On the modeling of ultrasound wave propagation in the frame of inverse problem solution

N S Novikov\textsuperscript{1,2}, D V Klyuchinskiy\textsuperscript{1,2}, M A Shishlenin\textsuperscript{1,2} and S I Kabanikhin\textsuperscript{1,2}

\textsuperscript{1} Institute of computational mathematics and mathematical geophysics, Ac. Lavrentiev ave., 6, Novosibirsk, 630090, Russia;
\textsuperscript{2} Novosibirsk State University, Pirogov str., 1, Novosibirsk, 630090, Russia

E-mail: novikov-1989@yandex.ru, dmitriy.klyuchinskiy@mail.ru, mshishlenin@ngs.ru, ksi52@mail.ru

Abstract. In this paper we consider the inverse problem of detecting the inclusions inside the human tissue by using the acoustic sounding wave. The problem is considered in the form of coefficient inverse problem for first-order system of PDE and we use the gradient descent approach to recover the coefficients of that system. The important part of the scheme is the solution of the direct and adjoint problem on each iteration of the descent. We consider two finite-volume methods of solving the direct problem and study their influence on the performance of recovering the coefficients.

Introduction
In this paper, we consider the problem of detecting inclusions in human soft tissues during ultrasound tomography [24, 25, 28, 31, 33, 34, 36].

The system of hyperbolic equations of the first order is considered as the mathematical model of the acoustic tomography, because these equations are obtained directly from the conservation laws of continuum mechanics. This allows us to control the preservation of the basic invariants when solving direct and inverse problems. This is important for solving unstable problems, since the conservation laws of basic invariants are the only criterion for the well-posedness of the solution. Some considerations on the choice of such a mathematical model based on a first-order system and a method for solving such a system can be found in [32, 35, 37, 38].

We apply the numerical algorithm for solving direct problems, based on the S. K. Godunov schemes [1]. Actually, one of the most popular second-order scheme has been introduced by van Leer [4]; namely the MUSCL scheme. It is a finite volume method where the flux approximation is second-order accurate. This scheme is considered and extended in a lot of applications [6, 8, 9, 13].

MUSCL–Hancock method was introduced in [5] as a variant of the MUSCL scheme. This scheme is full time and space second-order accurate. It differs from the other MUSCL variants [4, 10, 11, 15] in its second-order time of accuracy. Indeed, the time accuracy rises when considering a step based on a central-like scheme [7, 16]. The stability of the MUSCL–Hancock scheme was investigated in [20].

Numerical methods of an increased order of accuracy based on the Godunov scheme are widely used and there are a huge number of their variants and implementations [30].
Inverse problems for hyperbolic systems were investigated theoretically in [12].

The problem of modelling acoustic radiation pattern of source for the 2D first-order system of hyperbolic equations was considered in [39].

1. Problem formulation

Let us consider the direct problem of acoustic wave propagation through the 2D medium in the domain $\Omega = (x, y) \in [0, L] \times [0, L]$:

\[
\frac{\partial u}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0, \quad (x, y) \in \Omega, \quad 0 < t \leq T, \tag{1}
\]

\[
\frac{\partial p}{\partial t} + \rho c^2 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \theta_\Omega(x, y) I(t), \quad (x, y) \in \Omega, \tag{2}
\]

\[
u, v, p|_{(x, y) \in \partial \Omega} = 0, \tag{3}
\]

\[
u, v, p|_{t=0} = 0. \tag{4}
\]

Here $u = u(x, y, t)$ is the velocity vector with respect to $x$, $v = v(x, y, t)$ is the velocity vector with respect to $y$, $p = p(x, y, t)$ is the exceeded pressure, $\rho = \rho(x, y)$ is the density of the medium, $c = c(x, y)$ is the wave speed. Such system is often used to describe the propagation of the ultrasound through the fluid medium, and the acoustic parameters of the models, that were considered during the numerical experiments are close to fluid. $\theta_\Omega(x, y)$ is the characteristic function of the source location, $I(t)$ has the following form:

\[
I(t) = \sin \left( \frac{\pi \nu_0}{\nu_0} \left( t - \frac{1}{\nu_0} \right) \right) e^{-\pi \nu_0 \left( t - \frac{1}{\nu_0} \right)}. \tag{5}
\]

