Obstruction theory on 7-manifolds

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Abstract. This paper gives a uniform, self-contained, and direct approach to a variety of obstruction-theoretic problems on manifolds of dimension 7 and 6. We give necessary and sufficient cohomological criteria for the existence of various $G$-structures on vector bundles over such manifolds especially using low dimensional representations of $U(2)$.

1. Introduction

Let $M$ be a connected, closed, smooth, spin$^c$ manifold of dimension $m = 6$ or 7 and let $\xi$ be an $m$-dimensional oriented real vector bundle over $M$ admitting a spin$^c$-structure. For various homomorphisms $\rho : G \rightarrow SO(m)$ from a compact Lie group $G$ to $SO(m)$ we consider the problem of reducing the structure group $SO(m)$ of the vector bundle $\xi$ to $G$ via the representation $\rho$. Necessary and sufficient conditions for such reductions will be obtained in terms of the cohomology of $M$ and cohomology characteristic classes of $M$ and $\xi$. Thus as for methods and results the present paper is a continuation of [4].

Most of our results depend on the existence of 2-dimensional complex vector bundles over low dimensional manifolds. So we can provide more or less complete answers for all homomorphisms $\rho$ which are connected with low dimensional representations of the group $G = U(2)$.

Our results complete the characterization of $m$-dimensional vector bundles over $m$-dimensional complexes ($m = 6, 7$) given in [13] and the results on the existence of vector fields over $m$-dimensional manifolds in [12].

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We conclude this introduction by describing our results for 7-manifolds in the case that $G$ is the group $\text{Sp}(1)$ and the manifold $M$ and vector bundle $\xi$ are spin.

There are 4 irreducible real $\text{Sp}(1)$-modules of dimension at most 7: the Lie algebra $A_1$ of dimension 3, the defining 4-dimensional module $E$ ($= \mathbb{H}$), a module $A_2$ of dimension 5, and a module $A_3$ of dimension 7. We thus have, up to equivalence, the following seven 7-dimensional real $\text{Sp}(1)$-modules and associated representations $\rho$:

(i) $\mathbb{R}^7$, (ii) $E \oplus \mathbb{R}^3$, (iii) $A_1 \oplus \mathbb{R}^4$, (iv) $E \oplus A_1$, (v) $A_1 \oplus A_1 \oplus \mathbb{R}$, (vi) $A_2 \oplus \mathbb{R}^2$, (vii) $A_3$.

**Theorem 1.1.** Let $\xi$ be a 7-dimensional vector bundle with $w_1(\xi) = 0$ and $w_2(\xi) = 0$ over a 7-dimensional spin manifold $M$. Then the structure group of $\xi$ reduces from $\text{Spin}(7)$ to $\text{Sp}(1)$ through $\rho$ if and only if the spin characteristic class $q_1(\xi) \in H^4(M; \mathbb{Z})$ is divisible by

(i) 0, (ii) 1, (iii) 2, (iv) 3, (v) 4, (vi) 10, (vii) 28.

**Corollary 1.2.** A 7-dimensional spin bundle $\xi$ over a spin 7-manifold $M$ admits 4 linearly independent sections if and only if $w_4(\xi) = 0$.

**Proof.** This follows from case (iii), because $\rho_2(q_1(\xi)) = w_4(\xi)$. □

**Corollary 1.3.** For any 7-dimensional spin bundle $\xi$ over a spin 7-manifold $M$, the 8-dimensional vector bundle $\mathbb{R}^1 \oplus \xi$ admits the structure of a bundle of Cayley algebras and, hence, the structure group of $\xi$ reduces from $\text{SO}(7)$ to $G_2$.

See [11, Section 4.4] and the references cited there for the case of the tangent bundle.

**Proof.** According to case (ii), $\xi$ is isomorphic to $\mathbb{R}^3 \oplus \mu$ for some (left) $\mathbb{H}$-line bundle $\mu$. A Cayley multiplication can then be written down on $\mathbb{R}^1 \oplus \xi = \mathbb{H} \oplus \mu$ using the quaternionic inner product on $\mu$ satisfying $\langle au, bv \rangle = a\langle u, v \rangle b$:

$$(a, u) \cdot (b, v) = (ab - \langle v, u \rangle, a\overline{v} + bu),$$

where $a, b \in \mathbb{H}$ and $u, v$ are vectors in a fibre of $\mu$. There is an associated principal $G_2$-bundle with fibre the bundle of algebra isomorphisms from the standard Cayley algebra (with automorphism group $G_2$) to the fibre of $\mathbb{H} \oplus \mu$. □

### 2. The spin characteristic classes

In this section we recall some standard facts about spin and spin$^c$ vector bundles in low dimensions and their characteristic classes. The inclusion

$$\text{SU}(\infty) \to \text{Spin}(\infty) \quad (2.1)$$

induces an isomorphism $\pi_i(B\text{SU}(\infty)) \to \pi_i(B\text{Spin}(\infty))$ for $i \leq 5$. So for a spin vector bundle $\xi$ over a manifold $M$ we may define $q_1(\xi) \in H^4(M; \mathbb{Z})$ to be the characteristic class corresponding in the above isomorphism to the negative of the
second Chern class $-c_2$. In Section 2 of [4] it was shown that $q_1(\xi)$ is independent of the choice of the spin structure.

The inclusion (2.1) induces the inclusion

$$U(\infty) \cong SU(\infty) \times \{\pm 1\} U(1) \to \text{Spin}(\infty) \times \{\pm 1\} U(1) = \text{Spin}^c(\infty)$$

which yields again an isomorphism $\pi_i(BU(\infty)) \to \pi_i(B\text{Spin}^c(\infty))$ for $i \leq 5$. A vector bundle $\xi$ over $M$ with a fixed $\text{Spin}^c(\infty)$-structure has one characteristic class $l \in H^2(M; \mathbb{Z})$ which corresponds to the first Chern class $c_1$ and a second characteristic class $q_1 \in H^4(M; \mathbb{Z})$ corresponding to $-c_2$. However, in this case both classes depends on the choice of the $\text{Spin}^c(\infty)$-structures.

Let $\zeta$ and $\zeta'$ be two $\text{Spin}^c(\infty)$-structures of the vector bundle $\xi$. Over $M^{(5)}$ they correspond to complex vector bundles. Their difference $\zeta' - \zeta$ considered as a map to $M \to B\text{Spin}^c(\infty)$ lifts to the fiber $BU(1)$ of the fibration $B\text{Spin}^c(\infty) \to B\text{SO}(\infty)$. The inclusion $i : BU(1) \to B\text{Spin}^c(\infty)$ induces the multiplication by two

$$H^2(B\text{Spin}^c(\infty); \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\times 2} H^2(BU(1); \mathbb{Z}) \cong \mathbb{Z}. $$

Hence for fixed $\zeta$ and any $m \in H^2(M; \mathbb{Z})$ we can choose $\zeta'$ in such a way that $c_1(\zeta' - \zeta) = 2m$. Since the choice of $\text{Spin}^c(\infty)$-structure $\zeta$ is determined by the choice of $c_1(\zeta) = l$, we can now define

$$q_1(\xi; l) = -c_2(\xi).$$

These classes for different lifts are related by the formula

$$q_1(\xi; l + 2m) = q_1(\xi; l) - 2lm - 2m^2. $$

Moreover, $2q_1(\xi; l) = p_1(\xi) - l^2$ and $\rho_2(q_1(\xi; l)) = w_4(\xi)$.

3. **Spin$^c$ structures on 7-manifolds**

For spin$^c$ manifolds of dimension 7 we shall use the following general result on manifolds of dimension $m \equiv 3 \pmod{4}$.

