Asymptotic normality for eigenvalue statistics of a general sample covariance matrix when $p/n \to \infty$
and applications

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Abstract: The asymptotic normality for a large family of eigenvalue statistics of a general sample covariance matrix is derived under the ultra-high dimensional setting, that is, when the dimension to sample size ratio $p/n \to \infty$. Based on this CLT result, we first adapt the covariance matrix test problem to the new ultra-high dimensional context. Then as a second application, we develop a new test for the separable covariance structure of a matrix-valued white noise. Simulation experiments are conducted for the investigation of finite-sample properties of the general asymptotic normality of eigenvalue statistics, as well as the second test for separable covariance structure of matrix-valued white noise.

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1. Introduction

Let $y \in \mathbb{R}^p$ be a population of the form $y = \Sigma^{1/2} \epsilon$ where $\Sigma_p$ is a $p \times p$ positive definite matrix, $\epsilon \in \mathbb{R}^p$ a $p$-dimensional random vector with independent and identically distributed (i.i.d.) components with zero mean and unit variance. Given an i.i.d. sample $\{y_j = \Sigma^{1/2} \epsilon_j, 1 \leq j \leq n\}$ of $y$, the sample covariance matrix is $S_n = \frac{1}{n} \sum_{j=1}^{n} y_j y_j'$, where $X = (x_1, x_2, \ldots, x_n)$. We consider the ultra-high dimensional setting where $n \to \infty$, $p = p(n) \to \infty$ such that $p/n \to \infty$. The $p \times p$ matrix $S_n$ has only a small number of non-zero eigenvalues which are the same as those of its $n \times n$ companion matrix $S_{n,n} = \frac{1}{n} X' \Sigma_p X$. The limiting distribution of these non-zero eigenvalues is known (see Bai and Yin (1988), Wang and Paul (2014)). Precisely, consider the re-normalized sample covariance matrix

$$A_n = \frac{1}{\sqrt{npb_p}} \left( X' \Sigma_p X - pa_p I_n \right),$$

where $I_n$ is the identity matrix of order $n$, $a_p = \frac{1}{p} \text{tr}(\Sigma_p)$, $b_p = \frac{1}{p^2} \text{tr}(\Sigma_p^2)$. Denote the eigenvalues of $A_n$ as $\lambda_1, \ldots, \lambda_n$. According to Wang and Paul (2014), under the condition that $\sup_p \|\Sigma_p\| < \infty$, the eigenvalue distribution of $A_n$, i.e. $F^{A_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}$ converges to the celebrated semi-circle law. In this paper, we focus on the so-called linear spectral statistics (LSS) of $A_n$, i.e. $\frac{1}{n} \sum_{i=1}^{n} f(\lambda_i)$ where $f(\cdot)$ is any smooth function we are interested in. The main contribution of this paper is to establish the central limit theorem (CLT) for LSS of $A_n$ under the ultra-high dimensional setting. The study of fluctuations of LSS for different
types of random matrix models has received extensive attention in the past decades, see monographs Bai and Silverstein (2010a); Couillet and Debbah (2011); Yao et al. (2015). It plays a very important role in high dimensional data analysis because many well-established statistics can be represented as LSS of sample covariance or correlation matrix. In facing the curse of dimensionality, most asymptotic results are discussed under the Marchenko-Pastur asymptotic regime, where \( p/n \rightarrow c \in (0, \infty) \). However, this doesn’t fit the case of ultra-high dimension when \( p \gg n \). Hence in this paper we re-examined the asymptotic behavior of LSS of \( A_n \) when \( n \rightarrow \infty, p = p(n) \rightarrow \infty \) such that \( p/n \rightarrow \infty \).

A special version of \( A_n \) for the case where \( \Sigma_p = I_p \) has already been studied in the literature. The matrix becomes

\[
A_n^{\text{iden}} = \frac{1}{\sqrt{np}}(X'X - pI_n).
\]  

Bai and Yin (1988) is the first to study this matrix. They proved that the ultra-high dimensional limiting eigenvalue distribution of \( A_n^{\text{iden}} \) is the semi-circle law. Chen and Pan (2012) studied the behavior of the largest eigenvalue of \( A_n^{\text{iden}} \). Chen and Pan (2015) and Bao (2015) independently established the CLT for LSS of \( A_n^{\text{iden}} \), the limiting variance function of which coincides with that of a Wigner matrix given in Bai and Yao (2005). From these results, we know that some spectral properties of \( A_n^{\text{iden}} \) are similar to those of a \( n \times n \) Wigner matrix. Indeed, the general matrix \( A_n \) also has some spectral properties similar to those of a Wigner matrix. In particular, the eigenvalue distribution of \( A_n, F^{A_n} \), also converges to the semi-circle law. However, the second order fluctuations for LSS of \( A_n \) are quite different and worth further investigation.

In this paper, we establish the CLT for LSS of \( A_n \). The general strategy of the proof follows that of Bai and Yao (2005) for the CLT for LSS of a large Wigner matrix. However, the calculations are more involved here as the matrix \( A_n \) is a quadratic function of the independent entries \( (X_{ij}) \) while a Wigner matrix is a linear function of its entries. Similar to Chen and Pan (2015), a key step is to establish the CLT for some smooth integral of the Stieltjes transform \( M_n(z) \) of \( A_n \), see Proposition 6.1. To derive the limiting mean and covariance functions, we divide \( M_n(z) \) into two parts: a non-random part and a random part. Our approaches to handle these two parts are technically different. For the random part, we follow a method in Chen and Pan (2015) which depends heavily on an explicit expression for \( \text{tr}(M_k^{(1)})/(npb_p) \) (see Section 6.3 for more details). This explicit expression does not exist in our matrix model, so we need to provide a first-order approximation for it, which is given in Lemma 6.1. For the non-random part, we utilize the generalized Stein’s equation to find the asymptotic expansion of the expectation of Stieltjes transform, which provides some new enlightenment for conventional procedures.

To demonstrate the potential of our newly established CLT, we further studied two hypothesis testing problems about population covariance matrices. First, we examine the identity hypothesis \( H_0 : \Sigma_p = I_p \) under the ultra-high dimensional setting and compare it with cases of relatively low dimensions. Next, we consider the hypothesis that a matrix-valued noise has a separable covariance matrix. For a sequence of i.i.d. \( p_1 \times p_2 \) matrices \( \{E_t\}_{1 \leq t \leq T} \), we adopt a Frobenius-norm-type statistic to test whether the covariance matrix of \( \text{vec}(E_t) \) is separable, i.e. \( \text{Cov}(\text{vec}(E_t)) = \Sigma_1 \otimes \Sigma_2 \), where \( \Sigma_1 \) and \( \Sigma_2 \) are two given \( p_1 \times p_1 \) and \( p_2 \times p_2 \) nonnegative definite matrices. Here \( p_1, p_2 \) and \( T \) are of comparable magnitude. Our test statistic can be represented as a LSS of the sample covariance matrix with dimension \( p_1p_2 \) much larger than
the sample size $T$. Therefore our CLT can then be employed to derive the null distribution and perform power analysis of the test. Good numerical performance lends full support to the correctness of our CLT results.

The paper is organized as follows. Section 2 provides preliminary knowledge of some technical tools. Section 3 establishes our main CLT for LSS of $A_n$. Section 4 contains two hypothesis testing applications. Section 5 reports numerical studies. Technical proofs and lemmas are relegated to Section 6 and Appendices.

Throughout the paper, we reserve boldfaced symbols for vectors and matrices. For any matrix $A$, we let $A_{ij}$, $\lambda^A_j$, $A'$, $\text{tr}(A)$ and $\|A\|$ represent, respectively, its $(i, j)$-th element, its $j$-th largest eigenvalue, its transpose, its trace, and its spectral norm (i.e., the largest singular value of $A$). $\mathbb{1}_{\{\cdot\}}$ stands for the indicator function. For the random variable $X_{11}$, we denote the $a$-th moment of $X_{11}$ by $\nu_a$ and the $a$-th cumulant of $X_{11}$ by $\kappa_a$. We use $K$ to denote constants which may vary from line to line. For simplicity, we sometimes omit the variable $z$ when representing some matrices and functions (e.g. Stieltjes transforms) of $z$, provided that it does not lead to confusion.

2. Preliminaries

In this section, we introduce some useful preliminary results. For any $n \times n$ Hermitian matrix $B_n$, its empirical spectral distribution (ESD) is defined by

$$F_{B_n}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{\lambda^B_i \leq x\}}.$$  

If $F_{B_n}(x)$ converges to a non-random limit $F(x)$ as $n \to \infty$, we call $F(x)$ the limiting spectral distribution (LSD) of $B_n$.

As for the LSD of $A_n$ defined in (1.1), Wang and Paul (2014) derived the LSD of renormalized sample covariance matrices with more generalized form

$$C_n = \sqrt{\frac{p}{n}} \left( \frac{1}{p} T_n^{1/2} X_n^{*} \Sigma_p X_n T_n^{1/2} - \frac{1}{p} \text{tr}(\Sigma_p) T_n \right),$$  \hspace{1cm} (2.1)

where $X_n$ and $\Sigma_p$ are the same as those in (1.1). $T_n$ is a $n \times n$ nonnegative definite Hermitian matrix, whose ESD, $F_{T_n}$, converges weakly to $H$, a nonrandom distribution function on $\mathbb{R}^+$ which does not degenerate to zero. The LSD of $C_n$ is described in terms of its Stieltjes transform. The Stieltjes transform of any cumulative distribution function $G$ is defined by

$$m_G(z) = \int \frac{1}{\lambda - z} \, dG(\lambda), \quad z \in \mathbb{C}^+ := \{u + iv, u \in \mathbb{R}, v > 0\}.$$  

Wang and Paul (2014) proved that, when $p \land n \to \infty$ and $p/n \to \infty$, $F_{C_n}$ almost surely converges to a nonrandom distribution, whose Stieltjes transform $m_C(z)$ satisfies the following system of equations:

$$\begin{cases} m_C(z) = -\int \frac{dH(z)}{\pi + xg(z)}, \\ g(z) = -\int \frac{x \, dH(z)}{\pi + xg(z)}, \end{cases}$$  \hspace{1cm} (2.2)

for any $z \in \mathbb{C}^+$, where $\theta = \lim_{p \to \infty} (1/p) \text{tr}(\Sigma_p^2)$.  


Note that $A_n$ is a special case of $C_n$ with $T_n = I_n$. By (2.2) we can easily show that the Stieltjes transform $m_A(z)$ of LSD of $A_n$ satisfies

$$m_A(z) = -\frac{1}{z + m_A(z)},$$

which is exactly the Stieltjes transform of the semi-circle law with density function given by

$$F'(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbb{1}_{\{|x| \leq 2\}}.$$ (2.4)

Hereafter we use $m(z)$ to represent $m_A(z)$ for ease of presentation.

3. Main Results

Let $U$ denote any open region on the complex plane including $[-2, 2]$ and $M$ be the set of analytic functions defined on $U$. For any $f \in M$, we consider a LSS of $A_n$ of the form:

$$\int f(x) dF_{A_n}(x) = \frac{1}{n} \sum_{i=1}^{n} f(\lambda_{A_n}^i).$$

Since $F_{A_n}$ converges to $F$ almost surely, we have

$$\int f(x) dF_{A_n}(x) \to \int f(x) dF(x).$$

A question naturally arises: how fast does $\int f(x) d\{F_{A_n}(x) - F(x)\}$ converge to zero?

To answer this question, we consider a re-normalized functional:

$$G_n(f) = n \int_{-\infty}^{+\infty} f(x) d\{F_{A_n}(x) - F(x)\} - \frac{n}{2\pi i} \int_{|m| = \rho} f(-m - m^{-1}) \chi_n(m) \frac{1 - m^2}{m^2} dm,$$ (3.1)

where

$$\chi_n(m) = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \quad A = m - \sqrt{\frac{n}{p} \frac{c_p}{b_p}} (1 + m^2),$$

$$C = \frac{m^3}{n} \left\{ \frac{1}{1 - m^2} + \frac{(\nu_4 - 3)b_p}{b_p} \right\} - \sqrt{\frac{n}{p} \frac{c_p}{b_p}} m^4 + \frac{n}{p} \left( -\frac{c_p^2}{b_p^2} + \frac{d_p}{b_p^2} \right) m^5,$$

$$B = m^2 - 1 - \sqrt{\frac{n}{p} \frac{c_p}{b_p}} m (1 + 2m^2), \quad a_p = \frac{1}{p} \text{tr}(\Sigma_p),$$

$$b_p = \frac{1}{p} \text{tr}(\Sigma_p^2), \quad \tilde{b}_p = \frac{1}{p} \sum_{i=1}^{p} \sigma_{ii}^2, \quad c_p = \frac{1}{p} \text{tr}(\Sigma_p^3), \quad d_p = \frac{1}{p} \text{tr}(\Sigma_p^4),$$

here $\rho < 1$ and $\sqrt{B^2 - 4AC}$ is a complex number whose imaginary part has the same sign as that of $B$. In what follows, we have established the asymptotic normality of $G_n(f)$ and the main result is formulated in the theorem below.

**Theorem 3.1.** Suppose that
(A) \( \mathbf{X} = (X_{ij})_{p \times n} \) where \( \{X_{ij}, \ 1 \leq i \leq p, \ 1 \leq j \leq n\} \) are i.i.d. real random variables with 
\( \mathbb{E}X_{ij} = 0, \mathbb{E}X_{ij}^2 = 1, \mathbb{E}X_{ij}^3 = \nu_4 \) and \( \mathbb{E}|X_{ij}|^{\gamma + \varepsilon_0} < \infty \) for some small positive \( \varepsilon_0; \)

\[ 3.1 \]

(B) \( \{\Sigma_p, \ p \geq 1\} \) is a sequence of non-negative definite matrices, bounded in spectral norm, such that the following limits exist:

- \( \gamma = \lim_{p \to \infty} \frac{1}{p} \text{tr}(\Sigma_p), \)
- \( \theta = \lim_{p \to \infty} \frac{1}{p} \text{tr}(\Sigma_p^2), \)
- \( \omega = \lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^p \sigma_{ii}^2; \)

\[ C1 \]

Then, for any \( f_1, \cdots, f_k \in \mathcal{M}, \) the finite dimensional random vector \( (G_n(f_1), \cdots, G_n(f_k)) \) converges weakly to a Gaussian vector \( (Y(f_1), \cdots, Y(f_k)) \) with mean function \( \mathbb{E}Y(f) = 0 \) and covariance function

\[
\text{Cov}(Y(f_1), Y(f_2)) = \frac{\omega}{\theta}(\nu_4 - 3)\Psi_1(f_1)\Psi_1(f_2) + 2\sum_{k=1}^{\infty} k\Psi_k(f_1)\Psi_k(f_2) \tag{3.3}
\]

\[
= \frac{1}{4\pi^2} \int_{-2}^{2} \int_{-2}^{2} f_1'(x)f_2'(y) H(x,y) \, dx \, dy, \tag{3.4}
\]

where

\[
\Psi_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(2\cos \theta) e^{ik\theta} \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(2\cos \theta) \cos k\theta \, d\theta, \tag{3.5}
\]

\[
H(x,y) = \frac{\omega}{\theta}(\nu_4 - 3)\sqrt{4 - x^2}\sqrt{4 - y^2} + 2\log \left( \frac{4 - xy + \sqrt{(4 - x^2)(4 - y^2)}}{4 - xy - \sqrt{(4 - x^2)(4 - y^2)}} \right). \tag{3.6}
\]

The proofs of Theorem 3.1 is postponed to Section 6.

Remark 3.1. Note that we require \( p \geq Kn^2 \) asymptotically in Assumption \( (C1) \), while the situation of \( n \ll p \ll n^2 \) remains unknown.

Remark 3.2. If \( \Sigma_p = I_p \), we have \( a_p = b_p = \tilde{b}_p = c_p = d_p = 1 \) and \( \gamma = \theta = \omega = 1 \). Our Theorem 3.1 reduces to the CLT derived in Chen and Pan (2015).

Applying Theorem 3.1 to three polynomial functions, we obtain the following corollary.

Corollary 3.1. With the same notations and assumptions given in Theorem 3.1, consider three analytic functions \( f_1(x) = x, f_2(x) = x^2, f_3(x) = x^3 \), we have

\[
G_n(f_1) = \text{tr}(A_n) \xrightarrow{d} \mathcal{N}\left(0, \frac{\omega}{\theta}(\nu_4 - 3) + 2\right); \]

\[
G_n(f_2) = \text{tr}(A_n^2) - n - \left\{ \frac{\tilde{b}_p}{b_p}(\nu_4 - 3) + 1 \right\} \xrightarrow{d} \mathcal{N}(0, 4); \]

\[
G_n(f_3) = \text{tr}(A_n^3) - \frac{c_p}{b_p \sqrt{b_p}} \sqrt{n} \{ n + 1 + \frac{\tilde{b}_p}{b_p}(\nu_4 - 3) \} \xrightarrow{d} \mathcal{N}\left(0, \frac{9\omega}{\theta}(\nu_4 - 3) + 24\right). \]

The calculations in these applications are elementary, thus omitted. Note that the mean correction terms for \( G_n(f_1), G_n(f_2), \) and \( G_n(f_3) \) are 0, \( \frac{\tilde{b}_p}{b_p}(\nu_4 - 3) + 1 \), and \( \frac{c_p}{b_p \sqrt{b_p}} \sqrt{n} \{ n + 1 + \frac{\tilde{b}_p}{b_p}(\nu_4 - 3) \} \), respectively.
3.1. Case of \( p \geq Kn^3 \)

When \( p \geq Kn^3 \), the mean correction term in (3.1) can be further simplified, i.e.

\[
- \frac{n}{2\pi i} \int_{|m|=\rho} f(-m - m^{-1}) \varphi_n(m) \frac{1 - m^2}{m^2} \, dm
\]

\[
= - \left[ \frac{1}{4} (f(2) + f(-2)) - \frac{1}{2} \Psi_0(f) + \frac{b_p}{b_p} (\nu_4 - 3) \Psi_2(f) \right] - \sqrt{\frac{n^3}{p} \frac{c_p}{b_p \sqrt{b_p}}} \Psi_3(f) + o(1). \tag{3.7}
\]

For any function \( f \in \mathcal{M} \), we define a new normalization of the LSS:

\[
Q_n(f) = n \int_{-\infty}^{+\infty} f(x) \, d \left\{ F^{A_n}(x) - F(x) \right\} - \sqrt{\frac{n^3}{p} \frac{c_p}{b_p \sqrt{b_p}}} \Psi_3(f). \tag{3.8}
\]

Note that the last term in (3.8) makes no contribution if the function \( f \) is even \((\Psi_3(f) = 0)\) or \( n^3/p = o(1) \). Substituting (3.7) into Theorem (3.1), we obtain the following CLT for \( Q_n(f) \).

**Corollary 3.2.** Under assumptions (A), (B) in Theorem (3.1) and (C2) \( p \land n \to \infty \) and \( n^3/p = O(1) \).

For any \( f_1, \ldots, f_k \in \mathcal{M} \), the finite dimensional random vector \((Q_n(f_1), \ldots, Q_n(f_k))\) converges weakly to a Gaussian vector \((Y(f_1), \ldots, Y(f_k))\) with mean function

\[
\mathbb{E} Y(f_k) = \frac{1}{4} (f_k(2) + f_k(-2)) - \frac{1}{2} \Psi_0(f_k) + \frac{\omega}{\theta} (\nu_4 - 3) \Psi_2(f_k), \tag{3.9}
\]

and covariance function given in (3.3).

**Remark 3.3.** As a special case of Theorem 3.1, Corollary 3.2 is used in Li and Yao (2016) to derive the asymptotic power of two sphericity tests, John’s invariant test and Quasi-likelihood ratio test (QLRT), when the dimension \( p \) is much larger than sample size \( n \). Specifically, let \( \mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \) be a \( p \times n \) data matrix with \( n \) i.i.d. \( p \)-dimensional random vectors \( \{\mathbf{x}_i\}_{1 \leq i \leq n} \) with covariance matrix \( \Sigma = \text{Var}(\mathbf{x}_i) \). The goal is to test

\[
H_0 : \Sigma = \sigma^2 \mathbf{I}_p, \quad \text{vs.} \quad H_1 : \Sigma \neq \sigma^2 \mathbf{I}_p,
\]

where \( \sigma^2 \) is an unknown positive constant. John’s test statistic is defined by

\[
U = \frac{1}{p} \text{tr} \left[ \left( \frac{\mathbf{S}_n}{\text{tr}(\mathbf{S}_n)/p} - \mathbf{I}_p \right)^2 \right] = \frac{1}{p} \sum_{i=1}^{p} (l_i - \bar{l})^2,
\]

where \( \{l_i\}_{1 \leq i \leq p} \) are eigenvalues of \( p \)-dimensional sample covariance matrix \( \mathbf{S}_n = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i' = \frac{1}{n} \mathbf{X} \mathbf{X}' \) and \( \bar{l} = \frac{1}{p} \sum_{i=1}^{p} l_i \). The QLRT statistic is defined by

\[
\mathcal{L}_n = \frac{p}{n} \log \left( \frac{n^{-1} \sum_{i=1}^{n} l_i}{\Pi_{i=1}^{n} l_i} \right),
\]

where \( \{\bar{l}_i\}_{1 \leq i \leq n} \) are the eigenvalues of the \( n \times n \) matrix \( \frac{1}{p} \mathbf{X}' \mathbf{X} \). The main idea is that both \( U \) and \( \mathcal{L}_n \) can be expressed as functions of eigenvalues of \( \mathbf{A}_n \) in (1.1). Thus, asymptotic distributions of John’s statistic and QLRT statistic can be derived either using Theorem 3.1 or Corollary 3.2. Li and Yao (2016) used Corollary 3.2 to derive the limiting distributions of \( U \) and \( \mathcal{L}_n \) under the alternative hypothesis. Their power functions are proven to converge to \( 1 \) under the assumption \( n^3/p = O(1) \). More details can be found in Li and Yao (2016).
4. Applications to Hypothesis Testing about Large Covariance Matrices

4.1. The Identity Hypothesis “$\Sigma_p = I_p$”

Let $\mathbf{Y} = (y_1, \ldots, y_n)$ be a $p \times n$ data matrix with $n$ i.i.d. $p$-dimensional random vectors $\{y_i = \Sigma_p^{1/2} x_i\}_{1 \leq i \leq n}$ with covariance matrix $\Sigma_p = \text{Var}(y_i)$ and $x_i$ has $p$ i.i.d. components $\{X_{ij}, 1 \leq j \leq p\}$ satisfying $\text{E}X_{ij} = 0$, $\text{E}X^2_{ij} = 1$, $\text{E}X^4_{ij} = \nu_4$. We explore the identity testing problem

$$H_0 : \Sigma_p = I_p, \quad \text{vs.} \quad H_1 : \Sigma_p \neq I_p,$$

under two different asymptotic regimes: high-dimensional regime, “$p \wedge n \to \infty$, $p/n \to c \in (0, \infty)$” and ultra-high dimensional regime, “$p \wedge n \to \infty$, $p/n \to \infty$”. We will consider two well-known test statistics and discuss their limiting distributions under both regimes.

