alpha'(t)| \lesssim |\delta(t)| \quad \text{for all } t \in I_0, \]

which completes the proof of lemma. \( \Box \)

5. Precluding the compact solution

Throughout this section we assume that \( u \) is the solution constructed in Proposition 3.1. In particular, \( u \) satisfies (4.1), \( P_r(u_0) \geq 0 \) and

\[ \|u\|_{L^2_t(0, \infty) \times \mathbb{R}^3) = \infty. \]

Moreover,

\[ \{u(t, \cdot + x_0(t)) : t \in [0, \infty)\} \text{ is pre-compact in } H^1(\mathbb{R}^3). \]

From (2.13) we see that

\[ \delta(t) := \delta(u(t)) > 0 \quad \text{for all } t \in [0, \infty). \]  \( (5.1) \)

Lemma 5.1. If \( \delta_0 \) is small, then there exists a constant \( C > 0 \) so that

\[ |x_0(t) - y(t)| < C \quad \text{for } t \in I_0. \]

Here, the parameter \( y(t) \) is given in Proposition 4.4.

Proof. The proof is the same as the proof of [15, Lemma 4.2]. \( \Box \)

From Lemma 5.1 we infer that

\[ \{u(t, \cdot + x(t))\} \text{ is pre-compact in } H^1(\mathbb{R}^3), \]  \( (5.2) \)

where the spatial center \( x(t) \) is given by

\[ x(t) = \begin{cases} x_0(t) & t \in [0, \infty) \setminus I_0, \\ y(t) & t \in I_0. \end{cases} \]

Proposition 5.2. If the spatial center \( x(t) \) is bounded, then \( x(t) \) is unbounded.

Proposition 5.2 will be a consequence of the following lemmas.

Lemma 5.3. For any time sequence \( \{t_n\} \subset [0, \infty) \), we have

\[ |x(t_n)| \to \infty \quad \text{if and only if} \quad \int_{\mathbb{R}^3} V(x)|u(t_n, x)|^2 dx \to 0. \]  \( (5.3) \)
Proof. With Lemma 2.6 in hand, the proof is the same as in [15, Lemma 4.3].

**Lemma 5.4.** Suppose \( t_n \to \infty \). Then
\[
|x(t_n)| \to \infty \quad \text{if and only if} \quad \delta(t_n) \to 0.
\]  
(5.4)

**Proof.** If \( \delta(t_n) \to 0 \), then combining (5.3) and estimate (4.10) we see that \( |x(t_n)| \to \infty \).

Next, let \( t_n \to \infty \) and assume by contradiction that \( |x(t_n)| \to \infty \) but, possibly for a subsequence only,
\[
\delta(u(t_n)) \geq c > 0.
\]  
(5.5)

As \( \{ u(t_n, t_n(x(t_n)) \} \) is pre-compact in \( H^1(\mathbb{R}^3) \), we have that there exists \( v_0 \in H^1(\mathbb{R}^3) \) so that
\[
u(t_n, x(t_n)) \to v_0 \quad \text{in} \quad H^1(\mathbb{R}^3),
\]  
(5.6)
along some subsequence in \( n \). In particular, since \( |x(t_n)| \to \infty \), it follows from (5.5) and (5.3),
\[
M(v_0) = M(Q), \quad E_0(v_0) = E_0(Q) \quad \text{and} \quad \|\nabla v_0\|^2_{L^2} < \|\nabla Q\|^2_{L^2}.
\]

An application of [7, Theorem 3] implies that the solution \( v \) of the free NLS on \( \mathbb{R}^3 \) (i.e., \((\text{NLS}_a)\) with \( a = 0 \)) with initial data \( v_0 \) is global and either scatters as \( t \to \infty \) or as \( t \to -\infty \) (or both).

