FINITELY GENERATED INFINITE SIMPLE GROUPS
OF INFINITE SQUARE WIDTH AND VANISHING
STABLE COMMUTATOR LENGTH

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Abstract. It is shown that there exist finitely generated infinite simple
groups of infinite commutator width and infinite square width on which
there exists no stably unbounded conjugation-invariant norm, and in
particular stable commutator length vanishes. Moreover, a recursive
presentation of such a group with decidable word and conjugacy prob-
lems is constructed.

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1. INTRODUCTION

This article is an extension of [21], where finitely generated infinite sim-
ple groups of infinite commutator width were constructed using a small-
cancellation approach. It is explained here how to modify that construction
in order to make the square width of the constructed group $G$ also infinite
and to satisfy the additional condition that there exist no stably unbounded
conjugation-invariant norm on $G$. This additional condition implies in par-
ticular vanishing of stable commutator length, which in turn is equivalent
to injectivity of the natural homomorphism $H^2_b(G, \mathbb{R}) \rightarrow H^2(G, \mathbb{R})$ from
the second bounded cohomology to the second ordinary cohomology (see Propo-
sition 3.4 in [3] or Theorem 3.18 in [14]). Also an effort is made to explain
the technics of [21] more informally.

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Definition. Let $G$ be a group. The commutator of elements $x$ and $y$ of $G$ is $[x, y] = xyx^{-1}y^{-1}$. The commutator length of an element $g$ of the derived subgroup $[G, G]$, denoted $\text{cl}_G(g)$, is the minimal $n$ such that there exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in G$ such that $g = [x_1, y_1] \cdots [x_n, y_n]$. (Naturally, $\text{cl}(1) = 0$.) The stable commutator length of $g \in [G, G]$ shall be denoted by $\text{cl}_{[G,G]}(g)$ and is defined by

$$\text{cl}_{[G,G]}(g) = \lim_{n \to \infty} \frac{\text{cl}_G(g^n)}{n} = \inf_{n \in \mathbb{N}} \frac{\text{cl}_G(g^n)}{n}.$$

The commutator width of $G$ shall be denoted $\text{cw}(G)$ and is defined by

$$\text{cw}(G) = \sup_{[G,G]} \text{cl}_G.$$

Stable commutator length is related to the space of homogeneous quasi-morphisms on $G$, and hence to the kernel of the natural homomorphism $H^2_b(G, \mathbb{R}) \to H^2(G, \mathbb{R})$, which is isomorphic to the quotient of the space of all homogeneous quasi-morphisms $G \to \mathbb{R}$ by the subspace of all homomorphisms $G \to \mathbb{R}$, see Proposition 3.3.1(1) in [3] or Theorem 3.5 in [14].

Definition. Let $G$ be a group. A function $\phi : G \to \mathbb{R}$ is called a quasi-morphism if the function $(x, y) \mapsto \phi(xy) - (\phi(x) + \phi(y))$ is bounded on $G \times G$. A quasi-morphism is homogeneous if its restriction to every cyclic subgroup is a homomorphism to $(\mathbb{R}, +)$.

Every quasi-morphism $G \to \mathbb{R}$ is bounded on the set of all commutators of $G$. It is also relatively easy to see that if some homogeneous quasi-morphisms is non-zero on $g \in [G, G]$, then $\text{cl}_G(g) > 0$, and in particular $\text{cw}G = \infty$. Christophe Bavard [3, Proposition 3.4] proved that in fact stable commutator length vanishes on the whole of the derived subgroup if and only if so do all homogeneous quasi-morphisms. (Moreover, Bavard provided a formula for commutator length in terms of homogeneous quasi-morphisms.) Furthermore, observe (or see Proposition 3.3.1(2) in [3]) that a homogeneous quasi-morphism vanishes on the derived subgroup only if it is a homomorphism; therefore, the natural homomorphism $H^2_b(G, \mathbb{R}) \to H^2(G, \mathbb{R})$ is injective if and only if stable commutator length vanishes on $[G, G]$.

A comprehensive introduction to the theory of stable commutator length may be found in [6].

Until 1991 it was apparently not known that there exist simple groups of commutator width greater than 1. For finite simple groups, it was shown in 2008 by Martin W. Liebeck, Eamonn A. O’Brien, Aner Shalev, and Pham Huu Tiep [18] that every element of every non-abelian finite simple group is a commutator, and thus the long-standing conjecture of Oystein Ore [24] was proved. Jean Barge and Étienne Ghys [2, Theorem 4.3] showed that there are simple groups of symplectic diffeomorphisms of $\mathbb{R}^{2n}$ (kernels of Calabi homomorphisms) which possess nontrivial homogeneous quasi-morphisms, and thus their commutator width is infinite. Theorem 1.1 in [13] provides

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1A stronger conjecture of John Thompson that every non-abelian finite simple group $G$ has a conjugacy class $C$ such that $G = CC$ still remains open. Previously, important results towards resolution of the two conjectures were obtained by Erich Ellers and Nikolai Gordeev in [12] and by A. Shalev in [26].
other similar examples of simple groups of infinite commutator width. Existence of finitely generated simple groups of commutator width greater than 1 was proved in [21]. Pierre-Emmanuel Caprace and Koji Fujiwara [7] recently proved that there are finitely presented simple groups for which the space of homogeneous quasi-morphisms is infinite-dimensional, and in particular whose commutator width is infinite. Those groups are the quotients of certain non-affine Kac–Moody lattices by the center; they were defined by Jaques Tits [29] and their simplicity was proved by P.-E. Caprace and Bertrand Rémy [8].

Commutator length in a group $G$ is an example of a conjugation-invariant norm on the derived subgroup $[G,G]$, as defined by Dmitri Burago, Sergei Ivanov, and Leonid Polterovich in [5].

**Definition.** A conjugation-invariant norm on a group $G$ is a function $\nu: G \to [0, \infty)$ which satisfies the following five axioms:

1. $\nu(g) = \nu(g^{-1})$ for all $g \in G$,
2. $\nu(gh) \leq \nu(g) + \nu(h)$ for all $g, h \in G$,
3. $\nu(g) = \nu(hgh^{-1})$ for all $g, h \in G$,
4. $\nu(1) = 0$,
5. $\nu(g) > 0$ for all $g \in G \setminus \{1\}$.

For brevity, conjugation-invariant norms shall be sometimes called simply norms.

**Definition.** If $\nu$ is a norm on $G$, then its stabilization, which shall be denoted $\nu_\infty$, is defined by

$$\nu_\infty(g) = \lim_{n \to \infty} \frac{\nu(g^n)}{n} = \inf_{n \in \mathbb{N}} \frac{\nu(g^n)}{n}, \quad g \in G.$$ 

A norm $\nu$ is stably unbounded if $\nu_\infty(g) > 0$ for some $g \in G$.

Note that in general the stabilization of a norm is not a norm, as it has no reason to satisfy the axioms (2) and (5) of the definition.

The following question was raised by Burago, Ivanov, and Polterovich [5]:

Does there exist a group that does not admit a stably unbounded norm and yet admits a norm unbounded on some cyclic subgroup?

The main theorem of this article answers this question positively.

Similarly to commutator length, one can define square length.

**Definition.** Let $G$ be a group. The square length of an element $g$ of the subgroup $G^2 = \langle x^2 \mid x \in G \rangle$, denoted $sql_G(g)$, is the minimal $n$ such that there exist $x_1, \ldots, x_n \in G$ such that $g = x_1^2 \cdots x_n^2$. The square width of $G$ is

$$sqw(G) = \sup_{G^2} sql_G.$$ 

Observe that finite commutator width implies finite square width, since every commutator is the product of 3 squares (and since abelian groups have square width at most 1). Moreover, $sql_G(x) \leq 2cl_G(x) + 1$ for every $x \in [G,G]$, and this estimate cannot be improved unless $cl_G(x) = 0$ ($x = 1$), see [11, Section 2.5].
Main Theorem. There exists a torsion-free simple group $G$ generated by 2 elements $a$ and $b$ such that:

1. $a^2$ and $b^2$ freely generate a free subgroup $H$ such that
   \[
   \lim_{n \to \infty} \cl_G(h^n) = \infty \quad \text{for every } h \in H \setminus \{1\}
   \]
   (in particular, cw($G$) = $\infty$),
2. sq$G$ is unbounded on $H = \langle a^2, b^2 \rangle$ (in particular, sqw($G$) = $\infty$),
3. $G$ does not admit any stably unbounded conjugation-invariant norm
   (in particular, clf$G$ = 0),
4. $G$ is the direct limit of a sequence of hyperbolic groups with respect to a family of surjective homomorphisms,
5. the cohomological and geometric dimensions of $G$ are 2,
6. $G$ has decidable word and conjugacy problems.

This theorem is stronger than Theorem 5 in [21]. It provides positive answer to the question of Burago-Ivanov-Polterovich and also shows that stable commutator length can vanish on a simple group of infinite commutator width (and even of infinite square width).

2. The presentation

The construction presented here is similar to the one in [21, Section 2].

In what follows, if $w$ denotes a group word, then the group element represented by $w$ shall be denoted by $[w]$, or $[w]_G$ if $G$ is the group in question.

Let $\{a, b\}$ be a 2-letter alphabet (one could start here with any finite alphabet containing at least 2 letters). The set of all group words over $\{a, b\}$ shall be denoted by $\{a^{\pm 1}, b^{\pm 1}\}^*$ or $(\{a, b\}^{\pm 1})^*$. Let $F$ be the free group formally freely generated by $a$ and $b$, that is the group presented by $\langle a, b \rangle$.

To prove the main theorem, a group $G$ with desired properties shall be constructed as a quotient of $F$ by recursively constructing its presentation $\langle a, b \parallel R \rangle$.

First of all, fix an arbitrary nontrivial reduced group word $v$ over $\{a, b\}$ (or any group word which is not freely trivial), and let $C_1, C_2, \ldots$ be a list of conditions to be satisfied by $G$ composed as follows:

1. for every $w \in \{a^{\pm 1}, b^{\pm 1}\}^*$ and $x \in \{a, b\}$, the list contains the condition
   "if $[w] \neq 1$, then $[x]$ is the product of conjugates of $[w]$",
   this condition shall be called a condition of the first kind,
2. for every $w \in \{a^{\pm 1}, b^{\pm 1}\}^*$ and every $n \in \mathbb{N}$, the list contains the condition
   "there exist $p, q \in \mathbb{N}$ such that $p \geq nq$ and $[w]^p$ is the product of $q$ conjugates of $[w]$",
   this condition shall be called a condition of the second kind,
3. the list contains no other conditions.

Observe that all conditions in this list are preserved under passing to quotients.

Let $z_1 = aa = a^2$ and $z_2 = bb = b^2$. The elements $[z_1]_G, [z_2]_G$ are intended to freely generate a free subgroup $H$ of $G$ such as in the statement of the main theorem (there are many other possible choices for $z_1, z_2$). The
following properties of $z_1, z_2$ (some necessary, other just convenient) shall be noted:

(1) $[z_1], [z_2]$ freely generate a free subgroup of $F$,
(2) $z_1$ and $z_2$ are positive in the sense that they do not contain $a^{-1}$ or $b^{-1}$, and hence any concatenation of copies of $z_1, z_2$ is cyclically reduced,
(3) every element of the free subgroup $\langle [z_1], [z_2] \rangle$ of $F$ can be written as a reduced concatenation of copies of $z_1^{\pm 1}, z_2^{\pm 1}$, and is conjugate to an element that can be written as a cyclically reduced concatenation of copies of $z_1^{\pm 1}, z_2^{\pm 1}$,
(4) for every $\varepsilon > 0$ and every $L > 0$, there exists a non-periodic (and hence not representing a proper power in $F$) concatenation $w$ of copies of $z_1, z_2$ of length at least $L$ such that if $sq_1$ and $sq_2$ are two distinct cyclic shifts of $w$, then $|s| \leq \varepsilon |w|$.

To verify the last property, one can consider, for example,

$$w = \prod_{i=0}^{n} a^{2i}b^{2n-2i}$$

with $n$ sufficiently large.

The desired indexed family of defining relators $R = \{r_n\}_{n \in I}$ shall be constructed in two steps. First, a family $Q = \{r_n\}_{n \in \mathbb{N}}$ shall be constructed so that for all $n$, the condition $C_n$ be a consequence of the relation $r_n = 1$, and hence the group presented by $\langle a, b \parallel Q \rangle$ shall satisfy all of the conditions $C_1, C_2, \ldots$; however, this group will be the trivial group. Second, certain redundant relators $r_n$ corresponding to conditions $C_n$ of the first kind shall be removed from $Q$ to obtain $R = \{r_n\}_{n \in I}, I \subset \mathbb{N}$.

The construction of $Q$ can easily be carried out effectively, so that $Q$ be recursive. In the process of establishing decidability of the word and conjugacy problems in $G = \langle a, b \parallel R \rangle$, it shall be shown that $R$ can be made recursive as well (all depends on a good choice of $Q$).

The defining relators shall be chosen to satisfy certain small-cancellation conditions to ensure that the obtained group $G$ be nontrivial, of infinite commutator and square widths, with commutator length unbounded on non-trivial cyclic subgroups of $H = \langle [z_1]_G, [z_2]_G \rangle$, etc. Additionally, these small-cancellation conditions will cause all finite subpresentations of $\langle a, b \parallel R \rangle$ to define hyperbolic groups.

The idea of the proof that commutator and square lengths are unbounded is to find for every $N < 0$ some group word $w$ such that the Euler characteristic of every van Kampen diagram over $\langle a, b \parallel R \rangle$ with the boundary label $w$ be necessarily less than $N$. The defining relations shall be chosen so that “most” sufficiently long reduced concatenation of copies of $z_1^{\pm 1}$ and $z_2^{\pm 1}$ suit for the role of $w$ here.

Here follows the formal construction of $Q$ and $R$.

