Well-posedness of infinite-dimensional non-autonomous passive boundary control systems

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Abstract. We study a class of non-autonomous linear boundary control and observation systems that are governed by non-autonomous multiplicative perturbations. This class is motivated by fundamental partial differential equations, such as controlled wave equations and Timoshenko beams. Our main results give sufficient condition for well-posedness, existence and uniqueness of classical and mild solutions.

1. Introduction. Let $X, U$ and $Y$ be complex Hilbert spaces. The aim of this paper is to study linear systems with boundary control and observation described by

\[
\begin{align*}
\dot{x}(t) &= \mathfrak{A}(t)M(t)x(t), \quad x(0) = x_0, \quad t \geq 0, \\
\mathfrak{B}M(t)x(t) &= u(t), \\
y(t) &= \mathfrak{C}M(t)x(t).
\end{align*}
\]

Here $x(t) \in X$ is the state of the system, $\mathfrak{A}(t), t \geq 0$, is a family of unbounded linear operators on $X$ with a common domain $\mathfrak{D} \subset X$ and $M(t) \in \mathcal{L}(X)$ for every $t \geq 0$. Further, $u, y$ are the input- and the output functions of the system taking values in the input space $U$ and the output space $Y$, respectively. The time independent boundary operators $\mathfrak{B} : D(\mathfrak{B}) \subset X \to U$ and $\mathfrak{C} : \mathfrak{D} \subset X \to Y$ are linear and unbounded.

Systems of the form (1)-(3) are called non-autonomous boundary control systems. Several results for autonomous boundary control systems are available in the literature, we refer to e.g., [12, 17, 36] and the references therein.

A pair $(x, y)$ is a classical solution of (1)-(3) if $x \in C^1((0, \infty); X) \cap C([0, \infty); X)$, $y \in C([0, \infty); Y)$ and $x(t) \in D(\mathfrak{A}(t)M(t))$ for all $t \geq 0$ such that $x, y$ satisfy (1)-(3). The system (1)-(3) is called well-posed if for each (classical) solution $(x, y)$ and any...
final time $T > 0$, the operator mapping the input functions $u$ and the initial state $x_0$ to $x(T)$ and the output functions $y$ is bounded, i.e.

$$\|x(T)\|^2 + \int_0^T \|y(s)\|^2 ds \leq m_T (\|x_0\|^2 + \int_0^T \|u(s)\|^2 ds)$$

for some constant $m_T > 0$ independent of $x_0$ and $u$.

Our main results is to study solvability and well-posedness for non-autonomous linear boundary control and observation systems of the form (1)-(3). Moreover, we apply our results to port-Hamiltonian systems with time dependent parameters.

In order to study existence of classical solutions it is often useful to write the boundary control system as an abstract Cauchy problem. Generalizing an idea of Fattorini [12], see also [17, 9], to non-autonomous boundary control systems we associate to our system an abstract non-autonomous inhomogeneous Cauchy problem of the form

$$\dot{z}(t) = A(t)M(t)z(t) + \mathfrak{F}(t), \quad t \geq 0 \quad (4)$$

on the state space $X$, where $A(t) = \mathfrak{A}(t)|_{\ker \mathfrak{B}}$ for each $t \geq 0$ and $\mathfrak{F}(\cdot)$ is a function depending on $\mathfrak{B}$ and $u$. Under suitable conditions system (4) is solvable if and only if (1)-(2) is, see Section 3.1 where a general situation is studied.

The class of non-autonomous Cauchy problems (4) has been studied by Schnaubelt and Weiss in [31] in the case where $A(t) = A$ is constant (and $\mathfrak{F} = 0$). In this paper we prove, among other, a generalization to the case where $A$ is not constant. Indeed, exploiting some ideas of [31] we start the study of the classical solvability of (4) in a more general setting by extending the concept of similar $C_0$-semigroups to evolution families, see Section 2. Evolution families are a generalization of $C_0$-semigroups, and are often used to describe the solution of an abstract non-autonomous Cauchy problem. In Section 2, we therefore review the concept of evolution families and that of $C^1$-well posed non-autonomous Cauchy problems. Furthermore, we provide several abstract results which are crucial for the analysis of our class of non-autonomous boundary control and observation systems.

Well-posedness for non-autonomous boundary and observation systems is studied in Section 3.2 restricting ourselves to the case where for every $t \geq 0$ the autonomous system associated with $(\mathfrak{A}(t), \mathfrak{B}, \mathfrak{C})$ is $(R(t), P(t), J(t))$-scattering passive, i.e., when

$$2 \Re (\mathfrak{A}(t)x \mid P(t)x)_X \leq (R(t)u \mid \mathfrak{B}x)_U - (\mathfrak{C}x \mid J(t)\mathfrak{C}x)_V$$

for all $x$ in an appropriate subspace of $X \times U$ where $P(t)$, $R(t)$ and $J(t)$ are bounded linear operators. A precise definition and a characterization of scattering passive non-autonomous systems is the subject of sections 3.2 and 3.3.

In Section 4 we associates a mild solution to the control part (1)-(2) of the non-autonomous boundary control system. Mild solutions are important in control theory, as classical solvability usually require smooth input functions. The latter are often only Lebesgue functions in applications.
As mentioned above, we aim to apply our results to the following class of port-Hamiltonian systems

\[
\frac{\partial}{\partial t} x(t, \zeta) = \sum_{k=1}^{N} P_k(t) \frac{\partial^k}{\partial \zeta^k} [\mathcal{H}(t, \zeta) x(t, \zeta)] + P_0(t, \zeta) \mathcal{H}(t, \zeta) x(t, \zeta), \quad t \geq 0, \quad \zeta \in (a, b)
\]

\[
x(0, \zeta) = x(\zeta), \quad \zeta \in (a, b),
\]

\[
u(t) = W_{B, 1} \tau(\mathcal{H}x)(t), \quad t \geq 0,
\]

\[
y(t) = W_C \tau(\mathcal{H}x)(t), \quad t \geq 0.
\]

Here \( x(t, \zeta) \in \mathbb{K}^n, \tau \) denotes the trace operator \( \tau : H^N((a, b); \mathbb{K}^n) \to \mathbb{K}^{2Nn} \) defined by

\[
\tau(x) := (x(b), x'(b), \ldots, x^{(N-1)}(b), x(a), x'(a), \ldots, x^{(N-1)}(a)),
\]

\( P_k(t) \) is \( n \times n \) matrix for all \( t \geq 0, k = 1, \ldots, N \), \( P_0(t, \zeta), \mathcal{H}(t, \zeta) \in \mathbb{K}^{n \times n} \) for all \( t \geq 0 \) and almost every \( \zeta \in [a, b] \), \( W_{B, 1} \) is a \( m \times 2nN \)-matrix, \( W_{B, 2} \) is \((nN - m) \times 2nN\)-matrix and \( W_C \) is a \( d \times 2nN \)-matrix. Finally, \( u(t) \in \mathbb{K}^m \) denotes the input and \( y(t) \in \mathbb{K}^d \) is the output at time \( t \). Here \( H^N((a, b); \mathbb{K}^n) \) is the Sobolev space.

The class of non-autonomous port-Hamiltonian systems covers wave equations, transport equations, beam equations, coupled beam and wave equations as well as certain networks. Autonomous port-Hamiltonian systems, that is when \( \mathcal{H}, P_k \) are time-independent, have been intensively investigated, see e.g., [15, 16, 3, 2, 17, 23, 37, 42]. The existence of mild/classical solutions with non-increasing energy and well-posedness for autonomous port-Hamiltonian systems can in most cases be tested via a simple matrix condition [23, Theorem 4.1]. Well-posedness of linear systems in general is not easy to prove and a necessary condition is that the state operator generates a strongly continuous semigroup. For the class of autonomous port-Hamiltonian systems of first order i.e., \( N = 1 \), this condition is even sufficient under some weak assumptions on \( P_i \mathcal{H} \), see [23] or [17, Theorem 13.2.2].

In Section 5 we generalize these solvability and well-posedness results to the non-autonomous situation. In contrast to autonomous port-Hamiltonian systems, the non-autonomous counterpart has not been discussed so far. Setting

\[
\mathcal{A}(t)x = \sum_{k=0}^{N} P_k(t) \frac{\partial^k}{\partial \zeta^k} x, \quad \mathcal{B}x := \begin{bmatrix} W_{B, 1} \\ W_{B, 2} \end{bmatrix} \tau(x), \quad \mathcal{C}x := W_C \tau(x), \quad \text{and} \quad \mathcal{H}(t, \cdot) := M(t)
\]

we see that the non-autonomous port-Hamiltonian system fit in the framework of (1)-(3). In particular, we deduce in Theorem 5.5 that well-posedness for a large class of non-autonomous port-Hamiltonian systems can be checked via a simple matrix condition. Finally, a mild solution is associated to the non-autonomous port-Hamiltonian system when the matrices \( P_k, k = 1, \ldots, N \), are constant with respect to time variable.

In the literature most attention has been devoted to autonomous control systems. However, in view of applications, the interest in non-autonomous systems has been rapidly growing in recent years, see e.g., [22, 13, 27, 7, 31, 19, 6, 18, 30] and the references therein. In particular, a class of scattering passive linear non-autonomous
linear systems of the form
\[
\begin{align*}
\dot{x}(t) &= A_1M(t)x(t) + Bu(t), \quad t \geq 0, \quad x(0) = x_0, \\
y(t) &= CM(t)x(t) + Du(t)
\end{align*}
\] (5)
has been studied by Schnaubelt and Weiss in [31]. Here \((A,D)\) generates a strongly continuous semigroup on \(X\), \(A_1 \in \mathcal{L}(X,X_{-1})\) is a bounded extension of \((A,D)\), \(B \in \mathcal{L}(U,X_{-1})\), \(C \in \mathcal{L}(Z,Y)\) and \(D \in \mathcal{L}(U,Y)\), where \(X_{-1}\) is the extrapolation space corresponding to \(A\), and \(Z := D(A) + (\alpha - A)^{-1}B U\) for some \(\alpha \in \rho(A)\).

The control part (1)-(2) can be rewritten in the (standard) abstract formulation (5). The case where \(A(t) = \mathfrak{A}\) is constant correspond for non-autonomous port-Hamiltonian systems to the case where the matrices \(P_k, k = 1, \cdots, N\), are constant with respect to the time variable. On the other hand, when \(A(t) = \mathfrak{A}\) the output part (3) could be written into (6) using the concept of system nodes. Indeed, well-posed autonomous port-Hamiltonian system fit into the framework of compatible system nodes [38, Theorem 10]. This can be also easily generalized for boundary control and observation systems defined in Definition 3.2. Since we do not follow the approach of [31], this topic will not be discussed in this paper and we refer to [33, 36] for more details on system nodes. For the general case, that is if \(A(t) = \mathfrak{A}\) is not constant, then \(A_{-1}, B, C, D\) and \(Z\) will be time dependent. Thus, the abstract results in [31] cannot applied immediately to deduce classical solvability and well-posedness for (1)-(3). We expect that the results in [31] can be generalized to include this general case. However, for the class of boundary control systems defined in Definition 3.2 we deal directly with (1)-(2) in combination with Fattorini’s trick instead of its corresponding system (5)-(6). Our method is indeed much simpler. Moreover, in general it is not clear how the solution of (5)-(6) can be related to that of (1)-(3) even for the special case where \(A(t) = \mathfrak{A}\) is constant. In the autonomous case this relationship is quite simple [17]. The reason is that \(C_0\)-semigroups can be always extended to the extrapolation space. The situation is more delicate for the non-autonomous setting since a general extrapolation theory for evolution families is still missing.

This paper is organized as follows. In Section 2 we recall some abstract results on the theory of evolution families and the well-posedness for non-autonomous evolution equations and give several preliminary results. The results from Sections 2 are used in Section 3 and Section 4 to investigate well-posedness, existence and uniqueness of classical and mild solutions for non-autonomous boundary control and observation systems. An application to non-autonomous port-Hamiltonian systems is given in the last section.