For the identity testing problem (4.1), Nagao (1973) proposed a statistic based on the Frobenius norm:

$$V = \frac{1}{p} \text{tr}[(S_n - I)^2],$$

where $S_n = \frac{1}{n} \mathbf{X} \mathbf{X}'$ is the sample covariance matrix. Nagao’s test based on $V$ performs well when $n$ tends to infinity while $p$ remains fixed. However, Ledoit and Wolf (2002) showed that Nagao’s test has poor properties when $p$ is large. They made some modifications as

$$W = \frac{1}{p} \text{tr}[(S_n - I_p)^2] - \frac{1}{n} \left( \frac{1}{p} \text{tr}(S_n) \right)^2 + \frac{p}{n}. \quad (4.2)$$

When $p \wedge n \to \infty, p/n = c_n \to c \in (0, \infty)$, under normality assumption, Ledoit and Wolf (2002) proved that the limiting distribution of $W$ under $H_0$ is

$$nW - p - 1 \xrightarrow{d} \mathcal{N}(0, 4).$$

Wang and Yao (2013) further removed the normality assumption and show that under $H_0$, when $p \wedge n \to \infty, p/n = c_n \to c \in (0, \infty)$,

$$nW - p - (\nu_4 - 2) \xrightarrow{d} \mathcal{N}(0, 4). \quad (4.3)$$

Now we derive the limiting distribution of $W$ under both $H_0$ and $H_1$ when $p/n \to \infty$. We will show that the test based on $W$ is consistent under the ultra-high dimensional setting. The main results of the test based on $W$ is as follows.

**Theorem 4.1.** Assume that $\mathbf{Y} = (y_1, \ldots, y_n)$ is a $p \times n$ data matrix with $n$ i.i.d. $p$-dimensional random vectors $\{y_i = \Sigma_p^{1/2} x_i\}_{1 \leq i \leq n}$ with covariance matrix $\Sigma_p = \text{Var}(y_i)$ and $x_i$ has $p$ i.i.d. components $\{X_{ij}, 1 \leq j \leq p\}$ satisfying $\text{E}X_{ij} = 0$, $\text{E}X^2_{ij} = 1$, $\text{E}X^4_{ij} = \nu_4$ and $\text{E}|X_{ij}|^{6+\varepsilon_0} < \infty$ for some small positive $\varepsilon_0$. $W$ is defined as (4.2). Then under $H_0$, when $p \wedge n \to \infty$ and $n^2/p = O(1)$,

$$nW - p - (\nu_4 - 2) \xrightarrow{d} \mathcal{N}(0, 4). \quad (4.4)$$

Note that the asymptotic distribution (4.4) coincides with (4.3), which means $W$ has the same limiting null distribution in both high dimensional and ultra-high dimensional setting.
Therefore $W$ can be used to test (4.1) under the ultra-high dimensional setting. For nominal level $\alpha$, the corresponding rejection rule is

$$
\frac{1}{2}\left\{nW - p - (\nu_4 - 2)\right\} \geq z_\alpha, \quad (4.5)
$$

where $z_\alpha$ is the $\alpha$ upper quantile of standard normal distribution.

As for the case of $H_1$ when $\Sigma_p \neq I_p$, we have

**Theorem 4.2.** Under the same assumptions as in Theorem 4.1, further assume that $\{\Sigma_p, p \geq 1\}$ is a sequence of non-negative definite matrices, bounded in spectral norm such that the following limits exist:

$$
\gamma = \lim_{p \to \infty} \frac{1}{p} \text{tr}(\Sigma_p), \quad \theta = \lim_{p \to \infty} \frac{1}{p} \text{tr}(\Sigma_p^2), \quad \omega = \lim_{p \to \infty} \frac{1}{p} \sum_{i=1}^{p} (\Sigma_p)_{ii}^2,
$$

then when $p \land n \to \infty$ and $n^2/p = O(1)$,

$$
nW - p - \theta \left[\frac{\omega}{\theta} (\nu_4 - 3) + 1\right] + n (2\gamma - 1 - \theta) \xrightarrow{d} \mathcal{N}(0, 4\theta^2).
$$

Note that Theorem 4.2 reveals the limiting null distribution of $W$. Let $\Sigma_p = I_p$, then $\gamma = \theta = \omega = 1$, Theorem 4.2 reduces to Theorem 4.1, which states the limiting null distribution of $W$. With Theorem 4.1 and 4.2, asymptotic power of $W$ can be derived.

**Proposition 4.1.** With the same assumptions as in Theorem 4.2, when $p \land n \to \infty$ and $n^2/p = O(1)$, the testing power of $W$ for (4.1)

$$
\beta(H_1) \to 1 - \Phi\left(\frac{1}{2}\theta \left(2z_\alpha - \omega (\nu_4 - 3) - \theta + n (2\gamma - 1 - \theta) + (\nu_4 - 2)\right)\right).
$$

If $\gamma = \theta = 1$, then $\beta(H_1) \to 1 - \Phi(z_\alpha - \frac{\omega - 1}{2} (\nu_4 - 3))$; otherwise, $\beta(H_1) \to 1$.

The second test statistic of (4.1) we consider is the likelihood ratio test (LRT) statistic studied in Bai et al. (2009). Bai et al. (2009) assumed that $\nu_4 = 3$. The LRT statistic is defined as

$$
\mathcal{L}_0 = \text{tr}(S_n) - \log |S_n| - p. \quad (4.6)
$$

Bai et al. (2009) derived the limiting null distribution of $\mathcal{L}_0$ when $p \land n \to \infty$, $p/n \to c \in (0, 1)$. However, this LRT statistic is degenerate and not applicable when $p > n$ because $|S_n| = 0$. Thus for $p > n$ we introduce a quasi-LRT test statistic

$$
\mathcal{L} = \text{tr}(\hat{S}_n) - \log |\hat{S}_n| - n,
$$

where $\hat{S}_n = \frac{1}{p}\mathbf{Y}'\mathbf{Y}$. When $p \land n \to \infty$, $p/n = c_n \to c \in (1, \infty)$, the limiting null distribution of $\mathcal{L}$ is

$$
\mathcal{L}^* := \frac{\mathcal{L} - n F_1(c_n) - \mu_1}{\sigma_1} \xrightarrow{d} \mathcal{N}(0, 1), \quad (4.7)
$$

where

$$
F_1(c_n) = 1 - (1 - c_n) \log\left(1 - \frac{1}{c_n}\right), \quad \mu_1 = -\frac{1}{2} \log\left(1 - \frac{1}{c_n}\right), \quad \sigma_1^2 = -2 \log\left(1 - \frac{1}{c_n}\right) - \frac{2}{c_n}.
$$
Now we will show that this asymptotic distribution (4.7) still holds in the ultra-high dimensional setting. Note that
\[
\sigma_1 = \sqrt{-2 \log \left(1 - \frac{1}{c_n} \right) - \frac{2}{c_n}} = \sqrt{\frac{1}{c_n^2} + \frac{2}{3 c_n^3} + o \left(\frac{1}{c_n^3}\right)} = \frac{1}{c_n} + \frac{1}{3 c_n^2} + o \left(\frac{1}{c_n^2}\right),
\]
which implies that
\[
\frac{1}{\sigma_1} = c_n - \frac{1}{3} + o(1). \tag{4.8}
\]

Firstly, we consider the random part of \( L \). Let \( \hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_n \) be the eigenvalues of \( \hat{S}_n \) and \( \tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_n \) be the eigenvalues of \( \tilde{S}_n = \sqrt{\frac{n}{p}} (n\hat{X}'\hat{X} - \frac{p}{n} I_n) \). By using the basic identity
\[
\tilde{\lambda}_i = \frac{\lambda_i}{\sqrt{c_n}} + 1,
\]
we have
\[
\mathcal{L} = \sum_{i=1}^n \hat{\lambda}_i - n - \sum_{i=1}^n \log(\hat{\lambda}_i) = \sum_{i=1}^n \tilde{\lambda}_i - \sum_{i=1}^n \log \left(1 + \frac{\tilde{\lambda}_i}{\sqrt{c_n}}\right)
\]
\[
= \sum_{i=1}^n \frac{\tilde{\lambda}_i}{\sqrt{c_n}} - \sum_{i=1}^n \left(\frac{\tilde{\lambda}_i}{\sqrt{c_n}} - \frac{1}{2} \frac{\tilde{\lambda}_i^2}{c_n} + \frac{1}{3} \frac{\tilde{\lambda}_i^3}{c_n \sqrt{c_n}} - \frac{1}{4} \frac{\tilde{\lambda}_i^4}{c_n^2} + o \left(\frac{1}{c_n^2}\right)\right)
\]
\[
= \frac{1}{2 c_n} \text{tr}(\tilde{S}_n^2) - \frac{1}{3 c_n \sqrt{c_n}} \text{tr}(\tilde{S}_n^3) + \frac{1}{4 c_n^2} \text{tr}(\tilde{S}_n^4) + o \left(\frac{n}{c_n^2}\right). \quad (4.9)
\]

Taking \( \nu_4 = 3 \) (the assumption in Bai et al. (2009)) and \( \Sigma_p = I_p \) in Corollary 3.1, we have, under \( H_0 \),
\[
\text{tr}(\tilde{S}_n^2) - n - 1 \xrightarrow{d} \mathcal{N}(0, 4), \quad \text{tr}(\tilde{S}_n^3) - \frac{n + 1}{\sqrt{c_n}} \xrightarrow{d} \mathcal{N}(0, 24). \tag{4.10}
\]
\[
\text{tr}(\tilde{S}_n^4) - 2n - \left(\frac{n}{c_n} + \frac{1}{c_n} + 5\right) \xrightarrow{d} \mathcal{N}(0, 72). \tag{4.11}
\]
Combining (4.8) \sim (4.11) gives us that
\[
\frac{\mathcal{L}}{\sigma_1} = \frac{1}{2} \text{tr}(\tilde{S}_n^2) + o \left(\frac{n^2}{p}\right). \tag{4.12}
\]

Secondly, we consider the deterministic part of \( \mathcal{L}^* \). Note that
\[
n F_1(c_n) + \mu = n - \left[ n(1 - c_n) + \frac{1}{2} \right] \log \left(1 - \frac{1}{c_n}\right)
\]
\[
= n - \left[ n(1 - c_n) + \frac{1}{2} \right] \left[ - \frac{1}{c_n} - \frac{1}{2 c_n^2} - \frac{1}{3 c_n^3} + o \left(\frac{1}{c_n^3}\right)\right]
\]
\[
= \frac{n}{2 c_n} + \frac{1}{2 c_n} + \frac{n}{6 c_n^2} + o \left(\frac{n}{c_n^2}\right),
\]

together with (4.8) which implies that
\[
\frac{n F_1(c_n) + \mu_1}{\sigma_1} = \frac{n + 1}{2} + o \left(\frac{n^2}{p}\right). \tag{4.13}
\]
Therefore, from (4.10), (4.12) and (4.13), we conclude that, under $H_0$, as $p \wedge n \to \infty$, $n^2/p = O(1)$,

$$L^* = \frac{L}{\sigma_1} - \frac{nF_1(c_n) + \mu_1}{\sigma_1} = \frac{1}{2} \left( \text{tr}(S_n^2) - n - 1 \right) + o(1) \to \mathcal{N}(0, 1),$$

which is the same as the limiting distribution (4.7) when $p \wedge n \to \infty$, $p/n = c_n \to c \in (1, \infty)$. Finally, we summarize the discussion above in the following proposition.

**Proposition 4.2.** (1) (Bai et al. (2009)) Assume that $Y = (y_1, \ldots, y_n)$ is a $p \times n$ data matrix with $n$ i.i.d. $p$-dimensional random vectors $\{y_i = \Sigma_p^{1/2} x_i\}_{1 \leq i \leq n}$ with covariance matrix $\Sigma_p = \text{Var}(y_i)$ and $x_i$ has $p$ i.i.d. components $\{X_{ij}, 1 \leq j \leq p\}$ satisfying $EX_{ij} = 0$, $EX_{ij}^2 = 1$, $EX_{ij}^4 = \nu_4 = 3$. $L_0$ is defined as (4.6). Then under $H_0$, when $p \wedge n \to \infty$, $p/n \to c \in (0, 1)$, we have

$$\frac{L_0 - nF_0(c_n) - \mu_0}{\sigma_0} \to \mathcal{N}(0, 1),$$

where $c_n = p/n$ and

$$F_0(c_n) = 1 - \frac{c_n - 1}{c_n} \log(1 - c_n), \quad \mu_0 = -\frac{\log(1 - c_n)}{2}, \quad \sigma_0^2 = -2\log(1 - c_n) - 2c_n.$$

(2) Under the same assumptions as in (1) and the normalized quasi LRT statistic $L^*$ is defined in (4.7). Then under $H_0$, when $p \wedge n \to \infty$, $p/n \to c \in (1, \infty)$, we have

$$L^* \to \mathcal{N}(0, 1).$$

(3) Under the same assumptions as in (1) and the normalized quasi LRT statistic $L^*$ is defined in (4.7). Then under $H_0$, when $p \wedge n \to \infty$ and $n^2/p = O(1)$, we have

$$L^* \to \mathcal{N}(0, 1).$$

Note that the results (2) and (3) in this proposition are newly derived.

### 4.2. Separable Covariance Structure for Matrix-valued Noise

In this section, we develop a test for the structure of the covariance matrix of a matrix-valued white noise. Chen et al. (2021) proposed a matrix autoregressive model with the form

$$X_t = AX_{t-1}B^' + E_t, \quad t = 1, \cdots, T,$$

where $X_t$ is a $p_1 \times p_2$ random matrix observed at time $t$, $A$ and $B$ are $p_1 \times p_1$ and $p_2 \times p_2$ deterministic autoregressive coefficient matrices, $E_t = (e_{t,ij})$ is a $p_1 \times p_2$ matrix-valued white noise. It’s assumed that the error white noise matrix $E_t$ has a specific covariance structure

$$\text{Cov} (\text{vec}(E_t)) = \Sigma_1 \otimes \Sigma_2,$$

where $\text{vec}(\cdot)$ denotes the vectorization, $\Sigma_1$ and $\Sigma_2$ are $p_1 \times p_1$ and $p_2 \times p_2$ non-negative definite matrices. In other words, the noise $E_t$ has a separable covariance matrix.
Now for any observed matrix-valued time sequence, we aim to test whether it has a separable covariance matrix. Specifically, suppose that \( \{E_t\}_{1 \leq t \leq T} \) is an observed i.i.d. sequence of \( p_1 \times p_2 \) matrices and \( p_1, p_2, T \) are of comparable magnitude, we aim to test
\[
H_0 : \text{Cov} (\text{vec}(E_t)) = \Sigma_1 \otimes \Sigma_2, \quad \text{vs}., \quad H_1 : \text{Cov} (\text{vec}(E_t)) \neq \Sigma_1 \otimes \Sigma_2, \tag{4.14}
\]
where \( \Sigma_1 \) and \( \Sigma_2 \) are two prespecified \( p_1 \times p_1 \) and \( p_2 \times p_2 \) non-negative definite matrices. Testing \( H_0 : \text{Cov} (\text{vec}(E_t)) = \Sigma_1 \otimes \Sigma_2 \) is equivalent to testing
\[
H'_0 : \text{Cov} \left( (\Sigma_1 \otimes \Sigma_2)^{-1/2} \text{vec}(E_t) \right) = I_{p_1p_2}. \tag{4.15}
\]
To this end, we define a test statistic
\[
W^* = \frac{1}{p_1p_2} \text{tr} \left[ \left( B_T - I_{p_1p_2} \right)^2 \right] - \frac{p_1p_2}{T} \left[ \frac{1}{p_1p_2} \text{tr} (B_T) \right]^2 + \frac{p_1p_2}{T} \tag{4.16}
\]
where
\[
B_T = \frac{1}{T} Y_T Y_T', \quad Y_T = (\Sigma_1 \otimes \Sigma_2)^{-1/2} \left( \text{vec}(E_1), \ldots, \text{vec}(E_T) \right) : = (Y_{ij})_{p_1p_2 \times T}. \tag{4.17}
\]
Note that \( W^* \) measures the distance between sample covariance matrix of \( \text{vec}(E_t) \) and \( \Sigma_1 \otimes \Sigma_2 \). Naturally we reject \( H_0 \) when \( W^* \) is too large and the critical value is determined by the limiting null distribution of \( W^* \).

Since \( p_1, p_2, T \) are about the same order, we examine the asymptotic behavior of \( W^* \) under the high dimensional regime
\[
T \to \infty, \quad \frac{p_1}{T} = \frac{p_1(T)}{T} \to d_1 \in (0, \infty), \quad \frac{p_2}{T} = \frac{p_2(T)}{T} \to d_2 \in (0, \infty). \tag{4.17}
\]
The asymptotic null distribution of the test statistic \( W^* \) is given in the following Theorem. It is a direct implementation of Theorem 4.1.

**Theorem 4.3.** Assume that

1. \( \{E_t = (e_{t,ij})_{p_1 \times p_2}\}_{1 \leq t \leq T} \) is a sequence of i.i.d. sample matrices satisfying \( \text{vec}(E_t) = (\Sigma_1 \otimes \Sigma_2)^{1/2} \text{vec}(Z_t) \), where \( Z_t = (Z_{t,ij})_{p_1 \times p_2} \) is a \( p_1 \times p_2 \) matrix with i.i.d. real entries \( Z_{t,ij} \) satisfying \( \mathbb{E}Z_{t,ij} = 0, \mathbb{E}Z_{t,ij}^2 = 1, \mathbb{E}Z_{t,ij}^4 = \nu_4 \) and \( \mathbb{E}|Z_{t,ij}|^{6+\varepsilon_0} < \infty \) for some small positive \( \varepsilon_0 \);
2. \( p_1, p_2, T \) tend to infinity as in (4.17).

Then under the null hypothesis \( H_0 : \text{Cov} (\text{vec}(E_t)) = \Sigma_1 \otimes \Sigma_2 \), \( W^* \) is defined as in (4.15), we have
\[
TW^* - p_1p_2 - (\nu_4 - 2) \xrightarrow{d} \mathcal{N}(0, 4).
\]

According to the asymptotic normality of \( W^* \) presented in Theorem 4.3, we reject \( H_0 \) at nominal level \( \alpha \) if
\[
\frac{1}{2} \left\{ TW^* - p_1p_2 - (\nu_4 - 2) \right\} \geq z_\alpha.
\]

Moreover, the asymptotic power of the proposed test for (4.14) can be derived as follows.

**Proposition 4.3.** Suppose that assumptions (1) and (2) in Theorem 4.3 hold, and
(3) $\Sigma_1$ and $\Sigma_2$ are two $p_1 \times p_1$ and $p_2 \times p_2$ non-negative definite matrices with bounded spectral norm, $\overline{\Sigma} := (\Sigma_1 \otimes \Sigma_2)^{1/2}(\Sigma_1 \otimes \Sigma_2)^{-1}(\Sigma_1 \otimes \Sigma_2)^{1/2}$ and the following limits exist:

$$\gamma = \lim_{T \to \infty} \frac{1}{p_1 p_2} \text{tr}(\overline{\Sigma}), \quad \theta = \lim_{T \to \infty} \frac{1}{p_1 p_2} \text{tr}(\overline{\Sigma}^2), \quad \omega = \lim_{T \to \infty} \frac{1}{p_1 p_2} \sum_{i=1}^{p_1 p_2} (\overline{\Sigma})_{ii}^2.$$

Then when $p_1, p_2, T$ tend to infinity as in (4.17), the testing power of $W^*$ for (4.14)

$$\beta(H_1) \to 1 - \Phi\left(\frac{1}{2\theta}[2z_\alpha - \omega(\nu_4 - 3) - \theta + n(2\gamma - 1 - \theta) + (\nu_4 - 2)]\right).$$

If $\gamma = \theta = 1$, then $\beta(H_1) \to 1 - \Phi(z_\alpha - \frac{\omega}{2}(\nu_4 - 3))$; otherwise, $\beta(H_1) \to 1$.

5. Simulation results

In this section, we implement some simulation studies to examine

(1) finite-sample properties of some LSS for $A_n$ by comparing their empirical means and variances with theoretical limiting values;

(2) finite-sample performance of the separable covariance structure test in Section 4.2.

5.1. LSS of $A_n$

Firstly we compare the empirical mean and variance of normalized $\{G_n(f_i) = \text{tr}(A_n^i), \ i = 1, 2, 3\}$ with their theoretical limits in Corollary 3.1. Define

$$\overline{G}_n(f_1) := \frac{G_n(f_1)}{\sqrt{\text{Var}(Y(f_1))}} = \frac{\text{tr}(A_n)}{\sqrt{\theta(\nu_4 - 3) + 2}},$$

$$\overline{G}_n(f_2) := \frac{G_n(f_2)}{\sqrt{\text{Var}(Y(f_2))}} = \frac{1}{2}\left\{\text{tr}(A_n^2) - n - \frac{b_p}{b_p}(\nu_4 - 3) + 1\right\},$$

$$\overline{G}_n(f_3) := \frac{G_n(f_3)}{\sqrt{\text{Var}(Y(f_3))}} = \frac{\text{tr}(A_n^3) - \frac{c_p}{b_p} \sqrt{\frac{n + 1}{b_p}(\nu_4 - 3)}}{\sqrt{\frac{9\omega}{2}(\nu_4 - 3) + 24}}.$$

According to Corollary 3.1, $\{\overline{G}_n(f_i)\} \overset{d}{\to} \mathcal{N}(0, 1), \ i = 1, 2, 3$. Hence we directly compare the empirical distribution of $\{\overline{G}_n(f_i)\}$ with $\mathcal{N}(0, 1)$ under different scenarios. Specifically, we consider two data distributions of $\{X_{ij}\}$ and three types of covariance matrix $\Sigma_p$, i.e.

(1) **Gaussian data:** $\{X_{ij}, \ 1 \leq i \leq p, \ 1 \leq j \leq n\}$ i.i.d. $\mathcal{N}(0, 1)$, with $\mathbb{E}X^4_{ij} = \nu_4 = 3$.

(2) **Non-Gaussian data:** $\{X_{ij}, \ 1 \leq i \leq p, \ 1 \leq j \leq n\}$ i.i.d. Gamma$(4, 2) - 2$, with $\mathbb{E}X^4_{ij} = 0, \mathbb{E}X^2_{ij} = 1, \mathbb{E}X^4_{ij} = 4.5$.

As for $\Sigma_p$,

(A) $\Sigma_A = I_p$;

(B) $\Sigma_B$ is diagonal, $1/4$ of its diagonal elements are 0.5, and $3/4$ are 1.