Let $A$ be the set of all $n$ such that the condition $C_n$ is of the first kind. For every $n \in A$, let $w_n \in \{a^{\pm 1}, b^{\pm 1}\}^*$ and $x_n \in \{a, b\}$ be such that $C_n$ states:

“If $[w_n] \neq 1$, then $[x_n]$ is the product of conjugates of $[w_n]$.”
the relator \( r_n \) then shall be chosen in the form
\[
\begin{align*}
    r_n &= u_{n,1} w_n u_{n,1}^{-1} \cdots u_{n,k_n} w_n u_{n,k_n}^{-1} x_n^{-1} = w_n u_{n,1} \cdots u_{n,k_n} x_n^{-1},
\end{align*}
\]
where \( u_{n,1}, \ldots, u_{n,k_n} \in \{a^{\pm 1}, b^{\pm 1}\}^* \) and \( k_n \geq 3 \); such \( r_n \) shall be called a \textit{relator of the first kind}. For every \( n \in \mathbb{N} \setminus A \), let \( w_n \in \{a^{\pm 1}, b^{\pm 1}\}^* \) and \( m_n \in \mathbb{N} \) be such that \( C_n \) states:
\[
\begin{align*}
    \text{“there exist } p, q \in \mathbb{N} \text{ such that } p \geq m_n q \text{ and } [w_n]^p \text{ is the product of } q \text{ conjugates of } [w].”
\end{align*}
\]
the relator \( r_n \) then shall be chosen in the form
\[
\begin{align*}
    r_n &= u_{n,1} v_{n,1} u_{n,1}^{-1} \cdots u_{n,k_n} v_{n,k_n}^{-1} \left( w_n^{m_n k_n} \right)^{-1} = v_{u_{n,1}} u_{n,1} \cdots v_{u_{n,k_n}} u_{n,k_n}^{-1} w_n^{-m_n k_n},
\end{align*}
\]
where \( u_{n,1}, \ldots, u_{n,k_n} \in \{a^{\pm 1}, b^{\pm 1}\}^* \) and \( k_n \geq 3 \); such \( r_n \) shall be called a \textit{relator of the second kind}. Observe that the group presented by \( \langle a, b \parallel Q \rangle \) will be trivial because some of the defining relations of the first kind will make both generators \([a]\) and \([b]\) equal to products of conjugates of \( 1 \).

The family \( \mathcal{R} = \{r_n\}_{n \in I} \) shall be obtained from \( \mathcal{Q} = \{r_n\}_{n \in \mathbb{N}} \) by discarding certain redundant relators of the first kind. Namely, let the set of indices \( I \subset \mathbb{N} \) be defined inductively as follows:
\[
\text{for every } n \in \mathbb{N}, n \notin I \text{ if and only if } n \in A \text{ (i.e. } r_n \text{ is of the first kind) and } [w_n] = 1 \text{ in the group presented by } \langle a, b \parallel r_i, \ i \in I, i < n \rangle.
\]
Observe that \( I \) is infinite (because there are infinitely many relators of the second kind in \( \mathcal{Q} \)).

To finalize the construction of \( \mathcal{Q} \) and \( \mathcal{R} \), it is left to specify how to choose the sequence \( \{k_n\}_{n \in \mathbb{N}} \) and the indexed family \( \{u_{n,i}\}_{n \in \mathbb{N}, i = 1, \ldots, k_n} \).

Roughly speaking, the main requirements shall be that the integers \( k_n \) tend to infinity, that the words \( u_{n,i} \) be reduced, “very long,” and have “very short” common subwords, and also it is important that for every \( n \), the words \( u_{n,1}, \ldots, u_{n,k_n} \) be “more or less of the same length” (for further convenience they may be chosen of the same length: \( |u_{n,1}| = \cdots = |u_{n,k_n}| \)).

The sequence \( \{k_n\}_{n \in \mathbb{N}} \) and the family \( \{u_{n,i}\}_{n \in \mathbb{N}, i = 1, \ldots, k_n} \) shall be chosen simultaneously with four other sequences: a sequence of integers \( \{\lambda_n\}_{n \in \mathbb{N}} \) tending to \(-\infty\) (to be used as upper bounds on Euler characteristics) and three sequences of “small” positive reals \( \{\mu_n\}_{n \in \mathbb{N}}, \{\nu_n\}_{n \in \mathbb{N}}, \text{ and } \{\upsilon_n\}_{n \in \mathbb{N}} \). Also it will be convenient to use two auxiliary sequences \( \{\kappa_n\}_{n \in \mathbb{N}} \) and \( \{\gamma_n\}_{n \in \mathbb{N}} \) defined by
\[
\begin{align*}
    \kappa_n &= 2k_n \quad \text{and} \quad \gamma_n = \lambda_n + (3 + 2\kappa_n)\mu_n + 2\nu_n.
\end{align*}
\]
for all \( n \in \mathbb{N} \). The following 12 conditions shall be satisfied:

C1) for every \( n \in \mathbb{N} \), \( k_n \geq 3 \),

C2) for every \( n \in \mathbb{N} \) and \( i = 1, \ldots, k_n \), \( u_{n,i} \) is reduced,

C3) for every \( n \in \mathbb{N} \), \( 2(|u_{n,1}| + \cdots + |u_{n,k_n}|) \geq (1 - \lambda_n)|r_n| \),

C4) for every \( n \in \mathbb{N} \) and \( i = 1, \ldots, k_n \), \( |u_{n,i}| \leq \nu_n |r_n| \),

C5) for every \( n_1, n_2 \in \mathbb{N} \), \( i_1 = 1, \ldots, k_{n_1} \) and \( i_2 = 1, \ldots, k_{n_2} \), if \( u_{n_1,i_1}^{\sigma_1} = p_1 s q_1 \) and \( u_{n_2,i_2}^{\sigma_2} = p_2 s q_2 \) with \( \sigma_1, \sigma_2 \in \{\pm 1\} \), then either
\[
(n_1, i_1, \sigma_1, p_1, q_1) = (n_2, i_2, \sigma_2, p_2, q_2),
\]
or
\[ \mu_{n_1} |r_{n_1}| \geq |s| \leq \mu_{n_2} |r_{n_2}|, \]

(C6) for every \( n \in \mathbb{N} \) and \( i = 1, \ldots, k_n \), if \( s \) is a common subword of \( u_{n,i}^{\pm 1} \)
and of a concatenation of several copies of \( z_1^{\pm 1}, z_2^{\pm 1} \), then
\[ |s| \leq \mu_n |r_n|, \]

(C7)
\[ \lim_{n \to \infty} \chi_n = -\infty, \]

(C8) for every \( n \in \mathbb{N} \),
\[ \gamma_n < \frac{1}{2} \quad \text{and} \quad (3 - 3\chi_n)\mu_n + (1 - \chi_n)\nu_n < \frac{1}{2} - \gamma_n, \]

(C9) for every \( n \in \mathbb{N} \) and every \( i < n \) such that \( i \in A \) (i.e. \( r_i \) is of the first kind),
\[ |w_i| < (1 - 2\gamma_n)|r_n|, \]

(C10) for every \( n \in \mathbb{N} \),
\[ |v| < (1 - 2\gamma_n)|r_n|, \]

(C11) the normal closures of the elements \( [r_1], [r_2], \ldots \) in the free group \( F \)
(presented by \( \langle a, b \parallel \emptyset \rangle \)) are pairwise distinct,

(C12) the elements \( [r_1], [r_2], \ldots \) are not proper powers in \( F \).

In fact, conditions (C11) and (C12) can be deduced from the previous ones, but it is easier to deduce them from a suitable choice of \( \{k_n\}_{n \in \mathbb{N}} \), \( \{u_{n,i}\}_{n \in \mathbb{N}; i=1,...,k_n} \), etc.

All of these conditions may be fulfilled as follows. First, choose an arbitrary sequence of integers \( \{k_n\}_{n \in \mathbb{N}} \) such that \( k_n \geq 3 \) for all \( n \) and
\[ \lim_{n \to \infty} k_n = \infty. \]

Then, for every \( n \), set
\[ \nu_n = \frac{1}{\kappa_n} = \frac{1}{2k_n}, \quad \chi_n = 4 - k_n. \]

Observe that
\[ \lim_{n \to \infty} \chi_n = -\infty, \]

and that for every \( n \),
\[ 2\nu_n < \frac{1}{2} \quad \text{and} \quad (3 - \chi_n)\nu_n < \frac{1}{2}. \]

Now, for every \( n \), choose \( \lambda_n \) and \( \mu_n \) sufficiently small so that
\[ \gamma_n < \frac{1}{2} \quad \text{and} \quad \gamma_n + (3 - 3\chi_n)\mu_n + (1 - \chi_n)\nu_n < \frac{1}{2}. \]

Finally, given that \( z_1 = a^2 \) and \( z_2 = b^2 \), the family \( \{u_{n,i}\}_{n \in \mathbb{N}; i=1,...,k_n} \) may be chosen in the following form:
\[ u_{n,i} = \prod_{j=2M_n(i-1)+1}^{2M_n} a^j b^{2M_n k_n + 1-j}, \]

where \( \{M_n\}_{n \in \mathbb{N}} \) is a suitable sequence of “large” positive integers “rapidly”
growing to infinity (compare with formula (4) in [21]). Observe that:

(1) \( |u_{n,1}| = \cdots = |u_{n,k_n}| = 2M_n(2M_n k_n + 1) \) for every \( n \).
Lemma 2.2. Let the imposed relations of the second kind that for every \( \{C_n\} \), there exists \( a, b \in \mathbb{N} \) and the set \( I \) and the family \( R \) have been constructed. (Recall that their construction depended on the choice of the words \( v, z_1, z_2 \), and on the list of conditions \( C_1, C_2, \ldots \).) Let, from now on, \( G \) denote the group presented by \( \langle a, b \parallel R \rangle \).

**Proposition 2.1.** If \( G \) is not trivial, then it is simple. There is no stably unbounded conjugation-invariant norm on \( G \).

**Proof.** Because of the imposed relations of the first kind, the normal closure of every nontrivial element of \( G \) contains both generators \( [a] \) and \( [b] \), and hence is the whole of \( G \).

Consider an arbitrary conjugation-invariant norm \( \theta \) on \( G \). It follows from the imposed relations of the second kind that for every \( g \in G \) and every \( \varepsilon > 0 \), there exists \( p \in \mathbb{N} \) such that \( \theta(g^p) \leq \varepsilon \theta([v]) \). Hence for every \( g \in G \),

\[
\inf_{p \in \mathbb{N}} \frac{\theta(g^p)}{p} = 0.
\]

\[\square\]

In order for \( G \) to have decidable word and conjugacy problems, the presentation \( \langle a, b \parallel R \rangle \) shall be constructed in a less random way, so as in particular to be recursive.

**Lemma 2.2.** The construction described above in this section can be carried out in such a manner that the following additional properties be satisfied:

1. \( \{a_n\}_{n \in \mathbb{N}}, \{v_n\}_{n \in \mathbb{N}} \), and \( \{\lambda_n\}_{n \in \mathbb{N}}, \{\mu_n\}_{n \in \mathbb{N}}, \{\nu_n\}_{n \in \mathbb{N}} \) are recursive sequences of integer and rational numbers, respectively,
2. the set \( A \) and the families \( Q, \{w_n\}_{n \in A}, \{x_n\}_{n \in A}, \{w_n\}_{n \in \mathbb{N} \setminus A}, \{m_n\}_{n \in \mathbb{N} \setminus A}, \{u_{n,1}\}_{n \in \mathbb{N}, i=1, \ldots, k_n} \) are recursive, and the sequence \( \{(1 - 2\gamma_n) r_n\}_{n \in \mathbb{N}} \) is bounded from below by a recursive sequence of integers monotonically tending to \( \infty \),
3. if there is an algorithm that decides the word problem for all finite subpresentations of \( \langle a, b \parallel R \rangle \) (the input consisting of a finite part of the indexed family \( R \) and a pair of words to compare), then the set \( I \) and the family \( R \) are recursive.

**Proof.** To satisfy (1), it suffices to define \( k_n = 3 + n \) for all \( n \), and to choose the four other sequences accordingly in an effective manner.
To satisfy the rest, the list $C_1, C_2, \ldots$ should be enumerated so as to be recursive. Then the family $\{u_{n,i}\}_{n \in \mathbb{N}, i=1,\ldots,k_n}$ can be constructed effectively and so that $(1 - 2\gamma_n)|r_n| \geq n$ for all $n$, and hence (2) be satisfied.

Suppose there is a way to effectively solve the word problem for all finite subpresentations of $\langle a, b \mid R \rangle$. Then, once (2) is satisfied, and hence the indexed family $Q$ is recursive, the set $I$ can be constructed inductively by deciding for each $n \in \mathbb{N}$, whether $[w_n] = 1$ in the group presented by $\langle a, b \mid r_i, i \in I, i < n \rangle$, and hence concluding whether or not $n \in I$. If the family $Q$ and the set $I$ are recursive, then $R$ is recursive as well, and (3) is satisfied.

\begin{remark}
Note that in items (2) and (3) of Lemma 2.2, nothing is stated explicitly about recursiveness of $Q$ and $R$ as sets, i.e. whether the sets $\{r_n \mid n \in \mathbb{N}\}$ and $\{r_n \mid n \in I\}$ are recursive. However, if the sequence $\{|r_n|\}_{n \in \mathbb{N}}$ is bounded from below by a recursive sequence of integers monotonically tending to $\infty$, and if the indexed families $Q$ and $R$ are recursive (as functions $\mathbb{N} \to \{a \pm 1, b \pm 1\}^\ast$ and $I \to \{a \pm 1, b \pm 1\}^\ast$, respectively), then the corresponding sets $\{r_n \mid n \in \mathbb{N}\}$ and $\{r_n \mid n \in I\}$ are recursive as well.
\end{remark}

3. Maps and diagrams

The terminology related to graphs, combinatorial complexes, maps, and diagrams used in this paper shall be similar to the one in [21], but there shall be a few differences. Definitions of maps and diagrams in a more topological (less combinatorial) way may be found in [19, 22, 23].

3.1. Combinatorial complexes. Roughly speaking, a combinatorial complex shall be viewed as a combinatorial model of a CW-complex. As defined in [21], combinatorial complexes are somewhat restricted in their capability of representing CW-complexes, for example they cannot represent a CW-complex in which the whole characteristic boundary of some 2-cell is attached to a single 0-cell, but otherwise, at least in the 2-dimensional case, they adequately represent the combinatorics of CW-complexes. The CW-complex represented by a given combinatorial complex shall be called its geometric realization. Standard terminology of CW-complexes may also be used for combinatorial complexes, as the analogy is usually clear.

The structure of each cell in a combinatorial complex comprises its characteristic boundary and its attaching morphism. The characteristic boundary of a 0-cell is empty, the characteristic boundary of a 1-cell is a complex consisting of 2 vertices, the characteristic boundary of a 2-cell is a combinatorial circle. The attaching morphism of an $n$-cell $x$ of a combinatorial complex $\Delta$ is a morphism from the characteristic boundary of $x$ to the $(n-1)$-skeleton of $\Delta$. The set of all $n$-cells of a complex $\Delta$ shall be denoted by $\Delta(n)$.

In 2-dimensional combinatorial complexes, 0-cells shall be called vertices, 1-cells shall be called edges, and 2-cells shall be called faces.

A path in a combinatorial complex is an alternating sequence of vertices and oriented edges in which every oriented edges is immediately preceded by its initial vertex and immediately succeeded by its terminal vertex. A path of length 0 (consisting of a single vertex) shall be called trivial. For simplicity, a nontrivial path may be written as the sequence of its oriented
edges. A path is **closed** if its initial and terminal vertices coincide. A path is **reduced** if it does not contain a subpath of the form $ee^{-1}$ where $e$ is an oriented edge. A closed path is **cyclically reduced** if it is reduced and its first oriented edge is not inverse to its last oriented edge. A path is **simple** if it is nontrivial, reduced, and none of its intermediate vertices appears in it more than once (a closed path can be simple).

