CORRELATIONS OF WAVE FUNCTIONS IN DISORDERED SYSTEMS

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INTRODUCTION

Statistical properties of eigenfunction amplitudes in disordered and chaotic systems have attracted considerable research interest recently. Fluctuations of the wave function amplitudes are believed to determine statistical properties of conductance peaks in quantum dots in the Coulomb blockade regime\(^1, 2, 3, 4, 5\), and can be directly measured in the microwave cavity experiments\(^6, 7\). On the theoretical side, the recent progress is based on application of the supersymmetry method to the problem of eigenfunction statistics\(^8, 2\). It was found that in so-called zero-mode approximation the distribution of eigenfunction amplitudes is correctly described by formulas of the random matrix theory (RMT). Deviations from the RMT predictions were studied in Refs.\(^8, 9, 10, 11\).

The present article addresses a problem of correlations of eigenfunction amplitudes. A description of correlations in amplitudes of a wave function in relatively close points of a chaotic billiard was proposed by Berry\(^12\) within a RMT-like assumption that the wave function is a superposition of plane waves with random coefficients. More recently, these correlations were considered in a disordered system within the zero-mode approximation\(^13\); the result was later shown\(^14\) to be equivalent to that of Ref.\(^12\). This result is valid for small separations of the two points (less than the mean free path\( l\)). Here we consider such correlations for arbitrary distances. These correlations determine, in particular, fluctuations of matrix elements of the (Coulomb) interaction, which are in turn important for statistical properties of spectra of quantum dots. Another topic addressed in the present paper is that of correlation of amplitudes of different eigenfunctions. Such correlations, while absent in RMT, appear in a disordered system. They become especially strong near the Anderson metal-insulator transition, where they play a crucial role in supporting the RMT-like level repulsion.

This article is based on recent works done in collaboration with Ya. M. Blanter, Y. V. Fyodorov, and B. A. Muzykantskii\(^9, 15, 16, 17, 18, 19\).
In this section, we study the correlations of eigenfunctions in the regime of a good conductor \(^9,^{15,16}\). The correlation function of amplitudes of one and the same eigenfunction with energy \(E\) can be formally defined as follows:

\[
\alpha(r_1, r_2, E) = \left\langle |\psi_k(r_1)\psi_k(r_2)|^2 \right\rangle_E \equiv \Delta \left\langle \sum_k |\psi_k(r_1)\psi_k(r_2)|^2 \delta(E - \epsilon_k) \right\rangle,
\]

where \(\psi_k(r)\) and \(\epsilon_k\) are eigenfunctions and eigenvalues of the Hamiltonian in a particular disorder configuration \(U(r)\), the angular brackets \(\langle \ldots \rangle\) denote averaging over the disorder potential \(U(r)\), and \(\Delta = \langle \sum_k \delta(E - \epsilon_k) \rangle^{-1}\) is the mean level spacing. In the case of a real or numerical experiment, calculation of the correlation function (1) would include averaging over all states in a relatively narrow energy window around \(E\). An analogous correlation function for two different eigenfunctions is defined as

\[
\sigma(r_1, r_2, E, \omega) = \left\langle |\psi_k(r_1)\psi_l(r_2)|^2 \right\rangle_{E,\omega} \equiv \Delta^2 R^{-1}_2(\omega) \left\langle \sum_{k \neq l} |\psi_k(r_1)\psi_l(r_2)|^2 \delta(E - \epsilon_k)\delta(E + \omega - \epsilon_l) \right\rangle,
\]

where \(R_2(\omega)\) denotes the two-level correlation function,

\[
R_2(\omega) = \Delta^2 \left\langle \sum_{k,l} \delta(E - \epsilon_k)\delta(E + \omega - \epsilon_l) \right\rangle.
\]

Eq. (2) defines an overlap of the eigenfunctions \(\psi_k\) and \(\psi_l\) provided they have energies close to \(E\) with the energy difference equal to \(\omega\).

To evaluate \(\alpha(r_1, r_2, E)\) and \(\sigma(r_1, r_2, E, \omega)\) (Ref.16), we employ an identity

\[
2\pi^2 \left[ \Delta^{-1}\alpha(r_1, r_2, E)\delta(\omega) + \Delta^{-2}\tilde{R}_2(\omega)\sigma(r_1, r_2, E, \omega) \right] = \text{Re} \left[ \left\langle G^R(r_1, r_1, E)G^A(r_2, r_2, E + \omega) - G^R(r_1, r_1, E)G^R(r_2, r_2, E + \omega) \right\rangle \right],
\]

where \(G^{R,A}(r, r', E)\) are retarded and advanced Green’s functions and \(\tilde{R}_2(\omega)\) is non-singular part of the level-level correlation function: \(R_2(\omega) = \tilde{R}_2(\omega) + \delta(\omega/\Delta)\). A natural question, which arises at this point, is whether the r.h.s. of Eq.(1) cannot be simply found within the diffuson-Cooperon perturbation theory \(^{20}\). Such a calculation would, however, be justified only for \(\omega \gg \Delta\) (more precisely, one has to introduce an imaginary part of frequency: \(\omega \to \omega + i\Gamma\), and require that \(\Gamma \gg \Delta\)). Therefore, it would only allow to find a smooth in \(\omega\) part of \(\sigma(r_1, r_2, E, \omega)\) for \(\omega \gg \Delta\). Evaluation of \(\alpha(r_1, r_2, E)\), as well as of \(\sigma(r_1, r_2, E, \omega)\) at \(\omega \sim \Delta\) cannot be done within such a calculation. For this reason, we employ a non-perturbative supersymmetry approach below.

The r.h.s. of Eq.(1) can be expressed in terms of the supermatrix \(\sigma\)-model \(^{21}\), yielding:

\[
2\pi^2 \left[ \Delta^{-1}\alpha(r_1, r_2, E)\delta(\omega) + \Delta^{-2}\tilde{R}_2(\omega)\sigma(r_1, r_2, E, \omega) \right] = (\pi\nu)^2 \left[ 1 - \text{Re}\langle Q_{bb}^{11}(r_1)Q_{bb}^{22}(r_2) \rangle_F - k_d(r_1 - r_2)\text{Re}\langle Q_{bb}^{12}(r_1)Q_{bb}^{21}(r_1) \rangle_F \right],
\]

where \(k_d(r) = (\pi\nu)^{-2}\langle \text{Im}G^R(r) \rangle^2\) is a short-range function explicitly given by

\[
k_d(r) = \exp(-r/l) \begin{cases} J_0^2(p_F r), & 2D \leq r \\ (p_F r)^{-2}\sin^2 p_F r, & 3D \end{cases},
\]

