Cancellation in the Additive Twists of Fourier Coefficients for GL$_2$ and GL$_3$ over Number Fields

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Abstract. In this article, we study the sum of additively twisted Fourier coefficients of an irreducible cuspidal automorphic representation of GL$_2$ or GL$_3$ over an arbitrary number field. When the representation is unramified at all non-archimedean places, we prove the Wilton type bound for GL$_2$ and the Miller type bound for GL$_3$ which are uniform in terms of the additive character.

1. Introduction

1.1. Backgrounds. It is a classical problem to estimate sums involving the Fourier coefficients of a modular form. For example, when $f(z)$ is a holomorphic cusp form of weight $k$ for SL$_2(\mathbb{Z})$ with the Fourier expansion,

$$f(z) = \sum_{n=1}^{\infty} A_f(n) n^{(k-1)/2} e(nz),$$

with $e(z) = e^{2\pi i z}$, it is well known that for any real $\theta$ and $T \geq 1$,

$$\sum_{n \leq T} A_f(n) e(\theta n) \ll_f T^{1/2} \log(2T),$$

with the implied constant depending only on $f$ (see [Iwa] Theorem 5.3]). This is a classical estimate due to Wilton. Because of the square-root cancellation, which is essentially best possible, (1.1) shows that there is no correlation between Fourier coefficients $A_f(n)$ and additive characters $e(\theta n)$. In the work of Stephen Miller [Mil], for the Fourier coefficients $A_f(q, n)$ of a cusp form $f$ for GL$_3(\mathbb{Z})$ he proved

$$\sum_{n \leq T} A_f(q, n) e(\theta n) \ll_{\epsilon, q, f} T^{3/4 + \epsilon},$$

with the implied constant depending only on $\epsilon$, $q$ and the cusp form $f$. This is halfway between the trivial bound of $O(T)$ and the best-possible bound of $O(T^{1/2})$. Uniformity in $\theta$ is an important feature of these estimates of Wilton and Miller.

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Fourier coefficients of a cuspidal automorphic representation of \( \text{GL}_2 \) or \( \text{GL}_3 \) over an arbitrary number field. As an illustration, let \( F \) be a totally real field of degree \( N \) and consider a Hilbert modular form \( f(z) \) of weight \( k \) for the Hilbert modular group \( \text{SL}_2(\mathbb{C}) \) with the Fourier expansion,

\[
f(z) = \sum_{n \in \mathbb{C}'} A_f(n)N(n^{(k-1)/2}) e(\text{Tr}(nz)), \quad N(n^{(k-1)/2}) = \prod_{j=1}^{N} n_j^{(k_j-1)/2}, \text{Tr}(nz) = \sum_{j=1}^{N} n_j z_j.
\]

Here \( \mathbb{C} \) is the ring of integers in \( F \) and \( \mathbb{C}' \) is its dual. Then the function \(|f(z)|^N(\text{Im}
 z^{k/2})\) is \( \text{SL}_2(\mathbb{C}) \)-invariant, which, by setting \( z_j = \theta_j + i/T \), leads to an estimate analogous to (1.3),

\[
(1.4) \quad \sum_{n \in \mathbb{C}'} A_f(n)e(\text{Tr}(n\theta))N(n^{(k-1)/2}) \exp(-2\pi\text{Tr}(n)/T) \ll_f T^{(k-1)/2},
\]

for any \( \theta \in \mathbb{R}^N \). In order to generalize this further, we shall use the Voronoï summation formula over number fields in the paper of Ichino and Templier [11]. To estimate the Hankel transforms in the Voronoï summation, we shall apply the asymptotic formulae for Bessel kernels over real and complex numbers in the author’s recent work [Q11].

1.2. Statement of Results. Let \( F \) be a number field and let \( \mathbb{C} \) be its ring of integers. Let \( N \) be the degree of \( F \). Let \( S_\infty \) be the set of all archimedean places of \( F \). For \( v \in S_\infty \), let \( F_v \) be the corresponding local field. Define \( F_\infty = \prod_{v \in S_\infty} F_v = \mathbb{R}^N \) and the trace on \( F_\infty \) by \( \text{Tr}_x = \sum_{v \in S_\infty} \text{Tr}_{F_v/\mathbb{R}} x_v \). Let \( \Pi \subset F_\infty \) be a fixed fundamental parallelootope for \( F_\infty/\mathbb{C} \) which is symmetric about zero.

Let \( \pi \) be an irreducible cuspidal automorphic representation of \( \text{GL}_r \) over \( F \), with \( r = 2, 3 \). Suppose that \( \pi \) is unramified and has trivial central character at all non-archimedean places of \( F \). Let \( A_\pi(\gamma), \gamma \in \mathbb{C} \setminus \{0\} \), denote the Fourier coefficients of \( \pi \).

In this paper, we study the sums of Fourier coefficients twisted by additive characters

\[
(1.5) \quad S_\theta(T) = \sum_{\gamma \in \mathbb{C} \setminus \{0\} \cap T \cdot \Pi} A_\pi(\gamma) e(\text{Tr}(\theta\gamma)), \quad \theta \in F_\infty,
\]

and we shall be concerned with obtaining estimates for \( S_\theta(T) \) which are uniform in \( \theta \).
Thanks to the Rankin-Selberg theory, we know that $A_\pi(\gamma)$ obey the Ramanujan conjecture on average. More precisely, we have
\[ \sum_{\gamma \in \mathbb{C} \setminus \{0\} \cap T^{-\Pi}} |A_\pi(\gamma)|^2 = O(T^{N+\varepsilon}), \]
(the $\varepsilon$ in the exponent is due to the possible infinitude of the group of units in $\mathbb{C}$ and $T^\varepsilon$ may be reduced to a power of $\log T$.) and by Cauchy-Schwarz,
\[ \sum_{\gamma \in \mathbb{C} \setminus \{0\} \cap T^{-\Pi}} |A_\pi(\gamma)| = O(T^{N+\varepsilon}). \]
See §4. Thus the trivial estimate for $S_\delta(T)$ is $O(T^{N+\varepsilon})$, obtained by taking the absolute value of each term in (1.5). On the other hand, by Parseval’s identity,
\[ \int_0^1 |S_\delta(T)|^2 d\theta = \sum_{\gamma \in \mathbb{C} \setminus \{0\} \cap T^{-\Pi}} |A_\pi(\gamma)|^2, \]
Hence the best possible uniform estimate for $S_\delta(T)$ would be $O(T^{1/2+\varepsilon})$. Our main results are the following uniform estimates for $S_\delta(T)$ analogous to (1.1) and (1.2). In the GL2 case, we have the best-possible bound of Wilton type, whereas, in the GL3 case, we have Miller’s bound, which is halfway between the trivial bound and the best-possible bound.

**Theorem 1.1.** Let notations as above. Suppose that $T > 0$ is sufficiently large in terms of $\pi$ and $F$. Then, for any $\theta \in F_{\infty}$, we have
\[ \sum_{\gamma \in \mathbb{C} \setminus \{0\} \cap T^{-\Pi}} A_\pi(\gamma)e(\text{Tr}(\theta \gamma)) = O(T^{N/2}(\log T)^N), \]
if $r = 2$, and for any $\varepsilon > 0$,
\[ \sum_{\gamma \in \mathbb{C} \setminus \{0\} \cap T^{-\Pi}} A_\pi(\gamma)e(\text{Tr}(\theta \gamma)) = O(T^{3N/4 + \varepsilon}), \]
where the implied constants in (1.6) and (1.7) depend only on $\pi$, $F$, the choice of $\Pi$, and in addition on $\varepsilon$ for (1.7).

We shall deduce Theorem 1.1 from its smoothed version as follows.

**Theorem 1.2.** Let notations as above. Let $\Delta > 1$. Let $T > 0$ be sufficiently large in terms of $\pi$ and $F$. For each archimedean place $v$, let $w_v$ be a smooth function on $F_v$ such that
- for $F_v = \mathbb{R}$, $w_v(x)$ is supported on $\{x \in \mathbb{R} : |x| \in [T, AT]\}$ and
  \[ (d/dx)^j w_v(x) \ll_{j, \Delta} T^{-j}, \]
- for $F_v = \mathbb{C}$, $w_v(z)$ is supported on $\{z \in \mathbb{C} : |z| \in [T, AT]\}$ and
  \[ (\partial/\partial z)^j (\partial/\partial \bar{z})^j w_v(z) \ll_{j, \Delta} T^{-j}, \]
or equivalently, in the polar coordinates,
\[ (\partial/\partial r)^j (\partial/\partial \phi)^j w_v(r \phi) \ll_{j, \Delta} T^{-j} \]
Let $w$ be the product function of $w_v$ on $F_{\infty}$. Then, for any $\theta \in F_{\infty}$, we have
\[ \sum_{\gamma \in \mathbb{C} \setminus \{0\}} A_\pi(\gamma)e(\text{Tr}(\theta \gamma)) w(\gamma) = O(T^{N/2}), \]
if \( r = 2 \), and for any \( \varepsilon > 0 \),

\[
\sum_{\gamma \in \mathbb{C} \setminus \{0\}} A_{\varepsilon}(\gamma) e(\text{Tr}(\theta \gamma)) w(\gamma) = O_{\varepsilon}(T^{3N/4 + \varepsilon}),
\]

where the implied constants in (1.11, 1.12) depend only on \( \pi, F, \Delta \) and those in (1.8, 1.9, 1.10), and in addition on \( \varepsilon \) for (1.12).