2. Background on evolution families and preliminary results. Throughout this section \((X, \| \cdot \|)\) is a Banach space. Let \( \mathcal{A} := \{ A(t) \mid t \geq 0 \} \) be a family of linear, closed operators with domains \( \{ D(A(t)) \mid t \geq 0 \} \). Consider the non-autonomous Cauchy problem
\[
\dot{x}(t) = A(t)x(t) \quad \text{on } [s, \infty), \quad x(s) = x_s, (s > 0).
\] (7)
A continuous function \( x : [s, \infty) \to X \) is called a classical solution of (7) if \( x(t) \in D(A(t)) \) for all \( t \geq s, x \in C^1((s, \infty), X) \) and \( x \) satisfies (7).

**Definition 2.1.** The non-autonomous Cauchy problem (7) is called \(C^1\)-well posed if there is a family \( \{ Y_t \mid t \geq 0 \} \) of dense subspaces of \( X \) such that:
If we want to specify the regularity subspaces \( Y_t \), \( t \geq 0 \), we also say (7) is \( C^1 \)-well posed on \( Y_t \).

The following definition provides a natural generalization of operator semigroups for non-autonomous evolution equations.

**Definition 2.2.** A family \( \mathcal{U} := \{ U(t,s) \mid (t,s) \in \Delta \} \subset \mathcal{L}(X) \) where \( \Delta := \{ t, s \geq 0 \mid t \geq s \} \) is called an evolution family if:

1. \( U(t,t) = I \) and \( U(t,s) = U(t,r)U(r,s) \) for every \( 0 \leq s \leq r \leq t \),
2. \( U(\cdot, \cdot) : \Delta \rightarrow \mathcal{L}(X) \) is strongly continuous.

The evolution family \( \mathcal{U} \) is said to be generated by \( \mathcal{A} \), if there is a family \( \{ Y_t \mid t \geq 0 \} \) of dense subspaces of \( X \) with \( Y_t \subset D(A(t)) \), \( U(t,s)Y_s \subset Y_t \) for all \( (t,s) \in \Delta \) and

3. For every \( x_s \in Y_s \), the function \( t \rightarrow U(t,s)x_s \) is the unique classical solution of (7).

The Cauchy problem (7) is then \( C^1 \)-well posed if and only if \( A(t), t \geq 0 \), generates a unique evolution family, see [10, Proposition 9.3, pag. 478] or [25, Proposition 3.10].

### 2.1. Similar evolution families.

Let \( \mathcal{U} := \{ U(t,s) \mid (t,s) \in \Delta \} \) be an evolution family on \( X \) and let \( \{ Q(t) \mid t \geq 0 \} \subset \mathcal{L}(X) \) be a family of isomorphisms on \( X \) such that \( Q \) and \( Q^{-1} \) are strongly continuous on \([0, \infty)\). Define the two parameters operator family \( \mathcal{W} := \{ W(t,s) \mid (t,s) \in \Delta \} \) by

\[
W(t,s) = Q^{-1}(t)U(t,s)Q(s) \quad \text{for} \quad (t,s) \in \Delta. \tag{8}
\]

It is well known that if \( S \) is a \( C_0 \)-semigroup on \( X \) with generator \( A \) and \( Q \in \mathcal{L}(X) \) is an isomorphism, then \( T(\cdot) := Q^{-1}S(\cdot)Q \) is again a \( C_0 \)-semigroup on \( X \), called similar \( C_0 \)-semigroup to \( S \), and its generator is given by \( Q^{-1}AQ \), where

\[
D(Q^{-1}AQ) = D(AQ) = \{ x \in X \mid Qx \in D(A) \} = Q^{-1}D(A).
\]

In the following proposition we generalize the concept of similar semigroups to evolution families.

**Proposition 1.** The two parameters family \( \mathcal{W} \), defined by (8), defines an evolution family on \( X \). Assume that \( Q(\cdot) \) is in addition strongly \( C^1 \)-differentiable. Then \( \mathcal{U} \) is generated by a family \( \mathcal{A} \) with regularity spaces \( \{ Y_t \mid t \geq 0 \} \) if and only if \( \mathcal{W} \) is generated by \( \mathcal{A}_Q := \{ Q^{-1}(t)A(t)Q(t) - Q^{-1}(t)Q(t) \mid t \geq 0 \} \) with regularity spaces \( \{ \tilde{Y}_t \mid t \geq 0 \} \) where

\[
\tilde{Y}_t := \{ x \in X \mid Q(t)x \in Y_t \}.
\]

**Proof.** The proof of the first assertion is similar to that of [10, Lemma B. 15, page], and we omit it. Let us prove the second assertion.

Assume that \( \mathcal{U} \) is generated by \( \mathcal{A} \) with regularity spaces \( \{ Y_t \mid t \geq 0 \} \). We first remark that \( \tilde{Y}_t \) is a dense subspace of \( X \) and

\[
\tilde{Y}_t = Q^{-1}(t)Y_t \subset Q^{-1}(t)D(A(t)) = D(A(t)Q(t)) = D(A_Q(t)) \tag{9}
\]
for every \( t \geq 0 \), where \( A_Q(t) := Q^{-1}(t)A(t)Q(t) - Q^{-1}(t)\dot{Q}(t) \). Next, let \( x_s \in \tilde{Y}_s \). Then \( Q(s)x_s \in Y_s \) and by assumption \( U(\cdot, s)Q(s)x_s \) is the unique classical solution of

\[
\dot{x}(t) = A(t)x(t) \quad \text{on} \ [s, \infty), \ x(s) = Q(s)x_s, (s > 0).
\]

(10)

It follows that \( W(t, s)x_s \in Y_t \subset D(A_Q(t)) \) by (9) and

\[
\frac{d}{dt}W(t, s)x_s = \frac{d}{dt}Q(t)^{-1}U(t, s)Q(s)x_s + Q(t)^{-1}\frac{d}{dt}U(t, s)Q(s)x_s
\]

\[
= -Q(t)^{-1}\dot{Q}(t)Q^{-1}(t)U(t, s)Q(s)x_s + Q(t)^{-1}A(t)U(t, s)Q(s)x_s \quad (11)
\]

\[
= [Q^{-1}(t)A(t)Q(t) - Q^{-1}(t)\dot{Q}(t)]W(t, s)x_s. \quad (12)
\]

Since \( Q \) is strongly \( C^1 \)-differentiable, it now follows from (11)-(12) that \( W(\cdot, s)x_s \in C^1((s, \infty), X) \) and \( W(\cdot, s)x_s \) solves the non-autonomous problem

\[
\dot{x}(t) = A_Q(t)x(t) \quad \text{on} \ [s, \infty), \ x(s) = x_s.
\]

(13)

Clearly, \( W(\cdot, s)x_s \) is the unique classical solution of (13). We conclude that \( \mathcal{W} \) is generated by \( \{A_Q(t) \mid t \geq 0\} \) with regularity space \( \tilde{Y}_t \mid t \geq 0 \). Conversely, assume that \( A_Q \) generates the evolution family \( \mathcal{W} \) with some regularity spaces \( \tilde{Y}_t \mid t \geq 0 \).

Since \( Q^{-1} \) is \( C^1 \)-strongly continuous we obtain by (i) that the family \( \{A_Q(t)Q^{-1} = A \) generates the evolution \( \mathcal{W} \) defined by

\[
V(t, s) := Q(t)W(t, s)Q^{-1}(s) = U(t, s), \quad (t, s) \in \Delta
\]

with regularity space \( Y_t = Q(t)\tilde{Y}_t \). This completes the proof.

If \( A : D(A) \subset X \to X \) is the generator of a \( C_0 \)-semigroup and \( B \in \mathcal{L}(X) \), then the perturbed operator \( \tilde{A} := A + B \) is again the generator of a \( C_0 \)-semigroup, see e.g., [10, Section 1.3 pag. 158] or [28]. This perturbation results fails to be true in general for non-autonomous evolution equations [10, Example 9.21 pag. 489]. Thus one cannot conclude from Proposition 1 that the family \( \{Q^{-1}(t)A(t)Q(t) \mid t \geq 0\} \) generates an evolution family. Nevertheless, inspired by an idea of Schnaubelt and Weiss [31], using Proposition 1 we show that a positive answer can be given under some additional regularity assumptions.

For this we first need to introduce the following definition.

Definition 2.3. (Kato’s class)

1. A family \( \mathcal{A} \) is said to be Kato-stable if for each \( t \geq 0 \) there exists a norm \( \| \cdot \|_t \) on \( X \) equivalent to the original one such that for each \( T \geq 0 \) there exists a constant \( c_T \geq 0 \) with

\[
\|x\|_t - \|x\|_s \leq c_T|t - s||x||_s, \quad x \in X, t, s \in [0, T] \quad (14)
\]

and \( A(t) \) generates a contractive \( C_0 \)-semigroup on \( X_t := (X, \| \cdot \|_t) \) for all \( t \geq 0 \).

2. A family \( \mathcal{A} \) is said to belong to Kato’s class if it is Kato-stable and the operators \( A(t), t \geq 0 \), have a common domain \( D \) such that \( A(\cdot) : [0, \infty) \to \mathcal{L}(D, X) \) is strongly \( C^1 \)-differentiable, where \( D \) is equipped with the norm \( \| \cdot \|_D = \| \cdot \| + \| A(0) \cdot \| \).

It is known that Kato-stability is a sufficient condition for \( C^1 \)-well posedness of (hyperbolic) non-autonomous evolution equations [21, 28, 34]. In particular, each non-autonomous evolution equation that is governed by a Kato-class family is \( C^1 \)-well posed.
Obviously, $\mathcal{A}$ is Kato-stable if each operator $A(t)$ generates a contractive $C_0$-semigroup, as one can simply choose $\|x\|_t = \|x\|$, $t \geq 0$. In this case we say that $\mathcal{A}$ belongs to the elementary Kato class. Starting from this simple case many less trivial Kato-stable families can be constructed.

**Example 2.4.** Assume that $(\mathcal{H}, \| \cdot \|_\mathcal{H})$ is a Hilbert space. Let $M : \left[0, \infty \right) \rightarrow \mathcal{L}(\mathcal{H})$ be self-adjoint and uniformly coercive, i.e., $M(t)^* = M(t)$ and $(M(t)x)_H \geq \beta \|x\|^2_H$ for some constant $\beta > 0$ and all $t \geq 0$. If $M$ is strongly $C^1$-continuous and $M^{-1}$ is strongly continuous, then for each $t \in [0, \infty)$ the function

$$x \mapsto \|x\| := \sqrt{(M(t)x,x)} = \|M^{1/2}(t)x\|$$

defines a norm on $\mathcal{H}$ which is equivalent to the norm $\| \cdot \|_\mathcal{H}$ and satisfies (14). Moreover, if $\mathcal{A}$ has a common domain $D$ and for each $t \geq 0$ the operator $(A(t),D)$ generates a contraction $C_0$-semigroup in $\mathcal{H}$, then $(A(t)M(t),D(A(t)M(t)))$ and $(M(t)A,D(A(t)))$ generate contractive $C_0$-semigroups on $\mathcal{H}$, and thus both families

$$\{A(t)M(t) | t \geq 0\} \text{ and } \{M(t)A(t) | t \geq 0\}$$

are Kato-stable. We refer to [17, Lemma 7.2.3] and to the proof of [31, Proposition 2.3] for precise details. Finally, if $P : [0, \infty) \rightarrow \mathcal{L}(X)$ is a locally uniformly bounded function, then $\{M(t)A(t) + P(t) | t \geq 0\}$ and $\{A(t)M(t) + P(t) | t \geq 0\}$ are Kato-stable [34, Propositions 4.3.2 and 4.3.3].

