A DICHOTOMY BETWEEN TWISTED TENSOR PRODUCTS OF BIALGEBRAS AND FROBENIUS ALGEBRAS

PABLO S. OCAL AND AMREI OSWALD

Abstract. We endow twisted tensor products with a natural notion of counit and comultiplication, and we provide sufficient and necessary conditions making the twisted tensor product a counital coassociative coalgebra. We then characterize when the twisted tensor product of bialgebras is a bialgebra, and when the twisted tensor product of Frobenius algebras is a Frobenius algebra. Our methods are purely diagrammatic, so these results hold for (braided) monoidal categories. As an application, we recover that some quantum complete intersections are Frobenius algebras, and we construct families of noncommutative symmetric Frobenius algebras. Along the way, we also characterize when twisted tensor products of separable algebras are separable, and we prove that twisted tensor products of special Frobenius algebras are special Frobenius.

1. Introduction

The study of quantum symmetries is intimately related with the deformation theory of classical objects. It is useful to understand how the properties of the classical objects are inherited, or not, by their deformations. For example, a key idea in the connection between Hopf algebras and solutions of the Yang-Baxter equation is the fact that the representation theory of a quantized universal enveloping algebra of a (complex semisimple) Lie algebra outside roots of unity coincides with the representation theory of the original Lie algebra [Dri87, Jan96]. This can be exploited further to understand Topological Quantum Field Theories, where both Frobenius algebras and Hopf algebras play significant roles. Namely commutative Frobenius algebras correspond to 2-dimensional TQFTs [Abr97, Koc04], and quantum groups can be used to construct 3-dimensional TQFT’s [Maj90, TV17].

In this paper we consider the deformation of the tensor product of algebras over a field known as twisted tensor product [CSV95]. These were originally conceived as a noncommutative analogue of the product of topological spaces, but they have an accessible algebraic formulation that encompass vast families of interesting algebras (such as the aforementioned quantized universal enveloping algebras). We seek to describe how the structures of Hopf algebra and Frobenius algebra are inherited by twisted tensor products, if at all. Related ideas have appeared in [CIMZ00, CMZ04].

Our motivating goal is to understand the tensor triangular geometry of Hopf algebras. Given a Hopf algebra $H$, we would like to study the Balmer spectrum [Bal05] of its associated stable module category in terms of its Hopf subalgebras. In other words, we would like to find appropriate Hopf algebras $A$ and $B$ that are subalgebras of $H$ such that
the information provided by Spc(stmodA) and Spc(stmodB) can be used to reconstruct Spc(stmodH). The twisted tensor product serves our purpose as a precise way of encoding how the algebra structure of H is inherited by the algebra structures of A and B. To extend this to the coalgebra structure of H, we introduce natural candidates for the counit and the comultiplication in terms of the coalgebra structures of A and B. Since Frobenius structures also determine a compatibility between the algebra and coalgebra structures of a given vector space, it is natural to ask whether twisted tensor products could inherit a Frobenius algebra structure via the counit and comultiplication we introduce.

Techniques that rely on understanding a twisted tensor product $A \otimes \tau B$ in terms of the algebras A and B have been successfully used on numerous occasions. Results that can be proved using decompositions as twisted tensor products include the characterization of when the cohomology of quantum complete intersections is finitely generated over the Hochschild cohomology ring [BO08], the fact that Ore extensions preserve Artin-Schelter regularity [ZZ08], and that crossed products of the quantum plane with the quantized enveloping algebra of $\mathfrak{sl}_2$ admit PBW deformations at primitive third roots of unity [WW18].

Our first main result is that bialgebra structures are not inherited by twisted tensor products with our proposed counit and comultiplication, so Hopf algebra structures will not be inherited either.

**Theorem 1.1** (see Theorem 4.8). Let $A$ and $B$ be $k$-bialgebras, let $\tau : B \otimes A \to A \otimes B$ be a twisting map. Then $A \otimes \tau B$ inherits a $k$-bialgebra structure if and only if $\tau$ is trivial.

Our second main result is that Frobenius algebra structures are always inherited by twisted tensor products with our proposed counit and comultiplication.

**Theorem 1.2** (see Theorem 5.1). Let $A$ and $B$ be Frobenius algebras over $k$, let $\tau : B \otimes A \to A \otimes B$ be a twisting map. Then $A \otimes \tau B$ is a Frobenius algebra if and only if $A \otimes \tau B$ is a counital coassociative coalgebra.

Moreover, we determine when twisted tensor products of separable and special Frobenius algebras are separable or special, respectively. We also recover the fact that certain quantum complete intersections are symmetric Frobenius algebras.

**Corollary 1.3** (see Corollaries 6.6 and 6.7 and Lemma 3.1 in [Ber09]). Let $n \in \mathbb{N}$, $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$, $n, m_1, \ldots, m_n \geq 2$, and $q = (q_{ij}) \in M_n(k^\times)$ such that $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$ for all $1 \leq i, j \leq n$. If

1. $q_{ij}$ is a root of unity whose order divides $\gcd(m_i - 1, m_j - 1)$ for all $i, j = 1, \ldots, n$, or

2. $k$ is a field of characteristic $p > 0$ and $m = (p, \ldots, p)$,

then the quantum complete intersections $\Lambda^n_{q, m}$ are symmetric Frobenius algebras.

Finally, we construct noncommutative symmetric Frobenius algebras from finite groups.

**Corollary 1.4** (see Section 6). Let $G$ and $H$ be finite groups, let $\tau : kH \otimes kG \to kG \otimes kH$ be a non-trivial strongly graded twisting map and denote $\tau(h \otimes g) = \lambda_{h,g}^{-1} \otimes h$ for some $\lambda_{h,g} \in k^\times$. If $\lambda_{h,g} = \lambda_{s^{-1}, g}\lambda_{sh,g}$ and $\lambda_{h,g} = \lambda_{h,r}^{-1}\lambda_{h,rg}$ for all $g, r \in G$ and $h, s \in H$ then $kG \otimes \tau kH$ is a noncommutative symmetric Frobenius algebra.

We observe that our approach and techniques have potential implications for the study of twisted Segre products [HU22] in noncommutative geometry. More precisely, twisted
Segre products appear as a subalgebra of twisted tensor products when the twist is strongly graded, in which case diagrammatic proofs automatically carry over. In general, it would be interesting to know when the twisted tensor product of Noetherian algebras is Noetherian.

Throughout this paper, we actively emphasize the use of commutative diagrams. We intentionally give all definitions in terms of maps satisfying certain commutative diagrams, interpret all conditions and properties in terms of commutative diagrams involving maps, and the majority of our proofs rely on the commutativity of the necessary diagrams. Not only is this a remarkably efficient way of working with twisted tensor products, but it also guarantees that all results in Sections 2, 3, 4, and 5 hold at the level of Hopf objects and Frobenius objects in a (braided) monoidal category \((\mathcal{C}, \otimes, 1)\). In particular, we extend some of the results in [FFRS06, Section 3.2].

Outline. In Section 2, we establish the definitions, characterizations, and examples of Hopf algebras and Frobenius algebras to be used throughout the paper. In Section 3, we recall the definition of twisted tensor products of algebras, introduce natural candidates for counit and comultiplication on twisted tensor products of coalgebras, and establish useful computational tools. In Section 4, we prove that non-trivial twisted tensor products of bialgebras do not inherit a bialgebra structure. In Section 5, we characterize when twisted tensor products of Frobenius algebras inherit a Frobenius algebra structure, and we determine the induced pairing and co-pairing. In Section 6, we recover known Frobenius algebra structures on quantum complete intersections, and we provide novel families of noncommutative symmetric Frobenius algebras. Numerous illustrative examples and counterexamples are provided throughout the paper.

Notation. The following notational conventions will be used in this paper. We will denote by \(k\) a fixed field, of arbitrary characteristic unless otherwise stated. Unadorned tensor products are taken to be over \(k\), namely \(\otimes := \otimes_k\). The identity morphism will be denoted by \(1 : V \rightarrow V\) for all \(k\) vector spaces \(V\). The map \(\sigma_{ij} : \bigotimes_{l=1}^{m} V_l \rightarrow \bigotimes_{l=1}^{-1} V_l \otimes V_j \otimes \bigotimes_{l=i+1}^{j-1} V_l \otimes V_i \otimes \bigotimes_{l=j+1}^{m} V_l\) denotes the exchange of the \(i\)-th and \(j\)-th coordinates. Unless otherwise stated, vector spaces in Section 6 will be graded by additive abelian groups, and maps \(f : V \rightarrow W\) between graded \(k\) vector spaces are assumed to be homogeneous of some degree \(d \in \mathbb{Z}\). Namely, if \(f(v) = \sum_{i \in I} w_i\) for some indexing set \(I\), \(v \in V\), and \(w_i \in W\) for all \(i \in I\), then \(|v| + d = |w_i|\) for all \(i \in I\). The Kronecker delta will be denoted by \(\delta_{i,j}\). We will use Sweedler’s notation for the comultiplication. Finally, exclusively within diagrams, we omit tensor products to economize space and composition is denoted by \(\circ\). Outside commutative diagrams, composition is denoted by concatenation. Namely given \(k\) vector spaces \(X, Y, U, V\) and \(k\)-linear morphisms \(f : X \rightarrow Y\), \(g : U \rightarrow V\), \(h : Y \rightarrow U\), then within commutative diagrams \(UV := U \otimes V\) and \(fg := f \otimes g\). We use \(\square\) to indicate a diagram that has not yet been shown to commute.