1.3. Remarks.

In Miller’s paper [Mil], Hankel transforms in the Mellin-Barnes form are analyzed using Stirling’s asymptotic formula for the Gamma function. This approach is very restrictive from the analytic aspect in the complex case, as it only applies to spherical weight functions. Nevertheless, the spherical assumption is usually sufficient, for example, for deducing Theorem 1.1 from 1.2. In contrast, we use the asymptotic formulae for Bessel kernels, which turn Hankel transforms into oscillatory integrals so that we may directly use the method of stationary phase. Our method does not only yield better estimates for Hankel transforms but also enables us to treat complex Hankel transforms of the large class of compactly supported weight functions described in Theorem 1.2.

In this paper, same as [Mil], the cuspidal automorphic representation \( \pi \) is considered fixed. There however have been papers after [Mil], for example [LY, God, Li], on obtaining estimates which are also uniform in the archimedean parameters of \( \pi \). These bounds are useful when one considers varying automorphic forms, for example, in the subconvexity problem (see [HMQ]). With Ichino and Templier’s Voronoi summation formula, the author believes that they may also be generalized to an arbitrary number field. When the parameters are taken into account, one has to use the Mellin-Barnes form of Hankel transforms and Stirling’s asymptotic formula, or one needs to get an asymptotic formula for Bessel kernels at the transition range, which is currently only known for \( \text{GL}_2(\mathbb{R}) \).

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2. Notations for Number Fields

We now give a list of our most frequently used notations from algebraic number theory. See [Lan] for more details.

Let \( F \) be a number field of degree \( N \). Let \( \mathbb{O} \) be its ring of integers and \( \mathbb{O}^\times \) the group of units. Let \( \text{N} \) and \( \text{Tr} \) denote the norm and the trace for \( F \), respectively. Denote by \( \mathbb{A} = \mathbb{A}_F \) the adele ring of \( F \).

For any place \( v \) of \( F \), we denote by \( F_v \) the corresponding local field. When \( v \) is non-archimedean, let \( v \) be the corresponding prime ideal of \( \mathbb{O} \), and let \( \text{ord} \) denote the additive valuation. Let \( \mathcal{N}_v \) be the local degree of \( F_v \); in particular, \( \mathcal{N}_v = 1 \) if \( F_v = \mathbb{R} \) and \( \mathcal{N}_v = 2 \) if \( F_v = \mathbb{C} \). Let \( \| \cdot \|_v \) denote the normalized module of \( F_v \). We have \( \| \cdot \|_v = | \cdot | \) if \( F_v = \mathbb{R} \) and \( \| \cdot \|_v = | \cdot |^2 \) if \( F_v = \mathbb{C} \), where \( | \cdot | \) is the usual absolute value.

Fix the (non-trivial) additive character \( \psi = \mathbb{O}_\phi \psi_v \) on \( \mathbb{A}/F \) as in [Lan] §XIV.1 such that \( \psi_v(x) = e(-x) \) if \( F_v = \mathbb{R} \), \( \psi_v(z) = e(-(z + \mathbb{T})) \) if \( F_v = \mathbb{C} \), and that \( \psi_v \) is unramified for all non-archimedean \( F_v \). Let \( S_\infty \), respectively \( S_f \), denote the set of all archimedean,
respectively non-archimedean, places of $F$. Accordingly, we split $\psi = \psi_f \psi_r$. Note that $\psi_{r,\epsilon}(\gamma) = e(-\text{Tr} \gamma)$ for $\gamma \in F$.

We choose the Haar measure of $F_v$ as in [Lan §IV.1]; the Haar measure is the ordinary Lebesgue measure on the real line if $F_v = \mathbb{R}$ and twice the ordinary Lebesgue measure on the complex plane if $F_v = \mathbb{C}$.

For each $v \in S_\infty$, let $\sigma_v$ denote the embedding $F \hookrightarrow F_v$, then all the $\sigma_v$ yield an embedding $\sigma : F \hookrightarrow F_\infty = \prod_{v \in S_\infty} F_v = \mathbb{R}^N$. Let $r_1$ be the number of real and $r_2$ be the number of conjugate pairs of complex embeddings of $F$.

3. Voronoï Summation Formulae for $GL_2$ and $GL_3$ over Number Fields

Our main tool is the $GL_2$ and $GL_3$ Voronoï summation formulae over number fields in the work of Ichino and Templier [IT]. It is formulated in an adelic framework. We shall give a brief summary of the adelic formulae and then translate them into classical language.

3.1. Adelic Voronoï Summation Formulae [IT]. Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $GL_r(\mathbb{A})$, with $r = 2, 3$. Let $\tilde{\pi} = \otimes_v \tilde{\pi}_v$ be the contragradient representation of $\pi$.

Let $S$ be a finite set of places of $F$ including the ramified places of $\pi$ and $\psi$ and all archimedean places. Denote by $\mathbb{A}^S$ the subring of adeles with trivial component above $S$. Denote by $W^S_\pi = \prod_{v \in S} W_{\pi_v}$ the unramified Whittaker function of $\pi^S = \otimes_{v \not\in S} \pi_v$ above the complement of $S$. Let $\tilde{W}^S_\pi = \prod_{v \in S} \tilde{W}_{\pi_v}$ be the unramified Whittaker function of $\tilde{\pi}^S = \otimes_{v \not\in S} \tilde{\pi}_v$.

For any place $v$ of $F$, to a smooth compactly supported function $w_v \in C^\infty_c(F_v^\times)$ is associated a dual function $\tilde{w}_v$ of $w_v$ such that
\[
\int_{F_v^\times} \tilde{w}_v(x) \chi(x)^{-1} \|x\|_{F_v^\times}^{s - \frac{1}{2}} d^\times x = \chi(-1)^{-1} \gamma(1 - s, \pi_v \otimes \chi, \psi_v) \int_{F_v^\times} w_v(x) \chi(x) \|x\|_{F_v^\times}^{1 - s - \frac{1}{2}} d^\times x,
\]
for all $s$ of real part sufficiently large and all unitary multiplicative characters $\chi$ of $F_v^\times$. The equality (3.1) defines $\tilde{w}_v$ uniquely in terms of $\pi_v, \psi_v$ and $w_v$. For $S$ as above, we put $w_S = \prod_{v \in S} w_v, \tilde{w}_S = \prod_{v \in S} \tilde{w}_v$.

Let $v$ be an unramified place of $\pi$. For $\gamma, \zeta \in F_v^\times$, we define $K_v(\gamma, \zeta, \tilde{W}_{\pi_v})$ as follows. First, we put $K_v(\gamma, \zeta, \tilde{W}_{\pi_v}) = 0$ if $\|\gamma\|_v > \|\zeta\|_v^s$. Otherwise, when $r = 2$, for $\|\gamma\|_v \leq \|\zeta\|_v^{s_v}$ we let
\[
K_v(\gamma, \zeta, \tilde{W}_{\pi_v}) = \tilde{W}_{\pi_v} \left( -\gamma \zeta^{-1} \right) \psi_v (\gamma \zeta^{-1}),
\]
and, when $r = 3$, for $\|\gamma\|_v \leq \|\zeta\|_v^{3_v}$ we let
\[
(3.3) \quad K_v(\gamma, \zeta, \tilde{W}_{\pi_v}) = \|\zeta\|_v \sum_{\gamma' \in F_v^\times/\mathbb{C}_v^\times, \|\gamma\|_v < \|\zeta\|_v^2, \|\gamma\|_v < \|\gamma'\|_v, \|\gamma\|_v \leq \|\gamma'\|_v^2} \tilde{W}_{\pi_v} \left( -\gamma \gamma' \zeta^{-1} \right) \psi_v (\gamma \gamma' \zeta^{-1}) K_v(\gamma \gamma' \zeta^{-1}, \gamma'),
\]
with \( Kl_\nu(\gamma \xi^{-1}, \gamma') \) the Kloosterman sum defined by
\[
Kl_\nu(\gamma \xi^{-1}, \gamma') = \sum_{v \in \gamma' \mathfrak{C}_v^\times / \mathfrak{C}_v} \psi_v(v - \gamma \xi^{-1} v^{-1}) .
\]
In the quotient \( \gamma' \mathfrak{C}_v^\times / \mathfrak{C}_v \) above, the group \( \mathfrak{C}_v \) acts additively on \( \gamma' \mathfrak{C}_v^\times \) if \( \| \gamma' \|_v > 1 \) and \( \gamma' \mathfrak{C}_v^\times / \mathfrak{C}_v = \{1\} \) if \( \| \gamma' \|_v = 1 \), that is, imposing that \( v = 1 \). Now, let \( R \) be a finite set of places where \( \pi \) is unramified. We define \( K_R(\gamma, \zeta, \tilde{W}_o) = \prod_{\nu \in R} K_\nu(\gamma, \zeta, \tilde{W}_o) \) for \( \gamma, \zeta \in F_R^\times = \prod_{\nu \in R} F_\nu^\times \).