**Proposition 2.** Let $\mathcal{A}$ belong to the Kato-class and let $D$ denote the common domain of $A(t)$, $t \geq 0$. Assume that $Q(\cdot)$ is strongly $C^2$-continuous. Then the family

$$\{Q^{-1}(t)A(t)Q(t) | t \geq 0\}$$

generates a unique evolution family $\mathcal{W}$ with regularity spaces $Y_s = Q^{-1}(t)D$, $t \geq 0$. Moreover, for each $F \in C^1([0, \infty); X)$ and $x_s \in Q^{-1}(s)D$ the inhomogeneous non-autonomous Cauchy problem

$$\dot{z}(t) = Q^{-1}(t)A(t)Q(t)x(t) + F(t) \quad \text{a.e. on } [s, \infty), \quad x(s) = x_s, s > 0,$$

has a unique classical solution given by

$$x(t) = W(t,s)x_s + \int_s^t W(t,r)F(r) dr \quad t \geq s.$$  \hspace{1cm} (17)

**Proof.** It is not difficult to verify that (14) implies that $\|x\|_s \leq e^{\gamma |t-s|}\|x\|_s$ for all $x \in X$, $t,s \in [0,T]$ and $T > 0$. Using [34, Propositions 4.3.2 and 4.3.3] and [34, Corollary of Theorem 4.4.2] we obtain that $\{A(t) + Q(t)Q^{-1}(t) | t \geq 0\}$ generates a unique evolution family $\mathcal{U}$ on $X$. Thus the first assertion follows from Proposition 1. Next, let $F \in C^1([0, \infty); X)$ and $x_s \in Q(s)D$. By [34, Theorem 4.5.3] the inhomogeneous Cauchy problem

$$\dot{z}(t) = A(t)z(t) + \dot{Q}(t)Q^{-1}(t)z(t) + Q(t)F(t) \quad \text{a.e. on } [s, \infty),$$

$$z(s) = Q^{-1}(s)x_s, s > 0.$$  \hspace{1cm} (19)

has a unique classical solution $z$ given by

$$z(t) = U(t,s)Q^{-1}(s)x_s + \int_s^t U(t,r)Q(r)F(r) dr \quad t \geq s.$$  \hspace{1cm} (20)

On the other hand, arguing as in the proof of Proposition 1 we see that $x := Q^{-1}(\cdot)z$ is a classical solution of (16). The uniqueness of classical solutions of (16) follows from the uniqueness of classical solutions of (18). Finally, (17) follows from (20) and (8).
Using Example 2.4 and Proposition 2 one can formulate the following two corollaries.

**Corollary 1.** Assume that $X$ is a Hilbert space. Assume that $A$ belongs to the elementary Kato class and denote by $D$ the common domain of $A(t)$, $t \geq 0$. Let $M : [0, \infty) \to \mathcal{L}(X)$ and $P : [0, \infty) \to \mathcal{L}(X)$ be self-adjoint and uniformly coercive such that $M$ is strongly $C^2$-continuous while $P$ is strongly $C^1$-differentiable. Then \{\{A(t)M(t) + P(t)\mid t \geq 0\}\} generates a unique evolution family $W$ with regularity spaces $Y_t = M^{-1}(t)D$, $t \geq 0$. Moreover, for each $F \in C^1([0, \infty); X)$ and $x_s \in M^{-1}(s)D$ the inhomogeneous non-autonomous Cauchy problem
\[
\dot{x}(t) = A(t)M(t)x(t) + P(t)x(t) + F(t) \quad \text{a.e. on } [s, \infty), \quad x(s) = x_s, s > 0.
\] (21)
has a unique classical solution given by (17).

**Proof.** For the proof we just have to apply Proposition 2 for $M(t)A(t) + M(t)P(t)M^{-1}(t)$ instead of $A(t)$ and $M(t)$ instead of $Q(t)$. \qed

**Corollary 2.** Let $X$ be a Hilbert space and let $(A, D(A))$ be generator of a contractive $C_0$-semigroup on $X$. Let $M : [0, \infty) \to \mathcal{L}(X)$ and $P : [0, \infty) \to \mathcal{L}(X)$ be as in Corollary 1. Further, let $R : [0, \infty) \to \mathcal{L}(X)$ be self-adjoint and uniformly coercive such that $R$ is strongly $C^1$-continuous and commute with $M$ i.e.
\[
R(t)M(t) = M(t)R(t) \quad \text{for all } t \geq 0.
\] (22)
Then the family \{\{R(t)AM(t) + P(t)\mid t \geq 0\}\} generates a unique evolution family $W$ with regularity spaces $Y_t = M^{-1}(t)D(A)$, $t \geq 0$. Moreover, for each $F \in C^1([0, \infty); X)$ and $x_s \in M^{-1}(s)D(A)$ the inhomogeneous non-autonomous Cauchy problem
\[
\dot{x}(t) = R(t)AM(t)x(t) + P(t)x(t) + F(t) \quad \text{a.e. on } [s, \infty), \quad x(s) = x_s, s > 0.
\] (23)
has a unique classical solution given by (17).

**Proof.** From Example 2.4 we deduce that the family \{\{M(t)R(t)A\mid t \geq 0\}\}, and therefore \{\{M(t)R(t)A + M(t)P(t)M^{-1}(t)\mid t \geq 0\}\}, belongs to Kato’s class. In fact, using (23) we see that $M(\cdot)R(\cdot) : [0, \infty) \to \mathcal{L}(X)$ is selfadjoint and uniformly coercive. Now, applying Proposition 2 for $M(t)R(t)A + M(t)P(t)M^{-1}(t)$ instead of $A(t)$ and $M(t)$ instead of $Q(t)$ concludes the proof. \qed

**Remark 1.** Corollary 1 has been proved in [31, Proposition 2.8-(a)] using a slightly different method for $A(t) = A$ and $F = F = 0$.

2.2. Backward evolution families. Let $X$ be a Hilbert space over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

**Definition 2.5.** A family $V := \{V(t, s)\mid (t, s) \in \Delta\} \subset \mathcal{L}(X)$ is called a backward evolution family if
\[
(i) \ V(t, t) = I \quad \text{and} \quad V(r, s) V(t, r) = V(t, s) \quad \text{for every } 0 \leq s \leq r \leq t,
\]
\[
(ii) \ V(\cdot, \cdot) : \Delta \to \mathcal{L}(X) \quad \text{is strongly continuous.}
\]
A family $A(t) : D(A(t)) \subset X \to X$, $t \geq 0$, of linear operators generates a backward evolution equation $V$ if there is a family $\{Y_t\mid t \geq 0\}$ of dense subspaces of $X$ with $Y_t \subset D(A(t))$ and
\[
V(t, s)Y_t \subset Y_s \quad \text{for all } 0 \leq s \leq t,
\] (24)
$V(t, \cdot)x_t \in C^1([0, t], X)$ for every $x_t \in Y_t$ and $V(t, \cdot)x_t$ solves uniquely the backward non-autonomous problem
\[
\dot{x}(s) = -A(s)x(s) \quad \text{on } 0 \leq s \leq t, \quad x(t) = x_t, (t > 0).
\] (25)
Lemma 2.6. 1. Assume that $A = \{A(t) | t \geq 0\}$ belongs to the elementary Kato-class. Then $A$ generates a backward evolution family.

2. Assume that $A$ generates an evolution family $U$. If the adjoint operators $A^* := \{A^*(t) | t \geq 0\}$ generate a backward evolution family $U_* := \{U_*(t,s) | (t,s) \in \Delta\}$, then for $(t,s) \in \Delta$ we have

$$U(t,s) = [U_*(t,s)]^t.$$  

Proof. (i) Let $T > 0$ be fixed and set $A_T := \{A(T-t) | t \in [0,T]\}$. Then, obviously $A_T$ belongs to the Kato-class and thus generates an evolution family $U_T := \{U_T(t,s) | 0 \leq s \leq t \leq T\}$ \cite[Theorem 4.8]{28} such that for each $x \in D$ and $0 \leq s \leq t \leq T$

$$\frac{d}{dt}U_T(t,s)x = A_T(t)U_T(t,s)x,$$  

$$\frac{d}{ds}U_T(t,s)x = -U_T(t,s)A_T(s)x.$$  

It is easy to see that $S(t,s) := U_T(T-s,T-t)$ for each $0 \leq s \leq t \leq T$ defines a backward evolution family with generator $\{A(t) | t \in [0,T]\}$. This completes the proof since $T$ is arbitrary.

(ii) Denote by $Y_t$ and $Y_{t,s}, t \geq 0$ the regularity spaces corresponding to $A$ and $A^*$, respectively. Let $t > s \geq 0$ and let $x_s \in Y_s$ and $y_t \in Y_{t,s}$. Then for $s \geq r \geq t$ we have

$$\frac{d}{dr}(x_s | (U(r,s)y_t)|U_*(r,t)y_t) = \frac{d}{dr}(U(r,s)x_s | U_*(r,t)y_t)$$

$$= (A(r)U(r,s)x_s | U_*(r,t)y_t) - (U(r,s)x_s | A^*(r)U_*(r,t)y_t)$$

$$= 0.$$  

Integrating over $[s,t]$ and using that $Y_s$ and $Y_{t,s}$ are dense in $X$ yield the desired identity. \hfill \Box

3. Non-autonomous boundary and observation systems. Many systems governed by linear partial differential equations with inhomogeneous boundary conditions are described by an abstract time-dependent boundary system of the form

$$\dot{x}(t) = A(t)x(t), \quad t \geq s, \quad x(s) = x_s, \quad (s \geq 0)$$  

$$B(t)x(t) = u(t), \quad t \geq s,$$  

$$C(t)x(t) = y(t), \quad t \geq s,$$  

which we denote by $\Sigma_N(A, B, C)$. Here $A(t) : D(A(t)) \subset X \to X$, $B(t) : D(B(t)) \subset X \to U$ and $C(t) : D(C(t)) \subset X \to Y$ are linear unbounded operators such that $D(A(t)) \subset D(B(t))$ for each $t \geq 0$ and $X$, $U$ and $Y$ are Hilbert spaces. We shall call $X$ the state space, $U$ the input space and $Y$ the output space of the system.

Definition 3.1. Let $x_0 \in X$ and $u : [0, \infty) \to U$ be given.

(i) $x$ is called a classical solution of (29)-(30), if $x \in C^1([0, \infty), X)$, $x(t) \in D(A(t))$ for all $t \geq 0$ and $x$ satisfies (29)-(30).

(ii) A pair $(x, y)$ is called a classical solution of (29)-(31), if $x$ is a classical solution of (29)-(30), $y \in C([0, \infty) ; Y)$ and $y$ satisfies (31).

(iii) The system $\Sigma_N(A, B, C)$ is called well-posed, if for any final time $\tau > 0$ there exists $m_\tau > 0$ such that for all classical solution of (29)-(31) we have

$$\|x(\tau)\|^2 + \int_0^\tau \|y(s)\|^2ds \leq m_\tau(\|x(0)\|^2 + \int_0^\tau \|u(s)\|^2ds).$$  

(32)
In order to study existence of classical solutions it seems useful to write the boundary control system (29)-(30) as a $C^1$-well posed (inhomogeneous) non-autonomous Cauchy problem. For that we introduce the following definition which can be seen as a generalization of Curtain and Zwart [9, Definition 3.3.2] to the non-autonomous setting.

**Definition 3.2.** The linear non-autonomous system (29)-(31) is called a boundary control and observation non-autonomous system, and we write $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a NBCO-system, if the following assertions hold:

(i) The family of the main operators $A(t) : D(A(t)) \subset X \to X, t \geq 0$, defined by

\[ D(A(t)) : = D(\mathfrak{A}(t)) \cap \ker(\mathfrak{B}(t)) \]

\[ A(t)x : = \mathfrak{A}(t)x \quad \text{for } x \in D(A(t)) \]

generates an evolution family on $X$.

(ii) For each $t \geq 0$ there exists a linear operator $\tilde{B}(t) \in \mathcal{L}(U, X)$ such that for all $u \in U$ we have

\[ \tilde{B}(t)u \in D(\mathfrak{A}(t)), \mathfrak{A}(t)\tilde{B}(t) \in \mathcal{L}(U, X) \quad \text{and} \quad \mathfrak{B}(t)\tilde{B}(t)u = u. \]

(iii) $\mathfrak{C}(t) : D(\mathfrak{A}(t)) \subset X \to Y$ is a linear bounded operator for each $t \geq 0$, where $D(\mathfrak{A}(t))$ is equipped with the graph norm.

