SOME VIRTUALLY POLY-FREE ARTIN GROUPS

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ABSTRACT. In this short note we prove that a class of Artin groups of affine and complex types are virtually poly-free, answering partially the question if all Artin groups are virtually poly-free.

We recall the following definition.

Definition 1. Let $\mathcal{F}$ and $\forall\mathcal{F}$ denote the class of free groups and virtually-free groups, respectively. Let $\mathcal{C}$ be either $\mathcal{F}$ or $\forall\mathcal{F}$. A group $G$ is called virtually poly-$\mathcal{C}$, if $G$ contains a finite index subgroup $H$, and $H$ admits a normal series $1 = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_n = H$, such that $H_i+1/H_i \in \mathcal{C}$, for $i = 0, 1, \ldots, n - 1$. In this case, $H$ is called poly-$\mathcal{C}$ or that $H$ has a poly-$\mathcal{C}$ structure. The minimum such $n$ is called the length of the poly-$\mathcal{C}$ structure.

Poly-$\mathcal{F}$ groups have nice properties like, locally indicable and right orderable. In [2, Question 2] it was asked if all Artin groups are virtually poly-$\mathcal{F}$. Among the finite type Artin groups, the groups of types $\tilde{A}_n$, $B_n = C_n$, $D_n$, $F_4$, $G_2$ and $I_2(p)$ are already known to be virtually poly-$\mathcal{F}$ [3].

Here, we extend this class and prove the following theorem.

Theorem 2. Let $A$ be an Artin group of the affine type $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$ or of the finite complex type $G(de, e, r)$ ($d, r \geq 2$). Then, $A$ is virtually poly-$\mathcal{F}$.

The main idea behind the proof of Theorem [2] is the following result, which is easily deducible from [3, Theorem 2.2 and Remark 2.4].

Let $\mathbb{C}(m, k; q)$ be the orbifold, whose underlying space is the complex plane minus $m$ points $p_1, p_2, \ldots, p_m \in \mathbb{C}$, with $k$ cone points $x_1, x_2, \ldots, x_k \in \mathbb{C} - \{p_1, p_2, \ldots, p_m\}$ of orders $q_1, q_2, \ldots, q_k$, respectively. $q$ denotes the $k$-tuple $(q_1, q_2, \ldots, q_k)$. Let $PB_n(\mathbb{C}(m, k; q))$ be the configuration orbifold of $n$ distinct points of $\mathbb{C}(m, k; q)$. By convention $PB_1(\mathbb{C}(m, k; q)) = \mathbb{C}(m, k; q)$.

Theorem 3. The orbifold fundamental group $\pi_1^{orb}(PB_n(\mathbb{C}(m, k; q)))$ has a poly-$\forall\mathcal{F}$ structure, consisting of finitely presented subgroups in a normal series.

Proof. Recall that, in [3, Theorem 2.2 and Remark 2.4] we proved the following exact sequence. The second homomorphism is induced by the projection to the first $n - 1$ coordinates.

$$1 \longrightarrow K \longrightarrow \pi_1^{orb}(PB_n(S)) \longrightarrow \pi_1^{orb}(PB_{n-1}(S)) \longrightarrow 1.$$ 

Here, $S = \mathbb{C}(k, m; q)$, and $K$ is isomorphic to $\pi_1^{orb}(F)$, $F = S - \{(n - 1) - \text{regular points}\}$. By regular points we mean points which are not cone points, that is, points in $\mathbb{C} - \{x_1, \ldots, x_m, p_1, \ldots, p_k\}$. That is, $F = \mathbb{C}(k, m + n - 1; q)$. 

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The theorem now follows by induction on $n$, since $\pi_{1}^{\text{orb}}(C(k, m; q))$ is finitely presented and virtually free, for all $k$, $m$ and $q$. □

Note that the symmetric group $S_n$ acts on $PB_n(C(m, k; q))$ by permuting the coordinates. Hence, the quotient $PB_n(C(m, k; q))/S_n$ is again an orbifold, and it is denoted by $B_n(C(m, k; q))$.

Therefore, by Theorem 3 we have the following corollary, since if a group has a finite index normal poly-$VF$ subgroup, then the group is also poly-$VF$.

**Corollary 4.** The orbifold fundamental group $\pi_{1}^{\text{orb}}(B_n(C(m, k; q)))$ is finitely presented, and has a poly-$VF$ structure, consisting of finitely presented subgroups in a normal series.