Proposition 3.1. ([11 Theorem 1]). Let \( \zeta \in A^\times \), let \( R \) be the set of places \( v \) such that \( \| \zeta \|_v > 1 \), and for all \( v \in S \) let \( w_v \in \mathcal{C}^{\times}_v(F_v^\times) \). Then we have the identity
\[
\sum_{\gamma \in F^\times} \psi(\gamma \zeta) W_0^S \left( \gamma, 1_{r-1} \right) w_S(\gamma) = \sum_{\gamma \in F^\times} K_R(\gamma, \zeta, \tilde{W}_o \circ \gamma) \tilde{W}_0(\gamma) \left( \gamma, 1_{r-1} \right) \tilde{w}_S(\gamma).
\]

3.2. Voronoï Summation in Classical Formulation. Subsequently, we assume that \( \pi \) is unramified and has trivial central character at all non-archimedean places. Then one may choose \( S = S_{\infty} \). In practice, one usually let \( \zeta \in A^\times \) be the diagonal embedding of a fraction \( \alpha/\beta \in F \), and it is preferable to have a classical formulation of the Voronoï summation in terms of Fourier coefficients, exponential factors, Kloosterman sums and Hankel transforms. However, when \( F \) does not have class number one, there is some subtlety due to the fact that the fraction \( \alpha/\beta \) can not always be chosen so that \( \alpha \) and \( \beta \) are relatively prime integers. Therefore the look of our translation of Ichino and Templier’s Voronoï summation formula is slightly different from those in [13, 14].

When \( r = 2 \), for a nonzero ideal \( a \) of \( \mathfrak{C} \) we define the Fourier coefficient
\[
A_\pi(a) = (Na)^{1/2} W_0^S \left( a \begin{array}{c} 0 \\ 1 \end{array} \right).
\]
When \( r = 3 \), for nonzero ideals \( a, a' \) we define the Fourier coefficient
\[
A_\pi(a, a') = N(aa') W_0^S \left( \begin{array}{c} a \alpha' \\ a' \end{array} \right).
\]
For our convenience, we denote \( A_\pi(a) = A_\pi(1, a) \) when \( r = 3 \). Moreover, For \( \gamma, \gamma' \in \mathfrak{C} \setminus \{0\} \), Fourier coefficients \( A_\pi(\gamma), A_\pi(\gamma', \gamma) \) will be defined in the same manner.

It is known from [11, 2.1] that, when \( v \) is archimedean, \( R_v(y) = -w_v(-y) \| y \|_v^{-1/2} \) is the Hankel integral transform of \( f_v(x) = w_v(x) \| x \|_v^{-1/2} \) integrated against the Bessel kernel \( J_v \), associated with \( \pi_v \) (see [11, (2.12), (2.17), (2.18)]), namely
\[
R_v(y) = \int_{F_v^\times} J_v(x) f_v(x) \, dx.
\]
We shall postpone the discussions on \( J_v \) to §5.1 and §5.2.

For a nonzero ideal \( b \) of \( \mathfrak{C} \), we define the additive character \( \psi_b \) on \( b^{-1}/\mathfrak{C} \) by \( \psi_b(\gamma) = e(\text{Tr} \gamma) \), with \( \text{Tr} : F/\mathfrak{C} \to \mathbb{Q}/\mathbb{Z} \) the trace induced from that on \( F \) (see [13, §IV.1]). Moreover, let \( F_b \) denote the ring of elements \( \alpha \in F \) such that \( \| \alpha \|_b \leq 1 \) for \( p \| b \), then \( \psi_b \) extends to a character on \( b^{-1}F_b/F_b \) via the isomorphism \( F_b/bF_b \cong \mathfrak{C}/b \).
We now choose \( \beta \in F^\times \) such that ord\( \beta \) = ord\( b \) if \( v_\nu | b \). For \( \alpha \in F^\times \) we let \( \overline{\alpha} \) denote a representative in \( \mathbb{Q} \cap F^\times \) of the multiplicative inverse of \( \alpha \) in \( F^*/bF^* \cong \mathbb{Q}/b \) so that
\[
\psi_b(\gamma / \alpha \beta) = \psi_b(\overline{\alpha} \gamma / \beta)
\]
for all \( \gamma \in F_b \). For \( \gamma, \gamma' \in F_b \), we define the Kloosterman sum
\[
S_b(\gamma, \gamma'; \beta) = \sum_{\gamma \in \mathbb{Q} / b} \psi_b \left( \frac{\gamma \gamma' + \gamma'}{\beta} \right).
\]
We have Weil’s bound for the Kloosterman sum
\[
S_b(\gamma, \gamma'; \beta) = O_\varepsilon((Nb)^{1/2+\varepsilon}),
\]
provided that \( \gamma \) and \( \gamma' \) are relative prime in \( \mathbb{Q} \), if \( v_\nu | b \).

By some straightforward calculations, we may reformulate the Voronoï summation formula in Proposition 3.1 as follows.

**Proposition 3.2.** Let notations be as above. Let \( \alpha \in F \) and \( \beta \in F^\times \) be such that \( \|\alpha\|_v = 1 \) whenever \( \|\beta\|_v < \|\alpha\|_v \), with \( v \in S_f \). Define \( b = \prod_{\|\beta\|_v < \|\alpha\|_v} \nu_v^{\text{ord} \beta} \). For \( v \in S_f \), let \( f_\gamma \in C^\infty_c(F_v) \) and \( \hat{f}_\gamma \) be the Hankel transform of \( f_\gamma \). Put \( f = \prod f_\gamma \) and \( \hat{f} = \prod \hat{f}_\gamma \). Then, when \( r = 2 \), we have
\[
\sum_{\gamma \in \mathbb{C} \setminus \{0\}} A_\pi(\gamma) \psi_\alpha \left( -\frac{\alpha \gamma}{\beta} \right) f(\gamma) = \frac{1}{Nb} \sum_{\gamma \in F^\times} A_{\overline{\beta}} \left( \gamma b^2 \right) \psi_b \left( -\overline{\alpha} \beta^2 \gamma / \beta \right) \hat{f}(\gamma),
\]
and, when \( r = 3 \), we have
\[
\sum_{\gamma \in \mathbb{C} \setminus \{0\}} A_\pi(\gamma) \psi_\alpha \left( -\frac{\alpha \gamma}{\beta} \right) f(\gamma)
= \sum_{b \subset a' \subset \mathbb{C}} \frac{Na'}{(Nb)^{r/2}} \sum_{\gamma \in F^\times} A_{\overline{\beta}} \left( a', \gamma b^3 a'^{-2} \right) S_{\alpha \gamma' b^{-1}} \left( 1, \overline{\beta} \gamma^2 / \beta \right) \hat{f}(\gamma),
\]
in which \( \gamma' \) is an element in \( F \) such that ord\( \gamma' = \text{ord} \alpha' \) whenever \( v_\nu | b \).

**Remark 3.3.** When \( F \) is of class number one, then we may choose \( \alpha \) and \( \beta \) to be a pair of integers that are relatively prime and let \( b = (\beta), \alpha' = (\gamma') \). In this way, upon changing \( \gamma \) to \( \gamma / \beta^2 \) or \( \gamma' / \beta^2 \), we obtain the classical Voronoï summation formula as in [MS1, MS2].

### 4. Average of Fourier Coefficients

The Rankin-Selberg \( L \)-function
\[
L(s, \pi \otimes \overline{\pi}) = \begin{cases} \sum_{\alpha \neq 0} |A_\pi(\alpha)|^2 (Na)^{-s}, & \text{if } r = 2, \\ \sum_{\alpha, \alpha' \neq 0} |A_\pi(\alpha, \alpha')|^2 (\alpha a'^2)^{-s}, & \text{if } r = 3, \end{cases}
\]
initially convergent for \( s \) of real part large, has a meromorphic continuation to the whole complex plane with only a simple pole at \( s = 1 \) (see [JS]). The Wiener-Ikehara theorem
Hence, Cauchy-Schwarz implies
\[
\sum_{N \leq X} |A_\pi(a)|^2 = O(X),
\]
\[
\sum_{N(a^c)^{\frac{1}{2}}} |A_\pi(a')|^2 = O(X), \quad \sum_{N \leq X} |A_\pi(a', a)|^2 = O((N^a')^2 X).
\]
In addition, it follows from the Wiener-Ikehara theorem for the Dedekind zeta function of \( F \) that
\[
\sum_{N \leq X} 1 = O(X).
\]
Hence, Cauchy-Schwarz implies
\[
(4.1) \quad \sum_{N \leq X} |A_\pi(a)| = O(X),
\]
\[
(4.2) \quad \sum_{N \leq X} |A_\pi(a', a)| = O (N^a X).
\]
All the implied constants above depend only on \( F \) and \( \pi \). More generally, for \( 0 \leq c < 1 \), partial summation yields
\[
(4.3) \quad \sum_{N \leq X} \frac{|A_\pi(a)|}{(N^a)^c} = O_{c, \pi} \left(X^{1-c}\right),
\]
\[
(4.4) \quad \sum_{N \leq X} \frac{|A_\pi(a', a)|}{(N^a)^c} = O_{c, \pi} \left( (N^a X)^{1-c}\right).
\]
In the partial summation identity
\[
\sum_{n=1}^X a_n b_n = \sum_{n=1}^{X-1} S_n (b_n - b_{n+1}) + S_X b_X, \quad S_n = \sum_{m=1}^n a_m,
\]
if one chooses \( a_n = \sum_{N=a=n} |A_\pi(a)| \) and \( b_n = n^{-c} \), then (4.3) follows immediately from (4.1); one may prove (4.4) in the same way.