**Theorem 3.1.** Let $M$ be an orientable closed manifold of dimension $4k + 3$.

(i) Suppose that $\xi$ is a $(4k + 3)$-dimensional oriented real vector bundle over $M$ such that $w_2(\xi) = w_2(M)$. Then $\xi$ splits as a direct sum $\xi' \oplus \mathbb{R}$ with $e(\xi') = 0 \in H^{4k+2}(M; \mathbb{Z})$.

(ii) Suppose that $\xi'$ is a $(4k + 2)$-dimensional oriented vector bundle over $M$ such that $w_2(\xi') = w_2(M)$ and $e(\xi') = 0 \in H^{4k+2}(M; \mathbb{Z})$. Then $\xi'$ has a nowhere-zero section.

(iii) Suppose that $\xi''$ is a $(4k + 1)$-dimensional oriented vector bundle over $M$ such that $w_2(\xi'') = w_2(M)$ and $e(\xi'') = 0 \in H^{4k+1}(M; \mathbb{Z})$. Then $\xi''$ has a nowhere-zero section.
Most of this may be found in [7]; see also [12]. For the sake of completeness we include a proof using the $K$-theoretic methods introduced by Atiyah and Dupont [3] in Section 6.

Using the properties of the spin$^c$ characteristic class $q_1$ we apply Theorem 3.1 to a connected closed spin$^c$ manifold $M$ of dimension 7. If $\xi$ is an oriented 7-dimensional vector bundle over $M$ with $w_2(\xi) = w_2(M) = \rho_2 l$, then

$$
\delta^* w_4(\xi) = \delta^* \rho_2 q_1(\xi; l) = 0,
$$

where $\delta^* : H^4(M; \mathbb{Z}/2) \to H^5(M; \mathbb{Z})$ is the Bockstein homomorphism. By parts (i) and (ii), $\xi$ splits as $\xi'' \oplus \mathbb{R}^2$, and then we can apply part (iii) to $\xi''$, because $e(\xi'') = \delta^* w_4(\xi'') = 0$. So $\xi$ has three linearly independent cross sections, that is the structure group of $\xi$ reduces to $\text{SO}(4) < \text{SO}(7)$. In fact, under the same assumptions more is true.

**Proposition 3.2.** Let $\xi$ be a 7-dimensional vector bundle over a closed, connected, smooth, spin$^c$ 7-manifold $M$. If $w_2(\xi) = w_2(M) = \rho_2 l$ for an $l \in H^2(M; \mathbb{Z})$, then there is a 2-dimensional complex bundle $\eta$ over $M$ with $c_1(\eta) = l$ such that $\xi \cong \eta \oplus \mathbb{R}^3$. Moreover, $c_2(\eta) = -q_1(\xi; l)$.

**Proof.** The inclusion

$$
\text{SU}(2) \cong \text{Sp}(1) = \text{Sp}(1) \times 1 \hookrightarrow \text{Sp}(1) \times \text{Sp}(1) \cong \text{Spin}(4) \hookleftarrow \text{Spin}(5)
$$

induces an isomorphism on homotopy groups $\pi_i$ for $i \leq 5$ and an epimorphism on $\pi_6$. Indeed, it can be identified with the inclusion $\text{Sp}(1) \hookrightarrow \text{Sp}(2), \ g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$, with quotient $S^7$. The same applies to the induced inclusion

$$
\text{U}(2) \cong \text{SU}(2) \times_{\{\pm 1\}} \text{U}(1) \hookrightarrow \text{Spin}(5) \times_{\{\pm 1\}} \text{U}(1) = \text{Spin}^c(5).
$$

By (i) and (ii) of Theorem 3.1 the structure group $\text{Spin}^c(7)$ of $\xi$ can be reduced to $\text{Spin}^c(5)$. Since the inclusion $\text{U}(2) \hookrightarrow \text{Spin}^c(5)$ is a 6-equivalence there is a 2-dimensional complex bundle $\eta$ such that $\eta \oplus \mathbb{R}^3 \cong \xi$. In the previous section it was shown that the $\text{Spin}^c(5)$-structure of $\xi$ can be chosen in such a way that $c_1(\eta) = l$ since $w_2(\xi) = \rho_2 l$. Then according to the definition of the spin$^c$ characteristic class $c_2(\eta) = -q_1(\xi; l)$. \hfill \square

Notice that the above inclusion $\text{U}(2) \hookrightarrow \text{Spin}^c(5) \hookrightarrow \text{Spin}^c(7)$ is a lift of the standard inclusion

$$
\text{U}(2) \hookrightarrow \text{SO}(4) \hookrightarrow \text{SO}(7).
$$

**Remark 3.3.** Let us recall from [4] the notion of $\mathbb{H}_k$-bundle for a complex line bundle $\lambda$. This is a complex vector bundle which is a left module over the bundle $\mathbb{H}_k = \mathbb{C} \oplus \lambda$ of quaternion algebras. The structure groups for $\mathbb{H}_k$-bundles are $\text{Sp}(n) \times_{\{\pm 1\}} U(1)$. Since $\text{U}(2) \cong \text{SU}(2) \times_{\{\pm 1\}} \text{U}(1) \cong \text{Sp}(1) \times_{\{\pm 1\}} \text{U}(1)$ every 2-dimensional complex bundle $\eta$ is naturally an $\mathbb{H}_k$-line bundle, where $\lambda$ is the determinant bundle $\Lambda^2(\eta)$ (so that $c_1(\lambda) = c_1(\eta)$). Hence Proposition 3.2 can be read in the following way:
Let $\lambda$ be a complex line bundle over $M$ with $c_1(\lambda) = l$. Suppose that $w_1(\xi) = 0 = w_1(M)$ and $w_2(\xi) = \rho_2(l) = w_2(M)$. Then there is an $\mathbb{H}_\lambda$-line bundle $\eta$ with the Euler class $e(\eta) = -q_1(\xi; l)$ such that $\xi$ is isomorphic to $\eta \oplus \mathbb{R}^3$.

The crucial role for our obstruction theory on 7-manifolds is played by Propositions 3.4 and 3.5. The first completes the characterization of 7-dimensional vector bundles by characteristic classes in [13]. It has been already used in [11] to obtain results on the existence of multisymplectic 3-forms on 7-dimensional manifolds.

**Proposition 3.4.** Suppose that $M$ is a 7-dimensional connected closed manifold (not necessarily spin$^c$). Consider two orientable real 7-dimensional vector bundles $\xi$ and $\xi'$ with $w_2(\xi) = w_2(\xi') = \rho_2(l)$ for some $l \in H^2(M; \mathbb{Z})$. Then $\xi$ and $\xi'$ are isomorphic if and only if $q_1(\xi; l) = q_1(\xi'; l)$.

**Proof.** Suppose that $q_1(\xi; l) = q_1(\xi'; l)$. Then the proof of Proposition 2.5 in [4] shows that the restrictions of $\xi$ and $\xi'$ to the 6-skeleton $M^6$ of the manifold $M$ are stably isomorphic. Since the inclusion $\text{BSO}(7) \hookrightarrow \text{BSO}(\infty)$ induces isomorphisms $\pi_i(\text{BSO}(7)) \rightarrow \pi_i(\text{BSO}(\infty))$ for $i \leq 7$, the vector bundles $\xi$ and $\xi'$ are also unstably isomorphic over $M^6$. The vanishing of $\pi_6(\text{SO}(7))$ says that there is no obstruction to extending an isomorphism over the whole manifold $M$. □

**Proposition 3.5.** Suppose that $M$ is a connected closed smooth spin$^c$ 7-manifold with $w_2(M) = \rho_2(l)$ for some $l \in H^2(M; \mathbb{Z})$. Let $u \in H^4(M; \mathbb{Z})$. Then there is a 2-dimensional complex bundle $\eta$ such that $c_1(\eta) = l$, $c_2(\eta) = u$.

