Computing Linear Matrix Representations of Helton-Vinnikov Curves

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Dedicated to Bill Helton on the occasion of his 65th birthday.

Abstract. Helton and Vinnikov showed that every rigidly convex curve in the real plane bounds a spectrahedron. This leads to the computational problem of explicitly producing a symmetric (positive definite) linear determinantal representation for a given curve. We study three approaches to this problem: an algebraic approach via solving polynomial equations, a geometric approach via contact curves, and an analytic approach via theta functions. These are explained, compared, and tested experimentally for low degree instances.

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1. Introduction

The Helton-Vinnikov Theorem [16] gives a geometric characterization of two-dimensional spectrahedra. They are precisely the subsets of \( \mathbb{R}^2 \) that are bounded by rigidly convex algebraic curves, here called Helton-Vinnikov curves. These curves are cut out by hyperbolic polynomials in three variables, as discussed in [15]. This theorem is a refinement of a result from classical algebraic geometry which states that every homogeneous polynomial in three variables can be written as

\[
f(x, y, z) = \det(Ax + By + Cz)
\]

where \( A, B \) and \( C \) are symmetric matrices. Here the coefficients of \( f \) and the matrix entries are complex numbers. When the coefficients of \( f \) are real then it is desirable to find \( A, B \) and \( C \) with real entries. The representations relevant for spectrahedra

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are the real definite representations, which means that the linear span of the real matrices $A, B$ and $C$ contain a positive definite matrix. Such a representation is possible if and only if the corresponding curve $\{ (x : y : z) \in \mathbb{P}^2_\mathbb{R} : f(x, y, z) = 0 \}$ is rigidly convex. This condition means that the curve has the maximal number of nested ovals, namely, there are $d/2$ resp. $(d - 1)/2$ nested ovals when the degree $d$ of $f$ is even resp. odd. The innermost oval bounds a spectrahedron.

Two linear matrix representations $Ax + By + Cz$ and $A'x + B'y + C'z$ of the same plane curve are said to be equivalent if they lie in the same orbit under conjugation, i.e. if there exists an invertible complex matrix $U$ that satisfies

$$U \cdot (Ax + By + Cz) \cdot U^T = A'x + B'y + C'z.$$ 

We call an equivalence class of complex representations real (resp. real definite) if it contains a real (resp. real definite) representative. Deciding whether a given complex representation is equivalent to a real or real definite one is rather difficult. We shall see that the number of equivalence classes of complex representations \(^1\) is finite, and, for smooth curves, the precise number is known (Thm. 2.1). Using more general results of Vinnikov \([24]\), we also derive the number of real and real definite equivalence classes. If a Helton-Vinnikov curve is smooth then the number of real definite equivalence classes equals $2^g$, where $g = (d-1)/2$ is the genus.

This paper concerns the computational problem of constructing one representative from each equivalence class for a given polynomial $f(x, y, z)$. As a warm-up example, consider the following elliptic curve in Weierstrass normal form:

$$f(x, y, z) = (x + ay)(x + by)(x + cy) - xz^2.$$ 

Here $a, b, c$ are distinct non-zero reals. This cubic has precisely three inequivalent linear symmetric determinantal representations over $\mathbb{C}$, given by the matrices

$$\begin{bmatrix}
x + ay & z\sqrt{\frac{b}{b-c}} & z\sqrt{\frac{c}{c-a}} \\
z\sqrt{\frac{b}{b-c}} & x + by & 0 \\
z\sqrt{\frac{c}{c-a}} & 0 & x + cy
\end{bmatrix}, \quad \begin{bmatrix}
x + ay & z\sqrt{\frac{a}{a-c}} & 0 \\
z\sqrt{\frac{a}{a-c}} & x + by & z\sqrt{\frac{c}{c-a}} \\
z\sqrt{\frac{a}{a-c}} & 0 & x + cy
\end{bmatrix}, \quad \begin{bmatrix}
x + ay & 0 & z\sqrt{\frac{a}{a-b}} \\
z\sqrt{\frac{a}{a-b}} & x + by & z\sqrt{\frac{b}{b-a}} \\
z\sqrt{\frac{a}{a-b}} & 0 & x + cy
\end{bmatrix}$$

All three matrices are non-real if $a, b$ and $c$ have the same sign, and otherwise two of the matrices are real. For instance, if $a < 0$ and $0 < b < c$ then the first two matrices are real. In that case, the cubic is a Helton-Vinnikov curve, and its bounded region, when drawn in the affine plane $\{x = 1\}$, is the spectrahedron

$$\left\{ (y, z) \in \mathbb{R}^2 : \begin{bmatrix}
1 + ay & z\sqrt{\frac{a}{a-c}} & 0 \\
z\sqrt{\frac{a}{a-c}} & 1 + by & z\sqrt{\frac{c}{c-a}} \\
0 & z\sqrt{\frac{c}{c-a}} & 1 + cy
\end{bmatrix} \succeq 0 \right\}.$$ 

The symbol “$\succeq$” means that the matrix is positive semidefinite. This spectrahedron is depicted in Figure 1 for the parameter values $a = -1$, $b = 1$ and $c = 2$.  

This article is organized as follows. In Section 2 we translate (1) into a system of polynomial equations in the matrix entries of $A, B, C$, we determine the number of solutions (in Theorem 2.1), and we discuss practical aspects of computing these solutions using both symbolic and numeric software. Section 3 is devoted to geometric constructions for obtaining the representation (1). Following Dixon [10], these require finding contact curves of degree $d - 1$ for the given curve of degree $d$.

An explicit formula for (1) appears in the article of Helton and Vinnikov [16, Eq. 4.2]. That formula requires the numerical evaluation of Abelian integrals and theta functions. In Section 4, we explain the Helton-Vinnikov formula, and we report on our computational experience with the implementations of [7, 8, 9] in the Maple package algcurves. In Section 5 we focus on the case of quartic polynomials and relate our results in [21] to the combinatorics of theta functions. Smooth quartics have 36 inequivalent representations (1). In the Helton-Vinnikov case, twelve of these are real, but only eight are real definite. One of our findings is an explicit quartic in $\mathbb{Q}[x, y, z]$ that has all of its 12 real representations over $\mathbb{Q}$.

2. Solving Polynomial Equations

Our given input is a homogeneous polynomial $f(x, y, z)$ of degree $d$, usually over $\mathbb{Q}$. We assume for simplicity that the corresponding curve in the complex projective plane is smooth, we normalize so that $f(x, 0, 0) = x^d$, and we further assume that the factors of the binary form $f(x, y, 0) = \prod_{i=1}^{d}(x + \beta_i y)$ are distinct. Under these hypotheses, every equivalence class of representations (1) contains a representative where $A$ is the identity matrix and $B$ is the diagonal matrix with entries $\beta_1 < \beta_2 < \cdots < \beta_d$. This follows from the linear algebra fact that any two quadratic forms with distinct eigenvalues can be diagonalized simultaneously over $\mathbb{C}$. See Section IX.3 in Greub’s text book [13] or the proof of Theorem 4.3 in [21].
After fixing the choices $A = \text{diag}(1, 1, \ldots, 1)$ and $B = \text{diag}(\beta_1, \beta_2, \ldots, \beta_d)$ for the first two matrices, we are left with the problem of finding the $\binom{d+1}{2}$ entries of the symmetric matrix $C = (c_{ij})$. By equating the coefficients of all terms $x^\alpha y^\beta z^\gamma$ with $\gamma \geq 1$ on both sides of (1), we obtain a system of $\binom{d+1}{2}$ polynomial equations in the $\binom{d+1}{2}$ unknowns $c_{ij}$. More precisely, the coefficient of $x^\alpha y^\beta z^\gamma$ in (1) leads to an equation of degree $\gamma$ in the $c_{ij}$. We are thus faced with the problem of solving a square system of polynomial equations. The expected number of complex solutions of that system is, according to Bézout’s Theorem,

