Estimating the average causal effect of intervention in continuous variables using machine learning

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Abstract
The most widely discussed methods for estimating the Average Causal Effect/Average Treatment Effect are those for intervention in discrete binary variables whose value represents intervention/non-intervention groups. On the other hand, methods for intervening in continuous variables independent of data generating models have not been developed. In this study, we give a method for estimating the average causal effect for intervention in continuous variables that can be applied to data of any generating models as long as the causal effect is identifiable. The proposing method is independent of machine learning algorithms and preserves the identifiability of data.

1 Introduction
The causal effect is defined by Pearl’s do operation as a probability distribution over observed data in the way that it is altered from one which generates data originally [Pearl 1995; Pearl 2009]. When dealing with causal effects in real-world problems, it is also necessary to take into account unobserved variables that is not included in data. In general, causal effects are counterfactual probability distributions that differ from data generating systems in the real world. When we consider the existence of unobserved data, it becomes a problem if it can be determined by observed data available. That is, we need to consider the identifiability of causal effects in this case. This problem has recently been resolved to a certain extent [Tian and Pearl 2002; Shpitser and Pearl 2006; Shpitser and Pearl 2012].

If causal effects are identifiable, the next problems are how to estimate the (conditional) average causal effect, that is, how to calculate the (conditional)
expected value of causal effects. Recently some methods have been proposed for this estimation, such as one by reweighting probability [Jung, Tian, and Bareinboim 2020] or one by assuming a semiparametric model [Chernozhukov et al. 2018]. However, any method that is independent of models generating data and available for intervention in continuous variables have not been developed yet.

In addition, related to estimation of the average causal effects, it has been widely discussed in recent years to estimate of (Conditional) Average Treatment Effects that are the difference between the average causal effects of intervention / non-intervention groups. The major methods of these, for example, are the re-weighting method by propensity score such as Inverse propensity weighting (IPW) method [Chernozhukov et al. 2018; Rosenbaum and Rubin 1984], the matching method [Rosenbaum and Rubin 1983; Rosenbaum and Rubin 1985], the stratification method [Rosenbaum and Rubin 1984; Imbens 2004], the tree-based method such that Bayesian Additive Regression Trees (BART) [H. Chipman, E. George, and R. McCulloch 2006; H. A. Chipman, E. I. George, and R. E. McCulloch 2010; Hill 2011]. In addition, methods of meta-machine learning algorithm have been proposed in recent years such that T-Learner [Künzel et al. 2019], S-Learner [Künzel et al. 2019], X-Learner [Künzel et al. 2019], U-Learner [Nie and Wager 2021] and R-Learner [Nie and Wager 2021]. Currently, when estimating the average treatment effects of intervention in continuous variables, for example patient responses to drug dose, the above methods for discrete binary intervention are only used, by extending it to available for multi-values intervention [Schwab et al. 2020]. However, when estimating the average causal effects of continuous value intervention, it is necessary that estimators of them have to be continuous functions for values of intervention, whereas the estimators of above methods for discrete intervention is not.

In this paper, we propose a method that enables estimation of the average causal effect of intervention in continuous variables. The proposing method use supervised machine learning algorithms but is independent of algorithms to use. In addition, it preserves the identifiability of causal effects for models generating data, and is designed to make as few assumptions about training data as possible.

This article is divided to five parts. First, we introduce the background of study. Second, we define terminologies and concepts necessary for causal inference. Third, we state the Main Theorem and present the proposing method for estimating the average causal effects of intervention in continuous variables. Fourth, we report results of a numerical experiment for the proposing methods and discuss them. Lastly, we conclude this paper.

2 Notation and Definitions

We denote random variables by capital letters, $A$. Small letters, $a$, represents a values of random variables corresponding, $A$. Bold letters, $\mathbf{A}$ or $\mathbf{a}$, represent a set of variables or values of variables. In particular, we use $\mathbf{V} = \{V_1, V_2, \ldots, V_n\}$ for
observed variables and \( \mathbf{U} = \{ U_1, U_2, \ldots, U_m \} \) for unobserved variables. For sake of simplicity, we assume that any \( V \in \mathbf{V} \) is not determined by the others, that is, \( V \neq f(\mathbf{W}) \) for any function \( f \) and any \( \mathbf{W} \subset \mathbf{V} \setminus \{ V \} \). We denote the domain of a variable \( A \) by \( \mathcal{X}_A \). For sake of simplicity, we assume that \( P(v) > 0 \) for all \( V \in \mathbf{V} \) and \( v \in \mathcal{X}_V \). For a set of variables \( \mathbf{A} \), let \( \mathcal{X}_\mathbf{A} = \mathcal{X}_{A_1} \times \cdots \times \mathcal{X}_{A_n} \). In this paper, \( \mathbf{V} \cup \mathbf{U} \) is a semi-Markovian model, and a Directed Acyclic Graph (DAG) \( G = G_{\mathbf{V} \cup \mathbf{U}} \) is a causal graph for them. Each \( Pa(\mathbf{A})_G, Ch(\mathbf{A})_G, An(\mathbf{A})_G \) and \( De(\mathbf{A})_G \) represents the parents, children, ancestors and descendants of observed variables in \( G \) for \( \mathbf{A} \subset \mathbf{V} \). \( UPa(\mathbf{A})_G \) represents the parents of unobserved variables in \( G \). In this paper, \( Pa(\mathbf{A})_G, Ch(\mathbf{A})_G, An(\mathbf{A})_G, De(\mathbf{A})_G \) doesn’t include \( \mathbf{A} \) itself. The set of observed variables which has no paths to each \( \mathbf{A} \) in \( G \) is denoted by \( Ind(\mathbf{A})_G := \{ V \in \mathbf{V} | (V \perp \mathbf{A})_G \} \). \( \overline{G}(\mathbf{X}) \) is denoted the graph obtained from \( G \) by deleting all arrows emerging from variables to \( \mathbf{X} \). \( \overline{G}(\mathbf{X}_1, \mathbf{X}_2) \) is denoted the graph obtained from \( G \) by deleting all arrows emerging from variables to \( \mathbf{X}_1 \) and all arrows emerging from \( \mathbf{X}_2 \) to variables. \( G_{\mathbf{obs}} \) is denoted the graph obtained from \( G \) by remaining only arrows between observed variables.

According to \( G \), a probability distribution \( P \) of \( \mathbf{V} \cup \mathbf{U} \) can be decomposed into as below.

\[
P(V_1 = v_1, \ldots, V_n = v_n, U_1 = u_1, \ldots, U_m = u_m) = \prod_i P(V_i = v_i | Pa(V_i)_G = pa_i, UPa(V_i)_G = upa_i),
\]

where \( pa_i \in \mathcal{X}_{Pa(V_i)} \) is values of \( Pa(V_i)_G \), \( upa_i \in \mathcal{X}_{UPa(V_i)} \) is values of \( UPa(V_i)_G \).

The probability distribution for only observed variables is obtained by marginalized this distribution over all unobserved variables.

\[
P(V_1 = v_1, \ldots, V_n = v_n) = \sum_{u \in \mathcal{X}_U} \prod_i P(V_i = v_i | Pa(V_i)_G = pa_i, UPa(V_i)_G = upa_i)
\]

\[
= \prod_i P(V_i = v_i | Pa(V_i)_G = pa_i).
\]

Give two disjoint sets of \( \mathbf{X}, \mathbf{Y} \subset \mathbf{V} \), the causal effect of \( \mathbf{X} \) on \( \mathbf{Y} \), denoted by \( P(\mathbf{Y} = \mathbf{y} | do(\mathbf{X} = \mathbf{x})) \), is defined as the probability distribution as follows.

\[
P(\mathbf{Y} = \mathbf{y} | do(\mathbf{X} = \mathbf{x})) = \sum_{\mathbf{V}' \in \mathcal{X}_{\mathbf{V}'}} \frac{P(\mathbf{Y} = \mathbf{y}, \mathbf{X} = \mathbf{x}, \mathbf{V}' = \mathbf{v}')}{P(\mathbf{X} = \mathbf{x} | Pa(\mathbf{X})_G = pa_\mathbf{x})}.
\]

where, \( \mathbf{V}' = \mathbf{V} \setminus (\mathbf{X} \cup \mathbf{Y}) \) and \( pa_\mathbf{x} \) represents values of \( Pa(\mathbf{X})_G \).

Give disjoint sets of \( \mathbf{X}, \mathbf{Y}, \mathbf{Z} \subset \mathbf{V} \), the causal effect of \( \mathbf{X} \) on \( \mathbf{Y} \) under conditions \( \mathbf{Z} \), denoted by \( P(\mathbf{Y} = \mathbf{y} | do(\mathbf{X} = \mathbf{x}), \mathbf{Z} = \mathbf{z}) \), is defined as the probability distribution as follows.

\[
P(\mathbf{Y} = \mathbf{y} | do(\mathbf{X} = \mathbf{x}), \mathbf{Z} = \mathbf{z}) = \frac{P(\mathbf{Y} = \mathbf{y}, \mathbf{Z} = \mathbf{z} | do(\mathbf{X} = \mathbf{x}))}{P(\mathbf{Z} = \mathbf{z} | do(\mathbf{X} = \mathbf{x}))}
\]
3 Main Theorem

In this paper, we consider the conditional expectations of causal effects $P(Y = y|do(X = x), Z = z)$. That is, we consider the probability distribution $P'$ following

$$P' = P(Y|do(X), Z) \cdot P(X_1) \cdot P(X_2) \cdot \cdots \cdot P(X_n) \cdot P(Z),$$

and we consider the expectation

$$E_{P'}[Y = y|X = x, Z = z].$$

Here, $X$ are variables to intervene in, $Z$ are covariates and $Y$ are target variables to estimate.

3.1 Graph structure that can be used for modeling the average causal effects

We give a definition of a graph structure that can be used for modeling the average causal effects, that is, the conditional expectations for the probability distribution after intervention.

**Definition 1** (Availability for modeling the average causal effects). For disjoint sets $Y, X \subset V$, a DAG $G$ is said to be available for modeling the average causal effects $P(Y|do(X), Z)$, if $G$ satisfies following two conditions

1. $P(Y|do(X), Z)$ is identifiable in $G$.
2. $\overline{G_{obs}}(X) = G_{obs}$.

If a graph is available for modeling the average causal effects $E[Y = y|do(X) = x]$, then $P'$ of (3) is as follows.

$$P' = P(Y = y|do(X = x), Z = z) \times P(X_1 = x_1) \cdot P(X_2 = x_2) \cdot \cdots \cdot P(X_n = x_n) \cdot P(Z = z)$$

$$= P(Y = y, X = x, Z = z).$$

That is, there is no difference of joint probability distributions over $Y, X$ and $Z$ between before and after intervention in $X$. Thus, if a model generating a data has this structure, we can use the original data for modeling the average causal effects.

Figure 1 is an example of a graph available for modeling the average causal effects $E[Y|do(X), Z_1, Z_2]$. Figure 2 is an example of one not available for that. Comparing two graphs, the graph not available for modeling has the arrows (red) between $X$ and $Z_1, Z_2$ that hinder modeling the average causal effect $E[Y|do(X), Z_1, Z_2]$, but one available for modeling has been deleted them. If a data is generated by models of Figure 1, it is available for modeling the average causal effect $E[Y|do(X), Z_1, Z_2]$. On the other hand, if a data is not generated by models of Figure 2, it is not available for the modeling.
Figure 1: A graph available for modeling the average causal effects. If a data is generated by models of this, it is available for modeling the average causal effect \( E[Y|do(X), Z_1, Z_2] \) without any operations.

