A semi-group of Sturm-Liouville operators having the same spectrum

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Abstract. Given a function $p$ that is positive on $(0, 1)$, and, together with its reciprocal, Lebesgue integrable on this interval, we define its spectrum to be the set of eigenvalues of the Sturm-Liouville problem

$$(py')' + \lambda py = 0, \ y(0) = y(1) = 0,$$

and prove that its image under any one of the family of operators $T_\alpha$ defined by

$$(T_\alpha p)(x) = p(x) \left(1 + \alpha \int_0^x \frac{1}{p(t)} \, dt\right)^2, \ \alpha \geq 0,$$

has the same spectrum. We also discuss the corresponding inverse problem, and, in particular, prescribe a sufficient condition on $p$ that allows us to identify it as a multiple of $T_\alpha 1$, for some $\alpha$, if $\pi^2$ is a member of its spectrum.

1. Introduction

Consider the motion of a polytropic gas contained in an axisymmetric tube of finite length, closed at both ends, when the tube is oscillated parallel to its axis. How does the shape of the tube affect the resulting motion of the gas? In particular, in what tubes are shocks avoidable?

At the Muphys Conference in 2004, Michael Mortell summarised the contents of his joint paper with Brian Seymour [10], in which, inter alia, these questions are examined when the tube is that obtained by rotating the graph of a function $r : [0, 1] \rightarrow (0, \infty)$ about the horizontal axis, and the tube is oscillated at or near a resonant frequency. If $r$ is a non-zero constant, the tube is a right-circular cylinder, and shocks are known to occur in the corresponding output. In their paper, loc. cit., Mortell and Seymour exhibit three families of specific examples of functions $r$ that generate tubes with variable cross-section, in which shocks do not occur, thereby confirming analytically the experimental results that had been obtained previously for such families of tubes by others [8]; for details, the interested reader should consult [10].

The presence or absence of shocks in the motion of the gas is bound up with the geometry of the tube and a certain arithmetical property of the numbers $z$ for which the boundary-value problem

$$\frac{d}{dx} \left(\frac{1}{r^2(x)} \frac{d}{dx} y(x)\right) + \frac{zy(x)}{r^2(x)} = 0, \ 0 < x < 1, \ y(0) = y(1) = 0,$$

is satisfied.
has a non-trivial solution. The latter problem is a special case of a well-understood eigenvalue problem that is ubiquitous in Mathematical Physics ([3],[4],[15]): its study was initiated independently by Sturm and Liouville in a series of papers published in the period 1836–1837.

It is instructive to consider this when \( r = 1 \), in which case the gas is contained in a right-circular cylindrical tube. As is well-known in this case, the set of permissible values of \( z \)—the so-called eigenvalues of the problem—are equal to \( \{n^2\pi^2 : n = 1, 2, \ldots \} \). In the present context, the essential arithmetical feature to observe about these eigenvalues is that they are all integer multiples of the smallest one. It is this property that causes shocks to occur in the motion of the gas in a cylindrical tube, and is one of the reasons, but seemingly not the only one, why shocks may occur in similar closed cavities ([2],[7],[11]). Thus, in particular, axisymmetric tubes with variable cross-sectional area fall into two classes, one class which consists of tubes, which we call admissible tubes, whose geometry prevents shocks from being created, and its complement, which consists of inadmissible tubes whose geometry allows the creation of shocks. The specific tubes—conical, horn and bulb shaped resonators—considered in [10] are ones in which the non-linear interaction between the geometry and the gas leads to a shockless resonant output; these are, therefore, cogent examples of admissible tubes. On the other hand, among the inadmissible tubes are those (such as right-circular cylinders) for which the eigenvalues of the corresponding Sturm-Liouville problem described above are multiples of the smallest one.

This division of axisymmetric tubes into two classes raises a number of questions some of which are addressed here: for instance, can one characterise the admissible tubes intrinsically in terms of the function \( r \)? can one completely describe the inadmissible tubes which generate eigenvalues that are commensurate? are there any inadmissible tubes apart from right-circular cylindrical ones? The third question was raised by a member of the audience at Mortell’s talk in 2004, and sparked my interest in these issues; we answer it affirmatively. The first and second questions are much deeper, and we deal only with the second, though our treatment of it is incomplete. It lies on the periphery of a much more general question called “the inverse problem for an operator”: knowing its spectrum, can one recover the operator? This has been extensively studied for a variety of operators. The study of such problems for Sturm-Liouville operators was seemingly initiated by Ambartsyunyan in 1929 when he obtained the following result:

Suppose \( q \) is continuous on \([0,1]\) and \( \{n^2\pi^2 : n = 1, 2, \ldots \} \) is the set of eigenvalues of the boundary-value problem

\[
y'' + (\lambda - q)y = 0, \quad y'(0) = y'(1) = 0,
\]

then \( q = 0 \). For this, and much more, the reader should consult [9] and [13].

Our investigation of the second and third questions is governed by the action of an interesting semi-group of non-linear operators \( \{T_\alpha : \alpha \geq 0\} \), which leaves a certain family \( \Gamma \) of positive functions invariant. We show that if \( p \in \Gamma \), then \( p \), and any member in its orbit, \( \{T_\alpha p : \alpha \geq 0\} \), determine tubes with the same sets of eigenvalues. This is the key result in the paper, which is exploited to answer some of the afore-mentioned questions. It was motivated by a desire to obtain an analogue of the familiar fact that similar matrices have the same eigenvalues.