$$1^d \cdot 2^{d-1} \cdot 3^{d-2} \cdot 4^{d-3} \ldots (d-1)^2 \cdot d.$$  

This estimate overcounts the number of equivalence classes of representations (1) because we can conjugate the matrix $Ax + By + Cz$ by a diagonal matrix whose entries are $+1$ or $-1$. This conjugation does not change $A$ or $B$ but it leads to $2^{d-1}$ distinct matrices $C$ all of which are equivalent. Hence, we can expect the number of inequivalent linear matrix representations (1) to be bounded above by

$$3^{d-2} \cdot 4^{d-3} \ldots (d-1)^2 \cdot d.$$  

We shall refer to this number as the Bézout bound for our problem.

It is a result in classical algebraic geometry that the number of complex solutions to our equations is finite, and the precise number of solutions is in fact known as well. The following theorem summarizes both what is known for arbitrary smooth curves over $\mathbb{C}$ and what can be shown for Helton-Vinnikov curves over $\mathbb{R}$:

**Theorem 2.1.** The number of equivalence classes of linear symmetric determinantal representations (1) of a generic smooth curve of degree $d$ in the projective plane is

$$2^{(d-1)-1} \cdot \left(2^{\binom{d-1}{2}} + 1\right),$$  

unless $d \geq 11$ and $d$ is congruent to $\pm 3$ modulo 8, when the number drops by one. In the case of a Helton-Vinnikov curve, the number of real equivalence classes of symmetric linear determinantal representations (1) is either $2^{(d-1)-1}(2^{\binom{d}{2}}+1)$ or one less. The number of real definite equivalence classes is precisely $2^{\binom{d}{2}}$.

**Sketch of Proof.** The equivalence classes of representations (1) correspond to ineffective even theta characteristics [3] on a smooth curve of genus $g = (d-1)/2$. The number of even theta characteristics is $2^{g-1}(2^g + 1)$, and all even theta characteristics are ineffective for $d \leq 5$ and $d \equiv 0, 1, 2, 4, 6, 7 \mod 8$. In all other cases there is precisely one effective even theta characteristic, provided the curve is generic.

This was shown by Meyer-Brandis in his 1998 diploma thesis [19], and it refines results known classically in algebraic geometry [11] Chapters 4-5. The count of real and real definite representations will be proved at the end of Section 4.
The following table lists the numbers in (3) and (4) for small values of \(d\):

| genus \(g\) | 2  | 3  | 4  | 5  | 6  | 7  |
|-------------|----|----|----|----|----|----|
| Bézout bound| 1  | 3  | 36 | 2160 | 777600 | 1959552000 |
| True number | 1  | 3  | 36 | 2080 | 524800 | 536887296 |

This table shows that computing all solutions to our equations is a challenge when \(d \geq 6\). Below we shall discuss some computer experiments we conducted for \(d \leq 5\).

As before, we fix \(A\) to be the \(d \times d\) identity matrix, denoted \(\text{Id}_d\), and we fix \(B\) to be the diagonal matrix with entries \(\beta_1 < \cdots < \beta_d\). We also fix the diagonal entries of \(C\) since these are determined by solving the \(d\) linear equations that arise by comparing the coefficient of any of the \(d\) monomials \(x^iy^{d-i-1}z\) in (1). They are expressed in terms of \(f\) and the \(\beta_i\) by the following explicit formula:

\[
c_{ij} = \beta_i \cdot \frac{\partial f}{\partial y}(-\beta_i, 0, 1) \quad \text{for } i = 1, 2, \ldots, d.
\]

We are thus left with a system of \(\binom{d}{2}\) equations in the \(\binom{d}{2}\) off-diagonal unknowns \(c_{ij}\). In order to remove the extraneous factor of \(2^{d-1}\) in the Bézout bound \(2\) coming from sign changes on the rows and columns of \(C\), we can perform a multiplicative change of coordinates as follows: \(x_{ij} = c_{ij}^2\) for \(j = 1, 2, \ldots, d\) and \(x_{ij} = c_{i1}c_{ij}\) for \(2 \leq i < j \leq d\). This translates our system of polynomial equations in the \(c_{ij}\) into a system of Laurent polynomial equations in the \(x_{ij}\), and each solution to the latter encodes an equivalence class of \(2^{d-1}\) solutions to the former.

**Example 2.2.** Let \(d = 4\). We shall illustrate the two distinct formulations of the system of equations to be solved. We fix a quartic Helton-Vinnikov polynomial

\[
f(x, y, z) = \det \begin{bmatrix}
x + \beta_1 y + \gamma_{11} z & \gamma_{12} z & \gamma_{13} z & \gamma_{14} z \\
\gamma_{12} z & x + \beta_2 y + \gamma_{22} z & \gamma_{23} z & \gamma_{24} z \\
\gamma_{13} z & \gamma_{23} z & x + \beta_3 y + \gamma_{33} z & \gamma_{34} z \\
\gamma_{14} z & \gamma_{24} z & \gamma_{34} z & x + \beta_4 y + \gamma_{44} z
\end{bmatrix}
\]

where \(\beta_i\) and \(\gamma_{jk}\) are rational numbers. From the quartic \(f(x, y, z)\) alone we can recover the \(\beta_i\) and the diagonal entries \(\gamma_{jj}\) as described above. Our aim is now to compute all points \((c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34})\) \(\in \mathbb{C}^6\) that satisfy the identity

\[
\det \begin{bmatrix}
x + \beta_1 y + \gamma_{11} z & c_{12} z & c_{13} z & c_{14} z \\
c_{12} z & x + \beta_2 y + \gamma_{22} z & c_{23} z & c_{24} z \\
c_{13} z & c_{23} z & x + \beta_3 y + \gamma_{33} z & c_{34} z \\
c_{14} z & c_{24} z & c_{34} z & x + \beta_4 y + \gamma_{44} z
\end{bmatrix} = f(x, y, z).
\]

The coefficient of \(z^4\) gives one equation of degree 4 in the six unknowns \(c_{ij}\), the coefficients of \(xz^3\) and \(yz^3\) give two cubic equations, and the coefficient of \(x^2z^2\) and \(y^2z^2\) give three quadratic equations in the \(c_{ij}\). The number of solutions in \(\mathbb{C}^6\) to this system of equations is equal to \(2^63^4 = 288\). These solutions can be found using symbolic software, such as **Singular** \([\text{6}]\). However, the above formulation has the disadvantage that each equivalence class of solutions appears eight times.
We note that, for generic choices of $\beta_i, \gamma_{jk}$, all solutions lie in the torus $(\mathbb{C}^*)^6$ where $\mathbb{C}^* = \mathbb{C}\{0\}$, and we shall now assume that this is the case. Then the 8-fold redundancy can be removed by working with the following invariant coordinates:

$$x_{12} = c_{12}^2, \quad x_{13} = c_{13}^2, \quad x_{14} = c_{14}^2,$$

$$x_{23} = c_{12}c_{13}c_{23}, \quad x_{24} = c_{12}c_{14}c_{24}, \quad x_{34} = c_{13}c_{14}c_{34}.$$ 