2. Preliminaries

In this section, we introduce the definitions and background necessary to understand the majority of the statements and results. These are presented diagrammatically to facilitate the proofs in the following sections. We refer the reader to [Lam99] [Koc04] [Rad12] for the details.
Definition 2.1. An associative, unital $k$-algebra is a triple $(A, \nabla, \eta)$ where $A$ is a $k$ vector space, and $\nabla : A \otimes A \to A$ and $\eta : k \to A$ are $k$-linear maps making the following diagrams commute.

\[
\begin{array}{ccc}
A \otimes k & \xrightarrow{1 \otimes \eta} & A \otimes A \\
\cong && \Downarrow \nabla \\
A & & \\
\end{array}
\quad
\begin{array}{ccc}
k \otimes A & \xrightarrow{\eta \otimes 1} & A \otimes A \\
\cong && \Downarrow \nabla \\
A & & \\
\end{array}
\quad
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\nabla \otimes 1} & A \otimes A \\
\Downarrow 1 \otimes \nabla && \\
A \otimes A & \xrightarrow{\nabla} & A \\
\end{array}
\]

Definition 2.2. A $k$-algebra $(A, \nabla, \eta)$ is separable when the $A$ bimodule morphism $\nabla : A \otimes A \to A$ has a right inverse. That is, there is a $k$-linear map $\Gamma : A \to A \otimes A$ making the following diagrams commute.

\[
\begin{array}{ccc}
A & \xrightarrow{\Gamma} & A \otimes A \\
\Downarrow 1 && \Downarrow \nabla \\
A & & \\
\end{array}
\quad
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{1 \otimes \nabla} & A \otimes A \\
\Downarrow 1 \otimes \Gamma \otimes 1 && \Downarrow \nabla \otimes \nabla \\
A \otimes A \otimes A & \xrightarrow{1 \otimes \nabla} & A \otimes A \\
\Downarrow \nabla \otimes \nabla && \\
A & \xrightarrow{\Gamma} & A \\
\end{array}
\]

The right diagram above states that $\Gamma : A \to A \otimes A$ is an $A$ bimodule morphism.

Definition 2.3. A coassociative, counital $k$-coalgebra is a triple $(C, \Delta, \epsilon)$ where $C$ is a $k$ vector space, and $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ are $k$-linear maps making the following diagrams commute.

\[
\begin{array}{ccc}
A \otimes k & \xrightarrow{1 \otimes \epsilon} & A \otimes A \\
\cong && \Downarrow \Delta \\
A & & \\
\end{array}
\quad
\begin{array}{ccc}
A \otimes A & \xrightarrow{\epsilon \otimes 1} & k \otimes A \\
\cong && \Downarrow \Delta \\
A \otimes A & & \\
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \otimes A \\
\Downarrow \Delta \otimes 1 && \\
A \otimes A & \xrightarrow{\Delta} & A \otimes A \\
\Downarrow 1 \otimes \Delta && \\
A & & \\
\end{array}
\]

Definition 2.4. A $k$-bialgebra is a tuple $(A, \nabla, \eta, \Delta, \epsilon)$ where $(A, \nabla, \eta)$ is a $k$-algebra, $(A, \Delta, \epsilon)$ is a $k$-coalgebra, and the following diagrams commute.

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{\nabla} & A \\
\Downarrow \epsilon \otimes \epsilon && \Downarrow \eta \otimes \eta \\
k \otimes k & \xrightarrow{\eta} & k \\
\Downarrow \epsilon && \Downarrow \Delta \\
k & & \\
\end{array}
\quad
\begin{array}{ccc}
A \otimes A & \xrightarrow{\eta} & A \\
\Downarrow \Delta \otimes \epsilon && \Downarrow \nabla \otimes \nabla \\
A \otimes A \otimes A \otimes A & \xrightarrow{\sigma_{23}} & A \otimes A \otimes A \otimes A \\
\end{array}
\]

The examples in this paper will mostly use the following two bialgebra structures.

Example 2.5. Let $G$ be a finite group, the group algebra $kG$ with the usual unit and multiplication, and counit $\epsilon : kG \to k$ and comultiplication $\Delta : kG \to kG \otimes kG$ given by extending $\epsilon(g) = 1$ and $\Delta(g) = g \otimes g$ for all $g \in G$ is a bialgebra. The polynomial ring in one variable $k[x]$ with the usual unit and multiplication, and counit $\epsilon : k[x] \to k$ and comultiplication $\Delta : k[x] \to k[x] \otimes k[x]$ given by extending $\epsilon(1) = 1$, $\epsilon(x) = 0$, and $\Delta(x) = 1 \otimes x + x \otimes 1$ is a bialgebra.
Definition 2.6. A Frobenius algebra over $k$ is a tuple $(A, \nabla, \eta, \Delta, \epsilon)$ where $(A, \nabla, \eta)$ is a $k$-algebra, $(A, \Delta, \epsilon)$ is a $k$-coalgebra, and the following diagrams commute.

$\Delta \otimes 1 \Rightarrow 1 \otimes \Delta \Rightarrow \nabla \otimes 1$,

$A \otimes A \Rightarrow A \Rightarrow A \otimes A \Rightarrow A \otimes A$.

The examples in this paper will mostly use the following two Frobenius algebra structures.

Example 2.8. Let $G$ be a finite group, the group algebra $kG$ with the usual unit and multiplication, and counit $\epsilon : kG \to k$ and comultiplication $\Delta : kG \to kG \otimes kG$ given by extending $\epsilon(g) = \delta_{g, 1}$, $\Delta(g) = \sum_{r \in G} rg \otimes r^{-1}$ is a Frobenius algebra. Let $n \in \mathbb{N}$, the truncated polynomial ring in one variable $k[x]/(x^n)$ with the usual unit and multiplication, and counit $\epsilon : k[x]/(x^n) \to k$ and comultiplication $\Delta : k[x]/(x^n) \to k[x]/(x^n) \otimes k[x]/(x^n)$ given by extending $\epsilon(x^i) = \delta_{i, n-1}$, $\Delta(p(x)) = \sum_{j=0}^{n-1} x^j p(x) \otimes x^{n-1-j}$ for all $i = 1, \ldots, n-1$ and all $p(x) \in k[x]/(x^n)$ is a Frobenius algebra.

The characterization of Frobenius algebras in Definition 2.6 favors our approach seeking to emphasize the use of commutative diagrams. It is well known that there are many equivalent definitions, see for example [Lam99, Chapter 6] for an extensive review. Another predominant viewpoint in the literature of noncommutative algebra follows, see for example [Abr97, Section 2.1].

Definition 2.9. Let $(A, \nabla, \eta)$ be an algebra. A pairing is a $k$-linear map $\beta : A \otimes A \to k$. A co-pairing is a $k$-linear map $\alpha : k \to A \otimes A$. A pairing $\beta$ is said to be non-degenerate when there exists a co-pairing $\alpha$ such that the following diagram commutes.

$\alpha \otimes 1 \Rightarrow 1 \otimes \beta \Rightarrow \nabla \otimes 1$,

$A \otimes A \Rightarrow A \Rightarrow A \otimes A \Rightarrow A \otimes A$.

A pairing $\beta$ is said to be associative when the following diagram commutes.

$\alpha \otimes 1 \Rightarrow 1 \otimes \beta \Rightarrow \nabla \otimes 1$,

$A \otimes A \Rightarrow A \Rightarrow A \otimes A \Rightarrow A \otimes A$.

Proposition 2.12. A $k$-algebra $(A, \nabla, \eta)$ is a Frobenius algebra if and only if there exists an associative non-degenerate pairing $\beta : A \otimes A \to k$.

Remark 2.13. A Frobenius algebra $(A, \nabla, \eta, \Delta, \epsilon)$ has associative non-degenerate pairing $\beta$ and co-pairing $\alpha$ given by
A Frobenius algebra $(A, \nabla, \eta)$ with an associative non-degenerate pairing $\beta : A \otimes A \to k$ and co-pairing $\alpha : k \to A \otimes A$ has comultiplication(s) $\Delta$ and counit(s) $\epsilon$ given by the following commutative diagrams.

\[
\begin{array}{ccc}
A \otimes k & \xrightarrow{1 \otimes \alpha} & A \otimes A \otimes A \\
\cong & & \cong \\
A & \xrightarrow{\Delta} & A \otimes A \\
\downarrow & & \downarrow \\
k \otimes A & \xrightarrow{\alpha \otimes 1} & A \otimes A \otimes A \\
\end{array}
\quad
\begin{array}{ccc}
A \otimes k & \xrightarrow{1 \otimes \eta} & A \otimes A \\
\cong & & \cong \\
A & \xrightarrow{\epsilon} & k \\
\downarrow & & \downarrow \\
k \otimes A & \xrightarrow{\eta \otimes 1} & A \otimes A \\
\end{array}
\]

**Definition 2.14.** A Frobenius algebra $A$ with associative non-degenerate pairing $\beta : A \otimes A \to k$ is *symmetric* when the following diagram commutes.

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{\sigma_{12}} & A \otimes A \\
\downarrow_{\beta} & & \downarrow_{\beta} \\
k & & k \\
\end{array}
\]

**Definition 2.15.** A Frobenius algebra $(A, \nabla, \eta, \Delta, \epsilon)$ is *special* when the following diagram commutes.

\[
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \otimes A \\
\downarrow_{1} & & \downarrow_{\nabla} \\
A & & A \\
\end{array}
\]

**Proposition 2.16.** Let $(A, \nabla, \eta, \Delta, \epsilon)$ be a Frobenius algebra. Then $A$ is special if and only if $(A, \nabla, \eta)$ is a separable algebra where $\Delta$ is a right inverse of $\nabla$.

**Proof.** Clearly a special Frobenius algebra is separable as claimed. For the converse, it suffices to show that the comultiplication is a morphism of $A$ bimodules. The commutativity of the following diagram guarantees that.

\[
\begin{array}{ccc}
AAA & \xrightarrow{\nabla} & AA \\
\downarrow_{1 \Delta 1} & \swarrow_{\nabla 1} & \searrow_{1 \nabla} \\
AAA & & \Delta \\
\end{array}
\]

(2.17)

3. **Twisted tensor products as algebras and coalgebras**

In this section, we recall the definition of twisted tensor products in the sense of Čap, Schichl, and Vanžura [CSV95], and we endow them with natural notions of counit and comultiplication. Unless otherwise stated, the (co)algebra structure(s) on twisted tensor products will be the one(s) inherited from the twisting map for the remainder of this paper.
Definition 3.1. Let \((A, \nabla_A, \eta_A)\) and \((B, \nabla_B, \eta_B)\) be unital associative \(k\)-algebras. Let \(\tau: B \otimes A \to A \otimes B\) be a bijective \(k\)-linear map such that the following diagrams commute.

\[
\begin{array}{ccc}
B \otimes k & \cong & k \otimes B \\
\downarrow_{1 \otimes \eta_A} & & \downarrow_{\eta_B \otimes 1} \\
B \otimes A & \tau & A \otimes B \\
\end{array}
\]

\[
\begin{array}{ccc}
k \otimes A & \cong & A \otimes k \\
\downarrow_{\eta_B \otimes 1} & & \downarrow_{1 \otimes \eta_B} \\
B \otimes A & \tau & A \otimes B \\
\end{array}
\]

We say that \(\tau\) is a twisting map. The trivial twisting map is \(\sigma_{12}: B \otimes A \to A \otimes B\). When \(A\) and \(B\) are \(k\)-algebras graded by commutative groups \(F\) and \(G\) respectively, a twisting map \(\tau: B \otimes A \to A \otimes B\) is said to be strongly graded provided \(\tau(B_j \otimes A_i) \subseteq A_i \otimes B_j\) for all \(i \in F\) and \(j \in G\).

