Universality and Scale Invariance in Hourly Rainfall

Pankaj Jain, Suman Jain and Gauher Shaheen

Physics Department, I.I.T. Kanpur, India, 208016

Abstract: We show that the hourly rainfall rate distribution can be described by a simple power law to a good approximation. We show that the exponent of the distribution in tropics is universal and is equal to 1.13 ± 0.11. At higher latitudes the exponent increases and is found to lie in the range 1.3-1.6.

It is well known that daily rainfall over a localized region displays an almost chaotic behavior. This implies that it is very difficult to make even short term prediction of daily rainfall. The distribution of daily rainfall amount over a localized region is also well known to have a very large tail which implies that there is very high probability to have very large rainfall. The distribution shows considerable deviation from the normal distribution.

In the present paper we analyze the distribution of hourly rainfall amount. We are particularly interested in determining whether we can identify some universal features in the rainfall distribution which are observed at all spatial locations. The nature of distribution can also reveal some underlying features of the dynamical system. For example, it has been argued that dissipative systems with large number of variable tend to reach a critical state under the influence of an external perturbation. In the critical state the system displays an almost chaotic behavior with fluctuations displaying scale invariant power law distribution. This has been demonstrated numerically by postulating simplified models of the behavior of sand piles, earthquakes etc. The mechanism by which the dynamical systems achieve the critical state under the influence of external perturbation is called the self organized criticality. In the present paper we shall determine if the rainfall rate also displays this behavior by studying its distribution. Some preliminary results of this study have been presented in Ref. [4].

The data is taken from the web site and consists of SSM/I satellite data. The data consists of rainfall rate in units of mm/hour with the minimum rate equal to 0.1 mm/hour. It is available over the entire globe on a grid size of half a degree both in latitude and longitude. We extracted the data files from this website for the year 1997 for three different
time periods. At different times the data is available at different locations. We randomly selected a total of 97 different locations on the globe out of the extracted data. In order that a reasonable amount of data is available for a given region, we group the rainfall data in intervals of 5 degrees both in latitude and longitude. The typical distribution for several different regions are shown in Fig. 1 and 2. We represent the power law distribution as

\[ f(x) = \frac{a}{x^b} \]  

where \( x \) is equal to the hourly rainfall amount in mm and \( a \) and \( b \) are parameters. If the underlying dynamical system is in a critical state then we expect that certain observables in the system will follow a power law distribution. The rainfall amount per hour represents an interesting physical observable which we study in this paper.

We point out that while testing for a power law behaviour we are not testing the hypothesis that the distribution is a power law over the entire interval of rainfall per hour. The power law is expected to be valid only in some intermediate range of rainfall. At low as well as large rainfall we expect that the distribution will be distorted in comparison to a pure power behavior. At very large rainfall, for example, we expect the distribution to decay very rapidly since rainfall amounts larger than a certain value are physically not possible. We, therefore, cut off the distribution at some large value as well as at a low value of the rainfall. At the low rainfall value we make the distribution constant below a certain value \( x_{\min} \), which is treated as an additional parameter of the distribution. At the large rainfall end we put a cut on the data so as to exclude all data with \( x > x_{\max} = 10 \) mm/hr. The number of data points in the excluded region are typically found to be very small. The distribution is set equal to zero beyond \( x = x_{\max} \). The power distribution, therefore, contains two adjustable parameters \( b \) and \( x_{\min} \). The parameter \( a \) is obtained by normalization of the distribution.

We compare the fit obtained by the power law distribution with an alternative exponential distribution

\[ g(x) = \alpha \exp(-\beta x) \]  

In analogy with the power distribution, \( g(x) \) is also set equal to zero for \( x > x_{\max} \). Hence the normalization \( \alpha \) for this distribution is equal to \( \beta/(1 - \beta x_{\max}) \). We point out that we are not interested in finding the best possible distribution that describes the data. We are only interesting in demonstrating
that a power law provides a good description of the data over a wide range of rainfall amounts. Hence we have not made extensive comparisons of various distributions and compare the power law only to a simple exponential fit.