The paper is based on my talk at this year’s Murphys, the slides of which are accessible at the Workshop’s website [5]; it is organised as follows. To make our treatment as self-contained as possible, we begin by recapitulating some standard fare about the existence and uniqueness of solutions of the above boundary-value problem. Nothing essentially new emerges from our treatment of this venerable problem, though it has a few novel features. Thereafter, we define what we mean by the spectrum, \( \sigma(p) \), of a nonnegative function \( p \) which, together with its reciprocal, is Lebesgue integrable on \((0,1)\). Such functions comprise the family \( \Gamma \), previously mentioned. We define the reflection of a function \( f \) by \( \tilde{f}(x) = f(1-x) \), and prove that if \( p \in \Gamma \), then \( \sigma(\tilde{p}) = \sigma(p) \). Next, we introduce a collection of non-linear operators \( \{T_\alpha : \alpha \geq 0\} \) defined...
on $\Gamma$ by

$$(T_\alpha p)(x) = p(x) \left(1 + \alpha \int_0^x \frac{1}{p(t)} \, dt\right)^2, \quad p \in \Gamma, \; x \in (0, 1),$$

and show that it is a semi-group on $\Gamma$. We investigate the relationship between the members of the semi-group and the reflection operator. We also establish that if $p \in \Gamma$, then $p$ and $T_\alpha p$ have the same spectrum, and present examples to illustrate this fact. In particular, it ensues that, for all $\alpha > 0$, $\sigma(T_\alpha 1) = \sigma(1) = \{n^2 \pi^2 : n = 1, 2, \ldots\}$, which means that the oscillations of a gas in a tube other than a right-circular cylinder can also create shocks. It also emerges, for instance, that an exponential function and its reciprocal have the same spectrum. This begs the question: if $p \in \Gamma$ and $\sigma(p) = \sigma(1/p)$, must $p$ be an exponential? Imposing a mild regularity condition on $p$ we provide an affirmative answer to this. In the penultimate section, we show that, for some $p \in \Gamma$, $\pi^2 \in \sigma(p)$, and $p$ satisfies some additional requirements, then $p$ belongs to the orbit of 1. This result goes somewhere towards characterizing the family of admissible tubes. Another natural question also arises: if two distinct functions belonging to $\Gamma$ have the same spectrum, and are not multiples of each other, how are they related? Does one, perhaps, belong to the orbit of the other or its reflection? We are unable to answer this, and pose it as an open question.

2. A general initial-value problem

In this section, we outline the essential features of a particular Sturm-Liouville eigenvalue problem, but it’s convenient to begin by examining the existence and uniqueness of a general first-order matrix differential equation.

In what follows, if $A = [a_{ij}]$ is any matrix, we write $A^*$ for the transpose of its complex conjugate, and $||A||_2$ for the square-root of the sum of the entries in $||a_{ij}||^2$. Thus $||A||_2^2$ is the trace of the matrix product $A^*A$. If $M = [m_{ij}]$ is an $n \times n$ matrix,

$$||M||_2 = \sqrt{\sum_{i,j=1}^n |m_{ij}|^2},$$

whence $M \rightarrow ||M||_2$ is a multiplicative norm on the algebra of $n \times n$ matrices.

In the theorem that follows, all matrices are taken to be of order $n \times n$, for some natural number $n$.

**Theorem 1** Suppose each entry of the matrix-valued function $P$ is Lebesgue integrable on $[0, 1]$. Then there is a unique matrix-valued function $\Phi$ that is absolutely continuous on $[0, 1]$ and such that

$$\Phi' = P\Phi, \; a.e. \; on \; [0, 1], \; and \; \Phi(0) = C,$$

where the prime denotes differentiation with respect to the hidden variable, and $C$ is any preassigned matrix. Moreover,

$$||\Phi(t)||_2 \leq ||C||_2 \exp(\sqrt{n} \int_0^t ||P(s)||_2 \, ds), \; 0 \leq t \leq 1.$$  

Proof. Let $k > 1$. Denote by $E$ the vector space of $n \times n$ matrices $X$, with entries that are continuous on $[0, 1]$, and carry the norm defined by

$$||X|| = \max\{e^{-k \int_0^t ||P(s)||_2 \, ds} ||X(t)||_2 : 0 \leq t \leq 1\}.$$
It’s easy to see that $E$ is a Banach space under this norm. Consider the operator $T$ defined on $E$ by

$$ TX(t) = \int_0^t P(s)X(s)\, ds, \quad 0 \leq t \leq 1, \ X \in E. $$

Clearly, $T$ is linear and maps $E$ to $E$. Also,

$$ \|TX(t)\|_2 \leq \int_0^t \|P(s)X(s)\|_2\, ds \leq \int_0^t \|P(s)\|_2 \|X(s)\|_2\, ds = \int_0^t \|P(s)\|_2 \exp(k \int_0^s \|P(u)\|_2\, du) \left( \exp(-k \int_0^s \|P(u)\|_2\, du) \|X(s)\|_2 \right)\, ds \leq \|X\| \int_0^t \|P(s)\|_2 \exp(k \int_0^s \|P(u)\|_2\, du)\, ds = \frac{1}{k} \|X\| \left( \exp(k \int_0^t \|P(u)\|_2\, du) - 1 \right), $$

whence $\|TX\| \leq \|X\|/k$, for all $X \in E$, and so $T$ is a bounded linear operator on $E$, and its norm doesn’t exceed $k^{-1}$. Since $\|T\| < 1$, it follows that $I - T$ is invertible on $E$, and its inverse is given by the Neumann series

$$ (I - T)^{-1} = \sum_{n=0}^{\infty} T^n. $$

In particular, the equation $X = C + TX$ has a unique solution in $E$ which we denote by $\Phi$:

$$ \Phi = (I - T)^{-1}C = \sum_{n=0}^{\infty} T^n C. $$

In other words,

$$ \Phi(t) = C + \int_0^t P(s)\Phi(s)\, ds, \quad 0 \leq t \leq 1, $$

or equivalently, $\Phi$ satisfies the initial-value problem.