Figure 2: A graph not available for modeling the average causal effect. If a data is not generated by models of this, it is not available for modeling the average causal effect \( E[Y|do(X), Z_1, Z_2] \).
3.2 Main theorem

The following Theorem 2 shows how the proposing method builds models from original data. In addition, it shows that the method preserves for the identifiability of causal effects for models which generate data. This theorem includes cases with more than two variables to intervene in. That is, it includes the causal effects of joint intervention.

**Theorem 2.** For $Y \in V$ and $X = \{X_1, X_2, \ldots, X_n\} \subset V$, let $Z = V \setminus (\{Y\} \cup X)$. Suppose that $X \subset An(Y)_G$ and that $Z \cap De(X)_G \cap De(Y)_G = \emptyset$. Then, for $x_1 \in X_{X_1}$, $x_2 \in X_{X_2}$, $\ldots$, $x_n \in X_{X_n}$ and $z \in \mathcal{X}_Z$, let $\tilde{X}^{(k)} = \{X_1^{(k)}, X_2^{(k)}, \ldots, X_n^{(k)}\}$ ($1 \leq k \leq n$) be as follows.

If $k = 1$

$$
\tilde{X}_i^{(1)} = X_i - E[X_i|Z = z] \quad \text{for } 1 \leq i \leq n
$$

$$
\tilde{x}_i^{(1)} = x_i - E[X_i|Z = z] \quad \text{for } 1 \leq i \leq n.
$$

If $2 \leq k \leq n - 1$

$$
\tilde{X}_i^{(k+1)} = \begin{cases} 
\tilde{X}_i^{(k)} & \text{if } 1 \leq i \leq k \\
\tilde{X}_i^{(k)} - E[\tilde{X}_i^{(k)}|\tilde{X}_k^{(k)} = \tilde{x}_k^{(k)}] & \text{if } i \geq k + 1
\end{cases}
$$

$$
\tilde{x}_i^{(k+1)} = \begin{cases} 
\tilde{x}_i^{(k)} & \text{if } 1 \leq i \leq k \\
\tilde{x}_i^{(k)} - E[\tilde{x}_i^{(k)}|\tilde{x}_k^{(k)} = \tilde{x}_k^{(k)}] & \text{if } i \geq k + 1.
\end{cases}
$$

Using the above results, for $Y$ and $y \in \mathcal{X}_Y$, let $\tilde{Y}^{(k)}$ ($1 \leq k \leq n$) be as follows.

If $k = 1$

$$
\tilde{Y}^{(1)} = Y - E[Y|Z = z]
$$

$$
\tilde{y}^{(1)} = y - E[Y|Z = z].
$$

If $2 \leq k \leq n - 1$

$$
\tilde{Y}^{(k+1)} = \tilde{Y}^{(k)} - E[\tilde{Y}^{(k)}|\tilde{X}_k^{(k)} = \tilde{x}_k^{(k)}]
$$

$$
\tilde{y}^{(k+1)} = \tilde{y}^{(k)} - E[\tilde{y}^{(k)}|\tilde{x}_k^{(k)} = \tilde{x}_k^{(k)}]}
$$

Additionally, for each $U \in \mathcal{U}$, let $\tilde{U}^{(k)}$ ($1 \leq k \leq n$) be as follows.

$$
\tilde{U}^{(1)} = U - E[U|Z = z]
$$

$$
\tilde{U}^{(2)} = \tilde{U}^{(1)} - E[\tilde{U}^{(1)}|\tilde{X}_1^{(1)} = \tilde{x}_1^{(1)}]
$$

$$
\vdots
$$

$$
\tilde{U}^{(n)} = \tilde{U}^{(n-1)} - E[\tilde{U}^{(n-1)}|\tilde{X}_{n-1}^{(n-1)} = \tilde{x}_{n-1}^{(n-1)}]
$$

Let $\tilde{Y} = \tilde{Y}^{(n)}$, $\tilde{X} = \tilde{X}^{(n)} = \{\tilde{X}_1^{(1)}, \tilde{X}_2^{(2)}, \ldots, \tilde{X}_n^{(n)}\}$, $\tilde{U} = \{\tilde{U}^{(n)}|U \in \mathcal{U}\}$, and let $\tilde{V} = \{\tilde{Y}\} \cup \tilde{X} \cup Z$. Then, $\tilde{V} \cup \tilde{U}$ is a semi-Markovian model. Let $\tilde{G}$ be a DAG.
for \( \bar{V} \cup \bar{U} \), then \( \bar{X} \subset An(\bar{Y})_G \cup Ind(\bar{Y})_G \). If \( P(Y|do(X), Z) \) is identifiable in \( G \), then \( \bar{G} \) is available for modeling the average causal effect \( E[\bar{Y}|do(\bar{X})] \).

Moreover, it holds that

\[
P(Y = y|do(X_1 = x_1), \ldots, do(X_n = x_n), Z = z) = \prod_{i=1}^nP(\bar{Y} = y^{(n)}|do(\bar{X}_1^{(1)} = x_1^{(1)}), do(\bar{X}_2^{(2)} = x_2^{(2)}), \ldots, do(\bar{X}_n^{(n)} = \bar{z}_n^{(n)}))
\]

(4)

\[
P(\bar{Y} = y^{(n)}|X_1^{(1)} = x_1^{(1)}, X_2^{(2)} = x_2^{(2)}, \ldots, X_n^{(n)} = \bar{z}_n^{(n)})
\]

(5)

\[
P(\bar{Y} = y^{(n)}|X_1^{(1)} = \bar{x}_1^{(1)}, X_2^{(2)} = \bar{x}_2^{(2)}, \ldots, X_n^{(n)} = \bar{x}_n^{(n)})
\]

(6)

The following corollary shows how to estimate the average causal effects from each of the conditional expectations (predictions of supervised algorithms) obtained from Theorem 2.

**Corollary 1.** For \( \bar{Y}^{(1)}, \bar{Y}^{(2)}, \ldots, \bar{Y}^{(n)} \), \( \bar{Y}^{(1)}, \bar{Y}^{(2)}, \ldots, \bar{Y}^{(n)} \), \( \bar{X}^{(1)}, \bar{X}^{(2)}, \ldots, \bar{X}^{(n)} \) and \( \bar{X}^{(1)}, \bar{X}^{(2)}, \ldots, \bar{X}^{(n)} \) in Theorem 2 and for \( Y, Y, X_1, X_2, \ldots, X_n, x_1, x_2, \ldots, x_n, Z \) and \( z \), let

\[
P'(Y = y, X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n, Z = z) = \prod_{i=1}^nP(Y = y|do(X_1 = x_1), do(X_2 = x_2), \ldots, do(X_n = x_n), Z = z)
\]

\[
\times P(\bar{X}_1 = x_1) \cdot P(\bar{X}_2 = x_2) \cdots P(\bar{X}_n = x_n)
\]

\[
\times P(Z = z)
\]

and let

\[
P''(Y = y, \bar{X}_1 = \bar{x}_1^{(1)}, \bar{X}_2 = \bar{x}_2^{(2)}, \ldots, \bar{X}_n = \bar{x}_n^{(n)}, Z = z) = \prod_{i=1}^nP(Y = y|\bar{X}_1^{(1)} = \bar{x}_1^{(1)}, \bar{X}_2^{(2)} = \bar{x}_2^{(2)}, \ldots, \bar{X}_n^{(n)} = \bar{x}_n^{(n)})
\]

\[
\times P(\bar{X}_1 = \bar{x}_1^{(1)}) \cdot P(\bar{X}_2 = \bar{x}_2^{(2)}) \cdots P(\bar{X}_n = \bar{x}_n^{(n)})
\]

\[
\times P(Z = z)
\]

Then,

\[
E_{P''}[Y|X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n, Z = z] = E_{P''[Y|\bar{X}_1^{(1)} = \bar{x}_1^{(1)}, \bar{X}_2^{(2)} = \bar{x}_2^{(2)}, \ldots, \bar{X}_n^{(n)} = \bar{x}_n^{(n)}, Z = z]
\]

(7)

\[
= E_{P}[Y|Z = z] + E_P[Y^{(1)}|\bar{X}_1^{(1)} = \bar{x}_1^{(1)}] + E_P[Y^{(2)}|\bar{X}_2^{(2)} = \bar{x}_2^{(2)}] + \cdots + E_P[Y^{(n)}|\bar{X}_n^{(n)} = \bar{x}_n^{(n)}]
\]

(8)
3.3 Algorithms

From the theorem and its corollary, we propose a meta-algorithm to estimate the average causal effects of intervention in continuous variables. The algorithm is divided into two phases; the model building phase and the average causal effect estimating phase.

Algorithm 1 shows the meta-algorithm of the model building phase. In this phase, a original data is used as input, then models is built while transforming it. In Algorithm 1 “SupervisedLearn” represents training of an arbitrary supervised regression algorithm. “SupervisedPredict” represents computation of predicted values from a pair of the input data and the models obtained from training “SupervisedLearn”.

Algorithm 2 shows the meta-algorithm of the average causal effect estimating phase. In this phase, an estimator \( \hat{Y} \) of the average causal effect \( \mathbb{E}[Y \mid \text{do}(X_1 = x_1), \text{do}(X_2 = x_2), \ldots, \text{do}(X_n = x_n), Z = z] \) is computed using models of “SupervisedLearn” in Algorithm 1 values \( x_1, x_2, \ldots, x_n \) and values \( z \), where \( x_1, x_2, \ldots, x_n \) correspond to values of intervention in \( X \) for each individual and \( z \) correspond to values of each individual’s covariates \( Z \).

4 Simulation Results and Discussion

We conducted numerical experiments on the proposing method by generating 1000 data sets from a model such as \( Z \sim \mathcal{U}(0, 1), X \sim \sin(Z) + \varepsilon_X, Y = X \cdot Z + \varepsilon_Y, \varepsilon_X \sim \mathcal{U}(-0.5, 0.5) \) and \( \varepsilon_Y \sim \mathcal{N}(0, 0.05) \). Here, “~ \( \mathcal{U}(a, b) \)” represents sampling from the Uniform distribution on an interval \( [a, b] \). “~ \( \mathcal{N}(a, b) \)” represents sampling from the Normal distribution with mean \( a \) and standard deviation \( b \).

Two results are shown here. One is a result of using LightGBM and the other is RandomForest. Figure 3 is the result of of using LightGBM. Figure 4 is the result of of using RandomForest. The figures show the estimate or the theoretical values of the average causal effect \( \mathbb{E}[Y \mid \text{do}(X = x), Z] \), where \( Z \) is fixed to 0.5. The plots show the estimated values by the proposing method. The dashed line shows the theoretical values \( Y = 0.5X \) for the model which generates data. In the figures, the \( x \)-axis corresponds to values \( x \) of intervention, and the \( y \)-axis corresponds to values of the average causal effect.

The result of LightGBM (Figure 3) shows that the proposing method almost exactly estimates the theoretical values of the average causal effects. On the other hand, one of RandomForest (Figure 4) shows that it fails to estimate that values.