3.1. **Existence of classical solutions.** Let $\Sigma_N(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a NBCO-system. In this subsection, we study existence and uniqueness of classical solutions of $\Sigma_N(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ without output, i.e., classical solution of (29)-(30). More precisely, we will show that (29)-(30) can be equivalently written as a $C^1$-well-posed inhomogeneous Cauchy problem (in $X$) for sufficiently smooth initial data and inputs. This is known in the autonomous case, see [17, Theorem 11.1.2] or [9, Theorem 3.3.3]. Consider the time-dependent admissible spaces $\mathcal{V}(t), t \geq 0$, defined by

\[ \mathcal{V}(t) : = \{(x, u) \in X \times U \mid x \in D(\mathfrak{A}(t)) \text{ and } \mathfrak{B}(t)x = u\}. \]

**Remark 2.** Let $\Sigma_N(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a NBCO-system. Then for each $(x, u) \in X \times U$ and $t \geq 0$ we have

\[ x \in D(\mathfrak{A}(t)) \text{ and } \mathfrak{B}(t)x = u \iff x - \tilde{B}(t)u \in D(A(t)). \]

This is an easily consequence of Definition 3.2.

Since $\{A(t) | t \geq 0\}$ generates an evolution family $\mathcal{U}$ on $X$, for a given $f \in L^1_{\text{Loc}}([0, \infty); X)$ the inhomogeneous non-autonomous Cauchy problem

\[ \dot{v}(t) = A(t)v(t) + f(t), \quad t \geq s, \quad (s \geq 0), \quad (33) \]

\[ v(s) = v_s, \quad (34) \]

has at most one classical solution given by

\[ v(t) = U(t, s)v_s + \int_s^t U(t, r)f(r)dr, \]

see e.g., [28, Section 5.5.1]. Thus the following proposition provides a generalization of [9, Theorem 3.3.3].

**Proposition 3.** Assume that $u \in C^1([0, \infty); U)$, $\tilde{B}(\cdot)u_0 \in C^1([0, \infty); X)$ and $\mathfrak{A}(\cdot)\tilde{B}(\cdot)u_0 \in L^1([0, \infty); X)$ for each $u_0 \in U$. Let $x_s \in X$ such that $(x_s, u_s) \in \mathcal{V}(s)$.
Then $x$ is a classical solution of (29)-(30) if and only if $v := x - \hat{B}u$ is a classical solution of (33)-(34) with inhomogeneity

$$f(t) = \mathcal{F}_u(t) := \mathcal{A}(t)\hat{B}(t)u(t) - \frac{d}{dt}[\hat{B}(t)u(t)]$$

(35)

and initial data $v_s = x_s - \hat{B}(s)u(s)$. Therefore, (29)-(30) has at most one classical solution $x$ given by

$$x(t) = U(t,s)[x_s - \hat{B}(s)u(s)] + \hat{B}(t)u(t) + \int_s^t U(t,r)\mathcal{F}_u(r)dr$$

(36)

for each $t \geq s$.

**Proof.** Let $s \geq 0$. Clearly $x \in C^1([s,\infty); X)$ if and only if $v \in C^1([s,\infty); X)$. Assume now that $x$ is a classical solution of (29)-(30). Then $v(t) \in \mathcal{V}_t \subset D(A(t))$ for every $t \geq s$ by Remark 2 and

$$\dot{v}(t) = \mathcal{A}(t)x(t) - \mathcal{A}(t)\hat{B}(t)u(t) + \mathcal{A}(t)\hat{B}(t)u(t) - \frac{d}{dt}[\hat{B}(t)u(t)]$$

$$= A(t)[x(t) - \hat{B}(t)u(t)] + \mathcal{A}(t)\hat{B}(t)u(t) - \frac{d}{dt}[\hat{B}(t)u(t)]$$

$$= A(t)v(t) + \mathcal{A}(t)\hat{B}(t)u(t) - \frac{d}{dt}[\hat{B}(t)u(t)].$$

Thus $v$ is a classical solution of (33) with $f$ given by (35). The converse implication can be proved similarly. Finally, (36) follows by the remark above.  

3.2. Scattering passive NBCO-systems. Let $R : [0,\infty) \to \mathcal{L}(U)$, $P : [0,\infty) \to \mathcal{L}(X)$ and $J : [0,\infty) \to \mathcal{L}(Y)$ be continuous functions such that $P$ is strongly differentiable and $R(t) = R(t), P(t) = P(t), J(t) = J(t)$ for all $t \geq 0$.

**Definition 3.3.** Let $(x, y)$ be classical solution of (29)-(31). Then $\Sigma_N(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is called $(R, P, J)$-scattering passive if for all $t \geq s$

$$\frac{d}{dt}(P(t)x(t)|x(t)) + (y(t)|J(t)y(t)) \leq (u(t)|R(t)u(t)) + (\hat{P}(t)x(t)|x(t)).$$

(37)

Further, $\Sigma_N(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is called $(R, P, J)$-scattering energy preserving if equality holds in (37). If $P = I, R = I$ and $J = I$ then $\Sigma_N(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is called scattering passive, and scattering energy preserving if we have equality in (37).

The following proposition characterizes $(P, R, J)$-scattering passive NBCO-systems. A comparable results has been proved in the autonomous case in [24, Theorem 3.2, Proposition 5.2] for systems nodes.

**Proposition 4.** The following assertion are equivalent.

(i) $\Sigma_N(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is $(R, P, J)$-scattering passive.

(ii) For each $t \geq 0$ and all $(x, u) \in \mathcal{V}(t)$ we have

$$2\Re(\mathcal{A}(t)x|P(t)x) \leq (R(t)u|\mathcal{B}(t)x) - (\mathcal{C}(t)x|J(t)x).$$

(38)

**Proof.** Assume that (i) holds and let $s \geq 0$ and let $(x_s, u_s) \in \mathcal{V}(s)$. Let $u : [s, \infty) \to U$ such that $u(s) = u_s$. If $(x, y)$ is a classical solution of (29)-(31) corresponding to $(x_s, u)$ then $(x(t), u(t)) \in \mathcal{V}(t)$, $y(t) = \mathcal{C}(t)x(t)$ and

$$\frac{d}{dt}(P(t)x(t)|x(t)) - (\hat{P}(t)x(t)|x(t)) = 2\Re(\hat{x}(t)|P(t)x(t)) = 2\Re(\mathcal{A}(t)x(t)|P(t)x(t)).$$

(39)
for all $t \geq s$. Inserting this into (37) yields

$$2 \text{Re}(\mathfrak{A}(t)x(t) \mid P(t)x(t)) \leq (R(t)u(t) \mid \mathfrak{B}(t)x(t))_{U} - (\mathcal{C}(t)x(t) \mid J(t)\mathcal{C}(t)x(t))_{Y}$$

for all $t \geq s$. The last inequality (ii) by taking $t = s$. Conversely, assume that (ii) holds and let $(x, y)$ be a classical solution of (29)-(31). Then $(x(t), u(t)) \in \mathcal{V}(t)$ and (39) holds for all $t \geq s$. This together with (38) imply (37), which completes the proof.

**Lemma 3.4.** Let $\Sigma_{N}(\mathfrak{A}, \mathfrak{B}, \mathcal{C})$ be $(R, P, J)$-scattering passive such that $J \geq 0$. Assume that $P$ is strongly $C^{1}$-continuous and uniformly coercive with

$$(P(t)x|x) \geq \beta\|x\|^{2}, \text{ for all } t \geq 0, x \in X$$

for some constant $\beta > 0$. Then each classical solution of (29)-(31) satisfies the following inequality

$$\beta\|x(t)\|^{2} + \int_{s}^{t} (y(r) \mid J(r)y(r))dr \leq c_{t,s}e^{\frac{t-s}{\beta}}\int_{s}^{t} \|J(r)u(r)\|dr + \|x(s)\|^{2}$$

where $c_{t,s} = \max\{1, \max_{r \in [s,t]} \|P(r)\|\}$. Therefore, $\Sigma_{N}(\mathfrak{A}, \mathfrak{B}, \mathcal{C})$ is well-posed provided that $J$ is uniformly coercive and $R \in L_{Loc}^{\infty}([0, \infty); \mathcal{L}(U))$.

**Proof.** For the proof we follow a similar argument as in fourth steps of the proof of [31, Theorem 4.1]. Assume that $\Sigma_{N}(\mathfrak{A}, \mathfrak{B}, \mathcal{C})$ is $(R, P, J)$-scattering passive. Clearly (37) holds if and only if

$$\begin{align*}
(P(t)x(t) \mid x(t))_{X} + \int_{s}^{t} (y(r) \mid J(r)y(r))dr & \leq \int_{s}^{t} (u(r) \mid R(r)u(r))dr \\
& + \int_{s}^{t} (\dot{P}(r)x(r) \mid x(r))_{X} + (P(s)x_{s} \mid x_{s})_{X}
\end{align*}$$

for all $t \geq s \geq 0$. Thus using (40) and that $J \geq 0$ we obtain

$$\begin{align*}
\beta\|x(t)\|^{2} + \int_{s}^{t} (y(r) \mid J(r)y(r))dr & \leq \int_{s}^{t} (u(r) \mid R(r)u(r))dr + \|P(s)\|\|x(s)\|^{2} + \int_{s}^{t} \|\dot{P}(r)\|\|x(r)\|^{2}dr \\
& \leq \int_{s}^{t} (u(r) \mid R(r)u(r))dr + \|P(s)\|\|x(s)\|^{2} \\
& + \int_{s}^{t} \frac{1}{\beta} \|\dot{P}(r)\| \left[\beta\|x(r)\|^{2} + \int_{s}^{r} (y(\zeta) \mid J(\zeta)y(\zeta))d\zeta\right]dr.
\end{align*}$$

Now (41) follow by applying Gronwall’s Lemma.

3.3. **Multiplicative perturbations of NBCO-systems.** We will adopt the same notations of the previous sections. The main purpose of this section is to study two classes of NBCO-systems which are governed by a time-dependent multiplicative perturbation.

Throughout this section we assume that the following assumption holds:

**Assumption 3.5.**

1. $M : [0, \infty) \to \mathcal{L}(X)$ and $R : [0, \infty) \to \mathcal{L}(X)$ be two self-adjoint and uniformly coercive functions.
2. $M(\cdot)x \in C^{2}([0, \infty); X)$ and $M^{-1}(\cdot)x \in C([0, \infty); X)$ for each $x \in X$.
3. $L(\cdot)x \in C^{1}([0, \infty); X)$ for each $x \in X$ such that $L$ and $M$ commute.
We first consider the following perturbed system

\begin{align}
\dot{x}(t) &= \mathcal{A}(t)x(t), \quad x(0) = x_0, \\
\mathcal{B}M(t)x(t) &= u(t), \\
\mathcal{C}M(t)x(t) &= y(t),
\end{align}

which we denote by \( \Sigma_{N,M}(\mathcal{A}, \mathcal{B}, \mathcal{C}) = \Sigma_N(\mathcal{A}M, \mathcal{B}M, \mathcal{C}M) \) such that \( D(\mathcal{A}(t)) = \mathcal{D} \) for all \( t \geq 0 \) and the autonomous (unperturbed) system \( \Sigma_N(\mathcal{A}(t), \mathcal{B}, \mathcal{C}) \) is a boundary control and observation system for every fixed \( t \geq 0 \). Let \( \tilde{B}(t) \) and \( A(t) \) be the operators associated with \( \Sigma_N(\mathcal{A}, \mathcal{B}, \mathcal{C}) \) provided by Definition 3.2-(ii). Then it is easy to see that the corresponding operators for the perturbed system \( \Sigma_{N,M}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \) are respectively given by \( \tilde{B}_M(t) := M^{-1}(t)B(t) \) and \( \{A(t)M(t) \mid t \geq 0\} \) where \( D(A(t)M(t)) = M^{-1}(t)(\mathcal{D} \cap \ker(\mathcal{B})) \) for each \( t \geq 0 \).