If $\lambda$ is a complex line bundle over $M$ with $c_1(\lambda) = l$, the complex bundle $\eta$ admits the structure of an $\mathbb{H}_\lambda$-line bundle with Euler class $e(\eta) = u$.

**Proof.** We construct a 7-dimensional vector bundle $\xi$ with $w_1(\xi) = 0$, $w_2(\xi) = \rho_2(l)$ and $q_1(\xi; l) = -u$. Then we can apply Proposition 3.2 to obtain $\eta$ with the prescribed properties.

If we can construct $\xi$ over the 5-skeleton $M^{(5)}$ we are done, because $\pi_5(\text{BO}(\infty)) = 0$ and $\pi_6(\text{BO}(\infty)) = 0$, so that any stable bundle over $M^{(5)}$ can be extended over $M$. And, of course, any stable bundle over a 7-manifold can be reduced to dimension 7.

On $M^{(5)}$ we can take $\mu \oplus \lambda \oplus \mathbb{R}$ where $\lambda$ is a complex line bundle with $c_1(\lambda) = l$ and $\mu$ is an $\mathbb{H}$-line bundle such that $e(\mu) = u$. The existence of $\mu$ over $M^{(5)}$ follows from the fact that the Euler class $e : \text{BSU}(2) \rightarrow K(\mathbb{Z}, 4)$ induces isomorphism on $\pi_i$ for $i \leq 4$ and an epimorphism for $i = 5$. (Notice that $\mu$ on $M^{(5)}$ is a 2-dimensional complex bundle with $c_1(\mu) = 0$ and $c_2(\mu) = e(\mu) = u$.) Then $q_1(\xi; l) = -c_2(\mu \oplus \lambda) = -c_2(\mu) = -u$. □

We add also a characterization of three-dimensional complex vector bundles over manifolds of dimension less or equal to 7. To keep the notation from [4], we use the symbol $[X, Y]$ for pointed homotopy classes of maps from $X$ to $Y$ (although in our case the sets of pointed and unpointed homotopy classes are the same). In Section 2 of [4], it has been shown:
Proposition 3.6. Let $M$ be a manifold of dimension $\leq 7$. The image of the mapping determined by the first three Chern classes

$$(c_1, c_2, c_3) : [M; BU(3)] \to H^2(M; \mathbb{Z}) \oplus H^4(M; \mathbb{Z}) \oplus H^6(M; \mathbb{Z})$$

is the set $\{(l, u, v) \mid Sq^2 \rho_2(u) + \rho_2(lu) = \rho_2(v)\}$.

This Proposition can be also used to give an alternative proof of Proposition 3.5.

Another proof of Proposition 3.5. To prove Proposition 3.5 from Proposition 3.6 we show first that $l \in H^2(M; \mathbb{Z}), u \in H^4(M; \mathbb{Z})$ and $0 \in H^6(M; \mathbb{Z})$ are the Chern classes of a three-dimensional complex vector bundle, $\beta$ say, over $M$. For every $x \in H^1(M; \mathbb{Z}/2)$ we have

$$x Sq^2 \rho_2 u = Sq^2(x \rho_2(u)) = w_2(M)(x \rho_2(u)) = x \rho_2(lu),$$

which gives that $Sq^2 \rho_2 u = \rho_2(lu)$. By part (ii) of Theorem 3.1, $\beta$ has a nowhere-zero section and so splits as a direct sum $\beta = \mathbb{C} \oplus \eta$, where $\eta$ is a two-dimensional complex vector bundle.

\(\square\)

4. Representations of U(2) in dimension 7

Let $E$ be the standard 2-dimensional complex representation of $U(2)$ and write $L = \Lambda^2 E$. Then the irreducible complex representations of $U(2)$ can be listed as

$$V_{j,k} = S^j E \otimes \mathbb{C} L^k, \quad j \geq 0, \ k \in \mathbb{Z},$$

where $S^j$ is the $j$th symmetric power. The representation is real if and only if $j = 2i$ is even and $k = -i$. In that case we write

$$V_{2i,-i} = \mathbb{C} \otimes A_i, \quad i \geq 0.$$

Notice that dim $A_i = 2i + 1$ and that the centre $\mathbb{T}$ of $U(2)$ acts trivially on $A_i$, so that $A_i$ is a module over $U(2)/\mathbb{T} = SO(3)$. The 5-dimensional representation $A_2$ has attracted the special interest of geometers, see \cite{1} or \cite{5}.

Denote by $EU(2)$ the universal principal $U(2)$-bundle over $BU(2)$. For a real or complex representation $V$ of the group $U(2)$ we define characteristic classes of $V$ as the characteristic classes of the associated vector bundle $EU(2) \times_{U(2)} V$. It is straightforward to calculate characteristic classes of a representation $V$ in terms of $c_1(E) = l$ and $c_2(E) = e(E) = u$. If we restrict to the classifying space of a maximal torus, $E$ splits as the direct sum $L_1 \oplus L_2$ of two 1-dimensional representations and $L = L_1 \otimes \mathbb{C} L_2$.

For any real representation $V$ of $U(2)$ write

$$p_1(V) = -2a(V)u + b(V)l^2,$$
where $a(V), b(V) \in \mathbb{Z}$. Thus $a$ and $b$ are linear in $V$: $a(V \oplus W) = a(V) + a(W)$, $b(V \oplus W) = b(V) + b(W)$ for two representations $V$ and $W$. Since $\rho_2(p_1(V)) = w_2^r(V)$, we get that $w_2(V) = b(V)\rho_2(l)$. If $b(V)$ is even, then

$$q_1(V) = -a(V)u + \frac{1}{2}b(V)l^2; \quad (4.1)$$

if $b(V)$ is odd, then

$$q_1(V; l) = -a(V)u + \frac{1}{2}(b(V) - 1)l^2. \quad (4.2)$$

Using the fact that $S^j(E) = \bigoplus_{0 \leq r \leq j} L_1^{(j-r)} \otimes_{\mathbb{C}} L_2^{r}$, we get

$$V_{2t,-i} = \mathbb{C} \otimes A_i = \bigoplus_{i \leq r \leq i} (L_1 \otimes_{\mathbb{C}} L_2^*)^{|r|}$$

which yields $A_i = \mathbb{R} \oplus \bigoplus_{1 \leq r \leq i} (L_1 \otimes_{\mathbb{C}} L_2^*)^{|r|}$. Since $p_1(A_i) = -c_2(V_{2i,-i})$, a standard computation expresses the second Chern class as a symmetric polynomial in $x_1 = c_1(L_1)$ and $x_2 = c_1(L_2)$. In particular,

$$b(A_i) = 1^2 + 2^2 + \cdots + i^2 = \frac{1}{6}i(i + 1)(2i + 1), \quad a(A_i) = 2b(A_i)$$

and $w_2(A_i)$ equals to 0 for $i \equiv 0, 3 \pmod{4}$, and to $\rho_2 l$ for $i \equiv 1, 2 \pmod{4}$.