In Fig. 1 and 2 we show the fits in several representative cases. The figure shows the power as well as the exponential fit to the rainfall distribution. It is clear from these plots that a power distribution fits the data reasonably well. It can also be seen that the fit is much better at lower latitudes in comparison to the higher latitudes. The log likelihood difference, defined to be the difference of the log likelihood for the power fit and the exponential fit, is generally found to lie in the range 50 to 150 in tropics and between 10 to 30 at the higher latitudes. Hence the power distribution gives an overall much better description of the data in comparison to an exponential distribution. In Fig. 3 and 4 we show a scatter plot of the exponent of the power distribution as a function of the latitude and longitude respectively. The error in the exponent values ranges from 0.02 to 0.1 in most cases. Only rarely does the error exceed 0.1. The precise value depends on the number of data points obtained in each region which typically range from 100 to 2000. We also point out that the uneven distribution of the data points as a function of latitude and longitude is caused by our selection of times for which the data is extracted. However this does not have any influence on our results or conclusions since the data is spread over the entire globe.

We clearly see that the values of the exponent $b$ are different at lower latitudes in comparison to higher latitudes. The exponent close to equator is found to be close to unity whereas at higher latitudes the exponent is much larger. The exponent does not show very significant dependence on the longitude. Only a very marginal relationship is found after eliminating one outlier with a very large value of the exponent. In order to quantify the dependence of the exponent on the latitude we evaluate the correlation between the absolute value of the latitude and the exponent. For the entire set of 97 data points the correlation $\rho = 0.54$. If we delete two outliers the correlation goes up to $\rho = 0.608$. The probability $p$ that we can get this correlation from a random sample is very small. We find that for the entire set $p = 10^{-5}\%$ and after eliminating the two outliers $p = 3 \times 10^{-7}\%$.

The fact that at lower latitude the exponent values are clustered around unity irrespective of longitude is a clear indication of a universal behaviour in the rainfall distributions in the tropics. By putting a cut on the latitude in order to select only the region that lies between 20 N and 20 S we find
that the mean value of the exponent $b$, defined in Eq. 1, is given by,

$$b = 1.13 \pm 0.11$$  

(3)

The total number of data points contained within the latitude 20 N and 20 S are equal to 61. Two of these points were found to give an anomalously large values of the exponent compared to the rest and were treated as outliers. The mean value given in Eq. 3 is obtained after eliminating these outliers. The median value of the exponent is equal to 1.12 and remains unchanged with or without the inclusion of the outliers.

As this work was near completion we became aware of Ref. [5] where the authors have analyzed high-resolution rainfall data at Baltic coast Zingst in order to investigate its distribution. The authors find that the distribution is well described by a power law with exponent 1.36. Our results are in agreement with their findings since at higher latitudes we also find similar exponents.

In conclusion, we summarize the main results of the paper. We have shown that the hourly rainfall distribution is well described by a scale invariant power law. This is particularly true in the tropical region where the exponent is found to be $1.13 \pm 0.11$ independent of the longitude. As we go towards the higher latitudes the exponent increases and generally is found to lie in a range 1.3-1.6. The power dependence and its universal character in the tropics indicates that the underlying dynamical system may be best describable in terms of self organized criticality.

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All correspondence should be directed to Pankaj Jain (pkjain@iitk.ac.in).

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Figure 1: Hourly rainfall distributions $f(x)$ of a sample of data sets in the tropical region. The variable $x$ is the rainfall rate in mm/hour. The best fit power and exponential distributions are also shown. The longitude and latitude range from where the data was taken is also indicated in each graph.
Figure 2: Hourly rainfall distributions $f(x)$ of a sample of data sets at high latitudes. The variable $x$ is the rainfall rate in mm/hour. The best fit power and exponential distributions are also shown. The longitude and latitude range from where the data was taken is also indicated in each graph.
Figure 3: Scatter plot of the exponent of the power distribution fits as a function of the latitude.

Figure 4: Scatter plot of the exponent of the power distribution fits as a function of the longitude.