Suppose that \( v \) scatters as \( t \to \infty \). As \( |x(t_n)| \to \infty \), we can use a similar argument as in [15, Lemma 4.4] to find a solution \( v_0 \) to \((\text{NLS}_a)\) so that
\[
v_n(0) = v_0(x), \quad \text{and} \quad \|v_0\|_{L^1_x(\mathbb{R}^3)} \lesssim 1
\]
for large \( n \). Notice that by (5.6) we get \( \|u(t_n, x) - v_n(0)\|_{H^1} \to 0 \) as \( t \to \infty \). Then the stability result (cf. Lemma 2.4) applies and
\[
\|u(t_n + t)\|_{L^1_x(\mathbb{R}^3)} = \|u\|_{L^1_x(\mathbb{R}^3)} \lesssim 1
\]
for large \( n \), which contradicts that the \( L^1_x \)-norm of \( u \) is infinite.

Next, suppose that \( v \) scatters as \( t \to -\infty \). An argument similar to the one developed above shows that
\[
\|u(t_n + t)\|_{L^1_x((-\infty, 0) \times \mathbb{R}^3)} = \|u\|_{L^1_x((-\infty, t_n) \times \mathbb{R}^3)} \lesssim 1
\]
for large \( n \). This also contradicts that \( u \) does not scatter. Therefore, \( \delta(u(t_n)) \to 0 \) as \( n \to \infty \). This completes the proof of lemma.

Recall that \( F_{\infty, 0} \) is defined in Lemma 2.10. We have the following result.

**Lemma 5.5.** Fix \( a > 0 \). There exists \( c > 0 \) so that
\[
F_{\infty, 0}[u(t)] = 8\|\nabla u(t)\|^2_{L^2} - 6\|u(t)\|^4_{L^4} \geq c\delta(t).
\]  
(5.7)

**Proof.** Suppose (5.7) is false. Then there exists \( \{ t_n \}_{n \in \mathbb{N}} \) such that
\[
F_{\infty, 0}[u(t_n)] \leq \frac{1}{n}\delta(t_n).
\]  
(5.8)

Notice that \( \{ \delta(t_n) \} \) is bounded (cf. Lemma 2.8). We claim that
\[
\delta(t_n) \to 0 \quad \text{as} \quad n \to \infty.
\]
Indeed, by using the Pohozaev’s identities and \( E_V(u(t_n)) = E_0(Q) \) we have (cf. (2.7))
\[
\|Q\|^4_{L^4_x} - \|u(t_n)\|^4_{L^4_x} = 2\delta(t_n).
\]  
(5.9)

Thus, as \( F_{\infty, 0}[Q] = 0 \), we get
\[
F_{\infty, 0}[u(t_n)] = 4\delta(t_n) - 8 \int_{\mathbb{R}^3} V(x)|u(t_n, x)|^2dx.
\]  
(5.10)
By using sharp Gagliardo-Nirenberg inequality (2.5), (2.7) and (2.13) we deduce
\[ \|u(t_n)\|_{L^2}^2 \leq C_{GN} \|u\|_{L^2} \|\nabla u(t_n)\|_{L^2}^2 \]
\[ \leq \frac{\|\nabla u(t_n)\|_{L^2}}{\|\nabla Q\|_{L^2}} \cdot \frac{\|Q\|_{L^2}}{\|\nabla Q\|_{L^2}} \cdot \|\nabla u(t_n)\|_{L^2}^2 \]
\[ < \frac{\delta}{4} \|\nabla u(t_n)\|_{L^2}^2. \]
Since \( F_{\infty,0}[u(t_n)] \to 0 \) as \( n \to \infty \) we have
\[ 0 \leftarrow \|\nabla u(t_n)\|_{L^2}^2 \left( 4 - C_{GN} \|u\|_{L^2} \|\nabla u(t_n)\|_{L^2} \right) = \frac{\delta}{4} \|\nabla u(t_n)\|_{L^2}^2 \left( 1 - \frac{\|\nabla u(t_n)\|_{L^2}}{\|\nabla Q\|_{L^2}} \right) \]
as \( n \to \infty \). Here, it follows from compactness of \( u \) that there exists a positive constant \( A > 0 \) such that \( A \cdot M(t_0) \leq \|\nabla u(t)\|_{L^2}^2 \) for each \( t \in \mathbb{R} \) (e.g. see [18, Lemma 6.6]). So, we obtain
\[ \|\nabla u(t_n)\|_{L^2} \to \|\nabla Q\|_{L^2} \]
as \( n \to \infty \).
In particular,
\[ \delta(t_n) = -\int_{\mathbb{R}^3} V(x)|u(t_n, x)|^2 dx + o(1) \quad \text{as} \quad n \to \infty. \quad (5.11) \]
Combining (5.10) and (5.11) we obtain the claim.
Finally, by Proposition 4.4 we have
\[ \int_{\mathbb{R}^3} V(x)|u(t_n, x)|^2 dx \lesssim \delta(t_n)^2 \leq \frac{1}{4} \delta(t_n) \]
for \( n \) large.
Thus, by using (5.10) and (5.8) we get
\[ 2\delta(t_n) \leq \frac{1}{n} \delta(t_n) \quad \text{for} \quad n \text{ large}, \]
which is a contradiction with (5.1).