Now consider $V_{j,k}$ as a real representation. Then the computation starting with

$$p_1(V_{j,k}) = p_1 \left( \bigoplus_{0 \leq r \leq j} L_1^{(j-r+k)} \otimes_{\mathbb{C}} L_2^{(r+k)} \right)$$

$$= \sum_{r=0}^{j} p_1 \left( L_1^{(j-r+k)} \otimes_{\mathbb{C}} L_2^{(r+k)} \right) = \sum_{r=0}^{j} c_1^2 \left( L_1^{(j-r+k)} \otimes_{\mathbb{C}} L_2^{(r+k)} \right)$$

$$= \sum_{r=0}^{j} ((j-r+k)x_1 + (r+k)x_2)^2$$

leads to

$$b(V_{j,k}) = \sum_{r=0}^{j} (r+k)^2 \quad \text{and} \quad a(V_{j,k}) = \frac{1}{2} \sum_{r=0}^{j} (j-2r)^2 = \frac{1}{6}j(j+1)(2j+1).$$

The 7-dimensional real representations $V$ of $U(2)$ can be listed, for $r, s, t \in \mathbb{Z}$ (with some redundancy because $L^{\otimes r}$ and $L^{\otimes(-r)}$ are isomorphic over $\mathbb{R}$) as:

Notice that $A_1 \otimes_{\mathbb{R}} L^{\otimes s} \simeq V_{2s-1}$ as real vector bundles. The general statement is
The conditions in both cases are necessary since if \( \rho \) satisfies (4.1) or (4.2).

**Theorem 4.1.** Let \( \xi \) be a 7-dimensional spin\( ^c \) vector bundle over a closed, connected, smooth, spin\( ^c \) 7-manifold \( M \) with \( w_2(M) = w_2(\xi) \). Let \( V \) be a 7-dimensional real representation of \( U(2) \) and let \( l \) be a class in \( H^2(M; \mathbb{Z}) \).

1. If \( b(V) \) is odd, then \( \xi \) is isomorphic to \( P \times_{U(2)} V \) for some principal \( U(2) \)-bundle \( P \) over \( M \) with \( c_1 = l \) if and only if \( \rho_2(l) = w_2(\xi) \) (\( = w_2(M) \)) and

\[
q_1(\xi; l) - \frac{1}{2} (b(V) - 1)l^2 \in a(V)H^4(M; \mathbb{Z}).
\]

2. If \( b(V) \) is even and \( \rho_2(l) = w_2(\xi) = w_2(M) \), then \( \xi \) is isomorphic to \( P \times_{U(2)} V \) for some principal \( U(2) \)-bundle \( P \) over \( M \) with \( c_1 = l \) if and only if \( \rho_2(l) = 0 \) (so that \( \xi \) and \( M \) are spin) and

\[
q_1(\xi) - b(V)l^2 \in a(V)H^4(M; \mathbb{Z}).
\]

**Proof.** The conditions in both cases are necessary since if \( c_2 \) of a principal \( U(2) \)-bundle \( P \) is equal to an element \( u \in H^4(M; \mathbb{Z}) \), the spin\( ^c \) characteristic class satisfies (4.1) or (4.2).

Conversely, suppose that there is \( u \in H^4(M; \mathbb{Z}) \) such that

\[
q_1(\xi; l) = \frac{1}{2} (b(V) - 1)l^2 + a(V)u,
\]

if \( b(V) \) is odd, or

\[
q_1(\xi) = \frac{1}{2} b(V)l^2 + a(V)u,
\]

if \( b(V) \) is even and \( \rho_2(l) = 0 \).

According to Proposition 3.5 there is a 2-dimensional complex bundle \( \eta \) with \( c_1(\eta) = l \) and \( c_2(\eta) = u \). Since \( \eta = P \times_{U(2)} E \) for some principal \( U(2) \)-bundle \( P \), we get that the real vector bundle \( \xi' = P \times_{U(2)} V \) has \( w_2(\xi') = \rho_2(l) = w_2(\xi) \) and \( q_1(\xi'; l) = q_1(\xi; l) \). Hence Proposition 3.4 implies that \( \xi = \xi' \).

Notice that the assumption \( \rho_2(l) = w_2(\xi) \) in (2) is needed to apply Proposition 3.5.

In special cases we obtain results on SO\( (3) \)-structures. The real representations of \( U(2) \) trivial on the centre \( \mathbb{T} \) of the group \( U(2) \) determine representations of \( SO(3) = U(2)/\mathbb{T} \). The 7-dimensional representations of \( SO(3) \) are:

| Representation \( V \) | \( a(V) \) | \( b(V) \) |
|------------------------|----------|----------|
| \( L^\otimes r \oplus L^\otimes s \oplus L^\otimes t \oplus \mathbb{R} \) | 0 | \( r^2 + s^2 + t^2 \) |
| \( (L^\otimes s \otimes \mathbb{C})E \oplus L^\otimes t \oplus \mathbb{R} \) | 1 | \( s^2 + (s+1)^2 + t^2 \) |
| \( A_1 \oplus L^\otimes s \oplus L^\otimes t \) | 2 | \( 1 + s^2 + t^2 \) |
| \( A_1 \oplus (L^\otimes s \otimes \mathbb{C})E \) | 3 | \( 1 + s^2 + (s+1)^2 \) |
| \( (A_1 \otimes \mathbb{R})L^\otimes s \oplus \mathbb{R} \) | 4 | \( 2 + 3s^2 \) |
| \( A_2 \oplus L^\otimes s \) | 10 | \( 5 + s^2 \) |
| \( A_3 \) | 28 | 14 |
In view of the isomorphism between $U(2)$ and $\text{Spin}^c(3)$, given $l$ and $V$, Theorem 4.1 gives necessary and sufficient conditions for the existence of a $\text{Spin}^c(3)$ vector bundle $\alpha$ with $\text{Spin}^c$ characteristic class equal to $l$ such that $\xi$ is isomorphic to the bundle associated with $\alpha$ and $l$ by the representation $V$.

We have to distinguish two cases.

(i) If $b(V)$ is even, then the manifold $M$ and the bundle $\xi$ must be spin ($w_2(M) = w_2(\xi) = 0$). Theorem 4.1 gives necessary and sufficient conditions for the existence of a spin$^c$ bundle $\alpha$ with $\text{Spin}^c$ characteristic class equal to $l$ such that $q_1(\xi) - a(V)m^2 = q_1(\xi) - b(V)m^2 = q_1(\xi) - \frac{1}{2}b(V)(2m)^2 \in a(V)H^4(M; \mathbb{Z})$.

(ii) If $b(V)$ is odd, then $w_2\alpha = w_2\xi (= w_2(M))$ is a necessary condition for the existence of $\alpha$. For a representation $V$ factoring through $U(2)/\mathbb{T} = \text{SO}(3)$, this condition on $w_2\alpha$ is necessary even without assuming that $\alpha$ is spin$^c$, because the homomorphism $H^2(\text{BSO}(3); \mathbb{Z}/2) \to H^2(\text{BU}(2); \mathbb{Z}/2)$ is injective. So in this case Theorem 4.1 gives necessary and sufficient conditions for the existence of an $\text{SO}(3)$-bundle $\alpha$ such that $\xi$ is isomorphic to the bundle associated with $\alpha$ by the representation $\text{SO}(3) \to \text{O}(7)$ giving $V$: there exists $l \in H^2(M; \mathbb{Z})$ such that $w_2(M) = \rho_2(l)$ and $q_1(\xi; l) = \frac{1}{2}(b(V) - 1)l^2 \in a(V)H^4(M; \mathbb{Z})$. And, if such a bundle $\alpha$ exists it must satisfy $w_2\alpha = w_2\xi$ and so be spin$^c$.