**Lemma 3.6.** The perturbed system \( \Sigma_{N,M}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \) is \((R, P, J)\)-scattering passive if and only if \( \Sigma_N(\mathcal{A}, \mathcal{B}, \mathcal{C}) \) is \((R, PM^{-1}, J)\)-scattering passive.

**Proof.** For each \( t \geq 0 \) we set

\[ V_M(t) := \{(x, u) \in \mathcal{V} \times U \mid x \in D(\mathcal{A}M(t)) \text{ and } \mathcal{B}M(t)x = u\}. \]

Then, \((x, u) \in V_M(t)\) if and only if \((M(t)x, u) \in \mathcal{V}(t)\) for all \( t \geq 0 \). Assume now that \( \Sigma_N(\mathcal{A}, \mathcal{B}, \mathcal{C}) \) is \((R, M^{-1}P, J)\)-scattering passive and let \((x, u) \in V_M(t)\). Using Proposition 4 we obtain

\[
2 \text{Re} \left( \mathcal{A}M(t)x \mid P(t)x \right) = 2 \text{Re} \left( \mathcal{A}(t)M(t)x \mid P(t)M^{-1}(t)M(t)x(t) \right) \\
\leq (R(t)u \mid \mathcal{B}M(t)x(t)) - (\mathcal{C}M(t)x(t) \mid J(t)\mathcal{C}M(t)x(t))_Y
\]

This implies, again by Proposition 4, that \( \Sigma_{N,M}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \) is \((R, P, J)\)-scattering passive.

Conversely, assume that \( \Sigma_{N,M}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \) is \((R, P, J)\)-scattering passive. This means that \( \Sigma_N(\mathcal{A}M, \mathcal{B}M, \mathcal{C}M) \) is \((R, PM^{-1}, J)\)-scattering passive. Applying the first part of the proof yields that

\[
\Sigma_N(\mathcal{A}, \mathcal{B}, \mathcal{C}) = \Sigma_{N,M^{-1}}(\mathcal{A}M, \mathcal{B}M, \mathcal{C}M)
\]

is \((R, PM^{-1}, J)\)-scattering passive. This completes the proof. \( \square \)

In particular, the system \( \Sigma_{N,M}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \) is \((R, M, J)\)-scattering passive if and only if the unperturbed system \( \Sigma_N(\mathcal{A}, \mathcal{B}, \mathcal{C}) \) is \((R, J, J)\)-scattering passive. According to the above assumptions, we remark that \((x, y)\) is a classical solution of (43)-(45) if and only if \( x \) is a classical solution of (43)-(44).

Now we can formulate the first main result of this section.

**Theorem 3.7.** Assume that the following additional assumptions hold.

(a) \( \mathcal{A}(\cdot)x : [0, \infty) \to X \) is \( C^1 \)-continuous for each \( x \in D \cap \ker(\mathcal{B}) \).

(b) The main operators \( A(t) : D \cap \ker(\mathcal{B}) \to X, \; t \geq 0 \) generate contraction \( C_0 \)-semigroups.

(c) \( \tilde{B}(\cdot)u \in C^2([0, \infty); U) \) for each \( u \in U \).

Then the perturbed system \( \Sigma_{N,M}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \) is a NBCO-system on \((X, U, Y)\). Furthermore, if we denote by \( \mathcal{W} \) the associated evolution family, then for each \( s \geq 0 \) and \((x_s, u) \in X \times C^2([0, \infty); U)\) with \((M(s)x_s, u(s)) \in \mathcal{V}(s)\) the system (43)-(45)
has a unique classical solution $(x, y)$ given by
\[
x(t) = W(t, s) x_s + \int_s^t W(t, r) \mathfrak{A}(r) \tilde{B}(r) u(r) dr - \int_s^t W(t, r) \frac{d}{dr} [\tilde{B}_M(r) u(r)] dr, \quad t \geq s,
\]
\[
y(t) = \mathcal{C} M(t) W(t, s) x_s + \mathcal{C} M(t) \int_s^t W(t, r) \mathfrak{A}(r) \tilde{B}(r) u(r) dr
\]
\[
\quad - \mathcal{C} M(t) \int_s^t W(t, r) \frac{d}{dr} [\tilde{B}_M(r) u(r)] dr, \quad t \geq s.
\]

The system $\Sigma_{N, M}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is well-posed if in addition
\[
2 \Re(\mathfrak{A}(t)x_0 | x_0)_X \leq (R(t)u_0 | \mathfrak{B} x_0)_U - (\mathfrak{C} x_0 | J(t) \mathfrak{C} x_0)_Y
\]  
for all $t \geq 0$ and $(x_0, u_0) \in \mathcal{V}(t)$ where $R = R^* \in L^\infty_{\text{loc}}([0, \infty); L(U))$ and $J = J^*$ is uniformly coercive.

**Proof.** The first and the second assertion follow from Proposition 3 and Corollary 1, whereas the last assertion follows from Lemma 3.6, Proposition 4 and Lemma 3.4. \qed

Next we consider the case where $\mathfrak{A}(t) = L(t) \mathfrak{A}$ with $L(t)$ is as in Assumption 3.5 and such that $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is an autonomous BCO-system. If $\tilde{B}$ denotes the operator associated with the autonomous BCO-system $\Sigma(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, then it is easy to see that $\tilde{B}$ satisfies all properties listed in Definition 3.2-(ii) corresponding to $\Sigma_N(L(\cdot) \mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. We consider the following perturbed system
\[
\dot{x}(t) = L(t) \mathfrak{A} M(t) x(t), \quad x(0) = x_0, 
\]  
\[
\mathfrak{B} M(t) x(t) = u(t), 
\]  
\[
\mathcal{C} M(t) x(t) = y(t),
\]
which we denote by $\Sigma_{N, M, L}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) = \Sigma_N(L \mathfrak{A} M, \mathfrak{B} M, \mathcal{C} M)$. Clearly, the main operators associated with $\Sigma_{N, M, L}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ are given by $\{L(t) A M(t) | t \geq 0\}$.

**Theorem 3.8.** Assume that the main operators $A : \mathcal{D} \cap \ker(\mathfrak{B}) \to X$ generate a contraction $C_0$-semigroup on $X$. Then the perturbed system $\Sigma_{N, M, L}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is a NBCO-system on $(X, U, Y)$. Furthermore, if we denote by $W$ the associated evolution family, then for each $s \geq 0$ and $(x_s, u) \in X \times C^2([0, \infty); U)$ with $(M(s)x_s, u(s)) \in \mathcal{V}(s)$ the system (47)-(49) has a unique classical solution $(x, y)$ given by
\[
\begin{align*}
\dot{x}(t) &= W(t, s) x_s + \int_s^t W(t, r) L(r) \mathfrak{A} \tilde{B}_M(r) u(r) dr - \int_s^t W(t, r) \frac{d}{dr} [M^{-1}(r) \tilde{B}_M(r)] dr, \quad t \geq s, \\
y(t) &= \mathcal{C} M(t) W(t, s) x_s + \mathcal{C} M(t) \int_s^t W(t, r) L(r) \mathfrak{A} \tilde{B}_M(r) dr
\end{align*}
\]
\[
\quad - \mathcal{C} M(t) \int_s^t W(t, r) \frac{d}{dr} [M^{-1}(r) \tilde{B}_M(r)] dr, \quad t \geq s.
\]

The system $\Sigma_{N, M, L}(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is well-posed if in addition
\[
2 \Re(L(t) \mathfrak{A} x_0 | x_0)_X \leq (R(t)u_0 | \mathfrak{B} x_0)_U - (\mathfrak{C} x_0 | J(t) \mathfrak{C} x_0)_Y
\]  
for all $t \geq 0$ and $(x_0, u_0) \in \mathcal{V}$ where $R = R^* \in L^\infty_{\text{loc}}([0, \infty); L(U))$ and $J = J^*$ is uniformly coercive.

**Proof.** The first and the second assertion follow from Proposition 3 and Corollary 2, whereas the last assertion follows from Lemma 3.6, Proposition 4 and Lemma 3.4. \qed
Remark 3. Theorem 3.8 is not a special case of Theorem 3.7 since we do not assume that $P(t)A$ generates a contractive $C_0$-semigroup on $X$.

4. Mild solutions for NBC-systems. The main purpose of this section is to associate mild solutions to non-autonomous boundary control and observation systems $\Sigma_N(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. We will adopt here the notations of the previous sections. Let $\Sigma_N(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a NBCO-system. Then the main operators $\{A(t)\mid t \geq 0\}$ generate, by definition, an evolution family $U = \{U(t, s)\mid (s, t) \in \Delta\}$ with regularity space $Y_t$, $t \geq 0$. We restrict ourselves to case where $\{A(t)\mid t \geq 0\}$ have a common extrapolation space $X_{-1}$, i.e.,

$$X_{-1} := X_{-1}(t) = X_{-1}(s) \text{ for all } t, s \geq 0.$$  \hfill (51)

According to [35, Proposition 2.10.2], (51) holds if and only if $D(A^*(t)) = D(A^*(s))$ for $t, s \in [0, \infty)$ and the corresponding graph norms are locally uniformly equivalent. In fact, $X_{-1}(t)$ is the dual space of $D(A^*(t))$ with respect to the pivot space $X$. This condition holds, if for instance $A(t) = AM(t)$ or $A(t) = A + M(t)$ and $M \in C^1([0, \infty); \mathcal{L}(X))$.

In the following we denote $\mathfrak{D}_* := D(A^*(0))$ equipped with the graph norm and by $\langle \cdot, \cdot \rangle$ the duality between $X_{-1}$ and $\mathfrak{D}_*$ and

$$B(t) := \mathfrak{A}(t)\tilde{B}(t) - A_{-1}(t)\tilde{B}(t), \quad t \geq 0.$$ \hfill (52)

**Proposition 5.** Assume that $A^* := \{A^*(t)\mid t \geq 0\}$ generates a backward evolution family $\mathcal{U}$. Then $U(t, s)$ has a unique extension $\tilde{U}_{-1}(t, s) \in \mathcal{L}(X_{-1})$ for each $(t, s) \in \Delta$ and for each $T > 0$ there is $c_T > 0$ such that

$$\sup_{0 \leq s \leq t \leq T} \|U_{-1}(t, s)\|_{\mathcal{L}(X_{-1})} < c_T.$$ \hfill (53)

Moreover, if the assumptions of Proposition 3 hold, then each classical solution $x$ of the boundary control system (29)-(30) satisfies

$$x(t) = U(t, s)x_s + \int_s^t U_{-1}(t, r)B(r)u(r)dr, \quad t \geq s \geq 0.$$ \hfill (54)

**Proof.** By [35, Proposition 2.9.3-(b)] we obtain that for each $(t, s) \in \Delta$ the operator $U(t, s)$ has a unique extension $\tilde{U}_{-1}(t, s) \in \mathcal{L}(X_{-1})$ since $[U(t, s)]^\ast \mathfrak{D}_* = U_*(t, s)\mathfrak{D}_* \subset \mathfrak{D}_*$. Next, similar to the proof of [31, Proposition 2.7-(c)] we show the uniform boundedness of $\tilde{U}_{-1}$ on compact intervals. Next, we claim that for each $y \in \mathfrak{D}_*, x \in X_{-1}$ we have

$$\langle U_{-1}(t, s)x, y \rangle = \langle x, U_*(t, s)y \rangle.$$ \hfill (55)

In fact, this equality holds for $x \in X$ by Lemma 2.6-(ii) since

$$\langle x, U_*(t, s)y \rangle = \langle x \mid U_*(t, s)y \rangle = U(t, s)x \mid y = \langle U_{-1}(t, s)x, y \rangle.$$  

Remark that $U_*(t, s)y \in \mathfrak{D}_*$, thus the claim follows since $X$ is dense in $X_{-1}$.