**Proof of Proposition 5.2.** The proof is divided into 2 steps.

**Step 1.** Virial estimate. Let \( T > 0 \) and \( \varepsilon > 0 \), then there exists \( \rho_\varepsilon = \rho(\varepsilon) > 0 \) such that
\[ \int_0^T \delta(t) dt \lesssim \varepsilon T + [\rho_\varepsilon + \sup_{t \in [0, T]} |x(t)|] \|u\|_{L^\infty H^1}. \quad (5.12) \]
The proof of Step 1 is the same as for [15, Lemma 4.7], using our Lemma 5.5 instead of Lemma 4.5 of their paper.

**Step 2.** Conclusion. We argue by contradiction. If the spatial center \( x(t) \) is bounded, then by Step 1 above we have
\[ \frac{1}{T} \int_0^T \delta(t) dt \lesssim \varepsilon + \frac{1}{T} \rho_\varepsilon \quad \text{for all} \quad T > 0, \]
and for any \( \varepsilon > 0 \). Consider a sequence \( \varepsilon_n \to 0 \) as \( n \to \infty \). By choose appropriately times \( T_n \to \infty \), an application of the mean value theorem for integrals implies that there exists a time sequence \( t_n \to \infty \) such that that \( \delta(t_n) \to 0 \) as \( n \to \infty \) (recall that \( \delta(t) > 0 \) for \( t \geq 0 \)). But then Lemma 5.4 implies that \( |x(t_n)| \to \infty \), which is a contradiction. This completes the proof of proposition. □

**Proposition 5.6.** If the spacial center \( x(t) \) is unbounded, then \( x(t) \) is bounded.

**Proof.** The proof is essentially identical to that of [15, Proposition 4.8] and we omit the details. □

Now we are ready to give the proof of scattering result of Theorem 1.3.
Proof of Theorem 1.3 (i). If Theorem 1.3 (i) is not true, then there exists a critical element \( u_0 \in H^1(\mathbb{R}^3) \) and a spatial center \( x(t) \) such that the corresponding solution to \( (\text{NLS}_a) \) satisfies \( \{u(t, \cdot + x(t)) : t \geq 0\} \) is precompact in \( H^1(\mathbb{R}^3) \) (cf. Proposition 3.1). However, we have that this is impossible by Propositions 5.2 and 5.6. \( \square \)

6. Criteria for Blow-up

In this section we give the proof of the blow-up result of Theorem 1.3. Before the proof of Theorem 1.3 (ii), we need the following result.

**Proposition 6.1.** If the initial data \( u_0 \in H^1(\mathbb{R}^3) \) satisfies \( |x|u_0 \in L^2(\mathbb{R}^3), \)
\[
M(u_0) = M(Q), \quad E_V(u_0) = E_0(Q), \quad \text{and} \quad P_V(u_0) < 0, \tag{6.1}
\]
then the solution \( u \) to \( (\text{NLS}_a) \) with data \( u_0 \) blows up in both directions.

To prove the proposition above we need some preparation. We begin with the following lemma.