Two combinatorial complexes shall be called **combinatorially equivalent** if they possess isomorphic subdivisions.

A 2-dimensional combinatorial complex whose geometric realization is a surface (with or without boundary) shall be called a **combinatorial surface**.

If $\Delta$ is a combinatorial complex, then its **Euler characteristic** shall be denoted by $\chi(\Delta)$.

In what follows, the word “combinatorial” shall sometimes be omitted for brevity. For example, a sphere shall also mean a combinatorial sphere, that is a complex whose geometric realization is a sphere, or equivalently a complex combinatorially equivalent to any particular “sample” combinatorial sphere.

### 3.2. Maps

A **nontrivial connected map** $\Delta$ consists of the following data:

1. a nontrivial (i.e. not consisting of a single vertex) non-empty finite connected combinatorial complex, which shall also be denoted $\Delta$ if no confusion arises;
2. a finite collection of combinatorial circles together with their morphisms to $\Delta$, which shall be called the characteristic boundary of $\Delta$;
3. a choice of simple closed paths in the characteristic boundaries of all faces of $\Delta$ and in all components of the characteristic boundary of $\Delta$, called the characteristic contours of faces and characteristic contours of $\Delta$, respectively; i.e. to every face there should be assigned its characteristic contour, and in every component of the characteristic boundary of $\Delta$, there should be chosen one characteristic contour of $\Delta$; moreover, all characteristic contours of $\Delta$ shall be enumerated for further convenience.

The image of the characteristic contour of a face in the 1-skeleton of $\Delta$ (under the attaching morphism of that face) shall be called the contour of that face, and the image of a characteristic contour of $\Delta$ shall be called a contour of $\Delta$ and shall be enumerated with the same number. The only requirement to this structure is that attaching a new face along each of the contours of $\Delta$ must turn $\Delta$ into a closed combinatorial surface.

A **trivial disc map**, or simply a **trivial map**, is defined as a trivial (i.e. consisting of a single vertex) combinatorial complex together with another trivial complex mapped to it in the role of its characteristic boundary and together with the corresponding trivial characteristic contour and also trivial contour. (A trivial disc map cannot be turned into a combinatorial sphere by attaching a face along its contour because a face cannot be attached to a single vertex.)

In general a **map** consists of a finite number of connected components each of which is either a nontrivial connected map or a trivial disc map, with the assumption that all characteristic contours are enumerated by different numbers.
The contour of a face $\Pi$ shall be denoted by $\partial \Pi$. The contours of a map $\Delta$ shall be denoted by $\partial_1 \Delta, \partial_2 \Delta$, etc., but if $\Delta$ has only one contour, then it may be denoted $\partial \Delta$.

Note that it is more usual to fix contours and characteristic contours only up to cyclic shifts. The definition given above is chosen because it is convenient to have each contour defined as a specific path, rather than a “cyclic path,” to have a “base point” chosen in the characteristic boundary of each face and in each component of the characteristic boundary of the map, and because in the context of diagrams (defined below), it is convenient to be able to label contours with specific group words, rather than “cyclic group words.”

It is useful to note that a map can be reconstructed up to isomorphism and choice of characteristic contours from its characteristic boundary and the characteristic boundaries of its faces if for each pair of oriented edges taken from the characteristic boundary of the map and the characteristic boundaries of its faces is known whether they are mapped to the same oriented edge in the 1-skeleton of the map.

A map is closed if it has no contours, or, equivalently, if its underlying complex is a closed surface. If a map $\Delta$ has no trivial disc maps as connected components (which shall be called its trivial connected components), then the closure of $\Delta$ is the closed map $\overline{\Delta}$ obtained from $\Delta$ by naturally turning each component of the characteristic boundary of $\Delta$ into a new “outer” face, which may be viewed as attaching new faces along all contours of $\Delta$ and choosing the characteristic contours of the attached faces so that their contours coincide respectively with the contours of $\Delta$.

A map shall be called orientable if the closure of each of its nontrivial connected components is orientable. To orient a map shall mean to orient all of its faces, which means to choose positive and negative directions in their characteristic boundaries, and to choose positive and negative directions in all components of the characteristic boundary of the map so that this choice would induce an orientation of the closures of all nontrivial connected components. Nontrivial contours and characteristic contours shall be called positive or negative accordingly, and positive contours shall be said to agree with its orientation.

A singular combinatorial disc is any simply connected non-empty subcomplex of a combinatorial sphere. A disc map is a map whose underlying complex is a singular combinatorial disc. Alternatively, a disc map can be defined as a connected one-contour map of Euler characteristic 1. The closure of a nontrivial disc map is a spherical map.

A map is simple if its contours are pairwise disjoint simple closed paths.

An elementary spherical map is a spherical map with exactly 2 faces whose 1-skeleton is a combinatorial circle. Such maps are the least interesting ones.

The following lemma follows from the classification of compact (or finite combinatorial) surfaces.

**Lemma 3.1.** The maximal possible Euler characteristic of a closed connected combinatorial surface is 2, and among all such surfaces, only spheres

\[ \text{The word “singular” here apparently is not related in meaning to the same word as used in [10] or [19].} \]
have Euler characteristic 2, and only projective planes have Euler characteristic 1. The maximal possible Euler characteristic of a proper connected subcomplex of a combinatorial surface is 1, and every such subcomplex is a singular combinatorial disc.

**Corollary 3.2.** The maximal possible Euler characteristic of a connected map with \(n\) contours is \(2 - n\). A connected map of Euler characteristic 2 is spherical. A closed connected map of Euler characteristic 1 is projective-planar. A non-closed connected map of Euler characteristic 1 is disc.

Every subcomplex \(\Gamma\) of a map \(\Delta\) can be endowed with a structure of a submap, which is unique up to choice and enumeration of characteristic contours of \(\Gamma\) (characteristic contours of faces of \(\Gamma\) can be taken from \(\Delta\)). Note that the characteristic boundary of \(\Gamma\) is uniquely determined, and hence so is the underlying complex of its closure.

### 3.3. Diagrams.
Maps have been introduced to be used as underlying objects of diagrams over group presentations.

Recall that a group presentation \(\langle X \| R \rangle\) consists of an alphabet \(X\) and an indexed family \(R\) of group words over \(X\) (words over \(X^{\pm 1}\)) called defining relators. A distinction shall be made between elements of \(R\) and indexed members of \(R\): an indexed member is an element together with its index in \(R\), hence the same relator can be included in \(R\) multiple times as distinct indexed members. (Using indexed families of relators instead of sets is convenient in the case when a priori it is not known if there are any repetitions in the imposed relations, and also for constructing certain geometric object from a given group presentation.) The group presented by \(\langle X \| R \rangle\) is formally generated by \(X\) subject to the defining relations \(\langle r = 1 \rangle\), \(r \in R\).

If \(\langle X \| R \rangle\) is a group presentation where \(R\) is an indexed family of non-trivial group words, then the combinatorial realization \(K(X; R)\) of \(\langle X \| R \rangle\) shall be defined as a combinatorial complex with 1 vertex, \(|A|\) edges (loops), \(|R|\) faces, and additional structure consisting of

1. a bijection between the set of oriented edges of \(K(X; R)\) and the set \(X^{\pm 1}\) such that mutually inverse oriented edges be associated with mutually inverse group letters; the group letter associated to an oriented edge shall be called the label of that oriented edge;
2. a bijection between the set of faces of \(K(X; R)\) and the set of indexed members of \(R\); the indexed members of \(R\) associated to a face shall be called the label of that face;
3. a choice of a simple closed path in the characteristic boundary of each face of \(K(X; R)\), which shall be called the characteristic contour of that face, and whose image in the 1-skeleton of \(K(X; R)\) shall be called the contour of that face;

this structure shall satisfy the condition that the contour label of each face shall coincide with the label of that face viewed as a group word. The group presented by \(\langle X \| R \rangle\) is naturally isomorphic to the fundamental group of \(K(X; R)\).

A diagram over a group presentation \(\langle X \| R \rangle\) is a map in which all oriented edges are labeled with elements of \(X^{\pm 1}\) and all faces are labeled with indexed relators from \(R\) so that mutually inverse oriented edges are labeled
with mutually inverse group letters and if a face $\Pi$ is labeled with an indexed relator $r$, then the contour label of $\Pi$ is the group word $r$. Labeling of a diagram $\Delta$ over $\langle X \parallel R \rangle$ is equivalent to defining a morphism from its underlying map to $K(X; R)$ that preserves the characteristic contours of faces.

In a diagram or a combinatorial realization of a presentation, the labels of an oriented edge $e$, a path $p$, and a face $\Pi$ shall be denoted by $\ell(e)$, $\ell(p)$, and $\ell(\Pi)$, respectively.

3.3.1. Augmented diagrams. An augmented diagram over $\langle X \parallel R \rangle$, also called a 0-refined diagram in [22], is, similarly to a regular diagram, a kind of a labeled map, but in which edges and faces are divided into regular and auxiliary. Regular edges shall be called 1-edges, and regular faces shall be called 2-faces. Auxiliary edges shall be called 0-edges, and auxiliary faces shall be divided into 0-faces and 1-faces. Regular oriented edges shall be labeled with elements of $X^{\pm1}$, and regular faces shall be labeled with indexed members of $R$ following the same rules as for regular diagrams. All oriented 0-edges shall be labeled with the symbol 1. A 0-face can be incident only to 0-edges, and hence the contour label of every 0-face has to be of the form $1^k$, $k \in \mathbb{N}$. The contour label of every 1-face has to be a cyclic shift of $x^1kx^{-1}l$ for some $x \in X$, $k, l \in \mathbb{N} \cup \{0\}$. Thus the contour label of every face of an augmented diagram over $\langle X \parallel R \rangle$ is either an element of $R$ or freely trivial.

Regular diagrams shall be viewed as augmented diagrams without auxiliary edges or faces, but a diagram by default shall mean a regular diagram.

3.3.2. Regularization of augmented diagrams. There is a procedure to canonically transform any augmented diagram into a regular one which shall be called its regularization. Essentially, this procedure consists in collapsing all 0-edges and 0-faces into vertices, and all 1-faces onto incident 1-edges; there are however a few special cases. Here follows a more formal definition of this transformation.

Let $\Delta$ be an augmented diagram. To regularize $\Delta$, the following steps shall be done (in the given order):

(1) for each contour of $\Delta$ that lies entirely on 0-edges, this contour shall be filled in with a 0-face, and a trivial connected component shall be added, whose contour shall replace the filled-in contour;

(2) for each contour that does not lie entirely on 0-edges but not entirely on 1-edges either, a “collar” of 0- and 1-faces shall be “glued” to this contour so as to “shift” it to a new contour that lies entirely on 1-edges and whose label is obtained from the label of the original contour by canceling all 1 symbols; after this operation, all nontrivial contours lie entirely on 1-edges;

(3) for each nontrivial contour, this contour shall be temporarily filled in with an “outer” face, which shall be regarded as a 2-face, and which shall be removed at the end of the procedure;

(4) each maximal connected subcomplex without 1-edges (i.e. containing only 0-edges and 0-faces) shall be collapsed into a point, unless it forms a closed connected component, in which case it shall be entirely removed; after this operation, the star of each vertex looks like a
finite collection of discs with their centers “glued” to this vertex (equivalently, the link is a disjoint union of circles);
(5) split each vertex star which is “glued” from more than one disc into a disjoint union of these discs; after this operation, each nontrivial connected component is again a closed combinatorial surface;
(6) remove all connected components that consist entirely of 1-faces (observe that at this stage all such components are spherical);
(7) collapse each of the remaining 1-faces into an edge (observe that at this stage all 1-faces are digons);
(8) remove previously added “outer” faces.

There is an alternative way to define the regularization of a given augmented diagram $\Delta$ which shows its invariance. As was noted after the definition of a map, a map can be “essentially” reconstructed from the following data: its characteristic boundary and the collection of the characteristic boundaries of its faces together with the information which pairs of oriented edges shall be “identified.” To obtain the regularization of $\Delta$, take its characteristic boundary and the characteristic boundaries of all of its 2-faces, collapse all 0-edges in the characteristic boundary of $\Delta$ (replacing circles consisting entirely of 0-edges with trivial components), and reconstruct a new map by requiring that two oriented edges be mapped to the same oriented edge if their images in $\Delta$ either coincided or were connected by a “band” of 1-faces. The characteristic contours of $\Delta$ and of its faces shall be inherited by the regularization of $\Delta$ in the natural sense.

Lemma 3.3. Let $\Delta_0$ be a connected augmented diagram and let $\Delta$ be its regularization. Let $m$ be the number of contours of $\Delta$ (the same as for $\Delta_0$), and $n$ be the number of connected components of $\Delta$. Then

1. $-m \geq \chi(\Delta) - 2n \geq \chi(\Delta_0) - 2$,
2. if $\Delta_0$ is orientable, then so is $\Delta$,
3. suppose $\chi(\Delta) - 2n = \chi(\Delta_0) - 2$, and suppose that $\Psi$ is a nontrivial connected component of $\Delta$ such that all the other components either are trivial or have spherical closure; then the closures of $\Psi$ and $\Delta_0$ are combinatorially equivalent.

Outline of a proof. The first inequality of (1) follows from Corollary 3.2. To prove the second inequality, observe that collapsing a proper connected subcomplex of a closed connected combinatorial surface into a point increases the Euler characteristic by at least $k-1$ if $k$ is the number of disjoint circles in the link of the newly obtained vertex (see Corollary 3.2), while the subsequent splitting of the star of this vertex increases the Euler characteristic by $k-1$ and increases the number of connected components by at most $k-1$.

It is more or less straightforward to prove (2).

To prove (3), observe that if $\chi(\Delta) - 2n = \chi(\Delta_0) - 2$, then each operation of collapsing/splitting described above preserves (up to combinatorial equivalence) the connected sum of the closures of all nontrivial connected components. (Given a finite set of closed connected combinatorial surfaces, its connected sum can be defined up to combinatorial equivalence similarly to connected sum of topological surfaces. If $\Psi$ is a connected sum of $\Psi_1, \ldots, \Psi_k$, then $\chi(\Psi) - 2 = \chi(\Psi_1) + \cdots + \chi(\Psi_k) - 2k.$) \hfill $\square$
3.3.3. **Diamond move.** If $e_1$ and $e_2$ are two oriented edges of an augmented diagram $\Delta$ with a common terminal vertex and identical labels, then the diamond move along $e_1$ and $e_2$ is defined,\(^3\) see [10, Section 1.4] or [21, Section 3.3.3]. There are three kinds of diamond move: *proper*, *untwisting*, and *disconnecting*. In the case when none of the oriented edges $e_1$ and $e_2$ is a loop, all possible diamond moves are shown on Figure 1.