We now use the differential equation to derive a bound on the function $\|\Phi\|_2$. Since $\Phi$ satisfies the differential equation we see that

$$ \frac{d}{dt} \|\Phi(t)\|_2^2 = \frac{d}{dt} \text{trace} \left( \Phi^* (t) \Phi(t) \right) = \text{trace} \left( \Phi^* (t) \Phi'(t) + \Phi'^* (t) \Phi(t) \right) = \text{trace} \left( \Phi^* (t) P(t) \Phi(t) + \Phi^* (t) P'(t) \Phi(t) \right) \leq 2 \sqrt{n} \|\Phi(t)\|_2 \|P(t)\|_2, $$

whence the result follows by integrating this since $\|\Phi(0)\|_2 = \|C\|_2$.

We specialise this result as follows.

**Corollary 1** Assume $p, 1/p \in L(0, 1)$. Then, for every complex number $z$, there is one and only one $4 \times 4$ matrix $X$ such that

$$ X' = \begin{bmatrix} 0 & -1 & 1/p & 0 \\ z & 0 & 0 & 1/p \\ -zp & 0 & 0 & -1 \\ 0 & -zp & z & 0 \end{bmatrix} X, \ a.e \ on \ [0, 1] \ and \ X(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}. \quad (1) $$

Proof. Apply the theorem taking \( n = 4 \), and
\[
P = \begin{bmatrix}
0 & -1 & 1/p & 0 \\
 z & 0 & 0 & 1/p \\
-zp & 0 & 0 & -1 \\
0 & -zp & z & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

The conclusion follows by defining \( X \) to be the second column of the solution \( \Phi \) of the resulting initial-value problem just discussed.

Of course, the solution \( X \) depends on \( z \), and it can be proved that every entry of \( X(t) \) is an entire function of \( z \), but we gloss over this point. Moreover, from the bound on \( ||\Phi(t)||_{2} \), we may conclude that every entry of \( X(1) \) is an entire function of exponential type. We can sharpen the growth order. With this in mind, we build on a technique that Zettl attributes to F. V. Atkinson (see Theorem 2.5.3 of [15]), and proceed as follows. Suppose \( z \neq 0 \), and let \( W \) stand for the \( 4 \times 4 \) diagonal matrix, whose diagonal entries are \( w, 1, w^{-1}, \) in that order, where \( w = \sqrt{|z|} \). Let \( Y = WX \). Then \( Y' = WPW^{-1}Y \equiv QY \), say. Hence,
\[
(||Y||_{2})^2' = Y^*Y' + (Y^*)'Y
= 2\Re\{Y^*Y'\}
= 2\Re\{Y^*QY\}
\leq 2||Q||_{2}||Y||_{2}^2.
\]

Therefore
\[
||Y(t)||_{2} \leq ||Y(0)||_{2} \exp\left(\int_{0}^{t}||Q(x)||_{2} dx\right) = \sqrt{2} \exp\left(\int_{0}^{t}||Q(x)||_{2} dx\right), \quad 0 \leq x \leq 1.
\]

But
\[
Q = WPW^{-1} = \begin{bmatrix}
w & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1/w
\end{bmatrix} \begin{bmatrix}
0 & -1 & 1/p & 0 \\
z & 0 & 0 & 1/p \\
-zp & 0 & 0 & -1 \\
0 & -zp & z & 0
\end{bmatrix} \begin{bmatrix}
1/w & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & w
\end{bmatrix}
= \begin{bmatrix}
0 & -w & w/p & 0 \\
z & 0 & 0 & 1/p \\
-zp & 0 & 0 & -1 \\
0 & -zp/w & z/w & 0
\end{bmatrix} \begin{bmatrix}
1/w & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & w
\end{bmatrix}
= \begin{bmatrix}
0 & -w & w/p & 0 \\
z/w & 0 & 0 & w/p \\
-zp/w & 0 & 0 & -w \\
0 & -zp/w & z/w & 0
\end{bmatrix}.
\]

Hence
\[
||Q||_{2} = \sqrt{2||w||^{2}(1 + |p|^{-2}) + |zw^{-1}|^{2}(1 + |p|^{2})}
= \sqrt{2||z||(2 + |p|^{-2} + |p|^{2})}
= \sqrt{2||z||(|p| + |p|^{-1})}.
\]
and so
\[ ||Y(1)||_2 \leq \exp \left( \sqrt{2|z|} \int_0^1 (|p(x)| + \frac{1}{|p(x)|} \, dx) \right). \]

But
\[ Y(1) = \begin{bmatrix} wa(1) \\ b(1) \\ c(1) \\ w^{-1}d(1) \end{bmatrix}, \]
whence it follows that, as entire functions of \( z \), the entries of \( X(1) \) are at most of order 1/2.

3. A particular Sturm-Liouville eigenvalue problem
Here we consider non-zero solutions of the following differential equation
\[ (py')' + zpy = 0, \quad y(0) = 0, \tag{2} \]
on the interval \((0,1)\), when \( p \) and its reciprocal are Lebesgue integrable on \((0,1)\). In the first place, another application of the Theorem shows that the matrix equation
\[ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1/p \\ -zp & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]
admits of a unique solution vector whose value at the origin is the unit vector \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Clearly, its first component satisfies \( \text{(2)} \), and is not the zero function.

The model example for this problem occurs when \( p = 1 \), and, as is well-known, for every complex number \( z \), the general solution of the equation \( y'' + zy = 0 \) is a linear combination of the functions \( c_z, s_z \) defined by
\[ c_z(x) \equiv c(x, z) = \cos(\sqrt{zx}) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} z^n, \]
and
\[ s_z(x) \equiv s(x, z) = \frac{\sin(\sqrt{zx})}{\sqrt{z}} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} z^n. \]

Note that, for each \( z \), \( c_z, s_z \) are entire functions of order at most 1, and that \( c'_z = -zs_z, s'_z = c_z \). Moreover, for each \( x \neq 0 \), the functions \( z \to c(x, z), z \to s(x, z) \) are entire functions of order 1/2. (Information about entire functions can be found in [14].)