Subsequently, we shall restrict ourselves to integral ideals in a given ideal class, then both (4.3) and (4.4) hold if the sums are over these ideals. More precisely, given an ideal \( a \) of \( \mathfrak{O} \), we have
\[
(4.5) \quad \sum_{[\gamma] \in F^\times / \mathfrak{O}^\times \atop \ell \leq [Ny] \parallel Na \leq X} \frac{|A_\pi(\gamma a)|}{|Ny|^c} = O_{c, \pi} \left((Na)^c X^{1-c}\right),
\]
\[
(4.6) \quad \sum_{[\gamma] \in F^\times / \mathfrak{O}^\times \atop \ell \leq [Ny] \parallel Na \leq X} \frac{|A_\pi(a', \gamma a)|}{|Ny|^c} = O_{c, \pi} \left( (Na')(Na)^c X^{1-c}\right),
\]
where \([\gamma]\) is the orbit in \( F^\times \) that contains \( \gamma \), under the action of \( \mathfrak{O}^\times \), namely, \([\gamma] = \{\gamma \epsilon : \epsilon \in \mathfrak{O}^\times\}\).

**Lemma 4.1.** Let \( V > 0 \). For \( T \in \mathbb{R}^{1/r_2}_{+} \) define \( N(T) = \prod \ell^N_\ell \) and \( F_{X}(T) = \{ x \in F_X : |x|_r \leq T \} \). Suppose that \( VN(T) \geq 1 \). Then, for all \( \gamma \in F^\times \) with \( |\gamma a| \geq 1/V \), we have
\[
\text{Card} \{[\gamma] \cap F_{X}(T)\} \ll 1 + \log(VN(T))^{r_1 + r_2 - 1},
\]
with the implied constant depending only on $F$.

**Proof.** By Dirichlet’s unit theorem (see [Lan, §V.1]), the rank of $\mathbb{O}^\times$ is $r_1 + r_2 - 1$ and the torsion in $\mathbb{O}^\times$ is the group of all roots of unity in $F$, which form a finite cyclic group. Define the map $\log : F_\infty \to \mathbb{R}^{r_1 + r_2}$ by $\log x = (\log \|x\|)_{v \in S_F}$. Restricting on $\mathbb{O}^\times$, the image $\log(\mathbb{O}^\times)$ is an $(r_1 + r_2 - 1)$-dimensional lattice $\Lambda$ in the hyperplane $L_{r_1 + r_2 - 1}$ and $\log$ has a finite kernel, namely, the torsion of $\mathbb{O}^\times$.

We now want to count the number of points in the image of $[\gamma] \cap F_\infty(T)$ under the map $\log$. We have $\log[\gamma] = (\log \|\gamma\|) + \Lambda$ and it is a shifted lattice lying in the hyperplane $L_{r_1 + r_2 - 1}$, with the definition $L_y = \{y \in \mathbb{R}^{r_1 + r_2} : \gamma y = y\}$. On the other hand, $\log F_\infty(T) = \{y \in \mathbb{R}^{r_1 + r_2} : y \leq N_y \log T\}$. Note that $\mathbb{R}_{\log[N_y]}^{r_1 + r_2 - 1} \cap \log F_\infty(T)$ are $(r_1 + r_2 - 1)$-dimensional simplexes of the same shape. Their volumes do not exceed the volume of $L_{r_1 + r_2 - 1} \cap \log F_\infty(T)$ and hence are uniformly $O((\log V + \log N(T))^{r_1 + r_2 - 1})$. Therefore, the number of lattice points in $\log[\gamma]$ lying in the simplex $\mathbb{R}_{\log[N_y]}^{r_1 + r_2 - 1} \cap \log F_\infty(T)$ is $O(1 + \log(\log N(T))^{r_1 + r_2 - 1})$. Finally, taking account of the finite torsion of $\mathbb{O}^\times$, which disappears under $\log$, we find that the number of points in $[\gamma] \cap F_\infty(T)$ is still $O(1 + \log(\log N(T))^{r_1 + r_2 - 1})$.

**Lemma 4.2.** Let notations be as above. Let $0 < c < 1$. Suppose that $\mathrm{Na}(T) \gg 1$. Then

$$\sum_{\gamma \in F_\infty \cap F_\infty(T) \atop \gamma \leq \mathbb{O}} \frac{|A_\tau(\gamma)|}{|\mathbb{O}|} = O((\mathrm{Na})^{1 + \varepsilon}(\log N(T))^{1 - c + \varepsilon}),$$

and

$$\sum_{\gamma \in F_\infty \cap F_\infty(T) \atop \gamma \leq \mathbb{O}} \frac{|A_\tau(a') \gamma|}{|\mathbb{O}|} = O((\mathrm{Na}')(\log N(T))^{1 - c + \varepsilon}),$$

with implied constants depending only on $e$, $c$, $F$ and $\pi$.

**Proof.** We shall only prove the first estimate as the second follows in the same way. By Lemma 4.1 and (4.5),

$$\sum_{\gamma \in F_\infty \cap F_\infty(T) \atop \gamma \leq \mathbb{O}} \frac{|A_\tau(\gamma)|}{|\mathbb{O}|} = \sum_{[\gamma] \in F_\infty / \mathbb{O} \times 1/\mathrm{Na} \leq |\mathbb{O}| \leq |\log N(T)|} \frac{|A_\tau(\gamma)|}{|\mathbb{O}|} \cdot \text{Card } [\gamma] \cap F_\infty(T)$$

$$\ll (1 + \log(\log N(T))^{r_1 + r_2 - 1}) \sum_{[\gamma] \in F_\infty / \mathbb{O} \times 1/\mathrm{Na} \leq |\mathbb{O}| \leq |\log N(T)|} \frac{|A_\tau(\gamma)|}{|\mathbb{O}|}$$

$$\ll (1 + \log(\log N(T))^{r_1 + r_2 - 1}) \mathrm{NaN}(T)^{1 - c}$$

$$\ll (\mathrm{Na})^{1 + \varepsilon}(\log N(T))^{1 - c + \varepsilon}.$$
0 < \varepsilon < d - 1,

\sum_{\gamma \in F^* \cap F^S_\varepsilon(T)} \frac{|A_\varepsilon(y\gamma)|}{|Ny\gamma|^d \gamma S^d} = O \left( \frac{(Na)^{1+\varepsilon}N(T)^{1-c+\varepsilon}}{||T||^{d-c}} \right),

\sum_{\gamma \in F^* \cap F^S_\varepsilon(T)} \frac{|A_\varepsilon(a', \gamma)|}{|Ny\gamma|^d \gamma S^d} = O \left( \frac{(Na')^{1+\varepsilon}N(T)^{1-c+\varepsilon}}{||T||^{d-c}} \right),

with implied constants depending only on \varepsilon, c, d, F and \pi.

Proof. For \( j \in \mathbb{N}^{[S]} \) we define \( T_j \in \mathbb{R}^{[1]} \) by \( T_j^{(v)} = 2^j T_v \) if \( v \in S \) and \( T_j^{(v)} = T_v \) if \( v \in S \setminus S \). We introduce a dyadic partition of \( F^S_\varepsilon(T) \),

\[ F^S_\varepsilon(T) = \bigcup_{j \in \mathbb{N}^{[S]}} F_j^{S_\varepsilon}(T), \]

with \( F_j^{S_\varepsilon}(T) = \{ x \in F_{\varepsilon} : T_j^{(v)} < \|x\|_v < T_{j+1}^{(v)} \text{ if } v \in S \}, \|x\|_v < T_{j+1}^{(v)} \text{ if } v \in S \setminus S \} \). Then

\[ \sum_{\gamma \in F^* \cap F^S_\varepsilon(T)} \frac{|A_\varepsilon(y\gamma)|}{|Ny\gamma|^d \gamma S^d} = \sum_{j \in \mathbb{N}^{[S]}} \sum_{\gamma \in F^* \cap F^S_\varepsilon(T)} \frac{|A_\varepsilon(y\gamma)|}{|Ny\gamma|^d \gamma S^d} \]

\[ \leq \sum_{j \in \mathbb{N}^{[S]}} \frac{1}{||T_j||^{d-c}} \sum_{\gamma \in F^* \cap F^S_\varepsilon(T)} \frac{|A_\varepsilon(y\gamma)|}{|Ny\gamma|^d \gamma S^d} \]

\[ \leq (Na)^{1+\varepsilon} \sum_{j \in \mathbb{N}^{[S]}} \frac{N(T_{j+1})^{1-c+\varepsilon}}{||T_j||^{d-c}} \]

\[ = \frac{(Na)^{1+\varepsilon}N(T)^{1-c+\varepsilon}}{||T||^{d-c}} \sum_{j \in \mathbb{N}^{[S]}} \frac{2^{(1-c+\varepsilon)\sum_{N_j} N_j}}{2^{(d-1)\varepsilon} \sum_{N_j, N_j}} \]

\[ \leq \frac{(Na)^{1+\varepsilon}N(T)^{1-c+\varepsilon}}{||T||^{d-c}}, \]

where Lemma 4.3 is applied for the second inequality. This proves the first estimate and the second follows in the same way. Q.E.D.