In general, the diamond move is defined as follows:

1. Let $v$ be the common terminal vertex of $e_1$ and $e_2$, fix an orientation locally around $v$, and define *left* and *right* sides of $e_1$ and $e_2$ according to how they enter $v$ (and not how they leave $v$, if they are loops);
2. Cut $\Delta$ along $e_1$ and $e_2$, and glue the right side of $e_1$ with the left side of $e_2$, and the right side of $e_2$ with the left side of $e_1$; label the new 4 oriented edges naturally (with $\ell(e_1) = \ell(e_2)$ and $\ell(e_1^{-1}) = \ell(e_2^{-1})$).

\(^3\)Diamond moves in diagrams correspond to *bridge moves* in pictures, see [17, 25].
(if the component of $\Delta$ that contains $e_1$ and $e_2$ is not closed, and $e_1$ or $e_2$ is traversed by one of its contours, then start by temporarily closing that component, and remove the temporary “outer” faces at the end.)

Note that a diamond move does not change the number of edges or the number of faces, and it does not change the contour labels of faces. It cannot decrease the number of vertices or increase it by more than 2. Call a diamond move proper if it does not change the number of vertices (and in particular preserves the Euler characteristic), untwisting if it increases the number of vertices by 1, and disconnecting if it increases the number of vertices by 2.

Some useful properties of diamond moves are summarized by the following lemma.

**Lemma 3.4.** Let $\Delta_0$ be an arbitrary connected diagram and $\Delta$ be a diagram obtained from $\Delta_0$ by a diamond move. Then

1. $\Delta$ and $\Delta_0$ have the same number of contours and the same contour labels;
2. $\|\Delta_0(0)\| \leq \|\Delta(0)\| \leq \|\Delta_0(0)\| + 2$, $\|\Delta(1)\| = \|\Delta_0(1)\|$, $\|\Delta(2)\| = \|\Delta_0(2)\|$, and hence $\chi(\Delta_0) \leq \chi(\Delta) \leq \chi(\Delta_0) + 2$;
3. if $\|\Delta(0)\| = \|\Delta_0(0)\|$ (the move is proper), then the closures of $\Delta$ and $\Delta_0$ are combinatorially equivalent, and the move can be “undone” by another diamond move;
4. if $\|\Delta(0)\| = \|\Delta_0(0)\| + 1$ (the move is untwisting), then $\Delta$ is connected and $\Delta_0$ is non-orientable;
5. if $\|\Delta(0)\| = \|\Delta_0(0)\| + 2$ (the move is disconnecting), then
   a. $\Delta$ has at least 2 connected components,
   b. if $\Delta_0$ is orientable, then so is $\Delta$,
   c. if $\Delta$ has 2 connected components and $\Delta_0$ is non-orientable, then so is (at least one of the components of) $\Delta$,
   d. if $\Delta$ has 2 connected components and the closure of one of them is spherical, then the closure of the other is combinatorially equivalent to the closure of $\Delta_0$.

**Outline of a proof.** Let $e_1$ and $e_2$ be the oriented edges such that $\Delta$ is obtained from $\Delta_0$ by the diamond move along $e_1$ and $e_2$. The most difficult case is the one when both $e_1$ and $e_2$ are loops. (If $e_1$ is a loop and $e_2$ is not, then the move is proper.) Only this case shall be considered here.

Let $\Theta$ be the diagram obtained from $\Delta_0$ by cutting it along the loops $e_1$ and $e_2$, so that the common terminal vertex of $e_1$ and $e_2$ in $\Delta_0$ be split into 4 vertices in $\Theta$. Then, compared to $\Delta_0$, $\Theta$ has 1, 2, or 3 additional contours, and the sum of the lengths of these additional contours is 4. Let $a, b, c, d$ be the oriented edges of $\Theta$ corresponding to the left and right sides of $e_1$ and left and right sides of $e_2$, respectively; they all lie on the additional contours of $\Theta$. Then $\Delta$ is obtained from $\Theta$ by gluing $b$ with $c$ and $d$ with $a$.

Without loss of generality, it is enough to consider the following 3 cases and their 9 subcases (see Figure 2):

1. the orientation is preserved along both $e_1$ and $e_2$, and hence
   a. either $\Theta$ has 3 additional contours $a, bd, c$, and the move is proper, or
Figure 2. Oriented loops $e_1$ and $e_2$.

(b) $\Theta$ has 1 additional contour $acb^{-1}d^{-1}$, and the move is disconnecting but cannot change the number of connected components, or

(c) $\Theta$ has 3 additional contours $a, bc^{-1}, d$, and the move is disconnecting;

(2) the orientation is preserved along $e_1$ and inverted along $e_2$, and hence

(a) either $\Theta$ has 2 additional contours $a, bcd$, and the move is proper, or

(b) $\Theta$ has 1 additional contour $adbc^{-1}$, and the move is untwisting, or

(c) $\Theta$ has 2 additional contours $a, bc^{-1}d^{-1}$, and the move is untwisting;

(3) the orientation is inverted along both $e_1$ and $e_2$, and hence

(a) either $\Theta$ has 1 additional contour $abcd$, and the move is proper, or

(b) $\Theta$ has 2 additional contours $ad, bc^{-1}$, and the move is disconnecting, or

(c) $\Theta$ has 1 additional contour $abc^{-1}d^{-1}$, and the move is disconnecting but cannot change the number of connected components.
3.4. **Reduced diagrams.** A pair of distinct faces \( \{ \Pi_1, \Pi_2 \} \) in a diagram over \( \langle X \parallel R \rangle \) shall be called immediately strictly cancelable if the labels of \( \Pi_1 \) and \( \Pi_2 \) are identical as indexed members of \( R \), both faces are incident to some common edge, and their contours can be written as \( \partial \Pi_1 = p_1sq_1 \) and \( \partial \Pi_2 = p_2sq_2 \) with a common nontrivial subpath \( s \), \( |s| > 0 \), so that \( |p_1| = |p_2| \) (and \( |q_1| = |q_2| \)). A diagram shall be called weakly strictly reduced if it does not have immediately strictly cancelable pairs of faces.

A pair of distinct faces \( \{ \Pi_1, \Pi_2 \} \) in a diagram shall be called diagrammatically cancelable if there exists a sequence of diamond moves that separates these two faces into a spherical subdiagram (i.e. leads to a diagram in which the faces corresponding to \( \Pi_1 \) and \( \Pi_2 \) form a spherical connected component). A diagram shall be called diagrammatically reduced if it does not have diagrammatically cancelable pairs of faces.

Clearly, an immediately strictly cancelable pair is diagrammatically cancelable, and hence a diagrammatically reduced diagram is weakly strictly reduced.

A diagram shall be called diamond-move reduced if no sequence of diamond moves can increase its Euler characteristic.

By Lemma 3.4, every diagram can be transformed into a diamond-move reduced one by a sequence of diamond moves, and if a diagram is diamond-move reduced, then any sequence of diamond moves preserves its closure up to combinatorial equivalence (and in particular it preserves its Euler characteristic and orientability).

A diamond-move reduced diagram is diagrammatically reduced if and only if it does not have spherical components with exactly 2 faces.

**Lemma 3.5.** Let \( \Delta_0 \) be an augmented diagram over \( \langle X \parallel R \rangle \). Let \( \Delta_1 \) be the regularization of \( \Delta_0 \), and \( \Delta \) be a diamond-move reduced diagram obtained from \( \Delta_1 \) by some sequence of diamond moves. If \( \Delta \) is not closed, then let \( \Delta' \) be the diagram obtained from \( \Delta \) by discarding all closed connected components. Then

1. \( \Delta \) has the same number of contours as \( \Delta_0 \), and contour labels of \( \Delta \) are obtained from the corresponding contour labels of \( \Delta_0 \) by canceling the 1 symbols;
2. each component of \( \Delta \) either is diagrammatically reduced or is a spherical diagram with 2 faces;
3. if \( \Delta_0 \) is orientable, then so is \( \Delta \); moreover, any orientation of \( \Delta_0 \) naturally induces an orientation of \( \Delta \) so that positive contours which do not become trivial stay positive, and negative which do not become trivial stay negative;
4. if the closures of all nontrivial connected components of \( \Delta_0 \) are spherical, then so are the closures of all nontrivial components of \( \Delta \);
5. if \( \Delta_0 \) is disc, then \( \Delta' \) is disc;

There exist different non-equivalent definitions of cancelable pairs, reduced diagrams, etc. The modifiers such as “weakly,” “strictly,” “diagrammatically” are added here to distinguish among them. For example, diagrams reduced in the sense of [19] would be called weakly diagrammatically reduced here.
(6) if $\Delta_0$ is annular, then $\Delta'$ either is annular or consists of 2 disc components;

(7) if $\Delta_0$ is a connected one-contour diagram, then $\chi(\Delta') \geq \chi(\Delta_0)$, and if additionally $\chi(\Delta') = \chi(\Delta_0)$, then the closures of $\Delta'$ and $\Delta_0$ are combinatorially equivalent, and all closed components of $\Delta$ are spherical.

No proof shall be given as this lemma follows more or less directly from Corollary 3.2 and Lemmas 3.3, 3.4.

One of basic results of combinatorial group theory is the fact that a group word $w$ represents the trivial element in the group presented by $\langle X \parallel R \rangle$ if and only if there exists a disc diagram over $\langle X \parallel R \rangle$ with contour label $w$. The following lemma summarizes several results of this kind that shall be used in the proof of the main theorem.

Lemma 3.6. Let $G$ be the group presented by $\langle X \parallel R \rangle$. Let $w, w_1, w_2$ be arbitrary group words over $X$, and $n$ a positive integer. Then

(1) if there exists an augmented disc diagram over $\langle X \parallel R \rangle$ with contour label $w$, then $[w] = 1$ in $G$;

(2) if $[w] = 1$ in $G$, then there exists a diamond-move reduced disc diagram over $\langle X \parallel R \rangle$ with contour label $w$;

(3) if there exists an augmented oriented annular diagram over $\langle X \parallel R \rangle$ whose contours agree with the orientation and whose contour labels are $w_1$ and $w_2^{-1}$, then $[w_1]$ and $[w_2]$ are conjugate in $G$;

(4) if $[w_1]$ and $[w_2]$ are conjugate in $G$ and $[w_1] \neq 1$, then there exists an oriented diamond-move reduced annular diagram over $\langle X \parallel R \rangle$ whose contours agree with the orientation and whose contour labels are $w_1$ and $w_2^{-1}$;

(5) if there exists an augmented one-contour diagram over $\langle X \parallel R \rangle$ with contour label $w$ and whose closure is a combinatorial sphere with $n$ handles, then $[w] \in [G, G]$ and $cl_G([w]) \leq n$, $sqcl_G([w]) \leq 2n + 1$;

(6) if $cl_G([w]) = n > 0$, then there exists a diamond-move reduced one-contour diagram over $\langle X \parallel R \rangle$ with contour label $w$ and whose closure is a combinatorial sphere with $n$ handles;

(7) if there exists an augmented connected one-contour diagram over $\langle X \parallel R \rangle$ with contour label $w$ and whose closure is either a non-orientable combinatorial surface of Euler characteristic $2 - n$, then $[w] \in G^2$ and $sqcl_G([w]) \leq n$;

(8) if $sqcl_G([w]) = n > 0$, then there exists a diamond-move reduced connected one-contour diagram over $\langle X \parallel R \rangle$ with contour label $w$ and whose closure is either a non-orientable combinatorial surface of Euler characteristic $2 - n$ or an orientable combinatorial surface of Euler characteristic $3 - n$ (i.e. a combinatorial sphere with $(n-1)/2$ handles).

Outline of a proof. Most of these statements are usually considered well-known. Statements analogous or equivalent to (1), (2), (3), (4) may be found, for example, in Theorem V.1.1 and Lemmas V.1.2, V.5.1, V.5.2 in [19, Chapter V] and in Lemmas 11.1, 11.2 in [22, 23, Chapter 4]. A care should be
taken, however, since definitions of diagrams are not exactly equivalent everywhere, and because in the latter reference all results are stated only for augmented diagrams. Here only outlines of proofs of (5), (7), (8) shall be given. Statement (6) can be proved similarly to (8).

To prove (5), it suffices to notice that if \( \Delta \) is a one-contour map whose closure is a combinatorial sphere with \( n \) handles, then \( \partial \Delta \) is the product of \( n \) commutators in the fundamental group of \( \Delta \), and that every product of \( n \) commutators is also the product of \( 2n + 1 \) squares, see [11, Section 2.5].

Similarly, to prove (7), it suffices to notice that if \( \Delta \) is a one-contour map whose closure is a non-orientable combinatorial surface of Euler characteristic \( 2 - n \), then \( \partial \Delta \) is the product of \( n \) squares in the fundamental group of \( \Delta \).

Here follows an outline of a proof of (8). Assume \( \text{sql}_G([w]) = n > 0 \). Then there exists an augmented simple disc diagram \( \Phi \) over \( \langle X \parallel R \rangle \) with the contour label of the form \( v_1^2 v_2^2 \cdots v_n^2 w^{-1} \). After gluing together accordingly parts of the contour corresponding to distinct occurrences of each \( v_i \), obtain an augmented connected one-contour diagram \( \Delta_0 \) over \( \langle X \parallel R \rangle \) with contour label \( w \) and whose closure is a non-orientable combinatorial surface of Euler characteristic \( 2 - n \) (and \( \chi(\Delta_0) = 1 - n \)). Consider a diamond-move reduced diagram obtained from the regularization of \( \Delta_0 \) by some sequence of diamond moves, and let \( \Delta \) be its non-closed connected component (i.e. the component containing its contour). Then, by Lemma 3.5, \( \ell(\partial \Delta) = \ell(\partial \Delta_0) = w \), and either the closure of \( \Delta \) is combinatorially equivalent to \( \Delta_0 \), in which case \( \Delta \) is a desired diagram, or \( \chi(\Delta) \geq \chi(\Delta_0) + 1 = 2 - n \). It is left to show that in the second case \( \Delta \) is orientable and \( \chi(\Delta) = 2 - n \). Indeed, if \( \chi(\Delta) \geq 2 - n \) and \( \Delta \) is non-orientable, then \( \text{sql}_G([w]) \leq n - 1 \) by (7). If \( \Delta \) is orientable and \( \chi(\Delta) \geq 3 - n \), then \( \text{sql}_G([w]) \leq n - 1 \) again by (5). Thus (8) is proved. □

**Corollary 3.7.** Let \( G \) be the group presented by \( \langle X \parallel R \rangle \) and \( w \) a group word over \( X \). Then

1. \( [w] \in G^2 \) if and only if there exists a one-contour diagram over \( \langle X \parallel R \rangle \) with contour label \( w \);

2. \( [w] \in [G,G] \) if and only if there exists an orientable one-contour diagram over \( \langle X \parallel R \rangle \) with contour label \( w \).