These diagrams encode the compatibility conditions of a twisting map \(\tau\) with the unital associative structures of \(A\) and \(B\). Diagram (3.2) can be read as stating that \(\tau\) preserves the units \(\eta_A\) and \(\eta_B\) of \(A\) and \(B\) respectively. Diagram (3.3) is equivalent to \(\tau\) preserving the multiplications \(\nabla_A\) and \(\nabla_B\) of \(A\) and \(B\) respectively.

Lemma 3.4. Let \((A, \nabla_A, \eta_A)\) and \((B, \nabla_B, \eta_B)\) be unital associative \(k\)-algebras and let \(\tau: B \otimes A \to A \otimes B\) be a twisting map. Then the following diagrams commute.

\[
\begin{array}{ccc}
B \otimes A \otimes A & \tau \otimes 1 & A \otimes B \otimes A \\
\downarrow_{1 \otimes \nabla_A} & & \downarrow_{\nabla_A \otimes 1} \\
B \otimes A & \tau & A \otimes B \\
\end{array}
\]

\[
\begin{array}{ccc}
B \otimes B \otimes A \otimes A & \nabla_B \otimes \nabla_A & B \otimes A \\
\downarrow_{\nabla_B \otimes 1} & & \downarrow_{1 \otimes \nabla_B} \\
B \otimes A & \tau & A \otimes B \\
\end{array}
\]

Proof. The following commutative diagram shows the commutativity of (3.5).

The commutativity of (3.6) follows analogously. \(\square\)
In fact, the above shows that a bijective \( k \)-linear map \( \tau : B \otimes A \to A \otimes B \) is a twisting map if and only if diagrams (3.2), (3.5), and (3.6) commute.

**Definition 3.7.** Let \((A, \nabla_A, \eta_A)\) and \((B, \nabla_B, \eta_B)\) be unital associative \( k \)-algebras and let \( \tau : B \otimes A \to A \otimes B \) be a twisting map. The *twisted tensor product algebra* \( A \otimes_{\tau} B \) is the \( k \) vector space \( A \otimes B \) with the following multiplication and unit.

\[
\nabla_{A \otimes_{\tau} B} : (A \otimes B) \otimes (A \otimes B) \xrightarrow{1 \otimes 1 \otimes \tau} A \otimes A \otimes B \otimes B \xrightarrow{\nabla_A \otimes \nabla_B} A \otimes B
\]

\[
\eta_{A \otimes_{\tau} B} : k \xrightarrow{\cong} k \otimes k \xrightarrow{\eta_A \otimes \eta_B} A \otimes B
\]

**Proposition 3.8.** Let \((A, \nabla_A, \eta_A)\) and \((B, \nabla_B, \eta_B)\) be unital associative \( k \)-algebras and let \( \tau : B \otimes A \to A \otimes B \) be a twisting map. Then \((A \otimes_{\tau} B, \nabla_{A \otimes_{\tau} B}, \eta_{A \otimes_{\tau} B})\) is a unital associative \( k \)-algebra.

*Proof.* Left unitality follows from the following commutative diagram.

![Diagram](image)

Right unitality follows analogously. Associativity follows from the following commutative diagram.

![Diagram](image)

The original proof is [CSV95, Proposition/Definition 2.3]. □

As before, the above shows that given a bijective \( k \)-linear map \( \tau : B \otimes A \to A \otimes B \) then \((A \otimes_{\tau} B, \nabla_{A \otimes_{\tau} B}, \eta_{A \otimes_{\tau} B})\) is a unital associative algebra if and only if (3.2) and (3.3) commute, equivalently if and only if \( \tau \) is a twisting map.

**Example 3.9 (Twisting by a bicharacter).** [BO08, Definition 2.2] Let \( A \) and \( B \) be \( k \)-algebras graded by abelian groups \( F \) and \( G \) respectively, let \( t : F \otimes_{\mathbb{Z}} G \to k^\times \) be a homomorphism of abelian groups and denote \( t(f \otimes_{\mathbb{Z}} g) = t(f)^{t(g)} \) for all \( f \in F \) and \( g \in G \).
Then $\tau : B \otimes A \to A \otimes B$ given by linearly extending $\tau(b \otimes a) = t^{(||a||)}b \otimes a$ for all homogeneous $a \in A$ and $b \in B$ is a twisting map. We denote $A \otimes^\tau B := A \otimes_\tau B$.

So far, we have only required that $A$ and $B$ are unital associative algebras. When they are also counital coassociative coalgebras, a twisting map $\tau$ induces natural candidates for counit and comultiplication in $A \otimes B$.

**Definition 3.10.** Let $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ be counital coassociative $k$-coalgebras and let $\tau : B \otimes A \to A \otimes B$ be a $k$-linear map. We define $\Delta_{A \otimes \tau B}$ and $\epsilon_{A \otimes \tau B}$ as

$$\Delta_{A \otimes \tau B} : A \otimes B \xrightarrow{\Delta_A \otimes \Delta_B} A \otimes A \otimes B \otimes B \xrightarrow{1 \otimes \tau \otimes 1} (A \otimes B) \otimes (A \otimes B),$$

and

$$\epsilon_{A \otimes \tau B} : A \otimes B \xrightarrow{\epsilon_A \otimes \epsilon_B} k \otimes k \xrightarrow{\sim} k.$$  

**Proposition 3.11.** Let $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ be counital coassociative $k$-coalgebras and let $\tau : B \otimes A \to A \otimes B$ be a $k$-linear map. Then $(A \otimes \tau B, \Delta_{A \otimes \tau B}, \epsilon_{A \otimes \tau B})$ is a counital coassociative coalgebra if and only if the following diagrams commute.

![Diagram](3.12)

**Proof.** It follows from reversing the arrows in the proof of Proposition 3.8. \qed

Reversing the arrows in the proof of Lemma 3.4 gives that diagram (3.13) commutes if and only if diagrams (3.14) and (3.15) commute.

![Diagram](3.14)

For computational and technical purposes, it is useful to note that consecutive applications of twisting maps yield the same result. Namely setting

$$\tau_{2, A} : B \otimes B \otimes A \xrightarrow{1 \otimes \tau} B \otimes A \otimes B \xrightarrow{\tau \otimes 1} A \otimes B \otimes B,$$

$$\tau_{B, 2} : B \otimes A \otimes A \xrightarrow{\tau \otimes 1} A \otimes B \otimes A \xrightarrow{1 \otimes \tau} A \otimes A \otimes B,$$

and recursively considering

$$\tau_{i, A} : B^{\otimes i} \otimes A \xrightarrow{1 \otimes \tau_{i-1, A}} B \otimes A \otimes B^{\otimes (i-1)} \xrightarrow{\tau \otimes 1^{(i-1)}} A \otimes B^{\otimes i}$$
\[ \tau_{B,j} : B \otimes A \overset{\rho_{B,j-1}}\longrightarrow A \otimes (j-1) \otimes B \otimes A \overset{1 \otimes (j-1) \otimes \tau}{\longrightarrow} A \otimes j \otimes B \]

for all \( i, j \in \mathbb{N} \), we have the following commutative diagrams.

\[ B \otimes (i-1) \otimes A \overset{\tau_{i,j}}\longrightarrow A \otimes j \otimes B \overset{1 \otimes (i-2) \otimes \tau_{B,j-1}}\longrightarrow \cdots \overset{1 \otimes (i-j) \otimes \tau_{B,j-1}}\longrightarrow B \otimes A \otimes j \otimes B \otimes (i-1) \]

\[ B \otimes j \otimes A \overset{\tau_{i,j}}\longrightarrow A \otimes j \otimes B \overset{1 \otimes (i-j) \otimes \tau_{i,A}}\longrightarrow \cdots \overset{1 \otimes (i-j) \otimes \tau_{i,A}}\longrightarrow A \otimes (j-1) \otimes B \otimes (i-1) \]

In particular, any such consecutive application of twisting maps deserves to be denoted \( \tau_{i,j} : B \otimes A \otimes j \rightarrow A \otimes j \otimes B \). Of course, it is important to note that consecutive applications of twisting maps are compatible with the multiplications, namely the following diagrams commute.

\[ B \otimes A \otimes i \overset{\tau_{i,A}}\longrightarrow A \otimes i \otimes B \overset{1 \otimes (i-1) \otimes \nabla_A}{\longrightarrow} B \otimes A \otimes i \otimes B \overset{\tau_{i-1,A}}\longrightarrow A \otimes (i-1) \otimes B \overset{1 \otimes (i-1) \otimes \nabla_B}{\longrightarrow} B \otimes (i-1) \otimes A \overset{\tau_{i-1,A}}\longrightarrow A \otimes B \otimes (i-1) \]

More concisely, for all \( i, j \in \mathbb{N} \) we have the following commutative diagrams.

\[ B \otimes j \otimes A \overset{\tau_{i,j}}\longrightarrow A \otimes j \otimes B \overset{1 \otimes (i-j) \otimes \nabla_B}{\longrightarrow} B \otimes (i-1) \otimes A \otimes (j-1) \overset{\tau_{i-1,j-1}}\longrightarrow A \otimes (j-1) \otimes B \otimes (i-1) \]

When diagrams (3.14) and (3.15) commute then \( \tau \) is compatible with the comultiplications, and the corresponding diagrams also commute.