We rewrite our six equations in these coordinates by performing the substitution:

$$c_{12} = x_{12}^{-1/2}, \quad c_{13} = x_{13}^{-1/2}, \quad c_{14} = x_{14}^{-1/2},$$

$$c_{23} = \frac{x_{23}}{x_{12}^{-1/2} x_{13}}, \quad c_{24} = \frac{x_{24}}{x_{12}^{-1/2} x_{14}}, \quad c_{34} = \frac{x_{34}}{x_{13}^{-1/2} x_{14}}.$$ 

This gives six Laurent polynomial equations in six unknowns $x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$. They have precisely 36 solutions in $(\mathbb{C}^*)^6$, one for each equivalence class. □

While the solution of the above equations using symbolic Gröbner-based software is easy for $d = 4$, we found that this is no longer the case for $d \geq 5$. For $d = 5$, it was necessary to employ tools from numerical algebraic geometry, and we found that Bertini [4] works well for our purpose. The computation reported below is due to Charles Chen, an undergraduate student at UC Berkeley. This was part of Chen's term project in convex algebraic geometry during Fall 2010.

For a concrete example, let us consider the following polynomial which defines a smooth Helton-Vinnikov curve of degree $d = 5$:

$$f(x, y, z) = x^5 + 3x^4y - 2x^4z - 5x^3y^2 - 12x^3z^2 - 15x^2y^3 + 10x^2y^2z - 28x^2yz^2 + 14x^2z^3 + 4xy^4 - 6xy^2z^2 - 12xyz^3 + 26xz^4 + 12y^5 - 8y^4z - 32y^3z^2 + 16y^2z^3 + 48yz^4 - 24z^5.$$ 

The symmetric linear determinantal representation we seek has the form

$$\begin{bmatrix} x + y & 0 & 0 & 0 & 0 \\ 0 & x + 2y & 0 & 0 & 0 \\ 0 & 0 & x - y & 0 & 0 \\ 0 & 0 & 0 & x - 2y & 0 \\ 0 & 0 & 0 & 0 & x + 3y - 2z \end{bmatrix} + C \cdot z,$$

where $C = (c_{ij})$ is an unknown symmetric $5 \times 5$-matrix with zeros on the diagonal. This leads to a system of 10 polynomial equations in the 10 unknowns $c_{ij}$, namely, 4 quadrics, 3 cubics, 2 quartics and one quintic. The number of complex solutions equals $16 \cdot 2080 = 33280$, which is less than the Bézout bound of $2^4 \cdot 3^3 \cdot 4^2 \cdot 5 = 16 \cdot 2160 = 34560$. One of the 33280 solutions is the following integer matrix, which we had used to construct $f(x, y, z)$ in the first place:

$$C = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix}$$

Of course, the other 15 matrices in the same equivalence class have the same friendly integer entries. The other $16 \cdot 2079 = 33264$ complex solutions were found
metric linear determinantal representations of a plane curve of degree \( d \) in classical algebraic geometry states that the equivalence classes of symmetric linear determinantal representations of a plane curve of degree \( d \) are in one-to-one correspondence with certain systems of contact curves of degree \( d - 1 \). Following [11] Prop. 4.1.6], we now state this in precise terms. Suppose that our given polynomial is \( f = \det(M) \) where \( M = (t_{ij}) \) is a symmetric \( d \times d \)-matrix of linear forms in \( x, y, z \). We can then form the \( d \times d \) adjoint matrix, \( \text{adj}(M) \), whose entry \( m_{ij} \) is the \( (i, j) \)-th \( (d - 1) \)-minor of \( M \) multiplied by \((-1)^{i+j}\). For any vector of parameters \( u = (u_1, u_2, \ldots, u_d)^T \), we consider the degree \( d - 1 \) polynomial

\[
g_u(x, y, z) = u^T \text{adj}(M)u.
\]

The curve \( V(g_u) \) has degree \( d - 1 \), and it is a contact curve, which means that all intersection points of \( V(f) \) and \( V(g_u) \) have even multiplicity, generically multiplicity 2. To see this, we use [11] Lemma 4.1.7], which states that, for any \( u, v \in \mathbb{C}^d \),

\[
g_u(x, y, z) \cdot g_u(x, y, z) - (u^T \text{adj}(M)v)^2 \in (f) \quad \text{in } \mathbb{C}[x, y, z].
\]

In particular, for \( u = e_i, v = e_j \), this shows that both \( V(m_{ii}) \) and \( V(m_{jj}) \) are contact curves, and \( V(m_{ij}) \) meets \( V(f) \) in their \( d(d - 1) \) contact points.

We say that two contact curves \( V(g_1) \) and \( V(g_2) \) of degree \( r \) lie in the same system if there exists another curve \( V(h) \) of degree \( r \) that meets \( V(f) \) precisely in the \( r \cdot d \) points \( V(f, g_1) \cup V(f, g_2) \). A system of contact curves is called syzygetic if it contains a polynomial of the form \( \ell^2g \), where \( \ell \) is linear and \( g \) is a contact curve of degree \( r - 2 \), and azygetic otherwise. A contact curve of \( V(f) \) is called syzygetic, resp. azygetic, if it lies in a system that is syzygetic, resp. azygetic.

Dixon [10] proved that the contact curves \( V(g_u) \) are azygetic and all azygetic contact curves of degree \( d - 1 \) appear as \( g_u \) for some determinantal representation \( f = \det(M) \). In particular, he gives a method of constructing a determinantal representation \( M \) for \( f \) starting from one azygetic contact curve of degree \( d - 1 \).

The input to Dixon’s algorithm is a azygetic contact curve \( g \) of degree \( d - 1 \) of the given curve \( f \) of degree \( d \). Given the two polynomials \( f \) and \( g \), the algorithm constructs the matrix \( \bar{M} = \text{adj}(M) \) in [6]. It proceeds as follows. Since \( V(g) \) meets \( V(f) \) in \( d(d - 1)/2 \) points, the vector space of polynomials of degree \( d - 1 \) vanishing numerically using the software Bertini [4]. Of these, 16 · 63 are real. Chen’s code, based on Bertini, outputs one representative per class. One of the real solutions is

\[
C \approx \begin{bmatrix}
0 & 1.8771213868 & 0.1333876113 & 0.3369345269 & 0.2151885297 \\
1.8771213868 & 0 & 1.326201851 & 0.1725327846 & 1.0570303927 \\
0.1333876113 & 1.326201851 & 0 & -2.093203944 & -0.8767796987 \\
0.3369345269 & 0.1725327846 & -2.093203944 & 0 & -0.7659896773 \\
0.2151885297 & 1.0570303927 & -0.8767796987 & -0.7659896773 & 0
\end{bmatrix}.
\]
at these points (without multiplicity) has dimension $d$. Let $m_{11} = g$ and extend $m_{11}$ to a basis $\{m_{11}, m_{12}, \ldots, m_{1d}\}$ of this vector space. For $i, j \in \{2, 3, \ldots, d\}$, the polynomial $m_{ij}m_{ij}$ vanishes to order two on $\mathcal{V}(f, m_{11})$, so it lies in the ideal $(m_{11}, f)$. Using the Extended Buchberger Algorithm, one finds a degree $d-1$ polynomial $m_{ij}$ such that $m_{1i}m_{1j} - m_{11}m_{ij} \in (f)$. The $d \times d$-matrix $\hat{M} = (m_{ij})$ has rank 1 modulo $(f)$, therefore its $2 \times 2$-minors are multiples of $f$. This implies that the adjoint matrix $\text{adj}(\hat{M}) = \text{det}(\hat{M}) \cdot \hat{M}^{-1}$ has the form $\lambda f^{d-2} \cdot M$ where $\lambda \in \mathbb{C}\{0\}$ and $M$ is a symmetric matrix of linear forms with $\text{det}(M) = f$. One could run through this construction starting from a syzygetic contact curve, but the resulting matrix $\hat{M}$ would have determinant zero.