Here $\nu_0$ is a frequency. The data of inverse problem is the pressure, obtained in the receivers, located in $\Omega_k$, $k = 1, \ldots, N$:

\[
p(x, y, t) = f_k(x, y, t), \quad (x, y) \in \Omega_k, \quad k = 1, \ldots, N. \tag{6}
\]

We reformulate inverse problem in the (1)–(4), (6) in operator form

\[
A(q) = f, \quad q(x, y) = (q_1, q_2) = (\rho, c^2) \rightarrow f_k(x, y, t), \quad k = 1, \ldots, N. \tag{7}
\]

Let us reduce the inverse problem (1)–(4), (6) to minimization problem of the following cost functional:

\[
J(q) = ||A(q) - f||_2^2 = \sum_{k=1}^{N} \int_{0}^{T} \int_{\Omega_k} \left[ p(x, y, t; q) - f_k(x, y, t) \right]^2 dx dy dt \rightarrow \min_q. \tag{8}
\]

We use the gradient-based approach to minimize the functional (8) by considering the following iteration scheme:

\[
q^{(n+1)} = q^{(n)} - \alpha J'(q^{(n)}). \tag{9}
\]

The gradient of the functional can be obtained by solving the adjoint problem. Explicit formulas are presented in [27, 36, 37]. At each iteration of gradient descent, the solution of both direct and conjugate problems is required. In [40] it was presented an approach to save twice memory on the stage of adjoint problem and gradient calculation and compare it with usual approach in memory and CPU time cost. Convergence of gradient methods for hyperbolic equations was investigated in [14, 18, 21]. Work [29] presented a criterion for the termination of the iterative process in the gradient method, consistent with the accumulation of machine round-off errors.
What is of interest in this study is how the choice of a numerical method for solving a direct problem affects the solution of the inverse problem.

We consider two suitable methods, that are based on the approach of S.K. Godunov [1]. His approach utilises using the integral form of the problem, piecewise-constant approximation of the state variables inside the numerical cell and the solution of the Riemann problem - the initial problem with conditions represented by two constant states separated by a discontinuity. The MUSCL-Hancock scheme, first published by van Leer [4], extends the ideas behind the Godunov scheme in order to obtain higher accuracy of the method and nowadays is one of the common approaches for dealing with the hyperbolic systems. One can find more detailed review of papers related to both methods in [4, 20, 22].

The short description of the methods is presented below. Considering the direct problem (1)–(4), we use a generalized formulation of the equations (1)–(2) of the following form:

\[ \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} = 0. \]  

(9)

Here \( \mathbf{U} = (u, v, p) \) is the vector of state variables, and \( \mathbf{F}(\mathbf{U}), \mathbf{G}(\mathbf{U}) \) are the vectors of fluxes correspondingly. After introducing the discretization the computational domain into the finite number of cells, one can obtain:

\[ \mathbf{U}^{i-1/2,j-1/2} = \mathbf{U}^{i-1/2,j-1/2} - \frac{\tau}{h_x} (\mathbf{F}^{i,j-1/2} - \mathbf{F}^{i-1,j-1/2}) - \frac{\tau}{h_y} (\mathbf{G}^{i-1/2,j} - \mathbf{G}^{i-1/2,j-1}). \]  

(10)

The equation (10) corresponds to the numerical cell \((i - 1/2, j - 1/2)\), where the sub-indexes indicate the state values \( \mathbf{U} \) on the current time step, and sup-indexes - on the next time step, \( h_x, h_y, \tau \) are the grid steps with respect to the spatial coordinates and time correspondingly. The values \( \mathbf{F}, \mathbf{G} \) on the each boundary of the cell considered, are the solutions of the Riemann (or discontinuity decay) problem [2]. For example, the approximation (10) of the first of the equations (1) has the following form:

\[ (\rho u_{i-1/2,j-1/2}^i - \rho u_{i-1/2,j-1/2}^{i-1}) h_x h_y + \tau h_y (P_{i,j-1/2}^i - P_{i-1,j-1/2}^i) = 0, \]  