5. Asymptotic of Bessel Kernels and Estimates for Hankel Transforms

In this section, we shall exploit analytic results on the asymptotic behaviors of Bessel kernels in [QH] to study Hankel transforms, which arise as the analytic component of the Voronoi summation formula.

Throughout this section, we shall suppress \( v \) from our notations. Accordingly, \( F \) will be an archimedean local field, and \( \pi \) will be a unitary irreducible admissible generic representation of \( GL_v(F) \). With \( \pi \) are associated certain embedding parameters \((\mu, \delta) \in \mathbb{C}^r \times (\mathbb{Z}/2\mathbb{Z})^r \) if \( F = \mathbb{R} \) or \((\mu, m) \in \mathbb{C}^r \times \mathbb{Z}^r \) if \( F = \mathbb{C} \). Unitarity implies that \( \mu \in \mathbb{L}^{-1} = \{ \lambda \in \mathbb{C} : \sum_{l=1}^r \lambda_l = 0 \} \). With the notations of [QH], we shall denote Bessel kernels by \( J_{(\mu, \delta)}(x) \) or \( J_{(\mu, m)}(x) \).
5.1. Bounds for Bessel Kernels near Zero. The asymptotic behaviour of the Bessel kernel $J_\pi$ near zero is very close to its associated parameters, so it is worthwhile to first describe them in more details.

When $F = \mathbb{R}$, there are essentially two types of representations of $GL_r(\mathbb{R})$ (see the discussions in [Ma1] §2). First, $\pi$ may be a fully induced principal series representation, in which case $|\text{Re}\, \mu_i| < \frac{1}{2}$ for all $l = 1, ..., r$. Second, when $r = 2$, $\pi$ is the discrete series $\sigma(m)$ of $GL_2(\mathbb{R})$, $m = 1, 2, ..., \delta$, or, when $r = 3$, $\pi$ is an induced representation of $GL_3(\mathbb{R})$ constructed from a discrete series $\sigma(m)$ of $GL_2(\mathbb{R})$; in the former case one may choose

$$\mu_1 = \frac{m}{2}, \mu_2 = -\frac{m}{2}, \quad \delta_1 + \delta_2 \equiv m + 1 \pmod{2},$$

and in the latter,

$$\mu_1 = it + \frac{m}{2}, \mu_2 = -2it, \mu_3 = it - \frac{m}{2}, \quad \delta_1 + \delta_3 \equiv \delta_2 \equiv m + 1 \pmod{2}.$$ 

When $F = \mathbb{C}$, then $\pi$ must be a fully induced principal series representation and $|\text{Re}\, \mu_i| < \frac{1}{2}$ for all $l = 1, ..., r$.

**Lemma 5.1.** Let $C > 0$. When $F = \mathbb{R}$, for $|x| < C$, we have

$$J_{(\mu, \delta)}(x) \ll |x|^{-1/2}. $$

Similarly, when $F = \mathbb{C}$, for $|z| < C$, we have

$$J_{(\mu, m)}(z) \ll |z|^{-1}. $$

The implied constants above depend only on $C$ and the parameters $(\mu, \delta)$ or $(\mu, m)$.

We shall prove this lemma by bounding the Mellin-Barnes type integrals of certain gamma factors as in [Qi1] §2.4.

5.1.1. Bounds for Real Bessel Kernels near Zero. Suppose $F = \mathbb{R}$. According to [Qi1] §2.4, let

$$G_\delta(s) = i^\delta \pi^{\frac{1}{2} - \tau} \frac{\Gamma \left( \frac{1}{2} (s + \delta) \right)}{\Gamma \left( \frac{1}{2} (1 - s + \delta) \right)} = \begin{cases} 2(2\pi)^{-r} \Gamma(s) \cos \left( \frac{\pi s}{2} \right), & \text{if } \delta = 0, \\ 2i(2\pi)^{r} \Gamma(s) \sin \left( \frac{\pi s}{2} \right), & \text{if } \delta = 1, \end{cases}$$

and define

$$j_{(\mu, \delta)}(x) = \frac{1}{2\pi i} \int_{C_{(\mu, \delta)}} G_{(\mu, \delta)}(s) x^{-s} ds, \quad x > 0,$$

where $C_{(\mu, \delta)}$ is a curve obtained from a vertical line on the right of all the poles of $G_{(\mu, \delta)}(s)$ with two ends at infinity shifted to the left, say, to $\text{Re} \, s = -1$ (see [Qi1] Definition 2.4.2); the shift is needed only to guarantee convergence. The Bessel kernel $J_\pi = J_{(\mu, \delta)}$ is defined as

$$J_{(\mu, \delta)}(\pm x) = \frac{1}{2} \left( j_{(\mu, \delta)}(x) \pm j_{(\mu, \delta+\epsilon)}(x) \right),$$

where $\epsilon = (1, 1)$ if $r = 2$ and $\epsilon = (1, 1, 1)$ if $r = 3$. 

We remark that both \( G_{(\mu, \delta)}(s) \) and \( G_{(\mu, \delta+e)}(s) \) have no poles in the half plane \( \Re s \geq \frac{1}{2} \). This is clear when \( \pi \) is a principal series. When \( \pi \) is the discrete series \( \sigma(m) \) of \( \text{GL}_2(\mathbb{R}) \), by the duplication formula and Euler’s reflection formula of the Gamma function, both \( G_{(\mu, \delta)}(s) \) and \( G_{(\mu, \delta+e)}(s) \) are equal to

\[
p^{m+1}(2\pi)^{1-2s} \frac{\Gamma\left(s + \frac{1}{2}m\right)}{\Gamma\left(1 - s + \frac{1}{2}m\right)},
\]

which has no poles even in the larger region \( \Re s > -\frac{1}{2}m \). We are in a similar situation, if \( \pi \) is a representation of \( \text{GL}_3(\mathbb{R}) \) coming from the discrete series \( \sigma(m) \) of \( \text{GL}_2(\mathbb{R}) \). Therefore, we may and do choose the contours \( \mathcal{C}_{(\mu, \delta)} \) and \( \mathcal{C}_{(\mu, \delta+e)} \) to be contained in the half plane \( \Re s \leq \frac{1}{2} \). Then, some simple estimations of the integrals that define \( j_{(\mu, \delta)}(x) \) and \( j_{(\mu, \delta+e)}(x) \) yield the bound for \( J_{(\mu, \delta)}(\pm x) \) in Lemma 5.1.

5.1.2. Bounds for Complex Bessel Kernels near Zero. Now let \( F = \mathbb{C} \). Let

\[
G_m(s) = i^{|m|}(2\pi)^{1-2s} \frac{\Gamma\left(s + \frac{1}{2}|m|\right)}{\Gamma\left(1 - s + \frac{1}{2}|m|\right)},
\]

\[
G_{(\mu, m)}(s) = \prod_{l=1}^{n} G_m(s - \mu_l),
\]

and

\[
j_{(\mu, m)}(x) = \frac{1}{2\pi} \int_{\mathcal{C}_{(\mu, m)}} G_{(\mu, m)}(s)x^{-s}ds, \quad x > 0,
\]

with \( 2 \cdot \mathcal{C}_{(\mu, m)} \) defined similarly as \( \mathcal{C}_{(\mu, \delta)} \) such that all the poles of \( G_{(\mu, m)}(s) \) stay on the left (see [QH] Definition 2.4.2)). We define the Bessel kernel \( J_\pi = J_{(\mu, m)} \) in the polar coordinates,

\[
J_{(\mu, m)}(xe^{i\theta}) = \frac{1}{2\pi} \sum_{m = -\infty}^{\infty} j_{(\mu, m+me)}(x)e^{im\theta}.
\]

Now assume that \( \max\{|\Re \mu_l|\} < \frac{1}{4} \) and set \( M = \max\{|m_i|\} \). Let \( x < C^2 \). We truncate the Fourier series at \( |m| = M \). When \( |m| > M \), the estimate in [QH] Lemma 2.4.13] yields

\[
j_{(\mu, m+me)}(x) \ll \left(\frac{(2\pi e)|m|}{(|m| + 1)|m|-M}\right)^r \frac{1}{x^{|m|-M-1}}.
\]

When \( |m| \leq M \), we choose the contours \( \mathcal{C}_{(\mu, \delta+me)} \) in the half plane \( \Re s \leq \frac{1}{2} \), so

\[
j_{(\mu, m+me)}(x) \ll x^{-1}.
\]

Combining these estimates, we obtain

\[
J_{(\mu, m)}(xe^{i\theta}) \ll x^{-1}.
\]

All the implied constants above depend only on \( C \) and \( (\mu, m) \).

5.2. Asymptotic of Bessel Kernels at Infinity. We now collect some results from [QH] on the asymptotic of Bessel kernels at infinity. See [QH] Theorem 1.5.13, 1.9.3, 2.8.7).

**Proposition 5.2.** Let \( K \geq 0 \).
When \( F = \mathbb{R} \) for \( x > 0 \), we may write
\[
J_{(\mu, \delta)} (x^2) = \sum_{\pm} e^{\pm (2\pi x)} W_{(\mu, \delta)}^\pm (x) + E_{(\mu, \delta)}^\pm (x),
\]
if \( r = 2 \),
\[
J_{(\mu, \delta)} (-x^2) = E_{(\mu, \delta)}^- (x),
\]
with notations \( (5.1) \).