Using again Lemma 2.6 and (55), we obtain for each $y \in \mathfrak{D}_*$

$$\frac{d}{ds} \langle U(t, s)\tilde{B}(s)u(s), y \rangle = \frac{d}{ds} \langle \tilde{B}(s)u(s), U_*(t, s)y \rangle = \langle \frac{d}{ds} [\tilde{B}(s)u(s)], U_* (t, s)y \rangle - \langle \tilde{B}(s)u(s), A^*(s)U_* (t, s)y \rangle$$

$$= \langle U(t, s) \frac{d}{ds} [\tilde{B}(s)u(s)], y \rangle - \langle U_{-1}(t, s)A_{-1}(s)\tilde{B}(s)u(s), y \rangle.$$
Integrating over $[s,t]$, we obtain
\[
\int_s^t U_{-1}(t,r)A_{-1}(r)\dot{B}(r)u(r)dr = -\dot{B}(t)u(t) + U(t,s)\dot{B}(s)u(s) + \int_s^t U(t,r)\frac{d}{dr}[\dot{B}(r)u(r)]dr.
\] (56)

Inserting this equality in (36), we obtain that a classical solution $x$ of (29)-(30) satisfies (54).

If the assumptions of Proposition 5 hold, then for $x_s \in X$ and $u \in L^2_{\text{Loc}}([0, \infty); U)$ we see that (54) is well defined with value in $X_{-1}$ provided $B(\cdot)u(\cdot) \in L^1_{\text{Loc}}([0, \infty); X_{-1})$. In fact, (53) guarantees that the integral term on the right hand side of (57) is well defined. Thus the following definition makes sense.

**Definition 4.1.** Let $\Sigma_N(\mathfrak{M}, \mathfrak{N}, \mathcal{C})$ be a NBCO-system and let $U$ and $\{B(t) \mid t \geq 0\}$ be the associated evolution family and control operators, respectively. Let $(x_s, u) \in X \times L^2_{\text{Loc}}([0, \infty); U)$. If $U(t, s)$ has a unique extension $U_{-1}(t, s) \in \mathcal{L}(X_{-1})$ for each $(t, s) \in \Delta$ such that $U_{-1}(t, \cdot)B(\cdot)u(\cdot) \in L^1_{\text{Loc}}([0, \infty); X_{-1})$, then the function
\[
x(t) = U(t, s)x_s + \int_s^t U_{-1}(t, r)B(r)u(r)dr, \quad t \geq s \geq 0,
\] (57)
is called the *mild solution of* (29)-(31) in $X_{-1}$. Further, (57) is called a *mild solution of* (29)-(31) in $X$, if in addition
\[
\Phi_{t,s}u := \int_s^t U_{-1}(t, r)B(r)u(r)dr \in X, \quad \text{for all } (t, s) \in \Delta,
\] (58)
and $x \in C([s, \infty); X)$.

This definition is related to the notion of *admissibility* for non-autonomous linear systems. More precisely, recall that a family $\{B(t) \mid t \geq 0\} \subset \mathcal{L}(U, X_{-1})$ is $L^2$-admissible for a given evolution family $U$ that admits an extension to $\mathcal{L}(X_{-1})$ if $U_{-1}(t, \cdot)B(\cdot)u(\cdot) \in L^1_{\text{Loc}}([0, \infty); X_{-1})$, (58) holds and for each $T > 0$ there exists $c_T > 0$ such that
\[
\left\| \int_s^t U_{-1}(t, r)B(r)u(r)dr \right\|_X^2 \leq c_T \int_s^t ||u(r)||_U^2dr
\] (59)
for each $u \in L^2_{\text{Loc}}([0, \infty); U)$ and all $0 \leq s \leq t \leq T$ [30, Definition 3.3]. For $L^2$-admissible control operators we have that $(t, s) \mapsto \Phi_{t,s}u$ is continuous on $\Delta$ with values in $X$ [30, Proposition 3.5-(2)]

**Proposition 6.** Assume that $\mathcal{A}^* := \{A^*(t) \mid t \geq 0\}$ belongs to the Kato-class and $\{B(t) \mid t \geq 0\}$ is $L^2$-admissible. Then for each $(x_s, u) \in X \times L^2_{\text{Loc}}([0, \infty); U)$ with $B(\cdot)u(\cdot) \in L^1_{\text{Loc}}([0, \infty); X_{-1})$ the system (29)-(31) has a unique mild solution in $X$.

**Proof.** The proof follows from Lemma 2.6-(i) and Proposition 5. \hfill \square

If $\Sigma_N(\mathfrak{M}, \mathfrak{N}, \mathcal{C})$ is a well-posed NBCO-system and the classical solutions is given by (54), then the corresponding family $\{B(t) \mid t \geq 0\}$ is $L^2$-admissible provided
\[
U_{-1}(t, \cdot)B(\cdot)L^2_{\text{Loc}}([0, \infty); U) \subset L^1_{\text{Loc}}([0, \infty); X_{-1}).
\]
Thus the following corollary follows from Proposition 6, Lemma 3.4 and (53).

**Corollary 3.** Assume $\Sigma_N(\mathfrak{M}, \mathfrak{N}, \mathcal{C})$ is $(R, P, J)$-scattering passive such that $R \in L^\infty_{\text{Loc}}([0, \infty); \mathcal{L}(U))$ and $J, P$ are uniformly coercive. In addition, we assume that $\mathcal{A}^* := \{A^*(t) \mid t \geq 0\}$ belongs to the Kato-class. Then for each $(x_s, u) \in X \times
$L^2_{\text{Loc}}([0, \infty); U)$ with $B(\cdot)u(\cdot) \in L^1_{\text{Loc}}([0, \infty); X_{-1})$ the system $(29)-(31)$ has a unique mild solution in $X$.

Finally, if Assumption 3.5 holds such that $A(t)$ generates contractive $C_0$-semigroup on $X$ for each $t \geq 0$ then we can follow [31, Section 2, page 8] to deduce that the extrapolation spaces corresponding to $X$ and $X^{-1}$ have a common domain. Then the perturbed system \((44)\) has a unique mild solution in $X$ if the unperturbed system $\Sigma_{N, \text{id}}(A, B, C)$ is $(R, I, J)$-scattering passive with $R \in L^\infty_{\text{Loc}}([0, \infty); L(U))$ and $J, P$ are uniformly coercive.

Proof. The proof is an easy consequence of Corollary 3 and Lemma 3.6.

5. Application to non-autonomous Port-Hamiltonian systems. Let $N, n \in \mathbb{N}$ be fixed and let $X := L^2([a, b]; \mathbb{K}^n)$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. In this section we investigate the well-posedness of the linear non-autonomous $\text{port-Hamiltonian systems of order } N \in \mathbb{N}$, given by the boundary control and observation system

$$
\frac{\partial}{\partial t} x(t, \zeta) = \sum_{k=1}^N P_k(t) \frac{\partial^k}{\partial \zeta^k} [H(t, \zeta) x(t, \zeta)] + P_0(t, \zeta) H(t, \zeta) x(t, \zeta), \ t \geq 0, \ \zeta \in (a, b)
$$

(60)

$H(0, \zeta) x(0, \zeta) = x_0(\zeta), \ \zeta \in (a, b)$

(61)

$u(t) = W_B \tau(Hx)(t), \ \ t \geq 0,$

(62)

$0 = W_B \tau(Hx)(t), \ \ t \geq 0,$

(63)

$y(t) = W_C \tau(Hx)(t), \ \ t \geq 0.$

(64)

Here $\tau$ denotes the trace operator $\tau : H^N((a, b); \mathbb{K}^n) \to \mathbb{K}^{2Nn}$ defined by

$$
\tau(x) := (x(b), x'(b), \ldots, x^{N-1}(b), x(a), x'(a) \ldots, x^{N-1}(a)),
$$

$P_k(t)$ is $n \times n$ matrix for all $t \geq 0$, $k = 0, 1, \ldots, N$, $H(t, \zeta) \in \mathbb{K}^{n \times n}$ for all $t \geq 0$ and almost every $\zeta \in [a, b]$, $W_{B, 1}$ is a $n \times 2nN$-matrix, $W_{B, 2}$ is $(nN - m) \times 2nN$-matrix and $W_C$ is a $d \times 2nN$-matrix. Finally, $u(t) \in U := \mathbb{K}^m$ denotes the input and $y(t) \in Y := \mathbb{K}^d$ is the output at time $t$.

Set $W_B := \begin{bmatrix} W_{B, 1} \\ W_{B, 2} \end{bmatrix}$, $\Sigma := [0 \ I_0 \ I]$ and for each $t \geq 0$ we set

$$
Q(t) := \begin{bmatrix} P_1(t) & P_2(t) & \cdots & P_N(t) \\ -P_2(t) & -P_3(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (1)^{N-1} P_N(t) & 0 & \cdots & 0 \end{bmatrix},
$$

(60)

$R_{\text{ext}}(t) := \begin{bmatrix} Q(t)^{-1} Q(t) \\ 1 \end{bmatrix}$ and $W_B(t) := W_B R_{\text{ext}}^{-1}(t), W_C(t) := W_B R_{\text{ext}}^{-1}(t)$. In this section we assume the following assumption:

Assumption 5.1.

- $W_B$ has full rank and $W_B(t) \Sigma W_B(t) \geq 0$ for all $t \geq 0$.
- $P_N(t)$ is invertible and $P_k(t) = (-1)^{k-1} P_k(t)$ for all $k \geq 1$, $t \geq 0$.
- $P_k \in C^1([0, \infty); L^\infty([0, \infty]; C^{n \times n}))$ for all $k \geq 0$ and $k = 0, 1, \ldots, N$.
- $H \in C^2([0, \infty); L^\infty([0, \infty]; C^{n \times n}))$ and there exist $m, M \geq 0$ such that

$$
m \leq H(t, \xi) = H^*(t, \xi) \leq M, \ \ \text{a.e. } \xi \in [a, b], t \geq 0.
$$
On the Hilbert space $X$ we consider the (maximal) port-Hamiltonian operators

$$
\mathfrak{A}(t)x = \sum_{k=0}^{N} P_k(t) \frac{\partial^k}{\partial t^k} x \quad \text{with domain} \quad D(\mathfrak{A}(t)) = \left\{ H^N([a,b]; \mathbb{K}^n) \mid W_B \tau(x) = 0 \right\}
$$

(65)

Then $(\mathfrak{A}(t), D(\mathfrak{A}(t)))$ is a closed and densely defined operator and its graph norm $\| \cdot \|_{D(\mathfrak{A}(t))}$ is equivalent to the Sobolev norm $\| \cdot \|_{H^N((a,b); \mathbb{K}^n)}$ as $P_N(t)$ is invertible. Moreover, for each $t \geq 0$ the operator $A(t) : D(A(t)) \subset X \to X$ defined by

$$
A(t)x = \mathfrak{A}(t)x \quad x \in D(A(t))
$$

(66)

$$
D(A(t)) = \left\{ x \in H^N((a,b); \mathbb{K}^n) \mid W_B \tau(x) = 0 \right\}
$$

(67)

generates a contractive $C_0$-semigroup on $X$. Further, we define the input operator $\mathfrak{B}$ and output operator $\mathfrak{C}$ as follows

$$
\mathfrak{B} : H^N((a,b); \mathbb{K}^n) \to U, \quad \mathfrak{B}x := W_{B,1} \tau(x),
$$

and

$$
\mathfrak{C} : H^N((a,b); \mathbb{K}^n) \to Y, \quad \mathfrak{C}x := W_C \tau(x).
$$

The operator $\mathfrak{C}$ is a linear and bounded operator from $D(\mathfrak{A}(t))$ to $Y$, since the trace operator $\tau$ is bounded and the graph norm of $D(\mathfrak{A})$ is equivalent to the $H^N((a,b); \mathbb{K}^n)$-norm. Moreover, Lemma 5.2 below shows that there exists an operator $\tilde{B} \in \mathcal{L}(U, X)$ which is independent of $t \geq 0$ satisfying the assumption (ii) of Definition 3.2. The proof of this fact follows by a minor modification of the proof of [17, Theorem 11.3.2] and that of [2, Lemma 3.2.19] (see also the second step of the proof of [23, Theorem 4.2]).

