The geometry of electric charge:
The charge characteristic class

by

G.H.Gadiyar

Department of Mathematics, Indian Institute of Technology,
Madras 600 036, INDIA.

Abstract. It is well known that magnetic monopoles are related to the first Chern class. In this note electric charge is used to construct an analogous characteristic class: the charge class.
The intimate relationship between geometry and physics is now accepted by all. A prime example of this relationship is the Yang-Mills theory. As is well known the equations of this theory are

\[ F = D_A A, \quad D_A F = 0 \]

and

\[ D_A \ast F = \ast J, \quad D_A \ast J = 0 \]

where \( A \) is the connection, \( F \) is the curvature (for mathematicians) and \( A \) is the potential and \( F \) is the field strength (for physicists). \( J \) is called the source by the physicists. Mathematicians especially differential geometers set \( J \) to be zero thus simplifying the equations.

The question raised and answered in this article is what is the geometric significance of \( J \)? The answer to the question leads to the construction of a new characteristic class: the charge class.

The key idea is to use the second set of Yang-Mills equations and mimic the arguments usually carried out with the first set of equations.

Recall the usual arguments for proving properties of characteristic classes. These are outlined in [1]. One has to take a polynomial in \( F \) defined by

\[ \text{Det}(tI + i\frac{F}{2\pi}) = \sum_j t^j P_{m-j}(F). \]

Then \( C_i(P) = P_i(F) \) are called Chern classes.

\( F \) is a two-form. So \( P_1(F) \) is a \( 2i \) form. Then to show \( P_i \) belongs to \( H^{2i}(M, \mathbb{R}) \), two results have to be established.
(i) $P_1(F)$ is closed
and
(ii) $P_1(F)$ is independent of the connection $A$ used to compute $F$.

The proof that follows is modelled on the usual arguments. The main
difference is, to repeat, instead of using the first set set, the second set of
Yang-Mills equations is used.

The object considered in the case of the monopole is $Tr F$, the first Chern
class. Here by analogy the object considered in the case of electric charge
is $Tr * J$. This is called the charge class in this paper. Notice that $*J$ is a
three-form. So one should get an element of $H^3(M, R)$.

Now conservation of current reads

$$D_A * J = 0 .$$

Using the definition of $D_A * J = d * J + A \wedge * J - * J \wedge A$ and

$Tr A \wedge B = (-1)^{pq} Tr B \wedge A$ if $A$ is a $p$ form and $B$ is a $q$ form, it
follows that

$$d Tr * J = 0 .$$

Hence $Tr * J$ is closed.

The next result to be established is that $Tr * J$ is independent of the
connection $A$ chosen. Again the proof follows standard lines.

Let $Tr * J$ and $Tr * J'$ be the charge classes corresponding to connections
$A$ and $A'$. To prove independence of the charge class it has to be established

3
that

\[ Tr * J - Tr * J' = d\eta \]

for some \( \eta \), then it follows that \( Tr * J \) and \( Tr * J' \) belong to the same cohomology class in \( H^3(M, R) \).

Consider two connections \( A \) and \( A' \) and define a family of connections

\[ A^t = A + t(A' - A) \]

\[ = A + t a \]

where \( a = A' - A \).

Corresponding to \( A^t \) will be \( F^t \) and \( J^t \). Note

\[ Tr * J - Tr * J' = \int_0^1 \frac{d}{dt} Tr (\ast J^t) dt. \]

It is shown that \( \frac{d}{dt} Tr * J^t \) is exact for \( 0 \leq t \leq 1 \), that is,

\[ \frac{d}{dt} Tr * J^t = d\theta(t) , \quad 0 \leq t \leq 1. \]

Thus

\[ Tr * J - Tr * J' = \int_0^1 d\theta(t) dt \]

\[ = d \int_0^1 \theta(t) dt \]

\[ = d\eta \]

(1)

where \( \eta = \int_0^1 \theta(t) dt \).
The proof that $\frac{d}{dt} Tr \ast J^t$ is exact for $0 \leq t \leq 1$ is displayed below. It is sufficient to show $\frac{d}{dt} Tr \ast J^t|_{t=0}$ is exact.

For if $\frac{d}{dt} Tr \ast J(t)$ is exact for $t = 0$, it is also exact for any subsequent value of $t$, called $t'$. For the interval $[0,1]$ can be replaced by $[t,1]$ and replace $A$ and $A'$ by $A^t$ and $A'$.

$$\frac{d}{dt} Tr \ast J^t = Tr \ast \frac{d}{dt} J^t.$$ But $D_{A'} \ast F^t = \ast J^t$ and

$$F^t = dA^t + A^t \wedge A^t$$
$$= d(A + ta) + (A + ta) \wedge (A + ta)$$
$$= F + t(da + A \wedge a + a \wedge A) + t^2 a \wedge a .$$

So

$$\frac{d}{dt} Tr \ast J^t = \frac{d}{dt} Tr D_{A'} \ast F^t$$
$$= \frac{d}{dt} Tr (d \ast F^t + A^t \wedge F^t - F^t \wedge A^t)$$
$$= \frac{d}{dt} Tr d \ast F^t$$
$$= Tr d \ast \frac{dF^t}{dt}$$
$$= Tr d \ast (da + A \wedge a + a \wedge A)$$
$$= dTr \ast (da + A \wedge a + a \wedge A)$$
$$= dTr \ast da .$$
Thus \( \frac{d}{dt} Tr \ast J' = dTr \ast da \). Using (1) it follows that

\[
Tr \ast J - Tr \ast J' = d\eta .
\]

Thus we have established

(i) \( Tr \ast J \) is closed.

(ii) \( Tr \ast J \) is independent of connection \( A \) used to compute \( F \).

The situation is clear in the case of electro magnetism. The equations then collapse to

\[
F = dA, \quad dF = 0
\]

\[
J = d\ast F, \quad d\ast J = 0 .
\]

The idea outlined here is as follows: The second set of equations is the same as the first set, except that \( A \) is replaced by \( \ast F \) and \( F \) is replaced by \( \ast J \). Note however that these are forms of different degree.

A monopole is an appropriate configuration of \( A \) (with a Dirac ‘string’ or singularity) and is related to \( Tr F \) or the first Chern class.

Corresponding to \( Tr \ast J \) or charge class is an appropriate configuration of \( \ast F \) (with some singularity). This corresponds to nontrivial \( H^3(M, R) \). In short we have used the current conservation law instead of the Bianchi identity to prove the result. This gives some idea as how to interpret the current \( J \) in Yang-Mills theory.

Reference.

[1]. C.Nash and S.Sen, *Topology and Geometry for physicists*, Academic Press, 1983.