Inspired by the approach taken in [12] to solve partial differential equations with variable coefficients, we adapt this to show that, for every complex number \( z \), there are absolutely continuous functions \( a(x) \equiv a(x, z), b(x) \equiv b(x, z) \) such that \( y = ac_z + bs_z \) satisfies \( \text{(2)} \) a.e. on \((0,1)\). To do so, we apply Corollary 1, letting \( a, b \) equal to the first and second entries in the solution vector \( X \) of \( \text{(1)} \). A computation shows that the resulting \( y \) satisfies \( \text{(2)} \).

Indeed, \( y = \begin{bmatrix} c_z & s_z & 0 & 0 \end{bmatrix} X, \tag{3} \)
and so
\[ y' = \begin{bmatrix} -zs_z & c_z & 0 & 0 \end{bmatrix} X + \begin{bmatrix} c_z & s_z & 0 & 0 \end{bmatrix} X' \]
\[ = \begin{bmatrix} -zs_z & c_z & 0 & 0 \end{bmatrix} + \begin{bmatrix} c_z & s_z & 0 & 0 \end{bmatrix} P \]
\[ X \]
\[
\begin{bmatrix}
-zs & cz & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
-1 & 1/p & 0 & 0
\end{bmatrix} X
\]

so that

\[py' = \begin{bmatrix} 0 & 0 & cz & zs \end{bmatrix} X,\]

whence

\[
(py')' = \left( \begin{bmatrix} 0 & 0 & -zs & cz \end{bmatrix} + \begin{bmatrix} 0 & 0 & cz & zs \end{bmatrix} P \right) X
\]

\[= \begin{bmatrix} 0 & 0 & -zs & cz \end{bmatrix} \begin{bmatrix} -p & -ps & zs & -c \end{bmatrix} X
\]

\[= -zpy.\]

It’s clear, too, that the solutions of (2) are parameterised by the solutions of (1) via (3), and that the presence of the sinusoidal functions in the row vector appearing in (3) highlights the oscillatory nature of the solutions. Of course, since \(a(0) = 0, y(0) = 0\), for all \(z\). But also

\[
\lim_{x \to 0^+} py'(x) = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} X(0) = 1,
\]

so that \(y\) is not the zero function.

To guarantee that \(y(1) = 0\), \(z\) must satisfy the equation

\[a(1)cz(1) + b(1)sz(1) = 0.\]  (4)

Since \(a(1)\) and \(b(1)\) are entire functions of \(z\) of order at most 1/2, the solutions of (2) are likewise entire functions of \(z\) of order at most 1/2. In particular, as a function of \(z\), \(y(1)\) is an entire function of order at most 1/2. In fact, it can be proved that its exact order is equal to 1/2, which means that the solutions of (4), being the zero set of an entire function of order 1/2, form a countable set, whose only accumulation point is at infinity [14].

4. The spectrum of an element of \(\Gamma\)

As we’ve just seen, for any \(p \in \Gamma\), there is a countable number of values of \(z\), depending on \(p\), such that

\[(py')' + zpy = 0, \ y(0) = y(1) = 0,\]  (5)

admits of a non-trivial continuous solution on \([0, 1]\). Henceforth, we refer to this set of values of \(z\) as the spectrum of \(p\), and denote it by \(\sigma(p)\). The members of \(\sigma(p)\) are, of course, the eigenvalues of (5).

**Theorem 2** Let \(p \in \Gamma\). Then \(\sigma(p) \subset (0, \infty)\).

Proof. Suppose \(z \in \sigma(p)\). Then (5) has a non-trivial solution \(y\)—which may however be complex-valued—whence

\[
\int_0^1 \bar{y}(py')' dx + z \int_0^1 p|y|^2 dx = 0.
\]

But, by integration by parts,

\[
\int_0^1 \bar{y}(py')' dx = -\int_0^1 \bar{y}py' dx = -\int_0^1 p|y'|^2,
\]
on using the boundary conditions, in which case
\[ z \int_0^1 p|y|^2 \, dx = \int_0^1 p|y'|^2 \, dx. \]

Thus, \( z \) is real and nonnegative. If \( z = 0 \), then \( y' = 0 \), whence \( y \) is the zero function, which is false. Moreover, \( p \) is not the zero function. Hence, \( z > 0 \).

Now that we know that the elements in the spectrum of any function \( p \in \Gamma \) are positive, we revisit the equation (4) which they satisfy. Setting \( z = \lambda^2 \), we conclude that \( \lambda \) is a non-zero solution of the equation
\[ \tan \lambda = -\frac{\lambda a(1, \lambda^2)}{b(1, \lambda^2)}, \]
bearing in mind that \( a(1, z), b(1, z) \) are entire functions.

We'll close our summary of these basic facts about Sturm-Liouville equation by noting that if we denote the elements in \( \sigma(p) \) as the terms of an unbounded increasing positive sequence \( z_n(p), n = 1, 2, \ldots \) then
\[ \lim_{n \to \infty} z_n(p) = 1, \quad \forall p \in \Gamma. \]
In other words, surprisingly perhaps, the eigenvalues associated with all the functions in \( \Gamma \) exhibit the same asymptotic behaviour. This statement can be located in any of ([3], [4], [15]).

5. The reflection of a function

Given a measurable function \( f \) on \((0, 1)\), we define its reflection, \( \tilde{f} \), about the midpoint of this interval by
\[ \tilde{f}(x) = f(1 - x). \]

If \( S \) is a set of functions, we write \( \tilde{S} \) for \( \{ \tilde{f} : f \in S \} \). We note the following properties of this operator.

Theorem 3 The reflection operator has the following properties.

(a) It is an involution: \( \tilde{\tilde{f}} = f \).

(b) It is linear and multiplicative:
\[ \tilde{f + cg} = \tilde{f} + \tilde{cg}, \quad (\tilde{fg}) = \tilde{f} \tilde{g}. \]

(c) If \( f \) is differentiable on \((0, 1)\), then \( (\tilde{f})' = -\tilde{f}' \).