**Lemma 5.2.** There exists a linear operator $\tilde{B} \in \mathcal{L}(\mathbb{K}^m, X)$ such that $\tilde{B} \mathbb{K}^m \subset D(\mathfrak{A}(t)), \mathfrak{A}(t)\tilde{B} \in \mathcal{L}(\mathbb{K}^m, X)$ for each $t \geq 0$ and $\mathfrak{B}\tilde{B} = I_{\mathbb{K}^m} = I_U$.

**Proof.** Since the $nN \times 2nN$-matrix $W_B$ has full rank $nN$ there exists a $2nN \times nN$-matrix $S$ such that

$$
W_B S = \begin{bmatrix} W_{B,1} & W_{B,2} \end{bmatrix} S = \begin{bmatrix} I_{\mathbb{K}^m} & 0 \\ 0 & 0 \end{bmatrix}.
$$

(68)

In fact, one can choose $S$ as follows

$$
S = W_B^*(W_B W_B^*)^{-1} \begin{bmatrix} I_{\mathbb{K}^m} & 0 \\ 0 & 0 \end{bmatrix}.
$$

Let us write $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \\ \vdots & \vdots \\ S_{(2nN)1} & S_{(2nN)2} \end{bmatrix} =: [S_1, S_2]$, where $S_{j1}, j = 1, \cdots, 2nN$, are $1 \times m$ matrices.

Next, let $\{e_j\}_{j=1}^{2Nn}$ be the standard orthogonal basis in $\mathbb{K}^{2nN}$. For each $j = 1, 2, \cdots, 2nN$ we take $f_j \in H^N(a, b; \mathbb{K}^n)$ such that $\tau(f_j) = e_j$ [38, Lemma A.3], and we define the operator $\tilde{B} \in \mathcal{L}(\mathbb{K}^m, X)$ by

$$
\tilde{B}u := \sum_{j=1}^{2nN} S_{j1} u f_j \quad u \in \mathbb{K}^m.
$$

(69)
Thus $B \in \mathcal{L}([a,b]; \mathbb{K}^n))$. Furthermore, (68) implies that $W_{B,2} \tilde{S}_1 = 0$ and thus
\[
W_{B,2} \tau(\tilde{B}u) = W_{B,2} \sum_{j=1}^{2nN} S_{j1} u \tau(f_j) = W_{B,2} \sum_{j=1}^{2nN} S_{j1} u e_j = W_{B,2} \tilde{S}_1 u = 0
\]
for every $u \in \mathbb{K}^m$. We deduce that $\tilde{B} \mathbb{K}^m \subset D(\mathfrak{A}(t))$ for all $t \geq 0$. It follows that $\Sigma(\mathfrak{A}(t), \mathfrak{B}, \mathfrak{C})$ is for each $t \geq 0$ a BCO-system on $(L^2([a,b]; \mathbb{K}^n), \mathbb{K}^m, \mathbb{K}^d)$. Using (68) once more, we obtain that
\[
\mathfrak{B} \tilde{B}u = W_{B,1} \tau(\tilde{B}u) = W_{B,1} \tilde{S}_1 u = u
\]
for all $u \in \mathbb{K}^m$. This completes the proof.

Moreover, if in addition the following assumption holds

**Assumption 5.3.**

- $nN = m = d$ (and thus $W_B = W_{B,1}$ or equivalently $W_{B,2} = 0$),
- $R = \{R(t) \mid t \geq 0\}$ and $J = \{J(t) \mid t \geq 0\}$ are bounded and self adjoint operators on $\mathbb{K}^n$,
- $\text{Re} P_0(t, \zeta) \leq 0$ for all $t \geq 0$ and a.e. $\zeta \in [a, b]$,
- the matrix $W_C$ has full rank,

then we obtain:

**Lemma 5.4.** Assume that 5.3 and 5.1 hold. Then assertion (ii) in Proposition 4 holds if
\[
P_{W_B, W_C}(t) := \left( \begin{bmatrix} W_B(t) \\ W_C(t) \end{bmatrix} \Sigma \begin{bmatrix} W_B(t) & W_C(t) \end{bmatrix} \right)^{-1} \leq \begin{bmatrix} 2R(t) & 0 \\ 0 & -2J(t) \end{bmatrix}.
\]

**Proof.** Using [2, Lemma 3.2.13] we obtain
\[
\text{Re}(\mathfrak{A}(t)x \mid x) = \text{Re}(R_{\text{ext}}(t) \tau(x) \mid \Sigma R_{\text{ext}}(t) \tau(x)) + \text{Re}(P_0(t)x \mid x).
\]

Inserting
\[
\begin{bmatrix} W_B \tau(x) \\ W_C \tau(x) \end{bmatrix} = \begin{bmatrix} W_B \\ W_C \end{bmatrix} R_{\text{ext}}^{-1}(t) R_{\text{ext}}(t) \tau(x) = \begin{bmatrix} W_B(t) \\ W_C(t) \end{bmatrix} R_{\text{ext}}(t) \tau(x)
\]
into (71) we obtain that
\[
2 \text{Re}(\mathfrak{A}(t)x \mid x) \leq \left( \begin{bmatrix} W_B \tau(x) \\ W_C \tau(x) \end{bmatrix} \right) \left( \begin{bmatrix} W_B(t) \\ W_C(t) \end{bmatrix} \right)^{-1} \Sigma \left( \begin{bmatrix} W_B(t)^* \\ W_C(t)^* \end{bmatrix} \right)^{-1} \left( \begin{bmatrix} W_B \tau(x) \\ W_C \tau(x) \end{bmatrix} \right)_\mathbb{K}^{2nN},
\]
holds for every $t \geq 0$, $x \in H^N([a, b]; \mathbb{K}^n)$, since $\text{Re} P_0(t, \zeta) \leq 0$, which completes the proof.

Finally, the assumption on $\mathcal{H}$ ensures that the family of operators $M(t) := H(t)(\cdot) := \mathcal{H}(t, \cdot)$ as matrix multiplication operators on $L^2([a, b]; \mathbb{K}^n)$ satisfies all assumptions of Section 3.3.

Our abstract results in the previous sections hence yield the following main result.
Theorem 5.5. If Assumption 5.1 holds, then the port-Hamiltonian system (60)-(64) is a non-autonomous boundary control and observation system. Furthermore, there exists a unique evolution family $W$ in $L^2([a,b]; \mathbb{K}^n)$ such that for each $x_0 \in H^N((a,b); \mathbb{K}^n)$ and $u \in C^2([a,b]; \mathbb{K}^m)$ with, $W_{B,1} \tau(x_0) = u(0)$ and $W_{B,2} \tau(x_0) = 0$ we have

$$x(t) = W(t,0)H^{-1}(0,\zeta)x_0 + \int_0^t W(t,r)\mathcal{A}(r)\dot{B}u(r)dr$$

$$+ \int_0^t W(t,r)H^{-1}(r,\zeta)\dot{H}(r)H^{-1}(r)\dot{B}u(r)dr - \int_0^t W(t,r)H^{-1}(r)\dot{B}u(r)dr,$$

$$t \geq 0,$$

$$y(t) = \mathcal{C}(t)W(t,0)H^{-1}(0,\zeta)x_0 + \mathcal{C} \int_0^t \mathcal{H}(t)W(t,r)\left[\mathcal{A}(r) - H^{-1}(r)\dot{H}(r)H^{-1}(r)\right]\dot{B}u(r)dr$$

$$- \mathcal{C} \int_0^t \mathcal{H}(r)W(t,r)H^{-1}(r)\dot{B}u(r)dr, \quad t \geq 0,$$

is the unique classical solution of (60)-(64). If in addition Assumption 5.3 and (70) hold, then (60)-(64) is $(R, \mathcal{H}, J)$-scattering passive and the classical solution $(x, y)$ satisfies the balance inequality

$$m\|x(t)\|^2 + \int_s^t (y(r) | J(r)y(r))dr \leq c_{L,s} \int_s^t \|\mathcal{H}(r)\|^2 dr \left[ \|u(r) | R(r)u(r)\|dr + \|x(s)\|^2 \right]$$

(73)

where $c_{L,s} = \max\{1, \max_{r \in [s,t]} \|\mathcal{H}(r)\|\}$. Moreover, (60)-(64) is well posed if in addition $J$ is uniformly coercive and $R \in L_{\text{Loc}}^\infty([0,\infty); \mathcal{L}(\mathbb{K}^n))$.

Finally, we give a result on the existence of mild solution of the non-autonomous port-Hamiltonian system. For that we assume that $nN = m = d$. Then it is known [23, Lemma A1] (see also [17, Section 7.3]) that there exist a matrix $V \in \mathbb{K}^{nN \times nN}$ and an invertible matrix $S \in \mathbb{K}^{nN \times nN}$ such that

$$W_B = S \begin{bmatrix} I + V & I - V \end{bmatrix}$$

with $VV^* \geq I$. Further, we have $\ker W_B = \text{Ran} \left[ \frac{I-V}{I-V^*} \right]$. For each $t \geq 0$, the adjoint operator $A^*(t) : D(A^*(t)) \rightarrow X$ of (66)-(67) is given by

$$A^*(t)x = -\mathcal{A}(t)x \quad x \in D(A^*(t))$$

(74)

$$D(A^*(t)) = \left\{ x \in H^N((a,b); \mathbb{K}^n) \mid [I - V^* - I - V^*]\begin{bmatrix} Q(t) & 0 \\ 0 & -Q(t) \end{bmatrix} \tau(x) = 0 \right\}$$

(75)

see e.g., [37, Theorem 2.24], [2, Proposition 3.4.3]. We deduce that the domain of $A^*(t)$ are time-independent if for instance all matrices $P_k, k = 1, 2, \cdots N$ are constant. Thus using Corollary 4 we obtain the following proposition.

**Proposition 7.** Assume that Assumption 5.1 and Assumption 5.3 hold with $P_k, k = 1, 2, \cdots N$ are constant and $J$ is uniformly coercive and $R \in L_{\text{Loc}}^\infty([0,\infty); \mathcal{L}(\mathbb{K}^n))$. If (70) holds, then the non-autonomous system (60)-(63) has a unique mild solution.