The main reason for this is thought that (a) insufficient hyperparameter search in RandomForest and (b) differences in the ability between algorithms to estimate the conditional expectation. It is also important to choose an algorithm that can learn well patterns of data of interest and estimate well the conditional expectation of target.
Algorithm 1 Model building phase

Input: outcome \( Y \), treatments \( \{ X_1, X_2, \ldots, X_n \} \), covariates \( Z \)
Output: models \( \{ M_{Z \to Y}, M_{Z \to X_i}, M_{\bar{X}^{(i)} \to \bar{Y}^{(i)}}, M_{\bar{X}^{(i)} \to \bar{X}^{(i)}} \} \)

1. \( M_{Z \to Y} \leftarrow \text{SupervisedLearn}(Z, Y) \)
2. \( \bar{Y}^{(1)} \leftarrow Y - \text{SupervisedPredict}(M_{Z \to Y}, Z) \)
3. for \( i \) from 1 to \( n \) do
4. \( M_{Z \to X_i} \leftarrow \text{SupervisedLearn}(Z, X_i) \)
5. \( \bar{X}^{(1)}_i \leftarrow X_i - \text{SupervisedPredict}(M_{Z \to X_i}, Z) \)
6. end for
7. for \( i \) from 1 to \( n - 1 \) do
8. \( M_{\bar{X}^{(i)} \to \bar{Y}^{(i)}} \leftarrow \text{SupervisedLearn}(\bar{X}^{(i)}_i, \bar{Y}^{(i)}) \)
9. \( \bar{Y}^{(i+1)} \leftarrow \bar{Y}^{(i)} - \text{SupervisedPredict}(M_{\bar{X}^{(i)} \to \bar{Y}^{(i)}}, \bar{X}^{(i)}_i) \)
10. for \( j \) from \( i + 1 \) to \( n \) do
11. \( M_{\bar{X}^{(i)} \to \bar{X}^{(j)}} \leftarrow \text{SupervisedLearn}(\bar{X}^{(i)}_i, \bar{X}^{(j)}_j) \)
12. \( \bar{X}^{(i+1)}_j \leftarrow \bar{X}^{(i)}_j - \text{SupervisedPredict}(M_{\bar{X}^{(i)} \to \bar{X}^{(j)}}, \bar{X}^{(j)}_j) \)
13. end for
14. end for
15. \( M_{\bar{X}^{(n)} \to \bar{Y}^{(n)}} \leftarrow \text{SupervisedLearn}(\bar{X}^{(n)}_n, \bar{Y}^{(n)}) \)

Algorithm 2 Average causal effect estimating phase

Input: values of intervention \( \{ x_1, x_2, \ldots, x_n \} \), covariates \( z_i \), models \( \{ M_{Z \to Y}, M_{Z \to X_i}, M_{\bar{X}^{(i)} \to \bar{X}^{(j)}}, M_{\bar{X}^{(i)} \to \bar{Y}^{(i)}} \} \)
Output: the average causal effect \( \bar{Y} \)

1. for \( i \) from 1 to \( n \) do
2. \( \bar{x}^{(1)}_i \leftarrow x_i - \text{SupervisedPredict}(M_{Z \to X_i}, z) \)
3. end for
4. for \( i \) from 1 to \( n - 1 \) do
5. for \( j \) from \( i + 1 \) to \( n \) do
6. \( \bar{x}^{(i+1)}_j \leftarrow \bar{x}^{(i)}_j - \text{SupervisedPredict}(M_{\bar{X}^{(i)} \to \bar{X}^{(j)}}, \bar{x}^{(j)}_i) \)
7. end for
8. end for
9. \( \bar{Y} \leftarrow \text{SupervisedPredict}(M_{Z \to Y}, z) \)
10. for \( i \) from 1 to \( n \) do
11. \( \bar{Y} \leftarrow \bar{Y} + \text{SupervisedPredict}(M_{\bar{X}^{(i)} \to \bar{Y}^{(i)}}, \bar{x}^{(i)}_i) \)
12. end for
Figure 3: Results of estimating the Average Causal Effect on $Z = 0.5$ (LightGBM)

Figure 4: Results of estimating the Average Causal Effect on $Z = 0.5$ (RandomForestst)
5 Conclusion

We proposed a method to estimate the average causal effects of intervention in continuous variables using supervised regression algorithms of machine learning. We also have showed that the proposing method preserves for the identifiability of causal effects for models which generate original data. In addition, by simulation examines, it has been confirmed that the proposing method successfully estimates the average causal effect.

This method can estimate values of the average causal effect of intervention in continuous variables, whenever we can estimate it from original data. Furthermore, because this method is algorithm-free, it can be widely applied to supervised regression algorithms in machine learning.

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A Proof of Theorem

The following C-component, C-Forest and hedge are the important structures of a causal graph for the identifiability of causal effects. C-component is defined in [Tian and Pearl 2002]. C-Forest and hedge is first defined by [Shpitser and Pearl 2006].

**Definition 3 (C-Component).** Let \( G \) be a causal graph of semi-Markovian model such that a subset of its bidirected arcs forms a spanning tree over all vertices in \( G \). Then \( G \) is a C-Component(confounded Component).

**Definition 4 (C-Forest).** Let \( G \) be a causal graph of semi-Markovian model, where \( R \) is the root set. Then \( G \) is a \( R \)-rooted C-Forest if all nodes in \( G \) form a C-component.

**Definition 5 (hedge).** Let \( X, Y \) be sets of variables in \( G \). Let \( F, F' \) be \( R \)-rooted C-Forests such that \( F \cap X \neq \emptyset, F' \cap X = \emptyset, F' \subset F \) and \( R \subset \text{An}(Y) \cap \mathcal{X} \). Then \( F \) and \( F' \) form a hedge for \( P(Y|\text{do}(X)) \).

The following theorem gives a necessary and sufficient condition of the identifiability of causal effects for joint intervention. It is presented by [Shpitser and Pearl 2006].

**Theorem 6 (hedge criterion).** \( P(Y|\text{do}(X)) \) is identifiable from \( P \) in \( G \) if and only if there does not exist a hedge for \( P(Y'|\text{do}(X')) \) in \( G \), for any \( X' \subset X \) and \( Y' \subset Y \).

[Pearl 1995] has given do-calculus, the following rules R1-R3 of transformation between causal effects.

**do-calculus** [Pearl 1995]

R1. \( P(Y|\text{do}(X), Z, W) = P(Y|\text{do}(X), W) \) if \( (Y \perp \perp Z|X, W)_{\mathcal{G}(X)} \)
R2. \( P(Y|do(X), do(Z), W) = P(Y|do(X), Z, W) \) if \( (Y \perp Z \mid X, W)_{\overline{G}(X, Z)} \)

R3. \( P(Y|do(X), do(Z), W) = P(Y|do(X), W) \) if \( (Y \perp Z \mid X, W)_{\overline{G}(X, Z^*)} \)
where \( Z^* = Z \setminus An(W)_{\overline{G}(X)} \)

A necessary and sufficient condition of the identifiability of conditional causal effects \( P(Y = y|do(X = x), Z = z) \) is obtained by applying R2 above to it, which is presented by [Shpitser and Pearl 2012].

**Theorem 7** (Shpitser and Pearl [2012]). Let \( Z' \subset Z \) be the maximal set such that \( P(Y|do(X), Z) = P(Y, Z \setminus Z'|do(X), do(Z')) \). Then \( P(Y|do(X), Z) \) is identifiable in \( G \) if and only if \( P(Y, Z \setminus Z'|do(X), do(Z')) \) is identifiable in \( G \).

Hence, a necessary and sufficient condition of the identifiability of a conditional causal effect \( P(Y = y|do(X = x), Z = z) \) is that the \( P(Y, Z \setminus Z'|do(X), do(Z')) \) above satisfies the hedge criterion.

### A.1 Lemmas

First, we give some lemmas to prove the theorems.

**Lemma 8.** Let \( G \) be a DAG for \( V \) and \( U \). For disjoint sets of \( X, Y, Z \subset V \), let \( X \subset An(Y) \cup G \), and let \( P(Y|do(X), Z) \) be identifiable in \( G \). Let \( Z_{De} = Z \cap De(X) \cap De(Y) \) \( G \). Then,

\[
P(Y|do(X), Z) = \begin{cases} P(Y|X, Z) & \text{if } Z_{De} = \phi \\ \frac{P(Y|X, Z, Z_{De})P(Z_{De}|Y, X, Z_{De})}{P(Z_{De}|X, Z_{De})} & \text{if } Z_{De} \neq \phi. \end{cases}
\]  

(9)

**Proof of Lemma.** In this proof, each \( Z_i \in Z \) is assumed not independent of \( Y \). If there exists \( Z_i \in Z \) such that it is independent of \( Y \), remove such \( Z_i \)'s by using do-calculus R1 to them, then we can prove that case in the same manner below.

Firstly, note that \( Z \cap An(X) \cup G \cap De(Y) = \phi \). Let \( V' = V \setminus (Y \cup X \cup Z) \). If \( Z \cap An(X) \cup G \cap De(Y) = \phi \), then there exists a directed path through \( V' \) such that \( Y \leftarrow\rightarrow Z \leftarrow\rightarrow X \), which contradicts the assumption \( X \subset An(Y) \cup G \). Thus, let disjoint sets \( Z_1, Z_2, \) and \( Z_3 \) be

\[
\begin{align*}
Z_1 &= (Z \setminus De(X) \cup G) \cap An(Y) \cup G \\
Z_2 &= Z \cap De(X) \cup G \cap An(Y) \\
Z_3 &= Z \cap De(X) \cup G \cap De(Y) 
\end{align*}
\]

then \( Z \) can be divided such that \( Z = Z_1 \cup Z_2 \cup Z_3 \).

Secondly, note that there exist the directed paths through \( V' \) between \( Z_1, Z_2 \) and \( Z_3 \) in the only 3 cases as below.

P1. \( Z_1 \leftarrow\rightarrow Z_2 \)
In the case that  \( Z \), then there exists a path such that  \( X_i \xrightarrow{\varphi} Z_2 \xrightarrow{\varphi'} \cdots \xrightarrow{\varphi'} Z_1 \xrightarrow{\varphi'} X_j \), which contradicts the assumption  \( Z_1 \subset Z \setminus \text{De}(X)_G \). Similarly, the other paths than listed above are denied.