**Lemma 6.2.** Fix \( a > 0 \) and \( 0 < \mu < 2 \). Under assumption of Proposition 6.1 we have
\[
\left( \text{Im} \int_{\mathbb{R}^3} x \cdot \nabla u(t) u(t) dx \right)^2 \lesssim |P_V(u(t))|^2 \int_{\mathbb{R}^3} |x|^2 |u(t)|^2 dx, \tag{6.2}
\]
where \( u(t) \) is the corresponding solution to \( (\text{NLS}_a) \) with data \( u_0 \).

**Proof.** As \( P_V(u_0) < 0 \), from Lemma 2.9 we see that \( \delta(t) < 0 \) and \( P_V(u(t)) < 0 \) for all \( t \) in the existence time. Now, since
\[
\int_{\mathbb{R}^3} |\nabla u|^2 - \frac{3}{4} |u|^4 dx = 3E_V(u) - \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 + \frac{3}{2} V(x)|u|^2 dx 
\]
and \( E_V(u) = E_0(Q) \), by (2.7) we obtain
\[
P_V(u(t)) = 2 \left( \frac{1}{2} \delta(t) - \int_{\mathbb{R}^3} V(x)|u(t, x)|^2 dx \right) - \int_{\mathbb{R}^3} x \cdot \nabla V(x)|u(t, x)|^2 dx 
\]
\[
= \delta(t) - (2 - \mu) \int_{\mathbb{R}^3} V(x)|u(t, x)|^2 dx \leq \delta(t) < 0
\]
for all \( t \) in the existence time. Thus, \( |\delta(t)| \lesssim |P_V(u(t))|^2 \) and Lemma 2.13 implies (6.2). \( \square \)

**Lemma 6.3.** Under assumption of Proposition 6.1, if \( u(t) \) is global in positive time, then there exists positive constant \( C > 0 \) and \( c > 0 \) such that
\[
\int_0^\infty |\delta(\tau)|d\tau \leq C e^{-ct} \quad \text{for all} \ t \in (0, \infty). \tag{6.3}
\]

**Proof.** Writing \( f(t) = \|xu(t)\|_2^2 \), we see that (cf. Lemma 2.10)
\[
f'(t) = 4 \text{Im} \int_{\mathbb{R}^3} \overline{u}(t) \nabla u(t) \cdot x dx, \quad f''(t) = 4P_V(u(t)).
\]

Lemma 2.9 implies that
\[
f'(t_2) - f'(t_1) = \int_{t_1}^{t_2} f''(s)ds = 4 \int_{t_1}^{t_2} P_V(u(s))ds < 0 \quad \text{for} \ t_1 < t_2.
\]

We claim that \( f'(t) > 0 \) for all \( t \) in the existence time. Indeed, assume by contradiction that there exists \( t^* \in \mathbb{R} \) so that \( f'(t^*) \leq 0 \). Then inequality above shows that \( f'(t) < 0 \) for any \( t > t^* \), which is a contradiction because \( f(t) > 0 \) for all \( t \in [0, +\infty) \). Therefore,
\[
\frac{1}{4} f'(t) = \text{Im} \int_{\mathbb{R}^3} \overline{u}(t, x) \nabla u(t, x) \cdot x dx > 0, \quad \text{for all} \ t \in (0, \infty). \tag{6.4}
\]
Now we will show that \( f'(t) \leq Ce^{-ct} \) for all \( t \geq 0 \). Indeed, note that \( f > 0 \), \( f' > 0 \), and \( f'' < 0 \). Thus, thanks to Lemma 6.2 we see that
\[
|f'(t)|^2 \lesssim (f''(t))^2 f(t)
\]
for all \( t \) in the existence time, which implies
\[
\frac{f'(t)}{\sqrt{f(t)}} \leq -f''(t) \quad \text{for all } t \text{ in the existence time.} \tag{6.5}
\]
Integrating inequality above on \((0, t)\) we get
\[
\sqrt{f(t)} - \sqrt{f(0)} \lesssim -f'(t) + f'(0) \lesssim f'(0),
\]
i.e., \( \sqrt{f(t)} \) is bounded. From (6.5), it follows that \( f'(t) \lesssim -f''(t) \) for all \( t \) in the existence time, which shows \( f'(t) \leq Ce^{-ct} \) for some constants \( C > 0 \), \( c > 0 \). In particular, \( \lim_{t \to \infty} f'(t) = 0 \).