(11)

The value \( P_{i,j-1/2} \) is the solution of the discontinuity decay problem, that arises on the boundaries of the cell considered. The formula has the following form:

\[ p = P_{i,j-1/2} = \frac{p_{i-1/2,j-1/2}^i + p_{i+1/2,j-1/2}^i}{2} - \rho_0 c_0 \frac{u_{i+1/2,j-1/2}^i + u_{i-1/2,j-1/2}^i}{2}. \]  

(12)

The two others equations of the system can be considered in the same manner. The right-hand side of the equation can also be easily taken into account. We skip the rest of the formulas, yet one can find them, for example, in [2], as well as the study of the stability of the scheme.

Another approach that we applied for solving the direct problem is MUSCL-Hancock scheme [19, 23]. First, we summarize the basic Godunov approach as the combination of two following steps:

(i) Obtaining the flux values by solving the Riemann problem;
(ii) Updating the state variables on the next time step, using the solution of the Riemann problem.

One should mention, that the Godunov approach based on the piecewise-constant approximation of the parameters of the each cells. The MUSCL-Hancock scheme uses the piecewise-linear approximation of the parameters, based on the slope limited approximation. Another feature of the scheme is the solution of the Riemann problem on the half-step, corresponded to \( 0.5\tau \). Such update allows the described scheme to second order accuracy [22]. The workflow of the scheme can now be summarized as follows:
(i) Reconstruction of the state variables on the current time step, using piecewise-linear extrapolation;

(ii) Evolution of the reconstructed state variables by conservation laws with a time $0.5\tau$;

(iii) Obtaining the flux values for the "midpoint" time step by solving the Riemann problem for evolved state variables;

(iv) Updating the state values from the current time step on the next time step by conservation laws with a time $\tau$.

2. The influence of the method of the direct problem solution

In order to compare the usage of two methods, we consider the test model, illustrated by the figure 1. The large circle in the center corresponds to a human tissue, small objects inside are inclusions (values of the density and speed of sound distribution inside the inclusions are higher). As one can see, the model contains several inclusions of different shape and size. The acoustic parameters of human body are taken from [3, 17, 26].

![Density distribution](image1.png)  ![Speed of sound distribution](image2.png)

*Figure 1. Test — Exact model.*

We solve the inverse problems using the following settings. We use the uniform grid with 200 nodes for each space variable. We consider the system of 16 transducers, and use Pusyrev wavelet with frequency $\nu_0 = 100$ KHz. We use the simple gradient descent method with consecutive updates of the density and velocity. During first test we considered the same parameters of descent ($\alpha_\rho = 250$, $\alpha_c = 500$) for both variations of the method. The results are presented on the figure 2.

According to the results, the the error of the solution, when one uses the second order solver of direct problem, tends to decrease faster, but shortly after start the convergence of the iterations fails. Thus, we have to use different descent parameters when changing between the first order and the second order solvers of direct problem. We find the appropriate values of the descent parameters by trial and error approach. During the next test we considered $N_{Iter} = 1000$ iterations with descent parameters equal to $\alpha_\rho = 100$, $\alpha_c = 200$ when using the classic Godunov’s scheme, and $\alpha_\rho = 25$, $\alpha_c = 50$, when using the MUSCL scheme. The comparative results for the residual and relative errors using the mentioned settings are presented on the figure 3. One can notice the the usage of the second order solver provides the more accurate solution. The next figure 4 provides the result of inverse problems solution. The better performance of the descent that uses the second order scheme for the direct problem solution is noticeable. The smallest inclusion was recovered better, as was the complex structure on the leftmost inclusion.
Figure 2. Comparing the error behaviour (the same descent parameters)

Figure 3. Comparative analysis of the results: A - the errors of the methods, B - the residual functional

Conclusion
In this paper we considered the influence of the method of solving the direct problem on the behaviour of the gradient descent for the residual functional, that corresponds to the coefficient
Figure 4. Comparative analysis of the results: left - density restoration, right - the speed of sound restoration; a, b - using the Godunov scheme, c, d - using the MUSCL scheme.