**Theorem 3.16.** Let \( (A, \nabla_A, \eta_A) \) and \( (B, \nabla_B, \eta_B) \) be separable unital associative \( k \)-algebras, and denote by \( \Gamma_A : A \rightarrow A \otimes A \) and \( \Gamma_B : B \rightarrow B \otimes B \) the right inverses of the respective multiplications. Then, the following diagrams commute if and only if \( (A \otimes \tau, \nabla_{A \otimes \tau}, \eta_{A \otimes \tau}) \) is a separable unital associative \( k \)-algebra and \( \Gamma_{A \otimes \tau} : A \otimes \tau \rightarrow (A \otimes \tau) \otimes (A \otimes \tau) \) given by \( \Gamma_{A \otimes \tau} = (1 \otimes \tau^{-1} \otimes 1)(\Gamma_A \otimes \Gamma_B) \) is a right inverse of its multiplication.

\[ (3.17) \]

\[ B \otimes A \overset{\tau}{\longrightarrow} A \otimes B \]

\[ (3.18) \]

\[ B \otimes A \overset{\tau}{\longrightarrow} A \otimes B \]

**Proof.** Clearly \( 1 \otimes \tau^{-1} \otimes 1)(\Gamma_A \otimes \Gamma_B) \) is a right inverse of \( \nabla_{A \otimes \tau} \) since the diagram
commutes. Now $\Gamma_{A \otimes \tau B}$ is an $A \otimes \tau B$ bimodule morphism if and only if the diagram

\[
\begin{array}{c}
ABABAB \xrightarrow{11\Gamma_A \Gamma_B 11} ABABABAB \xrightarrow{111r^{-1}111} ABABABAB \\
\downarrow 1r 111 \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu
commutes, if and only if the diagram

\[
\begin{array}{ccc}
AAABBB & \xrightarrow{\nabla_A1111} & AABBB \\
\downarrow & & \downarrow \\
AABB & \xrightarrow{\nabla_A1111} & AABBB \\
\downarrow & & \downarrow \\
AABB & \xrightarrow{\nabla_A1111} & ABBB \\
\downarrow & & \downarrow \\
AAB & \xrightarrow{\eta_B1} & AAABBB \\
\end{array}
\]

commutes. This finishes the proof.

\[\square\]

4. Twisted tensor product of bialgebras

In this section, we determine when a twisted tensor product of bialgebras inherits a bialgebra structure from the twisting map. As an application, we obtain that twisted tensor products of Hopf algebras inherit the Hopf algebra structure if and only if the twisting map is trivial.

Let \((A, \nabla_A, \eta_A, \Delta_A, \epsilon_A)\) and \((B, \nabla_B, \eta_B, \Delta_B, \epsilon_B)\) be \(k\)-bialgebras for the remainder of this section, and let \(\tau : B \otimes A \rightarrow A \otimes B\) be a twisting map. Now \((A \otimes \tau B, \nabla_A \otimes \tau B, \eta_A \otimes \tau B, \Delta_A \otimes \tau B, \epsilon_A \otimes \tau B)\) is a unital associative algebra by Proposition 3.8, but \((A \otimes \tau B, \Delta_A \otimes \tau B, \epsilon_A \otimes \tau B)\) is not necessarily a counital coassociative coalgebra. The compatibility of \(\tau\) with the counit \(\epsilon_B\) is established by the commutative diagram

\[
\begin{array}{ccc}
AB & \xrightarrow{\tau} & BA \\
\downarrow^{1 \epsilon_B} & & \downarrow^{\epsilon_B1} \\
Ak & \xrightarrow{\eta_B} & kA \\
\downarrow & & \downarrow \\
Ak & \xrightarrow{\eta_B1} & kA \\
\end{array}
\]

and the compatibility of \(\tau\) with the counit \(\epsilon_A\) follows similarly. Hence diagrams (3.12) commute so by Proposition 3.11 the only condition imposing a restriction on \(\tau\) to obtain
the desired coalgebra structure is its compatibility with the comultiplications $\Delta_B$ and $\Delta_A$. The following examples show this cannot be guaranteed in general.

**Example 4.1** (Jordan plane). Let $k[x]$ and $k[y]$ be polynomial rings in one variable with the bialgebra structure of Example 2.5. Consider the twisting map $\tau : k[y] \otimes k[x] \to k[x] \otimes k[y]$ given by linearly extending $\tau(y \otimes x) = x \otimes y + x^2 \otimes 1$. Observe that

$$k[x] \otimes_\tau k[y] \cong k(x, y)/(xy - yx + x^2)$$

is the Jordan plane. The twisting map is not compatible with the comultiplications, namely diagram (3.13) does not commute.

This example can be extended effortlessly to $2n \in \mathbb{N}$ variables, obtaining

$$k(x_1, \ldots, x_n, y_1, \ldots, y_n) = \frac{(x_i x_j - x_j x_i, y_i y_j - y_j y_i, x_i y_j - y_j x_i - \delta_{i,j} x_i^2)_{i,j \in \{1, \ldots, n\}}}{(x_i x_j - x_j x_i, y_i y_j - y_j y_i, x_i y_j - y_j x_i - \delta_{i,j} x_i^2)_{i,j \in \{1, \ldots, n\}}}.$$

Again, the twisting map is not compatible with the comultiplications.

**Example 4.2** (Weyl algebra). Let $k[x]$ and $k[y]$ be polynomial rings in one variable with the bialgebra structure of Example 2.5. Consider the twisting map $\tau : k[y] \otimes k[x] \to k[x] \otimes k[y]$ given by linearly extending $\tau(y \otimes x) = x \otimes y - 1 \otimes 1$. Now

$$k[x] \otimes_\tau k[y] \cong k(x, y)/(xy - yx - 1)$$

is the Weyl algebra, and the twisting map is also not compatible with the comultiplications.

We can again extend this to

$$k(x_1, \ldots, x_n, y_1, \ldots, y_n) = \frac{(x_i x_j - x_j x_i, y_i y_j - y_j y_i, x_i y_j - y_j x_i - \delta_{i,j} x_i^2)_{i,j \in \{1, \ldots, n\}}}{(x_i x_j - x_j x_i, y_i y_j - y_j y_i, x_i y_j - y_j x_i - \delta_{i,j} x_i^2)_{i,j \in \{1, \ldots, n\}}}$$

where once more the twisting map is not compatible with the comultiplications.

**Example 4.3** (Quantum enveloping algebra). Let $q \in k^\times$, let $U_q(\mathfrak{sl}_2)$ be the free $k$-algebra generated by $E$, $F$, $K$, and $K^{-1}$ subject to the relations

$$KK^{-1} = 1 = K^{-1}K, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

This is a quantized enveloping algebra whose Borel subalgebra is the free $k$-algebra $U_q^2(\mathfrak{b})$ generated by $E$, $K$, and $K^{-1}$, subject to the same relations. Consider the map $\tau : U_q^2(\mathfrak{b}) \otimes k[F] \to k[F] \otimes U_q^2(\mathfrak{b})$ given by extending

$$\tau(K \otimes F) = q^{-2}F \otimes K, \quad \tau(E \otimes F) = F \otimes E - \frac{1 \otimes K - 1 \otimes K^{-1}}{q - q^{-1}}.$$

This gives a twisting map making $k[F] \otimes_\tau U_q^2(\mathfrak{b})$ isomorphic to $U_q(\mathfrak{sl}_2)$ as $k$-algebras. Considering $k[F]$ as a polynomial ring in one variable with the bialgebra structure of Example 2.5 and $U_q^2(\mathfrak{b})$ with its usual coalgebra structure, whose comultiplication is given by extending

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(K) = K \otimes K,$$

then the twisting map is not compatible with the comultiplications.

Consequently, we have to require that $\tau$ makes diagram (3.13) commute. This requirement is equivalent to $(A \otimes_\tau B, \Delta_{A \otimes_\tau B}, \epsilon_{A \otimes_\tau B})$ being a counital coassociative coalgebra. Independently of this, we always have the commutative diagrams
yielding the compatibility of $\nabla_{A \otimes^\tau B}$ with $\epsilon_{A \otimes^\tau B}$ and the compatibility of $\Delta_{A \otimes^\tau B}$ with $\eta_{A \otimes^\tau B}$. Moreover the commutative diagram

\[
\begin{array}{ccc}
kkk & \xrightarrow{\eta_{A \otimes^\tau B}} & kk \\
\downarrow{11\eta_{A \otimes^\tau B}} & \Downarrow{11\epsilon_{A \otimes^\tau B}} & \downarrow{11\eta_{A \otimes^\tau B}} \\
AB & \xrightarrow{\Delta_{A \otimes^\tau B}} & AABB
\end{array}
\]

establishes the compatibility of $\eta_{A \otimes^\tau B}$ with $\epsilon_{A \otimes^\tau B}$. However, this still does not guarantee that $A \otimes^\tau B$ is a bialgebra.

**Example 4.4** (Quantum plane or quantum affine space). Let $k[x]$ and $k[y]$ be polynomial rings in one variable with the bialgebra structure of Example 2.5. Consider the twisting map $\tau : k[y] \otimes k[x] \to k[x] \otimes k[y]$ given by linearly extending $\tau(y \otimes x) = qx \otimes y$ for some non-zero $q \in k$. Then

\[
k[x] \otimes^\tau k[y] \cong k[qx, y]/(qxy - yx)
\]

is the quantum plane. The twisting map $\tau$ is compatible with the comultiplications of $k[x]$ and $k[y]$, namely diagram (3.13) commutes. However, $k[x] \otimes^\tau k[y]$ is not a bialgebra. Following Examples 4.1 and 4.2 we can extend this to $n \in \mathbb{N}$ variables. Given $q = (q_{ij}) \in M_n(k^\times)$ a square matrix with non-zero entries such that $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$ for all $1 \leq i, j \leq n$, set

\[
k_q[x_1, \ldots, x_n] = \frac{k\langle x_1, \ldots, x_n \rangle}{(x_i x_j - q_{ij} x_j x_i)_{1 \leq i, j \leq n}}.
\]

Again, the twisting maps are compatible with the comultiplications, but the induced algebra and coalgebra structures on $k_q[x_1, \ldots, x_n]$ do not give a bialgebra.

For $A \otimes^\tau B$ to be a bialgebra we have to impose the compatibility of the multiplication and comultiplication. Requiring this completely determines $\tau$.