**Corollary 4.2.** Let $\xi$ be a 7-dimensional spin$^c$ vector bundle over a closed, connected, smooth, spin$^c$ 7-manifold $M$ with $w_2(M) = w_2(\xi)$. Then

1. $\xi$ is a trivial vector bundle if and only if $w_2(\xi) = 0$ and $q_1(\xi) = 0$;
2. $\xi$ always has 3 linearly independent sections;
3. $\xi$ has 4 linearly independent sections if and only if $w_4(\xi) = 0$;
4. there is a 3-dimensional spin vector bundle $\alpha$ such that $\xi \cong \alpha \oplus \alpha \oplus \mathbb{R}$ if and only if $w_2(\xi) = 0$ and $q_1(\xi) \in 4H^4(M; \mathbb{Z})$;
5. the structure group of $\xi$ reduces to $\text{SO}(3)$ through the homomorphism $\text{SO}(3) \to \text{SO}(5) \subseteq \text{SO}(7)$ corresponding to $A_2$ if and only if $w_4(\xi) = 0$ and $p_1(\xi) \in 5H^4(M; \mathbb{Z})$;
6. the structure group of $\xi$ reduces to Spin(3) through the homomorphism $\text{Spin}(3) \to \text{SO}(3) \to \text{SO}(7)$ corresponding to $A_3$ if and only if $w_2(\xi) = 0$ and $q_1(\xi) \in 28H^4(M; \mathbb{Z})$. 

Theorem 1.1 from the Introduction is an immediate consequence of this corollary.

Remark 4.3. In [1], Theorem 3.2 states that for the existence of the irreducible SO(3)-structure corresponding to the representation $A_2$ on tangent bundles of 5-dimensional oriented manifolds the conditions $w_4(M) = 0$ and $p_1(M) \in 5H^4(M; \mathbb{Z})$ are necessary. Part (4) of Corollary 4.2 shows that on 7-dimensional spin$^c$ manifolds these conditions are also sufficient.

Proof. The proofs of (1), (4) and (6) are covered by the considerations for $b(V)$ even in (i) preceding Corollary 4.2.

(2) follows directly from Proposition 3.2.

(3) If $\xi = \alpha \oplus \mathbb{R}^4$ for a 3-dimensional vector bundle $\alpha$, then $w_4(\xi) = 0$ and $w_2(\alpha) = w_2(\xi) = \rho_2 l$. Hence $\alpha$ has the Spin$^c(3)$-structure given by the class $l \in H^2(M; \mathbb{Z})$. The condition $w_4(\xi) = 0$ is equivalent to the condition

$$q_1(\xi; l) = q_1(\xi; l) - \frac{1}{2}(1 - 1)l^2 \in 2H^4(M; \mathbb{Z})$$

which is the sufficient and necessary condition for the existence of the Spin$^c(3)$-structure corresponding to $A_1$ according to Theorem 4.1.

(5) Let $\xi$ reduce to SO(3) via the representation $A_2$. Since $b(A_2) = 5$, according to (ii) preceding Corollary 4.2 the 3-dimensional real vector bundle $\alpha$ associated to the SO(3)-structure has $w_2(\alpha) = w_2(\xi) = \rho_2 l$. We can apply Theorem 4.1 to get as a necessary and sufficient condition

$$q_1(\xi; l) - \frac{1}{2}(5 - 1)l^2 \in 10H^4(M; \mathbb{Z}).$$

This condition is equivalent to $w_4(\xi) = \rho_2 q_1(\xi; l) = 0$ and $p_1(\xi) \in 5H^4(M; \mathbb{Z})$ since

$$p_1(\xi) = 2q_1(\xi; l) + l^2 = 2(q_1(\xi; l) - \frac{1}{2}(5 - 1)l^2) + 5l^2.$$ 

□

Remark 4.4. Let $M$ be a manifold as above. If we want to apply the previous results to the tangent of $M$ or the normal bundle of some immersion of $M$ into $\mathbb{R}^{14}$, we can use the following computation of $w_4$. Let $v_i(M)$ denote the Wu classes of $M$. Since $w_2(M) = \rho_2 l$, $w_3(M) = Sq^1 w_2(M) = 0$. Now $w(M) = Sq(v(M))$ and $v_j(M) = 0$ for $j \geq 4$. Further, $v_1(M) = w_1(M) = 0$, $v_2(M) = w_2(M)$, $v_3(M) + Sq^1 w_2(M) = w_3(M) = 0$, so that $v_3(M) = 0$, and $w_4(M) = Sq^1 v_3(M) + Sq^2 v_2(M) = v_2(M)^2 = w_2(M)^2$. This implies that:

(i) For the tangent bundle $w_4(M) = 0$ if and only if $w_2(M)^2 = 0$.

(ii) If $v$ is the normal bundle of an immersion, then $w_4(v) = 0$.

We conclude this section with a result on the existence of $U(3)$-structures.
Proposition 4.5. Suppose that $M$ is a connected, closed, smooth, spin$^c$ 7-manifold and $\xi$ an oriented 7-dimensional real vector bundle with $w_2(\xi) = w_2(M) = \rho_2 l$. Then for any $u \in H^4(M; \mathbb{Z})$ and $v \in 2H^6(M; \mathbb{Z})$ there is a 3-dimensional complex vector bundle $\xi$ such that $c_1(\xi) = l$, $c_2(\xi) = u$, $c_3(\xi) = v$. Moreover, $\xi$ is isomorphic to $\xi \oplus \mathbb{R}$ if and only if $q_1(\xi; l) = -u$.

Proof. According to Proposition 3.6 the condition

$$Sq^2 \rho_2(u) + \rho_2(lu) = w_2(M)\rho_2(u) + w_2(M)\rho_2(u) = 0 = \rho_2(v)$$

is sufficient for the existence of a 3-dimensional complex bundle $\xi$ with the Chern classes $c_1(\xi) = l$, $c_2(\xi) = u$ and $c_3(\xi) = v$.

Since $w_2(\xi) = \rho_2(l) = w_2(\xi)$, the real vector bundles $\xi$ and $\xi \oplus \mathbb{R}$ are isomorphic if and only if $q_1(\xi; l) = q_1(\xi \oplus \mathbb{R}; l) = -c_2(\xi) = -u$ by Proposition 3.4. \hfill \Box

5. Dimension 6

Now suppose that $M$ is a connected closed manifold of dimension 6. Let $\xi$ and $\xi'$ be two $m$-dimensional vector bundles over $M$ with $w_2(\xi) = w_2(\xi') = \rho_2(l)$ for some $l \in H^2(M; \mathbb{Z})$ and with $q_1(\xi; l) = q_1(\xi'; l)$. Consider the 7-dimensional manifold $N = M \times S^1$ and the 7-dimensional bundles $p^*(\xi) \oplus \mathbb{R}$ and $p^*(\xi') \oplus \mathbb{R}$ where $p : N \rightarrow M$ is the obvious projection. According to Proposition 3.4 these two bundles are isomorphic. Hence $\xi$ and $\xi'$ are stably isomorphic.

For even dimensions the Euler class will distinguish stably isomorphic bundles. This fact has been considered to be well known for a long time; see the sentence following Lemma 2 in [13]. Unfortunately, we have found only one reference for its proof, the relatively recent paper [9], (Theorem 3.9). This proof uses the Moore-Postnikov tower for the map $BSO(m) \rightarrow BSO(\infty)$ induced by inclusion. For the sake of completeness we include an alternative proof in the “Appendix”. Using this fact we get

Proposition 5.1. Suppose that $M$ is a connected closed manifold of dimension 6. Consider two orientable 6-dimensional real vector bundles $\xi$ and $\xi'$ with $w_2(\xi) = w_2(\xi') = \rho_2(l)$ for some $l \in H^2(M; \mathbb{Z})$. Then $\xi$ and $\xi'$ are isomorphic if and only if

$$q_1(\xi; l) = q_1(\xi'; l), \quad \text{and} \quad e(\xi) = \pm e(\xi').$$

Further we need the following analogue of Proposition 3.5.