\(L = 2D, 3D\)
Here $\langle \ldots \rangle_F$ denotes the averaging with the action of the supermatrix sigma-model $F[Q]$:

$$
\langle \ldots \rangle_F = \int DQ(\ldots) \exp(-F[Q]),
$$

$$
F[Q] = -\frac{\pi\nu}{4} \int d\mathbf{r} \text{Str}[D(\nabla Q)^2 + 2i(\omega + i0)\Lambda Q],
$$

where $D$ is the diffusion coefficient, $\nu$ is the density of states, $Q = T^{-1}\Lambda T$ is a 4×4 supermatrix, $\Lambda = \text{diag}(1, 1, -1, -1)$, and $T$ belongs to the supercoset space $U(1, 1) \times SU(2)$. The symbol $\text{Str}$ denotes the supertrace defined as $\text{Str}B = B_{11}^1 + B_{11}^2 - B_{11}^3$. The upper matrix indices correspond to the retarded-advanced decomposition, while the lower indices denote the boson-fermion one. The action (7) is written for the case of so-called unitary ensemble (broken time reversal symmetry), which we consider below. Generalization to a system with time reversal symmetry (orthogonal ensemble) is straightforward, and the results are presented in the end of the section. Evaluating the $\sigma$-model correlation functions in the r.h.s. of Eq. (3) and separating the result into the singular and regular parts, one can obtain the correlation functions $\alpha(\mathbf{r}_1, \mathbf{r}_2, E)$ and $\sigma(\mathbf{r}_1, \mathbf{r}_2, E, \omega)$. The two-level correlation function $R_2(\omega)$ entering Eq. (3) is given in this formalism by

$$
R_2(\omega) = \frac{1}{2V^2} \text{Re} \int d\mathbf{r}_1 d\mathbf{r}_2 \left[ 1 - \langle Q_{bb}^{11}(\mathbf{r}_1)Q_{bb}^{22}(\mathbf{r}_2) \rangle_F \right].
$$

(8)

In the metallic (weak localization) regime, the sigma-model correlation functions $\langle Q_{bb}^{11}(\mathbf{r}_1)Q_{bb}^{22}(\mathbf{r}_2) \rangle_F$ and $\langle Q_{bb}^{12}(\mathbf{r}_1)Q_{bb}^{21}(\mathbf{r}_2) \rangle_F$ can be calculated for relatively low frequencies $\omega \ll E_c$ with the use of a general method developed in Refs. 22, 9, which allows one to take into account spatial variations of the field $Q$. The results are obtained in form of expansions in $g^{-1}$, where $g$ is the dimensionless conductance. First, we restrict ourselves to the terms of order $g^{-1}$. Then, the result for the first correlation function reads as

$$
\langle Q_{bb}^{11}(\mathbf{r}_1)Q_{bb}^{22}(\mathbf{r}_2) \rangle_F = -1 - 2i\frac{\exp(i\pi s)\sin\pi s}{(\pi s)^2} - \frac{2i}{\pi s}\Pi(\mathbf{r}_1, \mathbf{r}_2),
$$

(9)

where $s = \omega/\Delta + i0$. Here the diffusion propagator $\Pi$ is the solution to the diffusion equation

$$
-D\nabla^2\Pi(\mathbf{r}_1, \mathbf{r}_2) = (\pi\nu)^{-1}[\delta(\mathbf{r}_1 - \mathbf{r}_2) - V^{-1}]
$$

(10)

with the Neumann boundary condition (normal derivative equal to zero at the sample boundary), which can be presented in the form

$$
\Pi(\mathbf{r}_1, \mathbf{r}_2) = (\pi\nu)^{-1}\sum_q (Dq^2)^{-1}\phi_q(\mathbf{r}_1)\phi_q(\mathbf{r}_2),
$$

(11)

with $\phi_q$ being the eigenfunction of the diffusion operator corresponding to the eigenvalue $Dq^2$, $q \neq 0$. The first two terms in Eq. (4) represent the result of the zero-mode approximation 21, which takes into account only the spatially constant configurations of the field $Q(\mathbf{r})$, so that the functional integral over $DQ(\mathbf{r})$ is reduced to an integral over a single matrix $Q$. The last term is the correction of order $g^{-1}$. An analogous calculation for the second correlator yields:

$$
\langle Q_{bb}^{12}(\mathbf{r}_1)Q_{bb}^{21}(\mathbf{r}_2) \rangle_F = -2 \left\{ \frac{i}{\pi s} + \left[ 1 + i\frac{\exp(i\pi s)\sin\pi s}{(\pi s)^2} \right]\Pi(\mathbf{r}_1, \mathbf{r}_2) \right\}.
$$

(12)

Now, separating regular and singular parts in r.h.s. of Eq. (5), we obtain the following result for the autocorrelations of the same eigenfunction:

$$
V^2\langle |\psi_k(\mathbf{r}_1)\psi_k(\mathbf{r}_2)|^2 \rangle_E - 1 = k_d(r)[1 + \Pi(\mathbf{r}_1, \mathbf{r}_1)] + \Pi(\mathbf{r}_1, \mathbf{r}_2),
$$

(13)
and for the correlation of amplitudes of two different eigenfunctions

\[ V^2 \langle |\psi_k(r_1)\psi_l(r_2)|^2 \rangle_{E,\omega} - 1 = k_d(r)\Pi(r_1, r_1), \quad k \neq l \]  

(14)

In particular, for \( r_1 = r_2 \) we have

\[ V^2 \langle |\psi_k(r)\psi_l(r)|^2 \rangle_{E,\omega} - 1 = \delta_{kl} + (1 + \delta_{kl})\Pi(r, r). \]  

(15)

Note that the result \([3]\) for \( r_1 = r_2 \) is the inverse participation ratio calculated in Ref. 9; on the other hand, neglecting the terms with the diffusion propagator (i.e. making the zero-mode approximation), we reproduce the result of Refs. 12, 13, 14.

Eqs. \([14], [15]\) show that the correlations between different eigenfunctions are relatively small in the weak disorder regime. Indeed, they are proportional to the small parameter \( \Pi(r, r) \), which is equal in the case of 2D geometry to \( (L/l)^{1/2} \) (the size of the system)

\[ \Pi(r, r) = (\pi g)^{-1} \ln L/l, \quad 2D, \]  

(16)

with \( g = 2\pi\nu D \). For a quasi-1D wire or strip of the length \( L \),

\[ \Pi(r, r) = \frac{2}{g} \left[ \frac{1}{6} + B_2 \left( \frac{r}{L} \right) \right], \quad 0 \leq r \leq L, \]  

(17)

where \( g = 2\pi\nu D/L \), and \( B_2(x) = x^2 - x + 1/6 \) is the Bernoulli polynomial. The correlations are enhanced by disorder; when the system approaches the strong localization regime, the relative magnitude of correlations, \( \Pi(r, r) \) ceases to be small. The correlations near the Anderson localization transition will be discussed in the next section of the paper.

Another correlation function, generally used for the calculation of the linear response of the system,

\[ \gamma(r_1, r_2, E, \omega) = \langle \psi_k^*(r_1)\psi_l(r_1)\psi_k(r_2)\psi_l^*(r_2) \rangle_{E,\omega} \]  

(18)

\[ \equiv \Delta^2 R^{-1}_2(\omega) \left\{ \sum_{k \neq l} \psi_k^*(r_1)\psi_l(r_1)\psi_k(r_2)\psi_l^*(r_2)\delta(E - \epsilon_k)\delta(E + \omega - \epsilon_l) \right\} \]

can be calculated in a similar way; the result reads

\[ V^2 \langle \psi_k^*(r_1)\psi_l(r_1)\psi_k(r_2)\psi_l^*(r_2) \rangle_{E,\omega} = k_d(r) + \Pi(r_1, r_2), \quad k \neq l. \]  

(19)