### 3.5. Asphericity

There exist different non-equivalent definitions of asphericity of a group presentation in the literature. Group presentations themselves may also be viewed differently, notable variations including the following: whether or not the relators are required to be reduced, or even cyclically reduced, and whether or not the relators form a set or an indexed family. Different notions of a group presentation necessitate different definitions of asphericity. In this paper, relators are allowed to be non-reduced, and even freely trivial as long as they are not trivial (i.e. not of length 0), and they may form an indexed family rather than just a set. Consistent definitions of different kinds of asphericity of a group presentation shall be borrowed from [9, 10]. Only diagrammatic and singular asphericities shall be used in this paper.

A group presentation is **diagrammatically aspherical** if there exist no diagrammatically reduced spherical diagrams over it (see [10]). Equivalently, a
group presentation is diagrammatically aspherical if and only if every spherical diagram over this presentation can be transformed by a sequence of diamond moves into a diagram whose all connected components (which are necessarily spherical by Lemma 3.4) have exactly 2 faces.

Observe that a diagrammatically aspherical presentation cannot have freely trivial relators because otherwise there would exist one-face spherical diagrams over it. A group presentation \langle X \parallel R \rangle is singularly aspherical if it is diagrammatically aspherical, no element of \( R \) represents a proper power in (the free group presented by) \langle X \parallel \emptyset \rangle, and no two distinct indexed members of \( R \) are conjugate or conjugate to each other’s inverses in \langle X \parallel \emptyset \rangle.

A group shall be called diagrammatically or singularly aspherical, accordingly, if it has a presentation which is such.

It is considered well-known that “aspherical in a certain sense groups are of cohomological and geometric dimension at most 2, and hence are torsion-free.” The following lemma states this result precisely using the chosen terminology.

**Lemma 3.8.** Every nontrivial singularly aspherical group either is free (hence of cohomological and geometric dimension 1) or has cohomological and geometric dimension 2; in both cases it is torsion-free.

*Outline of a proof.* Let \( G \) be an arbitrary nontrivial non-free singularly aspherical group, and let \langle X \parallel R \rangle be its singularly aspherical presentation. The goal is to show that in this case \( K(X; R) \) is an Eilenberg-MacLane complex of type \((G, 1)\), which is equivalent to showing that \( H_2(\tilde{K}(X; R)) = 0 \), where \( \tilde{K}(X; R) \) is the universal cover of \( K(X; R) \).

Every 2-cycle in the group of 2-dimensional cellular chains of \( \tilde{K}(X; R) \) can be represented by a morphism to \( \tilde{K}(X; R) \) of some (generally not connected) closed oriented combinatorial surface. Such a surface is naturally a closed diagram over \langle X \parallel R \rangle, the structure of a diagram is induced by the morphism. Since \( \tilde{K}(X; R) \) is simply connected, every such morphism of a closed oriented diagram can be transformed without changing the represented homology class to a morphism of an oriented diagram whose all connected components are spherical (here it may be helpful to pass to augmented diagrams first and then to use Lemma 3.3). Because applying a diamond move to the diagram and changing accordingly the morphology does not change the represented homology class, and because the presentation is diagrammatically aspherical, the morphology and the diagram can be transformed by diamond moves to a morphology of an oriented diagram whose all connected components are spherical with 2 faces each. Since the presentation is singularly aspherical, each morphology of an oriented spherical diagram with 2 faces represents 0 in the group of 2-dimensional chains (because the images of the two faces coincide). Hence every cellular 2-cycle in \( \tilde{K}(X; R) \) represents the trivial homology class, and thus \( H_2(\tilde{K}(X; R)) = 0 \).

Since \( K(X; R) \) is a 2-dimensional Eilenberg-MacLane complex of type \((G, 1)\), the geometric and cohomological dimensions of \( G \) are at most 2. The cohomological dimension of \( G \) cannot be 1 because by Stallings-Swan theorem \([27, 28]\), every group of cohomological dimension 1 is free. Thus both
the cohomological and the geometric dimensions of $G$ are 2 (the cohomological dimension is always less than or equal to the geometric dimension).

Corollary VIII.2.5 in [4, Chapter VIII] states that every group of finite cohomological dimension is torsion-free. □

It is worth mentioning here that there is also a notion of combinatorial asphericity, which is weaker than diagrammatic asphericity: a presentation is combinatorial aspherical if its Cayley complex (which is obtained from $\tilde{K}(X; R)$ by identifying certain faces) is aspherical, though in this form the definition only makes good sense for presentations in which all relators are cyclically reduced and hence the Cayley complex is well-defined. Relators of a combinatorially aspherical presentation may represent proper powers in the free group, and a combinatorially aspherical group may have torsion, but Theorem 3 in [16] shows that finite-order elements in such a group are only the “obvious” ones.

4. TECHNICAL LEMMAS

The idea of the proof of the main theorem is to use properties of the constructed presentation $\langle a, b \parallel R \rangle$ to show nonexistence of certain diagrams with certain contour labels. The desired nonexistence can be proved by contradiction, by using the imposed small-cancellation-type conditions and some combinatorics to show that all edges of a hypothetical diagram $\Delta$ can be distributed among its faces in such a way that sufficiently few edges be assigned to each face, and hence coming to a contradiction with the equality

$$\|\Delta(1)\| = \sum_{\Pi \in \Delta(2)} \frac{1}{2} |\partial \Pi| + \sum_{i} \frac{1}{2} |\partial_{i} \Delta|.$$ 

For example, to prove that the group $G$ is nontrivial, it is enough to show that there exist no disc diagram over $\langle a, b \parallel R \rangle$ with the contour of length 1, and hence $[a]_{G} \neq 1$. This can be done as follows. If there exists a disc diagram over $\langle a, b \parallel R \rangle$ with the contour of length 1, then it can be transformed into another such diagram which will be “convenient” in a certain sense, and all the edges of this new diagram can be distributed among its faces in such a way that the number of edges assigned to each face be strictly less than half of the contour length of that face, but this is impossible because of the aforementioned equality.

Lemmas 4.3, 4.4, and 4.5 (Estimating Lemmas and Inductive Lemma) proved in this section shall be the main tools in proving by induction that edges of the diagrams under consideration can be distributed among their faces as needed.

Some of the definitions and results of this section are simplified versions of the corresponding definitions and results of [21] and may be not exactly equivalent.

Definition. An $S_{1}$-map is a map together with a system of nontrivial reduced paths in the characteristic boundaries of its faces, called selected paths, satisfying the following conditions:

1. the inverse path of every selected path is selected,
2. every nontrivial subpath of every selected path is selected,
The image of every selected path in the 1-skeleton of the map is reduced, and

(4) distinct maximal selected paths do not overlap (meaning that no edge lies on both of them) unless they are mutually inverse, and maximal selected paths are simple (make at most one full circle).

(The last condition of this definition is not present in [21]; it is added here to slightly simplify the definition of \( \kappa \) and \( \kappa' \) and the proof of Lemma 4.3.)

Figure 3 shows two situations prohibited by the condition \( Z(2) \) relative to \( \Phi \), where \( q, q_1, q_2 \) are selected paths in the characteristic boundaries of \( \Pi, \Pi_1, \Pi_2 \), respectively.

**Definition.** Let \( \Delta \) be an \( S_1 \)-map, \( \Phi \) its simple disc submap. Then \( \Delta \) is said to satisfy the **condition \( Z(2) \) relative to \( \Phi \)** if no cyclic shift of \( \partial \Phi \) can be decomposed as the concatenation of 2 paths each of which either is trivial or is the image of a selected path in the characteristic boundary of a face that does not belong to \( \Phi \) (is outside of \( \Phi \)).

**Definition.** An indexed map is a map \( \Delta \) together with a function \( \iota : \Delta(2) \to I \), where \( I \) is an arbitrary index set. The value \( \iota(\Pi) \) is called the **index** of the face \( \Pi \).

**Definition.** An \( S_2 \)-map is an indexed map together with a system of non-overlapping internal arcs called **exceptional** arcs, such that if two faces are incident to the same exceptional arc, they have the same index. Exceptional arcs shall be assigned indices according to the indices of the incident faces.

If \( \Gamma \) is a submap of an \( S_2 \)-map \( \Delta \), then \( \Gamma \) naturally inherits the structure of an \( S_2 \)-map: an arc of \( \Gamma \) is exceptional if and only if it is exceptional in \( \Delta \) and internal in \( \Gamma \) (this is different from [21]).

**Definition.** Let \( \Gamma \) be a connected submap of an \( S_2 \)-map \( \Delta \), and for every \( i \), let \( A_i \) denote the set of all index-\( i \) exceptional arcs of \( \Gamma \) (i.e. exceptional arcs of \( \Delta \) that are internal in \( \Gamma \)), \( B_i \) denote the set of all index-\( i \) faces of \( \Gamma \), and \( K_i \) denote the set of all connected components of the subcomplex of \( \Gamma \) obtained by removing all the faces that are in \( B_i \) and all the arcs that are
in $A_i$. Then $\Gamma$ is said to satisfy the condition $Y$ relative to $\Delta$ if for every $i$ such that $A_i \neq \emptyset$, the number of elements of $K_i$ that either have Euler characteristic 1 or contain an index-$i$ exceptional arc of $\Delta$ incident to a face of $\Gamma$ (which has to be external in $\Gamma$) is less than or equal to $\|B_i\|$.

Note that in the last definition, the ambient map $\Delta$ only serves to distinguish certain external arcs of $\Gamma$ as exceptional in $\Delta$.

**Definition.** An $S$-map is a map together with structures of both an $S_1$-map and an $S_2$-map such that every exceptional arc is selected.

**Definition.** Let $\Gamma$ be a submap of an $S$-map $\Delta$, and let $\lambda, \mu, \nu$ be functions $\Gamma(2) \to [0, 1]$. Then $\Gamma$ is said to satisfy the condition $D(\lambda, \mu, \nu)$ relative to $\Delta$ if the following three conditions hold:

- $D_1(\lambda)$: if $\Pi$ is a face of $\Gamma$ and $L$ is the number of edges of the characteristic boundary of $\Pi$ that do not lie on any selected path, then
  $$L \leq \lambda(\Pi)|\partial\Pi|;$$
- $D_2(\mu)$: if $\Pi$ is a face of $\Gamma$, $u$ is a selected (internal) arc of $\Delta$ incident to $\Pi$, and $M$ is the number of those edges of $u$ that do not lie on any exceptional arc of $\Delta$, then
  $$M \leq \mu(\Pi)|\partial\Pi|;$$
- $D_3(\nu)$: if $\Pi$ is a face of $\Delta$, $p$ is a simple path in $\Gamma$ which is the image of a selected path in the characteristic boundary of $\Pi$, and $N$ is the sum of the lengths of all the exceptional arcs of $\Delta$ that lie on $p$, then
  $$N \leq \nu(\Theta)|\partial\Theta|$$
  for every face $\Theta$ of $\Gamma$ satisfying $\iota(\Theta) = \iota(\Pi)$.

In the context of proving the main theorem, diagrams over $\langle a, b \parallel R \rangle$ shall be given the structure of $S$-maps as shown on Figure 4. Namely, the maximal selected paths in the characteristic boundaries of faces in those diagrams shall correspond to the “$u$-subwords” of the defining relations, the index of a face shall be the index of its label (if $\ell(\partial\Pi) = r_n$, then $\iota(\Pi) = n$), and an internal arc shall be exceptional if and only if it corresponds “on both sides” to the the same part of the same “$u$-subword” (this can happen even in a weakly strictly reduced diagram since relators contain pairs of mutually inverse “$u$-subwords”). In fact, not all diagrams over $\langle a, b \parallel R \rangle$ shall be used, but only weakly strictly reduced ones in which each exceptional arc corresponds to an entire “$u$-subword”; such diagrams shall be called “convenient.” This restriction shall be needed in order for the defined above condition $Y$ to be satisfied. Note also that if $s$ is a selected internal arc incident to a face $\Pi$ in any such $S$-diagram over $\langle a, b \parallel R \rangle$, then $|s| \leq \nu(\Pi)|\partial\Pi|$, and if additionally $s$ is not exceptional, then $|s| \leq \mu(\Pi)|\partial\Pi|$, and hence such $S$-diagrams satisfy the condition $D$ with appropriate parameters.

The following notation shall be used starting from Lemma 4.3 (First Estimating Lemma). Let $\Pi$ be a face of an $S_1$-map $\Delta$. If $\Pi$ has at least one selected path in its characteristic boundary, then let $\kappa(\Pi)$, or $\kappa_\Delta(\Pi)$, denote the number of connected components obtained from the characteristic
boundary of $\Pi$ by removing all edges and all intermediate vertices of all its selected paths, and let $\kappa'(\Pi)$, or $\kappa'_{\Delta}(\Pi)$, denote the number of those components that either do not consist of a single vertex or consist of a single vertex whose image in the 1-skeleton of $\Delta$ has degree 1. If $\Pi$ has no selected paths, then let $\kappa(\Pi) = \kappa'(\Pi) = 0$. Observe that $\kappa'(\Pi) \leq \kappa(\Pi)$ and that $\kappa(\Pi)$ is also the number of maximal selected paths in the characteristic boundary of $\Pi$ going in the same direction. Note that if all nontrivial reduced paths in the characteristic boundary of $\Pi$ are selected, then $\kappa(\Pi) = \kappa'(\Pi) = 0$. In the context of proving the main theorem, $\kappa(\Pi) = \kappa_{\Pi}(\Pi)$.

Lemmas 4.3, 4.4, 4.5 are identical or slightly simplified versions of Lemmas 50, 54, 58 in [21]; however, somewhat less formal proofs shall be given below for convenience.

The proof of Lemma 4.3 relies on the lemma of Philip Hall stated below. If $R$ is a binary relation and $X$ is a set, then denote

$$R(X) = \{y \mid (\exists x \in X)(x R y)\}.$$

**Lemma 4.1 (Philip Hall [15]).** Let $A$ and $B$ be two finite sets, and $R$ be a relation from $A$ to $B$ (i.e. $R \subset A \times B$). Then the following are equivalent.

(I) There exists an injection $h: A \rightarrow B$ such that for every $x \in A$, $x R h(x)$.

(II) For every $X \subset A$, $\|R(X)\| \geq \|X\|$.

**Corollary 4.2.** Let $A$ and $B$ be finite sets, $R \subset A \times B$, $f: B \rightarrow \mathbb{N} \cup \{0\}$. Then the following are equivalent.

(I) There exists $h: A \rightarrow B$ such that:
(1) for every $x \in A$, $x \mathrel{R} h(x)$, and
(2) for every $y \in B$, $\|h^{-1}(y)\| \leq f(y)$.

(II) For every $X \subset A$,

$$\sum_{y \in R(X)} f(y) \geq \|X\|.$$ 

Proofs of the lemma and the corollary may be found, for example, in [20].