The main challenge with Dixon’s algorithm is to construct its input polynomial $g$. Suitable contact curves are not easy to find. A symbolic implementation of the algorithm may involve large field extensions, and we found it equally difficult to implement numerically. For further discussions see \cite{19, \S2.2} and \cite{21, \S2}.

**Remark 3.1.** Starting from a real azygetic contact curve $g$, one can use Dixon’s method to produce a real determinantal representation of $f$. A determinantal representation $M$ is equivalent to its conjugate $\overline{M}$ if and only if the system of contact curves $\{u^T\text{adj}(M)u : u \in \mathbb{C}^4\} \subset \mathbb{C}[x, y, z]_3$ is real, i.e., invariant under conjugation. The representation $M$ is equivalent to a real matrix if and only if this system contains a real contact curve. By \cite{14, Prop 2.2}, if the curve $\mathcal{V}(f)$ has real points, then these two notions of reality agree. However, this approach does not easily reveal whether an equivalence class contains a real definite representative.

**Example 3.2.** The following Helton-Vinnikov quartic was studied in \cite{21, Ex. 4.1}:

$$f(x, y, z) = 2x^4 + y^4 + z^4 - 3x^2y^2 - 3x^2z^2 + y^2z^2.$$
It is shown on the left in Figure 2. It has a symmetric determinantal representation
\[
f(x, y, z) = \det \begin{bmatrix} ux + y & 0 & az & bz \\ 0 & ux - y & cz & dz \\ az & cz & x + y & 0 \\ bz & dz & 0 & x - y \end{bmatrix},
\]
where \(a = -0.5746..., b = 1.0349..., c = 0.6997..., d = 0.4800...\) and \(u = \sqrt{2}\) are the coordinates of a real zero of the following maximal ideal in \(\mathbb{Q}[a, b, c, d, u]\):
\[
\langle u^2 - 2, 256d^8 - 384d^6u + 256d^4 - 384d^4u + 672d^4 - 336d^2u + 448d^2 - 84u + 121, \\
23c + 7584d^7u + 10688d^5u - 5872d^5u - 8384d^5u + 1806d^3u + 2452d^3u - 181du - 307d, \\
23b + 5760d^7u + 8192d^7u + 4688d^5u - 6512d^5u + 1452d^5u + 2200d^3u - 212du - 232d, \\
23a - 1440d^7u - 2048d^7u + 1632d^7u + 2272d^7u - 570d^5u - 872d^3u + 99du + 81d \rangle.
\]
The principal 3×3-minors of the 4×4-matrix in (7) are Helton-Vinnikov polynomials of degree 3. They are the four contact cubics shown on the right in Figure 2.

In summary, Dixon’s method furnishes an explicit bijection between equivalence classes of symmetric determinantal representations (1) of a fixed curve \(V(f)\) of degree \(d\) and azygetic systems of contact curves of \(V(f)\) of degree \(d - 1\).

For \(d = 3\), there is another geometric approach to finding representations (1). We learned this from Didier Henrion who attributes it to Frédéric Han. Suppose we are given a general homogeneous cubic \(f(x, y, z)\). We first compute the Hessian
\[
\text{Hes}(f) = \det \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}.
\]
This is also a cubic polynomial, and hence so is the linear combination \(t \cdot f + \text{Hes}(f)\), where \(t\) is a parameter. We now take the Hessian of that new cubic, with the aim of recovering \(f\). It turns out that we can do this by solving a cubic equation in \(t\).

**Proposition 3.3.** There exist precisely three points \((s, t) \in \mathbb{C}^2\) such that
\[
s \cdot f = \text{Hes}(t \cdot f + \text{Hes}(f)).
\]
The resulting three symmetric determinantal representations of \(f\) are inequivalent.

**Proof.** The statement is invariant under linear changes of coordinates in \(\mathbb{P}^2\), so, by [1] Lemma 1], we may assume that the given cubic is in Hesse normal form:
\[
f(x, y, z) = x^3 + y^3 + z^3 - mxyz.
\]
In that case, the result follows from the discussion in [1] page 139]. Alternatively, we can solve the equations obtained by comparing coefficients in (8). This leads to
\[
t^3 - (12m^4 + 2592m^2)t - 16m^6 + 8640m^3 + 93312 = 0 \quad \text{and} \quad
s = (12m^4 + 2592m^2)t^2 + (48m^8 - 25920m^3 - 279936)t + 48m^8 + 20736m^5 + 2239488m^2.
\]
This has three solutions \((s, t) \in \mathbb{C}^3\). The resulting representations (1) are inequivalent because the Hessian normal form of a \(\text{PGL}(3, \mathbb{C})\)-orbit of cubics is unique. □
4. Evaluating Theta Functions

The proof of the Helton-Vinnikov Theorem relies on a formula, stated in [16, Eq. 4.2], that gives a positive definite determinantal representation of a Helton-Vinnikov curve in terms of theta functions and the period matrix of the curve. Our aim in this section is to explain this formula and to report on computational experiments with it. Numerical algorithms for computing theta functions, period matrices and Abelian integrals have become available in recent years through work of Bobenko, Deconinck, Heil, van Hoeij, Patterson, Schnies, and others [7, 8, 9]. There exists an implementation in Maple, and we used that for our computations. Our Maple worksheet that evaluates the Helton-Vinnikov formula can be found at

\[
\text{www.math.uni-konstanz.de/~plaumann/theta.html}
\]

Before stating the Helton-Vinnikov formula, we review the basics on theta functions. Our emphasis will be on clearly defining the ingredients of the formula rather than explaining the underlying theory. For general background see [20]. Fix \( g \in \mathbb{N} \) and let \( \mathcal{H}^g \) be the Siegel upper half-space, which consists of all complex, symmetric \( g \times g \)-matrices whose imaginary part is positive definite. The Riemann theta function is the holomorphic function on \( \mathbb{C}^g \times \mathcal{H}^g \) defined by the exponential series

\[
\theta(u, \Omega) = \sum_{m \in \mathbb{Z}^g} \exp\left(\pi i (m^T \Omega m + 2m^T u)\right),
\]

where \( i = \sqrt{-1} \), \( u = (u_1, \ldots, u_g) \in \mathbb{C}^g \) and \( \Omega \in \mathcal{H}^g \). We will only need to consider \( \theta(u, \Omega) \) as a function in \( u \), for a fixed matrix \( \Omega \), so we may drop \( \Omega \) from the notation. In other words, we define \( \theta : \mathbb{C}^g \to \mathbb{C} \) by \( \theta(u) = \theta(u, \Omega) \). The theta function is quasi-periodic with respect to the lattice \( \mathbb{Z}^g + \Omega \mathbb{Z}^g \subset \mathbb{C}^g \), which means

\[
\theta(u + m + \Omega n) = \exp(\pi i (-2m^T u - n^T \Omega n)) \cdot \theta(u), \quad \text{for all } m, n \in \mathbb{Z}^g.
\]