Finally, since \( 0 < -\delta(t) \leq -P_V(u(t)) \), we get
\[
\int_\tau^\infty |\delta(s)|ds = \int_\tau^\infty [-\delta(s)]ds \leq \int_\tau^\infty [-P_V(u(s))]ds = \frac{1}{4} \int_\tau^\infty [-f''(s)]ds = \frac{1}{4} f'(t) \leq Ce^{-ct},
\]
for \( t \in (0, +\infty) \). This completes the proof. \( \square \)

Proof of Proposition 6.1. Assume that the initial data \( u_0 \in H^1(\mathbb{R}^3) \) satisfies \( |x|u_0 \in L^2(\mathbb{R}^3) \),
\[
M(u_0) = M(Q), \quad E_V(u_0) = E_0(Q), \quad \text{and} \quad P_V(u_0) < 0. \tag{6.6}
\]
From Lemma 2.9 we see that \( \delta(t) < 0 \) for all \( t \) in the existence time.

Step 1. The corresponding solution \( u(t) \) to (NLS) with initial data \( u_0 \) is not global in positive time. Indeed, by contradiction, assume that \( u \) global in positive time. By (6.3) we deduce that there exists \( \{t_n\}_{n \in \mathbb{N}} \) with \( t_n \to +\infty \) such that \( \lim_{n \to \infty} \delta(t_n) = 0 \). Fix such \( \{t_n\}_{n \in \mathbb{N}} \).

Notice that \( \lim_{t \to \infty} \delta(t) = 0 \). If not, there exists a sequence \( \{t'_n\}_{n \in \mathbb{N}} \) such that \( -\delta(t'_n) \geq \varepsilon \) for some \( \varepsilon \in (0, \delta_0) \). Extracting subsequences of \( \{t_n\}_{n \in \mathbb{N}} \) and \( \{t'_n\}_{n \in \mathbb{N}} \) if necessary, we can assume the following properties:
\[
t_n < t'_n, \quad -\delta(t'_n) = \varepsilon, \quad -\delta(t) < \varepsilon \quad \text{for all } t \in [t_n, t'_n). \tag{6.7}
\]
On \([t_n, t'_n)\) the parameter \( \alpha(t) \) is well defined. As \( |\alpha'(t)| \leq C|\delta(t)| \) (cf. (4.12)), estimate (6.3) implies
\[
\lim_{n \to \infty} |\alpha(t_n) - \alpha(t'_n)| = 0. \tag{6.9}
\]
Thus, as \( |\alpha| \sim |\delta| \) (cf. (4.11)), we deduce
\[
|\alpha(t_n)| \sim |\delta(t_n)| \to 0 \quad \text{and} \quad |\alpha(t'_n)| \sim |\delta(t'_n)| = \varepsilon > 0,
\]
which is a contradiction to (6.7). Therefore, \( \lim_{t \to \infty} \delta(t) = 0 \). Note that by estimate (4.10) we also have
\[
\frac{e^{-2|\delta(t)|}}{|y(t)|^2} \lesssim |\delta(t)| \to 0 \quad \text{as } t \to \infty. \tag{6.9}
\]

Now, by using (4.10) and (6.3) we get
\[
|y(t) - y(t_1)| \leq \int_{t_1}^t |y'(s)|ds \lesssim \int_{t_1}^t |\delta(s)|ds \lesssim e^{-ct_1} \quad \text{for all } t > t_1.
\]
This implies that there exists a sequence \( \{\tau_n\}_{n \in \mathbb{N}} \) such that \( \tau_n \to \infty \) with \( |y(\tau_n)| \to a \in \mathbb{R} \). In particular,
\[
\lim_{n \to \infty} \frac{e^{-2|y(\tau_n)|}}{|y(\tau_n)|^2} > 0,
\]
which is a contradiction to (6.8). Thus, \( u(t) \) is not global in positive time.