Lemma 4.3 (First Estimating Lemma). Let $\Delta$ be a connected $S_1$-map which either is not elementary spherical or has a face in whose characteristic boundary not all nontrivial reduced paths are selected. Let $A$ be the set of all maximal selected (internal) arcs of $\Delta$. Let $B$ be the set of all faces of $\Delta$ that are incident to selected arcs. Let $C$ be some set of faces of $\Delta$ such that $\Delta$ satisfies the condition $Z(2)$ relative to every simple disc submap that does not contain any faces from $C$ and does not contain at least one arc from $A$. Let $n$ be the number of contours of $\Delta$. Then either $A = \emptyset$, or

$$\|A\| \leq \sum_{\Pi \in B} (3 + \kappa(\Pi) + \kappa'(\Pi)) + 2\|C \setminus B\| - 3\chi(\Delta) - n.$$

Moreover, if $D \subset \Delta(2)$, then there exist a set $E \subset A$ and a function $f: A \setminus E \to D$ such that:

1. either $E = \emptyset$, or

$$\|E\| \leq \sum_{\Pi \in B \setminus D} (3 + \kappa(\Pi) + \kappa'(\Pi)) + 2\|C \setminus (B \setminus D)\| - 3\chi(\Delta) - n;$$

2. for every $u \in A \setminus E$, the arc $u$ is incident to the face $f(u)$;
3. for every $\Pi \in D$, $\|f^{-1}(\Pi)\| \leq 3 + \kappa(\Pi) + \kappa'(\Pi)$;
4. for every $\Pi \in C$, $\|f^{-1}(\Pi)\| \leq 1 + \kappa(\Pi) + \kappa'(\Pi)$.

Proof. It suffices to prove this lemma in the case when $\Delta$ is closed (if $\Delta$ is not closed, apply this lemma to the closure of $\Delta$ in which the added “outer” faces have no selected paths in their characteristic boundaries, to the set $C$ extended by the “outer” faces, and to the same set $D$). So assume $\Delta$ is closed. Without loss of generality, assume $D \subset B$.

If $A = \emptyset$, the proof is easy, so assume $A \neq \emptyset$. 
Consider an arbitrary non-empty subset $X$ of $A$, and let $Y$ be the set of all faces of $\Delta$ that are incident to arcs from $X$ (in particular, $Y \subset B$). Then

$$
\sum_{\Pi \in Y \cap (D \cap C)} (1 + \kappa(\Pi) + \kappa'(\Pi)) + \sum_{\Pi \in Y \cap (D \setminus C)} (3 + \kappa(\Pi) + \kappa'(\Pi))
+ \sum_{\Pi \in B \setminus D} (3 + \kappa(\Pi) + \kappa'(\Pi)) + 2\|C \setminus (B \setminus D)\| - 3\chi(\Delta)
= \sum_{\Pi \in (Y \cap D) \cap C} (1 + \kappa(\Pi) + \kappa'(\Pi)) + \sum_{\Pi \in (Y \cap D) \setminus C} (3 + \kappa(\Pi) + \kappa'(\Pi))
+ \sum_{\Pi \in (B \setminus D) \cap C} (1 + \kappa(\Pi) + \kappa'(\Pi)) + \sum_{\Pi \in (B \setminus D) \setminus C} (3 + \kappa(\Pi) + \kappa'(\Pi))
+ 2\|C\| - 3\chi(\Delta)
\geq \sum_{\Pi \in Y \cap C} (1 + \kappa(\Pi) + \kappa'(\Pi)) + \sum_{\Pi \in Y \setminus C} (3 + \kappa(\Pi) + \kappa'(\Pi))
+ 2\|C\| - 3\chi(\Delta)
= \sum_{\Pi \in Y} (3 + \kappa(\Pi) + \kappa'(\Pi)) + 2\|C \setminus Y\| - 3\chi(\Delta).
$$

Thus to complete the proof, it will suffice to show that

$$
\|X\| \leq \sum_{\Pi \in Y} (3 + \kappa(\Pi) + \kappa'(\Pi)) + 2\|C \setminus Y\| - 3\chi(\Delta),
$$

and afterwards to apply the implication (II) ⇒ (I) of Corollary 4.2 of Hall’s Lemma to the sets $A$ and $D \cup \{\varepsilon\}$, where $\varepsilon$ is some additional element whose preimage is to be taken as the set $E$, and to the relation $R$ defined as follows: for all $x \in A$ and $y \in D \cup \{\varepsilon\}$, $x \sim R y$ if and only if either $x$ is incident to $y \in D$, or $y = \varepsilon$.

Let $K$ be the set of all connected components of the complex obtained from $\Delta$ by removing all the faces that are in $Y$ and all the arcs that are in $X$. For every $\Psi \in K$, let $d(\Psi)$ denote the number of ends of arcs from $X$ that are attached to $\Psi$. (One can consider the graph obtained from $K$ and $X$ by collapsing each elements of $K$ into a point and replacing each arc from $X$ with a single edge—then $d(\Psi)$ will be the degree of the corresponding vertex of this graph.) Clearly,

$$
\sum_{\Psi \in K} \chi(\Psi) - \|X\| + \|Y\| = \chi(\Delta) \quad \text{and} \quad \sum_{\Psi \in K} d(\Psi) = 2\|X\|.
$$

These two equations imply that

$$
\|X\| = 3\|Y\| + 3 \sum_{\Psi \in K} \chi(\Psi) - 2\|X\| - 3\chi(\Delta)
= 3\|Y\| + \sum_{\Psi \in K} (3\chi(\Psi) - d(\Psi)) - 3\chi(\Delta).
$$

By Lemma 3.1 and Corollary 3.2, for every $\Psi \in K$, $\chi(\Psi) \leq 1$, and if $\chi(\Psi) = 1$, then $\Psi$ is the underlying complex of a disc submap of $\Delta$. Let $K'_i = \{ \Psi \in K \mid d(\Psi) = i \text{ and } \chi(\Psi) = 1 \}$ for $i = 0, 1, 2, \ldots$.
Observe that $K'_0 = \emptyset$. Then
\[ \|X\| \leq 3\|Y\| + 2\|K'_1\| + \|K'_2\| - 3\chi(\Delta). \]
It will suffice to show now that
\[ 2\|K'_1\| + \|K'_2\| \leq \sum_{\Pi \in Y} (\kappa(\Pi) + \kappa'(\Pi)) + 2\|C \setminus Y\|. \]
Here the condition $Z(2)$ shall be used.

Let $L$ be the set of all connected components obtained from the (disjoint) union of the characteristic boundaries of all faces from $Y$ by removing all edges and all intermediate vertices of all their selected paths. Let $L'$ be the set of those of these components that either do not consist of a single vertex or consist of a single vertex whose image in the 1-skeleton of $\Delta$ has degree 1. Then
\[ \|L\| = \sum_{\Pi \in Y} \kappa(\Pi), \quad \|L'\| = \sum_{\Pi \in Y} \kappa'(\Pi). \]
The image of each element of $L$ under the corresponding attaching morphism lies entirely in some element of $K$, hence to each element of $L$ there is associated an element of $K$. Because of the condition $Z(2)$, and because arcs in $X$ are maximal selected, to each element of $K'_1 \sqcup K'_2$ not containing any faces from $C \setminus Y$ there is associated at least 1 element of $L$, and furthermore, to each element of $K'_1$ not containing faces from $C \setminus Y$ there is associated either at least 2 elements of $L' \subset L$. (A care should be taken when verifying this last statement because elements of $K'_1 \sqcup K'_2$ are not necessarily underlying complexes of simple disc maps, and some may consist of a single vertex.) Hence
\[ 2\|K'_1\| + \|K'_2\| \leq \|L\| + \|L'\| + 2\|C \setminus Y\| = \sum_{\Pi \in Y} (\kappa(\Pi) + \kappa'(\Pi)) + 2\|C \setminus Y\|. \]

\[ \Box \]

**Lemma 4.4** (Second Estimating Lemma). Let $\Delta$ be a connected submap of an $S_2$-map, and suppose $\Delta$ satisfies the condition $Y$ relative to the ambient $S_2$-map. For every $i$, let $A_i$ be the set of all index-$i$ internal exceptional arcs of $\Delta$, and $B_i$ the set of all index-$i$ faces of $\Delta$. For every $i$, let $\delta_i = 1$ if $\Delta$ has an external arc incident to a face which is an exceptional arc of index $i$ in the ambient $S_2$-map, and let $\delta_i = 0$ otherwise. Then for every $i$, either $A_i = \emptyset$, or
\[ \|A_i\| \leq 2\|B_i\| - \chi(\Delta) - \delta_i. \]
Furthermore, there exists a set $E$ such that:
1. either $E = \emptyset$, or $\|E\| \leq -\chi(\Delta)$;
2. for every $i$, $\|A_i \setminus E\| \leq 2\|B_i\| - \delta_i$.

**Proof.** For every set $I$, denote $A_I = \bigcup_{i \in I} A_i$, $B_I = \bigcup_{i \in I} B_i$. Denote $K_I$ the set of all connected components of the subcomplex obtained from $\Delta$ by removing all faces that are in $B_I$ and all arcs that are in $A_I$. Denote $K'_I$ the set of all the elements of $K_I$ of Euler characteristic 1.
Let $I$ be an arbitrary set such that $A_I \neq \emptyset$. It is to be shown that
\[
\|A_I\| \leq 2\|B_I\| - \sum_{i \in I} \delta_i - \chi(\Delta).
\]

By Lemma 3.1 and Corollary 3.2, $\chi(\Psi) \leq 0$ for each $\Psi \in K_I \setminus K'_I$, and every element of $K'_I$ is the underlying complex of a disc submap of $\Delta$. Therefore,
\[
\chi(\Delta) = \sum_{\Psi \in K_I} \chi(\Psi) - \|A_I\| + \|B_I\| \leq \|K'_I\| - \|A_I\| + \|B_I\|,
\]
and
\[
\|A_I\| \leq \|K'_I\| + \|B_I\| - \chi(\Delta).
\]

Every element of $K_I$ which is the underlying complex of a one-contour submap of $\Delta$ is also an element of $K_{\{i\}}$ for some $i \in I$. In particular, every element of $K'_I$ is an element of $K_{\{i\}}$ for some $i \in I$. It follows from the condition $Y$ that for every $i$ such that $A_i \neq \emptyset$, $\|K'_{\{i\}}\| \leq \|B_i\| - \delta_i$. (Because if $\Psi \in K_{\{i\}}$ and $\Psi$ contains an index-$i$ exceptional arc of the ambient $S_2$-map which is incident to a face of $\Delta$, then $\Psi$ is the underlying complex of a submap with at least 2 contours, and hence $\Psi \notin K'_{\{i\}}$.) Hence
\[
\|K'_I\| \leq \|B_I\| - \sum_{i \in I} \delta_i,
\]
and
\[
\|A_I\| \leq 2\|B_I\| - \sum_{i \in I} \delta_i - \chi(\Delta).
\]

Now to prove the first part of the statement of the lemma, it suffices to apply the last inequality to the set $I = \{i\}$, and to prove the second part (the existence of the set $E$), it suffices to apply it to
\[
I = \{i \mid \|A_i\| > 2\|B_i\| - \delta_i\}.
\]

\[\square\]

**Lemma 4.5** (Inductive Lemma). *Let $\Delta$ be an $S$-map, and $\Phi$ a simple disc submap of $\Delta$. Assume that $\Phi$ satisfies the conditions $Y$ and $D(\lambda, \mu, \nu)$ relative to $\Delta$, where $\lambda, \mu, \nu \colon \Phi(2) \to [0,1]$, and that
\[
(\lambda + (3 + \kappa + \kappa')\mu + 2\nu)(\Pi) \leq \frac{1}{2}
\]
for every $\Pi \in \Phi(2)$. Suppose $\Delta$ satisfies the condition $Z(2)$ relative to every proper simple disc submap of $\Phi$. Then $\Delta$ satisfies $Z(2)$ relative to $\Phi$.*

**Proof.** This lemma shall be proved by contradiction, so suppose that $\Delta$ does not satisfy $Z(2)$ relative to $\Phi$.

Call an $S_1$-map "bad" if it is elementary spherical and every nontrivial reduced path in the characteristic boundary of each of its two faces is selected. One reason why "bad" $S_1$-maps are inconvenient is that distinct maximal selected arcs in them can overlap. More importantly, the First Estimating Lemma does not apply to them. Observe that $\Delta$ cannot be "bad." Indeed,
if $\Delta$ were “bad,” then $\Phi$ would consist of a single face, and for this face $\Pi$, the conditions $D_2(\mu)$ and $D_3(\nu)$ would imply that

$$|\partial \Pi| \leq (\mu(\Pi) + \nu(\Pi))|\partial \Pi| \leq \frac{1}{4}|\partial \Pi| < |\partial \Pi|. $$

Let $A$ be the set of all maximal selected arcs of $\Delta$ that are in $\Phi$, and $B$ the set of all exceptional arcs of $\Delta$ that are in $\Phi$. Then every element of $B$ is a subarc of an element of $A$. Let $L$ be the total number of edges of $\Phi$ that are not edges of elements of $A$, $M$ be the total number of edges of elements of $A$ that are not edges of elements of $B$, and $N$ be the total number of edges of all elements of $B$. To achieve a contradiction, it shall be shown that

$$L + M + N \leq \sum_{\Pi \in \Phi(2)} |\partial \Pi| < ||\Phi(1)||.$$ 

The condition $D_1(\lambda)$ gives immediately the following estimate on $L$:

$$L \leq \sum_{\Pi \in \Phi(2)} \lambda(\Pi)|\partial \Pi|.$$ 

To find a good estimate on $M$, consider the closure $\Phi$ of $\Phi$ (if $\Phi$ is a spherical map). Denote $\Theta$ the face of $\Phi$ that is not in $\Phi$ (the “outer” face). Extend the structure of an $S_1$-map from $\Phi$ to $\Phi$ as follows: select those paths in the characteristic boundary of $\Theta$ whose images in $\Phi$ coincide with the images of selected paths from the characteristic boundaries of faces that are in $\Delta(2) \setminus \Phi(2)$. Then $\Phi$ satisfies $Z(2)$ relative to every proper simple disc submap of $\Phi$, but not relative to $\Phi$ itself, since so does $\Delta$. In particular, $\kappa(\Theta) \leq 2$ and $\kappa'(\Theta) = 0$. Observe that $\Phi$ is not “bad” because otherwise $\Delta$ would be “bad” as well, and that $A$ is exactly the set of all maximal selected arcs of $\Phi$. Apply Lemma 4.3 (First Estimating Lemma) to $\Phi$, $\Theta$ (in the role of the “set $C$”), and $\Phi(2)$ (in the role of the “set $D$”). Let $f$ be a function $A \to \Phi(2)$ such that:

1. for every $u \in A$, the face $f(u)$ is incident to the arc $u$, and
2. for every $\Pi \in \Phi(2)$, $\|f^{-1}(\Pi)\| \leq 3 + \kappa(\Pi) + \kappa'(\Pi)$.