(C) $\Sigma_C$ is diagonal, one half of its diagonal elements are 0.5, and one half are 1.
Empirical mean and variance of \{G_n(f_i)\} are calculated for various combinations of \((p, n)\) under different model settings. For each pair of \((p, n)\), 5000 independent replications are used to obtain the empirical mean and variance. Table 1 reports the empirical values of \{G_n(f_i)\} when \(p = n^2\). Table 2 reports the case of \(p = n^2\).5. As shown in Tables 1 and 2, the empirical mean and variance of \{G_n(f_i)\} perfectly match their theoretical limits 0 and 1 under all scenarios, including all three types of \(\Sigma_p\), and for both Gaussian and non-Gaussian data.

### Table 1

| \(n\) | \(\Sigma_p = \Sigma_A\) | \(\Sigma_p = \Sigma_B\) | \(\Sigma_p = \Sigma_C\) |
|-------|----------------|----------------|----------------|
|       | mean | var | mean | var | mean | var |
| 50    | 0.0050 | 1.0992 | 0.0038 | 1.0074 | -0.0157 | 1.0292 |
| 100   | -0.0103 | 0.9962 | 0.0148 | 1.0073 | -0.0048 | 1.0252 |
| 150   | -0.0075 | 1.0293 | -0.0054 | 1.0372 | -0.0113 | 0.9915 |
| 200   | -0.0052 | 0.9989 | 0.0206 | 1.0140 | -0.0008 | 1.0012 |

### 5.2. Test for the Separable Covariance Structure

Empirical size and power of the separable structure test in Section 4.2 are examined to testify the asymptotic testing power of \(W^*\) given in Proposition 4.3. We compare the empirical power of \(W^*\) with its limits under various model settings. Specifically, the vectorization of data matrix \(E_t\) is \(\text{vec}(E_t) = (\Sigma_1 \otimes \Sigma_2)^{1/2}\text{vec}(Z_t)\). We consider two data distributions of \(Z_t = \{Z_{t,ij}\}\).
Table 2

Empirical mean and variance of $G_n(f_i)$, $i = 1, 2, 3$ from 5000 replications. Theoretical mean and variance are 0 and 1, respectively. Dimension $p = n^{2.5}$.

| $n$ | $\Sigma_p = \Sigma_A$ | $\Sigma_p = \Sigma_B$ | $\Sigma_p = \Sigma_C$ |
|-----|------------------------|------------------------|------------------------|
|     | mean       | var       | mean       | var       | mean       | var       |
| 50  | -0.0095    | 0.9870    | -0.0023    | 1.0067    | 0.0092     | 1.0233    |
| 100 | -0.0067    | 1.0274    | 0.0009     | 0.9991    | 0.0115     | 1.0150    |
| 150 | -0.0056    | 1.0164    | 0.0109     | 0.9772    | -0.0086    | 0.973     |
| 200 | 0.0139     | 0.9949    | 0.012      | 0.9907    | -0.0179    | 1.0002    |

$G_n(f_1)$ Gaussian

| $n$ | mean       | var       |
|-----|------------|-----------|
| 50  | 0.0087     | 1.0332    |
| 100 | 0.0016     | 0.9859    |
| 150 | 0.0093     | 1.0325    |
| 200 | 0.0109     | 0.9947    |

$G_n(f_1)$ Non-Gaussian

| $n$ | mean       | var       |
|-----|------------|-----------|
| 50  | 0.0044     | 1.0243    |
| 100 | 0.0191     | 0.9982    |
| 150 | 0.0010     | 1.0353    |
| 200 | 0.0039     | 1.0111    |

$G_n(f_2)$ Gaussian

| $n$ | mean       | var       |
|-----|------------|-----------|
| 50  | 0.0049     | 1.0585    |
| 100 | -0.017     | 1.04      |
| 150 | 0.0113     | 1.0449    |
| 200 | -0.0178    | 1.0492    |

$G_n(f_2)$ Non-Gaussian

| $n$ | mean       | var       |
|-----|------------|-----------|
| 50  | 0.0045     | 1.0491    |
| 100 | -0.0051    | 1.0387    |
| 150 | 0.0023     | 0.9959    |
| 200 | 0.0323     | 1.0186    |

$G_n(f_3)$ Gaussian

| $n$ | mean       | var       |
|-----|------------|-----------|
| 50  | 0.0551     | 1.1447    |
| 100 | 0.0342     | 1.0608    |
| 150 | 0.0281     | 1.0671    |
| 200 | 0.0347     | 1.0455    |

$G_n(f_3)$ Non-Gaussian

1. Gaussian matrix white noise: \{$Z_{t,ij}, 1 \leq i \leq p, 1 \leq j \leq n$\} i.i.d. $\mathcal{N}(0, 1)$, with $\nu_4 = \mathbb{E}Z_{t,ij}^4 = 3$.

2. Non-Gaussian matrix white noise: \{$Z_{t,ij}, 1 \leq i \leq p, 1 \leq j \leq n$\} i.i.d. Gamma(4,2)–2, with $\mathbb{E}Z_{t,ij} = 0, \mathbb{E}Z_{t,ij}^2 = 1, \nu_4 = \mathbb{E}Z_{t,ij}^4 = 4.5$.

As for covariance matrix $\Sigma_1 \otimes \Sigma_2$, we set $\Sigma_1$ as a $p_1 \times p_1$ tri-diagonal matrix, and $\Sigma_2$ as a $p_2 \times p_2$ symmetric Toeplitz matrix. More specifically,

$$\Sigma_1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ \vdots & \ddots & \ddots \\ 1 & \ddots & 1 \\ 1 & 2 \\ \end{pmatrix}_{p_1 \times p_1}$$
and \( \Sigma_2 = \left( \rho_{|i-j|} \right)_{p_2 \times p_2} \) with \( |\rho| < 1 \). We set \( \rho = 0.45, p_1 = p_2 = T \) and \( p_1 = 40, 60, 80, 100, 120 \). The nominal level of the test is \( \alpha = 0.05 \). To obtain the empirical power, we keep \( \Sigma_1 \) unchanged and replace \( \rho \) in \( \Sigma_2 \) with \( \rho(1 + \lambda) \) satisfying \( |\rho(1 + \lambda)| < 1 \). We vary \( \lambda = 0, 0.2, 0.3, 0.4, 0.5 \) to obtain different levels of testing power. For each pair of \( (p_1, p_2, T) \), 5000 independent replications are used to obtain the empirical size and power. Empirical values and theoretical limits are compared in Table 3. As shown in Table 3, the empirical power tends to 1 when either \( p_1, p_2, T \) or \( \lambda \) increases. Most importantly, the empirical power value is consistent with its theoretical limit under all scenarios.

### Table 3

| \( \lambda = 0 \) | \( \lambda = 0.2 \) | \( \lambda = 0.3 \) | \( \lambda = 0.4 \) | \( \lambda = 0.5 \) |
|------------------|------------------|------------------|------------------|------------------|
| \( p_1 \)       | \( p_2 \)       | \( T \)       | Emp   | Theo | Emp   | Theo | Emp   | Theo | Emp   | Theo | Emp   | Theo |
| 40              | 40              | 40              | 0.0490 | 0.05  | 0.0950 | 0.0880 | 0.2856 | 0.3087 | 0.8230 | 0.8354 | 0.9992 | 0.9992 |
| 60              | 60              | 60              | 0.0554 | 0.05  | 0.1650 | 0.1625 | 0.6484 | 0.6606 | 0.9974 | 0.9969 | 1      | 1     |
| 80              | 80              | 80              | 0.0520 | 0.05  | 0.2600 | 0.2699 | 0.8994 | 0.9084 | 1      | 1     | 1      | 1     |
| 100             | 100             | 100             | 0.0526 | 0.05  | 0.3916 | 0.4049 | 0.9864 | 0.9878 | 1      | 1     | 1      | 1     |
| 120             | 120             | 120             | 0.0542 | 0.05  | 0.5356 | 0.5524 | 0.9986 | 0.9992 | 1      | 1     | 1      | 1     |

Gaussian

| \( p_1 \)       | \( p_2 \)       | \( T \)       | Emp   | Theo | Emp   | Theo | Emp   | Theo | Emp   | Theo | Emp   | Theo |
| 40              | 40              | 40              | 0.0568 | 0.05  | 0.0716 | 0.0662 | 0.2214 | 0.2353 | 0.7008 | 0.7568 | 0.9942 | 0.9977 |
| 60              | 60              | 60              | 0.0610 | 0.05  | 0.1298 | 0.1277 | 0.5462 | 0.5752 | 0.9878 | 0.9930 | 1      | 1     |
| 80              | 80              | 80              | 0.0580 | 0.05  | 0.2202 | 0.2216 | 0.8356 | 0.8655 | 1      | 1     | 1      | 1     |
| 100             | 100             | 100             | 0.0530 | 0.05  | 0.3312 | 0.3464 | 0.9694 | 0.9785 | 1      | 1     | 1      | 1     |
| 120             | 120             | 120             | 0.0562 | 0.05  | 0.4886 | 0.4910 | 0.9974 | 0.9984 | 1      | 1     | 1      | 1     |

Non-Gaussian

### 6. Proof of Theorem 3.1

In Section 6.1 we first present the preliminary step of data truncation. The general strategy of the main proof of Theorem 3.1 is explained in Section 6.2. Three major steps of the general strategy are presented in Section 6.3, 6.4 and 6.5 respectively.

#### 6.1. Truncation, Centralization and Rescaling

We first truncate the elements of \( X \) without changing the weak limit of \( G_n(f) \). We choose a positive sequence \( \{\delta_n\} \) such that

\[
\delta_n^{-4} \mathbb{E}|X_{11}|^4 \mathbb{I}_{\{|X_{11}| \leq \delta_n \sqrt{np}\}} \to 0, \quad \delta_n \downarrow 0, \quad \delta_n \sqrt{np} \uparrow \infty, \tag{6.1}
\]

as \( n \to \infty \). Define

\[
\hat{X}_{ij} = X_{ij} \mathbb{I}_{\{|X_{11}| \leq \delta_n \sqrt{np}\}}, \quad \sigma^2 = \mathbb{E}|\hat{X}_{ij} - \mathbb{E}\hat{X}_{ij}|^2, \quad \hat{X} = (\hat{X}_{ij})_{p \times n},
\]

\[
\bar{X}_{ij} = (\hat{X}_{ij} - \mathbb{E}\hat{X}_{ij})/\sigma, \quad \bar{X} = (\bar{X}_{ij})_{p \times n},
\]

\[
\hat{A}_n = \left( \hat{X}' \Sigma_p \hat{X} - p \mu_p \mathbb{I}_n \right) / \sqrt{npb_p}, \quad \bar{A}_n = \left( \bar{X}' \Sigma_p \bar{X} - p \mu_p \mathbb{I}_n \right) / \sqrt{npb_p}.
\]
Define \( \tilde{G}_n(f) \) and \( \bar{G}(f) \) similarly by means of (3.1) with the matrix \( A_n \) replaced by \( \tilde{A}_n \) and \( \bar{A}_n \), respectively. First, observe that

\[
\mathbb{P}(G_n(f) \neq \bar{G}_n(f)) \leq \mathbb{P}(A_n \neq \tilde{A}_n) = o(1).
\]

Indeed,

\[
\mathbb{P}(A_n \neq \tilde{A}_n) \leq np\mathbb{P}(|X_{11}| \leq \delta_n \sqrt{np}) \leq K\delta_n^{-4}E|X_{11}|^4 f(\{X_{11} \geq \delta_n \sqrt{np}\}) = o(1).
\]

Now we consider the difference between \( \tilde{G}_n(f) \) and \( \bar{G}_n(f) \). For any analytic function \( f \) on \( \mathcal{U} \), we have

\[
E|\tilde{G}_n(f) - \bar{G}_n(f)| = \sum_{k=1}^n \left| f(\lambda_k^\ast) - f(\lambda_k^\tilde{\ast}) \right| \leq \frac{K_f}{\sqrt{np}} \sum_{k=1}^n \left| \lambda_k^\ast \Sigma_{p} \hat{X}_k - \lambda_k^\tilde{\ast} \Sigma_{p} \tilde{X}_k \right|
\]

\[
\leq \frac{K_f}{\sqrt{np}} \left| E(\tilde{X}_k - \hat{X}_k)^2 \right| \left| \Sigma_{p} \hat{X}_k - \Sigma_{p} \tilde{X}_k \right| \left| \left( \hat{X}_k \Sigma_{p} \hat{X}_k\right)^{\dagger} \left( \tilde{X}_k \Sigma_{p} \tilde{X}_k\right)^{\dagger} \right|^{1/2}
\]

\[
\leq \frac{2K_f}{\sqrt{np}} \left| E(\tilde{X}_k - \hat{X}_k)^2 \right| \left| \Sigma_{p} \hat{X}_k - \Sigma_{p} \tilde{X}_k \right| \left| \left( \hat{X}_k \Sigma_{p} \hat{X}_k\right)^{\dagger} \left( \tilde{X}_k \Sigma_{p} \tilde{X}_k\right)^{\dagger} \right|^{1/2},
\]

where \( K_f \) is a bound on \( |f'(x)| \).

It follows from (6.1) that

\[
|\sigma^2 - 1| \leq 2E X_{11}^2 f(\{X_{11} \geq \delta_n \sqrt{np}\})
\]

\[
\leq \frac{2}{\delta_n^2 \sqrt{np}} E|X_{11}|^4 f(\{X_{11} \geq \delta_n \sqrt{np}\}) = o((np)^{-1/2}),
\]

and

\[
|E \tilde{X}_{11}| = |E X_{11} f(\{X_{11} \geq \delta_n \sqrt{np}\})| \leq E|X_{11}| f(\{X_{11} \geq \delta_n \sqrt{np}\})
\]

\[
\leq \frac{1}{\delta_n^3 (np)^{-3/4}} E|X_{11}|^4 f(\{X_{11} \geq \delta_n \sqrt{np}\}) = o((np)^{-3/4}).
\]

These give us

\[
\frac{1}{\sqrt{np}} \left[ \left( \hat{X}_k \Sigma_{p} \hat{X}_k\right)^{\dagger} \left( \tilde{X}_k \Sigma_{p} \tilde{X}_k\right)^{\dagger} \right]^{1/2} \leq \sum_{i,j} \sigma_{ij} E|\tilde{X}_{ij} - \hat{X}_{ij}|^2 = \sum_{i,j} \sigma_{ij} \left| \frac{1 - \sigma}{\sigma} \hat{X}_{ij} + \frac{\tilde{X}_{ij}}{\sigma} \right|^2
\]

\[
\leq Kpn \left( \frac{(1 - \sigma)^2}{\sigma^2} E|\tilde{X}_{11}|^2 + \frac{1}{\sigma^2} E|\hat{X}_{11}|^2 \right) = o(1),
\]

and

\[
E tr(\tilde{X}' \tilde{X}) \leq \sum_{i,j} E|\tilde{X}_{ij}|^2 \leq Knp,
\]

\[
E tr(\hat{X}' \hat{X}) \leq \sum_{i,j} E|\hat{X}_{ij}|^2 \leq Knp.
\]

From the above estimates, we obtain

\[
G_n(f) = \bar{G}_n(f) + o_p(1).
\]

Thus, we only need to find the limit distribution of \( \{\tilde{G}(f_j), j = 1, \ldots, k\} \). Hence, in what follows, we assume that the underlying variables are truncated at \( \delta_n \sqrt{np} \), centralized, and
renormalized. For convenience, we shall suppress the superscript on the variables, and assume that, for any $1 \leq i \leq p$ and $1 \leq j \leq n$,

\[
|X_{ij}| \leq \delta_n \sqrt{np}, \quad \mathbb{E}X_{ij} = 0, \quad \mathbb{E}X_{ij}^2 = 1,
\]

\[
\mathbb{E}X_{ij}^a = \nu_a + o(1), \quad a = 4, 5, \quad \mathbb{E}|X_{ij}|^{6+\varepsilon_0} < \infty,
\]

where $\delta_n$ satisfies the condition (6.1).

6.2. Strategy of the proof

The general strategy of the proof follows the method established in Bai and Silverstein (2004) and Bai and Yao (2005).

Let $\mathcal{C}$ be the closed contour formed by the boundary of the rectangle with $(\pm u_1, \pm v_1)$ where $u_1 > 2, 0 < v_1 \leq 1$. Assume that $u_1$ and $v_1$ are fixed and sufficiently small such that $\mathcal{C} \subset \mathcal{U}$. By Cauchy theorem, with probability one, we have

\[
G_n(f) = \frac{1}{2\pi i} \oint_{\mathcal{C}} f(z)n[m_n(z) - m(z) - \chi_n(m(z))]dz,
\]

where $m_n(z)$ and $m(z)$ are the Stieltjes transforms of $F^{A_n}$ and $F$, respectively. The representation reduces our problem to finding the limiting process of

\[
M_n(z) = n[m_n(z) - m(z) - \chi_n(m(z))], \quad z \in \mathcal{C}.
\]

For $z \in \mathcal{C}$, we decompose $M_n(z)$ into a random part $M_n^{(1)}(z)$, and a determinist part $M_n^{(2)}(z)$, where

\[
M_n^{(1)}(z) = n[m_n(z) - \mathbb{E}m_n(z)]; \quad M_n^{(2)}(z) = n[\mathbb{E}m_n(z) - m(z) - \chi_n(m(z))].
\]

Throughout the paper, we set $\mathcal{C}_1 = \{z : z = u + iv, u \in [-u_1, u_1], |v| > v_1\}$. The limiting process of $M_n(z)$ on $\mathcal{C}_1$ is stated in the following proposition.

**Proposition 6.1.** Under the assumption $p \wedge n \to \infty, n^2/p = O(1)$ and after truncation of the data, the empirical process $\{M_n(z), z \in \mathcal{C}_1\}$ converges weakly to a centred Gaussian process $\{M(z), z \in \mathcal{C}_1\}$ with the covariance function

\[
\Lambda(z_1, z_2) = m'(z_1)m'(z_2)[\frac{\omega}{\theta}(\nu_4 - 3) + 2(1 - m(z_1)m(z_2))^{-2}].
\] (6.2)

Write the contour $\mathcal{C}$ as $\mathcal{C} = \mathcal{C}_\ell \cup \mathcal{C}_r \cup \mathcal{C}_u \cup \mathcal{C}_0$, where

\[
\mathcal{C}_\ell = \{z = -u_1 + iv, \xi_n/n < |v| < v_1\},
\]

\[
\mathcal{C}_r = \{z = u_1 + iv, \xi_n/n < |v| < v_1\},
\]

\[
\mathcal{C}_0 = \{z = \pm u_1 + iv, |v| \leq \xi_n/n\},
\]

\[
\mathcal{C}_u = \{z = u \pm iv_1, |u| \leq u_1\}
\]

and $\xi_n$ is a slowly varying sequence of positive constants and $v_1$ is a positive constant which is independent of $n$. Note that $\mathcal{C}_\ell \cup \mathcal{C}_0 \cup \mathcal{C}_u = \mathcal{C} \setminus \mathcal{C}_1$. To prove Theorem 3.1, we need to show that for $j = \ell, r, 0$ and some event $U_n$ with $\mathbb{P}(U_n) \to 1$,

\[
\lim \limsup_{v_1 \downarrow 0} \int_{\mathcal{C}_j} \mathbb{E} \left| M_n(z)I_{U_n} \right|^2 dz = 0 (6.3)
\]
The verification of (6.3) and (6.4) follows similar procedures developed in Sections 2.3, 3.1 and 4.3 of Bai and Yao (2005) and the details will be omitted here.

The calculation of the limiting covariance function of $Y(f)$ (see (3.3) & (3.4)) is quite similar to that given in Section 5 of Bai and Yao (2005), it is then omitted.

Next, we can prove Proposition 6.1 by the following three steps:

- Finite-dimensional convergence of the random part $M_n^{(1)}(z)$ in distribution on $\mathbb{C}_1$;
- Tightness of the random part $M_n^{(1)}(z)$.
- Convergence of the non-random part $M_n^{(2)}(z)$ to the mean function on $\mathbb{C}_1$.

Details of the three steps are presented in the coming sections 6.3, 6.4 and 6.5, respectively.

### 6.3. Finite dimensional convergence of $M_n^{(1)}(z)$ in distribution

We first decompose the random part $M_n^{(1)}(z)$ as a sum of martingale difference sequences, which is given in (6.19). Then, we apply the martingale CLT (Lemma A.4) to obtain the asymptotic distribution of $M_n^{(1)}(z)$. Note that we prove the finite dimensional convergence of $M_n^{(1)}(z)$ under the assumption $p/n \to \infty$, which is weaker than $n^2/p = O(1)$.

First, we introduce some notations. Define

$$X_k = (x_1, \ldots, x_{k-1}, x_k, \ldots, x_n), \quad A_k = \frac{1}{\sqrt{npb_p}}(X_k \Sigma_p X_k - p a_p I_{n-1}),$$

$$D = (A - z I_n)^{-1}, \quad D_k = (A_k - z I_{n-1})^{-1}, \quad M_k^{(s)} = \Sigma_p X_k D_k X_k' \Sigma_p, s = 1, 2,$$

$$a_k^{\text{diag}} = A_{kk} - z = \frac{1}{\sqrt{npb_p}}(x_k' \Sigma_p x_k - p a_p) - z, \quad q_k' = \frac{1}{\sqrt{npb_p}}(x_k' \Sigma_p x_k),$$

$$\beta_k = \frac{1}{a_k^{\text{diag}} + q_k' D_k q_k}, \quad \beta_k^{\text{tr}} = \frac{1}{z + (npb_p)^{-1} \text{tr} M_k^{(1)}},$$

$$\gamma_k = -\frac{1}{npb_p} \text{tr} M_k^{(s)} + q_k' D_k q_k, s = 1, 2, \quad \eta_k = \frac{1}{\sqrt{npb_p}}(x_k' \Sigma_p x_k - p a_p) - \gamma_k,$$

$$\ell_k = -\beta_k \beta_k^{\text{tr}} \eta_k (1 + q_k' D_k^2 q_k).$$

Note that $a_k^{\text{diag}}$ is the $k$-th diagonal element of $D^{-1}$ and $q_k'$ is the vector from the $k$-th row of $D^{-1}$ by deleting the $k$-th element. By applying Theorem A.5 in Bai and Silverstein (2010b), we obtain the equality

$$\text{tr} D - \text{tr} D_k = \frac{1 + q_k' D_k^2 q_k}{-a_k^{\text{diag}} + q_k' D_k q_k} = -\beta_k (1 + q_k' D_k^2 q_k). \quad (6.5)$$

Straightforward calculation gives:

$$\beta_k - \beta_k^{\text{tr}} = \beta_k \beta_k^{\text{tr}} \eta_k \quad (6.6)$$
and
\[
(\mathbb{E}_k - \mathbb{E}_{k-1})\beta_k^\text{tr}(1 + q_k^\text{tr}D_k^2q_k) = \mathbb{E}_k(\beta_k^\text{tr}\gamma_k2), \quad \mathbb{E}_{k-1}(\beta_k^\text{tr}\gamma_k2) = 0,
\] (6.7)
where \(\mathbb{E}_k(\cdot)\) is the expectation with respect to the \(\sigma\)-field generated by the first \(k\) columns of \(X\).