As is seen from Eqs. \([13], [14], [15]\), in the \( 1/g \) order the correlation functions \( \alpha(r_1, r_2, E) \) and \( \gamma(r_1, r_2, E, \omega) \) survive for the large separation between the points, \( r \gg l \), while \( \sigma(r_1, r_2, E, \omega) \) decays exponentially for the distances larger than the mean free path \( l \). This is, however, an artifact of the \( g^{-1} \) approximation, and the investigation of the corresponding tails requires the extension of the above calculation to the terms proportional to \( g^{-2} \). We find that the correlator \( \langle Q_{bb}^{11}(r_1)Q_{bb}^{22}(r_2) \rangle_F \) gets the following correction:

\[ \delta \langle Q_{bb}^{11}(r_1)Q_{bb}^{22}(r_2) \rangle_F = -f_1 + 2f_4 + \exp(2i\pi s)f_3 - 2i \frac{\exp(2i\pi s)}{\pi s}(f_2 - f_3) \]

\[ - \frac{\exp(2i\pi s)}{2(\pi s)^2}(f_1 - 4f_2 + 3f_3 - 2f_4). \]  

(20)

\( ^{\dagger} \) In the 3D geometry, the sum over the momenta \( q \) in Eq. \([11]\) determining \( \Pi(r, r) \) diverges at large \( q \) and is determined by the upper cut-off, \( q \sim 1/l \), yielding \( \Pi(r, r) \sim g^{-1}L/l \). This reflects the fact that in 3D geometry the truly local \( (r_1 = r_2) \) correlations may not be given correctly by the diffusion approximation and can depend on microscopic structure of the random potential. For this reason, we do not consider local correlations in 3D geometry here. Note, however, that this concerns the global geometry of the sample; locally the system can be either of 2D or 3D nature, which determines the form of the function \( k_d(r) \) (e.g., a wire is locally 3D, but has a quasi-1D geometry).
Here we defined the functions

\[
\begin{align*}
  f_1(r_1, r_2) &= \Pi^2(r_1, r_2), \\
  f_2(r_1, r_2) &= (2V)^{-1} \int dr \left[ \Pi^2(r, r_1) + \Pi^2(r, r_2) \right], \\
  f_3 &= V^{-2} \int dr dr' \Pi^2(r, r'), \\
  f_4(r_1, r_2) &= V^{-1} \int dr \Pi(r, r_1) \Pi(r, r_2). \\
\end{align*}
\]

Consequently, we obtain the following results for the correlations of different \((k \neq l)\) eigenfunctions at \(r > l\):

\[
\begin{align*}
  V^2 \langle |\psi_k(r_1)\psi_l(r_2)|^2 \rangle_{E, \omega} - 1 &= \frac{1}{2} (f_1 - f_3 - 2f_4) \\
  &+ 2(f_2 - f_3) \left( \frac{\sin^2 \pi s}{\pi^2 s^2} - \frac{\sin 2\pi s}{2\pi s} \right) \left( 1 - \frac{\sin^2 \pi s}{\pi^2 s^2} \right)^{-1}.
\end{align*}
\]

As it should be expected, the double integral over the both coordinates of this correlation function is equal to zero. This property is just the normalization condition and should hold in arbitrary order of expansion in \(g^{-1}\).

The quantities \(f_2, f_3, \) and \(f_4\) are proportional to \(g^{-2}\), with some (geometry-dependent) prefactors of order unity. On the other hand, \(f_1\) in 2D and 3D geometry depends essentially on the distance \(r = |r_1 - r_2|\). In particular, for \(l \ll r \ll L\) we find

\[
\begin{align*}
  f_1(r_1, r_2) &= \Pi^2(r_1, r_2) \approx \begin{cases} 
  \frac{1}{(\pi g)^2} \ln^2 \frac{L}{r}, & 2D \\
  \frac{1}{1 - (4\pi^2 \nu Dr)^2}, & 3D
\end{cases}.
\end{align*}
\]

Thus, for \(l < r \ll L\), the contributions proportional to \(f_1\) dominate in Eq.(22), and we get

\[
V^2 \langle |\psi_k(r_1)\psi_l(r_2)|^2 \rangle_{E, \omega} - 1 = \frac{1}{2} \Pi^2(r_1, r_2), \quad k \neq l.
\]

On the other hand, for the case of quasi-1D geometry (as well as in 2D and 3D for \(r \sim L\)), all quantities \(f_1, f_2, f_3, \) and \(f_4\) are of order of \(1/g^2\). Thus, the correlator \(\sigma(r_1, r_2, E, \omega)\) acquires a non-trivial (oscillatory) frequency dependence on a scale \(\omega \sim \Delta\) described by the second term in the r.h.s. of Eq.(22). In particular, in the quasi-1D case the function \(f_2 - f_3\) determining the spatial dependence of this term has the form

\[
f_2 - f_3 = -\frac{2}{3g^2} \left[ B_4 \left( \frac{r_1}{L} \right) + B_4 \left( \frac{r_2}{L} \right) \right],
\]

where \(B_4(x) = x^4 - 2x^3 + x^2 - 1/30\).

Let us remind the reader that the above derivation is valid for \(\omega \ll E_c\). In order to obtain the results in the range \(\omega \geq E_c\) one can calculate the sigma-model correlation functions entering Eqs. (5) by means of the perturbation theory. We find then

\[
\begin{align*}
  V^2 \langle |\psi_k(r_1)\psi_l(r_2)|^2 \rangle_{E, \omega} &= 1 + \text{Re} \{k_d(r)\Pi_\omega(r_1, r_2) \\
  &\quad + \frac{1}{2} \left[ \Pi_\omega^2(r_1, r_2) - \frac{1}{V^2} \int drr' \Pi_\omega^2(r, r') \right]\}, \\
  V^2 \langle \psi^*_k(r_1)\psi_l(r_2)\psi^*_k(r_2)\psi_l(r_1) \rangle_{E, \omega} &= k_d(r) + \text{Re} \Pi_\omega(r_1, r_2),
\end{align*}
\]
where $\Pi_\omega(r_1, r_2)$ is the finite-frequency diffusion propagator

$$
\Pi_\omega(r_1, r_2) = (\pi \nu)^{-1} \sum_{q, q'} \frac{\phi_q(r_1)\phi_{q'}(r_2)}{Dq^2 - i\omega},
$$

and the summation in Eq. (26) now includes $q = 0$. As was mentioned, the perturbation theory should give correctly the non-oscillatory (in $\omega$) part of the correlation functions at $\omega \gg \Delta$. Indeed, it can be checked that Eqs.(25) match the supersymmetric $\sigma$-model results in this regime. Furthermore, in the $1/g$ order [which means keeping only linear in $\Pi_\omega$ terms in (25)] and neglecting $-i\omega$ in denominator of Eq.(26), Eqs.(25) reproduce the exact results (13), (19) even at small frequencies $\omega \sim \Delta$. We stress however that the perturbative calculation is not justified in this region and only the supersymmetry method provides a rigorous derivation of these results.