Therefore, we can obtain  \( P(Y, X, Z) = \sum_{\varphi'} P(V) \) as the form

\[
P(Y, X, Z) = P(Y|X, Z_1, Z_2) \cdot P(X|Z_1) \cdot P(Z_1) \\
\quad \times P(Z_2|X, Z_1) \cdot P(Z_3|Y, X, Z_1, Z_2).
\]

By (1), we have

\[
P(Y, Z|\text{do}(X)) = P(Y|X, Z_1, Z_2) \cdot P(Z_1) \cdot P(Z_2|X, Z_1) \\
\quad \times P(Z_3|Y, X, Z_1, Z_2). \tag{10}
\]

In the case  \( Z_3 = \phi \), we obtain by marginalizing out  \( Y \) of (11),

\[
P(Z|\text{do}(X)) = \sum_{y \in \mathcal{Y}_V} P(Y = y, Z|\text{do}(X)) \\
= \sum_{y \in \mathcal{Y}_V} P(Y = y|X, Z_1, Z_2) \cdot P(Z_1) \cdot P(Z_2|X, Z_1) \\
= P(Z_1) \cdot P(Z_2|X, Z_1) \sum_{y \in \mathcal{Y}_V} P(Y = y|X, Z_1, Z_2) \\
= P(Z_1) \cdot P(Z_2|X, Z_1).
\]

On the other hand, in the case that  \( Z_3 \neq \phi \), we obtain

\[
P(Z|\text{do}(X)) = \sum_{y \in \mathcal{Y}_V} P(Y = y, Z|\text{do}(X)) \\
= \sum_{y \in \mathcal{Y}_V} P(Y = y|X, Z_1, Z_2) \cdot P(Z_1) \\
\quad \times P(Z_2|X, Z_1) \cdot P(Z_3|Y = y, X, Z_1, Z_2) \\
= P(Z_1) \cdot P(Z_2|X, Z_1) \sum_{y \in \mathcal{Y}_V} P(Z_3|Y = y, X, Z_1, Z_2) \\
\quad \times P(Y = y|X, Z_1, Z_2) \\
= P(Z_1) \cdot P(Z_2|X, Z_1) \cdot P(Z_3|X, Z_1, Z_2).
\]

Summarizing the above results,

\[
P(Z|\text{do}(X)) = \begin{cases} 
P(Z_1) \cdot P(Z_2|X, Z_1) & \text{if } Z_3 = \phi \\
P(Z_1) \cdot P(Z_2|X, Z_1) \cdot P(Z_3|X, Z_1, Z_2) & \text{if } Z_3 \neq \phi.
\end{cases} \tag{11}
\]
Inserting \((10)\) and \((11)\) into \((2)\),

\[
P(Y|do(X), Z) = \frac{P(Y, Z|do(X))}{P(Z|do(X))} = \begin{cases} 
P(Y|X, Z_1, Z_2) & \text{if } Z_3 = \phi \\
\frac{P(Y|X, Z_1, Z_2)P(Z_1|Y, X, Z_2)}{P(Z_2|X, Z_2)} & \text{if } Z_3 \neq \phi.
\end{cases}
\]

Note that \(Z_{De} = Z_3\) and \(Z \setminus Z_{De} = Z_1 \cup Z_2\), therefore we obtain \((9)\). \(\square\)

**Lemma 9.** Let \(G\) be a DAG for \(V\) and \(U\). For disjoint sets of \(X, Y \subset V\), let \(P(Y|do(X))\) be identifiable in \(G\). For fixed \(X' \in X\), let \(\tilde{G}\) a graph obtained from \(G\) by deleting all the arrows emerging from \(X'\) to \(V \setminus \{X'\}\) and all the arrows emerging from \(X'\) to \(U\). Then \(P(Y|do(X \setminus \{X'\}))\) is also identifiable in \(\tilde{G}\).

**proof of Lemma 9.** Let \(P\) be a model relative to \(G\) and let \(\tilde{P}\) be a model relative to \(\tilde{G}\). Assume that \(P(Y|do(X))\) is identifiable in \(G\) but that \(P(Y|do(X \setminus \{X'\}))\) is not identifiable in \(\tilde{G}\).

By theorem \((6)\) for some \(Y_0 \subset Y, X_0 \subset (X \setminus \{X'\})\) and \(R \subset An(Y_0)_{G(X_0)}^\perp\), there exist \(R\)-rooted \(C\)-Forests \(\tilde{F}, \tilde{F}'\) such that \(\tilde{F} \cap X_0 \neq \phi\) and \(\tilde{F}' \cap X_0 = \phi\), and \(\tilde{F}, \tilde{F}'\) form a hedge for \(\tilde{P}(Y_0|do(X_0))\).

Now let \(F = \tilde{F} \setminus \{X'\}\) and \(F' = \tilde{F}' \setminus \{X'\}\), then \(F\) and \(F'\) form a hedge for \(P(Y_0|do(X_0))\) in \(G\).

In fact, \(R \subset Y_0\) and \(F \cap X_0 \neq \phi\) and \(F' \cap X_0 = \phi\). In addition, \(G\) has the same arrows as \(\tilde{G}\) between variables of \(V \setminus \{X'\}\) and between \(U\) and \(V \setminus \{X'\}\). Since \(F, F' \subset V \setminus \{X'\}\), \(F\) and \(F'\) have same bidirected arcs and same arrows in \(G\) as in \(\tilde{G}\). Thus, \(F\) and \(F'\) are \(C\)-Forest in \(G\), and form a hedge for \(P(Y_0|do(X_0))\). Therefore, \(P(Y|do(X))\) is not identifiable in \(G\), which contradicts the assumption. Consequently, \(\tilde{P}(Y|do(X \setminus \{X'\}))\) is identifiable in \(\tilde{G}\).

This completes the proof of the lemma. \(\square\)

**Lemma 10.** Let \(G\) be a DAG for \(V\) and \(U\). For disjoint sets of \(X, Y, Z \subset V\), let \(P(Y|do(X), Z)\) be identifiable in \(G\). Let \(\tilde{G}\) a graph obtained from \(G\) by deleting all the arrows emerging from \(Z\) to \(V \setminus Z\) and all the arrows emerging from \(Z\) to \(U\). Then, \(P(Y|do(X), Z)\) is also identifiable in \(\tilde{G}\).

**proof of Lemma 10.** Let \(P\) be a model relative to \(G\) and let \(\tilde{P}\) be a model relative to \(\tilde{G}\). Assume that \(P(Y|do(X), Z)\) is identifiable in \(G\) but that \(\tilde{P}(Y|do(X))\) is not identifiable in \(\tilde{G}\).

By theorem \((7)\) there exists \(Z_1 \subset Z\) such that

\[
P(Y|do(X), Z) = P(Y, Z \setminus Z_1|do(X), do(Z_1)),
\]
and $P(Y, Z \setminus Z_1|do(X), do(Z_1))$ is identifiable in $G$. On the other hand, by theorem 8 for some $Y' \subset Y$, $X' \subset X$ and $R \subset An(Y'_{\overline{G}(X')}^{-})$ there exist $R$-rooted C-Forests $\tilde{F}, \tilde{F}'$ such that $\tilde{F} \cap X' \neq \phi$ and $\tilde{F}' \cap X' = \phi$, and $\tilde{F}, \tilde{F}'$ form a hedge for $P(Y|do(X'))$.

Now, let $F = \tilde{F} \setminus Z_1$ and $F' = \tilde{F}' \setminus Z_1$. Then $F$ and $F'$ form a hedge for $P(Y, Z \setminus Z_1|do(X), do(Z_1))$ in $G$. In fact, $R \subset Y'$ and $F \cap X' \neq \phi$ and $F' \cap X' = \phi$. In addition, $G$ has the same arrows as $\overline{G}$ between variables of $V \setminus Z_1$ and between $U$ and $V \setminus Z_1$. Since $F, F' \subset (V \setminus Z_1)$, $F$ and $F'$ have same bidirected arcs and same arrows in $G$ as in $\overline{G}$. Thus, $F$ and $F'$ are C-Forest in $G$, and form a hedge for $P(Y'|do(X'))$. Therefore, $P(Y, Z|Z_1|do(X), do(Z_1))$ is not identifiable in $G$, which contradicts the assumption. Consequently, $\tilde{P}(Y|do(X))$ is identifiable in $\overline{G}$.

By the way, by the definition of $\overline{G}$, it holds that $(Y \perp \perp Z)_{\overline{G}}$. Thus, $(Y \perp \perp Z|X)_{\overline{G}}$. Therefore, We can apply do-calculus R1 to $\tilde{P}(Y|do(X), Z)$, and we obtain

$$\tilde{P}(Y|do(X), Z) = \tilde{P}(Y|do(X)).$$

That is, the identifiability of $\tilde{P}(Y|do(X), Z)$ coincides that of $\tilde{P}(Y|do(X))$. As a result, $\tilde{P}(Y|do(X), Z)$ is identifiable in $\overline{G}$.

This completes the proof of the lemma. \qed

**Lemma 11.** Let $G$ be a DAG for $V$ and $U$. Assume that for disjoint sets $Y, X \subset V, X \subset An(Y)_G \cup Ind(Y)_G$. Let a $X' \in X$ be fixed. For $A \in V \setminus \{X'\} \cup U$, let

$$\tilde{A} = \begin{cases} A - E[A|X' = x'] & \text{if } A \neq X' \\ A & \text{if } A = X', \end{cases}$$

(12)

For $A \subset V \cup U$, let $\tilde{A} = \{\tilde{A}|A \in A\}$. Then, $\tilde{V} \cup \tilde{U}$ is a semi-Markovian model. Let $\overline{G}$ be a DAG for $\tilde{V} \cup \tilde{U}$, then $(X \setminus \{X'\}) \subset An(\tilde{Y})_{\overline{G}} \cup Ind(\tilde{Y})_{\overline{G}}$. Moreover, if $P(Y|do(X))$ is identifiable in $G$, then $P(\tilde{Y}|do(\tilde{X} \setminus \{X'\}))$ also is identifiable in $\overline{G}$.

**proof of Lemma 11** Initially, we will show the lemma in the case that $X \cap Ind(Y)_G = \phi$. Let $V_0 = V \setminus \{X'\}$. Note that, for any $V_0 \in V_0$,

$$\overline{V}_0 \perp \perp X'.$$

(13)