We closed this section by some examples of physical systems which can be modelled as a non-autonomous port-Hamiltonian system. Here we will present just two relevant examples, however various other control systems fit into the framework of port-Hamiltonian system and into the general class of NBCO-systems.
5.1. **Vibrating string.** Let us consider the model of vibrating string on the compact interval \([a, b]\). The string is fixed at the left end point \(a\) and at the right end point \(b\) a damper is attached. The Young’s modulus and the mass density of the string are assumed to be time- and spatial dependent. Let us denote by \(\omega(t, \zeta)\) the vertical position of the string at position \(\zeta \in [a, b]\) and time \(t \geq 0\). Then the evolution of the controlled vibrating string can be modelled by a non-autonomous wave equation of the form

\[
\frac{\partial}{\partial t} \left( \alpha(t) \rho(t, \zeta) \frac{\partial w}{\partial t}(t, \zeta) \right) = \frac{1}{\alpha(t)} \frac{\partial}{\partial \zeta} \left( T(t, \zeta) \frac{\partial w}{\partial \zeta}(t, \zeta) \right), \quad \zeta \in [a, b], \ t \geq 0,
\]

(76)

\[
T(b, t) \frac{\partial w}{\partial \zeta}(t, b) + k \alpha(t) \frac{\partial w}{\partial t}(t, b) = u_1(t),
\]

(77)

\[
\frac{\partial w}{\partial t}(t, a) = u_2(t).
\]

(78)

We assume that \(k \geq 0\) and \(T, \rho \in C^2([0, \infty); L^\infty(a, b)) \cap C^1([0, \infty); L^\infty(a, b))\) such that for some \(m > 0\), for a.e \(\zeta \in [a, b]\) and all \(t \geq 0\) we have \(m^{-1} \leq \rho(t, \zeta), T(t, \zeta) \leq m\), moreover, \(\alpha \in C^1([0, \infty))\) is strictly positive. We take as state variable the momentum-strain couple \(x := (\alpha \rho \frac{\partial w}{\partial t}, \frac{\partial w}{\partial \zeta})\). Then the first equation can be equivalently written as follows

\[
\frac{\partial}{\partial t} x(t, \zeta) = \mathcal{A}(t) \mathcal{H}(t, \zeta) x(t, \zeta)
\]

(79)

where

\[
\mathcal{A}(t) := \begin{bmatrix} 0 & 1/\alpha(t) \\ 1/\alpha(t) & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \quad \text{and} \quad \mathcal{H}(t, \zeta) := \begin{bmatrix} \frac{1}{\rho(t, \zeta)} & 0 \\ 0 & T(t, \zeta) \end{bmatrix}.
\]

Indeed, we have

\[
\mathcal{A}(t) \mathcal{H}(t, \zeta) x(t, \zeta) = \begin{bmatrix} 0 & 1/\alpha(t) \\ 1/\alpha(t) & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left[ \begin{bmatrix} \frac{1}{\rho(t, \zeta)} & 0 \\ 0 & T(t, \zeta) \end{bmatrix} \alpha(t) \rho(t, \zeta) \frac{\partial w}{\partial t}(t, \zeta) \right]
\]

\[
= \begin{bmatrix} 0 & 1/\alpha(t) \\ 1/\alpha(t) & 0 \end{bmatrix} \frac{\partial}{\partial \zeta} \left[ \alpha(t) \frac{\partial w}{\partial t}(t, \zeta) \frac{\partial w}{\partial \zeta}(t, \zeta) \right]
\]

\[
= \frac{1}{\alpha(t)} \left( \frac{\partial}{\partial \zeta} \left( T(t, \zeta) \frac{\partial w}{\partial \zeta}(t, \zeta) \right) \right)
\]

\[
= \left( \frac{\partial}{\partial \zeta} \right) \left( \alpha(t) \rho(t, \zeta) \frac{\partial w}{\partial t}(t, \zeta) \right) = \frac{\partial}{\partial t} x(t, \zeta).
\]

Moreover, the boundary conditions (78)-(77) with \(u = (u_1, u_2) = 0\) can be equivalently written as follows

\[
W_B \begin{bmatrix} \mathcal{H}(t, b) x(t, b) \\ \mathcal{H}(t, a) x(t, a) \end{bmatrix} := \begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{H}(t, b) x(t, b) \\ \mathcal{H}(t, a) x(t, a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
The $2 \times 4$ matrix $W_B$ has full rank. Next,

\[
W_B(t) = W_B \begin{bmatrix} 0 & \alpha(t) & 1 & 0 \\ \alpha(t) & 0 & 0 & 1 \\ 0 & -\alpha(t) & 1 & 0 \\ -\alpha(t) & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha(t) & k\alpha(t) & k & 1 \\ 0 & -\alpha(t) & 1 & 0 \end{bmatrix}
\]

and $W_B(t)\Sigma W_B^*(t) = \begin{bmatrix} 4k\alpha(t) & 0 \\ 0 & 0 \end{bmatrix} \geq 0$. The corresponding matrices $W_{B,1}, W_{B,2}$ and the corresponding boundary operator $\mathfrak{B}$ can be defined as follows:

**Case** $u_2 = 0 : W_{B,2} = [0 \ 0 \ 1 \ 0]$ and

\[
\mathfrak{B} : H^N((a,b); \mathbb{K}^2) \to U = \mathbb{K},
\]

\[
\mathfrak{B} x := W_{B,1}\tau(x) := \begin{bmatrix} k & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(b) \\ x(a) \end{bmatrix}.
\]

**Case** $u_2 \neq 0 : W_{B,2} = 0$ and

\[
\mathfrak{B} : H^N((a,b); \mathbb{K}^2) \to U = \mathbb{K}^2, \quad \mathfrak{B} x := \begin{bmatrix} k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x(b) \\ x(a) \end{bmatrix}
\]

For each $S, V \in \mathbb{K}^{2 \times 2}$ such that $S$ is invertible and $VV^* \geq I$ we can take

\[
y(t) = S \begin{bmatrix} I + V & I - V \end{bmatrix} \begin{bmatrix} \mathcal{H}(t,b)x(t,b) \\ \mathcal{H}(t,a)x(t,a) \end{bmatrix}
\]

as an output of (78)-(77). Thus, we are in the position to apply Theorem 5.5. However, Proposition 7 concerning mild solutions can be applied only if $\alpha(t) \equiv \alpha > 0$ is constant.

**Proposition 8.** Under the conditions on the physical parameters $T, \alpha, \rho, k$ listed above we have:

1. The abstract linear system associated with the controlled vibrating string (76), (78) with output (80) yields a non-autonomous boundary control and observation system on $(L^2([a,b]; \mathbb{K}^2), \mathbb{K}^1, \mathbb{K}^2)$ if $u_2 = 0$, i.e., when the string is clamped at the end point $a$, and in $(L^2([a,b]; \mathbb{K}^2), \mathbb{K}^2, \mathbb{K}^2)$ if $u_2 \neq 0$.

2. Let $\omega_0, \omega_1 \in H^1(a, b; \mathbb{K})$ be such that $k\omega_0(b) + \omega_1(b) = u_1(0)$ and $\omega_0(a) = u_2(0)$. Then (76)-(78) with output equation (80) and initial conditions

\[
\alpha(0)\rho(0,\cdot) \frac{\partial w}{\partial t}(0,\cdot) = \omega_0, \quad \frac{\partial \omega}{\partial t}(0,\cdot) = \omega_1
\]

has a unique solution $(\omega, y)$ such that $y \in C([0,\infty); \mathbb{K}^2)$ and

\[
t \mapsto \begin{bmatrix} \alpha(t) \frac{\partial w}{\partial t}(t,\cdot) \\ T(t,\cdot) \frac{\partial \omega}{\partial t}(t,\cdot) \end{bmatrix} \in C^1\left(\left[0,\infty\right); L^2(a,b;\mathbb{K}^2)\right) \cap C\left(\left[0,\infty\right); L^2(a,b;\mathbb{K}^2)\right).
\]

3. Let $u_2 \neq 0$. Let $R(t), J(t)$ be self-adjoint $2 \times 2$-matrices such that $R \in L^\infty_{loc}([0,\infty); C(\mathbb{K}^2))$ and $c_0^{-1} \leq J(t) \leq c_0$ for all $t \geq 0$ and some constant $c_0 > 0$. Choose $V, S$ in (80) such that (70) holds for all $t \geq 0$. Then the linear system associated with the non-autonomous controlled vibrating string (76)-(78) and (80) is a well-posed non-autonomous boundary control and observation system.
4. Assume that \( \alpha(t) \equiv \alpha > 0 \) is constant such that the assumptions in (3) hold. Let \( \omega_0, \omega_1 \in L^2(a, b; \mathbb{C}) \). Then (76)-(78) with initial conditions

\[
\alpha(0) \rho(0, \cdot) \frac{\partial w}{\partial t}(0, \cdot) = \omega_0, \quad \frac{\partial \omega}{\partial t}(0, \cdot) = \omega_1
\]

has a unique (mild) solution \( \omega \) such that

\[
t \mapsto \begin{bmatrix} \alpha(t) \frac{\partial w}{\partial \omega(t, \cdot)} \\ T(t, \cdot) \end{bmatrix} \in C\left([0, \infty); L^2(a, b; \mathbb{K}^2)\right).
\]

5.2. Timoshenko beam. Consider the following model of the Timoshenko beam with time-dependent coefficient and time dependent boundary control

\[
\frac{\partial}{\partial t} \left( \rho(t) \rho(t, \cdot) \frac{\partial w}{\partial t}(t, \cdot) \right) = \frac{1}{\rho(t)} \frac{\partial}{\partial \zeta} \left[ K(t, \zeta) \left( \frac{\partial}{\partial \zeta} w(t, \zeta) - \phi(t, \zeta) \right) \right],
\]

(81)

\[
\frac{\partial}{\partial t} \left( I_\rho(t) I_\rho(t, \cdot) \frac{\partial \phi}{\partial t}(t, \cdot) \right) = \frac{1}{I_\rho(t)} \frac{\partial}{\partial \zeta} \left( EI(t, \zeta) \frac{\partial^2}{\partial \phi \partial t} \phi(t, \zeta) \right) + \frac{1}{\rho(t)} K(t, \zeta) \left( \frac{\partial}{\partial \zeta} w(t, \zeta) - \phi(t, \zeta) \right),
\]

(82)

for some positive constants \( \alpha_1, \alpha_2 > 0 \) and input function \( u = (u_1, u_2, u_3, u_4) \). Here \( \zeta \in (a, b) \), \( t \geq 0 \), \( w(t, \zeta) \) is the transverse displacement of the beam and \( \phi(t, \zeta) \) is the rotation angle of the filament of the beam. We assume that \( K, \rho, EI, I_\rho \in C^2([0, \infty); L^\infty(a, b)) \cap C^0([0, \infty); L^\infty(a, b)) \) and there exists \( m > 0 \) such that for a.e. \( \zeta \in [a, b] \) and all \( t \geq 0 \) we have

\[
m^{-1} \leq \rho(t, \zeta), K(t, \zeta), EI, I_\rho \leq m,
\]

where \( \rho(t, \zeta) \) and \( I_\rho \) are strictly positive. Moreover, \( \tilde{\rho}, \tilde{I}_\rho \in C^1([0, \infty)) \) are strictly positive.

Indeed, taking as state variable \( x := (\frac{\partial w}{\partial \zeta} - \phi \rho \frac{\partial w}{\partial t}, \frac{\partial \phi}{\partial \zeta}, \tilde{I}_\rho I_\rho \frac{\partial \phi}{\partial t}) \) one can easily see that (81)-(82) can be written as a system of the form (60)-with

\[
P_1 = \begin{bmatrix} 0 & \tilde{\rho}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_\rho^{-1} & 0 \\ 0 & 0 & \tilde{I}_\rho^{-1} & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -I_\rho^{-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & EI & 0 \\ 0 & 0 & 0 & I_\rho^{-1} \end{bmatrix}.
\]

The boundary condition can be formulated as follows

\[
\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{\partial w}{\partial \zeta} + \frac{\partial \phi}{\partial \zeta} = \begin{bmatrix} H(t, b) \frac{\partial w}{\partial \zeta} + \frac{\partial \phi}{\partial \zeta} \\ H(t, a) \frac{\partial w}{\partial \zeta} + \frac{\partial \phi}{\partial \zeta} \end{bmatrix} = : W_B \begin{bmatrix} H(t, b) \frac{\partial w}{\partial \zeta} + \frac{\partial \phi}{\partial \zeta} \\ H(t, a) \frac{\partial w}{\partial \zeta} + \frac{\partial \phi}{\partial \zeta} \end{bmatrix}
\]
Thus $W_B$ has full rank and the corresponding $4 \times 8$ matrix $W_B(t)$ is given by

$$W_B(t) = W_B \begin{bmatrix} P_{-1}^{-1}(t) & I \\ -P_1^{-1}(t) & I \end{bmatrix} = \begin{bmatrix} -\tilde{\rho}(t) & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\tilde{I}_\rho(t) & 0 & 0 & 0 & 1 \\ \alpha_1 \tilde{\rho}(t) & 0 & 0 & 1 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 \tilde{I}_\rho(t) & \tilde{I}_\rho(t) & 0 & 0 & 1 & \alpha_2 \end{bmatrix}.$$

Thus $W_B(t)\Sigma W_B^*(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4\alpha_1 \tilde{\rho}(t) & 0 \\ 0 & 0 & 0 & \alpha_2 \tilde{I}_\rho(t) \end{bmatrix} \geq 0$. As in Example 5.1, the output equation can be choosing similarly as (80). Thus the above Timoshenko beam fit into the framework of port-Hamiltonian system and thus one obtain a similar results to that presented in Proposition 8.

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