As was mentioned, the case of broken time reversal symmetry (unitary ensemble) was considered above. Generalization to a system with unbroken time reversal symmetry is straightforward 16, in $1/g$-order Eqs.(13), (14), and (19) are modified as follows:

$$
V^2\langle|\psi_k(r_1)\psi_k(r_2)|^2\rangle_E = [1 + 2k_d(r)] [1 + 2\Pi_D(r_1, r_2)],
$$

$$
V^2\langle|\psi_k(r_1)\psi_l(r_2)|^2\rangle_{E,\omega} - 1 = 2k_d(r)\Pi_D(r_1, r_2).
$$

$$
V^2\langle|\psi_k^*(r_1)\psi_l(r_1)\psi_k(r_2)\psi_l^*(r_2)\rangle_{E,\omega} = k_d(r) + [1 + k_d(r)] \Pi_D(r_1, r_2), \quad k \neq l.
$$

Using the supersymmetry method, one can calculate also higher order correlation functions of eigenfunction amplitudes. In particular, the correlation function $\langle|\psi_k^*(r_1)||\psi_k^*(r_2)|\rangle_E$ determines fluctuations of the inverse participation ratio (IPR) $I_2 = \int dr |\psi^4(r)|$. Details of the corresponding calculation can be found in Ref.9; the result for the relative variance of IPR, $\delta(I_2) = \text{var}(I_2)/\langle I_2 \rangle^2$ being

$$
\delta(I_2) = \frac{8}{\beta^2} f_3 = \frac{32a_d}{\beta^2 g^2},
$$

with a numerical coefficient $a_d$ depending on the sample dimensionality $d$ and equal to $a_1 = 1/90$, $a_2 \approx 0.0266$ and $a_3 \approx 0.0527$ for quasi-1D, 2D, and 3D geometry respectively. The fluctuations (30) have the same relative magnitude as the famous universal conductance fluctuations. Note also that extrapolating Eq.(30) to the Anderson transition point, where $g \sim 1$, we find $\delta(I_2) \sim 1$, so that the magnitude of IPR fluctuations is of the order of its mean value (which is, in turn, much larger than in the metallic regime; see the next section).

Similarly, correlations of eigenfunction amplitudes determine fluctuations of matrix elements of an operator of some (say, Coulomb) interaction computed on eigenfunctions $\psi_k$ of the one-particle Hamiltonian in a random potential. Such a problem naturally arises, when one wishes to study the effect of interaction onto statistical properties of excitations in a mesoscopic sample (see below).

Finally, the above consideration can be generalized to a ballistic chaotic system, by applying a recently developed ballistic generalization of the $\sigma$-model 23, 24. The results are then expressed in terms of the (averaged over the direction of velocity) kernel $g(r_1, n_1; r_2, n_2)$ of the Liouville operator $\hat{K} = v_F n \nabla$ governing the classical dynamics in the system,

$$
\Pi(r_1, r_2) = \int dn_1 dn_2 g(r_1, n_1; r_2, n_2);
$$

$$
\hat{K} g(r_1, n_1; r_2, n_2) = (\pi \nu)^{-1} \left[ \delta(r_1 - r_2)\delta(n_1 - n_2) - V^{-1} \right].
$$
Here \( \mathbf{n} \) is a unit vector determining the direction of momentum, and normalization \( \int d\mathbf{n} = 1 \) is used. Equivalently, the function \( \Pi(\mathbf{r}_1, \mathbf{r}_2) \) can be defined as

\[
\Pi(\mathbf{r}_1, \mathbf{r}_2) = \int_0^\infty dt \int d\mathbf{n}_1 \tilde{g}(\mathbf{r}_1, \mathbf{n}_1, t; \mathbf{r}_2),
\]

where \( \tilde{g} \) is determined by the evolution equation

\[
\left( \frac{\partial}{\partial t} + v_F \mathbf{n}_1 \nabla_1 \right) \tilde{g}(\mathbf{r}_1, \mathbf{n}_1, t; \mathbf{r}_2) = 0, \quad t > 0
\]

with the boundary condition

\[
\tilde{g}|_{t=0} = (\pi \nu)^{-1} \left[ \delta(\mathbf{r}_1 - \mathbf{r}_2) - V^{-1} \right].
\]

Eq. (31) is a natural “ballistic” counterpart of Eq. (10). In particular, generalization of Eqs. (13), (27) for the correlations of an eigenfunction amplitudes in two different points onto the ballistic case reads

\[
\alpha(\mathbf{r}_1, \mathbf{r}_2, E) = 1 + \frac{2}{\beta} \Pi(\mathbf{r}_1, \mathbf{r}_2).
\]

Note that the ballistic \( \sigma \)-model approach is of semiclassical nature and thus valid on distances much larger than the wave length \( \lambda_F \). For this reason, Eqs. (33) represent the smoothed correlation function, which is not valid for \( |\mathbf{r}_1 - \mathbf{r}_2| \leq \lambda_F \). Indeed, the contribution to \( \Pi(\mathbf{r}_1, \mathbf{r}_2) \) from the straight line motion from \( \mathbf{r}_2 \) to \( \mathbf{r}_1 \) can be easily evaluated, yielding e.g. in the 2D case \( \Pi(\mathbf{r}_1, \mathbf{r}_2) = 1/(\pi p_F |\mathbf{r}_1 - \mathbf{r}_2|) \). This is nothing else but the smoothed version of the function \( k_2(|\mathbf{r}_1 - \mathbf{r}_2|) = J_0^2(p_F |\mathbf{r}_1 - \mathbf{r}_2|) \) giving the leading contribution to the short-scale correlations. A formula for the variance of matrix elements closely related to Eq. (35) was obtained in the semiclassical approach in Ref. 25. In a very recent paper 26 a similar generalization of the Berry formula for \( \langle \psi_k^*(\mathbf{r}_1) \psi_k(\mathbf{r}_2) \rangle \) was proposed.

Eq. (33) shows that correlations in eigenfunction amplitudes in remote points are determined by the classical dynamics in the system. It is closely related to the phenomenon of scarring of eigenfunctions by the classical orbits 27, 28. Indeed, if \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) belong to a short periodic orbit, the function \( \Pi(\mathbf{r}_1, \mathbf{r}_2) \) is positive, so that the amplitudes \( |\psi_k(\mathbf{r}_1)|^2 \) and \( |\psi_k(\mathbf{r}_2)|^2 \) are positively correlated. This is a reflection of the “scars” associated with this periodic orbits and a quantitative characterization of their strength in the coordinate space. Note that this effect gets smaller with increasing energy \( E \) of eigenfunctions. Indeed, for a strongly chaotic system and for \( |\mathbf{r}_1 - \mathbf{r}_2| \sim L \) (\( L \) being the system size), we have in the 2D case \( \Pi(\mathbf{r}_1, \mathbf{r}_2) \sim \lambda_F/L \), so that the magnitude of correlations decreases as \( E^{-1/2} \). The function \( \Pi(\mathbf{r}_1, \mathbf{r}_2) \) was explicitly calculated in Ref. 19 for a circular billiard with diffusive surface scattering.