A theta characteristic is a vector \( \varepsilon = a + \Omega b \in \mathbb{C}^g \) with \( a, b \in \{0, \frac{1}{2}\}^g \). The function

\[
\theta[\varepsilon](u) = \exp(\pi i (a^T \Omega a + 2a^T (u + b))) \cdot \theta(u + \Omega a + b)
\]

is the theta function with characteristic \( \varepsilon \). There are \( 2^{2g} \) different theta characteristics, indexed by ordered pairs \( (2a, 2b) \) of binary vectors in \( \{0, 1\}^g \). We also use the notation \( \theta\left[\begin{bmatrix} 2a \\ 2b \end{bmatrix}\right](u) \) for the function \( \theta[\varepsilon](u) \). For \( \varepsilon = 0 \) we simply recover \( \theta(u) \).

Let \( f \in \mathbb{C}[x, y, z]_d \) be a homogeneous polynomial of degree \( d \). Assume that the projective curve \( X = \mathcal{V}_C(f) \) is smooth and thus a compact Riemann surface of genus \( g = \frac{1}{2}(d-1)(d-2) \). Let \( (\omega_1, \ldots, \omega_g) \) be a basis of the \( g \)-dimensional complex vector space of holomorphic 1-forms on \( X \), and let \( \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \) be closed 1-cycles on \( X \) that form a symplectic basis of \( H_1(X, \mathbb{Z}) \cong \mathbb{Z}^g \times \mathbb{Z}^g \). This means that the intersection numbers of these cycles on \( X \) satisfy \( (\alpha_j, \alpha_k) = (\beta_j, \beta_k) = 0 \), and \( (\alpha_j, \beta_k) = \delta_{jk} \) for all \( j, k = 1, \ldots, g \). The period matrix of the curve \( X \) with respect to these bases is the complex \( g \times 2g \)-matrix \( (\Omega_1, \Omega_2) \) whose entries are

\[
(\Omega_1)_{jk} = \int_{\alpha_k} \omega_j^\prime \quad \text{and} \quad (\Omega_2)_{jk} = \int_{\beta_k} \omega_j^\prime, \quad \text{for } j, k = 1, \ldots, g.
\]
The $g \times g$-matrices $\Omega_1$ and $\Omega_2$ are invertible. Performing the coordinate change

$$(\omega_1, \ldots, \omega_g) = (\omega'_1, \ldots, \omega'_g) \cdot (\Omega_1^{-1})^T$$

leads to a basis in which the period matrix is of the form $(\text{Id}_g \mid \Omega_1^{-1} \Omega_2)$. The basis $\omega = (\omega_1, \ldots, \omega_g)$ is called a normalized basis of differentials and depends uniquely on the symplectic homology basis. The $g \times g$-matrix $\Omega = \Omega_1^{-1} \Omega_2$ is symmetric and lies in the Siegel upper half-space $H^g$. It is called the Riemann period matrix of the polynomial $f(x, y, z)$ with respect to the homology basis $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$.

With the given polynomial $f$ we have now associated a system $\{\theta[\varepsilon](\cdot, \Omega)\}$ of $2^g$ theta functions with characteristics. A theta characteristic $\varepsilon = a + \Omega b$ is called even (resp. odd) if the scalar product $(2a)^T(2b)$ of its binary vector labels is an even (resp. odd) integer. This is equivalent to $\theta[\varepsilon]$ being an even (resp. odd) function in $u$. In symbols, we have $\theta[\varepsilon](-u) = (-1)^{4a^Tb}\theta[\varepsilon](u)$. Changing symplectic bases of $H_1(X, \mathbb{Z})$ corresponds to the right-action of the symplectic group $\text{Sp}_{2g}(\mathbb{Z})$ on the period matrix $(\Omega_1 \mid \Omega_2)$. This action will permute the theta characteristics. In particular, there is no distinguished even theta characteristic 0.

Finally, we define the Abel-Jacobi map by $\phi(P) = (\int_{P_0}^P \omega_1, \ldots, \int_{P_0}^P \omega_g)^T$ for $P \in X$, where $P_0 \in X$ is any fixed base point. This is a holomorphic map, but it is well-defined only up to the period lattice $\Lambda = \mathbb{Z}^g + \Omega \mathbb{Z}^g \subset \mathbb{C}^g$. In other words, the Abel-Jacobi map is a holomorphic map $\phi : X \rightarrow \text{Jac}(X) = \mathbb{C}^g/\Lambda$.

We are now ready to state the formula for (1) in terms of theta functions.

**Theorem 4.1 (Helton-Vinnikov [16]).** Let $f \in \mathbb{R}[x, y, z]_d$ with $f(1, 0, 0) = 1$ and let $X$ denote $V_C(f) \subset \mathbb{P}^2$. We make the following two assumptions:

1. The curve $X$ is a non-rational Helton-Vinnikov curve with the point $(1 : 0 : 0)$ inside its innermost oval. The latter means that, for all $v \in \mathbb{R}^3 \setminus \{0\}$, the univariate polynomial $f(v + t \cdot (1, 0, 0)) \in \mathbb{R}[t]$ has only real zeros.
2. The $d$ real intersection points of $X$ with the line $\{z = 0\}$ are distinct non-singular points $Q_1, \ldots, Q_d$, with coordinates $Q_i = (-\beta_j : 1 : 0)$ where $\beta_j \neq 0$. Then $f(x, y, z) = \det(\text{Id}_x x + By + Cz)$ where $B = \text{diag}(\beta_1, \ldots, \beta_d)$ and $C$ is real symmetric with diagonal entries $c_{jj}$ as in [3]. The off-diagonal entries of $C$ are

$$c_{jk} = \frac{\beta_k - \beta_j}{\theta[\delta](0)} \cdot \frac{\theta[\varepsilon](\phi(Q_k) - \phi(Q_j))}{\theta[\varepsilon](\phi(Q_k) - \phi(Q_j))} \sqrt{\frac{\omega \cdot \nabla \theta[\varepsilon](0)}{-d(z/y)}}(Q_j) \sqrt{\frac{\omega \cdot \nabla \theta[\varepsilon](0)}{-d(z/y)}}(Q_k). \quad (10)$$

Here $\varepsilon$ is an arbitrary odd theta characteristic and $\delta$ is a suitable even theta characteristic with $\theta[\delta](0) \neq 0$. The theta functions are taken with respect to a normalized basis of differentials $\omega = (\omega_1, \ldots, \omega_d)$, and $\phi : X \rightarrow \text{Jac}(X)$ is the Abel-Jacobi map.

The remarkable expression for the constants $c_{jk}$ in (10) does not depend on the choice of the odd characteristic $\varepsilon$. If the curve $X$ is smooth, then all equivalence classes of symmetric determinantal representations are obtained when $\delta$ runs through all non-vanishing even theta characteristics. The proof of Theorem 4.1 given in [16] is only an outline. It relies heavily on earlier results on Riemann
surfaces due to Ball and Vinnikov in \cite{2, 24}. As we found these not easy to read, we were particularly pleased to be able to verify Theorem 4.1 with our experiments.