**Lemma 4.5.** Let $A$ and $B$ be $k$-bialgebras, let $\tau : B \otimes A \to A \otimes B$ be a twisting map. Then the following diagrams commute.

\[
\begin{array}{ccc}
AkkB & \xrightarrow{1111} & AkkB \\
\downarrow{1\eta_{B} \eta_{A}^{1}} & & \downarrow{1\eta_{A} \eta_{B}^{1}} \\
ABAB & \xrightarrow{1\tau_{1}} & AABB
\end{array}
\]

\[
\begin{array}{ccc}
AAkkkkBB & \xrightarrow{\cong} & AkAkkkBkB \\
\downarrow{11\eta_{B} \eta_{A} \eta_{A}^{1} \eta_{A}^{11}} & & \downarrow{1\eta_{B} \eta_{A} \eta_{A}^{1} \eta_{A}^{11}} \\
AABBAABB & \xrightarrow{1\tau_{1}^{-1} \tau_{1}^{-1} \tau_{1}^{-1} \tau_{1}^{-1}} & ABABABAB
\end{array}
\]
A DICHOTOMY BETWEEN TWISTED BIALGEBRAS AND TWISTED FROBENIUS ALGEBRAS

\[
\begin{array}{c}
AABB \\
\sigma_{23} \\
\cong \\
AkAkkkB \\
\downarrow \cong \\
AkkBAkkB \\
\downarrow \cong \\
ABABABAB \\
\sigma_{35} \circ \sigma_{46} \\
\cong \\
AABBABAB
\end{array}
\]

\textbf{Proof.} The diagram

\[
\begin{array}{c}
AkB \xrightarrow{1_{\eta_B}} ABk \xrightarrow{1_{\eta_A}} ABAB \\
\downarrow \cong \\
AkB \xrightarrow{1_{\eta_B}} AkB \xrightarrow{1_{\eta_A}} AABB
\end{array}
\]

commutes. The diagram

\[
\begin{array}{c}
AkkkBB \xrightarrow{1111_{\eta_B 111 \eta_A 111}} AkkBB \xrightarrow{1111_{\eta_B 111 \eta_A 111}} ABBABABB \\
\downarrow \cong \\
AkAkkkBk \xrightarrow{1111_{\eta_B 111 \eta_A 111}} AkABABkBk \xrightarrow{1_{\eta_B 1111 \eta_A 111}} ABABABAB \\
\downarrow \cong \\
AABBABAB \xrightarrow{1_{\eta_B 1111 \eta_A 111}} ABBABABB
\end{array}
\]

commutes, and the following diagram commutes.

\[
\begin{array}{c}
AABB \xrightarrow{\cong} AkkBBkB \xrightarrow{1_{\eta_B 1111 \eta_A 111}} ABkBkAk \xrightarrow{1111_{\eta_B 111 \eta_A 111}} ABABABAB \\
\downarrow \sigma_{23} \\
ABAB \xrightarrow{\cong} AkkBBkB \xrightarrow{1_{\eta_B 1111 \eta_A 111}} ABkBkAk \xrightarrow{1111_{\eta_B 111 \eta_A 111}} ABABABAB \\
\downarrow \sigma_{35} \circ \sigma_{46} \\
\downarrow \sigma_{35} \circ \sigma_{46}
\end{array}
\]

\textbf{Lemma 4.6.} Let $A$ and $B$ be $k$-bialgebras, let $\tau : B \otimes A \to A \otimes B$ be a twisting map. Then the following diagrams commute.

\[
\begin{array}{c}
BBAA \xrightarrow{\sigma_{23}} BABA \\
\cong \\
kBkBAkAk \xrightarrow{\eta_A \eta_B \eta_A \eta_B \eta_B} kBkBAkAk \\
\downarrow \cong \\
ABABABAB \xrightarrow{\sigma_{35} \circ \sigma_{46}} ABABABAB \xrightarrow{\eta_A \eta_B \eta_A \eta_B \eta_B} AABBAABB \xrightarrow{1_{\tau^{-1}} 1_{\tau^{-1}} 1_{\tau^{-1}}} ABABABAB
\end{array}
\]

\[
\begin{array}{c}
kBk \xrightarrow{1_{\tau^{-1}}} kBk \\
\eta_A \eta_B \\
\downarrow \cong \\
ABAB \xrightarrow{1_{\tau^{-1}}} AABB
\end{array}
\]

\textbf{Proof.} The diagram
Lemma 4.7. Let $A$ and $B$ be $k$-bialgebras, let $\tau : B \otimes A \to A \otimes B$ be a twisting map. If $(A \otimes \tau B, \nabla_{A \otimes \tau B}, \eta_{A \otimes \tau B}, \nabla_{A \otimes \tau B}, \epsilon_{A \otimes \tau B})$ is a $k$-bialgebra then the following diagrams commute.

Proof. In this proof we slightly abuse notation by omitting the field $k$ to simplify the diagrams. Observe that if $A \otimes \tau B$ is a bialgebra then it is a counital coassociative coalgebra, so diagram (3.13) commutes by Proposition 3.11. The outside of the following diagram commutes because $A \otimes \tau B$ is a $k$-bialgebra. The diagrams indicated by a $\otimes$ commute by Lemma 4.5. Thus, the outer diagrams commute, and so does the central square.
Similarly, the parts of the following diagram indicated by a $\circ$ commute by Lemma 4.6, and therefore, the inner square commutes.

\[
\begin{array}{cccccc}
ABAB & \xrightarrow{\eta_A^{11}\eta_B} & AABB & \xrightarrow{\tau} & AB & \xrightarrow{\Delta_A\Delta_B} & AABB \\
\downarrow{\Delta_A\Delta_B\Delta_A\Delta_B} & & \downarrow{\Delta_B\Delta_A} & & \downarrow{\Delta_B\Delta_A} & & \downarrow{1\tau_1} \\
AABBAABB & \xrightarrow{\eta_A\eta_A^{111}\eta_B\eta_B} & BBAA & \xrightarrow{\sigma_{23}} & BABA & \xrightarrow{\tau\tau^{-1}} & ABAB \\
\downarrow{1\tau^{-1}1\tau^{-1}1} & & \downarrow{\eta_A\eta_A^{111}\eta_B\eta_B} & & \downarrow{\eta_A\eta_B\eta_A^{111}\eta_B} & & \downarrow{1\tau_1} \\
ABABABAB & \xrightarrow{\sigma_{35}\sigma_{46}} & ABABABAB & \xrightarrow{1\tau_1} & AABBAABB & \xrightarrow{\nabla_A\nabla_B\nabla_A\nabla_B} & ABAB \\
\end{array}
\]

**Theorem 4.8.** Let $A$ and $B$ be $k$-bialgebras, let $\tau : B \otimes A \to A \otimes B$ be a twisting map. Then $(A \otimes_\tau B, \nabla_{A \otimes_\tau B}, \eta_{A \otimes_\tau B}, \Delta_{A \otimes_\tau B}, \epsilon_{A \otimes_\tau B})$ is a $k$-bialgebra if and only if $\tau$ is trivial.

**Proof.** Suppose that $\tau$ is trivial, then $A \otimes_\tau B \cong A \otimes B$ and the claim is clear. Suppose that $A \otimes_\tau B$ is a bialgebra, in particular it is a counital coassociative coalgebra, so diagram (3.13) commutes by Proposition 3.11. Note then that the following diagram commutes.

\[
\begin{array}{cccccc}
BBAA & \xrightarrow{\tau\tau} & ABAB & \xrightarrow{\sigma_{23}} & AABB \\
\downarrow{1\epsilon_A} & & \downarrow{1\epsilon_A^{11}} & & \downarrow{1\epsilon_A^{11}} \\
BABk & \xrightarrow{\tau\tau} & ABkB & \xrightarrow{\sigma_{23}} & AkkB \\
\downarrow{1\epsilon_B} & & \downarrow{1\epsilon_B^{11}} & & \downarrow{1\epsilon_B^{11}} \\
BBAk & \xrightarrow{\epsilon_B^{111}} & kBAk & \xrightarrow{\epsilon_B^{111}} & kkkB \\
\end{array}
\]

Then, the outside of the following diagram commutes, and the parts of the diagram that commute by Lemma 4.7 have been marked by a $\circ$.

\[
\begin{array}{cccccc}
BBAA & \xrightarrow{\tau_2,2} & AABB & \xrightarrow{1\tau^{-1}} & ABAB \\
\downarrow{\sigma_{23}} & & \downarrow{\epsilon_B^{111}} & & \downarrow{1\epsilon_A^{111}} \\
BBAA & \xrightarrow{\Delta_B\Delta_A} & BA & \xrightarrow{\epsilon_B^{111}} & BA \\
\downarrow{1\epsilon_A^{111}} & & \downarrow{1\epsilon_A^{111}} & & \downarrow{1\epsilon_A^{111}} \\
kBAk & \xrightarrow{\cong} & BA & \xrightarrow{\epsilon_B^{111}} & BA \\
\end{array}
\]

Thus, the center square of the above diagram also commutes, proving that $\tau$ is trivial. □

Alas, it is now clear that non-trivial Hopf algebras will not come from twisted tensor products with the counit and comultiplication induced by the twisting map.
Corollary 4.9. Let $A$ and $B$ be Hopf algebras, let $\tau : B \otimes A \to A \otimes B$ be a twisting map such that diagram (3.13) commutes. If $A \otimes B$ is a Hopf algebra, then it is the tensor product Hopf algebra $A \otimes B$.

Corollary 4.10. Let $A$ and $B$ be Hopf algebras, let $\tau : B \otimes A \to A \otimes B$ be a non-trivial twisting map such that diagram (3.13) commutes. Then $A \otimes B$ is not a Hopf algebra.

5. Twisted tensor product of Frobenius algebras

In this section, we show that a twisted tensor product of Frobenius algebras always inherits a Frobenius algebra structure from the twisting map, and we explicitly give the induced pairing and co-pairing. Let $(A, \nabla_A, \eta_A, \Delta_A, \epsilon_A)$ and $(B, \nabla_B, \eta_B, \Delta_B, \epsilon_B)$ be Frobenius algebras over $k$ for the rest of the section.

Theorem 5.1. Let $A$ and $B$ be Frobenius algebras over $k$, let $\tau : B \otimes A \to A \otimes B$ be a twisting map such that diagram (3.13) commutes. Then $(A \otimes B, \nabla_A \otimes B, \eta_A \otimes B, \Delta_A \otimes B, \epsilon_A \otimes B)$ is a Frobenius algebra.