In fact, for any variables $\xi$ and $\eta$, it holds that

$$\xi \perp \eta \iff E[\xi|\eta] = E[\xi],$$

and for $\overline{V}_0$ and $X'$,

$$E[\overline{V}_0|X'] = E[V_0 - E[V_0|X']|X'] = E[V_0|X'] - E[V_0|X'] = 0 = E[\overline{V}_0].$$
Thus,
\[ P(\tilde{V}_0, X') = P(\tilde{V}_0)P(X'). \]
Similarly, for \( U \in U \),
\[ P(\tilde{U}, X') = P(\tilde{U})P(X'). \]
By the way, for \( v \in \mathcal{A}_{V_0} \) and \( u \in \mathcal{A}_{U} \), let
\[ \tilde{v} = v - E[V_0|X' = x'], \quad \tilde{u} = u - E[U|X' = x'], \]
then according to the definition of \( \tilde{V}_0 \) and \( \tilde{U} \),
\[ P(\tilde{V}_0 = \tilde{x}, \tilde{U} = \tilde{u}) = P(\tilde{V}_0 = v, U = u|X' = x'). \]
Therefore,
\[
P(\tilde{V}_0 = \tilde{v}, \tilde{U} = \tilde{u}, \tilde{X}' = \tilde{x}') = \begin{align*}
P(\tilde{V}_0 = \tilde{v}, \tilde{U} = \tilde{u}, \tilde{X}' = \tilde{x}') &= P(\tilde{V}_0 = \tilde{v}, \tilde{U} = \tilde{u}|\tilde{X}' = \tilde{x}')P(\tilde{X}' = \tilde{z}) \\
&= P(\tilde{V}_0 = \tilde{v}, \tilde{U} = \tilde{u})P(\tilde{X}' = \tilde{x}') \\
&= P(\tilde{V}_0 = v, U = u|X' = x')P(X' = x') \\
&= P(\tilde{V}_0 = v, U = u, X' = x').
\end{align*}
\]
Similarly, for \( V' \subset V_0, U' \subset U, v \in \mathcal{A}_{V'} \) and \( u \in \mathcal{A}_{U'} \), let
\[ \tilde{v} = v - E[V'|X'], \quad \tilde{u} = u - E[U'|X'], \]
then,
\[ P(V' = v, U' = u, X' = x') = P(\tilde{X}' = \tilde{v}, \tilde{U}' = \tilde{u}, X' = x'). \]
Therefore
\[
P(V = v|Pa(V)_G = pa_V, UPa(V)_G = u_V) = P(\tilde{V} = \tilde{v}|\tilde{Pa}(V)_G = \tilde{pa}_V, \tilde{UP}(\tilde{V})_G = \tilde{u}_V), \tag{14}\]
where \( pa_V \) and \( u_V \) are values of \( Pa(V) \) and \( UPa(V) \).
Hence,
\[ Pa(\tilde{V})_G \subset \tilde{Pa}(V)_G \tag{15} \]
\[ UPa(\tilde{V})_G \subset \tilde{UP}(\tilde{V})_G. \tag{16} \]
Now, for \( V \in X, W \in Pa(V)_G \) and \( U \in UPa(V)_G \), the following equations \([17],[21]\) hold.
\[
\begin{align*}
V &= X' \quad \text{and} \quad W \neq X' \quad \implies \quad \tilde{V} \perp \tilde{W} \tag{17} \\
V \neq X' \quad \text{and} \quad W = X' \quad \implies \quad \tilde{V} \perp \tilde{W} \tag{18} \\
V \neq X' \quad \text{and} \quad W \neq X' \quad \implies \quad \tilde{W} \in Pa(\tilde{V})_G \tag{19} \\
V = X' \quad \implies \quad \tilde{V} \perp \tilde{U} \tag{20}
\end{align*}
\]
Therefore, since $V \neq X'$, it holds that $\tilde{U} \in UPa(\tilde{V})_G$. 

(17), (18) and (20) hold from (13). For (19), assume that $V \neq X'$ and $W \neq X'$ and $\tilde{W} \notin Pa(\tilde{V})_G$. From (13), $\tilde{W} \in Pa(V) \setminus Pa(\tilde{V})_G$. Thus, at the right hand side of (14),

$$P(\tilde{V}|Pa(V)_G, \overline{UPa(V)_G}) = P(\tilde{V}|\bar{Pa}(V)_G \setminus \tilde{W}, \overline{UPa(V)_G}).$$

Therefore,

$$\tilde{V} \perp \tilde{W}.$$ 

(22)

Now, by (13), it holds that $\tilde{V} \perp X'$ and $\tilde{W} \perp X'$. Thus,

$$\tilde{V} \perp \tilde{W} + E[W|X'] = W;$$

and

$$\tilde{W} \perp \tilde{V} + E[V|X'] = V.$$ 

Therefore, since

$$E[\tilde{V} + E[V|X']|\tilde{W}, X'] = E[\tilde{V}|\tilde{W}, X'] - E[E[V|X']|\tilde{W}, X'] = E[V|X'],$$

it holds that

$$E[V|W] = E[\tilde{V} + E[V|X']|\tilde{W}] + E[W|X'], \tilde{W}, X'] = E[E[V|X']|\tilde{W}] + E[W|X'], \tilde{W}, X'] = E[V|X']$$

Since any variables $A, B$ and $C$, if $E[A|B] = E[A|C]$, then there exists a function $f : X_B \to X_C$ such that

$$P(B = b) = P(C = f(b)),$$

there exists a function $f : X_{X'} \to X_W$ such that $W = f(X')$. Therefore, it contradicts that $W \neq X'$ because we assume that $V \neq f(W)$ for any function $f, V \in V$ and $W \subset V \setminus V$. Therefore, the proof of (19) is shown by replacing $Pa(V)_G$ with $UPa(V)_G$ and replacing $Pa(\tilde{V})_G$ with $UPa(\tilde{V})_G$ at the above proof of (19) because always $U \neq X'$.

Next, consider a DAG $G'$ obtained from $G$ by replacing each $A \in V \cup U$ with $\bar{A}$. Since $P(Y|do(X))$ is identifiable in $G$, $P(\bar{Y}|do(\bar{X}))$ is identifiable in $G'$. Because of (17)-(21), $G$ is just a graph obtained by deleting the arrows in $G'$ between $X'$ and $\bar{V}_0$ and between $X'$ and $U$. Therefore, $(\bar{X} \setminus \{X'\}) \subset An(\bar{Y})_{\bar{G}} \cup Ind(\bar{Y})_{\bar{G}}$. The remaining claims of the lemma is shown by lemma 9.

Next, we will show in the case that $X \cap Ind(Y)_G \neq \phi$. Let $X_1 = X \setminus Ind(Y)_G$ and let $X_2 = X \cap Ind(Y)_G$. Since $Y \perp X_2$, it holds that $(Y \perp X_2|X_1)_G$. Therefore, by do-calculus $R3$, $P(Y|do(X)) = P(Y|do(X_1))$. Hence, we can apply $P(Y|do(X_1))$ to the above proof of the case that $X \cap Ind(Y)_G = \phi$ in
$G$. In consequence, $\tilde{V} \cup \tilde{U}$ is a semi-Markovian model and let $\tilde{G}$ be a DAG for $\tilde{V} \cup \tilde{U}$, then $(\tilde{X}_1 \setminus \{X'\}) \subset An(\tilde{Y})_{\tilde{G}} \cup Ind(\tilde{Y})_{\tilde{G}}$. Moreover, $P(\tilde{Y}|do(\tilde{X}_1 \setminus \{X'\}))$ is identifiable in $\tilde{G}$.

Note that, $(\tilde{X}_2 \setminus \{X'\}) \not\to \tilde{Y}_{\tilde{G}}$. In fact, because of (15), let $W = Pa(Y)_{\tilde{G}} \cap (\tilde{X}_2 \setminus \{X'\})$, then

$$Pa(\tilde{Y})_{\tilde{G}} \cap (\tilde{X}_2 \setminus \{X'\}) \subset \tilde{P}a(Y)_{\tilde{G}} \cap (\tilde{X}_2 \setminus \{X'\}) = \tilde{W} = \phi.$$ 

Therefore, by $do$-calculus $R3$ to delete $do(\tilde{X}_2 \setminus \{X'\})$ of $do(\tilde{X} \setminus \{X'\})$, we obtain $P(\tilde{Y}|do(\tilde{X} \setminus \{X'\})) = P(\tilde{Y}|do(\tilde{X}_1 \setminus \{X'\}))$ in $\tilde{G}$. Recall that $P(\tilde{Y}|do(\tilde{X}_1 \setminus \{X'\}))$ is identifiable in $\tilde{G}$. Therefore, $P(\tilde{Y}|do(\tilde{X} \setminus \{X'\}))$ is identifiable in $\tilde{G}$. Finally, since $(\tilde{X}_2 \setminus \{X'\}) \subset An(\tilde{Y})_{\tilde{G}} \cup Ind(\tilde{Y})_{\tilde{G}}$ and since $(\tilde{X}_2 \setminus \{X'\}) \subset Ind(\tilde{Y})_{\tilde{G}}$, we obtain $\tilde{X} \setminus \{X'\} = (\tilde{X}_1 \cup \tilde{X}_2) \setminus \{X'\} \subset An(\tilde{Y})_{\tilde{G}} \cup Ind(\tilde{Y})_{\tilde{G}}$.

This completes the proof of the lemma. 

**Lemma 12.** Let $G$ be a DAG for $V$ and $U$. Assume that for disjoint sets $Y, X, Z \subset V, X \subset An(Y)_{\tilde{G}} \cup Ind(Y)_{\tilde{G}}$. Let $z \in A_Z$ be fixed. For $A \in V \cup U$, let

$$\tilde{A} = \begin{cases} A - E[A|Z = z] & \text{if } A \notin Z \\ A & \text{if } A \in Z. \end{cases}$$

For $A \subset V \cup U$, let $\tilde{A} = \{\tilde{A}|A \in A\}$. Then, $\tilde{V} \cup \tilde{U}$ is a semi-Markovian model. Let $\tilde{G}$ be a DAG for $\tilde{V} \cup \tilde{U}$, then $\tilde{X} \subset An(\tilde{Y})_{\tilde{G}} \cup Ind(\tilde{Y})_{\tilde{G}}$. Moreover, if $P(Y|do(X), Z)$ is identifiable in $G$, then $P(\tilde{Y}|do(\tilde{X}))$ is identifiable in $\tilde{G}$.

**proof of Lemma 12.** Initially, we will show the lemma in the case that $X \cap Ind(Y)_{\tilde{G}} = \phi$. Let $V_0 = V \setminus Z$. Note that, for any $V_0 \in V_0$,

$$\tilde{V}_0 \perp \perp Z.$$  

(23)

in fact, for any variables $\xi$ and $\eta$, it holds that

$$\xi \perp \perp \eta \iff E[\xi|\eta] = E[\xi],$$

and for $\tilde{V}_0$ and $Z$,

$$E[\tilde{V}_0|Z] = E[V_0 - E[V_0|Z]|Z] = E[V_0|Z] - E[V_0|Z] = 0 = E[\tilde{V}_0].$$

Thus,

$$P(\tilde{V}_0, Z) = P(\tilde{V}_0)P(Z).$$

Similarly, for $U \in U$,

$$P(\tilde{U}, Z) = P(\tilde{U})P(Z).$$

By the way, for $v \in X_{\tilde{V}_0}$ and $u \in X_{\tilde{U}}$, let

$$\tilde{v} = v - E[V_0|Z], \quad \tilde{u} = u - E[U|Z],$$

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then according to the definition of \( \tilde{V}_0 \) and \( \tilde{U} \),

\[
P(\tilde{V}_0 = \tilde{v}, \tilde{U} = \tilde{u}) = P(V = v, U = u | Z = z).
\]

Therefore,

\[
P(\tilde{V}_0 = \tilde{v}, \tilde{U} = \tilde{u}, \tilde{Z} = \tilde{z}) = P(\tilde{V}_0 = \tilde{v}, \tilde{U} = \tilde{u}, \tilde{Z} = \tilde{z}) P(\tilde{Z} = \tilde{z}) = P(\tilde{V}_0 = \tilde{v}, \tilde{U} = \tilde{u}) P(\tilde{Z} = \tilde{z}) = P(V_0 = v, U = u | Z = z) P(Z = z) = P(V_0 = v, U = u, Z = z).
\]

Similarly, for \( V' \subset V_0, U' \subset U_G, v \in X_{V'} \) and \( u \in X_{U'} \), let

\[
\tilde{v} = v - E[V'|Z], \quad \tilde{u} = u - E[U'|Z],
\]

then,

\[
P(V' = v, U' = u, Z = z) = P(\tilde{V}' = \tilde{v}, \tilde{U}' = \tilde{u}, Z = z).
\]

Therefore,

\[
P(V = v | Pa(V)_G = pa_V, U Pa(V)_G = u_V) = P(V = \tilde{v} | Pa(V)_G = \tilde{pa}_V, U \tilde{Pa}(V)_G = \tilde{u}_V),
\]

where \( pa_V \) and \( u_V \) are the values of \( Pa(V)_G \) and \( U Pa(V)_G \).