By the definition of \(D\) and \(D_k\), we obtain two basic identities:
\[
DX'\Sigma_pX = p\sigma_pD + \sqrt{np\sigma_p}(I_n + zD),
\] (6.8)
\[
D_kX'_k\Sigma_pX_k = p\sigma_pD_k + \sqrt{np\sigma_p}(I_{n-1} + zD_k).
\] (6.9)

If \(\Sigma_p = I_p\), it is straightforward to derive that the limit of \(\text{tr}(M_k^{(1)}(z))/(np\sigma_p)\) is \(m(z)\) by using (6.9). However, when \(\Sigma_p \neq I_p\), we need more detailed estimate.

**Lemma 6.1.** Under the assumption that \(p \land n \to \infty\) and \(p/n \to \infty\), we have, for \(z \in \mathbb{C}_1\),
\[
\mathbb{E}\left|\frac{1}{np\sigma_p}\text{tr}(M_k^{(1)}(z)) - m(z)\right|^2 \leq \frac{Kn}{p} + \frac{K}{n^2}.
\] (6.10)

**Proof.** Using Lemma A.1, we have
\[
\mathbb{E}(x'_p\Sigma_p^2x_i - pb_p)^2 \leq Kn_1\text{tr}(\Sigma_p^4) \leq K \cdot p\|\Sigma_p^4\| \leq Kp.
\] (6.11)

Note that \(\text{tr}(A^*B)\) is the inner product of \(\text{vec}(A)\) and \(\text{vec}(B)\) for any \(n \times m\) matrices \(A\) and \(B\). It follows from the Cauchy-Schwarz inequality that
\[
|\text{tr}(A^*B)|^2 \leq \text{tr}(A^*A) \cdot \text{tr}(B^*B).
\] (6.12)

By using (6.12), we have
\[
\mathbb{E}\left|\frac{1}{np\sigma_p}\text{tr}M_k^{(1)}(z) - \frac{1}{n}\text{tr}D_k(z)\right|^2
\]
\[
= \frac{1}{(np\sigma_p)^2}\mathbb{E}\left|\text{tr}\left(D_k(z)(X'_k\Sigma_p^2X_k - pb_pI_{n-1})\right)\right|^2
\]
\[
\leq \frac{1}{(np\sigma_p)^2}\mathbb{E}\left[\text{tr}(D_k(z)D_k(z)) \cdot \text{tr}(X'_k\Sigma_p^2X_k - pb_pI_{n-1})^2\right]
\]
\[
\leq \frac{1}{(np\sigma_p)^2}\mathbb{E}\left[n\|D_k(z)\| \cdot \text{tr}(X'_k\Sigma_p^2X_k - pb_pI_{n-1})^2\right]
\]
\[
\leq \frac{1}{n(p\sigma_p,v_1)^2}\mathbb{E}\left[\text{tr}(X'_k\Sigma_p^2X_k - pb_pI_{n-1})^2\right].
\]

Indeed, by using (6.11) and the fact \(\mathbb{E}(x'_p\Sigma_p^2x_j)^2 = \text{tr}(\Sigma_p^4)\), we have
\[
\mathbb{E}\left[\text{tr}(X'_k\Sigma_p^2X_k - pb_pI_{n-1})^2\right] = \sum_i\mathbb{E}(x'_p\Sigma_p^2x_i - pb_p)^2 + \sum_{i \neq j, i \neq k, j \neq k}\mathbb{E}(x'_p\Sigma_p^2x_j)^2
\]
\[
\leq (n - 1) \cdot pK + (n - 1)(n - 2) \cdot pK.
\]

Thus we have
\[
\mathbb{E}\left|\frac{1}{np\sigma_p}\text{tr}M_k^{(1)}(z) - \frac{1}{n}\text{tr}D_k(z)\right|^2 \leq \frac{Kn}{p},
\] (6.13)
Moreover, by (6.5) and (A.17), we have

\[
\left| \frac{1}{n} \text{tr} D(z) - \frac{1}{n} \text{tr} D_k(z) \right| \overset{(6.5)}{=} \frac{1}{n} | \beta_k (1 + q_k^2 D_k^2 q_k) | \overset{(A.17)}{\leq} \frac{1}{nv_1}, \tag{6.14}
\]

which, together with (6.13) and the fact that \( m_n(z) \overset{a.s.}{\to} \mu \), implies (6.10).

Applying (6.5) \sim (6.7), we have the following decomposition:

\[
M_n^{(1)}(z) = \text{tr} D - \mathbb{E} \text{tr} D = \sum_{k=1}^{n} (\mathbb{E}_k - \mathbb{E}_{k-1})(\text{tr} D - \text{tr} D_k)
\]

\[
= - \sum_{k=1}^{n} (\mathbb{E}_k - \mathbb{E}_{k-1}) \beta_k \left( 1 + q_k^2 D_k^2 q_k \right) \overset{(6.5)}{=} \sum_{k=1}^{n} (\mathbb{E}_k - \mathbb{E}_{k-1}) (-\beta_k \beta_k^{\text{tr}} \eta_k \gamma_{k2}) \left( 1 + q_k^2 D_k^2 q_k \right) - \sum_{k=1}^{n} (\mathbb{E}_k - \mathbb{E}_{k-1}) \beta_k^{\text{tr}} \left( 1 + q_k^2 D_k^2 q_k \right) \tag{6.16}
\]

By using (6.6), we can split \( \ell_k \) as

\[
\ell_k = - \left[ (\beta_k^{\text{tr}})^2 \eta_k + \beta_k (\beta_k^{\text{tr}})^2 \eta_k \right] \left( 1 + q_k^2 D_k^2 q_k \right)
\]

\[
= - (\beta_k^{\text{tr}})^2 \eta_k \left( 1 + \frac{1}{npb_p} \text{tr} M_k^{(2)} \right) - \beta_k^{\text{tr}} \eta_k \gamma_{k2} - \beta_k (\beta_k^{\text{tr}})^2 \eta_k \left( 1 + q_k^2 D_k^2 q_k \right)
\]

\[
=: \ell_{k1} + \ell_{k2} + \ell_{k3}. \tag{6.17}
\]

By Lemma A.9 and Lemma A.10, it is not difficult to verify that

\[
\mathbb{E} \left\| \sum_{k=1}^{n} (\mathbb{E}_k - \mathbb{E}_{k-1}) \ell_{k2} \right\|^2 = o(1), \quad \mathbb{E} \left\| \sum_{k=1}^{n} (\mathbb{E}_k - \mathbb{E}_{k-1}) \ell_{k3} \right\|^2 = o(1). \tag{6.18}
\]

These estimates, together with (6.16) and (6.17), imply that

\[
M_n^{(1)}(z) = \sum_{k=1}^{n} \mathbb{E}_k \left[ - (\beta_k^{\text{tr}})^2 \eta_k \left( 1 + \frac{1}{npb_p} \text{tr} M_k^{(2)} \right) - \beta_k^{\text{tr}} \gamma_{k2} \right] + o_{L_2}(1)
\]

\[
=: \sum_{k=1}^{n} Y_k(z) + o_{L_2}(1), \tag{6.19}
\]

where \( Y_k(z) \) is a sequence of martingale difference. Thus, to prove finite-dimensional convergence of \( M_n^{(1)}(z), z \in \mathbb{C}_1 \), we need only to consider the limit of the following term:

\[
\sum_{j=1}^{r} a_j M_n^{(1)}(z_j) = \sum_{j=1}^{r} a_j \sum_{k=1}^{n} Y_k(z_j) + o(1) = \sum_{k=1}^{n} \left( \sum_{j=1}^{r} a_j Y_k(z_j) \right) + o(1),
\]

where \( \{a_j\} \) are complex numbers and \( r \) is any positive integer.
By Lemma A.9 and Lemma A.10, we have
\[
\mathbb{E}|Y_j(z)|^4 \leq K \frac{\delta^4}{n} + K \left( \frac{1}{n^2} + \frac{n}{p^2} \right),
\]  
(6.20)
which implies that, for each \( \varepsilon > 0 \),
\[
\sum_{k=1}^{n} \mathbb{E} \left( \left| \sum_{j=1}^{r} a_j Y_k(z_j) \right|^2 \right)^2 \mathbb{1}_{\{ \sum_{j=1}^{r} a_j Y_k(z_j) \geq \varepsilon \}} \leq \frac{1}{\varepsilon^2} \sum_{k=1}^{n} \mathbb{E} \left| \sum_{j=1}^{r} a_j Y_k(z_j) \right|^4 = o(1),
\]
which implies that the second condition (A.3) of the martingale CLT (see Lemma A.4) is satisfied. Thus, to apply the martingale CLT, it is sufficient to verify that, for \( z_1, z_2 \in \mathbb{C}^+ \), the sum
\[
\Lambda_n(z_1, z_2) := \sum_{k=1}^{n} \mathbb{E}_{k-1} \left( Y_k(z_1) Y_k(z_2) \right)
\]
(6.21)
converges in probability to a constant (and to determine this constant).

Note that
\[-(\beta_k^{tr})^2 \eta_k \left( 1 + \frac{1}{npb_p} \text{tr} M_k^{(2)} \right) - \beta_k^{tr} \gamma k_2 = \frac{\partial}{\partial z} \left[ \beta_k^{tr}(z) \eta_k(z) \right],
\]
thus, we have
\[
\Lambda_n(z_1, z_2) = \frac{\partial^2}{\partial z_2 \partial z_1} \sum_{k=1}^{n} \mathbb{E}_{k-1} \left[ \mathbb{E}_k \left[ \beta_k^{tr}(z_1) \eta_k(z_1) \right] \cdot \mathbb{E}_k \left[ (\beta_k^{tr}(z_2) \eta_k(z_2)) \right] \right].
\]  
(6.22)
It is enough to consider the limit of
\[
\sum_{k=1}^{n} \mathbb{E}_{k-1} \left[ \mathbb{E}_k \left[ \beta_k^{tr}(z_1) \eta_k(z_1) \right] \cdot \mathbb{E}_k \left[ (\beta_k^{tr}(z_2) \eta_k(z_2)) \right] \right].
\]  
(6.23)
By (2.3), (6.1) and the dominated convergence theorem, we conclude that
\[
\mathbb{E} \left| \beta_k^{tr}(z) + m(z) \right|^2 = o(1).
\]  
(6.24)
Combining (6.24) into (6.23) yields that
\[
\sum_{k=1}^{n} \mathbb{E}_{k-1} \left[ \mathbb{E}_k \left[ \beta_k^{tr}(z_1) \eta_k(z_1) \right] \cdot \mathbb{E}_k \left[ (\beta_k^{tr}(z_2) \eta_k(z_2)) \right] \right]
\]
\[= m(z_1)m(z_2) \sum_{k=1}^{n} \mathbb{E}_{k-1} \left( \mathbb{E}_k \eta_k(z_1) \cdot \mathbb{E}_k \eta_k(z_2) \right) + o_p(1)
\]
\[=: m(z_1)m(z_2) \Lambda_n(z_1, z_2) + o_p(1).
\]  
(6.25)
In view of (6.21) \sim (6.25), it suffices to derive the limit of \( \Lambda_n(z_1, z_2) \), which further gives the limit of (6.21).

Since \( \mathbb{E}_k[\gamma_k(z)] = (1/\sqrt{npb_p}) (x'_k \Sigma_p x_k - p a_p) - \mathbb{E}_k[\gamma k_1(z)] \), we have
\[
\mathbb{E}_{k-1} \left[ \mathbb{E}_k \eta_k(z_1) \cdot \mathbb{E}_k \eta_k(z_2) \right] = \frac{1}{n} \left[ \frac{\tilde{b}_p}{b_p} (\nu_4 - 3) + 2 \right] + A_1^{(k)} + A_2^{(k)} + A_3^{(k)},
\]  
(6.26)
where
\[ A_1^{(k)} = \mathbb{E}_{k-1} \left[ \mathbb{E}_k \gamma_{k1}(z_1) \cdot \mathbb{E}_k \gamma_{k1}(z_2) \right], \quad A_2^{(k)} = -\mathbb{E}_{k-1} \left[ \frac{1}{\sqrt{npb_p}} (x_k^i \Sigma_p x_k - pa_p) \cdot \mathbb{E}_k \gamma_{k1}(z_1) \right], \]
\[ A_3^{(k)} = -\mathbb{E}_{k-1} \left[ \frac{1}{\sqrt{npb_p}} (x_k^i \Sigma_p x_k - pa_p) \cdot \mathbb{E}_k \gamma_{k1}(z_2) \right]. \]

First, we show that \( A_2^{(k)} \) and \( A_3^{(k)} \) are negligible. Denote \( M_k^{(1)}(z) = (a_{ij}(z))_{p \times p} \), using the independence between \( x_k \) and \( M_k^{(1)} \), we have
\[ A_2^{(k)} = -\frac{1}{npb_p \sqrt{npb_p}} \mathbb{E}_k \left[ \left( \sum_{i,j} \sigma_{ij} X_{ik} X_{jk} - pa_p \right) \left( \sum_{i \neq j} X_{ik} X_{jk} \mathbb{E}_k a_{ij}^{(1)}(z_1) + \sum_{i=1}^p (X_{ik}^2 - 1) \mathbb{E}_k a_{ii}^{(1)}(z_1) \right) \right] \]
\[ = -\frac{1}{npb_p \sqrt{npb_p}} \mathbb{E}_k \left[ \left( \sum_{i \neq j} \sigma_{ij} X_{ik} X_{jk}^2 \mathbb{E}_k a_{ij}^{(1)}(z_1) + \sum_{i=1}^p \sigma_{ii} X_{ik}^2 (X_{ik}^2 - 1) \mathbb{E}_k a_{ii}^{(1)}(z_1) \right) \right] \]
\[ = -\frac{1}{npb_p} \mathbb{E}_k \left[ \frac{1}{npb_p} \text{tr} \left( \Sigma_p M_k^{(1)} \right) - \frac{\nu - 2}{npb_p} \sum_{i=1}^p \sigma_{ii} a_{ii}^{(1)}(z_1) \right]. \] (6.27)

As for the first term in the bracket of (6.27), we can estimate it by using the similar argument as in the proof of Lemma 6.1. Replacing \( pb_p \) and \( M_k^{(1)} \) in the proof of Lemma 6.1 with \( \text{tr}(\Sigma_p^3) \) and \( \Sigma_p M_k^{(1)} \), we can prove that
\[ \mathbb{E} \left| \frac{1}{n \text{tr}(\Sigma_p^3)} \text{tr} \left( \Sigma_p M_k^{(1)} \right) - \frac{1}{n} \text{tr} D_k \right|^2 \leq \frac{Kn}{p}. \]

Moreover, by the fact \( \frac{b^2}{a_p} \leq \text{tr}(\Sigma_p^3) \leq Kp \), the first inequality of which follows from (6.12), we conclude that
\[ \frac{1}{npb_p} \text{tr} \left( \Sigma_p M_k^{(1)} \right) = \frac{\text{tr}(\Sigma_p^3)}{npb_p} \cdot \frac{1}{n \text{tr}(\Sigma_p^3)} \text{tr} \left( \Sigma_p M_k^{(1)} \right) = O_p(1). \]

As for the second term in the bracket of (6.27), we have
\[ \frac{1}{npb_p} \sum_{i=1}^p \sigma_{ii} a_{ii}^{(1)} \leq \left\| \Sigma_p \right\| \cdot \frac{1}{npb_p} \sum_{i=1}^p a_{ii}^{(1)} = \left\| \Sigma_p \right\| \cdot \text{tr} M_k^{(1)} = O_p(1). \]

Thus, we conclude that the term in the square bracket of (6.27) is bounded in probability. Thus \( \left| \sum_{k=1}^n A_2^{(k)} \right| \to 0 \). Similarly, we can show that \( \left| \sum_{k=1}^n A_3^{(k)} \right| \to 0 \).

Now we consider \( A_1^{(k)} \). We consider the second terms on the RHS of (6.26) with the notation \( \mathbb{E}_k M_k^{(1)}(z) = (a_{ij}(z))_{n \times n} \),
\[ A_1^{(k)} = \frac{1}{(npb_p)^2} \mathbb{E}_{k-1} \left[ \left( \sum_{i \neq j} X_{ik} X_{jk} \mathbb{E}_k a_{ij}^{(1)}(z_1) + \sum_{i=1}^p (X_{ik}^2 - 1) \mathbb{E}_k a_{ii}^{(1)}(z_1) \right) \right] \]
\[ \mathcal{A}_n(z_1, z_2) = 2 \frac{\tilde{b}_p}{(npb_p)^2} \sum_{k=1}^n \text{tr} \left( \mathbf{E}_k \mathbf{M}_k^{(1)}(z_1) \cdot \mathbf{E}_k \mathbf{M}_k^{(1)}(z_2) \right) + \left[ \frac{\tilde{b}_p}{b_p} (\nu_4 - 3) + 2 \right] + o_p(1), \]

where

\[ Z_k = \frac{1}{n(npb_p)^2} \text{tr} \left( \mathbf{E}_k \mathbf{M}_k^{(1)}(z_1) \cdot \mathbf{E}_k \mathbf{M}_k^{(1)}(z_2) \right). \]

In Lemma A.6, we derive the asymptotic expression of \( Z_k \). This asymptotic expression ensures that

\[ \frac{1}{n} \sum_{k=1}^n Z_k \rightarrow \int_0^1 \frac{tm(z_1)m(z_2)}{1 - tm(z_1)m(z_2)} \, dt = -1 - \frac{\log(1 - m(z_1)m(z_2))}{m(z_1)m(z_2)}. \]

By (6.22), (6.25), (6.28) and (6.30), we have

\[ \bar{\Lambda}_n(z_1, z_2) \xrightarrow{p} \frac{\omega}{\theta} (\nu_4 - 3) - \frac{2 \log(1 - m(z_1)m(z_2))}{m(z_1)m(z_2)}. \]

Therefore,

\[ \Lambda_n(z_1, z_2) \xrightarrow{p} \frac{\partial^2}{\partial z_1 \partial z_2} \left\{ \frac{\omega}{\theta} (\nu_4 - 3)m(z_1)m(z_2) - 2 \log \left( 1 - m(z_1)m(z_2) \right) \right\}. \]
\[ = m'(z_1)m'(z_2)\left[ \frac{\omega}{6}(\nu_4 - 3) + 2(1 - m(z_1)m(z_2))^{-2} \right]. \]

### 6.4. Tightness of \(M_n^{(1)}(z)\)

This subsection is to verify the tightness of \(M_n^{(1)}(z)\) for \(z \in \mathbb{C}_1\) by using Lemma A.5. Applying the Cauchy-Schwarz inequality, Lemma A.9 and Lemma A.10, we have

\[
\mathbb{E}\left[ \sum_{k=1}^{n} \sum_{j=1}^{r} a_j Y_k(z_j) \right]^2 = O(1),
\]

which shows that the condition (i) of Lemma A.5 holds. Condition (ii) of Lemma A.5 will be verified by showing

\[
\frac{\mathbb{E}[M_n^{(1)}(z_1) - M_n^{(1)}(z_2)]^2}{|z_1 - z_2|^2} \leq K, \quad z_1, z_2 \in \mathbb{C}_1.
\]

(6.31)

The proof of (6.31) exactly follow Chen and Pan (2015), it is then omitted.

### 6.5. Convergence of \(M_n^{(2)}(z)\)

In this section, we obtain the asymptotic expansion of \(n(\mathbb{E}m_n(z) - m(z))\) for \(z \in \mathbb{C}_1\) (see definition \(\mathbb{C}_1\) of in Section 6.2) and the result is stated in Lemma 6.2. This lemma, together with the finite dimensional convergence (see Section 6.3 and the tightness of \(M_n^{(1)}(z)\) (see Section 6.4), implies Proposition 6.1. To prove Lemma 6.2, we will follow the strategy in Khorunzhy et al. (1996) and Bao (2015). The main tool is the generalized Stein’s equation (see Lemma 6.3).

**Lemma 6.2.** With the same notations as in the previous sections, for \(z \in \mathbb{C}_1\),

(1) if \(p \land n \to \infty\) and \(n^2/p = O(1)\), we have

\[
M_n^{(2)} = n\left[ \mathbb{E}m_n(z) - m(z) - \mathcal{X}_n(m(z)) \right] = o(1),
\]

where \(\mathcal{X}_n(m)\) is defined by (3.2);

(2) if \(p \land n \to \infty\) and \(n^3/p = O(1)\), we have

\[
n\left[ \mathbb{E}m_n(z) - m(z) + \sqrt{\frac{n}{p}} c_p \frac{m^4}{1 - m^2} \right] = \frac{m^3}{1 - m^2} \left( \frac{m^2}{1 - m^2} + \frac{b_p}{m}(\nu_4 - 3) + 1 \right) + o(1).
\]

(6.33)

**Proof.** Let \(Y = (npb_p)^{-1/4}X\), then

\[
A = Y^\prime \Sigma_p Y - \sqrt{\frac{p}{n}} b_p \mathbf{I}_n.
\]

To simplify notations, we let

\[
\mathbf{E} := \Sigma_p YDY^\prime \Sigma_p = (E_{ij})_{p \times p}, \quad \mathbf{F} := \Sigma_p YD = (F_{ij})_{p \times n}.
\]
By the basic identity
\[ D = -\frac{1}{z} I_n + \frac{1}{z} DA = -\frac{1}{z} I_n + \frac{1}{z} \left( DY' \Sigma_p Y - \sqrt{\frac{p}{n}} \frac{a_p}{b_p} D \right), \]
we have
\[ \mathbb{E} m_n(z) = -\frac{1}{z^2} + \frac{1}{z} \cdot \frac{1}{n} \mathbb{E} \text{tr}(DA) \]
\[ = -\frac{1}{z} + \frac{1}{z} \sqrt{\frac{p}{n}} \frac{a_p}{b_p} \mathbb{E} \left( \frac{1}{n} \text{tr} D \right) + \frac{1}{zn} \mathbb{E} \text{tr} \left( Y' \Sigma_p Y D \right) \]
\[ = -\frac{1}{z} - \frac{1}{z} \sqrt{\frac{p}{n}} \frac{a_p}{b_p} \mathbb{E} m_n(z) + \frac{1}{zn} \sum_{j,k} \mathbb{E} (Y_{jk} F_{jk}). \tag{6.34} \]

The basic idea of the following derivation is regarding \( F_{jk} := (\Sigma_p Y D)_{jk} \) as an analytic function of \( Y_{jk} \), and then apply the generalized Stein’s equation (Lemma 6.3 below) to expand \( \mathbb{E} (Y_{jk} F_{jk}) \).