To prove our main theorem, furthermore, we need the following two results.

**Theorem 5.** All affine type Artin groups are torsion free.

*Proof.* This was recently proved in [5]. □

**Theorem 6.** (Π) Let $\mathcal{A}$ be an Artin group, and $\mathcal{O}$ be an orbifold as described in the following table. Then, $\mathcal{A}$ can be embedded as a normal subgroup in $\pi_{1}^{\text{orb}}(B_n(\mathcal{O}))$. The third column gives the quotient group $\pi_{1}^{\text{orb}}(B_n(\mathcal{O}))/\mathcal{A}$.

| Artin group of type | Orbifold $\mathcal{O}$ | Quotient group | $n$ |
|--------------------|------------------------|----------------|-----|
| $B_n$              | $C(1,0)$               | $<1>$          | $n > 1$ |
| $\tilde{A}_{n-1}$  | $C(1,0)$               | $\mathbb{Z}$   | $n > 2$ |
| $\tilde{B}_n$      | $C(1,1; (2))$          | $\mathbb{Z}/2$| $n > 2$ |
| $\tilde{C}_n$      | $C(2,0)$               | $<1>$          | $n > 1$ |
| $\tilde{D}_n$      | $C(0,2; (2,2))$        | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $n > 2$ |

*Table*

*Proof.* See [1]. □

We also need the following.

**Proposition 7.** If a torsion free, finitely presented poly-$VF$ group has a normal series with finitely presented subgroups, then the group is virtually poly-$VF$.

*Proof.* Let $H$ be a poly-$VF$ group of length $n$ satisfying the hypothesis of the statement. The proof is by induction on $n$. If $n = 1$, then $H$ is virtually free, and hence free, since it is torsion free ([Π]). Therefore, assume that the lemma is true for all poly-$VF$ groups of length $\leq n - 1$ satisfying the hypothesis. Consider a finitely presented normal series for $H$ giving the poly-$VF$ structure. Then, $H_{n-1}$ is finitely presented, torsion free and has a poly-$VF$ structure of length $n - 1$. Hence, by the induction hypothesis, there is a finite index subgroup $K \leq H_{n-1}$ and $K$ is poly-$F$. Since $H_{n-1}$ is finitely presented, we can find a finite index subgroup $K'$ of $K$ which is also a characteristic subgroup of $H_{n-1}$. Hence, $K'$ is also a poly-$F$ group, and is a normal subgroup of $H = H_{n}$ with quotient virtually free. Let $q : H \to H/K'$ be
the quotient map. Consider a free subgroup $L$ of $H/K'$ of finite index, then $q^{-1}(L)$ is a finite index poly-$\mathcal{F}$ subgroup of $H$.

This proves the Proposition. $\square$

Now, we are ready to prove our main theorem.

Proof of Theorem 2. From the Table of Theorem 6, we see that the Artin group of type $\tilde{A}_n$ is a subgroup of the finite type Artin group of type $B_{n+1}$. Hence, by $[3]$, the Artin group of type $\tilde{A}_n$ is virtually poly-$\mathcal{F}$.

Now, let $\mathcal{A}$ be an Artin group of type $\tilde{B}_n$, $\tilde{C}_n$ or $\tilde{D}_n$. Then, by Theorem 1, $\mathcal{A}$ can be embedded as a normal subgroup in $\pi_1^{\text{orb}}(B_n(C(k, m; q)))$ of finite index, for some suitable $k, m, q$ and $n$. Next, note that by Corollary 1, $\pi_1^{\text{orb}}(B_n(C(k, m; q)))$ is poly-$\mathcal{VF}$ by a normal series consisting of finitely presented subgroups. Since $\mathcal{A}$ is finitely presented and of finite index in $\pi_1^{\text{orb}}(B_n(C(k, m; q)))$, it follows that $\mathcal{A}$ is also poly-$\mathcal{VF}$ by a normal series consisting of finitely presented subgroups. But by Theorem 5, $\mathcal{A}$ is also torsion free. Hence, by Proposition 7, $\mathcal{A}$ is virtually poly-$\mathcal{F}$.

The $G(\text{de}, e, r)$ $(d, r \geq 2)$ type case is easily deduced from the fact that, this Artin group can be embedded as a subgroup in the finite type Artin group of type $B_r$. See $[4]$, Proposition 4.1.

Therefore, we have completed the proof of Theorem 2. $\square$
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