The Helton-Vinnikov formula \eqref{eq:HV_formula} remains valid when \(X\) is a singular curve. In that case the period matrix, the differentials, and the Abel-Jacobi map are meant to be defined on the desingularization of \(X\), a compact Riemann surface of genus \(g\) with \(g < \frac{1}{2}(d - 1)(d - 2)\). The formula holds as stated, but one no longer obtains all equivalence classes of symmetric determinantal representations.

The Riemann period matrix, the theta functions, their directional derivatives, and the Abel-Jacobi-map can all be evaluated numerically in recent versions of Maple. When computing the expressions under the square roots, note that both the numerator and denominator are 1-forms on \(X\). Every holomorphic 1-form on the curve \(X\) can be written as \(r \cdot du\), where \(u = z/x\), \(v = y/x\), and \(r\) is a rational function in \(u\) and \(v\). The \texttt{algcurves} package in Maple will compute \(\omega\) in this form, so we obtain \(\omega_j = r_j(u, v) \cdot du\). To evaluate the 1-form \(d(z/y)\), we set \(h(u, v) = f(1, v, u)\) and use the identity \(dh(u, v) = \frac{\partial h}{\partial u} du + \frac{\partial h}{\partial v} dv = 0\). This implies

\[
d(z/y) = d(u/v) = \frac{1}{v} du - \frac{u}{v^2} dv = \left(\frac{1}{v} + \frac{u \frac{\partial h}{\partial u}}{v^2 \frac{\partial h}{\partial v}}\right) du,
\]

so that \(d(z/y)(Q_j) = -\beta_j du\). Under the square root signs in \eqref{eq:HV_formula}, the factor \(du\) appears in the numerator \(\omega\) and also in the denominator \eqref{eq:1_form}, and we cancel it. Hence the expressions under the square roots are rational functions, namely \(r(u, v) \cdot \nabla \theta(z)(0)\) divided by the expression in parentheses on the right in \eqref{eq:1_form}, where \(r(u, v)\) is the vector of rational functions \(r_j(u, v)\).

While the evaluation of theta functions is numerically stable, we found the computation of the period matrix and the Abel-Jacobi map to be more fragile. Computing the \(d\) vectors \(\phi(Q_j)\) is also by far the most time-consuming step. Nonetheless, Maple succeeded in correctly evaluating the Helton-Vinnikov formula for a wide range of curves with \(d \leq 4\), and for some of degree \(d = 5\). However, the off-diagonal entries in \eqref{eq:HV_formula} we found in our computations were sometimes wrong by a constant factor (independent of \(j, k\)), for reasons we do not currently understand.

For a concrete example take the quartic in Example 3.2. Using the formula \eqref{eq:HV_formula} we obtained all eight definite determinantal representations \(\det(Id_d x + By + Cz)\). Our Maple code runs for a few minutes and finds all solutions accurately with a precision of 20 digits. We verified this using the prime ideal in Example 3.2.

The case of smooth quintics (genus 6) is already a challenge. With the help of Bernard Deconinck, we were able to compute a determinantal representation \eqref{eq:HV_formula} with an error of less than \(10^{-3}\) for the quintic polynomial at the end of Section 2. However, the representation we obtained was not real (see Remark 4.4).

We conclude this section with the proof of the second part of Thm. 2.1 and discuss what are the suitable choices for the even theta characteristic \(\delta\) in Thm. 4.1 that will lead to real and real definite equivalence classes of representations. Note that a representation obtained from the theorem is real definite if and only if the matrix \(C\) is real. The real non-definite equivalence classes of representations
correspond to the case when $C$ is a non-real matrix for which there exists a matrix $U \in \text{GL}_d(\mathbb{C})$ such that $UU^T$, $UBU^T$, and $UCU^T$ have all real entries. Whether such $U$ exists for given complex symmetric matrices $B$ and $C$ is not at all obvious. An explicit example is given in Ex. 5.2.

For the proof of Thm. 2.1, we repeat the relevant part of the statement:

**Theorem 4.2.** Let $f \in \mathbb{R}[x, y, z]_d$ be homogeneous and assume that the projective curve $X = \mathcal{V}_C(f)$ is a smooth Helton-Vinnikov curve. The number of real equivalence classes of symmetric determinantal representations is generically either $2^{g-1}(2^{k-1} + 1)$ or one less, where $k = \left\lceil \frac{d^2}{2} \right\rceil$ is the number of connected components of the set of real points $X(\mathbb{R})$. Of these real equivalence classes, exactly $2^g$ are definite.

**Proof.** The result follows from work of Vinnikov [24] on self-adjoint determinantal representations, which we apply here to our situation. By [24, Prop. 2.2], a symplectic basis of $H_1(X, Z)$ can be chosen in such a way that the Riemann period matrix $\Omega$ satisfies $\Omega + \overline{\Omega} = H$, where $H$ is a $g \times g$ block diagonal matrix of rank $r = g - k + 1$ with $r/2$ blocks $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in the top left corner and all other entries zero.

The linear symmetric determinantal representation obtained by Thm. 4.1 from an even theta characteristic $\delta$ is equivalent to a real one if and only if $\delta$ is real, i.e. invariant under the action of complex conjugation on the $g$-dimensional torus $\text{Jac}(X)$. Since $X(\mathbb{R}) \neq \emptyset$, any conjugation-invariant divisor class on $X$ contains a real divisor (see [14, Prop. 2.2]). From such a divisor, one can construct a symmetric determinantal representation (see [3] or [23]). When the symplectic basis of $H_1(X, Z)$ is chosen as above, the action of complex conjugation on $\text{Jac}(X)$ is given by $\zeta = u + \Omega v \mapsto \overline{\zeta} = u + \Omega(Hu - v)$ (see [24, Prop. 2.2]). For the even theta characteristic $\delta = a + \Omega b$, $a, b \in \{0, \frac{1}{2}\}^g$, the condition $\delta = \overline{\delta}$ in $\text{Jac}(X) = C^g/(Z^g + \Omega Z^g)$ thus becomes

$$Ha \equiv 2b \mod Z^g.$$ 

This happens if and only if $a_1 = \cdots = a_r = 0$. Counting the possible choices of $a_{r+1}, \ldots, a_g$ and $b$, we conclude that there are exactly $2^{g-1}(2^{k-1} + 1)$ even real theta characteristics. From the first part of Thm. 2.1, we know that when $d \equiv \pm 3 \mod 8$, exactly one even theta characteristic vanishes, i.e. $\theta[\delta](0) = 0$. All other even theta characteristics are non-vanishing and therefore correspond to determinantal representations. Furthermore, by [24, Thm. 6.1], an even theta characteristic $\delta = a + \Omega b$ will correspond to a real definite equivalence class if and only if $a = 0$, and all such $\delta$ are always non-vanishing [24, Cor. 4.3]. Thus there are exactly $2^g$ definite representations, since $b$ can be any element of $\{0, \frac{1}{2}\}^g$. □

**Example 4.3.** When $g = 3$ and a homology basis has been picked as above, the even real theta characteristics are given by the 12 binary labels

$$\begin{bmatrix} 000 & 000 & 000 & 000 & 000 & 000 & 000 & 000 & 000 & 001 & 001 & 001 \\ 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 & 000 & 010 & 100 & 110 \end{bmatrix}. \quad (12)$$
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The first eight labels correspond to the definite representations and the last four correspond to the non-definite real equivalence classes of representations.