**STRONG CORRELATIONS OF EIGENFUNCTIONS AND LEVEL REPULSION AT THE ANDERSON LOCALIZATION TRANSITION**

In the preceding section, we considered the metallic regime, where a typical wavefunction \( \psi_i(\mathbf{r}) \) is extended and covers uniformly all the sample volume. When system approaches the point of Anderson transition \( E_c \), these extended eigenfunctions become less and less homogeneous in space showing regions with larger and smaller amplitudes and eventually forming a multifractal structure in the vicinity of \( E_c \). To characterize the
degree of non-homogeneity quantitatively, it is convenient to use the inverse participation ratio (IPR) $\langle I_2(E) \rangle = \int d\alpha(r, r, E)$ For extended states this quantity is inversely proportional to the system volume: $\langle I_2(E) \rangle = A(E) L^{-d}$, with $L$ and $d$ standing for the system size and spatial dimension, respectively. The coefficient $A$ in this relation measures a fraction of the system volume where eigenfunction is appreciably non-zero. In the regime of a good conductor $A \approx 1 + 2/\beta \sim 1$, whereas close to the mobility edge $E = E_c$ it becomes large and diverges like $A(E) \propto |E - E_c|^{-\mu_2}$, with a critical index $\mu_2 > 0$ \textsuperscript{29}. This means that eigenfunctions become more and more sparse, when the system approaches the critical point of the Anderson transition. Just at the mobility edge eigenfunctions occupy a vanishing fraction of the system volume and IPR scales like $\langle I_2(E) \rangle \propto L^{-d+\eta}$, with $\eta > 0$. Such a behavior reflects multifractal \textsuperscript{30, 31, 32} structure of critical eigenstates. At last, in the insulating phase any eigenstate is localized in a domain of finite extension $\xi$ and IPR remains finite in the limit of infinite system size $L \to \infty$.

This transparent picture serves as a basis for qualitative understanding of spectral properties of disordered conductors. Indeed, as long as eigenstates are well extended and cover the whole sample, they overlap substantially and corresponding energy levels repel each other in the same way as in RMT. As a result, the Wigner-Dyson (WD) statistics describes well energy levels in a good metal \textsuperscript{21, 20, 22}. In contrast, in the insulating phase different eigenfunctions corresponding to levels close in energy are localized far apart from one another and their overlap is negligible. This is the reason for absence of correlations of energy levels in this regime – the so-called Poisson statistics.

However, a naive extrapolation of this argument to the vicinity of the transition point would lead to a wrong conclusion. Indeed, one might expect that sparse (multifractal in the critical point) eigenstates fail to overlap, that would result in essential weakening of level correlations close to the mobility edge and vanishing level repulsion at $E = E_c$. However, numerical simulations show \textsuperscript{33, 34, 35, 36, 37} that even at the mobility edge levels repel each other strongly, though the whole statistics is different from the WD one. The purpose of this section is to explain how this apparent contradiction is resolved. We will show that critical eigenstates for nearby levels are so strongly correlated that they overlap well in spite of their sparse structure.

To calculate the overlap function $\sigma(r, r, E, \omega)$ in the critical regime \textsuperscript{17}, we will exploit an exactly solvable model of the Anderson transition – so-called sparse random matrix (SRM) model \textsuperscript{38}. This model is essentially equivalent to a tight-binding model, which is locally of tree-like stricture (i.e has no small-scale loops), with matrix size $N$ playing a role of the system volume (number of sites of the tight-binding model). The SRM model can be used to construct an effective mean-field theory of Anderson localization \textsuperscript{39} corresponding to the limit $d = \infty$. The inverse participation ratio $\langle I_2(E) \rangle = N\alpha(r, r, E)$ is proportional to $1/N$ in the delocalized phase, as expected: $\langle I_2(E) \rangle = A(E)/N$. The coefficient $A(E)$ diverges close to the transition point, $|E - E_c| \ll E_c$, like $A(E) \propto \exp \left( \text{const} |E - E_c|^{-1/2} \right)$ \textsuperscript{39} which differs from the power-law behavior expected for conventional $d$-dimensional systems. The origin of such a critical dependence was explained in \textsuperscript{39, 40} and stems from the fact that $A(E)$ is determined essentially by the ”correlation volume” $V(\xi)$ (i.e. number of sites at a distance smaller than correlation length $\xi$), which is exponentially large, $V(\xi) \propto \exp(\text{const}\,\xi)$, for tree-like structures, whereas $V(\xi) \propto \xi^d$ for a $d$-dimensional lattice. Having this difference in mind, one can translate the results obtained in the framework of $d = \infty$ models to their finite-dimensional counterparts \textsuperscript{40}.

The calculation of the inverse participation ration $N\alpha(r, r, E)$ can be extended onto the overlap correlation function $\sigma(r, r, E, \omega)$. Evaluating the r.h.s. of Eq.(4), we
find the following relation 17:
\[
\sigma(r, r, E, \omega) = \frac{\beta}{\beta + 2} \alpha(r, r, E) .
\] (36)

This relation is valid everywhere in the phase of extended eigenstates, up to the mobility edge \(E = E_c\), provided the number of sites (the system volume) exceeds the correlation volume. In particular, it is valid in the critical region \(|E - E_c| \ll E_c\), where a typical eigenfunction is very sparse and \(\alpha(r, r, E)\) grows like \(\exp(\text{const}|E - E_c|^{-1/2})\).

Eq. (36) implies the following structure of eigenfunctions within an energy interval \(\delta E = \omega < A^{-1}(E)\). Each eigenstate can be represented as a product \(\Psi_i(r) = \psi_i(r)\Phi_E(r)\). Here the function \(\Phi_E(r)\) is an eigenfunction envelope of “bumps and dips” it is the same for all eigenstates around energy \(E\), reflects underlying gross (multifractal) spatial structure and governs the divergence of the inverse participation ratio (i.e. of the factor \(A(E)\)) at the critical point. In contrast, \(\psi_i(r)\) is Gaussian white-noise component fluctuating in space on the scale of lattice constant. It fills in the envelope function \(\Phi_E(r)\) in an individual way for each eigenfunction, but is not critical, i.e. is not sensitive to the vicinity of the Anderson transition. These Gaussian fluctuations are responsible for the factor \(\beta/(\beta + 2)\) (which is the same as in the corresponding Gaussian Ensemble) in Eq. (36).

As was already mentioned, this picture is valid in the energy window \(\delta E \sim A^{-1}(E)\) around the energy \(E\); the number of levels in this window being large as \(\delta E/\Delta \sim NA^{-1}(E) \gg 1\) in the thermodynamic limit \(N \to \infty\). These states form a kind of Gaussian Ensemble on a spatially non-uniform (multifractal for \(E \to E_c\)) background \(\Phi_E(r)\). Since the eigenfunction correlations are described by the formula (36), which has exactly the same form as in the Gaussian Ensemble, it is not surprising that the level statistics has the WD form everywhere in the extended phase 38.

We believe on physical grounds that the same picture should hold for a conventional \(d\)-dimensional conductor. First of all, the general mechanism of the transition is the same in \(d < \infty\) and \(d = \infty\) models. Furthermore, the sparsity (multifractality) of eigenstates near the transition point takes its extreme form for \(d = \infty\) models 40, so that since the strong correlations (36) take place at \(d = \infty\) it would be very surprising if they do not hold at finite \(d\) as well. Finally, Eq. (36) is supported by the results of the calculations in the weak localization regime. Keeping only the leading (in \(d \geq 2\)) terms proportional to the powers of \(\Pi(r, r)\) and considering the unitary ensemble for definiteness, we have up to the two-loop order (see Eq. (24) and Ref. 9)

\[
\sigma(r, r, E, \omega) = V^{-2} \left[ 1 + \Pi(r, r) + \frac{1}{2} \Pi^2(r, r) + \ldots \right] = \frac{1}{2} \alpha(r, r, E) ,
\] (37)
in full agreement with Eq. (36).