\textbf{Step 2.} The solution \( u \) is not global in negative time. Suppose by contradiction that \( u \) global in negative time. Writing \( v(t,x) := u(-t,x) \), we have that \( v \) is a global in positive time solution to (NLS).

But then, since \(|x|v(0) \in L^2(\mathbb{R}^3)\), \( E_v(v(0)) = E_0(Q) \), \( M(v(0)) = M(Q) \) and \( P_v(v(0)) < 0 \), it follows by Step 1 above that \( v \) blows-up in positive time, which is a contradiction.

This completes the proof of proposition. \( \square \)

\textbf{Proof of Theorem 1.3 (ii).} First, assume \(|x|u_0 \in L^2(\mathbb{R}^3)\). Then the proof is a direct consequence of Proposition 6.1 and the following claim:

\textbf{Claim 6.4.} Assume that Theorem 1.3 (ii) holds for \( a > 0 \) with the condition (1.5) replaced by (6.1). Then we have the same conclusion in Theorem 1.3 (ii) with the original hypothesis (1.5).

\textbf{Proof of Claim.} The proof is very similar to that given in Lemma 3.2. Let \( a > 0 \). Assume that Theorem 1.3 (ii) is true with the condition (6.1). Consider \( u_0 \in H^1(\mathbb{R}^3) \) such that \(|x|u_0 \in L^2(\mathbb{R}^3)\),

\[ E_v(u_0)M(u_0) = E_0(Q)M(Q) \quad \text{and} \quad P_v(u_0) < 0. \]

Writing \( V_\lambda(x) = \lambda^2 V(\lambda x) \), \( v_0(x) = \lambda u_0(\lambda x) \) and \( v(t,x) = \lambda u(\lambda t, \lambda x) \) with \( \lambda = \frac{M(u_0)}{M(Q)} \) we deduce

\[ E_{V_\lambda}(v_0) = E_0(Q), \quad M(v_0) = M(Q) \quad \text{and} \quad P_{V_\lambda}(v_0) = \lambda P_v(u_0) < 0. \]

As \( v \) satisfies the equation

\[ i\partial_t v + \Delta v - \lambda^{\nu-2} a |x|^{-\mu} v + |v|^2 v = 0 \]

and \( \lambda^{\nu-2}a > 0 \), Proposition 6.1 implies that \( v \) blow up in both directions. In particular, we see that \( u \) blow up in both directions. This completes the proof of claim. \( \square \)

Next, assume that \( u_0 \) is radially symmetric. The proof is based on [7, Subsection 5.2]. We consider only positive time. We assume for contradiction that the solution \( u \) exists on \([0, \infty)\) under the assumptions of Theorem 1.3 (ii). Then, we prove \(|x|u_0 \in L^2(\mathbb{R}^3)\).

We define a functional

\[ J_R[u(t)] := \int_{\mathbb{R}^3} w_R(x)|u(t,x)|^2 dx, \]

where \( w_R \) is defined by (2.17). We also assume \( \phi''(r) \leq 2 \). Then, we have

\[
\frac{d^2}{dt^2} J_R[u(t)] = F_{R,V}[u(t)]
\]

\[ = 4\delta(t) + \int_{\mathbb{R}^3} (-\Delta w_R)|u|^2 dx - \int_{|x| \geq R} (\Delta w_R - 6)|u|^4 dx
\]

\[ + 4 \int_{|x| \geq R} (\phi'' \left( \frac{\mu}{|x|} \right) - 2)|\nabla u|^2 dx + 2 \int_{\mathbb{R}^3} \left\{ \frac{w_R}{|x|} \phi' \left( \frac{\mu}{|x|} \right) - 4 \right\} \frac{a}{|x|^\mu} |u|^2 dx
\]

\[ = 4\delta(t) + A_R[u(t)] + 2 \int_{\mathbb{R}^3} \left\{ \frac{w_R}{|x|} \phi' \left( \frac{\mu}{|x|} \right) - 4 \right\} \frac{a}{|x|^\mu} |u|^2 dx.
\]