**Remark 4.4.** The characterization of real and real definite even theta characteristics provided by the proof of Thm. 4.2 depends on the choice of a particular symplectic homology basis. Unfortunately, the current Maple code for computing the period matrix does not give the user any control over the homology basis. This makes it hard to find real representations using Thm. 4.1 in any systematic way.

5. Quartic Curves Revisited

In this section we focus on the case of smooth quartic curves, studied in detail in [21], so we now fix \( d = 4 \) and \( g = 3 \). Quartic curves are special because they have contact lines, i.e. bitangents, and we can explicitly write down higher degree contact curves as products of bitangents. This was exploited in [21] [§2], where we used azygetic triples of bitangents as our input to Dixon’s algorithm (Section 3).

Plane quartics are canonical embeddings of genus 3 curves [11], and there is a close connection between contact curves and theta functions. The 28 bitangents of the curve are in bijection with the 28 odd theta characteristics \( \varepsilon = a + \Omega b \), and this will be made explicit in (15) below. The 36 azygetic systems of contact cubics correspond to the 36 even theta characteristics. As seen in Example 4.3 of the resulting 36 determinantal representations, 12 are real, but only 8 are definite. We can also derive the number 12 from the combinatorics of the bitangents as in [21].

**Proposition 5.1.** A smooth Helton-Vinnikov quartic \( \mathcal{V}(f) \) has exactly 12 inequivalent representations \( f = \det(Ax + By + Cz) \) with \( A, B, C \) symmetric and real.

**Proof.** This is a special case of Theorem 2.1; however, we here give an alternative proof using the setup of [21]. Let \( M \) be a symmetric linear determinantal representation of \( f \) and \( \mathcal{M} \) the system of contact cubics \( \{ u^T \text{adj}(M)u \} \subset \mathbb{C}[x, y, z]^3 \). The representation \( M \) is equivalent to its conjugate \( \overline{M} \) if and only if the system \( \mathcal{M} \) is real, i.e. invariant under conjugation. The representation \( M \) is equivalent to a real matrix if and only if \( \mathcal{M} \) contains a real cubic. The matrix \( M \) induces a labeling of the 28 = \( \binom{8}{2} \) bitangents, \( b_{ij} \), with \( 1 \leq i < j \leq 8 \). The system \( \mathcal{M} \) is real if and only if conjugation acts on the bitangents via this labeling, that is, there exists \( \pi \in S_8 \) such that \( \overline{b_{ij}} = b_{\pi(i)\pi(j)} \). Since \( f \) is a Helton-Vinnikov polynomial, this permutation will be the product of four disjoint transpositions (see [21] Table 1).

Suppose \( \mathcal{M} \) is real, with permutation \( \pi \in S_8 \). The other 35 representations correspond to the \( \binom{8}{4}/2 \) partitions of \( \{1, \ldots, 8\} \) into two sets of size 4. If \( I/F \) is such a partition then the corresponding system of contact cubics contains 56 products of three bitangents, namely \( b_{ij}b_{jk}b_{il} \) and \( b_{im}b_{jm}b_{kl} \) where \( i, j, k, l, m \) are distinct and \( \{ i, j, k, l \} = I \) or \( F \). This system is real if and only if \( \pi \) fixes the partition \( I/F \). There are exactly 11 such partitions: if \( \pi = (12)(34)(56)(78) \), they are

\[
1234|5678, \quad 1256|3478, \quad 1278|3456, \quad 1357|2468, \quad 1358|2467, \quad 1368|2457, \quad 1367|2458, \quad 1457|2368, \quad 1458|2367, \quad 1467|2358, \quad 1468|2357.
\]
Together with the system $\mathcal{M}$, there are 12 real systems of azygetic contact cubics. Next, we will show that each of these systems actually contains a real cubic. To do this, we use contact conics, as the product of a bitangent with a contact conic is a contact cubic. By Lemma 6.7, there exists a real bitangent $b \in \mathbb{R}[x, y, z]_1$ and a real system of contact conics $\mathcal{Q} \subset \mathbb{C}[x, y, z]_2$ such that their product $\{b \cdot q : q \in \mathcal{Q}\}$ lies in the system $\mathcal{M} \subset \mathbb{C}[x, y, z]_3$. Furthermore, by Prop. 6.6, since $\mathcal{V}_R(f)$ is nonempty, every real system of contact conics $\mathcal{Q}$ to $f$ contains a real conic $q$. The desired real contact cubic is the product $b \cdot q$.

The technique in the last paragraph of the above proof led us to the following result: There exists a smooth Helton-Vinnikov quartic $f \in \mathbb{Q}[x, y, z]_4$ that has 12 inequivalent matrix representations over the field $\mathbb{Q}$ of rational numbers.

Example 5.2. The special rational Helton-Vinnikov quartic we found is

$$f(x, y, z) = 93081 x^4 + 53516 x^3 y - 73684 x^2 y^2 + -31504 x y^3 + 9216 y^4 - 369150 x^2 z^2 - 159700 x y z^2 + 57600 y^2 z^2 + 90000 z^4.$$ 

This polynomial satisfies $f(x, y, z) = \det(M)$ where

$$M = \begin{bmatrix}
50x & -25x & -26x - 34y - 25z & 9x + 6y + 15z \\
-25x & 25x & 27x + 18y - 20z & -9x - 6y \\
-26x - 34y - 25z & 27x + 18y - 20z & 108x + 72y & -18x - 12y \\
9x + 6y + 15z & -9x - 6y & -18x - 12y & 6x + 4y
\end{bmatrix}.$$ 

This representation is definite because the matrix $M$ is positive definite at the point $(1 : 0 : 0)$. Hence $\mathcal{V}(f)$ is a Helton-Vinnikov curve with this point in its inner convex oval. Rational representatives for the other seven definite classes are found at our website (9), along with representatives for the four non-definite real classes. One of them is the matrix

$$M_{1468} = \begin{bmatrix}
25x & 0 & -32x + 12y & -60z \\
0 & 25x & 10z & 24x + 16y \\
-32x + 12y & 10z & 6x + 4y & 0 \\
-60z & 24x + 16y & 0 & 6x + 4y
\end{bmatrix}.$$ (14)

We have $\det(M_{1468}) = 4 \cdot f(x, y, z)$, and this matrix is neither positive definite nor negative definite for any real values of $x, y, z$. Any equivalent representation of a multiple of $f$ in the form $\det(\text{Id}_4 x + B y + C z)$ considered in Sections 2 and 4 cannot have all entries of $C$ real. One such representation, for a suitable $U \in \text{GL}_4(\mathbb{C})$, is

$$U^T M_{1468} U = \begin{bmatrix}
x + \frac{64}{71} y & 0 & -\frac{23}{134} \sqrt{26980} i z & -\frac{51}{1633} \sqrt{16330} i z \\
0 & x + \frac{2}{3} y & -\frac{7}{29} \sqrt{570} z & -\frac{2}{29} \sqrt{345} i z \\
-\frac{23}{134} \sqrt{26980} i z & -\frac{7}{29} \sqrt{570} z & x - \frac{4}{9} y & 0 \\
-\frac{51}{1633} \sqrt{16330} z & -\frac{2}{29} \sqrt{345} i z & 0 & x - \frac{18}{29} y
\end{bmatrix}.$$ 