**Lemma 6.3 (Generalized Stein’s Equation, Khorunzhy et al. (1996)).** For any real-valued random variable \( \xi \) with \( \mathbb{E} |\xi|^{p+2} < \infty \) and complex-valued function \( g(t) \) with continuous and bounded \( p+1 \) derivatives, we have
\[ \mathbb{E} \left[ \xi g(\xi) \right] = \sum_{a=0}^{p} \frac{\kappa_{a+1}}{a!} \mathbb{E} \left( g^{(a)}(\xi) \right) + \varepsilon, \]
where \( \kappa_a \) is the \( a \)-th cumulant of \( \xi \), and
\[ |\varepsilon| \leq C \sup_t |g^{(p+1)}(t)| \mathbb{E}(|\xi|^{p+2}), \]
where the positive constant \( C \) depends on \( p \).

Applying Lemma 6.3 to the last term in (6.34), we obtain the following expansion:
\[ \mathbb{E} m_n(z) = -\frac{1}{z} - \frac{1}{z} \sqrt{\frac{p}{n}} \frac{a_p}{b_p} \mathbb{E} m_n(z) + \frac{1}{zn} \sum_{a=0}^{p} \frac{1}{(a+1)/4} \sum_{j,k} \frac{\kappa_{a+1}}{a!} \mathbb{E} \left( \frac{\partial^a F_{jk}}{\partial Y_{jk}^a} \right) + \varepsilon_n, \tag{6.35} \]
where \( \kappa_a \) is the \( a \)-th cumulant of \( Y_{jk} \), \( \frac{\partial^a F_{jk}}{\partial Y_{jk}^a} \) denotes the \( a \)-th order derivative of \( F_{jk} \) w.r.t. \( Y_{jk} \), and
\[ \varepsilon_n \leq \frac{K}{n} \frac{1}{(npb_p)^{a/4}} \sum_{j,k} \sup_{j,k} \mathbb{E} \left| \frac{\partial^5 F_{jk}}{\partial Y_{jk}^5} \right|. \tag{6.36} \]

The explicit formula of the derivatives of \( F_{jk} \) are provided in Lemma 6.4. These derivatives can be derived by using the chain rule and Lemma A.14 repeatedly, and the details will be omitted here.

**Lemma 6.4 (Derivatives of \( F_{jk} \)).**
\[ \frac{\partial F_{jk}}{\partial Y_{jk}} = \sigma_{jj} D_{kk} - E_{jj} D_{kk} - F_{jk}^2; \]
\[ \frac{\partial^2 F_{j_1 k_1}}{\partial Y_{j_2 k_2}^2} = -6\sigma_{j_1 j_2} F_{j_2 k_2} D_{kk} + 6 E_{j_1 j_2} F_{j_2 k_2} D_{kk} + 2 F_{j_1 k_2}^3; \]

\[ \frac{\partial^3 F_{j_1 k_1}}{\partial Y_{j_2 k_2}^3} = -6\sigma_{j_1 j_2}^2 D_{kk} + 36\sigma_{j_1 j_2} F_{j_2 k_2}^2 D_{kk} + 12\sigma_{j_1 j_2} E_{j_1 j_2} F_{j_2 k_2}^2 D_{kk} - 36 E_{j_1 j_2} F_{j_2 k_2}^2 D_{kk} - 6 E_{j_1 j_2}^2 D_{kk}^2 - 6 F_{j_1 k_2}^4; \]

\[ \frac{\partial^4 F_{j_1 k_1}}{\partial Y_{j_2 k_2}^4} = 12\sigma_{j_1 j_2}^2 F_{j_2 k_2} D_{kk}^2 - 240\sigma_{j_1 j_2} F_{j_2 k_2}^3 D_{kk} - 240\sigma_{j_1 j_2} E_{j_1 j_2} F_{j_2 k_2}^3 D_{kk} + 240 E_{j_1 j_2} F_{j_2 k_2}^3 D_{kk} \]

\[ + 120 E_{j_1 j_2}^2 F_{j_2 k_2} D_{kk}^2 + 24 F_{j_1 k_2}^5; \]

\[ \frac{\partial^5 F_{j_1 k_1}}{\partial Y_{j_2 k_2}^5} = -120 F_{j_1 k_2}^6 - 1800 E_{j_1 j_2} F_{j_2 k_2}^4 D_{kk} - 1800 E_{j_1 j_2}^2 F_{j_2 k_2}^2 D_{kk}^2 - 120 E_{j_1 j_2}^3 D_{kk}^3 + 1800 \sigma_{j_1 j_2} F_{j_2 k_2}^4 D_{kk} \]

\[ + 3600 \sigma_{j_1 j_2} E_{j_1 j_2} F_{j_2 k_2}^3 D_{kk} - 1800 \sigma_{j_1 j_2} F_{j_2 k_2}^2 D_{kk}^2 - 360 \sigma_{j_1 j_2}^2 E_{j_1 j_2}^2 D_{kk}^3 + 120 \sigma_{j_1 j_2}^3 D_{kk}^3. \]

From (6.8) and Lemma A.13, it is not difficult to obtain the following estimates:

\[ D_{n_1}^2 F_{n_2}^a F_{n_3}^a \leq K n^{a_1/2} \left( \sum_{\alpha} \left[ \left( \Sigma Y \right)_{j_\alpha} \right]^2 \right)^{(a_2 + 2a_3)/2}, \quad (a_1, a_2, a_3 \geq 0) \]  

(6.37)

\[ \mathbb{E} \left[ \left( \Sigma_p^{-1/2} \mathbb{E} \Sigma_p^{-1/2} \right)_{j_1 j_2} \right] - \frac{\mathbb{E} m_n}{a_p \sqrt{p/(np)} + z + \mathbb{E} m_n} = O \left( \left( \frac{n}{p} \right)^2 \right) + O \left( \frac{1}{p} \right), \]  

(6.38)

\[ \left| \sum_{j,k} F_{j k} \right| = O \left( (np)^{3/4} \right), \]  

(6.39)

\[ \left| \sum_{j,k} F_{j k}^a \right| = O \left( p^{a_2/4} n^{1-a_2/4} \right), \quad (a_2 \geq 2). \]  

(6.40)

By (6.36) and (6.37), we have the following estimate for \( \varepsilon_n \) defined in (6.36):

\[ |\varepsilon_n| = o \left( \frac{1}{n} \right). \]  

(6.41)

By the fact

\[ \mathbb{E} \left[ D_{j k} \left( 1 \right) + \frac{1}{z + \mathbb{E} m_n} \right]^2 = O \left( \frac{1}{n} \right) + O \left( \frac{n}{p} \right), \quad 1 \leq \ell \leq n, \]  

(6.42)

which is verified in Lemma A.12, and the estimates above, we can extract the leading order terms in (6.35) to obtain

\[ \mathbb{E} m_n(z) = \frac{1}{z} - \frac{1}{z} \frac{1}{\sqrt{p}} \frac{1}{\sqrt{b_p}} \mathbb{E} m_n(z) + \frac{1}{zn} \frac{1}{npb_p} \sum_{j,k} \mathbb{E} \left( \sigma_{j j} D_{k k} - E_{j j} D_{k k} - F_{j k}^2 \right) \]

\[ - \frac{1}{zn} \frac{1}{npb_p} \sum_{j,k} \mathbb{E} \left( \sigma_{j j}^2 D_{k k}^2 \right) + o \left( \frac{1}{n} \right) \]

\[ = \frac{1}{z} - \frac{1}{zn} \frac{1}{\sqrt{p}} \sum_{j,k} \mathbb{E} \left[ \text{tr}(E) \text{tr}(D) \right] - \frac{1}{zn} \frac{1}{npb_p} \mathbb{E} \left[ \text{tr}(F F') \right] \]
− \frac{(\nu_4 - 3)\tilde{b}_p}{zn^2b_p}E\left(\sum_k D_{kk}^2\right) + o\left(\frac{1}{n}\right). \tag{6.43}

Using the same argument as in the proof of Lemma 6.1, if \(n^2/p = O(1)\), we can show that

\[E\left|\frac{1}{\sqrt{npb_p}}\text{tr}E - m(z)\right|^2 = O\left(\frac{1}{n}\right).\] \tag{6.44}

This, together with \(c_t\)-inequality, implies that

\[E\left|\frac{1}{\sqrt{npb_p}}\text{tr}E - E\left(\frac{1}{\sqrt{npb_p}}\text{tr}E\right)\right|^2 = o(1).\] \tag{6.45}

Together with the fact that \(\text{Var}(m_n) = O(n^{-2})\) (see Lemma A.11 for more details), we obtain

\[\text{Cov}\left(\frac{1}{\sqrt{npb_p}}\text{tr}E, \frac{1}{n}\text{tr}D\right) \leq \sqrt{(6.45)} \cdot \sqrt{\text{Var}(m_n)} = o\left(\frac{1}{n}\right).\] \tag{6.46}

Note that

\[\text{tr}(FF') = \text{tr}(\Sigma_p YD^2Y'\Sigma_p) = \frac{\partial}{\partial z}\text{tr}(\Sigma_p YDY'\Sigma_p) = \frac{\partial}{\partial z}\text{tr}E.\] \tag{6.47}

Applying (6.42), (6.46), and (6.47) to (6.43), we have

\[Em_n(z) = -\frac{1}{z} - \frac{1}{z} \cdot \frac{1}{\sqrt{npb_p}}E(\text{tr}E) \cdot \frac{1}{n} E(\text{tr}D) - \frac{1}{zn^2b_p}E\left(\frac{\partial}{\partial z}\text{tr}E\right) - \nu_4 - 3\frac{\tilde{b}_p}{zn^2b_p}(Em_n(z))^2 + o\left(\frac{1}{n}\right). \tag{6.48}

The problem reduces to estimate \((1/\sqrt{npb_p})E(\text{tr}E)\). To this end, we apply Lemma 6.3 again to the term \((1/\sqrt{npb_p})E(\text{tr}E)\) to find its expansion. Denote

\[\bar{E} := \Sigma^2_pYDY'\Sigma^2_p, \quad \bar{E} := \Sigma_p YDY'\Sigma_p, \quad \bar{F} := \Sigma^2_pYD, \quad \bar{F} := \Sigma^2_pYD,\]

and write

\[\frac{1}{\sqrt{npb_p}}E(\text{tr}E) = \frac{1}{\sqrt{npb_p}} \sum_{j,k} E(Y_{jk} \bar{F}_{jk}). \tag{6.49}\]

The first four derivatives of \(\bar{F}_{jk}\) w.r.t. \(Y_{jk}\) is presented in the following lemma.

**Lemma 6.5** (Derivatives of \(\bar{F}_{jk}\)).

\[
\frac{\partial \bar{F}_{jk}}{\partial Y_{jk}} = \sigma_{jj}D_{kk} - \bar{E}_{jj}D_{kk} - F_{jk} \bar{F};
\]

\[
\frac{\partial^2 \bar{F}_{jk}}{\partial Y_{jk}^2} = -2\sigma_{jj} \bar{F}_{jk} D_{kk} - 4\bar{E}_{jj} F_{jk} D_{kk} + 2F^2_{jk} \bar{F};
\]

\[
\frac{\partial^3 \bar{F}_{jk}}{\partial Y_{jk}^3} = -6\sigma_{jj} \bar{F}_{jk} D_{kk}^2 - 6F^3_{jk} \bar{F}_{jk} - 18 \bar{E}_{jj} F^2_{jk} D_{kk} - 18E_{jj} F_{jk} \bar{F}_{jk} D_{kk} - 6E_{jj} \bar{E}_{jj} D^2_{kk}.
\]
+ 18\sigma_{jj}F_{jk}\tilde{F}_{jk}D_{kk} + 6\sigma_{jj}\tilde{E}_{jj}D_{kk}^2 + 18\sigma_{jj}E_{jj}D_{kk}^2; \\
\frac{\partial^4 \tilde{F}_{jk}}{\partial Y_{jk}^4} = 24F_{jk}^{4}\tilde{F}_{jk} + 96\tilde{E}_{jj}F_{jk}\tilde{F}_{jk}D_{kk} + 144F_{jk}^2\tilde{F}_{jk}D_{kk} + 96E_{jj}\tilde{E}_{jj}F_{jk}D_{kk}^2 + 24\sigma_{jj}\tilde{F}_{jk}D_{kk}^2 \\
- 144\sigma_{jj}F_{jk}^2\tilde{F}_{jk}D_{kk} - 96\sigma_{jj}\tilde{E}_{jj}F_{jk}D_{kk}^2 - 48\sigma_{jj}E_{jj}\tilde{F}_{jk}D_{kk}^2 + 24\sigma_{jj}\tilde{F}_{jk}D_{kk}^2 \\
- 96\sigma_{jj}F_{jk}^3D_{kk} - 96\sigma_{jj}E_{jj}F_{jk}D_{kk}^2 + 96\sigma_{jj}\tilde{E}_{jj}F_{jk}D_{kk}^2.

Applying generalized Stein’s equation with the derivatives of \( \tilde{F}_{jk} \) (see Lemma 6.5) to the last term in (6.49), and using the similar estimates above, gives us

\[
\frac{1}{\sqrt{npb_p}} \mathbb{E} (\text{tr}E) \]

\[
= \frac{1}{\sqrt{npb_p}} \sum_{a=0}^{3} \frac{1}{(npb_p)^{(a+1)/4}} \sum_{j,k} \kappa_a + 1 \frac{1}{a!} \mathbb{E} \left( \frac{\partial^a \tilde{F}_{jk}}{\partial Y_{jk}^a} \right) + \tilde{\varepsilon}_n 
\]

\[
= \frac{1}{npb_p} \sum_{j,k} \left( \mathbb{E} (\tilde{\sigma}_{jj}D_{kk} - \tilde{E}_{jj}D_{kk} - F_{jk}\tilde{F}_{jk}) + \frac{1}{\sqrt{npb_p}} \nu_3 - 3 \frac{1}{npb_p} \sum_{j,k} \mathbb{E} (\tilde{\sigma}_{jj}\tilde{\sigma}_{jj} D_{kk}^2) + o\left( \frac{1}{n} \right) \right) 
\]

\[
= \mathbb{E} m_n - \frac{1}{npb_p} \mathbb{E} \left[ \text{tr}(\tilde{E})\text{tr}(D) \right] - \frac{1}{npb_p} \mathbb{E} \left[ \text{tr}(\tilde{F}\tilde{F}') \right] 
\]

\[
- \frac{1}{\sqrt{npb_p}} \nu_3 - 3 \frac{1}{npb_p} \mathbb{E} \left( \sum_j \tilde{\sigma}_{jj}\tilde{\sigma}_{jj} \right) \left( \sum_k D_{kk}^2 \right) + o\left( \frac{1}{n} \right) 
\]

\[
= \mathbb{E} m_n - \frac{\sqrt{n}}{p} \mathbb{E} (\text{tr}E) \cdot \frac{1}{n} \mathbb{E} (\text{tr}D) + o\left( \frac{1}{n} \right) \quad (6.50) 
\]

\[
= \mathbb{E} m_n - \frac{\sqrt{n}}{p} \frac{c_p}{b_p} \mathbb{E} m_n + o\left( \frac{\sqrt{n}}{p} \right) \cdot \mathbb{E} m_n + o\left( \frac{1}{n} \right) \quad (6.51) 
\]

\[
= \mathbb{E} m_n - \frac{\sqrt{n}}{p} \frac{c_p}{b_p} \left( \mathbb{E} m_n \right)^2 + o\left( \frac{1}{n} \right) + o\left( \frac{\sqrt{n}}{p} \right). \quad (6.52) 
\]

Plugging (6.52) into (6.48), we have

\[
\mathbb{E} m_n = -\frac{1}{z} - \frac{1}{z} \left[ \mathbb{E} m_n - \sqrt{\frac{n}{p}} \frac{c_p}{b_p} \mathbb{E} (m_n)^2 \right] \mathbb{E} m_n 
\]

\[
- \frac{1}{zn} \frac{\partial}{\partial z} \left[ \mathbb{E} m_n - \sqrt{\frac{n}{p}} \frac{c_p}{b_p} \mathbb{E} (m_n)^2 \right] 
\]

\[
- \nu_3 - 3 \frac{b_p}{b_p} \mathbb{E} m_n \left( \mathbb{E} m_n \right)^2 + o\left( \frac{1}{n} \right) + o\left( \frac{\sqrt{n}}{p} \right), 
\]

\[
= -\frac{1}{z} - \left( \mathbb{E} m_n \right)^2 + \frac{1}{z} \sqrt{\frac{n}{p}} \frac{c_p}{b_p} m_n 
\]

\[
- \frac{1}{zn} \left[ \frac{m_n^2}{1 - m_n^2} + \frac{\nu_3 - 3}{b_p} m_n^2 \right] + o\left( \frac{1}{n} \right) + o\left( \frac{\sqrt{n}}{p} \right). 
\]
This implies (6.33) under the assumption \( n^3/p = O(1) \).

Moreover, to obtain (6.32) under the assumption \( n^2/p = O(1) \), we need to figure out the remainder term \( o(\sqrt{n}/p) \) in (6.52) more carefully. Indeed, this remainder term comes from the estimate of \( \mathbb{E}(\text{tr}\hat{E})/(b_p\sqrt{np}) \) in (6.50). To get a more precise estimate, we use the similar argument above for calculating the asymptotic expansion of \( \mathbb{E}(\text{tr}\hat{E})/(b_p\sqrt{np}) \).

\[
\frac{1}{\sqrt{npb_p}} \mathbb{E}(\text{tr}\hat{E}) = \mathbb{E}m_n - \sqrt{\frac{n}{p}} \frac{c_p}{b_p\sqrt{b_p}} (\mathbb{E}m_n)^2 + \frac{n}{p} \frac{d_p}{b_p^2} (\mathbb{E}m_n)^3 + o\left(\frac{1}{n}\right) + o\left(\frac{n}{p}\right). \quad (6.53)
\]

Plugging (6.53) into (6.48), we have

\[
\mathbb{E}m_n = -\frac{1}{z} - \frac{1}{z} \left[ \mathbb{E}m_n - \sqrt{\frac{n}{p}} \frac{c_p}{b_p\sqrt{b_p}} (\mathbb{E}m_n)^2 \right] \mathbb{E}m_n - \frac{1}{zn} \cdot \frac{\partial}{\partial z} \left[ \mathbb{E}m_n - \sqrt{\frac{n}{p}} \frac{c_p}{b_p\sqrt{b_p}} (\mathbb{E}m_n)^2 \right] - \frac{\nu_z - 3}{zn} \frac{b_p}{b_p}(\mathbb{E}m_n)^2 + o\left(\frac{1}{n}\right) + o\left(\frac{n}{p}\right),
\]

\[
= -\frac{1}{z} - \frac{1}{z} (\mathbb{E}m_n)^2 + \frac{1}{z} \sqrt{\frac{n}{p}} \frac{c_p}{b_p\sqrt{b_p}} (\mathbb{E}m_n)^3 - \frac{1}{z} \frac{d_p}{b_p^2} m^4
\]

\[
- \frac{1}{zn} \left[ \frac{m^2}{1 - m^2} + \frac{(\nu_z - 3)b_p}{b_p} m^2 \right] + o\left(\frac{1}{n}\right) + o\left(\frac{n}{p}\right).
\]

Multiplying \(-z\) on both sides, we have

\[
-z\mathbb{E}m_n = 1 + (\mathbb{E}m_n)^2 - \sqrt{\frac{n}{p}} \frac{c_p}{b_p\sqrt{b_p}} (\mathbb{E}m_n)^2 \cdot \mathbb{E}m_n + \frac{n}{p} \frac{d_p}{b_p^2} m^4
\]

\[
+ \frac{1}{n} \left[ \frac{m^2}{1 - m^2} + \frac{(\nu_z - 3)b_p}{b_p} m^2 \right] + o\left(\frac{1}{n}\right) + o\left(\frac{n}{p}\right). \quad (6.54)
\]

This implies that

\[
(\mathbb{E}m_n)^2 = -1 - z\mathbb{E}m_n + \sqrt{\frac{n}{p}} \frac{c_p}{b_p\sqrt{b_p}} (\mathbb{E}m_n)^3 + O\left(\frac{1}{n}\right) + O\left(\frac{n}{p}\right). \quad (6.55)
\]

Plugging (6.55) into (6.54) yields that

\[
-z\mathbb{E}m_n = 1 + (\mathbb{E}m_n)^2 + \sqrt{\frac{n}{p}} \frac{c_p}{b_p\sqrt{b_p}} (1 + z\mathbb{E}m_n) \mathbb{E}m_n - \frac{n}{p} \frac{c_p^2}{b_p^3} m^4
\]

\[
+ \frac{n}{p} \frac{d_p}{b_p^2} m^4 + \frac{1}{n} \left[ \frac{m^2}{1 - m^2} + \frac{(\nu_z - 3)b_p}{b_p} m^2 \right] + o\left(\frac{1}{n}\right) + o\left(\frac{n}{p}\right).
\]

This equation can be written as a quadratic equation of \( \mathbb{E}m_n - m \):

\[
0 = \left( m - \sqrt{\frac{n}{p}} \frac{c_p}{b_p\sqrt{b_p}} (1 + m^2) \right) (\mathbb{E}m_n - m)^2
\]

\[
+ \left[ m^2 - 1 - \left( \sqrt{\frac{n}{p}} \frac{c_p}{b_p\sqrt{b_p}} m (1 + 2m^2) \right) (\mathbb{E}m_n - m) \right] (\mathbb{E}m_n - m).\]
\[
+ \frac{m^3}{n} \left[ \frac{1}{1-m^2} + \frac{(\nu_4 - 3)b_p}{b_p} \right] - \left( \sqrt{\frac{n}{p} \frac{c_p}{b_p \sqrt{b_p}}} \right) m^4 + \frac{n}{p} \left( - \frac{c_p^2}{b_p^2} + \frac{d_p}{b_p^3} \right) m^5
+ o \left( \frac{1}{n} \right) + o \left( \frac{n}{p} \right)
\]

=: A(\mathbb{E} m_n - m)^2 + B(\mathbb{E} m_n - m) + C + o \left( \frac{1}{n} \right) + o \left( \frac{n}{p} \right).