Therefore,

\[
Pa(\tilde{V})_G \subset \tilde{Pa}(V)_G
\]

\[
U \tilde{Pa}(\tilde{V})_G \subset \tilde{U \tilde{Pa}(V)}_G
\]

Now, for \( V \in V, W \in Pa(V)_G \) and \( U \in U Pa(V)_G \), the following equations \( (27) \) - \( (32) \) hold.

\[
V \in Z \text{ and } W \in Z \quad \Rightarrow \quad \tilde{W} \in Pa(\tilde{V})_G
\]

\[
V \in Z \text{ and } W \notin Z \quad \Rightarrow \quad \tilde{V} \perp \tilde{W}
\]

\[
V \notin Z \text{ and } W \in Z \quad \Rightarrow \quad \tilde{V} \perp \tilde{W}
\]

\[
V \notin Z \text{ and } W \notin Z \quad \Rightarrow \quad \tilde{W} \in Pa(\tilde{V})_G
\]

\[
V \in Z \quad \Rightarrow \quad \tilde{V} \perp \tilde{U}
\]

\[
V \notin Z \quad \Rightarrow \quad \tilde{U} \in U Pa(\tilde{V})_G.
\]

\( (27) \) is clear from \( \tilde{V} = V, \tilde{W} = W \). \( (28), (29) \) and \( (31) \) hold from \( (23) \). For \( (30) \), assume that \( V \notin Z \) and \( W \notin Z \) and \( W \notin Pa(V)_G \). From \( (24) \), \( \tilde{W} \in Pa(V)_G \setminus Pa(\tilde{V})_G \). Thus, at the right hand side \( (24) \),

\[
P(\tilde{V} | \tilde{Pa}(V)_G, U \tilde{Pa}(V)_G) = P(\tilde{V} | \tilde{Pa}(V)_G \setminus \tilde{W}, \tilde{U \tilde{Pa}(V)}_G).
\]
Therefore, 
\[ \widetilde{V} \perp \widetilde{W}. \]  

(33)

Now, according to (23), \( \widetilde{V} \perp \mathbf{Z} \) and \( \widetilde{W} \perp \mathbf{Z} \), thus \( \widetilde{V} \perp E[W|\mathbf{Z}] \) and \( \widetilde{W} \perp E[V|\mathbf{Z}] \).

Now, by (23), it holds that \( \widetilde{V} \perp \mathbf{Z} \) and \( \widetilde{W} \perp \mathbf{Z} \). Thus,
\[ \widetilde{V} \perp \widetilde{W} + E[W|\mathbf{Z}] = W, \]
and
\[ \widetilde{W} \perp \widetilde{V} + E[V|\mathbf{Z}] = V. \]

Therefore, since
\[ E[\widetilde{V} + E[V|\mathbf{Z}]|\widetilde{W}, \mathbf{Z}] = E[\widetilde{V}|\widetilde{W}, \mathbf{Z}] - E[E[V|\mathbf{Z}]|\widetilde{W}, \mathbf{Z}] = E[V|\mathbf{Z}], \]

it holds that
\[ E[V|W] = E[\widetilde{V} + E[V|\mathbf{Z}]|\widetilde{W} + E[W|\mathbf{Z}], \widetilde{W}, \mathbf{Z}] \]
\[ = E[E[V|\mathbf{Z}]|\widetilde{W} + E[W|\mathbf{Z}], \widetilde{W}, \mathbf{Z}] \]
\[ = E[V|\mathbf{Z}] \]

Since any variables \( A, B \) and \( C \), if \( E[A|B] = E[A|C] \), then there exists a function \( f : \mathcal{X}_B \to \mathcal{X}_C \) such that
\[ P(B = b) = P(C = f(b)), \]
there exists a function \( f : \mathcal{X}_Z \to \mathcal{X}_W \) such that \( W = f(Z) \). Then, \( W \in Z \)

because we assume that \( V \neq f(W) \) for any function \( f, V \in V \) and \( W \subset V \setminus V \).

Therefore, it contradicts that \( W \notin Z \). Therefore, (30) holds. For (32), by replacing \( Pa(V)_G \) with \( UPa(V)_G \) and replacing \( Pa(V)_{\bar{G}} \) with \( UPa(V)_{\bar{G}} \) at the above proof of (30) because always \( U \notin Z \).

Now, We consider a DAG \( G' \) obtained from \( G \) by replacing each \( A \in V \cup U \) with \( \bar{A} \). Since \( P(Y|do(X), Z) \) is identifiable in \( G \), \( P(\bar{Y}|do(\bar{X}), Z) \) is identifiable in \( G' \). Because of (27)-(32), \( \bar{G} \) is just a graph obtained by deleting the arrows in \( G' \) between \( Z \) and \( V \setminus Z \) and between \( Z \) and \( U \). Therefore, \( \bar{X} \subset An(\bar{Y})_\bar{G} \cup Ind(\bar{Y})_\bar{G} \). The remaining claims of the lemma is shown by lemma (10).

Next, we will show in the case that \( X \cap Ind(Y)_G \neq \phi \). Let \( X_1 = X \setminus Ind(Y)_G \) and let \( X_2 = X \cap Ind(Y)_G \). Since \( Y \perp X_2 \), it holds that \( Y \perp X_2 \). Therefore, by do-calculus R3, \( P(Y|do(X)) = P(Y|do(X_1)) \). Hence, we can apply \( P(Y|do(X_1)) \) to the above proof of the case that \( X \cap Ind(Y)_G = \phi \). In consequence, \( \bar{Y} \cup \bar{U} \) is a semi-Markovian model and let \( \bar{G} \) be a DAG for \( \bar{V} \cup \bar{U} \), then \( \bar{X}_1 \subset An(\bar{Y})_{\bar{G}} \cup Ind(\bar{Y})_{\bar{G}} \). Moreover, \( P(\bar{Y}|do(X_1)) \) is identifiable in \( \bar{G} \).

Note that, \( \bar{X}_2 \perp \bar{Y} \). In fact, because of (15), let \( \bar{W} = Pa(\bar{Y})_{\bar{G}} \cap \bar{X}_2 \), then
\[ Pa(\bar{Y})_{\bar{G}} \cap \bar{X}_2 \subset Pa(Y)_G \cap \bar{X}_2 = \bar{W} = \phi \]
Therefore, by do-calculus R3 to delete do(\(\tilde{X}_2\)) of do(\(\tilde{X}\)), \(P(\tilde{Y}|\text{do}(\tilde{X})) = P(\tilde{Y}|\text{do}(\tilde{X}_1))\) in \(\tilde{G}\). Recall that \(P(\tilde{Y}|\text{do}(\tilde{X}_1))\) is identifiable in \(\tilde{G}\). Therefore, \(P(\tilde{Y}|\text{do}(\tilde{X}))\) is identifiable in \(\tilde{G}\). Finally, since \(\tilde{X}_1 \subset \text{An}(\tilde{Y})_{\tilde{G}} \cup \text{Ind}(\tilde{Y})_{\tilde{G}}\) and \(\tilde{X}_2 \subset \text{Ind}(\tilde{Y})_{\tilde{G}}\), we obtain \(\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2 \subset \text{An}(\tilde{Y})_{\tilde{G}} \cup \text{Ind}(\tilde{Y})_{\tilde{G}}\).

This completes the proof of the lemma. \(\square\)

**proof of Theorem 2** Let \(\tilde{V}^{(k)} = \{\tilde{V}^{(k)}\} \cup \tilde{X}^{(k)} \cup Z\) and let \(\tilde{U}^{(k)} = \{\tilde{U}^{(k)}|U \in U\}\).

By lemma [12] \(\tilde{V}^{(1)} \cup \tilde{U}^{(1)}\) is a semi-Markovian model. Let \(\tilde{G}^{(1)}\) be a DAG for \(\tilde{V}^{(1)} \cup \tilde{U}^{(1)}\), then \(\tilde{X}^{(1)} \subset \text{An}(\tilde{Y}^{(1)})_{\tilde{G}^{(1)}} \cup \text{Ind}(\tilde{Y}^{(1)})_{\tilde{G}^{(1)}}\). In addition, from the identifiability of \(P(Y|\text{do}(X), Z)\) in \(G\), \(P(\tilde{Y}^{(1)}|\text{do}(\tilde{X}^{(1)}))\) is identifiable in \(\tilde{G}^{(1)}\). Moreover, it holds that

\[
\tilde{Y}^{(1)}, \tilde{X}^{(1)} \perp \perp Z.
\]

In fact, for any variables \(\xi\) and \(\eta\), it holds that

\[
\xi \perp \perp \eta \iff E[\xi|\eta] = E[\xi],
\]

and for \(\tilde{Y}^{(1)}\) and \(Z\),

\[
E[\tilde{Y}^{(1)}|Z] = E[Y - E[Y|Z]|Z] = E[Y|Z] - E[Y|Z] = 0 = E[\tilde{Y}^{(1)}].
\]

For \(\tilde{X}^{(1)}_i\) and \(Z\) \((1 \leq i \leq n)\), it holds that

\[
E[\tilde{X}^{(1)}_i|Z] = E[X_i - E[X_i|Z]|Z] = E[X_i|Z] - E[X_i|Z] = 0 = E[\tilde{X}^{(1)}_i].
\]

Additionally,

\[
E[\tilde{Y}^{(2)}|\tilde{X}^{(1)}_1] = E[\tilde{Y}^{(1)} - E[\tilde{Y}^{(1)}|\tilde{X}^{(1)}_1]|\tilde{X}^{(1)}_1] = E[\tilde{Y}^{(1)}|\tilde{X}^{(1)}_1] - E[\tilde{Y}^{(1)}|\tilde{X}^{(1)}_1] = 0 = E[\tilde{Y}^{(1)}]
\]

\[
E[\tilde{X}^{(2)}_2|\tilde{X}^{(1)}_1] = E[\tilde{X}^{(1)}_2 - E[\tilde{X}^{(1)}_2|\tilde{X}^{(1)}_1]|\tilde{X}^{(1)}_1] = E[\tilde{X}^{(1)}_2|\tilde{X}^{(1)}_1] - E[\tilde{X}^{(1)}_2|\tilde{X}^{(1)}_1] = 0 = E[\tilde{X}^{(1)}_1].
\]

Summarizing above results,

\[
\tilde{Y}^{(1)}, \tilde{X}^{(1)} \perp \perp Z
\]

\[
\tilde{Y}^{(2)}, \tilde{X}^{(2)}_2 \perp \perp \tilde{X}^{(1)}_1.
\]

In addition, it also holds that

\[
\tilde{Y}^{(2)}, \tilde{X}^{(2)} \perp \perp Z.
\]
In fact,
\[
E[\tilde{Y}^{(2)}|Z] = E[\tilde{Y}^{(1)} - E[\tilde{Y}^{(1)}|\tilde{X}_1^{(1)}]|Z] \\
= E[\tilde{Y}^{(1)}|Z] - E[E[\tilde{Y}^{(1)}|\tilde{X}_1^{(1)}]|Z] \\
= E[\tilde{Y}^{(1)}] - E[E[\tilde{Y}^{(1)}|\tilde{X}_1^{(1)}]] \\
= E[\tilde{Y}^{(1)}] - E[\tilde{Y}^{(1)}] \\
= 0 \\
= E[\tilde{Y}^{(2)}] \\
\]

\[
E[\tilde{X}_2^{(2)}|Z] = E[\tilde{X}_2^{(1)} - E[\tilde{X}_2^{(1)}|\tilde{X}_1^{(1)}]|Z] \\
= E[\tilde{X}_2^{(1)}|Z] - E[E[\tilde{X}_2^{(1)}|\tilde{X}_1^{(1)}]|Z] \\
= E[\tilde{X}_2^{(1)}] - E[E[\tilde{X}_2^{(1)}|\tilde{X}_1^{(1)}]] \\
= E[\tilde{X}_2^{(1)}] - E[\tilde{X}_2^{(1)}] \\
= 0 \\
= E[\tilde{X}_2^{(2)}].
\]

Since lemma [1], \(\tilde{V}^{(2)} \cup \tilde{U}^{(2)}\) is a semi-Markovian model. Let \(\tilde{G}^{(2)}\) be a DAG for \(\tilde{V}^{(2)} \cup \tilde{U}^{(2)}\), then \(\tilde{X}^{(2)} \subset An(\tilde{Y}^{(2)})_{\tilde{G}^{(2)}} \cup Ind(\tilde{Y}^{(2)})_{\tilde{G}^{(2)}}\). Because \(P(Y|do(X^{(1)}), Z^{(1)})\) is identifiable in \(\tilde{G}^{(1)}\), \(P(Y^{(2)}|do(X^{(2)}))\) is identifiable in \(\tilde{G}^{(2)}\).