When \( F = \mathbb{C} \), we may write
\[
J_{(\mu, m)} (z^2) = \sum_{\xi = 1} e^{i (\xi z + \bar{\xi} \bar{z})} W_{(\mu, m)} (z, \xi) + E_{(\mu, m)} (z),
\]
with notations \( [z] = z/|z| \) and \( |m| = \sum_{i=1}^r m_i \), and there is a constant \( C_{K, \mu, m} \), depending only on \( K \) and \( (\mu, m) \), such that for \( z \in \mathbb{C} \), with \( |z| \geq C_{K, (\mu, m)} \), we have
\[
W_{(\mu, m)} (z, \xi) = \sum_{k, k' = 0}^{K-1} \sum_{k + k' \leq K-1} i^{k-k'} e^{-i k \xi} B_{k,k'} z^{k-k'} + O_{K, (\mu, m)} ([z]^{-K}),
\]
with coefficients \( B_{k,k'} \) depending only on \( (\mu, m) \),
\[
E_{(\mu, m)} (z) = 0,
\]
if \( r = 2 \), and
\[
E_{(\mu, m)} (z) = O_{(\mu, m)} ([z]^{-2} \exp (-12\pi \sin (\frac{1}{12} \pi) \sin (\frac{1}{12} \pi) |z|)),
\]
if \( r = 3 \).

### 5.3. Estimates for Hankel Transforms
Recall that Hankel transforms are defined as
\[
\tilde{f}(y) = \int_{\mathbb{R}} J_{(\mu, \delta)} (xy) f(x) dx, \quad \tilde{f}(u) = \int_{\mathbb{C}} J_{(\mu, m)} (zu) f(z) i dz \wedge d\bar{z}.
\]
For a weight function \( w \) given as in Theorem 1.2 and a scalar \( \rho \in F \), our choice of test function will be
\[
f(x) = w(x) e (-\rho x), \quad \text{if } F = \mathbb{R},
\]
\[
f(z) = w(z) e (-\rho z - \bar{\rho} \bar{z}), \quad \text{if } F = \mathbb{C}.
\]
LEMMA 5.3. Let $C > 0$. Suppose that $0 < T|y|, T|u| \leq C$. Then,
\[ \tilde{f}(y) \ll \mathcal{C}_{\mu, \lambda} T^{1/2}|y|^{-1/2}, \]
if $F = \mathbb{R}$, and
\[ \tilde{f}(u) \ll \mathcal{C}_{(\mu, m), \lambda} T|u|^{-1}, \]
if $F = \mathbb{C}$.

PROOF. This lemma immediately follows from Lemma 5.1. Q.E.D.

Next, we consider $\tilde{f}$ of large argument. For this we have two lemmas for the cases $T|\rho| \gg 1$ and $T|\rho| \ll 1$ separately.

LEMMA 5.4. Let $K \geq 0$. Put $C = C_{K, \mu}$ or $C_{2K, (\mu, m)}$. Suppose that $T|\rho|$ is sufficiently large, say $T|\rho| > A^{r-1}C^{1/r}$, and that $T|y|, T|u| > C$.

When $F = \mathbb{R}$, we have
\[ \tilde{f}(y) \ll \mathcal{C}_{\mu, \lambda} T^{1/2}|y|^{-1/2}, \]
if $(T|\rho|)^r / A^{(r-1)} \leq T|y| \leq A^{(r-1)}(T|\rho|)^r$, and
\[ \tilde{f}(y) \ll \mathcal{C}_{K, \mu, \lambda} T(T|y|)^{-r}/2K/r, \]
in other cases.

When $F = \mathbb{C}$, we have
\[ \tilde{f}(u) \ll \mathcal{C}_{(\mu, m), \lambda} T|u|^{-1}, \]
if $(T|\rho|)^r / A^{(r-1)} \leq T|u| \leq A^{(r-1)}(T|\rho|)^r$, and
\[ \tilde{f}(u) \ll \mathcal{C}_{K, (\mu, m), \lambda} T^2(T|u|)^{-r}/2K/r, \]
in other cases.

PROOF. We first consider the real case. In view of Proposition 5.2, there are four similar integrals coming from the leading terms of the asymptotic expansions of $W_{(\mu, \delta)}^x(x)$ as in 5.1, two of them being of the following form, up to a constant multiple,
\[ \frac{1}{y^{(r-1)/2}} \int_{1/y}^{(y)^{1/2}} e\left( \pm ry^{1/r} x - \rho x^r \right) x^{(r-1)/2} w(x) \, dx, \]
in which we assume that $\rho, y > 0$. Upon changing the variable $x$ to $y^{1/(r-1)} x / \rho^{1/(r-1)}$, we obtain the integral
\[ h_{\pm}(\lambda) = \frac{T^{(r-1)/2r}}{\rho^{(r+1)/2} \lambda^{(r-2r-1)/2r}} \int_X e\left( \lambda p_{\pm}(x) \right) g(x, \lambda) \, dx, \]
with
\[ \lambda = \left( \frac{y}{\rho} \right)^{1/(r-1)}, \quad X = \left( \frac{T\rho}{\lambda} \right)^{1/r}, \]
\[ p_{\pm}(x) = \pm rx - x^r, \quad g(x, \lambda) = \left( \frac{\lambda x^{r}}{T \rho} \right)^{(r-1)/2} w\left( \frac{\lambda x^{r}}{\rho} \right), \]
and the weight function $g(x, \lambda)$ satisfying
\[ (\partial/\partial x)^k (\partial/\partial \lambda)^k g(x, \lambda) \ll X^{-i\lambda^{-k}}. \]
For \( h_-(\lambda) \), we have \( p'_-(x) = -r(x^{r-1} + 1) \leq -r \) for all \( x \in [X, A^{1/r}X] \), and hence by repeating partial integration \( K \) times, we have the estimate
\[
    h_-(\lambda) \leq \frac{T^{(r+1)/2r}}{\rho \lambda^{r-1} (r-1)/2r} \frac{1}{(X\lambda)^K} = \frac{T}{(T\lambda)^{(r-1)/2r+K/r}}.
\]
As for \( h_+(\lambda) \), since \( |p'_+(x)| > r(1-A^{-(r-1)/2}) \) on \([X, A^{1/r}X]\) if either \( X < 1/A \) or \( X > A \), repeating partial integration yields the same bound for \( h_+(\lambda) \) as above. When \( 1/A \leq X \leq A \), so that \( T^{r-1} \rho'/\lambda^{r-1} \leq y \leq A^{r-1} T^{r-1} \rho' \), we use the stationary phase estimate in \([\text{Sog Theorem 1.1.1}]\) to bound the integral in (5.6) by \( \lambda^{-1/2} \) (note that \( \lambda \geq T\rho \geq 1 \)), then
\[
    h_+(\lambda) \leq \frac{T^{(r-1)/2r}}{\rho^{r+1}/2} \frac{1}{\lambda^{1/2} y^{(r-1)/2r}} = \frac{T^{(r-1)/2r}}{\rho^{1/2} (r+1)/(2(1-1))} \leq \frac{T^{1/2}}{y^{1/2}}.
\]
Moreover, contributions from lower order terms may be treated in the same manner and those from the error term in (5.1) and \( E_{\mu, \theta}(x) \) can also be bounded by \( T(T\lambda)^{(r-1)/2r-1/S}/r \).

The complex case is similar. Thus we shall only consider the total leading-term contributions from the asymptotic expansions of all \( W_{(\mu, m)}(z, \xi) \) as in (5.2). Without loss of generality, we assume \( \rho > 0 \). The integral that we need to estimate is
\[
    \frac{1}{|u|^{(r-1)/2r} \rho^{1/r}} \sum_{\xi \in \mathbb{Z}} \int_{S_{\xi}} e(r(\xi u^{1/r}z + \xi u^{1/r}/\rho) - \rho (\zeta + \overline{\zeta})) |z|^{1-r} |\xi| \omega \, idz \wedge dz
\]
\[
    = \frac{1}{|u|^{(r-1)/2r} \rho^{1/r}} \int_{C} e(r(u^{1/r}z + u^{1/r}/\rho) - \rho (\zeta + \overline{\zeta})) |z|^{1-r} |\xi| \omega \, idz \wedge dz,
\]
where \( u^{1/r} \) is the principal branch of the \( r \)-th root of \( u \). In the polar coordinates, write \( z = xe^{i\phi} \) and \( u = ye^{i\omega} \), then the integral above turns into
\[
    \frac{2}{y^{(r-1)/2r} e^{i|\omega|/\rho}} \int_{T^1} e(2y^{1/r}x \cos(\phi + \omega/\rho) - 2\rho x^{r} \cos(\rho r)) \, x^{r} e^{-|\xi| \omega} w(\xi x^{r} e^{i\phi}) \, x^{r} d\phi.
\]
After changing the variable \( x \) to \( y^{1/r} x_{1/r}^{1/r} \), we obtain
\[
    h(\lambda, \omega) = \frac{2T}{\rho \lambda^{r-2} e^{i|\omega|/\rho}} \int_{X} e(4 \lambda x^{r} \cos(\omega/\rho) x^{r} d\phi),
\]
with
\[
    \lambda = \left( \frac{y}{\rho} \right)^{1/(r-1)}, \quad X = \left( \frac{T\rho}{\lambda} \right)^{1/r}, \quad \rho \lambda^{r-2} e^{i|\omega|/\rho} \omega \left( \frac{\lambda x^{r} e^{i\phi}}{\rho} \right),
\]
and the weight function \( g(x, \phi, \omega) \) satisfying
\[
    (\partial/\partial x)^{j}(\partial/\partial \phi)^{k}(\partial/\partial \lambda)^{i} g(x, \phi, \lambda) \approx X^{-i} \lambda^{-j} \rho^{-k}.
\]
We now apply the method for stationary phase for double integrals; a similar integral is also treated in \([\text{Qi2}]\) for \( r = 2 \). Since
\[
    p'(x, \phi, \omega) = (2r \cos(\phi + \omega/\rho) - x^{r-1} \cos(\rho r)), \quad -2r x \sin(\phi + \omega/\rho) - x^{r-1} \sin(\rho r)) \),
\]
there is a unique stationary point \( (x_0, \phi_0) = (1, \omega/\rho(r - 1)) \). First, in the case that either \( X < 1/A \) or \( X > A \), we repeatedly apply the elaborated partial integration of Hörmander
(see the proof of [Hor Theorem 7.7.1] and also [QI2 §5.1.3]). More precisely, we define
\[ q(x, \phi; \omega) = x^{1-r} (\partial_x p(x, \phi; \omega))^2 + x^{-1-r} (\partial_\phi p(x, \phi; \omega))^2 \]
\[ = 4r^2 \left( x^{-1} + x^{-1-r} - 2 \cos((r-1)\phi - \omega / r) \right). \]