(d) It leaves \( \Gamma \) unchanged.

(e) \( \sigma(\tilde{p}) = \sigma(p), \quad \forall p \in \Gamma. \)

Proof. Properties (a), (b), (c) and (d) are easily checked. We proceed to confirm (e). So, let \( p \in \Gamma \), and \( \lambda \in \sigma(p) \). Then there is a function \( y \), that is not identically zero, with \( y(0) = y(1) = 0 \), such that \( (py)' + \lambda py = 0 \). Hence, by repeated application of properties (b) and (c),
\[ 0 = (\tilde{py})' + \lambda \tilde{py} = -((\tilde{p}y)' + \lambda \tilde{p}y) = -(\tilde{p}(\tilde{y})')' + \lambda \tilde{p} \tilde{y} = \tilde{(p(\tilde{y}))}' + \lambda \tilde{p} \tilde{y}, \]
and clearly \( \tilde{y} \) satisfies the boundary conditions. Hence, \( \lambda \in \sigma(\tilde{p}) \), and (e) follows.
6. A semi-group of non-linear operators
We next turn our attention to the family of operators $T_\alpha$ defined on $\Gamma$ by
\[ T_\alpha p(x) = p(x) \left( 1 + \alpha \int_0^x \frac{1}{p(t)} \, dt \right)^2, \quad p \in \Gamma, \quad x \in (0, 1). \]
We call the set of images $\{ T_\alpha p : \alpha \geq 0 \}$ the orbit of $p$.

The properties of this family that are relevant to our discussion are contained in the following theorem.

**Theorem 4** The following statements are true.

(i) For each $\alpha \geq 0$, $T_\alpha : \Gamma \to \Gamma$;
(ii) $T_\beta T_\alpha = T_{\alpha + \beta}$, for all $\alpha, \beta \geq 0$;
(iii) $\sigma(T_\alpha p) = \sigma(p)$, for all $p \in \Gamma$, and for all $\alpha \geq 0$.

Proof. Fix $\alpha \geq 0$, let $p \in \Gamma$ and set
\[ \phi(x) = \left( 1 + \alpha \int_0^x \frac{1}{p(t)} \, dt \right)^2, \]
and $q = T_\alpha p = p\phi$.

Part 1: Clearly, $q > 0$. Being the product of the continuous function $\phi$, and the integrable function $p$, $q$ is integrable on $(0, 1)$. Also,
\[
\int_0^1 \frac{1}{q(x)} \, dx = \int_0^1 \frac{1}{p(x) \left( 1 + \alpha \int_0^x \frac{1}{p(t)} \, dt \right)^2} \, dx \\
= \left[ \int_0^1 \frac{1}{p(t)} \, dt \right]^{-1} \left[ \int_0^1 \frac{1}{\phi(t)} \, dt \right]^{-1} \\
= \int_0^1 \frac{1}{p(t)} \, dt \\
< \infty.
\]
Hence, $1/q$ is also integrable. It follows that $q \in \Gamma$.

Part 2: With the same notation, if $\beta \geq 0$, then, for almost all $x \in (0, 1),
\[
(T_\beta (T_\alpha p))(x) = (T_\beta q)(x) \\
= q(x)(1 + \beta \int_0^x \frac{1}{q(t)} \, dt)^2 \\
= q(x) \left( 1 + \beta \int_0^x \frac{1}{p(t)} \, dt \right)^2 \\
= q(x) \left( 1 + \alpha \int_0^x \frac{1}{p(t)} \, dt \right)^2 \\
= \frac{q(x)}{\phi(x)} \left( 1 + \alpha \int_0^x \frac{1}{p(t)} \, dt \right)^2.
\]
\[ \begin{align*}
  &= p(x) \left( 1 + (\alpha + \beta) \int_0^x \frac{1}{p(t)} \, dt \right)^2 \\
  &= (T_{\alpha+\beta}p)(x).
\end{align*} \]

Thus Property 2 holds, which confirms the semi-group action of the family.

Part 3: Let \( z \in \sigma(p) \). Then there is a non-zero function \( y \), with \( y(0) = y(1) = 0 \), such that \( (py')' + zpy = 0 \), a.e. on \((0, 1)\). Define \( u = y/\phi \). Then

\[ u' = \frac{y'}{\phi} - \frac{\phi' y}{\phi^2} \]
\[ = \frac{y'}{\phi} - \frac{\alpha y}{p\phi^2} \]
\[ = \frac{y'}{\phi} - \frac{\alpha y}{q}, \]

whence

\[ (qu')' = (p\phi y' - \alpha y)' \]
\[ = \phi(py')' + \phi' py' - \alpha y' \]
\[ = -z\phi py \]
\[ = -zqu. \]

Since \( u \) is not the zero function, and vanishes at 0 and 1, this implies that \( z \in \sigma(q) \). Thus \( \sigma(p) \subset \sigma(q) \), and its easy to see that the opposite inclusion holds as well. This establishes the third statement, and verifies that all functions in the orbit of \( p \) have the same spectrum.

**Corollary 2**

\[ \sigma(T_\alpha 1) = \sigma(1) = \{ n^2 \pi^2 : n = 1, 2, \ldots \}, \quad \forall \alpha \geq 0. \]

**Theorem 5** Suppose \( p \in \Gamma \), then, unless \( p = \tilde{p} \), the orbits of \( p \) and \( \tilde{p} \) are disjoint.