The correspondence between bitangents and odd theta characteristics can be understood abstractly via the isomorphism between the Jacobian of $X$ and the divisor class group $\text{Cl}^0(X)$, and can be turned into an explicit formula for the
bitangents. Let \( u = z/x \), \( v = y/x \) and write \( h(u, v) = f(1, v, u) \). Then a basis for the 3-dimensional complex vector space of holomorphic 1-forms on \( X \) is given by

\[
(\omega_1, \omega_2, \omega_3) = \left( du \frac{\partial h}{\partial v}, vdu \frac{\partial h}{\partial v}, udu \frac{\partial h}{\partial v} \right).
\]

Let \( (\Omega_1|\Omega_2) \) be the period matrix of \( f \) with respect to \( (\omega_1, \omega_2, \omega_3) \) and any symplectic basis of \( H_1(X, \mathbb{Z}) \). Then given an odd theta characteristic \( \varepsilon = a + \Omega b \), the corresponding bitangent is defined by the linear form

\[
b_\varepsilon(x, y, z) = \left( \nabla \theta[\varepsilon](0) \right)^T \cdot \Omega^{-1} \cdot (x, y, z)^T,
\]

where \( \nabla \theta[\varepsilon] \) is the gradient of \( \theta[\varepsilon] \) in the three complex variables \( z_1, z_2, z_3 \). For the proof, see Dolgachev [11, Section 5.5.4]. This holds independently of the symplectic basis of \( H_1(X, \mathbb{Z}) \), but a change of that basis will permute the bitangents.

The formula (15) can be evaluated using the Maple code described in Section 4. This allows us to compute the 8×8-bitangent matrix \( (b_{ij}) \) of [21, Eq. 3.4] directly from the Riemann period matrix \( \Omega \) of the curve \( X \), using a technique due to Riemann described in [15, §2]. In this manner, one computes the symmetric determinantal representations (1) of the curve \( X \) directly from the period matrix \( \Omega \). This computation seems to be a key ingredient in constructing explicit three-phase solutions of the Kadomtsev-Petviashvili equation [12], and we hope that the combinatorial tools developed here and in [21] will be useful for integrable systems.

One of the earliest papers on algorithms for theta functions in genus three was written by Arthur Cayley in 1897. In [5] he gives a concrete bijection between the bitangents \( b_{ij} \) of a plane quartic and the odd theta characteristics, and also between the classes \( I|I^c \) of determinantal representations and the even theta characteristics. We here reproduce a relabeled version of the table in Cayley’s article:

| 2b | 2a | 000 | 100 | 010 | 110 | 001 | 101 | 011 | 111 |
|----|----|-----|-----|-----|-----|-----|-----|-----|-----|
| 000 | 0 | 1238 | 1287 | 1245 | 1468 | 1578 | 1356 | 1347 |
| 100 | 1234 | 48 | 1235 | 35 | 1457 | 16 | 1378 | 27 |
| 010 | 1256 | 1247 | 57 | 46 | 1367 | 1345 | 23 | 18 |
| 110 | 1278 | 37 | 68 | 1236 | 1358 | 25 | 14 | 1567 |
| 001 | 1357 | 1346 | 1478 | 1568 | 12 | 38 | 67 | 45 |
| 101 | 1368 | 26 | 1456 | 17 | 34 | 1248 | 58 | 1235 |
| 011 | 1458 | 1678 | 13 | 28 | 56 | 47 | 1257 | 1246 |
| 111 | 1467 | 15 | 24 | 1348 | 78 | 1237 | 1268 | 36 |

Here a partition \( I|I^c \) of \( \{1, \ldots, 8\} \) is represented by the 4-tuple \( I \) which contains the index 1. For instance, the 4-tuple 1238 corresponds to the even theta characteristic \([100]_0\). Each partition \( I|I^c \) represents a Cremona transformation leading to a new representation \([1]_1\) as described in [21 §3]. The twelve 4-tuples marked in bold face are the real equivalence classes, and this gives a bijection between the lists in (12) and in (13). Likewise, the pairs \( ij \) in Cayley’s table represent bitangents \( b_{ij} \) and the corresponding odd theta characteristics. For instance, the odd characteristic...
Figure 3. Degeneration of a Helton-Vinnikov quartic into four lines.

\[
\begin{bmatrix}
100 \\
111
\end{bmatrix}
\]
represents the bitangent \(b_{15}\). In this manner, we can parametrize the 28 bitangents of all plane quartics explicitly with odd theta functions.

Experts in moduli of curves will be quick to point out that this parametrization should extend from smooth curves to all stable curves. This is indeed the case. For instance, four distinct lines form a stable Helton-Vinnikov quartic such as

\[
f(x, y, z) = xyz(x + y + z).
\]

The bitangent matrix \((b_{ij})\) of this reducible curve has 7 distinct non-zero entries:

\[
\begin{bmatrix}
0 & z & y & y+z & x & x+z & x+y & x+y+z \\
z & 0 & y+z & y & x+z & x & x+y+z & x+y \\
y & y+z & 0 & z & x+y & x+y+z & x & x+z \\
y+z & y & z & 0 & x+y+z & x+y & x+z & x \\
x & x+z & x+y & x+y+z & 0 & z & y & y+z \\
x+z & x & x+y+z & x+y & z & 0 & y+z & y \\
x+y & x+y+z & x & x+z & y & y+z & 0 & z \\
x+y+z & x+y & x+z & x & y+z & y & z & 0
\end{bmatrix}.
\]

All principal 4×4-minors of this 8×8-matrix are multiples of \(f(x, y, z)\), most of them non-zero. They are all in the same equivalence class, which is real but not definite. The entries in the bitangent matrix indicate a partition of the 28 odd theta characteristics into seven groups of four. For instance, the antidiagonal entry \(x + y + z\) corresponds to the four entries 18, 27, 36 and 45 in Cayley’s table, and hence to the four odd theta characteristics

\[
\begin{bmatrix}
001 \\
111
\end{bmatrix}, \begin{bmatrix}
100 \\
111
\end{bmatrix}, \begin{bmatrix}
100 \\
111
\end{bmatrix}, \begin{bmatrix}
001 \\
111
\end{bmatrix}.
\]

If we consider a family of smooth quartics that degenerates to the reducible quartic \(f(x, y, z)\), then its bitangent matrix will degenerate to the above 8×8-matrix, and hence the 28 distinct bitangents of the smooth curve bunch up in
seven clusters of four. This degeneration is visualized in Figure 3. Among the seven limit bitangents are the three lines spanned by pairs of intersection points.

Algebraically, such a degenerating family can be described as a curve over a field with a valuation, such as the field of real Puiseux series \( \mathbb{R}\{\{\varepsilon\}\} \). The notions of spectrahedra and Helton-Vinnikov curves makes perfect sense over the real closed field \( \mathbb{R}\{\{\varepsilon\}\} \). This has been investigated from the perspective of tropical geometry by David Speyer, who proved in [22] that tropicalized Helton-Vinnikov curves are precisely honeycomb curves. We believe that the tropicalization in [22] offers yet another approach to constructing linear determinantal representations \(^{1}\), in addition to the three methods presented here, and we hope to return to this topic.

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