Replacing \(A(E)\) by the \(d\)-dimensional correlation volume \(\sim \xi^d\), we conclude that for \(E\) close to \(E_c\), Eq. (36) should be valid for \(\omega < \Delta \xi\), where \(\Delta \xi \propto 1/\xi^d\) is the level spacing in the correlation volume. For larger \(\omega\), \(\sigma(r, E, \omega)\) is expected to decrease as \(\omega^{-\eta/d}\) according to the scaling arguments 31, 32, 41, so that we find \(\sigma(r, E, \omega)/\alpha(r, E) \sim (\omega/\Delta \xi)^{-\eta/d}\), up to a numerical coefficient of order of unity. Again, for any value of the energy \(E\) in the delocalized phase, taking the system size \(L\) large enough, \(L \gg \xi\), we have a large number of levels \(\delta E/\Delta \sim \Delta \xi/\Delta \propto (L/\xi)^d\) in the energy window \(\delta E\) where Eq. (36) holds, so that the level correlation will be of the WD form.

Finally, let us consider what happens when we go from the critical regime (\(\xi\) large, but \(L \gg \xi\) to the critical point \((\xi \gg L)\). For this purpose, let us keep the system size \(L\) fixed and change the energy toward \(E_c\), so that \(\xi\) increases. When \(\xi\) gets comparable
to the system size, $\xi \sim L$, we have $\Delta \xi \sim \Delta$. This is the border of applicability of the above consideration. Correspondingly, we find

$$\sigma(r, r, E, \omega)/\alpha(r, r, E) \sim 1, \quad \omega < \Delta$$

(38)

and $\sigma(r, r, E, \omega)/\alpha(r, r, E) \sim (\omega/\Delta)^{-n/d}$ for $\omega > \Delta$. When $E$ approaches further $E_c$, the correlation length $\xi \gg L$ gets irrelevant, so that these results will hold in the critical point ($\xi = \infty$). Of course, Eq. (38) is not sufficient to ensure the WD statistics in the critical point, since there is only of order of one level within its validity range $\delta E \sim \Delta$. Indeed, the numerical simulations show that the level statistics on the mobility edge is different from the WD one.

However, Eq. (38) allows us to make an important conclusion concerning the behavior of $R_2(\omega)$ at small $\omega < \Delta$, or, which is essentially the same, the behavior of the nearest neighbor spacing distribution $P(s) = s/\Delta$, at $s < 1$. For this purpose, it is enough to consider only two neighboring levels. Let their energy difference be $\omega_0 \sim \Delta$. Let us now perturb the system by a random potential $V(\mathbf{r})$ with $\langle V(\mathbf{r}) \rangle = 0$, $\langle V(\mathbf{r})V(\mathbf{r}') \rangle = \Gamma \delta(\mathbf{r} - \mathbf{r}')$. For the two-level system it reduces to a $2 \times 2$ matrix $\{V_{ij}\}$, $i, j = 1, 2$, with elements $V_{ij} = \int d^d \mathbf{r} V(\mathbf{r})\Psi_i^*(\mathbf{r})\Psi_j(\mathbf{r})$. The crucial point is that the variances of the diagonal and off-diagonal matrix elements are according to Eq. (38) equal to each other up to a factor of order of unity:

$$\langle V_{11}^2 \rangle/\langle |V_{12}|^2 \rangle = \sigma(r, r, E, \omega)/\alpha(r, r, E) \sim 1$$

(39)

The distance between the perturbed levels is given by $\omega = [(V_{11} - V_{22} + \omega_0)^2 + |V_{12}|^2]^{1/2}$. Choosing the amplitude of the potential in such a way that the typical energy shift $V_{11} \sim \Delta$ and using Eq. (39), we find $\langle |V_{12}|^2 \rangle \sim \Delta$. As a result, the probability density for the level separation $\omega$ is for $\omega \ll \Delta$ of the form $dP \sim (\omega/\Delta)^3 d\omega/\Delta$, with some prefactor of order of unity. We thus conclude that in the critical point $P(s) \sim c_\beta s^\delta$ for $s \ll 1$ with a coefficient $c_\beta$ of order of unity, in agreement with the numerical findings.

**FLUCTUATIONS IN THE ADDITION SPECTRA OF QUANTUM DOTS**

In this section we discuss effects of the eigenfunction statistics on the spectral properties of quantum dots. The electron levels of a quantum dot can be resolved if the temperature $T$ is less than the mean single-particle level spacing in a dot and can be studied in transport experiments (see the recent review and references therein). In small dots containing only few electrons these levels show a regular structure familiar from the atomic physics. What we have in mind here are however larger dots (containing in typical experiments from few hundred to few thousand conducting electrons). These dots are either disordered (diffusive) or, although being ballistic, are expected to have chaotic dynamics due to their irregular shape. As a result, it is natural to use a statistical description of the properties of energy levels and eigenfunctions in such dots.

There are two types of the quantum dot spectra studied experimentally via measuring their I–V characteristics: (i) excitation spectrum, when excited levels are probed in a dot with given number of electrons by increasing the source-drain voltage, and (ii) addition spectrum, when electrons are added one by one by changing the gate voltage. In the latter case, which will be the subject of our consideration here, one finds narrow conductance peaks separated the regions of (almost) zero current (Coulomb blockade). Statistics of the spacings between these peaks was studied in a number of recent experiments; we will return to the experimental results below.
The simplest theoretical model which may be used to study distribution of the spacings is as follows. One considers a dot as a fixed size diffusive mesoscopic sample and assumes that changing a gate voltage by an amount $\delta V_g$ simply reduces to a uniform change of the potential inside the dot by a constant $\gamma \delta V_g$, with certain numerical coefficient $\gamma$ ("lever arm"). Such a model was used for numerical simulations of the addition spectra in Refs. 44, 47. Below we consider the statistics of peak spacings within this model, and later return to the approximations involved. We will neglect the spin degree of freedom of electrons of first; inclusion of the spin will be also discussed in the end of the section.

The distance between the two consecutive conductance peaks is given by

$$S_N = (E_{N+2} - E_{N+1}) - (E_{N+1} - E_N)$$

where $E_N$ is the ground state of a sample with $N$ electrons. In the second line of Eq. (40) we rewrote $S_N$ in terms of the Hartree-Fock single electron energy levels, with $\mu_i^j$ denoting the energy of the state $#(i,j)$ in the dot containing $i$ electrons. It is convenient to decompose $S_N$ in the following way

$$S_N = (\mu_{N+1}^{N+2} - \mu_N^{N+2}) + (\mu_N^{N+2} - \mu_N^{N+1})$$

$$\equiv E_1 + E_2$$

The quantity $E_2$ is the distance between the two levels of the same one-particle (Hartree-Fock) Hamiltonian $\hat{H}_N$ (describing a dot with $N$ electrons) and is expected to obey RMT; in particular $\langle E_2 \rangle = \Delta$ and r.m.s. $\langle E_2 \rangle = a\Delta$ with a numerical coefficient $a$ of order of unity $|a| = 0.52 \ (0.42)$ for the orthogonal (resp. unitary) ensemble. On the other hand, $E_1$ is a shift of the level $#(N + 2)$ due to the change of the Hamiltonian $\hat{H}_N \rightarrow \hat{H}_{N+1}$ accompanying addition of the electron $#(N + 1)$ to the system. It can be in turn decomposed into the following three contributions

$$E_1 = \frac{e^2}{C} + \int \left( |\psi_{N+1}^2(r)| + |\psi_{N+2}^2(r)| \right) \delta U(r)$$