**Proof.** Suppose \( q \) is common to the orbits of \( p \) and \( \tilde{p} \). Then there are nonnegative numbers \( \alpha, \beta \) such that \( q = T_\alpha p = T_\beta \tilde{p} \). In other words, if

\[ \psi(x) = \int_0^x \frac{1}{p(t)} \, dt, \quad 0 \leq x \leq 1, \]

then

\[ q = p(1 + \alpha \psi)^2 = \tilde{p}(1 + \beta(\psi(1) - \tilde{\psi})^2, \]

whence

\[ (1 + \alpha \psi)(1 + \alpha \tilde{\psi}) = (1 + \beta(\psi(1) - \tilde{\psi})(1 + \beta(\psi(1) - \psi). \]

Evaluating this identity at \( x = 0 \), we deduce that \( 1 + \alpha \psi(1) = 1 + \beta \psi(1) \), and so \( \alpha = \beta \). Using this, and expanding and simplifying the last displayed equation, we arrive at the identity \( \psi + \tilde{\psi} = \psi(1) \). Equivalently, for all \( x \in [0, 1] \),

\[ \int_0^x \frac{1}{p(t)} \, dt = \int_0^1 \frac{1}{p(t)} \, dt - \int_0^{1-x} \frac{1}{p(t)} \, dt = \int_0^1 \frac{1}{p(t)} \, dt = \int_0^x \frac{1}{p(t)} \, dt. \]

It follows from this that \( p \) agrees with \( \tilde{p} \) except possibly on a set of Lebesgue measure zero, which is what we claimed.
Corollary 3 If \( p \in \Gamma \), and \( p \neq \tilde{p} \), then neither one of \( p, \tilde{p} \) belongs to the orbit of the other.

Corollary 4 Suppose \( p \in \Gamma \). Then, unless \( p = \tilde{p} \), the sets \( \{ T_\alpha p : \alpha \geq 0 \} \) and \( \{ \tilde{T}_\alpha \tilde{p} : \alpha \geq 0 \} \) are disjoint.

By contrast, note that if \( p \in \Gamma \) the sets \( \{ T_\alpha p : \alpha \geq 0 \} \), \( \{ \tilde{T}_\alpha \tilde{p} : \alpha \geq 0 \} \) may have elements in common. To see this, let

\[
q(t) = \sqrt{t(1-t)}, \quad 0 \leq t \leq 1,
\]

so that \( q = \tilde{q} \), and observe that

\[
\int_0^1 \frac{1}{q(t)} \, dt = \pi.
\]

Set

\[
p(x) = q(x) \left( 1 - \frac{1}{2\pi} \int_0^x \frac{1}{q(t)} \, dt \right)^2, \quad 0 \leq x \leq 1.
\]

Then

\[
\int_0^1 \frac{1}{p(t)} \, dt = \int_0^1 \frac{1}{q(x)} \left( 1 - \frac{1}{2\pi} \int_0^x \frac{1}{q(t)} \, dt \right)^2 \, dx
\]

\[
= \frac{\int_0^1 \frac{1}{q(x)} \, dx}{1 - \frac{1}{2\pi} \int_0^1 \frac{1}{q(x)} \, dx}
\]

\[
= 2\pi,
\]

whence \( p \in \Gamma \). Also,

\[
(T_{1/2\pi}p)(x) = p(x) \left( 1 + \frac{1}{2\pi} \int_0^x \frac{1}{p(t)} \, dt \right)^2
\]

\[
= p(x) \left( 1 + \frac{1}{2\pi} \int_0^x \frac{1}{q(t)} \, dt \right)^2
\]

\[
= \frac{p(x)}{(1 - \frac{1}{2\pi} \int_0^x \frac{1}{q(t)} \, dt)^2}
\]

\[
= q(x).
\]

Thus \( q = T_{1/2\pi}p \), and \( q = \tilde{q} \). Hence \( q \) belongs to the intersection of the two displayed sets.

The next result shows that the reflection operator and \( T_\alpha \) do not commute multiplicatively.

Theorem 6 Suppose \( p \in \Gamma \). If for some \( \alpha \geq 0 \), \( T_\alpha \tilde{p} = \tilde{T}_\alpha p \), then \( p = \tilde{p} \).

Proof. With \( \psi \) as in the previous theorem, the assumption is easily seen to be equivalent to the statement that

\[
\tilde{p}(1 + \alpha(\psi(1) - \psi))^2 = \tilde{p}(1 + \alpha\tilde{\psi})^2,
\]

on using property (b) from Theorem 3. Hence, \( 1 + \alpha(\psi(1) - \psi) = 1 + \alpha\tilde{\psi} \), and so, either \( \alpha = 0 \) or \( \psi(1) = \psi + \tilde{\psi} \). In either case, \( p = \tilde{p} \), as required.
7. A selection of examples

**Example 1** Since the orbit of the constant function 1 consists of functions of the form \((1 + \alpha x)^2\), we can deduce from Corollary 2 that tubes generated by rotating any one of the family of the curves \(x \rightarrow (1 + \alpha x)^{-1} : \alpha \geq 0\) about the \(x\)-axis are all inadmissible.

**Example 2** By contrast, the tubes generated by rotating any one of the family of the curves \(x \rightarrow (1 + kx) : k > 0\) about the \(x\)-axis contain some that are admissible [10].

Proof. Fix \(k > 0\). Let \(r(x) = 1 + kx\). The corresponding eigenvalue problem is to determine those values of \(z\) for which
\[
\left(\frac{y'}{r^2}\right)' + \frac{z}{r^2}y = 0, \quad y(0) = y(1) = 0
\]
has a non-trivial solution. In other words, we require the spectrum of \(p = 1/r^2\). A direct verification shows that if \(z = \lambda^2 \neq 0\) and
\[
\tan \lambda = \frac{k^2\lambda}{k^2 + \lambda^2(1 + k)},
\]
then
\[
y(x) = k^2 x c_z(x) - (k^2 + z(1 + kx))s_z(x),
\]
is a solution of the boundary-value problem. Thus
\[
\sigma(1/r^2) = \{\lambda^2 \neq 0 : \tan \lambda = \frac{k^2\lambda}{k^2 + \lambda^2(1 + k)}\}.
\]
And it seems clear that the elements of this set are not integer multiples of its smallest member. As well, in their paper [10], Mortell and Seymour confirm analytically that shocks do not occur when \(k = 7\), in which case, the corresponding tube—a frustum of a right-circular cone—is admissible. This raises the question: is there a critical value of \(k\) below which the corresponding tubes are inadmissible? We don’t have an answer to this.