Inverse problem for the system of acoustic equations. The better performance of the MUSCL scheme is related to the fact, that the second order method provides more information to the framework of the gradient descent of the functional, when dealing with inclusions of relatively small size and complex structure. In the future work we will study its influence on the quality of the inverse problems solution when using Newton-type methods for solving the inverse problem.

Acknowledgments
The work was supported by RSCF, project 19-11-00154 “Developing of new mathematical models of acoustic tomography in medicine. Numerical methods, HPC and software”.

References
[1] Godunov S K 1959 Differential method for numerical computation of noncontinuous solutions of hydrodynamics equations Matematicheskiy Sbornik 47 271–306 (In Russian)
[2] Godunov S K, Zabrodin A V, Ivanov M Y, Kraikov A N and Prokopov G P 1976 Numerical Solution for Multidimensional Problems of Gas Mechanics (Moscow: Nauka)
[3] Goss S A, Johnston R L and Dunn F 1978 Comprehensive compilation of empirical ultrasonic properties of mammalian tissues J Acoust Soc Am 64(2) 423–457
[4] van Leer B 1979 Towards the ultimate conservative difference scheme. V. A second-order sequel to Godunov’s method J. Comp. Phys. 32 101–136
[5] van Leer B 1984 On the relation between the upwind-differencing schemes of Godunov Engquist-Osher and Roe. SIAM J Sci Statist Comput. 5(1) 1–20
[6] Colella P 1990 Multidimensional upwind methods for hyperbolic conservation laws J. Comput. Phys. 87 171–200
[7] Nessyahu H and Tadmor E 1990 Nonoscillatory central differencing for hyperbolic conservation laws J Comput Phys 87(2) 408–463
[8] Durlofsky L J, Engquist B and Osher S 1992 Triangle based adaptive stencils for the solution of hyperbolic conservation laws J Comput Phys 98 64–73
[9] Sanders R and Weiser A 1992 High resolution staggered mesh approach for nonlinear hyperbolic systems of conservation laws J Comput Phys 101 314–329
[10] Khobalatte B and Perthame B 1994 Maximum principle on the entropy and second-order kinetic schemes Math of Comput 62(205) 119–131
[11] Perthame B and Qiu Y 1994 A variant of Van Leer’s method for multidimensional systems of conservation laws J Comput Phys 112(2) 370–381
[12] Romanov V G and Kabanihin S I 1994 Inverse Problems for Maxwell’s Equations(Utrecht : VSP)
[13] Godlewski E and Raviart P A 1995 Hyperbolic systems of conservations laws. Applied Mathematical Sciences 118 Berlin Heidelberg (New York: Springer)
[14] He S and Kabanikhin S I 1995 An optimization approach to a three-dimensional acoustic inverse problem in the time domain J. Math. Phys. 36 4028–4043
[15] Toro E F 1999 Riemann solvers and numerical methods for fluid dynamics. A practical introduction, 2nd edn. (Berlin Heidelberg New York: Springer)
[16] Bianco F, Puppo G and Russo G 1999 High-order central schemes for hyperbolic systems of conservation laws SIAM J Sci Comput 21(1) 294–322
[17] Douglas T 2000 Mast. Empirical relationships between acoustic parameters in human soft tissues Acoustics Research Letters Online 1 37
[18] Kabanihin S I, Scherzer O and Shishlenin M A 2011 Iteration methods for solving a two dimensional inverse problem for a hyperbolic equation J. Inverse Ill-Posed Probl. (11) 87–109
[19] Berthon C 2005 Stability of the MUSCL schemes for the Euler equations Commun Math Sci
[20] Berthon C 2006 Why the MUSCL–Hancock Scheme is Math of Comput 62(205) 119–131
[21] Kabanihin S I and Shishlenin M A 2008 Quasi-solution in inverse coefficient problems J. Inverse Ill-Posed Probl. 16(1) 705–713
[22] Toro E.F. (2009) Riemann Solvers and Numerical Methods for Fluid Dynamics. 3d Edition. Berlin: Springer-Verlag.
[23] van Leer B 2011 A historical overview: Vladimir P. Kolgan and his high-resolution scheme J. Comp. Phys. 230(7) 2378–2383.
[24] Duric N, Littrup P, Li C, Roy O, Schmidt S, Janer R, Cheng X, Goll J, Rama O, Bey-Knight L and Greenway W 2012 Breast ultrasound tomography: Bridging the gap to clinical practice. Proc. SPIE (8320) 83200O
[25] Jirik R, Peterlik I, Ruiter N, Fousek J, Dapp R, Zapf M and Jan J 2012 Sound-speed image reconstruction in sparse-aperture 3D ultrasound transmission tomography IEEE Trans. Ultrason. Ferroelectr. Freq. Control (59) 254–264
[26] Reis S F 2013 Characterisation of biological tissue: measurement of acoustic parameters for Ultrasound Therapy Dissertacao Mestrado Integrado em Engenharia Biomedica e Biofisica Perl de Sinais e Imagens Medicas
[27] Kabanihin S I, Nurseitov D B, Shishlenin M A and Sholpanbaev B B 2013 Inverse problems for the ground penetrating radar Journal of Inverse and Ill-Posed Problems 21(6) 885–892
[28] Burov V A, Zokov D I and Rumyantseva O D 2015 Reconstruction of the sound velocity and absorption spatial distributions in soft biological tissue phantoms from experimental ultrasound tomography data Acoust. Phys. (61) 231–248
[29] Wang Y, Lukyanenko D V and Yagola A G 2016 Regularized Inversion of Full Tensor Magnetic Gradient Data Numerical Methods and Programming (Vychislitel’nye Metody i Programmirovanie) 17 13–20
[30] Rodionov A V 2016 Correlation between the discontinuous Galerkin method and MUSCL type schemes Math Models Comput Simul 8(3) 285–300
[31] Wiskin J, Malik B, Natesan R and Lenox M 2019 Quantitative assessment of breast density using transmission ultrasound tomography Med. Phys. (46) 2610–2620
[32] Kabanihin S I, Klychinskiy D V, Kulikov I M, Novikov N S and Shishlenin M A 2020 Direct and Inverse Problems for Conservation Laws. In Continuum Mechanics, Applied Mathematics and Scientific Computing: Godunov’s Legacy Demidenko, G., Romenski, E., Toro, E., Dumbser, M., Eds. (Cham, Switzerland: Springer) 217–222
Imaging (13) 1367–1393