(Since $3 + \kappa(\Theta) + \kappa'(\Pi) - 3\chi(\Phi) \leq -1 \leq 0$, the “set $E$” is empty.) By the condition $D_2(\mu)$,

$$M = \sum_{\Pi \in \Phi(2)} \sum_{u \in f(u) = \Pi} |u| \leq \sum_{\Pi \in \Phi(2)} (3 + \kappa(\Pi) + \kappa'(\Pi))\mu(\Pi)|\partial \Pi|.$$ 

It is left to estimate $N$. Let $B'$ be the set of those arcs from $B$ that are internal in $\Phi$, and $B''$ the set of those arcs from $B$ that are external in $\Phi$. For every $i$, let $C_i$ be the set of all index-$i$ faces of $\Phi$, and $B_i$, $B'_i$, and $B''_i$ be the sets of all index-$i$ elements of $B$, $B'$, and $B''$, respectively. As in Lemma 4.4 (Second Estimating Lemma), for every $i$, let $\delta_i = 1$ if $B''_i \neq \emptyset$, and $\delta_i = 0$ otherwise. Let

$$I = \{ i \mid C_i \neq \emptyset \}. $$

By Lemma 4.4 applied to $\Phi$,

$$\|B'_i\| \leq 2\|C_i\| - 1 - \delta_i.$$
for every \( i \in I \). It follows now from the condition \( D_3(\nu) \) that

\[
\sum_{u \in B'_i} |u| \leq (2\|C_i\| - 1 - \delta_i) \min_{\Pi \in C_i} \nu(\Pi) |\partial \Pi|,
\]

and

\[
\sum_{u \in B''_i} |u| \leq 2\delta_i \min_{\Pi \in C_i} \nu(\Pi) |\partial \Pi|
\]

for every \( i \in I \). (The constant 2 in the second inequality comes from the assumption that \( \Delta \) does not satisfy \( Z(2) \) relative to \( \Phi \).) Therefore,

\[
\sum_{u \in B_i} |u| \leq 2\|C_i\| \min_{\Pi \in C_i} \nu(\Pi) |\partial \Pi| \leq \sum_{\Pi \in C_i} 2\nu(\Pi) |\partial \Pi|
\]

for every \( i \), and hence

\[
N = \sum_{u \in B} |u| \leq \sum_{\Pi \in \Phi(2)} 2\nu(\Pi) |\partial \Pi|.
\]

Thus,

\[
\|\Phi(1)\| = L + M + N \leq \sum_{\Pi \in \Phi(2)} (\lambda(\Pi) + (3 + \kappa(\Pi) + \kappa'(\Pi))\mu(\Pi) + 2\nu(\Pi)) |\partial \Pi| \leq \sum_{\Pi \in \Phi(2)} \frac{1}{2} |\partial \Pi| < \|\Phi(1)\|
\]

which gives a contradiction. \( \square \)

The following lemma follows from Lemmas 4.3, 4.4, 4.5 and is a slightly simplified version of Lemma 59 in [21].

**Lemma 4.6.** Let \( \Delta \) be a connected \( S \)-map with \( n \) contours such that \( n + 3\chi(\Delta) \geq 0 \). Suppose \( \Delta \) satisfies the conditions \( Y \) and \( D(\lambda, \mu, \nu) \) relative to itself, where \( \lambda, \mu, \nu: \Delta(2) \to [0, 1] \). Let \( \gamma = \lambda + (3 + \kappa + \kappa')\mu + 2\nu \), and suppose \( \gamma(\Pi) \leq 1/2 \) for every \( \Pi \in \Delta(2) \). Then

\[
\sum_{i} |\partial_i \Delta| \geq \sum_{\Pi \in \Delta(2)} (1 - 2\gamma(\Pi)) |\partial \Pi|.
\]

5. **Proof of the main theorem**

In this section, \( G \) is the (simple or trivial) group whose presentation \( \langle a, b \| R \rangle \) is constructed in Section 2.

Every diagram over \( \langle a, b \| R \rangle \) shall be automatically endowed with a structure of an \( S \)-map as mentioned in Section 4, see Figure 4. Namely, if \( \Delta \) is a diagram over \( \langle a, b \| R \rangle \), then

1. if \( \Pi \) is a face of \( \Delta \) and \( \ell(\partial \Pi) = r_n \), then the index of \( \Pi \) shall be \( n \), denoted \( \iota(\Pi) = n \), and a path in the characteristic boundary of \( \Pi \) shall be selected if and only if it is a nontrivial subpath of one of the \( 4k_n \) paths corresponding to the subwords \( u_{n,1}, \ldots, u_{n,k_n} \) of \( r_n \); note that \( \kappa(\Pi) = \kappa_n = 2k_n \);
(2) an arc of $\Delta$ shall be exceptional if and only if it is a maximal selected arc which corresponds on both sides to the same part of some word $u_{n,j}$ (in particular, the incident faces must have the same index $n$ and the label $r_n$).

Call a diagram over $\langle a, b \parallel R \rangle$ convenient if it is weakly strictly reduced and every exceptional selected arc in it corresponds to an entire word $u_{n,j}$ (on both sides).

**Lemma 5.1.** Every diagrammatically reduced diagram over $\langle a, b \parallel R \rangle$ can be transformed into a convenient one by a sequence of diamond moves.

**Proof.** Apply diamond moves to “extend” one by one all exceptional arcs that correspond to proper subwords of some $u_{n,j}$, and see Lemma 3.4. \[\square\]

If $\Delta$ is a diagram over $\langle a, b \parallel R \rangle$, $\overline{\Delta}$ is the closure of $\Delta$, and all contour labels of $\Delta$ are cyclically reduced concatenations of copies of $z_1^{\pm 1}$ and $z_2^{\pm 1}$, then the existing structure of an $S$-map on $\Delta$ shall be automatically extended to a structure of an $S$-map on $\overline{\Delta}$ as follows. If $\Theta$ is any of the “outer” faces of $\overline{\Delta}$ (a face that is not in $\Delta$), then assign to $\Theta$ the index of $-1$ (to distinguish it from “inner” faces) and choose all nontrivial reduced paths in the characteristic boundary of $\Theta$ as selected (note in particular that $\kappa(\Theta) = 0$). There shall be no additional exceptional arcs in $\overline{\Delta}$.

If $\Delta$ is a diagram over $\langle a, b \parallel R \rangle$, then let functions $\lambda, \mu, \nu, \gamma : \Delta(2) \rightarrow [0, 1]$ be defined as follows:

\[
\lambda(\Pi) = \lambda_{i(\Pi)}, \quad \mu(\Pi) = \mu_{i(\Pi)}, \quad \nu(\Pi) = \nu_{i(\Pi)}, \quad \gamma(\Pi) = \lambda(\Pi) + (3 + 2\kappa(\Pi))\mu(\Pi) + 2\nu(\Pi) = \gamma_{i(\Pi)}
\]

for every $\Pi \in \Delta(2)$. Observe that

\[
\gamma_{\Delta}(\Pi) < \frac{1}{2}
\]

for every $\Pi \in \Delta(2)$ by condition (C8) in Section 2.

**Lemma 5.2.** Every convenient diagram $\Delta$ over $\langle a, b \parallel R \rangle$ satisfies the condition $D(\lambda, \mu, \nu)$ relative to itself. If all contour labels of $\Delta$ are cyclically reduced concatenations of copies of $z_1^{\pm 1}$ and $z_2^{\pm 1}$, then $\Delta$ also satisfies $D(\lambda, \mu, \nu)$ relative to its closure.

**Proof.** This lemma follows from conditions (C3), (C4), (C5), (C6) in Section 2. \[\square\]

**Lemma 5.3.** Let $\Delta$ be a convenient connected subdiagram of some diagram over $\langle a, b \parallel R \rangle$, and let $J$ be the set of indices of faces of $\Delta$ ($J \subset I$). Suppose that for every proper subset $J' \subset J$, the presentation $\langle a, b \parallel r_i, \ i \in J' \rangle$ defines a torsion-free group. Then $\Delta$ satisfies the condition $Y$ relative to the ambient diagram.

**Proof.** Without loss of generality (or inducting on $J$), assume that for every proper subset $J' \subset J$, every convenient connected diagram over $\langle a, b \parallel r_i, \ i \in J' \rangle$ satisfies $Y$ relative to itself.

Let $i$ be the index of an arbitrary internal exceptional arc of $\Delta$. Let $A_i$ be the set of all internal exceptional arcs of $\Delta$ of index $i$, and $B_i$ be the set
of all faces of $\Delta$ of index $i$. Let $K_i$ be the set of all connected components of the subcomplex obtained from $\Delta$ by removing all faces that are in $B_i$ and all arcs that are in $A_i$. View each element of $K_i$ as a subdiagram of $\Delta$.

Let $J_i = J \setminus \{i\}$. Then the group presented by $\langle a, b \parallel r_i, i \in J_i \rangle$ is torsion-free, and every element of $K_i$ is a diagram over $\langle a, b \parallel r_i, i \in J_i \rangle$.

Let $L$ be the set of vertices of the characteristic boundaries of index-$i$ faces chosen as follows: if $\Pi \in B_i$, then let $s_1, \ldots, s_{k_i}, s'_{k_i}, t_1, \ldots, t_{k_i}, t_0$ be the paths in the characteristic boundary of $\Pi$ such that the concatenation $s_1t_1s'_1s_2t_2s'_2 \cdots s_{k_i}t_{k_i}s'_{k_i}t_0$ is the characteristic contour of $\Pi$, and the label of $s_j$ is $u_{i,j}$ and the label of $s'_j$ is $u_{i,j}^{-1}$ for $j = 1, \ldots, k_i$. Let $L$ contain the initial vertices of the $k_i$ paths $s_1t_1s'_1, \ldots, s_{k_i}t_{k_i}s'_{k_i}$ and no other vertices of the characteristic boundary of $\Pi$. Note that $\|L\| = k_i\|B_i\|$.

The image (under the corresponding attaching morphism) of each element of $L$ is a vertex of an element of $K_i$. It suffices to prove now that for every $\Psi \in K_i$ such that $\chi(\Psi) = 1$ and for every $\Psi \in K_i$ such that $\Psi$ contains an index-$i$ exceptional arc of the ambient diagram incident to a face of $\Delta$, the number of elements of $L$ whose images are in $\Psi$ is at least $k_i$.

It follows from the definition of exceptional arcs in a convenient diagram that for every $\Psi \in K_i$, the number of elements of $L$ whose images are in $\Psi$ is divisible by $k_i$. An example of a possible situation is shown on Figure 5 (under the assumption that $i = 1$ and $k_1 = 4$), elements of $L$ are marked there with thick dots.

It is left to show that if $\Psi \in K_i$ and either $\chi(\Psi) = 1$ or $\Psi$ contains an index-$i$ exceptional arc incident to a face of $\Delta$, then the set of elements of $L$ mapped to $\Psi$ is non-empty (and hence there are at least $k_i$ such elements).

Suppose $\Psi \in K_i$ and $\Psi$ contains an index-$i$ exceptional arc $u$ incident to a face $\Pi$ of $\Delta$. (This arc is an external arc of $\Delta$.) Let $s_1, \ldots, s_{k_i}, s'_1, \ldots, s'_{k_i}$,
Figure 6. A situation prohibited by the absence of torsion (if \( w^3 = 1 \), then \( w = 1 \)).
Lemma 5.4. Let $J$ be a finite subset of $I$. Then every convenient connected diagram over $\langle a, b \parallel r_i, i \in J \rangle$ satisfies the condition $Y$ relative to itself, the presentation $\langle a, b \parallel r_i, i \in J \rangle$ is singularly aspherical, and the group presented by $\langle a, b \parallel r_i, i \in J \rangle$ is torsion-free.

Proof. Without loss of generality (or inducting of $J$), assume that all proper subpresentations of $\langle a, b \parallel r_i, i \in J \rangle$ define torsion-free groups.

Then, by Lemma 5.3, every convenient connected diagram over $\langle a, b \parallel r_i, i \in J \rangle$ satisfies the condition $Y$ relative to itself.

To show that $\langle a, b \parallel r_i, i \in J \rangle$ is singularly aspherical, it suffices to show that it is diagrammatically aspherical and to recall conditions (C11) and (C12).

Suppose $\langle a, b \parallel r_i, i \in J \rangle$ is not diagrammatically aspherical. Then there exists a convenient spherical diagram over $\langle a, b \parallel r_i, i \in J \rangle$ with at least 3 faces (use Lemmas 3.4, 3.5, 5.1); let $\Delta$ be such a diagram. Then $\Delta$ satisfies the condition $Y$ relative to itself. By Lemma 5.2, $\Delta$ satisfies $D(\lambda, \mu, \nu)$ relative to itself. Hence, by Lemma 4.6 and condition (C8),

$$0 \geq \sum_{\Pi \in \Delta(2)} (1 - 2\gamma(\Pi))|\partial\Pi| > 0,$$

which gives a contradiction.

Thus $\langle a, b \parallel r_i, i \in J \rangle$ is singularly aspherical, and it defines a torsion-free group by Lemma 3.8.

Corollary 5.5. Every convenient connected diagram over $\langle a, b \parallel R \rangle$ satisfies the condition $Y$ relative to itself. The presentation $\langle a, b \parallel R \rangle$ is singularly aspherical. The group $G$ is torsion-free.

Proposition 5.6. The group $G$ is torsion-free and of cohomological and geometric dimension 2.

Proof. This follows from Lemma 3.8 and Corollary 5.5.

Lemma 5.7. If $\Delta$ is a convenient disc or annular diagram over $\langle a, b \parallel R \rangle$, then

$$\sum_{\Pi \in \Delta(2)} (1 - 2\gamma(\Pi))|\partial\Pi| \leq \sum_i |\partial_i \Delta|.$$

Proof. This lemma follows from Lemmas 4.6, 5.2, and Corollary 5.5.

Proposition 5.8. Every finite subpresentation of $\langle a, b \parallel R \rangle$ presents a hyperbolic group.

Proof. Let $J$ be a finite subset of $I$. Let $q = \min_{i \in J}(1 - 2\gamma_i)$. Then $q > 0$ by condition (C8).

A group is hyperbolic if and only if it has a finite presentation with a linear isoperimetric function, see Theorems 2.5, 2.12 in [1].