Proof. The diagram

\[
\begin{array}{cccccc}
ABAB & \xrightarrow{1\tau_1} & AABB & \xrightarrow{\nabla_A \nabla_B} & AB \\
\Delta_A \Delta_B & \xrightarrow{1\Delta_B} & ABBAB & \xrightarrow{1\tau_2 A} & AABBB & \xrightarrow{1\Delta_B} & \Delta_A \Delta_B \\
\Delta_A & \xrightarrow{1\nabla_B} & AABAB & \xrightarrow{11\tau_2 A} & AAABBB & \xrightarrow{1\nabla_B} & \Delta_A \\
ABABAB & \xrightarrow{1\tau^{-1}_1 11} & AABB & \xrightarrow{11\tau_1} & ABAABB & \xrightarrow{11\nabla_B} & ABAB
\end{array}
\]

commutes because the diagram

\[
\begin{array}{cccccc}
AABB & \xrightarrow{11\nabla_B} & AB & \xrightarrow{\nabla_A} & AB \\
\Delta_A & \xrightarrow{1\nabla_B} & AABBB & \xrightarrow{11\nabla_B} & ABB & \xrightarrow{\nabla_A} & AB \\
\Delta_A & \xrightarrow{1\nabla_B} & AABBB & \xrightarrow{11\nabla_B} & ABB & \xrightarrow{\nabla_A} & AB \\
AAABBB & \xrightarrow{11\nabla_B} & AAABBB & \xrightarrow{11\nabla_B} & ABB & \xrightarrow{\nabla_A} & AB \\
ABAABB & \xrightarrow{11\nabla_B} & ABABB & \xrightarrow{11\nabla_B} & ABB & \xrightarrow{\nabla_A} & AB
\end{array}
\]

commutes, giving one of the associativity conditions on $\nabla_{A \otimes B}$ and $\Delta_{A \otimes B}$. The remaining one follows analogously. \qed
This is an unexpected generalization of the fact that the tensor product of Frobenius algebras is a Frobenius algebra (see [Jan60, Theorem 2.1]).

Remark 5.2. Note that if $A \otimes B$ is a Frobenius algebra then in particular $A \otimes \tau B$ is a counital coassociative coalgebra, whence diagram (3.13) commutes because of Proposition 3.11. Namely, Theorem 5.1 gives a necessary and sufficient condition.

Example 5.3 (Skew group algebra). Let $G$ and $H$ be finite groups where $G$ acts on $H$ via $\varphi : G \to \text{Aut}(H)$. The map $\tau : kG \otimes kH \to kH \otimes kG$ given by $\tau(g \otimes h) = \varphi(g)(h) \otimes g$ for $g \in G$, $h \in H$ extends to a twisting map giving $kH \otimes_{\tau} kG \cong k(H \rtimes_{\varphi} G)$ as $k$-algebras. If we consider the coalgebra structures of $kG$ and $kH$ as in Example 2.5 then the twisting map is not compatible with the comultiplications, namely diagram (3.13) does not commute.

Alternatively, we can consider the Frobenius coalgebra structure on the group algebras where the counits $\epsilon_G : kG \to k$ and $\epsilon_H : kH \to k$ and comultiplications $\Delta_G : kG \to kG \otimes kG$ and $\Delta_H : kH \to kH \otimes kH$ are given by linearly extending $\epsilon_G(r) = \delta_{r,1}$ and $\epsilon_H(s) = \delta_{s,1}$ for $r \in G$ and $s \in H$, and

$$\Delta_G(g) = \sum r \in G gr \otimes r^{-1}, \quad \Delta_H(h) = \sum s \in H hs \otimes s^{-1}.$$

The above coalgebra structures coincides with the one in Example 2.8 by Remark 2.13. These comultiplications are compatible with the twisting map, namely diagram (3.13) commutes. Further, the Frobenius algebra structure obtained on $kG \otimes_{\tau} kH$ recovers exactly the Frobenius algebra structure on $k(H \rtimes_{\varphi} G)$ given by $\epsilon_{H \rtimes_{\varphi} G} : k(H \rtimes_{\varphi} G) \to k$ and $\Delta_{H \rtimes_{\varphi} G} : k(H \rtimes_{\varphi} G) \to k(H \rtimes_{\varphi} G) \otimes k(H \rtimes_{\varphi} G)$ as

$$\epsilon_{H \rtimes_{\varphi} G}(s, r) = \delta_{(s, r), (1, 1)}, \quad \Delta_{H \rtimes_{\varphi} G}(h, g) = \sum_{(s, r) \in H \rtimes_{\varphi} G} (h, g)(s, r) \otimes (s, r)^{-1}.$$

Corollary 5.4. Let $A$ and $B$ be separable Frobenius algebras over $k$, let $\Gamma_A : A \to A \otimes A$ and $\Gamma_B : B \to B \otimes B$ the right inverses of the respective multiplications. Let $\tau : B \otimes A \to A \otimes B$ be a twisting map such that diagram (3.13) commutes. Then $(A \otimes B, \nabla_{A \otimes B}, \eta_{A \otimes B}, \Delta_{A \otimes B}, \epsilon_{A \otimes B})$ is a separable Frobenius algebra if and only if diagrams (3.17) and (3.18) commute.

Proof. Apply Theorem 5.1 and Theorem 3.16.

Corollary 5.5. Let $A$ and $B$ be special Frobenius algebras over $k$. Let $\tau : B \otimes A \to A \otimes B$ be a twisting map such that diagram (3.13) commutes. Then, the Frobenius algebra $(A \otimes B, \nabla_{A \otimes B}, \eta_{A \otimes B}, \Delta_{A \otimes B}, \epsilon_{A \otimes B})$ is special.

Proof. By Theorem 5.1 $(A \otimes B, \nabla_{A \otimes B}, \eta_{A \otimes B}, \Delta_{A \otimes B}, \epsilon_{A \otimes B})$ is indeed a Frobenius algebra. Since $A$ and $B$ are special Frobenius algebras, by Proposition 2.16 $A$ and $B$ are both separable with $\Delta_A$ and $\Delta_B$ the right inverses of $\nabla_A$ and $\nabla_B$ respectively. The algebra $(A \otimes B, \nabla_{A \otimes B}, \eta_{A \otimes B}, \Delta_{A \otimes B}, \epsilon_{A \otimes B})$ is then separable and, by Theorem 3.16, a right inverse of its multiplication is $(\Delta_A \otimes \Delta_B)(1 \otimes \tau^{-1} \otimes 1) = \Delta_{A \otimes B}$. Hence, $(A \otimes B, \nabla_{A \otimes B}, \eta_{A \otimes B}, \Delta_{A \otimes B}, \epsilon_{A \otimes B})$ is a special Frobenius algebra by Proposition 2.16.

The previous results together with Theorem 3.16 are a noncommutative generalization of the fact that the tensor product of étale algebras is an étale algebra. Moreover, we achieved a very computationally efficient criterion for self-injectivity.
Corollary 5.6 (Criterion for self-injectivity). Let $\Lambda$ be a unital associative $k$-algebra, let $A$ and $B$ be $k$-subalgebras of $\Lambda$ via the injective $k$-algebra morphisms $\iota_A : A \to \Lambda$ and $\iota_B : B \to \Lambda$ such that $\nabla_\Lambda(\iota_A \otimes \iota_B) : A \otimes B \to \Lambda$ is an isomorphism of $k$ vector spaces. If $A$ and $B$ are Frobenius algebras and $\tau = (\nabla_\Lambda(\iota_A \otimes \iota_B))^{-1}\nabla_\Lambda(\iota_B \otimes \iota_A)$ makes diagram (3.13) commute, then $\Lambda$ is self-injective.

Proof. By [CSV95, Proposition 2.7] the hypothesis give $\Lambda \cong A \otimes B$ as $k$-algebras. Then $\Lambda$ is a Frobenius algebra by Theorem 5.1, so it is self-injective. □

This is an extremely useful criterion that has multiple applications, including to twisted tensor products of truncated polynomial rings and finite dimensional semisimple algebras. For example, it can be used to prove that some quantum complete intersections are self-injective (see Corollary 6.6 for a more general statement). We conclude the section by explicitly giving the pairing and co-pairing of a twisted tensor product of Frobenius algebras.

Proposition 5.7. Let $A$ and $B$ be Frobenius algebras over $k$ with pairings $\beta_A$ and $\beta_B$ and co-pairings $\alpha_A$ and $\alpha_B$, let $\tau : B \otimes A \to A \otimes B$ be a twisting map such that diagram (3.13) commutes. Then

\[
\beta_{A \otimes B} : A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes \tau \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{\beta_A \otimes \beta_B} k \otimes k \xrightarrow{\cong} k,
\]

\[
\alpha_{A \otimes B} : k \xrightarrow{\cong} k \otimes k \xrightarrow{\alpha_A \otimes \alpha_B} A \otimes A \otimes B \otimes B \xrightarrow{1 \otimes \tau^{-1} \otimes 1} A \otimes B \otimes A \otimes B,
\]

are an associative non-degenerate pairing and co-pairing of $A \otimes B$.

Proof. Since $A \otimes B$ is a Frobenius algebra by Theorem 5.1, it has an associative pairing $\beta_{A \otimes B}$ and co-pairing $\alpha_{A \otimes B}$ as in Remark 2.13. The claimed expression for $\alpha_{A \otimes B}$ is given by the following commutative diagram.

Similarly for $\beta_{A \otimes B}$ we have the following commutative diagram.
Although the pairing and co-pairing of $A \otimes_\tau B$ are readily attainable, it is not easy to compute the corresponding Nakayama automorphism. Some natural candidates to consider are $\Theta_A \otimes \Theta_B : A \otimes B \to A \otimes B$ and $(\tau)(\Theta_B \otimes \Theta_A)(\tau^{-1}) : A \otimes B \to A \otimes B$, but it is hard to see that either of the following diagrams commute.

\[ \begin{array}{ccc}
ABAB & \xrightarrow{\Theta_A \otimes \Theta_B} & ABAB \\
\sigma_{13\sigma_{24}} & \downarrow & \beta_{A \otimes \tau B} \\
ABAB & \xrightarrow{\tau^{-1} \otimes \tau} & ABAB \\
\sigma_{13\sigma_{24}} & \downarrow & \beta_{A \otimes \tau B} \\
ABAB & \xrightarrow{\beta_{A \otimes \tau B}} & k \\
\end{array} \]

At the end of Section 6 we prove that these candidates coincide in the case of group algebras and strongly graded twists, and that both these diagrams commute. It would be interesting to know whether either of the proposed automorphisms is the Nakayama automorphism $\Theta_{A \otimes \tau B}$ corresponding to the co-pairing $\beta_{A \otimes \tau B}$ in general.