Solving the equation yields two solutions:

\[
x_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A} + o \left( \frac{1}{n} \right) + o \left( \frac{n}{p} \right), \quad x_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A} + o \left( \frac{1}{n} \right) + o \left( \frac{n}{p} \right).
\]

When \(n^2/p = \mathcal{O}(1)\), from the definition of \(A, B, C\), we can verify that \(x_1 = o(1)\) while \(x_2 = (1 - m^2)/m^2 + o(1)\). Since \(\mathbb{E} m_n - m = o(1)\), we choose \(x_1\) to be the expression of \(\mathbb{E} m_n - m\), that is,

\[
\mathbb{E} m_n - m = \frac{-B + \sqrt{B^2 - 4AC}}{2A} + o \left( \frac{1}{n} \right) + o \left( \frac{n}{p} \right).
\]

This implies (6.32) under the assumption \(n^2/p = \mathcal{O}(1)\). \(\Box\)
Appendix A: Some Technical Lemmas

Lemma A.1 (Bai and Silverstein (2010b), Lemma B.26). Let $A = (a_{ij})$ be an $n \times n$ nonrandom matrix and $x = (X_1, \ldots, X_n)'$ be a random vector of independent entries. Assume that $E X_i = 0$, $E|X_i|^2 = 1$ and $E|X_i|^k \leq \nu_k$. Then, for any $k \geq 1$,
$$
E|x'Ax - \text{tr}A|^k \leq C_k \left( (\nu_1 \text{tr}(AA^*))^{k/2} + \nu_2 \text{tr}(AA^*)^{k/2} \right),
$$
where $C_k$ is a constant depending on $k$ only.

Lemma A.2 (Pan and Zhou (2011), Lemma 5). Let $A$ be a $p \times p$ deterministic complex matrix with zero diagonal elements. Let $x = (X_1, \ldots, X_p)'$ be a random vector of i.i.d. real entries. Assume that $E X_i = 0$, $E|X_i|^2 = 1$. Then, for any $k \geq 2$,
$$
E|x'Ax|^k \leq C_k \left( E|X_1|^k \right)^{k/2} (\text{tr}AA^*)^{k/2}, \quad (A.1)
$$
where $C_k$ is a constant depending on $k$ only.

Lemma A.3 (Burkholder’s inequality, Burkholder (1973)). Let $\{X_i\}$ be a complex martingale difference sequence with respect to the increasing $\sigma$-field $\{F_i\}$. Then for $k \geq 2$, the following inequality
$$
E \left| \sum_i X_i \right|^k \leq C_k E \left( \sum_i E(\left| X_i \right|^2 | F_{i-1}) \right)^{k/2} + C_k E \sum_i |X_i|^k
$$
holds, where $C_k$ is a constant depending on $k$ only.

Lemma A.4 (Martingale CLT, Billingsley (2008)). Suppose for each $n$, $Y_{n1}, Y_{n2}, \ldots, Y_{nr_n}$ is a real martingale difference sequence with respect to the $\sigma$-field $\{F_{nj}\}$ having second moments. If as $n \to \infty$,
$$
\sum_{j=1}^{r_n} E(Y_{nj}^2 | F_{n,j-1}) \overset{p}{\to} \sigma^2, \quad (A.2)
$$
where $\sigma^2$ is a positive constant, and for each $\varepsilon > 0$,
$$
\sum_{j=1}^{r_n} E(Y_{nj}^2 \mathbf{1}_{\{|Y_{nj}| \geq \varepsilon\}}) \to 0, \quad (A.3)
$$
then
$$
\sum_{j=1}^{r_n} Y_{nj} \overset{d}{\to} \mathcal{N}(0, \sigma^2).
$$

Lemma A.5 (Billingsley (1968), Theorem 12.3). The sequence $\{X_n\}$ is tight if it satisfies these two conditions:

(i) The sequence $\{X_n(0)\}$ is tight.

(ii) There exist constants $\gamma \geq 0$ and $\alpha > 1$ and a non-decreasing, continuous function $F$ on $[0, 1]$ such that
$$
P\left( |X_n(t_2) - X_n(t_1)| \geq \lambda \right) \leq \frac{1}{\lambda^\gamma} |F(t_2) - F(t_1)|^\alpha
$$
holds for all $t_1, t_2$ and $n$ and all positive $\lambda$. 
Lemma A.6. For $z_1, z_2 \in \mathbb{C}^+$,
\[
Z_k := \frac{1}{n(pb_p)^2} \text{tr}\left(\mathbb{E}_k M_k^{(1)}(z_1) \cdot \mathbb{E}_k M_k^{(1)}(z_2)\right) = \frac{k m(z_1)m(z_2)}{1 - \frac{k}{n} m(z_1)m(z_2)} + o_{L_1}(1).
\]

This lemma is used in Section 6.3 to derive the finite dimensional convergence of $M_n^{(1)}(z)$.

Proof. Let $\{e_i, i = 1, \ldots, k - 1, k + 1, \ldots, n\}$ be the $(n - 1)$-dimensional unit vectors with the $i$-th (or $(i - 1)$-th) element equal to 1 and the remaining equal to 0 according as $i < k$ (or $i > k$). Write $X_k = X_{k_1} + x_i e_i$. Let
\[
D_{ki,r}^{-1} = D_k^{-1} - e_i h'_i = \frac{1}{\sqrt{n}p b_p} \left( X'_{ki} \Sigma_p X_k - p a_p I_i \right) - z I_{n-1},
\]
\[
D_{k}^{-1} = D_{k}^{-1} - e_i h'_i - r_i e_i' = \frac{1}{\sqrt{n}p b_p} \left( X'_{ki} \Sigma_p X_k - p a_p I_i \right) - z I_{n-1},
\]
\[
h'_i = \frac{1}{\sqrt{n}p b_p} x'_i \Sigma_p x_{k_i} + \frac{1}{\sqrt{n}p b_p} x'_i \Sigma_p x_i - p a_p e_i, \quad r_i = \frac{1}{\sqrt{n}p b_p} X'_{ki} \Sigma_p x_i,
\]
\[
\zeta_i = \frac{1}{1 + \vartheta_i}, \quad \vartheta_i = h'_i D_{ki,r}(z)e_i, \quad M_{ki} = \Sigma_p x_{ki} D_{ki}(z)X_{k_i}' e_i
\]
We have some crucial identities,
\[
X_{ki} e_i = 0, \quad e_i' D_{ki,r} = e_i' D_k = -\frac{e_i'}{z}
\]
where 0 is a $p$-dimensional vector with all the elements equal to 0. By using (A.4) and some frequently used formulas about the inverse of matrices, we have two useful identitites,
\[
D_k - D_{ki,r} = -D_{ki,r}(D_k^{-1} - D_{ki,r}^{-1})D_k = -D_{ki,r}(e_i h'_i)^2 D_k
\]
\[
= -D_{ki,r}(e_i h'_i)(\zeta_i D_{ki,r}) = -\zeta_i D_{ki,r}(e_i h'_i)D_{ki,r}
\]
and
\[
D_{ki,r} - D_{ki} = -(D_{ki,r}^{-1} - D_{ki}^{-1})D_{ki,r} = -D_{ki}(r_i e_i'^2)D_{ki,r}
\]
\[
= -D_{ki} \left( \frac{1}{\sqrt{n}p b_p} X'_{ki} \Sigma_p x_i e_i' \right) D_{ki} = \frac{1}{z \sqrt{n}p b_p} D_{ki} X'_{ki} \Sigma_p x_i e_i'.
\]

Using (A.5) and (A.6), for $i < k$, we obtain the following decomposition $\mathbb{E}_k M_k^{(1)}(z)$,
\[
\mathbb{E}_k M_k^{(1)}(z) = \mathbb{E}_k \left( \Sigma_p (X_{ki} + x_i e_i') D_k (X_{ki} + x_i e_i') \Sigma_p \right)
\]
\[
= \mathbb{E}_k \left( \Sigma_p x_{ki} D_k X'_{ki} \Sigma_p + \Sigma_p x_{ki} D_k x_i x_i' \Sigma_p + \Sigma_p x_i e_i'^2 D_k x_i x_i' \Sigma_p + \Sigma_p x_i e_i'^2 D_k e_i x_i' \Sigma_p \right)
\]
\[
= \mathbb{E}_k M_{ki} - \mathbb{E}_k \left( \frac{\zeta_i(z)}{z \sqrt{n}p b_p} M_{ki} x_i x_i' \Sigma_p \right) + \mathbb{E}_k \left( \frac{\zeta_i(z)}{z \sqrt{n}p b_p} M_{ki} \right) x_i x_i' \Sigma_p
\]
\[
+ \Sigma_p x_i x_i' \mathbb{E}_k \left( \frac{\zeta_i(z)}{z \sqrt{n}p b_p} M_{ki} \right) - \mathbb{E}_k \left( \frac{\zeta_i(z)}{z} \right) \Sigma_p x_i x_i' \Sigma_p.
\]
\[ : B_1(z) + B_2(z) + B_3(z) + B_4(z) + B_5(z). \]

Write
\[
D_k^{-1} = \sum_{i=1(\neq k)}^n e_i h'_i - zI_{n-1}.
\]

Multiplying \( D_k \) on the right-hand side, we have
\[
zD_k = -I_{n-1} + \sum_{i=1(\neq k)}^n e_i h'_i D_k. \]

Multiplying \( \Sigma_p X_k \) on the left-hand side, \( X'_k \Sigma_p \) on the right-hand side, we get
\[
zM_k^{(1)}(z) = -\Sigma_p X_k X'_k \Sigma_p + \sum_{i=1(\neq k)}^n \Sigma_p X_k e_i h'_i D_k X'_k \Sigma_p.
\]

Thus,
\[
z\mathbb{E}_k(M_k^{(1)}(z)) = -\mathbb{E}_k(\Sigma_p X_k X'_k \Sigma_p) + \sum_{i=1(\neq k)}^n \mathbb{E}_k(\Sigma_p X_k e_i h'_i D_k X'_k \Sigma_p)
\]
\[
= -\Sigma_p \mathbb{E}_k \left( \sum_{i=1(\neq k)}^n x_i x'_i \right) \Sigma_p + \sum_{i=1(\neq k)}^n \mathbb{E}_k \left( \zeta_i \Sigma_p x_i h'_i D_{k_i, r} (X'_{k_i} + e_i x'_i) \Sigma_p \right)
\]
\[
= -(n - k) \Sigma_p^2 - \sum_{i<k} (\Sigma_p x_i x'_i \Sigma_p) + \sum_{i=1(\neq k)}^n \mathbb{E}_k \left( \zeta_i \Sigma_p x_i h'_i D_{k_i, r} e_i x'_i \Sigma_p \right)
\]
\[
= -(n - k) \Sigma_p^2 - \sum_{i<k} (\Sigma_p x_i x'_i \Sigma_p) + \sum_{i=1(\neq k)}^n \mathbb{E}_k \left( \zeta_i \Sigma_p x_i x'_i \Sigma_p \right)
\]
\[
+ \sum_{i=1(\neq k)}^n \mathbb{E}_k \left( \zeta_i \theta_i \Sigma_p x_i x'_i \Sigma_p \right), \quad \text{(A.8)}
\]

Applying (A.7) and (A.8) to \( \mathbb{E}_k M_k^{(1)}(z_1) \) (for \( i < k \)) and \( z_1 \mathbb{E}_k M_k^{(1)}(z_1) \), we get the following decomposition:

\[
z_1 Z_k = \frac{z_1}{n(npb_p)^2} \text{tr} \left( \mathbb{E}_k M_k^{(1)}(z_1) \cdot \mathbb{E}_k M_k^{(1)}(z_2) \right)
\]
\[
= \frac{1}{n(npb_p)^2} \text{tr} \left\{ \left[ -(n - k) \Sigma_p^2 - \sum_{i<k} (\Sigma_p x_i x'_i \Sigma_p) + \sum_{i=1(\neq k)}^n \mathbb{E}_k \left( \zeta_i \Sigma_p x_i x'_i \Sigma_p \right) \right] \times \mathbb{E}_k M_k^{(1)}(z_2) \right\}
\]
\[
= C_1(z_1, z_2) + C_2(z_1, z_2) + C_3(z_1, z_2) + C_4(z_1, z_2), \quad \text{(A.9)}
\]
where

\[ C_1(z_1, z_2) = -\frac{n-k}{n(pb)^2} \text{tr} \left( \Sigma_p^2 \cdot \mathbb{E}_k M_k^{(1)}(z_2) \right), \]

\[ C_2(z_1, z_2) = -\frac{1}{n(pb)^2} \sum_{i<k} x'_i \Sigma_p \left( \sum_{j=1}^5 B_j(z_2) \right) \Sigma_p x_i = \sum_{j=1}^5 C_{2j}, \quad (A.10) \]

\[ C_3(z_1, z_2) = \frac{1}{n(pb)^2} \sum_{i<k} \mathbb{E}_k \left[ \frac{\zeta_i(z_1)}{\sqrt{npb}} x'_i M_{ki}(z_1) \left( \sum_{j=1}^5 B_j(z_2) \right) \Sigma_p x_i \right] \]

\[ + \frac{1}{n(pb)^2} \sum_{i>k} \mathbb{E}_k \left[ \frac{\zeta_i(z_1)}{\sqrt{npb}} x'_i M_{ki}(z_1) \left( \mathbb{E}_k M_k^{(1)}(z_2) \right) \Sigma_p x_i \right] = \sum_{j=1}^6 C_{3j}, \quad (A.11) \]

\[ C_4(z_1, z_2) = \frac{1}{n(pb)^2} \sum_{i<k} \mathbb{E}_k \left[ \zeta_i(z_1) \vartheta_i(z_1) x'_i \Sigma_p \left( \sum_{j=1}^5 B_j(z_2) \right) \Sigma_p x_i \right] \]

\[ + \frac{1}{n(pb)^2} \sum_{i>k} \mathbb{E}_k \left[ \zeta_i(z_1) \vartheta_i(z_1) x'_i \Sigma_p \left( \mathbb{E}_k M_k^{(1)}(z_2) \right) \Sigma_p x_i \right] = \sum_{j=1}^6 C_{4j}. \quad (A.12) \]

Now we estimate all the terms in (A.9). We will show that these terms are negligible as \( n \to \infty \), expect \( C_{25}, C_{33}, C_{45} \) defined in (A.10) \sim (A.12). Before proceeding, we provide two useful lemmas.

For \( C_1(z_1, z_2) \), we have

\[ \mathbb{E} | C_1(z_1, z_2) | = \frac{n-k}{n(pb)^2} \left| \text{tr} \left( \Sigma_p^2 \cdot \mathbb{E}_k M_k^{(1)}(z_2) \right) \right| = O \left( \frac{1}{np^2} \right) \cdot O(np) = O \left( \frac{n}{p} \right), \]

where the second equality follows from the fact \( \left| \text{tr} \left( \Sigma_p^2 \cdot \mathbb{E}_k M_k^{(1)}(z_2) \right) \right| = O(np) \), which can be verified by using the similar argument in the proof of Lemma 6.1.

Applying Lemma A.7 and inequality (A.14) with \( B = I_p \), we have

\[ \mathbb{E} | C_{21} | \leq \frac{1}{n(pb)^2} \sum_{i<k} \mathbb{E} \left| x'_i \Sigma_p \cdot \mathbb{E}_k M_{ki}(z_2) \cdot \Sigma_p x_i \right| \]

\[ \leq \frac{1}{n(pb)^2} \sum_{i<k} \left( \mathbb{E} \left| x'_i \Sigma_p \cdot \mathbb{E}_k M_{ki}(z_2) \cdot \Sigma_p x_i \right|^2 \right)^{1/2} \leq \frac{Kn}{p}. \]

Applying Lemma A.7 and inequality (A.14) with \( B = \Sigma_p \), we have

\[ \mathbb{E} | C_{22} | \leq \frac{1}{n(pb)^2} \sum_{i<k} \mathbb{E} \left| x'_i \Sigma_p \cdot \mathbb{E}_k \left( \frac{\zeta_i(z_2)}{z_2 npb} M_{ki}(z_2) x_i x'_i M_{ki}(z_2) \right) \cdot \Sigma_p x_i \right| \]

\[ = \frac{K}{n^2(pb)^3} \sum_{i<k} \mathbb{E} \left| x'_i M_{ki}(z_2) \Sigma_p x_i \right|^2 \leq \frac{Kn}{p}. \]

Similarly, we obtain

\[ \mathbb{E} | C_{23} | = \mathbb{E} | C_{24} | \leq \frac{1}{n(pb)^2} \sum_{i<k} \mathbb{E} \left| x'_i \Sigma_p \cdot \mathbb{E}_k \left( \frac{\zeta_i(z_2)}{z_2 \sqrt{npb}} M_{ki}(z_2) \right) x_i x'_i \Sigma_p \cdot \Sigma_p x_i \right| \]
Applying Lemma A.7 and inequality (A.13) with $B = E_k \Sigma_k$, we have

$$E|C_{31}| = \frac{1}{n(p b_p)^2} \sum_{i < k} \left| E_k \left[ \frac{\zeta(z)\sqrt{np}}{\sqrt{n p b_p}} x_i^2 \Sigma_k \cdot E_k \left( \zeta(z)\sqrt{2n p b_p} M_k \right) x_i \right] \right|$$

$$\leq K \sqrt{\frac{np}{p}} \sum_{i < k} \left| E_k \left[ x_i^2 \Sigma_k \cdot E_k \left( \zeta(z)\sqrt{2n p b_p} M_k \right) x_i \right] \right| \leq K \sqrt{\frac{np}{p}}.$$  

Define $\tilde{\zeta}_i$ and $\tilde{\Sigma}_{k_i}$, the analogues of $\zeta_i(z)$ and $\Sigma_k(z)$ respectively, by $(x_1, \ldots, x_k, \bar{x}_{k+1}, \ldots, \bar{x}_n)$, where $\bar{x}_{k+1}, \ldots, \bar{x}_n$ are i.i.d. copies of $x_{k+1}, \ldots, x_n$ and independent of $x_1, \ldots, x_k$. Then,

$$E|C_{32}| = \frac{1}{n(p b_p)^2} \sum_{i < k} \left| E_k \left[ \frac{\zeta(z)\sqrt{np}}{\sqrt{n p b_p}} x_i^2 \Sigma_k \cdot E_k \left( \tilde{\zeta}(z)\sqrt{2n p b_p} \tilde{\Sigma}_{k} \right) x_i \right] \right|$$

$$= \frac{1}{n(p b_p)^2} \sum_{i < k} \left| E_k \left[ \frac{\zeta(z)\sqrt{np}}{\sqrt{n p b_p}} x_i^2 \Sigma_k \cdot E_k \left( \tilde{\zeta}(z)\sqrt{2n p b_p} \tilde{\Sigma}_{k} \right) x_i \right] \right|$$

$$\leq K \sqrt{\frac{np}{p}} \sum_{i < k} \left| E_k \left[ x_i^2 \Sigma_k \cdot \tilde{\Sigma}_{k} \Sigma x_i \right] \right|$$

$$\leq K \sqrt{\frac{np}{p}} \sum_{i < k} \left( E_k \left[ x_i^2 \Sigma_k \cdot \tilde{\Sigma}_{k} \Sigma x_i \right] \right) \leq K \sqrt{\frac{np}{p}}.$$  

Similarly, we have

$$E|C_{3j}| \leq K \frac{n}{p}, \quad j = 4, 5, 6.$$  

Applying Lemma A.7 and inequality (A.14) with $B = I_{n-1}$, we obtain

$$E|C_{4j}| \leq K \frac{n}{p}, \quad j = 1, 2, 3, 4, 6.$$  

Moreover, by using Lemma 6.1, Lemma A.7 and Lemma A.8, we obtain the following limits:

$$C_{25} = -\frac{1}{n(p b_p)^2} \sum_{i < k} \left\{ x_i^2 \Sigma \left[ -E_k \left( \frac{\zeta(z)\sqrt{np}}{\sqrt{np b_p}} \Sigma x_i x_i^2 \Sigma \right) \Sigma x_i \right] \right\}$$

$$= -\frac{1}{n(p b_p)^2} m(z_2) \sum_{i < k} \left( x_i^2 \Sigma_i^2 \right)$$

$$= -\frac{k}{n} m(z_2) + o_L(1),$$

$$C_{45} = \frac{1}{n(p b_p)^2} \sum_{i < k} E_k \left[ \zeta(z) \theta(z) x_i^2 \Sigma \left[ -E_k \left( \frac{\zeta(z)\sqrt{np}}{\sqrt{np b_p}} \Sigma x_i x_i^2 \Sigma \right) \Sigma x_i \right] \right]$$
Proof. This lemma can be proved by using the similar arguments in Section 5.2.2 of Chen and Pan (2015).

\textbf{Lemma A.7.} For $\vartheta_i(z)$ and $\zeta_i(z)$ defined in Lemma A.6, we have

$$E \left| \vartheta_i(z) - \frac{m(z)}{z} \right|^4 \rightarrow 0, \quad E \left| \zeta_i(z) + zm(z) \right|^4 \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty.$$  

Proof. This lemma can be proved by using the similar arguments in Section 5.2.2 of Chen and Pan (2015).

\textbf{Lemma A.8.} Let $\mathbf{B}$ be any matrix independent of $\mathbf{x}_i$.

\begin{align}
E \left| \mathbf{x}_i' \mathbf{M}_{ki} \mathbf{B} \mathbf{x}_i \right|^2 &\leq K p^2 n^2 E \| \mathbf{B} \|^2, \\
E \left| \mathbf{x}_i' \Sigma_p \mathbf{M}_{ki} \mathbf{B} \Sigma_p \mathbf{x}_i \right|^2 &\leq K p^2 n^2 E \| \mathbf{B} \|^2.
\end{align}

Proof. Note that $\mathbf{M}_{ki}$ and $\mathbf{x}_i$ are independent. By using Lemma A.1, we have

\begin{align}
E \left| \mathbf{x}_i' \mathbf{M}_{ki} \mathbf{B} \mathbf{x}_i - \text{tr} \mathbf{M}_{ki} \mathbf{B} \right|^2 &\leq K \left( \nu_4 \text{tr} (\mathbf{M}_{ki} \mathbf{B} \mathbf{B} \mathbf{M}_{ki}) \right) \leq K n p^2 \| \mathbf{B} \|^2,
\end{align}

where we use the fact that

\begin{align*}
\left| \text{tr} (\mathbf{M}_{ki} \mathbf{B} \mathbf{B} \mathbf{M}_{ki}) \right| &= \left| \text{tr} (\Sigma_p \mathbf{x}_{ki} ^\prime \mathbf{D}_{ki} \mathbf{x}_{ki} ^\prime \Sigma_p \mathbf{B} \mathbf{B} \Sigma_p \mathbf{x}_{ki} \mathbf{D}_{ki} \mathbf{x}_{ki} ^\prime \Sigma_p) \right| \\
&= \left| \text{tr} (\mathbf{D}_{ki}^{1/2} \mathbf{x}_{ki} ^\prime \Sigma_p \mathbf{B} \mathbf{B} \Sigma_p \mathbf{x}_{ki} \mathbf{D}_{ki}^{1/2}) \right| \\
&\leq n \cdot \| \mathbf{D}_{ki}^{1/2} \mathbf{x}_{ki} ^\prime \Sigma_p^{1/2} \| \cdot \| \Sigma_p^{1/2} \| \cdot \| \mathbf{B} \mathbf{B} \| \cdot \| \Sigma_p \| \cdot \| \Sigma_p^{1/2} \mathbf{x}_{ki} \mathbf{D}_{ki}^{1/2} \|
\end{align*}
\[ n \cdot \| \Sigma_p \|^2 \cdot \| B \|^2 \cdot \| \Sigma_p^{1/2} X_{ki} D_{ki} X'_{ki} \Sigma_p^{1/2} \|^2 \]
\[ = n \cdot \| \Sigma_p \|^2 \cdot \| B \|^2 \cdot \| D_{ki} X'_{ki} \Sigma_p X_{ki} \|^2 \]
\[ = n \cdot \| \Sigma_p \|^2 \cdot \| B \|^2 \cdot \| \sqrt{npb_p(I_{n-1} + zD_{ki}) + pa_pI_{(i)}D_{ki}} \|^2 \]
\[ \leq Kn^p \| B \|^2. \quad (A.16) \]

By (A.15) and the \( c_r \)-inequality, we have
\[ \mathbb{E}|x'_i M_{ki} B x_i|^2 \leq K \left( \mathbb{E}|x'_i M_{ki} B - trM_{ki} B|^2 + \mathbb{E}|trM_{ki} B|^2 \right) \leq Kp^2n^2 \| B \|^2, \]
which completes the proof of (A.13). By using the same argument, we get (A.14). \( \square \)

Lemma A.7 and A.8 are used in the proof of Lemma A.6.
Recall that \( C_1 = \{ z : z = u + iv, u \in [-u_1, u_1], |v| \geq v_1 \} \), where \( u_1 > 2 \) and \( 0 < v_1 \leq 1 \).