**Example 3** Let \(m\) be any real number, and let \(p(x) = e^{2mx}\). Then
\[
\sigma(p) = \{m^2 + n^2\pi^2 : n = 1, 2, \ldots\}.
\]

Proof. This is so, because if \(n\) is a natural number, a direct argument confirms that \(e^{-mx}\sin(n\pi x)\) is a non-zero solution of the equations
\[
(e^{2mx}y'(x))' + (m^2 + n^2\pi^2)e^{2mx}y(x) = 0, \quad y(0) = y(1) = 0.
\]

**Example 4** If \(-\pi/2 < m < \pi/2\), it’s easy to show that the spectrum of \(x \rightarrow \cos^2(mx)\) is the set \(\{n^2\pi^2 - m^2 : n = 1, 2, \ldots\}\). Hence, for instance, it’s probable that the family of tubes generated by rotating a member of the union of the families of curves \(x \rightarrow \sec(mx) : 0 < m < \pi/2\) and \(x \rightarrow 1/\sqrt{1 + \sin(2mx)} : 0 < m < \pi/2\) about the \(x\)-axis contains ones that are admissible.

**Example 5** Let \(x_0 > 0\), \(r(x) = \tanh(x + x_0)\) and define \(p(x) = 1/r^2(x)\). Then
\[
\sigma(p) = S = \{\lambda^2 \neq 0 : \tan \lambda = \frac{(r(1) - r(0))\lambda}{1 + r(0)r(1)\lambda^2}\}.
\]
Proof. For any complex number $z$ consider the function

$$y = (r - r(0))c_z - (1 + zr(0)r)s_z.$$ 

Since $r' = 1 - r^2$, it’s easy to see firstly that

$$py' = -(1 + \frac{zr(0)}{r})c_z + z(r(0) - \frac{1}{r})s_z,$$

and secondly that

$$(py')' = \left(\frac{r'zr(0)}{r^2} + z(r(0) - \frac{1}{r})\right)c_z + \left(\frac{zr'}{r^2} + z(1 + \frac{zr(0)}{r})\right)s_z$$

$$= \frac{z(r(0) - r)}{r^2}c_z + \frac{z(1 + zr(0)r)}{r^2}s_z$$

$$= pz ((r(0) - r)c_z + (1 + zr(0)r)s_z)$$

$$= -zpy.$$ 

Furthermore, it’s evident that $y(0) = 0$, and easy to verify that $y(1) = 0$ if and only if $z \in S$. The stated result follows.

In all of the examples considered so far, the function $p$ is continuous and strictly positive on $[0, 1]$, so that it’s easy to discern that it qualifies for membership of $\Gamma$. It may be of some interest, therefore, to consider examples of unbounded functions that belong to $\Gamma$.

**Example 6** Let $0 < \alpha < 1$ and let $p(x) = x^{1-2\alpha}$. Then $p \in \Gamma$. Its spectrum is the set of squares of the zeros of the Bessel function $J_\alpha$.

This is because $x^\alpha J_\alpha(\lambda x)$ satisfies the differential equation

$$\frac{d}{dx} (x^{1-2\alpha} \frac{dy}{dx}) + \lambda^2 x^{1-2\alpha}y = 0.$$ 

While the last displayed equation has solutions also when $\alpha \in \{0, 1\}$, namely, $J_0(\lambda x)$ when $\alpha = 0$, and $J_1(\lambda x)$ when $\alpha = 1$, these examples fall outside our scope since $1/x$ is not integrable on $(0, 1)$. However, they do suggest how one might broaden the concept of the spectrum of a nonnegative function to include them, but we leave that for another occasion.

8. Functions which have the same spectrum as their reciprocals

As Example 3 above shows, if $m$ is real, and $p(x) = e^{2mx}$, then $\sigma(p) = \sigma(1/p)$. What about the converse? We provide a partial solution to this problem: we impose a sufficient condition on a function $p \in \Gamma$ that forces the conclusion that such exponentials are essentially the only kinds of functions with this property.

**Theorem 7** Suppose $p \in \Gamma$, $\ln p$ is either convex or concave on $(0, 1)$, and $(\ln p)'' \in L(0, 1)$. Assume that $\min(\sigma(p)) = \min(\sigma(1/p))$. Then there are real numbers $a, A$, with $A > 0$, such that $p(x) = Ae^{ax}$.

Proof. Let $\lambda$ be the smallest element in $\sigma(p)$. Then $\lambda$ is the smallest element in $\sigma(1/p)$ also. Hence, there are continuous functions $f, g$, with $f(0) = f(1) = g(0) = g(1) = 0$, such that the following equations hold almost everywhere on $(0, 1)$:

$$(pf')' + \lambda pf = 0 = (\frac{g'}{p})' + \lambda \frac{g}{p}.$$
In other words,
\[ f'' + \frac{p'}{p} f' + \lambda f = 0 = g'' - \frac{p'}{p} g' + \lambda g, \]
and so
\[ g(f'') - f(g'') + \frac{p'}{p} (gf' + fg') = 0. \]

Now, by integration by parts,
\[ \int_0^1 g f'' \, dx = -\int_0^1 f' g' \, dx = \int_0^1 f g'' \, dx. \]
Hence
\[ 0 = \int_0^1 \frac{p'}{p} (fg)' \, dx = \int_0^1 \frac{p'}{p} fg \, dx = \int_0^1 (\ln p)'' fg \, dx. \]

But, because \( \lambda \) is the least element of both \( \sigma(p) \) and \( \sigma(1/p) \), it’s a standard fact [3] that neither of \( f \) and \( g \) have any zeros on the open interval \( (0, 1) \). But, by hypothesis, \((\ln p)''\) has constant sign on this interval. Hence this function is identically zero, whence the result follows.