$$+ \int \int \psi_{N+1}^2(r) \psi_{N+2}^2(r') U_\kappa(|r - r'|)$$

$$\equiv E_1^{(0)} + E_1^{(1)} + E_1^{(2)}$$

Here $C$ is the dot capacitance, $\delta U(r)$ is the change of the self-consistent potential due to addition of one electron (i.e. difference in the self-consistent potential in the dots with $N$ and $N+1$ electrons), and $U_\kappa(r)$ is the screened Coulomb interaction (with the subscript $\kappa$ denoting the inverse screening length). In particular, in the experimentally most relevant 2D case (which we will consider below) and assuming a circular form of the dot with radius $R$, we have

$$\delta U(r) = - \frac{e^2}{2\kappa R} (R^2 - r^2)^{1/2},$$

while $U_\kappa$ is given in the Fourier space by $\hat{U}_\kappa(q) = 2\pi e^2/\epsilon(q + \kappa)$ with $\kappa = 2\pi e^2\nu/\epsilon$ and $\epsilon$ being the dielectric constant. The first term in Eq. (42) (the charging energy) determines the average value $\langle E_1 \rangle$ and thus the average peak spacing $\langle S_N \rangle$ (since $e^2/C \gg \Delta$ for a large dot with $N \gg 1$). This is the only contribution to $E_1$ which is kept by so-called constant interaction model, which in addition neglects fluctuations of the capacitance $C$. Consequently, fluctuations of $S_N$ in the constant interaction model are determined...
solely by fluctuations of the single-particle level spacing $E_2$ and thus should be described by RMT: r.m.s.($S_N$) $= a\Delta$.

The term $E_1$ in Eq. (41) is however an additional source of fluctuations and is thus responsible for the enhancement of fluctuations in comparison with RMT. In principle, all three terms $E_1^{(0)}$, $E_1^{(1)}$, and $E_1^{(2)}$ in Eq. (42) contribute to this enhancement. Fluctuations of the first one, $E_1^{(0)} = e^2/C$ are due to the fact that the capacitance is slightly different from its purely geometric value because of a finite value of the screening length. The corresponding correction to $C$ can be expressed in terms of the polarization operator $P(r, r')^{16}$. The latter is a fluctuating quantity (because of fluctuations of the eigenfunctions in the Fermi sea) and contains a random part $P_r(r, r')$ leading to the following expression for the random part of the charging energy:

$$\left(\frac{e^2}{C}\right)_r = 2 \int drdr' \delta U(r)P_r(r, r')\delta U(r').$$  \hspace{1cm} (44)$$

Evaluating the fluctuations of the polarization operator $^{16}$, we find $^9$

$$\text{var}(E_1^{(0)}) = \frac{48}{\beta} e^2 \ln g \left[ \frac{1}{V} \int dr_1dr_2 \delta U(r_1)\Pi(r_1, r_2)\delta U(r_2) \right]^2$$

$$\propto \frac{1}{\beta} \ln g \left( \Delta g \right)^2$$ \hspace{1cm} (45)

Now we consider fluctuations of the last term, $E_1^{(2)}$, in Eq. (42). Using Eqs. (13), (27) for the correlations of eigenfunction amplitudes in two remote points, the variance of $E_1^{(2)}$ is found to be

$$\text{var}(E_1^{(2)}) = \frac{4}{\beta^2 V^4} \int dr_1dr_1'dr_2dr_2'U_\kappa(|r_1 - r_1'|)U_\kappa(|r_2 - r_2'|)\Pi(r_1, r_2)\Pi(r_1', r_2')$$

$$\approx \frac{4\Delta^2}{\beta^2 V^2} \int dr_1dr_2 \Pi^2(r_1, r_2)$$

$$\propto \frac{1}{\beta^2} \left( \Delta g \right)^2.$$ \hspace{1cm} (46)

Finally, fluctuations of the term $E_1^{(1)}$ can be also evaluated with help of Eqs. (13), (27), yielding

$$\text{var}(E_1^{(1)}) = \frac{4}{\beta V^2} \int dr_1dr_2 \delta U(r_1)\Pi(r_1, r_2)\delta U(r_2)$$

$$\propto \frac{1}{\beta} \Delta^2 g.$$ \hspace{1cm} (47)

It is seen that for $g \gg 1$ all the contributions Eqs. (45)–(47) are parametrically small compared to the RMT fluctuations (which are $\sim \Delta$). Fluctuations of the term $E_1^{(1)}$ related to the change $\delta U(r)$ of the self-consistent potential represent parametrically leading contribution to the enhancement of the peak spacing fluctuations with respect to RMT.

Let us now discuss approximations made in the course of the above derivation:

$^9$We use the obvious notations for the variance and the root mean square deviations of a quantity $X$: var($X$) $= \langle X^2 \rangle - \langle X \rangle^2$; r.m.s.($X$) $= [\text{var}(X)]^{1/2}$. 

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i) It was assumed that changing of the gate voltage results in a spatially uniform change of the potential in the sample, that led us to the expression Eq.(13) for the change of the self-consistent potential \( \delta U(\mathbf{r}) \) accompanying the addition of one electron to the dot. This result would correspond to a gate located far enough from the sample. In a more realistic situation, when the gate is relatively narrow and located close to the sample, the potential change \( \delta U(\mathbf{r}) \) (as well as the additional electron density) will be mainly located on the side of the dot facing the gate. Furthermore, the total area of the dot is not fixed, so that the dot gets larger (and slightly deformed) with adding each electron to it. These effects lead to some increase of the fluctuations of \( E_1^{(1)} \) in comparison with the model considered above. If the size of the gate and its distance to the dot are of the same order of magnitude as the dot size, then

\[
\text{r.m.s.}(E_1^{(1)}) \propto \Delta/\sqrt{\beta g},
\]

as in Eq.(17), with a geometry dependent-numerical prefactor. The upper bound for the magnitude of fluctuations of \( E_1^{(1)} \) is given by the opposite limiting case, when additional electron density occupies an area \( \sim \lambda_F \times \lambda_F \) near the gate, in which case one finds

\[
\text{r.m.s.}(E_1^{(1)}) \sim \Delta/\sqrt{\beta}.
\]

ii) The dot was supposed to be diffusive in the calculation. For a ballistic dot one should replace \( \Pi(\mathbf{r},\mathbf{r}') \) by \( \Pi_B(\mathbf{r},\mathbf{r}') \), as was explained above. This would mean that the parameter \( g \) is replaced by \( \sim N^{1/2} \sim L/\lambda_F \), where \( N \) is the number of electrons in the dot and \( L \) the characteristic linear dimension. The numerical coefficient would depend, however, on “how strongly chaotic” is the dot. Role of the eigenfunctions fluctuations and correlations (“scars”) in enhancement of the peak spacing fluctuations was studied in \(^{48}\) via numerical simulations of a dot with \( N \approx 100 \) electrons.

iii) It was assumed that the dot energy and the measured gate voltage are related through a constant (or smoothly varying) coefficient \( \gamma \). This “lever arm” \( \gamma \) depends, however on the dot-gate capacitance, which is also a fluctuating quantity. If the gate size and the distance to the gate is of the same order as the size of the dot, these fluctuations should be of the same order as fluctuations of the dot self-capacitance given by Eq.(15), and thus lead to additional fluctuations which are parametrically small compared to \( \Delta \). In a more general situation (thin gate located close to the sample) an additional analysis along the lines of Ref.16 is necessary.