Proposition 5.2. Suppose that $M$ is a connected closed spin$^c$ 6-manifold with $w_2(M) = \rho_2(l)$ for some $l \in H^2(M; \mathbb{Z})$. Let $u \in H^4(M; \mathbb{Z})$ be arbitrary. Then there is a 2-dimensional complex bundle $\eta$ such that

$$c_1(\eta) = l, \quad c_2(\eta) = u.$$

If $\lambda$ is a complex line bundle with $c_1(\lambda) = l$, then $\eta$ is an $\mathbb{H}_\lambda$-line bundle with $e(\eta) = u$. 

Proof. Let \( i : M \to M \times S^1 \) be the inclusion of \( M \times \{ \ast \} \) and \( p : M \times S^1 \to M \) the projection. According to Proposition 3.5 there is a 2-dimensional complex bundle \( \eta' \) over \( M \times S^1 \) with \( c_1(\eta') = p^*(l) \) and \( c_2(\eta') = p^*(u) \). Then \( \eta = i^*(\eta') \) over \( M \) has the prescribed properties. \( \square \)

Now we can follow the same lines as in Sect. 4 to obtain results on the existence of \( G \)-structures on vector bundles over 6-manifolds.

Consider a real 6-dimensional representation \( V \) of \( U(2) \). As in the previous section the first Pontrjagin class of the associated vector bundle over \( BU(2) \) is determined by integers \( a(V) \), \( b(V) \) and the Euler class is given by integers \( c(V) \), \( d(V) \) where

\[
e(V) = c(V)lu + d(V)l^3.
\]

The list of these representations is as follows:

| Representation \( V \) | \( a(V) \) | \( b(V) \) | \( c(V) \) | \( d(V) \) |
|------------------------|---------|---------|---------|---------|
| \( L^{\otimes s} \otimes L^{\otimes s} \otimes L^{\otimes t} \) | 0       | \( r^2 + s^2 + t^2 \) | 0       | \( rst \) |
| \( L^{\otimes s} \otimes \mathcal{E} \otimes L^{\otimes t} \) | 1       | \( s^2 + (s + 1)^2 + t^2 \) | \( t \)  | \( st(s + 1) \) |
| \( A_1 \oplus L^{\otimes s} \otimes \mathbb{R} \) | 2       | \( 1 + s^2 \) | 0       | 0       |
| \( A_1 \otimes \mathbb{R} L^{\otimes s} \) | 4       | \( 2 + 3s^2 \) | \( 4s \) | \( (s^2 - 1)s \) |
| \( A_2 \oplus \mathbb{R} \) | 10      | 5       | 0       | 0       |

**Theorem 5.3.** Let \( \xi \) be a 6-dimensional spin\( ^c \) vector bundle over a closed, connected, smooth, spin\( ^c \) 6-manifold \( M \) with \( w_2(M) = w_2(\xi) \). Let \( V \) be a 6-dimensional real representation of \( U(2) \), and let \( l \) be a class in \( H^2(M; \mathbb{Z}) \).

1. If \( b(V) \) is odd, then \( \xi \) is isomorphic to \( P \times_{U(2)} V \) for some principal \( U(2) \)-bundle \( P \) over \( M \) with \( c_1 = l \) if and only if \( \rho_2(l) = w_2(\xi) (= w_2(M)) \) and there is a class \( u \in H^4(M; \mathbb{Z}) \) such that

\[
q_1(\xi; l) = \frac{1}{2}(b(V) - 1)l^2 - a(V)u \quad \text{and} \quad \pm e(\xi) = c(V)lu + d(V)l^3.
\]

2. If \( b(V) \) is even and \( \rho_2(l) = w_2(\xi) = w_2(M) \), then \( \xi \) is isomorphic to \( P \times_{U(2)} V \) for some principal \( U(2) \)-bundle \( P \) over \( M \) with \( c_1 = l \) if and only if \( \rho_2(l) = 0 \) (so that \( \xi \) and \( M \) are spin) and there is a class \( u \in H^4(M; \mathbb{Z}) \) such that

\[
q_1(\xi) = \frac{1}{2}b(V)l^2 - a(V)u \quad \text{and} \quad \pm e(\xi) = c(V)lu + d(V)l^3.
\]

In special cases we obtain

**Corollary 5.4.** Suppose that \( M \) is a closed, connected, smooth, spin\( ^c \) manifold of dimension 6. Let \( \xi \) be a 6-dimensional spin\( ^c \) real vector bundle over \( M \) with \( w_2(\xi) = w_2(M) \).
There is always a 3-dimensional complex vector bundle $\mu$ over $M$ such that $\xi \cong \mu$ as real vector bundles. Moreover, $\mu$ can be chosen in such a way that $c_1(\mu) = 1$, where $\rho_2(l) = w_2(\xi)$, $c_2(\mu) = -q_1(\xi; l)$ and $c_3(\mu) = e(\xi)$.

2. There is a 2-dimensional complex vector bundle $\eta$ such that $\xi \cong \eta \oplus \mathbb{R}^2$ as real vector bundles if and only if $e(\xi) = 0$. Moreover, $\eta$ can be chosen in such a way that $c_1(\eta) = l$, where $\rho_2(l) = w_2(\xi)$, and $c_2(\eta) = -q_1(\xi; l)$.

3. There is a 3-dimensional real oriented vector bundle $\alpha$ such that $\xi \cong \alpha \oplus \mathbb{R}^3$ if and only if $e(\xi) = 0$ and $w_4(\xi) = 0$.

4. There is a 3-dimensional real spin vector bundle $\alpha$ such that $\xi \cong \alpha \oplus \alpha$ if and only if $w_2(\xi) = 0$, $e(\xi) = 0$ and $q_1(\xi) \in 4H^4(M; \mathbb{Z})$.

5. There is a 3-dimensional real vector bundle $\alpha$ such that $\xi \cong S^2\alpha$ (the structure group $SO(6)$ of $\xi$ reduces to $SO(3)$ via the representation $A_2 \oplus \mathbb{R}$) if and only if $e(\xi) = 0$, $w_4(\xi) = 0$ and $p_1(\xi) \in 5H^4(M; \mathbb{Z})$.

6. There is a complex line bundle $\lambda$ with $c_1(\lambda) = l$, where $\rho_2(l) = w_2(\xi)$, such that $\xi \cong \lambda \oplus \mathbb{R}^2$ as real vector bundles if and only if $e(\xi) = 0$ and $q_1(\xi; l) = 0$.

7. $\xi$ is a trivial bundle if and only if $w_2(\xi) = 0$, $q_1(\xi) = 0$ and $e(\xi) = 0$.

6. Obstruction theory on $(4k + 3)$-dimensional manifolds

In this section we prove Theorem 3.1 using the $K$-theoretic methods introduced by Atiyah and Dupont. A proof in the case that $M$ is simply connected is given in [7, Theorem 0.4], and a proof for the case that $\xi$ (or $\xi' \oplus \mathbb{R}$ or $\xi'' \oplus \mathbb{R}^2$) is the tangent bundle of $M$ is contained in [3,8].

We use $\mathbb{Z}/2$-equivariant $K$-theory and write $L$ in this section for the $\mathbb{Z}/2$-module $\mathbb{R}$ with the action $\pm 1$. The reduced real $K$-theory of the Thom space of a virtual vector bundle $\alpha$ over a compact Hausdorff space $X$ will be written as $KO^\ast(X; \alpha)$.

We may assume that $M$ is connected and write its dimension as $m = 4k + 3$; the tangent bundle of $M$ is denoted by $\tau M$. The proof hinges on the vanishing of the equivariant $K$-groups $KO_{\mathbb{Z}/2}(\ast; -nL)$ for any $n \in \mathbb{Z}$, see table 3.1 in [6]. This paper is probably the best source for a discussion of the $\mathbb{Z}/2$-equivariant $KO$-obstruction theory and the usage of the $KO_{\mathbb{Z}/2}$-Euler class.