6. Known and novel Frobenius algebra structures

In this section, we show that twisting by a bicharacter is compatible with graded Frobenius algebras having comultiplications of degree zero, and we establish sufficient and necessary conditions for the compatibility when the comultiplication is graded. We use this to show that certain quantum complete intersections inherit a symmetric Frobenius algebra structure when seen as twisted tensor products, and we construct noncommutative symmetric Frobenius algebras.

**Lemma 6.1.** Let $A$ and $B$ be $k$ vector spaces graded by abelian groups $F$ and $G$ respectively, let $\Delta_A : A \to A \otimes A$ and $\Delta_B : B \to B \otimes B$ be $k$-linear graded maps of degree $d_A$ and $d_B$ respectively, let $t : F \otimes G \to k^\times$ be a homomorphism of abelian groups and denote $t(f \otimes g) = t^f g$ for all $f \in F$ and $g \in G$, let $\tau : B \otimes A \to A \otimes B$ be given by linearly extending $\tau(b \otimes a) = t^{[a][b]} a \otimes b$ for all homogeneous $a \in A$ and $b \in B$. Then

(1) diagram (3.14) commutes if and only if $t^{[a][b]} = t^{[a]+d_A}[b]$ for all homogeneous $a \in A$ and $b \in B$,

(2) diagram (3.15) commutes if and only if $t^{[a][b]} = t^{[a][b]+d_B}$ for all homogeneous $a \in A$ and $b \in B$.

**Proof.** We first consider diagram (3.14). For homogeneous $a \in A$ and $b \in B$, a computation yields

\[(1 \otimes \Delta_B)(\tau)(b \otimes a) = t^{[a][b]} a \otimes \sum_b b(1) \otimes b(2),\]

\[(\tau \otimes 1)(1 \otimes \tau)(\Delta_B \otimes 1)(b \otimes a) = a \otimes \sum_b t^{[a][b(1)]} t^{[a][b(2)]} b(1) \otimes b(2).\]

Since $t$ is a homomorphism of abelian groups and $\Delta_B$ is graded of degree $d_B$ the above are equal if and only if $t^{[a][b]} = t^{[a][b(1)]+d_A} = t^{[a][b]+d_B}$, as desired. It follows analogously for diagram (3.15).

**Theorem 6.2.** Let $A$ and $B$ be Frobenius algebras over $k$ graded by abelian groups $F$ and $G$ respectively, with comultiplications $\Delta_A : A \to A \otimes A$ and $\Delta_B : B \to B \otimes B$ being graded maps of degree $d_A$ and $d_B$ respectively. Let $t : F \otimes G \to k^\times$ be a homomorphism of
abelian groups. Then, \( A \otimes^t B \) is a Frobenius algebra if and only if \( t^{(|a||b|)} = t^{(|a|+d_A||b|)} \) and \( t^{(|a||b|) + d_B} \) for all homogeneous \( a \in A \) and \( b \in B \).

**Proof.** Diagram (3.13) commutes because of Lemma 6.1 and Proposition 3.11. This is a sufficient and necessary condition for \( A \otimes^t B \) to be a Frobenius algebra by Theorem 5.1 and Remark 5.2.

Note that if \( A \) and \( B \) are connected self-injective \( k \)-algebras graded by abelian groups \( F \) and \( G \) then they are finite dimensional and Frobenius by [SZ97]. Hence if their comultiplications are graded maps of degree \( d_A \) and \( d_B \) respectively, and \( t : F \otimes \mathbb{Z} G \to k^\times \) is a homomorphism of abelian groups, then \( A \otimes^t B \) is a Frobenius algebra if and only if \( t^{(|a||b|)} = t^{(|a|+d_A||b|)} \) and \( t^{(|a||b|) + d_B} \) for all homogeneous \( a \in A \) and \( b \in B \) by Theorem 6.2.

**Remark 6.3.** When \( F = \langle f \rangle_{\mathbb{Z}} \) or \( G = \langle g \rangle_{\mathbb{Z}} \) are generated by a single element, say of degree \(|f| = 1 = |g|\), the bicharacters \( t \) such that \( A \otimes^t B \) is Frobenius can be characterized as the ones satisfying \( (t^{(1|1)})^{d_A} = 1 = (t^{(1|1)})^{d_B} \), namely the order of \( t^{(1|1)} \) divides the greatest common divisor of \( d_A \) and \( d_B \).

In particular, when \( d_A = 0_F \) and \( d_B = 0_G \) we have the following.

**Corollary 6.4.** Let \( A \) and \( B \) be Frobenius algebras over \( k \) graded by abelian groups \( F \) and \( G \) respectively, with comultiplications \( \Delta_A : A \to A \otimes^t A \) and \( \Delta_B : B \to B \otimes^t B \) being graded maps of degree 0. Let \( t : F \otimes \mathbb{Z} G \to k^\times \) be a homomorphism of abelian groups. Then, \( A \otimes^t B \) is a Frobenius algebra.

We now use these results on quantum complete intersections.

**Example 6.5 (Quantum complete intersection).** [AGP97, Section 2] Let \( m_1, m_2 \in \mathbb{N}, m_1, m_2 \geq 2 \), and consider the truncated polynomial rings in one variable \( k[x_1]/(x_1^{m_1}) \) and \( k[x_2]/(x_2^{m_2}) \) with the twisting map \( \tau : k[x_1]/(x_1^{m_2}) \otimes k[x_1]/(x_1^{m_1}) \to k[x_1]/(x_1^{m_1}) \otimes k[x_2]/(x_2^{m_2}) \) given by linearly extending \( \tau(x_2 \otimes x_1) = qx_1 \otimes x_2 \) for some non-zero \( q \in k \).

The algebra
\[
\begin{align*}
  k[x_1]/(x_1^{m_1}) & \otimes \tau k[x_2]/(x_2^{m_2}) \cong k[x,y]/(x_1^{m_1}, x_2^{m_2}) = k\langle x_1, x_2 \rangle/(x_1^{m_1}, x_2^{m_2}, qx_1 x_2 - x_2 x_1)
\end{align*}
\]

called a quantum complete intersection. Note that \( k[x_i]/(x_i^{m_i}) \) for \( i = 1, 2 \) is a Frobenius algebra as in Example 2.8. More generally, given \( n \in \mathbb{N} \), \( \mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n \), \( n, m_1, \ldots, m_n \geq 2 \), and \( \mathbf{q} = (q_{ij}) \in M_n(k^\times) \) as in Example 4.4 the above can be extended to
\[
\Lambda^n_{\mathbf{q}, \mathbf{m}} := k\mathbf{q}[x_1, \ldots, x_n]/(x_1^{m_1}, \ldots, x_n^{m_n}) = \frac{k\langle x_1, \ldots, x_n \rangle}{(x_i^{m_i}, x_ix_j - q_{ij}x_jx_i)_{i,j \in \{1, \ldots, n\}}}
\]

which account for all quantum complete intersections.

Note that \( \Lambda^n_{\mathbf{q}, \mathbf{m}} \) can be obtained via twists by a bicharacter in two equivalent ways. To interpret \( \Lambda^n_{\mathbf{q}, \mathbf{m}} \) as an iterated twisted tensor product (see [JMLPnPvO08, Section 2]) of twists by a bicharacter we use the \( \mathbb{Z} \)-grading on \( k[x_i]/(x_i^{m_i}) \) given by setting \( |x_i| = 1 \) for all \( i = 1, \ldots, n \). Consider \( t^i : \mathbb{Z} \otimes \mathbb{Z} \to k^\times \) given by \( t^i(r \otimes s) = q_{ij}^s \) for all \( r, s \in \mathbb{Z} \) and all \( i, j = 1, \ldots, n \), which yield twisting maps \( \tau_{ij} : k[x_i]/(x_i^{m_i}) \otimes k[x_j]/(x_j^{m_j}) \to k[x_j]/(x_j^{m_j}) \otimes k[x_i]/(x_i^{m_i}) \) as in Example 3.9. Now
\[
\Lambda^n_{\mathbf{q}, \mathbf{m}} \cong k[x_1]/(x_1^{m_1}) \otimes t^1_k k[x_2]/(x_2^{m_2}) \otimes t^2 \cdots \otimes t^{n-1}_k k[x_{n-1}]/(x_{n-1}^{m_{n-1}}) \otimes t^n_k k[x_n]/(x_n^{m_n})
\]
denotes the \( k \)-algebra with the expected unit and multiplication. In particular
\[
\nabla_{\Lambda_{q,m}^n} := (\nabla_1 \otimes \cdots \otimes \nabla_n) \tau_{n,n-1} \tau_{n,n-2} \cdots \tau_{n,2} \tau_{n,1}
\]
where \( \nabla_i : k[x_i]/(x_i^{m_i}) \otimes k[x_i]/(x_i^{m_i}) \to k[x_i]/(x_i^{m_i}) \) is the usual multiplication and
\[
\tau_{n,n,i} := (1 \otimes 1 \otimes \cdots \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1)
\]
for all \( i = 1, \ldots, n \). To interpret \( \Lambda_{q,m}^n \) as a single twisting by a bicharacter, observe
\[
\Lambda_{q,m}^n \cong \Lambda_{(q_{ij}),i,j \in \{1,\ldots,n-1\}}^{n-1}(m_1,\ldots,m_{n-1}) \otimes t_{i_1,\ldots,i_{n-1},-1}^{m_1,\ldots,m_{n-1}} k[x_i]/(x_i^{m_i})
\]
where \( t_{i_1,\ldots,i_{n-1}}^{m_1,\ldots,m_{n-1}}(x_1 \cdots x_{i_{n-1}} \otimes x_{i_n}) := \prod_{j=1}^{n-1} q_{j_i}^{m_j} \), see Bergh and Oppermann [BO08, Lemma 5.1]. We can now partially recover the fact that quantum complete intersections are Frobenius [Ber09, Lemma 3.1].