Consider an arbitrary $w \in \{a^{\pm 1}, b^{\pm 1}\}^*$ such that $|w| = 1$ in the group presented by $\langle a, b \parallel r_i, i \in J \rangle$. Take an arbitrary diamond-move reduced disc diagram over $\langle a, b \parallel r_i, i \in J \rangle$ with contour label $w$ and transform it by diamond moves into a convenient disc diagram $\Delta$ with contour label $w$ (see Lemmas 3.4, 3.6, 5.1). Then, by Lemma 5.7,

$$\|\Delta(2)\| \leq \sum_{\Pi \in \Delta(2)} |\partial\Pi| \leq \frac{1}{q} |\partial\Delta|.$$
Thus the isoperimetric function of $\langle a, b \parallel r_i, i \in J \rangle$ is linear. □

**Proposition 5.9.** The elements $[z_1]$ and $[z_2]$ freely generate a free subgroup $H$ of $G$ such that for every $h \in H \setminus \{1\}$,

$$\lim_{n \to \infty} \text{cl}_G(h^n) = \infty.$$  

**Proof.** Let $w$ be a nontrivial reduced concatenation of several copies of $z_1^{\pm 1}$ and $z_2^{\pm 1}$, and $n$ a positive integer. To prove this lemma, it suffices to show that $\text{cl}_G([w^m]) > n$ for every large enough $m$. Without loss of generality, assume that $w$ is cyclically reduced.

Denote $c_n = 2 - 2^{-2n}$ (the Euler characteristic of a sphere with $n$ handles).

Let $J_n = \{ i \in I \mid \chi_i > c_n \}$.

The set $J_n$ is finite by condition (C7). Let

$$p_n = \max_{i \in J_n} \mu_i |r_i|, \quad q_n = \max_{i \in J_n} \nu_i |r_i|.$$

Let $m$ be an arbitrary integer such that

$$(3 - 3c_n) \max\{p_n, |w| - 1\} + (1 - c_n)q_n < \frac{1}{2} |w^m|.$$  

It shall now suffice to show that $\text{cl}_G([w^m]) > n$. This can be done by contradiction.

Suppose $\text{cl}_G([w^m]) \leq n$. Then, by Lemmas 3.4, 3.6, 5.1, there exists a convenient one-contour diagram $\Delta$ over $\langle a, b \parallel R \rangle$ with contour label $w^m$ and whose closure is a combinatorial sphere with $n$ or fewer handles. Let $\bar{\Delta}$ be the closure of $\Delta$. Then $\chi(\bar{\Delta}) \geq c_n$.

Let $\Theta$ be the “outer” face of $\bar{\Delta}$ (the face that is not in $\Delta$). Then $\kappa(\Theta) = 0$ (recall that a structure of an $S$-map is defined on $\Delta$).

By Lemma 5.2, $\Delta$ satisfies $D(\lambda, \mu, \nu)$ relative to $\bar{\Delta}$. By Corollary 5.5, $\Delta$ satisfies $Y$ relative to itself and hence relative to $\bar{\Delta}$ as well. By induction and Lemma 4.5 (Inductive Lemma), using condition (C8), obtain that $\bar{\Delta}$ satisfies $Z(2)$ relative to every simple disc subdiagram of $\bar{\Delta}$.

Let $N$ be the sum of the lengths of all exceptional arcs of $\bar{\Delta}$, $M$ the sum of the lengths of all non-exceptional maximal selected arcs of $\bar{\Delta}$, and $L$ the number of edges of $\bar{\Delta}$ that do not lie on any selected arc. To obtain a contradiction, it suffices to show that

$$L + M + N < \frac{1}{2} \sum_{\Pi \in \Delta(2)} |\partial \Pi|.$$  

The following upper estimate on $L$ follows from the condition $D_1(\lambda)$ satisfied by $\Delta$ relative to $\bar{\Delta}$:

$$L \leq \sum_{\Pi \in \Delta(2)} \lambda(\Pi) |\partial \Pi|$$  

(because every edge of the characteristic boundary of $\Theta$ lies on a selected path).

Let $A$ be the set of all maximal selected arcs of $\bar{\Delta}$. Then

$$\sum_{u \in A} |u| = M + N.$$
Apply Lemma 4.3 (First Estimating Lemma) to \( \bar{\Delta} \) and the sets \( C = \{ \Theta \} \) and \( D = \Delta(2) \). Let \( E \) be a subset of \( A \) and \( h \) a function \( A \setminus E \to \Delta(2) \) such that:

1. either \( E = \emptyset \), or \( \|E\| \leq 3 - 3\chi(\bar{\Delta}) \leq 3 - 3c_n \);
2. for every arc \( u \in A \setminus E \), the face \( h(u) \) is incident to \( u \);
3. for every face \( \Pi \in \Delta(2) \), \( \|h^{-1}(\Pi)\| \leq 3 + 2\kappa(\Pi) \).

Let \( M_1 \) be the sum of the lengths of all non-exceptional arcs from \( A \setminus E \), and \( M_2 \) the sum of the lengths of all non-exceptional arcs from \( E \). Then \( M_1 + M_2 = M \), and, by the condition \( D_2(\mu) \),

\[
M_1 \leq \sum_{\Pi \in \Delta(2)} (3 + 2\kappa(\Pi))\mu(\Pi)|\partial\Pi|,
\]

while

\[
M_2 \leq (3 - 3c_n)p_\Delta,
\]

where \( p_\Delta \) is the maximal length of a non-exceptional maximal selected arc of \( \Delta \).

Apply Lemma 4.4 (Second Estimating Lemma) to \( \Delta \). Let \( F \) be a set of exceptional arcs of \( \Delta \) such that:

1. either \( F = \emptyset \), or \( \|F\| \leq -\chi(\Delta) \leq 1 - c_n \), and
2. for every \( i \), the number of exceptional arcs of index \( i \) that are not in \( F \) is at most twice the number of faces of \( \Delta \) of index \( i \).

Let \( N_1 \) be the sum of the lengths of all exceptional arcs that are not in \( F \), and \( N_2 \) the sum of the lengths of all arcs in \( F \). Then \( N_1 + N_2 = N \), and, by the condition \( D_3(\nu) \),

\[
N_1 \leq \sum_{\Pi \in \Delta(2)} 2\nu(\Pi)|\partial\Pi|,
\]

while

\[
N_2 \leq (1 - c_n)q_\Delta,
\]

where \( q_\Delta \) is the maximal length of an exceptional arc of \( \Delta \).

Adding up the estimates for \( L \), \( M_1 \), and \( N_1 \) gives:

\[
L + M_1 + N_1 \leq \sum_{\Pi \in \Delta(2)} \gamma(\Pi)|\partial\Pi|.
\]

Now \( M_2 + N_2 \) is to be estimated by estimating \( p_\Delta \) and \( q_\Delta \).

The length of every arc of \( \Delta \) that is incident to \( \Theta \) and not incident to any other face is at most \( |w| - 1 \). Indeed, the label of each of the two oriented arcs associated with such an arc is a common subword of \( w^m \) and \( w^{-m} \) (because \( \bar{\Delta} \) is orientable), and any such word of length \( |w| \) would be simultaneously a cyclic shift of \( w \) and of \( w^{-1} \), but in a free group, no nontrivial element can be conjugate to its own inverse (if it was, it would commute with the square of the conjugating element, and hence with the conjugating element itself).

Let

\[
Q_{\Delta,n} = \{ \Pi \in \Delta(2) \mid \iota(\Pi) \notin J_n \} = \{ \Pi \in \Delta(2) \mid \chi_{i(\Pi)} \leq c_n \}.
\]
Then, as follows from conditions (C4), (C5), (C6), and the estimate \(|w| - 1\) on the lengths of arcs incident only to \(\Theta\),

\[
P_\Delta \leq \max\{\mu(\Pi) | \partial \Pi| \mid \Pi \in Q_{\Delta,n} \} \cup \{p_n, |w| - 1\},
\]

\[
q_\Delta \leq \max\{\nu(\Pi) | \partial \Pi| \mid \Pi \in Q_{\Delta,n} \} \cup \{\eta_n\}.
\]

Therefore,

\[
M_2 \leq \sum_{\Pi \in Q_{\Delta,n}} (3 - 3c_n)\mu(\Pi) | \partial \Pi| + (3 - 3c_n) \max\{p_n, |w| - 1\},
\]

\[
N_2 \leq \sum_{\Pi \in Q_{\Delta,n}} (1 - c_n)\nu(\Pi) | \partial \Pi| + (1 - c_n)\eta_n,
\]

and hence, by the choice of \(n\),

\[
M_2 + N_2 < \sum_{\Pi \in \Delta(2)} ((3 - 3\chi(\Pi)\mu(\Pi)) + (1 - \chi(\Pi)\nu(\Pi)) | \partial \Pi| + \frac{1}{2} |\partial \Theta|).
\]

Adding up the obtained estimates for \(L + M_1 + N_1\) and \(M_2 + N_2\) and using condition (C8) gives a contradiction:

\[
\|\Delta(1)\| = L + M + N < \sum_{\Pi \in \Delta(2)} \frac{1}{2} |\partial \Pi| + \frac{1}{2} |\partial \Theta| = \|\Delta(1)\|.
\]

\[\square\]

**Proposition 5.10.** The function \(\text{sql}_G\) is unbounded on the free subgroup \(H = \langle [z_1], [z_2] \rangle\).

**Proof.** Let \(n\) be an arbitrary positive integer, \(c_n = 2 - n\), and

\[
J_n = \{ i \in I \mid \chi_i > c_n \}.
\]

The set \(J_n\) is finite by condition (C7). Let

\[
p_n = \max_{i \in J_n} \mu_i |r_i|, \quad \eta_n = \max_{i \in J_n} \nu_i |r_i|.
\]

Let \(w\) be a cyclically reduced non-periodic concatenation of several copies of \(z_1\) and \(z_2\) such that if \(s\) is a common prefix of two distinct cyclic shifts of \(w\), then

\[
(3 - 3c_n) \max\{p_n, |s|\} + (1 - c_n)\eta_n < \frac{1}{2} |w|
\]

(see properties of \(z_1, z_2\) in Section 2.) Note that \(w\) is a positive word, and hence \(w\) and \(w^{-1}\) have no nontrivial common subwords.

Suppose \(\text{sql}_G([w]) \leq n\). Then there exists a convenient connected one-contour diagram \(\Delta\) over \(\langle a, b \parallel R \rangle\) such that \(\chi(\Delta) \geq c_n - 1\) and \(\ell(\partial \Delta) = w\), see Lemmas 3.4, 3.6, 5.1.

Let \(\overline{\Delta}\) be the closure of \(\Delta\) and \(\Theta\) be the “outer” face. It follows from the choice of \(w\) that if \(u\) is an arc of \(\overline{\Delta}\) which is incident to \(\Theta\) and not incident to any other face, then

\[
(3 - 3c_n)|u| + (1 - c_n)\eta_n < \frac{1}{2} |\partial \Theta|.
\]

Now virtually the same argument of counting all edges of \(\Delta\) in two different ways as in the proof of Proposition 5.9 leads to a contradiction and completes the proof. \[\square\]
Proposition 5.11. The construction of the presentation \( \langle a, b \parallel R \rangle \) in Section 2 can be carried out in such a way that \( G \) have decidable word and conjugacy problems.

Proof. Let the construction of the presentation \( \langle a, b \parallel R \rangle \) be carried out in such a way that the conditions in the statement of Lemma 2.2 be satisfied (use Lemma 2.2).

It shall be shown first that there exists an algorithm that solves the word problem for all finite subpresentations of \( \langle a, b \parallel R \rangle \). More precisely, it needs to be shown that there exists an algorithm which for every input \( (S, w) \) with \( S = \{ r_n \}_{n \in J}, J \subset I, J \) finite, and \( w \in \{ a^{\pm 1}, b^{\pm 1} \}^\ast \), decides correctly whether \([w] = 1\) in the group presented by \( \langle a, b \parallel S \rangle \). The following algorithm does it:

1. input: \( S, w \);
2. construct all disc diagrams \( \Delta \) over \( \langle a, b \parallel S \rangle \) satisfying
   \[ \sum_{\Pi \in \Delta(2)} (1 - 2\gamma(\Pi))|\partial \Pi| \leq |\partial \Delta| = |w| \]
   (there are only finitely many such diagrams, and they all can be effectively constructed);
3. if among the constructed diagrams there is one with contour label \( w \), then the output is "yes" (meaning \([w] = 1\) in the group presented by \( \langle a, b \parallel S \rangle \)); otherwise the output is "no."

It follows from Lemmas 3.4, 3.6, 5.1, 5.7 that the described algorithm works correctly. Hence, by Lemma 2.2, the set \( I \) and the family \( \mathcal{R} \) are recursive.

The following algorithm decides the word problem in \( \langle a, b \parallel \mathcal{R} \rangle \), i.e., given \( w \in \{ a^{\pm 1}, b^{\pm 1} \}^\ast \), it decides whether \([w] = 1\) in the group presented by \( \langle a, b \parallel \mathcal{R} \rangle \):

1. input: \( w \);
2. find an \( n \) such that \((1 - 2\gamma_i)|r_i| > |w| \) for every \( i > n \) (see condition (2) of the statement of Lemma 2.2);
3. construct all disc diagrams \( \Delta \) over \( \langle a, b \parallel r_i, i \in I, i \leq n \rangle \) satisfying
   \[ \sum_{\Pi \in \Delta(2)} (1 - 2\gamma(\Pi))|\partial \Pi| \leq |\partial \Delta| = |w| \];
4. if among the constructed diagrams there is one with contour label \( w \), then the output is "yes" (meaning \([w] = 1\) in the group presented by \( \langle a, b \parallel \mathcal{R} \rangle \)); otherwise the output is "no."

The following algorithm decides the conjugacy problem in \( \langle a, b \parallel \mathcal{R} \rangle \), i.e., given \( w_1 \) and \( w_2 \), whether \([w_1] \) and \([w_2] \) are conjugate in the group presented by \( \langle a, b \parallel \mathcal{R} \rangle \):

1. input: \( w_1, w_2 \);
2. decide, using the previous algorithm, whether \([w_1] = 1\) and whether \([w_2] = 1\) in the group presented by \( \langle a, b \parallel \mathcal{R} \rangle \);
3. if \([w_1] = 1\) or \([w_2] = 1\), then output the appropriate answer (whether or not \([w_1]\) and \([w_2]\) are conjugate) and stop here; otherwise continue;
4. find an \( n \) such that \((1 - 2\gamma_i)|r_i| > |w_1| + |w_2| \) for every \( i > n \);
5. construct all oriented annular diagrams $\Delta$ over $\langle a,b \parallel r_i, i \in I, i \leq n \rangle$ whose contours agree with the orientation and which satisfy

$$|\partial_1 \Delta| = |w_1|, \quad |\partial_2 \Delta| = |w_2|,$$

$$\sum_{\Pi \in \Delta(2)} (1 - 2\gamma(\Pi))|\partial \Pi| \leq |\partial_1 \Delta| + |\partial_2 \Delta|;$$

6. if among the constructed diagrams there is one with contour labels $w_1$ and $w_2^{-1}$, respectively, then the output is “yes” (meaning $[w_1]$ and $[w_2]$ are conjugate in the group presented by $\langle a,b \parallel R \rangle$); otherwise the output is “no.”

(Lemmas 3.4, 3.6, 5.1, 5.7 shall be used to verify correctness of the last two algorithms.)

□

The main theorem follows from Propositions 2.1, 5.6, 5.8, 5.9, 5.10, 5.11.

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