An important observation is that \( q(x, \phi; \omega) \) has a uniform positive lower bound,
\[ q(x, \phi; \omega) \geq 4 r^2 \left( A^{(r-1)/2r} - A^{-2(r-1)/2r} \right)^2, \quad x \in [X, A^{1/r}X]. \]

Then, writing the integral in (5.7) as
\[ \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(x, \phi, \lambda)}{q(x, \phi; \omega)} \left( \frac{\partial_x p(x, \phi; \omega)}{x^{1-1}} \partial_x (e (\lambda p(x, \phi; \omega))) \right. \]
\[ \left. + \frac{\partial_\phi p(x, \phi; \omega)}{x^{r+1}} \partial_\phi (e (\lambda p(x, \phi; \omega))) \right) dx d\phi, \]
and integrating by parts, we obtain
\[ -\frac{1}{2\pi i} \int_0^{2\pi} \frac{g(x, \phi, \lambda)}{q(x, \phi; \omega)} \left( \frac{\partial_x}{x} \left( \frac{\partial_x p(x, \phi; \omega)}{x^{1-1}} q(x, \phi; \omega) \right) \right. \]
\[ \left. + \frac{\partial_\phi}{x^{r+1}} \left( \frac{\partial_\phi p(x, \phi; \omega)}{x^{r+1}} q(x, \phi; \omega) \right) \right) e (\lambda p(x, \phi; \omega)) dx d\phi. \]

By calculating the derivatives in the integrand and estimating each resulted integral, it is easy to see that we get a saving of \( 1/X \lambda \). On repeating this elaborated partial integration \( 2K \) times, some calculations yield
\[ h(\lambda, \omega) \leq \frac{T^{(r+1)/r}}{\rho^{(r-1)/r} \lambda^{r(r-1)/2r} (X \lambda)^{2K}} = \frac{T^2}{(T \rho^{1/r} \lambda^{(r-1)/2r}) (X \lambda)^{2K}}. \]

Second, when \( 1/A \leq X \leq A \), we use the stationary phase estimate in [Sog Theorem 1.1.4] to bound the integral in (5.7) by \( \lambda^{-1} \), then
\[ h(\lambda, \omega) \leq \frac{T}{\rho^{K+2}} \frac{1}{\lambda} = \frac{T}{y}. \]

Q.E.D.

When \( T |\rho| \ll 1 \), we are in a much easier situation. Since \( e(-\rho x) \) and \( e(-\rho z - \rho \bar{z}) \) are non-oscillatory, they may be absorbed into the weight functions, and we may save any power of \( (T |y|)^{1/r} \) or \( (T |u|)^{1/r} \) by repeating partial integration.

**Lemma 5.5.** Let \( K \geq 0 \). Put \( C = C_{K, \rho} \) or \( C_{2K, \mu, \omega} \). Suppose that \( T |\rho| \) is bounded, say by \( A^{-1} C^{1/r} \), and that \( T |y|, T |u| > C \). Then,
\[ \tilde{f}(y) \leq C_{K, \mu, \omega} T (T |y|)^{-(r-1)/2r - K/r}, \]
and
\[ \tilde{f}(u) \leq C_{K, \mu, \omega, \Delta} T^2 (T |u|)^{-(r-1)/r - 2K/r}. \]

In conclusion, combining Lemma 5.3, 5.4, and 5.5 we have the following corollary.

**Corollary 5.6.** Let \( T, Y > 0 \) be sufficiently large in terms of \( \pi \) and \( A \). Then, for any \( \rho \in F \) with \( |\rho| \leq Y/A^{-1}T \), we have uniform bounds
\[ \tilde{f}(y) \leq \frac{T}{y}^{1/2}, \quad \tilde{f}(u) \leq \frac{T}{u}^{1/2}, \]
and, for any $A > \frac{1}{2}$, we have

$$\tilde{f}(y) \leq \|T\|^{-A}\|y\|^{-A}, \quad \tilde{f}(u) \leq \|T\|^{-A}\|u\|^{-A},$$

if $T|y|, T|u| > Y'$. All the implied constants depend only on $\pi, A$, and in addition on $A$ for the last two inequalities.

6. Proof of Theorem 1.2

Let $T \gg 1$, $\theta \in F_x$, and let $\omega$ be a smooth product function on $F_x$ as in Theorem 1.2. Let $Q = \sqrt{T}$ and let $\hat{\alpha} \in \mathcal{O}, \hat{\beta} \in \mathcal{O} \setminus \{0\}$ be chosen as in Lemma A.2. By the Chinese remainder theorem, there is $\delta \in \mathcal{O} \setminus \{0\}$ such that $\alpha = \hat{\alpha}/\delta$ and $\beta = \hat{\beta}/\delta$ are relatively prime in $\mathcal{O}_v$ for all non-archimedean places $v$ with $\|\beta\|_v < \|\alpha\|_v$. Let $\rho_v = \alpha/\beta - \theta_v$. Then, in view of (A.1), we have

$$|\hat{\beta}|_v \leq Q = \sqrt{T}, \quad |\rho_v| \leq \frac{1}{Q|\hat{\beta}|_v} = \frac{1}{\sqrt{T}|\hat{\beta}|_v}$$

for all $v \in S_x$. Set $Y_v = C_v \sqrt{T}/|\hat{\beta}|_v$, for some constants $C_v$ depending only on $F, \pi$ and $A$, which are expressible in terms of those implied constants in Lemma A.2 and Corollary 5.6.

Now, let $S_{\theta, \omega}$ denote the sum in (1.11) or (1.12), and define $f_v$ as in (5.4) and (5.5), then

$$S_{\theta, \omega} = \sum_{y \in \mathbb{C}} A_x(y) e \left( Tr \left( \frac{\alpha y}{\beta} \right) \right) f(y),$$

which is the left hand side of the Voronoï summation formula (3.11) or (3.12) in Proposition 3.2. In the following, we shall estimate the right hand side.

When $r = 2$, applying Corollary 5.6 to bound $\tilde{f}(y)$ on the right hand side of (3.11), we obtain

$$S_{\theta, \omega} \leq \frac{T^{N/2}}{\text{Nb}} \sum_{y \in F^0 \setminus F_x} |A_x(yb^2)| \left| \frac{T^{N/2}}{\text{Nb}} \sum_{S \subseteq \mathbb{C}} \frac{1}{|T|} \sum_{y \in F^0 \setminus F_x} \frac{|A_x(yb^2)|}{|Ny|^{1/2} \|\gamma\|_S} \right|. $$

Note that $Y_v^2/T = C_v^2/|\hat{\beta}|_v^2$. Let us assume momentarily that $N(Y^2/T)\text{Nb}^2 \geq 1$. Then, by Lemma 4.2 and 4.3 along with $\text{Nb} \leq |N\hat{\beta}|$ (note that both $\hat{\alpha}$ and $\hat{\beta}$ are integral) and $|\hat{\beta}|_v \leq \sqrt{T}$, we find that the first and the second sum are bounded by

$$\frac{T^{N/2}}{\text{Nb}} \frac{\text{Nb}^{2+2r}}{|N\hat{\beta}|^{1+2r}} \leq T^{N/2},$$

and

$$\frac{T^{N/2}}{\text{Nb}} \sum_{S \subseteq \mathbb{C}} \frac{1}{|T|} \frac{|\hat{\beta}|_v^2 \text{Nb}^{2+2r}}{|N\hat{\beta}|^{1+2r}} \leq T^{N/2}.$$