We see

\[ \frac{w_R}{|x|} \phi' \left( \frac{\mu}{|x|} \right) - 4 \leq 0 \]
from simple calculation (recall that $1 < \mu < 2$ and $\phi'(r) \leq 2r$). The argument in [7, Subsection 5.2] with Proposition 4.4 deduces that there exists $R_0 > 0$ such that

$$A_R[u(t)] \leq -2\delta(t) \quad (6.9)$$

for any $R \geq R_0$ and any $t \in [0, \infty)$. In particular, we see that $\frac{d^2}{dt^2} J_R[u(t)] \leq 2\delta(t) < 0$ for any $R \geq R_0$ and any $t \in [0, \infty)$.

From now on, we prove that $|x|u_0 \in L^2(\mathbb{R}^3)$ holds. Fix $R \geq R_0$, where $R_0$ is taken above.

**Step 1:** First, we prove $\frac{d}{dt} J_R > 0$ for any $t \in [0, \infty)$. If not, then there exists $\varepsilon > 0$ and $t_0 \in [0, \infty)$ such that $\frac{d}{dt} J_R[u(t)] < -\varepsilon$ for any $t \geq t_0$ from $\frac{d^2}{dt^2} J_R[u(t)] < 0$. This contradicts the fact that $J_R[u(t)] \geq 0$ for any $t \in [0, \infty)$.

**Step 2:** We show that $u$ has finite variance. Since $\frac{d}{dt} J_R[u(t)]$ is positive and decreasing, we have $\frac{d}{dt} J_R[u(t)] \to c$ as $t \to \infty$ for some $c \geq 0$ and hence,

$$-\infty < c - \frac{d}{dt} J_R[u_0] = \int_0^\infty \frac{d^2}{dt^2} J_R[u(s)] ds \leq 2 \int_0^\infty \delta(s) ds \leq 0.$$ 

This inequality implies that there exists a sequence $\{t_n\} \subset [0, \infty)$ with $t_n \to \infty$ such that $\delta(t_n) \to 0$ as $n \to \infty$. Now, since $u$ is radially symmetric, we can take a sequence $\{\theta_n\} \subset \mathbb{R}$ such that $e^{i\theta_n} u(t_n) \to Q$ in $H^1$ as $n \to \infty$ by Lemma 4.1. Therefore, as $J_R[u(t)]$ is increasing, we obtain

$$J_R[u_0] = \int_{\mathbb{R}^3} w_R(x)|u_0(x)|^2 dx \leq \int_{\mathbb{R}^3} w_R(x)|Q(x)|^2 dx \leq \int_{\mathbb{R}^3} |x|^2|Q(x)|^2 dx < \infty.$$

Letting $R \to \infty$, monotone convergence theorem deduces

$$J_R[u_0] \to \int_{\mathbb{R}^3} |x|^2|u_0(x)|^2 dx \leq \int_{\mathbb{R}^3} |x|^2|Q(x)|^2 dx < \infty.$$

Therefore, $|x|u_0 \in L^2(\mathbb{R}^3)$. It follows that $u$ blows up. However, this is contradiction. This completes the proof of theorem.\qed

7. FAILURE OF UNIFORM SPACE-TIME BOUNDS AT THRESHOLD

In this section, we prove Theorem 1.2. We follow the proof of Theorem 1.5 in [12].

**Proof of Theorem 1.2.** Consider $\varphi_n = (1-\varepsilon_n)Q(x-x_n)$ with $\varepsilon_n \to 0$ and $|x_n| \to \infty$. Since $\mu \in (1, 2)$, by Lemma 2.7 in [8], we see that

$$Ev(\varphi_n) M(\varphi_n) \geq E_0(Q) M(Q) \quad \text{and} \quad P_V(\varphi_n) \to 0$$

as $n \to \infty$. Moreover, combining (2.7) and $P_0(Q) = 0$, it is not hard to show that $P_V(\varphi_n) > 0$ for all $n \in \mathbb{N}$. Therefore, from Theorem 1.1 we get that the corresponding solution $u_n$ to (NLS$_a$) with initial data $\varphi_n$ exists globally and scatters.