Now, assume the following as inductive assumptions for \(k \geq 1\).

1. \(\tilde{Y}^{(k)} \perp \{\tilde{X}_1^{(1)}, \tilde{X}_2^{(2)}, \ldots, \tilde{X}_{k-1}^{(k-1)}\}\)
2. \(\tilde{X}_i^{(k)} \perp \{\tilde{X}_1^{(1)}, \tilde{X}_2^{(2)}, \ldots, \tilde{X}_{k-1}^{(k-1)}\} \quad (\forall i \geq k)\)
3. \(\tilde{Y}^{(k)}, \tilde{X}^{(k)} \perp Z\)
4. \(\tilde{V}^{(k)} \cup \tilde{U}^{(k)}\) is a semi-Markovian model, let \(\tilde{G}^{(k)}\) be a DAG for \(\tilde{V}^{(k)} \cup \tilde{U}^{(k)}\), then \(\tilde{X}^{(k)} \subset An(\tilde{Y}^{(k)})_{\tilde{G}^{(k)}} \cup Ind(\tilde{Y}^{(k)})_{\tilde{G}^{(k)}}\) and \(P(\tilde{Y}^{(k)}|do(\tilde{X}^{(k)}))\) is identifiable in \(\tilde{G}^{(k)}\).

Firstly,
\[
E[\tilde{Y}^{(k+1)}|\tilde{X}_k^{(k)}] = E[\tilde{Y}^{(k)} - E[\tilde{Y}^{(k)}|\tilde{X}_k^{(k)}]|\tilde{X}_k^{(k)}] \\
= E[\tilde{Y}^{(k)}|\tilde{X}_k^{(k)}] - E[\tilde{Y}^{(k)}|\tilde{X}_k^{(k)}] \\
= 0 = E[\tilde{Y}^{(k+1)}].
\]

In addition, for \(1 \leq i < k\), it also holds that
\[
\{x \in \mathcal{X}_{\tilde{X}_k^{(k)}}|\mathcal{X}_{\tilde{X}_i^{(i)}} = 2^{(i)}\} = \mathcal{X}_{\tilde{X}_k^{(k)}}.
\]
Therefore, for fixed $\bar{x}_i^{(i)} \in X_i^{(i)}$,

$$E[\bar{Y}^{(k+1)} | \bar{x}_i^{(i)}] = E[\bar{Y}^{(k)}] - E[\bar{Y}^{(k)} | \bar{x}_i^{(i)}]$$

$$= E[\bar{Y}^{(k)}] - E[E[\bar{Y}^{(k)} | \bar{x}_i^{(i)}]] = E[\bar{Y}^{(k)}] - E[\bar{Y}^{(k)} | \bar{x}_i^{(i)}]$$

Hence, the assumption $\text{[1]}$ holds when $k + 1$.

Next, it holds that

$$E[\bar{X}_{k+1}^{(k+1)} | \bar{x}_i^{(i)}] = E[\bar{X}_{k+1}^{(k)}] - E[\bar{X}_{k+1}^{(k)} | \bar{x}_i^{(i)}]$$

In addition, for $j \geq k + 1$, $i < k$ it also holds that

$$E[\bar{X}_j^{(k+1)} | \bar{x}_i^{(i)}] = E[\bar{X}_j^{(k)}] - E[\bar{X}_j^{(k)} | \bar{x}_i^{(i)}]$$

Hence, the assumption $\text{[2]}$ holds when $k + 1$.

As for the assumption $\text{[3]}$ it holds that

$$E[\bar{Y}^{(k+1)} | Z] = E[\bar{Y}^{(k)}] - E[\bar{Y}^{(k)} | \bar{x}_i^{(i)}]$$

In addition, for $i \geq k + 1$, it holds that

$$E[\bar{X}_i^{(k+1)} | Z] = E[\bar{X}_i^{(k)}] - E[\bar{X}_i^{(k)} | \bar{x}_i^{(i)}]$$

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Hence, the assumption 3 holds when \( k + 1 \).

Finally, we show that the assumption 4 holds when \( k + 1 \). Note that, \( \tilde{X}_1^{(k+1)} = \tilde{X}_1^{(1)}, \tilde{X}_2^{(k+2)} = \tilde{X}_2^{(2)}, \ldots, \tilde{X}_k^{(k+1)} = \tilde{X}_k^{(k)} \). Since the assumption II we can apply do-calculus R3 to \( \tilde{X}_1^{(k)}, \tilde{X}_2^{(k)}, \ldots, \tilde{X}_{k-1}^{(k)} \) of \( P(\tilde{Y}^{(k)}|do(\tilde{X}^{(k)})) \), and we obtain

\[
P(\tilde{Y}^{(k)}|do(\tilde{X}^{(k)})) = P(\tilde{Y}^{(k)}|do(\tilde{X}_1^{(k)}), do(\tilde{X}_2^{(k)}), \ldots, do(\tilde{X}_n^{(k)}))
\]

\[
= P(\tilde{Y}^{(k)}|do(\tilde{X}_k^{(k)}), do(\tilde{X}_{k+1}^{(k)}), \ldots, do(\tilde{X}_n^{(k)}))
\]

Therefore, from the assumption III, \( P(\tilde{Y}^{(k)}|do(\tilde{X}_1^{(k)}), do(\tilde{X}_{k+1}^{(k)}), \ldots, do(\tilde{X}_n^{(k)})) \) is identifiable in \( \tilde{G}^{(k)} \). Moreover, in a setting that \( \tilde{V} = \tilde{V}_n^{(k)} \tilde{U} = \tilde{U}_n^{(k)} \), \( \tilde{X} = \{ \tilde{X}_k^{(k)}, \tilde{X}_{k+1}^{(k)}, \ldots, \tilde{X}_n^{(k)} \} \) and \( \tilde{X}' = \tilde{X}_k^{(k)} \), by the lemma II \( \tilde{V}^{(k+1)} \cup \tilde{U}^{(k+1)} \) is a semi-Markovian model. Let \( \tilde{G}^{(k+1)} \) be a DAG for \( \tilde{V}^{(k+1)} \cup \tilde{U}^{(k+1)} \), then \( \tilde{X}^{(k+1)} \subset An(\tilde{Y}^{(k+1)})_{\tilde{G}^{(k+1)}} \cup \text{Ind}(\tilde{Y}^{(k+1)})_{\tilde{G}^{(k+1)}} \). Moreover, \( P(\tilde{Y}^{(k+1)}|do(\tilde{X}_{k+1}^{(k+1)}), do(\tilde{X}_{k+2}^{(k+1)}), \ldots, do(\tilde{X}_n^{(k+1)})) \) is identifiable in \( \tilde{G}^{(k+1)} \). Now, note that it holds that

\[
\tilde{X}_i^{(k)} = E[\tilde{X}_i^{(k)}|\tilde{X}_k^{(k)}] = \tilde{X}_i^{(k)} - E[\tilde{X}_i^{(k)}|\tilde{X}_k^{(k)}] = \tilde{X}_i^{(k)},
\]

because \( \tilde{X}_1^{(k)}, \tilde{X}_2^{(k)}, \ldots, \tilde{X}_{k-1}^{(k)} \) are independent of \( \tilde{X}_k^{(k)} \) for \( i < k - 1 \) from II in lemma II. Namely, \( \tilde{X}_i^{(k)}(i \leq k - 1) \) are the same before and after operation of II.

By the way, we already have shown the assumption II when \( k + 1 \). Therefore, we can use do-calculus R3 to \( P(\tilde{Y}^{(k+1)}|do(\tilde{X}^{(k+1)})) \) in \( \tilde{G}^{(k+1)} \), and we obtain

\[
P(\tilde{Y}^{(k+1)}|do(\tilde{X}^{(k+1)}))
\]

\[
= P(\tilde{Y}^{(k+1)}|do(\tilde{X}_1^{(k+1)}), do(\tilde{X}_{k+1}^{(k+1)}), \ldots, do(\tilde{X}_n^{(k+1)}))
\]

Recall that the right hand side of above equations is identifiable in \( \tilde{G}^{(k+1)} \). As a result, \( P(\tilde{Y}^{(k+1)}|do(\tilde{X}^{(k+1)})) \) is identifiable in \( \tilde{G}^{(k+1)} \).

Summarizing above results, the assumptions III holds when \( k + 1 \).