[34] Klibanov M V 2019 On the travel time tomography problem in 3D J. Inverse Ill-Posed Probl. (27) 591–607

[35] Kabaniikhin S I, Kulikov I M and Shishlenin M A 2020 An Algorithm for Recovering the Characteristics of the Initial State of Supernova Comp. Math. and Math. Phys. (60) 1008–1016

[36] Kabaniikhin S I, Klyuchinskiy D V, Novikov N S and Shishlenin M A 2020 Numerics of acoustical 2D tomography based on the conservation laws J. Inverse Ill-Posed Probl. (28) 287–297

[37] Klyuchinskiy D, Novikov N and Shishlenin M 2020 A Modification of Gradient Descent Method for Solving Coefficient Inverse Problem for Acoustics Equations Computation 8 73

[38] Klyuchinskiy D, Novikov N and Shishlenin M 2021 Recovering Density and Speed of Sound Coefficients in the 2D Hyperbolic System of Acoustic Equations of the First Order by a Finite Number of Observations Mathematics 9 199

[39] Kabaniikhin S I, Klyuchinskiy D V, Novikov N S and Shishlenin M A 2021 On the problem of modeling the acoustic radiation pattern of source for the 2D first-order system of hyperbolic equations Journal of Physics: Conference Series 1715 (1) 012038

[40] Klyuchinskiy D V, Novikov N S and Shishlenin M A 2021 CPU-time and RAM memory optimization for solving dynamic inverse problems using gradient-based approach Journal of Computational Physics 439 110374