When $N(Y^2/T)\text{Nb}^2 < 1$, the argument actually simplifies because the first sum has no terms and the second can be taken over a smaller range by rescaling $Y$. 
When \( r = 3 \), we apply similar arguments. We estimate the right hand side of (3.12) using Weil’s bound (3.10) for the Kloosterman sum \( S_{bd^{-1}} \left( 1, \pi (\beta^3 \gamma / \gamma') ; \beta / \gamma' \right) \) and Corollary 5.8 for \( \tilde{f}(\gamma) \). Then

\[
S_{\theta,w} \ll \sum_{b \subset c' \subset \mathbb{C}} \frac{(N a')^{1/2 - \epsilon}}{(N b)^{3/2 - \epsilon}} \left( M(a', b; \bar{\beta}) + E(a', b; \bar{\beta}) \right)
\]

with

\[
M(a', b; \bar{\beta}) = T^{N/2} \sum_{\gamma \in F' \cap F (Y / T)} |A_2(a', \gamma b^3 a^{-2})| |N \gamma|^{1/2}
\]

\[
E(a', b; \bar{\beta}) = T^{N/2} \sum_{S \subset S_{x \not\in \emptyset}} \frac{1}{\|T\|_S} \sum_{\gamma \in F' \cap F (Y / T)} |A_2(a', \gamma b^3 a^{-2})| |N \gamma|^{1/2} |\gamma S|
\]

Note that \( Y^3 / T = C^T \sqrt{7} / |\beta|^3 \). Again, we shall assume that \( N(Y^3 / T) N(b^3 a^{-2}) \gg 1 \); otherwise, the analysis is simpler as it was above when \( r = 2 \). It follows from Lemma 4.2 and 4.3 that

\[
M(a', b; \bar{\beta}) \ll T^{N/2} \frac{(N b)^{3+3\epsilon} T^{(1/4+\epsilon/2)N}}{(N a')^{1+2\epsilon} |N \bar{\beta}|^{3/2+3\epsilon}} \ll \frac{(N b)^{3/2} T^{3N/4+\epsilon}}{(N a')^{1+\epsilon}}.
\]

\[
E(a', b; \bar{\beta}) \ll T^{N/2} \sum_{S \subset S_{x \not\in \emptyset}} \frac{1}{\|T\|_S} \frac{|\bar{\beta}|^3}{\|T\|_S^{1/2}} \frac{(N b)^{3+3\epsilon} T^{(1/4+\epsilon/2)N}}{(N a')^{1+2\epsilon} |N \bar{\beta}|^{3/2+3\epsilon}} \ll \frac{(N b)^{3/2} T^{3N/4+\epsilon}}{(N a')^{1+\epsilon}}.
\]

Combining these, we have

\[
S_{\theta,w} \ll \sum_{b \subset c' \subset \mathbb{C}} \frac{(N a')^{1/2 - \epsilon}}{(N b)^{3/2 - \epsilon}} \left( \frac{(N b)^{3/2} T^{3N/4+\epsilon}}{(N a')^{1+\epsilon}} \right) = T^{3N/4+\epsilon} \sum_{b \subset c' \subset \mathbb{C}} \frac{(N b)^{\epsilon}}{(N a')^{1/2+\epsilon}} \ll T^{3N/4+\epsilon}.
\]

### 7. Proof of Theorem 1.1

In this final section, we show how to deduce Theorem 1.1 from Theorem 1.2 following [LY] §8. First, we have the following higher dimensional generalization of [LY] Lemma 8.1.

**Lemma 7.1.** Let \( \Pi \subset F_{\infty} \) be a fundamental parallelootope for \( F_{\infty} / \mathcal{O} \) which is symmetric about zero. For \( X \gg 2 \) there exists a function \( h_X(x) \) on \( F_{\infty} \) such that

\[
\int_{F_{\infty}} |h_X(x)| dx \ll (\log X)^N,
\]

with the implied constant depending only on \( F \), and that for \( \gamma \in \mathcal{O} \),

\[
\int_{F_{\infty}} h_X(x) e(\text{Tr}(xy)) dx = \begin{cases} 1, & \text{if } \gamma \in X \cdot \Pi, \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof.** By a linear transform \( A \), we may transform \( \mathcal{O} \) to the lattice \( \mathbb{Z}^n \) and \( \Pi \) to the unit hypercube centered at zero. Define a function on \( \mathbb{R}^n \),

\[
g_X(t) = \begin{cases} \min\{1, 1 + (X - |t|)/\lambda\}, & \text{for } |t| \leq X/2 + 1, \\ 0, & \text{for } |t| > X/2 + 1. \end{cases}
\]
We form the product \( g_N^N(y) = \prod_{n=1}^N g_N(y_n) \) and let \( h_N(x) \) be the Fourier transform (with Fourier kernel \( e(-\text{Tr}(xy)) \)) of \( g_N^N(Ax) \).

Note that \( g_N^N(Ay) = 1 \) if \( y \in X \cdot \Pi \) and vanishes otherwise, giving (7.2).

As for (7.1), we first prove that the \( L^1 \) norm of the Fourier transform \( \hat{g}_N^N(t) \) is \( O(\log X) \).

The proof is similar to that of [DFI Lemma 9]. Since \( |\det A| h_N(I'Ax) \) splits into the product \( 2^n \prod_{n=1}^N \hat{g}_N(x_n) \), the \( L^1 \) norm of \( h_N(x) \) is of size \( O((\log X)^n) \). Here \( I \) is the diagonal matrix that represents the bilinear form \( \text{Tr}(xy) \) so that \( \text{Tr}(xy) = 'xy \).

Q.E.D.

Now we prove Theorem 11. We choose suitable parameters \( \Delta \gg 1 \gg \Delta' \) such that \( M \cdot \Pi \setminus (M/2) \Pi \subset A[2\Delta'M, \Delta' \Delta'M/2] \), with the definition \( A[Y, Z] = \{ x \in F_{\infty} : |x_i| \in [Y, Z] \} \). We choose a weight function satisfying the conditions in Theorem 12 with \( T = \Delta'M \) and \( w(x) \equiv 1 \) on the annulus \( A[2\Delta'M, \Delta' \Delta'M/2] \). Thus

\[
\sum_{\gamma \in \mathbb{C} \cap \{ M \cdot \Pi \setminus (M/2) \Pi \}} A_x(\gamma) e(\text{Tr}(\theta \gamma)) = \sum_{\gamma \in \mathbb{C} \cap \{ M \cdot \Pi \setminus (M/2) \Pi \}} A_x(\gamma) e(\text{Tr}(\theta \gamma)) w(\gamma).
\]

Write the right hand side as \( S_w(M) - S_w(M/2) \). Then apply Lemma 7.1 with \( X = M, M/2 \) to get

\[
S_w(X) \leq \int_{F_{\infty}} |h(x)| \left| \sum_{\gamma \in \mathbb{C}} A_x(\gamma) e\left(\text{Tr}\left((\theta + x)\gamma\right)\right) w(\gamma) \right| dx.
\]

Theorem 12 applies to the sum over \( \gamma \), the uniformity in \( \theta \) being critical, and (7.1) controls the integral over \( F_{\infty} \), giving Theorem 11.

### Appendix A. Dirichlet Approximation for Number Fields

In this appendix, we give a generalization of Dirichlet’s approximation theorem to an arbitrary number field. First, we have the following result of simultaneous Dirichlet approximation for lattices in \( \mathbb{R}^N \) (compare [Sch Theorem 1E]).

**Proposition A.1.** Let \( \Lambda \subset \mathbb{R}^N \) be a lattice and let \( \Pi \) be a closed fundamental parallelotope for \( \Lambda \). Suppose that \( a_{jk}, j = 1, ..., M, k = 1, ..., N \), are \( nm \) many real numbers and that \( Q > 1 \) is a real number. Then there exist integers \( q_1, ..., q_M \) and a lattice point \( \lambda \in \Lambda \) with

\[
1 \leq \max\{|q_1|, ..., |q_M|\} < Q^{N/M},
\]

\[
\sum_{j=1}^M q_j a_{jk} - \lambda_j \leq 2Q^{-1} \cdot \Pi, \quad k = 1, ..., N.
\]

As a consequence, we have the following lemma.

**Lemma A.2.** Let \( \theta \in F_{\infty} \), and let \( Q > 1 \) be real. Then there exist \( \alpha, \beta \in \mathbb{Q} \), with \( \beta \neq 0 \), such that

\[
|\beta|_v \ll Q, \quad |\beta, \theta - \alpha|_v \ll 1 \quad Q,
\]

for all \( v \in S_{\infty} \), where the implied constants depends only on the field \( F \).
Proof. In the notations of Proposition [Λ,1], let \( \Lambda = \mathbb{Q}^r \), for a fixed integer basis \( \beta_1, \ldots, \beta_N \) for \( \mathbb{Q} \), let \( \Pi = \left\{ \sum_{j=1}^{N} t_j \beta_j^r : t_j \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \right\} \), and let \( a_j = \theta_v \beta_j^r \). Note that \( M = N \). Then Proposition [Λ,1] implies the existence of integers \( q \) and \( \beta \in \mathbb{Q} \) such that

\[
1 \leqslant \max \{ |q_v| \} < Q,
\]

\[
\theta_v \sum_{j=1}^{N} q \beta_j^r - \alpha_v \in 2Q^{-1} \cdot \Pi, \quad v \in \mathbb{S}^\infty.
\]

Therefore follows the lemma by setting \( \beta = \sum_{j=1}^{N} q \beta_j \). Q.E.D.

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