**Lemma A.9.** For \( z \in C_1 \), we have
\[ |\beta_k(z)| \leq 1/v_1, \quad |\beta_k(z)| \leq 1/v_1. \]
\[ 1 + \frac{1}{npb_p} \text{tr} M_k^{(s)}(z) \leq 1 + \frac{1}{v_1^s}, \quad s = 1, 2, \]
\[ \left| \beta_k \left( 1 + q_k^i D_k^2(z)q_k \right) \right| \leq \frac{1}{v_1}. \quad (A.17) \]

Proof. The proof exactly follows Chen and Pan (2015), so is omitted. \( \square \)

**Lemma A.10.** Under the assumption \( p \land n \to \infty, p/n \to \infty \) and truncation, for \( z \in C_1 \),
\[ \mathbb{E}|\gamma_{ks}|^2 \leq \frac{K}{n}, \]
\[ \mathbb{E}|\gamma_{ks}|^4 \leq K \left( \frac{1}{n^2} + \frac{n}{p^2} \right), \]
\[ \mathbb{E}|\eta_k|^2 \leq \frac{K}{n}, \]
\[ \mathbb{E}|\eta_k|^4 \leq K \frac{\delta_n^4}{n} + K \left( \frac{1}{n^2} + \frac{n}{p^2} \right). \]

Proof. By Lemma A.1 and taking \( B = I_p \) in the inequality (A.16), we have
\[ \mathbb{E}|\gamma_{ks}|^2 \leq \frac{K}{n^2p^2} \text{tr}(M_k^{(s)} \overline{M_k^{(s)}}) \leq \frac{K}{n}. \]
Similarly, we can prove that \( \mathbb{E}|\eta_k|^2 \leq K/n \).

Now, we prove the bounds for the 4-th moments of \( \gamma_{ks} \) and \( \eta_{ks} \). Let \( H \) be \( M_k^{(s)} \) with all diagonal elements replaced by zeros, then we have
\[ \mathbb{E}|x'_k H x_k|^4 \leq K (\mathbb{E}X_{11}^4)^2 (\text{tr} HH^*)^2 \leq K \mathbb{E}(\text{tr} M_k^{(s)} \overline{M_k^{(s)}})^2 \leq Kn^2p^4. \quad (A.22) \]
The first inequality follows from Lemma A.2, and the last inequality follows from (A.16). Denote \( E_j(\cdot) \) be the conditional expectation with respect to \( (X_{1k}, X_{2k}, \ldots, X_{jk}) \), where \( j = \)
Since \( E_{j-1}(X_{jk}^2 - 1)a_{jj}^{(s)} = 0 \), then \((X_{jk}^2 - 1)a_{jj}^{(s)}\) can be expressed as a martingale difference

\[
(X_{jk}^2 - 1)a_{jj}^{(s)} = (E_j - E_{j-1})[(X_{jk}^2 - 1)a_{jj}^{(s)}].
\] (A.23)

Applying the Burkholder’s inequality (Lemma A.3) to (A.23) yields that

\[
E\left|\sum_{j=1}^{p} (X_{jk}^2 - 1)a_{jj}^{(s)}\right|^4 \leq K E\left(\sum_{j=1}^{p} |E_j - E_{j-1}|(X_{jk}^2 - 1)a_{jj}^{(s)}|^2\right)^2 + KE\left(\sum_{j=1}^{p} |(X_{jk}^2 - 1)a_{jj}^{(s)}|^4\right)
\]

\[
\leq K\left(\sum_{j=1}^{p} E[X_{j1}]^4|a_{jj}^{(s)}|^2\right) + K\sum_{j=1}^{p} E|X_{j1}|^8|a_{jj}^{(s)}|^4
\]

\[
\leq Kn^5p^2 + Kn^3p^3,
\] (A.24)

where we use the fact that, with \( e_j \) be the \( j \)-th \( p \)-dimensional standard basis vector and \( y \) be an \((n-1)\)-dimensional random vector with \( E y_i = 0 \) and \( E y_i^2 = 1 \),

\[
E|a_{jj}^{(s)}|^4 = E|e_j^T \Sigma_p X_k \mathbf{D} \Sigma_p e_j|^4
\]

\[
\leq v_1^{-4}\|e_j^T \Sigma_p X_k \mathbf{D} \Sigma_p e_j\|^4 = v_1^{-4}\bar{\sigma}_{jj}^4 \|y\|^8 \leq Kn^4 + Kn^2p,
\] (A.25)

where \( \bar{\sigma}_{jj} = \sum \sigma_{jj}^2 \) is the \( j \)-th diagonal elements of \( \Sigma_p^2 \). By Rayleigh-Ritz Theorem, we know that \( \bar{\sigma}_{jj} \leq \lambda_{\text{max}}(\Sigma_p^2) \leq K \). Combining (A.22) and (A.24) yields that

\[
E|\gamma_{kj}|^4 \leq \frac{1}{(np^2)^4} E\left|\sum_{j=1}^{p} (X_{jk}^2 - 1)a_{jj}^{(s)} + x_k^T H x_k\right|^4
\]

\[
\leq \frac{K}{n^4p^4} E\left|\sum_{j=1}^{p} (X_{jk}^2 - 1)a_{jj}^{(s)}\right|^4 + \frac{K}{n^4p^4} E|x_k^T H x_k|^4
\]

\[
\leq K\left(\frac{1}{n^2} + \frac{n}{p^2}\right).
\]

Moreover, by Lemma A.1, we have

\[
E|\eta_k|^4 \leq \frac{K}{n^2p^2} E|x_k^T \Sigma_p x_k - pa_p|^4 + K E|\gamma_{k1}|^4 \leq \frac{K \delta_k^4}{n} + K\left(\frac{1}{n^2} + \frac{n}{p^2}\right).
\]

Lemmas (A.11), (A.12) and (A.13) below are used in Section 6.5 to derive the convergence of the non-random part \( M_n^2(z) \). We will prove them following the strategy in Bao (2015).

**Lemma A.11.** Under the assumption \( p \land n \to \infty, p/n \to \infty \), for \( z \in \mathbb{C}_1 \), we have

\[
\text{Var}(m_n) = O\left(\frac{1}{n^2}\right).
\] (A.26)

**Proof.** By the identity \( m_n - EM_n = -\sum_{k=1}^{n} (E_{k-1} m_n - E_k m_n) \), we have

\[
\text{Var}(m_n) = \sum_{k=1}^{n} E \left|\sum_{k=1}^{n} \left(E_{k-1} m_n - E_k m_n\right)^2 + 2 \sum_{1 \leq s < t \leq 1} E \left(E_{s-1} m_n - E_s m_n \right)\left(E_{t-1} m_n - E_t m_n\right)\right.
\]

\[
\left. - \sum_{1 \leq s < t \leq 1} E \left(E_{s-1} m_n - E_s m_n \right)\left(E_{t-1} m_n - E_t m_n\right)\right|.
\]

\[
\text{Var}(m_n) = O\left(\frac{1}{n^2}\right).
\]
Since each term in the second sum on the RHS of the above identity is zero, we write

\[
\begin{align*}
\Var(m_n) &= \sum_{k=1}^{n} \E \left| \E_{k-1} m_n - \E_k m_n \right|^2 \\
&= \sum_{k=1}^{n} \E \left| \E_{k-1} \left( m_n - \E_{(k)} m_n \right) \right|^2 \\
&\leq \sum_{k=1}^{n} \E \left| m_n - \E_{(k)} m_n \right|^2,
\end{align*}
\]

where \( \E_{(k)}(\cdot) \) denotes the expectation w.r.t. the \( \sigma \)-field generated by \( x_k \). To prove (A.26), it suffices to show

\[
\E \left| m_n - \E_{(k)} m_n \right|^2 = O \left( \frac{1}{n^3} \right), \quad 1 \leq k \leq n.
\]

(A.27)

Now we deal with the case \( k = 1 \), and the remaining cases are analogous and omitted.

Denote \( \tilde{Y} = (\tilde{Y}_{ij})_{p \times n} := \Sigma^{1/2} \tilde{Y} \) where \( \tilde{Y} = (nwp_{b})^{-1/4} \tilde{x} \), and let \( \tilde{y}_k \) be the \( k \)-th column of \( \tilde{Y} \). Let \( \tilde{Y}_k \) be the \( p \times (n-1) \) matrix extracted from \( \tilde{Y} \) by removing \( \tilde{y}_k \), then the matrix model (1.1) can be written as

\[
A_n = \begin{pmatrix}
\tilde{y}_1' \tilde{y}_1 - \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}} & (\tilde{y}_1' \tilde{y}_1)' \\
\tilde{y}_1' \tilde{y}_1 & \tilde{y}_1' \tilde{y}_1 - \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}} \text{I}_{n-1}
\end{pmatrix}.
\]

With notations \( A_k = \tilde{y}_k' \tilde{y}_k - \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}} \text{I}_{n-1} \) and \( D_k = (A_k - z \text{I}_n)^{-1} \), we have

\[
\begin{align*}
\text{trD} - \text{trD}_1 &= 1 + \tilde{y}_1' \left( \tilde{y}_1' \tilde{y}_1 - \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}} \text{I}_{n-1} - z \text{I}_{n-1} \right)^{-2} \tilde{y}_1 \\
&= \frac{1 + \tilde{y}_1' \left( \tilde{y}_1' \tilde{y}_1 - \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}} \text{I}_{n-1} - z \text{I}_{n-1} \right)^{-1} \tilde{y}_1}{\tilde{y}_1' \tilde{y}_1 - \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}} \text{I}_{n-1} - z \text{I}_{n-1}} \tilde{y}_1 \\
&=: \frac{1 + U}{V},
\end{align*}
\]

where the second “=“ comes from the identity

\[
B(AB - \alpha \text{I})^{-n} A = B A (BA - \alpha \text{I})^{-n}.
\]

Moreover, with notations \( U \) and \( V \), we can write \( D_{11} = 1/V \) and

\[
\E \left| m_n - \E_{(1)} m_n \right|^2 = \frac{1}{n^2} \E \left| (\text{trD} - \text{trD}_1) - \E_{(1)}(\text{trD} - \text{trD}_1) \right|^2 \quad (\because \E_{(1)} \text{trD}_1 = \text{trD}_1)
\]
\[
\begin{align*}
&= \frac{1}{n^2} \mathbb{E} \left| \frac{1+U}{V} - \mathbb{E}_1 \left( \frac{1+U}{V} \right) \right|^2 \\
&\leq \frac{2}{n^2} \left\{ \mathbb{E} \left( \frac{1}{V} - \mathbb{E}_1 \left( \frac{1}{V} \right) \right)^2 + \mathbb{E} \left( \frac{U}{V} - \mathbb{E}_1 \left( \frac{U}{V} \right) \right)^2 \right\}.
\end{align*}
\]

By the same arguments as those on Page 196 of Bao (2015), it is sufficient to prove that
\[
\mathbb{E}_1 |U - \mathbb{E}_1 U|^2 = O \left( \frac{1}{n} \right), \quad \mathbb{E}_1 |V - \mathbb{E}_1 V|^2 = O \left( \frac{1}{n} \right). \tag{A.28}
\]

For simplicity of presentation, we define
\[
\mathbf{H}^{[\ell]} = \left( \mathbf{H}^{[\ell]}_{jk} \right)_{p \times p} := \mathbf{Y}_1 \mathbf{Y}_1' - \frac{p}{n} \frac{a_p}{b_p} \mathbf{I}_{n-1} - z \mathbf{I}_{n-1} \right)^{-\ell}, \quad \ell = 1, 2.
\]

Then, we write
\[
\begin{align*}
U - \mathbb{E}_1 U &= \sum_{i \neq j} H_{ij}^{[2]} \tilde{Y}_i \tilde{Y}_j + \sum_{i=1}^{p} H_{ii}^{[2]} \left( \tilde{Y}_i^2 - \mathbb{E} \tilde{Y}_i^2 \right), \\
V - \mathbb{E}_1 V &= \tilde{Y}_1 \tilde{Y}_1 - \frac{p}{n} \frac{a_p}{b_p} - \sum_{i \neq j} H_{ij}^{[1]} \tilde{Y}_i \tilde{Y}_j - \sum_{i=1}^{p} H_{ii}^{[1]} \left( \tilde{Y}_i^2 - \mathbb{E} \tilde{Y}_i^2 \right). \tag{A.29}
\end{align*}
\]

Now we proceed to prove (A.28). From (A.30), we have
\[
\begin{align*}
\mathbb{E}_1 |V - \mathbb{E}_1 V|^2 \\
&\leq K \left\{ \mathbb{E}_1 \left| \tilde{Y}_1 \tilde{Y}_1' - \frac{p}{n} \frac{a_p}{b_p} \right|^2 + \mathbb{E}_1 \left| \sum_{i \neq j} H_{ij}^{[1]} \tilde{Y}_i \tilde{Y}_j \right|^2 + \mathbb{E}_1 \left| \sum_{i=1}^{p} H_{ii}^{[1]} \left( \tilde{Y}_i^2 - \mathbb{E} \tilde{Y}_i^2 \right) \right|^2 \right\} \\
&= K \left\{ \mathbb{E} \left| \tilde{Y}_1 \tilde{Y}_1' - \frac{p}{n} \frac{a_p}{b_p} \right|^2 + \sum_{i \neq j} \left| H_{ij}^{[1]} \right|^2 \mathbb{E} \left( \tilde{Y}_i^2 \tilde{Y}_j^2 \right) + \sum_{i=1}^{p} \left| H_{ii}^{[1]} \right|^2 \mathbb{E} \left( \tilde{Y}_i^2 - \mathbb{E} \tilde{Y}_i^2 \right)^2 \right\}. \tag{A.31}
\end{align*}
\]

After some straightforward calculations, we obtain some estimates:
\[
\mathbb{E} \tilde{Y}_i^2 = O \left( \frac{1}{np} \right), \quad \mathbb{E} \tilde{Y}_i^4 = O \left( \frac{1}{np} \right), \quad \mathbb{E} \left( \tilde{Y}_i \tilde{Y}_1' - \sqrt{\frac{p}{n}} \frac{a_p}{b_p} \right)^2 = O \left( \frac{1}{n} \right). \tag{A.32}
\]

Combining (A.32) and (A.31), we obtain
\[
\mathbb{E}_1 |V - \mathbb{E}_1 V|^2 \leq \frac{K}{n} + \frac{K}{np} \text{tr} |\mathbf{H}^{[1]}|^2. \tag{A.33}
\]

Similarly, we can show that
\[
\mathbb{E}_1 |U - \mathbb{E}_1 U|^2 \leq \frac{K}{np} \text{tr} |\mathbf{H}^{[2]}|^2. \tag{A.34}
\]

To get (A.28), it suffices to show that
\[
\text{tr} |\mathbf{H}^{[\ell]}|^2 = O(p), \quad \ell = 1, 2.
\]
Let \( \{ \mu_i^{(k)} , i = 1, 2, \ldots, n - 1 \} \) be eigenvalues of \( A_k \), then the eigenvalues of \( H^{[\ell]} ( \ell = 1, 2 ) \) are
\[
\frac{\left( \mu_i^{(1)} + ap \sqrt{p/(nb_p)} \right)^2}{|\mu_i^{(1)} - z|^{2\ell}}, \quad i = 1, 2, \ldots, n - 1,
\]
and a zero eigenvalue with algebraic multiplicity \((p-n+1)\). Using the fact \( \mu_1^{(1)} > -ap \sqrt{p/(nb_p)} \), we conclude that
\[
\text{tr}|H^{[\ell]}|^2 = \sum_{i=1}^{n-1} \frac{\left( \mu_i^{(1)} + ap \sqrt{p/(nb_p)} \right)^2}{|\mu_i^{(1)} - z|^{2\ell}} = O(p), \quad \ell = 1, 2. \tag{A.36}
\]

**Proof.** We only provide the estimation of \( D_{11} \), since others are analogous. Note that
\[
D_{11} = V^{-1} \left( \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}} - z - \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}} H[1] \right) V^{-1}.
\]

Let \( \mathbf{v}_i^{(1)} = (v_{i1}^{(1)}, \ldots, v_{ip}^{(1)}) \), \((i = 1, 2, \ldots, n-1)\) be the unit eigenvector of \( A_1 \) corresponding to the eigenvalue \( \mu_i^{(1)} \), and let
\[
w_i^{(1)} = \sqrt{\frac{npd_p}{b_p}} |\mathbf{v}_i^{(1)}|^2.
\]
Applying spectral decomposition to \( H^{[1]} \) yields
\[
D_{11} = \left[ \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}} - z - \sum_{i=1}^{n-1} \left( \frac{\mu_i^{(1)} + \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}}}{\mu_i^{(1)} - z} \right) |\mathbf{v}_i^{(1)}|^2 \right]^{-1}.
\]
\[
= \left[ \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}} - z - \frac{1}{\sqrt{np}} \frac{b_p}{ap} \sum_{i=1}^{n-1} \left( \frac{\mu_i^{(1)} + \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}}}{\mu_i^{(1)} - z} \right) \right]^{-1}
\]
\[
=: (z - m_n(z) + h_1)^{-1}, \tag{A.36}
\]
where
\[
h_1 = \left[ m_n - \frac{1}{n} \sum_{i=1}^{n-1} \left( \frac{\sqrt{\frac{p}{n}} \frac{b_p}{ap} \mu_i^{(1)} + 1}{\mu_i^{(1)} - z} \right) \right] + \left[ \sqrt{\frac{p}{n}} \frac{ap}{\sqrt{b_p}} - \frac{1}{n} \sum_{i=1}^{n-1} \left( \frac{\sqrt{\frac{p}{n}} \frac{b_p}{ap} \mu_i^{(1)} + 1}{\mu_i^{(1)} - z} \right) \right] (w_i^{(1)} - 1).
\]

By (A.36), we obtain
\[
\left| D_{11} + \frac{1}{z + E m_n} \right| = \left| \frac{E m_n - m_n + h_1}{(z - m_n + h_1)(z + E m_n)} \right| \leq K \left| (E m_n - m_n) + h_1 \right|,
\]
which implies that
\[
\mathbb{E}
\left|
D_{11} + \frac{1}{z + \mathbb{E}m_n}
\right|^2
\leq K \left\{ \mathbb{E} \left| \mathbb{E}m_n - m_n \right|^2 + \mathbb{E} \left| m_n - \left( \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{\mu_i^{(1)} - z} \right) \right| \right.
\left. \sqrt{n \frac{\sqrt{b_p}}{a_p} \left( \frac{1}{n} \sum_{i=1}^{n-1} \frac{\mu_i^{(1)}}{\mu_i^{(1)} - z} \right)} \right| ^2
\]
\[
+ \mathbb{E} \left| \frac{1}{n} \sum_{i=1}^{n-1} \left( \sqrt{\frac{\sqrt{b_p}}{a_p} \mu_i^{(1)}} + 1 \right) \left( w_i^{(1)} - 1 \right) \right| ^2
\] + \mathbb{E} \left| \mathbb{Y}_1^2 - \sqrt{\frac{p}{n} \frac{a_p}{\sqrt{b_p}}} \right|^2 \right\}
\]
\[=: K(I + II + III + IV) \]
\[= O \left( \frac{1}{n^2} \right) + \left[ O \left( \frac{1}{n^2} \right) + O \left( \frac{n}{p} \right) \right] + O \left( \frac{1}{n} \right) + O \left( \frac{1}{n} \right) \] (A.37)
\[= O \left( \frac{1}{n} \right) + O \left( \frac{n}{p} \right). \] (A.38)

Below we explain (A.37) in more detail:

(I) Follows from Lemma A.11.

(II) Use the fact
\[
\sqrt{\frac{n \frac{\sqrt{b_p}}{a_p}}{p}} \left| \frac{1}{n} \sum_{i=1}^{n-1} \frac{\mu_i^{(1)}}{\mu_i^{(1)} - z} \right| = O \left( \sqrt{\frac{n}{p}} \right)
\]
and
\[
\left| m_n - \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{\mu_i^{(1)} - z} \right| = \left| \frac{1}{n} \text{tr} D - \frac{1}{n} \text{tr} D_k \right| = O \left( \frac{1}{n} \right). \] (6.14)

(III) Use (A.32).

(IV) Analogous to the estimation of \( \mathbb{E} \left| V - \mathbb{E}(1) V \right|^2. \) \( \Box \)

The following lemma is used to prove (6.38). Define
\[
\tilde{D} = (D_{ij})_{p \times p} = \left( \Sigma_p^{1/2} Y Y^\top \Sigma_p^{1/2} - \sqrt{\frac{p}{n} \frac{a_p}{\sqrt{b_p}}} I_p - z I_p \right)^{-1}.
\]

Lemma A.13. Under the assumption \( p \wedge n \to \infty, p/n \to \infty, \) for \( z \in \mathbb{C}, \) and \( 1 \leq \ell \leq p, \)
\[
\mathbb{E} \left| \tilde{D}_{\ell \ell} + \frac{1}{a_p \sqrt{p/(nb_p)} + z + \mathbb{E}m_n} \right|^2 = O \left( \left( \frac{n}{p} \right)^3 \right) + O \left( \frac{n}{p^2} \right). \] (A.39)

Proof. We only provide the estimation of \( \tilde{D}_{11}, \) since the others are analogous.