Proof of (i). Since $e(\xi) = 0$, the sphere bundle $S(\xi)$ has a section and so $\xi$ splits as $\xi' \oplus \mathbb{R}$ for some oriented vector bundle $\xi'$ of dimension $4k + 2 = m - 1$. We show first that $w_{m-1}(\xi') = 0$ and do this by proving that $(x \cdot w_{m-1}(\xi'))[M] = 0$ for any class $x \in H^1(M; \mathbb{Z}/2)$. Let $\nu$ be a smooth real line bundle over $M$ with $w_1(\nu) = x$. Choose a generic smooth section of $\nu$ with zero-set a submanifold $N \subseteq M$ with normal bundle $\nu|N$. We have to show that $w_{m-1}(\xi' | N)[N] = 0$. Now there is a section $s'$ of $\xi' | N$ with only finitely many zeros (or, indeed, with at most one zero in each component of $N$). To each zero of $s'$ we can assign an index in $\mathbb{Z}/2$ as a degree (mod 2), and the sum of these local indices is equal, by a cohomology Hopf theorem, to $w_{m-1}(\xi' | N)[N]$. To show that the sum is zero, we shall repeat the computation using $KO$-theory.
Consider the commutative diagram

\[
\begin{align*}
KO^0_{\mathbb{Z}/2}(N; -L \otimes \xi') & \rightarrow KO^0(N; -\xi') \\
\cdot \eta(v) \downarrow & \downarrow \cdot \eta(v) \\
KO^0_{\mathbb{Z}/2}(N; v - L \otimes \xi') & \rightarrow KO^0(N; v - \xi') \\
\simeq \downarrow & \downarrow \simeq \\
KO^{-m}_{\mathbb{Z}/2}(N; -(m-1)L - \tau N) & \rightarrow KO^{-m}(N; -(m-1)\mathbb{R} - \tau N) \\
\pi_1 \downarrow & \downarrow \pi_1 \\
KO^{-m}(\ast; -(m-1)L) = 0 & \rightarrow \mathbb{Z}/2 = KO^{-m}(\ast; -(m-1)\mathbb{R}),
\end{align*}
\]

in which the Hopf element \(\eta(v) \in KO^0(N; v)\) is constructed from a universal class \(\eta(L)\) generating \(KO^0_{\mathbb{Z}/2}(\ast; L) \simeq \mathbb{Z}\) and restricting to the Hopf generator \(\eta\) of \(KO^0(\ast; \mathbb{R}) = (\mathbb{Z}/2)\eta\), the vertical isomorphisms are given by Bott isomorphisms determined by the orientation of \(\xi\) and a choice of spin structure for \(\xi - \tau M\) and so for \((\xi' - v)|N - \tau N\), and \(\pi_1\) is the Umkehr homomorphism for the projection \(\pi : N \rightarrow \ast\) to a point. The horizontal maps restrict from the \(\mathbb{Z}/2\)-equivariant to the non-equivariant theory. The \(K\)-theory Euler class \(\gamma(\xi' | N) \in KO^0(N; -\xi')\) maps to the sum of the local indices in \(\mathbb{Z}/2\), because the relative Euler class gives an isomorphism

\[
\pi_{m-2}(S^{m-2}) = \mathbb{Z} \rightarrow KO^0(D^{m-1}, S^{m-2}; -\mathbb{R}^{m-1}) = \mathbb{Z}
\]

and multiplication by \(\eta\)

\[
KO^0(D^{m-1}, S^{m-2}; -\mathbb{R}^{m-1}) = \mathbb{Z} \rightarrow KO^{-1}(D^{m-1}, S^{m-2}; -\mathbb{R}^{m-1}) = \mathbb{Z}/2
\]

is reduction (mod 2). Since \(\gamma(\xi' | N)\) lifts in the commutative diagram to the equivariant Euler class \(\gamma(L \otimes \xi' | N) \in KO^0_{\mathbb{Z}/2}(N; -L \otimes \xi')\), the sum of the obstructions in \(\mathbb{Z}/2\) must be zero, and we have proved that \(w_{m-1}(\xi') = 0\).

Now if \(s_0\) and \(s_1\) are two sections of \(S(\xi)\) with complementary \((m - 1)\)dimensional bundles \(\xi_0'\) and \(\xi_1'\), there is an associated difference class \(d(s_0, s_1) \in H^{m-1}(M; \mathbb{Z})\) with the property, because \(m - 1\) is even, that \(e(\xi_1') - e(\xi_0') = 2d(s_0, s_1)\). Moreover, because the Hurewicz homomorphism \(\pi_j(S^{m-1}) \rightarrow H_j(S^{m-1}; \mathbb{Z})\) is an isomorphism for \(j < m\) and an epimorphism for \(j = m\), for every section \(s_0\) and class \(y \in H^{m-1}(M; \mathbb{Z})\) there is a section \(s_1\) such that \(d(s_0, s_1) = y\).

We conclude from the vanishing of \(w_{m-1}(\xi)\) that, for each \(y \in H^{m-1}(M; \mathbb{Z})\), there is a section \(s\) of \(S(\xi)\) such that \(e(\xi') = 2y\).

\(\square\)

Remark 6.1. When \(k = 0\), \(\xi'\) is a complex line bundle with \(c_1(\xi') = 0\) and so is trivial. It follows that the bundle \(\xi\) is trivial. For the special case \(\xi = \tau M\) we have reproved the classical result that an oriented 3-manifold is parallelizable. In (ii) and (iii), the bundles \(\xi'\) and \(\xi''\) are clearly trivial if \(k = 0\).
Proof of (ii). The argument that follows is a reformulation of the method used in [7, Theorem 0.4].

Consider the diagram
\[
\begin{array}{ccc}
K^0_{\mathbb{Z}/2}(M; -L \otimes \xi') & \rightarrow & K^0(M; -\xi') \\
\simeq \downarrow & & \downarrow \simeq \\
K^{-m}_{\mathbb{Z}/2}(M; -(m-1)L - \tau M) & \rightarrow & K^{-m}(M; -(m-1)\mathbb{R} - \tau M) \\
\pi_1 \downarrow & & \downarrow \pi_1 \\
K^{-m}_{\mathbb{Z}/2}(\ast; -(m-1)L) = 0 & \rightarrow & \mathbb{Z}/2 = K^{-m}(\ast; -(m-1)\mathbb{R}),
\end{array}
\]
in which the vertical Bott isomorphisms are again determined by the orientation of \(\xi'\) and a choice of spin structure for \(\xi' - \tau M\) and \(\pi_1\) is the Umkehr for the map \(\pi : M \rightarrow \ast\).

The vanishing of \(e(\xi')\) guarantees that there is a section \(s\) of \(S(\xi')\) over the complement of the interior of an embedded disc \(D^m \subseteq M\). The obstruction to extending \(s\) over \(M\) lies in \(\pi_{m-1}(S^{m-2}) = (\mathbb{Z}/2)\eta\) if \(k > 0\). (If \(k = 0\), the obstruction group is zero.) It is detected by the \(K\)-theory Euler class \(\gamma(\xi') \in K^0(M; -\xi')\), because the relative Euler class gives an isomorphism
\[
\pi_{m-1}(S^{m-2}) \rightarrow K^0(D^m, S^{m-1}; -(m-1)\mathbb{R}) = \mathbb{Z}/2.
\]
Since \(\gamma(\xi')\) lifts in the commutative diagram above to the \(\mathbb{Z}/2\)-equivariant Euler class \(\gamma(L \otimes \xi')\), the obstruction vanishes and \(s\) extends to a section of \(S(\xi')\) over \(M\). \(\square\)

Proof of (iii). We may assume that \(k > 0\). Choose a triangulation of the manifold \(M\). The vanishing of \(e(\xi'')\) guarantees that there is a section \(s\) of \(S(\xi'')\) over the \((m - 2)\)-skeleton \(M^{(m-2)}\). Since \(\pi_{m-2}(S^{m-3}) = (\mathbb{Z}/2)\eta\), there is an associated obstruction class \(\sigma(s) \in H^{m-1}(M; \mathbb{Z}/2)\) to extending \(s\) over \(M\); if \(\sigma(s) = 0\), there is a section of \(S(\xi'')\) over \(M^{(m-1)}\) (coinciding with \(s\) on \(M^{(m-3)}\)).