iv) The calculation was done within the random phase approximation, which assumes that \( r_s \equiv e^2/\epsilon v_F \ll 1 \), with \( v_F \) being the Fermi velocity. In realistic dots however \( r_s \sim 1 \). Since this value is still considerably lower than the Wigner crystallization threshold, the calculations should be still valid, up to a numerical factor \( \alpha(r_s) \) [depending on \( r_s \) only and such that \( \alpha(r_s \ll 1) = 1 \)].

v) We considered the model of spinless electrons up to now. Let us briefly discuss the role of the spin degree of freedom. Within the constant interaction model, it would lead to a bimodal distribution \(^{47}\) of peak spacings

\[
P(S_N) = \frac{1}{2} \left[ \delta(S_N - e^2/C) + \frac{1}{2\Delta} P_{WD} \left( \frac{S_N - e^2/C}{2\Delta} \right) \right], \quad (48)
\]
where \( P_{WD}(s) \) is the Wigner-Dyson distribution and \( \Delta \) denotes the level spacing in the absence of spin degeneracy. The value of the coefficient \( a \) in the relation \( \text{r.m.s.}(S_N) = a \Delta \) is then increased (compared to the spinless case) and is equal to 1.24 (1.16) for the orthogonal (resp. unitary) ensemble. Taking into account fluctuations of eigenfunctions (and thus of \( E_1 \)) however modifies the form of the distribution \(^{18}\). The value of the term \( E_1^{(2)} \) representing the interaction between two electrons is larger in the case when \( \psi_{N+2} \) and \( \psi_{N+1} \) correspond to two spin-degenerate states (i.e. have the same spatial dependence of the wave function), since

\[
\langle \int \text{d}r \text{d}r' |\psi^2_i(r)||\psi^2_i(r')|U_\kappa(|r-r'|)\rangle - \langle \int \text{d}r \text{d}r' |\psi^2_i(r)||\psi^2_j(r')|U_\kappa(|r-r'|)\rangle = \frac{2}{\beta V^2} \int \text{d}r \text{d}r' k_r(|r-r'|) \sim \Delta
\]

for \( r_s \sim 1 \) (the coefficient depends on \( r_s \), see \(^{18}\)). Therefore, filling a state \( \psi_{i\uparrow} \) pushes up the level \( \psi_{i\downarrow} \) (with respect to other eigenstates) by an amount of order of \( \Delta \). This removes a bimodal structure of the distribution of peak spacings and slightly modifies the value of the coefficient \( a \).

Basing on the above analysis, we can make the following general statement. Imaging that we fix \( r_s \sim 1 \) (i.e. fix the electron density and thus the Fermi wave length) and the system geometry, and then start to increase the linear dimension \( L \) of the system. Then, while the average value of the peak spacing \( S_N \) scales as \( \langle S_N \rangle \approx e^2/C \propto 1/L \), its fluctuations will scale differently: \( \text{r.m.s.}(S_N) \sim \Delta \propto 1/L^2 \). This result is not at all trivial, since in an analogous problem for classical particles \(^{49,50}\) the fluctuations are proportional to the mean value \( \langle S_N \rangle \). The physical reason for smaller fluctuations in the quantum case is in the delocalized nature of the electronic wave functions, which are spread roughly uniformly over the system.

The above prediction was confirmed by a recent experiment \(^{46}\), where a thorough study of the peak spacing statistics was carried out. It was found that the low-temperature value of \( \text{r.m.s.}(S_N) \), as well the typical temperature scale for its change are approximately given by the mean level spacing \( \Delta \) (while in units of \( E_c \) the magnitude of fluctuations was very small, typically 2–4%). We note also that in recent numerical simulations \(^{48}\) fluctuations of the addition energies were found to be approximately 0.7\( \Delta \), in agreement with our results.

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REFERENCES

1. R. A. Jalabert, A. D. Stone, and Y. Alhassid, Phys. Rev. Lett. 68, 3468 (1992).
2. V. N. Prigodin, K. B. Efetov, and S. Iida, Phys. Rev. Lett. 71, 1230 (1993).
3. E. R. Mucciolo, V. N. Prigodin, and B. L. Altshuler, Phys. Rev. B 51, 1714 (1995).
4. A. M. Chang, H. U. Baranger, L. N. Pfeiffer, K. W. West, and T. Y. Chang, Phys. Rev. Lett. 76, 1695 (1996).
5. J. A. Folk, S. R. Patel, S. F. Godijn, A. G. Huibers, S. M. Cronenwett, C. M. Marcus, K. Camp-
man, and A. C. Gossard, Phys. Rev. Lett. 76, 1699 (1996).
6. H. J. Stöckmann and J. Stein, Phys. Rev. Lett. 64, 2215 (1990); J. Stein and H. J. Stöckmann,
Phys. Rev. Lett. 68, 2867 (1992).
7. S. Sridhar, Phys. Rev. Lett. 67, 785 (1991); A. Kudrolli, V. Kidambi, and S. Sridhar, Phys.
Rev. Lett. 75, 822 (1995).
8. A. D. Mirlin and Y. V. Fyodorov, J. Phys. A: Math. Gen. 26, L551 (1993); Y. V. Fyodorov
and A. D. Mirlin, Int. Journ. Mod. Phys. B 8, 3795 (1994).
9. Y. V. Fyodorov and A. D. Mirlin, Phys. Rev. B 51, 13403 (1995).
10. Y. V. Fyodorov and A. D. Mirlin, J. Math. Phys. 38, 1888 (1997).
11. M. V. Berry, J. Phys. A 10, 2083 (1977).
12. M. V. Berry, J. Phys. A 10, 2083 (1977).
13. V. N. Prigodin, Phys. Rev. Lett. 74, 1566 (1995); V. N. Prigodin, N. Taniguchi, A. Kudrolli,
V. Kidambi, and S. Sridhar, Phys. Rev. Lett. 75, 2392 (1995).
14. M. Srednicki, Phys. Rev. E 54, 954 (1996); M. Srednicki and F. Stiernelof, J. Phys. A 29,
5817 (1996).
15. Ya. M. Blanter and A. D. Mirlin, Phys. Rev. B 52, 17413 (1995).
16. Ya. M. Blanter, A. D. Mirlin, and B. A. Muzykantskii, Phys. Rev. Lett. 78, 2449 (1997).
17. Ya. M. Blanter and A. D. Mirlin, to appear in Phys. Rev. B.
44. U. Sivan, R. Berkovits, Y. Aloni, O. Prus, A. Auerbach, and G. Ben-Joseph, Phys. Rev. Lett. 77, 1123 (1996).
45. F. Simmel, T. Heinzel, and D. A. Wharam, Europhys. Lett. 38, 123 (1997).
46. S. R. Patel, S. M. Cronenwett, D. R. Stewart, A. G. Huibers, C. M. Marcus, C. I. Duruöz, J. S. Harris, K. Campman, and A. C. Gossard, preprint cond-mat/9708090.
47. O. Prus, A. Auerbach, Y. Aloni, U. Sivan, and R. Berkovits, Phys. Rev. B 54, R14289 (1996).
48. M. Stopa, preprint cond-mat/9709119.
49. J. R. Morris, D. M. Deaven, and K. M. Ho, Phys. Rev. B 53, R1740 (1996).
50. A. A. Kouitakov, F. G. Pikus, and B. I. Shklovskii, Phys. Rev. B 55, 9223 (1997).