**Corollary 6.6.** Let \( n \in \mathbb{N} \), \( m = (m_1, \ldots, m_n) \in \mathbb{N}^n \), \( n, m_1, \ldots, m_n \geq 2 \), and \( q = (q_{ij}) \in M_n(k^\times) \) such that \( q_{ii} = 1 \) and \( q_{ij}q_{ji} = 1 \) for all \( 1 \leq i, j \leq n \). If \( q_{ij} \) is a root of unity whose order divides \( \gcd(m_i - 1, m_j - 1) \) for all \( i, j = 1, \ldots, n \), then the quantum complete intersections \( \Lambda_{q,m}^n \) are Frobenius algebras.

**Proof.** Considering Remark 6.3 for the \( \mathbb{Z} \)-grading on \( k[x_i]/(x_i^{m_i}) \), that \( t_1^I(1 \otimes \mathbb{Z}) = q_{ii} \), and that the comultiplication provided in Example 6.5 is graded of degree \( m_i - 1 \), then \( k[x_i]/(x_i^{m_i}) \otimes t_1^{j_i} k[x_j]/(x_j^{m_j}) \) is a Frobenius algebra for all \( i, j = 1, \ldots, n \) by Theorem 6.2.

An induction argument finishes the proof.

The co-pairing \( \beta_{\Lambda_{q,m}^n} \) can be computed by successively applying the twisting maps as
\[
\beta_{\Lambda_{q,m}^n}(x_1^{a_1} \cdots x_n^{a_n} \otimes x_1^{b_1} \cdots x_n^{b_n}) = \prod_{j=2}^{n} \prod_{i=1}^{j-1} q_{ji}^{a_i b_j} \epsilon_{\Lambda_{q,m}^n}(x_1^{a_1+b_1} \cdots x_n^{a_n+b_n})
\]
whence it will be non-zero exactly when \( a_l + b_l = m_l - 1 \) for all \( l = 1, \ldots, n \). In that case
\[
\beta_{\Lambda_{q,m}^n}(x_1^{b_1} \cdots x_n^{b_n} \otimes x_1^{a_1} \cdots x_n^{a_n}) = \prod_{j=2}^{n} \prod_{i=1}^{j-1} q_{ji}^{b_i a_j}
\]
and since \( b_i a_j = (m_i - 1 - a_i)(m_j - 1 - b_j) \) then \( q_{ji}^{b_i a_j} = q_{ji}^{a_i b_j} \). Thus
\[
\beta_{\Lambda_{q,m}^n}(x_1^{a_1} \cdots x_n^{a_n} \otimes x_1^{b_1} \cdots x_n^{b_n}) = \beta_{\Lambda_{q,m}^n}(x_1^{b_1} \cdots x_n^{b_n} \otimes x_1^{a_1} \cdots x_n^{a_n})
\]
and \( \Lambda_{q,m}^n \) is symmetric. These Frobenius structures are exactly the ones given by Bergh structures are exactly the ones given by [Ber09, Lemma 3.1], since when the order of \( q_{ij} \) divides \( \gcd(m_i - 1, m_j - 1) \) for all \( i, j = 1, \ldots, n \) then their Nakayama automorphism is the identity map, whence their Frobenius algebras are also symmetric.

**Corollary 6.7.** Let \( k \) be a field of characteristic \( p > 0 \). Let \( n \in \mathbb{N} \), \( m = (p, \ldots, p) \), and \( q = (q_{ij}) \in M_n(k^\times) \) such that \( q_{ii} = 1 \) and \( q_{ij}q_{ji} = 1 \) for all \( 1 \leq i, j \leq n \). Then the quantum complete intersections \( \Lambda_{q,m}^n \) are Frobenius algebras.
Proof. Since \( k \) has characteristic \( p \) then \( k[x_i]/(x_i^p) \cong kC_p \), where \( C_p \) is the cyclic group of order \( p \). Now \( kC_p \) has a Frobenius algebra structure where the comultiplications are graded maps of degree 0, whence \( k[x_i]/(x_i^{m_i}) \otimes^{t_1} k[x_j]/(x_j^{m_j}) \) is a Frobenius algebra for all \( i, j = 1, \ldots, n \) by Corollary 6.4. An induction argument finishes the proof. \( \square \)

All the quantum complete intersections encompassed by Corollaries 6.6 and 6.7 satisfy the finite generation hypothesis by [HO08, Theorem 5.5]. Moreover, if \( q_{ij} \) are not roots of unity whose order divides \( \text{gcd}(m_i - 1, m_j - 1) \), then the twists giving \( \Lambda^n_{q,m} \) are not compatible with the coproducts of Example 6.5. Thus our specific description of quantum complete intersections as twists by a bicharacter does not impose a Frobenius algebra structure on \( \Lambda^n_{q,m} \). It would be interesting to know if there are other interpretations of quantum complete intersections as twisted tensor products that give the Frobenius algebra structures found by Bergh [Ber09, Lemma 3.1].

We conclude with a brief study of the Frobenius algebra structures on twisted tensor products of group algebras with strongly graded twists. Let \( G \) and \( H \) be finite groups, let \( \tau : kH \otimes kG \to kG \otimes kH \) be a strongly graded twisting map. Consider the gradings \( kG = \bigoplus_{r \in G} k_r \) and \( kH = \bigoplus_{s \in H} k_s \), since \( \tau : kH \otimes kG \to kG \otimes kH \) is strongly graded then \( \tau(h \otimes g) = \lambda_{h,g} h \otimes g \) for some \( \lambda_{h,g} \in k^\times \). Then
\[
(1 \otimes \Delta_H)(\tau)(h \otimes g) = \lambda_{h,g} h \otimes \sum_{s \in H} s h \otimes s^{-1},
\]
\[
(\tau \otimes 1)(1 \otimes \tau)(\Delta_H \otimes 1)(h \otimes g) = g \otimes \sum_{s \in H} \lambda_{s^{-1},g} \lambda_{s h,g} s h \otimes s^{-1},
\]
\[
(\Delta_G \otimes 1)(\tau)(h \otimes g) = \lambda_{h,g} \sum_{r \in G} r g \otimes r^{-1} \otimes h,
\]
\[
(1 \otimes \tau)(\tau \otimes 1)(1 \otimes \Delta_G)(h \otimes g) = \sum_{r \in G} \lambda_{h,g} \lambda_{h,r^{-1}} r g \otimes r^{-1} \otimes h.
\]
If \( \lambda_{h,g} = \lambda_{s^{-1},g} \lambda_{s h,g} \) and \( \lambda_{h,g} = \lambda_{h,r^{-1}} \lambda_{h,r g} \) for all \( g, r \in G \) and \( h, s \in H \) then the first two expressions above are equal, and the last two are also equal. Thus diagram (3.13) commutes and \( kG \otimes \tau kH \) is Frobenius by Theorem 5.1. This is slightly more general than Corollary 6.4 applied to \( kG \) and \( kH \), since \( \tau \) does not have to come from a bicharacter. Let \( a, b \in G \) and \( c, d \in H \), we can explicitly compute the co-pairing \( \beta_{kG \otimes kH} = \Delta_k(\beta_{kG} \otimes \beta_{kH})(1 \otimes \tau \otimes 1) \) given in Proposition 5.7.

\[
\beta_{kG \otimes kH}(a \otimes c \otimes b \otimes d) = \Delta_k(\beta_{kG} \otimes \beta_{kH})(1 \otimes \tau \otimes 1)(a \otimes c \otimes b \otimes d)
\]
\[
= \Delta_k(\beta_{kG} \otimes \beta_{kH})(\lambda_{c,b} a \otimes b \otimes c \otimes d) = \lambda_{c,b} \delta_{ab,1} \delta_{cd,1}
\]

In particular it is non-zero exactly when \( ab = 1_G \) and \( cd = 1_H \), in which case \( \lambda_{c,b} = \lambda_{d^{-1},a^{-1}} = \lambda_{d,a} \). Thus
\[
\beta_{kG \otimes kH}\sigma_{24}\sigma_{13}(a \otimes b \otimes c \otimes d) = \lambda_{d,a} \delta_{ba,1} \delta_{dc,1} = \lambda_{c,b} \delta_{ab,1} \delta_{cd,1} = \beta_{kG \otimes kH}(a \otimes c \otimes b \otimes d)
\]
and \( kG \otimes \tau kH \) is a symmetric Frobenius algebra.

**Example 6.8** (Symmetric twisted tensor product of Frobenius algebras). Let \( C_2 \) be the cyclic group of order 2 with multiplicative generator \( g \). Now \( kC_2 \) is a symmetric Frobenius
algebra with co-pairing \( \beta : kC_2 \otimes kC_2 \to k \) given by

\[
\beta \left( \sum_{a,b \in C_2} \lambda_{a,b} a \otimes b \right) = \lambda_{1,1} + \lambda_{g,g}.
\]

Let \( A = B = kC_2 \), let \( \tau : B \otimes A \to A \otimes B \) be the twisting map induced by the bicharacter \( t : C_2 \otimes Z \to k \times k \) defined as \( t(1 \otimes 1) = t(g \otimes 1) = t(1 \otimes g) = 1 \) and \( t(g \otimes g) = -1 \). Now \( A \otimes \tau B \) is a symmetric Frobenius algebra with co-pairing \( \beta_{kC_2 \otimes kC_2} : kC_2 \otimes kC_2 \to k \) given by \( \beta_{kC_2 \otimes kC_2}(1 \otimes 1 \otimes 1 \otimes 1) = \beta_{kC_2 \otimes kC_2}(1 \otimes g \otimes 1 \otimes g) = \beta_{kC_2 \otimes kC_2}(g \otimes 1 \otimes g \otimes 1) = 1 \), \( \beta_{kC_2 \otimes kC_2}(g \otimes g \otimes g \otimes g) = -1 \), and \( \beta_{kC_2 \otimes kC_2}(a \otimes c \otimes b \otimes d) = 0 \) for other choices of \( a, b, c, d \in kC_2 \).

It is clear that commutative Frobenius algebras are always symmetric, but the converse is not true. The above provides a systematic construction of noncommutative symmetric Frobenius algebras.

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UCLA MATHEMATICS DEPARTMENT, LOS ANGELES, CA 90095-1555, USA

Email address: socal@math.ucla.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195, USA

Email address: amreio@uw.edu