Next, we show that \( \tilde{G} \) is available for modeling the average causal effect \( E[\tilde{Y}|do(\tilde{X})] \). Note that \( \tilde{G} = \tilde{G}^{(n)} \). From the inductive assumptions when \( k = n \), it holds that for each \( \tilde{X}_i^{(i)} \),

\[
\tilde{X}_i^{(i)} \perp \tilde{X} \setminus \tilde{X}_i^{(i)} \quad \text{and} \quad \tilde{X}_i^{(i)} \not\perp \not\tilde{X}_i^{(i)} \quad \text{and} \quad \tilde{X}_i^{(i)} \in An(\tilde{Y})_{\tilde{G}} \cup \text{Ind}(\tilde{Y})_{\tilde{G}}.
\]

That is, there exist no arrows emerging from \( \tilde{V} \) to each \( \tilde{X}_i^{(i)} \). Hence,

\[
\tilde{G}^{\text{obs}}(\tilde{X}) = \tilde{G}^{\text{obs}}
\]

As the above result, \( \tilde{G} \) is available for modeling the average causal effect \( E[\tilde{Y}|do(\tilde{X})] \).
Finally, we show equations (5)-(6). From (1),

\[
P(\tilde{Y}^{(n)}) = g^{(n)}|do(\tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}), do(\tilde{X}_{n-1}^{(n-1)} = \tilde{x}_{n-1}^{(n-1)}), \ldots, do(\tilde{X}_1^{(1)} = \tilde{x}_1^{(1)})
\]

\[
= P(\tilde{Y}^{(n)}) = g^{(n)}|\tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, \tilde{X}_{n-1}^{(n-1)} = \tilde{x}_{n-1}^{(n-1)}, \ldots, \tilde{X}_1^{(1)} = \tilde{x}_1^{(1)}
\]

\[
= P(\tilde{Y}^{(n)}) = g^{(n)}|\tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, \tilde{X}_{n-1}^{(n-1)} = \tilde{x}_{n-1}^{(n-1)}, \ldots, \tilde{X}_1^{(1)} = \tilde{x}_1^{(1)}
\]

\[
= P(\tilde{Y}^{(n)}) = g^{(n)}|\tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, \tilde{X}_{n-1}^{(n-1)} = \tilde{x}_{n-1}^{(n-1)}, \ldots, \tilde{X}_1^{(1)} = \tilde{x}_1^{(1)}
\]

Now that, from the definitions of $\tilde{Y}^{(k)}$ and $\tilde{X}^{(k)}$ for $2 \leq k \leq n$, it holds that

\[
P(\tilde{Y}^{(k)}) = g^{(k)} = P(\tilde{Y}^{(k-1)}) = \tilde{y}^{(k-1)}|\tilde{X}_{k-1}^{(k-1)} = \tilde{x}_{k-1}^{(k-1)}
\]

\[
P(\tilde{X}_i^{(k)}) = \tilde{x}_i^{(k)} = P(\tilde{X}_i^{(k-1)}) = \tilde{x}_i^{(k-1)}|\tilde{X}_{k-1}^{(k-1)} = \tilde{x}_{k-1}^{(k-1)}
\]

In addition, since we already have shown that

\[\tilde{Y}^{(k)} \perp \{\tilde{X}_1^{(1)}, \tilde{X}_2^{(2)}, \ldots, \tilde{X}_{k-1}^{(k-1)}\} \quad \text{and} \quad \tilde{X}_i^{(k)} \perp \{\tilde{X}_1^{(1)}, \tilde{X}_2^{(2)}, \ldots, \tilde{X}_{k-1}^{(k-1)}\} \quad (\forall i \geq k),
\]

we obtain

\[
P(\tilde{Y}^{(n)}) = g^{(n)}, \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, \tilde{X}_{n-1}^{(n-1)} = \tilde{x}_{n-1}^{(n-1)}, \ldots, \tilde{X}_1^{(1)} = \tilde{x}_1^{(1)}
\]

\[
= P(\tilde{Y}^{(n)}) = g^{(n)}, \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, \tilde{X}_{n-1}^{(n-1)} = \tilde{x}_{n-1}^{(n-1)}
\]

\[
\times P(\tilde{X}_{n-2}^{(n-2)} = \tilde{x}_{n-2}^{(n-2)}) \times \cdots P(\tilde{X}_1^{(1)} = \tilde{x}_1^{(1)})
\]

\[
= P(\tilde{Y}^{(n-1)}) = g^{(n-1)}, \tilde{X}_n^{(n-1)} = \tilde{x}_n^{(n-1)}, \tilde{X}_{n-1}^{(n-1)} = \tilde{x}_{n-1}^{(n-1)}
\]

\[
\times P(\tilde{X}_{n-2}^{(n-2)} = \tilde{x}_{n-2}^{(n-2)}) \times \cdots P(\tilde{X}_1^{(1)} = \tilde{x}_1^{(1)})
\]

\[
= P(\tilde{Y}^{(n-1)}) = g^{(n-1)}, \tilde{X}_n^{(n-1)} = \tilde{x}_n^{(n-1)}, \tilde{X}_{n-1}^{(n-1)} = \tilde{x}_{n-1}^{(n-1)}
\]

\[
\times P(\tilde{X}_{n-2}^{(n-2)} = \tilde{x}_{n-2}^{(n-2)}) \times \cdots P(\tilde{X}_1^{(1)} = \tilde{x}_1^{(1)})
\]

By repeating this operation until $k=n$, we obtain

\[
P(\tilde{Y}^{(n)}) = g^{(n)}, \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, \tilde{X}_{n-1}^{(n-1)} = \tilde{x}_{n-1}^{(n-1)}\ldots, \tilde{X}_1^{(1)} = \tilde{x}_1^{(1)}
\]

\[
P(\tilde{Y}^{(1)}) = g^{(1)}, \tilde{X}_n^{(1)} = \tilde{x}_n^{(1)}, \tilde{X}_{n-1}^{(1)} = \tilde{x}_{n-1}^{(1)}\ldots, \tilde{X}_1^{(1)} = \tilde{x}_1^{(1)}.
\]
Similarly, we can also obtain
\[ P(\tilde{Y}^{(1)} = \tilde{y}^{(1)}) = P(Y = y | Z = z) \]
\[ P(\tilde{X}_i^{(1)} = \tilde{x}_i^{(1)}) = P(X_i = x_i | Z = z) \quad (1 \leq k \leq n). \]
Thus,
\[ P(\tilde{Y}^{(1)} = \tilde{y}^{(1)}, \tilde{X}_n^{(1)} = \tilde{x}_n^{(1)}, \tilde{X}_{n-1}^{(1)} = \tilde{x}_{n-1}^{(1)}, \ldots, \tilde{X}_1^{(1)} = \tilde{x}_1^{(1)}) = P(Y = y, X_n = x_n, \ldots, X_1 = x_1 | Z = z). \]
From the above results,
\[ P(\tilde{Y}^{(n)} = \tilde{y}^{(n)}, \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, \tilde{X}_{n-1}^{(n)} = \tilde{x}_{n-1}^{(n)}, \ldots, \tilde{X}_1^{(n)} = \tilde{x}_1^{(n)}) = P(Y = y, X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1 | Z = z). \]
(36)
Similarly, we can also obtain
\[ P(\tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, \tilde{X}_{n-1}^{(n)} = \tilde{x}_{n-1}^{(n)}, \ldots, \tilde{X}_1^{(n)} = \tilde{x}_1^{(n)}) = P(X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1 | Z = z). \]
(37)
By inserting (36) and (37) to (34),
\[ P(\tilde{Y}^{(n)} = \tilde{y}^{(n)}, \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, \tilde{X}_{n-1}^{(n)} = \tilde{x}_{n-1}^{(n)}, \ldots, \tilde{X}_1^{(n)} = \tilde{x}_1^{(n)}) = P(Y = y, X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1 | Z = z). \]
(38)
On the other hand, by lemma 8
\[ P(Y = y | do(X_1 = x_1), \ldots, do(X_n = x_n), Z = z) = P(Y = y | X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1, Z = z). \]
(39)
Thus, from (38) and (39), (4) holds.
Finally, by (37) when \( k = n \),
\[ P(\tilde{Y}^{(n)} = \tilde{y}^{(n)} | \tilde{X}_1^{(n)} = \tilde{x}_1^{(n)}, \tilde{X}_2^{(n)} = \tilde{x}_2^{(n)}, \ldots, \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}) = \frac{P(\tilde{Y}^{(n)} = \tilde{y}^{(n)}, \tilde{X}_1^{(n)} = \tilde{x}_1^{(n)}, \tilde{X}_2^{(n)} = \tilde{x}_2^{(n)}, \ldots, \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)})}{P(\tilde{X}_1^{(n)} = \tilde{x}_1^{(n)}, \tilde{X}_2^{(n)} = \tilde{x}_2^{(n)}, \ldots, \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)})} \]
\[ = \frac{P(\tilde{Y}^{(n)} = \tilde{y}^{(n)} | \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, \tilde{X}_{n-1}^{(n)} = \tilde{x}_{n-1}^{(n)}, \ldots, \tilde{X}_1^{(n)} = \tilde{x}_1^{(n)}) \cdot P(\tilde{X}_n^{(n)} = \tilde{x}_n^{(n)} | \tilde{X}_{n-1}^{(n)} = \tilde{x}_{n-1}^{(n)}, \ldots, \tilde{X}_1^{(n)} = \tilde{x}_1^{(n)})}{P(\tilde{X}_n^{(n)} = \tilde{x}_n^{(n)})} \]
\[ = \frac{P(\tilde{Y}^{(n)} = \tilde{y}^{(n)} | \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, \tilde{X}_{n-1}^{(n)} = \tilde{x}_{n-1}^{(n)}, \ldots, \tilde{X}_1^{(n)} = \tilde{x}_1^{(n)})}{P(\tilde{X}_n^{(n)} = \tilde{x}_n^{(n)})} \]
\[ = P(\tilde{Y}^{(n)} = \tilde{y}^{(n)} | \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}). \]
Thus, (6) holds.
This completes the proof of the theorem. \( \square \)
proof of Corollary 1. In the proof of Theorem 2 we obtain the following results.

\[ \tilde{Y} \perp \tilde{X} \]  \hspace{1cm} (40)

\[ \tilde{X}_i^{(i)} \perp \tilde{X}_j^{(j)} \quad (1 \leq i < j \leq n) \]  \hspace{1cm} (41)

\[ \tilde{Y}, \tilde{X} \perp Z \]  \hspace{1cm} (42)

\[ P'(Y = y | X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n, Z = z) = P(Y = y | X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n, Z = z) \]  \hspace{1cm} (43)

\[ P(\tilde{X}_1^{(1)} = \tilde{x}_1^{(1)}, \tilde{X}_2^{(2)} = \tilde{x}_2^{(2)}, \ldots, \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, Z = z) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n | Z = z) \]  \hspace{1cm} (44)

From this, we obtain

\[ P''(Y = y | \tilde{X}_1^{(1)} = \tilde{x}_1^{(1)}, \tilde{X}_2^{(2)} = \tilde{x}_2^{(2)}, \ldots, \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, Z = z) = P''(Y = y | X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n, Z = z) \]

Thus, (4) holds.

Now that, it holds that for each \( \tilde{X}_i^{(i)} (1 \leq i \leq n) \), \( Z \) and all continuous functions \( f(x) \) and \( g(z) \),

\[ EP''[f(\tilde{X}_i^{(i)}) | \tilde{X}_1^{(1)} = \tilde{x}_1^{(1)}, \tilde{X}_2^{(2)} = \tilde{x}_2^{(2)}, \ldots, \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, Z = z] = f(\tilde{x}_i^{(i)}) \]  \hspace{1cm} (45)

\[ EP''[g(Z) | \tilde{X}_1^{(1)} = \tilde{x}_1^{(1)}, \tilde{X}_2^{(2)} = \tilde{x}_2^{(2)}, \ldots, \tilde{X}_n^{(n)} = \tilde{x}_n^{(n)}, Z = z] = g(z) \]  \hspace{1cm} (46)

In fact, for \( P'' \), it holds that for \( 1 \leq i < j \leq n \),

\[ \tilde{X}_i^{(i)} \perp \tilde{X}_j^{(j)} \]

In addition, for \( P'' \), it hold that for \( 1 \leq i \leq n \),

\[ \tilde{X}_i^{(i)} \perp Z \]
From the above results, we obtain $\tilde{\eta}^2$. Thus, (8) holds.

Now, we obtain (47) from (45) and (46), and we obtain (48) from (6) in Theorem and

Therefore, this completes the proof of the corollary.