Following the method used to establish (i), we shall show that \(\sigma(s) = 0\). Given \(x \in H^1(M; \mathbb{Z}/2)\) we choose, as in (i), a line bundle \(\nu\) and submanifold \(N\). If we fix a triangulation of \(N\), the inclusion of \(N\) in \(M\) is homotopic to a cellular map \(f : N \rightarrow M\) and \(f^*s\) on \(N^{(m-2)}\) extends to a section of \(f^*\xi''\) with zeros at the barycentres of the \((m-1)\)-cells. Then \((x \cdot \sigma(s))[M] = \sigma(f^*s)[N].\)

The KO-theory Hopf theorem shows that \(\sigma(f^*s)[N] = 0\). This time the relative Euler class gives an isomorphism
\[
\pi_{m-2}(S^{m-3}) = \mathbb{Z}/2 \rightarrow K^0(D^{m-1}, S^{m-2}; -\mathbb{R}^{m-2}) = \mathbb{Z}/2
\]
and multiplication by \(\eta\) is an isomorphism
\[
K^0(D^{m-1}, S^{m-2}; -\mathbb{R}^{m-2}) = \mathbb{Z}/2 \rightarrow K^{-1}(D^{m-1}, S^{m-2}; -\mathbb{R}^{m-2}) = \mathbb{Z}/2.
\]

The vanishing of \(\sigma(s)\) implies that \(S(\xi'')\) admits a section on the complement of the interior of an embedded disc \(D^m \subseteq M\). One shows finally, by the method
used in (ii) (and [7]), that the obstruction in \( \pi_{m-1}(S^{m-3}) = (\mathbb{Z}/2)\eta^2 \) to extending this section over the disc vanishes, because the local obstruction map

\[
\pi_{m-1}(S^{m-3}) \to KO^0(D^m, S^{m-1}; -\mathbb{R}^{m-2}) = \mathbb{Z}/2
\]
is an isomorphism.

\[\Box\]

**Remark 6.2.** Massey showed in [10] that \( w_{4k}(M) \) is the reduction of an integral class. (We have \( w_{4k}(M) = Sq^{2k}v_{2k}(M) + Sq^{2k-1}v_{2k+1}(M) = v_{2k}(M)^2 + Sq^1(Sq^{2k-2}v_{2k+1}(M)). \) But \( v_{2k}(\tau M)^2 = v_{4k}(\mathbb{C} \otimes \tau M) \) is integral, as is any class in the image of \( Sq^1 \).) It follows that, if \( \xi'' \) is stably equivalent to \( \tau M \), then \( e(\xi'') = \delta^*w_{4k}(M) = 0 \). In particular, as is noted in [8], the manifold \( M \) admits 3 linearly independent vector fields.

### 7. Appendix

In this final section we give an alternative proof of the following statement which is considered to be well known:

**Theorem 7.1.** Let \( M \) be a connected oriented closed manifold of even dimension \( m \) and let \( \xi \) and \( \xi' \) be two oriented real vector bundles over \( M \) of dimension \( m \) which are stably isomorphic. Then they are isomorphic if and only if

\[
e(\xi) = \pm e(\xi').
\]

For the proof we need some preparatory considerations. Let \( \zeta \) be an oriented real vector bundle of dimension \( n+1 \) over a finite complex \( X \). Given two sections \( s_0, s_1 \) of the sphere bundle \( S(\zeta) \), the difference class \( d(s_0, s_1) \in H^n(X; \mathbb{Z}) \) is defined as follows. Choose a homotopy \( \tilde{s}_t \), \( 0 \leq t \leq 1 \), from \( s_0 \) to \( s_1 \) through sections of the disc bundle \( D(\zeta) \). Regarding \( \tilde{s} \) as a map \( ([0, 1], [0, 1]) \times X \to (D(\zeta), S(\zeta)) \), we define the difference class \( d(s_0, s_1) \) to be \( \tilde{s}^*(u) \in H^n(X; \mathbb{Z}) \cong H^{n+1}(([0, 1], [0, 1]) \times X; \mathbb{Z}) \), the pullback of the Thom class \( u \in H^{n+1}(D(\zeta), S(\zeta); \mathbb{Z}) \).

Clearly, \( d(s_0, s_1) = 0 \) if \( s_0 \) and \( s_1 \) are homotopic through sections of \( S(\zeta) \). By elementary obstruction theory, the converse is true if \( \dim X \leq n \).

Now, if \( s_2 \) is a third section, it is elementary to check that \( d(s_0, s_2) = d(s_0, s_1) + d(s_1, s_2) \). So \( d(s_0, s_0) = 0 \) and \( d(s_1, s_0) = -d(s_0, s_1) \).

For a section \( s \) of \( S(\zeta) \), let \( \zeta(s) \) be the complementary \( n \)-dimensional vector bundle; it is oriented by the orientation of \( \zeta \). It follows from the definition of the Euler class of \( \zeta(s) \) as the pullback of the Thom class in \( H^n(D(\zeta(s)), S(\zeta(s)); \mathbb{Z}) \) by the zero-section \( (X, \emptyset) \to (D(\zeta(s)), S(\zeta(s))) \) that \( e(\zeta(s)) = d(-s, s) \in H^n(X; \mathbb{Z}) \) (with the correct choice of sign convention).

Now \( d(-s_0, s_0) + d(s_0, s_1) + d(s_1, -s_1) + d(-s_1, -s_0) = d(-s_0, -s_0) = 0 \).

From the definitions, we have \( d(s_1, -s_1) = -d(-s_1, s_1) \) and \( d(-s_0, -s_1) = (-1)^{n+1}d(s_0, s_1) \). Hence \( e(\zeta(s_0)) + d(s_0, s_1) = e(\zeta(s_1)) + (-1)^nd(s_0, s_1) = 0 \), that is,

\[
e(\zeta(s_1)) - e(\zeta(s_0)) = (1 + (-1)^n)d(s_0, s_1).
\]
Proof of Theorem 7.1. Put \( \zeta = \xi \oplus \mathbb{R} \cong \xi' \oplus \mathbb{R} \). The bundle \( \xi \) can be considered as the complementary bundle \( \zeta(s_0) \) of a section \( s_0 \) of the sphere bundle \( S(\zeta) \) and \( \xi' \) can be considered up to orientation as the complementary bundle \( \zeta(s_1) \) of another section \( s_1 \). Hence

\[
e(\xi) = e(\zeta(s_0)) \quad \text{and} \quad e(\xi') = \pm e(\zeta(s_1)).
\]

Using the formula derived above we get that \( e(\xi') \pm e(\xi') = e(\zeta(s_0)) - e(\zeta(s_1)) = 0 \) if and only if \( d(s_0, s_1) = 0 \). This means that \( s_0 \) and \( s_1 \) are homotopic through sections of \( S(\zeta) \). Such a homotopy determines, up to homotopy, an isomorphism between \( \zeta(s_0) \) and \( \zeta(s_1) \) as oriented vector bundles. So the vanishing of \( d(s_0, s_1) \) implies that \( \xi \) and \( \xi' \) are isomorphic as vector bundles. \( \square \)

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