9. Functions whose spectrum is the set \( \{n^2 \pi^2 : n = 1, 2, \ldots\} \)

According to Example 1, every function of the form \((1 + \alpha x)^2\), that is non-zero on \([0, 1]\), has the set \( \{n^2 \pi^2 : n = 1, 2, \ldots\} \) as its spectrum. Are there any other functions with the same spectrum?

To deal with this, we prove a slight improvement of a variant of a well-known integral inequality due to G. H. Hardy [6].

**Lemma 1** Suppose \( f : [0, 1] \rightarrow (-\infty, \infty) \), \( f' \in L^2(0, 1) \) and \( f(0) = f(1) = 0 \). Then \( x \rightarrow f(x)/\sin(\pi x) \in L^2(0, 1) \), and

\[ \int_0^1 \frac{f^2(x)}{\sin^2(\pi x)} \, dx \leq \frac{4}{\pi^2} \int_0^1 (f'(x))^2 \cos^2(\pi x) \, dx. \]

Proof. Since \( f \) is absolutely continuous and, for all \( x \in [0, 1] \),
\[ f(x) = \int_0^x f'(t) \, dt = -\int_x^1 f'(t) \, dt, \]

it’s easy to see that
\[ f(x) = o(\sqrt{x}) \ (x \to 0^+), \text{ and } f(x) = o(\sqrt{1-x}) \ (x \to 1^-). \]

Hence, integrating by parts and applying the Cauchy-Schwartz inequality—and skipping over some minor technicalities—we deduce that
\[ \int_0^1 \frac{f^2(x)}{\sin^2(\pi x)} \, dx = -\int_0^1 f^2(x) \frac{d}{dx} \frac{1}{\pi \tan(\pi x)} \, dx \]
\[ = \frac{2}{\pi} \int_0^1 \left( \frac{f(x)}{\sin(\pi x)} \right) (f'(x) \cos(\pi x)) \, dx \]
\[ \leq \frac{2}{\pi} \sqrt{\int_0^1 \frac{f^2(x)}{\sin^2(\pi x)} \, dx \int_0^1 (f'(x))^2 \cos^2(\pi x) \, dx}, \]
whence the result follows.
Theorem 8 Suppose \( p \in \Gamma, \sqrt{\psi} \) is convex on \((0, 1),\) and \((\sqrt{\psi})''/\sqrt{\psi} \in L(0, 1)\). Assume that \( \pi^2 \in \sigma(p) \). Then \( p(x) = M(1 + \alpha x)^2 \) for some constants \( M > 0 \) and \( \alpha > -1 \). Hence \( \sigma(p) = \{n^2\pi^2 : n = 1, 2, \ldots\} \).

Proof. Suppose \( y \) is a non-zero solution of the boundary-value problem

\[
(py')' + \pi^2 py = 0, \quad y(0) = y(1) = 0.
\]

Let \( f = \sqrt{\psi}y, \) and \( g(x) = \sin(\pi x) \). A straightforward argument shows that \( f'' + (\pi^2 - \hat{\alpha})f = 0, \) where \( \hat{\alpha} = (\sqrt{\psi})''/\sqrt{\psi}, \) and \( f(0) = f(1) = 0, \) so that

\[
\int_0^1 f'^2 dx = -\int_0^1 f'' dx = \int_0^1 (\pi^2 - \hat{\alpha}) f^2 dx.
\]

Hence \( f \) satisfies the assumptions of the lemma, and so \( f/g \in L^2(0, 1), \) whence \( f' - g'f/g \in L^2(0, 1). \)

Therefore

\[
\int_0^1 (f' - g'f/g)^2 dx = \int_0^1 \left( f'^2 - 2f'f''g + f^2 g'^2 \right) / g \right) dx
\]

\[
= \int_0^1 \left( f'^2 - 2f'f''g \right) dx + \left[ -f^2 g' \right]_0^1 + \int_0^1 \frac{f(2f'g' + f g'')}{g} dx
\]

\[
= \int_0^1 f'^2 dx + \int_0^1 \frac{f^2 g''}{g} dx
\]

\[
= \int_0^1 f'^2 dx - \pi^2 \int_0^1 f^2 dx
\]

\[
= -\int_0^1 \hat{\alpha} f^2 dx.
\]

But, by hypothesis, \( \hat{\alpha} \geq 0, \) and \( f \) is not identically zero. Hence, it follows from this identity that \( (\sqrt{\psi})'' = 0, \) from which the conclusion results.

While it lies outside our range, nevertheless it may be of some interest to point out that the eigenvalues of the problem

\[
(\tan^2(\pi x)y')' + \lambda \tan^2(\pi x)y = 0, \quad y(0) = y(1) = 0,
\]

comprise the set \( \{n^2\pi^2 : n = 2, 3, \ldots\} \) [1]. Thus the spectra of two totally different functions, in this case \( 1 \) and \( \tan^2(\pi x), \) can differ by precisely one element, something which suggests that knowledge of the spectrum alone is not enough to determine the function. This is indeed the situation ([9], [13]).

10. An Open isospectrum problem

If two functions belonging to \( \Gamma \) have the same spectrum, how are they related? According to the results of Sections 5 and 6, if \( p \in \Gamma, \) the four sets

\[
\{T_\alpha p : \alpha \geq 0\}, \quad \{T_\alpha \hat{\alpha} p : \alpha \geq 0\}, \quad \{\overline{T_\alpha p} : \alpha \geq 0\}, \quad \{\overline{T_\alpha \hat{\alpha} p} : \alpha \geq 0\}
\]

consist of functions which have the same spectrum as \( p. \) Following on from this, is it possible that, if \( p, q \in \Gamma \) and \( \sigma(q) = \sigma(p), \) \( q \neq \hat{\alpha} \) and \( q \) is not a positive multiple of \( p, \) then \( q \) is a member of one of the displayed sets? Or, is this too much to hope for?
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