Let \( \tilde{r}_k^\ell \) be \( k \)-th row of \( \tilde{Y} \) and let \( B_k \) be the \( (p-1) \times n \) matrix extracted from \( \tilde{Y} \) by deleting \( \tilde{r}_k. \)

With notations defined above, we can write
\[
\tilde{A} = \begin{pmatrix}
\tilde{r}_1^\top \tilde{r}_1 - \sqrt{\frac{p}{n} \frac{a_p}{\sqrt{b_p}}}
& \tilde{r}_1^\top B_1^\top \\
B_1 \tilde{r}_1 & B_1 B_1^\top - \sqrt{\frac{p}{n} \frac{a_p}{\sqrt{b_p}}} I_{p-1}
\end{pmatrix}.
\]
Denote
\[ \tilde{A}_k = B_k' B_k - \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} I_n, \]
and
\[ W = \tilde{r}_1' B_1 B_1 \left( B_1' B_1 - \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} I_n \right)^{-1} \tilde{r}_1. \]

Let \( \{ \tilde{\mu}_i^{(k)}, i = 1, 2, \ldots, n - 1 \} \) be the eigenvalues of \( \tilde{A}_k \), and let \( \tilde{v}_i^{(1)} = (\tilde{v}_i^{(1)}_1, \ldots, \tilde{v}_i^{(1)}_p) \), \( i = 1, 2, \ldots, n \) be the unit eigenvector of \( \tilde{A}_1 \) corresponding to the eigenvalue \( \tilde{\mu}_i^{(1)} \), and set
\[ \tilde{w}_i^{(1)} = \frac{\sqrt{npa_p}}{\sqrt{b_p}} |\tilde{r}_1' \tilde{v}_i^{(1)}|^2, \]
then we have
\[ W = \frac{1}{n} \sum_{i=1}^n \left( \frac{\sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} \tilde{\mu}_i^{(1)} + 1}{\tilde{\mu}_i^{(1)} - z} \right) \tilde{w}_i^{(1)}, \]
and
\[ \tilde{D}_{11} = \left( \tilde{r}_1' \tilde{r}_1 - \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} - z - W \right)^{-1} \left( -\sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} - z - m_n + \tilde{h}_1 \right)^{-1}, \]
where
\[ \tilde{h}_1 = \tilde{r}_1' \tilde{r}_1 + m_n - W \]
\[ = \tilde{r}_1' \tilde{r}_1 + m_n - \frac{1}{n} \sum_{i=1}^n \left( \frac{\sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} \tilde{\mu}_i^{(1)} + 1}{\tilde{\mu}_i^{(1)} - z} \right) - \frac{1}{n} \sum_{i=1}^n \left( \frac{\sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} \tilde{\mu}_i^{(1)} + 1}{\tilde{\mu}_i^{(1)} - z} \right) \left( \tilde{w}_i^{(1)} - 1 \right). \] (A.40)

We define the set of events
\[ \Omega_0 = \left\{ |\mathbb{E} m_n - m_n + \tilde{h}_1| \geq \frac{1}{2} \sqrt{\frac{p}{n}} \right\}, \]
then the inequality
\[ \left| \left( \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} + z + \mathbb{E} m_n \right) \left( \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} + z + m_n - \tilde{h}_1 \right) \right| \geq K \frac{p}{n} \]
holds on \( \Omega_0 \). Thus we obtain
\[ \mathbb{E} \left| \tilde{D}_{11} + \frac{1}{a_p \sqrt{p/(nb_p)} + z + \mathbb{E} m_n} \right|^2 \leq \mathbb{E} \left| \frac{\mathbb{E} m_n - m_n + \tilde{h}_1}{a_p \sqrt{p/(nb_p)} + z + \mathbb{E} m_n} \right|^2 \leq K \left[ \left( \frac{n}{p} \right)^2 \cdot \mathbb{P}(\Omega_0) + \frac{n}{p} \cdot \mathbb{P}(\Omega_0) \right] \cdot \mathbb{E} \left| \mathbb{E} m_n - m_n + \tilde{h}_1 \right|^2, \]
where we use the inequality
\[
\left| \left( \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} + z + E m_n \right) \left( \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} + z + m_n - \tilde{h}_1 \right) \right| \geq K \sqrt{\frac{p}{n}}, \tag{A.41}
\]
that holds on the full set \( \Omega \). The inequality (A.41) follows from the facts
\[
\sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} + z + m_n - \tilde{h}_1
= \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} + z - \tilde{r}_1 \tilde{r}_1 + \tilde{r}_1 \tilde{B}_1 \left( B'_1 B_1 - \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} I_n \right)^{-1} \tilde{r}_1
= \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} + z + \tilde{r}_1 \left[ B'_1 B_1 \left( B'_1 B_1 - \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} I_n \right)^{-1} - I_n \right] \tilde{r}_1
= \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} + z + \frac{1}{\sqrt{n p}} \sqrt{b_p} a_p \sum_{i=1}^{n} \left( \frac{\tilde{\mu}^{(1)}_i + \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}}}{\tilde{\mu}^{(1)}_i - z} - 1 \right) \tilde{w}_i^{(1)}
= \left( \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} + z \right) \left[ 1 + \frac{1}{\sqrt{n p}} \sqrt{b_p} a_p \sum_{i=1}^{n} \frac{\tilde{w}_i^{(1)}}{\tilde{\mu}^{(1)}_i - z} \right]
=: \left( \sqrt{\frac{p}{n}} \frac{a_p}{\sqrt{b_p}} + z \right) (1 + S),
\]
and
\[
|1 + S| \geq K \sqrt{\frac{n}{p}}.
\]
We now proceed to complete the proof of (A.39). Note that we have
\[
\mathbb{P}(\Omega_0) \leq \frac{4n}{p} \mathbb{E} \left| E m_n - m_n + \tilde{h}_1 \right|^2,
\]
thus it is sufficient to prove that
\[
\mathbb{E} \left| E m_n - m_n + \tilde{h}_1 \right|^2 = O \left( \frac{1}{n} \right) + O \left( \frac{n}{p} \right). \tag{A.42}
\]
Applying (A.40) gives us
\[
\mathbb{E} \left| E m_n - m_n + \tilde{h}_1 \right|^2
\leq K \left\{ \mathbb{E} \left| E m_n - m_n \right|^2 + \mathbb{E} \left| m_n - \left( \frac{1}{n} \sum_{i=1}^{n-1} \frac{1}{\tilde{\mu}^{(1)}_i} - \frac{1}{\tilde{\mu}^{(1)}_i - z} \right) \right|^2 - \sqrt{n} \frac{1}{p} \sqrt{b_p} \left( \frac{1}{n} \sum_{i=1}^{n-1} \frac{\tilde{w}_i^{(1)}}{\tilde{\mu}^{(1)}_i} \right) ^2 \right\}.
\tag{A.43}
\]
Combining the similar method used for (A.38) with (A.43) and the fact
\[
\mathbb{E} \left( \tilde{r}_1 \tilde{r}_1 \right)^2 = \mathbb{E} \left[ \sum_{j=1}^{n} \left( \sum_{i=1}^{p} \tilde{\sigma}_{ij} Y_{ij} \right) \right]^2.
\]
For any Lemma A.14. we obtain \((\frac{\partial F}{\partial Y})_j = O(n/p)\).

The following two lemmas about the derivatives of some quantities with respect to \(Y_{jk}\), which can be used to obtain the derivatives of \(F_{jk}\) (Lemma 6.4) and \(\tilde{F}_{jk}\) (Lemma 6.5).

Recall that

\[
E := \Sigma_p YDY' \Sigma_p = (E_{ij})_{p \times p}, \quad F := \Sigma_p YD = (F_{ij})_{p \times n}.
\]

**Lemma A.14.** For any \(\alpha, j \in \{1, 2, \ldots, p\}\) and \(\beta, k \in \{1, 2, \ldots, n\}\), we have

\[
\frac{\partial D_{\alpha\beta}}{\partial Y_{jk}} = -F_{j\alpha} D_{\beta k} - F_{j\beta} D_{\alpha k};
\]

\[
\frac{\partial F_{\alpha\beta}}{\partial Y_{jk}} = \sigma_{\alpha j} D_{k\beta} - E_{j\alpha} D_{\beta k} - F_{j\beta} F_{\alpha k};
\]

\[
\frac{\partial (E_{jj} D_{kk})}{\partial Y_{jk}} = 2\sigma_{jj} F_{jk} D_{kk} - 4E_{jj} F_{jk} D_{kk}.
\]

**Proof.** (1) By using chain rule and the results in the Table 4, we have

\[
\frac{\partial D_{\alpha\beta}}{\partial Y_{jk}} = \sum_{1 \leq s \leq t \leq p} \frac{\partial D_{\alpha\beta}}{\partial A_{st}} \frac{\partial A_{st}}{\partial Y_{jk}} \left[ \frac{\partial A_{st}}{\partial Y_{jk}} = \frac{\partial (Y' \Sigma_p Y)_{st}}{\partial Y_{jk}} \right]
\]

\[
= \sum_{s=1}^{p} \frac{\partial D_{\alpha\beta}}{\partial A_{ss}} \frac{\partial A_{ss}}{\partial Y_{jk}} + \sum_{1 \leq s < t \leq p} \frac{\partial D_{\alpha\beta}}{\partial A_{st}} \frac{\partial A_{st}}{\partial Y_{jk}}
\]

\[
= \sum_{s=1}^{p} \left( -D_{\alpha s} D_{t\beta} \right) \sum_{r, \ell} \left( \sigma_{rt} \frac{\partial (Y_{rs} Y_{t\ell})}{\partial Y_{jk}} \right) + \sum_{s < t} \left( -D_{\alpha s} D_{t\beta} - D_{at} D_{\beta s} \right) \sum_{r, \ell} \left( \sigma_{rt} \frac{\partial (Y_{rs} Y_{t\ell})}{\partial Y_{jk}} \right)
\]

\[
= \left( -D_{\alpha k} D_{t\beta} \right) \cdot \left( 2\sigma_{jj} Y_{jk} + \sum_{\ell \neq j} \sigma_{j\ell} Y_{tk} + \sum_{r \neq j} \sigma_{rj} Y_{rk} \right)
\]

\[
+ \sum_{k < t} \left( -D_{\alpha k} D_{t\beta} - D_{at} D_{\beta s} \right) \cdot \left( \sigma_{jj} Y_{jt} + \sum_{\ell \neq j} \sigma_{j\ell} Y_{lt} \right)
\]

\[
+ \sum_{s < k} \left( -D_{\alpha s} D_{k\beta} - D_{ak} D_{s\beta} \right) \cdot \left( \sigma_{jj} Y_{js} + \sum_{r \neq j} \sigma_{rj} Y_{rs} \right)
\]
\[= \sum_{s=1}^{p} \left( -D_{\alpha s} D_{k\beta} - D_{\alpha k} D_{s\beta} \right) \left( \sum_{r=1}^{p} \sigma_{rj} Y_{rs} \right) \]
\[= \sum_{s, t} \left[ -\left( \sigma_{jr} Y_{rs} D_{\alpha s} \right) D_{\beta k} - \left( \sigma_{jr} Y_{rs} D_{s\beta} \right) D_{\alpha k} \right] \]
\[= -E_{j\alpha} D_{\beta k} - F_{j\beta} D_{\alpha k}, \]
where the third equality follows from the formula (II. 18) in Khorunzhy et al. (1996);

(2)
\[
\frac{\partial F_{\alpha\beta}}{\partial Y_{jk}} = \frac{\partial}{\partial Y_{jk}} \sum_{s, t} \left( \sigma_{\alpha s} Y_{st} D_{t\beta} \right) = \sum_{s, t} \sigma_{\alpha s} \left( \frac{\partial Y_{st}}{\partial Y_{jk}} \cdot D_{t\beta} + Y_{st} \cdot \frac{\partial D_{t\beta}}{\partial Y_{jk}} \right)
\]
\[= \sigma_{\alpha j} D_{k\beta} - \sum_{s, t} \sigma_{\alpha s} Y_{st} \left( F_{jt} D_{\beta k} + F_{j\beta} D_{tk} \right) \]
\[= \sigma_{\alpha j} D_{k\beta} - E_{j\alpha} D_{\beta k} - F_{j\beta} F_{\alpha k}; \]

(3)
\[
\frac{\partial E_{jj}}{\partial Y_{jk}} = \frac{\partial}{\partial Y_{jk}} \sum_{r} \left( \Sigma_{p} YD \right)_{jr} \left( Y' \Sigma_{p} \right)_{rj}
\]
\[= \sum_{r} \frac{\partial F_{jr}}{\partial Y_{jk}} \cdot \left( Y' \Sigma_{p} \right)_{rj} + \sum_{r} F_{jr} \cdot \frac{\partial \left( Y' \Sigma_{p} \right)_{rj}}{\partial Y_{jk}}
\]
\[= \sum_{r} \left( \sigma_{jj} D_{kr} - E_{jj} D_{rk} - F_{jr} F_{jk} \right) \cdot \left( Y' \Sigma_{p} \right)_{rj} + \sigma_{jj} F_{jk}
\]
\[= 2\sigma_{jj} F_{jk} - 2E_{jj} F_{jk}, \]

\[
\frac{\partial (E_{jj} D_{kk})}{\partial Y_{jk}} = \frac{\partial E_{jj}}{\partial Y_{jk}} \cdot D_{kk} + \frac{\partial D_{kk}}{\partial Y_{jk}} \cdot E_{jj}
\]
\[= \left( 2\sigma_{jj} F_{jk} - 2E_{jj} F_{jk} \right) \cdot D_{kk} - 2F_{jk} D_{kk} \cdot E_{jj}
\]
\[= 2\sigma_{jj} F_{jk} D_{kk} - 2E_{jj} F_{jk} D_{kk}. \]

Recall that \( \Sigma_{p}^2 = (\tilde{\sigma}_{ij}) \), and

\[\tilde{E} := \Sigma_{p}^2 YDY' \Sigma_{p}^2, \quad \tilde{F} := \Sigma_{p} YDY' \Sigma_{p}^2, \quad \tilde{\Sigma} := \Sigma_{p} YD.\]

**Lemma A.15.** For any \( \alpha, j \in \{1, 2, \ldots, p\} \) and \( \beta, k \in \{1, 2, \ldots, n\} \), we have

\[
\frac{\partial \tilde{F}_{\alpha\beta}}{\partial Y_{jk}} = \tilde{\sigma}_{\alpha j} D_{k\beta} - \tilde{E}_{j\alpha} D_{\beta k} - \tilde{F}_{j\beta} F_{\alpha k};
\]

\[
\frac{\partial (\tilde{E}_{jj} D_{kk})}{\partial Y_{jk}} = \sigma_{jj} \tilde{F}_{jk} D_{kk} + \tilde{\sigma}_{jj} \tilde{F}_{jk} \tilde{D}_{kk} - \tilde{E}_{jj} \tilde{F}_{jk} \tilde{D}_{kk} - 3\tilde{E}_{jj} \tilde{F}_{jk} \tilde{D}_{kk}.
\]

**Proof.**

\[
\frac{\partial \tilde{F}_{\alpha\beta}}{\partial Y_{jk}} = \frac{\partial}{\partial Y_{jk}} \sum_{s, t} \left( \tilde{\sigma}_{\alpha s} Y_{st} D_{t\beta} \right) = \sum_{s, t} \tilde{\sigma}_{\alpha s} \left( \frac{\partial Y_{st}}{\partial Y_{jk}} \cdot D_{t\beta} + Y_{st} \cdot \frac{\partial D_{t\beta}}{\partial Y_{jk}} \right)
\]
\[ \Sigma_{\alpha j} D_{k \beta} - \sum_{s, t} \Sigma_{\alpha s} Y_{st} \left( F_{jt} D_{s \beta k} + F_{j s} D_{t \beta k} \right) \]
\[ = \Sigma_{\alpha j} D_{k \beta} - \hat{E}_{j \alpha} D_{s \beta k} - F_{j \beta} \hat{F}_{\alpha k}; \]

\[ \frac{\partial E_{jr}}{\partial Y_{jk}} = \frac{\partial}{\partial Y_{jk}} \sum_{\ell} F_{j \ell} \left( Y' \Sigma_p \right)_{\ell r} = \sum_{\ell} \frac{\partial F_{j \ell}}{\partial Y_{jk}} \left( Y' \Sigma_p \right)_{\ell r} + \sum_{\ell} F_{j \ell} \frac{\partial (Y' \Sigma_p)_{\ell r}}{\partial Y_{jk}} \]
\[ = \sum_{\ell} \left( \sigma_{jj} D_{k \ell} - E_{jj} F_{jk} + F_{j \ell} F_{jk} \right) \left( Y' \Sigma_p \right)_{\ell r} + \sigma_{jr} F_{jk} \]
\[ = \sigma_{jj} F_{rk} + \sigma_{jr} F_{jk} - E_{jj} F_{rk} - F_{jk} E_{jr}. \]

\[ \frac{\partial (F_{ij} D_{kk})}{\partial Y_{jk}} = \left( \frac{\partial}{\partial Y_{jk}} \sum_{\ell} E_{jr} \sigma_{r j} \right) D_{kk} + \hat{E}_{jj} \left( -2F_{jk} D_{kk} \right) \]
\[ = D_{kk} \sum_{\ell} \sigma_{r j} \left( \sigma_{jj} F_{rk} + \sigma_{jr} F_{jk} - E_{jj} F_{rk} - F_{jk} E_{jr} \right) - 2\hat{E}_{jj} F_{jk} D_{kk} \]
\[ = \sigma_{jj} F_{jk} D_{kk} + \sigma_{jj} F_{jk} D_{kk} - E_{jj} F_{jk} D_{kk} - 3\hat{E}_{jj} F_{jk} D_{kk}. \]

\[ \Box \]

**Appendix B: Proofs in applications**

**Proof of Theorem 4.2.**

*Proof.* For notational simplicity, we denote \( a_p = \text{tr}(\Sigma_p)/p \) and \( b_p = \text{tr}(\Sigma_p^2)/p \). Let \( \tilde{A}_n = \frac{1}{\sqrt{npb_p}} (Y'Y - pa_p I_n) = \frac{1}{\sqrt{npb_p}} (X'\Sigma_p X - pa_p I_n) \). By some elementary calculations, we obtain two identities:

\[ \text{tr}(S_n) = \frac{\sqrt{pb_p}}{n} \text{tr}(\tilde{A}_n) + pa_p, \quad \text{tr}(S_n^2) = \frac{pb_p}{n} \text{tr}(\tilde{A}_n^2) + \frac{2pa_p}{n} \sqrt{pb_p} \text{tr}(\tilde{A}_n) + \frac{(pa_p)^2}{n}. \]

Then \( W \) can be written as

\[ W = \frac{b_p}{n} \text{tr}(\tilde{A}_n^2) - \frac{2}{p} \sqrt{pb_p} \frac{\sqrt{pb_p}}{n} \text{tr}(\tilde{A}_n) - \frac{b_p}{n^2} \left[ \text{tr}(\tilde{A}_n) \right]^2 + \frac{p}{n} - 2a_p + 1. \]

Li and Yao (2016) derived the limiting joint distribution of \((\text{tr}(\tilde{A}_n^2)/n, \text{tr}(\tilde{A}_n)/n)\) (see their Lemma 3.1) as follows:

\[ n \left( \frac{1}{n} \text{tr}(\tilde{A}_n^2) - 1 - \frac{1}{n} \left( \frac{\omega}{\theta} (\nu_4 - 3) + 1 \right) \right) \xrightarrow{d} \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & \frac{\omega}{\theta} (\nu_4 - 3) + 2 \end{pmatrix} \right). \]  

(B.1)

Define the function

\[ g(x, y) = b_p x - \frac{2n}{p} \sqrt{pb_p} y - b_p y^2 + \frac{p}{n} - 2a_p + 1, \]

then \( W = g(\text{tr}(\tilde{A}_n^2)/n, \text{tr}(\tilde{A}_n)/n) \), we have

\[ \frac{\partial g}{\partial x} \left( 1 + \frac{1}{n} \left( \frac{\omega}{\theta} (\nu_4 - 3) + 1 \right), 0 \right) = b_p. \]
\[
\frac{\partial g}{\partial y} \left( 1 + \frac{1}{n} \left( \frac{\omega}{\theta} (\nu_4 - 3) + 1 \right), 0 \right) = -\frac{2n}{p} \sqrt{\frac{p b_p}{n}},
\]
\[
g \left( 1 + \frac{1}{n} \left( \frac{\omega}{\theta} (\nu_4 - 3) + 1 \right), 0 \right) = b_p + \frac{b_p}{n} \left( \frac{\omega}{\theta} (\nu_4 - 3) + 1 \right) + \frac{p}{n} - 2a_p + 1.
\]

By (B.1), we have
\[
n \left( W - g \left( 1 + \frac{1}{n} \left( \frac{\omega}{\theta} (\nu_4 - 3) + 1 \right), 0 \right) \right) \xrightarrow{d} \mathcal{N}(0, \lim A),
\]
where
\[
A = \left( \frac{\partial^2 g}{\partial x \partial y} \left( 1 + \frac{1}{n} \left( \frac{\omega}{\theta} (\nu_4 - 3) + 1 \right), 0 \right) \right) \xrightarrow{d} \begin{pmatrix} 4 & 0 \\ 0 & (\nu_4 - 3) + 2 \end{pmatrix} \left( \frac{\partial g}{\partial x} \left( 1 + \frac{1}{n} \left( \frac{\omega}{\theta} (\nu_4 - 3) + 1 \right), 0 \right) \right) \xrightarrow{d} 4\theta^2.
\]

Thus,
\[
n \left( W - b_p - \frac{b_p}{n} \left( \frac{\omega}{\theta} (\nu_4 - 3) + 1 \right) + 2a_p - 1 - \frac{p}{n} \right) \xrightarrow{d} \mathcal{N}(0, 4\theta^2),
\]
that is,
\[
nW - p - \theta \left( \frac{\omega}{\theta} (\nu_4 - 3) + 1 \right) + n(2\gamma - 1 - \theta) \xrightarrow{d} \mathcal{N}(0, 4\theta^2).
\]

**Proof of Proposition 4.1.**

Proof. For the test based on statistic \( W \), by Theorem 4.1 and 4.2, we have
\[
\beta(H_1) = \mathbb{P} \left( \frac{1}{2} \left( nW - p - (\nu_4 - 2) \right) \geq z_a \mid H_1 \right)
\]
\[
= \mathbb{P} \left( nW - p - \theta \left( \frac{\omega}{\theta} (\nu_4 - 3) + 1 \right) + n(2\gamma - 1 - \theta)
\]
\[
\geq 2z_a - \theta \left( \frac{\omega}{\theta} (\nu_4 - 3) + 1 \right) + n(2\gamma - 1 - \theta) + (\nu_4 - 2) \mid H_1 \right)
\]
\[
= 1 - \Phi \left( \frac{1}{2\theta} \left( 2z_a - \omega(\nu_4 - 3) - \theta + n(2\gamma - 1 - \theta) + (\nu_4 - 2) \right) \right),
\]

since \( 2\gamma - 1 \leq \gamma^2 \leq \theta \), Proposition 4.1 follows.

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