We want to apply Lemma 2.4 over $[-T, T] \times \mathbb{R}^3$. With this in mind, for each $n$, let $\chi_n$ be a smooth function obeying

$$\chi_n(x) = \begin{cases} 0 & |x_n + x| < \frac{1}{2}|x_n|, \\
1 & |x_n + x| > \frac{1}{2}|x_n|, \end{cases} \quad \text{with} \quad \sup_x |\partial^k \chi_n(x)| \lesssim \left( \frac{n}{|x_n|} \right)^{|k|}$$

uniformly in $x$. Notice that $\chi_n(x) \to 1$ as $n \to \infty$ for each $x \in \mathbb{R}^3$.

Now, fix $T > 0$. We put

$$\tilde{v}_n(t, x) = (1-\varepsilon_n)e^{it}[\chi_n Q](x-x_n).$$
We need to estimate \( \langle \nabla \hat{
abla}^2 e_n \rangle_{N([-T, T])} \), where
\[
e_n = (i \partial_t - H) \tilde{v}_n + |\tilde{v}_n|^2 \tilde{v}_n
= e^{it}[|e_n|^3 \chi_n^3(x - x_n) - (1 - e_n)\chi_n(x - x_n)]Q^3(x - x_n)
+ (1 - e_n)e^{it} [Q \Delta \chi_n + 2 \nabla \chi_n \cdot \nabla Q](x - x_n)
- \frac{a}{|x|^\alpha}(1 - e_n)e^{it} [\chi_n Q](x - x_n).
\]

(7.1)

For (7.1), we apply Hölder’s inequality, Sobolev embedding and dominated convergence theorem to estimate (recall that \( Q \in S(R^3) \))
\[
\langle \nabla(7.1) \rangle_{L^1_t L^4_x} \lesssim T \langle \nabla \chi_n \rangle_{L^2_x} \langle Q \rangle_{L^2_x} + \langle \nabla Q \rangle_{L^2_x} \langle \chi_n \rangle_{L^\infty_x}
\lesssim T \langle \chi_n \rangle_{L^\infty_x} + \langle x_n^{-2} \rangle_{L^\infty_x} \to 0.
\]

Similarly,
\[
\langle \nabla(7.2) \rangle_{L^1_t L^4_x} + \langle \nabla(7.2) \rangle_{L^1_t L^4_x} \lesssim T \langle \nabla \chi_n \rangle_{L^2_x} + \langle \nabla \chi_n \rangle_{L^2_x} + \langle \nabla \chi_n \rangle_{L^\infty_x} \langle Q \rangle_{H^1_x}
\lesssim T \langle \chi_n \rangle_{L^\infty_x} + \langle x_n^{-2} \rangle_{L^\infty_x} \to 0.
\]

as \( n \to \infty \). Therefore, for any \( T > 0 \) fixed, by interpolation we get
\[
\langle \nabla \hat{
abla}^2 e_n \rangle_{N([-T, T])} \to 0 \quad \text{as } n \to \infty.
\]

But then, since
\[
\langle \tilde{v}_n(0) \cdot \varphi_n \rangle_{\dot{H}^{\frac{1}{2}}} = \langle (1 - e_n)(\chi_n - 1)Q \rangle_{\dot{H}^{\frac{1}{2}}} \to 0,
\]
\[
\langle \tilde{v}_n \rangle_{\dot{L}^{\infty}_{x,t}([-T, T] \times R^3)} \gtrsim Q T \quad \text{for any } T > 0,
\]
Lemma 2.4 implies that
\[
\langle u_n \rangle_{\dot{L}^{\infty}_{x,t}([-T, T] \times R^3)} \gtrsim Q T
\]
which finished the proof because \( T > 0 \) is arbitrary. \(\square\)

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